On the number of Birch partitions

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Abstract

Birch and Tverberg partitions are closely related concepts from discrete geometry. We show two properties for the number of Birch partitions: Evenness, and a lower bound. This implies the first non-trivial lower bound for the number of Tverberg partitions that holds for arbitrary \( q \), where \( q \) is the number of partition blocks. The proofs are based on direct arguments, and do not use the equivariant method from topological combinatorics.

1 Introduction

Our starting point is the following theorem due to B. J. Birch [3] from 1959.

**Theorem 1.** Given \( 3N \) points in \( \mathbb{R}^2 \), we can divide them into \( N \) triads such that their convex hulls contain a common point.

The proof of Theorem 1 is based on a lemma on partitioning a general measure which is due to Richard Rado, nowadays known as the center point theorem. See e.g. Matoušek’s textbook [9], or Tverberg and Vrećica [13] for more details.

**Definition.** Let \( X \) be a set of \( k(d+1) \) points in \( \mathbb{R}^d \) for some \( k \geq 1 \). A point \( p \in \mathbb{R}^d \) is a Birch point of \( X \) if there is a partition of \( X \) into \( k \) subsets of size \( d+1 \), each containing \( p \) in its convex hull. The partition of \( X \) is a Birch partition for \( p \). For fixed \( p \in \mathbb{R}^d \), let \( B_p(X) \) be the number of unordered Birch partitions for \( p \).

From now on, we fix \( p \) to be the origin, and we write *Birch partition* instead of *Birch partition for the origin* for short. A set of points in \( \mathbb{R}^d \) is in general position if no \( k+2 \) points are on a common \( k \)-dimensional affine subspace. A set \( X \) of points in \( \mathbb{R}^d \) is in general position with respect to a point \( p \) if \( X \cup \{ p \} \) is in general position.

Our first main result is the following theorem on the number of Birch partitions.

**Theorem 2.** Let \( d \geq 1 \) and \( k \geq 2 \) be integers, and \( X \) be a set of \( k(d+1) \) points in \( \mathbb{R}^d \) in general position with respect to the origin 0. Then the following properties hold for \( B_0(X) \):

i) \( B_0(X) \) is even.

ii) \( B_0(X) > 0 \Rightarrow B_0(X) \geq k! \)

If the origin is not in the convex hull of \( X \), then one has \( B_0(X) = 0 \) which is even. If there is a Birch partition then the lower bound given in Property ii) is tight. Based on computer experiments, we moreover conjecture:

\[
B_0(X) \leq (k!)^d. \tag{1}
\]

B. J. Birch proved Theorem 1 to obtain the following statement for \( d = 2 \). Helge Tverberg then settled the problem for arbitrary dimension \( d \) in 1966.
Theorem 3 (Tverberg’s theorem). Let $d$ and $q$ be integers. Any $(q - 1)(d + 1) + 1$ points in $\mathbb{R}^d$ can be partitioned into $q$ subsets such that their convex hulls have a point in common.

Partitions as in Theorem 3 are Tverberg partitions (into $q$ blocks). From now on, we implicitly assume that Tverberg partitions are partitions into $q$ blocks given a set of $(q - 1)(d + 1) + 1$ points in $\mathbb{R}^d$. The point in common is a Tverberg point.

A wave of excitement started in 1981 when Bárány et al. [2] were able to prove a more general topological version known as the Topological Tverberg Theorem when $q$ is a prime number using Borsuk-Ulam’s theorem from algebraic topology. This has then been extended to prime powers $q$ by many authors, e.g. Özaydin [10], Volovikov [14], Sarkaria [11]. The general case for arbitrary $q$ is still open; see Matoušek’s textbook [8] for more background.

The number of Tverberg partitions has been studied by Vučić and Živaljević [15], and Hell [6]. Using the equivariant method from topological combinatorics they have obtained:

Theorem 4. Let $q = p^r$ be a prime power and $d \geq 1$. For any continuous map $f : \sigma^N \to \mathbb{R}^d$, where $N = (d + 1)(q - 1)$, the number of unordered $q$-tuples $\{F_1, F_2, \ldots, F_q\}$ of disjoint faces of the $N$-simplex with $\bigcap_{i=1}^q f(||F_i||) \neq \emptyset$ is at least

$$\frac{1}{(q - 1)!} \cdot \left( \frac{q}{r + 1} \right)^{\left\lceil \frac{N}{2} \right\rceil}.$$

Restricting $f$ to an affine map, unordered $q$-tuples as in Theorem 4 are in bijection with Tverberg partitions of the set $\{f(v_0, v_1, \ldots, v_N) \mid v_0, v_1, \ldots, v_N \text{ vertices of } \sigma^N\}$ of $N + 1 = (d + 1)(q - 1) + 1$ many points in $\mathbb{R}^d$.

Using Theorem 2 we obtain our second main result: The first non-trivial lower bound for the number of Tverberg partitions that holds for arbitrary $q$.

Theorem 5. Let $X$ be a set of $(d + 1)(q - 1) + 1$ points in general position in $\mathbb{R}^d$. Then the following properties hold for the number $T(X)$ of Tverberg partitions:

i) $T(X)$ is even for $q > d + 1$.

ii) $T(X) \geq (q - d)!$

Property ii) improves the result of Theorem 4 for $d = 2$ and $q \geq 7$. Sierksma conjectured in 1979 that $T(X)$ is bounded from below by $((q - 1))!^d$. Combining Theorem 5 and methods from topological combinatorics, we have been able to confirm this conjecture for $d = 2$ and $q = 3$ in Hell [7], see also Hell [5].

In Section 2 we prove Theorem 2. Section 3 comes with a proof of Theorem 5.

2 On the number of Birch partitions

Figure 1 shows a Birch partition for the origin denoted as +. Each triangle corresponds to a partition block. There is another way to obtain a Birch partition for the origin in this example.

For $d = 1$, a Birch partition of a set $X$ of $2k$ points corresponds to $k$ intervals containing 0. Therefore $k$ points of $X$ are in $\mathbb{R}^+$, and $k$ many in $\mathbb{R}^-$. It is easy to check that there are exactly $k!$ ways to obtain a Birch partition. Hence we have settled Theorem 2 for $d = 1$. 
We now prove Theorem 2 for \( d \geq 2 \) in two steps: We first prove Property i), then we prove that Property i) implies Property ii).

In our proof, we make use of the following basic lemma; see e. g. Bárány and Matoušek \[1\], or Deza et al. \[4\] for a proof.

**Lemma 6.** If \( X \subset \mathbb{R}^d \) is a set of points in general position with respect to the origin 0 and \( p \in X \), then \( 0 \in \text{conv}(X) \) if and only if \( -p \in \text{cone}(X \setminus \{p\}) \).

The following lemma is an easy consequence of Lemma 6.

**Lemma 7.** Let \( X \) be a set of \( d+2 \) points in \( \mathbb{R}^d \) that is in general position with respect to the origin. Then the number of \( d \)-simplices with vertices in \( X \) that contain the origin is even. In fact, this number is either 0, or 2.

See Figure 2 for a configuration of four points in dimension \( d = 2 \) such that two triangles contain the origin +.

**Proof.** (of Theorem 2) We first prove Property i) for arbitrary \( d \geq 2 \), by induction on \( k \geq 2 \). The base case \( k = 2 \) is the key part.

\( k = 2 \): If all rays of the \( 2d+2 \) points of \( X \) intersect \( S^{d-1} \subset \mathbb{R}^d \) close to the north pole then \( B_0(X) = 0 \), as \( 0 \notin \text{conv}(X) \). We move one point \( p \) of \( X \) at a time while all other points remain fixed. The point \( p \) can be moved on its ray without changing \( B_0(X) \). Instead of following \( p \), we look at its antipode \( -p \) as for any \( d \)-element subset \( S \) of \( X \setminus \{p\} \) one has due to Lemma 6:

\[ 0 \in \text{conv}(S \cup \{p\}) \quad \text{iff} \quad -p \in \text{cone}(S). \]

Every \( d \)-element subset of \( X \setminus \{p\} \) defines a cone, and these cones define a decomposition of the sphere \( S^{d-1} \subset \mathbb{R}^d \) into cells. The boundary of a cell is defined by hyperplanes spanned by \( (d-1) \)-element subsets of \( X \setminus \{p\} \) and the origin. At some point we are forced to move \( -p \) transversally from one side of a boundary hyperplane defined by a \( (d-1) \)-element subset \( T \) to the other side. When \( -p \) crosses
such a hyperplane then \( B_0(X) \) might change. We show in the case distinction below that for every change the parity of \( B_0(X) \) does not change. The number \( B_0(X) \) is thus even as we can move every point of \( X \) to its position while fixing all other points. The cell decomposition during this process is nice: We can move \(-p\) to every position on the sphere while crossing hyperplanes in a transversal way.

Let us first look at the set of all \( d\)-simplices \( S \) spanned by \( d+1 \) points from \( X \) that contain the origin. If \(-p\) crosses the hyperplane through \( T \) transversally, this set might change. For this, put \( \tilde{T} = T \cup \{p\} \). For all simplices that do not contain \( T \) as a face of \( B \), this property switches: If \( S \) is of the form \( \tilde{T} \cup \{x\} \) for some \( x \in X \setminus \tilde{T} \), then this property switches:

\[
0 \in \text{conv}(S) \text{ before the crossing iff } 0 \notin \text{conv}(S) \text{ afterwards.}
\]

A Birch partition consists of a \( d\)-simplex \( S \) and its complement \( \tilde{S} \) in \( X \) – which is again a \( d\)-simplex – such that both contain the origin. The change of \( B_0(X) \) coming from the crossing of \(-p\) can thus only be affected by partitions that contain \( \tilde{T} \) as a face of \( S \), or of \( \tilde{S} \).

Case 1: The complements of all simplices using \( \tilde{T} \) do not contain the origin. \( B_0(X) \) does not change as the set of all Birch partition remains the same.

Case 2: Assume that \( \tilde{T} \) is not part of a \( d\)-simplex \( S \) such that \( \{S, \tilde{S}\} \) is a Birch partition, and that after the crossing of \(-p\) a Birch partition comes up. We show that Birch partitions come up in pairs.

Suppose there is a new Birch partition of the form \( S = \tilde{T} \cup \{x_1\} \) together with its complement \( \tilde{S} \). Due to Lemma 7, there is exactly two \( d\)-simplices in \( \tilde{S} \cup \{x_1\} \) such that both contain the origin. One of them is \( S \), let \( S^* \) be the other. By assumption \( 0 \notin S^* \) before the crossing of \(-p\). In fact, \( S^* = \tilde{T} \cup \{x_2\} \) for some \( x_2 \). The set \( \{S^*, S^*\} \) is thus our second Birch partition as \( 0 \in \text{conv}(S^*) \) afterwards.

Case 3: This is the inverse case of Case 2. Assume that there are exactly two Birch partitions of the form \( \tilde{T} \cup \{x_1\} \) and \( \tilde{T} \cup \{x_2\} \), with \( x_1, x_2 \in X \setminus \tilde{T} \), plus their complements before the crossing. Both of them vanish after crossing of \(-p\). New Birch partitions do not come up as for this we needed another \( \tilde{T} \cup \{x_3\} \) such that its complement contains the origin. This cannot exist due to Lemma 7.

Case 4: Assume there is exactly one Birch partition of the form \( S = \tilde{T} \cup \{x\} \), with \( x \in X \setminus \tilde{T} \), together with its complement before the crossing. This Birch partition vanishes, and a new one comes up.

One has \( 0 \notin S \) after the crossing of \(-p\) so that \( \{S, \tilde{S}\} \) vanishes. As in Case 2, there are exactly two \( d\)-simplices in \( \tilde{S} \cup \{x\} \) such that each contains the origin. One of them is \( \tilde{S} \), let \( S^* \) be the other. By assumption \( 0 \notin S^* \) before the crossing of \(-p\). In fact, \( S^* = \tilde{T} \cup \{x'\} \) for some \( x' \). The set \( \{S^*, S^*\} \) is thus the new Birch partition as \( 0 \in \text{conv}(S^*) \) afterwards.

Let now \( k \geq 3 \), and let \( p \) be a point in \( X \). Let \( F_1^{(1)}, F_1^{(2)}, \ldots, F_1^{(i)} \) be all \( d\)-simplices containing \( p \) that can be completed to a Birch partition of the origin into \( k \) subsets. For every \( F_i \), omitting \( F_i \) leads to a Birch partition into \( k-1 \) subsets. By induction hypothesis, there is an even number of Birch partitions into \( k-1 \) subsets for the restriction of every \( F_i \).

Now we assume Property 3, and derive Property 5 by induction on \( k \geq 2 \). The
case $k = 2$ is due to Property [1]: $B_0(X)$ is even, so

$$B_0(X) > 0 \implies B_0(X) \geq 2 = k!$$

Let $k \geq 3$ and $B_0(X) > 0$. Then there is a Birch partition $F_1, F_2, \ldots, F_k$. If we take any $k - 1$ of the $F_i$, they form again a Birch partition. By induction hypothesis, the union of $k - 1$ many $F_i$ has at least $(k - 1)!$ Birch partitions. In particular, there are $(k - 1)!$ many Birch partitions of $X$ into $k$ subsets that start with $F_1$. Let $p$ be an element of $F_1$.

For every pair $F_1, F_i$, for $i \in \{2, 3, \ldots, k\}$, one has again $B_0(F_1 \cup F_i) > 0$ and so there is a second Birch partition $\tilde{F}_1, \tilde{F}_i$ of $F_1 \cup F_i$. Assume without loss of generality $p \in \tilde{F}_1$. The $k$ sets $F_1, \tilde{F}_1, \tilde{F}_2, \ldots, \tilde{F}_k$ are pairwise distinct by construction. Every one of them contributes $(k - 1)!$ many Birch partitions of $X$ by induction hypothesis.

\[ \square \]

**Remark 8.** In the induction of our second step, we didn’t make use of convexity. The key is the base case $k = 2$:

$$B_0(X) > 0 \implies B_0(X) \geq 2.$$ 

**Remark.** Sierksma’s configuration shown for $d = 2$ and $q = 4$ in Figure 3 attains the conjectured upper bound $[1]$ for $B_0(X)$. Hence it would be maximal for the number of Birch partitions. At the same time, Sierksma conjectured it to be minimal for the number of Tverberg partitions.

\[ \text{Figure 3: A planar configuration with } 36 = (3!)^2 \text{ Birch resp. Tverberg partitions.} \]

\[ \text{3 On the number of Tverberg partitions} \]

In this section, we prove Theorem [3]. The proof is based on the fact that Birch partitions come up while studying Tverberg partitions.

Figure 3 shows a set $X$ of $(d + 1)(q - 1) + 1 = 10$ points in the plane for $q = 4$. A Tverberg partition can be read off as follows: Each triangle corresponds to a partition block. The point in the center is the forth block, and at the same time a Tverberg point.

In our proof, we need the following reformulation of Lemma 2.7 from Schöeneborn and Ziegler [12].

**Lemma 9.** Let $X$ be set of $(d + 1)(q - 1) + 1$ in general position in $\mathbb{R}^d$. Then a Tverberg partition consists of:

- **Type I:** One vertex $v$, and $(q - 1)$ many $d$-simplices containing $v$.
- **Type II:** $k$ intersecting simplices of dimension less than $d$, and $(q - k)$ $d$-simplices containing the intersection point for some $1 < k \leq \min\{d, q\}$.
For \( d = 2 \), a type II partition consist of two intersecting segments, and \( q - 2 \) many triangles containing their intersection point.

**Proof.** (of Theorem 5) Tverberg’s Theorem 3 implies the existence of a Tverberg partition together with a Tverberg point \( p \). The set \( X \) is in general position such that the partition is either of type I, or type II.

For type I, \( q - 1 \) disjoint \( d \)-simplices contain a point \( p \) of \( X \). The \( q - 1 \) disjoint \( d \)-simplices make up a Birch partition for \( p \). Theorem 2 implies that there are at least \( (q - 1)! \) many Birch partitions. Hence there are at least \( (q - 1)! \) many Tverberg partitions.

For type II, the Tverberg point \( p \) is the intersection of the convex hull of \( k \leq d \) many sets of cardinality at most \( d \). The remaining points are partitioned into \( q - k \) many \( d \)-simplices containing \( p \). For \( q > d + 1 \), this makes up a Birch partition for \( p \) into \( q - k \geq 2 \) sets. Again by Theorem 2 there are at least \( (q - k)! \) Tverberg partitions.

Properties i) and ii) follow from the corresponding results on the number of Birch partitions from Theorem 6. For \( q > d + 1 \), both types of Tverberg partitions correspond bijectively to Birch partitions so that the number of Tverberg partitions is even. As we can not predict the type of the Tverberg partition, the lower bound is equal to \( (q - d)! \).

**Remark.** 1. Our proof shows a bit more than a lower bound of \( (q - d)! \). If we knew what type of Tverberg partition showed up, then we would obtain \( (q - k)! \) for some \( k \in \{1, 2, \ldots, d\} \). If there is a Tverberg partition of type I then the lower bound equals \( (q - 1)! \).

2. In Hell [7], we improve the result of Theorem 5 by proving a lower bound for the number of Tverberg points, and by using Tverberg’s theorem with constraints.

## 4 Further directions

Motivated by recent work of Schöneborn and Ziegler [12], and Remark 8 we have also studied the concept of winding Birch partitions to obtain lower bounds in the topological setting, see Hell [5] for more details. The properties of Theorem 2 do not carry over to the topological setting. Hence a lower bound for the number of Tverberg partitions cannot be derived. A computer project led to many examples of piecewise linear maps that have exactly one winding Birch partition for \( k = 2 \); a smoothed version of one of them is shown in Figure 4. There the only winding Birch partition – shown in broken lines – is \{1, 2, 3\} and \{4, 5, 6\} with winding numbers \( \pm 1 \) resp. \( \pm 2 \). For arbitrary dimension \( d \geq 2 \), note that any example for dimension \( d \) can be extended to an example in dimension \( d + 1 \) using a construction from Schöneborn and Ziegler [12].

Let us end with two problems. Both are promising starting points for future research.

**Problem.** Relate the properties on the number \( B_p(X) \) of Birch partitions to polytope theory. Birch partitions show up while studying Gale diagrams; see Ziegler’s textbook [16] for an introduction to Gale diagrams. In fact, a set \( X \) of \( k(d + 1) \) points in \( \mathbb{R}^d \) with \( B_0(X) > 0 \) corresponds to a Gale diagram of a \( k \)-neighborly \( (k - 1)(d + 1) \)-dimensional simplicial polytope on \( k(d + 1) \) vertices.

**Problem.** It is well-known that Radon’s, Helly’s, and Carathéodory’s theorem are closely related. Do the results on the number of Birch partitions imply new Helly-type, or Carathéodory-type results?
Figure 4: $K_6$ with exactly one winding Birch partition.

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