THE BRST COMPLEX OF HOMOLOGICAL POISSON REDUCTION

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Abstract. So-called BRST complexes associated to a coisotropic ideal $J$ of a Poission algebra $P$ provide a description of the Poisson algebra $(P/J)^J$ as their cohomology in degree zero. Using the notion of stable equivalence introduced in [5], we prove that any two BRST complexes associated to the same coisotropic ideal are quasi-isomorphic in the symplectic case $P = \mathbb{R}[x_i, y_j]$ with $[x_i, y_j] = \delta_{ij}$. As a corollary, the cohomology of the BRST complexes is canonically associated to the coisotropic ideal $J$ in the symplectic case. We do not require any regularity assumptions on the constraints generating the ideal $J$.

CONTENTS

1. Introduction 2
2. BFV Models 2
3. Existence 3
4. Properties 3
5. Uniqueness 6
6. Cohomology 11
7. An Example 14

Appendix A. Graded Poisson Algebras 15
References 20

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1. Introduction

In the quantization of gauge systems, the so-called BRST complex plays a prominent role \[9\]. In the Hamiltonian formalism, the theory is called BFV-theory and goes back to Batalin, Fradkin, Fradkina and Vilkovisky \[12\].

In a gauge theory, the presence of gauge freedom yields constraints in the phase space. The gauge group still acts on the resulting constraint surface. The physical observables are the functions on the quotient $\widetilde{M}$ of the constraint surface $M_0$ by this action. In order to quantize these observables, one introduces variables of non-zero degree to the Poisson algebra $P$ of functions on the original phase space. One then defines the so-called BRST-differential on the resulting complex and recovers the functions on the subquotient $\widetilde{M}$ as the cohomology in degree zero. One may then attempt to quantize the system by quantizing the BRST-complex instead of the algebra of functions on $M$.

The quantization procedure involves the construction of gauge invariant observables from the cohomology of the BRST complex \[3\].\[13\]. Kostant and Sternberg gave a mathematical description of this theory \[10\] in the case where the constraints arise from a group action on phase space. They make certain assumptions that allow the BRST complex to be constructed as a double complex combining a Koszul resolution of the vanishing ideal of the constraint surface $M_0$ with the Lie algebra cohomology of the gauge group. More recently, Felder and Kazhdan formalized the corresponding construction in the Lagrangian formulation of the theory \[5\].

The aim of this note is to perform a similar formalization in the context of Poisson algebras, which arise in the Hamiltonian viewpoint. We define the notion of a BFV-model for a coisotropic ideal $J$ of a general Poisson algebra $P$. We use techniques from \[3\],\[12\] to prove existence of BFV-models and show that they model the Poisson algebra $(P/J)^J$ cohomologically. This latter Poisson algebra is the physically interesting one since in the case where $P$ are the functions on phase space and $J$ is the vanishing ideal of the constraint surface, it corresponds to the function on the subquotient $\widetilde{M}$, which are the true observables of the system. Those statements about the existence of what we call BFV-models and their cohomology are known \[9\]. Under certain local regularity assumptions on the constraint functions, which for instance imply that the constraint surface $M_0$ is smooth, a construction for a uniqueness proof for the BRST-cohomology was given in \[6\]. Stasheff considers the problem from the perspective of homological perturbation theory \[12\] and gives further special cases under which such uniqueness theorems hold. For instance, he considers the case, where a proper subset of the constraints satisfy a regularity condition.

Using the notion of stable equivalence from \[9\], we show that, for a symplectic polynomial algebra $P = \mathbb{R}[x_i, y_j]$ with $[x_i, y_j] = \delta_{ij}$, any two BRST complexes for the same coisotropic ideal $J \subset P$ are quasi-isomorphic. Hence, we rigorously prove uniqueness of the BRST-cohomology for such $P$. However, the assumption on $P$ does not force the constraint surface to be smooth. Moreover, we do not assume a subset of the constraints to be regular.

Also recently, Paugam introduced the language of derived geometry into the subject \[11\]. However, we do not use this language.

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2. BFV Models

We work over $\mathbb{K} = \mathbb{R}$. Let $P$ be a unital, Noetherian Poisson algebra and $J \subset P$ a coisotropic ideal. Then the Poisson structure on $P$ induces one on $(P/J)^J$. The purpose of the BRST complex is to model this Poisson algebra cohomologically.

Let $M$ be a negatively graded real vector space with finite dimensional homogeneous components $M^j$. Denote its component-wise dual by $M^* = \bigoplus_{j \geq 0} (M^*)^j$. Define a Poisson bracket on $\text{Sym}(M \oplus M^*)$ via the natural pairing between $M$ and $M^*$. For details of the construction we refer to chapter A in the appendix.

Form the tensor product $X_0 = P \otimes \text{Sym}(M \oplus M^*)$ of the two Poisson algebras defined above. Let $F^p X_0$ denote the ideal generated by all elements in $X_0$ of degree at least $p$. Using the filtration defined by the $F^p X_0$, complete the space $X_0$ to a graded commutative algebra $X$ with homogeneous components

$$X^j = \lim_{\to p} \frac{X_0^j}{F^p X_0 \cap X_0^j}.$$ 

Extend the bracket on $X_0$ to $X$, thus turning $X$ into a graded Poisson algebra. Again we refer to chapter A for details. Denote the bracket on $X$ by $[-, -]$. 

Set $I \subset X$ to be the homogeneous ideal with homogeneous components

$$I^j = \lim_{\nu} \frac{F^1 X_0 \cap X^j_0}{F^{p+1} X_0 \cap X^j_0} \subset X^j.$$

An element $R \in X$ of odd degree which solves $[R, R] = 0$ defines a differential $d_R = [R, -]$ on $X$ by the Jacobi identity. If $R \in X^1$, the differential $d_R$ induces a differential on $X/I$ since it preserves $I$.

**Definition 1.** A BFV model for $P$ and $J$ is a pair $(X, R)$ where $(X, [-,-])$ is a graded Poisson algebra constructed as above and $R \in X^1$ is such that the following conditions hold:

1. $[R, R] = 0$.
2. $H^j(X/I, d_R) = 0$ for $j \neq 0$.
3. $H^0(X/I, d_R) = P/J$.

The first equation is called the classical master equation and the element $R$ is called a BRST charge.

The aim of this note is to prove

**Theorem 1.** Let $P$ be a Poisson algebra and $J \subset P$ a coisotropic ideal. BFV models exist and in the case of $P = \mathbb{R}[x_i, y_j]$ with $[x_i, y_j] = \delta_{ij}$ the complexes of any two BFV-models for the same ideal $J$ are quasi-isomorphic, whence the cohomology $H(X, d_R)$ is uniquely determined by $J$ up to isomorphism.

The existence of BFV models is known [6, 9]. The problem of uniqueness has been dealt with under certain regularity assumptions [6, 9]. These assumptions imply that the constraint surface is smooth. The novel part is the statement that any two BRST-complexes are quasi-isomorphic, which gives uniqueness of the BRST-cohomology as a corollary. We prove this without assuming that the constraint surface is smooth. For completeness we also include proofs of the already known facts in our framework.

### 3. Existence

#### 3.1. Tate Resolutions

In order to construct BFV models, we first have to construct a suitable commutative graded algebra $X$. The odd variables are obtained via Tate resolutions.

Let $P$ be a unital, Noetherian Poisson algebra and $J \subset P$ be a coisotropic ideal. Tate constructed resolutions of Noetherian rings by adding certain odd variables to the ring [13]. Consider a Tate resolution $T = P \otimes \text{Sym}(M)$ of $P/J$ given by a negatively graded vector space $M$ with finite dimensional homogeneous components together with a differential $\delta$ of degree 1. Define the dual $M^*$ degree-wise. Extend $\delta$ to $X_0 := P \otimes \text{Sym}(M \oplus M^*) = P \otimes \text{Sym}(M) \otimes \text{Sym}(M^*)$ by tensoring with the identity. Endow $X_0$ with the natural extension of the Poisson bracket, define the filtration $F^p X_0$, and extend the bracket to the completion $X$ as described in section [A] in the appendix. We will frequently refer to statements from that chapter.

**Remark 1.** The natural isomorphism of lemma [57] identifies the differential $\delta$ on the associated graded with $1 \otimes \delta$ on $B \otimes_p T$.

We now discriminate elements in $\text{gr}^p X^n$ according to how many positive factors they contain by defining $A^n_{p,q} := \{v \in \text{gr}^p X^n : v \text{ has representative in } I^{(q)}\}$. From the proof of lemma [57] we see that $A^n_{p,q}$ can be identified with $(B^p \cap I^{(q)}_0) \otimes_p T$. We now use remark [1] to see that $A^*_{p,q}$ is a subcomplex and bound its cohomology:

**Lemma 2.** Fix $p$ and $q$. We have $H^j(A^*_{p,q}, \delta) = 0$ for $j < p$.

**Proof.** From remark [1] we have $H^j(A^*_{p,q}, \delta) \cong H^j((B^p \cap I^{(q)}_0) \otimes_p T, 1 \otimes \delta)$. Now we factor this space into $(B^p \cap I^{(q)}_0) \otimes_p H^{j-p}(T, \delta)$. For $j < p$, the second factor vanishes, since $T$ is a resolution of $P/J$. \qed
3.1.2. Contracting homotopy. From the Tate resolution construct a contracting homotopy $s : T \to T$ of degree $-1$. Then there exists a $\mathbb{K}$-linear split $P/J \to P$ and a map $\pi : T \to T$ which is defined as the composition $P \otimes \text{Sym}(\mathcal{M}) \to P \to P/J \to P \to P \otimes \text{Sym}(\mathcal{M})$ such that
\[
\delta s + s \delta = 1 - \pi
\]
Extend $\delta$, $s$ and $\pi$ to $X_0$ by tensoring with the identity on $\text{Sym}(\mathcal{M}^*)$. From the definition of $\pi$ we find

**Remark 2.** $\pi : X_0 \to X_0$ is zero on monomials which contain a factor of negative degree.

The homotopy $s$ does not act on elements in $\text{Sym}(\mathcal{M}^*)$ and hence preserves the filtration. For the same reason $\pi$ preserves the filtration. Both $s$ and $\pi$ hence naturally extend to the extension and equation $\square$ is valid in $X$ too. Moreover,

**Remark 3.** $s$ preserves $I^{(2)}$.

3.2. Constructing The BRST Charge.

3.2.1. First Approximation.

**Definition 2.** Let $Q_0$ be the differential $\delta$ on $X/I$ considered as an element of $X$.

Hence the cohomological conditions to be a BRST charge are satisfied. However, $Q_0$ does not in general satisfy the classical master equation. We are going to prove the existence part of theorem $\square$ by adding correction terms to $Q_0$.

An explicit description of $Q_0$ is the following: Let $e^i$ be a homogeneous basis of $\mathcal{M}$, $e^i_*$ its dual basis. Set $d_i := \deg e^i = - \deg e^i_*$ $\equiv \deg e^i$ (mod 2). Assume that $i \leq j$ implies $d_i \geq d_j$. Define $Q_0 := \sum_j e^*_j \delta(e^j)$. By lemma $\square$ this defines an element of $X^1$. For each $p$, let $L_p$ be an integer with $\{j \in \mathbb{N} : -d_j \leq p - 1\} = \{1, \ldots, L_p\}$ so that $(q_0)_p := \sum_{j=1}^{L_p} e^*_j \delta(e^j)$ defines a representative of the $p$-th component of $Q_0$. Of course, the element $Q_0$ is independent of the choice of basis $e^j$ of $\mathcal{M}$.

**Lemma 3.** We have $\delta = \sum_j \delta(e^j)[e^*_j, -]$ on $X$ where the operator on the right hand side is well-defined.

**Proof.** Set $\delta' = \sum_j \delta(e^j)[e^*_j, -]$. This defines a map on $X$: for $x \in X^n$, the elements $[e^*_j, x]$ are in $\mathcal{F}^{-d_j+n} X_0$. Hence the sum converges by lemma $\square$. By linearity, $\delta'$ is defined on all of $X$. We claim that $\delta'$ is continuous on each $X^n$. Let $x^j = (x^j_0 + \mathcal{F}^p X^0)_p \in X^n$ be a sequence converging to zero. Fix $p$. Then there exists a $K$, independent of $j$, such that a $p$-th representative of $\delta'(x^j)$ is given by
\[
\sum_{k=1}^{K} \delta(e^k)[e^*_k, x^j_{-d_k+n(p)}]
\]

since the bracket is in $\mathcal{F}^{-d_k+n}$. Now let $j_0$ be such that for $j \geq j_0$ and for all $k \in \{1, \ldots, K\}$ we have $x^j_{-d_k+n(p)} \in \mathcal{F}^{p-d_k+n(p)} X_0^n$. Then the above representative vanishes modulo $\mathcal{F}^p X_0^n$. Hence $\delta'$ is continuous on $X^n$. The map $\delta'$ restricts to a map on $X_0$ since the sum is then effectively finite since $[e^*_j, x]$ becomes zero for $j$ large enough, depending on $x \in X_0$. This restriction agrees with $\delta$. Hence $\delta' = \delta$ on each $X^n$ by continuity. Hence $\delta = \delta'$.

**Lemma 4.** For $L_0 := [Q_0, -] - \delta$ we have $L_0(\mathcal{F}^p X) \subset \mathcal{F}^{p+1} X$.

**Proof.** Fix $x \in \mathcal{F}^p X^n$. Then, by lemma $\square$
\[
[Q_0, x] = \lim_{m \to \infty} \left(\sum_{j=1}^{m} \delta(e^j)[e^*_j, x] + \sum_{j=1}^{m} e^*_j [\delta(e^j), x]\right).
\]
The first part converges to $\delta(x)$ by lemma $\square$. The second part converges by lemma $\square$ and hence equals $L_0$. Fix $j$. By lemma $\square$ it suffices to prove that $e^*_j [\delta(e^j), x] \in \mathcal{F}^{p+1} X$. By the derivation property it suffices to consider $x = e^*_j$ for some $l$. Then $\delta(e^j)$ is a sum of monomials whose factors have degrees in $\{-d_j + 1, \ldots, 0\}$. Hence all elementary factors in $\delta(e^j)$ that could possibly kill $e^*_j$ have degree $-d_j$ and get compensated by a factor $e^*_j$ with $\deg(e^*_j) > \deg(-d_j)$.

Moreover, we have

**Lemma 5.** $[Q_0, Q_0] \in X^2 \cap I^{(2)} \subset \mathcal{F}^2 X \cap I^{(2)}$. 

Proof. We compute \( \deg [Q_0, Q_0] = 2 \deg Q_0 = 2 \) and hence \( [Q_0, Q_0] \in \mathcal{F}^2 X \). For the last statement we need to calculate. By lemma [38]

\[
[Q_0, Q_0] = \lim_{m \to \infty} \sum_{j,k=1}^{m} [\delta(e_j^*) c_k^*, \delta(e_k^*) c_j^*] = \lim_{m \to \infty} \sum_{j,k=1}^{m} \left( 2 \delta(e_j^*) [c_j^*, \delta(e_k^*)] c_k^* + e_j^* [\delta(e^*_j), \delta(e_k^*)] c_k^* \right)
\]

By lemma [37] the first term is a sum in \( k \) with summands that contain factors \( \delta(\delta(e_k^*)) = 0 \) hand hence the first term vanishes. By lemma [39] \([Q_0, Q_0] = \sum_{j,k} e_j^* [\delta(e^*_j), \delta(e_k^*)] c_k^* \in I(2)\).

**Corollary 6.** \( \delta[Q_0, Q_0] \in X^3 \subset \mathcal{F}^3 X \).

3.2.2. Recursive Construction. We now inductively construct out of \( Q_0 \) a sequence of elements \( R_n \in \mathcal{F}X \) by setting

\[
R_n = \sum_{j=0}^{n} Q_j, \quad Q_0 \text{ as defined above}, \quad Q_{n+1} = -\frac{1}{2} \delta s[R_n, R_n].
\]

The elements \( R_n \) have degree 1 since \( Q_0 \) has and \( s \) is of degree \(-1\). The idea for the construction is taken from [12]. Also the proof of the following theorem is adapted from that paper.

**Theorem 7.** For all \( n \), \([R_n, R_n] \in \mathcal{F}^{n+2} X \cap I(2)\), and \( \delta[R_n, R_n] \in \mathcal{F}^{n+3} X \).

**Proof.** The base step was done in lemma [35] and corollary [6]. We assume the statement is true for \( 0 \leq j \leq n \) and consider

\[
[R_{n+1}, R_{n+1}] = [R_n, R_n] + 2[R_n, Q_{n+1}] + [Q_{n+1}, Q_{n+1}]
\]

By construction and assumption \( Q_{n+1} = -\frac{1}{2} \delta s[R_n, R_n] \in \mathcal{F}^{n+2} X \). Hence, by corollary [38]

\[
[R_{n+1}, R_{n+1}] \equiv [R_n, R_n] + 2[R_n, Q_{n+1}] \pmod{\mathcal{F}^{n+3} X}.
\]

Expand \([R_n, Q_{n+1}] = \sum_{j=0}^{n} [Q_j, Q_{n+1}] + [Q_0, Q_{n+1}]\). We have, for \( j \in \{1, \ldots, n + 1\} \) by inductive hypothesis, \( Q_j = -\frac{1}{2} \delta s[R_{j-1}, R_{j-1}] \in \mathcal{F}^{j+1} X \cap I(2)\). Hence, by lemma [39]

\[
[R_{n+1}, R_{n+1}] \equiv [R_n, R_n] + 2[Q_0, Q_{n+1}] \pmod{\mathcal{F}^{n+3} X}.
\]

We split \([Q_0, Q_{n+1}] = \delta Q_{n+1} + \delta L_0 Q_{n+1} + \delta L_0 Q_{n+1}\) and, by lemma [3]

\[
[R_{n+1}, R_{n+1}] \equiv [R_n, R_n] + 2 \delta Q_{n+1} \pmod{\mathcal{F}^{n+3} X}.
\]

Commuting \( \delta \) and \( s \),

\[
2 \delta Q_{n+1} = -\delta s[R_n, R_n] = s \delta [R_n, R_n] - [R_n, R_n] + \pi[R_n, R_n].
\]

Since \([R_n, R_n] \in \mathcal{F}^{n+2} X \), we have that \( \pi[R_n, R_n] = 0 \) for \( n > 0 \) by lemma [2]. For \( n = 0 \) we obtain \( \pi[R_0, R_0] = 0 \) from \([Q_0, Q_0] = \sum_{j,k} e_j^* [\delta(e_j^*), \delta(e_k^*)] c_k^* \) and the fact, that \( \pi \) is zero on \([J, J] \subset J\). Hence

\[
[R_{n+1}, R_{n+1}] \equiv s \delta [R_n, R_n] \pmod{\mathcal{F}^{n+3} X}.
\]

which vanishes modulo \( \mathcal{F}^{n+3} X \) by the assumption on \( \delta[R_n, R_n] \).

Next, by the graded Jacobi identity we have \( 0 = [R_{n+1}, [R_n, R_n]] \). From lemma [4] and lemma [50] we find that \( L_{n+1} := [R_{n+1}, -\delta] = L_{n+1} + \sum_{j=1}^{n+1} [Q_j, -] \) increases filtration degree. Hence \( L_{n+1} [R_{n+1}, R_{n+1}] \in \mathcal{F}^{n+4} X \) and thus \( \delta[R_{n+1}, R_{n+1}] \in \mathcal{F}^{n+4} X \).

Finally, we prove \([R_{n+1}, R_{n+1}] = [R_n, R_n] + 2[R_n, Q_{n+1}] + Q_{n+1}, Q_{n+1}] \in I(2)\). By hypothesis \([R_n, R_n] \in I(2)\). Next \( [Q_{n+1}, Q_{n+1}] \in I(2), I(2) \subset I(2) \) by lemma [50]. Now, by the same lemma, for \( j \in \{1, \ldots, n\}, [Q_j, Q_{n+1}] \in [I(2), I(2)] \subset I(2) \) and \([Q_0, Q_{n+1}] \in [I, I(2)] \subset I(2) \) which concludes the proof.

From \( Q_{n+1} = -\frac{1}{2} \delta s[R_n, R_n] \in \mathcal{F}^{n+2} X \) it follows that the \( R_n = \sum_{j=0}^{n} Q_j \) converge to an element \( R \in X^1 \) by lemma [51]. From lemma [38] we obtain \([R_n, R_n] \to [R, R] \) as \( n \to \infty \). We obtain

**Corollary 8.** \( [R, R] = 0 \).

**Proof.** We have \([R_{n+1}, R_{n+1}] \in \mathcal{F}^{n+2} X \) for all \( l \geq 0 \). Hence \([R, R] \in \mathcal{F}^{n+2} X \) for all \( n \) by lemma [39]. Hence \([R, R] = 0 \).

We also remark that \( R \) as defined above satisfies \( \equiv Q_0 \) (mod \( I(2) \)) since for \( j > 0 \), we have \( Q_j \in I(2) \). We are left to consider the cohomology of \( d_R = [R, -] \) on \( X/I \).
Lemma 9. The action of $d_R$ preserves the filtration and hence defines a differential on $\text{gr} X$, which is identified with $1 \otimes \delta$ under the natural isomorphism of lemma 7.

Proof. $R \in X^1$ and lemma 7 imply that $d_R$ preserves the filtration and hence descends to the associated graded. We have $Q_0 = L_0 + \delta$. Since $L_0$ increases filtration degree by lemma 4 we have that $Q_0$ and $\delta$ induce the same maps on $X$. Moreover, $K := R - Q_0 \in I^{(2)} \cap X^1$ by the remark above. Hence by lemma 5 $R$ and $Q_0$ induce the same maps on the associated graded. □

Corollary 10. $H^1(X/I, d_R) \cong P/J$ if $j = 0$ and zero otherwise.

Proof. $H^1(X/I, d_R) = H^1(\text{gr}^0 X, d_R) \cong H^1(B^0 \otimes_p T, 1 \otimes \delta) \cong H^1(T, \delta)$.

Lemma 11. The induced map $\delta : T \to T$ is a derivation and a differential of degree 1.

Proof. The derivation property follows immediately. For $a \in T$ we have $j(\pi(d_R(j(a)))) - d_R(j(a)) \in I$ and hence $d_R(j(\pi(d_R(j(a)))) - d_R(j(a))) \in I$ is in the kernel of $\pi$. The statement about the degree is obvious. □

Lemma 12. Under the identification of lemma 7, the differential $d_R$ induced on $\text{gr} X$ corresponds to the differential $1 \otimes \delta$ on $B \otimes_p T$.

Proof. Let $x \in \text{gr}^0 X$ and pick a representative $a \otimes b \in B^p \otimes_p T$. Then $d_R(ab) = d_R(a)b + (-1)^p ad_R(b)$. The first summand is in $F^{p+1} X$ and the second is equivalent to $1 \otimes (a \otimes b)$ modulo $F^{p+1} X$. □

Let $Q_0$ be the differential $d_R$ on $X/I$ as an element of $X$.

Remark 4. The complex $(X/I, d_R) = (T, \delta)$ is a Tate resolution of $P/J$. Hence the results from appendix A and sections 3.1 and 3.2.1 apply.

Lemma 13. We have $R \equiv Q_0 \mod (I^{(2)})$. Moreover, $[R, -] \equiv [Q_0, -] \mod (I)$. 

Proof. We have $[R, -] \equiv [Q_0, -] \mod (I)$ by construction. Expand $R - Q_0 = \sum_{j \geq 0} h_j$ with $h_j \in B^j \otimes_p T^{1-j}$. Such a decomposition exists by lemma 52. Decompose $h_j = \alpha_j + \beta_j$ with $\alpha_j \in B^j \otimes_p T^{1-j} \cap I^{(2)}_0$ and $\beta_j \in B^j \otimes_p T^{1-j} \setminus I^{(2)}_0$. Let $\{e^{(i)}_k\}_k$ be a basis of $\mathcal{M}^{-1}$ with dual basis $\{e^{(i)}_k\}^*$. By the Leibnitz rule, $\sum_j [\beta_j, e^{(i)}_k] = [R - Q_0, e^{(i)}_k] - \sum_j [\alpha_j, e^{(i)}_k] \in I$. Expand each $\beta_j = \sum_s a_{j,s} e^{(i)}_k$ with $a_{j,s} \in T$. We obtain $a_k[l] = [\beta_j, e^{(i)}_k] \in I_0$, hence all $a_{l,k}$ vanish. □

5. Uniqueness

Fix a unital, Noetherian Poisson algebra $P$ and a coisotropic ideal $J$. In a first step, we prove that two BFV models for $(P, J)$ related to the same Tate-resolutions have isomorphic cohomologies. This is a known fact [9] and is presented in sections 5.1 to 5.2. The key tool will be the notion of gauge equivalences. In a second step, we prove that BFV models for $(P = \mathbb{R}[x, y], J)$ on different spaces have isomorphic cohomologies too. We present this result in sections 5.3 to 5.5. Here, the key tool will be the notion of stable equivalence, introduced in the corresponding Lagrangian setting in 3. The novel part is that we do not require regularity assumptions, that imply that the constraint surface is smooth.
5.1. Gauge equivalences. We adapt the language of [5] and call the elements in \( g = X^0 \cap I^{(2)} \) gauge equivalences. Different BRST-charges for the same space will be related by these equivalences.

**Lemma 14.** The set of gauge equivalences \( g \) is a closed subset which forms a Lie algebra acting nilpotently on \( X/FPX \). The Lie algebra \( g \) exponentiates to a group \( G \) acting on \( X \) by Poisson automorphisms.

**Proof.** By lemma [13] the set is closed. By lemma [50] and the fact that \([X^0, X^0] \subset X^0\), this is a Lie algebra. By lemma [58] \( g \) acts on \( X/FPX \). By lemma [50] this action is nilpotent.

Fix \( a \in g \). Let \( x \in X^0 \). By lemma [50] \( ad_a x \in FPX^n \). Hence, by lemma [51] the sum \( \sum_{i=0}^{\infty} \frac{1}{i!} ad_a^i x \) converges to an element in \( X \) which we denote by \( \exp(ad_a)x \). Hence \( G \) acts on \( X \) by automorphisms. We are left to show that they preserve the bracket. By the graded Jacobi identity, \( ad \) is a derivation for the bracket. Thus, \( ad_a^k[x, y] = \sum_{i=0}^{k} \binom{k}{i} [ad_a^i x, ad_a^{k-i} y] \), and hence

\[
\exp(ad_a)[x, y] = \lim_{n \to \infty} \sum_{k=0}^{n} \frac{1}{k!} ad_a^k[x, y]
\]

\[
= \lim_{n \to \infty} \sum_{k=0}^{n} \frac{1}{k!} \sum_{i=0}^{k} \binom{k}{i} [ad_a^i x, ad_a^{k-i} y]
\]

\[
= \lim_{n \to \infty} \sum_{k=0}^{n} \frac{1}{k!} \sum_{i=0}^{k} \left( \frac{1}{i!} ad_a^i x, \frac{1}{(k-i)!} ad_a^{k-i} y \right)
\]

\[
= \lim_{n \to \infty} \sum_{j=0}^{n} \frac{1}{j!} ad_a^j x, \frac{1}{j!} ad_a^j y = [\exp(ad_a)x, \exp(ad_a)y].
\]

\( \square \)

**Lemma 15.** For \( x \in X^1 \) and a gauge equivalence \( g \) we have \( gx \equiv x \pmod{I^{(2)}} \).

**Proof.** Let \( c \in X^0 \cap I^{(2)} \) be a generator. Then \( gx - x = \sum_{j>0} \frac{1}{j!} ad_a^j x \equiv 0 \pmod{I^{(2)}} \) by lemma [50] and [49]. \( \square \)

5.2. Uniqueness for fixed \( X \). In this section we prove that given two solutions \( R, R' \) of the classical master equation in the same space \( X \) which induce the same map on \( X/I \) are related by a gauge equivalence. Since by lemma [13] gauge equivalences are Poisson automorphisms, this implies that they have isomorphic cohomologies. We use well-known techniques, which are adapted from [5].

**Remark 5.** If \( R \) solves \( [R, R] = 0 \) and \( g \in G \) is a gauge equivalence, then also \([gR, gR] = 0\).

**Lemma 16.** Fix \( p \geq 2 \). Let \( R, R' \in X^1 \) be two solutions of the classical master equation which induce the same maps on \( X/I \). Then, for \( 2 \leq q \leq p \), we have that \( R \equiv R' \pmod{I^{(q)} \cap FPX^1 + FP^{p+1} X^1} \) implies the existence of a gauge equivalence \( g \) with generator \( c \in FPX^0 \cap I^{(2)} \) such that \( gr \equiv R' \pmod{I^{(q+1)} \cap FPX^1 + FP^{p+1} X^1} \).

**Proof.** Let \( \delta \) be the common differential on \( X/I \) and \( Q_0 \) be the map \( \delta \) as an element of \( X \). Hence \( R \equiv Q_0 \equiv R' \pmod{I^{(2)}} \) by lemma [13]. Define \( v := R - R' \in I^{(q)} \cap FPX^1 + FP^{p+1} X^1 \subset FPX^1 \). We have \( 0 = [R + R', R - R] = [R + R', v] = 2[Q_0, v] + [R - Q_0, v] + [R' - Q_0, v] \equiv 2[Q_0, v] \pmod{FP^{p+1} X^2} \) by lemma [32]. By lemma [12] the maps \( d_R \) and \( d_{R'} \) induce the same map \( \delta \) on all of \( grX \). Since \( \delta v = [Q_0, v] = L_0 v \equiv [Q_0, v] \pmod{FP^{p+1} X^2} \) by lemma [4] the above implies \( \delta v = 0 \pmod{FP^{p+1} X^2} \). Hence \( v \) defines a cocycle \( \bar{v} \) in \( grpX^1 \). By lemma [2] there exists \( \bar{c} \in grpX^0 \) with \( d \bar{c} = \bar{v} \) since \( p > 1 \) and a corresponding representative \( c \in FPX^0 \cap I^{(q)} \), so that \( d\bar{c} = v \pmod{FP^{p+1} X^1} \). We have \( v + [c, R] = \delta(c) - [R, c] \equiv 0 \pmod{FP^{p+1} X^1} \)

since by construction, \( \delta \) is the map \( d_R \) on the associated graded. Set \( g := \exp c \). Calculate

\[
g \cdot R - R' = v + [c, R] + \sum_{j=2}^{\infty} \frac{1}{j!} ad_{\bar{c}}^j R \equiv \sum_{j=2}^{\infty} \frac{1}{j!} ad_{\bar{c}}^j R \pmod{FP^{p+1} X^1}.
\]

From lemma [37] we find that this sum is in \( FPX^1 \). We are left to show that the sum is in \( I^{(q+1)} \). By lemma [50] we have \( ad_{\bar{c}} R \in I^{(q)} \). By lemma [50] we obtain \( ad_{\bar{c}}^j R \in I^{(q+1)} \) for all \( j \geq 2 \) since \( q \geq 2 \). \( \square \)

**Theorem 17.** Let \( R, R' \in X^1 \) be solutions of the classical master equation with differentials inducing the same maps on \( X/I \). Then there exists a gauge equivalence \( g \in G \) with \( R' = gR \).
Proof. First we prove by induction that for all $p \geq 2$ there exist gauge equivalences $g_p \in G$ such that $g_p \cdots g_3 g_2 g_1 \equiv R' \pmod{F^{p+1} X^1 \cap I^{(2)}}$. By lemma 13 it suffices to prove $g_p \cdots g_3 g_2 g_1 \equiv R' \pmod{F^{p+1} X^1}$.

For $p = 2$ we may apply lemma 14 with $q = p$ by lemma 13 and since $I^{(2)} \subset F^2 X^1$. Now, assume the $g_p$ have been constructed to fulfill
\[ R'' := g_p \cdots g_2 R \equiv R' \pmod{F^{p+1} X^1 \cap I^{(2)}}. \]

By remark 5, $R''$ solves the classical master equation. Moreover $R'' \equiv Q_0 \pmod{I^{(2)}}$ by lemmas 15 and 13. Hence the pair $(R'', R')$ satisfies the requirements of lemma 16 with $q = 2$. We obtain a gauge equivalence $g_{p+1,2} \in F^{p+1} X^0$ and
\[ g_{p+1,2} R'' \equiv R' \pmod{I^{(3)} \cap F^{p+1} X^1 + F^{p+2} X^1}. \]

By remark 5 $R''' := g_{p+1,2} R''$ still satisfies the classical master equation and $R''' \equiv Q_0 \pmod{I^{(2)}}$ by lemma 15. Hence the pair $(R''' \circ R')$ again satisfies the requirements of lemma 16.

Applying the lemma for $q = 2, \ldots, p + 1$ we obtain gauge equivalences $g_{p+1,2}, \ldots, g_{p+1,p+1}$ with generators $g_{p+1,2, \ldots, p+1,p+1} \in F^{p+1} X^0$ such that
\[ g_{p+1,2} \cdots g_{p+1,p+1} R''' \equiv R' \pmod{F^{p+2} X^1 \cap I^{(2)}}. \]

Set $g_{p+1} := g_{p+1,2} \cdots g_{p+1,p+1}$. The induction step is complete.

We claim that $\lim_{m \to \infty} g_m g_{m-1} \cdots g_2$ converges point-wise to a gauge equivalence $g$. Since all $c_{m,j}$ are in $F^m X^0$ and this set is closed under the bracket, the Campbell-Baker-Hausdorff formula implies that the generator $c_m$ of $g_m$ is also in $F^m X^0$. Now denote the generator of $g_m \cdots g_2$ by $\gamma_m$. Then, the Campbell-Baker-Hausdorff formula implies that the generator $\gamma_{m+1}$ of $g_{m+1} g_m \cdots g_2$ satisfies
\[ \gamma_{m+1} = c_{m+1} + \gamma_m + \text{higher terms} \]
where “higher terms” are terms involving commutators of $c_{m+1}$ and $\gamma_m$ where each contains at least one instance of $c_{m+1} \in F^{m+1} X^0$. Since $\gamma_m \in X^0$ all these terms are in $F^{m+1} X^0$. Hence
\[ \gamma_{m+1} \equiv \gamma_m \pmod{F^{m+1} X^0}. \]

Hence there exists $\gamma \in X^0$ with $\gamma_m \to \gamma$ as $m \to \infty$. We set $g := \exp \gamma$. By lemma 10 this element defines a gauge equivalence. We claim that $\exp \gamma_m \to g$ point-wise. Let $x \in X^n$. Then
\[ \exp \gamma_m x - \exp \gamma x = [...]. \]

Modulo a fixed $F^k X$, this sum is finite and the number of terms does not depend on $m$ since all $\gamma_m$ are at least in $I^{(2)}$. Since $\gamma_m \to \gamma$ and the bracket is continuous in fixed degree by lemma 18 we obtain the claim.

Finally it follows that $\exp \gamma_{m+1} R - R' \in F^m X^1$ and thus $g R - R' \in F^m X^1$ for all $m$ which implies $g R = R'$.

5.3. Trivial BFV models. The key construction in the proof of uniqueness for different spaces $X$ in theorem 1 is the notion of stable equivalence. The idea of adding variables that do not change the cohomology was already present in [9]. It was first explicitly formalized in [5] in a similar situation in the Lagrangian setting. Roughly speaking, one proves that different BRST complexes for the same pair $(P, J)$ are quasi-isomorphic by adding more variables of non-zero degree. This is formalized by taking products with so-called trivial BFV models.

Let $P = K$ with zero bracket and $J = 0$. Then $P$ is a unital, Noetherian Poisson algebra and $J$ is a coisotropic ideal. Let $\mathcal{N}$ be a negatively graded vector space and $\mathcal{N}[1]$ the same space with degree shifted by $-1$. Define the differential $\delta$ on $\mathcal{M} = \mathcal{N} \oplus \mathcal{N}[1]$ by $\delta(a \oplus b) = b \oplus 0$. Set $T = P \otimes \text{Sym}(\mathcal{M})$ and extend $\delta$ to an odd derivation on $T$.

Lemma 18. The complex $(T, \delta)$ has trivial cohomology and hence defines a Tate resolution of $P/J = K$. 

Proof. On $\mathcal{M}$ there is a map $s(a \oplus b) = 0 \oplus a$ with $s\delta + \delta s = \text{id}_\mathcal{M}$. Extend also $s$ as an odd derivation to $T$. Then $s\delta + \delta s$ is an even derivation on $T$ and hence
\[ s\delta + \delta s = k \text{id} \quad \text{on } P \otimes \text{Sym}^k(\mathcal{M}). \]

Since both $s$ and $\delta$ preserve the $k$-degree, we have
\[ H^j(T, \delta) = \bigoplus_k H^j(P \otimes \text{Sym}^k(\mathcal{M}), \delta) = H^j(P \otimes \text{Sym}^0(\mathcal{M}), \delta) = \begin{cases} P, & \text{if } j = 0 \\ 0, & \text{otherwise} \end{cases} \]

□
Complete the space $Y_0 = P \otimes \text{Sym}(M \oplus M^*)$ to the space $Y$. Let $e_j$ be a homogeneous basis of $M$ such that $\delta(e_j) = e_k$ for some $k$ depending on $j$. Define $Q_0 = \sum_j e_j^* \delta(e^j)$ as in section 4. Since $Q_0 = \sum_{j,k} e_j^* \delta(e^j) = 0$, the construction of section 5 yields the BRST charge $S = Q_0$. Hence $(Y, S)$ is a BFV model for $(P, J) = (\mathbb{K}, 0)$. BFV models arising from this construction are called trivial.

Lemma 19. For trivial BFV models, $d_S$ equals the induced map of $\delta \oplus \delta^*$ on $Y$, where $\delta^* : Y \to Y$ is induced from the dual differential $\mathcal{M}^* \to \mathcal{M}^*$ of $\delta : \mathcal{M} \to \mathcal{M}$.

Proof. By acting on the generators $e_j$ and $e_j^*$ defined above one sees that the induced differential $d_S$ on $Y_0$ equals $\delta \oplus \delta^*$. Since $Y_0$ is dense in $Y$ and both maps are continuous the claim follows.

Lemma 20. We have $H^j(Y, d_S) = 0$ for $j \neq 0$ and $H^0(Y, d_S) = \mathbb{K}$.

Proof. Let $x \in Y^n$ with $d_Sx = 0$. By lemma 59 there are $x_j \in Y^n$ of form degree $j$ with $\sum_j x_j = x$. By continuity of the bracket, $0 = \sum d_Sx_j$. By lemma 60 $d_Sx_j = 0$ for all $j$, since $d_S$ reduces filtration degree by one. The proof of lemma 18 shows that there is a map $s$ on $\mathcal{M} \oplus \mathcal{M}^*$ whose extension to $Y_0$ satisfies

$$s(\delta \oplus \delta^*) + (\delta \oplus \delta^*)s = j \text{id} \quad \text{on} \quad Y_0^{n,j}.$$ 

Lift all maps to $Y$. The equation is still valid on $Y_0^{n,j}$. For $j > 0$, there are $y_j \in Y_0^{n-1,j+1}$ with $d_Sy_j = x_j$. By lemma 58 the element $y = \sum_{j>0} y_j$ is well defined and

$$x = \sum_{j>0} x_j + x_0 = d_Sy + x_0$$

with $x_0 \in Y_0^{n,0}$. For $n \neq 0$ this is the empty set and hence $x$ is exact. For $n = 0$ this set is $\mathbb{K}$. We are left to show that two distinct $d_S$-closed elements of $\mathbb{K}$ always define distinct cohomology classes. This follows from the fact that $d_Sy = \sum_j (\delta(e_j)[e^j, \delta(e^j)])$ is zero or has nonzero filtration degree since $\delta(e_j) = e_k$ for some $k$ depending on $j$.

5. Stable Equivalence. Let $P$ be a unital, Noetherian Poisson algebra and $J \subset P$ a cotegratic ideal. Let $(X, R)$ be a BFV model for $(P, J)$ and $(Y, S)$ be a trivial BFV model. Let $\mathcal{M}$ and $\mathcal{N}$ be the corresponding vector spaces. Define $Z$ as the completion of $Z_0 := X_0 \otimes Y_0 = \text{Sym}_P(\mathcal{U} \oplus \mathcal{U}^* \mathcal{N})$ where $\mathcal{U} = \mathcal{M} \oplus \mathcal{N}$ and $L = R \otimes 1 + 1 \otimes \mathcal{S}$. The derivation property of the bracket implies $d_L = d_R + (-1)^k d_S$ on $X^k \otimes Y$. Both $X$ and $Y$ naturally sit inside $Z$ as Poisson subalgebras since the inclusions $X_0 \to Z_0$ and $Y_0 \to Z_0$ preserve the respective filtrations.

Lemma 21. The pair $(Z, L)$ defines another BFV model for $(P, J)$.

Proof. Since the bracket between elements of $X$ and elements of $Y$ is zero, the element $L$ solves the master equation. The K"unneth formula implies the conditions on the cohomology.

We call $Z$ the product of $X$ and $Y$ and write $Z = X \otimes Y$. Adding the new variables in $\mathcal{N}$ does not change the cohomology of the BRST complex $X$.

Lemma 22. The natural map $X \to Z$ defines a quasi-isomorphism of differential graded commutative algebras.

Proof. From lemma 20 we know that there is a decomposition $Y = V \oplus \mathbb{R}$ such that the differential $d_S$ splits as a differential $\delta : V \to V$ with trivial cohomology and the zero map. Hence there is a map $s : V \to V$ with $\delta s + s \delta = \text{id}_V$. If we denote $\sigma = s \circ 0$ and $\pi : Y \to Y$ the projection onto $V$ we have $d_S \sigma + \sigma d_S = \pi$.

The natural map is injective: Let $x \in X^j$ be a $d_P$ closed element which is $d_{R+S}$ exact. Hence

$$x = d_{R+S} \sum_k x_k \otimes y_k = \sum_k (d_R x_k \otimes y_k + (-1)^{\deg y_k} x_k \otimes d_S y_k)$$

By applying $(1 - \pi)$ to both sides, we arrive at

$$x = \sum_k (d_R x_k \otimes (1 - \pi) y_k + (-1)^{\deg y_k} x_k \otimes (1 - \pi) d_S y_k).$$

We have $(1 - \pi) d_S y_k = d_S (1 - \pi) y_k = 0$. Hence $x = d_R \sum_k \lambda_k x_k$ for $\lambda_k = (1 - \pi) y_k \in \mathbb{R}$.

The natural map is also surjective: Consider an element $z$ that is $d_{R+S}$-closed:

$$z = \sum_k x_k \otimes y_k \quad 0 = d_{R+S} z = \sum_k (d_R x_k \otimes y_k + (-1)^{\deg y_k} x_k \otimes d_S y_k)$$
Applying $\sigma$ to the second equation, we obtain
\[ \sum_k (-1)^{\deg x_k} d_R x_k \otimes \sigma y_k = \sum_k x_k \otimes \sigma d_S y_k \]

Set $\lambda_k = (1 - \pi) y_k \in \mathbb{R}$. Then we conclude
\[ z - \sum_k \lambda_k x_k = \sum_k (x_k \otimes y_k - x_k \otimes (1 - \pi) y_k) = \sum_k x_k \otimes \pi y_k \]
\[ = \sum_k (x_k \otimes d_S \sigma y_k + x_k \otimes \sigma d_S y_k) \]
\[ = \sum_k (x_k \otimes d_S \sigma y_k + (-1)^{\deg x_k} d_R x_k \otimes y_k) = d_R + S \sum_k (-1)^{\deg x_k} x_k \otimes \sigma y_k \]

Now we are ready to formulate the notion of stable equivalence introduced in [3]:

**Definition 3.** Let $(X, R)$ and $(X', R')$ be two simple BFV models for $(P, J)$. We say that $(X, R)$ and $(X', R')$ are stably equivalent if there exist trivial BFV models $(Y, S)$ and $(Y', S')$ and a Poisson isomorphism $X \otimes Y \to X' \otimes Y'$ taking $R + S$ to $R' + S'$.

5.5. **Relating Tate Resolutions.** Now we want to consider BFV models $(R, X)$ and $(R', X')$ whose Tate resolutions $(X/I, d_R)$ and $(X'/I', d_{R'})$ are not equal. We have the notion of stable equivalence. Our aim is to prove that any two such BFV models are stably equivalent and that stably equivalent BFV models are quasi-isomorphic. As a tool we need

**Lemma 23.** Let $P = \mathbb{R}[x, y]$ with $[x, y] = \delta_{ij}$ and $J \subset P$ be a coisotropic ideal. Consider two Tate resolutions $(T, \delta)$ and $(T, \delta')$ of $P/J$ with $T = P \otimes \text{Sym}(\mathcal{M})$. Assume there is an isomorphism $\phi : (T, \delta) \to (T, \delta')$ of differential graded commutative algebras. Let $X$ be the completion of $X_0 = P \otimes \text{Sym}(\mathcal{M} \otimes \mathcal{M}^*)$. Then $\phi$ lifts to a Poisson automorphism $\Phi : X \to X$.

**Proof.** Let $\{e_j^{(i)}\}$ be a basis of $\mathcal{M}^{ij}$ and $\{e_j^{(i)*}\}$ be the respective dual bases. Then there are elements $a_{i, j, k}^{(l)}(x, y) \in \mathbb{R}[x, y]$ and invertible matrices $a_{jk}^{(l)} \in \mathbb{R}[x, y]$ such that
\[ \phi(e_j^{(i)}) = \sum_k a_{jk}^{(l)}(x, y) e_k^{(l)} + \sum_{i,j,k} a_{i,j,k}^{(l)}(x, y) e_i^{(l)} e_j^{(l)} e_k^{(l)} \]
where the sum runs over all integers $k \geq 2$ and $(j, l), \ldots, (j, k)$ with $l_1 + \cdots + l_k = l$ and is thus finite.

Consider indeterminants $Y_i, E_j^{(l)*} \in X_0$ of degree 0 and l respectively, defining
\[ S(x_i, Y_i, e_j^{(l)}), E_j^{(l)*} = \sum_i x_i Y_i + \sum_{j, k, l} a_{j, k}^{(l)}(x_i, Y_i) E_j^{(l)*} e_k^{(l)} + \sum_{j, l} a_{i, j, k}^{(l)}(x_i, Y_i) E_j^{(l)*} e_i^{(l)} e_j^{(l)} e_k^{(l)} \]

Consider the equations
\[ \frac{\partial S}{\partial x_i} = y_i, \quad \frac{\partial S}{\partial Y_i} = X_i, \quad \frac{\partial S}{\partial e_j^{(l)}} = (-1)^l e_j^{(l)*}, \quad \frac{\partial S}{\partial E_j^{(l)*}} = E_j^{(l)} \]
which read
\[ y_i = Y_i + \sum_{j, k, l} \frac{\partial a_{j, k}^{(l)}(x_i, Y_i)}{\partial x_i} E_j^{(l)*} e_k^{(l)} + \sum_{j, l} \frac{\partial a_{i, j, k}^{(l)}(x_i, Y_i)}{\partial x_i} E_j^{(l)*} e_i^{(l)} e_j^{(l)} e_k^{(l)} \]
\[ X_i = x_i + \sum_{j, k, l} \frac{\partial a_{j, k}^{(l)}(x_i, Y_i)}{\partial Y_i} E_j^{(l)*} e_k^{(l)} + \sum_{j, l} \frac{\partial a_{i, j, k}^{(l)}(x_i, Y_i)}{\partial Y_i} E_j^{(l)*} e_i^{(l)} e_j^{(l)} e_k^{(l)} \]
\[ e_j^{(l)*} = \sum_k a_{j, k}^{(l)}(x_i, Y_i) E_k^{(l)*} + \sum_{j', l'} a_{i, j', k}^{(l)}(j', l')(x_i, Y_i)(-1)^{l l'} E_j^{(l)*} \frac{\partial (e_j^{(l)} e_i^{(l)} e_k^{(l)})}{\partial e_j^{(l)}} \]
\[ E_j^{(l)} = \sum_k a_{j, k}^{(l)}(x_i, Y_i) e_k^{(l)} + \sum_{j, l} a_{i, j, k}^{(l)}(j, l)(x_i, Y_i) e_i^{(l)} e_j^{(l)} e_k^{(l)} \]

The linear part is invertible. Hence we can solve the equations for $(X_i, Y_i, E_j^{(l)*}, E_j^{(l)})$ in terms of $(x_i, y_i, e_j^{(l)}, e_j^{(l)*})$ (and vice versa) and hence also for $(x_i, Y_i, e_j^{(l)}, E_j^{(l)*})$ in terms of $(x_i, y_i, e_j^{(l)}, e_j^{(l)*})$ (and vice versa) in the completion $X$. Hence the function $S$ generates a Poisson automorphism $\Phi : X \to X$. 
by lemma[61] Let $I$ be the ideal of positive elements as defined previously. We have $\Phi(x_i) = X_i \equiv x_i = \phi(x_i) \mod I$ and $\Phi(y_i) = Y_i \equiv y_i = \phi(y_i) \mod I$. Thus also $\Phi(e_j^{\{l\}}) = E_j^{\{l\}} \equiv \phi(e_j^{\{l\}}) \mod I$. Hence $\Phi$ is a lift of $\phi$.

**Theorem 24.** Consider $P = \mathbb{R}[x_i, y_j]$ with $[x_i, y_j] = \delta_{ij}$. Any two BFV models for $(P, J)$ are stably equivalent.

**Proof.** Let $(X, R)$ and $(X', R')$ be BFV models with associated Tate resolutions $T := X/I \cong P \otimes \text{Sym}(M)$ and $T' := X'/I' \cong P \otimes \text{Sym}(M')$. By [5] theorem A.2, there exist negatively graded vector spaces $N$ and $N'$ with finite dimensional homogeneous components, differentials $\delta_N : \text{Sym}(N) \to \text{Sym}(N'), \delta_{N'} : \text{Sym}(N') \to \text{Sym}(N')$ with cohomology $\mathbb{K}$, and an isomorphism $\phi$ of differential graded commutative algebras

$$P \otimes \text{Sym}(M \oplus N) \to P \otimes \text{Sym}(M' \oplus N')$$

restricting to $\text{id}_P : P \to P$ in degree 0. Let $Y$ and $Y'$ be the trivial BFV models corresponding to $N$ and $N'$ with BRST charges $S$ and $S'$, respectively. Consider the spaces $Z = X \otimes Y$ and $Z' = X' \otimes Y'$. Together with the operators $L = R + S$ and $L' = R' + S'$ they form BFV models $(Z, L)$ and $(Z', L')$ for $(P, J)$ by lemma[61]

We now construct a Poisson isomorphism $\Phi : X \otimes Y \to X' \otimes Y'$ inducing $\phi$ and sending $R + S$ to $R' + S'$. By lemma[61] the map $\phi$ lifts to a Poisson automorphism $\Psi : X \otimes Y \to X' \otimes Y'$. Now $L'' = \Psi(L)$ solves $[-, -] = 0$ in $X' \otimes Y'$. Moreover, $[L'', -]$ induces $\delta'$ on $P \otimes \text{Sym}(M' \oplus N')$. By lemma[61] there exists a Poisson isomorphism $\chi$ of $X \otimes Y$ with $L' = \chi(L'')$. Set $\Phi = \chi \circ \Psi$.

We are now in the situation

$$X \otimes Y \xrightarrow{\Phi} X' \otimes Y'$$

where the vertical arrows represent natural maps which are quasi-isomorphisms by lemma[62] $\Box$

**Lemma 25.** The complexes of two stably equivalent BFV models are quasi-isomorphic. In particular, they have cohomologies which are isomorphic as graded commutative algebras.

**Proof.** Let $(X, R)$ and $(X', R')$ be two stably equivalent BFV models. Hence we are in the situation

$$X \otimes Y \xrightarrow{\Phi} X' \otimes Y'$$

where the downward arrows are quasi-isomorphisms of differential graded commutative algebras by lemma[62] and the bottom arrow is a Poisson isomorphism $X \otimes Y \to X' \otimes Y'$ sending $R + S$ to $R' + S'$. $\Box$

From theorem[24] and lemma[25] we obtain analogously to the treatment of the Lagrangian case in [5]

**Corollary 26.** Let $P = \mathbb{R}[x_i, y_j]$ with $[x_i, y_j] = \delta_{ij}$. Any two BRST-complexes arising from BFV-models for the same coisotropic ideal $J \subset P$ are quasi-isomorphic. Hence, the BRST cohomology is uniquely determined by $(P = R[x_i, y_j], J)$ up to an isomorphism of graded commutative algebras.

6. Cohomology

Let $P$ be a unital, Noetherian Poisson algebra and $J$ a coisotropic ideal. Let $(X, R)$ be a BFV model for $J \subset P$. In this section we analyze the cohomology of the complex $(X, d_R)$. We follow the strategy from [5].

6.1. Cohomology and Filtration. Recall that the associated graded is defined as $\text{gr}^p X = \mathcal{F}^p X / \mathcal{F}^{p+1} X$. The differential $d_R$ induces a map $\delta$ on $X/I = T = P \otimes \text{Sym}(M)$ and the results from section 4 apply.

**Lemma 27.** $H^j(\text{gr}^p, d_R) \cong B^p \otimes_P P/J$ for $j = p$ and $H^j(\text{gr}^p, d_R) \cong 0$ for $j \neq p$.

**Proof.** Fix $p$. By lemma[12] we have

$$H^j(\mathcal{F}^p X / \mathcal{F}^{p+1} X, d_R) \cong H^j(B^p \otimes_P T^{*-p}, 1 \otimes \delta) \cong B^p \otimes_P H^{j-p}(T, \delta) \cong B^p \otimes_P H^{j-p}(X/I, d_R)$$

$\Box$
Next, we want to prove that, in order to compute the cohomology in a fixed degree, one may disregard elements of high filtration degree.

**Lemma 28.** Let $j < p$ be integers with $p \geq 0$. Then $H^j(F^p X, d_R) = 0$.

**Proof.** Let $x \in F^p X^j$ be a cocycle representing a cohomology class in $H^j(F^p X, d_R)$. Then $x + F^{p+1} X^j$ defines a cocycle in $H^j(F^p X / F^{p+1} X, d_R)$. By lemma 27 there is $y_0 \in F^p X^{j-1}$ with $x - d_R y_0 \in F^{p+1} X^j$. Hence this element defines a cocycle in $H^j(F^{p+1} X / F^{p+2} X, d_R) = 0$. Hence there is $y_1 \in F^{p+1} X^{j-1}$ with $x - d_R y_0 - d_R y_1 \in F^{p+2} X^j$. Iterating this procedure we find a sequence $y_0, y_1, \ldots$ of elements $y_j \in F^{p+j} X^{j-1}$ with $x - d_R y_0 - \cdots - d_R y_j \in F^{p+j+1} X^j$. By lemma 31 the element $y := y_0 + \cdots + y_j \in X^{j-1}$ is well-defined and $y_0 + \cdots + y_j \to y$. Since all $y_j$ are in $F^p X^{j-1}$ and this set is closed by lemma 19 we have $y \in F^p X^{j-1}$. Finally, for $n$ fixed, and all $j$,

$$d_R y_0 + \cdots + d_R y_n + \cdots + d_R y_{n+j} - x \in F^{n+1} X^j$$

Since $d_R = [R, -]$ is continuous (lemma 18), we have $d_R y - x \in F^{n+1} X^j$. Since $n$ was arbitrary, $d_R y = x$. \qed

**Corollary 29.** The cohomology of $(X, d_R)$ is concentrated in non-negative degree.

**Corollary 30.** The natural map $H^j(X, d_R) \to H^j(X/F^{p+1} X, d_R)$ is an isomorphism for $j < p$ and injective for $j = p$.

**Proof.** The short exact sequence $0 \to F^{p+1} X \to X \to X/F^{p+1} X \to 0$ defines the long exact sequence

$$\cdots \to H^j(F^{p+1} X, d_R) \to H^j(X, d_R) \to H^j(X/F^{p+1} X, d_R) \to H^{j+1}(F^{p+1} X, d_R) \to \cdots$$

For $j < p$ the first term is zero and for $j < p$ both the first and the last terms are zero by lemma 28. \qed

### 6.2. Spectral Sequences

**Lemma 31.** Let $E^p_{a,q}$ be the spectral sequence corresponding to the filtered complex $F^p X^{p+q}$ with differential $d_R$. We have $H^*(X, d_R) \cong E^2_{2,0}$ as graded commutative algebras.

**Proof.** Begin with $E^0_{p,q} := F^p X^{p+q}/F^{p+1} X^{p+q}$. It is concentrated in degree $p \geq 0, q \leq 0$. By lemma 22 we have the following isomorphism of differential bigraded algebras:

$$E^p_{1,q} = H^q(E^p_0, d_R) = H^q(F^p X^{p+q}/F^{p+1} X^{p+q}, d_R) = H^{p+q}(F^p X / F^{p+1} X, d_R)$$

where

$$d_R : E^p_{1,q} \to E^p_{2,q+1} \cong B^p \otimes_{P/J} P/J, \quad \text{if } q = 0$$

$$0, \quad \text{if } q \neq 0.$$

Hence $E^p_{1,q}$ is concentrated in degree $p \geq 0$ and $q = 0$. Moreover, $d_R^{p,q}$ maps $E^p_{1,q}$ to $E^{p+1}_{1,q}$. Hence also $E^p_{2,q}$ is concentrated in $p \geq 0, q = 0$. Since $d_R$ maps $E^2_{2,0}$ to $E^2_{2,2}$ it is zero for degree reasons and hence the spectral sequence degenerates at $E_2$.

We are left to prove that the spectral sequence converges to the cohomology. By [3] chapter XV, proposition 4.1, this follows from lemma 28. \qed

We could use this lemma to prove $H^0(X, d_R) \cong (P/J)^J$ as algebras. However, we want to consider an additional structure on the latter space.

### 6.3. The Poisson Algebra Structure on $(P/J)^J$

**We have a Poisson algebra structure on $(P/J)^J$.**

**Lemma 32.** The Poisson algebra structure on $P$ induces a Poisson algebra structure on $(P/J)^J$.

**Proof.** Let $p + J, q + J \in (P/J)^J$, i.e. $p, q \in P$ with $[p, J], [q, J] \subset J$ and $a \in K$. $J$ is a linear subspace of $P$ and $ap + J \subset J$ hence $(P/J)^J$ is a vector space. $J$ is an ideal in $P$ and $[pq, J] \subset J$ by the Leibniz rule. Hence $(P/J)^J$ is an algebra. Finally, we have to show that the Poisson bracket descends to $(P/J)^J$.

We have $[p, q] \in J$ if $p \in J$. Hence the definition $[p + J, q + J] := [p, q] + J$ does not depend on the choice of representative. Moreover, $[[p, q], J] \subset J$ by the Jacobi identity. Hence the bracket is well-defined. \qed

We also have a Poisson algebra structure on $H^0(X, d_R)$.

**Lemma 33.** The graded Poisson algebra structure on $X$ induces a Poisson algebra structure on $H^0(X, d_R)$.

**Proof.** The cohomology of a differential graded commutative algebra is naturally a graded commutative algebra. In particular, the cohomology in degree $0$ is a commutative algebra. We have to show that the bracket descends to $H^0(X, d_R)$. Let $x, y \in X^0$ be representatives of cohomology classes in $H^0(X, d_R)$. Let $a \in K$. Then $x, y \in X^0$ is closed: $[R, [x, y]] = 0$ by the graded Jacobi identity. Moreover, if $x = d_R x'$ is exact, then $d_R[x', y] = [R, [x', y]] = -[x', [y, R]] = [y, [R, x']] = -[y, x] = [x, y]$. \qed
Those two structures are in fact isomorphic. We will explicitly construct a Poisson isomorphism. By lemma \[30\] we have \(H^0(X, d_R) \cong H^0(X/ F^2 X, d_R)\) as vector spaces.

**Lemma 34.** Representatives in \(X^0\) of cocycles in \(X^0/ F^2 X^0\) defining elements in \(H^0(X/ F^2, d_R)\) may be taken of the form

\[x = x_0 + \sum_{i,j \in L} a_{ij} e^*_i e^j\]

where \(L = \{n \in \mathbb{N} : \deg(e^*_i) = 1\}\), \(x_0 \in P\) and the \(a_{ij} \in P\) are chosen such that

\[\delta(e^j)\] \(x_0\) = \(\sum_{i \in L} a_{ji} \delta(e^i)\)

Conversely, every such element defines a cohomology class.

**Proof.** We have

\[X^0/ F^2 X^0 = P \oplus (P \otimes (M^*)^{-1} \otimes M^{-1})\]

\[X^{-1}/ F^2 X^{-1} = (P \otimes M^{-1}) \oplus (P \otimes (M^*)^1 \otimes M^{-2}) \oplus (P \otimes (M^*)^2 \otimes (M^{-1} \wedge M^{-1}))\]

Hence an arbitrary cochain may be taken to be of the form

\[x = x_0 + \sum_{i,j \in L} a_{ij} e^*_i e^j\]

for some \(x_0, a_{ij} \in P\). We compute with the help of lemma \[13\]

\[d_R x = [R, x_0] + \sum_{i,j \in L} ([R, a_{ij}] e^*_i e^j + [R, e^*_i] a_{ij} - [R, e^j] a_{ij})\]

\[= \sum_{j \in L} e^*_j [\delta(e^j), x_0] - \sum_{i,j \in L} \delta(e^j) a_{ij} e^*_i = \sum_{j \in L} \left( [\delta(e^j), x_0] - \sum_{i \in L} a_{ji} \delta(e^i) \right) e^*_i = (\mod F^2 X^1)\]

**Theorem 35.** \(H^0(X, d_R) \cong (P/ J)^d\) as Poisson algebras.

**Proof.** Let \(\pi : X \to P\) denote the projection onto all monomials which contain no factors of nonzero degree. Define the map \(\Phi : H^0(X, d_R) \to P/ J\) by \(\Phi([x]) := \pi(x) + J\). This map is well defined: For \(x = d_R y\) we obtain \(\pi(d_R y) = \pi(\delta(y))\) since \(d_R\) and \(\delta\) agree up to elements in \(I\). Hence

\[\pi(x) = \pi(\sum_j \delta(e^j)[e^*_j, y]) = \sum_j \delta(e^j) \pi([e^*_j, y]) \in J.\]

By lemma \[30\] we have \(H^0(X, d_R) \cong H^0(X/ F^2 X, d_R)\) as vector spaces. Hence we have a corresponding linear map \(X/ F^2 X \to P/ J\).

The image of either of those maps is \(J\)-invariant: Let \([x] \in H^0(X, F^2 X, d_R)\). According to lemma \[30\] we may pick a representative \(x_0 = \pi(x_0) + \sum_{i,j \in L} a_{ij} e^*_i e^j\) of \(x\) where \(a_{ij} \in P\) satisfy \([\delta(e^j), \pi(x_0)] = \sum_{i \in L} a_{ji} \delta(e^i)\). In particular \([\delta(e^j), \pi(0)] \in J\). Fix \(b \in J\). Then there exist \(b_j \in P\) with \(b = \sum_{i \in L} b_i \delta(e^i)\) and thus \([b, \pi(x_0)] = \sum_{i \in L} (b_i \delta(e^i), \pi(x_0)) + \delta(e^i) [b_i, \pi(x)] \in J\).

Hence we have two linear maps

\[\phi : H^0(X/ F^2 X, d_R) \to (P/ J)^d\]

\[\Phi : H^0(X, d_R) \to (P/ J)^d\]

given by projection onto the \(P\) component followed by modding out \(J\), which correspond to each other under the isomorphism \(H^0(X, d_R) \cong H^0(X/ F^2 X, d_R)\).

The map \(\phi\) is surjective: Let \(p \in P\) with \([J, p] \subset J\). By lemma \[30\] the element \(x = p + \sum_{i \in L} a_{ij} e^*_i e^j\) is a cocycle if \([\delta(e^j), p] = \sum_{i \in L} a_{ij} \delta(e^i)\). But those \(a_{ij} \in P\) exist since the \([\delta(e^j)]_{j \in L}\) generate \(J\). Hence also the map \(\Phi\) is surjective.

The map \(\Phi\) is injective: Let \(x \in X^0\) represent \([x] \in H^0(X, d_R)\) with \(\pi(x) \in J\). We claim that there exist \(y_j \in F^j X^{-1}\) with \(x - d_R (y_0 + \cdots + y_n) \in F^{n+1} X^0\). By lemma \[24\] we know that \(H^j(F^p X/ F^{p+1} X, d_R)\) is concentrated in degree zero with \(H^0(F^p X/ F^{p+1} X, d_R)\) \(\cong P/ J\) via the natural map. Now \(x + F^2 X^0\) defines the zero cohomology class in \(H^0(F^1 X/ F^2 X, d_R)\) since \(\pi(x) \in J\). Hence there exists \(y_0 \in F^{-1} X^0\) with \(x - d_R y_0 \in F^1 X^0\). Again, \(x - d_R (y_0 + F^2 X^0)\) defines the zero cohomology class in \(H^0(F^1 X/ F^2 X, d_R) = 0\). Hence there exists \(y_1 \in F^1 X^{-1}\) with \(x - d_R (y_0 + y_1) \in F^2 X^0\) and so on. Hence the \(y_j\) exist and their sum converges to an element \(y \in X^{-1}\) by lemma \[51\] which satisfies \(x - d_R y = 0\) by lemma \[49\].
Hence the map $\Phi$ is an isomorphism of vector spaces. This map also respects the product structure
\[
\Phi([x][y]) = \Phi([xy]) = \pi(xy) + J = \pi(x)\pi(y) + J = \Phi([x])\Phi([y])
\]
and is hence an isomorphism of algebras. Finally, map $\Phi$ respects the bracket:
\[
\Phi([x, y]) = \Phi([x, y]) = \pi([x, y]) + J = \pi(\pi(x), \pi(y)) + J
\]
since $[\pi(x) - x, X^0] \subset [\ker \pi \cap X^0, X^0] \subset \ker \pi$ where $\ker \pi \subset X$ is the ideal generated by all elements of nonzero degree.

7. An Example

Here we present an example, where the cohomology in degree zero has a nontrivial bracket and the cohomology in degree 1 does not vanish. It is obtained by considering the symplectic lift of the rotations of the punctured plane to the cotangent bundle of the punctured plane.

Consider $P = \mathbb{R}[x_1, x_2, y_1, y_2]$ with $[x_i, y_j] = \delta_{ij}$. The ideal $J \subset P$ generated by $\mu = x_1p_2 - x_2p_1$ is coisotropic. A Tate resolution of $J$ is given by
\[
0 \to P \cdot e \to P \to P/J \to 0
\]
where the differential $\delta$ is the derivation given by $\delta(e) = \mu$ and is zero on $P$. Indeed, this complex is a Koszul complex which is exact since $\mu \neq 0$ defines a regular sequence. We now apply the construction from section 3. We obtain $Q_0 = e^*\mu$ and $R = Q_0$ since $[Q_0, Q_0] = 0$. Hence $X = (P \cdot e) \oplus (P \oplus P \cdot e^*e) \oplus (P \cdot e^*)$. One easily calculates
\[
H^0(X, d_R) = \{a + be^*e : [\mu, a] = \mu b, a, b \in P\}
\]
Notice, that the isomorphism $H^0(X, d_R) \to (P/J)^J$ given by projection onto $P$ is evident here. Moreover, the bracket on this space does not vanish: $x_1^2 + x_2^2$ and $p_1^2 + p_2^2$ define cohomology classes, where $[x_1^2 + x_2^2, p_1^2 + p_2^2] = 4(x_1p_1 + x_2p_2)$ is not in $J$. Furthermore,
\[
H^1(X, d_R) = \{ae^*: a \in P\}/\{d_R(a + be^*e) : a, b \in P\} \cong \frac{P}{[\mu, a] + \mu b : a, b \in P} \cong P
\]
does not vanish since $\deg_0[\mu, a] \geq 1$ and $\deg_0(\mu b) \geq 2$. Here, $\deg_0$ denotes the degree in $P = \mathbb{R}[x, y]$. 
Appendix A. Graded Poisson Algebras

Let \((P, [\cdot, \cdot])\) be a unital, Poisson algebra over \(\mathbb{K} = \mathbb{R}\). Let \(M\) be a negatively graded vector space with finite dimensional homogeneous components. Let \(M^*\) be the positively graded vector space with homogeneous components \((M^*)^i = (M^{-i})^*\). Define the graded algebra \(X_0 = P \otimes \text{Sym}(M \oplus M^*)\).

**Lemma 36.** The bracket \([-\cdot, -]\) on \(P\) naturally extends to a skew-symmetric, bilinear map \([-\cdot, -]\) on \(X_0\) via the natural pairing of \(M\) and \(M^*\). This map has degree zero. Moreover, it is a derivation for the product on \(X_0\) and satisfies the Jacobi identity. Thus, it turns \(X_0\) into a graded Poisson algebra.

**Proof.** First we define a bracket on \(\text{Sym}(M \oplus M^*)\). For \(x \in M\) and \(a \in M^*\) we set

\[
[x, x]_1 = 0, \quad [a, a]_1 = 0, \quad [x, a]_1 = a(x), \quad [a, x]_1 = (-1)^{\deg a \deg x} a(x)
\]

and extend this definition as a bi-derivation to all of \(\text{Sym}(M \oplus M^*)\). It is then a bilinear, skew-symmetric map \([-\cdot, -] : \text{Sym}(M \oplus M^*) \times \text{Sym}(M \oplus M^*) \rightarrow \text{Sym}(M \oplus M^*)\) of degree zero which is by definition a derivation for the product. The expression

\[
\zeta(a, b, c) := (-1)^{\deg a \deg c}[a, [b, c]] + \text{cyclic permutations}
\]

satisfies \(\zeta(a_1 a_2, b, c) = (-1)^{\deg a_1 \deg c} a_1 \zeta(a_2, b, c) + (-1)^{\deg b \deg c} \zeta(a_1, b, c) a_2\) and similar derivation-like statements for the other entries. Let \(e_j\) be a homogeneous basis of \(M\) and \(e^*_j\) its dual basis. Then the bracket of any two of those generators is a scalar, whence \(\zeta\) is zero on generators. By the above derivation-type property, \(\zeta\) vanishes identically, proving the graded Jacobi identity.

Now set \([-\cdot, -] = [-\cdot, -]_0 + [-\cdot, -]_1\) on \(X_0\). \(\square\)

The grading on \(M\) induces a grading \(X_0\). We obtain a filtration: \(F^n X_0\) is defined to be the ideal generated by elements of \(X_0\) of degree at least \(n\). We set \(F^n X_0^n = F^n X_0 \cap X_0^n\). We also define \(I_0 := F^1 X_0\) and \(I^{(n)}_0 := I_0 \cdots I_0\) to be the \(n\)-fold product ideal.

A.1. Compatibility of Filtration and Bracket on \(X_0\). We use the derivation properties of the bracket on \(X_0\) to derive compatibility relations between the filtration and the bracket.

**Lemma 37.** For \(m, n \in \mathbb{Z}\) and \(p, q \in \mathbb{N}_0\) we have \([F^p X_0^q, F^p X_0^q] \subset F^{m, n}(p-q) X_0^{m+n}\) where

\[
\{r, m, p, q\} = \max\{m + n, \min\{\max\{p, q + n\}, \max\{q, p + m\}\}\}.
\]

**Proof.** Let \(a, b, u, v \in X_0\) be homogeneous elements with \(\deg a + \deg u = n, \deg b + \deg v = m, \deg u = p, \) and \(\deg v = q\). Suppose without loss of generality that \(p \geq n\) and \(q \geq m\). Then

\[
[au, bv] = \pm ab[u, v] + \pm av[u, b] + \pm ub[a, v] \pm uv[a, b].
\]

We now combine different factors to construct elements of high degree using the fact that the bracket has degree zero. The first summand is in \(F^{p+q} X_0\). The second summand is in \(F^{\max(p, p+m)} X_0\). The third summand is in \(F^{\max(p, p+n)} X_0\). Finally, the last summand is again in \(F^{p+q} X_0\). So the whole sum is in \(F^r X_0\) where \(r = \min\{p + q, \max\{p, q + n\}, \max\{q, p + m\}\} = \min\{\max\{p, q + n\}, \max\{q, p + m\}\}\). \(\square\)

**Corollary 38.** We obtain for \(p, q \in \mathbb{N}_0\) and \(m, n \in \mathbb{Z}\),

1. \([F^p X_0^q, F^p X_0^q] \subset F^{l(p, q)} X_0\), where \(l(p, q) = \max\{p, q\}, \quad \text{if } p \neq q\), \(p + 1, \quad \text{if } p = q\).

2. \([F^p X_0^m, X_0^m] \subset F^p X_0\), provided \(m \geq 0\).

3. \([F^p X_0^m, F^p X_0^m] \subset F^{l(n, m)} X_0^{m+n}\), where \(l(n, m)(p) = p - \max\{|n|, |m|\}\).

**Lemma 39.** We have \([X_0^1 \cap I_0^{(2)}, F^m X_0] \subset F^{m+1} X_0\).

**Proof.** Let \(a, u_1, u_2 \in X_0\) with \(\deg(a) = 1 - n, \deg(u_1) + \deg(u_2) = n, \) and \(\deg(u_1), \deg(u_2) > 0\). Let \(b, v \in X_0\) with \(\deg(v) = m\). \([au_1 u_2, b v] = au_1 u_2 b v \pm au_2 u_1 b v \pm u_1 u_2 a b v \pm [au_1 u_2, b] v \in F^{m+1} X_0\). \(\square\)

**Lemma 40.** The ideal \(I_0\) is closed under the bracket.

**Proof.** Let \(a, u, b, v \in X_0\) with \(\deg(a) = \deg(v) = 1\). Then \([au, bv] = \pm (ab)[u, v] \pm (a[u, b]) v \pm ([a, v] b) a \pm ([a, b] a) v \in I_0\). \(\square\)
A.2. Completion. For each \( j \), we use the filtration on \( X^j_0 \) to complete this space to the space

\[
X^j = \lim_{\to \pm} \frac{X^j_0}{F^p X_0 \cap X^j_0}.
\]

The sum and scalar multiplication on \( X^j_0 \) extend to this space, turning \( X = \bigoplus_j X^j \) into a graded vector space. The product of two elements \((x_p + F^p X^j_0)_p \in X^j \) and \((y_p + F^p X^k_0)_p \in X^k \) is defined to be \((x_p y_p + F^p X^{j+k})_p \in X^{j+k} \). This definition does not depend on the choice of representatives since the product is compatible with the filtration. Moreover, it defines an element of \( X^{j+k} \) since for \( p \leq q \) we have \( x_p y_p \equiv x_q y_q \) (mod \( F^p X^{j+k}_0 \)), since we may shift the representatives of \( x \) and \( y \). The multiplication is compatible with the addition turning \( X \) into a graded commutative algebra.

Endow \( X^j_0/F^p X^j_0 \) with the discrete topology and \( \prod_p X^j_0/F^p X^j_0 \) with the product topology. Equip \( \lim_{\to \pm} X^j_0/F^p X^j_0 \subset \prod_p X^j_0/F^p X^j_0 \) with the subspace topology. Finally, equiv \( X = \bigoplus_j X^j \) with the product topology.

Hence a sequence \( \{x_i\}_i \subset X^j \), with \( x_i = (x_{i,p} + F^p X^j_0)_p \), converges to an element \( x = (x_p + F^p X^j_0)_p \in X^j \) if and only if for all \( p \in \mathbb{N}_0 \) there exists a \( l_0 \) such that for all \( l \geq l_0 \) we have \( x_{p,l} \equiv x_p \) (mod \( F^p X^j_0 \)). A sequence \( \{x_i\}_i \subset X \) converges to an element \( x \in X \) if and only if all homogeneous components converge. Since \( X \) is first-countable, continuity is characterized by the convergence of sequences. We immediately obtain:

**Lemma 41.** The sum \( X \times X \to X \) is continuous.

For the product, only a weaker statement holds in general:

**Lemma 42.** The product \( X \to X \) is continuous in each entry. For each pair \((j,k)\) in \( \mathbb{Z}^2 \), the product \( X^j \times X^k \to X^{j+k} \) is continuous.

**Proof.** Consider a sequence \( \{x_i\}_i \) in \( X \) converging to \( x \in X \) and \( y \in Y \). Denote the homogeneous components of \( x_i \) by \( x_i = (x_{i,p} + F^p X^j_0)_p \) and similarly for \( x \) and \( y \). Fix \( i \in \mathbb{Z} \) and \( p \in \mathbb{N}_0 \). The \( l \)-th homogeneous component of \( x_i y \) has a \( p \)-th component with representative \( \sum_{j \in C} x_{i,p} x_{j,p} y_j \) where \( C \subset \mathbb{Z} \) is the finite set for which \( y_j \neq 0 \). It does not depend on \( i \). (Such a finite set which is independent of \( i \) only exists in general when one entry of the product remains fixed.) We have

\[
\sum_{j \in C} x_{i,p} x_{j,p} y_j = \sum_{j \in C} \left( (x_{i,p} - x_{i,j}) y_j + x_{i,j} y_j \right)
\]

For each \( j \in C \) pick a number \( i_{0,j} \) such that for \( i_j \geq i_{0,j} \) we have \( x_{i,j} \equiv x_{i_{0,j}} \) (mod \( F^p X^{j-}\)) and let \( i_0 \) be their maximum. Now, for \( i \geq i_0 \), we have \( \sum_{j \in C} x_{i,j} y_j = \sum_{j \in C} x_{i_{0,j}} y_j \) (mod \( F^p X^0_0 \)). The second statement follows similarly.

Next, we approximate elements in \( X \) by elements in \( X^0_0 \).

**Lemma 43.** The map \( \iota : X^0_0 \to X^n \) sending \( x \in X^0_0 \) to \((x + F^p X^0_0)_p \) is injective.

**Proof.** Since \( \iota \) is linear, the claim follows from \( \bigcap_p F^p X^0_0 = \{0\} \).

**Lemma 44.** For \( x = (x_p + F^p X^j_0)_p \in X^n \) we have \( \lim_{m \to \infty} \iota(x_m) = x \).

**Proof.** Fix \( p \). For \( m \geq p \) we have that the \( p \)-th component of \( \iota(x_m) - x \) is \( x_m - x_p \in F^m X^0_0 \).

**Corollary 45.** \( X^0 \) can be considered a dense subset of \( X \).

Now, we turn to the extension of the bracket to the completion. Let \( x = (x_p + F^p X^j_0)_p \in X^j \) and \( y = (y_p + F^p X^k_0)_p \in X^k \). We define \([x,y] \in X^{j+k}\) to be the element

\[
([s_{x,k}(p), s_{y,k}(p)] + F^p X^{j+k}_0)_p,
\]

where \( s_{x,k}(p) = p + \max\{|j|, |k|\} \). This definition does not depend on the representatives of \( x \) and \( y \) by corollary 42 since \( t_{j,k}(s_{x,k}(p)) = s_{j,k}(t_{j,k}(p)) = p \). Moreover, it defines an element of \( X^{j+k} \). For \( p \leq q \) we have \([s_{x,k}(p), s_{y,k}(q)] \equiv [s_{x,k}(q), s_{y,k}(p)] \) (mod \( F^{s_{x,k}(p)} \)) since we may shift the representatives of \( x \) and \( y \). We extend this bracket as a bilinear map to \( X \times X \).

**Lemma 46.** The extension of the bracket on \( X^0 \) is a skew-symmetric, bilinear, degree zero map on \( X \) that satisfies the graded Jacobi identity (i.e. the bracket is an odd derivation for itself).
Proof. It is trivial that the extended bracket is skew-symmetric, bilinear degree zero map. These properties follow directly from the definitions.

We prove the graded Jacobi identity. Consider elements
\[ x = (x_0 + \mathcal{F}^n X^n_0)_p \in X^j \]
\[ y = (y_0 + \mathcal{F}^n X^n_0)_p \in X^k \]
\[ z = (z_0 + \mathcal{F}^n X^n_0)_p \in X^l. \]
The \( p \)-th element of \([y, z]\) has representative \([y_{s_0, l}(p), z_{s_0, l}(p)]\). Hence the \( p \)-th element of \([x, [y, z]]\) has representative \([x_{s_0, l}(p), y_{s_0, l}(p), z_{s_0, l}(p)]\). We now want to bound the indices from above by a term which is invariant under cyclic permutations of \((j, k, l)\). The function \( r_{j, k, l}(p) := p + 2|\{}j\} + |\{k\}| \) does the job. So, \((-1)^j[x_{r_{j, k, l}(p)}, [y_{r_{j, k, l}(p)}, z_{r_{j, k, l}(p)}]] + \) cyclic permutations which vanishes by the graded Jacobi identity on \( X_0 \).

\[ \square \]

Lemma 47. The bracket on \( X \) is a derivation for the product.

Proof. Let
\[ x = (x_0 + \mathcal{F}^n X^n_0)_p \in X^j \]
\[ y = (y_0 + \mathcal{F}^n X^n_0)_p \in X^k \]
\[ z = (z_0 + \mathcal{F}^n X^n_0)_p \in X^l. \]
The \( p \)-th element of \([x, y, z] - (xy, z) + (-1)^{|y|}|x, z\) has representative
\[ [x_{s_0, l}(p), y_{s_0, l}(p), z_{s_0, l}(p)] = \left( \mathcal{F}^p [y_{s_0, l}(p), z_{s_0, l}(p)] + (-1)^{|y|} (x_{s_0, l}(p), y_{s_0, l}(p)) \right) \equiv [x_{s_0, l}(p), y_{s_0, l}(p)] + (-1)^{|y|} (x_{s_0, l}(p), y_{s_0, l}(p)) \pmod{\mathcal{F}^p X^{n_j + k + l}} \]
where \( q := p + |\{n\}| + |\{l\}| + |\{k\}| \) is a common upper bound of all indices appearing in the formula. The last line vanishes by the derivation property of the bracket on \( X_0 \).

\[ \square \]

Lemma 48. For each pair \((j, k) \in \mathbb{Z}^2\), the map \([-,-] : X^j \times X^k \to X \) is continuous in each entry.

Proof. Let \( x_n = (x_n + \mathcal{F}^n X^n_0)_p \in X^j \) and \( y_n = (y_n + \mathcal{F}^n X^n_0)_p \in X^k \) define two sequences converging to the respective elements \( x = (x_0 + \mathcal{F}^n X^n_0)_p \in X^j \) and \( y = (y_0 + \mathcal{F}^n X^n_0)_p \in X^k \). Fix \( p \), set \( s = s_{j,k}(p) \), and pick \( n_0 \) such that for \( n \geq n_0 \),
\[ x_n, s \equiv x_s \pmod{\mathcal{F}^s X^n_0} \]
\[ y_n, s \equiv y_s \pmod{\mathcal{F}^s X^n_0}. \]
Let \( n \geq n_0 \). The \( p \)-th element of \([x_n, y_n] - [x, y]\) has the representative \([x_n, s, y_n, s] - [x, s, y, s] \in \mathcal{F}^{s_{j,k}(p)} \subset \mathcal{F}^p X^{n_j + k} \) by corollary [23].

Now consider a sequence \( \{x_i\} \) in \( X \) converging to \( x \) in \( X \) and fix \( y \in X \). Denote the homogeneous components of \( x_i \) by \( x^j_{i} = (x^j_{i, p} + \mathcal{F}^n X^n_0)_p \) and similarly for \( x \) and \( y \). Fix \( l \in \mathbb{Z} \) and \( p \in \mathbb{N}_0 \). Set \( C = \{j \in \mathbb{Z} : y^j \neq 0\} \). This is a finite set. The \( l \)-th homogeneous component of \([x, y]\) has a \( p \)-th component with representative
\[ \sum_{j \in C} [x^j_{i, s_{j, l}(p)}, y^j_{s_{j, l}(p)}] \]
Set \( s = \max\{s_{j, l}(p) : j \in C\} \) and pick \( n_0 \) such that for \( n \geq n_0 \) and all \( j \in C \) we have \( x^j_{n, s} \equiv x^j_{s} \pmod{\mathcal{F}^s X^{n_j}} \). For such \( n \),
\[ \sum_{j \in C} [x^j_{s_{j, l}(p)}, y^j_{s_{j, l}(p)}] \equiv \sum_{j \in C} [x^j_{s}, y^j_{s}] \equiv \sum_{j \in C} [x^j_{s}, y^j_{s}] \pmod{\mathcal{F}^p X^n_0}. \]

\[ \square \]

A.3. The Filtration and the Bracket on the Completion. The filtration on \( X_0 \) induces a filtration on the completion with homogeneous components
\[ \mathcal{F}^p X^n := \lim_{\leftarrow q} \frac{\mathcal{F}^p X^n_0}{\mathcal{F}^p X^q_0} = \{ (x_q + \mathcal{F}^q X^n_0)_q \subset X^n : x_q \in \mathcal{F}^p X^n_0 \} \]
This defines a homogeneous ideal \( \mathcal{F}^p X = \bigoplus_n \mathcal{F}^p X^n \) in \( X \). We set \( I := \mathcal{F}^1 X \) and \( I^{(n)} := \bigoplus_m \lim_{\leftarrow p} \frac{X^n_0 \cap I^{(n)}}{\mathcal{F}^p X^n_0 \cap I^{(n)}} \). Those are homogeneous ideals in \( X \).

Lemma 49. For each \( j \in \mathbb{Z} \), the sets \( \mathcal{F}^p X^j \) and \( I^{(2)} \cap X^j \) are closed.
Proof. Consider the first statement. Since $X$ is first-countable, it suffices to consider sequences $x_n = (x_{n,q} + F^q X^j_0)_q$ converging to an $x = (x_q + F^q X^j_0)_q$ in $X$ with $x_{n,q} \in F^q X_0^j$ and show that $x \in F^q X^j$. So, fix $q \geq p$. Let $n$ be an integer with $x_{n,q} \equiv x_q \pmod{F^q X^j_0}$. Then $x_q \equiv x_{n,q} \equiv 0 \pmod{F^q X_0^j}$.

Now let $x_n = (x_{n,p} + F^p X_0^n)_p$ be a sequence converging to $x = (x_p + F^p X_0^n)_p$ in $X$ with $x_{n,p} \in I_0^{(2)}$. Fix $p$. For $n$ large enough we may replace $x_p$ by $x_{n,p}$ in $I_0^{(2)}$. □

Lemma 50. Fix $p \in \mathbb{N}_0$.
1. $[I^{(2)}, I^{(2)}] \subset I^{(2)}$.
2. $[I^{(2)}, I^{(p)}] \subset I^{(p+1)} \subset F^{p+1} X$.
3. $[I^{(p)}, X^1] \subset I^{(p)}$.

Proof. The first statement: Consider elements $u = (u_p + F^p X_0^n)_p$ and $v = (v_p + F^p X_0^n)_p$ of $X$ with $u_p, v_p \in I_0^{(2)}$. Then the $p$-th element of $[u, v]$ has the representative $[u_{p,q}, v_{p,q}] \in [I_0^{(2)}, I_0^{(2)}] \subset I_0^{(2)}$ by the Leibnitz rule.

Now the second statement: First consider $p = 0$. The by the Leibnitz rule, $[I_0^{(2)}, X] \subset I_0[I_0, X] \subset I_0$.
Now consider $p > 0$. By repeated use of the Leibnitz rule

$$[I_0^{(2)}, I_0^{(p)}] \subset [I_0^{(2)}, I_0^{(p-1)}] \subset I_0[I_0, I_0]I_0^{(p-1)} \subset I_0^{(p+1)}$$

by lemma 40. The statement generalizes to the completion, as in the case above.

The third statement follows analogously by picking representatives. □

Lemma 51. Let $l \mapsto q(l)$ define an unbounded non-decreasing function $\mathbb{N} \to \mathbb{N}$. Let $x_l = (x_{l,p} + F^p X_0^n)_p \in F^{q(l)} X^n$ define a sequence of elements in $X^n$. Then $\sum_{l=1}^{\infty} x_l$ converges to an element $x \in X^n$.

Proof. We may suppose $q(l) = l$. Define $x_p := \sum_{l=1}^{p-1} x_{l,p}$. Then $x := (x_p + F^p X_0^n)_p$ defines an element of $X^n$ since, for $p \leq k$, we have

$$x_q - x_p = \sum_{l=0}^{q-1} x_{l,q} - \sum_{l=0}^{p-1} x_{l,p} = \sum_{l=0}^{p-1} (x_{l,q} - x_{l,p}) + \sum_{l=p}^{q-1} x_{l,q} \in F^{p+1} X_0^n.$$  

We claim that $\sum_{l=0}^{k} x_l$ converges to $x$ as $k \to \infty$. Fix $p$. Let $k \geq k_0 := p$. Then the $p$-th element of $\sum_{l=0}^{k} x_l - x$ has representative $\sum_{l=0}^{k} x_{l,p} - x_p = \sum_{l=p}^{k} x_{l,q} \in F^{q} X_0^n$. □

Lemma 52. Each $H \in X^n$ can be expanded as $H = \sum_{p \geq 0} h_p$ with $h_p \in B^p \otimes_p T^{n-p}$.

Proof. Write $H = (x_p + F^p X_0^n)_p$ with $x_0 = 0$. Redefine $x_p$ such that $x_p$ does not contain a summand in $F^p X_0^n$. Set $h_p = x_{p+1} - x_p \in F^p X_0^n \setminus F^{p+1} X_0^n = B^p \otimes_p T^{n-p}$. Then by lemmas 44 and 51

$$\sum_{p} h_p = H.$$  

□

Lemma 53. All statements from section A.7 are valid for $X_0$ replaced by $X$.

Proof. The bracket on $X$ is defined by acting on representatives with the bracket of $X_0$ where the statements hold. □

A.4. Extension of Maps. Next, we consider the problem of extending maps on $X_0$ to $X$.

Remark 6. A linear map on $X_0$ of a fixed degree preserving the filtration naturally extends to a linear map on $X$ preserving the filtration. This extension is continuous.

Lemma 54. Let $f$ be a derivation on $X_0$ of a fixed degree which preserves the filtration. Then its extension preserves $I_0^{(2)}$.

Proof. Such a derivation preserves $I_0^{(2)}$. The statement follows. □

Next, we want to consider separate gradings by positive and negative elements. We define $T = P \otimes \text{Sym}(M)$ and $B = P \otimes \text{Sym}(M^*)$. Then $X_0 = T \otimes_p B$.

Lemma 55. A linear map on $X_0$ of a fixed degree preserving the filtration naturally extends to $X$. If the map on $X_0$ is of the form $1 \otimes g : B \otimes_p T \to B \otimes_p T$ for some linear map $g : T \to T$, then its extension preserves $I_0^{(2)}$.

Proof. The map $1 \otimes g$ preserves $I_0^{(2)}$. □
A.5. The Associated Graded. The associated graded \( \text{gr} X \) of \( X \) is defined as the graded algebra with homogeneous components \( \text{gr}^p X = F^p X / F^{p+1} X \). We have \( \text{gr}^0 X = X / I \).

**Lemma 56.** \( X / I \) is naturally identified with \( P \otimes \text{Sym}(M) \).

**Proof.** Let \( x = (x_p + F^p X_0^p) \in X^n \). Let \( u_p \in I_0 \) and \( z_p \in X^n \) such that \( x_p = u_p + z_p \) and \( z_p \) does not contain a summand in \( I_0 \) (or is zero), i.e. \( z_p \in \text{Sym}_p(M) \). Then \( z_p - z_1 = x_p - x_1 - (u_p - u_1) \in I_0 \).

Hence \( x = z_1 \) for all \( p \). Hence \( x := (z_1 + F^p X_0^p) \in X^n \) and \( x \) define the same equivalence class in \( X / I \).

It is clear that different values of \( z_1 \) yield different equivalence classes. \( \square \)

**Lemma 57.** There is a natural isomorphism \( \text{gr}^* X \cong B^* \otimes P T \) of graded algebras.

**Proof.** The inclusions \( B^p \hookrightarrow F^p X \) induce a \( P \)-linear map \( B \rightarrow \text{gr} X \). From this, we obtain a map \( B \otimes P T \rightarrow \text{gr} X \) via \( B^p \otimes P T \ni b \otimes t \mapsto bt \in F^p X \). The claim follows since the monomials in \( B^p \) span the free \( T \)-module \( F^p X \). The image of the monomials in \( B^p \) under the above map is obviously \( T \)-linearly independent. Now for a given \( x \in F^p X \) decompose it into homogeneous elements \( x^n = (x_{n,q} + F^q X_0^n)q \in F^p X^n \). Split \( x_{n,q} = b_{n,q} + y_{n,q} \) with \( y_{n,q} \in F^{q+1} X_0^n \) and \( b_{n,q} \in F^p X_0^n \) does not contain a summand in \( F^{q+1} X_0^n \). Then \( b_{n,q} - b_{n,p+1} \in F^{p+1} X_0^n \) for \( q > p \), and hence this difference vanishes. Set \( b^n = (b_{n,p+1} + F^q X_0^n)q \). We have that \( b^n \) and \( x^n \) define the same equivalence class in \( \text{gr}^p X^n \) and hence \( b = \sum b^n \) and \( x \) define the same equivalence class in \( F^p X \).

Each \( b^n \) is in the image of \( B^p \otimes P T \rightarrow F^p X \). \( \square \)

A.6. Form Degree. We can filter the algebra \( X \) by form degree. For \( n \in \mathbb{Z} \) and \( j \in \mathbb{N}_0 \), we set \( X_0^{n,j} = P \otimes \text{Sym}^j(M \oplus M^*) \cap X^n_0 \) and define the homogeneous components of \( X^{(j)} = \bigoplus_n X_0^{n,j} \) to be

\[
X_0^{n,j} = \lim_{\leftarrow n} \frac{X_0^{n,j}}{X_0^{n+1,j}}
\]

We have

**Lemma 58.** If \( x_j \in X^n \) have form degree \( j \) then \( \sum x_j \) converges in \( X \).

**Proof.** Fix \( n \). Let \( g \) denote the ghost degree and \( a \) the anti-ghost degree. Hence \( g \geq 0 \), \( a \leq 0 \) and \( a + g = \deg \). We decompose \( x_j = x_{j,1} \ldots x_{j,j} \) according to form degree.

Let \( l_j \) be the number of factors of positive degree in this decomposition. Then \( g(x_j) \geq l_j \) and \( a(x_j) \leq -(l_j - j) \).

\[
g(x_j) = (a(x_j) + (g(x_j) - a(x_j))) \geq a(x_j) + (l_j + (j - l_j)) = a(x_j) + j = n + j - g(x_j)
\]

So \( g(x_j) \geq \frac{1}{2}(n + j) \). Set \( p(j) = \max\{k \in \mathbb{Z} : k \leq \frac{n+j}{2}\} \). We obtain \( x_j \in F^{p(j)} X^n \).

Now apply lemma 5\( 1 \) \( \square \)

**Lemma 59.** If \( x \in X^n \). Then there are \( x_j \in X^n \) of form degree \( j \) with \( x = \sum x_j \).

**Proof.** Write \( x = \sum x_i \) with \( x_i \in \mathbb{F}_l X_0^n \). Expand each \( x_i = \sum x_j^l \) where the sum is finite with \( x_j^l \) being of form degree \( j \). By lemma 5\( 1 \), \( x_j = \sum x_j^l \) converges to an element of \( X^n \) of form degree \( j \). We have \( x = \sum \sum x_j^l = \sum \sum x_j^l \), which can be verified evaluating both sides modulo \( F^p \) for general \( p \).

**Lemma 60.** For \( \xi_j \in X^n \) of form degree \( j \) with \( \sum \xi_j = 0 \) we have \( \xi_j = 0 \).

**Proof.** No cancellations in the sum are possible. \( \square \)

A.7. Symplectic Case.

**Lemma 61.** Consider the graded commutative algebra \( A = \mathbb{R}[x_i, y_i, \xi_i^{(l)}, \epsilon_i^{(l)}] \) where \( x_i, y_i \) are of degree zero and \( \xi_i^{(l)} \) and \( \epsilon_i^{(l)} \) are of degree \( -l \) and \( l \), respectively. Define a Poisson structure by setting \( [x_i, y_i] = \delta_{ij}, [\xi_i^{(l)}, \epsilon_m^{(n)}] = \delta_{lm} \delta_{in} \) and setting all other brackets between generators to zero. Let \( X_i, Y_i, E_i^{(l)}, E_i^{(h)} \in A \) such that the maps

\[
(x_i, y_i, \xi_i^{(l)}, \epsilon_i^{(l)}) \mapsto (x_i, Y_i, \xi_i^{(l)}, E_i^{(h)})
\]

both define automorphisms \( A \rightarrow A \) of graded commutative algebras. Then the latter map is a Poisson automorphism if there exists an element \( S(x_i, Y_i, E_i^{(h)}) \in X \) such that

\[
\frac{\partial S}{\partial x_i} = y_i \quad \frac{\partial S}{\partial Y_i} = X_i \quad \frac{\partial S}{\partial E_i^{(l)}} = (-1)^l \epsilon_i^{(l)} \quad \frac{\partial S}{\partial E_i^{(h)}} = E_i^{(l)}
\]
Proof. Set $\xi_i^{(l)} = x_i$ and $\eta_i^{(l)} = e_i^{(l)}$ for $l > 0$ and similarly $\eta_i^{(0)} = y_i$ and $\eta_i^{(l)} = e_i^{(l)*}$. We have

$$[f, g] = \sum_l (-1)^{l \deg f} \sum_j ((-1)^l \frac{\partial f}{\partial \xi_j^{(l)}} \frac{\partial g}{\partial \eta_j^{(l)}} - \frac{\partial f}{\partial \eta_j^{(l)}} \frac{\partial g}{\partial \xi_j^{(l)}})$$

since both sides define derivations which agree on generators. The sums converge by lemma 51. We have

$$\frac{\partial S}{\partial \xi_j^{(l)}} = (-1)^l \eta_j^{(l)} \text{ and } \frac{\partial S}{\partial H_j^{(l)}} = \Xi_j^{(l)}$$

so also

$$\frac{\partial^2 S}{\partial \xi_j^{(l)} \partial H_m^{(n)}} = \frac{\partial^2 S}{\partial \eta_j^{(l)} \partial \eta_m^{(n)}}$$

There are functions $f_{j,l}$ and $g_{j,l}$ realizing the change of coordinates with

$$f_{j,l}(\xi, \eta, H) = \frac{\partial S}{\partial H_j^{(l)}} \quad \text{and} \quad g_{j,l}(\xi, \eta, H) = H_j^{(l)}$$

We obtain in the variables $(\xi_j^{(l)}, H_j^{(l)})$

$$\frac{\partial f_{m,n}}{\partial \xi_j^{(l)}} + \sum_{p,q} \frac{\partial f_{m,n}}{\partial \eta_j^{(p)}} \frac{\partial g_{m,n'}}{\partial \eta_j^{(q)}} = \frac{\partial^2 S}{\partial \xi_j^{(l)} \partial H_m^{(n)}}$$

These expressions make sense in the completion by lemma 51 since $(j, l, n, m)$ are fixed and the $\eta_j^{(p)}$ derivatives are of non-decreasing and unbounded degree. Using those equalities, we calculate in the variables $(\xi_j^{(l)}, H_j^{(l)})$ the bracket $[f_{m,n}, g_{m',n'}]$ as

$$\sum_l (-1)^{ln} \sum_j ((-1)^l \frac{\partial f_{m,n}}{\partial \eta_j^{(l)}} \frac{\partial g_{m',n'}}{\partial \eta_j^{(l)}} - \frac{\partial f_{m,n}}{\partial \eta_j^{(l)}} \frac{\partial g_{m',n'}}{\partial \eta_j^{(l)}})$$

$$= \sum_l (-1)^{l(n+1)} \frac{\partial^2 S}{\partial \xi_j^{(l)} \partial H_m^{(n)}} \frac{\partial g_{m',n'}}{\partial \eta_j^{(l)}} - \sum_l (-1)^{ln} \left( \sum_{pq} (-1)^l \frac{\partial f_{m,n}}{\partial \eta_j^{(l)}} \frac{\partial g_{m',n'}}{\partial \eta_j^{(l)}} \frac{\partial f_{m,n}}{\partial \eta_j^{(l)}} \frac{\partial g_{m',n'}}{\partial \eta_j^{(l)}} \right)$$

$$= \sum_l \frac{\partial f_{m,n}}{\partial H_m^{(n)}} \frac{\partial g_{m',n'}}{\partial \eta_j^{(l)}} - \sum_l (-1)^{ln} \left( \sum_{pq} (-1)^l \frac{\partial f_{m,n}}{\partial \eta_j^{(l)}} \frac{\partial g_{m',n'}}{\partial \eta_j^{(l)}} \frac{\partial f_{m,n}}{\partial \eta_j^{(l)}} \frac{\partial g_{m',n'}}{\partial \eta_j^{(l)}} \right)$$

$$= \delta_{nm'} \delta_{n'n'} - \sum_{jlpq} \left( (-1)^{l+n} \frac{\partial f_{m,n}}{\partial \eta_j^{(l)}} \frac{\partial g_{m',n'}}{\partial \eta_j^{(l)}} - (-1)^{l+n+q} \frac{\partial f_{m,n}}{\partial \eta_j^{(l)}} \frac{\partial g_{m',n'}}{\partial \eta_j^{(l)}} \right) = \delta_{nm'} \delta_{n'n'}$$

$\square$

REFERENCES

[1] I. Batalin and E Fradkin. A generalized canonical formalism and quantization of reducible gauge theories. Physics Letters, 122B(2):157–164, 1983.
[2] I. Batalin and G. Vilkovisky. Relativistic s-matrix of dynamical systems with boson and fermion constraints. Physics Letters, 69B(5):309–312, 1977.
[3] B. Becchi, A. Rouet, and R. Stora. Renormalization of gauge theories. Annals of Physics, 98:287–321, 1976.
[4] H. Cartan and S. Eilenberg. Homological Algebra. Princeton University Press, 1956.
[5] G. Felder and D. Kazhdan. The classical master equation. ArXiv e-prints, December 2012.
[6] J. Fisch, M. Henneaux, J. Stasheff, and C. Teitelboim. Existence, uniqueness and cohomology of the classical brst charge with ghosts of ghosts. Communications in Mathematical Physics, 120(3):379–407, 1989.
[7] E. Fradkin and T. Fradkin. Quantization of relativistic systems with boson and fermion first- and second-class constraints. Physics Letters, 72B(3):343–348, 1978.
[8] E. Fradkin and G Vilkovisky. Quantization of relativistic systems with constraints. Physics Letters, 55B(2):224–226, 1975.
[9] M. Henneaux and C. Teitelboim. Quantization of Gauge Systems. Princeton University Press, 1994.
[10] B. Kostant and S. Sternberg. Symplectic reduction, brs cohomology, and infinite-dimensional clifford algebras. Annals of Physics, 176(1):49–113, 1987.
[11] F. Paugam. Homotopical Poisson reduction of gauge theories. ArXiv e-prints, June 2011.
[12] J. Stasheff. Homological reduction of constrained poisson algebras. *Journal of Differential Geometry*, 45(1):221–240, 1997.

[13] J. Tate. Homology of noetherian rings and local rings. *Illinois Journal of Mathematics*, 1(1):14–27, 03 1957.

[14] I. Tyutin. *FIAN preprint*, (39), 1975.