Local structure of the moduli space of vector bundles over curves

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0. Introduction. —

Let $X$ be a smooth, projective and connected curve (over an algebraically closed field of characteristic zero) of genus $g(X) \geq 2$. Let $x$ be a (closed) point of $X$ and $SU(r,d)$ the moduli space of semi-stable vector bundles on $X$ of rank $r \geq 2$ and determinant $\mathcal{O}(dx)$. As usual, the geometric points of $SU(r,d)$ correspond to $\mathcal{S}$-equivalence classes $[E]$ where $E$ is a semi-stable rank $r$ bundle of determinant $\mathcal{O}(dx)$ (another semi-stable bundle $F$ is said to be $\mathcal{S}$-equivalent to $F$ if the graded objects $gr(E)$ and $gr(F)$ are isomorphic).

The singular locus of $SU(r,d)$ consists exactly of the non stable points (except if $r = g(X) = 2$ and $d = 0$. In this case, $SU(r,d) = \mathbb{P}^3$ [N-R1]). In particular, except in the exceptional case above, $SU(r,d)$ is smooth if and only if $r$ and $d$ are relatively prime. General facts about the action of reductive groups ensure that $SU(r,d)$ is Cohen-Macaulay [E-H], normal and that the singularities are rational [B]. The principal aim of this paper is to give additional information about the singularities, essentially the description of the completion of the local ring at a non smooth point of $SU(r,d)$ and to compute the multiplicity and the tangent cones at those singular points $[E]$ which are not too bad, i.e. the corresponding graded object $gr(E)$ of $[E]$ has only two non isomorphic stable summands (or equivalently $\text{Aut}(gr(E)) = \mathbb{G}_m \times \mathbb{G}_m$). Further, we give a complete description in the rank 2 case (proposition II.2, corollary II.3 and theorem III.4).

As a corollary, we get the local form of the so called Coble quartic and prove that the Kummer variety of the Jacobian of a genus 3 non hyperelliptic curve is schematically defined by 8 cubics, the partials derivatives of the Coble quartic (theorem III.6).

One could also give partial information at least if $\text{Aut}(gr(E))$ is a torus, or by using results of [P], if $\text{Aut}(gr(E)) = \text{GL}_r(k)$ (the latter case essentially means that $gr(E)$ is the trivial bundle). But it seems to be difficult and somewhat messy to calculate for instance the multiplicity. In the remaining part of the paper, we compute the multiplicity of a generalized theta divisor of $SU_X(2,\mathcal{O})$ at a point $[L \oplus L^\vee]$.

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where $L^2 \neq O$. In fact, this computation could be done with only minor changes for a point $[E]$ of any rank with $\det(gr(E)) = O$ and $\text{Aut}(gr(E)) = G_m \times G_m$.

Let us also mention that similar results could be obtained exactly in the same way for certain surfaces. But, all the future applications that we have in mind as well as the applications that we have in our hand are for curves. Therefore, we have restricted ourselves to the case of curves.

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**Notations and conventions.** All the schemes are of finite type over $k$, by point we mean closed point. If $(X_i)_{1 \leq N}$ (resp. $(n_i)_{1 \leq N}$) are indeterminates (resp. non negative integers), let me denote by $\underline{X}$, $\underline{n}$ and $\underline{X^\mathbb{N}}$ the $N$-tuple $\underline{X} = (X_i)$, the multi-index $\underline{n} = (n_i)$ and the product $X^\mathbb{N} = \prod_{i=1}^N x_i^{n_i}$ respectively. For $V$ a finite dimensional vector space with dual $V^\vee$, the ring $k[V]$ (resp. $k[[V]]$) is the polynomial ring $\text{Sym}V^\vee$ (resp. its completion at the origin). The scheme $\mathbf{P}(V)$ is the projective space $\text{Proj}(\text{Sym}V^\vee)$ of lines of $V$ and $V$ will also denote the pointed affine space $\text{Spec}(\text{Sym}V^\vee) = \text{Spec} k[V]$ (notice that $k[V]$ is the coordinate ring of $V$). Let $(E_i)$ be a set of vector bundles over $X$, the kernel of the trace map

$$\text{Ker}(\oplus \text{Ext}^1(E_i, E_i) \rightarrow \oplus \text{Tr}_i \rightarrow H^1(X, O))$$

will be denoted by

$$\left( \oplus \text{Ext}^1(gr_i, gr_i) \right) _0.$$

Finally, $E$ will always denote a rank $r$ semi-stable bundle on $X$ of degree $d$. Let $\text{Fil}$ be a strictly increasing Jordan-Hölder filtration of $E$ by stable bundles $\text{Fil}^i(E)$ (with slope $\frac{d}{r}$).

$$0 = \text{Fil}^0(E) \subset \text{Fil}^1(E) \ldots \subset \text{Fil}^N(E) = E.$$

Then, the graded object

$$\oplus gr_i$$

with

$$gr_i = \text{Fil}^i(E)/\text{Fil}^{i-1}(E)$$

is well defined (up to isomorphism!) and is denoted by $gr$. 

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I. Local structure of $\text{SU}(r,d)$ and classical invariant theory. —

It is well known (see [S] for instance) that the key ingredient to analyse the local structure of $\text{SU}(r,d)$ is the étale slice theorem of Luna. Let us recall this analysis.

Take $n$ big enough such that $E(nx)$ is globally generated and has no $H^1$ for every semi-stable rank $r$ vector bundle $E$ of degree $d$ (every $n$ such that $rn + d > r(2g - 1)$ has this property). Let $\chi = \chi(E(nx))$ be the corresponding Euler-characteristic. In Grothendieck’s scheme $\text{Quot}$ which parametrizes quotients

$$\mathcal{O}(-nx)^{\oplus \chi} \rightarrow E,$$

let $Q$ be the open set whose closed points correspond to such quotients with the following properties:

(i) $E$ is semi-stable of rank $r$ and degree $d$.
(ii) $H^1(X, E(nx)) = 0$ and the natural map $H^0(X, \mathcal{O}^{\oplus \chi}) \rightarrow H^0(X, E(nx))$ is onto.

Let $E$ be the universal quotient bundle on $Q$. The scheme $Q$ is smooth and thesemi-simple group $G = \text{PGL}_\chi$ acts on it. The moduli space $\text{SU}(r,d)$ is the GIT quotient of $Q//G$.

Let $q = [\mathcal{O}(-nx)^{\oplus \chi} \rightarrow E]$ a point of $Q$ and $gr$ the corresponding graded object. By the very definition of semi-stability, there exists a $G$-stable open affine neighborhood $\Omega$ and the fibre of $\Omega \rightarrow \Omega/G$ at $[E]$ contains a unique closed orbit $G(q)$. This orbit is either characterized as being of minimal dimension, or as having an isotropy group $G_q = \text{Aut}(E)$ of maximal dimension. Note that the scalar matrices act trivially. Inside the isotropy groups of the elements of the $S$-equivalence class of $E$, the isotropy group $\text{Aut}(gr)$ is of maximal dimension, and therefore the corresponding orbit is closed. The closedness of the orbit corresponding to $gr$ allows us to use the Luna étale slice theorem [Lu] which gives precisely the:

**Theorem 1 (Luna).** — *There exists a closed subscheme $V$ of $\Omega$ such that:*

(i) $V$ is stable under the isotropy group $G_q$ and the tangent space $T_qV$ is $G_q$-isomorphic to $T_q\Omega/T_qG(q) = \text{Ext}^1_0(gr, gr)$.

(ii) The $G$-morphism

$$G \times_{G_q} V \rightarrow G.V$$

is étale and onto the $G$-saturated open subset $G(V)$ of $\text{SU}(r,d)$.

(iii) The induced morphism $V/G_q \rightarrow \Omega/G$ is étale at the image $\bar{q}$ of $q$ in $V/G_q$.

Let $G$ be a reductive group acting on a $k$-algebra of finite type $R$ and $\mathfrak{m}$ an invariant ideal. Let $R_0$ be the ring of invariant and $\mathfrak{m}_0 = \mathfrak{m} \cap R_0$ the corresponding
ideal. The group $G$ acts on the completion $\hat{R} = \varprojlim R/m^n$. Finally let $\hat{R}_0$ be the $m_0$-adic completion of $R_0$.

The following easy lemma is certainly well known, but by of lack of reference let me prove the

**Lemma 2.** — *With the previous notations, one has a canonical isomorphism*

$$\hat{R}^G \sim \rightarrow \hat{R}_0$$

**Remark :** the lemma says that that passing to the invariants commutes with completion.

**Proof :** one has to compare $(\varprojlim R/m^n)^G$ and $\varprojlim R_0/m_0^n$. Let $S$ be the coordinate ring of $G$ and denote by $\sigma$ the morphism

$$\sigma : R \rightarrow S \otimes_k R$$

defining the action. From [M], assertion (3) of theorem 1.2, the ring of invariants of $R/m^n$ is $R_0/m_0^n$. One therefore has to prove that the injection

$$\varprojlim (R/m^n)^G \hookrightarrow (\varprojlim R/m^n)^G$$

is onto. Let $(r_n) \in \varprojlim R/m^n$ an $G$-invariant sequence, that is

$$\sigma(r_n) - 1 \otimes r_n \in S \otimes m_u^n$$

for a sequence $u_n$ which goes to $\infty$. After renormalization, one can assume that $u_n \geq n$ which implies $\sigma(r_n) - 1 \otimes r_n \in S \otimes m^n$ and therefore $(r_n)$ defines an element of $\varprojlim (R/M^n)^G$. 

Let $G_E$ denote the subgroup of $G_q$ given by

$$G_E = \text{Ker}\{\text{Aut}(gr) \rightarrow \text{det} \rightarrow G_m\}.$$ 

One can either prove by a direct calculation or by using Luna’s result that $G_E$ is reductive. Let

$$A_E = k[\text{Ext}_0^1(gr, gr)]^{G_E}$$

be the ring of polynomial maps on $\text{Ext}_0^1(gr, gr)$ invariant under $G_E$ (the group $G_E$ acts by functoriality on both arguments of $\text{Ext}_0^1(gr, gr)$). Let $\hat{A}_E$ be the completion at the origin. Using twice the previous lemma and using Luna’s theorem, one obtains
Theorem 3. — There is an isomorphism of complete local $k$-algebras

$$\hat{O}_{SU(r,d),[E]} \sim \hat{A}_E.$$ 

Corollary. — The local ring $\hat{O}_{SU(r,d),[E]}$ depends only on the numerical invariants of $X$ and $gr(E)$.

Suppose once for all that $E$ is non stable.

One has of course the inequalities

$$1 \leq \dim G_E \leq r^2 - 1$$

with equality on the left hand side (resp. right hand side) of (1) if $G_E = G_m$ (Case 1) (resp. $G_E = Sl_r$ (Case 2)). Let’s examine these 2 cases.

II. Case 1 : $G_E = G_m$. —

In this case, the graduate $gr$ of $E$ is a direct sum

$$gr = gr_1 \oplus gr_2$$

where $gr_i$ is stable of slope $\frac{d}{r}$, rank $r_i \neq 0$ and

$$gr_1 \neq gr_2.$$ 

Each element $(\alpha_1, \alpha_2) \in G_E(k)$ acts by multiplication by $\alpha_j.\alpha_i^{-1}$ on each factor $Ext^1(gr_i, gr_j)$ of

$$Ext^1(gr, gr) = \oplus Ext^1(gr_i, gr_j).$$

Let

$$d_{i,j} = \dim Ext^1(gr_i, gr_j) = \begin{cases} 
  r_ir_j(g - 1) & \text{if } i \neq j \\
  r_ir_j(g - 1) + 1 & \text{if } i = j 
\end{cases}$$

and $X^k_{i,j}$, $k = 1, \ldots, d_{i,j}$ a basis of $Ext^1_0(gr_i, gr_j)^\vee$.

The ring

$$A_E \subset k[Ext^1(gr, gr)] = k[X^k_{i,j}, \ 1 \leq i,j \leq n, \ 1 \leq k \leq d_{i,j}]$$

is the ring generated by $(Ext^1(gr_1, gr_1) \oplus Ext^1(gr_2, gr_2))^\vee_0$ and the products

$$< X^k_{1,2}X^l_{2,1}, \ 1 \leq k, l \leq d_{i,j} >.$$ 

Let $S$ be the the cone over the Segre variety

$$P(Ext^1(gr_1, gr_2)) \times P(Ext^1(gr_2, gr_1)) \subset P(Ext^1(gr_1, gr_2) \otimes_k Ext^1(gr_2, gr_1)).$$
Proposition 1. — There is an isomorphism
\[
\text{Spec}(A_E) \xrightarrow{\sim} (\text{Ext}^1(gr_1, gr_1) \oplus \text{Ext}^1(gr_2, gr_2))_0 \times S.
\]

Note that \(\text{Spec}(A_E)\) is a (homogeneous) cone. Using the theorem I.3 and the previous proposition, one therefore obtains the

Proposition 2. — With the previous notations, the completion of \(SU(r, d)\) at \([E]\) is the completion at the origin of the cone \(\text{Spec}(A_E)\).

Using furthermore that the multiplicity at the origin of the affine cone of a projective variety is just its degree, one gets the

Corollary 3. — The completion of the tangent cone of \(SU(r, d)\) at \([E]\) is the completion at the origin of
\[
(\text{Ext}^1(gr_1, gr_1) \oplus \text{Ext}^1(gr_2, gr_2))_0 \times S.
\]
The Zariski tangent space is
\[
(\text{Ext}^1(gr_1, gr_1) \oplus \text{Ext}^1(gr_2, gr_2))_0 \oplus (\text{Ext}^1(gr_1, gr_2) \otimes_k \text{Ext}^1(gr_2, gr_1)).
\]

Moreover the multiplicity of \(SU(r, d)\) at \([E]\) is
\[
\text{mult}_{[E]}(SU(r, d)) = \begin{pmatrix}
2d_{1,2} - 2 \\
d_{1,2} - 1
\end{pmatrix}
\]
(recall that \(d_{1,2} = r_1r_2(g - 1)\)).

Remark 4 : Note that \(\begin{pmatrix}
2d_{1,2} - 2 \\
d_{1,2} - 1
\end{pmatrix} = 1\) if and only if \(g = 2\) and \(r_1 = r_2 = 1\). Using that the singular locus of \(\text{Sing} \ SU(r, d)\) is closed in \(SU(r, d)\), one obtains easily in this way another proof of the fact that \(\text{Sing} \ SU(r, d)\) is the non stable locus, except if \(g = r = 2\) and \(d = 0\).

III Case 2 : \(G_E = \text{Sl}_r\). —

In this case, \(gr = L^\oplus r\) where \(L^\oplus r = \mathcal{O}\). Using a translation by \(L^{-1}\) which induces an automorphism of \(SU(r, \mathcal{O})\), one may assume \(L = \mathcal{O}\). The ring \(A_E\) is the ring of polynomial maps on \(M_0(r)^g\) invariant under \(\text{Sl}_r\). This group \(\text{Sl}_r\) acts diagonally by conjugation on each factor \(M_0(r)\), which is the space of traceless matrices of size \(r\). For general \(r\), Procesi [P] and Rasmyslev [Ras] have obtained the following description of the first 2 syzigies of \(A_E\) :
Generators: for every sequence \( i = (i_1, \ldots, i_{N(i)}) \) of integers of \([1, \ldots, g]\), let \( t_i \) be the invariant polynomial map

\[
t_i : \begin{cases}
M_0(r)^g &\to k \\
(X_1, \ldots, X_g) &\mapsto Tr(X_{i_1} \cdots X_{i_N})
\end{cases}
\]

Then the \( t_i \) with \( N(i) \leq 2^g - 1 \) form a system of generators.

Relations between all the \( t_i \)'s: Let \( P_X \) be the characteristic polynomial of the general matrix \( X \). The homogeneous polynomial \( X \mapsto \text{Tr}(X.P_X(X)) \) gives by polarization (namely by taking the total differential of order \( g + 1 \)) a multilinear map \( F(H_1, \ldots, H_{g+1}) \). Then the relations are generated by \( tr_1, \ldots, tr_n \) and the relations \( F(H_1, \ldots, H_{g+1}) \) where the \( H_i \)'s runs in the set of all possible monomials in the \( X_i \).

Although this description is quite explicit, it looks difficult to obtain a complete finite set of relations between the (finite) set of generators constructed above.

As far as I know, the only case where such a finite description is available is for \( r = 2 \).

(*)

III Case 2: \( G_E = \text{Sl}_r \) and \( r = 2 \).

In this case, \( A_E \) can be described by using classical results of the geometric invariant theory of \( \text{SO}_3(k) \). Following [LeB], I.4, let me briefly explain this description.

For \( X \in M_0(2) \), let \( u(X) = (u_1(X), u_2(X), u_3(X)) \in k^3 \) be defined by the equality

\[
X = \begin{pmatrix}
u_1(X) \\
u_2 + \sqrt{-1}u_3(X) \\
u_2(X) - \sqrt{-1}u_3(X) \\
u_3(X)
\end{pmatrix}
\]

By theorem 4.1 of [LeB] the isomorphism

\[
\begin{cases}
M_0(2) &\to k^3 \\
X &\mapsto u(X)
\end{cases}
\]

induces an identification of \( A_E \) with the polynomial maps of \( (k^3)^{\oplus g} \) invariant under the canonical diagonal action of \( \text{SO}_3(k) \).

Let \( T_{i,j} \) be the invariant function corresponding to \( (u_1, \ldots, u_n) \mapsto (u_i.u_j) \) (scalar product), namely \( T_{i,j}(X_1, \ldots, X_g) = \frac{1}{2} \text{Tr}(X_iX_j) \).

Let \( T_{i,j,k} \) be the invariant function corresponding to \( (u_1, \ldots, u_n) \mapsto u_i \wedge u_j \wedge u_k \) (the wedge product lives in \( \wedge^3 k^3 \cong k \)), namely \( T_{i,j,k}(X_1, \ldots, X_g) = \text{Tr}(X_iX_jX_k) \).

With some abuse of notation, one can now use the results of H. Weyl [W], theorem (2.9 A) and (2.17 B) and its sequel on page 77 which says the following :

(*) According to some experts of invariant theory, it is more or less hopeless to obtain such a finite description of \( A_E \) in the general case.
-Generators for the invariants maps under $O_3(k)$ : the set
$$< T_{i,j} >$$

-Relations between the generators : the 4-minors of the $g \times g$-symmetric matrix

$$
\begin{pmatrix}
... & & & & \\
... & T_{i,j} & & \\
... & & & & \\
... & & & & \\
\end{pmatrix}
$$

and the relations $T_{i,j} = T_{j,i}$. One recognizes the coordinate ring of the (affine) cone $C$ of symmetric matrices of rank $\leq 3$. This scheme is well understood : it is integral and normal [easy], Cohen-Macaulay [H-R], its multiplicity at the origin (or the degree of the projectivization $PC$) is known [H-T]...

-Generators for the $SO_3(k)$-invariants maps are : the $T_{i,j}$’s and the $T_{i,j,k}$’s.
-Relations : the previous 4-minors and :

(1) $T_{i_1,i_2,i_3}T_{j_1,j_2,j_3} = \det([T_{i_n,j_m}]_{1 \leq n,m \leq 3})$

(2) $T_{i_0,i_4}T_{i_1,i_2,i_3} = T_{i_1,i_4}T_{i_0,i_2,i_3} + T_{i_2,i_4}T_{i_0,i_1,i_3} - T_{i_3,i_4}T_{i_0,i_1,i_2}$

and the relations given by the symmetry of $T_{i,j}$ and the skew symmetry of $T_{i,j,k}$ in the indices.

Remark 1 : The relation (2) comes from the vanishing of the $4 \times 4$-determinant $\det((u_{i_n},v_m))_{0 \leq n,m \leq 3}$ where $v_0 = e_4$ and $v_m = e_m$, $m = 1,2,3$ (the vectors of the canonical basis of $k^3$).

Let $\bar{C}$ be the tangent cone of $\text{Spec}(A_E)$ at the origin. It is the subscheme of

$$C \times \text{Spec} k[T_{i,j,k}]$$

whose ideal is generated by

(3) $T_{i_1,i_2,i_3}T_{j_1,j_2,j_3}$

and

(4) $T_{i_0,i_4}T_{i_1,i_2,i_3} = T_{i_1,i_4}T_{i_0,i_2,i_3} + T_{i_2,i_4}T_{i_0,i_1,i_3} - T_{i_3,i_4}T_{i_0,i_1,i_2}$

and relations given by the skew symmetry of $T_{i,j,k}$ in the indices. The ideal described above is the ideal of initial forms of the ideals given by (1) and (2).
Let \( k(C) \) be the function field of \( C \) and \( K \) its algebraic closure. Note that, according to (3), the ideal \( I_{C/\bar{C}} \) of \( C \) in \( \bar{C} \) is nilpotent. This implies by [F] example 4.3.4, the formula

\[
\text{mult}_0 \bar{C} = \text{length}_{O_{C,\bar{C}}} \cdot \text{mult}_0 (C).
\]

The next formula is clear:

\[
\text{length}_{O_{C,\bar{C}}} = 1 + \dim \kappa(C) \otimes O_{C,\bar{C}} = 1 + \dim_K \frac{I_{C/\bar{C}} \otimes O_{\bar{C}}}{I_{C/\bar{C}} \otimes O_{\bar{C}} K}.
\]

One therefore has to compute the dimension over \( K \) of the sub-vector space \( V_T \) of the dual space of \( W = \oplus K T_{i,j,k} \) of equations given by (4) and the skew symmetry condition for \( T_{i,j,k} \). This vector space is isomorphic to \( I_{C/\bar{C}} \otimes O_{\bar{C}} K \).

**Lemma 2.** — The dimension of \( V_T \) depends only on the conjugation class of \( T \).

**Proof:** The symmetric matrix

\[
T = [T_{i,j}] \in M_g(K)
\]

acts on the dual vector space \( V \) of \( K^g \). Let \((e_i)_{1 \leq i \leq g}\) be the canonical basis and \((e_i^\vee)_{1 \leq i \leq g}\) its dual basis. The map

\[
\begin{cases}
W & \to \wedge^3 V \\
T_{i,j,k} & \mapsto e_i \wedge e_j \wedge e_k
\end{cases}
\]

identifies \( W \) and \( \wedge^3 V^\vee \). With this identification, the relations (4) become

\[
-T(e_i^\vee) \bigwedge e_{i_0} \wedge e_{i_1} \wedge e_{i_2} \wedge e_{i_3}
\]

and \( \dim V_T \) is the corank of

\[
\begin{cases}
V^\vee \otimes \wedge^4 V & \to \wedge^3 V \\
 x^\vee \otimes y & \mapsto T(x^\vee) \bigwedge y
\end{cases}
\]

This map depends only on the conjugation class of \( T \). \( \blacksquare \)

One can therefore assume that \( T \) is diagonal of eigenvalues \( \lambda_i \) with \( \lambda_i = 0 \) for \( 3 < i \leq g \) and \( \lambda_i \neq 0 \) if \( i \leq 3 \).

Let us prove this simple lemma

**Lemma 3.** — The dimension of \( V_T \) is \( \frac{g(g-1)(g-2)}{6} - \frac{(g-3)(g-4)(g-5)}{6} \) if \( g \geq 3 \) and 0 if \( g \leq 2 \).

**Proof:** If \( g \leq 2 \), the vector space \( \wedge^3 V \) is zero and so is \( V_T \). Suppose \( g \geq 3 \). Consider an equation

\[
T_{i_0,i_4} T_{i_1,i_2,i_3} - T_{i_1,i_4} T_{i_0,i_2,i_3} + T_{i_2,i_4} T_{i_0,i_1,i_3} - T_{i_3,i_4} T_{i_0,i_1,i_2}
\]
defining $V_T$. If $i_4 \notin \{1, 2, 3\}$ or $i_4 \notin \{i_0, i_1, i_2, i_3\}$ then the equation (4) is trivial. Let me suppose that $i_4 \in \{1, 2, 3\}$ and for instance that $i_4 = i_0$. If $i_4 \in \{i_1, i_2, i_3\}$, the equation is just a consequence of the skew symmetry of $T_{i_1, i_2, i_3}$. To get a new relation, one has therefore further to suppose further that $i_4 \notin \{i_1, i_2, i_3\}$ and the equation become

$$
\lambda_{i_0} T_{i_1, i_2, i_3} = 0.
$$

Of course, the other cases are obtained by symmetry. One has proved the following: the equations (4) are non trivial if and only if

$$
\{i_0, i_1, i_2, i_3\} = \{i, j, k\} \sqcup \{i_4\} \text{ and } i_4 \notin \{i, j, k\}.
$$

In this case the relation (4) becomes

$$
T_{i, j, k} = 0,
$$

or equivalently

$$
T_{i, j, k} = 0 \text{ if } \{i, j, k\} \cap \{1, 2, 3\} \neq \emptyset.
$$

In particular this corank is

$$
\dim \wedge^3 V - A^3_{g-3} = \frac{g(g-1)(g-2)}{6} - \frac{(g-3)(g-4)(g-5)}{6}.
$$

The degree $d_g^r$ of the locus (*) of $g \times g$-symmetric matrices of corank $\geq r$ is computed in [H-T], proposition 12.b:

$$
d_g^r = \deg PC = \prod_{\alpha=0}^{r-1} \frac{g + \alpha}{r - \alpha} \cdot \frac{2\alpha + 1}{\alpha}.
$$

Using the formulas (5) and (6) and the lemma 3, one obtains the

**Theorem 4.** — The multiplicity of $[O \oplus O]$ in $SU(2, O)$ is

$$
1 + \frac{g(g-1)(g-2)}{6} - \frac{(g-3)(g-4)(g-5)}{6}.d_g^{g-3}
$$

if $g \geq 3$ and $1$ if $g = 2$.

(*) In [H-T], this locus is endowed with the reduced scheme structure. But it is known in full generality that the natural scheme structure given by the vanishing of the $(g - r + 1)$-minors is Cohen-Macaulay [J] and generically reduced [easy] and therefore reduced. In our case ($r = g - 3$), this reduceness is obvious, because $C$ is a ring of invaraiants.
Remarks 5:
1.- The preceding discussion gives a precise description of the tangent cone $\bar{C}$ of $\text{SU}(2, \mathcal{O})$. In particular, the Zariski tangent space at the trivial bundle is
\[ T_{[\mathcal{O} \oplus \mathcal{O}]} \text{SU}(2, \mathcal{O}) = \text{Sym}^2 V \oplus \wedge^3 V. \]

2.- One recovers the smoothness of $\text{SU}(2, \mathcal{O})$ if $g = 2$.

In the case of an a non hyperelliptic genus 3 curve, the generalized $\Theta$ divisor embeds $\text{SU}(2, \mathcal{O})$ as the Cobble quartic $\text{SU}(2, \mathcal{O})$ in $\text{PH}^0(J^2, 2, \Theta_{J^2})$ (see [N-R2] and [D-O] pages 184-185). Let $q_X$ be an equation of the Coble quartic.

Theorem 6. — Suppose that $X$ is a non hyperelliptic genus 3 curve.
(i) The local equation of the Coble quartic at the trivial bundle is
\[ T^2 = \det([T_{i,j}]_{1 \leq i,j \leq 3}) \]
in the affine space $\mathbb{A}^7$ with coordinates $T, T_{i,j}$ with $T_{i,j} = T_{j,i}$.
(ii) The local equation of the Coble quartic at $gr = gr_1 \oplus gr_2$ with $gr_1 \neq gr_2$ is a rank 4 quadric in $\mathbb{A}^7$.
(iii) The ideal of the Kummer $K(X)$ variety of $J(X)$ in $\text{PH}^0(J^2, 2, \Theta_{J^2})$ is generated by the 8 cubic equations which are the partials derivatives of the Coble quartic.

Proof: The first 2 points are clear from proposition II.2 and (1), (2). Let me prove (iii). Let $\tilde{K}$ the scheme defined by the partials of $q_X$. The Kummer variety $K(X)$ is the reduced scheme of $\tilde{K}$. It is therefore enough to prove that the completion of $\tilde{K}$ at each non stable class $[gr]$ of $K(X) \subset \tilde{K}$ is reduced.

Because of the invariance of the Coble quartic under the Theta group of $2, \Theta_{J^2}$, there are 2 cases: either $gr$ is trivial, or $gr = gr_1 \oplus gr_2$ with $gr_1 \neq gr_2$. In the first case, by (i), the equations in $k[[T, T_{i,j}]]$ of the completion of $\tilde{K}$ at $[\mathcal{O} \oplus \mathcal{O}]$ are $T$ and the $2 \times 2$-minors of $[T_{i,j}]_{1 \leq i,j \leq 3}$. It is precisely the (completion at the origin) of the cone over the Veronese surface in $\mathbb{P}^5$ (with homogenous coordinates $T_{i,j}$) and $K$ is therefore reduced. The second case is even simpler, $\tilde{K}$ being (the completion of) a 3-plane in $\mathbb{A}^7$ (the tangent space of $K(X)$).

III The case $\text{SU}(3, \mathcal{O})$ for of a genus 2 curve. —

Suppose in this section that $X$ has genus 2 and let $\mathcal{M} = \text{SU}(3, \mathcal{O})$ be the moduli space of rank 3 semi-stable vector bundles on $X$ with trivial determinant. Consider a non stable point of $\mathcal{M}$ defining by the graded object $gr$ of a semi-stable bundle.
The case $G_{gr} = G_m$ has been treated in section II: in this case, the completion of $\mathcal{M}$ at $gr$ is the completion at the origin of a rank 4 quadric in $\mathbf{A}^9$.

Suppose now that $G_{gr} = G_m \times G_m$ which means $gr = gr_1 \oplus gr_2 \oplus gr_3$ with $gr_i \neq gr_j$ for $i \neq j$ and $\deg(gr_i) = 0$.

Let $X_{i,j}$ be a basis of $\text{Ext}^1(gr_i, gr_j)^\vee$ for $i \neq j$. Let

$$A_{gr} = k[X_{i,j}]^{G_{gr}}$$

be the ring of invariants of $k[X_{i,j}]$ under $G_{gr}$ with the action defined by the following rule

$$\alpha \cdot X^n = \prod (\frac{\alpha_i}{\alpha_j})^{n_{i,j}} X^n$$

for $(\alpha_1, \alpha_2, \alpha 3) \in G_{gr}(k) = (k^*)^3/k^*$. The following equality is easy

$$A_{gr} = k[(\oplus \text{Ext}^1(gr_i, gr_i))_0] \otimes A_{gr}.$$ 

A polynomial $\sum p_n X^n$ is in $A_{gr}$ if and only if

$$(1) \quad \sum_j n_{j,i} = \sum_j n_{i,j} \text{ for } i = 1, 2, 3$$ 

if $p_n \neq 0$. Therefore, $A_{gr}$ is generated by the monomials

$$X^n$$ 

such that $n$ satisfies (1).

**Lemma 1.** — The ring $A_{gr}$ is generated by

$$X_{3,2}X_{2,1}X_{1,3}, \, X_{1,2}X_{2,3}X_{3,1} \text{ and } X_{i,j}X_{j,i}, \, i < j.$$ 

Proof: put $\delta_{i,j} = n_{i,j} - n_{j,i}$ and let $\underline{n}$ be a multi-index satisfying (1). The relations (1) become

$$\delta_{1,2} + \delta_{1,3} = 0, \, \delta_{1,2} = \delta_{2,3}, \, \delta_{1,3} + \delta_{2,3} = 0.$$ 

If $\delta^+ = \delta_{1,2} \geq 0$, we write

$$\underline{n} = (n_{2,1} + \delta^+, n_{2,1} + n_{1,3}, n_{1,3} + \delta^+, n_{3,2} + \delta^+, n_{3,2})$$

and use the monomial

$$X_{1,2}X_{2,3}X_{3,1}$$
corresponding to
\[ n_0 = (1, 0, 0, 1, 1, 0) \]
to write \( n = m + \delta^+. n_1 \). This allows us to write
\[ X^n = (X_{1,2}X_{2,3}X_{3,1})^{\delta^+} \prod_{i<j} (X_{i,j}X_{j,i})^{m_{i,j}} \]
with \( m_{i,j} \geq 0 \). In the same way, when \( \delta^- = -\delta_{1,2} > 0 \), one has an equality
\[ X^n = (X_{3,2}X_{2,1}X_{1,3})^{\delta^-} \prod_{i<j} (X_{i,j}X_{j,i})^{m_{i,j}} \]
with \( m_{i,j} \geq 0 \).

For \( i \in I = \{1, 2, 3\} \), put
\[ x_i = X_{j,k}, \ x_{i+3} = X_{k,j}, \ \zeta_i = x_i x_{i+3} \]
with \( I = \{i, j, k\} \) and \( j < k \). Let \( \zeta_4 = X_{3,2}X_{2,1}X_{1,3} \) and \( \zeta_5 = X_{1,2}X_{2,3}X_{3,1} \). There is an equality
\[ \zeta_4 \zeta_5 = \zeta_1 \zeta_2 \zeta_3. \]

PROPOSITION 2. — The natural morphism
\[ f : \left\{ \begin{array}{c} k[X_i] \rightarrow A_{gr} \\ X_i \mapsto \zeta_i \end{array} \right. \]
gives an isomorphism
\[ k[X_i]/(X_4X_5 - X_1X_2X_3) \sim A_{gr}. \]

Proof: Let \( p \) be the (prime) ideal generated by \( X_4X_5 - X_1X_2X_3 \). Let
\[ P = \sum \alpha_n X^n \]
be an element of \( \text{Ker}(f) \). Then, one finds by simple expansion
\[ f(P) = \sum \alpha_n \prod x^{\phi(n)} = \sum m \sum_{\phi(n)=m} \alpha_n \]
with
\[ \phi(n) = (n_1 + n_5, n_2 + n_4, n_3 + n_5, n_1 + n_4, n_2 + n_5, n_3 + n_5) \]
which implies
\[ \sum_{\phi(n)=m} \alpha_n = 0. \]
(Here \( X^u = \prod_i X_i^{n_i} \) and \( x^m = \prod_i x_i^{m_i} \), \( n = (n_i)_{1 \leq i \leq 5} \) and \( m = (m_i)_{1 \leq i \leq 6} \) are multi-indices).

The kernel of \( \phi \) is generated by \( (1, 1, 1, -1, -1) \) :
if \( \phi(m') = \phi(m) \), there exists \( \alpha \in \mathbb{Z}_+ \) such that
\[
\pm \alpha(1, 1, 1, 0, 0) + m' = \pm \alpha(0, 0, 0, 1, 1) + m.
\]

In particular, one has the congruence
\[
(X_4.X_5)^{\alpha}.X^{m'} \equiv (X_4.X_5)^{\alpha}.X^m \mod p.
\]

According to (2) and (3), we get the existence of a positive integer \( a \) such that
\[
(X_4.X_5)^a.P \equiv 0 \mod p.
\]

Since the ideal \( p \) is prime and \( (X_4.X_5) \not\in p \) and therefore \( P \in p. \quad \blacksquare \)

**Corollary 3.** — *The completion of \( \mathcal{M} \) at a point \([gr]\) satisfying \( G_{gr} = G_m \times G_m \) is the completion at the origin of*
\[
\left( \bigoplus \text{Ext}^1(gr_i, gr_i) \right) \times \text{Spec}(k[X_i]/(X_4X_5 - X_1X_2X_3)).
\]

*Its tangent cone is a rank 2 quadric in the Zariski tangent space \( T_{[E]} \mathcal{M} = A^9 \).*

In particular, there exists a family \( E \) of semi-stable bundles of trivial determinant over a germ of curve such that :
(i) The group \( G_{E_\eta} \) of the generic bundle \( E_\eta \) is \( G_m \otimes_k k(\eta) \).
(ii) The group of \( G_{E_s} \) the special bundle \( E_s \) is \( G_m \times G_m \).
(iii) The multiplicity of \( \mathcal{M} \) at \([E_\eta]\) and \([E_s]\) are the same.

This shows that 2 points of \( \mathcal{M} \) can have the same multiplicity without having the same group of automorphisms.

When the \( gr \) has 3 summands for which at least 2 are isomorphic, or equivalently if \( G_{gr} \) is not a torus, the calculations are very intricate (but seem to be possible). In fact, one can in spite of this obtain the following

**Proposition 4.** — *The tangent cone at each non stable point \( gr \) such that \( G_{gr} \) is not a torus is a quadric in \( A^9 \) of rank \( \leq 2 \).*

Proof : thanks to the corollary 1.7.4 of [Ray], for *every* semi-stable vector bundle \( E \) of rank 3 and determinant \( O \), the determinantal locus \( \Theta_E \) in \( \text{Pic}^1(X) \)
\[
\Theta_E = \{ L \in \text{Pic}^1(X) \text{ such that } H^0(X, E \otimes L) \neq 0 \}.
\]
is a divisor in $|\mathcal{O}(3\Theta_{J1})|$ where $\Theta_{J1}$ is the canonical theta divisor on $\text{Pic}^1(X)$. The Picard group of $\mathcal{M}$ is cyclic with ample generator $\mathcal{O}(\Theta)$ [D-N]. By [B-N-R], the inverse image of $\mathcal{O}(1)$ by the morphism

$$\pi : \begin{cases} \mathcal{M} \to |\mathcal{O}(3\Theta_{J1})| \\ [E] \mapsto \Theta_E \end{cases}$$

is $\mathcal{O}(\Theta)$ and $\pi^*$ is a (canonical) isomorphism

$$|\Theta| \sim |3\Theta_{J1}|.$$  

(In particular, $|\Theta|$ has no base point in this case !)

Using this isomorphism, $\pi$ becomes the morphism given by the complete linear system $|\Theta|$. There are various ways to prove this simple lemma, but the following one can be generalized for the higher rank case.

**Lemma 5 (\*)**. — *The morphism $\pi$ is finite of degree 2 over $\mathbf{P}^8$.*

Proof of the lemma : using the isomorphism $\pi^*\mathcal{O}(1) \sim \mathcal{O}(\Theta)$, we get that $\pi$ is finite of degree $c_1(\Theta)^8$ onto $\mathbf{P}^8$. One therefore has to compute the degree of $\pi$. Although there exists a general beautiful formula due to Witten to evaluate the volume

$$\frac{c_1(\Theta)^{\dim SU(r,\mathcal{O})}}{(\dim SU(r,\mathcal{O}))!},$$

we give a simplest (and elementary) method to get this volume for $\mathcal{M}$. One has to prove that the leading term of the Hilbert polynomial

$$n \mapsto P(n) = \chi(X, \Theta^n)$$

is $\frac{2}{8!}$. The canonical divisor of $\mathcal{M}$ si $\Theta^{-6}$ [D-N]. Serre duality implies therefore the symmetry

$$(4) \quad P(n) = P(-6 - n).$$

The Grauert-Reimenschneider vanishing theorem (recall that $\mathcal{M}$ has rational singularities) gives the equality

$$P(n) = \dim H^0(\mathcal{M}, \Theta^n) \text{ for } n \geq -5.$$  

One therefore obtains the values

$$(5) \quad P(n) = 0 \text{ for } n = -5, \ldots, -1, \ P(0) = 1 \text{ and } P(1) = 9.$$  

By (4) and (5), one obtains

$$P(X) = \lambda(X + 5)(X + 2)(X + 3)^2(X + 2)(X + 1)(X - \alpha)(X + 6 + \alpha).$$

The equalities $P(0) = 1$ and $P(1) = 9$ imply

$$\alpha = -3 \pm \sqrt{-47} \text{ and } \lambda = \frac{2}{8!}.$$  

\[\blacksquare\]

(*) This fact, which is due to I. DOLGACEV, was pointed out to me by A. BEAUVILLE.
One has proved that the morphism \( \pi \) is finite of degree 2 onto \( |\Theta|^\vee = \mathbb{P}^8 \). Since \( \mathcal{M} \) is Cohen-Macaulay and \( \mathbb{P}^8 \) smooth, this double covering is flat (E.G.A. IV.15.4.2) and is given locally by an equation
\[
t^2 = f(x)
\]
where \( x \) are local coordinates on \( \mathbb{P}^8 \). This implies that the multiplicity of each point of \( \mathcal{M} \) is \( \leq 2 \).

Take a point \([gr] \in \mathcal{M}\) such that \( G_{gr} \) is not a torus: it is a non smooth point of \( \mathcal{M} \), therefore the tangent cone is a quadric (the initial term of \( f \) is not linear). But, such a point is a specialization of a point \( gr_\eta \) such that \( G_{gr_\eta} = G_m \times G_m \): by the (obvious) semi-continuity of the rank of the quadric cone of \([gr]\), the inequality \( \text{rank} \leq 2 \) follows from the corollary 3.

\textbf{Remark 6}: There exists by the way a nice 3-dimensional family of sextics in \( \mathbb{P}^8 = \text{PH}^0(J^1, \mathcal{O}(3, \Theta_{J^1})) \) (given by the ramification of \( \pi \)), which could be called the family of Dolgacev-Coble sextics associated to a genus 2 curve. It would be interesting to know if these sextics share the same kind of properties as the Coble quartics.

\textbf{IV. Multiplicity of the theta divisor (rank 2 case).} —

Recall that there exists a (Cartier) divisor \( \Theta \) on \( SU(2, \omega_X) \) which is characterized by the following universal property [D-N]: let \( S \) be a \( k \)-scheme and \( \mathbf{E} \) a family of semi-stable vector bundles over \( X_S = X \times_k S \) of determinant \( \omega_X \). Let \( \pi: S \to SU_X(2, \omega_X) \) be the classifying map corresponding to \( \mathbf{E} \). Then, one has the equality
\[
\pi^*(\Theta) = \text{div} \left( \det R^*_p \mathbf{E} \right)^\vee.
\]
By construction, the geometric points of \( \Theta \) are the classes \([\mathbf{E}] \in SU(2, \omega_X)\) such that \( H^0(X, \mathbf{E}) \neq 0 \).

\textit{Remark}: The vanishing of \( H^0(X, \mathbf{E}) \) depends only on the \( S \)-equivalence class \([\mathbf{E}]\) of \( \mathbf{E} \).

Let \( \mathbf{E} \) be a semi-stable vector bundle of determinant \( \omega_X \). Recall ([La], theorem III.3) that for \( \mathbf{E} \) stable
\[
\text{mult}_{[\mathbf{E}]} \Theta = \dim H^0(X, \mathbf{E})
\]
and that the tangent cone is defined in \( \text{Ext}^1_0(\mathbf{E}, \mathbf{E}) \) by the ideal of the determinant of linear forms defined by the cup-product
\[
H^0(X, \mathbf{E}) \otimes \text{Ext}^1_0(\mathbf{E}, \mathbf{E}) \to H^1(X, \mathbf{E}).
\]
These facts can be generalized formally (using the universal property of $\Theta$) as follows. Let $[E] \in SU(2, \omega_X)$ a non stable point of $\Theta$ of graded object $gr = gr_1 \oplus gr_2$ and let

$$h = \frac{1}{2} \dim H^0(X, gr) = \dim H^0(X, gr_1) = \dim H^0(X, gr_2)$$

(note that $gr_1 \otimes gr_2 = \omega_X$ which implies by Serre duality and Riemann-Roch the equality

$$(1) \quad \dim H^0(X, gr_2) = \dim H^1(X, gr_2) = \dim H^0(X, gr_1)).$$

With the notations of I, let $V = \text{Spec } k[[\text{Ext}^1_0(gr, gr)]]$ be a (formal) étale slice of $Q$ at $gr$ and

$$\pi : V \to V/\text{Aut}(gr) \rightarrow SU(2, \omega_X)$$

the canonical morphism. Then, the induced map

$$\pi^*(\Theta)/\text{Aut}(gr) \to \Theta$$

is étale. The tangent cone of $\pi^*(\Theta)$ is given by the determinant

$$d_{gr} \in \text{Sym}^{2h} \text{Ext}^1_0(gr, gr)^\vee$$

defined (up to a non zero scalar) by the cup-product

$$H^0(X, gr) \otimes \text{Ext}^1_0(gr, gr) \to H^1(X, gr)$$

In particular, a point $e \in \text{Ext}^1_0(gr, gr)$ is in the tangent cone of $\pi^*(\Theta)$ at $[E]$ if and only if the cup-product

$$\cup e : H^0(X, gr) \to H^1(X, gr)$$

is non onto.

**Proposition 1.** — Assume that $gr_1 \neq gr_2$. Then, the multiplicity of $\Theta$ at $[gr]$ is

$$\text{mult}_{[gr]} \Theta = \frac{1}{2} \dim H^0(X, gr). \text{mult}_{[gr]} SU(2, \mathcal{O}).$$

Proof: with the notation of the second section, the completion of $SU(2, \omega_X)$ at $[gr]$ is the completion at the origin of

$$(\text{Ext}^1(gr_1, gr_1) \oplus \text{Ext}^1(gr_2, gr_2))_0 \times S$$
where \( S \) is the cone over the Segre variety

\[
P(\text{Ext}^1(gr_1, gr_2)) \times P(\text{Ext}^1(gr_2, gr_1)) \subset P(\text{Ext}^1(gr_1, gr_2) \otimes_k \text{Ext}^1(gr_2, gr_1)).
\]

Fix coordinates

\[
X = (X^k_{i,j}) \text{ on } \text{Ext}^1(gr_i, gr_j) \text{ for } i \neq j \text{ and } (Y_i) = Y^k_i \text{ on } \text{Ext}^1(gr_i, gr_i).
\]

The equation \( F \) of \( \pi^*(\Theta) \) is of the form

\[
F = d_{gr} + G_{2h+1}
\]

where \( G_{2h+1} \) vanishes at the origin with order \( \geq 2h + 1 \). The polynomials \( d_{gr} \) and \( G_{2h+1} \) are \( G_E \) invariant and therefore (see section 2) can be written in terms of \( Y \) and \( z^{k,l} = X^k_{1,2}X^l_{2,1} \). Let me decompose \( d_{gr} \) as

\[
d_{gr} = \sum_{i=0}^{2h} Q_i(X)P_{2r-i}(Y)
\]

where the degree of \( Q_i \) (resp. \( P_{2r-i} \)) is \( i \) (resp. \( 2r - i \)) and \( P_0 = 1 \). Using the degrees, one finds the following properties:

- If \( i \) is odd, then \( Q_i(X)P_{2r-i}(Y) = 0 \).
- If \( P_{2r-2i} \neq 0 \), then \( Q_{2i} \) is invariant and therefore

\[
Q_{2i}(X) = R_i(z)
\]

with \( R_i \) is a polynomial in \( z \) of degree \( i \) which is defined up to the ideal of the Segre cone \( S \).

It follows that the equation of \( \pi^*(\Theta) \) can therefore be written as

\[
(2) \quad R_h(z) + S(z, Y)
\]

where \( S \) vanishes at the origin with order \( \geq h + 1 \) at the origin.

**Lemma 2.** — *The polynomial \( Q_{2h}(X) \) is non zero.*

Proof of the lemma: according to the previous discussion, one just has to prove the existence of

\[
e \in \text{Ext}^1(gr_1, gr_2) \oplus \text{Ext}^1(gr_2, gr_1) \subset \text{Ext}^1_0(E, E)
\]

such that the cup product

\[
\cup e : H^0(X, gr) \rightarrow H^1(X, gr)
\]
is onto. By symmetry, one only has to prove the existence of $e_1 \in \text{Ext}^1(gr_1, gr_2)$ such that the cup product

$$\cup e_1 : H^0(X, gr_1) \to H^1(X, gr_2)$$

is onto. This is classical (see [La], lemma II.8) : let $\Gamma$ be the variety

$$\Gamma = \{(k.s, k.e) \in \text{PH}^0(X, gr_1) \times \text{P Ext}^1(gr_1, gr_2) \text{ such that } s \cup e = 0\}$$

and $p$ (resp. $q$) the first (resp. second) projection. Let $0 \neq s \in H^0(X, gr_1)$ and $D = \text{div}(s)$ its zero divisor. The canonical surjection

$$\cup s : \text{Hom}(gr_1, gr_2) \to gr_2(-D)$$

gives a surjection

(3) $$\cup s : \text{Ext}^1(gr_1, gr_2) \to H^1(X, gr_2).$$

By (3) the dimension of $p^{-1}(k.s)$ is

$$\dim p^{-1}(k.s) = \dim \text{P Ext}^1(gr_1, gr_2) - \text{P dim } H^0(X, gr_1) - 1$$

Therefore

$$\dim \Gamma = \dim \text{P Ext}^1(gr_1, gr_2) - 1$$

and $q(\Gamma) \neq \Gamma$. ■

The polynomial $R_h$ can be thought of as an element of

$$H^0(\text{P Ext}^1(gr_1, gr_2) \times \text{P Ext}^1(gr_2, gr_1), \mathcal{O}(h, h))$$

which by the lemma 2 is non zero. According to (2), the completion of the tangent cone is therefore the hypersurface of

$$\left(\text{Ext}^1(gr_1, gr_1) \oplus \text{Ext}^1(gr_2, gr_2)\right)_0 \times \mathcal{S}$$

given by $R_h$. The proposition follows. ■
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