SCALING LIMITS OF CORRELATIONS OF CHARACTERISTIC POLYNOMIALS
FOR THE GAUSSIAN $\beta$-ENSEMBLE WITH EXTERNAL SOURCE

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Abstract. We study the averaged product of characteristic polynomials of large random matrices in the Gaussian $\beta$-ensemble perturbed by an external source of finite rank. We prove that at the edge of the spectrum, the limiting correlations involve two families of multivariate functions of Airy and Gaussian types. The precise form of the limiting correlations depends on the strength of the nonzero eigenvalues of the external source. A critical value for the latter is obtained and a phase transition phenomenon similar to that of [2] is established. The derivation of our results relies mainly on previous articles by the authors, which deal with duality formulas [18] and asymptotics for Selberg-type integrals [22].

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1. Introduction

1.1. Gaussian ensembles with source. Let $X$ and $F$ be $N \times N$ hermitian matrices with either real ($\beta = 1$), complex ($\beta = 2$) or quaternion real ($\beta = 4$) entries. We say that $X$ belongs to the Gaussian ensemble with external source $F$ if it is randomly distributed according to a probability density function proportional to

$$\exp \left\{ -\frac{\beta}{2} \text{Tr}(X - F)^2 \right\}. \tag{1.1}$$

Note that the Gaussian ensemble with external source is also called the shifted mean Gaussian ensemble. Obviously, when $F$ is the null matrix, the three classical Gaussian ensembles -- that is GOE ($\beta = 1$), GUE ($\beta = 2$), and GSE ($\beta = 4$) -- are recovered.

Now let $x = (x_1, \ldots, x_N)$ and $f = (f_1, \ldots, f_N)$ respectively denote the eigenvalues of $X$ and $F$. Then, the use of standard techniques in random matrix theory and Jack polynomial theory (see [24, Chapters 1 & 13], [31] Chapter VII) allows to show that the probability density for the eigenvalues of $X$ is equal to

$$\frac{1}{G_{\beta,N}} \exp \left\{ -\frac{\beta}{2} \sum_{i=1}^{N} (x_i^2 + f_i^2) \right\} \prod_{1 \leq j < k \leq N} |x_j - x_k|^\beta \, \phi_{0}^{(2/\beta)}(\beta x; f). \tag{1.2}$$

The normalization constant is a special case of Selberg’s celebrated formula and is given in Appendix A while $\phi_{0}^{(2/\beta)}$ is a multivariate hypergeometric function of exponential type whose exact expansion in terms of the Jack polynomials is known explicitly (see Section 2). When $f = (0, \ldots, 0)$, the density (1.2) defines the now standard Gaussian $\beta$-ensemble of random matrices (see for instance, [12, 20, 23, 34, 35] and [24, Section 1.9]).

We stress that for all $f \in \mathbb{R}^N$ and $\beta > 0$, Eq. (1.2) provides a well-defined density, even if $\beta \neq 1, 2, 4$. It is thus natural to define, like in [18, 26, 39], the Gaussian $\beta$-ensemble with external source as the set of real random variables $x = (x_1, \ldots, x_N)$ distributed according to the density (1.2).

A typical problem in random matrix theory is to determine the influence of $f_1, \ldots, f_N$ on the distribution of $x_1, \ldots, x_N$ as $N \to \infty$. This was first addressed in physics in the case where the matrices are real ($\beta = 1$) and all the entries of $F$ are equal to $\mu$, so that $f = (N \mu, 0, \ldots, 0)$. Indeed, in the mid 1960s, Lang [31] gave theoretical arguments in favor of a phenomenon first noticed by Porter with the help of numerical simulations: if $\mu$ is large enough, then one eigenvalue of $X$ separates from the main support for the eigenvalues, which is $[-\sqrt{2N}, \sqrt{2N}]$. Jones et al. (see [27]) later proved the existence of critical value $\mu_c = (2N)^{-1/2}$ that splits the statistical behavior of the eigenvalues of $X$ into two separate phases: if $\mu < \mu_c$, then the eigenvalues are distributed as if $\mu = 0$, while if $\mu > \mu_c$, then one eigenvalue completely separates from the others.

Few progresses were made on the distribution of the eigenvalue $x_1, \ldots, x_N$ in the real case ($\beta = 1$) until the very recent works of Bleumendal, Virág, Mo, and Wang [10, 11, 12, 35]. In fact, the latter references were motivated, to a large extent, by the recent breakthroughs in the complex case ($\beta = 2$) [2].

The ensembles with external source in the $\beta = 2$ case are indeed very special, since the Harish-Chandra-Itzykson-Zuber integral formula provides the following compact expression:

$$i^{N(N-1)/2} \frac{1}{1!2!\cdots(N-1)!} \phi_{0}^{(1)}(s; iw) = \frac{\det(e^{is_j w_k})}{\det(s_j^{k-1}) \det(w_j^{k-1})}. \tag{1.3}$$

Thus, when $\beta = 2$, the Gaussian ensemble with external source is a determinantal process. It was extensively studied by several authors since 1996, starting with Brézin and Hikami [13, 15], Zinn-Justin [37, 38], and Bleher and Kuijlaars [8, 9]. In the latter references, the eigenvalue correlation functions (marginal densities) were shown to be exactly computable in terms of multiple Hermite polynomials [17, 21, 29]. A physical interpretation for this ensemble was also proposed (see [16] and references therein): setting

$$\lambda_i = \sqrt{\frac{2t(1-t)}{N}} x_i, \quad \pi_i = \sqrt{\frac{2(1-t)}{Nt}} f_i, \quad 1 \leq i \leq N, \quad 0 < t < 1,$$
the $\beta = 2$ eigenvalue density, given by (1.2) and (1.3), becomes equal to the density at time $t$ for $N$ independent non-intersecting Brownian motions on the line, $(\lambda_1, \ldots, \lambda_N)$, such that each $\lambda_i$ starts $(t = 0)$ at the origin and ends $(t = 1)$ at the point $\pi_i$. If we suppose that the external source $F$ is only of finite rank $r$, which means

$$f_1 \neq 0, \ldots, f_r \neq 0, \quad f_{r+1} = 0, \ldots, f_N = 0, \quad \lim_{N \to \infty} \frac{r}{N} = 0, \quad (1.4)$$

then we obtain a particularly beautiful model of Brownian motions with a few outliers [1].

The interest in ensembles with finite rank external source was prompted by the work of Baik, Ben Arous, and Pêché [2]. While analyzing the distribution function for the largest eigenvalue, $x_1$ say, of the spiked complex Wishart model, the authors discovered (completely independent from [27, 30]) a phase transition phenomenon and obtained the limiting distributions for $x_1$, both at the critical point and away from the critical point. This analysis was almost immediately adapted by Pêché [33] to the GUE with finite rank external source, which in our notation, is defined by the eigenvalue density (1.2) with $\beta = 2$ and equation (1.4). It is actually more convenient to rescale the variables as follows:

$$x_i = \sqrt{\frac{N}{2}} \lambda_i, \quad f_i = \sqrt{\frac{N}{2}} \pi_i, \quad i = 1, \ldots, N. \quad (1.5)$$

For a rank $r = 1$ perturbation, Pêché found three phases for the distribution of the largest eigenvalue $\lambda_1$ in the neighborhood of the soft edge of the spectrum. Following the nomenclature used in [33], the phases are divided as follows:

**Subcritical regime:** If $\pi_1 < 1$, then $\lim_{N \to \infty} \text{Prob} \left( N^{2/3} (\lambda_1 - 2) \leq y \right) = F_2(y)$.

**Critical regime:** If $\pi_1 = 1$, then $\lim_{N \to \infty} \text{Prob} \left( N^{2/3} (\lambda_1 - 2) \leq y \right) = F_{2+1}(y)$.

**Supercritical regime:** If $\pi_1 > 1$, $c = \pi_1 + 1/\pi_1$, and $\sigma^2 = \pi_1^2/(\pi_1^2 - 1)$, then

$$\lim_{N \to \infty} \text{Prob} \left( \sigma^2 \sqrt{N} (\lambda_1 - c) \leq y \right) = \frac{1}{2} \left( 1 + \text{erf} \left( \frac{y}{\sigma \sqrt{2}} \right) \right).$$

See [33] for the definition of the distribution functions $F_{2+k}$. Ensembles of complex hermitian matrices with finite rank external source were subsequently studied by many authors, see for instance [1, 3, 5–7, 10, 11, 36, 39, 40].

1.2. Goals. We are interested in studying correlations of characteristic polynomials for the Gaussian $\beta$-ensemble with external source, when $\beta$ is any positive real and the finite rank condition (1.4) is satisfied. So far, few authors have worked on $\beta$-ensembles with external source for generic values of $\beta$. Dualities relating expectation values of products of characteristic polynomials were studied in [18, 26]. The latter reference also contains the limiting expectation value of a single characteristic polynomial in the critical regime. In [25, 26], different approaches used for defining $\beta$-ensembles with external source were shown to be equivalent. Finally, the distribution of the largest eigenvalue was studied in [10], [11], [36], where a phase transition phenomenon, completely similar to that described previously, was also revealed.

More specifically, we aim to get exact closed-form expressions for the asymptotic limit of the expectation value of a product of $n$ characteristic polynomials. We moreover want to prove that the phase transition at the soft edge is observable not only for the distribution of the largest eigenvalue, as noticed in [10], but also at the level of correlations of characteristic polynomials.

Thus, throughout the article, we want to determine the large $N$ behavior, under the assumption of the finite rank condition (1.4), of the following expectation value of products of characteristic polynomials:

$$K_{\beta,N}(s_1, \ldots, s_n; f_1, \ldots, f_r) = \frac{1}{G_{\beta,N}} \int_{\mathbb{R}^N} \prod_{j=1}^{n} \prod_{i=1}^{N} \left( s_j - x_i \right) e^{-\frac{\beta}{2} \sum_{i=1}^{N} (x_i^2 + f_i^2)} |\Delta_N(x)|^\beta 0^{(2/\beta)}(\beta x; f) d^N x, \quad (1.6)$$
where we have used a shorthand notation for the Vandermonde determinant, that is,
\[ \Delta_N(x) = \prod_{1 \leq j < k \leq N} (x_k - x_j). \quad (1.7) \]

Obviously, the integral representation (1.6) is not suitable for studying the asymptotic limit of \( K_{\beta,N}(s; f) \) as \( N \to \infty \). However, as was shown in [18, Proposition 7], there exists a duality formula that provides an alternative \( n \)-dimensional integral representation for (1.6):
\[ K_{\beta,N}(s; f) = D_{\beta,N,n} e^{\sum_{j=1}^{n} s_j^2} \int_{\mathbb{R}^n} \prod_{j=1}^{n} \prod_{k=1}^{N} (i f_k - y_j) e^{-\sum_{j=1}^{n} y_j^2} |\Delta_n(y)|^{4/\beta} \phi_0(\beta/2)(y; 2is) d^m y, \quad (1.8) \]
where \( D_{\beta,N,n} \) denotes a constant, which is given in \( \Delta.4 \). Notice that one essentially goes from (1.6) to (1.8) by changing \((\beta, N, n, f, s)\) into \((4/\beta, n, N, s, f)\).

1.3. Main results. We now give the asymptotic limits for the averaged product of \( n \) characteristic polynomials. It is actually more convenient to display the results for the weighted expectation
\[ \varphi_{\beta,N}(s; f) = e^{-\frac{1}{2} \sum_{j=1}^{n} s_j^2} K_{\beta,N}(s; f). \quad (1.9) \]

1.3.1. Multivariate functions. The asymptotic results are written in terms of new multivariate functions, which are of Airy and Gaussian types. They are defined below, but will be further studied in Section 2. Note that in the following definitions, it is understood that \( \prod_{k=1}^{m} (iw_j + f_k) = 1 \) whenever \( m = 0 \).

**Definition 1.1.** For \( s \in \mathbb{R}^n_+ \) and \( f \in \mathbb{C}^m \), the incomplete multivariate Airy function is
\[ \text{Ai}^{(\alpha)}_{n,m}(s; f) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{\sum_{j=1}^{n} w_j^2/3} \prod_{j=1}^{n} \prod_{k=1}^{m} (iw_j + f_k) |\Delta_n(w)|^{2/\alpha} \phi_0(\alpha/2)(s; iw) d^m w. \quad (1.10) \]

For \( s \in \mathbb{C}^n \setminus \mathbb{R}^n_+ \), the function \( \text{Ai}^{(\alpha)}_{n,m} \) is defined similarly, except that each variable \( w_j \) follows a complex path going from \(-\infty + i\delta\) to \(\infty + i\delta\) for some \(\delta > 0\).

It is worth mentioning that for \( \alpha = 1 \) and \( m = 0 \), the above function is equivalent to Kontsevich’s version of the matrix Airy function [28]. Moreover, for all \( \alpha > 0 \),
\[ \text{Ai}^{(\alpha)}_{n,m}(s; f) \bigg|_{m=0} = \text{Ai}^{(\alpha)}(s), \]
where the function on the RHS is the multivariate Airy function in one set of variables, which previously appeared in [18,22]. Asymptotic series of \( \text{Ai}^{(\alpha)}_{n,m}(s; f) \) as \( s_j \to \pm \infty \) will be given later in Proposition 2.2.

**Definition 1.2.** For \( s \in \mathbb{C}^n \) and \( f \in \mathbb{C}^m \), the multivariate Gaussian function is
\[ G^{(\alpha)}_{n,m}(s; f) = \frac{1}{\Gamma_{2/\alpha,n}} \int_{\mathbb{R}^n} e^{-\sum_{j=1}^{n} w_j^2/2} \prod_{j=1}^{n} \prod_{k=1}^{m} (iw_j + f_k) |\Delta_n(w)|^{2/\alpha} \phi_0(\alpha)(s; iw) d^m w, \quad (1.11) \]
where the constant \( \Gamma_{2/\alpha,n} \) is given in \( \Delta.3 \).

Let \( \beta = 2/\alpha \). Then, \( G^{(\alpha)}_{n,m}(s; f) \) is proportional to the expectation of a product of \( m \) characteristic polynomials for matrices of size \( n \) in the Gaussian \( \beta \)-ensemble with external source. As will be proved in Section 2, this multivariate Gaussian function can be written explicitly as a series involving Jack polynomials and multivariate Hermite polynomials.
1.3.2. Soft edge limits. In order to get the correlations of characteristic polynomials at the soft edge of the spectrum, we have to rescale the spectral variables either as

\[ s_j = \sqrt{\frac{N}{2}} \mu + \frac{\tilde{s}_j}{\sqrt{2N^{1/3}}}, \quad j = 1, \ldots, n, \]

or as

\[ s_j = \sqrt{\frac{N}{2}} \mu + \frac{\tilde{s}_j}{\sqrt{2\sigma}}, \quad j = 1, \ldots, n. \]

(1.13)

Note that \( \mu \) and \( \sigma \) are real positive parameters. The sources must also be rescaled as

\[ f_k = \sqrt{\frac{N}{2}} \pi_k, \quad k = 1, \ldots, N. \]

(1.14)

To avoid any confusion, we rewrite the finite rank criterion (1.4) as follows:

\[ \pi_1 \geq \cdots \geq \pi_r, \quad \pi_{r+1} = \cdots = \pi_N = 0. \]

(1.15)

We stress that the order of \( \pi_j \)'s is not essential and the key point is whether they equal the critical values or not.

The next theorems describe the phase transition phenomenon. If \( \pi_1 < 1 \), then the correlation \( \varphi_{\beta,N}(s; f) \) is asymptotically the same as the correlation for a \( \beta \)-ensemble without external source, which was obtained in [22]. For \( \pi_1 = 1 \) and nearby, new asymptotic correlations occur, but are still of Airy type. For \( \pi_1 > 1 \), the correlations become those that one would normally observe for matrices of size \( n \) in a Gaussian 4/\( \beta \)-ensemble. We stress that the scalings (1.12) and the critical value, which is \( \pi_1 = 1 \), do not depend on \( \beta \), in accordance with what was found for the distribution of the largest eigenvalue [10].

**Theorem 1.3** (Subcritical regime). Assume (1.12), (1.14), (1.15), \( \mu = 2 \), and \( \pi_1 < 1 \). Then, as \( N \to \infty \),

\[ \frac{\varphi_{\beta,N}(s_1, \ldots, s_n; f_1, \ldots, f_r)}{\Phi_{\beta,N,n}} \sim \prod_{k=1}^r (1 - \pi_k)^n A_{1}(\beta/2)(\tilde{s}_1, \ldots, \tilde{s}_n). \]

(1.16)

The constant \( \Phi_{\beta,N,n} \) is given in Appendix A.

**Theorem 1.4** (Critical regime). Assume (1.12), (1.14), (1.15), \( \mu = 2 \). Suppose moreover that

\[ \pi_1 = 1 + \frac{\pi_1}{N^{1/3}}, \quad \pi_2 = 1 + \frac{\pi_2}{N^{1/3}}, \quad \ldots, \quad \pi_m = 1 + \frac{\pi_m}{N^{1/3}}, \]

while \( \pi_k \) belongs to a compact subset of \( (-\infty, 1) \) for all \( m + 1 \leq k \leq r \). Then, as \( N \to \infty \),

\[ \frac{\varphi_{\beta,N}(s_1, \ldots, s_n; f_1, \ldots, f_r)}{\Phi_{\beta,N,n,m}} \sim \prod_{k=m+1}^r (1 - \pi_k)^n A_{1}(\beta/2)(\tilde{s}_1, \ldots, \tilde{s}_n; \tilde{\pi}_1, \ldots, \tilde{\pi}_m). \]

(1.17)

The constant \( \Phi_{\beta,N,n,m} \) is given in Appendix A.

**Theorem 1.5** (Supercritical regime). Assume (1.13), (1.14), (1.15), \( \mu > 2 \). Let \( \sigma^2 = \nu^2/(\nu^2 - 1) \), where \( \nu > 1 \) is such that \( \mu = \nu + \nu^{-1} \). Suppose moreover that

\[ \pi_1 = \nu + \frac{\sigma \pi_1}{N^{1/2}}, \quad \pi_2 = \nu + \frac{\sigma \pi_2}{N^{1/2}}, \quad \ldots, \quad \pi_m = \nu + \frac{\sigma \pi_m}{N^{1/2}}, \]

while \( \pi_k \) belongs to a compact subset of \( (-\infty, \nu) \) for all \( m + 1 \leq k \leq r \). Then, as \( N \to \infty \),

\[ \prod_{j=1}^n e^{2(\nu-\nu^2)N \tilde{s}_j} \frac{\varphi_{\beta,N}(s_1, \ldots, s_n; f_1, \ldots, f_r)}{\Phi_{\beta,N,n,m}^{\sup}} \sim \prod_{k=m+1}^r (\nu - \pi_k)^n \prod_{j=1}^n e^{\frac{1}{\nu_2} s_j^2} G_{n,m}^{(\beta/2)}(\tilde{s}_1, \ldots, \tilde{s}_n; \tilde{\pi}_1, \ldots, \tilde{\pi}_m). \]

(1.18)

The constant \( \Phi_{\beta,N,n,m}^{\sup} \) is given in Appendix A.
1.3.3. **Bulk limits.** Different scalings are required for this part of the spectrum. For \( u \in (-1, 1) \), \( 1 \leq j \leq n \), and \( 1 \leq k \leq r \), let

\[
s_j = \sqrt{2Nu} + \frac{\pi \bar{s}_j}{\sqrt{2N(1-u^2)}} \quad \text{and} \quad f_k = \sqrt{\frac{N}{2}(u + \sqrt{1-u^2}r_k}).
\]

The scaling correlations in the bulk are given below. In contradistinction with the soft edge case, the limit correlations in the bulk can be continuously deformed into that for the ensemble without source [22] – here corresponding to \( \pi_j = 0, j = 1, \ldots, r \) – so no phase transition has been found in this part of the spectrum.

**Theorem 1.6** (Bulk limit). Assume \((1.19)\) and \((1.15)\). Then as \( N \to \infty \)

\[
(\Psi_{N,2m})^{-1} \varphi_{\beta,N}(s; f) \sim \gamma_{m}(\frac{N}{\beta}) \prod_{k=1}^{r} (1 + \pi_k^2)^m e^{-i\pi \sum_{j=1}^{r} \bar{s}_j} F_1^{(\beta/2)}(2m/\beta; 2n/\beta; 2i\pi \bar{s})
\]

for \( n = 2m \) while

\[
\frac{1}{\Psi_{N,2m-1}^{(0)}} \left\{ \frac{\varphi_{\beta,N}(s; f) \varphi_{\beta,N-1}(s'; f) - \varphi_{\beta,N}(s'; f) \varphi_{\beta,N-1}(s; f)}{2\pi \sum_{j=1}^{r} (\bar{s}_j' - \bar{s}_j)} \right\} \sim \prod_{k=1}^{r} (1 + \pi_k^2)^{2m-1}
\]

for \( n = 2m-1 \). The coefficients are given in Appendix A.

**Remark 1.7** (Slowly growing rank case). When the rank \( r = r_N \) depends on \( N \), but grows slowly with \( N \), our main results remain true. To be more precise, let us replace the finite rank condition \((1.15)\) by

\[
\pi_1 \geq \cdots \geq \pi_r, \quad \pi_{r+1} = \cdots = \pi_N = 0, \quad \lim_{N \to \infty} \frac{r}{N} = 0 \quad \text{for some } b \in (0, 1].
\]

We suppose moreover that the integer \( m \) of Theorems 1.4 and 1.5 is finite, and also that each \( \pi_k \) belongs to a compact subset of \((-\infty, 1)\) (resp. \( \mathbb{R} \)) for all \( 1 \leq k \leq r \) in Theorem 1.3 (resp. Theorem 1.6). Then, Theorems 1.3 to 1.6 hold respectively for \( b = 1/3, 1/3, 1/2, 1/2 \). See Section 5.4 for more details. This kind of slowly growing rank perturbation was studied by Pêché for the largest eigenvalue of the Gaussian Unitary Ensemble [33].

The rest of the paper is devoted to the study of the multivariate functions in Section 2, followed by the proof of Theorems 1.3 to 1.6 and 1.9.

2. **Jack polynomials and hypergeometric functions**

This section first provides a brief review of Jack polynomial theory and the associated multivariate hypergeometric functions. A recent textbook presentation of these multivariate functions can be found in Chapters 12 and 13 of Forrester’s book [24]; a classical reference for the Jack polynomials is Macdonald’s book [31, Chap. VI 10]. A few results stated here have been proved in the previous paper [22] and will be used later. We also derive new results on the multivariate functions of Gaussian type and of Airy type.

2.1. **Partitions.** A partition \( \kappa = (\kappa_1, \ldots, \kappa_i, \ldots) \) is a sequence of non-negative integers \( \kappa_i \) such that

\[
\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_r \geq \cdots
\]

and only a finite number of the terms \( \kappa_i \) are non-zero. The number of non-zero terms is referred to as the length of \( \kappa \), and is denoted by \( l(\kappa) \). We shall not distinguish between two partitions that differ only by a string of zeros. The weight of a partition \( \kappa \) is the sum

\[
|\kappa| := \kappa_1 + \kappa_2 + \cdots
\]

of its parts, and its diagram is the set of points \((i, j) \in \mathbb{N}^2 \) such that \( 1 \leq j \leq \kappa_i \). Reflection with respect to the diagonal produces the conjugate partition \( \kappa' = (\kappa_1', \kappa_2', \ldots) \).
The set of all partitions of a given weight is partially ordered by the dominance order: $\kappa \leq \sigma$ if and only if $\sum_{i=1}^{k} \kappa_i \leq \sum_{i=1}^{k} \sigma_i$ for all $k$.

2.2. Jack polynomials. Let $\Lambda_\kappa(x)$ denote the algebra of symmetric polynomials in $n$ variables $x_1, \ldots, x_n$ and with coefficients in the field $\mathbb{F}$. In this article, $\mathbb{F}$ is assumed to be the field of rational functions in the parameter $\alpha$. As a ring, $\Lambda_\kappa(x)$ is generated by the power-sums:

$$p_k(x) := x_1^k + \cdots + x_n^k. \quad (2.1)$$

The ring of symmetric polynomials is naturally graded: $\Lambda_n(x) = \oplus_{k \geq 0} \Lambda_n^k(x)$, where $\Lambda_n^k(x)$ denotes the set of homogeneous polynomials of degree $k$. As a vector space, $\Lambda_n^k(x)$ is equal to the span over $\mathbb{F}$ of all symmetric monomials $m_\kappa(x)$, where $\kappa$ is a partition of weight $k$ and

$$m_\kappa(x) := x_1^{\kappa_1} \cdots x_n^{\kappa_n} + \text{distinct permutations.}$$

Note that if the length of the partition $\kappa$ is larger than $n$, we set $m_\kappa(x) = 0$.

The whole ring $\Lambda_\kappa(x)$ is invariant under the action of homogeneous differential operators related to the Calogero-Sutherland models $[4]$:

$$E_k = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}, \quad D_k = \sum_{i=1}^{n} x_i^2 \frac{\partial^2}{\partial x_i^2} + \frac{2}{\alpha} \sum_{1 \leq i \neq j \leq n} \frac{x_i^k}{x_i - x_j} \frac{\partial}{\partial x_j} \quad (k \geq 0). \quad (2.2)$$

The operators $E_k$ and $D_k$ are special since they also preserve each $\Lambda_\kappa^k(x)$. They can be used to define the Jack polynomials. Indeed, for each partition $\kappa$, there exists a unique symmetric polynomial $P_\kappa^{(\alpha)}(x)$ that satisfies the following two conditions:

$$P_\kappa^{(\alpha)}(x) = m_\kappa(x) + \sum_{\mu < \kappa} c_{\kappa \mu} m_\mu(x) \quad \text{(triangularity)} \quad (2.3)$$

and

$$(D_2 - \frac{2}{\alpha} (n-1) E_1) P_\kappa^{(\alpha)}(x) = \epsilon_\kappa P_\kappa^{(\alpha)}(x) \quad \text{(eigenfunction)} \quad (2.4)$$

where the coefficients $c_\kappa$ and $c_{\kappa \mu}$ belong to $\mathbb{F}$. Because of the triangularity condition, $\Lambda_\kappa(x)$ is also equal to the span over $\mathbb{F}$ of all Jack polynomials $P_\kappa^{(\alpha)}(x)$, with $\kappa$ a partition of length less than or equal to $n$.

2.3. Hypergeometric series. Recall that the arm-lengths and leg-lengths of the box $(i, j)$ in the partition $\kappa$ are respectively given by

$$a_\kappa(i, j) = \kappa_i - j \quad \text{and} \quad l_\kappa(i, j) = \kappa_j' - i. \quad (2.5)$$

We define the hook-length of a partition $\kappa$ as the following product:

$$h_\kappa^{(\alpha)} = \prod_{(i, j) \in \kappa} \left( 1 + a_\kappa(i, j) + \frac{1}{\alpha} l_\kappa(i, j) \right), \quad (2.6)$$

and closely related is the following $\alpha$-deformation of the Pochhammer symbol:

$$[x]_{\kappa}^{(\alpha)} = \prod_{1 \leq i \leq \ell(\kappa)} \left( x - \frac{i-1}{\alpha} \right)_{\kappa_i} = \prod_{(i, j) \in \kappa} \left( x + a_\kappa'(i, j) - \frac{1}{\alpha} l_\kappa'(i, j) \right). \quad (2.7)$$

In the middle of the last equation, $(x)_j \equiv x(x+1) \cdots (x+j-1)$ stands for the ordinary Pochhammer symbol, to which $[x]^{(\alpha)}_{\kappa}$ clearly reduces for $\ell(\kappa) = 1$. The right-hand side of (2.7) involves the co-arm-lengths and co-leg-lengths box $(i, j)$ in the partition $\kappa$, which are respectively defined as

$$a_\kappa'(i, j) = j - 1 \quad \text{and} \quad l_\kappa'(i, j) = i - 1. \quad (2.8)$$

We are now ready to give the precise definition of the hypergeometric series used in the article. Fix $p, q \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and let $a_1, \ldots, a_p, b_1, \ldots, b_q$ be complex numbers such that $(i-1)/\alpha - b_j \notin \mathbb{N}_0$.
for all \( i \in \mathbb{N}_0 \). The \((p, q)\)-type hypergeometric series refers to
\[
pFq_p\left(\begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{array} ; x \right) = \sum_{k=0}^{\infty} \sum_{|\alpha|=k} \frac{1}{h_{\alpha}(k)} \frac{[a_1]_{\alpha}}{[b_1]_{\alpha}} \cdots \frac{[a_p]_{\alpha}}{[b_q]_{\alpha}} P_\alpha(x).
\]
(2.9)

Similarly, the hypergeometric series in two sets of \( n \) variables \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \) is given by
\[
pFq_p\left(\begin{array}{c} a_1, \ldots, a_p \\ b_1, \ldots, b_q \\ x \\ y \end{array} ; x \right) = \sum_{\kappa} \frac{1}{h_{\kappa}(\kappa)} \frac{[a_1]_{\alpha}}{[b_1]_{\alpha}} \cdots \frac{[a_p]_{\alpha}}{[b_q]_{\alpha}} F_\kappa(x) F_\kappa(y).
\]
(2.10)

where we have used the shorthand notation \( 1^n \) for \( 1, \ldots, 1 \). Note that when \( p \leq q \), the above series converge absolutely for all \( x \in \mathbb{C}^n, y \in \mathbb{C}^n \) and \( \alpha \in \mathbb{R}_+ \). In the case where \( p = q + 1 \), then the series converge absolutely for all \( |x_i| < 1, |y_i| < 1 \) and \( \alpha \in \mathbb{R}_+ \).

Now we give a translation property of \( _0F_0^{(\alpha)} \), which proves to be of practical importance. For convenience, we write
\[
(a^n) = (a, \ldots, a), \quad b + ax = (b + ax_1, \ldots, b + ax_n),
\]
where \( a, b \) are complex numbers and \( x = (x_1, \ldots, x_n) \). Then we have
\[
_0F_0^{(\alpha)}(a + x; b + y) = \exp\{nab + ap_1(y) + bp_1(x)\} \ _0F_0^{(\alpha)}(a; y)
\]
(2.11)

and
\[
_0F_0^{(\alpha)}(x_1, \ldots, x_n; a^k, b^{n-k}) = e^{bp_1(x)} \ _1F_1^{(\alpha)}(k/\alpha; n/\alpha; (a-b)x_1, \ldots, (a-b)x_n)
\]
\[
= e^{(a-b)x_1+bp_1(x)} \ _1F_1^{(\alpha)}(k/\alpha; n/\alpha; (a-b)(x_2-x_1), \ldots, (a-b)(x_n-x_1)).
\]
(2.12)

See [22] for the proofs of (2.11)–(2.13).

2.4. **Airy type functions.** The incomplete multivariate Airy function \( Ai_{n,m}^{(\alpha)} \) has been introduced in Definition 1.1. In the case of \( m = 0 \) and general \( \alpha \), this definition coincides with that of the multivariate Airy function of [18, 22]. For the special values of \( \alpha = 2, 1, 1/2 \), the RHS of (1.10) is proportional to the following matrix Airy function:
\[
\int \exp\{i\text{Tr}(\frac{1}{3}W^3 + SW)\} \prod_{j=1}^{m} \det(iW + f_j) dW,
\]
(2.14)

where both \( S \) and \( W \) are either symmetric \( (\alpha = 2) \), Hermitian \( (\alpha = 1) \) or self-dual quaternion \( (\alpha = 1/2) \) \( n \times n \) matrices. Actually, in order to ensure convergence of (2.14), one first assumes that \( S \) is positive definite. The latter matrix can be further restricted to be of diagonal form because of the orthogonal, unitary or symplectic conjugate invariance for the integral. One then extends the integral to the general case by imposing that the matrix entries of \( W \) to follow some contour in the complex plane. When \( m = 0 \) and \( \alpha = 1 \), the above integral definition was first given by Kontsevich [28].

**Proposition 2.1** (Closed-form expression for \( \alpha = 1 \)). Define the operators \( L_k \), for \( k = 1, 2, \ldots, \), as
\[
L_k h(x) = (\frac{d}{dx})^{k-1} \prod_{i=1}^{m} (\frac{d}{dx} + f_i) h(x),
\]
(2.15)

for some smooth function \( h(x) \). Then
\[
Ai_{n,m}^{(1)}(s; f) = \prod_{k=1}^{n} k! \frac{\det(L_k Ai(s_j))}{\det(s_{j-1}^k)}.
\]
(2.16)
Proof. The formula of the Harish-Chandra-Itzykson-Zuber integral gives
\[ \mathcal{F}_0^{(1)}(s; i\omega) = c \frac{\det(e^{is_j \omega_k})}{\det(s_j^{k-1})} \]
for some constant \( c \), see Section 4 of [28]. We next determine \( c \) by expanding the determinant \( \det(e^{is_j \omega_k}) \) for the series
\[ e^{is_j \omega_k} = \sum_{l \geq 0} \frac{l!}{l!} (s_j \omega_k)^l, \]
then set \( s_j = \omega_j = 0 \), and get \( c = i^{-n(n-1)/2} \prod_{j=1}^{n-1} j! \). Simple manipulations give the desired result (2.16).

We conclude this subsection with a proposition devoted to the asymptotic behavior of the incomplete multivariate Airy function. The proof will be given in Section 3. Notice the following shorthand notation: \((A + Bs)\) stands for \((A + Bs_1, \ldots, A + Bs_n)\) for \( A, B \in \mathbb{C} \).

**Proposition 2.2.** As the real positive variable \( x \to \infty \), the following hold.
(i) For some \( 0 \leq k \leq r \), let
\[ f_j = 1 + (2x^{3/2})^{-1/2} f_j \quad (1 \leq j \leq k) \text{ and } f_l = f_l \neq 1 \quad (k < l \leq r), \]
then
\[ A_{n,r}^{(\alpha)}(x + x^{-1/2} s; x^{1/2} f) \sim \frac{\Gamma_2(\alpha, n) \prod_{i=k+1}^r (f_i - 1)^n}{(2\pi)^n 2((1+k)n+n(n-1)/\alpha)^{1/2} x^{(1+2r-3k)n+n(n-1)/\alpha}/4} e^{-\sum_{j=1}^k s_j} G_{n,k}^{(\alpha)}(0; f). \]  

(ii) For \( n = 2m \),
\[ A_{n,r}^{(\alpha)}(-x + x^{-1/2} s; x^{1/2} f) \sim (2\pi)^{-n} (2m)! (\Gamma_2(\alpha, m))^2 (2\sqrt{x})^{-m+m(m+1)/\alpha} x^m \]
\[ \times \prod_{i=1}^r (1 + f_i^2)^m e^{-\sum_{j=1}^k s_j} F_1^{(\alpha)}(m/\alpha; n/\alpha; 2i). \]

**Remark 2.3.** Another type of multivariate function defined by
\[ \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i \sum_{j=1}^n w_j^3/3} \prod_{j=1}^n \prod_{k=1}^m (iw_j + f_k)^{-1/\alpha} |\Delta_n(w)|^{2/\alpha} \mathcal{F}_0^{(\alpha)}(s; i\omega) d^n w \]
may also be useful in Random Matrix Theory, where all \( f_k \) lie in the domain \( \{ z \in \mathbb{C} : \Re z \neq 0 \} \), although it is not used in the present paper.

2.5. Gaussian type functions. We have introduced the function \( G_{n,m}^{(\alpha)} \) in Definition 1.2. Obviously, for the cases \( \alpha = 2, 1, 1/2 \), this function can be interpreted as the expectation of a product of \( m \) characteristic polynomials:
\[ G_{n,m}^{(\alpha)}(s; f) = c_{\beta,n} \int \exp\left\{-\frac{1}{2} \text{Tr} W^2 + \text{Tr} SW\right\} \prod_{j=1}^m \det(iW + f_j) dW, \]
where \( c_{\beta,n} \) is a normalization constant while both \( S \) and \( W \) are either symmetric (\( \alpha = 2 \)), Hermitian (\( \alpha = 1 \)) or self-dual quaternion (\( \alpha = 1/2 \)) \( n \times n \) matrices.

Similarly to the Airy functions, there are determinantal formulas in the unitary (\( \alpha = 1 \)) case.

**Proposition 2.4** (Closed-form expression for \( \alpha = 1 \)). Let \( L_k \) be the operator in (214). Then,
\[ G_{n,m}^{(1)}(s; f) = \frac{\det(L_ke^{-s_j^2/2})}{\det(s_j^{k-1})}. \]

Other explicit formulas are given below.
Proposition 2.5 (Closed-form expression for \( m = 1 \)). Let \( f_1 = z, a = \sqrt{2/\alpha} \) and \( H_k(z) = \sum_{l=0}^{[k/2]} c_{e-l} z^{k-2l} \) denote the standard Hermite polynomial of degree \( k \). Define \( \bar{H}_k(z) = \sum_{l=0}^{[k/2]} (-1)^l c_{e-l} z^{k-2l} \). Then

\[
G_{n,1}(a; s; z) = \frac{e^{-p_2(s)/\alpha}}{\sqrt{2\pi}} \sum_{k=0}^{n} (-2)^n k e_{n-k}(s) \bar{H}_k(z),
\]

where \( e_k(s) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} s_{i_1} \cdots s_{i_k} \) denotes the elementary symmetric function of degree \( k \).

**Proof.** This directly follows from: (1) the duality formula \( \|N \), which gives

\[
G_{n,1}(s; z) = \frac{1}{\sqrt{2\pi}} e^{-\sum_{j=1}^{n} y_j^2/2} \int_{-\infty}^{\infty} e^{-y^2/2} \prod_{j=1}^{n} (y/\sqrt{\alpha} + z - s_j) dy;
\]

(2) the generating function \( \prod_{j=1}^{n} (z + s_j) = \sum_{k=0}^{n} \alpha^k e_{n-k}(s) \); and (3) the following integral representation of the Hermite polynomials:

\[
H_n(z) = \frac{\sqrt{n!}}{\sqrt{\pi}} \int_{\mathbb{R}} (z + iv)^n e^{-v^2} dv.
\]

\( \square \)

Proposition 2.6 (Series expansion: general case). Let \( D_0(s) \) denote the operator defined in \( 2.22 \), but this time for the set of variables \( s \). Then

\[
G_{n,m}(s; f) = e^{-\frac{1}{2} p_2(s)} e^{-\frac{1}{2} D_0(s)} \prod_{j=1}^{n} \prod_{k=1}^{m} (f_k - s_j).
\]

Equivalently, if \( H_\lambda^{(s)}(s) \) is the multivariate Hermite polynomial (with monic normalization), then

\[
G_{n,m}^{(\alpha)}(\sqrt{2}s; \sqrt{2} f) = 2^{nm} e^{-p_2(s)} \sum_\lambda H_\lambda^{(s)}(s) P_\lambda^{(\alpha)}(f).
\]

**Proof.** The first equation is a consequence of the following generalized Fourier transform, which is valid for any holomorphic function \( f : \mathbb{C}^n \to \mathbb{C} \): \( H_\lambda^{(s)} \)

\[
\frac{1}{\Gamma_{2/\alpha, n}} \int_{\mathbb{R}^n} e^{-\frac{1}{2} \sum_{j=1}^{n} y_j^2} f(iy) |\Delta_n(y)|^{2/\alpha} 0_{\alpha}^{(\alpha)}(\Delta_0^{(\alpha)}(-iy; s)) d^n y = e^{-\frac{1}{2} D_0(s)} f(s).
\]

The second equation follows from the first, the series expansion

\[
\prod_{i,j} (1 + x_i y_j) = \sum_\lambda P_\lambda^{(\alpha)}(x) P_\lambda^{(1/\alpha)}(y),
\]

and Lassalle’s formula (see \( \| \))

\[
H_\lambda^{(s)}(s) = e^{-\frac{1}{2} D_0(s)} P_\lambda^{(\alpha)}(s).
\]

\( \square \)

3. Proofs for the scaling limits

In this section, we are going to prove Theorems 1.3, 1.4, 1.5, 1.6 and Proposition 2.2 through some detailed computations based on Corollaries 3.11 and 3.12 of \( 22 \).
3.1. **General procedure.** We now want to asymptotically evaluate the weighted expectation \( \varphi_{\beta,N}(s;f) \), defined in \((1.9)\), for the finite rank perturbation case, which means \( f_{r+1} = \cdots = f_N = 0 \). For this, we first use the duality formula \((1.8)\) and introduce the scaled variables \( y_j = \sqrt{2N} t_j \) on its RHS. We also introduce a spectral parameter \( u \) that allows us to select the part of the spectrum we are going to study. This allows us to rescale the spectral variables \( s \) and \( f \) as follows:

\[
s_j = \sqrt{2N} \left( u + \frac{s_j}{\rho N} \right), \quad f_k = \sqrt{2N} f_k, \quad \text{for all} \quad j = 1, \ldots, n, \quad k = 1, \ldots, r, \tag{3.1}
\]

where \( \rho \) denotes a real parameter whose value will depend on the spectral parameter \( u \). Given that the spectrum of the Gaussian ensemble is symmetrical, we restrict ourselves to \( u \geq 0 \). We then apply \((2.11)\) and get, for \( \beta' = 4/\beta \),

\[
\varphi_{\beta,N-1}(s;f) = (-i\sqrt{2N})^{n(N-1)} (2\sqrt{N})^{\beta' n(n-1)/2 + n} (\gamma_{\beta',n})^{-1} e^{nN \rho^2 + \rho^2 / (\rho^2 N)} I_N(s;\tilde{f}), \tag{3.2}
\]

where

\[
I_N(s;\tilde{f}) = \int_{\mathbb{R}^n} \exp\left\{ -N \sum_{j=1}^{n} p(t_j) \right\} |\Delta_n(t)|^{\beta'} Q(t) d^n t
\]

and

\[
p(t_j) = 2t_j^2 - 4iu t_j - \ln t_j, \quad Q(t) = \prod_{j=1}^{n} \prod_{k=1}^{r} (t_j - i\tilde{f}_k) \prod_{j=1}^{n} t_j^{-1-r} - 4\mathcal{F}_0^{(2/\beta')}(4i\bar{s}/\rho; t - iu/2). \tag{3.4}
\]

In the last equations, \( l \) denotes a fixed non-negative integer.

In order to evaluate \( I_N \) as \( N \to \infty \), we will make use of the results obtained in \([22]\) which are based on the steepest descent method. We recall that according to the latter method, when considering a single integration over a complex variable \( z \), one first finds complex numbers \( z_0 \) satisfying \( p'(z_0) = 0 \). If \( p^{(d)}(z_0) \neq 0 \) for all \( d = 1, \ldots, d-1 \) but \( p^{(d)}(z_0) = 0 \), then we say that \( z_0 \) is a saddle point of degree \( d-1 \). In a second time, one checks if the original path of integration (in our case, the real line) can be deformed into the path of steepest descent, which must pass through the saddle point \( z_0 \) and be such that the phase of \( \{(z - z_0)^d p^{(d)}(z_0)\} \) is zero. In our case, since

\[
p'(z) = 4z - 4iu - 1/z,
\]

there are at most two saddle points \( z_{\pm} \), which satisfy

\[
z_{\pm} = (iu \pm \sqrt{1 - u^2})/2.
\]

The nature of the saddle points depends on the value of \( u \). We distinguish three cases:

1. Two complex saddle points of degree one when \( u \in [0,1) \):

\[
z_+ = (iu + \sqrt{1 - u^2})/2, \quad z_- = (iu - \sqrt{1 - u^2})/2.
\]

2. One imaginary saddle point of degree two when \( u = 1 \):

\[
z_0 = i/2.
\]

3. Two imaginary saddle points of degree one but only one accessible when \( u \in (1, \infty) \):

\[
z_+ = i(u + \sqrt{u^2 - 1})/2, \quad z_- = i(u - \sqrt{u^2 - 1})/2.
\]

For convenience, we list Corollaries 3.11 and 3.12 of \([22]\) as the following propositions. Note that the assumptions mentioned below have all been verified in Subsection 4.1 of \([22]\) for the special case of integral \((2.23)\). Let us mention the most significant assumptions: \( p(z) \) and \( q(t) \) are analytic in some domains \( T \subseteq \mathbb{C} \) and \( T^n \), respectively; the saddle points \( z_0 \) and \( z_{\pm} \) belong to \( T \); the integration path along the real axis can be deformed into a path that contains straight lines passing through the saddle points and such that \( \text{Re}\{p(z) - p(z_0)\} > 0 \) or \( \text{Re}\{p(z) - p(z_{\pm})\} > 0 \) along these straight lines, except possibly at the saddle points.
Proposition 3.1. Under the assumptions (i)-(v) of Section 3.2 and (iv) of Section 3.4 in [22], let

$$I_N = \int_{(a,b)^n} \exp\{-N \sum_{j=1}^n p(t_j)\}|\Delta_n(t)|^\beta q(t) g(N^{1/d}(t-t_0)) \, d^n t$$

where $g(t)$ is analytic in $\mathbb{C}^n$, $p(z)$ admits one saddle point $z_0$ of order $d-1$, and $t_0 = (z_0, \ldots, z_0)$. Then, as $N \to \infty$,

$$I_N \sim \frac{e^{-N p(z_0)}}{N^{(n-1)\beta}} A_0 q(t_0)$$

where $n_\beta = n(n-1)\beta/2$ and

$$A_0 = \int_{\mathbb{R}^n} \exp\{-\frac{p(d)(z_0)}{d!} \sum_{j=1}^n w_j^d\} \, g(w) |\Delta_n(w)|^\beta \, d^n w.$$ 

Proposition 3.2. Under the assumptions (i)-(v) of Section 3.3 and (iv) of Section 3.4 in [22], let

$$I_{N,n} = \int_{(a,b)^n} \exp\{-N \sum_{j=1}^n p(t_j)\}|\Delta_n(t)|^\beta q(t) \, d^n t$$

where $p(z)$ admits two simple saddle points $z_+, z_-$, and $\text{Re}\{z_+ - z_-\} \geq 0$. Moreover, let $p_\pm = p''(z_\pm)$ and $\Gamma_{\beta,m}$ be given in Appendix A. If $\text{Re}\{p(z_+), p(z_-)\}$, then as $N \to \infty$,

$$I_{N,2m} \sim \binom{2m}{m} (\Gamma_{\beta,m})^2 \frac{(z_+ - z_-)^\beta m^2}{(\sqrt{p_+} + \sqrt{p_-})^m (m-1)!/2} \frac{e^{-m N(p(z_+)+p(z_-))}}{N^{m+(m-1)/2}} q(z_+^m, z_-^m)$$

while

$$I_{N,2m-1} \sim \binom{2m-1}{m} \Gamma_{\beta,m-1} \Gamma_{\beta,m} \frac{(z_+ - z_-)^\beta m(m-1)}{(\sqrt{p_+} + \sqrt{p_-})^m (m-1)!/2} \frac{e^{-m N(p(z_+)+p(z_-))}}{N^{2m-1+(m-1)/2}}$$

$$\times \left( e^{N p(z_+)} (\sqrt{p_+})^{1+\beta(m-1)} q(z_+^{m-1}, z_-^m) + e^{N p(z_-)} (\sqrt{p_-})^{1+\beta(m-1)} q(z_+^m, z_-^{m-1}) \right).$$

3.2. Bulk limit.

Proof of Theorem 1.6 We will use Proposition 3.2 in order to establish the $N \to \infty$ asymptotic limit of the integral (3.3) in the bulk regime, which is given by the first of the three cases enumerated on page 11.

The method used here is almost identical to bulk regime of the Gaussian $\beta$-ensemble without external source, which was analyzed in Section 4.1 of [22]. To avoid repetition, we will only sketch the proof.

First, we go back to the scaling (3.1) and set

$$u \in [0,1), \quad \rho = \frac{2}{\pi} \sqrt{1-u^2}, \quad \text{and} \quad \bar{f}_k = \frac{u}{2} + \frac{1-u^2}{2} \pi_k.$$ 

We stress that the bulk scaling used in the Introduction, which is given in (1.19), immediately follows from the substitution of (3.4) into (3.1).

Secondly, we recall that in the bulk regime, we are giving two points, $z_\pm = (iu \pm \sqrt{1-u^2})/2$, such that $p'(z_\pm) = 0$ and $p''(z_\pm) \neq 0$. Since $u \in [0,1)$, we can write

$$u = \sin \theta, \quad \theta \in (-\pi/2, \pi/2),$$

which implies that

$$z_+ = e^{i\theta}/2 \quad \text{and} \quad z_- = e^{i(\pi-\theta)/2}.$$ 

(3.6)
Some simple manipulations then lead to the following equations:

\[ p(z_+) = -\frac{1}{2} \cos 2\theta + (1 + \ln 2) - i(\theta + \frac{1}{2} \sin 2\theta) \quad (3.7) \]

\[ p(z_-) = -\frac{1}{2} \cos 2\theta + (1 + \ln 2) + i(\theta + \frac{1}{2} \sin 2\theta - \pi) \quad (3.8) \]

\[ p_\pm := p''(z_\pm) = 8\epsilon^{\mp \theta} \cos \theta \quad (3.9) \]

We are now ready to evaluate the asymptotic limit of \( \varphi_{\beta,N-l}(s; f) \), which is related to that of \( I_N(s; f) \) via \( \Psi_{2m} \). We start with the case \( n = 2m \) and \( l = 0 \). By substituting Eqs. \( (3.6)-(3.9) \) and the equality \( q(t) = Q(t) \) into Proposition \( 3.2 \) we obtain

\[
I_{N,2m}(s; f) \sim (8N)^{-\beta m(m-1)/2-m} \exp\{-nN u^2 - nN(1 + 2 \ln 2 - i\pi)/2\}
\times \left( \sqrt{1 - u^2} \right)^{\beta m(m+1)/2+(2r-1)m} (\Gamma_{m} \Gamma_{m})^2 \prod_{k=1}^{r} \left( 1 + \pi_0^2 \right)^m \Theta_0^{(2/\beta)}((-1)^m, 1^m; i\pi s). \quad (3.10)
\]

Here the notation \((-1)^m \) (resp. \(1^m \)) means that \(-1 \) (resp. \(1 \)) is repeated \( m \) times. As a consequence, we have that

\[
\varphi_{\beta,N}(s; f) \sim \Psi_{N,2m} \gamma_{m}(\beta) \prod_{k=1}^{r} \left( 1 + \pi_0^2 \right)^m \Theta_0^{(2/\beta)}((-1)^m, 1^m; i\pi s). \quad (3.11)
\]

The factors \( \Psi_{N,2m} \) and \( \gamma_{m}(\beta) \) are given in Appendix \( A \). Application of formula \( (2.12) \) then establishes equation \( (1.20) \) of Theorem \( 1.0 \).

The case \( n = 2m - 1 \) and \( l = 1 \) is very similar. Combining Eqs. \( (3.6)-(3.9) \), the equality \( q(t) = Q(t) \), and Proposition \( 3.2 \) we get

\[
I_{N,2m-1}(s; f) \sim (8N)^{-\beta (m-1)^2/2-n/2} \exp\{-nN u^2 - nN(1 + 2 \ln 2 - i\pi)/2\}
\times \left( \sqrt{1 - u^2} \right)^{\beta (m^2-1)/2-n^2+rn} (\Gamma_{m} \Gamma_{m-1}) \Gamma_{m} (-2i)^{(2m-1)l}
\times \left( e^{-i\theta_N + il(\theta + \pi)/2} + i\theta \prod_{k=1}^{r} (1 - i\pi k)^{m-1} (1 + i\pi k)^m \Theta_0^{(2/\beta)}((-1)^{m-1}, 1^m; -i\pi s) + (i \to -i) \right), \quad (3.12)
\]

where

\[
\theta_N = N(2\theta + \sin 2\theta - \pi)/2 + \theta(1 + (m-1)\beta'), \quad \theta = \arcsin u.
\]

Hence,

\[
\varphi_{\beta,N-l}(s; f) \sim \Psi_{N,2m-1}^{(j)} \frac{1}{2\sqrt{\cos \theta}}
\times \left( e^{-i\theta_N + il(\theta + \pi)/2} + i\theta \prod_{k=1}^{r} (1 - i\pi k)^{m-1} (1 + i\pi k)^m \Theta_0^{(2/\beta)}((-1)^{m-1}, 1^m; -i\pi s) + (i \to -i) \right). \quad (3.13)
\]

From this and Eq. \( (2.12) \), one easily derives Eq. \( (1.24) \) of Theorem \( 1.0 \). The coefficient \( \Psi_{N,2m-1}^{(j)} \) is given in Appendix \( A \). \( \square \)

3.3. Edge limit: sub-critical regime.

Proof of Theorem \( 1.3 \) We will prove the asymptotic limit of \( \varphi_{\beta,N}(s; f) \) in the sub-critical regime of the soft-edge. Our method relies mainly on Proposition \( 3.1 \).

We start with Eq. \( (3.1) \) and set

\[
u = 1, \quad \rho = 2N^{-1/3}, \quad \bar{f}_k = \pi_k/2.
\]

We also suppose that \( \pi_k \) belongs to a compact subset of \((-\infty, 1)\).
Next, we go to Eqs. (3.2)–(3.4). We set $l = 0$. Given that $u = 1$, we know that the soft edge is reached. This situation corresponds to the second case of page 11 for which $p(z)$ admits one saddle point $z_0 = i/2$ of degree $d = 1 = 2$. Simple calculations then yield
\[ p(z_0) = \ln 2 + (3 - i\pi)/2, \quad p'''(z_0) = -16i. \]
Moreover, the choice of $\rho = 2N^{-1/3}$ allows us to factorize $Q(t)$ as follows:
\[ Q(t) = q(t)g(N^{1/3}(t - t_0)) \]
where
\[ q(t) = \prod_{j=1}^{n} \prod_{k=1}^{r} (t_j - i\pi_k/2) \prod_{j=1}^{n} t_j^{-r}, \quad g(N^{1/3}(t - t_0)) = 0_{\mathcal{F}_0^{(2/3)}}(2i\bar{s}; N^{1/3}(t - i/2)). \]

We are now ready to take the asymptotic limit. According to the above equations and Proposition 3.1 we have as $N \to \infty$,
\[ I_{N}(\bar{s}; \bar{f}) \sim \frac{e^{-nN(\ln 2 + (3 - i\pi)/2)}}{N^{(n+n,m)/3}} \prod_{k=1}^{r} (1 - \pi_k)^n \int_{\mathbb{R}^n} \exp \left\{ \frac{8i}{3} \sum_{j=1}^{n} w_j^3 \right\} 0_{\mathcal{F}_0^{(2/3)}}(2i\bar{s}; w) |\Delta_{n}(w)|^\beta \, d^n w. \]

Theorem 1.3 follows from this result together with Eqs. (3.2) and (1.10). \hfill \Box

### 3.4. Edge limit: critical regime.

**Proof of Theorem 1.4** This case is similar to the previous one. Once again, in Eqs. (3.1)–(3.4), we let $u = 1$, $\bar{f}_k = \pi_k/2$, $l = 0$. The point $z_0 = i/2$ is still a double saddle point and is such that
\[ p(z_0) = \ln 2 + (3 - i\pi)/2, \quad p'''(z_0) = -16i. \]

This time however, we keep $\pi_l$ fixed in a compact subset of $(-\infty, 1)$ only for all $m + 1 \leq l \leq r$, while we set
\[ \pi_k = 1 + \frac{\pi_k}{N^{1/3}}, \quad \text{for all } 1 \leq k \leq m. \]
This allows a slightly different factorization of $Q(t)$ in Eq. (3.4):
\[ Q(t) = q(t)g(N^{1/3}(t - t_0)) \]
where
\[ q(t) = \prod_{j=1}^{n} \prod_{k=m+1}^{r} (t_j - i\pi_k/2) \prod_{j=1}^{n} t_j^{-r} \]
and
\[ g(N^{1/3}(t - t_0)) = N^{-nm/3} \prod_{j=1}^{m} \prod_{k=1}^{r} (N^{1/3}(t_j - i/2) - i\pi_k/2) \, 0_{\mathcal{F}_0^{(2/3)}}(2i\bar{s}; N^{1/3}(t - i/2)). \]

Finally, making use of Proposition 3.1 we get
\[ I_{N}(\bar{s}; \bar{f}) \sim \frac{e^{-nN(\ln 2 + (3 - i\pi)/2)}}{N^{(n+n,m)/3}} \left( \frac{\beta}{\beta} \right)^{nm} \prod_{k=m+1}^{r} (1 - \pi_k)^n \]
\[ \times \int_{\mathbb{R}^n} \exp \left\{ \frac{8i}{3} \sum_{j=1}^{n} w_j^3 \right\} \prod_{j=1}^{m} \prod_{k=1}^{r} (w_j - i\pi_k/2) \, 0_{\mathcal{F}_0^{(2/3)}}(2i\bar{s}; w) |\Delta_{n}(w)|^\beta \, d^n w. \]

Obvious simplifications and the comparison with Eq. (1.10) complete the proof. \hfill \Box
3.5. Edge limit: supercritical regime.

Proof of Theorem 3.5. The third case of page 11 gives the supercritical regime. More explicitly, when the spectral parameter \( u > 1 \), the saddle points of \( p \) are

\[
z_+ = i(u + \sqrt{u^2 - 1})/2, \quad z_- = i(u - \sqrt{u^2 - 1})/2.
\]

One easily verifies that

\[
p''(z_+) = \frac{u^2 + u \sqrt{u^2 - 1} - 1}{(u + \sqrt{u^2 - 1})^2}, \quad p''(z_-) = \frac{u^2 - u \sqrt{u^2 - 1} - 1}{(u - \sqrt{u^2 - 1})^2},
\]

so that

\[
0 < p''(z_+) < 4, \quad -\infty < p''(z_-) < 0.
\]

This implies that for the first saddle point, the angles of steepest descent are 0 and \( \pi \), while for the second, they are \( \pm \pi/2 \). Given that the original path of integration of each variable \( t_j \) follows the real line, we see that the path of integration cannot be deformed into a path of steepest descent that would go through both saddle points. Consequently, we consider \( z_0 = z_+ \) as a single saddle point of degree one.

Before evaluating the integral, let us simplify the notation by introducing new variables:

\[
\nu = u + \sqrt{u^2 - 1}, \quad \mu = \nu + \frac{1}{\nu}, \quad \sigma^2 = \frac{\nu^2}{\nu^2 - 1}.
\]

Thus,

\[
u = u + \sqrt{u^2 - 1}, \quad \mu = \nu + \frac{1}{\nu}, \quad \sigma^2 = \frac{\nu^2}{\nu^2 - 1}.
\]

In Eq. (3.11), we also set

\[
\rho = \frac{2\sigma}{N^{1/2}}, \quad \bar{\rho}_j = \frac{\nu}{2} + \frac{\sigma \bar{\pi}_j}{2N^{1/2}}, \quad 1 \leq j \leq m,
\]

and suppose that the other spectral variables \( \bar{\rho}_{m+1} = \pi_{m+1}/2, \ldots, \bar{\rho}_r = \pi_r/2 \) belong to a compact subset of \( (-\infty, \nu/2) \).

The function \( Q(t) \), which appears in the integrand of \( I_N(\bar{s}; \bar{f}) \), can now be factorized as

\[
Q(t) = q(t) g(N^{1/2}(t - t_0)),
\]

where

\[
q(t) = \prod_{j=1}^{n} \prod_{k=m+1}^{r} (t_j - i\pi_k/2) \prod_{j=1}^{n} t_j^{-r}
\]

and

\[
g(N^{1/2}(t - t_0)) = N^{-nm/2} \exp \left\{ -\frac{2\nu - \mu}{2\sigma} N^{1/2} p_1(\bar{s}) \right\}
\]

\[
\times \prod_{j=1}^{n} \prod_{k=1}^{m} (N^{1/2}(t_j - z_0) - i\sigma \bar{\pi}_k/2) F_0^{(2/\beta)}(2i\bar{s}/\sigma; N^{1/2}(t - t_0)).
\]

The use of Proposition 3.3 then leads to

\[
I_N(\bar{s}; \bar{f}) \sim \frac{e^{-nN(\mu/2 - \nu/2 + ln(1+i\pi)/2)(\bar{\pi}/2)^n}}{N(n+n_m+n_m/2)^{nm/2}} (\frac{\sigma}{2})^{nm} \nu^{-rn} \exp \left\{ -\frac{2\nu - \mu}{2\sigma} N^{1/2} p_1(\bar{s}) \right\}
\]

\[
\times \int_{\mathbb{R}^n} \exp \left\{ -\frac{2}{\sigma^2} \sum_{j=1}^{n} w_j^2 \right\} \prod_{j=1}^{n} \prod_{k=1}^{m} (w_j - i\sigma \bar{\pi}_k/2) F_0^{(2/\beta)}(2i\bar{s}/\sigma; w) |\Delta_n(w)|^{\beta} d^n w.
\]

Comparing with the definition of the multivariate Gaussian function (1.11), we get

\[
I_N(\bar{s}; \bar{f}) \sim \frac{(-1)^{nm} e^{nN(\mu/2 - \nu/2 + ln(1+i\pi)/2)(\bar{\pi}/2)^n}}{2^n(N-m)^2N(n+n_m+n_m/2)^{nm/2}} (\frac{\sigma}{2})^{nm+n_m+n_m} e^{-\frac{2\nu - \mu}{2\sigma} \sqrt{n} p_1(\bar{s})} \prod_{k=m+1}^{r} (\nu - \pi_k)^n G_n^{(2/\beta)}(\bar{s}; \bar{\pi}).
\]
Finally, the substitution of the latter equation into Eq. (3.2) leads to
\[
\varphi_{\beta,N}(s; f) \sim (-1)^{nm} 2^{-nN/2} \sigma^{n(n+1)} \beta \rho(n-r) N^{h} N(N-m)^{2} e^{-nN(1+(\nu-\mu)/2)/2} \times \\
\exp \left\{ -\frac{2\nu - \mu}{2\sigma} N^{1/2} p_1(\bar{s}) \right\} \prod_{k=m+1}^{r} (\nu - \pi_k)^{n} e^{\frac{1}{2\pi} p_2(\bar{s})(\bar{s}; \bar{\pi})} \tag{3.15}
\]
which is equivalent to the expected result.

\[\square\]

Remark 3.3. We stress that on the RHS of (3.15), the factor \(e^{-\frac{2\nu - \mu}{2\sigma} N^{1/2} p_1(\bar{s})}\) is not negligible, even when \(N \to \infty\). This differs considerably from what we have observed for the limiting correlations in the subcritical and the critical regimes. Indeed, in these regimes, the asymptotic limit of \(\varphi_{\beta,N}\) factorizes as product of one function depending on \(\beta, N, n, m\) and another function depending the \(\bar{s}_j\)'s and \(\bar{\pi}_j\)'s, but independent of \(N\).

One reason of causing the difference in the asymptotic behaviors may come from the weighted factor \(e^{-\frac{1}{2\pi} p_2(\bar{s})(\bar{s}; \bar{\pi})}\) in Eq. (3.14). As a matter of fact, if we replace the weighted quantity by \(e^{-\frac{1}{2\pi} p_2(s)}\varphi_{\beta,N}(s; f)\) in the supercritical regime, then with the same scalings, we get
\[
\hat{\varphi}_{\beta,N}(s; f) \sim (-1)^{nm} 2^{-nN/2} \sigma^{n(n+1)} \beta \rho(n-r) N^{h} N(N-m)^{2} e^{-nN/2} \times \\
\prod_{k=m+1}^{r} (\nu - \pi_k)^{n} e^{\frac{1}{2\pi} p_2(\bar{s})(\bar{s}; \bar{\pi})} \tag{3.16}
\]

Thus, in order to rewrite the three regimes in a consistent way, we could introduce the following function:

\[
\nu(u) = \begin{cases} 
1, & |u| \leq 1, \\
\frac{u + \sqrt{u^2 - 1}}{2}, & |u| > 1.
\end{cases}
\]

This would allow us to write
\[
\varphi_{\beta,N}(s; f) = e^{-\frac{1}{2\pi} p_2(s)} K_{\beta,N}(s; f).
\]

The asymptotic behavior of \(\varphi_{\beta,N}(s; f)\) in the subcritical and critical regimes would be the same as in Theorems 1.3 and 1.4. However, in the supercritical regime, Eq. (1.18) of Theorem 1.5 would be replaced by Eq. (3.16).

3.6. Slowly growing rank case. In Remark 1.7 we claim that Theorems 1.3 to 1.6 still hold in the case where \(r\) grows sufficiently slowly with \(N\). This can be understood as follows.

Suppose first that in Eq. (3.2), the function \(p(z)\) has a saddle point \(z_0\) of order \(d - 1\). Recall that \(d = 3\) in the subcritical and critical regimes, while \(d = 2\) in the supercritical regime and in the bulk. In the neighborhood of the saddle point, let
\[
t_j = z_0 + \frac{1}{N^b} w_j, \quad b = \frac{1}{d}.
\]

Then,
\[
N(p(t_j) - p(z_0)) = \frac{p^{(d)}(z_0)}{d!} w_j^d + O(N^{-b}) \tag{3.18}
\]

It is worth stressing that Eqs. (3.17) and (3.18) are basic steps in the proof of Propositions 3.1 and 3.2. Moreover, under the change (3.17), the factor \(t_j^{-r} \prod_{k=1}^{r} (t_j - i\bar{f}_k)\) coming from the function \(Q(t)\) of Eq. (3.4) becomes
\[
(z_0 + \frac{w_j}{N^b})^{-r} \prod_{k=1}^{r} \left( z_0 - i\bar{f}_k + \frac{w_j}{N^b} \right). \tag{3.19}
\]

Now, suppose the following asymptotic growth of \(r\) as \(N \to \infty\):
\[
r \sim R N^a \tag{3.20}
\]
for some non-negative constants $R$ and $a$. For fixed values of the variables $w_j$ and $\bar{f}_k$, the product \( \lim_{N \to \infty} \frac{r}{N^a} = 0 \)

(3.21)
is a sufficient condition that guarantees the non-growing behavior of \( \frac{r}{N^a} \) as $N \to \infty$. Given that \( \frac{r}{N^a} \) is well defined whenever \( \lim_{N \to \infty} \frac{r}{N^a} = 0 \), one can apply Propositions 3.1 and 3.2 as if the rank $r$ was finite.

3.7. Proof of Proposition 2.2.

Proof of Proposition 2.2. (2.17) and (2.18) originate from integrals evaluated around one simple saddle point and two simple saddle points, respectively. For (2.17), set $N = x^{3/2}$. Simple manipulations and the use of (2.11) lead to

$$
\text{Ai}^{(a)}(x + x^{-1/2} s; x^{1/2} f) = (2\pi)^{-n} N^{((1+r)n+n(n-1)/a)/3} 
\times \int_{\mathbb{R}^n} e^{-N \sum_j p(t_j)} |\Delta_n(t)|^{2/\alpha} \prod_{j=1}^n \prod_{l=1}^r (it_j + \bar{f}_l) \text{Ai}^{(a)}(s; it) d^n t
$$

(3.22)

where $p(t_j) = -it_j^3/3 - it_j$.

The function $p(t_j)$ has two simple saddle points at $\pm i$. With $z_0 = i$, we have $p''(z_0) = 2$ which implies that the steepest descent path near $z_0$ would follow the horizontal line, as desired. We thus have an integral like in Proposition 3.1 with

$$
q(t) = \prod_{j=1}^n \prod_{l=1}^r (it_j + f_l) \text{Ai}^{(a)}(s; it), 
\quad g(N^{1/2}(t - t_0)) = \prod_{j=1}^n \prod_{l=1}^k (it_j - z_0 + f_l).
$$

We may thus apply Proposition 3.1 to the case $\mu = 2$, $z_0 = i$, $p(z_0) = 2/3$ and (2.17) follows immediately.

For (2.18), we also let $N = x^{3/2}$. In the definition of $\text{Ai}^{(a)}(s)$, substitute $t_j$ by $N^{1/3} t_j$ and apply (2.11), so we get

$$
\text{Ai}^{(a)}(-x + x^{-1/2} s; x^{1/2} f) = (2\pi)^{-n} N^{((1+r)n+n(n-1)/a)/3} 
\times \int_{\mathbb{R}^n} e^{-N \sum_j p(t_j)} |\Delta_n(t)|^{2/\alpha} \prod_{j=1}^n \prod_{l=1}^r (it_j + f_l) \text{Ai}^{(a)}(s; it) d^n t
$$

(3.23)

where $p(t_j) = -it_j^3/3 + it_j$. This function has 2 simple saddle points, namely $x_\pm = \pm 1$. This time we have to consider both of them because they are already on the path of integration. We have $p(x_\pm) = \pm 2i/3$, $p_\pm = p''(x_\pm) = \mp 2$. This means that the steepest descent path is given by

$$
\mathcal{P} = \{-1 + \tau e^{-i\pi/4} : \tau \in (-\infty, \sqrt{2})\} \cup \{1 + \tau e^{i\pi/4} : \tau \in [-\sqrt{2}, \infty)\}.
$$

Thus (2.18) follows from Proposition 3.2.

4. Conclusion

The scaling limits of correlations of characteristic polynomials for the Gaussian $\beta$-ensemble, perturbed by a finite rank matrix source, have been computed. In particular, at the soft edge of the spectrum, two distinct families of multivariate functions have been proved to be the scaling limits in the (sub)critical and supercritical regimes, so a phase transition phenomenon has been observed. To our knowledge, even in the case of $\beta = 1, 4$ the results obtained in this paper are new.

The duality formula (1.8) for the Gaussian ensemble plays a key role in our asymptotic analysis. A similar formula holds for the chiral Gaussian ensemble (13), so a future challenging problem is to compute the corresponding limit of characteristic polynomials and show that it is the same as in the Gaussian case (some universal pattern). As a matter of fact, Forrester (26) has obtained the soft-edge limit of one single characteristic polynomial not only for the Gaussian case (any finite rank $r$), but also for the chiral case (only rank 1). Moreover, it has been proved in the previous paper (22) that both ensembles without
source indeed share the same scaling limit – there the duality formula for the chiral case is not used. Those observations suggest that the same phase transition phenomenon might still hold for the chiral case.

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**Appendix A. Notation and Constants**

First of all, the normalization constant for the Gaussian $\beta$-ensemble is equal to (see e.g., [24])

$$G_{\beta,N} = \beta^{-N/2-\beta N(N-1)/4} (2\pi)^{N/2} N \prod_{j=0}^{N-1} \frac{\Gamma(1 + \beta/2 + j\beta/2)}{\Gamma(1 + \beta/2)}.$$  \hspace{1cm} (A.1)

This constant is in fact a special case of the following integral:

$$\int_{\mathbb{R}^n} \prod_{i=1}^{n} e^{-x_i^2/2} \prod_{1 \leq i < j \leq n} |x_i - x_j|^\beta \, dx_1 \cdots dx_n = \frac{1}{z(n+\beta(n-1)/2)} \Gamma_{\beta,n}, \quad \text{Re}\{z\} > 0,$$ \hspace{1cm} (A.2)

where

$$\Gamma_{\beta,n} = (2\pi)^{n/2} \prod_{j=1}^{n} \frac{\Gamma(1 + j\beta/2)}{\Gamma(1 + \beta/2)}.$$  \hspace{1cm} (A.3)

The duality formula [18] involves the latter factor as follows:

$$D_{\beta,N,n} = \frac{i^{nN/2 + n(n-1)/\beta}}{\Gamma_{4/\beta,n}}.$$  \hspace{1cm} (A.4)

We now consider the constants related to the asymptotic behavior of the average product of characteristic polynomials at the soft edge. In the sub-critical regime, we have

$$\Phi_{\beta,N,n} = \frac{\pi^n N^{n(3N/2+\beta+2n/\beta)} \frac{n!}{n!}}{\Gamma_{4/\beta,n} N^{n/2} n^{N(N-2)/2}}.$$  \hspace{1cm} (A.5)

For the critical regime, the constant is

$$\Phi_{\beta,N,n,m} = (-1)^n m^m N^{-m/3} \Phi_{\beta,N,n}.$$  \hspace{1cm} (A.6)

For the supercritical regime, new positive parameters are needed: $\nu$, $\mu = \nu + \nu^{-1}$, and $\sigma^2 = \nu^2/(\nu^2 - 1)$. The constant then reads:

$$\Phi_{\beta,N,n,m}^{\sup} = (-1)^n m^m \nu^{n(N^2-2n+2)/4} \sigma^{n(2N+\beta-2m+\beta)} N^{n(N-m)/2} \frac{\nu^{n(N-1)/4}}{\sqrt{1 - \nu^2}}.$$  \hspace{1cm} (A.7)

In the bulk of the spectrum, the constant is, for $n = 2m$,

$$\Psi_{N,m} = 2^\beta m(m+1)/2 - m(N+1) \frac{\beta^m m^m N \Gamma_{\beta,N,n}}{\sqrt{1 - \nu^2}}.$$  \hspace{1cm} (A.8)

while for $n = 2m - 1$,

$$\Psi_{N,m-1} = (2m-1) \frac{\Gamma_{\beta,N,n,m} \Gamma_{\beta,N,n,m-1} \beta^m m^m N^{(N+1-\beta)/2}}{\Gamma_{\beta,N,n,m} \Gamma_{\beta,N,n,m} \beta^m m^m N^{(N+1-\beta)/2}} \times \frac{\beta^m m^m N^{(N+1-\beta)/2}}{\sqrt{1 - \nu^2}}.$$ \hspace{1cm} (A.9)

Finally, the universal coefficient is

$$\gamma_m(\beta) = \frac{2m^m}{m} \prod_{j=1}^{m} \Gamma(1 + \beta j/2) \Gamma(1 + \beta (m + j)/2).$$  \hspace{1cm} (A.10)
References

[1] M. Adler, J. Delépine and P. van Moerbeke, Dyson's nonintersecting Brownian motions with a few outliers, Commun. Pure and Applied Math. 62 (2009), 334–395.

[2] J. Baik, G. Ben Arous, and S. Péché, Phase transition of the largest eigenvalue for non-null complex sample covariance matrices, Ann. Prob. 33 (2005), no. 5, 1643–1697.

[3] J. Baik and D. Wang, On the Largest Eigenvalue of a Hermitian Random Matrix Model with Spiked External Source I, Rank 1 Case, Int. Math. Res. Not. 2011, No. 22, 5164–5240.

[4] T. H. Baker and P. J. Forrester, The Calogero-Sutherland model and generalized classical polynomials, Commun. Math. Phys. 188 (1997), 175–216.

[5] K. E. Bassler, P. J. Forrester and N. E. Frankel, Eigenvalue separation in some random matrix models, J. Math. Phys. 5 (2009) 033302, 1–25.

[6] F. Benaych-Georges and R. Rao, The eigenvalues and eigenvectors of finite, low rank perturbations of large random matrices, Adv. Math. 227 (2011), 494–521.

[7] M. Bertola, R. Buckingham, S. Y. Lee, V. Pierce, Spectra of Random Hermitian Matrices with a Small-Rank External Source: The Critical and Near-Critical Regimes, J. Stat. Phys. 146 (2012), 475–518.

[8] P. M. Bleher and A. B. J. Kuijlaars, Large n limit of Gaussian random matrices with external source I, Comm. Math. Phys. 252 (2004), 43–76.

[9] P. M. Bleher and A. B. J. Kuijlaars, Random matrices with external source and multiple orthogonal polynomials, Int. Math. Res. Notices 2004: 3 (2004), 109–129.

[10] A. Bloemendal and B. Virág, Limits of spiked random matrices I, Probab. Theory Relat. Fields 156 (2013), 795–825.

[11] A. Bloemendal and B. Virág, Limits of spiked random matrices II, arXiv:1109.3704v1.

[12] P. Bourgade, L. Erdős, and H. Yau, Bulk universality of general β-ensembles with non-convex potential, J. Math. Phys. 53 (2012), 095218, 1–19.

[13] E. Brézin, S. Hikami, Extension of level-spacing universality, Phys. Rev. E 56 (1997), 264–269.

[14] E. Brézin, S. Hikami, Level spacing of random matrices in an external source, Phys. Rev. E 58 (1998), no. 6, 7176–7185.

[15] E. Brézin, S. Hikami, Intersection theory from duality and replica, Comm. Math. Phys. 283 (2008), 507–521.

[16] E. Daems, A. B. J. Kuijlaars and W. Veys, Asymptotics of non-intersecting Brownian motions and a 4 × 4 Riemann-Hilbert problem, J. Approx. Theory 153 (2008), 225–256.

[17] S. Delvaux, Average characteristic polynomials for multiple orthogonal polynomial ensembles, J. Approx. Theory 162 (2010), 1053–1067.

[18] P. Desrosiers, Duality in random matrix ensembles for all β, Nucl. Phys. B 817 (2009), 224–251.

[19] P. Desrosiers and P. J. Forrester, Asymptotic correlations for Gaussian and Wishart matrices with external source, Int. Math. Res. Notices (2006), ID 27395, 1–43.

[20] P. Desrosiers and P. J. Forrester, Hermite and Laguerre β-ensembles: Asymptotic corrections to the eigenvalue density, Nucl. Phys. B 746 (2006), 307–332.

[21] P. Desrosiers and P. J. Forrester, A note on biorthogonal ensembles, J. Approx. Theory 152 (2008), 167–187.

[22] P. Desrosiers, D. Z. Liu, Asymptotics for products of characteristic polynomials in classical β-ensembles, Constructive Approximation (2013), 50 pages, doi:10.1007/s00365-013-9206-2, arXiv:1112.1119v3.

[23] I. Dumitriu and A. Edelman, Matrix models for beta ensembles, J. Math. Phys. 43 (2002), 5830–5847.

[24] P. J. Forrester, Log-gases and Random Matrices, London Mathematical Society Monographs 34, Princeton University Press (2010).

[25] P. J. Forrester, Probability densities and distributions for spiked Wishart β-ensembles, arXiv:1101.2261, 17 pages.

[26] P. J. Forrester, The averaged characteristic polynomial for the Gaussian and chiral Gaussian ensembles with a source, J. Phys. A: Math. Theor. 46 (2013) 345204, 17pp.

[27] R. C. Jones, J. M. Kosterlitz, and D. J. Thouless, The eigenvalue spectrum of a large symmetric random matrix with a finite mean, J. Phys. A 11 (1978), 3, L45–L48.

[28] M. Kontsevich, Intersection theory on the moduli space of curves and the matrix Airy function, Commun. Math. Phys. 147 (1992), 1–23.

[29] A. B. J. Kuijlaars, Multiple orthogonal polynomial ensembles, Contemporary Mathematics 507 (2010), 155–176

[30] D. W. Lang, Isolated Eigenvalue of a Random Matrix, Phys. Rev. 135 (1965), 4B, B1082–B1084

[31] I. G. Macdonald, Symmetric Functions and Hall Polynomials (2nd ed.), Oxford University Press Inc, New York, 1995.

[32] M. Y. Mo, Rank 1 real Wishart spiked model, Commun. Pure and Applied Math. LXV (2012), 1528–1638.

[33] S. Péché, The largest eigenvalue of small rank perturbations of Hermitian random matrices, Probab. Theory and Related Fields 134 (2006), 127–173.

[34] J. A. Ramírez, B. Rider, and B. Virág, Beta ensembles, stochastic Airy spectrum, and a diffusion, J. Amer. Math. Soc. 24 (2011), 919–944.

[35] B. Valkó and B. Virág, Continuum limits of random matrices and the Brownian carousel, Inventiones Mathematicae Vol. 177 (2009), 463–508.

[36] D. Wang, The largest eigenvalue of real symmetric, Hermitian and Hermitian self-dual random matrix models with rank one external source, part I, J. Stat. Phys. 146 (2012), no. 4, 719–761.
[37] P. Zinn-Justin, *Random Hermitian matrices in an external field*, Nuclear Phys. B 497 (1997), no. 3, 725–732.
[38] P. Zinn-Justin, *Universality of correlation functions of Hermitian random matrices in an external field*, Comm. Math. Phys. 194 (1998), no. 3, 631–650.

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