Another construction of edge-regular graphs with regular cliques

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Abstract

We exhibit a new construction of edge-regular graphs with regular cliques that are not strongly regular. The infinite family of graphs resulting from this construction includes an edge-regular graph with parameters $(24, 8, 2)$. We also show that edge-regular graphs with 1-regular cliques that are not strongly regular must have at least 24 vertices.

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1 Introduction and definitions

We begin with various definitions of regularity. A $v$-vertex graph $\Gamma$ is called $k$-regular if there exists a $k$ such that each vertex of $\Gamma$ has degree $k$. A $v$-vertex, $k$-regular non-empty graph is called edge-regular with parameters $(v,k,\lambda)$ if every pair of adjacent vertices $x \sim y$ have $\lambda$ common neighbours. A $(v,k,\lambda)$-edge-regular graph is called strongly regular with parameters $(v,k,\lambda,\mu)$ if every pair of distinct nonadjacent vertices $x \not\sim y$ have $\mu$ common neighbours. A clique $C$ is called regular (or $e$-regular) if every vertex not in $C$ is adjacent to a constant number $e > 0$ of vertices in $C$.

Recently, the authors [7] provided an infinite family of non-strongly-regular, edge-regular graphs having regular cliques, thus answering a question of Neumaier [8, Page 248]. The smallest graph in this family has parameters $(28,9,2)$. In this note we offer a new construction that gives rise to a non-strongly-regular, edge-regular graph $G$ with parameters $(24,8,2)$ having a 1-regular clique. In fact, $G$ is isomorphic to one of the four examples found by Goryainov and Shalaginov [6]. Recently, however, Evans et al. [4] discovered an edge-regular, but not strongly regular graph on 16 vertices that has 2-regular cliques with order 4, and proved that, up to isomorphism, this is the unique edge-regular, but not strongly regular graph on at most 16 vertices having a regular clique.

A graph $\Gamma$ of diameter $d$ is called $a$-antipodal if the relation of being at distance $d$ or distance 0 is an equivalence relation on the vertices of $\Gamma$ with equivalence classes of size $a$. A set of cliques of a graph $\Gamma$ that partition the vertex set of $\Gamma$ is called a spread in $\Gamma$. Distance regular graphs and generalised quadrangles are edge-regular graphs satisfying further regularity conditions, for their definitions, see Brouwer and Haemers [3] (see below for the definition of a distance regular graph). Brouwer [1] gave a construction for antipodal distance regular graphs from generalised quadrangles having a spread of regular cliques. Inspired by Brouwer, our construction is a generalisation of the other direction, i.e., we construct graphs from antipodal distance-regular graphs of diameter three.

In Section 2 we present our construction and in Section 3 we show that non-strongly-regular, edge-regular graphs having 1-regular cliques must have at least 24 vertices.
2 Construction

Let Γ be a graph with diameter \( d \). We call Γ \textit{distance-regular} if, for any two vertices \( x, y \in V(\Gamma) \), the number of vertices at distance \( i \) from \( x \) and distance \( j \) from \( y \) depends only on \( i, j \), and the distance from \( x \) to \( y \). It is clear that distance regular graphs are edge-regular.

Let \( \Gamma \) be a graph of diameter \( d \). For a given \( t \), make \( t \) copies \( \Gamma^{(1)}, \ldots, \Gamma^{(t)} \) of \( \Gamma \). For each vertex \( x \in V(\Gamma) \), denote by \( \Gamma^i(x) \) the set of vertices at distance \( i \) from \( x \) and denote by \( x_j \) the corresponding copy of \( x \) in the graph \( \Gamma^{(j)} \). Define the sets

\[
E_1(\Gamma) = \{ \{x_i, y_j\} : x \in V(\Gamma), y \in \Gamma_d(x), \text{ and } i, j \in \{1, \ldots, t\} \}
\]

\[
E_2(\Gamma) = \{ \{x_i, x_j\} : x \in V(\Gamma) \text{ and } i \neq j \}.
\]

Let \( \hat{\Gamma} \) denote the disjoint union of the graphs \( \Gamma^{(1)}, \ldots, \Gamma^{(t)} \). Then define \( F_t(\Gamma) \) to be the graph with vertex set \( V(F_t(\Gamma)) = V(\hat{\Gamma}) \) and edge set

\[
E(F_t(\Gamma)) = E(\hat{\Gamma}) \cup E_1(\Gamma) \cup E_2(\Gamma).
\]

**Theorem 2.1.** Let \( \Gamma \) be an \( a \)-antipodal distance-regular graph of diameter 3 with edge-regular parameters \( (v, k, \lambda) \) such that \( a \) is a proper divisor of \( \lambda + 2 \). Then

1. \( F_{\lambda + 2}(\Gamma) \) has a spread of 1-regular cliques each of size \( \lambda + 2 \);
2. \( F_{\lambda + 2}(\Gamma) \) is \( (v(\lambda + 2)/a, k + \lambda + 1, \lambda) \)-edge-regular;
3. \( F_{\lambda + 2}(\Gamma) \) is not strongly regular.

**Proof.** Let \( t = (\lambda + 2)/a \).

1. Since \( \Gamma \) is \( a \)-antipodal, its vertex set can be partitioned into \( v/a \) \( a \)-subsets of \( V(\Gamma) \) such that each subset contains a vertex and all its antipodes. For each part \( P \) in the partition, take a vertex \( x \in P \) and define the set

\[
C_x = \{x_i : i \in \{1, \ldots, t\}\} \cup \bigcup_{y \in \Gamma_3(x)} \{y_i : i \in \{1, \ldots, t\}\} \subset V(F(\Gamma)).
\]

It is clear that each \( C_x \) is a clique in \( F(\Gamma) \) of size \( \lambda + 2 \) and that these cliques partition the vertex set of \( F(\Gamma) \). To see that these cliques are
1-regular, consider a vertex $z$ not in the clique $C_x$. Let $\Gamma^{(i)}$ be the copy of $\Gamma$ containing $z$. Since $\Gamma$ is distance-regular and has diameter 3, the vertex $z$ must be adjacent to precisely one vertex in the set $\{x_i\} \cup \{y_i : y \in \Gamma_3(x)\}$. Therefore $C_x$ is 1-regular.

2. It is clear that $F(\Gamma)$ has $vt = v(\lambda + 2)/a$ vertices. Let $x$ be a vertex of $F(\Gamma)$ inside the $i$th copy $\Gamma^{(i)}$ of $\Gamma$. Then $x$ is adjacent to $k + a - 1$ vertices inside $\Gamma^{(i)}$ and $x$ is adjacent to the vertices $x_j$ and $y_j$ for all $j \neq i$ and $y \in \Gamma_3(x)$. Hence $x$ has valency $k + a - 1 + a(t - 1) = k + \lambda + 1$. Now suppose that $y$ is adjacent to $x$. If $y$ is in the clique $C_x$ then, since $C_x$ is 1-regular, $y$ and $x$ have $\lambda$ common neighbours. Otherwise $y$ must be a vertex of $\Gamma^{(i)}$, where the number of common neighbours of $y$ and $x$ are equal to the number of common neighbours of adjacent vertices in $\Gamma$, which is $\lambda$.

3. Let $x$ be a vertex of $F_t(\Gamma)$ inside the $i$th copy $\Gamma^{(i)}$ of $\Gamma$. Form the set $\mu(\Gamma) = \{\nu_{y,z} : y, z \in V(\Gamma) and y \not\sim z\}$, where $\nu_{y,z}$ denotes the number of common neighbours of $y$ and $z$. Since $\Gamma$ is not strongly regular, the set $\mu(\Gamma)$ must have at least 2 elements. Furthermore, since $\Gamma$ has diameter 3, we see that $0 \in \mu(\Gamma)$. Let $\eta \in \mu(\Gamma)$ with $\eta \neq 0$. Consider the vertices $y$ and $z$, where $y$ is in $\Gamma^{(i)}$ such that $y \not\sim x$ and $\nu_{x,y} = \eta$ and $z$ is in $\Gamma^{(j)}$ with $j \neq i$ and $z \not\sim x$. The number of common neighbours of $x$ and $y$ is $\eta + 2$ and the number of common neighbours of $x$ and $z$ is 2. Hence $F_t(\Gamma)$ is not strongly regular.

A Taylor graph is a 2-antipodal distance-regular graph of diameter 3 (for a proper definition, see Brouwer, Cohen, and Neumaier [2, Page 13]).

Example 1. Let $\Gamma$ be a Taylor graph with edge-regular parameters $(v, k, \lambda)$. It is known [2, Theorem 1.5.3] that $\lambda$ is even. By Theorem 2.1, the graph $F_{t/2+1}(\Gamma)$ is a non-strongly-regular $(v(\lambda + 2)/2, k + \lambda + 1, \lambda)$-edge-regular graph having a 1-regular clique. The smallest example of this family is the icosahedral graph $P$, which has parameters $(12, 5, 2)$. The graph $F_2(P)$ is a non-strongly-regular $(24, 8, 2)$-edge-regular graph having a 1-regular clique, furthermore, $F_2(P)$ is isomorphic to one of the four examples of Goryainov and Shalaginov [6].

We can also use other constructions of antipodal distance-regular graphs of diameter 3 due to Brouwer, Hensel, and Mathon (see Godsil and Hensel [5].
or Brouwer, Cohen, and Neumaier [2, Page 385]). These constructions produce $a$-antipodal edge-regular graphs satisfying $\lambda + 2 \equiv 0 \pmod{a}$ with $a \geq 3$.

Note added in proof: Sergey Goryainov observed that one can generalise the construction given in this section. One does not need an isomorphism between the antipodal graphs, a bijection between the fibre classes suffices.

3 At least 24 vertices for 1-regular cliques

In the remainder of this note, we prove the following result.

Theorem 3.1. Let $\Gamma$ be an edge-regular graph with a 1-regular clique that is not strongly regular. Then $\Gamma$ has at least 24 vertices.

First we establish a lower bound on the vertex degree. For a graph $\Gamma$ and a vertex $x \in V(\Gamma)$, let $\Gamma(x)$ denote the set of neighbours of $x$. The $q \times q$ grid (also known as the square lattice graph) is defined to be the Cartesian product of two complete graphs of order $q$. It is well-known [2] that the $q \times q$ grid is strongly regular with parameters $(q^2, 2(q-1), q-2, 2)$.

Lemma 3.2. Let $\Gamma$ be a non-complete $k$-regular edge-regular graph having a 1-regular clique of order $c$. Then $k \geq 2(c - 1)$. In the case of equality, $\Gamma$ is the $c \times c$ grid and is thus strongly regular.

Proof. Set $m = k/(c-1)$. Since $\Gamma$ is not complete, we have $m > 1$. Let $C$ be a regular clique of order $c$. Let $x$ be a vertex in $C$. Since there are no edges between $\Gamma(x) \cap C$ and $\Gamma(x) \setminus C$, we find that $\lambda = c - 2$.

Now suppose $y$ is a vertex adjacent to $x$ but not in $C$. Note that $x$ has $k - (c-1) = (m-1)(c-1)$ neighbours outside of $C$. Hence the number of common neighbours of $x$ and $y$ is at most $(m-1)(c-1) - 1$. Therefore $c - 2 \leq (m-2)(c-1) - 1$. Again, since $\Gamma$ is not complete, we have $c - 1 \geq 1$. Hence we must have $m \geq 2$.

In the case of equality, we see that $y$ is adjacent to every neighbour of $x$ outside $C$. Furthermore, the subgraph induced on the neighbourhood of $x$ is the disjoint union of two complete graphs each of order $c - 1$. Therefore, $\Gamma$ is the Cartesian product of two complete graphs of order $c$, i.e., the $c \times c$ grid.

Next, we need a lower bound on the size of a regular clique.
Proposition 3.3 ([7, Proposition 5.2]). Let \( \Gamma \) be an edge-regular graph having a regular clique. Suppose that \( \Gamma \) is not strongly regular. Then \( \Gamma \) has a regular clique of order at least 4.

The final ingredient is a nonexistence result for a graph on 20 vertices.

Proposition 3.4. There does not exist an edge-regular graph with parameters \((20, 7, 2)\) and a 1-regular clique.

Proof. Suppose, for a contradiction, there does exist such a graph \( \Gamma \). By Proposition 3.3, \( \Gamma \) must have a regular clique \( \mathcal{C} \) of size at least 4. Moreover, since \( \lambda = 2 \), the clique \( \mathcal{C} \) must have size 4. Let \( x \in \mathcal{C} \). The subgraph induced on \( \Gamma(x) \) is the disjoint union of a 3-cycle \( T \) and a 4-cycle \( C \). Set \( K = \{x\} \cup V(T) \) and let \( \Delta \) be the subgraph induced on the vertices \( V(\Gamma) \setminus K \). Observe that, since \( K \) is a 1-regular clique of \( \Gamma \), the subgraph \( \Delta \) is 6-regular. Furthermore, observe that each of the 16 pairs of adjacent vertices in \( \bigcup_{x \in K} \Gamma(x) \setminus K \) has a common vertex in \( K \). Hence there are 16 edges in \( \Delta \) that are each contained in precisely one triangle and the remaining 32 edges are in precisely 2 triangles. Therefore \( \Delta \) has \((16 + 2 \cdot 32)/3\) triangles. Since this number is not an integer, we establish a contradiction. \( \square \)

Now the proof of Theorem 3.1

Proof of Theorem 3.1. By Proposition 3.3, \( \Gamma \) must have a regular clique \( \mathcal{C} \) of size \( c \geq 4 \). Furthermore, by Lemma 3.2, the degree of the vertices of \( \Gamma \) is at least 7. If \( k \geq 8 \) then, since \( \mathcal{C} \) is 1-regular, \( \Gamma \) must have at least 24 vertices. It therefore suffices to consider the case when \( k = 7 \) and \( c = 4 \). In this case, \( \Gamma \) must be edge-regular with parameters \((20, 7, 2)\). But by Proposition 3.4, no such graph exists. \( \square \)

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