Wrapping corrections for long range spin chains

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The long range spin chains play an important role in the gauge/string duality. The aim of this paper is to generalize the recently introduced transfer matrices of integrable medium range spin chains to long range models. These transfer matrices define a large set of conserved charges for every length of the spin chain. These charges agree with the original definition of long range spin chains for infinite length. However, our construction works for every length, providing the definition of integrable finite size long range spin chains whose spectrum already contains the wrapping corrections.

I. INTRODUCTION

In the early studies of the planar limit of the $N = 4$ super Yang-Mills theory it turned out that the anomalous dimensions of single-trace operators can be obtained from the spectrum of an integrable Hamiltonian with long range interaction. At one-loop the dilatation operator corresponds to an integrable nearest neighbor interacting model [1]. For higher loops the interaction range increases, more precisely, the interaction range is $\ell + 1$ at $\ell$-loop.

In the region where the spin chain length $J$ is bigger than the loop order (asymptotic region), the Hamiltonian can be written as a sum of local densities. For these local operators the integrability condition can be generalized and it was showed that the Hamiltonian of the SU(2) sector preserves integrability for higher loops [2]. The asymptotic Hamiltonians can be diagonalized with the asymptotic Bethe Ansatz [3] and the result can be generalized to the full $\mathfrak{psu}(2, 2|4)$ spectrum [4]. However, this result is correct only in the asymptotic region. In the region where the spin chain length $J$ is smaller than the loop order (wrapping region), wrapping corrections appear [5]. So far, it was not clear whether good spin chain toy models, which mimic the wrapping corrections, could be found, i.e., even if an asymptotic Hamiltonian was given, we could not define the corresponding finite size Hamiltonian.

The solution for the wrapping corrections came from holographic duality. In the string theory side the scaling dimensions correspond to the energy spectrum of strings which can be described as a 1+1 dimensional integrable field theory [6]. In the field theory if we know the dispersion relation and the scattering matrix at infinite volume then we can calculate the finite volume spectrum as well (at least in principle). The finite size corrections can be obtained from the thermodynamic Bethe Ansatz [7–10] and it was showed that they agree with the wrapping corrections [11–13].

Since the asymptotic data of the string theory (dispersion relation, scattering matrix) completely defines the finite size corrections, a natural conclusion is, that the asymptotic data on the spin chain side should also define the wrapping corrections. In other words, there must be a procedure that gives the finite size Hamiltonians from the asymptotic ones. The aim of this paper to present such a method.

Recently, an algebraic framework was developed for integrable medium range spin chains (interaction range bigger than two but finite) [14]. This method gives a recipe how to define transfer matrices which are the generating functions of the conserved quantities, including the Hamiltonians. An interesting observation is that, this transfer matrix is well defined even when the length of the spin chain is smaller than the interaction range therefore generalizing this method to long range spin chains, we obtain transfer matrices which define the finite length Hamiltonians even for the lengths where the wrapping corrections appear.

II. PRELIMINARIES

In this section we summarize the definition of the long range spin chain following [3, 15] and specify our goals.

An integrable long range spin chain has a tower of coupling constant $\lambda$ dependent commuting charges $Q_r(\lambda) \equiv Q_r$ [16] which have the following series expansions

$$Q_r = Q_r^{(0)} + \sum_{j=1}^{\infty} \lambda^j Q_r^{(j)},$$

where $r \geq 2$ and the $\lambda$ independent operators $Q_r^{(\ell)}$ are sum of local operators with range $r + \ell$

$$Q_r^{(\ell)} = \sum_{j=-\infty}^{\infty} q_{r,\ell}^{j,j+\ell} = \sum_{j=-\infty}^{\infty} q_{j,j+1,...,j+r+\ell-1}^{r,\ell}$$

where the local densities $q_{r,\ell}^{j,j+\ell} \equiv q_{j,j+1,...,j+r+\ell-1}^{r,\ell}$ act on the sites $j, j+1, \ldots, j + r + \ell - 1$. The Hamiltonian is the charge $Q_2$.

It turned out that, for a fixed nearest neighbor model $Q_k(\lambda = 0)$, a large class of integrable deformations exists. The moduli space is given by four sets of parameters $\alpha_\ell(\lambda), \beta_\ell(\lambda), \gamma_{\ell,\alpha}(\lambda), \epsilon_\ell(\lambda)$. The last two sets are unphysical parameters and they correspond to the linear
combinations of the charges $Q_r \to \sum \gamma_{r,s}(\lambda) Q_s$ and the similarity transformations

$$Q_r \to e^X Q_r e^{-X}, \quad X = \sum_{j=-\infty}^{\infty} \sum_{k} \epsilon_k(\lambda) X_j^k,$$  

where $X_j^k \equiv X_{j+\ell_k}^k$ are local operators with range $\ell_k$. The remaining parameters are the physical ones. The $\alpha_r$ and $\beta_{r,s}$ appear in the rapidity map and the scattering phase [15].

It is clear that the operators $Q^{(k)}_k$ can also be defined on finite length $J$ for $J > \ell + k$. More concretely, the Hamiltonian $\mathcal{H}$ on size $J$ is defined up to order $\lambda^{J-2}$ (asymptotic region). Our goal is the find an integrability preserving method that defines the finite volume version of the asymptotic Hamiltonians even for higher orders than $\lambda^{J-2}$ (wrapping region).

### III. MEDIUM RANGE TO LONG RANGE

In this section we generalize the construction of [14] (the basics appeared first in [17]) to obtain transfer matrices for perturbative long range spin chains [3]. In [14] an algebraic framework was introduced for integrable spin chains with interaction range $\ell + 2$ which is defined by the Hamiltonian

$$H^{(\ell)} = \sum_{j=1}^{J} h_{jj+1,...,j+\ell+1} = \sum_{j} h_{j}^{(\ell)},$$  

where $h_{j}^{(\ell)}$ is the Hamiltonian density which acts on the sites $j, j+1, \ldots, j+\ell+1$. We use periodic boundary condition. The construction of [14] is based on the existence of the Lax- and the $R$-operators

$$\hat{L}_j^{(\ell)}(u) = \hat{L}_j^{(\ell)}(u) + 1 + u h_{j}^{(\ell)} + \mathcal{O}(u^2),$$  

$$\hat{R}_j^{(\ell)}(u) = \hat{R}_j^{(\ell)}(u,v),$$  

which satisfy the RLL-relation

$$\hat{R}_2^{(\ell)}(u,v)\hat{L}_1^{(\ell)}(u)\hat{L}_{\ell+2}^{(\ell)}(v) = \hat{L}_1^{(\ell)}(v)\hat{L}_{\ell+2}^{(\ell)}(u)\hat{R}_1^{(\ell)}(u,v).$$  

In this letter we chose to write these Lax in the "checked" form (the $R$-matrix is multiplied by a permutation), which might be less familiar to some readers [18], although it has the advantage that the Lax-operator has a simpler expansion in the spectral parameter (5). In the alternative "unchecked" convention the quantum and auxiliary spaces are separated. The figure 1 shows graphical presentations of Lax-operators and RLL-relations and the colored legs denote the auxiliary spaces of the "unchecked" convention.

The consequence the RLL-relation is that the following transfer matrix

$$\hat{T}^{(\ell)}(u) = \hat{T}_{J+1}(u) \hat{L}_1^{(\ell)}(u) \hat{L}_{\ell+2}^{(\ell)}(u) \hat{R}_1^{(\ell)}(u,v).$$  

defines commuting quantities $[\hat{T}^{(\ell)}(u), \hat{T}^{(\ell)}(v)] = 0$ [18]. In (8) we defined the twisted trace operator $\hat{T}_{J+1}$ which acts on an operator $X$ as

$$\hat{T}_{J+1}(X) = \hat{T}_{J+1}(X \hat{P}_{\ell_1} \hat{P}_{\ell_1+1} \ldots \hat{P}_{J+1})$$  

where $P_{\ell}$ is the permutation operator and $\hat{T}_{J+1}(X \hat{P}_{\ell_1} \hat{P}_{\ell_1+1} \ldots \hat{P}_{J+1})$ is the usual trace on the sites $J+1, \ldots, J+\ell$. The transfer matrix generates the local conserved charges

$$Q_{k}^{(\ell)} = \frac{\partial^{k}}{\partial u^{k}} \log \hat{T}^{(\ell)}(u) \bigg|_{u=0}.$$  

The interaction range of $Q^{(\ell)}_k$ is $(k-1)\ell + k$ and $H^{(\ell)} = Q^{(\ell)}_1$.

Let us turn to the long range spin chains. At first we have to introduce the coupling constant $\lambda$ dependent truncated operators $\hat{L}_1^{(\ell)}(u,\lambda) \equiv L_1^{(\ell)}(u)$ and $\hat{R}_1^{(\ell)}(u,v,\lambda) \equiv R_1^{(\ell)}(u,v)$ with range $\ell + 2$ and $2\ell + 2$ as

$$\hat{L}_1^{(\ell)}(u) = L_1^{(\ell)}(u) + \sum_{j=1}^{\ell} \lambda^j \hat{L}_j^{(\ell)}(u),$$  

$$\hat{R}_1^{(\ell)}(u,v) = R_1^{(\ell)}(u,v) + \sum_{j=1}^{\ell} \lambda^j \hat{R}_j^{(\ell)}(u,v),$$  

which satisfy the RLL-relation up to order $O(\lambda^{\ell+1})$:

$$\hat{R}_2^{(\ell)}(u)\hat{L}_1^{(\ell)}(u)\hat{L}_{\ell+2}^{(\ell)}(v) = \hat{L}_1^{(\ell)}(v)\hat{L}_{\ell+2}^{(\ell)}(u)\hat{R}_1^{(\ell)}(u,v) + O(\lambda^{\ell+1}),$$  

where we used the shorthand notation $\hat{R}_j := \hat{R}_j^{(\ell)}(u,v)$. We also require that

$$\hat{L}_1^{(\ell)}(u) = 1 + u h_1^{(\ell)} + O(u^2), \quad h_1^{(\ell)} = \sum_{j=0}^{\ell} \lambda^j h_{j}^{(\ell)}.$$  

where $h_{j}^{(\ell)}$ are $\lambda$ and $u$ independent operators with interaction range $\ell + 2$.

At the first sight we might think that the truncated RLL-relation (13) and the matrices $\hat{R}_1^{(\ell)}$ are completely independent for every order $\ell$ but it is not true. It turns out that the equation (13) up to order $O(\lambda^{\ell})$ is equivalent with the truncated RLL-relation for $\ell - 1$. We can show that

$$\hat{R}_1^{(\ell)} + O(\lambda^{\ell}) = \hat{L}_{\ell+1}^{(\ell)}(u)\hat{R}_1^{(\ell)} \hat{J}_{\ell+1}^{(\ell-1)}(v),$$  

where we defined the perturbative inverse $\hat{J}_{\ell+1}^{(\ell-1)}(u)$ as

$$\hat{J}_{\ell+1}^{(\ell-1)}(u)\hat{R}_1^{(\ell-1)}(u) = 1 + O(\lambda^{\ell}) [18].$$  

The consequence of the equation (15) is that the matrices $\hat{R}_1^{(\ell,j)}$ are determined by $\hat{R}_1^{(\ell-1,j)}$ for $j = 1, \ldots, \ell - 1$ therefore the full truncated $R$-matrix $\hat{R}_1^{(\ell)}$ is completely determined by the matrices $\hat{R}_1^{(\ell,j)}$ for $j = 1, \ldots, \ell$. Fixing
the leading order \( \ell \leq 1, 2, 3 \) to already known \( L \) and \( R \)-matrices of a nearest neighbor interacting model, we can calculate the matrices \( \mathcal{L}_1^{(1)}(u), \mathcal{R}_1^{(1)}(u,v) \) order by order from the highest order \( \lambda^2 \) of the truncated RLL-relation (13).

As in the medium range case, the transfer matrix

\[
\mathcal{T}^{(\ell)}(u) = \mathcal{Tr}_{\ell,\ell+1}[\mathcal{L}_j^{(\ell)}(u)\ldots\mathcal{L}_1^{(\ell)}(u)] + \mathcal{O}(\lambda^{\ell+1}) \tag{16}
\]
defines commuting quantities up to order \( \lambda^{\ell+1} \):

\[
[\mathcal{T}^{(\ell)}(u), \mathcal{T}^{(\ell)}(v)] = \mathcal{O}(\lambda^{\ell+1}).
\]
The transfer matrix generates the conserved charges up to order \( \mathcal{O}(\lambda^{\ell+1}) \)

\[
Q_k^{(\ell)} = \frac{\partial k}{\partial u} \log \mathcal{T}^{(\ell)}(u) \bigg|_{u=0} + \mathcal{O}(\lambda^{\ell+1}), \tag{17}
\]
where

\[
Q_k^{(\ell)} = Q_k^{(0)} + \sum_{j=1}^{\ell} \lambda^j Q_j^{(0)}(u) + \mathcal{O}(\lambda^{\ell+1}). \tag{18}
\]

It turns out that the charge \( Q_k^{(\ell)} \) has interaction range \( \ell + k \).

Since the Lax-operators has the property (14) the Hamiltonian reads as

\[
Q_2^{(\ell)} = \sum_{j=1}^{J} \mathcal{Tr}_{\ell,\ell+1}[h_j^{(\ell)}] + \mathcal{O}(\lambda^{\ell+1}). \tag{19}
\]

Since \( h_j^{(\ell)} = \sum_{k=0}^{\ell} \lambda^k h_j^{(k)} \) we obtain that

\[
Q_2^{(\ell)} = \sum_{j=1}^{J} \mathcal{Tr}_{\ell,\ell+1}[h_j^{(\ell)}]. \tag{20}
\]

For the asymptotic region i.e. \( J > \ell + 1 \), we have the identity \( \mathcal{Tr}_{\ell,\ell+1}[h_1^{(\ell)}] = h_1^{(\ell)} \) therefore this charge has the usual form \( Q_2^{(\ell)} = \sum_{j=1}^{J} h_j^{(\ell)} \).

Above we showed that the solutions of the RLL-relations (13) define long range charges (1) in the asymptotic limit. An important question is that whether the reverse statement is also true i.e. do there exist Lax-operators for every integrable long range charges \( Q_2^{(\ell)} \)? At this point we do not know the answer. However I investigated the long range \( \mathfrak{gl}(N) \) spin chains of [3] up to order \( \mathcal{O}(\lambda^3) \). After fixing the unphysical parameters \( \gamma(\lambda), \epsilon(\lambda) \), I found the matrices \( \mathcal{L}_1^{(2)}(u) \) which give the \( \mathcal{Q}_2^{(2)} \) for every physical parameters \( \alpha(\lambda), \beta(\lambda) \) [18].

IV. LONG RANGE SPIN CHAINS AT THE WRAPPING REGION

The main advantage of the algebraic construction of the previous section is that, the transfer matrix is well defined and satisfies \( [\mathcal{T}^{(\ell)}(u), \mathcal{T}^{(\ell)}(v)] = \mathcal{O}(\lambda^{\ell+1}) \) [19] even for \( J < \ell + 2 \) i.e. the wrapping region. So far it was not clear how to define the Hamiltonian in the wrapping region in an integrability preserving way but our transfer matrix gives a recipe. We emphasis that Lax-operator (11) is an asymptotic, density-like quantity (since it is defined on an infinite chain and it contains the asymptotic Hamiltonian density) therefore it describes the elementary physical interaction. The transfer matrix is a consistent way to "put" this interaction to finite size in a translation invariant and integrability preserving way.

To obtain the integrable Hamiltonian for the wrapping region \( J \leq \ell + 1 \), we only have to use the definition (17). We can repeat the previous calculation up to (20) i.e., the Hamiltonian reads as \( Q_2^{(\ell)} = \sum_{j=1}^{J} h_j^{(\ell)} \) where we introduced a "wrapped" Hamiltonian density (see figure...
2) \( \tilde{h}^{(J,\ell)}_1 \equiv \tilde{h}^{(J,\ell)}_{12-j} := \hat{\text{Tr}}_{J,\ell+2-J} \left( h^{(J)}_1 \right) \), (21)

and the periodic boundary condition is prescribed i.e. \( \tilde{h}^{(J,\ell)}_1 = \tilde{h}^{(J,\ell+1)}_{J+1,1,2-J-1} \). We saw that the twisted trace acts as identity for the asymptotic region but for the wrapping region it defines a new operator which "fits" with the length of the chain.

Let us summarize what we learned from this analysis. Let us take an asymptotic integrable long range Hamiltonian

\[
\mathcal{H}^\infty = \sum_{j=-\infty}^{\infty} b_j^\infty = \sum_{j=-\infty}^{\infty} \sum_{\ell=0}^{\infty} \lambda_j^{(\ell)},
\]

where \( \lambda_j^{(\ell)} \equiv \lambda_{\ell,\ell+2-j}^{(\ell)} \) is a coupling constant dependent operator with interaction range \( \ell + 2 \). We saw that our method defines a unique integrability preserving Hamiltonian for every finite length \( J \) as

\[
\mathcal{H}^J = \sum_{j=1}^{J} \sum_{\ell=0}^{\infty} \lambda_j^{(\ell)},
\]

\[
\tilde{\lambda}_1^{(J,\ell)} = \left\{ \lambda_1^{(\ell)}, \frac{1}{\hat{\text{Tr}}_{J,\ell+2-J}} \left( \lambda_1^{(\ell)} \right) \right\}, \quad \ell + 2 > J.
\]

V. INOZEMTSEV’S SPIN CHAIN

In this section we demonstrate that our finite volume Hamiltonian is consistent with a naive physical argument. Let us take a long range interaction and assume that we already know its manifestation for every length \( J \), i.e., for every length \( J \) we have the Hamiltonians \( \mathcal{H}^J \) which correspond to the same physical interaction. Since we know the Hamiltonians for every length we can obtain the asymptotic model by the limit

\[
\mathcal{H}^\infty = \lim_{J\to\infty} \mathcal{H}^J.
\]

A natural requirement is that our procedure (23) should return to the original models \( \mathcal{H}^J \).

In the following we validate this requirement on the Inozemtsev’s spin chain [20] for which the finite volume Hamiltonian reads as

\[
\mathcal{H}^J = \sum_{1 \leq j, k \leq J} \left( \varphi(k-j) + \frac{2}{\omega} \zeta(\frac{\omega}{2}) \right) P_{j,k},
\]

where \( \varphi(z) \), \( \zeta(z) \) are the Weierstrass functions defined on the torus \( \mathbb{C}/\mathbb{Z}+\mathbb{Z}\omega \), \( \omega = \frac{i\pi}{2} \) and the local Hilbert spaces are \( \mathbb{C}^N \). The \( J \) is length of the spin chain and \( \kappa \in \mathbb{R} \) is the coupling. In the asymptotic limit we obtain the hyperbolic Inozemtsev’s spin chain [20]

\[
\mathcal{H}^\infty = \sum_{-\infty < j < k < \infty} V(k-j)P_{j,k}, \quad V(j) = \left( \frac{\kappa}{\sinh(j\kappa)} \right)^2.
\]

After a renormalization of the coupling constant \( \kappa(\lambda) \), this Hamiltonian is compatible with the perturbative long range description [21]. Let us rewrite the Hamiltonian as

\[
\mathcal{H}^\infty = \sum_{-\infty < j < \infty} h_j^\infty = \sum_{\ell=0}^{\infty} \sum_{\ell=0}^{\infty} \lambda_j^{(\ell)},
\]

\[
\lambda_1^{(\ell)} = V(\ell+1)P_{1,\ell+2}.
\]

Now let us apply (23) on \( \lambda_1^{(\ell)} \). At first let us wrap the permutations

\[
\check{P}_{1,k} = \begin{cases} N, & k \equiv 1 \mod J \\ P_{1,k}, & k \equiv 2, \ldots, J \mod J, \end{cases}
\]

where \( 1 < k \leq J \) and \( k \equiv 2 \mod J \). Now we can wrap the full \( b_j^\infty = \sum_{\ell=0}^{\infty} \lambda_1^{(\ell)} \) as

\[
b_j^\ell := \sum_{\ell=0}^{\infty} \check{\lambda}_1^{(J,\ell)} = \sum_{1 < k \leq J} \sum_{1 \leq j < \ell} V(k + j\ell - 1) + c,
\]

where \( c = N \sum_{1 \leq \ell < \infty} V(j\ell) \). The full finite volume Hamiltonian is

\[
\mathcal{H}^J = \sum_{1 \leq j \leq J} b_j^\ell = \sum_{1 \leq j \leq J} \sum_{1 \leq k \leq J} \sum_{-\infty < l < \infty} V(k-j+l\ell) + Jc.
\]

The infinite sum can be written in the following closed form (23.8.3. in [22])

\[
\sum_{-\infty < l < \infty} V(k + l\ell) = \varphi(k) + \frac{2}{\omega} \zeta(\frac{\omega}{2}).
\]

Substituting back and dropping the irrelevant identity operator we just obtained the original Inozemtsev’s Hamiltonian (25). We can see that our wrapping method gives the finite volume Inozemtsev’s spin chain from the infinite volume hyperbolic Inozemtsev’s spin chain which is an expectation for a consistent wrapping procedure.

VI. WRAPPING CORRECTIONS IN ADS/CFT

In this section we summarize some properties of the wrapping corrections in the planar \( N = 4 \) SYM. We show that our finite volume Hamiltonians are compatible with these requirements.

Argument 1. In the string theory side (1+1 dimensional field theory description) we know that the asymptotic data (dispersion relation and scattering matrix) defines uniquely the wrapping corrections. This fact is in agreement with our method which uniquely defines finite size Hamiltonians (23) for a given asymptotic Hamiltonian (22).
Argument 2 In the dilation operator of the $N = 4$ SYM there are unfixed parameters coming from the free choice of the renormalization scheme [23, 24]. These are unphysical parameters which disappear from the spectrum. On the asymptotic level these parameters correspond to $\epsilon_k(\lambda)$ i.e. the global rotations (3) therefore it is clear that they have no effect on the spectrum. The disappearance on finite volume is a non-trivial condition for the physical finite size Hamiltonians. It turns out, the spectrum of our finite volume Hamiltonians is free from $\epsilon_k(\lambda)$ as well [18].

Argument 3 In the asymptotic limit the spectrum of the closed sectors are completely independent from the full theory. To be more concrete, let us consider three asymptotic Hamiltonians $\mathcal{H}_{N=4}^\infty$, $\mathcal{H}_{SU(N)}^\infty$ and $\mathcal{H}_{SU(2)}^\infty$ which correspond to the $N = 4$ SYM, one of the $SU(N)$ and the $SU(2)$ long range models for which the restriction to the $SU(2)$ sector are the same i.e.

$$\mathcal{H}_{N=4}^\infty \bigg|_{SU(2)} = \mathcal{H}_{SU(N)}^\infty \bigg|_{SU(2)} = \mathcal{H}_{SU(2)}^\infty.$$ 

Clearly, the spectrum of a closed sector does not know about the full model in which it is embedded. However, we know that for proper wrapping corrections we have to consider contributions from the full spectrum (for Lüscher corrections we have to sum for all virtual particles of the mirror model [11]) therefore

$$\mathcal{H}_{N=4}^\infty \bigg|_{SU(2)} \neq \mathcal{H}_{SU(2)}^\infty.$$ 

This is an important requirement for the definition of the finite size long range Hamiltonians. Let us take our definition (23). We can see that the wrapped operator $\mathcal{X}_N^{(J,I)}$ contains a sum for a tensor product of the full local Hilbert spaces! Therefore these wrapped operators, even in the closed sub-sectors, explicitly depend on the full asymptotic Hamiltonian therefore our definition satisfies that

$$\mathcal{H}_{SU(N)}^\infty \bigg|_{SU(2)} \neq \mathcal{H}_{SU(2)}^\infty.$$ 

Argument 4 We also know that, in the wrapping corrections, extra transcendental numbers appear. For example, let us consider the Konishi operator (length 4 operator in the $SU(2)$ sector). At four loop, the asymptotic dilatation operator of the $SU(2)$ sector contains only one transcendental number $\zeta(3)$ [23, 24]. However, the length 4 Hamiltonian at four loop [23, 24] contains an extra $\zeta(5)$ compared to the asymptotic Hamiltonian. We already mentioned that our finite volume Hamiltonian includes a sum for the full one-site Hilbert space in the wrapping region, therefore extra transcendental numbers can appear in the finite size Hamiltonians if the one-site Hilbert space is infinite dimensional, which is the case for the $N = 4$ SYM. We note that transcendental numbers appear in the spectrum of higher chargers of nearest neighbor spin chains with infinite dimensional local Hilbert spaces [25].

VII. CONCLUSIONS

In this paper we generalized the algebraic framework of medium range spin chains [14] to perturbative long range spin chains [3]. Using this method we were able to define finite volume Hamiltonians (23) for every asymptotic long range models. We demonstrated that this definition is physically relevant by showing that our definition is in agreement with several physical requirements coming form the Inozemtsev’s spin chain and AdS/CFT.

We saw that our wrapping procedure (23) leads to wrapping corrections with similar properties than what we expect from the $N = 4$ SYM. This is an important result, because so far, the finite size corrections under simpler conditions could be studied using integrable field theories. From now on, the wrapping corrections can be also tested on spin chains, which can be simpler in many ways.

I believe that this result could open up a number of new research directions. One possible direction is to generalize the integrable boundary states [26, 27] for long range spin chains as well. Combining this to the method of this paper we could investigate the wrapping corrections of the overlaps between boundary and Bethe states which describes certain one- and three-point functions in $N = 4$ SYM [28–34] and ABJM theories [35, 36].

It would be interesting to apply the algebraic Bethe Ansatz, although it is not clear how this should be done due to the increasing number of auxiliary spaces. However, there are other ways to diagonalize the transfer matrices e.g. functional techniques [37] (quantum spectral curve [38] for simpler long range models?) and the separation of variables [39–41].

Another interesting directions would be to give some non-perturbative definitions of the quantities appearing in this paper (Lax-operators, transfer matrix); derivation of the Yangian symmetry [42] from our framework; connection for the $TT$-deformations of spin chains [43].

Finally, we need to address a major shortcoming of our method. The spin chain which appears in the perturbation theory of $N = 4$ SYM is dynamical which means that the Hamiltonian can change the length of the spin chain. Our method in its present form is not suitable for describing such models. In the future, we plan to extend the process to dynamic spin chains, but in the meantime, the non-dynamical Hamiltonians like (23) can serve as good toy models of wrapping effects.

It is worth to mention that a parallel research is also started on the topic of long range spin chains [44].

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Supplemental Materials: Wrapping corrections for long range spin chains

S-I. REVIEW FOR THE INTEGRABILITY OF MEDIUM RANGE SPIN CHAINS

In this section we review the algebraic construction of integrable medium range spin chains Gombor and Pozsgay [14]. Some readers might be more familiar to the “unchecked” notations for Lax- and $R$-operators therefore we start with this convention. Assuming that the interaction range is $\ell + 2$ we have to introduce $\ell + 1$ auxiliary spaces labeled by $a_1, a_2, \ldots, a_{\ell+1}$ and the tensor product of all of them is labeled by $A = (a_1, a_2, \ldots, a_{\ell+1})$. The Lax operator acts on the auxiliary space $A$ and one site of the quantum space $j$ (see figure S-1):

$$L^{(\ell)}_{A,j}(u) \equiv L^{(\ell)}_{(a_1, a_2, \ldots, a_{\ell+1}),j}(u). \quad (S-1)$$

We also define the $R$-matrix $R^{(\ell)}_{A,B}(u,v)$ for which the usual $RLL$-relation is satisfied

$$R^{(\ell)}_{A,B}(u,v)L^{(\ell)}_{A,j}(v)L^{(\ell)}_{B,j}(u) = L^{(\ell)}_{B,j}(v)L^{(\ell)}_{A,j}(u)R^{(\ell)}_{A,B}(u,v), \quad (S-2)$$

where the $A, B$ are the auxiliary space and $j$ is one site of the quantum space. The associativity of the algebra of $L^{(\ell)}(u)$-s requires the Yang-Baxter equation

$$R^{(\ell)}_{A,B}(u,v)R^{(\ell)}_{A,C}(u,w)R^{(\ell)}_{B,C}(v,w) = R^{(\ell)}_{B,C}(v,w)R^{(\ell)}_{A,C}(u,w)R^{(\ell)}_{A,B}(u,v). \quad (S-3)$$

We can also define the transfer matrix

$$T^{(\ell)}(u) = \text{Tr}_A \left( L^{(\ell)}_{A,j}(u)L^{(\ell)}_{A,j-1}(u) \ldots L^{(\ell)}_{A,1}(u) \right). \quad (S-4)$$

The commutativity of the transfer matrix

$$[T^{(\ell)}(u), T^{(\ell)}(v)] = 0 \quad (S-5)$$

is a simple consequence of the $RLL$-relation. To obtain local conserved charges we introduce the regularity condition

$$L^{(\ell)}_{A,j}(u) \equiv L^{(\ell)}_{(a_1, a_2, \ldots, a_{\ell+1}),j}(u) = P_{a_1,j}P_{a_2,j} \ldots P_{a_{\ell+1},j} \left( 1 + u h^{(\ell)}_{a_1,a_2,\ldots,a_{\ell+1},j} + O(u^2) \right). \quad (S-6)$$
The first consequence of the regularity condition is that the transfer matrix is a \( \ell + 1 \) site shift operator (see figure S-3)

\[
T^{(\ell)}(0) = U^{\ell+1},
\]

(S-7)

where \( U \) is the one site shift operator

\[
U = P_{1,2}P_{2,3} \ldots P_{J-1,J}.
\]

(S-8)

The second consequence of the regularity condition is that the transfer matrix gives local Hamiltonian with interaction range \( \ell + 2 \) (assuming \( J \geq \ell + 2 \):

\[
\frac{d}{du} \log T^{(\ell)}(u) \bigg|_{u=0} = \sum_{j=1}^{J} h^{(\ell)}_j = H^{(\ell)}.
\]

(S-9)

We can see that series expansion is more simpler if we introduce the "checked" operators as (see figure S-4)

\[
L^{(\ell)}_{(a_1,a_2,\ldots,a_{\ell+1}),j}(u) = P_{a_1,j}P_{a_2,j} \ldots P_{a_{\ell+1},j} \tilde{L}^{(\ell)}_{a_1,a_2,\ldots,a_{\ell+1},j}(u),
\]

(R-10)

\[
R^{(\ell)}_{(a_1,\ldots,a_{\ell+1}),(b_1,\ldots,b_{\ell+1})}(u,v) = P_{a_1,b_1} \ldots P_{a_{\ell+1},b_{\ell+1}} \tilde{R}^{(\ell)}_{a_1,\ldots,a_{\ell+1},b_1,\ldots,b_{\ell+1}}(u,v).
\]

(S-11)

Substituting back to (S-2) we obtain the RLL-relation in the "checked" convention

\[
\tilde{R}^{(\ell)}_{a_2,\ldots,a_{\ell+1},b_1,\ldots,b_{\ell+1},j}(u,v) \tilde{L}^{(\ell)}_{a_1,a_2,\ldots,a_{\ell+1},b_1}(u) \tilde{L}^{(\ell)}_{b_1,b_2,\ldots,b_{\ell+1},j}(v) = \tilde{L}^{(\ell)}_{a_1,a_2,\ldots,a_{\ell+1},b_1}(v) \tilde{L}^{(\ell)}_{b_1,b_2,\ldots,b_{\ell+1},j}(u) \tilde{R}^{(\ell)}_{a_1,\ldots,a_{\ell+1},b_1,\ldots,b_{\ell+1}}(u,v).
\]

(S-12)

Introducing new labeling for the sites as \( a_1, \ldots, a_{\ell+1} \leftrightarrow 1, \ldots, \ell + 1 \) and \( b_1, \ldots, b_{\ell+1} \leftrightarrow \ell + 2, \ldots, 2\ell + 2 \) and \( j \leftrightarrow 2\ell + 3 \) we obtain that

\[
\tilde{R}^{(\ell)}_{2,\ldots,2\ell+3}(u,v) \tilde{L}^{(\ell)}_{1,\ldots,\ell+2}(u) \tilde{L}^{(\ell)}_{\ell+2,\ldots,2\ell+3}(v) = \tilde{L}^{(\ell)}_{1,\ldots,\ell+2}(v) \tilde{L}^{(\ell)}_{\ell+2,\ldots,2\ell+3}(u) \tilde{R}^{(\ell)}_{1,\ldots,2\ell+2}(u,v).
\]

(S-13)

We can see that the "checked" operators always act on neighboring sites therefore, it is unnecessary to write out all the sites where the "checked" operators act, it is sufficient to label only the one on the left as

\[
\tilde{L}^{(\ell)}_{j}(u) = \tilde{L}^{(\ell)}_{j,j+\ell+1}(u),
\]

(S-14)

\[
\tilde{R}^{(\ell)}_{j}(u,v) = \tilde{R}^{(\ell)}_{j,j+2\ell+1}(u,v).
\]

(S-15)

We can also introduce the "checked" version of the transfer matrix

\[
\tilde{T}^{(\ell)}(u) = \tilde{T}_{J,J+1} \left( \tilde{L}^{(\ell)}_{J}(u) \ldots \tilde{L}^{(\ell)}_{1}(u) \right).
\]

(S-16)
Since the checked Lax-operator has the regularity property \( \tilde{L}_j^{(\ell)}(0) = 1 \) the checked transfer matrix is identity at \( u = 0 \), i.e.,

\[
\tilde{T}^{(\ell)}(0) = 1. \tag{S-17}
\]

We can also show the following connection between the two conventions for the transfer matrices (see figure S-2)

\[
T^{(\ell)}(u) = \tilde{T}^{(\ell)}(u)U^{\ell+1}. \tag{S-18}
\]

The commutativity of the unchecked transfer matrix (S-5) can be derived in the usual way:

\[
T^{(\ell)}(u)T^{(\ell)}(v) = \text{Tr}_{A,B} \left( L_A^{(\ell)}(u)L_B^{(\ell)}(v) \cdots L_{A,1}^{(\ell)}(u)L_{B,1}^{(\ell)}(v) \right) = \\
\text{Tr}_{A,B} \left( R_A^{(\ell)}(u,v)^{-1} L_A^{(\ell)}(u,v)L_{A,J}^{(\ell)}(u) \cdots L_{A,1}^{(\ell)}(u)L_{B,1}^{(\ell)}(v) \right) = \\
\text{Tr}_{A,B} \left( R_A^{(\ell)}(u,v)^{-1} L_B^{(\ell)}(v)L_{A,J}^{(\ell)}(u) \cdots L_{A,1}^{(\ell)}(u)R_{A,B}^{(\ell)}(u,v) \right) = \\
\text{Tr}_{A,B} \left( L_A^{(\ell)}(u) \cdots L_{B,1}^{(\ell)}(u)R_{A,B}^{(\ell)}(u,v) \right) = T^{(\ell)}(v)T^{(\ell)}(u), \tag{S-19}
\]

where we used the RLL-relation in the third line. From (S-18) it is clear that the unchecked transfer matrices are also commuting. We note that the proof can be also done using the checked notations [45]. It is clear that the derivation (S-19) is independent of the length of the spin chain \( J \) therefore the transfer matrix is well defined and gives commuting charges even when \( J < \ell + 2 \).

**S-II. PROPERTIES OF THE TRUNCATED RLL-RELATION**

In this section we investigate the truncated RLL-relation

\[
\tilde{R}_2^{(\ell)} \tilde{L}_1^{(\ell)}(u) \tilde{L}_{\ell+2}^{(\ell)}(v) = \tilde{L}_1^{(\ell)}(v) \tilde{L}_{\ell+2}^{(\ell)}(u) \tilde{R}_1^{(\ell)} + O(\lambda^{\ell+1}). \tag{S-20}
\]

This relation is equivalent with \( \ell + 1 \) different equations

\[
\sum_{k=0}^{r} \sum_{l=0}^{r-k} \tilde{R}_2^{(\ell,r-k-l)} \tilde{L}_1^{(k)}(u)\tilde{L}_{\ell+2}^{(l)}(v) = \sum_{k=0}^{r} \sum_{l=0}^{r-k} \tilde{L}_1^{(k)}(v)\tilde{L}_{\ell+2}^{(l)}(u)\tilde{R}_1^{(r-k-l)}, \tag{S-21}
\]

for \( r = 0,1, \ldots, \ell \). We can see that the operators \( \tilde{R}_1^{(\ell,r)} \), \( \tilde{L}_1^{(\ell)}(u) \) appear only in the equation \( r = \ell \). In the following we will show that the remaining equations \( r = 0,1, \ldots, \ell - 1 \) are equivalent to the equations coming from the previous order

\[
\tilde{R}_2^{(\ell-1)} \tilde{L}_1^{(\ell-1)}(u) \tilde{L}_{\ell+1}^{(\ell-1)}(v) = \tilde{L}_1^{(\ell-1)}(v) \tilde{L}_{\ell+1}^{(\ell-1)}(u) \tilde{R}_1^{(\ell-1)} + O(\lambda^{\ell}). \tag{S-22}
\]

At first, let us define the operator

\[
\lambda_1^{(\ell)} = \tilde{L}_{\ell+1}^{(\ell-1)}(u)\tilde{R}_1^{(\ell-1)} + O(\lambda^{\ell}), \tag{S-23}
\]

with range \( 2\ell + 1 \). In the following, we prove that, it satisfies the RLL-relations (S-21) for \( r = 0, \ldots, \ell - 1 \), i.e.,

\[
\lambda_2^{(\ell)} \tilde{L}_1^{(\ell)}(u)\tilde{L}_{\ell+2}^{(v)} = \tilde{L}_1^{(v)}(v)\tilde{L}_{\ell+2}^{(u)}(u)\lambda_1^{(\ell)} + O(\lambda^{\ell}). \tag{S-24}
\]

Since \( \tilde{L}_1^{(\ell)}(u) = \tilde{L}_1^{(\ell-1)}(u) + O(\lambda^{\ell}) \), the equation (S-24) is equivalent to

\[
\lambda_2^{(\ell)} \tilde{L}_1^{(\ell-1)}(u)\tilde{L}_{\ell+2}^{(\ell-1)}(v) = \tilde{L}_1^{(\ell-1)}(v)\tilde{L}_{\ell+2}^{(\ell-1)}(u)\lambda_1^{(\ell)} + O(\lambda^{\ell}). \tag{S-25}
\]

Let us substitute (S-23) to (S-25).

\[
\tilde{L}_{\ell+2}^{(\ell-1)}(u)\tilde{R}_2^{(\ell-1)} \tilde{L}_1^{(\ell-1)}(u)\tilde{L}_{\ell+1}^{(\ell-1)}(v) = \tilde{L}_1^{(\ell-1)}(v)\tilde{L}_{\ell+2}^{(\ell-1)}(u)\tilde{L}_{\ell+1}^{(\ell-1)}(u)\tilde{R}_1^{(\ell-1)} \tilde{J}_{\ell+1}^{(\ell-1)}(v) + O(\lambda^{\ell}). \tag{S-26}
\]
Since \( \hat{L}^{(t-1)}_1 = \hat{L}^{(t-1)}_{1,2,\ldots,t+1} \), we have commuting operators
\[
\left[ \hat{L}^{(t-1)}_1(u), \hat{L}^{(t-1)}_{\ell+2}(v) \right] = 0,
\]
therefore we can obtain that
\[
\hat{R}^{(t-1)}_2 \hat{L}^{(t-1)}_1(u) = \hat{L}^{(t-1)}_1(v) \hat{L}^{(t-1)}_{\ell+1} \hat{R}^{(t-1)}_1 \hat{J}^{(t-1)}_{\ell+1}(v) + O(\lambda^\ell),
\]
where we used that the \( \hat{J} \) is the inverse of \( \hat{L} \) i.e.
\[
\hat{J}^{(t-1)}_1(u) \hat{L}^{(t-1)}_1(u) = 1 + O(\lambda^\ell).
\]

Multiplying \( \hat{L}^{(t-1)}_{\ell+1}(v) \) from the right we obtain that
\[
\hat{R}^{(t-1)}_2 \hat{L}^{(t-1)}_1(u) \hat{L}^{(t-1)}_{\ell+1}(v) = \hat{L}^{(t-1)}_1(v) \hat{L}^{(t-1)}_{\ell+1}(u) \hat{R}^{(t-1)}_1 + O(\lambda^\ell),
\]
which is satisfied by the initial condition (S-22) therefore we just proved (S-24). Since (S-24) is the defining equation of \( \hat{R}^{(t)} \) up to order \( O(\lambda^\ell) \) we can identify \( \hat{R}^{(t)} \) with \( \lambda^{(t)} \) up to this order i.e.
\[
\hat{R}^{(t)}_1 = \lambda^{(t)} + O(\lambda^\ell) = \hat{L}^{(t-1)}_{\ell+1}(u) \hat{R}^{(t-1)}_1 \hat{J}^{(t-1)}_{\ell+1}(v) + O(\lambda^\ell).
\]

Let us summarize what we obtained. If we have operators
\[
\hat{R}^{(t-1)}_1 = \sum_{k=0}^{t-1} \lambda^k \hat{R}^{(t-1,k)}_1, \quad \hat{L}^{(t-1)}_1 = \sum_{k=0}^{t-1} \lambda^k \hat{L}^{(k)}_1, \quad \hat{J}^{(t-1)}_1 = \sum_{k=0}^{t-1} \lambda^k \hat{J}^{(k)}_1,
\]
which satisfy the truncated RLL-equations (S-22) and inversion relation (S-29) then the operators
\[
\hat{R}^{(t,r)}_1 = \sum_{k=0}^{r} \sum_{l=0}^{r-k} \hat{R}^{(t-1,r-k-l)}_1 \hat{J}^{(t)}_{\ell+1}(v),
\]
for \( r = 0, \ldots, t-1 \) satisfy the next level truncated RLL-equations (S-21) for \( r = 0, \ldots, t-1 \). From the remaining \( r = t \) equation
\[
\sum_{k=0}^{t} \sum_{l=0}^{t-k} \hat{R}^{(t,k)}_1(u) \hat{L}^{(l)}_{\ell+2}(v) = \sum_{k=0}^{t} \sum_{l=0}^{t-k} \hat{L}^{(k)}_1(v) \hat{L}^{(l)}_{\ell+2}(u) \hat{R}^{(t,k-l)}_1,
\]
we can obtain the operators \( \hat{R}^{(t)}_1 \) and \( \hat{L}^{(t)}_1 \) which are the only remaining terms of the operators
\[
\hat{R}^{(t)}_1 = \sum_{k=0}^{t} \lambda^k \hat{R}^{(t,k)}_1, \quad \hat{L}^{(t)}_1 = \sum_{k=0}^{t} \lambda^k \hat{L}^{(k)}_1.
\]

Defining the operator
\[
\hat{J}^{(t)}_1(u) = -\left( \sum_{k=0}^{t-1} \hat{J}^{(k)}_1(u) \hat{L}^{(t-k)}_1(u) \right) \hat{J}^{(0)}_1(u),
\]
we also obtain the inverse of the Lax in the next level
\[
\hat{J}^{(t)}_1 = \sum_{k=0}^{t} \lambda^k \hat{J}^{(k)}_1.
\]

We can see that we obtained a recursion procedure to solve the truncated RLL-relations (see the left figure of S-5 for a summary).
Figure S-5. Two strategy for finding the Lax-operators and R-matrices.

Figure S-6. The graphical proof of identities (S-38) and (S-41) for $J = 4$, $\ell = 3$.

**S-III. IDENTITIES OF THE TWISTED TRACE**

In this section, we show some useful identities of the twisted trace $\widehat{\text{Tr}}_{J,\ell}$. Let $X_1^{(J+\ell-2)}$ be a range $J + \ell$ operator. We can easily show that (see also figure S-6)

$$\widehat{\text{Tr}}_{J,\ell+k} \left( X_1^{(J+\ell-2)} \right) = \widehat{\text{Tr}}_{J,\ell} \left( X_1^{(J+\ell-2)} \right),$$

(S-38)

for $k \geq 0$. We can obtain analogous identities for the shifted operators

$$\widehat{\text{Tr}}_{J,\ell+k} \left( X_1^{(J+\ell-2)} \right) = \widehat{\text{Tr}}_{J,\ell+1} \left( X_1^{(J+\ell-2)} \right),$$

(S-39)

for $k \geq \ell - 1$. We can see that the action of $\widehat{\text{Tr}}_{J,\ell}$ does not depend on $\ell$ for large enough $\ell$ therefore in the following we use a shorter notation

$$\widehat{\text{Tr}}_J \left( X_k^{(J)} \right) := \widehat{\text{Tr}}_{J,n} \left( X_k^{(J)} \right),$$

(S-40)

for $n \geq \max(1, k + \ell + 2 - J)$.

Another useful identity is (see also figure S-6)

$$\widehat{\text{Tr}}_J \left( X_k^{(J)} \right) = U_{J}^{k-1} \widehat{\text{Tr}}_J \left( X_1^{(J)} \right) U_{J}^{1-k},$$

(S-41)

where we used the shift operator

$$U_J = P_{1,2}P_{2,3} \ldots P_{J-1,J}.$$  

(S-42)
Using these definitions, the wrapped Hamiltonians have the following equivalent forms

\[ \mathcal{H}^J = \sum_{j=1}^{J} \sum_{\ell=0}^{\infty} \mathcal{A}_j^{(J,\ell)} = \sum_{j=1}^{J} \sum_{\ell=0}^{\infty} \mathcal{A}_j^{(J)} = \sum_{j=1}^{J} \sum_{\ell=0}^{\infty} U_j^{\ell-1} \mathcal{A}_j^{(J)} U_j^{1-\ell}. \]  
\[ \text{(S-43)} \]

Now let us continue with the identities corresponding to the twisted trace of products of operators:

\[ \mathcal{A}_j^{(m)} = \mathcal{A}_j^{(1)} \mathcal{A}_j^{(m)} = \mathcal{A}_j^{(1)} \mathcal{A}_j^{(m)} \]

for \( k = 1, \ldots, m + 2 \) where \( \mathcal{A}_j^{(m)} \) and \( \mathcal{A}_j^{(m)} \) are range \( m + 2 \) operators. In the following we assume that \( m + 2 > J \)
but our formulas work for shorter operators as well, since a shorter operator \( \mathcal{A}_j^{(n)} = \mathcal{A}_j^{(n+1)} \) for \( n < m \) always can be considered as a range \( m + 2 \) operator \( \mathcal{A}_j^{(n)} = \mathcal{A}_j^{(m+1)} \) which acts identically on the sites \( n + 2, \ldots, m + 1 \) i.e. \( \mathcal{A}_j^{(m)} = \mathcal{A}_j^{(n)} \).

The first class of identities is

\[ \mathcal{A}_j^{(m)} = \mathcal{A}_j^{(1)} \mathcal{A}_j^{(m)} = \mathcal{A}_j^{(1)} \mathcal{A}_j^{(m)} \]

where \( k = 1, \ldots, J \).

The second class of identities is

\[ \mathcal{A}_j^{(m)} = \mathcal{A}_j^{(1)} \mathcal{A}_j^{(m)} = \mathcal{A}_j^{(1)} \mathcal{A}_j^{(m)} \]

for \( k = J + 1, \ldots, m + 2 \).

The third class of identities is

\[ \mathcal{A}_j^{(m)} = \mathcal{A}_j^{(1)} \mathcal{A}_j^{(m)} = \mathcal{A}_j^{(1)} \mathcal{A}_j^{(m)} \]

for \( k = 1, \ldots, m + 2 - J \).

The fourth class of identities is

\[ \mathcal{A}_j^{(m)} = \mathcal{A}_j^{(1)} \mathcal{A}_j^{(m)} = \mathcal{A}_j^{(1)} \mathcal{A}_j^{(m)} \]

for \( k = m + 3 - J, \ldots, m + 2 \).

The graphical proof of these identities are showed in figure S-7 for \( J = 4, m = 4 \).

S-IV. THE \( \epsilon \)-DEPENDENCE OF THE FINITE VOLUME HAMILTONIANS

Let us take an asymptotic Hamiltonian \( \mathcal{H}^{a,\infty} \) and a local operator

\[ \mathcal{V} = \sum_{j=-\infty}^{\infty} \mathcal{V}_j^{(\ell)}, \]  
\[ \text{(S-48)} \]

where \( \mathcal{V}_j^{(\ell)} \) is a \( \lambda \) dependent range \( \ell + 2 \) operator. We can obtain a new integrable asymptotic Hamiltonian by the rotation

\[ \mathcal{H}^{a,\infty} = \exp(a \mathcal{V}) \mathcal{H}^{0,\infty} \exp(-a \mathcal{V}) = \sum_{j=-\infty}^{\infty} \sum_{r=0}^{\infty} \mathcal{A}_j^{a,(r)} = \sum_{r=0}^{\infty} \mathcal{X}_j^{a,(r)}, \]  
\[ \text{(S-49)} \]

where \( \mathcal{X}_j^{a,(r)} \) is a range \( r + 2 \) operator. In this section we investigate the connection between the wrapped Hamiltonians \( \mathcal{H}^{a,J} \) and \( \mathcal{H}^{0,J} \) corresponding to the asymptotic ones \( \mathcal{H}^{a,\infty} \) and \( \mathcal{H}^{0,\infty} \), respectively. The finite volume Hamiltonian reads as

\[ \mathcal{H}^{a,J} = \sum_{j=1}^{J} \sum_{\ell=0}^{\infty} \mathcal{A}_j^{a,(\ell)} \]  
\[ \text{(S-50)} \]
Figure S-7. The graphical proof of identities (S-44)-(S-47) for \( J = 4 \), \( m = 4 \).

At first, let us take the derivative of the asymptotic Hamiltonian with respect to \( a \).

\[
\frac{d}{da} \mathcal{H}_a^{a, \infty} = [\mathcal{Y}, \mathcal{H}_a^{a, \infty}] = \sum_{r=0}^{\infty} \left[ \mathcal{Y}, \mathcal{X}^{a,(r)} \right].
\]  

(S-51)

Let us calculate a term from the r.h.s.

\[
\left[ \mathcal{Y}, \mathcal{X}^{a,(r)} \right] = \sum_{j} \left[ \mathcal{Y}_{j}^{(\ell)}, \mathcal{X}_{j}^{a,(r)} + \mathcal{X}_{j+1}^{a,(r)} + \cdots + \mathcal{X}_{j+r+1}^{a,(r)} + \cdots + \mathcal{X}_{j-r-1}^{a,(r)} \right] = \sum_{j} \left[ \mathcal{Y}_{j}^{(\ell)}, \mathcal{X}_{j}^{a,(r)} + \mathcal{X}_{j+1}^{a,(r)} + \cdots + \mathcal{X}_{j+r+1}^{a,(r)} \right] + \left[ \mathcal{Y}_{j+1}^{(\ell)} + \cdots + \mathcal{Y}_{j+r+1}^{(\ell)}, \mathcal{X}_{j}^{a,(r)} \right].
\]  

(S-52)
Introducing the operator
\[
\mathcal{M}_j^{(r+\ell+1)} = \left[ Y_j^{(\ell)}, X_j^{a,(r)} + X_{j+1}^{a,(r)} + \ldots + X_{j+\ell}^{a,(r)} \right] + \left[ Y_{j+1}^{(\ell)} + \ldots + Y_{j+r+1}^{(\ell)}, X_j^{a,(r)} \right],
\]  
(S-53)
the equation (S-51) can be written as
\[
\sum_{j=-\infty}^{\infty} \sum_{r=0}^{\infty} \frac{d}{da} X_j^{a,(r)} = \sum_{j=-\infty}^{\infty} \sum_{r=0}^{\infty} \mathcal{M}_j^{(r+\ell+1)},
\]  
(S-54)
from which we can obtain that
\[
\sum_{r=0}^{\infty} \frac{d}{da} X_j^{a,(r)} = \sum_{r=0}^{\infty} \mathcal{M}_j^{(r+\ell+1)} + \sum_{r=0}^{\infty} \left( \mathcal{F}_j^{(r)} - \mathcal{F}_{j+1}^{(r)} \right),
\]  
(S-55)
where \( \mathcal{F}_j^{(r)} \) is a range \( r + 2 \) operator. Now let us calculate the derivative of the wrapped Hamiltonian
\[
\frac{d}{da} \mathcal{H}^{a,J} = \sum_{j=1}^{J} \hat{\text{Tr}} J \left( \sum_{r=0}^{\infty} \frac{d}{da} X_j^{a,(r)} \right) = \sum_{j=1}^{J} \sum_{r=0}^{\infty} \hat{\text{Tr}} J \left( \mathcal{M}_j^{(r+\ell+1)} \right) + \sum_{j=1}^{J} \sum_{r=0}^{\infty} \hat{\text{Tr}} J \left( \mathcal{F}_j^{(r)} - \mathcal{F}_{j+1}^{(r)} \right),
\]  
(S-56)
where we used (S-50) and (S-55). Since \( U_j^J = 1 \), the second sum is vanishing
\[
\sum_{j=1}^{J} \sum_{r=0}^{\infty} \hat{\text{Tr}} J \left( \mathcal{F}_j^{(r)} - \mathcal{F}_{j+1}^{(r)} \right) = \sum_{j=1}^{J} \sum_{r=0}^{\infty} \hat{\text{Tr}} J \left( \mathcal{M}_j^{(r+\ell+1)} \right) + \sum_{j=1}^{J} \sum_{r=0}^{\infty} \hat{\text{Tr}} J \left( \mathcal{F}_j^{(r)} - \mathcal{F}_{j+1}^{(r)} \right),
\]  
(S-57)
therefore the derivative simplifies as
\[
\frac{d}{da} \mathcal{H}^{a,J} = \sum_{r=0}^{\infty} \sum_{j=1}^{J} \mathcal{M}_j^{(r+\ell+1)},
\]  
(S-58)
where we defined that
\[
\mathcal{M}_j^{(r+\ell+1)} = \hat{\text{Tr}} J \left( \mathcal{M}_j^{(r+\ell+1)} \right).
\]  
(S-59)
Let us fix the number \( r \) and let \( m = \max(r, \ell) \). We can also enlarge the ranges of operators \( \mathcal{V} \) and \( \mathcal{X} \) to \( m \) as
\[
\mathcal{V}_1^{(m)} := \mathcal{V}_1^{(\ell)}, \quad \mathcal{X}_1^{a,(m)} := \mathcal{X}_1^{a,(r)},
\]  
(S-60)
and we can obtain that
\[
\mathcal{M}_1^{(r+\ell+1)} = \mathcal{M}_1^{(2m+1)} := \left[ \mathcal{V}_1^{(m)}, \mathcal{X}_1^{a,1,(m)} + \mathcal{X}_2^{a,1,(m)} + \ldots + \mathcal{X}_{m+1,2}^{a,1,(m)} \right] + \left[ \mathcal{V}_2^{(m)} + \ldots + \mathcal{V}_m^{(m)}, \mathcal{X}_1^{a,1,(m)} \right].
\]  
(S-61)
Let us rearrange the operators as
\[
\mathcal{M}_1^{(2m+1)} = A^1 - A^2 + B^1 - B^2 + C^1 - C^2 + D^1 - D^2,
\]  
(S-62)
where
\[
A^1 = \sum_{k=1}^{J} \mathcal{V}_k^{(m)} \mathcal{X}_k^{a,1,(m)}, \quad A^2 = \sum_{k=2}^{J+1} \mathcal{X}_k^{(m)} \mathcal{V}_1^{(m)},
\]  
(B-63)
\[
B^1 = \sum_{k=J+1}^{m+2} \mathcal{V}_k^{(m)} \mathcal{X}_k^{a,1,(m)}, \quad B^2 = \sum_{k=1}^{m+2} \mathcal{X}_k^{a,1,(m)} \mathcal{V}_1^{(m)},
\]  
(C-63)
\[
C^1 = \sum_{k=2}^{m+2} \mathcal{V}_k^{(m)} \mathcal{X}_k^{a,1,(m)}, \quad C^2 = \sum_{k=J+2}^{m+2} \mathcal{X}_k^{a,1,(m)} \mathcal{V}_1^{(m)},
\]  
(D-63)
\[
D^1 = \sum_{k=m+3-J}^{m+2} \mathcal{V}_k^{(m)} \mathcal{X}_k^{a,1,(m)}, \quad D^2 = \sum_{k=m+3-J}^{m+2} \mathcal{X}_k^{a,1,(m)} \mathcal{V}_1^{(m)}.
\]  

Using the identities (S-45)-(S-47) we can obtain that
\[
\tilde{\text{Tr}}_J (B^1) = \tilde{\text{Tr}}_J (B^2), \quad (S-64)
\]
\[
\tilde{\text{Tr}}_J (C^1) = \tilde{\text{Tr}}_J (C^2), \quad (S-65)
\]
\[
\tilde{\text{Tr}}_J (D^1) = \sum_{k=1}^J \tilde{\text{Tr}}_J \left( \mathcal{Y}^1(m)_k \right) \tilde{\text{Tr}}_J \left( \mathcal{X}^r(m)_k \right), \quad (S-66)
\]
\[
\tilde{\text{Tr}}_J (D^2) = \sum_{k=1}^J \tilde{\text{Tr}}_J \left( \mathcal{X}^a(m)_k \right) \tilde{\text{Tr}}_J \left( \mathcal{Y}^m(m)_k \right). \quad (S-67)
\]

Using the identities
\[
\tilde{\text{Tr}}_J \left( \mathcal{X}^a(m)_k \right) = \tilde{\text{Tr}}_J \left( \mathcal{X}^a(r)_k \right), \quad \tilde{\text{Tr}}_J \left( \mathcal{Y}^m(m)_k \right) = \tilde{\text{Tr}}_J \left( \mathcal{Y}^\ell(m)_k \right), \quad (S-68)
\]
we can obtain that
\[
\tilde{\text{Tr}}_J \left( D^1 \right) = \sum_{k=1}^J \tilde{\mathcal{Y}}^{(J,\ell)}_k \tilde{\mathcal{X}}^{a,(J,r)}_k, \quad \tilde{\text{Tr}}_J \left( D^2 \right) = \sum_{k=1}^J \tilde{\mathcal{X}}^{a,(J,r)}_1 \tilde{\mathcal{Y}}^{(J,\ell)}_k, \quad (S-69)
\]
where we defined the wrapped operators
\[
\tilde{\mathcal{X}}^{a,(J,\ell)}_k = \tilde{\text{Tr}}_J \left( \mathcal{X}^{a,(\ell)}_k \right), \quad \tilde{\mathcal{Y}}^{(J,\ell)}_k = \tilde{\text{Tr}}_J \left( \mathcal{Y}^{(\ell)}_k \right). \quad (S-70)
\]

Using the identity (S-44), we can obtain that
\[
\mathcal{A}^1 = \sum_{k=1}^J U^{1-k}_{J-k} \tilde{\text{Tr}}_J \left( \mathcal{X}^{a,(m)}_k \mathcal{Y}^{1(m)}_1 \right) U^{1-k}_{J-k} = \sum_{k=2}^{J+1} U^{1-k}_{J-k} \tilde{\text{Tr}}_J \left( \mathcal{X}^{a,(m)}_k \mathcal{Y}^{1(m)}_1 \right) U^{1-k}_{J-k}. \quad (S-71)
\]
Substituting back to (S-62), we obtain that
\[
\tilde{\mathcal{M}}^{l+1} \sum_{k=1}^J \left( \mathcal{Y}^{(J,\ell)}_k \tilde{\mathcal{X}}^{a,(J,r)}_k - \tilde{\mathcal{X}}^{a,(J,r)}_k \mathcal{Y}^{(J,\ell)}_k \right) \right) + \sum_{k=2}^{J+1} \left( U^{1-k}_{J-k} \tilde{\text{Tr}}_J \left( \mathcal{X}^{a,(m)}_k \mathcal{Y}^{1(m)}_1 \right) U^{1-k}_{J-k} - \tilde{\text{Tr}}_J \left( \mathcal{X}^{a,(m)}_k \mathcal{Y}^{1(m)}_1 \right) \right), \quad (S-72)
\]
therefore whole sum simplifies as
\[
\sum_{j=1}^J \tilde{M}^{(J,\ell)}_j = \sum_{j,k=1}^J \left[ \tilde{\mathcal{Y}}^{(J,\ell)}_j, \tilde{\mathcal{X}}^{a,(J,r)}_k \right] = \sum_{j=1}^J \left[ \mathcal{Y}^{J,\ell}_j, \tilde{\mathcal{X}}^{a,(J,r)}_j \right], \quad (S-73)
\]
where we defined that
\[
\mathcal{Y}^{J,\ell}_j = \sum_{j=1}^J \tilde{\mathcal{Y}}^{(J,\ell)}_j. \quad (S-74)
\]
Substituting back to (S-58) we just obtained a differential equation for the finite volume Hamiltonian
\[
\frac{d}{da} \mathcal{H}^{a,J} = \sum_{r=0}^\infty \sum_{j=1}^J \left[ \mathcal{Y}^{J,\ell}_j, \tilde{\mathcal{X}}^{a,(J,r)}_j \right] = \left[ \mathcal{Y}^{J,\ell}_r, \mathcal{H}^{a,J} \right]. \quad (S-75)
\]
Clearly, the solution is
\[
\mathcal{H}^{a,J} = \exp(a\mathcal{Y}^{J,\ell}_r) \mathcal{H}^{0,J} \exp(-a\mathcal{Y}^{J,\ell}_r). \quad (S-76)
\]
We can see that the spectrum of the Hamiltonian is independent from the rotations (parameters \(\epsilon_n(\lambda)\)) even for the finite volume Hamiltonians!
S-V. LAX-OPERATORS OF THE GL(N) LONG RANGE SPIN CHAIN

In this section we demonstrate that, the Lax operators exist for the GL(N) long range spin chains. Now, we follow a slightly different method than that described in the first section. At first, we fix the integrable charge densities \( q_k \) which define the charges

\[
\tilde{Q}_k^{(\ell)} = \sum_{r=0}^{\ell} \lambda^r \tilde{Q}_k^{(r)}, \quad \tilde{Q}_k^{(0)} = \sum_{j=1}^{J} k_{j} \frac{1}{\ell+k}
\]

in the asymptotic region \( J \geq \ell + k \). We derive the corresponding Lax-operators from the commutation relations

\[
\left[ \tilde{Q}_k^{(\ell)}, \tilde{T}^{(\ell)}(u) \right] = O(\lambda^{\ell+1}). \tag{S-78}
\]

These relations are equivalent with the equations

\[
\sum_{k=0}^{\ell} \left[ \tilde{Q}_k^{(k)}, \tilde{T}^{(\ell-k)}(u) \right] = 0. \tag{S-79}
\]

We know that, the long range charges have \( \gamma \)-ambiguities i.e. the linear combinations of the charges \( \tilde{Q}_k^{(\ell)} \) define equivalent long range models. Since we only require the commutativity (S-78), the charges coming from the transfer matrix can be in different \( \gamma \) convention. Let us introduce the charges from the derivatives of the transfer matrix

\[
\tilde{Q}_k^{(\ell)} = \left. \frac{\partial^k}{\partial u^k} \log \tilde{T}^{(\ell)}(u) \right|_{u=0} + O(\lambda^{\ell+1}). \tag{S-80}
\]

Since (S-78) is satisfied, the new charges \( \tilde{Q}_k^{(\ell+1)} \) can be expressed by the original ones as [3]

\[
\tilde{Q}_k^{(\ell+1)} = \sum_{s=2}^{\ell+r} \gamma_r(\lambda) \tilde{Q}_s^{(\ell)} + O(\lambda^{\ell+1}), \tag{S-81}
\]

where

\[
\gamma_r(\lambda) = \sum_{k=\max(s-r,0)}^{\infty} \lambda^k \gamma_r^{(k)}. \tag{S-82}
\]

Substituting back, we can obtain an equivalent form

\[
\tilde{Q}_k^{(\ell)} = \gamma_r(\lambda) \tilde{Q}_s^{(\ell)} + \sum_{s=2}^{\ell+r} \sum_{k=\max(s-r,0)}^{\ell} \gamma_r^{(k)} \tilde{Q}_s^{(\ell-k)}. \tag{S-83}
\]

Let us write down the first three order of the Hamiltonian explicitly

\[
\tilde{Q}_2^{(0)} = \gamma_2^{(0)} \tilde{Q}_2^{(0)} + \gamma_2^{(1)} \tilde{Q}_2^{(1)} + \gamma_2^{(2)} \tilde{Q}_2^{(2)}, \tag{S-84}
\]

\[
\tilde{Q}_2^{(1)} = \gamma_2^{(1)} \tilde{Q}_2^{(1)} + \gamma_2^{(2)} \tilde{Q}_2^{(2)} + \gamma_2^{(3)} \tilde{Q}_2^{(3)} + \gamma_2^{(4)} \tilde{Q}_2^{(4)} + \gamma_2^{(5)} \tilde{Q}_2^{(5)} + \gamma_2^{(6)} \tilde{Q}_2^{(6)} + \gamma_2^{(7)} \tilde{Q}_2^{(7)} + \gamma_2^{(8)} \tilde{Q}_2^{(8)} + \gamma_2^{(9)} \tilde{Q}_2^{(9)} + \gamma_2^{(10)} \tilde{Q}_2^{(10)}. \tag{S-85}
\]

Our strategy is summarized on the right figure of S-5. Assuming that we have the operators \( \tilde{L}_1^{(r)}, \tilde{j}_1^{(r)}, \tilde{R}_1^{(r)} \) for \( r = 0, \ldots, \ell - 1 \), we can calculate the transfer matrices \( T^{(\ell)}(u) \) for \( r = 0, \ldots, \ell - 1 \). Having an ansatz for \( \tilde{L}_1^{(\ell)} \) we can also calculate \( T^{(\ell)}(u) \), and using equation (S-79), we can obtain the explicit form of \( \tilde{L}_1^{(\ell)} \). From the equations (S-33), we can calculate \( \tilde{R}_1^{(\ell,r)} \) for \( r = 0, \ldots, \ell - 1 \). After that, we can obtain \( \tilde{R}_1^{(\ell,\ell)} \) from (S-34). Finally we can calculate \( \tilde{j}_1^{(\ell)} \) from (S-36) therefore we just have the next level operators \( \tilde{L}_1^{(\ell)}, \tilde{j}_1^{(\ell)}, \tilde{R}_1^{(\ell,\ell)} \) for \( r = 0, \ldots, \ell \).
In this work we only calculate the first two order. We use the following conventions for the previously calculated integrable charges \[3\]

\[
\bar{q}^{2,0}_1 = () - (12), \quad \text{(S-87)}
\]

\[
\bar{q}^{2,1}_1 = \alpha^{(1)}_0 (-3() + 4(12) - (13)), \quad \text{(S-88)}
\]

\[
\bar{q}^{2,2}_1 = \left(\alpha^{(1)}_0\right)^2 (20() - 29(12) + 10(13) - (14) - (1234) + (1243) + (1342) - (1432)) +
\]

\[
+ \alpha^{(2)}_0 (-3() + 4(12) - (13)) +
\]

\[
+ \frac{i}{2} \beta^{(2)}_{1,2} (6(123) - (124) - 6(132) - (134) + (142) + (143)) +
\]

\[
+ \frac{1}{2} \beta^{(2)}_{2,3} (-4() + 8(12) - 2(123) + 124) - 2(132) + (134) + (142) + (143) -
\]

\[
- 2(1234) + 2(1243) + 2(1342) - 2(1432) - 2(12)(34) - 2(13)(24)). \quad \text{(S-89)}
\]

where \((j_1 \ldots j_k)\) denotes the cycles of the permutation group which act on the Hilbert space as

\[
(j_1 \ldots j_k) \equiv P_{j_1,j_2} P_{j_2,j_3} \ldots P_{j_k-1,j_k}, \quad \text{(S-90)}
\]

and the empty cycle is just the identity () \(\equiv 1\). In (S-87)-(S-89) we already fixed the unphysical parameters \(\gamma_{r,s}\) and \(\epsilon_k\). Our goal is to find the Lax-operator \(\hat{L}^{(2)}\) which defines the transfer matrix which generates a class of conserved charges corresponding to the physical parameters \(\alpha^{(1)}_0, \alpha^{(2)}_0, \alpha^{(2)}_1, \beta^{(2)}_{2,3}\). We will also need the leading order of the higher charges

\[
\bar{q}^{3,0}_1 = \frac{i}{2} ((132) - (123)), \quad \text{(S-91)}
\]

\[
\bar{q}^{4,0}_1 = -(12) + (13) - \frac{1}{2}(1234) + \frac{1}{2}(1243) + \frac{1}{2}(1342) - \frac{1}{2}(1432). \quad \text{(S-92)}
\]

**Order 0**

In the zeroth order we have the usual nearest neighbor \(GL(N)\) spin chain which has the following Lax operator

\[
\hat{L}^{(0)}_1(u) = () - (12)u. \quad \text{(S-93)}
\]

We also have the zeroth order \(R-\) and \(J-\)operators as

\[
\hat{R}^{(0,0)}_1 = \hat{L}^{(0)}_1(u - v), \quad \hat{J}^{(0)}_1(u) = \frac{1}{1 - u^2} \hat{L}^{(0)}_1(-u). \quad \text{(S-94)}
\]

This Lax operator satisfies the regularity condition \(\hat{L}^{(0)}_1(0) = 1\) and the first derivative is

\[
\frac{d}{du} \hat{L}^{(0)}_1(u) \bigg|_{u=0} = -(12) = () + \bar{q}^{2,0}_1, \quad \text{(S-95)}
\]

therefore we obtained the following \(\gamma\) parameters

\[
\gamma^{(0)}_{2,0} = -1, \quad \gamma^{(0)}_{2,2} = 1. \quad \text{(S-96)}
\]

**Order 1**

Having a general ansatz for \(\hat{L}^{(1)}_1(u)\) as

\[
\hat{L}^{(1)}_1(u) = x_1(u)() + x_2(u)(12) + x_3(u)(123) + x_4(u)(132) + x_5(u)(13),
\]

\[
\frac{d}{du} \hat{L}^{(1)}_1(u) \bigg|_{u=0} = -(12) = () + \bar{q}^{2,0}_1, \quad \text{(S-95)}
\]
the equation (S-79) for \( \ell = 1 \) fixes only two unknown functions

\[
x_4(u) = -x_3(u) - \frac{\alpha_0^{(1)}}{u^2 - 1} u^2, \quad x_5(u) = \frac{\alpha_0^{(1)}}{u^2 - 1}.
\]

We can see that, the Lax operator contains three unfixed functions which correspond to the \( \gamma \)-ambiguities of the definition of charges. For the simplicity, let us fix these ambiguities as \( x_1(u) = x_2(u) = x_3(u) = 0 \) therefore the Lax-operator simplifies as

\[
\hat{L}^{(1)}_1(u) = \frac{\alpha_0^{(1)}}{u^2 - 1}(13) - \frac{\alpha_0^{(1)}}{u^2 - 1}(123). \tag{S-97}
\]

The first order \( R \)-matrix \( \hat{R}^{(1,1)} \) can be obtained from the equation (S-34):

\[
\hat{R}^{(1,1)} = -\frac{(u-v)(3u^2v^2 - u^2 - v^2 - 1)\alpha_0^{(1)}}{(u^2 - 1)(v^2 - 1)^2}(13) - \frac{u^2(u-v)\alpha_0^{(1)}}{(u^2 - 1)(v^2 - 1)}(123) \tag{S-98}
\]

\[
- u^2(u-v)\alpha_0^{(1)}(v^2 - 1)^2(132) + \frac{u(u-v)\alpha_0^{(1)}}{(u^2 - 1)(v^2 - 1)}(124) + \frac{2u(u-v)\alpha_0^{(1)}}{(u^2 - 1)(v^2 - 1)}(132) + \frac{u(u-v)\alpha_0^{(1)}}{(u^2 - 1)(v^2 - 1)}(243) + \frac{uv(u-v)\alpha_0^{(1)}}{(u^2 - 1)(v^2 - 1)}(1234) + \frac{uv(u-v)\alpha_0^{(1)}}{(u^2 - 1)(v^2 - 1)}(1423) + \frac{uv(u-v)^2(uv+1)\alpha_0^{(1)}}{(u^2 - 1)(v^2 - 1)^2}(13)(24) - \frac{(u-v)(u^2v^2-1)\alpha_0^{(1)}}{(u^2 - 1)(v^2 - 1)^2}(24) - \frac{2v(u-v)\alpha_0^{(1)}}{(u^2 - 1)(v^2 - 1)^2}(123) - \frac{v(u-v)\alpha_0^{(1)}}{(u^2 - 1)(v^2 - 1)^2}(142) - \frac{v(u-v)\alpha_0^{(1)}}{(u^2 - 1)(v^2 - 1)^2}(234).
\]

This Lax operator satisfies the regularity condition \( \hat{L}^{(1)}_1(0) = 0 \) and the first derivative is

\[
\left. \frac{d}{du} \hat{L}^{(1)}_1(u) \right|_{u=0} = q_1^{2,1} + 4q_0^{(1)}q_1^{2,0} - \alpha_0^{(1)}1, \tag{S-99}
\]

therefore the we obtained the following \( \gamma \) parameters

\[
\gamma^{(1)}_{2,0} = -\alpha_0^{(1)}, \quad \gamma^{(1)}_{2,2} = 4\alpha_0^{(1)}, \quad \gamma^{(1)}_{2,3} = 0. \tag{S-100}
\]

Order 2

The following Lax operator

\[
\hat{L}^{(2)}_1(u) = y_1(u)(142) + y_2(u)(14) + y_3(u)(132) + y_4(u)(13) + y_5(u)(124) + y_6(u)(143) + y_7(u)(134) + y_8(u)(1432) + y_9(u)(1324) + y_{10}(u)(1423) + y_{11}(u)(1243) + y_{12}(u)(12)(34) + y_{13}(u)(1234) + y_{14}(u)(14)(23) + y_{15}(u)(13)(24), \tag{S-101}
\]
is a solution of the equation (S-79) for \( \ell = 2 \), where

\[
y_1(u) = \frac{-2u^2(u^2 - 1)(a_0^{(1)})^2 + u\beta_{2,3}^{(2)} + iu\alpha_1^{(2)}}{2(u^2 - 1)^2},
\]
\[
y_2(u) = \frac{2u(u^2 - 1)(a_0^{(1)})^2 - u^2\beta_{2,3}^{(2)} - iu^2\alpha_1^{(2)}}{2(u^2 - 1)^2},
\]
\[
y_3(u) = \frac{-u\beta_{2,3}^{(2)} + iu\alpha_1^{(2)}}{2(u^2 - 1)},
\]
\[
y_5(u) = \frac{-u\beta_{2,3}^{(2)} - iu\alpha_1^{(2)}}{2(u^2 - 1)},
\]
\[
y_6(u) = \frac{u\beta_{2,3}^{(2)} + IU(2u^2 - 1)\alpha_1^{(2)}}{2(u^2 - 1)^2},
\]
\[
y_7(u) = \frac{-u^2\beta_{2,3}^{(2)} - iu^2\alpha_1^{(2)}}{2(u^2 - 1)^2},
\]
\[
y_8(u) = \frac{-u^2\beta_{2,3}^{(2)} + iu^2\alpha_1^{(2)}}{2(u^2 - 1)^2},
\]
\[
y_9(u) = \frac{u^2\beta_{2,3}^{(2)} + iu^2\alpha_1^{(2)}}{2(u^2 - 1)^2},
\]
\[
y_{10}(u) = \frac{u^2\beta_{2,3}^{(2)} + iu^2\alpha_1^{(2)}}{2(u^2 - 1)^2},
\]
\[
y_{11}(u) = \frac{u^2(2u^2 - 1)\beta_{2,3}^{(2)} - iu^4\alpha_1^{(2)}}{2(u^2 - 1)^2},
\]
\[
y_{13}(u) = \frac{-u^3\beta_{2,3}^{(2)} - iu^3\alpha_1^{(2)}}{2(u^2 - 1)^2},
\]
\[
y_{14}(u) = \frac{u(u^2 - 2)\beta_{2,3}^{(2)} - iu^4\alpha_1^{(2)}}{2(u^2 - 1)^2},
\]
\[
y_{15}(u) = \frac{u(u^2 - 2)\beta_{2,3}^{(2)} - iu^3\alpha_1^{(2)}}{2(u^2 - 1)^2},
\]

and

\[
y_3(u) = \frac{2u^2(2u^2 - 5)(a_0^{(1)})^2 + u(-2u^3 + 3u^2 + 2u - 4)\beta_{2,3}^{(2)} - 2u^2(u^2 - 1)\alpha_0^{(2)} + iu^3(-2u^2 + 5)\alpha_1^{(2)}}{2(u^2 - 1)^2},
\]
\[
y_4(u) = \frac{u(-4u^2 + 10)(a_0^{(1)})^2 - u(u^3 - 2u^2 - 2u + 2)\beta_{2,3}^{(2)} + 2u(u^2 - 1)\alpha_0^{(2)} + iu(u^2 - 4)\alpha_1^{(2)}}{2(u^2 - 1)^2}.
\]

The \( R \)-matrix \( \hat{R}^{(2,2)} \) (which satisfies the relation (S-34)) also exists but it has very complicated form. It can be written as a linear combination of 375 cycles therefore we omit to describe the explicit form.

This Lax operator satisfies the regularity condition \( \hat{L}_1^{(2)}(0) = 0 \) and the first derivative is

\[
\left. \frac{d}{du} \hat{L}_1^{(2)}(u) \right|_{u=0} = \bar{q}_1^{2,2} + (2i\beta_{2,3}^{(2)} + 6\alpha_1^{(2)})q_1^{3,0} - \left( (a_0^{(1)})^2 + \beta_{2,3}^{(2)} \right)q_1^{4,0} + 4\alpha_0^{(1)}q_1^{2,1} + (-12(a_0^{(1)})^2 + 5\beta_{2,3}^{(2)} + 4\alpha_0^{(2)})q_1^{3,0} - (4(a_0^{(1)})^2 - 3\beta_{2,3}^{(2)} - \alpha_0^{(2)}) \mathbf{1}.
\]

therefore we obtained the following \( \gamma \) parameters

\[
\gamma_{2,0}^{(2)} = 4(a_0^{(1)})^2 - 3\beta_{2,3}^{(2)} - \alpha_0^{(2)},
\]
\[
\gamma_{2,2}^{(2)} = -12(a_0^{(1)})^2 + 5\beta_{2,3}^{(2)} + 4\alpha_0^{(2)},
\]
\[
\gamma_{2,3}^{(2)} = 2i\beta_{2,3}^{(2)} + 6\alpha_1^{(2)},
\]
\[
\gamma_{2,4}^{(2)} = - (a_0^{(1)})^2 - \beta_{2,3}^{(2)}.
\]