Commutative Families of the Elliptic Macdonald Operator

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Abstract

In the paper [1], using the Ding-Iohara algebra and the trigonometric Feigin-Odesskii algebra, Feigin, Hashizume, Hoshino, Shiraiishi, and Yanagida constructed two families of commuting operators which contain the Macdonald operator (commutative families of the Macdonald operator). In the previous paper [3], the author constructed the elliptic Ding-Iohara algebra and the free field realization of the elliptic Macdonald operator. In this paper, we show that by using the elliptic Ding-Iohara algebra and the elliptic Feigin-Odesskii algebra, we can construct commutative families of the elliptic Macdonald operator.

In Appendix, we will show a relation between the elliptic Macdonald operator and its kernel function by the free field realization.

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In this paper, we use the following symbols.

\[ \mathbb{Z} : \text{The set of integers}, \quad \mathbb{Z}_{\geq 0} := \{0, 1, 2, \ldots\}, \quad \mathbb{Z}_0 := \{1, 2, \ldots\}, \]
\[ \mathbb{Q} : \text{The set of rational numbers}, \quad \mathbb{C} : \text{The set of complex numbers}, \quad \mathbb{C}^\times := \mathbb{C} \setminus \{0\}, \]
\[ \mathbb{C}[[z, z^{-1}]] : \text{The set of formal power series of } z, z^{-1} \text{ over } \mathbb{C}. \]

If a sequence \( \lambda = (\lambda_1, \cdots, \lambda_N) \in (\mathbb{Z}_{\geq 0})^N \) satisfies the condition \( \lambda_i \geq \lambda_{i+1} (1 \leq i \leq N) \), \( \lambda \) is called a partition. We denote the set of partitions by \( \mathcal{P} \).

Let \( q, p \in \mathbb{C} \) be complex parameters satisfying \( |q| < 1, |p| < 1 \). We define the \( q \)-infinite product as \( (x; q)_\infty := \prod_{n \geq 0} (1 - qx^n) \) and the theta function as
\[ \Theta_p(x) := (p; p)_\infty (x; p)_\infty (px^{-1}; p)_\infty. \]

We set the double infinite product as \((x; q, p)_\infty := \prod_{m,n \geq 0} (1 - qx^mp^n)\) and the elliptic gamma function as
\[ \Gamma_{q,p}(x) := \frac{(qx^{-1}; q, p)_\infty}{(x; q, p)_\infty}. \]

**Organization of this paper.**

In section 1, we review the trigonometric case treated in the paper [1]. In section 2, first we recall related materials of the elliptic Ding-Iohara algebra and the free field realization of the elliptic Macdonald operator. Second we show that using the elliptic Ding-Iohara algebra and the elliptic Feigin-Odesskii algebra, we can obtain commutative families of the elliptic Macdonald operator.

In Appendix, by the free field realization of the elliptic Macdonald operator, we show a functional equation of the elliptic kernel function.

## 1 Trigonometric case

In this section, we review the construction of the commutative families of the Macdonald operator by Feigin, Hashizume, Hoshino, Shiraishi, Yanagida [1].

### 1.1 Ding-Iohara algebra \( \mathcal{U}(q, t) \)

The Ding-Iohara algebra is a quantum group obtained from the free field realization of the elliptic Macdonald operator [1]. Here we define the Ding-Iohara algebra and collect some basic facts.

**Definition 1.1 (Ding-Iohara algebra \( \mathcal{U}(q, t) \)).** Let us define the structure function \( g(x) \) as
\[ g(x) := \frac{(1 - qx)(1 - t^{-1}x)(1 - q^{-1}tx)}{(1 - q^{-1}x)(1 - tx)(1 - qt^{-1}x)}. \]
Let $\gamma$ be a central, invertible element and $x^{\pm}(z) := \sum_{n \in \mathbb{Z}} x_n^{\pm} z^{-n}, \psi^{\pm}(z) := \sum_{n \in \mathbb{Z}} \psi_n^{\pm} z^{-n}$ be currents satisfying the relations:
\[
[\psi^{\pm}(z), \psi^{\pm}(w)] = 0, \quad \psi^{\pm}(z)\psi^{-}(w) = \frac{g(\gamma z/w)}{g(\gamma^{-1} z/w)} \psi^{-}(w)\psi^{\pm}(z),
\]
\[
\psi^{\pm}(z)x^{\pm}(w) = g\left(\gamma^{\pm\frac{1}{2}} \frac{z}{w}\right)x^{\pm}(w)\psi^{\pm}(z), \quad \psi^{\pm}(z)x^{-}(w) = g\left(\gamma^{\pm\frac{1}{2}} \frac{z}{w}\right)^{-1} x^{-}(w)\psi^{\pm}(z),
\]
\[
x^{\pm}(z)x^{\pm}(w) = g\left(\frac{z}{w}\right)^{\pm1} x^{\pm}(w)x^{\pm}(z),
\]
\[
[x^{\pm}(z), x^{-}(w)] = \frac{(1-q)(1-t^{-1})}{1-qt^{-1}} \left\{ \delta\left(\frac{w}{z}\right) \psi^{\pm}(\gamma^{1/2} w) - \delta\left(\gamma^{-1} \frac{w}{z}\right) \psi^{-}(\gamma^{-1/2} w) \right\}. \tag{1.2}
\]

Here we set the delta function by $\delta(x) := \sum_{n \in \mathbb{Z}} x^n$. We define the Ding-Iohara algebra $U(q, t)$ to be an associative $\mathbb{C}$-algebra generated by $\{x_n^{\pm}\}_{n \in \mathbb{Z}}, \{\psi_n^{\pm}\}_{n \in \mathbb{Z}},$ and $\gamma$.

The free field realization of the Ding-Iohara algebra is stated as follows. In the following, let $q, t \in \mathbb{C}$ be parameters and we assume $|q| < 1$. First we define the algebra $\mathcal{B}$ of boson to be generated by $\{a_n\}_{n \in \mathbb{Z}\setminus\{0\}}$ and the relation:
\[
[a_m, a_n] = m\frac{1 - q^{|m|}}{1 - t^{|m|}} \delta_{m+n, 0}. \tag{1.3}
\]

We set the normal ordering $:\cdot:\$ as
\[
:\cdot:\ a_m a_n := \begin{cases} 
  a_m a_n & (m < n), \\
  a_n a_m & (m \geq n).
\end{cases}
\]

Let $|0\rangle$ be the vacuum vector which satisfies $a_n|0\rangle = 0 (n > 0)$. For a partition $\lambda$, we set $a_{-\lambda} := a_{-\lambda_1} \cdots a_{-\lambda_{\ell(\lambda)}}$ and define the boson Fock space $\mathcal{F}$ as a left $\mathcal{B}$ module:
\[
\mathcal{F} := \text{span}\{a_{-\lambda}|0\rangle : \lambda \in \mathcal{P}\}.
\]

**Proposition 1.2 (Free field realization of the Ding-Iohara algebra $U(q, t)$).** Set $\gamma := (qt^{-1})^{-1/2}$ and define operators $\eta(z), \xi(z), \varphi^{\pm}(z) : \mathcal{F} \to \mathcal{F} \otimes \mathbb{C}[z, z^{-1}]$ as follows:
\[
\eta(z) := \exp \left( - \sum_{n \neq 0} (1-t^n) a_n \frac{z^{-n}}{n} \right), \quad \xi(z) := \exp \left( \sum_{n \neq 0} (1-t^n) \gamma^n a_n \frac{z^{-n}}{n} \right),
\]
\[
\varphi^{+}(z) := \eta(\gamma^{1/2} z) \xi(\gamma^{-1/2} z), \quad \varphi^{-}(z) := \eta(\gamma^{-1/2} z) \xi(\gamma^{1/2} z).
\]

Then the map
\[
x^{\pm}(z) \mapsto \eta(z), \quad x^{-}(z) \mapsto \xi(z), \quad \psi^{\pm}(z) \mapsto \varphi^{\pm}(z)
\]
gives a representation of the Ding-Iohara algebra $U(q, t)$.

Set the $q$-shift operator as $T_{q,x}f(x) := f(qx)$. We define the Macdonald operator $H_N(q, t) (N \in \mathbb{Z}_{>0})$ as
\[
H_N(q, t) := \sum_{i=1}^{N} \prod_{j \neq i}^{N} \frac{t x_i - x_j}{x_i - x_j} T_{q,x}.
\]

In the following $[f(z)]_1$ denotes the constant term of $f(z)$ in $z$. 
Proposition 1.3 (Free field realization of the Macdonald operator). Set the operator $\phi(z) : F \to F \otimes \mathbb{C}[z, z^{-1}]$ as follows.

$$\phi(z) := \exp \left( \sum_{n>0} \frac{1-t^n}{1-q^n} a_{-n} z^n \right).$$

We use the symbol $\phi_N(x) := \prod_{j=1}^N \phi(x_j)$.

1. The operator $\eta(z)$ reproduces the Macdonald operator $H_N(q, t)$ as follows.

$$[\eta(z)]_1 \phi_N(x)|0\rangle = t^{-N}\{(t - 1)H_N(q, t) + 1\} \phi_N(x)|0\rangle.$$  

2. The operator $\xi(z)$ reproduces the Macdonald operator $H_N(q^{-1}, t^{-1})$ as follows.

$$[\xi(z)]_1 \phi_N(x)|0\rangle = t^{N}\{(t^{-1} - 1)H_N(q^{-1}, t^{-1}) + 1\} \phi_N(x)|0\rangle.$$  

We also have the dual version of the proposition 1.3. Let $\langle 0 |$ be the dual vacuum vector which satisfies the condition $\langle 0 | a_n = 0$ $(n < 0)$ and define the dual boson Fock space $F^*$ as a right $B$ module:

$$\mathcal{F}^* := \text{span}\{\langle 0 | a_\lambda : \lambda \in \mathcal{P} \} \quad (a_\lambda := a_{\lambda_1} \cdots a_{\ell(\lambda)}).$$

Proposition 1.4 (Dual version of the proposition 1.3). Let us define an operator $\phi^*(z) : F^* \to F^* \otimes \mathbb{C}[z, z^{-1}]$ as

$$\phi^*(z) := \exp \left( \sum_{n>0} \frac{1-t^n}{1-q^n} a_{n} z^n \right).$$

We use the symbol $\phi^*_N(x) := \prod_{j=1}^N \phi^*(x_j)$.

1. The operator $\eta(z)$ reproduces the Macdonald operator $H_N(q, t)$ as follows.

$$\langle 0 | \phi^*_N(x) [\eta(z)]_1 = t^{-N}\{(t - 1)H_N(q, t) + 1\} \langle 0 | \phi^*_N(x).$$

2. The operator $\xi(z)$ reproduces the Macdonald operator $H_N(q^{-1}, t^{-1})$ as follows.

$$\langle 0 | \phi^*_N(x) [\xi(z)]_1 = t^{N}\{(t^{-1} - 1)H_N(q^{-1}, t^{-1}) + 1\} \langle 0 | \phi^*_N(x).$$

Remark 1.5. The kernel function of the Macdonald operator $\Pi(q, t)(\{x_i\}_{i=1}^M, \{y_j\}_{j=1}^N)$ $(M, N \in \mathbb{Z}_{>0})$ is defined by

$$\Pi(q, t)(\{x_i\}_{i=1}^M, \{y_j\}_{j=1}^N) := \prod_{1 \leq i \leq M, 1 \leq j \leq N} \frac{(tx_i y_j; q)_\infty}{(x_i y_j; q)_\infty},$$

Then the kernel function $\Pi(q, t)(\{x_i\}_{i=1}^M, \{y_j\}_{j=1}^N)$ is reproduced from the operators $\phi^*_M(x)$, $\phi_N(y)$ as

$$\langle 0 | \phi^*_M(x) \phi_N(y)|0\rangle = \Pi(q, t)(\{x_i\}_{i=1}^M, \{y_j\}_{j=1}^N).$$
1.2 Trigonometric Feigin-Odesskii algebra $\mathcal{A}$

In this subsection, we review basic facts of the trigonometric Feigin-Odesskii algebra [1].

**Definition 1.6 (Trigonometric Feigin-Odesskii algebra $\mathcal{A}$).** Let $\varepsilon_n(q; x) (n \in \mathbb{Z}_{>0})$ be a function defined as

$$
\varepsilon_n(q; x) := \prod_{1 \leq a < b \leq n} \frac{(x_a - q x_b)(x_a - q^{-1} x_b)}{(x_a - x_b)^2}.
$$

(1.12)

We also define $\omega(x, y)$ as

$$
\omega(x, y) := \frac{(x - q^{-1} y)(x - ty)(x - qt^{-1} y)}{(x - y)^3}.
$$

(1.13)

For a $N$-variable function $f(x_1, \cdots, x_N)$, we define the action of the $N$-th symmetric group $\mathfrak{S}_N$ on $f(x_1, \cdots, x_N)$ by $\sigma \cdot (f(x_1, \cdots, x_N)) := f(x_{\sigma(1)}, \cdots, x_{\sigma(N)}) (\sigma \in \mathfrak{S}_N)$. We define the symmetrizer as

$$
\text{Sym}[f(x_1, \cdots, x_N)] := \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} \sigma \cdot (f(x_1, \cdots, x_N)).
$$

(1.14)

For a $m$-variable function $f(x_1, \cdots, x_m)$ and a $n$-variable function $g(x_1, \cdots, x_n)$, we define the star product $\ast$ as follows.

$$(f \ast g)(x_1, \cdots, x_{m+n}) := \text{Sym} \left[ f(x_1, \cdots, x_m)g(x_{m+1}, \cdots, x_{m+n}) \prod_{\substack{1 \leq \alpha \leq m \\atop m+1 \leq \beta \leq m+n}} \omega(x_\alpha, x_\beta) \right].
$$

(1.15)

For a partition $\lambda$, we define $\varepsilon_\lambda(q; x)$ as

$$
\varepsilon_\lambda(q; x) := \varepsilon_{\lambda_1}(q; x) \ast \cdots \ast \varepsilon_{\lambda(\lambda)}(q; x).
$$

(1.16)

Set $\mathcal{A}_0 := \mathbb{C}$, $\mathcal{A}_n := \text{span}\{\varepsilon_\lambda(q; x) : |\lambda| = n\} (n \geq 1)$. We define the trigonometric Feigin-Odesskii algebra $\mathcal{A} := \bigoplus_{n \geq 0} \mathcal{A}_n$ whose algebraic structure is given by the star product $\ast$.

**Remark 1.7.** The definition of the trigonometric Feigin-Odesskii algebra $\mathcal{A}$ above is a reduced version of the paper [1]. For instance, there would be a question why the function $\varepsilon_n(q; x)$ appears. For more detail of the trigonometric Feigin-Odesskii algebra $\mathcal{A}$, see [1].

In the paper [1], the following fact is shown.

**Proposition 1.8.** The trigonometric Feigin-Odesskii algebra $(\mathcal{A}, \ast)$ is unital, associative, and commutative.
1.3 Commutative families $\mathcal{M}, \mathcal{M}'$

Here we give an overview of the construction of the commutative families of the Macdonald operator by using the Ding-Iohara algebra and the trigonometric Feigin-Odesskii algebra.

**Definition 1.9 (Map $\mathcal{O}$).** Define a linear map $\mathcal{O} : \mathcal{A} \to \text{End}(\mathcal{F})$ as

$$ \mathcal{O}(f) := \left[ f(z_1, \ldots, z_n) \prod_{1 \leq i < j \leq n} \omega(z_i, z_j)^{-1}\eta(z_1) \cdots \eta(z_n) \right]_1 \quad (f \in \mathcal{A}_n). \quad (1.17) $$

Here $[f(z_1, \ldots, z_n)]_1$ denotes the constant term of $f(z_1, \ldots, z_n)$ in $z_1, \ldots, z_n$, and we extend the map $\mathcal{O}$ linearly.

**Proposition 1.10.** The map $\mathcal{O}$ and the star product $\ast$ are compatible: for $f, g \in \mathcal{A}$, we have $\mathcal{O}(f \ast g) = \mathcal{O}(f)\mathcal{O}(g)$.

**Proof.** To prove the proposition, for $f \in \mathcal{A}_m$ and $g \in \mathcal{A}_n$, we show $\mathcal{O}(f \ast g) = \mathcal{O}(f)\mathcal{O}(g)$. First we have

$$ \mathcal{O}(f \ast g)(x_1, \ldots, x_{m+n}) = \left[ \text{Sym} \left( f(x_1, \ldots, x_m)g(x_{m+1}, \ldots, x_{m+n}) \prod_{1 \leq \alpha \leq m, m+1 \leq \beta \leq m+n} \omega(x_{\alpha}, x_{\beta}) \right) \right. $$

$$ \times \prod_{1 \leq i < j \leq m+n} \omega(x_i, x_j)^{-1}\eta(x_1) \cdots \eta(x_{m+n}) \left. \right]_1. $$

Then from the relation

$$ \eta(z)\eta(w) = g\left( \frac{z}{w} \right)\eta(w)\eta(z), $$

we have the following:

$$ \frac{1}{\omega(z, w)}\eta(z)\eta(w) = \frac{1}{\omega(w, z)}\eta(w)\eta(z). \quad (1.18) $$

The relation shows that the operator-valued function

$$ \prod_{1 \leq i < j \leq N} \omega(x_i, x_j)^{-1}\eta(x_1) \cdots \eta(x_N) \quad (1.19) $$

is symmetric in $x_1, \ldots, x_N$. Further we have

$$ [\text{Sym}(F(x_1, \ldots, x_N))]_1 = [F(x_1, \ldots, x_N)]_1. $$

This follows from the fact that the constant term is invariant under the action of the symmetric group. In addition for a symmetric function $f(x_1, \ldots, x_N)$, we have

$$ \sigma(f(x_1, \ldots, x_N)g(x_1, \ldots, x_N)) = f(x_1, \ldots, x_N)\sigma(g(x_1, \ldots, x_N)) \quad (\sigma \in \mathfrak{S}_N). $$

Hence we have

$$ \text{Sym}(f(x_1, \ldots, x_N)g(x_1, \ldots, x_N)) = f(x_1, \ldots, x_N)\text{Sym}(g(x_1, \ldots, x_N)). $$
1.3 Commutative families $\mathcal{M}, \mathcal{M}'$

From them we have the following.

\[
\mathcal{O}(f \ast g) = \left[ \text{Sym} \left( f(x_1, \ldots, x_m)g(x_{m+1}, \ldots, x_{m+n}) \prod_{1 \leq \alpha \leq m, m+1 \leq \beta \leq m+n} \omega(x_\alpha, x_\beta) \right) \prod_{1 \leq i < j \leq m+n} \omega(x_i, x_j)^{-1} \eta(x_1) \cdots \eta(x_{m+n}) \right]_1 
\]

\[
= \left[ f(x_1, \ldots, x_m) \prod_{1 \leq i < j \leq m} \omega(x_i, x_j)^{-1} \eta(x_1) \cdots \eta(x_m) \right. 
\]

\[
\times g(x_{m+1}, \ldots, x_{m+n}) \prod_{m+1 \leq i < j \leq m+n} \omega(x_i, x_j)^{-1} \eta(x_{m+1}) \cdots \eta(x_{m+n}) \left. \right]_1 
\]

\[= \mathcal{O}(f)\mathcal{O}(g). \quad \square \]

The trigonometric Feigin-Odesskii algebra $\mathcal{A}$ is commutative by means of the star product $\ast$, therefore we have the following corollary.

Let $V$ be a $\mathbb{C}$-vector space and $T : V \rightarrow V$ be a $\mathbb{C}$-linear operator. Then for a subset $W \subset V$, the symbol $T|_W$ denotes the restriction of $T$ on $W$. For a subset $M \subset \text{End}_\mathbb{C}(V)$, we use the symbol $M|_W := \{T|_W : T \in M\}$ ($W \subset V$).

**Corollary 1.11 (Commutative family $\mathcal{M}$).**

1. Set $\mathcal{M} := \mathcal{O}(\mathcal{A})$. The space $\mathcal{M}$ consists of operators commuting with each other : $[\mathcal{O}(f), \mathcal{O}(g)] = 0$ ($f, g \in \mathcal{A}$).

2. The space $\mathcal{M}|_{\mathcal{O}_\phi_N(x)|0}$ consists of commuting $q$-difference operators which contains the Macdonald operator $H_N(q, t)$ (commutative family of the Macdonald operator $H_N(q, t)$).

**Proof.**

1. This statement follows from the commutativity of $\mathcal{A}$ and the compatibility of the star product $\ast$ and the map $\mathcal{O}$. We have $\mathcal{O}(f \ast g) = \mathcal{O}(f)\mathcal{O}(g)$ and also have $\mathcal{O}(f \ast g) = \mathcal{O}(g \ast f) = \mathcal{O}(g)\mathcal{O}(f)$. This shows $[\mathcal{O}(f), \mathcal{O}(g)] = 0$.

2. Due to the free field realization of the Macdonald operator $H_N(q, t)$, the operator $\mathcal{O}(\varepsilon_r(q; z))$ ($r \in \mathbb{Z}_{>0}$) acts on $\phi_N(x)|0$ ($N \in \mathbb{Z}_{>0}$) as a $r$-th order $q$-difference operator. By the fact that $\mathcal{M} = \mathcal{O}(\mathcal{A})$ is generated by $\{\mathcal{O}(\varepsilon_r(q; z))\}_{r \in \mathbb{Z}_{>0}}$ and the relation

\[\mathcal{O}(\varepsilon_1(q; z))\phi_N(x)|0 = [\eta(z)]_1 \phi_N(x)|0 = t^{-N}\{(t-1)H_N(q, t) + 1\}\phi_N(x)|0\]

and (1) in the corollary 1.11, the restriction $\mathcal{M}|_{\mathcal{O}_\phi_N(x)|0}$ is a space of commuting $q$-difference operators which contains the Macdonald operator $H_N(q, t)$. \quad \square

The Macdonald operator $H_N(q^{-1}, t^{-1})$ is reproduced from the operator $\xi(z)$. By this fact, we can construct another commutative family of the Macdonald operator.
Definition 1.12 (Map $O'$). Set a function $\omega'(x, y)$ as follows.

$$
\omega'(x, y) := \frac{(x - qy)(x - t^{-1}y)(x - q^{-1}ty)}{(x - y)^3}. \quad (1.20)
$$

Define a linear map $O' : A \to \End(\mathcal{F})$ as

$$
O'(f) := \left[ f(z_1, \cdots, z_n) \prod_{1 \leq i < j \leq n} \omega'(z_i, z_j)^{-1} \xi(z_1) \cdots \xi(z_n) \right]_1 (f \in \mathcal{A}_n). \quad (1.21)
$$

We extend the map $O'$ linearly.

Lemma 1.13. Define another star product $\ast'$ as follows :

$$(f \ast' g)(x_1, \cdots, x_{m+n}) := \text{Sym} \left[ f(x_1, \cdots, x_m) g(x_{m+1}, \cdots, x_{m+n}) \prod_{1 \leq \alpha \leq m, m+1 \leq \beta \leq m+n} \omega'(x_{\alpha}, x_{\beta}) \right].$$

In the trigonometric Feigin-Odesskii algebra $\mathcal{A}$, we have $\ast' = \ast$.

Proof. From the relation $\omega(x, y) = \omega'(y, x)$, for $f \in \mathcal{A}_m$, $g \in \mathcal{A}_n$ we have the following.

$$(f \ast' g)(x_1, \cdots, x_{m+n}) = \text{Sym} \left[ f(x_1, \cdots, x_m) g(x_{m+1}, \cdots, x_{m+n}) \prod_{1 \leq \alpha \leq m, m+1 \leq \beta \leq m+n} \omega'(x_{\alpha}, x_{\beta}) \right]$$

$$= \text{Sym} \left[ f(x_1, \cdots, x_m) g(x_{m+1}, \cdots, x_{m+n}) \prod_{1 \leq \alpha \leq m, m+1 \leq \beta \leq m+n} \omega(x_{\beta}, x_{\alpha}) \right]$$

$$= \text{Sym} \left[ g(x_1, \cdots, x_n) f(x_{n+1}, \cdots, x_{n+m}) \prod_{1 \leq \alpha \leq n, n+1 \leq \beta \leq n+m} \omega(x_{\alpha}, x_{\beta}) \right]$$

$$= (g \ast f)(x_1, \cdots, x_{m+n})$$

$$= (f \ast g)(x_1, \cdots, x_{m+n}) \quad (\because \mathcal{A} \text{ is commutative by means of } \ast).$$

Since the relation holds for any $f \in \mathcal{A}_m$, $g \in \mathcal{A}_n$, we have $\ast' = \ast$. \hfill \square

We can check the map $O'$ and the star product $\ast'$ are compatible in the similar way of the proof of the proposition 1.10. Furthermore by the lemma 1.13 as $\ast' = \ast$, we have the following corollary.

Corollary 1.14 (Commutative family $\mathcal{M}'$). (1) Set $\mathcal{M}' := O'(\mathcal{A})$. The space $\mathcal{M}'$ consists of operators commuting with each other.

(2) The space $\mathcal{M}'|_{c\psi_N(z)[0]}$ consists of commuting q-difference operators which contains the Macdonald operator $H_N(q^{-1}, t^{-1})$ (commutative family of the Macdonald operator $H_N(q^{-1}, t^{-1})$).

From the relation $[[\eta(z)], [\xi(w)]] = 0$, we have the following proposition.

Proposition 1.15. The commutative families $\mathcal{M}$, $\mathcal{M}'$ satisfy $[\mathcal{M}, \mathcal{M}'] = 0$. For any $a \in \mathcal{M}$ and $a' \in \mathcal{M}'$, we have $[a, a'] = 0$. 

**Proof.** This proposition follows from the existence of the Macdonald symmetric functions. That is, elements of the commutative families are simultaneously diagonalized by the Macdonald symmetric functions. □

From the proposition 1.15, commutative families \( M_{\mathbf{C}^{\phi_N}(x)_{\{0\}}} \), \( M'_{\mathbf{C}^{\phi_N}(x)_{\{0\}}} \) also commute with each other : \([M_{\mathbf{C}^{\phi_N}(x)_{\{0\}}}, M'_{\mathbf{C}^{\phi_N}(x)_{\{0\}}}] = 0\).

## 2 Elliptic case

In this section, we are going to construct a commutative family of the elliptic Macdonald operator by using the elliptic Ding-Iohara algebra and the elliptic Feigin-Odesskii algebra.

In the following, for complex parameters \( q, p \in \mathbb{C} \) we assume \(|q| < 1, |p| < 1\).

### 2.1 Elliptic Ding-Iohara algebra \( U(q, t, p) \)

The elliptic Ding-Iohara algebra is an elliptic analog of the Ding-Iohara algebra introduced by the author [3]. First we recall the definition of the elliptic Ding-Iohara algebra and its free field realization.

**Definition 2.1 (Elliptic Ding-Iohara algebra \( U(q, t, p) \)).** Set the structure function \( g_p(x) \) as

\[
g_p(x) := \frac{\Theta_p(qx)\Theta_p(t^{-1}x)\Theta_p(q^{-1}tx)}{\Theta_p(q^{-1}x)\Theta_p(tx)\Theta_p(q^{-1}x)}.
\]

Let \( x^\pm(p; z) := \sum_{n \in \mathbb{Z}} x_n^\pm(p)z^{-n}, \psi^\pm(p; z) := \sum_{n \in \mathbb{Z}} \psi_n^\pm(p)z^{-n} \) be currents and \( \gamma \) be central, invertible element satisfying the following relations:

\[
[\psi^+(p; z), \psi^+(p; w)] = 0, \quad \psi^+(p; z)\psi^-(p; w) = \frac{g_p(\gamma z/w)}{g_p(\gamma^{-1}z/w)}\psi^-(p; w)\psi^+(p; z),
\]

\[
\psi^+(p; z)x^+(p; w) = g_p(\gamma^{1/2}z/w)x^+(p; w)\psi^+(p; z),
\]

\[
\psi^+(p; z)x^-(p; w) = g_p(\gamma^{1/2}z/w)^{-1}x^-(p; w)\psi^+(p; z),
\]

\[
x^+(p; z)x^+(p; w) = g_p(z/w)^{\pm 1}x^+(p; w)x^+(p; z),
\]

\[
[x^+(p; z), x^-(p; w)] = \frac{\Theta_p(q)\Theta_p(t^{-1})}{(p; p)^3}\left\{\delta(x/w)\psi^+(p; \gamma^{1/2}w) - \delta(\gamma^{-1}w/z)\psi^-(p; \gamma^{-1/2}w)\right\}.
\]

Here we define the delta function as \( \delta(x) := \sum_{n \in \mathbb{Z}} x_n^\pm \). We define the elliptic Ding-Iohara algebra \( U(q, t, p) \) to be an associative \( \mathbb{C} \)-algebra generated by \( \{x_n^\pm(p)\}_{n \in \mathbb{Z}}, \{\psi_n^\pm(p)\}_{n \in \mathbb{Z}} \) and \( \gamma \).

**Theorem 2.2 (Free field realization of the elliptic Ding-Iohara algebra \( U(q, t, p) \)).** Set an algebra of boson \( B_{\alpha} \) generated by \( \{a_n\}_{n \in \mathbb{Z}\setminus\{0\}}, \{b_n\}_{n \in \mathbb{Z}\setminus\{0\}} \) and the following relations:

\[
[a_m, a_n] = m(1 - p|m|)\frac{1 - q|n|}{1 - t|n|}\delta_{m+n, 0}, \quad [b_m, b_n] = m\frac{1 - p|m|}{(qt^{-1})|m|}\frac{1 - q^{-1}t|m|}{1 - t^{-1}|m|}\delta_{m+n, 0},
\]

\[
[a_m, b_n] = 0.
\]
Let $|0\rangle$ be the vacuum vector which satisfies the condition $a_n|0\rangle = b_n|0\rangle = 0 \ (n > 0)$ and set the boson Fock space $\mathcal{F}$ as a left $\mathcal{B}_{a,b}$ module.

$$\mathcal{F} = \text{span}\{a_\lambda b_\mu|0\rangle : \lambda, \mu \in \mathcal{P}\}.$$ 

We also define the normal ordering $\bullet$ as usual:

$$a_m a_n := \begin{cases} a_m a_n & (m < n), \\ a_n a_m & (m \geq n), \\ b_m b_n & (m < n), \\ b_n b_m & (m \geq n). \end{cases}$$ 

Set $\gamma := (qt^{-1})^{-1/2}$ and operators $\eta(p; z)$, $\xi(p; z)$, $\varphi^\pm(p; z) : \mathcal{F} \to \mathcal{F} \otimes \mathbb{C}[z, z^{-1}]$ as

$$\eta(p;z) := \exp \left( - \sum_{n \neq 0} \frac{1 - t^{-n}}{1 - p|n|} b_n \frac{z^n}{n} \right) \exp \left( - \sum_{n \neq 0} \frac{1 - t^{-n}}{1 - p|n|} b_n \frac{z^{-n}}{n} \right) :;$$

$$\xi(p;z) := \exp \left( \sum_{n \neq 0} \frac{1 - t^{-n}}{1 - p|n|} \gamma|n| p|n| b_n \frac{z^n}{n} \right) \exp \left( \sum_{n \neq 0} \frac{1 - t^{-n}}{1 - p|n|} \gamma|n| a_n \frac{z^{-n}}{n} \right) ;$$

$$\varphi^+(p;z) := \eta(p;\gamma^{1/2}z) \xi(p;\gamma^{-1/2}z) ;, \quad \varphi^-(p;z) := \eta(p;\gamma^{-1/2}z) \xi(p;\gamma^{1/2}z) .$$

Then the map

$$x^+(p;z) \mapsto \eta(p;z), \quad x^-(p;z) \mapsto \xi(p;z), \quad \phi^\pm(p;z) \mapsto \varphi^\pm(p;z)$$

gives a representation of the elliptic Ding-Iohara algebra $\mathcal{U}(q, t, p)$.

The elliptic Macdonald operator $H_N(q, t, p) \ (N \in \mathbb{Z}_{>0})$ is defined as follows.

$$H_N(q,t,p) := \sum_{i=1}^N \prod_{j \neq i} \Theta_p(t x_i/x_j) T_{q,x_i} . \quad (2.2)$$ 

By the operators $\eta(p;z)$, $\xi(p;z)$ which are in the theorem 2.2, we can reproduce the elliptic Macdonald operator as follows [3].

**Theorem 2.3 (Free field realization of the elliptic Macdonald operator).** Let $\phi(p;z) : \mathcal{F} \to \mathcal{F} \otimes \mathbb{C}[z, z^{-1}]$ be an operator defined as follows.

$$\phi(p;z) := \exp \left( \sum_{n > 0} \frac{(1 - t^n)(qt^{-1}p)^n}{(1 - q^n)(1 - p^n)} b_n \frac{z^n}{n} \right) \exp \left( \sum_{n > 0} \frac{1 - t^n}{(1 - q^n)(1 - p^n)} a_n \frac{z^{-n}}{n} \right) . \quad (2.3)$$ 

We use the symbol $\phi_N(p;x) := \prod_{j=1}^N \phi(p;x_j)$.

(1) The elliptic Macdonald operator $H_N(q,t,p)$ is reproduced by the operator $\eta(p;z)$ as follows.

$$[\eta(p;z) - t^{-N}(\eta(p;z)) - (\eta(p;p^{-1}z))]_+ \phi_N(p;x)|0\rangle = \frac{t^{-N+1}\Theta_p(t^{-1})}{(p;p)^{N+1}_3} H_N(q,t,p) \phi_N(p;x)|0\rangle . \quad (2.4)$$

Here we use the notation $(\eta(p;z))_\pm$ as

$$(\eta(p;z))_\pm := \exp \left( - \sum_{\pm n > 0} \frac{1 - t^{-n}}{1 - p|n|} b_n \frac{z^n}{n} \right) \exp \left( - \sum_{\pm n > 0} \frac{1 - t^n}{1 - p|n|} a_n \frac{z^{-n}}{n} \right) .$$
2.1 Elliptic Ding-Iohara algebra $U(q,t,p)$

(2) The elliptic Macdonald operator $H_N(q^{-1}, t^{-1}, p)$ is reproduced by the operator $\xi(p; z)$ as follows.

$$[\xi(p; z) - t^n(\xi(p; z)) - (\xi(p; p^{-1}z))_+]_1 \phi_N(p; x)|0\rangle = \frac{t^{N-1}\Theta_p(t)}{(p; p)^3_\infty} H_N(q^{-1}, t^{-1}, p)\phi_N(p; x)|0\rangle.$$  \hspace{1cm} (2.5)

Here we use the notation $(\xi(p; z))_\pm$ as

$$(\xi(p; z))_\pm := \exp \left( \sum_{n>0} \frac{1-t^{-n}}{1-p^{n}}\gamma_{-n|p^n}b_n z^n \right) \exp \left( \sum_{n>0} \frac{1-t^{-n}}{1-p^{n}}\gamma_{|n}a_n z^{-n} \right).$$

The theorem 2.3 is also stated as follows. First we set zero mode generators $a_0$, $Q$ which satisfy the relation:

$$[a_0, Q] = 1, \quad [a_n, a_0] = [b_n, a_0] = 0, \quad [a_n, Q] = [b_n, Q] = 0 \quad (n \in \mathbb{Z} \setminus \{0\}).$$  \hspace{1cm} (2.6)

We also set the condition $a_0|0\rangle = 0$. For a complex number $\alpha \in \mathbb{C}$ we define $|\alpha\rangle := e^{\alpha Q}|0\rangle$. Then we can check $a_0|\alpha\rangle = \alpha|\alpha\rangle$. For $\alpha \in \mathbb{C}$, we set $F_\alpha := \text{span}\{a_{-\lambda}b_{-\mu}|\alpha\rangle : \lambda, \mu \in \mathcal{P}\}$.

**Theorem 2.4.** Set $\tilde{\eta}(p; z) := (\eta(p; z)) - (\eta(p; p^{-1}z))_+,$ $\tilde{\xi}(p; z) := (\xi(p; z)) - (\xi(p; p^{-1}z))_+$. Using these symbols we define operators $E(p; z)$, $F(p; z)$ as follows:

$$E(p; z) := \eta(p; z) - \tilde{\eta}(p; z)t^{-a_0}, \quad F(p; z) := \xi(p; z) - \tilde{\xi}(p; z)t^{a_0}.$$ \hspace{1cm} (2.7)

Then the elliptic Macdonald operators $H_N(q, t, p)$, $H_N(q^{-1}, t^{-1}, p)$ are reproduced by the operators $E(p; z)$, $F(p; z)$ as follows.

$$[E(p; z)]_1 \phi_N(p; x)|N\rangle = \frac{t^{-N+1}\Theta_q(t^{-1})}{(p; p)^3_\infty} H_N(q, t, p)\phi_N(p; x)|N\rangle,$$ \hspace{1cm} (2.8)

$$[F(p; z)]_1 \phi_N(p; x)|N\rangle = \frac{t^{N-1}\Theta_p(t)}{(p; p)^3_\infty} H_N(q^{-1}, t^{-1}, p)\phi_N(p; x)|N\rangle.$$ \hspace{1cm} (2.9)

The dual versions of the theorem 2.3, 2.4 are also available. Let $|0\rangle$ be the dual vacuum vector which satisfies the condition $\langle 0|a_n = \langle 0|b_n = 0 \quad (n < 0)$ and $\langle 0|a_0 = 0$. We define the dual boson Fock space as a right $\mathcal{B}_{a,b}$ module:

$$\mathcal{F}^* := \text{span}\{\langle 0|a_\lambda b_\mu : \lambda, \mu \in \mathcal{P}\}.$$  \hspace{1cm}

For a complex number $\alpha \in \mathbb{C}$, set $\langle \alpha \rangle := \langle 0|e^{-\alpha Q}$. Then we have $\langle \alpha|a_0 = \alpha \langle \alpha|$. For $\alpha \in \mathbb{C}$, we set $\mathcal{F}_\alpha^* := \text{span}\{\langle \alpha|a_\lambda b_\mu : \lambda, \mu \in \mathcal{P}\}$.

**Theorem 2.5 (Dual versions of the theorem 2.3, 2.4).** Let us define an operator $\phi^*(p; z) : \mathcal{F}^* \rightarrow \mathcal{F}^* \otimes \mathbb{C}[[z, z^{-1}]]$ as follows.

$$\phi^*(p; z) := \exp \left( \sum_{n>0} \frac{(1-t^n)(qt^{-1}p)^n}{(1-q^n)(1-p^n)}b_n z^n \right) \exp \left( \sum_{n>0} \frac{1-t^n}{(1-q^n)(1-p^n)}a_n z^{-n} \right).$$ \hspace{1cm} (2.10)

We use the symbol $\phi^*_N(p; x) := \prod_{j=1}^N \phi^*(p; x_j)$.
(1) The operators $\eta(p; z), \xi(p; z)$ reproduce the elliptic Macdonald operators $H_N(q, t, p), H_N(q^{-1}, t^{-1}, p)$ as follows.

\[
\langle 0| \phi_N^*(p; x) | \eta(p; z) - t^{-N}(\eta(p; z))_+ \rangle = t^{-N + 1} \Theta_p(t^{1/3}) H_N(q, t, p) \langle 0| \phi_N^*(p; x),
\]

\[
(2.11)
\]

\[
\langle 0| \phi_N^*(p; x) | \xi(p; z) - t^N(\xi(p; z))_+ \rangle = t^{-N + 1} \Theta_p(t^{1/3}) H_N(q^{-1}, t^{-1}, p) \langle 0| \phi_N^*(p; x).
\]

\[
(2.12)
\]

(2) The operators $E(p; z), F(p; z)$ reproduce the elliptic Macdonald operators $H_N(q, t, p), H_N(q^{-1}, t^{-1}, p)$ as follows.

\[
\langle N| \phi_N^*(p; x) | E(p; z) \rangle = t^{-N + 1} \Theta_p(t^{1/3}) H_N(q, t, p) \langle N| \phi_N^*(p; x),
\]

\[
(2.13)
\]

\[
\langle N| \phi_N^*(p; x) | F(p; z) \rangle = t^{-N + 1} \Theta_p(t^{1/3}) H_N(q^{-1}, t^{-1}, p) \langle N| \phi_N^*(p; x).
\]

\[
(2.14)
\]

**Remark 2.6.** Let $\Pi(q, t, p)(\{x_i\}_{i=1}^M, \{y_j\}_{j=1}^N)$ ($M, N \in \mathbb{Z}_>$) be the kernel function of the elliptic Macdonald operator defined as

\[
\Pi(q, t, p)(\{x_i\}_{i=1}^M, \{y_j\}_{j=1}^N) := \prod_{1 \leq i < j \leq M} \frac{\Gamma_{q,p}(x_i, y_j)}{\Gamma_{q,p}(x_j, y_i)}.
\]

Then the kernel function $\Pi(q, t, p)(\{x_i\}_{i=1}^M, \{y_j\}_{j=1}^N)$ is reproduced from the operators $\phi_M^*(p; x), \phi_N(p; y)$ as

\[
\langle 0| \phi_M^*(p; x) \phi_N(p; y) | 0 \rangle = \Pi(q, t, p)(\{x_i\}_{i=1}^M, \{y_j\}_{j=1}^N).
\]

### 2.2 Elliptic Feigin-Odesskii algebra $\mathcal{A}(p)$

The elliptic Feigin-Odesskii algebra is defined in the similar way of the trigonometric case except the emergence of elliptic functions [1].

**Definition 2.7 (Elliptic Feigin-Odesskii algebra $\mathcal{A}(p)$).** Define a $n$-variable function $\varepsilon_n(q, p; x) (n \in \mathbb{Z}_>$) as follows.

\[
\varepsilon_n(q, p; x) := \prod_{1 \leq a < b \leq n} \frac{\Theta_p(q x_a / x_b) \Theta_p(q^{-1} x_a / x_b)}{\Theta_p(x_a / x_b)^2}.
\]

\[
(2.15)
\]

Set a function $\omega_p(x, y)$ as

\[
\omega_p(x, y) := \frac{\Theta_p(q^{-1} y / x) \Theta_p(t y / x) \Theta_p(q t^{-1} y / x)}{\Theta_p(y / x)^3}.
\]

\[
(2.16)
\]

Define the star product $*$ as

\[
(f * g)(x_1, \cdots, x_{m+n}) := \text{Sym} \left[ f(x_1, \cdots, x_m) g(x_{m+1}, \cdots, x_{m+n}) \prod_{1 \leq a \leq m} \omega_p(x_a, x_a) \right].
\]

\[
(2.17)
\]
For a partition \( \lambda \), we set \( \varepsilon_\lambda(q, p; x) \) as

\[
\varepsilon_\lambda(q, p; x) := \varepsilon_{\lambda_1}(q, p; x) \ast \cdots \ast \varepsilon_{\lambda_m}(q, p; x).
\]  

\( (2.18) \)

Set \( \mathcal{A}_0(p) := \mathbb{C} \), \( \mathcal{A}_n(p) := \text{span}\{\varepsilon_\lambda(q, p; x) : |\lambda| = n\} \) \((n \geq 1)\). We define the elliptic Feigin-Odesskii algebra as \( \mathcal{A}(p) := \bigoplus_{n \geq 0} \mathcal{A}_n(p) \) whose algebraic structure is given by the star product \( \ast \).

Similar to the trigonometric case, the following is shown [1].

**Proposition 2.8.** The elliptic Feigin-Odesskii algebra \( (\mathcal{A}(p), \ast) \) is unital, associative, and commutative algebra.

### 2.3 Commutative families \( \mathcal{M}(p), \mathcal{M}'(p) \)

For the operators \( E(p; z), F(p; z) \) which are used in the theorem 2.4, we have the following proposition [3].

**Proposition 2.9.** (1) Operators \( E(p; z), F(p; z) \) satisfy the relation as

\[
E(p; z)E(p; w) = g_p\left(\frac{z}{w}\right)E(p; w)E(p; z),
\]

\( (2.19) \)

\[
F(p; z)F(p; w) = g_p\left(\frac{z}{w}\right)^{-1}F(p; w)F(p; z).
\]

\( (2.20) \)

Due to the relations operator-valued functions as

\[
\prod_{1 \leq i < j \leq N} \omega_p(x_i, x_j)^{-1}E(p; x_1) \cdots E(p; x_N), \quad \prod_{1 \leq i < j \leq N} \omega'_p(x_i, x_j)^{-1}F(p; x_1) \cdots F(p; x_N)
\]

are symmetric in \( x_1, \cdots, x_N \).

(2) Commutator of \( E(p; z) \) and \( F(p; z) \) takes the following form.

\[
[E(p; z), F(p; w)] = \frac{\Theta_p(q)\Theta_p(t^{-1})}{(p; p)_\infty^2} \delta\left(\frac{w}{z}\right) \{\varphi^+(p; \gamma^{1/2}w) - \varphi^+(p; \gamma^{1/2}p^{-1}w)\}.
\]

\( (2.21) \)

From the relation \( (2.21) \) we have \([E(p; z)]_1, [F(p; w)]_1 = 0\). This corresponds to the commutativity of the elliptic Macdonald operators \([H_N(q, t), H_N(q^{-1}, t^{-1}, p)] = 0\).

**Definition 2.10 (Map \( \mathcal{O}_p \)).** We define a linear map \( \mathcal{O}_p : \mathcal{A}(p) \rightarrow \text{End}(\mathcal{F}_\alpha) \) \((\alpha \in \mathbb{C})\) as follows.

\[
\mathcal{O}_p(f) := \left[ f(z_1, \cdots, z_n) \prod_{1 \leq i < j \leq n} \omega_p(z_i, z_j)^{-1}E(p; z_1) \cdots E(p; z_n) \right]_1 \quad (f \in \mathcal{A}_n(p)).
\]

\( (2.22) \)

Here \([f(z_1, \cdots, z_n)]_1\) denotes the constant term of \( f(z_1, \cdots, z_n) \) in \( z_1, \cdots, z_n \). We extend the map linearly.

In the similar way of the trigonometric case, we can check the following.

**Proposition 2.11.** The map \( \mathcal{O}_p \) and the star product \( \ast \) are compatible : for \( f, g \in \mathcal{A}(p) \), we have \( \mathcal{O}_p(f \ast g) = \mathcal{O}_p(f)\mathcal{O}_p(g) \).
Lemma 2.14. Set another star product

\[ (14) \] Elliptic Case

We define a linear map

\[ O \]

We extend the map linearly.

In the elliptic Feigin-Odesskii algebra

Theorem 2.15 (Elliptic Macdonald operator

H_\[ \alpha \]

A commutative family of the elliptic Macdonald operator

H_\[ \alpha \]

Assume that a theorem 2.16 in a direct way. For the proof we prepare the following lemma.

Definition 2.13 (Map \( O'_p \)). Set a function \( \omega'_p(x, y) \) as

\[ \omega'_p(x, y) := \frac{\Theta_p(qy/x)\Theta_p(t^{-1}y/x)\Theta_p(q^{-1}ty/x)}{\Theta_p(y/x)^3}. \] (2.23)

We define a linear map \( O'_p : A(p) \to \text{End}(\mathcal{F}_\alpha) (\alpha \in \mathbb{C}) \) as follows.

\[ O'_p(f) := \left[ f(z_1, \cdots, z_n) \prod_{1 \leq i < j \leq n} \omega'_p(z_i, z_j)^{-1}F(p; z_1) \cdots F(p; z_n) \right] (f \in A_n(p)). \] (2.24)

We extend the map linearly.

In the same way of the trigonometric case, we have the following lemma.

Lemma 2.14. Set another star product \( *' \) as

\[ (f *' g)(x_1, \cdots, x_{m+n}) := \text{Sym}\left[ f(x_1, \cdots, x_m)g(x_{m+1}, \cdots, x_{m+n}) \prod_{1 \leq \alpha \leq m, m+1 \leq \beta \leq m+n} \omega'_p(x_\alpha, x_\beta) \right]. \] (2.25)

In the elliptic Feigin-Odesskii algebra \( A(p) \), we have \( *' = * \).

Theorem 2.15 (Commutative family \( \mathcal{M}'(p) \)). (1) Set \( \mathcal{M}'(p) := O'_p(A(p)) \). The space is a commutative algebra of boson operators.

(2) The space \( \mathcal{M}'(p)[\phi_{\mathcal{N}(p,x)}]N \) consists of commuting elliptic \( q \)-difference operators which contains the elliptic Macdonald operator \( H_N(q^{-1}, t^{-1}, p) \) (commutative family of the elliptic Macdonald operator \( H_N(q^{-1}, t^{-1}, p) \)).

Similar to the proposition 1.15, we can show that the commutative families \( \mathcal{M}(p) \), \( \mathcal{M}'(p) \) commute with each other.

Theorem 2.16. For the commutative families \( \mathcal{M}(p) \), \( \mathcal{M}'(p) \), we have the commutativity \( [ \mathcal{M}(p), \mathcal{M}'(p) ] = 0 \).

The theorem 2.16 is the elliptic analog of the proposition 1.15. But we can’t prove the theorem 2.16 in the similar way of the proof of the proposition 1.15, because we don’t have an elliptic analog of the Macdonald symmetric functions. Hence we will show the theorem 2.16 in a direct way. For the proof we prepare the following lemma.

Lemma 2.17. Assume that a \( r \)-variable function \( A(x_1, \cdots, x_r) \) and a \( s \)-variable function \( B(x_1, \cdots, x_s) \) have a period \( p \), i.e. \( T_{p,x_i}A(x_1, \cdots, x_r) = A(x_1, \cdots, x_r) (1 \leq i \leq r), \)

\[ T_{p,x_i}B(x_1, \cdots, x_s) = B(x_1, \cdots, x_s) (1 \leq i \leq s). \]

Then we have

\[ [[A(z_1, \cdots, z_r)E(p; z_1) \cdots E(p; z_r)], [B(w_1, \cdots, w_s)F(p; w_1) \cdots F(p; w_s)]] = 0. \] (2.26)
2.3 Commutative families \( \mathcal{M}(p), \mathcal{M}'(p) \)

**Proof.** First we have

\[
[A_1 \cdots A_r, B_1 \cdots B_s] = \sum_{i=1}^{r} \sum_{j=1}^{s} A_1 \cdots A_{i-1} B_1 \cdots B_{j-1} [A_i, B_j] B_{j+1} \cdots B_s A_{i+1} \cdots A_r.
\]  

(2.27)

Set \( c(q, t, p) := \Theta_p(q)\Theta_p(t^{-1})/(p/p) \Theta_p(q t^{-1}) \) and we denote the \( p \)-difference of \( f(z) \) by \( \Delta_p f(z) := f(pz) - f(z) \). By the equation (2.27) and the proposition 2.9 (2) (2.21), we have the following.

\[
[A(z_1, \cdots, z_r) E(p; z_1) \cdots E(p; z_r), B(w_1, \cdots, w_s) F(p; w_1) \cdots F(p; w_s)]
\]

\[
= \sum_{i=1}^{r} \sum_{j=1}^{s} A(z_1, \cdots, z_r) B(w_1, \cdots, w_s) E(p; z_1) \cdots E(p; z_{i-1}) F(p; w_1) \cdots F(p; w_{j-1})
\]

\[
\times [E(p; z_i), F(p; w_j)] F(p; w_{j+1}) \cdots F(p; w_s) E(p; z_{i+1}) \cdots E(p; z_r)
\]

\[
= c(q, t, p) \sum_{i=1}^{r} \sum_{j=1}^{s} E(p; z_1) \cdots E(p; z_{i-1}) F(p; w_1) \cdots F(p; w_{j-1})
\]

\[
\times A(z_1, \cdots, \gamma w_j, \cdots, z_r) B(w_1, \cdots, w_j, \cdots, w_s) \delta \left( \frac{w_j}{z_i} \right) \Delta_p \varphi^+ (p; \gamma^{1/p-1} w_j)
\]

\[
\times F(p; w_{j+1}) \cdots F(p; w_s) E(p; z_{i+1}) \cdots E(p; z_r).
\]  

(2.28)

By picking up the constant term of \( z_i, w_j \) dependence part of (2.28), we have

\[
\left[ A(z_1, \cdots, \gamma w_j, \cdots, z_r) B(w_1, \cdots, w_{i-1}, w_j, \cdots, w_s) \delta \left( \frac{w_j}{z_i} \right) \Delta_p \varphi^+ (p; \gamma^{1/p-1} w_j) \right]_{z_i, w_j, 1}
\]

\[
= \left[ A(z_1, \cdots, \gamma w_j, \cdots, z_r) B(w_1, \cdots, w_{i-1}, w_j, \cdots, w_s) \Delta_p \varphi^+ (p; \gamma^{1/p-1} w_j) \right]_{w_j, 1}
\]

\[
= \left[ A(z_1, \cdots, \gamma w_j, \cdots, z_r) B(w_1, \cdots, w_{i-1}, w_j, \cdots, w_s) \varphi^+ (p; \gamma^{1/p} w_j) \right]_{w_j, 1}
\]

\[
- \left[ A(z_1, \cdots, \gamma w_j, \cdots, z_r) B(w_1, \cdots, w_{i-1}, w_j, \cdots, w_s) \varphi^+ (p; \gamma^{1/p} w_j) \right]_{w_j, 1}.
\]  

(2.29)

We recall \( [f(z)]_1 = [f(az)]_1 \) \((a \in \mathbb{C})\) and the functions \( A(z_1, \cdots, z_r) \) and \( B(w_1, \cdots, w_s) \) have a period \( p \). Hence we have

\[
\left[ A(z_1, \cdots, \gamma w_j, \cdots, z_r) B(w_1, \cdots, \gamma w_j, \cdots, w_s) \delta \left( \frac{w_j}{z_i} \right) \Delta_p \varphi^+ (p; \gamma^{1/p-1} w_j) \right]_{z_i, w_j, 1} = 0.
\]

The equation holds for any \( i, j \), hence we have the lemma 2.17. \( \square \)

**Proof of the theorem 2.16.** For the proof what we have to show is

\[
[\mathcal{O}_p(\varepsilon_r(q, p; z)), \mathcal{O}'_p(\varepsilon_s(q, p; w))] = 0 \quad (r, s \in \mathbb{Z}_{>0}).
\]
By the definition of \( O_p, O'_p \), operators \( O_p(\varepsilon_r(q, p; z)) \), \( O'_p(\varepsilon_s(q, p; w)) \) are the constant terms of the following operators.

\[
\varepsilon_r(q, p; z) \prod_{1 \leq i < j \leq r} \omega_p(z_i, z_j)^{-1} E(p; z_1) \cdots E(p; z_r), \tag{2.30}
\]

\[
\varepsilon_s(q, p; w) \prod_{1 \leq i < j \leq s} \omega'_p(w_i, w_j)^{-1} F(p; w_1) \cdots F(p; w_s). \tag{2.31}
\]

Then their functional parts take the following forms.

Functional part of (2.30)

\[
\varepsilon_r(q, p; z) \prod_{1 \leq i < j \leq r} \omega_p(z_i, z_j)^{-1} = \prod_{1 \leq i < j \leq r} \frac{\Theta_p(z_i/z_j)\Theta_p(q^{-1}z_i/z_j)}{\Theta_p(t^{-1}z_i/z_j)\Theta_p(q^{-1}t z_i/z_j)}, \tag{2.32}
\]

Functional part of (2.31)

\[
\varepsilon_s(q, p; w) \prod_{1 \leq i < j \leq s} \omega'_p(w_i, w_j)^{-1} = \prod_{1 \leq i < j \leq s} \frac{\Theta_p(w_i/w_j)\Theta_p(q w_i/w_j)}{\Theta_p(t w_i/w_j)\Theta_p(q t^{-1} w_i/w_j)}. \tag{2.33}
\]

We can check (2.32), (2.33) have a period \( p \). By the lemma 2.17, we have the theorem 2.16. \( \square \)

By the theorem 2.16, commutative families \( M(p)|_{C_\phi(p;x)|N}, M'(p)|_{C_\phi(p;x)|N} \) also commute with each other : \([M(p)|_{C_\phi(p;x)|N}, M'(p)|_{C_\phi(p;x)|N}] = 0\).

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3 Appendix

3.1 Trigonometric kernel function and its functional equation

By Komori, Noumi, and Shiraishi [2], the following theorem is shown.

**Theorem 3.1** ([2]). Define the Macdonald operator \( H_N(q, t) \) \((N \in \mathbb{Z}_{>0})\) as

\[
H_N(q, t) := \sum_{i=1}^{N} \prod_{j \neq i} \frac{t x_i - x_j}{x_i - x_j} T_{q, x_i} \quad (T_{q, x} f(x) := f(qx)) \tag{3.1}
\]

and its kernel function \( \Pi(q, t)(\{x_i\}_{i=1}^{M}, \{y_j\}_{j=1}^{N}) \) \((M, N \in \mathbb{Z}_{>0})\) as

\[
\Pi(q, t)(\{x_i\}_{i=1}^{M}, \{y_j\}_{j=1}^{N}) := \prod_{1 \leq i \leq M, 1 \leq j \leq N} \frac{(tx_i y_j; q)_{\infty}}{(x_i y_j; q)_{\infty}}. \tag{3.2}
\]

Then we have the following functional equation.

\[
\{H_M(q, t)x - t^{M-N}H_N(q, t)y\} \Pi(q, t)(\{x_i\}_{i=1}^{M}, \{y_j\}_{j=1}^{N}) = 1 - t^{M-N} \Pi(q, t)(\{x_i\}_{i=1}^{M}, \{y_j\}_{j=1}^{N}) \tag{3.3}
\]

Here we denote the Macdonald operator which acts on functions of \(x_1, \cdots, x_M\) by \(H_M(q, t)x\).

In the following, we will show the elliptic analog of the theorem 3.1 by the free field realization of the elliptic Macdonald operator.
3.2 Recollection : free field realization of the elliptic Macdonald operator

The elliptic Macdonald operator $H_N(q, t, p)$ ($N \in \mathbb{Z}_{>0}$) is defined as

$$H_N(q, t, p) := \sum_{i=1}^{N} \prod_{j \neq i} \frac{\Theta_p(tx_i/x_j)}{\Theta_p(x_i/x_j)} T_{q,x_i}.$$  (3.4)

First we review the free field realization of the elliptic Macdonald operator. In the following we use the notations in section 2.1.

**Theorem 3.2 (Free field realization of the elliptic Macdonald operator).** Let $\phi(p; z) : \mathcal{F} \to \mathcal{F} \otimes \mathbb{C}[[z, z^{-1}]]$ be an operator defined as follows.

$$\phi(p; z) := \exp \left( \sum_{n>0} \frac{(1-t^n)(qt^{-1}p)^n}{(1-q^n)(1-p^n)} b_n z^n \right) \exp \left( \sum_{n>0} \frac{1-t^n}{(1-q^n)(1-p^n)} a_n z^n \right).$$  (3.5)

We use the symbol $\phi_N(p; x) := \prod_{j=1}^{N} \phi(p; x_j)$.

The elliptic Macdonald operator $H_N(q, t, p)$ is reproduced by the operator $\eta(p; z)$ as follows.

$$[\eta(p; z) - t^{-N} (\eta(p; z)) - (\eta(p; p^{-1}z))]_1 \phi_N(p; x)|0\rangle = \frac{t^{-N+1} \Theta_p(t^{-1})}{(p;p)_\infty^3} H_N(q, t, p) \phi_N(p; x)|0\rangle.$$  (3.6)

**Theorem 3.3 (Dual version of the theorem 3.2).** Let $\phi^*(p; z) : \mathcal{F}^* \to \mathcal{F}^* \otimes \mathbb{C}[[z, z^{-1}]]$ be an operator defined as follows.

$$\phi^*(p; z) := \exp \left( \sum_{n>0} \frac{(1-t^n)(qt^{-1}p)^n}{(1-q^n)(1-p^n)} b_n z^n \right) \exp \left( \sum_{n>0} \frac{1-t^n}{(1-q^n)(1-p^n)} a_n z^n \right).$$  (3.7)

We use the symbol $\phi^*_N(p; x) := \prod_{j=1}^{N} \phi^*(p; x_j)$.

The elliptic Macdonald operator $H_N(q, t, p)$ is reproduced by the operator $\eta(p; z)$ as follows.

$$\langle 0|\phi^*_N(p; x)[\eta(p; z) - t^{-N} (\eta(p; z)) - (\eta(p; p^{-1}z))]_1|0\rangle \phi^*_N(p; x).$$  (3.8)

3.3 Elliptic kernel function and its functional equation

For a partition $\lambda$, set $n_\lambda(a) := \sharp \{i : \lambda_i = a\}$, $z_\lambda := \prod_{a>1} a^{n_{\lambda}(a)} n_\lambda(a)!$ and define $z_\lambda(q, t, p)$, $z_\lambda(q, t, p)$ by

$$z_\lambda(q, t, p) := z_\lambda \prod_{i=1}^{\ell(\lambda)} (1 - p^{\lambda_i}) \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}}, \quad \overline{z}_\lambda(q, t, p) := \overline{z}_\lambda \prod_{i=1}^{\ell(\lambda)} (qt^{-1}p)^{\lambda_i} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}}.$$  

We define a bilinear form $\langle \bullet | \bullet \rangle : \mathcal{F}^* \times \mathcal{F} \to \mathbb{C}$ by the following conditions.

$$\langle 0|0\rangle = 1, \quad \langle 0|a_{\lambda_1} b_{\lambda_2} a_{-\mu_1} b_{-\mu_2}|0\rangle = \delta_{\lambda_1 \mu_1} \delta_{\lambda_2 \mu_2} z_\lambda(q, t, p) \overline{z}_\lambda(q, t, p).$$

By the free field realization, we can show the following theorem.
Theorem 3.4 (Functional equation of the elliptic kernel function). Define the elliptic kernel function \( \Pi(q, t, p)(\{x_i\}^M_{i=1}, \{y_j\}^N_{j=1}) \) by

\[
\Pi(q, t, p)(\{x_i\}^M_{i=1}, \{y_j\}^N_{j=1}) := \prod_{1 \leq i \leq M \atop 1 \leq j \leq N} \frac{\Gamma_{q, p}(x_i y_j)}{\Gamma_{q, p}(t x_i y_j)}.
\] (3.9)

We also define \( C_{MN}(p; x, y) \) as

\[
C_{MN}(p; x, y) := \frac{\langle 0| \phi^*_M(p; x)[(\eta(p; z))_+ (\eta(p; p^{-1}z))_+) \phi_N(p; y) |0 \rangle}{\Pi(q, t, p)(\{x_i\}^M_{i=1}, \{y_j\}^N_{j=1})} = \left[ \prod_{i=1}^{M} \frac{\Theta_p(t^{-1}x_i z)}{\Theta_p(x_i z)} \prod_{j=1}^{N} \frac{\Theta_p(z/y_j)}{\Theta_p(t^{-1}z/y_j)} \right]_{1}.
\] (3.10)

For the elliptic Macdonald operator and the function \( \Pi(q, t, p)(\{x_i\}^M_{i=1}, \{y_j\}^N_{j=1}) \), we have the following functional equation.

\[
\{H_M(q, t, p)_x - t^{M-N} H_N(q, t, p)_y\} \Pi(q, t, p)(\{x_i\}^M_{i=1}, \{y_j\}^N_{j=1}) = \frac{(1 - t^{M-N})(p; p)^3}{\Theta_p(t)} C_{MN}(p; x, y) \Pi(q, t, p)(\{x_i\}^M_{i=1}, \{y_j\}^N_{j=1}).
\] (3.11)

**Proof.** The proof is straightforward. What we have to do is to calculate the matrix element \( \langle 0| \phi^*_M(p; x)[(\eta(p; z))_+ \phi_N(p; y) |0 \rangle \) by the theorem 3.2, 3.3 in different two ways as follows :

\[
\langle 0| \phi^*_M(p; x)[(\eta(p; z))_+ \phi_N(p; y) |0 \rangle = \frac{t^{-M+1} \Theta_p(t^{-1})}{(p; p)^3} H_M(q, t, p)_x \Pi(q, t, p)(\{x_i\}^M_{i=1}, \{y_j\}^N_{j=1})
\]

\[+ t^{-M} C_{MN}(p; x, y) \Pi(q, t, p)(\{x_i\}^M_{i=1}, \{y_j\}^N_{j=1})
\]

\[= \frac{t^{-N+1} \Theta_p(t^{-1})}{(p; p)^3} H_N(q, t, p)_y \Pi(q, t, p)(\{x_i\}^M_{i=1}, \{y_j\}^N_{j=1})
\]

\[+ t^{-N} C_{MN}(p; x, y) \Pi(q, t, p)(\{x_i\}^M_{i=1}, \{y_j\}^N_{j=1}).
\]

Therefore we obtain the theorem 3.4. \( \Box \)

**Remark 3.5.** We can check the following :

\[
C_{MN}(p; x, y) = \left[ \prod_{i=1}^{M} \frac{\Theta_p(t^{-1}x_i z)}{\Theta_p(x_i z)} \prod_{j=1}^{N} \frac{\Theta_p(z/y_j)}{\Theta_p(t^{-1}z/y_j)} \right]_{1}
\]

\[\longrightarrow_{p \to 0} \left[ \prod_{i=1}^{M} \frac{1 - t^{-1}x_i z}{1 - x_i z} \prod_{j=1}^{N} \frac{1 - z/y_j}{1 - t^{-1}z/y_j} \right]_{1} = 1.
\]

Hence by taking the limit \( p \to 0 \) the equation (3.11) reduces to the equation (3.3).
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