Self-similar spherically symmetric solutions of the massless Einstein-Vlasov system

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We construct the general spherically symmetric and self-similar solution of the Einstein-Vlasov system (collisionless matter coupled to general relativity) with massless particles, under certain regularity conditions. Such solutions have a curvature singularity by construction, and their initial data on a Cauchy surface to the past of the singularity can be chosen to have compact support in momentum space. They can also be truncated at large radius so that they have compact support in space, while retaining self-similarity in a central region that includes the singularity. However, the Vlasov distribution function can not be bounded. As a simpler illustration of our techniques and notation we also construct the general spherically symmetric and static solution, for both massive and massless particles.

I. INTRODUCTION

In an astrophysical process of gravitational collapse of matter there is always some physical effect opposing the force of gravity, so that not all the matter necessarily ends up in the final black hole. Since the initial work of Choptuik [1], it has become clear that in many physical systems it is possible to fine tune the initial conditions so that collapse can result in black holes of arbitrarily small mass. This gives rise to interesting effects which are now studied as a particular branch (called type II) of Critical Phenomena in Gravitational Collapse (see [2] for a review). In a perfect fluid it is the pressure that opposes collapse. In field theories, it appears to be the fact that the matter field propagates at the speed of light.

Here we study another type of matter, a cloud of point particles that do not interact directly through collisions but only through their averaged gravitational field. This system is described by the coupled Einstein and Vlasov equations. In collisionless matter it is the tendency of particles in the cloud to miss each other that acts as the source of dispersion. In a spherically symmetric situation, such as we shall study here, one can also think of their angular momentum as the force opposing gravity. (In spherical symmetry, each particle has angular momentum, but the total angular momentum of all particles is zero.)

In numerical simulations of the gravitational collapse of asymptotically flat, spherically symmetric collisionless matter configurations Rein, Rendall and Schaeffer [3] have not found any sign of critical phenomena at the black hole threshold. Olabarrieta and Choptuik [1, 4] have not found type II critical phenomena either, but have found some evidence of type I critical phenomena, where a metastable static solution acts as an intermediate attractor at the black hole threshold. They have found the expected scaling of the lifetime of the intermediate attractor with distance to the black hole threshold.

The critical exponent, however, varies from 5.0 to 5.9 between families of initial data, for a quoted uncertainty of 0.2. The Vlasov distribution function of the critical solution was not found to be universal, and the metric of the critical solution was found to be universal only up to a rescaling of space.

We want to explore the problem of existence of critical phenomena in the Einstein-Vlasov system by looking for both static and continuously self-similar (from now on CSS) solutions of the system and then testing for their stability against linear perturbations. A static solution with exactly one growing mode is a potential critical solution for type I critical collapse, and a self-similar solution with exactly one growing mode is a potential critical solution for type II critical collapse. In this paper, we construct the general static and the general self-similar solutions, assuming some regularity conditions. In order to avoid introducing a scale, we further restrict ourselves to massless particles in the self-similar case. Their perturbations and the extension to massive particles will be studied elsewhere. The static solutions of the Einstein-Vlasov system have been extensively studied already. We deal with them here partly as an introduction to the methods we use later in the CSS case but also for the light they shed on type I critical phenomena. To our knowledge little is known about CSS solutions: The CSS equations were derived in [6], but no CSS solutions were found. Spherically symmetric CSS solutions for rotating dust, where $L(r)$ can be specified freely in the initial data on top of $\rho(r)$ and $v(r)$, were constructed recently by Gair [5].

We begin by fixing notation and stating the field equations for the Einstein-Vlasov system in spherical symmetry. We then introduce the idea that the general solution of the Vlasov equation on a fixed spacetime can be given in terms of an arbitrary function of three variables that are conserved along trajectories and which can be given as quadratures if the spacetime has an additional continuous symmetry, besides spherical symmetry. This is true in particular for both massive and massless particles in static spacetime, and for massless particles in a CSS spacetime.

The further requirement that the resulting stress-
energy of the collisionless matter is consistent with the assumption of staticity or CSS reduces the free function of three variables to one of two variables. We show this first for the static case, and discuss an interesting subtlety. We also construct some explicit solutions numerically. This section can be omitted by the reader who is only interested in the self-similar case, but we feel that it serves as a useful warmup.

In discussing the CSS case, we first discuss the conserved quantities, and classify the possible particle trajectories on a CSS spacetime. We then find the general solution on a fixed background spacetime, and reduce it to the general self-gravitating solution. We again construct some explicit solutions numerically, and discuss the general features of these solutions. The implications of our results for critical phenomena in gravitational collapse are discussed in the Conclusions section.

II. THE EINSTEIN-VLASOV SYSTEM IN SPHERICAL SYMMETRY

A. Collisionless matter

In this section we briefly review the essential ideas and equations needed in the rest of the Article. For a longer introduction to the Einstein-Vlasov system see [8]. We add some considerations about the integration of the Vlasov equation and its relation to the symmetries of the spacetime.

The Einstein-Vlasov system describes the evolution of a statistical ensemble of non-interacting particles coupled to gravity through their averaged stress-energy. It is possible to work with particles of different masses, but in this work we restrict ourselves to a single particle species, with mass $m \geq 0$. We keep $m$ arbitrary, so that all the equations in this section are valid both for massless and massive particles.

As in classical statistical mechanics we describe the state of a many body system with a positive distribution function over the phase space of the system. As the particles do not interact except through their averaged stress-energy, we can use a distribution function $f(x^\mu, p^\nu)$ on the one-particle phase space, where $x^\mu$ are coordinates in spacetime and the coordinates $p^\mu$ are constructed using the metric at the point $x^\mu$ from the coordinates $p_\mu$ in its cotangent space at that point. As we always have $g_{\mu\nu} p^\mu p^\nu = -m^2$ and $p^\mu > 0$, we can consider the distribution $f$ as a function $f(x^\mu, p^\nu)$. Greek indices denote the range 0-3 and Latin indices the range 1-3.

The free-fall trajectories of the particles define a congruence of curves on phase space, tangent to the Liouville operator (or “geodesic spray”)

$$\mathcal{L} = \frac{d}{d\sigma} - \frac{dx^\mu}{d\sigma} \frac{\partial}{\partial x^\mu} + \frac{dp^\nu}{d\sigma} \frac{\partial}{\partial p^\nu} - \Gamma^\nu_{\lambda\mu} p^\mu \frac{\partial}{\partial p^\nu}. \tag{1}$$

In this article $\sigma$ is an affine parameter related to proper time $s$ through $ds = m \, d\sigma$ for massive particles. This allows us to discuss both massive and massless particles simultaneously. As the particles do not interact directly, the evolution of $f$ is governed by the Vlasov equation

$$\mathcal{L} f = 0. \tag{2}$$

The stress-energy tensor at each point is obtained by integrating the distribution function over momentum space:

$$T^\mu\nu(x) = \int_{P(x)} \frac{d^3p}{p_0} \sqrt{-g} f(x, p) p^\mu p^\nu, \tag{3}$$

where $P(x)$ is the 3-momentum space at the point $x^\mu$ and $p_0$ is determined from $p^\mu$ and the metric. The Vlasov equation is a sufficient condition for stress-energy conservation. The particle number current given by

$$N^\mu(x) = \int_{P(x)} \frac{d^3p}{p_0} \sqrt{-g} f(x, p) p^\mu. \tag{4}$$

is also conserved. Note that the distribution function $f(x, p)$ is a scalar on phase space even though there is a factor $\sqrt{-g}$ in the integrals. The natural volume measure on the phase space is $-dx^{0123} \wedge dp_{0123}$, which can be rewritten as $(\sqrt{-g} \, dx^{0123}) \wedge (\sqrt{-g} \, dp_{0123})$. It is the second factor $\sqrt{-g}$ that appears in the integrals (3) and (4), which are themselves tensors on spacetime and would therefore be integrated over spacetime using the measure $(\sqrt{-g} \, dx^{0123})$.

A comment on the dimensions of the variables. It would seem natural to measure the mass $m$ and energy-momentum $p$ of individual particles in the same units: one measure gravitational mass and energy-momentum in, for example in the stress-energy tensor. But particle momentum appears in the stress-energy momentum tensor only under the integration over momentum space. Dimensional analysis is therefore less restrictive than one might assume.

It is consistent to assign particle mass and energy-momentum a dimension $P$ that is independent of the dimension $M$ of gravitational mass. In units in which $c = G = 1$, the unit of gravitational mass $M$ is equal to the unit of length $L$, but $P$ remains independent. For example, $m$ and $p^\mu$ have dimension $P$, and angular momentum squared $F = |\vec{x} \wedge \vec{p}|^2$ has dimensions $L^2 P^2$, but the stress-energy tensor has dimension $L^{-2}$, which can be thought of as $ML^{-3}$. The distribution function relates both kinds of masses and has dimensions $L^{-2} P^{-4}$. The particle current has dimension $P^{-1} L^{-2}$, which can be thought of as $MP^{-1} L^{-3}$; it has the dimension $L^{-3}$ of a number density only if gravitational and particle mass are measured in the same units.

B. Spherical symmetry

Now we impose spherical symmetry. We describe the metric using polar-radial coordinates,

$$ds^2 = -a^2(t, r)dt^2 + a^2(t, r)dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2). \tag{5}$$

In this case the coordinate $\theta$ is independent of time, and $p_\theta$ is conserved. The distribution function $f$ is independent of $\theta$, and we can replace it by a function on the two-dimensional phase space $\mathbb{R}^{2+1}$ with coordinates $x^\mu$. The Liouville operator is then a function of $x^\mu$ and $p^\nu$, and the Vlasov equation can be written as a Hamiltonian system on this phase space. The solutions of the Vlasov equation are then obtained by solving the Hamiltonian equations of motion.

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The Hamiltonian equations of motion are then

$$\frac{dx^\mu}{dt} = \frac{\partial H}{\partial p^\mu}, \quad \frac{dp^\mu}{dt} = -\frac{\partial H}{\partial x^\mu}, \tag{6}$$

where $H$ is the Hamiltonian function given by

$$H = \int \sqrt{-g} \, f(x, p) \, p^\mu p^\nu T^\mu\nu. \tag{7}$$

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In this coordinate choice there is still the gauge freedom \( t \to t'(t) \), which changes the lapse \( \alpha \). We fix this freedom by setting \( \alpha = 1 \) either at \( r = 0 \) or at \( r = \infty \).

The Einstein equations give the following equations for the first derivatives of the metric coefficients:

\[
\frac{\alpha_r}{\alpha} = \frac{a^2 - 1}{2} + \frac{ra^2}{2} 8\pi T^r_r, \tag{6}
\]
\[
\frac{a_r}{a} = -\frac{a^2 - 1}{2} - \frac{ra^2}{2} 8\pi T^t_t, \tag{7}
\]
\[
\frac{a_t}{a} = \frac{ra^2}{2} 8\pi T^r_t. \tag{8}
\]

The fourth Einstein equation, involving \( T_{\theta \theta} \), is a combination of derivatives of these three. The third of these equations, for \( a_t \), becomes an identity when the other two equations are obeyed at each \( t \).

In spherical symmetry, the distribution function has to be of the form \( f(t, r, p^r, |p|) \), where \(|p|^2 \equiv g_{\alpha\beta}p^\alpha p^\beta \). We can simplify the Vlasov equation using the fact that angular momentum is a constant of motion. Following [2] we define the variables

\[
w = ap^r, \quad F = r^2(|p|^2 - w^2) = r^4(p^2 + \sin^2 \theta \rho^2). \tag{10}
\]

We also introduce the shorthand

\[W \equiv \sqrt{m^2 + w^2 + F/r^2} = \alpha \rho^t. \tag{11}\]

In these variables the Vlasov equation in a spherically symmetric spacetime becomes

\[\frac{\partial f}{\partial t} + \frac{w}{W} \frac{\partial f}{\partial r} + \left( \frac{F\alpha}{r^3 aW} - W\frac{\alpha_r}{\alpha} - \frac{a_t}{a} \right) \frac{\partial f}{\partial w} = 0, \tag{12}\]

where \( f \) is now a function \( f(t, r, w, F) \). There is no derivative with respect to \( F \) because it is a constant of motion.

The nonvanishing components of the particle number current are

\[N^t = \frac{\pi}{r^2 \alpha} \int_0^\infty dF \int_{-\infty}^\infty dw \ f, \tag{13}\]
\[N^r = \frac{\pi}{r^2 a} \int_0^\infty dF \int_{-\infty}^\infty dw \ f \frac{w}{W}. \tag{14}\]

The distribution function is always nonnegative and therefore we have \( \alpha N^t \geq |a|N^r \), which means that \( N^t \) is always timelike or lightlike. It is lightlike only for distributions where all particles are massless and moving radially (\( F = 0 \)) in the same direction, so that either \( w = W \) or \( w = -W \). The non-vanishing components of the stress-energy tensor are

\[T^r_r = \frac{\pi}{r^2} \int_0^\infty dF \int_{-\infty}^\infty dw \ f \frac{w^2}{W} \geq 0, \tag{15}\]
\[T^t_t = -\frac{\pi}{r^2} \int_0^\infty dF \int_{-\infty}^\infty dw \ f \ W \leq 0, \tag{16}\]

\[T^\theta_\theta = -\frac{\pi}{2r^2} \int_0^\infty dF \int_{-\infty}^\infty dw \ f \ W \geq 0. \tag{17}\]

This tensor is conserved and satisfies the dominant and strong energy conditions. We assume that \( f \) behaves in such a way that all integrals (13-18) converge and are finite at every point.

Assuming \( a \geq 1 \) the following inequalities follow from the Einstein equations:

\[\alpha_r \geq 0, \tag{19}\]
\[(aa)_r \geq |(a^2)_t|, \tag{20}\]
\[\frac{\alpha}{a} (\frac{a}{\alpha},_r) \geq 1 - a^2. \tag{21}\]

Finally we have that

\[T^\mu_\nu = -m^2 \pi \int_0^\infty dF \int_{-\infty}^\infty dw \ f \ W \leq 0, \tag{22}\]

and therefore the Ricci scalar \( R \) is always nonnegative. For plotting results we shall also use the Hawking (or Misner-Sharp) mass function \( M(t, r) \), which is defined by

\[a^{-2}(t, r) \equiv 1 - \frac{2M(t, r)}{r}. \tag{23}\]

\[C. \text{ Further symmetries and conserved quantities}\]

The Vlasov equation (8) expresses the fact that \( f \) is constant along trajectories of particles. Therefore the solution of that equation is, formally, an arbitrary function of the constants of motion of the problem. In the general case \( f \) would be a function of eight constants of motion, but when we impose spherical symmetry \( f \) can only depend on some of these. Spherical symmetry divides the four degrees of freedom of a point particle into two radial and two angular degrees of freedom. The angular part gives rise to four constants of motion: the three components of the angular momentum vector, and the initial angle with respect to the axis given by this vector. If the averaged stress-energy of the particles is to be spherically symmetric, however, the distribution function \( f \) can only depend on the modulus (squared) \( F \) of the angular momentum. As we saw in the previous subsection, this can be used to reduce the Vlasov and Einstein equations to spherical symmetry. In the same way, additional symmetries can be used to further simplify the equations.

The reduced, radial, system has two degrees of freedom \([t(\sigma), r(\sigma)]\) with conjugate momenta \([p_t(\sigma), p_r(\sigma)]\), where \( \sigma \) is an affine parameter along the particle trajectories. The Hamiltonian of the reduced system is

\[H = \frac{1}{2} \left( -\frac{p_t^2}{a^2} + \frac{p_r^2}{a^2} + \frac{F}{r^2} \right), \tag{24}\]
where $F$ is now a given constant. This Hamiltonian system has four independent constants of motion $c_1, ..., c_4$. One of these is the value of the Hamiltonian itself, $H = -\frac{m^2}{2}$. The general solution of the equations of motion can formally be written as

$$
\begin{align*}
t(\sigma) &= t(\sigma, c_1, ..., c_4, F) \\
r(\sigma) &= r(\sigma, c_1, ..., c_4, F) \\
p_t(\sigma) &= p_t(\sigma, c_1, ..., c_4, F) \\
p_r(\sigma) &= p_r(\sigma, c_1, ..., c_4, F)
\end{align*}
$$

(25)

where we now consider the constants $c_i$ as functions of $t, r, p_t, p_r$ and $\sigma$. Let $c_4 = m$. In the following we do not refer to $m$ as a conserved quantity, but consider it a parameter of the equations. At least one of the four constants of motion, say $c_3$, must depend on $\sigma$, but because $f$ does not depend explicitly on $\sigma$, $f$ can not depend on this constant of motion. Therefore, the general solution $f$ in spherical symmetry will be an arbitrary function of $F$ and two nontrivial constants of motion $c_1$ and $c_2$.

Although the two nontrivial constants of motion $c_1$ and $c_2$ always exist formally (and we can find them by integrating the equations of motion numerically), finding their analytical expressions in terms of the variables of the problem is only possible when the reduced radial system is integrable. Using the Liouville theorem, we need to find only one constant of motion in terms of $(t, r, p_t, p_r)$, because then we will have two constants in involution for a system with two degrees of freedom. The easiest way of finding such a constant is imposing an additional continuous symmetry. In the following we analyze two possible symmetries: staticity and self-similarity. We will then be able to give analytic expressions (at least quadratures) for the trajectories of the particles, simplifying the calculation of the energy-momentum tensor components. This is equivalent to giving a formal solution of the Vlasov equation in terms of its characteristics in phase space.

Even though we are interested in self-similarity, we first review the static case to show some important ideas in a well-known context.

### III. REVIEW OF STATIC SPACETIMES

#### A. Field equations

A static spacetime has an additional Killing vector

$$\xi = \partial_t,$$

(26)

so that the metric functions $\alpha$ and $\alpha$ are just functions of $r$. The particles have an additional constant of motion

$$E \equiv -\xi^t p_t = -p_t = \alpha W = \alpha \sqrt{m^2 + \omega^2 + F/r^2}.\quad (27)$$

The four constants of motion (of the full 4-dimensional system) $p_t^2, F, L_z$ and $E$ are in involution and therefore, by the Liouville theorem, we can integrate the system.

The radial equations of motion in terms of the constants of motion $E$ and $F$ are

$$
\begin{align*}
\frac{dr}{d\sigma} &= p_r^\prime = \pm \frac{1}{a(r)} \sqrt{\frac{E^2}{\alpha(r)^2} - m^2 - \frac{F}{r^2}}, \\
\frac{dt}{d\sigma} &= p_t^\prime = \frac{E}{\alpha^2(r)}.
\end{align*}
$$

(28)

(29)

Defining $v \equiv p_r^\prime / p_t^\prime$ we can eliminate $\sigma$ and integrate the equations to obtain the quadrature

$$
\int_{r_0}^r \frac{dr'}{v(r')} = t - t_0 = \int_{r_0}^{r(t)} \frac{dr'}{v(r')} ,
$$

(30)

where

$$
\frac{dr}{dt} = v(r) = \pm \frac{\alpha(r)}{a(r)} \sqrt{1 - \frac{m^2 \alpha^2(r)}{E^2} - \frac{\alpha^2(r)F}{r^2 E^2}}.\quad (31)
$$

We have the new constant of motion

$$t_0(t, r, E, F) = t - \int_{r_0}^r \frac{dr'}{v(r')} ,
$$

(32)

which is the time when the particle will be at $r_0$, given that it is at $r$ at time $t$. After $E$, this is the second of the nontrivial constants of motion $c_1$ and $c_2$ referred to in the general discussion.

Therefore, the most general solution to the Vlasov equation on a static background can be given as an arbitrary function of three variables

$$\text{Vlasov in static spacetime } \Rightarrow f(t, r, w, F) = g(t_0, E, F).\quad (33)$$

As $t_0$ depends explicitly on $t$, this allows for a solution $f$ that is explicitly time-dependent.

If particles with a given $E$ and $F$ have bound orbits, $g$ must be periodic in $t_0$ with the period of that orbit. The period is determined as the integral between the two turning points, and depends on $E$ and $F$. We can avoid this complicated periodicity of $g(t_0, E, F)$ by inverting equation (32) to obtain a constant of motion

$$r_0(t, r, E, F) \quad \text{which gives the position } r_0 \quad \text{of a particle with } E, F \quad \text{at some canonical time } t_0 \quad \text{given that it is at } r \quad \text{at time } t:
$$

$$\text{Vlasov in static spacetime } \Rightarrow f(t, r, w, F) = \tilde{g}(r_0, E, F).\quad (34)$$

This means that we can freely specify $f(r, w, F)$ at $t = t_0$, and still obtain an automatic solution of the Vlasov equation.

So far we have considered Vlasov matter moving on a given spacetime. But to be consistent with staticity, we need the stress-energy momentum to be independent of $t$. Therefore, in principle, $f$ should not depend on $t_0$ or $r_0$, which depend on $t$. The most general self-consistent static solution of Einstein-Vlasov in spherical symmetry would then have the form

$$\text{Einstein-Vlasov + staticity } \Rightarrow f(t, r, w, F) = h(E, F).\quad (35)$$
This result is referred to as Jeans’ theorem and it is known to hold in Newtonian physics (that is, in the Vlasov-Poisson system), but there are some counterexamples in the Einstein-Vlasov system. That is why we have put the implication arrow in quotes. In order to understand this subtlety we first give some explicit solutions where that result is valid.

B. Some explicit solutions

We choose a function $h$ of two variables. The spacetime can then be determined by solving the Einstein equations, which become a system of integral-differential equations. The integrals occur in forming the stress-energy tensor, which depends on the metric as well as on the arbitrary function $h(E,F)$.

In a static spherically symmetric solution we have two nontrivial Einstein equations. They are just the Einstein equations, which become a system of integral-differential equations, that result is valid. In order to understand this subtlety we first give some explicit solutions where that result is valid.

The particular class of functions $h(E,F)$ where $h(E,F) = \theta(E_0 - E)\theta(F - F_0)\phi(E)(F - F_0)^l$ (called a “polytrope”), with upper cutoff $E_0$ and lower cutoff $F_0$, and $l > -1/2$, is analyzed in detail in [11] for massive particles, where it is shown that solutions exist that describe matter distributions with finite total mass and compact radial support.

If $h(E,F)$ is restricted to a function of $E$ alone, one finds by direct comparison of the integrals in the stress-energy tensor that $T_r = T_\theta = T_\phi = T_\varphi$, so that the pressure is isotropic. This means that the stress-energy content in this case (spherically symmetric and static) has perfect fluid form. If $\rho = -T_t$ and $p = T_r = T_\theta = T_\phi$, then $\rho$ and $p$ are monotonic, one could also read off a, fairly meaningless, formal “equation of state”.

It is interesting to solve a simple example in closed form, namely the ansatz $h(E,F) = c\theta(E - E_0)$, with $c$ and $E_0$ positive constants. This is a “polytrope” with $l = 0$ and no cutoff in angular momentum. For convenience of notation we define the dimensionless constant $\bar{m} = m/E_0$, and the constant $\bar{c} = cE_0^l$. With the gauge choice $\alpha(r) = 1$ we have $E \geq m$, and therefore $\bar{m}$ must be in the range from 0 to 1. The parameter $\bar{c}$ can always be set to 1 by a choice of length units, and is therefore trivial. The integrals in equations (33) and (34) can be evaluated explicitly, and give

$$v^2 = \frac{\bar{c}^2}{a^2 E^2}(E^2 - E_{\text{min}}^2).$$ (42)

When the function $h(E,F)$ has been specified, the stress-energy integrals become given functions of $r$ and $\alpha(r)$. The Einstein equations then become two coupled ordinary differential equations (ODEs) that determine $a(r)$ and $\alpha(r)$. Conversely we can think of $a(r)$ and $\alpha(r)$ as given functions. We then have an integral equation for $h(E,F)$. By function counting, this integral equation does not determine $h(E,F)$ uniquely. There is an infinite number of functions $h(E,F)$ that give rise to the same static spacetime, with the same stress-energy tensor. The change of integration variables to $E$ and $F$ shows that, for given $h(E,F)$, the integrals in (33) and (34) are functions only of the two combinations $r/\alpha$ and $ma$. In particular, when $m = 0$, those integrals for fixed $h(E,F)$ are functions only of the combination $r/\alpha$.
\[
\rho = -T^t_t = \frac{\pi \bar{c}}{2\alpha^4} \left[ \sqrt{1 - \bar{m}^2 \alpha^2} (2 - \bar{m}^2 \alpha^2) + \bar{m}^4 \alpha^4 \log \left( \frac{\bar{m} \alpha}{1 + \sqrt{1 - \bar{m}^2 \alpha^2}} \right) \right],
\]
\[
p = T^r_r = \frac{\pi \bar{c}}{2\alpha^4} \left[ \frac{1}{3} \sqrt{1 - \bar{m}^2 \alpha^2} (2 - 5\bar{m}^2 \alpha^2) - \bar{m}^4 \alpha^4 \log \left( \frac{\bar{m} \alpha}{1 + \sqrt{1 - \bar{m}^2 \alpha^2}} \right) \right],
\]

for \( \bar{m} \alpha \leq 1 \), and \( \rho = p = 0 \) otherwise. For massless particles, these expressions simplify to
\[
\rho = 3p = \frac{\pi \bar{c}}{\alpha},
\]
so that we have the formal equation of state \( p = \rho/3 \).

\( \alpha(r) \) is an increasing function, defining the surface of the “star” at \( \bar{m} \alpha = 1 \). Note that lighter particles give larger stars.

An explicit numerical example for the resulting metric is shown in Fig. 1, both for massive (\( \bar{m} = 1/2 \), continuous line) and massless (\( \bar{m} = 0 \), dashed line) particles. As we expected, the massive case gives a star of finite mass and finite radius, while the massless case gives an infinite mass, infinite size distribution of matter. This can be easily understood in terms of the motion of the particles: in this example massive particles follow bounded orbits, so that we have a static situation with a finite number of particles in the system, but massless particles follow unbounded orbits, so that we need an infinite number of them to get a static distribution.

C. Counterexamples to Jeans’ Theorem

We shall now review some static solutions that do, after all, depend on the third constant of motion \( r_0 \). Assume a static distribution of matter and its corresponding static gravitational field. Two different test particles can have the same energy \( E \) and angular momentum \( F \) at different values of \( r \). It has been proved that in Newtonian physics (the Vlasov-Poisson system) the distribution function must be the same at both points for those values of \( E \) and \( F \) \cite{rein}. This result is referred to as Jeans’ theorem. We can understand it physically in the following way. In Newtonian physics it is always possible to find a trajectory for one of the particles on which it reaches the position of the other particle without changing \( E \) or \( F \). Because \( f \) is constant along particle trajectories it must be the same at both positions. This result is not true in general relativity: it is possible to show that two particles with the same \( E \) and \( F \) can be separated by a potential barrier. \( f \) can then be consistently different at those positions, and therefore it is not just a function of \( E \) and \( F \), but of \( r_0 \) as well.

The key observation is that the Newtonian minimum energy
\[
E_{\text{min}}(r, F) = U(r) + \frac{F^2}{2mr^2},
\]
where the gravitational potential \( U(r) \) obeys the Poisson equation, always has a single minimum and no maxima, while the relativistic function may have several extrema.

Rein has given a nice example in \cite{rein}. Given a Schwarzschild black hole of mass \( M \), it is impossible to set up a static, finite-mass distribution of Vlasov particles very close to it (for \( r < 3M \ldots 6M \), depending on their angular momentum) because it is not possible to have bound stable orbits in that region. Therefore we must set \( h = 0 \) in that region. However we can have a static distribution of particles with the same energy and angular momentum further away from the black hole. The two regions are separated by a potential barrier. Therefore \( h \) may be different from zero for the same values of \( E \) and \( F \). Again, \( h \) depends on \( r_0 \), as well as on \( E \) and \( F \). The left side of Fig. 2 gives an explicit numerical example of a self-gravitating shell of collisionless matter surrounding a black hole.

A second example has been given by Schaeffer \cite{schaeffer}. He shows that a shell of matter generates a well in the potential which can sustain another shell, if the inner shell is concentrated near its own Schwarzschild radius. We have an example in Fig. 3 on the right. The second shell can have a different value of the distribution function \( h \) for particles with the same energy and angular momentum. In the example in Fig. 3 \( h \) in the second shell is half of \( h \) in the first shell. The same values of \( E \) and \( F \) are possible in a third region stretching to infinity (unbound particles), and \( h \) has been chosen to vanish there.

We now state the Einstein-Vlasov form of “Jeans’ theorem”. In general relativity particles with given values of \( E \) and \( F \) can exist in disjoint regions \( n = 1, 2, \ldots \) for the particles. Each region is bounded by turning points \( r^{(n)}_\pm (E, F) \). We assume that \( r^{(n)}_+ < r^{(n+1)}_- \). If the first allowed region contains the center then \( r^{(1)}_- = 0 \) and if the last allowed region is not bounded then \( r^{(\text{last})}_+ = \infty \).

The general form of the distribution is
\[
h(r_0, E, F) = \sum_n \theta(r_0 - r^{(n)}_-) \theta(r^{(n)}_+ - r_0) h^{(n)}(E, F).
\]

In order to find self-gravitating solutions of this form, we shall usually need a convergent iterative procedure. For example the shells in the previous counterexample given by Schaeffer can be constructed iteratively using the fact that in spherical symmetry the interior shells are not affected by the exterior shells (Birkhoff theorem).

The dependence of \( h \) on \( r_0 \) in (47) is admittedly rather restricted, but we have discussed it in some detail, be-
FIG. 1: Mass distribution \( M'(r) \) (left) and lapse function \( \alpha(r) \) (right) corresponding to the solution given in Eqs. (43) and (44) for \( \bar{m} = 1/2 \) (continuous line) and \( \bar{m} = 0 \) (dashed line).

FIG. 2: Graphs of the effective potential \( E_{\text{min}}(r, F_0) = \alpha(r) \sqrt{m^2 + F_0/r^2} \), where \( F_0 \) is a lower cutoff in angular momentum. The horizontal dashed line gives the upper cutoff \( E_0 \) in energy. We can have particles between both lines (shaded regions). In both examples, \( m = 1 \), and \( h \) is independent of \( E \) and \( F \) for \( F > F_0 \) and \( E < E_0 \), but takes different values in different potential wells. In both examples we have set \( \alpha = 1 \) at \( r = \infty \), and so \( E_{\text{min}}/m \to 1 \) from below as \( r \to \infty \). In the first graph we have a black hole with \( M = 1 \) and we plot just the exterior region. \( F_0 = 17 \). We could have particles in the region \( 2M < r < 3M \), but not a static distribution of them. In the second graph we have two shells of matter, and \( F_0 = 2 \). Both of them contain particles, but \( h \) is half in the outer well of what it is in the inner well.

due the dependence of the equivalent of \( h \) in self-similar solutions on all three constants of motion will be non-trivial, and in fact no solutions can be found that depend only on \( F \) and the equivalent of \( E \): solutions must also depend on the equivalent of \( r_0 \).

D. Massless particles

Massless particles are not very interesting when one looks for static solutions because it is difficult to form asymptotically flat solutions, but the system with massless particles has an additional symmetry: the trajectory of a massless particle, in a given spacetime, for a given initial position, is completely determined by the direction of its initial four-momentum, independently of the modulus. In order to make a given contribution to the stress-energy tensor, we can therefore use \( N \) particles of four-momentum \( p^\mu \) or one particle of four-momentum \( N p^\mu \). We shall see that this allows us to state explicitly which part of the Vlasov function \( h(E, F) \) is determined by the spacetime metric and which is arbitrary. This is interesting as a toy model for the CSS solutions (where we are interested in massless particles anyway).

In a spherically symmetric static solution in particular, the trajectory of a particle is determined not by \( E \) and \( F \) separately, but only by the combination \( E^2/F \), which is invariant under rescalings of the four-momentum. We rewrite the free function \( h(E, F) \) as

\[
h(E, F) = k(y, F),
\] (48)
\[ y \equiv \frac{E}{\sqrt{F}} = \frac{\alpha W}{\sqrt{F}} = \alpha \left( \frac{w^2 + 1}{F} \right)^{\frac{1}{2}}. \]  

(49)

With \( m = 0 \), the limit \( E_{\text{min}}(r, F) \) in the integration over \( E \) becomes independent of \( F \), and the two integrations can be interchanged. We find

\[
\frac{\alpha'}{\alpha} = \frac{a^2 - 1}{2r} + \frac{4\pi a^2}{r\alpha^2} \times \int_{\alpha/r}^\infty \left( 1 - \frac{a^2}{r^2 y^2} \right)^{\frac{1}{2}} \tilde{k}(y) 2y dy, \quad (50)
\]

\[
\frac{a'}{a} = \frac{a^2 - 1}{2r} + \frac{4\pi a^2}{r\alpha^2} \times \int_{\alpha/r}^\infty \left( 1 - \frac{a^2}{r^2 y^2} \right)^{-\frac{1}{2}} \tilde{k}(y) 2y dy, \quad (51)
\]

where \( \tilde{k} \) is the integral of the Vlasov function \( k \) over all particles with the same trajectory:

\[
\tilde{k}(y) \equiv \int_0^\infty k(y, F) F dF. \quad (52)
\]

The integration limit \( y = \alpha(r)/r \) is the turning point (perihelion) of all particles with a given \( y \). Note that \([f] = [g] = [h] = [k] = L^{-2}P^{-4}\) (for both massive and massless particles), and \([\tilde{k}] = L^2\).

For massless particles the Ricci scalar vanishes, and as a consequence \( a(r) \) and \( \alpha(r) \) are not independent but are related by a second order ODE. This ODE has a unique solution for either \( a(r) \) given \( \alpha(r) \) and the boundary condition \( a(r) = 1, a'(r) = 0 \), or the other way around. \( a(r) \) and \( \alpha(r) \) are therefore related one-to-one.

Furthermore, Eqs. (50,51) can be turned into linear Fredholm equations of the second kind by a change of variable. This means that either of these equations can be solved for \( \tilde{k}(y) \) for given \( a(r) \) and \( \alpha(r) \). The relationship between \( a(r) \) and \( \alpha(r) \) on the one hand and \( \tilde{k}(y) \) on the other is therefore one-to-one.

IV. CSS SPACETIME

A. Spacetime geometry

Because the particle mass \( m \) introduces a scale into the field equations, it is not clear that massive particles are compatible with exact self-similarity. In this paper we therefore construct CSS solutions of the massless Einstein-Vlasov system. We defer the question of massive particles allowing CSS solutions or asymptotically CSS solutions to another paper.

We start by defining CSS and stating some general properties of CSS spacetimes. Then we analyze the possible trajectories of massless particles in those spacetimes. We find two nontrivial conserved quantities for massless particles, \( J \) and \( \tau_0 \), which are the equivalents of \( E \) and \( t_0 \) in the static case. In contrast to the static case, we will find that the Vlasov distribution function \( f \) can not just depend on \( J \) and \( F \) but must also depend on \( \tau_0 \) in order to construct a regular self-similar stress-energy tensor.

A CSS spacetime is one that possesses a homothetic vector field

\[
\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = -2g_{\mu\nu}. \quad (53)
\]

In spherical symmetry, in the coordinates \( r \) and \( t \), this vector field is

\[
\xi = -t\partial_t - r\partial_r. \quad (54)
\]

\( \xi \) is the infinitesimal generator of scale transformations \( r \to sr, \ t \to st \). It is customary to define the following coordinates, which are adapted to self-similarity in the region \( r \geq 0, t < 0 \):

\[
x \equiv \frac{r}{-t}, \quad \tau \equiv -\log \left( \frac{-t}{L_0} \right), \quad (55)
\]

where \( L_0 \) is a constant put in to get dimensions right. It is helpful to assign \( t, r \), and \( L_0 \) dimension \( L \), while \( x \) and \( \tau \) are dimensionless. Fig. 3 gives a graphical description of the new coordinates. In these coordinates the homothetic vector is

\[
\xi = \partial_r, \quad (56)
\]

and the metric functions \( a \) and \( \alpha \) are functions of \( x \) only. (We could write the metric explicitly in terms of coordinates \( x \) and \( \tau \), but it is often more helpful to refer back to coordinates \( t \) and \( r \) as they are more familiar.)

Geometrically, the self-similar spacetime singles out the point \( r = t = 0 \), which in fact is a curvature singularity. It will be useful to introduce two related auxiliary functions,

\[
G(x) \equiv \frac{a(x)}{\alpha(x)}, \quad H(x) \equiv \frac{\alpha^2(x)}{x^2} - a^2(x). \quad (57)
\]

The past light cone of the singularity (from now on also referred to as “the” light cone) is the set of points \( (\tau, x_{lc}) \) with \( x_{lc} \) the solution to the equation

\[
G(x_{lc}) = 1. \quad (58)
\]

(Of course \( G = 1 \) is equivalent to \( H = 0 \).) In the following, we assume that there is only a single light cone \( G = 1 \) in the region \( t < 0 \).

Once we know the behavior of the metric under scale transformations we can derive the behavior of the Einstein tensor, and therefore of the stress-energy tensor. The components of the stress-energy tensor in coordinates \((t, r)\) obey

\[
T^\nu_\mu(t, r) = t^{-2}T^\nu_\mu(x). \quad (59)
\]

Using the rescaled components \( \tilde{T}^\nu_\mu \), the Einstein equations become ODEs in \( x \):

\[
\frac{a'}{a} + \frac{a^2 - 1}{2x} = -4\pi x a^2 \tilde{T}^t_t(x), \quad (60)
\]
ter, as time with a negative mass naked singularity at the center, there is no central vacuum region, but no vacuum region can be added to the energy-momentum tensor. Thus, we only have to solve for one of the metric functions, or equivalently, we only have to know one of the components of the energy-momentum tensor.

\[
\frac{a'}{\alpha} + \frac{1-a^2}{2x} = 4\pi xa^2 T_r^r(x),
\]

(61)

\[
\frac{a'}{a} = 4\pi a^2 T_r^r(x).
\]

(62)

Because in CSS a is a function of x only, the equations for \(a_t\) and \(a_r\) have both become equations for \(a'\), and can be combined to eliminate \(a'\) and obtain an algebraic equation for \(a\),

\[
a^{-2}(x) - 1 = 8\pi [x^2 T_t^t(x) + x T_r^r(x)]
\]

(63)

In a CSS spacetime, the Hawking mass \(M(t,r)\) is of the form \(M(t,r) = e^{-\tau} M(x)\). At the regular center, \(\bar{M}(0) = 0\), and from the Einstein equations it follows that \(\bar{M}'(x) \geq 0\). Suppose now that a spherically symmetric CSS spacetime contained a vacuum region, so that no matter was present in some interval of \(x\). The only spherically symmetric vacuum solution is the Schwarzschild. But a Schwarzschild solution has a finite Hawking mass \(M(t,r) = M\), which is incompatible with \(M(t,r) = e^{-\tau} M(x)\), unless \(M(t,r) = 0\). This means that a spherically symmetric regular CSS solution can contain a central vacuum region, but no vacuum region surrounding a matter region. In particular, it can not be asymptotically flat. (The exception would be a spacetime with a negative mass naked singularity at the center, as \(\bar{M}(0) < 0\) would make \(\bar{M}(\infty) = 0\) compatible with \(\bar{M}'(x) \geq 0\).)

We note in passing that for massless particles the energy-momentum tensor is traceless, and therefore both the Einstein and Ricci tensors are traceless. This gives an additional relation between the metric functions \(a\) and \(\alpha\):

\[
2a \left(\frac{\alpha}{a}\right)' - \frac{a^2-1}{x} + \frac{xa}{\alpha} \left(\frac{a'}{a} - \frac{a''}{\alpha}\right)' = 0.
\]

(64)

Thus, we only have to solve for one of the metric functions, or equivalently, we only have to know one of the components of the energy-momentum tensor.

\[\frac{\xi}{\alpha} = -\alpha^2 p^t + a^2 r p^r, \]

(69)

\[F = \frac{\pi^2}{\alpha} p^i p^i - \alpha^2 r^2 - m^2.\]

(70)

We now discuss test particle trajectories in a fixed CSS spacetime in some detail, as their properties are unfamiliar. Starting from the homothetic vector \(\xi\) is natural to construct a “homothetic energy”;

\[ J = -\xi^\mu p_\mu = -p_t = t p_t + r p_r. \]

(65)

In a self-similar spacetime it obeys the equation of motion

\[\frac{dJ}{d\sigma} = -m^2, \]

(66)

where \(\sigma\) is the affine parameter along particle trajectories. For massless particles \(J\) is a constant of motion. For massive particles, \(J + m^2 \sigma\) is constant, but to use this constant of motion one would have to solve for \(\sigma\) as a function of \((t, r, w, F)\).

In a CSS background, it is natural to rewrite the Vlasov equation in terms of \(f(t, r, J, F)\), obtaining

\[\frac{p_r}{\partial t} + p^r \frac{\partial f}{\partial r} - m^2 \frac{\partial f}{\partial \bar{J}} = 0, \]

(67)

where \(p^r\) and \(p^t\) are now functions of \((t, r, J, F)\).

In order to integrate trajectories of the reduced system in \(r\) and \(t\), we define

\[\frac{dr}{dt} = \frac{p^r}{p^t} \equiv v. \]

(68)

\(p^r\) and \(p^t\) can be obtained from \(F\) and \(J\) by solving the system

\[J = -\alpha^2 t p^t + a^2 r p^r, \]

(69)

\[F = \frac{\pi^2}{\alpha} p^i p^i - \alpha^2 r^2 - m^2. \]

(70)

We know that \(F \geq 0\), and by convention \(p^t > 0\). The radial momentum \(p_r\) can be positive, negative or zero,
and therefore may also have either sign. For \( a(x)x > a(x) \), that is outside the light cone, a unique solution exists for all \( J \). For \( a(x)x < a(x) \), that is inside the light cone, one solution exists for \( J = J_c \), and two solutions exist for \( J > J_c \), where

\[
J_c(x, F, m, r) \equiv \sqrt{H(x)(F + m^2r^2)}. \tag{71}
\]

The condition \( J \geq J_c \) means that we are outside or at the minimum radius that a particle with given \( x \) and \( F \) can have at a given \( t \). On the light cone, \( a(x)x = a(x) \), we have a unique solution for \( J > J_c = 0 \). On the light cone \( J = 0 \) is only possible with \( F = m = 0 \) and applies to massless particles of arbitrary momentum moving radially into the singularity along its past light cone.

C. Massless particle trajectories

We now discuss the trajectories of massless particles in more detail. \( J \) is then a constant of the motion. From the definitions of \( v \), \( J \) and \( F \) it is clear that for \( m = 0 \), \( v \) depends only on \( x \) and the combination

\[
Y \equiv \frac{J}{\sqrt{F}}. \tag{72}
\]

With the convention \( p^t > 0 \) we find that

\[
Y = \frac{\alpha^2/x + a^2v}{\sqrt{a^2 - a^2v^2}}. \tag{73}
\]

To solve this equation for \( v(x, Y) \), we need to square it first. This gives rise to a sign ambiguity that we discuss below. The result is

\[
v = \left( \frac{\alpha}{ax} \right) -\frac{aY \sqrt{-a^2 + (a^2 + Y^2)x^2}}{Y^2} = v_\pm(x, Y). \tag{74}
\]

The allowed range of \( v \) for particles with \( F > 0 \) is \(-a/\alpha < v < \alpha/a\). The rates of change of \( x \) and \( r \) are related by

\[
\frac{dx}{d\tau} \equiv V = v + x \tag{75}
\]

In coordinates \( x \) and \( \tau \), the particle equation of motion is autonomous:

\[
\frac{dx}{d\tau} = v_\pm(x, Y) + x \equiv V_\pm(x, Y). \tag{76}
\]

Trajectories of massless particles are therefore determined by \( Y \), up to a translation in \( \tau \). A translation in \( \tau \) corresponds to a simultaneous rescaling of \( t \) and \( r \). A second symmetry of massless particle trajectories arises because \( Y \) is invariant under a simultaneous rescaling of all components of \( p^\mu \). This leaves the trajectory in spacetime completely invariant: photons of different energy can have the same trajectory. The overall momentum scale, at constant \( Y \) and constant \( r \) and \( t \), is set by \( \sqrt{F} \). Here we assume that the contribution of particles with \( F = 0 \) to the stress-energy is zero, and we now concentrate on particles with \( F > 0 \). (If they made a finite contribution, this would constitute null dust. In our numerical examples we avoid this simply by imposing a cutoff.)

The projection into the \((r, t)\) plane of a the trajectory of a massless particle with \( F > 0 \) is timelike. Such particles come in from infinity, reach a minimum radius \( r \), and go out to infinity again. At large \( r \), the tangential velocity can be neglected compared to the radial velocity, and the radial projection of the trajectory is asymptotically null. Fig. 6 shows several trajectories in the \((r, t)\) plane that share the same value of \( Y \) but are related by a rescaling of \( r \) and \( t \). In the \((x, \tau)\) plane these would be related by a translation in \( \tau \). Fig. 3 shows typical particle trajectories in the \((r, t)\) plane with different values of \( Y \). Fig. 4 shows the same trajectories in \((x, \tau)\) coordinates, and Fig. 5 shows the equivalent trajectories in \((x, V)\) space. One should keep in mind that the turning point of smallest \( x \) (if one exists) does not coincide with the turning point of smallest \( r \).

Fig. 4 illustrates that there are two types of particles: type I particles are initially inside the light cone but leave it, while type II particles are always outside the light cone. Both types of particles start at \( x = x_{lc} \) in the asymptotic past as \( \tau \to -\infty \). Type I particles peel off from \( x = x_{lc} \) towards smaller \( x \), reach a minimum value \( x_{\text{min}}(Y) \) of \( x \) and turn round. A finite \( \tau \)-time later they cross the light cone \( x = x_{lc} \) and continue to large \( x \). Type II particles peel off from \( x = x_{lc} \) towards larger \( x \). Although they reach a minimum value of \( r \), their value of \( x \) always increase. Both types of particles reach \( x = \infty \) at \( \tau = \infty \). This is just the surface \( t = 0 \), where our coordinate system breaks down. Here we are only interested in the region \( t < 0 \).

In order to see when \( dx/d\tau \) vanishes, and which of the two values \( V_\pm \) is the correct one in a given situation, it is helpful to factorize \( V_\pm \) as

\[
V_\pm(x, Y) = x(a^2+Y^2)^{-1}(Y^2-H)^{1/2} \left[ (Y^2 - H)^{1/2} \pm G^{-1}Y \right], \tag{77}
\]

where \( G(x) \) and \( H(x) \) were defined in (57). Clearly both \( V_+ \) and \( V_- \) vanish at \( x = x_{\text{min}}(Y) \) which is defined by

\[
H(x_{\text{min}}) \equiv Y^2, \tag{78}
\]

One can show that at these points \( d^2x/d\tau^2 \) does not vanish and has the same sign as \( -dH/dx \). Therefore these points are true turning points. The square bracket in (77) can vanish only at the light cone \( x = x_{lc} \), where \( H = 0 \) and \( G = 1 \). In particular \( V_+ \) vanishes at \( x = x_{lc} \) for all \( Y < 0 \) (but not for \( Y > 0 \)), and \( V_- \) vanishes for all \( Y > 0 \) (but not for \( Y < 0 \)). This corresponds to \( V \to 0 \) as \( \tau \to -\infty \) (compare Fig. 3), and one can show that \( d^2x/d\tau^2 = 0 \) there.

To see how the sign choice \( V_\pm \) relates to our classification of particles, we note first that \( Y \) and \( -Y \) give rise to
FIG. 4: Trajectories of particles in the self-similar spacetime described in Sect. IV below, all with the same value $Y = 5$ but different overall scales. The past light cone of the singularity at $x_{lc} = 1.3642$ is denoted with a short-dashed line, and for orientation $x = 2$ has been marked with a long-dashed line.

FIG. 5: $(t, r)$-trajectories of particles in the self-similar space-time with $Y = 10, 5, 3, 2, 1, 0, -1$ (from left to right). The trajectory with $Y = 0$ has been indicated by a thicker line. The overall scale of each trajectory has fixed so that all trajectories meet in one point.

FIG. 6: The same trajectories in coordinates $(x, \tau)$. From left to right we have $Y = 10, 5, 3, 2, 1, 0, -1$. The thicker line is $Y = 0$. The origin in the $\tau$ axis has been arbitrarily chosen (corresponding to a choice of units $L_0$).

FIG. 7: The equivalent trajectories (from left to right $Y = 10, 5, 3, 2, 1, 0, -1$) in variables $x$ and $V$.

Inequivalent trajectories. With $t < 0$ and $p^t > 0$, it follows from (69) that $J > 0$ for any particle inside the light cone. Therefore type I particles have $Y > 0$. Inside the light cone, the velocity of type I particles is $V = V_- < 0$ while they are ingoing, and $V = V_+ > 0$ while they are outgoing. By continuity, $V = V_+$ also outside the light cone.

From Fig. 7 we see that type II particles have $v < 0$, and therefore $p^r < 0$. (To be precise, this holds for the region $t < 0$ covered by our coordinates. Presumably these particles have an $r$-turning point for some $t > 0$. $Y = 0$ particles have an $r$-turning point at $t = 0$.) Therefore type II particles have $J < 0$, and so $Y < 0$. This is summarized in Table I.

It seems plausible that CSS spacetimes exist where $H(x)$ has a local minimum, so that for some range of $Y^2$ the turning point equation $H = Y^2$ would have two so-

| TABLE I: Type I and type II particles |
|---------------------------------------|
| inside light cone | $V = V_-$ | never there |
| outside light cone | $V = V_+$ | $V = V_+$ |
| Type I | Type II |
| $Y > 0$ | $Y < 0$ |
lutions \( x_{\text{min}} \) and \( x_{\text{max}} \) for each value of \( Y \). Particles with \( Y \) in that range would oscillate between the two turning points. Such "type III" particles would be trapped within a finite range of \( x \), and would therefore hit the singularity. However, from the definition \( J \) of \( J \) it follows that \( p^r \) and \( p^t \) at constant \( x \) and \( J \) scale as \( t^{-1} \). As a type III particle returns to the same value of \( x \) again and again, its energy and radial momentum as measured by a constant \( r \) observer increase as \( t^{-1} \). A similar argument holds for the tangential momenta. A finite number of such particles would therefore constitute an energy-momentum scaling as \( t^{-1} \) confined to a region of a physical size scaling as \( t^{-1} \), so that the energy and other frame components of the stress-energy would scale as \( t^{-4} \). This is incompatible with self-similarity, where these stress-energy components scales as \( t^{-2} \). Type III particles are allowed as test particles, but can not be used as the matter content of a CSS spacetime. From now on we restrict consideration to type I and II particles.

### D. Solutions with massless particles

For massless particles, \( J \) is a constant of motion, and therefore \( \partial / \partial J \) disappears from the Vlasov equation \( \Theta \). We then obtain an automatic solution of the Vlasov equation by the ansatz \( f(t, r, J, F) = h(J, F) \). However, we shall see that this ansatz leads to a divergent stress-energy tensor. Therefore we have to resort to the most general situation on a CSS background, by allowing for the dependence of \( f \) on the spacetime coordinates \( r \) and \( t \) through a dependence on a third non-trivial constant of the motion. (With \( c_1 = J \), this is the constant \( c_2 \) of our general discussion of conserved quantities.) We rewrite the Vlasov equation for massless particles in coordinates \( x \) and \( \tau \) as

\[
\left( \frac{\partial}{\partial \tau} + V(x, Y) \frac{\partial}{\partial x} \right) f(\tau, x, J, F) = 0, \tag{79}
\]

We can solve this by the method of characteristics as

\[
f(\tau, x, J, F) = g(\tau_0, J, F), \tag{80}
\]

where

\[
\tau_0(\tau, x, Y) \equiv \tau - \int_{x_0(Y)}^{x} \frac{dx'}{V(x', Y)}. \tag{81}
\]

Clearly \( \tau_0 \) is the time \( \tau \) when the particle with a given \( Y \) and \( F \) was at \( x = x_0 \). Here we have specified \( x_0 \) as a given function of \( Y \) (which could be a constant). A priori \( x_0 \) could also depend on \( F \), but that would break self-similarity of the resulting stress-energy. For \( x_0(Y) < x_c \) (only possible for type I particles) we must specify if the spacetime point \( (x_0, \tau_0) \) is on the ingoing or outgoing part of the trajectory. A natural choice then is \( x_0(Y) = x_{\text{min}}(Y) \), but this does not work for type II particles, which do not have a turning point. For either type of particle we could use a fixed value \( x_0 = \text{const} > x_c \).

As we are mainly working in coordinates \( t \) and \( r \) in this paper, we shall use \( \tau_0 \) in the form of

\[
t_0 \equiv -L_0 e^{-\tau_0} = t \exp \int_{x_0(Y)}^{x} \frac{dx'}{V(x', Y)} \equiv t Q(x, Y). \tag{82}
\]

(Note again that \( Q(x, Y) \) depends on the choice of \( x_0(Y) \).)

We have just shown that the general spherically symmetric solution of the Vlasov equation in a spherically symmetric CSS spacetime is

\[
\text{Vlasov in CSS spacetime} \Rightarrow f(t, r, w, F) = g(t_0, J, F). \tag{83}
\]

One could think, by analogy with the static case, that the requirement of a self-similar energy-momentum tensor eliminates the dependence on \( t_0 \). However, that is not the case. In the static case, the constants of motion \( E \) and \( F \) depend only on the variables \( r, p^r \) and \( p^t \). Only the constant of motion \( t_0 \) depends also on \( t \), and that is why \( g \) can not depend on \( t \) (except in the piecewise constant way of Eq. \((47)) \). In the self-similar case, not just \( t_0 \) but all three constants \( t_0, J \), and \( F \) change under a scale transformation. Imposing consistency with a CSS spacetime is therefore less straightforward.

In order to derive the appropriate form of \( g(t_0, J, F) \), it is helpful to change the integration over \( F \), which has dimension \( P^2 L^2 \), temporarily to an integration over \( F' = F / r^2 \), which has dimension \( P^2 \). This completely separates the integration over momentum space from the remaining dependence on \( r \) and \( t \). The rescaled stress-energy tensor components are

\[
\begin{align*}
\bar{T}_{rr} & = \pi \int_0^\infty d\bar{F} \int_{-\infty}^\infty dw t^2 g(t, J, F) \frac{w^2}{W}, \\
\bar{T}_{rt} & = -\pi \int_0^\infty d\bar{F} \int_{-\infty}^\infty dw t^2 g(t, J, F) W, \\
\bar{T}_{tt} & = -\pi \frac{\alpha}{a} \int_0^\infty d\bar{F} \int_{-\infty}^\infty dw t^2 g(t, J, F) w, \\
\bar{T}_{\theta\theta} & = \frac{\pi}{2} \int_0^\infty d\bar{F} \int_{-\infty}^\infty dw t^2 g(t, J, F) F W .
\end{align*}
\tag{84}
\]

The functions \( t_0, J, F, W \) are homogeneous functions of \( r \) and \( t \) of degree 1, 1, 2 and 0, respectively. If we now demand that \( \bar{T}_{\mu\nu}(st, sr) = \bar{T}_{\mu\nu}(t, r) \) for any scaling constant \( s \), so that the stress-energy tensor is compatible with CSS, we find that \( g \) must be the following homogeneous function of its arguments:

\[
\text{Vlasov + CSS} \Rightarrow g(st, J, s^2 F) = s^{-2} g(t, J, F). \tag{88}
\]

A homogeneous function of three variables can be re-expressed as an arbitrary function of two variables. We define

\[
g(t_0, J, F) = \frac{1}{F} k(Y, Z), \quad \text{where} \quad Z \equiv \frac{F}{t_0^2} \tag{89}
\]

and where \( Y \) was defined above in \((73)\). The situation is similar to static solutions of Einstein-Vlasov in that we
can freely specify a function $k(Y, Z)$ of two variables that automatically solves the Vlasov equation, and obtain the spacetime by solving the Einstein equations, which become integral-differential equations. It differs in that $g(t_0, J, F)$ is not simply the form $h(J, F)$. Because we need $t_0$ in order to obtain the general solution, we need to determine $Q(x, Y)$, which is equivalent to integrating the equations of motion. Note that $f(t, r, w, F)$ is determined by $h(Y, Z)$ only when the function $x_0(Y)$ that appears in $Q$ has been specified.

It is interesting to note that the homogeneity condition (88) arises from dimensional analysis in the length dimension $L$ alone. $Y$ has neither $P$ nor $L$ dimension, but $Z$ has dimension $P^2$. This is not an obstruction to self-similarity of the spacetime. If we had measured particle energy-momentum in units of length, dimensional analysis would have confused us in this point. Note that $[f] = [g] = -L^{-2}P^{-4}$, $[k] = P^{-2}$, and $[\bar{k}] = L^2$.

The Einstein equations are now once again a set of integral-differential equations that we can solve by iteration, once $k(Y, Z)$ and $x_0(Y)$ have been specified. The next step in our program would be to change the integration variables from $F$ and $w$ to $Y$ and $Z$. By analogy with the massless static case, we expect the integration over $Z$ to decouple from that over $Y$. This is correct, but going to the variables $Y$ and $Z$ directly gives rise to rather complicated expressions. When calculating the stress-energy components, we find it helpful to integrate over the variables $Z$ and $u = w/\sqrt{F}$. This already decouples the integrals:

$$\bar{T}_r^r = \frac{\pi}{x^4} \int_{-\infty}^{\infty} Q^2(x, Y) \bar{k}(Y) \frac{u^2}{\sqrt{u^2 + 1}} du,$$  
(90)  
$$\bar{T}_t^t = -\frac{\pi}{x^2} \int_{-\infty}^{\infty} Q^2(x, Y) \bar{k}(Y) \sqrt{u^2 + 1} du,$$  
(91)  
$$\bar{T}_r^t = -\frac{\pi \alpha}{x^3 u} \int_{-\infty}^{\infty} Q^2(x, Y) \bar{k}(Y) u du,$$  
(92)  
$$\bar{T}_\theta^\theta = \frac{\pi}{2x^4} \int_{-\infty}^{\infty} Q^2(x, Y) \bar{k}(Y) \frac{1}{\sqrt{u^2 + 1}} du,$$  
(93)  
where

$$\bar{k}(Y) \equiv \int_0^\infty k(Y, Z) dZ.$$  
(94)

The auxiliary variable $u$ is related to $Y$ by

$$Y = Y(x, u) = \frac{a(x)}{x} \sqrt{u^2 + 1} + a(x) u.$$  
(95)

We can now see clearly why $g$ must depend on $t_0$. If it did not, $k(Y, Z)$ could not depend on $Z$, and therefore $\bar{k}(Y)$ would diverge. The dependence on $t_0$ is necessary to create a finite CSS stress-energy tensor.

### E. Some explicit solutions

We begin with a CSS solution of the Vlasov system in flat spacetime, which is trivially self-similar with $x_{lc} = 1$. $Q_\pm(x, Y)$ can then be calculated in closed form. From the definition of $Q$ it follows that it obeys the differential equation

$$\frac{\partial \ln Q_\pm(x, Y)}{\partial x} = \frac{1}{V_\pm(x, Y)}.$$  
(96)

Integration of this is straightforward when $a(x) = \alpha(x) = 1$. We now consider the boundary conditions. We make the assumption that the reference point $(x_0, \tau_0)$ is on the outgoing branch of the trajectories. This gives rise to the boundary condition $Q_+(x_0(Y), Y) = 1$ for outgoing particles. For ingoing particles we have to piece $Q$ together from an ingoing and an outgoing piece, integrating first from $x$ to $x_{min}$ using $V_-$ and then from $x_{min}$ to $x_0$ using $V_+$, respectively. Here the turning point $x_{min}$ of type I particles is given by

$$x_{min}(Y) = (1 + Y^2)^{-1/2}$$  
(97)

for $Y > 0$. (Type II particles, with $Y < 0$, have no turning point.) Our final result is

$$Q_\pm(x, Y) = \frac{Y \pm \sqrt{x^2(1 + Y^2) - 1}}{Y + \sqrt{x_0^2(Y)(1 + Y^2) - 1}}$$  
(98)

which holds for type I and type II particles as long as $x$ and $x_0(Y)$ are both in the physical range: $x > x_{min}(Y)$ for $Y > 0$, and $x > 1$ for $Y < 0$. Note that $Q_- \to 0$ as $x \to 1_-$ for type I particles and $Q_+ \to 0$ as $x \to 1_+$ for type II particles: this is the limit in which the trajectory peels off the light cone at $\tau \to -\infty$. Note also that $Q_+$ is finite for type I particles as $x \to 1$ because the reference point is the point where the trajectory crosses the light cone while going out. By contrast, $Q_+$ diverges for type II particles as $x \to 1_+$ because this pushes the reference point to $\tau \to -\infty$ where the trajectory peels off the light cone.

On a flat spacetime, neglecting gravity, we are just dealing with the Vlasov equation, which is linear. It is therefore sufficient to consider the following ansatz for $\bar{k}(Y)$:

$$\bar{k}(Y) = \bar{k}_+ \delta(Y - Y_0) + \bar{k}_- \delta(Y + Y_0),$$  
(99)

with $Y_0 > 0$. We can then integrate the resulting stress-energy tensor over $Y_0$, with arbitrary functions $\bar{k}_\pm(Y_0)$, in order to obtain the general result. We shall see later why it is useful to consider the contributions $Y = Y_0$ and $Y = -Y_0$ together.

We first calculate the stress-energy inside the light cone. Because type II particles are always outside the light cone, a solution with both type I and type II particles is independent of its type II particle content for $0 \leq x < 1$. Inside the light cone we find that the stress-energy tensor and particle number current components are

$$\bar{T}_t^t(x) = -\frac{2\pi K_+}{x_{\min}^3} \frac{1}{x \sqrt{x^2 - x_{\min}^2}},$$  
(100)
\[ T^r_r(x) = \frac{2\pi K_+ x_{\text{min}}^2}{x^3 \sqrt{x^2 - x_{\text{min}}^2}} + \frac{(1 - x_{\text{min}}^2)(x^2 - x_{\text{min}}^2)}{x^3 \sqrt{x^2 - x_{\text{min}}^2}}, \quad (101) \]

\[ T^r_t(x) = \frac{2\pi K_+}{x^2 \sqrt{x^2 - x_{\text{min}}^2}}, \quad (102) \]

\[ T^0_\theta(x) = \frac{\pi K_+}{x^2 \sqrt{x^2 - x_{\text{min}}^2}} x^3 \sqrt{x^2 - x_{\text{min}}^2}, \quad (103) \]

\[ \bar{T}^0_t(x) = \frac{2\pi K_+}{x_{\text{min}}^2} \sqrt{1 - x_{\text{min}}^2} \ \frac{x^2 - x_{\text{min}}^2}{x^3 \sqrt{x^2 - x_{\text{min}}^2}}, \quad (104) \]

\[ \bar{N}^t(x) = \frac{2\pi K_+}{x_{\text{min}}^2} \sqrt{1 - x_{\text{min}}^2} \ \frac{x^2 - 2x_{\text{min}}^2}{x^3} \quad (105) \]

for \( x > x_{\text{min}} \), and vanish for \( x < x_{\text{min}} \). Here \( x_{\text{min}} \) is shorthand for \( x_{\text{min}}(Y_0) \), and we have introduced the shorthand

\[ K_\pm = \frac{\sqrt{1 + x_{\text{min}}^2(1 + Y_0^2)} - 1}{Y_0 \pm \sqrt{1 + x_{\text{min}}^2(1 + Y_0^2)}}, \quad (106) \]

\[ \bar{T}^t_t(x) = -\frac{\pi}{x_{\text{min}}^2} \frac{K_+ + K_-}{x \sqrt{x^2 - x_{\text{min}}^2}}, \quad (107) \]

\[ \bar{N}^t(x) = \frac{\pi}{x_{\text{min}}^2} \left( (K_+ + K_-) \frac{x_{\text{min}}^2}{x^3 \sqrt{x^2 - x_{\text{min}}^2}} - 2(K_+ - K_-) \frac{x_{\text{min}}^2 \sqrt{1 - x_{\text{min}}^2}}{x^3} \right), \quad (108) \]

\[ \bar{T}^r_r(x) = \frac{\pi}{x_{\text{min}}^2} \left( (K_+ + K_-) \frac{x_{\text{min}}^2}{x^3 \sqrt{x^2 - x_{\text{min}}^2}} - (K_+ - K_-) \frac{\sqrt{1 - x_{\text{min}}^2}}{x^2 x_{\text{min}}^2} \right), \quad (109) \]

\[ \bar{T}^r_t(x) = \frac{\pi}{x_{\text{min}}^2} \left( (K_+ + K_-) \frac{1}{x^2 \sqrt{x^2 - x_{\text{min}}^2}} - (K_+ - K_-) \frac{\sqrt{1 - x_{\text{min}}^2}}{x^2 x_{\text{min}}^2} \right), \quad (110) \]

\[ \bar{N}^r(x) = \frac{\pi}{x_{\text{min}}^2} \left( (K_+ + K_-) \frac{1 + x^2 - 2x_{\text{min}}^2}{x^3 \sqrt{x^2 - x_{\text{min}}^2}} + 2(K_+ - K_-) \frac{\sqrt{1 - x_{\text{min}}^2}}{x^2} \right), \quad (111) \]

\[ \bar{N}^r(x) = \frac{\pi}{x_{\text{min}}^2} \left( (K_+ + K_-) \frac{1 - 2x_{\text{min}}^2}{x^3} + (K_+ - K_-) \frac{\sqrt{1 - x_{\text{min}}^2}}{x^2} \right), \quad (112) \]

The constant \( K_- \) was defined above. Here, as above, we use \( x_{\text{min}} \) as shorthand for the turning point \( x_{\text{min}}(Y_0) \) of the type I particles. (This number also appears in the contribution of the type II particles, but is then just used as a shorthand.)

Based on these results, we can make two important general points. The first is that for general values of \( K_\pm \) the stress-energy tensor is not continuous at the light cone. However, if and only if \( K_+ = K_- \) the stress-energy is continuous at the light cone, and in fact analytic. On the other hand, the particle-current is always continuous, but never analytic. In the general flat-space case, where \( k_\pm \) are promoted to functions of \( Y > 0 \), the stress-energy is analytic if \( K_+(Y) = K_-(Y) \) for all \( Y \).

The second important point is that while we can freely specify both \( k(Y, Z) \) and \( x_0(Y) \), the location of the reference point \( x_0(Y) \) is not really physical. We have seen that the stress-energy in our example does not depend on the two numbers \( x_0 \) and \( k_+ \) (or \( k_- \)) separately but only on the combination \( K_+ \) (or \( K_- \)). Generally, the Vlasov density \( f(t, r, w, F) \) is completely determined by one free function of \( Y \) and \( Z \). In flat spacetime, and assuming that \( x_0(Y) = x_0(|Y|) \), we have shown that this free function is

\[ K(Y, Z) = \frac{k(Y, Z)}{|Y| + \sqrt{x_0^2(Y)(1 + Y^2)} - 1}. \quad (113) \]

In the self-gravitating case we cannot give this function in closed form. Therefore, in the self-gravitating case we must just “fix the gauge” by fixing \( x_0(Y) \) arbitrarily.
We shall now smear the $\delta$-function in $Y$, and then couple this distribution to gravity. In the following we restrict our ansatz to type I particles. Inside the light cone this is no restriction anyway. It allows us to use $x_0(Y) = x_{\text{min}}(Y)$, which simplifies the numerical calculations. We also want to start with a matter distribution whose stress-energy tensor in flat spacetime is discontinuous, because we shall see that the gravitational back-reaction makes it continuous. As an example, the jump in $-\hat{T}^t_t$ is shown in Fig. 8 for the case $Y_0 = 6$, $k_+ = 10^{-4}\sqrt{2\pi}$ and $k_- = 0$, using $x_0(Y) = x_{\text{min}}(Y)$.

Still in flat spacetime, we smear the $\delta$-function into a Gaussian:

$$\bar{k}(Y) = \frac{k_+}{\sqrt{2\pi}\sigma} e^{-(Y-Y_0)^2/2\sigma^2} \quad (114)$$

with a minimum cutoff just above $Y = 0$ and a maximum cutoff far from $Y_0$. It is not possible to calculate the results analytically, so that we calculate them with a C code. The divergence of the stress-energy at $x_{\text{min}}(Y)$ is then smoothed out by the integration over $Y$, and the stress-energy components are very smooth apart from at the light cone. The results for the stress-energy tensor are shown in Fig. 9 for the case $\sigma = 1$ with $Y_0 = 6$ and $k_+ = 10^{-4}\sqrt{2\pi}$ as in the $\delta$-function example.

Once we couple the matter to gravity we must solve the integral-differential equations by iteration. That is, we start with some initial metric, say flat spacetime, and calculate the stress-energy tensor of a set of particles for a certain function $\bar{k}(Y)$. Then we integrate the metric corresponding to this stress-energy distribution using Eqs. (110, 111) and calculate a new stress-energy tensor, and so on, until the process converges. In Fig. 10 we show the metric that results from coupling the Gaussian $\bar{k}$ of the previous example, with parameters $Y_0 = 6$, $\sigma = 1$ and $\ell_0 = 10^{-4}\sqrt{2\pi}$, to gravity. The convergence of the numerical method is demonstrated in Fig. 11. Fig. 12 compares the radial pressure profile of the Gaussian ansatz in flat spacetime with the Gaussian coupled to gravity. Even though we started with a discontinuous stress-energy tensor on flat spacetime, we now find that the self-consistent stress-energy is now $C^0$ (but not $C^1$). This is illustrated in Fig. 13. As a consequence, the metric is $C^1$. Another check of the procedure is given by the Eq. (113).

With very low values of $\bar{k}_+$ we can form solutions with a metric which is very close to flat spacetime. On the other hand, with larger values of $\bar{k}_+$ we get solutions which are very close to horizon formation. For example with $\bar{k}_+ = 3 \cdot 10^{-4}\sqrt{2\pi}$, the maximum value of $a$ is greater than 1.45 (giving $2M/r \simeq 0.53$), which is even bigger than the maximum value of the same function in the Choptuik or Evans-Coleman spacetimes.

We can understand analytically why the coupling to gravity makes the matter stress-energy more regular. For particles that just peel off the light cone, that is for $x = x_{lc}$, and $V = V_-$ for type I particles and $V = V_+$ for type II particles, we can expand $V$ as

$$V_{\pm}(x, \mp |Y|) = c(x - x_{lc}) + O(x - x_{lc})^2, \quad (115)$$

$$c = x_{lc}G'(x_{lc}) = 1 + x_{lc} \left(\frac{a'}{a} - \frac{a''}{a}\right) |_{x_{lc}}. \quad (116)$$

Note that $c$ is independent of $Y$, and depends only on the spacetime curvature at the light cone. (In Fig. 7 we see that all trajectories approach the light cone with the same slope $c$.) Integrating this we find that

$$Q_{\pm}(x, \mp |Y|) \simeq C \left[ x_0(Y), Y \right] |x - x_{lc}|^2, \quad x \simeq x_{lc} \quad (117)$$

We see that the decay of $Q$ towards the light cone depends on $c$. When the metric is flat we have $c = 1$. This combines with a factor $(x - x_{lc})$ arising elsewhere in the integrals to give a finite discontinuity at the light cone. However, if $c < 1$ the decay is faster and particles peeling off the light cone make no contribution at the light cone, giving a continuous stress-energy and particle current. Typically we find that $c \simeq 0.8$ in our examples.
when gravity comes into play. Therefore gravity makes the
correlating type I with type II particles
that was required for this in flat spacetime. On the other
hand, generically the stress-energy is not analytic at the
light cone because of the non-integer power of \(|x - x_{lc}|\),
but we could have a situation where \(c^{-1}\) is an integer and
then a suitable arrangement of the particle distributions
could render an analytic stress-energy tensor. We have not
been able to construct any particular example.

F. Properties of the distribution function

Vlasov matter is considered to be a good matter model
both in Newtonian gravity and in general relativity be-
cause it does not develop singularities in flat spacetime.
For massive particles it has been proved that singulari-
ties are also absent in the Vlasov-Poisson system even for
large initial data \([14]\), and for small data in the Einstein-
Vlasov system \([15]\).

Even though it is clear that we cannot apply this sec-
ond result to massless particles, it is interesting to com-
pare the assumptions of that theorem with the properties
of our solutions, because we can also construct solutions
which are arbitrarily close to flat spacetime. The main
assumptions of the theorem are that 1. the initial data
has compact support in both momentum space and physical
space, 2. \(f\) is small, and in particular bounded, 3. \(f\)
is \(C^1\) and the metric is \(C^2\), and 4. the spacetime has a
regular center.

Using only type I particles, our metric solutions are
genERICALLY only \(C^1\) at the light cone, but we believe that
correlating type I with type II particles solutions can be
made \(C^2\) there. We clearly have a regular center before
the formation of the singularity, so we only need to an-
alyze the boundedness of \(f\) and the compactness of its
support at the initial time.

Our CSS solutions are infinitely extended in space, but
we can match initial data for a CSS solution for \(r < r_0\)
smoothly to initial data of compact support in space. As
long as \(r_0 > x_{lc}(-t)\), the domain of dependence of the
CSS part of the data includes the singularity. We then
have a solution with compact support in space at the
initial time which is nevertheless CSS in a central region
including the singularity.

We choose a function \(k(Y, Z)\) that has support only
in a neighborhood of \((Y_0, Z_0)\) not including \(Y = 0\) or
\(Z = 0\). For given \(t < 0\) and \(r\), \(f(t, r, w, F)\) then has
support in momentum space only in a neighborhood of the
two points \((w_\pm(t, r, Y_0, Z_0), F_\pm(t, r, Y_0, Z_0))\). The
two signs correspond to ingoing and outgoing particles, for
\(x < x_{lc}\). For \(x > x_{lc}\), only the positive sign applies. We
find immediately that

\[
F_\pm(t, r, Y_0, Z_0) = Z_0^2 Q_{\pm}^2(x, Y_0),
\]

which is finite for finite \(r\). Because angular momentum
is conserved along particle trajectories, we conclude that
the support of \(f\) in angular momentum is always com-
compac.

\[
Y = \frac{\alpha(x)}{x} \sqrt{1 + \frac{(rw)^2}{F}} + a(x) \frac{rw}{\sqrt{F}},
\]

for fixed \(r, t\). \(Y = Y_0\) and \(F = F_k(t, r, Y_0, Z_0)\). This is
also finite. However, now \(w\) is not a constant of motion
and therefore the support in \(w\) does not stay constant.
The typical time evolution of the radial momentum \(w\)
of a type I particle is the following: an ingoing particle
starting with certain negative \(w\) accelerates towards the
center, reaching a maximum modulus, and then decel-
erates having a turning point and being ejected off the
light cone with positive \(w\). Eqs. \((118)\) and \((119)\) to-
gether show us that along lines of constant \(x\), the radial
momentum \(w\) of the particles that happen to be there
at time \(t\) is independent of \(t\). (The value of \(F\) of those
particles even decays as \(F^2\). Note that at each \(t\) these
are different particles.) Therefore the support in radial
momentum remains compact during the evolution.

We now show that in our CSS solutions \(f(t, r, w, F)\) can
not be finite on the light cone \(r = x_{lc}(-t)\) at zero par-
cle momentum \(w = F = 0\). Consider again a function
\(k(Y, Z)\) that has support only in a small neighborhood
of \((Y_0, Z_0)\). Clearly if \(f\) in a CSS solution is bounded
at one value of \(t < 0\), it is bounded for all \(t < 0\). Therefore
we consider a fixed value \(t < 0\). In order to find a limit
in \((r, w, F)\) space in which

\[
f(t, r, w, F) = \frac{1}{F} k(Y, Z)
\]

blows up, we need to find a limit in which \(F \to 0\) while
\(Y \to Y_0\) and \(Z \to Z_0\) simultaneously. Consider therefore
the limit

\[
F \to 0, \quad w \to A(-t)^{-1} F^{\frac{1}{2}}, \quad x \to x_{lc} + B F^{\frac{1}{2}},
\]

where \(c\) is the constant defined in \((117)\), and \(A\) and \(B\)
are real constants that will be determined. From \((119)\)
we find that in this limit

\[
Y \to \alpha(x_{lc}) \left(\frac{A^2 + x_{lc}^{-2}}{2} + A\right) \equiv Y_0(A)
\]

and therefore

\[
Z \to C \sqrt{|x_0(Y_0), Y_0|^2} \frac{|B|}{2} \equiv Z_0(A, B).
\]

We can therefore always arrange the required limit for
any values of \(Y_0 > 0\) and \(Z_0\) by a suitable choice of
the constants \(A\) and \(B\). We have therefore shown that
\(F\) is infinite on the light cone at zero momentum un-
less it is identically zero. From the fact that the stress-
energy tensor and the particle current are both finite if
\(k(y) = \int k(Y, Z) dZ\) exists, this blowup is not a phys-
ical problem. Furthermore, for the ansatz of \(k(Y, Z)\) with
compact support in \(Y\) and \(Z\), bounded away from \(Y = 0\)
and \(Z = 0\), \(f\) is finite and can be made arbitrarily small
(everywhere but at the light cone at zero momentum).
FIG. 10: Metric function $a(x)$ and corresponding (dimensionless) mass function $\bar{M}(x)$. Note that the spacetime is not asymptotically flat, as we expected. The vertical lines give the position of the light cone $x_{lc} = 1.3642$.

FIG. 11: On the left we show the different iterations of the metric function $a(x)$, starting with the flat case in the iteration $i = 0$. The convergence is fast and starting from $i = 3$ or $i = 4$ it is not possible to resolve different iterations in the figure. On the right, we show the decay of differences between successive iterations. We have defined $\Delta_i a = a_i - a_{i-1}$. The continuous line represents the 2-norm and the dotted line the $\infty$-norm, both integrated between $x = 0$ and $x = 2$.

V. CONCLUSIONS

Type II critical phenomena, in which the black hole mass vanishes as a power of distance from the black hole threshold, have been found (in some region of parameter space) for almost all Einstein-matter systems in spherical symmetry. These include real and complex scalar fields with arbitrary potential terms, conformal couplings, and coupling to a Maxwell field, perfect fluids, sigma models and Yang-Mills fields. The only exception seems to be the spherically symmetric Einstein-Vlasov system. This raises the question what distinguishes this system from the other ones, and the wider question if the existence of type II critical phenomena is the rule or the exception.

Critical phenomena at the black hole threshold require the existence of a critical solution. This is a solution that has precisely one unstable linear perturbation mode, with the additional property that a fully nonlinear evolution starting with a finite amplitude of this perturbation mode results in a black hole, while a finite amplitude of the opposite sign results in dispersion (or another outcome, such as a star or a naked singularity). In type II critical phenomena the critical solution is also self-similar, either continuously (CSS) or discretely (DSS). As a first step into understanding the absence of type II critical phenomena we have therefore investigated the existence and regularity of CSS solutions.

The rest mass of the particles of the collisionless matter introduces a scale into the coupled Vlasov and Einstein equations, which could be incompatible with exact self-similarity. However, there are many examples in other matter models where such a scale can be treated as a small perturbation in a class of solutions that are asymptotically self-similar. Type II critical phenomena are then not affected by the presence of the scale in the field equations. We have therefore focussed in this paper on spherically CSS solutions with a regular center of the Einstein-Vlasov system with massless particles, hoping to generalize these later to asymptotically CSS solutions with massive particles.

The main result of this paper is the explicit construction of a family of spherically symmetric CSS solutions
with massless particles that is parameterized by an arbitrary function of two conserved quantities $k(Y,Z)$. By function counting we have constructed the most general spherically symmetric CSS solution with a regular center before the formation of the singularity, but there may be particular solutions that we have overlooked. We have also assumed the presence of certain cutoffs for very low and very large $Y,Z$, in order to avoid divergences.

In order to make this result more transparent, we have also rederived the well-known general static spherically symmetric solution, which is parameterized by an arbitrary function $h(E,F)$ of the conserved particle energy $E$ and angular momentum $F$. In both cases, static and CSS, we have first found the general solution for Vlasov test particles on a fixed spacetime of that symmetry, which is parameterized by a free function of three conserved quantities. We have then shown that when we demand that the resulting stress-energy tensor is compatible with the symmetry, the general solution depends only on two conserved quantities. In the static case, these are the obvious ones $E$ and $F$, while the choice in the CSS case is much less obvious.

Initial data for our solutions can be given compact support in momentum space, and can be truncated in space without affecting a central CSS region that includes the usual CSS singularity. However, the Vlasov function $f(x^\mu,p^\nu)$ diverges as $p^\nu \to 0$ on the past light cone of the singularity. This divergence is integrable, so that both the stress-energy and the particle current are finite everywhere in spacetime except at the CSS singularity. Furthermore, our solutions can be constructed arbitrarily close to flat spacetime.

The stress-energy tensor and particle current are less differentiable at the light cone than elsewhere. The stress-energy tensor of a generic test particle distribution is discontinuous at the light cone, but by imposing a relation between $k(Y,Z)$ and $k(-Y,Z)$ it can be made $C^0$. When the Vlasov matter is coupled to gravity, we gain one order of differentiability: the stress-energy tensor is now generically $C^0$, and can be made $C^1$. The metric is generically $C^1$ at the light cone, and can be made $C^2$. (Naively, one would expect the metric to be two orders up from the stress-energy, but in polar-radial coordinates the two metric coefficients $a$ and $\alpha$ can be determined from Einstein equations that contain only first derivatives of $a$ and $\alpha$.)

Because our CSS solutions can be constructed with arbitrarily weak curvature, their curvature singularity is essentially kinematic: particles are “aimed” at the spacetime point where the singularity will occur, rather than being focussed by gravity. In this context, the relation between regularity at the past light cone and the coupling to gravity is worth commenting on. Any spherically symmetric CSS solution of a massless scalar test field that is regular at the center for $t < 0$ is necessarily singular at the past light cone. Coupling the scalar field to gravity, there exists an isolated strong field solution which is analytic both at the center and the light cone. The same is true for other field theories and a for a perfect fluid. By contrast, coupling collisionless matter to gravity adds one order of differentiability at the light cone to all matter configurations, but no solution can be analytic at the light cone. This behavior is mathematically more similar to the behavior of the preferred scalar field solution at its future light cone.

We shall consider the implications of our results for type II critical phenomena in detail elsewhere. However, we have seen that the spacetime metric depends on the free function $k(Y,Z)$ only through the integral $k(Y) = \int k(Y,Z) dZ$. This means that there are infinitely many matter configurations that give rise to the same spacetime. It is clear that a similar result will hold for the linearized perturbations of these solutions. Therefore there will be an infinite number of linear perturbation modes with the same eigenvalue $\lambda$. None of our solutions can therefore have a single growing mode. This seems to be related to the fact that Einstein-Vlasov is not a field theory. (While the phase space of the spherically sym-
metric scalar field consists of pairs \( \phi(r), \dot{\phi}(r) \), the phase space of the spherically symmetric collisionless matter consists of functions \( f(r, w, F) \): this phase space is much bigger.)

On the other hand, due to the fact that \( f \) is conserved along particles trajectories, we cannot expect to get a very close approach to a self-similar solution during the evolution of initial data with bounded \( f \). We have seen that in the self-similar solutions \( f \) is unbounded for low momentum near the light cone, while numerical simulations typically work with finite \( f \).

Any, or both, of those two reasons could explain why the collapse simulations that have been carried out did not find any sign of type II critical phenomena. We must not forget, however, that those simulations worked with massive particles, while here we have assumed massless particles.

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