KÄHLER CUTS

D. BURNS, V. GUILLEMIN, AND E. LERMAN

Abstract. A symplectic cut of a manifold $M$ with a Hamiltonian circle action is a symplectic quotient of $M \times \mathbb{C}$. If $M$ is Kähler then, since $\mathbb{C}$ is Kähler, the cut space is Kähler as well. The symplectic structure on the cut is well understood. In this paper we describe the complex structure (and hence the metric) on the cut. We then generalize the construction to the case where $M$ has a torus action and $\mathbb{C}$ is replaced by a toric Kähler manifold.

1. Introduction

Let $(M, \omega)$ be a symplectic manifold and $\tau : S^1 \times M \to M$ a Hamiltonian action of $S^1$ with moment map $\phi : M \to \mathbb{R}$. We will assume that, for $\lambda \in \mathbb{R}$, $S^1$ acts freely on $\phi^{-1}(\lambda)$. In particular, $\lambda$ is then a regular value of $\phi$ and the symplectic quotient

$$M_{\lambda} = \phi^{-1}(\lambda)/S^1$$

is well-defined and non-singular. In [8] it was shown that the disjoint union of this quotient with the open subset

$$M_{\lambda}^\circ = \{ p \in M \mid \phi(p) < \lambda \}$$

of $M$ can be given the structure of a smooth symplectic manifold $(M_{\lambda}, \omega_{\lambda})$ which was called the symplectic cut of $M$ at $\lambda$. A number of applications of this cutting operation were also given to problems in symplectic geometry, and since then many other applications have been found. (See, for instance, [4], [9], [14], [12], and [11].)

In this article we will look at this cutting operation from the Kählerian perspective. Our motivation is a basic problem in Kähler geometry: what happens to the Kähler metric on a non-singular projective variety when one blows this variety up along a non-singular subvariety? The symplectic cutting operation turns out to have some bearing on this problem. Suppose that the moment map above takes its maximum value $\lambda_o$ on a subset $W$ of $M$. Then $W$ is a symplectic submanifold of $M$, and, for $\lambda = \lambda_o - \epsilon, \epsilon \approx 0$, $M_{\lambda}$ can be obtained from $M$ by blowing up $M$ symplectically along $W$. (See [10] and [6].)

Suppose now that $M$ is a complex manifold and $\omega$ a Kähler form. In addition, suppose that $\tau$ extends to a holomorphic action of $\mathbb{C}^*$ on $M$. Then $M^\lambda$ is also Kähler, so the set (1.2) has two Kähler structures: the Kähler structure it acquires as an open subset of $M^\lambda$ and the Kähler structure it acquires as an open subset of $M$. In [8] it was shown that symplectically these two structures coincide: $\omega = \omega^\lambda$ on $M_{\lambda}^\circ$. However, it was also pointed out that the two complex structures on $M_{\lambda}^\circ$ don’t coincide. The main result of this paper is an amplification of this remark, a global description of the complex structure on $M_{\lambda}^\circ$ coming from $M^\lambda$. Let

$$M^\# = \mathbb{C}^* \cdot M_{\lambda}^\circ.$$
For $M$ compact this is a Zariski open subset of $M$, and we will construct below a canonical diffeomorphism of $M^\lambda_\circ$ onto $M^\#_\circ$ which is biholomorphic with respect to the $M^\lambda$ complex structure on $M^\lambda_\circ$ and the $M$ complex structure on $M^\#_\circ$. 

Since $M^\#$ is Zariski open, its complement is a complex subvariety of $M$, and we will show that this variety is the union of unstable manifolds for the $\mathbb{C}^*$-action, or, in other words, “generalized Schubert varieties”. Thus the cutting operation compactifies the complement of these Schubert varieties by adjoining the non-singular hypersurface $M^\lambda_\circ$ to $M$.

We give now a brief summary of the contents of this article. In §2 we review the definition and elementary properties of the cutting operation and prove the assertions above. In §3 we generalize the version of the cutting operation given in [8] by replacing the standard Kähler potential $|z|^2$ on $\mathbb{C}$ by an arbitrary radial potential $F$, and show that, with minor modifications, the results of §2 are still true. Then, in §4 we apply these results to the Kähler potential $F(z) = \frac{2}{c} \log(1 + |z|^2)$, $c > 0$ and show that if the Kähler structure on $M$ is Kähler-Einstein with structure constant $c$, the Kähler form on $M^\lambda$ satisfies a modified version of the Kähler-Einstein equation. Finally, in §5 we explain how to generalize the results of this paper to the “toric variety” version of symplectic cutting developed in [8] and [9].

### 2. Symplectic cuts

Let $(M, \omega)$ be a Kähler manifold and $\tau : S^1 \times M \to M$ a Hamiltonian action of $S^1$ on $M$ with moment map $\phi : M \to \mathbb{R}$. We will assume that this action extends to a holomorphic action of $\mathbb{C}^*$ on $M$ which we will continue to denote by $\tau$. Suppose that $S^1$ acts freely on the level set $\phi^{-1}(\lambda)$. Let $W = M \times \mathbb{C}$, equipped with the Kähler form

$$\omega + \sqrt{-1} dz \wedge d\bar{z}$$

and let $S^1$ act on $W$ by its product action. This is a Hamiltonian action with moment map

$$\psi = \phi + \bar{z}|^2.$$ 

Moreover, $S^1$ acts freely on the level set, $\psi^{-1}(\lambda)$, so the reduced space

$$M^\lambda = \psi^{-1}(\lambda)/S^1$$

is well-defined, and this symplectic manifold is by definition the manifold “$M$ cut at $\lambda$” (see [8]). The level set $\psi^{-1}(\lambda)$ is the disjoint union of the two $S^1$-invariant sets:

$$\{(p, 0) \mid \phi(p) = \lambda\}$$

and

$$\{(p, z) \mid \phi(p) = \lambda - |z|^2 < \lambda\},$$

so the quotient $\psi^{-1}(\lambda)/S^1$ is the disjoint union of the quotients of these two sets. The quotient of (2.4) is by definition the reduced space

$$M_\lambda = \phi^{-1}(\lambda)/S^1$$

and the quotient of (2.3) can be identified with the open subset

$$M^\lambda_\circ = \{p \in M \mid \phi(p) < \lambda\}$$

of $M$. In fact this identification is given explicitly by the map

$$M^\lambda_\circ \ni p \to (p, \sqrt{\lambda - \phi(p)})$$
which maps \( M^\lambda_{\phi} \) onto a global cross-section for the action of \( S^1 \) on the set (2.3). Thus \( M^\lambda \) is the disjoint union of the open subset \( M^\lambda_{\phi} \) of \( M \) and a codimension two symplectic submanifold, the reduced space \( M_\lambda \). In particular, \( M^\lambda_{\phi} \) has two Kähler structures: the restriction of the Kähler structure on \( M^\lambda \) and the restriction of the Kähler structure on \( M \). We propose to show below how these Kähler structures are related.

From the symplectic point of view the situation is extremely simple. Let \( \omega^\lambda \) be the symplectic form on \( M^\lambda \). The pull-back by the map (2.7) of the form (2.1) is just \( \omega \), and so on \( M^\lambda_{\phi} \), \( \omega = \omega^\lambda \). However, as is pointed out in [8], section 1.1, \((M^\lambda_{\phi}, \omega)\) and \((M^\lambda_{\phi}, \omega^\lambda)\) are not identical as Kähler manifolds: the \( M \)-complex structure on \( M^\lambda_{\phi} \) doesn’t coincide with the \( M^\lambda \)-complex structure. How these complex structures are related is the question we will investigate below.

Let \( v = \nabla \phi \) be the infinitesimal generator of the one parameter group \( \tau^\phi : M \to M \), and for \( q \in M^\lambda_{\phi} \) let
\[
\kappa(q) = -1/2 \log(\lambda - \phi(q)).
\]
We define a map \( g : M^\lambda_{\phi} \to M \) by setting
\[
g(q) = \exp(\kappa(q)v)(q).
\]
Let \( M_{\text{stable}} = \mathbb{C}^* \cdot \phi^{-1}(\lambda) \). We claim that \( g \) is a diffeomorphism of \( M^\lambda_{\phi} \) onto the open subset
\[
M^\# = M^\lambda_{\phi} \cup M_{\text{stable}}
\]
of \( M \). That \( g \) is a diffeomorphism onto its image is clear. To see that its image is (2.8) we first note that if \( q \) is in \( M^\lambda_{\phi} \), then \( p = g(q) \) if and only if
\[
\exp(tv)(p) = q
\]
for \( t \) satisfying
\[
\lambda - e^{2t} = \phi(q).
\]
Now let \( p \) be in \( M_{\text{stable}} \) with \( \phi(p) > \lambda \), and let \( \gamma(t) = \exp(tv)(p) \). Then
\[
a = \lim_{t \to -\infty} \phi(\gamma(t)) < \lambda,
\]
so the curves, \( y = \lambda - e^{2t} \) and \( y = \phi(\gamma(t)) \) must intersect at some point on the negative \( t \) axis; and, at that point \( q = \gamma(t) \) satisfies \( p = g(q) \) by (2.11) and (2.12). Similarly if \( p \) is in \( M^\lambda_{\phi} \) and \( \gamma(t) = \exp(tv)(p) \), then if \( \phi(p) < \lambda - 1 \), the curve, \( y = \phi(\gamma(t)) \), has to intersect the curve, \( y = \lambda - e^{2t} \) at some point on the positive \( t \) axis, and if \( \lambda - 1 < \phi(p) < \lambda \), these curves must intersect at some point on the negative \( t \) axis. Thus the image of \( g \) contains \( M^\lambda_{\phi} \cup M_{\text{stable}} \), and the inclusion the other way is obvious. If \( M \) is compact the set (2.10) doesn’t depend on \( \lambda \) but only on the critical values of \( \phi \) lying above \( \lambda \). More explicitly, let \( F_i, i = 1, ..., k \), be the connected components of the set of critical points of \( \phi \), i.e., the connected components of the fixed point set of the action \( \tau \). Let \( W_i \) be the unstable manifold of \( \nabla \phi \) at \( F_i \). Thus
\[
q \in W_i \iff \lim_{z \to 0} \tau_z(q) \in F_i
\]
These manifolds are complex submanifolds of positive codimension. Let \( S \) be the set of \( i \)'s for which \( \phi(F_i) \) is greater than \( \lambda \). We claim
\[
M - M^\# = \bigcup_{i \in S} W_i.
\]
Proof: By the Whitney decomposition theorem $M$ is the disjoint union of the $W_i$’s so every point $p \in M$ is in some $W_i$, and if $p$ is in $W_i$ and $\phi(F_i)$ is less than $\lambda$, then either $\phi(p) < \lambda$ or the $\mathbb{C}^*$ orbit through $p$ intersects $\phi^{-1}(\lambda)$. We will now prove:

**Theorem 2.1.** The map $g : M^\lambda_o \to M^\#$ is a biholomorphism of $M^\lambda_o$ with its $M^\lambda$ complex structure onto $M^\#$ with its $M$ complex structure.

**Proof.** The biholomorphic map

\[(2.16) \quad f : M \times \mathbb{C}^* \to M \times \mathbb{C}^*\]

defined by

\[(2.17) \quad f(p, z_o) = (\tau_{z_o} p, z_o)\]

intertwines the action

\[(2.18) \quad z(p, z_o) = (p, z z_o)\]

of $\mathbb{C}^*$ on $M \times \mathbb{C}^*$ with the diagonal action

\[(2.19) \quad z(p, z_o) = (\tau_{z} p, z z_o)\]

so the pullback by $f$ of the Kähler form (2.1) is invariant under the action of $S^1$ on the second factor of $M \times \mathbb{C}^*$. Moreover this action is Hamiltonian with moment map

\[(2.20) \quad \tilde{\psi}(p, z) = \phi(\tau_z p) + |z|^2\]

In particular, $f$ maps the level set, $\tilde{\psi}^{-1}(\lambda)$ onto the level set $\psi^{-1}(\lambda)$, and induces an isomorphism of Kähler manifolds

\[(2.21) \quad h : \tilde{\psi}^{-1}(\lambda)/S^1 \to \psi^{-1}(\lambda)/S^1\]

To describe this isomorphism more explicitly note that the set

\[(2.22) \quad \{\text{Re } z > 0, \text{Im } z = 0\}\]

is a global cross-section for both of the $S^1$ actions above, so $\tilde{\psi}^{-1}(\lambda)/S^1$ can be identified with the set

\[(2.23) \quad \{(p, e^t) \mid \phi(\tau_{e^t} p) + e^{2t} = \lambda\}\]

and $\psi^{-1}(\lambda)$ with the set

\[(2.24) \quad \{(q, e^t) \mid \phi(q) + e^{2t} = \lambda\}\]

and, modulo these identifications, $h$ maps the point $(p, e^t)$ in the set (2.23) onto the point $(q, e^t)$ in the set (2.24) where

\[q = \tau_{e^t} p\]

\[\lambda - e^{2t} = \phi(q).\]

However, $\tau_{e^t} p = \exp(tv)(p)$, so by (2.12) $p = g(q)$, i.e., $h$ is just $g^{-1}$. In particular, the domain of $h$ is the open subset $M^\#$ of $M$ defined by (2.10). Identifying $M^\#$ with $\tilde{\psi}^{-1}(\lambda)/S^1$, we note that since $\mathbb{C}^*$ is acting on $M \times \mathbb{C}^*$ by the action (2.11) the geometric invariant theory quotient

\[M \times \mathbb{C}^*/\mathbb{C}^* = \tilde{\psi}^{-1}(\lambda)/S^1\]

is not just equal to the open set $M^\#$ set theoretically, but is this open set with its $M$-complex structure. (See, for instance, [3, § 4.) Thus $g$ intertwines the $M^\lambda$-complex structure on $M^\lambda_o$ with the complex structure on $M$. 

□
Example: Let $\lambda_o$ be the maximum value of $\phi$ and let $W = \phi^{-1}(\lambda_o)$. Then $W$ is a complex submanifold of $M$, and if $\lambda = \lambda_o - \epsilon, \epsilon \approx 0$, then

$$M^\# = M - W$$

by (2.13). On the other hand, it is not hard to show that $M_\lambda$ is the projective normal bundle, $\mathbb{P}(NW)$, of $W$. Therefore, the assertion that $M^\lambda - M_\lambda$ is biholomorphic to $M^\#$ implies that $M^\lambda$ can be obtained from $M$ by deleting $W$ and adjoining $\mathbb{P}(NW)$, i.e., by blowing up $M$ along $W$.

Next we will describe some equivariance properties of the mapping $g$. The holomorphic action of $\mathbb{C}^*$ on $M \times \mathbb{C}^*$ defined by (2.19) extends to a holomorphic product action of $\mathbb{C}^* \times \mathbb{C}^*$

(2.25) $$(z_1, z_2) \cdot (p, z) = (\tau z_1 p, z_2 z)$$

and from this one gets a residual holomorphic action of $\mathbb{C}^*$ on the quotient space $M^\lambda$. The submanifold $M_\lambda$ of $M^\lambda$ is fixed by this action, and hence one gets an induced action of $\mathbb{C}^*$ on its complement $M^\lambda_\circ$. Restricted to $S^1$ this action coincides with the $M$-action of $S^1$ on $M^\lambda$; however, this action itself can’t coincide with the $M$-action of $\mathbb{C}^*$ since the $M$-action of $\mathbb{C}^*$ doesn’t leave $M^\lambda_\circ$ fixed.\footnote{This is one of the more compelling ways of seeing that the $M$-complex structure on $M^\lambda_\circ$ can’t be identical with the $M^\lambda$-complex structure.} As we noted above, however, the action of $\mathbb{C}^*$ on $M$ does leave fixed the submanifold $M^\#$. We claim

**Theorem 2.2.** The mapping $g$ intertwines the $M$-action of $\mathbb{C}^*$ on $M^\#$ with the $M^\lambda$ action of $\mathbb{C}^*$ on $M^\lambda_\circ$.

**Proof.** Since the function (2.7) and the vector field $v$ are $S^1$-invariant, the map $g$ intertwines the two $S^1$ actions. Therefore, since $g$ is biholomorphic, it intertwines the two $\mathbb{C}^*$ actions as well. $\square$

Let $U$ be a $\mathbb{C}^*$-invariant open subset of $M$, and let $(z_1, ..., z_n)$ be a holomorphic coordinate system on $U$. We will say that say that $(z_1, ..., z_n)$ is a $\mathbb{C}^*$-adapted coordinate system if $z_n \neq 0$ on $U$ and if the action of $\mathbb{C}^*$ on $U$ is given by

(2.26) $$\tau_a(z_1, ..., z_n) = (z_1, ..., z_{n-1}, az_n).$$

Then, for $p \in M^\lambda_\circ \cap U$

(2.27) $$g(p) = \tau_{e^t}(p), t = \kappa(p).$$

But $\kappa(p) = -1/2 \log(\lambda - \phi(p))$ so

(2.28) $$p = (z_1, ..., z_n) \Leftrightarrow g(p) = (z_1, ..., z_{n-1}, (\lambda - \phi(z))^{-1/2}z_n).$$

In other words

(2.29) $$g^*z_i = z_i, i = 1, ..., n - 1,$$

and

$$g^*z_n = (\lambda - \phi(z))^{-1/2}z_n$$

Thus from theorem 2.1 we conclude

**Theorem 2.3.** If $U$ is a $\mathbb{C}^*$-invariant open subset of $M$ and $(z_1, ..., z_n)$ a $\mathbb{C}^*$-adapted coordinate system on $U$, then (2.23) is a complex coordinate system on $U \cap M^\lambda_\circ$ compatible with the $M^\lambda$-complex structure.
3. Cuts with invariant potentials

In the construction of the cut space $M^\lambda$ described in § 2 above one began by equipping $W = M \times \mathbb{C}$ with the Kähler form the sum of that on $M$ and the form $\sqrt{-1}dz \wedge d\bar{z}$ on $\mathbb{C}$, and then reduced with respect to the diagonal action of $S^1$. There is, of course, nothing sacrosanct about the form $\sqrt{-1}dz \wedge d\bar{z}$, and in fact, it is sometimes advantageous to consider more general Kähler forms on the $\mathbb{C}$ factor, e.g., when $M$ is Kähler-Einstein, (cf., § 4). In this section we will replace $\sqrt{-1}dz \wedge d\bar{z}$ by an arbitrary $S^1$-invariant Kähler form on $\mathbb{C}$ and show that most of the results of § 2 are still true, modulo small changes detailed below.

Let $s$ be a real valued function on $\mathbb{C}$ invariant under the standard action of $S^1$: $\lambda \cdot z = \lambda z$. By a theorem of G. Schwarz there exists a function $F$ on $\mathbb{R}$ such that $s(z, \bar{z}) = F(|z|^2)$. Now

$$\partial \bar{\partial} F(|z|^2) = (F''(|z|^2)|z|^2 + F'(|z|^2)) \, dz \wedge d\bar{z}.$$  

Let $H(t) = tF(t)$.

**Lemma 3.1.** Let $F$ and $H$ be as above. Then

1. $\omega_F := \sqrt{-1} \partial \bar{\partial} F(|z|^2)$ is a symplectic form on $\mathbb{C}$ iff $H'(t) > 0$ for all $t \geq 0$.
2. $\phi = H(|z|^2)$ is a moment map for the standard action of $S^1$ on $(\mathbb{C}, \omega_F)$.

**Proof.** The form $\omega_F$ is nondegenerate iff $F''(|z|^2)|z|^2 + F'(|z|^2) > 0$ for all $z \in \mathbb{C}$ iff $H'(t) = F''(t)t + F'(t) > 0$ for all $t \geq 0$. This proves the first claim. Our proof of the second claim is a computation:

$$t \left( \sqrt{-1} \left( \frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}} \right) \right) \omega_F = -(F''(|z|^2)|z|^2 + F'(|z|^2))(zd\bar{z} + \bar{z}dz) = -dH(|z|^2).$$

Note that by definition $H(0) = 0$. Also, since $H'(t) > 0$ on $[0, \infty)$, $H(t)$ is strictly increasing on $[0, \infty)$. Let $a = \lim_{t \to +\infty} H(t)$, so that $0 < a \leq \infty$. Then $H$ is invertible and its inverse $K$ is a strictly increasing function $K : [0, a) \to [0, \infty)$.

Now let $(M, \omega)$ be a symplectic manifold with a Hamiltonian action $\tau$ of $S^1$ and let $\phi : M \to \mathbb{R}$ be an associated moment map. For $\lambda \in \mathbb{R}$ we define the cut of $(M, \omega)$ with respect to the potential $F$ at $\lambda$ to be the symplectic quotient at $\lambda$ of $(M \times \mathbb{C}, \omega + \omega_F)$ under the diagonal action of $S^1$. We denote the cut by $M_{cut}(\lambda, F)$. Of course $M^\lambda$ (2.3) of § 2 above is just $M_{cut}(\lambda, |z|^2)$.

**Theorem 3.2.** Let $(M, \omega)$ be a symplectic manifold with a Hamiltonian action of $S^1$ and let $\phi : M \to \mathbb{R}$ be an associated moment map. Suppose $S^1$ acts freely on $\phi^{-1}(\lambda)$. Then the cut $M_{cut}(\lambda, F)$ is naturally a smooth symplectic manifold.

Moreover, the natural embedding of the reduced space $M_\lambda := \phi^{-1}(\lambda)/S^1$ into $M_{cut}(\lambda, F)$ is symplectic, and the complement $M_{cut}(\lambda, F) - M_\lambda$ is symplectomorphic to the open subset $M_\delta := \{m \in M \mid \lambda - a < \phi(m) < \lambda\}$ of $(M, \omega)$, where as above $a = \lim_{|z| \to +\infty} \phi(z)$.

Furthermore, if $(M, \omega)$ is a Kähler manifold and the action of $S^1$ is holomorphic, then $M_{cut}(\lambda, F)$ is Kähler and $M_{cut}(\lambda, F) - M_\lambda$ is biholomorphic to $M^\# := \mathbb{C}^* \cdot M_\lambda = \mathbb{C}^* \cdot \{m \in M \mid \lambda - a < \phi(m) < \lambda\} \subset M$.

**Proof.** The moment map $\psi$ for the diagonal action of $S^1$ on $(M \times \mathbb{C}, \omega + \omega_F)$ is given by $\psi(m, z) = \phi(m) + H(|z|^2)$ and so

$$\psi^{-1}(\lambda) = \{(m, z) \in M \times \mathbb{C} \mid \phi(m) + H(|z|^2) = \lambda\}.$$
Now $\phi(m) + H(|z|^2) = \lambda$ iff $|z|^2 = K(\lambda - \phi(m))$, where as before $K = H^{-1}$. Consider the map

$$\sigma: \{m \in M \mid \lambda - a < \phi(m) \leq \lambda\} \rightarrow \psi^{-1}(\lambda), \sigma(m) = (m, (K(\lambda - \phi(m)))^{1/2}).$$

Let $\pi: \psi^{-1}(\lambda) \rightarrow \psi^{-1}(\lambda)/S^1 = M_{cut}(\lambda, F)$ denote the orbit map. It is easy to see that $\pi \circ \sigma$ is onto and that $\pi \circ \sigma$ is an open embedding on $M_o^\lambda$. Moreover, since $\sigma^*(\omega + \omega_F) = \omega$, $\pi \circ \sigma: M_o^\lambda \rightarrow M_{cut}(\lambda, F)$ is a symplectic embedding.

Similarly one checks that $\pi \circ \sigma$ induces a symplectic embedding $j$ of $M_\lambda$ into $M_{cut}(\lambda, F)$. Clearly, $M_{cut}(\lambda, F) \setminus j(M_\lambda) \simeq M_o^\lambda$.

Assume now that $(M, \omega)$ is Kähler. Then, since the symplectic quotient of a Kähler manifold is Kähler ([3, 3]), the cut is Kähler as well. We now argue that $M_{cut}(\lambda, F) - M_\lambda$ is biholomorphic to $M^\# := \mathbb{C}^* \cdot \{m \in M \mid \lambda - a < \phi(m) < \lambda\} \subset M$.

Consider the map $f: M \times \mathbb{C}^* \rightarrow M \times \mathbb{C}^*$ as in (2.16). We assume again as in § 2 that the action of $S^1$ extends to an action $\tau$ of $\mathbb{C}^*$. Recall also that $\frac{d}{dt}\tau_t(z) = \nabla \phi(\tau_t(z))$ for all $t \in \mathbb{R}$, where $\nabla \phi$ is the gradient of $\phi$ with respect to the Kähler metric. The map $f$ is biholomorphic, and intertwines the right action (2.18) of $\mathbb{C}^*$ on $M \times \mathbb{C}^*$ with the diagonal action (2.19). Consequently the pull-back form

$$\tilde{\omega} := f^*(\omega + \omega_F)$$

is invariant under the action of $S^1$ on the second factor. Moreover, this action of $S^1$ is Hamiltonian with moment map $\tilde{\psi}$ satisfying

$$\tilde{\psi} = \psi_0 \circ f,$$

where $\psi_0$ denotes the restriction of $\psi$ from $M \times \mathbb{C}$ to $M \times \mathbb{C}^*$. Consequently $f$ induces a biholomorphic map $h$ between quotients:

$$h: \tilde{\psi}^{-1}(\lambda)/S^1 \rightarrow \psi_0^{-1}(\lambda)/S^1.$$

Note that $\psi^0_0^{-1}(\lambda)/S^1 \subset M_{cut}(\lambda, F)$ is precisely the subset symplectomorphic to $M_o^\lambda$. We claim that $\tilde{\psi}^{-1}(\lambda)/S^1$ is naturally isomorphic to $M^\# = \mathbb{C}^* \cdot M_o^\lambda$.

Now $\tilde{\psi}^{-1}(\lambda) = \{(m, z) \in M \times \mathbb{C}^* \mid \phi(\tau_t(m)) + H(|z|^2) = \lambda\}$. Since $\frac{d}{dt}\tau_t(z) = \nabla \phi(\tau_t(z)), \frac{d}{dt}\phi(\tau_t(m)) = |\nabla \phi(\tau_t(m))|^2 \geq 0$. Hence

$$\frac{d}{dt} (\phi(\tau_t(m)) + H(|e^{t}|^2)) = |\nabla \phi(\tau_t(m))|^2 + 2H'(e^{2t})e^{2t} > 0.$$

It follows that every $\mathbb{C}^*$ orbit intersects the level set $\tilde{\psi}^{-1}(\lambda)$ in precisely one $S^1$ orbit. Consequently

$$\tilde{\psi}^{-1}(\lambda)/S^1 \simeq \{m \in M \mid \phi(\tau_t(m)) + H(e^{2t}) = \lambda \text{ for some } t \in \mathbb{R}\}.$$

We claim that the set on right hand side is $\mathbb{C}^* \cdot M_o^\lambda$. Indeed, if for $m \in M$ there is $t \in \mathbb{R}$ such that $\phi(e^t \cdot m) + H(e^{2t}) = \lambda$, then, since $H([0, \infty)) = [0, a)$ and $H$ is strictly increasing,

$$\lambda - a < \phi(\tau_t(m)) < \lambda.$$

Therefore $\tau_t(m) \in \phi^{-1}((\lambda - a, \lambda)) = M_o^\lambda \Rightarrow m \in \tau_t^{-1}(M_o^\lambda) \subset \mathbb{C}^* \cdot M_o^\lambda$.

Conversely, suppose $m \in \mathbb{C}^* \cdot M_o^\lambda$. There are three cases to consider: $\phi(m) \leq \lambda - a$, $\lambda - a < \phi(m) < \lambda$ and $\lambda \leq \phi$. Recall that $\lim_{t \to -\infty} H(e^{2t}) = H(0) = 0$ and that $\lim_{t \to +\infty} H(e^{2t}) = a$.

If $m \in \mathbb{C}^* \cdot M_o^\lambda$ and $\phi(m) \geq \lambda$ then there is a $t \in \mathbb{R}$ such that $\tau_t(m) \in M_o^\lambda$. So $\lim_{t \to -\infty} \phi(\tau_t(m)) < \lambda$. Therefore $\lim_{t \to -\infty} (\phi(\tau_t(m)) + H(e^{2t})) < \lambda + 0$ while $\lim_{t \to +\infty} (\phi(\tau_t(m)) + H(e^{2t})) > \phi(e^0 \cdot m) + H(e^0) > \lambda$. Consequently there is $t_1 \in \mathbb{R}$ with $\phi(e^{t_1} \cdot m) + H(e^{2t_1}) = \lambda$. 


If $m \in \mathbb{C}^* \cdot M^\lambda_o$ and $\phi(m) \leq \lambda - a$ then there is $t \in \mathbb{R}$ with $\tau_{e^t}(m) \in M^\lambda_o$. Hence $\lim_{t \to +\infty}(\phi(\tau_{e^t}(m)) + H(e^{2t})) > \phi(m) + a = \lambda - a + a = \lambda$. On the other hand $\lim_{t \to -\infty}(\phi(\tau_{e^t}(m)) + H(e^{2t})) \leq \phi(m) + 0 \leq \lambda - a$. Consequently there is $t_1 \in \mathbb{R}$ with $\phi(e^{t_1} \cdot m) + H(e^{2t_1}) = \lambda$.

Finally, if $\lambda - a < \phi(m) < \lambda$ then $\lim_{t \to +\infty}(\phi(\tau_{e^t}(m)) + H(e^{2t})) \geq \phi(m) + a > \lambda - a + a = \lambda$ while $\lim_{t \to -\infty}(\phi(\tau_{e^t}(m)) + H(e^{2t})) < \phi(m) + 0 < \lambda$. Consequently there is $t_1 \in \mathbb{R}$ with $\phi(e^{t_1} \cdot m) + H(e^{2t_1}) = \lambda$. \hfill \Box

4. Kähler-Einstein manifolds

Let $F$ be the function

$$F(z) = \frac{2}{\kappa} \log(1 + |z|^2), \kappa > 0.$$ 

Then $\omega_F = \sqrt{-1} \partial \overline{\partial} F(|z|^2)$ is a Kähler-Einstein form on $\mathbb{C}$ with structure constant $\kappa$. Hence if the symplectic form $\omega$ on $M$ is Kähler-Einstein with structure constant $\kappa$, so is the form $\omega + \omega_F$ on $M \times \mathbb{C}$. Let $\psi$ be the moment map associated to the action of $S^1$ on $(M \times \mathbb{C}, \omega + \omega_F)$, and let $M_{cut}(\lambda, F)$ be the cut space, $\psi^{-1}(\lambda)/S^1$. Since

$$a = \lim_{t \to +\infty} t \log(1 + t) = +\infty,$$

$M_{cut}(\lambda, F) - M^\lambda$ is symplectomorphic to the open subset $M^\lambda_o = \{m \in M, -\infty < \phi(m) < \lambda\}$ of $M$ and biholomorphic to the open subset $M^\# = \mathbb{C}^* \cdot M^\lambda_o$ of $M$ by theorem 3.1.

The level set $Y = \psi^{-1}(\lambda)$ can be regarded as a principal $S^1$-bundle over the cut space:

$$\pi : Y \to M_{cut}(\lambda, F),$$

and from the restriction of the Kähler metric to $Y$ one gets a connection on this bundle and an associated curvature form $\mu$. For $p \in M_{cut}(\lambda, F)$, $\pi^{-1}(p)$ is an embedded circle in $Y$, and we will denote the length of this circle measured with respect to the Kähler metric by $V_{eff}(p)$. For the following we refer to [3], § 11.

**Theorem 4.1.** If $\omega^\lambda$ is the Kähler form and $\mu^\lambda$ the Ricci form on $M_{cut}(\lambda, F)$, then

$$(4.1) \quad \mu^\lambda - 2\sqrt{-1} \partial \overline{\partial} \log V_{eff} + c\mu = \kappa(\omega^\lambda + \lambda \mu)$$

where $c$ is a constant satisfying

$$(4.2) \quad \kappa \psi = -\text{div} Z + c,$$

and $Z$ is the complex vector field on $M \times \mathbb{C}$ generating the $\mathbb{C}^*$-action.

Notice by the way that the “$\lambda \mu$” on the right hand side of (4.1) has to be present for (4.1) to be compatible with the Duistermaat-Heckman theorem. However, the “$c\mu$” on the left is an artifact of (4.2) which fixes the ambiguous additive constant in the definition of the moment map. Finally the “$V_{eff}$” on the left occurs for much the same reason that “effective potentials” occur for reduced Hamiltonian systems in classical mechanics: as the contribution to the downstairs metric of the vertical piece of the Kähler metric on $Y$ (see [1], § 4.5). (Note, by the way, that the case $\kappa = 0$ can be treated similarly using $F = a|z|^2$, $a > 0$, as in § 2 above.)

Here are some elementary examples of the Kähler cut construction.

**Example 1.** Take $M = \mathbb{C}^n, \omega = \sqrt{-1} \sum dz_j \wedge d\overline{z}_j$, with $\tau_{e^{i\theta}}(z) = e^{i\theta} \cdot z$. Then $\phi = |z|^2, \psi = |z|^2 + |w|^2, w \in \mathbb{C}$, and the “cut” of $M$ at $\lambda > 0$ is just $\mathbb{C}P^n$ with $\lambda$ times the Fubini-Study metric:

$$\omega^\lambda = \sqrt{-1} \lambda \partial \overline{\partial} \log(|z|^2 + |w|^2).$$
Example 2. Take the same $M, \omega$, but with $\tau_{e^{i\theta}}(z) = e^{-i\theta} \cdot z$. Now $\phi = -|z|^2, \psi = -|z|^2 + |w|^2$. For $\lambda < 0$, $M^\lambda = \mathbb{C}^n = \mathbb{C}^n$ with the origin blown up. Here, in coordinates $\zeta = w \cdot z$ on $\mathbb{C}^n - \{0\}$,

$$\rho_\lambda = \sqrt{\lambda^2 + 4|\zeta|^2} - \lambda \log(\lambda + \sqrt{\lambda^2 + 4|\zeta|^2}),$$

and thus,

$$\omega_\lambda = \sqrt{-1} \partial \bar{\partial} (\sqrt{\lambda^2 + 4|\zeta|^2} - \lambda \log(\lambda + \sqrt{\lambda^2 + 4|\zeta|^2})).$$

Furthermore, $V_{\text{eff}} = 2\pi \sqrt{\lambda^2 + 4|\zeta|^2}, c = -n + 1$, and $\mu_{L, \lambda} = \sqrt{-1} \partial \bar{\partial} \log(\lambda + \sqrt{\lambda^2 + 4|\zeta|^2})$, so that finally

$$\text{Ric}_\lambda = (n - 1)\mu_{L, \lambda} + 2\sqrt{-1} \partial \bar{\partial} \log V_{\text{eff}}.$$

If we “cut” $\mathbb{C}^n$ at level $\lambda > 0$, then $M^\lambda = \mathbb{C}^n$, with coordinates $\zeta = w \cdot z$. We find the same formulas for $\rho_\lambda, \omega_\lambda$ as in (1.3), (4.4), though in this case the logarithmic term in $\rho_\lambda$ extends smoothly across $\zeta = 0$. The singular case $\lambda = 0$ has a Kähler metric on $\mathbb{C}^n - \{0\}$ with coordinates $\zeta = w \cdot z$ as above, and $\rho_0 = 2|\zeta|$. This metric has non-negative Ricci form which vanishes in the complex radial directions in $\mathbb{C}^n$.

Example 3. Take $M = \mathbb{CP}^n$, with $\omega = (n + 1) \times$ the Fubini-Study Kähler form, so the Einstein constant $\kappa = 1$. We will work in an affine coordinate patch $\equiv \mathbb{C}^n$, with coordinates $z = (z_1, \ldots, z_n)$, so that $\omega = \sqrt{-1} (n + 1) \partial \bar{\partial} \log(1 + |z|^2)$. We will blow up the origin in the Fubini-Study metric by considering $\tau_{e^{i\theta}}(z) = e^{-i\theta} \cdot z$, with Hamiltonian

$$\phi = (n + 1) \times \frac{-|z|^2}{1 + |z|^2},$$

and cutting it with $(\mathbb{C}, 2\sqrt{-1} \partial \bar{\partial} \log(1 + |w|^2))$. Thus

$$\psi = -(n + 1) \frac{|z|^2}{1 + |z|^2} + 2 \frac{|w|^2}{1 + |w|^2},$$

and so $-(n + 1) < \psi < 2$ on $\mathbb{C}^n \times \mathbb{C} \ni (z, w)$. As in example 2, for $\lambda \in (-n - 1, 0), M(\lambda, F), F = 2 \log(1 + |w|^2)$, as in § 3, is $\mathbb{CP}^n$ with the origin blown up as a complex manifold, while for $\lambda \in (0, 2)$, it is just $\mathbb{CP}^n$ itself. The case $\lambda = 0$ gives a singular metric, smooth on $\mathbb{CP}^n - \{0\}$. In order to begin calculating these forms, it is convenient to do this in terms of coordinates $\zeta = w \cdot z, w$ on $\mathbb{C}^n - \{0\} \times \mathbb{C}$. From (4.6) we solve $\psi = \lambda$ for $|w|^2$ restricted to $\psi^{-1}(\lambda)$, and get

$$2(2 - \lambda)|w|^2 = A + \sqrt{A^2 + B},$$

where

$$A = \lambda + (\lambda + n - 1)|\zeta|^2, \quad \text{and} \quad B = 4(2 - \lambda)(\lambda + n + 1)|\zeta|^2.$$

Note, curiously, that for $\lambda = -n + 1$, this reduces to a solution seen above in the case of cutting with Euclidean space. Then

$$\mu_{L, \lambda} = \sqrt{-1} \partial \bar{\partial} \log |w|^2,$$

where $|w|^2$ is substituted using (4.7). The qualitative behavior of these forms at $\zeta = 0$ is the same as for the analogous Euclidean case in example 2 above. The form $\omega_\lambda$ and the effective potential are more complicated and we do not write them down here.
5. Cuttings by toric manifolds

In this section we generalize the construction of section 3 from $\mathbb{C}$ with a Kähler form defined by an $S^1$ invariant potential to an arbitrary Kähler toric manifold. The construction is the Kähler counterpart of a symplectic cut with respect to a polyhedral set (cf. § 2 in [14]).

Let us briefly review the symplectic construction. Let $(M, \omega_M)$ be a symplectic manifold with a Hamiltonian action of a torus $G$ and let $\varphi : M \to g^*$ denote an associated moment map. Let $(X, \omega_X)$ be a toric $G$-manifold, i.e., a (connected) symplectic manifold with a completely integrable Hamiltonian action of the torus $G$, i.e., $2 \dim_G(G) = \dim_G(M)$. Let $\Psi : X \to g^*$ denote an associated moment map. We are mostly interested in the cases where $X$ is a compact projective toric manifold or $\mathbb{C}^n$. We therefore assume that there is a convex open set $U \subset g^*$ such that $\Psi(X) \subset U$ and $\Psi : X \to U$ is proper. Then $g^*$

1. The moment image $\Delta := \Psi(X)$ is locally a rational polyhedral set. That is for any $\eta \in \Delta$ there is a neighborhood $W$ in $g^*$ and $N_1, \ldots, N_\ell \in \mathbb{Z}_G$ ($\mathbb{Z}_G$ is the integral lattice of the torus $G$) such that

$$W \cap \Delta = W \cap \bigcap_{i=1}^\ell \{ \eta \in g^* \mid \langle \eta, N_j \rangle \geq c_j \}$$

for some $c_1, \ldots, c_\ell \in \mathbb{R}$.

2. The fibers of $\Psi$ are connected.

Consequently, by dimension count, the fibers of $\Psi$ are $G$-orbits.

Assume next that $(S)$ $\Psi : X \to \Delta$ has a continuous Lagrangian section which is smooth over the interior $\Delta^\circ$ of $\Delta$. This assumption holds in a variety of contexts, for example:

1. If $X$ is compact, or
2. If $X$ is $\mathbb{C}^n$ with the standard action of the $n$-torus $\mathbb{T}^n$ preserving the standard symplectic form, or
3. If $X = \mathbb{C}$ and the symplectic form is defined by an invariant Kähler potential, or
4. If $X$ is the symplectization of a contact toric manifold of Reeb type [3], or
5. If $X$ is obtained from $\mathbb{C}^n$ by repeated symplectic cuts using various circle subgroups of $\mathbb{T}^n$.

The diagonal action of $G$ on $(M \times X, \omega_M - \omega_X)$ is Hamiltonian with moment map $\Phi = \varphi - \Psi$. We define the cut of $(M, \omega_M, \varphi)$ with respect to $(X, \omega_X, \Psi)$ to be the symplectic quotient $\overline{M}_X := \Phi^{-1}(0)/G$. If $G$ acts freely on $\Phi^{-1}(0)$, then $\overline{M}_X$ is a smooth symplectic manifold. If $\theta$ is only a regular value, then the cut space is an orbifold. More generally it is a symplectic stratified space in the sense of [10].

Note that since $G$ is abelian, the choice of $\varphi$ and $\Psi$ involve arbitrary choices of constant vectors in $g^*$. Note also that the trivial extension of the action of $G$ on $M$ to $M \times X$ commutes with the diagonal action of $G$. Consequently $\overline{M}_X$ is a Hamiltonian $G$-space. Finally observe that the space we called $M_{\text{cut}}(\lambda, F)$ is the cut of $M$ with respect to $(\mathbb{C}, -\sqrt{-1} \partial \bar{\partial} F(|z|^2), \lambda - H(|z|^2))$, the notation as in § 3 above.

Next we give a topological description of the cut space $\overline{M}_X$. But first, a bit more notation. Since $X$ is toric the isotropy group $G_x$ of a point $x \in X$ is connected. Thus $G_x$ is a subtorus of $G$. Its Lie algebra $g_x$ can be read off from $\Psi(x)$ as follows: Let $E$ be the open face of $\Delta$ containing $\Psi(x)$. Then $g_x$ is the annihilator of $\text{span}_{\mathbb{R}} \{ E - \Psi(x) \}$. In particular, if $\Psi(x), \Psi(x')$ are in the same face $E$ then $G_x = G_{x'}$. We denote this torus by $G_{E}$. We now collect a few facts about the cut space $\overline{M}_X$ (cf. [3], [2]).
Lemma 5.1. Let \((M, \omega_M, \varphi), (X, \omega_X, \Psi)\), \(G_E\) and \(\overline{M}_X\) be as above.

(1) As a topological space \(\overline{M}_X := \varphi^{-1}(\Delta)/\sim\) where \(m \sim m' \iff \varphi(m) = \varphi(m')\) and \(m = g \cdot m'\) for some \(g \in G_E\) where \(E\) is the open face of \(\Delta\) containing \(\varphi(m)\).

(2) The natural embedding of \(\varphi^{-1}(\Delta^o)\) into \(\overline{M}_X\) is symplectic (as above, \(\Delta^o\) denotes the interior of \(\Delta\).

(3) The cut space \(\overline{M}_X\) is smooth \(\iff\) for any \(m \in M\) with \(\varphi(m) \in E \subset \Delta\) (\(E\) is an open face) the group \(G_E\) acts freely at \(m\), i.e., \(G_E \cap G_m\) is trivial, where \(G_m\) denotes the isotropy group of \(m\).

Proof. Let \(s : \Delta \to X\) denote a Lagrangian section of \(\Psi : X \to \Delta\), the existence of which was assumed, see (S) above. Naturally \(\Phi^{-1}(0) = \{(m, x) \in M \times X \mid \varphi(m) - \Psi(x) = 0\}\). Hence \((m, x) \in \Phi^{-1}(0) \iff G \cdot x \ni s(\Psi(x)) = s(\varphi(m))\). Consider the map

\[
\sigma : \varphi^{-1}(\Delta) \to \Phi^{-1}(0), \quad \sigma(m) = (m, s(\varphi(m))).
\]

Its composition with the orbit map \(\pi : \Phi^{-1}(0) \to \Phi^{-1}(0)/G = \overline{M}_X\) is surjective. Moreover \(\pi \circ \sigma\) descents to a homeomorphism

\[
h : \varphi^{-1}(\Delta)/\sim \to \overline{M}_X.
\]

Since \(s^* \omega_X = 0\),

\[
(\sigma|_{\varphi^{-1}(\Delta^o)})^*(\omega_M - \omega_X) = \omega_M|_{\varphi^{-1}(\Delta^o)}.
\]

Consequently,

\[
(5.1) \quad h : \varphi^{-1}(\Delta^o) \to \overline{M}_X
\]

is a symplectic embedding.

The cut space \(\overline{M}_X\) is smooth \(\iff\) for all \((m, x) \in \Phi^{-1}(0)\) the group \(G\) acts freely at \((m, x)\). But for \((m, x) \in \Phi^{-1}(0)\) we have \(\varphi(m) = \Psi(x)\), and so \(G_E = G_x\). Thus \(G\) acts freely at \((m, x)\) \(\iff\) \(G_E \cap G_m\) is trivial. \(\square\)

Remark 5.2. For every open face \(E\) of \(\Delta\) the set \(M_E := \varphi^{-1}(E)/\sim \subset \overline{M}_X\) is a submanifold fixed by \(G_E\) (here \(E\) denotes the closure of \(E\) in \(\Delta\), hence is symplectic. In particular if \(E = \overline{E} = \{\mu\}\) is a vertex of \(\Delta\), then \(M_E = \varphi^{-1}(\mu)/G\) is a symplectic quotient of \(M\) at \(\mu\); it is a component of the fixed point set \((\overline{M}_X)^G\).

Let \((M, \omega_M, \varphi)\) and \((X, \omega_X, \Psi)\) be as before. Assume further that \((M, \omega_M)\) and \((X, \omega_X)\) are Kähler and that the actions of \(G\) are holomorphic. Then the cut \(\overline{M}_X\), being a symplectic quotient, is also Kähler. However, as we already observed in § 2, the embedding \(h : \varphi^{-1}(\Delta^o) \to \overline{M}_X\) (cf. (5.1)) cannot be holomorphic relative to the complex structures on \(M\) and \(\overline{M}_X\). Indeed, the complement of \(h(\varphi^{-1}(\Delta^o))\) in \(\overline{M}_X\) is the union of submanifolds of the form \(M_E = \varphi^{-1}(E)/\sim\) where \(E \subset \Delta\) is a proper open face.

Each manifold \(M_E\) is a component of the fixed point set \((\overline{M}_X)^G_E\), hence is \(G\)-invariant and Kähler. Therefore \(M_E\)'s are preserved by the action of the complexified group \(G^C\). Consequently \(h(\varphi^{-1}(\Delta^o)) = \overline{M}_X \setminus \bigcup_{E \subset \Delta} M_E\) is \(G^C\) invariant. On the other hand there is no reason for \(M^o := \varphi^{-1}(\Delta^o)\) to be \(G^C\) invariant, and usually it is not. Clearly

\[
(5.2) \quad M^# := G^C \cdot M^o = G^C \cdot \varphi^{-1}(\Delta^o)
\]

is the smallest \(G^C\) invariant subset of \(M\) containing \(M^o\). We remark:

Lemma 5.3. Let \(M, X\), etc. be as above. If \(G\) acts freely on \(\Phi^{-1}(0)\) then \(M^#\) defined by (5.2) is an open subset of \(M\).

Following [15] we recall the notion of semistablility for Hamiltonian group actions on (not necessarily integral) Kähler manifolds.
Definition 5.4. Let $N$ be a Kähler manifold with a holomorphic Hamiltonian action of a compact Lie group $G$ and associated moment map $\Phi : N \to g^*$. A point $x \in N$ is \emph{analytically semistable} if the closure of the $G^C$ orbit through $x$ intersects the zero level set $\Phi^{-1}(0)$ nontrivially ($G^C$ denotes the complexification of $G$). We denote the set of semistable points in $N$ by $N_{ss}$.

Remark 5.5. One can show that if the action of $G$ on $\Phi^{-1}(0)$ is locally free, then $N_{ss}$ is simply $G^C \cdot \Phi^{-1}(0)$. In this case one refers to the points of $N_{ss}$ as \emph{stable} points and denotes it by $N_{stable}$ (cf. (2.11) above).

We will need the following property of the set of semistable points.

Lemma 5.6. Let $N$ be a Kähler manifold with a holomorphic Hamiltonian action of a compact Lie group $G$ and associated moment map $\Phi : N \to g^*$. Assume that for every $x \in N$ the forward flowline of $-\nabla ||\Phi||^2$, the negative gradient flow of the norm squared of the moment map, is contained in a compact set. Then the set of semistable points $N_{ss}$ is the smallest $G^C$-invariant open subset of $N$ containing $\Phi^{-1}(0)$. Its complement $N \setminus N_{ss}$ is a complex-analytic subset.

Proof. See [1], § 4. Compare [13], pp. 109–110. \hfill $\Box$

Remark 5.7. Sjamaar (op. cit.) refers to moment maps $\Phi : N \to g^*$ with the property that for every $x \in N$ the forward flowline of $-\nabla ||\Phi||^2$ is contained in a compact set as \emph{admissible}. All proper moment maps are admissible. Other examples include moment maps for finite dimensional unitary representations (Example 2.3 in [15]).

We are now in a position to state and prove the main result of the section, which is the generalization of Theorem 2.3 and of Theorem 3.2.

Theorem 5.8. Let $(M, \omega_M, \varphi)$, $(X, \omega_X, \Psi)$, $G_E$ and $\overline{M}_X$ be as above with $G$ acting freely on $\Phi^{-1}(0)$. Assume further that $M$ and $X$ are Kähler and that $\Phi : M \times X \to g^*$ is admissible. Then $M^\# = G^C \cdot \varphi^{-1}(\Delta^\#) \subset M$ is biholomorphic to $h(\varphi^{-1}(\Delta^\#)) \subset \overline{M}_X$.

Proof. Fix a point $x^* \in \Psi^{-1}(\Delta^\#)$. Since $X$ is connected, the orbit $G^C \cdot x^*$ is all of $X^\# := \Psi^{-1}(\Delta^\#)$.

There are three actions of $G^C$ on $M \times X$:

1. The trivial extension of the action of $G^C$ on $M$; we denote its image in $\text{Diff}(M \times X)$ by $G^C_1$.
2. The trivial extension of the action of $G^C$ on $X$; we denote its image in $\text{Diff}(M \times X)$ by $G^C_d$.
3. The diagonal action of $G^C$; we denote its image in $\text{Diff}(M \times X)$ by $G^C_2$.

Since $G$ acts freely on $\Phi^{-1}(0)$,

$$\overline{M}_X = (M \times X)_{ss}/G^C_d,$$

where $(M \times X)_{ss} = G^C_2 \Phi^{-1}(0)$ is the set of analytically semi-stable points. Note that by Lemma 5.3 $(M \times X)_{ss}$ is $G^C_1$-invariant!

The actions of $G^C_1$ and $G^C_d$ commute. Hence the induced action of $G$ on $\overline{M}_X$ is holomorphic and therefore extends to an action of $G^C$. But the action of $G^C$ on $\overline{M}_X$ is induced by the action of $G^C_1$ on $(M \times X)_{ss}$.

Consequently $(M \times X)_{ss}$ is $G^C_1 \times G^C_2$-invariant. Since $M \times X^\#$ is also $G^C_1 \times G^C_2$-invariant, the set $(M \times X^\#)_{ss} = (M \times X^\#) \cap (M \times X)_{ss}$ is $G^C_1 \times G^C_2$-invariant as well. We conclude (5.3)

$$(M \times X^\#)_{ss} = ((G^C_1 \times G^C_2) \cdot \Phi^{-1}(0)) \cap (M \times X^\#).$$
The manifold \( M \times \{x^*\} \) is a cross-section for the action of \( G^C_d \) on \( M \times X^o \). Hence
\[
(M \times \{x^*\}) \cap (M \times X^o)_{ss} \simeq (M \times X^o)_{ss}/G^C_d.
\]
Thus
\[
(M \times X^o)_{ss}/G^C_d \simeq ((G^C_1 \times G^C_2) \cdot \Phi^{-1}(0)) \cap (M \times \{x^*\}).
\]
On the other hand,
\[
M^\# = \{ m \in M \mid \varphi(g_1 \cdot m) = \Psi(x) \text{ for some } g_1 \in G^C, x \in X^o \}
= \{ m \in M \mid \varphi(g_1 \cdot m) - \Psi(g_2 \cdot x^*) = 0 \text{ for some } (g_1, g_2) \in G^C_1 \times G^C_2 \}
= \{ m \in M \mid (G^C_1 \times G^C_2) \cdot (m, x^*) \cap \Phi^{-1}(0) \neq \emptyset \}
= \{ m \in M \mid (m, x^*) \in (G^C_1 \times G^C_2) \cdot \Phi^{-1}(0) \}
= ((G^C_1 \times G^C_2) \cdot \Phi^{-1}(0)) \cap (M \times \{x^*\}).
\]

We conclude with a few very simple examples of this construction.

**Example 1.** One may replace, in Example 3, § 4 above, \((\mathbb{C}, 2\sqrt{-1} \partial \bar{\partial} \log(1 + |z|^2))\) by the compact toric manifold \((\mathbb{CP}^1, \omega_{FS})\) and obtain the same cut space.

**Example 2.** We note that [3] has, in effect, a large number of examples of this construction, and relates them to the question of Grauert tubes of infinite radius. In [2], one has a distinguished Kähler structure on the tangent bundle \(TM\) of a real analytic Riemannian manifold \((M, g)\), usually acted upon by a compact group \(G\) of isometries. For every invariant metric on \(G\), one gets such a distinguished Kähler structure on \(TG\) as well, and [3] studies the reduction of the product \(TM \times TG\), which is the same as cutting \(TM\) by \(TG\). Indeed, even in the case \(G = S^1\), varying the symplectic form on \(TG\) by a positive scaling constant gives interesting new examples of Riemannian manifolds \((M, g')\) giving such a Kähler structure on \(TM\).

In fact, when \(G = S^1\), it is easy to see that the Kähler structure on \(TG\) is, in the notation of this section, \(X = \mathbb{C}^*\) with
\[
\omega_X = c \sqrt{-1} \frac{dz \wedge d\bar{z}}{|z|^2}, c > 0.
\]
In this case, cutting any \((M, \omega_M)\) by \((X, \omega_X)\) changes neither the topology nor the complex structure on \(M\), but only alters the resultant symplectic form \(\omega = \omega(c)\). This is because the moment map on \(X = \mathbb{C}^*\) is proper, and the action of \(G^C\) is free and proper on \(X\).

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M.I.T., Cambridge, MA 02139 and University of Michigan, Ann Arbor, MI 48109
E-mail address: dburns@umich.edu

M.I.T., Cambridge, MA 02139
E-mail address: vwg@math.mit.edu

University of Illinois, Urbana, IL 61801
E-mail address: lerman@math.uiuc.edu