A single mode realization of SU(1,1) algebra is proposed. This realization makes it possible to reduce the problem of two nonlinear interacting oscillators to the problem of a free moving particle.

I. INTRODUCTION

The Lie algebra of SU(1,1) group is used in many branches of physics (see, e.g., [1] [2] [3] [4] [5] [6]). These algebras consist of the three hyperbolic operators \( K_0, K_+, K_- \) which satisfy the following commutation relations:

\[
[K_0, K_\pm] = \pm K_\pm, \quad [K_+, K_-] = -2K_0. \tag{1}
\]

The Casimir invariant for this algebra is

\[
C = K_0^2 - \frac{1}{2} (K_+ K_- + K_- K_+). \tag{2}
\]

The relations \( (1) \) and \( (2) \) resemble, respectively, the spin operator commutations

\[
[S_\pm, S_\pm] = \pm S_\pm, \quad [S_+, S_-] = 2S_z, \tag{3}
\]

where \( S_\pm = S_x \pm iS_y \), and the following invariant

\[
S(S+1) = S_z^2 + \frac{1}{2}(S_+ S_- + S_- S_+). \tag{4}
\]

This similarity can be continued. The SU(1,1) algebra has a single-bosonic realization

\[
K_- = (2k + a^+ a)^{1/2} a, \quad K_+ = a^+(2k + a^+ a)^{1/2}, \quad K_0 = k + a^+ a, \quad C = k(k-1). \tag{5}
\]

Here \( a^+ \) and \( a \) are the creation and annihilation boson operators, respectively, with the commutation rule \([a, a^+] = 1\). This representation was considered first by Mlodinov and Papanicolaou [7], and it is an analog of the Holstein-Primakoff [8] representation for spin operators:

\[
S_- = (2S - a^+ a)^{1/2} a, \quad S_+ = a^+(2S + a^+ a)^{1/2}, \quad S_z = -S + a^+ a. \tag{6}
\]

There is another important single-mode representation for spin proposed by Villain [8]. In this Letter we shall find an analog of this single mode representation for the hyperbolic operators and consider its application.

II. A SINGLE MODE REPRESENTATION

If \( X \) is a coordinate and \( P = -i d/dX \) is the momentum operator, the Villain representation can be written as

\[
S_- = \sqrt{(S+1/2)^2 - (P-1/2)^2} \exp(-iX), \quad S_+ = \exp(iX) \sqrt{(S+1/2)^2 - (P-1/2)^2}, \quad S_z = P. \tag{7}
\]

Due to the similarity between spin and hyperbolic operators mentioned above, one can construct the following single mode representation:

\[
K_- = (P + P_0) \exp(-iX), \quad K_+ = \exp(iX) (P + P_0^*), \quad K_0 = P + (P_0 + P_0^*)/2 - 1/2 \tag{8}
\]

with

\[
C = -1/4 + (P_0 - P_0^*)^2/4. \tag{9}
\]

Here \( P_0 \) is a complex number. Taking into account that \([X, P] = i\) and

\[
\exp(i\beta X) P^n \exp(-i\beta X) = (P - \beta)^n, \tag{10}
\]

where \( \beta \) is a formal parameter and \( n = 1, 2, \ldots \), it is simple to check out the validity of commutations \([1] \) for \( \beta \).

Another form of representation \([3] \) may be obtained if we change \( X \rightarrow -P \) and \( P \rightarrow X \).

It should be noted that Perelomov [8] mentioned a single mode realization of SU(1,1) in the form:

\[
K_0 = -i d/d\theta, \quad K_\pm = -i \exp(\pm i\theta) d/d\theta \mp (1/2 + i\lambda) \exp(\pm i\theta) \tag{11}
\]

with
\[ C = -1/4 - \lambda^2/4. \quad \lambda > 0. \quad (12) \]

After simple algebra, one can find that (11) can be transformed to the form (8) with \( \theta = X \) and \( P_0 = 1/2 + i\lambda \).

Substituting
\[ Q = \frac{\alpha + \alpha^+}{\sqrt{2}}, \quad P = \frac{\alpha - \alpha^+}{i\sqrt{2}}, \quad (13) \]

we obtain the following representations of the hyperbolic operators in terms of boson operators \( \alpha \) and \( \alpha^+ \):
\[ K_- = \left( \frac{\alpha - \alpha^+}{i\sqrt{2}} + P_0 \right) \exp \left( -\frac{\alpha + \alpha^+}{\sqrt{2}} \right), \]
\[ K_+ = \exp \left( \frac{\alpha + \alpha^+}{\sqrt{2}} \right) \left( \frac{\alpha - \alpha^+}{i\sqrt{2}} + P_0^* \right), \]
\[ K_0 = \frac{\alpha - \alpha^+}{i\sqrt{2}} + (P_0 + P_0^*)/2 - 1/2, \quad (14) \]

or,
\[ K_- = \left( \frac{\alpha + \alpha^+}{\sqrt{2}} + P_0 \right) \exp \left( \frac{\alpha - \alpha^+}{\sqrt{2}} \right), \]
\[ K_+ = \exp \left( -\frac{\alpha - \alpha^+}{\sqrt{2}} \right) \left( \frac{\alpha + \alpha^+}{i\sqrt{2}} + P_0^* \right), \]
\[ K_0 = \frac{\alpha + \alpha^+}{\sqrt{2}} + (P_0 + P_0^*)/2 - 1/2. \quad (15) \]

Note that a classical analog of the representation (15) has been derived in [6].

### III. EXAMPLE

Let us consider now the following Hamiltonian:
\[ \mathcal{H} = \varepsilon(a^+a + b^+b) + \Phi_2a^+b^+ba \]
\[ + \Phi_1(a^+a^+aa + b^+b^+bb). \quad (16) \]

It describes a system of two nonlinear interacting oscillators with the Bose operators \( a^+ \), \( a \) and \( b^+ \), \( b \). The two mode representation of SU(1,1) is defined by
\[ K_- = ab, \quad K_+ = a^+b^+, \]
\[ K_0 = \frac{1}{2} (a^+a + b^+b + 1) \quad (17) \]

with
\[ C = -1/4 + (a^+a - b^+b)^2/4. \quad (18) \]

The Hamiltonian (16) in terms of (17) has the form:
\[ \mathcal{H} = 2\Phi_1 - \varepsilon + (2\varepsilon - 6\Phi_1)K_0 \]
\[ + 4\Phi_1K_0^2 + (\Phi_2 - 2\Phi_1)K_+K_- \quad (19) \]

Let us consider the pair states when the numbers of ‘a’ and ‘b’ bosons are equal (formally, \( a^+a = b^+b \)). In

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