Fekete–Szegö Inequalities for a New Subclass of Bi-Univalent Functions Associated with Gegenbauer Polynomials

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Abstract: We introduce and investigate in this paper a new subclass of bi-univalent functions associated with the Gegenbauer polynomials which satisfy subordination conditions defined in a symmetric domain, which is the open unit disc. For this new subclass, we obtain estimates for the Taylor–Maclaurin coefficients |a2|, |a3| and the Fekete–Szegö inequality |a3 − μa22|.

Keywords: Gegenbauer polynomial; subordination; bi-univalent functions; analytic functions; Fekete–Szegö problem

MSC: 30C45; 30C50

1. Introduction

Let A represent the class of functions whose members are of the form

\[ f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in \Delta), \]  

which are analytic in \( \Delta = \{ z \in \mathbb{C} : |z| < 1 \} \).

A subclass of A with members that are univalent in \( \Delta \) is indicated by the symbol S. The Koebe one-quarter theorem [1] guarantees that a disk with a radius of 1/4 exists in the image of every univalent function \( f \in A \). As a result, each univalent function \( f \) has a satisfied inverse function \( f^{-1} \)

\[ f^{-1}(f(z)) = z, \quad (z \in \Delta) \quad \text{and} \quad f\left(f^{-1}(\omega)\right) = \omega, \quad (|\omega| < r_0(f), r_0(f) \geq \frac{1}{4}). \]

If \( f \) and \( f^{-1} \) are univalent in \( \Delta \), then we say that \( f \in A \) is bi-univalent in \( \Delta \). The class of bi-univalent functions defined on the unit disk \( \Delta \) is denoted by \( \Sigma \). Due to the fact that \( f \in \Sigma \) has the Maclaurin series described by (1), a calculation reveals that \( g = f^{-1} \) has the expansion

\[ g(\omega) = f^{-1}(\omega) = \omega - a_2 \omega^2 + \left(2a_2^2 - a_3\right) \omega^3 + \ldots \]  

We know that the class \( \Sigma \) is not empty. For example, the functions

\[ f_1(z) = \frac{z}{z - 1}, \quad f_2(z) = \frac{1}{2} \log \frac{1 + z}{1 - z}, \quad f_3(z) = - \log (1 - z) \]

with their respective inverses

\[ f_1^{-1}(\omega) = \frac{\omega}{1 + \omega}, \quad f_2^{-1}(\omega) = \frac{\omega^2 - 1}{\omega^2 + 1}, \quad f_3^{-1}(\omega) = \frac{\omega - 1}{\omega} \]
belong to $\Sigma$.

In addition, the Koebe function does not belong to $\Sigma$.

The research of analytical and bi-univalent functions is reintroduced in [2]; previous studies include those of [3–8]. Several authors introduced new subclasses of bi-univalent functions and obtained bounds for the initial coefficients (see [2–4,6,9–11]).

Let $f$ and $g$ be the analytic functions in $\Delta$. We say that $f$ is subordinate to $g$ and denoted by

$$f(z) \prec g(z) \quad (z \in \Delta),$$

if there exists a Schwarz function $w$, which is analytic in $\Delta$ with $w(0)=0$ and $|w(z)|<1$ such that

$$f(z) = g(w(z)) \quad (z \in \Delta).$$

If $g$ is a univalent function in $\Delta$, then

$$f(z) \prec g(z) \iff f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta).$$

In [6], by means of Loewner’s method, the Fekete–Szegö inequality for the coefficients of $f \in S$ is that

$$|a_3 - \mu a_2^2| \leq 1 + 2 \exp \left( \frac{-2\mu}{1-\mu} \right) \quad \text{for } 0 \leq \mu < 1.$$

As $\mu \to 1^-$, the elementary inequality $|a^3 - a_2^2| \leq 1$ is obtained. Moreover, the coefficient functional

$$F_\mu(f) = a_3 - \mu a_2^2$$

on the normalized analytic functions $f$ in the open unit disk $\Delta$ plays an important role in geometric function theory. The problem of maximizing the absolute value of the functional $F_\mu(f)$ is called the Fekete–Szegö problem.

The Fekete–Szegö inequalities introduced in 1933, see [12], preoccupied researchers regarding different classes of univalent functions [13–16]; hence, it is obvious that such inequalities were obtained regarding bi-univalent functions too and very recently published papers can be cited to support the assertion that the topic still provides interesting results [17–19].

In recent years orthogonal polynomials have been explored from a variety of angles. We know their relevance in mathematical physics, mathematical statistics, probability theory and engineering. The classical orthogonal polynomials are the most typically encountered orthogonal polynomials in applications (Laguerre polynomials, Jacobi polynomials and Hermite polynomials). For more details about the classical orthogonal polynomials we mention the papers: [17,18,20–24].

The Gegenbauer polynomials [25] are defined in terms of the Jacobi polynomials $P^{(u,v)}_n$, with $v = u = \lambda - \frac{1}{2}$, $(\lambda > -\frac{1}{2}, \lambda \neq 0)$, which are described by

$$B^\lambda_n(l) = \frac{\Gamma(n+2)\Gamma\left(\lambda + \frac{1}{2}\right)}{\Gamma(2\lambda)\Gamma\left(n + \lambda + \frac{1}{2}\right)} \frac{P^{\lambda,\frac{1}{2},\lambda - \frac{1}{2}}_n}{P^{n+\lambda,\lambda}_n}(l)$$

$$= \frac{(n-1+2\lambda)}{n} \sum_{k=0}^{n} \frac{(nk)(2\lambda+n)_{k}}{(\lambda+\frac{1}{2})_{k}} \left(\frac{l-1}{2}\right)^k. \quad (3)$$

From (3), it follows that $B^\lambda_n(l)$ is a polynomial of degree $n$ with real coefficients and $B^\lambda_n(1) = \left(\frac{n-1+2\lambda}{n}\right)$, while the leading coefficient of $B^\lambda_n(l)$ is $2^n \left(\frac{n+\lambda-1}{n}\right)$. According to Jacobi polynomial theory, for $\mu = v = \lambda - \frac{1}{2}$, with $\lambda > -\frac{1}{2}$, and $\lambda \neq 0$, we have

$$B^\lambda_n(-1) = (-1)^n B^\lambda_n(l).$$
In [25,26], the Gegenbauer polynomials’ generating function is provided by
\[
\frac{2^{\lambda - \frac{1}{2}}}{(1 - 2lz + z^2)^{\frac{\lambda}{2}}(1 - lz + \sqrt{1 - 2lz + z^2})^{\lambda - \frac{1}{2}}} = \frac{(\lambda - \frac{1}{2})_n B_n^\lambda(l) t^n}{(2\lambda)_n},
\]
and this equivalence may be deduced from the Jacobi polynomial generating function. From (4), we obtain
\[
\phi^\lambda_l(z) = \frac{1}{(1 - 2lz + z^2)^{\lambda}} = \sum_{n=0}^{\infty} B_n^\lambda(l) z^n, z \in \Delta, l \in [-1,1], \lambda \in \left(\frac{-1}{2}, +\infty\right) \setminus \{0\},
\]
and the proof is given in [6,23,25].

When \(\lambda = 1\), the relation 5 yields the ordinary generating function for the Chebyshev polynomials, and when \(\lambda = \frac{1}{2}\), we obtain the ordinary generating function for the Legendre polynomials (see [27]).

The Taylor–Maclaurin series expansion for the function \(\phi^\lambda_l(z)\) is as follows:
\[
\phi^\lambda_l(z) = z + B^\lambda_1(l)z^2 + B^\lambda_2(l)z^3 + B^\lambda_3(l)z^4 + \cdots + B^\lambda_{n-1}(l)z^2(l)z^n + \cdots,
\]
where
\[
B^\lambda_0(l) = 1, \quad B^\lambda_1(l) = 2\lambda l, \quad B^\lambda_2(l) = 2\lambda(\lambda + 1)l^2 - \lambda = 2(\lambda)z^2 - \lambda.
\]
and \((\lambda)_n\) is the Pochhammer symbol defined by
\[
(\lambda)_n = \begin{cases} 1, & n = 0 \\ \lambda(\lambda + 1)\cdots(\lambda + n - 1), & n \in \mathbb{N}. \end{cases}
\]

Many researchers have recently explored bi-univalent functions associated with Gegenbauer polynomials [18,20–22,24].

First, we define a new subclass of bi-univalent functions associated with Gegenbauer polynomials.

**Definition 1.** We say that \(f\) of the form (1) is in the class \(M_{\Sigma}(\tau, \theta; \phi^\lambda_l)\), for \(\tau \in \mathbb{C}\setminus\{0\}\), \(0 \leq \theta \leq 1\) and \(l \in \left(\frac{1}{2}, 1\right]\), if the following subordinations hold:
\[
1 + \frac{1}{\tau}(f'(z) + \theta zf''(z) - 1) \prec \phi^\lambda_l(z)
\]
and
\[
1 + \frac{1}{\tau}(g'(\omega) + \theta \omega g''(\omega) - 1) \prec \phi^\lambda_l(\omega),
\]
z, \(\omega \in \Delta, \phi^\lambda_l\) is given by (7), and \(g = f^{-1}\) is given by (2).

Upon allocating the parameters \(\tau\) and \(\theta\), one can obtain several new subclasses of \(\Sigma\), as illustrated in the following two examples.

**Example 1.** We say that \(f\) of the form (1) is in the class \(M_{\Sigma}(\tau, 0; \phi^\lambda_l) = M_{\Sigma}(\tau; \phi^\lambda_l)\), for \(\tau \in \mathbb{C}\setminus\{0\}\), and \(l \in \left(\frac{1}{2}, 1\right]\), if the following subordinations hold:
\[
1 + \frac{1}{\tau}(f'(z) - 1) \prec \phi^\lambda_l(z)
\]
and
\[
1 + \frac{1}{\tau}(g'(\omega) - 1) \prec \phi^\lambda_l(\omega),
\]
z, \(\omega \in \Delta, \phi^\lambda_l\) is given by (7), and \(g = f^{-1}\) is given by (2).
Example 2. We say that $f$ of the form (1) is in the class $M_{\Sigma}(1,0; \phi^1_l) = M_{\Sigma}(\phi^1_l)$, for $l \in \left(\frac{1}{2}, 1\right]$, if the following subordinations hold:

$$f'(z) < \phi^1_l(z)$$

and

$$g'(\omega) < \phi^1_l(\omega),$$

$z, \omega \in \Delta, \phi^1_l$ is given by (7), and $g = f^{-1}$ is given by (2).

2. Initial Taylor Coefficients Estimates for the Class $M_{\Sigma}(\tau, \theta; \phi^1_l)$

Lemma 1 ([28] p. 172). Suppose $w(z) = \sum_{n=1}^{\infty} w_n z^n$, $z \in \Delta$, is an analytic function in $\Delta$ such that $|w(z)| < 1$, $z \in \Delta$. Then,

$$|w_1| \leq 1, |w_n| \leq 1 - |w_1|^2, n = 2, 3, \ldots$$

For the functions belonging to a class $M_{\Sigma}(\tau, \theta; \phi^1_l)$, we will obtain upper bounds for the modulus of coefficients $a_2$ and $a_3$.

Theorem 1. If the class $M_{\Sigma}(\tau, \theta; \phi^1_l)$ contains all the functions $f$ given by (1), then

$$|a_2| \leq \frac{2\lambda |\tau| \sqrt{2\lambda l}}{\sqrt{3|\tau(1+2\theta)(2\lambda l)^2 - 4(1+\theta)^2(2\lambda)^2 - \lambda}|},$$

and

$$|a_3| \leq \frac{2\lambda |\tau|}{3|1+2\theta|} + \frac{|\tau|^2 (\lambda l)^2}{|1+\theta|^2}.$$ (11)

Proof. Let $f \in M_{\Sigma}(\tau, \theta; \phi^1_l)$ and $g = f^{-1}$. We have the following from the definition in Formulas (8) and (9)

$$1 + \frac{1}{\tau} (f'(z) + \theta zf''(z) - 1) = \phi^1_l(v(z))$$

and

$$1 + \frac{1}{\tau} (g'(\omega) + \theta \omega g''(\omega) - 1) = \phi^1_l(v(\omega)),$$

where the analytical functions $v$ and $v$ have the form

$$v(z) = c_1 z + c_2 z^2 + \ldots,$$

$$v(\omega) = d_1 \omega + d_2 \omega^2 + \ldots,$$

and $v(0) = 0 = v(0), |v(z)| < 1, |v(\omega)| < 1, z, \omega \in \Delta$.

From Lemma 1, it follows that

$$|c_j| \leq 1, |d_j| \leq 1, \text{ where } j \in \mathbb{N}.$$ (16)

If we replace (14) and (15) in (12) and (13), respectively, we obtain

$$1 + \frac{1}{\tau} (f'(z) + \theta zf''(z) - 1) = 1 + B^1_1(l) v(z) + B^2_1(l) v^2(z) + \ldots,$$

and

$$1 + \frac{1}{\tau} (g'(\omega) + \theta \omega g''(\omega) - 1) = 1 + B^1_1(l) v(\omega) + B^2_1(l) v^2(\omega) + \ldots.$$ (18)
In view of (1) and (2), from (17) and (18), we obtain
\[
1 + \frac{1}{\tau}(2a_2(1 + \theta)z + 3a_3(1 + 2\theta)z^2) = 1 + B_1^1(l)c_1z + \left[B_1^1(l)c_2 + B_2^1(l)c_1^2\right]z^2
\]
and
\[
1 + \frac{1}{\tau}(-2a_2(1 + \theta)\omega + 3\left(2a_2^2 - a_3\right)(1 + 2\theta)\omega^2) = 1 + B_1^1(l)d_1\omega + \left[B_1^1(l)d_2 + B_2^1(l)d_1^2\right]\omega^2.
\]
It gives rise to the following relationships:
\[
2a_2(1 + \theta) = \tau B_1^1(l)c_1, \quad \text{(19)}
\]
\[
3a_3(1 + 2\theta) = \tau B_1^1(l)c_2 + \tau B_2^1(l)c_1^2, \quad \text{(20)}
\]
and
\[
-2a_2(1 + \theta) = \tau B_1^1(l)d_1, \quad \text{(21)}
\]
\[
3\left(2a_2^2 - a_3\right)(1 + 2\theta) = \tau B_1^1(l)d_2 + \tau B_2^1(l)d_1^2. \quad \text{(22)}
\]
From (19) and (21), it follows that
\[
c_1 = -d_1, \quad \text{(23)}
\]
and
\[
a_2^2 = \frac{\tau^2[B_1^1(l)]^2(c_1^2 + d_1^2)}{8(1 + \theta)^2}. \quad \text{(24)}
\]
Adding (20) and (22), using (24), we obtain
\[
a_3 = \frac{\tau B_1^1(l)(c_2^2 - d_1^2) + \tau B_1^1(l)(c_2 - d_2)}{6(2\theta + 1)} + a_2^2 = \frac{\tau B_1^1(l)(c_2 - d_2) + \tau B_2^1(l)(c_1^2 - d_1^2)}{6(1 + 2\theta)} + \frac{\tau^2[B_1^1(l)]^2(c_1^2 + d_1^2)}{8(1 + \theta)^2}. \quad \text{(26)}
\]
Once again applying (16) and using (7), for the coefficients \(c_1, d_1, c_2, d_2\), we deduce (11). Thus, the proof is completed.

In Theorem 1, we obtain the following result for \(\tau = \theta = 1\).

**Corollary 1.** Let \(f \in M_{\Sigma}(1, 1; \phi_1^1)\). Then, we have
\[
|d_2| \leq \frac{2\lambda l \sqrt{2\lambda l}}{\sqrt{9(2\lambda l)^2 - 16(2(\lambda)l^2 - \lambda)}}.
\]
and

$$|a_3| \leq \frac{2\lambda l}{9} + \frac{(\lambda l)^2}{4}.$$  

For $\tau = 1$ and $\theta = 0$ in Theorem 1, we obtain the following corollary.

**Corollary 2.** Let $f \in M_\Sigma(1, 0; \phi_\lambda^1)$. Then, we have

$$|a_2| \leq \frac{2\lambda l \sqrt{2\lambda l}}{\sqrt{[3(2\lambda l)^2 - 4(2\lambda l)l^2 - \lambda]}}$$

and

$$|a_3| \leq (\lambda l)^2 + \frac{2\lambda l}{3}.$$  

For $\lambda = \frac{1}{2}$ and $l = 1$ in Corollary 2, we obtain the following corollary.

**Corollary 3.** Let $f \in M_\Sigma\left(1, 0; \phi_\lambda^2\right)$. Then, we have

$$|a_2| \leq 1,$$

and

$$|a_3| \leq \frac{7}{12}.$$  

3. The Fekete–Szegö Problem for the Function Class $M_\Sigma\left(\tau, \theta; \phi_\lambda^\mu\right)$

Due to the Zaprawa result, which is discussed in [19], we obtain the Fekete–Szegö inequality for the class $M_\Sigma(\tau, \theta; \phi_\mu^1)$.

**Theorem 2.** If $f$ given by (1) is in the class $M_\Sigma(\tau, \theta; \phi_\mu^1)$ where $\mu \in \mathbb{R}$, then we have

$$|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{2\lambda l|\tau|}{6|1+2\theta|}, & \text{if } |h(\mu)| \leq \frac{1}{6|1+2\theta|}, \\
\frac{(1 - \mu)\tau^2(4\lambda l)^3}{3\tau(1+2\theta)(2\lambda l)^2 - 4(1+\theta)^2(2\lambda l)l^2 - 1}, & \text{if } |h(\mu)| \geq \frac{1}{6|1+2\theta|},
\end{cases}$$

where

$$h(\mu) = \frac{(1 - \mu)\tau^2(4\lambda l)^3}{3\tau(1+2\theta)(2\lambda l)^2 - 4(1+\theta)^2(2\lambda l)l^2 - 1}.$$  

**Proof.** If $f \in M_\Sigma(\tau, \theta; \phi_\mu^1)$ is given by (1), from (25) and (26), we have

$$a_3 - \mu a_2^2 = \frac{\tau B_1^\lambda(l)(c_2 - d_2)}{6(1+2\theta)} + (1 - \mu)a_2^2$$

$$= \frac{\tau B_1^\lambda(l)(c_2 - d_2)}{6(1+2\theta)} + \frac{(1 - \mu)\tau^2[B_1^\lambda(l)]^3(c_2 + d_2)}{6\tau(1+2\theta)[B_1^\lambda(l)]^2 - 8(1+\theta)^2B_2^\lambda(l)}$$

$$= \tau B_1^\lambda(l) \left[ \frac{c_2}{6(1+2\theta)} - \frac{d_2}{6(1+2\theta)} + \frac{(1 - \mu)\tau^2[B_1^\lambda(l)]^3 c_2}{6\tau(1+2\theta)[B_1^\lambda(l)]^2 - 8(1+\theta)^2B_2^\lambda(l)} \right]$$

$$+ \frac{(1 - \mu)\tau^2[B_1^\lambda(l)]^3 d_2}{6\tau(1+2\theta)[B_1^\lambda(l)]^2 - 8(1+\theta)^2B_2^\lambda(l)}$$

$$= \tau B_1^\lambda(l) \left[ \left( h(\mu) + \frac{1}{6(1+2\theta)} \right)c_2 + \left( h(\mu) - \frac{1}{6(1+2\theta)} \right)d_2 \right],$$
where

\[ h(\mu) = \frac{(1 - \mu)\tau^2 [B_1^\lambda(l)]^3}{6\tau(1 + 2\theta)[B_1^\lambda(l)]^2 - 8(1 + \theta)^2 B_2^\lambda(l)} \]

Now, by using (7)

\[ a_3 - \mu a_2^2 = \tau 2\lambda [\left( h(\mu) + \frac{1}{6(1 + 2\theta)} \right) c_2 + \left( h(\mu) - \frac{1}{6(1 + 2\theta)} \right) d_2], \]

where

\[ h(\mu) = \frac{(1 - \mu)\tau^2 4(\lambda l)^3}{3\tau(1 + 2\theta)(2\lambda l)^2 - 4(1 + \theta)^2 (2(\lambda + 1)l^2 - 1)} \]

Therefore, given (7) and (16), we conclude that the required inequality holds. Thus, the proof is completed. \( \square \)

In Theorem 2, we have the following result for \( \tau = \vartheta = 1 \).

**Corollary 4.** Let \( f \in M_\Sigma (1, 1; \phi_1^\lambda) \) and \( \mu \in \mathbb{R} \). Then, we have

\[ |a_3 - \mu a_2^2| \leq \begin{cases} \frac{2\lambda l}{4\lambda l |h(\mu)|}, & \text{if } |h(\mu)| \leq \frac{1}{16l}, \\ \frac{4(1 - \mu)(\lambda l)^3}{9(2\lambda l)^2 - 16(2(\lambda + 1)l^2 - 1)}, & \text{if } |h(\mu)| \geq \frac{1}{16l}, \end{cases} \]

where

\[ h(\mu) = \frac{4(1 - \mu)(\lambda l)^3}{9(2\lambda l)^2 - 16(2(\lambda + 1)l^2 - 1)}. \]

In Theorem 2, we have the following result for \( \tau = 1 \) and \( \vartheta = 0 \).

**Corollary 5.** Let \( f \in M_\Sigma (1, 0; \phi_1^\lambda) \) and \( \mu \in \mathbb{R} \). Then, we have

\[ |a_3 - \mu a_2^2| \leq \begin{cases} \frac{2\lambda l}{4\lambda l |h(\mu)|}, & \text{if } |h(\mu)| \leq \frac{1}{16l}, \\ \frac{4(1 - \mu)(\lambda l)^3}{3(2\lambda l)^2 - 4(2(\lambda + 1)l^2 - 1)}, & \text{if } |h(\mu)| \geq \frac{1}{16l}, \end{cases} \]

where

\[ h(\mu) = \frac{4(1 - \mu)(\lambda l)^3}{3(2\lambda l)^2 - 4(2(\lambda + 1)l^2 - 1)}. \]

In Corollary 5, we obtain the next result for \( \lambda = \frac{1}{2} \) and \( l = 1 \).

**Corollary 6.** Let \( f \in M_\Sigma \left( 1, 0; \phi_1^\frac{1}{2} \right) \) and \( \mu \in \mathbb{R} \). Then, we have

\[ |a_3 - \mu a_2^2| \leq \frac{1}{3}. \]

4. Conclusions

In this paper, we introduced and investigated a new subclass of bi-univalent functions in the open unit disk defined by Gegenbauer polynomials which satisfy subordination conditions. Furthermore, we obtain upper bounds for \( |a_2|, |a_3| \) and the Fekete–Szegö inequality \( |a_3 - \mu a_2^2| \) for functions in this subclass.

In addition, the approach presented here has been extended to establish new subfamilies of bi-univalent functions with other special functions. The related outcomes may be left to researchers for practice.
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