Gaussification through decoherence

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We investigate the loss of nonclassicality and non-Gaussianity of a single-mode state of the radiation field in contact with a thermal reservoir. The damped density matrix for a Fock-diagonal input is written using the Weyl expansion of the density operator. Analysis of the evolution of the quasiprobability densities reveals the existence of two successive characteristic times of the reservoir which are sufficient to assure the positivity of the Wigner function and, respectively, of the \( P \) representation. We examine the time evolution of non-Gaussianity using three recently introduced distance-type measures. They are based on the Hilbert-Schmidt metric, the relative entropy, and the Bures metric. Specifically, for an \( M \)-photon-added thermal state, we obtain a compact analytic formula of the time-dependent density matrix that is used to evaluate and compare the three non-Gaussianity measures. We find a good consistency of these measures on the sets of damped states. The explicit damped quasiprobability densities are shown to support our general findings regarding the loss of negativities of Wigner and \( P \) functions during decoherence. Finally, we point out that Gaussification of the attenuated field mode is accompanied by a nonmonotonic evolution of the von Neumann entropy of its state conditioned by the initial value of the mean photon number.

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I. INTRODUCTION

In quantum optics, a nonclassical state is defined as having either a negative or a highly singular \( P \) representation (more singular than Dirac’s \( \delta \)). Otherwise, we term it as a classical state. Cahill ¹ and later on Hillery ² proved that the only pure states that are classical are the coherent ones: all other classical states are mixtures. Accordingly, a pure state whose Wigner function has also negative values is neither classical nor Gaussian. On the other hand, Hudson’s theorem ³ implies that the Wigner function of a non-Gaussian pure state is not pointwise non-negative. Thus, for pure states, negativity of the Wigner function could be interpreted as an indicator of nonclassicality as prominent as the negativity of the \( P \) representation. Moreover, signatures of nonclassicality were identified by means of the negativity of the Wigner function for some mixed non-Gaussian states as well ⁴. Interest in the non-Gaussian states has interestingly emerged in quantum information processing. Very recently it was shown that continuous-variable states and operations represented by non-negative (Gaussian or non-Gaussian) Wigner functions can be efficiently simulated on a classical computer ⁵  ⁶. This result is a continuous-variable generalization of the Gottesman-Knill theorem for qubits ⁷. At the same time, the negativity of the discrete Wigner function defined for odd-dimensional discrete-variable systems was identified to be a quantum computational resource ⁸  ⁹. This means that the states with negative Wigner functions cannot be efficiently simulated on a classical computer or, otherwise said, a quantum computer outperforms a classical one. This property was extended to the infinite-dimensional case ¹⁰  and examples of useful resource states for quantum computation were given ¹ⁱ. Some time ago, it was also realized that non-Gaussian resources and operations could be more efficient in some quantum protocols such as teleportation ¹²  ¹³ and cloning ¹⁴ or even indispensable in entanglement distillation ¹⁵. To quantify the non-Gaussianity as a resource in such cases, some distance-type measures of this property were proposed ¹⁶  ¹⁷. However, in order to avoid the complications encountered in extremization procedures of the previously defined distance-type degrees of entanglement ¹⁶, another pattern was followed in the case of non-Gaussianity degrees ¹⁴  ¹⁵. Let us denote by \( \hat{\rho}_G \) the Gaussian state having the same average displacement and covariance matrix as the given state \( \hat{\rho} \). This state was reasonably used as a reference one in defining a degree of non-Gaussianity as the distance between the given state and the Gaussian state \( \hat{\rho}_G \). In Refs. ¹⁴  ¹⁵  ¹⁷, Genoni et al. employed the Hilbert-Schmidt metric to define a corresponding non-Gaussianity measure,

\[
\delta_{\text{HS}}[\hat{\rho}] := \frac{d^{2}_{\text{HS}}(\hat{\rho}, \hat{\rho}_G)}{2\text{Tr}(\hat{\rho}^2)} = \frac{1}{2} \left[ 1 + \frac{\text{Tr}(\hat{\rho}_G^2) - 2\text{Tr}(\hat{\rho}_G\hat{\rho})}{\text{Tr}(\hat{\rho}^2)} \right],
\]

(1.1)

and the relative entropy to introduce the entropic measure of non-Gaussianity:

\[
\delta_{\text{RE}}[\hat{\rho}] := S(\hat{\rho} | \hat{\rho}_G).
\]

(1.2)

On the one hand, Eq. (1.1) gives an easily computable expression. On the other hand, despite its not being a...
true distance, the relative entropy is acceptable and used as a measure of distinguishability between two quantum states. It is remarkable that the relative entropy \( \delta \mathcal{RE}[\hat{\rho}] \) reduces to a difference of von Neumann entropies:

\[
\delta \mathcal{RE}[\hat{\rho}] = S(\hat{\rho}_G) - S(\hat{\rho}),
\]

where \( S(\hat{\rho}) := -\text{Tr}(\hat{\rho} \ln \hat{\rho}) \) is the von Neumann entropy of the state \( \hat{\rho} \). General properties of the non-Gaussianity measures \( \delta \mathcal{RE}_1 \) and \( \delta \mathcal{RE}_2 \) were discussed in detail in Refs. \[14, 15, 17\]. Another approach to quantifying non-Gaussianity is based on the \( Q \) function and leads to a measure expressed by the difference between the Wehrl entropies of the Gaussian state \( \hat{\rho}_G \) and the given non-Gaussian state \( \hat{\rho} \). It is also worth noting that the non-Gaussianity measures \( \delta \mathcal{RE}_1 \) and \( \delta \mathcal{RE}_2 \) were used to check on the relation between non-Gaussianity and the possibility of extending Hudson’s theorem to mixed states \[19, 20\]. In the two-mode case, the relation between non-Gaussianity and entanglement both quantified by the relative entropy \( \delta \mathcal{RE}_3 \) was measured for single-photon-added coherent states \[22\]. An experimental study on the same measure was recently reported for phase-averaged coherent states \[23\].

The aim of the present paper is threefold. First, we reinforce the Bures distance to the associate Gaussian state \( \hat{\rho}_G \) as a measure of non-Gaussianity of an arbitrary one-mode state \( \hat{\rho} \). We have recently introduced it on general grounds in Ref. \[24\] by using this well-known metric related to the fidelity between two quantum states \[25\]. Our motivation to use this measure was given by the valuable distinguishability properties of the fidelity described in Refs. \[26, 27\]. Application of this measure for \( M \)-photon-added thermal states showed us its good agreement with previously defined distance-type measures \[14, 15, 17\]. Note that fidelity-based metrics have proven to be fruitful in quantum optics and quantum entanglement \[29–31\]. Second, we use the three above-mentioned distance-type measures to investigate the loss of non-Gaussianity for \( M \)-photon-added thermal states of a field coupled to a heat bath. To this end, we give a new derivation of the Fock-basis solution of the quantum optical master equation by employing very conveniently the Weyl expansion of the density operator. Third, we write the damped quasiprobability densities and examine their evolution towards the equilibrium thermal state imposed by the reservoir. This evolution is marked by the loss of negativity of both Wigner and \( P \) functions. Colaterally, we notice the nonmonotonic evolution of the von Neumann entropy during decoherence as depending on the mean photon number of the input state.

The plan of the paper is as follows. In Sec. II we recapitulate the principal features of the Bures degree of non-Gaussianity arising from some general properties of the fidelity. We here concentrate on an easily workable case, namely, that of mixed states having diagonal density matrix in the Fock basis. A comparison between various degrees of non-Gaussianity is further intended. Therefore, we consider in Sec. III a set of states generated as solutions of the quantum optical master equation \[32\]. The density matrix of the damped field mode and the quasiprobability densities at the moment \( t \) are derived for an arbitrary input state which is diagonal in the Fock basis. The evolution of the quasiprobability densities allows us the formulation of some general properties. Section IV is dedicated to a specific damped state which is important in applications: An \( M \)-photon-added thermal state evolving under the influence of a thermal reservoir. With this explicit input density matrix at the moment \( t = 0 \) and exploiting some results of Sec. III, we here find the damped quasiprobability distributions and density matrix in a closed analytic form. Further, the time evolutions of the Hilbert-Schmidt, entropic, and Bures measures of non-Gaussianity are analyzed and compared. Special attention is paid to the entropy production \( S(\hat{\rho}) \) as another tool to examine Gaussianization of single-mode states due to the field-reservoir interaction. Section V summarizes our results focused on the consistency of the analyzed measures of non-Gaussianity and the principal features of nonclassicality decay considered in the present paper.

II. BURES MEASURE OF NON-GAUSSIANITY

Let us consider two arbitrary states, \( \hat{\rho} \) and \( \hat{\sigma} \), of a given quantum system. According to Uhlmann, when both states are mixed, a good measure of the closeness between their properties is the maximal quantum-mechanical transition probability between their purifications in an enlarged Hilbert space \[23, 26\]. This is an extended notion of transition probability between quantum states which is now called fidelity \[24\] and has the intrinsic expression

\[
\mathcal{F}(\hat{\rho}, \hat{\sigma}) = \left\{ \text{Tr} \left[ \left( \sqrt{\hat{\rho}} \sqrt{\hat{\sigma}} \right)^{1/2} \right] \right\}^2. \tag{2.1}
\]

In Ref. \[24\] we have defined a fidelity-based degree of non-Gaussianity

\[
\delta F[\hat{\rho}] := \frac{1}{2} d_B^2(\hat{\rho}, \hat{\rho}_G) = 1 - \sqrt{\mathcal{F}(\hat{\rho}, \hat{\rho}_G)}. \tag{2.2}
\]

As seen in Eq. \( \langle 2.2 \rangle \), the fidelity is tightly related to the Bures metric \( d_B \) introduced in Ref. \[33\] on mathematical grounds. Several general properties of fidelity are listed below together with the features they transfer to our definition \( \langle 2.2 \rangle \).

(P1) \( 0 \leq \mathcal{F}(\hat{\rho}, \hat{\sigma}) \leq 1 \), and \( \mathcal{F}(\hat{\rho}, \hat{\sigma}) = 1 \) if and only if \( \hat{\rho} = \hat{\sigma} \). This property implies

\[
\delta F[\hat{\rho}] = 0, \quad \text{iff} \quad \hat{\rho} \quad \text{is Gaussian}, \tag{2.3}
\]

\[
0 < \delta F[\hat{\rho}] \leq 1, \quad \text{otherwise}. \tag{2.4}
\]
Properties (2.3) and (2.4) justify the interpretation of the quantity $\delta_{FE}[\hat{P}]$ as a degree of non-Gaussianity.

(P2) $F(\hat{P}, \hat{\sigma}) = F(\hat{\rho}, \hat{\sigma})$ (symmetry).

(P3) $F(\hat{\rho}, \hat{\sigma}) \geq \text{Tr}(\hat{\rho} \hat{\sigma})$. When at least one of the states is pure, then the inequality is saturated: $F(\hat{\rho}, \hat{\sigma}) = \text{Tr}(\hat{\rho} \hat{\sigma})$. Accordingly, for any pure state $|\Psi\rangle\langle\Psi|$, Eq. (2.2) reads

$$\delta_{FE}[|\Psi\rangle\langle\Psi|] = 1 - \sqrt{|\langle\Psi|\hat{\rho}_{G}|\Psi\rangle|}. \quad (2.5)$$

Unlike $\delta_{RE}$, which has the same value for all pure states possessing the same covariance matrix, $\delta_{FE}$ is state dependent.

(P4) $F(\hat{U} \hat{\rho} \hat{U}^\dagger, \hat{U} \hat{\sigma} \hat{U}^\dagger) = F(\hat{\rho}, \hat{\sigma})$ (invariance under unitary transformations). As shown in Refs. [14, 15, 17], when $\hat{U}$ are the unitary operators of the metaplectic representation on the Hilbert space of states, then the property

$$\hat{\rho}' = \hat{U} \hat{\rho} \hat{U}^\dagger \implies (\hat{\rho}')_G = \hat{U} \hat{\rho}_G \hat{U}^\dagger$$

holds, and therefore

$$\delta_{FE}[\hat{U} \hat{\rho} \hat{U}^\dagger] = \delta_{FE}[\hat{\rho}]. \quad (2.6)$$

It follows that, in the one-mode case, $\delta_{FE}[\hat{\rho}]$ does not depend on one-mode squeezing and displacement operations.

(P5) $F(\Phi(\hat{\rho}), \Phi(\hat{\sigma})) \geq F(\hat{\rho}, \hat{\sigma})$ (monotonicity under any trace-preserving, convex-linear, and completely positive map $\Phi$). This means that fidelity does not decrease strictly under any trace-preserving quantum operation, including an arbitrary nonselective measurement. Therefore, by virtue of definition (2.2), the Bures degree of non-Gaussianity does not increase under any such quantum operation.

(P6) $F(\hat{\rho}_1 \otimes \hat{\rho}_2, \hat{\sigma}_1 \otimes \hat{\sigma}_2) = F(\hat{\rho}_1, \hat{\sigma}_1) F(\hat{\rho}_2, \hat{\sigma}_2)$ (multiplicativity). Let us consider a bipartite product state $\hat{\rho} := \hat{\rho}_1 \otimes \hat{\rho}_2$. If $\hat{\rho}_2$ is a Gaussian state, we get

$$F(\hat{\rho}_1 \otimes \hat{\rho}_2, (\hat{\rho}_1)_G \otimes \hat{\rho}_2) = F(\hat{\rho}_1, (\hat{\rho}_1)_G)$$

and therefore

$$\delta_{FE}[\hat{\rho}_1 \otimes \hat{\rho}_2] = \delta_{FE}[\hat{\rho}_1]. \quad (2.7)$$

(P7) For commuting density operators, $[\hat{\rho}, \hat{\sigma}] = 0$, Eq. (2.1) simplifies to

$$F(\hat{\rho}, \hat{\sigma}) = \left[ \text{Tr} \left( \sqrt{\hat{\rho} \sqrt{\hat{\sigma}}} \right) \right]^2. \quad (2.8)$$

Properties (2.3), (2.6), and (2.7) of $\delta_{FE}[\hat{\rho}]$ are shared by the non-Gaussianity measures (1.1) and (1.3) as well [17]. In general, the Hilbert-Schmidt measure (1.1) is easier to compute than $\delta_{FE}[\hat{\rho}]$ or $\delta_{RE}[\hat{\rho}]$ and one could wonder why these last two were in fact considered? The answer is given by the monotonicity property under any trace-preserving quantum operation shared by the relative entropy and the fidelity, unlike the Hilbert-Schmidt distance [27]. One of our goals in this work is to notice the role the monotonicity plays in the evolution of non-Gaussianity under damping (see Sec. IV).

In the following we concentrate on an interesting computable case. The Gaussian reference state of a Fock-diagonal non-Gaussian one,

$$\hat{\rho} = \sum_{l=0}^{\infty} p_l |l\rangle\langle l| \quad \text{with} \quad \sum_{l=0}^{\infty} p_l = 1, \quad (2.9)$$

is a thermal state with the same mean photon occupancy $\langle N \rangle = \sum_l l p_l$:

$$\hat{\rho}_G = \sum_{l=0}^{\infty} s_l |l\rangle\langle l| \quad \text{with} \quad s_l := \frac{\langle N \rangle^l}{\langle N \rangle^{l+1}}, \quad (2.10)$$

The corresponding Hilbert-Schmidt and entropic degrees of non-Gaussianity were written in Refs. [14, 15] as:

$$\delta_{HS}[\hat{\rho}] = \frac{1}{2} \left[ 1 + \frac{1}{\sum_l p_l} \left( \frac{1}{2\langle N \rangle + 1} - 2 \sum_l p_l s_l \right) \right], \quad (2.11)$$

where $\text{Tr}(\hat{\rho}_G^2) = (2\langle N \rangle + 1)^{-1}$ is the purity of the state $\hat{\rho}_G$, and

$$\delta_{RE}[\hat{\rho}] = \sum_l p_l \ln p_l + (\langle N \rangle + 1) \ln(\langle N \rangle + 1) - \langle N \rangle \ln \langle N \rangle. \quad (2.12)$$

Here we have employed the von Neumann entropy $S(\hat{\rho}_G)$ of a Gaussian state. In this special case, we notice the commutation relation $[\hat{\rho}, \hat{\rho}_G] = 0$, which allows one the use of Eq. (2.8) to get

$$\delta_{FE}[\hat{\rho}] = 1 - \sum_{l=0}^{\infty} \sqrt{p_l s_l}. \quad (2.13)$$

An important example of states having the density operator of the form (2.9) is the class of phase-averaged (or phase-randomized) coherent states. A complete theoretical and experimental characterization of such states including measurements of the degrees of non-Gaussianity (2.11), (2.12), and (2.13) was recently reported in Ref. [34]. Note finally that various excitations on a thermal state of the type $\hat{\rho} \sim (\hat{a}^\dagger)^k \hat{a}^k \hat{\rho}_{th} (\hat{a}^\dagger)^l \hat{a}^l$ have the density operator of the type (2.9). Here $\hat{a}$ and $\hat{a}^\dagger$ are the amplitude operators of the field mode.

III. MASTER EQUATION OF ONE-MODE FIELD DAMPING

Loss of nonclassical properties of the single-mode field states (Gaussian and non-Gaussian as well) during the interaction with a dissipative environment has been intensively studied in the last decades [35–40]. Quite recently, more general properties of damped non-Gaussian
states such as the evolution of mixing measured by linear-entropy production \cite{11, 13}, or the evolution of various measures of nonclassicality \cite{44} were investigated by employing the quantum optical master equation in the interaction picture \cite{32}:

\[
\frac{\partial \hat{\rho}}{\partial t} = \frac{\gamma}{2} (2\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger) + \gamma\bar{n}_R (\hat{a}\hat{\rho}\hat{a}^\dagger + \hat{\rho}\hat{a}^\dagger\hat{a}) - \gamma\bar{n}_R (\hat{a}^\dagger\hat{\rho}\hat{a} + \hat{\rho}\hat{a}^\dagger\hat{a}) - \gamma\bar{n}_R (\hat{a}\hat{\rho}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{\rho} - \hat{\rho}\hat{a}\hat{a}^\dagger). \tag{3.1}
\]

In Eq. (3.1), \(\hat{\rho}\) is the reduced density operator of the field, \(\gamma\) is the coupling constant between field and bath, and \(\bar{n}_R\) stands for the mean occupancy of the reservoir. This popular master equation is of the Lindblad type and thus preserves both the positivity and normalization of the density operator. It has the clear physical significance of describing decoherence of a field mode coupled to a heat bath. For a comprehensive list of references regarding solutions arising from Eq. (3.1) for the expectation values of field operators, characteristic functions, and quasiprobability densities, we refer the reader to some recent work of Dodonov \cite{45, 46}.

### A. Damped density matrix

We here intend to write the evolving damped-mode state generated by this master equation from an input non-Gaussian one. Then we will compare the time evolution of the above-discussed degrees of non-Gaussianity. To go on this programme, we need a workable solution of Eq. (3.1) in the Fock basis. In what follows our main tool is the Weyl expansion of the density operator:

\[
\hat{\rho} = \frac{1}{d^2} \int d^2\lambda \chi(\lambda) \hat{D}(-\lambda). \tag{3.2}
\]

In Eq. (3.2), \(\chi(\lambda)\) is the characteristic function (CF) of the state \(\hat{\rho}\), defined as the expectation value of the displacement operator \(\hat{D}(\lambda) = \exp(\lambda\hat{a}^\dagger - \lambda^*\hat{a})\),

\[
\chi(\lambda) := \text{Tr}[\hat{\rho}\hat{D}(\lambda)]. \tag{3.3}
\]

The CF \(\chi(\lambda, t)\) of the damped field state is found to be determined by its initial form \(\chi(\lambda, 0)\) \cite{42, 43, 44}:

\[
\chi(\lambda, t) = \chi(\lambda e^{-\gamma t}, 0) \times \exp \left[ -\left(\bar{n}_R + \frac{1}{2}\right)(1 - e^{-\gamma t})|\lambda|^2 \right]. \tag{3.4}
\]

The mean photon number in the damped mode is

\[
\langle \hat{N} \rangle_t = \langle \hat{N} \rangle_0 e^{-\gamma t} + \bar{n}_T(t), \tag{3.5}
\]

where \(\bar{n}_T(t) := \bar{n}_R(1 - e^{-\gamma t})\) denotes the thermal mean occupancy in the field mode at time \(t\) and \(\langle \hat{N} \rangle_0\) is the initial mean photon number.

We expect that the damping master equation (3.1), which is phase insensitive, preserves the diagonal form of an evolving state whose initial density matrix is Fock-diagonal. Indeed, the CF of an input Fock-diagonal state \(\hat{\rho}(0)\) is

\[
\chi(\lambda, 0) = \exp \left( -\frac{1}{2} |\lambda|^2 \right) \sum_{l=0}^{\infty} \rho_{ll}(0) L_l(|\lambda|^2), \tag{3.6}
\]

where \(\rho_{ll}(0)\) is the photon-number distribution and \(L_l(x)\) is a Laguerre polynomial of degree \(l\). Equation (3.4) gives further

\[
\chi(\lambda, t) = \exp \left\{ -\left[\bar{n}_T(t) + \frac{1}{2}\right] |\lambda|^2 \right\} \times \sum_{l=0}^{\infty} \rho_{ll}(0) L_l(|\lambda|^2 e^{-\gamma t}). \tag{3.7}
\]

The density matrix \(\rho_{jk}(t)\) of the damped field state is obtained from the Weyl expansion (3.2) after here inserting the CF (3.7). Use is also made of the matrix elements of the displacement operator in the Fock basis. After a simple calculation using the polar coordinates and an obvious change of variable in the integral (3.2) we are left with the following series expansion:

\[
\rho_{jk}(t) = \delta_{jk} \sum_{l=0}^{\infty} \rho_{ll}(0) \times \int_0^{\infty} dx e^{-[\bar{n}_T(t)+1]x} L_l(xe^{-\gamma t}) L_j(x). \tag{3.8}
\]

The integral in the above equation can be routinely performed \cite{48} yielding the formula

\[
\rho_{jk}(t) = \delta_{jk} \sum_{l=0}^{\infty} \rho_{ll}(0) \times \frac{[\bar{n}_T(t)+1]}{\bar{n}_T(t)+1} \sum_{l=0}^{\infty} \rho_{ll}(0) \times 2F_1 \left[ -j, -l; 1; \frac{e^{-\gamma t}}{\bar{n}_R+1(1-e^{-\gamma t})\bar{n}_T(t)} \right]. \tag{3.9}
\]

where \(2F_1\) is a Gauss hypergeometric function, Eq. (A.1). Equation (3.9) is the general solution for the damped density matrix as function of the input one in the Fock-diagonal case. Let us take the limit \(t \to \infty\) in Eq. (3.9). The result is a thermal state with the Bose-Einstein mean occupancy \(\bar{n}_R\). We thus deal with a Gaussianization process, namely, evolution under this master equation eventually destroys the non-Gaussianity and also the non-classicality properties of any input state. A special case of Eq. (3.9) arises for the thermal contact with a zero-temperature bath (\(\bar{n}_R = 0\)), i.e., for the field coupling to the vacuum. The evolving state (3.9) then becomes:

\[
\rho_{jk}(t) = \delta_{jk} \sum_{l=j}^{\infty} \left( \frac{1}{l} \right) \rho_{ll}(0) e^{-\gamma l t} (1 - e^{-\gamma t})^{-j}. \tag{3.10}
\]

Equation (3.10) describes both dissipation by contact with a zero-temperature reservoir and photon counting for which the exponential \(e^{-\gamma t}\) should be simply replaced by the quantum efficiency \(\eta\) of the detector \cite{38}. 
B. Damped quasiprobability densities

We recall the most useful s-ordered CFs [42]:
\[ \chi(\lambda; s) = \exp \left( \frac{s}{2} |\lambda|^2 \right) \text{Tr}[\hat{\rho} \hat{D}(\lambda)], \]
(s = −1, 0, 1). \hspace{1cm} (3.11)

Their corresponding normalized Fourier transforms
\[ W(\beta; s) = \frac{1}{\pi} \int d^2 \lambda \exp(\beta \lambda^* - \beta^* \lambda) \chi(\lambda; s) \hspace{1cm} (3.12) \]
can be interpreted as quasiprobability densities and are important tools for understanding the nonclassical features of quantum states. Specifically, they are: for s = 1, P(β) := W(β; 1) (Glauber’s P function), s = 0, W(β) := W(β; 0) (Wigner function), and s = −1, Q(β) := W(β; −1) (Husimi function).

Our aim here is to write a general expression for the quasiprobability densities describing a damped state. Therefore we insert the CF (3.7) into Eq. (3.12) and get
\[ W(\beta, t; s) = \frac{1}{\pi} \sum_{l=0}^{\infty} \rho_l(0) \int d^2 \lambda \exp(\beta \lambda^* - \beta^* \lambda) \chi(\lambda; s) \]
\[ \times \exp \left\{ - \left( \bar{n}_T(t) + \frac{1}{2}(1-s) \right) |\lambda|^2 \right\} L_l(|\lambda|^2 e^{-\gamma t}). \hspace{1cm} (3.13) \]

We first perform the integration over the polar angle, then we are left with a known integral over |λ| whose evaluation [49] leads us to the following double summation:
\[ W(\beta, t; s) = \frac{1}{\pi} \sum_{l=0}^{\infty} \rho_l(0) \frac{|A(t; s)|^l}{[\bar{n}_T(t) + \frac{1}{2}(1-s)]^{l+1}} \]
\[ \times \sum_{k=0}^{\infty} \frac{(-1)^k |\beta|^{2k}}{l!} \left[ \bar{n}_T(t) + \frac{1}{2}(1-s) \right]^k \]
\[ \times 2 \sum \left\{ -k, -l; 1; -e^{-\gamma t} \right\} A(t; s). \hspace{1cm} (3.14) \]

In Eq. (3.14) we have introduced a bath-dependent parameter whose relevance for the evolution of the quasiprobability densities will arise further:
\[ A(t; s) := \bar{n}_T(t) + \frac{1}{2}(1-s) - e^{-\gamma t} \]
\[ = \bar{n}_R + \frac{1}{2}(1-s) - (\bar{n}_R + 1) e^{-\gamma t}. \hspace{1cm} (3.15) \]

Now, the sum over k is of the type [3.3] and leads to a rather simple result:
\[ W(\beta, t; s) = \frac{1}{\pi} \exp \left[ - \left( \bar{n}_T(t) + \frac{1}{2}(1-s) \right) \right] \]
\[ \times \sum_{l=0}^{\infty} \rho_l(0) \frac{|A(t; s)|^l}{[\bar{n}_T(t) + \frac{1}{2}(1-s)]^{l+1}} \]
\[ \times L_l \left\{ - \frac{|\beta|^{2} e^{-\gamma t}}{[\bar{n}_T(t) + \frac{1}{2}(1-s)] A(t; s)} \right\}. \hspace{1cm} (3.16) \]

Any Laguerre polynomial \( L_l(x) \) displays, via Eqs. (A.4) and (A.2), its positivity for negative values of the argument x. Equation (3.16) presents two main advantages. First, it makes possible a simultaneous evaluation of all quasiprobability densities \( W(\beta, t; s) \). Second, its structure implies a general statement regarding the positivity of the quasiprobability densities independent of the initial state \( \hat{\rho}(0) \). Indeed, at any \( \beta \) all the quasiprobability densities \( W(\beta, t; s) \) [Eq. (3.16)] are positive for positive \( A(t; s) \). This allows us to write the ultimate times at which any quasiprobability distribution displaying initially some negativities becomes positive due to the field interaction with the thermal bath.

(i) \( s = -1 \Rightarrow A(t; -1) = (\bar{n}_R + 1)(1 - e^{-\gamma t}) > 0 \). As expected from its definition as the average value of the density operator in a coherent state, the \( Q \) function is always positive.

(ii) \( s = 0 \Rightarrow A(t; 0) = \bar{n}_R + \frac{1}{2} - (\bar{n}_R + 1) e^{-\gamma t} \) is positive for
\[ t \geq t_c := \frac{1}{\gamma} \ln \left( 1 + \frac{1}{2\bar{n}_R + 1} \right). \hspace{1cm} (3.17) \]

This is the time at which the Wigner function completely loses any negativity.

(iii) \( s = 1 \Rightarrow A(t; 1) = \bar{n}_R - (\bar{n}_R + 1) e^{-\gamma t} \) is positive for
\[ t \geq t_w := \frac{1}{\gamma} \ln \left( 1 + \frac{1}{\bar{n}_R} \right). \hspace{1cm} (3.18) \]

Beyond the time \( t_c \), the Glauber-Sudarshan \( P \) representation exists as a genuine probability density. The threshold times (3.17) and (3.18) are constants of the bath valid for any input Fock-diagonal state. Note also that \( t_c > t_w \), that is, the Wigner function is more fragile than the \( P \) function under damping. Let us now remark that the characteristic time (3.18) was previously obtained for particular input states such as even coherent states in Refs. [42, 44] and Fock states in [42]. The limit time (3.17) was found in Refs. [42, 43] in describing the mixing process during the damping of an input pure state. As an onset of positivity of the Wigner function for damped photon-added thermal states, the time \( t_w \) was written in Refs. [50, 51]. In the case of a zero-temperature reservoir, \( \bar{n}_R = 0 \), we get an unique state-independent time \( t_w = \frac{1}{\gamma} \ln 2 \) for disappearance of any negativity of the Wigner function, while, according to Eq. (3.18), the \( P \) representation could have negative domains at any time.

What happens at earlier times \( t < t_w \)? We cannot make general predictions using our Eq. (3.16). The behavior of the distribution functions \( P(\beta, t) \) and \( W(\beta, t) \) will depend on the initial state \( \hat{\rho}(0) \).

IV. A CASE STUDY: DAMPED PHOTON-ADDED THERMAL STATES

According to a pertinent remark of Lee [52], adding photons to a classical state implies removal of the vac-
uum state from its Fock expansion. A nonclassical output is obtained in every case. Addition of photons to a classical Gaussian state, such as a coherent or a thermal one, generates a non-classical output which is no longer Gaussian. We can say that photon-added states are both non-classical and non-Gaussian. An M-photon-added thermal state \([\xi, \xi]\) is an interesting example of a Fock-diagonal state whose non-classicality was recently investigated experimentally \([53, 54]\). The density matrix of the state obtained by adding M photons to a thermal one \(\hat{\rho}_{th}\) is

\[
\hat{\rho}^{(M)} = \frac{1}{M! (\bar{n} + 1)^M} (\hat{a}^{\dagger})^M \hat{\rho}_{th} \hat{a}^M. \tag{4.1}
\]

The spectral decomposition of a given thermal state \(\hat{\rho}_{th}\) is of the type (2.10), with a mean photon number denoted \(\bar{n}\). The nonclassicality of the state (4.1) was first discussed by Agarwal and Tara \([53]\) in terms of its nonpos-

give non-Gaussianity of the state (4.1) was recently evaluated down the density matrix:

The spectral decomposition of a given thermal state \(\hat{\rho}_{th}\) is like the Wehrl-entropy measure \([18]\). Our evaluations ability distributions with negativities towards positive

test bed to examine the evolution of the initial quasiprobability distributions with negativities towards positive representations following our general findings in Sec. III B. We evaluate first the time development of quasiprobability densities \(W(\beta, t; s)\) by inserting the density matrix (4.2) into Eq. (3.10). By applying Eq. (A.9) we finally get after minor rearrangements:

\[
W(\beta, t; s) = \frac{1}{\pi} \exp \left[ -\frac{|\beta|^2}{B(t; s)} \right] \frac{|A(t; s)|^M}{L_M} \times \left[ -\frac{(\bar{n} + 1) e^{-\gamma t} |\beta|^2}{B(t; s) A(t; s)} \right], \tag{4.4}
\]

where we have denoted

\[
B(t; s) := \bar{n}_T(t) + \frac{1}{2} (1 - s) + \bar{n} e^{-\gamma t}, \tag{4.5}
\]

and \(A(t; s)\) is the universal bath parameter given by Eq. (4.13). Let us now write the quasiprobability densities of the input state \([Eq. (4.1)]\). By specializing \(\bar{n}_R = 0, t = 0\) in Eq. (4.3), we easily get:

\[
Q(\beta, 0) = \frac{1}{\pi M! (\bar{n} + 1)^{M+1}} \exp \left( -\frac{|\beta|^2}{\bar{n} + 1} \right), \tag{4.6}
\]

\[
P(\beta, 0) = \frac{1}{\pi} \frac{(-1)^M}{\bar{n}^{M+1}} \exp \left( -\frac{|\beta|^2}{\bar{n}} \right) \times L_M \left( \frac{\bar{n} + 1}{\bar{n}} |\beta|^2 \right), \tag{4.7}
\]

\[
W(\beta, 0) = \frac{2}{\pi} \frac{(-1)^M}{(2\bar{n} + 1)^{M+1}} \exp \left( -\frac{2|\beta|^2}{2\bar{n} + 1} \right) \times L_M \left( 4(\bar{n} + 1) |\beta|^2 \right). \tag{4.8}
\]

We can see that the distributions \(P(\beta, 0)\) and \(W(\beta, 0)\) have a similar functional form which guarantees the existence of some negativity regions. Indeed, the Laguerre polynomial \(L_M(x)\) has precisely \(M\) distinct positive roots. According to Eq. (A.4), \(L_M(0) = 1\), so that \(L_M(x) > 0\) for \(x \leq 0\). On the contrary, a polynomial \(L_M(x)\) is negative on a finite number of subintervals of \(\mathbb{R}^+\): this number is either \((M-1)/2\), if \(M\) is odd, or \(\frac{M}{2}\), if \(M\) is even. Therefore, a product of the type \((-1)^M L_M(x), x \geq 0\), occurring in both expressions (4.7) and (4.8) presents negative values. Hence the M-photon-added thermal states are nonclassical for any values of the parameters \(\bar{n}\) and \(M\).

As regards the distributions of the damped field, we remark that the state-dependent quantity \(B(t; s)\) is positive at any time. Therefore, the discussion on the characteristic times at which the distributions \(P(\beta, t)\) and \(W(\beta, t)\) lose their initial negativities as presented in Sec. III is equally valid here. There is no alteration of the limit times \(t_c\) [Eq. (3.18)] and \(t_w\) [Eq. (3.17)] produced by the specific input state [Eq. (4.1)] we are dealing with. The photon-added thermal states with their limit case

**A. Decay of negativities**

The damped photon-added thermal states offer a good test bed to examine the evolution of the initial quasiprobability distributions with negativities towards positive
\( \bar{n} = 0 \), that is, a number state \( |M\rangle\langle M| \), represent a perfect example of nonclassicality lost during Gaussionisation by damping. They develop in time through three distinct stages (regimes) of decoherence which can be related to their efficiency in continuous-variable quantum computation \( R\mathbb{G} S \) as follows.

(1) For \( 0 \leq t < t_w \), the damped states are nonclassical from both points of view of quantum optics (negative \( P \) function) and quantum information processing (negative \( W \) function) and could be used for quantum computational speed-up.

(2) For \( t_w \leq t < t_c \), the damped states have positive Wigner function, while their \( P \) representation still displays negativities (region of bound universal states \( K \)). According to the recent results in Refs. [5, 6, 8], such states admit an efficient classical simulation due to the positivity of their Wigner distribution, in spite of not having a well-behaved \( P \) representation.

(3) For \( t \geq t_c \), the damped states are classical and can be efficiently simulated on a classical computer.

The discussion simplifies when dealing with a zero-temperature bath. We have \( t_w = \frac{\gamma}{2} \ln 2 \) and \( t_c \to \infty \). This case was recently tackled in Ref. [6] for \( M = 1 \). The fidelity between the damped states and the vacuum was there considered as a measure of the closeness to the convex hull of Gaussian states. This was found to be small for \( t > t_w \), namely, for the set of bound universal states created by dissipation. We shall further reconsider this problem for an arbitrary thermal reservoir by using the more rigourous measures of non-Gaussianity presented in Secs. I and II.

### B. Measures of non-Gaussianity

Insertion of the input photon-number distribution \( \rho_G^{(M)} \) into Eq. (4.12) gives the Fock-diagonal density matrix of a damped photon-added thermal state. Technically we have to evaluate the sum of a power series involving Gauss hypergeometric functions which is discussed in the Appendix. We apply Eq. (A.8) and get

\[
\rho^{(M)}_{jk}(t) = \delta_{jk} \left[ \frac{(\bar{n}_R + 1)(1 - e^{-\gamma t})}{\bar{n} e^{-\gamma t} + \bar{n}_T(t) + 1} \right]^M \\
\times \left[ \frac{\bar{n} e^{-\gamma t} + \bar{n}_T(t)}{(\bar{n} e^{-\gamma t} + \bar{n}_T(t) + 1)^{j+1}} \right] \\
\times 2F_1 \left\{ -M, -j; 1; \frac{(\bar{n} + 1) e^{-\gamma t}}{(\bar{n}_R + 1)(1 - e^{-\gamma t})[\bar{n} e^{-\gamma t} + \bar{n}_T(t)]} \right\}.
\]

(4.9)

The reference Gaussian state associated with this density matrix is a thermal one whose mean occupancy equals the average photon number of the damped field state. According to Eq. (3.5), this is

\[
\langle \hat{N} \rangle_t = [\bar{n}(M + 1) + M] e^{-\gamma t} + \bar{n}_T(t).
\]

(4.10)

We have used the time-dependent density matrix \( \rho^{(M)} \) and the mean occupancy \( \langle \hat{N} \rangle_t \) to evaluate numerically two of the distance-type measures of non-Gaussianity we are interested in: \( \delta_F[\hat{\rho}^{(M)}] \) and \( \delta_{\text{RE}}[\hat{\rho}^{(M)}] \) via Eqs. (2.13) and (2.12), respectively. Our results are displayed in Figs. 1 and 2, where the time evolutions of these non-Gaussianity measures are presented for weak damping \( (\bar{n}_R = 0.1) \) and for a large thermal mean occupancy of the reservoir \( (\bar{n}_R = 5) \), respectively. On both figures we have marked the three regions delimited by the bath parameters \( \gamma t_w \) and \( \gamma t_c \). In Fig. 1 the two regions of non-classicality are large enough \( (\gamma t_w = 0.606 \text{ and } \gamma t_c = 2.398) \). They contain states with higher non-Gaussianity, while the third (classical) region displays states slowly evolving to the Gaussian equilibrium state of the reservoir. Figure 2 shows that in a noisy bath the nonclassicality rapidly disappears \( (\gamma t_w = 0.87 \text{ and } \gamma t_c = 0.182) \), while non-Gaussianity is still large at the classicality threshold \( \gamma t_c \). The functions \( \delta_F[\hat{\rho}^{(M)}(t)] \) and \( \delta_{\text{RE}}[\hat{\rho}^{(M)}(t)] \) are monotonically decreasing, in accordance with the corresponding property discussed in Sec. II. Both Figs. 1 and 2 display a very good agreement between the two measures. They show consistency by inducing the same ordering of non-Gaussianity when considering specific sets of damped states continuously generated by the field-reservoir interaction.

The last degree of non-Gaussianity we are interested in is evaluated using Eq. (2.11). Fortunately, as in the input-state case [24], we are able to give nice analytic expressions for both the scalar product \( \text{Tr}[\hat{\rho}^{(M)}(t)\hat{\rho}_G^{(M)}(t)] \) and the purity \( \text{Tr}\{[\hat{\rho}^{(M)}(t)]^2\} \) of the damped state. We first write

\[
\text{Tr} \left[ \hat{\rho}^{(M)}(t)\hat{\rho}_G^{(M)}(t) \right] = \frac{1}{\langle \hat{N} \rangle_t} \sum_{j=0}^{\infty} \left( \frac{\langle \hat{N} \rangle_t}{\langle \hat{N} \rangle_t + 1} \right)^j \times \rho_{jj}^{(M)}(t),
\]

(4.11)

where the expectation value \( \langle \hat{N} \rangle_t \) is given in Eq. (4.10). By inserting Eq. (4.9) for the damped density matrix into Eq. (4.11), we eventually get a sum of hypergeometric polynomials of the type \( (A.6) \). After a little algebra we obtain the formula

\[
\text{Tr} \left[ \hat{\rho}^{(M)}(t)\hat{\rho}_G^{(M)}(t) \right] = \frac{\langle \hat{N} \rangle_t + \bar{n}_T(t) + 1 - e^{-\gamma t}M}{[\langle \hat{N} \rangle_t + \bar{n}_T(t) + \bar{n} e^{-\gamma t} + 1]^{M+1}}.
\]

(4.12)

Now, the purity of the damped state,

\[
\text{Tr} \left\{ [\hat{\rho}^{(M)}(t)]^2 \right\} = \sum_{j=0}^{\infty} \rho_{jj}^{(M)}(t)^2,
\]

(4.13)

is proportional to the sum of a power series whose coefficients are squared Gauss hypergeometric polynomials. We apply the summation formula \( (A.11) \) and find after
FIG. 1: Time evolution of the fidelity-based non-Gaussianity [Eq. (2.13)] (left plot) and the entropic measure [Eq. (2.12)] (right plot), for input $M$-photon-added thermal states with $M = 1, 3, 5, 10$ (from bottom to top) and $\bar{n} = 1$ in contact with a heat bath having $\bar{n}_R = 0.1$. The nonclassicality threshold parameters are $\gamma_{t_w} = 0.606$ and $\gamma_{t_c} = 2.398$ and delimitate the three regions of decay.

FIG. 2: As in Fig.1, but for input $M$-photon-added thermal states with $M = 1, 3, 5, 10$ (from bottom to top) and $\bar{n} = 0.5$ in contact with a heat bath having $\bar{n}_R = 5$. The characteristic parameters of this noisy bath are $\gamma_{t_w} = 0.087$ and $\gamma_{t_c} = 0.182$.

some minor rearrangements:

$$
\text{Tr} \left\{ \left[ \hat{\rho}(M)(t) \right]^2 \right\} = \frac{[\bar{n} + 1] e^{-\gamma t} [2\bar{n} + 2\bar{n} e^{-\gamma t} + 1]^{2M+1}}{[\bar{n} + 1] e^{-\gamma t}}.
$$

Note that we can express the right-hand side of Eq. (4.14) in terms of a Legendre polynomial via Eq. (A.5). The equilibrium values of both the scalar product (4.12) and the purity (4.14) are equal to the purity of the thermal state eventually imposed by the reservoir, namely, $(2\bar{n}_R + 1)^{-1}$. The interesting case of a number state $|M\rangle\langle M|$ is given by setting $\bar{n} = 0$ in the above formulas. To compute the Hilbert-Schmidt degree of non-Gaussianity, we have to substitute into Eq. (2.11) the overlap (4.12), the purity (4.14) of the damped state, and the purity of the reference Gaussian state,

$$
\text{Tr} \left\{ \left[ \hat{\rho}(M)(t) \right]^2 \right\} = \frac{1}{2\langle N \rangle_{t} + 1}.
$$

In Fig. 3 we point out the time evolution of the Hilbert-Schmidt degree of non-Gaussianity for the same sets of states as in Figs. 1 and 2. The agreement between the Hilbert-Schmidt degree of non-Gaussianity and the other two distance-type measures we discuss here is more accurate during the interaction with noisier reservoirs, as shown in Figs. 2 and 3. All plots showing the time development of $\delta_F$ and $\delta_{RE}$ in Figs. 1 and 2 have a similar aspect which indicates a good consistency of these two measures coming from their monotonicity under the damping map. We expect they induce the same ordering of non-Gaussianity. This property is not shared by the Hilbert-Schmidt degree of non-Gaussianity and the relative entropy or Bures distance. In Fig.4 (left) we have plotted the entropic measure as a function of the fidelity-based degree for the set of damped states with $\bar{n}_R = 0.1$, $\bar{n} = 1$, and different values of $M$. The two measures show here a good consistency by preserving the same ordering of non-Gaussianity. Also plotted in Fig.4 is $\delta_{HS}$ as a function of $\delta_{RE}$ and $\delta_F$ for the same sets of states as in the left panel. We notice that both of them have the same concavity but they lack con-
FIG. 3: Time evolution of the Hilbert-Schmidt degree of non-Gaussianity for contact with a heat bath when $\bar{n}_R = 0.1$ (left plot) and $\bar{n}_R = 5$ (right plot). The input states are $M$-photon-added thermal states with $\bar{n} = 1$ (left) and $\bar{n} = 0.5$ (right), and $M = 1, 3, 5, 10$ (from bottom to top). For the noisier bath (right plot), the characteristic times are very small and the non-Gaussianity is rapidly going to zero in the classicality zone even it is appreciable at the threshold $\gamma t_c = 0.182$.

FIG. 4: Displaying a full consistency between $\delta_{\text{RE}}$ and $\delta_F$ (left panel) and a partial one between these two measures and the Hilbert-Schmidt one for input $M$-photon added thermal states ($\bar{n} = 1$) evolving in contact with a thermal bath when $\bar{n}_R = 0.1$. We used $M = 1, 3, 5, 10$ (from bottom to top).

Consistency for some values of the parameters. Indeed, as shown by Figs. 1-3, at a given time the all three measures are increasing with the number $M$. This is fairly displayed by the left plot in Fig. 4 but not entirely by the other two. We see that the non-Gaussianity of the states with $M = 1$ appears to be larger than in the case of $M = 10$.

C. Gaussification and entropy production

The monotonic decay of all non-Gaussianity degrees under damping does not tell us too much about the loss of nonclassicality of the states \[1, 1\]. Decreasing of negativity regions of quasiprobability densities seems to be a faithful indicator of nonclassicality decay as shown on very general grounds in Sec. III. Some time ago, another interesting evolution was found for two functions related to the von Neumann entropy, namely, the linear entropy and the 2-entropy \[42, 43\]. In Ref. 38, it was shown that a nonclassical Gaussian state under damping displays a maximum in its linear entropy provided that its initial mean occupancy $\langle \hat{N} \rangle_0$ exceeds the mean occupancy of the reservoir $\bar{n}_R$. See also a more recent analysis in Ref. 58. For some pure nonclassical input states, a number state in Ref. 42 and an even coherent state in Ref. 43, it was found analytically that a maximum in the evolution of the 2-entropy exists under the same mean-occupancy condition. It appears to us that the existence of a maximum in the evolution of the entropy under damping is primarily conditioned by the nonclassicality (expressed in terms of a not well-behaved $P$ representation) of the input state. For instance, the entropy of a classical Gaussian state does not exhibit a maximum under damping in any case. But this statement was not proven for mixed non-Gaussian states so far.

The mixed states we are dealing with in this paper are both nonclassical and non-Gaussian. We thus find it interesting to check on their entropic evolution under damping because we can take advantage of having evaluated the von Neumann entropy for both the time-dependent state $\hat{\rho}^{(M)}(t)$ and the reference Gaussian one $\hat{\rho}_G^{(M)}(t)$. In the latter case, the field-reservoir interaction preserves the feature of $\hat{\rho}_G^{(M)}(t)$ being a thermal state. Its mean photon number $\langle \hat{N} \rangle_t$ [Eq. (4.10)] varies monotonically from the input value $\langle \hat{N} \rangle_0 = M(\bar{n} + 1) + \bar{n}$ to the reservoir mean occupancy $\bar{n}_R$. Therefore, the von Neumann entropy $S(\hat{\rho}_G^{(M)}(t))$ is a monotonic function of time between the corresponding imposed limits at $t = 0$ and $t \to \infty$. Good examples of this monotonic evolution are shown on the left-hand side of Fig. 5 for a noisy reservoir.
with $\bar{n}_R = 5$. The input mean occupancy $\langle \hat{N} \rangle|_0$ for the states with $\bar{n} = 0.5$ and various $M$ are placed in Table I together with the corresponding values of the entropy for the reference Gaussian states $\hat{\rho}^{(M)}_G(0)$ and the given ones $\hat{\rho}^{(M)}(0)$. To calculate the latter ones, we made use of the density matrix \cite{9}. Notice that in all the chosen cases the entropy of the input $\hat{\rho}^{(M)}(0)$ is smaller than that of the reservoir. The von Neumann entropy of the damped state $\hat{\rho}^{(M)}(t)$ is plotted on the right-hand side of Fig. 5 for the same parameters.

| $M$ | $\langle \hat{N} \rangle|_0$ | $S(\hat{\rho}^{(M)}_G)$ | $S(\hat{\rho}^{(M)})$ | $\gamma t_{max}$ |
|-----|-----------------|-----------------|-----------------|----------------|
| 1   | 2               | 1.909           | 1.372           | -              |
| 3   | 5               | 2.703           | 1.824           | -              |
| 5   | 8               | 3.140           | 2.078           | 0.5562         |
| 10  | 15.5            | 3.770           | 2.429           | 0.5538         |

TABLE I: Entropic characterization of the input states whose Gaussification is analyzed in Figs.2 and 3, namely having $\bar{n} = 0.5$. The thermal mean occupancy of the reservoir is $\bar{n}_R = 5$. In the last column we have inserted the position of the maximum of the entropy.

By contrast to its monotonic evolution for $\langle \hat{N} \rangle|_0 \leq \bar{n}_R$ (cases $M = 1, 3$), we point out the existence of a short-time maximum of the von Neumann entropy $S(\hat{\rho}^{(M)}(t))$ when $\langle \hat{N} \rangle|_0 > \bar{n}_R$ (cases $M = 5, 10$). The maximum is reached at a time depending slightly on $M$ and situated quite deeply in the classicality region. Therefore, the evolution of the von Neumann entropy follows the same pattern as in the case of mixed Gaussian states and obeys the same mean-occupancy and nonclassicality conditions.

V. DISCUSSION AND CONCLUSIONS

To sum up, in this work we have investigated the evolution of non-Gaussian one-mode states during the interaction of the field mode with a thermal reservoir resulting in two processes: Loss of nonclassicality indicated by the time development of the Wigner and $P$ functions and loss of non-Gaussianity shown by some recently introduced distance-type measures. We have analytically proved that the evolution of all input Fock-diagonal states is characterized by three decoherence regions that can be related to some recent findings in quantum computation \cite{8}. The end of the first one at the time $t_w$ [Eq. (3.17)] represents the onset of the positivity of the Wigner function. The damped states belonging to the first region are nonclassical and can be used for quantum computational speed-up. The second region ends at the time $t_c$ [Eq. (3.18)] when the $P$ representation becomes positive. The states in this region were described in quantum computation as bound universal states. The third one is a classicality domain towards the limit $t \to \infty$ when the mode shares the thermal state of the reservoir. States damped after the moment $t_c$ can be efficiently simulated on a classical computer because they possess a positive Wigner function \cite{3, 8}. The $M$-photon-added thermal states perfectly supported these general findings on the nonclassicality decay. For these specific states we have also found that decaying of non-Gaussianity was consistently described by the three distance-type measures used in this paper. Even with some alterations of ordering, the fidelity-based degree, the Hilbert-Schmidt one, and the entropic measure evolve monotonically, as expected for any good measure of non-Gaussianity \cite{24}. We have also marked on all time-development plots in Figs.1-3 the three decoherence zones discussed in Sec. IV A, which are delimited by the thresholds $\gamma t_w$ [Eq. (3.17)] for the onset of the positivity of the Wigner function and $\gamma t_c$ [Eq. (3.18)] representing the onset of classicality. In the case of weak damping (small $\bar{n}_R$), the non-Gaussianity is appreciable for $t < t_c$ and slowly decays for $t > t_c$. For noisier baths, the characteristic times are very small and the non-Gaussianity is rapidly going to zero in the classicality zone.

As a side effect, we pointed out a nonmonotonic time development of the von Neumann entropy whose behavior depends on the relation between the input mean occupancy $\langle \hat{N} \rangle|_0$ of the state and the thermal mean occupancy $\bar{n}_R$ of the reservoir. States with input mean occupancies greater than $\bar{n}_R$, but with input entropy not exceeding that of the reservoir, were found to present a transient mixing enhancement, visible as a maximum in the evolution of their entropy.

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Appendix: Some power series involving hypergeometric functions

A Gauss hypergeometric function is the sum of the corresponding hypergeometric series,

$$ _2F_1(a, b; c; z) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n z^n}{(c)_n n!} \quad (|z| < 1), \quad (A.1) $$

where $(a)_n := \Gamma(a + n)/\Gamma(a)$ is Pochhammer’s symbol standing for a rising factorial. This definition is extended by analytic continuation. The confluent (Kummer) hypergeometric function has the Maclaurin series:

$$ _1F_1(a; c; z) := \sum_{n=0}^{\infty} \frac{(a)_n z^n}{(c)_n n!}. \quad (A.2) $$
In the body of the paper we need to evaluate the sum structure, because the right-hand side of Eq. (A.6) has a simple ating function using Cauchy’s integral formula. Second, we have applied an elegant method described in the recent class of hypergeometric polynomials [60]:

\begin{equation}
\sum_{l=0}^{\infty} \binom{l}{n} u^l \frac{\partial^n}{\partial u^n} G_0(u, z) = \frac{1}{n!} u^n \frac{\partial^n}{\partial u^n} G_0(u, z).
\end{equation}

(A.7)

We have performed the sum (A.7) in two ways. First, we have applied an elegant method described in the recent paper [61] to find the nth-order derivative of such a generating function using Cauchy’s integral formula. Second, because the right-hand side of Eq. (A.6) has a simple structure, the nth-order derivative can be put in a closed form after some routine algebra. With both methods we got the formula

\begin{equation}
G_n(u, z) = u^n (1 - u)^{a-n-1} (1 - u + uz)^{-a}
\end{equation}

(A.8)

In evaluating the quasiprobability distributions \( W(\beta; s) \), we employed the sum of a nontrivial expansion in terms of Laguerre polynomials whose derivation can be found in Ref. [61]:

\begin{equation}
\sum_{l=0}^{\infty} \binom{l}{n} u^l L_l(z) = \frac{u^n}{(1 - u)^{n+1}} \exp \left( -\frac{uz}{1 - u} \right)
\end{equation}

(A.9)

The known summation of a power series whose coefficients are proportional to the product of two Gauss hypergeometric polynomials [62] was also employed in Sec. IV:

\begin{equation}
\sum_{n=0}^{\infty} \binom{c}{n} \frac{(-v)^n}{n!} \binom{b}{n} \frac{(-1)^n}{n!} \binom{b'}{n} \frac{(-1)^n}{n!} 2F_1(-n, b; c) 2F_1(-n, b'; c; z') = (1 + v)^{-c+b'+b}(1 + v - vz')^{-b'} \times 2F_1 \left[ b, b'; c; \frac{-vz'}{(1 + v - vz)(1 + v - vz')^2} \right].
\end{equation}

(A.10)

In our case, the parameters take particular values,

\[ b' = b, c = 1, z' = z, \]

so that Eq. (A.10) reduces to a slightly simpler one:

\begin{equation}
\sum_{n=0}^{\infty} v^n \left[ 2F_1(-n, b; 1; z) \right]^2 = (1 - v)^{-1+2b}(1 - v + vz)^{-2b} \times 2F_1 \left[ b, b; 1; \frac{vz^2}{(1 - v + vz)^2} \right].
\end{equation}

(A.11)
