Characterization of split distinguished triangles

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Abstract. Let \( A \) be a finite dimensional \( k \)-algebra over an algebraically closed field \( k \). The aim of this paper is to show an equivalent condition on parameterizations of complexes of projective \( A \)-modules for existence of short exact sequences of complexes of projective \( A \)-modules. Moreover, we show an equivalent condition for split distinguished triangle \( U \to W \to V \to U[1] \) related to the difference between dimension of \( \text{Hom}(U[i] \oplus V[i], X) \) and dimension of \( \text{Hom}(W[i], X) \) for every \( X \) object in triangulated category.

1. Introduction

Representation theory has been successfully studied parametrization of modules over \( k \)-algebras. One can thereby regard left \( A \)-modules structure on vector space \( k^d \) are nothing but sequences of \( d \times d \)-matrices which satisfy some relations. When view as elements of affine space \( \mathbb{A}^N \), that sequences of matrices are roots of some polynomials simultaneously. That variety is named Module Variety which denote by \( \text{mod}_d^A \). To this variety one can define an action of general linear group \( GL_d(k) \) by conjugation whose orbits corresponding to isomorphism classes of \( d \)-dimensional left \( A \)-modules.

The framework of \( A \)-modules was extended to objects of derived category of \( A \)-modules. Jensen, Su, and Zimmermann [2] defined a topological space parameterizing right bounded complexes of projective \( A \)-modules \( \text{comproj}_d^A \) with fixed dimension array \( d \). The topological group \( G_d \cong \prod_{i \in \mathbb{Z}} \text{Aut}_A \left( \bigoplus_{j=1}^{d_{ij}} P^d_j \right) \) acting on it so that orbits correspond to quasi-isomorphism classes of right bounded complexes of projective \( A \)-modules in the category of complexes.

In module category, Zwara [4] showed that the existence of short exact sequence \( 0 \to U \to V \to W \to 0 \) if \( \begin{pmatrix} u & z \\ 0 & v \end{pmatrix} \) is a block matrix. In this paper, we generalize it in the frame of derived category. Moreover, we show an equivalent condition for split distinguished triangle \( U \to W \to V \to U[1] \) related to the difference between dimension of \( \text{Hom}(U[i] \oplus V[i], X) \) and dimension of \( \text{Hom}(W[i], X) \) for every \( X \) object in triangulated category.

2. Basic Definitions

Let \( A \) be a finite dimensional algebra over an algebraically closed field \( k \). Let \( \text{mod}_d^A \) denote the affine variety of \( d \)-dimensional left \( A \)-modules. To this variety one can define an action of general linear group \( GL_d(k) \) by conjugation whose orbits corresponding to isomorphism classes
of $d$-dimensional left $A$-modules.

Let $P_1, \ldots, P_l$ be a complete set of representatives of isomorphism classes of indecomposable projective $A$-modules. For an element $\mathbf{d} = (d_1, \ldots, d_l) \in \mathbb{N}^l$ define $\alpha(\mathbf{d}) = \sum_{i=1}^{l} d_i \dim_k P_i$ the dimension of $\bigoplus_{i=1}^{l} P_{ij}^{d_{ij}} \in \text{mod}_A(\mathbf{d})$. Now for a matrix $\mathbf{d} = (d_{ij})$ with $l$ columns and coefficients in $\mathbb{N}$ defined $\mathbf{d}_i$ to be $i$-th row of $\mathbf{d}$ and defined $\text{comproj}_A^{\mathbf{d}}$ to be the subset of

$$
\left( \prod_{i \in \mathbb{Z}} \text{mod}_A^{\mathbf{d}_{ii}} \right) \times \left( \prod_{i \in \mathbb{Z}} \text{Hom}_k \left( k^{\alpha(\mathbf{d})}, k^{\alpha(\mathbf{d}_{i-1})} \right) \right)
$$

consisting elements $M_\bullet = (M_i, \partial_i)_{i \in \mathbb{Z}}$ with the properties $\partial_i$ is an $A$-homomorphism as a map from $M_i$ to $M_{i-1}$ and $\partial_{i-1} \partial_i = 0$. Such matrix $\mathbf{d}$ is called a dimension array.

Jensen, Su and Zimmermann [2] define a topological space

$$
\text{comproj}_A^\mathbf{d} := \pi_M^{-1} \left( \prod_{i \in \mathbb{Z}} \bigoplus_{j=1}^l P_{ij}^{d_{ij}} \right)
$$

with $\pi_M : \text{comproj}_A^{\mathbf{d}} \longrightarrow \prod_{i \in \mathbb{Z}} \text{mod}_A^{\mathbf{d}_{ii}}$ the natural projection. That topological space parameterize objects in bounded derived category $D^b(A)$. They use the fact that $D^b(A) \cong K^-(A-\text{proj})$.

Concerning conventions for derived categories we shall follow [3].

**Theorem 1.** Let $M_\bullet$ and $N_\bullet$ be two right bounded complexes with same dimension array $\mathbf{d}$. Any short exact sequence $0 \rightarrow N_\bullet \rightarrow M_\bullet \oplus Z_\bullet \rightarrow Z_\bullet \rightarrow 0$ gives arise a distinguished triangle $N_\bullet \rightarrow M_\bullet \oplus Z_\bullet \rightarrow Z_\bullet \rightarrow N[1]_\bullet$ in derived category $D^b(A)$.

**Proof.** See [1].

3. Results

The following theorems characterize existence of short exact sequences of complexes of projective $A$-modules in $\text{comproj}_A^\mathbf{d}$. Let $\mathbf{d}, \mathbf{d}', \mathbf{d}''$ be dimension arrays and let $\partial^U, \partial^V, \partial^W$ be points in $\text{comproj}_A^{\mathbf{d}}, \text{comproj}_A^{\mathbf{d}'}, \text{comproj}_A^{\mathbf{d}''}$ corresponding to complexes $U_\bullet, V_\bullet, W_\bullet$. The following theorems hold.

**Theorem 2.** Let $\partial^U \in \text{comproj}_A^{\mathbf{d}}(k)$ and $\partial^V \in \text{comproj}_A^{\mathbf{d}''}(k)$. Define $\mathbf{d} = \mathbf{d}' + \mathbf{d}''$. Then $\partial^W \in \text{comproj}_A^{\mathbf{d}''}(k)$ with

$$
\partial^W = \begin{bmatrix}
\partial^U & \nu_i \\
0 & \partial^V
\end{bmatrix}
$$

for some $\nu_i$ in $\text{Hom}_A \left( k^{\alpha(\mathbf{d}')} , k^{\alpha(\mathbf{d}_{i-1})} \right)$ if and only if there exists a short exact sequence $\Sigma$ of complexes of $A$-modules

$$
0 \rightarrow U_\bullet \xrightarrow{i} W_\bullet \xrightarrow{p} V_\bullet \rightarrow 0,
$$

where $i$ is a natural injection and $p$ is a natural projection. Moreover, the short exact sequence $\Sigma$ is split if and only if there exist $t_i$ in $\text{Hom}_A \left( k^{\alpha(\mathbf{d}')}, k^{\alpha(\mathbf{d}_{i-1})} \right)$, $i \in \mathbb{Z}$, such that $\nu_i = t_{i-1} \partial^V - \partial^U t_i$.

**Proof.** For all $i \in \mathbb{Z}$, let $W_i := U_i \oplus V_i$ and $\partial^W_i = \begin{bmatrix}
\partial^U_i & \nu_i \\
0 & \partial^V_i
\end{bmatrix}$. By introducing the injection $i$ of $U_i$ to the first factor of $W_i$ and the projection $p$ of the second factor of $W_i$ to $V_i$, we get a short exact sequence of complexes $\Sigma$ as follows

$$
\Sigma : 0 \rightarrow U_\bullet \xrightarrow{i} W_\bullet \xrightarrow{p} V_\bullet \rightarrow 0.
$$
The verification that \( i \) is a morphism of complexes:

\[
\begin{bmatrix}
  x \\
  0
\end{bmatrix} \mapsto \left[ \begin{array}{cc}
  \partial^U_i & \alpha_i \\
  0 & \partial^V_i
\end{array} \right] \begin{bmatrix}
  x \\
  0
\end{bmatrix} = \begin{bmatrix}
  \partial^U_i(x) \\
  0
\end{bmatrix}
\]

The verification that \( p \) is a morphism of complexes:

\[
\begin{bmatrix}
  x \\
  y
\end{bmatrix} \mapsto \left[ \begin{array}{cc}
  \partial^U_i & \alpha_i \\
  0 & \partial^V_i
\end{array} \right] \begin{bmatrix}
  x \\
  y
\end{bmatrix} = \begin{bmatrix}
  \partial^U_i(x) + \alpha_i(y) \\
  \partial^V_i(y)
\end{bmatrix}
\]

Conversely suppose that we have an exact sequence \( \Sigma \). Since \( i \) and \( p \) are morphisms of complexes, for \( u \in U_i \) we obtain

\[
p_{i-1}\partial^W_i \begin{bmatrix}
  u \\
  0
\end{bmatrix} = \partial^V_i p_i \begin{bmatrix}
  u \\
  0
\end{bmatrix} = 0.
\]

Thus

\[
\partial^W_i \begin{bmatrix}
  u \\
  0
\end{bmatrix} = \begin{bmatrix}
  \psi_i(u) \\
  0
\end{bmatrix}
\]

for some \( A \)-module homomorphism \( \psi_i : U_i \to U_{i-1} \). On the other side

\[
u_{i-1} \partial^U_i (u) = \partial^W_i \nu_i (u) = \partial^W_i \begin{bmatrix}
  u \\
  0
\end{bmatrix} = \begin{bmatrix}
  \psi_i(u) \\
  0
\end{bmatrix}.
\]

Then \( \partial^U_i (u) = \psi_i(u) \) for every \( u \in U_i \). Hence \( \psi_i = \partial^U_i \). This yields \( \partial^W_i \begin{bmatrix}
  u \\
  0
\end{bmatrix} = \begin{bmatrix}
  \partial^U_i (u) \\
  0
\end{bmatrix} \) for every \( u \in U_i \). For every \( v \in V_i \)

\[
p_{i-1}\partial^W_i \begin{bmatrix}
  0 \\
  v
\end{bmatrix} = \partial^V_i p_i \begin{bmatrix}
  0 \\
  v
\end{bmatrix} = \partial^V_i (v).
\]

Hence \( \partial^W_i \begin{bmatrix}
  0 \\
  v
\end{bmatrix} = \begin{bmatrix}
  u_{i-1} \\
  \partial^V_i (v)
\end{bmatrix} \) for some \( u_{i-1} \) in \( U_{i-1} \). It determines an \( A \)-module homomorphism \( \nu_i : V_i \to U_{i-1} \), given by \( v \mapsto u_{i-1,k} \) for all \( v \in V_i \). Then we obtain

\[
\partial^W_i = \begin{bmatrix}
  \partial^U_i & \nu_i \\
  0 & \partial^V_i
\end{bmatrix}.
\]

Now we are going to see when the short exact sequence \( \Sigma \) splits. Since \( p : W_\bullet \to V_\bullet \) just projects onto the second factor of \( W_\bullet \), such sections are precisely the chain map \( V_\bullet \to W_\bullet \) of the form
The stipulation that family $t$ is a chain map of complexes gives the equality

$$
\begin{align*}
\left[ \partial_i U\right]_i \left[ t_i \right]_i \in \mathcal{Z}, t_i: V_i \to U_i \text{ an } A\text{-module homomorphism.}
\end{align*}
$$

$$
\begin{align*}
\left[ \partial_i U\right]_i \left[ t_i \right]_i \in \mathcal{Z}, t_i: V_i \to U_i \text{ an } A\text{-module homomorphism.}
\end{align*}
$$

so $\nu_i = t_{i-1} \partial_i V_i - \partial_i t_i$.  

The dual of the above theorem is as follows.

**Theorem 3.** $\partial_i W = \left( \left[ \partial_i U\right]_i \left[ \mu_i \right]_i \right) \in \text{comproj}^A_d(k)$ for some $\mu_i$ in $Hom_A \left( k^\alpha(d'_i), k^\alpha(d''_i) \right)$ if and only if there exists an exact sequence of complexes $\Sigma'$ of the form

$$
0 \to V_* \xrightarrow{i} W' \xrightarrow{p} U_* \to 0.
$$

where $i$ is a natural injection and $p$ is a natural projection. Moreover $\Sigma'$ is split if and only if $\mu_i = t'_{i-1} \partial_i U_i - \partial_i V_i$ for some $t' \in Hom_A \left( k^\alpha(d'_i), k^\alpha(d''_i) \right)$.

From Theorem 1, Theorem 2, and Theorem 3 we get the following corollaries.

**Corollary 4.** Let $\partial_i U \in \text{comproj}^A_d(k)$ and $\partial_i V \in \text{comproj}^A_d'(k)$. Define $d = d' + d''$. Then $\partial_i V \in \text{comproj}^A_d(k)$ with

$$
\begin{align*}
\partial_i W = \left[ \partial_i U\right]_i \mu_i \left[ \partial_i V\right]_i
\end{align*}
$$

for some $\mu_i$ in $Hom_A \left( k^\alpha(d'_i), k^\alpha(d''_i) \right)$ if and only if there exists a distinguished triangle

$$
\begin{align*}
U_* \xrightarrow{i} W_* \xrightarrow{p} V_* \to U[1]*
\end{align*}
$$

in derived category $D^b(A)$.

**Corollary 5.** $\partial_i W = \left( \left[ \partial_i U\right]_i \left[ \mu_i \right]_i \right) \in \text{comproj}^A_d(k)$ for some $\mu_i$ in $Hom_A \left( k^\alpha(d'_i), k^\alpha(d''_i) \right)$ if and only if there exists a distinguished triangle

$$
\begin{align*}
V_* \xrightarrow{i} W_* \xrightarrow{p} U_* \to V[1]*
\end{align*}
$$

in derived category $D^b(A)$.

Let $\Sigma$ be a triangulated category. Assume that for every $Y$ and $Z$ objects in $\Sigma$, $Hom_{\Sigma}(Y, Z)$ is a finite dimensional $k$-vector space. Let

$$
\Gamma: U \xrightarrow{f} W \xrightarrow{g} V \xrightarrow{h} U[1]
$$

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be a triangle in the triangulated category $\mathfrak{T}$. We define an additive function $\vartheta_T$ from all objects of $\mathfrak{T}$ to the set of sequences with coefficients in integers,
\[
\vartheta_T(X) = (\dim_k \text{Hom}(V[i] \oplus U[i], X) - \dim_k \text{Hom}(W[i], X))_{i \in \mathbb{Z}}
\]
for any object $X$ in $\mathfrak{T}$.

The following theorem shows an equivalent condition for split distinguished triangle $U \to W \to V \to U[1]$ related to the difference between dimension of $\text{Hom}(U[i] \oplus V[i], X)$ and dimension of $\text{Hom}(W[i], X)$ for every $X$ object in triangulated category.

**Theorem 6.** Let $\Gamma : U \xrightarrow{f} W \xrightarrow{g} V \xrightarrow{h} T(U)$ be a distinguished triangle in the triangulated category $\mathfrak{T}$. Then $\vartheta_T(X)$ is a sequence with coefficients in $\mathbb{N} \cup \{0\}$ for any object $X$ in $\mathfrak{T}$. Moreover, the distinguished triangle $\Gamma$ is split if and only if $\vartheta_T(X) = 0$ for any object $X$ in $\mathfrak{T}$.

**Proof.** By applying the functor $\text{Hom}_\mathfrak{T}(-, X)$ to the distinguished triangle $\Gamma$, we obtain the exact sequence
\[
\cdots \to \text{Hom}_\mathfrak{T}(T(U), X) \xrightarrow{h^\ast} \text{Hom}_\mathfrak{T}(V, X) \xrightarrow{g^\ast} \text{Hom}_\mathfrak{T}(W, X) \xrightarrow{f^\ast} \text{Hom}_\mathfrak{T}(U, X) \to \cdots
\]
Without loss of generality, we will see only the 0-th position of the sequence $\vartheta_T(X)$
\[
\dim_k \text{Hom}_\mathfrak{T}(V \oplus U, X) = \dim_k \text{Hom}_\mathfrak{T}(V, X) + \dim_k \text{Hom}_\mathfrak{T}(U, X)
\]
\[
= \text{Rank} f^\ast + \text{Null} g^\ast + \dim_k \text{Hom}_\mathfrak{T}(U, X)
\]
\[
= \text{Null} f^\ast + \text{Null} g^\ast + \dim_k \text{Hom}_\mathfrak{T}(U, X)
\]
\[
= \dim_k \text{Hom}_\mathfrak{T}(W, X) - \text{Rank} f^\ast + \text{Null} g^\ast + \dim_k \text{Hom}_\mathfrak{T}(U, X)
\]
\[
= \dim_k \text{Hom}_\mathfrak{T}(W, X) + \text{Null} g^\ast + \varepsilon, \quad \varepsilon \geq 0.
\]
with $f^\ast$ defined by $f^\ast(\alpha) = \alpha \circ f$, $\alpha \in \text{Hom}_\mathfrak{T}(W, X)$, similarly with $g^\ast$ and $h^\ast$. Hence the entries of $\vartheta_T(X)$ are in $\mathbb{N} \cup \{0\}$.

Moreover if we let $\Gamma$ be split then $h = 0$, so $(T^i(h))^\ast = 0$ for any object $X$ in $\mathfrak{T}$ and any $i \in \mathbb{Z}$. It yields $(T^i(f))^\ast$ is an epimorphism and in particular $(T^i(g))^\ast$ is a monomorphism so $\text{Null} (T^i(g))^\ast = 0$. Thus
\[
\dim_k (T^i(V) \oplus T^i(U), X) = \dim_k (T^i(W), X)
\]
for all $i \in \mathbb{Z}$, so $\vartheta_T(X) = (0, \ldots, 0)$.

Conversely if $\vartheta_T(X) = (0, \ldots, 0)$ then
\[
\dim_k \text{Hom}_\mathfrak{T}(T(U), X) - \text{Rank} (T^i(f))^\ast + \text{Null} (T^i(g))^\ast = 0.
\]
It implies $\dim_k \text{Hom}_\mathfrak{T}(T(U), X) = \text{Rank} (T^i(f))^\ast$ and $\text{Null} (T^i(g))^\ast = 0$, since $\text{Rank} (T^i(f))^\ast \leq \dim_k (T^i(U), X)$ and $\text{Null} (T^i(g))^\ast \geq 0$, for every $i \in \mathbb{Z}$. Thus $\text{Ker} (T^i(g))^\ast = 0$ for every $i \in \mathbb{Z}$. This condition implies $(T^i(h))^\ast = 0$ for every $i \in \mathbb{Z}$. In particular, if we put $X = T^{i+1}(U)$ and evaluate the identity morphism $id_{T^{i+1}(U)} \in \text{Hom}_\mathfrak{T}(T^{i+1}(U), T^{i+1}(U))$,
\[
0 = (T^i(h))^\ast (id_{T^{i+1}(U)}) = id_{T^{i+1}(U)} \circ T^i(h) = T^i(h)
\]
so that $h = 0$. Hence the distinguished triangle splits. 

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