Spin angular momentum of the electron: one-loop studies

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We combine bare perturbation theory with imaginary time evolution technique to study one-loop radiative corrections to spin angular momentum of the electron. We use for this purpose quantum electrodynamics in covariant and Coulomb gauges. We discuss intricacies of proper implementation of imaginary time evolution as well as the difference between results obtained with Pauli-Villars and three-dimensional cutoff regularizations.

I. INTRODUCTION

The electron, undoubtedly one of the most fundamental constituents of matter, is characterized by a set of physical properties such as the mass, charge, magnetic moment, and spin.

Experimental studies of its mass and charge, \( m \) and \( e \) below, started in the late nineteenth century in a series of experiments conducted by Thomson [1]. They have been successfully continued ever since. By contrast, progress in theoretical characterization of these parameters is rather uninspiring, if we notice that dimensionless quantities involving them—such as the fine structure constant or ratios of the electron mass to other lepton masses—have never been convincingly estimated.

The electron’s intrinsic magnetic moment was introduced by Uhlenbeck and Goudsmit [2] about a century ago in an attempt to explain the anomalous Zeeman effect, which was discovered by Preston at the same time Thomson conducted his electron experiments [3]. Its understanding rapidly progressed soon after thanks to Dirac [4], whose theory predicted

\[
\frac{e}{2m}
\]

for the electron’s magnetic moment. Two decades later [5], Schwinger found a more accurate approximation through a one-loop quantum electrodynamics (QED) calculation replacing (1) with

\[
\frac{e}{2m} \left( 1 + \frac{\alpha}{2\pi} \right),
\]

where

\[
\alpha = \frac{e^2}{4\pi}
\]

is the fine structure constant written here in the Heaviside-Lorentz system of units combined with \( \hbar = c = 1 \) (we use such units throughout this work). This prediction immediately explained spectroscopic “anomalies” found in measurements of Nafe and Nelson [6] and Foley and Kusch [7] that were done concurrently with Schwinger’s calculations. Ever since perturbative calculations of the electron’s magnetic moment have gone hand in hand with various experimental measurements reaching astonishing accuracy [8]. These efforts allowed for some of the most stringent tests of QED.

The electron’s spin was introduced together with its intrinsic magnetic moment in [2]. It was then put on a firm theoretical basis by Dirac [4], whose relativistic quantum mechanics leads to the following expression for the angular momentum operator [9]

\[
\frac{1}{2} \int d^3z :\bar{\psi} \Sigma^i \psi: - i \int d^3z :\bar{\psi} \left( z \times \nabla \right)^i \psi:,
\]

(4)
where $\psi$ is the Dirac field operator, $::$ denotes normal ordering,

$$\Sigma^i = \imath \varepsilon^{imn} \gamma^m \gamma^n / 2,$$

and $\gamma$ are Dirac matrices. The first (second) operator in (4) is the fermionic spin (orbital) angular momentum operator. Consider now the electron at rest, whose spin is polarized in the $\pm z$ direction. The expectation value of operator (4), in the corresponding quantum state $|\Psi\rangle$, is $s_z \delta^{i3}$, where

$$s_z = \pm \frac{1}{2}$$

reflects the fact that the electron’s spin equals one-half. The orbital component of the angular momentum operator does not contribute to such an expectation value and so one finds

$$\langle \frac{1}{2} \int d^3z :\psi^\dagger \Sigma^i \psi : \rangle_{\Psi} = s_z \delta^{i3}.$$  

We will refer to the left-hand side of this equation as spin angular momentum of the electron.

The situation is considerably more complex in QED, where the total angular momentum operator, besides fermionic components (4), contains also electromagnetic spin and orbital angular momentum operators. More importantly, expectation values of all these operators receive radiative corrections. It is the purpose of this work to compute such corrections to the right-hand side of (7).

Primary motivation for these studies comes from our interest in fundamental properties of the electron. The expectation value of the fermionic spin angular momentum operator should be in principle measurable due to gauge invariance of such an operator. As a result, it should provide one more insight into the nature of the electron just as, e.g., the electron’s magnetic moment does (2).

There are, of course, other reasons for pursuing this line of research. One of them is the angular momentum controversy [10] widely discussed in the context of the so-called nucleon spin crisis [10, 11]. While challenges related to nucleon’s angular momentum are much bigger on the technical level and quite different from the physical perspective than those encountered in the corresponding electron studies, it seems to us reasonable to make sure that QED calculations are properly done and understood in the first place.

Spin angular momentum of the electron was studied not long ago in [12, 13]. These calculations were done in the light-front formalism, employed the light-cone gauge, and used either Pauli-Villars or dimensional regularization. They are, on the technical level, very different from our studies as we use imaginary time evolution formalism, work in covariant and Coulomb gauges, and use either Pauli-Villars or three-dimensional (3D) cutoff regularizations. Therefore, we see our work as complementary to the previous efforts. Among other things, this paper presents some non-trivial results on implementation of imaginary time evolution, it supports gauge independence of the earlier results, and it raises some questions about regularization (in)dependence of QED calculations. Its outline is the following.

We explain in Sec. II the approach that we use to carry out computations. Next, we discuss in Sec. III different contributions to spin angular momentum of the electron. Then, in Sec. IV, we compute one-loop radiative corrections to this quantity in Pauli-Villars-regularized QED. After that, we show in Sec. V that a different result is obtained in the theory regularized with the 3D cutoff. The discussion of these findings is presented in Sec. VI. Several appendices are added to this paper to make its main body better readable and to facilitate verification of our results. We explain our notation in Appendix A and collect all bispinor matrix elements, which we use throughout this work, in Appendix B. The intricacies associated with implementation of imaginary time evolution are discussed in Appendix C, while adaptation of the Pauli-Villars regularization technique to our problem is presented in Appendix D.
II. BASICS

The starting point for our considerations is the QED Lagrangian density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi} (i \gamma^\mu \partial_\mu - m_o) \psi - e_o \bar{\psi} \gamma^\mu \gamma_5 A_\mu, \tag{8}$$

where the bare mass and charge of the electron are denoted by $m_o$ and $e_o$ and the remaining symbols follow all standard conventions.

The fermionic spin angular momentum operator, which we have already introduced in Sec. I, can be conveniently written as

$$J_{\text{spin}}^i = \int d^3 z \bar{\psi}(z) \Gamma^i \psi(z), \quad \Gamma^i = \frac{i}{4} \varepsilon^{imn} \gamma^0 \gamma^m \gamma^n. \tag{9}$$

This operator is gauge invariant. Therefore, we assume that its expectation value is independent of the gauge choice used in the computations (a brief discussion of subtle points associated with gauge independence can be found in Sec. 2.5.2. of [10]). Moreover, as the ground-state expectation value of this operator does not depend on time, we set

$$z = (0, z) \tag{10}$$

to simplify the discussion in the intermediate steps (as a self-consistency check, we have verified that $z^0$ drops out from the final result if it is not set to zero).

We will compute expectation value of (9) in the QED ground state with one net electron at rest, which we denote as $|\Omega s\rangle$. The term net refers to the fact that besides electrons in electron-positron pairs, there is one electron in such a state. For this purpose, we will use imaginary time evolution technique, which we will briefly summarize below.

It starts with the one-electron ground state of the free Hamiltonian

$$|0_s\rangle = a^\dagger_{0s} |0\rangle, \tag{11}$$

where $|0\rangle$ is the vacuum state of the free theory and the $a_{0s}$ operator is introduced in Appendix A. State (11) describes the zero-momentum electron in such a spin state $s$ that $\langle J_{\text{spin}}^s \rangle_{0s} = s_z \delta^3$. It is then evolved in time (its non-trivial dynamics is induced by the interaction Hamiltonian $\int d^3 x \mathcal{H}_{\text{int}}$). Enforcement of the imaginary time limit leads to [14]

$$\langle J_{\text{spin}}^s \rangle_{\Omega s} = \lim_{T \to \infty (1-i0)} \langle J_{\text{spin}}^s \rangle_{\Omega s}^T, \tag{12a}$$

$$\langle J_{\text{spin}}^s \rangle_{\Omega s}^T = \frac{\langle 0_s | T \mathcal{J}_{\text{spin}}^I \exp(-i \int_T d^4 x \mathcal{H}_{\text{int}}^I) |0_s \rangle_{\Omega s}}{\langle 0_s | T \exp(-i \int_T d^4 x \mathcal{H}_{\text{int}}^I) |0_s \rangle_{\Omega s}}, \tag{12b}$$

$$\int_T d^4 x = \int_T^\infty dx^0 \int d^3 x, \tag{12c}$$

where the interaction-picture operators are labeled with the index $I$, $T$ is the time-ordering operator, and the calculations will be performed with $T > 0$ until the limit

$$T \to \infty (1-i0) \tag{13}$$

will be taken. The important thing to remember in the following discussion is that total angular momentum in states $|0_s\rangle$ and $|\Omega s\rangle$ is the same and equals $s_z \delta^3$, which we will comprehensively discuss in [15].
To compute (12), one needs electromagnetic and fermionic propagators and the interaction Hamiltonian density, all expressed in terms of interaction-picture fields.

The fermionic propagator is gauge invariant and reads

$$S(x - y) = \langle 0 | \mathcal{T} \psi_I(x) \bar{\psi}_I(y) | 0 \rangle = i \int \frac{d^4p}{(2\pi)^4} \frac{\gamma p + m_o}{p^2 - m_o^2 + i0} e^{-ip(x - y)},$$

(14)

On the contrary, expressions for electromagnetic propagators and interaction Hamiltonian densities are gauge non-invariant. We will do calculations in the covariant and Coulomb gauges.

In the general covariant gauge, the gauge-fixing term

$$-\frac{\xi}{2} (\partial_\mu A^\mu)^2$$

(15)

is added to the Lagrangian density (the term general refers to the arbitrary non-zero value of $\xi$). Subsequent quantization of the theory leads to

$$D_{\mu\nu}(x - y) = \langle 0 | \mathcal{T} A^I_\mu(x) A^I_\nu(y) | 0 \rangle = -i \int \frac{d^4p}{(2\pi)^4} \frac{d_{\mu\nu}(p)}{p^2 + i0} e^{-ip(x - y)},$$

(16a)

$$d_{\mu\nu}(p) = \eta_{\mu\nu} + \frac{1 - \xi}{\xi} \frac{p_\mu p_\nu}{p^2 + i0},$$

(16b)

which is discussed in [9, 16]. Moreover,

$$\mathcal{H}^I_{\text{int}}(x) = e_o :\bar{\psi}_I(x)\gamma^\mu \psi_I(x) : A^I_\mu(x).$$

(17)

In the Coulomb gauge, the condition $\partial_i A^i = 0$ is imposed and one ends up with [9]

$$D^C_{ij}(x - y) = \langle 0 | \mathcal{T} A^I_i(x) A^I_j(y) | 0 \rangle = i \int \frac{d^4p}{(2\pi)^4} \frac{\Delta_{ij}(p)}{p^2 + i0} e^{-ip(x - y)},$$

(18a)

$$\Delta_{ij}(p) = \delta_{ij} - \frac{p_i p_j}{\omega^2_p},$$

(18b)

and

$$\mathcal{H}^I_{\text{int}}(x) = e_o :\bar{\psi}_I(x)\gamma^m \psi_I(x) : A^I_m(x) + \delta \mathcal{H}^I_{\text{int}}(x),$$

(19a)

$$\delta \mathcal{H}^I_{\text{int}}(x) = \frac{e^2}{2} \int d^3y \frac{\bar{\psi}_I(x)\gamma^0 \psi_I(y)\bar{\psi}_I(x^0, y)\gamma^0 \psi_I(x^0, y)}{4\pi |x - y|}.$$  

(19b)

Since we will be doing the perturbation theory around the one-electron state, we will be encountering the normalizing constant

$$V = \langle 0s | 0s \rangle = \int \frac{d^3x}{(2\pi)^3}. $$

(20)

While such a constant is formally infinite, it gets unambiguously cancelled during computations. This happens because all expressions that contribute to the final result describe processes that happen homogeneously in space. As a result, the outermost spatial integral in every such expression is done over a function that is constant in space and so it exactly cancels down normalizing constant (20) appearing in the denominator of such an expression. Needless to say, factors like (20) are frequently encountered in studies involving delocalized states (see e.g. above-cited [10]).
III. PERTURBATIVE EXPANSION

We will derive here an unregularized expression for spin angular momentum of the electron in the general covariant gauge. As will be shown in Secs. IV and V, it is then an easy exercise to see what form it takes in either the Pauli-Villars or 3D cutoff regularization. Moreover, such an expression will be also reused in Coulomb-gauge calculations. It is thus convenient to work initially without any particular regularization scheme.

To proceed, we use the bare perturbation theory expanding (12b) in the series in $e_o$

$$\langle J_{\text{spin}}^T \rangle_{\Omega_s} = \frac{\langle 0s | J_{\text{spin}} | 0s \rangle}{V} + \frac{1}{2!} \frac{\langle 0s | J_{\text{spin}} | 0s \rangle \int d^4x \mathcal{H}_{\text{int}}^i \int d^4y \mathcal{H}_{\text{int}}^i | 0s \rangle}{V} + O(e_o^4). \quad (21a)$$

$$- \frac{1}{2!} \frac{\langle 0s | J_{\text{spin}} | 0s \rangle \int d^4x \mathcal{H}_{\text{int}}^i \int d^4y \mathcal{H}_{\text{int}}^i | 0s \rangle}{V} + \frac{1}{2!} \frac{\langle 0s | J_{\text{spin}} | 0s \rangle \int d^4x \mathcal{H}_{\text{int}}^i \int d^4y \mathcal{H}_{\text{int}}^i | 0s \rangle}{V} + O(e_o^4). \quad (21b)$$

In (21c) the analytical expressions will be worked out.

Zeroth-order contribution (21a) is illustrated in Fig. 1. We obtain after using (A4) and (A5)

$$\frac{\langle 0s | J_{\text{spin}}^i | 0s \rangle}{V} = \frac{\pi_s \Gamma^i u_s}{V} \int \frac{d^3z}{(2\pi)^3} = s_z \delta^3, \quad (22)$$

which has been already mentioned below (11).

Since we will be discussing diagrams depicted in different figures, we introduce the following notation. The unregularized contribution of the diagram from Fig. X to $\langle J_{\text{spin}}^i \rangle_{\Omega_s}$ will be referred to as Diag. X. For example, a trivial illustration of this notation is

$$\text{Diag. 1} = \frac{\langle 0s | J_{\text{spin}}^i | 0s \rangle}{V}. \quad (23)$$

The rules for drawing position-space Feynman diagrams, which we present in Figs. 1–3, can be deduced without much effort by comparing those diagrams to the analytical expressions that we list for them. There is no need to dwell on these rules because all diagrams will be drawn only after the analytical expressions will be worked out.

To compute (21b), we need the following matrix element that can be obtained through Wick’s theorem in combination with (14) and (A4)

$$\langle 0s | \Gamma : \bar{\psi}_I(z) \Gamma^i \psi_I(z) \Gamma^j \psi_I(x) : \bar{\psi}_I(y) \gamma^\mu \psi_I(y) : | 0s \rangle =$$

$$\frac{e^{i\theta(x-y)}}{(2\pi)^3} u_s \gamma^\mu S(x - y) \Gamma^i S(z - y) \gamma^\nu u_s \quad (24a)$$

$$+ \frac{e^{i\theta(z-y)}}{(2\pi)^3} u_s \gamma^\mu S(z - x) \gamma^\nu S(x - y) \Gamma^i u_s \quad (24b)$$

$$+ \frac{e^{i\theta(x-z)}}{(2\pi)^3} u_s \gamma^\mu S(x - y) \gamma^\nu S(y - z) \Gamma^i u_s \quad (24c)$$

$$- \frac{1}{2(2\pi)^3} \text{Tr} [S(y - x) \gamma^\mu S(x - y) \gamma^\nu] u_s \Gamma^i u_s \quad (24d)$$

$$- \frac{1}{(2\pi)^3} \text{Tr} [S(y - z) \Gamma^i S(z - y) \gamma^\nu] u_s \gamma^\mu u_s \quad (24e)$$

$$- V \text{Tr} [S(x - z) \Gamma^i S(z - y) \gamma^\nu S(y - x) \gamma^\mu] \quad (24f)$$

$$+(x, \mu \leftrightarrow y, \nu \text{ on all terms}), \quad (24g)$$
where
\[ f = (m_0, 0) \]  
(25)
comes from contractions on external lines. Matrix element (24) can be additionally simplified with (10) and (25) leading to \( e^{i \hat{J}_z} = 1 \). Its contractions with the photon propagator are diagrammatically depicted in Fig. 2.

To compute (21c), we proceed similarly as in (24) getting

\[
\langle 0_s | : \bar{\psi}_I(z) \Gamma^i \psi_I(z) : | 0_s \rangle = \frac{e^{i f(z-y) \frac{m_0}{2} \gamma^\mu \bar{\Gamma}_s \gamma^\nu \bar{\psi}_I(z) \psi_I(z) : | 0_s \rangle}{(2\pi)^6 \frac{m_0}{2} \gamma^\mu \bar{\Gamma}_s \gamma^\nu \bar{\psi}_I(z) \psi_I(z) : | 0_s \rangle} (-1)^2 \frac{(m_0)^2}{2(2\pi)^3} \text{Tr} [S(y-x) \gamma^\mu S(x-y) \gamma^\nu \bar{\Gamma}_s \Gamma^i \bar{\psi}_I(x) \psi_I(x) : | 0_s \rangle + (x, \mu \leftrightarrow y, \nu \text{ on all terms}), \]
(26a)

whose contractions with the photon propagator are diagrammatically shown in Fig. 3. Replacements (24g) and (26c) produce a factor of 2 during evaluation of the diagrams, which cancels 1/2! prefactors from (21b) and (21c).

To correctly compute contributions of different diagrams to spin angular momentum of the electron, one must properly enforce limit (13). To illustrate the subtle point here, we note that integration over time leads to expressions of the form

\[
\int_{-T}^{T} dx \int\frac{dx}{2\pi} e^{i x^0 P_0} = \frac{\sin(T P_0)}{\pi P_0}.
\]
(27)

Limit (13) cannot be taken on (27). The standard textbook solution of this complication is to transfer the \(-i0\) from the limit to the imaginary part of propagators’ denominators. After that, the limit \( T \rightarrow \infty \) is taken. This leads to the Dirac delta function due to the following well-known identity

\[
\delta(P_0) = \lim_{T \rightarrow \infty} \frac{\sin(T P_0)}{\pi P_0}.
\]
(28)

Such a procedure, greatly simplifying calculations, leads to incorrect results when Diags. 2b, 2c, and 3a are considered. To overcome this difficulty, we will simplify expressions containing (27) up to the point, where limit (13) can be taken. Key technical results regarding this procedure are explained in Appendix C. They are used in this section under tacit assumption that some infrared regularization will be implemented later on to facilitate enforcement of imaginary time limit (13).

Moreover, to make equations a bit more compact, we introduce the following notation

\[
\text{Diag. } X = \lim_{T \rightarrow \infty} \text{Diag. } X|_T, \quad \tilde{q} = (q^0, \mathbf{p}), \quad \tilde{k} = (k^0, -\mathbf{p}).
\]
(29)

(30)
FIG. 2: The (a)–(f) panels illustrate photon-propagator contractions with expressions (24a)–(24f), respectively.

Diagram 3a. We start computation of diagrams with

\[
\text{Diag. } 3a \bigg|_T = \frac{e_o^2}{V^2} \int \frac{d^3z}{(2\pi)^3} \Gamma^i u_s \int \frac{d^4x}{(2\pi)^4} \frac{e^{ij(x-y)}(x-y)\gamma^\mu S(x-y)\gamma^\nu u_s}{(2\pi)^3} D_{\mu\nu}(x-y) \bar{\pi}_s \gamma^\mu S(x-y) \gamma^\nu u_s
\]

\[
= \frac{e_o^2 s_z \delta^{i3}}{(2\pi)^3 V} \int_0^\infty d^4p \frac{d^4k}{(2\pi)^4 (2\pi)^4} \frac{e^{i(x-y)(f-k-p)} \bar{\pi}_s \gamma^\mu \gamma(\gamma p + m_o) \gamma^\nu u_s}{k^2 + i0} \frac{p^2 - m_o^2 + i0}{p^2 - m_o^2 + i0}
\]

\[
= 2e_o^2 s_z \delta^{i3} \int \frac{d^3p}{(2\pi)^4} dk^0 F(k^0, p^0) \frac{\sin^2[T(k^0 + p^0 - m_o)]}{(k^0 + p^0 - m_o)^2}
\]

where

\[
F(k^0, p^0) = \frac{2}{\pi} \frac{2m_o - p^0}{(k^2 + i0)(p^2 - m_o^2 + i0)} + \frac{1}{\pi \xi} \frac{2k^0 \bar{k}p + \bar{k}^2(m_o - p^0)}{(k^2 + i0)(p^2 - m_o^2 + i0)^2},
\]

and we have employed identities (B1) and (B2). Note that we only list those arguments of the
function $F$ that are most relevant for enforcement of the imaginary time limit. Using (C8), we get

$$\text{Diag. 3a} = 2\pi e_\sigma^2 s_\sigma^2 \delta^{33} \int \frac{d^4p}{(2\pi)^4} F(m_o - p^0, p^0) \lim_{T \to \infty (1-i\delta)} T$$

$$+ \frac{e_\sigma^2 s_\sigma^2 \delta^{33}}{2} \int \frac{d^4p}{(2\pi)^4} dk^0 \left[ \frac{F(k^0, p^0)}{(k^0 + p^0 - m_o + i0)^2} + \frac{F(k^0, p^0)}{(k^0 + p^0 - m_o - i0)^2} \right].$$

(33a)

(33b)

We mention in passing that term (33b) cannot be obtained without careful treatment of factors like (27); see the discussion in Appendix C. The procedure described between (27) and (28) produces only ill-defined term (33a).

**Diagrams 2b and 2c.** Now, we compute

$$\text{Diag. 2b}|_T = -\frac{e_\sigma^2}{V} \int T d^4x d^4y \int \frac{d^3z}{(2\pi)^3} D_{\mu\nu}(x-y) e^{i f(z-y)} \overline{u}_s \Gamma S(z-x) \gamma^\mu S(x-y) \gamma^\nu u_s$$

$$= -\frac{ie_\sigma^2}{V} \int T d^4x d^4y \int \frac{d^3z}{(2\pi)^3} \int \frac{d^4k}{(2\pi)^4} \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4}$$

$$\frac{e^{ix(q-k-p)+iy(k+p-f)+iz(f-q)}}{k^2 + i0}$$

$$\overline{u}_s \Gamma^i(\gamma q + m_o) \gamma^\mu(\gamma p + m_o) \gamma^\nu u_s$$

$$\frac{(q^2 - m_o^2 + i0)(p^2 - m_o^2 + i0)}{(q^2 - m_o^2 + i0)(p^2 - m_o^2 + i0)} d_{\mu\nu}(k).$$

(34)

Before moving on, we note that the procedure outlined between (27) and (28) leads to $\delta(f-q)$ producing the meaningless factor of $1/0$ in the above expression due to (25).

Employing (B3)–(B4), (34) can be written as

$$\text{Diag. 2b}|_T = -2ie_\sigma^2 s_\sigma^3 \int \frac{d^4p}{(2\pi)^4} \frac{d^4q}{(2\pi)^4} \frac{F(k^0, p^0)}{2\pi} \frac{\sin[T(k^0 + p^0 - m_o)]}{q^0 - m_o + i0} \frac{\sin[T(k^0 + p^0 - m_o)]}{k^0 + p^0 - m_o}.$$
follows now straightforwardly as through formal manipulations one can show that

\[ \text{Diag. 2c} = \text{Diag. 2b} \]

if (10) holds.

**Diagram 2a.** We compute here

\[
\text{Diag. 2a}_1 = -\frac{e^2}{V} \int d^4x \, d^4y \int \frac{d^3z}{(2\pi)^3} D_{\mu\nu}(x-y) e^{i\xi(x-y)} \gamma^\mu \gamma^\nu \gamma^\nu \gamma^\nu u_s
\]

\[
= -\frac{ie^2}{V} \int d^4x \, d^4y \int \frac{d^3z}{(2\pi)^3} \frac{d^4k \, d^4p \, d^4q}{(2\pi)^4 (2\pi)^4} \frac{e^{i\xi(f(k-p)+ip(k-q)+f)} + i0}{k^2 + i0}
\]

\[
\times \frac{\gamma^\mu (\gamma^\mu + m_0) \Gamma^4 (\gamma^\nu + m_0) \gamma^\nu \gamma^\nu \gamma^\nu \gamma^\nu u_s}{(p^2 - m_0^2 + i0)(q^2 - m_0^2 + i0)}
\]

\[
d_{\mu\nu}(k)
\]

With the help of (B5), (B6), and (C21) we arrive at

\[
\text{Diag. 2a} = -ie^2 \int \frac{d^4p}{(2\pi)^4} \left[ \frac{2\delta^{i3}(p^2 + m_0^2)}{(p^2 - m_0^2 + i0)^2} \right] + \frac{1 - \xi}{\xi} \frac{\delta^{i3}}{[(p - f)^2 + i0]^2}
\]

We mention in passing that the procedure discussed between (27) and (28) gives a correct result here.

There are no other one-loop contributions to spin angular momentum of the electron in covariantly quantized QED. Indeed, the disconnected vacuum bubble Diags. 2d and 3b immediately cancel out due to the difference in overall signs of (21b) and (21c). Therefore, there is no need to write down expressions for them. Moreover,

\[
\text{Diag. 2e} = \lim_{T \to \infty(1 - i0)} \frac{e^2}{V} \int d^4x \, d^4y \int \frac{d^3z}{(2\pi)^3} D_{\mu\nu}(x-y) \text{Tr} \left[ \gamma^\mu \gamma^\nu \gamma^\nu \gamma^\nu u_s \right]
\]

and

\[
\text{Diag. 2f} = \lim_{T \to \infty(1 - i0)} \frac{e^2}{V} \int d^4x \, d^4y \int d^3z D_{\mu\nu}(x-y) \text{Tr} \left[ \gamma^\mu \gamma^\nu \gamma^\nu \gamma^\nu S(x-z) \gamma^\nu S(y-x) \gamma^\nu \right]
\]

both vanish. We will skip the discussion of these two not-so-interesting computations as they can be straightforwardly carried out along the lines of other calculations from this work.

The final unregularized result for spin angular momentum of the electron comes from Diags. 1, 2a–2c and 3a

\[
\langle J_{\text{spin}}^i \rangle_\pi = \text{Diag. 1} + \text{Diag. 2a} + \text{Diag. 2b} + \text{Diag. 2c} + \text{Diag. 3a} + O(e_0^4),
\]

which can be obtained by adding (22) and (40) to

\[
\text{Diag. 2b} + \text{Diag. 2c} + \text{Diag. 3a} = -2ie_0^2 s_0 \delta^{i3} \int \frac{d^4p}{(2\pi)^4} \left[ \frac{2(p^0 - m_0)(2m_0 - p^0)}{(p^2 - m_0^2 + i0)[(p - f)^2 + i0]^2} \right] + \frac{1 - \xi}{\xi} \frac{\delta^{i3}}{[(p - f)^2 + i0]^2} + \frac{\omega_p^2}{[(p - f)^2 + i0]^2}.
\]

Note that the offending, linearly divergent in \( T \), terms cancel out in (44).

It is now important to stress that the resulting expression for spin angular momentum of the electron does not have a definite value until some regularization scheme is specified. We will illustrate this fact by showing that distinct finite values for (43) can be obtained in Pauli-Villars (Secs. IV) and 3D cutoff (Sec. V) regularizations.
IV. PAULI-VILLARS REGULARIZATION

We will determine in this section spin angular momentum of the electron in the Pauli-Villars-regularized theory. Application of this regularization to our problem is discussed in Appendix D. The small photon mass $\lambda$ is introduced there to regulate the infrared (IR) sector of the calculation while the large mass $\Lambda$ of Pauli-Villars ghost particles is used to take care of the ultraviolet (UV) part of the problem

$$\lambda \ll m_o, \; \Lambda \gg m_o.$$  \hfill (45)

The calculations will be done in the Feynman gauge

$$\xi = 1,$$  \hfill (46)

which is something one must keep in mind when comparing the following expressions to the ones from Sec. III. Such a covariant gauge choice simplifies the discussion a bit (arbitrary-$\xi$ studies will be presented in the next section).

Next, we introduce

$$\text{Diag. } X|_{\lambda \Lambda}$$  \hfill (47)

as the Pauli-Villars-regularized version of unregularized Diag. X from Sec. III. Note that limit (13) is already taken in (47).

It now follows from the discussion in Sec. III and Appendix D that

$$\text{Diag. } 2a|_{\lambda \Lambda} = -ie_o^2s_z \int \frac{d^4p}{(2\pi)^4} \frac{2\delta^3 (p^2 + m_o^2) + 4p_ip_i}{(p^2 - m_o^2 + i0)^2} \left[ \frac{1}{(p - f)^2 - \lambda^2 + i0} - (\lambda \to \Lambda) \right].$$  \hfill (48)

Moreover, we obtain

$$\text{Diag. } 2b|_{\lambda \Lambda} + \text{Diag. } 2c|_{\lambda \Lambda} + \text{Diag. } 3a|_{\lambda \Lambda}$$

$$= -4ie_o^2s_z \delta^3 \int \frac{d^4p}{(2\pi)^4} \frac{(p^0 - m_o)(2m_o - p^0)}{p^2 - m_o^2 + i0} \left[ \frac{1}{[(p - f)^2 - \lambda^2 + i0]^2} - (\lambda \to \Lambda) \right].$$  \hfill (49)

Note that one does not end up here with

$$\left[ \frac{1}{(p - f)^2 - \lambda^2 + i0} - (\lambda \to \Lambda) \right]^2$$

under the integral. Besides trivial (22), these are the only contributions to the final result. They can be compared to (40) and (44).

We can now cast these expressions into a familiar form using the standard procedure. To this aim, we introduce

$$\Delta \chi = (1 - s)^2 + s\left( \frac{\chi}{m_o} \right)^2,$$  \hfill (51)

employ

$$\frac{1}{AB^2} = \int_0^1 da \, db \, (a + b - 1) \frac{2b}{(aA + bB)^3},$$  \hfill (52)
and make use of the Wick rotation technique to get
\[
\text{Diag. } 2a|_{\Lambda} = \frac{e_0^2 s z \delta^{i3}}{8\pi^2} \int_0^1 ds (1 - s) \left[ \ln \frac{\Delta_{\Lambda}}{\Delta_{\lambda}} + (1 + s^2) \left( \frac{1}{\Delta_{\Lambda}} - \frac{1}{\Delta_{\lambda}} \right) \right], \tag{53}
\]
and
\[
\text{Diag. } 2b|_{\Lambda} + \text{Diag. } 2c|_{\Lambda} + \text{Diag. } 3a|_{\Lambda} = \frac{e_0^2 s z \delta^{i3}}{8\pi^2} \int_0^1 ds \left[ s \ln \frac{\Delta_{\Lambda}}{\Delta_{\lambda}} + 2(2 - s)(1 - s) \left( \frac{1}{\Delta_{\Lambda}} - \frac{1}{\Delta_{\lambda}} \right) \right]. \tag{54}
\]

We start discussing these expressions by noting that it can be now easily verified through \[14\] that (54) is equal to \(s z \delta^{i3}(Z_2 - 1)\), where \(Z_2\) is defined in any textbook on QED.

Next, we take the limits of \(\lambda \to 0\) and \(\Lambda \to \infty\) on the sum of (22), (53), and (54), which leads to
\[
\langle J_{\text{spin}} \rangle_{\Omega s} = s z \delta^{i3} \left( 1 - e_0^2 \frac{8}{\pi^2} \right) + O(e_0^4) = s z \delta^{i3} \left( 1 - \frac{\alpha}{2\pi} \right) + O(\alpha^2), \tag{59}
\]
where the right-most value comes from
\[
e_0 = e + O(e^3) \tag{60}
\]
replacing the bare electron charge by the physical one. Two remarks are in order now.

First, we would like to stress that (59) can be obtained in the bare perturbation theory only after careful implementation of imaginary time evolution (Appendix C).

Second, the same result was derived not long ago in a very different way in \[12, 13\].
V. 3D CUTOFF REGULARIZATION

We will compute in this section spin angular momentum of the electron in the 3D cutoff regularization. The calculations will be done in the general covariant gauge in Sec. VA and in the Coulomb gauge in Sec. VB.

The IR cutoff $\lambda_c$ and the UV cutoff $\Lambda_c$ will be imposed on 3-momenta in either electromagnetic or fermionic propagators. We have the following options for replacements in expressions for propagators

$$d^4p \rightarrow [d^4p] = dp^0 [d^3p] = dp^0 \theta(\omega_p - \lambda_c),$$  
(61)

$$d^4p \rightarrow [d^4p] = dp^0 [d^3p] = dp^0 \theta(\Lambda_c - \omega_p),$$  
(62)

$$d^4p \rightarrow [d^4p] = dp^0 [d^3p] = dp^0 \theta(\Lambda_c - \omega_p) \theta(\omega_p - \lambda_c),$$  
(63)

where $\theta$ is the Heaviside step function.

Replacements (61) and (63) will not be applied to the fermionic propagator. It is so because they exclude the zero-momentum fermionic mode, which is crucially important in our calculations (imaginary time evolution starts with the electron at rest). As can be easily checked, the IR cutoff imposed on the fermionic propagator would lead to vanishing contributions from Diags. 2b and 2c, which would have disastrous consequences for the whole calculation. Therefore, the IR cutoff will be imposed on the electromagnetic propagator.

We adopt below the notation from Sec. IV, so that the cutoff-regularized version of unregularized Diag. X from Sec. III will be denoted as

$$\text{Diag. } X|_{\lambda_c, \Lambda_c}.$$  
(64)

The regularization will be eventually removed by taking the limits

$$\lambda_c \to 0, \quad \Lambda_c \to \infty.$$  
(65)

A. Covariant gauge

The following calculations are regularized by imposing IR and UV cutoffs on the electromagnetic propagator and keeping the fermionic propagator unregularized (the additional UV cutoff applied to the fermionic propagator would lead to the same final result, but it is unnecessary). Alternatively, one may impose the IR (UV) cutoff on the electromagnetic (fermionic) propagator. Both choices lead to the same results.

As can be quickly checked by going through the calculations in Sec. III, the cutoff-regularized versions of (40) and (44) are obtained by replacing $d^4p$ with $[d^4p]$ in these expressions. Integrating them over $p^0$, we end up with

$$\text{Diag. } 2a|_{\lambda_c, \Lambda_c} = \frac{e^2 s_\sigma \delta^{i3}}{2} \int \frac{[d^3p]}{(2\pi)^3} \frac{1}{\omega_p^3} \left[ \frac{\varepsilon_p (\omega_p - \varepsilon_p)}{m_0^2} + \frac{(p_1)^2 + (p_2)^2}{m_0^2} \left( 1 - \frac{\omega_p^2}{\varepsilon_p^2} \right) + \frac{1 - \xi}{2\xi} \right],$$  
(66)

and

$$\text{Diag. } 2b|_{\lambda_c, \Lambda_c} + \text{Diag. } 2c|_{\lambda_c, \Lambda_c} + \text{Diag. } 3a|_{\lambda_c, \Lambda_c}$$

$$= \frac{e^2 s_\sigma \delta^{i3}}{2} \int \frac{[d^3p]}{(2\pi)^3} \frac{1}{\omega_p^3} \left[ \left( 1 - \frac{\omega_p^2}{m_0^2} \right) \left( 1 - \frac{\omega_p}{\varepsilon_p} \right) - \frac{1 - \xi}{2\xi} \right].$$  
(67)
Using these results, we get upon taking (65)

\[ \langle J_{\text{spin}}^i \rangle_{\Omega s} = s_z \delta^{i3} \left( 1 - \frac{e_o^2}{12\pi^2} \right) + O(e_o^4) = s_z \delta^{i3} \left( 1 - \frac{\alpha}{3\pi} \right) + O(\alpha^2), \tag{68} \]

where again (60) has been used. Quite importantly, this result is \( \xi \)-independent. Several remarks are in order now.

First, after adding (66) and (67) the resulting integral is IR finite and so \( \lambda_c \) can be set to zero then. (68) is obtained after doing the angular and radial integrations first and then taking the limit \( \Lambda_c \rightarrow \infty \). Alternatively, one may start with angular integrations and then carry out the radial integration with \( \Lambda_c = \infty \). The resulting expression, after rescaling the frequency \( \omega_p \) by \( m_o \), reads

\[ \langle J_{\text{spin}}^i \rangle_{\Omega s} = s_z \delta^{i3} - \frac{e_o^2}{6\pi^2} s_z \delta^{i3} \int_0^\infty dx \frac{2\pi(1 + x^2)^3/2 - 3x^2 - 2x^4}{(1 + x^2)^{3/2}} + O(e_o^4). \tag{69} \]

It can be easily computed after the following change of variables

\[ x + \sqrt{1 + x^2} = \frac{1}{\sqrt{y}} \tag{70} \]

leading to

\[ \langle J_{\text{spin}}^i \rangle_{\Omega s} = s_z \delta^{i3} - s_z \delta^{i3} \frac{e_o^2}{12\pi^2} \int_0^1 dy \left[ \frac{4}{(y + 1)^2} - 1 \right] + O(e_o^4), \tag{71} \]

where the integral is trivially equal to one. As a curiosity, we mention that substitution (70) can be found in the Ramanujan’s first quarterly report prepared for the University of Madras [17]. Naturally, it has been invoked in other contexts as well (see e.g. Sec. 2.25 of [18]). We find it quite useful in our studies of angular momentum of the electron.

Second, unless \( \xi \) is fine-tuned, (66) and (67) are (without regularization) logarithmically divergent in both UV and IR. For \( \xi = \infty \), the Landau gauge, these expressions are still IR divergent but UV finite. Moreover, after removing the terms proportional to \( \lim_{T \rightarrow \infty} (1 - i0)^T \), all individual diagrams are UV finite in such a gauge. For \( \xi = 1/3 \), the Fried-Yennie gauge, (66) and (67) are IR finite but UV divergent, which reflects the well-known fact that QED in such a gauge has significantly improved IR properties with respect to other covariant gauge choices. In fact, for \( \xi = 1/3 \) all individual diagrams are IR finite (the same can be said about integrands in these diagrams).

Last but not least, (68) does not agree with (59). It is thus reasonable to double check it by doing a calculation in a different gauge.

### B. Coulomb gauge

We will impose here the IR cutoff on the electromagnetic propagator and the UV one on the fermionic propagator. The following calculations will be done in two independent steps because the two terms in Hamiltonian density (19) will never come together in the one-loop perturbative expansion. So, we write

\[ \langle J_{\text{spin}}^i \rangle_{\Omega s} = s_z \delta^{i3} + S^i_\perp + S^i_\parallel + O(e_o^4), \tag{72} \]

where \( S_\perp \) and \( S_\parallel \) will be called the transverse and longitudinal contributions to \( \langle J_{\text{spin}} \rangle_{\Omega s} \).

To obtain the regularized expression for \( S_\perp \), we need to replace \( \mathcal{H}_{\text{int}}^I(x) \) in (21b) and (21c) by \( e_o : \overline{\psi}_I(x)\gamma^m \psi_I(x) : A^I_m(x) \), compute resulting expressions with regularized propagators, and take limit (13). All these steps can be easily accomplished with results from Sec. III.
On the other hand, a new calculation has to be done for $S_\parallel$, which after regularization reads

$$S_\parallel|_{\Lambda_c} = \lim_{T \to \infty (1-i0)} S_\parallel|_{T\Lambda_c},$$

where

$$S_\parallel|_{T\Lambda_c} = -\frac{i}{V} \int_T d^4x \langle 0s | \Gamma(J^I_{\text{spin}}) \delta \mathcal{H}_{\text{int}}^I(x) | 0s \rangle$$

is computed with the UV-modified expression for the fermionic propagator. Note that such an expression does not depend on the electromagnetic propagator and so it does not rely on the IR cutoff $\lambda_c$. Lack of IR regularization does not cause problems here because the longitudinal contribution is IR finite and imaginary time evolution does not need the IR cutoff for its evaluation (Appendix C).

**Transverse contribution.** The Coulomb-gauge results are obtained by replacing covariant-gauge electromagnetic propagator (16) with Coulomb-gauge propagator (18) in all equations in Sec. III. For example, the Coulomb-gauge version of Diag. 3a is

$$\lim_{T \to \infty (1-i0)} \frac{e_0^2}{V^2} \int_{T} \frac{d^4z}{(2\pi)^4} \bar{u}_s \Gamma^i u_s \int_{T} d^4x d^4y \frac{e^{i f(x-y)}}{(2\pi)^3} D^C_{ab}(x-y) \gamma^a S(x-y) \gamma^b u_s.$$  (75)

If we now use cutoff-modified propagators to compute such an expression, we will get Diag. 3a$^{\text{Coulomb}}_{\Lambda_c\Lambda_c}$ (the similar notation will be used below).

With the help of matrix elements (B7) and (B8), it is then easy to show that

$$\text{Diag. 2b, 2c, 3a}^{\text{Coulomb}}_{\Lambda_c\Lambda_c} = \text{Diag. 2b, 2c, 3a with } F \to \mathcal{F} \text{ and } d^4p \to [d^4p],$$

$$\mathcal{F}(k^0, p^0) = \frac{2}{\pi (k^2 + i0)(p^2 - m_o^2 + i0)},$$

where $k$ is given by (30) and (76b) can be compared to (32). This simple modification of covariant gauge results leads to

$$\text{Diag. 2b}^{\text{Coulomb}}_{\Lambda_c\Lambda_c} + \text{Diag. 2c}^{\text{Coulomb}}_{\Lambda_c\Lambda_c} + \text{Diag. 3a}^{\text{Coulomb}}_{\Lambda_c\Lambda_c} = 4ie_0^2 \delta^{ij} \int \frac{[d^4p]}{(2\pi)^4} \frac{(p^0 - m_o)^2}{(p - f)^2 + i0} \frac{m_o}{p^2 - m_o^2 + i0}.$$  (77)

Moreover, using (B9), we find that

$$\text{Diag. 2a}^{\text{Coulomb}}_{\Lambda_c\Lambda_c} = -2ie_0^2 \delta^{ij} \int \frac{[d^4p]}{(2\pi)^4} \frac{p_ip_3}{\omega_p^2} \frac{(p^0 - m_o)^2}{(p - f)^2 + i0}.$$  (78)

As other diagrams do not contribute to $S_\perp$, it is given in the regularized form by

$$S_\perp|_{\Lambda_c\Lambda_c} = \text{Diag. 2a}^{\text{Coulomb}}_{\Lambda_c\Lambda_c} + \text{Diag. 2b}^{\text{Coulomb}}_{\Lambda_c\Lambda_c} + \text{Diag. 2c}^{\text{Coulomb}}_{\Lambda_c\Lambda_c} + \text{Diag. 3a}^{\text{Coulomb}}_{\Lambda_c\Lambda_c}.$$  (79)

After simple calculations, we get

$$S_\perp|_{\Lambda_c\Lambda_c} = e_0^2 \delta^{ij} \int \frac{[d^4p]}{(2\pi)^4} \frac{1}{m_o^2} \left( \frac{1}{\omega_p^2} - \frac{(p_1)^2 + (p_2)^2}{2} \left( \frac{1}{\omega_p^2} - \frac{1}{\omega_p^2面临} \right) \right),$$

which is UV divergent but IR finite when (65) is enforced. In fact, all individual diagrams in (79) are IR finite after removing regularization (integrand expressions defining them also have such
a property). This is rather unsurprising as many other examples of IR finite QED calculations in the Coulomb gauge can be found in literature (see e.g. [19, 20]).

**Longitudinal contribution.** We evaluate now $S^i_t\big|_{\Lambda_c}$ starting with

$$S^i_t\big|_{\Lambda_c} = \frac{i e^2}{2 V} \int d^4 x \int d^3 z \frac{\langle 0 s | :T \bar{\psi}_I(z) \Gamma^i \psi_I(\bar{z}) : \bar{\psi}_I(x) \gamma^0 \psi_I(x) \bar{\psi}_I(y) \gamma^0 \psi_I(y) : | 0 s \rangle}{4 \pi |x - y|},$$

where $y = (x^0, y)$ and

$$\langle 0 s | :T \bar{\psi}_I(z) \Gamma^i \psi_I(\bar{z}) : \bar{\psi}_I(x) \gamma^0 \psi_I(x) \bar{\psi}_I(y) \gamma^0 \psi_I(y) : | 0 s \rangle = \frac{\delta^{ij}(x - y)}{(2\pi)^3} \bar{u}_s \gamma^0 S(x - z) \Gamma^i S(z - y) \gamma^0 u_s$$

$$- \frac{1}{(2\pi)^3} \text{Tr} [S(y - z) \Gamma^i S(z - y) \gamma^0] \bar{\pi}_s \gamma^0 u_s$$

$$+ (x \leftrightarrow y \text{ on all terms})$$

with cutoff (62) imposed on the fermionic propagator. (82) is trivially obtained from (24).

Spin-independent term (82b) does not contribute to (81). This can be shown, by proceeding with the calculations similarly as below, with

$$\text{Tr} [(\gamma p + m_o) \Gamma^i (\gamma q + m_o) \gamma^0] = 0,$$

where $\bar{q}$ is defined in (30).

Thus, we are left with

$$S^i_t\big|_{\Lambda_c} = \frac{i e^2}{V} \int d^4 x \int d^3 z d^3 y \frac{(d^4 p) [d^4 q]}{(2\pi)^4} \frac{e^{i (f \cdot p) + i (q \cdot f) + i (p \cdot q)}}{4 \pi |x - y|} \cdot \bar{\pi}_s \gamma^0 (\gamma p + m_o) \Gamma^i (\gamma q + m_o) \gamma^0 u_s$$

$$\frac{1}{(p^2 - m_o^2 + i0)(q^2 - m_o^2 + i0)}.$$ (84)

To integrate out the Coulomb potential, we need the standard trick to justify commutation of spatial and momentum integrals. Namely,

$$\int d^3 y \int d^3 p \frac{e^{ip(x - y)}}{4 \pi |x - y|} = \lim_{\epsilon \rightarrow 0^+} \int d^3 p \cdot \int d^3 y e^{-\epsilon |x - y|^2} \frac{e^{ip(x - y)}}{4 \pi |x - y|} = \lim_{\epsilon \rightarrow 0^+} \int d^3 p \frac{\cdot}{\omega_p + \epsilon^2},$$

(85)

where the placeholder stands for the $p$-dependent function in our calculations. Two remarks are in order now.

First, the above $\epsilon$-regularization can be used to justify commutation of spatial and momentum integrals that we routinely do in all our calculations integrating out the exponential terms to get the Dirac delta functions. This simplifies expressions and takes care of 3-momentum conservation in every vertex. Second, the final result of these calculations is the same if we set $\epsilon = 0$ in the last expression in (85), which we do below to simplify equations. Such a conclusion follows from the fact that $S^i_t$ is IR finite with $\epsilon = 0$.

Moving on, we arrive after simple manipulations at

$$S^i_t\big|_{\Lambda_c} = 2 i e^2 \int \frac{(d^4 p)}{(2\pi)^4} \frac{1}{\omega_p} \frac{\bar{\pi}_s \gamma^0 (\gamma p + m_o) \Gamma^i (\gamma q + m_o) \gamma^0 u_s \sin [T(p^0 - q^0)]}{(p^2 - m_o^2 + i0)(q^2 - m_o^2 + i0)} \frac{1}{p^0 - q^0}.$$ (86)

We get with the help of (B10) and (C25)

$$S^i_t\big|_{\Lambda_c} = i e^2 s_3 \int \frac{(d^4 p)}{(2\pi)^4} \frac{\delta^{ij}(p + f)^2 + 2 p_i p_j}{\omega_p (p^2 - m_o^2 + i0)^2} = e^2 s_3 \delta^{ij} \int \frac{(d^4 p)}{(2\pi)^4} \frac{(p_1)^2 + (p_2)^2}{\omega_p (p^2 - m_o^2 + i0)^2}.$$ (87)
It is now easy to see that the sum of (80) and (87) is equal to the sum of (66) and (67) after taking limit (65). This means that we have obtained in the 3D cutoff regularization the same result for spin angular momentum of the electron in Coulomb and general covariant gauges.

VI. DISCUSSION

We have teamed bare perturbative expansion with the imaginary time evolution technique to study radiative corrections to spin angular momentum of the electron. This required careful implementation of the latter procedure, which we have comprehensively discussed in Appendix C containing results that can be useful in other studies as well.

Using Pauli-Villars regularization, we have rederived in a very different way recent results for spin angular momentum of the electron [12, 13], thereby showing equivalence of the light-cone and covariant (Feynman) gauge calculations of this observable. As is discussed in Sec. 2.5.2 of [10], the issue of gauge independence is quite non-trivial and so such explicit verification of the earlier results shall be of interest.

We have then found that a different result is obtained when the 3D cutoff regularization is used. This raises the questions of which result is correct and what is the fundamental reason for disagreement.

Having two different results, (59) and (68), one may try to eliminate one of them with some self-consistency check. The obvious choice here is to consider total angular momentum of the electron, whose expectation value should be exactly equal to

\[ s_z \delta^{i3}. \] (88)

Such a check can verify the overall consistency of the approach used for getting spin angular momentum of the electron. We computed total angular momentum of the electron in 3D cutoff-regularized QED that was covariantly quantized [15]. We found that one-loop radiative corrections to its different components cancel out so that (88) is obtained. The same value of total angular momentum of the electron was reported in [12]. These computations, however, were criticized in [13], and so their recalculation may be appropriate.

We are thus left with the obvious option that the choice of regularization is responsible for the difference between (59) and (68). The former result is obtained in either Pauli-Villars or dimensional regularization ([12, 13] and Sec. IV), while the later one in the 3D cutoff regularization (Sec. V). Therefore, it is reasonable to compare these regularizations in the context of our studies.

We start by noting that each result was obtained in different gauges. Indeed, (59) was obtained in light-cone and covariant (Feynman) gauges, while (68) was obtained in Coulomb and general covariant gauges. As a result, none of the above-mentioned regularizations seems to break gauge independence of QED.

The main advantage of the 3D cutoff regularization is that it is physically motivated unlike dimensional and Pauli-Villars regularizations. Such motivation comes from the observation that QED cannot describe particles having arbitrarily high energy. Therefore, it seems reasonable to stay within the regime of its validity by bounding their energies with the 3D UV cutoff (the 3D IR cutoff could be also physically justified if one assumes that the Universe is finite). This is particularly well seen in the framework of the old-fashioned perturbation theory, where contributions of excited states, which are always on-shell, are “weighted” by the magnitude of 3-momentum.

The main technical incentive to use the 3D cutoff regularization instead of the dimensional regularization comes from the fact that employment of the latter is problematic in our studies. It is so because one has to deal with objects such as \[ \epsilon^{\mu\nu\rho\sigma} \] resisting straightforward extensions to arbitrarily dimensional space-times (Appendix B and e.g. Appendix B.2 of [21]). As a result,
calculations are more appealing in the four-dimensional space-time, where all fields and operations on Dirac matrices are unambiguously defined.

The 3D cutoff regularization, just as any other regularization scheme, has its own problems. They can be labeled as either technical or physical and the question is whether they influence computations of spin angular momentum of the electron.

The technical problems follow from the fact that it breaks translational invariance in momentum space, which can complicate evaluation of integrals. Moreover, it can lead to boundary terms during partial integrations. Neither the dimensional nor Pauli-Villars regularization seems to face such problems. In our calculations in Sec. V, however, we do not experience them.

The physical problem with the 3D cutoff regularization is that it breaks Lorentz symmetry unlike Pauli-Villars and dimensional regularizations. The lack of Lorentz symmetry during calculations, however, does not necessarily mean that the final result, which is obtained after removing the regularization, is incorrect. Nonetheless, we suspect that this could be the fundamental reason for the disagreement between (59) and (68). Therefore, it is our educated guess that the former of these two results is correct.

On account of all these remarks, we hope that our work will trigger some discussion about regularization (in)dependence of QED calculations in general and the 3D cutoff regularization in particular. We also hope that our work will raise interest in the experimental studies of spin angular momentum of the electron. Given the fact that components of angular momentum of nucleons are experimentally studied [10, 11], we are hopeful that such a quantity can be also measured.

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Appendix A: Conventions and all that

We use the Minkowski metric $\eta = \text{diag}(+ -- -)$ and choose $\epsilon^{0123} = +1 = \epsilon^{123}$. Greek and Latin indices take values $0, 1, 2, 3$ and $1, 2, 3$, respectively, when they refer to the components of 4- and 3-vectors. We use the Einstein summation convention. 3-vectors are written in bold, e.g. $x = (x^\mu) = (x^0, x)$. Electron’s bare and physical charges are both negative.

We frequently use $\langle \cdots \rangle = \langle \Psi | \cdots | \Psi \rangle$, $\omega_q = |q|$, $\epsilon_p = \sqrt{m_o^2 + \omega_p^2}$, and write the interaction-picture Dirac field operator as

$$\psi_I(x) = \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{m_o}{\epsilon_p}} \sum_s \left[ a_{ps} u(p, s) e^{-ipx} + b_{ps}^\dagger v(p, s) e^{ipx} \right] , \ (p^\mu) = (\epsilon_p, p),$$

and

$${\{a_{qs}, a_{qr}^\dagger\}} = {\{b_{qs}, b_{qr}^\dagger\}} = \delta_{sr} \delta(p-q),$$

where $a_{ps}$ annihilates the electron and $b_{ps}$ annihilates the positron (both of momentum $p$ and the spin state $s$). All other anticommutators involving those operators are equal to zero. We choose bispinors $u(p, s)$ and $v(p, s)$, in the standard representation of $\gamma$ matrices that we use, so that

$$u(p, s) = \frac{1}{\sqrt{2m_o(\epsilon_p + m_o)}} \begin{pmatrix} (\epsilon_p + m_o) \phi^s \\ p \sigma \phi^s \end{pmatrix}, \ v(p, s) = \frac{1}{\sqrt{2m_o(\epsilon_p - m_o)}} \begin{pmatrix} (\epsilon_p - m_o) \phi^s \\ p \sigma \phi^s \end{pmatrix},$$

where $\phi^s = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$. 

(A3b)
We define contractions of $\psi_f$ on zero-momentum external lines as

$$
\psi_f(x)|0s\rangle = \frac{u_s}{(2\pi)^{3/2}} e^{-ix}, \quad \langle 0s|\psi_f(x) = \frac{\bar{u}_s}{(2\pi)^{3/2}} e^{ix}, \quad u_s = u(0,s),
$$

(A4)

where $|0s\rangle$ and $f$ are given by (11) and (25), respectively. The $u_s$ bispinors are eigenstates of the $z$-component of the one-particle fermionic spin angular momentum operator

$$
\frac{1}{2} \Sigma_3 u_s = s_z u_s,
$$

(A5)

Finally, we mention that there is no summation over $s$ in matrix elements $\bar{u}_s \cdots u_s$.

Appendix B: Bispinor matrix elements

The results presented below are obtained in the standard (Dirac) representation of $\gamma$ matrices. It is then a simple exercise to show that the same results are obtained in all representations unitarily similar to the standard one (Weil, Majorana, etc.). This statement is equivalent to saying that they are invariant under $\gamma^\mu \rightarrow U^\mu U^\dagger$ and $u_s \rightarrow \bar{U} u_s$ transformations, where $U$ is an arbitrary unitary matrix of dimension four (see [23, 24] for the discussion of representation-independence of various results associated with the Dirac equation).

The following expressions are used in our computations:

$$
\bar{u}_s \gamma^\mu (\gamma p + m_o) \gamma_\mu u_s = 4m_o - 2p^0,
$$

(B1)

$$
\bar{u}_s \gamma k (\gamma p + m_o) \gamma_\mu u_s = 2k^0 k p + k^2 (m_o - p^0),
$$

(B2)

$$
\bar{u}_s \Gamma^i (\gamma^0 q^0 + m_o) \gamma^\mu (\gamma p + m_o) \gamma_\mu u_s = s_z \delta^{i3} (m_o + q^0) \bar{u}_s \gamma^\mu (\gamma p + m_o) \gamma_\mu u_s,
$$

(B3)

$$
\bar{u}_s \Gamma^i (\gamma^0 q^0 + m_o) \gamma k (\gamma p + m_o) \gamma_\mu u_s = s_z \delta^{i3} (m_o + q^0) \bar{u}_s \gamma k (\gamma p + m_o) \gamma_\mu u_s,
$$

(B4)

$$
\bar{u}_s \gamma^\mu (\gamma p + m_o) \Gamma^i (\gamma p + m_o) \gamma_\mu u_s = 2s_z \delta^{i3} (p^2 + m_o^2) + 2p_i p_3,
$$

(B5)

$$
\bar{u}_s \gamma^0 (m_o - \gamma p) (\gamma p + m_o) \Gamma^i (\gamma p + m_o) \gamma_\mu u_s = s_z \delta^{i3} (p^2 - m_o^2)^2,
$$

(B6)

$$
\bar{u}_s \gamma^0 (\gamma p + m_o) \Gamma^i (\gamma p + m_o) \gamma^0 u_s = 2p^0 - 2m_o
$$

(B7)

$$
\bar{u}_s \Gamma^i (\gamma^0 q^0 + m_o) \gamma^a (\gamma p + m_o) \gamma^b u_s \Delta_{ab}(p) = s_z \delta^{i3} (m_o + q^0) \bar{u}_s \gamma^a (\gamma p + m_o) \gamma^b u_s \Delta_{ab}(p),
$$

(B8)

$$
\bar{u}_s \gamma^0 (\gamma p + m_o) \Gamma^i (\gamma p + m_o) \gamma^0 u_s \Delta_{ab}(p) = -2s_z p_i p_3 [(p^0 - m_o)^2 + \omega_p^2] / \omega_p^2,
$$

(B9)

$$
\bar{u}_s \gamma^0 (\gamma p + m_o) \Gamma^i (\gamma p + m_o) \gamma^0 u_s = s_z \delta^{i3} (p + f)^2 + 2p_i p_3,
$$

(B10)

where $\tilde{q}$ is given by $\tilde{q}$.

It is interesting to note that $s_z$-dependence in all the above $\bar{u}_s \cdots u_s$ matrix elements comes from expressions that critically depend on the 4-dimensional Levi-Civita symbol, whose extension to a $d \neq 4$ dimensional space-time, used in the dimensional regularization, is problematic (see e.g. Appendix B.2 of [21]). This can be proved by combining the following easy-to-verify identities

$$
\bar{u}_s \gamma^\nu u_s = \eta^{\nu 0},
$$

(B11)

$$
\bar{u}_s \gamma^\mu \gamma^\nu u_s = \eta^{\mu \nu} - 2i s_z \varepsilon^{\nu \mu \nu 3},
$$

(B12)

$$
\bar{u}_s \gamma^\mu \gamma^\sigma \gamma^\nu u_s = \eta^{\mu \nu} \eta^{\sigma 0} + \eta^{\mu \sigma} \eta^{\nu 0} - \eta^{\mu \nu} \eta^{\sigma 0} - 2i s_z \varepsilon^{\mu \sigma \nu 3},
$$

(B13)

$$
\bar{u}_s \gamma^0 \gamma^1 \gamma^2 \gamma^3 u_s = 0
$$

(B14)

with the observation that any product of $\gamma$ matrices can be always reduced to the single term containing at most four $\gamma$ matrices, whose indices are distinct.
Appendix C: Implementation of imaginary time evolution

In the following, we work out integrals that are necessary for implementation of imaginary time evolution. While doing so, we will frequently use the Sochocki-Plemelj formula

$$\int dx \frac{f(x)}{x - x_0} = \int dx \left[ \pm i\pi \delta(x - x_0) + \frac{1}{x - x_0 \pm i\epsilon} \right] f(x),$$

(C1)

where \( f \) stands for the Cauchy principal value. Several things have to be kept in mind in the following discussion.

First, as we have mentioned in Sec. II, \( T \) will be greater than zero during evaluation of integrals and then the limit \( T \to \infty (1 - i0) \) will be taken.

Second, we will use below function

$$G(k^0, p^0, \ldots),$$

(C2)

which will be assumed to have poles at

$$k^0 = \pm \sqrt{\omega_p^2 + M^2} \pm i0, \quad p^0 = \pm \sqrt{\omega_p^2 + M'^2} \pm i0,$$

(C3)

eq etc. Masses \( M, M' \), etc. will be greater than or equal to zero. In other words, the poles of (C2) will come from propagators denominators: \((k^0)^2 - \omega_p^2 - M^2 + i0, (p^0)^2 - \omega_p^2 - M'^2 + i0, etc.\) For example, computation of Diagram 3a \( \chi_{c\Lambda_c}^{0,0} \) in Sec. V B, requires identification of function (C2) with \( \mathcal{F}(k^0, p^0) \) given by (76b) and then \( M = 0 \) and \( M' = m_o \).

Third, as (C2) will vanish for large arguments in our studies, there will be no problems with convergence of the contour integrations that we will discuss. Integrations over \( k^0, p^0, etc. \) will be unbounded, i.e. from \(-\infty \) to \( \infty \), which is consistent with the regularizations that we use.

Diagram 3a. The integral of interest is

$$\chi_{3a} = \lim_{T \to \infty (1-i0)} \int dp^0 dk^0 G(k^0, p^0) \frac{\sin^2[T(k^0 + p^0 - m_o)]}{(k^0 + p^0 - m_o)^2}. \quad (C4)$$

We rewrite it as

$$\chi_{3a} = \frac{1}{4} \lim_{T \to \infty (1-i0)} \int dp^0 dk^0 G(k^0, p^0) \frac{1 - e^{2iT(k^0 + p^0 - m_o)}}{k^0 + p^0 - m_o} \frac{1}{k^0 + p^0 - m_o}$$

\( + \frac{1}{4} \lim_{T \to \infty (1-i0)} \int dp^0 dk^0 G(k^0, p^0) \frac{1 - e^{-2iT(k^0 + p^0 - m_o)}}{k^0 + p^0 - m_o} \frac{1}{k^0 + p^0 - m_o}. \quad (C5)$$

Using now (C1), we arrive at

$$\chi_{3a} = \pi \int dp^0 G(m_o - p^0, p^0) \lim_{T \to \infty (1-i0)} T$$

\( + \frac{1}{4} \lim_{T \to \infty (1-i0)} \int dp^0 dk^0 G(k^0, p^0) \frac{1 - e^{2iT(k^0 + p^0 - m_o)}}{k^0 + p^0 - m_o} \frac{1}{k^0 + p^0 - m_o + i0} \quad (C6a)$$

\( + \frac{1}{4} \lim_{T \to \infty (1-i0)} \int dp^0 dk^0 G(k^0, p^0) \frac{1 - e^{-2iT(k^0 + p^0 - m_o)}}{k^0 + p^0 - m_o} \frac{1}{k^0 + p^0 - m_o - i0}. \quad (C6b)$$

Term (C6a) is what one obtains after taking the limit of \( T \to \infty \) under integral (C4), which formally leads to the ill-defined \([\pi \delta(k^0 + p^0 - m_o)]^2\) after employment of (28). Suppose now that we
evaluate integrals (C6b) and (C6c) on the semicircular contours in the upper and lower half-planes of complex $k^0$ and $p^0$, respectively. This turns the exponential terms in (C6b) and (C6c) into
\[
e^{\pm 2i T (k^0 + p^0 - m_o)} \left[ G_{\text{contour}} \right] e^{-2i T \gamma_{\pm}},
\]
where
\[
\gamma_{\pm} = \sqrt{\omega_p^2 + M^2 + \omega_p^2 / 2} \mp m_o.
\]

Next, we look at what $\gamma_{\pm}$ we get in different cases. In Sec. IV, we have $M = \lambda, \Lambda$ and $M' = m_o$, which implies that $\gamma_{\pm} > 0$ if we use condition (45). For calculations in Sec. V, we have $M = 0, M' = m_o$, and $\omega_p \geq \lambda_c > 0$, which again means that $\gamma_{\pm} > 0$. Therefore, if we take the limit $T \to \infty (1 - i0)$, the exponential terms can be dropped from (C6b) and (C6c) if we properly shift the poles of $1/(k^0 + p^0 - m_o)$, which amounts to
\[
\chi_{3a} = \pi \int dp^0 G(m_o - p^0, p^0) \lim_{T \to \infty (1 - i0)} T
\]
\[
= \frac{1}{4} \int dp^0 dk^0 \left[ \frac{G(k^0, p^0)}{(k^0 + p^0 - m_o + i0)^2} + \frac{G(k^0, p^0)}{(k^0 + p^0 - m_o - i0)^2} \right].
\]

Diagram 2b. Next, we introduce
\[
\tilde{G}(k^0, p^0, q^0) = \frac{G(k^0, p^0)}{q^0 - m_o + i0},
\]
where $G(k^0, p^0)$ is the same as in $\chi_{3a}$, and consider
\[
\chi_{2b} = \lim_{T \to \infty (1 - i0)} \int dp^0 dq^0 G(k^0, p^0, q^0) \frac{\sin[T(k^0 + p^0 - q^0)]}{k^0 + p^0 - q^0} \frac{\sin[T(k^0 + p^0 - m_o)]}{k^0 + p^0 - m_o}.
\]

We rewrite it as
\[
\chi_{2b} = \frac{1}{4} \lim_{T \to \infty (1 - i0)} \int dp^0 dq^0 \frac{G(k^0, p^0, q^0) e^{-i T (q^0 - m_o)}}{k^0 + p^0 - m_o} \frac{1}{k^0 + p^0 - q^0} \frac{1}{k^0 + p^0 - m_o - i0} - \frac{e^{2i T (k^0 + p^0 - m_o)}}{k^0 + p^0 - m_o} \frac{1}{k^0 + p^0 - q^0} \frac{1}{k^0 + p^0 - m_o + i0}
\]
\[
= \frac{1}{4} \lim_{T \to \infty (1 - i0)} \int dp^0 dq^0 \frac{G(k^0, p^0, q^0) e^{-i T (q^0 - m_o)}}{k^0 + p^0 - m_o} \frac{1}{k^0 + p^0 - q^0 + i0} \frac{1}{k^0 + p^0 - m_o - i0}
\]
\[
- \frac{1}{4} \lim_{T \to \infty (1 - i0)} \int dp^0 dq^0 \frac{G(k^0, p^0, q^0) e^{-i T (q^0 - m_o)}}{k^0 + p^0 - m_o} \frac{1}{k^0 + p^0 - q^0 - i0} \frac{1}{k^0 + p^0 - m_o + i0}.
\]

Employing (C1), we obtain
\[
\chi_{2b} = -\frac{1}{2} \lim_{T \to \infty (1 - i0)} \int dp^0 dq^0 \frac{G(k^0, p^0, q^0) e^{-i T (q^0 - m_o)}}{k^0 + p^0 - m_o} \frac{1}{k^0 + p^0 - q^0 - i0} \frac{1}{k^0 + p^0 - m_o + i0}
\]
\[
- \frac{1}{2} \lim_{T \to \infty (1 - i0)} \int dp^0 dq^0 \frac{G(k^0, p^0, q^0) e^{-i T (q^0 - m_o)}}{k^0 + p^0 - m_o} \frac{1}{k^0 + p^0 - q^0 + i0} \frac{1}{k^0 + p^0 - m_o - i0}.
\]
Using again (C1), we obtain
\[
\chi_{2b} = -i\pi^2 \int dp^0 G(m_o - p^0, 0) \lim_{T \to \infty(1-i0)} T
\frac{1}{k^0 + p^0 - m_o + i0} \frac{e^{iT(k^0+p^0-m_o)} - e^{2iT(k^0+p^0-m_o)}}{k^0 + p^0 - m_o + i0}
\]
(C14)
\[
\frac{-i\pi}{2} \lim_{T \to \infty(1-i0)} \int dp^0 dk^0 G(k^0, p^0) \frac{e^{iT(k^0+p^0-m_o)} - e^{2iT(k^0+p^0-m_o)}}{k^0 + p^0 - m_o + i0} \frac{1}{k^0 + p^0 - m_o - i0}.
\]

Repeating now the steps around (C7), we note that exponential terms in (C14) vanish upon taking the limit, which after proper shifting of the pole of \(1/(k^0 + p^0 - m_o)\) leaves us with
\[
\chi_{2b} = -i\pi^2 \int dp^0 G(m_o - p^0, 0) \lim_{T \to \infty(1-i0)} T - \frac{i\pi}{2} \int dp^0 dk^0 \frac{G(k^0, p^0)}{(k^0 + p^0 - m_o - i0)^2}.
\]
(C15)

**Diagram 2a.** Now, we consider
\[
\chi_{2a} = \lim_{T \to \infty(1-i0)} \int dp^0 dk^0 dq^0 G(k^0, p^0, q^0) \sin[T(k^0+p^0-m_o)] \sin[T(k^0+q^0-m_o)] \frac{1}{k^0 + p^0 - m_o + i0} \frac{1}{k^0 + q^0 - m_o}. \]
(C16)

where the poles of \(G(k^0, p^0, q^0)\) are characterized by \(M = \lambda, \Lambda\), and \(M' = M'' = m_o\) in Sec. IV and by \(M = 0\) and \(M' = M'' = m_o\) in Sec. V.

We rewrite it as
\[
\chi_{2a} = \frac{1}{2i} \lim_{T \to \infty(1-i0)} \int dp^0 dk^0 dq^0 G(k^0, p^0, q^0) \sin[T(k^0+p^0-m_o)] \frac{1}{k^0 + p^0 - m_o + i0} \frac{1}{k^0 + q^0 - m_o} \frac{e^{iT(k^0+p^0-m_o)}}{k^0 + p^0 - m_o}
\]
\[
- \frac{1}{2i} \lim_{T \to \infty(1-i0)} \int dp^0 dk^0 dq^0 G(k^0, p^0, q^0) \sin[T(k^0+p^0-m_o)] \frac{1}{k^0 + p^0 - m_o + i0} \frac{1}{k^0 + q^0 - m_o} \frac{e^{-iT(k^0+q^0-m_o)}}{k^0 + q^0 - m_o}
\]
(C17)

which after using (C1) leads to
\[
\chi_{2a} = \pi \lim_{T \to \infty(1-i0)} \int dp^0 dq^0 G(m_o - q^0, p^0, q^0) \sin[T(p^0-q^0)] |p^0 - q^0|
\]
\[
\frac{1}{2i} \lim_{T \to \infty(1-i0)} \int dp^0 dk^0 dq^0 G(k^0, p^0, q^0) \sin[T(k^0+p^0-m_o)] \frac{e^{iT(k^0+q^0-m_o)}}{k^0 + p^0 - m_o} - \frac{e^{-iT(k^0+q^0-m_o)}}{k^0 + q^0 - m_o + i0}
\]
(C18)

After splitting the integrals over the sinuses into the Cauchy principal value integrals and then
one more employment of (C1), we obtain

\[ \chi_{2a} = \pi^2 \int dp^0 G(m_o - p^0, p^0, p^0) \]

\[ + \frac{\pi i}{2T} \lim_{T \to \infty(1-i0)} \int dp^0 dq^0 [G(m_o - q^0, p^0, q^0) + G(m_o - p^0, q^0, p^0)] \frac{e^{iT(p^0 - q^0)}}{p^0 - q^0 + i0} \]

\[ + \frac{\pi i}{2T} \lim_{T \to \infty(1-i0)} \int dp^0 dq^0 [G(m_o - q^0, p^0, q^0) + G(m_o - p^0, q^0, p^0)] \frac{e^{iT(q^0 - p^0)}}{q^0 - p^0 + i0} \]

\[ - \frac{1}{4} \lim_{T \to \infty(1-i0)} \int dp^0 dk^0 dq^0 G(k^0, p^0, q^0) \frac{e^{iT(2k^0 + p^0 + q^0 - 2m_o)}}{(k^0 + p^0 - m_o + i0)(k^0 + q^0 - m_o + i0)} \]

\[ - \frac{1}{4} \lim_{T \to \infty(1-i0)} \int dp^0 dk^0 dq^0 G(k^0, p^0, q^0) \frac{e^{-iT(2k^0 + p^0 + q^0 - 2m_o)}}{(k^0 + p^0 - m_o - i0)(k^0 + q^0 - m_o - i0)} \]

\[ + \frac{1}{4} \lim_{T \to \infty(1-i0)} \int dp^0 dk^0 dq^0 G(k^0, p^0, q^0) \frac{e^{iT(p^0)}}{(k^0 + p^0 - m_o + i0)(k^0 + q^0 - m_o + i0)} \]

\[ + \frac{1}{4} \lim_{T \to \infty(1-i0)} \int dp^0 dk^0 dq^0 G(k^0, p^0, q^0) \frac{e^{-iT(q^0)}}{(k^0 + p^0 - m_o - i0)(k^0 + q^0 - m_o - i0)}. \]

Integrands in terms (C19b)–(C19g) involve factors

\[ \frac{e^{\pm iT k^0 + \cdots}}{\cdots + h^0 \pm i0} \]

where \( h^0 \) variables are timelike components of 4-momenta appearing in expressions for propagators. If we now integrate each term on the semicircular contours in the upper (+) and lower (−) half-planes of complex \( h^0 \), we will see that poles of (C20) do not contribute to such contour integrals. Thus, only poles of the \( G \) function contribute, but they turn the exponential terms into the form similar to (C7). Just as below (C7), one can argue then that (C19b)–(C19g) are removed by the limit \( T \to \infty(1 - i0) \).

All in all, we get

\[ \chi_{2a} = \pi^2 \int dp^0 G(m_o - p^0, p^0, p^0). \]

The limit \( T \to \infty \) under integral (C16) produces the same result.

**Coulomb-gauge \( S_\parallel \).** Finally, we discuss

\[ \chi_\parallel = \lim_{T \to \infty(1-i0)} \int dk^0 dp^0 G(k^0, p^0) \frac{\sin[T(k^0 - p^0)]}{k^0 - p^0}. \]

We rewrite it as

\[ \chi_\parallel = \frac{1}{2i} \lim_{T \to \infty(1-i0)} \int dk^0 dp^0 G(k^0, p^0) \frac{e^{iT(k^0 - p^0)}}{k^0 - p^0} \]

\[ - \frac{1}{2i} \lim_{T \to \infty(1-i0)} \int dk^0 dp^0 G(k^0, p^0) \frac{e^{-iT(k^0 - p^0)}}{k^0 - p^0}, \]
which, with the help of (C1), can be cast into the following form

\[ \chi || = \pi \int dk^0 G(k^0, k^0) + \frac{1}{2i} \lim_{T \to \infty(1-i0)} \int dk^0 dp^0 G(k^0, p^0) \frac{e^{iT(k^0-p^0)}}{k^0-p^0+i0} \]

\[ - \frac{1}{2i} \lim_{T \to \infty(1-i0)} \int dk^0 dp^0 G(k^0, p^0) \frac{e^{-iT(k^0-p^0)}}{k^0-p^0-i0} \] \hspace{1cm} (C24)

Adopting the procedure described around (C7), and noting that this time \( G(k^0, p^0) \) has poles parameterized by \( M = M' = m_o \) (C3), we readily realize that the last two integrals vanish as \( T \to \infty(1-i0) \). So, we find that

\[ \chi || = \pi \int dk^0 G(k^0, k^0). \] \hspace{1cm} (C25)

Using (28), one immediately finds that the same result is obtained by taking \( T \to \infty \) under integral (C22).

**Appendix D: Pauli-Villars regularization**

We will discuss here implementation of Pauli-Villars regularization [25] in our covariant gauge calculations (see e.g. [26, 27] for textbook introduction to this technique). We will work in Feynman gauge (46), where calculations are most efficient. The Lagrangian density that we consider reads

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (\partial_\mu A^\mu)^2 + \frac{\lambda^2}{2} A_\mu A^\mu + \bar{\psi} (i\gamma^\mu \partial_\mu - m_0) \psi \]

\[ + \frac{1}{4} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} + \frac{1}{2} (\partial_\mu \tilde{A}^\mu)^2 - \frac{\Lambda^2}{2} \tilde{A}_\mu \tilde{A}^\mu + \bar{\tilde{\psi}} (i\gamma^\mu \partial_\mu - \Lambda) \tilde{\psi} \]

\[ - e_o (\bar{\psi} \gamma^\mu \psi + \bar{\tilde{\psi}} \gamma^\mu \tilde{\psi}) (A_\mu + \tilde{A}_\mu), \] \hspace{1cm} (D1)

where both the photon mass \( \lambda \) and the tilded fields are introduced to impose regularization. \( \tilde{A}_\mu \) and \( \tilde{\psi} \) are known as Pauli-Villars ghost fields. They are both assumed to be bosonic operators. Their interaction-picture propagators read [27]

\[ \tilde{S}(x-y) = \langle \hat{0} | \mathcal{T} \tilde{\psi}_I(x) \tilde{\psi}_I(y) | \hat{0} \rangle = i \int \frac{d^4p}{(2\pi)^4} \frac{\gamma p + \Lambda}{p^2 - \Lambda^2 + i0} e^{-ip(x-y)}, \] \hspace{1cm} (D2)

\[ \tilde{D}_{\mu\nu}(x-y) = \langle \hat{0} | \mathcal{T} \tilde{A}^I_\mu(x) \tilde{A}^I_\nu(y) | \hat{0} \rangle = i \int \frac{d^4p}{(2\pi)^4} \frac{\eta_{\mu\nu}}{p^2 - \Lambda^2 + i0} e^{-ip(x-y)}, \] \hspace{1cm} (D3)

where \( | \hat{0} \rangle \) is the vacuum state of the free Pauli-Villars theory.

The interaction-picture propagator for the Dirac field is still given by (14) while the one for the 4-potential \( A_\mu \) needs to be redefined so that it now reads

\[ D_{\mu\nu}(x-y) = \langle 0 | \mathcal{T} A^I_\mu(x) A^I_\nu(y) | 0 \rangle = -i \int \frac{d^4p}{(2\pi)^4} \frac{\eta_{\mu\nu}}{p^2 - \Lambda^2 + i0} e^{-ip(x-y)} \] \hspace{1cm} (D4)

(see e.g. Sec. 7.6 of [9] for a shortcut to evaluations of propagators). Note that (D3) and (D4) differ not only in masses but also in overall signs.
The calculations of spin angular momentum of the electron in Pauli-Villars-regularized QED proceed now similarly as in Sec. III. As a result, we minimally modify the notation used there. The analog of (12a) is

$$\langle J_{\text{spin}} \rangle_{\Omega_s} = \lim_{A \to \infty} \lim_{\lambda \to 0} \langle J_{\text{spin}} \rangle_{\Omega_s}$$

(D5)

where \( \langle J_{\text{spin}} \rangle_{\Omega_s} \) is obtained from (12b) after obvious replacements

\[
|0_s \rangle \rightarrow |\bullet \rangle = |0_s \rangle \otimes |0\rangle,
\]

(D6)

\[
\mathcal{H}_{\text{int}}^I \rightarrow \mathcal{H}_{\text{int}}^{I*} = e_o (\ddot{\psi}_I \gamma^\mu \psi_I + \dddot{\psi}_I \gamma^\mu \psi_I) (A_{\mu}^I + \bar{A}_{\mu}^I),
\]

(D7)

which then “propagate” to (21).

Subsequent calculations involve the following easy-to-verify identity

\[
\langle \bullet | T \gamma^I \mathcal{H}_{\text{int}}^I (x) \mathcal{H}_{\text{int}}^{I*} (y) | \bullet \rangle = e_o^2 \langle 0_s | T \gamma^I \ddot{\psi}_I (x) \dddot{\psi}_I (x) + \dddot{\psi}_I (y) \gamma^\nu \psi_I (y) : |0_s \rangle D_{\mu \nu}^{\text{reg}} (x - y) + e_o^2 \langle 0_s | T |0_s \rangle \text{Tr} \left[ \overline{S} (y - x) \gamma^\nu \tilde{S} (y - x) \gamma^\rho \right] D_{\rho \mu}^{\text{reg}} (x - y),
\]

(D8)

where \( T \) stands for either \( J_{\text{spin}}^I \) or the unit operator and

\[
D_{\mu \nu}^{\text{reg}} (x - y) = D_{\mu \nu} (x - y) + \tilde{D}_{\mu \nu} (x - y) = -i \int \frac{d^4 p}{(2\pi)^4} \left[ \eta_{\mu \nu} \right] \frac{1}{p^2 - \lambda^2 + i0} - (\lambda \to \Lambda) e^{-i p (x - y)}.
\]

(D9)

One can readily show with the help of these results that \( \langle J_{\text{spin}} \rangle_{\Omega_s}^{T \Lambda A} \) can be obtained from (21) by replacing \( \mathcal{H}_{\text{int}}^I \) with

\[
\mathcal{H}_{\text{int}}^{I*} = e_o \overline{\psi}_I \gamma^\mu \psi_I : A_{\mu}^{I*}
\]

(D10)

and using \( D_{\mu \nu}^{\text{reg}} (x - y) \) as the propagator for the \( A_{\mu}^{I*} \) field.

[1] J. J. Thomson, Philos. Mag. 44, 293 (1897).
[2] G. E. Uhlenbeck and S. Goudsmit, Nature 117, 264 (1926).
[3] T. Preston, The Scientific Transactions of the Royal Dublin Society 6, 385 (1898).
[4] P. A. M. Dirac, Proc. Royal Soc. A 117, 610 (1928).
[5] J. Schwinger, Phys. Rev. 73, 416 (1948).
[6] J. E. Nafe and E. B. Nelson, Phys. Rev. 73, 718 (1948).
[7] H. M. Foley and P. Kusch, Phys. Rev. 73, 412 (1948).
[8] E. D. Commins, Annu. Rev. Nucl. Part. Sci. 62, 133 (2012).
[9] R. Greiner and J. Reinhardt, Field Quantization (Springer-Verlag, 1996).
[10] E. Leader and C. Lorcé, Phys. Rep. 541, 163 (2014) [Erratum ibid 802, 23 (2019)].
[11] A. Deur, S. J. Brodsky, and G. F. de Téramond, Rep. Prog. Phys. 82, 076201 (2019).
[12] T. Liu and B.-Q. Ma, Phys. Rev. D 91, 017501 (2015).
[13] X. Ji, A. Schäfer, F. Yuan, J.-H. Zhang, and Y. Zhao, Phys. Rev. D 93, 054013 (2016).
[14] M. E. Peskin and D. V. Schroeder, An Introduction to Quantum Field Theory (Westview Press, 1995).
[15] B. Damski, Total angular momentum of the electron: one-loop studies, in preparation.
[16] B. Lautrup, Mat. Fys. Medd. Dan. Vid. Selsk. 35, No. 11 (1967).
[17] B. C. Berndt, Ramanujan’s Notebooks. Part I (Springer-Verlag, 1985).
[18] I. S. Gradshteyn, I. M. Ryzhik, D. Zwillinger, and V. Moll, Table of Integrals, Series, and Products (Academic Press, 2014), 8th ed.
[19] G. S. Adkins, Phys. Rev. D 27, 1814 (1983).
[20] G. S. Adkins, Phys. Rev. D 34, 2489 (1986).
[21] H. K. Dreiner, H. E. Haber, and S. P. Martin, Phys. Rep. 494, 1 (2010).
[22] D. Binosi, J. Collins, C. Kaufhold, and L. Theussl, Comput. Phys. Commun. 180, 1709 (2009).
[23] P. B. Pal, arXiv e-prints (2007), physics/0703214.
[24] M. Arminjon and F. Reifler, Braz. J. Phys. 38, 248 (2008).
[25] W. Pauli and F. Villars, Rev. Mod. Phys. 21, 434 (1949).
[26] M. D. Schwartz, Quantum Field Theory and the Standard Model (Cambridge University Press, 2015).
[27] S. N. Gupta, Quantum Electrodynamics (Gordon and Breach Science Publishers, 1977).