External Sources in Field–Antifield Formalism

Igor A. Batalin\textsuperscript{a} and Klaus Bering\textsuperscript{b}

\textsuperscript{a}I.E. Tamm Theory Division
P.N. Lebedev Physics Institute
Russian Academy of Sciences
53 Leninsky Prospect
Moscow 119991
Russia

\textsuperscript{b}Institute for Theoretical Physics & Astrophysics
Masaryk University
Kotlářská 2
CZ–611 37 Brno
Czech Republic

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Abstract

We introduce external sources $J_A$ directly into the quantum master action $W$ of the field–antifield formalism instead of the effective action. The external sources $J_A$ lead to a set of BRST-invariant functions $W^A$ that are in antisymplectic involution. As a byproduct, we encounter quasi-groups with open gauge algebras.

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\textsuperscript{a}E–mail: batalin@lpi.ru \hspace{1cm} \textsuperscript{b}E–mail: bering@physics.muni.cz
1 Introduction

Historically, several authors have worked on a formalism with external sources, e.g., in Yang-Mills theories [1], or e.g., for the effective action $\Gamma$ [2, 3, 4, 5]. Here we shall not consider the effective action, but rather work directly in terms of the quantum master action $W^0$.

At first sight, it seems tempting to try to introduce external sources $J_\alpha$ in the field–antifield formalism in a naive manner by simply modifying the standard quantum master action

$$ W^0 \rightarrow W^0 + J_\alpha \Phi^\alpha, $$

(1.1)

where $\Phi^\alpha$ denote the fundamental field variables, $\alpha \in \{1, 2, \ldots, N\}$. However, consistency (as we shall see in next Section) requires $\Phi^\alpha$ to be BRST–invariant. This is in general not the case, so a more sophisticated approach is clearly needed.

2 BRST–Invariant $W^A$ Functions

In this paper, we suggest to use BRST–invariant functions $W^A= W^A(\Gamma; \hbar)$ to multiply the external sources $J^A$:

$$ W^J = W^0 + J^A W^A. $$

(2.1)

The actions $W^J$ and $W^0$ denote the quantum master action with and without external sources $J^A$, respectively. The index $A \in \{1, 2, \ldots, 2N\}$ runs over twice as many values as the index $\alpha \in \{1, 2, \ldots, N\}$ to reflect the full antisymplectic phase space $\Gamma^A = \{\Phi^\alpha; \Phi^*_\alpha\}$. The Grassmann parity and ghost number are

$$ \varepsilon_A : = \varepsilon(\Gamma^A) = \varepsilon(W^A) = \varepsilon(J^A), $$

(2.2)

$$ \text{gh}_A : = \text{gh}(\Gamma^A) = \text{gh}(W^A) = -\text{gh}(J^A), $$

(2.3)

$$ \varepsilon(W^J) = 0, \quad \text{gh}(W^J) = 0. $$

(2.4)

REMARK: The action $W^J$ could more generally be a power series expansion in the sources $J^A$, but we shall for simplicity assume in this paper that $W^J$ only depends affinely on the $J^A$ sources, as indicated in eq. (2.1). (An affine function is a function with first–order terms and zero-order terms.)

GENERAL REMARKS ABOUT NOTATION: The superscript “0” on a quantity means the source–free limit $J=0$ of that quantity. For example, $W^0 = W^J|_{J=0}$.

3 $\Delta$ operator and Antibracket $(\cdot, \cdot)$

To set up the field-antifield formalism [6, 7, 8] one needs the $\Delta$ operator

$$ \Delta = \Delta_\rho + \nu_\rho, \quad \Delta^2 = 0, \quad \varepsilon(\Delta) = 1, $$

(3.1)

where

$$ \Delta_\rho : = \frac{(-1)^{a^A}}{2\rho} \frac{\partial}{\partial \Gamma^A} \rho E^{AB} \frac{\partial}{\partial \Gamma^B} $$

(3.2)

is the odd Laplacian, and $\nu_\rho$ is an Grassmann–odd scalar function, see Refs. [9, 10, 11, 12, 13] for details.
The antibracket is given as
\[
(f, g) := (-1)^{\varepsilon_f} [[\Delta, f], g]_1 = (f \frac{\partial}{\partial \Gamma^A}) E^{AB} (\frac{\partial}{\partial \Gamma^B} g) = -(-1)^{[\varepsilon_f+1][\varepsilon_g+1]} (f \leftrightarrow g) .
\]

4 Quantum Master Equation

The quantum master equation with external sources
\[
\Delta e^{\frac{i}{\hbar} W^J} = 0 \iff \frac{1}{2} (W^J, W^J) = i \hbar \Delta \rho W^J + \hbar^2 \nu_\rho
\]
is equivalent to the following \(J\)-independent conditions (4.2)–(4.4).

1. The standard quantum master equation:
\[
\Delta e^{\frac{i}{\hbar} W^0} = 0 \iff \frac{1}{2} (W^0, W^0) = i \hbar \Delta \rho W^0 + \hbar^2 \nu_\rho
\]

2. The functions \(W^A = W^A(\Gamma; \hbar)\) are BRST–invariant,
\[
\sigma_{W^0}(W^A) = 0 ,
\]
where \(\sigma_{W^0} := (W^0, \cdot) + \frac{\hbar}{i} \Delta \rho\) is the quantum BRST operator.

3. The functions \(W^A = W^A(\Gamma; \hbar)\) are\(^*\) mutually in involution with respect to the antibracket \((\cdot, \cdot)\),
\[
(W^A, W^B) = 0 .
\]

The third condition (4.4) shows that the \(2N\) function \(W^A\) can carry at most \(N\) independent functions, so in other words the set \(W^A\) will always be redundant. The redundant description is sometimes necessary for relativistic quantum field theories to preserve symmetry, such as, \(e.g.,\) Lorentz symmetry, and locality.

The second condition (4.3) immediately illustrates that one cannot pick \(W^\alpha = \Phi^\alpha\) and \(W^*_\alpha = 0\), cf eq. (1.1). This does not work because the fundamental field variables \(\Phi^\alpha\) are in general not BRST invariant.

5 Classical Master Equation

Let us next consider the classical limit
\[
W^J = S^J + \mathcal{O}(h) , \quad W^0 = S^0 + \mathcal{O}(h) , \quad W^A = S^A + \mathcal{O}(h) ,
\]
where
\[
S^J = S^0 + J_A S^A .
\]
The classical master equation with external sources
\[
(S^J, S^J) = 0
\]
is equivalent to the following \(J\)-independent conditions
\[
(S^0, S^0) = 0 , \quad (S^0, S^A) = 0 , \quad (S^A, S^B) = 0 ,
\]
which, in turn, are the classical limit of the the conditions (4.2)–(4.4), respectively.

\(^*\)To see eq. (4.4), differentiate the quantum master eq. (4.1) twice with respect to the external sources \(J_A\) and \(J_B\) to get \((W^A, W^B) - (-1)^{[\varepsilon_A+1][\varepsilon_B+1]} (A \leftrightarrow B) = 0\). Recalling the symmetry (3.3) of the antibracket then leads to eq. (4.4).
Table 1: Multiplicity, Grassmann parity and ghost number of the fundamental variables $\Gamma^A$ and the BRST-invariant $W^A$ functions for irreducible theories in the minimal sector.

| Variables | $\Gamma^A$ | Fields $\Phi^i$ | Antifields $\Phi^*_a$ |
|-----------|------------|-----------------|---------------------|
| Multiplicity | $\text{rank}(\Gamma^A) = 2N$ | $n$ | $m$ |
| Grassmann Parity | $\varepsilon_A$ | $\varepsilon_i$ | $\varepsilon^+_a + 1$ |
| Ghost Number | $\text{gh}_A$ | $0$ | $1$ | $-1$ | $-2$ |
| Rank of $W^A = S^A + O(h)$ | $N = n + m$ | $n - m$ | $m$ | $0$ |
| Classical BRST–Invariants | $S^A$ | $S^i$ | $S^a$ | $S^*_i$ | $S^*_a = 0$ |
| Quantum BRST–Invariants | $W^A$ | $W^i$ | $W^a$ | $W^*_i$ | $W^*_a = 0$ |
| | | | | | $W^* = S^* + O(h)$ | $\overline{W^*_a} = S^*_a + O(h)$ |

6 Existence of $W^J$

Existence of the source–free classical master action $S^0$ for reducible theories was proven in Ref. [14] and further elaborated in Ref. [15]. The presence of external sources $J_A$ does not change the proof in other respect that pertinent quantities now depend on the external sources $J_A$. A sufficient condition for the existence of the quantum master action $W^J$ is that the cohomology of the classical BRST operator $s^J$ vanishes in the sector with ghost number equal to 1.

7 Irreducible Theories

We shall only consider the irreducible case from now on. According to Theorem 3.4 of Ref. [15], to prove the existence of the external source formalism at the classical level, it remains to prove the existence of a $J$-dependent acyclic, nilpotent Koszul–Tate operator $s^J_{-1}$. Here the nilpotency of $s^J_{-1}$ is just the $J$-dependent Noether identities

$$
\left( S^J_o \left( \frac{\partial}{\partial \varphi^a} \right) \right) R^J_{i a} = 0 ,
$$

(7.1)

where $S^J_o = S^J_o(\varphi)$ is the $J$-dependent action in the original field sector, and $R^J_{i a} = R^J_{i a}(\varphi)$ are the $J$-dependent gauge–generators.

Thus we imagine that we are given a source–free theory that satisfies the Noether identity (7.1) for $J_A = 0$, and we are seeking solutions to these identities for non–vanishing external sources $J_A \neq 0$.

8 Irreducible and Closed Theories

In the irreducible and closed case, the proper solution can be taken on the form in the minimal sector

$$
S^J = S^J_o + \varphi^*_i R^J_{i a} c^a + \frac{1}{2} e^c c^c U^{Jc} U^{a} e^b (-1)^{i_a} ,
$$

(8.1)
with \( J \)-dependent structure functions \( U^{Jc}_{ab} = U^{Jc}_{ab}(\varphi) \). Besides the Noether identity (7.1), the classical master equation (5.3) contains the gauge algebra relation

\[
(R^{ji}_{a} \frac{\partial}{\partial \varphi^j}) R^{kj}_{b} - (-1)^{\varepsilon_a \varepsilon_b}(a \leftrightarrow b) = R^{ji}_{c} U^{Jc}_{ab} .
\]

and a six–term Jacobi identity

\[
\sum_{\text{cycl. } a,b,c} (-1)^{\varepsilon_a \varepsilon_c} \left( U^{Jd}_{ae} U^{Je}_{bc} - (U^{Jd}_{ab} \frac{\partial}{\partial \varphi^i}) R^{ji}_{c} \right) = 0 .
\]

\section{Groupoid/Quasi–group}

The above set of eqs. (7.1), (8.2) and (8.3) has an interpretation in terms of a (closed) groupoid/quasi–group \cite{16}. The fields \( \varphi^i \) are coordinates on the quasi–group. We shall use the quasi–group construction to deduce BRST–invariants \( S^i \) associated with the (transversal) original fields \( \varphi^i \), cf. Section 10. (Differences in notation as compared with Ref. \cite{6} and Ref. \cite{16} are for most parts obvious, except for the subtle fact that the structure functions \( U^{Jc}_{ab} = -t^{Jc}_{ab} \) have precisely the opposite sign there.) In general, the quasi–group construction could in principle also works with external sources \( J_A \), as we will indicate in this Section 9. However for the applications that we will present in this paper in the next couple of Sections 10–13, the external sources \( J_A \) will actually not enter into the quasi–group construction itself, but only have an organizing rôle (in the sense of splitting the master equation in various sections).

Recall that the main idea of the quasi–group is to generalize Sophus Lie’s original work for transformation groups, such that the composition law \( \Theta^J(\theta, \theta'; \varphi) \) for transformations (and hence the structure “constants” \( U^{Jc}_{ab} \)) depend on the point \( \varphi \). The transformations (=arrows) are the (finite) gauge transformations

\[
\varphi^i \rightarrow \varphi^i_{Ji} = f^{ji}(\varphi, \theta) ,
\]

where \( \theta^a \) are the gauge parameters. The composition law reads

\[
f^{ji}(f^j(\varphi, \theta), \theta') = f^{ji}(\varphi, \Theta^J(\theta, \theta'; \varphi)) .
\]

The modified law of associativity reads

\[
\Theta^J a(\Theta^J(\theta, \theta'; \varphi), \theta''; \varphi) = \Theta^J a(\theta, \Theta^J(\theta', \theta''; f^J(\varphi, \theta)); \varphi) .
\]

The gauge transformation (9.1) is assumed to have an inverse gauge transformation

\[
\varphi^i \rightarrow \varphi^{ji} = ((f^J)^{-1})^i(\varphi, \theta) = f^{ji}(\varphi, \theta(\theta; \varphi)) .
\]
Define

\[
R^{Ji}_{a}(\varphi) := \left( f^{Ji}(\varphi, \theta) \frac{\partial}{\partial \theta^a} \right)_{\varphi=0},
\]

\[
U^{Jc}_{ab}(\varphi) := \left( \Theta^{Jc}(\theta, \theta'; \varphi) \frac{\partial}{\partial \theta'^b} \right)_{\varphi=0} - (-1)^{\varepsilon_a \varepsilon_b} (a \leftrightarrow b),
\]

\[
\mu^J_{ab}(\theta, \varphi) := \left( \Theta^J(\theta, \theta'; \varphi) \frac{\partial}{\partial \theta'^a} \right)_{\varphi, \theta'=0}, \quad \lambda^J := (\mu^J)^{-1},
\]

\[
\tilde{\mu}^J_{ab}(\theta, \varphi) := \left( \Theta^J(\theta', \theta; \varphi) \frac{\partial}{\partial \theta'^b} \right)_{\varphi, \theta'=0}, \quad \tilde{\lambda}^J := (\tilde{\mu}^J)^{-1},
\]

\[
\Sigma^{Ji}_{\lambda}(\varphi, \theta) := \left( f^{Ji}(\varphi, \theta) \frac{\partial}{\partial \varphi} \right)_{\theta'=0},
\]

\[
E^J(\varphi, \theta) := \text{sdet}(\Sigma^J(\varphi, \theta)) \frac{\text{sdet}(\mu^J(\theta, \varphi))}{\text{sdet}(\tilde{\mu}^J(\theta, \varphi))}.
\]

It is assumed that the matrices (9.7), (9.8) and (9.9) are invertible. The Lie equation follows from

\[
(\varphi^i \frac{\partial}{\partial \varphi^a} \varphi^J)_{\varphi=0} = R^{Ji}_{a}(\varphi^J) \lambda^J_{ab}(\theta, \varphi),
\]

\[
\text{(9.12)}
\]

\[
(\varphi^i \frac{\partial}{\partial \varphi^a} \varphi^J)_{\varphi=0} = (f^{Ji}(\varphi, \theta') \frac{\partial}{\partial \theta^a})_{\varphi=0} = (f^{Ji}(\varphi, \Theta^J(\theta, \theta'; \varphi)) \frac{\partial}{\partial \theta'^a})_{\varphi=0} = \text{(9.1)+}(9.2)
\]

\[
\text{(9.7)}
\]

The inverse Lie equation can be deduced as follows

\[
-(\varphi^i \frac{\partial}{\partial \varphi^a} \varphi^J)_{\varphi=0} = (\varphi^i \frac{\partial}{\partial \varphi^J})_{\theta} \left( \varphi^J \varphi^i \frac{\partial}{\partial \varphi^J} \varphi \right)_{\varphi=0} = \left( (\Sigma^J)^{-1} \right)^i_{\lambda}(\varphi, \theta) \left( \varphi^J \varphi^i \frac{\partial}{\partial \varphi^J} \varphi \right)_{\varphi=0} = \text{(9.9)}
\]

\[
\text{(9.14)}
\]

In the last equality of eq. (9.13) we used that

\[
\Sigma^{Ji}_{\lambda}(\varphi, \theta) R^{Jj}_{ab}(\varphi) \quad \text{(9.5)+}(9.9) \quad \left( \varphi^i \frac{\partial}{\partial \varphi^J} \right)_{\theta} \left( f^{Ji}(\varphi, \theta') \frac{\partial}{\partial \theta^a} \right)_{\varphi=0} = f^{Ji}(f^J(\varphi, \theta'), \theta) \frac{\partial}{\partial \theta^a} \left( \varphi^i \frac{\partial}{\partial \varphi^J} \right)_{\varphi=0} = \text{(9.1)+}(9.2)
\]

\[
\text{(9.2)}
\]

\[
(\varphi^i \frac{\partial}{\partial \varphi^a} \varphi^J)_{\varphi=0} = \left( \varphi^i \frac{\partial}{\partial \varphi^J} \right)_{\theta} \left( \varphi^J \varphi^i \frac{\partial}{\partial \varphi^J} \varphi \right)_{\varphi=0} = \text{(9.1)+}(9.8)
\]

\[
\text{(9.14)}
\]
Using similar arguments and, in particular, associativity (9.3), it is possible to deduce the Maurer–Cartan equation and the inverse Maurer–Cartan equation

\[
\begin{align*}
(\lambda^a_{j a} \frac{\partial}{\partial \theta^c}) &= -(1)^{\epsilon_{b c}}(b \leftrightarrow c) = U^{Ja}_{de}(\varphi^I) \lambda^{fe}_{b} \lambda^{je}_{c}(-1)^{\epsilon_{b d}} , \\
(\bar{\lambda}^a_{j a} \frac{\partial}{\partial \theta^c}) &= -(1)^{\epsilon_{b c}}(b \leftrightarrow c) = -U^{Ja}_{de}(\varphi) \bar{\lambda}^{fe}_{b} \bar{\lambda}^{je}_{c}(-1)^{\epsilon_{b d}} .
\end{align*}
\]  

(9.15)  

(9.16)

It will become important when discussing quantum corrections in Section 11 that the $E^I$-function (9.10) satisfies an initial value problem [16]

\[
\begin{align*}
(ln E^I(\varphi, \theta) \frac{\partial}{\partial \theta^b}) \varphi &= A^I_a(f^I(\varphi, \theta)) \lambda^a_{j a}(\theta, \varphi) , \quad E^I(\varphi, \theta = 0) = 1 ,
\end{align*}
\]  

(9.17)

which in turn satisfies pertinent consistency relations. Here we have defined the formal anomaly function

\[
A^I_a := (-1)^{\epsilon_i} \frac{\partial}{\partial \varphi^i} R^j_{i a} + (-1)^{\epsilon_k} U^{j a}_{ba} .
\]

Locally, eq. (9.17) leads to an integral representation

\[
\ln E^I(\varphi, \theta) = \int_0^\theta A^I_a(f^I(\theta', \varphi)) \lambda^a_{j a}(\theta', \varphi) d\theta^b ,
\]

(9.19)

where the integral (9.19) is independent of the integration contour.

## 10 Construction of BRST–invariants $S^i$

Let the original action $S^0$ be invariant under gauge transformations (9.1). We will for simplicity restrict our search to solutions $R^{j i}_{a} = R^{0 i}_{a}$ and $U^{j c}_{a b} = U^{0 c}_{a b}$ that are independent of the external sources $J_A$, so that the external sources only enter through the action

\[
S^I = S^0 + J_A S^A = S^0 + J_i S^i , \quad S^0 = S^0_0 + \varphi_i^a R^{0 i}_{a} c^a + \frac{1}{2} c^a c^b U^{0 c}_{a b} c^a(1)^{-\epsilon_a} ,
\]

(10.1)

in the original field sector, i.e., via $J_i$. Here we will focus on constructing the BRST–invariants $S^i$ associated with the original fields $\varphi^i$ (or more precisely the transversal parts thereof). The idea is to gauge–fix the $m$ quasi–group gauge–parameters $\theta^a$ to be a function $\theta^a = \theta^a(\varphi)$ of $\varphi$ in precisely such a way that

\[
S^i := \varphi^0_i(\varphi, \theta(\varphi))
\]

(10.2)

become $n$ gauge–invariants, of which $n – m$ are independent. Total differentiation with respect to $\varphi^j$ yields

\[
\begin{align*}
\frac{\partial}{\partial \varphi^j} (10.2) &= \frac{\partial}{\partial \varphi^0_i} \theta(\varphi^0_i) \frac{\partial}{\partial \theta^b} (\varphi^0_i \frac{\partial}{\partial \theta^b}) \frac{\partial}{\partial \varphi^j} (9.9) = \Sigma^0_{j i} + (\varphi^0_i \frac{\partial}{\partial \theta^b}) \varphi(\theta^b \frac{\partial}{\partial \varphi^j}) .
\end{align*}
\]

(10.3)

Let $\chi^a = \chi^a(S^0)$ be the $m$ independent gauge-fixing conditions, in the sense that we impose $\chi^a = 0$ for all possible values of $\varphi$. This determines implicitly $m$ functions $\theta^a = \theta^a(\varphi)$ if we assume that the matrix

\[
D^a_{b} := (\chi^a \frac{\partial}{\partial S^i}) (\varphi^0_i \frac{\partial}{\partial \theta^b}) \varphi
\]

(10.4)
is invertible. (Note that unlike ordinary gauge–fixing, the BRST–invariants \( S^i \) will depend on gauge-fixing conditions \( \chi^a = 0 \) by construction.) Then
\[
0 = \left( \chi^a \frac{\partial}{\partial \varphi^j} \right) = \left( \chi^a \frac{\partial}{\partial S^i} \right) \left( S^i \frac{\partial}{\partial \varphi^j} \right) \quad (10.3) + (10.4) = \left( \chi^a \frac{\partial}{\partial S^i} \right) \sum_i^{0j} + D_a^b \left( \theta^b \frac{\partial}{\partial \varphi^j} \right).
\]
Now we can use eq. (10.5) to rewrite eq. (10.3) as
\[
\left( \frac{\partial}{\partial \varphi^k} \right) = P^i_j \Sigma^{0j}_k,
\]
where we have defined the idempotent
\[
Q^i_j := \left( \varphi^i \frac{\partial}{\partial \theta^a} \right) \varphi \left( D^{-1} \right)^a_b \left( \chi^b \frac{\partial}{\partial S^j} \right), \quad Q = Q^2,
\]
and its complementary idempotent
\[
P := 1 - Q = P^2, \quad PQ = 0 =QP.
\]
This in turn implies
\[
P^i_j \left( \varphi^i \frac{\partial}{\partial \theta^a} \right) \varphi = 0,
\]
and
\[
\left( \chi^a \frac{\partial}{\partial S^i} \right) P^i_j = 0.
\]
It follows that \( S^i \) is gauge–invariant,
\[
\left( \frac{\partial}{\partial \varphi^k} \right) R^{0k}_a(\varphi) = P^i_j \Sigma^{0j}_k R^{0k}_a(\varphi) \quad (10.10) = P^i_j \left( \varphi^i \frac{\partial}{\partial \theta^b} \right) \varphi \mu^{0b}_a.
\]
All together, we have solved the \( J \)-dependent Noether identities (7.1) in the original field sector with the help of the inverse Lie eq. (9.14). It is easy to check that the other conditions in the \( J \)-dependent classical master eq. (5.4) are satisfied as well.

11 Quantum Corrections

In this Section 11 we look for a solution to the quantum master eq. (4.1) with a truncated one-loop Ansatz of the form
\[
W^J = S^J + \hbar M^J.
\]
Besides the classical master eq. (5.3), the quantum master eq. (4.1) becomes
\[
(M^J, S^J) + \Delta_{\rho} S^J = 0, \quad (11.2)
\]
\[
\frac{1}{2}(M^J, M^J) + \Delta_{\rho} M^J + \nu_{\rho} = 0. \quad (11.3)
\]
We now assume for simplicity Darboux coordinates \( \Gamma^A = \{ \Phi^\alpha, \Phi^{*}_\alpha \} \) with trivial density \( \rho = 1 \) and trivial odd scalar \( \nu_{\rho} = 0 \). We furthermore assume that the one–loop contribution
\[
M^J = M^0(\varphi)
\]

8
only depends on the original fields \( \varphi^i \), and in particular, that the one–loop contribution is independent of all the external sources \( J_A \) and all the antifields \( \Phi^*_a \). Then eq. (11.3) is automatically satisfied. The eq. (11.2) reads in the sector proportional to \( c^a \)

\[
(M^0(\varphi) \frac{\partial}{\partial \varphi^i}) R^0{a_5}{a_0} (\varphi) + A^0_a(\varphi) = 0 ,
\]

(11.5)

where the formal anomaly function \( A^0_a \) is defined in eq. (9.18). Therefore

\[
(M^0(\varphi)) \frac{\partial}{\partial \varphi^i}) \varphi = (M^0(\varphi)) \frac{\partial}{\partial \varphi^i}) (\varphi^0) = (M^0(\varphi)) \frac{\partial}{\partial \varphi^i}) R^0{a_5}{a_0} (\varphi) \lambda^0_b (\theta, \varphi)
\]

(11.11)

\[
\Rightarrow - A^0_a(\varphi) \lambda^0_{a}{b} (\theta, \varphi) .
\]

(11.6)

Comparing with the differential eq. (9.17), we conclude that a solution to the differential eq. (11.6) is

\[
M^0(\varphi) = M^0(\varphi) - \ln E^0(\varphi, \theta) .
\]

(11.7)

The partition function reads

\[
Z^J[J] = \int [d\Phi] \exp \left[ M^0(\varphi) + \frac{i}{\hbar} S^J(\Phi, \Phi^* = \frac{\partial \Psi}{\partial \Phi}) \right] ,
\]

(11.8)

where it is implicitly understood in eq. (11.8) that the field multiplet

\[
\Phi^a = \{ \varphi^i; c^a; \tau^a; \pi^a \}
\]

(11.9)

now includes non-minimal fields for gauge-fixing purposes; namely a Faddeev–Popov antighost \( \tau^a \) and a Nakanishi–Lautrup Lagrange multiplier \( \pi^a \); and it is furthermore implicitly understood that the minimal \( S^J \) action (10.1) in eq. (11.8) has been replaced with the non-minimal action

\[
S^J \rightarrow S^J + \epsilon^{a}{b} \pi^a .
\]

(11.10)

The partition function is independent of the gauge fermion \( \Psi = \Psi(\Phi) \), where the Faddeev-Popov matrix

\[
\Delta^a_{b} := (\frac{\partial}{\partial c^a}) R^0{a_5}{b} (\varphi)
\]

(11.11)

is invertible; and where \( \epsilon(\Psi) = 1 \) and \( gh(\Psi) = -1 \).

12 Orbit Method

In this Section 12, we introduce the gauge parameter \( \theta^a \) into the antisymplectic phase space. Let us consider irreducible (possibly open) theories in the minimal sector of the antisymplectic phase space

\[
\Gamma_{\text{min}} := \{ \varphi^i, \varphi^*_i; c^a; \tau^a \}
\]

(12.1)

The action in the minimal sector is of the form

\[
S^0_{\text{min}} = S^0_0(\varphi) + \varphi^*_i R^0{i}{a} (\varphi) c^a + \ldots .
\]

(12.2)

We assume for simplicity from now on that the underlying groupoid structure is independent of the external sources \( J_A \). The main new feature in this Section 12 is that the gauge parameters \( \theta^a \) and their antifields \( \theta^*_a \) are included into the total antisymplectic phase space as active participants

\[
\Gamma_{\text{tot}} := \{ \Gamma_{\text{min}}; \theta^a, \theta^*_a \}
\]

(12.3)
The action in the total sector is of the form
\[ S^0_{\text{tot}} = S^0_{\text{min}} - \theta^a \bar{\mu}^a_b(\theta, \varphi) c^b + \ldots . \] (12.4)

The \( S^i \) functions are of the form
\[ S^i = \varphi^i(\varphi, \theta) + \varphi^* j K^{ij}_a(\varphi, \theta) c^a - \theta^a K^{ia}_b(\varphi, \theta) c^b + \ldots . \] (12.5)

The set of classical master eqs. (5.4) leads to a hierarchy of equations: (i) The Noether identity (7.1) in the sector proportional to \( c^a \):
\[ (S^0 \frac{\partial}{\partial \varphi^i}) R^0_i(\varphi) = 0 . \] (12.6)

(ii) An open version of inverse Lie eq. (9.14) in the sector proportional to \( J_i c^a \):
\[ (\varphi^i \frac{\partial}{\partial \varphi^j}) \theta R^{0j}_i(\varphi) - (\varphi^i \frac{\partial}{\partial \theta^k}) \varphi \bar{\mu}^{0b}_a + (S^0 \frac{\partial}{\partial \varphi^j}) K^{ij}_a(-1)^{\varepsilon_i} = 0 . \] (12.7)

Or equivalently, if one multiplies eq. (12.7) from left with the matrix \((\varphi^i \frac{\partial}{\partial \varphi^j}) \theta\), one gets
\[ R^{0i}_a(\varphi) + (\varphi^j \frac{\partial}{\partial \theta^b}) \varphi \bar{\mu}^{0b}_a = -(\varphi^i \frac{\partial}{\partial \theta^k}) \theta \frac{\partial}{\partial \varphi^j}(K^{j}k_a(-1)^{\varepsilon_k}) \]
\[ = -(S^0 \frac{\partial}{\partial \varphi^j}) \theta (\varphi^i \frac{\partial}{\partial \theta^b}) \varphi K^{jb}_a(1)^{(\varepsilon_i+\varepsilon_k)}(i \leftrightarrow j) . \] (12.8)

(iii) In the sector proportional to \( J_i J_j c^a \), one gets
\[ (\varphi^i \frac{\partial}{\partial \theta^j}) \varphi K^{jk}_a + (\varphi^j \frac{\partial}{\partial \theta^b}) \varphi K^{jb}_a \]
\[ = (\varphi^i \frac{\partial}{\partial \theta^k}) \varphi K^{jk}_a - (\varphi^i \frac{\partial}{\partial \theta^b}) \varphi K^{jb}_a \]
\[ = (-1)^{(\varepsilon_i+1)(\varepsilon_j+1)}(i \leftrightarrow j) . \] (12.9)

Firstly, note that the replacement of the closed inverse Lie eq. (9.14) with the open inverse Lie eq. (12.7) still allows for essentially the same construction of the BRST–invariant \( S^i \) from Section 10. The only difference is that the off-shell BRST–invariance (10.11) turns into an on-shell BRST–invariance
\[ (S^i \frac{\partial}{\partial \varphi^i}) R^{0k}_a(\varphi) = -P^i_j (S^0 \frac{\partial}{\partial \varphi^i}) K^{jk}_a(-1)^{\varepsilon_j} . \] (12.10)

Secondly, let us now consider a gauge orbit
\[ \varphi^0_i = f^0_i(\varphi, \theta) \quad \iff \quad \varphi^i = ((f^0)^{-1})^i(\varphi^0, \theta) , \] (12.11)
and composed action
\[ \overline{S}_0(\varphi^0, \theta) := S^0((f^0)^{-1}(\varphi^0, \theta)) . \] (12.12)

Multiplying eq. (12.7) with \((S^0 \frac{\partial}{\partial \varphi^i}) \theta\) yields
\[ 0 \overset{(12.6)+(12.7)}{=} (S^0 \frac{\partial}{\partial \theta^b}) \varphi \bar{\mu}^{0b}_a + (S^0 \frac{\partial}{\partial \varphi^j}) \theta (S^0 \frac{\partial}{\partial \varphi^j}) K^{ij}_a(-1)^{\varepsilon_i} . \]
\[ (12.9) \quad \mu^{0b}_a - \left( S_0 \frac{\partial}{\partial \theta^b} \right)_{\varphi^0} \mu^{0b}_a = \left( S_0 \frac{\partial}{\partial \varphi^0} \right)_{\varphi^0} \left( \varphi^j \frac{\partial}{\partial \theta^b} \right)_{\varphi^0} K^{ib}_a (-1)^{\varepsilon_i} \]

\[ = \left( S_0 \frac{\partial}{\partial \theta^b} \right)_{\varphi^0} \left( \mu^{0b}_a - \left( S_0 \frac{\partial}{\partial \varphi^0} \right)_{\varphi^0} K^{ib}_a (-1)^{\varepsilon_i} \right) . \quad (12.13) \]

Assuming that the \( \mu^{0b}_a \) matrix is an invertible matrix, we deduce that the action (12.12) is gauge invariant

\[ \left( S_0 \frac{\partial}{\partial \theta^b} \right)_{\varphi^0} (12.13) = 0 , \quad (12.14) \]

at least sufficiently close to the classical trajectories \( S_0 \frac{\partial}{\partial \varphi^0} \approx 0 \). We next introduce shifted structure functions

\[ \tilde{K}^{ij}_a := \left( \varphi^i \frac{\partial}{\partial \varphi^k} \right)_{\varphi^0} K^{kj}_a + \left( \varphi^j \frac{\partial}{\partial \theta^b} \right)_{\varphi^0} K^{ib}_a (-1)^{\varepsilon_i + \varepsilon_k + \varepsilon_j} = -(-1)^{\varepsilon_i} (i \leftrightarrow j) . \quad (12.15) \]

Eqs. (12.8), (12.14) and (12.15) imply an \((i \leftrightarrow j)\) symmetric version of eq. (12.8):

\[ R^{0i}_a (\varphi) + \left( \varphi^i \frac{\partial}{\partial \varphi^0} \right)_{\varphi^0} \mu^{0b}_a = -\left( S_0 \frac{\partial}{\partial \varphi} \right)_{\varphi^0} \tilde{K}^{ij}_a . \quad (12.16) \]

We stress that eq. (12.8), or equivalently eq. (12.16), can be viewed as an open version of the inverse Lie eq. (9.13) for quasi–groups.

### 13 Gauge–Invariants

In this Section 13, we construct on-shell gauge–invariants \( \xi^I \). See also Section 4.1 in Ref. [15]. Let \( \chi^a = \chi^a (\varphi^0) \) be \( m \) gauge-fixing conditions, in the sense that we impose \( \chi^a = 0 \) for all possible values of \( \varphi^0 \). The gauge-fixing conditions leave \( n - m \) gauge–invariants \( \xi^I \) unconstrained:

\[ \chi^a (\varphi^0) = 0 \quad \iff \quad \varphi^0 = g^i (\xi) . \quad (13.1) \]

(Again, note that unlike ordinary gauge–fixing, the gauge–invariants \( \xi^I \) will depend on gauge-fixing conditions \( \chi^a = 0 \) by construction.) Recalling that

\[ \varphi^0 = f^0 (\varphi, \theta) , \quad (13.2) \]

we can now reparametrize the original variable \( \varphi^i \) as

\[ (\xi^I, \theta^a) \quad \longrightarrow \quad \varphi^i = ((f^0)^{-1})^i_j g^j (\xi, \theta) . \quad (13.3) \]

In words, the coordinates \((\xi^I, \theta^a)\) represent the decomposition of the \( n \) original fields \( \varphi \) in \( n - m \) physical gauge–invariants \( \xi^I \) and \( m \) gauge variables \( \theta^a \). The following rank conditions are assumed:

\[ \text{rank}(\varphi^0)_{\xi^I} = n - m , \quad \text{rank}(\varphi^0)_{\varphi^0} = m . \quad (13.4) \]

We assume that there exists an inverse map to the reparametrization (13.3)

\[ \varphi^i \quad \longrightarrow \quad \xi^I = \xi^I (\varphi) , \quad \theta^a = \theta^a (\varphi) . \quad (13.5) \]
The fact that $\xi^I$ are independent of $\theta^a$ is encoded via the relations

$$0 = (\xi^I \frac{\partial}{\partial \varphi^j}) (\varphi^i \frac{\partial}{\partial \theta^a}) \varphi^0 .$$

(13.6)

Next, let us consider a source-dependent master action of the form

$$S^J = S^{0 \text{min}}_0 + J_I \Xi^I$$

(13.7)

in the minimal sector (12.1). Here the BRST–invariants

$$\Xi^I = \xi^I + \varphi^*_i K^i a c^a + \ldots$$

(13.8)

are deformations of the gauge invariants $\xi^I$. The set of classical master eqs. (5.4) in the minimal sector reads

$$(S^{0 \text{min}}_0, S^{0 \text{min}}_0) = 0 , \quad (S^{0 \text{min}}_0, \Xi^I) = 0 , \quad (\Xi^I, \Xi^I) = 0 .$$

(13.9)

The second and third involution eqs. (13.9) imply, among other things, that

$$\xi^I \frac{\partial}{\partial \varphi^j} R^{ij} a (\varphi) + (S^0 \frac{\partial}{\partial \varphi^j} K^{ij} a (-1)^{\epsilon_i} = 0 ,$$

(13.10)

and

$$\xi^I \frac{\partial}{\partial \varphi^j} K^{ij} a = (-1)^{\epsilon_i(I+1)(\epsilon_L+1)} (I \longleftrightarrow L) ,$$

(13.11)

respectively.

Moreover, we can also derive eqs. (13.10) and (13.11) from the orbit method of Section 12. If we use the open version of the inverse Lie eq. (12.16), we get

$$\left(\xi^I \frac{\partial}{\partial \varphi^j} \right) \left( R^{ij} a (\varphi) + (S^0 \frac{\partial}{\partial \varphi^j} K^{ij} a \right) \overset{(12.16)+(13.6)}{=} 0 .$$

(13.12)

Eq. (13.12) shows that $\xi^I$ are gauge–invariant on–shell. If we now identify

$$K^{ij} a = - (\xi^I \frac{\partial}{\partial \varphi^j} K^{ij} a (-1)^{\epsilon_j(I+1)} (\varphi^i \frac{\partial}{\partial \varphi^j} \theta) (\xi^I \frac{\partial}{\partial \varphi^j} K^{ij} a (-1)^{\epsilon_j(I+1)(\epsilon_L+1)} ,$$

(13.13)

then eqs. (13.12) and (12.15) become the classical master eqs. (13.10) and (13.11), respectively.

Finally, let us use the $g^i$ functions from eq. (13.1) and the BRST–invariants (13.8) to define a new set of BRST–invariants

$$S^i = g^i (\Xi) \overset{(13.8)}{=} g^i (\xi) + (g^i (\xi) \frac{\partial}{\partial \xi^I} \varphi^*_j K^{ij} a c^a + \ldots ,$$

(13.14)

which we pair with the minimal action $S^{0 \text{min}}_0$. It follows that $\{S^{0 \text{min}}_0, S^i\}$ satisfies the set of classical master eqs. (5.4), because $\{S^{0 \text{min}}_0, \Xi^I\}$ does, cf. eq. (13.9). This means that $S^i$ is a minimal analogue to the $S^i$ function (12.5) without the $\{\theta^a; \theta^*_a\}$ dependence (so that, e.g., the $K^{ia} b$ structure functions are absent)

$$S^i = \varphi^0 a (\varphi) + \varphi^*_j K^{ij} a (\varphi) c^a + \ldots .$$

(13.15)
Here $\varphi^i = g^i(\xi)$ and

$$K^{ij}_a = (g^i(\xi) \frac{\partial}{\partial \xi^i}) K^{ij}_a (-1)^{i(\varepsilon_i+1)}(\varepsilon_j+1).$$

Moreover $S^i$ and $K^{ij}_a$ satisfy minimal versions of the corresponding formulas from Section 12.

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