Proof of the cosmic no-hair conjecture in the $T^3$-Gowdy symmetric Einstein-Vlasov setting

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Abstract

The currently preferred models of the universe undergo accelerated expansion induced by dark energy. One model for dark energy is a positive cosmological constant. It is consequently of interest to study Einstein’s equations with a positive cosmological constant coupled to matter satisfying the ordinary energy conditions; the dominant energy condition etc. Due to the difficulty of analysing the behaviour of solutions to Einstein’s equations in general, it is common to either study situations with symmetry, or to prove stability results. In the present paper, we do both. In fact, we analyse, in detail, the future asymptotic behaviour of $T^3$-Gowdy symmetric solutions to the Einstein-Vlasov equations with a positive cosmological constant. In particular, we prove the cosmic no-hair conjecture in this setting. However, we also prove that the solutions are future stable (in the class of all solutions). Some of the results hold in a more general setting. In fact, we obtain conclusions concerning the causal structure of $T^2$-symmetric solutions, assuming only the presence of a positive cosmological constant, matter satisfying various energy conditions and future global existence. Adding the assumption of $T^3$-Gowdy symmetry to this list of requirements, we obtain $C^0$-estimates for all but one of the metric components. There is consequently reason to expect that many of the results presented in this paper can be generalised to other types of matter.

1 Introduction

At the end of 1998, two research teams studying supernovae of type Ia announced the unexpected conclusion that the universe is expanding at an accelerating rate; cf. [24, 17]. After the observations had been corroborated by other sources, there was a corresponding shift in the class of solutions to Einstein’s equations used to model the universe. In particular, physicists attributed the acceleration to a form of matter they referred to as ‘dark energy’. However, as the nature of the dark energy remains unclear, there are several models for it. The simplest one is that of a positive cosmological constant (which is the one we use in the present paper), but there are several other possibilities; cf., e.g., [21, 22, 23] and references cited therein for some examples. Combining the different observational data, the currently preferred model of the universe is spatially homogeneous and isotropic (i.e., the cosmological principle is assumed to be valid), spatially flat, and has matter of the following forms: ordinary matter (usually modelled by a radiation fluid and dust), dark matter (often modelled by dust), and dark energy (often modelled by a positive cosmological constant). In the present paper, we are interested in the Einstein-Vlasov system. This corresponds to a different description of the matter than the one usually used. However, this system can also be used in order to obtain models consistent with observations; cf., e.g., [27, Chapter 28]. In fact, Vlasov matter has the property that it naturally behaves as radiation close to the singularity and as dust in the expanding direction, a desirable feature which is usually put in by hand when using perfect fluids to model the matter.

The cosmic no-hair conjecture. The standard starting point in cosmology is the assumption of spatial homogeneity and isotropy. However, it is preferable to prove that solutions generally
isotropise and that the spatial variation (as seen by observers) becomes negligible. This is expected to happen in the presence of a positive cosmological constant; in fact, solutions are in that case expected to appear de Sitter like to observers at late times. The latter expectation goes under the name of the cosmic no-hair conjecture; cf. Conjecture 11 for a precise formulation. The main objective when studying the expanding direction of solutions to Einstein’s equations with a positive cosmological constant is to verify this conjecture.

**Spatial homogeneity.** Turning to the results that have been obtained in the past, it is natural to begin with the spatially homogeneous setting. In 1983, Robert Wald wrote a short, but remarkable, paper [37], in which he proves results concerning the future asymptotic behaviour of spatially homogeneous solutions to Einstein’s equations with a positive cosmological constant. In particular, he confirms that the cosmic no-hair conjecture holds. What is remarkable about the paper is the fact that he is able to obtain conclusions assuming only that certain energy conditions hold and that the solution does not break down in finite time. Concerning the symmetry type, the only issue that comes up in the argument is whether it is compatible with the spatial hypersurfaces of homogeneity having positive scalar curvature or not; positive scalar curvature of these hypersurfaces sometimes leads to recollapse. The results should be contrasted with the case of Einstein’s vacuum equation in the spatially homogeneous setting, where the behaviour is strongly dependent on the symmetry type. Since Wald does not prove future global existence, it is necessary to carry out a further analysis in order to confirm the picture obtained in [37] in specific cases. In the case of the Einstein-Vlasov system, this was done in [12]. It is also of interest to note that it is possible to prove results analogous to those of Wald for more general models for dark energy; cf., e.g., [21, 22, 23, 13].

**Surface symmetry.** Turning to the spatially inhomogeneous setting, there are results in the surface symmetric case with a positive cosmological constant; cf. [34, 35, 36, 14], and see [19] for a definition of surface symmetry. In this case, the isometry group (on a suitable covering space) is 3-dimensional. Nevertheless, the system of equations that result after symmetry reduction is 1 + 1-dimensional. However, the extra symmetries do eliminate some of the degrees of freedom. Again, the main results are future causal geodesic completeness and a verification of the cosmic no-hair conjecture.

**T^2-symmetry.** A natural next step to take after surface symmetry is to consider Gowdy or T^2-symmetry. That is the purpose of the present paper. In particular, we prove future causal geodesic completeness of solutions to the T^3-Gowdy symmetric Einstein-Vlasov equations with a positive cosmological constant (note, however, the caveat concerning global existence stated in Subsection 1.1). Moreover, we verify that the cosmic no-hair conjecture holds. It is of interest to note that most of the arguments go through under the assumption of T^2-symmetry. However, in order to obtain the full picture in this setting, it is necessary to prove one crucial inequality, cf. Definition 1 which we have not yet been able to do in general.

**Stability.** A fundamental question in the study of cosmological solutions is that of future stability: given initial data corresponding to an expanding solution, do small perturbations thereof yield maximal globally hyperbolic developments which are future causally geodesically complete and globally similar to the future? In the case of a positive cosmological constant, the first result was obtained by Helmut Friedrich; he proved stability of de Sitter space in 3 + 1 dimensions in [9]. Later, he and Michael Anderson generalised the result to higher (even) dimensions and to include various matter fields; cf. [10, 11]. Moreover, results concerning radiation fluids were obtained in [15]. However, conformal invariance plays an important role in the arguments presented in these papers. As a consequence, there seems to be a limitation of the types of matter models that can be dealt with using the corresponding methods. The paper [25] was written with the goal of developing methods that are more generally applicable. The papers [26, 33, 29, 31, 32, 11], in which the methods developed in [25] play a central role, indicate that this goal was achieved. In fact, a general future global non-linear stability result for spatially homogeneous solutions to the Einstein-Vlasov equations with a positive cosmological constant was obtained in [27], the ideas developed in [25] being at the core of the argument. In the present paper, we not only derive
detailed future asymptotics of $\mathbb{T}^3$-Gowdy symmetric solutions to the Einstein-Vlasov equations with a positive cosmological constant. We also prove that all the resulting solutions are future stable in the class of all solutions (without symmetry assumptions).

**Outlook.** As we describe in the next subsection, some of the results concerning $\mathbb{T}^3$-Gowdy symmetric solutions hold irrespective of the matter model (as long as it satisfies the dominant energy condition and the non-negative pressure condition). As a consequence, we expect that it might be possible to derive detailed asymptotics in the case of the Einstein-Maxwell equations (with a positive cosmological constant), and in the case of the Einstein-Euler system (though the issue of shocks may be relevant in the latter case). Due to the stability results demonstrated in [33, 29, 31, 32], it might also be possible to prove stability of the corresponding solutions.

1.1 General results under the assumption of $\mathbb{T}^2$-symmetry

**$\mathbb{T}^2$-symmetry.** In the present paper, we are interested in $\mathbb{T}^2$-symmetric solutions to Einstein’s equations. There are various geometric ways of imposing this type of symmetry (cf., e.g., [6, 31]), but for the purposes of the present paper, we simply assume the topology to be of the form $I \times \mathbb{T}^3$, where $I$ is an open interval contained in $(0, \infty)$. If $\theta, x$ and $y$ are ‘coordinates’ on $\mathbb{T}^3$ and $t$ is the coordinate on $I$, we also assume the metric to be of the form

$$g = t^{-1/2}e^{J/2}(-dt^2 + \alpha^{-1}d\theta^2) + te^P(dx + Qdy + (G + QH)d\theta)^2 + te^{-P}(dy + Hd\theta)^2,$$

(1)

where the functions $\alpha > 0$, $\lambda$, $P$, $Q$, $G$ and $H$ only depend on $t$ and $\theta$; cf., e.g., [30]. Note that translation in the $x$ and $y$ directions defines a smooth action of $\mathbb{T}^2$ on the spacetime (as well as on each constant $t$-hypersurface). Moreover, the metric is invariant under this action, and the corresponding orbits are referred to as the symmetry orbits, given by $\{t\} \times \{\theta\} \times \mathbb{T}^2$. Note that the area of the symmetry orbits is proportional to $t$. For this reason, the foliation of the spacetime corresponding to the metric form (1) is referred to as the constant areal time foliation. The case of $\mathbb{T}^3$-Gowdy symmetry corresponds to the functions $G$ and $H$ being independent of time; again, there is a more geometric way of formulating this condition: the spacetime is said to be Gowdy symmetric if the so-called twist quantities, given by

$$J = \epsilon_{\alpha\beta\gamma\delta}X^\alpha Y^\beta \nabla^\gamma X^\delta, \quad K = \epsilon_{\alpha\beta\gamma\delta}X^\alpha Y^\beta \nabla^\gamma Y^\delta,$$

(2)

vanish, where $X = \partial_x$ and $Y = \partial_y$ are Killing fields of the above metric and $\epsilon$ is the volume form. A basic question to ask concerning $\mathbb{T}^2$-symmetric solutions to Einstein’s equations is whether the maximal globally hyperbolic development of initial data admits a constant areal time foliation which is future global. There is a long history of proving such results. The first one was obtained by Vincent Moncrief, cf. [16], in the case of vacuum solutions with $\mathbb{T}^3$-Gowdy symmetry. The case of $\mathbb{T}^2$-symmetric vacuum solutions with and without a positive cosmological constant have also been considered in [7] and [5], respectively. Turning to Vlasov matter, [2] contains an analysis of the existence of foliations in the $\mathbb{T}^3$-Gowdy symmetric Einstein-Vlasov setting. The corresponding results were later extended to the $\mathbb{T}^2$-symmetric case in [3]. However, from our point of view, the most relevant result is that of [30]. Due to the results of this paper, there is, given $\mathbb{T}^2$-symmetric initial data to the Einstein-Vlasov equations with a positive cosmological constant, a future global foliation of the spacetime of the form (1). In other words $I = (t_0, \infty)$. Moreover, if the distribution function is not identically zero, then $t_0 = 0$. Finally, if the initial data have Gowdy symmetry, then the same is true of the development. Strictly speaking, the future global existence result in [30] is based on the observation that the argument should not be significantly different from the proofs in [5, 7, 3]. It would be preferable to have a complete proof of future global existence in the case of interest here, but we shall not provide it in this paper.

**Results.** Turning to the results, it is of interest to note that some of the conclusions can be obtained without making detailed assumptions concerning the matter content. For that reason, let us, for the remainder of this subsection, assume that we have a solution to Einstein’s equations
with a positive cosmological constant, where the metric is of the form \( \mathbb{I} \), the existence interval \( I \) is of the form \((t_0, \infty)\) and the matter satisfies the dominant energy condition and the non-negative pressure condition; recall that the matter is said to satisfy the dominant energy condition if \( T(u, v) \geq 0 \) for all pairs \( u, v \) of future directed timelike vectors (where \( T \) is the stress energy tensor associated with the matter); and that it is said to satisfy the non-negative pressure condition if \( T(w, w) \geq 0 \) for every spacelike vector \( w \). To begin with, there is a constant \( C > 0 \) such that \( \alpha(t, \theta) \leq Ct^{-3} \) for all \((t, \theta) \in [t_0 + 2, \infty) \times S^1\); cf. Proposition \( \text{32} \). In fact, this conclusion also holds if we replace the cosmological constant with a non-linear scalar field with a positive lower bound; cf. Remark \( \text{33} \). One particular consequence of this estimate for \( \alpha \) is that the \( \theta \)-coordinate of a causal curve converges. Moreover, observers whose \( \theta \)-coordinates converge to different \( \theta \)-values are asymptotically unable to communicate. In this sense, there is asymptotic silence. In the case of Gowdy symmetry, more can be deduced. In fact, for every \( \epsilon > 0 \), there is a \( T > t_0 \) such that

\[
\lambda(t, \theta) \geq -3 \ln t + 2 \ln \left( \frac{3}{4\Lambda} \right) - \epsilon
\]

for all \((t, \theta) \in [T, \infty) \times S^1\); cf. Proposition \( \text{34} \). This estimate turns out to be of crucial importance also in the general \( T^2 \)-symmetric case. For this reason, we introduce the following terminology.

**Definition 1.** A metric of the form \( \mathbb{I} \) which is defined for \( t > t_0 \) for some \( t_0 \geq 0 \) is said to have \( \lambda \)-asymptotics if there, for every \( \epsilon > 0 \), is a \( T > t_0 \) such that

\[
\lambda(t, \theta) \geq -3 \ln t + 2 \ln \left( \frac{3}{4\Lambda} \right) - \epsilon
\]

for all \((t, \theta) \in [T, \infty) \times S^1\).

**Remark 2.** All Gowdy solutions have \( \lambda \)-asymptotics under the above assumptions.

**Proposition 3.** Consider a \( T^2 \)-symmetric solution to Einstein’s equations with a positive cosmological constant. Assume that the matter satisfies the dominant energy condition and the non-negative pressure condition. Assume, moreover, that the corresponding metric admits a foliation of the form \( \mathbb{I} \), on \( I \times T^3 \), where \( I = (t_0, \infty) \) and \( t_0 \geq 0 \). Finally, assume that the solution has \( \lambda \)-asymptotics and let \( t_1 = t_0 + 2 \). Then there is a constant \( C > 0 \) such that

\[
\begin{align*}
\left\| \lambda(t, \cdot) + 3 \ln t - 2 \ln \left( \frac{3}{4\Lambda} \right) \right\|_{C^0} &\leq Ct^{-1/2}, \\
\left\| Q(t, \cdot) \right\|_{C^0} + \left\| P(t, \cdot) \right\|_{C^0} &\leq C, \\
\left\| H_i(t, \cdot) \right\|_{L^1} + \left\| G_i(t, \cdot) \right\|_{L^1} &\leq Ct^{-3/2}
\end{align*}
\]

for all \((t, \theta) \in [t_1, \infty) \times S^1\).

**Remark 4.** The choice \( t_1 = t_0 + 2 \) may seem unnatural. However, we need to stay away from \( t_0 \) (since we do not control the solution close to \( t_0 \)). Moreover, in some situations we need to know that \( \ln t \) is positive and bounded away from zero. Since \( t_0 = 0 \) for most solutions, it is therefore natural to only consider the interval \( t \geq t_0 + 2 \) in the study of the future asymptotics.

**Remark 5.** If \( h \) is a scalar function on \( S^1 \), we use the notation

\[
\langle h \rangle = \frac{1}{2\pi} \int_{S^1} h \, d\theta.
\]

Sometimes, we shall use the same notation for a scalar function \( h \) on \( I \times S^1 \). In that case, \( \langle h \rangle \) is the function of \( t \) defined by \( \langle h(t, \cdot) \rangle \). Finally, if \( \bar{p} \in \mathbb{R}^3 \), we shall also use the notation \( \langle \bar{p} \rangle \). However, in that case, \( \langle \bar{p} \rangle = (1 + |\bar{p}|^2)^{1/2} \); cf. Remark \( \text{18} \).

**Proof.** The statement is a consequence of Lemmas \( \text{96} \), \( \text{37} \) and \( \text{68} \).

In particular, in the case of a \( T^3 \)-Gowdy symmetric solution, there is asymptotic silence in the sense that the \( \theta xy \)-coordinates of a causal curve converge, and causal curves whose asymptotic \( \theta xy \)-coordinates differ are asymptotically unable to communicate; cf. Proposition \( \text{99} \).
1.2 Results in the Einstein-Vlasov setting

In order to be able to draw detailed conclusions, we need to restrict our attention to a specific type of matter. In the present paper, we study the Einstein-Vlasov system.

A general description of Vlasov matter. Intuitively, Vlasov matter gives a statistical description of an ensemble of collections of particles. In practice, the matter is described by a distribution function defined on the space of states of particles. The possible states are given by the future directed causal vectors (here and below, we assume the Lorentz manifolds under consideration to be time oriented). Usually, one distinguishes between massive and massless particles. In the former case, the distribution function is defined on the future light cone, and in the latter case, it is defined on the interior. In the present paper, we are interested in the massive case, and we assume all the particles to have unit mass (under reasonable circumstances, it is possible to reduce the case of varying masses to the case of all particles having unit mass). As a consequence, the distribution function is a non-negative function on the mass shell \( P \), defined to be the set of future directed unit timelike vectors. In order to connect the matter to Einstein’s equations, we need to associate a stress energy tensor with the distribution function. It is given by

\[
T^{Vl}_{\alpha\beta}(\xi) = \int_{P_\xi} f p_\alpha p_\beta \mu_{P_\xi}.
\]  

In this expression, \( P_\xi \) denotes the set of future directed unit timelike vectors based at the spacetime point \( \xi \). Moreover, the Lorentz metric \( g \) induces a Riemannian metric on \( P_\xi \), and \( \mu_{P_\xi} \) denotes the corresponding volume form. Finally, \( p_\alpha \) denotes the components of the one form obtained by lowering the index of \( p \in P_\xi \) using the Lorentz metric \( g \). Clearly, it is necessary to demand some degree of fall off of the distribution function \( f \) in order for the integral (3) to be well defined. In the present paper, we shall mainly be interested in the case that the distribution function has compact support in the momentum directions (for a fixed spacetime point). However, in Subsection 1.3 we shall consider a somewhat more general situation. Turning to the equation the distribution function has to satisfy, it is given by

\[
\mathcal{L} f = 0.
\]  

Here \( \mathcal{L} \) denotes the vector field induced on the mass shell by the geodesic flow. An alternate way to formulate this equation is to demand that \( f \) be constant along \( \dot{\gamma} \) for every future directed unit timelike geodesic \( \gamma \). The intuitive interpretation of the Vlasov equation (4) is that collisions between particles are neglected. It is of interest to note that if \( f \) satisfies the Vlasov equation, then the stress energy tensor is divergence free. To conclude, the Einstein-Vlasov equations with a positive cosmological constant consist of (4) and

\[
G + \Lambda g = T,
\]  

where \( T \) is given by the right hand side of (3) and \( \Lambda \) is a positive constant. Moreover, \( G \) is the Einstein tensor of a Lorentz manifold \( (M, g) \). The above description is somewhat brief, and the reader interested in more details is referred to, e.g., [8, 20] [1] [27].

Vlasov matter under the assumption of \( T^2 \)-symmetry. In the case of \( T^2 \)-symmetry, it is convenient to use a symmetry reduced version of the distribution function. Introduce, to this end, the orthonormal frame

\[
\begin{align*}
  e_0 &= t^{1/4} e^{-\lambda/4} \partial_t, & e_1 &= t^{1/4} e^{-\lambda/4} a^{1/2} (\partial_\theta - G \partial_x - H \partial_y), \\
  e_2 &= t^{-1/2} e^{-P/2} \partial_x, & e_3 &= t^{-1/2} e^{P/2} (\partial_y - Q \partial_x).
\end{align*}
\]  

Since the distribution function \( f \) is defined on the mass shell, it is convenient to parametrise this set; note that the manifolds we are interested in here are parallelisable. An element in \( P \) can be written \( v^\alpha \epsilon_\alpha \), where

\[
v^0 = [1 + (v^1)^2 + (v^2)^2 + (v^3)^2]^{1/2}.
\]
As a consequence, we can think of \( f \) as depending on \( v^i, i = 1, 2, 3, \) and the base point. However, due to the symmetry requirements, the distribution function only depends on the \( t\theta\)-coordinates of the base point. As a consequence, the distribution function can be considered to be a function of \((t, \theta, v)\), where \( v = (v^1, v^2, v^3)\). In what follows, we shall abuse notation and denote the symmetry reduced function, defined on \( I \times S^1 \times \mathbb{R}^3 \), by \( f \). A symmetry reduced version of the equations is to be found in Section 2.

**Remark 6.** In the \( \mathbb{T}^2 \)-symmetric setting, we always assume the distribution function to have compact support when restricted to constant \( t \)-hypersurfaces.

The first question to ask concerning \( \mathbb{T}^2 \)-symmetric solutions is that of existence of constant areal time foliations for an interval of the form \((t_0, \infty)\). However, due to previous results, cf. \( \text{[30]} \), we know that \( \mathbb{T}^2 \)-symmetric solutions to the Einstein-Vlasov equations with a positive cosmological constant are future global in this setting (keeping the caveat stated in Subsection 1.1 in mind). In other words, there is a \( t_0 \geq 0 \) such that the solution admits a foliation of the form \( \text{[1]} \) on \( I \times \mathbb{T}^3 \), where \( I = (t_0, \infty) \). Consequently, the issue of interest here is that of the asymptotics. Unfortunately, we are unable to derive detailed asymptotics for all \( \mathbb{T}^2 \)-symmetric solutions. However, we do obtain results for solutions with \( \lambda \)-asymptotics; recall that all \( \mathbb{T}^3 \)-Gowdy symmetric solutions fall into this class.

**Theorem 7.** Consider a \( \mathbb{T}^2 \)-symmetric solution to the Einstein-Vlasov system with a positive cosmological constant. Choose coordinates so that the corresponding metric takes the form \( \text{[1]} \) on \( I \times \mathbb{T}^3 \), where \( I = (t_0, \infty) \). Assume that the solution has \( \lambda \)-asymptotics and let \( t_1 = t_0 + 2 \). Then there are smooth functions \( \alpha_{\infty} > 0, P_{\infty}, Q_{\infty}, G_{\infty} \) and \( H_{\infty} \) on \( S^1 \), and, for every \( 0 \leq N \in \mathbb{Z} \), a constant \( C_N > 0 \) such that

\[
\begin{align*}
&\|H_t(t, \cdot)\|_{C_N} + t\|G_t(t, \cdot)\|_{C_N} + \|H(t, \cdot) - H_{\infty}\|_{C_N} + \|G(t, \cdot) - G_{\infty}\|_{C_N} \leq C_N t^{-3/2}, \\
&\|P_t(t, \cdot)\|_{C_N} + t\|Q_t(t, \cdot)\|_{C_N} + \|P(t, \cdot) - P_{\infty}\|_{C_N} + \|Q(t, \cdot) - Q_{\infty}\|_{C_N} \leq C_N t^{-1}, \\
&\left\|\frac{\alpha}{t} + \frac{3}{t}\right\|_{C_N} + \left\|\lambda t + \frac{3}{4} \frac{3}{4A_{\infty}}\right\|_{C_N} \leq C_N t^{-2},
\end{align*}
\]

for all \( t \geq t_1 \). Define \( f_{\text{sc}} \) via \( f_{\text{sc}}(t, \theta, v) = f(t, \theta, t^{-1/2}v) \). Then there is an \( R > 0 \) such that

\[
\text{supp}\, f_{\text{sc}}(t, \cdot) \subseteq S^1 \times B_R(0)
\]

for all \( t \geq t_1 \), where \( B_R(0) \) is the ball of radius \( R \) in \( \mathbb{R}^3 \). Moreover, there is a smooth, non-negative function with compact support, say \( f_{\text{sc}, \infty} \), on \( S^1 \times \mathbb{R}^3 \), such that

\[
\begin{align*}
&\|\partial_t f_{\text{sc}}(t, \cdot)\|_{C_N(S^1 \times \mathbb{R}^3)} + \|f_{\text{sc}}(t, \cdot) - f_{\text{sc}, \infty}\|_{C_N(S^1 \times \mathbb{R}^3)} \leq C_N t^{-1}
\end{align*}
\]

for all \( t \geq t_1 \). Turning to the geometry, let \( \bar{g}(t, \cdot) \) and \( \bar{k}(t, \cdot) \) denote the metric and second fundamental form induced by \( g \) on the hypersurface \( \{t\} \times \mathbb{T}^3 \), and let \( \bar{g}_{ij}(t, \cdot) \) denote the components of \( \bar{g}(t, \cdot) \) with respect to the vectorfields \( \partial_1 = \partial_\theta, \partial_2 = \partial_x \) and \( \partial_3 = \partial_y \) etc. Then

\[
\begin{align*}
&\|t^{-1} \bar{g}_{ij}(t, \cdot) - \bar{g}_{\infty, ij}\|_{C_N} + \|t^{-1} \bar{k}_{ij} - H \bar{g}_{\infty, ij}\|_{C_N} \leq C_N t^{-1} \\
&\text{for all } t \geq t_1, \text{ where } H = (\chi/3)^{1/2}
\end{align*}
\]

and

\[
\bar{g}_{\infty} = \frac{3}{4A_{\infty}} \, d\theta^2 + e^{P_{\infty}} \left[ dx + Q_{\infty} dy + (G_{\infty} + Q_{\infty} H_{\infty}) d\theta \right]^2 + e^{-P_{\infty}} (dy + H_{\infty} d\theta)^2.
\]

Moreover, the solution is future causally geodesically complete.
The proof of the above theorem is to be found in Section 10.

It is of interest to record how the spacetime appears to an observer. In particular, we wish to prove the cosmic no-hair conjecture in the present setting. The rough statement of this conjecture is that the spacetime appears de Sitter like to late time observers. However, in order to be able to state a theorem, we need a formal definition. Before proceeding to the details, let us provide some intuition. Let

\[ g_{\text{dS}} = -dt^2 + e^{2rt}g_E, \]

where \( H = (\Lambda/3)^{1/2} \) and \( g_E \) denotes the standard flat Euclidean metric. Then \((\mathbb{R}^3, g_{\text{dS}})\) corresponds to a part of de Sitter space. It may seem more reasonable to consider de Sitter space itself. However, as far as the asymptotic behaviour of de Sitter space is concerned, \((\mathbb{R}^4, g_{\text{dS}})\) is as good a model as de Sitter space itself. Consider a future directed and inextendible causal curve in \((\mathbb{R}^4, g_{\text{dS}})\), say \( \gamma = (\gamma^0, \gamma) \), defined on \((s_-, s_+)\). Then \( \gamma(s) \) converges to some \( \bar{x}_0 \in \mathbb{R}^3 \) as \( s \to s_- \). Moreover, \( \gamma(s) \in C_{\bar{x}_0,\Lambda} \) for all \( s \), where

\[ C_{\bar{x}_0,\Lambda} = \{(t, \bar{x}) : |\bar{x} - \bar{x}_0| \leq H^{-1}e^{-Ht}\}. \]

In practice, it is convenient to introduce a lower bound on the time coordinate and to introduce a margin in the spatial direction. Moreover, it is convenient to work with open sets. We shall therefore be interested in sets of the form

\[ C_{\Lambda,K,T} = \{(t, \bar{x}) : t > T, \ |\bar{x}| < KH^{-1}e^{-HT}\}; \]

note that \( \bar{x}_0 \) can be translated to zero by an isometry. Since we are interested in the late time behaviour of solutions, it is natural to restrict attention to sets of the form \( C_{\Lambda,K,T} \) for some \( K \geq 1 \) and \( T > 0 \).

**Definition 8.** Let \((M, g)\) be a time oriented, globally hyperbolic Lorentz manifold which is future causally geodesically complete. Assume, moreover, that \((M, g)\) is a solution to Einstein’s equations with a positive cosmological constant \( \Lambda \). Then \((M, g)\) is said to be future asymptotically de Sitter like if there is a Cauchy hypersurface \( \Sigma \) in \((M, g)\) such that for every future oriented and inextendible causal curve \( \gamma \) in \((M, g)\), the following holds:

- there is an open set \( D \) in \((M, g)\), such that \( J^-(\gamma) \cap J^+(\Sigma) \subset D \), and \( D \) is diffeomorphic to \( C_{\Lambda,K,T} \) for a suitable choice of \( K \geq 1 \) and \( T > 0 \);
- using \( \psi : C_{\Lambda,K,T} \to D \) to denote the diffeomorphism; letting \( R(t) = KH^{-1}e^{-HT} \); using \( \bar{g}_{\text{dS}}(t, \cdot) \) and \( \bar{k}_{\text{dS}}(t, \cdot) \) to denote the metric and second fundamental form induced on \( S_t = \{t\} \times B_{R(t)}(0) \) by \( g_{\text{dS}} \); using \( \bar{g}(t, \cdot) \) and \( \bar{k}(t, \cdot) \) to denote the metric and second fundamental form induced on \( S_t \) by \( \psi^*g \); and letting \( N \in \mathbb{N} \), the following holds:

\[
\lim_{t \to \infty} \left( \|\bar{g}_{\text{dS}}(t, \cdot) - \bar{g}(t, \cdot)\|_{C^N_{\text{dS}}(S_t)} + \|\bar{k}_{\text{dS}}(t, \cdot) - \bar{k}(t, \cdot)\|_{C^N_{\text{dS}}(S_t)} \right) = 0. \tag{15}
\]

**Remark 9.** In the definition, we use the notation

\[
\|h\|_{C^N_{\text{dS}}(S_t)} = \left( \sup_{S_t} \sum_{i=0}^{N} \bar{g}_{\text{dS},s_{i_1j_{i_1}}} \cdots \bar{g}_{\text{dS},s_{i_Nj_{i_N}}} \bar{g}_{\text{dS}} \bar{g}_{\text{dS}} \nabla_{\text{dS}}^{j_1} \cdots \nabla_{\text{dS}}^{j_N} h_{ij} \nabla_{\text{dS}}^{j_1} \cdots \nabla_{\text{dS}}^{j_N} h_{mn} \right)^{1/2}
\]

for a covariant 2-tensor field \( h \) on \( S_t \), where \( \nabla_{\text{dS}} \) denotes the Levi-Civita connection associated with \( \bar{g}_{\text{dS}}(t, \cdot) \).

**Remark 10.** In some situations it might be more appropriate to adapt the Cauchy hypersurface \( \Sigma \) to the causal curve \( \gamma \); i.e., to first fix \( \gamma \) and then \( \Sigma \).

The above definition leads to a formal statement of the cosmic no-hair conjecture.
Conjecture 11 (Cosmic no-hair). Let \( \mathcal{A} \) denote the class of initial data such that the corresponding maximal globally hyperbolic developments (MGHD’s) are future causally geodesically complete solutions to Einstein’s equations with a positive cosmological constant \( \Lambda \) (for some fixed matter model). Then generic elements of \( \mathcal{A} \) yield MGHD’s that are future asymptotically de Sitter like.

Remark 12. It is probably necessary to exclude certain matter models in order for the statement to be correct. Moreover, the statement, as it stands, is quite vague; there is no precise definition of the notion generic. However, what notion of genericity is most natural might depend on the situation.

Remark 13. The Nariai spacetimes, discussed, e.g., in [25, pp. 126-127], are time oriented, globally hyperbolic, causally geodesically complete solutions to Einstein’s vacuum equations with a positive cosmological constant that do not exhibit future asymptotically de Sitter like behaviour. They are thus potential counterexamples to the cosmic no-hair conjecture. There is a similar example in the Einstein-Maxwell setting (with a positive cosmological constant) in [25, p. 127]. However, both of these examples are rather special, and it is natural to conjecture them to be unstable. Nevertheless, they constitute the motivation for demanding genericity.

Finally, we are in a position to phrase a result concerning the cosmic no hair conjecture in the \( \mathbb{T}^3 \)-Gowdy symmetric setting. The proof of the theorem below is to be found in Section 10.

Theorem 14. Consider a \( \mathbb{T}^3 \)-symmetric solution to the Einstein-Vlasov system with a positive cosmological constant. Choose coordinates so that the corresponding metric takes the form (1) on \( I \times \mathbb{T}^3 \), where \( I = (t_0, \infty) \). Assume that the solution has \( \lambda \)-asymptotics. Then the solution is future asymptotically de Sitter like; i.e., the cosmic no-hair conjecture holds.

Remark 15. Recall that all \( \mathbb{T}^3 \)-Gowdy symmetric solutions have \( \lambda \)-asymptotics.

Remark 16. In the particular case of interest here, the equality (15) can actually be improved to the estimate
\[
\| \mathcal{g}_{ds}(\tau, \cdot) - \mathcal{g}(\tau, \cdot) \|_{C^1_{ds}(S_\tau)} + \| \mathcal{K}_{ds}(\tau, \cdot) - \mathcal{K}(\tau, \cdot) \|_{C^1_{ds}(S_\tau)} \leq C_N e^{-2H\tau}
\]
for all \( \tau > T \) and a suitable constant \( C_N \).

1.3 Stability of \( \mathbb{T}^3 \)-Gowdy symmetric solutions

Let us now turn to the subject of stability. Combining Theorem 4 with the results of [27], it turns out to be possible to prove that the solutions to which Theorem 2 applies are also future stable. Let us begin by introducing the terminology necessary in order to make a formal statement of this result. Let \( (M, g) \) be a time oriented \( n + 1 \)-dimensional Lorentz manifold. We say that \( (x, U) \) are canonical local coordinates if \( \partial_{\phi_i} \) is future oriented timelike on \( U \) and \( g(\partial_{\phi_i} |_{\xi}, \partial_{\phi_j} |_{\xi}) \), \( i, j = 1, \ldots, n \), are the components of a positive definite metric for every \( \xi \in U \); cf. [27, p. 87]. If \( p \in P_{\xi} \) for some \( \xi \in U \), we then define
\[
\Xi_x(p) = \Xi_x(p^a \partial_{\phi_i} |_{\xi}) = [x(\xi), \bar{p}],
\]
where \( \bar{p} = (p^1, \ldots, p^n) \). Note that \( \Xi_x \) are local coordinates on the mass shell. If \( f \) is defined on the mass shell, we shall use the notation \( f_\xi = f \circ \Xi^{-1}_x \). Let us recall [27, Definition 7.1, p. 87]:

Definition 17. Let \( 1 \leq n \leq \infty \), \( \mu \in \mathbb{R} \), \( (M, g) \) be a time oriented \( n + 1 \)-dimensional Lorentz manifold and \( P \) be the set of future directed unit timelike vectors. The space \( \mathcal{D}_\mu^\infty(P) \) is defined to consist of the smooth functions \( f : P \to \mathbb{R} \) such that, for every choice of canonical local coordinates \( (x, U) \), \( n + 1 \)-multiindex \( \alpha \) and \( n \)-multiindex \( \beta \), the derivative \( \partial_{\phi_i}^\alpha \partial_{\phi_j}^\beta f_\xi \) (where \( x \) symbolises the first \( n + 1 \) and \( \bar{p} \) the last \( n \) variables), considered as a function from \( x(U) \) to the set of functions from \( \mathbb{R}^n \) to \( \mathbb{R} \), belongs to
\[
C^\infty_{x(U), \mathcal{D}_\mu^\infty([\mathbb{R}^n])}.
\]
Remark 18. The space $L^2_\mu(\mathbb{R}^n)$ is the weighted $L^2$-space corresponding to the norm

$$\|h\|_{L^2_\mu} = \left( \int_{\mathbb{R}^n} (|\bar{\rho}|^{2\mu} |h(\bar{\rho})|^2 d\bar{\rho}) \right)^{1/2}, \tag{18}$$

where $(\bar{\rho}) = (1 + |\bar{\rho}|^2)^{1/2}$; recall the comments made in Remark 5.

Remarks 19. If $f \in \mathcal{D}^\infty_\mu(P)$ for some $\mu > n/2 + 1$, then the stress energy tensor is a well defined smooth function; cf. [27, Proposition 15.37, p. 246]. Moreover, the stress energy tensor is divergence free if $f$ satisfies the Vlasov equation.

It is worth pointing out that it is possible to introduce more general function spaces, corresponding to a finite degree of differentiability; cf. [27, Definition 15.1, p. 234]. However, the above definition is sufficient for our purposes. The above function spaces are suitable when discussing functions on the mass shell. However, we also need to introduce function spaces for the initial datum for the Vlasov–non–linear scalar field case, the required definition was given in [27, Definition 7.11, pp. 93–94]. Here we are interested in the special case corresponding to the Einstein–Vlasov equations with a positive cosmological constant. This leads to the following simplification of [27, Definition 7.11, pp. 93–94]:

Let us now recall the definition of initial data in the present setting. In the case of the Einstein–Vlasov–non–linear scalar field case, the required definition was given in [27, Definition 7.11, pp. 93–94]. Here we are interested in the special case corresponding to the Einstein–Vlasov equations with a positive cosmological constant. This leads to the following simplification of [27, Definition 7.11, pp. 93–94]:

Remark 21. According to the criteria appearing in Definitions 17 and 20 we need to verify continuity conditions for every choice of local coordinates. However, it turns out to be sufficient to consider a collection of local coordinates covering the manifold of interest; cf. [27, Lemma 15.9, p. 235] and [27, Lemma 15.19, p. 237].

Finally, in order to be able to state a stability result, we need a norm. Recall, to this end, [27, Definition 7.7, pp. 89–90]:

Definition 20. Let $1 \leq n \in \mathbb{Z}$, $\mu \in \mathbb{R}$ and $\Sigma$ be an $n$-dimensional manifold. The space $\mathcal{D}^\infty_\mu(T\Sigma)$ is defined to consist of the smooth functions $f : T\Sigma \to \mathbb{R}$ such that, for every choice of local coordinates $(\bar{x}, U)$, $n$-multiindex $\alpha$ and $n$-multiindex $\beta$, the derivative $\partial_\alpha \partial_\beta^\beta \bar{f}_\bar{x}$ (where $\bar{x}$ symbolises the first $n$ and $\bar{p}$ the last $n$ variables), considered as a function from $\bar{x}(U)$ to the set of functions from $\mathbb{R}^n$ to $\mathbb{R}$, belongs to $C[\bar{x}(U), L^2_{\mu+|\beta|}(\mathbb{R}^n)]$.

Remark 23. Clearly, the norm depends on the choice of partition of unity and on the choice of coordinates. However, different choices lead to equivalent norms. Here, we are mainly interested in the case $\Sigma = T^3$, in which case it is neither necessary to introduce local coordinates nor a partition of unity.

Let us now recall the definition of initial data in the present setting. In the case of the Einstein–Vlasov–non–linear scalar field case, the required definition was given in [27, Definition 7.11, pp. 93–94]. Here we are interested in the special case corresponding to the Einstein–Vlasov equations with a positive cosmological constant. This leads to the following simplification of [27, Definition 7.11, pp. 93–94]:

Remark 22. Let $1 \leq n \in \mathbb{Z}$, $0 \leq l \in \mathbb{Z}$, $\mu \in \mathbb{R}$ and $\Sigma$ be a compact $n$-dimensional manifold. Let, moreover, $\bar{\chi}_i$, $i = 1, \ldots, j$, be a finite partition of unity subordinate to a cover consisting of coordinate neighbourhoods, say $(\bar{x}_i, U_i)$. Then $\| \cdot \|_{H^{l,\mu}_{1,\mu}}$ is defined by

$$\| \bar{f} \|_{H^{l,\mu}_{1,\mu}} = \left( \sum_{i=1}^{j} \sum_{|\alpha|+|\beta| \leq l} \int_{\bar{x}_i(U_i) \times \mathbb{R}^n} \langle \bar{\rho} \rangle^{2\mu+2|\beta|} \bar{\chi}_i(\bar{\xi})(\partial_\alpha \partial_\beta^\beta \bar{f}_{\bar{x}_i}(\bar{\xi}, \bar{\rho}))^2(\bar{\xi}, \bar{\rho}) d\bar{\xi} d\bar{\rho} \right)^{1/2}$$

for each $\bar{f} \in \mathcal{D}^\infty_\mu(T\Sigma)$. 

Remark 23. Clearly, the norm depends on the choice of partition of unity and on the choice of coordinates. However, different choices lead to equivalent norms. Here, we are mainly interested in the case $\Sigma = T^3$, in which case it is neither necessary to introduce local coordinates nor a partition of unity.
**Definition 24.** Let $1 \leq n \in \mathbb{Z}$ and $n/2 + 1 < \mu \in \mathbb{R}$. *Initial data for* $(4)$ and $(5)$ *consist of an oriented $n$-dimensional manifold $\Sigma$, a non-negative function $\tilde{f} \in \mathcal{D}_\mu^\infty(T\Sigma)$, a Riemannian metric $\bar{g}$ and a symmetric covariant 2-tensor field $\bar{k}$ on $\Sigma$, all assumed to be smooth and to satisfy*

\[
\bar{r} - \bar{k}_{ij}\bar{k}^{ij} + (\text{tr}\bar{k})^2 = 2\Lambda + 2\rho^\mu,
\]

where $\bar{\nabla}$ is the Levi-Civita connection of $\bar{g}$, $\bar{r}$ is the associated scalar curvature, indices are raised and lowered by $\bar{g}$, and $\rho^\mu$ and $\bar{J}^\mu$ are given by $(21)$ and $(22)$ below respectively. Given initial data, the *initial value problem* is that of finding a solution $(M, g, f)$ to $(4)$ and $(5)$ (in other words, an $n + 1$-dimensional manifold $M$, a smooth time oriented Lorentz metric $g$ on $M$ and a non-negative function $f \in \mathcal{D}_\mu^\infty(P)$ such that $(4)$ and $(5)$ are satisfied), and an embedding $i : \Sigma \to M$ such that

\[
i^*g = \bar{g}, \quad \tilde{f} = i^*(f \circ pr_{i(\Sigma)}^{-1})
\]

and if $\kappa$ is the second fundamental form of $i(\Sigma)$, then $i^*\kappa = \bar{k}$. Such a triple $(M, g, f)$ is referred to as a *development* of the initial data, the existence of an embedding $i$ being tacit. If, in addition to the above conditions, $i(\Sigma)$ is a Cauchy hypersurface in $(M, g)$, the triple is said to be a *globally hyperbolic development*.

**Remark 25.** The map $pr_{i(\Sigma)}$ is the diffeomorphism from the mass shell above $i(\Sigma)$ to the tangent space of $i(\Sigma)$ defined by mapping a vector $v$ to its component perpendicular to the normal of $i(\Sigma)$.

The *energy density* and *current* induced by the initial data are given by

\[
\rho^\mu(\xi) = \int_{T^1\Sigma} \tilde{f}(\bar{p})(1 + \bar{g}(\bar{p}, \bar{p}))^{1/2} \bar{\mu}_{\bar{g}, \xi},
\]

\[
\bar{J}^\mu(\bar{X}) = \int_{T^1\Sigma} \tilde{f}(\bar{p})\bar{g}(\bar{X}, \bar{p})\bar{\mu}_{\bar{g}, \xi}.
\]

In these expressions, $\xi \in \Sigma$, $\bar{X} \in T^1\Sigma$, $\bar{\mu}_{\bar{g}, \xi}$ is the volume form on $T^1\Sigma$ induced by $\bar{g}$ and $\bar{p} \in T^1\Sigma$. It is important to note that under the assumptions of the above definition, the energy density is a smooth function and the current is a smooth one-form field on $\Sigma$; cf. [27] Lemma 15.40, p. 246].

**Theorem 27.** Consider a $T^2$-symmetric solution to the Einstein–Vlasov system with a positive cosmological constant. Choose coordinates so that the corresponding metric takes the form $(27)$ on $I \times T^3$, where $I = (t_0, \infty)$. Assume that the solution has $\lambda$-asymptotics. Choose a $t \in I$ and let $i : T^3 \to I \times T^3$ be given by $i(\bar{x}) = (t, \bar{x})$. Let $\bar{g}_{bg} = i^*g$ and let $\bar{k}_{bg}$ denote the pullback (under $i$) of the second fundamental form induced on $i(T^3)$ by $g$. Let, moreover,

\[
\bar{f}_{bg} = i^*(f \circ pr_{i(\Sigma)}^{-1}).
\]
Make a choice of $\mu > 5/2$, a choice of norms as in Definition 22 and a choice of Sobolev norms on tensorfields on $T^3$. Then there is an $\epsilon > 0$ such that if $(T^3, \bar{g}, k, \bar{f})$ are initial data for (4) and (5), with $\bar{f} \in \mathcal{D}_\mu^{\infty}(T\Sigma)$, satisfying
\[
\|\bar{g} - \bar{g}_{bg}\|_{H^5} + \|\bar{k} - \bar{k}_{bg}\|_{H^4} + \|\bar{f} - \bar{f}_{bg}\|_{H^4_{1,\mu}} \leq \epsilon,
\]
then the maximal globally hyperbolic development of the initial data is future causally geodesically complete.

Remarks 28. The conclusions are rather weak in the sense that we only obtain future causal geodesic completeness. However, the argument is based on an application of [27, Theorem 7.16, pp. 104–106], the conclusions of which contain a detailed description of the asymptotics. However, since we do not wish to repeat the conclusions here, the interested reader is referred to [27]. It should also be mentioned that up to the point where we appeal to [27], Cauchy stability applies, so that it should be possible to obtain detailed control over the perturbed solutions for the entire future. The interested reader is encouraged to write down the details. Finally, let us note that the conclusions obtained in [27] Theorem 7.16, pp. 104–106 are strong enough to confirm the cosmic no-hair conjecture for the perturbed solutions.

Remark 29. The function $\bar{f}_{bg}$ has compact support, but $\bar{f}$ need not have compact support.

The proof is to be found in Section 10.

1.4 Outline

Finally, let us give an outline of the paper. In Section 2 we write down the equations in the case that the metric takes the form (1). In Section 3 we then collect the conclusions which are not dependent on the particular type of matter model (as long as it satisfies the dominant energy condition and the non-negative pressure condition). The section ends with conclusions concerning the causal structure of $T^3$-Gowdy symmetric spacetimes. Turning to the more detailed conclusions, we specialise to the case of Vlasov matter. The natural first step is to derive light cone estimates; i.e., to consider the behaviour along characteristics. This is the subject of Section 4. As opposed to the vacuum case, we need to control the characteristics associated with the Vlasov equation at the same time as the first derivatives of the metric components. Fortunately, the $e_2$- and $e_3$-components of the momentum are controlled automatically due to the symmetry. However, an argument is required in the case of the $e_1$-component. In order to obtain control of higher order derivatives, we need to take derivatives of the characteristic system (associated with the Vlasov equation; i.e. with the geodesic flow). Naively, this should require control of second order derivatives of the metric functions, something we do not have. Nevertheless, by an appropriate choice of variables, controlling first order derivatives turns out to be sufficient. It is of interest to note that a similar choice was already suggested in [2] Lemma 3, p. 363; cf. also [3] Lemma 3, p. 257. However, in the present setting, it is not sufficient to derive a system involving only first order derivatives of the metric functions. We also need to be able to use the system to derive the desired type of asymptotics for the derivatives of the characteristic system. It turns out to be possible to do this, and we write down the required arguments in Section 6. After we have obtained this conclusion, it turns out to be possible to proceed inductively in order to derive higher order estimates for the characteristic system and the metric components. The required arguments are written down in Sections 7 and 8. In order to obtain the desired conclusions concerning the distribution function, it turns out to be convenient to consider $L^2$-based energies. This subject is treated in Section 9. Finally, in Section 10 we prove the main theorems of the paper.
2 Symmetry assumptions and equations

In this paper, we study $\mathbb{T}^2$-symmetric solutions of Einstein’s equations. Since it will turn out to be convenient to express the equations using the orthonormal frame $\{e_i\}$, let us introduce the notation
\[
\rho = T(e_0, e_0), \quad J_i = -T(e_0, e_i), \quad P_i = T(e_i, e_i), \quad S_{ij} = T(e_i, e_j),
\]
where we do not sum over any indices; here, and below, we tacitly assume Latin indices to range from 1 to 3 and Greek indices to range from 0 to 3. It is also convenient to introduce the notation
\[
J = -t^{5/2} \alpha^{1/2} e^{-\rho/2} (G_t + QH_t), \quad K = QJ - t^{5/2} \alpha^{1/2} e^{-\rho/2} H_t.
\]
Note that these objects are the twist quantities introduced in (2). Using the above notation, the 00 and 11-components of Einstein’s equations can be written
\[
\lambda_t - 2 \frac{\alpha_t}{\alpha} = t \left[ P_t^2 + \alpha P_0^2 + e^{2\rho} (Q_0^2 + \alpha Q_0^3) \right] + \frac{e^{\lambda/2 - P} J_t^2}{t^{5/2}} + \frac{e^{\lambda/2 + P} (K - QJ)^2}{t^{5/2}} \tag{24}
\]
\[
\lambda_t = t \left[ P_t^2 + \alpha P_0^2 + e^{2\rho} (Q_0^2 + \alpha Q_0^3) \right] - \frac{e^{\lambda/2 - P} J_t^2}{t^{5/2}} - \frac{e^{\lambda/2 + P} (K - QJ)^2}{t^{5/2}} \tag{25}
\]
respectively. The 22-component minus the 33-component can be written
\[
\begin{align*}
\partial_t (\alpha^{-1/2} P_t) &= \partial_\theta (\alpha^{1/2} P_\theta) + t \alpha^{-1/2} e^{2\rho} (Q_0^2 - \alpha Q_0^3) + \frac{\alpha^{-1/2} e^{\lambda/2 - P} J_t^2}{2t^{5/2}} - \frac{\alpha^{-1/2} e^{\lambda/2 + P} (K - QJ)^2}{2t^{5/2}} + t^{1/2} e^{\rho/2} \alpha^{-1/2} (P_2 - P_3), \tag{26}
\end{align*}
\]
The 22-component plus the 33-component can be written
\[
\begin{align*}
\partial_t \left[ \alpha^{-1/2} \left( \lambda_t - 2 \frac{\alpha_t}{\alpha} - \frac{3}{t} \right) \right] &= \partial_\theta \left( \alpha^{1/2} \lambda_\theta \right) - t \alpha^{-1/2} \left[ P_t^2 + e^{2\rho} Q_0^2 - \alpha (P_\theta^2 + e^{2\rho} Q_\theta^3) \right] - 2t \alpha^{-1/2} \left( \frac{e^{\lambda/2 - P} J_t^2}{t^{5/2}} + \frac{e^{\lambda/2 + P} (K - QJ)^2}{t^{5/2}} \right) + \alpha^{-1/2} \lambda_\theta + 2t^{1/2} e^{\rho/2} \alpha^{-1/2} (2\Lambda - P_2 - P_3). \tag{27}
\end{align*}
\]
The 01, 02, 03, 12 and 13-components read
\[
\begin{align*}
\lambda_\theta &= 2t (P_t P_\theta + e^{2\rho} Q_t Q_\theta) - 4t^{1/2} e^{\rho/2} \alpha^{-1/2} J_1, \tag{28}
J_\theta &= 2^{5/4} \alpha^{-1/2} e^{P/2+\lambda/4} J_2, \tag{29}
K_\theta &= 2^{5/4} \alpha^{-1/2} e^{-P/2+\lambda/4} J_3 + 2^{5/4} \alpha^{-1/2} e^{P/2+\lambda/4} Q J_2, \tag{30}
J_t &= -2t^{5/4} e^{\rho/4+P/2} S_{12}, \tag{31}
K_t &= -2t^{5/4} e^{\rho/4+P/2} Q S_{12} - 2t^{5/4} e^{P/2+\lambda/4} S_{13}, \tag{32}
\end{align*}
\]
respectively. Finally, the 23-component reads
\[
\begin{align*}
\partial_t (\alpha^{-1/2} e^{2\rho} Q_t) - \partial_\theta (\alpha^{1/2} e^{2\rho} Q_\theta) &= t^{5/2} \alpha^{-1/2} e^{\lambda/2+P} J (K - QJ) + 2t^{1/2} \alpha^{-1/2} e^{\lambda/2+P} S_{23}, \tag{33}
\end{align*}
\]
For future reference, it is also of interest to note that
\[
\begin{align*}
\frac{\alpha_t}{\alpha} &= -\frac{e^{-P+\lambda/2} J_t^2}{t^{5/2}} - \frac{e^{P+\lambda/2} (K - QJ)^2}{t^{5/2}} - 4t^{1/2} e^{\rho/2} \alpha^{-1/2} (\rho - P_1), \tag{34}
\lambda_t - \frac{\alpha_t}{\alpha} &= t \left[ P_t^2 + \alpha P_0^2 + e^{2\rho} (Q_0^2 + \alpha Q_0^3) \right] + 2t^{1/2} e^{\rho/2} (\rho + P_1). \tag{35}
\end{align*}
\]
2.1 Preliminary calculations

Since the metric components only depend on two variables, it is natural to derive estimates by integrating along characteristics. In the present subsection, we record a general calculation which is of interest in that context. To begin with, let us define

\[ \partial_\pm = \partial t \pm \alpha^{1/2} \partial \theta, \quad A_\pm = (\partial_\pm P)^2 + e^{2P}(\partial_\pm Q)^2. \]

We then have the following result.

**Lemma 30.** Consider a \( \mathbb{T}^2 \)-symmetric solution to Einstein’s equations with a cosmological constant \( \Lambda \) such that the metric takes the form (1). Then

\[
\partial_\pm A_\mp = - \left( \frac{2}{t} - \frac{\alpha t}{\alpha} \right) A_\pm \mp \frac{2}{t} \alpha^{1/2} (P_0 \partial_\mp P + e^{2P} Q_0 \partial_\mp Q) \\
+ \frac{e^{2P}(\partial_\mp P)^2}{t^{\gamma/2}} - \frac{2}{t^{\gamma/2}} \partial_\mp P + 2 e^{\gamma/2} (K - QJ) \}
+ 2t^{-1/2} e^{\gamma/2} (P_2 - P_3) \partial_\mp P + 4t^{-1/2} e^{\gamma/2} S_{23} \]

Similarly, due to (33), we obtain

\[
\partial_\pm \partial_\mp Q = - \frac{1}{t} Q_1 + \frac{\alpha t}{2\alpha} \partial_\mp Q - 2(Q_1 P_t - \alpha Q_0 P_0) + \frac{e^{2P} (K - QJ)}{t^{\gamma/2}} + 2t^{-1/2} e^{\gamma/2} - P S_{23}. \]

Combining (37) and (38) with the fact that

\[-4(Q_1 P_t - \alpha P_0 Q_0) \partial_\mp Q + 2\partial_\mp P(\partial_\mp Q)^2 = -2\partial_\mp P(Q_1^2 - \alpha Q_0^2),\]

a calculation yields the conclusion of the lemma.

\[ \square \]

2.2 Vlasov matter

The equations (24)–(35) hold in general. However, we are here particularly interested in matter of Vlasov type. Recall the conventions concerning \( f \) introduced in Subsection 1.2. Recall, moreover, the fact that the Vlasov equation is equivalent to \( f \) being constant along future directed unit timelike geodesics. As a consequence, it can be calculated that the Vlasov equation takes the form

\[
\frac{\partial f}{\partial t} + \frac{\alpha^{1/2} v_1}{v_0} \frac{\partial f}{\partial \theta} - \left[ \frac{1}{4} \alpha^{1/2} \lambda \theta \right] v_0^3 + \frac{1}{4} \left( \lambda t - \frac{2\alpha t}{\alpha} - \frac{1}{t} \right) v_1^3 - \alpha^{1/2} e^{P} Q_0 \left( v_0^2 v_1^2 + v_0^3 v_1^3 \right) \\
+ \frac{1}{2} \alpha^{1/2} P_0 \left( v_1^3 v_0^2 - v_0^5 \right) - \frac{\alpha^{1/2}}{2} \partial_\mp P \left[ v_1^2 v_0^2 \right] \frac{\partial f}{\partial v_1} \\
- \left[ \frac{1}{2} \left( P_t + \frac{1}{t} \right) v_1^3 - \frac{1}{2} \alpha^{1/2} P_0 \left( v_1^3 \frac{v_0}{v_0^3} \right) + e^{P} v_2^2 \left( Q_t + \alpha^{1/2} Q_0 \frac{v_1}{v_0} \right) \right] \frac{\partial f}{\partial v_3} = 0. \]

(39)
Turning to the stress energy tensor, it satisfies
\[ T(e_\mu, e_\nu) = \int_{\mathbb{R}^3} v_\mu v_\nu f \frac{1}{v_0} dv, \] (40)
where the indices of \( v \) are raised and lowered with the Minkowski metric. In particular, in the case of Vlasov, we thus have
\[ \rho = \int_{\mathbb{R}^3} v_0 f dv, \quad P_k = \int_{\mathbb{R}^3} (v^k)^2 v_0 f dv, \quad J_k = \int_{\mathbb{R}^3} v^j v^k v_0 f dv, \quad S_{jk} = \int_{\mathbb{R}^3} v^j v^k v_0 f dv, \]
where \( j, k = 1, 2, 3 \).

3 Preliminary conclusions concerning the asymptotics

In the present section, we are interested in \( T^2 \)-symmetric solutions to Einstein’s equations such that the corresponding metric admits a foliation of the form \( [1] \) on \( I \times \mathbb{T}^3 \), where \( I = (t_0, \infty) \) and \( t_0 \geq 0 \). For the sake of brevity, we shall below refer to solutions of this form as future global, and we shall speak of \( t_0 \) and \( t_1 = t_0 + 2 \) without further introduction.

**Proposition 32.** Given a future global solution to Einstein’s equations with a cosmological constant \( \Lambda > 0 \), \( T^3 \)-symmetry and a stress energy tensor satisfying the dominant energy condition and the non-negative pressure condition, there is a constant \( C > 0 \) such that
\[ \alpha(t, \theta) \leq Ct^{-3} \]
for all \((t, \theta) \in [t_1, \infty) \times S^1\).

**Remark 33.** The same conclusion holds if we replace the cosmological constant with a non-linear scalar field with a potential with a positive lower bound; in other words, if we set \( \Lambda = 0 \) and consider stress energy tensors of the form \( T = T^o + T^{nf} \), where \( T^o \) is the stress energy tensor associated with matter fields satisfying the dominant energy condition and the non-negative pressure condition, and \( T^{nf} \) is the stress energy tensor associated with a non-linear scalar field with a potential \( V \) having a positive lower bound.

**Proof.** Due to (35) and the fact that the matter satisfies the dominant energy condition and the non-negative pressure condition, we conclude that \( \lambda - \alpha_i / \alpha \geq 0 \). There is thus a \( c_0 > 0 \) such that
\[ (\alpha^{-1/2} e^{\lambda/2})(t, \theta) \geq c_0 \]
for all \((t, \theta) \in [t_1, \infty) \times S^1\). Combining this observation with (34) and the fact that the matter satisfies the dominant energy condition (so that \( \rho - P_1 \geq 0 \)), we obtain
\[ \partial_t \alpha^{-1/2} = \frac{\alpha t}{2\alpha} \alpha^{-1/2} \geq 2t^{1/2} \alpha^{-1/2} e^{\lambda/2} \Lambda \geq c_1 t^{1/2} \]
for some constant \( c_1 > 0 \) and all \((t, \theta) \in [t_1, \infty) \times S^1\). Integrating this inequality, we obtain the conclusion of the proposition. \( \square \)

In the Gowdy case, the second and third terms on the right hand side of (25) are zero, and as a consequence, we can extract more information. In fact, we have the following observation.

**Proposition 34.** Consider a future global solution to Einstein’s equations with a cosmological constant \( \Lambda > 0 \), \( T^3 \)-Gowdy symmetry and matter satisfying the non-negative pressure condition. Then there is, for every \( \epsilon > 0 \), a \( T > t_0 \) such that
\[ \lambda(t, \theta) \geq -3 \ln t + 2 \ln \left( \frac{3}{4\Lambda} \right) - \epsilon \]
for all \((t, \theta) \in [T, \infty) \times S^1\).
Proof. Let

$$
\hat{\lambda} = \lambda + 3 \ln t - 2 \ln \left( \frac{3}{4\Lambda} \right).
$$

(41)

Then

$$
\partial_t \hat{\lambda} = t \left[ P_t^2 + \alpha P_\theta^2 + e^{2P}(Q_t^2 + \alpha Q_\theta^2) \right] + 4t^{1/2}e^{\lambda/2}P_t + \frac{3}{t}(1 - e^{\lambda t}).
$$

Due to the non-negative pressure condition, we conclude that

$$
\partial_t \hat{\lambda} \geq \frac{3}{t}(1 - e^{\lambda t}).
$$

For every $\epsilon > 0$, there is thus a $T$ such that $\hat{\lambda}(t, \theta) \geq -\epsilon$ for all $(t, \theta) \in [T, \infty) \times S^1$. The proposition follows.

In order to proceed, it is convenient to introduce an energy:

$$
E = \int_{S^1} t^{-1/2} \left( \lambda_t - 2 \frac{\alpha t}{\alpha} - 4t^{1/2}e^{\lambda/2} \Lambda \right) d\theta
= \int_{S^1} \left( t^{1/2} \alpha^{-1/2} \left[ P_t^2 + \alpha P_\theta^2 + e^{2P}(Q_t^2 + \alpha Q_\theta^2) \right] + \frac{\alpha^{-1/2}e^{\lambda/2-P}J^2}{t^{5/2}} + \frac{\alpha^{-1/2}e^{\lambda/2+P}(K - QJ)^2}{t^{5/2}} + 4t^{3/2}\alpha^{-1/2}e^{\lambda/2} \rho \right) d\theta.
$$

In case the metric has $\lambda$-asymptotics (recall Definition 3), it turns out to be possible to estimate this energy.

**Lemma 35.** Consider a future global solution to Einstein’s equations with a cosmological constant $\Lambda > 0$, $T^2$-symmetry, $\lambda$-asymptotics, and a stress energy tensor satisfying the dominant energy condition and the non-negative pressure condition. Then for every $a > 1/2$, there is a constant $C_a > 0$ such that $E(t) \leq C_a t^a$ for all $t \geq t_1$.

Proof. Due to (27), we obtain

$$
\partial_t \left[ t^{1/2} \left( \lambda_t - 2 \frac{\alpha t}{\alpha} - 4t^{1/2}e^{\lambda/2} \Lambda \right) \right] = \partial_\theta (t^{1/2} \lambda_\theta) + 2t^{1/2} \left( P_\theta^2 + e^{2P}Q_\theta^2 \right)
- \frac{3}{2} \frac{\alpha^{-1/2}e^{\lambda/2-P}J^2}{t^{5/2}} - \frac{3}{2} \frac{\alpha^{-1/2}e^{\lambda/2+P}(K - QJ)^2}{t^{5/2}}
- 2t^{5/2} \alpha^{-1/2}e^{\lambda/2} \Lambda [P_t^2 + e^{2P}Q_t^2 + \alpha(P_\theta^2 + e^{2P}Q_\theta^2)]
+ t^{1/2} \alpha^{-1/2}e^{\lambda/2} (3\rho + P_t - 2P_\theta - 2P_\theta)
+ 4t^2 \alpha^{-1/2}e^{\lambda} \Lambda (\rho + P_t).
$$

(42)

Thus, since the matter satisfies the non-negative pressure condition, we have

$$
\frac{dE}{dt} \leq \int_{S^1} 2t^{1/2}(P_\theta^2 + e^{2P}Q_\theta^2) d\theta - \int_{S^1} t^{5/2} \alpha^{1/2}e^{\lambda/2} \Lambda (P_t^2 + e^{2P}Q_t^2) d\theta
+ \int_{S^1} t^{1/2} \alpha^{-1/2}e^{\lambda/2} (3\rho + P_t) d\theta - \int_{S^1} 4t^2 \alpha^{-1/2}e^{\lambda} \Lambda (\rho + P_t) d\theta.
$$

Using the fact that the solution has $\lambda$-asymptotics, we conclude that for every $a > 1/2$, there is a $T \geq t_1$ such that

$$
\frac{dE}{dt} \leq \frac{a}{t} E
$$

for all $t \geq T$. As a consequence, $E(t) \leq Ct^a$ for $t \geq t_1$. □
Using the above estimate for $E$, it is possible to extract more information concerning the asymptotics.

**Lemma 36.** Consider a future global solution to Einstein’s equations with a cosmological constant $\Lambda > 0$, $T^2$-symmetry, $\lambda$-asymptotics, and a stress energy tensor satisfying the dominant energy condition and the non-negative pressure condition. Then there is a constant $C > 0$ such that

$$
\| \langle \lambda(t, \cdot) + 3 \ln t - 2 \ln \frac{3}{4\Lambda} \rangle \|_{C^0} \leq Ct^{-1/2},
$$

$$
E(t) \leq Ct^{1/2}
$$

for all $t \geq t_1$.

**Proof.** Due to the estimate $E(t) \leq C_a t^a$, the fact that $\alpha^{1/2} \leq Ct^{-3/2}$, and (44), we conclude that

$$
\langle \lambda_t \rangle = -4t^{1/2} \langle e^{\lambda/2} \rangle \Lambda + O(t^{a-5/2}),
$$

where we have used the dominant energy condition; recall that the notation $\langle \lambda_t \rangle$ was introduced in Remark 5. Due to (28) and the dominant energy condition, we also have

$$
|\lambda_{\theta}| \leq t \alpha^{-1/2} [P_t^2 + \alpha P_\theta^2 + e^{2P} (Q_t^2 + \alpha Q_\theta^2)] + 4t^{1/2} \alpha^{-1/2} e^{\lambda/2} \rho.
$$

Due to the above estimate for the energy, we thus obtain

$$
\int_{S^1} |\lambda_{\theta}| d\theta \leq Ct^{a-1}.
$$

(44)

Note that, due to (43),

$$
\langle \lambda_t \rangle = \frac{3}{t} (1 - (e^{\lambda/2})^2) + O(t^{a-5/2}).
$$

(45)

Let us first prove that $\langle \lambda \rangle$ converges to zero. Let, to this end, $\epsilon > 0$. Since the solution has $\lambda$-asymptotics, we know that there is a $T$ such that $\langle \lambda \rangle(t) \geq -\epsilon$ for all $t \geq T$. In order to prove that there is a $T$ such that $\langle \lambda \rangle(t) \leq \epsilon$ for all $t \geq T$, let us assume that $\langle \lambda \rangle(t) \geq \epsilon$ for some $t$. Due to (44), we conclude that $\lambda(t, \theta) \geq \epsilon/2$ for all $\theta \in S^1$ (assuming $t$ to be large enough). Inserting this information into (45), we conclude that

$$
\langle \lambda_t \rangle \leq \frac{2}{t} (1 - e^{\epsilon/4}),
$$

assuming $t$ to be large enough. Since the right hand side is negative and non-integrable, we conclude that $\langle \lambda \rangle$ has to decay until it is smaller than $\epsilon$ (assuming the starting time $t$ to be large enough). Moreover, $\langle \lambda \rangle$ cannot exceed $\epsilon$ at a later time. In order to obtain a quantitative estimate, note that

$$
\langle \lambda_t \rangle = \frac{3}{t} (1 - e^{\langle \lambda \rangle/2}) + O(t^{a-2} \langle \lambda \rangle),
$$

where we have used the fact that $\lambda$ is bounded to the future as well as (44). As a consequence,

$$
\partial_t \langle \lambda \rangle^2 = 2 \langle \lambda \rangle \langle \lambda_t \rangle = \frac{6}{t} \langle \lambda \rangle \left[ 1 - \left( 1 + \frac{1}{2} \langle \lambda \rangle + O(\langle \lambda \rangle^2) \right) \right] + O(t^{a-2} \langle \lambda \rangle)
$$

$$
= -\frac{3}{t} \langle \lambda \rangle^2 + \frac{1}{t} O(\langle \lambda \rangle^3) + O(t^{a-2} \langle \lambda \rangle).
$$

Let $0 < b < 1 - a$ and define

$$
E = t^{2b} \langle \lambda \rangle^2.
$$

Then

$$
\frac{dE}{dt} = \frac{2b}{t} E - \frac{3}{t} E + \frac{1}{t} O(\langle \lambda \rangle E) + t^{-1} O(t^{b+a-1} E^{1/2}).
$$
As a consequence, there is a constant $C > 0$ such that $\partial_t E \leq 0$ when $E \geq C$ and $t$ is large enough. In particular, $E$ is thus bounded to the future. For every $0 < b < 1/2$, there is thus a constant $C_b$ such that
\[
\left\| \lambda(t, \cdot) + 3 \ln t - 2 \ln \frac{3}{4\Lambda} \right\|_{\mathcal{C}_0} \leq C_b t^{-b}
\]
for all $t \geq t_1$. Due to this estimate, we can return to the argument presented in the proof of Lemma 35 and obtain the improvement $E(t) \leq Ct^{1/2}$ for $t \geq t_1$. As a consequence, we can go through the above arguments with $a = 1/2$ and $b = 1/2$. The lemma follows.

**Lemma 37.** Consider a future global solution to Einstein’s equations with a cosmological constant $\Lambda > 0$, $T^2$-symmetry, $\lambda$-asymptotics, and a stress energy tensor satisfying the dominant energy condition and the non-negative pressure condition. Then there is a constant $C > 0$ such that
\[
t^{-3/2} \langle \alpha^{-1/2}(t, \cdot) \rangle + \|Q(t, \cdot)\|_{\mathcal{C}_0} + \|P(t, \cdot)\|_{\mathcal{C}_0} \leq C
\]
for all $t \geq t_1$.

**Proof.** Compute, using (34) and the above estimates,
\[
\partial_t (\alpha^{-1/2}) = -\frac{1}{2} \left\langle \alpha^{-1/2} \frac{\alpha_t}{\alpha} \right\rangle = \frac{1}{2\pi} \int_{S^3} \left[ \frac{\alpha^{-1/2} e^{-\lambda/2} J^2}{2t^{5/2}} + \frac{\alpha^{-1/2} e^{\lambda/2}(K - Q J)^2}{2t^{5/2}} \right] d\theta
\]
\[
+ \frac{1}{2\pi} \int_{S^3} [2t^{1/2} \alpha^{-1/2} e^{\lambda/2} \Lambda + t^{1/2} \alpha^{-1/2} e^{\lambda/2}(\rho - P_t)] d\theta
\]
\[
\leq Ct^{-1/2} + \frac{1}{2\pi} \int_{S^3} 2t^{1/2} \alpha^{-1/2} e^{\lambda/2} \Lambda d\theta
\]
\[
\leq Ct^{-1/2} + \frac{3}{2t} e^{\lambda/2} \alpha^{-1/2} \leq \frac{3}{2t} (\langle \alpha^{-1/2} \rangle + Ct^{-3/2} \langle \alpha^{-1/2} \rangle + Ct^{-1/2}).
\]
Let $A = \langle \alpha^{-1/2} \rangle + t$. Then
\[
\frac{dA}{dt} = \partial_t (\alpha^{-1/2}) + 1 \leq \frac{3}{2t} A + Ct^{-3/2} A.
\]
Consequently,
\[
\ln \frac{A(t)}{A(t_1)} \leq \frac{3}{2} \ln t + C_0,
\]
so that $\langle \alpha^{-1/2} \rangle \leq Ct^{3/2}$ for $t \geq t_1$. In order to proceed, note that
\[
\int_{S^3} |P_t| d\theta \leq \left( \int_{S^3} \alpha^{1/2} P_t^2 d\theta \right)^{1/2} \left( \int_{S^3} \alpha^{-1/2} d\theta \right)^{1/2} \leq Ct^{-3/4} \rho^{3/4} \leq C
\]
for $t \geq t_1$. On the other hand
\[
|\partial_t (P)| = |\{P_t\}| \leq \frac{1}{2\pi} \int_{S^3} |P_t| d\theta \leq \frac{1}{\sqrt{2\pi}} \left( \int_{S^3} P_t^2 d\theta \right)^{1/2} \leq Ct^{-3/2}.
\]
Consequently, $\langle P \rangle$ is bounded to the future. Combining these two observations, we conclude that
\[
\|P(t, \cdot)\|_{\mathcal{C}_0} \leq C
\]
for all $t \geq t_1$. As a consequence, we obtain the same conclusion concerning $Q$. The lemma follows.
Lemma 38. Consider a future global solution to Einstein’s equations with a cosmological constant \( \Lambda > 0 \), \( T^2 \)-symmetry, \( \lambda \)-asymptotics, and a stress energy tensor satisfying the dominant energy condition and the non-negative pressure condition. Then there is a constant \( C > 0 \) such that

\[
\left\| \frac{e^{\lambda/2-P} J^2}{t^{5/2}} \right\|_{C^0} \leq C t^{-2}, \tag{46}
\]

\[
\left\| \frac{e^{P+\lambda/2}(K - Q J)^2}{t^{5/2}} \right\|_{C^0} \leq C t^{-2} \tag{47}
\]

for all \( t \geq t_1 \). Moreover, for \( t \geq t_1 \),

\[
\| H_i \|_{L^1} + \| G_i \|_{L^1} \leq C t^{-3/2}.
\]

Proof. Combining (29), the dominant energy condition and the above estimates, we conclude that

\[
\int_{S^1} |J_t| d\theta \leq C t^{5/4} \int_{S^1} \alpha^{-1/2} e^{\lambda/4} \rho d\theta \leq C t^{5/4} t^{-3/2} \int_{S^1} \alpha^{-1/2} e^{\lambda/2} \rho d\theta \leq C t.
\]

The spatial variation of \( J \) is consequently not greater than \( Ct \). Combining (31), the dominant energy condition and the above estimates, we conclude that

\[
\int_{S^1} |J| d\theta \leq C t^{5/4} \int_{S^1} e^{\lambda/4} \rho d\theta \leq C t^{5/4} t^{-3/2} \int_{S^1} \alpha^{-1/2} e^{\lambda/2} \rho d\theta \leq C t^{-1/2}.
\]

As a consequence,

\[
\| J \|_{C^0} \leq C t^{1/2}.
\]

Combining the above two estimates, we conclude that

\[
\| J(t, \cdot) \|_{C^0} \leq C t.
\]

Thus (40) holds.

Let us now turn to \( K - Q J \). Note that

\[
\int_{S^1} |K_\theta - Q_\theta J - Q J_\theta| d\theta \leq \int_{S^1} |K_\theta - Q_\theta J| d\theta + \int_{S^1} |Q_\theta J| d\theta \leq \int_{S^1} |K_\theta - Q_\theta J| d\theta + C t,
\]

where we have used (18) and the fact that the \( L^1 \)-norm of \( Q_\theta \) is bounded (cf. the proof of Lemma 37). On the other hand, (30) yields

\[
K_\theta - Q J_\theta = 2 t^{5/4} \alpha^{-1/2} e^{-P/2+\lambda/4} J_3.
\]

We can thus argue as above in order to conclude that

\[
\int_{S^1} |K_\theta - Q_\theta J - Q J_\theta| d\theta \leq C t.
\]

In particular, the spatial variation of \( K - Q J \) is bounded by \( C t \). On the other hand,

\[
\int_{S^1} |K_t - Q t J - Q J_t| d\theta \leq \int_{S^1} |K_t - Q J_t| d\theta + \int_{S^1} |Q t J| d\theta \leq \int_{S^1} |K_t - Q J_t| d\theta + C t^{-1/2}.
\]

Moreover, due to (32),

\[
K_t - Q J_t = -2 t^{5/4} e^{-P/2+\lambda/4} S_{13}.
\]

Proceeding as above, we thus obtain

\[
\int_{S^1} |K_t - Q J_t| d\theta \leq C t^{-1/2}.
\]
Combining the above estimates, we conclude that the mean value of $K - QJ$ cannot grow faster than $Ct^{1/2}$. Thus

$$\|K - QJ\|_{C^0} \leq Ct. $$

We obtain (47). Using the fact that (23) holds, we conclude that

$$\int_{S^1} |H_t|d\theta \leq \int_{S^1} t^{-5/2} \alpha^{-1/2} e^{P + \lambda/2} |K - QJ|d\theta.$$

Combining this inequality with earlier estimates, we obtain the desired $L^1$-estimate for $H_t$. A similar argument for $G_t$ yields the remaining conclusion of the lemma.

3.1 Causal structure of $T^3$-Gowdy symmetric solutions

It is of interest to note that in the $T^3$-Gowdy symmetric case, it is sufficient to assume future global existence and energy conditions in order to conclude that there is asymptotic silence. In fact, we have the following result.

**Proposition 39.** Consider a future global $T^3$-Gowdy symmetric solution to Einstein’s equations with a cosmological constant $\Lambda > 0$ and a stress energy tensor satisfying the dominant energy condition and the non-negative pressure condition. Then there is a constant $C$, depending only on the solution, such that if

$$\gamma(s) = [s, \theta(s), x(s), y(s)] = [s, \bar{\gamma}(s)]$$

is a causal curve, then

$$|\dot{\bar{\gamma}}(s)|^2 \leq Cs^{-3}.$$

In particular, there is a point $\bar{x}_0 \in T^3$ such that

$$d[\gamma(s), \bar{x}_0] \leq Cs^{-1/2}$$

for all $s \geq t_1$, where $d$ is the standard metric on $T^3$.

**Proof.** The causality of the curve is equivalent to the estimate

$$\alpha^{-1/2} \dot{\theta}^2 + t^{3/2} e^{P-\lambda/2} [\dot{x} + QJ + (G + QH)\dot{\theta}]^2 + t^{3/2} e^{-P-\lambda/2} (\dot{y} + H\dot{\theta})^2 \leq 1.$$

Note that in the case of Gowdy symmetry, $G$ and $H$ are time-independent. In particular, they are thus bounded. As a consequence, it is sufficient to appeal to previous estimates in order to obtain the conclusion of the proposition. $\square$

4 Light cone estimates

In the presence of matter of Vlasov type, it is necessary to consider the characteristic system in parallel with the light cone estimates for the metric components. Let us therefore begin by writing down the characteristic system. It is given by

$$\frac{d\Theta}{ds} = \alpha^{1/2} \frac{V_1}{V_0},$$

$$\frac{dV_1}{ds} = -\frac{1}{4} \alpha^{1/2} \lambda_0 V_1^0 - \frac{1}{4} \left( \lambda_1 - \frac{\alpha_1}{s} \right) V_1^1 + \alpha^{1/2} e^P Q_0 \frac{V_2^2 V_3^3}{V_0^3},$$

$$\frac{dV_2}{ds} = \frac{1}{2} \left( P_1 + \frac{1}{s} \right) V_2^2 - \frac{1}{2} \alpha^{1/2} P_0 \frac{V_1^1 V_2}{V_0^3},$$

$$\frac{dV_3}{ds} = -\frac{1}{2} \left( \frac{1}{s} - P_1 \right) V_3^3 + \frac{1}{2} \alpha^{1/2} P_0 \frac{V_1^1 V_3^3}{V_0^2} - e^P Q_1 V_2^2 - \alpha^{1/2} e^P Q_0 \frac{V_1^1 V_2}{V_0^3}. $$

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Remark 41. As mentioned in Remark 6, we tacitly assume \( s \) for all \( P \) are conserved along characteristics. Since we know \( \Theta \), \( V \) is a solution to (51)–(52) with initial data \( \Theta(t_1), V(t_1) \) such that \( [t_1, \Theta(t_1), V(t_1)] \) is in the support of \( f \), then

\[
|V^2(s)| + |V^3(s)| \leq C s^{-1/2}
\]

for all \( s \geq t_1 \).

Remark 42. As mentioned in Remark 6, we tacitly assume \( f(t_1, \cdot) \) to have compact support.

Proof. Due to (51) and (52), it can be verified that

\[
s^{1/2} e^{P/2} V^2, \quad s^{1/2} Q e^{P/2} V^2 + s^{1/2} e^{-P/2} V^3
\]

are conserved along characteristics. Since we know \( P \) and \( Q \) to be uniformly bounded, cf. Lemma 47, we obtain the conclusion of the lemma.

Let us now turn to \( V^1 \). To begin with, we have the following estimate.

Lemma 40. Consider a \( T^2 \)-symmetric solution to the Einstein-Vlasov equations with a cosmological constant \( \Lambda > 0 \) and existence interval \( (t_0, \infty) \), where \( t_0 \geq 0 \). Assume that the solution has \( \lambda \)-asymptotics and let \( t_1 = t_0 + 2 \). Then there is a constant \( C > 0 \), depending only on the solution, such that if \( \Theta, V \) is a solution to (49)–(52) with initial data \( \Theta(t_1), V(t_1) \) such that \( [t_1, \Theta(t_1), V(t_1)] \) is in the support of \( f \), then

\[
|V^1|^2 \leq -\frac{1}{s} (V^1)^2 + C s^{-1/2} e^{\lambda/2} (Q^1)^2 \frac{|V^1|^2}{V^0} + C s^{1/2} F V^1 |V^1| + C s^{-1/2} (V^1)^2 + C s^{-1/2} |V^1|
\]

for all \( s \geq t_1 \), where \( Q^1(t) := \sup_{t \in [s, t]} |v^1| : (s, \theta, v^1, v^2, v^3) \in \text{supp} f, \ s \in [t_1, t] \) and

\[
F(t) = \sup_{\theta \in S^2} A_+(t, \theta) + \sup_{\theta \in S^2} A_-(t, \theta).
\]

Proof. Due to (51), we have

\[
\frac{d(V^1)^2}{ds} = -\frac{1}{2} \alpha^{1/2} V^0 V^1 - \frac{1}{2} \left( \frac{q}{\alpha} - 1 \right) (V^1)^2 + 2 \alpha^{1/2} e^{P/2} Q e^{P/2} Q V^1 V^2 V^3
\]

\[
- \alpha^{1/2} P V^1 (V^3)^2 - \frac{(V^2)^2}{V^0} + 2 e^{-P/2 + \lambda/4} s^{3/4} V^1 V^2 + 2 e^{-P/2 + \lambda/4} s^{3/4} (K - Q J) V^1 V^2.
\]

However, due to Lemmas 48 and 49, we can estimate the last two terms by \( C s^{-2} |V^1| \). We thus have

\[
\frac{d(V^1)^2}{ds} \leq -\frac{1}{2} \alpha^{1/2} V^0 V^1 - \frac{1}{2} \left( \frac{q}{\alpha} - 1 \right) (V^1)^2 + C s^{-1/2} |V^1| + C s^{-2} |V^1|,
\]
where we have used Lemma [33]. Due to (24) and (28), the sum of the first and the second term on the right hand side can be written
\[ 2s^{1/2}e^{\lambda/2}(J_1V^0 - \rho V^1)V^1 - 2s^{1/2}e^{\lambda/2} \Lambda(V^1)^2 - \frac{e^{-P+\lambda/2}J^2}{2s^{5/2}}(V^1)^2 - \frac{e^{P+\lambda/2}(K - QJ)^2}{2s^{5/2}}(V^1)^2 - sa^{1/2}(P_tP_\theta + e^{2P}Q_tQ_\theta)V^0V^1 - \frac{1}{2}s [P_t^2 + \alpha P_\theta^2 + e^{2P}(Q_t^2 + \alpha Q_\theta^2)] (V^1)^2 + \frac{1}{2} s(V^1)^2. \]

Note that
\[ -sa^{1/2}P_tP_\theta V^0V^1 - \frac{1}{2}s[P_t^2 + \alpha P_\theta^2](V^1)^2 \leq sa^{1/2}|P_tP_\theta||V^0||V^1| - \frac{1}{2}s[P_t^2 + \alpha P_\theta^2](V^1)^2 \leq sa^{1/2}|P_tP_\theta||V^1||V^0| - |V^1|) - \frac{1}{2}s|P_t| - \frac{1}{2}s|P_\theta|^2 (V^1)^2. \]

Combining this estimate with a similar estimate for \( Q \), we conclude that the second and third last terms in (55) can be estimated by
\[ sa^{1/2}(|P_tP_\theta| + e^{2P}|Q_tQ_\theta||V^1| |V^1| = CsF|V^1| V^0. \]

Due to Lemma [36] we have
\[ -2s^{1/2}e^{\lambda/2} \Lambda + \frac{1}{2}s = -\frac{1}{s} + O(s^{-3/2}). \]

Combining the above observations with Lemma [38] we conclude that
\[ \frac{d(V^1)^2}{ds} \leq -\frac{1}{s}(V^1)^2 + 2s^{1/2}e^{\lambda/2}(J_1V^0 - \rho V^1)V^1 + CsF\frac{|V^1|}{V^0} + Cs^{-1}F^{1/2}\frac{|V^1|}{V^0} + Cs^{-3/2}(V^1)^2 + Cs^{-2}|V^1|. \]

Let us estimate the term
\[ T_1 := 2s^{1/2}e^{\lambda/2}(J_1V^0 - \rho V^1)V^1 = 2s^{1/2}e^{\lambda/2}(I^- + I^+)V^1, \quad (56) \]
where
\[ I^-(s) = \int_{\mathbb{R}^2} \int_{-\infty}^{0} [v^1V^0(s) - v^0V^1(s)]f(s, \Theta(s), v)dv^1dv^2dv^3, \]
\[ I^+(s) = \int_{\mathbb{R}^2} \int_{0}^{\infty} [v^1V^0(s) - v^0V^1(s)]f(s, \Theta(s), v)dv^1dv^2dv^3. \]

There are two cases to be distinguished, \( V^1(s) > 0 \) and \( V^1(s) < 0 \). When \( V^1(s) > 0 \), \( I^- \) is non-positive and can be dropped. Furthermore, for \( v^1 \geq 0 \),
\[ v^1V^0 - v^0V^1 = \frac{(v^1)^2(V^0)^2 - (v^0)^2(V^1)^2}{v^1V^0 + v^0V^1} = \frac{(v^1)^2(1 + (V^2)^2 + (V^3)^2)}{v^1V^0 + v^0V^1} - \frac{(V^1)^2(1 + (V^2)^2 + (V^3)^2)}{v^1V^0 + v^0V^1} \leq \frac{v^1(1 + (V^2)^2 + (V^3)^2)}{V^0}. \]
Letting \( E = \{(u^2, v^3) : |v^2| \leq Cs^{-1/2}, |v^3| \leq Cs^{-1/2}\} \), where \( C \) is the constant appearing in Lemma\(^\text{[10]}\) we obtain

\[
T_1 \leq 2\|f(t_1, \cdot)\|_{\infty} s^{1/2} e^{\lambda/2} \int_0^1 \int_0^{Q^1} \frac{v^1(1 + (V^2)^2 + (V^3)^2)}{V_0^1} dv_1 dv_2 dv_3 \leq Cs^{-1/2} e^{\lambda/2} \int_0^{Q^1} v_1 dv_1 \leq Cs^{-1/2} e^{\lambda/2} (Q^1)^2 \frac{V^1}{V_0^1}.
\]

(57)

where \( Q^1(t) \) is defined as in the statement of the lemma. When \( V^1 < 0 \), an analogous argument can be given and it follows that in both cases

\[
T_1 \leq Cs^{-1/2} e^{\lambda/2} (Q^1)^2 \frac{|V^1|}{V_0^1}.
\]

(58)

Combining this observation with previous estimates, we obtain the conclusion of the lemma. \(\square\)

**Lemma 43.** Consider a \( T^2 \)-symmetric solution to the Einstein-Vlasov equations with a cosmological constant \( \Lambda > 0 \) and existence interval \((t_0, \infty)\), where \( t_0 \geq 0 \). Assume that the solution has \( \lambda \)-asymptotics and let \( t_1 = t_0 + 2 \). Then there is a constant \( C > 0 \), depending only on the solution, such that

\[
t^3 \|P_t^2 + \alpha P_\theta^2 + e^{2\lambda/2} Q_\theta^2\|_{C^0} + t |Q^1(t)|^2 \leq C
\]

for all \( t \geq t_1 \).

**Proof.** Let us use \(\text{[59]}\) to derive an estimate for \( F \). Note, to begin with, that

\[
-(\frac{2}{t} - \frac{\alpha}{\Lambda}) \leq -(\frac{2}{t} + 4t^{1/2} e^{\lambda/2} \Lambda) \leq -\frac{5}{t} + O(t^{-3/2}).
\]

Note also that the second term on the right hand side of \(\text{[59]}\) can be written

\[
\frac{1}{2t} (A_+ - A_{\pm}) + \frac{2}{t} \alpha (P_\theta^2 + e^{2\lambda} Q_\theta^2) \leq \frac{1}{t} A_+ + \frac{1}{2t} (A_+ + A_-).
\]

Combining these estimates with Lemma \(\text{[58]}\) we conclude that

\[
\partial_\pm A_+ \leq -\frac{4}{t} A_+ + Ct^{-3/2} A_+ + \frac{1}{2t} (A_+ + A_-) + Ct^{-3} A_+^{1/2} + 2t^{-1/2} e^{\lambda/2} (P_2 - P_3) \partial_+ P + 4t^{-1/2} e^{\lambda/2} S_{23} e^\theta \partial_+ Q.
\]

Moreover,

\[
|P_k| \leq Ct^{-2} \ln(1 + Q^1), \quad |S_{23}| \leq Ct^{-2} \ln(1 + Q^1)
\]

for \( k = 2, 3 \). As a consequence,

\[
\partial_\pm A_+ \leq -\frac{4}{t} A_+ + Ct^{-3/2} A_+ + \frac{1}{2t} (A_+ + A_-) + Ct^{-3} A_+^{1/2} + Ct^{-4} A_+^{1/2} \ln(1 + Q^1).
\]

Defining \( \hat{A}_\pm = t^4 A_\pm + t \), we obtain

\[
\partial_\pm \hat{A}_+ \leq \frac{1}{2t} (A_+ + \hat{A}_-) + Ct^{-3/2} \hat{A}_+ + Ct^{-2} \hat{A}_+^{1/2} \ln(1 + Q^1).
\]

Introducing

\[
\hat{F}(t) = \sup_{\theta \in S^3} \hat{A}_+(t, \theta) + \sup_{\theta \in S^3} \hat{A}_-(t, \theta),
\]

we obtain

\[
\hat{F}(t) \leq \hat{F}(t_1) + \int_{t_1}^t \left( \frac{1}{s} \hat{F}(s) + Cs^{-3/2} \hat{F}(s) + Cs^{-2} \hat{F}^{1/2}(s) \ln(1 + Q^1) \right) ds.
\]

(59)
Introducing
\[ R^1(s) = [s(V^1(s))^2 + 1]^{1/2}; \quad \hat{Q}^1(s) = [s(Q^1(s))^2 + 1]^{1/2}; \]
Lemma 44 implies that
\[ \frac{d(R^1)^2}{ds} \leq C s^{-2}(\hat{Q}^1)^2 + C s^{-2} \hat{F} + C s^{-5/2} \hat{F} + C s^{-3/2}(\hat{Q}^1)^2. \]
Integrating this inequality from \( t_1 \) to \( t \) and taking the supremum over initial data belonging to the support of \( f \), we obtain
\[ [\hat{Q}^1(t)]^2 \leq [\hat{Q}^1(t_1)]^2 + \int_{t_1}^t \left( C s^{-2} \hat{F} + C s^{-3/2} (\hat{Q}^1)^2 \right) ds. \] (60)
Adding (59) and (60) and introducing \( G = \hat{F} + (\hat{Q}^1)^2 \), we obtain
\[ G(t) \leq G(t_1) + \int_{t_1}^t \left( \frac{1}{8} G(s) + C s^{-3/2} G(s) \right) ds. \]
In particular, \( G(t) \leq Ct \), so that \( \hat{F}(t) \leq Ct \) and \( Q^1 \) is bounded. Returning to Lemma 44 with this information in mind, we conclude that
\[ \frac{d(R^1)^2}{ds} \leq C s^{-3/2}(\hat{Q}^1)^2. \]
By arguments similar to ones given above, we conclude that \( \hat{Q}^1 \) is bounded. The lemma follows. \( \square \)

5 Intermediate estimates

Before proceeding, it is useful to collect the estimates that follow from the above arguments.

Lemma 44. Consider a \( T^2 \)-symmetric solution to the Einstein-Vlasov equations with a cosmological constant \( \lambda > 0 \) and existence interval \((t_0, \infty)\), where \( t_0 \geq 0 \). Assume that the solution has \( \lambda \)-asymptotics and let \( t_1 = t_0 + 2 \). Then there is a constant \( C > 0 \), depending only on the solution, such that
\[ \left\| \lambda(t, \cdot) + 3 \ln t - 2 \ln \frac{3}{4\lambda} \right\|_{C^0} \leq Ct^{-1}, \]
\[ \left\| \frac{\alpha}{\alpha + 3} \right\|_{C^0} + \left\| \frac{\lambda t + 3}{\alpha + 3} \right\|_{C^0} \leq Ct^{-2}, \]
\[ \left\| \alpha^{1/2} \lambda \right\|_{C^0} \leq Ct^{-2} \]
for all \( t \geq t_1 \). Moreover,
\[ \left\| e^{P + \lambda/2} (K - QJ) e^{-P + \lambda/2} \right\|_{C^0} \leq Ct^{-4}, \]
\[ \left\| \alpha \right\|_{C^0} + \left\| K \right\|_{C^0} \leq C, \]
\[ \left\| \partial_\theta \left( \frac{e^{P + \lambda/2} (K - QJ) e^{-P + \lambda/2}}{t_0} \right) \right\|_{C^0} \leq Ct^{-4} \]
for all \( t \geq t_1 \). Finally,
\[ \left\| \rho \right\|_{C^0} + t^{1/2} \left\| J_1 \right\|_{C^0} + t \left\| P_1 \right\|_{C^0} + t \left\| S_{im} \right\|_{C^0} \leq Ct^{-3/2} \]
for all \( t \geq t_1 \).
Remark 45. Note that as a consequence of (62),
\[ \left\| \partial_t (\alpha^{-1/2} e^{\lambda/2} \right\|_{C^0} \leq Ct^{-2} \]
for all \( t \geq t_1 \).

Proof. Combining Lemmas 33, 10 and 43 with (25), we conclude that
\[ \partial_t \dot{\lambda} = \frac{3}{t} - \frac{3}{t} e^{\lambda/2} + O(t^{-2}), \]
where we have used the notation (41). Combining this observation with Lemma 36, we conclude that there is a constant \( C \) such that
\[ \partial_t \dot{\lambda}^2 \leq -\frac{3}{t} \dot{\lambda}^2 + Ct^{-2} |\dot{\lambda}|. \]
Introducing \( L = t^2 \dot{\lambda}^2 \), we obtain
\[ \partial_t L \leq -\frac{1}{t} L + \frac{C}{t} L^{1/2}. \]
In particular, it is clear that \( L \) decreases once it exceeds a certain value. As a consequence, \( L \) is bounded, and we obtain (61). Combining (61) with (31) and (25), we then obtain (62).

Returning to (31) and (32), we conclude that (64) holds. As a consequence, it is clear that \( J \) and \( K \) are bounded. Thus (65) holds. Due to (20), (30) and previously derived estimates, we also obtain (66). Combining this observation with Lemma 36, we conclude that (67) holds. Moreover, due to (28) and estimates already derived, we obtain (68). Finally, due to Lemmas 11 and 43, we conclude that (63) holds.

6 Derivatives of the characteristic system

Solutions to the Vlasov equation can be expressed in terms of the initial datum for the distribution function and appropriate solutions to the characteristic system (19)–(22). In fact, let \( \Theta, V \) be the solution to (49)–(52) corresponding to the initial data
\[ \Theta(t; t, \theta, v) = \theta, \quad V(t; t, \theta, v) = v. \]
Given a fixed \( \tau \in (t_0, \infty) \), where \((t_0, \infty)\) is the existence interval of the solution to the Einstein-Vlasov system under consideration, we know that
\[ f(t, \theta, v) = f(\tau, \Theta(\tau; t, \theta, v), V(\tau; t, \theta, v)). \]
Since \( f(\tau, \cdot) \) is a smooth function with compact support, it is sufficient to estimate the derivatives of solutions to the characteristic system in order to estimate the derivatives of \( f \). Unfortunately, differentiating the characteristic system leads to second order derivatives of \( P, Q \) etc., quantities over which we have no control. However, using the ideas introduced in [2], this problem can be circumvented. In fact, let \( \partial \) be a shorthand for \( \partial_t, \partial_\theta \) or \( \partial_v \), and let
\[ \Psi = \alpha^{-1/2} e^{\lambda/2} \partial \Theta, \]
\[ Z^1 = \partial V^1 + \left[ \frac{1}{4} \alpha^{-1/2} \left( \lambda_v - 2 \frac{\alpha}{\alpha} - 4 s^{1/2} e^{\lambda/2} \right) V^0 - \frac{1}{2} \alpha^{-1/2} P \frac{V^0 (V^2)^2 - (V^3)^2}{(V^0)^2 - (V^1)^2} \right. \]
\[ + \left. \frac{1}{2} P \frac{V^0 (V^2)^2 - (V^3)^2}{(V^0)^2 - (V^1)^2} - \alpha^{-1/2} e^P Q \right] \frac{V^0 V^2 V^3}{(V^0)^2 - (V^1)^2} \partial \Theta, \]
\[ Z^2 = \partial V^2 + \frac{1}{2} P \partial V^2 \partial \Theta, \]
\[ Z^3 = \partial V^3 - \left( \frac{1}{2} P \partial V^3 - e^P Q_v V^2 \right) \partial \Theta. \]
It is then possible to derive an ODE for \((\Psi, Z^1, Z^2, Z^3)\) such that the coefficients are controlled due to previous arguments. The definitions \((70)-(73)\) differ slightly from those of \([2]\). The reason for this is that in the present context, it is not sufficient to know that no second order derivatives of \(P, Q\) etc. occur; we need to analyse, in detail, all the terms that appear, and to use the resulting system in order to derive specific estimates for \(\partial \Theta\) and \(\partial V^i\). The relevant result is the following.

**Lemma 46.** Consider a \(\mathbb{T}^2\)-symmetric solution to the Einstein-Vlasov equations with a cosmological constant \(\lambda > 0\) and existence interval \((t_0, \infty)\), where \(t_0 \geq 0\). Assume that the solution has \(\lambda\)-asymptotics and let \(t_1 = t_0 + 2\). Then there is a constant \(C > 0\), depending only on the solution, such that

\[
\begin{align*}
\frac{dZ^1}{ds} &= -\frac{1}{2s} Z^1 + c_{1,\theta} \Psi + c_{1,j} Z^j, \\
\frac{dZ^2}{ds} &= -\frac{1}{2s} Z^2 + c_{2,2} Z^2, \\
\frac{dZ^3}{ds} &= -\frac{1}{2s} Z^3 + c_{3,2} Z^2 + c_{3,3} Z^3, \\
\frac{d\Psi}{ds} &= c_{\theta,\theta} \Psi + c_{\theta,i} Z^i,
\end{align*}
\]

where Einstein’s summation convention applies to \(i\) and \(j\),

\[
|c_{i,j}(s; t, \theta, v)| + |c_{\theta,i}(s; t, \theta, v)| + |c_{i,\theta}(s; t, \theta, v)| + s^{1/2}|c_{\theta,\theta}(s; t, \theta, v)| \leq C s^{-3/2},
\]

and the estimate holds for all \((t, \theta, v) \in [t_1, \infty) \times S^1 \times \mathbb{R}^3\) in the support of \(f\) and for all \(s \in [t_1, t]\).

**Proof.** Let us begin by noting that

\[
\begin{align*}
s^{1/2} e^{-P/2} Z^2 &= \partial (s^{1/2} e^{P/2} V^2), \\
s^{1/2} e^{-P/2} Z^3 + e^{P/2} Q s^{1/2} Z^2 &= \partial (s^{1/2} Q e^{P/2} V^2 + s^{1/2} e^{-P/2} V^3).
\end{align*}
\]

Since the quantities appearing in \((53)\) are preserved along characteristics, we obtain

\[
\frac{d}{ds} \left( e^{P/2} s^{1/2} Z^2 \right) = 0, \quad \frac{d}{ds} \left( s^{1/2} e^{-P/2} Z^3 + e^{P/2} Q s^{1/2} Z^2 \right) = 0.
\]

We also have

\[
\frac{d\Psi}{ds} = \frac{1}{2} \left( \lambda_i - \frac{\alpha_i}{\alpha} \right) \Psi + \frac{1}{2} \alpha^{1/2} \lambda_0 \frac{V^1}{V^0} \Psi + e^{\lambda/2} \partial \left( \frac{V^1}{V^0} \right).
\]

In the end, we shall express \(\partial (V^1/V^0)\) in terms of \(Z^i\) and \(\Psi\). However, there is no immediate gain in doing so here. The most cumbersome part of the argument is to compute the derivative of \(Z^1\). This calculation can be divided into several parts. Let us begin by considering

\[
\frac{d}{ds} (\partial V^1) = 0 \left( \frac{dV^1}{ds} \right).
\]

When calculating the right hand side, it is convenient to divide the result into terms which include a \(\partial V^i\) factor, \(i = 1, 2, 3\), and terms which do not. Due to estimates already derived, the terms which include such a factor can be written

\[
-\frac{1}{2s} \partial V^1 + c_i \partial V^i,
\]

where \(c_i(s) = O(s^{-2})\); note that the factor in front of \(V^1\) is given by

\[
-\frac{1}{2s} + O(s^{-2}).
\]
It is straightforward to calculate the remaining terms, and we conclude that

\[
\frac{d}{ds}(\partial V^1) = -\frac{1}{4} \partial_\theta (\alpha^{1/2}\lambda_\theta)V^0 \partial \Theta - \frac{1}{4} \partial_\theta \left( \lambda_\theta - 2\frac{\alpha_\theta}{\alpha} \right) V^1 \partial \Theta + \partial_\theta (\alpha^{1/2} e^P Q_\theta) \frac{V^2 V^3}{V^0} \partial \Theta \\
- \frac{1}{2} \partial_\theta (\alpha^{1/2} P_\theta) \frac{(V^3)^2 - (V^2)^2}{V^0} \partial \Theta + t^{-7/4} \partial_\theta (e^{\lambda^{1/4} e^{-P/2} J}) V^2 \partial \Theta \\
+ t^{-7/4} \partial_\theta [e^{\lambda^{1/4} e^{-P/2} (K - Q J)}] V^3 \partial \Theta - \frac{1}{2s} \partial V^1 + c_i \partial V^i,
\]

where \( c_i(s) = O(s^{-2}) \). Due to previous estimates, we can estimate the factors multiplying \( \partial \Theta \) in the third and fourth last terms on the right hand side. In fact, we obtain

\[
\frac{d}{ds}(\partial V^1) = -\frac{1}{2s} \partial V^1 - \frac{1}{4} \partial_\theta (\alpha^{1/2} \lambda_\theta)V^0 \partial \Theta - \frac{1}{4} \partial_\theta \left( \lambda_\theta - 2\frac{\alpha_\theta}{\alpha} \right) V^1 \partial \Theta + \partial_\theta (\alpha^{1/2} e^P Q_\theta) \frac{V^2 V^3}{V^0} \partial \Theta \\
- \frac{1}{2} \partial_\theta (\alpha^{1/2} P_\theta) \frac{(V^3)^2 - (V^2)^2}{V^0} \partial \Theta + c_\theta \partial \Theta + c_i \partial V^i,
\]

where \( c_i(s) = O(s^{-2}) \) and \( c_\theta(s) = O(s^{-3}) \). As a next step, it is of interest to consider the terms that arise when \( d/ds \) hits a \( V^n \) in the second term in the definition of \( Z^1 \). Before writing down the result, let us note that

\[
\frac{dV^i}{ds} = -\frac{1}{2s} V^i + O(s^{-2}), \quad \frac{dV^0}{ds} = O(s^{-2}). (77)
\]

Due to these observations, estimates already derived and the definition of \( Z^1 \), we conclude that when \( d/ds \) hits a \( V^n \) in the second term in \( Z^1 \), the resulting expression can be written \( c_\theta \partial \Theta \), where \( c_\theta(s) = O(s^{-2}) \).

Note that every term appearing in the second term in the definition of \( Z^1 \) can be written in the form

\[
h(\cdot, \Theta) \psi(V) \alpha^{-1/2}(\cdot, \Theta) \partial \Theta.
\]

We have already estimated the terms that arise when \( d/ds \) hits \( \psi \). Let us therefore consider the terms that arise when the derivative hits the remaining factors. Omitting the arguments, we need to consider

\[
\left( h_t + \alpha^{1/2} h_\theta \frac{V^1}{V^0} \right) \psi \alpha^{-1/2} \partial \Theta - \frac{\alpha_\theta}{2\alpha^{3/2}} h \psi \partial \Theta \\
+ h \psi \left[ -\frac{\alpha_\theta}{2\alpha^{3/2}} \alpha^{1/2} \frac{V^1}{V^0} \partial \Theta + \alpha^{-1/2} \frac{\alpha_\theta}{2\alpha^{1/2}} \frac{V^1}{V^0} \partial \Theta + \partial \left( \frac{V^1}{V^0} \right) \right] (78)
\]

\[
= \left( \partial_t (\alpha^{-1/2} h) + h_\theta \frac{V^1}{V^0} \right) \psi \partial \Theta + h \psi \partial \left( \frac{V^1}{V^0} \right).
\]

In all the terms of interest, \( h \psi = O(s^{-2}) \). As a consequence, this expression can be written

\[
\left( \partial_t (\alpha^{-1/2} h) + h_\theta \frac{V^1}{V^0} \right) \psi \partial \Theta + c_i \partial V^i,
\]
where \( c_i(s) = O(s^{-2}) \). Adding up the above observations, we conclude that
\[
\frac{dZ^1}{ds} = -\frac{1}{2s} \partial V^1 + \left[ \frac{1}{4} \partial_t \left( \lambda t - 2 \frac{\alpha t}{\alpha} - 4s^{1/2} \epsilon^{\lambda/2} \Lambda \right) \right] V^0 - \frac{1}{4} \partial_t \left( \alpha \lambda t - 2 \frac{\alpha t}{\alpha} - 4s^{1/2} \epsilon^{\lambda/2} \Lambda \right) V^0
\]
\[+ \frac{1}{2} \lambda \theta s^{1/2} \epsilon^{\lambda/2} \Lambda V^1 - \frac{1}{2} \partial_t \left( \alpha \lambda t - 2 \frac{\alpha t}{\alpha} - 4s^{1/2} \epsilon^{\lambda/2} \Lambda \right) V^0 \]
\[\frac{V^0 \partial^2 V^1}{(V^0)^2 - (V^1)^2}\]
\[\frac{V^0 V^2 V^3}{(V^0)^2 - (V^1)^2}\]
\[\frac{V^1 V^2 V^3}{(V^0)^2 - (V^1)^2}\]
\[\frac{V^1 V^2 V^3}{(V^0)^2 - (V^1)^2}\]
\[\frac{V^1 V^2 V^3}{(V^0)^2 - (V^1)^2}\]
where \( c_0(s) = O(s^{-2}) \) and \( c_i(s) = O(s^{-2}) \). This expression can be simplified somewhat. In fact, we have
\[
\frac{dZ^1}{ds} = -\frac{1}{2s} \partial V^1 + \frac{1}{4} \partial_t \left( \lambda t - 2 \frac{\alpha t}{\alpha} - 4s^{1/2} \epsilon^{\lambda/2} \Lambda \right) V^0 - \frac{1}{4} \partial_t \left( \alpha \lambda t - 2 \frac{\alpha t}{\alpha} - 4s^{1/2} \epsilon^{\lambda/2} \Lambda \right) V^0
\]
\[\frac{V^0 \partial^2 V^1}{(V^0)^2 - (V^1)^2}\]
\[\frac{V^0 V^2 V^3}{(V^0)^2 - (V^1)^2}\]
\[\frac{V^1 V^2 V^3}{(V^0)^2 - (V^1)^2}\]
\[\frac{V^1 V^2 V^3}{(V^0)^2 - (V^1)^2}\]
\[\frac{V^1 V^2 V^3}{(V^0)^2 - (V^1)^2}\]
\[\frac{V^1 V^2 V^3}{(V^0)^2 - (V^1)^2}\]
where \( c_0(s) = O(s^{-2}) \) and \( c_i(s) = O(s^{-2}) \). Due to (20), (33), (42) and previous estimates, we conclude that
\[
\frac{dZ^1}{ds} = -\frac{1}{2s} \partial V^1 + c_0 \partial V^1 + c_i \partial V^i,
\]
where \( c_0(s) = O(s^{-3/2}) \) and \( c_i(s) = O(s^{-2}) \). Since
\[
Z^i = \partial V^i + c_i \partial \Psi,
\]
where \( c_{i,\partial}(s) = O(s^{-1/2}) \), we can use this equation, together with (74), (76), in order to obtain the conclusion of the lemma.

**Lemma 47.** Consider a \( T^2 \)-symmetric solution to the Einstein-Vlasov equations with a cosmological constant \( \Lambda > 0 \) and existence interval \( (t_0, \infty) \), where \( t_0 \geq 0 \). Assume that the solution has \( \lambda \)-asymptotics and let \( t = t_0 + 2 \). Then there is a constant \( C > 0 \), depending only on the solution, such that
\[
(\ln s)^2 |\partial \theta \Theta(s; t, \theta, v)| + s^{1/2} |\partial \theta V^i(s; t, \theta, v)| \leq C(\ln t)^2
\]
for all \( s \in [t_1, t] \) and \( (t, \theta, v) \in [t_1, \infty) \times S^1 \times \mathbb{R}^3 \) in the support of \( f \).

**Remark 48.** It is possible to use arguments similar to the ones given below in order to derive estimates for \( \partial \theta \Theta, \partial_v \Theta \) etc.

**Proof.** Let
\[
\hat{Z}^i(s; t, \theta, v) = s^{1/2} Z^i(s; t, \theta, v), \quad \hat{\Psi}(s; t, \theta, v) = (\ln s)^2 \Psi(s; t, \theta, v).
\]
Then, due to Lemma 46,
\[
\frac{d\hat{Z}^1}{ds} = c_{1,\theta}s^{1/2}(\ln s)^{-2}\hat{\Psi} + c_{1,j}\hat{Z}^j,
\]
\[
\frac{d\hat{Z}^2}{ds} = c_{2,2}\hat{Z}^2,
\]
\[
\frac{d\hat{Z}^3}{ds} = c_{3,2}\hat{Z}^2 + c_{3,3}\hat{Z}^3,
\]
\[
\frac{d\hat{\Psi}}{ds} = \frac{2}{s\ln s}\hat{\Psi} + c_{\theta,\theta}\hat{\Psi} + c_{\theta,i}s^{-1/2}(\ln s)^2\hat{Z}^i
\]
for \(s \in [t_1, t]\), with coefficients as in Lemma 46. Introducing
\[
\hat{E} = \sum_{i=1}^{3}(\hat{Z}^i)^2 + (\hat{\Psi})^2,
\]
we conclude that there is a constant \(C > 0\), depending only on the solution, such that
\[
\frac{d\hat{E}}{ds} \geq -C\frac{1}{s(\ln s)^2}\hat{E}
\]
for \(s \in [t_1, t]\). As a consequence,
\[
\hat{E}(s; t, \theta, v) \leq C\hat{E}(t; t, \theta, v)
\]
for \(s \in [t_1, t]\). Let us now assume \(\partial = \partial_{\theta}\). Then
\[
\hat{\Psi}(t; t, \theta, v) = O(\ln t)^2.
\]
Moreover,
\[
\hat{Z}^i(t; t, \theta, v) = [t^{1/2}\partial_{\theta}V^i + O(1)\Psi](t; t, \theta, v) = O(1).
\]
As a consequence, \(\hat{E}(t; t, \theta, v) = O(\ln t)^4\). Thus (80) implies (79); note that the estimate for \(\partial_{\theta}\Theta\) is immediate and that
\[
|\partial_{\theta}V^i| \leq |Z^i| + C\, s^{-1/2}|\partial_{\theta}\Theta|.
\]
The lemma follows.

**7 Higher order light cone estimates**

Before proceeding to the higher order light cone estimates, let us record some consequences of the estimates obtained in Lemma 47.

**Lemma 49.** Consider a \(T^2\)-symmetric solution to the Einstein-Vlasov equations with a cosmological constant \(\Lambda > 0\) and existence interval \((t_0, \infty)\), where \(t_0 \geq 0\). Assume that the solution has \(\lambda\)-asymptotics and let \(t_1 = t_0 + 2\). Then there is a constant \(C > 0\), depending only on the solution, such that
\[
\|\partial_{\theta}\rho\|_{C^0} + t^{1/2}\|\partial_{\theta}J_i\|_{C^0} + t\|\partial_{\theta}S_{ij}\|_{C^0} + t\|\partial_{\theta}P_k\|_{C^0} \leq Ct^{-3/2}(\ln t)^2,
\]
\[
\left\|\partial_{\theta}\left(\frac{\alpha(t)}{\alpha}\right)\right\|_{C^0} \leq C t^{-3/2},
\]
\[
\left\|\frac{\alpha(t)}{\alpha}\right\|_{C^0} \leq C
\]
for all \(t \geq t_1\).
Remark 50. Note that $\partial_\theta (\alpha f) = \partial_t (\alpha_\theta f) = \partial_t \partial_\theta \ln \alpha$.

Proof. The estimate (S1) follows from the fact that (69) and (71) hold and the fact that $|v| \leq C t^{-1/2}$ in the support of $f(t, \cdot)$. Consider (S1). Since $J, K, Q$ and $P$ are bounded in $C^1$ and $\lambda_\theta$ is bounded, the first two terms are $O(t^{-4})$ in $C^1$. Since $\lambda_\theta$ is $O(t^{-1/2})$, the $\theta$-derivative of the third term on the right hand side of (S1) is $O(t^{-3/2})$. Due to (S1), the $\theta$-derivative of the last term is better. Thus (S2) holds. Integrating this estimate yields (S3).

In what follows, we shall proceed inductively in order to derive estimates for higher order derivatives. Let us therefore assume that we have a $T^2$-symmetric solution to the Einstein-Vlasov equations with a cosmological constant $\Lambda > 0$ and existence interval $(t_0, \infty)$, where $t_0 \geq 0$. Assume, moreover, that the solution has $\lambda$-asymptotics and let $t_1 = t_0 + 2$. Let us make the following inductive assumption.

**Inductive assumption 51.** For some $1 \leq N \in \mathbb{Z}$, there are constants $0 \leq m_j \in \mathbb{Z}$ and $C_j$, $j = 1, \ldots, N$, (depending only on $N$ and the solution) such that

$$s^{1/2} \left| \frac{\partial^j V}{\partial \theta^j} (s; t, \theta, v) \right| + \left| \frac{\partial^j \Theta}{\partial \theta^j} (s; t, \theta, v) \right| \leq C_j (\ln t)^{m_j},$$

$$\|P_0\|_{C^N} + \|Q_0\|_{C^N} + t^{3/2} \|P_1\|_{C^N} + t^{3/2} \|Q_1\|_{C^N} \leq C_{N-1},$$

for all $j = 1, \ldots, N$, $(t, \theta, v) \in [t_1, \infty) \times S^1 \times \mathbb{R}^3$ in the support of $f$ and $s \in [t_1, t]$.

**Remarks 52.** The induction hypothesis holds for $N = 1$. In what follows, $C_j$ and $m_j$ will change from line to line. However, they are only allowed to depend on $N$ and the solution.

In this section we prove that, given that Inductive assumption 51 holds, then (S5) holds with $N$ replaced by $N + 1$. In the next section, we close the induction argument by proving that (S4) holds with $j$ replaced by $N + 1$.

We shall need the following consequences of the inductive assumption.

**Lemma 53.** Consider a $T^2$-symmetric solution to the Einstein-Vlasov equations with a cosmological constant $\Lambda > 0$ and existence interval $(t_0, \infty)$, where $t_0 \geq 0$. Assume that the solution has $\lambda$-asymptotics and let $t_1 = t_0 + 2$. Assume, moreover, that Inductive assumption 51 holds for some $1 \leq N \in \mathbb{Z}$. Then there are constants $C_j$, $j = 0, \ldots, N$, and $m_N$, depending only on $N$ and the solution, such that

$$\|\rho\|_{C^N} + t^{1/2} \|J_0\|_{C^N} + t \|P_0\|_{C^N} + t \|S_{im}\|_{C^N} \leq C_N t^{-3/2} (\ln t)^{m_N}$$

$$\|\alpha^{-1} \partial^2_\theta \alpha\|_{C^0} \leq C_j,$$

$$\|J_0\|_{C^N} + \|K_0\|_{C^N} \leq C_N (\ln t)^{m_N},$$

$$\left| \frac{\partial^2 f}{\partial \theta^2} \right| \leq C_j (\ln t)^{m_j};$$

for $t \geq t_1$, $0 \leq j \leq N$, $0 \leq l \leq N - 1$ and $i, m = 1, 2, 3$.

Proof. For $N = 1$, the conclusions follow from Lemmas 14, 19, and the equations (20) and (30). We may thus, without loss of generality, assume that $N \geq 2$. An immediate consequence of the inductive assumption is that, for $0 \leq j \leq N$ and $t \geq t_1$,

$$\left| \frac{\partial^2 f}{\partial \theta^2} \right| \leq C_j (\ln t)^{m_j};$$

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Lemma 54. Consider a $\mathbb{T}^2$-symmetric solution to the Einstein-Vlasov equations with a cosmological constant $\Lambda > 0$ and existence interval $(t_0, \infty)$, where $t_0 \geq 0$. Assume that the solution has $\lambda$-asymptotics and let $t_1 = t_0 + 2$. Assume, moreover, that Inductive assumption 51 holds for some $1 \leq N \in \mathbb{Z}$. Then there is a constant $C_N > 0$, depending only on $N$ and the solution, such that
\[
i^{3/2}\|\partial_\theta^N P_t\|_{C^0} + \|\partial_\theta^N P_\theta\|_{C^0} + t^{3/2}\|e^P \partial_\theta^N Q_t\|_{C^0} + \|e^P \partial_\theta^N Q_\theta\|_{C^0} \leq C_N
\]
for all $t \geq t_1$. As a consequence, (55) holds with $N$ replaced by $N + 1$.

Proof. Let us begin by pointing out that if $N = 1$, then the lower bound is larger than the upper bound in some of the sums below. In that case, the corresponding sum should be equated with zero. Moreover, terms which are bounded by $Ct^{-3}$ for $t \geq t_1$ will sometimes be written $O(t^{-3})$. Let us compute
\[
\partial_\pm[\partial_\theta^N P_t \mp \partial_\theta^N (\alpha^{1/2} P_\theta)] = \partial_\theta^N P_t \mp \partial_\theta^N \left(\frac{\alpha t}{2\alpha} + \alpha^{1/2} P_\theta + \alpha^{1/2} P_{1\theta}\right) \pm \alpha^{1/2} \partial_\theta^{N+1} P_t - \alpha^{1/2} \partial_\theta^{N+1} (\alpha^{1/2} P_\theta) \\
= \partial_\theta^N P_t \mp \frac{1}{2} \sum_{j=0}^{N-1} \beta_j \partial_\theta^{N-j} \left(\frac{\alpha t}{\alpha}\right) \partial^{(j+1)}(\alpha^{1/2} P_\theta) \mp \frac{1}{2} \alpha \partial_\theta^N (\alpha^{1/2} P_\theta) \\
\mp \frac{N\alpha}{2\alpha^{1/2}} \partial_\theta^N P_t \mp \sum_{j=0}^{N-2} \beta_j \partial_\theta^{N-j} (\alpha^{1/2}) \partial^{(j+1)}(P_t) - \partial_\theta^N (\alpha^{1/2} \partial_\theta (\alpha^{1/2} P_\theta)) \\
+ \frac{N\alpha}{2\alpha^{1/2}} \partial_\theta^N (\alpha^{1/2} P_\theta) + \sum_{j=0}^{N-2} \beta_j \partial_\theta^{N-j} (\alpha^{1/2}) \partial^{(j+1)}(\alpha^{1/2} P_\theta),
\]
for $1 \leq j \leq l < N$. Note that we know the inductive hypothesis to be true for $l = 1$. Differentiating (28) $l$ times with respect to $\theta$ and applying to (85), (86), (91) and (92), we conclude that (92) holds with $j$ replaced by $l + 1$. In order to improve our knowledge concerning $\alpha$, let us begin by improving our estimates for the $\theta$-derivatives for $J$ and $K$. Differentiating (29) and (30) $0 \leq j \leq l$ times and using the above information, we conclude that
\[
\|\partial_\theta^j J\|_{C^0} + \|\partial_\theta^j K\|_{C^0} \leq C_j (\ln t)^m,
\]
for $t \geq t_1$ and $0 \leq j \leq l$. Differentiating (34) $l + 1$ times with respect to $\theta$, using the above information, we obtain
\[
\|\partial_\theta^{l+1} \left(\frac{\alpha t}{\alpha}\right)\|_{C^0} = \|\partial_\theta \partial_\theta^l \left(\frac{\alpha t}{\alpha}\right)\|_{C^0} \leq Ct^{-3/2}
\]
for $t \geq t_1$. Thus
\[
\|\partial_\theta^j \left(\frac{\alpha t}{\alpha}\right)\|_{C^0} \leq C_l
\]
for $t \geq t_1$. Combining this estimate with the inductive hypothesis, we conclude that (91) holds with $j$ replaced by $l + 1$. Thus (92) and (91) hold for $1 \leq j \leq N$. We thus conclude that (87), (88) and (89) hold. In addition, the above estimates imply that (90) holds.

We are now in a position to derive higher order light cone estimates.
where the $\beta_j$ are binomial coefficients. Note that all the sums are $O(t^{-3})$ due to Lemma 23. Let us use (26) in order to compute

$$\partial_0^N [P_{tt} - \alpha^{1/2} \partial_0(\alpha^{1/2} P_0)] = \partial_0^N \left( P_{tt} - \alpha P_{t0} - \frac{\alpha_0}{2} P_0 \right)$$

$$= -\frac{1}{t} \partial_0^N P_t + \frac{\alpha_1}{2\alpha} \partial_0^N P_t + \sum_{j=0}^{N-1} \beta_j \partial_0^{N-j} \left( \frac{\alpha_1}{2\alpha} \right) \partial_0^j P_t$$

$$+ \sum_{j=0}^{N-1} \beta_j \partial_0^{N-j} N \partial_0^j (\alpha^{1/2} Q_0)$$

$$+ \partial_0^N \left( \frac{e^{\omega + \lambda/2} (K - QJ)^2}{2t^{1/2}} \right) + t^{-1/2} \partial_0^N [e^{\lambda/2} (P_2 - P_3)].$$

Again, the sums are $O(t^{-3})$, as well as the last three terms on the right hand side. We thus obtain

$$\partial_{\pm} [\partial_0^N P_t + \partial_0^N (\alpha^{1/2} P_0)] = -\frac{1}{t} \partial_0^N P_t + \frac{\alpha_1}{2\alpha} \partial_0^N P_t + \frac{N \alpha_0}{2\alpha^{1/2}} \partial_0^N P_t + \partial_0^N (\alpha^{1/2} P_0)$$

$$+ 2e^{2P} [Q_t \partial_0^N Q_t - \alpha^{1/2} Q_0 \partial_0^N (\alpha^{1/2} Q_0)] - \partial_0^N \left( \frac{e^{\omega + \lambda/2} (K - QJ)^2}{2t^{1/2}} \right)$$

Introducing

$$\mathcal{A}_{\lambda+} = [\partial_0^N P_t \pm \partial_0^N (\alpha^{1/2} P_0)]^2 + e^{2P} [\partial_0^N Q_t \pm \partial_0^N (\alpha^{1/2} Q_0)]^2,$$

we conclude that

$$\partial_{\pm} [\partial_0^N P_t \mp \partial_0^N (\alpha^{1/2} P_0)]^2 \leq -\frac{5}{t} [\partial_0^N P_t \mp \partial_0^N (\alpha^{1/2} P_0)]^2 + \frac{2}{t} \partial_0^N (\alpha^{1/2} P_0) \partial_0^N P_t \pm \partial_0^N (\alpha^{1/2} P_0)$$

$$+ C_N t^{-3} \mathcal{A}_{\lambda+} + C_N t^{-3/2} (\mathcal{A}_{N+1, +} + \mathcal{A}_{N+1, -}),$$

where we have used (62). Let us now consider

$$\partial_{\pm} [\partial_0^N Q_t \mp \partial_0^N (\alpha^{1/2} Q_0)] = \partial_0^N Q_{tt} + \frac{1}{2} \sum_{j=0}^{N-1} \beta_j \partial_0^{N-j} \left( \frac{\alpha_1}{\alpha} \right) \partial_0^j (\alpha^{1/2} Q_0)$$

$$+ \frac{N \alpha_0}{2\alpha^{1/2}} \partial_0^N Q_t + \sum_{j=0}^{N-2} \beta_j \partial_0^{N-j} (\alpha^{1/2} Q_0) \partial_0^j Q_{tt} - \partial_0^N (\alpha^{1/2} Q_0) \partial_0^N Q_t$$

$$+ \frac{N \alpha_0}{2\alpha^{1/2}} \partial_0^N (\alpha^{1/2} Q_0) + \sum_{j=0}^{N-2} \beta_j \partial_0^{N-j} (\alpha^{1/2} Q_0) \partial_0^j (\alpha^{1/2} Q_0).$$

Compute, using (33),

$$\partial_0^N [Q_{tt} - \alpha^{1/2} \partial_0 (\alpha^{1/2} Q_0)] = -\frac{1}{t} \partial_0^N Q_t + \frac{\alpha_1}{2\alpha} \partial_0^N Q_t + \sum_{j=0}^{N-1} \beta_j \partial_0^{N-j} \left( \frac{\alpha_1}{2\alpha} \right) \partial_0^j Q_t - 2(\partial_0^N P_t) Q_t$$

$$- 2P_t (\partial_0^N Q_t) + 2\partial_0^N (\alpha^{1/2} P_0) \alpha^{1/2} Q_0 + 2\alpha^{1/2} \partial_0^N (\alpha^{1/2} Q_0)$$

$$- 2 \sum_{j=1}^{N-1} \beta_j (\partial_0^{N-j} P_t) (\partial_0^j Q_t) + 2 \sum_{j=1}^{N-1} \beta_j (\partial_0^{N-j} (\alpha^{1/2} P_0) (\partial_0^j (\alpha^{1/2} Q_0))$$

$$+ \partial_0^N \left( \frac{e^{\omega - \lambda/2} J (K - QJ)}{t^{1/2}} \right) + 2t^{-1/2} \partial_0^N (e^{\lambda/2 - P} S_{23}).$$
Thus
\[
\partial_\pm[\partial_0^N Q_t + \partial_0^N (\alpha^{1/2} Q_0)] = -\frac{1}{t} \partial_0^N Q_t + \frac{\alpha t}{2\alpha} \partial_0^N (\alpha^{1/2} Q_0) + \frac{N_\alpha^2}{2\alpha^{1/2}} \partial_\pm^N Q_t + \partial_\pm^N (\alpha^{1/2} Q_0) \\
- 2(\partial_\pm^N P_\pm) Q_t - 2P_\pm(\partial_\pm^N Q_t) + 2\partial_\pm^N (\alpha^{1/2} P_\pm) \alpha^{1/2} Q_0 \\
+ 2\alpha^{1/2} P_\pm \partial_\pm^N (\alpha^{1/2} Q_0) + O(t^{-3}).
\]
Consequently,
\[
\partial_\pm[\partial_0^N Q_t + \partial_0^N (\alpha^{1/2} Q_0)]^2 \leq -\frac{5}{t} [\partial_0^N Q_t + \partial_0^N (\alpha^{1/2} Q_0)]^2 + \frac{2}{t} \partial_0^N (\alpha^{1/2} Q_0) [\partial_0^N Q_t + \partial_0^N (\alpha^{1/2} Q_0)] \\
+ C_N t^{-3} A_{N+1,\mp} + C_N t^{-3/2} (A_{N+1,\mp} + A_{N+1,-})
\]
Adding up the above estimates, we conclude that
\[
\partial_\pm A_{N+1,\mp} \leq -\frac{5}{t} A_{N+1,\mp} + \frac{1}{2t}(A_{N+1,\mp} - A_{N+1,\pm}) + \frac{1}{t}(A_{N+1,\mp} + A_{N+1,-}) \\
+ C_N t^{-3/2} (A_{N+1,\mp} + A_{N+1,-}) + C_N t^{-3} A_{N+1,\mp}.
\]
Let us introduce
\[
\bar{A}_{N+1,\pm} = t^{7/2} A_{N+1,\pm} + t^{1/2}, \quad \bar{F}_{N+1,\pm} = \sup_{\theta \in \delta \theta \in \delta \mathbb{S}} \bar{A}_{N+1,\pm}, \quad \bar{F}_{N+1} = \bar{F}_{N+1,\mp} + \bar{F}_{N+1,\pm}
\]
Then
\[
\partial_\pm \bar{A}_{N+1,\pm} \leq \frac{1}{2t} \bar{A}_{N+1,\pm} + C_N t^{-3/2} (\bar{A}_{N+1,\mp} + \bar{A}_{N+1,-}).
\]
Integrating this differential inequality, taking the supremum etc., we obtain
\[
\bar{F}_{N+1}(t) \leq \bar{F}_{N+1}(t_1) + \int_{t_1}^{t} \left( \frac{1}{2s} \bar{F}_{N+1}(s) + C_N s^{-3/2} \bar{F}_{N+1}(s) \right) ds.
\]
Thus \( \bar{F}_{N+1}(t) \leq C_N t^{1/2} \). Combining this observation with previous estimates, we obtain \([\text{**}]\).

8 **Higher order derivatives of the characteristic system**

In the previous section we showed that \([\text{**}]\) holds with \( N \) replaced by \( N + 1 \); i.e., that
\[
\|P_\theta \|_{C^N} + \|Q_\theta \|_{C^N} + t^{3/2} \|P_\pm \|_{C^N} + t^{3/2} \|Q_\pm \|_{C^N} \leq C_N
\]
holds for all \( t \geq t_1 \). We also need to prove that \([\text{**}]\) holds with \( j \) replaced with \( N + 1 \). Before stating the relevant result, let us make the following preliminary observation.

**Lemma 55.** Consider a \( T^2 \)-symmetric solution to the Einstein-Vlasov equations with a cosmological constant \( \Lambda > 0 \) and existence interval \( (t_0, \infty) \), where \( t_0 \geq 0 \). Assume that the solution has \( \lambda \)-asymptotics and let \( t_1 = t_0 + 2 \). Assume, moreover, that Inductive assumption \([\text{**}]\) holds for some \( 1 \leq N \in \mathbb{Z} \). Then there are constants \( C_j \) and \( m_j \), \( j = 0, \ldots, N \), depending only on \( N \) and the solution, such that

\[
\|\rho\|_{C^N} + t^{1/2} \|J_\pm\|_{C^N} + t \|P_\pm\|_{C^N} + t \|S_{im}\|_{C^N} \leq C_N t^{-3/2} (\ln t)^{m_N}, \quad (95)
\]

\[
\left\| \alpha^{-1} \partial_0^\lambda \right\|_{C^0} + t^{1/2} \left\| \partial_0^{\lambda+1} \right\|_{C^0} \leq C_j, \quad (96)
\]

\[
\left\| J_\theta \right\|_{C^N} + \left\| K_\theta \right\|_{C^N} \leq C_N (\ln t)^{m_N}, \quad (97)
\]

\[
\left\| \partial_0^{\lambda+1} \left( \frac{\alpha t}{\alpha} \right) \right\|_{C^0} + \left\| \partial_0^{\lambda+1} \right\|_{C^0} \leq C_N t^{-3/2}, \quad (98)
\]

\[
\left\| \lambda_t - 2 \frac{\alpha t}{\alpha} - 4t^{1/2} e^{\lambda/2} A \right\|_{C^N} \leq C_N t^{-2}, \quad (99)
\]

\[
\left\| \lambda_t - \frac{\alpha t}{\alpha} \right\|_{C^N} \leq C_N t^{-2} \quad (100)
\]
for \( t \geq t_1, 0 \leq j \leq N, 0 \leq l \leq N - 1 \) and \( i,m = 1,2,3 \). Moreover, using the notation
\[
\Psi_j = \partial^j_{\theta} \Psi, \quad Z^i_j = \partial^j_{\theta} Z^i, \quad V^i_j = \partial^j_{\theta} V^i, \quad \Theta_j = \partial^j_{\theta} \Theta
\]
(where the \( \partial \)-operator used to define \( Z \) and \( \Psi \) is given by \( \partial_\theta \)), there are functions \( c_{i,\theta}, i = 1,2,3 \), such that the following estimates hold:
\[
|\Psi_j(s;t,\theta,v)| + s^{1/2}|Z^i_j(s;t,\theta,v)| \leq C_j(\ln t)^{m_j}, \quad (101)
\]
\[
|\Psi_j(s;t,\theta,v) - (\alpha^{-1/2}e^{\lambda/2}Q_{j+1})(s;t,\theta,v)| \leq C_j(\ln t)^{m_j}, \quad (102)
\]
\[
|Z^i_j(s;t,\theta,v) - V^i_j(s;t,\theta,v) - (c_{i,\theta}\Psi_j)(s;t,\theta,v)| \leq C_j s^{-1/2}(\ln t)^{m_j}, \quad (103)
\]
\[
|c_{i,\theta}(s;t,\theta,v)| \leq C_0 s^{-1/2} \quad (104)
\]
for all \( (t,\theta,v) \in [t_1,\infty) \times S^1 \times \mathbb{R}^3 \) in the support of \( f \), \( 0 \leq j \leq N \), \( 0 \leq l \leq N - 1 \) and \( s \in [t_1,t] \).

**Remark 56.** Due to (100),
\[
|\partial^j_{\theta} \alpha^p| \leq C_{p,j} \alpha^p
\]
for all \( (t,\theta) \in [t_1,\infty) \times S^1, p \in \mathbb{R} \) and \( 0 \leq j \leq N \).

**Proof.** Combining Lemma 53 with (14), (26) and (28), we obtain (14)–(18). The estimate (19) is a consequence of the fact that
\[
\lambda_t - 2 \frac{\alpha_t}{\alpha} - 4t^{1/2} e^{\lambda/2} \Lambda = t \left[ P^2 + \alpha P^2 + e^{2P} (Q_1^2 + \alpha Q_0^2) \right] + \frac{e^{P+\lambda/2} f^2}{t^{3/2}} + \frac{e^{P+\lambda/2} (K - QJ)^2}{t^{3/2}} + 4t^{1/2} e^{\lambda/2} \rho.
\]
For similar reasons, (100) holds; cf. (35). Finally, combining the above estimates with the inductive hypothesis, we obtain the remaining estimates.

We now finish the induction argument by proving that (34) holds with \( j \) replaced by \( N + 1 \).

**Lemma 57.** Consider a \( T^2 \)-symmetric solution to the Einstein-Vlasov equations with a cosmological constant \( \Lambda > 0 \) and existence interval \( (t_0,\infty) \), where \( t_0 > 0 \). Assume that the solution has \( \lambda \)-asymptotics and let \( t_3 = t_0 + 2 \). Assume, moreover, that Inductive assumption 51 holds for some \( 1 \leq N \in \mathbb{Z} \). Then (34) holds with \( j \) replaced by \( N + 1 \).

**Proof.** Before proceeding to a proof of the statement, it is of interest to introduce some notation. Let \( b \) be a \( C^1 \) function on \( M = (t_0,\infty) \times S^1 \). Evaluating this function along a characteristic, we obtain
\[
B(s;t,\theta,v) = b[s,\Theta(s;t,\theta,v)].
\]
Differentiating \( B \) with respect to \( \theta \), we obtain
\[
\frac{\partial B}{\partial \theta}(s;t,\theta,v) = \frac{\partial b}{\partial \theta}[s,\Theta(s;t,\theta,v)] \frac{\partial \Theta}{\partial \theta}(s;t,\theta,v). \quad (105)
\]
On the other hand, distinguishing between \( B \) and \( b \) is quite cumbersome in the arguments that we are about to carry out. As a consequence, we shall write \( b \) when we mean \( B \). Moreover, we shall use the notation \( \partial_\theta b \) as a shorthand for \( \partial_\theta B \), whereas \( \partial_\theta b \) should be interpreted as the first factor on the right hand side of (105) and \( \partial b \) should be interpreted as the function mapping \( (s;t,\theta,v) \) to
\[
\frac{\partial b}{\partial \theta}[s,\Theta(s;t,\theta,v)].
\]
In particular, we thus have \( b_\theta = b_\theta \Theta_1 \). Finally, let us point out that if in some expression a \( \partial \)-derivative hits a \( V \) or a \( \Theta \), it is to be interpreted as an ordinary \( \theta \)-derivative.
Note, to begin with, that
due to (76). Differentiating this equality \( N \) times, we obtain
\[
\frac{d\Psi_N}{ds} = \frac{1}{2} \left( \lambda_t - \frac{\alpha_t}{\alpha} \right) \Psi_N + \frac{1}{2} \alpha^{1/2} \lambda_0 \frac{V^1}{V_0} \Psi + e^{\lambda/2} \frac{\partial V^1}{V_0} - e^{\lambda/2} \frac{V^1}{(V_0)^3} \sum_{i=1}^3 V^i \partial V^i
\]
due to (76). Differentiating this equality \( N \) times, we obtain
\[
\frac{d\Psi_N}{ds} = \frac{1}{2} \left( \lambda_t - \frac{\alpha_t}{\alpha} \right) \Psi_N + \frac{1}{2} \alpha^{1/2} \lambda_0 \frac{V^1}{V_0} \Psi_N + e^{\lambda/2} \frac{V^1}{V_0} - e^{\lambda/2} \frac{V^1}{(V_0)^3} \sum_{i=1}^3 V^i \partial V^i + O[s^{-2} (\ln t)^m N].
\]
As a consequence,
\[
\text{Since we already have the estimate (107), we can proceed as above in order to obtain}
\]
\[\ldots\]
\[
\text{Note, to begin with, that}
\]
\[
\frac{d\Psi_N}{ds} = \frac{1}{2} \left( \lambda_t - \frac{\alpha_t}{\alpha} \right) \Psi_N + \frac{1}{2} \alpha^{1/2} \lambda_0 \frac{V^1}{V_0} \Psi_N + e^{\lambda/2} \frac{V^1}{V_0} - e^{\lambda/2} \frac{V^1}{(V_0)^3} \sum_{i=1}^3 V^i \partial V^i + O[s^{-2} (\ln t)^m N].
\]
Due to (103), this equation can be written
\[
\frac{d\Psi_N}{ds} = c_N^N \Psi_N + c_N^N Z^i_N + O[s^{-2} (\ln t)^m N],
\]
where \( c_N^N = O(s^{-2}) \) and \( c_N^N = O(s^{-3/2}) \) and we sum over \( i \) but not \( N \). Turning to \( Z^2 \), we have
\[
\frac{dZ^2}{ds} = -\frac{1}{2s} Z^2 - \frac{1}{2} \left( P_t + \alpha^{1/2} P_0 \frac{V^1}{V_0} \right) Z^2;
\]
cf. (74). Differentiating this equality \( N \) times with respect to \( \theta \), we obtain
\[
\frac{dZ^2}{ds} = -\frac{1}{2s} Z^2_N - \frac{1}{2} \left( P_t + \alpha^{1/2} P_0 \frac{V^1}{V_0} \right) Z^2_N + O[s^{-2} (\ln t)^m N].
\]
Integrating this equality from \( s \) to \( t \), we obtain (assuming \( N \geq 1 \))
\[
-\left( s^{-1/2} e^{\lambda/2} Z^2_N \right) (s; t, \theta, v) = O[(\ln t)^m N];
\]
\[
\text{note that}
\]
\[
\left( t^{-1/2} e^{\lambda/2} Z^2_N \right) (t; t, \theta, v) = t^{-1/2} e^{\lambda/2} \frac{1}{2} \partial^N_\theta P_0^2 = O(1)
\]
due to (74). In particular, we thus have
\[
|Z^2_N(s; t, \theta, v)| \leq C_N s^{-1/2} (\ln t)^m N
\]
for \( s \in [t_1, t] \). Turning to \( Z^3 \), we have
\[
\frac{dZ^3}{ds} = -\frac{1}{2s} Z^3 + \frac{1}{2} \left( P_t + \alpha^{1/2} P_0 \frac{V^1}{V_0} \right) Z^3 - e^{\lambda} \left( Q_t + \alpha^{1/2} Q_0 \frac{V^1}{V_0} \right) Z^2;
\]
cf. (75). Differentiating \( N \) times with respect to \( \theta \), we obtain
\[
\frac{dZ^3}{ds} = -\frac{1}{2s} Z^3_N + \frac{1}{2} \left( P_t + \alpha^{1/2} P_0 \frac{V^1}{V_0} \right) Z^3_N - e^{\lambda} \left( Q_t + \alpha^{1/2} Q_0 \frac{V^1}{V_0} \right) Z^2_N + O[s^{-2} (\ln t)^m N].
\]
Since we already have the estimate (107), we can proceed as above in order to obtain
\[
|Z^3_N(s; t, \theta, v)| \leq C_N s^{-1/2} (\ln t)^m N
\]
for \( s \in [t_1, t] \). Finally, we need to derive an equation for \( Z^1_N \). Just as in the derivation of the equation for \( Z^1 \), it is natural to divide the analysis into several steps. Consider, to begin with,
for \(0 \leq j \leq N\). All the terms appearing in \(dV^1/ds\) can be written \(h \psi \circ V\). When differentiating an expression of this form, the terms that arise are (up to numerical factors) of the form \(\partial^k_\vartheta h \partial^l_\theta \psi \circ V\). If both \(k\) and \(l\) are \(\geq 1\), the resulting term is \(O(s^{-2}(\ln t)^{m_j})\). If all the derivatives hit \(\psi\), we obtain (after summing over all the terms appearing in \(dV^1/ds\))

\[-\frac{1}{2s}V^1_{j+1} + c_i^j V^i_{j+1} + O(s^{-2}(\ln t)^{m_j})\]

where \(c_i^j = O(s^{-2})\) and we sum over \(i\) but not \(j\). If all the derivatives hit \(h\), we obtain (after summing over all the terms appearing in \(dV^1/ds\))

\[-\frac{1}{4} \partial^{j+1}(\alpha^{1/2}\lambda\theta)V^0 - \frac{1}{4}(\lambda_j - 2\alpha_l\alpha) V^1 + \partial^{j+1}(\alpha^{1/2}\epsilon P\theta) \frac{V^2 V^3}{V^0} \]

\[-\frac{1}{2} \partial^{j+1}(\alpha^{1/2}\epsilon P\theta) \frac{(V^3 - V^2)^2}{V^0} + c_j^i \Theta_{j+1} + O(s^{-3}(\ln t)^{m_j})\]

where \(c^j_i = O(s^{-2})\); note that, due to the above estimates, we control \(N + 1\) \(\theta\)-derivatives of the first factor in each of the last two terms appearing on the right hand side of (108). Adding up, we conclude that

\[
\partial^{j+1}\left(\frac{dV^1}{ds}\right) = -\frac{1}{2s}V^1_{j+1} + c_i^j V^i_{j+1} + c_j^i \Theta_{j+1} - \frac{1}{4} \partial^{j+1}(\alpha^{1/2}\lambda\theta)V^0 - \frac{1}{4}(\lambda_j - 2\alpha_l\alpha) V^1
+ \partial^{j+1}(\alpha^{1/2}\epsilon P\theta) \frac{V^2 V^3}{V^0} - \frac{1}{2} \partial^{j+1}(\alpha^{1/2}\epsilon P\theta) \frac{(V^3 - V^2)^2}{V^0} + O(s^{-2}(\ln t)^{m_j}),
\]

(109)

where \(c^j_i = O(s^{-2})\) and \(c^j_i = O(s^{-3})\) and we sum over \(i\) but not \(j\).

The second term in the definition of \(Z^1\) consists of a sum of terms of the form

\[h \psi \circ V \alpha^{-1/2} \Theta_1.\]

(110)

The relevant \(h\)'s are

\[h_1 = \frac{1}{4}(\lambda_j - 2\alpha_l\alpha - 4s^{1/2}\epsilon^{1/2}\lambda),\]
\[h_2 = -\frac{1}{2} P_t,\]
\[h_3 = \frac{1}{2} \alpha^{1/2} P\theta,\]
\[h_4 = -\epsilon P\theta,\]
\[h_5 = \alpha^{1/2} \epsilon P\theta,\]

and the relevant \(\psi\)'s are

\[\psi_1 = V^0,\]
\[\psi_2 = V^0\frac{(V^2)^2 - (V^3)^2}{(V^0)^2 - (V^1)^2},\]
\[\psi_3 = V^1\frac{(V^2)^2 - (V^3)^2}{(V^0)^2 - (V^1)^2},\]
\[\psi_4 = \frac{V^0 V^2 V^3}{(V^0)^2 - (V^1)^2},\]
\[\psi_5 = \frac{V^1 V^2 V^3}{(V^0)^2 - (V^1)^2}.\]

We want to differentiate (111) with respect to \(s\) and then \(N\) times with respect to \(\vartheta\). Before turning to the details, let us record the following estimate:

\[s^{1/2} |(\partial^j_\vartheta h_1)(s; t, \theta, v)| + \sum_{i=2}^{5} |(\partial^j_\vartheta h_i)(s; t, \theta, v)| \leq C_{N}s^{-3/2}(\ln t)^{m_N} \]

(111)

for \(0 \leq j \leq N, (t, v, \theta) \in [t_1, \infty) \times S^1 \times \mathbb{R}^3\) in the support of \(f\) and \(s \in [t_1, t]\). In the case of \(h_i, i = 2, \ldots, 5,\) (111) is an immediate consequence of the inductive hypothesis, (94) and Lemma (56) and in the case of \(h_1,\) it is a consequence of (93). We also have

\[|\psi_1(s; t, \theta, v)| + \sum_{i=2}^{5} |\psi_i(s; t, \theta, v)| \leq C,\]
\[\sum_{i=1}^{5} |(\partial^j_\vartheta \psi_i \circ V)(s; t, \theta, v)| \leq C_j s^{-1}(\ln t)^{m_j} \]

(112)
for $0 \leq j \leq N-1$, $(t, v, \theta) \in [t_1, \infty) \times \mathbb{S}^1 \times \mathbb{R}^3$ in the support of $f$ and $s \in [t_1, t]$; this is an immediate consequence of the inductive hypothesis.

Let us consider the term that arises when $d/ds$ hits the $\psi$-factor in (110). Note, to this end, that

$$
\left| \frac{\partial_i^j \left( \frac{dV_i}{ds} \right) }{c_f} \right| \leq C_N s^{-3/2}(\ln t)^{m_N}
$$

for all $i = 1, 2, 3$ and all $0 \leq j \leq N$; for $j = 0$, the estimate is a consequence of (77); for $j \geq 1$ and $i = 1$, it is a consequence of (109); and in the case of $i = 2, 3$, it follows immediately from (51), (52) and the induction hypothesis. Due to the above observations,

$$
\partial_i^0 \left( h \frac{d\psi}{ds} V_{\alpha^{-1/2}} \right) = c_f^N \Theta_{N+1} + O[s^{-2}(\ln t)^{m_N}],
$$

where $c_f^N = O(s^{-2})$. When the $s$-derivative hits the remaining terms in (110) (not $\psi$), we obtain

$$
\partial_i^0 \left[ \left( \partial_i (\alpha^{-1/2} h_{2i}) + h_{2i} \frac{V_1}{V_0} \right) \psi \Theta_1 + h \psi \frac{\partial_i V_1}{V_0} - h \psi \frac{\partial_i V_1}{V_0^3} \right],
$$

cf. (78). Due to (111) and (112), this expression can be written

$$
\partial_i^0 \left[ \left( \partial_i (\alpha^{-1/2} h_{2i}) + h_{2i} \frac{V_1}{V_0} \right) \psi \Theta_1 \right] + \sum_{i=1}^3 c_i^N V_{N+1}^i + O[s^{-2}(\ln t)^{m_N}],
$$

where $c_i^N = O(s^{-2})$. Differentiating the second term appearing in the definition of $Z^1$ once with respect to $s$ and $N$ times with respect to $\theta$, we obtain (by adding up the above)

$$
\sum_{i=1}^5 \partial_i^0 \left[ \left( \partial_i (\alpha^{-1/2} h_{2i}) + h_{2i} \frac{V_1}{V_0} \right) \psi \Theta_1 \right] + \sum_{i=1}^3 c_i^N V_{N+1}^i + c_f^N \Theta_{N+1} + O[s^{-2}(\ln t)^{m_N}],
$$

where $c_f^N = O(s^{-2})$ and $c_i^N = O(s^{-2})$. In order to obtain the desired equation we need to add this expression to (109) with $j = N$. However, before doing so, note that

$$
-\frac{1}{4} \partial_i^0 (\alpha^{1/2} P_i) V^0 = -\frac{1}{4} \partial_i^0 [\partial_i (\alpha^{1/2} \lambda_0) V^0 \Theta_1] + O[s^{-2}(\ln t)^{m_N}]
$$

etc. Due to this observation, we can argue as in the proof of Lemma [40] in order to obtain

$$
\frac{dZ^1_i}{ds} = -\frac{1}{2s} V_{N+1}^i + \partial_i^0 \left[ \frac{\partial_i}{\partial t} \left( \alpha^{-1/2} \left( \lambda_i - \frac{2\alpha_t}{\alpha} - 4s^{1/2} e^{\lambda/2} \Lambda \right) \right) \right] V^0 \Theta_1 - \frac{1}{4} \partial_i^0 (\alpha^{1/2} \lambda_0) V^0 \Theta_1
$$

$$
- \frac{1}{2} \lambda_0 s^{1/2} e^{\lambda/2} \Lambda V^1 \Theta_1 - \frac{1}{2} [\partial_t (\alpha^{-1/2} P_i) - \partial_i (\alpha^{1/2} \lambda_0)] V^0 \frac{(V^2)^2 - (V^3)^2}{(V_0)^2} \Theta_1
$$

$$
- \frac{1}{2} \left[ \partial_i (\alpha^{-1/2} e^P Q_i) - \partial_i (\alpha^{1/2} e^P Q_0) \right] \frac{V^0 V^2 V^3}{(V_0)^2} \Theta_1
$$

$$
+ c_f^N \Theta_{N+1} + c_f^N V_{N+1}^i + O[s^{-2}(\ln t)^{m_N}],
$$

where $c_f^N = O(s^{-2})$, $c_i^N = O(s^{-2})$ and we sum over $i$ but not over $N$. The term of importance is the second one on the right hand side. If all the $\theta$-derivatives hit $\Theta_1$, the resulting term can be dealt with as in the proof of Lemma [40] and we obtain a $c_f^N \Theta_{N+1}$-term, where $c_f^N = O(s^{-3/2})$. In the case of all the remaining terms, it is possible to use the equations (as in the proof of Lemma [40]) together with previous estimates in order to obtain terms of the form $O[s^{-3/2}(\ln t)^{m_N}]$. Thus

$$
\frac{dZ^1_i}{ds} = -\frac{1}{2s} V_{N+1}^i + c_f^N \Theta_{N+1} + c_f^N V_{N+1}^i + O[s^{-3/2}(\ln t)^{m_N}],
$$

36
where \( c_\phi^N = O(s^{-3/2}) \), \( c_i^N = O(s^{-3/2}) \) and we sum over \( i \) but not over \( N \). Due to (102) and (103), we conclude that
\[
\frac{dZ_N^1}{ds} = -\frac{1}{2s}Z_N^1 + \frac{c_\phi^N \Psi_N + c_i^N Z_N^1}{3} + O[s^{-3/2}(\ln t)^{m_N}],
\]
where \( c_\phi^N = O(s^{-3/2}) \), \( c_i^N = O(s^{-3/2}) \) and we sum over \( i \) but not over \( N \). Combining this equation with (106), (107) and (108), we conclude that
\[
\frac{d\hat{\Psi}_N}{ds} = \frac{2}{s\ln s} \hat{\Psi}_N + \frac{c_\phi^N \hat{\Psi}_N + c_i^N s^{-1/2}(\ln s)^2 \hat{Z}_N^1}{s} + O[s^{-2}(\ln s)^2(\ln t)^{m_N}],
\]
\[
\frac{d\hat{Z}_N^1}{ds} = \frac{c_i^N s^{1/2}(\ln s)^{-2} \hat{\Psi}_N + c_i^N s \hat{Z}_N^1}{s} + O[s^{-1}(\ln t)^{m_N}],
\]
where \( c_\phi^N = O(s^{-2}) \), \( c_i^N = O(s^{-2}) \), \( c_i^N = O(s^{-2}) \), \( c_i^N = O(s^{-2}) \), there is no summation over \( N \) and we have used the notation
\[
\hat{Z}_N^1(s; t, \theta, v) = s^{1/2}Z_N^1(s; t, \theta, v), \quad \hat{\Psi}_N(s; t, \theta, v) = (\ln s)^2\Psi_N(s; t, \theta, v).
\]

Introducing the energy
\[
\hat{E}_N = (\hat{\Psi}_N)^2 + (\hat{Z}_N^1)^2,
\]
we conclude that
\[
\frac{d\hat{E}_N}{ds} \geq -\frac{C_N}{s(\ln s)^2} \hat{E}_N - C_N s^{-1}(\ln t)^{m_N} \hat{E}_N^{1/2}.
\]

Letting \( r_N \) be such that \( r_N(t_1) = 0 \) and its derivative is the first factor in the first term on the right hand side, we obtain
\[
\frac{d\mathcal{E}_N}{ds} \geq -C_N s^{-1}(\ln t)^{m_N} \mathcal{E}_N^{1/2},
\]
where \( \mathcal{E}_N = \exp(-r_N)\hat{E}_N \). Dividing by \( \mathcal{E}_N^{1/2} \) and integrating from \( s \) to \( t \), we obtain
\[
\mathcal{E}_N^{1/2}(s; t, \theta, v) \leq \mathcal{E}_N^{1/2}(t; t, \theta, v) + C_N(\ln t)^{m_N}.
\]

However, the first term on the right hand side can be estimated by \( C_N(\ln t)^2 \). Combining the resulting estimate with (102) and (103), we conclude that (54) holds with \( j = N + 1 \).

**Corollary 58.** Consider a \( T^2 \)-symmetric solution to the Einstein-Vlasov equations with a cosmological constant \( \Lambda > 0 \) and existence interval \( (t_0, \infty) \), where \( t_0 \geq 0 \). Assume that the solution has \( \lambda \)-asymptotics and let \( t_1 = t_0 + 2 \). Let \( 0 \leq k \in \mathbb{Z} \). Then there is a constant \( C_k \), depending only on \( k \) and the solution, such that
\[
\|P_t\|_{C^k} + \|Q_t\|_{C^k} + t^{1/2}\|\alpha_{1/2}\lambda_{\theta}\|_{C^k} \leq C_k t^{-2}
\]
for all \( t \geq t_1 \). Moreover,
\[
\left\| \frac{\alpha_k}{\alpha} + \frac{3}{t} \right\|_{C^k} + \left\| \lambda_{t} + \frac{3}{t} \right\|_{C^k} \leq C_k t^{-2}
\]
for all \( t \geq t_1 \).

**Proof.** Due to (53), (50) and previous estimates, we conclude that
\[
\|\partial_t(t^{\alpha_{1/2}}P_t)\|_{C^k} + \|\partial_t(t^{\alpha_{1/2}}Q_t)\|_{C^k} \leq C_k t^{-2/\alpha}.
\]
As a consequence,
\[
\|t^{\alpha_{1/2}}P_t\|_{C^k} + \|t^{\alpha_{1/2}}Q_t\|_{C^k} \leq C_k t^{1/2}.
\]
Due to this and previous estimates, we can proceed inductively in order to conclude that
\[
\|P_t\|_{C^k} + \|Q_t\|_{C^k} \leq C_k t^{-2}.
\]
Combining this estimate with (28) and previous estimates, we obtain (113). Combining (113) with (24), (25) and (22), we obtain the final estimate stated in the corollary. \( \square \)
9 Energy estimates for the distribution function

In the proof of the existence of \( f_{sc,\infty} \), cf. Theorem 7, a natural first step is to estimate \( L^2 \)-based energies for \( f \). In the process of deriving such estimates, it is useful to consider equations for the derivatives of the distribution function. Such equations take the following general form:

\[
\frac{\partial h}{\partial t} + \alpha^{1/2} v^1 \frac{\partial h}{\partial \theta} - \frac{1}{2t} v^i \frac{\partial h}{\partial v^i} = R.
\]  

(114)

In case \( h = f \), \( R \) is given by

\[
R = L^i \frac{\partial f}{\partial v^i},
\]

(115)

where

\[
L^1 = \frac{1}{4} \alpha^{1/2} \lambda_0 v^0 + \frac{1}{4} \left( 3 - \alpha \right) v^1 - \frac{\alpha}{t} e^{P/2} \frac{v^2 v^3}{v^0} + \frac{1}{2} \frac{v^3}{v^0} \left( Q_0 - Q J \right),
\]

(116)

\[
L^2 = \frac{1}{2} P_0 v^2 + \frac{1}{2} \alpha^{1/2} P_0 \frac{v^3}{v^0},
\]

(117)

\[
L^3 = - \frac{1}{2} P_0 v^3 - \frac{1}{2} \alpha^{1/2} P_0 \frac{v^3}{v^0} + e^{P/2} \left( Q_t + \frac{\alpha}{2} Q_0 \frac{v^1}{v^0} \right).
\]

(118)

The energies we shall consider are

\[
E_k[h](t) = \sum_{t + |\beta| \leq k} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} t^{-|\beta|} |\partial_\theta^\beta \alpha h(t, \theta, v)|^2 \alpha^{-1/2} t^{-3/2} dv d\theta.
\]

(119)

We shall also use the notation \( E = E_0 \).

**Remarks 59.** The purpose of the factor \( \alpha^{-1/2} t^{-3/2} \) is to simplify some of the terms that result upon carrying out partial integrations. We could equally well consider energies of the form

\[
H_k[f](t) = \sum_{t + |\beta| \leq k} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} t^{-|\beta|} (t^{1/2} v)^{2\mu + 2|\beta|} |\partial_\theta^\beta \alpha f(t, \theta, v)|^2 dv d\theta
\]

for \( \mu \geq 0 \); cf. [27]. However, there is a constant \( C > 1 \), depending only on the solution, \( \mu \) and \( \beta \), such that

\[
C^{-1} \leq (t^{1/2} v)^{2\mu + 2|\beta|} \leq C
\]

for \( t \geq t_1 \) (where \( t_1 \) is defined as in the statement of previous lemmas) and \( (t, \theta, v) \) in the support of \( f \). As a consequence, the corresponding weight is of no practical importance.

**Lemma 60.** Consider a \( T^2 \)-symmetric solution to the Einstein-Vlasov equations with a cosmological constant \( \lambda > 0 \) and existence interval \( (t_0, \infty) \), where \( t_0 \geq 0 \). Assume that the solution has \( \lambda \)-asymptotics and let \( t_1 = t_0 + 2 \). Let \( h \) be a smooth solution to (114) (where the function \( \alpha \) is the object appearing in the Einstein-Vlasov equations and \( R \) is some function) which has compact support when restricted to compact time intervals. Then there is a constant \( C > 0 \), depending only on the solution to the Einstein-Vlasov equations, such that

\[
\frac{dE[h]}{dt} \leq -\frac{3}{2t} E[h] + 2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} h R \alpha^{-1/2} t^{-3/2} dv d\theta + C t^{-2} E[h]
\]

for all \( t \geq t_1 \).

**Remark 61.** It is important to note that the constant \( C \) does not depend on \( h \). Moreover, \( R \) should be thought of as being defined by (114). In particular, due to the assumptions concerning \( h \), the function \( R \) is smooth and has compact support when restricted to compact time intervals.
Proof. Differentiating $E$ with respect to time, we obtain

$$\frac{dE}{dt} = 2 \int_{S^1} \int_{R^3} h \partial_t h \alpha^{-1/2} t^{-3/2}dv d\theta + \int_{S^1} \int_{R^3} h^2 \left( - \frac{3}{2t} - \frac{\alpha^4}{2\alpha} \right) \alpha^{-1/2} t^{-3/2} dv d\theta. \quad (120)$$

Due to (60), we can estimate the second term on the right hand side. Consider the first term on the right hand side of (120). Using (114), it can be written

$$2 \int_{S^1} \int_{R^3} h \left( - \frac{\alpha^{1/2} v^1}{v^0} \frac{\partial h}{\partial \theta} + \frac{1}{2t} v^1 \frac{\partial h}{\partial v^1} + R \right) \alpha^{-1/2} t^{-3/2} dv d\theta.$$

The term involving $\partial_\theta h$ can be integrated to zero. The term involving $R$ we leave as it is. What remains is to estimate the term

$$\frac{1}{2t} \int_{S^1} \int_{R^3} v^1 \frac{\partial h}{\partial v^1} \alpha^{-1/2} t^{-3/2} dv d\theta = - \frac{3}{2t} \int_{S^1} \int_{R^3} h^2 \alpha^{-1/2} t^{-3/2} dv d\theta.$$

The lemma follows. \hfill \Box

Let us turn to the higher order derivatives of the distribution function.

Lemma 62. Consider a $T^2$-symmetric solution to the Einstein-Vlasov equations with a cosmological constant $\Lambda > 0$ and existence interval $(t_0, \infty)$, where $t_0 \geq 0$. Assume that the solution has $\lambda$-asymptotics and let $t_1 = t_0 + 2$. Fix $0 \leq k \in \mathbb{Z}$. Then there is a constant $C_k > 0$, depending only on $k$ and the solution to the Einstein-Vlasov equations, such that

$$\frac{dE_k[f]}{dt} \leq - \frac{3}{2t} E_k[f] + C_k t^{-3/2} E_k[f]$$

for all $t \geq t_1$. In particular, $t^{3/2} E_k[f]$ is bounded to the future.

Proof. Differentiating (114) with $h = f$, we obtain

$$\frac{\partial f_{\beta,l}}{\partial t} + \alpha^{1/2} v^1 \frac{\partial f_{\beta,l}}{\partial \theta} - \frac{1}{2t} v^1 \frac{\partial f_{\beta,l}}{\partial v^1} = \partial^\beta_o \partial^l_o f + \left[ \frac{\alpha^{1/2} v^1}{v^0} \partial_{\theta} \partial^\beta_o \partial^l_o f \right] - \frac{1}{2t} \left[ v^1 \partial^\beta_o \partial^l_0 f \right], \quad (121)$$

where we use the notation $f_{\beta,l} = \partial^\beta_o \partial^l_o f$ and assume that $|\beta| + l \leq k$. Let us denote the right hand side of (121) by $R_{\beta,l}$. Due to Lemma 60, it is of interest to estimate

$$2 \int_{S^1} \int_{R^3} t^{-|\beta|} f_{\beta,l} R_{\beta,l} \alpha^{-1/2} t^{-3/2} dv d\theta. \quad (122)$$

By an inductive argument, it can be proven that the third term on the right hand side of (121) is given by $|\beta| f_{\beta,l} / 2t$. The corresponding contribution to (122) is thus

$$\frac{|\beta|}{t} t^{-|\beta|} E[f_{\beta,l}].$$

Turning to the second term on the right hand side of (121), it can (up to numerical factors) be written as a sum of terms of the form

$$\partial^\beta_v \partial^l_0 \left( \frac{\alpha^{1/2} v^1}{v^0} \right) \partial^\beta \partial^l_0 f,$$

where $\beta_1 + \beta_2 = \beta$, $l_1 + l_2 = l$ and $|\beta_1| + l_1 \geq 1$. Note that the first factor can always be estimated by $C_k t^{-3/2}$. In case $\beta_1 = 0$, it can be estimated by $C_k t^{-2}$ (on the support of $f$). Due to these observations, we have

$$t^{-|\beta|/2} \left[ \frac{\alpha^{1/2} v^1}{v^0} \partial_{\theta} \partial^\beta \partial^l_0 f \right] t^k \leq C_k t^{-2} \sum_{l_1 + |\beta_1| \leq k} t^{-|\beta|/2} f_{\beta_1,l_1}.$$
The corresponding contribution to (122) can thus be estimated by
\[ C_k t^{-2} E_k[f]. \]

Finally, let us consider the first term on the right hand side of (121). Since \( R \) is given by (115), the expression \( \partial^\beta_v \partial^\beta_s R \) is given by the sum of
\[ L^i \partial_v \partial^\beta_s \partial^\beta_s f \quad (123) \]
and terms which (up to numerical factors) can be written
\[ \partial^\beta_1 \partial^\beta_2 L^i \partial_v \partial^\beta_s \partial^\beta_s f, \quad (124) \]
where \( |\beta_1| + l_1 \geq 1 \). The contribution to (122) from (123) can be written
\[ \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} t^{-|\beta|} L^i \partial_v \partial^\beta_s f \beta,l \alpha^{-1/2} t^{-3/2} dv d\theta. \]
Integrating partially with respect to \( v^i \) and keeping in mind that the \( L^i \) are given by (116)–(118), we conclude that this expression can be estimated by
\[ C_k t^{-3/2} E_k[f]. \quad (125) \]

Let us now consider the contribution arising from terms of the form (124). It is natural to divide these terms into two different categories. Either \( \beta_1 = 0 \), or \( \beta_1 \neq 0 \). In case \( \beta_1 = 0 \), the expression (124) can be estimated by
\[ C_k t^{-3/2} |\partial_v \partial^\beta_2 \partial^\beta_2 f|. \]
In case \( \beta_1 \neq 0 \), the expression (124) can be estimated by
\[ C_k t^{-3/2} |\partial_v \partial^\beta_1 \partial^\beta_2 \partial^\beta_2 f|. \]
As a consequence, the contribution to (122) from terms of the form (124) can be estimated by (125). Adding up the above observations, we conclude that
\[ \frac{d}{dt} \left( t^{-|\beta|} E[f_{\beta,l}] \right) = - \frac{|\beta|}{t} t^{-|\beta|} E[f_{\beta,l}] + t^{-|\beta|} \frac{dE[f_{\beta,l}]}{dt} \leq - \frac{|\beta|}{t} t^{-|\beta|} E[f_{\beta,l}] - \frac{3}{2t} t^{-|\beta|} E[f_{\beta,l}]
+ 2 \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} t^{-|\beta|} R_{\beta,l} f_{\beta,l} \alpha^{-1/2} t^{-3/2} dv d\theta + C t^{-2} t^{-|\beta|} E[f_{\beta,l}]
\leq - \frac{3}{2t} t^{-|\beta|} E[f_{\beta,l}] + C_k t^{-3/2} E_k[f]. \]

Summing over \( \beta \) and \( l \), we obtain
\[ \frac{dE_k[f]}{dt} \leq - \frac{3}{2t} E_k[f] + C_k t^{-3/2} E_k[f]. \]
The lemma follows.

In order to obtain a better understanding for the asymptotics, it is convenient to rescale the distribution function according to
\[ f_{\text{sc}}(t, \theta, v) = f(t, \theta, t^{-1/2} v). \]
We have the following conclusions concerning \( f_{\text{sc}} \).
Lemma 63. Consider a $T^2$-symmetric solution to the Einstein-Vlasov equations with a cosmological constant $\Lambda > 0$ and existence interval $(t_0, \infty)$, where $t_0 \geq 0$. Assume that the solution has $\lambda$-asymptotics and let $t_1 = t_0 + 2$. Fix $0 \leq k \in \mathbb{Z}$. Then there is a constant $C$, depending only on the solution, such that in order for $(t, \theta, v) \in [t_1, \infty) \times S^1 \times \mathbb{R}^3$ to be in the support of $f_{sc}$, $v$ has to satisfy $|v| \leq \Lambda$. Moreover, there is a constant $C_k > 0$, depending only on $k$ and the solution to the Einstein-Vlasov equations, such that

$$\|\partial_t f_{sc}(t, \cdot)\|_{C^k(S^1 \times \mathbb{R}^3)} \leq C_k t^{-2}$$

for all $t \geq t_1$. In particular, there is thus a smooth, non-negative function with compact support, say $f_{sc, \infty}$, on $S^1 \times \mathbb{R}^3$ such that

$$\|f_{sc}(t, \cdot) - f_{sc, \infty}\|_{C^k(S^1 \times \mathbb{R}^3)} \leq C_k t^{-1}$$

for all $t \geq t_1$.

Proof. The statement concerning the support is an immediate consequence of earlier observations. Introducing $\lambda$-asymptotics and let $t_1 = t_0 + 2$. Fix $0 \leq k \in \mathbb{Z}$.

Using the Vlasov equation, we conclude that

$$\partial_t f_{sc} = -\alpha^{1/2}t^{-1/2}v \frac{1}{(t^{-1/2}v)} \partial_\theta f_{sc} + R_{sc},$$

where

$$R_{sc}(t, \theta, v) = L^i(t, \theta, t^{-1/2}v)(\partial_\theta^i f_{sc})(t, \theta, t^{-1/2}v) = t^{1/2}L^i(t, \theta, t^{-1/2}v)(\partial_\theta^i f_{sc})(t, \theta, t^{-1/2}v).$$

Introducing

$$L_{sc}^i(t, \theta, v) = t^{1/2}L^i(t, \theta, t^{-1/2}v),$$

we thus have

$$\partial_t f_{sc} = -\alpha^{1/2}t^{-1/2}v \frac{1}{(t^{-1/2}v)} \partial_\theta f_{sc} + L_{sc}^i(\partial_\theta^i f_{sc}). \tag{126}$$

Due to the properties of the support of $f_{sc}$, Corollary [58] and previous estimates, there is a constant $C_k$ for each $0 \leq k \in \mathbb{Z}$ such that

$$\sum_{l+|\beta| \leq k} |(\partial_\theta^\beta \partial_\theta^\beta L_{sc}^i)(t, \theta, v)| \leq C_k t^{-2} \tag{127}$$

for all $i = 1, 2, 3$ and $(t, \theta, v) \in [t_1, \infty) \times S^1 \times \mathbb{R}^3$ in the support of $f_{sc}$. In order to estimate the derivatives of $f_{sc}$ in $C^k$, it is convenient to translate the estimate $E_k \leq C_k t^{-3/2}$ into an estimate for $f_{sc}$. However,

$$\int_{S^1} \int_{\mathbb{R}^3} |(\partial_\theta^\beta \partial_\theta^\beta f_{sc})(t, \theta, v)|^2 d\theta dv = \int_{S^1} \int_{\mathbb{R}^3} t^{-|\beta|}(\partial_\theta^\beta \partial_\theta^\beta f)(t, \theta, t^{-1/2}v)|^2 d\theta dv$$

$$= t^{3/2} \int_{S^1} \int_{\mathbb{R}^3} t^{-|\beta|}(\partial_\theta^\beta \partial_\theta^\beta f)(t, \theta, v)|^2 d\theta dv \leq C t^{3/2} E_k(t) \leq C_k,$$

assuming $l+|\beta| \leq k$. Due to this estimate and Sobolev embedding, we conclude that all derivatives of $f_{sc}$ are bounded for $t \geq t_1$. Combining this observation with (126) and (127), we conclude that

$$\sum_{l+|\beta| \leq k} |\partial_\theta^\beta \partial_\theta^\beta \partial_t f_{sc}| \leq C_k t^{-2}$$

for $t \geq t_1$. The lemma follows. \qed
10 Proof of the main theorems

Finally, we are in a position to prove the main theorems. Let us begin with Theorem 7.

Theorem 7. To begin with, the conclusions concerning the distribution function are direct consequences of Lemma 63. Turning to prove that (129) holds. Let us define $\bar{\kappa}_{ij}$ where we have used the fact that $\partial_t$ and $\partial_i$ are perpendicular. In what follows, we would like to prove that

$$\|\bar{k}_{ij} - \mathcal{H}\bar{g}_{ij}\|_{C^N} \leq C_N,$$  (129)

where $\mathcal{H} = (\Lambda/3)^{1/2}$. Considering the metric and previous estimates, it is clear that when a time derivative hits $P$, $Q$, $G$ or $H$, the resulting term is bounded in $C^N$. As a consequence, what we need to consider are the components of the tensor field

$$\frac{1}{2}t^{1/4}e^{-\lambda/4} \left[ \left( -\frac{1}{2t} + \frac{1}{2} \lambda - \frac{\alpha}{\alpha} \right) t^{-1/2}e^{\lambda/2} - e^P \right] \right] d\theta^2 \right] + e^{-P}(dy + Hd\theta)^2 = \frac{1}{2}t^{-3/4}e^{-\lambda/4} \bar{g} + \frac{1}{2}t^{-1/4}e^{-\lambda/4} \left[ \frac{1}{2} \left( \lambda_t + \frac{3}{t} \right) - \left( \frac{\alpha_t}{\alpha} + \frac{3}{t} \right) \right] d\theta^2,$$

where $\bar{g}$ is the spatial part of the metric. Since the components of the second term on the right hand side are bounded in $C^N$, and since

$$\frac{1}{2}t^{-3/4}e^{-\lambda/4} = \mathcal{H}e^{-\lambda/4},$$

previous estimates lead to the conclusion that (129) holds. Let us define $\bar{g}_\infty$ by (12); note that this is a smooth Riemannian metric on $\mathbb{T}^3$. Moreover,

$$\|t^{-1}\bar{g}_{ij}(t, \cdot) - \bar{g}_{\infty, ij}\|_{C^N} \leq C_N t^{-1}. \quad (130)$$

Combining this estimate with (129), we obtain (11). The proof of future causal geodesic completeness is not very complicated, given the above estimates. It is, e.g., possible to proceed as in the proof of [25, Propositions 3 and 4, pp. 189–191]. However, we shall not write down the details, since the result follows from the proof of Theorem 27.

Let us now turn to the proof of the cosmic no-hair conjecture.

Theorem 7. We need to verify that the conditions stated in Definition 8 are fulfilled. To begin with, note that $\Sigma_t = \{t\} \times \mathbb{T}^3$ is a Cauchy hypersurface for each $t \in (t_0, \infty)$ (an argument is required in order to justify this statement, but since the details are quite standard, we have omitted the details). Let $\gamma = (\gamma^0, \gamma)$ be a future directed and indeextendible causal curve, defined
on $I_\gamma = (s_-, s_+)$. Reparametrising the curve, if necessary, we can assume that $\gamma^0(s) = s$ and that $I_\gamma = (t_0, \infty)$. Due to the causality of the curve, there is a constant $K_0 > 1$ (independent of the curve $\gamma$, as long as $\gamma^0(t) = t$) such that

$$\bar{g}_{\infty, ij}[\bar{\gamma}(t)]\dot{\bar{\gamma}}^i(t)\dot{\bar{\gamma}}^j(t) \leq \frac{1}{4}K_0^2\mathcal{H}^{-2}t^{-3}$$

for all $t \geq t_1$, where $t_1 = t_0 + 2$. In particular, there is thus an $\bar{x}_0 \in \mathbb{T}^3$ such that $d_\infty[\bar{\gamma}(t), \bar{x}_0] \leq K_0\mathcal{H}^{-1}t^{-1/2}$ for all $t \geq t_1$, where $d_\infty$ is the topological metric on $\mathbb{T}^3$ induced by $\bar{g}_\infty$. Let $\epsilon_{\bar{e}m} > 0$ denote the injectivity radius of $(\mathbb{T}^3, \bar{g}_\infty)$. Then, given $\bar{x} \in \mathbb{T}^3$, there are geodesic normal coordinates on $B_{\epsilon_{\bar{e}m}}(\bar{x})$, where distances are computed using $d_\infty$. Fix $t_- > K_0^2\mathcal{H}^{-2}\epsilon_{\bar{e}m}^{-2} + 1$ (note that $t_-$ is independent of the curve). Due to the above arguments and definitions,

$$J^-(\gamma) \cap J^+(\Sigma_{t_-}) \subseteq \{ (t, \bar{x}) \in I \times \mathbb{T}^3 : t \geq t_-, \ d_\infty(\bar{x}, \bar{x}_0) \leq K_0\mathcal{H}^{-1}t^{-1/2} \}$$

(131)

and the closed ball of radius $K_0\mathcal{H}^{-1}t_-^{-1/2}$ (with respect to $d_\infty$) and centre $\bar{x}_0$ is contained in the domain of definition of geodesic normal coordinates $\bar{x}$ with centre at $\bar{x}_0$. Denote the set appearing on the right hand side of (131) by $D_{t_-}$, $\bar{x}_0$. Let us define

$$\psi(t, \xi) = \left[e^{2\mathcal{H}t}, \bar{x}^{-1}(\xi)\right].$$

Then

$$\psi^{-1}(D_{t_-}, \bar{x}_0) = \{ (t, \xi) \in J \times \mathbb{R}^3 : \tau \geq T_0, \ |\xi| \leq K_0\mathcal{H}^{-1}e^{-\mathcal{H}\tau} \},$$

where $T_0 = \mathcal{H}^{-1}\ln{t_-}/2$, and $J = (\tau_0, \infty)$, where $\tau_0 = \mathcal{H}^{-1}\ln{t_0}/2$. Letting $T$ be slightly smaller than $T_0$ and $K$ be slightly larger than $K_0$, the map $\psi$ is still defined on $C_{\Lambda,K,T}$; cf. (11). In analogy with Definition 8 let $D = \psi(C_{\Lambda,K,T})$ and $R(\tau) = K\mathcal{H}^{-1}e^{-\mathcal{H}\tau}$. Due to the above arguments, we have already verified all of the requirements of Definition 8 (with $\Sigma = \Sigma_{t_-}$ etc.) but the last one; i.e., that (15) hold.

In order to proceed, let $\bar{g}_{\infty, ij}$ denote the components of $\bar{g}_\infty$ with respect to the coordinates $\bar{x}$. Let $\bar{g}_{ij}(\tau, \cdot)$ and $\bar{k}_{ij}(\tau, \cdot)$ denote the components of $\bar{g}(e^{2\mathcal{H}t} \cdot)$ and $\bar{k}(e^{2\mathcal{H}t} \cdot)$, respectively, with respect to the coordinates $\bar{x}$. Moreover, consider $\bar{g}_{\infty, ij}$, $\bar{g}_{ij}(\tau, \cdot)$ and $\bar{k}_{ij}(\tau, \cdot)$ to be functions on the image of $\bar{x}$; i.e., on $B_{\epsilon_{\bar{e}m}}(\bar{x})$, with the origin corresponding to $\bar{x}_0$. Note that the estimates (129) and (130) hold with $\bar{g}_{ij}$ replaced by $\bar{g}_{\infty, ij}$ etc., assuming the domain on which the $C^N$ norm is computed is suitably restricted. In particular, letting $S_\tau$ be as in Definition 8 the following estimate holds:

$$||e^{-2\mathcal{H}\tau}\bar{k}_{ij}(\tau, \cdot) - \mathcal{H}\bar{g}_{\infty, ij}||_{C^N(S_\tau)} + ||e^{-2\mathcal{H}\tau}\bar{g}_{ij}(\tau, \cdot) - \bar{g}_{\infty, ij}||_{C^N(S_\tau)} \leq C_N e^{-2\mathcal{H}\tau}$$

for all $\tau \geq T$. Note that

$$\bar{g}_{\infty, ij}(0) = \delta_{ij}, \quad (\partial_t \bar{g}_{\infty, ij})(0) = 0$$

by definition of the coordinates $\bar{x}$. As a consequence, if $\xi \in S_\tau$, then

$$|\partial_t \bar{g}_{\infty, ij}(\xi)| = \left| \int_0^1 \frac{d}{ds}[(\partial_t \bar{g}_{\infty, ij})(\xi)] ds \right| \leq C e^{-\mathcal{H}\tau}.$$

Moreover,

$$|\bar{g}_{\infty, ij}(\xi) - \delta_{ij}| \leq C e^{-2\mathcal{H}\tau}$$

for $t \geq T$ and $\xi \in S_\tau$. In particular, we thus have

$$||e^{-2\mathcal{H}\tau}\bar{k}_{ij}(t, \cdot) - \mathcal{H}\bar{g}_{\infty, ij}||_{C^N(S_\tau)} + ||e^{-2\mathcal{H}\tau}\bar{g}_{ij}(t, \cdot) - \delta_{ij}||_{C^N(S_\tau)} \leq C e^{-2\mathcal{H}\tau}$$

for all $t \geq T$. Letting $\bar{g}_{\mathcal{H}S}(\tau, \cdot)$ and $\bar{k}_{\mathcal{H}S}(\tau, \cdot)$ be defined as in Definition 8 we conclude, in particular, that

$$||\bar{g}_{\mathcal{H}S}(\tau, \cdot) - \bar{g}(\tau, \cdot)||_{C^N_{\mathcal{H}S}(S_\tau)} + ||\bar{k}_{\mathcal{H}S}(\tau, \cdot) - \bar{k}(\tau, \cdot)||_{C^0_{\mathcal{H}S}(S_\tau)} \leq C e^{-2\mathcal{H}\tau}$$

for all $t \geq T$. In fact, due to the above estimates, we have

$$||\bar{g}_{\mathcal{H}S}(\tau, \cdot) - \bar{g}(\tau, \cdot)||_{C^0_{\mathcal{H}S}(S_\tau)} + ||\bar{k}_{\mathcal{H}S}(\tau, \cdot) - \bar{k}(\tau, \cdot)||_{C^0_{\mathcal{H}S}(S_\tau)} \leq C_N e^{-2\mathcal{H}\tau}$$

for all $t \geq T$. The theorem follows.
Finally, we are in a position to prove Theorem 27.

**Theorem** [27] The idea of the proof is to demonstrate that for late enough \( t \), there is a neighbourhood of each point in \( \{ t \} \times \mathbb{T}^3 \) such that [27] Theorem 7.16, p. 104] applies in the neighbourhood; combining this observation with Cauchy stability, cf. [27 Corollary 24.10, p. 432], then yields the desired result.

Fixing \( N \), there is an \( \epsilon > 0 \) and a constant \( C_N \) such that for every \( \bar{x} \in \mathbb{T}^3 \), there are geodesic normal coordinates \( \bar{x} \) on \( V = B_1(\bar{x}) \) with respect to \( g_\infty \), where distances on \( \mathbb{T}^3 \) are measured using the topological metric induced by \( g_\infty \). Moreover, the derivatives of \( g_\infty \) with respect to \( \bar{x} \) and \( \bar{g}^\infty_{ij} \) are the components of the inverse, then the derivatives of \( g_\infty_{ij} \) and \( \bar{g}^\infty_{ij} \) up to order \( N \) with respect to the coordinates \( \bar{x} \) on \( V \) are bounded by \( C_N \). Moreover, the derivatives of \( \bar{x} \), considered as functions of \( (\theta, x, y) \), up to order \( N + 1 \) are bounded by \( C_N \). Similarly, the derivatives of \( \bar{x}^{-1} \) up to order \( N + 1 \) are bounded by \( C_N \). The arguments required in order to prove the above statements are similar to those presented in the proof of [27 Lemma 34.9, p. 650].

The important point is that we obtain uniform bounds which hold regardless of the base point.

Define \( K \) by the condition \( e^K = 4/\delta \) and define the coordinates \( \bar{y} = e^{-K/2} \bar{x} \) on \( V \). Note that the range of \( \bar{y} \) is \( B_{e^{-K/2}}(0) \). For \( t \) large enough (the bound being independent of the base point \( \bar{x} \)), we consequently have \( e^{-Kt/2} > 1 \). From now on, we assume \( t \) to be large enough that this is the case. Moreover, we assume the coordinates \( \bar{y} \) to be defined on the image of \( B_1(0) \) under \( \bar{y}^{-1} \).

Let \( \bar{g}_{ij} \) denote the components of \( \bar{g}(t, \cdot) \) with respect to the coordinates \( \bar{y} \). Moreover, let \( \bar{g}_\infty_{ij} \) denote the components of \( \bar{g}_\infty \) with respect to the coordinates \( \bar{x} \). Due to (130), we have

\[
|\partial^\alpha (e^{-2K} \bar{g}_{ij} - \bar{g}_\infty_{ij}) \circ \bar{y}^{-1} | \leq C_N t^{-1-|\alpha|/2}
\]
on \( B_1(0) \) for \( |\alpha| \leq N \); note that

\[
\bar{y}^{-1}(\xi) = \bar{x}^{-1}(e^{K} t^{-1/2} \xi).
\]

Since \( \bar{g}_\infty_{ij} \circ \bar{y}^{-1}(0) = \delta_{ij} \),

\[
|e^{-2K} \bar{g}_{ij} \circ \bar{y}^{-1} - \delta_{ij} | \leq C_N t^{-1/2}
\]
on \( B_1(0) \). Similarly,

\[
|\partial_m \bar{g}_\infty_{ij} \circ \bar{y}^{-1}(u)| = \sum_{i=1}^n \int_0^1 \partial_i \partial_m \bar{g}_\infty_{ij} \circ \bar{y}^{-1}(su) u^i ds \leq C t^{-1}
\]
on \( B_1(0) \). To conclude, we thus have

\[
||e^{-2K} \partial_h \bar{g}_{ij} \circ \bar{y}^{-1}||_{C^{N-1}|B_1(0)|} \leq C_N t^{-1}.
\]

Due to (129), we also have

\[
||(\bar{k}_{ij} - \mathcal{H} \bar{g}_{ij}) \circ \bar{y}^{-1}||_{C^N|B_1(0)|} \leq C_N t^{-1},
\]

where \( \bar{k}_{ij} \) denotes the components of the second fundamental form \( \bar{k} \) calculated using the coordinates \( \bar{y} \).

Let us turn to the distribution function. First of all, recall that if \( \Sigma \) is a spacelike hypersurface in a Lorentz manifold, and \( \bar{f} \) is a distribution function defined on the mass shell, then the initial datum for the distribution function (denoted \( \check{f} \) and defined on \( T\Sigma \)) induced by \( \bar{f} \) on \( \Sigma \) is given by

\[
\check{f} = \bar{f} \circ \text{pr}_\Sigma^{-1},
\]

where \( \text{pr}_\Sigma \) is the projection from the mass shell over \( \Sigma \) to \( T\Sigma \); the projection is orthogonal to the future directed unit normal. In our case, we are interested in the hypersurface \( \Sigma = \{ t \} \times \mathbb{T}^3 \). If \( z = (t, \theta, x, y) \),

\[
\check{f}(\bar{p}^a e_{\alpha} | z) = \bar{f}(p^a e_{\alpha} | z) = f(t, \theta, \bar{p}),
\]

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where \( \bar{p} = (\bar{p}^1, \bar{p}^2, \bar{p}^3) \),
\[
p^0 = [1 + (\bar{p}^1)^2 + (\bar{p}^2)^2 + (\bar{p}^3)^2]^{1/2}
\]
and \( p^i = \bar{p}^i \). However, in the application of [31, Theorem 7.16, p. 104], we need to express \( \bar{f} \) with respect to the coordinates \( \bar{y} \). Consequently, we are interested in
\[
\bar{f}(\bar{z}, \bar{p}) = \bar{f} \left( \bar{p}^i \partial_{\bar{p}^i} \big|_{\bar{z}} \right) = \bar{f} \left( \bar{p}^i A_i^J(\bar{z}) e_j \big|_{\bar{z}} \right) = f[t, \theta, v(\bar{z}, \bar{p})],
\]
where \( \bar{z} = (t, \bar{z}) \),
\[
v(\bar{z}, \bar{p}) = (\bar{p}^i A_i^1(\bar{z}), \bar{p}^i A_i^2(\bar{z}), \bar{p}^i A_i^3(\bar{z}))
\]
and \( A_i^J \) is defined by the requirement that
\[
\partial_{\bar{p}^i} \big|_{\bar{z}} = A_i^J(\bar{z}) e_j \big|_{\bar{z}}.
\]
Thus
\[
A_i^J(\bar{z}) = \left( \partial_{\bar{p}^i} \big|_{\bar{z}}, e_j \big|_{\bar{z}} \right) = e^K t^{-1/2} \left( \partial_{\bar{x}^i} \big|_{\bar{z}}, e_j \big|_{\bar{z}} \right) = e^K t^{-1/2} \frac{\partial f^J}{\partial \bar{x}^i}(\bar{z}) \left( \partial_{\bar{x}^i} \big|_{\bar{z}}, e_j \big|_{\bar{z}} \right),
\]
where \( \bar{z} \) correspond to the standard coordinates on the torus (which are locally well defined). In particular, \( \partial_{\bar{z}^1} = \partial_{\theta} \), \( \partial_{\bar{z}^2} = \partial_\lambda \) and \( \partial_{\bar{z}^3} = \partial_\eta \). Due to previous observations and the form of the metric, it is clear that all derivatives of \( A_i^J \) up to order \( \mathcal{O} \) are uniformly bounded on the domain of \( \bar{y} \), the bound being independent of the base point \( \bar{x} \) and the time \( t \) (assuming \( t \) to be sufficiently large). What we need to estimate is
\[
\sum_{|\alpha| + |\beta| \leq \mathcal{O}} \int_{\mathbb{R}^3} \int_{\bar{y}(U)} (e^{-w})^{2|\beta|} (e^{-w})^{2\mu + 2|\beta|} |\partial_{\bar{x}^i} \partial_{\bar{p}^j} \psi^2(\bar{z}, \bar{p})| d\bar{xd}\bar{p},
\]
cf. [31, (7.34), p. 102, and (7.37), p. 104], where
\[
\bar{f}_\gamma(\bar{z}, \bar{p}) = \bar{f}[y^{-1}(\bar{z}), \bar{p}] = \bar{f}[\bar{x}(e^K t^{-1/2} \bar{z}), \bar{p}] = f[t, \bar{z}^1 \circ \bar{x}^{-1}(e^K t^{-1/2} \bar{z}), v(\bar{x}^{-1}(e^K t^{-1/2} \bar{z}), \bar{p})]
\]
and the constant \( w \) remains to be specified. Note that all derivatives of \( \bar{x}^{-1} \) and \( \bar{z}^1 \circ \bar{x}^{-1} \) up to order \( \mathcal{O} \) are uniformly bounded. As a consequence,
\[
\partial_{\bar{x}^i} \bar{f}_\gamma(\bar{z}, \bar{p}) = \sum_{|\gamma| = |\beta|} \partial_{\bar{x}^i} f[t, \bar{z}^1 \circ \bar{x}^{-1}(e^K t^{-1/2} \bar{z}), v(\bar{x}^{-1}(e^K t^{-1/2} \bar{z}), \bar{p})] \psi_\gamma(t^{-1/2} \bar{z})
\]
for functions \( \psi_\gamma \) with bounded derivatives; note that
\[
\frac{\partial f^i}{\partial \bar{p}^j}(\bar{x}^{-1}(e^K t^{-1/2} \bar{z}), \bar{p}) = A_j^1 \circ \bar{x}^{-1}(e^K t^{-1/2} \bar{z}).
\]
As a consequence, \( \partial_{\bar{x}^i} \partial_{\bar{p}^j} \bar{f}_\gamma(\bar{z}, \bar{p}) \) consists of sums of terms of the form
\[
t^{-|\alpha|/2} \partial_{\bar{x}^i} \phi_{\gamma, \delta, \ell}(t^{-1/2} \bar{z}) \bar{p}^\lambda,
\]
where \( |\alpha| = |\delta|, \| \delta \| \leq |\alpha|, \phi_{\gamma, \delta, \ell} \) are bounded functions and \( \bar{p}^\lambda = (\bar{p}^1)^{\lambda_1} (\bar{p}^2)^{\lambda_2} (\bar{p}^3)^{\lambda_3} \). On the other hand, we know that
\[
f(t, \theta, v) = f_{sc}(t, \theta, t^{1/2} v),
\]
where \( f_{sc} \) converges to a smooth function with compact support with respect to every \( C^k \) norm. Moreover, we know that \( f_{sc} \) has uniformly compact support. As a consequence,
\[
\partial_{\bar{x}^i} \partial_{\bar{p}^j} f(t, \theta, v) = t^{(|\gamma| + |\delta|)/2} \partial_{\bar{x}^i} \partial_{\bar{p}^j} f_{sc}(t, \theta, t^{1/2} v).
\]
Since there is a uniform constant \( C > 1 \) (independent of \( t \) large enough) and the base point \( \bar{x} \) such that
\[
C^{-1} \bar{p} \leq |v(\bar{x}^{-1}(e^K t^{-1/2} \bar{z}), \bar{p})| \leq C \bar{p},
\]
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we conclude that
\[ t^{-|\alpha|/2} |\partial^\alpha \partial^\delta \int [t, \bar{Z}^1 \circ x^{-1} (e^K t^{-1/2} \bar{\xi}), v(x^{-1} (e^K t^{-1/2} \bar{\xi}), \bar{p})]|\phi_{t, \delta, \beta}(t^{-1/2} \bar{\xi}) \bar{p}^\lambda| \]
\[ \leq C_{\alpha, \beta} t^{(|\gamma|-|\alpha|)/2} \chi(t^{1/2} \bar{p}), \]
where \( \chi \) is a smooth function with compact support. As a consequence,
\[ |\partial^\alpha \partial^\beta \bar{f}_y(\bar{\xi}, \bar{p})| \leq C_{\alpha, \beta} t^{(|\beta|-|\alpha|)/2} \chi(t^{1/2} \bar{p}) \]
for \( \bar{\xi} \in B_1(0) \). Let us now define \( w = K + K_{\text{VI}}, \) where \( K_{\text{VI}} = \ln t/2 \). Then
\[ \int_{\mathbb{R}^3} \int_{\mathcal{Y}(U)} (e^{-w})^{2|\beta|} (e^{w})^{2|\mu|+2|\beta|}|\partial^\alpha \partial^\beta \bar{f}_y|^2(\bar{\xi}, \bar{p}) d\bar{\xi} d\bar{p} \]
\[ \leq C_{\alpha, \beta}^2 \int_{\mathbb{R}^3} \int_{B_1(0)} e^{-2|\beta| K} t^{-|\beta|} (e^{K t^{1/2} \bar{p}})^{2|\mu|+2|\beta|} t^{2|\beta|-|\alpha|} \chi^2(t^{1/2} \bar{p}) d\bar{\xi} d\bar{p} \leq C_{\mu, \alpha, \beta} t^{-|\alpha|-3/2} \]
The square root of the right hand side of this expression should be compared with the right hand side of [27] (7.37), p. 104:
\[ \mathcal{H} \leq e^{5/2} e^{-3K/2 - K_{\text{VI}}} = \mathcal{H} e^{5/2} e^{-3K/2} t^{-1/2}. \]
Clearly, we have a margin. As a consequence, for \( t \) large enough there is, for every \( \bar{x} \in \mathbb{T}^3 \), a neighbourhood of \( \bar{x} \) satisfying the requirements of [27] Theorem 7.16, p. 104]. In the application of the theorem, we are allowed to choose \( \mu > 5/2 \) freely and \( k_0 \) to equal 4. Moreover, we can choose the function \( V \) to be given by \( V(\phi) = \Lambda + \Lambda \phi^2 \). In addition, the covering of \( \mathbb{T}^3 \) obtained by taking the neighbourhoods \( \mathcal{Y}^{-1}[B_{1/4}(0)] \) corresponding to varying base points \( \bar{x} \) has a finite subcovering. Appealing to Cauchy stability, we conclude that there is an \( \epsilon > 0 \) with the properties stated in the theorem; cf. [27] Corollary 24.10, p. 432].

References

[1] Anderson, M. T.: Existence and Stability of even-dimensional asymptotically de Sitter spaces. Ann. Henri Poincaré 6, 801–820 (2005)

[2] Andréasson, H.: Global Foliations of Matter Spacetimes with Gowdy Symmetry. Commun. Math. Phys. 206, 337-366 (1999)

[3] Andréasson, H.; Rendall, A.D.; Weaver, M.: Existence of CMC and constant areal time foliations in \( T^3 \) symmetric spacetimes with Vlasov matter. Comm. Partial Differential Equations 29, 237-262 (2004)

[4] Andréasson, H.: The Einstein-Vlasov System/Kinetic Theory, Living Rev. Relativity 14, 4. URL (cited on 2013-04-23): http://www.livingreviews.org/lrr-2011-4 (2011)

[5] Berger, B. K.; Chruściel, P. T.; Isenberg, J.; Moncrief, V.: Global foliations of vacuum space-times with \( T^2 \) isometry. Ann. Physics 260, no. 1, 117148 (1997)

[6] Chruściel, P. T. On space-times with \( U(1) \times U(1) \) symmetric compact Cauchy surfaces. Ann. Physics 202, no. 1, 100150 (1990)

[7] Clausen, A.; Isenberg, J.: Areal foliation and asymptotically velocity-term dominated behavior in \( T^2 \) symmetric space-times with positive cosmological constant. J. Math. Phys. 48, no. 8, 082501 (2007)

[8] Ehlers, J.: Survey of general relativity theory, in Israel, W., ed., Relativity, Astrophysics, and Cosmology. Proceedings of the summer school held 14–26 August 1972 at the Banff Centre, Banff, Alberta, Astrophysics and Space Science Library, vol. 38, pp. 1–125, (Reidel, Dordrecht, Netherlands; Boston, U.S.A., 1973).
[9] Friedrich, H.: On the existence of $n$-geodesically complete or future complete solutions of Einstein’s field equations with smooth asymptotic structure. Commun. Math. Phys. 107, 587–609 (1986)

[10] Friedrich, H.: On the global existence and the asymptotic behavior of solutions to the Einstein–Maxwell–Yang–Mills equations. J. Differential Geom. 34, no. 2, 275–345 (1991)

[11] Isenberg, J.; Luo, X.: Power Law Inflation with Electromagnetism. arXiv:1210.7566

[12] Lee, H.: Asymptotic behaviour of the Einstein–Vlasov system with a positive cosmological constant. Math. Proc. Camb. Phil. Soc. 137, 495–509 (2004)

[13] Lee, H.: The Einstein–Vlasov system with a scalar field. Ann. Henri Poincaré 6, 697–723 (2005)

[14] LeFloch, P. G.; Tchapada, S. B: Plane-symmetric spacetimes with positive cosmological constant. The case of stiff fluids. Adv. Theor. Math. Phys. 15, no. 4, 11151140 (2011)

[15] Lübbe, C.; Valiente Kroon, J. A.: A conformal approach for the analysis of the non-linear stability of pure radiation cosmologies. arXiv:1111.4691

[16] Moncrief, V.: Global properties of Gowdy spacetimes with $T^3 \times \mathbb{R}$ topology. Ann. Physics 132, no. 1, 87-107 (1981)

[17] Perlmutter, S. et. al.: Measurements of $\Omega$ and $\Lambda$ from 42 high-redshift supernovae. Astrophys. J. 517, 565–586 (1999)

[18] Rein, G.: On future geodesic completeness for the Einstein-Vlasov system with hyperbolic symmetry. Math. Proc. Cambridge Philos. Soc. 137, 237–244 (2004)

[19] Rendall, A. D.: Crushing singularities in spacetimes with spherical, plane and hyperbolic symmetry. Class. Quantum Grav. 12 1517 (1995)

[20] Rendall, A.D.: An introduction to the Einstein–Vlasov system, in Chruściel, P.T., ed. Mathematics of Gravitation, Part I: Lorentzian Geometry and Einstein Equations, Proceedings of the Workshop on Mathematical Aspects of Theories of Gravitation, held in Warsaw, February 29 - March 30, 1996, Banach Center Publications, vol. 41, pp. 35–68, (Polish Academy of Sciences, Institute of Mathematics, Warsaw, Poland, 1997).

[21] Rendall, A. D.: Accelerated cosmological expansion due to a scalar field whose potential has a positive lower bound. Class. Quant. Grav. 21, 2445–2454 (2004)

[22] Rendall, A. D.: Intermediate inflation and the slow-roll approximation. Class. Quant. Grav. 22, 1655–1666 (2005)

[23] Rendall, A. D.: Dynamics of $k$-essence. Class. Quant. Grav. 23, 1557–1570 (2006)

[24] Riess, A. G. et al.: Observational Evidence from Supernovae for an Accelerating Universe and a Cosmological Constant. Astron. J. 116, 1009–1038 (1998)

[25] Ringström, H.: Future stability of the Einstein non-linear scalar field system. Invent. math. 173, 123–208 (2008)

[26] Ringström, H.: Power law inflation. Commun. Math. Phys. 290, 155–218 (2009)

[27] Ringström, H.: On the Topology and Future Stability of the Universe, Oxford University Press, Oxford (2013)

[28] Rudin, W. P.: Real and Complex Analysis, McGraw Hill, Singapore (1986)
[29] Rodnianski, I., Speck, J.: The Stability of the Irrotational Euler–Einstein System with a Positive Cosmological Constant. arXiv:0911.5501v1

[30] Smulevici, J.: On the area of the symmetry orbits of cosmological spacetimes with toroidal or hyperbolic symmetry. Anal. PDE 4, no. 2, 191-245 (2011)

[31] Speck, J.: The nonlinear future stability of the FLRW family of solutions to the Euler-Einstein system with a positive cosmological constant. Selecta Math. (N.S.) 18, no. 3, 633-715 (2012)

[32] Speck, J.: The Stabilizing Effect of Spacetime Expansion on Relativistic Fluids With Sharp Results for the Radiation Equation of State. arXiv:1201.1963

[33] Svedberg, C.: Future Stability of the Einstein–Maxwell–Scalar Field System. Ann. Henri Poincaré 12, No. 5, 849–917 (2011)

[34] Tchapnda, S.B.; Rendall, A. D.: Global existence and asymptotic behaviour in the future for the Einstein-Vlasov system with positive cosmological constant. Classical Quantum Gravity 20, 3037–3049 (2003)

[35] Tchapnda, S. T., Noutchegueme, N.: The surface–symmetric Einstein–Vlasov system with cosmological constant. Math. Proc. Cambridge Phil. Soc. 138, 541–553 (2005)

[36] Tchapnda, S. B.: The plane symmetric Einstein-dust system with positive cosmological constant. J. Hyperbolic Differ. Equ. 5, no. 3, 681692 (2008)

[37] Wald, R.: Asymptotic behaviour of homogeneous cosmological models in the presence of a positive cosmological constant. Phys. Rev. D 28, 2118–2120 (1983)