A class of non-matchable distributive lattices

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Abstract

The set of all perfect matchings of a plane (weakly) elementary bipartite graph equipped with a partial order is a poset, moreover the poset is a finite distributive lattice and its Hasse diagram is isomorphic to $Z$-transformation directed graph of the graph. A finite distributive lattice is matchable if its Hasse diagram is isomorphic to a $Z$-transformation directed graph of a plane weakly elementary bipartite graph, otherwise non-matchable. We introduce the meet-irreducible cell with respect to a perfect matching of a plane (weakly) elementary bipartite graph and give its equivalent characterizations. Using these, we extend a result on non-matchable distributive lattices, and obtain a class of new non-matchable distributive lattices.

Key words: plane (weakly) elementary bigraph, $Z$-transformation digraph, meet-irreducible cell, non-matchable distributive lattice, planarity

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1 Introduction

Zhang et al. [9] introduced a concept of $Z$-transformation graph (called by some authors resonance graph) on the set of perfect matchings (or 1-factors) of hexagonal system; in addition, Zhang and Zhang [17] extended the concept to a general plane bipartite graph with a perfect matching and obtained some results on a plane (weakly) elementary bipartite graph. Let $G$ be a graph with a perfect matching, denote by $\mathcal{M}(G)$ the set of all perfect matchings of $G$. The $Z$-transformation directed graph $\vec{Z}(G)$ is an orientation of $Z$-transformation graph by orientating all the edges [16]. Lam and Zhang [4] proved that $\mathcal{M}(G)$ equipped with a partial order is a finite distributive lattice and its Hasse diagram is isomorphic to $\vec{Z}(G)$. There are some results on finite distributive lattices and $Z$-transformation directed graphs [14] [10] [11]. Recently, Zhang et al. [12] introduced the concept of matchable distributive lattice and got some consequences on matchable distributive lattices, Yao and Zhang [8] obtained some results on non-matchable distributive lattices.

In the paper we first obtain Proposition 3.1 from the Proof of Lemma 3.7 in [13]. In a finite lattice, an element is meet-irreducible if and only if it is covered by exactly one element. From a graphical point of view, if and only if there is exactly one arc (directed edge) to the vertex (element) in $\vec{Z}(G)$. Consider the arc $f$ with its tail $M$, since $M$ and $f$ are perfect matching of $G$ and proper $M$-alternating cell, respectively, thus we call the cell meet-irreducible cell with respect to $M$. Furthermore, we have Theorem 3.2 that is analogous to a lemma in [6]. However, our method is completely different from their proof. Finally, by Theorem 3.2 we extend Theorem 4.8 in [8], and obtain a class of non-matchable distributive lattices by Kuratowski’s Theorem.

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2 Preliminaries

A set $P$ equipped with a partial order relation $\leq$ is said to be a partially ordered set (poset for short). Given any poset $P$, the dual $P^*$ of $P$ by defining $x \leq y$ to hold in $P^*$ if and only if $y \leq x$ holds in $P$. A poset $P$ is a chain if any two elements of $P$ are comparable, and we write $n$ to denote the chain obtained by giving $\{0,1,\ldots,n-1\}$ the order in which $0 < 1 < \cdots < n-1$. The set of all filters of a poset $P$ is denoted by $\mathcal{F}(P)$, and carries the usual anti-inclusion order; and the filter lattice $\mathcal{F}(P)$ is a distributive lattice. A lattice is nontrivial if it has at least two elements and a finite distributive lattice is irreducible if it cannot be decomposed into a direct product of two nontrivial finite distributive lattices.

The symmetric difference of two finite sets $A$ and $B$ is defined as $A \oplus B := (A \cup B) \setminus (A \cap B)$. If $M$ is a perfect matching of a graph and $C$ is an $M$-alternating cycle of the graph, then the symmetric difference of $M$ and edge-set $E(C)$ is another perfect matching of the graph, which is simply denoted by $M \oplus C$. Let $G$ be a plane bipartite graph with a perfect matching, and the vertices of $G$ are colored properly black and white such that the two ends of every edge receive different colors. An $M$-alternating cycle of $G$ is said to be proper, if every edge of the cycle belonging to $M$ goes from white end-vertex to black end-vertex by the clockwise orientation of the cycle; otherwise improper [15]. An inner face of a graph is called a cell if its boundary is a cycle, and we will say that the cycle is a cell too.

For some concepts and notations not explained in the paper, refer to [2,7] for poset and lattice, [11,3] for graph theory.

Obverse that the $M$-alternating cycle intersecting an improper $M$-alternating cycle must be proper, vice versa. Obviously, we have the following result.

**Lemma 2.1** ([14]) If $G$ be a plane bipartite graph with a matching $M$, then any two proper (resp. improper) $M$-alternating cells are disjoint.

**Definition 2.1** ([17]) Let $G$ be a plane bipartite graph. The $Z$-transformation graph $Z(G)$ is defined on $\mathcal{M}(G)$: $M_1, M_2 \in \mathcal{M}(G)$ are joined by an edge if and only if $M_1 \oplus M_2$ is a cell of $G$. And $Z$-transformation digraph $\bar{Z}(G)$ is the orientation of $Z(G)$: an edge $M_1M_2$ of $Z(G)$ is oriented from $M_1$ to $M_2$ if $M_1 \oplus M_2$ form a proper $M_1$-alternating (thus improper $M_2$-alternating) cell.

An edge of graph $G$ is allowed if it lies in a perfect matching of $G$. A graph $G$ is said to be elementary if its allowed edges form a connected subgraph of $G$, then $G$ is connected and every edge of $G$ is allowed. A subgraph $H$ of $G$ is said to be nice if $G - V(H)$ has a perfect matching [5]. Let $G$ be a bipartite graph, from Theorem 4.1.1 in [5], we have that $G$ is elementary if and only if $G$ is connected and every edge of $G$ is allowed.

**Definition 2.2** ([17]) A bipartite graph $G$ is weakly elementary if the subgraph of $G$ consisting of $C$ together with its interior is elementary for every nice cycle $C$ of $G$.

Let $G$ be a plane bipartite graph with a perfect matching, a binary relation $\leq$ on $\mathcal{M}(G)$ is defined as: for $M_1, M_2 \in \mathcal{M}(G)$, $M_1 \leq M_2$ if and only if $\bar{Z}(G)$ has a directed path from $M_2$ to $M_1$ [17], thus $(\mathcal{M}(G); \leq)$ is a poset [4]. For convenient, we write $\mathcal{M}(G)$ for poset $(\mathcal{M}(G), \leq)$.

**Theorem 2.2** ([4]) If $G$ is a plane (weakly) elementary bipartite graph, then $\mathcal{M}(G)$ is a finite distributive lattice and its Hasse diagram is isomorphic to $\bar{Z}(G)$.

**Definition 2.3** ([12]) A finite distributive lattice $L$ is matchable if there is a plane weakly elementary bipartite graph $G$ such that $L \cong \mathcal{M}(G)$; otherwise it is non-matchable.
3 Meet-irreducible cell

The Proof of Lemma 3.7 in [13] implies the following proposition.

Proposition 3.1 If $G$ is a plane elementary bipartite graph with a perfect matching $M$, then there exists a hypercube in $\bar{Z}(G)$ generated by some pairwise disjoint $M$-alternating cells. In particular, $M$ is the top (resp. bottom) of the hypercube in $\mathcal{M}(G)$ if these $M$-alternating cells are proper (resp. improper).

It is obvious that the dimension of the hypercube is equal to the number of these pairwise disjoint $M$-alternating cells. In particular, the hypercube is a quadrilateral if and only if it is generated by exactly two disjoint $M$-alternating cells in $G$ [13].

Definition 3.1 Let $G$ be a plane (weakly) elementary bipartite graph with a perfect matching $M$. A meet-irreducible cell $f$ with respect to $M$ is a proper $M$-alternating cell if and only if $M \oplus f$ is meet-irreducible in $\mathcal{M}(G)$.

Theorem 3.2 Let $G$ be a plane (weakly) elementary bipartite graph $G$ with perfect matching $M$ and let $f$ be a proper $M$-alternating cell.

1. If $G$ has no improper $M$-alternating cell (namely, $M$ is the top of $\mathcal{M}(G)$), then every (proper) $M$-alternating cell is a meet-irreducible cell with respect to $M$;
2. If $G$ has some improper $M$-alternating cells, then the following are equivalent:
   (a) the cell $f$ is a meet-irreducible cell with respect to $M$;
   (b) the cell $f$ intersects every improper $M$-alternating cell;
   (c) there is no perfect matching $M'$ in $V(Q) \setminus \{M\}$ such that $f$ is a proper $M'$-alternating cell, where $Q$ is a hypercube generated by all improper $M$-alternating cells.

Proof 1. It is trivial by the definition of $Z$-transformation directed graph.

2. Firstly suppose that the cell $f$ is a meet-irreducible cell with respect to $M$, but there is at least one improper $M$-alternating cell $f'$ such that $f$ and $f'$ are disjoint. Thus $M \oplus f = ((M \oplus f') \oplus f) \oplus f'$, i.e. $G$ has two improper $M \oplus f$-alternating cells, hence $M \oplus f$ is not meet-irreducible, contradicting the supposition that $f$ is a meet-irreducible cell with respect to $M$.

Next suppose that the cell $f$ intersects every improper $M$-alternating cell, but there is a perfect matching $M'$ in $V(Q) \setminus \{M\}$ such that $f$ is a proper $M'$-alternating cell. In fact, by Proposition 3.1, there is at least one improper $M$-alternating cell $f'$ is a proper $M'$-alternating cell. Hence $f$ and $f'$ are disjoint by Lemma 2.1, a contradiction.

Finally, suppose that there is no perfect matching $M'$ in $V(Q) \setminus \{M\}$ such that $f$ is a proper $M'$-alternating cell, but $f$ is not a meet-irreducible cell with respect to $M$. Thus $G$ has at least one improper $M \oplus f$-alternating cell $f'$ except $f$, by Lemma 2.1, hence $f$ and $f'$ are disjoint. Therefore $f'$ is an improper $M$-alternating cell, this means that $f$ is a proper $M \oplus f'$-alternating cell, i.e. there is a perfect matching $M' = M \oplus f'$ in $V(Q) \setminus \{M\}$ such that $f$ is a proper $M'$-alternating cell, a contradiction. \qed

If every proper $M$-alternating cell is a meet-irreducible cell with respect to $M$, then $M$ is a top of $\mathcal{M}(G)$ if $G$ has no improper $M$-alternating cell, otherwise cut vertex in $Z(G)$. Moreover we obtain the following corollary as a consequence of Theorem 3.2.
Corollary 3.3 ([16][14]) If $G$ is a plane elementary bipartite graph with a perfect matching $M$, then $M$ is a cut vertex of $Z(G)$ if and only if $G$ has both proper and improper $M$-alternating cells and every proper $M$-alternating cell is a meet-irreducible cell with respect to $M$; i.e. every proper $M$-alternating cell intersects every improper $M$-alternating cell.

Note that duality of lattice, meet-irreducible cell, Theorem 3.2 and Corollary 3.3 could be treated in dual.

4 Non-matchable distributive lattice

Subdivide an edge $e$ is to delete $e$, add a new vertex $v$, and join $v$ to the ends of $e$. Any graph derived from a graph $G$ by a sequence of edge subdivisions is called a subdivision of $G$.

Theorem 4.1 (Kuratowski’s Theorem) A graph is planar if and only if it contains no subdivision of either $K_5$ or $K_{3,3}$.

From the proof of Lemma 4.2 in [8] and Theorem 3.2, the following theorem is immediate.

Theorem 4.2 Let $L$ be a finite distributive lattice and $x \in L$. If $x$ is covered by at least three elements and covers at least three meet-irreducible elements, then $L$ is non-matchable.

![Figure 1: Two non-matchable distributive lattices](image)

For instance, it is easy to see that each distributive lattice in Figure 1 is non-matchable by Theorem 4.2, but it is difficult to determine only by Theorem 4.3 in [8].

Obviously, theorem 4.2 could be obtained in dual.

Corollary 4.3 If $L$ is a matchable distributive lattice, then for every element of $L$, it either is covered by at most two elements or covers at most two meet-irreducible elements in both $L$ and $L^*$.

Given a plane graph $G$, its (geometric) dual $G^*$ is constructed as follows: place a vertex in each face of $G$ (including the exterior face) and, if two faces have an edge $e$ in common, join the corresponding vertices by an edge $e^*$ crossing only $e$. It is easy to see that the dual $G^*$ of a plane graph $G$ is itself a planar graph [1].

Theorem 4.4 The distributive lattice $\mathcal{F}(\Delta)$ is non-matchable, where $\Delta$ is a poset as shown in Figure 2(a).
Figure 2: (a) The poset Δ and (b) a part of \( \mathcal{F}(\Delta) \)

\[ \begin{array}{c}
(a) \\
(b)
\end{array} \]

Figure 3: Proof of Theorem 4.4

**Proof** Recall that \( \mathcal{F}(\Delta) \) is a finite distributive lattice. Suppose that \( \mathcal{F}(\Delta) \) is matchable, since \( \mathcal{F}(\Delta) \) is irreducible, then there exists a plane elementary bipartite graph \( G \) such that \( \bar{Z}(G) \cong \mathcal{F}(\Delta) \). Consider a part of \( \mathcal{F}(\Delta) \) as drawn in Figure 2(b), the vertices correspond to the perfect matchings \( M_\emptyset, M_0, M_1, \ldots, M_a \) of \( G \), respectively. Let \( f_0 = M_\emptyset \oplus M_0, f_1 = M_0 \oplus M_1, f_5 = M_12 \oplus M_5, f_6 = M_13 \oplus M_6, \ldots, \) and \( f_a = M_34 \oplus M_a \). By definition of \( Z \)-transformation graph, then \( f_0 \) is a nice cell, so are \( f_1, \ldots, f_a \). Since the cells \( f_0, f_1, \ldots, f_a \) are meet-irreducible cells, by Theorem 3.2(2), the cell \( f_0 \) intersects \( f_1, f_2, f_3 \) and \( f_4 \); the cell \( f_5 \) intersects \( f_1 \) and \( f_2 \), but it does not intersect \( f_3 \) or \( f_4 \), because \( f_5, f_3 \) and \( f_4 \) are proper \( M_12 \)-alternating cells. Thus \( f_0 \) and \( f_5 \) are distinct; analogously, \( f_0 \) and \( f_i \) \((i \in \{6, 7, 8, 9, a\})\) are distinct too.

Next, consider the dual \( G^* \) of \( G \), as drawn in Figure 3(a), vertex \( f_0^* \) is adjacent with \( f_1^*, f_2^*, f_3^* \) and \( f_4^* \), and \( f_5^* \) is adjacent with \( f_1^* \) and \( f_2^* \), etc. Therefore, let \( V' = \{f_0^*, \ldots, f_a^*\} \), thus \( G^* \) contains a subgraph \( S^* := G^*[V'] \). Clearly \( S^* \) (see Figure 3(b)) is a subdivision of \( K_5 \). By Kuratowski’s Theorem, hence \( S^* \) is non-planar, contradicting the planarity of \( G \). □

As a straightforward consequence of Theorem 4.4, we have the following result.

**Corollary 4.5** If a poset \( P \) contains \( \Delta \) as a convex sub-poset, then distributive lattice \( \mathcal{F}(P) \) is non-matchable.

Clearly, for any finite distributive lattice \( L \), the Cartesian product, linear sum and vertical sum of \( \mathcal{F}(P) \) and \( L \) are non-matchable. In particular, the following corollary is immediate.

**Corollary 4.6** The distributive lattice \( \mathcal{F}(2^4) \) is non-matchable. In addition, the distributive lattice \( \mathcal{F}\left(\prod_{j=1}^{k} \mathbf{n}_j\right) \) is non-matchable, where \( k \geq 4, \mathbf{n}_j \) is a chain of length \( n_j \) and \( n_j \geq 2 \) for every \( j = 1, 2, \ldots, k \).
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