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Non-monotonicity of Trace Distance Under Tensor Products

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Abstract The trace distance (TD) possesses several of the good properties required for a faithful distance measure in the quantum state space. Despite its importance and ubiquitous use in quantum information science, one of its questionable features, its possible non-monotonicity under taking tensor products of its arguments (NMuTP), has been hitherto unexplored. In this article, we advance analytical and numerical investigations of this issue considering different classes of states living in a discrete and finite dimensional Hilbert space. Our results reveal that although this property of TD does not show up for pure states and for some particular classes of mixed states, it is present in a non-negligible fraction of the regarded density operators. Hence, even though the percentage of quartets of states leading to the NMuTP drawback of TD and its strength decrease as the system’s dimension grows, this property of TD must be taken into account before using it as a figure of merit for distinguishing mixed quantum states.

Keywords Quantum distance measures · Trace distance · Monotonicity under tensor products

1 Introduction

Quantifiers for the distance (distinguishability) between two density operators in the quantum state space $\mathcal{D}(\mathcal{H})$—the space formed by positive semidefinite matrices with trace equal to one—are an essential and frequently used item in the quantum information scientist toolkit [1–3]. For instance, how well a quantum system was prepared [4], manipulated [5], or protected [6] in an experiment is usually evaluated via how close (how indistinguishable) its real state is from what one would ideally expect. Distance measures in $\mathcal{D}(\mathcal{H})$ naturally appear also in the contexts of quantum foundations [7–9], quantum processes [10, 11], quantum cryptography [12, 13], quantum phase transitions [14], quantum speed limits [15–20], quantum channel capacities [21], and also in the theories of quantum entanglement [22, 23], quantum discord [24–26], and quantum coherence [27–30].

Several distance (distinguishability) measures in the quantum state space have been proposed in the literature in the last decades. A partial list is provided in ref. [31]. A few examples are the Bures’ distance, that is defined in terms of a similarity measure known as Uhlmann’s fidelity [32, 33] (for a critical assessment regarding the use of this function in quantum information science see Ref. [34]), the $p-$norm distance, with the trace distance (or $1-$norm distance) and the Hilbert-Schmidt distance (or 2-$norm distance) being used more frequently (see refs. [35–38] and references therein), the quantum relative entropy [39–42], and the quantum Chernoff bound [43, 44], with these last two distinguishability measures being defined operationally, respectively, in the contexts of asymmetric and symmetric quantum hypothesis testing.

In this article, we are interested mainly in one of the most popular distance measures, the trace distance, that is defined using the trace norm. For a Hermitian matrix $A$, the trace norm is defined and given as follows:

$$||A||_1 := \text{Tr}\sqrt{A^* A} = \text{Tr}\sqrt{A^2} = \sum_j |a_j|,$$ (1)
with \(|a_j|\) being the absolute value of the real eigenvalues of \(A\). We can quantify how dissimilar two density operators \(\rho\) and \(\zeta\) are by their trace distance (TD), which is defined as the trace norm of their subtraction,

\[
d_{tr}(\rho, \zeta) := \|\rho - \zeta\|_1, \tag{2}
\]

and assumes values between zero and two \([2]\).

This mathematical function possesses several of those properties required for a faithful distance (distinguishability) measure in the quantum state space \([1, 2]\). For the density operators \(\rho\), \(\zeta\), and \(\alpha\), the trace distance is, e.g., positive semidefinite \((d_{tr}(\rho, \zeta) \geq 0)\), it is zero if and only if the two density operators are equal \((d_{tr}(\rho, \zeta) = 0 \iff \rho = \zeta\)\), it is symmetric \((d_{tr}(\rho, \zeta) = d_{tr}(\zeta, \rho))\), it obeys the triangle inequality \((d_{tr}(\rho, \zeta) \leq d_{tr}(\rho, \alpha) + d_{tr}(\alpha, \zeta))\), and it is invariant under unitary transformations \((d_{tr}(U\rho U^\dagger, U\zeta U^\dagger) = d_{tr}(\rho, \zeta))\) for \(U U^\dagger = I_d\), where \(I_d\) is the \(d \times d\) identity matrix, it leads to the equality \(d_{tr}(\rho \otimes \alpha, \zeta \otimes \alpha) = d_{tr}(\rho, \zeta)\), it is monotonic under discarding subsystems \((d_{tr}(\rho_1, \zeta_1) \leq d_{tr}(\rho_{12}, \zeta_{12})\) with \(x_1 = \text{Tr}_2(x_{12})\)), and it is consequently also monotonic under trace-preserving quantum operations \((d_{tr}(\rho, \zeta) \geq d_{tr}(\hat{\Phi}(\rho), \hat{\Phi}(\zeta))\) with \(\hat{\Phi}(x) = \sum_j K_j x K_j^\dagger + \sum_j K_j^\dagger K_j = I_d\).

Notwithstanding, it was mentioned in ref. \([44]\) that the trace distance lacks monotonicity under taken tensor products of its arguments. That is to say, we can find four density operators \(\rho\), \(\zeta\), \(\xi\), and \(\eta\) such that the following inequalities are satisfied:

\[
d_{tr}(\rho, \zeta) \leq d_{tr}(\xi, \eta), \tag{3}
\]

and

\[
d_{tr}(\rho \otimes \xi, \zeta \otimes \eta) \leq d_{tr}(\xi \otimes \xi, \eta \otimes \eta). \tag{4}
\]

This non-monotonicity under tensor products (NMuTP) does not seem to be a desirable property for a distance measure in \(D(H)\). If a pair of states of a quantum system is more distinguishable than another pair of states, one would expect the same to hold for two identical and uncorrelated copies of the system prepared in those states.

Two relevant questions to answer regarding this issue are (i) for what kind of state and (ii) how often the inequalities in (3) and (4) can simultaneously hold. The remainder of this article will be devoted to answer these questions for the cases of general state vectors (Section 2.1), for one-qubit states (Section 2.2), and also for high-dimensional quantum systems (Section 2.3).

2 The Non-monotonicity of Trace Distance Under Tensor Products

This section is dedicated to investigate such an issue considering some particular classes of states. Though we present some analytical results, much of the work should be numeric. We will start using general pure states and a two-level quantum system to address the question (i). In the sequence, the question (ii) will be studied mainly with regard to its dependence with the system’s dimension.

2.1 Arbitrary Pure States

Let \(\rho\), \(\zeta\), \(\xi\), and \(\eta\) be arbitrary state vectors on the discrete Hilbert space \(H\) of dimension \(d\). The trace distance between the pair of states \(x = \rho(\xi)\) and \(y = \xi(\eta)\) can be written as \([2]\):

\[
d_{tr}(x, y) = 2\sqrt{1 - \text{Tr}(xy)}. \tag{5}
\]

Given that \(0 \leq \text{Tr}(xy) \leq 1\), \(d_{tr}(\rho, \zeta) > d_{tr}(\xi, \eta)\) implies \(\text{Tr}(\rho \xi) < \text{Tr}(\xi \eta)\). Using this inequality and the fact that, in the present case,

\[
d_{tr}(x \otimes^2, y \otimes^2) = 2\sqrt{1 - (\text{Tr}(xy))^2}, \tag{6}
\]

we see that \(d_{tr}(\rho \otimes^2, \zeta \otimes^2) > d_{tr}(\xi \otimes^2, \eta \otimes^2)\). Thus, if all the states involved are pure states, the trace distance does not suffer from the NMuTP drawback under analysis here.

2.2 One-Qubit States

2.2.1 Collinear States

Let us consider the special case in which the pairs of density operators \((\rho, \zeta)\) and \((\xi, \eta)\) are, individually, collinear. That is to say, let, e.g.,

\[
\rho = 2^{-1}(I_2 + r \cdot \sigma) \quad \text{and} \quad \zeta = 2^{-1}(I_2 + z \cdot \sigma) \tag{7}
\]

with the two Bloch’s vectors being \(r = r \hat{n}\) and \(z = \pm \hat{z}\hat{n}\), where \(\hat{n}\) is any unit vector in \(\mathbb{R}^3\) and \(\sigma\) is the Pauli’s vector. One can readily show that

\[
d_{tr}(\rho, \zeta) = |r \mp z|. \tag{8}
\]

For the tensor products, we have

\[
\rho \otimes^2 - \zeta \otimes^2 = 2^{-2}(r \mp z)(I_2 \otimes \hat{n} \cdot \sigma + \hat{n} \cdot \sigma \otimes I_2)
+ (r^2 - z^2)\hat{n} \cdot \sigma \otimes \hat{n} \cdot \sigma
+ 2^{-2}(r \mp z)(r \pm z + 2)P_+ \otimes P_+
+ (r \mp z)(r \pm z - 2)P_- \otimes P_-
- (r^2 - z^2)(P_+ \otimes P_- + P_- \otimes P_+)) \tag{9}
\]

where we used \(\hat{n} \cdot \sigma = P_+ - P_-\) and \(I_2 = P_+ + P_-\). It is straightforward applying (10) to get

\[
d_{tr}(\rho \otimes^2, \zeta \otimes^2) = d_{tr}(\rho, \zeta)2^{-1}(2 + |r \mp z|). \tag{11}
\]

We see that \(d_{tr}(\rho \otimes^2, \zeta \otimes^2)\) is a monotonically increasing function of \(d_{tr}(\rho, \zeta)\). Thus, for this particular set of states, the inequalities in (3) and (4) cannot be satisfied simultaneously.
appearing in these equations can assume values in the ranges of maximally mixed states. For the sake of illustration, of the Bloch’s vector equal to one (zero), we obtain pure to produce these numbers. By setting the Euclidean norm Twister pseudo-random number generator \[48\] is applied \[\frac{2}{3}x = 2^{-1}\left(\sum_{j=1}^{3} x_j \sigma_j\right),\] with \(x = (x_1, x_2, x_3)\), where \(x_1 = \|x\|_2 \sin \theta \cos \phi, x_2 = \|x\|_2 \sin \theta \sin \phi,\) and \(x_3 = \|x\|_2 \cos \theta\). The parameters appearing in these equations can assume values in the ranges [46, 47]: \(\|x\|_2 \in [0, 1], \theta \in [0, \pi],\) and \(\phi \in [0, 2\pi]\). In order to obtain an uniform distribution of points (states) in the Bloch’s ball, each one of the quantum states is generated setting \[\|x\|_2 = (t_1)^{1/3}, \theta = \arccos(-1 + 2t_2), \phi = 2\pi t_3\] with \(t_j (j = 1, 2, 3)\) being a pseudorandom number with uniform distribution in the interval \([0, 1]\. The Mersenne Twister pseudo-random number generator [48] is applied to produce these numbers. By setting the Euclidean norm of the Bloch’s vector equal to one (zero), we obtain pure (maximally mixed) states. For the sake of illustration, the probability distribution for the values of TD between pairs of randomly-generated one-qubit states is presented in Fig. 1.

Even this apparently simple one-qubit case is not easily tamable for analytical computations. Hence, we recourse to numerical calculations via Monte Carlo (random) sampling of the quartets of states to be used. The computations of eigenvalues involved in this article are done utilizing the LAPACK subroutines (see ref. [45]). Let us start by using the Fano’s parametrization [46] to write an one-qubit density matrix \(\rho, \zeta, \xi, \eta\) in the form:

\[
x = 2^{-1}\left(\sum_{j=1}^{3} x_j \sigma_j\right),
\]

with \(x = (x_1, x_2, x_3)\), where \(x_1 = \|x\|_2 \sin \theta \cos \phi, x_2 = \|x\|_2 \sin \theta \sin \phi,\) and \(x_3 = \|x\|_2 \cos \theta\). The parameters appearing in these equations can assume values in the ranges [46, 47]: \(\|x\|_2 \in [0, 1], \theta \in [0, \pi],\) and \(\phi \in [0, 2\pi]\). In order to obtain an uniform distribution of points (states) in the Bloch’s ball, each one of the quantum states is generated setting \[\|x\|_2 = (t_1)^{1/3}, \theta = \arccos(-1 + 2t_2), \phi = 2\pi t_3\] with \(t_j (j = 1, 2, 3)\) being a pseudorandom number with uniform distribution in the interval \([0, 1]\:]. The Mersenne Twister pseudo-random number generator [48] is applied to produce these numbers. By setting the Euclidean norm of the Bloch’s vector equal to one (zero), we obtain pure (maximally mixed) states. For the sake of illustration, the probability distribution for the values of TD between pairs of randomly-generated one-qubit states is presented in Fig. 1.

It is worthwhile mentioning at this point that we have made several tests from which we found that the numerical and analytical results for the TD coincide up to the fifteenth digit when applied to random states in those classes considered in Sections 2.2.1 and 2.1. More specifically, we generated 1 million pairs of random collinear states (\(d = 2\))

### Table 1

| States generated | Percentage | \(\langle G \rangle\) | \(\Delta G\) | \(G_{\text{max}}\) |
|------------------|------------|----------------------|--------------|------------------|
| \((\rho, \zeta), (\xi, \eta)\) | 7.28 | 0.161 | 0.083 | 0.475 |
| \((\rho, \zeta), (\xi, |n|)\) | 7.86 | 0.186 | 0.091 | 0.486 |
| \((\rho, |z|), (\xi, \eta)\) | 3.16 | 0.071 | 0.038 | 0.190 |
| \((\rho, \zeta), (|\xi|, \eta)\) | 4.06 | 0.134 | 0.072 | 0.333 |
| \((\rho, |z|), (|\xi|, |n|)\) | 3.00 | 0.083 | 0.044 | 0.194 |
| \((\rho, \zeta), (|\xi|, I_2/2)\) | 8.49 | 0.192 | 0.098 | 0.488 |
| \((\rho, |z|), (|\xi|, I_2/2)\) | 20.75 | 0.226 | 0.095 | 0.500 |
| \((\rho, \zeta), (|\xi|, I_2/2)\) | 3.20 | 0.101 | 0.045 | 0.248 |
| \((\rho, |z|), (|\xi|, I_2/2)\) | 7.67 | 0.123 | 0.037 | 0.177 |
| \((\rho, |z|), (|\xi|, I_2/2)\) | 2.63 | 0.107 | 0.043 | 0.177 |
| \((\rho, |z|), (|\xi|, I_2/2)\) | 8.85 | 0.169 | 0.004 | 0.177 |

In the last three columns are presented the average value, standard deviation, and maximum value of the strength of the NMuTP drawback of TD, as defined in (14), for each case study.
Fig. 2 Example of a quartet of states for which the trace distance is not monotonic under taking tensor products of its arguments. Here is the size of the corresponding Bloch’s vector, the angles are given in radians, and $d_{tr}(\rho, \zeta) = 0.80$, $d_{tr}(\xi, \eta) = 0.76$, $d_{tr}(\rho \otimes^2, \zeta \otimes^2) = 0.87$, and $d_{tr}(\xi \otimes^2, \eta \otimes^2) = 1.07$

and 1 million pairs of random pure states (see, e.g., ref. [49]) for each value of the system dimension (with $d = 2, \cdots, 20$). The error, in each case, is computed by comparing the trace distance obtained via diagonalization with the LAPACK subroutines and the value of TD obtained using its analytical expression. Then the precision is established via the worst case error.

We proved in the previous subsections that if the pairs $(\rho, \xi)$ and $(\eta, \eta)$ are, individually, collinear or if all the four states are pure, we shall have no NMuTP drawback of trace distance. However, as is shown in Table 1, for all the other possibilities a significant fraction of the 1 million one-qubit quartets of states randomly generated presented this unwanted property of TD. In Fig. 2, we draw an example of such a quartet of states.

For the sake of measuring the strength of the NMuTP drawback of TD, when applicable, we will define the following quantity:

$$G(\rho, \xi, \eta) := \left| d_{tr}(\rho, \xi) - d_{tr}(\xi, \eta) \right| + \left| d_{tr}(\rho \otimes^2, \xi \otimes^2) - d_{tr}(\xi \otimes^2, \eta \otimes^2) \right|. \quad (14)$$

The quantity $G$ measures how far the TD is from been monotonic under tensor products. As $G$ is defined only for those quartets of states leading to the NMuTP of TD, its lower bound is zero. In order to access more details about the distribution of $G$, we shall use its mean value $\langle G \rangle$, standard deviation $\Delta G$, and maximum value $G_{\text{max}}$. These quantities are also shown in Table 1. A sample of the values of $G$ for the case study $(\rho, \xi), (\xi, \eta)$ is presented in Fig. 3.

Even with these additional informations, as can be seem in Table 1, in the general case the relationship between the existence of the NMuTP drawback of TD and the classes of states involved is not an easy matter. For instance, starting with general states and then restricting one of them to be pure, we pass from a percentage of 7.28 to 7.86 %. But then the addition of the same restriction for one state of the other pair reduces the percentage with the undesired property of TD to 3.16 %. Several other similar nontrivial changes in the percentages can be identified. One striking one is that in the last line of the table. For four pure states, there is no drawback; however, just by putting one of the states in the center of the Bloch’s ball, we get a percentage of 8.85 %, the
Fig. 5 Samples with 5000 values of the strength of the NMuTP drawback of trace distance for some values of the system’s dimension \(d\). The NMuTP average percentage of the whole sample with ten sets of \(10^6\) quartets of states is shown at the side of \(d\). The blue line indicates \(\langle G \rangle\) second higher among the classes of one-qubit states studied. These results stress the richness and complexity of the quantum state space, already for the composition of two of its simplest systems. Thus, in order to simplify the analysis, we will investigate in the next section the general dependence of the NMuTP drawback of TD with the dimension of the system.

### 2.3 General One-Qudit States

In this subsection, we shall study the NMuTP drawback of trace distance for \(d\)-level quantum systems, known as qudits. As there is no explicit parametrization for density matrices with \(d \geq 3\) [47], we will proceed as follows. Let us first look for the spectral decomposition of a given density operator \(x = \rho, \zeta, \xi, \eta\):

\[
x = \sum_{j=1}^{d} x_j |x_j\rangle\langle x_j|.
\]

Once the eigenvalues of \(x\) form a probability distribution, i.e.,

\[
x_j \geq 0 \text{ and } \sum_{j=1}^{d} x_j = 1,
\]

we can use a geometric parametrization for them [50]:

\[
x_j = \sin^2 \theta_{j-1} \prod_{k=j}^{d-1} \cos^2 \theta_k
\]

with \(\theta_0 = \pi/2\). The details about the numerical generation of \(|x_j\rangle\) using this parametrization can be found in ref. [49].

The basis formed by the eigenvectors of a density operator \(x, \{|x_j\rangle\}\), can be obtained from the computational basis, \(|j\rangle\), using an unitary matrix \(U\), i.e.,

\[
|x_j\rangle = U|j\rangle, \quad \text{with } j = 1, \ldots, d.
\]

There are several parametrizations for unitary matrices [47]. Here, we use the Hurwitz’s parametrization with Euler’s angles. For details, see, e.g., ref. [51].

By creating pseudo-random probability distributions \(|x_j\rangle\) and pseudo-random unitary matrices \(U\), we did ten numerical experiments for each value of the system’s dimension \(d\) generating 1 million quartets of states in each experiment. The mean, minimum, and maximum values of the percentages of the quartets of states leading to NMuTP of TD are shown in Fig. 4. In the Fig. 5, we present samples of the distribution of values of the drawback’s strength for some values of \(d\). We can see in these figures a steady decreasing of such a proportion and strength as the system dimension \(d\) grows.

### 3 Final Remarks

The trace distance has several good properties that rank it as one of the major distance measures between quantum states. Nevertheless, it also presents a potential drawback, the possibility of being non-monotonic under taking tensor products of its arguments, that was shown in this article to exist for a non-negligible fraction of the density matrices investigated. Thus, although such issue seems not to be much relevant for high-dimensional quantum systems, it must be taken into account when dealing with few qubits.

The important question that yet remains is if, in the cases were the NMuTP of TD is significant, it has some undesirable consequence for important functions in quantum information science. The possible implications of this issue regarding, for instance, the quantification of quantum entanglement, of quantum discord, and of quantum coherence is an appealing topic for further researches. It would also be fruitful analyzing the NMuTP drawback considering other quantum distance measures. The obtention of a more precise operational and/or physical interpretation of \(G\) and its upper bound are also left as open problems.
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