Coherence resonance near blowout bifurcation in nonlinear dynamical systems

Bambi Hu$^{1,2}$, Changsong Zhou$^1$

$^1$ Department of Physics and Center for Nonlinear Studies, Hong Kong Baptist University, Hong Kong, China
$^2$ Department of Physics, University of Houston, Houston, Texas 77204

Previous studies have shown that noise can induce coherence resonance in some nonlinear dynamical systems close to a bifurcation of a periodic motion, such as in excitable systems. We demonstrate that coherence resonance can be observed in systems close to a blowout bifurcation. It is shown that for dynamical systems with an invariant subspace in which there is a phase-coherent chaotic attractor, the interplay among the oscillation of local transverse stability, noise and nonlinearity can lead to coherence resonance phenomenon. The mechanism of coherence resonance in this type of system is different from that in previously studied systems.

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I. INTRODUCTION

The behavior of nonlinear dynamical systems subjected to noise has been an interesting subject of recent investigation. A great deal of work has been devoted to stochastic resonance [1], where an optimal amount of additive noise can generate the maximal response of the system to a weak external periodic or aperiodic signal.

Coherent motion can be induced purely by noise in some dynamical systems without an external signal. In nonlinear dynamical systems near the onset of bifurcations of periodic orbits, the periodicity is visible even before the bifurcation actually occurs if there is noise present, a phenomenon called noisy precursor of the bifurcation [2]. Similar phenomenon has also been observed in excitable systems [3], such as in various neural models [4-6] and laser system [7], in the fixed point regime close to a saddle-node bifurcation of a periodic orbit. A feature common to the systems close to the bifurcation of a periodic orbit is that during the relaxation to stable orbit below the bifurcation, the transient possesses the periodicity above the bifurcation, and the effect of external noise is to continually kick the system off of the stable orbit, so that the transient behavior displays coherent motion.

More interestingly, recent investigations have shown that the coherence of the noise-induced motion achieves a maximum at an optimal noise intensity [8-10]. For example in an excitable system, when the system is kicked away from the fixed state to overcome a certain threshold, it will come back to the fixed point only after a large excursion (noise-induced limit cycle, or spike in neural systems). When noise is weak, the system is rarely excited, and the motion is quite irregular. Increasing noise kicks the system over the threshold more often, and the system fires more and more spikes. The interspike interval becomes the most regular at an optimal noise level. After that, too high level of noise distorts greatly the near limit cycle, rendering the motion irregular again. Similar to the conventional stochastic resonance, this phenomenon of resonance without an external signal is called coherence resonance (CR). In a recent paper, Ohira and Sato showed that a simple two state model with time delay can display CR [11]. Indeed, the time delay introduces an intrinsic periodic oscillation into the system: if noise induces a spike at a certain moment, another spike is mostly expected after the time delay. More recent work has also shown noise-enhanced synchronization [11] and array-enhanced CR [12] in coupled or extended excitatory systems.

It is interesting and practically meaningful to see whether CR exists in other type of system. In this paper, we demonstrate CR in noisy chaotic dynamical systems close to a blowout bifurcation [13] which occurs in dynamical systems with an invariant subspace $S$ in the phase space. The transverse stability of the subspace is determined by the motion within the subspace. When the largest transverse Lyapunov exponent $\Lambda$ is negative, $S$ is stable; while unstable when $\Lambda$ is positive, and the critical point of the transverse stability is the blowout bifurcation point. However, the local stability of the subspace may fluctuate greatly when the motion within the subspace is chaotic. The finite time Lyapunov exponent $\Lambda_T$ measuring expansion or contraction of a transverse perturbation during a period of finite time $T$ may oscillate greatly around the average value $\Lambda$. This fluctuation of local stability can lead to interesting and unusual behaviors, such as bubbling [14], on-off intermittency [15] and additional complexity in the system by unstable dimension variability [16]. When there is a quasiperiodic torus in $S$, the system may undergo a transition to strange nonchaotic attractors via blowout bifurcation [17].
II. SYSTEM AND RESULTS

CR observed in this type of system is the most appreciable when the motions within the subspace are oscillations with property of being phase-wise, although it may be chaotic in the amplitude. A typical example of this type of motion is the Rössler chaotic attractor [Fig. 1]. In this work, we focus on the case where there is such a motion in the subspace, and study the system behavior in the presence of noise. Since the noise prevents the dynamics from approaching the subspace indefinitely, the system may be repelled far away from the subspace due to the local instability. Previous investigations on the effects of noise in this type of system focused on the change of the universal behavior of the laminar phase distribution [18] of the blowout motion in the context of on-off intermittency. Here we focus on the coherence of the blowout motion. Our results will show that, with the increase of the noise level, the output (e.g. the distance from the subspace) displays increasing coherence till too high level of noise dominates the dynamics and destroys the coherence, exhibiting typical CR phenomenon. To illustrate our findings, we consider the case where there is a Rössler chaotic attractor in the invariant subspace,

\[
\begin{align*}
\dot{x}_1 &= \alpha(-x_2 - x_3), \\
\dot{x}_2 &= \alpha(x_1 + ax_2), \\
\dot{x}_3 &= \alpha(0.4 + (x_1 - 8.5)x_3), \\
\dot{y} &= b(x_1 - \bar{x}_1) + c\sin y - y + \sigma \xi,
\end{align*}
\]

(1) (2) (3) (4)

where \(\alpha\) controls the time scale of the Rössler system, and \(\bar{x}_1\) is the time average of \(x_1\). Clearly, \(y = 0\) defines the invariant subspace in the absence of noise \(\xi\) which is a Gaussian white noise with \(\langle \xi(t) \xi(t') \rangle = \delta(t - t')\). The introduction of \(\bar{x}_1\) in Eq. (4) is for the convenience of discussion, because the largest transverse Lyapunov exponent

\[
\Lambda = \lim_{t \to \infty} \frac{1}{t} \int_0^t [b(x_1 - \bar{x}_1) + c - 1] d\tau = c - 1,
\]

(5)

and the transverse stability is only controlled by the parameter \(c\). Eqs. (1-4) are modified version of a physical model of superconducting quantum interference device (SQUID) [19]. Similar model with a torus in the subspace has been studied in the context of nonchaotic strange attractors [17]. In fact, the specific form of the nonlinearity in Eq. (4) is of no importance for the CR phenomenon. It serves to keep the system bounded.

We note that for parameter \(a = 0.15\), the Rössler system possesses a chaotic attractor with some degree of phase coherence [see Fig. 1(a)]; the cycling time \(T_R\) has a rather sharp distribution, or equivalently, there is a pronounced peak at frequency \(\omega_0\) on the broadband spectrum of the chaotic signal. Phase synchronization [20, 21] and lag synchronization [22] occur in coupled Rössler systems due to this phase coherence. For the parameter \(c\) near the blowout bifurcation point \(c = 1.0\), the local stability undergoes large chaotic oscillations. The subspace is only temporally attracting in about half period of the loops when \(b(x_1 - \bar{x}_1) + \Lambda < 0\), while is temporally repelling in the other half period when \(b(x_1 - \bar{x}_1) + \Lambda > 0\). In the following, we study system behavior in the presence of noise both below and above the blowout bifurcation point \(c = 1.0\), with parameters \(\alpha = 0.1, \sigma = 0.15, b = \sqrt{0.02}\).

A. Below the blowout bifurcation: \(c < 1\)

Typical behaviors of the output \(|y|\) in the presence of noise with different levels are shown in Fig. 2 for \(c = 0.9\). When the noise is very weak, the system stays very close to the invariant space \(y = 0\), giving no appreciable output [Fig. 2(a)]. If the noise is larger than a certain level, the system begins to produce large output, but the signal is quite irregular [Fig. 2(b)]. At a certain range of noise levels, the output become very regular; it is almost periodic, with the amplitude fluctuating only slightly [Fig. 2(c)]. However, when noise goes to even higher levels, it begins to deform the near periodic signal, and degrades the regularity [Fig. 2(d)]. The response property of the system to additive noise exhibits typical feature of CR: the system motion becomes the most coherent at an optimal noise level.

To characterize the degree of coherence, we compute the spectra of the output signals \(|y|\). It has a pronounced peak at the same frequency \(\omega_0\) as the driving signal \(x_1\). Fig. 3(a) shows the peak height \(p_m\) at \(\omega_0\) as a function of the noise level \(D_1 = \log_{10} \sigma\). Below a certain value of \(D_1\), \(p_m\) increases exponentially and quickly; then it comes to a slow increasing region covering many orders of the noise level, \(p_m\) reaches a maximal value and decreases at higher noise levels.

In the following, let us look into the mechanism of CR in the system. In the absence of noise, the subspace is stable, and from a random initial condition, \(|y|\) decreases exponentially on average as \(|y| \sim \exp(-\Lambda t)\), but with a local chaotic oscillation. The additive noise, however, prevents the system from approaching the subspace much deeper than the noise level, and the trajectory can be repelled away from the noise level to produce relatively
large output. Thus the noise sets a reflection boundary to the transverse motion, as seen in the corresponding semilog plots of the output in the right panel of Fig. 2. Due to this boundary, the transverse motion is attracted to and repelled away from the noise level alternately when the motion within the subspace comes into the local stable region \( b(x_1 - \bar{x}_1) + \Lambda < 0 \) and local unstable region \( b(x_1 - \bar{x}_1) + \Lambda > 0 \). The intrinsic coherence of the chaotic motion within the subspace is thus manifested by the noise-induced blowout motion. This boundary effect of noise is quite different from that in excitable systems and the systems with delay, where noise acts to kick the system over a threshold.

The calculation of \( p_m \) is not easy from solving the complete stochastic system in Eq. (4) which is driven multiplicatively by chaotic signal \( x_1 \). We choose to estimate it with more intuitive but quite accurate approximations based on the above observation that the effect of noise can be modeled by a reflection boundary. For weak enough noise, \(|y| \gg \sigma\), the dynamics is governed by the linear approximation \( \dot{y} = b(x_1 - \bar{x}_1) + \Lambda|y| \), which gives

\[
|y| \approx \exp \left\{ \int_0^t \left[ b(x_1 - \bar{x}_1) + \Lambda \right] d\tau \right\}. \tag{6}
\]

Now it is not surprising that the spectra of \(|y|\) have a peak at the same frequency \( \omega_0 \) as \( x_1 \). Note the intrinsic coherence of the chaotic signal \( x_1 \), we can roughly approximate \( x_1 \) by a periodic signal with frequency \( \omega_0 \). Take into account the boundary of the noise, the right panel of Fig. 2 suggests that we may approximate

\[
\int_0^t \left[ b(x_1 - \bar{x}_1) + \Lambda \right] d\tau \approx A(\cos \omega_0 t + 1) + D,
\]

and it follows from Eq. (6) that

\[
|y| \approx \exp [A(\cos \omega_0 t + 1) + D] \quad (7)
\]

with a proper shift of the time origin. The amplitude

\[
A = \left[ b(\bar{x}_m - \bar{x}_1) + \Lambda \right]/\omega_0, \quad \text{where} \quad \bar{x}_m \text{ is the average of the maxima of } x_1.
\]

\( 2A \) is the average of the maximal magnitude that \(|y|\) can depart from the boundary at the noise level \( D = \ln \sigma \) during a cycle of \( x_1 \). With the above system parameters, we numerically estimate \( \bar{x}_m = 11.42, \bar{x}_1 = 0.1324 \) and \( \omega_0 = 1.0350 \), thus giving \( A = 12.40 \). As the noise level increases, the boundary is moving to a higher order [Fig. 2]. If \( -D \leq 2A \), the maximal value of \(|y|\) comes to the order of unit, and the system begins to produce relatively large output, and the nonlinearity begins to set in. However, due to chaotic fluctuation of the amplitude of \( x_1 \), the maxima of \(|y|\) may not be as large during those small cycles of \( x_1 \) [Fig. 2(b)], and the behavior is quite irregular, as is familiar in the context of on-off intermittency. With the boundary going to even higher order, the system can also produce quite large output \(|y|\) during those small cycles of \( x_1 \) and the amplitude is confined to some saturated values by the nonlinearity [Fig. 2(c)]. The system now performs very coherently. In this nonlinear regime, Eq. (6) based on linear approximation is no longer valid. However, the fact that \(|y|\) always begins to rise from the noise level when \( x_1 \) comes into local unstable region \( b(x_1 - \bar{x}_1) + \Lambda > 0 \) shows that the blowout motion still possesses the frequency \( \omega_0 \) in the strong nonlinear regime. In addition, Fig. 2(c) suggests that we can still approximate \( \ln |y| \) by a periodic function whose amplitude is between the boundary of noise level and the confinement of the nonlinearity, i.e.,

\[
|y| \approx \exp [B(\cos \omega_0 t + 1) + D]. \quad (8)
\]

Now the amplitude

\[
B = (D_m - D)/2, \quad \text{where} \quad D_m = \ln \max(|y|) \approx 0.8 \text{ with the above system parameters.}
\]

With the approximations in Eq. (7) and Eq. (8), we can estimate the peak height at \( \omega_0 \) as a function of \( D \) in different dynamical regimes as

\[
\begin{align*}
p_m &= \left[ \frac{\omega_0}{2\pi} \int_{-\pi/\omega_0}^{\pi/\omega_0} |y| \cos \omega_0 t \, dt \right]^2 \nonumber \\
&\approx \left\{ \begin{array}{ll}
F^2(A)e^{2Ae^{2D}}, & D \geq 2A, \\
F^2(B)e^{2Be^{2D}}, & D < 2A.
\end{array} \right.
\end{align*} \tag{9}
\]

Here, \( F(A) = \frac{\omega_0}{2\pi} \int_{-\pi}^{\pi} \cos t \exp(A \cos t) \, dt \). The analytical result explains the exponential increase of the peak height in the linear regime observed in numerical simulation [Fig. 3(a), cycles]. With \( A = 12.80 \), Eq. (9) fits the exponential region very well [Fig. 3(a), solid line]. Note that the fitting parameter \( A = 12.80 \) is quite close to \( A = 12.40 \) estimated from the system parameters, showing that the above simple approximations give a good account for the system behavior. The analysis also reproduces qualitatively the result in the nonlinear regime. The discrepancy in the crossover region \((-D \sim 2A)\) is due to the fact that in this region the maximal value of

FIG. 2. Typical behavior of noise induced blowout motion (left panel) at different noise levels \( \sigma \). The right panel is the corresponding semilog plots of the same quantity. For the purpose of clear illustration of the boundary moving with the noise level, we use the same scale in the right panel. The dotted lines in the right panel indicate the noise level. (a) \( \sigma = 10^{-13} \), (b) \( \sigma = 10^{-11} \), (c) \( \sigma = 10^{-2} \), and (d) \( \sigma = 10^{-0.2} \).
$|y|$ is not saturated and $D_m = 0.8$ used in fitting overestimates the maximal values of $|y|$ in this region. One should note that in the above analysis, the specific form of nonlinearity is of no importance, and only the confinement property of the nonlinearity is employed, indicating that the phenomenon is universal in this type of systems. Now we understand that CR occurs in the system due to the interplay among the chaotic while somewhat coherent oscillation of the local stability of the subspace, the confinement of the nonlinearity to the transverse motion and the boundary effects of the additive noise. This mechanism is quite different from that in previously investigated systems [3–13].

FIG. 3. Illustration of coherence resonance below the blowout bifurcation. (a) Peak height $p_m$ as a function of noise level $D_1 = \log_{10} \sigma$. The solid line is the analytical estimation with $A = 12.80$. (b) Coherence factor $\beta$ as a function of noise level. The solid line is the result of the chaotic signal $x_1$.

Based on the above understanding of the behavior, we can characterize CR by another quantity, the “coherence factor” defined by the relative fluctuation of the output amplitude, e.g. the time average of the amplitude $A_y$ divided by its standard deviation

$$\beta = \langle A_y \rangle / \sqrt{\text{Var}(A_y)}.$$  \hspace{1cm} (10)

For relatively weak noise, $|y| \gg \sigma$ most of the time, and $|y|$ is smooth and $A_y$ is well defined. For relatively large noise, noise-induced short time fluctuation of $|y|$ can be rather strong. In numerical simulations $A_y$ is defined as follows: firstly a smooth series $|y|m$ is obtained from the noisy function $|y|$ using a moving average method, then $A_y$ is taken as the value of $|y|$ at the moment when $|y|m$ is maximal. This definition captures the noise-induced short time fluctuations of $|y|$ at large noise levels. The result is shown in Fig. 3(b) (cycles). $\beta$ reaches a maximum and decreases quickly when too high level of noise begins to dominate the fluctuation of the amplitude. In linear and weak nonlinear region ($D_1 < -7.5$), the chaotic fluctuation of the amplitude of $x_1$ is augmented by the exponential relationship between $|y|$ and $x_1$, as seen in Eq 4, so that the coherence degree of $|y|$ is lower than that of $x_1$, as can be seen by the comparison of the coherence factors of $|y|$ (cycles) and $x_1$ (solid line). It is very interesting to see that in the nonlinearity dominant region, the coherence of the noise-induced motion is much higher than the intrinsic coherence of $x_1$ in a wide range of the noise level, because the confinement of the nonlinearity smoothes the fluctuation of the amplitude of $|y|$. Thus the combination of the noise and nonlinearity enhances greatly the intrinsic coherence. Such pronounced CR phenomenon is able to be observed when other chaotic attractors possessing similar intrinsic coherence are within the subspace, such as the electronic circuit in Ref. [23] and the hybrid laser system in Ref. [24] or the ecological system in Ref. [25]. It is also clear that CR will occur for periodic and quasiperiodic motion within the subspace, i.e., the peaks in the spectra possess a maximal value at an optimal noise level.

In the Rössler system, the topology of the chaotic attractor changes if $a$ is large than 0.21: there are large and small loops [see Fig. 1(b), $a=0.25$]. This topology is quite typical in many low dimensional chaotic systems. Now, due to the large fluctuation of the amplitude and returning time, there are no pronounced peaks in the broadband spectra of the chaotic signals. However, CR is still observable. Fig. 4 shows the coherence factor $\beta$ for $|y|$ and $x_1$. Again we see the enhancement of the intrinsic coherence by noise, albeit with less intensity. Similar behavior should be observable in general systems of this type.

FIG. 4. Coherence factor $\beta$ as a function of the noise level for chaotic motion with funnel Rössler attractor in Fig. 1(b).

B. Above the blowout bifurcation: $c > 1$

Typical behavior of the system above the blowout bifurcation is similar to that below the bifurcation. The difference is that, for $c > 1$, the subspace is transversely unstable, and blowout motion has already existed even without noise, and the confinement of the nonlinearity has already taken place. For $c < 1$, the dynamics can always access the noise level, however weak it may be; while for $c > 1$, the dynamics can only come to the subspace no closer than $|y| \sim \exp(-2A)$, where $A = \sqrt{\bar{m}(x_m - x_1)} - \Lambda \pi / \omega_0$ for the chaotic attractor in Fig. 1(a). As a result, weak noise with $-D > 2A$ will have no discernible effects on the system behavior. However, stronger noise will act as a reflection boundary simi-
lar to the case below the blowout bifurcation, preventing the system from approaching the subspace to the closest level $|y| \sim \exp(-2A)$. The system behavior is now very similar to that in the nonlinear regime below the bifurcation. This analysis is demonstrated by numerical simulations with $c = 1.1$ in Fig. 5. It is seen that both $p_m$ and $\beta$ keep unchanged for very weak noise, and the coherence increases once the noise becomes effective. While $p_m$ increases monotonically till the noise dominates the dynamics and destroys the coherence of the blowout motion, $\beta$ also exhibits another peak at rather weak noise level ($\sigma \sim 10^{-10}$). This behavior is related to the topology of the chaotic attractor in the subspace. As seen in Fig. 1(a), the attractor always flips to the smallest loops from the largest ones. For weak noise without significant effects, the maxima of $|y|$ associated with those small loops have relatively small values, which contributes mainly to the fluctuation of the amplitude of $|y|$. The broad distribution of the amplitude is clearly illustrated by the return map of the maxima of $|y|$ for $\sigma = 10^{-14}$ in Fig. 6. For $\sigma \sim 10^{-10}$, the system has access to the noise level when $x_1$ cycles along the largest loops. The reflecting property of the noise has the effect to increase greatly the maximal values of $|y|$ for those smallest loops following those largest ones, while only slightly for others. Most of the maxima are confined to a small neighborhood around $A_y = 2.0$, as seen by the crosses in Fig. 6. This reduces the fluctuation of the amplitude of $|y|$ and enhances the coherence greatly. For intermediate noise level $\sigma \sim 10^{-7}$, the fluctuation becomes a little larger again when the maxima of $|y|$ are pushed to slightly larger values by the reflection of the noise. Further increase of noise pushes more maxima of $|y|$ to fluctuate slightly around a saturated value till too high level of noise destroys the coherence, resulting in another peak. The return map close to the peak is shown in Fig. 6 for $\sigma = 10^{-2}$.

FIG. 5. Illustration of coherence resonance above the blowout bifurcation. (a) Peak height $p_m$ as a function of noise level $D_1 = \log_{10} \sigma$. Unlike Fig. 3(a), linear scale is used for $p_m$ here. (b) Coherence factor $\beta$ as a function of noise level. The solid line is the result of the chaotic signal $x_1$.

CR phenomenon in the system is most appreciable around the blowout bifurcation, i.e., in the on-off intermittency regime. In general, the maximal coherence is higher for the system above the bifurcation point. If $c$ is far below the critical point, only large enough noise can induce blowout motion from the subspace, and the coherence of the motion may have already destroyed by the noise. The peaks of $p_m$ and $\beta$ become lower and narrower as $c$ decreases from $c = 1.0$ and disappear when the subspace becomes stable almost everywhere. For $c$ far above the critical point, the system has only seldom close access to the subspace, and weak noise has no significant effects on the motion, while strong noise degrades the coherence of the motion away from the subspace. Typically, one observes that both $p_m$ and $\beta$ keep unchanged for weak noise and begin to decrease for high enough noise when the subspace becomes unstable almost everywhere.

III. CONCLUSION

In summary, we have shown that the phenomenon of coherence resonance can be naturally observed in some nonlinear dynamical systems possessing an invariant subspace, close to the blowout bifurcation where previous studies were often in very different context of on-off intermittency. A link between these two distinct dynamical phenomena, CR and on-off intermittency, is the fluctuation of the local transverse stability of the subspace due to the oscillatory motion within the subspace. The noise-induced blowout motion manifests the coherent oscillation within the subspace, while the confinement of the nonlinearity reduces the chaotic fluctuation of the amplitude. The additive noise and the confinement of the nonlinearity to the transverse motion combine together
to manifest and enhance the intrinsic coherence of the motion within the subspace to the maximal degree. CR in this class of system with mechanism different from that in previously studied systems extends our understanding on nontrivial and positive effects of noise on nonlinear dynamical systems, and could be physically and practically meaningful.

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