MAXIMUM PRINCIPLE FOR SPDES AND ITS APPLICATIONS

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ABSTRACT. The maximum principle for SPDEs is established in multi-dimensional \( C^1 \) domains. An application is given to proving the Hölder continuity up to the boundary of solutions of one-dimensional SPDEs.

The maximum principle is one of the most powerful tools in the theory of second-order elliptic and parabolic partial differential equations. However, until now it did not play any significant role in the theory of SPDEs. In this paper we show how to apply it to one-dimensional SPDEs on the half line \( \mathbb{R}_+ = (0, \infty) \) and prove the Hölder continuity of solutions on \([0, \infty)\). This result was previously known when the coefficients of the first order derivatives of solution appearing in the stochastic term in the equation obeys a quite unpleasant condition. On the other hand, if they just vanish, then the Hölder continuity was well known before (see, for instance, [6] and the references therein).

To the best of our knowledge the maximum principle was first proved in [12] (see also [14] for the case of random coefficients) for SPDEs in the whole space by the method of random characteristics introduced there and also in [15]. Later the method of random characteristics was used in many papers for various purposes, for instance, to prove smoothness of solutions (see, for instance, [1], [2], [3], [17] and the references therein). It was very tempting to try to use this method for proving the maximum principle for SPDEs in domains. However, the implementation of the method turns out to become extremely cumbersome and inconvenient if the coefficients of the equation are random processes. Also, it requires more regularity of solutions than actually needed.

Here in Section 1 we state the maximum principle in domains under minimal assumptions. We prove it in Section 3 by using methods taken from PDEs after we prepare some auxiliary results in Section 2.

Section 5 contains an application of the maximum principle to investigating the Hölder continuity up to the boundary of solutions of one-dimensional SPDEs. Note that, for instance, in [1], [2] and in many other papers that can be found from our list of references the regularity properties are proved.
only inside domains. Quite sharp regularity for solutions of SPDEs in multi-
dimensional domains is established in [5], it is stated in terms of appropriate
weighted Sobolev spaces and, unfortunately, do not imply even the pointwise
continuity up to the boundary. It is worth saying that we only deal with
one-dimensional case and coefficients independent of the space variable. In
a subsequent paper we intend to treat the general case. In Section 4 we
introduce some auxiliary functions used in Section 5.

We denote by $\mathbb{R}^d$ the Euclidean space of points $x = (x^1, ..., x^d)$,
$$D_i = \frac{\partial}{\partial x^i}.$$ 

For a domain $D \subset \mathbb{R}^d$ and we set $W^1_2(D)$ to be the closure of the set of
infinitely differentiable functions $\phi$ having finite norm
$$\|\phi\|_{W^1_2(D)}^2 = \|\phi\|_{L^2(D)}^2 + \|\phi_x\|_{L^2(D)}^2$$
with respect to this norm. Here $\phi_x$ is the gradient of $\phi$. By $W^1_2(\partial D)$ we
denote the closure of $C^\infty_0(D)$ with respect to the norm $\|\cdot\|_{W^1_2(D)}$. Our
way to say that $u \leq v$ on $\partial D$ is that $(u - v)^+ \in W^1_2(D)$. As usual, the
summation convention is enforced and writing $N(...)$ is to say that the
constant $N$ depends and depends only on the contents of the parentheses.
Such constants may change from line to line.

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1. The maximum principle

Let $D$ be a domain in $\mathbb{R}^d$ of class $C^1_{loc}$ and let $(\Omega, \mathcal{F}, P)$ be a complete
probability space with a given filtration $(\mathcal{F}_t, t \geq 0)$ of $\sigma$-fields $\mathcal{F}_t \subset \mathcal{F}$
complete with respect to $\mathcal{F}, P$.

We are investigating some properties of a function $u_t(x) = u_t(\omega, x)$ satisfying

$$\begin{align*}
(\phi, u_t) &= (\phi, u_0) + \int_0^t (\phi, \sigma^i_s D_i u_s + \nu^k_s u_s + g^k_s) \, dm^k_s \\
&\quad + \int_0^t (\phi, D_i(a^i_s D_j u_s) + b^i_s D_i u_s + D_i(a^i_s u_s) - c_s u_s + f_s + D_i f^i_s) \, dV_s.
\end{align*} \tag{1.1}$$

for all $t \in [0, \infty)$ and any $\phi \in C^\infty_0(D)$. Here $m^k_t$, $k = 1, 2, ..., $ are one-
dimensional continuous local $\mathcal{F}_t$-martingales, starting at zero, $V_t$ is a nonde-
creasing continuous $\mathcal{F}_t$-adapted process starting at zero, $(\phi, \cdot)$ is the pairing
between a generalized function on $D$ and a test function $\phi$, the summation
convention over repeated indices is enforced, and the meaning of the
remaining objects and further assumptions are described below. We need
some real-valued functions $\xi_i(x)$, $K_1(t) > 0$, and $K_2(t) \geq 0$ defined for
$i = 1, ..., d$, $t \in [0, \infty)$, $x \in \mathbb{R}^d$ and also depending on $\omega$. 
We assume that $a^i_j(x), b^i_j(x), a^i_t(x), c_t(x), \sigma^i_k(x), \nu^k_t(x), \text{and } g^k_t$ are real-valued functions defined for $i, j = 1, \ldots, d, k = 1, 2, \ldots, t \in [0, \infty), x \in \mathbb{R}^d$

and also depending on $\omega \in \Omega$.

**Assumption 1.1.** We suppose that, for any $\omega$, $\langle m^i, m^j \rangle_t = 0$ if $i \neq j$, and for any $k$ we have $d(m^k)_t \leq dV_t$.

**Assumption 1.2.** For all values of the arguments

(i) $\sigma^i := (\sigma^{i1}, \sigma^{i2}, \ldots), \nu := (\nu^1, \nu^2, \ldots), g := (g^1, g^2, \ldots) \in \ell_2$;

(ii) for all $\lambda \in \mathbb{R}^d$

\[
\left| \sum_i \lambda^i \xi^i \right|^2 \leq K_1(2a^{ij} - \alpha^{ij})\lambda^i \lambda^j,
\]

where $\alpha^{ij} = (\sigma^i, \sigma^j)_{\ell_2}$.

The case $\xi \equiv 0$ is not excluded and in this case Assumption 1.2(ii) is just the usual parabolicity assumption.

**Assumption 1.3.** (i) The functions $a^i_j(x), b^i_j(x), a^i_t(x), c_t(x), \sigma^i_k(x), \nu^k_t(x)$, $\xi^j_t(x)$, $\bar{K}_1(t)$, and $\bar{K}_2(t)$ are measurable with respect to $(\omega, t, x)$ and $\mathcal{F}_t$-adapted for each $x$;

(ii) the functions $a^i_j(x), b^i_j(x), a^i_t(x), c_t(x), \sigma^i_k(x), \nu^k_t(x)$, and $\xi^j_t(x)$ are bounded;

(iii) for each $\omega, t$ the functions

\[
\eta^i_t := a^i_t - b^i_t - (\sigma^i, \nu_t)_{\ell_2} - \xi^j_t
\]

are once continuously differentiable on $D$, have bounded derivatives, and satisfy

\[
D_i \eta^i_t - 2c + |\nu_{ij}^2|_2 \leq K_2 \tag{1.2}
\]

for all values of arguments;

(iv) for each $\phi \in C_0^\infty(D)$ the processes $\phi f_t, \phi f^i_1, \ldots, \phi f^d_t$ are $L_2(D)$-valued and $\phi g_t$ is an $L_2(D, \ell_2)$-valued $\mathcal{F}_t$-adapted and jointly measurable; for all $t \in [0, \infty)$ and $\omega \in \Omega$

\[
\int_0^t (\|\phi f_s\|^2_{L_2(D)} + \sum_i \|\phi f^i_t\|^2_{L_2(D)} + \|\phi g_t\|^2_{L_2(D, \ell_2)} + K_1(s) + K_2(s)) dV_s < \infty.
\]

**Assumption 1.4.** For each $\phi \in C_0^\infty(D)$

(i) the process $\phi u_t = \phi u_t(\omega)$ is $L_2(D)$-valued, $\mathcal{F}_t$-adapted, and jointly measurable;

(ii) for any $\omega$

\[
\phi u_t \in W^1_2(D) \quad (dV_t \text{-a.e.});
\]

(iii) for each $t \in [0, \infty)$ and $\omega$

\[
\int_0^t \|\phi u_s\|^2_{W^1_2(D)} dV_s < \infty.
\]
The above assumptions are supposed to hold throughout this section. Here is the maximum principle saying, in particular, that if \( g^k = f^i = 0, \ f \leq 0 \) and \( u \leq 0 \) on the parabolic boundary of \([0,T] \times D\), then \( u \leq 0 \) in \([0,T]\). By the way, our solutions are \( L_{2,\text{loc}}(D)\)-valued functions of \( \omega \) and \( t \), so that for each \( \omega \) and \( t \) an equivalence class is specified. Naturally, if we write \( u_t(\omega) \leq 0 \), or \( u_t \leq 0 \) we mean that in the corresponding class there is a nonpositive function.

**Theorem 1.1.** Let \( \tau_2 \geq \tau_1 \) be stopping times, \( \tau_1 < \infty \) for any \( \omega \). Suppose that, for any \( \omega \), \( i = 1, \ldots, d, \ k = 1, 2, \ldots, \)

\[
I_{u_t>0} g_t^k = I_{u_t>0} f_t^i = 0, \quad u_t^+ \in W^1_{2}(D), \quad I_{u_t>0} f_t \leq 0
\]
d\( V_t \)-almost everywhere on \((\tau_1, \tau_2)\) and suppose that \( u_{\tau_1} \leq 0 \) for any \( \omega \). Then almost surely \( u_t \leq 0 \) for all \( t \in [\tau_1, \tau_2] \cap [\tau_1, \infty) \).

The following comparison principle is a generalization of Theorem 1.1.

**Theorem 1.2.** Let \( \tau_2 \geq \tau_1 \) be stopping times, \( \tau_1 < \infty \) for any \( \omega \). Let \( \rho_i \geq 0, \ t \in [0, \infty) \), be a nondecreasing continuous \( \mathcal{F}_t\)-adapted process and let \( f_t, \ f_t^1, \ldots, f_t^d, \) and \( \bar{g}_t \) satisfy Assumption 1.3 (iv). Let \( \bar{u}_t \) be a process satisfying Assumption 1.4 and such that equation (1.1) holds for all \( t \in [0, \infty) \) and any \( \phi \in C^\infty_0(D) \) with \( f_t, \ f_t^1, \ldots, f_t^d, \) and \( \bar{g}_t \) in place of \( f_t, f_t^1, \ldots, f_t^d, \) and \( g_t \), respectively.

Assume that, for any \( \omega \), (d\( V_t \)-a.e.) on \([\tau_1, \tau_2]\) we have

\[
I_{\rho_t \bar{u}_t} (a_t - \rho_t \bar{g}_t) = I_{\rho_t \bar{u}_t} (f_t^i - \rho_t \bar{f}_t^i) = 0, \quad i = 1, \ldots, d,
\]

\[
I_{\rho_t \bar{u}_t} (b_t - \rho_t \bar{f}_t) \leq 0, \quad I_{\rho_t \bar{u}_t} (\bar{u}_t - \rho_t \bar{u}_t)^+ \in W^1_2(D).
\]

Finally, assume that \( u_{\tau_1} \leq \rho \bar{u}_{\tau_1} \) for any \( \omega \).

Then almost surely \( u_t \leq \rho \bar{u}_t \) for all \( t \in [\tau_1, \tau_2] \cap [\tau_1, \infty) \).

**Corollary 1.3.** Assume that, for any \( \omega \), (d\( V_t \)-a.e.) on \((\tau_1, \tau_2) \times D\) we have

\[
I_{u_t>1} (\nu_t^k + g_t^k) = I_{u_t>1} (f_t^i + a_t^i) = 0, \quad i = 1, \ldots, d, \ k = 1, 2, \ldots
\]

\[
I_{u_t>1} f \leq I_{u_t>1} c, \quad (u_t - 1)^+ \in W^1_2(D).
\]

Also assume that \( u_{\tau_1} \leq 1 \) for any \( \omega \). Then almost surely \( u_t \leq 1 \) for all \( t \in [\tau_1, \tau_2] \cap [\tau_1, \infty) \).

Indeed, it suffices to take \( \bar{u}_t \equiv 1, \ \rho_t \equiv 1 \) and observe that \( \bar{u}_t \) satisfies 1.11 with \( \bar{f}_t^i = -a_t^i, \ \bar{f} = c, \) and \( \bar{g} = -\nu_t \) in place of \( f_t^i, f_t, \) and \( g_t \), respectively.

This corollary generalizes the corresponding results of [12] and [14], where \( \nu^k = g^k = f^i = a^i = 0 \).

**Remark 1.4.** Our equation has a special structure, which may look quite restrictive. In particular, we assume that the martingales \( m_t^k \) are mutually orthogonal. The general case, actually, reduces to this particular one after using the fact that one can always orthogonalize the martingales by using, for instance, the Gramm-Schmidt procedure. This, of course, would change
\( \sigma, \nu, \) and \( g, \) and writing the corresponding general conditions would only obscure the matter. Then passing from \( m^k_t \) to \( m^k_t \) to \( m^k_t \)(no summation in \( k \))

\[
\int_0^t \rho_s^k \, dm_s^k, \quad \rho_s^k = \left( \frac{dt}{dt + d(m^k)_t} \right)^{1/2}
\]

allows one to have \( d(m^k)_t \leq dt \) and adding after that \( t \) to \( V_t \) allows one to have \( d(m^k)_t \leq dV_t. \) Again we should modify our coefficients but we will see in the proof of Theorem 1.2 that this modification does not affect Assumption 1.2 which is an assumption about parabolicity of our equation and not strict nondegeneracy.

2. Auxiliary results

In this section the notation \( u_t \) is sometimes used for different objects than in Section 1.

Denote by \( \mathcal{R} \) the set of real-valued functions convex \( r(x) \) on \( \mathbb{R} \) such that

(i) \( r \) is continuously differentiable, \( r(0) = r'(0) = 0, \)
(ii) \( r' \) is absolutely continuous, its derivative \( r'' \) is bounded and left continuous, that is usual \( r'' \) which exists almost everywhere is bounded and there is a left-continuous function with which \( r'' \) coincides almost everywhere.

For \( r \in \mathcal{R} \) by \( r'' \) we will always mean the left-continuous modification of the usual second-order derivative of \( r. \)

**Remark 2.1.** For each \( r \in \mathcal{R} \) there exists a sequence \( r_n \in \mathcal{R} \) of infinitely differentiable functions such that \( |r_n(x)| \leq N|x|^2, \) \( |r_n'(x)| \leq N|x|, \) and \( |r''_n| \leq N \) with \( N < \infty \) independent of \( x \) and \( n, \) \( r_n, r_n', r_n'' \to r, r', r'' \) on \( \mathbb{R}. \)

Indeed, let \( \zeta \in C^\infty_0(\mathbb{R}) \) be a nonnegative function with support in \( (0,1) \) and unit integral. For \( \varepsilon > 0 \) define \( \zeta_\varepsilon(x) = \varepsilon^{-1} \zeta(x/\varepsilon) \) and \( r_\varepsilon(x) = r \ast \zeta_\varepsilon(x) - r \ast \zeta_\varepsilon(0) - x r' \ast \zeta_\varepsilon(0). \) Then \( r_\varepsilon \) is infinitely differentiable, \( r_\varepsilon(0) = r'_\varepsilon(0) = 0, \)

\[
|r''_\varepsilon| = |r'' \ast \zeta_\varepsilon| \leq \sup |r''| < \infty.
\]

In particular,

\[
|r'_\varepsilon(x)| = \left| \int_0^x r''_\varepsilon(y) \, dy \right| \leq N|x|, \quad |r_\varepsilon(x)| = \left| \int_0^x r'_\varepsilon(y) \, dy \right| \leq N|x|^2.
\]

Finally, the convergences \( r_\varepsilon \to r \) and \( r'_\varepsilon \to r' \) follow by the continuity of \( r \) and \( r' \) and the convergence \( r''_\varepsilon \to r'' \) follows from the dominated convergence theorem, the left continuity of \( r'' \) and the formula

\[
r''_\varepsilon(x) = \int_0^1 r''(x - \varepsilon y) \zeta(y) \, dy.
\]

In the following lemma the assumption that \( D \) is a locally smooth domain is not used.

**Lemma 2.2.** Let \( u_t = u_t(\omega) \) be an \( L^2(D) \)-valued process such that \( u_0 \) is \( \mathcal{F}_0 \)-measurable. Let \( f_t \) and \( g_t = (g^1_t, g^2_t, \ldots) \) be \( \mathcal{F}_t \)-adapted and jointly measurable
processes with values \(L_2(D)\) and \(L_2(D, \ell_2)\), respectively. Assume that for each \(t \in [0, \infty)\) we have
\[
\int_0^t (\|f_s\|_{L_2(D)}^2 + \|g_s\|_{L_2(D, \ell_2)}^2) \, dV_s < \infty \quad (2.1)
\]
and for any \(\phi \in C_0^\infty(D)\)
\[
(\phi, u_t)_{L_2(D)} = (\phi, u_0)_{L_2(D)} + \int_0^t (\phi, f_s)_{L_2(D)} \, dV_s + \int_0^t (\phi, g^k_s)_{L_2(D)} \, dm^k_s. \quad (2.2)
\]
Then (i) \(u_t\) is a continuous \(L_2(D)\)-valued function (a.s.); (ii) for any \(r \in \mathcal{R}\) (a.s.) for all \(t \in [0, \infty)\)
\[
\|r^{1/2}(u_t)\|_{L_2(D)}^2 = \|r^{1/2}(u_0)\|_{L_2(D)}^2 + \int_0^t h_s \, dV_s + m_t, \quad (2.3)
\]
where
\[
h_s := (r'(u_s), f_s)_{L_2(D)} + (1/2)\|r''(u_s)\|_{L_2(D, \ell_2)}^2,
\]
\[
g^k_s := \left(\frac{d(m^k_s)}{dV_s}\right)^{1/2} g^k_s, \quad m_t := \int_0^t (r'(u_s), g^k_s)_{L_2(D)} \, dm^k_s
\]
and \(m_t\) is a local martingale;
(iii) (a.s.) for \(t \in [0, \infty)\)
\[
\|u^+_t\|_{L_2(D)}^2 = \|u^+_0\|_{L_2(D)}^2 + \int_0^t h_s \, dV_s + m_t, \quad (2.5)
\]
where
\[
h_s := 2(u^+_s, f_s)_{L_2(D)} + \|g^k_s I_{u_s > 0}\|_{L_2(D, \ell_2)}^2,
\]
\[
m_t := 2 \int_0^t (u^+_s, g^k_s)_{L_2(D)} \, dm^k_s
\]
and \(m_t\) is a local martingale.

Proof. (i) Recall that the operation of stochastic integration of Hilbert space valued processes is well defined. Therefore, the process
\[
\hat{u}_t = u_0 + \int_0^t f_s \, dV_s + \int_0^t g^k_s \, dm^k_s
\]
is well defined as a continuous \(L_2(D)\)-valued process. We also recall how the scalar product interacts with integrals. Then it is seen that for any \(t\) and \(\phi \in C_0^\infty(D)\) we have \((\phi, u_t) = (\phi, \hat{u}_t)\) (a.s.). Since both parts are continuous in \(t\), the equality holds for all \(t\) at once (a.s.), and since \(C_0^\infty(D)\) is dense in \(L_2(D)\), we have that \(u_t = \hat{u}_t\) for all \(t\) (a.s.). This proves (i). As a corollary we obtain that
\[
\sup_{t \leq T} \|u_t\|_{L_2(D)} < \infty, \quad \forall T < \infty \quad (a.s.). \quad (2.6)
\]
(ii) It suffices to prove (2.3) for infinitely differentiable \(r \in \mathcal{R}\). Indeed, for \(r_n\) from Remark 2.1, passing to the limit in all term in (2.3) apart from \(m_t\)
presents no problem at all in light of (2.6) and the dominated convergence theorem. Also
\[ m_t(n) := \int_0^t (r_n'(u_s), g_s^k)_{L_2(D)} \, dm_s^k \rightarrow m_t \]
uniformly in \( t \) on finite intervals in probability because
\[ \langle m(n) - m \rangle_t = \int_0^t \sum_k ((r_n' - r')(u_s), g_s^k)_{L_2(D)}^2 \, d\langle m^k \rangle_s \]
\[ \leq \int_0^t \sum_k ((r_n' - r')(u_s), g_s^k)_{L_2(D)}^2 \, dV_s \]
\[ \leq \int_0^t \|r_n' - r'(u_s)\|^2_{L_2(D)} \|g_s\|^2_{L_2(D, \ell_2)} \, dV_s \rightarrow 0 \]
again owing to (2.6) and the dominated convergence theorem.

Thus, we may concentrate on the case that \( r \) is infinitely differentiable. Take a symmetric \( \zeta \in C_0^\infty(\mathbb{R}^d) \) with support in the unit ball centered at the origin and unit integral. For \( \varepsilon > 0 \) set \( \zeta_\varepsilon(x) = \varepsilon^{-d} \zeta(x/\varepsilon) \) and for functions \( v = v(x) \) define \( v(\varepsilon) = v \ast \zeta_\varepsilon \). Also set
\[ D_\varepsilon = \{ x \in D : \text{dist}(x, \partial D) < \varepsilon \}. \]
According to (2.2) for any \( x \in D_\varepsilon \) and \( t \geq 0 \) we have
\[ u_t(\varepsilon)(x) = u_0(\varepsilon)(x) + \int_0^t f_s(\varepsilon)(x) \, dV_s + \int_0^t g_s^k(\varepsilon)(x) \, dm_s^k. \]

By Itô’s formula we have that on \( D_\varepsilon \)
\[ r(u_t(\varepsilon)) = r(u_0(\varepsilon)) + \int_0^t [r'(u_s(\varepsilon))f_s(\varepsilon) + (1/2)r''(u_s(\varepsilon))g_s^k(\varepsilon)_{\ell_2^2}] \, dV_s \]
\[ + \int_0^t r'(u_s(\varepsilon))g_s^k(\varepsilon) \, dm_s^k. \quad (2.7) \]
Here, for each \( \varepsilon > 0 \), the integrands are smooth functions of \( x \) and their magnitudes along with the magnitudes of each of their derivatives in \( D_\varepsilon \) are majorated by a constant (possibly depending on \( \varepsilon \)) times
\[ \|f_s\|_{L_2(D)} \sup_{s \leq t} \|u_s\|_{L_2(D)} \quad \text{or} \quad \|g_s\|^2_{L_2(D, \ell_2)} \quad \text{or} \quad \|g_s^k\|_{L_2(D)} \sup_{s \leq t} \|u_s\|_{L_2(D)}. \]
This and (2.3) and (2.6) allow us to use Fubini’s theorem while integrating through (2.7) and conclude
\[ \|r^{1/2}(u_t(\varepsilon))\|^2_{L_2(D_\varepsilon)} = \|r^{1/2}(u_0(\varepsilon))\|^2_{L_2(D_\varepsilon)} + \int_0^t \|[r'(u_s(\varepsilon)), f_s(\varepsilon)]_{L_2(D_\varepsilon)} \]
\[ + (1/2)\|r''(u_s(\varepsilon))g_s^k(\varepsilon)\|^2_{L_2(D_\varepsilon, \ell_2)} \, dV_s \]
\[ + \int_0^t (r'(u_s(\varepsilon)), g_s^k(\varepsilon))_{L_2(D_\varepsilon)} \, dm_s^k. \quad (2.8) \]
Now we let $\varepsilon \downarrow 0$. We use that for any function $v \in L_2(D)$
\[
\|v^{(\varepsilon)}\|_{L_2(D,\varepsilon)} \leq \|(vI_D)^{(\varepsilon)}\|_{L_2(\mathbb{R}^d)} \leq \|vI_D\|_{L_2(\mathbb{R}^d)} = \|v\|_{L_2(D)}
\]
and $v^{(\varepsilon)}I_{D_\varepsilon} \to v$ in $L_2(D)$. In particular, $g^{k(\varepsilon)}_s I_{D_\varepsilon} \to g^k_s$ and $u^{(\varepsilon)}_s I_{D_\varepsilon} \to u_s$ implying that $r'(u^{(\varepsilon)}_s)I_{D_\varepsilon} \to r'(u_s)$ in $L_2(D)$ and
\[
(r'(u^{(\varepsilon)}_s), g^{k(\varepsilon)}_s)_{L_2(D_\varepsilon)} \to (r'(u_s), g^k_s)_{L_2(D)}
\]
for each $k$ and $dP \times dV_\varepsilon$-almost all $(\omega, s)$.

We also use (2.4) and (2.6) to assert that
\[
\sum_{k=1}^\infty \int_0^t \sup_{\varepsilon \in (0,1)} \|r'(u^{(\varepsilon)}_s), g^{k(\varepsilon)}_s\|_{L_2(D_\varepsilon)}^2 dV_s \leq N \sup_{s \leq t} \|u_s\|_{L_2(D)}^2 \int_0^t \|g_s\|_{L_2(D,\ell_2)}^2 dV_s < \infty.
\]
As is easy to see this implies that the local martingale part in (2.8) converges to $m_t$ as $\varepsilon \downarrow 0$ in probability locally uniformly with respect to $t$.

Similar manipulations with other terms in (2.8) allow us to get (2.9). Since (2.5) is just a particular case of (2.3), the lemma is proved.

Remark 2.3. Lemma 2.2 remains true if in the definition of $\mathcal{R}$ instead of requiring $r''$ to have a left-continuous modification we required it to have a right-continuous one, and of course, in (2.4) used this right-continuous modification. This is seen after replacing $u$ with $-u$.

In case $r(x) = (x^+)^2$ the function $r''$ has both right- and left-continuous modifications, so that in the definition of $m_t$ one can use $2I_{u_s \geq 0}$ or $2I_{u_s \geq 0}$. It follows that (a.s.) for any $t$
\[
\int_0^t \|\tilde{g}_s I_{u_s = 0}\|^2_{L_2(D,\ell_2)} dV_s = 0.
\]
Furthermore, since, for any $v \in L_2(D)$, $u_t + v$ has the same form as $u_t$,
\[
\int_0^t \|\tilde{g}_s I_{u_s = v}\|^2_{L_2(D,\ell_2)} dV_s = 0.
\]

Lemma 2.4. Let $D$ be an arbitrary domain. Let $u_t$ be an $L_2(D)$-valued $\mathcal{F}_0$-measurable process such that for any $\omega$
\[
u_t \in W_2^1(D)
\]
(d$V_t$-a.e.) and for each $T \in [0, \infty)$ and $\omega$
\[
\int_0^T \|u_t\|^2_{W_2^1(D)} dV_t < \infty. \tag{2.9}
\]
Let $f_1, f_1^1, ..., f_1^d$, and $g_t = (g_1^1, g_1^2, ...)$ be $\mathcal{F}_t$-adapted and jointly measurable processes with values in $L_2(D)$ and $L_2(D,\ell_2)$, respectively. Assume that for
each $t \in [0, \infty)$ we have

$$\int_0^t (\|f_s\|_{L^2(D)}^2 + \sum_i \|f_s^i\|_{L^2(D)}^2 + \|g_s\|_{L^2(D, \ell^2)}^2) \, dV_s < \infty, \quad (2.10)$$

and for each $t \in [0, \infty)$, $\phi \in C_0^\infty(D)$, and $\omega$

$$\langle \phi, u_t \rangle = \langle \phi, u_0 \rangle + \int_0^t \langle \phi, f_s + D_i f_s^i \rangle \, dV_s + \int_0^t \langle \phi, g_s^k \rangle \, dm_s^k. \quad (2.11)$$

Finally, assume that there is a compact set $G \subset D$ such that

$$u_t(x) = f_t(x) = f_t^i(x) = g_t^k(x) = 0$$

outside $G$. Then

(a) $u_t$ is a continuous $L^1(D)$-valued function (a.s.); (b) (a.s.) for all $t \in [0, \infty)$

$$\|u_t^+\|^2_{L^2(D)} = \|u_0^+\|^2_{L^2(D)} + \int_0^t h_s \, dV_s + m_t, \quad (2.12)$$

where

$$h_s := 2(u_s^+, f_s)_{L^2(D)} - 2(I_{u_s > 0} D_i u_s, f_s^i)_{L^2(D)} + \|\tilde{g}_s I_{\phi u_s > 0}\|^2_{L^2(D, \ell^2)},$$

$$m_t := 2 \int_0^t (u_s^+, g_s^k)_{L^2(D)} \, dm_s^k.$$

Proof. Observe that (2.11) holds for all infinitely differentiable functions $\phi$. Furthermore, since $u_t \in W^{1,1}_D(D) \, (dV_t$-a.e.) assertion (a) is well known (see, for instance, [13], the references therein, and Remark [14].

To prove (b), take $\varepsilon$ smaller than the distance between $G$ and $\partial D$. Notice that, owing to the symmetry of $\zeta$, for $\phi \in C_0^\infty(D)$

$$\langle \phi, u_t \rangle = \langle \phi, u_0 \rangle + \int_0^t \langle \phi, \tilde{f}_s^\varepsilon \rangle \, ds + \int_0^t \langle \phi, g_s^k(\varepsilon) \rangle \, dm_s^k, \quad (2.13)$$

where

$$\tilde{f}_s^{\varepsilon} := f_s^{(\varepsilon)} + D_i f_s^{i(\varepsilon)}$$

is an $L^2(D)$-valued function with norm that is locally square integrable against $dV_s$. By Lemma [22] for any $r \in \mathcal{R}$

$$\|r^{1/2}(u_t^{(\varepsilon)})\|^2_{L^2(D)} = \|r^{1/2}(u_0^{(\varepsilon)})\|^2_{L^2(D)} + \int_0^t h_s^\varepsilon \, dV_s + m_t^\varepsilon, \quad (2.14)$$

where

$$m_t^\varepsilon := \int_0^t (r'(u_s^{(\varepsilon)}), g_s^k(\varepsilon))_{L^2(D)} \, dm_s^k,$$

$$h_s^\varepsilon := (r'(u_s^{(\varepsilon)}), \tilde{f}_s^{(\varepsilon)})_{L^2(D)} + (1/2)\|r''(u_s^{(\varepsilon)})\|_{L^2(D, \ell^2)}$$

$$= (r'(u_s^{(\varepsilon)}), f_s^{(\varepsilon)})_{L^2(D)} - (r''(u_s^{(\varepsilon)}))_{L^2(D, \ell^2)} + (1/2)\|r''(u_s^{(\varepsilon)})\|_{L^2(D, \ell^2)}.$$
If $r$ is infinitely differentiable, then by using (2.14) and (2.10) one easily passes to the limit in (2.13) as $\varepsilon \to 0$. The argument is quite similar to the corresponding argument in the proof of Lemma 2.2 and, for smooth $r \in \mathcal{R}$, yields

$$\|r^{1/2}(u_t)\|_{L^2(D)}^2 = \|r^{1/2}(u_0)\|_{L^2(D)}^2 + \int_0^t h_s \, dV_s + m_t, \quad (2.15)$$

where

$$m_t = \int_0^t (r'(u_s), g^k_s)_{L^2(D)} \, dm_s^k,$$

$$h_s = (r'(u_s), f_s)_{L^2(D)} - (r''(u_s)D_iu_s, f^i_s)_{L^2(D)} + (1/2)\|r''(u_s)\|_{L^2(D, \ell_2)}^2.$$

Finally, as in the proof of Lemma 2.2 one easily passes from smooth $r \in \mathcal{R}$ to arbitrary ones and gets (2.5) by taking $r(x) = (x^+)^2$. The lemma is proved.

Lemma 2.4 serves as an auxiliary tool to prove a deeper result.

**Lemma 2.5.** Let $D$ be an arbitrary domain. Assume that for each $\phi \in C_0^\infty(D)$

(i) $\phi u_t$ is an $L^2(D)$-valued process such that $\phi u_0$ is $\mathcal{F}_0$-measurable;

(ii) for any $\omega$

$$\phi u_t \in W^1_2(D) (dV_t\text{-a.e.})$$

and for each $T \in [0, \infty)$ and $\omega$

$$\int_0^T \|\phi u_t\|_{W^1_2(D)}^2 \, dV_t < \infty. \quad (2.16)$$

(iii) Let $f_t, f^i_t, \ldots, f^d_t$, and $g_t = (g^1_t, g^2_t, \ldots)$ be $\mathcal{F}_t$-adapted and jointly measurable processes with values in $L^2(D)$ and $L^2(D, \ell_2)$, respectively. Assume that for each $t \in [0, \infty)$ and $\phi \in C_0^\infty(D)$ we have

$$\int_0^t (\|\phi f_s\|_{L^2(D)}^2 + \sum_i \|\phi f^i_s\|_{L^2(D)}^2 + \|\phi g_s\|_{L^2(D, \ell_2)}^2) \, dV_s < \infty, \quad (2.17)$$

$$(\phi, u_t) = (\phi, u_0) + \int_0^t (\phi, f_s + D_i f^i_s) \, dV_s + \int_0^t (\phi, g^k_s) \, dm_s^k. \quad (2.18)$$

Then, for any $\phi \in C_0^\infty(D)$,

(a) $\phi u_t$ is a continuous $L^2(D)$-valued function (a.s.); (b) (a.s.) for all $t \in [0, \infty)$

$$\|\phi u_t\|_{L^2(D)}^2 = \|\phi u_0\|_{L^2(D)}^2 + \int_0^t h_s \, dV_s + m_t, \quad (2.19)$$

where

$$h_s := 2(\phi u_s)^+ \phi f_s - f^i_s D_i(\phi)_{L^2(D)} - 2(I_{\phi u_s > 0}D_i(\phi u_s), \phi f^i_s)_{L^2(D)}$$

$$+ \|\phi g_s I_{\phi u_s > 0}\|_{L^2(D, \ell_2)}^2, \quad m_t := 2 \int_0^t (\phi u_s^+, \phi g^k_s)_{L^2(D)} \, dm_s^k.$$
Proof. Clearly, for any $\phi, \eta \in C_0^\infty(D)$ we have
\begin{align}
(\phi, \eta u_t) &= (\phi, \eta u_0) + \int_0^t (\phi, \eta f_s - f_s^i D_i \eta + D_i (\eta f_s^i)) \, dV_s + \int_0^t (\phi, \eta g_s^k) \, dm_s^k.
\end{align}

Therefore, $\eta u_t$ satisfies the assumptions of Lemma 2.4 with $\eta f_s - f_s^i D_i \eta$, $\eta f_s^i$, and $\eta g_s^k$ in place of $f_s$, $D_i f_s^i$, and $g_s^k$, respectively.

By applying Lemma 2.4 to $\eta u_t$ in place of $u_t$ we get the result with $\eta$ in place of $\phi$. This certainly proves the lemma.

3. Proof of Theorems 1.1 and 1.2

In this section the assumptions stated in Section 1 are supposed to be satisfied. We use the fact that due to our hypothesis that $D \in C^1$, there exist sequences $\zeta_n$ and $\tilde{\zeta}_n$ of nonnegative $C_0^\infty(D)$-functions such that $0 \leq \zeta_n, \tilde{\zeta}_n \leq 1$, $\zeta_n, \tilde{\zeta}_n \rightarrow 1$ in $D$ as $n \rightarrow \infty$ and for any $v \in W_2^1(D)$, $i = 1, \ldots, d$,
\begin{align}
\|v D_i \zeta_n\|_{L_2(D)} \leq N \| (1 - \tilde{\zeta}_n) v \|_{L_2(D)} + \| (1 - \tilde{\zeta}_n) Dv \|_{L_2(D)}),
\end{align}
where $N$ is independent of $n$ and $v$ (see, for instance, the proof of Theorem 5.5.2 in [4]). We also know (see, for instance, the proof of Lemma 2.3.2 in [6] or Problem 17, Chapter 5 of [1]) that if $v \in W_2^1(D)$, then $v^+ \in W_2^1(D)$ and $D_i v^+ = I_{v^+} D_i v$.

Proof of Theorem 1.1 Set
\begin{align}
K = K_1 + K_2, \quad \varphi_t = \int_0^t K(s) \, ds.
\end{align}
Take the sequences of nonnegative $\zeta_n, \tilde{\zeta}_n \in C_0^\infty(D)$ from above. By Itô’s formula and Lemma 2.5
\begin{align}
\|\zeta_n \varphi_t^\mu\|^2_{L_2(D)} e^{\varphi_t} = \|\zeta_n \varphi_t^\mu\|^2_{L_2(D)} + \int_0^t h_s^\mu \, dV_s + m_t(n),
\end{align}
where
\begin{align}
e^{\varphi_t} h_s^\mu &= I_{1s} + I_{2s} + I_{3s} - K(s) \|\zeta_n \varphi_t^\mu\|^2_{L_2(D)},
I_{1s} &= 2 \langle \zeta_n u_s^+, \zeta_n [f_s + b_s^i D_i u_s - c_s u_s] - [u_s a_s^i + a_s^j D_j u_s] f_s^i \rangle_{L_2(D)},
I_{2s} &= -2 \langle I_{\zeta_n u_s > 0} D_i \zeta_n, \zeta_n [u_s a_s^i + a_s^j D_j u_s] f_s^i \rangle_{L_2(D)},
I_{3s} &= \|\zeta_n I_{\zeta_n u_s > 0} [\sigma_s^i D_i u_s + \nu_s u_s + \tilde{g}_s] \|_{L_2(D, \mu)}^2,
\end{align}
and
\begin{align}
m_t(n) &= 2 \int_0^t e^{-\varphi_s} \langle \zeta_n u_s^+, \zeta_n [\sigma_s^i D_i u_s + \nu_s u_s + \tilde{g}_s] \rangle_{L_2(D)} \, dm_s^k.
\end{align}
Since $u_s^+ = 0$ we have
\begin{align}
e^{-\varphi_t} \|\zeta_n \varphi_t^\mu\|^2_{L_2(D)} = \int_0^t I_{\tau_2 > \tau_1} h_s^\mu \, dV_s + \tilde{m}_t(n),
\end{align}
where
\begin{align}
\tilde{m}_t(n) := m_{\tau_2 \wedge \tau_1} (n) - m_{\tau_1} (n)
\end{align}
is a local martingale.

Next we use the assumptions of the theorem and see that for \(dV_s\)-almost all \(s \in (\tau_1, \tau_2)\) we have

\[
I_{1s} \leq 2(\zeta_n u_s^+, \zeta_n b_i^j D_i u_s - c_s u_s) - [u_s a_i^j + a_i^j D_j u_s] D_i \zeta_n)_{L^2(D)}
\]

\[
= 2(\zeta_n^2 u^+_s, b_i^j D_i u^+_s - c_s u^+_s)_{L^2(D)} + I_{4s}
\]

with

\[
I_{4s} = -2(\zeta_n u^+_s D_i \zeta_n, u^+_s a_i^j + a_i^j D_j u^+_s)_{L^2(D)}.
\]

At this moment we recall (3.1) and observe that for

\[
e \in (I_3, s) \leq \|u^+_s\|_{L^2(D)} \|D_i \zeta_n\|_{L^2(D)}.
\]

Then we see that

\[
I_{4s} \leq N(\|(1 - \zeta_n) u^+_s\|_{L^2(D)} + \|(1 - \zeta_n) D u^+_s\|_{L^2(D)}) \|u^+_s\|_{W^2_2(D)},
\]

where and below by \(N\) we denote various finite constants.

In \(I_{2s}\)

\[
I_{\zeta_n u_s > 0} D_i (\zeta_n u_s) = D_i (\zeta_n u^+_s) = u^+_s D_i \zeta_n + \zeta_n D_i u^+_s,
\]

so that

\[
I_{2s} = -2(\zeta_n^2 D_i u^+_s, u^+_s a_i^j + a_i^j D_j u^+_s)_{L^2(D)} + I_{4s}.
\]

Next,

\[
\zeta_n I_{\zeta_n u_s > 0} = \zeta_n I_{u_s > 0},
\]

\[
I_{3s} \leq \|\zeta_n I_{u_s > 0} [\sigma^j_i D_i u_s + \nu_s u_s]\|^2_{L^2(D, \ell^2)} = (\zeta_n^2 D_i u^+_s, \alpha^j_i D_j u^+_s)_{L^2(D)}
\]

\[
+ 2(\zeta_n^2 D_i u^+_s, u^+_s (\sigma^j_i, \nu_s)_{\ell^2})_{L^2(D)} + \|\zeta_n |\nu_s|_{\ell^2} u^+_s\|^2_{L^2(D)}.
\]

Also observe that certain parts of \(I_{2s}\) and \(I_{3s}\) can be combined if we use that

\[
-2(\zeta_n^2 D_i u^+_s, a_i^j D_j u^+_s)_{L^2(D)} + (\zeta_n^2 D_i u^+_s, \alpha^j_i D_j u^+_s)_{L^2(D)}
\]

\[
\leq -K^{-1}_1(s) \|\zeta_n \xi_i^j D_i u^+_s\|^2_{L^2(D)}.
\]

It follows that for \(dV_s\)-almost all \(s \in (\tau_1, \tau_2)\)

\[
e^{\varphi_s} h_s^+ \leq \int_D \left[ \zeta_n^2 (b^j_i - a^j_i + (\sigma^j_i, \nu_s)_{\ell^2}) 2u^+_s D_i u^+_s + \zeta_n^2 (u^+_s)^2 (|\nu_s|_{\ell^2}^2 - 2c_s) \right] dx
\]

\[
+ N(\|(1 - \zeta_n) u^+_s\|_{L^2(D)} + \|(1 - \zeta_n) D u^+_s\|_{L^2(D)}) \|u^+_s\|_{W^2_2(D)}
\]

\[
- K^{-1}_1(s) \|\zeta_n \xi_i^j D_i u^+_s\|^2_{L^2(D)} - K(s) \|\zeta_n u^+_s\|^2_{L^2(D)}.
\]

Here

\[
b^j_i - a^j_i + (\sigma^j_i, \nu_s)_{\ell^2} = -\zeta^j_i - \eta^j_i
\]

and we transform the integral of

\[
\zeta_n^2 (-\eta^j_i) 2u^+_s D_i u^+_s = -\eta^j_i \zeta_n^2 D_i (u^+_s)^2
\]

by integrating by parts. Then we get that for \(dV_s\)-almost all \(s \in (\tau_1, \tau_2)\)

\[
e^{\varphi_s} h_s^+ \leq \int_D \left[ \zeta_n^2 \xi_i^j 2u^+_s D_i u^+_s + \zeta_n^2 (u^+_s)^2 (|\nu_s|_{\ell^2}^2 - 2c_s + D_i \eta^j_i) \right] dx
\]
\[ +2 \int_D (u_s^+)^2 \zeta_n \eta^s_D t \zeta_n \text{d}x + N \left( \| (1 - \zeta_n) u_s^+ \|_{L_2(D)} + \| (1 - \zeta_n) Du_s^+ \|_{L_2(D)} \right) \| u_s^+ \|_{W_0^2(D)} \]

We also use the fact that

\[ | - \zeta_n^s \eta^s_D t \zeta_n u_s^+ | \leq K_1^{-1}(s) \zeta_n^s \eta^s_D t \zeta_n u_s^+^2 + K_1(s) \zeta_n^2(u_s^+)^2, \]

then we easily see that for \( \eta_D \)-almost all \( s \in (\tau_1, \tau_2) \)

\[ e^{\varphi_h^s} h^s \leq N \left( \| (1 - \zeta_n^s) u_s^+ \|_{L_2(D)} + \| (1 - \zeta_n^s) Du_s^+ \|_{L_2(D)} \right) \| u_s^+ \|_{W_0^2(D)} \]

\[ + 2 \int_D (u_s^+)^2 \zeta_n \eta^s_D t \zeta_n \text{d}x \leq N \left( \| (1 - \zeta_n^s) u_s^+ \|_{L_2(D)} + \| (1 - \zeta_n^s) Du_s^+ \|_{L_2(D)} \right) \| u_s^+ \|_{W_0^2(D)}. \]

Now (3.2) yields

\[ e^{-\varphi_{\tau_2} \land \tau_1} \| u_s^+ \|_{L_2(D)}^2 \leq \tilde{m}_t(n) \]

\[ + N \int_0^t \left( \| (1 - \zeta_n^s) u_s^+ \|_{L_2(D)} + \| (1 - \zeta_n^s) Du_s^+ \|_{L_2(D)} \right) \| u_s^+ \|_{W_0^2(D)} \text{d}V_s. \quad (3.3) \]

The integrals against \( \text{d}V_s \) in (3.3) tend to zero as \( n \to \infty \) by the dominated convergence theorem. Since the sum of them with continuous local martingales is nonnegative, the local martingales and the right-hand side of (3.3) tend to zero uniformly on finite time intervals in probability (see, for instance, [7]). So does the left-hand side and the theorem is proved.

**Proof of Theorem 1.2**

Obviously, \( \hat{u}_t = \rho_t \tilde{u}_t \) satisfies

\[ (\phi, \hat{u}_t) = (\phi, \tilde{u}_0) + \int_0^t (\phi, \sigma^s D_i \tilde{u}_s + \nu^s \tilde{u}_s + \rho_s g^k) \text{d}m^k_s + \int_0^t (\phi, \tilde{u}_s) \text{d}\rho_s + \]

\[ + \int_0^t (\phi, D_i (a_i a_j \tilde{u}_s) + b_i D_j \tilde{u}_s + D_i (a_i \tilde{u}_s) - c_s \tilde{u}_s + \rho_s \tilde{f}_s + \rho_s D_i \tilde{f}^i_s) \text{d}V_s. \quad (3.4) \]

We rewrite this equation introducing

\[ \tilde{V}_t = V_t + \rho_t, \quad p_t = \frac{d\rho_t}{dV_t}, \quad q_t = \frac{dV_t}{\tilde{V}_t}, \]

\[ (\tilde{a}^i, \tilde{a}^j, \tilde{b}^i, \tilde{c}_t) = q_t (a_i, a_j, b^i, c_t), \quad (\tilde{\sigma}^i, \tilde{\nu}^k) = q_t^{1/2} (\sigma^i, \nu^k), \]

\[ \tilde{f}_t = q_t \rho_t \tilde{f}_t + p_t u_t, \quad \tilde{f}^i_t = q_t \rho_t \tilde{f}^i_t, \quad \tilde{g}^k_t = q_t^{1/2} \rho_t g^k. \]

We also set

\[ \tilde{m}_t^k = \int_0^t q_s^{-1/2} \text{d}m^k_s \quad (0^{-1/2} := 0). \]

Notice that since \( \text{d} \tilde{m}_t^k \leq \text{d}V_t = q_t \text{d}\tilde{V}_t \) the last integral makes sense.

In this notation (1.11) and (3.14) are rewritten as

\[ (\phi, u_t) = (\phi, u_0) + \int_0^t (\phi, \tilde{a}^i D_i u_s + \tilde{\nu}^k u_s + q_s^{1/2} g^k) \text{d}\tilde{m}_s^k \]
Lemma 4.1. For $m = 0, 1, 2, \ldots$, $t \geq 0$ and $x \in (0, 2^{-m/2})$ introduce
\[ r_m(t, x) = r_m(x, t, x) = P(x_t + x + w_t \sqrt{\delta} = x_{t-\tau} + 2^{-m/2}), \]
where $\tau = \inf \{ s > 0 : (t - s, x_t + x + w_s \sqrt{\delta}) \notin Q_m \}$. Then
\[ r_m(t, x) \leq [\gamma(c, d, \delta)]^{M_{m+n}(x, c, t) - M_{m-1}(x, c, t) - k}, \tag{4.1} \]
where \( n = n(2^{m/2}x/d) \), \( k = k(c + d) \), and 
\[
n(y) = [(-2 \log_2 y)_+] , \quad k(d) = 2 + [(2 \log_2 d)_+].
\]

Proof. Define 
\[
\bar{t} = 2^m t, \quad \bar{x} = 2^{m/2} x, \quad \bar{w}_s = 2^{m/2} w_{s2^{-m}}, \quad \bar{x}_s = 2^{m/2} x_{s2^{-m}}.
\]

Then as is easy to see \( r_m(t, x) \) is rewritten as 
\[
P(\bar{x}_t + \bar{x} + \bar{w}_s \sqrt{\delta} = \bar{x}_{\bar{t} - \bar{r}} + 1),
\]
where 
\[
\bar{r} = \inf\{s > 0 : (\bar{t} - s, \bar{x}_t + \bar{x} + \bar{w}_s \sqrt{\delta}) \notin Q_0(\bar{x})\} = 2^m \tau.
\]

Since \( \bar{w} \) is a Wiener process, by Corollary 3.4 of \cite{9} expression (4.2) is less than 
\[
\left[\gamma(c, d, \delta)\right] M^n_{\bar{\bar{\bar{n}}}}(\bar{x}, c, \bar{t})^{-k},
\]
where \( \bar{\bar{\bar{n}}} = n(\bar{x}/d) \). Here 
\[
M^n_{\bar{\bar{\bar{n}}}}(\bar{x}, c, \bar{t}) = \#\{j = 0, \ldots, \bar{n} : 2^{j/2} \text{osc}_{\bar{t} - 2^{-j}, \bar{t}} \bar{x} \leq c\}
\]
\[
= \#\{j = 0, \ldots, \bar{n} : 2^{(j+m)/2} \text{osc}_{t - 2^{-j-m}, t} x \leq c\}
\]
\[
= \#\{j = m, \ldots, m + \bar{n} : \Delta_j(x) \leq c\} = M^n_{m+n}(x, c, t) - M^n_{m-1}(x, c, t)
\]
and the result follows. The lemma is proved.

**Lemma 4.2.** Let \( T \in (0, \infty) \). Assume that 
\[
\lim_{m \to \infty} \frac{1}{m + 1} \inf_{t \in [0, T]} M^n_{\bar{\bar{\bar{n}}}}(x, c, t) > \alpha > 0.
\]

Take constants \( p > 0 \) and \( \nu \) so that 
\[
1 < \nu p < \rho \chi + 1 < 0,
\]
where \( \chi = -2a \log_2 \gamma(c, d, \delta) \). Then, for \( r_m \) from Lemma 4.1 it holds that 
\[
\sup_{m \geq 0} \sup_{t \in [0, T]} \frac{1}{x^{v(p-1)/(2a)}} \int_0^{2^{-m/2}} r_m(t, x) dx < \infty.
\]

Proof. By Lemma 4.1 for a constant \( N \) and \( \gamma = \gamma(c, d, \delta) \) 
\[
r_m(t, x) \leq N \gamma^{M^n_{m+n}-m},
\]
where \( x \leq 2^{-m/2}, n = n(2^{m/2}x/d), M^n_{m+n} = M^n_{m+n}(x, c, t) \). Furthermore, 
\[
n(2^{m/2}x/d) \geq (-2 \log_2(2^{m/2}x) + 2 \log_2 d)_+ - 1
\]
\[
\leq (-2 \log_2(2^{m/2}x))_+ - N = -m - 2 \log_2 x - N,
\]
where \( N \) is a constant. Hence, \( m + n \geq -2 \log_2 x - N \). Since obviously \( r_m \leq 1 \) we have that 
\[
r_m(t, x) \leq 1 \wedge (N \gamma^{-m+M^n_{-2 \log_2 x-N}}).
\]
By the assumption if \( x \) is small enough

\[
M_{-2\log_2 x-N} > \alpha(-2\log_2 x).
\]

Therefore, for \( x \in (0, 2^{-m/2}] \)

\[
r_m(t, x) \leq 1 \wedge (N\gamma^{-m-2\alpha\log_2 x}) = 1 \wedge (N\gamma^{-m}x^\chi).
\]

Next,

\[
\int_0^{2^{-m/2}} \frac{1}{x^{\nu_p}} r_m(t, x) \, dx \leq \int_0^{\infty} \frac{1}{x^{\nu_p}} (1 \wedge (N\gamma^{-m}x^\chi))^\nu \, dx
\]

\[
= \gamma^{m(1-\nu_p)/\chi} \int_0^{\infty} \frac{1}{x^{\nu_p}} (1 \wedge (x^\chi))^\nu \, dx,
\]

where the last integral is finite owing to (4.3). This proves the lemma.

Let \( w_t \) be a Wiener process with respect to a filtration \( \{ \mathcal{F}_t, t \geq 0 \} \) of complete \( \sigma \)-fields and let \( a_t \) and \( \sigma_t \) be bounded real-valued processes predictable with respect to \( \{ \mathcal{F}_t, t \geq 0 \} \) and such that \( a_t - \sigma_t^2 \geq \delta \sigma_t^2 \), where \( \delta \in (0, \infty) \) is a constant, \( a_t - \sigma_t^2 > 0 \) for all \( (\omega, t) \) and for all \( \omega \)

\[
\int_0^\infty [a_t - \sigma_t^2] \, dt = \infty.
\]

Set \( D_x = \partial/\partial x \). For \( m = 0, 1, 2, ... \) we will be dealing with the SPDE

\[
dv(t, x) = (1/2)a_tD_x^2v(t, x) \, dt + \sigma_tD_xv(t, x) \, dw_t
\]

in \( B_m = (0, \infty) \times (0, 2^{-m/2}) \) with boundary conditions

\[
v(t, 0) = 0, \quad v(t, 2^{-m/2}) = 1, \quad t > 0,
\]

\[
v(0, x) = 0, \quad 0 < x < 1.
\]

Recall that by Theorem 2.1 of [9] there is a deterministic function \( \alpha_0(c) \), \( c > 0 \), such that \( \alpha_0(c) \to 1 \) as \( c \to \infty \) and with probability one for any \( T \in (0, \infty) \)

\[
\lim_{n \to \infty} \inf_{t \in [0, T]} \frac{1}{n+1} M_n(w, c, t) = \alpha_0(c).
\]

**Theorem 4.3.** For each \( m = 0, 1, 2, ... \) there is a function \( v_m(t, x) = v_m(\omega, t, x) \) defined on \( \Omega \times B_m \) such that

(i) \( v_m(t, x) \) is \( \mathcal{F}_t \)-measurable for each \( (t, x) \in B_m \),

(ii) \( v_m(t, x) \) is bounded and continuous in \( B_m \setminus \{(0, 2^{-m/2})\} \) for each \( \omega \),

(iii) derivatives of \( v_m(t, x) \) of any order with respect to \( x \) are continuous in \( B_m \cup \{(0, 0, 2^{-m/2})\} \) for each \( \omega \),

(iv) equations (4.6) and (4.7) hold for each \( \omega \),

(v) almost surely, for any \( (t, x) \in B_m \)

\[
v_m(t, x) = \int_0^t (1/2)a_sD_x^2v_m(s, x) \, ds + \int_0^t \sigma_sD_xv_m(s, x) \, dw_s,
\]

\[
\lim_{n \to \infty} \inf_{t \in [0, T]} \frac{1}{n+1} M_n(w, c, t) = \alpha_0(c).
\]
(vi) for any \( T \in (0, \infty) \), \( c, d > 0 \), \( p > 0 \), \( \alpha > 0 \) such that \( \alpha_0(c\sqrt{\delta}) > \alpha \), and \( \nu \) satisfying
\[
1 < \nu p < \chi p + 1,
\]
where \( \chi = -2\alpha \log_2 \gamma(c, d, 1) \), we have that with probability one
\[
\pi_T := \sup_{m \geq 0} \sup_{t \in [0, T]} 2^{-m(\nu p - 1)/(2\alpha)} \int_0^{2^{-m/2}} \frac{1}{x^p} r_m^p(t, x) \, dx < \infty. \tag{4.9}
\]

Proof. In Lemma 4.1 take \( \delta = 1 \) and set \( \tilde{v}_m(x, x_t + x, t) = r_m(x, t, x) \), where \( r_m \) is introduced in that lemma. Set
\[
\psi_t = \int_0^t (a_s - \sigma_s^2) \, ds, \quad \xi_t = \int_0^{\phi_t} \sigma_s \, dw_s, \quad \tilde{F}_t = \tilde{F}_{\psi_t},
\]
\[
\tilde{v}_m(t, x) = \tilde{v}_m(\omega, t, x) = \tilde{v}_m(\xi, t, x), \quad v_m(t, x) = v_m(\omega, t, x) = \tilde{v}_m(\psi_t, x + \xi),
\]
where \( \phi_t = \inf\{s \geq 0 : \psi_s \geq t\} \) is the inverse function to \( \psi_t \).

It is proved in Theorem 4.1 of [11] that \( v_0 \) possesses properties (i)-(v). The proof that this is also true for any \( m \) is no different.

Furthermore, it is well known that
\[
\sqrt{\delta} \int_0^t \sigma_s \, dw_s = \tilde{w}_t(\tilde{\psi}_t),
\]
where \( \tilde{w}_t \) is a Wiener process and
\[
\tilde{\psi}_t = \delta \int_0^t \sigma_s^2 \, ds.
\]
Hence \( \xi_t = \delta^{-1/2}\tilde{w}_t(\phi_t) \) with
\[
(\tilde{w}_t(\phi_t))' = \delta \sigma_s^2/(a_s - \sigma_s^2)|_{s = \phi_t} \leq 1.
\]

It follows that for \( n = 0, 1, 2, \ldots \) we have
\[
M_n^- (\xi, c, t) \geq M_n^- (\tilde{w}, c\sqrt{\delta}, \tilde{\psi}(\phi_t)),
\]
\[
\inf_{t \leq T} M_n^- (\xi, c, t) \geq \inf_{t \leq T} M_n^- (\tilde{w}, c\sqrt{\delta}, t),
\]
and with probability one
\[
\lim_{n \to \infty} \inf_{t \leq T} \frac{1}{n + 1} M_n^- (\xi, c, t) \geq \alpha_0(c\sqrt{\delta}) > \alpha.
\]

Finally, for \( M = \sup_{a_t - \sigma_t^2} \) we have
\[
\sup_{t \leq T} \int_0^{2^{-m/2}} \frac{1}{x^p} \tilde{r}_m^p(t, x) \, dx \leq \sup_{\tilde{a} \leq MT} \int_0^{2^{-m/2}} \frac{1}{x^p} r_m^p(t, x + \xi_t) \, dx
\]
\[
= \sup_{t \leq MT} \int_0^{2^{-m/2}} \frac{1}{x^p} \tilde{r}_m^p(\xi, t, x + \xi_t) \, dx = \sup_{t \leq MT} \int_0^{2^{-m/2}} \frac{1}{x^p} r_m^p(\xi_t, t, x) \, dx.
\]

After this it only remains to use Lemma 4.2. The theorem is proved.
Remark 4.4. Obviously, for any \( \varepsilon \in (0, 2^{-(m+2)/2}) \) we have
\[
v_m(t, \cdot) \in W^1_2(\varepsilon, 2^{-m/2} - \varepsilon)
\]
for any \( t \in [0, \infty) \) and for any \( T \in (0, \infty) \) we have
\[
\int_0^T \|v_m(t, \cdot)\|_{W^1_2(\varepsilon, 2^{-m/2} - \varepsilon)} \, dt < \infty.
\]
Furthermore, by using the deterministic and stochastic versions of Fubini’s theorem one easily proves that for any \( \phi \in C_0^\infty(0, 2^{-m/2}) \) with probability one for all \( t \in [0, \infty) \)
\[
(\phi, v_m(t, \cdot)) = (1/2) \int_0^t (\phi, a_s D^2_x v_m(s, \cdot)) \, ds + \int_0^t (\phi, \sigma_s D_x v_m(s, \cdot)) \, dw_s.
\]

5. CONTINUITY OF SOLUTIONS OF SPDES

We take the processes \( a_t, \sigma_t \) as before Theorem 4.3 but impose stronger assumptions on them.

Assume that there exist constants \( \delta_0, \delta_1 \in (0, 1] \) such that, for every \((\omega, t)\)
\[
\delta_0 \leq \delta_1 a_t \leq a_t - \sigma_t^2 \leq \delta_0^{-1},
\]
We will be dealing with solutions \( u_t(x) \) of
\[
du_t = \left((1/2) a_t D^2_x u_t + f_t\right) \, dt + (\sigma_t D_x u_t + g_t) \, dw_t	ag{5.1}
\]
on \( \mathbb{R}_+ \) with zero initial condition. To specify the assumptions on \( f, g \) and the class of solutions we borrow the Banach spaces \( \mathbb{H}^{\gamma}_{p, \theta}(\tau) \) and \( \mathbb{L}_{p, \theta}(\tau) \) from \( [10] \). We also denote by \( M \) the operator of multiplying by \( x \). Recall that, for \( p \geq 2, 0 < \theta < p \), the norms in \( \mathbb{H}^{\gamma}_{p, \theta}(\tau) \), \( \gamma = 1, 2 \), and \( \mathbb{L}_{p, \theta}(\tau) \) are given by
\[
\|v\|_{\mathbb{L}_{p, \theta}(\tau)}^p = E \int_0^\tau \int_0^\infty x^{\theta-1} |v(t, x)|^p \, dx \, dt,
\]
\[
\|v\|_{\mathbb{H}^1_{p, \theta}(\tau)} = \|v\|_{\mathbb{L}_{p, \theta}(\tau)} + \|MD_x v\|_{\mathbb{L}_{p, \theta}(\tau)},
\]
\[
\|v\|_{\mathbb{H}^2_{p, \theta}(\tau)} = \|v\|_{\mathbb{H}^1_{p, \theta}(\tau)} + \|M^2 D^2_x v\|_{\mathbb{L}_{p, \theta}(\tau)}.
\]

Given \( p \geq 2, \theta \in [p-1, p) \), any stopping time \( \tau, f \in M^{-1}\mathbb{L}_{p, \theta}(\tau) \), and \( g \in \mathbb{H}^1_{p, \theta}(\tau) \) by Theorem 3.2 of \( [10] \) equation (5.1) with zero initial condition has a unique solution \( u \in M^2_{\mathbb{H}^2_{p, \theta}(\tau)} \) and
\[
\|M^{-1} u\|_{\mathbb{H}^2_{p, \theta}(\tau)} \leq N(\|M f\|_{\mathbb{L}_{p, \theta}(\tau)} + \|g\|_{\mathbb{H}^1_{p, \theta}(\tau)}),
\]
where \( N = N(p, \theta, \delta_0, \delta_1) \).

We will also use Theorem 4.7 of \( [8] \), which implies that if \( u \) is a solution of (5.1) of class \( M^2_{\mathbb{H}^2_{p, \theta}(\tau)} \) with zero initial condition and \( f \in M^{-1}\mathbb{L}_{p, \theta}(\tau) \), and \( g \in \mathbb{H}^1_{p, \theta}(\tau) \) and if there are numbers \( T \in (0, \infty) \) and \( \beta \) such that
\[
2/p < \beta \leq 1, \quad \tau \leq T,
\]
then for almost any $\omega$ the function $u_t(x)$ is continuous in $(t, x)$ (that is, has a continuous modification) and

$$E \sup_{t \leq \tau} \sup_{x > 0} |x^{\beta-1+\theta/p} u_t(x)|^p \leq N T^{3p/2} \left( \| M^{-1} u \|_{L^p} + \| M f \|_{L^p} + \| g \|_{L^p} \right),$$

where $N = N(p, \theta, \beta, \delta_0)$.

Everywhere below we take $p > 2$.

**Theorem 5.1.** Let $T \in (0, \infty)$, $c > 0$, $\alpha \in (0, 1)$, $\theta > 0$, $\mu$ be some constants such that $\alpha_0(c \sqrt{\delta_1}) > \alpha$,

$$\theta_0 < \theta < p, \quad \mu < p(1 + 2 \log_2 \gamma(c)) - 2 = \theta_0 - 2 + 2p(1 - \alpha) \log_2 \gamma(c),$$

where

$$\gamma(c) = \gamma(c, 1, 1), \quad \theta_0 = p(1 + 2\alpha \log_2 \gamma(c)) \quad (> 0).$$

Let $f \in M^{-1} L_{p,1}(T)$, $g \in H^1_{p,0}(T)$, and let $u \in M H^2_{p,1}(T)$ be a solution of (5.1) with zero initial condition.

Finally, assume that $f_t(x) = g_t(x) = 0$ for $x \geq 1$ and $f \in M^{-1} L_{p,\mu}(T)$, $g \in H^1_{p,\mu}(T)$. Then there exist stopping times $\tau_n \uparrow T$, defined independently of $f$ and $g$ such that, for each $n$, $u \in M H^2_{p,\theta}(\tau_n)$ and

$$\| M^{-1} u \|_{H^2_{p,\theta}(\tau_n)} \leq n \left( \| M f \|_{L_{p,\mu}(\tau_n)} + \| g \|_{L_{p,\mu}(\tau_n)} \right) \quad (5.2)$$

Here is the result about the continuity of $u_t(x)$ we were talking about in the introduction.

**Remark 5.2.** By Theorem 4.7 of [8] and Theorem 5.1 if we have a number $\beta \in (2/p, 1]$, then there exists a sequence of stopping times $\tau_n \uparrow T$ such that

$$E \sup_{t \leq \tau_n} \sup_{x > 0} |x^{-\varepsilon} u_t(x)|^p < \infty,$$

where $\varepsilon = 1 - \beta - \theta/p$. Due to the freedom of choosing $\alpha$, $\beta$, and $\theta$, the number $\varepsilon$ can be made as close from the right as we wish to

$$1 - \lim_{\alpha \to \alpha_0(c \sqrt{\delta_1})} (2 + \theta_0)/p = -2\mu - 2\alpha_0(c \sqrt{\delta_1}) \log_2 \gamma(c).$$

If we allow arbitrary $p$, then the rate of convergence of $u_t(x)$ to zero as $x \downarrow 0$ is almost

$$x^{\varepsilon_0}, \quad \varepsilon_0 = -2\alpha_0(c \sqrt{\delta_1}) \log_2 \gamma(c) > 0,$$

which is the same as we obtained for $v_m(t, x)$ (see (4.30)). Hence, the presence of $f$ and $g$ does not spoil the situation too much.

It is also worth noting that if $f$ and $g$ still may blow up near zero even if $p$ is large. When $p$ is large we can take $(\mu - 1)/p$ as close to $1 + 2 \log_2 \gamma(c)$ as we wish and then the integral

$$\int_0^1 x^{\mu-1} |x f_t(x)|^p \, dx$$
converges if $|f_t(x)|$ blows up near $x = 0$ slightly slower than $x^{-2(1 + \log_2 \gamma(c))}$. Here $\log_2 \gamma(c) \to 0$ as $c \to \infty$ and one can allow $|f_t(x)|$ to blow up almost as $x^{-2}$.

However, when $f$ and $g$ become more irregular near 0, the rate at which the solution goes to zero at 0 deteriorates. In connection with this it is interesting to investigate what happens with $\varepsilon_0$ as $\delta_1 \downarrow 0$. Take an $m$ so large that $\alpha_0(m) > 1/2$ and set $c = m\delta_1^{-1/2} - 1/2$. Then for $\delta_1$ small we have $\alpha_0(c\sqrt{\delta_1}) > 1/2$ and

$$\varepsilon_0 \geq - \log_2 \left[ 1 - P(\min_{s \leq 1/2} w_s \leq -c - 1/\sqrt{2}, \max_{s \leq 1/2} w_s \leq 1 - 1/\sqrt{2}) \right],$$

$$\varepsilon_0 \ln 2 \geq - \ln \left[ 1 - P(\min_{s \leq 1/2} w_s \leq -c - 1/\sqrt{2}, \max_{s \leq 1/2} w_s \leq 1 - 1/\sqrt{2}) \right]$$

$$\sim P(\min_{s \leq 1/2} w_t \leq -c - 1/\sqrt{2}, \max_{s \leq 1/2} w_s \leq 1 - 1/\sqrt{2})$$

$$= P(\min_{s \leq 1/2} w_t \leq -c - 1/\sqrt{2}) - P(\min_{s \leq 1/2} w_t \leq -c - 1/\sqrt{2}, \max_{s \leq 1/2} w_s \geq 1 - 1/\sqrt{2})$$

and

$$P(\min_{s \leq 1/2} w_t \leq -c - 1/\sqrt{2}, \max_{s \leq 1/2} w_s \geq 1 - 1/\sqrt{2}) \leq 2P(\min_{s \leq 1/2} w_t \leq -c - 1),$$

so that

$$P(\min_{s \leq 1/2} w_t \leq -c - 1/\sqrt{2}, \max_{s \leq 1/2} w_s \leq 1 - 1/\sqrt{2})$$

$$\geq P(\min_{s \leq 1/2} w_t \leq -c - 1/\sqrt{2}) - 2P(\min_{s \leq 1/2} w_t \leq -c - 1).$$

Next, as $a \to \infty$

$$P(\min_{s \leq 1/2} w_s \leq -a) = P(|w_{1/2}| \geq a) = \frac{2}{\sqrt{\pi}} \int_{a}^{\infty} e^{-x^2} \, dx \sim \frac{1}{\sqrt{\pi}} a^{-1} e^{-a^2}$$

and

$$\lim_{\delta_1 \downarrow 0} \left[ \int_{s \leq 1/2} P(\min_{s \leq 1/2} w_t \leq -c - 1/\sqrt{2}) \right]$$

$$-2P(\min_{s \leq 1/2} w_t \leq -c - 1)(c + 1/\sqrt{2})e^{(c+1/\sqrt{2})^2} = \frac{1}{\sqrt{\pi}}.$$

Hence

$$\lim_{\delta_1 \downarrow 0} \left[ m\delta_1^{-1/2} e^{m^2/\delta_1} \right] \geq \frac{1}{\sqrt{\pi} \ln 2}.$$

This result may seem unsatisfactory since the guaranteed value of $\varepsilon_0$ is extremely small when $\delta_1$ is small. However, recall that by Remark 4.2 of [9] the best possible rate with which the solutions go to zero for small $\delta_1$ is less than

$$(1 + \kappa)(2\pi \delta_1)^{-1/2} e^{-1/(2\delta_1)},$$

where $\kappa > 0$ is any number.

To prove Theorem 5.1 first we prove the following.
Lemma 5.3. Assume that, for an m = 0, 1, 2, ... we have \( f_t(x) = g_t(x) = 0 \) if \( x \leq 2^{-m/2} \). Then almost surely for all \( t \leq T \) and \( x \in (0, 2^{-m/2}] \)

\[
|u_t(x)| \leq v_m(t, x) \sup_{s \leq t} |u_s(2^{-m/2})|.
\]

(5.3)

Proof. By Theorem 4.7 of \([8]\) the function \( u_t(x) \) is continuous in \([0, T] \times (0, 2^{-m/2}] \) (a.s.) and therefore to prove (5.3) it suffices to prove that for each \( \varepsilon \in (0, 2^{-(m+2)/2}] \) almost surely for all \( t \leq T \) and \( x \in D := (\varepsilon, 2^{-m/2} - \varepsilon) \)

\[
|u_t^\varepsilon(x)| \leq v_m(t, x) \sup_{s \leq t} |u_s^\varepsilon(2^{-m/2} - \varepsilon)| =: v_m(t, x) \rho_t^\varepsilon,
\]

(5.4)

where \( u_t^\varepsilon(x) = u_t(x - \varepsilon) \). The function \( u_t^\varepsilon \) satisfies (5.1) with \( f = g = 0 \) in \((0, T) \times (\varepsilon, 2^{-m/2} + \varepsilon) \) and in \((0, T) \times D \). Furthermore, (a.s.) for almost any \( t \in (0, T) \) we have \( D_x u_t \in L_p(D) \) implying that the limit of \( u_t^\varepsilon(x) \) as \( x \downarrow \varepsilon \) exists. Since (a.s.) for almost all \( t \in (0, T) \) also \((x - \varepsilon)^{-1} u_t^\varepsilon \in L_p(D) \), the limit is zero. As \( x \uparrow 2^{-m/2} - \varepsilon \) the situation is simpler and we see that (a.s.) for almost all \( t \in (0, T) \) we have

\[
\lim_{D \ni x \to 0D} (u_t^\varepsilon(x) - v_m(t, x) \rho_t^\varepsilon)^+ = 0.
\]

Furthermore, (a.s.) for almost all \( t \in (0, T) \) it holds that \( u_t^\varepsilon \in W^1_2(D) \) and

\[
\int_0^T \|u_t^\varepsilon\|^2_{W^1_2(D)} \, dt < \infty.
\]

Combining this with Remark 4.4 we see that (a.s.) for almost all \( t \in (0, T) \) we have \((u_t^\varepsilon - v_m(t, \cdot) \rho_t^\varepsilon)^+ \in W^1_2(D)\), \((u_t^\varepsilon - v_m(t, \cdot) \rho_t^\varepsilon)^+ \in W^1_2(D)\) and

\[
\int_0^T \|(u_t^\varepsilon - v_m(t, \cdot) \rho_t^\varepsilon)^+\|^2_{W^1_2(D)} \, dt < \infty.
\]

By Theorem 12 we conclude that almost surely for all \( t \leq T \) and \( x \in D \)

\[
u_t^\varepsilon(x) \leq v_m(t, x) \rho_t^\varepsilon.
\]

By combining this with similar inequality for \(-u_t^\varepsilon \) we obtain (5.4). The lemma is proved.

**Proof of Theorem 5.1** Clearly, we only need prove Theorem 5.1 for \( f \) and \( g \) such that \( f_t(x) = g_t(x) = 0 \) for all \( \omega, t \) if \( x \) is small. Then

\[
f \in M^{-1} \mathbb{L}_{p, \vartheta}(T), \quad g \in H^1_{p, \vartheta}(T)
\]

(5.5)

for any \( \vartheta \).

According to Lemma 3.6 of \([10]\), for each stopping time \( \tau_n \leq T \), we have \( u \in M^{H^2_{p, \vartheta}}(\tau_n) \) if \( u \in M_{p, \vartheta}(\tau_n) \) and under this condition the left-hand side of (5.2) is dominated by a constant \( N = N(\theta, p, \delta_0, \delta_1) \) times

\[
\|M^{-1} u\|^p_{\mathbb{L}_{p, \vartheta}(\tau_n)} + E \int_0^{\tau_n} \int_0^\infty x^{\theta-1} |F_t(x)|^p \, dx \, dt,
\]

(5.6)

where

\[
F_t(x) := |xf_t(x)| + |g_t(x)| + |xD_x g_t(x)|.
\]
Observe that obviously $(\alpha, \gamma(c) \leq 1)$
\[ \theta > \theta_0 > \mu \] (5.7)
and since $f_t(x) = g_t(x) = 0$ for $x \geq 1$, the integral involving $F_t$ will increase if we replace $\theta$ with $\mu$. It follows that to prove the theorem, it suffices to estimate only the lowest norm of $u$, that is to prove the existence of $\tau_n \uparrow T$ such that
\[ \|M^{-1}u\|^p_{L_{p,0}(\tau_n)} \leq n[\|Mf\|^p_{L_{p,0}(\tau_n)} + \|g\|^p_{H^1_{p,0}(\tau_n)}]. \] (5.8)

Next, take a $\vartheta \in [p-1, p)$ such that $\vartheta > \theta$. For any stopping time $\tau \leq T$, by Lemma 4.3 of [10] we have $u \in M^H_{2,p,\vartheta}(\tau)$ and by Theorem 3.2 of [10]
\[ E\int_0^\tau \int_0^\infty \frac{|u_t(x)/x|^p}{\vartheta} \, dx \, dt \leq N[\|Mf\|^p_{L_{p,0}(\tau)} + \|g\|^p_{H^1_{p,0}(\tau)}], \] (5.9)
where $N = N(p, \vartheta, \delta_0, \delta_1)$. As before on the right we can replace $\theta$ with $\mu$. On the left one can replace $\vartheta$ with $\theta$ if one restricts the domain of integration with respect to $x$ to $x \geq 1$. Therefore (5.8) will be proved if we prove the existence of appropriate stopping times $\tau_n$ such that
\[ E\int_0^{\tau_n} \int_0^1 \frac{|u_t(x)/x|^p}{\theta} \, dx \, dt \leq n[\|Mf\|^p_{L_{p,0}(\tau_n)} + \|g\|^p_{H^1_{p,0}(\tau_n)}]. \] (5.10)

Take a nonnegative $\eta \in C_0^\infty(\mathbb{R}_+)$ with support in $(1, 4)$ such that the $(1/2)$-periodic function on $\mathbb{R}$
\[ \sum_{k=-\infty}^{\infty} \eta(2^{x+k/2}) \]
is identically equal to one. Introduce,
\[ \eta_{m}(x) = \eta(2^{m/2}x), \quad (f_{mt}, g_{mt}) = (f_t, g_t)\eta_m. \]

Also introduce $u_{mt}$ as solutions of class $M^H_{2,p-1}(T)$ of (5.11) with zero initial condition and $f_{mt}$ and $g_{mt}$ in place of $f_t$ and $g_t$, respectively. Since only finitely many $f_{mt}$ and $g_{mt}$ are not zero, we have
\[ u_t(x) = \sum_{m=1}^{\infty} u_{mt}(x) = I_1(t, x) + I_2(t, x), \]
where
\[ I_1(t, x) := \sum_{m=1}^{\infty} u_{mt}(x)I_{x \leq 2^{-m/2}}, \quad I_2(t, x) := \sum_{m=1}^{\infty} u_{mt}(x)I_{x > 2^{-m/2}}. \]

Estimating $I_2$. Take a $\vartheta$ as above, set $\varepsilon = (\vartheta - \theta)/(2p)$ and use Hölder’s inequality to obtain
\[ |I_2(t, x)|^p \leq \sum_{m=1}^{\infty} 2^{\varepsilon m n} u_{mt}^p(x)J^{p/q}(x), \]
where
\[
J(x) := \sum_{m=1}^{\infty} 2^{-\varepsilon q m} I_{x > 2^{-m/2}} \leq N x^{2\varepsilon q}, \quad J^{p/q}(x) \leq N x^{\theta - \varepsilon q q}.
\]

Then use (5.9) again to get
\[
E \int_{\tau}^{0} \int_{0}^{1} x^{\theta-1} |I_{2}(t, x)| x^p \, dx \, dt
\leq N \sum_{m=1}^{\infty} E \int_{\tau}^{0} \int_{0}^{\infty} 2^{m(\theta-\varepsilon q q)/2} x^{\varepsilon q q} \, dx \, dt
\leq N \sum_{m=1}^{\infty} E \int_{\tau}^{0} \int_{0}^{\infty} 2^{m(\theta-\varepsilon q q)/2} x^{\varepsilon q q} \, dx \, dt,
\]
where
\[
F_{mt}(x) = |x f_{mt}(x)| + |g_{mt}(x)| + |x D_x g_{mt}(x)|.
\]

Here we notice few facts, which will be also used in the future, that on the supports of \( f_{mt}(x) \) and \( g_{mt}(x) \) we have \( x \sim 2^{-m/2}, 2^{m(\theta-\varepsilon q q)/2} F_{mt}(x) \sim x^{\theta-\varepsilon q q} F_{mt}(x) \) and
\[
F_{mt}(x) \leq F_{t}(x) \tilde{\eta}_m(x)
\]
where \( \tilde{\eta}_m(x) = \eta_m(x) + x^{2m/2} |\eta'(2^{m/2} x)| \). Notice that the (1/2)-periodic function
\[
\sum_{m=-\infty}^{\infty} \tilde{\eta}_m^p(2y)
\]
is bounded on \( \mathbb{R} \). Then we see that
\[
E \int_{\tau}^{0} \int_{0}^{1} x^{\theta-1} |I_{2}(t, x)| x^p \, dx \, dt
\leq N E \int_{\tau}^{0} \int_{0}^{\infty} x^{\theta-1} |F_{t}(x)|^p \sum_{m=1}^{\infty} \tilde{\eta}_m^p(x) \, dx \, dt
\leq N [||M f||_{L_p, \theta}^p + ||g||_{H^1_{p, \theta}}^p]
\]
for any \( \tau \leq T \) with a constant \( N \) under control. As above we can reduce \( \theta \) in the last expression to \( \mu \).

**Estimating \( I_1 \).** Here we will see how \( \tau_n \) appear and how we get a substantial drop from \( \theta \) to \( \mu \). We have seen above that the smaller \( \mu \) is the weaker the statement of the theorem becomes. Therefore, we may concentrate on \( \mu \) so close to \( p(1 + 2 \log_2 \gamma(c)) - 2 \) from below that
\[
2 < \beta p := p(1 + 2 \log_2 \gamma(c)) - \mu \leq p.
\]
Then
\[
2/p < \beta \leq 1.
\]
Observe that
\[
|J_1(t, x)|^p \leq \left( \sum_{m=1}^{\infty} m^{-q} \right)^{p/q} \sum_{m=1}^{\infty} m^p |u_{mt}(x)|^p I_{x \leq 2^{-m/2}} \leq N |\log_2 x|^p \sum_{m=1}^{\infty} |u_{mt}(x)|^p I_{x \leq 2^{-m/2}}.
\]

It follows that for any \( \theta' < \theta \)
\[
E \int_0^\tau \int_0^1 x^{\theta'-1} |J_1(t, x)|/x^p \, dx \, dt 
\leq N \sum_{m=1}^{\infty} E \int_0^\tau \int_0^{2^{-m/2}} x^{\theta'-1} |u_{mt}(x)/x|^p \, dx \, dt 
\leq NJ(\tau),
\]
where
\[
J(\tau) := \sum_{m=1}^{\infty} E \int_0^\tau \int_0^{2^{-m/2}} x^{\theta'-1} |u_{mt}(x)/x|^p \, dx \, dt.
\]

By Theorem 4.7 of [8] and Theorem 3.3 of [11], for any \( \tau \leq T \)
\[
E \sup_{t \leq \tau, x>0} |x^\varepsilon u_{mt}(x)|^p \leq NT^{\beta p/2} E \int_0^\tau \int_0^{\infty} x^{\theta_1-1} |F_{mt}(x)|^p \, dx \, dt,
\]
where \( N = N(p, \delta_0, \delta_1, \beta) \) and
\[
\theta_1 := p-1, \quad \varepsilon := \beta - 1 + \theta_1/p = \beta - 1/p > 0.
\]

Therefore,
\[
E \sup_{t \leq \tau} |u_{mt}(2^{-m/2})|^p \leq 2^{m(\beta p - p + \theta_1)/2} E \sup_{t \leq \tau, x>0} |x^\varepsilon u_{mt}(x)|^p
\leq NE \int_0^\tau \int_0^{\infty} 2^{m(\beta p - p + \theta_1)/2} x^{\theta_1-1} |F_{mt}(x)|^p \, dx \, dt
\leq N 2^{m(\beta p + 1)/2} E \int_0^\tau \int_0^{\infty} |F_{mt}(x)|^p \, dx \, dt.
\]

Next, observe that, by Lemma 5.3 for \( x \in [0, 2^{-m/2}] \) and \( t \leq T \),
\[
|u_{mt}(x)| \leq v_m(t, x) \sup_{s \leq t} |u_{mst}(2^{-m/2})|.
\]

Hence
\[
J(\tau) \leq \sum_{m=1}^{\infty} E \sup_{t \leq \tau} |u_{mt}(2^{-m/2})| \int_0^\tau \int_0^{2^{-m/2}} x^{\theta'-1} |v_m(t, x)/x|^p \, dx \, dt
\leq \sum_{m=1}^{\infty} 2^m (\nu p - 1)/(2\alpha) E \sup_{t \leq \tau} |u_{mt}(2^{-m/2})| \int_0^\tau \pi_t \, dt,
\]
where \( \nu \) is defined according to
\[
\nu p = p - \theta' + 1
\]
and \( \pi_t \) is introduced in Theorem \[4.3\]. So far \( \theta' < \theta \), so that \( \nu p > 1 \). Due to the assumption that \( \theta > \theta_0 \) one can satisfy \( \theta_0 < \theta' < \theta \) in which case \[4.3\] holds. Then in light of Theorem \[4.3\] one can find stopping times \( \tau_n \uparrow T \) such that

\[
\int_0^{\tau_n} \pi_t \, dt \leq n.
\]

Then

\[
J(\tau_n) \leq nN \sum_{m=1}^{\infty} 2^{m(\nu p - 1)/(2\alpha)} 2^{m(\beta p - p + 1)/2} E \int_0^{\tau_n} \int_0^\infty |F_{mt}(x)|^p \, dx \, dt.
\]

As is easy to see the inequalities \( \theta' > \theta_0 \) and

\[
\beta p - p + (\nu p - 1)/\alpha < -\mu
\]

are equivalent. Hence,

\[
E \int_0^{\tau_n} \int_0^1 x^{\theta-1} |I_1(t,x)/x|^p \, dx \, dt \leq NJ(\tau_n)
\]

\[
\leq nN \sum_{m=1}^{\infty} 2^{m(1-\mu)/2} E \int_0^{\tau_n} \int_0^\infty |F_{mt}(x)|^p \, dx \, dt
\]

\[
\leq nN \sum_{m=1}^{\infty} E \int_0^{\tau_n} \int_0^\infty x^{\mu-1} |F_{mt}(x)|^p \, dx \, dt
\]

\[
\leq nNE \int_0^{\tau_n} \int_0^\infty x^{\mu-1} |F_{1}(x)|^p \, dx \, dt.
\]

By combining this estimate with the estimate of \( I_2 \), noticing that the above constants \( N \) are independent of \( f \) and \( g \) and, if necessary, renumbering the sequence \( \tau_n \) we come to \[5.10\]. This proves the theorem.

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