Quantum backflow in the presence of a purely transmitting defect

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Abstract

We analyse the quantum backflow effect and extend it, as a limiting constraint to its spatial extent, for scattering situations in the presence of a purely transmitting discontinuous jump-defect. Analytical and numerical comparisons are made with a situation in which a defect is represented by a δ function potential. Furthermore, we make the analysis compatible with conservation laws.

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1 Introduction

Quantum theory certainly has different mathematical formulations and significant conceptual differences that go beyond a relativistic extension of quantum mechanics (QM) to quantum field theory (QFT) [1]. Nevertheless, quantum theory shares some bedrock ideas such as the Heisenberg’s uncertainty principle among any of its formulations or extensions. Other effects related to the uncertainty principle may arise as inequalities. That is the case of the “quantum inequalities” in QFT [2,3], which are lower bound restrictions on the fluxes and energy densities of physical systems, and the quantum backflow phenomenon [4] for the probability current in QM.

Similarly to the quantum energy inequalities, which are limitations on the magnitude and duration of negative energy densities, the backflow inequality can be stated in its time-averaged or spatial-averaged versions. The total energy of a physical system being bounded below is a fact related to the existence of a stable ground state. Nonetheless, there is an incompatibility between positive energy density conditions and local quantum fields [5]. The lower bound on the backflow effect, however, does not seem to have an immediately clear physical interpretation. Consideration of both effects in a common framework such as a free relativistic theory may provide some insight on their relationship. In fact, whilst most of the work on quantum backflow considered only the situation without any internal degree of freedom and non-relativistic, the case for a free Dirac particle was studied in [6], for instance. Moreover, as the energy is usually considered in connection with a conservation law, it is reasonable to do the same for the backflow analysis and associate a conservation law with it when possible. That is not possible for an interaction described by a $\delta$ potential function, but a jump-defect provides us with this possibility.

Interaction-free situations present a playground for numerous discussions, but more realistically one has to consider the effect of interaction. In [7], the backflow effect was extended to scattering situations in short-range potentials. It reinforced the universality of quantum backflow beyond a free theory and also stated that the lower bound feature, the constraint on how negative it can be, is stable under the inclusion of interaction. Although their work has proved the existence of lower bound estimates for a particular class of short-range potentials, they also noticed that a very short-range $\delta$ potential, although outside their theorem’s validity, has a limited backflow effect. A special particularity of the $\delta$ is that it can be seen as a potential function, but it can also be seen as a point-defect that is characterised by some sewing conditions at the defect’s location. Knowing that, we ask ourselves about the possibility of including other kinds of point-defects described by a set of sewing conditions in the discussion of the quantum backflow effect. Defects were previously considered in scattering situations [8] in the context of the Yang-Baxter equation. Generally, both transmission and reflection terms might be consistently included in an integrable theory by weakening some constraining assumptions [9]. However, integrable defects are generally categorised as purely transmitting [10]. In particular, we consider a jump-defect that is purely transmitting.

This paper is composed of eight sections and one appendix. In Sec. 2, 3 and 4 we present the quantum backflow setting for both the free case and in the presence of a scattering potential. Sec. 5 focuses attention on a particular integrable defect in our case of interest, namely the discontinuous jump-defect in the linear Schrödinger equation. Sec. 6 introduces the calculations for the backflow effect in the presence of the $\delta$-defect and the jump-defect. In Sec. 7 numerical details are provided along with the results in two-dimensional plots for both the delta-case and the jump-case. We summarise the work in the concluding remarks, Sec. 8. Finally, the Appendix contains three-dimensional plots encapsulating the behaviour of the lowest backflow eigenvalue under changes of the defect parameter and the position of measurement.
2 Quantum backflow

In non-relativistic quantum mechanics, the continuity equation for the probability density in one space dimension is

$$\partial_t \rho = -\partial_x j,$$  \hfill (2.1)

where $\rho = |\psi|^2$ is the probability density, $j$ is the probability current density and $\psi$ the square-integrable wavefunction of the system. The Schrödinger equation for the wavefunction of a quantum system is simply

$$i\hbar \partial_t \psi = H\psi,$$  \hfill (2.2)

where $H$ is the Hamiltonian operator associated with the system. The state vector is commonly denoted by $|\psi\rangle \in \mathcal{H}$, as an abstract vector in the Hilbert space of the physical system. Not all solutions of this equation are elements of the space of (equivalence classes of) square-integrable functions $L^2(\mathbb{R})$, but these solutions are crucial for scattering theory. As a consequence of the Schrödinger equation, in the free case, one has

$$j_\psi(x) = \frac{i\hbar}{2m} (\partial_x \psi^*(x) \psi(x) - \psi^*(x) \partial_x \psi(x)) := \langle \psi, J(x)\psi \rangle$$  \hfill (2.3)

where now the $\psi$-dependence is explicitly indicated, and $j_\psi(x)$ can be expressed in terms of the associated quadratic form $J(x)$. The space average of (2.3) with a test function, generally $f \in \mathcal{S}(\mathbb{R})$ in Schwartz-class is given by

$$j_\psi(f) = \langle \psi, J(f)\psi \rangle = \int dx f(x) j_\psi(x),$$  \hfill (2.4)

and is understood as the spatial-averaged probability current measured by a spatially extended apparatus. The corresponding smeared operator is the integration $J(f) = \int f(x) J(x) dx$, understood in the sense of quadratic forms. This operator is Hermitian for a real function $f$ and is written as

$$J(f) = \frac{1}{2} \left( \hat{P} f(\hat{X}) + f(\hat{X}) \hat{P} \right),$$  \hfill (2.5)

with position operator $\hat{X}$ and momentum operator $\hat{P}$.

The effect that, for a particle with positive momentum, the probability of finding it to the right of some reference point may decrease with time is simply called quantum ‘backflow’ [7,11–14]. This means that given a wavefunction $\tilde{\psi}$ with support in momentum space restricted by $\text{supp} \left( \tilde{\psi} \right) \subset \mathbb{R}_+$, right-moving wave function, it is not guaranteed at all that the probability current density fulfills the positivity condition $j_\psi(x) > 0$ with $x \in \mathbb{R}$. The backflow effect has been discussed in both time-averaged and spatial-averaged versions.

3 Free case

The maximal amount of backflow, spatially averaged with a positive test function $f$, i.e. the lowest bound, is defined [7] by

$$\beta_0(f) := \inf \langle E_+ J(f) E_+ \rangle_\psi,$$  \hfill (3.1)

where the infimum is understood as

$$\inf \langle A \rangle := \inf_{\|\psi\|=1} \langle \psi, A\psi \rangle \in (-\infty, \infty).$$
According to the minimax principle \[15\], \( \beta_0(f) \) is the minimum eigenvalue of the averaged current evaluated in right-moving states, \( E_+J(f)E_+ \). The orthogonal projection \( E_+ \) of the momentum operator makes sure that the momentum is positive \( (k > 0) \). The question of how negative this quantity can be, and if it is actually bounded below, is answered by the following theorem \[11\].

**Theorem 1.** For every positive test function \( f \in S(\mathbb{R}) \), \( \exists C_f \geq 0 \) such that \( \langle J(f) \rangle_\psi \geq -C_f \), where \( \psi \) is taken to be normalised and right-moving, i.e. \( \psi \in R = \{ \psi \in L^2(\mathbb{R}) | \bar{\psi}(k) = 0, \text{ for } k < 0 \} \).

This theorem describes a quantum inequality which is state-vector independent. Moreover, whilst this quantity \( \beta_0(f) \) is bounded below, it is unbounded above, for positive \( f \), exactly as the non-smeared version \( E_+J(x)E_+ \). Effectively, the unboundedness is a high momentum effect \[7\].

### 4 Interaction in scattering situations

As usually done, we consider the effect of an interaction with a potential term \( V \), external and time-independent for simplicity, added to the free Hamiltonian so that

\[
H = \frac{\hat{p}^2}{2m} + V(\hat{X}). \tag{4.1}
\]

As a physical requirement, the potential is Hermitian. While the concept of right-movers is clear in a free case, the time evolution associated with an interacting Hamiltonian does not commute with the projector \( E_+ \), meaning that the space of right-movers \( E_+L^2(\mathbb{R}) \) is not invariant under time evolution transformations. As an alternative equivalent concept, we adopt the asymptotic right-movers in the sense of scattering theory, as used before in \[7\]. In this way, we consider a state such that its incoming asymptote is a right-mover. The incoming Møller operator is given by

\[
\Omega^{(\text{IN})} = \Omega_V := \text{s-lim}_{t \to -\infty} e^{+iHt} e^{-iH_0t}, \tag{4.2}
\]

with \( \text{s-lim} \) denoting the strong operator limit and \( H_0 \) the free Hamiltonian. Whilst we have the isometry property \( ||\Omega_V|| = 1 \), the unitarity property \( (\Omega_V \Omega_V^\dagger = 1) \) is not valid in the presence of bound states, which do not admit a scattering description. Our quantity of interest is now dependent on the potential and defined as

\[
\beta_V(f) := \inf \langle E_+ \Omega_V^\dagger J(f) \Omega_V E_+ \rangle_\psi, \tag{4.3}
\]

which is called the “asymptotic backflow constant” and it is the lowest eigenvalue of the operator \( E_+\Omega_V^\dagger J(f) \Omega_V E_+ \). In the future, we will refer to that as the “asymptotic current operator” or simply the “interacting current”.

To ensure the existence of the scattering theory, we work with potentials which vanish sufficiently fast at spatial infinity. This is based on the fact that the fall-off properties of the potential are related to smoothness properties of the scattering data. Specifically, we require the fulfillment of the condition \[16\]

\[
||V||_{1+} := \int (1 + |x|) |V(x)| \, dx < \infty, \tag{4.4}
\]

and we say that \( V \in L^{1+}(\mathbb{R}) \). In the stationary scattering theory, one has the time-independent Schrödinger equation (TISE) for a wavefunction \( \varphi(x) \), an eigenvalue equation for the operator \( H \), written, in position basis, as

\[
\left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right) \varphi(x) = \frac{(\hbar k)^2}{2m} \varphi(x),
\]
which can be notationally simplified by denoting $U(x) = \frac{2m}{\hbar}V(x)$ and taking $\hbar = m = 1$. In a more compact form, it is

$$(-\partial_x^2 + U(x) - k^2)\varphi(x) = 0, \quad k \in \mathbb{R}. \quad (4.5)$$

We focus on solutions $x \to \varphi_k(x)$ with $k > 0$ and asymptotics

$$\varphi_k(x) = \begin{cases} T_V(k)e^{ikx} + o(1) & \text{for } x \gg 0, \\
e^{ikx} + R_V(k)e^{-ikx} + o(1) & \text{for } x \ll 0, \end{cases} \quad (4.6)$$

where $R_V(k)$ is the reflection coefficient, and $T_V(k)$ is the transmission coefficient for the potential function $V$. In the scattering context, the Schrödinger equation together with boundary conditions (4.6) is equivalent to a Lippmann-Schwinger equation

$$\varphi_k(x) = T_V(k)e^{ikx} + \int dy G_k(x-y)U(y)\varphi_k(y). \quad (4.7)$$

For this choice of complementary function (the inhomogeneous term of the integral equation), the Green function for the differential equation (4.5) is

$$G_k(x) = -\frac{\sin(kx)}{k} \theta(-x),$$

where $\theta$ is the Heaviside function: $\theta(x) = 0$ for $x < 0$ and $\theta(x) = 1$ for $x > 0$. Another possible choice would be to take

$$\varphi_k(x) = g_k(x) + \int dy G_k(x-y)U(y)\varphi_k(y), \quad (4.8)$$

where $g_k(x) = \exp(ikx)$ was chosen as the complementary function, solution of the free equation $(V = 0)$, or incident wave. Given the boundary conditions (4.6), we employ the Green function

$$G_k(x-y) = \frac{1}{2ik} e^{ik|x-y|}, \quad (4.9)$$

which gives the transmission amplitude

$$T_V = 1 + \int_{-\infty}^{+\infty} dy \frac{e^{-iky}}{2ik} U(y)\varphi_k(y), \quad (4.10)$$

and the reflection amplitude

$$R_V = \int_{-\infty}^{+\infty} dy \frac{e^{iky}}{2ik} U(y)\varphi_k(y). \quad (4.11)$$

This choice of Green function corresponds to the $+i\epsilon$-prescription on the poles of the free Hamiltonian’s resolvent, that is

$$\lim_{\epsilon \to 0} (H_0 - (E_k + i\epsilon))^{-1} = G(E_k + i\epsilon) \equiv G_k,$$  

$$\quad (4.12)$$

where the Green function is a kernel of this Green operator; commonly written as $\langle x | G_k | y \rangle = G_k(x, y)$. Although the transmission and the reflection coefficients for a particular potential can be worked out by means of Lippmann-Schwinger equation in general, as long as we are not committed to work with perturbation theory, we may use the exact solution without any approximation. Either working with perturbation approximations or the exact solution, a key ingredient for our analysis is the use of an expansion, in the scattering setting, of the Møller wave operator in the following integral form; see, for example, [17] for the Lemma below.
Lemma 1. Let $V \in L^{1+}(\mathbb{R})$. Then the operator $\Omega_V$ defined in (4.2) exists. Further, the solution $x \mapsto \varphi_k(x)$ ($k > 0$) of (4.5) with the asymptotics (4.6) exists and is unique, and for any $\tilde{\psi} \in C_0^\infty(\mathbb{R})$,

$$
\langle x | \Omega_V E_+ | \psi \rangle = (\Omega_V E_+ \psi)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty dk \varphi_k(x)\tilde{\psi}(k) .
$$

(4.13)

By the use of some estimates, e.g., [16,18], the following theorem [7] is a result on the existence of backflow in scattering situations and also on its lower bound.

Theorem 2. Let the potential function $V$ be a $L^{1+}(\mathbb{R})$-class potential, i.e.

$$
\|V\|_{1+} \doteq \int_{-\infty}^{+\infty} dx (1+ | x |) |V(x)| < \infty ,
$$

for any $f \in C_c(\mathbb{R})$, $f > 0$, $\exists C_{V,f} > 0$ such that

$$
\langle \psi | E_+ \Omega_V^* J(f) \Omega_V E_+ | \psi \rangle \geq -C_{V,f} \text{ for } \|\psi\| = 1.
$$

(4.14)

We denote the expectation value of the interacting operator in a general state vector $| \psi \rangle$ by

$$
\langle J_V(f) \rangle_\psi := \langle \psi | E_+ \Omega_V^* J(f) \Omega_V E_+ | \psi \rangle .
$$

(4.15)

The expansion of this expectation value relies on the use of the Lemma of the Lemma. The existence of backflow and the boundness (below) of backflow are stable under the addition of a scattering potential to the Hamiltonian. This means that, even in the presence of reflection, the effect is bounded below.

As is the case for the Hamiltonian, we expect that the asymptotic current operator has a spectrum composed of pure point and absolutely continuous parts. Thus, we have some eigenvalues, with the lowest one denoted by $\beta_V(f)$, and at some point a continuum of “generalized” eigenvalues. It is important to stress our interest in this lowest eigenvalue in the context of quantum inequalities.

5 Integrable defects

Either in classical or quantum theory, partial differential equations come in to describe the dynamics of the systems we want to study. The same physical idea can be implemented in different ways, depending on what one wants to describe. Integrable defects [8,10,20,22] can be treated both in classical and quantum contexts in linear and nonlinear theories. In a linear theory, integrability is certainly redundant, but the underlying motivation is the same: to preserve conservation laws.

In the Schrödinger equation for a wavefunction $\varphi$, one explicitly writes down a potential term, usually a function of the position, in the Hamiltonian of the system. In case the potential is only a function of the space coordinate (and possibly of time), but not of $\varphi$ itself, the equation is still a linear partial differential equation. Additionally to working with an explicit potential term, there is another way of implementing interactions in the presence of point-like impurities or defects, a kind of internal boundary at a point. Rather than written as an external potential function, the defect can be described by a set of sewing conditions. In 1 + 1 dimensions, these conditions relate the field and its derivatives on the left to the field and its derivatives on the right of the defect’s location.

The $\delta$-type defect has the pedagogical advantage of allowing both descriptions; it can be written as the usual delta potential $\delta(x)$ or as a set of two sewing conditions. In particular, for the $\delta$-type defect, one condition is a statement of the continuity of the field at the defect location and the other one describes the discontinuity of the spatial derivative of the field. Although interesting and more familiar, the $\delta$-type of impurity may spoil the integrability of a nonlinear integrable system. For instance, that is the case for the sine-Gordon equation [23]. However, some years ago it was shown

\footnote{The Lemma requires $\tilde{\psi}$ to be smooth of compact support $C^\infty_c$. However, $C^\infty_c(\mathbb{R})$ is dense in $L^2(\mathbb{R})$ and through the use of Friedrichs extensions [19] the discussion applies to a general $\psi$ in the domain $D(J)$ of our operator.}
that there exist two types of defects that are integrable, proved by constructing Lax pairs, and they were categorised as type I and type II [22]. The former is simpler in the sense that only the field has dynamics, and the latter is a generalization with an extra function defined on the defect; it has an extra internal degree of freedom.

In this work, we focus on the type I integrable defects. While the $\delta$-type defect has continuous solutions at the defect location, we can have a defect that allows a discontinuity of the field at the same location. Such a defect, with a particular set of sewing conditions, is called a “jump-defect”. In the context of the mechanics of the continuum, such defects are very similar to shock waves, for example, which have sewing conditions expressed by the Rankine-Hugoniot conditions [24].

### 5.1 Jump-defect in non-relativistic context

Although a Lagrangian description is not necessary for the set-up of the situation we are interested in, we can start from a Lagrangian in $1 + 1$ dimensions. From a non-relativistic Schrödinger Lagrangian

$$
\mathcal{L} = \frac{i\hbar}{2} (\bar{\psi} \psi_t - \bar{\psi}_t \psi) - \frac{\hbar^2}{2m} |\psi_x|^2, 
$$

for the field $\psi(x,t)$. By a suitable rescaling of the physical units and denoting $\psi_t$ and $\psi_x$ as time and space derivatives, respectively, the Euler-Lagrange equation gives the linear Schrödinger equation

$$
i\psi_t + \psi_{xx} = 0.
$$

The defect can be placed at the position $x_D = 0$ on the real line, for example. This means that the bulk region, $-\infty < x < \infty$, will effectively split in two parts. The field on the left of the defect ($x < 0$) will be denoted $u = u(x,t)$ and the field on the right ($x > 0$) will be denoted $v = v(x,t)$. For the complete set-up of the system [21], the Lagrangian density has now three contributions coming from $u$, $v$ and the defect

$$
\mathcal{L} = \theta(x_D - x)\mathcal{L}_u + \theta(x - x_D)\mathcal{L}_v + \delta(x - x_D)\mathcal{B}(u,v),
$$

where $\mathcal{B}(u,v)$ comes from the defect and it is chosen so that the system remains integrable. For different interactions, one may have different $\mathcal{B}(u,v)$. The full action is

$$
A = \int dt \left[ \int_{-\infty}^{x_D} dx \mathcal{L}(u) + \mathcal{B} + \int_{x_D}^{\infty} dx \mathcal{L}(v) \right],
$$

where $\mathcal{B}$ is the Lagrangian part of the defect itself. The defect conditions at $x = x_D = 0$, when $\mathcal{B}(u,v)$ does not depend on the spatial derivatives, are

$$
u_x = \frac{\partial \mathcal{B}}{\partial \bar{v}} - \frac{\partial}{\partial t} \frac{\partial \mathcal{B}}{\partial \bar{u}_t}, \quad u_x = \frac{\partial \mathcal{B}}{\partial \bar{u}} + \frac{\partial}{\partial t} \frac{\partial \mathcal{B}}{\partial \bar{v}_t},
$$

and similarly for the complex conjugate fields. Note, the values of the fields at the defect, $u(x_D,t)$ and $v(x_D,t)$, are defined by limits from the left ($x < x_D$) and the right ($x > x_D$), respectively.
In [21], the set of sewing conditions were obtained for a nonlinear Schrödinger equation of the form
\[ iu_t + u_{xx} + 2u(\bar{u}u) = 0. \]
In particular,
\[ B = \Omega \left[ \frac{i}{2} \frac{\partial}{\partial t} \ln \left( \frac{u - v}{\bar{u} - \bar{v}} \right) + B \right], \quad \Omega = (\alpha^2 - |u - v|^2)^{1/2}, \quad (5.6) \]
with
\[ B = \frac{1}{3} (\alpha^2 - |u - v|^2) + (|u|^2 + |v|^2), \quad (5.7) \]
where \( \alpha \) is a real parameter. From the nonlinear situation, we can particularise to our linear case where \( u \) and \( v \) obey the linear Schrödinger equation. The linearisation of conditions (5.5) is given by
\[ u_x = -\frac{i}{2\alpha} (u - v)_t + \frac{\alpha}{2} (u + v), \]
\[ v_x = -\frac{i}{2\alpha} (u - v)_t - \frac{\alpha}{2} (u + v), \quad (5.8) \]
which can immediately be rearranged as
\[ u_x - v_x = \alpha (u + v), \]
\[ v_x + u_x = -\frac{i}{\alpha} (u - v)_t, \quad (5.9) \]
both valid at the defect’s position. The first thing to notice is that the difference of the spatial derivatives is proportional to the average (arithmetic mean) value of the fields meeting at the defect’s location, and the parameter \( \alpha \) works as a strength of that difference. The second is the discontinuity at the defect, namely we can have \( u \neq v \). For reference, we shall mention that in the \( \delta \) potential function case, with potential \( V(x) = \lambda \delta(x) \), the conditions are
\[ u = v, \quad (v_x - u_x) = \lambda u, \quad x = x_D = 0, \quad (5.10) \]
both evaluated at the \( \delta \)-defect’s position, and \( \lambda \) is the associated defect parameter. Note also the similarity to the Bäcklund transformations [25] for the linear Schödinger equation. It would actually be one if the relations were valid for all positions instead of being frozen at the defect’s location.

These sewing conditions [5.9] allow the following pair of traveling wave solutions [21]
\[ u = u_0 \exp(-ik^2 t + ikx), \quad v = v_0 \exp(-ik^2 t + ikx), \quad v_0 = \frac{k + i\alpha}{k - i\alpha} u_0, \quad (5.11) \]
where \( k \) is real, and the frequency \( \omega = k^2 \) obeys the usual quadratic dispersion relation from the non-relativistic theory. Of course, a particular solution is to take the constant \( u_0 = 1 \). From (5.11), it is possible to see the existence of bound states associated with the jump-defect for either \( k = i\alpha \) or \( k = -i\alpha \). The respective bound states can then be described by
\[ u = 0, v = v_0 \exp(i\alpha^2 t - \alpha x), \quad (k = i\alpha); \quad u = u_0 \exp(i\alpha^2 t + \alpha x), v = 0, \quad (k = -i\alpha), \quad (5.12) \]
which are clearly square integrable solutions (provided \( \alpha > 0 \)) [21].

The jump-defect has the property of being a purely transmitting defect. In this sense, it is similar to the Pöschl-Teller potential [26] given by
\[ V(x) = -\frac{\mu(\mu + 1)}{2 \cosh^2 x}, \quad \mu > 0. \quad (5.13) \]
However, the latter is only reflectionless when the parameter $\mu$ is taken to be an integer while the jump-defect is always purely transmitting.

Our analysis will be restricted to the linear case so that we directly look at conservation laws as our guiding principle for the construction of the jump-defect. Specifically, the jump-defect is designed in order to keep valid some conservation laws, which are true in the free case. In other words, we ask that the implementation of the jump-defect does not cause a breakdown of the conservation laws we have in the free Schrödinger theory.

5.2 Conservation laws

In the free Schrödinger case, we know that quantities such as energy, probability and momentum are conserved. For that, we check how a point-defect may affect these conservation laws, and how (if possible) we can fix the quantity so that it remains conserved in the presence of the defect. In particular, we compare the $\delta$-defect to the jump-defect and analyse energy, momentum and total probability.

Taking only the $u$ contribution, the energy density derived from the Lagrangian density given by (5.1) is

$$E = |u_x|^2,$$

and similarly for $v$. The total energy is therefore

$$E = \int_{-\infty}^{0} \bar{u}_x u_x dx + \int_{0}^{\infty} \bar{v}_x v_x dx,$$

where we have split up the integral taking into consideration that the defect is located at the origin $x = 0$. For checking conservation, we calculate the time derivative

$$E_t = \int_{-\infty}^{0} (\bar{u}_x u_x + \bar{u}_x u_{xt}) dx + \int_{0}^{\infty} (\bar{v}_x v_x + \bar{v}_x v_{xt}) dx$$

$$= \int_{-\infty}^{0} \frac{\partial}{\partial x} (-i\bar{u}_x u_x + i\bar{u}_x u_{xx}) dx + \int_{0}^{\infty} \frac{\partial}{\partial x} (-i\bar{v}_x v_x + i\bar{v}_x v_{xx})$$

$$= (\bar{u}_t u_x + \bar{u}_x u_t) |_{x=0} - (\bar{v}_t v_x + \bar{v}_x v_t) |_{x=0},$$

where we used the Schrödinger equation and threw away the zero contributions at $\pm$ infinity as usual. At this stage, we ask ourselves when $E_t$ is zero, if possible. For the $\delta$-defect, using (5.10) we obtain

$$E_t = -\lambda \bar{u}_t u - \lambda \bar{uu}_t$$

$$= -\lambda \frac{\partial}{\partial t} (\bar{uu}) |_{x=0},$$

which may not be zero, but we can guarantee the total energy is conserved by fixing it with an extra contribution. That means the fixed conserved energy is

$$E_c := E + \lambda \bar{uu} |_{x=0}. \tag{5.16}$$

For the jump-defect, we can check that the choice (5.8) produces

$$E_t = \left[ \bar{u}_t \left( -\frac{i}{2\alpha} (u - v)_t + \frac{\alpha}{2} (u + v) \right) + u_t \left( \frac{i}{2\alpha} (\bar{u} - \bar{v})_t + \frac{\alpha}{2} (\bar{u} + \bar{v}) \right) \right]_{x=0}$$

$$- \left[ \bar{v}_t \left( -\frac{i}{2\alpha} (u - v)_t - \frac{\alpha}{2} (u + v) \right) + v_t \left( \frac{i}{2\alpha} (\bar{u} - \bar{v})_t - \frac{\alpha}{2} (\bar{u} + \bar{v}) \right) \right]_{x=0}$$

$$= \frac{\alpha}{2} \frac{\partial}{\partial t} ((u + v)(\bar{u} + \bar{v})) |_{x=0},$$

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which depends on the parameter $\alpha$ and may not be zero, but we can guarantee the conservation by fixing it with the following redefinition of the energy

$$E_c := E - \frac{\alpha}{2} |u + v|^2 \Big|_{x=0} \ .$$

(5.17)

Hence, the energy $E_c$ is conserved.

Now, let us analyse the momentum. The momentum density associated with $u$ is given by

$$\mathcal{P}(u) = i (\bar{u}u_x - \bar{u}_x u) \ ,$$

(5.18)

so that the total momentum is

$$P = \int_{-\infty}^{0} i (\bar{u}u_x - \bar{u}_x u) \, dx + \int_{0}^{\infty} i (\bar{v}v_x - \bar{v}_x v) \, dx .$$

(5.19)

We take the time derivative

$$P_t = \int_{-\infty}^{0} i(\bar{u}_t u_x + \bar{u}u_{xt} - \bar{u}_x u - \bar{u}_x u_t) \, dx + \int_{0}^{\infty} i(\bar{v}_t v_x + \bar{v}v_{xt} - \bar{v}_x v - \bar{v}_x v_t) \, dx$$

$$= \int_{-\infty}^{0} (2(\bar{u}_x u_x) - (\bar{u}u_{xx} + u\bar{u}_{xx})) \, dx + \int_{0}^{\infty} (2(\bar{v}_x v_x) - (\bar{v}v_{xx} + v\bar{v}_{xx})) \, dx$$

$$= \left[ 2(\bar{u}_x u_x) - (\bar{u}u_{xx} + u\bar{u}_{xx}) \right]_{x=0} - \left[ 2(\bar{v}_x v_x) - (\bar{v}v_{xx} + v\bar{v}_{xx}) \right]_{x=0} ,$$

where we used the Schrödinger equation and threw away the zero contributions at ± infinity. For the $\delta$-defect, with (5.10),

$$P_t = (2(\bar{v}_x - \lambda \bar{v})(v_x - \lambda v) - (-i \bar{v}_t v + i v\bar{v}_t)) |_{x=0} - 2\bar{v}_x v_x |_{x=0} + (\bar{v}_x v_x + v\bar{v}_{xx}) |_{x=0}$$

$$= (-2\lambda(v\bar{v})_x + 2\lambda^2 v\bar{v}) |_{x=0} ,$$

which cannot be written as a time derivative by the use of the sewing conditions. Hence, we are not able to fix this conservation law without any other extra considerations. The momentum $P$ highlights the difference between the $\delta$ and the jump-defect because the same calculation applied to the jump-defect, using (5.8), yields

$$P_t \left[ 2 \left( + \frac{i}{2\alpha}(\bar{u} - \bar{v})(u - v)_t + \frac{\alpha}{2}(\bar{u} + \bar{v}) \left( - \frac{i}{2\alpha}(u - v)_t + \frac{\alpha}{2}(u + v) \right) \right) \right.$$

$$- \left( \bar{u} \right. \left( - \frac{i}{2\alpha}(u - v)_t + \frac{\alpha}{2}(u + v) \right) + u \left( + \frac{i}{2\alpha}(\bar{u} - \bar{v})_x + \frac{\alpha}{2}(\bar{u} + \bar{v}) \right) \left. \right|_{x=0}$$

$$- \left[ 2 \left( \frac{i}{2\alpha}(\bar{u} - \bar{v})_t - \frac{\alpha}{2}(\bar{u} + \bar{v}) \left( - \frac{i}{2\alpha}(u - v)_t - \frac{\alpha}{2}(u + v) \right) \right) \right.$$

$$- \left( \bar{v} \right. \left( - \frac{i}{2\alpha}(u - v)_t + \frac{\alpha}{2}(u + v) \right) + v \left( + \frac{i}{2\alpha}(\bar{u} - \bar{v})_t + \frac{\alpha}{2}(\bar{u} + \bar{v}) \right) \left. \right|_{x=0} ,$$

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which can be simplified to
\[
P_t = \left[ \frac{i}{2} (u + v)(\bar{u} - \bar{v})_t - i(\bar{u} + \bar{v})(u - v)_t + \frac{i}{2} (\bar{u} - \bar{v})(u + v)_t \right. \\
\left. + \frac{i}{2} (\bar{u} + \bar{v})(u - v)_t - \frac{i}{2} (u - v)(\bar{u} + \bar{v})_t \right]_{x=0} \\
= i \left[ \frac{1}{2} ((u + v)(\bar{u} - \bar{v}))_t - \frac{1}{2} (u + v)_t(\bar{u} - \bar{v}) - (u - v)_t(\bar{u} + \bar{v}) + \frac{1}{2} (u + v)_t(\bar{u} - \bar{v}) \\
+ \frac{1}{2} (u - v)_t(\bar{u} + \bar{v}) - \frac{1}{2} ((u - v)(\bar{u} + \bar{v}))_t + \frac{1}{2} (u - v)_t(\bar{u} + \bar{v}) \right]_{x=0} \\
= i \left[ \frac{\partial}{\partial t} (\bar{u}v - \bar{v}u) \right]_{x=0} ,
\]
which may not be zero, but we can guarantee the conservation if we redefine \( P \) in order to take in consideration the contribution at the defect’s location by
\[
P_c := P - i(\bar{u}v - \bar{v}u)|_{x=0}. \tag{5.20}
\]
Then, the momentum \( P_c \) is conserved.

The probability density for \( u \) is given by
\[
N(u) = \bar{u}u, \tag{5.21}
\]
so that the total probability is
\[
\mathcal{N} = \int_{-\infty}^{0} N(u)dx + \int_{0}^{\infty} N(v)dx = \int_{-\infty}^{0} \bar{u}udx + \int_{0}^{\infty} \bar{v}vdx. \tag{5.22}
\]
To examine its conservation, we consider
\[
N_t = \int_{-\infty}^{0} (\bar{u}_t u + \bar{u}u_t)dx + \int_{0}^{\infty} (\bar{v}_t v + \bar{v}v_t)dx \\
= \int_{-\infty}^{0} \frac{\partial}{\partial x} (-i\bar{u}_x u + i\bar{u}u_x) dx + \int_{0}^{\infty} \frac{\partial}{\partial x} (-i\bar{v}_x v + i\bar{v}v_x) dx \\
= (\bar{u}_x u + i\bar{u}u_x)|_{x=0} = (\bar{v}_x v + i\bar{v}v_x)|_{x=0} ,
\]
where we used the Schrödinger equation and threw away the zero contributions at infinities. For the \( \delta \)-defect, using \([5.10]\),
\[
N_t = -i\bar{u}(\bar{v}_x - \lambda \bar{u})|_{x=0} = (\bar{u}v_x - \lambda \bar{u})|_{x=0} = (\bar{u}_x v + i\bar{u}v_x)|_{x=0} = 0 ,
\]
which means this is automatically conserved. For the jump-defect, with the choice \([5.8]\)
\[
N_t = \left[ \frac{u(\bar{u} - \bar{v})_t + \bar{u}(u - v)_t}{2\alpha} + \frac{i\alpha}{2} (\bar{u}(u + v) - u(\bar{u} + \bar{v})) \right]_{x=0} \\
= \left[ \frac{v(\bar{u} - \bar{v})_t + \bar{v}(u - v)_t}{2\alpha} - \frac{i\alpha}{2} (\bar{v}(u + v) - v(\bar{u} + \bar{v})) \right]_{x=0} \\
= \left[ \frac{(\bar{u} - \bar{v})(u - v) + (\bar{u} - \bar{v})(u - v)_t}{2\alpha} \right]_{x=0} \\
= \frac{1}{2\alpha} \left[ \frac{\partial}{\partial t} ((u - v)(\bar{u} - \bar{v})) \right]_{x=0} ,
\]
and
\[
\mathcal{N} = \int_{-\infty}^{0} N(u)dx + \int_{0}^{\infty} N(v)dx = \int_{-\infty}^{0} \bar{u}udx + \int_{0}^{\infty} \bar{v}vdx.
\]

which depends on the parameter \( \alpha \) and may not be zero. However, we can guarantee conservation if we redefine \( N \) to take into account a contribution at the defect by defining

\[
N_c := N - \frac{1}{2\alpha} \langle u - v \rangle^2 \bigg|_{x=0}.
\]  

(5.23)

We have shown how both the \( \delta \) and the jump-defect affect some conservation laws and we have seen how we can fix these conservation laws by redefining quantities with an extra contribution which comes from the defect. However, it is clear that the jump-defect allows conservation of \( P \) without any further extra information, but the \( \delta \)-defect does not. When we treat these defects in the context of quantum mechanics, the momentum \( P \) is actually related to the probability current. Moreover, we will see how this extra term associated with the defect affects the calculation of the quantum backflow and how it significantly differs from the \( \delta \)-defect case. Strikingly interesting, in the jump-case, is that the fixing term, to restore the conservation of \( P \), has a substantial contribution to the lowest backflow eigenvalue.

6 Backflow in the presence of a defect

A local defect can sometimes be understood as an interacting potential term added to a physical system playing the role of interaction. Nevertheless, whilst local defects add interaction to a physical system, they are not restricted by an explicit potential function. More generally, they are defined by a set of sewing conditions rather than a function. As mentioned before, the \( \delta \)-defect can be characterised both ways, as a potential term or as a set of sewing conditions. Because of its importance, we first consider the backflow calculation in the presence of a \( \delta \)-defect. Before the actual calculation, we need to set the general structure of our quantities of interest.

In abstract Dirac notation, we can write the general structure of our operator of interest \( E_+ \Omega_{\delta}^1 J(f) \Omega_V E_+ \). For that, let us first write (4.8) formally as

\[
| \varphi_k \rangle = | g_k \rangle + G_k U | \varphi_k \rangle,
\]  

(6.1)

where we have made use of the relation \( \langle x|k \rangle = e^{ikx}/\sqrt{2\pi} = g_k(x)/\sqrt{2\pi} = \langle x|g_k \rangle /\sqrt{2\pi} \), that is \( |g_k \rangle = \sqrt{2\pi} |k \rangle \), and expand our quantity of interest in abstract notation

\[
\langle \psi | E_+ \Omega_{\delta}^1 J(f) \Omega_V E_+ | \psi \rangle = \frac{1}{(\sqrt{2\pi})^2} \int_0^\infty dk' \langle \psi | |k' \rangle \langle \varphi_{k'} | J(f) \int_0^\infty dk | \varphi_k \rangle \langle k | | \psi \rangle,
\]  

(6.2)

where we used (4.13) in its abstract notation; \( \Omega_V E_+ \equiv \int_0^\infty dk | \varphi_k \rangle \langle k | \) and \( E_+ \Omega_{\delta}^1 \equiv \int_0^\infty dk' | \varphi_{k'} \rangle \langle \varphi_{k'} | \). For extracting the operator, we need the trick of inserting Dirac delta distributions \( \delta(k' - q') \) and \( \delta(k - q) \) as

\[
\langle \psi | E_+ \Omega_{\delta}^1 J(f) \Omega_V E_+ | \psi \rangle = \frac{1}{2\pi} \langle \psi | \int_0^\infty dk \int_0^\infty dk' |k' \rangle \left[ \int dq' \delta(k' - q') \times \langle \varphi_{q'} | J(f) \left( \int dq \delta(q - k) | \varphi_q \rangle \right) \right] \langle k | | \psi \rangle,
\]

but the delta functions \( \delta(k' - q') = \langle k'|q' \rangle \) and \( \delta(q - k) = \langle q|k \rangle \) can be “factored” out as

\[
\langle \psi | E_+ \Omega_{\delta}^1 J(f) \Omega_V E_+ | \psi \rangle = \frac{1}{2\pi} \langle \psi | \int_0^\infty dk \int_0^\infty dk' |k' \rangle \int dq' \langle \varphi_{q'} | J(f) \left( \int dq | \varphi_q \rangle \right) \langle k | | \psi \rangle,
\]

\[\times \left( \int dq | \varphi_q \rangle \langle q | \right) \langle k | | \psi \rangle,
\]

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and we can isolate the abstract operator by the use of the completeness relation to obtain
\[
E_+\Omega_V^\dagger J(f)\Omega_V E_+ = J_V(f) = \frac{1}{2\pi} \left[ E_+ \int dq' |q'\rangle \langle \varphi_{q'} | J(f) \int dq |\varphi_q\rangle \langle q| E_+ \right].
\] (6.3)

This expression is the operator expanded in terms of the interacting vector state. It highlights how the interacting operator differs from the free case. Having the linear operator is also the starting point for analytical perturbation theory. For a practical calculation such as the lowest eigenvalue, we will work with a basis of the Hilbert space.

For a general interaction, either exactly solvable or not, we can write our expectation value quantity, in position space, as
\[
\langle \psi | E_+\Omega_V^\dagger J(f)\Omega_V E_+ | \psi \rangle = \int dx \int dx' (\Omega_V E_+ \psi)^\ast (x') \left[ J(f)(x', x) \right] (\Omega_V E_+ \psi)(x),
\] (6.4)

which is expanded by
\[
\frac{1}{2\pi} \int_0^\infty dk \int_0^\infty dk' \tilde{\psi}^\ast (k') \tilde{\psi}(k) \int dx \int dx' \left( g^\ast_{k'}(x') + \int U(y')G^\ast_{k'}(x' - y')\varphi_{y'}^\ast(y')dy' \right)
\times J(f)(x', x) \left( g_k(x) + \int G_k(x - y)U(y)\varphi_k(y)dy \right),
\]
with the kernel \(J(f)(x', x)\) in position space. In order to solve this equation, we need the expression for \(J(f)(x', x)\), which can be obtained starting from \(\langle \psi | J(f) | \psi \rangle\) as
\[
\langle \psi | J(f) | \psi \rangle = \frac{1}{2} \langle \psi | Pf(X) + f(X)P | \psi \rangle = -\frac{i}{2} \int dy \psi^\ast (y) \left( f(y) \frac{\partial \psi(y)}{\partial y} + \frac{\partial f(y)}{\partial y} (f(y)\psi(y)) \right),
\]
which can also be rewritten as
\[
-\frac{i}{2} \int dy \psi^\ast (y) \left[ \int dy' f(y) \frac{\partial \delta(y - y')}{\partial y} \psi(y') + \frac{\partial f(y)}{\partial y} \delta(y - y') \psi(y') + f(y) \frac{\partial \delta(y - y')}{\partial y} \psi(y') \right],
\]
where we have used the trick of rewriting \(\psi(y) = \int \delta(y - y')\psi(y')dy'\). Hence, since \(\langle \psi | J(f) | \psi \rangle = \int dy \int dy'\psi^\ast (y') [J(f)(y', y)] \psi(y')\), we obtain
\[
J(f)(y, y') = -\frac{i}{2} \left[ 2f(y) \frac{\partial \delta(y - y')}{\partial y} + \frac{\partial f(y)}{\partial y} \delta(y - y') \right].
\] (6.5)

Our expectation value expression can, therefore, be written as
\[
\langle \psi | J_V(f) | \psi \rangle = \frac{1}{2\pi} \int_0^\infty dk \int_0^\infty dk' \tilde{\psi}^\ast (k') \tilde{\psi}(k) \int dx \int dx' \left( \varphi_{k'}^\ast(x')J(f)(x', x)\varphi_k(x) \right),
\] (6.6)
where we will denote the inner integrals by
\[
L(k', k) = \int dx \int dx' \left( \varphi_{k'}^\ast(x')J(f)(x', x)\varphi_k(x) \right).
\] (6.7)

For the lowest backflow eigenvalue expression, we do need to take the minimum of the (6.6) as
\[
\beta_V(f) = \frac{1}{2\pi} \int_0^\infty dk \int_0^\infty dk' \tilde{\varphi}^\ast (k') \tilde{\varphi}(k) L(k', k),
\] (6.8)
where we assume the existence of the lowest eigenvector \(|J_{\text{min}}\rangle\) of the operator \(J_V(f)\), for which the associated wavefunction, in momentum space, is denoted by \(\tilde{\varphi}(k)\). At the present moment, however, an explicit analytical solution for the lowest eigenvector is not known even in the free case [27,28].
6.1 Backflow in the presence of a delta-defect

Although the $\delta$ potential function is not a $L^{1+}(\mathbb{R})$-class potential (it is not a locally integrable function), it was shown in [7] that one can have a (rough) estimate of the lowest backflow eigenvalue, and the numerical results show that the $\delta$ potential is indeed a special case that also has a lower bound for its $\beta_V(f)$. The delta impurity (as a local defect) has the particularity of being a transition from the implementation of interaction by means of an explicit potential function to the the sewing conditions at its location. Specifically, the delta can either be explicitly used as $V(x) = \lambda \delta(x)$ in the calculations or through the use of the matching sewing conditions which divide (in one spatial dimension) the real line in left ($u$) and right ($v$) semi-infinite lines;

$$u(0) = v(0) := \psi(0), \quad v_x(0) - u_x(0) = \lambda \psi(0), \tag{6.9}$$

where the $u'$ and $v'$ are derivatives with respect to the spatial coordinate and the evaluation of the wavefunction $\psi$ at zero is understood in the right and left limit sense. As we want to introduce other type of defect, namely the jump-defect, which has no explicit expression as a potential function, but only a set of sewing conditions, the delta is the most natural and inspiring example to spend some efforts on before we analyse the jump-defect.

When our interacting model is not exactly solvable, or we want to use perturbation theory to expand the potential in powers of its associated parameter, it is a usual procedure to iterate the Lippmann-Schwinger equation multiple times up to the order we wish to analyse. However, as the $\delta$ is one of the few completely solvable examples, we are now interested in the calculation involving the full solution of the TISE without approximations.

Let $\varphi_k$ denote the solution for the TISE in the presence of a $\delta$-defect, we can work with derivatives in the weak sense as both $\varphi_k$ and its derivative $\partial_x \varphi_k$ are both locally integrable functions $\varphi_k \in L^1_{loc}(\mathbb{R}), \partial_x \varphi_k \in L^1_{loc}(\mathbb{R})$. The full time-dependent solution to the Schrödinger equation is denoted by

$$\varphi(x,t) = \int_{-\infty}^{\infty} dk \, \hat{g}(k) \exp(-iwt) \varphi_k(x) = \int_{-\infty}^{\infty} dk \, \hat{g}(k) \exp(-iwt) \left( \theta(-x) u_k(x) + \theta(x) v_k(x) \right), \tag{6.10}$$

where $\hat{g}$ is an arbitrary non-zero smoothly varying function used for producing the wave packet (superposition of improper states) as a proper square-integrable $L^2(\mathbb{R})$-solution. As we established before, we denote the solution at the left of the defect by $u$ and at the right by $v$. The time-independent scattering states in position basis are

$$u_k(x) = \exp(ikx) + \frac{\lambda}{2ik - \lambda} \exp(-ikx), \quad x < 0$$
$$v_k(x) = \left( \frac{2ik}{2ik - \lambda} \right) \exp(ikx), \quad x > 0, \tag{6.11}$$

where the reflection $R(k)$ and the transmission coefficient $T(k)$ for the $\delta$-defect are explicitly written. We want to concentrate our attention on the time-independent part $\varphi_k(x)$ composed of [6.11] and, for that, the inner integral (6.7) reads

$$L(k', k) = \int dx \int dx' \left[ \left( \theta(-x') u_{k'}^* + \theta(x') v_{k'}^* \right) J(f)(x', x) \theta(-x) u_k + \theta(x) v_k \right]. \tag{6.12}$$

Since the expression (6.5) for $J(f)(x', x)$ has a factor of $-i/2$, we absorb it by working with $2iL(k', k)$ instead. Each term is expanded by the insertion of the $J(f)(x', x)$ and simplified after integration.
Let us focus only on the spatial integrals, namely the kernel $2iL(k', k)$. There are four contributions which we denote by $uu, uv, vu$ and $vv$. The first contribution $(uu)$ to the kernel $2iL(k', k)$ is

$$\int dx \int dx' \theta(-x')u_{k'}^*(x') \left(2f(x') \frac{\partial \delta(x' - x)}{\partial x'} + \frac{\partial f(x')}{\partial x'} \delta(x' - x)\right) \theta(-x)u_k(x),$$

which, on integrating by parts, becomes

$$\int dx \int dx' \left\{-\frac{\partial}{\partial x'} \left(\theta(-x')u_{k'}^*(x')2f(x')\right) \delta(x' - x)\theta(-x)u_k(x) + \theta(-x')u_{k'}^*(x') \frac{\partial f(x')}{\partial x'} \delta(x' - x)\theta(-x)u_k(x)\right\},$$

and, after one integration is carried out, becomes

$$\int dx' \left\{\theta(-x')u_{k'}^*(x')2f(x') \frac{\partial}{\partial x'}(\theta(-x')u_k(x')) - \frac{\partial}{\partial x'} \left(\theta(-x')u_{k'}^*(x')\right) f(x')\theta(-x')u_k(x') - \theta(-x')u_{k'}^*(x')f(x')(\theta(-x')u_k(x'))\right\},$$

which, written in terms of $R(k)$, yields the following expression

$$\int dx' \left\{\theta(-x')u_{k'}^*(x')2f(x') \left(-\delta(x')\right) u_k(x') + \theta^2(-x')u_{k'}^*(x')2f(x')ik \left(\exp(ikx') - R(k) \exp(-ikx')\right)ight.$$  
$$+ \left(\delta(x')\right)u_{k'}^*(x')f(x')\theta(-x')u_k(x')$$  
$$- \theta^2(-x')ik\left(-\exp(-ikx') + R^*(k') \exp(ikx')\right) f(x')u_k(x')$$  
$$- \theta(-x')u_{k'}^*(x')f(x')(\delta(x')u_k(x'))$$  
$$+ \theta^2(-x')u_{k'}^*(x')f(x')(\exp(ikx') - R(k) \exp(-ikx'))\right\},$$

where we have used $\delta(x) = \delta(-x)$. Noting that $\theta^2(x') = \theta(x')$ and simplifying terms, it becomes the first contribution

$$i(k + k') \int dx' f(x')\theta(-x') \exp(ikx'(k - k')) + i(k + k') \int dx' f(x')\theta(-x')R(k) \exp(-ikx'(k + k'))$$  
$$+ i(k - k') \int dx' f(x')\theta(-x')R^*(k') \exp(ikx'(k + k'))$$  
$$- i(k + k') \int dx' f(x')\theta(-x')R^*(k')R(k) \exp(-ikx'(k - k')).$$

The second contribution $(uv)$ to the kernel $2iL(k', k)$ is

$$\int dx \int dx' \theta(-x') \left(\exp(-ikx') + R^*(k') \exp(ikx')\right) \left(2f(x') \frac{\partial \delta(x' - x)}{\partial x'} + \frac{\partial f(x')}{\partial x'} \delta(x' - x)\right) \times (\theta(x)T(k) \exp(ikx)), $$

which, integrated by parts, gives

$$\int dx' \left\{-\frac{\partial}{\partial x'} \left(\theta(-x') \left(\exp(-ikx') + R^*(k') \exp(ikx')\right) 2f(x')\right) \theta(x')T(k) \exp(ikx')$$  
$$+ \left(\theta(-x') \left(\exp(-ikx') + R^*(k') \exp(ikx')\right)\right) \frac{\partial f(x')}{\partial x'} \theta(x')T(k) \exp(ikx')$$  
$$- \theta(-x') \left(\exp(-ikx') + R^*(k') \exp(ikx')\right) f(x') \frac{\partial}{\partial x'} \left(\theta(x')T(k) \exp(ikx')\right)\right\}.$$
Now, it can then be written as the sum of three terms
\[
\int dx' \theta(-x') \left( \exp(-ik'x') + R^*(k') \exp(ik'x') \right) 2f(x') \delta(x'-x) + R(k) \exp(-ikx') \\
+ \int dx' \delta(x') \left( \exp(-ik'x') + R^*(k') \exp(ik'x') \right) f(x') \theta(x') T(k) \exp(ikx') \\
- \int dx' \theta(-x') \left( \exp(-ik'x') + R^*(k') \exp(ik'x') \right) f(x') \delta(x') T(k) \exp(ikx'),
\]
where we use \( \theta(-x') + \theta(x') = 1 \), without any problem at the origin as \( \varphi_k \) is continuous, to obtain the second term given by
\[
\int dx' f(x') \left( \exp(-ik'x') + R^*(k') \exp(ik'x') \right) T(k) \exp(ikx') \delta(x') \\
= \left( \frac{-2ik'}{-2ik' - \lambda} \right) \left( \frac{2ik}{2ik - \lambda} \right) f(0).
\]

Now, the third contribution \((vu)\) to the kernel \( 2iL(k', k) \) is
\[
\int dx \int dx' \theta(x') \left( T^*(k') \right) \exp(-ik'x') \left( 2f(x') \frac{\partial \delta(x' - x)}{\partial x'} + \frac{\partial f(x')}{\partial x'} \delta(x' - x) \right) \\
\times \left( \theta(-x') \left( \exp(ikx) + R(k) \exp(-ikx) \right) \right),
\]
which, on integrating by parts, becomes
\[
\int dx' \left\{ -\frac{\partial}{\partial x'} \left( \theta(x') \left( T^*(k') \exp(-ik'x') \right) \right) 2f(x') \theta(x') \left( \exp(ikx') + R(k) \exp(-ikx') \right) \\
+ \theta(x') \left( T^*(k') \exp(-ik'x') \right) \frac{\partial f(x')}{\partial x'} \left( \theta(-x') \left( \exp(ikx') + R(k) \exp(-ikx') \right) \right) \\
- \theta(x') \left( T^*(k') \exp(-ik'x') \right) f(x') \frac{\partial}{\partial x'} \left( \theta(-x') \left( \exp(ikx') + R(k) \exp(-ikx') \right) \right) \right\}
\]
and it is simplified to become the third term given by
\[
\int dx' \left( -f(x') \right) \left( \exp(ikx') + R(k) \exp(-ikx') \right) \delta(x') \\
= \left( \frac{2ik'}{-2ik' - \lambda} \right) \left( \frac{2ik}{2ik - \lambda} \right) f(0).
\]

That is exactly the second contribution \((6.14)\) with opposite sign. Thus, the second and the third term cancel out. The fourth contribution \((vv)\) to the kernel \( 2iL(k', k) \) is
\[
\int dx \int dx' \theta(x') \left( T^*(k') \right) \exp(-ik'x') \left( 2f(x') \frac{\partial \delta(x' - x)}{\partial x'} + \frac{\partial f(x')}{\partial x'} \delta(x' - x) \right) \\
\times \left( \theta(x) \left( T(k) \right) \exp(ikx) \right),
\]
which, integrated by parts, gives
\[
\int dx \int dx' \left\{ -\frac{\partial}{\partial x'} \left( \theta(x') \left( T^*(k') \right) \exp(-ik'x') \right) 2f(x') \delta(x' - x) \left( \theta(x) \left( T(k) \right) \exp(ikx) \right) \\
+ \theta(x') \left( T^*(k') \right) \exp(-ik'x') \frac{\partial f(x')}{\partial x'} \delta(x' - x) \theta(x) \left( T(k) \right) \exp(ikx) \right\}
\]
Finally, we can write down the lowest backflow eigenvalue as

\[ i(k + k') T^*(k') T(k) \int dx' f(x') \theta(x') \exp(ix'(k - k')). \]  

(6.16)

Finally, we can write down the lowest backflow eigenvalue as

\[ \beta_V(f) = \frac{1}{2\pi} \int_0^\infty dk \int_0^\infty dk' \tilde{\delta}^*(k') \tilde{\delta}(k) L(k', k), \]

with the Hermitian kernel

\[ 2L(k', k) = (k + k') \int_{-\infty}^0 dx' f(x') \exp(ix'(k - k')) \]
\[ + \frac{\lambda(k' - k)}{(2ik - \lambda)} \int_{-\infty}^0 dx' f(x') \exp(-ix'(k + k')) \]
\[ - \frac{\lambda(k - k')}{(2ik') + \lambda} \int_{-\infty}^0 dx' f(x') \exp(ix'(k + k')) \]
\[ + \frac{\lambda^2(k + k')}{(2ik' + \lambda)(2ik - \lambda)} \int_{-\infty}^0 dx' f(x') \exp(-ix'(k - k')) \]
\[ - \frac{4(k + k')}{(2ik' + \lambda)(2ik - \lambda)} \int_{-\infty}^0 dx' f(x') \exp(ix'(k - k')). \]  

(6.17)

Interestingly, the term proportional to \( f(0) \), which involves the evaluation of the test function at the defect’s location, was cancelled out. That term would be a contribution coming purely from the defect. In the case of the jump-defect, we will see in Sec. 6.2 that, if we insist on assigning a value to the wavefunction at the origin, this term is non-zero, due to discontinuity, and it has a direct connection with the conservation law of the total momentum. In the \( \delta \)-defect case, as mentioned before in Sec. 5 there was no additional term we could have added for redefining the total momentum in order to keep it conserved. Coincidentally, what would be a possibly equivalent additional fixing term, coming purely from the defect, for that purpose is zero. The calculation above considered the full time-independent solution \( \varphi_k \) including its value at the origin, where the defect was placed, but the jump-defect case will be treated differently.

For the calculation of the lowest backflow eigenvalue \( \beta_V(f) \), as the eigenfunction \( \tilde{\delta}(k) \) is not analytically known, we need to rely upon numerical calculations in order to plot the result. Some graphs for the \( \delta \)-defect case can be found in Sec. 7 along with some details of the numerical methods.

### 6.2 Backflow in the presence of a jump-defect

Now we consider the backflow calculation for the the jump-defect. However, we have to keep in mind that now our wavefunction \( \varphi_k \) has a jump discontinuity at the origin. Specifically, in the \( \delta \)-defect case, the wavefunction and its derivative are locally integrable, that is \( \varphi_k \in L^1_{\text{loc}}(\mathbb{R}) \) and \( \partial_x \varphi_k \in L^1_{\text{loc}}(\mathbb{R}) \). In the jump-defect case, just the wavefunction is locally integrable \( \varphi_k \in L^1_{\text{loc}}(\mathbb{R}) \) but not its derivative. Such discontinuities may cause the presence of undefined terms when multiplied by distributions. By avoiding the origin, we avoid this undesirable problem.

Given that now \( \varphi_k \) denotes the the solution for the TISE in the presence of a jump-defect. Let us write the full time-dependent jump solution to the Schrödinger equation as

\[ \varphi(x, t) = \int_{-\infty}^\infty dk \tilde{g}(k) \exp(-iwt) \varphi_k(x) \]  

(6.18)
with \( \tilde{g} \) an arbitrary non-zero smoothly varying function, and the time-independent scattering states in position basis are given by

\[
\varphi_k(x) = \begin{cases}
  u_k(x) = \exp(ikx), & x < 0 \\
  v_k(x) = \frac{k + i\alpha}{k - i\alpha} \exp(ikx), & x > 0,
\end{cases}
\] (6.19)

where the reflection coefficient \( R(k) = 0 \) and the transmission coefficient \( T(k) \) for the jump-defect is explicit. While the jump-defect connects the theory on the left of the origin \((x = 0)\) with the theory on the right, we do not assign a definite value for the wavefunction at the defect’s position. Because of that, we do not have all the corresponding four contributions calculated in Sec. 6.1 but only two of them, \( uu \) and \( vv \). In fact, if we do a similar calculation as in Sec. 5.2, we only split our integration \((\text{6.7})\) in left part \(-\infty < x < 0\) and right part \(0 < x < \infty\), corresponding to contributions purely from \( u \) and purely from \( v \), respectively, and we avoid crossing the discontinuity.

We concentrate our attention to the time-independent part \( \varphi_k(x) \) composed of \((6.19)\) and, for that, the asymptotic backflow constant of the jump-defect is

\[
\beta_v(f) = \frac{1}{2\pi} \int_0^\infty dk \int_0^\infty dk' \tilde{f}(k') \tilde{f}(k) \int dx \int dx' \left[ \theta(-x')u_{k'}^*(x')J(f)(x'), x)\theta(-x)u_k(x) + \theta(x')v_{k'}^*(x')J(f)(x', x)\theta(x)v_k(x) \right].
\] (6.20)

Thus we have two contributions where each term is expanded by the insertion of the \( J(f)(x', x) \) and simplified after integration. Let us focus only on the spatial integrals, namely the kernel \( \beta_v(f) \).

The first contribution \((uu)\) is

\[
\int dx \int dx' \theta(-x'u_{k'}^*(x') \left( 2f(x') \frac{\partial\delta(x' - x)}{\partial x'} + \frac{\partial f(x')}{\partial x'} \delta(x' - x) \right) \theta(-x)u_k(x),
\]

which is integrated by parts to become

\[
\int dx \int dx' \left\{ -\frac{\partial}{\partial x'} \left[ \theta(-x'u_{k'}^*(x')2f(x') \delta(x' - x)\theta(-x)u_k(x) \right. \\
- \theta(x'u_{k'}^*(x') \frac{\partial f(x')}{\partial x'} \delta(x' - x)\theta(-x)u_k(x) \right] \right. \\
+ \theta(x'u_{k'}^*(x') \frac{\partial f(x')}{\partial x'} \delta(x' - x)\theta(-x)u_k(x) \left. \right\}.
\]

After one integration is carried over, it gives

\[
\int dx' \left\{ \theta(-x'u_{k'}^*(x')2f(x')u_k(x) \left[ -\delta(x') + \theta(-x')ik \right] - \theta(-x')(ik' - i\delta(x')f(x')u_k(x') \\
+ \left( \delta(x')u_{k'}^*(x')f(x')\theta(-x')u_k(x') \right) - \theta(-x')u_{k'}^*(x')f(x')u_k(x') \left( -\delta(x') + ik\theta(-x') \right) \right\},
\]

where we have used \( \partial\theta(-x')/\partial x' = -\delta(x') \). Hence, the first contribution term is

\[
i(k + k') \int_{-\infty}^0 dx' \exp(ix'(k - k')) f(x') \). (6.21)
\]

The second contribution \((vv)\) to \(2iL(k', k)\) will involve similar calculations. It is given by

\[
\int dx \int dx' \theta(x')v_{k'}^*(x') \left( 2f(x') \frac{\partial\delta(x' - x)}{\partial x'} + \frac{\partial f(x')}{\partial x'} \delta(x' - x) \right) \theta(x)v_k(x),
\]
which, on integrating by parts, yields

$$\int dx \int dx' \left\{ -\frac{\partial}{\partial x'} (\theta(x') T^*(k') \exp(-ik'x') 2f(x') \delta(x' - x) \theta(x) T(k) \exp(ikx) \\
+ \theta(x') T^*(k') \exp(-ik'x') \frac{\partial f(x')}{\partial x'} \delta(x' - x) \theta(x) T(k) \exp(ikx) \right\}. $$

After one integration is carried out, it becomes

$$i(k + k') \int_0^\infty dx' f(x') T^*(k') T(k) \exp(ikx(k - k')). \tag{6.22}$$

Finally, we can write the lowest backflow eigenvalue of the operator $J_V(f)$ as

$$\beta_V(f) = \frac{1}{2\pi} \int_0^\infty dk \int_0^\infty dk' \tilde{J}^*(k') \tilde{J}(k) L(k', k), \tag{6.23}$$

with the kernel

$$2L(k', k) = (k + k') \int_{-\infty}^0 dx' f(x') \exp(ikx(k - k')) \\
+ \frac{(kk' + i\alpha(k' - k) + \alpha^2)(k + k')}{(k' + i\alpha)(k - i\alpha)} \int_0^\infty dx' f(x') \exp(ikx(k - k')),$$

which is a Hermitian kernel, and $\tilde{J}(k)$ is the eigenfunction, in momentum space, associated with the lowest eigenvalue of the integral operator $J_V(f)$. This expression (6.23) was worked out for the non-conserved situation where we have not introduced any fixing term to conserve the probability current. In physical situations, we are interested in conserved quantities, and our jump-defect was specially devised for allowing conservation laws.

In Sec. 5, we have established the condition for having a conserved total momentum $P_c$ associated with a particular momentum density. It is easily shown that the probability current is intimately related to the total momentum since

$$\int j\psi(x) dx = \frac{\langle \hat{P} \rangle}{m}, \tag{6.24}$$

where $\hat{P}$ is the momentum operator, and we can, therefore, interchangeably, refer to either momentum density or, equivalently, probability current density. However, we need to point out the fact that, from the Lagrangian description to the quantum mechanics, we have a minus sign and a constant factor of difference between the momentum density (5.18) and the probability current density (2.3). That is, of course, fixed by suitable re-scaling of the physical units. For the conservation analysis, we adjust the definition of the momentum density derived from the Lagrangian to match the common definition used in usual quantum mechanics which is a probability current density. Such an adjustment is necessary in order to find the precise fixing term that shall be added to our non-conserved probability current to guarantee its conservation, since the numerical calculations are sensitive to that. In particular, Eq. (5.20) becomes

$$P_c := P + \frac{i}{2}(\bar{u}v - \bar{v}u) \bigg|_{x=0}, \tag{6.25}$$
which needs to be written in terms of a kernel, in momentum space, such that the extra term can be added to the kernel \( L(k', k) \) in (6.7). From (6.18), we can write the time-independent solution at the left of the defect as

\[
|u\rangle = \frac{1}{\sqrt{2\pi}} \int dk \, |u_k\rangle \langle k|,
\]

\[
\langle u| = \frac{1}{\sqrt{2\pi}} \int dk' \langle g|k'\rangle \langle u_{k'}|,
\]

(6.26)

and similarly for the solution \( v \) at the right of the defect. Hence, (6.25) will give, after introducing the required projectors \( E_+ \) for right-movers,

\[
\frac{i}{2} E_+ (\bar{u}v - \bar{v}u) E_+ \bigg|_{x=0} = \frac{i}{4\pi} \int_0^\infty \int_0^\infty dk' dk \, \bar{\psi}(k') \bar{\psi}(k) \left( \frac{2i\alpha(k + k')}{(k - i\alpha)(k' + i\alpha)} \right) f(0).
\]

(6.27)

The term in parenthesis has equal absolute value to the corresponding one in (6.30) but different sign. Note that Sec. 5 has no reference to the smearing process with a positive test function \( f \) for producing spatial averaged quantities as introduced in our discussion of the quantum backflow. The term in parenthesis has equal absolute value to the corresponding one in (6.30) but different sign.

From this one only needs to take the infimum over the functions \( \psi \), as in (3.1), in order to obtain the lowest backflow eigenvalue \( \beta_V(f) = \inf \langle J_{cV}(f) \rangle_\psi \) of the probability current operator \( J_{cV}(f) \) in the presence of the jump-defect. Once we have simplified the kernel, we again need to rely upon the numerical calculations as the eigenfunction \( \tilde{\psi}(k) \) is not analytically known.

As we mentioned, non-removable discontinuities pose difficulties to the products with Dirac measure \( \delta \), which is a Radon measure rather than a locally integrable function. In particular, a distributional product such as

\[
\langle \varphi \delta, f \rangle = \langle \delta, \varphi f \rangle = \varphi(0) f(0),
\]

with a function \( \varphi \) discontinuous at the origin and test function \( f \), is undefined. Moreover, if \( \varphi \) was at least such that the product \( \varphi f \in C^0_c(\mathbb{R}) \), one could work with functionals on the space of continuous functions with compact support \( C^0_c(\mathbb{R}) \), but would have problem with derivatives. Nevertheless, if we had insisted on following the same calculations as done in the \( \delta \)-defect case, crossing the origin
as a range of integration, and assigning a particular value for the Heaviside function at the origin, such as \( \theta(0) = 1/2 \), in order to have a meaningful way to interpret the difficulties caused by the discontinuity, we would get an extra non-zero term, corresponding to the combination of \( uv \) and \( vu \) contributions,

\[
\frac{2i\alpha(k + k')}{(k' + i\alpha)(k - i\alpha)} f(0)
\]

to the kernel \( 2tL(k', k) \). In the \( \delta \)-defect case with a continuous wavefunction, the corresponding contribution is zero as previously discussed. This term \( (6.30) \) would come after a delta \( \delta(x') \) is integrated out and it has, therefore, support at the origin. In the Lagrangian \( (5.3) \), we have made it clear that what is defined only at the defect’s position is a contribution purely from the defect itself rather than the theory of the left or the theory of the right semi-infinite lines. Incidentally, it is curious that this term \( (6.30) \), apart from a sign, is exactly the fixing term we need for the conserved probability current.

7 Results

7.1 Numerical calculations

We have adapted the basic numerical methods of \[7\] (where one can find the essential numerical description with a Java program) for a FORTRAN 90 program with some changes in regards to the method of integration and the calculation of the lowest eigenvalue of a complex Hermitian matrix \( M \). For that, the libraries used were QUADPACK \[29\] and EISPACK \[30\], respectively. Here is a summary of the meaning of each relevant variable to understand the plots presented in this work.

For the numerical calculations, the discretization of an infinite-dimensional operator \( T \) on \( L^2(\mathbb{R}_+, dk) \) with kernel \( K \) given by

\[
K(k', k) = \frac{1}{2\pi} L(k', k) = \frac{i}{4\pi} \int dx f(x) (\partial_x \varphi_k^*(x) \varphi_k(x) - \varphi_k^*(x) \partial_x \varphi_k(x))
\]

into a \( N \times N \)-matrix \( M \) is characterized by the parameter \( N \), the number of equally spaced steps which divide the momentum interval \([0, P_{\text{cutoff}}]\), where the upper-limit cutoff of the integrations \( (6.6) \) in \( k \) and \( k' \) is denoted by \( P_{\text{cutoff}} \). The components of such a matrix can be written as

\[
M_{ij} = \langle \psi_i, T \psi_j \rangle = \int dk' \int dk \tilde{\psi}_i(k') K(k', k) \tilde{\psi}_j(k) \approx \frac{P_{\text{cutoff}}}{N} K(k_i, k_j),
\]

where \( \tilde{\psi}_i (i \in \mathbb{N} | i = (0, \ldots, N - 1)) \) are orthonormal step functions supported on the corresponding interval, and mid-points \( k_i = (i + 1/2) (P_{\text{cutoff}}/N) \). The adoption of a cutoff \( P_{\text{cutoff}} \) is consistent with the fact that the lowest backflow eigenvector decays at large momentum.

The positive test function chosen for the spatial average of the probability current was a Gaussian

\[
f(x) = \frac{1}{\sigma \sqrt{2\pi}} \exp \left( -\frac{(x - x_0)^2}{2\sigma^2} \right),
\]

with width \( \sigma = 0.1 \) centered at the position \( x_0 \) of measurement where a spatially extended detector is located and supported on the interval \( x \in [x_0 - 8\sigma, x_0 + 8\sigma] \). Therefore, for each \( x_0 \) we have a matrix \( M \) for which the lowest eigenvalue needs to be calculated. We have restricted our position of measurement to \( x_0 \in [-2, 2] \) in the case of the delta-defect and to \( x_0 \in [-1, 1] \) in the jump-defect case because, as we move away from the jump-defect’s location, the lowest eigenvalue approaches the free case value \( \beta_0(f) \approx -0.241 \) for this choice of test function.
Although essentially the same, the numerical analysis done for the conserved probability current involves an extra step, which is the addition of a fixing term to the non-conserved one such that the fixing term allows the conservation law to hold. Specifically, the fixing term in the presence of a jump-defect,

\[- \frac{1}{2\pi} \frac{\alpha(k + k')}{(k - i\alpha)(k' + i\alpha)} f(0), \tag{7.4}\]

is added to \(K(k',k)\) to compose a new kernel denoted by \(K_c(k',k)\), which is associated with a conserved quantity. The discretization process now involves that new kernel, and the FORTRAN program is asked to calculate the lowest eigenvalue \(\beta_V(f)\) of the corresponding \(N \times N\)-matrix \(M\).

### 7.2 δ-defect case

The backflow calculation in the presence of a δ-defect was analysed in Sec. 6.1, and the corresponding kernel was analytically simplified to (6.17). Here, we present some numerical results for it. All the graphs refer to the probability current operator smeared with a Gaussian test function. Specifically, the graphs show the lowest eigenvalue against the position \(x_0\) where the Gaussian function is centered at, see the following Fig. 7.1 and Fig. 7.2 where we vary the parameter \(\lambda\) for displaying its behaviour under the strengthening or weakening of the interaction. In particular, in the limit \(\lambda \to \pm \infty\), it becomes a purely reflecting situation, equivalent to a boundary theory. Naturally, when \(\lambda \to 0\) the interaction-free case is obtained. For the free case, the lowest eigenvalue is represented by the line \(\beta_0(f) \approx -0.241\). As shown by Fig. 7.1, there is a maximum of the lowest eigenvalue, in the attractive case, close to the defect’s location when \(|\lambda| < 1\). Increasing its absolute value \((|\lambda| > 1)\) causes the attractive and repulsive cases to approach each other. In order to make comparisons with the numerical results reported in 7, we replace \(\lambda\) by \(2\lambda\) in (6.9), obtaining the reflection and transmission coefficients (6.11) given by \(R(k) = (\lambda/(ik - \lambda))\) and \(T(k) = (ik/(ik - \lambda))\).

Additionally to the two-dimensional plots, we have varied the parameter \(\lambda\) for displaying a three-dimensional picture, Figure 8.1, of how the lowest backflow eigenvalue is affected in the presence of the δ-defect. This can be compared to the jump-defect case in the Appendix.
Figure 7.1: Lowest backflow eigenvalue of the current operator. For which (a) $|\lambda| = 0.5$. (b) $|\lambda| = 1.0$. 
Backflow in delta defect: $N = 2000$, $P_{\text{cutoff}} = 200$, $|\lambda| = 5.00$

(a)

Backflow in delta defect: $N = 2000$, $P_{\text{cutoff}} = 200$, $|\lambda| = 10.00$

(b)

Figure 7.2: Lowest backflow eigenvalue of the current operator. For which (a) $|\lambda| = 5.0$. (b) $|\lambda| = 10.0$. 

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7.3 Jump-defect case

For being a purely-transmitting defect, the solution $\varphi_k$ with asymptotic incoming right-mover maintains itself as a right-mover also after scattering off the defect. This is not the case of the $\delta$-defect that has a mixture of right-mover and left-mover as a result of being scattered by the defect. In this sense, the reflectionless Pöschl-Teller potential is more similar to the jump-defect than the $\delta$. However, for the Pöschl-Teller, the backflow effect is smaller inside the interaction region than in the free case [7]. That this is not true in the jump-case, can be seen from the figures in this section. In fact, at the defect’s location, the effect can be either smaller or bigger.

Several graphs of the backflow lowest eigenvalue in the presence of the jump-defect were plotted below. All the graphs refer to the probability current operator smeared with a Gaussian test function. Specifically, as mentioned in the $\delta$-defect case, the graphs show the lowest eigenvalue against the position $x_0$ where the Gaussian function is centered at. Our main freedom to be tuned is the parameter $\alpha$ corresponding to the strength of the defect. Unlike the Dirac $\delta$-defect, or other explicit potential functions, the jump-defect has a parameter that can not be clearly distinguished as attractive or repulsive according to its sign, being either positive or negative, respectively. As particular cases, $\alpha = 0$ gives the expected free case represented by a constant horizontal line $\beta_0(f) \approx -0.241$ and the limiting cases $\alpha \to \pm \infty$ also approach the free backflow eigenvalue $\beta_V(f) \to \beta_0(f)$. We already expected this as the solutions $\varphi_k$ for the limiting cases $\alpha \to \pm \infty$ are related to the free case by only a global phase, but the probability current density has products of the solution wavefunction with its complex conjugated spatial derivative. Thus, in the limit, their lowest backflow eigenvalue is the same as the free case.

Initially, for small absolute values of the parameter $\alpha$, the lowest backflow eigenvalues has some symmetry between the positive and negative parameter values, Fig. 7.3. Slightly increasing $|\alpha|$, $\beta_V(f)$ of the associated conserved probability current starts to show a distinctly different behaviour between the positive and the negative values of $\alpha$, see Fig. 7.4. As its absolute value sufficiently increases, the graphs become more similar in terms of the magnitude of the lowest backflow eigenvalue. However, as indicated by the plots, both positive $\alpha > 0$ and negative $\alpha < 0$ seem to unveil some stationary points, and, in some cases while a positive parameter shows three of these points, the corresponding negative parameter can show up to five stationary points, Fig. 7.5. With successive increases of the parameter’s absolute value $|\alpha|$, the graphs tend to become more similar again. In particular, both positive and negative values show the same number of stationary points, though when one has a minimum the other one has a maximum and vice-versa, Fig. 7.6. Whilst the non-conserved current develops a persistent trough for both positive and negative parameters, the conserved one develops a mixture of trough and bumps as shown by Fig. 7.4, 7.5 and 7.6.

Additionally to the two-dimensional plots, we have varied the parameters for displaying a three-dimensional picture of how the lowest backflow eigenvalue is affected in the presence of the jump-defect. For comparison, we have plotted both cases, that is, before conservation Fig. 8.2 and Fig. 8.3 and after fixing the conservation Fig. 8.4 and Fig. 8.5 of the probability current operator. All these can be found in the Appendix.
Figure 7.3: Lowest backflow eigenvalue of the current operator. Red/blue refer to the non-conserved probability current. Yellow/green refer to the conserved one. (a) $|\alpha| = .10$. (b) $|\alpha| = .20$. 
Figure 7.4: Lowest backflow eigenvalue of the current operator. Red/blue refer to the non-conserved probability current. Yellow/green refer to the conserved one. (a) $|\alpha| = 1.0$. (b) $|\alpha| = 4.0$. 

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Figure 7.5: Lowest backflow eigenvalue of the current operator. Red/blue refer to the non-conserved probability current. Yellow/green refer to the conserved one. (a) $|\alpha| = 9.0$. (b) $|\alpha| = 10.0$. 

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Figure 7.6: Lowest backflow eigenvalue of the current operator. Red/blue refer to the non-conserved probability current. Yellow/green refer to the conserved one. (a) $|\alpha| = 20$. (b) $|\alpha| = 50$. 

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Figure 7.7: Lowest backflow eigenvalue of the current operator. Red/blue refer to the non-conserved probability current. Yellow/green refer to the conserved one. (a) $|\alpha| = 200$. (b) $|\alpha| = 1000$.  

(a) $|\alpha| = 200$

(b) $|\alpha| = 1000$. 

Table: 

| α < 0 | α > 0 | α < 0 | α > 0 |
|-------|-------|-------|-------|
|       |       |       |       |

Diagram: 

- Plot (a): $|\alpha| = 200$
- Plot (b): $|\alpha| = 1000$
8 Concluding remarks

The quantum energy inequalities, both in QM and QFT, are conditions upon how much spatially and temporally averaged energy densities and fluxes are bounded below. Inequalities for free theories were explored before, but there is no general result for interacting models even in the QM theory. The case of the probability flux inequality is also a quantum inequality and it is called quantum backflow. For the interaction-free situation, that the backflow effect is limited in space can be seen from the sharp Gårding inequalities as shown in [11]. The extension of backflow to scattering situations in short-range potentials was established in [7], where the interacting potential function is assumed to be in the $L^{1+}(\mathbb{R})$-class.

This work focused on the spatial average probability current operator in the case of a quantum mechanical system in the presence of a local impurity, the discontinuous and purely transmitting jump-defect. As a similar interesting case, we also presented the $\delta$-defect results for comparison. The lowest averaged backflow eigenvalue in the presence of a jump-defect was shown to be spatially constrained even though it has no explicit potential function to be classified in the $L^{1+}(\mathbb{R})$-class. In particular, there is a low bound on the spatial extent of the backflow effect for both the non-conserved and conserved current operator. Whilst the maximum amount of backflow in the presence of a $\delta$-defect was also found to be bounded, a striking difference between the $\delta$-defect case and the jump-defect is that the lowest eigenvalue can get increasingly negative as the parameter $\lambda \to \pm \infty$ on the left of a $\delta$-defect and tends to zero on the right of it, but, in the case of the jump-defect, the lowest eigenvalue on regions sufficiently far from its location, both on the left and on the right, is simply equal to the interaction-free situation with asymptotic backflow constant $\beta_0(f) \approx -0.241$.

In a field theory of physical interest, we want to keep not only the energy conserved but also the momentum. Equivalently in the quantum mechanics setup, we want to keep the probability current conserved together with the energy and the probability density. In a massless relativistic theory, the generators with respect to space and time translation, momentum and energy, respectively, are proportional to each other. A right-moving field at speed of light has both Hamiltonian $H$ and momentum operator $\hat{P}$ positive. Any positive condition, inequality, on the former is, therefore, also the same on the latter. In our non-relativistic massive case in the presence of a defect, an energy inequality is not the same as the backflow inequality, but they may be intimately related.

While the energy is conserved even in the presence of interaction, the conservation of probability current in QM is easily violated by an interaction which breaks the translational symmetry. The jump-defect, in contrast to the $\delta$-defect, which breaks the conservation of the probability current, allows a fixing term, a contribution purely from the defect, to be added in order to maintain its conservation. In this sense, it might be a good model for investigating the relation between these different quantum inequalities.

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Appendix: Three-dimensional plot

Here you find three-dimensional plots displaying the lowest eigenvalue of the corresponding current operator as the defect parameter and the position of measurement $x_0$, which is the center of the averaging Gaussian function $f$, change. In each case, for the jump-defect, we have plotted a version which runs over a large range of the defect parameter and another one which runs over a smaller range for capturing some local details. In the $\delta$-defect case, this was not necessary and we only plotted it once.
Figure 8.1: Probability current lowest eigenvalue for δ-defect, $P_{\text{cutoff}} = 200$, $N = 2000$. 
Figure 8.2: Probability current lowest eigenvalue, $P_{\text{cutoff}} = 200$, $N = 2000$. 

$\gamma_{\sigma}$
Figure 8.3: Probability current lowest eigenvalue, $P_{\text{cutoff}} = 200$, $N = 2000$. 
Figure 8.4: Conserved probability current lowest eigenvalue, $P_{\text{cut-off}} = 200$, $N = 2000$.
Figure 8.5: Conserved probability current lowest eigenvalue, $P_{\text{mod}} = 200, N = 2000$. 

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