Effects of Violation of Equivalence Principle on UHE Neutrinos at IceCube in 4 Flavour Scenario

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Abstract

If weak equivalence principle is violated then different types of neutrinos would couple differently with gravity and that may produce a gravity induced oscillation for the neutrinos of different flavour. We explore here the possibility that very small violation of the principle of weak equivalence (VEP) can be probed by ultra high energy neutrinos from distant astrophysical sources. The very long baseline length and the ultra high energies of such neutrinos could be helpful to probe very small VEP. We consider a 4-flavour neutrino scenario (3 active + 1 sterile) with both mass-flavour and gravity induced oscillations and compare the detection signatures for these neutrinos (muon tracks and shower events) with and without gravity induced oscillations at a kilometer scale detector such as IceCube. We find that the muon track to shower ratios vary considerably (by a factor of $\sim 3.6$) when compared the estimation without any gravity induced oscillation (no VEP case).
1 Introduction

The oscillation of neutrinos \[1\] from one flavour to another are now established by several terrestrial experiments with neutrinos of natural origin such as solar and atmospheric neutrinos and man-made neutrinos that include reactor \[2, 3, 4\] or accelerator neutrinos. Due to the mass-flavour mixing of the neutrino eigenstates, a flavour eigenstate after traversing a distance can oscillate into a different flavour due to the phase difference acquired by the mass eigenstates on its propagation. This phase depends on the mass squared difference and the baseline length. Thus discovery of the oscillation ensures the neutrinos are massive. The neutrinos should also therefore undergo gravitational interactions. In case the weak interaction eigenstates of neutrinos are not the same as those of their gravity eigenstates, neutrino oscillation can again be induced if different neutrinos interact with gravity with different strengths, i.e. the gravitational constant $G$ is different for different types of neutrinos. This situation may occur if the principle of weak equivalence is violated \[5, 6\].

General consequence of the weak equivalence principle is that there is no difference between the gravitational mass and the inertial mass. This is to say that the force experienced by an object grounded on earth is the same as the force experienced by the same object at the floor of a spaceship which is moving with an acceleration same as that of the acceleration due to gravity in a no gravity environment. This can lead to the phenomenon of gravitational redshift \[−\] under the influence of which the wavelength of a radiation suffers a widening (or the energy of a particle is shifted towards a lower energy) while traversing through a gravitational field. The neutrinos too from a distant astrophysical object such as Gamma Ray Bursts would experience such a gravitational redshift on travelling to the earth. The shifted energy is given by $E' = (1 - \phi)E$ \[6, 7\], where $\phi(= GM/R)$ is the gravitational potential \[8\] through which the neutrino is propagating. If the equivalence principle is not violated then the energy shifts are equal and this will not induce any phase difference between two types of neutrinos during its propagation. But if the equivalence is violated then the energy shifts will be different for different types of neutrinos (since the gravitational coupling $G_i(= G\alpha_i$, say) of the neutrino species $i$ is different from $G_j(= G\alpha_j)$, the coupling for species $j$. As a result, a pair of neutrino species $(i, j)$ will acquire a phase $\sim \Delta EL$ ($\Delta E = |E_i - E_j|$, $E_i$ and $E_j$ being the redshifted energies of the species $i$ and $j$ respectively) while traversing a distance $L$ (baseline length) from a distant GRB, say, to earth. Note that $E_i, E_j$ are the energy eigenstates in gravity basis. This would lead to a gravity induced oscillation between neutrinos of different flavour with the oscillatory part given by $\sim |\Delta EL| = \ldots$
\[ |\Delta f_{ij}| LE \ (|\Delta f_{ij}| = |f_i - f_j|) \text{ with } f_i = (GM/R)|\Delta \alpha_i| = \phi \alpha_i. \]

In general there are no specific signatures of violation of weak equivalence principle (VEP) in nature. But in case this is very weakly violated (\(|\Delta f_{ij}| \text{ very small}\)) then depending on length of the baseline, neutrinos may probe such small VEP. For distant ultra high energy (UHE) neutrino sources such as GRBs, since the baseline length can be of the order of tens or hundreds of \(\sim\) Mpc or more, gravity induced neutrino oscillations can be effective for very small violation of equivalence principle (such that \(|\Delta f_{ij}| LE \text{ is not very small or very very large}\)). The mass induced oscillations however, for such a long baseline of astronomical length, will be averaged to a reduced value producing an overall suppression of flux for a particular flavour.

In this work, we consider the UHE neutrinos from a GRB and estimate its flux on reaching the earth if they suffer both mass induced oscillations/suppressions and gravity induced oscillation. We then estimate the number of muon track events as well as shower events for these neutrinos at a kilometer square detector such as IceCube [9] and compare our results with similar estimation when no oscillations are considered. For our estimation we consider a four flavour scenario where an extra sterile [10] neutrino is added with the three flavour families.

The paper is organised as follows. In Section 2 we give the formalism of gravity and mass induced oscillations in the 4 flavour and three flavour scenario. In Section 3 we furnish calculational details and results. A discussion and summary is given in Sect. 4.

2 Formalism

2.1 Astrophysical neutrino fluxes from diffuse GRBs

Neutrinos and antineutrinos are produced from the diffuse GRB sources with the flavour proportion

\[ \nu_e : \nu_\mu : \nu_\tau : \nu_s = 1 : 2 : 0 : 0. \]

The possibility of detecting UHE neutrinos associated with the gamma ray bursts has been claimed by Waxman-Bahcall [11] [12]. By summing over all the sources we can estimate the diffused isotropic flux for both \(\nu_\mu\) and \(\bar{\nu}_\mu\) as [13]

\[ F(E_\nu) = \frac{dN_{\nu_\mu+\bar{\nu}_\mu}}{dE_\nu} = \mathcal{N} \left( \frac{E_\nu}{1\text{GeV}} \right)^{-n} \text{cm}^{-2}\text{s}^{-1}\text{sr}^{-1}\text{GeV}^{-1}, \quad (1) \]

with

\[ \mathcal{N} = 4.0 \times 10^{-13} \quad n = 1 \text{ for } E_\nu < 10^5 \text{ GeV}, \]
\(N = 4.0 \times 10^{-8}\) \(n = 2\) for \(E_\nu > 10^5\) GeV.

In the absence of the CP violation the fluxes for both neutrinos and antineutrinos are same \((\phi_{\nu_\mu} = \phi_{\bar{\nu}_\mu})\). So at the source the fluxes for the corresponding flavours can be written as

\[
\phi_{\nu_e}^s = \frac{1}{4} F(E_\nu), \phi_{\nu_\mu}^s = \frac{1}{2} F(E_\nu) = 2\phi_{\nu_e}^s, \phi_{\nu_\tau}^s = 0, \phi_{\nu_s}^s = 0.
\]  

(2)

In this 4-flavour framework, the neutrinos experience flavour oscillations upon reaching the terrestrial detector from the astronomical extragalactic sources. The flux of neutrino flavours on reaching the Earth can be expressed as

\[
\begin{align*}
F_{\nu_e} &= P_{ee}\phi_{\nu_e}^s + P_{\mu e}\phi_{\nu_\mu}^s, \\
F_{\nu_\mu} &= P_{\mu\mu}\phi_{\nu_\mu}^s + P_{e\mu}\phi_{\nu_e}^s, \\
F_{\nu_\tau} &= P_{e\tau}\phi_{\nu_e}^s + P_{\mu\tau}\phi_{\nu_\mu}^s, \\
F_{\nu_s} &= P_{e s}\phi_{\nu_e}^s + P_{\mu s}\phi_{\nu_\mu}^s,
\end{align*}
\]

(3)

where \(P_{\alpha\beta}(\alpha, \beta = e, \mu, \tau, s)\) is the oscillation probability and \(F_{\nu_\alpha}\) is the flux for the neutrinos \(\nu_\alpha(\alpha = e, \mu, \tau, s)\) on reaching the Earth for the four flavour case.

### 2.2 Contribution of VEP to the neutrino oscillation probability in 4-flavour framework

In the case of a nonvanishing rest mass of the neutrino the weak and mass eigenstates are not necessarily identical, a fact wellknown in the quark sector where both types of state are connected by the Cabibbo-Kobayashi-Maskawa (CKM) matrix. This non zero mass nature of the neutrino allows for the phenomenon of neutrino oscillations, first given by Pontecorvo \[14, 15\], and it can be described by pure quantum mechanics. They are observable as long as the neutrino wave packets from a coherent superposition of states.

Such oscillations among the different neutrino flavours don’t conserve individual lepton numbers only total lepton number. So that neutrino oscillation can be expressed as a quantum mechanical phenomenon whereby a neutrino created with a specific lepton family number (“lepton flavour”) can later be measured to have a different lepton family number.

The \(n\) flavour eigenstate \(|\nu_\alpha\rangle\) (with \(\langle \nu_\beta | \nu_\alpha \rangle = \delta_{\alpha\beta}\)), where \(n\) is an arbitrary number of orthonormal eigenstates, are connected to the \(n\)th mass eigenstate (with \(\langle \nu_i | \nu_j \rangle = \delta_{ij}\)) via a unitary matrix \(U\)

\[
|\nu_\alpha\rangle = \sum_i U_{\alpha i} |\nu_i\rangle,
\]

(4)
with

$$U^+ U = 1, \sum \alpha U_{\alpha i}^* U_{\beta i} = \delta_{\alpha \beta}, \sum \alpha U_{\alpha i}^* U_{\alpha j} = \delta_{ij}. \quad (5)$$

For the 4 (3 active +1 sterile) flavour scenario, the relation between the neutrino flavour eigenstates and mass eigenstates through a unitary matrix which can be parameterized as

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \\ \nu_s \end{pmatrix} = \begin{pmatrix} \tilde{U}_{e1} & \tilde{U}_{e2} & \tilde{U}_{e3} & \tilde{U}_{e4} \\ \tilde{U}_{\mu1} & \tilde{U}_{\mu2} & \tilde{U}_{\mu3} & \tilde{U}_{\mu4} \\ \tilde{U}_{\tau1} & \tilde{U}_{\tau2} & \tilde{U}_{\tau3} & \tilde{U}_{\tau4} \\ \tilde{U}_{s1} & \tilde{U}_{s2} & \tilde{U}_{s3} & \tilde{U}_{s4} \end{pmatrix} \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \\ \nu_4 \end{pmatrix}, \quad (6)$$

where $\tilde{U}_{ij}$ etc. indicate the elements of the Pontecorvo-Maki-NAkigawa-Sakata (PMNS) matrix \[16\]. In 4-flavour framework, where an extra sterile neutrino ($\nu_s$) is considered in addition to the usual three active neutrino families ($\nu_e, \nu_\mu, \nu_\tau$), the PMNS matrix $\tilde{U}_{(4\times4)}$ can be generated by considering the successive rotations ($R$) in terms of mixing angles $\theta_{14}, \theta_{24}, \theta_{34}, \theta_{13}, \theta_{12}, \theta_{23}$ \[17\]

$$\tilde{U}_{(4\times4)} = R_{34}(\theta_{34}) R_{24}(\theta_{24}) R_{14}(\theta_{14}) R_{23}(\theta_{23}) R_{13}(\theta_{13}) R_{12}(\theta_{12}), \quad (7)$$

There is no CP violation in the neutrino sector and so that the CP phases are absent. The successive rotation terms ($R$) in 4-flavour case can be written as

$$R_{34}(\theta_{34}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c_{34} & s_{34} \\ 0 & 0 & -s_{34} & c_{34} \end{pmatrix}, \quad R_{24}(\theta_{24}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{24} & 0 & s_{24} \\ 0 & 0 & 1 & 0 \\ 0 & -s_{24} & 0 & c_{24} \end{pmatrix}, \quad (8)$$

$$R_{14}(\theta_{14}) = \begin{pmatrix} c_{14} & 0 & 0 & s_{14} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -s_{14} & 0 & 0 & c_{14} \end{pmatrix}, \quad R_{12}(\theta_{12}) = \begin{pmatrix} c_{12} & s_{12} & 0 & 0 \\ -s_{12} & c_{12} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$R_{13}(\theta_{13}) = \begin{pmatrix} c_{13} & 0 & s_{13} & 0 \\ 0 & 1 & 0 & 0 \\ -s_{13} & 0 & c_{13} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad R_{23}(\theta_{23}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & c_{23} & s_{23} & 0 \\ 0 & -s_{23} & c_{23} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$
Now $\tilde{U}_{(4 \times 4)}$ can be expressed as

$$
\tilde{U}_{(4 \times 4)} = \begin{pmatrix}
  c_{14} & 0 & 0 & s_{14} \\
  -s_{14}s_{24} & c_{24} & 0 & c_{14}s_{24} \\
  -c_{24}s_{14}s_{34} & -s_{24}s_{34} & c_{14}c_{24}s_{34} \\
  -c_{24}s_{14}c_{34} & -s_{24}c_{34} & -s_{34} & c_{14}c_{24}c_{34}
\end{pmatrix}
\begin{pmatrix}
  U_{e1} & U_{e2} & U_{e3} & 0 \\
  U_{\mu1} & U_{\mu2} & U_{\mu3} & 0 \\
  U_{\tau_1} & U_{\tau_2} & U_{\tau_3} & 0 \\
  0 & 0 & 0 & 1
\end{pmatrix}
$$

(9)

where $U_{ai}$ are the elements of 3 flavour mixing matrix $U$, which can be expressed as

$$
U = \begin{pmatrix}
  c_{12}c_{13} & s_{12}s_{13} & s_{13} \\
  -s_{12}c_{23} - c_{12}s_{23}s_{13} & c_{12}c_{23} - s_{12}s_{23}s_{13} & s_{23}c_{13} \\
  s_{12}s_{23} - c_{12}c_{23}s_{13} & -s_{12}s_{23} - c_{12}c_{23}s_{13} & c_{23}c_{13}
\end{pmatrix}
\begin{pmatrix}
  s_{14} \\
  -c_{24}s_{14}s_{34}U_{e1} & -c_{24}s_{14}s_{34}U_{e2} & -c_{24}s_{14}s_{34}U_{e3} \\
  -s_{24}s_{34}U_{\mu1} & -s_{24}s_{34}U_{\mu2} & -s_{24}s_{34}U_{\mu3} \\
  +c_{34}U_{\tau_1} & +c_{34}U_{\tau_2} & +c_{34}U_{\tau_3}
\end{pmatrix}
$$

(10)

In the above Eqs. (9-11), $c_{ij}$ and $s_{ij}$ are referred as $\cos \theta_{ij}$ and $\sin \theta_{ij}$, where $\theta_{ij}$ is the mixing angle between $i$th and $j$th neutrinos having mass eigenstates $|\nu_i \rangle$ and $|\nu_j \rangle$.

The time evolution equation of neutrinos in mass basis is given by

$$
\frac{d}{dt}\begin{pmatrix}
  \nu_1 \\
  \nu_2 \\
  \nu_3 \\
  \nu_4
\end{pmatrix} = \begin{pmatrix}
  E_1 & 0 & 0 & 0 \\
  0 & E_2 & 0 & 0 \\
  0 & 0 & E_3 & 0 \\
  0 & 0 & 0 & E_4
\end{pmatrix}\begin{pmatrix}
  \nu_1 \\
  \nu_2 \\
  \nu_3 \\
  \nu_4
\end{pmatrix},
$$

(12)

$$
= H_m \begin{pmatrix}
  \nu_1 \\
  \nu_1 \\
  \nu_3 \\
  \nu_4
\end{pmatrix},
$$

(13)
By considering Eq. (14) we can describe the evolution equation of neutrino oscillations in the case of flavour basis in 4-flavour scenario as

$$i \frac{d}{dt} \begin{pmatrix} \nu_e \\ \nu_{\mu} \\ \nu_\tau \\ \nu_s \end{pmatrix} = H' \begin{pmatrix} \nu_e \\ \nu_{\mu} \\ \nu_\tau \\ \nu_s \end{pmatrix}$$ (14)

where

$$H' = U H_m U^+ .$$ (15)

$H_m$ represents the vacuum Hamiltonian, i.e. the Hamiltonian in mass basis

$$H_m = \begin{pmatrix} E_1 & 0 & 0 & 0 \\ 0 & E_2 & 0 & 0 \\ 0 & 0 & E_3 & 0 \\ 0 & 0 & 0 & E_4 \end{pmatrix} .$$ (16)

For the case of relativistic neutrinos of the momentum $p$, the energy eigen value can be expressed as

$$E_i = \sqrt{p_i^2 + m_i^2} \simeq p_i + \frac{m_i^2}{2p_i} \simeq p + \frac{m_i^2}{2E} ,$$ (17)

where $p_i = p \simeq E, i = 1, 2, 3, 4$. Now by using Eq. (17) we can write $H_m$ as

$$H_m = \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} + \frac{1}{2E} \begin{pmatrix} m_1^2 & 0 & 0 & 0 \\ 0 & m_2^2 & 0 & 0 \\ 0 & 0 & m_3^2 & 0 \\ 0 & 0 & 0 & m_4^2 \end{pmatrix} .$$ (18)

In the above Eq. (18), we can neglect the matrix diag($p, p, p, p$) as it does not create any phase differences between the neutrinos and hence does not contribute to neutrino oscillations. In addition to this, we subtract the term $m_i^2$ from all the diagonal elements of the matrix diag($m_1^2, m_2^2, m_3^2, m_4^2$). Eq. (18) now takes the form

$$H_m = \frac{1}{2E} \text{diag}(0, \Delta m_{21}^2, \Delta m_{31}^2, \Delta m_{41}^2) .$$ (19)

The oscillation of neutrinos can also be induced in case the equivalence principle is violated in nature. In such a scenario, the gravitational coupling to different types of...
neutrinos will be different. Therefore in this case the gravitational constant $G$ is no more remain same for different types of neutrinos. If the neutrino eigenstates in gravity basis $|\nu_{Gi}\rangle$ are not the same as the flavour eigenstates $|\nu_\alpha\rangle$ of neutrinos then this can lead to neutrino oscillations even though neutrinos are massless. In the present work however we consider $|\nu_\alpha\rangle \neq |\nu_i\rangle \neq |\nu_{Gi}\rangle$ such that both the mass flavour oscillations and gravity induced oscillations are explored in a single framework.

In general, no positive signatures have been found for the violation of the weak equivalence principle. In the event that the equivalence principle is violated by a very small account then this may be detected by studying this gravity induced oscillations of neutrino. The effect can be manifested for the neutrinos with the very long basline ($\sim Mpc$). The UHE neutrinos from distant high energy extragalactic sources can well be a possibility to test the VEP.

In the theory of general relativity the equivalence principle is the equivalence of gravitational and inertial mass. The gravitational “force” as experienced locally while standing on a massive body (such as the Earth) is the same as the pseudo force experienced by an observer in a noninertial (accelerated) frame of reference. Therefore equivalence principle is violated if the universality of the gravitational constant $G$ is no more valid. A consequence of equivalence principle is that an object with an energy $E$ in a gravitational field will suffer a shift in energy in the same way as would be observed in an accelerated frame of reference in a no gravity environment. If we assume a weak and static gravitational field then this can be shown that for such a field, the metric is diagonal with $g_{00} = (1+2\phi)$ with $E' = E(1 - \frac{GM}{R}) = E(1 + \phi)$, where $R$ is the distance over which the gravitation field is operational and $M$ is the mass of the source of the gravitational field. Here $\phi$ is the gravitational potential. In such a field the energy from an object will be redshifted by an amount given by $E' = \sqrt{g_{00}}E = E(1 - \frac{GM}{R}) = E(1 + \phi)$.

Suppose in several neutrino oscillation based experiments, which are performed in the laboratory, the neutrinos can propagate through a given gravitational field in addition to the vacuum. With respect to the vacuum, the neutrino energies are redshifted (due to the Doppler effect) by an amount $E \rightarrow E' = \sqrt{g_{00}}E$. But because of the universality

The relation $E' = \sqrt{g_{00}}E$ can be realized by noting that the proper time in a curved manifold (presence of gravitation) is $d\tau = \sqrt{g_{\mu\nu}dx^\mu dx^\nu}$. Now the proper time is related to the coordinate time by $d\tau = \sqrt{g_{00}}dt$ (clock is at rest). If $N$ number of waves are emitted from a distant star with frequency $\nu_{\text{star}}$, proper time interval $\Delta \tau_{\text{star}}$ and the same are detected at Earth with frequency $\nu_{\text{Earth}}$, proper time interval $\Delta \tau_{\text{Earth}}$ then

$$\frac{\nu_{\text{Earth}}}{\nu_{\text{star}}} = \frac{\Delta \tau_{\text{star}}}{\Delta \tau_{\text{Earth}}} = \frac{\sqrt{g_{00}(x_{\text{Earth}})}}{\sqrt{g_{00}(x_{\text{star}})}} = \sqrt{\frac{1 + 2\phi_{\text{Earth}}}{1 + 2\phi_{\text{star}}}} = 1 + |\Delta \phi|.$$
nature of the gravitational coupling, the equivalence principle indicates that for all the neutrino flavours the energy shift should be the same and therefore it cannot lead to any neutrino flavour oscillations. Only the non-universality of the coupling of gravity to the neutrino field, which means the presence of the violation of equivalence principle, can contribute to the neutrino flavour oscillations.

In the presence of the gravitational field, the flavour eigenstates $|\nu_{\alpha}\rangle (\alpha = e, \mu, \tau, s)$ can be expressed as the superpositions of the gravitational eigenstate $|\nu_{\alpha i}\rangle (i = 1, 2, 3, 4)$ through the mixing angle $\theta'_{ij} (i \neq j), i, j = 1, 2, 3, 4$ in the 4-flavour framework.

$$|\nu_{\alpha}\rangle = \tilde{U}_{4\times 4}^{\prime} |\nu_{\alpha i}\rangle,$$

where $\tilde{U}_{4\times 4}^{\prime}$ represents the flavour-gravity mixing matrix in 4-flavour scenario

$$\tilde{U}_{4\times 4}^{\prime} = \begin{pmatrix}
  c_{14}^{\prime} U_{e1}^{\prime} & c_{14}^{\prime} U_{e2}^{\prime} & c_{14}^{\prime} U_{e3}^{\prime} & s_{14}^{\prime} \\
  -s_{14}^{\prime} s_{24} U_{\mu 1}^{\prime} + c_{24}^{\prime} U_{\mu 1}^{\prime} & -s_{14}^{\prime} s_{24} U_{\mu 2}^{\prime} + c_{24}^{\prime} U_{\mu 2}^{\prime} & -s_{14}^{\prime} s_{24} U_{\mu 3}^{\prime} + c_{24}^{\prime} U_{\mu 3}^{\prime} & s_{14}^{\prime} c_{24}^{\prime} U_{\mu 3}^{\prime} \\
  -c_{24}^{\prime} s_{34} U_{e1}^{\prime} + c_{34}^{\prime} U_{e1}^{\prime} & -c_{24}^{\prime} s_{34} U_{e2}^{\prime} + c_{34}^{\prime} U_{e2}^{\prime} & -c_{24}^{\prime} s_{34} U_{e3}^{\prime} + c_{34}^{\prime} U_{e3}^{\prime} & c_{14}^{\prime} c_{24}^{\prime} s_{34}^{\prime} \\
  -s_{24}^{\prime} s_{34} U_{\mu 1}^{\prime} + c_{34}^{\prime} U_{\mu 1}^{\prime} & -s_{24}^{\prime} s_{34} U_{\mu 2}^{\prime} + c_{34}^{\prime} U_{\mu 2}^{\prime} & -s_{24}^{\prime} s_{34} U_{\mu 3}^{\prime} + c_{34}^{\prime} U_{\mu 3}^{\prime} & c_{14}^{\prime} c_{24}^{\prime} c_{34}^{\prime}
\end{pmatrix} \quad (21)$$

Now the evolution equation for $|\nu_{G}\rangle$ is can be written as

$$i \frac{d}{dt} |\nu_{G}\rangle = H_{G} |\nu_{G}\rangle,$$  

where $H_{G} = \text{diag}(E_{G1}, E_{G2}, E_{G3}, E_{G4})$ for 4-flavour. Therefore the evolution equation for the flavour eigenstate ($|\nu_{\alpha}\rangle$) for the case of massless neutrinos is written as

$$i \frac{d}{dt} |\nu_{\alpha}\rangle = U H_{G} U^{\dagger} |\nu_{G}\rangle,$$  

Now in the absence of any violation of equivalence principle all the gravitational energy eigenvalues ($E_{G} = \sqrt{g_{00}} E = (1 - \frac{GM}{R}) E$) will not induce any phase difference to the neutrino eigenstate after the propagation. But if the equivalence principle is violated, the gravitational coupling $G$ is different for different types of neutrinos and in that case we have $H_{G} = \text{diag} \left( (1 - \phi_{\alpha 1}) E, (1 - \phi_{\alpha 2}) E, (1 - \phi_{\alpha 3}) E, (1 - \phi_{\alpha 4}) E \right)$, where $\frac{GM}{r} = \frac{GM}{R} \alpha_{i} = \phi_{\alpha i}$. Therefore this will induce the phase differences $\Delta E_{ij,G}$, where
\[ \Delta E_{ij,G} = \frac{G M}{r} \Delta \alpha_{ij} E, \quad (24) \]

where \( \Delta \alpha_{ij} = \alpha_i - \alpha_j \). In what follows we use \( U' \) and \( U \) to signify the mixing matrix \( \tilde{U}_{4 \times 4}' \) and \( \tilde{U}_{4 \times 4} \) respectively. The effective Hamiltonian of the system, which includes the contribution of both mass and gravitational mixing terms, can be written as

\[ H'' = U H_m U^+ + U' H_g U'^+. \quad (25) \]

It may be noted that for the UHE neutrinos of TeV and above the MSW oscillations for the neutrino passing through the matter has no effect at all. In our formalism, we assume that the mixing angle between mass and the flavour states and the mixing angle between the flavour and the gravitational eigenstate are same, i.e. \( \tilde{U}_{(4 \times 4)} = \tilde{U}_{(4 \times 4)}' \). Then the Hamiltonian with this assumption is given as

\[ H'' = U (H_m + H_g) U^+, \quad (26) \]

where

\[ H_g = \text{diag}(0, \Delta f_{21}, \Delta f_{31}, \Delta f_{41}). \quad (27) \]

In Eq. (25) \( \Delta f_{ij} - \Delta \alpha_{ij} \phi, i, j = 1, 2, 3, 4 \). \( H_m \) ia already defined in Eq. (19). So finally \( H'' \) can be written as

\begin{align*}
H'' &= U \text{diag}(0, \frac{\Delta m_{21}^2}{2E}, \frac{\Delta m_{31}^2}{2E} + \Delta f_{31} E, \frac{\Delta m_{41}^2}{2E} + \Delta f_{41} E) U^+ \\
&= \frac{1}{2E} U \text{diag}(0, \Delta \mu_{21}^2, \Delta \mu_{31}^2, \Delta \mu_{41}^2) U^+ , \quad (28)
\end{align*}

where

\begin{align*}
\Delta \mu_{21}^2 &= \Delta m_{21}^2 + 2\Delta f_{21} E^2 \\
\Delta \mu_{31}^2 &= \Delta m_{31}^2 + 2\Delta f_{31} E^2 \\
\Delta \mu_{41}^2 &= \Delta m_{41}^2 + 2\Delta f_{41} E^2 . \quad (29)
\end{align*}

Generally, the oscillation probability from a neutrino \( |\nu_\alpha\rangle \) of flavour \( \alpha \) to a neutrino \( |\nu_\beta\rangle \) of flavour \( \beta \) can be expressed as [20]

\[ P_{\nu_\alpha \rightarrow \nu_\beta} = \delta_{\alpha\beta} - 4 \sum_{j>i} U_{\alpha i} U_{\beta i} U_{\alpha j} U_{\beta j} \sin^2 \left( \frac{\pi L}{\lambda_{ij}} \right). \quad (30) \]
Gravitational induced oscillation probabilities are expressed as
\[
\lambda_{ij} = \frac{4\pi E}{\Delta m_{ij}^2} = \frac{4\pi E}{(\Delta m_{ij}^2 + 2\Delta f_{ij}E^2)}.
\]

We have already discussed that \(P_{4}^{ij}\) is given by
\[
P_{4}^{ij} = 1 - 4\left|\langle U_{e2}\rangle^2|U_{e1}\rangle^2 S_{21}^2 + \left(\langle U_{e3}\rangle^2|U_{e1}\rangle^2 + |U_{e3}\rangle^2|U_{e2}\rangle^2\right)S_{32}^2 + \left(\langle U_{e4}\rangle^2|U_{e1}\rangle^2 + |U_{e4}\rangle^2|U_{e2}\rangle^2\right)S_{42}^2 + |U_{e4}\rangle^2|U_{e3}\rangle^2 S_{43}^2\right).
\]

Now Eq. (28) looks like
\[
P_{\alpha\beta} = \delta_{\alpha\beta} - 4 \sum_{j>i} U_{\alpha j}U_{\beta i}U_{\alpha j}S_{ij}^2.
\]

where \(S_{ij}^2 = \sin^2\left(\frac{\Delta m_{ij}^2}{4E} \cdot \frac{\Delta f_{ij}E}{2}\right)\). In the 4-flavour scenario, the mass and the gravitational induced oscillation probabilities are expressed as
\[
\begin{align*}
P_{4}^{ee} & = 1 - 4\left|\langle U_{e2}\rangle^2|U_{e1}\rangle^2 S_{21}^2 + \left(\langle U_{e3}\rangle^2|U_{e1}\rangle^2 + |U_{e3}\rangle^2|U_{e2}\rangle^2\right)S_{32}^2 + \left(\langle U_{e4}\rangle^2|U_{e1}\rangle^2 + |U_{e4}\rangle^2|U_{e2}\rangle^2\right)S_{42}^2 + |U_{e4}\rangle^2|U_{e3}\rangle^2 S_{43}^2\right), \\
P_{4}^{e}\mu & = 4\left|\langle U_{e2}\rangle||U_{\mu 2}\rangle||U_{e1}\rangle S_{21}^2 + \left(\langle U_{e3}\rangle||U_{\mu 3}\rangle||U_{e2}\rangle||U_{\mu 2}\rangle + |U_{e3}\rangle||U_{\mu 3}\rangle||U_{e1}\rangle||U_{\mu 1}\rangle\right)S_{32}^2 + \left(\langle U_{e4}\rangle||U_{\mu 4}\rangle||U_{e2}\rangle||U_{\mu 2}\rangle + |U_{e4}\rangle||U_{\mu 4}\rangle||U_{e1}\rangle||U_{\mu 1}\rangle\right)S_{42}^2 + \left(\langle U_{e4}\rangle||U_{\mu 4}\rangle||U_{e3}\rangle||U_{\mu 3}\rangle S_{43}^2\right), \\
P_{4}^{e}\tau & = 4\left|\langle U_{e2}\rangle||U_{\tau 2}\rangle||U_{e1}\rangle S_{21}^2 + \left(\langle U_{e3}\rangle||U_{\tau 3}\rangle||U_{e2}\rangle||U_{\tau 2}\rangle + |U_{e3}\rangle||U_{\tau 3}\rangle||U_{e1}\rangle||U_{\tau 1}\rangle\right)S_{32}^2 + \left(\langle U_{e4}\rangle||U_{\tau 4}\rangle||U_{e2}\rangle||U_{\tau 2}\rangle + |U_{e4}\rangle||U_{\tau 4}\rangle||U_{e1}\rangle||U_{\tau 1}\rangle\right)S_{42}^2 + \left(\langle U_{e4}\rangle||U_{\tau 4}\rangle||U_{e3}\rangle||U_{\tau 3}\rangle S_{43}^2\right), \\
P_{4}^{\mu}\mu & = 1 - 4\left|\langle U_{\mu 2}\rangle^2|U_{\mu 1}\rangle^2 S_{21}^2 + \left(\langle U_{\mu 3}\rangle^2|U_{\mu 1}\rangle^2 + |U_{\mu 3}\rangle^2|U_{\mu 2}\rangle^2\right)S_{32}^2 + \left(\langle U_{\mu 4}\rangle^2|U_{\mu 1}\rangle^2 + |U_{\mu 4}\rangle^2|U_{\mu 2}\rangle^2\right)S_{42}^2 + |U_{\mu 1}\rangle^2|U_{\mu 3}\rangle^2 S_{43}^2\right), \\
P_{4}^{\mu}\tau & = 4\left|\langle U_{\mu 2}\rangle||U_{\tau 2}\rangle||U_{\mu 1}\rangle S_{21}^2 + \left(\langle U_{\mu 3}\rangle||U_{\tau 3}\rangle||U_{\mu 2}\rangle||U_{\tau 2}\rangle + |U_{\mu 3}\rangle||U_{\tau 3}\rangle||U_{\mu 1}\rangle||U_{\tau 1}\rangle\right)S_{32}^2 + \left(\langle U_{\mu 4}\rangle||U_{\tau 4}\rangle||U_{\mu 2}\rangle||U_{\tau 2}\rangle + |U_{\mu 4}\rangle||U_{\tau 4}\rangle||U_{\mu 1}\rangle||U_{\tau 1}\rangle\right)S_{42}^2 + \left(\langle U_{\mu 4}\rangle||U_{\tau 4}\rangle||U_{\mu 3}\rangle||U_{\tau 3}\rangle S_{43}^2\right), \\
P_{4}^{\tau}\tau & = 1 - 4\left|\langle U_{\tau 2}\rangle^2|U_{\tau 1}\rangle^2 S_{21}^2 + \left(\langle U_{\tau 3}\rangle^2|U_{\tau 1}\rangle^2 + |U_{\tau 3}\rangle^2|U_{\tau 2}\rangle^2\right)S_{32}^2 + \left(\langle U_{\tau 4}\rangle^2|U_{\tau 1}\rangle^2 + |U_{\tau 4}\rangle^2|U_{\tau 2}\rangle^2\right)S_{42}^2 + |U_{\tau 1}\rangle^2|U_{\tau 3}\rangle^2 S_{43}^2\right), \\
P_{4}^{\tau}\mu & = 4\left|\langle U_{\tau 2}\rangle||U_{\mu 2}\rangle||U_{\tau 1}\rangle S_{21}^2 + \left(\langle U_{\tau 3}\rangle||U_{\mu 3}\rangle||U_{\tau 2}\rangle||U_{\mu 2}\rangle + |U_{\tau 3}\rangle||U_{\mu 3}\rangle||U_{\tau 1}\rangle||U_{\mu 1}\rangle\right)S_{32}^2 + \left(\langle U_{\tau 4}\rangle||U_{\mu 4}\rangle||U_{\tau 2}\rangle||U_{\mu 2}\rangle + |U_{\tau 4}\rangle||U_{\mu 4}\rangle||U_{\tau 1}\rangle||U_{\mu 1}\rangle\right)S_{42}^2 + \left(\langle U_{\tau 4}\rangle||U_{\mu 4}\rangle||U_{\tau 3}\rangle||U_{\mu 3}\rangle S_{43}^2\right), \\
P_{4}^{\mu}\mu & = 1 - 4\left|\langle U_{\mu 2}\rangle^2|U_{\mu 1}\rangle^2 S_{21}^2 + \left(\langle U_{\mu 3}\rangle^2|U_{\mu 1}\rangle^2 + |U_{\mu 3}\rangle^2|U_{\mu 2}\rangle^2\right)S_{32}^2 + \left(\langle U_{\mu 4}\rangle^2|U_{\mu 1}\rangle^2 + |U_{\mu 4}\rangle^2|U_{\mu 2}\rangle^2\right)S_{42}^2 + |U_{\mu 1}\rangle^2|U_{\mu 3}\rangle^2 S_{43}^2\right).
\]

\[(33)\]
### 2.3 Detection of secondary muons produced from neutrino-nucleon interactions of diffuse GRB sources / Detection of UHE neutrinos from diffuse GRB sources

Upward going muons observed by the Super-Kamiokande detector are produced by the interactions between high energy atmospheric neutrinos, such as UHE neutrinos from distant extragalactic sources namely GRBs and the rock around the detector. For the case of detection of UHE neutrinos at a km$^2$ detector like IceCube, we are looking for these upward going muons, whose production depends on neutrino ($\nu_\mu$) charge current interactions ($\nu_\mu + N \rightarrow \mu + X$). The most promising advantage of considering upward going muons is that it cannot be misidentified as muons created in cosmic ray showers in the atmosphere.

The rate of upward going muon events from diffuse GRB neutrinos depends on $\nu_\mu N$ cross-sections in two different ways -

1. The interaction length, which is a function of the total cross-section, that leads the attenuation of the neutrino flux due to interactions in the Earth.
2. The probability that the neutrino induced muon arriving at the detector with an energy larger than the threshold energy $E_{\mu}^{\text{min}}$.

For the isotropic flux, we can represent the attenuation of the neutrinos, reaching the terrestrial detector being unabsorbed by the Earth, by a shadow factor ($S_{\text{shadow}}(E_\nu)$). This shadow factor is equivalent to the effective solid angle divided by $2\pi$ for upward going muons, which is given by

$$S_{\text{shadow}}(E_\nu) = \frac{1}{2\pi} \int_{-1}^{0} d\cos\theta_z \int d\phi \exp[-z(\theta_z)/L_{\text{int}}(E_\nu)] ,$$  \hspace{1cm} (34)

where $z(\theta_z)$ is the column depth for the incident zenith angle $\theta_z$ of the neutrinos

$$z(\theta_z) = \int \rho(r(\theta_z, l))dl .$$  \hspace{1cm} (35)

In Eq. (35) $\rho(r(\theta_z, l))$ ($l$ is the path length of neutrino in the Earth) indicates the matter density profile inside the Earth. We have taken Preliminary Earth Model (PREM) \cite{21} to express the matter density profile of the Earth in a more convenient way as we consider Earth as a spherically symmetric ball in our work (dense inner and outer core and a lower mantle having medium density).

The interaction length ($L_{\text{int}}(E_\nu)$) in Eq. (34) can be expressed as

$$L_{\text{int}} = \frac{1}{\sigma^{\text{tot}}(E_\nu)N_A} ,$$  \hspace{1cm} (36)
where $\sigma^{\text{tot}}$ corresponds to the total (charge current ($\sigma_{\text{CC}}$) + neutral current ($\sigma_{\text{NC}}$)) cross-section and $N_A$ represents the Avogadro number $N_A$ ($= 6.023 \times 10^{23}\text{mol}^{-1} = 6.023 \times 10^{23}\text{gm}^{-1}$).

The probability $P_{\mu}(E_\nu; E_{\mu}^{\text{min}})$ for a muon, produced due to charge current interactions of neutrinos, reaching the detector having energy above $E_{\mu}^{\text{min}}$ is expressed as

$$P_{\mu}(E_\nu; E_{\text{thr}}) = N_A \sigma_{\text{CC}}(E_\nu) \langle R(E_\nu; E_{\mu}^{\text{min}}) \rangle, \quad (37)$$

where the average range of muon in rock ($\langle R(E_\nu; E_{\mu}^{\text{min}}) \rangle$) is given as [22]

$$\langle R(E_\nu; E_{\mu}^{\text{min}}) \rangle = \frac{1}{\sigma_{\text{CC}}} \int_0^{(1-E_{\mu}^{\text{min}}/E_\nu)} dy R(E_\nu(1-y); E_{\mu}^{\text{min}}) \times \frac{d\sigma_{\text{CC}}(E_\nu, y)}{dy}. \quad (38)$$

We can write $E_{\mu}$ in the place of $E_\nu(1-y)$ in Eq. (38) as $y = (E_\nu - E_{\mu})/E_\nu$, defines the fraction of energy lost by a neutrino having energy $E_\nu$ in the production of secondary muons having energy $E_{\mu}$ via charge current interactions. The muon range $R(E_{\mu}; E_{\mu}^{\text{min}})$ in Eq. (38) can be written as

$$R(E_{\mu}; E_{\mu}^{\text{min}}) = \int_{E_{\mu}^{\text{min}}}^{E_{\mu}} \frac{dE_{\mu}}{\langle dE_{\mu}/dX \rangle} \approx \frac{1}{\beta} \ln \left( \frac{\alpha + \beta E_{\mu}}{\alpha + \beta E_{\mu}^{\text{min}}} \right). \quad (39)$$

The energy loss rate of muon having energy is expressed as [23]

$$\langle \frac{dE_{\mu}}{dX} \rangle = -\alpha - \beta E_{\mu}, \quad (40)$$

where the constant $\alpha$ stands for the energy losses and $\beta$ describes the catastrophic losses (namely bremsstrahlung, pair production and hadron production) respectively. Now these two constants we have considered in our work are for $E_{\mu} \leq 10^6$ GeV [24]

$$\alpha = 2.033 + 0.077 \ln[E_{\mu}(\text{GeV})] \times 10^3 \text{ GeV cm}^2 \text{ gm}^{-1},$$
$$\beta = 2.033 + 0.077 \ln[E_{\mu}(\text{GeV})] \times 10^{-6} \text{ GeV cm}^2 \text{ gm}^{-1}, \quad (41)$$

and otherwise [25]

$$\alpha = 2.033 \times 10^{-3} \text{ GeV cm}^2 \text{ gm}^{-1},$$
$$\xi = 3.9 \times 10^{-6} \text{ GeV cm}^2 \text{ gm}^{-1}. \quad (42)$$

As we already mentioned, the detection of $\nu_{\mu}$'s from diffuse GRB sources can be estimated from the tracks of the secondary muons. The total number of secondary muon
yields, which is a function of both \( S_{\text{shadow}}(E_\nu) \) and \( P_\mu(E_\nu; E_\mu^{\text{min}}) \), can be detected in a detector such as IceCube of unit area is \([26, 23, 27]\)

\[
S = \int_{E_\nu^{\text{min}}}^{E_\nu^{\text{max}}} dE_\nu S_{\text{shadow}}(E_\nu) P_\mu(E_\nu; E_\mu^{\text{min}}) \frac{dN_\nu}{dE_\nu} .
\]  

(43)

We replace \( \frac{dN_\nu}{dE_\nu} \) in Eq. (43) by \( F_\nu \), mentioned in Eqs. (3), (33). We also consider the production of muons via the decay channel \( \nu_\tau \rightarrow \tau \rightarrow \nu_\mu \nu_\tau \) with probability 0.18. In such cases we can compute the muon events by solving Eqs. (34)-(43) numerically, where \( \frac{dN_\nu}{dE_\nu} \) in Eq. (43) is equivalent to \( F_\nu \) (Eqs. (3),(33)).

We also consider the shower events, produced via the CC interaction of \( \nu_e \) and NC interactions of all three flavours \( (\nu_e, \nu_\mu, \nu_\tau) \). The track events have been neglected and we consider the whole detector volume \( V \) for the calculations of the shower event rate, which is given by

\[
S_{\text{sh}} = V \int_{E_\nu^{\text{min}}}^{E_\nu^{\text{max}}} dE_\nu \frac{dN_\nu}{dE_\nu} \int dy \frac{1}{\sigma^i} \frac{d\sigma^i}{dy} P_{\text{int}}(E_\nu, y) .
\]  

(44)

For the electromagnetic shower \( \sigma^i = \sigma^{\text{CC}} \) and \( \sigma^i = \sigma^{\text{NC}} \) for \( \nu_e \nu_\mu \) NC interactions. The probability \( P_{\text{int}} \), by which the neutrino interactions produces the shower, is given by

\[
P_{\text{int}} = \rho N_\rho \sigma^i L ,
\]  

(45)

where \( \rho \) represents the matter density and the length of the detector is \( L \). For the case of the shower events \( \frac{dN_\nu}{dE_\nu} \) in Eq. (44) is replaced by \( F_\nu \), \( F_\nu \), \( F_\nu \) from Eqs. (3), (33).

3 Calculations and Results

In this section we are trying to explore the fact that how the presence of the violation of equivalence principle effects the possible neutrino induced muon yields and the shower yields detected at a Km\(^2\) detector. Now to estimate this effect, we consider a 4-flavour framework, where an extra sterile neutrino \( (\nu_s) \) is added to the usual three active neutrino families \( (\nu_e, \nu_\mu, \nu_\tau) \) in the UHE regime. We also have taken diffuse GRBs as extragalactic sources of UHE neutrinos and a kilometer square Cherenkov detector such as IceCube for the detection of the possible secondary muon and shower yields.

In the presence of the gravity induced oscillations with the usual mass flavour oscillations, we can calculate the neutrino induced secondary muon yield and the shower
yield at a Km$^2$ IceCube detector for 4-flavour UHE neutrinos by using Eqs. (1) - (33) in section 1.1 and 1.2 and Eqs. (34) - (43) in section 1.3. For this purpose, we consider upward going muons, which are produced due to charge-current interactions of UHE neutrinos with the rock around the detector. By considering Waxman-Bahcall flux as a diffuse flux from several GRBs, the threshold energy of the detector has been taken as $E_{\text{min}}^\mu = 1 \text{ TeV}$.

In our present work, we consider a ratio $R$ between the muon and the shower yield as

$$R = \frac{T_{\mu}}{T_{\text{sh}}} ,$$

where

$$T_{\mu} = S(\text{for } \nu_\mu) + S(\text{for } \nu_\tau)$$

$$T_{\text{sh}} = S_{\text{sh}}(\text{for } \nu_e \text{ CC interaction}) + S_{\text{sh}}(\text{for } \nu_e \text{ NC interaction}) + S_{\text{sh}}(\text{for } \nu_\mu \text{ NC interaction}) + S_{\text{sh}}(\text{for } \nu_\tau \text{ NC interaction}) .$$

The quantities $S$ and $S_{\text{sh}}$, which are mentioned in the above Eq. (47), has been already discussed in section 1.3. In the 4-flavour framework the ratio $R$ is signified as $R_4$.

For the purpose of the further calculations, we have chosen the best fit values of the three active mixing angles as $\theta_{12} = 33.48^\circ, \theta_{23} = 45^\circ, \theta_{13} = 8.5^\circ$. Different neutrino experimental groups such as MINOS \[28\]-[39], Daya Bay \[40\]-[46], Bugey \[47\], NOVA \[48\]-[53] have given some limits on the flavour mixing angles ($\theta_{14}, \theta_{24}, \theta_{34}$) in 4-falvour scheme. The upper limits on the four flavour neutrino mixing angles have obtained by NOVA as $\theta_{24} \leq 20.8^0$ and $\theta_{34} \leq 31.2^0$ for $\Delta m^2_{41} = 0.5 \text{ eV}^2$. For the same value of $\Delta m^2_{41}$, MINOS has proposed the upper limits on $\theta_{34}$ and $\theta_{24}$ as $\theta_{24} \leq 7.3^0$ and $\theta_{34} \leq 26.6^0$. In addition to the above mentioned experimental groups the IceCube-Deepcore results \[54\] suggest $\theta_{24} \leq 19.4^0$ and $\theta_{34} \leq 22.8^0$ for $\Delta m^2_{41} = 1 \text{ eV}^2$. In the present work we have taken the ranges over which $\theta_{24}, \theta_{34}$ and $\theta_{14}$ vary are $2^0 \leq \theta_{24} \leq 20^0, 2^0 \leq \theta_{34} \leq 20^0$ and $1^0 \leq \theta_{14} \leq 4^0$ respectively and the limits on $\theta_{14}$ is consistent with the combined results obtained from MINOS, Daya Bay and Bugey experiments.

The main motivation of our work is to show the effects of the gravity induced oscillations in addition to the mass flavour oscillations on the ratio $R_4$ between the muon and the shower events and compare them with the ratio obtained for the only mass flavour oscillations without any violation of the equivalence principle. Her we like
Table 1: Comparison of the muon to shower ratio ($R_4$) for a diffuse GRB neutrino flux (Waxman-Bahcall flux) for with VEP (violation of equivalence principle) case compared to the same for without VEP case for two sets of active-sterile neutrino mixing angles.

| $\theta_{14}$ | $\theta_{24}$ | $\theta_{34}$ | $R_4$ (with VEP) | $R_4$ (without VEP) |
|---------------|---------------|---------------|-----------------|-------------------|
| 3°            | 5°            | 20°           | 0.79            | 2.83              |
| 4°            | 6°            | 15°           | 0.78            | 2.85              |

to mention that we have made our calculations for the representative values of $\Delta f_{ij}$ to be $\Delta f_{21} = 10^{-43}$, $\Delta f_{32} = 10^{-42}$, $\Delta f_{41} = 10^{-43}$, $\Delta f_{43} = 10^{-42}$ and we have considered $\Delta m_{22}^2$ and $\Delta m_{21}^2$ as $\Delta m_{32}^2 = 7.0 \times 10^{-5}\text{eV}^2$ (from solar neutrino oscillations) and $\Delta m_{32}^2 = 2.4 \times 10^{-3}\text{eV}^2$ (from atmospheric neutrino oscillations) respectively. The values of the mass square differences in the 4-flavour cases such as $\Delta m_{41}^2$ lies within the range $0.2\text{eV}^2 \leq \Delta m_{41}^2 \leq 2\text{eV}^2$ and we assume that $\Delta m_{32}^2 \simeq \Delta m_{31}^2 \simeq 2.4 \times 10^{-3}\text{eV}^2$ and $\Delta m_{32}^2 \simeq \Delta m_{31}^2 \simeq 1\text{eV}^2$.

Figure 1: Variation of $R_4$ with $\theta_{24}$ and $\theta_{34}$ for (a) $\theta_{14} = 1^\circ$ and (b) $\theta_{14} = 4^\circ$. See text for details.

In Table 1, by considering the Waxman-Bahcall flux as the diffuse flux we furnish the calculated values of $R_4$ with and without the violation of equivalence principle for the two different sets of representative values for $\theta_{14}, \theta_{24}, \theta_{34}$ in the 4-flavour framework. From Table 1 it is observed that the muon-to-shower ratio with VEP decreases by a factor of 3.5 compared to the ratio without VEP.
Table 2: Same as Table 1, but here we consider the recent analysis of the IceCube (HESE) data for the case of diffuse flux of UHE neutrinos.

| $\theta_{14}$ | $\theta_{24}$ | $\theta_{34}$ | $R_4$ (with VEP) | $R_4$ (without VEP) |
|--------------|--------------|--------------|----------------|---------------------|
| 3°           | 5°           | 20°          | 0.62           | 2.03                |
| 4°           | 6°           | 15°          | 0.61           | 2.06                |

Figure 1 indicates the variations of $R_4$ in the purpose of VEP in addition to the usual mass flavour oscillations with $\theta_{24}$ and $\theta_{34}$ for two fixed values of $\theta_{14}$. It is quite evident from Figure 1 that $R_4$ with VEP is $\sim 3$ times lower than $R_4$ without VEP.

Figure 2: Variation of $R_4$ with $\theta_{24}$ and $\theta_{34}$ for (a) $\theta_{14} = 1^\circ$ and (b) $\theta_{14} = 4^\circ$ (UHE neutrino diffused flux has been taken from the recent analysis of the IceCube HESE data). See text for details.

Till now we have discussed the diffuse flux of UHE neutrinos obtained from the theoretical considerations, which is given by Waxman-Bahcall. Recently the IceCube Collaboration has updated their high energy search results for events with interaction vertices inside the detector fiducial volume. They have published their six year high energy starting events (HESE) data and the analysis based on this data [55]. The search results of the IceCube Collaboration, performed on six years of detector data, led to the discovery of the astrophysical neutrino flux above atmospheric background. For this neutrino flux they calculated the best fit power law as $E^2 \phi(E) = 2.46 \pm 0.8 \times$
$10^{-8} \left( \frac{E}{100 \text{TeV}} \right)^{-0.92} \text{GeV cm}^{-2} \text{s}^{-1} \text{sr}^{-1}$. So the neutrino flux for no broken power law case (that is, one component fit) can be expressed as $\phi(E) \sim E^{-\gamma_{\text{astro}}}$, where $\gamma_{\text{astro}}$ indicates the astrophysical spectral index and the best fit value of the spectral index is $\gamma_{\text{astro}} = 2.92^{+0.33}_{-0.29}$. By considering the aforesaid HESE data, we have also computed the muon-to-shower ratio for both with VEP and without VEP case in the 4-flavour regime and for this calculations the threshold energy is to be considered as $E_{\text{min}}^\mu = 60 \text{ TeV}$. These computed results are shown in Table 2 and it is obvious from the table that $R_4$ with VEP is decreased by an amount 3.43 from $R_4$ without VEP case for the chosen values of $\theta_{14}, \theta_{24}$ and $\theta_{34}$, which is same as Table 1. The track-to-shower ratio $R_4$ with VEP is reduced by a factor 2.7 from $R_4$ without VEP.

4 Summary and Discussions

In this work, we explore the possibility that very small violation of equivalence principle can be probed via the gravity induced neutrino oscillation. We demonstrate such a possibility by calculating the muon neutrino flux and consequently muon track events as well as the shower events for such neutrinos at a kilometer square detector such as IceCube. We then compare our results for no oscillation case. We consider here a 4 neutrino scenario and calculated the oscillation formalism with mass induced and gravity induced oscillations. We compare our results for gravity induced oscillations with a representative value of the VEP with those where no VEP but only mass-flavour oscillation (suppression) is considered. From comparisons of the muon track to shower ratios we find that UHE neutrinos from distant sources could be an effective way to probe very small violation of weak equivalence principle.

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