Chiral Thirring-Wess Model

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The vector type of interaction of the Thirring-Wess model was replaced by the chiral type and a new model was presented which was termed as chiral Thirring-Wess model in [26]. The model was studied there with a Faddeevian class of regularization that contained few ambiguity parameters with the apprehension that unitarity might be threatened like the chiral generation of the Schwinger model. In the present work it has been shown that no counter term containing the regularization ambiguity is needed for this model to be physically sensible. So the chiral Thirring-Wess model is studied here without the presence of any ambiguity parameter and it has been found that the model not only remain exactly solvable but also does not lose the unitarity like the chiral generation of the Schwinger model. The phase space structure and the theoretical spectrum of this new model has been determined in the present scenario through Dirac’s method of quantization of constraint system. The theoretical spectrum is found to contain a massive boson with ambiguity free mass and a massless boson.

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I. INTRODUCTION

Schwinger model is an interesting and exactly solvable lower dimensional field theoretical model and it has been studied over the years in different perspective, e.g., dynamical mass generation, confinement aspect of fermion, charge shielding etc. [1–6]. Here photon acquires mass via a kind of dynamical symmetry breaking keeping the gauge symmetry of the model intact. The model in the non commutating pace time setting also has been found to render interesting results [7–9]. A novel back ground interaction appears when noncommutative space-time setting is incorporated in place of its standard commutating framework [10]. After few years of the birth of Schwinger model, Thirring and Wess jointly presented a two dimensional field theoretical model where also photon acquired mass but the gauge symmetry of the model failed to preserve at the classical level [10]. Recently, an attempt has been made in [11], for systematic functional integral bosonization of this mode. After few years of presentation of the Thirring-Wess model, chiral generation of Schwinger model was perused in [12], however the model remaind less attractive over a long period of time because of its non-unitary problem. But it attracted huge attentions and gradually acquired a significant position in lower dimensional field theory after the removal of the non-unitary problem by Jackiw and Rajaraman [13] taking into account anomaly into consideration. The welcome entry of the anomaly and a suitable exploitation of the ambiguity involved therein made Jackiw-Rajaraman version of Chiral Schwinger model [13–18] along with the other independent regularized version of that model [19–22] interesting as well as attractive in lower dimensional field theory regime.

Thirring-Wess model describes an interacting theory of massless fermion with massive vector field (Proca) in two dimension. It can be thought of as a study of QED, viz., Schwinger model [1,2] replacing Maxwell’s field by Proca and that very replacement breaks the gauge symmetry at the classical level but a consistent field theoretical model gets birth. It is true that the so called non-confining Schwinger model [23,24] is a structurally equivalent gauge non-invariant model to the Thirring-Wess model but there lies a crucial difference between these two. In the Thirring-Wess mode the masslike term for the gauge field was added at the classical level however in the the so called non-confining Schwinger model the same type of masslike term gets involved through one loop correction which contains an ambiguity parameter like Jackiw-Rajaraman version of Chiral Schwinger model.

An attempt has been made in [26] to get chiral generation of the Thirring-Wess mode in the similar way the chiral generation of the Schwinger model was made in [12] and it was studied with a Faddeevian class of regularization where few ambiguity parameter entered there due to the one loop correction during the process of removal of divergence of the fermionic determinant. Let us now explore a bit related to entry of the ambiguity parameter in the QED and chiral QED.

It is known that Schwinger model remains unitary and exactly solvable in absence of any ambiguity parameter both in the fermionic as well as bosonized version, however the chiral generation of this model [12] faced a severe
The dynamics of the A field with It is known that the vector and axial vector current that couple with the gauge field are defined by

\[ J^\mu = \bar{\psi} \gamma^\mu [i \partial_\mu + e \sqrt{\pi} A_\mu (1 - \gamma_5)] \psi \]

\[ = \bar{\psi} R \gamma^\mu [i \partial_\mu \psi_R + \psi_L \gamma^\mu (i \partial_\mu + 2e \sqrt{\pi} A_\mu)] \psi \]

Here dynamics of the \( A_\mu \) field is governed by the Proca field and the lagrangian of which is given by

\[ \mathcal{L}_{Proca} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu \]

It is known that the vector and axial vector current that couple with the gauge field are defined by \( J_\mu = \bar{\psi} \gamma_\mu \psi \) and \( J_5 = \bar{\psi} \gamma_\mu \gamma_5 \psi \). In order to make chiral generation we replaced the vector type of interaction \( \bar{\psi} \gamma_\mu \psi A^\mu \) by the chiral type \( \bar{\psi} \gamma_\mu (1 + \gamma_5) \psi A^\mu \). If we now focus on the fermionic part of the lagrangian density we find that the right handed fermion remains uncoupled for this type of chiral interaction. As a result, integration over this right handed part leads to a field independent counter part which can be absorbed within the normalization. However, the integration over the left handed fermion is little bit tricky and that leads to

\[ Z[A] = \int d\psi d\bar{\psi} e^{\int d^2x \mathcal{L}_f} \]

\[ = \exp \frac{i e^2}{2} \int d^2x A_\mu [m^2 g_{\mu \nu} - (\partial^\mu + \partial^\nu) \frac{1}{2} (\partial^\nu + \partial^\nu)] A_\nu, \]

The above generating functional \( \mathcal{L}_B \) can be expressed in terms of some auxiliary field and when it is done so in terms of the an auxiliary field \( \phi(x) \) it turns out to the following

\[ Z[A] = \int d\phi e^{i \int d^2x \mathcal{L}_B}, \]

with

\[ \mathcal{L}_B = \frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi) + e (g^{\mu\nu} - \epsilon^{\mu\nu}) \partial_\nu \phi A_\mu + \frac{1}{2} m^2 A_\mu A^\mu \]

\[ = \frac{1}{2} (\phi^2 - \phi'^2) + e (\phi + \phi')(A_0 - A_1) + \frac{1}{2} (A_1^2 - A_0^2) \]

So the total lagrangian density, i.e., the bosonized lagrangian density along with the Proca background reads

\[ \mathcal{L} = \mathcal{L}_B + \mathcal{L}_{Proca} \]

\[ = \frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi) + e (g^{\mu\nu} - \epsilon^{\mu\nu}) \partial_\nu \phi A_\mu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu \]

\[ = \frac{1}{2} (\phi^2 - \phi'^2) + e (\phi + \phi')(A_0 - A_1) + \frac{1}{2} (A_1^2 - A_0^2) + \frac{1}{2} m^2 (A_1^2 - A_0^2) \]
Let us recall that when this type of chiral generalization was attempted in the Schwinger model in [12], the model though did not lose its solvability nevertheless unitarity failed to be maintained and the model remained non-attractive because of its inability to provide the necessary condition for being physically sensible and was left un-studied till this severe non-unitary problem was removed in [13]. So a question may automatically appear whether the chiral generalization of Thirring-Wess model face the same problem or it is free from that severe un-physical situation. To see it let us begin our analysis for the model proposed here. A careful look reveals that the lagrangian (7) gives the following electromagnetic current that takes part in the interacting with matter field.

\[ J^\mu = e(g^{\mu\nu} - \epsilon^{\mu\nu}) \partial_\nu \phi + m^2 A^\mu. \]  

Note that this current is non-conserved since \( \partial_\mu J^\mu \neq 0 \). The Euler Lagrange equations that flow from the same lagrangian (7) are

\[ \partial_\mu F^{\mu\nu} + (g^{\mu\nu} - \epsilon^{\mu\nu}) \partial_\nu \phi + m^2 A^\mu = 0, \]  

\[ \Box \phi + e(g^{\mu\nu} - \epsilon^{\mu\nu}) \partial_\nu A^\mu = 0. \]

Solving the above equations one may obtain the free field operator solutions, however we will follow the hamiltonian formulation provided by Dirac for quantization of constrained system to get the theoretical spectrum because this hamiltonian formulation helps to understand directly whether the Hamiltonian for this system remains positive definite or not during the course of its quantization to which we now turn. Needless to mention that positive definiteness is the first step of ensuring the unitary property of a field theoretical model. To investigate the fate of this model in the present scenario one needs to quantize the theory. So, in the following section an attempt has been made to quantize the theory.

### III. CONSTRAINT ANALYSIS AND DETERMINATION OF THE THEORETICAL SPECTRUM

Applications of Dirac’s formalism involves the identification of the constraints of the theory that remains embedded in its phase space. We, therefore, proceed to identify the constraint of the theory at the early stage of our analysis. To this end, we require to calculate the canonical momenta of the fields with which model is constituted. The momentum corresponding to the field \( \phi, A_0 \) and \( A_1 \) respectively are

\[ \pi_\phi = \dot{\phi} + e(A_0 - A_1), \]  

\[ \pi_0 = 0, \]  

\[ \pi_1 = \dot{A}_1 - A'_0. \]

The above three equations (11), (12), and (13) help to obtain the Hamiltonian through the Legendre transformation

\[ H_B = \int dx [\pi_\phi \dot{\phi} + \pi_1 \dot{A}_1 + \pi_0 \dot{A}_0 - L], \]

which gives the following hamiltonian density for the system we are interested in.

\[ H_B = \frac{1}{2} (\pi_\phi^2 + \pi_0^2 + \pi_1^2) + \pi_1 A'_0 - e(\pi_\phi + \phi')(A_0 - A_1) + \frac{1}{2} e^2 (A_0 - A_1)^2 - \frac{1}{2} m^2 (A_0^2 - A_1^2). \]

Equation (12), is independent of \( \dot{A}_0 \). So it is the primary constraint of the theory. According to Dirac’s prescription [27] the analysis at this stage has to be done using the effective hamiltonian in stead of using the canonical hamiltonian obtained directly from the Legendre transformation and the effective hamiltonian in this situation is

\[ H_{e_{ff}} = H + \int dx u_0 \pi_0. \]

The Lagrangian multiplier (velocity) \( u_0 \) is yet to be determined. It will be fixed later. The primary constraint \( \pi_0 \approx 0 \), under its essential physical requirement which is its preservation of that for all time leads to the secondary constraint

\[ G = \pi'_1 + 2e(\pi_\phi + \phi') + (m^2 - e^2) A_0 + e^2 A_1 \approx 0. \]
It is known as the Gauss law of the theory. The preservation of the constraint (17), however does not give rise to any new constraint. It fixes the velocity \( u_0 \). It indicates that these two second class constraints are embedded within the phase space of the system and these two are the initial input to calculate the Dirac brackets.

According to the Dirac terminology (27), the constraints (12) and (17) both are weak conditions up to this stage. If it is now attempted to impose these into the hamiltonian treating these two as strong condition, the hamiltonian will be then be reduced to

\[
H_R = \int dx \left[ \frac{1}{2} \pi_1^2 + \frac{1}{2} \frac{1}{m^2 - e^2} \pi_1^2 + \frac{m^2}{2 m^2 - e^2} (\pi_1^2 + \phi^2) + \frac{e^2}{m^2 - e^2} \pi_1^2 A_1 \right.
\]
\[
+ \frac{e^2}{m^2 - e^2} \pi_1 \phi' + \frac{m^2}{2 m^2 - e^2} \pi_1^2 \phi' + \frac{e^2}{m^2 - e^2} \pi_1 \phi' + \frac{e^2}{m^2 - e^2} A_1 \phi' + \frac{1}{2} \frac{m^4}{m^2 - e^2} A_1^2 \right].
\]

(18)

But the price that has to be paid for this is to replace the canonical Poisson brackets by the corresponding Dirac bracket (27) because the Poission brackets become inadequate when the constraints are plugged in strongly into the Hamiltonian. It is known that Dirac bracket between the two variables \( A(x) \) and \( B(y) \) is defined by

\[
[A(x), B(y)]^* = [A(x), B(y)] - \int [A(x) \omega_i(\eta)] C^{-1}_{ij}(\eta, z) [\omega_j(\eta), B(y)] d\eta dz,
\]

(20)

where \( C^{-1}_{ij}(x, y) \) is given by

\[
\int C^{-1}_{ij}(x, z)[\omega_i(z), \omega_j(y)] dz = 1.
\]

(21)

Here \( \omega_i \)'s represents the second class constraints that remains embedded within the phase space of the theory. The matrix \( C^{-1}(x, y) \) for this theory is given by

\[
C^{-1}(x, y) = \frac{1}{m^2 - e^2} \begin{pmatrix} 0 & \delta(x - y) \\ -\delta(x - y) & 0 \end{pmatrix},
\]

(22)

With the help of equation (20), the Dirac brackets among the fields \( A_1, \pi_1, \phi \) and \( \pi_\phi \) are calculated:

\[
[A_1(x), A_1(y)]^* = 0
\]

(23)

\[
[A_1(x), \pi_1(y)]^* = \delta(x - y)
\]

(24)

\[
[\phi(x), \phi(y)]^* = \delta(x - y)
\]

(25)

\[
[\phi(x), \pi_\phi(y)]^* = 0
\]

(26)

Note that the above equations imply that the Dirac brackets retain its own Poission bracket structures here which was not the case when it was studied in [26] with the Faddeevian class of regularization. Making use of the Dirac brackets (23), (24), (25) and (26) the equations of motion for the fields through the hamiltonian (19) is constituted with are computed and it is found that the first order equations of motion result in

\[
\dot{A}_1 = \pi_1 - \frac{e^2}{m^2 - e^2} A_1' - \frac{e}{m^2 - e^2} \pi_\phi' - \frac{e}{m^2 - e^2} \phi'' - \frac{1}{m^2 - e^2} \pi_1''
\]

(27)

\[
\dot{\pi}_1 = -\frac{e^2}{m^2 - e^2} \pi_1' - \frac{em^2}{m^2 - e^2} \pi_\phi - \frac{em^2}{m^2 - e^2} \phi' - \frac{m^4}{m^2 - e^2} A_1
\]

(28)

\[
\dot{\phi} = \frac{m^2}{m^2 - e^2} \pi_\phi + \frac{e^2}{m^2 - e^2} \phi' + \frac{e}{m^2 - e^2} \pi_1' + \frac{em^2}{m^2 - e^2} A_1
\]

(29)
\[ \dot{\pi}_\phi = \frac{m^2}{m^2 - e^2} \phi'' + \frac{e^2}{m^2 - e^2} \pi'_\phi + \frac{e}{m^2 - e^2} \pi''_1 + \frac{em^2}{m^2 - e^2} A'_1 \]  

(30)

The above first order equations of motion gets simplified into the following second order differential equations after a little algebra.

\[ (\Box + \frac{m^4}{m^2 - e^2})\pi_1 = 0, \]  

(31)

\[ (\Box + \frac{m^4}{m^2 - e^2})(A_1 + \frac{e}{m^2} \phi) = 0 \]  

(32)

\[ \Box(\phi + \frac{e}{m^2} \pi_1) = 0 \]  

(33)

\[ \Box(\pi_\phi + \frac{e}{m^2} \pi'_1) = 0 \]  

(34)

The above four equations (31), (32), (33) and (34) suggest that the field \( A_1 + \frac{e}{m^2} \phi \) describe a massive boson with square of the mass \( \tilde{m}^2 = \frac{m^4}{m^2 - e^2} \) and the field \( \phi + \frac{e}{m^2} \pi_1 \) represents a boson with vanishing mass. The field \( \pi_1 \) and \( \pi_\phi + \frac{e}{m^2} \pi'_1 \) may be considered as the momenta corresponding to the field \( A_1 + \frac{e}{m^2} \phi \) and \( \phi + \frac{e}{m^2} \pi_1 \) respectively because the pair of fields describing equations (31) and (32) satisfy canonical poisson bracket among themselves and the pairs describing the equations (33) and (34) also satisfy the canonical poisson bracket. Note that \( m^2 \) must be greater than \( e^2 \) in order to get the mass of the massive boson a physical (positive) one. A careful look also reveals that the Hamiltonian (19) also demands the condition \( m^2 > e^2 \) in order to be positive definite. The massless bosons are equivalent to massless fermion in two dimensions so it can be taught of as fermion in a de-confined state [13, 19, 20, 23].

IV. INCLUSION OF COUNTER TERM CONTAINING THE AMBIGUITY PARAMETERS

In Sec. II, Chiral Thirring-Wess model has been defined by the following generating functional

\[ Z[A] = \int d\psi d\bar{\psi} e^{\int d^2 x \mathcal{L}_f} \]  

(35)

where the definition of \( \mathcal{L}_f \) is available from the equation (2) of Sec. II. As mentioned in Sec. II, the dynamics of the \( A_\mu \) field is governed by the Proca field and the lagrangian of which is given in equation (3) In Sec. II, it has been already mentioned that in the lagrangian (2), right handed fermion remains uncoupled when vector interaction is replaced by the chiral interaction. So integration over this right handed part leads to field independent counter part which can be absorbed within the normalization. However the integration over the left handed fermion is much involved because one needs to regularize the fermionic determinant during the process of integration since it has a diverging nature, and after a careful calculation one arrives at the following zenanaing functional [19, 21, 22, 26]

\[ Z[A] = \int d\psi_L d\bar{\psi}_L \gamma^\mu (i\partial_\mu + 2e\sqrt{\pi}A_\mu) \psi_L = \exp i \frac{e^2}{2} \int d^2 x A_\mu [M_{\mu\nu} - (\partial_\mu + \tilde{\partial_\mu})^1 (\partial_\nu + \tilde{\partial_\nu})] A_\nu, \]  

(36)

In general, the elements of the \( M_{\mu\nu} \) can take any arbitrary values. However, the model looses both its solvability and Lorentz invariance in that situation [26]. In [26], we considered a symmetric form of \( M_{\mu\nu} \):

\[ M_{\mu\nu} = \left( \begin{array}{ccc} \tilde{a} & \alpha & \gamma \\ \alpha & \gamma & \gamma \\ \gamma & \gamma & \gamma \end{array} \right) \delta(x - y). \]  

(37)

where regularization ambiguity got involved within the parameters \( \tilde{a}, \alpha \) and \( \gamma \). These parameters entered there in order to remove the divergence in the fermionic determinant since the evaluation of the determinant needs a one loop correction [19, 21, 22]. It was found in [26] that all the parameters did remain independent with each other for the model to be physically sensible.
This generating functional \([41]\), when written there in terms of the auxiliary field \(\phi(x)\) it turned out to the following

\[
Z[A] = \int d\phi e^{i \int d^2x L_B},
\]

with

\[
L_B = \frac{1}{2} (\dot{\phi}^2 - \phi'^2) + e(\phi + \phi')(A_0 - A_1) + \frac{1}{2} e^2(\hat{a} A_0^2 + 2 \alpha A_0 A_1 + \gamma A_0^2).
\]

So the total lagrangian density with which we dealt there was

\[
\mathcal{L} = \frac{1}{2} (\dot{\phi}^2 - \phi'^2) + e(\phi + \phi')(A_0 - A_1) + \frac{1}{2} (A_1^2 - A_0^2) + \mathcal{L}_{\text{mass}}
\]

where the term \(\mathcal{L}_{\text{mass}}\) was

\[
\mathcal{L}_{\text{mass}} = \frac{1}{2} e^2[(\hat{a} + \frac{m^2}{e^2}) A_0^2 + 2\alpha A_0 A_1 + (\gamma - \frac{m^2}{e^2}) A_0^2].
\]

This Lagrangian in general failed to provide Poincaré invariant equations of motion. Ambiguity in the regularization allowed us to set two conditions \(\hat{a} + \frac{m^2}{e^2} = 1\) and \(m^2 = e^2(1 + \gamma - 2\alpha)\) without violating any physical principle. In our previous work \([26]\), we showed that the condition \(\hat{a} + \frac{m^2}{e^2} = 1\) helped us to fit the model within the Faddeevian class \([28, 31]\) and the theory rendered an interesting Lorentz covariant in variant theoretical spectrum though there was no lorentz covariance in the starting lagrangian provided the constraint \(m^2 = e^2(1 + \gamma - 2\alpha)\) among the ambiguity parameters were maintained.

Some other choices may also lead to physically sensible theory. Indeed we are free to choose \(\text{Jjackiw-Rajaraman}\) type of regularization for this model. In that case we have to choose the matrix \(M_{\mu\nu} = a g_{\mu\nu}\). It keeps the model Lorentz covariant to start with and the lagrangian density for this situation reads

\[
\mathcal{L} = \frac{1}{2} (\dot{\phi}^2 - \phi'^2) + e(\phi + \phi')(A_0 - A_1) + \frac{1}{2} (A_1^2 - A_0^2) + \frac{1}{2} e^2(a + \frac{m^2}{e^2})(A_0^2 - A_1^2)
\]

Note that the only difference between the model given in equation \([12]\) and the model considered in Sec.II lies in is in the masslike terms for gauge fields. To be more precise the difference between ambiguity (one loop correction) less situation and the situation where ambiguity (one loop correction) is taken into consideration lies in the masslike terms for gauge fields. The masslike term in the lagrangian \([12]\) now turns into \(\frac{1}{2} e^2(a + \frac{m^2}{e^2})(A_0^2 - A_1^2)\). A careful look reveals that \(m^2\) of Sec. II, is shifted to \(e^2(a + \frac{m^2}{e^2})\). All the other terms remains unaltered. So the analysis will also follow the same direction as it has been made in Sec. II. Like the case studied in \([26]\) the constraint structure will also be remain identical except the shifting of the parameter \(m^2\) to \(e^2(a + \frac{m^2}{e^2})\). It can be easily understood that there will be a shift in the mass of the massive field too because of this shifting of the parameter \(m^2\) in the masslike term of \([12]\). The square of the mass of the massive boson that will follow from the lagrangian \([12]\) will certainly be \(\tilde{m}^2 = e^2 \frac{m_0^2 + a^2 e^4}{m_0^2 + a^2 e^4 (a -1)}\). However the massless excitation remains as usual and which can be thought of as a de-confined fermion in \((1 + 1)\) dimension as usual.

V. CONCLUSION

In this paper we have studied the Thirring-Wess model replacing its vector interaction by the chiral one. Using the standard method of quantization of constrained system by Dirac \([27]\), we have quantized the model and has obtained an interesting Lorentz invariant theoretical spectrum. The spectrum contains a massive boson like the usual vector Schwinger model. A mass less boson has also been found to appear like chiral Schwinger model, however the model does not remain gauge symmetric like the Schwinger model. The point on which we would like to emphasize is that the chiral generation made here does not need any counter term in order to make it physically sensible like the chiral generation of the Schwinger model made in \([12]\). It is found from our analysis that unlike the previously proposed chiral generation of Schwinger model \([12]\) this model does not fail to loose unitarity and exactly solvable nature without any counter term containing ambiguity parameter, and as result, mass of the massive boson does not acquire any ambiguity unless a one loop correction is made. It is also contrary to the fact as it was found to happen in the chiral generation of Schwinger model \([12, 13]\) which was later termed as Chiral Schwinger model (jackiw -Rajaraman, version).
It is already mentioned that the gauge symmetry is absent here like its ancestor. Though the absence of gauge symmetry does not violate any physical principle nevertheless one may argue that a gauge symmetric theory is advantageous because it reflects an increased symmetry of the lagrangian, albeit it has to be remembered that what is increased is not a physical symmetry of the states but only a symmetry of the effective action which has to be broken by gauge fixing. Even though, if some emphasis is given on the symmetry of the effective action that also can be met here. There is certainly room for converting this gauge non-invariant model into a gauge invariant version and that too within the ambit of its own physical phase space. The method of converting a gauge non-invariant model to a gauge invariant one as suggested by of Mitra and Rajaraman \cite{32,33} may be useful in this context. The method of bringing back gauge symmetry of a gauge non-invariant model in the extend phase spacer is also well known. But here one needs to introduce the appropriate Wess-Zumino term \cite{34}. In this situation some extra fields enter within the model, however that extra fields allocate themselves within the un-physical sector of the theory.

The hamiltonian of the model under present consideration is positive definite that gives the signature of absence of non-unitary problem. However, the the formal proof of unitarity certainly comes from the BRST quantization and there is no problem to carry it out if the model provides positive definite hamiltonian. Therefore, this model satisfy the necessary requirements that seem to be essential for being a physically sensible model from whatever aspects it have been looked for.

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