Determining holographic wave functions from Wilsonian renormalization group

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We show a possible way to build the AdS/CFT correspondence starting from the quantum field theory side based on renormalization group approach. An extra dimension is naturally introduced in our scheme as the renomalization scale. The holographic wave equations are derived, with the potential term being determined by the QFT properties. We discover that only around the fixed point, i.e. the conformal limit, the potential in the bulk equations can be fully constrained, and upon this foundation, the correspondence is build. We demonstrate this fact using a 3d scalar theory in which, besides the trivial fixed point, there exists the Wilson-Fisher fixed point. From the energy scalings around those fixed points, we determine the behavior of the potential in the bulk equations.

Introduction—The AdS/CFT (or gauge/gravity) correspondence [1] provides a possible way to understanding nonperturbative nature of gauge theories. The correspondence conjectures that strongly correlated gauge theories in $d$ dimensional spacetime can be mapped onto weak gravitational theories in $d+1$ dimensional spacetime.

The studies of the AdS/CFT correspondence may include mainly two manners: One is to start from a string theory, choosing the background in such a way as to reproduce essential ingredients of, for instance, quantum chromodynamics (QCD) as confinement [2] or matter in fundamental representation [3–5], and to study the resulting QCD-like theories. The other, bottom-up approach is to begin with QCD and attempt to determine resulting QCD-like theories. The studies of the AdS/CFT correspondence conjectures that strongly correlated gauge theories. The correspondence [1] provides a possible way to understanding nonperturbative nature of gauge theories. The correspondence [1] provides a possible way to understanding nonperturbative nature of gauge theories. The correspondence [1] provides a possible way to understanding nonperturbative nature of gauge theories.

The purpose of this work is to propose a way to determine $U(z)$ from the QFT side. All information about the QFT dynamics is encoded in the effective action $\Gamma_k$ and can be studied via the functional renormalization group (fRG) [8]. There, the effective average action $\Gamma_k$ is defined such that high momentum modes $p > k$ for a cutoff scale $k$ are integrated out. The fRG equation follows the change of $\Gamma_k$ by sliding $k$ as a functional differential equation and one obtains the full effective action as $\Gamma = \Gamma_k \rightarrow 0$. In addition to the fact that fRG has been elucidated the nonperturbative nature in QFT (see e.g. Ref. [9]), an important feature of the method is the existence of fixed point at which $\Gamma_k$ is independent of $k$, i.e. the system becomes scale invariant. The perturbation of the effective action around the fixed point gives universal critical exponents. Heuristically, the dynamics of the fixed point is related to the conformal property of the theory and presumably also to the dynamics in AdS spacetime.

In this work, we show that the fRG equation can be reformed in a form of the wave equation in a holographic theory by a certain change of variables. We demonstrate using the 3d scalar field theory that energy scaling at fixed points uniquely determines the form of $U(z)$, and illustrate how the conformal property of QFT is encoded in the AdS dynamics in this correspondence.

Holographic wave equations from functional renormalization group—The AdS metric $g_{MN}$ is defined by

$$ds^2 = g_{MN}(dx^M dx^N) = \frac{R^2}{z^2}(dz^2 + \eta_{\mu\nu} dx^\mu dx^\nu),$$

where $M, N = 0,\ldots,d$, while $\mu, \nu = 0,\ldots,d-1$. Here, $x^M$ is $d+1$ dimensional spacetime coordinate which is decomposed into $d$ dimensional spacetime coordinate $x_\mu$ and an extra dimensional space coordinate $z$. The $d$ dimensional flat metric is denoted by $\eta_{\mu\nu} = \text{diag}(-1,1,\ldots,1)$ and $R$ is the radius of the additional dimension. The wave equations can then read as:

$$[ar{z}^2\partial_z^2 - (d - 1 - 2 J)z \partial_z + z^2 M^2 - (\mu R)^2 - z^2 U(z)]\Phi = 0,$$

where $(\mu R)^2 = (\Delta - J)(\Delta - d + J)$ with conformal dimension $\Delta$ and spin $J$.

Through redefining the field as $\psi = z^{-(d-1-2 J)/2} \Phi$, it can be changed to the Schrödinger equation on the bulk:

$$\left(-\frac{d^2}{dz^2} - \frac{L^2}{4z^2} + U(z)\right)\psi(z) = M^2 \psi(z),$$

where $L$ is the Casimir representation of orbital angular momentum and $L^2 = (J - d/2)^2 + (\mu R)^2$, which has been shown in Ref. [4] for the equivalence between Eq. (2) and the Schrödinger equation (3). If $U(z) \sim z^2$ for a large $z$, its solution reproduces $m_n^2 \sim n$ as eigenvalues [7].

We show from now that Eq. (2) can be derived from the QFT language and $U(z)$ is related to the beta function. To this end, we start with introducing the fRG equation...
(or simply the flow equation). One of its forms is given by [10]:

\[
k \partial_k \Gamma_k = \frac{1}{2} \text{Tr} \left[ k \partial_k R_k(p) \cdot G_k(p) \right] \equiv \beta \Gamma ,
\]

where \( k \) is the energy scale, \( \partial_k = \partial / \partial k \) and Tr is the functional trace acting on all internal spaces in which field variables are defined. Here \( R_k(p) \) is the regulator function to realize the coarse-graining procedure, and \( G_k(p) \) is the regulated full propagator whose explicit form is given by \( G_k(p) = (\Gamma_k^{(2)}(p) + R_k(p))^{-1} \) with the full two-point function, i.e. the second-order functional derivative with respect to field variables, \( \Gamma_k^{(2)}(p) = \delta^2 \Gamma_k / \delta \phi(p) \delta \phi(-p) \).

The flow equation for the full n-point function \( \Gamma_k^{(n)} \) can be obtained by taking the n-th order functional derivative for both sides of Eq. (4) with respect to field variables. The regulator function satisfies \( R_k(p) > 0 \) for \( p^2 / k^2 \to 0 \) and behaves as \( R_k(p) \to 0 \) for \( k^2 / p^2 \to 0 \) and as \( R_k(p) \to \infty \) for \( k^2 \to \infty \). There is an infinite number of possible forms for such a function. A specific choice of its form corresponds to the renormalization scheme.

An important fact is that the functional renormalization group equation is derived without any approximations. In other words, solving the flow equation without making approximations means exactly performing the path integral and obtaining full information about the QFT dynamics.

Now we show that the flow equation can be written in the form of the bulk wave equation. A similar attempt to build relations between these two conceptions has been made in Refs. [11–14]. In this work, we focus on the flow equation for the two-point function \( \Gamma_k^{(2)} \) and gives the exact correspondence. We start with writing the scale derivative as \( \partial_k \) instead of \( k \partial_k \) in Eq. (4) and then taking a derivative with respect to \( k^2 \) on the flow equation for \( \Gamma_k^{(2)} \) and rewrite the flow equation as

\[
\partial_k^2 \Gamma_k^{(2)} = \partial_k \left( \frac{1}{2k^2} \frac{\delta^2}{\delta \phi(p) \delta \phi(q)} \beta \right) ,
\]

where \( 1/(2k^2) \) on the right-hand side arises from \( \partial_k \) of \( k \partial_k \) in Eq. (4). Then, we perform the following change of variables:

\[
k = 1/z , \quad \Gamma_k^{(2)} = [z^{d-2-\eta} \Phi_\eta(z)]^{-1} ,
\]

where \( \eta \) is a constant. For the left-hand side of the flow equation (5), one finds

\[
(\text{LHS of Eq. (5)}) = z^{d-2-\eta} \left( \frac{\Gamma_k^{(2)}}{4} \right)^2 \left[ z^2 \partial_z^2 - (2d - 7 - 2\eta)z \partial_z - 2z^2 \left( \frac{\partial \log \Phi_\eta}{\partial z} \right)^2 - (d - \eta - 4)(d - \eta - 2) \right] \Phi_\eta(z) .
\]

From the right-hand side of the flow equation (5), one can recognize the potential \( U(z) \) as

\[
U(z) = -z^{-3} \frac{\Gamma_k^{(2)}}{\Gamma_k^{(2)}_1} \partial_z \left( \frac{z^2 \frac{\delta^2 \beta_k}{\delta \phi(p) \delta \phi(q)}}{\delta \phi(p) \delta \phi(q)} \right) ,
\]

where \( \Gamma_k^{(2)} = \Gamma_k^{(2)}_1 = \Gamma_k^{(2)} \). Together with the identification of variables as

\[
\eta = J + d/2 - 3 , \quad M^2 = -2 \left( \frac{\partial \log \Phi_\eta}{\partial z} \right)^2 ,
\]

we can see that the flow equation (5) is equivalent to the bulk wave equations (2). For a purpose in the later discussion we note here that the last term in Eq. (8) with the redefinition (9) is

\[
(\mu R)^2 = (d - \eta - 4)(d - \eta - 2) = \frac{1}{4}(d - 2J)^2 - 1 .
\]

This quantity corresponds to the angular momentum contribution, and eventually leads to \( L^2 = 2(J - d/2)^2 - 1 \) in Eq. (3).

We see from the derivation of the wave equation presented above that the dynamics in AdS spacetime can be fully captured by the fRG equation, with all information about the QFT dynamics in \( d \) dimensional spacetime now being included in \( U(z) \). The interesting question now is what the potential \( U(z) \) looks like from the functional renormalization group analysis, i.e. directly evaluating the right-hand side of Eq. (8). First of all, it becomes clear now how the conformal property of QFT is encoded in the AdS dynamics. The key point is that only around the fixed point, the potential \( U(z) \) is fully determined, otherwise the potential \( U(z) \) depends on the artificial regularization term \( R_k(p) \) which then leads the mass term \( M^2 \) to be arbitrary. Note that Eq. (5) is still satisfied in the latter case, but is only fixed at \( z \to \infty \) typically, with setting \( R_k(z=k\to0(p)) = 0 \). In a word, the fixed point leads to a unique correspondence from QFT to AdS dynamics.

Before discussing the evaluation of \( U(z) \) within a specific model, we briefly review the notion of fixed points and scaling (critical) exponents in the functional renormalization group. Fixed points characterize the scale invariance of a theory, i.e. certain points so as to be \( \beta_k = 0 \).
Let us here denote $\Gamma$, a scale invariant action. For such a fixed point, consider a small perturbation $\delta \Gamma_k = \Gamma_k - \Gamma$, around $\Gamma^*$. In general, the effective action is expanded by an infinite number of effective operators such that

$$
\Gamma_k = \int d^3x \sum_i \frac{g_k^i}{k^{d_i-s}} \mathcal{O}_i[\phi],
$$

where $g_k^i$ are dimensionless couplings depending on $k$ and $d_i$ are dimensions of operators $\mathcal{O}_i$. The flow equation (4) is reduced to a coupled differential equation for $g_k^i$, i.e. $k\partial_k g_k^i = \beta_i\{g_k\}$ where $\{g_k\}$ stands for a set of couplings. Within such an operator expansion scheme, a fixed action $\Gamma_k$ infers vanishing beta functions $\beta_i = 0$ for all $i$ which gives a set of fixed couplings $\{g_k\}$ and the small perturbation of the effective action $\delta \Gamma_k$ is given in terms of couplings:

$$
g^i_k = g^i_k + \sum_i C^i_j \left( \frac{k_0}{k} \right)^{\theta_j},
$$

where $k_0$ is a reference scale and $C^i_j$ is a matrix representing mixing effects between different couplings $i \neq j$. When the mixing effects are negligible, one has $C^i_j \approx c_i \delta^i_j$ with $c_i$ small parameters. Here, $\theta_j$ are associated critical exponents characterizing the energy scalings of operators around the fixed point. Couplings with a positive scaling exponent are amplified for $k \to 0$ and then are called “relevant”, while “irrelevant” couplings have a negative $\theta_i$ and thus decrease for the IR limit. In particular, at the trivial (or Gaussian) fixed point $g^i_k = 0$, the critical exponents correspond to the canonical mass dimension of coupling constants, i.e. $\theta_i = d - d_i$.

Now with all these knowledge, one can see that the analysis of the potential $U(z)$ can be carried on by analyzing the fixed point of FRG equations.

**Computation of $U(z)$ in 3d scalar theory**— We here demonstrate the determination of $U(z)$ from the fixed point structure. To this end, we employ a scalar theory in 3 dimensional spacetime. Hence, we consider an AdS$_3$/CFT$_3$ correspondence in a scalar field theory.

The scaling property of the 3d scalar field theory has been well-studied by various methods and it is known that there are two fixed points: One is the trivial fixed point which corresponds to a UV fixed point and characterizes asymptotic freedom of the theory. Another is the so-called Wilson-Fisher fixed point [15] being an IR fixed point at which there is only a relevant coupling. The flow equation interpolates these fixed points as the renormalization group trajectory. Below, we give $z$-dependencies of $U(z)$ from the scaling behaviors around the fixed points.

At trivial fixed point: For a theory with the asymptotically free property, the trivial fixed point is located at the UV scale (corresponding to $z = 0$) while for the others are at $z = \infty$. Around the trivial fixed point, the system can be analyzed perturbatively. Let us here assume the polynomial expansion (derivative expansion) of $\Gamma_k$ into $\phi^2$, namely

$$
\Gamma_k = \int d^3x \left[ \frac{Z_\phi}{2} (\partial_\mu \phi)^2 + \frac{m^2}{2} \phi^2 + \frac{\lambda}{4} \phi^4 + \cdots \right],
$$

where $Z_\phi$ is the field renormalization factor; $m^2$ is the dimensionful mass parameter; $\lambda$ is the quartic coupling; and $\cdots$ include higher dimensional operators such as $\phi^6$.

The two-point Green function for Eq. (13) is calculated as

$$
\Gamma^{(2)}(p, q) = (Z_\phi p^2 + m^2) \delta^3(p - q) + \cdots
$$

in momentum space. We are interested in properties of static scalar particle rather than their dynamics, so we could neglect vertex corrections to $\phi^4$ etc. This approximation allows us to focus on the flow equation for the mass parameter:

$$
k \partial_k m^2 = (2 - \eta_\phi - A) m^2,
$$

where $m^2 = m^2 / (Z_\phi k^2)$ is the dimensionless mass parameter and $\eta_\phi = -k \partial_k Z_\phi / Z_\phi$ is the anomalous dimension arising from the field renormalization. We denote loop effects by $\mathcal{A} = -\frac{1}{m^2} \frac{1}{\Omega_3} \int \delta^3(p) \phi(p) \phi(p) \delta^3(p) \phi(p)$ with $\Omega_3 = \int d^3x$ 3 dimensional spacetime volume. Here the first term on the right-hand side in Eq. (14) corresponds to the canonical scaling of $m^2$ with the anomalous dimension from the field renormalization. Those quantities ($\eta_\phi$ and $\mathcal{A}$) depend on couplings such as $\lambda$. Neglecting vertex corrections mentioned above is equivalent to $\eta_\phi$ and $\mathcal{A}$ to be constant under RG evolution and thus the solution to Eq. (14) reads

$$
m^2(k) = m^2_0 + \eta_\phi (k_0 / k)^{\theta_m} \sim m^2_0 + c_m z^{\theta_m},
$$

where $m^2_0$ is a constant given at $k_0$. For the trivial fixed point $m^2_0 = \lambda = \cdots = 0$, one has $\theta_m = 2$ for which the potential $U(z)$ around the trivial fixed point is found to be

$$
U(z) = 4z^2 \frac{\partial^2_z (\tilde{m}^2)}{\tilde{m}^2} = 0,
$$

where we have used the trivial fixed point $\tilde{m}^2_0 = 0$. A perturbation around the trivial fixed point gives a small deviation from $\theta_m = 2$ to $\theta_m = 2 - a_m$ with $a_m = \eta_\phi + \mathcal{A} < 1$. In such a case, one obtains $U(z) = -2a_m / z^2$ if we neglect the $z$-dependence of $\eta_\phi$ and $\mathcal{A}$.

At nontrivial fixed point: For the 3d scalar theory, there exists the Wilson-Fisher fixed point which gives a nontrivial energy scaling behavior. The elaborated studies on the determination of the scaling exponent [16, 17] have shown that $\theta_m \sim 1.59$. We find

$$
U(z) = 4z^2 \frac{\partial^2_z (\tilde{m}^2 + c_m z^{\theta_m})}{\tilde{m}^2 + c_m z^{\theta_m}} \sim z^{-2 + \theta_m},
$$

where we have assumed that $c_m$ is a small perturbative parameter around the nontrivial fixed point and have taken the lowest order of the expansion in terms of $c_m$. The Schrödinger equation with the potential $\sim 1 / z^{2 - \theta_m}$ could entail some unusual bound states. The numerical
computations actually have suggested some bound states exist in 3d $\phi^4$ theory [18–20], which coincides with what we found here. The additional bound states can survive between $d = 2$ to $d = 4$ as the interval of the dimension for occurring the nontrivial fixed point, which also verifies the numerical computation [19].

To demonstrate our approach, we show a few eigenvalues by solving the Schrödinger equation (3) with the potential (17) for $J = 0$ which yields $L^2 = 7/2$ in $d = 3$. To do this, we need to fix two parameters. One of them is the coefficient, denoted here by $\gamma$, in the potential, i.e. $U(z) = \gamma z^{2 - \theta_m}$. Although we cannot determine the exact value of $\gamma$, we here would infer from the typical order of the fixed point value $m^2_{\nu} \sim O(1)$ and a small perturbation parameter $c_m$ that $\gamma$ is of order of 0.1. The potential (17) is valid around the fixed point. Therefore, there must be a cutoff scale $z_{\text{cut}} = 1/k_{\text{cut}}$ where the mass parameter starts to deviate from the scaling regime with $\theta_m = 1.59$ under the RG evolution. In this study, we set $\gamma = -0.15$ and similarly to the hard wall model [7, 21], we set $z_{\text{cut}} = 10$ and solve the Schrödinger equation with the boundary condition $\psi(z_{\text{cut}}) = 0$.

We compare energy eigenvalues obtained between in Ref. [18] and in our work. In order to avoid mismatches of the energy scale, the ratios between excited states and the ground state are shown in Fig. 1. It seems that our approach captures the property of bound states in 3d scalar field theory despite the simple and crude calculation. We also give our prediction for the higher excited states ($n = 3, 4$). These mass eigenvalues should be tested by other methods. More accurate fRG computations could provide precise values for $\gamma$ and $z_{\text{cut}}$ without any ambiguities. We would leave such an analysis for future works.

**Discussion on gauge theory**— We have shown that the holographic wave function can be derived from the functional renormalization group equation which includes information about the QFT dynamics. The derivation can be naturally generalized to gauge theory.

Firstly, the trivial fixed point can be related to the anomalous dimensions and analyzed perturbatively. Generally, for a gauge theory, one may have $U(z) = -2a_m/z^2$ with $a_m$ the anomalous dimension (cf. see below Eq. (16)), and again coincides with the angular momentum part (10) in the wave equation is corrected so as to be:

$$-\frac{1 - 4L^2}{4} \rightarrow -\frac{1 - 4L^2}{4} - 2a_m. \quad (18)$$

This in general shows how the anomalous dimensions are introduced into the AdS dynamics.

Now, it is especially interesting to study the nontrivial fixed point for gauge theory. In particular, it is expected that for QCD, there is presumably an infrared fixed point [22–26]. This fixed point and strong coupling nature of QCD lead to a large conformal window for studying hadrons, which is the standing point of the holographic QCD [27, 28]. However, due to the complexity of the gauge theory, here we will not do the computation directly but instead, we just draw some conclusions from the correspondence.

Based on the derivation in this work, we can confirm the infrared fixed point exists in order to reproduce the potential $U(z) \sim z^2$ at $z \rightarrow \infty$, and consequently, the Regge trajectory. In another word, the Regge trajectory reveals that in QCD there must exist an infrared fixed point at which an associate critical exponent reproduces $U(z) \sim z^2$ via Eq. (8). More interestingly, the $z^2$ dependence of the potential in turn leads:

$$\frac{\delta^2 \Gamma_k}{\delta J_\mu \delta J_\nu} \sim \frac{1}{k^4},$$

with $J_\mu \sim \bar{\psi}\gamma_\mu \psi$ the vector meson field or in QCD, the vector current of quarks, respectively. The relation implies a linear confining potential between the vector current of quarks, which is associated with the Regge trajectory.

**Summary**— In this work, we show that the holographic wave equation can be obtained from the quantum field theory via the renormalization group method. One may start from the flow equation and then obtain the potential $U(z)$, the metric and the dilaton background introduced in the AdS space are then fixed. In this sense, the flow equation describes the dynamics in different AdS spaces by equipping with different fixed points which characterize conformalities of the system. This picture seems natural. The additional dimension in the AdS spacetime is naturally interpreted as the renormalization scale, and the conformality is encoded through the fixed point of the renormalization group flow, to guarantee the uniqueness of the correspondence.

Practically, we show that the anomalous dimensions can be introduced in the AdS dynamics through the trivial fixed point in perturbative region. After that, we
verify that the Wilson-Fisher fixed point brings in some bound states in 3d scalar theory as discovered from some numerical studies. Due to the complexity of QCD dynamics, it is hard to directly compute the potential $U(z)$ from QCD. Nevertheless, what we discovered here is that the Regge trajectory naturally entails a nontrivial fixed point to occur in the infrared of QCD. In turn, the critical exponent at this fixed point uniquely determines the correspondence of AdS spacetime and greatly simplifies the low-energy QCD dynamics.

**Acknowledgements** We thank J. M. Pawlowski and A. Pastor-Gutiérrez for valuable discussions. F. G. also thanks all the other members in fQCD collaboration [29]. We are supported by the Alexander von Humboldt Foundation. The work of M. Y. is also supported by the DFG Collaborative Research Centre “SFB 1225 (ISO-QUANT)” and Germany’s Excellence Strategy EXC-2181/1-390900948 (the Heidelberg Excellence Cluster STRUCTURES).

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