Modules identification by a Dynamical Clustering algorithm based on chaotic Rössler oscillators

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Abstract. A new dynamical clustering algorithm for the identification of modules in complex networks has been recently introduced [1]. In this paper we present a modified version of this algorithm based on a system of chaotic Rössler oscillators and we test its sensitivity on real and computer generated networks with a well known modular structure.

Keywords: Networks, Dynamical Clustering, Synchronization

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INTRODUCTION

An important property common to many networks is the presence of modules or community structures, that can be roughly defined as subsets of network nodes within which the network connections are dense, but between which they are sparser. Since nodes belonging to tightly connected clusters of nodes are most likely to have other properties in common, the detection of these structures in complex networks is potentially very useful.

We have recently proposed a dynamical clustering (DC) method for the modules identification based on the properties of a dynamical system associated to the graph [1]. Such a method combines topological and dynamical information in order to devise an algorithm that is able to identify the modular structure of a graph with a precision and a computational cost (O(N^2)) competitive with the best techniques based solely on the topology. The method is based upon the well-known phenomenon of synchronization clusters of non identical phase oscillators [2], each one associated to a node, and interacting through the edges of the graph. Clusters of synchronized oscillators represent an intermediate regime between global phase locking and full absence of synchronization, thus implying a division of the whole graphs into groups of elements which oscillate at the same (average) frequency. The key idea is that, starting from a fully synchronized state of the network, a dynamical change in the weights of the interactions, retaining information of the original betweenness distribution, yields a progressive hierarchical clustering that fully detects modular structures.

In this paper we implement our algorithm on a systems of N identical (three-dimensional) chaotic Rössler oscillators and we test the precision (sensitivity) obtained
by the algorithm on real and computer generated networks with well known modular structures.

**DYNAMICS OF A WEIGHTED NETWORK OF RÖSSLER OSCILLATORS**

The dynamics of a network of $N$ coupled identical oscillators is described by:

$$\dot{x}_i = F(x_i) - \sigma \sum_{j=1}^{N} G_{ij} H[x_i - x_j], \quad i = 1, \ldots, N,$$

(1)

where $F(x)$ governs the dynamics of each individual oscillator, $H(x)$ is a linear vectorial function, $\sigma$ is the overall coupling strength and the $G = G_{ij}$ is the coupling matrix. The rows of matrix $G$ have zero sum and this ensures that the completely synchronized state $\{x_i(t) = s(t), \forall i \mid \dot{s} = F(s)\}$ is a solution of Eq. (1). By means of the so called Master Stability Function approach, it is possible to study the conditions under which such a state is stable [3], i.e. the propensity for synchronization (PFS) of a given network.

In Ref.[3] it was shown that an enhancement in the PFS can be achieved by exploiting the information contained in the overall topology of the network. This can be done through an opportune choice of the coupling matrix $G$ that makes use of the load concept and by scaling the coupling strength $\sigma$ in Eq. (1) to the load of each link. The load $l_{ij}$ of the link connecting nodes $i$ and $j$ is quantified by the so called edge betweenness, i.e. the fraction of shortest paths that are making use of that link. By means of this weighting procedure, that clearly reflects the network structure at a global scale, Eq. (1) reads:

$$\dot{x}_i = F(x_i) - \sigma \sum_{j \in K_i} \frac{l_{ij}^G}{\sum_{j \in K_i} l_{ij}^G} H[x_i - x_j] \quad i = 1, \ldots, N,$$

(2)

where $\alpha$ is a real tunable parameter, and $K_i$ is the set of neighbors of node $i^{th}$. For a given dynamical system $F(x_i)$, for a given value of $\sigma$ and for a given network topology it is possible to find, by means of the Master Stability Function approach [3], what is the value of $\alpha_{best}$ providing the best PFS of the system. In practice, it is more convenient to put $\alpha = 0$ and to find a value of the coupling parameter $\sigma$ which would ensure a fully synchronized state for the oscillators network.

In Ref.[1] we used as dynamical system the so-called Opinion Changing Rate (OCR) model [4], with an Heigselmann-Krause dynamics [5], and we showed that, starting from a perfectly synchronized state for $\alpha = 0$, if $\alpha$ is let to decrease from 0 to $-\infty$, the links with the higher load will be weighted less and less with respect to the other links, thus induing a progressive desynchronization of the system in clusters of frequencies (dynamical clustering) corresponding to different modules, or communities, of a given network. Here we apply our analysis to real or trial networks using a system of chaotic Rössler oscillators and we study again the dynamical clustering process as a function of decreasing values of $\alpha$, identifying a likely community subdivision of the networks by looking to local or global maxima of the modularity $Q$ [1]. The latter simply quantifies the degree of correlation between the probability of having an edge joining two sites and
the fact that the sites belong to the same community \[6\], thus in general it makes sense to look for large values of \(Q\). In fact we get \(Q = 0\) if we consider the whole network as a single community or if we consider a completely random network. On the other hand, for networks with an appreciable subdivision in classes, \(Q\) usually falls in the range between 0.2 and 0.7.

The dynamics of a system of \(N\) identical (three-dimensional) chaotic Rössler oscillators, defined over the nodes of a given network, is ruled by Eq. (2), with \(x_i = (x_i, y_i, z_i)\), \(F(x_i) = [-\omega y_i - z_i, \omega x_i + 0.165y_i, 0.2 + z_i(x_i - 10)]\) and \(H(x) = [x, 0, 0]\) (thus the coupling acts only on the \(x\) variable). In other words we have the following equations of motion:

\[
\begin{align*}
\dot{x}_i &= -\omega y_i - z_i - \frac{\sigma}{\sum_{j \in K_i} l_{ij} \sum_{j \in K_i} l_{ij}} (x_i - x_j) \\
\dot{y}_i &= \omega x_i + 0.165y_i \\
\dot{z}_i &= 0.2 + z_i(x_i - 10) \\
\end{align*}
\]

Here \(\omega\) is a common natural frequency associated at each oscillator that, without loss of generality, we put equal to 1.0. The load matrix \(l_{ij}\) (the matrix of the edge betweennesses) is calculated once forever for the chosen network with a computational cost of \(O(KN)\), \(K\) being the total number of links.

In order to evaluate the degree of synchronization of the Rössler system (3) one has to calculate the order parameter \(\Psi = \langle \frac{1}{N} | \sum_{i=1}^{N} e^{i \Phi_i(t)} | \rangle_t\), where \(\Phi_i(t) = \arctan\left(\frac{y_i(t)}{x_i(t)}\right)\) indicates the instantaneous phase of the \(i\)-th oscillator and \(\langle ... \rangle_t\) stays for a time average. If all the oscillators rotate independently, no clusters exist and we have \(\Psi \sim \frac{1}{\sqrt{N}}\). On the contrary, if their motions are synchronized in phase, only one cluster exists and we obtain \(\Psi \sim 1\). Once a network is fixed, the first task is to find the value of the coupling parameter \(\sigma\) providing a fully synchronized starting state for the Rössler oscillators at \(\alpha = 0\) (i.e. at

\[\text{FIGURE 1.}\] The Karate Club network, with the two Zachary's communities identified by circles and squares.
Asymptotic Rössler order parameter (at $\alpha = 0$) versus the coupling $\sigma$ for the Karate Club network. See text for further details.

Then, one can let $\alpha$ to decrease in time and study the dynamical clustering process acting on the instantaneous phases $\Phi_i(t)$'s of the oscillators. Notice that these phases play here the same role played by the instantaneous frequencies in the OCR model [1]: in this case we call "cluster" a group of contiguous phases in the $\Phi$'s interval (usually $[-3,3]$) separated by a distance of more than 0.02 units. For each value of $\alpha$ a different configuration of clusters (corresponding to a given network structure) will appear and one has to calculate the corresponding modularity and select the configuration with the best modularity score.

In the following we will show in detail this process by putting the Rössler system over different real and trial networks.

**Zachary's karate club**

As first example, in order to test our algorithm for finding community structures, we consider a real network, the well-known Karate Club network analyzed by Zachary [7]. It consists of $N = 34$ individuals (nodes), whose mutual friendship relations (expressed by $K = 78$ edges) have been carefully investigated over a period of two years. Due to contrasts between a teacher and the administrator of the club, the club splitted into two smaller communities. The corresponding network is presented in Fig.1, where squares and circles label the members of the two groups. The 'circles' community has 18 elements (corresponding to nodes 9, 10, 15, 16, 19, 21, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34) while the 'squares' community has 16 elements. (nodes 5, 6, 7, 11, 17, 1, 2, 3, 4, 8, 12, 13, 14, 18, 20, 22). The *a-priori* modularity of such a configuration results to be equal to $Q_Z \sim 0.37$.

Firstly, in Fig.2 we plot the behavior of the asymptotic Rössler order parameter (averaged over 10 events) versus the coupling strength $\sigma$ for the Karate Club network with $\alpha = 0$. It results that, above $\sigma \sim 0.6$, the system lies in the fully synchronized phase, thus in the following we reasonably set $\sigma = 1.3$. Using such a value in the equations of motion (), we can now integrate them numerically and study the dynamical clustering...
FIGURE 3. Typical run for the Karate Club Network: time-evolution of both the Rössler’s phases configurations (top panel) and the corresponding modularity (bottom panel) as a function of $\alpha(t)$. See text for further details.

TABLE 1. Clusters configuration with the best modularity score $Q_{\text{best}} = 0.42$ for the Karate Club Network (at $\alpha_{\text{best}} \sim -1.76$). See text for further details.

| Cluster   | Nodes   | Members          |
|-----------|---------|------------------|
| cluster 1 | 11 nodes| 1, 2, 3, 4, 8, 12, 13, 14, 18, 20, 22 |
| cluster 2 | 12 nodes| 9, 10, 15, 16, 19, 21, 23, 27, 29, 30, 31, 32, 33, 34 |
| cluster 3 | 4 nodes | 24, 25, 26, 28 |
| cluster 4 | 2 nodes | 27, 30 |
| cluster 5 | 5 nodes | 5, 6, 7, 11, 17 |

desynchronization process acting on the oscillators’ phases. The task of our algorithm is to extract the best modular structure of the network by using only the information expressed by the edge betweenness of its links, which are calculated only once for the given network.

The system starts in a perfectly synchronized state for $\alpha_{\text{start}} = 0$, i.e. $x_i(0) = y_i(0) = z_i(0) = 0 \ \forall i$, thus $\Phi_i(0) = 0 \ \forall i$ and only one cluster exists (choosing a different starting value for $x_i, y_i, z_i$ does not change significantly the conclusions shown in the following); then we decrease $\alpha$ during a single run as a function of time with a rate of 2 time steps and a given constant decrement $\delta \alpha = 0.0008$. Simultaneously, for each value of $\alpha(t)$ and for each configuration of clusters, the modularity $Q$ is calculated.

In the top panel of Fig.3 we show the phases time evolution as a function of $\alpha(t)$ in a typical run with $\alpha$ going from 0 to $-3$. In this figure 34 points are plotted for each value of $\alpha(t)$, corresponding to the instantaneous phases of the oscillators (from which the average instantaneous phase of the system has been subtracted in order to have a symmetric plot). Correspondingly, in the bottom panel of Fig.3, the modularity $Q$ is also plotted as a function of $\alpha(t)$. Even if the system strongly oscillates during the
desynchronization process (at variance with the much more stable behavior of the OCR system in [1]), clusters configurations (i.e. community structures of the underlying network) with very large values of modularity appear.

In Table 1 we report the detailed clusters configuration corresponding to the maximum value of modularity, $Q_{\text{best}} = 0.42$, found for $\alpha_{\text{best}} \sim -1.76$ (see the arrow in the bottom panel). One can see that five clusters have been found: the first and the last ones, respectively made of 11 and 5 nodes (oscillators), if considered together, correspond to the first community of 16 nodes (squares in the left panel of Fig.1) observed by Zachary, while the sum of the remaining three clusters corresponds to tightly connected subgroups of the second community of 18 nodes (circles in the left panel of Fig.1). On the other hand, for several values of $\alpha$ in Fig.3 (e.g. for $\alpha = -0.538$ or $\alpha = -1.075$) the clusters configuration corresponding to the two Zachary communities has been also recovered, but - as previously seen -, being its modularity smaller than 0.42 (in fact $Q_{\text{Z}} \sim 0.37$), the algorithm favours the five clusters configuration shown in Table 1 (in other words, the Zachary community subdivision corresponds only to a local maximum of modularity for this particular network and not to a global one).

In conclusion, at least in so far as it concerns the Zachary Club network, the dynamical clustering algorithm based on the Rössler system seems to work very well, even if compared with the analogous results presented in [1]. In fact it results to be at the same time very fast (it extracts the best clusters configuration of the network versus $\alpha$ in a single run) and very sensitive (it is able to recover community structures of the Zachary network with a great modularity). In the next section we will apply the same algorithm to another real network, the Chesapeake Bay food web.

**Chesapeake Bay Food Web**

Another classical benchmark for the community identification algorithms is a food web of marine organisms living in the Chesapeake Bay, situated on the Atlantic coast of the United States. This ecosystem was originally studied by Baird and Ulanowicz [8], who carefully investigated the predatory interactions between the most important taxa (species or groups of species) and constructed a network of 33 vertex and 71 links. We will consider here (as usually done in many papers [1, 9, 10]) its non-directed and non-valued version and calculate first (only once before each simulation using the same network) the edge betweenness of each link.

In Fig.4 we show the result of a typical event obtained in a simulation performed with the same procedure described in the previous section and for a value of the interaction strength $\sigma = 1$ (such that the Rössler’s system would lie in its synchronized phase for $\alpha = 0$). The system starts in a fully synchronized state (we set again $x_i(0) = y_i(0) = z_i(0) = 0 \forall i$) at $\alpha_{\text{start}} = 0$ and evolves through decreasing values of $\alpha(t)$ (with a decrement $\delta \alpha = 0.0008$), up to the value $\alpha_{\text{end}} = -2$. The clusters evolution and the corresponding modularity $Q(t)$ are plotted as a function of $\alpha$ and, again, look very oscillating in time. The detailed configuration with the highest modularity peak (see the arrow in the bottom panel) is reported in Table 2. It consists of 6 clusters with a $Q_{\text{max}} = 0.43$, obtained for $\alpha_{AVT} = -1.62$, and it is quite consistent with the correspond-
FIGURE 4. Typical run for the Chesapeake Bay food web: time-evolution of both the Rössler’s phases configurations (top panel) and the corresponding modularity (bottom panel) as a function of $\alpha(t)$.

TABLE 2. Clusters configuration with the best modularity score $Q_{\text{best}} = 0.43$ for the Chesapeake Bay food web (at $\alpha_{\text{best}} \sim -1.62$)

| Cluster | Nodes |
|---------|-------|
| Cluster 1 (10 nodes) | 3,14,15,16,18,25,26,27,28,29 |
| Cluster 2 (3 nodes) | 4,17,19 |
| Cluster 3 (1 nodes) | 30 |
| Cluster 4 (3 nodes) | 22,31,32 |
| Cluster 5 (14 nodes) | 1,2,7,8,9,10,11,12,13,20,21,23,24,33 |
| Cluster 6 (2 nodes) | 5,6 |

Such a result corroborates the good performance of the Rössler algorithm in the identification of community structures in real networks. The next step will be to test this method on "ad hoc" trial networks with a well known fixed community structure [1,6], in order to explore in deeper detail its sensitivity.

**Sensitivity test for ad hoc trial networks**

Typical trial networks are generated with $N = 128$ nodes and split into four communities containing 32 nodes each. Pairs of nodes belonging to the same community are linked with probability $p_{\text{in}}$, whereas pairs belonging to different communities are joined
with probability $p_{out}$. The value of $p_{out}$ is taken so that the average number of links a node has to members of any other community, i.e. $z_{out}$, can be controlled. While $p_{out}$ (and therefore $z_{out}$) is varied freely, the value of $p_{in}$ is chosen to keep the total average node degree $k$ constant, and set to 16. As $z_{out}$ is increased from 2 (very well defined structures) to 8 (bad defined structures), the communities become more and more diffuse and harder to detect. Since the “real” community structure is well known in this case, it is possible to measure the number of nodes correctly classified by our method of community identification (see for example [1]).

We apply our algorithm to 10 different sets of trial networks. For each network, as done in the previous sections, after having calculated the load matrix $l_{ij}$, we integrate numerically the equations of motion (\ref{eq:odes}). Every time we start from a perfectly synchronized state for $\alpha_{\text{start}} = 0$ and we analyze the desynchronization process when $\alpha(t)$ decreases in time (with a given constant decrement $\delta \alpha$) during a single run. In Fig. 5 we plot the number of correctly classified nodes as a function of $z_{out}$, averaged over the set of 10 networks and for three increasing values of $\delta \alpha$. The error bars (standard deviations) for each point is also reported. As one can see, the smaller is $\delta \alpha$, the better is the result. However, in any case the sensitivity abruptly falls above $z_{out} = 5$, staying around the 40% of correctly identified nodes up to $z_{out} = 8$. Such a performance of the Rössler system is surely worse than that of the OCR – HK system shown in [1], but in any case it confirms the possibility of extending the dynamical clustering algorithm presented in [1] to other dynamical systems (also three-dimensional, like in this case) with quite good results.

**CONCLUSION**

Summarizing, even if the global sensitivity of the dynamical clustering (DC) algorithm for the a system of chaotic Rössler oscillators seems to be not competitive with respect
to those of other methods (see Refs. [1, 11, 12]), on the other hand the results here presented confirm that (i) the DC algorithm is robust with respect to the change of the adopted dynamical system; (ii) the system of chaotic Rössler oscillators works quite well if applied to the real networks considered; (iii) the DC algorithm is quite fast and needs only few runs of integration for each network (after the calculation of the corresponding load matrix $l_{ij}$) [1, 2, 3, 6].

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