Diameters of Graphs with Spectral Radius at most $\frac{3}{2}\sqrt{2}$

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Abstract

The spectral radius $\rho(G)$ of a graph $G$ is the largest eigenvalue of its adjacency matrix. Woo and Neumaier discovered that a connected graph $G$ with $\rho(G) \leq \frac{3}{2}\sqrt{2}$ is either a dagger, an open quipu, or a closed quipu. The reverse statement is not true. Many open quipus and closed quipus have spectral radius greater than $\frac{3}{2}\sqrt{2}$. In this paper we proved the following results. For any open quipu $G$ on $n$ vertices ($n \geq 6$) with spectral radius less than $\frac{3}{2}\sqrt{2}$, its diameter $D(G)$ satisfies $D(G) \geq \frac{2n-4}{3}$. This bound is tight. For any closed quipu $G$ on $n$ vertices ($n \geq 13$) with spectral radius less than $\frac{3}{2}\sqrt{2}$, its diameter $D(G)$ satisfies $\frac{n}{3} < D(G) \leq \frac{2n-2}{3}$. The upper bound is tight while the lower bound is asymptotically tight.

Let $G_{min}^{n,D}$ be a graph with minimal spectral radius among all connected graphs on $n$ vertices with diameter $D$. We applied the results and found $G_{min}^{n,D}$ for some range of $D$. For $n \geq 13$ and $D \in \left[\frac{n}{2}, \frac{2n-7}{3}\right]$, we proved that $G_{min}^{n,D}$ is the graph obtained by attaching two paths of length $D - \left\lfloor \frac{n}{2} \right\rfloor$ and $D - \left\lceil \frac{n}{2} \right\rceil$ to a pair of antipodal vertices of the even cycle $C_{2(n-D)}$. Thus we settled a conjecture of Cioab-van Dam-Koolen-Lee [2], who previously proved a special case $D = \frac{n + 2e}{2}$ for $e = 1, 2, 3, 4$.

1 Introduction

The spectral radius of a graph $G$, denoted by $\rho(G)$, is the largest eigenvalue of its adjacency matrix. Hoffman and Smith [7] [8] [11] determined all connected graphs $G$ with $\rho(G) \leq 2$. The graphs $G$ with $\rho(G) < 2$ are simple Dynkin Diagrams $A_n$, $D_n$, $E_6$, $E_7$, and $E_8$, while the graphs $G$ with $\rho(G) = 2$ are simple extended Dynkin Diagrams $\tilde{A}_n$, $\tilde{D}_n$, $\tilde{E}_6$, $\tilde{E}_7$, and $\tilde{E}_8$. Cvetković et al. [4] gave a nearly complete description of all graphs $G$ with $2 < \rho(G) \leq \sqrt{2 + \sqrt{5}}$. Their description was completed by Brouwer and Neumaier [1]. Wang et al. [13] studied some graphs with spectral radii close to $\frac{3}{2}\sqrt{2}$. Woo and Neumaier [15] proved that any connected graph $G$ with $\sqrt{2 + \sqrt{5}} < \rho(G) < \frac{3}{2}\sqrt{2}$ is one of the following graphs.

1. If $G$ has maximum degree at least 4, then $G$ is a dagger (i.e., a tree obtained by attaching a path to a leaf of the star $S_5$).

2. If $G$ is a tree with maximum degree at most 3, then $G$ is an open quipu (see Figure 1).

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3. If \( G \) contains a cycle, then \( G \) is a closed quipu (see Figure 2).

No (finite) graph has spectral radius exactly \( \frac{3}{2}\sqrt{2} \). The spectral radii of daggers are always in the interval \((\sqrt{2} + \sqrt{5}, \frac{3}{2}\sqrt{2})\). However, some open quipus (and closed quipus) have spectral radii greater than \( \frac{3}{2}\sqrt{2} \).

Either an open quipu or a closed quipu can be determined by the lengths of its internal paths and pendent paths (see Figures 1 and 2). Here an internal path of a graph \( G \) is a path whose internal vertices have degree 2 and the two end vertices have degree at least 3. An internal path is called closed if its two end vertices coincide. The length of an internal path is the number of its edges. The internal path with \( k \) internal vertices has length \( k + 1 \).

Denote by \( P^{(m_0,m_1,...,m_r)}_{(k_0,k_1,...,k_r,k_{r+1})} \) the open quipu with \( r \) internal paths of lengths \( k_1 + 1, ..., k_r + 1 \) and \( r + 3 \) pendent paths of lengths \( k_0, m_0, m_1, ..., m_r, k_{r+1} \). Without loss of generality, we assume \( m_0 \leq k_0 \) and \( m_r \leq k_r + 1 \) through the paper. Denote by \( C^{(m_1,...,m_r)}_{(k_1,...,k_r)} \) the closed quipu with \( r \) internal paths of lengths \( k_1 + 1, ..., k_r + 1 \) and \( r \) pendent paths of lengths \( m_1, ..., m_r \). Here for \( 1 \leq i \leq r \), \( k_i \) measures the number of internal vertices of the \( i \)-th internal path.

For convenience, a T-shape graph is viewed as an open quipu with \( r = 0 \) (see Figure 12). The graph \( P^{(m+1,m,...,m,m+1)}_{(m+1,0,...,0,m+1)} \) (or \( C^{(m,...,m)}_{(0,...,0)} \)) is called the \( m \)-Laundry graph (or the \( m \)-Urchin graph) respectively.

Suppose that \( G \) is a connected graph. The diameter of \( G \), denoted by \( D(G) \), is the maximum distance among all pairs of vertices. We have the following theorems.

**Theorem 1.1** Suppose that \( T \) is an open quipu on \( n \) vertices \( (n \geq 6) \) with \( \rho(T) < \frac{3}{2}\sqrt{2} \). Then the diameter of \( T \) satisfies \( D(T) \geq \frac{2n-4}{3} \). The equality holds if and only if \( T = P^{(1,m)}_{(1,m-2,m)} \) (for \( m \geq 2 \)) as shown by Figure 3.
Theorem 1.2 Suppose that $L$ is a closed quipu on $n$ vertices ($n \geq 13$) with $\rho(L) < \frac{2}{3}\sqrt{2}$. Then the diameter of $L$ satisfies $\frac{2}{3} < D(L) \leq \frac{2n-2}{3}$. Moreover, if $L$ is neither $C_{(2m+3)}^{(m)}$ nor $C_{(2m+5)}^{(m)}$ (see Figure 4), then $D(L) \leq \frac{2n-4}{3}$.

Remark 1.1 The coefficient $\frac{1}{3}$ in the lower bound for $D(L)$ in Theorem 1.2 cannot be improved. Consider the special closed quipus $C_{m,2m+3,r}$ with $m \geq 2$ and even $r \geq 2$ (see Figure 10). Corollary 4.1 implies $\rho(C_{m,2m+3,r}) < \frac{2}{3}\sqrt{2}$ for all $m$. It has order $n = (3m+4)r$ and diameter $D = (m+2)r$. So $\frac{D}{n} = \frac{m+2}{3m+4} \to \frac{1}{3}$ as $m$ goes to infinity.

In 2007, van Dam and Kooij [3] asked an interesting question “which connected graph of order $n$ with a given diameter $D$ has minimal spectral radius?”. A minimizer graph, denoted by $G^{\text{min}}_{n,D}$, is a graph which has the minimal spectral radius among all connected graphs of order $n$ and diameter $D$. Van Dam and Kooij [3] determined $G^{\text{min}}_{n,D}$ for $D \in \{1,2,\lfloor n/2 \rfloor, n-3, n-2, n-1\}$. The minimizer graph $G^{\text{min}}_{n,D}$ is also determined for $D = n-4$ (Yuan-Shao-Liu [5]), for $D = n-5$ (Cioabă-van Dam-Koolen-Lee [2]), and for $D = n-6, n-7, n-8$ (Lan-Lu-Shi [12]). Note $G^{\text{min}}_{n,D}$ is not unique in general.

Cioabă-van Dam-Koolen-Lee [2] posed the following conjecture for $D = \frac{n+e}{2}$ and proved it for $e = 1, 2, 3, 4$.

Conjecture 1.1 (Cioabă-van Dam-Koolen-Lee [2]) For any $e \geq 1$ and sufficiently large $n$ with $n + e$ even, $C^{(\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor)}_{(n/2, n/2-e-2)}$ is the unique minimizer graph $G^{\text{min}}_{n,\frac{n+e}{2}}$.

We settle this conjecture by proving the statement holds for all $n \geq 3e + 14$. It is implied by the following theorem.

Theorem 1.3 For $n \geq 13$ and $\frac{n}{2} \leq D \leq \frac{2n-7}{3}$, $C^{(D-\lfloor \frac{n}{2} \rfloor, D-\lfloor \frac{n}{2} \rfloor)}_{(n-D-1,n-D-1)}$ is the unique minimizer graph $G^{\text{min}}_{n,D}$.

Remark 1.2 It has been observed by van Dam and Kooij [3] and was finally proved by Sun [13] that $C^{(m,m)}_{(2m+2,2m+2)}$ and $P^{(m+1,m+1)}_{(m+1,2m,m+1)}$ have the same spectral radius (see Lemma 2.7). Both graphs have the same $n$ and $D$. Thus $C^{(D-\lfloor \frac{n}{2} \rfloor, D-\lfloor \frac{n}{2} \rfloor)}_{(n-D-1,n-D-1)}$ can not be the unique minimizer graph for $n = 6m + 6$ and $D = 3m + 3$. For $D \geq \frac{2n-2}{3}$, Sun [13] proved that $G^{\text{min}}_{n,D}$ is always a tree.

This paper is organized as follows. We present some useful lemmas in section 2 and determine the spectral radius of a family of special quipus in section 3. The proofs of Theorems 1.1 and 1.2 are given in section 4 while the proof of Theorem 1.3 is given in the last section.
2 Basic notation and Lemmas

2.1 Preliminary results

For a vertex \( v \) in \( G \), the neighborhood of \( v \) in \( G \), denoted by \( N(v) \), is the set \( \{ u : uv \in E(G) \} \). Denote by \( G - v \) the remaining graph of \( G \) after deleting the vertex \( v \) (and all edges incident to \( v \)). Similarly, \( G - u - v \) is the remaining graph of \( G \) after deleting two vertices \( u, v \). We need the following basic facts (see \([6, 7, 9, 13]\)).

Lemma 2.1 \([10]\) Let \( G \) be a graph, \( v \in V(G) \), and \( C(v) \) be the set of all cycles containing \( v \). Let \( e = uv \) be an edge of \( G \), and \( C(e) \) be the set of all cycles containing \( e \). Then the characteristic polynomial \( \phi(G) \) satisfies

\[
\phi(G) = \lambda \phi(G - v) - \sum_{w \in N(v)} \phi(G - w - v) - 2 \sum_{C \in C(v)} \phi(G - C),
\]

\[
\phi(G) = \phi(G - e) - \phi(G - u - v) - 2 \sum_{C \in C(e)} \phi(G - C).
\]

Lemma 2.2 \([6]\) Let \( G_1 \) and \( G_2 \) be two graphs. Then the following statements hold.

1. If \( G_2 \) is a proper subgraph of \( G_1 \), then \( \rho(G_1) > \rho(G_2) \).
2. If \( \phi_{G_2}(\lambda) > \phi_{G_1}(\lambda) \) for all \( \lambda \geq \rho(G_1) \), then \( \rho(G_2) < \rho(G_1) \).
3. If \( \phi_{G_1}(\rho(G_2)) < 0 \), then \( \rho(G_1) > \rho(G_2) \).

![Figure 5: The graphs \( H_1 \) and \( H_2 \)](image)

Lemma 2.3 \([13]\) Let \( G_1 \) and \( G_2 \) be two (possibly empty) graphs with \( a \in V(G_1) \) and \( b \in V(G_2) \), and let \( H_1 \) and \( H_2 \) be two graphs shown in Figure 5. Then \( \rho(H_1) = \rho(H_2) \).

Lemma 2.4 \([7]\) Let \( uv \) be an edge of a connected graph \( G \) of order \( n \), and denote by \( G_{u,v} \) the graph obtained from \( G \) by subdividing the edge \( uv \) once, i.e., adding a new vertex \( w \) and edges \( wu, wv \) in \( G - uv \). Then the following two properties hold.

1. If \( uv \) does not belong to an internal path of \( G \) and \( G \neq C_n \), then \( \rho(G_{u,v}) > \rho(G) \).
2. If \( uv \) belongs to an internal path of \( G \) and \( G \neq P^{(1,1)}_{(1, n-6, 1)} \), then \( \rho(G_{u,v}) < \rho(G) \).

Lemma 2.5 \([14]\) For any positive integer \( m \), we have

\[
\rho(P^{(m,m)}_{(m,0,m)}) < \lim_{m \to \infty} \rho(P^{(m,m)}_{(m,0,m)}) = \sqrt{5}.
\]
Lemma 2.6 ([14]) For any integers \(m_1, m_2 \geq 1\) and \(k_1, k_2\) with \(0 \leq k_1 \leq k_2 - 2\), we have

\[
\rho(C((m_1, m_2)_{(k_1, k_2)}) > \rho(C((m_1, m_2)_{(k_1+1, k_2-1)}).
\]

Lemma 2.7 ([13]) For any integers \(k \geq 2, r_1, r_2 \geq 1\), we have

\[
\rho(C((r_1-1, r_2-1)_{(k, k)}) = \rho(P((r_1, r_2)_{(r_1, k-2, r_2)}), (1)
\]

The two graphs in Lemma 2.7 are shown in Figure 6.

2.2 Our approach

Let \(v\) be a vertex of a graph \(G\). In [12], we introduced two functions (of \(\lambda\)) \(p(G, v)\) and \(q(G, v)\), which satisfy

\[
\begin{pmatrix}
\phi_G \\
\phi_{G-v}
\end{pmatrix} = \begin{pmatrix} 1 & 1 \\ x_2 & x_1 \end{pmatrix} \begin{pmatrix} p(G, v) \\ q(G, v) \end{pmatrix}.
\]

(2)

Here \(x_1, x_2\) are two roots of the equation \(x^2 - \lambda x + 1 = 0\). In this paper, we always assume \(\lambda \geq 2\) and \(x_1 \leq 1 \leq x_2\). The fact \(x_1 + x_2 = \lambda, x_1 x_2 = 1\) will be used later deliberately. Solving \(p(G, v)\) and \(q(G, v)\), we get

\[
\begin{pmatrix} p(G, v) \\ q(G, v) \end{pmatrix} = \frac{1}{x_2 - x_1} \begin{pmatrix} -x_1 & 1 \\ x_2 & -1 \end{pmatrix} \begin{pmatrix} \phi_G \\ \phi_{G-v} \end{pmatrix}.
\]

(3)

For example, let \(v\) be the center of the odd path \(P_{2k+1}\) for \(k \geq 0\). For simplification, we denote \(p(P_{2k+1}, v)\) and \(q(P_{2k+1}, v)\) by \(p_{2k+1}\) and \(q_{2k+1}\) respectively. We have

\[
\begin{pmatrix} p_{2k+1} \\ q_{2k+1} \end{pmatrix} = \frac{x_2^{k+1} - x_1^{k+1}}{(x_2 - x_1)^3} \begin{pmatrix} x_2^{k-1} - 2x_1^{k+1} + x_1^{k+3} \\ x_2^{k-1} - 2x_1^{k+1} + x_1^{k+3} \end{pmatrix}.
\]

(4)

Lemma 2.8 For \(m \geq 0\), let \(G_m\) be a graph constructed from \(H\) by appending a path \(P_{m+1}\) to vertex \(v'\) (see Figure 7). We have

\[
\begin{pmatrix} p(G_0, v) \\ q(G_0, v) \end{pmatrix} = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} \begin{pmatrix} p(H, v') \\ q(H, v') \end{pmatrix}.
\]
Figure 7: Graph $G_m$ ($m \geq 0$).

Generally, for $m \geq 1$, we have
\[
\left( \begin{array}{c}
p(G_m, v) \\ q(G_m, v)
\end{array} \right) = \frac{1}{x_2 - x_1} \left( \begin{array}{cc}
\phi_{P_m} - x_1^{m+2} & x_1 \phi_{P_{m-1}} \\
-x_2 \phi_{P_{m-1}} & x_2^{m+2} - \phi_{P_m}
\end{array} \right) \left( \begin{array}{c}
p(H, v') \\ q(H, v')
\end{array} \right),
\]
where $\phi_{P_m} = \frac{x_2^{m+1} - x_1^{m+1}}{x_2 - x_1}$.

**Proof** For $m = 0$, by Lemma 2.1, we have
\[
\left( \begin{array}{c}
\phi_{G_0} \\ \phi_{G_0-v}
\end{array} \right) = \left( \begin{array}{cc}
\lambda & -1 \\ 1 & 0
\end{array} \right) \left( \begin{array}{c}
\phi_H \\ \phi_{H-v'}
\end{array} \right).
\]
Combining it with Equations (2) and (3), we get
\[
\left( \begin{array}{c}
p(G_1, v) \\ q(G_1, v)
\end{array} \right) = \left( \begin{array}{cc}
1 & 1 \\ x_2 & x_1
\end{array} \right)^{-1} \left( \begin{array}{cc}
\lambda & -1 \\ 1 & 0
\end{array} \right) \left( \begin{array}{cc}
1 & 1 \\ x_2 & x_1
\end{array} \right) \left( \begin{array}{c}
p(H, v') \\ q(H, v')
\end{array} \right) = \left( \begin{array}{cc}
x_1 & 0 \\ 0 & x_2
\end{array} \right) \left( \begin{array}{c}
p(H, v') \\ q(H, v')
\end{array} \right).
\]

For $m \geq 1$, by Lemma 2.1, we have
\[
\left( \begin{array}{c}
\phi_{G_m} \\ \phi_{G_m-v}
\end{array} \right) = \left( \begin{array}{cc}
\phi_{P_{m+1}} & -\phi_{P_m} \\ \phi_{P_m} & 0
\end{array} \right) \left( \begin{array}{c}
\phi_H \\ \phi_{H-v'}
\end{array} \right).
\]
Similarly we get
\[
\left( \begin{array}{c}
p(G_m, v) \\ q(G_m, v)
\end{array} \right) = \left( \begin{array}{cc}
1 & 1 \\ x_2 & x_1
\end{array} \right)^{-1} \left( \begin{array}{cc}
\phi_{P_{m+1}} & -\phi_{P_m} \\ \phi_{P_m} & 0
\end{array} \right) \left( \begin{array}{cc}
1 & 1 \\ x_2 & x_1
\end{array} \right) \left( \begin{array}{c}
p(H, v') \\ q(H, v')
\end{array} \right) = \frac{1}{x_2 - x_1} \left( \begin{array}{cc}
\phi_{P_m} - x_1^{m+2} & x_1 \phi_{P_{m-1}} \\
-x_2 \phi_{P_{m-1}} & x_2^{m+2} - \phi_{P_m}
\end{array} \right) \left( \begin{array}{c}
p(H, v') \\ q(H, v')
\end{array} \right).
\]
The proof is completed.\[\square\]
We define $B_m$, $d^{(1)}_m$, and $d^{(2)}_m$ as follows,
\[
B_m = \frac{1}{x_2 - x_1} \left( \begin{array}{cc}
\phi_{P_m} - x_1^{m+2} & x_1 \phi_{P_{m-1}} \\
-x_2 \phi_{P_{m-1}} & x_2^{m+2} - \phi_{P_m}
\end{array} \right),
\]
\[
d^{(1)}_m = \phi_{P_m} - x_1^{m+2} = \frac{x_1^{m+3} - 2x_1^{m+1} + x_2^{m+1}}{x_2 - x_1},
\]
\[
d^{(2)}_m = x_2^{m+2} - \phi_{P_m} = \frac{x_2^{m+3} - 2x_2^{m+1} + x_1^{m+1}}{x_2 - x_1}.
\]
By a simple calculation, we have

\[ x_2^{m+2} \phi P_m - d_{m+1}^{(1)} x_1^{m+1} = (x_2^{m+2} - x_1^{m+2}) d_{m}^{(1)}, \tag{5} \]

\[ x_1^{m+2} \phi P_m + d_{m+1}^{(2)} x_2^{m+1} = (x_2^{m+2} - x_1^{m+2}) d_{m}^{(2)}, \tag{6} \]

and

\[ d_m^{(1)} x_2 - d_m^{(2)} x_1 = 2 \phi P_{m-1}. \tag{7} \]

**Remark 2.1** The following equations are equivalent to each other:

\[ d_m^{(2)} = \frac{2 \phi P_{m-1} x_1^k}{1 - x_1^{k+1}}, \]

\[ d_m^{(2)} x_2^k - d_m^{(1)} x_1^k = 2 \phi P_{m-1}, \]

\[ d_m^{(2)} = 2 \phi P_{m-1} x_1^k + d_m^{(1)} x_1^{2k}, \]

\[ d_m^{(2)} = d_m^{(1)} x_1^{k-1}, \]

\[ d_m^{(2)} x_2^{k-1} = d_m^{(1)} x_1^{k-1}. \]

If “=” is replaced by “\( \geq \)”, then these inequalities are still equivalent to each other.

These equivalences can be proved by Equation (7). The details are omitted.

### 3 Special Quipus

![Figure 8: A family of special trees: \( P_{m,k,r} \) for \( r \geq 2 \) and \( k \geq 1 \).](image)

![Figure 9: A family of special trees: \( P_{m,k,1} \) for \( k \geq 2 \).](image)
It has been already known that $\rho(P_{(1,1)}^{(1,1)}{(1,n-6,1)}) = 2$ and $\rho(C_n) = 2$ for all $n \geq 6$. This is actually a trivial case ($m = 0$). For any positive $s$, we define $A^s = \begin{pmatrix} x_1^s & 0 \\ 0 & x_2^s \end{pmatrix}$. For $k = 2s + 1$, we have the following equation to use later

$$A^sB_mA^{s+1} = \frac{1}{x_2 - x_1} \begin{pmatrix} d_m^{(1)} x_1^{k} & \phi_{P_{m-1}} \\ -\phi_{P_{m-1}} & d_m^{(2)} x_2^{k} \end{pmatrix}.$$  

(8)

**Lemma 3.1** For any integers $r, m, k \geq 1$ (except for $r = k = 1$), the spectral radius of the open quipu $P_{m,k,r}$ is the largest root $\rho_{m,k}$ of the equation $d_m^{(2)} = \frac{2\phi_{P_{m-1}} x_1^k}{1-x_1^k}$.

**Proof** Let $v$ be the leftmost vertex of $P_{m,k,r}$ and $s = (k-1)/2$. For $r \geq 2$, by Lemma 2.1 and Lemma 2.8 we have

$$\phi_{P_{m,k,r}} = (1,1) \begin{pmatrix} p(P_{m,k,r,v}) \\ q(P_{m,k,r,v}) \end{pmatrix} = (1,1)A^{m+1}B_{m+1}A^{k-1}B_mA^kB_mA^{k-1}B_mA^{m+1}A^m \begin{pmatrix} p_1 \\ q_1 \end{pmatrix}$$

$$= \frac{1}{(x_2 - x_1)^3} (d_m^{(1)} x_1^{m+1} - x_2^{m+2} \phi_{P_m} + x_1^{m+2} \phi_{P_m} + d_m^{(2)} x_2^{m+1})$$

$$A^s(A^sB_mA^{s+1})^{-1}A^s \begin{pmatrix} x_2^{m+2} \phi_{P_m} - d_m^{(1)} x_1^{m+1} \\ x_1^{m+2} \phi_{P_m} + d_m^{(2)} x_2^{m+1} \end{pmatrix}$$

$$= \frac{(x_2^{m+2} - x_1^{m+2})^2}{(x_2 - x_1)^3} (-d_m^{(1)} x_1^s, d_m^{(2)} x_2^s)(A^sB_mA^{s+1})^{-1} \begin{pmatrix} d_m^{(1)} x_1^k \\ d_m^{(2)} x_2^k \end{pmatrix}.$$  

In the last step, we applied Equations (5) and (6).

Now we prove that $\rho_{m,k}$ is a root of $\phi_G$. At $\lambda = \rho_{m,k}$, by Remark 2.1 we have

$$d_m^{(2)} x_2^s = d_m^{(1)} x_1^s \quad \text{and} \quad d_m^{(2)} x_2^k - \phi_{P_{m-1}} = d_m^{(1)} x_1^k + \phi_{P_{m-1}}.$$  

Thus, by Equation (8) we get

$$(A^sB_mA^{s+1}) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{x_2 - x_1} \begin{pmatrix} d_m^{(1)} x_1^k & \phi_{P_{m-1}} \\ -\phi_{P_{m-1}} & d_m^{(2)} x_2^k \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{d_m^{(1)} x_1^k + \phi_{P_{m-1}}}{x_2 - x_1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.  $$

At the point $\lambda = \rho_{m,k}$, we have

$$\phi_{P_{m,k,r}}(\rho_{m,k}) = \frac{(x_2^{m+2} - x_1^{m+2})^2}{(x_2 - x_1)^{r+2}} (d_m^{(1)} x_1^k + \phi_{P_{m-1}})^{r-1}(d_m^{(1)} x_1^k)^2 (1,1) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0.$$
It remains to prove \( \phi_G(\lambda) > 0 \) for all \( \lambda > \rho_{m,k} \).

By Remark 2.1 for \( \lambda > \rho_{m,k} \), we have \( d_m^{(2)} x_2^k - \phi_{P_{m-1}} > d_m^{(1)} x_1^k + \phi_{P_{m-1}} \) (and \( d_m^{(2)} x_2^s > d_m^{(1)} x_1^s \)). Observe that \( A^s B_m A^{s+1} \) maps the region \( \{(z_1, z_2): z_2 \geq z_1 > 0\} \) into itself. By induction on \( r \), \( (A^s B_m A^{s+1})^{r-1} \) maps the region \( \{(z_1, z_2): z_2 \geq z_1 > 0\} \) into itself.

Let
\[
\left( \frac{z_1}{z_2} \right) = (A^s B_m A^{s+1})^{r-1} \left( \frac{d_m^{(1)} x_1^s}{d_m^{(2)} x_2^s} \right).
\]

Since \( d_m^{(2)} x_2^s > d_m^{(1)} x_1^s > 0 \) for all \( \lambda > \rho_{m,k} \), we have \( z_2 > z_1 > 0 \). Thus,
\[
\phi_{P_{m,k,r}}(\rho_{m,k}) = \frac{(x_2^{m+2} - x_1^{m+2})^2}{(x_2 - x_1)^3} (d_m^{(2)} x_2^s - d_m^{(1)} x_1^s) > 0.
\]

For \( r = 1 \), by the similar calculation, we have
\[
\phi_{P_{m,k,1}} = (1, 1) A^{m+1} B_{m+1} A^{k-2} B_{m+1} A_m \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} = \frac{(x_2^{m+2} - x_1^{m+2})^2}{(x_2 - x_1)^3} (-d_m^{(1)}, d_m^{(2)}) A^{k-1} \begin{pmatrix} d_m^{(1)} \\ d_m^{(2)} \end{pmatrix} = \frac{(x_2^{m+2} - x_1^{m+2})^2}{(x_2 - x_1)^3} \left( (d_m^{(2)})^2 x_2^{k-1} - (d_m^{(1)})^2 x_1^{k-1} \right).
\]

So, \( \lambda = \rho(P_{m,k,1}) \) is the largest root of \( d_m^{(2)} x_2^{k-1} - d_m^{(1)} x_1^{k-1} = 0 \), which is equivalent to \( d_m^{(2)} = \frac{2 \phi_{P_{m-1}} x_1^k}{1-x_1^k} \) by Remark 2.1.

The proof of the lemma is finished. \( \square \)

Figure 10: \( C_{m,k,1} \) and \( C_{m,k,r} \) (\( r \geq 2 \)
Lemma 3.2 For \( m \geq 1, k \geq 2, \) and \( r \geq 1, \) the spectral radius of the closed quipu \( C_{m,k,r} \) is also \( \rho_{m,k}. \)

**Proof** We observe that \( C_{m,k,r} \) is a graph covering of \( C_{m,k,1}. \) The spectrum of \( C_{m,k,1} \) is a subset of \( C_{m,k,r}. \) The Perron-Frobenius vector of \( C_{m,k,1} \) can be lifted as the Perron-Frobenius vector of \( C_{m,k,r}. \) Hence, \( \rho(C_{m,k,1}) = \rho(C_{m,k,r}) \) for all \( r \geq 2. \) By Lemmas 2.7 and 3.1 we have \( \rho(C_{m,k,2}) = \rho(P_{m,k,1}) = \rho_{m,k} \) for \( k \geq 2. \) Hence, \( \rho(C_{m,k,r}) = \rho_{m,k} \) for all \( r \geq 1 \) and \( k \geq 2. \)

4 Quipus with spectral radii bounded by \( \frac{3\sqrt{2}}{2} \)

In this section, we will describe those open quipus and closed quipus with spectral radii less than \( \frac{3\sqrt{2}}{2}. \)

4.1 A Lemma

**Lemma 4.1** For \( i, j, m, m' \geq 1 \) and \( k \geq 0, \) we have the following results on the spectral radius of the tree \( P^{(m,m')}_{(i,k,j)} \) (shown in Figure 11).

1. \( \lim_{i,j \to \infty} \rho(P^{(m,m')}_{(i,k,j)}) = \begin{cases} > \frac{3\sqrt{2}}{2} & \text{if } m, m' \geq 2 \text{ and } k \leq m + m' \text{ or one of } m \text{ and } m' \text{ is } 1, \\ \frac{3\sqrt{2}}{2} & \text{if } (m, m', k) = (1, 1, 1), \\ < \frac{3\sqrt{2}}{2} & \text{otherwise.} \end{cases} \)

2. \( \lim_{j \to \infty} \rho(P^{(m,m')}_{(m,k,j)}) = \begin{cases} > \frac{3\sqrt{2}}{2} & \text{if } m \geq 2 \text{ and } k \leq m + m' - 1, (m, m', k) \neq (2, 1, 2), (2, 2, 3); \\ \frac{3\sqrt{2}}{2} & \text{if } m = 1 \text{ and } k \leq m' - 2, \\ < \frac{3\sqrt{2}}{2} & \text{otherwise.} \end{cases} \)

3. \( \rho(P^{(m,m')}_{(m,k,m')}) = \begin{cases} > \frac{3\sqrt{2}}{2} & \text{if } m, m' \geq 2 \text{ and } k \leq m + m' - 2, (m, m', k) \neq (2, 2, 2); \\ \frac{3\sqrt{2}}{2} & \text{if } m = 1 \text{ and } k \leq m' - 3, \\ < \frac{3\sqrt{2}}{2} & \text{otherwise.} \end{cases} \)

![Figure 11: The graph \( P^{(m,m')}_{(i,k,j)}. \)](image)

**Proof** Similar to the computation in Lemma 3.1, we have

\[
\phi_{P^{(m,m')}_{(i,k,j)}} = (1, 1) A^i B_m A^k B_{m'} A^{j-1} \left( \begin{array}{c} p_1 \\ q_1 \end{array} \right) \\
= \frac{x_2^{i+j+1}}{x_2 - x_1} (x_1^2, 1) B_m A^k B_{m'} \left( \begin{array}{c} -x_1^{2(j+1)} \\ 1 \end{array} \right).
\]
By Lemma 2.5 the spectral radii of all graphs considered in the lemma are in $[2, \sqrt{5}]$. We can restrict $\lambda$ to this interval.

For item 1, let $\rho = \lim_{i,j \to \infty} \rho(P^{(m,m')}_{(i,k,j)})$. Observe that $\rho$ is the largest root of the function

$$(0, 1)B_mA^kB_{m'} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{(x_2-x_1)^2}(-x_2\phi_{P_{m-1}}',d_m^{(2)})A^k \left( \frac{x_1\phi_{P_{m'}-1}}{d_m^{(2)}} \right) = \frac{x_2^{m+m'+k+2}}{(x_2-x_1)^4} f_{m,m',k}(\lambda).$$

Here $f_{m,m',k}(\lambda) = (x_2^2 - 2 + x_1^{2m+2})(x_2^2 - 2 + x_1^{2m'+2}) - x_1^{2k+2}(1-x_1^2)(1-x_1^{2m'}).$ The dominating term in $f_{m,m',k}(\lambda)$ is $(x_2^2 - 2)^2$. We have $\lim_{\lambda \to \infty} f_{m,m',k}(\lambda) = \infty.$

On one hand, to prove $\rho > \frac{3}{2} \sqrt{2}$, we will show $f_{m,m',k}(\frac{3}{2} \sqrt{2}) < 0$. On the other hand, to prove $\rho < \frac{3}{2} \sqrt{2}$, we will show $f_{m,m',k}(\lambda) > 0$ for all $\lambda \geq \frac{3}{2} \sqrt{2}$.

We assume $m,m' \geq 2$. Note that $x_2$ takes the value $\sqrt{2}$ at $\lambda = \frac{3}{2} \sqrt{2}$. If $k \leq m + m'$, then

$$f_{m,m',k} \left( \frac{3}{2} \sqrt{2} \right) = \frac{1}{2m+1} \frac{1}{2m'+1} - \frac{1}{2k+1} \left( 1 - \frac{1}{2m} \right) \left( 1 - \frac{1}{2m'} \right)$$

$$\leq \frac{1}{2m+m'+1} \left( \frac{1}{2} - \left( 1 - \frac{1}{2m} \right) \left( 1 - \frac{1}{2m'} \right) \right) < 0,$$

because of $(1 - \frac{1}{2m})(1 - \frac{1}{2m'}) \geq \frac{3}{4} \cdot \frac{3}{4} > \frac{1}{4}$. If $k \geq m + m' + 1$, then for $\lambda \geq \frac{3}{2} \sqrt{2}$ we have

$$f_{m,m',k}(\lambda) > x_1^{2(m-m')+4} - x_1^{2k+2} \geq 0.$$

Here we applied the fact $x_2^2 \geq 2$ and $0 < 1 - \frac{1}{2m}, 1 - \frac{1}{2m'} < 1$ for $\lambda > \frac{3}{2} \sqrt{2}$.

Now we assume one of $m$ and $m'$ is 1, say $m = 1$. If $k \leq m + m' - 1 = m'$, then

$$f_{1,m',k} \left( \frac{3}{2} \sqrt{2} \right) = \frac{1}{2m+3} - \frac{1}{2k+2} \left( 1 - \frac{1}{2m} \right) \leq \frac{1}{2m'+2} \left( \frac{1}{2m'} - \frac{1}{2} \right) \leq 0.$$

The equality $f_{m,m',k} \left( \frac{3}{2} \sqrt{2} \right) = 0$ holds if and only if $m = m' = k = 1$.

If $k \geq m + m' = m' + 1$, then for $\lambda \geq \frac{3}{2} \sqrt{2}$ we have

$$f_{1,m',k}(\lambda) = (x_2^2 - 2 + x_1^4)(x_2^2 - 2 + x_1^{2m'+2}) - x_1^{2k+2}(1-x_1^2)(1-x_1^{2m'})$$

$$> (x_2^2 - 2 + x_1^4)x_1^{2m'+2} - x_1^{2m'+4}(1-x_1^2)$$

$$= x_1^{2m'+2}(x_2^2 - 2)(1-x_1^4)$$

$$\geq 0.$$

Overall, the proof of item 1 is completed.

For item 2, let $\rho' = \lim_{j \to \infty} \rho(P^{(m,m')}_{m,k,j})$. A similar calculation shows that $\rho'$ is the largest root of the following function

$$(x_1^{2m}, 1)B_mA^kB_{m'} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{(x_2^{m+1} - x_1^{m+1})x_2^{m'+k+1}}{(x_2 - x_1)^4} g_{m,m',k}(\lambda),$$

11
Then we have the following statements hold.

**Theorem 4.1** Suppose an open quipu \( P^{(m_0, \ldots , m_r)}_{(m_0, k_1, \ldots , k_r, m_r)} \) has spectral radius less than \( \frac{3}{2} \sqrt{2} \). Then the following statements hold.

1. For \( 2 \leq i \leq r - 1 \), we have \( k_i \geq m_{i-1} + m_i \). Moreover if \( m_{i-1}, m_i \geq 2 \), then \( k_i \geq m_{i-1} + m_i + 1 \).

2. We have \( k_1 \geq m_0 + m_1 \) if \( m_0 \geq 2 \); and \( k_1 \geq m_1 - 1 \) if \( m_0 = 1 \).

3. We have \( k_r \geq m_r + m_{r-1} \) if \( m_r \geq 2 \); and \( k_r \geq m_{r-1} - 1 \) if \( m_r = 1 \).

The necessary conditions for \( \rho(P^{(m_0, \ldots , m_r)}_{(m_0, k_1, \ldots , k_r, m_r)}) < \frac{3}{2} \sqrt{2} \) are quite good as evidenced by the following theorem.

**Theorem 4.1** Suppose that an open quipu \( P^{(m_0, \ldots , m_r)}_{(m_0, k_1, \ldots , k_r, m_r)} \) satisfies

1. \( m_0, m_r \geq 2 \);

2. \( k_i \geq m_{i-1} + m_i + 3 \) for \( 2 \leq i \leq r - 1 \);

3. \( k_j \geq m_{j-1} + m_j + 1 \) for \( j = 1, r \).

Then we have \( \rho(P^{(m_0, \ldots , m_r)}_{(m_0, k_1, \ldots , k_r, m_r)}) < \frac{3}{2} \sqrt{2} \).
Proof. Denote \( l_1 = m_0 + m_1 + 1, l_r = m_{r-1} + m_r + 1 \) and \( l_i = m_{i-1} + m_i + 3 \) for \( 2 \leq i \leq r - 1 \). Let \( G = P_{(m_0, l_1, \ldots, l_r)}^{(m_1, m_2, \ldots, m_r)} \). By Lemma 2.4, we get \( \rho(P_{(m_0, k_1, \ldots, k_r)}^{(m_1, m_2, \ldots, m_r)}) \leq \rho(G) \). We have

\[
\phi_G = (1, 1)A^{m_0}B_{m_1}A^{l_1}B_{m_1}A^{l_2} \cdots A^{l_{r-1}}B_{m_{r-1}}A^{l_r}B_{m_r}A^{m_r - 1}\left( \begin{array}{c} p_1 \\ q_1 \end{array} \right)
\]

\[
= \frac{1}{(x_2 - x_1)^3}(d_{ma-1}^{(1)}x_1^{m_0} - x_2^{m_0+1}\phi_{P_{ma-1}}x_1^{m_0+1} + d_{ma}^{(2)}x_1^{m_0})
\]

\[
A^{l_1}B_{m_1}A^{l_2} \cdots A^{l_{r-1}}B_{m_{r-1}}A^{l_r+1}\left( \begin{array}{c} x_2^{m_{r+1}} - d_{ma-1}^{(1)}x_1^{m_{r+1}} + d_{ma}^{(2)}x_1^{m_r} \\ x_1^{m_{r+1}} - d_{ma-1}^{(1)}x_1^{m_{r}+1} + d_{ma}^{(2)}x_1^{m_r} \end{array} \right)
\]

\[
= \frac{(x_2^{m_{r+1}} - x_1^{m_{r+1}})(x_2^{m_r+1} - x_1^{m_r+1})}{(x_2 - x_1)^3}(-d_{ma-1}^{(1)}, d_{ma}^{(2)})A^{l_1}B_{m_1}A^{l_2} \cdots A^{l_{r-1}}B_{m_{r-1}}A^{l_r+1}\left( \begin{array}{c} d_{ma-1}^{(1)}x_1^{m_r} \\ d_{ma}^{(2)}x_1^{m_r} \end{array} \right)
\]

Since \( m_0, m_r \geq 2 \), by Corollary 4.1, we get

\[
d_{ma-1}^{(1)}x_2^{m_0} > d_{ma-1}^{(2)}x_1^{m_0} \quad \text{and} \quad d_{ma}^{(1)}x_2^{m_r} > d_{ma}^{(2)}x_1^{m_r}
\]

for all \( \lambda \geq \frac{3\sqrt{2}}{2} \).

Observe that \( A^{m+1}B_mA^{m+2} (m \geq 1) \) maps the region \( \{(x, y) | x < y\} \) to itself. Now repeatedly apply this fact for \( m = m_{r-1}, \ldots, m_1 \). We get \( \phi_G(\lambda) \geq 0 \) for all \( \lambda \geq \frac{3\sqrt{2}}{2} \). Thus,

\[
\rho(P_{(m_0, k_1, \ldots, k_r)}^{(m_1, m_2, \ldots, m_r)}) \leq \rho(G) \leq \frac{3\sqrt{2}}{2}.
\]

The proof is completed. \( \square \)

4.2 Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. Note that all the T-shape trees (see Figure 12) have spectral radii less than \( \frac{3\sqrt{2}}{2} \) and satisfy \( 3D > 2n - 4 \).

![Figure 12: The T-shape trees](image)

Now assume \( r \geq 1 \). Write \( T \) as \( P_{(k_0, k_1, \ldots, k_r)}^{(m_0, m_1, \ldots, m_r)} \) with \( m_i \geq 1 \) for \( i = 0, 1, \ldots, r \), \( k_0 \geq m_0 \), \( k_{r+1} \geq m_r \), \( k_j \geq 0 \) for \( j = 1, \ldots, r \).

Case 1: \( r = 1 \). Here \( T = P_{(k_0, k_1, k_2)}^{(m_0, m_1)} \). Without loss of generality, we assume \( m_0 \leq m_1 \).
Note that $T$ contains the subgraph $P_{(m_0,m_1)}^{(m_0,m_1)}$. Since $\rho(P_{(m_0,k_1,m_1)}^{(m_0,m_1)}) \leq \rho(T) < \frac{3}{2} \sqrt{2}$, by Item 3 of Lemma 4.1, we must have $k_1 \geq m_0 + m_1 - 3$; the equality holds if and only if $m_0 = 1$. Together with $k_0 \geq m_0$ and $k_2 \geq m_1$, we get

$$3D - (2n - 4) \geq 3(k_0 + k_1 + k_2 + 1) - (2(k_0 + k_1 + k_2 + m_0 + m_1) - 4) = k_0 + k_1 + k_2 - 2m_0 - 2m_1 + 3 \geq m_0 + (m_0 + m_1 - 3) + m_1 - 2m_0 - 2m_1 + 3 = 0.$$ When the equality $3D - (2n - 4) = 0$ holds, we must have $k_0 = m_0 = 1$, $k_1 = m_1 - 2$, and $k_2 = m_1$. In this case, we get the graph $P_{(1,m_1-2,m_1)}^{(1,m_1)}$, whose spectral radius is less than $\frac{3}{2} \sqrt{2}$.

**Case 2:** $r = 2$. Here $T = P_{(k_0,k_1,k_2,k_3)}^{(m_0,m_1,m_2)}$. Assume $m_0 \leq m_2$ without loss of generality.

Note that $T$ contains the subgraph $P_{(m_0,m_1,m_2)}^{(m_0,m_1,m_2)}$. For any $i$ we have

$$\frac{3}{2} \sqrt{2} > \rho(T) \geq \rho(P_{(m_0,m_1,m_2)}^{(m_0,m_1,m_2)}) \geq \lim_{i \to \infty} \rho(P_{(m_0,m_1)}^{(m_0,m_1)}).$$

By Item 2 of Corollary 4.2, we have $k_1 \geq m_0 + m_1 - 2$ with the equality if and only if $m_0 = 1$. By symmetry, we also have $k_2 \geq m_1 + m_2 - 2$ with the equality if and only if $m_2 = 1$. Thus, we get

$$3D - (2n - 4) \geq 3(k_0 + k_1 + k_2 + k_3 + 2) - 2(m_0 + m_1 + m_2 + k_0 + k_1 + k_2 + k_3 + 3) + 4 = k_0 + k_1 + k_2 + k_3 - 2(m_0 + m_1 + m_2) + 4 \geq m_0 + (m_0 + m_1 - 2) + (m_1 + m_2 - 2) - 2(m_0 + m_1 + m_2) + 4 = 0.$$ When the equality holds, we must have $k_3 = m_2 = k_0 = m_0 = 1$ and $k_1 = k_2 = m_1 - 1$. We get the graph $T = P_{(1,m_1-1,m_1-1,1)}^{(1,m_1)}$, which has spectral radius greater than $\frac{3}{2} \sqrt{2}$ shown as follows. For $m_1 = 1$, we can get $\rho(P_{(1,0,0,1)}^{(1,1,1)}) > \frac{3}{2} \sqrt{2}$ by a straight calculation. For $m_1 \geq 2$, by Lemma 2.3 and Item 3 of Lemma 4.1 we have

$$\rho(P_{(1,m_1,1)}^{(1,m_1,1)}) = \rho(P_{(1,m_1-1,m_1-1)}^{(1,m_1)}) > \frac{3}{2} \sqrt{2}.$$ **Case 3:** $r \geq 3$. Here $T = P_{(k_0,k_1,\ldots,k_r, k_{r+1})}^{(m_0,m_1,\ldots,m_r)}$. Since $\rho(T) < \frac{3}{2} \sqrt{2}$, by Item 1 of Corollary 4.2, we must have

$$m_{l-1} + m_l \leq k_l.$$ By Items 2 and 3 of Corollary 4.2, we have

$$m_0 + m_1 \leq k_1 + 2,$$ $m_{r-1} + m_r \leq k_r + 2.$$

Recall $m_0 \leq k_0$ and $m_r \leq k_{r+1}$. Summing up these two inequalities and equations (9) (for $2 \leq l \leq r - 1$), (10), (11), we get

$$2 \sum_{l=0}^{r} m_l \leq \sum_{i=0}^{r+1} k_i + 4.$$
Hence, we have

\[ 3D - (2n - 4) \geq 3 \left( \sum_{i=0}^{r+1} k_i + r \right) - 2 \left( \sum_{j=0}^{r} m_j + \sum_{i=0}^{r+1} k_i + r + 1 \right) + 4 \]

\[ = \sum_{i=0}^{r+1} k_i - 2 \sum_{j=0}^{r} m_j + r + 2 \]

\[ \geq r - 2 \]

\[ > 0. \]

The proof of Theorem 1.1 is completed. □

**Proof of Theorem 1.2.** Let \( L = C_{(k_1, \ldots, k_r)}^{(m_1, \ldots, m_r)} \), where \( k_i \geq 0 \) and \( m_i \geq 1 \) for \( i = 1, \ldots, r \). For convenience, we write \( m_0 = m_r \).

First, we prove the lower bound of \( D(L) \). Denote \( m = \max\{m_1, \ldots, m_r\} \). We have

\[ n = r + \sum_{i=1}^{r} m_i + \sum_{i=1}^{r} k_i, \]

\[ D \geq m + \left\lfloor \frac{1}{2} (r + \sum_{i=1}^{r} k_i) \right\rfloor > \frac{1}{2} (r + \sum_{i=1}^{r} k_i). \] (14)

By the condition \( \rho(L) \leq \frac{3}{2} \sqrt{2} \) and Item 1 of Corollary 4.2, we have \( m_{i-1} + m_i \leq k_i \) for all \( 1 \leq i \leq r \). We get

\[ 2 \cdot \sum_{i=1}^{r} m_i \leq \sum_{i=1}^{r} k_i. \] (15)

Let \( \bar{m} = \frac{\sum_{i=1}^{r} m_i}{r} \). Combining the inequalities (13), (14), and (15), we get

\[ \frac{n}{n - 2D} > \frac{r + 3 \cdot \sum_{i=1}^{r} m_i}{\sum_{i=1}^{r} m_i} = 3 + \frac{1}{\bar{m}}. \]

Solving for \( D \), we get

\[ D > \frac{2 + \frac{1}{\bar{m}}}{3 + \frac{1}{\bar{m}}} n > \frac{n}{3}. \]

Now we prove the upper bound \( \frac{2n}{3} \) for \( D(L) \).

If \( r = 1 \), then \( L = C_{(k)}^{(m)} \). We have \( \rho(L) = \rho_{m,k} \). By Corollary 4.1 and \( n \geq 13 \), we have \( k \geq 2m + 3 \) and

\[ 3D - 2n + 4 = 3 \left( m + \left\lfloor \frac{k+1}{2} \right\rfloor \right) - 2(m + k + 1) + 4 \]

\[ = m + 3 \left\lfloor \frac{k+1}{2} \right\rfloor - 2k + 2. \]

When \( k = 2t \) even, since \( 2t = k \geq 2m + 3 \), we get

\[ 3D - 2n + 4 = m - t + 2 \leq 0. \]
When $k = 2t + 1$ odd, since $2t + 1 \geq 2m + 3$, we get

$$3D - 2n + 4 = m - t + 3 \leq 2.$$ 

Here we get two exception cases to $3D \leq 2n - 4$: $k = 2m + 3$ and $k = 2m + 5$ (the graphs are shown in Figure 3).

Now we consider the case $r \geq 2$. Let $m$ (or $m'$) be the first (or the second) largest number in $\{m_1, \ldots, m_r\}$ respectively. Let $L'$ be the graph obtained from $L$ by removing all pendant paths other than the two longest ones. Let $g$ denote the length of the unique cycle in $L$. Let $L'' = C^{(m,m')}_{(\lfloor \frac{g}{2}\rfloor - 1, \lfloor \frac{g}{2}\rfloor - 1)}$. By Lemma 2.1 and Lemma 2.6 we have

$$\rho(L) \geq \rho(L') \geq \rho(L'').$$

We observe that $n(L) \geq n(L'')$ and $D(L) \leq D(L')$. It suffices to prove $3D(L'') \leq 2n(L'') - 4$.

If $g = 2k$ even, then $\rho(L'') = \rho(C^{(m,m')}_{(k-1,k-1)}) = \rho(P_{m+1,m'+1}^{m+1,m'+1})$. Since $\rho(P_{m+1,k-3,m'+1}^{m+1,k-3,m'+1}) = \rho(L'') < \frac{3}{2}\sqrt{\lambda}$, by Item 3 of Lemma 4.1 we get $m + m' + 2 \leq k - 2$ unless $m = m' = 1$ and $k = 5$. We will consider the special case later. For general case, we have

$$3D(L'') - 2n(L'') + 4 = 3(m + m' + k) - 2(m + m' + 2k) + 4 = m + m' - k + 4 \leq 0.$$ 

When $m = m' = 1$ and $k = 5$, we have $L'' = C^{(1,1)}_{(4,4)}$ and $3D(L'') = 2n(L'') - 3$. Since $n(L) \geq 13 > 12 = n(L'')$, we have

$$3D(L) - 2n(L) + 4 < 3D(L'') - 2n(L'') + 4 = 1.$$ 

When $g = 2k + 1$ is odd, let $L''' = C^{(m-1,m')}_{(k,k)}$. Since $L'''$ can be obtained from $L''$ by deleting a leaf vertex and subdividing an internal edge, by Lemma 2.4 we have $\rho(L''') < \rho(L'')$. We also observe that $n(L''') = n(L'')$ and $D(L''') = D(L'')$. By the previous cases, we have $3D(L''') \leq 2n(L''') - 4$. Thus $3D(L'') \leq 2n(L'') - 4$. We are done. \hfill \Box

5 Application to diameter $\frac{n}{2} \leq D \leq \frac{2n-4}{3}$

We have the following lemma.

**Lemma 5.1** For $m \geq 1$, let $\rho_m = \lim_{k \to \infty} \rho_{m,k}$. We have $\rho_{m+1} > \rho_{m,k}$ holds for $k \geq 2m + 5$.

**Proof** Recall $\rho_{m,k}$ is the largest root of $\frac{d^{(2)}_{m}}{d_{m}} = x_{1}^{k-1}$. Thus $\rho_{m+1}$ is the largest roots of $d_{m+1}^{(2)} = 0$ while $\rho_{m,2m+5}$ is the largest roots of $\frac{d^{(2)}_{m}}{d_{m}} = x_{1}^{2m+4}$. Let $f = f(\lambda)$ be a function of $\lambda$. The notation $f|_{\lambda_0}$ means the value of $f$ at $\lambda = \lambda_0$. We have

$$\left(x_2^2 - 2\right) + x_1^{2m+4}\bigg|_{\rho_{m+1}} = 0$$

and

$$\left(x_2^2 - 2\right) + x_1^{2m+4}\left(x_2^2 + 1 + x_1^{2m+2}(2-x_1^2)\right)\bigg|_{\rho_{m,2m+5}} = 0.$$ 

16
We get
\[
(x_2^2 - 2) + x_1^{2m+4} \mid_{\rho_{m,2m+5}} = x_1^{2m+4} - x_1^{2m+4} (x_2^2 - 1 + x_1^{2m+2}(2 - x_1^2)) \mid_{\rho_{m,2m+5}} \\
= x_1^{2m+4} (2 - x_2 - x_1^{2m+2}(2 - x_1^2)) \mid_{\rho_{m,2m+5}} \\
= x_1^{2m+4} [x_1^{2m+4} (x_2^2 - 1 + x_1^{2m+2}(2 - x_1^2)) - x_1^{2m+2}(2 - x_1^2)] \mid_{\rho_{m,2m+5}} \\
= x_1^{2m+6} (2 - x_1^2) \mid_{\rho_{m,2m+5}} \\
\leq x_1^{2m+6} (x^6(2 - x_1^2) - 1) \mid_{\rho_{m,2m+5}} \\
< x_1^{2m+6} \left( \frac{3x_1^6}{2} - 1 \right) \mid_{\rho_{m,2m+5}} \\
< 0.
\]

In the last step we use \(x_1^2 \mid_{\rho_{m,2m+5}} < \frac{\sqrt{5} - 1}{2}\), since \(\rho_{m,2m+5} > \sqrt{2 + \sqrt{5}}\). Thus, we have \(\rho_{m+1} > \rho_{m,2m+5} \geq \rho_m\). The proof is completed. \(\Box\)

Lemma 5.2 of [2] can be generalized to the following lemma. The proof is similar and will be omitted.

**Lemma 5.2** If a minimizer graph with \(n\) vertices and diameter \(D\) with \(n \geq D + 2\) and \(\frac{n}{2} \leq D \leq \frac{2n-4}{3}\) is a subgraph of an \((D - \lfloor \frac{n}{2} \rfloor\)-Urchin graph but not of an \((D - \lfloor \frac{n}{2} \rfloor\)-Laundry graph, then it is \(C_{n-D-1,n-D-1}\).

**Proof of Theorem 1.3** To apply Lemma 5.2 it suffices to prove the following two claims for \(n \geq 13\).

**Claim 1.** \(G_{min}^{n,D}\) must be a closed quipu.

**Claim 2.** The longest pendant path of \(G_{min}^{n,D}\) has length at most \(D - \lfloor \frac{n}{2} \rfloor\).

First we prove Claim 1. Consider the graph \(C_{(n-D-1,n-D-1)}^{D - \lfloor \frac{n}{2} \rfloor}\). Let \(m = D - \lfloor \frac{n}{2} \rfloor\) and \(k = n - D - 1\). Since \(n \geq 13\) and \(\frac{n}{2} \leq D \leq \frac{2n-4}{3}\), we have \(k \geq 4\) and \(k > 2m + 3\). By Corollary 4.1 we have
\[
\rho(G_{min}^{n,D}) \leq \rho \left(C_{(n-D-1,n-D-1)}^{D - \lfloor \frac{n}{2} \rfloor}\right) \leq \rho \left(C_{(k,k)}^{m,m}\right) < \frac{3\sqrt{2}}{2}.
\]

So, \(G_{min}^{n,D}\) is either a dagger, an open quipu, or a closed quipu. The minimizer graph \(G_{min}^{n,D}\) can not be a dagger since all daggers with \(n \geq 6\) do not satisfy \(3D \leq 2n - 4\). By Theorem 1.1, \(G_{min}^{n,D}\) can not be an open quipu either. Hence, \(G_{min}^{n,D}\) must be a closed quipu.

Now we prove Claim 2. Since \(3D \leq 2n - 7\), we have \(n - D - 1 \geq 2(D - \lfloor \frac{n}{2} \rfloor) + 5\). Suppose that \(G_{min}^{n,D}\) has a pendant path of length \(m' > D - \lfloor \frac{n}{2} \rfloor\). By Lemma 5.1 we have
\[
\rho \left(C_{(n-D-1,n-D-1)}^{D - \lfloor \frac{n}{2} \rfloor}\right) < \rho_{m'} < \rho(G_{min}^{n,D}).
\]

Contradiction! The proof of two claims are finished. Applying Lemma 5.2 we are done. \(\Box\)

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