Abstract
An analogue of the Moyal star product is presented for the deformed oscillator algebra. It contains several homotopy-like additional integration parameters in the multiplication kernel generalizing the differential Moyal star product formula \(\exp[i\epsilon_{\alpha\beta}\partial_{\alpha}\partial_{\beta}]\). Using Pochhammer formula Pochhammer (1890 Math. Ann. 35 495–526), integration over these parameters is carried over a Riemann surface associated with the expression of the type \(z^x(1-z)^y\) where \(x\) and \(y\) are arbitrary real numbers. Procedure for perturbative expansion in deformation parameter for the product is provided.

Keywords: Star product, deformed oscillator algebra, higher spin algebra, Lone-Star product, Riemann surface

1. Introduction
Possible deformations of the standard oscillator commutation relations
\[
[y_\alpha, y_\beta] = 2i\epsilon_{\alpha\beta}, \quad \alpha, \beta = 1, 2
\]
(1.1)
that nonetheless lead to the equally spaced energy spectrum of the deformed oscillator was studied by Wigner [2]. He found that there should be a one-parameter family of the deformed commutation relations. With the help of additional anticommuting operator Wigner’s deformed oscillator algebra can be represented in the form [3]
\[
[y_\alpha, y_\beta] = 2i\epsilon_{\alpha\beta}(1 + \nu\mathcal{K}), \quad \{y_\alpha, \mathcal{K}\} = 0, \quad \mathcal{K}^2 = 1.
\]
(1.2)
Here \(\nu \in \mathbb{C}\) is an arbitrary parameter and \(\mathcal{K}\) is the so called Klein operator. Algebra \(A_{q}(2, \nu)\) is the associative algebra generated as universal enveloping algebra of these (anti)commutation

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relations. Generic element of $\mathcal{A}q(2, \nu)$ can be written in the form of formal power series as

$$f(y, K) = \sum_{n=0}^{\infty} \sum_{A=0}^{1} f^{\alpha_1 \cdots \alpha_n}_{A} y_{\alpha_1} \cdots y_{\alpha_n} K^A,$$

(1.3)

where tensors $f^{\alpha_1 \cdots \alpha_n}_{A}$ are totally symmetric in upper indices. From now on Weyl ordering of oscillators is assumed. Product of two generic elements should be again written as formal power series of oscillators contracted with totally symmetric in upper indices coefficients, i.e.

$$f(y, K) \ast g(y, K) = h(y, K) = \sum_{n=0}^{\infty} \sum_{A=0}^{1} h^{\alpha_1 \cdots \alpha_n}_{A} y_{\alpha_1} \cdots y_{\alpha_n} K^A,$$

(1.4)

Here symmetrization is supposed to be performed using (anti)commutation relations (1.2). To compute rhs of (1.4) one should use structure constants $H(m, n, p, \nu K)$ for monomials found in [4]

$$f^{\alpha_1 \cdots \alpha_n}_{A} y_{\alpha_1} \cdots y_{\alpha_n} K^A \ast g^{\beta_1 \cdots \beta_n}_{B} y_{\beta_1} \cdots y_{\beta_n} K^B = f^{\alpha_1 \cdots \alpha_n}_{A} y_{\alpha_1} \cdots y_{\alpha_n} g^{\beta_1 \cdots \beta_n}_{B} y_{\beta_1} \cdots y_{\beta_n} (-1)^{\nu} K^A K^B$$

$$= f^{\alpha_1 \cdots \alpha_n}_{A} y_{\alpha_1} \cdots y_{\alpha_n} g^{\beta_1 \cdots \beta_n}_{B} y_{\beta_1} \cdots y_{\beta_n} \sum_{p=0}^{\min(m,n)} i^p \epsilon_{\alpha_1 \beta_1} \cdots \epsilon_{\alpha_n \beta_p}$$

$$\times y_{(\alpha_1} \cdots y_{\alpha_m-p} y_{\beta_1} \cdots y_{\beta_n-p})H(m, n, p, \nu K)(-1)^{\nu} K^{A+B},$$

(1.5)

where oscillators on the rhs are totally symmetrized, i.e.

$$y_{(\alpha_1} \cdots y_{\alpha_m-p} y_{\beta_1} \cdots y_{\beta_n-p}) = \frac{1}{(m + n - 2p)!} \left( y_{(\alpha_1} \cdots y_{\alpha_m-p} y_{\beta_1} \cdots y_{\beta_n-p} + \text{all permutations} \right).$$

(1.6)

Structure constants depend on parities of monomials in the product. Explicit formulas for each of parities are given in the next section.

Algebra $\mathcal{A}q(2, \nu)$ plays important role in field theory in $2 + 1$ dimensions since it is the symmetry algebra of the free fields. Consider for example the unfolded versions of Klein–Gordon and Dirac equation in AdS$_3$ obtained in [5] and their operator realizations derived in [6, 7]. To obtain the latter one should add to algebra $\mathcal{A}q(2, \nu)$ additional Clifford-like element $\psi^1$

$$\psi^2 = 1, \quad [\psi, y_\alpha] = 0, \quad [\psi, K] = 0.$$

(1.7)

We will denote product in this enlarged algebra (generated as universal enveloping of (1.2) and (1.7)) as $\oplus$ to avoid confusion with product in $\mathcal{A}q(2, \nu)$. Generating function for scalar, spinor and all their descendants (derivatives) can be written as

$$C(y, K, \psi | x) = \sum_{A=0}^{1} \sum_{B=0}^{1} \sum_{n=0}^{\infty} \frac{1}{n!} \lambda^{-[\frac{3}{2}]} C_{\alpha_1 \cdots \alpha_n}^{AB} (x) K^A \psi^B y^{\alpha_1} \cdots y^{\alpha_n}.$$

(1.8)

This element effectively doubles the number of oscillators. It is required due to the well-known fact that $\mathfrak{o}(2, 2) \cong \mathfrak{o}(1, 2) \oplus \mathfrak{o}(1, 2)$. 

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With dreibein $h^{\alpha\beta}$ and spin connection $\omega^{\alpha\beta}$ we define connection on AdS$_3$ isometry algebra according to

$$W = \frac{1}{8i} \omega^{\alpha\beta} \{ y^{\alpha}, y^{\beta} \} \otimes + \lambda \psi \frac{1}{8i} h^{\alpha\beta} \{ y^{\alpha}, y^{\beta} \} \otimes. \quad (1.9)$$

The operator analog of the unfolded equations from [5] now reads

$$dW(y, K, \psi|x) = W(y, K, \psi|x) \otimes W(y, K, \psi|x),$$

$$dC(y, K, \psi|x) = W(y, K, \psi|x) \otimes C(y, K, \psi|x) - C(y, K, \psi|x) \otimes W(y, K, -\psi|x). \quad (1.10)$$

First equation implies zero torsion and AdS$_3$ flatness conditions

$$dh^{\alpha\beta} = \omega^{\alpha\gamma} h^{\beta\gamma} + \omega^{\beta\gamma} h^{\alpha\gamma}, \quad d\omega^{\alpha\beta\gamma} = \omega^{\alpha\gamma}\omega^{\beta\gamma} + \lambda^2 h^{\alpha\gamma} h^{\beta\gamma}. \quad (1.11)$$

Second equation of (1.10) describes free scalar and spinor but compared to the system analysed in [5] it has twice as big number of fields (due to presence of Klein operator $K$). One can project system (1.10) onto individual independent subsystems with the pair of projectors

$$\Pi_{\pm} = \frac{1 \pm K}{2}. \quad (1.12)$$

Masses of scalar and spinor subsystems are then the following

$$M^2_{\pm} = \lambda^2 \nu(\nu \mp 2), \quad M^2_{\pm} = \lambda^2 \nu^2. \quad (1.13)$$

Equation (1.10) being written in the unfolded way are invariant under gauge transformation

$$\delta C = \varepsilon(y, K, \psi|x) \otimes C - C \otimes \varepsilon(y, K, -\psi|x), \quad \delta W = d\varepsilon(y, K, \psi|x) + [\varepsilon(y, K, \psi|x), W] \otimes. \quad (1.14)$$

Here $\varepsilon(y, K, \psi|x)$ is a spatial zero-form playing the role of infinitesimal parameter of gauge transformation. It can be written in the following way

$$\varepsilon(y, K, \psi|x) = \varepsilon_0(y, K|x) + \psi\varepsilon_1(y, K|x). \quad (1.15)$$

Here $\varepsilon_0$ and $\varepsilon_1$ take values in $Aq(2, \nu)$.

(Anti)commutation relations (1.2) on their own play an important role in HS theory since they determine the form of the full nonlinear system of equations that acquires HS symmetry as a gauge symmetry of the theory [8, 9]. Moreover in case of 2 + 1 dimensions nonlinear system can be naturally reformulated in terms of deformed oscillators [7].

In purely two dimensional conformal theory context the algebra was studied in [10–12]. Development of AdS$_3$/CFT$_2$ correspondence showed importance of the deformed oscillator algebra from another perspective [13–15]. In AdS$_3$/CFT$_2$ correspondence algebras $hs[\lambda]$ and $shs[\lambda]$ turn out to be important. They are the (super)Lie algebras constructed by taking the quotient of universal enveloping algebras $U(sp(2))$ and $U(osp(2|1))$ over the ideal generated by the quadratic Casimir of $sp(2)$ [16] and $osp(2|1)$ [12], respectively. There is also another version of AdS$_3$/CFT$_3$ holography that relies completely on the unfolded formulation which for AdS$_4$/CFT$_3$ case was studied in [17]. One starts from the unfolded equations for the free
fields like (1.10) and then by proper rescaling of oscillator variables\(^2\) obtains the unfolded equations on conformal currents. Counterpart of this analysis for AdS\(_3\)/CFT\(_2\) massive case is still unknown partially due to the absence of convenient version for the product in \(Aq(2, \nu)\) algebra. The aim of this paper is to provide another version of the product which might be applied to these kind of problems.

For the first time the deformed oscillator algebra was interpreted as a higher spin algebra in [3, 6]. In a slightly different realization the higher spin algebra was studied in [18]. The associative algebra underlying \(h\mathfrak{s}[\lambda]\) and \(sh\mathfrak{s}[\lambda]\) is \(Aq(2, \nu)\) restricted to the case of even powers in \(y\) or unrestricted, respectively. The associative product underlying \(hs[\lambda]\) was for the first time introduced in [10]. Even though the algebra is associative by construction the associativity of the Lone-Star product was only conjectured and numerically verified to some extent, the proof was given in [4]. In [19] product for two element of \(h\mathfrak{s}[\lambda]\) was obtained. This product was written in terms of generating functions making it possible to multiply any elements of \(h\mathfrak{s}[\lambda]\). However the algebra itself was represented in the different from the deformed oscillators way. From the deformed oscillators point of view \(sh\mathfrak{s}[\lambda]\) was considered in [20]. Algebra \(Aq(2, \nu)\) in a different from the deformed oscillators point of view was studied in [21].

There is also another approach to higher spin (super)algebras called the factorization by projector [22–24]. The main idea is to start from a larger Weyl algebra

\[ [y^\alpha_1, y^\beta_1] = 2\epsilon_{\alpha_1\beta_1} Y^{AB}, \]  

(1.16)

where \(\alpha, \beta\) are \(\mathfrak{sp}(2)\) indices and \(A, B\) are indices of some (pseudo)orthogonal algebra in \(M\) dimensions (as in [22]). Generic element of this algebra can be written as

\[ f(Y) = \sum_m f^{\alpha_1...\alpha_m} Y^{A_1...A_m}, \]

(1.17)

and product is defined as

\[ f(Y) \star g(Y) = \frac{1}{(2\pi)^M} \int d^M U d^M V f(Y + U) g(Y + V) \exp \left\{ i U^A V^A \right\}. \]

(1.18)

Here Greek and Latin indices are raised and lowered by \(\epsilon_{\alpha\beta}\) and \(\eta^{AB}\) and their inverted versions respectively. The algebra defined above is not simple and one can find projector \(\Delta\)\(^3\) that acts trivially on the elements from the ideal

\[ \Delta \circ a = a \circ \Delta = 0, \quad a \in \mathcal{I}. \]

(1.19)

Then the product in the quotient algebra can be written in the form

\[ g_1 \circ g_2 = f_1 \star f_2 \star \Delta, \quad g_i = f_i \Delta. \]

(1.20)

Even though such expressions do appear in papers they should be treated formally as \(g_i\) belongs to quotient algebra, and one actually has to invent some other method of finding elements of quotient algebra due to divergence of \(\Delta^2\) (see [19]). Equation for the projector for the \(h\mathfrak{s}[\lambda]\) case was found in [25]. In this paper Latin indices of oscillators from (1.16) are \(\mathfrak{o}(1, 2)\) indices and the ideal to be quotient is the one generated by quadratic Casimir operator of \(\mathfrak{o}(1, 2)\). Similar approach was used in [19] but the authors used it to compute traces on algebra.

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\(^{2}\) Two pairs of oscillators \(y\) and \(\bar{y}\) which commutation relations are not deformed as it was done in [17] get stretched in \(\sqrt{z}\) times where \(z\) is the radial Poincare coordinate in AdS.

\(^{3}\) Technically \(\Delta\) found in [22] is not a projector since \(\Delta^2\) diverges.
In the absence of deformation ($\nu = 0$), the product of two elements can be written in the well-known Moyal form

$$f(y) \ast_M g(y) \equiv f(y) \sum_{p=0}^\infty \frac{\nu^p}{p!} \left( \frac{\partial}{\partial y_\alpha} \to \epsilon_{\alpha\beta} \frac{\partial}{\partial y_\beta} \right)^p g(y). \quad (1.21)$$

Main result of this paper is the analogue of the Moyal product for the generating commutation relations (1.2).

The paper is organized as follows: in section 2 structure constants for the associative product in $Aq(2, \nu)$ are presented, in the section 3 some difficulties of integral representation are discussed and possible way to overcome them with the help of Pochhammer representation for Euler beta-function is presented. In section 4 star product for generic formal power series is given. Section 5 deals with analytic expansion in deformation parameter $\nu$ of the product. Conclusion contains discussion of the obtained result, its possible simplifications and future directions.

2. Structure constants for $Aq(2, \nu)$

Structure constants $H(m, n, p, \nu K)$ in (1.5) obtained in [4] depend on the parity of product factors, i.e.

$$H(m, n, p, \nu K) = \begin{cases} A(m, n, p, \nu K), & m \text{ is even}, n \text{ is even}, \\ B(m, n, p, \nu K), & m \text{ is odd}, n \text{ is odd}, \\ C(m, n, p, \nu K), & m \text{ is even}, n \text{ is odd}, \\ D(m, n, p, \nu K), & m \text{ is odd}, n \text{ is even}. \end{cases} \quad (2.1)$$

Since all structure constants in (2.1) are expressible in terms of $A(m, n, p, \nu K)$ in the sequel of this section we assume that $m$ and $n$ are even

$$A(m, n, p, \nu K) = \frac{i^p m! n!}{(m-p)!(n-p)!} \quad _4F_3 \left[ \begin{array}{c} -p, -1, -\nu K, -\nu K \\ 1, 1, 1 \end{array} \right] \left[ \begin{array}{c} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \\ \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \end{array} \right] \left[ \begin{array}{c} 1 - \frac{\nu K}{2} \\ 1 - \frac{\nu K}{2} \\ 1 - \frac{\nu K}{2} \\ 1 - \frac{\nu K}{2} \end{array} \right]. \quad (2.2)$$

Here $\quad _4F_3 \left[ \begin{array}{c} a, b, c, d \\ e, f, g \end{array} \right] \left[ z \right] = \sum_{q=0}^{\infty} \frac{(a)_q (b)_q (c)_q (d)_q}{(e)_q (f)_q (g)_q q!} z^q,$

where $(a)_q$ is the so called ascending Pochhammer symbol defined as

$$(a)_q \equiv \frac{\Gamma(a + q)}{\Gamma(a)}. \quad (2.4)$$
Structure constants for other parities namely odd × odd, even × odd and odd × even cases are given below

\[ B(m+1,n+1,p,\nu K) = A(m,n,p,\nu K) + i(m+n-2p+3+\nu K)A(m,n,p-1,-\nu K) + i^2(m+2-p)(n+2-p) \frac{m+n-2p+3+\nu K}{m+n-2p+3} \frac{m+n-2p+5\nu K}{m+n-2p+3} \times A(m,n,p-2,-\nu K), \] (2.5)

\[ C(m,n+1,p,\nu K) = A(m,n,p,\nu K) + i(m+1-p)m+n-2p+3+\nu K \frac{m+n-2p+3+\nu K}{m+2-2p+3}A(m,n,p-1,-\nu K), \] (2.6)

\[ D(m+1,n,p,\nu K) = A(m,n,p,\nu K) + i(n+1-p)m+n-2p+3-\nu K \frac{m+n-2p+3-\nu K}{m+n-2p+3} \times A(m,n,p-1,\nu K). \] (2.7)

Let us stress that structure constants (2.2) and (2.5)–(2.7) are unambiguously determined by (anti)commutation relations (1.2) and associativity of algebra \( A_{q(2,\nu)} \).

Using structure constants one can multiply any formal power series. But suppose one has to solve an equation of the form

\[ f(y) \ast X(y) = g(y) \] (2.8)

where function \( f \) and \( g \) are known and \( X \) is to be defined. If there is an integral representation of the product as

\[ f(y) \ast X(y) = \int d\tau_1 d\tau_2 f(\tau_1 y) \left[ \sum_{p=0}^{\infty} \frac{i^p}{p!} \left( \frac{\partial}{\partial y_\alpha} \epsilon_{\alpha\beta} \frac{\partial}{\partial y_\beta} \right)^p K(\tau_1, \tau_2; p) \right] X(\tau_2 y), \] (2.9)

then equation (2.8) can be turned into integro-differential equation which might be easier to solve. With the help of Pochhammer representation for the Euler beta-function such representation for the product is constructed.

3. Pochhammer representation for structure constants

3.1 Even × even case

For positive values of \( a, b, c, d, e, f, g \) and for upper arguments bounded by the lower ones, hypergeometric function (2.3) admits integral representation due to integral representation for the Euler beta-function, i.e.

\[ B(x, y) = \int_0^1 dt t^{x-1}(1-t)^{y-1}, \quad x > 0, \; y > 0. \] (3.1)
Figure 1. Pochhammer contour.

For example if \( d > g > 0 \) and \( z \in \mathbb{R} \) one can express \( {}_4F_3 \) as an integral of \( {}_3F_2 \)

\[
{}_4F_3\left[\begin{array}{cccc}
a & b & c & d \\ e & f & g & \cdot \end{array}\right|z\right] = \frac{\Gamma(g)}{\Gamma(d)\Gamma(g-d)} \int_0^1 dt t^{d-1}(1-t)^{g-d-1} {}_3F_2\left[\begin{array}{cc}
a & b & c \\ e & f & \cdot \end{array}\right| tz\right] \quad (3.2)
\]

One can proceed further and express \( {}_3F_2 \) as an integral of \( {}_2F_1 \) provided there is another positive pair of lower and upper arguments where the upper one is bounded by the lower one.

However, since some arguments of hypergeometric function (2.2) that depend on powers of the product factor monomials are negative, the standard formula like (3.2) is not applicable.

\[
\begin{vmatrix}
1 - \nu K & \nu K \\
1 - m & 1 - n \\
\frac{2}{<0} & \frac{2}{<0}
\end{vmatrix}
\]

(3.3)

However there is a remarkable alternative representation for the Euler beta-function, the so-called Pochhammer representation [1]

\[
\int_C dz \, z^{x-1}(1-z)^{y-1} = (1 - e^{2\pi i x})(1 - e^{2\pi i y}) B(x, y), \quad (3.4)
\]

where integration is carried on the Riemann surface defined by the integrand along the contour on figure 1.

Moreover formula (3.4) gives analytic continuation of Euler beta-function to the complex plane \((x, y) \in \mathbb{C}\). Note that the integral in (3.4) is well defined for all values of \( x \) and \( y \), however rhs is not. Indeed suppose \( x \) is some non-integer number and \( y = -N \) for \( N \in \mathbb{N} \) in this case rhs contains uncertainty \( 0 \times \infty \). Computing the integral one obtains

\[
\int_C dz \, z^{x-1}(1-z)^{-N-1} = (-1)^{N+1}(1 - e^{2\pi i x}) \frac{2\pi i}{N!} \frac{\Gamma(x)}{\Gamma(x-N)}. \quad (3.5)
\]

This result might be obtained also as the limit of rhs in (3.4), i.e.

\[
\lim_{\varepsilon \to 0} (1 - e^{2\pi i x})(1 - e^{2\pi i y}) B(x, -N + \varepsilon) = (-1)^{N+1}(1 - e^{2\pi i y}) \frac{2\pi i}{N!} \frac{\Gamma(x)}{\Gamma(x-N)}. \quad (3.6)
\]

\(^4\)Whenever integration contour crosses region \([1, +\infty)\) or \((-\infty, 0]\) it goes to another sheet on the Riemann surface. For rational values of \( x \) and \( y \) the Riemann surface has finite amount of sheets. For generic values in deformation parameter \( \nu \) it may be infinite (as for the complex valued log). Note nonetheless that contour crosses each region twice: once upwards and once downwards thus the contour is closed for any values of \( x \) and \( y \).
Even though the last lower argument of hypergeometric function, namely \( \frac{m+n-2+q+3}{2} \), is positive there is no upper positive argument that fulfills requirements for using Euler representation (3.1). I.e. real part of the upper argument should be bounded by \( \frac{m+n-2q+1}{2} \) which is not the case for generic values of \( \nu \).

Because of the phase factors on the rhs of (3.4) one should use Pochhammer representation twice. Indeed, suppose we want to obtain \( \left[ \left( \frac{1-m}{2} \right)_q \right]^{-1} \) (recall that \( m \) is even) which is by definition
\[
\frac{1}{\left( \frac{1-m}{2} \right)_q} = \frac{\Gamma \left( \frac{1-m}{2} \right)}{\Gamma \left( \frac{1-m}{2} + q \right)}.
\]

(3.7)

Consider an integral
\[
I_1 = \oint_C dz \, z^{-\frac{1-m}{2} - \xi} (1 - z)^{\mu + \xi - 1} = -2i \sin(2\pi \xi) \frac{\Gamma \left( \frac{1-m}{2} - \xi \right)}{\Gamma \left( \frac{1-m}{2} + q \right)} \Gamma (q + \xi). 
\]

(3.8)

Here \( \xi \) is some non-integer number and the phase factor was simplified because \( m \) is even and \( q \) is integer. Note that rhs of (3.8) contains the same gamma-function in denominator as rhs of (3.7). To obtain proper gamma-function in the numerator consider the following integral
\[
I_2 = \oint_C dz \, z^{-\frac{1-m}{2} - 1} (1 - z)^{-\xi - 1} = 2 \left( 1 - e^{-2\pi \xi} \right) \frac{\Gamma \left( \frac{1-m}{2} \right) \Gamma (-\xi)}{\Gamma \left( \frac{1-m}{2} - \xi \right)}.
\]

(3.9)

Product of (3.8) and (3.9) gives
\[
I_1 I_2 = -4i \left( 1 - e^{-2\pi \xi} \right) \sin(2\pi \xi) \Gamma (\xi) \Gamma (-\xi) \frac{\xi_q \Gamma (-\xi)}{\Gamma \left( \frac{1-m}{2} \right)}.
\]

(3.10)

Using the following gamma-function identities
\[
\Gamma (1 - \xi) = (-\xi) \Gamma (-\xi), \quad \Gamma (\xi) \Gamma (1 - \xi) = \frac{\pi}{\sin (\pi \xi)}, 
\]

(3.11)

the prefactor can be simplified and expression (3.10) turns into
\[
I_1 I_2 = -\frac{8\pi}{\xi} \sin (2\pi \xi) e^{-i\pi \xi} \frac{(\xi_q \Gamma (-\xi)}{\Gamma \left( \frac{1-m}{2} \right)}.
\]

(3.12)

Analogously one can represent \( \left[ \left( \frac{1-m}{2} \right)_q \right]^{-1} \) and \( \left[ \left( \frac{m+n-2+q+3}{2} \right)_q \right]^{-1} \) by introducing new non-integer variables \( \eta \) and \( \zeta \) respectively. To reproduce structure constants (2.2) we define the following functions
\[
\mathcal{F}_{\eta,\zeta}(p, \nu K, s_1, t_1, u_1) = \binom{1 - \nu K}{\nu K, \eta, \zeta} \frac{p}{2} \frac{1 - p}{2} \frac{1 - s_1}{2} \frac{1 - t_1}{2} \frac{1 - u_1}{2}.
\]

(3.13)

\[
\mathcal{R}_{\eta,\zeta}(s_1, s_2, t_1, t_2, u_1, u_2) = \left( \frac{1 - s_1}{1 - t_1} \right)^{\nu - 1} \left( \frac{1 - t_1}{1 - u_1} \right)^{\nu - 1} \left( \frac{1 - u_1}{1 - s_2} \right)^{\nu - 1}.
\]

(3.14)
Also we introduce a shorthand notation for the integrals

$$
\int d\Gamma \equiv \oint_{C_1} ds_1 \oint_{C_2} ds_2 \oint_{C_3} dt_1 \oint_{C_4} dt_2 \oint_{C_5} du_1 \oint_{C_6} du_2,
$$

(3.15)

where the integration contours are the Pochhammer contours (figure 1). To see how structure constants are reproduced consider the following expression

$$
\int d\Gamma \left( \frac{u_1 u_2}{s_1 s_2} \right)^n R_{\zeta n}(\nu, \eta, \zeta, t_1, t_2, u_1, u_2) \mathcal{F}_{\zeta n}(p, \nu \zeta, s_1, t_1, u_1) \frac{1}{(u_1 u_2)^p} \left( \frac{u_1 u_2}{t_1 t_2} \right)^n
$$

$$
= \sum_{q=0}^{\infty} \frac{(1 - \frac{\zeta}{\eta})_q (1 - \frac{\eta}{\zeta})_q (1 - \frac{\zeta}{\nu})_q (1 - \frac{\nu}{\zeta})_q}{(\zeta)_q (\eta)_q (\zeta)_q (\nu)_q q!} \int d\Gamma \left( \frac{u_1 u_2}{s_1 s_2} \right)^n \left( \frac{1}{(1 - s_1)^{p+\zeta-1}} \right) \times \\
\times \left( \frac{1}{(1 - s_2)^{p+\eta-1}} \right) \times \left( \frac{1}{(1 - t_1)^{p+\eta-1}} \right) \left( \frac{1}{(1 - t_2)^{p+\eta-1}} \right) \times \\
\times \left( \frac{1}{(1 - u_1)^{p+\zeta-1}} \right) \left( \frac{1}{(1 - u_2)^{p+\zeta-1}} \right)
$$

$$
= \frac{(8\pi)^3 \sin(2\pi \xi) \sin(2\pi \eta) \sin(2\pi \zeta)}{\xi \eta \zeta \cos(\pi \nu + \pi \zeta)}
$$

$$
\times \sum_{q=0}^{\infty} \frac{(1 - \frac{\zeta}{\eta})_q (1 - \frac{\eta}{\zeta})_q (1 - \frac{\zeta}{\nu})_q (1 - \frac{\nu}{\zeta})_q}{(\zeta)_q (\eta)_q (\zeta)_q (\nu)_q q!} \left[ \frac{1}{(1 - m)^{2}} \right] \left[ \frac{\nu \zeta}{2} \right] \left[ \frac{\nu \zeta}{2} \right] \left[ \frac{m + n - 2\eta + 3}{2} \right] \left[ \frac{1 - p}{2} \right]
$$

(3.16)

The additional factor is cancelled by the following constant

$$
C_{\xi, \eta, \zeta} = -\frac{\xi \eta \zeta \cos(\pi \nu + \pi \zeta)}{(8\pi)^3 \sin(2\pi \xi) \sin(2\pi \eta) \sin(2\pi \zeta)}.
$$

(3.17)

Recall that non-integer parameters \( \xi, \eta, \zeta \) define the Riemann surface on which integration in carried out.

Expanding even functions \( f \) and \( g \) in power series of \( y \) and performing all the integration one can show that formula

$$
f(y) \ast g(y) = C_{\xi, \eta, \zeta} \int d\Gamma \left( \frac{u_1 u_2}{s_1 s_2} \right)^n \sum_{p=0}^{\infty} \frac{i^p}{p! (u_1 u_2)^p} \left( \frac{\partial}{\partial y_\alpha} \frac{\partial}{\partial y_\beta} \right)^p
$$

$$
\times \mathcal{R}_{\zeta n}(s_1, s_2, t_1, t_2, u_1, u_2) \mathcal{F}_{\zeta n}(p, \nu \zeta, s_1, t_1, u_1)
$$

(3.18)

gives the same result as if the product was computed with structure constants (2.2). Here notation \( \frac{\partial}{\partial y_\alpha} \) means that derivative acts only on \( f \) and \( \frac{\partial}{\partial y_\beta} \) acts only on \( g \).
In the even case the role of the Klein operator in the decomposition

\[ f(y, \mathcal{K}) = f_0(y) + f_1(y) \mathcal{K} \quad (3.19) \]

is trivial: for functions even in \( y \) the star product is simply the sum of products, i.e.

\[
f(y, \mathcal{K}) * g(y, \mathcal{K}) = (f_0(y) + f_1(y) \mathcal{K}) * (g_0(y) + g_1(y) \mathcal{K}) = f_0(y) * g_0(y) + f_0(y) * g_1(y) \mathcal{K} \\
+ f_1(y) * g_0(y) \mathcal{K} + f_1(y) * g_1(y),
\]

where each product can be computed with the help of (3.18). Note that if either function \( f \) or \( g \) is odd the rhs of (3.18) vanishes because of the factors that appear upon integration over \( s_1 \) or \( t_1 \). Indeed, suppose function \( f \) contains a monomial of power \( m + 1 \) (recall that \( m \) and \( n \) are even positive numbers) then the integral over \( s_1 \) gives

\[
\int_{s_1} ds_1 \frac{\sin(\pi s_1)}{\pi s_1} = (1 - e^{-2\pi i s_1})(1 - e^{2\pi i s_1}) \frac{\Gamma(-\frac{q}{2})}{\Gamma(-\frac{m}{2} + q)} \quad (3.21)
\]

Here \( q \) is not arbitrary and in fact bounded by \( \frac{2}{\pi} \) or \( \frac{\pi}{2} \) due to the structure of the \( \mathcal{F} \)-function (3.13). One can differentiate considered odd monomial \( m + 1 \) times only. Thus the maximal possible value of \( q \) is \( \frac{m}{2} \) which corresponds to the pole in the denominator of (3.21). Hence the integral over \( s_1 \) is equal to zero. The same is of course valid for any odd monomial that might appear in \( g \).

We want to obtain product for all parities in the same fashion as in even \( \times \) even case. To this end we single out a part that in the final expression is obtained by differentiation \( \frac{\partial}{\partial \varphi_y} \) and then rewrite the remaining expression in the way that \( m \) and \( n \) appear only as Pochhammer symbols in the corresponding power series. To compensate the difference from even \( \times \) even case the additional factors of the form

\[
s_1^{\mu_1}(1 - s_1)^{\nu_1}s_2^{\mu_2}(1 - s_2)^{\nu_2}t_1^{\rho_1}(1 - t_1)^{\sigma_1}t_2^{\rho_2}(1 - t_2)^{\sigma_2}(1 - u_1)^{\xi_1}(1 - u_2)^{\xi_2} \quad (3.22)
\]

are inserted. Numbers \( \mu_i \) are to be defined from the transformed versions of structure constants.

### 3.2. Odd \( \times \) odd case

In this section the star product of two \( y \)-odd functions is obtained. The form of the final expression is similar to (3.18). To proceed with integral representation for structure constants (2.5) we rewrite this expression as

\[
\mathcal{B}(m + 1, n + 1, p, \nu \mathcal{K}) = \frac{i^p(m + 1)(n + 1)!}{(m + 1 + p)!((n + 1 + p)! p!)^{\frac{m}{2}}} \left[ \frac{(m + 1 - n)(n + 1 - p)}{(m + 1)(n + 1)} F(m, n, p, -\nu \mathcal{K}) \right. \\
+ \left. \frac{p(m + n - 2p + 3 + \nu \mathcal{K})}{(m + 1)(n + 1)} F(m, n, p - 1, -\nu \mathcal{K}) \right. \\
+ \left. \frac{p(p - 1)}{(m + 1)(n + 1)} \frac{m + n - 2p + 3 + \nu \mathcal{K}}{m + n - 2p + 5 + \nu \mathcal{K}} \frac{m + n - 2p + 5}{m + n - 2p + 3} \mathcal{K} \right] F(m, n, p - 2, -\nu \mathcal{K}). \tag{3.23}
\]
Here the underbraced prefactor was singled out because in the final expression it is obtained by differentiation
\[
\frac{\partial}{\partial \theta} \left( \frac{\theta}{\eta} \right)^n
\]
and the following notation is used for brevity
\[
F(m, n, p, \nu K) = \frac{1 - \nu K}{2} \frac{\nu K}{1 - m} \frac{-p}{2} \frac{1 - p}{2} \binom{m + n - 2p + 3}{2}.
\] (3.24)

Let us now transform each term in square brackets of (3.23) as follows.

3.2.1. \(F(m, n, p, -\nu K)\). The prefactor before \(F(m, n, p, -\nu K)\) can be rewritten as
\[
\frac{(m + 1 - p)(n + 1 - p)}{(m + 1)(n + 1)} = \left(1 + \frac{\nu K}{(\frac{1}{2} -\nu K)}\right) \left(1 + \frac{\nu K}{(\frac{1}{2} -1)}\right).
\] (3.25)

Then using the definition of hypergeometric function (2.3), Pochhammer symbol (2.4) and the gamma-function identities the whole expression can be represented in the form
\[
\frac{(m + 1 - p)(n + 1 - p)}{(m + 1)(n + 1)} F(m, n, p, -\nu K)
= F(m, n, p, -\nu K) + \frac{p}{2} \sum_{q=0}^{\infty} \binom{1 + \nu K}{\frac{1}{2} - q} \binom{\nu K - \nu}{\frac{1}{2} - q} \binom{-p}{\frac{1}{2} - q} \binom{1}{\frac{1}{2} - q} q!
+ \frac{p^2}{4} \sum_{q=0}^{\infty} \binom{1 + \nu K}{\frac{1}{2} - q} \binom{-p}{\frac{1}{2} - q} \binom{-p}{\frac{1}{2} - q} \binom{1}{\frac{1}{2} - q} \binom{1}{\frac{1}{2} - q} q!.
\] (3.26)

A slight modification of Pochhammer symbols from even \(\times\) even case like
\[
\frac{1}{\binom{1}{\frac{1}{2} - q}} \rightarrow \frac{1}{\binom{1}{\frac{1}{2} - 1}}
\] (3.27)
in the integral representation can be easily compensated by introducing additional factors of \(\frac{1}{\nu K}\) for the expression above or \(\frac{1}{\nu K}\) for
\[
\frac{1}{\binom{1}{\frac{1}{2} - q}} \rightarrow \frac{1}{\binom{1}{\frac{1}{2} - 1}}
\] (3.28)
or both like in the last term of (3.26).

3.2.2. \(F(m, n, p - 1, -\nu K)\). Analogously, part with \(F(m, n, p - 1, -\nu K)\) can be represented as
\[
\frac{p(m + n - 2p + 3 + \nu K)}{(m + 1)(n + 1)} F(m, n, p - 1, -\nu K)
= \frac{p}{2} \frac{(m + n - 2p + 3 + \nu K)}{(\frac{1}{2} -1)} \binom{1}{\frac{1}{2} -1} F(m, n, p - 1, -\nu K).
\] (3.29)
Again using definitions and identities (2.3), (2.4) and (3.11) it can be rewritten as

\[
\frac{p(m+n-2p+3+\nu K)}{(m+1)(n+1)} F(m, n, p-1, -\nu K) = \frac{1}{2} \sum_{q=0}^{\infty} \frac{(1 + \frac{\nu K}{2})_q}{(\frac{1}{2}m - 1)_{q+1}} \frac{(\frac{3-p}{2})_q}{(\frac{1}{2}n - 1)_{q+1}} \frac{1}{q!} \left( \frac{1}{(m+n-2p+5)_q} \right). \tag{3.30}
\]

Similarly to the previous case the modified Pochhammer symbols \( \frac{1}{(m+n-2p+3)_q} \) and \( \frac{1}{(m+n-2p+5)_q} \) can be compensated by introducing additional powers of \( u_1, u_2 \), \((1-u_1)\) or \((1-u_2)\).

\[3.2.3. \ F(m, n, p-2, -\nu K). \] Since the procedure is analogous we present below only the chain of transformations

\[
\frac{p(p-1)}{4} \left( \frac{m+n-2p+3}{2} \right) \left( \frac{m+n-2p+5}{2} \right) + \nu K \left( \frac{m+n-2p+3}{2} \right) + \nu K \left( \frac{m+n-2p+3}{2} \right) \sum_{q=0}^{\infty} \frac{(1 + \frac{\nu K}{2})_q}{(\frac{1}{2}m - 1)_{q+1}} \frac{(\frac{3-p}{2})_q}{(\frac{1}{2}n - 1)_{q+1}} \frac{1}{q!} \left( \frac{1}{(m+n-2p+3)_q} \right) \tag{3.31}
\]

To proceed we expand the enumerator

\[
\frac{p(p-1)}{4} \left[ \left( \frac{m+n-2p+3}{2} \right) \left( \frac{m+n-2p+5}{2} \right) + \nu K \left( \frac{m+n-2p+3}{2} \right) + \nu K \left( \frac{m+n-2p+3}{2} \right) \sum_{q=0}^{\infty} \frac{(1 + \frac{\nu K}{2})_q}{(\frac{1}{2}m - 1)_{q+1}} \frac{(\frac{3-p}{2})_q}{(\frac{1}{2}n - 1)_{q+1}} \frac{1}{q!} \left( \frac{1}{(m+n-2p+3)_q} \right) \right] + \nu K \left( \frac{m+n-2p+5}{2} \right)_{q+1} + \nu K \left( \frac{m+n-2p+5}{2} \right)_{q+1} \left( \frac{1 + \nu K}{2} \right) \frac{1}{(m+n-2p+5)_q}. \tag{3.32}
\]

Now we are in a position to write the product of two odd functions with the help of integration. To simplify formulas we introduce a pair of projectors

\[
\Pi_\pm \equiv \frac{1 \pm K}{2}. \tag{3.33}
\]
With the help of projectors the product of two odd functions can be written as

\[
f(y) * g(y)\Pi_\pm = C_{\xi,\eta,\zeta} \int d\Gamma f \left( \sqrt{\frac{u_1u_2}{s_1s_2}} \right) \sum_{p=0}^{\infty} \frac{i^p}{p!} \frac{1}{(u_1u_2)^p} \left( \frac{\partial}{\partial y} \right)^p \left( \frac{1 - s_2}{s_2} \right) \times \mathcal{R}_{\xi,\eta,\zeta}(s_1, s_2, t_1, t_2, u_1, u_2) \frac{1}{u_1u_2} \left\{ 1 - \frac{p}{2\xi} \left( \frac{1 - s_2}{s_2} \right) \right\} \times \left[ 1 - \frac{p}{2\eta} \left( \frac{1 - t_2}{t_2} \right) \right] F_{\xi,\eta,\zeta}(p, \mp, s_1, t_1, u_1) - \frac{p}{2\xi\eta} \left( \frac{1 - s_2}{s_2} \right) \left( \frac{1 - t_2}{t_2} \right) u_1u_2 \left[ \frac{\eta(1 + \zeta)}{1 - u_2} + \frac{\mu}{2} \right] \times \mathcal{F}_{\xi,\eta,\zeta}(p - 1, \mp, s_1, t_1, u_1) + \frac{p(p - 1)}{4\xi\eta} \left( \frac{1 - s_2}{s_2} \right) \left( \frac{1 - t_2}{t_2} \right) \times (u_1u_2)^2 \left[ 1 + \frac{\mu}{\zeta} \left( \frac{1 - u_2}{u_2} \right) + \frac{\nu(2 + \mu)}{4(1 - \zeta)\eta} \left( \frac{1 - u_2}{u_2} \right) \right] \times \mathcal{F}_{\xi,\eta,\zeta}(p - 2, \mp, s_1, t_1, u_1) \right] g \left( \sqrt{\frac{u_1u_2}{t_1t_2}} \right) \Pi_\pm. \tag{3.34}
\]

The product free from projector can be obtained simply as the sum

\[
f(y) * g(y)\Pi_+ + f(y) * g(y)\Pi_- = f(y) * g(y). \tag{3.35}
\]

Note that rhs of (3.34) vanishes due to integration over \(s_1\) or \(t_1\) provided either function \(f\) or \(g\) is even.

3.3. Even \(\times\) odd case

Since transformation of structure constants (2.6) is analogous, we present the final result only

\[
C(m, n + 1, p, \nu\mathcal{K}) = \frac{i^p m!(n + 1)!}{(m - p)!(n + 1 - p)!p!} \left\{ F(m, n, p, -\nu\mathcal{K}) + \frac{\nu\mathcal{K}}{2} \sum_{q=0}^{\infty} \left( \frac{1 + \frac{\nu\mathcal{K}}{2}}{\left( \frac{1 - \frac{\nu\mathcal{K}}{2}}{2} \right)_q} \right) q! \left[ \left( \frac{1 - p}{m + n - 2p + 3} \right)_q - \left( \frac{2 - p}{m + n - 2p + 3} \right)_q \right] \right\}. \tag{3.36}
\]
Star product for even function $f$ and odd function $g$ has the form

$$f(y) \ast g(y)\Pi_{\pm} = C_{\xi,\nu,\zeta} \int d\Gamma \; f\left(\sqrt{\frac{u_1u_2}{s_1s_2}}y\right) \sum_{p=0}^{\infty} \frac{i^p}{p!(u_1u_2)^p} \left(\frac{\partial}{\partial y_0} \partial_{y_0}\right)^p \left(\frac{1}{t_1t_2} \frac{1}{u_1u_2}\right)$$

$$\times R_{\xi,\nu,\zeta}(s_1, s_2, t_1, t_2, u_1, u_2) \left\{ 1 - \frac{p}{2\eta} \left(\frac{1}{t_1} - \frac{1}{t_2}\right) \right\}$$

$$\times F_{\xi,\nu,\zeta}(p, \pm \nu, s_1, t_1, u_1) + \frac{p}{2\eta} \left(\frac{1}{t_1} - \frac{1}{t_2}\right) (u_1u_2) \left[ 1 \pm \frac{\nu}{2\zeta} \left(\frac{1}{u_1} - \frac{1}{u_2}\right) \right]$$

$$\times F_{\xi,\nu,\zeta}(p - 1, \pm \nu, s_1, t_1, u_1) \right) g\left(\sqrt{\frac{u_1u_2}{t_1t_2}}y\right) \Pi_{\pm}, \quad (3.37)$$

rhs of (3.37) vanishes due to integration over $s_1$ or $t_1$ provided either $f$ is not even or $g$ is not odd.

### 3.4. Odd $\times$ even case

Analogously to previous section the transformed structure constants are

$$D(m + 1, n, p, \nu K) = \frac{i^p(m + 1)!n!}{(m + 1 - p)!(n - p)!p!} \{ F(m, n, p, \nu K)$$

$$+ \frac{p}{2} \sum_{q=0}^{\infty} \left(\frac{1}{m - 1}\right)_q \left(\frac{1}{n - 1}\right)_q \left(\frac{m + n - 2p + 1}{2}\right)_q + \left(\frac{m + n - 2p + 3}{2}\right)_q \right\}, \quad (3.38)$$

And the product for odd and even functions is

$$f(y) \ast g(y)\Pi_{\pm}$$

$$= C_{\xi,\nu,\zeta} \int d\Gamma \; f\left(\sqrt{\frac{u_1u_2}{s_1s_2}}y\right) \sum_{p=0}^{\infty} \frac{i^p}{p!(u_1u_2)^p} \left(\frac{\partial}{\partial y_0} \partial_{y_0}\right)^p \left(\frac{1}{t_1t_2} \frac{1}{u_1u_2}\right)$$

$$\times R_{\xi,\nu,\zeta}(s_1, s_2, t_1, t_2, u_1, u_2) \left\{ 1 - \frac{p}{2\xi} \left(\frac{1}{s_1} - \frac{1}{s_2}\right) \right\}$$

$$\times F_{\xi,\nu,\zeta}(p, \pm \nu, s_1, t_1, u_1) + \frac{p}{2\xi} \left(\frac{1}{s_1} - \frac{1}{s_2}\right) (u_1u_2) \left[ 1 \pm \frac{\nu}{2\zeta} \left(\frac{1}{u_1} - \frac{1}{u_2}\right) \right]$$

$$\times F_{\xi,\nu,\zeta}(p - 1, \pm \nu, s_1, t_1, u_1) \right) g\left(\sqrt{\frac{u_1u_2}{t_1t_2}}y\right) \Pi_{\pm}, \quad (3.39)$$

rhs of (3.39) vanishes due to integration over $s_1$ or $t_1$ provided either $f$ is not odd or $g$ is not even.
4. Full star product

Products for different parities (3.18), (3.34), (3.37) and (3.39) schematically have the form

\[ f(y) \ast g(y) \Pi_{\pm} = \int d\Gamma f \left( \sqrt{\frac{u_{1}u_{2}}{s_{1}s_{2}}y} \right) \text{Ker}^{s} \left( \frac{\partial}{\partial y_{\alpha}}\epsilon_{\alpha\beta}, \frac{\partial}{\partial y_{\beta}}, s_{1,2}, t_{1,2}, u_{1,2} \right) g \left( \frac{u_{1}u_{2}}{t_{1}t_{2}}y \right) \Pi_{\pm}, \]

(4.1)

where \text{Ker} for even \times even case from (3.18) is

\[ \text{Ker}^{E} \left( \frac{\partial}{\partial y_{\alpha}}\epsilon_{\alpha\beta}, \frac{\partial}{\partial y_{\beta}}, s_{1,2}, t_{1,2}, u_{1,2} \right) = C_{\xi,\eta,\zeta} \sum_{p=0}^{\infty} \frac{i^{p}}{p!(u_{1}u_{2})^{p}} \left( \frac{\partial}{\partial y_{\alpha}}\epsilon_{\alpha\beta} \frac{\partial}{\partial y_{\beta}} \right)^{p} \times \mathcal{R}_{\xi,\eta,\zeta}(s_{1}, s_{2}, t_{1}, t_{2}, u_{1}, u_{2}) \mathcal{F}_{\xi,\eta,\zeta}(p, \nu, s_{1}, t_{1}, u_{1}) \]

(4.2)

And if functions do not obey certain parity requirement the integral with corresponding kernel \text{Ker} vanishes. Hence the product of two functions can be written as an integral with the sum of kernels for all possible cases, i.e. for generic functions \( f \) and \( g \) the product has the form

\[ f(y) \ast g(y) \Pi_{\pm} = C_{\xi,\eta,\zeta} \int d\Gamma f \left( \sqrt{\frac{u_{1}u_{2}}{s_{1}s_{2}}y} \right) \sum_{p=0}^{\infty} \frac{i^{p}}{p!(u_{1}u_{2})^{p}} \left( \frac{\partial}{\partial y_{\alpha}}\epsilon_{\alpha\beta} \frac{\partial}{\partial y_{\beta}} \right)^{p} \times \mathcal{R}_{\xi,\eta,\zeta}(s_{1}, s_{2}, t_{1}, t_{2}, u_{1}, u_{2}) \mathcal{F}_{\xi,\eta,\zeta}(p, \nu, s_{1}, t_{1}, u_{1}) \]

(4.3)
Here for brevity we choose the same $\xi, \eta$ and $\zeta$ for each of the parities multiplied, though is not necessary while this freedom can be used for simplification in some cases.

Consider case when $\nu \neq 2k, \ k \in \mathbb{Z}$. For even $\times$ even and odd $\times$ even multiplications one can choose

$$\xi = 1 \mp \frac{\nu}{2}, \quad \eta = \pm \frac{\nu}{2}. \quad (4.4)$$

Recall that signs flip due to the projector $\Pi_\pm$. This specific choice allows one to simplify $\mathcal{F}_{\xi,\eta,\zeta}(p, \pm \nu; s_1, t_1, u_1)$ defined by (3.13) and move from $\mathcal{F}_\pm$ to $\mathcal{F}_1$, i.e.

$$\mathcal{F}_{\mp \pm \mp \zeta}(p, \pm \nu; s_1, t_1, u_1) = _2F_1\left[\frac{p}{2}, \frac{1-\nu}{2}, \frac{1}{2} \left(1 - s_1 \right) \left(1 - t_1 \right) \left(1 - u_1 \right)\right]$$

$$= \mathcal{F}_\zeta(p; s_1, t_1, u_1). \quad (4.5)$$

For the remaining cases, namely odd $\times$ odd and even $\times$ even, one can also simplify $\mathcal{F}_{\xi,\eta,\zeta}(p, \mp \nu; s_1, t_1, u_1)$ to (4.5) however keeping the Riemann surfaces the same as in even $\times$ even and odd $\times$ even cases. Indeed this can be obtained by choosing

$$\xi' = \xi - 1 = \mp \frac{\nu}{2}, \quad \eta' = 1 + \eta = 1 \pm \frac{\nu}{2} \quad (4.6)$$

since the Riemann surfaces are defined by non-integer parts of $\xi, \eta$ and $\xi', \eta'$. To completely compensate the difference note that

$$C_{\xi',\eta',\zeta} = \frac{(\xi - 1)(\eta + 1)}{\xi \eta} C_{\xi,\eta,\zeta},$$

$$R_{\xi',\eta',\zeta}(s_1, s_2; t_1, t_2; u_1, u_2) = \frac{s_1(1 - s_2)}{(1 - s_1)} \frac{(1 - t_1)}{t_1(1 - t_2)} R_{\xi,\eta,\zeta}(s_1, s_2; t_1, t_2; u_1, u_2). \quad (4.7)$$

To simplify expression for the product further one can choose $\zeta = \pm \frac{\nu}{2}$. For this specific choice of parameters expression for the product reads

$$f(y) \ast g(y) \Pi_\pm$$

$$= C_{\xi,\eta,\zeta} \int \mathrm{d}^\Gamma f \left( \sqrt{\frac{u_1u_2}{s_1s_2}} \sum_{p=0}^\infty \frac{i^p}{p!} (\frac{\partial}{\partial y_\alpha})^p \frac{\partial}{\partial y_\beta} \right) p$$

$$\times R_{\xi,\eta,\zeta}(s_1, s_2; t_1, t_2; u_1, u_2) \left( \mathcal{F}_\zeta(p; s_1, t_1, u_1) \right.$$}

$$\left. + \sqrt{\frac{s_1s_2}{u_1u_2}} \left( 1 - \frac{p}{2\zeta} \left( \frac{1 - s_2}{s_2} \right) \right) \mathcal{F}_\zeta(p; 1 - s_1, t_1, u_1) \right.$$}

$$\left. + \frac{p}{2\zeta} \left( \frac{1 - s_2}{s_2} \right) \left( u_1 \right) (2u_2 - 1) \mathcal{F}_\zeta(p; 1 - s_1, 1 - t_1, u_1) \right.$$}

$$\left. + \frac{\xi' \eta'}{\xi \eta} \frac{s_1(1 - s_2)}{(1 - s_1)} \frac{(1 - t_1)}{t_1(1 - t_2)} \sqrt{\frac{t_1t_2}{u_1u_2}} \left( 1 - \frac{p}{2\zeta} \left( \frac{1 - t_2}{t_2} \right) \right) \mathcal{F}_\zeta(p; 1 - s_1, 1 - t_1, u_1) \right.$$}

$$\left. + \frac{p}{2\zeta} \left( \frac{1 - t_2}{t_2} \right) u_1 \mathcal{F}_\zeta(p; 1 - s_1, 1 - t_1, u_1) \right)$$

$$16$$
This potentially dangerous pole is cancelled because analytic expansion of the $\mathcal{F}$-function and we leave parameter $\zeta$ arbitrary and $\nu$-independent. With this choice all of the $\nu$-dependence travels from the $\mathcal{F}$-function to the $\mathcal{R}$-function.

In the previous section we required $\nu$ not to be even. However it may be relaxed. Note that all the integrals along Pochhammer contours are well defined for any values of $\nu$. The only trouble comes from the overall constant $C_{\xi,\eta,\zeta}$ which is not analytic and contains the pole of the first order, i.e.

$$C_{1 \mp \nu, \pm \nu \zeta} = -\frac{1}{(8\pi)^3} \frac{1 \mp \frac{1}{\zeta}}{\sin (2\pi (1 \mp \frac{1}{\zeta}))} \frac{\left( \pm \frac{1}{\eta} \right)}{\sin(2\pi \left( \pm \frac{1}{\eta} \right))} \zeta e^{i\pi(1+\zeta)}$$

$$= \frac{1}{(8\pi)^3} \frac{\zeta e^{i\pi \zeta}}{\sin(2\pi \zeta)} \left[ \mp \frac{1}{2\pi \nu} + \frac{1}{4\pi \nu} \nu + \frac{\nu^2}{12} + \cdots \right]. \quad (5.1)$$

This potentially dangerous pole is cancelled because analytic expansion of the $\mathcal{R}$-function starts from first power in $\nu$. Indeed, with (4.4) choice for $\xi$ and $\eta$ it looks as follows

$$\mathcal{R}_{1 \mp \nu, \pm \nu \zeta}(s_1, s_2, t_1, t_2, u_1, u_2)$$

$$= s_1^{{\frac{1}{2}} \mp \frac{1}{\zeta}} (1 - s_1)^{{\frac{1}{2}} \mp \frac{1}{\eta}} s_2^{{\frac{1}{2}} \mp \frac{1}{\eta}}$$

$$\times t_1^{{\frac{1}{2}} \mp \frac{1}{\zeta}} (1 - t_1)^{{\frac{1}{2}} \mp \frac{1}{\eta}} t_2^{{\frac{1}{2}} \mp \frac{1}{\eta}} u_1^{{\frac{1}{2}} \mp \frac{1}{\eta}} u_2^{{\frac{1}{2}} \mp \frac{1}{\eta}} (1 - u_1)^{{\frac{1}{2}} \mp \frac{1}{\eta}}. \quad (5.2)$$

For future convenience we introduce function $r_\zeta(s_1, s_2, t_1, t_2, u_1, u_2)$ defined as follows

$$r_\zeta(s_1, s_2, t_1, t_2, u_1, u_2) = \frac{s_1^{{\frac{1}{2}} \mp \frac{1}{\zeta}} t_1^{{\frac{1}{2}} \mp \frac{1}{\zeta}} u_1^{{\frac{1}{2}} \mp \frac{1}{\zeta}} (1 - u_1)^{{\frac{1}{2}} \mp \frac{1}{\zeta}}}{(1 - s_2^2)(1 - t_1)(1 - t_2)(1 - u_2)^{1+\zeta}}. \quad (5.3)$$

Note that compared to (4.3) in this version of product the kernel is expressed in terms of one function, namely $\mathcal{F}_\zeta(p; s_1, t_1, u_1)$, thus bringing all the parities on equal footing in this sense.

5. Analytic expansion of the star product in $\nu$

In this section we confine ourselves to even $\times$ even case for simplicity. As in previous section we choose $\xi$ and $\eta$ as in (4.4) to simplify the $\mathcal{F}$-function and we leave parameter $\zeta$ arbitrary and $\nu$-independent. With this choice all of the $\nu$-dependence travels from the $\mathcal{F}$-function to the $\mathcal{R}$-function.
which is the $\nu$-independent part of the $R$-function. Each term of the $R$-function that is raised to the $\nu$-th power can be expressed as an exponential, i.e.

$$R_1^{\nu,\nu+\nu_1} = r_1 e^{\pm \nu \log s_1} e^{\pm \nu_1 \log (1-s_1)} e^{\pm \nu_2 \log (1-s_2)} e^{\pm \nu_3 \log (1-s_3)} e^{\pm \nu_4 \log (1-t_1)} e^{\pm \nu_5 \log (1-t_2)}.$$  \hspace{1cm} (5.4)

Here we suppress the arguments for brevity. Now one can expand the exponential as a power series. To show that this expansion starts from the first power one should consider the integrals that appear in computation of star product (3.16).

Integral over $s_1$ looks as follows

$$I_1 = \int_{C_1} e^{\pm \nu \log t_1} e^{\pm \nu_1 \log (1-s_1)} \frac{I_{\nu-2}}{s_1} (1-s_1) \, ds_1$$

$$= \int_{C_1 (1)} \left( \frac{-\nu}{2} \log s_1 \right) \frac{I_{\nu-2}}{s_1} (1-s_1) + \frac{\nu^2}{8} \log^2 s_1 + \frac{\nu^2}{4} \log (1-s_1)$$

$$- \frac{\nu^2}{4} \log (1-s_1) \log s_1 + \ldots \right) \frac{I_{\nu-2}}{s_1} (1-s_1) \, ds_1.$$  \hspace{1cm} (5.5)

Here underlined terms vanish after integration since the integration contour collapses to a point. Hence, up to the second order the integral over $s_1$ is

$$I_1 = (1 \pm \nu \frac{1}{s_1} \log (1-s_1)) \frac{I_{\nu-2}}{s_1} (1-s_1) \, ds_1 + \ldots$$  \hspace{1cm} (5.6)

Thus the pole in (5.1) is cancelled!

Below we provide the results of the expansion for other integrals

$$I_2 = \int_{C_2} \frac{I_{\nu-1}}{s_2} (1-s_2)^{-1} (1-s_2)^{-2} \, ds_2 = \int_{C_2} \frac{I_{\nu-1}}{s_2} (1-s_2)^{-1} \, ds_2 + \ldots$$  \hspace{1cm} (5.7)

$$I_3 = \int_{C_1} \frac{I_{\nu-1}}{s_1} \log (1-s_1) (1-t_1)^{-1} \, ds_1$$

$$= \int_{C_1} \left[ 1 \pm \nu \frac{1}{t_1} \log (1-t_1) \right] \frac{I_{\nu-1}}{t_1} (1-t_1) \, ds_1 + \ldots$$  \hspace{1cm} (5.8)

$$I_4 = \int_{C_2} \frac{I_{\nu-1}}{s_2} (1-s_2)^{-1} \, ds_2 = \int_{C_2} \frac{I_{\nu-1}}{s_2} (1-s_2)^{-1} \, ds_2 + \ldots$$  \hspace{1cm} (5.9)

Note that the zeroth power of $\nu$ in (5.8) survives only for $q = 0$. Hence only the first term of the $F$-function contributes to the underformed star product. Even though integrals over $u_1$ and $u_2$ do not contain $\nu$ we provide expressions for them nonetheless

$$I_{u_1} = \int_{C_1} \frac{I_{\nu-1}}{u_1} (1-u_1)^{\nu-1} \, ds_1, \quad I_{u_2} = \int_{C_2} \frac{I_{\nu-1}}{u_2} (1-u_2)^{\nu-1} \, ds_2.$$  \hspace{1cm} (5.10)
Computing integrals up to the required order in $\nu$ for $q = 0$ one is able to show that

$$I_n I_{n_1} I_{n_2} I_{n_3} C_{\alpha \beta \gamma} \zeta = 1,$$

which means that zeroth order in $\nu$ in the product expansion is precisely the Moyal star product. It is possible to write the product of two even functions in the following form

$$f(y) * g(y) I_\alpha = f(y) * M g(y) I_\alpha + (\pm \nu) \sqrt{215 \pi^3 / 2} \frac{\zeta e^{i\nu\zeta}}{\sin(2\pi \zeta)} \int d\Gamma f \left( \sqrt{\frac{u_1 u_2}{s_1 s_2}} \right)$$

$$\times \sum_{p=0}^{\infty} \frac{i^p}{p! (u_1 u_2)^p} \left( \frac{\partial}{\partial y_1} \frac{\partial}{\partial y_2} \right)^p r_c(s_1, s_2, t_1, t_2, u_1, u_2)$$

$$\times \log(1 - s_1) \left\{ 1 - \log(1 - s_2) - \log \left( \frac{1 - t_1}{t_1} \right) + \log(1 - t_2) \right\}$$

$$+ \frac{1}{2} \log \left( \frac{1 - s_1}{s_1} \right) F_c(p, s_1, t_1, u_1) g \left( \sqrt{\frac{u_1 u_2}{t_1 t_2}} \right) I_\alpha + O(\nu^2).$$

Here $f(y) * M g(y)$ is the usual Moyal product defined by (1.21).

6. Conclusion

The analogue of Moyal differential star product formula (1.21) is obtained. It requires six additional integration parameters. They appear only as certain combinations in arguments of multiplied functions, namely as $\sqrt{\frac{215 \pi^3}{\eta^3}}$ and $\sqrt{\frac{215 \pi^3}{\eta^2}}$. This fact suggests that there should be a proper change of integration variables that decreases the number of integration parameters while the product can be written in the form (2.9).

Even though the structure constants for all parities (2.2) and (2.5)–(2.7) are expressed in terms of structure constants for even $\times$ even case one might need in general more integration parameters to represent (2.5)–(2.7) due to coefficients that depend on $n$ and $m$. The fact that all structure constants can be represented in terms of the same six integration parameters (3.34), (3.37) and (3.39) looks mysterious at this stage. It may signal however that there should be some relations for structure constants or corresponding integration kernels that can lead to simplification of the product formula (4.3).

The case the deformation parameter $\nu$ takes even values is very special for algebra $Aq(2, \nu)$ and should be analysed separately since in this case the amount of terms in structure constants expansion is bounded by $\nu$. When $\nu$ is not even one can choose non-integer parameters $\xi, \eta$ and $\zeta$ in the way that the kernel may be written in terms of hypergeometric function $_2F_1$ (4.8) which is subject to numerous identities and might lead to some simplifications.

The analytic expansion in deformation parameter $\nu$ of the product was analyzed. It is shown that it starts with the Moyal star product with the first order correction is provided (5.12). The proposed procedure is general and may be applied to obtain corrections to the product of any order. This procedure in particular allows one computing corrections in $\nu$ to the structure constants which is generally a highly involved problem already for second order.

Another approach to simplification relies on application of integral identities on $F_{c, p, \alpha}(p, \nu, s_1, t_1, u_1)$ that result from associativity of the algebra $Aq(2, \nu)$. To obtain those identities one can try for example to rewrite the associativity condition on bosonic structure.
constants

\[
A(m, n, p, \nu K) = A(m, n - 2, p) + 2i(m - p + 1)A(m, n - 2, p - 1, \nu K) \\
+ i^2(m - p + 2)(m - p + 1) \frac{m + n - 2p + 3 - \nu K}{m + n - 2p + 3} \\
\times \frac{m + n - 2p + 1 + \nu K}{m + n - 2p + 1} A(m, n - 2, p - 2, \nu K)
\] (6.1)

as dΓ integral (3.15). However it cannot be done as straightforward as for structure constants in sections 3.2–3.4.

As mentioned in introduction the algebra of the deformed oscillators naturally appears in 3D HS gravity [7]. Two main approaches were used for computations in this theory: either realization of the deformed commutation relations with the doubled number of oscillators as in the original paper [7] or using the Lone-Star product directly [26]. Even though formula (4.3) does not look particularly promising for practical computations it can be useful for computation of products of functions that are not just formal power series. Hopefully it can also be used to prove some general results when explicit form of functions to be multiplied is unknown. For example it may be applied to the problem addressed in introduction (counterpart of AdS 4/CFT 3 holography via unfolded formulation of [17]) where exact result for product is not much in use, but maximally flexible expression for the product is required to perform the before mentioned rescaling.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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