Abstract

In a recent paper, Dixit et al. [Acta Arith. 177 (2017) 1–37] posed two open questions whether the integral

\[ \hat{J}_k(\alpha) = \int_0^\infty \frac{xe^{-\alpha x^2}}{e^{2\pi x} - 1} \, _1F_1(-k, \frac{3}{2}; 2\alpha x^2) \, dx \]

for \( \alpha > 0 \) could be evaluated in closed form when \( k \) is a positive even and odd integer. We establish that \( \hat{J}_k(\alpha) \) can be expressed in terms of a Gauss hypergeometric function and a ratio of two gamma functions, together with a remainder expressed as an integral. An upper bound on the remainder term is obtained, which is shown to be exponentially small as \( k \) becomes large when \( \alpha = O(1) \).

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1. Introduction

In the first of his letters to Hardy [5], Ramanujan gave the formula

\[ I(\alpha) := \alpha^{-1/4} \left( 1 + 4\alpha \int_0^\infty \frac{xe^{-\alpha x^2}}{e^{2\pi x} - 1} \, dx \right) = \beta^{-1/4} \left( 1 + 4\beta \int_0^\infty \frac{xe^{-\beta x^2}}{e^{2\pi x} - 1} \, dx \right), \]

where \( \alpha\beta = \pi^2 \), and in [6] obtained the approximate evaluation

\[ I(\alpha) \simeq \left( \frac{1}{\alpha} + \frac{1}{\beta} + \frac{2}{3} \right)^{1/4}. \quad (1.1) \]

This approximation is found to be good for small and large values of \( \alpha \). A proof of this result was given in [1], where the asymptotic expansion

\[ I(\alpha) \sim \frac{1}{\alpha^{1/4}} + \frac{\alpha^{3/4}}{6} - \frac{\alpha^{7/4}}{60} + \cdots \quad (\alpha \to 0) \]

was obtained.

In a recent paper, Dixit, Roy and Zaharescu [2] established an analogous formula for the integral

\[ \hat{J}_k(\alpha) := \int_0^\infty \frac{xe^{-\alpha x^2}}{e^{2\pi x} - 1} \, _1F_1(-k, \frac{3}{2}; 2\alpha x^2) \, dx, \]
where $\text{F}_1$ denotes the confluent hypergeometric function and $k$ is a positive integer. They showed that [2, (1.25), (1.27)]

$$\alpha^{-1/4} \text{F}_1(-2k, 1; \frac{3}{2}; 2) + 4\alpha^{3/4} \hat{J}_{2k}(\alpha) = \beta^{-1/4} \text{F}_1(-2k, 1; \frac{3}{2}; 2) + 4\beta^{3/4} \hat{J}_{2k}(\beta)$$

and

$$\alpha^{-1/4} \text{F}_1(-2k-1, 1; \frac{3}{2}; 2) + 4\alpha^{3/4} \hat{J}_{2k+1}(\alpha) = -\beta^{-1/4} \text{F}_1(-2k-1, 1; \frac{3}{2}; 2) - 4\beta^{3/4} \hat{J}_{2k+1}(\beta)$$

when $\alpha\beta = \pi^2$, where $\text{F}_1$ denotes the Gauss hypergeometric function. In the particular case $\alpha = \beta = \pi$, (1.2) yields the beautiful exact evaluation [2, Cor. 1.8]

$$\hat{J}_{2k+1}(\pi) := \int_0^{\infty} \frac{xe^{-\pi x^2}}{e^{2\pi x} - 1} \text{F}_1(-2k-1; \frac{3}{2}; 2\pi x^2) dx = -\frac{1}{4\pi} \text{F}_1(-2k-1, 1; \frac{3}{2}; 2)$$

(1.3)

for $k = 0, 1, 2, \ldots$. In addition, they gave the approximation [2, (1.26)]

$$4\alpha^{3/4} \int_0^{\infty} \frac{xe^{-\alpha x^2}}{e^{2\pi x} - 1} \text{F}_1(-2k; \frac{3}{2}; 2\alpha x^2) dx \simeq \text{F}_1(-2k, 1; \frac{3}{2}; 2) \left\{ -\alpha^{-1/4} + \left( \frac{1}{\alpha} + \frac{1}{\beta} + \frac{2}{3 \cdot \text{F}_1(-2k, 1; \frac{3}{2}; 2)} \right)^{1/4} \right\}.$$ 

(1.4)

which reduces to (1.1) when $k = 0$.

At the end of their paper, Dixit et al. posed the following two open questions, namely:

**Question 1.** Find the exact evaluation of the integral

$$\hat{J}_{2k}(\pi) := \int_0^{\infty} \frac{xe^{-\pi x^2}}{e^{2\pi x} - 1} \text{F}_1(-2k; \frac{3}{2}; 2\pi x^2) dx$$

(1.5)

for positive integer $k$.

**Question 2.** Find the exact evaluation of or at least an approximation to, the integral

$$\hat{J}_{2k+1}(\alpha) = \int_0^{\infty} \frac{xe^{-\alpha x^2}}{e^{2\pi x} - 1} \text{F}_1(-2k-1; \frac{3}{2}; 2\alpha x^2) dx$$

(1.6)

when $\alpha \neq \pi$ is a positive real number and $k$ is a non-negative integer. In this note we partially answer the above two questions by obtaining simple closed-form expressions for these integrals which, although not exact, approximate the given integrals to within exponentially small accuracy when $k$ is large and $a = O(1)$. In addition, we extend the scope of Question 1 by considering the integral $\hat{J}_{2k}(\alpha)$ with $\alpha > 0$ and, as a by-product of the analysis pertaining to Question 2, we supply an alternative proof of the result (1.3).

2. The analysis of $J_{2k}(a)$

Throughout we shall find it convenient to replace the parameter $\alpha$ by $\pi a$ and define the integral $J_{2k}(a)$ by

$$J_{2k}(a) = \int_0^{\infty} \frac{xe^{-\pi ax^2}}{e^{2\pi x} - 1} \text{F}_1(-2k; \frac{3}{2}; 2\pi ax^2) dx$$

(2.1)
for \( a > 0 \) and positive integer \( k \). Then \( J_{2k}(1) = \hat{J}_{2k}(\pi) \) in (1.5). The confluent hypergeometric function terminates and we have [4, p. 322]

\[
1F_1(-2k; \frac{3}{2}; 2\pi x^2) = \sum_{r=0}^{2k} \frac{(-2k)_r}{(\frac{3}{2})_r} (2\pi ax^2)^r.
\]

Substitution of this series into the left-hand side of the above yields

\[
J_{2k}(a) = \sum_{r=0}^{2k} \frac{(-2k)_r}{(\frac{3}{2})_r} (2\pi a)^r 
\int_0^\infty \frac{x^{2r+1} e^{-\pi ax^2}}{e^{2\pi x} - 1} \, dx
\]

upon reversal of the order of summation and integration.

Now

\[
\int_0^\infty \frac{x^{2r+1} e^{-\pi ax^2}}{e^{2\pi x} - 1} \, dx = \sum_{n\geq 1} \int_0^\infty x^{2r+1} e^{-\pi ax^2} (2\pi n x) dx = \frac{a^{1/2} r! \Gamma(r + \frac{3}{2})}{2(\pi a)^{r+3/2}} U_r,
\]

where

\[
U_r := \sum_{n\geq 1} U(r + 1, \frac{1}{2}, \pi n^2/a)
\]

with \( U(a, b, z) \) being the confluent hypergeometric function of the second kind [4, p. 322]. Then we obtain

\[
J_{2k}(a) = \frac{1}{4\pi a} \sum_{r=0}^{2k} (-2k)_r 2^r U_r.
\]

From the integral representation [4, p. 326]

\[
U(a, b, z) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-zt} t^{a-1} (1 + t)^{b-a-1} \, dt \quad (a > 0, \Re(z) > 0),
\]

we find

\[
U_r = \sum_{n\geq 1} \frac{1}{r!} \int_0^\infty e^{-\pi n^2 t/a} t^r (1 + t)^{-r-3/2} \, dt = \frac{1}{r!} \int_0^\infty \frac{\psi(t) t^r}{(1 + t)^{r+3/2}} \, dt,
\]

where we have defined

\[
\psi(t) := \sum_{n\geq 1} e^{-\pi n^2 t/a}.
\]

Hence

\[
J_{2k}(a) = \frac{1}{4\pi a} \int_0^\infty \frac{1}{r!} \left( \frac{2t}{1 + t} \right)^r \frac{\psi(t)}{(1 + t)^{3/2}} \, dt
\]

\[
= \frac{1}{4\pi a} \int_0^\infty \frac{\psi(t)(1 - t)^{2k}}{(1 + t)^{2k+3/2}} \, dt,
\]

where the finite sum has been evaluated as [4, (15.4.6)]

\[
1F_0(-2k; \frac{2t}{1 + t}) = \left( \frac{1 - t}{1 + t} \right)^{2k}.
\]

We now divide the integration path into \([0, 1]\) and \([1, \infty)\) and make the change of variable \( t \to 1/t \) in the integral over \([0, 1]\). This yields

\[
J_{2k}(a) = \frac{1}{4\pi a} \int_1^\infty \left\{ t^{-1/2} \psi(1/t) + \psi(t) \right\} \frac{(t - 1)^{2k}}{(1 + t)^{2k+3/2}} \, dt.
\]
For the sum
\[ \Psi(\tau) = \sum_{n \geq 1} e^{-\pi n^2 \tau}, \]  
we have the well-known Poisson transformation given by [7, p. 124]
\[ \Psi(\tau) + \frac{1}{2}(1 - \tau^{-1/2}) = \tau^{-1/2}\Psi(1/\tau). \]  
With \( \tau = at \), this yields
\[ t^{-1/2}\psi(1/t) = a^{1/2}(\phi(t) + \frac{1}{2}(1 - (at)^{-1/2})), \quad \phi(t) := \sum_{n \geq 1} e^{-\pi n^2 at}. \]
Hence
\[
J_{2k}(a) = \frac{1}{4\pi a} \int_{1}^{\infty} \left\{ \psi(t) + a^{1/2}\phi(t) + \frac{1}{2}a^{1/2}(1 - (at)^{-1/2}) \right\} \frac{(t - 1)^{2k}}{(1 + t)^{2k+3/2}} dt 
= -\frac{1}{8\pi a} \int_{0}^{1} (1 - a^{1/2}t^{-1/2}) \frac{(1 - t)^{2k}}{(1 + t)^{2k+3/2}} dt 
+ \frac{1}{4\pi a} \int_{\infty}^{1} \left\{ \psi(t) + a^{1/2}\phi(t) \right\} \frac{(t - 1)^{2k}}{(1 + t)^{2k+3/2}} dt.
\]
For positive integer \( k \), we have the integrals
\[
\int_{0}^{1} \frac{t^{-1/2}(1 - t)^{2k}}{(1 + t)^{2k+3/2}} dt = \sqrt{\frac{\pi}{2}} \frac{\Gamma(2k + 1)}{\Gamma(2k + \frac{3}{2})}
\]
and
\[
\int_{0}^{1} \frac{(1 - t)^{2k}}{(1 + t)^{2k+3/2}} dt = \frac{1}{2k + 1} \ {}_{2}F_{1}(1, 2k + \frac{3}{2}; 2k + 2; -1) 
= \ {}_{2}F_{1}(-2k, 1; \frac{3}{2}; 2) - \sqrt{\frac{\pi}{2}} \frac{\Gamma(2k + 1)}{\Gamma(2k + \frac{3}{2})}
\]
by application of the transformation [4, p. 390]
\[
\ {}_{2}F_{1}(a, b; c; z) = \frac{\Gamma(a)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} z^{-a} \ {}_{2}F_{1}(a, a - c + 1; a + b - c + 1; 1 - z^{-1}) 
+ \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)} z^{a - c} (1 - z)^{c - a - b} \ {}_{2}F_{1}(c - a, 1 - a; c - a - b + 1; 1 - z^{-1}).
\]  
Hence we obtain

**Theorem 1.** Let \( a > 0 \) and \( k \) be a positive integer. Then the integral \( J_{2k}(a) \) defined in (2.1) satisfies
\[
J_{2k}(a) = T_{2k}(a) + \epsilon_{2k}(a),
\]
where
\[
T_{2k}(a) = \frac{1}{4\pi a} \left\{ \left( \frac{1 + a^{1/2}}{2} \right) \sqrt{\frac{\pi}{2}} \frac{\Gamma(2k + 1)}{\Gamma(2k + \frac{3}{2})} - \ {}_{2}F_{1}(-2k, 1; \frac{3}{2}; 2) \right\}
\]
and
\[
\epsilon_{2k}(a) = \frac{1}{4\pi a} \int_{1}^{\infty} \left\{ \psi(t) + a^{1/2}\phi(t) \right\} \frac{(t - 1)^{2k}}{(1 + t)^{2k+3/2}} dt
\]
with the sums \( \psi(t) \) and \( \phi(t) \) defined in (2.3) and (2.7).
It will be found subsequently that \( \epsilon_{2k}(a) \) is small for \( k \geq 1 \) when \( a = O(1) \) and so we shall refer to it as the remainder term. We observe that when \( a = 1 \), we have \( \phi(t) = \psi(t) \) and hence that
\[
\epsilon_{2k}(1) = \frac{1}{2\pi} \int_1^\infty \psi(t) \frac{(t - 1)^{2k}}{(1 + t)^{2k+3/2}} dt.
\]

3. The analysis of \( J_{2k+1}(a) \)

A similar treatment for the integral
\[
J_{2k+1}(a) = \int_0^\infty \frac{xe^{-\pi ax^2}}{e^{2\pi x} - 1} \, \, _1F_1(-2k-1; \frac{3}{2}; 2\pi ax^2) \, dx \quad (k = 0, 1, 2, \ldots)
\]
shows that
\[
J_{2k+1}(a) = \frac{1}{4\pi a} \sum_{r=0}^{2k+1} \{(-2k-1), 2U_r \} = \frac{1}{4\pi a} \int_0^\infty \psi(t) \frac{(1 - t)^{2k+1}}{(1 + t)^{2k+5/2}} dt.
\]

Dividing the integration path as in Section 2, we find
\[
J_{2k+1}(a) = \frac{1}{4\pi a} \int_1^\infty \{ t^{-1/2} \psi(1/t) - \psi(t) \} \frac{(t - 1)^{2k+1}}{(1 + t)^{2k+5/2}} dt.
\]

Application of (2.7) and some straightforward algebra then produces
\[
J_{2k+1}(a) = -\frac{1}{8\pi a} \int_0^1 (1 - a^{1/2}t^{-1/2}) \frac{(1 - t)^{2k+1}}{(1 + t)^{2k+5/2}} dt
\]
\[
+ \frac{1}{4\pi a} \int_1^\infty \{ a^{1/2} \phi(t) - \psi(t) \} \frac{(t - 1)^{2k+1}}{(1 + t)^{2k+5/2}} dt.
\]

Now
\[
\int_0^1 (1 - a^{1/2}t^{-1/2}) \frac{(1 - t)^{2k+1}}{(1 + t)^{2k+5/2}} dt = \frac{1}{2k+2} \, _2F_1(1, 2k + \frac{5}{2}; 2k + 3; -1) - \sqrt{\frac{\pi a}{2}} \frac{\Gamma(2k + 2)}{\Gamma(2k + \frac{5}{2})}
\]
\[
= 2 \, _2F_1(-2k-1, 1; \frac{3}{2}; 2) + (1 - a^{1/2}) \sqrt{\frac{\pi}{2}} \frac{\Gamma(2k + 2)}{\Gamma(2k + \frac{5}{2})}
\]
by (2.8). Hence we obtain

**Theorem 2.** Let \( a > 0 \) and \( k \) be a non-negative integer. Then the integral \( J_{2k+1}(a) \) defined in (3.1) satisfies
\[
J_{2k+1}(a) = -T_{2k+1}(a) + \epsilon_{2k+1}(a),
\]
where
\[
T_{2k+1}(a) = \frac{1}{4\pi a} \left\{ \left(1 - a^{1/2}\right) \sqrt{\frac{\pi}{2}} \frac{\Gamma(2k + 2)}{\Gamma(2k + \frac{5}{2})} + 2 \, _2F_1(-2k-1, 1; \frac{3}{2}; 2) \right\}
\]
and
\[
\epsilon_{2k+1}(a) = \frac{1}{4\pi a} \int_1^\infty \{ a^{1/2} \phi(t) - \psi(t) \} \frac{(t - 1)^{2k+1}}{(1 + t)^{2k+5/2}} dt
\]
with the sums \( \psi(t) \) and \( \phi(t) \) defined in (2.3) and (2.7).
When \( a = 1 \), we have \( \psi(t) = \phi(t) \) and hence \( \epsilon_{2k+1}(1) = 0 \). It then follows from (3.2) that
\[
J_{2k+1}(1) = \frac{1}{4\pi} 2F_1(-2k-1,1;\frac{3}{2};2),
\]
which supplies another proof of the result stated in (1.3) obtained in [2].

4. Estimation of the remainder terms

We examine the remainder terms \( \epsilon_{2k}(a) \) and \( \epsilon_{2k+1}(a) \) appearing in (2.11) and (3.4) and determine bounds and an estimate of their large-\( k \) behaviour. We consider first the term \( \epsilon_{2k}(a) \) which can be written as
\[
\epsilon_{2k}(a) = \frac{a^{-3/4}}{4\pi} \int_1^\infty \left\{ a^{1/4} \phi(t) + a^{-1/4} \psi(t) \right\} \frac{(t-1)^{2k}}{(1+t)^{2k+3/2}} dt
\]
With the change of variable \( t \to 1 + u \), we have
\[
\epsilon_{2k}(a) = \frac{a^{-3/4}}{4\pi} \left\{ a^{1/4} \sum_{n \geq 1} e^{-\pi n^2 a} \int_0^\infty e^{-\pi n^2 u} h(u) du + a^{-1/4} \sum_{n \geq 1} e^{-\pi n^2 a} \int_0^\infty e^{-\pi n^2 u} h(u) du \right\}
< \frac{a^{-3/4}}{4\pi} \left\{ a^{1/4} \Psi(a) \int_0^\infty e^{-\pi a u} h(u) du + a^{-1/4} \Psi(1/a) \int_0^\infty e^{-\pi a u} h(u) du \right\},
\]
where \( \Psi(a) \) is defined in (2.5) and \( h(u) = u^{2k}/(2 + u)^{2k+3/2} \). Evaluation of the integrals appearing in (4.1) in terms of the confluent hypergeometric function \( U(a,b,z) \) by (2.2), we then obtain the upper bound in the form

**Theorem 3.** The remainder term \( \epsilon_{2k}(a) \) defined in (2.11) satisfies the upper bound
\[
\epsilon_{2k}(a) < B_{2k}(a), \quad B_{2k}(a) := \frac{a^{-3/4}(2k)!}{4\sqrt{2\pi}} \{ E_{2k}(a) + E_{2k}(1/a) \},
\]
where
\[
E_{2k}(a) := a^{1/4} \Psi(a) U(2k + 1, \frac{1}{2}, 2\pi a).
\]
and \( \Psi(a) \) is given by (2.5).

The behaviour of this bound as \( k \to \infty \) with \( a \) fixed can be obtained by making use of the result [4, (13.8.8)]
\[
U(2k + 1, \frac{1}{2}, 2\pi a) \sim \frac{e^{\pi a}}{(2k)!} \sqrt{\frac{\pi}{2k}} e^{-4\sqrt{\pi}ak} \quad (k \to \infty, \ a \ll 2k/\pi).
\]
For values of \( a \approx 1 \), we can bound the sum \( \Psi(a) \) by
\[
\Psi(a) := \sum_{n \geq 1} e^{-\pi n^2 a} = e^{-\pi a} \left( 1 + e^{-3\pi a} + e^{\pi a} \sum_{n \geq 3} e^{-\pi n^2 a} \right) < \lambda(a)e^{-\pi a},
\]
where
\[
\lambda(a) := 1 + e^{-3\pi a} + e^{\pi a} \sum_{n \geq 3} e^{-\pi n a} = 1 + e^{-3\pi a} + \frac{e^{-2\pi a}}{1 - e^{-\pi a}}.
\]
This then yields the estimate as $k \to \infty$

$$B_{2k}(a) \sim \frac{a^{-3/4}k^{-1/2}}{8\sqrt{\pi}} \left\{ a^{1/4} \lambda(a)e^{-4\sqrt{\pi}ak} + a^{-1/4} \lambda(1/a)e^{-4\sqrt{\pi}k/a} \right\}$$  \hspace{1cm} (4.4)

provided $a \gg \pi/(2k)$ and $a \ll 2k/\pi$ (that is, when $a$ is neither too small nor too large). In the case $a = 1$ we have

$$B_{2k}(1) \sim \frac{\lambda(1)}{4\sqrt{\pi}} k^{-1/2} e^{-4\sqrt{\pi}k} \quad (k \to \infty).$$

The remainder term $\epsilon_{2k+1}(a)$ may be written as

$$\epsilon_{2k+1}(a) = \frac{a^{-3/4}}{4\pi} \int_1^\infty \left\{ a^{1/4} \phi(t) - a^{-1/4} \psi(t) \right\} \frac{(t-1)^{2k+1}}{(1+t)^{2k+5/2}} dt.$$

It is straightforward to show (we omit these details) that $a^{1/2} \phi(t) - \psi(t)$ has opposite signs in the intervals $a \in (0, 1)$ and $a \in (1, \infty)$ when $t \in [1, \infty)$, being negative in $a \in (1, \infty)$. Hence it follows that $\epsilon_{2k+1}(a) < 0$ when $a \in (1, \infty)$ and $\epsilon_{2k+1}(a) > 0$ when $a \in (0, 1)$. The same procedure employed for $\epsilon_{2k}(a)$ shows that

$$|\epsilon_{2k+1}(a)| < \frac{a^{-3/4}}{4\pi} \int_1^\infty \left\{ a^{1/4} \phi(t) + a^{-1/4} \psi(t) \right\} \frac{(t-1)^{2k+1}}{(1+t)^{2k+5/2}} dt$$

and therefore we obtain

**Theorem 4.** The remainder term $\epsilon_{2k+1}(a)$ defined in (3.4) satisfies the upper bound

$$|\epsilon_{2k+1}(a)| < B_{2k+1}(a), \quad B_{2k+1}(a) := \frac{a^{-3/4}(2k+1)!}{4\sqrt{2\pi}} \left\{ E_{2k+1}(a) + E_{2k+1}(1/a) \right\},$$  \hspace{1cm} (4.5)

where

$$E_{2k+1}(a) := a^{1/4}\Psi(a)U(2k + 2, \frac{1}{2}, 2\pi a).$$

and $\Psi(a)$ is given by (2.5). The leading behaviour of $B_{2k+1}(a)$ for large $k$ and finite $a$ is given by the right-hand side of (4.4).

**5. Numerical results**

To demonstrate the smallness of the remainder terms $\epsilon_{2k}(a)$ and $\epsilon_{2k+1}(a)$ we define the quantities

$$J_{2k}(a) := J_{2k}(a) - \frac{1}{4\pi a} \left\{ \frac{1 + a^{1/2}}{2} \sqrt{\frac{\pi}{2}} \frac{\Gamma(2k + 1)}{\Gamma(2k + \frac{3}{2})} - 2F_1(-2k, 1; \frac{3}{2}; 2) \right\}$$

and

$$J_{2k+1}(a) := J_{2k+1}(a) + \frac{1}{4\pi a} \left\{ \frac{1 - a^{1/2}}{2} \sqrt{\frac{\pi}{2}} \frac{\Gamma(2k + 2)}{\Gamma(2k + \frac{5}{2})} + 2F_1(-2k - 1; 1; \frac{3}{2}; 2) \right\}.$$

\hspace{1cm} ^{1}\text{It is clear that this bound will not be sharp in the neighbourhood of } a \approx 1.
In Tables 1–3 we present numerical values of these quantities compared with their bounds $B_{2k}(a)$ and $B_{2k+1}(a)$ for a range of $k$ and three values of the parameter $a = O(1)$. It is seen that this bound agrees very well with the computed values of $J_{2k}(a)$ and $J_{2k+1}(a)$. The estimates in (4.4) and (4.5) show that the remainder terms are exponentially small for large $k$ when $a = O(1)$. Consequently, the terms $T_{2k}(a)$ and $T_{2k+1}(a)$ in (2.10) and (3.3) approximate $J_{2k}(a)$ and $J_{2k+1}(a)$, respectively, to exponential accuracy in the large-$k$ limit.

Now

$$J_{2k}(0) = J_{2k+1}(0) = \int_0^{\infty} \frac{x}{e^{2\pi x} - 1} \, dx = \frac{1}{24};$$

but it is easily seen that $T_{2k}(a)$ and $T_{2k+1}(a)$ in (2.10) and (3.3) are $O(a^{-1})$ as $a \to 0$ and $O(a^{-1/2})$ as $a \to \infty$. Routine calculations show that the bounds $B_{2k}(a)$ and $B_{2k+1}(a)$ also possess the same behaviour in these limits. Consequently, the approximations $T_{2k}(a)$ and $T_{2k+1}(a)$ will not be good for small or large values of the parameter $a$, although it is worth pointing out that the range of validity in $a$ will increase as $k$ increases.

Table 1: Values of $J_{2k}(a)$ and the bound $B_{2k}(a)$ for $\epsilon_{2k}(a)$ in (4.2) as a function of $k$ when $a = 1$.

| $k$  | $J_{2k}(1)$          | $B_{2k}(1)$          | $k$  | $J_{2k}(1)$          | $B_{2k}(1)$          |
|------|----------------------|----------------------|------|----------------------|----------------------|
| 1    | $1.250 \times 10^{-5}$ | $1.253 \times 10^{-5}$ | 10   | $3.905 \times 10^{-12}$ | $3.913 \times 10^{-12}$ |
| 2    | $8.571 \times 10^{-7}$ | $8.588 \times 10^{-7}$ | 20   | $3.186 \times 10^{-16}$ | $3.193 \times 10^{-16}$ |
| 3    | $9.818 \times 10^{-8}$ | $9.838 \times 10^{-8}$ | 30   | $2.305 \times 10^{-19}$ | $2.309 \times 10^{-19}$ |
| 5    | $2.883 \times 10^{-9}$ | $2.888 \times 10^{-9}$ | 50   | $2.433 \times 10^{-24}$ | $2.438 \times 10^{-24}$ |

Table 2: Values of $J_{2k}(a)$ and the bound $B_{2k}(a)$ as a function of $k$ when $a = 2$ and $a = 0.50$.

| $k$  | $J_{2k}(2)$          | $B_{2k}(2)$          | $k$  | $J_{2k}(2)$          | $B_{2k}(2)$          |
|------|----------------------|----------------------|------|----------------------|----------------------|
| 1    | $5.987 \times 10^{-5}$ | $6.364 \times 10^{-5}$ | 10   | $9.509 \times 10^{-10}$ | $1.011 \times 10^{-9}$ |
| 2    | $7.856 \times 10^{-6}$ | $8.355 \times 10^{-6}$ | 20   | $1.075 \times 10^{-12}$ | $1.143 \times 10^{-12}$ |
| 3    | $1.563 \times 10^{-6}$ | $1.662 \times 10^{-6}$ | 30   | $6.016 \times 10^{-15}$ | $6.398 \times 10^{-15}$ |
| 5    | $1.162 \times 10^{-7}$ | $1.236 \times 10^{-7}$ | 50   | $1.668 \times 10^{-18}$ | $1.774 \times 10^{-18}$ |

$J_{2k}(\frac{1}{2})$  $B_{2k}(\frac{1}{2})$  $J_{2k}(\frac{1}{2})$  $B_{2k}(\frac{1}{2})$
|------|----------------------|----------------------|------|----------------------|----------------------|
| 1    | $1.693 \times 10^{-4}$ | $1.800 \times 10^{-4}$ | 10   | $2.689 \times 10^{-9}$ | $2.860 \times 10^{-9}$ |
| 2    | $2.222 \times 10^{-5}$ | $2.363 \times 10^{-5}$ | 20   | $3.040 \times 10^{-12}$ | $3.234 \times 10^{-12}$ |
| 3    | $4.420 \times 10^{-6}$ | $4.700 \times 10^{-6}$ | 30   | $1.702 \times 10^{-14}$ | $1.810 \times 10^{-14}$ |
| 5    | $3.287 \times 10^{-7}$ | $3.496 \times 10^{-7}$ | 50   | $4.719 \times 10^{-18}$ | $5.019 \times 10^{-18}$ |

The approximation in (1.4) when $\alpha = a\pi$ (with $\alpha/\beta = \pi^2$) yields

$$J_{2k}(a) \approx -\frac{F}{4\pi a} \left\{1 - \left(1 + a^2 + \frac{2\pi a}{3F}\right)^{1/4}\right\}, \quad F := 2F_1(-2k, 1; \frac{3}{2}; 2). \quad (5.1)$$
Table 3: Values of $J_{2k+1}(a)$ and the bound $B_{2k+1}(a)$ as a function of $k$ when $a = 2$ and $a = 0.50$. 

| $k$ | $J_{2k+1}(2)$     | $B_{2k+1}(2)$   | $k$ | $J_{2k+1}(2)$     | $B_{2k+1}(2)$   |
|-----|------------------|-----------------|-----|------------------|-----------------|
| 0   | $2.230 \times 10^{-4}$ | $2.376 \times 10^{-4}$ | 10  | $6.340 \times 10^{-10}$ | $6.743 \times 10^{-10}$ |
| 1   | $2.018 \times 10^{-5}$ | $2.147 \times 10^{-5}$ | 20  | $8.067 \times 10^{-13}$ | $8.579 \times 10^{-13}$ |
| 2   | $3.376 \times 10^{-6}$ | $3.591 \times 10^{-6}$ | 30  | $4.760 \times 10^{-15}$ | $5.063 \times 10^{-15}$ |
| 5   | $6.603 \times 10^{-8}$ | $7.022 \times 10^{-8}$ | 40  | $6.287 \times 10^{-17}$ | $6.686 \times 10^{-17}$ |

This yields the limiting behaviours

$$J_{2k}(a) \simeq \frac{1}{24} + a \left( \frac{F}{16\pi} - \frac{\pi}{96F} \right) + O(a^2) \quad (a \to 0)$$

and

$$J_{2k}(a) \simeq \frac{F}{4\pi \sqrt{a}} \left\{ 1 - \frac{1}{\sqrt{a}} + \frac{\pi}{6aF} + O(a^{-3/2}) \right\} \quad (a \to \infty).$$

The approximation (5.1) is found to be quite accurate in the limits of small and large $a$, with $k$ finite. However, the accuracy is not good when $a = O(1)$. For example, when $a = 1$ and $k = 5$, the approximation (5.1) yields absolute relative errors of 8.8% and 19.2%, respectively; and this error increases as $k$ increases, in marked contrast to the approximations in (2.10) and (3.3).

As a final remark, it is doubtful that the remainder terms $\epsilon_{2k}(a)$ and $\epsilon_{2k+1}(a)$ can be expressed in simple closed forms.

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