DISTANCE DISTRIBUTION TO RECEIVED WORDS IN REED-SOLOMON CODES

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Abstract. In this paper, we obtain an asymptotic formula for the number of codewords with a fixed distance to a given received word of degree $k+m$ in the standard Reed-Solomon code $[q, k, q-k+1]_q$. Previously, explicit formulas were known only for the cases $m = 0, 1, 2$.

1. Introduction

In this paper we investigate the following question:

Problem 1.1. Let $C$ be a linear code over a finite field. Given a received word $u$, determine the distance distribution having $u$ as the center. That is, computing the number $N_i(u)$ of codewords in $C$ whose distance to $u$ is exactly $i$, where $i$ is a non-negative integer.

When the received word $u$ is a codeword, this is the classical weight distribution problem, which is generally hard and only well understood for certain special codes such as MDS codes. When the received word is not a codeword, this is the counting version of list decoding problem and is much harder and widely open even for standard Reed-Solomon codes. In this paper, we make the first attempt to study this problem and obtain an asymptotic formula for standard Reed-Solomon codes.

A significantly weaker version of our problem is computing the error distance from a received word $u$, that is, finding the smallest non-negative integer $i$ such that $N_i(u) > 0$. This is the decision version of the maximal likelihood decoding problem in coding theory. As Reed Solomon codes are constructed using polynomials, all such problems on Reed-Solomon codes can be reduced to polynomial factorization problems. Details will be explained in Section 2. To be more precise in this introduction, we now introduce some notations.

Let $F_q$ be the finite field of $q = p^t$ elements. Let $D = \{x_1, \cdots, x_n\} \subset F_q$ be a subset of cardinality $|D| = n > 0$. For $1 \leq k \leq n$, the Reed-Solomon code $RS_{n,k}$ has the codewords of the form

$$(f(x_1), \cdots, f(x_n)) \in F_q^n,$$

where $f$ runs over all polynomials in $F_q[x]$ of degree at most $k-1$. It is well known that the minimum distance of the Reed-Solomon code is $n-k+1$. If $D = F_q$ (or $F_q^*$), then the code $RS_{n,k}$ is called the standard (respectively the primitive) Reed-Solomon codes. All our results for standard Reed-Solomon codes extend to primitive Reed-Solomon codes with minor modification. For this reason, we shall focus on the standard Reed-Solomon codes in this paper.

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For any word $u = (u_1, u_2, \cdots, u_n) \in \mathbb{F}_q^n$, one can efficiently compute a unique polynomial $u(x) \in \mathbb{F}_q[x]$ of degree at most $n - 1$ such that $u(x_i) = u_i$, for all $1 \leq i \leq n$.

Explicitly, the polynomial $u(x)$ is given by the Lagrange interpolation formula

$$u(x) = \sum_{i=1}^{n} u_i \frac{\prod_{j \neq i}(x - x_j)}{\prod_{j \neq i}(x_i - x_j)}.$$ 

The degree $d(u)$ of $u$ is then defined as the degree of the associated polynomial $u(x)$. It is easy to see that $u$ is a codeword if and only if $d(u) \leq k - 1$.

For a given $u \in \mathbb{F}_q^n$, the distance from $u$ to the code $RS_{n,k}$ is defined to be

$$d(u, D_{n,k}) := \min_{v \in RS_{n,k}} d(u, v).$$

The maximum likelihood decoding of $u$ is to find a codeword $v \in RS_{n,k}$ such that $d(u, v) = d(u, RS_{n,k})$. Thus, computing $d(u, RS_{n,k})$ is essentially the decision version for the maximum likelihood decoding problem, which is NP-complete for general subset $D \subset \mathbb{F}_q$, see Guruswami-Vary [7] and Cheng-Murray [2]. For standard Reed-Solomon code with $D = \mathbb{F}_q$, the complexity of the maximum likelihood decoding is unknown to be NP-complete. This is an important open problem. It was shown by Cheng-Wan [3][4] that decoding the standard Reed-Solomon code is at least as hard as the discrete logarithm problem in a large extension of the finite field $\mathbb{F}_q$.

When $d(u) \leq k - 1$, then $u$ is a codeword and thus $d(u, RS_{n,k}) = 0$. We shall assume that $k \leq d(u) \leq n - 1$. The following simple result gives an elementary bound for $d(u, RS_{n,k})$.

**Theorem 1.2.** [11] Let $u \in \mathbb{F}_q^n$ be a word such that $k \leq d(u) \leq n - 1$. Then,

$$n - d(u) \leq d(u, D_{n,k}) \leq n - k.$$ 

The word $u$ is called a deep hole if $d(u, D_{n,k}) = n - k$, that is, it achieves the covering radius. When $d(u) = k$, $u$ is clearly a deep hole. This gives $(q - 1)q^k$ deep holes. For a general Reed-Solomon code $D_{n,k}$, it is already difficult to determine if a given word $u$ is a deep hole. In the special case that $d(u) = k + 1$, the deep hole problem is equivalent to the $(k + 1)$-subset sum problem over $\mathbb{F}_q$ which is NP-complete.

For the standard Reed-Solomon code, that is, $D = \mathbb{F}_q$ and thus $n = q$, there is the following interesting conjecture of Cheng-Murray [2].

**Conjecture** For the standard Reed-Solomon code with $D = \mathbb{F}_q$ and $p > 2$, the set $\{u \in \mathbb{F}_q^n | d(u) = k\}$ gives the set of all deep holes.

Many results were proved towards this conjecture. Please refer to [17][8] and the references there.

The deep hole problem is to determine when the upper bound in the above theorem agrees with $d(u, RS_{n,k})$. One is also interested in the situations when the lower bound $n - d(u)$ agrees with $d(u, RS_{n,k})$. It turns out that the lower bound agrees with $d(u, RS_{n,k})$ much more often. We call $u$ ordinary if $d(u, RS_{n,k}) = n - d(u)$. A basic problem is then to determine for a given word $u$, when $u$ is ordinary. This is equivalent to determining if $N_{n-d(u)}(u) > 0$. 

Since $k \leq d(u) \leq n - 1$, we can write $d(u) = k + m$ for some non-negative integer $m \leq n - k - 1$. Then, the word $u$ is represented uniquely by a polynomial $f(x) \in \mathbb{F}_q[x]$ of degree $k + m$. For $0 \leq r \leq k + m$, let $N_D(f(x), r)$ denote the number of polynomials $g(x) \in \mathbb{F}_q[x]$ with $\deg g(x) \leq k - 1$ such that $f(x) + g(x)$ has exactly $r$ distinct roots in $D$. It is clear that $N_i(u) = N_D(f(x), n - i)$. Thus, it is enough to study $N_D(f(x), r)$.

From now on, we only work with the standard Reed-Solomon codes $\mathcal{R}\mathcal{S}_{q,k}$. Since $D = \mathbb{F}_q$, we can write $N(f(x), r) = N_{\mathcal{R}\mathcal{S}}(f(x), r)$. In this case, $N_i(u) = N(f(x), q - i)$. Our distance distribution problem for the standard Reed-Solomon code is reduced the following number theoretic problem.

**Problem 1.3.** For a given polynomial $f(x) \in \mathbb{F}_q[x]$ of degree $k + m$, integer $0 \leq r \leq k + m$, count $N(f(x), r)$, the number of polynomials $g(x) \in \mathbb{F}_q[x]$ with $\deg g(x) \leq k - 1$ such that $f(x) + g(x)$ has exactly $r$ distinct roots in $\mathbb{F}_q$.

One may ask the decision version of this problem as follows:

**Problem 1.4.** For a given polynomial $f(x) \in \mathbb{F}_q[x]$ of degree $k + m$, and for an integer $0 \leq r \leq k + m$, determine if there is a polynomial $g(x) \in \mathbb{F}_q[x]$ with $\deg g(x) \leq k - 1$ such that $f(x) + g(x)$ has exactly $r$ distinct roots in $\mathbb{F}_q$.

Not many results are known about this problem. When $k$ is very small (say logarithmic in $q$), one can use the Chebotarev density theorem. This approach will be explained in Section 2. However, in coding theory applications, $k$ is the code dimension which can be as large as a linear function of $q$. The problem then becomes more difficult. We do expect nontrivial results for large $k$ if $m$ is not too large. This has been studied in the literature mostly in the extreme case $r = m + k$.

Some explicit formulas for $m \leq 2$ were known before.

### 1.1 Known Cases for $m \leq 2$

When $m < 0$, $f(x)$ represents a codeword and thus we may assume $f \equiv 0$, or equivalently $u = 0$. By a celebrating theorem of Mac Williams, for $0 \leq r \leq k - 1$, we have

$$N(0, r) = \begin{pmatrix} q \\ r \end{pmatrix} q^{k-r-1}(q - 1) \left( \sum_{j=0}^{k-r-1} (-1)^j \begin{pmatrix} q - r - 1 \\ j \end{pmatrix} q^{-j} \right).$$

When $m = 0$, then $\deg(f) = k$ and in this case $u$ is a deep hole. It was proved by A. Knopfmacher and J. Knopfmacher [10] that, for $0 \leq r \leq k$, we have

$$N(x^k, r) = \begin{pmatrix} q \\ r \end{pmatrix} q^{k-r} \left( \sum_{j=0}^{k-r} (-1)^j \begin{pmatrix} q - r \\ j \end{pmatrix} q^{-j} \right).$$

When $m = 1$, we have $\deg(f) = k + 1$. We may assume $f(x) = x^{k+1} + ax^k$. It turns out that $N(x^{k+1} + ax^k, r)$ depends on $a$. It was proved by Zhou, Wang and Wang [16] that when $p \nmid k + 1$, then for $0 \leq r \leq k + 1$,

$$N(x^{k+1} + ax^k, r) = \begin{pmatrix} q \\ r \end{pmatrix} q^{k-r} \left( \sum_{j=0}^{k-r} (-1)^j \begin{pmatrix} q - r \\ j \end{pmatrix} q^{-j} \right).$$
and when \( p \mid k + 1 \), then for \( 0 \leq r \leq k + 1 \), the number \( N(x^{k+1} + ax^k, r) \) is given by

\[
\binom{q^r}{r} q^{k-r} \left( \sum_{j=0}^{k-r} (-1)^j \binom{q-r}{j} q^{-j} \right) + (-1)^{k+1} \frac{v(a)}{q} \frac{k+1}{r} \left( \frac{q/p}{(k+1)/p} \right),
\]

where \( v(a) \) is a small constant depending on \( a \).

When \( m = 2 \), then \( \text{deg}(f) = k + 2 \) and an explicit but complicated counting formula is also given in [16]. Details are omitted. When \( m > 2 \), it is no longer reasonable to expect an explicit formula for \( N(f(x), r) \), but we can hope for an asymptotic formula. This is the aim of the present paper.

### 1.2. Main Result

We obtain the following counting formula which holds for a large range of \( k, m, \) and \( r \).

**Theorem 1.5.** Let \( f(x) \in \mathbb{F}_q[x] \) be a polynomial of \( \text{deg}(f) = k + m \leq n - 1 \). Let \( N(f, r) \) be the number of polynomials \( g(x) \in \mathbb{F}_q[x] \) with \( \text{deg} g(x) = k - 1 \) such that \( f(x) + g(x) \) has exactly \( r \) distinct roots in \( \mathbb{F}_q \). For an integer \( d \geq 0 \), define \( \mu_d = \sum_{j=0}^{d} (-1)^j \binom{q-r}{j} q^{-j} \). Then for \( 0 \leq r \leq k + m \), we have the following asymptotic formula:

\[
\left| N(f, r) - \mu_{k+m-r} \binom{q^r}{r} q^{k-r} \right| \leq \sum_{j=k+1}^{k+m} \binom{j}{r} \left( \frac{q/p + m\sqrt{q} + j}{j} \right) \left( \frac{m-1}{k+m-j} \right) \sqrt{q}^{k+m-j}.
\]

The number of non-zero error terms in the error estimate is \( k + m - \max\{k+1, r\} \). This means that if either \( m \) is small or if \( k + m - r \) is small, then there are only a few terms in the error estimate. The theorem also becomes stronger in the case \( q = p \) is a prime since then the number \( q/p - 1 \) becomes zero. We now derive a few corollaries and explain how they are related to previous results.

When \( m = 0 \), we may suppose \( f(x) = x^k \). In this case, there is no error term in our asymptotic formula and we obtain the following corollary. As reported in the above known cases, this corollary was first proved by A. Knopfmacher and J. Knopfmacher [10].

**Corollary 1.6.**

\[
N(x^k, r) = \binom{q^r}{r} q^{k-r} \left( \sum_{j=0}^{k-r} (-1)^j \binom{q-r}{j} q^{-j} \right).
\]

When \( r = k + m \), there is only one term in the error estimate and we obtain the following corollary, which was first proved by Li and Wan in 2010 [12].

**Corollary 1.7.** Let \( f(x) \in \mathbb{F}_q[x] \) be a polynomial of \( \text{deg}(f) = k + m \leq n - 1 \). Then,

\[
\left| N(f(x), k + m) - \frac{1}{q^m} \left( \frac{q}{k + m} \right) \right| \leq \left( \frac{q/p + m\sqrt{q} + k + m}{k + m} \right).
\]

When \( r = k + m - 1 \), there are two terms in the error estimate and we obtain the following corollary, which is already a new result.
Corollary 1.8. Let \( f(x) \in \mathbb{F}_q[x] \) be a polynomial of \( \deg(f) = k + m \leq n - 1 \). Then,
\[
N(f(x), k + m - 1) - \frac{k + m - 1}{q^m} \binom{q}{k + m - 1} \\
\leq \left( \frac{q/p + m\sqrt{q} + k + m}{k + m} \right) ((m-1)\sqrt{q} + k + m).
\]

From the definition of RS codes, we immediately obtain the following result.

Corollary 1.9. Let \( D_{q,k} \) be the standard RS code. Let the received word \( u \) be of degree \( k + m \), where \( 0 \leq m \leq q - k - 1 \). Let \( N_i(u) \) denote the number of codewords \( v \) with distance \( d(u,v) = i \). If \( i \notin [q - (k + m), q - k] \), then \( N_i(u) = 0 \).

If \( i \in [q - (k + m), q - k] \), then
\[
N_i(u) - \mu_{k+m+i-q} \binom{q}{i} q^{k+i-q} \\
\leq \sum_{k+1 \leq j \leq k+m} \binom{j}{q-i} \left( \frac{q/p + m\sqrt{q}}{j} \right) \binom{m-1}{k+m-j} \sqrt{q}^{k+m-j},
\]

where \( \mu_d = \sum_{j=0}^{d} (-1)^j \binom{i}{j} q^{-j} \).

Beside coding theory, our result has other applications in number theory and graph theory. In number theory, it is a classical problem to understand the factorization pattern of a family of polynomials. In graph theory, it is related to the spectrum distribution of Wenger type graphs, see [1].

2. Main Results for Small \( k + m \)

Without loss of generality, we may assume that the polynomial \( f(x) \) representing the received word \( u \) is monic. That is,
\[
f(x) = x^{k+m} + a_1 x^{k+m-1} + \cdots + a_m x^k + \cdots \in \mathbb{F}_q[x].
\]

Let
\[
g_b(x) = b_{k-1} x^{k-1} + b_{k-2} x^{k-2} + \cdots + b_0
\]
be a polynomial representing a typical codeword. We view
\[
F(x) = f(x) + g_b(x) \in \mathbb{F}_q(b_0, b_2, \ldots, b_{k-1})\langle x \rangle
\]
as a polynomial in \( x \) over the rational function field \( \mathbb{F}_q(b_0, b_2, \ldots, b_{k-1}) \) in \( k \) variables of degree \( k + m \). For \( k \geq 2 \), it is well known by Cohen [4] and Katz [3] that the geometric Galois group
\[
G_F := \text{Galois}(F(x)/\mathbb{F}_q(b_0, b_2, \ldots, b_{k-1})) = S_{k+m}
\]
is the full symmetric group. By the effective Chebotarev density theorem as in Cohen [4], one deduces
\[
N(f(x), r) = \nu_r(k + m) q^k + O((k + m)! q^{k-\frac{1}{2}}),
\]
where
\[
\nu_r(k + m) = \frac{\text{number of permutations in } S_{k+m} \text{ having exactly } r \text{ fixed points}}{(k + m)!}.
\]
Applying Lemma 3.1, we have
\[
\nu_r(k + m) = \frac{1}{r!} \left( \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^{k+m-r} \frac{1}{(k+m-r)!} \right).
\]
This shows that \( N(f(x), r) > 0 \) if \( k + m = O(\sqrt{\log q}) \). The last condition implies that \( k + m \) is very small compared to \( q \). As we mentioned before, in coding theory applications, we would like to have \( k \) as large as a linear function of \( q \). Our main result of this paper is a non-trivial asymptotic formula when \( m \) is small but \( k \) can be suitably large.

3. Proof of Theorem 1.4

The proof of the following key lemma is given in Section 7.

**Lemma 3.1.** Let \( f(x) \in \mathbb{F}_q[x] \) be a monic polynomial of \( \deg(f) = k + m \leq q - 1 \). Let \( M(f(x), r) \) denote the number of pairs \((D_r, g(x))\) with \( D_r \) being a \( r \)-subset in \( \mathbb{F}_q \), and \( g(x) \in \mathbb{F}_q[x] \) of degree at most \( k - 1 \) satisfying
\[
(f(x) + g(x))|_{D_r} \equiv 0.
\]
Then for \( k + 1 \leq r \leq k + m \), we have
\[
M(f(x), r) - \left( \frac{q}{r} \right) q^{k-r} \leq \left( \frac{q}{p} + m \sqrt{q} + r \right) \left( \frac{m - 1}{k + m - r} \right) \sqrt{q}^{k+m-r}.
\]

**Proof of Theorem 1.4.** We shall let \( g(x) \in \mathbb{F}_q[x] \) denote a polynomial of degree at most \( k - 1 \). For \( c \in \mathbb{F}_q \), let \( P_c \) denote the property that \( f(x) + g(x) \) has \( c \) as its root. For a subset \( C \subseteq \mathbb{F}_q \), let \( N_C \) be the number of \( g(x) \) such that \( f(x) + g(x) \) has property \( P_c \) for each \( c \in C \). The \(|C| \times |C|\) Vandermonde matrix formed using the elements of \( C \) is non-singular. It follows by linear algebra that for \(|C| \leq k \), we have \( N_C = q^{k-|C|} \). In the case \( r = 0 \), the inclusion-exclusion sieving [14] implies that
\[
N(f(x), 0) = q^k - \sum_{c \in \mathbb{F}_q} N_{\{c\}} + \cdots + (-1)^{k+m} \sum_{c_1, c_2, \ldots, c_{k+m} \in \mathbb{F}_q} N_{\{c_1, c_2, \ldots, c_{k+m}\}}
\]
\[
= q^k - \left( \frac{q}{1} \right) q^{k-1} + \left( \frac{q}{2} \right) q^{k-2} - \cdots + (-1)^k \left( \frac{q}{k} \right) q^0 + \sum_{k+1 \leq j \leq k+m} (-1)^j N_j,
\]
where \( N_j \) is the number of pairs \((D_j, g(x))\) with \( D_j \) being a \( j \)-subset in \( \mathbb{F}_q \) satisfying
\[
(f(x) + g(x))|_{D_j} \equiv 0.
\]
Applying Lemma 3.1 we have
\[
\left| N(f(x), 0) - \sum_{i=0}^{k+m} (-1)^i q^{k-i} \left( \frac{q}{i} \right) \right| \leq \sum_{k+1 \leq j \leq k+m} \left( \frac{q/p + m \sqrt{q} + j}{j} \right) \left( \frac{m - 1}{k + m - j} \right) \sqrt{q}^{k+m-j}.
\]
For general \( 0 \leq r \leq k + m \), using the weighted inclusion-exclusion sieving formula, we deduce
\[
N(f(x), r) = \sum_{c_1, c_2, \ldots, c_r \in \mathbb{F}_q} N_{\{c_1, c_2, \ldots, c_r\}} - \left( \frac{r + 1}{r} \right) \sum_{c_1, c_2, \ldots, c_{r+1} \in \mathbb{F}_q} N_{\{c_1, c_2, \ldots, c_{r+1}\}} + \cdots
\]
\[
= \sum_{j=r}^{k} (-1)^{j-r} \left( \frac{j}{r} \right) \left( \frac{q}{j} \right) q^{k-j} + \sum_{k+1 \leq j \leq k+m} (-1)^{j-r} \left( \frac{j}{r} \right) N_j.
\]
Applying Lemma 4.1 again, we have
\[ N(f(x), r) - \sum_{j=r}^{k+m} \binom{j}{r} \left( \frac{q}{j} (-1)^{j-r} q^{k-j} \right) \leq \sum_{k+1 \leq j \leq k+m} \binom{j}{r} \left( \frac{q}{j} + m\sqrt{q} + j \right) \binom{m-1}{k+m-j} \sqrt{q}^{k+m-j}. \]

By the elementary properties of binomials, the main term can be rewritten and we obtain the following final form
\[ N(f(x), r) - \left( \frac{q}{r} \right)^{k-r} \left( \sum_{j=0}^{k-m-r} (-1)^j \binom{q-r}{j} q^{-j} \right) \leq \sum_{k+1 \leq j \leq k+m} \binom{j}{r} \left( \frac{q}{j} + m\sqrt{q} + j \right) \binom{m-1}{k+m-j} \sqrt{q}^{k+m-j}. \]

\[ \square \]

4. A DISTINCT COORDINATE SIEVING FORMULA

In this section we introduce a sieving formula, which is a useful technique and might have its own interests. Roughly speaking, this formula significantly improves the classical inclusion-exclusion sieve in the distinct coordinates counting problems. We cite it here without proof. For details and related applications please refer to [12, 13].

Let \( \Omega \) be a finite set, and let \( \Omega^k \) be the Cartesian product of \( k \) copies of \( \Omega \). Let \( X \) be a subset of \( \Omega^k \). Define \( \overline{X} = \{(x_1, x_2, \ldots, x_k) \in X \mid x_i \neq x_j, \forall i \neq j\} \). Let \( f(x_1, x_2, \ldots, x_k) \) be a complex valued function defined over \( X \) and
\[ F = \sum_{x \in \overline{X}} f(x_1, x_2, \ldots, x_k). \]

Let \( S_k \) be the symmetric group on \( \{1, 2, \ldots, k\} \). Each permutation \( \tau \in S_k \) factorizes uniquely as a product of disjoint cycles and each fixed point is viewed as a trivial cycle of length 1. Two permutations in \( S_k \) are conjugate if and only if they have the same type of cycle structure (up to the order). For \( \tau \in S_k \), define the sign of \( \tau \) to be \( \text{sign}(\tau) = (-1)^{k-l(\tau)} \), where \( l(\tau) \) is the number of cycles of \( \tau \) including the trivial cycles. For a permutation \( \tau = (i_1 i_2 \cdots i_{a_1})(j_1 j_2 \cdots j_{a_2}) \cdots (l_1 l_2 \cdots l_{a_s}) \) with \( 1 \leq a_i, 1 \leq i \leq s \), define
\[ X_{\tau} = \{(x_1, \ldots, x_k) \in X, x_{i_1} = \cdots = x_{i_{a_1}}, \ldots, x_{i_s} = \cdots = x_{i_{a_s}}\}. \] (4.1)

Similarly, for \( \tau \in S_k \), define \( F_{\tau} = \sum_{x \in X_{\tau}} f(x_1, x_2, \ldots, x_k) \). Now we can state our sieve formula. We remark that there are many other interesting corollaries of this formula. For interested reader we refer to [12].

**Theorem 4.1.** Let \( F \) and \( F_{\tau} \) be defined as above. Then
\[ F = \sum_{\tau \in S_k} \text{sign}(\tau) F_{\tau}. \] (4.2)

Note that the symmetric group \( S_k \) acts on \( \Omega^k \) naturally by permuting coordinates. That is, for \( \tau \in S_k \) and \( x = (x_1, x_2, \ldots, x_k) \in \Omega^k \), \( \tau \circ x = (x_{\tau(1)}, x_{\tau(2)}, \ldots, x_{\tau(k)}) \).
A subset $X$ in $\Omega^k$ is said to be symmetric if for any $x \in X$ and any $\tau \in S_k$, $\tau \circ x \in X$. For $\tau \in S_k$, denote by $\mathcal{C}$ the conjugacy class determined by $\tau$ and it can also be viewed as the set of permutations conjugate to $\tau$. Conversely, for given conjugacy class $\mathcal{C} \in C_k$, denote by $\tau$ a representative permutation of this class. For convenience we usually identify these two symbols.

In particular, if $X$ is symmetric and $f$ is a symmetric function under the action of $S_k$, we then have the following simpler formula than (4.2).

**Corollary 4.2.** Let $C_k$ be the set of conjugacy classes of $S_k$. If $X$ is symmetric and $f$ is symmetric, then

$$F = \sum_{\tau \in C_k} \text{sign}(\tau) C(\tau) F_\tau,$$

where $C(\tau)$ is the number of permutations conjugate to $\tau$.

5. Bounds on Exponential Sums

Let $f(x) \in \mathbb{F}_q[x]$ be a monic polynomial of degree $n > 0$. Let $\chi$ be a group homomorphism from $(\mathbb{F}_q[x]/(f(x)))^*$ to $\mathbb{C}^*$. We extend this definition to $\mathbb{F}_q[x]/(f(x))$ by defining $\chi(g) = 0$ for $(g, f) \neq 1$. Define

$$M_k(\chi) = \sum_{g \in \mathbb{F}_q[x], \text{monic}, \deg(g) = k} \chi(g).$$

**Lemma 5.1.** Assume that $\chi$ is non-trivial. Then for $k \geq 0$,

$$|M_k(\chi)| \leq \binom{n-1}{k} \sqrt{q^k}.$$

Furthermore, if $\chi(\mathbb{F}_q^*) = 1$, then for $n \geq 2$, we have

$$\left| \sum_{g \in \mathbb{F}_q[x], \text{monic}, \deg(g) \leq k} \chi(g) \right| \leq \binom{n-2}{k} \sqrt{q^k}.$$

**Proof.** The Dirichlet L-function of $\chi$ is

$$L(\chi, t) = \sum_{g \in \mathbb{F}_q[x], \text{monic}} \chi(g) t^{\deg(g)}$$

$$= \sum_{k=0}^{\infty} M_k(\chi) t^k \in 1 + t\mathbb{C}[t].$$

If $k \geq n$, for monic $g$ of degree $k$, we can write uniquely $g = g_1 f + h$, where $g_1$ is monic in $\mathbb{F}_q[x]$, $\deg(g_1) = k - n$ and $h \in \mathbb{F}_q[x], \deg(h) \leq n - 1$. Thus in this case,

$$M_k(\chi) = \sum_{g_1 \in \mathbb{F}_q[x], \text{monic}, \deg(g_1) = k-n} \sum_{\deg(h) \leq n-1} \chi(h)$$

$$= q^{k-n} \sum_{h \in \mathbb{F}_q[x]/(f(x))} \chi(h)$$

$$= 0.$$

This implies

$$L(\chi, t) = \prod_{i=1}^{r} (1 - \rho_i t)$$
is a polynomial of degree \( \leq n - 1 \), i.e., \( r \leq n - 1 \). By the Weil bound \([15]\),

\[ |\rho_i| \leq \sqrt{q}. \]

It follows that for \( 0 \leq k \leq n - 1 \),

\[ |M_k(\chi)| \leq \binom{r}{k} \sqrt{q}^k \leq \left( \frac{n - 1}{k} \right) \sqrt{q}^k. \]

Now, note that \( \sum_{g \in \mathbb{F}_q[x], \text{monic, deg}(g) \leq k} \chi(g) \) is the coefficient of \( T^k \) in \( L(\chi, T)/(1 - T) \). Since \( \chi \) is a non-trivial character but trivial on \( \mathbb{F}_q^* \), \( L(\chi, T) \) has the trivial factor \( (1 - T) \), which means that \( L(\chi, T)/(1 - T) \) is a polynomial of degree \( n - 2 \), see \([15]\). It follows that

\[ \sum_{g \in \mathbb{F}_q[x], \text{monic, deg}(g) \leq k} \chi(g) \leq \left( \frac{n - 2}{k} \right) \sqrt{q}^k. \]

\[ \square \]

6. Proof of Lemma 6.1

Fix a polynomial \( f(x) \in \mathbb{F}_q[x] \) with degree \( d \) and assume \( f(0) = 1 \). Let \( r, s \) be integers such that \( 0 \leq r \leq d \) and \( 0 \leq s < d \). For \( k \geq 0 \), let \( P_k \) denote the set of all polynomials \( h(x) \in \mathbb{F}_q[x] \) of degree at most \( k \) with \( h(0) = 1 \). Let \( N_2 \) be defined by

\[ N_2 = \# \{(x_1, \ldots, x_r, h) \in \mathbb{F}_q^r \times P_{d-r}| (1 - x_1 x) \cdots (1 - x_r x) h(x) \equiv f(x)(\text{mod } x^{s+1}) \}, \]

where we require that the \( x_i \)'s are distinct. Let \( \chi \) be a character from \((\mathbb{F}_q[x]/(x^{s+1}))^*\) to \( \mathbb{C}^* \). We extend this definition to \( \mathbb{F}_q[x]/(x^{s+1}) \) by defining \( \chi(g) = 0 \) for \( g, x^{s+1} \neq 1 \). Let \( G \) denote the group of all characters \( \chi \) such that \( \chi(\mathbb{F}_q^*) = 1 \). This is an abelian group of order \( |G| = q^s \).

**Lemma 6.1.** Then

\[ |N_2 - q^{d-s-r}(q)_r| \leq (q/p + s\sqrt{q} + r - 1)_r \left( \frac{s - 1}{d - r} \right) \sqrt{q}^{l-r}. \]

**Proof.**

\[ N_2 = \frac{1}{q^s} \sum_{(x_1, \ldots, x_r) \in \mathbb{F}_q^r, x_i \neq x_j, h \in P_{d-r}} \sum_{\chi \in G} \chi((1 - x_1 x) \cdots (1 - x_r x) h(x)/f(x)) \]

\[ = (q)_r q^{d-s-r} \cdot \frac{1}{q^s} \sum_{1 \neq \chi \in G} \chi^{-1}(f(x)) \sum_{(x_1, \ldots, x_r) \in \mathbb{F}_q^r, x_i \neq x_j, h \in P_{d-r}} \chi((1 - x_1 x) \cdots (1 - x_r x) h(x)). \]

For each \( \chi \), the function \( \chi((1 - x_1 x) \cdots (1 - x_r x)) \) is clearly symmetric in the \( x_i \)'s. Recall that for a permutation \( \tau = (i_1 i_2 \cdots i_s)(j_1 j_2 \cdots j_s) \cdots (l_1 l_2 \cdots l_s) \) in the symmetric group \( S_r \) with \( 1 \leq a_i, 1 \leq i \leq s \), the subset \( X_\tau \) of \( X = \mathbb{F}_q^r \) is defined as

\[ X_\tau = \{(x_1, \ldots, x_r) \in \mathbb{F}_q^r, x_i = \cdots = x_{i_{a_i}}, \ldots, x_1 = \cdots = x_{i_s}\}. \] (6.1)

Then a complex function \( F_\tau(\chi) \) is defined:

\[ F_\tau(\chi) = \sum_{(x_1, \ldots, x_r) \in X_\tau} \sum_{h \in P_{d-r}} \chi((1 - x_1 x) \cdots (1 - x_r x) h(x)). \]

Thus by the sieving formula \([12]\), one has

\[ N_2 = (q)_r q^{d-s-r} + \frac{1}{q^s} \sum_{1 \neq \chi \in G} \chi^{-1}(f(x)) \sum_{\tau \in S_r} \text{sign}(\tau) F_\tau(\chi). \]
Thus it suffices to estimate $F_\tau(\chi)$ for non-trivial $\chi$, where

$$F_\tau(\chi) = \left( \sum_{(x_1, \ldots, x_r) \in X_r} \prod_{i=1}^r \chi(1 - x_i) \right) \cdot \left( \sum_{h \in \mathbb{F}_q^*} \chi(h) \right).$$

We first estimate the second factor. Since $\chi$ is non-trivial, $\chi(\mathbb{F}_q^*) = 1$ and $\chi(x) = 0$, by Lemma 5.1 we deduce

$$| \sum_{h \in \mathbb{F}_q^*} \chi(h(x)) | = \sum_{h \in \mathbb{F}_q^*[x], \text{monic}, \deg(h) \leq d-r} \chi(h(x)) \leq (s-r) \sqrt{q}^{d-r}.$$

To estimate the first factor, we suppose $\tau$ is of type $(c_1, c_2, \ldots, c_r)$, where $c_i$ is the number of $i$-cycles in $\tau$ for $1 \leq i \leq r$. Then the first factor is

$$G_\tau(\chi) = \left( \sum_{(x_1, \ldots, x_r) \in X_r} \prod_{i=1}^r \chi(1 - x_i) \right)$$

$$= \sum_{a \in \mathbb{F}_q} \left( \sum_{i=1}^r \chi(a(1 + at)) c_i \left( \sum_{a \in \mathbb{F}_q} \chi^2(a(1 + at))^c \prod_{i=1}^r \chi^i(a(1 + at))^c \right) \right)$$

$$= \prod_{i=1}^r \sum_{a \in \mathbb{F}_q} \chi^i(a(1 + at)) c_i.$$

Define $m_i(\chi) = 1$ if $\chi^i = 1$ and $m_i(\chi) = 0$ if $\chi^i \neq 1$. By the Weil bound [15], we deduce that

$$|G_\tau| \leq q \sum_{i=1}^r c_i m_i(\chi) (s \sqrt{q}) \sum_{i=1}^r c_i (1 - m_i(\chi)).$$

Since $F_\tau(\chi)$ is symmetric, we have

$$N_2 - (q)_r q^{d-r-s} = \frac{1}{q^s} \sum_{1 \neq \chi \in G} \chi^{-1}(f(x)) \sum_{\tau \in S_r} \text{sign} (\tau) F_\tau(\chi)$$

$$= \frac{1}{q^s} \sum_{1 \neq \chi \in G} \chi^{-1}(f(x)) \sum_{\tau \in C_r} \text{sign} (\tau) C(\tau) F_\tau(\chi)$$

$$= \frac{1}{q^s} \sum_{\chi \neq 1, \chi^d = 1, \text{for some} 2 \leq d \leq r} \chi^{-1}(f(x)) \sum_{\tau \in C_r} \text{sign} (\tau) C(\tau) F_\tau(\chi)$$

$$+ \frac{1}{q^s} \sum_{\chi \neq 1, \chi^d = 1, \text{for some} 2 \leq d \leq r} \chi^{-1}(f(x)) \sum_{\tau \in C_r} \text{sign} (\tau) C(\tau) F_\tau(\chi).$$

Let $S = \# \{ 1 \neq \chi \in G | \chi^d = 1 \text{ for some} 2 \leq d \leq r \}$. The last two terms were estimated by a combinatorial counting argument, see Li-Wan [13]. We obtain

$$|N_2 - (q)_r q^{d-r-s}| \leq \left( \frac{q^s - S}{q^s}((s-1)\sqrt{q} + r - 1)_r + \frac{S}{q^s} (q/p + (s-1)\sqrt{q} + r - 1)_r \right) \left( \frac{s-1}{d-r} \right) \sqrt{q}^{d-r}.$$

If $S$ is empty, we have the stronger estimate

$$|N_2 - (q)_r q^{d-r-s}| \leq ((s-1)\sqrt{q} + r - 1)_r \left( \frac{s-1}{d-r} \right) \sqrt{q}^{d-r}.$$
In general, we have the weaker estimate
\[ |N_2 - (q)_r q^{d-r-s}| \leq \left( \frac{q/p + (s - 1)\sqrt{q} + r - 1}{d-r} \right) \left( \frac{s-1}{d-r} \right) \sqrt{q^{d-r}}. \]

\[ \square \]

Similarly, if we consider the counting problem in \( \mathbb{F}_q^* \), then we will have a slightly different formula.

**Lemma 6.2.** Let \( N_2^* \) be defined by
\[ N_2^* = \#\{(x_1, \ldots, x_r), h) \in \mathbb{F}_q^r \times P_{d-r}, (1 - x_1) \cdots (1 - x_r)h(x) \equiv f(x)(mod x^{s+1}) \}, \]
where we require that the \( x_i \)’s are distinct. Then
\[ |N_2^* - q^{d-s-r}(q-1)_r| \leq \left( \frac{(q-1)/p + s\sqrt{q} + r - 1}{d-r} \right) \left( \frac{s-1}{d-r} \right) \sqrt{q^{d-r}}. \]

Now, we assume that \( f(x) \in \mathbb{F}_q[x] \) is a monic polynomial of degree \( d \). Suppose the top \( s \) coefficients of \( f \) are \( \alpha = (a_{d-1}, \ldots, a_{d-s}) \), i.e.,
\[ f^\alpha(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_{d-s}x^{d-s} + \cdots. \]
For integer \( k \geq 0 \), let \( \mathbb{F}_q[x]/k \) denote the set of polynomials \( g \in \mathbb{F}_q[x] \) of degree at most \( k \).

**Theorem 6.3.** Let \( N_\alpha^* \) be the number of pairs \( (D_r, g(x)) \), where \( D_r \) is an \( r \)-subset of \( \mathbb{F}_q \) and \( \mathbb{F}_q[x]r-d-s \) satisfying \( (f^\alpha + g)|_{D_r} \equiv 0 \). If \( r \geq d - s \), then
\[ |N_\alpha^* - q^{d-r-s}(q\alpha_1)| \leq 2\left( \frac{q/p + s\sqrt{q} + r}{d-r} \right) \left( \frac{s-1}{d-r} \right) \sqrt{q^{d-r}}. \]

**Proof.** Note that \( N_\alpha^* = N_\alpha^*(q, s) \) equals the number of pairs \( (D_r, g(x)) \), where \( D_r = (x_1, \ldots, x_r) \) is an \( r \)-subset of \( \mathbb{F}_q \) and \( g \in \mathbb{F}_q[x]r-d-s \) such that there is a unique monic \( w(x) \in \mathbb{F}_q \) of degree \( d - r \) satisfying
\[ x^d + a_{d-1}x^{d-1} + \cdots + a_{d-s}x^{d-s} + g(x) = (x - x_1) \cdots (x - x_r)w(x). \quad (6.2) \]

Clearly \( N_\alpha^*(q, s) = N_\alpha^* - (d - 1, s) \), where \( N_\alpha^* \) equals the number of such pairs \( (D_r, g(x)) \) with \( D_r \subseteq \mathbb{F}_q^* \) and \( N_\alpha^* \) equals the number of such pairs \( (D_r, g(x)) \) with \( D_r \) containing 0.

Suppose \( x_1 = 0 \), by dividing \( x \) on both sides of (6.2), it is easy to check \( N_\alpha^* - (d, s) = N_\alpha^*(d, s) \). It then suffices to count \( N_\alpha^*(d, s) \).

Since now we have \( x_1 \in \mathbb{F}_q^* \), Substitute \( x \) by \( 1/x \) one has
\[ \frac{1}{x^d} + a_{d-1}\frac{1}{x^{d-1}} + \cdots + a_{d-s}\frac{1}{x^{d-s}} + g\left( \frac{1}{x} \right) = (1/x - x_1)(1/x - x_2) \cdots (1/x - x_r)w(1/x). \]

Multiplying \( x^d \) on both sides we then have
\[ 1 + a_{d-1}x + \cdots + a_{d-s}x^s + x^dg\left( \frac{1}{x} \right) = (1 - x_1)(1 - x_2) \cdots (1 - x_r)x^{d-r}w(\frac{1}{x}). \]

Note that \( h(x) = x^{d-r}w\left( \frac{1}{x} \right) \) is a polynomial of degree \( \leq d - r \), \( x^d g\left( \frac{1}{x} \right) \) is a polynomial divisible by \( x^{s+1} \) and degree bounded by \( d \). It suffices to count the number of pairs \( (D_r, h(x)) \), where \( D_r = (x_1, \ldots, x_r) \) is an \( r \) subset of \( \mathbb{F}_q^* \) and \( h(x) \in \mathbb{F}_q[x] \) of degree \( \leq d - r \) such that
\[ 1 + a_{d-1}x + \cdots + a_{d-s}x^s \equiv (1 - x_1)(1 - x_2) \cdots (1 - x_r)h(x)(mod x^{s+1}). \]
Thus, if we let $N^*_2$ be defined as in Lemma 6.2, then

$$N_{r}^{α,1}(d, s) = \frac{1}{r!}N^*_2.$$  

By the estimate in Lemma 6.2 we have

$$\left| N_{r}^{α,1}(d, s) - q^{d-r-s}\binom{q-1}{r} \right| \leq \left( \frac{(q-1)/p + s\sqrt{q} + r - 1}{r} \right) \left( \frac{s-1}{d-r} \right) \sqrt{q}^{d-r}.$$  

We conclude

$$\left| N_{r}^{α} - q^{d-r-s}\binom{q-1}{r} + (q-1)\binom{q-1}{r-1} \right| \leq \left( \frac{(q-1)/p + s\sqrt{q} + r - 1}{r} \right) \left( \frac{s-1}{d-r} \right) \sqrt{q}^{d-r} + \left( \frac{(q-1)/p + s\sqrt{q} + r - 2}{r-1} \right) \left( \frac{s-1}{d-r} \right) \sqrt{q}^{d-r}.$$  

$$\leq \left( \frac{(q-1)/p + s\sqrt{q} + r}{r} \right) \left( \frac{s-1}{d-r} \right) \sqrt{q}^{d-r}.$$

□

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