ON THIRD-ORDER JACOBSTHAL POLYNOMIALS AND THEIR PROPERTIES

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Abstract. Third-order Jacobsthal polynomial sequence is defined in this study. Some properties involving this polynomial, including the Binet-style formula and the generating function are also presented. Furthermore, we present the modified third-order Jacobsthal polynomials, and derive adaptations for some well-known identities of third-order Jacobsthal and modified third-order Jacobsthal numbers.

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1. INTRODUCTION

The Jacobsthal numbers have many interesting properties and applications in many fields of science (see, [1]). The Jacobsthal numbers \( J_n \) \( n \geq 0 \) are defined by the recurrence relation

\[
J_0 = 0, \quad J_1 = 1, \quad J_{n+2} = J_{n+1} + 2J_n, \quad n \geq 0.
\]

(1.1)

Another important sequence is the Jacobsthal–Lucas sequence. This sequence is defined by the recurrence relation \( j_{n+2} = j_{n+1} + 2j_n \), where \( j_0 = 2 \) and \( j_1 = 1 \).

In Cook and Bacon’s work [5] the Jacobsthal recurrence relation is extended to higher order recurrence relations and the basic list of identities provided by A. F. Horadam [9] is expanded and extended to several identities for some of the higher order cases. In fact, the third-order Jacobsthal numbers, \( \{J_n^{(3)}\}_{n \geq 0} \), and third-order Jacobsthal–Lucas numbers, \( \{j_n^{(3)}\}_{n \geq 0} \), are defined by

\[
J^{(3)}_{n+3} = J^{(3)}_{n+2} + J^{(3)}_{n+1} + 2J^{(3)}_n, \quad J^{(3)}_0 = 0, \quad J^{(3)}_1 = J^{(3)}_2 = 1, \quad n \geq 0,
\]

(1.2)

and

\[
j^{(3)}_{n+3} = j^{(3)}_{n+2} + j^{(3)}_{n+1} + 2j^{(3)}_n, \quad j^{(3)}_0 = 2, \quad j^{(3)}_1 = 1, \quad j^{(3)}_2 = 5, \quad n \geq 0,
\]

(1.3)

respectively.

Some of the following properties given for third-order Jacobsthal numbers and third-order Jacobsthal–Lucas numbers are used in this paper (for more details, see...
Note that Eqs. (1.7) and (1.11) have been corrected in [3], since they have been wrongly described in [5]. Then, we have

\[ 3J_n^{(3)} + J_n^{(3)} = 2^{n+1}, \]
\[ J_n^{(3)} - 3J_n^{(3)} = 2J_{n-3}, \quad n \geq 3, \] (1.4)
\[ J_{n+2}^{(3)} - 4J_n^{(3)} = \begin{cases} -2 & \text{if } n \equiv 1 \pmod{3} \\ 1 & \text{if } n \neq 1 \pmod{3} \end{cases}, \]
\[ J_{n+1}^{(3)} + J_n^{(3)} = 3J_{n+2}^{(3)}, \] (1.5)
\[ J_n^{(3)} - J_{n+2}^{(3)} = \begin{cases} 1 & \text{if } n \equiv 0 \pmod{3} \\ -1 & \text{if } n \equiv 1 \pmod{3} \\ 0 & \text{if } n \equiv 2 \pmod{3} \end{cases}, \] (1.6)
\[ (J_{n-3}^{(3)})^2 + 3J_n^{(3)}J_n^{(3)} = 4^n, \] (1.7)
\[ \sum_{k=0}^n J_k^{(3)} = \begin{cases} J_{n+1}^{(3)} & \text{if } n \neq 0 \pmod{3} \\ J_{n+1}^{(3)} - 1 & \text{if } n \equiv 0 \pmod{3} \end{cases}, \] (1.8)
\[ (J_n^{(3)})^2 - 9(J_n^{(3)})^2 = 2^{n+2}J_{n-3}^{(3)}, \quad n \geq 3. \] (1.9)

Using standard techniques for solving recurrence relations, the auxiliary equation, and its roots are given by

\[ x^3 - x^2 - x - 2 = 0; \quad x = 2, \quad \text{and } x = -\frac{1 \pm i\sqrt{3}}{2}. \]

Note that the latter two are the complex conjugate cube roots of unity. Call them \( \omega_1 \) and \( \omega_2 \), respectively. Thus the Binet formulas can be written as

\[ J_n^{(3)} = \frac{2}{7}2^n - \frac{3 + 2i\sqrt{3}}{21} \omega_1^n - \frac{3 - 2i\sqrt{3}}{21} \omega_2^n \] (1.10)

and

\[ J_n^{(3)} = \frac{8}{7}2^n + \frac{3 + 2i\sqrt{3}}{7} \omega_1^n + \frac{3 - 2i\sqrt{3}}{7} \omega_2^n, \] (1.11)

respectively. Now, we use the notation

\[ Z_n = \frac{A\omega_1^n - B\omega_2^n}{\omega_1 - \omega_2} = \begin{cases} 2 & \text{if } n \equiv 0 \pmod{3} \\ -3 & \text{if } n \equiv 1 \pmod{3} \\ 1 & \text{if } n \equiv 2 \pmod{3} \end{cases}, \] (1.12)

where \( A = -3 - 2\omega_2 \) and \( B = -3 - 2\omega_1 \). Furthermore, note that for all \( n \geq 0 \) we have

\[ Z_{n+2} = -Z_{n+1} - Z_n, \quad Z_0 = 2, \quad Z_1 = -3. \] (1.13)
From the Binet formulas (1.12), (1.13) and Eq. (1.14), we have

\[ J_n^{(3)} = \frac{1}{7} (2^{n+1} - Z_n) \] and \[ j_n^{(3)} = \frac{1}{7} (2^{n+3} + 3Z_n) . \] (1.16)

A systematic investigation of the incomplete generalized Jacobsthal numbers and the incomplete generalized Jacobsthal–Lucas numbers was featured in [6]. In [7], Djordjević and Srivastava introduced the generalized incomplete Fibonacci polynomials and the generalized incomplete Lucas polynomials. In [8], the authors investigated some properties and relations involving generalizations of the Fibonacci numbers. In [10], Raina and Srivastava investigated the a new class of numbers associated with the Lucas numbers. Moreover they gave several interesting properties of these numbers.

In this paper, we introduce the third-order Jacobsthal polynomials and we give some properties, including the Binet-style formula and the generating functions for these sequences. Some identities involving these polynomials are also provided.

2. THE THIRD-ORDER JACOBSTHAL POLYNOMIAL, BINET’S FORMULA AND THE GENERATING FUNCTION

The principal goals of this section will be to define the third-order Jacobsthal polynomial and to present some elementary results involving it.

For any variable quantity \( x \) such that \( x^3 \neq 1 \). We define the third-order Jacobsthal polynomial, denoted by \( \{ J_n^{(3)}(x) \} \), This sequence is defined recursively by

\[ J_{n+3}(x) = (x-1)J_{n+2}(x) + (x-1)J_{n+1}(x) + xJ_n^{(3)}(x), \quad n \geq 0, \quad (2.1) \]

with initial conditions \( J_0^{(3)}(x) = 0, \ J_1^{(3)}(x) = 1 \) and \( J_2^{(3)}(x) = x-1 \).

In order to find the generating function for the third-order Jacobsthal polynomial, we shall write the sequence as a power series where each term of the sequence correspond to coefficients of the series. As a consequence of the definition of generating function of a sequence, the generating function associated to \( \{ J_n^{(3)}(x) \} \), denoted by \( \{ j(t) \} \), is defined by

\[ j(t) = \sum_{n \geq 0} J_n^{(3)}(x)t^n. \]

Consequently, we obtain the following result:

**Theorem 1.** The generating function for the third-order Jacobsthal polynomials \( \{ J_n^{(3)}(x) \} \) is

\[ j(t) = \frac{t}{1-(x-1)t-(x-1)t^2-xt^3}. \]

**Proof.** Using the definition of generating function, we have

\[ j(t) = J_0^{(3)}(x) + J_1^{(3)}(x)t + J_2^{(3)}(x)t^2 + \cdots + J_n^{(3)}(x)t^n + \cdots . \]

Multiplying both sides of this identity by \(-(x-1)t, -(x-1)t^2\) and by \(-xt^3\), and then from Eq. (2.1), we have
\[
(1 - (x - 1)t - (x - 1)t^2 - xr^3)j(t)
\]
\[
= J_0^{(3)}(x) + (J_1^{(3)}(x) - (x - 1)J_0^{(3)}(x))t + (J_2^{(3)}(x) - (x - 1)J_1^{(3)}(x) - (x - 1)J_0^{(3)}(x))t^2
\]
\]
\[
(2.2)
\]

and the result follows.

The following result gives the Binet-style formula for \(J_n^{(3)}(x)\).

**Theorem 2.** For \(n \geq 0\), we have

\[
J_n^{(3)}(x) = \frac{x^{n+1}}{x^2 + x + 1} - \frac{\omega_1^{n+1}}{(x - \omega_1)(\omega_1 - \omega_2)} + \frac{\omega_2^{n+1}}{(x - \omega_2)(\omega_1 - \omega_2)},
\]

where \(\omega_1, \omega_2\) are the roots of the characteristic equation associated with the respective recurrence relations \(T^3 + \lambda + 1 = 0\).

**Proof.** Since the characteristic equation has three distinct roots, the sequence \(J_n^{(3)}(x) = a(x)x^n + b(x)x_1^n + c(x)x_2^n\) is the solution of the Eq. (2.1). Considering \(n = 0, 1, 2\) in this identity and solving this system of linear equations, we obtain a unique value for \(a(x), b(x)\) and \(c(x)\), which are, in this case, \((x^2 + x + 1)a(x) = x, (x - \omega_1)(\omega_1 - \omega_2)b(x) = -\omega_1\) and \((x - \omega_2)(\omega_1 - \omega_2)c(x) = \omega_2\). So, using these values in the expression of \(J_n^{(3)}(x)\) stated before, we get the required result.

We define the modified third-order Jacobsthal polynomial sequence, denoted by \(\{K_n^{(3)}(x)\}_{n \geq 0}\). This sequence is defined recursively by

\[
K_{n+3}^{(3)}(x) = (x - 1)K_{n+2}^{(3)}(x) + (x - 1)K_{n+1}^{(3)}(x) + xK_n^{(3)}(x),
\]

with initial conditions \(K_0^{(3)}(x) = 3, K_1^{(3)}(x) = x - 1\) and \(K_2^{(3)}(x) = x^2 - 1\).

We give their versions for the third-order Jacobsthal and modified third-order Jacobsthal polynomials.

For simplicity of notation, let

\[
Z_n(x) = \frac{1}{\omega_1 - \omega_2} \left( (x - \omega_2)\omega_1^{n+1} - (x - \omega_1)\omega_2^{n+1} \right),
\]

\[
Y_n = \omega_1^n + \omega_2^n.
\]

Then, we can write

\[
J_n^{(3)}(x) = \frac{1}{x^2 + x + 1} \left( x^{n+1} - Z_n(x) \right)
\]

and

\[
K_n^{(3)}(x) = x^n + Y_n.
\]

Then, \(Z_n(x) = -Z_{n-1}(x) - Z_{n-2}(x), Z_0(x) = x\) and \(Z_1(x) = -(x + 1)\).

Furthermore, we easily obtain the identities stated in the following result:
Proposition 1. For a natural number \( n \) and \( m \), if \( J_n^{(3)}(x) \) and \( K_n^{(3)}(x) \) are, respectively, the \( n \)-th order Jacobi and modified third-order Jacobi polynomials, then the following identities are true:

\[
K_n^{(3)}(x) = (x - 1)J_n^{(3)}(x) + 2(x - 1)J_{n-1}^{(3)}(x) + 3xJ_{n-2}^{(3)}(x), \quad n \geq 2, \quad (2.5)
\]

\[
J_n^{(3)}(x)J_m^{(3)}(x) + J_{n+1}^{(3)}(x)J_{m+1}^{(3)}(x) + J_{n+2}^{(3)}(x)J_{m+2}^{(3)}(x) = \frac{1}{(x^2 + x + 1)^2} \left\{ \begin{array}{l}
- x^{n+1} \left( (1 - x^2)Z_n(x) + x(1 - x)Z_{n+1}(x) \right) \\
- x^{m+1} \left( (1 - x^2)Z_m(x) + x(1 - x)Z_{m+1}(x) \right) \\
+ x^2 + x + 1 (\omega_1^2 \omega_2^2 + \omega_1^2 \omega_2^3)
\end{array} \right\}, \quad (2.6)
\]

\[
\left( J_n^{(3)}(x) \right)^2 + \left( J_{n+1}^{(3)}(x) \right)^2 + \left( J_{n+2}^{(3)}(x) \right)^2 = \frac{1}{(x^2 + x + 1)^2} \left\{ \begin{array}{l}
- 2x^{n+2} \left( (1 - x^2)Z_n(x) + x(1 - x)Z_{n+1}(x) \right) \\
+ 2(x^2 + x + 1)
\end{array} \right\}, \quad (2.7)
\]

and \( Z_n(x) \) as in Eq. (2.4).

Proof. (2.5): To prove Eq. (2.5), we use induction on \( n \). Let \( n = 2 \), we get

\[
(x - 1)J_2^{(3)}(x) + 2(x - 1)J_1^{(3)}(x) + 3xJ_0^{(3)}(x) = (x - 1)(x - 1) + 2(x - 1)
\]

\[
= x^2 - 1 = K_2^{(3)}(x).
\]

Let us assume that \( K_n^{(3)}(x) = (x - 1)J_n^{(3)}(x) + 2(x - 1)J_{n-1}^{(3)}(x) + 3xJ_{n-2}^{(3)}(x) \) is true for all values of \( m \) less than or equal to \( n \) for \( n \geq 2 \). Then,

\[
K_{n+1}^{(3)}(x) = (x - 1)K_n^{(3)}(x) + (x - 1)K_{n-1}^{(3)}(x) + xK_{n-2}^{(3)}(x)
\]

\[
= (x - 1) \left\{ (x - 1)J_n^{(3)}(x) + 2(x - 1)J_{n-1}^{(3)}(x) + 3xJ_{n-2}^{(3)}(x) \right\}
\]

\[
+ (x - 1) \left\{ (x - 1)J_{n-1}^{(3)}(x) + 2(x - 1)J_{n-2}^{(3)}(x) + 3xJ_{n-3}^{(3)}(x) \right\}
\]

\[
+ x \left\{ (x - 1)J_{n-2}^{(3)}(x) + 2(x - 1)J_{n-3}^{(3)}(x) + 3xJ_{n-4}^{(3)}(x) \right\}
\]

\[
= (x - 1)J_{n+1}^{(3)}(x) + 2(x - 1)J_n^{(3)}(x) + 3xJ_{n-1}^{(3)}(x).
\]

(2.6): Using the Binet formula of \( J_n^{(3)}(x) \) in Theorem 2, we have

\[
J_n^{(3)}(x)J_m^{(3)}(x) + J_{n+1}^{(3)}(x)J_{m+1}^{(3)}(x) + J_{n+2}^{(3)}(x)J_{m+2}^{(3)}(x)
\]

\[
= \frac{1}{(x^2 + x + 1)^2} \left\{ \begin{array}{l}
(x^{n+1} - Z_n(x)) (x^{m+1} - Z_m(x)) \\
+ (x^{n+2} - Z_{n+1}(x)) (x^{m+2} - Z_{m+1}(x)) \\
+ (x^{n+3} - Z_{n+2}(x)) (x^{m+3} - Z_{m+2}(x))
\end{array} \right\}.
\]
Then, we obtain
\[
J_{n}^{(3)}(x)J_{m}^{(3)}(x) + J_{n+1}^{(3)}(x)J_{m+1}^{(3)}(x) + J_{n+2}^{(3)}(x)J_{m+2}^{(3)}(x)
= \frac{1}{(x^2 + x + 1)^2} \left\{ \begin{array}{l}
(1 + x^2 + x^4) \cdot x^{n+m+2} \\
- x^{n+1} \left( Z_m(x) + xZ_{m+1}(x) + x^2Z_{m+2}(x) \right) \\
- x^{m+1} \left( Z_n(x) + xZ_{n+1}(x) + x^2Z_{n+2}(x) \right) \\
+ Z_n(x)Z_m(x) + Z_{n+1}(x)Z_{m+1}(x) + Z_{n+2}(x)Z_{m+2}(x) \\
\end{array} \right\}
= \frac{1}{(x^2 + x + 1)^2} \left\{ \begin{array}{l}
(1 + x^2 + x^4) \cdot x^{n+m+2} \\
- x^{n+1} \left( (1 - x^2)Z_m(x) + x(1 - x)Z_{m+1}(x) \right) \\
- x^{m+1} \left( (1 - x^2)Z_n(x) + x(1 - x)Z_{n+1}(x) \right) \\
+ (x^2 + x + 1)(\omega_0^m\omega_1^n + \omega_0^n\omega_1^m) \\
\end{array} \right\}
\]

Then, we obtain the Eq. (2.7) if \( m = n \) in Eq. (2.6).

3. Some identities involving this type of polynomials

In this section, we state some identities related with these type of third-order polynomials. As a consequence of the Binet formula of Theorem 2, we get for this sequence the following interesting identities.

**Proposition 2** (Catalan-like identity). For a natural numbers \( n, s \), with \( n \geq s \), if \( J_n^{(3)}(x) \) is the \( n \)-th third-order Jacobsthal polynomials, then the following identity
\[
J_{n+s}^{(3)}(x)J_{n-s}^{(3)}(x) - \left( J_n^{(3)}(x) \right)^2
= \frac{1}{(x^2 + x + 1)^2} \left\{ \begin{array}{l}
x^{n+1} \left( x^s - x^{-s} \right) X_sZ_{m+1}(x) \\
-x^{n+1} \left( 2 + x^sX_{s+1} - x^{-s}X_{s-1} \right) Z_n(x) \\
- (x^2 + x + 1)^2X_s^2 \\
\end{array} \right\}
\]
is true, where \( Z_n(x) \) as in Eq. (2.4), \( X_n = \frac{\omega_0^n - \omega_1^n}{\omega_0 - \omega_1} \) and \( \omega_1, \omega_2 \) are the roots of the characteristic equation associated with the recurrence relation \( x^2 + x + 1 = 0 \).

**Proof.** Using the Eq. (2.4) and the Binet formula of \( J_n^{(3)}(x) \) in Theorem 2, we have
\[
J_{n+s}^{(3)}(x)J_{n-s}^{(3)}(x) - \left( J_n^{(3)}(x) \right)^2
= \frac{1}{(x^2 + x + 1)^2} \left\{ \begin{array}{l}
\left( x^{n+s+1} - Z_{n+s}(x) \right) \left( x^{n-s+1} - Z_{n-s}(x) \right) \\
- (x^{n+1} - Z_n(x))^2 \\
\end{array} \right\}
= \frac{1}{(x^2 + x + 1)^2} \left\{ \begin{array}{l}
- x^{n+1} \left( x^sZ_{n-s}(x) + x^{-s}Z_{n+s}(x) - 2Z_n(x) \right) \\
+ Z_{n+s}(x)Z_{n-s}(x) - (Z_n(x))^2 \\
\end{array} \right\}
\]
Using the following identity for the sequence \( Z_n(x) \):
\[
Z_{n+s}(x) = X_sZ_{n+1}(x) - X_{s-1}Z_n(x),
\]
where \( X_s = \frac{\alpha_s - \omega^2_s}{\omega_3 - \omega_2} \) and \( X_{-s} = -X_s \). Then, we obtain
\[
J_{n+3}(x)J_{n-x}(x) - \left( J_n^{(3)}(x) \right)^2 = \frac{1}{(x^2 + x + 1)^2} \left\{ x^{x+1} (x^3 - x^{-s}) X_s Z_{n+1}(x) - x^{x+1} (x^1 X_{x+1} - x^{-1} X_{x-1} - 2) Z_n(x) \right\}.
\]
Hence the result holds.

Note that for \( s = 1 \) in the Catalan-like identity obtained, we get the Cassini-like identity for the third-order Jacobsthal polynomial. Furthermore, for \( s = 1 \), the identity stated in Proposition 2, yields
\[
J_{n+1}^{(3)}(x)J_{n-1}^{(3)}(x) - \left( J_n^{(3)}(x) \right)^2 = \frac{1}{(x^2 + x + 1)^2} \left\{ x^{x+1} (x^3 - x^{-1}) X_1 Z_{n+1}(x) - x^{x+1} (x^1 X_{1+1} - x^{-1} X_{1-1} - 2) Z_n(x) \right\}.
\]
and using \( X_0 = 0 \) and \( X_1 = 1 \) in Proposition 2, we obtain the following result.

**Proposition 3** (Cassini-like identity). For a natural numbers \( n \), if \( K_n^{(3)} \) is the \( n \)-th third-order Jacobsthal numbers, then the identity
\[
J_{n+1}^{(3)}(x)J_{n-1}^{(3)}(x) - \left( J_n^{(3)}(x) \right)^2 = \frac{1}{(x^2 + x + 1)^2} \left\{ x^n \left( (x^2 - 1) Z_{n+1}(x) + x(x+2) Z_n(x) \right) \right\}
\]
is true.

The d’Ocagne-like identity can also be obtained using the Binet formula and in this case we obtain

**Proposition 4** (d’Ocagne-like identity). For a natural numbers \( m, n \), with \( m \geq n \) and \( J_n^{(3)}(x) \) is the \( n \)-th third-order Jacobsthal polynomial, then the following identity
\[
J_{m+1}^{(3)}(x)J_{n-x}(x) - J_m^{(3)}(x)J_{n+1}^{(3)}(x) = \frac{1}{(x^2 + x + 1)^2} \left\{ x^{m+1} (Z_{m+1}(x) - xZ_m(x)) - x^{n+1} (Z_{n+1}(x) - xZ_n(x)) + (x^2 + x + 1) X_{m-n} \right\}
\]
is true.

**Proof.** Using the Eq. (2.4) and the Theorem 2, we get the required result.

In addition, some formulae involving sums of terms of the third-order Jacobsthal polynomial sequence will be provided in the following proposition.
Proposition 5. For a natural numbers $m$, $n$, with $n \geq m$, if $J^{(3)}_n(x)$ and $K^{(3)}_n(x)$ are, respectively, the $n$-th third-order Jacobsthal and modified third-order Jacobsthal polynomials, then the following identities are true:

\[
\sum_{s=m}^{n} J^{(3)}_s(x) = \frac{1}{3(x-1)} \left\{ (3x-2)J^{(3)}_n(x) + (2x-1)J^{(3)}_{n-1}(x) + xJ^{(3)}_{n-2}(x) - J^{(3)}_{m+2}(x) + (x-2)J^{(3)}_{m+1}(x) + (2x-3)J^{(3)}_m(x) \right\}, \quad (3.1)
\]

\[
\sum_{s=0}^{n} K^{(3)}_s(x) = \frac{1}{x-1} \left\{ \begin{array}{ll}
\sum_{s=0}^{n} x^{s+1} + \omega_1 x^{s+1} - 1 & \text{if } n \equiv 0 \pmod{3} \\
\omega_1 x^{s+1} + \omega_2 x^{s+1} - 2 & \text{if } n \equiv 1 \pmod{3} \\
\omega_1 x^{s+1} - \omega_2 x^{s+1} - 1 & \text{if } n \equiv 2 \pmod{3} 
\end{array} \right., \quad (3.2)
\]

Proof. (3.1): Using Eq. (2.1), we obtain

\[
\sum_{s=m}^{n} J^{(3)}_s(x) = J^{(3)}_m(x) + J^{(3)}_{m+1}(x) + J^{(3)}_{m+2}(x) + \sum_{s=m+3}^{n} J^{(3)}_s(x)
\]

\[
= J^{(3)}_m(x) + J^{(3)}_{m+1}(x) + J^{(3)}_{m+2}(x) + (x-1) \sum_{s=m+2}^{n} J^{(3)}_s(x)
\]

\[
+ (x-1) \sum_{s=m+1}^{n-2} J^{(3)}_s(x) + x \sum_{s=m}^{n-3} J^{(3)}_s(x)
\]

Then,

\[
\sum_{s=m}^{n} J^{(3)}_s(x) = (3x-2) \sum_{s=m}^{n} J^{(3)}_s(x) - (x-2)J^{(3)}_{m+1}(x) - (2x-3)J^{(3)}_m(x)
\]

\[- (3x-2)J^{(3)}_n(x) - (2x-1)J^{(3)}_{n-1}(x) - xJ^{(3)}_{n-2}(x).
\]

Finally, the result in Eq. (3.1) is completed.

(3.2): As a consequence of the Eq. (2.4) of Theorem 2 and

\[
\sum_{s=0}^{n} Y_s = \sum_{s=0}^{n} \left( \omega_1^s + \omega_2^s \right)
\]

\[
= \frac{\omega_1^{n+1} - 1}{\omega_1 - 1} + \frac{\omega_2^{n+1} - 1}{\omega_2 - 1}
\]

\[
= \frac{1}{3} (Y_n - Y_{n+1}) + 1
\]

we have

\[
\sum_{s=0}^{n} K^{(3)}_s(x) = \sum_{s=0}^{n} x^{s+1} + \sum_{s=0}^{n} Y_s
\]

\[
= \frac{x^{n+1} - 1}{x - 1} + \frac{1}{3} (Y_n - Y_{n+1}) + 1
\]
\[ = \frac{1}{x-1} \begin{cases} 
    x^n + 2x - 3 & \text{if } n \equiv 0 \pmod{3} \\
    x^n + x - 2 & \text{if } n \equiv 1 \pmod{3} \\
    x^n + 1 & \text{if } n \equiv 2 \pmod{3}
\end{cases}. \]

Hence, we obtain the result. \(\square\)

For example, if \( n \equiv 0 \pmod{3} \) we have that \( x^n + 2x - 3 \) is divisible by \( x - 1 \).

For negative subscripts terms of the sequence of modified third-order Jacobsthal polynomial we can establish the following result:

**Proposition 6.** For a natural number \( n \) and \( x^3 \neq 0 \) the following identities are true:

\[ K^{(3)}_{-n}(x) = K^{(3)}_n(x) + x^{-n} - x^n, \quad (3.3) \]

\[ \sum_{s=0}^{3n} K^{(3)}_{-s}(x) = \frac{1}{x-1} (3x - 2 - x^{-3n}). \quad (3.4) \]

**Proof.** (3.3): Since \( Y_{-n} = Y_n \), using the Binet formula stated in Theorem 2 and the fact that \( \omega_1 \omega_2 = 1 \), all the results of this Proposition follow. In fact,

\[ K^{(3)}_{-n}(x) = x^{-n} + Y_{-n} = x^{-n} + x^n + Y_n - x^n = K^{(3)}_n(x) + x^{-n} - x^n. \]

So, the proof is completed.

(3.4): The proof is similar to the proof of Eq. (3.1) using Eq. (3.3). \(\square\)

4. Conclusion

Sequences of polynomials have been studied over several years, including the well-known Tribonacci polynomial and, consequently, on the Tribonacci-Lucas polynomial. In this paper, we have also contributed for the study of third-order Jacobsthal and modified third-order Jacobsthal polynomials, deducing some formulae for the sums of such polynomials, presenting the generating functions and their Binet-style formula. It is our intention to continue the study of this type of sequences, exploring some their applications in the science domain. For example, a new type of sequences in the quaternion algebra with the use of these polynomials and their combinatorial properties.

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