CONTINUITY OF UTILITY MAXIMIZATION UNDER WEAK CONVERGENCE

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Abstract. In this paper we find sufficient conditions for the continuity of the value of the utility maximization problem from terminal wealth with respect to the convergence in distribution of the underlying processes. We provide several examples which illustrate that without these conditions, we cannot generally expect continuity to hold. Finally, we apply our results to the computation of the minimum shortfall in the Heston model by building an appropriate lattice approximation.

1. Introduction

This paper deals with the following question: Given a utility function and a sequence of financial markets with underlying assets \((S^{(n)})_{n \in \mathbb{N}}\) that are converging weakly to \(S\), under which conditions do the values of the utility maximization problems (from terminal wealth) converge to the corresponding value for the model given by \(S\)? Although the utility maximization problems enjoyed a considerable attention in the literature (see, for instance, [22, 23, 16, 17, 15, 30, 3]), to the best of our knowledge, the continuity under weak convergence was studied only in [29, 31] in a complete market setup. In this work we consider this convergence question for general incomplete market model and a continuous state dependent utility.

We divide the proof into two main steps identifying when we have lower and upper-semi-continuity respectively. We show that for the lower semi–continuity to hold, it is sufficient that the approximating sequence \((S^{(n)})_{n \in \mathbb{N}}\) has bounded jump activity. The formal condition is given in Assumption 3.1. The main idea is to prove that an admissible integral of the form \(\int \gamma dS\) can be approximated in the weak sense by admissible integrals of the form \(\int \gamma^{(n)} dS^{(n)}, n \in \mathbb{N}\). The assumption on the jump activity is essential for the admissibility of the approximating sequence. We demonstrate the necessity of this assumption with an example. We should also note that the concavity of the utility function is not necessary in this step.

The second step, namely, the upper semi–continuity is more delicate. Roughly speaking, we prove that if the utility function is concave and the state price densities in the limit model can be approximated by state price densities in the approximating sequence (see Assumption 4.1) then upper semi–continuity holds. The proof relies on the optional decomposition theorem. We provide two examples which illustrate that these assumptions are crucial.

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We apply our continuity results in order to construct an approximating sequence for the Heston model. For technical reasons we truncate the model in such a way that the volatility is bounded. The novelty of our construction is that the approximating sequence lies on a grid and satisfies the assumptions required for the continuity of the value of the utility maximization problem from terminal wealth. The grid structure enables efficient numerical computations for stochastic control problems via dynamic programming.

Our last contribution is the implementation of the constructed approximating models for the numerical computations of the shortfall risk measure in the Heston model. We focus on European call options. It is well known (see [6, 12, 8, 27]) that in the Heston model the super–replication price is prohibitively high and lead to buy–and–hold strategies. Namely, the cheapest way to super–hedge a European call option is to buy one stock at the initial time and keep that position till maturity. For a given initial capital which is less than the initial stock price we want to compute the corresponding shortfall risk. This cannot be done analytically and so numerical schemes come into picture.

It is important to mention the series of papers [20, 2, 21, 25, 24] where the authors studied the stability of utility maximization and the corresponding asymptotic expansion of the utility maximization problem in terms of a perturbations of the model parameters. The main difference is that in these papers the stochastic base is fixed while in our setup each financial model is defined on its own probability space. As a result, while their approach deals with the stability of the models with respect to small perturbations, we are able to obtain numerical approximations using discrete models.

The rest of the paper is organized as follows. In the next section we introduce the setup and formulate the continuity result. In Section 3 we prove the lower semi–continuity. In Section 4 we prove the upper semi–continuity. In Section 5 we provide auxiliary results which can be applied for the verification of some of the assumptions. Section 6 is devoted to the construction of an approximating sequence for the Heston model. In Section 7 we provide a detailed numerical analysis for shortfall risk minimization.

2. Preliminaries and Main Results

We consider a model of a security market which consists of d risky assets which we denote by $S = (S_t^{(1)}, \ldots, S_t^{(d)})_{0 \leq t \leq T}$, where $T < \infty$ is the time horizon. We assume that the investor has a bank account that for simplicity bears no interest. The process $S$ is assumed to be a continuous semi–martingale on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t^S)_{0 \leq t \leq T}, \mathbb{P})$ where the filtration $(\mathcal{F}_t^S)_{0 \leq t \leq T}$ is the usual filtration generated by $S$. Without loss of generality we take $\mathcal{F} = \mathcal{F}_T^S$.

A (self–financing) portfolio $\pi$ is defined as a pair $(x, \gamma)$ where the constant $x$ is the initial value of the portfolio and $\gamma = (\gamma_i(t))_{1 \leq i \leq d}$ is an adapted left–continuous process specifying the amount of each asset held in the portfolio. The corresponding portfolio value process is given by

$$V_t^\pi = x + \int_0^t \gamma_u dS_u, \quad t \in [0, T].$$
Observe that the continuity of $S$ implies that the wealth process $\{V^\pi_t\}_{t=0}^T$ is continuous as well. We say that a trading strategy $\pi$ is admissible if $V^\pi_t \geq 0$, $\forall t \geq 0$. For any $x > 0$ we denote by $\mathcal{A}(x)$ the set of all admissible trading strategies.

Denote by $\mathcal{M}(S)$ the set of all equivalent (to $\mathbb{P}$) local martingale measures.

**Assumption 2.1.** $\mathcal{M}(S) \neq \emptyset$.

This condition is intimately related to the absence of arbitrage opportunities on the security market. See [7] for a precise statement and references.

Next, we introduce our state dependent utility maximization problem. Let $U : (0, \infty) \times \mathbb{D}([0, T]; \mathbb{R}^d) \to \mathbb{R}$ be a continuous function. As usual, $\mathbb{D}([0, T]; \mathbb{R}^d)$ denotes the space of all RCLL (right continuous with left limits) functions $f : [0, T] \to \mathbb{R}^d$ equipped with the Skorokhod topology (for details see [4]).

**Assumption 2.2.**

(i) For any $s \in \mathbb{D}([0, T]; \mathbb{R}^d)$ the function $U(\cdot, s)$ is non-decreasing.

(ii) For any $x > 0$ we have $\mathbb{E}_x[U(x, S)] > -\infty$.

We extend $U$ to $\mathbb{R}_+ \times \mathbb{D}([0, T]; \mathbb{R}^d)$ by $U(0, s) := \lim_{u \downarrow 0} U(u, s)$. In view of Assumption 2.2(i) the limit exists (might be $-\infty$).

For a given initial capital $x > 0$ consider the optimization problem

$$u(x) = \sup_{\pi \in \mathcal{A}(x)} \mathbb{E}_x[U(V^\pi_T, S)],$$

where we set $-\infty+\infty = -\infty$. Namely, for a random variable $X$ with $\mathbb{E}[\max(-X, 0)] = \infty$ we set $\mathbb{E}[X] = -\infty$.

Let us notice that Assumption 2.2(ii) implies $u(x) > -\infty$.

**Assumption 2.3.** The function $u : (0, \infty) \to \mathbb{R} \cup \{\infty\}$ is continuous. Namely, for any $x > 0$ we have $u(x) = \lim_{y \to x} u(y)$ where a priori the joint value can be equal to $\infty$.

Next, for any $n$, let $S^{(n)} = (S^{n, 1}_t, \ldots, S^{n, d}_t)_{t=0}^T$ be a RCLL semi-martingale defined on some filtered probability space $(\Omega_n, \mathcal{F}^{(n)}, (\mathcal{F}^{(n)}_t)_{0 \leq t \leq T}, \mathbb{P}_n)$ where the filtration $(\mathcal{F}^{(n)}_t)_{0 \leq t \leq T}$ satisfies the usual assumptions. For the $n$-th model we define $\mathcal{A}_n(x)$ as the set of all simple predictable integrands $\gamma^{(n)}$ such that the resulting portfolio value process

$$x + \int_0^t \gamma_u^{(n)} dS_u^{(n)} \geq 0, \quad t \in [0, T],$$

is non negative. As usual, a simple predictable integrand is of the form

$$\gamma_{t}^{(n)} = \sum_{i=1}^N \mathbb{I}_{\tau_i < t \leq \tau_{i+1}} \beta_i,$$

where $N \in \mathbb{N}$ and $0 = \tau_1 \leq \tau_2 \leq \ldots \leq \tau_{N+1} = T$ are stopping times. Set,

$$u_n(x) = \sup_{\pi \in \mathcal{A}_n(x)} \mathbb{E}_{\mathbb{P}_n}[U(V^\pi_T, S^{(n)})].$$

We assume the following.

**Assumption 2.4.**

(i) For any $x > 0$ the family of random variables $\{U^-(x, S^{(n)})\}_{n \in \mathbb{N}}$ is uniformly integrable where $U^- = \max(-U, 0)$.

(ii) We have the weak convergence $S^{(n)} \Rightarrow S$ on the space $\mathbb{D}([0, T]; \mathbb{R}^d)$ equipped
with the Skorokhod topology.

(iii) For any $x > 0$ the family of random variables $\{U^+(V^T_{T'}, S^{(n)})\}_{n \in \mathbb{N}, \pi \in A_n(x)}$ is uniformly integrable, where $U^+ = \max(U, 0)$.

**Remark 2.1.** The verification of Assumption 2.3 and Assumption 2.4 (iii) can be a difficult task. In Section 5 we provide quite general and easily verifiable conditions which are sufficient for the above assumptions to hold true.

Now, we are ready to formulate our main result.

**Theorem 2.1.** Under Assumptions 2.1–2.4 and Assumptions 3.1, 4.1–4.2 below, we have

$$u(x) = \lim_{n \to \infty} u_n(x), \quad \forall x > 0.$$  

(2.1)

The proof of this theorem will be done in Sections 3–4. In the next section we treat the inequality $u(x) \leq \liminf_{n \to \infty} u_n(x)$ (lower semi–continuity). In Section 4 we study the upper semi–continuity $u(x) \geq \limsup_{n \to \infty} u_n(x)$.

**Remark 2.2.** Observe that in view of Assumptions 2.4(i),(iii) we have

$$-\infty < \liminf_{n \to \infty} u_n(x) \leq \limsup_{n \to \infty} u_n(x) < \infty, \quad \forall x > 0.$$

We conclude that the joint value in (2.1) is finite.

**Remark 2.3.** A natural question is whether we have some kind of convergence for the optimal portfolios as well. Following [10] we say that a sequence of semi–martingales $S^{(n)}$, $n \in \mathbb{N}$ is good if the weak convergence $(H^{(n)}, S^{(n)}) \Rightarrow (H, S)$ implies the weak convergence $(H^{(n)}, S^{(n)}, \int H^{(n)}dS^{(n)}) \Rightarrow (H, S, \int HdS)$. In [10] the authors provide several applicable conditions for the “goodness” of a sequence of semi–martingales.

For a good sequence of semi–martingales we have the following. If for any $n$, $\pi_n = (x, \gamma^{(n)})$ in an optimal portfolio for the $n$–th model and we have the weak convergence $(S^{(n)}, \gamma^{(n)}) \Rightarrow (S, \gamma)$ for some predictable process $\gamma$, then $\pi = (x, \gamma)$ is an optimal portfolio for the model given by $S$. Indeed, for $\pi := (x, \gamma)$ we have $(S^{(n)}, V^{\pi_n}) \Rightarrow (S, V^\pi)$ and so $\pi$ is admissible. Moreover, from the continuity of $U$, Assumptions 2.4(ii)-(iii) and Theorem 2.1 we obtain

$$\mathbb{E}_\pi[U(V^T_{T'}, S)] \geq \lim_{n \to \infty} \mathbb{E}_{\pi_n}[U(V^T_{T'}, S^{(n)})] = \lim_{n \to \infty} u_n(x) = u(x).$$

There are two main challenges in establishing a limit theorem for the optimal portfolios. First, we need to obtain a tightness results for the optimal portfolios in the approximating models. Due to the fact that in incomplete markets we do not have an explicit characterization of the optimal portfolios this is far from obvious. The second challenge is related to the measurability of the limit portfolio with respect to the filtration which is generated by the underlying $S$. It seems that this requires (in addition to the convergence of the underlying assets) assuming weak convergence of the corresponding filtrations. This type of assumptions was made in the recent paper [28] where the authors studied the stability of martingale representation under weak convergence. It will be interesting to explore whether their stability results can be applied in our setup. This is left for future research.
3. LOWER SEMI–CONTINUITY UNDER WEAK CONVERGENCE

In this section we will assume (in addition to Assumptions 2.1–2.4) the following condition which bounds the uncertainty of the jump activity and needed for the admissibility requirements.

**Assumption 3.1.** For any \( n \in \mathbb{N} \) consider the non-decreasing RCLL process given by \( D_t^{(n)} = \sup_{0 \leq u \leq t} |S_u^{(n)} - S_{u-}^{(n)}|, \quad t \in [0, T] \) where \( | \cdot | \) denotes the Euclidean norm in \( \mathbb{R}^d \).

For any \( n \in \mathbb{N} \) there exists an adapted (to \( \{ F_t^{(n)} \}_{t=0}^T \) non-decreasing left continuous process \( \{ J_t^{(n)} \}_{t=0}^T \) such that \( \inf_{0 \leq u \leq t} J_t^{(n)} - D_t^{(n)} \geq 0 \) a.s. and \( J_T^{(n)} \to 0 \) in probability.

Let us explain this assumption with the following example.

**Example 3.1.** First, we notice that if \( S^{(n)} \) is a pure jump process of the form

\[
S_t^{(n)} = \sum_{i=1}^{m_n} S_{t_{i+1}^{(n)}}^{(n)} I_{t_{i}^{(n)} \leq t < t_{i+1}^{(n)}} + S_T^{(n)} \quad \text{for } t \in [0, T]
\]

for some deterministic partition \( 0 = t_1^{(n)} < t_2^{(n)} < \ldots < t_{m_n+1}^{(n)} = T \). Then, Assumption 3.1 is equivalent to

\[
\max_{i=1, \ldots, m_n} \mathrm{ess} \sup |S_{t+1}^{(n)} - S_t^{(n)}| \to 0 \quad \text{in probability}.
\]

As usual \( Z := \mathrm{ess} \sup (Y|G) \) is the minimal random variable (may take the value \( \infty \)) which is \( G \) measurable and satisfy \( Z \geq Y \) a.s.

Next, consider the Black–Scholes model

\[
S_t = S_0 e^{\sigma W_t - \frac{1}{2} \sigma^2 t}, \quad t \in [0, T]
\]

where \( \sigma > 0 \) is a constant volatility and \( W = \{ W_t \}_{t=0}^T \) is a Brownian motion.

If we take the naive discretization and define the processes \( S^{(n)}, \quad n \in \mathbb{N} \) by

\[
S_t^{(n)} := \frac{S_{kT/n}^{(n)} - S_{(k-1)T/n}^{(n)}}{T/n}, \quad kT/n \leq t < (k+1)T/n
\]

then for all \( k < n, \)

\[
\mathrm{ess} \sup \left( \left| S_{(k+1)T/n}^{(n)} - S_{kT/n}^{(n)} \right|, S_{(k+1)T/n}^{(n)} - S_{kT/n}^{(n)} \right) = \infty \quad \text{a.s.}
\]

and so Assumption 3.1 is not satisfied.

On the other hand, take the binomial models (which converge weakly to the Black–Scholes model)

\[
\tilde{S}_t^{(n)} = S_0 e^{\sigma \sqrt{T/n} \sum_{i=1}^{k} \xi_i}, \quad kT/n \leq t < (k+1)T/n
\]

where \( \xi_i = \pm 1, \quad i \in \mathbb{N} \) are i.i.d. with

\[
\mathbb{P}_n(\xi_k = 1) = 1 - \mathbb{P}_n(\xi_k = -1) = \frac{1}{1 + e^{\sigma \sqrt{T/n}}}
\]

Then for all \( k < n, \)

\[
\mathrm{ess} \sup \left( \left| \tilde{S}_{(k+1)T/n}^{(n)} - \tilde{S}_{kT/n}^{(n)} \right|, \tilde{S}_{(k+1)T/n}^{(n)} - \tilde{S}_{kT/n}^{(n)} \right) = \left( e^{\sigma \sqrt{T/n}} - 1 \right) \tilde{S}_{kT/n}^{(n)}
\]

so Assumption 3.1 holds true.
Now, we formulate our lower semi-continuity result.

**Proposition 3.1.** Under Assumptions 2.1–2.4 and Assumption 3.1 we have

\[ u(x) \leq \liminf_{n \to \infty} u_n(x), \quad \forall x > 0. \]

Before giving the proof of the above proposition, we establish the following general result.

**Lemma 3.1.** Let \( \gamma \) be an adapted left-continuous process with \(|\gamma| \leq M\) for some constant \(M\). Then there exists a sequence of simple predictable integrands \(\gamma^{(n)} = \{\gamma_t^{(n)}\}_{t=0}^T, n \in \mathbb{N}\) such that \(|\gamma^{(n)}| \leq M\) and we have the weak convergence

\[ (3.3) \quad \left( \{S_i^{(n)}\}_{i=0}^T, \left\{ \int_0^t \gamma_i^{(n)} \, dS_i\right\}_{t=0}^T \right) \Rightarrow \left( \{S_i\}_{i=0}^T, \left\{ \int_0^t \gamma_i \, dS_i\right\}_{t=0}^T \right) \]

on the space \(\mathbb{D}([0, T]; \mathbb{R}^d) \times \mathbb{D}([0, T]; \mathbb{R})\).

**Proof.** On the space \((\Omega, \mathcal{F}, (\mathcal{F}_t^S)_{0 \leq t \leq T}, \mathbb{P})\), introduce the set \(\Gamma_M\) of all simple predictable integrands of the form

\[ \beta_i = \sum_{i=1}^k \mathbb{1}_{t_i < t \leq t_{i+1}} \beta_i, \]

where \(0 = t_1 < t_2, ..., < t_{k+1} = T\) is a deterministic partition and

\[ \beta_i = \psi_i(S_{a_{i,1}}, ..., S_{a_{i,m_i}}), \quad i = 1, ..., k \]

for deterministic partition \(0 = a_{i,1} < \ldots < a_{i,m_i} = t_i\) and a continuous function \(\psi_i : (\mathbb{R}^d)^{m_i} \to \mathbb{R}^d\) which satisfy \(|\psi_i| \leq M\). Since \(\gamma\) is predictable with respect to the filtration generated by \(S\), then from standard density arguments it follows that for any \(\epsilon > 0\) we can find \(\hat{\gamma} \in \Gamma_M\) which satisfy

\[ \mathbb{P} \left( \sup_{0 \leq t \leq T} \left| \int_0^t \gamma_i \, dS_i - \int_0^t \hat{\gamma}_i \, dS_i \right| > \epsilon \right) < \epsilon. \]

Hence, without loss of generality we can assume that \(\gamma \in \Gamma_M\).

For any \(n \in \mathbb{N}\) define a trading strategy (with respect to \(S^{(n)}\)) by

\[ \gamma_t^{(n)} = \sum_{i=1}^k \mathbb{1}_{t_i < t \leq t_{i+1}} \psi_i \left( S_{a_{i,1}}^{(n)}, ..., S_{a_{i,m_i}}^{(n)} \right), \quad t \in [0, T]. \]

From the weak convergence \(S^{(n)} \Rightarrow S\) and the Skorokhod representation theorem (see [9]) it follows that we can redefine the stochastic processes \(S^{(n)}, n \in \mathbb{N}\) and \(S\) on the same probability space such that

\[ (3.4) \quad \sup_{0 \leq t \leq T} |S_t^{(n)} - S_t| \to 0 \text{ a.s.} \]

We have a uniform convergence in (3.4) because \(S\) is continuous.

Next, recall the partition \(0 = t_1 < t_2, ..., < t_{k+1} = T\). From (3.4) and the continuity of \(\psi_i, i = 1, ..., k\) we get that \(\sup_{0 \leq t \leq T} |\gamma_t^{(n)} - \gamma_t| \to 0 \text{ a.s.}\) Thus,

\[ \sup_{0 \leq t \leq T} \left| \int_0^t \gamma_i^{(n)} \, dS_i^{(n)} - \int_0^t \gamma_i \, dS_i \right| \leq \sup_{0 \leq t \leq T} \left| \int_0^t \gamma_i^{(n)} \, dS_i^{(n)} - \int_0^t \gamma_i \, dS_i \right| + \sup_{0 \leq t \leq T} \left| \int_0^t (\gamma_i^{(n)} - \gamma_i) \, dS_i \right| \leq 2Mk \sup_{0 \leq t \leq T} |S_t^{(n)} - S_t| + 2kd \sup_{0 \leq t \leq T} |S_t| \sup_{0 \leq t \leq T} |\gamma_t^{(n)} - \gamma_t| \to 0 \text{ a.s.} \]
and the proof is completed. □

Now, we are ready to prove Proposition 3.1.

Proof. Proof of Proposition 3.1.
The proof will be done in two steps.

Step I: In this step we show that for any $x_1 > x_2 > 0$
\begin{equation}
(3.5) \quad u(x_2) \leq \sup_{\pi=(x_1, \gamma) \in A(x_1)} \mathbb{E}_p[U(V_T^\pi, S)].
\end{equation}
A priori the left hand side and the right hand side of (3.5) can be equal to $\infty$.

Let $\pi = (x_2, \gamma) \in A(x_2)$ be an arbitrary portfolio. Define the stopping times
$$\theta_n = T \wedge \inf\{t : |\gamma_t| > n\}, \quad n \in \mathbb{N}$$
and the trading strategies
$$\gamma^{(n)}_t := 1_{t \leq \theta_n} \gamma_t, \quad t \in [0, T].$$
Clearly, for $\pi_n = (x_1, \gamma^{(n)})$ we have $V_t^{\pi_n} = x_1 - x_2 + V_{t \wedge \theta_n}^{\pi_n}$, $t \in [0, T]$. Hence, $\pi_n \in A(x_1)$. Since $\gamma$ is left–continuous, we have that $|\gamma^{(n)}| \leq n$ and $\theta_n \uparrow T$ a.s., and so,
$$\lim_{n \to \infty} V_t^{\pi_n} = \lim_{n \to \infty} V_{t \wedge \theta_n}^{\pi_n} = V_T^\pi + x_1 - x_2.$$
This together with the Fatou Lemma, Assumption 2.2 (observe that $V_T^{\pi_n} \geq x_1 - x_2 > 0$) and the fact that $U$ is continuous gives
$$\mathbb{E}_p[U(V_T^\pi, S)] \leq \lim_{n \to \infty} \inf \mathbb{E}_p[U(V_T^{\pi_n}, S)]$$
and (3.5) follows.

Step II: In view of (3.5) and Assumption 2.3 in order to prove Proposition 3.1 it is sufficient to show that for any initial capital $x > 0$, $0 < \epsilon < \frac{x}{2}$ and admissible portfolio $\pi = (x - 2\epsilon, \gamma)$ with uniformly bounded $\gamma$ we have
\begin{equation}
(3.6) \quad \lim_{n \to \infty} \inf \mathbb{E}_p[U(V_T^\pi, S)] = \mathbb{E}_p[U(V_T^\pi, S)].
\end{equation}
Thus, let $0 < \epsilon < \frac{x}{2}$ and $\pi = (x - 2\epsilon, \gamma)$ admissible with $|\gamma| \leq M$ for some $M$. Lemma 3.1 provides an existence of a sequence $\gamma^{(n)}$, $n \in \mathbb{N}$ which satisfy $|\gamma^{(n)}| \leq M$ and (3.3). From the Skorokhod representation theorem it follows that we can redefine the stochastic processes $\gamma^{(n)}, S^{(n)}$, $n \in \mathbb{N}$ and $\gamma, S$ on the same probability space such that (3.4) holds true and
\begin{equation}
(3.7) \quad \sup_{0 \leq t \leq T} \left| \int_0^t \gamma^{(n)}_u dS^{(n)}_u - \int_0^t \gamma_u dS_u \right| \to 0 \text{ a.s.}
\end{equation}
The uniform convergence is due to the fact that $\int_0^t \gamma_u dS_u$, $t \in [0, T]$ is a continuous process.

For a given $n$, the portfolio $(x, \gamma^{(n)})$ might fail to be admissible and so, modification is needed. Recall Assumption 3.1 and the stochastic process $J^{(n)}$. Introduce the stopping time
$$\Theta_n = T \wedge \inf \left\{ t : x + \int_0^{t-} \gamma^{(n)}_t dS^{(n)}_t > \epsilon + MdJ^{(n)} \right\}.$$
We denote by $\mathbb{E}$ the expectation on the common probability space which supports $S^{(n)}$ and $S$. Consider the portfolios $\pi_n = (x, \hat{\gamma}^{(n)})$, $n \in \mathbb{N}$ where $\hat{\gamma}^{(n)}_t = 1_{t \leq \Theta_n} \gamma^{(n)}_t$.
First, let us show that $V^{\pi_n} \geq \epsilon$. Indeed,

$$V_t^{\pi_n} = x + \int_0^t \gamma_{\pi_t}^{(n)} dS_t^{(n)} \geq x + \int_0^t \gamma_{\pi_t}^{(n)} dS_t^{(n)} - Md|s_{t \wedge \Theta_n} - S_t^{(n)}| \geq$$

$$\epsilon + MdJ_t^{(n)} - Md|S_t^{(n)} - S_{\Theta_n}^{(n)}| \geq \epsilon$$

as required. The first inequality follows from the fact that on the time interval $[0, \Theta_n)$ we have $x + \int_0^\infty \gamma_{\pi_t}^{(n)} dS_t^{(n)} \geq \epsilon + MdJ_t^{(n)}$. The second inequality follows from $|\gamma^{(n)}| \leq M$. The last inequality is due to $J_t^{(n)} \geq |S_t^{(n)} - S_{\Theta_n}^{(n)}|$.

Next, we argue that

$$\lim_{n \to \infty} \mathbb{P}(\Theta_n = T) = 1.$$ (3.8)

Recall, the admissible portfolio $\pi = (x - 2\epsilon, \gamma)$. From (3.7) it follows that

$$\lim \inf_{n \to \infty} \inf_{0 \leq t \leq T} \left( x + \int_0^t \gamma_{u_t}^{(n)} dS_t^{(n)} \right) = x + \inf_{0 \leq t \leq T} \int_0^t \gamma_u dS_u \geq 2\epsilon.$$ (3.9)

In particular

$$\lim_{n \to \infty} \mathbb{P} \left( \inf_{0 \leq t \leq T} \left( x + \int_0^t \gamma_{u_t}^{(n)} dS_t^{(n)} \right) > \epsilon \right) = 1.$$ (3.10)

From Assumption 3.1 we have $J_t^{(n)} \to 0$ in probability. This together with (3.9) gives (3.8).

Finally, form the Fatou Lemma, the continuity of $U$, Assumption 2.2(i), Assumption 2.4(i) (recall that $V^{\pi_n} \geq \epsilon$), (3.4), and (3.7)–(3.8) we obtain

$$\lim \inf_{n \to \infty} u_n(x) \geq \mathbb{E} \left[ U \left( x + \int_0^T \gamma_t dS_t, S \right) \right] \geq \mathbb{E}[U(V_T^x, S)]$$

and (3.6) follows.

We end this section with illustrating that Assumption 3.1 is essential for Proposition 3.1 to hold.

**Example 3.2.** Let $d = 1$. Consider a state dependent utility which corresponds to shortfall risk minimization for a call option with strike price $K > 0$. Namely, we set

$$U(v, s) = -((s_T - K)^+ - v)^+.$$ (3.10)

We have,

$$u(x) = -\inf_{\pi \in \mathcal{A}(x)} \mathbb{E}_\mathbb{P}[(S_T - K)^+ - V_T^x].$$

Recall Example 3.1 and consider the Black–Scholes model given by (3.1). In [26] (see Section 6.1.2) it was proved that for the processes $S^{(n)}$, $n \in \mathbb{N}$ given by (3.2) and initial capital $x := \mathbb{E}[(S_T - K)^+]$ (i.e. the Black–Scholes price) we have

$$\lim \inf_{n \to \infty} \inf_{\pi \in \mathcal{A}_n(x)} \mathbb{E}_\mathbb{P}[(S_T - K)^+ - V_T^x] > 0.$$ (3.11)

Clearly, the fact that $x$ is the Black–Scholes price implies that

$$\inf_{\pi \in \mathcal{A}(x)} \mathbb{E}_\mathbb{P}[(S_T - K)^+ - V_T^x] = 0.$$
We get
\[ u(x) = 0 > \lim \sup_{n \to \infty} u_n(x) \]
It is straightforward to check that Assumptions 2.1–2.4 hold true (see Remark 5.1 for the verification of Assumption 2.3).
We conclude that without Assumption 3.1, we can not expect (in general) that Proposition 3.1 will hold true.

4. Upper Semi–Continuity under Weak Convergence

Recall the set \( \mathcal{M}(S) \) of all equivalent local martingale measures.

**Assumption 4.1.** Denote by \( \mathcal{M}(S^{(n)}) \), \( n \in \mathbb{N} \) the set of all equivalent local martingale measures for the \( n \)-th model. For any \( Q \in \mathcal{M}(S) \) there exists a sequence of probability measures \( Q_n \in \mathcal{M}(S^{(n)}) \), \( n \in \mathbb{N} \) such that under \( P_n \) the joint distribution of \( \left( \{ S_t^{(n)} \}_{t=0}^T, \frac{dQ_n}{dP_n} \right) \) on the space \( \mathcal{D}([0,T]; \mathbb{R}^d) \times \mathbb{R} \) converges to the joint distribution of \( \left( \{ S_t \}_{t=0}^T, \frac{dQ}{dP} \right) \) under \( P \). We denote this relation by
\[
(4.1) \quad \left( \left( S^{(n)}, \frac{dQ_n}{dP_n} \right); P_n \right) \Rightarrow \left( \left( S, \frac{dQ}{dP} \right); P \right).
\]

**Assumption 4.2.** For any \( s \in \mathcal{D}([0,T]; \mathbb{R}^d) \), the function \( U(\cdot, s) \) is concave.

**Remark 4.1.** Let us notice that in order to verify Assumption 4.1 it is sufficient to establish (4.1) for a dense subset of \( \{ \frac{dQ}{dP} : Q \in \mathcal{M}(S) \} \). This simplification will be used in Section 6 where we introduce lattice approximations of the Heston model.

We are now ready to state the upper semi–continuity result.

**Proposition 4.1.** Under Assumptions 2.1–2.4 and Assumptions 4.1–4.2 we have
\[ u(x) \geq \lim \sup_{n \to \infty} u_n(x), \quad \forall x > 0. \]

**Proof.** The proof will be done in two steps.

**Step I:** Choose \( \epsilon > 0 \). In view of Assumption 2.3 it sufficient to prove that for any sequence of portfolios \( \pi_n \in \mathcal{A}_n(x) \), \( n \in \mathbb{N} \), there exists \( \pi \in \mathcal{A}(x+\epsilon) \) such that
\[ \mathbb{E}_\pi[U(V_T^\pi, S)] \geq \lim_{n \to \infty} \mathbb{E}_{\pi_n}[U(V_{T}^{\pi_n}, S^{(n)})]. \]
Without loss of generality (by passing to a sub-sequence) we assume that the limit \( \lim_{n \to \infty} \mathbb{E}_{\pi_n}[U(V_{T}^{\pi_n}, S^{(n)})] \) exists.

Choose \( Q \in \mathcal{M}(S) \) and denote \( Z = \frac{dQ}{dP} \). Let \( Q_n \in \mathcal{M}(S^{(n)}) \), \( n \in \mathbb{N} \) be a sequence such that (4.1) holds true. For any \( n \), \( \{ V_{T}^{\pi_n} \}_{t=0}^T \) is a \( Q_n \) super–martingale. Hence,
\[ \mathbb{E}_{\pi_n} \left( V_{T}^{\pi_n} \frac{dQ_n}{dP_n} \right) = \mathbb{E}_{Q_n} [V_{T}^{\pi_n}] \leq V_{0}^{\pi_n} = x. \]
We conclude that the sequence \( V_{T}^{\pi_n} \frac{dQ_n}{dP_n}, n \in \mathbb{N} \) is tight. This together with Assumption 4.1 yields that the sequence \( \left( \left( S^{(n)}, \frac{dQ_n}{dP_n}, V_{T}^{\pi_n} \frac{dQ_n}{dP_n} \right); P_n \right), n \in \mathbb{N} \) is tight on the space \( \mathcal{D}([0,T]; \mathbb{R}^d) \times \mathbb{R}^2 \). From Prohorov’s theorem it follows that there
exists a sub-sequence \( (S^{(n)}, \frac{dQ_n}{d\mathbb{P}_n}, V_T^{\pi_n}\frac{d\mathbb{Q}_n}{d\mathbb{P}_n}) ; \mathbb{P}_n ) \) (for simplicity the sub-sequence is still denoted by \( n \) which converge weakly. From Assumption 4.1 we obtain that
\[
\left( S^{(n)}, \frac{dQ_n}{d\mathbb{P}_n}, V_T^{\pi_n}\frac{d\mathbb{Q}_n}{d\mathbb{P}_n} \right) ; \mathbb{P}_n \Rightarrow (S, Z, Y),
\]
where \( Y \) is some random variable. In particular we have the weak convergence
\[
\left( \left( S^{(n)}, V_T^{\pi_n} \right) ; \mathbb{P}_n \right) \Rightarrow \left( S, \frac{Y}{Z} \right).
\]
The random vector \( (S, Z, Y) \) defined on a new probability space, \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\) which might be different from the original probability space \((\Omega, \mathcal{F}, \mathbb{P})\). We redefine the filtration \( \mathcal{F}_S \) and the sets \( \mathcal{M}(S), \mathcal{A}(\cdot) \) on the new probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})\). Define the random variable
\[
V = \mathbb{E}_{\tilde{\mathbb{P}}} \left( \frac{Y}{Z} \mid \mathcal{F}_T^{S} \right).
\]
From the Jensen inequality, the continuity of \( U \), Assumption 2.4 (iii), Assumption 4.2 and (4.3) we obtain
\[
\mathbb{E}_{\tilde{\mathbb{P}}} [U(V, S)] \geq \mathbb{E}_{\tilde{\mathbb{P}}} \left[ U \left( \frac{Y}{Z}, S \right) \right] \geq \lim_{n \to \infty} \mathbb{E}_{\mathbb{P}_n} [U(V_T^{\pi_n}, S^{(n)})].
\]
Thus, the final step is to show that there exists \( \pi \in \mathcal{A}(x + \epsilon) \) such that \( V_T^{\pi_n} \geq V \). In other words we need to show that the super–hedging price of \( V \) is smaller or equal than \( x \). This brings us to the second step.

**Step II:** From the optional decomposition theorem (see [18] and Remark 4.2 below) the super hedging price is given by \( \sup_{\hat{Q} \in \mathcal{M}(S)} \mathbb{E}_{\hat{Q}}[V] \). From (4.4) we obtain
\[
\sup_{\hat{Q} \in \mathcal{M}(S)} \mathbb{E}_{\hat{Q}}[V] = \sup_{\hat{Q} \in \mathcal{M}(S)} \mathbb{E}_{\hat{\mathbb{P}}} \left[ \frac{Y}{Z} \frac{d\hat{Q}}{d\hat{\mathbb{P}}} \right].
\]
Hence, it remains to prove that for any \( \hat{Q} \in \mathcal{M}(S) \)
\[
(4.5) \quad x \geq \mathbb{E}_{\hat{\mathbb{P}}} \left[ \frac{Y}{Z} \frac{d\hat{Q}}{d\hat{\mathbb{P}}} \right].
\]
From Assumption 4.1 we get a sequence \( \hat{Q}_n \in \mathcal{M}(S^{(n)}), n \in \mathbb{N} \) for which
\[
(4.6) \quad \left( S^{(n)}, \frac{d\hat{Q}_n}{d\mathbb{P}_n}, V_T^{\pi_n}\frac{d\hat{Q}_n}{d\mathbb{P}_n} \right) ; \mathbb{P}_n \Rightarrow \left( S, \frac{d\hat{Q}}{d\mathbb{P}} \right) \mathbb{P}.
\]
This together with (4.2) yields that the sequence
\[
\left( S^{(n)}, \frac{d\hat{Q}_n}{d\mathbb{P}_n}, V_T^{\pi_n}\frac{d\hat{Q}_n}{d\mathbb{P}_n} \right) ; \mathbb{P}_n, n \in \mathbb{N},
\]
is tight on the space \( \mathcal{D}([0, T]; \mathbb{R}^d) \times \mathbb{R}^3 \). From the Prohorov theorem and (4.2) there is a sub-sequence which converge weakly
\[
(4.7) \quad \left( S^{(n)}, \frac{d\hat{Q}_n}{d\mathbb{P}_n}, V_T^{\pi_n}\frac{d\hat{Q}_n}{d\mathbb{P}_n} \right) ; \mathbb{P}_n \Rightarrow (S, Z, Y, X)
\]
for some random variable \( X \).
Once again, the random vector \((S,Z,Y,X)\) defined on a new probability space, \((\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})\), on which we redefine the filtration \(\mathcal{F}^S\) and the sets \(\mathcal{M}(S), \mathcal{A}(\cdot)\). From (4.6) it follows that the distribution of \((S,X)\) equals to \(\left((S, \frac{d\hat{Q}}{d\mathbb{P}}); \mathbb{P}\right)\). Since \(\frac{d\hat{Q}}{d\mathbb{P}}\) determined by \(S\) we conclude that \(X = \frac{d\hat{Q}}{d\mathbb{P}}\). Thus, from the Fatou lemma, (4.7) and the fact that \(\{V_t^{\pi_n}\}_{t=0}^T\) is a \(\hat{Q}_n\) super–martingale it follows
\[
\mathbb{E}_{\hat{\mathbb{P}}_n} \left( \frac{\frac{d\hat{Q}}{d\mathbb{P}}}{X} \right) = \mathbb{E}_{\hat{\mathbb{P}}_n} \left( \frac{YX}{Z} \right) \leq \liminf_{n \to \infty} \mathbb{E}_{\mathbb{P}_n} \left( V_T^{\pi_n} \frac{d\hat{Q}}{d\mathbb{P}} \frac{d\mathbb{P}}{d\mathbb{P}_n} Z \right) = \liminf_{n \to \infty} \mathbb{E}_{\mathbb{Q}_n} [V_T^{\pi_n}] \leq x
\]
and (4.5) follows. \(\square\)

Remark 4.2. The optional decomposition given by Theorem 3.2 in [18] says that for a non–negative random variable \(X\) the super–hedging price is \(\sup_{\hat{Q} \in \mathcal{M}(S)} \mathbb{E}_{\hat{Q}}[X]\). Namely, if this price is finite, then for any \(\epsilon > 0\) we can find a predictable process \(\gamma = \{\gamma_t\}_{t=0}^T\) such that
\[
\epsilon + \frac{\epsilon}{2} + \sup_{\hat{Q} \in \mathcal{M}(S)} \mathbb{E}_{\hat{Q}}[X] + \int_0^T \gamma_t dS_t \geq 0, \quad t \in [0,T]
\]
and
\[
\epsilon + \frac{\epsilon}{2} + \sup_{\hat{Q} \in \mathcal{M}(S)} \mathbb{E}_{\hat{Q}}[X] + \int_0^T \gamma_t dS_t \geq X.
\]
Let us notice that a priory, \(\gamma\) does not have to be left continuous. Recall that our set of strategies given by \(\mathcal{A}(\cdot)\) contains only left continuous processes. Still, by applying the density argument given by Theorem 3.4 in [1] we obtain that there exists a predictable, continuous process \(\hat{\gamma} = \{\hat{\gamma}_t\}_{t=0}^T\) such that
\[
\sup_{0 \leq t \leq T} \left| \int_0^t \gamma_u dS_u - \int_0^t \hat{\gamma}_u dS_u \right| \leq \frac{\epsilon}{2}.
\]
Thus, there exists \(\pi \in \mathcal{A} \left( \sup_{\hat{Q} \in \mathcal{M}(S)} \mathbb{E}_{\hat{Q}}[X] + \epsilon \right)\) such that \(V_T^T \geq X\). By taking \(\epsilon \downarrow 0\) we conclude that the super–hedging price in our setup (i.e. where the trading strategies are left continuous) equals to \(\sup_{\hat{Q} \in \mathcal{M}(S)} \mathbb{E}_{\hat{Q}}[X]\).

Next, we provide two examples which demonstrate the importance of Assumptions 4.1–4.2.

Example 4.1. Let \(d = 1\). Assume that the investor utility function is given by \(U(v,s) = \min(2, \max(v,1))\) and depends only on the wealth. We notice that the function \(U\) does not satisfy Assumption 4.2.

For any \(n \in \mathbb{N}\) consider the binomial model given by
\[
S_t^{(n)} = \prod_{i=1}^k \left( 1 + \frac{\xi_i}{n^2} \right), \quad \frac{KT}{n} \leq t < \frac{(k+1)T}{n}
\]
where \(\xi_i = \pm 1, i \in \mathbb{N}\) are i.i.d. and symmetric. Namely, \(\mathbb{P}_n\) is the unique martingale measure for the \(n\)–th model. Clearly, for the constant process \(S \equiv 1\) we have the weak convergence \(S^{(n)} \Rightarrow S\). Thus, Assumptions 2.1–2.4 and Assumption 4.1 are satisfied (notice that Lemma 5.1 implies Assumption 2.3).
Next, consider the initial capital \( x = 1 \). Observe that for any \( n \), there is a set \( A_n \in \sigma\{\xi_1, \ldots, \xi_n\} \) with \( \mathbb{P}_n(A_n) = 1/2 \). Thus, from the completeness of the binomial models we get that there exists \( \pi_n \in A_n(1) \) such that \( V_T^{\pi_n} = 2I_{A_n} \). In particular,
\[
  u_n(1) \geq \mathbb{E}_{P_n}[\min(2, \max(2I_{A_n}, 1))] = 3/2, \quad n \in \mathbb{N}.
\]
On the other hand, trivially \( u(1) = 1 \), which means that Proposition 4.1 does not hold true.

**Example 4.2.** Take \( d = 1 \) and consider the Black–Scholes model
\[
S_t = e^{\sqrt{2}W_t - t}, \quad t \in [0, T].
\]
Clearly, \( \mathcal{M}(S) = \{\mathbb{P}\} \). We start with constructing a sequence of binomial models \( S^{(n)}, n \in \mathbb{N} \) such that
\[
\text{(4.8)} \quad \left(S^{(n)}; \mathbb{P}_n\right) \Rightarrow S
\]
and
\[
\text{(4.9)} \quad \left(S^{(n)}; \mathbb{Q}_n\right) \Rightarrow e^{W_t - t/2}
\]
where \( \mathbb{Q}_n \) is the unique martingale measure for \( S^{(n)} \).

For any \( n \in \mathbb{N} \) we set
\[
S_t^{(n)} = \exp\left(\sqrt{\frac{T}{n}} \sum_{i=1}^{k} \xi_i\right), \quad kT/n \leq t < (k+1)T/n
\]
where \( \xi_i = \pm 1, i \in \mathbb{N} \). Clearly, under the unique martingale measure \( \mathbb{Q}_n \) we have (4.9).

Next, we construct the sequence \( \mathbb{P}_n \) by following the ideas from [19]. Define the pure jump stochastic process
\[
M_t^{(n)} := e^{\frac{1}{2} \sqrt{T/n} \sum_{i=1}^{k} \xi_i} S_t^{(n)}, \quad kT/n \leq t < (k+1)T/n
\]
where we set \( \xi_0 \equiv 1 \). Notice that
\[
\frac{M_{kT/n}^{(n)}}{M_{(k-1)T/n}^{(n)}} = e^{\frac{1}{2} \sqrt{T/n} \xi_k - \frac{1}{2} \sqrt{T/n} \xi_{k-1}}.
\]
Let \( \mathbb{P}_n \) be the (unique) probability measure such that \( M^{(n)} \) is a martingale. This is equivalent to,
\[
\mathbb{E}_{\mathbb{P}_n}\left(e^{\frac{1}{2} \sqrt{T/n} \xi_k} \mid \xi_1, \ldots, \xi_{k-1}\right) = e^{\frac{1}{2} \sqrt{T/n} \xi_k - \frac{1}{2} \sqrt{T/n} \xi_{k-1}}, \quad k = 1, \ldots, n.
\]
Since \( \xi = \pm 1 \), we get
\[
\mathbb{P}_n(\xi_k = 1|\xi_1, \ldots, \xi_{k-1}) = 1 - \mathbb{P}_n(\xi_k = -1|\xi_1, \ldots, \xi_{k-1}) = e^{\frac{1}{2} \sqrt{T/n} \xi_{k-1} - \frac{1}{2} \sqrt{T/n} \xi_k} - e^{-\frac{1}{2} \sqrt{T/n} \xi_{k-1}} \in (0, 1),
\]
and so, \( \mathbb{P}_n \) is indeed a probability measure. Taylor’s expansion yields that
\[
\frac{M_{kT/n}^{(n)}}{M_{(k-1)T/n}^{(n)}} - 1 = \sqrt{\frac{T}{n}} \frac{3\xi_k - \xi_{k-1}}{2} + O(1/n).
\]
Hence, $\mathbb{E}_{P_n}(\xi_k|\xi_1, \ldots, \xi_{k-1}) = \frac{1}{3}\xi_{k-1} + O(1/\sqrt{n})$. This together with the fact that $\xi^2 \equiv 1$ gives

$$
\mathbb{E}_{P_n}\left(\left(\frac{M(n)}{M'(n)T/n} - 1\right)^2 |\xi_1, \ldots, \xi_{k-1}\right) = \frac{T}{4n} \left(10 - 6\xi_{k-1}\frac{1}{3}\xi_{k-1}\right) + O(n^{-3/2}).
$$

From the martingale invariance principle we obtain that $(M(n); P_n) \Rightarrow \{e^{\frac{3}{2}W_t - t}\}_{t=0}^T$, and from the estimate $|S(n) - M(n)| = O(1/\sqrt{n})$ we conclude (4.8).

Finally, let $K > 0$ and consider a call option with strike price $K$. For any $n \in \mathbb{N}$ let $V_n$ be the unique arbitrage free price of this call option in the (complete) model given by $S^{(n)}$. From (4.9) we get

$$
\lim_{n \to \infty} V_n = \mathbb{E}\left(\left(e^{\frac{3}{2}W_t - t} - K\right)^+\right) < \mathbb{E}\left(\left(e^{\sqrt{2}W_t - t} - K\right)^+\right)
$$

where the last inequality follows from the fact that the call option price in a Black–Scholes model is strictly increasing in volatility. Thus, for sufficiently large $n$,

$$
V_n < x := \mathbb{E}\left(\left(e^{\sqrt{2}W_t - t} - K\right)^+\right).
$$

In particular, for the utility function given by (3.10), for sufficiently large $n$, $u_n(x) = 0$. On the other hand, since $\mathbb{P}$ is a martingale measure and the call option price is increasing in volatility we obtain

$$
u(x) \leq -\inf_{\pi \in A(x)} \mathbb{E}_{\pi}[S_T - K]^+ - V_{\pi}]
$$

$$= x - \mathbb{E}_{\pi}[S_T - K]^+] < 0.
$$

We conclude that although Assumptions 2.1–2.4 and Assumption 4.2 holds true, Proposition 4.1 in not satisfied.

We finish this section with the following remark.

**Remark 4.3.** The paper [31] studies the continuity of the value of the utility maximization problem from terminal wealth (under convergence in distribution) for a state independent utility function in a complete market. The author does not assume that the utility function is concave. The main result says that if the limit probability space is atomless and the atoms in approximating sequence of models are vanishing (see Assumption 2.1 in [31]) then continuity holds. Clearly, this is not satisfied in the Example 4.1 above where the filtration generated by the limit process is trivial.

An open question is to understand whether the continuity result from [31] can be extended to the incomplete case.

5. Auxiliary Results

The following result provides a simple and quite general condition which implies Assumption 2.3.

**Lemma 5.1.** Assume that Assumption 2.2 holds true and there exists continuous functions $m_1, m_2 : [0, 1) \to \mathbb{R}_+$ with $m_1(0) = m_2(0) = 0$ (modulus of continuity) and a non negative random variable $\zeta \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ such that for any $\lambda \in (0, 1)$ and $v > 0$

$$U((1 - \lambda)v, S) \geq (1 - m_1(\lambda))U(v, S) - m_2(\lambda)\zeta.$$
Then Assumption 2.3 holds true.

Proof. In view of the fact that $u$ is a non-decreasing function (follows from Assumption 2.2 (i)) it sufficient to prove that for any $x > 0$

$$\lim_{\alpha \downarrow 0} u((1 - \alpha)x) \geq \lim_{\alpha \downarrow 0} u((1 + \alpha)x).$$

For any $\beta, y > 0$ the map $(y, \{\gamma\}_{t=0}^T) \rightarrow (\beta y, \{\beta \gamma\}_{t=0}^T)$ is a bijection between $A(y)$ and $A(\beta y)$. Thus,

$$\lim_{\alpha \downarrow 0} u((1 - \alpha)x) \geq \lim_{\alpha \downarrow 0} \left( (1 - m_1 \left( 1 - \frac{1-\alpha}{1+\alpha} \right) ) u((1 + \alpha)x) - m_2 \left( 1 - \frac{1-\alpha}{1+\alpha} \right) \mathbb{E}[\xi] \right) = \lim_{\alpha \downarrow 0} u((1 + \alpha)x).$$

\[\square\]

Remark 5.1. We notice that the power and the log utility satisfy the assumptions of Lemma 5.1. On the other hand for these utility functions Assumption 2.3 is straight forward.

A “real” application of Lemma 5.1 is the case which corresponds to the utility function given by (3.10). In this case, if $v > 0$ then $U((1 - \lambda)v, S) = U(v, S) = 0$. If $v < \frac{S_T}{1-\lambda}$ then $|U((1 - \lambda)v, S) - U(v, S)| \leq \lambda v \leq \frac{1}{1-\lambda} S_T$. Thus, for $m_1(\lambda) \equiv 0$, $m_2(\lambda) = \frac{1}{1-\lambda}$ and $\xi = S_T$ the assumptions of Lemma 5.1 hold true (provided that $\mathbb{E}[S_T] < \infty$).

Next, we treat Assumption 2.4 (iii).

Lemma 5.2. Suppose there exists constants $C > 0$, $0 < \gamma < 1$ and $q > \frac{1}{1-\gamma}$ which satisfy the following.

(I) For all $(v, s) \in (0, \infty) \times \mathbb{D}([0, T]; \mathbb{R}^d)$,

$$U(v, s) \leq C(1 + v^\gamma).$$

(II) For any $n$ there exists a local martingale measure $Q_n \in \mathcal{M}(S^n)$ such that

$$\sup_{n \in N} \mathbb{E}_{Q_n} \left[ \left( \frac{dP_n}{dQ_n} \right)^q \right] < \infty.$$

Then Assumption 2.4 (iii) holds true.

Proof. Let $p = \frac{q}{q - 1}\gamma$. Clearly $\frac{1}{p} > \gamma$. Thus in view of (5.1), in order to prove that Assumption 2.4 (iii) holds true, it suffices to show that for any $x > 0$

$$\sup_{n \in N} \sup_{\pi \in A_n(x)} \mathbb{E}_{P_n} [(V_T^x)^{1/p}] < \infty.$$

Indeed, from the Holder inequality (observe that $\frac{1}{p} + \frac{1}{q} = 1$) we get

$$\sup_{n \in N} \sup_{\pi \in A_n(x)} \mathbb{E}_{P_n} [(V_T^x)^{1/p}]$$

$$= \sup_{n \in N} \sup_{\pi \in A_n(x)} \mathbb{E}_{Q_n} \left[ (V_T^x)^{1/p} \frac{dP_n}{dQ_n} \right]$$

$$\leq \sup_{n \in N} \sup_{\pi \in A_n(x)} \left( \mathbb{E}_{Q_n} [V_T^x]^1 \right)^{1/p} \sup_{n \in N} \left( \mathbb{E}_{Q_n} \left[ \left( \frac{dP_n}{dQ_n} \right)^q \right] \right)^{1/q}$$

$$\leq x^{1/p} \sup_{n \in N} \left( \mathbb{E}_{Q_n} \left[ \left( \frac{dP_n}{dQ_n} \right)^q \right] \right)^{1/q} < \infty,$$

and the result follows. \[\square\]
6. Lattice Based Approximations of the Heston Model

Consider the Heston model \[14\] given by

\[
dS_t = \hat{S}_t(\mu dt + \sqrt{\nu_t}dW_t),
\]
\[
d\nu_t = \kappa(\theta - \nu_t)dt + \sigma\sqrt{\nu_t}d\hat{W}_t,
\]
where \(\mu \in \mathbb{R}, \kappa, \theta, \sigma > 0\) are constants and \(W, \hat{W}\) are two standard Brownian motions with a constant correlation \(\rho \in (-1, 1)\). The initial values \(\hat{S}_0, \tilde{\nu}_0 > 0\) are given. We assume the condition \(2\kappa \theta > \sigma^2\) which guarantees that \(\nu\) does not touch zero (see \[5\]).

For technical reasons our approximations require that the volatility will lie in an interval of the form \([a, \sigma]\) for some \(0 < a < \sigma\). Thus, we modify the Heston model as following. Fix two barriers \(0 < a < \sigma\) and define the function \(h(z) = \max(\sigma^2, \min(z, \sigma^2)), z \in \mathbb{R}\). Consider the SDE

\[
dS_t = S_t(\mu dt + \sqrt{h(\nu_t)}dW_t)
\]
\[
d\nu_t = \kappa(\theta - h(\nu_t))dt + \sigma\sqrt{h(\nu_t)}d\hat{W}_t
\]
where the initial values are \(S_0 = \hat{S}_0, \nu_0 = \tilde{\nu}_0\). Observe that \(\sqrt{h}, h\) are Lipschitz continuous, and so (6.1) has a unique solution.

We expect that if \(\sigma\) is small and \(\sigma\) is large then the value of the utility maximization problem in the Heston model will be close to the one in the model given by (6.1). For the shortfall risk measure we provide an error estimate in Lemma 7.1.

6.1. Discretization. In this section we construct discrete time lattice based approximations for the model given by (6.1). The novelty of our constructions is that the approximating sequence satisfies Assumptions 3.1, 4.1.

It is more convenient to work with a transformed system of equations driven by independent Brownian motions. Therefore, we set

\[
\Phi_t = \ln S_t, \quad \Psi_t = \frac{\nu_t}{\sigma} - \rho \Phi_t.
\]

From the Itô formula we obtain that

\[
d\Phi_t = \mu_\Phi(\Phi_t, \Psi_t)dt + \sigma_\Phi(\Phi_t, \Psi_t)dW_t,
\]
\[
d\Psi_t = \mu_\Psi(\Phi_t, \Psi_t)dt + \sigma_\Psi(\Phi_t, \Psi_t)d\hat{W}_t
\]
where

\[
\mu_\Phi(y, z) = \mu - h(\sigma(\rho y + z))/2, \quad \sigma_\Phi(y, z) = \sqrt{h(\sigma(\rho y + z))},
\]
\[
\mu_\Psi(y, z) = \frac{\kappa}{\sigma}(\theta - h(\sigma(\rho y + z))) - \rho \mu_\Phi(y, z), \quad \sigma_\Psi = \sqrt{(1 - \rho^2)}\sigma_\Phi.
\]

and \(\hat{W}^2 := \frac{\hat{W} - \rho W}{\sqrt{1 - \rho^2}}\) is a Brownian motion independent of \(W\).

Next, we define lattice based approximations for the process \((\Phi, \Psi)\). Choose \(\hat{\sigma} \geq \sigma\). For any \(n \in \mathbb{N}\) define the stochastic processes \(\Phi_t^{(n)}, \Psi_t^{(n)}, t \in [0, T]\) by

\[
\Phi_t^{(n)} = \Phi_0 + \hat{\sigma}\sqrt{\frac{T}{n}} \sum_{i=1}^{k} \xi_i, \quad \frac{kT}{n} \leq t < \frac{(k+1)T}{n},
\]
\[
\Psi_t^{(n)} = \Psi_0 + \hat{\sigma}\sqrt{\frac{T}{n}} \sum_{i=1}^{k} \xi_i, \quad \frac{kT}{n} \leq t < \frac{(k+1)T}{n}
\]
where \( \xi, \hat{\xi} \in \{-1, 0, 1\} \). Observe that the processes \( \Phi^{(n)} - \Phi_0, \Psi^{(n)} - \Psi_0 \) lie on the grid \( \bar{\sigma} \sqrt{\frac{T}{n}} \{-n, 1 - n, ..., n\} \).

Let \( \mathcal{F}^{(n)}_t, t \leq T \) be the piece wise constant filtration generated by the processes \( \Phi^{(n)}, \Psi^{(n)} \). It remains to define the probability measure \( P_n \). First since \( W \) and \( \tilde{W} \) are independent Brownian motions we require that for all \( a, b \in \{-1, 0, 1\} \) and \( k \geq 1 \)

\[
P_n(\xi_k = a, \hat{\xi}_k = b|\mathcal{F}^{(n)}_t) = P_n(\xi_k = a|\mathcal{F}^{(n)}_t)P_n(\hat{\xi}_k = b|\mathcal{F}^{(n)}_t).
\]

In order to match the drift and the volatility, we set,

\[
P_n(\xi_k = \pm 1|\mathcal{F}^{(n)}_t) = \frac{\sqrt{\frac{T}{n}}}{\bar{\sigma}} \left( \Phi^{(n)}_{(k-1)T}, \Psi^{(n)}_{(k-1)T} \right) \pm \sqrt{\frac{T}{n}} \frac{\mu \Phi^{(n)}_{(k-1)T}, \Psi^{(n)}_{(k-1)T}}{\bar{\sigma}^2},
\]

\[
P_n(\xi_k = 0|\mathcal{F}^{(n)}_t) = 1 - \frac{\sqrt{\frac{T}{n}}}{\bar{\sigma}} \left( \Phi^{(n)}_{(k-1)T}, \Psi^{(n)}_{(k-1)T} \right)
\]

and

\[
P_n(\xi_k = \pm 1|\mathcal{F}^{(n)}_t) = \frac{\sqrt{\frac{T}{n}}}{\bar{\sigma}} \left( \Phi^{(n)}_{(k-1)T}, \Psi^{(n)}_{(k-1)T} \right) \pm \sqrt{\frac{T}{n}} \frac{\mu \Phi^{(n)}_{(k-1)T}, \Psi^{(n)}_{(k-1)T}}{\bar{\sigma}^2},
\]

\[
P_n(\xi_k = 0|\mathcal{F}^{(n)}_t) = 1 - \frac{\sqrt{\frac{T}{n}}}{\bar{\sigma}} \left( \Phi^{(n)}_{(k-1)T}, \Psi^{(n)}_{(k-1)T} \right).
\]

Observe that for sufficiently large \( n \), the right hand side of the above equation lies in the interval \([0, 1]\).

**Proposition 6.1.** For any \( n \in \mathbb{N} \) (sufficiently large) consider the financial market given by \( S^{(n)} := e^{\Phi^{(n)}} \) and the filtration \( \mathcal{F}^{(n)} \) defined above. Then, the following holds true.

(I) We have the weak convergence \( S^{(n)} \Rightarrow S \) to the modified Heston model.

(II) Assumption 3.1 holds true.

**Proof.**

(I) Let us prove that

\[
(\Phi^{(n)}, \Psi^{(n)}) \Rightarrow (\Phi, \Psi).
\]

Clearly, (6.2) implies that \( S^{(n)} \Rightarrow S \).

From the definition of \( P_n \) we have

\[
\mathbb{E}_{P_n} \left( \Phi^{(n)}_{(k-1)T} - \Phi_0 \mathcal{F}^{(n)}_{(k-1)T} \right) = \frac{T}{n} \mu \Phi^{(n)}_{(k-1)T}, \Psi^{(n)}_{(k-1)T} \),
\]

\[
\mathbb{E}_{P_n} \left( \Psi^{(n)}_{(k-1)T} - \Psi_0 \mathcal{F}^{(n)}_{(k-1)T} \right) = \frac{T}{n} \mu \Phi^{(n)}_{(k-1)T}, \Psi^{(n)}_{(k-1)T} \),
\]

\[
\mathbb{E}_{P_n} \left( (\Phi^{(n)}_{(k-1)T} - \Phi_0)^2 \mathcal{F}^{(n)}_{(k-1)T} \right) = \frac{T}{n} \sigma_0^2 \Phi^{(n)}_{(k-1)T}, \Psi^{(n)}_{(k-1)T} \),
\]

\[
\mathbb{E}_{P_n} \left( (\Psi^{(n)}_{(k-1)T} - \Psi_0)^2 \mathcal{F}^{(n)}_{(k-1)T} \right) = \frac{T}{n} \sigma_0^2 \Phi^{(n)}_{(k-1)T}, \Psi^{(n)}_{(k-1)T} \).
\]
and
\[
\mathbb{E}_{\mathbb{P}_n} \left( \frac{(\Phi^{(n)}_{kT} - \Phi^{(n)}_{(k-1)T})(\Psi^{(n)}_{kT} - \Psi^{(n)}_{(k-1)T})}{n} \right) = O(n^{-2}).
\]

Thus, (6.2) follows from the martingale convergence result Theorem 7.4.1 in [11].

(II) Since the drift and the volatility are uniformly bounded we have a uniform bound on the exponential moments
\[
\sup_{n \in \mathbb{N}} \mathbb{E}_{\mathbb{P}_n} \left[ \sup_{0 \leq t \leq T} e^{\Phi^{(n)}_t} \right] < \infty, \ \forall \alpha > 0.
\]

In particular (for \( a = 1 \))
\[
\sup_{n \in \mathbb{N}} \mathbb{E}_{\mathbb{P}_n} \left[ \sup_{0 \leq t \leq T} S^{(n)}_t \right] < \infty.
\]

Similarly to Example 3.1, we have
\[
\text{ess sup} \left( \left| S^{(n)}_{kT} - S^{(n)}_{(k-1)T} \right| \right) \leq \left( e^{\sqrt{T}} - 1 \right) S^{(n)}_{(k-1)T}.
\]

Thus,
\[
\lim_{n \to \infty} \mathbb{E}_{\mathbb{P}_n} \left( \max_{1 \leq k \leq n} \text{ess sup} \left( \left| S^{(n)}_{kT} - S^{(n)}_{(k-1)T} \right| \right) \right) = 0
\]
as required.

\[\square\]

Next, we want to establish Assumption 4.1. We start with some preparations.

Denote by \( \mathcal{D} \) the set of all stochastic processes \( \Upsilon = \{ \Upsilon_t \}_{t=0}^T \) of the form \( \Upsilon = F(\Phi, \Psi) \) where \( F : \mathbb{D}[0, T] \times \mathbb{D}[0, T] \to \mathbb{D}[0, T] \) is a bounded, continuous function (we take the Skorokhod topology on the space \( \mathbb{D}[0, T] \)) and \( F \) is a progressively measurable map. Namely, for any \( t \in [0, T] \) and \( f^{(1)}, g^{(1)}, f^{(2)}, g^{(2)} \in \mathbb{D}[0, T] \), \( f^{(1)}_{[0, t]} = f^{(2)}_{[0, t]} \) if and only if \( f^{(1)}_t = f^{(2)}_t \).

Define the set
\[
\mathcal{M}^d(S) := \left\{ Q : \exists \Upsilon \in \mathcal{D}, \frac{dQ}{dP} |S_T^\Upsilon = e^\int_0^T \frac{\Theta}{\sqrt{\Theta(t)}} dW_t + f^{(1)}_T \sqrt{\Theta(t)} d\Upsilon_t - f^{(2)}_T \Theta(t) dt - f^{(2)}_T \Upsilon(t) dt \right\}
\]

From the Girsanov theorem it follows that \( \mathcal{M}^d(S) \subset \mathcal{M}(S) \). Moreover, the filtration \( F^S \) is generated by the processes \( \Phi, \Psi \), hence standard arguments yield that \( \mathcal{M}^d(S) \subset \mathcal{M}(S) \) is dense.

Choose an arbitrary \( \Upsilon = F(\Phi, \Psi) \in \mathcal{D} \) and denote
\[
Z_t = e^{\int_0^T \frac{\Theta}{\sqrt{\Theta(t)}} dW_t + f^{(1)}_T \sqrt{\Theta(t)} d\Upsilon_t - f^{(2)}_T \Theta(t) dt - f^{(2)}_T \Upsilon(t) dt}, \quad t \in [0, T].
\]

It is sufficient to prove that (recall Remark 4.1) there exists a sequence of probability measures \( Q_n \in \mathcal{M}(S^{(n)}) \), \( n \in \mathbb{N} \), such that for the processes \( Z^{(n)}_t := \frac{dQ_n}{dP_n} |F^{(n)}_t \), \( t \in [0, T] \), we have the weak convergence
\[
(S^{(n)}, Z^{(n)}) \Rightarrow (S, Z).
\]

For any \( n \in \mathbb{N} \) (sufficiently large) define the probability measure \( Q_n \) by the following relations
\[
Q_n(\xi_k = a, \hat{\xi}_k = b|F^{(n)}_{(k-1)T}) = Q_n(\xi_k = a|F^{(n)}_{(k-1)T}) Q_n(\hat{\xi}_k = b|F^{(n)}_{(k-1)T}),
\]
We have the weak convergence

$$Q_n(\xi_k = \pm 1|\mathcal{F}_{(k-1)T}^{(n)}) = \frac{\sigma_0^2 \left( \Phi_{(k-1)T}^{(n)} \Psi_{(k-1)T}^{(n)} \right)}{\sigma^2 \left( 1 + e^{-2\sigma} \sqrt{\frac{1}{n}} \right)},$$

$$Q_n(\xi_k = 0|\mathcal{F}_{(k-1)T}^{(n)}) = 1 - \frac{\sigma_0^2 \left( \Phi_{(k-1)T}^{(n)} \Psi_{(k-1)T}^{(n)} \right)}{\sigma^2},$$

and

$$Q_n(\hat{\xi}_k = \pm 1|\mathcal{F}_{(k-1)T}^{(n)}) = \frac{\sigma_0^2 \left( \Phi_{(k-1)T}^{(n)} \Psi_{(k-1)T}^{(n)} \right)}{2\sigma^2}$$

$$\pm \sqrt{\frac{T}{n}} \frac{F_{(k-1)T}^{(n)}(\Phi^{(n)},\Psi^{(n)})\sigma \Phi_{(k-1)T}^{(n)} \Psi_{(k-1)T}^{(n)} + \mu \Phi_{(k-1)T}^{(n)} \Psi_{(k-1)T}^{(n)}}{2\sigma},$$

$$Q_n(\hat{\xi}_k = 0|\mathcal{F}_{(k-1)T}^{(n)}) = 1 - \frac{\sigma_0^2 \left( \Phi_{(k-1)T}^{(n)} \Psi_{(k-1)T}^{(n)} \right)}{\sigma^2}.$$

Observe that (6.7) implies $Q_n \in \mathcal{M}(S^{(n)})$.

**Lemma 6.1.** We have the weak convergence

$$(\Phi^{(n)}, \Psi^{(n)}, Z^{(n)}) \Rightarrow (\Phi, \Psi, Z).$$

**Proof.** In order to prove the lemma it suffices to show that for any sub-sequence $\{n_i\}_{i=1}^{\infty}$ there exists a sub-sub-sequence $\{n_{i,j}\}_{j=1}^{\infty} \subset \{n_i\}_{i=1}^{\infty}$ such that

$$Q_n(\Phi^{(n_{i,j})}, \Psi^{(n_{i,j})}, Z^{(n_{i,j})}) \Rightarrow (\Phi, \Psi, Z).$$

Fix $n \in \mathbb{N}$. By applying Taylor’s expansion we obtain that there exist uniformly bounded (in $n$) processes $E_k^{n,1}, E_k^{n,2}, k = 0, 1, ..., n$ such that

$$Q_n(\xi_k|\mathcal{F}_{(k-1)T}^{(n)}) = 1 - \hat{\sigma}_k \sqrt{\frac{T}{n}} \left( 1 + \frac{\mu \Phi_{(k-1)T}^{(n)} \Psi_{(k-1)T}^{(n)}}{\sigma_0^2} \right) + \frac{E_k^{n,1}}{n} + o(1/n)$$

and

$$Q_n(\hat{\xi}_k|\mathcal{F}_{(k-1)T}^{(n)}) = 1 + \hat{\sigma}_k \sqrt{\frac{T}{n}} \frac{F_{(k-1)T}^{(n)}(\Phi^{(n)}, \Psi^{(n)})}{\sigma_0 \Phi_{(k-1)T}^{(n)} \Psi_{(k-1)T}^{(n)}} + \frac{E_k^{n,2}}{n} + o(1/n).$$
We conclude that there exists a uniformly bounded process $E^{(n)}_k, k = 0, 1, \ldots, n$ such that

\begin{equation}
\frac{Z^{(n)}_{(k-1)T} - Z^{(n)}_{kT}}{Z^{(n)}_{(k-1)T}} = \frac{Q_n(\xi_k | \mathcal{F}^{(n)}_{(k-1)T})}{P_n(\xi_k | \mathcal{F}^{(n)}_{(k-1)T})} - 1 = \frac{Q_n(\xi_k | \mathcal{F}^{(n)}_{(k-1)T})}{P_n(\xi_k | \mathcal{F}^{(n)}_{(k-1)T})}
\end{equation}

\begin{equation}
= -\left( \Phi^{(n)}_{kT} - \Phi^{(n)}_{(k-1)T} \right) \left( \frac{1}{2} + \frac{\mu_{\hat{\Phi}}}{\sigma^2_{\hat{\Phi}}(\Phi^{(n)}_{(k-1)T}, \Psi^{(n)}_{(k-1)T})} \right)
\end{equation}

\begin{equation}
= \frac{\Psi^{(n)}_{(k-1)T} - \Psi^{(n)}_{kT}}{\sigma_{\hat{\Phi}}(\Phi^{(n)}_{(k-1)T}, \Psi^{(n)}_{(k-1)T})} + \frac{E^{(n)}_{kT}}{n} + o(1/n).
\end{equation}

Introduce the process $\Xi^{(n)} = \int_0^1 E^{(n)}_{[n']T} \, dt', \quad t \in [0, T]$ where $[\cdot]$ is the integer part of $\cdot$. Since $E^{(n)}$, $n \in \mathbb{N}$, are uniformly bounded, the sequence $\Xi^{(n)}$, $n \in \mathbb{N}$, is tight. This yields that the sequence $(\Phi^{(n)}, \Psi^{(n)}, \Xi^{(n)})$, $n \in \mathbb{N}$, is tight as well. Thus, from the Prohorov theorem it follows that for any sub-sequence $\{n_i\}_{i=1}^\infty$ there exists a sub-sub-sequence $\{n_{i_j}\}_{j=1}^\infty$ such that

\begin{equation}
(\Phi^{(n_{i_j})}, \Psi^{(n_{i_j})}, \Xi^{(n_{i_j})}) \Rightarrow (\Phi, \Psi, \Xi)
\end{equation}

for some absolutely continuous process $\Xi = \{\Xi_t\}_{t=0}^T$. From the stability result Theorem 4.4 in [10] and (6.9)–(6.10) we obtain that

\begin{equation}
(\Phi^{(n_{i_j})}, \Psi^{(n_{i_j})}, \Xi^{(n_{i_j})}, Z^{(n_{i_j})}) \Rightarrow (\Phi, \Psi, \Xi, M)
\end{equation}

where $M$ the solution of the SDE

\begin{equation}
\frac{dM_t}{M_t} = -\left( \frac{1}{2} + \frac{\mu_{\hat{\Phi}}(\Psi_t, \Psi_t)}{\sigma^2_{\hat{\Phi}}(\Phi_t, \Psi_t)} \right) \, d\Phi_t + \frac{\Upsilon_t}{\sigma_{\Psi}(\Phi_t, \Psi_t)} \, d\Psi_t + \frac{d\Xi_t}{T}
\end{equation}

with the initial condition $M_0 = 1$.

Finally, since for any $n$, $Z^{(n)}$ is a uniformly integrable martingale with respect to the filtration generated by $\Phi^{(n)}, \Psi^{(n)}, \Xi^{(n)}, Z^{(n)}$, we have that $M$ is a martingale with respect to the filtration generated by $\Phi, \Psi, \Xi, M$. Moreover, from (6.3)–(6.4) we get that $\{ \Phi_t - \int_0^t \mu_{\Phi}(\Phi_t, \Psi_t) \, dt' \}_{t=0}^T$ and $\{ \Psi_t - \int_0^t \mu_{\Psi}(\Phi_t, \Psi_t) \, dt' \}_{t=0}^T$ are martingales with respect to the filtration generated by $\Phi, \Psi, \Xi, M$. In particular, from the Lévy theorem it follows that the stochastic processes $W$ and $\hat{W}$ which we redefine by

\begin{equation}
W_t = \Phi_t - \int_0^t \mu_{\Phi}(\Phi_t, \Psi_t) \, dt', \quad \hat{W}_t = \Psi_t - \int_0^t \mu_{\Psi}(\Phi_t, \Psi_t) \, dt'
\end{equation}

are (independent) Brownian motions with respect to the filtration generated by $\Phi, \Psi, \Xi, M$. We conclude that the drift of the right hand side of (6.11) is equal to zero. Namely,

\begin{equation}
\frac{dM_t}{M_t} = -\left( \frac{1}{2} + \frac{\mu_{\Phi}(\Phi_t, \Psi_t)}{\sigma^2_{\Phi}(\Phi_t, \Psi_t)} \right) \, d\Phi_t + \frac{\Upsilon_t}{\sigma_{\Psi}(\Phi_t, \Psi_t)} \, d\Psi_t + \frac{d\Xi_t}{T}
\end{equation}

where the last equality follows from (6.5). Hence, $M = Z$ and (6.8) follows.

Clearly, Lemma 6.1 implies (6.6). This gives us the following result.

\textbf{Proposition 6.2.} Consider the set-up of Proposition 6.1. Assumption 4.1 holds true.
We end this section by addressing condition (II) in Lemma 5.2.

**Remark 6.1.** Consider the martingale measures \( Q_n \in \mathcal{M}(S^{(n)}) \), \( n \in \mathbb{N} \) which were defined before Lemma 6.1 for \( Y \equiv 0 \). Since \( \mu^f, \sigma^f, \frac{1}{\sigma^2} \) are uniformly bounded, then standard arguments yield that for any \( q > 0 \) (5.2) holds true.

7. Numerical Results for the Shortfall Risk in the Heston Model

In this section we focus on shortfall risk minimization for European call options (which corresponds to \( U \) given by (3.10)) in the Heston model.

7.1. Computation of the shortfall risk in the approximating model. We start with the following estimate.

**Lemma 7.1.** For an initial capital \( x \) let \( \hat{R}(x) \) be the shortfall risk in the Heston model and let \( R(x) \) be the shortfall risk in the model given by (6.1). Then for any \( m \in \mathbb{N} \)

\[
|\hat{R}(x) - R(x)| \leq O(\sigma^{2\theta/\sigma^2 - 1}) + O(1/\sigma^m)
\]

where the \( O \) terms do not depend on \( x \).

**Proof.** Define the stopping time

\[
\Theta = T \wedge \inf \{ t : \hat{\nu}_t \notin [\sigma, \bar{\sigma}] \}.
\]

Observe that on the event \( \Theta = T \) the processes \( \hat{S} \) and \( S \) coincide. Hence,

\[
|\hat{R}(x) - R(x)| \leq E_P[(\hat{S}_T + S_T)I_{\Theta < T}] \leq 2e^{\mu T}E_P[e^{-\mu \theta \hat{S}_\theta}I_{\Theta < T}] (7.1)
\]

where the last inequality is due to the fact that the processes \( e^{-\mu t} \hat{S}_t, e^{-\mu t} S_t, t \in [0, T] \) are martingales.

Next, introduce the probability measure \( P \) by \( \frac{dP}{dP} |_{F_T} = e^{\mu t} \tilde{S}_t \). Then by Girsanov theorem the process \( W_t := \tilde{W}_t - \rho \int_0^{t \wedge \Theta} \sqrt{\hat{S}_u} du, t \in [0, T] \), is a Brownian motion with respect to \( P \). Clearly,

\[
d\hat{\nu}_t = (\kappa(\theta - \hat{\nu}_t) + \sigma \rho \hat{\nu}_t) dt + \sigma \sqrt{\hat{\nu}_t} dW_t.
\]

Standard comparison arguments yield that for any \( m \in \mathbb{N} \) we have

\[
E_P \left( \sup_{0 \leq t \leq T} \hat{\nu}_t^m \right) < \infty.
\]

Thus, from the Markov inequality we get

\[
P(\sup_{0 \leq t \leq T} \hat{\nu}_t \geq \sigma) = O(1/\sigma^m), \quad \forall m \in \mathbb{N}.
\]

Moreover, from Theorem 2 in [13] it follows that

\[
P(\inf_{0 \leq t \leq T} \hat{\nu}_t \leq \sigma) = O(\sigma^{2\theta/\sigma^2 - 1}).
\]

By combining (7.1)–(7.3) we conclude that

\[
|\hat{R}(x) - R(x)| \leq 2S_0 e^{\mu T} \left( P(\inf_{0 \leq t \leq T} \hat{\nu}_t \leq \sigma) + P(\sup_{0 \leq t \leq T} \hat{\nu}_t \geq \sigma) \right) \leq O(\sigma^{2\theta/\sigma^2 - 1}) + O(1/\sigma^M)
\]

as required. \( \square \)
Next, we focus on approximating the shortfall risk in the model given by (6.1). In view of Theorem 2.1, Lemma 6.1 and Corollary 6.2 we know that the shortfall risk for the discrete time models (constructed in the previous section) converges to the shortfall risk in the model given by (6.1). Observe that for the shortfall risk measure Assumption 2.4 (iii) is straightforward, and so we do not need Lemma 5.2.

Thus, fix $n \in \mathbb{N}$. The discrete time stochastic process $\{S_k^{(n)}, \nu_k^{(n)}\}_{k=0}^n$ is a Markov chain. Hence, we introduce the functions $J_k^{(n)}(s, v, z)$, $k = 0, 1, \ldots, n$ which measures the shortfall risk at time $k$ given that $S_k^{(n)} = s$, $\nu_k^{(n)} = v$ and the portfolio value is $z \geq 0$. Observe that $S_k^{(n)}$, $\nu_k^{(n)}$ are determined by $\sum_{i=1}^k \xi_i$ and $\sum_{i=1}^k \hat{\xi}_i$. Thus the variables $s, v$ can be discretized efficiently. Clearly, if $z \geq s$ the shortfall risk is zero because we can buy the stock and hold in until maturity. Namely, $J_k^{(n)}(s, v, z) = 0$ for $z \geq s$. Hence, we assume that $z = \lambda s$ where $\lambda \in [0, 1]$.

Next, we describe the dynamic programming principle to solve the discrete control-problem. At time $k$ the investor decides about his investment policy. Assume that the investor portfolio value is $\lambda S_k^{(n)}$. We have a trinomial model with growth rates $\{e^{-\hat{\sigma} \sqrt{T/n}}, 0, e^{\hat{\sigma} \sqrt{T/n}}\}$. From the binomial representation theorem we easily deduce that the set of replicable portfolios at time $k + 1$ are of the form $\Lambda(\xi_{k+1}) S_{k+1}^{(n)}$ where $\Lambda : \{-1, 0, 1\} \to \mathbb{R}$ satisfies $\Lambda(0) = \lambda$ and

$$
\Lambda(-1) + \Lambda(1)e^{\hat{\sigma} \sqrt{T/n}} = \lambda.
$$

Thus, if $\Lambda(-1)$ is known then we set

$$
(7.4) \quad \Lambda(1) = 1 \land \left( \lambda(1 + e^{-\hat{\sigma} \sqrt{T/n}}) - \Lambda(-1)e^{-\hat{\sigma} \sqrt{T/n}} \right).
$$

We take a truncation in order to have $\Lambda(1) \in [0, 1]$. In view of our admissibility condition, we denote by $\mathcal{A}(\lambda)$ the set of all $\Lambda(-1) \in [0, 1]$ for which the right hand side of (7.4) is non-negative.

We arrive to the following recursive relations. Define

$$
J_k^{(n)}(i, j, \lambda) = \begin{cases} 
\lambda S_0 \exp \left( i\hat{\sigma} \sqrt{\frac{T}{n}} \right), S_0 \exp \left( i\hat{\sigma} \sqrt{\frac{T}{n}} \right), & \text{if } k < n,
\end{cases}
$$

and for $k = n$,

$$
(7.5) \quad J_n^{(n)}(i, j, \lambda) = \sup_{\Lambda(-1) \in \mathcal{A}(\lambda)} \mathbb{E}_{\hat{\sigma}} \left( J_{k+1}^{(n)} \left( i + \xi_{m+1}, j + \hat{\xi}_{m+1}, \Lambda(\xi_{m+1}) \right) \right)
$$

where $\Lambda(0) = \lambda$ and $\Lambda(1)$ is given by (7.4). For $k = 0$ we have $J_0^{(n)}(x/S_0) = U_n(x)$.

Observe that the functions $J_k^{(n)}(i, j, \lambda)$ are piece wise linear and continuous in $\lambda$, and so they can be represented by an array which consists of the points where the slope has a jump and the slope values. This together with the condition
$J_k^{(n)}(i, j, 1) = 0$ is sufficient to recover the function. Of course the array will depend on time $k$ and the states $i, j$. Thus, theoretically, the dynamic programming given by (7.5) can be implemented using a computer. However, from practical point of view the number of the slope points of the function $J_k^{(n)}$ grows exponentially (in $n - k$), and so for large $n$ it cannot be implemented. Hence, we need to introduce a grid structure for the portfolio value as well.

Thus, choose $M \in \mathbb{N}$ and consider the grid

$$GR = \left\{ 0, \frac{1}{M}, \frac{2}{M}, \ldots, 1 \right\}.$$  \hspace{1cm} (7.6)

For a given $\Lambda(-1) \in GR$ we define two grid values for $\Lambda(1)$. The first value is

$$\Lambda^-(1) = 1 \land \left\lfloor \frac{\left( \lambda(1 + e^{-\sigma\sqrt{\frac{T}{n}}} - \Lambda(-1)e^{-\sigma\sqrt{\frac{T}{n}}} \right) M}{M} \right\rfloor$$  \hspace{1cm} (7.7)

where $\lfloor \cdot \rfloor = \max\{n \in \mathbb{Z} : n \leq \cdot \}$ is the integer part of $\cdot$. The second value is

$$\Lambda^+(1) = 1 \land \left\lceil \frac{\left( \lambda(1 + e^{-\sigma\sqrt{\frac{T}{n}}} - \Lambda(-1)e^{-\sigma\sqrt{\frac{T}{n}}} \right) M}{M} \right\rceil + 1$$  \hspace{1cm} (7.8)

where $\lceil \cdot \rceil = \min\{n \in \mathbb{Z} : n \geq \cdot \}$.

Define two value functions

$$J_k^{(n)}(\pm, i, j, \lambda) : \{-k, 1 - k, \ldots, k\} \times \{-k, 1 - k, \ldots, k\} \times GR \to \mathbb{R}_+, \; k = 0, 1, \ldots, n$$

as following. The terminal condition is

$$J_n^{(n)}(\pm, i, j, \lambda) = u \left( \lambda S_0 \exp \left( i\sigma \sqrt{\frac{T}{n}} \right), S_0 \exp \left( i\sigma \sqrt{\frac{T}{n}} \right) \right).$$

For $k < n$,

$$J_k^{(n)}(\pm, i, j, \lambda) = \max_{\Lambda(-1) \in A(\lambda) \cap GR} \mathbb{E}_{\mathbb{F}_n} \left( J_{k+1}^{(n)}(\pm, i + \xi_{m+1}, j + \xi_{m+1}, \Lambda^\pm(\xi_{m+1})) \right)$$

where $\Lambda^\pm(-1) = \Lambda(-1)$, $\Lambda^\pm(0) = \lambda$ and $\Lambda^\pm(1)$ is given by (7.7)-(7.8).

For $k = 0$ we obtain two values $J_0^{(n)}(\pm, x/S_0)$ and $J_0^{(n)}(\pm, x/S_0)$. Observe that the complexity of the above dynamic programming is polynomial in $M, n$. For the exact value $U_n(x) = J_0^{(n)}(x/S_0)$ we have the following simple lemma.

**Lemma 7.2.** Assume that $\frac{x}{S_0} \in GR$. Then

$$J_n(x/S_0) \in [J_0^{(n)}(-, x/S_0), J_0^{(n)}(+, x/S_0)]$$

**Proof.** The inequality $J_0^{(n)}(-, x/S_0) \leq J_0^{(n)}(+, x/S_0)$ is obvious. Let us prove that $J_0^{(n)}(x/S_0) \leq J_0^{(n)}(+, x/S_0)$. Choose $\lambda \in GR$ and $\hat{\Lambda}(-1), \hat{\Lambda}(1) \in [0, 1]$ which satisfy (7.4). Define $\Lambda(-1) = 1 \land \frac{[\hat{\Lambda}(-1)M]}{M}$ and let $\Lambda^+(1)$ be given by (7.8). Then it is straightforward to check that $\Lambda(-1) \geq \Lambda(-1)$ and $\Lambda^+(1) \geq \Lambda(1)$. Hence, by applying backward induction (on $k$) and the fact that $J_k^{(n)}(i, j, \lambda)$ is non-decreasing in $\lambda$ we get that for any $k$, $J_k^{(n)}(\cdot, \cdot) \leq J_k^{(n)}(+, \cdot)$ where we take the restriction
of $J_k^{(n)}(\cdot)$ to $\{-k, 1 - k, ..., k\} \times \{-k, 1 - k, ..., k\} \times \mathbb{G}$. For $k = 0$, we obtain $J_0^{(n)}(x/S_0) \leq J_0^{(n)}(+, x/S_0)$ as required. □

Remark 7.1. By using the fact that $U$ is Lipschitz continuous in the first variable, it can be shown that the difference $J_0^{(n)}(+, x/S_0) - J_0^{(n)}(-, x/S_0)$ is of order $O(n/M)$. In practice this difference goes to zero much faster (in $M$). As we will see in the following numerical results, already for $M$ ”close” to $n$ the difference $J_0^{(n)}(+, x/S_0) - J_0^{(n)}(-, x/S_0)$ becomes very small.

7.2. Numerical Results. In this section we implement numerically the procedure described in Section 7.1. In Table 1 and in the corresponding Figure 1 we compute the functions defined in (7.9). To serve as a reference we also evaluate the function $\pi(x) = -\mathbb{E}_P((S_T - K)^+ - x)^+$, a lower bound, which corresponds to the value of spending no extra effort in reducing the shortfall.
### Table 1. Shortfall risk minimization for call options.

Parameters used in computation: $K = 90$, $\bar{\sigma} = 1$, $\bar{\sigma} = 5$, $\bar{\sigma} = 0.0001$; $\sigma = 0.39$, $\rho = -0.64$, $\kappa = 1.15$, $\theta = 0.348$, $\mu = 0.05$, $S_0 = 100$, $T = 1$, $\nu_0 = 0.09$, $n = 400$, $M = 400$.

| $x$ | $J_0^{(n)}(-, x/S_0)$ | $J_0^{(n)}(+, x/S_0)$ | $u_0^{(n)}(x)$ |
|-----|------------------------|------------------------|-----------------|
| 0   | -24.5421               | -24.0371               | -24.6095        |
| 5   | -18.4702               | -17.7050               | -21.4086        |
| 10  | -12.3159               | -11.6165               | -18.2077        |
| 15  | -7.0529                | -6.3398                | -16.3018        |
| 20  | -2.7913                | -2.2453                | -14.3959        |
| 25  | -0.6802                | -0.4201                | -12.4901        |
| 30  | -0.0825                | -0.0274                | -10.5842        |
| 35  | -0.0043                | -0.0004                | -8.6783         |
| 40  | 0                      | 0                      | -7.1540         |
| 45  | 0                      | 0                      | -6.4423         |
| 50  | 0                      | 0                      | -5.7306         |
| 55  | 0                      | 0                      | -5.0190         |
| 60  | 0                      | 0                      | -4.3073         |
| 70  | 0                      | 0                      | -2.8840         |
| 80  | 0                      | 0                      | -2.0045         |
| 90  | 0                      | 0                      | -1.6418         |
| 100 | 0                      | 0                      | -1.2792         |
In the next table we analyze the sensitivity of the problem to $\bar{\sigma}$. The smaller this parameter, the faster the algorithm takes. Although, Lemma 7.1 indicates an error bound for large $\sigma$ (which was obtained by an application of Markov’s inequality), we observe that we can in practice take $\bar{\sigma} = 1$ for our parameters.

| $\bar{\sigma}$ | $\sigma = 0.4(0.1757)$ | $\sigma = 0.6(0.8085)$ | $\sigma = 0.8(0.9939)$ | $\bar{\sigma} = 1$ | $\bar{\sigma} = 2$ |
|-----------------|-------------------------|-------------------------|-------------------------|------------------|------------------|
| $x=0$           | -15.3139                | -23.1077                | -22.0861                | -24.5421         | -24.5421         |
| $x=10$          | -4.1129                 | -9.6884                 | -10.9334                | -12.3159         | -12.3159         |
| $x=20$          | -0.1435                 | -4.5287                 | -1.9145                 | -2.7913          | -2.7913          |

Table 2. Variation with respect to $\bar{\sigma}$. Parameters are the same as in Table 1. The values in the parentheses represent $\mathbb{P}(\sigma^2 \leq \nu_T \leq \bar{\sigma}^2)$ rounded to 4 decimals points. We did not indicate these values when this probability is extremely close to 1.

In Table 3 we analyze the sensitivity of solution to the grid size of the control variable defined in (7.6). We observe as stated in Remark 7.1, that the we can actually take $M = kn$, where $k < 1$. In this table, we determine the range of $k$ we can choose. We observe that choosing $n$ larger leads to more error reduction than choosing $k$ larger. We have also checked this for values of $k > 1$. 
Table 3. Variation with respect to $M$. $x = 20$. Other parameters are the same as in Table 1.

Table 4 and the corresponding Figure 4 demonstrate the convergence with respect to $n$. We observe that the convergence rate is a power of $n$. We leave the rigorous demonstration of this result for future work.

\[
\begin{array}{cccc}
M=n/4 & M=n/2 & M=n \\
n=50 & -9.2138 & -6.6971 & -6.6586 \\
n=100 & -5.4667 & -5.4282 & -5.4238 \\
n=200 & -3.7184 & -3.6541 & -3.6448 \\
n=400 & -2.9834 & -2.8392 & -2.7913 \\
n=800 & -2.6675 & -2.5299 & -2.4833 \\
\end{array}
\]

Table 4. $x = 20$. Other parameters are the same as in Table 1.

![Figure 2. Plot of the values in Table 4.](image-url)
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