Essential norm estimates for weighted composition operator on the logarithmic Bloch space

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Abstract

In this article, we estimate the essential norm of weighted composition operator $W_{u,\varphi}$, acting on the logarithmic Bloch space $B^{\text{log}}$, in terms of the $n$-power of the analytic function $\varphi$ and the norm of the $n$-power of the identity function. Also, we estimate the essential norm of the weighted composition operator from $B^{\text{log}}$ into the growth space $H^{\infty}_{\text{log}}$. As a consequence of our result, we estimate the essential norm of the composition operator $C_{\varphi}$ acting on the Logarithmic-Zygmund space.

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1 Introduction

Let $\mathbb{D}$ be the unit disk of the complex plane $\mathbb{C}$ and let $H(\mathbb{D})$ be the space of all holomorphic functions on $\mathbb{D}$ endowed with the topology of the uniform convergence on compact subsets of $\mathbb{D}$. For fixed holomorphic functions $u : \mathbb{D} \to \mathbb{C}$ and $\varphi : \mathbb{D} \to \mathbb{D}$, we can define the linear operator $W_{u,\varphi} : H(\mathbb{D}) \to H(\mathbb{D})$ by

$$W_{u,\varphi}(f) := u \cdot (f \circ \varphi).$$
Which is known as the weighted composition with symbols $u$ and $\varphi$. Clearly, if $u \equiv 1$ we have $W_{1,\varphi}(f) = f \circ \varphi = C_\varphi(f)$, the composition operator $C_\varphi$, and if $\varphi(z) = id(z) = z$ for all $z \in \mathbb{D}$, we have obtain $W_{u,id}(f) = u \cdot f = M_u(f)$, the multiplication operator $M_u$. Furthermore, we can see that $W_{u,\varphi}$ is 1-1 on $H(\mathbb{D})$ unless that $u \equiv 0$ or $\varphi$ is a constant function. However, if we wish to study properties like as continuity, compactness, essential norm, etc. of this operator, we need restrict the domain and target space $H(\mathbb{D})$ to a normed and complete subspace of $H(\mathbb{D})$. In this article we consider the restriction to the growth space of all analytic functions $f$ on $\mathbb{D}$ such that
\[ \|f\|_{H_\infty} = \sup_{z \in \mathbb{D}} v(z)|f(z)| < \infty, \] (1)
where $v : \mathbb{D} \to \mathbb{R}^+$ is a weight function, that is, a bounded, continuous and positive function defined on $\mathbb{D}$. Also, we consider the Bloch-type space $B^v$ of all analytic function $f$ on $\mathbb{D}$ such that $f' \in H_\infty^v$. It is known that $H_\infty^v$ is a Banach space with the norm defined in (1) and $B^v$ is a Banach space with the norm
\[ \|f\|_{B^v} = |f(0)| + \|f'\|_{H_\infty^v} = |f(0)| + \|f\|_{B^v}, \]
where,
\[ \|f\|_{B^v} = \sup_{z \in \mathbb{D}} v(z)|f'(z)|. \]

The properties of $W_{u,\varphi}$ acting between growth-type spaces were studied by Hyv"arinen et al. [7] and by Malavé-Ramírez and Ramos-Fernández [9], for very general weights $v$; however, properties of $W_{u,\varphi}$ acting on Bloch-type spaces are still in develops. About this last, we can mention the works of Hyvärinen and Lindström in [8]. Also, there are no much works about the properties of $W_{u,\varphi}$ between $H_\infty^v$ and $B^v$, we can mention the work of Stević in [11].

In this note, we estimate the essential norm of $W_{u,\varphi}$ acting on the logarithmic Bloch space $B^{v_{log}}$ (also known as the weighted Bloch space), where the weight consider here is defined by
\[ v_{log}(z) = (1 - |z|) \log \left( \frac{2}{1 - |z|} \right) \]
with $z \in \mathbb{D}$ which clearly is radial and typical ($\lim_{|z| \to 1^-} v_{log}(z) = 0$). This space appears in the literature when we study properties of certain operators acting on certain spaces of analytic functions on the unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$. For instance, in 1991, Brown and Shields [2] showed that an analytic function $u$ is a multiplier on the Bloch space $B$ if and only if $u \in B^{v_{log}}$. Also, in 1992, K. Atele [11]...
showed that the Hankel operator induced by a function \( f \in H(D) \) in the Bergman space \( L^1_a \) (for the definition of Bergman space and the Hankel operator see [16]) is bounded if and only if \( \|f\|_{B^\text{log}} < \infty \). The study of composition operators acting on the weighted Bloch space began with the work of Yoneda [15], where he characterized the continuity and compactness of composition operators acting on the weighted Bloch space \( B^\text{log} \). These last results were extended by Galanopoulos [6] and Ye [13] for weighted composition operators acting on \( B^\text{log} \). More recently, Malavé-Ramírez and Ramos-Fernández [9], following similar ideas used by Hyvärinen and Lindström in [8], characterized the continuity and compactness of \( W_{u,\varphi} \) acting on \( B^\text{log} \) in terms of certain expression involving the \( n \)-th power of \( \varphi \) and the log-Bloch norm of the \( n \)-th power of the identity function on \( \mathbb{D} \). The results obtained by Malavé-Ramírez and Ramos-Fernández can be enunciated as follows:

**Theorem 1.1** ([9]). Suppose that \( u : \mathbb{D} \to \mathbb{C} \) and \( \varphi : \mathbb{D} \to \mathbb{D} \) are holomorphic functions.

1. The operator \( W_{u,\varphi} \) is continuous on \( B^\text{log} \) if and only if
   \[
   \max \left\{ \sup_{n \in \mathbb{N}} (n+1) \left\| J_u(\varphi^n) \right\|_{B^\text{log}}, \sup_{n \in \mathbb{N}} \frac{\left\| I_u(\varphi^n) \right\|_{B^\text{log}}}{\left\| g_n \right\|_{B^\text{log}}} \right\} < \infty.
   \]

2. The operator \( W_{u,\varphi} \) is compact on \( B^\text{log} \) if and only if
   \[
   \max \left\{ \lim_{n \to \infty} (n+1) \left\| J_u(\varphi^n) \right\|_{B^\text{log}}, \lim_{n \to \infty} \frac{\left\| I_u(\varphi^n) \right\|_{B^\text{log}}}{\left\| g_n \right\|_{B^\text{log}}} \right\} = 0,
   \]
   where \( \mathbb{N} = \{0,1,2,\cdots\}, \ g_0 \equiv 1, \ g_n(z) = z^n \) for \( n \in \mathbb{N} = \{1,2,3,\cdots\} \) and \( z \in \mathbb{D} \),
   \[
   w_{\text{log}}(z) = \left[ \log \log \left( \frac{4}{1 - |z|^2} \right) \right]^{-1}, \tag{2}
   \]
   and the functionals \( I_u, J_u : H(\mathbb{D}) \to \mathbb{C} \) are defined by
   \[
   I_u(f(z)) = \int_0^z f'(s)u(s)ds, \quad \text{and} \quad J_u(f(z)) = \int_0^z f(s)u'(s)ds.
   \]

The main goal of the present article is to find an estimation of the essential norm of the operator \( W_{u,\varphi} : B^\text{log} \to B^\text{log} \) which implies the result, about compactness, mentioned in the item (2) of Theorem 1.1 above. Allow us recall that the essential norm of a continuous operator \( T : X \to Y \), between Banach spaces \( X \) and \( Y \), denoted by \( \|T\|_e^{X \to Y} \), is its distance to the class of the compact operators, that is,
\[ \|T\|^{X \rightarrow Y} = \inf \{\|T - K\|^{X \rightarrow Y} : K : X \rightarrow Y \text{ is compact} \}, \]
where \( \|T\|^{X \rightarrow Y} \) denotes the norm of the operator \( T : X \rightarrow Y \). Notice that \( T : X \rightarrow Y \) is compact if and only if \( \|T\|^{X \rightarrow Y} = 0 \).

Recent results about essential norm estimates on log-Bloch space can be found in [3] for the composition operator \( C_\phi : B^{v_3} \rightarrow B^v \), with \( v_3(z) = (1 - |z|) \log \left( \frac{3}{1 - |z|} \right) \) and in [14], where Ye estimated the essential norm of the operator \( DC_\phi : B^{v_e} \rightarrow H^{\infty}_v \) defined by \( DC_\phi(f) := W_\phi(f') \) with \( v_e(z) = (1 - |z|) \log \left( \frac{2e}{1 - |z|} \right) \).

In this article we are going to show the following result:

**Theorem 1.2.** Suppose that \( u : D \rightarrow C \) and \( \phi : D \rightarrow D \) are holomorphic functions and that \( W_{u, \phi} : B^{v_{\log}} \rightarrow B^{v_{\log}} \) is continuous. Then

\[ \|W_{u, \phi}\|^{B^{v_{\log}} \rightarrow B^{v_{\log}}} \simeq \max \left\{ \limsup_{n \to \infty} \frac{(n + 1)\|J_u(\phi^n)\|^{B^{v_{\log}}}}{\|g_{n+1}\|^{B^{v_{\log}}}}, \limsup_{n \to \infty} \frac{\|I_u(\phi^n)\|^{B^{v_{\log}}}}{\|g_n\|^{B^{v_{\log}}}} \right\}. \]

Above, and in what follows, for two positive quantities \( A \) and \( B \), we write \( A \simeq B \) and say that \( A \) is equivalent to \( B \) if and only if there is a positive constant \( K \), independent on \( A \) and \( B \), such that \( \frac{1}{K} A \leq B \leq KA \). To show Theorem 1.2 we establish, in Section 2, a triangle inequality which reduce our problem to estimate the essential norm of \( W_{u', \phi} : B^{v_{\log}} \rightarrow H^{\infty}_{v_{\log}} \). Such estimation is found in Section 3 in terms of the essential norm of \( W_{u, \phi} : H^{\infty}_{v_{\log}} \rightarrow H^{\infty}_{v_{\log}} \). Finally, in Section 4, we show Theorem 1.2. As a consequence of our results, in Section 5, we characterize continuity, compactness and we estimate the essential norm of the composition operator \( C_\phi \) acting on the logarithmic-Zygmund space.

We want finish this introduction by mentioning that throughout this paper, constants are denoted by \( C \) or \( C_v \) (if depending only on \( v \)), they are positive and may differ from one occurrence to the other.

## 2 A triangle inequality for the essential norm

Let \( \mu_1 \) and \( \mu_2 \) be two weights defined on \( D \). In this section we find upper bound for the essential norm of the operator \( W_{u, \phi} : B^{\mu_1} \rightarrow B^{\mu_2} \) in terms of the essential norm of the operators \( W_{u', \phi} : B^{\mu_1} \rightarrow H^{\infty}_{\mu_2} \) and \( W_{u, \phi', \phi} : H^{\infty}_{\mu_1} \rightarrow H^{\infty}_{\mu_2} \). To this end, for a weight \( \mu \) we set the class

\[ \tilde{B}^\mu = \{ f \in B^\mu : f(0) = 0 \}, \]

which is a closed subspace of \( B^\mu \). With this notation, we have the following result:
Lemma 2.1. If the operator $W_{u,\varphi} : \tilde{B}^{\mu_1} \to \mathcal{B}^{\mu_2}$ is continuous, then
\[
\|W_{u,\varphi}\|_{e}^{\tilde{B}^{\mu_1} \to \mathcal{B}^{\mu_2}} \leq \|W_{u',\varphi'}\|_{e}^{\tilde{B}^{\mu_1} \to H^{\mu_2}_{\mu_2}} + \|W_{u,\varphi'}\|_{e}^{H^{\mu_1}_{\mu_1} \to H^{\mu_2}_{\mu_2}}. \tag{3}
\]

Proof. Let us suppose that the operator $W_{u,\varphi} : \tilde{B}^{\mu_1} \to \mathcal{B}^{\mu_2}$ is continuous, then the continuous composition $D_{\mu_2}W_{u,\varphi}D_{\mu_1}^{-1}$ maps the space $H^{\mu_2}_{\mu_1}$ into $H^{\mu_2}_{\mu_2}$, where $D_{\mu_1} : \tilde{B}^{\mu_1} \to H^{\mu_1}_{\mu_1}$ and $D_{\mu_2} : \tilde{B}^{\mu_2} \to H^{\mu_2}_{\mu_2}$ denote the linear operators which transforms each $f \in H(\mathbb{D})$ into its derivative $f'$. Clearly, the operators $D_{\mu_1}$ and $D_{\mu_2}$ are isometry and therefore they are invertibles with norms equal to 1. Furthermore, for every a $f \in H^{\mu_1}_{\mu_1}$ we have the relation
\[
D_{\mu_2}W_{u,\varphi}D_{\mu_1}^{-1}(f) = W_{u',\varphi}D_{\mu_1}^{-1}(f) + W_{u,\varphi'}D_{\mu_1}(f).
\]

From the above relation, we deduce an expression for the operator $W_{u,\varphi} : \tilde{B}^{\mu_1} \to \mathcal{B}^{\mu_2}$ in terms of $D_{\mu_1}$ and $D_{\mu_2}$, that is,
\[
W_{u,\varphi} : \tilde{B}^{\mu_1} \to \mathcal{B}^{\mu_2}
W_{u,\varphi}(f) = D_{\mu_2}^{-1}W_{u',\varphi}(f) + D_{\mu_2}^{-1}W_{u,\varphi'}D_{\mu_1}(f). \tag{4}
\]

Now, by definition of essential norm, given $\epsilon > 0$, we can find compact operators $\mathcal{K}_1 : \tilde{B}^{\mu_1} \to H^{\mu_2}_{\mu_2}$ and $\mathcal{K}_2 : H^{\mu_1}_{\mu_1} \to H^{\mu_2}_{\mu_2}$ such that
\[
\|W_{u',\varphi}\|_{e}^{\tilde{B}^{\mu_1} \to H^{\mu_2}_{\mu_2}} + \|W_{u,\varphi'}\|_{e}^{H^{\mu_1}_{\mu_1} \to H^{\mu_2}_{\mu_2}}
\geq \frac{1}{1 + \epsilon} \left( \|W_{u',\varphi} - \mathcal{K}_1\|_{\tilde{B}^{\mu_1} \to H^{\mu_2}_{\mu_2}} + \|W_{u,\varphi'} - \mathcal{K}_2\|_{H^{\mu_1}_{\mu_1} \to H^{\mu_2}_{\mu_2}} \right).
\]

Thus, since the operators $D_{\mu_1}$ and $D_{\mu_2}$ are isometries, we can write
\[
\|W_{u',\varphi}\|_{e}^{\tilde{B}^{\mu_1} \to H^{\mu_2}_{\mu_2}} + \|W_{u,\varphi'}\|_{e}^{H^{\mu_1}_{\mu_1} \to H^{\mu_2}_{\mu_2}}
\geq \frac{1}{1 + \epsilon} \left( \|D_{\mu_2}^{-1}W_{u',\varphi} - D_{\mu_2}^{-1}\mathcal{K}_1\|_{\tilde{B}^{\mu_1} \to \mathcal{B}^{\mu_2}} + \|D_{\mu_2}^{-1}W_{u,\varphi'} - D_{\mu_2}^{-1}\mathcal{K}_2\|_{H^{\mu_1}_{\mu_1} \to \mathcal{B}^{\mu_2}} \right)
\geq \frac{1}{1 + \epsilon} \left( \|D_{\mu_2}^{-1}W_{u',\varphi} - D_{\mu_2}^{-1}\mathcal{K}_1\|_{\tilde{B}^{\mu_1} \to \mathcal{B}^{\mu_2}} + \|D_{\mu_2}^{-1}W_{u,\varphi'}D_{\mu_1} - D_{\mu_2}^{-1}\mathcal{K}_2D_{\mu_1}\|_{\tilde{B}^{\mu_1} \to \mathcal{B}^{\mu_2}} \right)
\geq \frac{1}{1 + \epsilon} \|D_{\mu_2}^{-1}W_{u',\varphi} + D_{\mu_2}^{-1}W_{u,\varphi'}D_{\mu_1} - (D_{\mu_2}^{-1}\mathcal{K}_1 + D_{\mu_2}^{-1}\mathcal{K}_2D_{\mu_1})\|_{\tilde{B}^{\mu_1} \to \mathcal{B}^{\mu_2}}
\geq \frac{1}{1 + \epsilon} \|W_{u,\varphi} - \mathcal{K}\|_{\tilde{B}^{\mu_1} \to \mathcal{B}^{\mu_2}} \geq \frac{1}{1 + \epsilon} \|W_{u,\varphi}\|_{e}^{\tilde{B}^{\mu_1} \to \mathcal{B}^{\mu_2}},
\]

where we have used (4) in the last equality and the fact that the operator $\mathcal{K} = D_{\mu_2}^{-1}\mathcal{K}_1 + D_{\mu_2}^{-1}\mathcal{K}_2D_{\mu_1} : \tilde{B}^{\mu_1} \to \mathcal{B}^{\mu_2}$ is compact since the composition of compact and continuous operators is a compact operator. The result follows because $\epsilon > 0$ was arbitrary. \hfill \Box
Now, we show that the essential norm of the weighted composition operator, acting on Bloch-type spaces, does not change if we restrict the domain to $\tilde{B}^\mu$.

**Lemma 2.2.** If $W_{u,\varphi} : B^\mu_1 \to B^\mu_2$ is continuous, then

$$\|W_{u,\varphi}\|_{B^\mu_1 \to B^\mu_2} = \|W_{u,\varphi}\|_{\tilde{B}^\mu_1 \to \tilde{B}^\mu_2}.$$  

**Proof.** This proof uses similar arguments given in [5], (see Lemma 3.1). Since $\tilde{B}^\mu_1 \subseteq B^\mu_1$, it is clear that all compact operator from $B^\mu_1$ into $B^\mu_2$ is also a compact operator from $\tilde{B}^\mu_1$ into $B^\mu_2$; hence we have

$$\|W_{u,\varphi}\|_{B^\mu_1 \to B^\mu_2} \geq \|W_{u,\varphi}\|_{\tilde{B}^\mu_1 \to \tilde{B}^\mu_2}.$$  

To show the reverse inequality, we observe that if $T : B^\mu_1 \to B^\mu_2$ is any compact operator, then we can write

$$\|W_{u,\varphi} - T\|_{B^\mu_1 \to B^\mu_2} = \sup_{\|f\|_{B^\mu_1} \leq 1} \|W_{u,\varphi}(f) - T(f)\|_{B^\mu_2}$$

$$= \sup_{\|f\|_{B^\mu_1} \leq 1} \|W_{u,\varphi}(f) + f(0)W_{u,\varphi}(1) - T(f - f(0)1 + f(0)1)\|_{B^\mu_2}$$

$$\leq \sup_{\|f\|_{B^\mu_1} \leq 1} \|W_{u,\varphi}(f - f(0)1) - T(\tilde{B}^\mu_1 (f - f(0)1))\|_{B^\mu_2} + \sup_{\|f\|_{B^\mu_1} \leq 1} \|W_{u,\varphi}(f(0)1) - T(f(0)1)\|_{B^\mu_2}$$

$$\leq \sup_{g \in \tilde{B}^\mu_1, \|g\|_{B^\mu_1} \leq 1} \|W_{u,\varphi}(g) - T(\tilde{B}^\mu_1 (g))\|_{B^\mu_2} + \|W_{u,\varphi}(h) - T(h)\|_{B^\mu_2},$$

where $1(z) = 1$ for all $z \in \mathbb{D}$, $A$ denotes the space of all constant functions in $B^\mu_1$, $P = T|_{\tilde{B}^\mu_1}$ and $Q_1 = T|_A$.

On the other hand, by definition of essential norm, given $\epsilon > 0$, we can find compact operators $P : \tilde{B}^\mu_1 \to B^\mu_2$ and $Q : A \to B^\mu_2$ such that

$$\|W_{u,\varphi} - P\|_{\tilde{B}^\mu_1 \to B^\mu_2} + \|W_{u,\varphi} - Q\|_{A \to B^\mu_2} \leq (1+\epsilon) \left( \|W_{u,\varphi}\|_{\tilde{B}^\mu_1 \to B^\mu_2} + \|W_{u,\varphi}\|_{A \to B^\mu_2} \right).$$

But each $f \in B^\mu_1$ can be written as $f = f(0) \cdot 1 + g$, where $g \in \tilde{B}^\mu_1$. Furthermore, we can see that if $f \in A$ then $g$ is the null function and if $f \in \tilde{B}^\mu_1$ then $f = g$. Thus, we can define the operator $T : B^\mu_1 \to B^\mu_2$ by

$$Tf = f(0)Q(1) + P(g), \quad (f = f(0) \cdot 1 + g \in B^\mu_1).$$

Clearly, $T$ is linear and compact operator from $B^\mu_1$ into $B^\mu_2$. Indeed, if $\{f_k\}$ is a bounded sequence in $B^\mu_1$, then there exists a bounded sequence $\{g_k\}$ in $\tilde{B}^\mu_1$ such that

$$f_k(z) = f_k(0) + g_k(z)$$
for all \( z \in \mathbb{D} \). Bolzano-Weierstrass’s theorem tell us that the numerical sequence \( \{f_k(0)\} \) has a convergent subsequence and since the operator \( P : \tilde{B}^{\mu_1} \rightarrow B^{\mu_2} \) is compact, the sequence \( \{P(g_k)\} \) also has a convergent subsequence in \( B^{\mu_2} \). Hence \( \{T(f_k)\} \) has a convergent subsequence in \( B^{\mu_2} \) and \( T : B^{\mu_1} \rightarrow B^{\mu_2} \) is compact as was claimed.

Now, since \( T|_{\tilde{B}^{\mu_1}} = P \) and \( T|_{A} = Q \). We obtain that
\[
\|W_{u,\varphi}\|_{e}^{B^{\mu_1} \rightarrow B^{\mu_2}} \leq \|W_{u,\varphi} - T\|_{B^{\mu_1} \rightarrow B^{\mu_2}} + \|W_{u,\varphi} - Q\|_{A \rightarrow B^{\mu_2}} \leq (1 + \epsilon) \left( \|W_{u,\varphi}\|_{e}^{\tilde{B}^{\mu_1} \rightarrow B^{\mu_2}} + \|W_{u,\varphi}\|_{e}^{A \rightarrow B^{\mu_2}} \right).
\]
and since \( \epsilon > 0 \) was arbitrary, we conclude that
\[
\|W_{u,\varphi}\|_{e}^{B^{\mu_1} \rightarrow B^{\mu_2}} \leq \|W_{u,\varphi}\|_{e}^{\tilde{B}^{\mu_1} \rightarrow B^{\mu_2}} + \|W_{u,\varphi}\|_{e}^{A \rightarrow B^{\mu_2}}
\]
and the result follows since the operator \( W_{u,\varphi} : A \rightarrow B^{\mu_2} \) is compact and therefore its essential norm is zero.

As a consequence of Lemmas 2.1 and 2.2 we have the following result:

**Theorem 2.3.** If the operator \( W_{u,\varphi} : \tilde{B}^{\mu_1} \rightarrow B^{\mu_2} \) is continuous, then
\[
\|W_{u,\varphi}\|_{e}^{B^{\mu_1} \rightarrow B^{\mu_2}} \leq \|W_{u,\varphi} - P\|_{\tilde{B}^{\mu_1} \rightarrow B^{\mu_2}} + \|W_{u,\varphi} - Q\|_{A \rightarrow B^{\mu_2}} \leq (1 + \epsilon) \left( \|W_{u,\varphi}\|_{e}^{\tilde{B}^{\mu_1} \rightarrow B^{\mu_2}} + \|W_{u,\varphi}\|_{e}^{A \rightarrow B^{\mu_2}} \right).
\]

### 3 Estimation of the essential norm of \( W_{u',\varphi} : B^{v_{\log}} \rightarrow H_{v_{\log}}^{\infty} \)

From the conclusion of Theorem 2.3 we see that we have to estimate the essential norm of the operators \( W_{u',\varphi} : H_{\mu_1}^{\infty} \rightarrow H_{\mu_2}^{\infty} \) and \( W_{u',\varphi} : \tilde{B}^{\mu_1} \rightarrow H_{\mu_2}^{\infty} \). However, in the case of the weights \( v_{\log} \) and \( w_{\log} \), the first one can be estimate using the results of Malavé-Ramírez and Ramos-Fernández in [9], since theses weights are radial and typical. Hence, in this section, we look for an upper bound for \( \|W_{u',\varphi}\|_{e}^{\tilde{B}^{v_{\log}} \rightarrow H_{v_{\log}}^{\infty}} \).

To this end, recall that for all \( f \in B^{v_{\log}} \) the following relation holds:
\[
|f(z)| \leq \left[ 1 + \log \log \left( \frac{2}{1 - |z|} \right) - \log \log(2) \right] \|f\|_{B^{v_{\log}}}.
\]
This fact allow us to show the following result, where \( w_{\log} \) is the weight defined in [2].
Theorem 3.1. Suppose that \( u : \mathbb{D} \to \mathbb{C} \) and \( \varphi : \mathbb{D} \to \mathbb{D} \) are holomorphic functions and that the operator \( W_{u, \varphi} : \mathcal{B}^{\log} \to H^\infty_{\log} \) is continuous. Then there exists a constant \( C > 0 \) such that

\[
\|W_{u, \varphi}\|_{\mathcal{B}^{\log} \to H^\infty_{\log}} \leq C \lim_{r \to 1^-} \sup_{|\varphi(z)| > r} \frac{v_{\log}(z)}{w_{\log}(\varphi(z))} |u(z)|.
\]

Proof. For each \( r \in (0, 1) \), we consider \( K_r : \mathcal{B}^{\log} \to \mathcal{B}^{\log} \) defined by \( K_r(f) = f_r \), where \( f_r \) is the dilatation of \( f \) given by \( f_r(z) = f(rz) \). It is known that (see \([3]\)) for each \( r \in (0, 1) \) the operator \( K_r \) is continuous and compact on \( \mathcal{B}^{\log} \).

Let \( \{r_n\} \) any sequence in \((0, 1)\) and consider the compact operators \( K_n : \mathcal{B}^{\log} \to \mathcal{B}^{\log} \) given by \( K_n = K_{r_n} \), then the operators \( W_{u, \varphi} K_n : \mathcal{B}^{\log} \to H^\infty_{\log} \) also are compacts for every \( n \in \mathbb{N} \), since \( W_{u, \varphi} : \mathcal{B}^{\log} \to H^\infty_{\log} \) is a continuous operator.

Hence, by definition of essential norm we can write

\[
\|W_{u, \varphi}\|_{\mathcal{B}^{\log} \to H^\infty_{\log}} \leq \limsup_{n \to \infty} \|W_{u, \varphi} - W_{u, \varphi} K_n\|_{\mathcal{B}^{\log} \to H^\infty_{\log}}.
\]

Furthermore, for any \( f \in \mathcal{B}^{\log} \) such that \( \|f\|_{\mathcal{B}^{\log}} \leq 1 \), we have

\[
\|(W_{u, \varphi} - W_{u, \varphi} K_n)(f)\|_{H^\infty_{\log}} = \|u(f - f_{r_n}) \circ \varphi\|_{H^\infty_{\log}} = \sup_{z \in \mathbb{D}} v_{\log}(z) |u(z) f(\varphi(z)) - u(z) f_{r_n}(\varphi(z))| = \sup_{z \in \mathbb{D}} v_{\log}(z) |f(\varphi(z)) - f_{r_n}(\varphi(z))| u(z)|.
\]

Now, we fix \( N \in \mathbb{N} \) and consider, for \( z \in \mathbb{D} \), the cases \(|\varphi(z)| \leq r_N \) and \(|\varphi(z)| > r_N \).

Case 1: \(|\varphi(z)| \leq r_N \)

Since the operator \( W_{u, \varphi} : \mathcal{B}^{\log} \to H^\infty_{\log} \) is continuous, then \( u = W_{u, \varphi}(1) \in H^\infty_{\log} \).

Hence

\[
\sup_{|\varphi(z)| \leq r_N} v_{\log}(z) |f(\varphi(z)) - f_{r_n}(\varphi(z))| u(z)| \leq \|u\|_{H^\infty_{\log}} \sup_{|\varphi(z)| \leq r_N} |f(\varphi(z)) - f_{r_n}(\varphi(z))| \leq \|u\|_{H^\infty_{\log}} \sup_{|u| \leq r_N} |f(u) - f_{r_n}(u)| \to 0,
\]

as \( n \to \infty \), since is a known fact that \( f_r \to f \) uniformly on compact subsets of \( \mathbb{D} \) as \( r \to 1^- \).

Case 2: \(|\varphi(z)| > r_N \)

By triangle inequality, we have

\[
\sup_{|\varphi(z)| > r_N} v_{\log}(z) |f(\varphi(z)) - f_{r_n}(\varphi(z))| u(z)| \leq \sup_{|\varphi(z)| > r_N} v_{\log}(z) |f(\varphi(z))| |u(z)| + \sup_{|\varphi(z)| > r_N} v_{\log}(z) |f_{r_n}(\varphi(z))| |u(z)|.
\]
Thus, it is enough to find upper bounds for the expression in the right side of the above inequality. Put

\[
L_1 = \sup_{|\varphi(z)|>r_N} \log (z) |f(r_N \varphi(z))| |u(z)|
\]

\[
L_2 = \sup_{|\varphi(z)|>r_N} \log (z) |f(r_N \varphi(z))| |u(z)|.
\]

By the inequality (6), the fact that \( \|f\|_{E^{\log}} \leq 1 \) and since \( \log \log \left( \frac{2}{1-|\varphi(z)|} \right) \leq \log \log \left( \frac{4}{1-|\varphi(z)|^2} \right) \), we obtain

\[
L_1 \leq \sup_{|\varphi(z)|>r_N} \log (z) |u(z)| \left[ 1 + \log \log \left( \frac{2}{1-|\varphi(z)|} \right) - \log (2) \right]
\]

\[
\leq \sup_{|\varphi(z)|>r_N} \frac{\log (z)}{\log (\varphi(z))} |u(z)| \left[ 1 + \log \log \left( \frac{2}{1-|\varphi(z)|} \right) - \log (2) \right] \frac{\log \log (4)}{\log \log (1-|\varphi(z)|^2)}
\]

\[
\leq \sup_{|\varphi(z)|>r_N} \frac{\log (z)}{\log (\varphi(z))} |u(z)| \left[ \log \log (2) - \log \log (\varphi(z)) + 1 \right]
\]

\[
\leq \sup_{|\varphi(z)|>r_N} \frac{\log (z)}{\log (\varphi(z))} |u(z)| \left[ \log (0) - \log (2) \log (\varphi(z)) + 1 \right]
\]

\[
\leq \sup_{|\varphi(z)|>r_N} \frac{\log (z)}{\log (\varphi(z))} |u(z)| \left[ 2 \log (0) + 1 \right]
\]

\[
\leq C \sup_{|\varphi(z)|>r_N} \frac{\log (z)}{\log (\varphi(z))} |u(z)|,
\]

where \( C = 2 \log (0) + 1 > 0 \) and we have used that \( \log (r) \) is decreasing on \([0, 1]\).

In a similar way, we have that

\[
L_2 \leq \sup_{|\varphi(z)|>r_N} \frac{\log (z)}{\log (r_N \varphi(z))} |u(z)| \left[ 2 \log (0) + 1 \right]
\]

\[
\leq \sup_{|\varphi(z)|>r_N} \frac{\log (z)}{\log (\varphi(z))} |u(z)| \left[ 2 \log (0) + 1 \right].
\]

Hence, we can say that there exists a constant \( C > 0 \) such that

\[
\|(W_{u,\varphi} - W_{u,\varphi} K_n) f\|_{H^\infty_{\log}} \leq C \sup_{|\varphi(z)|>r_N} \frac{\log (z)}{\log (\varphi(z))} |u(z)|.
\]

Therefore, taking \( N \to \infty \) we have \( r_N \to 1^- \),

\[
\|W_{u,\varphi}\|_{E^{\log}} \to H^\infty_{\log} \leq C \lim_{r_N \to 1^-} \sup_{|\varphi(z)|>r_N} \frac{\log (z)}{\log (\varphi(z))} |u(z)|.
\]

This shows the result. \( \square \)
As a consequence of the above result we have the following estimation:

**Corollary 3.2.** Suppose that \( u : \mathbb{D} \to \mathbb{C} \) and \( \varphi : \mathbb{D} \to \mathbb{D} \) are holomorphic functions and that the operator \( W_{u, \varphi} : \mathcal{B}^{v_{\log}} \to \mathcal{H}^{\infty}_{v_{\log}} \) is continuous. Then there exists a constant \( C > 0 \) such that
\[
\| W_{u, \varphi} \|_{\mathcal{B}^{v_{\log}} \to \mathcal{H}^{\infty}_{v_{\log}}} \leq C \limsup_{n \to \infty} \frac{\| u \varphi^n \|_{\mathcal{H}^{\infty}_{v_{\log}}}}{\| g_n \|_{\mathcal{H}^{\infty}_{w_{\log}}}},
\]
where \( g_n \) are the functions defined in Theorem 1.1.

**Proof.** Clearly, the weight \( w_{\log} \) is radial, typical and decreasing on \((0, 1)\) and hence (see [7] or [9]), this weight is essential, that is, \( w_{\log} \simeq \tilde{w}_{\log} \), where, for a weight \( v \), \( \tilde{v} \) denotes its associated weight given by
\[
\tilde{v}(z) = \left( \sup_{\| f \|_{\mathcal{H}^{\infty}_v} \leq 1} |f(z)| \right)^{-1}
\]
with \( z \in \mathbb{D} \). Also (see [4]), it is easy to see that \( v_{\log} \simeq v_3 \), where \( v_3 \) is given by
\[
v_3(z) = (1 - |z|) \log \left( \frac{3}{1 - |z|} \right)
\]
with \( z \in \mathbb{D} \). Thus, \( \mathcal{H}^{\infty}_{v_{\log}} \) is equal to \( \mathcal{H}^{\infty}_{v_3} \) with norms equivalents. Hence, from Theorem 3.1 we can write
\[
\| W_{u, \varphi} \|_{\mathcal{B}^{v_{\log}} \to \mathcal{H}^{\infty}_{v_{\log}}} \leq C \lim_{r \to 1^-} \sup_{|\varphi(z)| > r} \frac{v_{\log}(z)}{w_{\log}(\varphi(z))} \| u(z) \|
\]
\[
\leq C \lim_{r \to 1^-} \sup_{|\varphi(z)| > r} \frac{v_3(z)}{w_{\log}(\varphi(z))} \| u(z) \|
\]
\[
= C \| W_{u, \varphi} \|_{\mathcal{H}^{\infty}_{w_{\log}} \to \mathcal{H}^{\infty}_{v_3}}
\]
\[
\leq C \lim_{n \to \infty} \sup_{n+1} \frac{\| u \varphi^n \|_{\mathcal{H}^{\infty}_{v_{\log}}}}{\| g_n \|_{\mathcal{H}^{\infty}_{w_{\log}}}},
\]
where we have used Theorem 2.1 in [10] in the equality and the last inequality is due to Theorem 2.4 in [7] (see also [9], Theorem 4.3).

Now, using the definition of the functions \( g_n \), the definition of the functional \( J_u \), given in Theorem 1.1 and the fact that \( \| f \|_{\mathcal{B}^v} = \| f' \|_{\mathcal{H}^{\infty}_v} \) we can conclude:

**Corollary 3.3.** Suppose that \( u : \mathbb{D} \to \mathbb{C} \) and \( \varphi : \mathbb{D} \to \mathbb{D} \) are holomorphic functions and that the operator \( W_{u, \varphi} : \mathcal{B}^{v_{\log}} \to \mathcal{H}^{\infty}_{v_{\log}} \) is continuous. Then there exists a constant \( C > 0 \) such that
\[
\| W_{u, \varphi} \|_{\mathcal{B}^{v_{\log}} \to \mathcal{H}^{\infty}_{v_{\log}}} \leq C \limsup_{n \to \infty} \frac{(n + 1) \| J_u (\varphi^n) \|_{\mathcal{B}^{v_{\log}}}}{\| g_{n+1} \|_{\mathcal{B}^{v_{\log}}}}.
\]
4 The essential norm of $W_{u,\varphi} : B^{\log} \rightarrow B^{\log}$. Proof of Theorem 1.2

Now we can give the proof of Theorem 1.2. From Theorem 2.3 we have

$$
\|W_{u,\varphi}\|_{e}^{B^{\log} \rightarrow B^{\log}} \leq \|W_{u,\varphi}'\|_{e}^{B^{\log} \rightarrow H_{\log}^{1}} + \|W_{u,\varphi}'\|_{e}^{H_{\log}^{1} \rightarrow H_{\log}^{1}}.
$$

Thus, by Corollary 3.3 and Theorem 4.4 in [9], we can find a constant $C > 0$ such that

$$
\|W_{u,\varphi}'\|_{e}^{B^{\log} \rightarrow H_{\log}^{1}} + \|W_{u,\varphi}'\|_{e}^{H_{\log}^{1} \rightarrow H_{\log}^{1}} \leq C \left( \limsup_{n \rightarrow \infty} \frac{(n + 1) \|J_{u}(\varphi^{'n})\|_{e}^{B^{\log}} + \limsup_{n \rightarrow \infty} \|J_{u}(\varphi^{'n})\|_{e}^{B^{\log}}} {\|g_{n+1}\|_{e}^{B^{\log}} \|g_{n}\|_{e}^{B^{\log}}} \right).
$$

Now we go to give a lower bound for $\|W_{u,\varphi}\|_{e}^{B^{\log} \rightarrow B^{\log}}$. To this end, let $K : B^{\log} \rightarrow B^{\log}$ be a compact operator and let $\{z_{n}\}_{n \in \mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_{n})| \rightarrow 1^{-}$ as $n \rightarrow \infty$. The following sequence was defined in [13],

$$
f_{n}(z) = \frac{3}{a_{n}} \left[ \log \log \left( \frac{4}{1 - \varphi(z_{n})z} \right) ^{2} - \frac{2}{a_{n}^{2}} \left[ \log \log \left( \frac{4}{1 - \varphi(z_{n})z} \right) \right] ^{3} \right],
$$

where $a_{n} = \log \log \left( \frac{4}{1 - |\varphi(z_{n})|^{2}} \right)$. In [13] the author shown that $\{f_{n}\}$ is a bounded sequence in $B^{\log}$, that is, there exists a constant $M > 0$ such that $\|f_{n}\|_{e}^{B^{\log}} \leq M$ for all $n \in \mathbb{N}$. This sequence converges to zero uniformly on compact subsets of $\mathbb{D}$. The derivatives of $f_{n}$ is given by

$$
f_{n}'(z) = \frac{6\varphi(z_{n})}{a_{n}} \log \log \left( \frac{4}{1 - \varphi(z_{n})z} \right) - \frac{6\varphi(z_{n})}{a_{n}^{2}} \left[ \log \log \left( \frac{4}{1 - \varphi(z_{n})z} \right) \right] ^{2}.
$$

Hence, we have $f_{n}'(\varphi(z_{n})) = 0$, $f_{n}(\varphi(z_{n})) = a_{n}$ and

$$
M \|W_{u,\varphi} - K\|_{e}^{B^{\log} \rightarrow B^{\log}} \geq \limsup_{n \rightarrow \infty} \| (W_{u,\varphi} - K)(f_{n}) \|_{e}^{B^{\log}} \geq \limsup_{n \rightarrow \infty} \|W_{u,\varphi}(f_{n})\|_{e}^{B^{\log}} - \limsup_{n \rightarrow \infty} \|Kf_{n}\|_{e}^{B^{\log}}
$$

$$
= \limsup_{n \rightarrow \infty} \|W_{u,\varphi}(f_{n})\|_{e}^{B^{\log}}.
$$
where we have used the known fact (see [12]) that if $K : \mathcal{B}^{\log} \to \mathcal{B}^{\log}$ is compact then
\[
\lim_{n \to \infty} \|Kf_n\|_{\mathcal{B}^{\log}} = 0
\]
for all bounded sequence $\{f_n\} \subset \mathcal{B}^{\log}$ converging to zero uniformly on compact subsets of $\mathbb{D}$.

On the other hand,
\[
\|W_{u,\varphi}(f_n)\|_{\mathcal{B}^{\log}} = \|uf_n(\varphi)\|_{\mathcal{B}^{\log}} \\
\geq \limsup_{n \to \infty} v_{\log}(z_n) |u'(z_n)f_n(\varphi(z_n)) + u(z_n)\varphi'(z_n)f_n'(\varphi(z_n))| \\
= \limsup_{n \to \infty} v_{\log}(z_n) |u'(z_n)f_n(\varphi(z_n))| \\
= \limsup_{n \to \infty} v_{\log}(z_n) |u'(z_n)| \log \log \left(\frac{4}{1 - |\varphi(z_n)|^2}\right) \\
\geq C \limsup_{n \to \infty} \frac{v_{\log}(z_n)}{\tilde{w}_{\log}(\varphi(z_n))} |u'(z_n)| \geq C \limsup_{n \to \infty} \frac{v_{\log}(z_n)}{\tilde{w}_{\log}(\varphi(z_n))} |u'(z_n)|,
\]
where, in the last inequality, we have used that the weight $w_{\log}$ is essential. Thus, since the sequence $\{z_n\}$ such that $|\varphi(z_n)| \to 1$ was arbitrary, we can deduce that
\[
\|W_{u,\varphi}\|_{e}^{\mathcal{B}^{\log} \to \mathcal{B}^{\log}} \geq C' \limsup_{|\varphi(z)| \to 1} \frac{v_{\log}(z)}{\tilde{w}_{\log}(\varphi(z))} |u'(z)|,
\]
where we have used the argument in the proof of Corollary 3.2.

Now, we go to show that there exists a constant $C > 0$ such that
\[
\|W_{u,\varphi}\|_{e}^{\mathcal{B}^{\log} \to \mathcal{B}^{\log}} \geq C \limsup_{n \to \infty} \frac{\|I_u(\varphi^n)\|_{\mathcal{B}^{\log}}}{\|g_n\|_{\mathcal{B}^{\log}}}.
\]
As before, we consider a sequence $\{z_n\} \subset \mathbb{D}$ such that $|\varphi(z_n)| \to 1$ and we define the functions
\[
h_n(z) = \frac{1}{\varphi(z_n)a_n^2} \left[\log \log \left(\frac{4}{1 - \varphi(z_n)z}\right)\right]^3 - \frac{1}{\varphi(z_n)a_n} \left[\log \log \left(\frac{4}{1 - \varphi(z_n)z}\right)\right]^2.
\]
Then, clearly \( h_n (\varphi(z_n)) = 0 \) for all \( n \in \mathbb{N} \), \( h_n \) converges to zero uniformly on compact subsets of \( \mathbb{D} \),

\[
h'_n(z) = \frac{3}{a_n} \left[ \log \log \left( \frac{4}{1 - \varphi(z_n)z} \right) \right]^2 - \frac{2}{a_n} \log \log \left( \frac{4}{1 - \varphi(z_n)z} \right),
\]

and hence

\[
h'_n(\varphi(z_n)) = \left[ (1 - |\varphi(z_n)|^2) \log \left( \frac{4}{1 - |\varphi(z_n)|^2} \right) \right]^{-1}.
\]

Furthermore, \( \{h_n\} \) is a bounded sequence in \( \mathcal{B}^{\varphi_{\log}} \); that is, there exists a constant \( M > 0 \) such that \( \|h_n\|_{\mathcal{B}^{\varphi_{\log}}} \leq M \) for all \( n \in \mathbb{N} \). Indeed,

\[
\|h_n\|_{\mathcal{B}^{\varphi_{\log}}} = \sup_{z \in \mathbb{D}} v_{\log}(z) |h'_n(z)| \leq \sup_{z \in \mathbb{D}} v_{\log}(z) \left[ \frac{3}{a_n} \left| \log \log \left( \frac{4}{1 - \varphi(z_n)z} \right) \right|^2 + \frac{2}{a_n} \left| \log \log \left( \frac{4}{1 - \varphi(z_n)z} \right) \right| \right],
\]

where we applied the triangle inequality and the fact that \( |\log(w)| \geq \log(|w|) \) for each \( w \in \mathbb{D} \). Furthermore, since

\[
\lim_{x \to 0^+} \frac{\sqrt{\log^2 \left( \sqrt{\log^2 \left( \frac{4}{x} \right) + 4\pi^2} + 4\pi^2 \right)}}{\log \log \left( \frac{2}{x} \right)} = 1,
\]

then we can deduce that there exists a constant \( M > 0 \) such

\[
\frac{3}{a_n} \left| \log \log \left( \frac{4}{1 - \varphi(z_n)z} \right) \right|^2 + \frac{2}{a_n} \left| \log \log \left( \frac{4}{1 - \varphi(z_n)z} \right) \right| \leq M
\]

for all \( n \in \mathbb{N} \) and all \( z \in \mathbb{D} \). Hence, it is enough to find upper bound for

\[
\sup_{z \in \mathbb{D}} \frac{v_{\log}(z)}{|1 - \varphi(z_n)z| \log \left( \frac{4}{|1 - \varphi(z_n)z|} \right)}
\]

for all \( n \in \mathbb{N} \). Clearly, the above expression is uniformly bounded if \( \frac{2}{e} \leq |1 - \varphi(z_n)z| < 2 \). Now, if \( |1 - \varphi(z_n)z| < \frac{2}{e} \), then since the function \( h_2(t) = t \log \left( \frac{2}{t} \right) \) is increasing on \((0, \frac{2}{e})\), we have

\[
\frac{v_{\log}(z)}{|1 - \varphi(z_n)z| \log \left( \frac{4}{|1 - \varphi(z_n)z|} \right)} \leq \frac{h_2(1 - |z|)}{h_2 \left( \frac{1}{|1 - \varphi(z_n)z|} \right)} \leq 1.
\]
Hence, Inequality (4) is bounded and there exists a constant $L > 0$ such that $\| h_n \|_{B^{v \log}} \leq L$ for all $n \in \mathbb{N}$. Thus, we have

$$
\| W_{u, \varphi}(h_n) \|_{B^{v \log}} \geq \limsup_{n \to \infty} v \log (z_n) \frac{|u(z_n)\varphi'(z_n)|}{(1 - |\varphi(z_n)|^2) \log \left( \frac{4}{1 - |\varphi(z_n)|^2} \right)}
$$

$$
\geq \limsup_{n \to \infty} v \log (z_n) \frac{|u(z_n)\varphi'(z_n)|}{2(1 - |\varphi(z_n)|) \log \left( \frac{4}{1 - |\varphi(z_n)|} \right)}
$$

$$
\geq C \limsup_{n \to \infty} v \log (z_n) \frac{|u(z_n)\varphi'(z_n)|}{(1 - |\varphi(z_n)|) \log \left( \frac{2}{1 - |\varphi(z_n)|} \right)}
$$

where we have used the fact that $v \log \simeq v_3$ and that $v_3$ is an essential weight ($v_3$ is the weight defined in (7)).

Since the sequence \{z_n\} such that $|\varphi(z_n)| \to 1^-$ was arbitrary, we obtain that there exists a constant $C > 0$ such that

$$
\| W_{u, \varphi} \|_{e \to B^{v \log}} \geq C \limsup_{|\varphi(z)| \to 1^-} \frac{v_3(z)}{v_3(\varphi(z))} |u(z)\varphi'(z)|
$$

$$
\geq C \limsup_{|\varphi(z)| \to 1^-} \frac{v_3(z)}{v_3(\varphi(z))} |u(z)\varphi'(z)|
$$

$$
\geq C \limsup_{|\varphi(z)| \to 1^-} \frac{v_3(z)}{\tilde{v}_3(\varphi(z))} |u(z)\varphi'(z)|
$$

(10)

Therefore, from the inequalities (8) and (10), we can conclude that there exists a constant $C > 0$ such that

$$
\| W_{u, \varphi} \|_{e \to B^{v \log}} \geq C \max \left\{ \limsup_{n \to \infty} \frac{(n + 1) \| J_u(\varphi^n) \|_{B^{v \log}}}{\| g_n \|_{B^{v \log}}}, \limsup_{n \to \infty} \frac{\| I_u(\varphi^n) \|_{B^{v \log}}}{\| g_n \|_{B^{v \log}}} \right\}.
$$

This finishes the proof of Theorem 1.2.
5 An application. Composition operators on Zygmund-Logarithmic space

As an application of our results, in this section we study continuity, compactness and we estimate the essential norm of composition operators acting on Zygmund-logarithmic space. Recall that the Zygmund-logarithmic space $\mathcal{Z}^\log$, consists of all holomorphic functions $f \in \mathcal{H}(\mathbb{D})$ such that $f' \in \mathcal{B}^\log$. More precisely,

$$\mathcal{Z}^\log := \{ f \in \mathcal{H}(\mathbb{D}) : \| f \|_{\mathcal{Z}^\log} = \sup_{z \in \mathbb{D}} v^\log(z) |f''(z)| < \infty \}$$

endowed with the norm $\| f \|_{\mathcal{Z}^\log} := |f(0)| + |f'(0)| + \| f \|_{\mathcal{Z}^\log}$. $\mathcal{Z}^\log$ is a Banach space.

As before, for a holomorphic function $u : \mathbb{D} \to \mathbb{C}$, we define the functionals

$$I'_u f(z) = \int_0^z I_u(f(s))ds \quad \text{and} \quad J'_u f(z) = \int_0^z J_u(f(s))ds,$$

where $f \in \mathcal{H}(\mathbb{D})$ and $I_u, J_u$ are the functionals defined in Theorem 1.1. We have the relations

$$\| f \|_{\mathcal{Z}^\log} = \| f' \|_{\mathcal{B}^\log}$$

$$\| C_\varphi(f) \|_{\mathcal{Z}^\log} = \| \varphi' C_\varphi(f) \|_{\mathcal{B}^\log}.$$ 

Hence, the operator $C_\varphi : \mathcal{Z}^\log \to \mathcal{Z}^\log$ is continuous if and only if the weighted composition operator $W_{\varphi, \varphi}$ is continuous on $\mathcal{B}^\log$. Thus, by Theorem 1.1, and the relations

$$\sup_{n \in \mathcal{W}} \frac{(n+1) \| J'_{\varphi'}(\varphi^n) \|_{\mathcal{Z}^\log}}{\| g_{n+2} \|_{\mathcal{Z}^\log}^{n+2}} = \sup_{n \in \mathcal{W}} \frac{(n+1) \| J'_{\varphi'}(\varphi^n) \|_{\mathcal{B}^\log}}{\| g_{n+1} \|_{\mathcal{B}^\log}^{n+1}} < \infty$$

and

$$\sup_{n \in \mathcal{N}} \frac{\| I'_{\varphi'}(\varphi^n) \|_{\mathcal{Z}^\log}}{\| g_{n+1} \|_{\mathcal{Z}^\log}^{n+1}} = \sup_{n \in \mathcal{N}} \frac{\| I'_{\varphi'}(\varphi^n) \|_{\mathcal{B}^\log}}{\| g_n \|_{\mathcal{B}^\log}^{n+1}} < \infty.$$

we have the following result:

**Corollary 5.1.** Suppose that $\varphi : \mathbb{D} \to \mathbb{D}$ is a holomorphic function. The composition operator $C_\varphi$ is continuous on $\mathcal{Z}^\log$ if and only if

$$\max \left\{ \sup_{n \in \mathcal{W}} \frac{(n+1) \| J'_{\varphi'}(\varphi^n) \|_{\mathcal{Z}^\log}}{\| g_{n+2} \|_{\mathcal{Z}^\log}^{n+2}}, \sup_{n \in \mathcal{N}} \frac{(n+1) \| I'_{\varphi'}(\varphi^n) \|_{\mathcal{Z}^\log}}{\| g_{n+1} \|_{\mathcal{Z}^\log}^{n+1}} \right\} < \infty.$$
Now, we go to estimate the essential norm of the continuous operator $C_\varphi : \mathcal{Z}^{n_{\log}} \to \mathcal{Z}^{n_{\log}}$, in fact, we go to show the following relation:

$$\|C_\varphi\|_{\mathcal{Z}^{n_{\log}} \to \mathcal{Z}^{n_{\log}}} = \|W_{\varphi', \varphi}\|_{\mathcal{B}^{n_{\log}} \to \mathcal{B}^{n_{\log}}}.$$  \hfill (11)

The argument in the proof of Lemma 2.2 shows that

$$\|C_\varphi\|_{\mathcal{Z}^{n_{\log}} \to \mathcal{Z}^{n_{\log}}} = \|C_\varphi\|_{\tilde{\mathcal{Z}}^{n_{\log}} \to \mathcal{Z}^{n_{\log}}},$$

where $\tilde{\mathcal{Z}}^{n_{\log}} = \{f \in \mathcal{Z}^{n_{\log}} : f(0) = f'(0) = 0\}$. Hence, it is enough to show that

$$\|C_\varphi\|_{\tilde{\mathcal{Z}}^{n_{\log}} \to \mathcal{Z}^{n_{\log}}} = \|W_{\varphi', \varphi}\|_{\tilde{\mathcal{B}}^{n_{\log}} \to \mathcal{B}^{n_{\log}}}.$$  

To see this, we proceed as in the proof of Lemma 2.1, that is, we consider the derivative operator $D : \tilde{\mathcal{Z}}^{n_{\log}} \to \mathcal{B}^{n_{\log}}$. Since this operator is an isometry, we have

$$C_\varphi(g) := D^{-1}W_{\varphi', \varphi}D(g),$$  \hfill (12)

for all $g \in \mathcal{Z}^{n_{\log}}$. Thus, for $\epsilon > 0$ we can find a compact operator $T : \tilde{\mathcal{B}}^{n_{\log}} \to \mathcal{B}^{n_{\log}}$ such that

$$\|W_{\varphi', \varphi}\|_{\tilde{\mathcal{B}}^{n_{\log}} \to \mathcal{B}^{n_{\log}}} \geq \frac{1}{1 + \epsilon} \|W_{\varphi', \varphi} - T\|_{\tilde{\mathcal{B}}^{n_{\log}} \to \mathcal{B}^{n_{\log}}} \geq \frac{1}{1 + \epsilon} \|D^{-1}W_{\varphi', \varphi} - D^{-1}T\|_{\tilde{\mathcal{Z}}^{n_{\log}} \to \mathcal{Z}^{n_{\log}} \to \mathcal{B}^{n_{\log}}} \geq \frac{1}{1 + \epsilon} \|C_\varphi - K\|_{\tilde{\mathcal{Z}}^{n_{\log}} \to \mathcal{Z}^{n_{\log}}} \geq \frac{1}{1 + \epsilon} \|C_\varphi\|_{\tilde{\mathcal{Z}}^{n_{\log}} \to \mathcal{Z}^{n_{\log}}}.$$

where we have used that $D$ is an isometry and the fact that $K := D^{-1}TD$ is a compact operator on $\tilde{\mathcal{Z}}^{n_{\log}}$. Therefore, since $\epsilon$ was arbitrary, we conclude

$$\|C_\varphi\|_{\tilde{\mathcal{Z}}^{n_{\log}} \to \mathcal{Z}^{n_{\log}}} \leq \|W_{\varphi', \varphi}\|_{\tilde{\mathcal{B}}^{n_{\log}} \to \mathcal{B}^{n_{\log}}}.$$

Similarly, given $\epsilon > 0$ we can find a compact operator $\widehat{T} : \tilde{\mathcal{Z}}^{n_{1}} \to \mathcal{Z}^{n_{2}}$ such that

$$\|C_\varphi\|_{\tilde{\mathcal{Z}}^{n_{1}} \to \mathcal{Z}^{n_{2}}} \geq \frac{1}{1 + \epsilon} \|C_\varphi - \widehat{T}\|_{\tilde{\mathcal{Z}}^{n_{1}} \to \mathcal{Z}^{n_{2}}}.$$  

Thus, using the fact that $D$ is an isometry and the relation (12), we obtain

$$\frac{1}{1 + \epsilon} \|C_\varphi - \widehat{T}\|_{\tilde{\mathcal{Z}}^{n_{1}} \to \mathcal{Z}^{n_{2}}} = \frac{1}{1 + \epsilon} \|DC_\varphi - D\widehat{T}\|_{\tilde{\mathcal{Z}}^{n_{1}} \to \mathcal{Z}^{n_{2}}} \geq \frac{1}{1 + \epsilon} \|DC_\varphi D^{-1} - D\widehat{T}D^{-1}\|_{\tilde{\mathcal{Z}}^{n_{1}} \to \mathcal{Z}^{n_{2}}} \geq \frac{1}{1 + \epsilon} \|W_{\varphi', \varphi} - K\|_{\tilde{\mathcal{B}}^{n_{2}} \to \mathcal{B}^{n_{2}}} \geq \frac{1}{1 + \epsilon} \|W_{\varphi', \varphi}\|_{\tilde{\mathcal{B}}^{n_{2}} \to \mathcal{B}^{n_{2}}}.$$

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Hence, since $\varepsilon$ was arbitrary, we conclude
\[
\|C_\varphi\|_{e}^{Z_{v}^{\log} \rightarrow Z_{v}^{\log}} = \|W_{\varphi', \varphi}\|_{e}^{E_{v}^{\log} \rightarrow E_{v}^{\log}}.
\]
Therefore, we have shown the following result:

**Theorem 5.2.** Suppose that $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ is a holomorphic function and that $C_\varphi$ is continuous on $Z_{v}^{\log}$. Then
\[
\|C_\varphi\|_{e}^{Z_{v}^{\log} \rightarrow Z_{v}^{\log}} \asymp \max \left\{ \limsup_{n \to \infty} \frac{(n + 2)(n + 1)\|J_\varphi'(\varphi^n)\|_{Z_{v}^{\log}}}{\|g_{n+2}\|_{Z_{v}^{\log}}}, \limsup_{n \to \infty} \frac{(n + 1)\|I_\varphi'(\varphi^n)\|_{Z_{v}^{\log}}}{\|g_{n+1}\|_{Z_{v}^{\log}}} \right\}
\]

As an immediate consequence of the above result, we have the following corollary:

**Corollary 5.3.** Suppose that $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ is a holomorphic function. The composition operator $C_\varphi$ is compact on $Z_{v}^{\log}$ if and only if
\[
\max \left\{ \lim_{n \to \infty} \frac{(n + 2)(n + 1)\|J_\varphi'(\varphi^n)\|_{Z_{v}^{\log}}}{\|g_{n+2}\|_{Z_{v}^{\log}}}, \lim_{n \to \infty} \frac{(n + 1)\|I_\varphi'(\varphi^n)\|_{Z_{v}^{\log}}}{\|g_{n+1}\|_{Z_{v}^{\log}}} \right\} = 0.
\]

**References**

[1] Attele K. Toeplitz and Hankel operators on Bergman one space. Hokkaido Math J 1992; 21: 279–293.

[2] Brown L, Shields AL. Multipliers and cyclic vectors in the Bloch space. Michigan Math J 1991; 38: 141–146.

[3] Castillo R, Clahane D, Farías-López J, Ramos-Fernández JC. Composition operators from logarithmic Bloch spaces to weighted Bloch spaces. Appl Math Comput 2013; 219: 6692–6706.

[4] Castillo R, Marrero-Rodríguez CE, Ramos-Fernández JC. On a Criterion for Continuity and Compactness of Composition Operators on the Weighted Bloch Space. To appear in Mediterr J Math.

[5] Esmaeili K, Lindström M. Weighted composition operators between Zygmund type spaces and their essential norms. Integr Equ Oper Theory 2013; 75: 473–490.

[6] Galanopoulos P. On $B_{log}$ to $Q_{log}$ pullbacks. J Math Anal Appl 2008; 337: 712-725.
[7] Hyvärinen O, Kemppainen M, Lindström M, Rautio A, Saukko E. The essential norm of weighted composition operators on weighted Banach spaces of analytic functions. Integr Equ Oper Theory 2012; 72: 151–157.

[8] Hyvärinen O, Lindström M. Estimates of essential norms of weighted composition operators between Bloch-type spaces. J Math Anal Appl 2012; 393: 38–44.

[9] Malavé-Ramírez MT, Ramos-Fernández JC. The associated weight and the essential norm of weighted composition operators. To appear in Banach J Math Anal.

[10] Montes-Rodríguez A. Weighted composition operators on weighted Banach spaces of analytic functions. J. London Math Soc 2000; 61: 872–884.

[11] Stević S. Norm of some operators from logarithmic Bloch-type spaces to weighted-type spaces. Appl Math Comput 2012; 218: 11163–11170.

[12] Tjani M. Compact composition operators on Besov spaces. Trans Amer Math Soc 2003; 355: 4683–4698.

[13] Ye S. A weighted composition operator on the logarithmic Bloch space. Bull Korean Math Soc 2010; 47: 527-540.

[14] Ye S. Norm and essential norm of composition followed by differentiation from logarithmic Bloch spaces to $H^\infty$. Abstr Appl Anal 2014; Art ID 725145.

[15] Yoneda R. The composition operators on weighted Bloch space. Arch Math 2002; 78: 310-317.

[16] Zhu K. Operator theory in function spaces. New York: Marcel Dekker, 1990.