TWO PRIMALITY TESTS BASED ON
BINOMIAL COEFFICIENTS

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Abstract. In this paper, we report on two primality tests based on the divisibility properties of binomial coefficients. We enunciated and proved these properties in previous work.

1. Introduction

In a previous work [1] we introduced some results related to the divisibility properties of binomial coefficients. In particular, we proved a result that is equivalent to the following proposition:

Proposition 1.1. Given a prime number $p$ and positive integers $n$ and $j$, we have
\[
\binom{n-1}{p^j-1} \equiv \begin{cases} 
1 \pmod{p} & \text{if } p^j | n \\
0 \pmod{p} & \text{if } p^j \nmid n.
\end{cases} \tag{1.1}
\]

From this congruence relation we defined a Boolean operator related to whether or not $p^j$ divides $n$. We have also shown that the primality of $p$, as it appears in (1.1), is a necessary condition if this formula is expected to hold for all $n \in \mathbb{N}^*$. In fact, we shall prove in what follows that, for a composite number $q > 0$, divisible by a given prime number $\hat{p}$, $(q + \hat{p} - 1)_{q-1} \not \equiv 0 \pmod{q}$ even though $q \nmid q + \hat{p}$.

Theorem 1.2. Let $q > 0$ be a composite number divisible by the prime $\hat{p}$. If $n_q = q + \hat{p}$, then $(n_q - 1)_{q-1} \not \equiv 0 \pmod{q}$ even though we have $q \nmid n_q$.

Proof. Initially, we have:
\[
\binom{q + \hat{p} - 1}{q - 1} = \binom{q + \hat{p} - 1}{\hat{p}} - \binom{q + \hat{p} - 1}{\hat{p} - 1} \frac{q}{\hat{p}}, \tag{1.2}
\]

Since $(q + \hat{p} - 1)_{\hat{p}-1} \equiv 1 \pmod{\hat{p}}$, which can be inferred by (1.1), and is also established, for instance, in [1 2] as an application of Lucas’ theorem [4] (see Appendix), we have
\[
\binom{q + \hat{p} - 1}{q - 1} = (M\hat{p} + 1) \frac{q}{\hat{p}}, \tag{1.3}
\]

where $M$ is a positive integer. Thus, given that $q \nmid q/\hat{p}$, we conclude that
\[
\binom{q + \hat{p} - 1}{q - 1} \not \equiv 0 \pmod{q}. \tag{1.4}
\]

Remark 1.3. Theorem 1.2 is a particular case of a more general statement proved in our previous work [1].
Remark 1.4. If a prime $p$ does not divide the composite $q$, it can be shown that \( \binom{q+p-1}{q-1} \equiv 0 \pmod{q} \). We leave it to the reader to verify this fact.

In the next section we shall present and prove two primality tests which are based on the results above.

2. Primality Tests.

With the use of Proposition 1.1 and Theorem 1.2 we can express the first primality test as follows:

**Theorem 2.1.** Let $N > 3$ be an integer, let $p_i$ be the $i$-th prime number, and let $\pi(x)$ represent the prime counting function, which yields the number of primes that are less than or equal to $x$. The primality of $N$ will be confirmed if and only if

\[
\sum_{i=1}^{\pi(\sqrt{N})} \left( \frac{N + p_i - 1}{N - 1} \right) \mod N = 0 \tag{2.1}
\]

**Proof.** Since $p_i < N$, it follows that $N \nmid N + p_i$. If $N$ is prime, all terms in the sum will be zero as implied by Proposition 1.1. On the other hand, if $N$ is composite, at least one of its prime factors must be in the set $\mathcal{P}_N = \{p_i \mid 1 \leq i \leq \pi(\sqrt{N})\}$. Therefore, according to Theorem 1.2 whenever $p_i \in \mathcal{P}_N$ is a prime factor of $N$ a positive term will be generated and the sum will then be greater than zero. \(\square\)

Our second primality test is slightly simpler to prove, following directly from Proposition 1.1.

**Theorem 2.2.** Let $N > 3$ be an integer and $p_i$ be the $i$-th prime number. Let $\pi(x)$ be the prime counting function. The primality of $N$ will be confirmed if and only if

\[
\sum_{i=1}^{\pi(\sqrt{N})} \left( \frac{N - 1}{p_i - 1} \right) \mod p_i = 0 \tag{2.2}
\]

**Proof.** It is clear that each term of the sum in (2.2) is an application of Proposition 1.1 with $j = 1$, for a given prime which is less than or equal to $\sqrt{N}$. \(\square\)

**Remark 2.3.** Different algorithms can be used to evaluate the sums in (2.1) and (2.2), each of which presenting a different level of computational complexity. In fact, a detailed analysis of the efficiency of these two tests remains to be made. However, preliminary estimates point to asymptotic time complexities which depend on the exponential of the number of bits required to represent $N$ in binary. \(3\).

**Remark 2.4.** Sequential evaluation, in some randomized order, of the terms of any of the two sums above is a way to improve the efficiency of these tests. In fact, for any given integer $N > 3$ and for any randomly chosen prime number $p_i \leq \sqrt{N}$, if the corresponding term in either of the two sums is different from zero, one can immediately conclude that $N$ is composite. On the other hand, $N$ will be prime if and only if every such term vanishes.
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Remark 2.5. We believe these two tests are of theoretical interest since: (a) they are essentially combinatorial, i.e., they are based exclusively on the divisibility properties of binomial coefficients, (b) they are deterministic, (c) for \( N \) integer and greater than 3, they are unconditionally correct, and (d) they provide the list of prime divisors of \( N \) which are less than or equal to \( \sqrt{N} \).

3. Appendix

Lucas’ Theorem. Let \( p \) be a prime number and let \( R \) and \( S \) be positive integers such that
\[
R = r_0 + r_1 p + r_2 p^2 + \ldots + r_m p^m, \quad r_i \in \{0, 1, 2, \ldots, p-1\},
\]
and
\[
S = s_0 + s_1 p + s_2 p^2 + \ldots + s_l p^l, \quad s_i \in \{0, 1, 2, \ldots, p-1\},
\]
where \( r_i \) and \( s_i \) are, respectively, the digits of \( R \) and \( S \) when written in base \( p \). We then have
\[
\binom{R}{S} \equiv \prod_{i=0}^{\max\{m,l\}} \binom{r_i}{s_i} \pmod{p}.
\]
We adopt here the convention \( \binom{r}{s} = 0 \) if \( s \) is either greater than \( r \) or smaller than zero.

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