Abstract

We study a family of orthogonal polynomials which satisfy (apart from a 3-term recurrence relation) an eigenvalue equation involving a third order differential operator of Dunkl-type. These polynomials can be obtained from a Geronimus transformation of the little $q$-Jacobi polynomials in the limit $q = -1$.

Keywords: Jacobi polynomials, little $q$-Jacobi polynomials, Geronimus transformation.
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1 Introduction

Significant advances have been realized in the characterization of recently discovered families of $-1$ orthogonal polynomials (OPs). The striking feature of these OPs is that they are classical or bispectral and that they satisfy eigenvalue equations involving Dunkl-type operators in addition to the mandatory 3-term recurrence relation. They have arisen already in a number of physical problems and are connected with Jordan algebras.

At the top of the emerging $-1$ scheme are the Bannai-Ito polynomials and their kernel partners, the complementary Bannai-Ito polynomials. Both sets depend on 4 parameters. The Bannai-Ito polynomials are the eigensolutions of the most general operator which is of first order in Dunkl shifts, (i.e., first order in the operators $T$ and $R$ defined by $Tf(x) = f(x+1)$, $Rf(x) = f(-x)$ on a function $f(x)$) and which stabilizes polynomials of given degrees. The Bannai-Ito polynomials are positive definite and orthogonal on a finite set of $N + 1$ points. In the limits where $N \to \infty$, they tend to the big $-1$ Jacobi polynomials which are orthogonal on $(-1, -c) \cup [c, 1]$. When $c = 0$, the little $-1$ Jacobi polynomials arise as a special case. A Bochner-type theorem establishes that the big and little $-1$ polynomials are the only families of OPs satisfying a differential-difference eigenvalue equation which is of first order in Dunkl-type operators.

A significant property that justifies the nomenclature is that these $-1$ orthogonal polynomials can be obtained from appropriate $q \to -1$ limits of $q-$ polynomials.

The Bannai-Ito polynomials and their descendants all possess the Leonard duality property. (In fact, this led to their initial identification.) The dual $-1$ Hahn polynomials together with the generalized Gegenbauer and Hermite polynomials are also bispectral but, obeying eigenvalue equations of second order in Dunkl operators, they fall beyond the scope of Leonard duality.

We continue here the exploration of $-1$ orthogonal polynomials in that vein and look for another class of $-1$ OPs verifying a higher differential-difference equation.

In the wake of Krall’s classification of OPs satisfying a fourth order differential equations, it is appreciated that the addition of discrete masses to the measure leads to OPs verifying higher order equations.

In this connection, we present here a generalization of the little $-1$ Jacobi polynomials with the following features: these orthogonal polynomials obey a differential-difference equation of third order in Dunkl operators and a mass is located at the middle of the orthogonality interval.

The outline of the paper is the following. In section 2, we offer a brief review of useful results on little $q$-Jacobi polynomials. In section 3, we introduce the generalized little $q$-Jacobi polynomials that are eigensolutions of higher order $q$-difference equations. In section 4, focusing for definiteness on one of the
simpler cases, we obtain and characterize a set of generalized little $-1$ Jacobi polynomials by taking an appropriate $q \to -1$ limit of certain polynomials of the proceeding section. The paper ends with concluding remarks.

## 2 Little $q$-Jacobi polynomials

The monic little $q$-Jacobi polynomials are defined as

$$P_n(x; a, b) = (-1)^n \frac{q^{n(n-1)/2}(aq; q)_n}{(abq^{n+1}; q)_n} \Phi_1 \left( q^{-n}, abq^{n+1}; q \right)$$

where $(a; q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ is the $q$-shifted factorial and $\Phi_1$ denotes the $q$-hypergeometric function.

The orthogonality relation is

$$\sum_{k=0}^{\infty} w_k P_n(q^k; a, b) P_m(q^k; a, b) = h_n \delta_{nm},$$

where $h_n$ are appropriate normalization constants, and the normalized weight function is

$$w_k = \frac{(aq; q)_\infty}{(abq^2; q)_\infty} \frac{(bq; q)_k (aq)_k}{(q; q)_k}.$$

It is assumed that $0 < aq < 1, b < q^{-1}$. The expansion coefficients of the little $q$-Jacobi polynomials in

$$P_n(x; a, b) = \sum_{s=0}^{n} B_n^{(s)} x^{n-s}$$

are

$$B_n^{(s)} = b^{-s} \frac{(q^{-n}, a^{-1}q^{-n}; q)_s}{(q, a^{-1}b^{-1}q^{-2n}; q)_s}.$$

It is known that the little $q$-Jacobi polynomials satisfy a second-order difference equation \cite{23}.

Introduce the functions of the second kind,

$$Q_n(z) = \int_a^b \frac{P_n(x)w(x)}{z - x} dx,$$

where $w(x)$ is assumed to be the normalized weight function, i.e.

$$\int_a^b w(x) dx = 1.$$

The values of the functions $Q_n(z)$, at $z = 0$ (an accumulation point of the orthogonality measure), are:

$$Q_n(0; a, b) = \sum_{k=0}^{\infty} \frac{P_n(q^k; a, b) w_k}{q^k}.$$

Using the $q$-binomial theorem and the $q$-Saalschütz formula (see, e.g., \cite{24}) we have

$$Q_n(0; a, b) = (-1)^{n+1} a^n q^{n(n-1)/2} \frac{1 - abq}{1 - a} \frac{(q; q)_n (bq; q)_n}{(abq; q)_n (abq^{n+1}; q)_n}.$$

We note that if $a = q^j, j = 1, 2, 3, \cdots$, then

$$\Phi_n = Q_n(0; q^j, b) + MP_n(0; q^j, b)$$

$$= (-1)^n q^{n(n-1)/2} \frac{(q^j+1; q)_n}{(bq^{j+1}; q)_n} \left( M - q^{n+1} \frac{(1 - bq^{j+1})(bq; q)_j (q; q)_j}{(1 - q^j)(q^{n+1}; q)_j (bq^{n+1}; q)_j} \right).$$

2
3 Transformed $q$–Jacobi polynomials

Let $P_n(x)$ be orthogonal polynomials with measure localized on the interval $[a, b]$. Let $w(x)$ be the corresponding normalized unit weight function and $Q_n(x)$ be defined by (2.5). Let finally $c$ be a point beyond the orthogonality interval $[a, b]$ such that $Q_n(c)$ exists.

Consider the Geronimus transformation [25, 26] of the polynomials $P_n(x)$ at the point $x = c$

$$
\tilde{P}_n(x) = \mathcal{G}P_n(x) = P_n(x) - \Phi_n \Phi_{n-1}^{-1}P_{n-1}(x), \quad n = 1, 2, \ldots, \tilde{P}_0(x) = 1,
$$

(3.1)

where

$$
\Phi_n = Q_n(c) + MP_n(c).
$$

(3.2)

The weight function $\tilde{w}(x)$ of the polynomials $\mathcal{G}\{P_n(x)\}$ is

$$
\tilde{w}(x) = \kappa \left( \frac{w(x)}{x - c} - M\delta(x - c) \right),
$$

(3.3)

where $\kappa$ is an appropriate normalization constant. The Geronimus transformation thus inserts a concentrated mass at the point $x = c$. The value of this mass depends on the parameter $M$. Now take for $P_n(x)$ the little $q$-Jacobi polynomials with $a = q^j$, $j = 1, 2, 3, \ldots$, and perform the Geronimus transformation (3.1) with $\Phi_n$ given by (2.8). (In this case $c = 0$.)

The weight function $\tilde{w}(x)$ for the polynomials $\mathcal{G}(c)\{P_n(x)\}$ is

$$
\tilde{w}(x) = \kappa \left( \sum_{k=0}^{\infty} \tilde{w}_k \delta(x - q^k) - M\delta(x) \right),
$$

(3.4)

where

$$
\tilde{w}_k = \frac{(q^{j+1}; q)_{\infty}}{(bq^k; q)_{\infty}} \frac{(bjq^k; q)_{\infty}}{(q; q)_k}.
$$

(3.5)

The coefficients $B_n^{(s)}$ in the expansion

$$
\tilde{P}_n(x) = \mathcal{G}(0)\{P_n(x; q^j, b)\} = \sum_{s=0}^{n} B_n^{(s)} x^{n-s}
$$

(3.6)

have been given in [22]. Let us introduce the operators

$$
L_q = \sum_{k=0}^{2N} a_k(q^{-N}x)T^{-N}D_q^k,
$$

(3.7)

where

$$
a_k(x) = \sum_{s=0}^{k} \alpha_{ks} x^s, \quad k = 0, 1, \ldots, 2N.
$$

(3.8)

$T$ is the $q$–shift operator and $D_q$ the $q$–derivative operator. The operator $L_q$ is seen to be very practical in searching for orthogonal polynomials $P_n(x)$ satisfying eigenvalue equations of the kind

$$
L_q P_n(x) = \lambda_n P_n(x).
$$

(3.9)

Consider the action of the operator $L_q$ upon the monomials $x^n$. From (3.7) and (3.8) we get

$$
L_q x^n = \sum_{s=0}^{n} A_n^{(s)} x^{n-s},
$$

(3.10)

where

$$
A_n^{(s)} = q^{N(s-n)}[n][n-1] \cdots [n-s+1] \pi_n(q^n),
$$

(3.11)
and
\[
\pi_s(q^n) = \alpha_s + \sum_{i=1}^{2N-s} \alpha_{s+i} [n-s][n-s-1]\cdots[n-s-i+1]
\] (3.12)
are polynomials in \(z = q^n\) of degree not exceeding \(2N-s\). It is clear that
\[
A_n^{(s)} = 0, \quad s > 2N
\] (3.13)
(see [22]). The coefficients \(A_n^{(s)}\) completely characterize the operator \(\mathcal{L}_q\) and \(A_n^{(s)}\) are called the representation coefficients of the operator \(\mathcal{L}_q\).

The coefficients \(A_n^{(s)}\) for the \(q\)-difference operator \(\mathcal{L}_q\) that has the polynomials \(\tilde{P}_n(x)\) [3.6] as eigenfunctions have been constructed in [22], they are the following
\[
A_n^{(0)} = \lambda_n = \frac{M(q-1)q^{-2n(j+1)-1}(q^n;q)_{j+1}(bq^n;q)_{j+1}}{1-q^{-2}}
\]
\[
- (q^n - 1)(1-bq^{n+j})(bq^n;q)_{j+1}(q;q)_{j-1},
\]
(3.14)
\[
A_n^{(1)} = (1-q^{-n}) \left( Mq^{(1-n)}(q^n; q)_j(bq^n;q)_j(1-q^{-n-1}) - (q;q)_{j-1}(bq^n;q)_{j+1}(1-q^{n+j-1}) \right)
\]
(3.15)
\[
A_n^{(2)} = M(q-1)q^{(2-n)(j+1)-1}(1-q^{-j})(q^n-2)(q^n;q)_j(1-bq^{n+j})(bq^n;q)_{j+1},
\]
(3.16)
\[
A_n^{(s)} = 0, \quad \text{if } s \geq j + 2.
\]
(3.17)

4 Limit of the Krall-Jacobi polynomials as \(q \to -1\)

In this section we construct the \(q \to -1\) limit of the coefficients \(A_n^{(s)}\). We take \(j = 2\) and put
\[
q = -e^\epsilon, \quad b = -e^{\beta \epsilon},
\]
(4.1)
Substituting into the \(A_n^{(s)}\) as given in [3.1-3.3] and taking the limit \(\epsilon \to 0\), we have
\[
\frac{A_n^{(0)}}{e^3} \to \begin{cases} 
-8Mn(n+2)(n+1+\beta) + 8n(\beta + 1)(\beta + 3) & \text{n even} \\
8M(n+1)(\beta + 3) - 8n(n+2+\beta)(\beta + 1)(\beta + 3) & \text{n odd}
\end{cases}
\]
(4.2)
\[
\frac{A_n^{(1)}}{e^3} \to \begin{cases} 
8M(n+2)(n+1+\beta) - 8n(\beta + 1)(\beta + 3), & \text{n even} \\
8(\beta + 1)(\beta + 3)(n+1) - 8M(n^2 - 1)(n+\beta), & \text{n odd}
\end{cases}
\]
(4.3)
\[
\frac{A_n^{(2)}}{e^3} \to \begin{cases} 
-8M(n+2)(n-2), & \text{n even} \\
-8M(n+1)(n-1)(n+\beta), & \text{n odd}
\end{cases}
\]
(4.4)
\[
\frac{A_n^{(3)}}{e^3} \to \begin{cases} 
-8M(n+2)(n-2), & \text{n even} \\
8M(n+1)(n-1)(n-3), & \text{n odd}
\end{cases}
\]
(4.5)

Consider the form of the \(q\)-difference equation [3.5] in this limit. We divide both sides of [3.5] by \(e^3\) and introduce the operator \(L_\epsilon\) which acts on the polynomials \(\tilde{P}_n(x)\) as
\[
L_\epsilon \tilde{P}_n(x) = e^{-\epsilon} \lambda_n \tilde{P}_n(x)
\]
(4.6)
For monomials \(x^n\), from [3.10] and [4.2-4.5] we have in the limit \(\epsilon \to 0\),
\[
L_0x^n = \lim_{\epsilon \to 0} \frac{L_\epsilon}{e^3} x^n = \lim_{\epsilon \to 0} \frac{L_\epsilon}{e^3} x^n = -8Mn(n+2)(n+1+\beta) + 8n(\beta + 1)(\beta + 3)
\]
\[
+ \theta_n(16Mn^3 + 24\beta M + 48M)n^2 + (32M - 16\beta^2 + 8\beta^2 M - 48 + 48\beta M - 64\beta)n
\]
\[
- 48\beta^2 - 8\beta^3 + 16\beta M + 8\beta^2 M - 48 - 88\beta]
\]
\[
x^n + [8Mn^2 + 24M + 8\beta M]n^2 + [16M + 16\beta M - 8\beta^2 - 32\beta - 24]n
\]
\[
+ \theta_n(-16Mn^3 - 24M + 16\beta M)n^2 + (64\beta + 48 + 16\beta^2 - 16\beta M - 8M)n
\]
\[
+ 32\beta + 24 + 8\beta^2 + 8\beta M]
\]
\[
x^{n-1}
\]
\[
+ [8Mn^3 - 32Mn + \theta_n(-16Mn^3 - 8\beta Mn^2 + 40Mn + 8\beta M)]x^{n-2}
\]
\[
+ [-8Mn^3 + 32Mn + \theta_n(16Mn^3 - 24Mn^2 - 40Mn + 24M)]x^{n-3}
\]
(4.7)
where \( \theta_n = \frac{1-(-1)^n}{2} \). This allows one to present the operator

\[
L_0 = (-8M + 8Mx + 8Mx^2 - 8Mx^3) \partial_x^3 R
+ \left[ -12M/x + 24M + 4\beta M + (36M + 8\beta M)x - (12\beta M + 48M)x^2 \right] \partial_x^2 R
+ (12Mx + 4\beta Mx^2 - 12M/x - 4M) \partial_x^2 + \left[ (24M + 16\beta M - 8\beta^2 - 32\beta - 24) + 24M/x^2 + (4\beta - 12)M/x + (8\beta^2 - 36\beta M - 48M - 4\beta^2 M + 32\beta + 24)x \right] \partial_x R
+ \left[ (4\beta^2 M + 12\beta M)x - (12 + 4\beta)M/x + 24M + 8\beta M \right] \partial_x
+ \left[ 12M/x^3 + 4\beta M/x^2 + (12 + 4\beta^2 + 4\beta M + 16\beta)/x \right.
+ (8\beta M + 4\beta^2 M - 44\beta - 24 - 4\beta^3 - 24\beta^2) \left. \right] (1 - R),
\]

(4.8)

where \( R \) is the reflection operator \( Rf(x) = f(-x) \). We thus have that the polynomials \( \tilde{P}_n^{(-1)}(x) \) are classical and satisfy the eigenvalue equation

\[
L_0 \tilde{P}_n^{(-1)}(x) = \tilde{\lambda}_n \tilde{P}_n^{(-1)}(x),
\]

(4.9)

where

\[
\tilde{\lambda}_n = \begin{cases} 
-8Mn(n+2)(n+1+\beta) + 8n(\beta+1)(\beta+3) & n \text{ even } \\
8M(n+1)(n+2+\beta) - 8(n+2+\beta)(\beta+1)(\beta+3) & n \text{ odd }.
\end{cases}
\]

(4.10)

The lower degree eigensolutions of (4.9) can be obtained directly as a check and as examples. We take

\[
\tilde{\lambda}_1 = (\beta+1)(\beta+3)(16M - 8(\beta + 3)),
\]

(4.11)

and find the first order polynomial solution

\[
\tilde{P}_1^{(-1)}(x) = x - 1 + \frac{2\beta - 1 - \alpha}{2\beta - 3 - \alpha}.
\]

(4.12)

The second-order polynomial solution is

\[
\tilde{\lambda}_2 = (\beta+3)(16\beta + 16 - 64M),
\]

\[
\tilde{P}_2^{(-1)}(x) = x^2 - \frac{2(4M - \beta - 1)}{(5 + \beta)(2M - \beta - 1)} x + \frac{2(\beta + 1)}{(5 + \beta)(2M - \beta - 1)},
\]

(4.13)

(4.14)

and the third-order polynomial solution

\[
\tilde{\lambda}_3 = (3 + \beta)(5 + \beta)(32M - 8 - 8\beta),
\]

\[
\tilde{P}_3^{(-1)}(x) = x^3 - \frac{4(\beta - 2M + 1 + \beta)}{(7 + \beta)(-4M + \beta + 1)} x^2 - \frac{4}{7 + \beta} x + \frac{8(1 + \beta)}{(7 + \beta)(5 + \beta)(-4M + \beta + 1)}.
\]

(4.15)

(4.16)

Other polynomial eigensolutions can be obtained in the \( q = -1 \) limit of (3.1). The weight function (3.1) has the following moments

\[
\tilde{c}_n = k \left( \frac{(q^2; q)_n}{(bq^4; q)_n} - \frac{1 - q^2}{1 - bq^3} \delta_{n,0} \right).
\]

(4.17)

Using the parametrization (4.11) and taking the limit \( c \to 0 \), we can directly obtain from the above relation the moments corresponding to the polynomials \( \tilde{P}_n^{(-1)} \).

\[
\mu_{2n} = \mu_{2n-1} = k \left[ \frac{1}{(\beta/2 + 3/2)_n} - \frac{2}{3 + \beta} \delta_{n,0} \right], \quad n = 1, 2, \cdots,
\]

(4.18)

where \( (x)_n = x(x+1)(x+2) \cdots (x+n-1) \) is the ordinary Pochhammer symbol. It is then easily verified that

\[
\mu(x) = k \left( |x|(1 - x^2)^{(\beta - 1)/2} (1 + x) - \frac{4}{(1 + \beta)(3 + \beta)} \delta(x) \right),
\]

(4.19)
where
\[ \tilde{k} = \frac{\Gamma(\beta/2 + 3/2)}{\Gamma(\beta/2 + 1/2)} \tilde{k} = \frac{\beta + 1}{2} k \]  
(4.20)
is the orthogonality measure for these polynomials. Indeed we see that
\[ \int_{-1}^{1} w(x)x^n dx = \mu_n, \quad n = 0, 1, 2, \cdots, \]  
(4.21)
with \( \mu_n \) given by (4.18).

In the following we determine the three term recurrence relation that the polynomials \( \tilde{P}_n^{(-1)}(x) \) verify. It is already known that the little \( q \)-Jacobi polynomials satisfy the relation
\[ P_{n+1} + b_n P_n + u_n P_{n-1} = x P_n, \]  
(4.22)
and that the recurrence coefficients are defined by
\[ u_n = A_{n-1} C_n, \quad b_n = A_n + C_n, \]  
where \( A_n, C_n \) are given as
\[ A_n = q^n \frac{(1 - aq^{n+1})(1 - abq^{n+1})}{(1 - abq^{2n+1})(1 - abq^{2n+2})}, \quad C_n = aq^n \frac{(1 - q^n)(1 - bq^n)}{(1 - abq^{2n+1})(1 - abq^{2n+2})}. \]  
(4.23)

Under the Geronimus transformation
\[ \tilde{P}_n(x) = P_n(x) - B_n P_{n-1}(x), \]  
(4.24)
with \( B_n = \frac{\Phi_n}{\Phi_{n-1}} \) and \( \Phi_n \) defined by (2.3), these result the three term recurrence relation
\[ \tilde{P}_{n+1} + \tilde{b}_n \tilde{P}_n + \tilde{u}_n \tilde{P}_{n-1} = x \tilde{P}_n, \]  
(4.25)
with coefficients
\[ \tilde{u}_1 = \frac{\phi_1}{\phi_0}, \quad \tilde{u}_n = \frac{u_{n-1} B_n}{B_{n-1}}, \quad n = 2, 3, \cdots \]  
(4.26)
\[ \tilde{b}_0 = b_0 + \frac{\phi_1}{\phi_0}, \quad \tilde{b}_n = b_n + B_{n+1} - B_n, \quad n = 1, 2, \cdots \]  
(4.27)

When we set \( a = \tilde{q}^2, q = -e^\epsilon, b = -e^{2\epsilon} \) and take the limit \( \epsilon \to 0 \), Eq. (4.25) reduces to
\[ \tilde{P}_{n+1}^{(-1)} + \tilde{b}_n^{(-1)} \tilde{P}_n^{(-1)} + \tilde{u}_n^{(-1)} \tilde{P}_{n-1}^{(-1)} = x \tilde{P}_n^{(-1)}. \]  
(4.28)

Here the coefficients are
\[ \tilde{b}_n^{(-1)} = \lim_{\epsilon \to 0} (b_n + B_{n+1} - B_n) = b_n^{(-1)} + \lim_{\epsilon \to 0} (B_{n+1} - B_n) \]  
(4.29)
\[ \tilde{u}_n^{(-1)} = \lim_{\epsilon \to 0} \frac{u_{n-1} B_n}{B_{n-1}} = u_{n-1}^{(-1)} \lim_{\epsilon \to 0} \frac{B_n}{B_{n-1}}, \]  
(4.30)
where
\[ u_n^{(-1)} = -\frac{n(n+2)}{(2n+1+\beta)(2n+3+\beta)}, \quad b_n^{(-1)} = 1 \]  
(4.31)
when \( n \) is even, and
\[ u_n^{(-1)} = -\frac{(n+\beta)(n+2+\beta)}{(2n+1+\beta)(2n+3+\beta)}, \quad b_n^{(-1)} = -1 \]  
(4.32)
when \( n \) is odd. Note that
\[ \lim_{\epsilon \to 0} B_n = \lim_{q \to -1} \frac{\Phi_n}{\Phi_{n-1}} = \begin{cases} \frac{n+2}{2n+1+\beta} \frac{M-[(3+\beta)(1+\beta)]/[n(n+1+\beta)]}{n(n+1+\beta)} & \text{n even} \\ \frac{n+2+\beta}{2n+1+\beta} \frac{M-[(3+\beta)(1+\beta)]/[n(n+1+\beta)]}{n(n+1+\beta)} & \text{n odd} \end{cases} \]  
(4.33)
As a final observation, let us identify the matrix orthogonal polynomials that the even (or odd) part of the $P_{n}(x)$ define. Split the polynomials $P_{n}^{(-1)}(x)$ into its even ($E_{n}$) and odd ($O_{n}$) parts:

$$
P_{n}^{(-1)}(x) = E_{n}(x) + O_{n}(x).$$  \hspace{1cm} (4.34)

From the recurrence relation \[(4.34)\] we have

$$x^{2}E_{n} = E_{n+2} + \left(\tilde{b}_{n+1}^{(-1)} + \tilde{b}_{n}^{(-1)}\right)E_{n+1} + \left(\tilde{a}_{n+1}^{(-1)} + \tilde{a}_{n}^{(-1)}\right) + \left(\tilde{b}_{n}^{(-1)}\right)^{2}E_{n}$$

$$+ \left(\tilde{b}_{n+1}^{(-1)} + \tilde{b}_{n}^{(-1)}\right)\tilde{u}_{n+1}^{(-1)}E_{n+1} + \tilde{u}_{n}^{(-1)}\tilde{u}_{n}^{(-1)}E_{n-2}.$$ \hspace{1cm} (4.35)

With the redefinition $E_{n} = \sigma_{n}F_{n}$, $\sigma_{n} = \sqrt{\tilde{a}_{1}^{(-1)}\tilde{a}_{2}^{(-1)}\cdots\tilde{a}_{n}^{(-1)}}$, it is easy to see that the polynomial $F_{n}$ satisfies the five-term recurrence relation,

$$x^{2}F_{n}(x) = c_{n,0}F_{n} + c_{n,1}F_{n-1} + c_{n+1,1}F_{n+1} + c_{n,2}F_{n-2} + c_{n+2,2}F_{n+2}$$ \hspace{1cm} (4.36)

where the coefficients are

$$c_{n,0} = \left(\tilde{a}^{(-1)}_{n+1} + \tilde{u}^{(-1)}_{n} + \left(\tilde{b}^{(-1)}_{n}\right)^{2}\right),$$

$$c_{n,1} = \left(\tilde{b}^{(-1)}_{n+1} + \tilde{b}^{(-1)}_{n}\right)\sqrt{\tilde{u}^{(-1)}_{n}},$$

$$c_{n,2} = \sqrt{\tilde{u}^{(-1)}_{n}\tilde{u}^{(-1)}_{n-1}}.$$ \hspace{1cm} (4.37)

From the theorem in \[27\], the matrix polynomials \(P_{n}(x)\) defined by

$$P_{n}(x) = \begin{pmatrix} R_{2,0}(F_{2n})(x) & R_{2,n}(F_{2n})(x) \\ R_{0,0}(F_{2n+1})(x) & R_{2,1}(F_{2n+1})(x) \end{pmatrix}$$ \hspace{1cm} (4.38)

satisfy the matrix three term recurrence relation

$$xP_{n}(x) = D_{n+1}P_{n+1}(x) + E_{n}P_{n}(x) + D_{n}^{*}P_{n-1}(x)$$ \hspace{1cm} (4.39)

where

$$D_{n} = \begin{pmatrix} c_{2n,2} & 0 \\ c_{2n,1} & c_{2n+2,2} \end{pmatrix},$$

$$E_{n} = \begin{pmatrix} c_{2n,0} & c_{2n+1,1} \\ c_{2n+1,1} & c_{2n+2,2} \end{pmatrix}. $$ \hspace{1cm} (4.40)

The polynomials \(R_{N,m}(p)(x)\) are defined by

$$R_{N,m}(p)(x) = \sum_{n} P_{(nN+m)}^{(nN+m)}(0)x^{n}.$$ \hspace{1cm} (4.41)

5 Conclusion

To sum up, we have added to the exploration of $-1$ polynomials by introducing some $-1$ Krall-Jacobi polynomials. We focused on the simplest positive definite case. They have been obtained from generalized $q$–Jacobi polynomials through a limiting procedure and have remarkable features. Noteworthy is the fact that they obey a third-order differential-difference eigenvalue equation involving the reflection operator. The 3-term recurrence relation has also been determined. Finally let us stress that this present an interesting example of OPs whose measure involves a discrete mass at the center of the orthogonality interval (rather than at its boundary which is more common).

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