The coupling of an enhanced pseudo-3D model for hydraulic fracturing with a proppant transport model

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Abstract

This paper presents the coupled model of a hydraulic fracturing and proppant transport. The former is described in terms of enhanced pseudo-3D model that considers height growth across two symmetric stress barriers, while the latter is given by two-dimensional transport model, stemming from the solution of an elliptical equation for fluid pressure and advection equation for the proppant transport. These two sub-modules are solved numerically using implicit time integration in hydraulic part and explicit time stepping in the transport part. In addition to that, interpolation is used to couple the two models with different grids. Results of several numerical simulations are presented for different configurations to demonstrate the interplay between these two modules. In particular, the developed coupled scheme allows us to study phenomena associated with complex fluid flow within the fracture, such as for the case of Saffman-Taylor instability.

Keywords: hydraulic fracture, leak-off, Saffman-Taylor instability, proppant transport

1. Introduction

Hydraulic fracturing is a phenomenon of breaking a material by injecting fluid into it. It can occur naturally, for instance in the form of magma-driven dikes that can propagate in the Earth’s crust and have length up to tens of kilometers [20]. In addition, such fractures can propagate along glacier beds [35]. Most commonly, however, hydraulic fracturing is associated with a technique that is used to improve recovery of hydrocarbon reservoirs. This paper addresses this phenomenon in view of the latter application.

Typical approaches for hydraulic fracture modeling include the balance of injected fluid volume, fluid flow description, elasticity relation for rock equilibrium, and a propagation criterion, that describes evolution of the fracture front. Numerical solution for such a problem is not an easy task since the problem is nonlinear, non-local, and has a moving boundary. Models with simple geometries have been developed in early days. One of the first developed models is Khristianovich-Zheltov-Geertsma-De Klerk (KGD) model [18], in which the fracture propagates under the plane strain assumption. Then, there is a model for penny-shaped or radial crack [33], which is applicable for homogeneous rocks and can be used as an initial or local solution for a more sophisticated model. Perkins-Kern-Nordgren (PKN) model [23] considers a vertical planar fracture that is bounded in the height direction by impenetrable stress-barriers, in which case the fracture propagates horizontally. This model was later improved to pseudo-3D (P3D) [16, 32] by allowing vertical propagation of the fracture through the stress-barriers. The Enhanced Pseudo-3D (EP3D) model was developed in [9], which addressed the weaknesses of the classical P3D model in predicting the height growth and inclusion of lateral toughness. All the above models are essentially one-dimensional and are therefore computationally efficient. There is another class of models, which are multidimensional, and this class is able to capture the phenomena occurring in fractures with fewer simplifications. One of such models is a fully planar-3D model [50], in which the fracture is discretized using a two-dimensional grid. Despite this model requires more computational resources, its ability to accurately capture propagation within the plane makes this model useful in situations when such an accuracy is necessary. There is another approach to model a planar fracture, so-called Implicit Level Set Algorithm (ILSA) [29, 10], which uses accurate near-tip logic to advance the fracture tip. It is worth mentioning that there are even more complicated models that consider fully 3D propagation of fractures [4], propagation of fractures in naturally fractured reservoirs, or simultaneous propagation of multiple hydraulic fractures, see e.g. [28]. This paper, however, does not focus on...
geometrical complexity of fractures, but rather on the phenomena associated with fluid and proppant flow within the fracture and therefore we focus on the planar geometry for simplicity.

To maintain hydraulic conductivity of fractures after pumping is finished, proppant is added to fracturing fluids. Transport of proppant particles is influenced by multiple parameters, which includes particle size, density, base fluid rheological properties, and the effect of different additives, such as fibers, that dramatically alter particle settling behavior. One of the earliest proppant transport studies [6] and [24] consider one-dimensional transport in a straight crack (Hele-Shaw cell). More sophisticated models, developed later, consider 2D transport that describe the slurry flow using lubrication theory, however, the particles have the same velocity as the carrier fluid (single-velocity model). In such approaches there is typically a semi-empirical relation for the slurry viscosity versus particle volume concentration (effective viscosity) [26, 1]. The effective viscosity can also be computed on the basis of frictional rheology of suspensions [14, 8]. The two-velocity model in the context of the frictional rheology of suspensions was obtained in [8]. This model not only captures shear-induced particle migration, but also captures the transition from Poiseuille flow at low particle concentrations to Darcy’s law at high volume fractions of particles and thus resolves the problem of effective viscosity singularity by using physical principles. Proppant transport model based on the kinetic approach was proposed in [12]. A more thorough review of proppant transport models can be found in the review paper [25].

Practical hydraulic fracture modeling requires a coupled model for hydraulic fracture propagation and proppant transport. One of the difficulties in implementing such a coupled model is the need for simultaneous calculation of two processes that may require different time and length scales. For instance, a relatively coarse geomechanics grid can be taken, but, at the same time, a finer mesh can be used for transport. Such a modeling requires different computational grids and time steps, which makes coupled numerical schemes complicated. Note that typically, the coupled problem is solved in a pragmatic way by using the same relatively coarse grid for geomechanics and proppant transport, which significantly oversimplifies and smears the resultant proppant distribution within the fracture.

The primary goal of this study is to develop a coupled model for hydraulic fracturing with proppant transport. In this study we select enhanced Pseudo-3D model (EP3D) [9] for the geomechanics part, which is shown to be computationally efficient and yet accurate for the specific geometry that the model considers. The original EP3D model did not account for fluid leak-off, which is an essential component of hydraulic fracturing that is especially important for proppant transport. To overcome this issue, we first introduce leak-off according to Carter model [3] and test accuracy of the simulator without proppant against reference solutions. We then couple the geomechanics part with the proppant transport module, which, in contrast to the fracture model, is fully two-dimensional. In addition to that, the proppant transport is computed using high resolution mesh, which allows us to capture effects such as “fingering” or Saffman-Taylor instability, which can occur when a low viscosity fluid displaces a high viscosity fluid. Such situations can occur for instance when the injection schedule consists of alternations between the pulses of proppant and clean fluid.

This paper is organized as follows. First, the governing equations for the coupled problem of hydraulic fracture growth and proppant transport are described in Section 2. Brief description of the EP3D model, its modification with leak-off and verification against analytical and fully planar model are given in Section 3. Then, in Section 4, the adopted proppant transport model is described and verification of the numerical scheme is presented. Coupling between the hydraulic fracture and transport models is discussed in details in Section 5, while the numerical results are presented and discussed in Section 6, which is followed by summary.

2. Hydraulic fracturing model

We first outline assumptions underlying the mathematical model that is used to model hydraulic fracturing. In particular, the aim is to capture the dominant phenomena that occur in the rock formation, such as deformation of the rock caused by the presence of the fracture, the mechanism for fracture growth, flow of a viscous fluid inside the crack, and fluid leak-off into the surrounding porous rock formation. For the purpose of this study it is further assumed that:

1. Hydraulic fracture is single and planar and there are no other natural fractures.
2. The rock behavior is captured by the theory of linear elasticity and is characterized by Youngs modulus $E$ and Poissons ratio $\nu$.
3. The fracture growth is governed by linear elastic fracture mechanics (LEFM); for the single planar fracture under consideration it is sufficient to consider only mode I fracture with the associated rock fracture toughness denoted by $K_{Ic}$.
4. The rock is assumed to be homogeneous (i.e. all rock properties, such as fracture toughness $K_{Ic}$, Youngs modulus $E$, Poissons ratio $\nu$, and leak-off coefficient $C_L$ are all spatially constant).
5. Hydrostatic pressure change in the crack due to gravity is neglected.
6. We neglect fluid lag, so that the fluid front is assumed to coincide with the crack front.
7. The fluid is assumed to be incompressible and Newtonian (with a dynamic viscosity $\mu$), while the flow follows lubrication theory.
8. The proppant particles are identical and small compared to a characteristic length-scale (i.e. the fracture width).
9. Absence of settling – the proppant and fluid move at the same velocity at any given point.
10. Proppant dispersion is ignored.
11. The leak-off is described by Carter’s model [3], in which one-dimensional diffusion is assumed in the direction perpendicular to the fracture surfaces and is quantified by a single leak-off coefficient $C_L$.
With the reference to the above assumptions, schematics of the fracture is shown in Fig. 1. Central reservoir layer with thickness $H$ is surrounded by two symmetric stress barriers with magnitude $\Delta \sigma$. The injection interval is located in the middle of the reservoir layer, as depicted in Fig. 1. The coordinate system is introduced such that its origin is located at the wellbore, the $x$-coordinate is in the horizontal direction, while the $z$-coordinate is in the vertical direction. Various pumping schedules are considered, they will be further described in Section 6.

To reduce lengthy mathematical expressions, we adopt the following short-hand notation

$$\mu' = 12\mu, \ E' = \frac{E}{1 - \nu^2}, \ K' = 4\left(\frac{2}{\pi}\right)^{1/2}K_c, \ C' = 2C_L,$$  

where $\mu$ is the fluid viscosity, $E$ is Young’s modulus, $\nu$ is Poisson’s ratio, $K_c$ is fracture toughness, and $C_L$ is leak-off coefficient.

Multiphase flow of the suspension consisting of proppant particles mixed with fracturing fluid is governed by the volume balance for both fluid and solid components, and can be written as

$$\frac{\partial(c_f w)}{\partial t} + \nabla \cdot (c_f q) + Q_{\text{leak}} = c_{f,0}Q_0\psi(z),$$  

$$\frac{\partial(c_p w)}{\partial t} + \nabla \cdot (c_p q) = c_{p,0}Q_0\psi(z),$$  

where $w$ is the fracture width, $q$ is the slurry flux, $Q_0$ is the injection rate, $c_f$ and $c_p$ are the volumetric fractions of fluid and proppant inside the fracture such that $c_f + c_p = 1$, $c_{f,0}$ and $c_{p,0}$ are the volumetric fractions of fluid and proppant at the injection point, the term $Q_{\text{leak}}$ describes leak-off, the operator $\nabla \cdot (\cdot)$ acts in the $(x, z)$ plane, and the function $\psi(z)$ describes the injected slurry distribution along perforation interval:

$$\psi(z) = \begin{cases} \frac{1}{2a}, & |z| \leq a, \\ 0, & |z| > a. \end{cases}$$  

The perforation interval is symmetric and is located along the z axis from $-a$ to $a$, where $a$ is a parameter.

As it was already mentioned in the assumptions, Carter’s model is used to describe leak-off, in which case it is computed as

$$Q_{\text{leak}} = \frac{C'}{\sqrt{1 - t_0(x, z)}},$$  

where $C_L$ is leak-off coefficient, $t$ is time and the function $t_0(x, z)$ represents the time instants at which the fracture front was located at a given location $(x, z)$. The leak-off term $Q_{\text{leak}}$ is present only in \((\cdot)\), since proppant is not allowed to escape the fracture.

Flow of viscous fluid is described according to laminar flow of a Newtonian fluid between two parallel plates, in which case the flux is expressed as

$$q = -wu = -\frac{w^3}{\mu'(c_p)}\nabla p,$$  

where $u$ is the average slurry velocity and $\mu(c_p)$ is the effective viscosity of the suspension. There are numerous models that aim to capture the dependence of the effective viscosity on particle concentration, see e.g. [15, 19, 30]. However, in this study we consider Nolte empirical equation [22] for simplicity, so that

$$\mu'(c_p) = \mu'_0 \left(1 - \frac{c_p}{c_{\text{max}}}ight)^{-2.5},$$  

where $\mu'_0$ is the intrinsic scaled fluid viscosity, while the maximum allowed proppant concentration is taken as $c_{\text{max}} = 0.65$.

$^{1}$The value $c_{\text{max}} = 0.65$ is very close to 0.64, which corresponds to the max-
The coupled fluid-solid problem under investigation also requires equilibrium of rock that surrounds the fracture. For the planar crack under consideration, the governing elasticity equation [5] is

\[ p(x, z, t) - \sigma_v(x, z) = -\frac{E'}{8\pi} \int_{\Omega(t)} \frac{w(x', z', t)dx'dz'}{((x' - x)^2 + (z' - z)^2)^{3/2}}, \]  

(8)

where \( p(x, z, t) \) is the fluid pressure inside the crack, \( \sigma_v(x, z) \) corresponds to the confining stress, \( w(x, z, t) \) is the fracture width, while the integration is taken along the surface of the crack \( \Omega(t) \).

As was indicated in the list of assumptions, LEFM is used for propagation condition. In particular, fracture propagates once the stress intensity factor at the fracture tip reaches its critical value, which is a material parameter that is called fracture toughness. It is well known, that the relation (9) has a limited range of applicability for hydraulic fractures since fracture toughness. It is well known, that the relation (9) has a limited range of applicability for hydraulic fractures since

\[ w(s) = \frac{K'}{E' s^{1/2}}, \quad s \to 0, \]  

(9)

where \( s \) is the distance to the fracture front and \( K' \) is the scaled fracture toughness. It is well known, that the relation (9) has a limited range of applicability for hydraulic fractures since the fluid viscosity and leak-off may significantly affect solution near the tip [13, 9]. To deal with this problem, we are going to use the tip asymptotic solution, which is an extension of (9) that accounts for fluid viscosity and leak-off. Such a criterion can be summarized as

\[ w(s) = w_a(s), \quad s \to 0, \]  

(10)

where accurate approximation for the function \( w_a \) is specified in [9]. Note that the function \( w_a \) depends also on material parameters, such as toughness and Young’s modulus, as well as the fluid viscosity, leak-off, and fracture front velocity. Finally, the propagation condition is supplemented by zero flux condition along the fracture front.

3. Enhanced Pseudo-3D Model (EP3D)

One of the parts of the coupled model is the fracture propagation model. Here we utilize the Enhanced Pseudo-3D model (EP3D) [9], which is an extension of the classical Pseudo-3D (P3D) model [16, 32, 1]. In particular, the model assumes that the stress barriers that surround the reservoir layer lead to dominance of the horizontal fracture growth in comparison to the height growth. As a result, the primary assumptions for the geomechanics model can be formulated as:

1. The vertical component of the flux is negligible compared to the horizontal component (i.e. \( q_z / q_h \ll 1 \));
2. The previous item implies that the pressure is constant along the \( z \) axis in each vertical cross-section;
3. Due to the elongated nature of the fracture, it is assumed that the state of plane strain prevails in any vertical \((y, z)\) cross-section.

The primary computational advantage of the P3D model lies in the fact that it can be formulated in terms of an effective width and averaged flux. This averaging over the vertical direction allows us to reformulate the two-dimensional problem into a one-dimensional problem, in which the numerical solution is computed only along the lateral or \( x \) direction.

The vertical averaging for an arbitrary quantity \( f \) is defined as follows

\[ \overline{f}(x) = \frac{1}{H} \int_{-\frac{H}{2}}^{\frac{H}{2}} f(x, z)dz, \]  

(11)

where \( h(x) \) is the fracture height and \( H \) is the reservoir thickness. Applying this averaging to sum of the initial governing equations (3)-(4), we can write the vertically integrated lubrication equation as

\[ \frac{\partial \overline{\sigma}}{\partial t} + \frac{\partial \overline{q}}{\partial x} + \frac{1}{H} \int_{-\frac{H}{2}}^{\frac{H}{2}} \frac{C' dz}{\sqrt{l - l_0(x, z)}} = \frac{Q_0}{H} \delta(x), \]  

(12)

where the averaged flux is given as

\[ \overline{q} = -\frac{1}{H} \frac{\partial p}{\partial x} \int_{-\frac{H}{2}}^{\frac{H}{2}} \frac{w^3(x, z)}{\mu'(c_p(x, z))}dz. \]  

(13)

To have an ability to reconstruct solution in the vertical direction, the assumptions of uniform pressure and plane strain are used to compute the fracture width solution in the form [11]

\[ w(x, z) = \frac{K'}{2E'} \frac{\sqrt{h^2 - 4z^2}}{\sqrt{h}} + \frac{4\Delta \sigma}{\pi E'} \left( - \ln \frac{H \sqrt{h^2 - 4z^2} + 2z \sqrt{h^2 - H^2}}{H \sqrt{h^2 - 4z^2} + 2z \sqrt{h^2 - H^2}} \right) \]  

(14)

\[ + \frac{H}{2} \ln \frac{\sqrt{h^2 - 4z^2} + \sqrt{h^2 - H^2}}{\sqrt{h^2 - 4z^2} - \sqrt{h^2 - H^2}}, \]

where \( \Delta \sigma \) denotes the magnitude of the confining stress. By applying averaging (11) to the above equation, we get

\[ \overline{w} = \frac{H}{E'} \left( \frac{\pi K'}{8H} \frac{h^{3/2}}{\sqrt{H}} \right) + \Delta \sigma \sqrt{\frac{h^2}{H^2} - 1}. \]  

(15)

Equations (14) and (15) apply in the regions where \( h \geq H \). On the other hand, in the near tip region, in which \( h < H \), we utilize the following elliptic fracture width profile

\[ w(x, z) = \frac{4H}{nh^2} \sqrt{h^2 - 4z^2} \overline{w}. \]  

(16)
In order to provide the relationship between the fracture height and width in the near-tip region, solution for the radial fracture is used, which leads to

\[ \bar{w} = \frac{\pi K' h^2}{8 E' H (\min\{2l, H\})^{1/2}}, \]  

(17)

where \( l \) is the half-length of the fracture, see [9] for more details. Once the averaged width \( \bar{w} \) is known, equations (15) and (17) allow us to compute fracture height \( h \) for every cell, and then equations (14) and (16) are used to determine fracture width profile. Note that in order to incorporate the effect of fluid viscosity on the fracture height growth, the concept of apparent toughness is used. That is, the fracture toughness in the above equations is replaced with an effective toughness, which depends on the velocity of the fracture tip, as well as fluid viscosity, see [9] for more details.

Classical P3D model uses so-called local elasticity assumption, which is consistent with using the plane strain solution for each vertical cross-section. In this assumption, the pressure for each cell depends only on the width of this particular cell and is not influenced by the neighbouring cells. This, however, does not work near the fracture tip due to strong fracture height variation. At the same time, similar situation with rapid height change occurs for a radial fracture propagating in the toughness regime, but yet, each vertical cross-section is still elliptical. This demonstrates that it is appropriate to use the elliptical shape of the solution near the tip, see [16]. However, the elastic interactions need to be changed from local to non-local, so that the neighbouring cells are able to influence each other.

In order to proceed with the non-local elasticity, we need to substitute the fracture width (14) or (16) into the elasticity equation for a planar fracture (8) to obtain

\[ p(x) = -\frac{E'}{8\pi} \int_{-h(t)}^{0} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{W(x', z') dz' dx'}{((x'-x)^2 + z'^2)^{3/2}}, \]  

(18)

Here the pressure \( p(x) = p(x, z = 0) \) varies only with respect to the \( x \) coordinate because of the assumption that it is constant along the \( z \) direction. This elasticity equation needs to be evaluated numerically to achieve non-local elasticity. This introduces a computationally expensive procedure into an algorithm since Eq. (18) contains double integration, while the rest of the algorithm for fracture growth is one-dimensional. To overcome this problem, one should note that the integral over the height direction in (18) can be computed analytically for the elliptical fracture profile, see [2]. This reduces the two-dimensional integral to its one-dimensional analog, which can be computed significantly faster. Since the fracture width in (14) is not elliptical, it cannot be integrated precisely. However, it can be well approximated by two ellipses [9], in which case the two-dimensional integral is replaced with two one-dimensional integrals, which still brings significant computational time reduction.

The use of the non-local elasticity in the model allows us to incorporate a propagation condition in the lateral direction that is similar to that in planar fracture models. Thus, it enables us to account for the fracture toughness in the lateral direction, something that was not possible with the classical P3D model. In particular, the tip asymptotic solution (10) is used in the last element to advance the fracture length. Additional information about the numerical scheme can be found in [9].

3.1. Verification of EP3D model with leak-off

The primary difference between the current implementation of EP3D model in comparison with [9] is the presence of leak-off. As a result, it is necessary to perform a series of simulations to benchmark the developed fracture propagation model against reference solutions. In this study we consider a comparison with radial symmetric hydraulic fracture, as well as with the solution for the three-layer medium that is computed using fully planar 3D hydraulic fracturing simulator ILSA [29, 27, 10]. Since this study focuses on the coupling between the one-dimensional fracture propagation model and the two-dimensional proppant transport model, two-dimensional distribution of leak-off is computed according to (5) and is then used for both models. In EP3D, however, the computed leak-off is averaged over the height direction, as indicated in (12).

We first consider the case of a penny-shaped fracture that occurs in the absence of confining stress barriers (i.e. \( \Delta \sigma = 0 \) or \( H > l(t)/2 \)), in which case the fracture is contained within the reservoir layer. In order to reduce the number of problem parameters, which determine the behavior of the solution, it is useful to utilize the following normalization [21]

\[ L = \left( \frac{Q_0^3 E' P_{mk}}{\mu^*} \right)^{1/9}, \quad t_{mk} = \left( \frac{\mu^* E' P_{mk}^3 Q_0}{K_{18}} \right)^{1/2}, \]  

(19)

where \( L \) is the characteristic length scale, while \( t_{mk} \) is the time scale. In this setting, the following dimensionless parameters are introduced:

\[ \gamma = \frac{R}{L}, \quad \tau = \frac{t}{t_{mk}}, \quad \varphi = \frac{\mu^* E' P_{mk}^3 Q_0}{K_{18}^4}. \]  

(20)

Figure 2: Parametric space for a radial hydraulic fracture. Green, magenta, red and blue regions respectively indicate the applicability zones of \( \tilde{M}, \tilde{K}, K \) and \( M \) limiting solutions. Black thick lines indicate paths of the the six cases that are considered for benchmarking the EP3D model.
where $R$ is the radius (length) of the fracture, $\gamma$ is the corresponding dimensionless radius, $t$ is the injection time, $\tau$ is the corresponding dimensionless time, while $\phi$ is the leak-off parameter. The solution for the radial hydraulic fracture depends on two dimensionless parameters $\tau$ and $\phi$, i.e. the fracture radius is $\gamma(\tau, \phi)$. Once the dimensionless solution is obtained, the result can be “unscaled” back using (19) and (20). In addition to the fracture radius, it is useful to introduce efficiency $\eta$, where the latter is a dimensionless quantity that is defined as the ratio between the current fracture volume and the total volume of the injected slurry:

$$
\eta(t) = \frac{1}{Q_0t} \int_{\Omega(t)} w(x', y', t) \, dx' \, dy'.
$$

This parameter is an indicator of leak-off intensity: in case of small leak-off the efficiency is close to unity, while for large leak-off this parameter is small and approaches zero.

The problem of a radial hydraulic fracture has four limiting regimes of propagation [7]: storage viscosity ($M$), storage
toughness \((K)\), leak-off viscosity \((\bar{M})\) and leak-off toughness \((\bar{K})\), that correspond to the competition between two dissipative (toughness vs. viscosity) and two fluid storage (small vs. large leak-off) mechanisms. Fig. 2 shows the parametric map that indicates location of these limiting propagation regimes in the parametric space \((\tau, \phi)\). The black thick lines indicate \(\tau\)-intervals (for fixed \(\phi\)-values) at which calculations were performed using the reference solution for the radial fracture \([11]\) and EP3D model. The problem parameters that were used for computations are shown in Table 1. All the parameters are kept the same except the leak-off coefficient, whose values are

\[ C' = \{5.21 \times 10^{-7}, 0.00521, 0.0165, 0.0521, 0.165, 0.521\} \text{ m} \cdot \text{s}^{-1/2}, \quad (22) \]

which corresponds to the following dimensionless leak-off parameters

\[ \phi = \{10^{-20}, 10^{-4}, 10^{-2}, 1, 10^2, 10^4\}. \quad (23) \]

The calculations are selected such that they originate at the \(M\) regime and propagate towards \(M, \bar{K}\), or \(K\)-regime. In this case, we cover the majority of the parametric space.

Table 1: Problem parameters for the comparison between EP3D and radial fracture models.

| Parameter | Value |
|-----------|-------|
| \(K_{IC}\) | 1.0 MPa \cdot m^{1/2} |
| \(E\) | 9.5 GPa |
| \(\gamma\) | 0.2 |
| \(\mu\) | 0.1 Pa s |
| \(Q_0\) | 0.01 m^3 \cdot s^{-1} |

Fig. 5(a) shows evolution of the normalized fracture radius with time for all sets of leak-off levels that are considered. Colored lines represent the reference solution for radial fracture model \([11]\), while the dashed black lines depict the corresponding numerical solution computed using EP3D model. One can clearly observe that EP3D model is capable of capturing the solution accurately for all problem parameters considered. In addition, larger values of leak-off (i.e. \(\phi\)) lead to shorter fractures, which is expected since the fracture volume becomes smaller.

Furthermore, Fig. 5(b) plots the efficiency (defined in (21)) for the same six cases against the normalized time. The colored lines correspond to the radial solution, and the dashed lines represent the numerical EP3D solution. As can be seen from the display, both models agree well for all problem parameters. In addition, the calculation with \(\phi = 10^{-20}\) corresponds to almost no leak-off and the whole injected fluid stays within the fracture. All other cases, on the other hand, demonstrate decrease of the efficiency with respect to time. These results are consistent with their respective locations in the parametric space shown in Fig. 2. The regimes \(M\) and \(K\) correspond to zero leak-off or \(\eta = 1\), while the regimes \(\bar{M}\) and \(\bar{K}\) correspond to large leak-off or \(\eta = 0\). For instance, if the solution transitions from \(M\) to \(\bar{M}\), then the efficiency transitions from 1 to 0, which is observed in Fig. 5(b). On the other hand, if the solution transitions from \(M\) to \(\bar{K}\), then the efficiency stays close to 1.

To provide further benchmarking of the algorithm, another comparison is made, but now with a fully planar HF model \([10]\). This model solves the governing equations described in Section 2, it allows arbitrary variation of fracture geometry within a plane that can be caused by a layered rock structure. However, for the purpose of comparison with EP3D, which is restricted to the three-layer geometry, we consider the case with symmetric stress barriers. Problem parameters are summarized in Table 2.

Three different values of leak-off are considered

\[ C' = \{0.521, 1.65, 5.21\} \times 10^{-5} \text{ m} \cdot \text{s}^{-1/2}, \quad (24) \]

which correspond to \(\phi = \{10^{-4}, 10^{-2}, 1\}\). Fig. 6 shows the fracture footprints that are calculated using EP3D model and ILSA (results are taken from \([10]\)). Colored lines indicate the footprint from ILSA solution for the three considered cases, while the dashed black lines represent the numerical solution from EP3D for the same cases. Solid black line at the \(z = 10\) m indicates the boundary between the reservoir and the stress barrier layers. Results indicate that the fracture geometries are very similar for all values of leak-off, especially given the one-dimensional nature of EP3D and the underlying assumptions. Fig. 5 shows the corresponding results for fluid efficiency. Results are similar again, which indicates consistency in computing leak-off between two different algorithms.

Table 2: Problem parameters for the comparison between EP3D and ILSA.

| Parameter | Value |
|-----------|-------|
| \(K_{IC}\) | 1.0 MPa \cdot m^{1/2} |
| \(\Delta\sigma\) | 0.75 MPa |
| \(H\) | 20 m |
| \(E\) | 9.5 GPa |
| \(\gamma\) | 0.2 |
| \(\mu_f\) | 0.1 Pa s |
| \(Q_0\) | 0.01 m^3 \cdot s^{-1} |
| \(t\) | 3600 s |

4. Proppant Transport

Productivity of a hydraulically fractured reservoir depends on having a fracture that is effectively propped along its length and height. It is therefore important to simulate the proppant distribution within the fracture. In this section, we consider exclusively the proppant transport problem. In particular, the goal is to compute the updated values of proppant concentration for every transition from time \(t\) to \(t+\Delta t\) during the computation of EP3D simulator.

First, since the transport is described by a two-dimensional model (while EP3D is essentially a one-dimensional model), we have to compute the full two-dimensional pressure field for a given fracture width change and leak-off distribution within the time step. The computed pressure field allows us to calculate the slurry velocity components. Finally, the transport equations are solved to move proppant in space according to the slurry
velocity values. These tasks are performed sequentially and are described in the following subsections 4.1 and 4.2. To ensure accuracy of the developed algorithm, Section 4.3 presents verification of the numerical scheme against reference solutions.

4.1. Elliptic equation for pressure calculation

The two-dimensional model for the hydraulic fracture propagation was summarized in Section 2. The system of equations (2) and (3) can be solved with respect to the fracture width, $w$. The combination of balance equations (2) and (3) allows us to obtain the stationary elliptical equation in the form

$$
\frac{\partial}{\partial x} \left( \frac{w^3}{12\mu} \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial z} \left( \frac{w^3}{12\mu} \frac{\partial p}{\partial z} \right) = Q_{\text{ leak}} + \frac{\partial \psi}{\partial t}.
$$

Consequently, the velocity components can be computed as

$$
u = \begin{bmatrix} u_x \\ u_z \end{bmatrix} = \begin{bmatrix} \frac{w(x,z)^2}{12\mu(x,z)} \frac{\partial \psi}{\partial x} \\ \frac{\partial \psi}{\partial z} \end{bmatrix},
$$

where $u_x, u_z$ are the x and z-components of the velocity field, while the corresponding fluxes are $q_x = w u_x$ and $q_z = w u_z$. The governing equation (25) is supplemented with the following boundary conditions

$$
q_x|_{z=0} = Q_0 \psi(z), \quad q \cdot n|_{\partial G} = 0,
$$

where the first condition represents the source, while the second one is the no-flux condition at the fracture boundary. For the purpose of computations, the second boundary condition is extended to the external boundary of the whole computational domain, see Appendix B for details of the numerical scheme.

4.2. Transport equation

Once the slurry velocity components are computed, the particle transport is calculated by solving (3), which can be rewritten as

$$
\frac{\partial (c_p w)}{\partial t} + \frac{\partial (c_p w u_x)}{\partial x} + \frac{\partial (c_p w u_z)}{\partial z} = c_{p,0} Q_0 \delta(x),
$$

where $c_p$ is the relative proppant concentration, $w$ is the fracture width, $u_x, u_z$ are velocity components that are computed from (26), and the function $\psi(z)$ is given in (4). The governing transport equation (28) is solved in the same domain as pressure and is supplemented by the zero-flux condition at the outer boundaries of the domain except the perforation interval.

Details of the numerical methods for solving both the pressure (25) and the advection (28) parts of the transport model are given in Appendix B.

4.3. Verification of proppant transport

Many finite difference schemes have been developed for solving hyperbolic equations such as (28). The upwind and the Lax-Wendroff methods are probably among the most well-known. It is important to note that in the case of a solution with sharp discontinuity the first order accurate upwind method leads to smoothing due to numerical diffusion, while the Lax-Wendroff scheme is second-order accurate but produces numerical oscillations due to numerical dispersion. A family of high order Total Variation Diminishing (TVD) schemes have been developed to take the best from the two approaches. These
schemes successfully capture shock waves by providing non-
oscillatory solutions, however, they retain second order accu-
ry away from the discontinuities.

Numerical solution of the system of equations (25)-(28) on
a uniform rectangular grid is carried out to verify the prop-
 pant transport solver. Solution for the elliptic equation (25) is
constructed by a conservative finite volume method (see Ap-
pendix A for details). To solve the transport equation (28), sec-
ond order TVD scheme in space and the first order scheme in
time is used (see Appendix B for details).

To test the proppant transport module for the moving frac-
ture, we consider the following fracture solution
\[ w(r,t) = A \cdot (1 + t) \left( 1 - \frac{r}{R} \right), \quad R(t) = R_0 \cdot t^\alpha, \] (29)
where \( r = \sqrt{x^2 + y^2} \) is the radial coordinate, \( R \) is the fracture
radius, while \( A, R_0 \) and \( \alpha \) are the constants that correspond to
the width factor, the initial fracture radius, and the degree of
fracture growth with time. The perforation interval is located
within the initial fracture radius \( R_0 \), and the pumping rate \( Q_0 \)
is adjusted to match dynamics of the fracture volume. No leak-o
ff is assumed for the test problem.

Given the fracture solution, the clean fluid is first pumped
for \( t = 1 \) s, and then proppant with constant volume fraction of
0.25 is introduced. Fig. 6(a) shows the calculated proppant dis-
tribution within the fracture that was calculated using Superbee
limiter. As can be observed from the figure, proppant concen-
tration remains constant since there is no leak-off and there is a
sharp transition to clean fluid.

Fig. 6(b) shows the cross section of the solution for \( z = 0 \)
in terms of the proppant concentration for the given test prob-
lem. Colored dashed lines represent the numerical solution for
different flux limiters and meshes, while the black solid line
shows the exact solution. One can clearly observe that Super-
bee limiter performs better for the given advection problem. It
is therefore chosen for further computations.

5. Numerical solution for the coupled model

This section describes coupling between the EP3D hydraulic
fracture model and the proppant transport solver. One of the
challenges is that the models utilize different meshing strate-
gies: EP3D uses a moving mesh that conforms into the fracture
shape, while the proppant transport solver uses a fixed grid. In
addition, it is necessary to use a fine grid for proppant trans-
port if one wants to capture small scale effects such as Saffman-
Taylor instability.

Schematics of the algorithm is shown in Fig. 7. Simulation
starts with an initial solution for hydraulic fracture model,
which is obtained by using the radial fracture solution [11] for
the corresponding input parameters and small initial time, so
that the fracture does not grow through the stress barriers. The
numerical scheme then proceeds with solving the geomechan-
ics part, i.e. volume balance (12) together with elasticity (18),
fluid flow (13) and leak-off (5), (11). This occurs during pump-
ing pad, or clean fluid before initialing proppant pumping at
time instant \( t^* \). During this initial stage the viscosity remains
constant throughout the fracture. The leak-off is computed within
EP3D using two-dimensional fixed mesh of transport module.
As the front reaches a new cell with coordinates \((x, z)\), the ex-
posure time \( t_0(x, z) \) is set to the current simulation time. Note
that despite two-dimensional exposure time is calculated, the
fracture propagation algorithm uses the height-averaged one-
dimensional leak-off for calculations.

As soon as the simulation time reaches proppant injection
moment \( t^* \), the proppant transport module becomes active. To
couple the two modules, it is first necessary to pass the data
from one module to another. In particular, we need to pass
fracture width at the previous and current time steps, time step
value, and leak-off flux data. Given the fact that EP3D uses a
moving mesh, while the transport module uses a fixed mesh,
an interpolation is required. To help understanding the inter-
polation process, Fig. 8(a) shows fracture solution using EP3D

![Figure 7: Procedure of the coupled model.](image-url)
mesh and the fixed mesh used for transport in the background. In order to interpolate the fracture width, we first interpolate the average width \( \bar{w}(x) \) from the moving EP3D mesh to the fixed grid, as shown in Fig. 8(b). We use Piecewise Cubic Hermite Interpolating Polynomial (PCHIP) for interpolation. Then, we apply similar procedure for fracture height \( h(x) \) and use equations (14)–(16) to reconstruct two-dimensional width profile on the fixed grid of Poisson solver. Since the exposure time function \( t_0(x, z) \) is updated every time step already on the desired grid, no additional effort is needed for converting leak-off to the transport module.

Having the input data in the proppant transport module, the pressure field is first computed by solving (25). Then, velocity components are computed via (26), and finally the transport equation (28) is solved to compute an updated proppant distribution. Since EP3D uses implicit time integration, while the transport utilizes explicit time integration, the latter algorithm may require time step subdivision to satisfy the Courant-Friedrichs-Lewy (CFL) condition to ensure stability. That is the relation

\[
u_s \frac{\Delta t}{\Delta x} + u_i \frac{\Delta t}{\Delta z} \leq C_{CFL},
\]

should hold within the computational domain. Typically, the transport module uses a smaller time step, and therefore it computes several times before providing the output result, which is updated concentration and viscosity profile.

The influence of proppant transport on fracture propagation is incorporated through the viscosity that depends on proppant volume fraction \( \varepsilon \). To couple the two models, the corresponding viscosity values at the fixed grid need to be interpolated back to EP3D moving mesh. In order to do that, we first compute average viscosity in each vertical cross-section and then interpolate it to the moving EP3D mesh. After that, EP3D is called again to advance the fracture by next time step and the cycle repeats until the final simulation time is reached.

6. Results and discussion

To design an optimal hydraulic fracture treatment, a suitable carrier fluid, proppant and pumping schedule should be selected depending on the targeting layer properties, as well as operational constraints. For example, proppant type is one of the design parameters and should be selected to achieve a balance between asymmetry caused by settling and fracture conductivity. It is therefore necessary to perform numerical simulations in order to test various scenarios.

In this section, a series of numerical simulations is performed to illustrate capabilities of the developed algorithm and to understand the effect of pumping schedule on the resulting fracture geometry and proppant placement. The problem parameters that are common for the considered cases are shown in Table 3. Injection schedules are different and are shown in Table 4. Four different cases of the hydraulic fracturing design are compared: i) clean water pumping, ii) constant proppant concentration after pad, iii) increasing proppant concentration after pad, and iv) pulsed injection with alternation of clean fluid and proppant slurry after pad. The injection rate, total pumping time, and total proppant volume are kept the same for each case, except for the reference case without proppant.

To provide a reference, the first case assumes no proppant and it is used to estimate the extent of fracture propagation in the absence of particles. The input schedule parameters are summarized in Table 3. The fracture is driven by clean fracturing fluid until it reaches the boundary of the provided numerical domain at \( t = 2800 \) s. Figs. 9(a)–9(c) illustrate the fracture footprint evolution at different time instants: \( t = 500 \) s, 1800 s, and 2700 s.

The case with constant proppant injection is considered next. The fracture is driven by clean fracturing fluid until \( t = 400 \) s, and then solid particles with constant volume fraction are introduced to produce a slurry, see Table 4. Figs. 10(a)–10(d)
show results of simulations for different time instants: $t = 600$ s, $1800$ s, $2700$ s, and $t = 3600$ s. It can be seen that proppant was able to reach fracture tip after one hour of pumping. Apart from the near tip region, the proppant concentration is almost uniform within the whole fracture. Figs. [12(a), 12(d)] show the simulation results for the case of pulsed proppant injection, see schedule parameters in Table 4. The pad duration is $500$ s, and then proppant injection begins. Proppant is pumped at constant concentration for $400$ s, and then clean fluid is pumped for $100$ s. These pulses continue until the end of the schedule. Results of simulations demonstrate the development of a non-regular proppant distribution caused by Saffman-Taylor instability. Proppant accumulates near the vertical fracture tips, but the lateral tip is almost proppant-free and therefore the fracture continues to grow.

Figs. [13(a), 13(d)] plot the fluid pressure at the inlet, width and height at the borehole of hydraulic fracture, as well as its length as a function of time for all four cases considered. As can be clearly seen, for the case of no proppant, all the fracture characteristics evolve monotonically with time. In this case, the largest values of fracture height and length are obtained, while the pressure and the width are the lowest among the considered cases. All the cases with proppant, on the other hand show more similar behavior. Once proppant reaches the fracture tip ($2000$-$2800$ s for different cases), the fluid pressure and width start to increase rapidly, while the height and length growth practically diminishes. Clearly, the use of pulsed injection leads to the longest fracture and larger proppant area, compared to other cases with proppant. In addition, it may reduce the total amount of proppant that is needed for a given treatment.

7. Conclusion

The purpose of this paper is to develop a hydraulic fracturing model that is coupled with proppant transport. The fracture propagation part is solved using Enhanced Pseudo-3D (EP3D) model, in which a moving mesh, that conforms to the fracture shape for every time step, is used. On the other hand, the transport equations are solved on a fixed grid and using higher resolution. The resolution is the key since it allows us to capture effects such as Saffman-Taylor instability when a fluid with lower viscosity displaces a fluid with higher viscosity. Such a situation can occur if the proppant is pumped in pulses that alternate...
Figure 10: Simulation results for the case with constant proppant loading at different time instants $t = \{600, 1800, 2700, 3600\}$ s. Proppant concentration is shown by color.

with clean fluid pumping.

Firstly, leak-off was introduced in the EP3D fracture model. Then, the fracturing module was tested against solutions for a radial fracture in different regimes and for the three-layer formations with symmetric stress barriers. The proppant transport module was also tested to select the optimal flux limiter for numerical computations. The coupling procedure between the modules is described, which in particular involved data interpolation between the two domains.

A series of numerical examples for the coupled model is presented. These examples consider the same rock lithology, but various pumping schedules to investigate the effect of pumping schedule on the resultant fracture growth. Results demonstrate that the schedule in which proppant is pumped in pulses leads to the largest fracture length since the development of Saffman-Taylor instability effectively creates a pathway for slurry flow by washing out the zone of high proppant concentration near the tip.

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Appendix A. Pressure: Numerical Scheme

This appendix presents details of the numerical scheme for solving the pressure equation (25). Computational domain is covered with a uniform grid $(x_i, x_j)$:

$$(x_i, z_j), \quad i = 0, ..., N, \quad j = 0, ..., M. \quad (A.1)$$
Grid steps in the $x$ and $z$ directions are denoted by $h_x = x_{i+1} - x_i$ and $h_z = z_{j+1} - z_j$ correspondingly, and central nodes are then defined as

$$
x_{i+\frac{1}{2}} = (x_{i+1} + x_i)/2, \quad i = 0, \ldots, N - 1,
\quad z_{j+\frac{1}{2}} = (z_{j+1} + z_j)/2, \quad j = 0, \ldots, M - 1.
$$

(A.2)

For brevity, let us introduce the mobility of the fluid as

$$
\Lambda(x, y) = \frac{w(x, y)^3}{12\mu(x, y)}.
$$

Then, the left part of (25) can be integrated over one element to obtain

$$
\int_{z_{j-\frac{1}{2}}}^{z_{j+\frac{1}{2}}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left( \partial \frac{\partial p}{\partial x} + \partial \frac{\partial p}{\partial z} \right) dx dz = \left. \Lambda \frac{\partial p}{\partial x} \right|_{z=\frac{1}{2}}^{z=\frac{1}{2}} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} dx + \left. \Lambda \frac{\partial p}{\partial z} \right|_{x=\frac{1}{2}}^{x=\frac{1}{2}} \int_{z_{j-\frac{1}{2}}}^{z_{j+\frac{1}{2}}} dz 
\approx \left( \Lambda \frac{\partial p}{\partial x} \right)_{(x,z) = (x_{i+\frac{1}{2}}, z_j)} h_x + \left( \Lambda \frac{\partial p}{\partial z} \right)_{(x,z) = (x_{i}, z_{j+\frac{1}{2}})} h_z,
$$

(A.3)

where the fluxes through boundaries of the element are approx-
Figure 12: Simulation results for the pumping schedule with pulses or proppant at different time instants $t = \{600, 1800, 2700, 3600\}$ s. Proppant concentration is shown by color.

imated as follows

$$
(\Lambda \frac{\partial p}{\partial x})_{(x_i^2, z_j^2)} \approx \frac{1}{2} \left( \frac{p_{i+1,j} - p_{i,j}}{h_x} \right) \left( \frac{1}{\Lambda_{i+1,j}} + \frac{1}{\Lambda_{i,j}} \right)^{-1},
$$

$$
(\Lambda \frac{\partial p}{\partial x})_{(x_i^2, z_j^2, \frac{1}{2})} \approx \frac{1}{2} \left( \frac{p_{i,j+1} - p_{i,j}}{h_z} \right) \left( \frac{1}{\Lambda_{i,j+1}} + \frac{1}{\Lambda_{i,j}} \right)^{-1},
$$

If the pressure solution is smooth, the approximation (A.3)-(A.4) has the second order of accuracy, but it may lose the level of accuracy if the mobility $\Lambda$ is not smooth.

By utilizing the above discretization, computation of the pressure at the inner nodes can be written in terms of an implicit finite-difference scheme using the five-point stencil as

$$
-p_{i,j} \left[ \left( \Lambda_{i-\frac{1}{2},j} + \Lambda_{i+\frac{1}{2},j} \right) \frac{h_z}{h_x} + \left( \Lambda_{i,j-\frac{1}{2}} + \Lambda_{i,j+\frac{1}{2}} \right) \frac{h_x}{h_z} \right] +
$$

$$
p_{i+1,j} \Lambda_{i+\frac{1}{2},j} \frac{h_z}{h_x} + p_{i,j+1} \Lambda_{i,j+\frac{1}{2}} \frac{h_x}{h_z} + p_{i-1,j} \Lambda_{i-\frac{1}{2},j} \frac{h_z}{h_x} \right]
$$

$$
= \bar{f}_{i,j},
$$

where the function $\bar{f}_{i,j}$ represents the right hand side of (25).

The discretized equation (A.5) can be rewritten as a system of linear algebraic equations $Ap = f$, in which all the values $p_{i,j}$ are written as the unknown vector $p$, and the matrix $A$ of
Figure 13: Time evolution of fluid pressure, fracture width, fracture height, and length for the considered four pumping schedules, see Table 4 for schedule details.

The resulting system of linear algebraic equations is solved with respect to the pressure vector \( p \). Once the pressure is computed, the fluxes and velocities are computed based on the central difference of (26) for further use in proppant transport.

Appendix B. Transport: Numerical Scheme

To proceed with the transport equation, let us introduce \( q = c_p w \) as a quantity that is being transported, and \( f(q) \), \( g(q) \) are the corresponding flow functions along the \( x \) and \( z \)-directions. In this case, equation (28) can be rewritten as

\[
\frac{\partial q}{\partial t} + \frac{\partial}{\partial x} f(q) + \frac{\partial}{\partial z} g(q) = 0, \quad (B.1)
\]

where \( f(q) = uc_w \) and \( g(q) = vc_w \). This equation is solved by a finite volume scheme [17], which is conservative and has the second order of accuracy.

By focusing on the following control volume

\[
[x_i, x_{i+1}] \times [z_j, z_{j+1}] \times [t^n, t^{n+1}],
\]

the transport equation (B.1) can be discretized as

\[
q^n_{i,j} - q^n_{i,j} = \frac{\Delta t}{h_x} \left( F_{i+\frac{1}{2},j} - F_{i-\frac{1}{2},j} \right) - \frac{\Delta t}{h_z} \left( G_{i,j+\frac{1}{2}} - G_{i,j-\frac{1}{2}} \right), \quad (B.3)
\]

where \( F_{i+\frac{1}{2},j} = f(q(x_{i+\frac{1}{2}}, z_j, t^n)) \) and \( G_{i,j+\frac{1}{2}} = g(q(x_i, z_{j+\frac{1}{2}}, t^n)) \) are the fluxes in the vertical and horizontal directions respectively. The primary difference between various schemes is the way these fluxes \( F_{i,j+\frac{1}{2}} \) and \( G_{i,j+\frac{1}{2}} \) are computed for every node \((x_i, z_j)\).

By assuming \( u, v > 0 \), the two basic methods considered are...
here are the first-order upwind scheme
\[ F_{i+1/2,j}^{up} = u_{i+1/2,j}^{n+1} q_{i,j} - q_{i-1,j}, \]  
(4.4)
and the second-order Lax-Wendroff scheme, in which
\[ F_{i+1/2,j}^{LW} = F_{i+1/2,j}^{up} + \frac{1}{2} u_{i+1/2,j}^{n+1} \left( 1 - \frac{\Delta t}{h_x} \right) \left( q_{i+1,j} - q_{i,j} \right), \]  
(4.5)
\[ G_{i,j+1/2}^{LW} = G_{i,j+1/2}^{up} + \frac{1}{2} v_{i,j+1/2}^{n+1} \left( 1 - \frac{\Delta t}{h_y} \right) \left( q_{i,j+1} - q_{i,j} \right), \]  
(4.5)
where \( \Delta t \) is time step, \( h_x \) and \( h_y \) define the grid spacing. Both of these methods alone cannot provide satisfactory approximation of the solution because the first one features numerical dissipation and the second one numerical dispersion (see, Fig. 6(b)). A better result can be achieved by limiting the influence of the Lax-Wendroff scheme. By introducing the flux limiter \( \Phi(\theta_{i,j}) \), that depends on the local nature of the solution, final representation for both horizontal and vertical fluxes has following form
\[ F_{i+1/2,j} = F_{i+1/2,j}^{up} + \frac{1}{2} u_{i+1/2,j}^{n+1} \left( 1 - \frac{\Delta t}{h_x} \right) \left( q_{i+1,j} - q_{i,j} \right) \Phi(\theta_{i,j}^F), \]  
(4.6)
\[ G_{i,j+1/2} = G_{i,j+1/2}^{up} + \frac{1}{2} v_{i,j+1/2}^{n+1} \left( 1 - \frac{\Delta t}{h_y} \right) \left( q_{i,j+1} - q_{i,j} \right) \Phi(\theta_{i,j}^G), \]  
(4.6)
delete{while 
\[ \theta_{i,j}^F = \min \left( \frac{q_{i+1,j} - q_{i,j}}{q_{i,j} - q_{i-1,j}}, 1 \right), \]  
(4.7)
\[ \theta_{i,j}^G = \min \left( \frac{q_{i,j+1} - q_{i,j}}{q_{i,j} - q_{i,j-1}}, 1 \right), \]  
(4.7) denote ratios of the flux slopes at the points around the interface. Among all possible implementations for the function \( \Phi \) (see for instance [22], [31], [34]), the following two were used in Section 4.3 to see their effect on the solution of the transport problem:
\[ \minmod : \Phi(\theta_{i,j}) = \max \left( 0, \min(1, \theta_{i,j}) \right), \]  
(4.8)
\[ \superbee : \Phi(\theta_{i,j}) = \max \left( 0, \min(1, 2\theta_{i,j}), \min(2, \theta_{i,j}) \right). \]  
(4.8)
To obtain the numerical solution, we use (4.4), (4.6), and (4.8) to compute fluxes and then solve (B.3) to update the values of \( q \) in time.

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