Diophantine approximation with perfect squares and the solvability of an inhomogeneous wave equation

Victor Beresnevich* York Maurice Dodson† York Simon Kristensen‡ Edinburgh Jason Levesley York

August 22, 2018

2000 Mathematics Subject Classification: Primary 35L05; Secondary 11J83, 11J13, 11K60

Keywords and phrases: Diophantine approximation, Hausdorff dimension, wave equation, small denominators problem

1 Introduction

Diophantine criteria occur naturally in the theory of partial differential equations through the notorious problem of small denominators. An extensive treatment of such problems in the theory of PDEs can be found, e.g., in [6]. In this paper, we are interested in a Diophantine problem related to an inhomogeneous wave equation in $n$ spatial and one temporal dimension with periodic boundary conditions. In brief, to ensure the convergence of a formal solution to the equation certain conditions on the periods should be satisfied. These conditions normally leave a small set of exceptional periods for which the convergence of the series is problematic, though the solution might exist. It is therefore of interest to measure the ‘size’ of the exceptional set of periods. Regarding the wave equation we will discuss the problem in more details and derive the associated Diophantine problem in $\S$.

An analogous problem for the wave equation in one spatial dimension is considered in [5]. Even further, a more general class of one dimensional PDEs is studied by Gramchev and Yoshino in [2]. However, their methods does not seem to work in higher dimensions. In [3], the corresponding problem in two spatial dimensions is resolved for the Schrödinger equation, for which the corresponding Diophantine problem is partly linear, and it is settled by making use of a result of Rynne [7].

The rest of the paper is structured as follows. The results of the paper are stated in $\S$ 3. In $\S\S$ 4–5 we prove the results for the case when $n = 2$ and in $\S$ 6 we outline how the proofs
can be adapted to obtain the \( n \)-dimensional versions.

Throughout we will use the Vinogradov notation: Given two real valued functions \( f \) and \( g \), write \( f \ll g \) if there is a constant \( c > 0 \) such that \( f \leq cg \). If \( f \ll g \) and \( g \ll f \), write \( f \asymp g \).

## 2 The solubility of the wave equation and a related Diophantine problem

Let \( n \in \mathbb{N} \), \( \alpha_i > 0 \) for \( i = 1, \ldots, n \), \( \beta > 0 \) and \( f : \mathbb{R}^{n+1} \rightarrow \mathbb{R} \) be periodic in all variables with period \( \alpha_i \) in the \( i \)’th variable and period \( \beta \) in the \( n + 1 \)’st. We denote the \( n \) first variables by \( x_1, \ldots, x_n \) and the \( n + 1 \)’st by \( t \). Suppose furthermore that \( f \) is a smooth function of any of the variables \( x_i, t \), i.e., \( f \) has continuous partial derivatives of all orders. We will consider the partial differential equation given by

\[
\frac{\partial^2 u(x,t)}{\partial t^2} - \Delta u(x,t) = f(x,t), \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n, t \in \mathbb{R},
\]

under the additional condition that the solution \( u \) is smooth and periodic with the same periods. Here \( \Delta \) denotes the usual Laplacian, i.e.,

\[
\Delta u(x,t) = \sum_{i=1}^{n} \frac{\partial^2 u(x,t)}{\partial x_i^2}.
\]

The periodicity and smoothness conditions on \( f \) are well-known to be equivalent to the condition that \( f \) has an expansion into a Fourier series

\[
f(x,t) = \sum_{(a,b) \in \mathbb{Z}^{n+1}} f_{a,b} \exp \left( 2\pi i \left[ \sum_{i=1}^{n} \frac{a_i}{\alpha_i} x_i + \frac{b}{\beta} t \right] \right),
\]

where \( a = (a_1, \ldots, a_n) \), such that the coefficients \( f_{a,b} \) decay faster than the reciprocal of any polynomial in \( a_1, \ldots, a_n, b \) as \( \max\{|a_1|, \ldots, |a_n|, |b|\} \) tends to infinity.

Suppose for the moment that (1) has a solution \( u \) satisfying the periodicity and smoothness conditions. Clearly, \( u \) must also have the following Fourier expansion

\[
u(x,t) = \sum_{(a,b) \in \mathbb{Z}^{n+1}} u_{a,b} \exp \left( 2\pi i \left[ \sum_{i=1}^{n} \frac{a_i}{\alpha_i} x_i + \frac{b}{\beta} t \right] \right),
\]

Inserting this into (1) and identifying coefficients, we obtain

\[
u_{a,b} = \frac{\beta^2}{4\pi^2} \frac{f_{a,b}}{\sum_{i=1}^{n} a_i^2 \alpha_i^2 - b^2}.
\]

Now, since \( \alpha_1, \ldots, \alpha_n, \beta \) are fixed, and since \( f_{a,b} \) decays faster than the reciprocal of any polynomial, for \( u \) to be smooth it suffices to verify that

\[
\left| \sum_{i=1}^{n} a_i^2 \beta^2 \frac{1}{\alpha_i^2} - b^2 \right| \geq C \max\{|a_1|, \ldots, |a_n|\}^{-w},
\]
for some \( C > 0, w > 1 \) for all \((a, b) \in \mathbb{Z}^{n+1}\) with \(a \neq 0\). It is easy to see that this condition can only fail if for any \( w > 1 \) the inequality

\[
\left| \sum_{i=1}^{n} a_i^2 \beta^2_i - b^2 \right| < \max\{|a_1|, \ldots, |a_n|\}^{-w}
\]

holds for infinitely many \((a, b) \in \mathbb{Z}^{n+1}\) with \(a \neq 0\).

Note that the condition given in (3) is sufficient for the solubility of (1), but not necessary. The Diophantine problem considered in this paper is a natural generalisation of the one of equation (3).

3 Statement of results

Throughout \(\mathbb{Z}_{\geq 0}\) will denote the set of non-negative integer numbers and \(|A|\) the Lebesgue measure of a set \(A\). Given an \(n\)-tuple \(a \in \mathbb{Z}_{\geq 0}^2\), define the height \(h_a\) of \(a\) by setting

\[ h_a := \max(|a_1|, \ldots, |a_n|), \]

that is \(h_a\) is the highest coefficient of \(a\) in absolute value.

Let \(\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+\) be a function such that \(\psi(h) \rightarrow 0\) as \(h \rightarrow \infty\) and define the set \(W_n(\psi)\) to be

\[ W_n(\psi) := \{x \in [0, 1]^n : |a^2 \cdot x - b^2| < \psi(h_a), \] holds for infinitely many \((a, b) \in \mathbb{Z}_{\geq 0}^{n+1}\} \],

where \(a^2 := (a_1^2, \ldots, a_n^2)\).

The following statements constitute the main results of this paper.

**Theorem 1** Let \(\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+\) be monotonic. Then

\[
|W_n(\psi)| = \begin{cases} 
0, & \sum_{h=1}^{\infty} h^{n-2} \psi(h) < \infty, \\
1, & \sum_{h=1}^{\infty} h^{n-2} \psi(h) = \infty. 
\end{cases}
\]

**Theorem 2** Let \(\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+\) be a monotonic. Given any positive \(s < n\), the \(s\)-dimensional Hausdorff measure of \(W_n(\psi)\) satisfies the relation

\[
\mathcal{H}^s(W_n(\psi)) = \begin{cases} 
0, & \sum_{h=1}^{\infty} \psi(h)^{s-(n-1)h} 3^{n-2-2s} < \infty, \\
\infty, & \sum_{h=1}^{\infty} \psi(h)^{s-(n-1)h} 3^{n-2-2s} = \infty. 
\end{cases}
\]

**Corollary 1** Let \(\psi : \mathbb{R}^+ \rightarrow \mathbb{R}^+\) be a monotonic function such that \(\lim_{h \rightarrow \infty} \psi(h) = 0\). Define \(\lambda_\psi\), the lower order of \(1/\psi(2^r)\) at infinity, by setting

\[
\lambda_\psi = \lim_{r \rightarrow \infty} \inf \frac{-\log \psi(2^r)}{r \log 2}.
\]
Note that $\lambda_\psi$ is always non-negative, but can be infinity. If $n - 1 \leq \lambda_\psi < \infty$ then

$$\dim W_n(\psi) = (n - 1) + \frac{n + 1}{2 + \lambda_\psi}.$$ 

In particular, if $\psi(r) = r^{-v}$ for some $v > n - 1$ then

$$\dim W_n(r \mapsto r^{-v}) = (n - 1) + \frac{n + 1}{2 + v}.$$ 

In terms of the wave equation, we may derive the following corollary:

**Corollary 2** Let $\alpha_1, \ldots, \alpha_n, \beta > 0$ and consider the partial differential equation (1). Let $\delta_i = \beta^2 / \alpha_i^2$ for $i = 1, \ldots, n$. If $f$ is smooth and periodic in $x_1, \ldots, x_n, t$ with periods $\alpha_1, \ldots, \alpha_n, \beta$ respectively, then (1) is soluble with $u$ smooth and periodic with the same periods whenever $(\delta_1, \ldots, \delta_n)$ does not belong to

$$\bigcup_{v > 1} W_n(r \mapsto r^{-v}),$$

a null set of Hausdorff dimension $n - 1$.

4 Proof of Theorem 1

We first prove the result for the case $n = 2$ as the argument is easiest to follow in this dimension.

4.1 The case of convergence

For every triple $(a, b, c) \in \mathbb{Z}_{\geq 0}^3$ define the sets

$$\sigma_{a,b}(c) := \{(x, y) \in [0, 1]^2 : |a^2 x + b^2 y - c^2| < \psi(h_{a,b})\}$$

$$\sigma_{a,b} := \bigcup_{c \in \mathbb{Z}} \sigma_{a,b}(c).$$

Without loss of generality we can assume that $a + b > 0$. It is easy to verify that

$$|\sigma_{a,b}(c)| \ll \frac{\psi(h_{a,b})}{h_{a,b}^2}.$$ 

Given a pair $(a, b) \in \mathbb{Z}_{\geq 0}^2 \setminus \{0\}$, $\sigma_{a,b}(c) \neq \emptyset$ implies that $c \ll h_{a,b}$. It follows that

$$|\sigma_{a,b}| \ll \sum_{c \in \mathbb{Z}_{\geq 0} : \sigma_{a,b}(c) \neq \emptyset} \frac{\psi(h_{a,b})}{h_{a,b}^2} \ll \frac{\psi(h_{a,b})}{h_{a,b}}.$$
Now assume that \( \sum_{h=1}^{\infty} \psi(h) < \infty \). Then,
\[
\sum_{h=1}^{\infty} \sum_{(a,b) \in \mathbb{Z}_+^2 \setminus \{0\}; \ h_{a,b}=h} |\sigma_{a,b}| \ll \sum_{h=1}^{\infty} \sum_{(a,b) \in \mathbb{Z}_+^2 \setminus \{0\}; \ h_{a,b}=h} \frac{\psi(h)}{h} \ll \sum_{h=1}^{\infty} \psi(h) < \infty.
\]

As the set \( W_2(\psi) \) is exactly the set of points \((x, y)\) in the unit square that fall into infinitely many sets \( \sigma_{a,b} \), we can apply the Borel-Cantelli Lemma to (4) to conclude that the set \( W_2(\psi) \) has zero Lebesgue measure.

### 4.2 The case of divergence: Auxiliary Lemmas

It should be noted that the main difficulty in proving Theorem 1 is in the case of divergence, to be considered in sections 4.3 and 4.4. The line of investigation of this case will rely on the following standard auxiliary measure theoretic statements.

**Lemma 1** Let \( \Omega \) be an open subset of \( \mathbb{R}^n \) and let \( |A| \) be the Lebesgue measure of \( A \). Let \( E \) be a Borel subset of \( \mathbb{R}^n \). Assume that there are constants \( r_0, c > 0 \) such that for any ball \( B \) of radius \( r(B) < r_0 \) in \( \Omega \) we have
\[
|E \cap B| \geq c \ |B|.
\]

Then \( E \) has full measure in \( \Omega \), i.e. \( |\Omega \setminus E| = 0 \).

**Lemma 2** Let \((\Omega, A, \mu)\) be a probability space and \( E_n \) be a sequence of \( \mu \)-measurable sets such that \( \sum_{n=1}^{\infty} \mu(E_n) = \infty \). Then
\[
\mu(\limsup_{n \to \infty} E_n) \geq \limsup_{Q \to \infty} \frac{\left( \sum_{s=1}^{Q} \mu(E_s) \right)^2}{\sum_{s,t=1}^{Q} \mu(E_s \cap E_t)}.
\]

In our particular problem we will take \( E_n \) to be a subsequence of the sequence of sets \( \sigma_{a,b} \). More precisely, we will estimate pairwise intersections of \( \sigma_{a,b} \) restricted to a fixed ball \( B \) on average. The corresponding limsup set will be contained in \( W_2(\psi) \cap B \). On applying Lemma 2 we will arrive at a lower bound of the form \( |W_2(\psi) \cap B| \geq c |B| \) for some positive absolute constant. Lemma 1 will complete the proof.

Further, to avoid painful and unnecessary calculation we will restrict \( B \) to be a ball lying inside \( \Omega = [\varepsilon, 1]^2 \) for some arbitrarily small \( \varepsilon > 0 \). The corresponding probability measure \( \mu \) will be taken to be the normalized Lebesgue measure in \( \Omega \).

### 4.3 Estimates for the measure of \( \sigma_{a,b} \cap B \) and their pairwise intersections

Fix an arbitrary positive number \( \varepsilon < 1 \) and set \( \Omega := [\varepsilon, 1]^2 \). Take any ball \( B \) in \( \mathbb{R}^2 \) lying in \( \Omega \).
4.3.1 Restrictions on \( c \)

Assume that \( \sigma_{a,b}(c) \cap B \neq \emptyset \). Then there is a point \((x, y) \in B \subset [\varepsilon, 1]^2\) satisfying \(|a^2x + b^2y - c^2| < \psi(h_{a,b})\). If \( h_{a,b} \) is sufficiently large then \( \psi(h_{a,b}) < \varepsilon \). Therefore, \( c^2 < \varepsilon + a^2x + b^2y \leq 1 + 2h_{a,b}^2 \). Hence,

\[
|c| < 2h_{a,b}.
\]

On the other hand,

\[
c^2 > a^2x + b^2y - \psi(h) > \varepsilon(a^2 + b^2) - \varepsilon \geq \varepsilon(h^2 - 1).
\]

Therefore,

\[
|c| > \varepsilon h_{a,b}/2
\]

if \( h_{a,b} \) is sufficiently large. Therefore, for all \((a, b) \in \mathbb{Z}^2_{\geq 0}\) with sufficiently large \( h_{a,b} \) and all positive \( c \) with \( \sigma_{a,b}(c) \cap B \neq \emptyset \) we have

\[
\frac{\varepsilon}{2} h_{a,b} < |c| < 2 h_{a,b}.
\] (5)

4.3.2 The amount of different \( c \)

Define the line \( R_{a,b,c} := \{(x, y) \in \mathbb{R}^2 : a^2x + b^2y - c^2 = 0\} \). It is readily verified that \( \sigma_{a,b}(c) \cap B \neq \emptyset \) is equivalent to \( R_{a,b,c} \cap B \neq \emptyset \), except possibly for 2 ‘extremal’ cases when \( \sigma_{a,b}(c) \cap B \neq \emptyset \) but the corresponding lines do not hit the ball \( B \) but lie sufficiently close to \( B \).

To evaluate the number of different \( c \) such that \( \sigma_{a,b}(c) \neq \emptyset \) we will estimate the number of lines \( R_{a,b,c} \) that hit the ball \( B \) and then add 2 to the upper estimate.

Let \( x_0, y_0 \) be the center of \( B \) and \( r \) be the radius of \( B \). Any point \((x, y)\) in \( B \) can be written as

\[
x = x_0 + \theta r \cos \phi, \quad y = y_0 + \theta r \sin \phi, \quad 0 \leq \theta < 1, \quad 0 \leq \phi < 2\pi.
\] (6)

Clearly, \( R_{a,b,c} \cap B \neq \emptyset \) if and only if there is a choice of \((x, y)\) subject to (6) such that

\[
a^2(x_0 + \theta r \cos \phi) + b^2(y_0 + \theta r \sin \phi) - c^2 = 0.
\]

In such a case we have that

\[
c^2 = a^2x_0 + b^2y_0 + \theta r(a^2 \cos \phi + b^2 \sin \phi) =
\]

\[
a^2x_0 + b^2y_0 + \theta r \sqrt{a^4 + b^4} \left( \frac{a^2}{\sqrt{a^4 + b^4}} \cos \phi + \frac{b^2}{\sqrt{a^4 + b^4}} \sin \phi \right) =
\]

\[
a^2x_0 + b^2y_0 + \theta r \sqrt{a^4 + b^4} (\sin \phi_0 \cos \phi + \cos \phi_0 \sin \phi) =
\]

\[
a^2x_0 + b^2y_0 + \theta r \sqrt{a^4 + b^4} \sin(\phi + \phi_0),
\]
where $\phi_0 = \arcsin \frac{a^2}{\sqrt{a^4 + b^4}}$. Therefore, $c^2$ varies in the interval
\[ [a^2x_0 + b^2y_0 - r\sqrt{a^4 + b^4}, a^2x_0 + b^2y_0 + r\sqrt{a^4 + b^4}]. \quad (7) \]

Moreover, on taking $\phi = \pm \pi/2 - \phi_0$, $\sin(\phi + \phi_0) = \sin(\pm \pi/2) = \pm 1$ we see that any perfect squares in this interval do contribute to a line $R_{a,b,c}$ which hits the ball $B$. Clearly $c^2$ lies in (7) if and only if $c$ is in the interval
\[ \left[ \sqrt{a^2x_0 + b^2y_0 - r\sqrt{a^4 + b^4}}, \sqrt{a^2x_0 + b^2y_0 + r\sqrt{a^4 + b^4}} \right]. \quad (8) \]

The length of interval (8) is
\[ \xi_{a,b,B} = \frac{\sqrt{a^2x_0 + b^2y_0 + r\sqrt{a^4 + b^4}} - \sqrt{a^2x_0 + b^2y_0 - r\sqrt{a^4 + b^4}}}{\sqrt{a^2x_0 + b^2y_0 + r\sqrt{a^4 + b^4}} + \sqrt{a^2x_0 + b^2y_0 - r\sqrt{a^4 + b^4}}} = \frac{2r\sqrt{a^4 + b^4}}{\sqrt{a^2x_0 + b^2y_0 + r\sqrt{a^4 + b^4}} + \sqrt{a^2x_0 + b^2y_0 - r\sqrt{a^4 + b^4}}} \]

Taking into account that $\varepsilon \leq x_0, y_0 \leq 1$ and $r < 1$, it follows that
\[ \frac{1}{2} r h_{a,b} \leq \xi_{a,b,B} \leq \frac{8}{\varepsilon} r h_{a,b}. \]

Now, the number of possible values for $c$ lies between $\xi_{a,b,B}$ and $\xi_{a,b,B} + 3$ and is therefore $\asymp r h_{a,b}$.

### 4.3.3 The measure of $\sigma_{a,b} \cap B$

Given a $c$, it is easily verified that $|\sigma_{a,b}(c) \cap B| \leq 4r\psi(h_{a,b})/\sqrt{a^4 + b^4} \leq 4r\psi(h_{a,b})/h_{a,b}^2$, where $r$ is the radius of $B$.

The number of possible values of $c$ such that $\sigma_{a,b}(c) \cap B \neq \emptyset$ is bounded above by $\xi_{a,b,B} + 3 \leq \frac{10}{\varepsilon} r h_{a,b}$ if $h_{a,b}$ is sufficiently large. Therefore,
\[ |\sigma_{a,b} \cap B| \leq 4r\psi(h_{a,b})/h_{a,b}^2 \times \frac{10}{\varepsilon} r h_{a,b} = c_2 |B| \frac{\psi(h_{a,b})}{h_{a,b}}, \]
where $c_2 = \frac{40}{\varepsilon^2}$ and $h_{a,b}$ is sufficiently large.

Let $\frac{1}{2}B$ be the ball centred at the same point as $B$ of radius $r/2$. Then it is an elementary geometric task to compute that $|\sigma_{a,b}(c) \cap B| \geq r\psi(h_{a,b})/h_{a,b}^2$ whenever $\sigma_{a,b}(c) \cap \frac{1}{2}B \neq \emptyset$ and $h_{a,b}$ is sufficiently large.

The number of possible values of $c$ such that $\sigma_{a,b}(c) \cap \frac{1}{2}B \neq \emptyset$ is bounded below by $\xi_{a,b,\frac{1}{2}B} \geq \frac{1}{4} r h_{a,b}$. Therefore,
\[ |\sigma_{a,b} \cap B| \geq r\psi(h_{a,b})/h_{a,b}^2 \times \frac{1}{4} r h_{a,b} = c_1 |B| \frac{\psi(h_{a,b})}{h_{a,b}}, \]
where \( c_1 = \frac{1}{4\pi} \).

The upshot of the above is that

\[
c_1 |B| \frac{\psi(h_{a,b})}{h_{a,b}} \leq |\sigma_{a,b} \cap B| \leq c_2 |B| \frac{\psi(h_{a,b})}{h_{a,b}}
\]  

for all sufficiently large \( h_{a,b} \), where \( c_1, c_2 \) are absolute positive constants.

### 4.3.4 Additional conditions on \((a, b)\)

Throughout the remainder of the proof of Theorem 1 we will assume that the following conditions on \((a, b)\) hold:

\[
\gcd(a, b) = 1,
\]

where \( \gcd \) means the greatest common divisor, and

\[
1/2 \leq a/b \leq 2.
\]

The above conditions sift elements of the sequence of sets \( \sigma_{a,b} \) which prevent us from having sufficiently good estimates for the measures of pairwise intersections of these sets. On the other hand, the remaining ‘thinned out’ part of the sequence \( \sigma_{a,b} \) is still rich enough to ensure that the sum

\[
\sum |\sigma_{a,b}|
\]

diverges over this restricted sequence. Such a condition as that of Equation (12) is necessary to apply Lemma 2. Indeed, to verify that (12) diverges over \((a, b) \in \mathbb{Z}^2 \geq 0\) satisfying (10) and (11) define \( N_k \) to be the number of \((a, b)\) satisfying (10) and (11) with \( 2^k \leq h_{a,b} < 2^{k+1} \). Then in view of symmetry of the set of \((a, b)\) of interest we get

\[
N_k = 2 \sum_{2^k \leq a < 2^{k+1}} \sum_{b < a} 1 = 2 \sum_{2^k \leq a < 2^{k+1}} \left( \varphi(a) - \varphi([a/2]) \right),
\]

where \( \varphi \) is the Euler function. It is well known that

\[
\sum_{1 \leq q \leq Q} \varphi(q) = \frac{3}{\pi^2} Q^2 + O(Q \log Q).
\]

Then

\[
2 \sum_{2^k \leq a < 2^{k+1}} \varphi(a) = \frac{6}{\pi^2} \left( (2^{k+1})^2 - (2^k)^2 \right) + O(k2^k) = \frac{18}{\pi^2} 2^{2k} + O(k2^k)
\]

and

\[
2 \sum_{2^k \leq a < 2^{k+1}} \varphi([a/2]) = 4 \sum_{2^{k-1} \leq x < 2^k} \varphi(x) = \frac{12}{\pi^2} \left( (2^k)^2 - (2^{k-1})^2 \right) + O(k2^k) = \frac{9}{\pi^2} 2^{2k} + O(k2^k).
\]

It follows that

\[
N_k = \frac{9}{\pi^2} 2^{2k} + O(k2^k).
\]
Now the estimated sum is
\[
\sum_{(a,b) \in \mathbb{Z}^2_+} |\sigma_{a,b} \cap B| = \sum_{k=0}^{\infty} \sum_{2^k \leq h < 2^{k+1}} \sum_{(a,b) \in \mathbb{Z}^2_+: h_{a,b}=h} |\sigma_{a,b}| \gg \\
\gg |B| \sum_{k=0}^{\infty} \sum_{2^k \leq h < 2^{k+1}} \sum_{(a,b) \in \mathbb{Z}^2_+: h_{a,b}=h} \psi(2^{k+1}) \times \\
\times |B| \sum_{k=0}^{\infty} 2^k \psi(2^k) \times |B| \sum_{h=1}^{\infty} \psi(h) = \infty.
\]

Finally, note that the limsup set for the ‘thinned out’ sequence \(\sigma_{a,b}\) is contained in the limsup set for the complete sequence \(\sigma_{a,b}\), which is \(W_2(\psi)\). Therefore, it will be sufficient to prove that the thinned out limsup set is of full Lebesgue measure in order to ensure that \(W_2(\psi)\) is also of full measure.

An immediate consequence of condition (10) is that for any two pairs \((a,b)\) and \((a',b')\) satisfying (10) the assumption \((a,b) \neq (a',b')\) implies that \((a,b)\) and \((a',b')\) are not collinear. Moreover, \((a^2, b^2)\) and \((a'^2, b'^2)\) are not collinear. Therefore we can assume that the (smaller) angle between \((a^2, b^2)\) and \((a'^2, b'^2)\), which will be denoted by \(\alpha = \alpha(a,b,a',b')\), is not zero.

The analysis of the measures of intersections \(\sigma_{a,b} \cap \sigma_{a',b'} \cap B\) will rely on the behaviour of this angle and is given in the following sections.

### 4.3.5 The measure of intersections in the case of a big angle

We will assume that \((a,b) \neq (a',b')\). Within this subsection we set \(h = h_{a,b}\) and \(h' = h_{a',b'}\).

For simplicity we will assume that \(h \geq h'\). Now
\[
\sigma_{a,b} \cap \sigma_{a',b'} \cap B = \bigcup_{c' \in \mathbb{Z}_+} \sigma_{a,b} \cap \sigma_{a',b'}(c') \cap B
\]
(13)

For a fixed \(c'\) the set \(\sigma_{a',b'}(c') \cap B\) is covered with a strip of length \(2r\) (recall that \(r\) is the radius of \(B\)) and width \(\psi(h')/h'^2\). This strip is a piece of the \(\psi(h')/h'^2\)-neighbourhood of the line
\[
a^2x + b^2y - c^2 = 0.
\]
(14)

To estimate the measure in (13) we first estimate the measure of the intersection of \(\sigma_{a,b}\) with such a strip.

The angle \(\alpha = \alpha(a,b,a',b')\) introduced in the previous section is the (smaller) angle between the line defined in (14) and the family of parallel lines
\[
a^2x + b^2y - c^2 = 0, \text{ where } c \in \mathbb{Z}_+.
\]
(15)

Using (15) it is readily verified that the distance between two consecutive lines in the family (15) is \(\asymp h^{-1}\).
Now if \( A \) and \( B \) are two consecutive points on the line \( (14) \), obtained as a result of its intersection with two consecutive lines in \( (15) \), say \( L_1 \) and \( L_2 \), it is easy to calculate that the distance between \( A \) and \( B \) is the distance between \( L_1 \) and \( L_2 \) divided by \( \sin \alpha \), that is
\[
\frac{1}{h \sin \alpha}.
\]
Since the piece of the line \( (14) \) of interest is of length at most \( 2r \), there are at most
\[
\ll rh \sin \alpha + 1
\]
on-empty intersections \( \sigma_{a,b}(c) \cap \sigma_{a',b'}(c') \cap B \) when \( c \) runs over all integers.

As the set \( \sigma_{a,b}(c) \cap \sigma_{a',b'}(c') \) is a parallelepiped with area
\[
\ll \frac{\psi(h)}{h^2} \frac{\psi(h')}{h'^2} \frac{1}{\sin \alpha} \times (rh \sin \alpha + 1).
\]
Further, since there are \( \ll rh' \) values of \( c' \) that need to be considered, we have that
\[
|\sigma_{a,b} \cap \sigma_{a',b'} \cap B| \ll \frac{\psi(h)}{h^2} \frac{\psi(h')}{h'^2} \frac{1}{\sin \alpha} \times (rh \sin \alpha + 1) rh' \times
\]
\[
\ll |B| \frac{\psi(h)}{h} \frac{\psi(h')}{h'} \left( 1 + \frac{1}{rh \sin \alpha} \right).
\] (16)

Assuming that \( \frac{1}{rh \sin \alpha} \leq 1 \), or equivalently that
\[
\sin \alpha \geq \frac{1}{rh},
\] (17)
gives
\[
|\sigma_{a,b} \cap \sigma_{a',b'} \cap B| \ll |B| \frac{\psi(h)}{h} \frac{\psi(h')}{h'}.
\] (18)

Finally, since there are \( \asymp h \) integer vectors \((a,b)\) with \( h_{a,b} = h \) and \( \asymp h' \) integer vectors \((a',b')\) with \( h_{a',b'} = h' \), summing the measures of intersections \( |\sigma_{a,b} \cap \sigma_{a',b'} \cap B| \) in the case under consideration results in
\[
\sum_{h_{a,b} \leq H, h_{a',b'} \leq H \atop (a,b) \neq (a',b')} |\sigma_{a,b} \cap \sigma_{a',b'} \cap B| \ll |B| \left( \sum_{h=1}^{H} \psi(h) \right)^2.
\]

4.3.6 The measure of intersections in the case of a small angle

In this section we will deal with the case of
\[
\sin \alpha \leq \frac{1}{rh}.
\] (19)

Again we will assume that \((a,b) \neq (a',b')\) and given a matrix \( A \), \(|A|\) will denote its determinant and \( \|A\| \) the absolute value of its determinant.
Since $\alpha$ is the angle between the vectors $(a^2, b^2)$ and $(a'^2, b'^2)$ it follows that

$$h^2 h'^2 \sin \alpha \propto \sqrt{a^4 + b^4} \sqrt{a'^4 + b'^4} \sin \alpha = \left\| \begin{array}{cc} a^2 & b^2 \\ a'^2 & b'^2 \end{array} \right\| = \left\| \begin{array}{cc} a & b \\ a' & b' \end{array} \right\| \times \left\| \begin{array}{cc} a & -b \\ a' & b' \end{array} \right\| \ . \quad (20)$$

If $\beta$ denotes the (smaller) angle between $(a, b)$ and $(a, -b)$ then

$$\sin \beta = \frac{1}{a^2 + b^2} \left\| \begin{array}{cc} a & b \\ a & -b \end{array} \right\| = \frac{2|ab|}{a^2 + b^2} \geq \frac{1}{2} .$$

Hence, $\beta \geq \pi/6$ and the angle between $(a', b')$ and at least one of the vectors $(a, b)$ and $(a, -b)$ is at least $\pi/12$. Without loss of generality we can assume that such an angle is between $(a, -b)$ and $(a', b')$. Then

$$\left\| \begin{array}{cc} a' & -b \\ a' & b' \end{array} \right\| = \sqrt{a^2 + b^2} \sqrt{a'^2 + b'^2} \sin \pi/12 \gg h h' .$$

It now follows from (20) that

$$1 \leq \left\| \begin{array}{cc} a & b \\ a' & b' \end{array} \right\| \ll h h' \sin \alpha \leq \frac{h'}{r} . \quad (21)$$

This means that for every fixed $a', b', a$ there are at most $\ll \frac{1}{r}$ possible values for $b$. Indeed, $|ab' - a'b| \ll h'r^{-1}$, that is $|b - ab'/a'| \ll h'r^{-1}/a' \ll r^{-1}$. Moreover, (21) implies that

$$\sin \alpha \gg \frac{1}{h h'} . \quad (22)$$

To complete the analysis for this case we consider two specific subcases.

**Subcase (i) – moderately small angle.**

Assume for the moment that

$$\sin \alpha \geq \frac{1}{r^2 h h'} . \quad (23)$$

Using (16), (19) and (23) it follows that

$$|\sigma_{a,b} \cap \sigma_{a',b'} \cap B| \ll |B| \psi(h) \psi(h') \frac{1}{r h \sin \alpha} \ll |B| \frac{\psi(h)}{h} \psi(h') \frac{1}{r} .$$

Now the sum of intersections for this subcase can be estimated as follows,

$$\sum_{h_a, b \leq H, \ h_{a', b'} \leq H} |\sigma_{a,b} \cap \sigma_{a',b'} \cap B| \ll$$

$$\sum_{h=1}^{H} \sum_{h'=1}^{h-1} \sum_{h_a, b \leq h} \sum_{h_a', b' \leq h'} |\sigma_{a,b} \cap \sigma_{a',b'} \cap B| \ll$$

$$\sum_{h=1}^{H} \sum_{h'=1}^{h-1} |B| \frac{\psi(h)}{h} \psi(h') \frac{1}{r} \ll$$

11
The upshot of the above computations is the following estimates:

\[
S_1(H) = \sum_{(a,b) \in \mathcal{Z}_H} |\sigma_{a,b} \cap B| \gg |B| \left( \sum_{h=1}^H \psi(h) \right)^2.
\]

**Subcase (ii) – ultra small angle.**

To complete the analysis of all possible values of \( \alpha \) it remains to consider the case when

\[
\sin \alpha < \frac{1}{r^2 \, h \, h'}.
\]

Then

\[
\left\| \begin{array}{ll}
a & b \\
a' & b'
\end{array} \right\| \leq h' \sin \alpha \leq \frac{1}{r^2};
\]

and

\[
|\sigma_{a,b} \cap \sigma_{a',b'} \cap B| \ll |B| \frac{\psi(h) \psi(h')}{h \, h'} \frac{1}{r \sin \alpha} \ll |B| \frac{\psi(h) \psi(h')}{h} \frac{1}{r}. \tag{26}
\]

Now we estimate the number of quadruples \((a, b, a', b')\) satisfying (10), (11), (25), \(2^k \leq h_{a,b} < 2^{k+1}\) and \(2^l \leq h_{a',b'} < 2^{l+1}\). Given fixed \(a\) and \(b'\), (25) means that \(a', b\) can only be chosen to satisfy \(|ab' - a'b| \ll r^{-2}\). This means that there are \(\ll r^{-2}\) possible values for \(t = a'b\). In turn, for a fixed \(t\) there are at most \(d(t)\) possible values for \(a'\) and \(b\), where \(d(t)\) is the number of divisors of \(t\). It is well known that for any \(\delta > 0\) there is a constant \(c_{\delta} > 0\) such that \(d(t) \leq c_{\delta} t^{\delta}\) for all \(t\). Taking \(\delta = 1/4\) we get that the number of possible quadruples \(a, b, a', b'\) is \(\ll (2^k \, 2^l)^{5/4} r^{-2}\).

Without loss of generality we assume that \(\psi(h) \leq h^{-1}\). Then the sum of intersections for this subcase is estimate as follows

\[
\sum_{h_{a,b} \leq H, \, h_{a',b'} \leq H} |\sigma_{a,b} \cap \sigma_{a',b'} \cap B| = \sum_{k=1}^{[\log H]} \sum_{l=1}^{[\log H]} \sum_{2^k \leq h_{a,b} < 2^{k+1}, \, 2^l \leq h_{a',b'} < 2^{l+1}} |\sigma_{a,b} \cap \sigma_{a',b'} \cap B| \ll \sum_{k=1}^{[\log H]} \sum_{l=1}^{[\log H]} |B| \frac{\psi(2^k)}{2^k} \frac{1}{r} \times (2^k \, 2^l)^{5/4} r^{-2} \ll \frac{1}{r} \sum_{k=1}^{[\log H] + 1} \sum_{l=1}^{[\log H] + 1} 2^{k/4} \psi(2^k) 2^{l/4} \psi(2^l) \ll \frac{1}{r} \sum_{k=1}^{[\log H] + 1} \sum_{l=1}^{[\log H] + 1} 2^{3k/4} \psi(2^k) 2^{3l/4} \psi(2^l) \ll \frac{1}{r} \sum_{k=1}^{[\log H]} \sum_{l=1}^{[\log H]} 2^{-k/4} 2^{-l/4} < \infty.
\]

We are now in a position to complete the proof of Theorem 1 for the divergence case.

### 4.4 Completion of the proof of Theorem 1

The upshot of the above computations is the following estimates:

\[
S_1(H) = \sum_{(a,b) \in \mathcal{Z}_H} |\sigma_{a,b} \cap B| \gg |B| \left( \sum_{h=1}^H \psi(h) \right).
\]
\[ S_2(H) = \sum_{(a,b) \in \mathbb{Z}_H^2} \sum_{(a',b') \in \mathbb{Z}_H^2} |\sigma_{a,b} \cap \sigma_{a',b'} \cap B| \ll |B| \left( \sum_{h=1}^{H} \psi(h) \right)^2 \]

where \( \mathbb{Z}_H = \{(a,b) \in \mathbb{Z}_2^2 \geq 0, (10) \) and (11) hold and \( h_{a,b} \leq H \). Therefore,

\[
\frac{S_1(H)^2}{S_2(H)} \gg |B|
\]

for all sufficiently large \( H \). Since \( \lim \sup_{h_{a,b} \to \infty} \sigma_{a,b} \cap B \subset W_2(\psi) \cap B \), by Lemma 2

\[
|W_2(\psi) \cap B| \geq \lim \sup_{h_{a,b} \to \infty} |\sigma_{a,b} \cap B| \gg |B|.
\]

This holds for any ball \( B \) in \( \Omega \) with the implied constant independent of \( B \). Therefore, by Lemma 11 \( W_2(\psi) \) has full measure in \( \Omega = (\varepsilon, 1)^2 \). Since \( \varepsilon > 0 \) is arbitrary, \( W_2(\psi) \) has full measure in \([0,1]^2\). This completes the proof of Theorem 11.

5 Proof of Theorem 2

5.1 Hausdorff measures and dimension

In this section we give a very brief introduction to the theory of Hausdorff measures and dimension. For further details consult [1].

Let \( s \) be a positive real number. The Hausdorff \( s \)-measure will be denoted throughout by \( \mathcal{H}^s \) and is defined as follows. Suppose \( F \) is a non-empty subset of \( \mathbb{R}^k \). Suppose that \( \rho > 0 \). A \( \rho \)-cover of \( F \) is a countable collection \( \{B_i\} \) of balls in \( \mathbb{R}^k \) with radii \( r_i \leq \rho \) for each \( i \) such that

\[
F \subset \bigcup_i B_i.
\]

Define the function \( \mathcal{H}^s_{\rho} \) by

\[
\mathcal{H}^s_{\rho}(F) := \inf \left\{ \sum_i r_i^s \right\}
\]

where the infimum is taken over all possible \( \rho \)-covers of \( F \). Then \( \mathcal{H}^s(F) \) of the set \( F \) is defined by

\[
\mathcal{H}^s(F) := \lim_{\rho \to 0} \mathcal{H}^s_{\rho}(F) = \sup_{\rho > 0} \mathcal{H}^s_{\rho}(F).
\]

Let \( F \) be an infinite set. The Hausdorff dimension of \( F \) is the (unique) number

\[
\dim F = \inf \{ s > 0 : \mathcal{H}^s(F) = 0 \} = \sup \{ s > 0 : \mathcal{H}^s(F) = +\infty \}.
\]

Note that \( \mathcal{H}^k \) is a multiple of the \( k \)-dimensional Lebesgue measure in \( \mathbb{R}^k \) when \( k \in \mathbb{N} \).
5.2 Proof of Theorem 2. The case of convergence

The proof of convergence is straightforward. Recall from above that \( W_2(\psi) \) can be expressed as a limsup set of the form

\[
W_2(\psi) = \bigcap_{h=1}^{\infty} \bigcup_{(a,b) \in \mathbb{Z}^2, c \in \mathbb{Z}, h_{a,b}=h} (\sigma_{a,b}(c)).
\]

Each \( \sigma_{a,b}(c) \) can be covered by a family \( C_{a,b}^c \) of balls each of radius \( \psi(h_{a,b})/h_{a,b}^2 \) where

\[
\sharp C_{a,b}^c \ll \frac{h_{a,b}}{\psi(h_{a,b})}.
\]

By assumption \( \psi(h) \to 0 \) as \( h \to \infty \). Therefore, given any \( N \in \mathbb{N} \), \( \psi(h)/h^2 \leq 1/N \) for sufficiently large \( h \). It follows that

\[
\mathcal{H}^{s}_{1/(2N)}(W_2(\psi)) \ll \sum_{(a,b) \in \mathbb{Z}^2, h_{a,b} \geq N} \left( \frac{\psi(h_{a,b})}{h_{a,b}^2} \right)^s \frac{h_{a,b}^2}{\psi(h_{a,b})} h_{a,b} \ll \sum_{h \geq N} \left( \frac{\psi(h)}{h^2} \right)^s \psi(h)^{-1} h^2 hh
\]

\[
= \sum_{h \geq N} \psi(h)^{s-1} h^{4-2s} \to 0 \quad \text{as} \quad N \to \infty.
\]

Therefore \( \mathcal{H}^s(W_2(\psi)) = 0 \), as required.

5.3 Proof of Theorem 2. The case of divergence

To prove the divergence case of Theorem 2 we appeal to a recent result of Beresnevich & Velani \cite{Ber} in which a mass transference principle for linear forms based on a technique called ‘slicing’ is established. The result allows one to transfer statements about the Lebesgue measure of general limsup sets occurring in Diophantine approximation to ones involving Hausdorff measure.

The ideas outlined below are specialised to suit the particular Diophantine approximation problems posed in this paper and are therefore simplified versions of those given in \cite{Ber}. The general framework of \cite{Ber} is far richer and allows one to address Diophantine problems involving systems of linear forms, inhomogeneous approximation and general measure functions in one consuming package.

Let \( \mathcal{R} = (R_\alpha)_{\alpha \in J} \) be a family of lines in \( \mathbb{R}^2 \) indexed by an infinite countable set \( J \). For every \( \alpha \in J \) and \( \delta \geq 0 \) define the \( \delta \)-neighborhood \( \Delta(R_\alpha, \delta) \) of \( R_\alpha \) by

\[
\Delta(R_\alpha, \delta) := \{ x \in \mathbb{R}^2 : \text{dist}(x, R_\alpha) < \delta \}.
\]

Next, let

\[
\Upsilon : J \to \mathbb{R}^+ : \alpha \mapsto \Upsilon(\alpha) := \Upsilon_\alpha
\]
be a non-negative, real valued function on \( J \). Further, assume that for every \( \epsilon > 0 \) the set \( \{ \alpha \in J : \Upsilon_\alpha > \epsilon \} \) is finite. This condition implies that \( \Upsilon_\alpha \to 0 \) as \( \alpha \) runs through \( J \). Now define the following ‘lim sup’ set,

\[
\Lambda(\Upsilon) = \{ x \in \mathbb{R}^2 : x \in \Delta(R_\alpha, \Upsilon_\alpha) \text{ for infinitely many } \alpha \in J \}.
\]

**Theorem 3** Let \( \mathcal{R} \) and \( \Upsilon \) as above be given. Let \( V \) be a line in \( \mathbb{R}^2 \) and

(i) \( V \cap R_\alpha \neq \emptyset \) for all \( \alpha \in J \),

(ii) \( \sup_{\alpha \in J} \diam(V \cap \Delta(R_\alpha, 1)) < \infty \).

Let \( f \) and \( g : r \to g(r) := r^{-1} f(r) \) be dimension functions such that \( r^{-2} f(r) \) is monotonic and let \( \Omega \) be a ball in \( \mathbb{R}^2 \). Suppose for any ball \( B \) in \( \Omega \)

\[
\mathcal{H}^2(B \cap \Lambda(g(\Upsilon))) = \mathcal{H}^2(B)
\]

Then

\[
\mathcal{H}^f(B \cap \Lambda(\Upsilon)) = \mathcal{H}^f(B).
\]

Now, let \( f : r \to r^s \). As \( 1 < s < 2 \) it follows that \( r^{-2} f(r) \) is monotonic and \( f \) and \( g \), defined as above, are both dimension functions. Further, let \( \Omega \) to be the unit square \( [0, 1]^2 \), \( J := \{ (a, b, c) \in \mathbb{Z}^3 : h_{a,b} = |a| \} \),

\[
R_{(a,b,c)} = \{ (x, y) \in \mathbb{R}^2 : a^2 x + b^2 y = c^2 \}
\]

and \( \Upsilon_{(a,b,c)} := \psi(h_{a,b})/h_{a,b}^2 \). Define \( S_2(\psi) \) to be

\[
S_2(\psi) := \Lambda(\Upsilon) \cap [0, 1)^2.
\]

Note that \( S_2(\psi) \subset W_2(\psi) \) and \( |S_2(\psi)| = 1 \) whenever \( |W_2(\psi)| = 1 \). To complete the proof of Theorem 2 it is sufficient to prove the divergence case for \( S_2(\psi) \). With this in mind, let \( V := \{ (x_1, x_2) \in \mathbb{R}^2 : x_2 = 0 \} \). It is straightforward to verify that conditions (i) and (ii) of Theorem 3 hold in this case. From the divergence case of Theorem 1 it follows that \( \mathcal{H}^2(S_2(\psi)) = 1 = \mathcal{H}^2([0, 1]^2) \). Therefore, \( \mathcal{H}^s(S_2(\psi)) = \mathcal{H}^s([1]) = \infty \) and Theorem 2 is proved.

### 5.4 Proof of Corollary 1

By the definition of the lower order for any \( \delta > 0 \) the inequality \( \lambda_\psi + \delta \geq \frac{\log \frac{1}{\psi(2r)}}{\log 2} \) for infinitely many \( r \). It follows that

\[
\psi(2^r) \geq (2^r)^{-\lambda_\psi - \delta} \quad \text{for infinitely many } r.
\]

(27)

Take \( s = 1 + \frac{3}{2 + \lambda_\psi + \delta} - \delta \). Then

\[
\psi(2^r)^{s-1}(2^r)^{5-2s} \geq (2^r)^{-(\lambda_\psi + \delta)(s-1)+5-2s} = (2^r)^{-(\lambda_\psi + 2+ \delta)(s-1)+3} = (2^r)^{\delta(\lambda_\psi + 2+ \delta)} > 1
\]
for infinitely many \( r \). Therefore,
\[
\sum_{r=1}^{\infty} \psi(2^r)^{s-1}(2^r)^{5-2s} = \infty.
\]
Since \( \psi \) is monotonic, using a simple ‘condensation’ argument it is easy to verify that
\[
\sum_{h=1}^{\infty} \psi(h)^{s-1}h^{4-2s} = \infty.
\]
Hence, by Theorem \( \ref{thm1} \)
\[
\mathcal{H}^{s}(W_2(\psi)) = \infty \quad \text{and} \quad \dim W_2(\psi) \geq s = 1 + \frac{3}{2 + \lambda_{\psi} + \delta} - \delta.
\]
Since \( \delta > 0 \) is arbitrary, we have \( \dim W_2(\psi) \geq 1 + \frac{3}{2 + \lambda_{\psi}} \).

Again, by the definition of the lower order, for any \( \delta > 0 \) the inequality \( \lambda_{\psi} - \delta \leq \frac{\log \frac{1}{\psi(2^r)}}{\log 2^r} \) holds for all sufficiently large \( r \). It follows that
\[
\psi(2^r) \leq (2^r)^{-\lambda_{\psi} + \delta} \quad \text{for all sufficiently large} \quad r.
\] (28)
Take \( s = 1 + \frac{3}{2 + \lambda_{\psi} - \delta} + \delta \). Then
\[
\psi(2^r)^{s-1}(2^r)^{5-2s} \leq (2^r)^{-\lambda_{\psi} - \delta(s-1)+5-2s} = (2^r)^{-\delta(\lambda_{\psi}+2-\delta)}
\]
for infinitely many \( r \). Therefore,
\[
\sum_{r=1}^{\infty} \psi(2^r)^{s-1}(2^r)^{5-2s} < \sum_{r=1}^{\infty} (2^r)^{-\delta(\lambda_{\psi}+2-\delta)} < \infty.
\]
Since \( \psi \) is monotonic, using the ‘condensation’ argument it is easy to verify that
\[
\sum_{h=1}^{\infty} \psi(h)^{s-1}h^{4-2s} < \infty.
\]
Hence
\[
\mathcal{H}^{s}(W_2(\psi)) < \infty \quad \text{and} \quad \dim W_2(\psi) \leq s = 1 + \frac{3}{2 + \lambda_{\psi} - \delta} + \delta.
\]
Since \( \delta > 0 \) is arbitrary, we have \( \dim W_2(\psi) \leq 1 + \frac{3}{2 + \lambda_{\psi}} \). Therefore, we have the equality
\[
\dim W_2(\psi) = 1 + \frac{3}{2 + \lambda_{\psi}}.
\]

5.5 Proof of Corollary \( \ref{cor1} \)

The proof that Equation \( \ref{eq1} \) has a solution in \( \mathbb{H}^{n+2}(\alpha, \beta, \gamma) \) whenever \( (\delta_1, \delta_2) \notin W_2(r \mapsto r^{-2}) \), which is a set of dimension \( 7/4 \), is an immediate consequence of Corollary \( \ref{cor1} \).

Assume now that \( f \) is required to be smooth. As \( W_2(r \mapsto r^{-\tau'}) \subset W_2(r \mapsto r^{-\tau}) \) for \( \tau' > \tau \). It follows by continuity of \( \dim(\cdot) \) that
\[
\dim \left( \bigcap_{\nu > 1} W_2(r \mapsto r^{-\nu}) \right) = \lim_{v \to \infty} \dim \left( W_2(r \mapsto r^{-v}) \right) = \lim_{v \to \infty} \left( 1 + \frac{3}{2 + v} \right) = 1.
\]
This establishes Corollary \( \ref{cor2} \).
6 Outline of the General case \( n \geq 3 \)

The convergence case of Theorem 1 for \( n \geq 3 \) is almost immediate. For every \((n + 1)\)-tuple \((a, b) \in \mathbb{Z}^{n+1}_{\geq 0}\), let

\[
\sigma_a(b) := \{x \in [0, 1] : \|a^2 \cdot x - b^2\| < \psi(h_a)\}
\]

and

\[
\sigma_a := \bigcap_{b \in \mathbb{Z}} \sigma_a(b)
\]

where \(a^2\) is the vector \((a_1^2, a_2^2, \ldots, a_n^2)\). It is easy to see that each set \(\sigma_a(b)\) is an \(n - 1\)-dimensional hyperplane with area \(|\sigma_a(b)| \ll \psi(h_a)/h_a^2\). Fix an \(a \in \mathbb{Z}^{n}_{\geq 0}\), \(\sigma_a \neq \emptyset\) implies that \(b \ll h_a\). Note that the number of vectors \(a\) for which \(h_a = h\) is \(\ll h^{n-1}\). Now

\[
\sum_{h=1}^{\infty} \sum_{a \in \mathbb{Z}^{n}_{\geq 0}} \sum_{b \in \mathbb{Z} : |\sigma_a(b)| \ll \sum_{h=1}^{\infty} h^{n-2}\psi(h) < \infty
\]

by assumption. It follows that \(|W_n(\psi)| = 0\) and we are done.

Assuming for a moment the validity of the divergence part of Theorem 1 when \(n \geq 3\). Establishing Theorem 2 is relatively straightforward.

In the convergence case we note that

\[
W_n(\psi) = \bigcap_{h=1}^{\infty} \bigcup_{a \in \mathbb{Z}^{n}, b \in \mathbb{Z}} (\sigma_a(b) \cap \mathbb{I}^n)
\]

and each \(\sigma_a(b)\) can be covered by a family \(C_a^b\) of balls each of radius \(\psi(h_a)/h_a^2\) such that

\[
\#C_a^b \ll (h_a^2/\psi(h_a))^{n-1}.
\]

It is then a simple matter to amend the proof in the case when \(n = 2\) for \(n \geq 3\) and deduce that \(\mathcal{H}^n(W_n(\psi)) = 0\).

The divergence case of Theorem 2 can be proved with only minor modifications of the proof for the case when \(n = 2\). The main changes to be made to the general framework of Theorem 2 are that \(\mathcal{R}\) is now a countable family of \((n - 1)\)-dimensional hyperplanes, \(x \in \mathbb{R}^n\), \(V\) is a linear subspace of \(\mathbb{R}^n\), \(f\) is a dimension function such that \(r^{-n}f(r)\) is monotonic and \(g: r \rightarrow r^{-(n-1)}f(r)\) is a dimension function.

Now, let \(f : r \rightarrow r^8\), \(\Omega\) be the unit hypercube \([0, 1]^n\), \(J := \{(a, b) \in \mathbb{Z}^{n+1}_{\geq 0} : h_a = |a_1|\},

\[
R_{(a,b)} := \{x \in \mathbb{R}^n : a^2 \cdot x = b^2\}
\]

and \(\Upsilon_{(a,b)} := \psi(h_a)/h_a^2\). The rest of the argument is essentially the same as that given above with 2 replaced by \(n\) and \(V := \{x \in \mathbb{R}^n : x_n = 0\}\).

It remains to establish the divergence part of Theorem 1 for the cases when \(n \geq 3\). As noted above, the family of lines that we considered in §4 have now been replaced by \((n - 1)\)-dimensional hyperplanes, but the analysis again hinges on the angle between the members of
two non-collinear families. It is relatively easy to see that the restrictions that applied to \( c \) in §4.3.1 must also apply to \( b \) in the above argument and further, that the number of such \( b \) must also be \( \asymp rh_a \). This follows from the fact that the geometry in the \( n \)-dimensional case can be reduced to the same problem as that of the 2-dimensional case by projecting the ball \( B \) and the \((n-1)\)-dimensional hyperplanes onto a 2-dimensional plane perpendicular to the family of hyperplanes defined by the equations

\[
a^2 \cdot x - b^2 = 0
\]

where \( b \in \mathbb{Z} \). A simple geometric argument implies that \( |\sigma_a(b) \cap B| \ll r^{n-1} \frac{\psi(h_a)}{h_a^2} \) where \( r \) is the radius of \( B \). As the number of possible \( b \) such that \( \sigma_a(b) \cap B \neq \emptyset \) is \( \ll rh_a \) it follows that

\[
|\sigma_a \cap B| \ll r^n \frac{\psi(h_a)}{h_a} \ll |B| \frac{\psi(h_a)}{h_a},
\]

and by an analogous argument to that in §4.3.3 it can be shown that

\[
|\sigma_a \cap B| \gg |B| \frac{\psi(h_a)}{h_a}
\]

where the constants implied by the \( \ll \) and \( \gg \) are absolute. Recall that conditions (10) and (11) were imposed on \( a \) and \( b \) in the 2-dimensional cases. For the higher dimensional cases the corresponding conditions become

\[
\gcd(a_1, a_2, \ldots, a_n) = 1 \quad (29)
\]

and

\[
1/2 \leq a_1/a_2 \leq 2, \quad (30)
\]

with the same consequences as in §4.3.4, namely a sufficient quantity of vectors to maintain divergence of our sum and non-collinearity of any two vectors satisfying (29).

As in the 2-dimensional case considered above, take any two vectors \( a \) and \( a' \) with \( a \neq a' \), which must be linearly independent by (20). The upshot of linear independence is that the angle between the normals to the two hyperplanes, and therefore the hyperplanes themselves, is non-zero. Strictly speaking there are two angles, but we shall take the smaller of the two and call this \( \alpha \). The result of §4.3.5 also holds in this case. It is a simple geometric argument to show that the volume of the parallelepiped obtained by intersecting any two members of the two families is now

\[
\ll r^{n-2} \frac{\psi(h_a)}{h_a^2} \frac{\psi(h_{a'})}{h_{a'}^2},
\]

An analogous argument to that presented in §4.3.5 with the restriction that \( \sin \alpha \geq \frac{1}{rh} \) yields the desired estimate for the sum of the measures of the intersections subject to the above restriction on \( \alpha \).

To complete the proof requires taking care of the cases when the angle \( \alpha \) becomes small. Recall that in the 2-dimensional case, §4.3.6 this naturally split into two cases; that of a moderately small angle and an ultra-small angle. It was shown in the former case that the same estimate as that of the big angle case could be deduced and in the latter, that the sum...
of the intersections over the class of vectors with ultra small angle was in fact convergent and
could therefore be neglected. It is precisely these conclusions that can be shown to hold in
the general case and the divergence part of Theorem 1 will follow in exactly the same manner
as in the 2-dimensional case.

The analysis in § 4.3.6 relied on a key observation that the angle, $\alpha$, couldn’t get too
small. More precisely that $\sin \alpha \gg 1/h_a h_a'$. This was a consequence of the assumption that
$1/2 \leq a_1/a_2 \leq 2$. To establish this fact we used the standard result from elementary geometry
that $|a \times b| = |a||b|\sin \beta$ where $\beta$ is the angle between $a$ and $b$. In higher dimensions the
cross product $\times$ is replaced by the wedge product $\wedge$ where

$$a \wedge b = \left\{ \begin{array}{ccc} a_i & a_j \\ b_i & b_j \end{array} : 1 \leq i < j \leq n \right\}.$$  

Note without any loss of generality we can assume that the first two coordinates give the
biggest determinant by reordering if necessary and it is this observation, coupled with the
assumption that $1/2 \leq a_1/a_2 \leq 2$ that allows us to conclude that $\sin \alpha \gg 1/h_a h_a'$. The
argument for the case when the angle is moderately small is exactly the same as for the
2-dimensional case. Leaving only the case when

$$\sin \alpha < \frac{1}{r^2 h_a h_a'}$$  \hspace{1cm} (31)

to take care of. As there is a free choice in all but the first two components of either of
the vectors $a$ and $a'$ the number of pairs of vectors that we need to consider is $h_a^{n-2} h_{a'}^{n-2} \cdot
\#\{(a_1, a_2, a'_1, a'_2)\}$. Using the estimate we deduced in § 4.3.6 it follows that the sum we are
estimating is convergent and can therefore be neglected.

The final steps in proving the divergence part of Theorem 1 follow in exactly the same
manner as that of the 2-dimensional case.

There are only minor modifications needed to the proofs of Corollaries 1 and 2 to establish
them in the general case and the details are left to the reader.

References

[1] V. Beresnevich and S. Velani. Schmidt’s Theorem, Hausdorff Measure and Slicing. Pre-
print (20pp): [arXiv:math.NT/0507369], submitted.

[2] T. Gramchev and M. Yoshino. WKB analysis to global solvability and hypoellipticity.
Publ. Res. Inst. Math. Sci., 31(3):443–464, 1995.

[3] S. Kristensen. Diophantine approximation and the solubility of the Schrödinger equation.
Phys. Lett. A, 314(1-2):15–18, 2003.

[4] P. Mattila : Geometry of sets and measures in Euclidean space, CUP, Cambridge studies
in advance mathematics 44 (1995)

[5] B. Novák. Remark on periodic solutions of a linear wave equation in one dimension.
Comm. Math. Uni. Carolinae, 15:513–519, 1974.
[6] B. I. Ptashnik. *Improper boundary problems for partial differential equations*. Naukova Dumka, 1984.

[7] B. P. Rynne. The Hausdorff dimension of certain sets arising from Diophantine approximation by restricted sequences of integer vectors. *Acta Arith.*, 61(1):69–81, 1992.