Reduced Variations in Earth’s and Mars’ Orbital Inclination and Earth’s Obliquity from 58 to 48 Myr ago due to Solar System Chaos

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Abstract

The dynamical evolution of the solar system is chaotic with a Lyapunov time of only ∼5 Myr for the inner planets. Due to the chaos it is fundamentally impossible to accurately predict the solar system’s orbital evolution beyond ∼50 Myr based on present astronomical observations. We have recently developed a method to overcome the problem by using the geologic record to constrain astronomical solutions in the past. Our resulting optimal astronomical solution (called ZB18a) shows exceptional agreement with the geologic record to ∼58 Ma (Myr ago) and a characteristic resonance transition around 50 Ma. We show that ZB18a and integration of Earth’s and Mars’ spin vector based on ZB18a yield reduced variations in Earth’s and Mars’ orbital inclination and Earth’s obliquity (axial tilt) from ∼58 to ∼48 Ma—the latter being consistent with paleoclimate records. The changes in the obliquities have important implications for the climate histories of Earth and Mars. We provide a detailed analysis of solar system frequencies (g and s modes) and show that the shifts in the variation in Earth’s and Mars’ orbital inclination and obliquity around 48 Ma are associated with the resonance transition and caused by changes in the contributions to the superposition of s modes, plus g–s mode interactions in the inner solar system. The g–s mode interactions and the resonance transition (consistent with geologic data) are unequivocal manifestations of chaos. Dynamical chaos in the solar system hence not only affects its orbital properties but also the long-term evolution of planetary climate through eccentricity and the link between inclination and axial tilt.

Unified Astronomy Thesaurus concepts: Celestial mechanics (211); Orbital theory (1182); Dynamical evolution (421); Solar system (1528); Planetary climates (2184)

1. Introduction

The chaotic behavior of the solar system imposes an apparently firm limit of ∼50 Myr (past and future) on identifying a unique astronomical (orbital) solution, as small differences in the initial conditions/parameters cause astronomical solutions to diverge around that time interval (Lyapunov time ∼5 Myr; e.g., Laskar 1990; Sussman & Wisdom 1992; Morbidelli 2002; Varadi et al. 2003; Batygin & Laughlin 2008; Zeebe 2015a; Abbot et al. 2021). The dynamical chaos constitutes a fundamental physical barrier that cannot be overcome by, say, further refinement of current astronomical observations or improvement of the physical model (e.g., Laskar et al. 2011; Zeebe 2017). To constrain the solar system’s history beyond ∼50 Ma, for instance, alternative approaches are now required. Zeebe & Lourens (2019) recently developed a new approach that allows identifying an optimal astronomical solution based on the geologic record. Briefly, the approach uses deep-sea sediment records to select an optimal astronomical solution (dubbed ZB18a), which shows exceptional agreement with the geologic record to ∼58 Ma and a characteristic resonance transition around 50 Ma (see Section 4), consistent with geologic data (Zeebe & Lourens 2019, 2022b). The geologic evidence hence corroborates the validity of the orbital solution ZB18a from 58 to 0 Ma. In turn, the astronomical solution provides highly accurate geologic ages, including a revised age for the Paleocene–Eocene boundary, with small margins of error. The details are provided in Zeebe & Lourens (2019, 2022b) and shall not be repeated here.

Beyond astronomical applications, astronomical solutions are now used as an indispensable and highly accurate dating tool in disciplines such as geology, geophysics, and paleoclimatology, and represent the backbone of cyclostratigraphy and astrochronology (e.g., Montenari 2018). Furthermore, astronomical solutions form the basis for studying the astronomical forcing of climate. The astronomical theory of climate (Milanković 1941) has been impressively confirmed by explaining the pacing of long-term climate change on Earth (e.g., Paillard 2021), has been applied to other planets in our solar system such as Mars (e.g., Pollack 1979; Toon et al. 1980), and represents an element of exoplanet climatology (e.g., Spiegel et al. 2010; Shields 2019). Milanković forcing of Earth’s climate is primarily expressed as three major cyclicities related to the orbital eccentricity, obliquity (axial tilt), and precession. Astronomical solutions naturally provide orbital eccentricity, which directly affects climate through total insolation and indirectly through amplitude modulation of precession (Zeebe & Lourens 2019; Paillard 2021). A related but separate question is how the characteristics of the orbital solution affect precession and obliquity and, in turn, their associated planetary climate cycles.

Here we investigate the astronomical properties of the solution ZB18a and its consequences for the chaotic evolution of the solar system, including the orbital and climatic history of the inner planets, specifically Earth and Mars. We show that ZB18a and integration of Earth’s and Mars’ spin axis based on ZB18a yield reduced variations in Earth’s and Mars’ orbital inclination and Earth’s obliquity from ∼58 to ∼48 Ma. Below, we first describe the methods used to compute changes in the
planetary spin axis to obtain precession and obliquity solutions in the past and briefly summarize the solar system integrations (Section 2). Next, we present the results of the integrations, including orbital eccentricity and inclination, and obliquity for Earth and Mars (Section 3). A detailed analysis of solar system frequencies and the resonance transition, as well as a signal reconstruction based on key eigenmodes, or proper modes, is provided in Section 4. The implications of our results are discussed in Section 5, while a few details on geodetic precession and frequency uncertainties are given in Appendices A and B.

2. Methods

2.1. Precession and Obliquity

The change in the spin axis (unit vector \( \mathbf{s} \)) may be calculated from (e.g., Goldreich 1966; Ward 1974, 1979; Bills 1990; Quinn et al. 1991):

\[
\dot{s} = \alpha (\mathbf{n} \cdot s)(s \times n),
\]

where \( \alpha \) is the precession constant (see below), and \( \mathbf{n} \) the orbit normal (unit vector normal to the orbit plane; see Figure 1). The obliquity (polar) angle, \( \epsilon \), is given by:

\[
\cos \epsilon = \mathbf{n} \cdot \mathbf{s}.
\]

The precession (azimuthal) angle, \( \phi \), measures the motion of \( \mathbf{s} \) in the orbit plane (see Figure 1). Importantly, the accuracies of our numerical computations described below are designed for multimillion year integrations and do not take into account several second-order effects (see Capitaine et al. 2003).

2.1.1. Earth: Precession Constant and Lunisolar Precession

For Earth, we write \( \alpha \) as (e.g., Quinn et al. 1991):

\[
\alpha = K(\kappa + \beta),
\]

where \( \kappa = (1 - e^2)^{-3/2} \), and \( e \) is the orbital eccentricity. \( K \) and \( \beta \) relate to the torque due to the Sun and Moon, respectively:

\[
K = \frac{3}{2} \frac{C - A}{C} \omega^3 GM,
\]

\[
\beta = g_L \frac{a^3 m_L}{R^3 M},
\]

where \( A \) and \( C \) are the planet’s equatorial and polar moments of inertia, \( (C - A)/C = E_d \) is the dynamical ellipticity, \( \omega \) is the planet’s angular speed, \( a \) the semimajor axis of its orbit, \( R \) is the Earth–Moon distance parameter, and \( GM \) is the gravitational parameter of the Sun (see Table 1). The index “\( L \)” refers to the lunar properties, where \( g_L \) is a correction factor related to the lunar orbit (Kinoshita 1975, 1977; Quinn et al. 1991), and \( m_L/M \) is the lunar to solar mass ratio. The parameter values used for Earth are given in Table 1. The lunisolar precession \( \Psi \) at \( t_0 \) is given by:

\[
\Psi_0 = -\dot{\phi}_0 = -d\phi_0/dt = K(\kappa + \beta)\epsilon_0 + \gamma_{gp}.
\]

where \( \phi_0 \) and \( \epsilon_0 \) are the precession and obliquity angle at \( t_0 \) \((d\phi_0/dt < 0, \) retrograde precession along the ecliptic\), and \( \gamma_{gp} \) is the geodetic precession (see Table 1). The geodetic precession was included in our numerical routines for Earth as described in Appendix A. Earth’s dynamical ellipticity \( E_d \) at \( t = t_0 \) was determined from Equations (4) and (6) by setting the
Table 1
Notation and Values used in this Paper

| Symbol | Meaning | Value I/A | Unit | Note |
|--------|---------|-----------|------|------|
| $e$    | Obliquity angle | 0.0192 | °/yr | Capitaine et al. (2003) |
| $e_0$  | Obliquity Earth $t_0$ | 23.4392911 | deg | Fränz & Harper (2002) |
| $e_{OM}$ | Obliquity Mars $t_0$ | 25.189417 | deg | Folkner et al. (1997) |
| $\phi$ | Precession angle | 0 | | |
| $s$    | Spin vector | 0 | | |
| $n$    | Orbit normal | 0 | | |
| $e$    | Orbital eccentricity | 0 | | |
| $\varpi$ | Orbit LPa | 0 | | |
| $I$    | Orbital inclination | 0 | | |
| $\Omega$ | Orbit LANb | 0 | | |
| $\gamma_{GP}$ | Geodetic precession | 0 | | |
| $\alpha$ | Astronomical unit | 0 | | |
| $GM$   | Sun GPc | 0 | | |
| $M/(m_E + m_L)$ | Mass ratiod | 0 | | |
| $m_E/m_L$ | Mass ratio | 0 | | |
| $\omega$ | Earth’s angular speed | 0 | | |
| $R_0$  | Earth–Moon DPf | 0 | | |
| $A, C$ | Moments of inertia | 0 | | |
| $E_{db} = (C - A)/C$ | Earth’s dyn. ellipticity | 0 | | |
| $S_L$  | Lunar orbit factor | 0 | | |
| $\Psi_0$ | $\phi_0$ Earth | 0 | | |
| $\Psi_{OM}$ | $\phi_0$ Mars | 0 | | |

Notes.

a LP = Longitude of Perihelion.

b LAN = Longitude of Ascending Node.

c GP = Gravitational Parameter.

d Index $E$ = Earth, $L$ = Lunar.

e DP = Distance Parameter.

f Earth’s equatorial (A) and polar (C) moment of inertia.

g Value for $\Psi_0$ (Capitaine et al. 2003; see Table 1). The value of $E_d$ in a particular precession model depends on the choice of $\Psi_0$ (see, e.g., Quinn et al. 1991; Laskar et al. 1993; Chen et al. 2015).

2.1.2. Mars: Precession Constant

For Mars $\beta = 0$, while $K$ was determined from the observed $\Psi_{OM}$ and $e_{OM}$ (Folkner et al. 1997; Yoder et al. 2003; see Table 1):

$$K = \Psi_{OM}/(\cos e_{OM} \cdot k_{OM}),$$

(7)

where $k_{OM} = (1 - e_{OM}^2)^{-3/2}$, and $e_{OM}$ is Mars’ orbital eccentricity at $t_0$. To first order, there is no averaged torque on Mars from its moons Phobos and Deimos (Laskar et al. 2004). Our numerical integrations (see Section 3.3) confirmed that Mars’ obliquity is chaotic (e.g., Touma & Wisdom 1993; Laskar & Robutel 1993), i.e., is unpredictable on timescales beyond $\sim 10^7$ yr (although, see Bills & Keane 2019). As the present study focuses on timescales $> 10^7$ yr, no attempt was made to include second-order effects on Mars’ computed precession and obliquity such as relativistic corrections.

2.1.3. Coordinate Systems and Initial Conditions

Orbital motion, spin axis motion, precession, etc. may be described in different coordinate systems. For example, the orbital motion may be described in an inertial frame defined by Earth’s mean orbit at J2000 (hereafter ECLIPJ2000), or in the heliocentric inertial (HCI) frame, etc. (see, e.g., Fränz & Harper 2002; 1). The initial spin axis position, the precession angle, etc. may be conveniently described in a noninertial frame defined by the instantaneous orbit plane (IOP) with the z-axis parallel to the orbit normal and the x-axis along the line of the ascending node (see Figure 1). Our integrations for the orbital motion of the solar system were performed in a coordinate system equivalent to the HCI frame to conveniently account for the solar quadrupole moment (see Section 2.3 and Zeebe 2017). The transformation between the different frames is accomplished by some form of Euler transformation (rotation matrix). For example, let $s$ and $s'$ be the spin vectors in the inertial and IOP frames, respectively. Then (e.g., Ward 1974):

$$s' = \mathbf{A}(I, \Omega) s,$$

(8)

where $I$ and $\Omega$ are the orbital inclination and longitude of ascending node, respectively, and $A(I, \Omega)$ is the time-dependent Euler transformation:

$$A(I, \Omega) = \begin{pmatrix} \cos \Omega & \sin \Omega & 0 \\ -\cos I \sin \Omega & \cos I \cos \Omega & \sin I \\ \sin I \sin \Omega & -\sin I \cos \Omega & \cos I \end{pmatrix}.$$  

(9)

The static transformation matrix from ECLIPJ2000 to our HCI frame is given by $A(I, \Omega)$, where $I_0 = 7^\circ 155$ and $\Omega_0 = 75^\circ 594$ (see Zeebe 2017). The static transformation matrix from Earth’s mean equator frame at J2000 to ECLIPJ2000 is given by $A(\epsilon_0, \Omega_0)$, where $\epsilon_0 = 23^\circ 4392911$.  

1 naij.jpl.nasa.gov
The initial position of the spin vector in Earth’s mean equator frame at $t = 0$ (J2000) was set to $s_0 = [0, 0, 1]$. The numerical spin axis integration (see Section 2.1.4) is carried out in our inertial HCI frame, in which $s_0$ is given by $s_0 = \mathcal{A}_{(i,\Omega)}\mathcal{A}_{(\Omega,0)}s_0^0$.

The inclinations of Earth’s and Mars’ orbits are referenced below in the invariable plane, i.e., relative to the invariable plane (perpendicular to the total angular momentum vector), a natural, common reference frame for solar system bodies. For example, the transformation of a state vector $X$ from ECLIJP2000 to the invariable plane is given by $X_{ip} = \mathcal{A}_{(i,\Omega)}X$, where $I_{ip} = 1^\circ5787$ and $\Omega_{ip} = 107^\circ5823$ (Souami & Souachy 2012). The usual conversion is applied to switch between state vectors and orbital (Keplerian) elements.

### 2.1.4. Spin Vector Integration

The numerical integration of the spin vector $s = [s_x, s_y, s_z]$ employed here follows Ward (1979) and Bills (1990). Rewriting the orbit normal vector $n$ in terms of $p$ and $q$ as defined in Equation (16) and substituting into Equation (1) leads to:

$$
\dot{s}_x = A(c_1 s_y + c_2 \tilde{q} s_z),
\dot{s}_y = A(-c_1 s_y + c_2 \tilde{q} s_z),
\dot{s}_z = A(-c_1 (\tilde{q} s_y + \tilde{p} s_z)),
$$

(10)

where $A = \alpha(c_2 (\tilde{p} s_x - \tilde{q} s_y) + c_1 s_z)/(1 - \tilde{q}^2)^{3/2}$, $c_1 = \cos(I)$, $c_2 = \cos(I/2)$, $\tilde{q} = 2q$ and $\tilde{p} = 2p$, where $p$ and $q$ are supplied by our astronomical solution ZB18a as (see Section 2.3). The numerical spin vector integration is straightforward and fast, and provides a simple check on accuracy. As $s$ is a unit vector, $s = |s| - 1$ may be used to track the numerical error during the integration. A 100 Myr integration typically takes ~3 sec (Linux, Intel i7-10875H, 2.30 GHz) with $|\delta| < 1 \times 10^{-7}$. Our numerical routine in C is available at.

The obliquity $\epsilon$ may be calculated from Equation (2) at any given time step, once the solution for $s$ has been obtained. The precession angle $\phi$ is measured in the IOP frame ($\phi^* = \atan(s_y^*, s_x^*)$) with $\phi_0^* = 90^\circ$ at $t = 0$; hence we apply the transformation:

$$
\mathcal{A}_{(i,\Omega)}^{-1} \mathcal{A}_{(j,\Omega)} \mathcal{A}_{(0,\dot{\Omega})} s^* = \mathcal{A}_{(i,\Omega)}^{-1} \mathcal{A}_{(j,\Omega)} \mathcal{A}_{(0,\dot{\Omega})} s,
$$

(11)

which gives $\phi^*$ relative to the moving equinox. To obtain $\phi$ relative to the fixed equinox at J2000, we further apply $\mathcal{A}_{(i,0)}^{-1} \mathcal{A}_{(0,\dot{\Omega})}$, where the $\pi/2$ rotation accounts for the angle between the $x$-axis and the component of $s^*$ in the $x$–$y$ plane at $t = 0$, which points along the $y^*$-axis ($s_y^* = 0$; see Figure 1).

### 2.2. Earth: Tidal Dissipation and Dynamical Ellipticity

Tidal dissipation, $T_d$, refers to the energy dissipation in the earth and ocean, which reduces Earth’s rotation rate and increases the length of day and the Earth–Moon distance. The parameter relevant here for the precession–obliquity solution is the change in the lunar mean motion $n$, which is presently $(t = t_0)$ decreasing at a rate:

$$
T_{d0} = (\dot{n}/n_0) = -4.6 \times 10^{-18} \text{ s}^{-1}
$$

(12)

(Quinn et al. 1991). Given $n \propto R^{-3/2}$, where $R$ is the Earth–Moon distance parameter (see Table 1), it follows $\dot{R}/R = -\frac{2}{3} (\dot{n}/n)$. Dynamical ellipticity, $E_d$, refers to Earth’s gravitational shape, largely controlled by the hydrostatic response to its rotation rate. $E_d$ is proportional to $\omega^2$, where $\omega$ is Earth’s spin (see Table 1). Hence from Equation (4) follows $\dot{K}/K = \omega^2/\omega$ and from Equation (5) $\dot{\beta}/\beta = -3 \dot{R}/R = 2(\dot{n}/n)$. Note that the input arguments for our C routine are nondimensional, effective parameters, relative to the modern values, i.e., $\theta = T_d/T_{d0}$ and $\eta = (E_d/\omega^2)(E_d//\omega^0)$.

Changes in $T_d$ and $E_d$ over time cause slow changes in $K$ and $\beta$ (see Equations (4) and (5)). Following Quinn et al. (1991), these may be approximated to vary linearly with time (insert $\dot{K}/K$ and $\dot{\beta}/\beta$ from above):

$$
K = K_0 [1 + (\dot{n}/n_0)(t - t_0)]
$$

(13)

$$
\beta = \beta_0 [1 + 2(\dot{n}/n_0)(t - t_0)],
$$

(14)

where $(\dot{n}/n_0)$ is given by Equation (12) and $(\dot{n}/n_0)$ $\approx 51(\dot{n}/n_0)$ (Lambeck 1980). While $T_d$ and $E_d$ have significant effects on Earth’s precession and obliquity frequencies (Zeebe & Lourens 2022a), their effect on, for instance, Earth’s obliquity amplitude (which is relevant here) is small. In this study, the default values $\theta = 1$ and $\eta = 1$ were used, and the precession constant $\alpha$ as a function of time was calculated using Equations (3), (13), and (14). Parameters $\theta$ and $\eta$ may be varied for other purposes such as geologic dating (Zeebe & Lourens 2022a). By default, additional long-term effects on tidal dissipation on obliquity (secular trend) were not included here. However, our C code provides this option (available at).

### 2.3. Solar System Integration

For the present study, we use our astronomical solution ZB18a (see below), described in detail in Zeebe & Lourens (2019). Hence we only provide a brief summary of the integrations methods here. Solar system integrations were performed following our earlier work (Zeebe 2015a, 2015b, 2017; Zeebe & Lourens 2019, 2022b) with the integrator package HNBody (Rauch & Hamilton 2002; v1.0.10) using the sympletic integrator and Jacobi coordinates (Zeebe 2015a). All simulations include contributions from general relativity (Einstein 1916), available in HNBody as post-Newtonian effects due to the dominant mass. The Earth–Moon system was modeled as a gravitational quadrupole (Quinn et al. 1991; lunar option), shown to be consistent with expensive Bulirsch–Stoer integrations up to 63 Ma (Zeebe 2017). Initial conditions for the positions and velocities of the planets and Pluto were generated from the JPL DE431 ephemeris (Folkner et al. 2014), using the SPICE toolkit for Matlab. We have recently also tested the latest JPL ephemeris DE441 (Park et al. 2021), which has no effect on the current results because the divergence time relative to ZB18a (based on DE431) is $\sim 66$ Ma. The integrations for ZB18a (Zeebe & Lourens 2019) included 10 asteroids, with initial conditions generated at (for a list of asteroids, see Zeebe 2017). Coordinates were obtained

1. www2.hawaii.edu/~zeebe/Astro.html
2. naif.jpl.nasa.gov/pub/naif/generic_kernels/spk/planets
3. naif.jpl.nasa.gov/naif/toolkit.html
4. ssd.jpl.nasa.gov/x/spk.html
at JD2451545.0 in the ECLIPJ2000 reference frame and subsequently rotated to account for the solar quadrupole moment ($J_2$) alignment with the solar rotation axis (Zeebe 2017). Earth’s orbital eccentricity for the ZB18a solution is available at 7 and 8. We provide our solutions over the time interval from 100 to 0 Ma. However, as only the interval 58–0 Ma is constrained by geologic data (Zeebe & Lourens 2019), we solely focus on this particular interval here and caution that the interval prior to 58 Ma is unconstrained due to solar system chaos.

3. Results

In the following, we report the results of our numerical solar system and spin vector integrations. We focus on Earth and Mars, for which changes in orbital inclination around 48 Ma (see below) appear most pronounced and relevant to possible effects on planetary climate evolution. The results for Mercury’s orbit suggest only moderate changes in the inclination pattern (not shown), while the results for Venus’ orbital inclination are similar to those for Earth.

3.1. Earth’s Orbital Inclination and Obliquity

Numerical integration of the spin vector using our orbital solution ZB18a provides up-to-date solutions for Earth’s precession and obliquity as a function of tidal dissipation and dynamical ellipticity for geologic analyses (Zeebe & Lourens 2022a). In terms of solar system dynamics, the results are consistent with expectations over the past ~48 Ma (see, e.g., Quinn et al. 1991; Laskar et al. 1993; Zeebe & Lourens 2022a). However, from ~58 to ~48 Ma the obliquity shows significantly reduced variations (Figure 2). The reduced obliquity variations are a direct result of the damped inclination amplitude in ZB18a across the same time interval (Figure 2(a)). None of the spin vector integrations using any of the orbital solutions ZB17a-f (Zeebe 2017), for instance, shows a similar behavior. The reason for the damped inclination amplitude prior to ~48 Ma is a resonance transition (see Section 4) that occurs between ~53 and ~45 Ma in ZB18a (Zeebe & Lourens 2022b), but at different times in other solutions such as ZB17a-f. The magnitude of the change in the obliquity variation around 48 Ma may be illustrated by calculating the obliquity envelope (Hilbert transform, $\mathcal{H}(\epsilon)$; green). (c) $2\mathcal{H}(\epsilon - \tau)$ indicates a ~50% increase at ~48 Ma in the average amplitude around the mean obliquity value ($\bar{\tau}$).

3.2. Earth’s and Mars’ Orbital Eccentricity and Inclination

While the change in Earth’s orbital eccentricity and inclination across ~48 Ma may appear somewhat subtle, the change in Mars’ inclination variation is large (see Figure 3). Mars’ inclination in the invariant frame varies between $3^\circ.1$ and

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7 www2.hawaii.edu/~zeebe/Astro.html
8 www.ncdc.noaa.gov/paleo/study/35174
7.1 from 58 to 48 Ma, but between almost 0° and 7.4° from 48 to 0 Ma, showing a distinct M pattern (Figure 3(d)). The M pattern with near-zero values continues until the present. The M pattern is also apparent in Mars’ eccentricity (Figure 3(c)), albeit at twice the period than the inclination from 48 to 0 Ma. Prior to 48 Ma, the period ratio is ~1:1, with maxima in eccentricity roughly coinciding with minima in inclination, and illustrating the resonance transition in the solution ZB18a (see Section 4). Thus, our analysis suggests large changes in Mars’ orbital inclination and hence in the pattern of climate forcing on Mars around 48 Ma (see Section 3.3 and discussion, Section 5).

### 3.3. Mars’ Obliquity

As mentioned above, our integrations confirmed that Mars’ obliquity is chaotic (e.g., Touma & Wisdom 1993; Laskar & Robutel 1993); that is, the details of Mars’ obliquity evolution are unpredictable on timescales beyond ~10^7 yr (although, see Bills & Keane 2019). However, irrespective of the details, our orbital solution suggests a major shift in Mars’ inclination and hence in the pattern of Mars’ obliquity around 48 Ma. For example, we integrated sets of solutions with small differences in Mars’ precession constant (equal to reported error bounds), which all showed the same obliquity pattern (see Figure 4). We tested several values for Ψ_0M (see Table 1 and Equation (7)), including 7°597 ± 0°025 yr⁻¹ and 7°6083 ± 0°0021 yr⁻¹ (Yoder et al. 2003; Konopliv et al. 2016). Even using the small uncertainty of 0°0021 yr⁻¹, the obliquity solutions are different prior to ~14 Ma due to chaos. However, the pattern before and after ~48 Ma is the same (Figure 4). Before ~48 Ma, Mars’ obliquity varies continuously around a mean value with an approximate amplitude of ~15°–20°. After ~48 Ma, Mars’ obliquity shows bundling into amplitude modulation (AM) “couples” with strong nodes (reduced variation) at a period of ~2.4 Myr (arrows, Figure 4) that are absent from ~58 to 48 Ma. The timing of the nodes corresponds to the near-zero values in Mars’ inclination (see Figure 3). Around the nodes, Mars’ obliquity stays nearly constant for hundreds of thousands of years with variations < 2°.
4. Solar System Frequency Analysis

To investigate the shift in Earth’s and Mars’ orbital inclination and changes in the fundamental proper modes, or eigenmodes, of the solar system around 48 Ma in ZB18a, we performed spectral analysis of the classic variables (e.g., Nobili et al. 1989; Laskar et al. 2011; Zeebe 2017):

\[ h = e \sin(\varpi) \quad k = e \cos(\varpi) \]
\[ p = \sin(1/2) \sin \Omega \quad q = \sin(1/2) \cos \Omega \]

where \( e, I, \varpi, \) and \( \Omega \) are the eccentricity, inclination, longitude of perihelion, and longitude of ascending node, respectively. The quantities \( (h, k) = e \) and \( (p, q) = i \) may be referred to as the eccentricity and inclination vectors, respectively. For the frequency analysis, we use variables \( k \) and \( q \) (equivalent to using \( h \) and \( p \)) for Earth and Mars, and two time windows, one before and one after the transition around 48 Myr: Interval 1: [60 50] Ma and Interval 2: [46 36] Ma (see Figures 5 and 6).

Spectral analysis of \( k \) and \( q \) yield the fundamental frequencies of the solar system’s eigenmodes, \( \{ g, s \} \) modes, respectively. The \( g \) modes are loosely related to the perihelion precession of the planetary orbits, e.g., \( g_3 \) and \( g_4 \) to Earth’s and Mars’ orbits (\( s \) modes correspondingly to the nodes). The \( g \)’s and \( s \)’s are constant in quasiperiodic systems but vary over time in chaotic systems. It is critical to recall that there is no simple one-to-one relation between planet and eigenmode, particularly for the inner planets. The system’s motion is a superposition of all eigenmodes, although some modes represent the single dominant term for some (mostly outer) planets. For the current problem, analysis of changes in the frequency bands around \( g_3 \) and \( g_4 \), and \( s_3 \) and \( s_4 \) is most instructive to examine, for instance, \( (g_4 - g_3) \) and \( (s_4 - s_3) \). Changes in other important frequencies such as \( (g_2 - g_3) \) were found to be small on this timescale and across the transition around 48 Ma, consistent with earlier work (e.g., Laskar et al. 2011; Zeebe 2017; Spalding et al. 2018).

The \( g \) and \( s \) modes are key to understanding secular resonances and the resonance transition. In simple orbital configurations, secular resonances refer to the commensurability of apsidal and nodal precessional frequencies, directly involving the orbital \( \varpi \)’s and \( \Omega \)’s (for a reviews, see, e.g., Murray & Dermott 1999; Murray & Holman 2001; Morbidelli 2002). In the solar system, planetary secular resonances involve the \( g \) and \( s \) modes (see above) obtained through, e.g., \( e \) and \( I \) from numerical solutions. For example, in our orbital solution ZB18a, the ratio \( (g_4 - g_3) \) is \( \sim 1:1 \) before \( \sim 53 \) Ma (one resonance state) and \( \sim 1:2 \) after \( \sim 45 \) Ma (another resonance state; see Section 4.3 and Zeebe & Lourens 2019, 2022b). Hence during the interval from \( \sim 53 \) to \( \sim 45 \) Ma the system switches from one secular resonance state to another, aka resonance transition. A resonance transition represents an unmistakable expression of chaos and does not exist in periodic and quasiperiodic systems. For instance, if the mutual planet–planet perturbations in the solar system were sufficiently small (all eccentricities and inclinations small), then the full dynamics could be described by linear secular perturbation theory (Laplace–Lagrange solution; e.g., Murray & Dermott 1999; Morbidelli 2002; Laskar et al. 2011; Zeebe 2017). In the linear theory, the \( g \) and \( s \) modes are independent (do not interact) and resonance transitions are absent, which is hence insufficient for describing the chaotic nature of the solar system (see below; e.g., Batygin et al. 2015; Mogavero & Laskar 2022).

4.1. Changes in the \( g \) and \( s \) modes

Spectral analysis of \( k \) from ZB18a shows that the relative power of \( g_3 \) and \( g_4 \) and their frequency difference \( (g_4 - g_3) \) change significantly across the transition around 48 Ma (Figure 5). Going forward in time, \( g_4 \)’s relative power increases in both Earth’s and Mars’ \( k \) spectra, while the frequency difference \( (g_4 - g_3) \) drops by \( \sim 36\% \). The changes are most
Figure 5. Time-series analysis of $k = e \cos(\omega t)$ for Earth and Mars (see text) to extract solar system g modes from ZB18a. Int=Interval, $\mathcal{F}$=Fast-Fourier Transform (FFT). The vertical dashed lines in (c)–(h) indicate the frequencies of the g modes (see Zeebe 2017).
Figure 6. Time-series analysis of $q = \sin(\ell/2)\cos\Omega$ for Earth and Mars (see text) to extract solar system $s$ modes from ZB18a. Int = Interval, $\mathcal{F} =$ Fast-Fourier Transform (FFT). The arrows in (d) indicate the nodes (reduced amplitude; see Figure 4). The vertical dashed lines in (e)–(h) indicate the frequencies of the $s$ modes (see Zeebe 2017).
Interval 2, i.e., a resonance transition from $P_g = (q_4 - g_3)^{-1} \approx 1.5$ Myr in Interval 1 to $\sim 2.4$ Myr in Interval 2, i.e., a resonance transition (see Section 4.3).

Conversely, $s_3$’s power increases relative to $s_4$ in both Earth’s and Mars’ $q$ spectra, while the frequency difference $(s_4 - s_3)$ rises by $\sim 24\%$. The changes are again most apparent in Mars’ $q$ (Figure 6(d)), showing a decrease in the beat period from $\sim 1.5$ Myr in Interval 1 to $\sim 1.1$ Myr in Interval 2.

4.1.1. Earth

The maximum amplitude in both Earth’s $k$ and $q$ increases across the transition, although the increase is more pronounced in $q$ (Figures 5 and 6, top panels). The change in Earth’s $q$ is related to the $s$ modes and hence to inclination and was analyzed in more detail (see also Section 4.2). Most obviously, $s_3$’s power in Earth’s $q$ spectrum almost quadruples (Figures 6(e) and (f)), which should substantially increase $q$’s amplitude (everything else being equal). In addition, however, the power associated with $s_2$ and $s_4$ drops by about 70% and 25%, respectively (Figures 6(e) and (f)) and a peak of discernible power appears in Earth’s $q$ spectrum between $s_2$ and $s_3$ in Interval 2 (Figure 6(f), arrow). The peak can be identified as $s_3 - (q_4 - g_3)$, illustrating the interaction of $(q_4 - g_3)$ and $(s_4 - s_3)$: a feature almost certainly involved in the chaotic behavior of the system (e.g., Laskar 1990; Sussman & Wisdom 1992; Zeebe 2017; Mogavero & Laskar 2022). It turns out that the amplitude changes in $s_1$ through $s_4$ and the $s_3 - (q_4 - g_3)$ peak are critical to reconstruct the overall rise in Earth’s $q$ amplitude (see Section 4.2). As a result, the shift in the variation in Earth’s orbital inclination and obliquity around 48 Ma is largely due to the contribution change in the superposition of $s$ modes 1–4 and the $g$–$s$ mode interaction in the inner solar system. The $g$–$s$ mode interaction is also key to understanding the AM shift in Mars’ inclination vector (Section 4.1.2).

4.1.2. Mars

Across the transition, the AM in Mars’ $q$ intensifies, displaying bundling into AM “couples” with strong nodes (reduced amplitude) at twice the AM beat period (Figure 6(d), arrows). Remarkably, $s_3$ appears negligible for Mars’ $q$ in Interval 1. The spectral power at $s_3$’s frequency does not rise above the background level (Figure 6(g)). Instead, some power is concentrated in one of $s_4$’s side peaks at lower frequency, identified as $s_4 - (q_4 - g_3)$ (see Figure 6(g), arrow and Section 4.2). Interestingly, the combination of (difference between) $s_4$ and $s_4 - (g_4 - g_3)$ effectively leads to the same AM period as $(s_4 - s_3)$ because $(s_4 - s_3)$ and $(q_4 - g_3)$ are indistinguishable in Interval 1 within errors (see Section 4.3). Thus, the $\sim 1.5$ Myr beat in Mars’ $q$ in Interval 1 is actually due to $g$ modes, not $s$ modes, again illustrating the interaction of $(q_4 - g_3)$ and $(s_4 - s_3)$ and its likely involvement in the system’s resonances and chaos (Section 4.3).

The nodes in Mars’ $q$ at twice the AM beat period in Interval 2 (Figure 6(d), arrows) correspond to the minima near zero in inclination (see Figure 3(d)) and to the nodes in Mars’ obliquity (see arrows in Figure 4). As a result, the change in the variation of Mars’ orbital inclination and obliquity across the transition around 48 Ma can be traced back to the changes in amplitude and frequency of the $g$ and $s$ modes. That is, here largely to a stronger expression of $s_3$ in Mars’ orbit, causing a stronger AM in Mars’ inclination vector due to $(s_4 - s_3)$. The effect of changes in the $g$ and $s$ modes on Earth’s and Mars’ inclination vectors as inferred above are corroborated by a basic model of signal reconstruction using only key eigenmodes (Section 4.2).
the five modes in Interval 2 include $s_3 - (g_4 - g_3)$, i.e., the peak between $s_4$ and $s_3$, highlighting the $g$–$s$ mode interaction. For Mars’ $q$, a set of three and four modes, respectively, turned out to be essential in Intervals 1 and 2 (see Table 3 and Figure 7). The presence of $s_4 - (g_4 - g_3)$ is essential for the AM in Interval 1, while the rise in $s_3$’s power across the transition is the most important change to explain the substantial increase in AM and the strong nodes in Mars’ $q$ in Interval 2. In summary, the characteristic features of Earth’s and Mars’ inclination vectors before and after the transition around 48 Ma can be reconstructed using a few key eigenmodes (compare Figures 6a and 7). The reconstruction confirms that the shifts in the variation in Earth’s and Mars’ orbital inclination and obliquity around 48 Ma are due to contribution changes in the superposition of $s$ modes, plus the $g$–$s$ mode interaction in the inner solar system.

4.3. $(g_4 - g_3)$: $(s_4 - s_3)$ Resonance

The results of the spectral analysis for $k$ and $q$ (Figures 5 and 6) suggest ratios for $(g_4 - g_3)$: $(s_4 - s_3)$ very close to 1:1 and 1:2 in Intervals 1 and 2, respectively. However, examining whether or not these ratios represent exact resonances requires evaluation of the uncertainties in the frequency differences and hence uncertainties in the individual $g$’s and $s$’s. Based on literature estimates and several tests performed here (see Appendix B), we take $\Delta f \approx 6.5 \times 10^{-3}$ yr$^{-1}$ as an estimated uncertainty in determining the individual frequencies $g_3$, $g_4$, $s_3$, and $s_4$ from our solar system integrations. Note that the present uncertainty estimates generally do not apply to noisy geologic data. In the following, we focus on the periods of the amplitude modulation (or beats, e.g., $P_3 = (g_4 - g_3)^{-1}$), rather than frequencies, as the beats can be identified in the geologic record ($P_3 \approx P_1 \approx 1.5$ Myr in Interval 1 and $P_3 \approx 2.4$ Myr, $P_1 \approx 1.2$ Myr in Interval 2). Error propagation then yields the uncertainties ($\Delta P$) in the beat periods:

$$\Delta P_3 = \sqrt{\frac{\Delta f}{(g_4 - g_3)^2}}; \quad \Delta P_1 = \sqrt{\frac{\Delta f}{(s_4 - s_3)^2}}.$$  \hspace{1cm} (18)

The largest uncertainties are expected in Interval 2 with the smallest $(g_4 - g_3)$, which gives $\Delta P_3 \approx 40$ kyr and $\Delta P_1 \approx 10$ kyr.
Given the period ratio of $\sim 2:1$ in Interval 2, the uncertainty bound for $P_g - 2P_q$ is hence $\sim 40 + 2 \times 10 \approx 60$ kyr. Spectral analysis using FFT and the multitaper method (MTM) in Interval 1 (Earth’s $k$ and $q$) yielded $P_g - P_q \approx 0$ and 5 kyr, respectively. In Interval 2 (Earth’s $k$ and $q$), FFT and MTM yielded $P_g - 2P_q \approx 80$ and 6 kyr, respectively (30 and 6 kyr using Mars’ $k$ and $q$).

Within errors, the results of the spectral analysis for $k$ and $q$ (Figures 5 and 6) therefore indeed suggest a 1:1 and 2:1 resonance in Intervals 1 and 2, respectively. The slightly larger $P_g - 2P_q$ from FFT in Interval 2 (80 kyr, Earth’s $k$ and $q$) is unlikely to be significant. First, the result is not confirmed using FFT and Mars’ $k$ and $q$, or MTM. Second, extending Interval 2 by, say, 0.5 Myr toward the present, yields $P_g - 2P_r \approx 6$ kyr, indicating additional sensitivity to window selection and length. Third, the uncertainty in the individual frequencies ($\Delta f$) could be somewhat larger than the $5 \times 10^{-6}$ kyr$^{-1}$ assumed here (see Appendix B).

5. Summary and Discussion

Analysis of our optimal orbital solution ZB18a shows that solar system chaos caused reduced variations in Earth’s and Mars’ orbital inclination and Earth’s obliquity from $\sim 58$ to $\sim 48$ Ma. We applied time-series analyses and signal reconstruction using key eigenmodes to extract and investigate changes in the solar system frequencies. Both approaches highlight changes in the superposition of $s$ modes and the involvement and interaction of $(g_3 - g_2)$ and $(s_3 - s_2)$ in explaining changes in inclination and obliquity around 48 Ma. The $g$–$s$ mode interactions in the inner solar system (e.g., Laskar 1990; Sussman & Wisdom 1992; Zeebe 2017; Mogavero & Laskar 2022) and the resonance transition (Zeebe & Lourens 2019) represent unmistakable expressions of chaos in the solar system. Dynamical chaos hence not only affects the solar system’s orbital properties but also the long-term evolution of planetary climate through eccentricity and the link between inclination and axial tilt.

For Earth’s climate even small changes in obliquity are relevant because obliquity controls the seasonal contrast through changes in insolation—particularly important in high latitudes. For instance, over the past few million years, obliquity was a major forcing factor and pacemaker for the ice ages (e.g., Hays et al. 1976; Paillard 2021). Hence reduced variations in Earth’s obliquity from $\sim 58$ to $\sim 48$ Ma should also have affected Earth’s climate across this time interval (aka the late Paleocene—early Eocene; LPEE). Remarkably, a nearly ubiquitous phenomenon in long-term geologic records across the LPEE is a very weak or absent obliquity signal (e.g., Lourens et al. 2005; Westerhold et al. 2007; Littler et al. 2014; Zeebe et al. 2017; Barnet et al. 2019). We do not rule out that other factors such as the greenhouse climate at the time, the absence of large ice sheets, etc. may have contributed to a weak expression of obliquity (high-latitude) forcing as well. However, strong obliquity signals have been identified during other greenhouse episodes such as the mid-Cretaceous climate optimum in mid-latitude/equatorial sites (e.g., Meyers 2012), indicating that more than just high temperatures were necessary to suppress the obliquity signal in LPEE records. Notably, based on the expression of orbital cycles in the sedimentary record, Wahlenkamp et al. (2020) tuned their astronomical age model solely to eccentricity cycles during the early Eocene ($\sim 56$ to $\sim 47$ Ma) but to a mix of eccentricity and obliquity cycles during the middle Eocene ($\sim 48$ to $\sim 40$ Ma), indicating the onset of a stronger obliquity component around 48 Ma. We propose here that the reduced amplitude in Earth’s obliquity, as predicted by our astronomical solution ZB18a, contributed to the weak/absent obliquity signal in geologic records from $\sim 58$ to $\sim 48$ Ma.

As for Earth, astronomical theories of climate (Milankovitch 1941) also have a long history for Mars (e.g., Pollack 1979; Toon et al. 1980). Of particular interest here is the primary control of obliquity on the exchange of CO$_2$ and H$_2$O between Mars’ surface reservoirs and polar caps on timescales of $10^6 - 10^7$ yr (e.g., Armstrong et al. 2004; Levrard et al. 2007; Vos et al. 2019; Buhler & Piqueux 2021). Modeling suggests that CO$_2$ fluxes may be assumed in equilibrium on obliquity timescales (Buhler & Piqueux 2021). Hence the instantaneous, absolute value of obliquity would be the critical control variable for, e.g., the mass of CO$_2$ stored in Mars’ atmosphere, polar cap, and regolith; memory effects would be small. On the contrary, memory effects are significant for, e.g., water ice stored in tropical/mid-latitude surface reservoirs and Mars’ polar layered deposits. For example, estimates for the buildup time of the north-polar layered deposits to its current size are of the order of 4 Myr, with accelerated growth during intervals of small and relatively constant obliquity (Levrard et al. 2007; Vos et al. 2019). Hence for the mass of H$_2$O stored in Mars’ surface reservoirs, both the absolute value of obliquity, as well as its temporal evolution (pattern) are the critical control variables.

Our optimal astronomical solution suggests significant changes in Mars’ orbital inclination and obliquity pattern around 48 Ma (see Figures 3 and 4). For example, intervals of relatively constant obliquity (or nodes; see arrows, Figure 4) are absent from $\sim 58$ to $\sim 48$ Ma. Thus, rapid growth periods of polar layered deposits at low obliquity would not exist during this time period, which should be recorded in Mars’ climate archives (although not in the current north-polar layered deposits if the maximum age is $\sim 4$ Ma). Note that while the detailed long-term evolution of Mars’ obliquity is unknown prior to $\sim 10$–15 Ma, low-obliquity states are likely throughout its history (e.g., Armstrong et al. 2004; Laskar et al. 2004; Fassett et al. 2014; Holo et al. 2018; Bills & Keane 2019; Jakosky 2021). If suitable long-term climate/obliquity records exist on Mars, they should also show the onset of bundling into amplitude modulation “couples” with strong nodes (reduced obliquity variation) around 48 Ma with a period of $\sim 2.4$ Myr; and their absence from $\sim 58$ to $\sim 48$ Ma (Figure 4). Interestingly, Smith et al. (2020) recently laid out a road map for unlocking the climate record stored in Mars’ north-polar layered deposits, including a final mission to analyze $\sim 500$ m of vertical section. The section would allow accessing $\sim 1$ Myr of Martian climate history (if feasible, probably at most $\sim 4$ Myr for longer sections). Thus, unlocking Mars’ climate history on 10–100 Myr timescales to reveal the workings of chaos in the solar system would require different strategies.

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Software: NHBody (Rauch & Hamilton 2002).
Appendix A
Geodetic Precession

For Earth, we also include the relatively small contribution from geodetic precession (GP) to the total precession $\phi$, say, $\dot{\phi}_p = \gamma gp$, which represents a differential equation in terms of $\phi$. However, we integrate here a differential equation for $s$ (see Equation (1)), GP acts along the direction of $(s \times n)$; hence we use an ansatz in the form of Equation (1) to include GP:

$$\dot{s} = \gamma gp \cdot (n \cdot s) (s \times n), \quad (A1)$$

where the factor $\gamma gp$ may be determined as follows. Let $s_t$ and $s_s$ be the $s$ components in the orbit plane at $t_0$ (see Figure 1, omit asterisks). At $t_0$, $s_x = 0$, and $s_y = \sin \epsilon_0$. Then:

$$ds = -s_y d\phi_0, \quad \text{or} \quad \dot{s}_t = -s_y \dot{\phi}_0, \quad (A2)$$

Inserting Equation (6) into Equation (A2) and using $\alpha = K(\kappa + \beta)$ yields:

$$\dot{s}_t = \sin \epsilon_0 (\alpha \cos \epsilon_0 + \gamma gp). \quad (A3)$$

Evaluating Equation (A1) at $t_0$ gives $\dot{s} = \gamma gp \alpha \cos \epsilon_0 [\sin \epsilon_0, 0, 0]$ because $(n \cdot s)_0 = \cos \epsilon_0$ and $(s \times n)_0 = [\sin \epsilon_0, 0, 0]$. Thus,

$$\dot{s}_t = \gamma_{gp} \alpha \cos \epsilon_0 \sin \epsilon_0. \quad (A4)$$

By equating Equations (A3) and (A4), $\gamma gp$ can be calculated:

$$\gamma_{gp} = \frac{\alpha \cos \epsilon_0 + \gamma gp}{\alpha \cos \epsilon_0} = 0.999619. \quad (A5)$$

Appendix B
Frequency Uncertainties

The Rayleigh resolution $f_R = (N \Delta t)^{-1}$ may be considered an upper error bound for frequency uncertainties from spectral analyses but often greatly overestimates the error (e.g., Montgomery & O’Donoghue 1999; Kallinger et al. 2008; Zeebe et al. 2017). Several estimates for minimum errors are available in the literature (see below) that were usually derived for the frequency extraction of a single sinusoid from a noisy data set. The time series of our solar system integrations do not contain actual noise but produce a certain level of local background power in the spectra (see, e.g., Figure 5). Below, we treat the ratio of the local signal-to-background power equivalent to the signal-to-noise ratio. In the following, $T = N \Delta t$ is the total time interval, and $f = \omega/2 \pi$ and $A$ are the sinusoid frequency and amplitude, respectively; $\sigma^2$ and $\sigma$ are the variance and standard deviation, and index “W” indicates noise (uniform white noise is used below).

Rife & Boorstyn (1974) derived Cramér–Rao lower estimation error bounds (we use $N^2 - 1 \simeq N^2$):

$$\sigma^2(\omega) \geq \frac{12 \sigma^2_W}{A^2 N^2 \Delta t^2 \cdot N}; \quad \sigma(f) \geq \sqrt{\frac{6}{2N} \frac{1}{\pi T} \frac{1}{A}}, \quad (B1)$$

Similarly, Thomson (2009) gave:

$$\sigma(f) \geq \frac{6}{\rho} \frac{1}{2 \pi T} \geq \frac{6}{T} \frac{1}{\pi T} \frac{\sqrt{S_c}}{A}, \quad (B2)$$

where $\rho = (A^2/4) T/S_c$ is the signal-to-noise ratio ($S/N$) for a sinusoid, and $S_c$ is the noise spectrum. Based on a least-squares fit, Montgomery & O’Donoghue (1999) analytically derived:

$$\sigma(f) \geq \frac{6}{\nu} \frac{1}{\pi T} \frac{1}{A}. \quad (B3)$$

For example, the spectral analysis (here FFT) of $q = \sin(1/2) \cos \Omega$ for Earth in Interval 1 (see Figure 5) gives $\rho \simeq 66$ for $s_q$, corresponding to $A \simeq 0.1$ at $\sigma_w = 1$. For $T = 10$ Myr and $N = 25,001$, Equations (B1)–(B3) then yield $\sigma(f) \geq 3.4, 4.8, \text{and } 4.8 \times 10^{-6} \text{ kyr}^{-1}$, representing minimum uncertainty estimates. We also applied the multitaper method (MTM) using F-test values to obtain S/N estimates (Thomson 2009). However, the results were highly variable and depend on the selected time–bandwidth product and zero padding.

Next, we evaluated the applicability of the above minimum uncertainty estimates to the current problem using several tests. First, we ran 10,000 Monte Carlo simulations with a single sinusoid of known frequency $f$ plus random noise using the parameters above and extracted an estimated frequency $f$ for each run using FFT. Taking the error as $[f - f]$, the 10,000 simulations resulted in $\sigma(f) = 4.8 \times 10^{-6} \text{ kyr}^{-1}$, consistent with Equation (B3). Note that resolving such uncertainties requires sufficient zero padding. Including zero padding, the frequency spacing is $\Delta f = 2 f_0 / N$, where $f_0 = 1/(2 \Delta \omega)$ is the Nyquist (highest detectable) frequency, and $N$ is the total number of FFT points. For example, resolving $\Delta f = 5 \times 10^{-6} \text{ kyr}^{-1}$ requires $N \geq (1/\Delta f)/5 \times 10^{-6}$, i.e., here $N \geq 5 \times 10^5$, or $N \geq 21^5$, or $N \geq N \times 20$. Second, we ran 10,000 Monte Carlo simulations with two sinusoids plus random noise; the sinusoid periods were separated by only 2 kyr, similar to the smallest difference in fundamental modes ($g_4 - g_3$) in Interval 2 (see Section 4.1). The larger error for the two frequencies yielded $\sigma(f) = 4.5 \times 10^{-6} \text{ kyr}^{-1}$. Third, we generated artificial time series (see Equation (17)) using the $g$ and $s$ frequencies and amplitudes obtained from spectral analysis of the ZB18a solution for Earth (see Figures 5 and 6). For the moment, consider these frequencies as “known” $g$ and $s$. Next, we extracted estimated $g$ and $s$ frequencies from the time series using FFT. The largest error was found for $|g_3 - g_4|$ in Interval 2, i.e., $\Delta f = 2.7 \times 10^{-6} \text{ kyr}^{-1}$. Thus, our tests yielded frequency uncertainties similar to the minimum uncertainty estimates (Equations (B1)–(B3)). In summary, based on the analysis above, we take $\Delta f \simeq 5 \times 10^{-6} \text{ kyr}^{-1} = 6.5 \times 10^{-3} \mu \text{yr}^{-1}$ as an estimated uncertainty in determining the individual frequencies $g_3, g_4, s_3$, and $s_4$ from our solar system integrations. The present uncertainty estimates generally do not apply to noisy geologic data.

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