Tail estimates for random variables from interrelation between corresponding moments inequalities.

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Abstract.

We derive the tail inequalities between two random variables starting from inequalities between its moment, or more generally between its Lebesgue - Riesz norms, which holds true on certain sets of parameters.
We consider some applications into the martingale theory.

Key words and phrases. Probability space, tail of distribution, random variable (r.v.), moment, Young - Fenchel, or Legendre, transform; generating function, natural function, Lebesgue - Riesz and Grand Lebesgue Spaces and norms, martingale, Burkholder and Burkholder - Davis - Gundy inequalities, estimations, Lyapunov’s inequality, examples.

1 Introduction. Notations. Statement of problem.

Let \( \Omega = \{\omega\}, \mathcal{B}, \mathbb{P} \) be certain non-trivial probability space with an expectation \( \mathbb{E} \) and the classical Lebesgue - Riesz norm defined on the appropriate random variable (r.v.) (measurable function) \( \xi : \Omega \to \mathbb{R} \)

\[
|\xi|_p \stackrel{def}{=} \left[ \mathbb{E}[|\xi|^p] \right]^{1/p}, \quad 1 \leq p \leq \infty,
\]
where as ordinary

$$|\xi|_\infty := \text{vraisup}_{\omega \in \Omega} |\xi(\omega)|.$$  

Correspondingly,  $L_p = L_p(\Omega, P) = \{ \xi : |\xi|_p < \infty \}$.  

We will start from the inequality of the form

$$|\xi|_p \leq g(p, r, |\eta|_r), \ (p, r) \in D.$$  \hspace{1cm} (1)

and intent to deduce the exact tail estimate for the first r.v. $\xi$ via ones of the second one $\eta$.

Here $D$ is certain non-empty domain inside the quarter plane

$$D \subset [1, \infty) \otimes [1, \infty),$$

and $g(p, r, z), \ p, r \geq 1, \ z \geq 0$ is fixed numerical valued deterministic non-negative measurable function.

A famous example: let $(\epsilon_k, F_k), k = 1, 2, \ldots, n$ be a centered martingale:

$$\mathbb{E}\epsilon_m/F_k = \epsilon_k, \ k \leq m.$$  

Set $M^* = \max_k \epsilon_k, \ k \leq n$; then for the values $p > 1$

$$|M^*|_p \leq \frac{p}{p-1} \cdot |\epsilon_n|_p,$$  \hspace{1cm} (2)

Doob’s inequality.

This case was investigated in a previous article in this direction [20]. Many other inequalities of the form (1) arises in the martingale theory: Burkholder - Davis - Gundy (BDG) inequality at so one, see e.g. [2], [3], [4], [13], [18], [23], [24] and so one.

Another notations.

$$R(p) = \{ r : (p, r) \in D \}, \ U = \{ p, \ \exists r \geq 1, \Rightarrow (p, r) \in D \}.$$  

Further, let $p_0$ arbitrary number from the set $U$. Define the following important function generated by the random variable $\eta$ and by the values $p \in U$:

$$\psi(p) = \psi_{p_0}[\eta](p) \overset{def}{=} \inf_{r \in R(p)} g(p, r, |\eta|_r), \ p > p_0;$$  \hspace{1cm} (3)

$$\psi(p) = \psi_{p_0}[\eta](p) \overset{def}{=} \inf_{r \in R(p)} g(p_0, r, |\eta|_r), \ p \leq p_0.$$  \hspace{1cm} (4)

We have from (1) and from the Lyapunov’s inequality again for the variables $p \in U$.
Proposition 1.1.

\[ |\xi|_p \leq \psi(p) = \psi_{\psi_0[p]}(p), \quad p \in U. \]  

(5)

2 Grand Lebesgue Spaces estimates.

The relation (5) may be rewritten (and used) by means of the theory of the so-called Grand Lebesgue Spaces (GLS). These spaces are investigated in many works, see e.g. [1], [5], [6], [7], [8], [9], [10], [11], [12], [14], [15], [16], [17], [19], [21], [22].

We recall briefly the definition and some important properties of these GLS spaces, adapted for this report. Let \( \kappa = \kappa(p), \quad p \in U \) be numerical valued positive measurable function, which may be named as generating function for introduced space. Denote as ordinary by \( G\kappa \) the set of all the numerical valued random variables \( \zeta \) having a finite norm

\[ ||\zeta||_{G\kappa} \overset{\text{def}}{=} \sup_{p \in U} \left\{ \frac{|\zeta|_p}{\kappa(p)} \right\}. \]

(6)

For instance, the generating function \( \kappa(\cdot) \) for this space may be introduced by the natural way:

\[ \kappa_0(p) = \kappa_0[\zeta](p) := |\zeta|_p, \]

if of course it is finite for certain interval \( p \in (a,b), \quad 1 \leq a < b \leq \infty \).

These spaces are Banach functional complete and rearrangement invariant. They are closely related with the tail behavior of the r.v. \( \zeta \):

\[ T[\zeta](t) \overset{\text{def}}{=} \mathbf{P}(|\zeta| > t), \quad t \geq 0. \]

Namely, introduce the auxiliary function

\[ S(x) = S[\psi](x) \overset{\text{def}}{=} \exp \left( -h^*[\psi](\ln x) \right), \quad x \geq e, \]

where \( h(p) = h[\psi](p) = p \ln \psi(p), \) and

\[ h^*(y) \overset{\text{def}}{=} \sup_{p \in \text{Dom}[\psi]} \left( py - h(p) \right) - \]

is the famous Young - Fenchel, or Legendre, transform for the function \( h = h(p) \).

It is known [15], [15], [19], chapters 1,2 that if \( \zeta(\cdot) \in G\psi \) and \( ||\zeta|| = 1 \), then

\[ T[\zeta](x) \leq S[\psi](x); \quad (8) \]
and moreover the conversely proposition holds true. In detail, if the r.v. \( \zeta \) satisfies the estimate (8), then under simple appropriate natural conditions on the function \( \psi \Rightarrow \zeta \in G\psi \):

\[
\exists C = C[\psi] \in (0, \infty) \Rightarrow ||\zeta||G\psi \leq C(\psi).
\]  

(9)

For instance, the estimate for some r.v. \( \zeta \) of the form

\[
\sup_{p \geq 1} \left\{ \frac{|\zeta|_p}{p^{1/m}} \right\} < \infty,
\]

where \( m = \text{const} > 0 \) is completely equivalent to the following tail estimate

\[
\exists c(m) \in (0, \infty) \Rightarrow T[\zeta](t) \leq \exp(-c(m) t^m), \ t \geq 0.
\]

Consider in addition some comparison assertions. Let \( \nu_1 = \nu_1(p), \nu_2 = \nu_2(p), \ p \in U \) be two functions from the set \( G\Psi \). Suppose

\[
\exists C \in (0, \infty) \Rightarrow \forall p \in U \Rightarrow \nu_1(p) \leq C\nu_2(p);
\]

then evidently

\[
||\zeta||G\nu_2 \leq C ||\zeta||G\nu_1.
\]

Further, suppose that for some r.v. \( \zeta \), certain \( \psi \) function \( \nu = \nu(p), \ p \in U \) and some fixed positive finite constant \( C \in (0, \infty) \) there holds \( ||\zeta||G\nu \leq C \), or equally

\[
|\zeta|_p \leq C \nu(p), \ p \in U.
\]

Then

\[
T[\zeta](t) \leq \exp\left\{ -h^*[\nu](\ln(t/C)) \right\}, \ t \geq C e.
\]

(10)

To summarize:

**Proposition 2.1.** It follows under our notations and assumptions on the basis of the estimation (5)

\[
T[\xi](t) \leq \exp\left( -h^*[\psi](\ln t) \right), \ t \geq e,
\]

(11)

with correspondent (exponential) tail estimate.
3 Linear and multilinear cases.

Let us consider in this section a particular case of the inequality of the form

$$|\xi|_p \leq v(p,r) |\eta|_r, \ (p,r) \in D.$$  \hfill (12)

a "linear" case. This possibility appears in particular in the mentioned above works devoted to the theory of martingales.

Assume that the r.v. \( \eta \) belongs to some Grand Lebesgue Space \( G\beta \):

$$|\eta|_r \leq \beta(r) ||\eta||_{G\beta}, \ r \in (a,b).$$

Introduce the new \( \psi \)-function

$$\tau(p) := \inf_r [v(p,r) \beta(r)], \ p \in U.$$  

We have \( |\eta|_r \leq \beta(r) ||\eta||_{G\beta}, \)

$$|\xi|_p \leq v(p,r) \beta(r) ||\eta||_{G\beta},$$

following

$$|\xi(p)|_p \leq \tau(p) \cdot ||\eta||_{G\beta}.$$  

To summarize:

**Proposition 3.1.**

$$||\xi||_{G\tau} \leq ||\eta||_{G\beta},$$  \hfill (13)

with correspondent tail estimate.

**Remark 3.1.** The essential non-improvability in general case of the last estimate (13) holds true for example in the case of Doob's maximal inequality, see e.g. [20]; more simple example: \( \xi = \eta \).

**Remark 3.2.** The cases when \( |\xi|_p \leq v_1(p,r) |\eta|_r^\alpha \) or when

$$|\xi|_p \leq v(p,\bar{r}) \prod_k |\eta_k|_{r(\bar{k})}^{\alpha(k)}$$

may be investigated completely alike, as well as the case of the mixed (anisotropic) Lebesgue - Riesz norms

$$|\eta|_{\bar{r}} = || |||\eta||_{r_1,X_1} ||_{r_2,X_2} \cdots ||_{r_d,X_d},$$

where

$$\eta = \eta(x_1, x_2, \ldots, x_d), \ d = 2, 3, \ldots; \ x_j \in X_j,$$
$(X_j, \mu_j), j = 2, 3, \ldots, d$ are measurable spaces equipped correspondingly with the measures $\mu_j$.

4 Examples.

A. Doob’s inequality, see (2). Suppose for certain martingale $(\epsilon_k, F_k), 0 \leq k \leq n, \epsilon_0 = 0$

$$\exists C_2, m \in (0, \infty) \Rightarrow T[\xi_n](t) \leq \exp(-C_2 t^m), \ t \geq 0. \quad (14)$$

Then

$$\exists C_3 = C_3(m) \in (0, \infty) \Rightarrow T[M^*n](t) \leq \exp(-C_3(m) t^m), \ t \geq 0, \quad (15)$$

and conversely statement is also true.

B. Burkholder - Davis - Gundy inequality. Let $(\xi(t), F_t), t \geq 0$ be continuous time separable right - continuous martingale for which $\xi(0) = \xi(0+) = 0$, (cadlag). Denote as ordinary

$$M^*_\infty \overset{def}{=} \sup_{t \geq 0} \xi(t),$$

and by $\langle M, M \rangle$ its quadratic variation, if both this variables there exists (and are finite). The famous Burkholder - Davis - Gundy upper, (as well as lower,) inequality has a form

$$| M^*_\infty |_p \leq \sqrt{e} \cdot | \langle M, M \rangle^{1/2}_{p/2}, \ p \geq 2.$$

One can conclude as before that if

$$T[\sqrt{\langle M, M \rangle}](t) \leq \exp(-c_4 t^m), \ t > 0, \ \exists c_4 \in (0, \infty),$$

then alike

$$T[M^*_\infty](t) \leq \exp(-c_5 t^m), \ t > 0, \ \exists c_5 = c_5(m) \in (0, \infty),$$

and conversely statement is also true.

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