AN INFINITE FAMILY OF EXCLUDED MINORS FOR STRONG BASE-ORDERABILITY

JOSEPH E. BONIN AND THOMAS J. SAVITSKY

ABSTRACT. We discuss a conjecture of Ingleton on excluded minors for base-orderability, and, extending a result he stated, we prove that infinitely many of the matroids that he identified are excluded minors for base-orderability, as well as for the class of gammoids. We prove that a paving matroid is base-orderable if and only if it has no $M(K_4)$-minor. For each $k \geq 2$, we define the property of $k$-base-orderability, which lies strictly between base-orderability and strong base-orderability, and we show that $k$-base-orderable matroids form what Ingleton called a complete class. By generalizing an example of Ingleton, we construct a set of matroids, each of which is an excluded minor for $k$-base-orderability, but is $(k-1)$-base-orderable; the union of these sets, over all $k$, is an infinite set of base-orderable excluded minors for strong base-orderability.

1. INTRODUCTION

Basis-exchange properties are of long-standing interest in matroid theory (see Kung’s survey [14].) Condition (BE) in the following definition of a matroid is a simple basis-exchange property: a matroid $M$ is an ordered pair $(\mathcal{B}, E(M))$ where $E(M)$ is a finite set and $\mathcal{B}$ is a non-empty collection of subsets of $E(M)$ (the bases) such that

\begin{equation}
\text{(BE) } \text{if } B_1, B_2 \in \mathcal{B} \text{ and } x \in B_1 - B_2, \text{ then there is some } y \in B_2 - B_1 \text{ so that } (B_1 - x) \cup y \in \mathcal{B}.
\end{equation}

Brualdi [4] showed that property (BE) is equivalent to the following, seemingly stronger, symmetric basis-exchange property:

\begin{equation}
\text{if } B_1, B_2 \in \mathcal{B} \text{ and } x \in B_1 - B_2, \text{ then there is some } y \in B_2 - B_1 \text{ so that both } (B_1 - x) \cup y \text{ and } (B_2 - y) \cup x \text{ are in } \mathcal{B}.
\end{equation}

Brylawski [7], Greene [9], and Woodall [24] showed that the multiple symmetric exchange property holds for all matroids:

\begin{equation}
\text{if } B_1, B_2 \in \mathcal{B} \text{ and } X \subseteq B_1 - B_2, \text{ then there is some } Y \subseteq B_2 - B_1 \text{ so that both } (B_1 - X) \cup Y \text{ and } (B_2 - Y) \cup X \text{ are in } \mathcal{B}.
\end{equation}

In [4], Brualdi also showed that the bijective exchange property holds for all matroids:

\begin{equation}
\text{if } B_1, B_2 \in \mathcal{B}, \text{ then there is a bijection } \sigma: B_1 \to B_2 \text{ so that, for all } x \in B_1, \text{ the set } (B_1 - x) \cup \sigma(x) \text{ is in } \mathcal{B}.
\end{equation}

In contrast, our work here is motivated by the following basis-exchange properties that are not possessed by all matroids.

**Definition 1.1.** A matroid $M$ is base-orderable if, given any two bases $B_1$ and $B_2$, there is a bijection $\sigma: B_1 \to B_2$ such that for every $x \in B_1$, both $(B_1 - x) \cup \sigma(x)$ and $(B_2 - \sigma(x)) \cup x$ are bases.
A matroid $M$ is strongly base-orderable if, given any two bases $B_1$ and $B_2$, there is a bijection $\sigma: B_1 \to B_2$ such that for every $X \subseteq B_1$,

(*) $(B_1 - X) \cup \sigma(X)$ is a basis, and

(**) $(B_2 - \sigma(X)) \cup X$ is a basis.

To the best of our knowledge, the notion of base-orderability first appeared in both [6] and [4] at about the same time. Brualdi and Scrimger [6] showed that all transversal matroids are strongly base-orderable (and hence base-orderable). The property of base-orderability appeared (without the term) in Brualdi [4] as a natural strengthening of the basis-exchange properties discussed there.

Not all matroids are base-orderable; in particular, the cycle matroid $M(K_4)$ is not. We denote the class of base-orderable matroids by $BO$ and that of strongly base-orderable matroids by $SBO$. (In this paper, by a class of matroids we mean a set of matroids that is closed under isomorphism.) Clearly, $SBO \subseteq BO$. Ingleton [11] gave an example that shows that this containment is proper. In Section 9 we generalize his example; we construct an infinite collection of excluded minors for strong base-orderability, each of which is base-orderable.

It is easy to show that the class of base-orderable matroids is minor-closed, but describing its excluded minors remains an open problem. In Section 5, we discuss a conjecture of Ingleton on the excluded minors. Much of our work arose by exploring ideas in Ingleton’s paper [11], to which we owe a great debt. A number of our results and constructions grew from seeds in that paper, which, while providing a wealth of intriguing ideas, contains few proofs. To give a more complete account of this topic, we also offer proofs of some of the assertions that Ingleton made, either without proof or with a minimal sketch of the proof. In Section 3 we lay the groundwork for Section 5 and also prove that a paving matroid is base-orderable if and only if it has no $M(K_4)$-minor. In Section 4 we review cyclic flats, which we use extensively thereafter. In Section 8 we prove a special case of Ingleton’s conjecture. There we describe an infinite family of excluded minors for the class $BO$, each of which has precisely six cyclic flats; these matroids are also excluded minors for the class of gammoids. (Recall that a gammoid is a minor of a transversal matroid.)

Ingleton defined complete classes of matroids in [12].

Definition 1.2. A class of matroids is complete if it is closed under the operations of minors, duals, direct sums, truncations, and induction by directed graphs.

It is known that $BO$ and $SBO$ are complete classes. Ingleton [12] noted that the class of gammoids is complete and that each non-empty complete class contains all gammoids. In particular, $SBO$ contains all gammoids. This containment is proper since, for instance, the Vamos matroid is strongly base-orderable, but it is not a gammoid since gammoids are representable over the real numbers. These three complete classes form part of a hierarchy that Ingleton, again in [12], introduced. We quote:

“There is scope for introducing an infinity of complete classes between $BO$ and $SBO$ by appropriate limitations on the cardinals of subsets $X$ for which (*) is to hold, but these do not seem to have been studied.”

In Section 2 we begin to study some of these classes; we introduce the class $k$-$BO$ of $k$-base-orderable matroids, where $k$ is a fixed positive integer, and address the first three operations in Definition 1.2. The last two operations are treated in Section 7 using a reformulation of completeness that we prove in Section 6.

We assume familiarity with basic matroid theory. For notation, we follow Oxley [17]. We use $2^S$ to denote the set of subsets of a set $S$. For a family $\mathcal{F}$ of sets, we shorten
It is easy to check that
If \( a \) is not a coloop, then \( a \) is also a
The next two lemmas will be useful when discussing excluded minors.
Lemma 2.3. If \( M \) is not \( k \)-base-orderable, then it has a minor \( N \) whose ground set is the union of two disjoint bases of \( N \) that have no \( k \)-exchange ordering.
Proof. Let \( A \) and \( B \) be bases of \( M \) that have no \( k \)-exchange-ordering. Take
\[
N = M / (A \cap B) \setminus (E(M) - (A \cup B)).
\]
Lemma 2.4. If a rank-\( r \) matroid \( M \) with \( |E(M)| = 2r \) is not in \( k \)-BO but either
(1) all single-element contractions $M/x$ are in $k\cdot BO$, or
(2) all single-element deletions $M \setminus x$ are in $k\cdot BO$,
then $M$ is an excluded minor for $k\cdot BO$. The same is true if we replace $k\cdot BO$ by $SBO$.

**Proof.** By duality, it suffices to treat the case in which condition (1) holds. Since $M$ is not in $k\cdot BO$ but all of its single-element contractions are, $M$ has no coloops. Fix $y \in E(M)$. Let $B_1$ and $B_2$ be bases of $M \setminus y$. Since $M \setminus y$ has $2r - 1$ elements and rank $r$, the bases $B_1$ and $B_2$ cannot be disjoint. Fix $x \in B_1 \cap B_2$. In $M/x$, there is a $k$-exchange-ordering $\sigma$: $B_1 - x \rightarrow B_2 - x$ by condition (1). Extending $\sigma$ by setting $\sigma(x) = x$ gives a $k$-exchange-ordering for $B_1$ and $B_2$ in $M \setminus y$. Thus, $M$ is an excluded minor for $k\cdot BO$. □

### 3. The Basis-Exchange Digraph

The following construction is often helpful when examining basis-exchange properties.

**Definition 3.1.** Let $A$ and $B$ be bases of $M$. The basis-exchange digraph of $A$ and $B$ with respect to $M$ is the directed bipartite graph $\Omega_{M,A,B}$ on $2r(M)$ vertices with bipartition $\{A, B\}$ (using disjoint copies of $A$ and $B$ if $A \cap B \neq \emptyset$) where, for $a \in A$ and $b \in B$,

- (1) $(a, b) \in E(\Omega_{M,A,B})$ if and only if $(B - b) \cup a$ is not a basis of $M$, and
- (2) $(b, a) \in E(\Omega_{M,A,B})$ if and only if $(A - a) \cup b$ is not a basis of $M$.

We shorten $\Omega_{M,A,B}$ to $\Omega_{A,B}$ when $M$ is clear from the context. Figure 1 illustrates the definition.

![Figure 1](image)

**Figure 1.** The cycle matroid $M(K_4)$ and its basis-exchange digraph for the bases $A = \{a, b, c\}$ and $B = \{d, e, f\}$.

While some authors use this term for a different graph, our definition is consistent with the critical graphs that Ingleton defined in [11] (see Definition 5.1 below). The following proposition is well-known and easy to prove.

**Proposition 3.2.** Let $A$ and $B$ be bases of a matroid $M$. For $a \in A - B$, the fundamental circuit of $a$ with respect to $B$, denoted by $C(a, B)$, is

$$\{a\} \cup \{b \in B : (a, b) \text{ is not an edge of } \Omega_{A,B}\}.$$

We now recall Hall’s Theorem on matchings in a bipartite graph. It was originally stated and proved for systems of distinct representatives by Philip Hall in [10].

**Theorem 3.3.** Let $G$ be a bipartite graph with bipartition $\{X, Y\}$. There is a matching that covers $X$ if and only if $|X| \leq |N(X')|$ for all sets $X' \subseteq X$.

The next lemma, from [11], is of crucial importance, so we fill in the sketch of the proof that was given there.
Lemma 3.4. Let $A$ and $B$ be bases of a matroid $M$. There is no exchange-ordering between $A$ and $B$ if and only if some subgraph of $\Omega_{A,B}$ is an orientation of a complete bipartite graph $K_{s,t}$ for some $s \geq 2$ and $t \geq 2$ with $s + t = r(M) + 1$.

Proof. Let $\Omega$ be the undirected bipartite graph with the same bipartition as $\Omega_{A,B}$, in which $ab$, with $a \in A$ and $b \in B$, is an edge if and only if neither $(a, b)$ nor $(b, a)$ is an edge of $\Omega_{A,B}$. In other words, $ab$ is in $E(\Omega)$ exactly when both $(A - a) \cup b$ and $(B - b) \cup a$ are bases of $M$. Thus, $A$ and $B$ have an exchange-ordering if and only if $\Omega$ has a perfect matching.

By Hall’s Theorem, $\Omega$ has no perfect matching if and only if there is a subset $X \subseteq A$ with $|X| > |N(X)|$, where $N(X) \subseteq B$ is the neighborhood of $X$ in $\Omega$. Now

$$|X| + |B - N(X)| = r(M) + |X| - |N(X)|,$$

so the inequality $|X| > |N(X)|$ is equivalent to $|X| + |B - N(X)| \geq r(M) + 1$. Also, for every $x \in X$ and $y \in B - N(X)$, either $(x, y)$ or $(y, x)$ is an edge of $\Omega_{A,B}$. It follows that $\Omega$ has no perfect matching if and only if $\Omega_{A,B}$ has a restriction that is an orientation of $K_{s,t}$ for some $s$ and $t$ with $s + t = r(M) + 1$. By the symmetric basis-exchange property, neither $s$ nor $t$ can be $r$, so $s \geq 2$ and $t \geq 2$. □

Recall that a matroid $M$ is paving if it contains no circuit of size less than $r(M)$; it is sparse-paving if both $M$ and $M^*$ are paving. It is well-known that the classes of paving matroids and sparse-paving matroids are minor-closed. From Figure 1 and Lemma 3.4 we see that $M(K_4)$ is not base-orderable. In [8], de Sousa and Welsh proved that a binary matroid is base-orderable if and only if it has no $M(K_4)$-minor. We next prove that $M(K_4)$ is also the only obstacle to base-orderability for paving matroids.

Theorem 3.5. A paving matroid $M$ is base-orderable if and only if $M$ has no $M(K_4)$-minor.

Proof. If $M$ has an $M(K_4)$-minor, then $M$ is not base-orderable since $BO$ is minor-closed. We now show that if $M$ is not base-orderable, then it has an $M(K_4)$-minor. By Lemma 2.3, $M$ has a minor $N$ whose ground set is the union of two disjoint bases of $N$, say $A$ and $B$, that have no exchange-ordering. By Lemma 3.4, the basis-exchange digraph $\Omega^N_{A,B}$ has a subgraph that is an orientation of $K_{s,t}$ where $s + t = r(N) + 1$. Since $N$ is paving and $A \cap B = \emptyset$, Proposition 3.2 implies that the out-degree of any vertex in $\Omega^N_{A,B}$ is at most one. Therefore $|V(K_{s,t})| \geq |E(K_{s,t})|$, i.e., $s + t \geq st$. The only solution to this inequality with $s, t > 1$ is $s = t = 2$, so $r(N) = s + t - 1 = 3$, and so $|E(N)| = 6$. Transversal matroids are base-orderable, and the only rank-3 matroid on six elements that is not transversal is $M(K_4)$, so $N$ is $M(K_4)$. □

This theorem is interesting in light of the recent work of Pendavingh and van der Pol [18] that the number of sparse-paving matroids with no $M(K_4)$-minor is asymptotic to the best-known lower bound on the number of sparse-paving matroids. It is conjectured that asymptotically almost all matroids are sparse-paving, so it seems reasonable to also conjecture that almost all matroids are base-orderable.

The next two results are implicit in Ingleton [11].

Proposition 3.6. If $A$ and $B$ are disjoint bases of a matroid $M$ with $E(M) = A \cup B$, then $\Omega_{A,B}^{M^*}$ is obtained from $\Omega_{A,B}^{M}$ by reversing the orientation of each edge.

Proof. Let $a \in A$ and $b \in B$. Then $(A - a) \cup b$ is a basis of $M$ (or $M^*$) if and only if $(B - b) \cup a$ is a basis of $M^*$ (or $M$). □
The next proposition limits the structure of basis-exchange digraphs of excluded minors for $\mathcal{BO}$.

**Proposition 3.7.** Assume that bases $A$ and $B$ of a matroid $M$ have no exchange-ordering. Let $\Gamma$ be a subgraph of $\Omega_{A,B}$ that is an orientation of $K_{s,t}$ with $s + t = r(M) + 1$ and $s, t \geq 2$. If $\Gamma$ has either a source or a sink, then $M$ is not an excluded minor for base-orderability.

**Proof.** By Lemma 2.3, there is nothing to show unless $A \cap B = \emptyset$ and $E(M) = A \cup B$. Let $A = \{a_1, a_2, \ldots, a_r\}$ and $B = \{b_1, b_2, \ldots, b_t\}$, with $\{a_1, a_2, \ldots, a_s\} \cup \{b_1, b_2, \ldots, b_t\}$ being the vertex set of $\Gamma$. By Propositions 2.2 and 3.6, it suffices to treat the case in which $\Gamma$ has a source, say $a_1$. By the symmetric basis-exchange property, we may assume that $B' = (B - b_r) \cup a_1$ is a basis of $M$. We claim that $\Omega_{A,B'}$ has a subgraph that is an orientation of $K_{s,t}$. Clearly if $(b_j, a_i) \in E(\Omega_{A,B})$, then $(b_j, a_i) \in E(\Omega_{A,B'})$ as well. Let $(a_i, b_j)$, with $i > 1$, be an edge of $\Gamma$. Now $(a_1, b_j) \in E(\Omega_{A,B})$ since $a_1$ is a source of $\Gamma$. Proposition 3.3 gives $\{a_1, a_i\} \subseteq cl(B - b_j)$. Therefore

$$r((B - b_j) \cup \{a_1, a_i\}) = r(M) - 1.$$  

Thus, $(B - \{b_j, a_i\}) \cup \{a_1, a_i\}$ is not a basis of $M$, so $(a_1, b_j) \in E(\Omega_{A,B'})$. Lastly note that $(a_1, b_j) \in E(\Omega_{A,B'})$ since $a_1$ is itself a member of $B'$. Thus, $\Omega_{A,B'}$ has a subgraph that is an orientation of $K_{s,t}$, so $M/a_1$ is not base-orderable by Lemma 3.4. \hfill $\square$

### 4. BACKGROUND ON CYCLIC FLATS OF MATROIDS

The rest of this paper makes heavy use of cyclic flats, which we briefly review in this section. For a fuller account, see [3].

Let $M$ be a matroid with rank function $r$. A set $X \subseteq E(M)$ is cyclic if $X$ is a (possibly empty) union of circuits; equivalently, $X$ is cyclic if the restriction $M|X$ has no coloops. The collection of cyclic flats of $M$, denoted $Z(M)$, is a lattice under set-inclusion, with the same join as in the lattice of flats, namely, $X \vee Y = cl(X \cup Y)$. An attractive feature of cyclic flats is that they are well-behaved under duality.

**Proposition 4.1.** For a matroid $M$, we have $Z(M^*) = \{E(M) - X : X \in Z(M)\}$.

A matroid $M$ is determined by $E(M)$ along with the pairs $(A, r(A))$ for $A \in Z(M)$. The following result from [23] formulates matroids in these terms.

**Theorem 4.2.** Let $Z$ be a set of subsets of a set $S$ and let $r$ be an integer-valued function on $Z$. There is a matroid $M$ with $S = E(M)$ for which $Z = Z(M)$ and $r(X) = r_M(X)$ for all $X \in Z$ if and only if

1. $Z$ is a lattice under inclusion,
2. $r(0_Z) = 0$, where $0_Z$ is the least element of $Z$,
3. $0 < r(Y) - r(X) < |Y - X|$ for all sets $X, Y \in Z$ with $X \subseteq Y$, and
4. for all incomparable sets $X, Y \in Z$ (i.e., neither contains the other),

$$r(X) + r(Y) \geq r(X \vee Y) + r(X \wedge Y) + |(X \cap Y) - (X \wedge Y)|.$$  

The rank of a set $X \subseteq E(M)$, in terms of the ranks of cyclic flats, is given by

$$r_M(X) = \min \{r(A) + |X - A| : A \in Z(M)\}. \quad (1)$$

We also require information about the cyclic flats of minors.

**Lemma 4.3.** Let $M$ be a matroid, and let $x \in E(M)$. If $F \in Z(M\setminus x)$, then either $F$ or $F \cup x$ is a cyclic flat of $M$. The same conclusion holds if $F \in Z(M/x)$.
Proof. If a cyclic flat $F$ of $M\setminus x$ is also a flat of $M$, then $F \in \mathcal{Z}(M)$; otherwise $x$ is not a coloop of $M|F \cup x$ and $cl_M(F) = F \cup x$, so $F \cup x \in \mathcal{Z}(M)$. The second assertion follows by duality. □

We say that a matroid $N$ is freer than $M$ if $E(M) = E(N)$ and $r_M(X) \leq r_N(X)$ for all $X \subseteq E(M)$. We next formulate this order (the weak order) in terms of cyclic flats.

Lemma 4.4. A matroid $N$ is freer than $M$ if and only if for all $F \in \mathcal{Z}(N)$, there is some $A \in \mathcal{Z}(M)$ with $r_M(A) + |F - A| \leq r_N(F)$.

Proof. The necessity of the condition is clear. We focus on the converse. For $X \subseteq E(M)$, we have $r_N(X) = r_N(F) + |X - F|$ for some $F \in \mathcal{Z}(N)$. Now $r_M(A) + |F - A| \leq r_N(F)$ for some $A \in \mathcal{Z}(M)$ by assumption. Since $|X - A| \leq |X - F| + |F - A|$, we have

$$
r_M(X) \leq r_M(A) + |X - A|
\leq r_M(A) + |F - A| + |X - F|
\leq r_N(F) + |X - F|
= r_N(X).
$$

The first and last terms are the required inequality. □

We will use the Mason-Ingleton characterization of transversal matroids.

Theorem 4.5 (The Mason-Ingleton condition). A matroid $M$ is transversal if and only if for all nonempty antichains $\mathcal{F}$ of $\mathcal{Z}(M)$,

$$
r(\cap \mathcal{F}) \leq \sum_{\mathcal{F}' \subseteq \mathcal{F}} (-1)^{|\mathcal{F}'|+1} r(\cup \mathcal{F}').
$$

(2)

For a proof of this theorem, see [2]. Inequality (2) trivially holds when $|\mathcal{F}| = 1$, and it reduces to submodularity when $|\mathcal{F}| = 2$. We will use the following corollary.

Corollary 4.6. Let $M$ be a matroid. Fix $G \subseteq 2^{E(M)}$ with $\mathcal{Z}(M) \subseteq G$. If inequality (2) holds for all nonempty antichains $\mathcal{F}$ of $G$ with $|\mathcal{F}| \geq 3$, then $M$ is transversal.

5. INGLETON’S CONJECTURE

In [11], Ingleton discussed an infinite set of matroids that he conjectured to be excluded minors for BO. His conjectured excluded minors are associated to what he called critical graphs; however, he gave the definition of these matroids only for critical graphs that lack a structure that we call an obstruction. For a critical graph with no obstructions, he gave two families of sets and said that the bases of the associated matroid are their common transversals; in Sections 5.1 and 5.2, we develop a view of these matroids in terms of cyclic flats and show that Ingleton’s description of the bases applies precisely when obstructions are absent. The properties we prove about obstructions show that they are relatively well-behaved, and in Section 5.3 we define a likely candidate for the conjectured excluded minors that are associated to critical graphs with obstructions (that material is not used in the rest of the paper). In Section 5.4, we present data that supports the conjecture about excluded minors.
5.1. Critical graphs and pairs of transversal matroids. We start with a fundamental definition due to Ingleton.

**Definition 5.1.** Let $A$ and $B$ be disjoint sets of size $r$, where $r \geq 3$. A bipartite digraph $\Delta$ with bipartition $\{A, B\}$ is a critical graph if there are subsets $X$ of $A$ and $Y$ of $B$ such that

1. $|X| + |Y| = r + 1$,
2. for all $x \in X$ and $y \in Y$, exactly one of $(x, y)$ and $(y, x)$ is in $E(\Delta)$,
3. if $(u, v) \in E(\Delta)$, then $\{u, v\} \subseteq X \cup Y$, and
4. no element of $X \cup Y$ is a source or sink of $\Delta$.

Thus, a critical graph on $2r$ vertices is an orientation of $K_{s,t}$, for some $s \geq 2$ and $t \geq 2$ with $s + t = r + 1$, having neither a source nor a sink, with $r - 1$ isolated vertices adjoined.

For example, the digraph in Figure 1 is a critical graph. Definition 5.1 is motivated largely with Ingleton said that for each critical graph $\Delta$, he could construct a matroid $M(\Delta)$ on $A \cup B$ in which $A$ and $B$ are bases and the basis-exchange digraph $\Omega^{\Delta}_{A,B}$ is $\Delta$; furthermore, all excluded minors for $BO$ occur among what he called the good specializations of these matroids $M(\Delta)$. (One property of good specializations is that they can have more dependent sets.) Thus, the idea is to construct, for each critical graph $\Delta$, a matroid $M(\Delta)$ that has $\Delta$ as a basis-exchange digraph (so $M(\Delta) \not\in BO$ by Lemma 3.4) and whose dependent sets are, as much as possible, just those that are forced by $\Delta$.

To see what structure $\Delta$ imposes on $M(\Delta)$, let $M$ be a rank-$r$ matroid, with $r \geq 3$, where $E(M)$ is the disjoint union of two bases, $A$ and $B$, and $\Omega^{M}_{A,B}$ is a critical graph $\Delta$. Let $X$ and $Y$ be as in Definition 5.1. Using fundamental circuits, as in Proposition 3.2, we recast what $\Delta$ gives us. We have

1. proper subsets $X$ of $A$ and $Y$ of $B$ with $|X| + |Y| = r + 1$,
2. a fundamental circuit $C(y, A)$, for each $y \in Y$, with $A - X \subseteq C(y, A) - y \subseteq A$ and
   \[ A = \bigcup_{y \in Y} C(y, A) - y, \]
3. a fundamental circuit $C(x, B)$, for each $x \in X$, with $B - Y \subseteq C(x, B) - x \subseteq B$ and
   \[ B = \bigcup_{x \in X} C(x, B) - x, \]
4. whenever $x \in X$ and $y \in Y$, exactly one of the statements $x \in C(y, A)$ and $y \in C(x, B)$ holds, and
5. $C(b, A) = A \cup b$ for $b \in B - Y$, and $C(a, B) = B \cup a$ for $a \in A - X$.

For a subset $A'$ of the basis $A$, the flat $cl_{M}(A') = A' \cup \{b \in B : C(b, A) - b \subseteq A'\}$ has rank $|A'|$. This flat is cyclic if for each $a \in A'$, there is a $b \in cl_{M}(A') \cap B$ with $a \in C(b, A)$. The counterparts of these conclusions for subsets $B'$ of $B$ also hold. There may, of course, be circuits of $M$ besides the fundamental circuits that $\Delta$ gives.

As we will see, in many cases the minimal structure that $\Delta$ imposes on $M(\Delta)$ is enough to determine $M(\Delta)$. Let $\Delta$ be a critical graph with $r$, $A$, $B$, $X$, and $Y$ as in Definition 5.1. We begin to describe a candidate for $M(\Delta)$ by specifying some of its cyclic flats and their ranks. From the observations above, we see that in order to have $\Delta = \Omega^{\Delta}_{A,B}$, certain cyclic flats must be present in $Z(M(\Delta))$. For $b \in B$, we define

\[ C_{\Delta}(b, A) = \{b\} \cup \{a \in A : (b, a) \notin E(\Delta)\}. \]
We extend this notation as follows: for \( B' \subseteq B \), we define
\[
C_\Delta(B', A) = \bigcup_{b \in B'} C_\Delta(b, A).
\] (4)

Now we define \( Z_A \) and the ranks of its sets as follows: for each \( B' \subseteq B \), we adjoin the set
\[
D_\Delta(B') = C_\Delta(B', A) \cup \{ b \in B : C_\Delta(b, A) - b \subseteq C_\Delta(B', A) \}
\] (5)
to \( Z_A \) and set \( r(D_\Delta(B')) = |D_\Delta(B') \cap A| \). Note that \( \emptyset \) is in \( Z_A \) with rank 0 (take \( B' = \emptyset \)), and \( A \cup B \) is in \( Z_A \) with rank \( r \) (take \( B' = B \), say). Also note that
\[
D_\Delta(B') = C_\Delta(D_\Delta(B') \cap B, A)
\]
and thus
\[
D_\Delta(B') \cap Y = \{ y \in Y : C_\Delta(y, A) - y \subseteq D_\Delta(B') \}. \] (6)

Likewise construct \( Z_B \). Specifically, for \( a \in A \), we define
\[
C_\Delta(a, B) = \{ a \} \cup \{ b \in B : (a, b) \not\in E(\Delta) \}.
\]

For \( A' \subseteq A \), we define
\[
C_\Delta(A', B) = \bigcup_{a \in A'} C_\Delta(a, B),
\] (7)
and we adjoin the set
\[
D_\Delta(A') = C_\Delta(A', B) \cup \{ a \in A : C_\Delta(a, B) - a \subseteq C_\Delta(A', B) \}
\] (8)
to \( Z_B \) with rank \( |D_\Delta(A') \cap B| \).

Set \( Z_\Delta = Z_A \cup Z_B \). Since \( Z_A \cap Z_B = \{ \emptyset, A \cup B \} \), there is no ambiguity as to the ranks of the sets in \( Z_\Delta \).

In the proof of the next result, we use the following observations about the transversal matroid \( M \) that arises from a bipartite graph. By Theorem 3.3, the circuits of \( M \) are the subsets \( W \) of \( E(M) \) for which \( |N(W)| < |W| \) while \( |N(Z)| \geq |Z| \) whenever \( Z \subseteq W \). Thus, if \( W \) is a circuit and \( w \in W \), then \( |N(W)| = |N(W - \{ w \})| = r(W) \), and so \( N(W) = N(W - \{ w \}) \). Also, if \( |N(U)| = r(U) \), then \( cl(U) = \{ x : N(x) \subseteq N(U) \} \).

**Proposition 5.2.** The set \( Z_A \), with the rank of each set in \( Z_A \) as given above, is the set of cyclic flats, along with their ranks, of a transversal matroid on \( A \cup B \), and likewise for \( Z_B \). The sets \( A \) and \( B \) are bases of both of these transversal matroids.

**Proof.** By symmetry, it suffices to treat the assertions about \( Z_A \). For \( a \in A \), let
\[
S_a = \{ a \} \cup \{ b \in B : (b, a) \not\in E(\Delta) \}.
\]

Let \( \Gamma \) be the bipartite graph with bipartition \( \{ A \cup B, \{ S_a : a \in A \} \} \) and with edge set \( \{ xS_a : x \in S_a \} \). Let \( M \) be the transversal matroid on \( A \cup B \) defined by \( \Gamma \). It is easy to see that \( A \) is a basis of \( M \), that \( |N_\Gamma(A')| = r(A') \) for all subsets \( A' \) of \( A \), and that, for each \( b \in B \), the set \( C_\Delta(b, A) \) in equation 3 is the fundamental circuit \( C_M(b, A) \). From these conclusions, equation 5, and the observations above, it follows that all sets in \( Z_A \) are in \( Z(M) \).

To show that each set \( Z \) in \( Z(M) \) is in \( Z_A \) and that \( \{ A \cap Z \} \) is a basis of \( M|Z \), we start with a circuit \( W \) of \( M \). As noted above, \( |N_\Gamma(W)| = r(W) \). Thus, for \( b \in W \cap B \), we have \( C_M(b, A) \subseteq cl_M(W) \). Therefore \( cl_M(W) \cap A \) is a basis of \( M|cl_M(W) \). For any \( Z \in Z(M) \), there are circuits \( W_1, W_2, \ldots, W_t \) of \( M \) with \( Z = W_1 \cup W_2 \cup \cdots \cup W_t \), so, since \( cl_M(W_i) \cap A \) is a basis of \( M|cl_M(W_i) \) for each \( i \), and \( cl_M(W_i) \subseteq Z \), it follows that \( Z \cap A \) is a basis of \( M|Z \), and, furthermore, \( Z = D_\Delta(Z \cap B) \).
Finally, to show that $B$ is a basis of $M$, assume, instead, that $B$ contains a circuit $W$. We use $X$, $Y$, and $r$ as in Definition 5.1. Since $|N\Gamma(W)| = r(W) < r$, no element of $B - Y$ is in $W$, so $W \subseteq Y$. Now $N\Gamma(W) = \{S_a: a \in (A - X) \cup X'\}$ for some subset $X'$ of $X$. Since $W$ is a circuit, $|W|-1 = |N\Gamma(W)|$, so $|W|-1 = |A - X| + |X'|$. Adding $|X'|$ to both sides gives $|X - X'| + |W|-1 = |A|$, so $|X - X'| + |W| = r + 1$. Since $|X| + |Y| = r + 1$, we get $W = Y$ and $X' = \emptyset$, but $W = Y$ gives the contradiction $|N\Gamma(W)| = r$. (Having $X' = \emptyset$ also gives a contradiction: each vertex in $W$ would be a source of $\Delta$.) Thus, $B$ is indeed a basis of $M$. □

5.2. When $Z_\Delta$ is the lattice of cyclic flats of $M(\Delta)$. In this section, we show that the matroid $M(\Delta)$ associated to a critical graph $\Delta$ can have $Z_\Delta$ as its lattice of cyclic flats if and only if $\Delta$ does not contain a structure that we call an obstruction.

We start with some examples in which $Z(M(\Delta)) = Z_\Delta$. The digraph in Figure 1 is a critical graph, $\Delta_3$, where $A = \{a, b, c\}$ and $B = \{d, e, f\}$; the associated matroid $M(\Delta_3)$ is $M(K_4)$. For a more complex example, let $\Delta_5$ be the digraph in Figure 2 where $A = \{a_1, \ldots, a_5\}$ and $B = \{b_1, \ldots, b_5\}$. Then $M(\Delta_5)$ has the lattice of cyclic flats shown in Figure 3.

![Figure 2. The critical graph $\Delta_5$.](image)

![Figure 3. The lattice of cyclic flats of $M(\Delta_5)$.](image)
It follows from equation (1) that if \( Z(M(\Delta)) = \Omega_{\Delta} \), then the bases of \( M(\Delta) \) are those that are common to the two transversal matroids in Proposition 5.5. In this case, \( M(\Delta) \) is definitely the matroid that Ingleton intended since he said, in (11), “For a large class of \( \Delta \) it is possible to define the bases of \( M(\Delta) \) as the common transversals of two families of sets,” and he then gave the set system \( \{ S_a : a \in A \} \) that we used in the proof of Proposition 5.2, and its counterpart for \( Z_B \). In particular, \( A \) and \( B \) are both bases of \( M(\Delta) \). Also, \( \Omega_{\Delta,B}^{M(\Delta)} = \Delta \). Having \( Z(M(\Delta)) = \Omega_{\Delta} \) also implies that \( M(\Delta) \) is freest among the matroids \( \{ X : X \in M(\Delta) \} \). To see this, take \( A = F \) in Lemma 4.3.

Ingleton identified the structures in the following definition.

**Definition 5.3.** Given a critical graph \( \Delta \), a pair \((K, L)\) is an obstruction if

1. \( \emptyset \subseteq K \subseteq X \) and \( \emptyset \subseteq L \subseteq Y \),
2. \((k, y) \in E(\Delta)\) for every \( k \in K \) and \( y \in Y - L \), and
3. \((l, x) \in E(\Delta)\) for every \( l \in L \) and \( x \in X - K \).

The inclusions \( C_\Delta(K, B) - K \subseteq L \cup (B - Y) \) and \( C_\Delta(L, A) - L \subseteq K \cup (A - X) \) are equivalent to conditions (2) and (3), respectively.

Fortunately, as the next four results show, obstructions are rather well-behaved. The proof of the following lemma is immediate from the definition.

**Lemma 5.4.** Let \( \Delta \) be a critical graph, and let \( \Delta' \) be the digraph obtained by reversing the orientation of every edge of \( \Delta \). The pair \((K, L)\) is an obstruction of \( \Delta \) if and only if \((X - K, Y - L)\) is an obstruction of \( \Delta' \).

The next result shows that obstructions form a lattice.

**Proposition 5.5.** If \((K_1, L_1)\) and \((K_2, L_2)\) are obstructions of a critical graph \( \Delta \), then both \((K_1 \cap K_2, L_1 \cap L_2)\) and \((K_1 \cup K_2, L_1 \cup L_2)\) are obstructions.

**Proof.** First observe that if \( k \in K_1 - K_2 \) and \( l \in L_2 - L_1 \), then both \((k, l)\) and \((l, k)\) would be edges of \( \Delta \), which is impossible. Thus, at least one of \( K_1 - K_2 \) and \( L_2 - L_1 \) is empty, and likewise for the pair \( K_2 - K_1 \) and \( L_1 - L_2 \). That is, (i) either \( K_1 \subseteq K_2 \) or \( L_2 \subseteq L_1 \) and (ii) either \( K_2 \subseteq K_1 \) or \( L_1 \subseteq L_2 \). If \( K_1 \neq K_2 \) and \( L_1 \neq L_2 \), then conclusions (i) and (ii) imply that either (a) \( K_1 \subseteq K_2 \) and \( L_1 \subseteq L_2 \), or (b) \( K_2 \subseteq K_1 \) and \( L_2 \subseteq L_1 \): in these cases, the conclusion of the proposition is immediate.

By symmetry, we may now assume that \( K_1 = K_2 \). Thus, \((k, y) \in E(\Delta)\) for each \( k \in K_1 \) and \( y \in Y - (L_1 \cap L_2) \); likewise, \((l, x) \in E(\Delta)\) for each \( l \in L_1 \cup L_2 \) and \( x \in X - K_1 \). It cannot be that \( L_1 \cap L_2 = \emptyset \), for then each \( k \in K_1 \) would be a source of \( \Delta \). We also cannot have \( L_1 \cup L_2 = Y \), since then each \( x \in X - K_1 \) would be a sink of \( \Delta \). Thus, both \((K_1, L_1 \cap L_2)\) and \((K_1, L_1 \cup L_2)\) are obstructions, as needed. \( \square \)

Thus, if a critical graph has an obstruction, then it has a minimum obstruction and a maximum obstruction.

The next result shows that there are no obstructions if \( r < 7 \). However, obstructions can and do occur if \( r \geq 7 \). Figure 4 shows \( \Delta_7 \), the smallest critical digraph, up to isomorphism, that has an obstruction.

**Proposition 5.6.** If \((K, L)\) is an obstruction of a critical graph \( \Delta \), then each of the sets \( K, L, X - K, \) and \( Y - L \) has at least two elements.

**Proof.** Assume for a contradiction that \( Y - L = \{ y \} \). Since \( K \subseteq X \) and \( y \) is not a sink, there is some \( x \in X - K \) with \((y, x) \in E(\Delta)\). This implies that \( x \) is then a sink, contrary
to property (4) of $\Delta$. Thus $|Y - L| \geq 2$. By symmetry $|X - K| \geq 2$. Now Lemma 5.4 implies that $|K|, |L| \geq 2$.

**Figure 4.** The pair $\{a_3, a_4\}$ is an obstruction in this critical graph, $\Delta_7$. The edges with gray arrows show that conditions (2) and (3) in Definition 5.3 hold.

**Lemma 5.7.** If $(K, L)$ is the minimum obstruction of $\Delta$, then $C_\Delta(L, A) - L = K \cup (A - X)$ and $C_\Delta(K, B) - K = L \cup (B - Y)$.

Proof. For $k \in K$, if $C_\Delta(L, A) - L \subseteq (K - k) \cup (A - X)$, then $(K - k, L)$ would also be an obstruction, contrary to $(K, L)$ being the minimum. The second equality follows by symmetry.

We now treat a key result.

**Proposition 5.8.** Let $\Delta$ be a critical graph. The collection $Z_{\Delta}$, with the ranks given before Proposition 5.2, satisfies conditions (Z0)–(Z3) in Theorem 4.2 (and so defines a matroid) if and only if $\Delta$ has no obstructions.

Proof. From equation (5), each $I \in Z_A - \{A \cup B, \emptyset\}$ is a proper superset of $A - X$ that is disjoint from $B - Y$, while from equation (6), each $J \in Z_B - \{A \cup B, \emptyset\}$ is a proper superset of $B - Y$ that is disjoint from $A - X$; thus, $I \cap J = \emptyset$ and $I \cup J = A \cup B$. Also, if $I, J \in Z_A$, then $I \cup J$ and $I \cap J$ are as in $Z_A$, and likewise if $I, J \in Z_B$. Thus, condition (Z0) holds.

Each of the other conditions follows from its counterpart in $Z_A$ or $Z_B$ with one exception: we must check whether condition (Z3) holds for all $I, J$ with $I \in Z_A - \{A \cup B, \emptyset\}$ and $J \in Z_B - \{A \cup B, \emptyset\}$. Condition (Z3) for such an $I$ and $J$ is equivalent to

$$r + |I \cap J| \leq |I \cap A| + |J \cap B|,$$

which, since $|A - X| + |B - Y| = r - 1$, is equivalent to $1 + |I \cap J| \leq |I \cap X| + |J \cap Y|$. Assume that this inequality fails, that is,

$$|I \cap J| > |I \cap X| + |J \cap Y|.$$ 

Since $I \cap J$ is the disjoint union of $I \cap J \cap X$ and $I \cap J \cap Y$, the last inequality gives both $I \cap J \cap X = I \cap X$ and $I \cap J \cap Y = J \cap Y$, from which we get $I \cap X \subseteq J \cap X$ and $J \cap Y \subseteq I \cap Y$. Equation (6) and the inclusion $J \cap Y \subseteq I \cap Y$ give

$$C_\Delta(J \cap Y, A) - (J \cap Y) \subseteq I \cap A = (I \cap X) \cup (A - X).$$

Likewise, $C_\Delta(I \cap X, B) - (I \cap X) \subseteq (J \cap Y) \cup (B - Y)$ follows from $I \cap X \subseteq J \cap X$, so $(I \cap X, J \cap Y)$ is an obstruction. Thus, if condition (Z3) fails, then $\Delta$ has an obstruction.
Now assume that \( \Delta \) has an obstruction. Let \( (K, L) \) be the minimum obstruction. Set \( I = D_\Delta(L) \), so \( I \in \mathcal{Z}_A \), and set \( J = D_\Delta(K) \), so \( J \in \mathcal{Z}_B \). By Lemma 5.7:

\[
r(I) + r(J) = |A - X| + |K| + |B - Y| + |L| = r - 1 + |K| + |L|.
\]

On the other hand, \( I \wedge J = \emptyset \) and \( K \cup L \subseteq I \cap J \), so

\[
r(I \vee J) + r(I) + r(J) + |(I \cap J) - (I \wedge J)| \geq r + |K| + |L|.
\]

Thus condition (Z3) of Theorem 4.2 fails. \( \square \)

Thus, when \( \Delta \) has no obstructions, we take \( M(\Delta) \) to be the matroid with lattice of cyclic flats equal to \( \mathcal{Z}_\Delta \). We next reformulate a conjecture that Ingleton made in [11]. We prove a special case in Theorem 8.1.

**Conjecture 5.9.** If \( \Delta \) is a critical graph with no obstructions, then \( M(\Delta) \) is an excluded minor for \( \mathcal{B}_0 \).

### 5.3. A candidate for \( M(\Delta) \) when \( \Delta \) has obstructions.**

The only thing that Ingleton said in [11] about \( M(\Delta) \) when \( \Delta \) has an obstruction is that “the set of bases of \( M(\Delta) \) has to be a suitably chosen proper subset of the set of common transversals” of \( \{S_a : a \in A\} \) and its counterpart for \( \mathcal{Z}_B \). Thus, we cannot be sure that what we present below is what he had in mind. As we note below, our candidate for \( M(\Delta) \) has a property that Ingleton asserted for the matroids he had in mind. Also, the computational evidence cited in Section 5.4 lines up with Ingleton’s conjecture. The material in this section is not used in the rest of the paper.

Proposition 5.5 justifies the following notation. When \( \Delta \) has an obstruction, let \( P \) be the set \( K_0 \cup L_0 \) where \( (K_0, L_0) \) is the minimum obstruction of \( \Delta \), and let \( Q \) be the set \( K_1 \cup L_1 \cup (A - X) \cup (B - Y) \) where \( (K_1, L_1) \) is the maximum obstruction of \( \Delta \).

**Proposition 5.10.** Let \( \Delta \) be a critical graph having an obstruction. Set \( r(P) = |P| - 1 \) and \( r(Q) = r - 1 \). Each of the following collections of sets, with the ranks defined above, satisfies the conditions of Theorem 4.2 and so defines a matroid:

\[
\mathcal{Z}_A^P = \mathcal{Z}_\Delta \cup \{P\}, \quad \mathcal{Z}_A^Q = \mathcal{Z}_\Delta \cup \{Q\}, \quad \text{and} \quad \mathcal{Z}_A^{P,Q} = \mathcal{Z}_\Delta \cup \{P, Q\}.
\]

**Proof.** Each of the sets \( P \cap X, P \cap Y, A - Q, \) and \( B - Q \) has at least two elements by Proposition 5.6. Recall that \( \mathcal{C}_\Delta \) is given by equations (4) and (7), and \( \mathcal{D}_\Delta \) by (5) and (6).

We first treat \( \mathcal{Z}_A^P \). We begin with the lattice structure. Consider \( I, J \in \mathcal{Z}_A^P - \{A \cup B, \emptyset\} \). By symmetry, we may take \( I \in \mathcal{Z}_A \).

Assume first that \( J \in \mathcal{Z}_A \). Clearly \( I \cup J \) is the same as in \( \mathcal{Z}_A \), as is \( I \wedge J \) if \( P \not\subseteq I \cap J \). Sets in \( \mathcal{Z}_A \) that contain \( P \) also contain \( D_\Delta(L_0) \), so if \( P \not\subseteq I \cap J \), then \( D_\Delta(L_0) \subseteq I \) and \( D_\Delta(L_0) \subseteq J \), so \( I \wedge J \) is again the same as in \( \mathcal{Z}_A \).

If \( J \in \mathcal{Z}_B \), then \( I \cap J = A \cup B \) and

\[
I \wedge J = \begin{cases} P & \text{if } P \not\subseteq I \cap J, \\ \emptyset & \text{otherwise.} \end{cases}
\]

Comparable sets trivially have a meet and a join, so we may assume that the remaining sets to treat, \( I \) and \( P \), are incomparable, in which case it is easy to see that \( I \wedge P = \emptyset \) and

\[
I \vee P = D_\Delta(L_0 \cup (I \cap B)).
\]

Thus, condition (Z0) holds. Note that \( r(I \vee P) = |(I \cap A) \cup K_0| \); we will use this below.

Next, we check condition (Z3) for \( P \) and an incomparable set \( I \in \mathcal{Z}_A \). The following statements are equivalent:
(1) \( r(I \lor P) + r(I \land P) + |(I \cap P) - (I \land P)| \leq r(I) + r(P), \)
(2) \( |(I \cap A) \cup K_0| + |I \cap K_0| + |I \cap L_0| \leq |I \cap A| + |K_0| + |L_0| - 1, \)
(3) \( |K_0 - I| + |I \cap K_0| + |I \cap L_0| \leq |K_0| + |L_0| - 1, \)
(4) \( |I \cap L_0| \leq |L_0| - 1, \)
(5) \( I \cap L_0 \subseteq L_0. \)

Assume statement (5) fails, so \( L_0 \subseteq I. \) Lemma \[5.7\] gives \( C_\Delta(L_0, A) - L_0 = K_0 \cup (A - X). \) The inclusion \( L_0 \subseteq I \) gives \( C_\Delta(L_0, A) - L_0 \subseteq I \cap A. \) Thus, \( K_0 \subseteq I, \) so \( P \subseteq I, \) contrary to the assumption that they are incomparable.

We next check condition (Z3) for \( I \in Z_A \) and \( J \in Z_B. \) If the inequality in condition (Z3) holds for \( I \) and \( J \) in the lattice \( Z_\Delta, \) then its counterpart clearly holds in the lattice \( Z_\Delta^Q. \)

Thus, the inequality fails for \( I \) and \( J \) in \( Z_\Delta. \) The proof of Proposition \[5.8\] shows that \( P \subseteq I \cap J, \) so \( I \land J = P. \) Therefore the inequality in condition (Z3) amounts to

\[
|I \cap A| + |J \cap B| \geq r + |I \cap J| - 1, \quad \text{and so to}
\]

\[
|I \cap X| + |J \cap Y| \geq |I \cap J|.
\]

This last inequality holds because \( X \) and \( Y \) are disjoint and \( I \cap J \subseteq X \cup Y. \)

Condition (Z3) for the remaining incomparable pairs follows by symmetry and Proposition \[5.2\]. We now check condition (Z2). If \( I \subseteq J, \) with either \( I, J \in Z_A \) or \( I, J \in Z_B, \) then the condition holds by Proposition \[5.2\]. If \( P \subseteq I \) and \( J \in Z_A, \) then

\[
r(I) - r(P) < |I \cap A| - |P \cap X| = |I \cap A| - |P \cap A| \leq |I - P|,
\]

as required. Since \( |A \cup B - P| \geq r, \) the inequality in condition (Z2) holds for \( P \) and \( A \cup B. \) Checking the condition is trivial if one of the sets is \( \emptyset, \) and it follows for the remaining pairs of sets by symmetry. Thus, the assertion about \( Z_{\Delta}^Q \) holds.

For \( Z_{\Delta}^Q, \) let \( \Delta' \) be the critical graph obtained by reversing the orientation of each edge of \( \Delta. \) Let \((K_0', L_0')\) be the minimum obstruction of \( \Delta', \) and let \( P' = K_0' \cup L_0'. \) Lemma \[5.4\] gives \( P' = (A \cup B) - Q. \) Let \( M' \) be the matroid associated to \( Z_{\Delta}^{P'Q}. \) We claim that the cyclic flats and their ranks for the dual \( M'^* \) are precisely those of \( Z_{\Delta}^{P'Q}. \) We use Proposition \[4.1\].

We have

\[
r_{M'^*}(Q) = |Q| + r_{M'}(P') - r(M') = (2r - |P'|) + (|P'| - 1) - r = r - 1,
\]

as required. Now suppose \( I' \in Z(M') - \{A \cup B, P', \emptyset\} \) where \( (A - X) \subseteq I'. \) By checking the effect of reversing the orientation, we get that

\[
(A \cup B) - I' = D_\Delta(A - I'),
\]

which is assigned rank \( |B - I'| \) in \( Z_{\Delta}^{P'Q}. \) By symmetry, it follows that \( Z(M'^*) = Z_{\Delta}^{P'Q}. \) Also,

\[
r_{M'^*}((A \cup B) - I') = |(A \cup B) - I'| + r_{M'}(I') - r(M')
\]

\[
= 2r - |I'| + |I' \cap A| - r
\]

\[
= r - |I' \cap B|
\]

\[
= |B - I'|.
\]

Before treating \( Z_{\Delta}^{P'Q}, \) we make a general observation. Let \( M \) be a matroid with rank function \( \rho, \) and fix a set \( Z' \in 2^E(M) - Z(M). \) Let \( Z' \) be the collection \( Z(M) \cup \{Z'\} \) and suppose a function \( r': Z' \to \mathbb{N} \) agrees with \( \rho \) on \( Z(M). \) Now assume that \( r' \) and \( Z' \) satisfy conditions (Z0) and (Z2) of Theorem \[4.2\]. We claim that the inequality in condition (Z3) for \( r' \) and \( Z' \) follows for all sets \( I, J \in Z(M). \) To see this, note that the presence
Let \( Z \) exist. Let \( I \) follow from our work on this preserves the validity of condition (Z3). Also, the meet of \( (Z2) \) gives \( r'(Z') - r'(I \wedge Z(M)) \). In all other cases, neither the meet nor the join changes.

Now \( (A \cup B) - (X \cup Y) \subseteq Q - P \), so \(|Q - P| \geq r - 1 = r(Q) > r(Q) - r(P)\), so condition (Z2) holds for \( P \) and \( Q \). By what we noted above, condition (Z3) for \( Z'_A \) once we prove condition (Z0), which we do next.

Since \( Z'_{P,Q} \) is finite and has a greatest member, \( A \cup B \), it suffices to show that meets exist. Let \( \wedge_P \) and \( \vee_P \) denote the operations of the lattice \( Z'_{P,Q} \), and similarly for the others. Let \( I \in Z_A - \{A \cup B, \emptyset\} \). Let \( W = I \wedge_Q Q \), which, being contained in \( I \), is in \( Z_A \). The only possible candidates for \( I \wedge_{P,Q} Q \) are \( W \) and \( P \), and the latter is a candidate only if \( P \subseteq I \wedge Q \), so assume this inclusion holds. Now \( W \subseteq P \) if and only if \( W = \emptyset \), in which case \( I \wedge_{P,Q} Q = P \), so assume \( W \neq \emptyset \). Thus, \( P \vee_P W \in Z_A \). From \( W \subseteq I \) and \( P \subseteq I \) we get \( P \vee_P W \subseteq I \). From \( W \in Z_A \) and \( W \subseteq Q \) we get \( W \cap B \subseteq L_1 \). By equation (10),

\[
P \vee_P W = D_{\Delta}(L_0 \cup (W \cap B)) \subseteq D_{\Delta}(L_1) \subseteq Q.
\]

Thus, \( I \wedge_Q Q = W \subseteq P \vee_P W \subseteq I \wedge Q \), which, since \( P \vee_P W \in Z_A \), gives \( P \vee_P W = W \).

Thus, \( P \subseteq W \), so \( I \wedge_{P,Q} Q = W \). By symmetry, if \( J \in Z_B \), then \( J \wedge_{P,Q} Q \) exists. It is easy to see that all other meets exist since \( Z'_{P,Q} \) and \( Z^Q_{\Delta} \) are lattices, so conditions (Z0)–(Z3) hold for \( Z'_{P,Q} \).

Both \( A \) and \( B \) are bases of the matroids whose lattices of cyclic flats are \( Z'_{P,Q} \), \( Z^P_{\Delta} \), and \( Z^Q_{\Delta} \), as we see from the inequalities \(|P| - 1 + |A - P| > |A| \) and \( r - 1 + |A - Q| > |A| \), their counterparts for \( B \), equation (1), and Proposition 5.2. Also, Proposition 5.6 ensures that, for all three matroids, the basis-exchange digraph of \( A \) and \( B \) is \( \Delta \). Neither \( Z'_{\Delta} \) nor \( Z^Q_{\Delta} \) is a suitable choice for \( M(\Delta) \), where \( \Delta \) is the digraph in Figure 3 because both of the resulting matroids have non-base-orderable proper minors. We define \( M(\Delta) \) to be the matroid with \( Z(M(\Delta)) = Z'_{P,Q} \), and we know of no such \( M(\Delta) \) that is not an excluded minors for base-orderability. Another reason for choosing this definition of \( M(\Delta) \) is to make following proposition true.

**Proposition 5.11.** Let \( \Delta \) be a critical graph. If \( \Delta' \) is the digraph obtained by reversing the orientation of every edge of \( \Delta \), then \( M(\Delta) = M(\Delta') \).

While Ingleton did not state his construction of \( M(\Delta) \) when \( \Delta \) has an obstruction, he did state this duality result. We think it likely, but cannot be certain, that the matroid \( M(\Delta) \) defined above is the one he intended. With that caution, we state the next conjecture.

**Conjecture 5.12.** If \( \Delta \) has an obstruction, then \( M(\Delta) \), the matroid with lattice of cyclic flats equal to \( Z'_{P,Q} \), is an excluded minor for \( \mathcal{BO} \).

5.4. **Evidence for the conjectures.** Using a computer, we have verified Conjectures 5.9 and 5.12 for all critical graphs with \( r \leq 9 \). We first note that it is straightforward to write a program to test if a matroid is \( k \)-base-orderable: just check all possible bijections between all pairs of bases. To test the conjectures, we first generated, up to isomorphism, all orientations of each \( K_{s,t} \), where \( s + t = r + 1 \) and \( s, t \geq 2 \), using the `direct` command distributed with Brendan McKay’s `nauty` program [16]. Next we rejected orientations that had sources or sinks, checked if an obstruction was present, and then constructed \( M(\Delta) \) with the help of the matroid package for SageMath [19]. Finally, we checked
whether the single-element deletions and contractions of \( M(\Delta) \) were base-orderable using a program we wrote in the C programming language. We discovered that, for \( r \leq 9 \), if \( \Delta \) has no obstruction, then \( M(\Delta) \) is an excluded minor for \( \text{BO} \) and \( \text{SBO} \), while if \( \Delta \) has an obstruction, then \( M(\Delta) \) is an excluded minor for \( \text{BO} \) but not \( \text{SBO} \). Table 1 gives the number of matroids \( M(\Delta) \) checked this way.

| \( r \) | \( K_{s,t} \) | orientations with no obstructions | orientations with obstructions | total |
|---|---|---|---|---|
| 3 | \( K_{2,2} \) | 1 | 0 | 1 |
| 4 | \( K_{2,3} \) | 1 | 0 | 1 |
| 5 | \( K_{2,4} \) | 2 | 0 | 2 |
| 5 | \( K_{3,3} \) | 3 | 0 | 3 |
| 6 | \( K_{2,5} \) | 2 | 0 | 2 |
| 6 | \( K_{3,4} \) | 15 | 0 | 15 |
| 7 | \( K_{2,6} \) | 3 | 0 | 3 |
| 7 | \( K_{3,5} \) | 34 | 0 | 34 |
| 7 | \( K_{4,4} \) | 43 | 1 | 44 |
| 8 | \( K_{2,7} \) | 3 | 0 | 3 |
| 8 | \( K_{3,6} \) | 68 | 0 | 68 |
| 8 | \( K_{4,5} \) | 331 | 3 | 334 |
| 9 | \( K_{2,8} \) | 4 | 0 | 4 |
| 9 | \( K_{3,7} \) | 120 | 0 | 120 |
| 9 | \( K_{4,6} \) | 1111 | 8 | 1119 |
| 9 | \( K_{5,5} \) | 1203 | 10 | 1213 |

Table 1. The number of matroids \( M(\Delta) \) checked by computer.

6. COMPLETE CLASSES OF MATROIDS

Recall Definition 1.2: a class of matroids is complete if it is closed under the operations of minors, duals, direct sums, truncations, and induction by directed graphs. In Section 6.1 we justify the equivalent formulation of completeness given in Theorem 6.1 which better suits our work in Section 7. In Section 6.2 we discuss some properties of complete classes, including additional operations under which they are closed.

6.1. A reformulation of complete classes. We prove the following theorem.

**Theorem 6.1.** A class of matroids is complete if and only if it is closed under the operations of minors, duals, direct sums, and principal extension.

As we justify this theorem, largely by collecting known results, we discuss principal extension, as well as induction by both directed and bipartite graphs. Additional information on these topics can be found in [17, Sections 7.1 and 11.2].

We first review the two notions of inducing matroids. First let \( \Gamma \) be a directed graph. Let \( M \) be a matroid with \( E(M) \subseteq V(\Gamma) \). In the *induced matroid* \( \Gamma(M) \) on \( V(\Gamma) \), a subset of \( V(\Gamma) \) is independent if and only if it can be linked to an independent set of \( M \).
Now let $M$ be a matroid, let $T$ be a set disjoint from $E(M)$, and let $\Delta$ be a bipartite graph with bipartition \{\(T, E(M)\)\}. In the induced matroid $\Delta(M)$ on $T$, a subset of $T$ is independent if and only if it can be matched in $\Delta$ to an independent set of $M$. (We caution the reader to not confuse $\Delta(M)$ with the matroid $M(\Delta)$ defined in Section 5.) Thus, transversal matroids are those that can be induced from free matroids by bipartite graphs. For $X \subseteq T$, its rank in the induced matroid $\Delta(M)$ is

$$r_{\Delta(M)}(X) = \min \{r_M(N(Y)) + |X - Y| : Y \subseteq X\}. \quad (12)$$

ingleton and Piff \cite{13} showed that if $M$ is induced from $N$ by a directed graph, then $M^*$ is induced from $N^*$ by a bipartite graph (see their proof of their Theorem 3.7). This gives the next result.

**Theorem 6.2.** If a class of matroids is closed under induction by bipartite graphs and under duality, then it is also closed under induction by directed graphs.

Thus, a dual-closed, minor-closed class of matroids is closed under induction by directed graphs if and only if it is closed under induction by bipartite graphs.

Intuitively, we get a principal extension of a matroid by adding a point freely to a flat. To be precise, let $M$ be a matroid with rank function $r$, let $Y \subseteq E(M)$, and let $e$ be an element not in $E(M)$. The principal extension of $M$ into $Y$, denoted $M +_Y e$, is the matroid on the set $E(M) \cup e$ whose rank function is given as follows: for $X \subseteq E(M)$, we have $r_{M+_Ye}(X) = r(X)$, and

$$r_{M+_Ye}(X \cup e) = \begin{cases} r(X) & \text{if } r(X \cup Y) = r(X), \\ r(X) + 1 & \text{otherwise}, \end{cases}$$

or, more compactly,

$$r_{M+_Ye}(X \cup e) = \min \{r(X) + 1, r(X \cup Y)\}. \quad (13)$$

We also say that $M +_Y e$ is the matroid obtained by adding $e$ freely to the set $Y$. Note that $M +_Y e = M + e(Y) e$. The free extension of $M$ is the principal extension $M + E(M) e$. Also, the truncation of $M$ is $(M + E(M) e)/e$. Thus, in order to prove Theorem 6.1, it suffices to prove the next result.

**Theorem 6.3.** A class of matroids is closed under deletion and principal extension if and only if it is closed under induction by bipartite graphs.

Let $\phi : E(M) \to E(M')$ be an isomorphism of $M$ onto $M'$, where $E(M)$ and $E(M')$ are disjoint. Fix a subset $Y$ of $E(M)$ and element $e \not\in E(M) \cup E(M')$, and define a bipartite graph $\Delta$ with bipartition \{\(E(M) \cup e, E(M')\)\} and edge set

$$\{x \phi(x) : x \in E(M)\} \cup \{e \phi(y) : y \in Y\}.$$  

From equation (12) and those above giving the rank function of $M +_Y e$, it is routine to show that the matroid that $M'$ induces on $E(M) \cup e$ via $\Delta$ is the principal extension $M +_Y e$. It is easy to realize deletion via induction by a bipartite graph, so one direction of Theorem 6.3 follows.

The justification of the other direction, given in Lemma 6.5 uses the next lemma, which gives the rank function of a sequence of principal extensions of $M$ into subsets of $E(M)$. This lemma implies that the result of a sequence of principal extensions of $M$ into subsets of $E(M)$ does not depend on the order. Thus we may say that these principal extensions are performed simultaneously. To make this precise, let $M$ be a matroid with rank function $r$, and let $e_1, e_2, \ldots, e_n$ be distinct elements not in $E(M)$. For $1 \leq i \leq n$, choose a set
$F_i \subseteq E(M)$. Define $M_0 = M$, and for $0 \leq i < n$, define $M_{i+1} = M_i + F_{i+1} e_{i+1}$.

For simplicity, we define $r_i = r_M$ for $1 \leq i \leq n$. Thus, $M_n$ is the matroid obtained by consecutively adding $e_i$ freely to the set $F_i$ for $1 \leq i \leq n$. In Lemma 6.4, we use $[n]$ for the set $\{1, 2, \ldots, n\}$. For $I \subseteq [n]$, we define

$$e_I = \{e_i : i \in I\} \quad \text{and} \quad F_I = \bigcup_{i \in I} F_i.$$  

**Lemma 6.4.** Using the notation above, if $i \in [n]$, $X \subseteq E(M)$, and $J \subseteq [i]$, then

$$r_i(X \cup e_J) = \min_{I \subseteq J} \{r(X \cup F_I) + |J - I|\}. \quad (14)$$

**Proof.** Equation (13) gives the case $i = 1$. Assume that equation (14) holds for some $i \in [n-1]$. To deduce case $i + 1$, let $X \subseteq E(M)$ and $J \subseteq [i + 1]$. If $i + 1 \notin J$, then we have $r_{i+1}(X \cup e_J) = r_i(X \cup e_J)$, from which the needed equality for $r_{i+1}(X \cup e_J)$ follows. Now assume $i + 1 \in J$. Set $J' = J - \{i + 1\}$, so $e_J = e_{J'} \cup \{e_{i+1}\}$. By equation (14),

$$r_{i+1}(X \cup e_J) = r_{i+1}(X \cup e_{J'} \cup e_{i+1})$$

$$= \min_{I \subseteq J'} \{r(X \cup F_{J'}) + |J' - I'| + 1\} = \min_{I \subseteq J : i + 1 \notin I} \{r(X \cup F_I) + |J - I|\}. \quad (15)$$

By the induction hypothesis, the first term, $r_i(X \cup e_{J'}) + 1$, is

$$\min_{I \subseteq J'} \{r(X \cup F_{J'}) + |J' - I'| + 1\} = \min_{I \subseteq J : i + 1 \notin I} \{r(X \cup F_I) + |J - I|\},$$

and the second, $r_i(X \cup e_{J'} \cup F_{i+1})$, is

$$\min_{I \subseteq J'} \{r(X \cup F_{J'} \cup F_{i+1}) + |J' - I'|\} = \min_{I \subseteq J : i + 1 \notin I} \{r(X \cup F_I) + |J - I|\}.$$ 

With these equalities, equation (15) gives equation (14) for $r_{i+1}(X \cup e_J)$. \hfill \Box

**Lemma 6.5.** Let $M$ be a matroid, let $T$ be a set disjoint from $E(M)$, and let $\Delta$ be a bipartite graph with bipartition $\{T, E(M)\}$. The induced matroid $\Delta(M)$ is obtained from $M$ by first adding each $t \in T$ freely to the set $N_\Delta(t)$ and then deleting $E(M)$.

**Proof.** Write $T = \{e_1, e_2, \ldots, e_n\}$, set $F_n = N_\Delta(e_1)$, and define $M_n$ as above. Comparing equation (14) with $X = \emptyset$ to equation (12) gives $M_n[T] = \Delta(M)$. \hfill \Box

This completes the proofs of Theorems 6.3 and 6.1. When $M$ is free, Lemma 6.5 gives the geometric description of transversal matroids, as in [17 Proposition 11.2.26]. A simple variation on these ideas justifies the remark by Mason [15] that simultaneous principal extensions may be realized by induction from a bipartite graph.

### 6.2. Further properties of complete classes.

The class of gammoids is complete. As Ingleton [12] observed, since transversal matroids are those induced from free matroids via bipartite graphs, the class of gammoids is the smallest complete class.

**Proposition 6.6.** Every non-empty complete class of matroids contains all gammoids.

Other examples of complete classes of matroids include: $BO$, $SBO$, $k$-$BO$ (treated in Section 7); matroids representable over fields of a given characteristic (see [20]); and the class of matroids with no $M(K_4)$-minor (see [22]). Complete classes are closed under all operations that arise by combining those under which they are already known to be closed. For example, since the matroid union, $M \uplus N$, is obtained by induction from the direct sum $M \oplus N$ by a certain bipartite graph (see [17 Theorem 11.3.1]), complete classes are closed under matroid union. The same holds
for the free product and all principal sums (which are special matroid unions; see [11]). Closure under parallel connections follows from the description Mason [15] gave of this operation, which we recall. Let \( M \) and \( N \) be matroids with \( E(M) \cap E(N) = \{ p \} \). To obtain their parallel connection, \( P(M, N) \), in \( N \) replace \( p \) by \( p_N \), giving \( N' \), where \( p_N \notin E(M) \cup E(N) \); then \( P(M, N) \) is

\[
\left( (M \oplus N') +_{\{p,p_N\}} e \right) / e \setminus p_N.
\]

This applies even if \( p \) is a loop or coloop of either \( M \) or \( N \). It now follows that complete classes are also closed under series connection (the dual of parallel connection) and 2-sums.

While they play no role in this paper, we conclude this section with two observations.

Recall that a 2-connected matroid is not 3-connected if and only if it is a 2-sum of two of its proper minors. Thus, the results above imply that any excluded minor for a complete class of matroids must be 3-connected.

Second, we note that in Theorem 6.1, we may replace closure under principal extensions by closure under principal extensions into sets of at most two elements. This result, which is used in [20] and [22], follows by repeatedly applying Lemma 6.7, whose straightforward proof we omit.

**Lemma 6.7.** Let \( M \) be a matroid with rank function \( r \). Let \( F \subseteq E(M) \) and \( G \subseteq F \), and let \( e_1 \) and \( e_2 \) be points not in \( E(M) \). Then

\[
M +_F e_1 = ((M + G e_2) +_{(F \setminus G) \cup e_2} e_1) \setminus e_2.
\]

7. \( k\text{-BO} \) IS A COMPLETE CLASS

The main result of this section is the following theorem.

**Theorem 7.1.** For a fixed \( k \geq 1 \), the class of \( k \)-base-orderable matroids is complete.

This theorem implies the known results that \( \text{BO} \) and \( \text{SBO} \) are complete, and combining it with Proposition 6.6 gives another proof that gammoids are strongly base-orderable. (For a short direct proof, see [21] Theorem 42.11.) Also, \( k\text{-BO} \) is closed under all the operations discussed in Section 6.2. In particular, we recover the result of Brualdi [5] that \( \text{BO} \) is closed under induction by directed graphs.

By Proposition 7.2, the class \( k\text{-BO} \) is closed under direct sums, duals, and minors. To complete the proof of Theorem 7.1, we address closure under principal extensions in Proposition 7.3. The following well-known result (see, e.g., [17] Problem 7.2.4(a))) gives the bases of a principal extension.

**Lemma 7.2.** Let \( M \) be a matroid, with \( B(M) \) its set of bases, let \( F \) be a flat of \( M \), and let \( e \) be an element not in \( E(M) \). The set of bases of \( M +_F e \) is

\[
B(M) \cup \{(B - f) \cup e : B \in B(M) \text{ and } f \in B \cap F \}.
\]

**Proposition 7.3.** If \( M \) is \( k \)-base-orderable, then so is any principal extension \( M +_F e \).

**Proof.** Let \( B_1 \) and \( B_2 \) be bases of \( M +_F e \). If \( e \notin B_1 \cup B_2 \), then there is a \( k \)-exchange-ordering \( \sigma : B_1 \to B_2 \) by assumption.

If \( e \in B_1 - B_2 \), then there is a basis \((B_1 - e) \cup x \) of \( M \) with \( x \in F \), and a \( k \)-exchange-ordering \( \sigma : (B_1 - e) \cup x \to B_2 \). Define \( \tau : B_1 \to B_2 \) by

\[
\tau(z) = \begin{cases} 
\sigma(z) & \text{if } z \neq e, \\
\sigma(x) & \text{if } z = e.
\end{cases}
\]
Since $e$ can replace $x$ in any basis of $M$ to yield a basis of $M + e$, it follows that $\tau$ is a $k$-exchange-ordering for $M + e$. The case with $e \in B_2 - B_1$ follows by symmetry.

Now suppose $e \in B_1 \cap B_2$. There are bases $B_{1x} = (B_1 - e) \cup x$ and $B_{2y} = (B_2 - e) \cup y$ of $M$ with $x, y \in F$, and a $k$-exchange-ordering $\sigma : B_{1x} \to B_{2y}$. If $\sigma(x) = y$, then the bijection $\tau : B_1 \to B_2$ given by

$$\tau(z) = \begin{cases} 
\sigma(z) & \text{if } z \neq e, \\
e & \text{if } z = e,
\end{cases}$$

is a $k$-exchange-ordering for $M + e$.

The rest of the proof treats the case with $\sigma(x) \neq y$ and uses the following notation: $B_1 = \{e, a_2, a_3, \ldots, a_r\}$ and $B_2 = \{e, b_1, b_3, \ldots, b_r\}$, and $\sigma$ is given by $\sigma(x) = b_1$, $\sigma(a_2) = y$, and $\sigma(a_j) = b_j$ for $j \geq 3$; we abbreviate this by

$$\sigma = \begin{pmatrix} x & a_2 & a_3 & \ldots & a_r \\
 & b_1 & y & b_3 & \ldots & b_r
\end{pmatrix}.$$ 

Define $\tau : B_1 \to B_2$ to fix $e$, map $a_2$ to $b_1$, and agree with $\sigma$ on $a_3, \ldots, a_r$, that is,

$$\tau = \begin{pmatrix} e & a_2 & a_3 & \ldots & a_r \\
e & b_1 & b_3 & \ldots & b_r
\end{pmatrix}.$$ 

We claim that $\tau$ is a $k$-exchange-ordering for $M + e$. Let $X \subseteq B_1$ with $|X| \leq k$. The case with $X \cap \{e, a_2\} = \emptyset$ is immediate. If both $e \in X$ and $a_2 \in X$, then exchanging $(X - e) \cup x$ and $\sigma((X - e) \cup x)$ in $B_{1x}$ and $B_{2y}$, and then using Lemma 2.2 shows that both $(B_1 - X) \cup \tau(X)$ and $(B_2 - \tau(X)) \cup X$ are bases of $M + e$. We get the same conclusion if $e \in X$ and $a_2 \notin X$ by exchanging $X - x$ and $\sigma(X - x)$ in $B_{1x}$ and $B_{2y}$, and then using Lemma 2.2 (note the inequality $|X| \leq k$ in Definition 2.1).

Finally, assume that $e \notin X$ and $a_2 \in X$. By relabeling, we may assume that $X$ is $\{a_2, a_3, \ldots, a_i\}$, so $i \leq k + 1$. First assume $x = b_1$. Since $\sigma$ is a $k$-exchange-ordering, $\{x, y, b_3, \ldots, b_i, a_{i+1}, \ldots, a_r\}$ must be a basis of $M$, and we may replace $y$ with $e$ to get that $\{e, x, b_3, \ldots, b_i, a_{i+1}, \ldots, a_r\}$ is a basis of $M + e$. To show that $\{e, a_2, \ldots, a_i, b_{i+1}, \ldots, b_r\}$ is also a basis of $M + e$, note that

$$(B_{2y} - \sigma(X)) \cup X = \{b_1, a_2, \ldots, a_i, b_{i+1}, \ldots, b_r\} = \{x, a_2, \ldots, a_i, b_{i+1}, \ldots, b_r\}$$

is a basis of $M$, and then replace $x$ with $e$. The case where $y = a_2$ is similar, though not symmetric, to that of $x = b_1$. To provide the details, assume $y = a_2$. Now let $X' = (X - a_2) \cup x$. Since $\sigma$ is a $k$-exchange-ordering, we have that

$$(B_{1x} - X') \cup \sigma(X') = \{b_1, y, b_3, \ldots, b_i, a_{i+1}, \ldots, a_r\}$$

is a basis of $M$, and replacing $y$ with $e$ gives that $\{e, b_1, b_3, \ldots, b_i, a_{i+1}, \ldots, a_r\}$ is a basis of $M + e$. We also have that the following set is a basis of $M + e$:

$$(B_{2y} - \sigma(X')) \cup X' = \{x, y, a_3, \ldots, a_i, b_{i+1}, \ldots, b_r\}.$$ 

Thus replacing $x$ by $e$ and recalling that $y = a_2$ gives that $\{e, a_2, \ldots, a_i, b_{i+1}, \ldots, b_r\}$ is a basis of $M$.

So we assume that $x \neq b_1$ and $y \neq a_2$ for the rest of the proof. We next show that

$$(B_1 - X) \cup \tau(X) = \{e, b_1, b_3, \ldots, b_i, a_{i+1}, \ldots, a_r\}$$

is a basis of $M + e$. Assume the contrary. Then, since $x \neq b_1$ and $\sigma$ is a $k$-exchange-ordering, it follows that both

$\{x, b_1, b_3, \ldots, b_i, a_{i+1}, \ldots, a_r\}$ and $\{y, b_1, b_3, \ldots, b_i, a_{i+1}, \ldots, a_r\}$
have size \( r \), and, hence, they are dependent in \( M \). Therefore they contain circuits, say \( C_x \) and \( C_y \), respectively. Since \( \sigma \) is a \( k \)-exchange-ordering, it follows that both

\[
B' = \{x, y, b_3, \ldots, b_l, a_{i+1}, \ldots, a_r\} \quad \text{and} \quad \{b_1, a_2, b_3, \ldots, b_l, a_{i+1}, \ldots, a_r\}
\]

are bases of \( M \). Therefore, it must be that \( \{x, b_1\} \subseteq C_x \) and \( \{y, b_1\} \subseteq C_y \). By circuit elimination, there is a circuit \( C \) of \( M \) such that \( C \subseteq (C_x \cup C_y) - b_1 \). Since this last set is a subset of \( B' \), we have reached a contradiction. Thus, \( (B_1 - X) \cup \tau(X) \) is a basis of \( M + e \). A similar argument shows that \( (B_2 - \tau(X)) \cup X \) is also a basis of \( M + e \).

We next point out that there are other complete classes along these lines.

**Definition 7.4.** Let \( M \) be a matroid, and let \( 0 \leq k \leq r(M) \) and \( 0 \leq l \leq r(M) \) with \( k + l > 0 \). We say \( M \) is \((k, l)\)-base-orderable if, given any two bases \( B_1 \) and \( B_2 \), there is a bijection \( \sigma : B_1 \to B_2 \) so that for every \( X \subseteq B_1 \) with \( |X| \leq k \) or \( |X| \geq r(M) - l \), the set \( (B_1 - X) \cup \sigma(X) \) is a basis.

It follows from the definition that for \( k \geq 1 \), a matroid is \( k \)-base-orderable if and only if it is \((k, k)\)-base-orderable. It also follows that a matroid \( M \) is \((k, l)\)-base-orderable if and only if it is \((l, k)\)-base-orderable. Note that every matroid is \((1, 0)\)-base-orderable by the bijective-exchange property. However, matroids do not in general satisfy a multiple bijective-exchange property. For example, since \( r(M(K_4)) = 3 \) and \( M(K_4) \) is not base-orderable, it follows that it is not \((2, 0)\)-base-orderable. It is easy to modify the proofs in this section to show the following strengthening.

**Theorem 7.5.** For fixed \( k \) and \( l \), the class of \((k, l)\)-base-orderable matroids is complete.

In a different direction, we close this section by showing that \( BO \) is closed under circuit-hyperplane relaxation. We do not yet know whether the same holds for \( SBO \), \( k \)-BO, or all complete classes of matroids.

**Proposition 7.6.** Let \( M' \) be the relaxation of a matroid \( M \) by a circuit-hyperplane \( X \) of \( M \). If \( M \) is base-orderable, then so is \( M' \).

**Proof.** Let \( B_1 \) and \( B_2 \) be bases of \( M' \). It suffices to consider the case where \( B_2 = X \). By the bijective exchange property, there is a bijection \( \sigma : B_1 \to X \) such that for all \( y \in B_1 \), the set \( (B_1 - y) \cup \sigma(y) \) is also a basis. Clearly \( \sigma \) fixes any element of \( B_1 \cap X \). Let \( y \in B_1 \).

To show that \( \sigma \) is an exchange-ordering, we must show that \((X - \sigma(y)) \cup y \) is a basis of \( M' \). This holds because \( X - \sigma(y) \) is an independent hyperplane of \( M' \).

8. An Infinite Family of Excluded Minors for gammoids and for BO

Ingleton [11] stated (without giving his proof) that if \( \Delta \) is a critical graph and (in the notation of Definition 5.1) either \( |X| \) or \( |Y| \) is two, then \( M(\Delta) \) is an excluded minor for \( BO \). Since \( \Delta \) has neither a source nor a sink, Ingleton’s hypothesis implies that each of \( X \) and \( Y \) can be partitioned into two sets so that all edges between a given block of \( X \) and one of \( Y \) are oriented the same way. In this section we extend Ingleton’s result to the case where there are such partitions of \( X \) and \( Y \), even if \( \min\{|X|, |Y|\} > 2 \), as in Figure 5. We show more: all single-element contractions of such matroids \( M(\Omega) \) are transversal, and all single-element deletions are cotransversal. Besides verifying infinitely many more cases of Conjecture 5.9, this shows that these matroids are also excluded minors for the class of gammoids, and for \( SBO \) and \( k \)-BO. Such critical graphs \( \Delta \) look like generalizations of the graph in Figure 11 so we may view the matroids \( M(\Delta) \) as generalizations of \( M(K_4) \).
In this section, unlike Sections 3 and 5, the sets $A$ and $B$ are not bases; rather they are two sets in a 6-tuple of sets. Specifically, for an integer $r \geq 3$, let $\alpha = (A, B, C, D, E, F)$ be a 6-tuple of disjoint nonempty sets with

$$r = |A \cup B \cup C| = |D \cup E \cup F| \quad \text{and} \quad r + 1 = |A \cup B \cup D \cup E|.$$

Let $\Delta_\alpha$ be the directed bipartite graph with bipartition $\{A \cup B \cup C, D \cup E \cup F\}$ having the following edges:

1. $(a, d)$ for all $a \in A$ and $d \in D$,
2. $(e, a)$ for all $a \in A$ and $e \in E$,
3. $(d, b)$ for all $b \in B$ and $d \in D$,
4. $(b, e)$ for all $b \in B$ and $e \in E$.

Thus, $\Delta_\alpha$ is a critical graph with no obstructions. From Proposition 5.8, in the associated rank-$r$ matroid $M(\Delta_\alpha)$, which we shorten to $M_\alpha$, the proper nonempty cyclic flats are $C \cup B \cup E$, $C \cup A \cup D$, $F \cup E \cup A$, and $F \cup D \cup B$, and their ranks are given by

$$r(C \cup B \cup E) = |C| + |B|,$$

$$r(C \cup A \cup D) = |C| + |A|,$$

$$r(F \cup E \cup A) = |F| + |E|,$$

$$r(F \cup D \cup B) = |F| + |D|.$$  

(16)

Figure 5 gives an example.

![Figure 5](image)

**Figure 5.** An example, with $r = 8$, of the digraph $\Delta_\alpha$ described above. An arrow from block $U$ to block $V$ means that there is a directed edge $(u, v)$ for every $u \in U$ and $v \in V$.

**Theorem 8.1.** The matroid $M_\alpha$ defined above has the following properties:

1. It is not base-orderable,
2. Each of its single-element contractions is transversal,
3. Each of its single-element deletions is cotransversal, and
4. It is an excluded minor for the following classes of matroids: gammoids, BO, SBO, and $k$-BO for any $k \geq 1$.

**Proof.** Note that it suffices to prove the first three assertions since they imply the last. Also, assertion (1) follows from our work in Section 5.

We now prove assertion (2). For an element $x \in E(M_\alpha)$ and antichain $F$ of $Z(M_\alpha)$ with $|F| \geq 3$, let $F_x = \{F - x : F \in F\}$. Some sets in $F_x$ might not be cyclic flats of $M_\alpha/x$, but by Lemma 4.3 any antichain of at least three cyclic flats of $M_\alpha/x$ is equal to some $F_x$. To show that $M_\alpha/x$ is transversal, by Corollary 4.6 it suffices to check inequality
for all sets $F_x$; to do this efficiently, we simplify inequality (2) for each antichain $F$ of $\mathcal{Z}(M_\alpha)$, and compare the result to its counterpart for $F_x$ in $M_\alpha/x$.

First note that the union of any two proper, non-empty cyclic flats of $M_\alpha$ has rank $r$, and that the intersection of any three is empty. Thus, $\cap F = \emptyset$, so $\cap F_x = \emptyset$.

We first let $\{F_1, F_2, F_3\}$ consist of all four proper, nonempty cyclic flats of $M_\alpha$. From the ranks given in equations (16), the alternating sum in inequality (2) simplifies to

$$2|C| + |A| + |B| + 2|F| + |D| + |E| - 3r;$$

which is $|C| + |F| - r = -1$. In $M_\alpha/x$, the term $-3r$ is replaced by $-3(r - 1) = -3r + 3$, and, since $x$ is in at most two cyclic flats of $M_\alpha$, the rank of at most two sets in $F_x$ goes down by 1 compared to their counterparts in $M$; so the counterpart, in $M_\alpha/x$, of sum (17) is nonnegative, as needed. By the symmetry between the proper, nonempty cyclic flats of $M_\alpha$, to check triples, it suffices to consider the antichain

$$F = \{C \cup B \cup E, C \cup A \cup D, F \cup E \cup A\}.$$  

The alternating sum in inequality (2) for $F$ in $M_\alpha$ simplifies to

$$2|C| + |A| + |B| + |F| + |E| - 2r,$$

that is, $|C| + |F| + |E| - r$, or $|C| - |D|$. Since $|C| + 1 = |D| + |E|$, sum (18) equals $|E| - 1$, which is nonnegative. In $M_\alpha/x$, the term $-2r$ is replaced by $-2(r - 1) = -2r + 2$, and the rank of at most two sets in $F_x$ goes down by 1 compared to their counterparts in $M$; so the counterpart of sum (19) in $M_\alpha/x$ is nonnegative, as needed. Thus, statement (2) holds.

Assertion (3) follows by applying assertion (2) to the dual, which is $M(\Delta_\alpha')$ where $\Delta_\alpha'$ reverses the orientation of each edge of $\Delta$. \hfill $\Box$

In contrast, single-element deletions of an $M_\alpha$ need not be transversal. For example, if

$$\alpha = (\{a_1, a_2\}, \{b_1\}, \{c_1, c_2\}, \{d_1, d_2\}, \{e_1\}, \{f_1, f_2\}),$$

then $M_\alpha \setminus a_1$ is not transversal.

Note that for a given integer $r \geq 3$, this construction yields at least as many distinct excluded minors $M_\alpha$ as integer partitions of $r + 1$ with four parts, for which $\binom{r}{3}/4!$ is a crude lower bound.

9. An Infinite Family of Excluded Minors for $SBO$

Ingleton, in [11], was the first to exhibit a matroid that is in $BO$ but not in $SBO$. Here we generalize his construction: for a fixed integer $k \geq 2$, we construct a family of matroids that are in $(k - 1)$-$BO$ but are excluded minors for $k$-$BO$ and $SBO$. When $k = 2$, we recover Ingleton’s example. Taken together, i.e., as $k$ ranges over all integers exceeding one, these matroids form an infinite antichain of base-orderable matroids that are excluded minors for $SBO$.

Let $k \geq 2$ be an integer and let $\beta = (A, B, C, D, E, F)$ be a 6-tuple of disjoint nonempty sets with $k = |C| = |F| = |A \cup B| = |D \cup E|$. We will define a rank-$2k$ matroid $M_\beta$ on the union of these sets, which we (prematurely) denote $E(M_\beta)$, in terms
of cyclic flats and their ranks. Define a function \( r \) on seven subsets of \( E(M_\beta) \) as follows:

\[
\begin{align*}
    r(E(M_\beta)) &= 2k, \\
    r(A \cup B \cup D \cup E) &= 2k - 1, \\
    r(C \cup B \cup E) &= k + |B|, \\
    r(C \cup A \cup D) &= k + |A|, \\
    r(F \cup E \cup A) &= k + |E|, \\
    r(F \cup D \cup B) &= k + |D|, \quad \text{and} \\
    r(\emptyset) &= 0.
\end{align*}
\] (19)

Let \( Z \) consist of the sets on which \( r \) has been defined.

**Proposition 9.1.** Let \( r, \beta, \) and \( Z \) be as above. The function \( r \) can be extended to all subsets of \( E(M_\beta) \) to be the rank function of a matroid \( M_\beta \) with \( Z(M_\beta) = Z \).

**Proof.** We check the conditions in Theorem 4.2. Condition (Z1) holds by construction. Fix \( c \in C \) and \( f \in F \), and set \( \alpha = (A, B, C - c, D, E, F - f) \). Note that \( \alpha \) satisfies the assumptions in Section 8 with \( r = 2k - 1 \), so we can let \( M_\alpha \) be the matroid defined in that section. The sets and ranks in equation (19), apart from \( A \cup B \cup D \cup E \), are obtained from the cyclic flats of \( M_\alpha \) by adjoining \( c \) to the sets that contain \( C - c \), and \( f \) to the sets that contain \( F - f \), and increasing the rank of each augmented set by 1. From this observation and the fact that \( A \cup B \cup D \cup E \) is comparable only to \( \emptyset \) and \( E(M_\beta) \), it follows that condition (Z0) holds, and that conditions (Z2) and (Z3) hold for all pairs that do not include \( A \cup B \cup D \cup E \). It is routine to check the remaining requirements, namely, that conditions (Z2) and (Z3) hold for the pairs that include \( A \cup B \cup D \cup E \). \( \square \)

It follows from equations (1) and (19) that both \( A \cup B \cup C \) and \( D \cup E \cup F \) are bases of \( M_\beta \). Their basis-exchange digraph has the form illustrated in Figure 6.

\[ \text{Figure 6. An example, with } k = 4, \text{ of a basis-exchange digraph of a matroid } M_\beta. \]

To prove the next lemma, we use the same idea as in the proof of Theorem 8.1.

**Lemma 9.2.** Every single-element contraction of \( M_\beta \) is transversal.

**Proof.** For an element \( x \in E(M_\beta) \) and antichain \( \mathcal{F} \) of cyclic flats of \( M_\beta \) with \( |\mathcal{F}| \geq 3 \), let \( \mathcal{F}_x = \{ F - x : F \in \mathcal{F} \} \). To prove that \( M_\beta/x \) is transversal, it suffices to verify inequality (2) for all such \( \mathcal{F}_x \) in \( M_\beta/x \); we do this by comparing that inequality to its counterpart for \( \mathcal{F} \) in \( M_\beta \). Symmetry reduces the argument to the seven cases for \( \mathcal{F} \) treated below. In each case, we use the equality \( r = 2k = |A \cup B \cup D \cup E| \) as well as equations (19). We also
use the observations that the union of any two proper, non-empty cyclic flats of \( M_\beta \) has rank \( r = 2k \), and if \(|F| > 3\), then \( \cap F = \emptyset \), and so \( \cap F_x = \emptyset \).

Let \( F \) consist of all five proper nonempty cyclic flats of \( M_\beta \). The alternating sum in inequality (2) simplifies to

\[
6k + |A| + |B| + |E| + |D| - 1 - 4r, \tag{20}
\]

which equals \(-1\). In \( M_\beta/x \), the counterpart of sum (20) is nonnegative, as needed, since \(-4(r - 1) = -4r + 4\) replaces \(-4r\), and, with \( x \) in at most three cyclic flats of \( M_\beta \), the rank of at most three sets in \( F_x \) goes down by 1 compared to their counterparts in \( M \).

Next consider \( F = \{C \cup B \cup E, C \cup A \cup D, F \cup E \cup A, F \cup D \cup B\} \). The alternating sum in inequality (2) simplifies to

\[
4k + |A| + |B| + |E| + |D| - 3r,
\]

which equals 0. The counterpart in \( M_\beta/x \) is also nonnegative since \(-3(r - 1) \) replaces \(3r\) and, with \( x \) in two cyclic flats of \( M_\beta \), the rank of two sets in \( F_x \) goes down by 1 compared to their counterparts in \( M \).

If \(|F| = 4\) and \( A \cup B \cup D \cup E \in F \), then by symmetry it suffices to consider

\[
F = \{A \cup B \cup D \cup E, C \cup B \cup E, C \cup A \cup D, F \cup E \cup A\}. \tag{21}
\]

The alternating sum in inequality (2) for \( F \) in \( M_\beta \) simplifies to \( 5k - 1 + |B| + |A| + |E| - 3r \). This equals \(|E| - 1\), which is nonnegative. This case is completed as above by noting that \( x \) is in at most three cyclic flats of \( M_\beta \).

If \(|F| = 3\) but \( A \cup B \cup D \cup E \notin F \), then by symmetry it suffices to consider

\[
F = \{C \cup B \cup E, C \cup A \cup D, F \cup E \cup A\}. \tag{21}
\]

In this case, \( \cap F = \emptyset \), and the alternating sum in inequality (2) for \( F \) in \( M_\beta \) simplifies to

\[
3k + |A| + |B| + |E| - 2r, \tag{21}
\]

which equals \(|E|\). In \( M_\beta/x \), the term \(-2(r - 1) \) replaces \(-2r\), and the rank of at most two sets in \( F_x \) goes down by 1 compared to their counterparts in \( M \); so the counterpart of sum (21) in \( M_\beta/x \) is positive.

When \(|F| = 3\) and \( A \cup B \cup D \cup E \in F \), then by symmetry, it suffices to assume that \( C \cup A \cup D \in F \) and to examine the three remaining cases.

First let \( F = \{A \cup B \cup D \cup E, C \cup A \cup D, C \cup B \cup E\} \). In this case, \( \cap F = \emptyset \), and the alternating sum in inequality (2) for \( F \) in \( M_\beta \) simplifies to \( 2r - 1 + |A| + |B| - 2r \), which equals \( k - 1 \). This case follows as above by noting that \( x \) is in at most two cyclic flats of \( F_x \).

Next, let \( F = \{A \cup B \cup D \cup E, C \cup A \cup D, F \cup E \cup A\} \). In this case, \( \cap F = A \), which is independent, and the alternating sum in inequality (2) for \( F \) in \( M_\beta \) simplifies to

\[
4k + |A| + |E| - 1 - 2r,
\]

that is, \(|A| + |E| - 1\). In \( M_\beta/x \), the term \(-2(r - 1) \) replaces \(-2r\). If \( x \notin A \), then \( x \) is in at most two cyclic flats of \( F \), so the rank of at most two sets in \( F_x \) goes down by 1; also, \( r_{M_\beta/x}(\cap F_x) = r_{M_\beta/x}(A) \leq |A| \). If \( x \in A \), then the rank of all three sets in \( F_x \) goes down by 1; also, \( r_{M_\beta/x}(\cap F_x) = r_{M_\beta/x}(A - x) = |A| - 1 \). Either way, the counterpart of inequality (2) for \( F \) in \( M_\beta/x \) holds because \(|E| - 1 \geq 0 \).

Finally, the case that \( F = \{A \cup B \cup D \cup E, C \cup A \cup D, F \cup D \cup B\} \) is similar to the previous case, with \( D \) playing the role of \( A \).

\[ \square \]

**Theorem 9.3.** For \( k \geq 2 \), the matroid \( M_\beta \) defined above has the following properties:
(1) $M_β$ is neither $k$-base-orderable nor strongly base-orderable,
(2) every single-element contraction of $M_β$ is transversal,
(3) $M_β$ is an excluded minor for $\text{SBO}$ and $k$-BO,
(4) $M_β$ is $(k-1)$-base-orderable, and
(5) if $|A| = |B| = |D| = |E| = k/2$, then $M_β$ has exactly two pairs of bases that have no $k$-exchange-ordering; otherwise, there is only one such pair of bases.

Proof. We first show that $M_β$ is not $k$-base-orderable, and so not strongly base-orderable. As noted above, both $A∪B∪C$ and $D∪E∪F$ are bases of $M_β$. Assume for a contradiction that $σ: D∪E∪F → A∪B∪C$ is a $k$-exchange-ordering. Since $C∪B$ is a basis of $C∪B∪E$, we get $σ(E) ⊆ C∪B$. Likewise, the fact that $C∪A$ is a basis of $C∪A∪D$ forces $σ(D) ⊆ C∪A$. However, it cannot be that $σ(D∪E) = C$ since $A∪B∪D∪E$ is a dependent set. So either $σ(D) ∩ A \neq \emptyset$ or $σ(E) ∩ B \neq \emptyset$. The former cannot occur because $F∪E$ is a basis of $F∪E∪A$, but the latter is also impossible because $F∪D$ is a basis of $F∪D∪B$. Therefore, we have reached a contradiction.

Assertion (2) is Lemma 2.4. Since transversal matroids are strongly base-orderable, Lemma 2.4 implies assertion (3).

Next we prove assertion (4). Since proper minors of $M_β$ are in $\text{SBO}$, we need only show that there is a $(k-1)$-exchange-ordering between every disjoint pair of bases.

Note that nothing distinguishes elements that are in the same set in $β$. We will use the following consequence of that observation: for distinct members $x, y$ of the same set in $β$, if $B_1$ is a basis of $M_β$ with $x \in B_1$ and $y \notin B_1$, then $(B_1 - x) \cup y$ is also a basis of $M_β$.

Let $B_1$ and $B_2$ be disjoint bases of $M_β$. First assume that some set $X$ in $β$ contains some element, say $x$, in $B_1$ and some element, say $y$, in $B_2$. Now $y \notin B_1$. By the observation in the previous paragraph, $(B_1 - x) \cup y$ is a basis of $M_β$. Since $M_β/y$ is strongly base-orderable, there is a $k$-exchange-ordering $σ: (B_1 - x) \cup y → B_2$ with respect to $M_β$. It must be that $σ(y) = y$. Defining $τ: B_1 → B_2$ by

$$τ(e) = \begin{cases} σ(e) & \text{if } e \neq x \\ y & \text{if } e = x, \end{cases}$$

gives a $k$-exchange-ordering with respect to $M_β$.

Now assume that no set in $β$ has elements in both $B_1$ and $B_2$. Thus, $B_1$ is a union of sets in $β$, as is $B_2$. The size constraints on the sets imply that $B_1$ is a union of at least two sets, as is $B_2$. The only union of two sets that is a basis is $C∪F$, but its complement, $A∪B∪D∪E$, is not a basis, so $B_1$ is a union of three sets, as is $B_2$. Since bases have $2k$ elements and $|C| = k$, the fact that $C∪B∪E, C∪A∪D, F∪E∪A$, and $F∪D∪B$ are cyclic flats implies that there are at most two pairs of disjoint bases. Namely, one of $B_1$ and $B_2$ must be either $A∪B∪C$ or $C∪D∪E$. If $B_1 = A∪B∪C$, then $B_2 = D∪E∪F$, and any bijection $σ: B_1 → B_2$ with $σ(A∪B) = F$ and $σ(C) = D∪E$ is a $(k-1)$-exchange-ordering. If $B_1 = C∪D∪E$, then $B_2 = A∪B∪F$, and any bijection $σ: B_1 → B_2$ with $τ(C) = A∪B$ and $τ(D∪E) = F$ is a $(k-1)$-exchange-ordering. This proves assertion (4).

As we now show, it is possible for both $A∪B∪F$ and $C∪D∪E$ to be bases of $M_β$ only if $|A| = |B| = |D| = |E| = k/2$. If this equality does not hold, then the larger of $|A|$ and $|B|$ must be strictly greater than the smaller of $|D|$ and $|E|$. By symmetry, we may assume that $|B| ≥ |A|$. If $|B| > |D|$, then since $r(F∪D∪B) = |F| + |D|$, it follows that $F∪B$ is dependent. If instead $|B| > |E|$, then we have $|D| > |A|$ since $|A| + |B| = |D| + |E|$. Now since $r(C∪A∪D) = |C| + |A|$, it follows that $C∪D$ is dependent.
Furthermore, if $A \cup B \cup F$ and $C \cup D \cup E$ are indeed both bases of $M_\beta$, then the basis-exchange digraphs $\Omega_{A\cup B\cup C, D\cup E\cup F}$ and $\Omega_{C\cup D\cup E, A\cup B\cup F}$ are isomorphic, and, by symmetry, there is no $k$-exchange-ordering between $A \cup B \cup F$ and $C \cup D \cup E$. This proves assertion (5). \hfill $\square$

Note that, for a given $k \geq 2$, the number matroids $M_\beta$, up to isomorphism, is the number of 4-cycles $(p, q, r, s)$ of positive integers (allowing repetitions) with $p + r = k$ and $q + s = k$. (We get $p, q, r, s$ from $|A|, |D|, |B|, |E|$ by some cyclic shift.) First let $k = 2h + 1$, so $k$ is odd. There are $h$ choices for the smaller of $p$ and $r$, and likewise for $q$ and $s$. These two smallest integers must be adjacent in the cycle. If the two smallest integers differ, then the cycle is determined by deciding which follows the other, so there are $h(h - 1)$ such cycles. If the two smallest integers are equal, then each of the $h$ choices of that integer yields only one cycle. Thus, there are $h(h - 1) + h^2$ matroids $M_\beta$, up to isomorphism, when $k = 2h + 1$. Now let $k = 2h$, so $k$ is even. The analysis above applies if the two smallest integers are less than $h$, so there are $(h - 1)^2$ such cycles. If either of the two smallest integers is $h$, then the cycle is determined by the other smallest integer, so there are $h$ such cycles. Thus, there are $(h - 1)^2 + h$ matroids $M_\beta$, up to isomorphism, when $k = 2h$.

By Proposition $[2,2]$ the dual matroid $M_\beta^*$ is also an excluded minor for $k$-BO and $SBO$ that is $(k - 1)$-base-orderable. We note that $M_\beta^*$ may be thought of as a variation on $M_\beta$ in the following sense. Given $\beta = (A, B, C, D, E, F)$, modify the construction of $M_\beta$ by replacing the circuit-hyperplane $A \cup B \cup D \cup E$ with the circuit-hyperplane $C \cup F$, giving the matroid $M_\beta^*$, say. Now let $\beta' = (A, B, C, E, D, F)$. One can show that $M_\beta^* = M_\beta'$ by using equation (1) and Proposition $[2,1]$.

We showed that the single-element contractions of $M_\beta$ are transversal and hence gammoids. Perhaps the single-element deletions of $M_\beta$ are also gammoids, but showing that would require a different type of argument. To see why, for $k = 5$, let $|A| = |D| = 2$, $|B| = |E| = 3$, and $|C| = |F| = 5$. Let $c \in C$. Using the Mason-Ingleton condition, one can check that neither $M_\beta \setminus c$ nor $(M_\beta \setminus c)^*$ is transversal. Testing whether $M_\beta \setminus c$ is a gammoid therefore would require a different approach.

REFERENCES

[1] Joseph E. Bonin and Joseph P.S. Kung. Semidirect sums of matroids. Ann. Comb., 19(1):7–27, 2015.
[2] Joseph E. Bonin, Joseph P.S. Kung, and Anna de Mier. Characterizations of transversal and fundamental transversal matroids. Electron. J. Combin., 18(1):Paper 106, 16, 2011.
[3] Joseph E. Bonin and Anna de Mier. The lattice of cyclic flats of a matroid. Ann. Comb., 12(2):155–170, 2008.
[4] Richard A. Brualdi. Comments on bases in dependence structures. Bull. Austral. Math. Soc., 1:161–167, 1969.
[5] Richard A. Brualdi. Induced matroids. Proc. Amer. Math. Soc., 29:213–221, 1971.
[6] Richard A. Brualdi and Edward B. Scrimger. Exchange systems, matchings, and transversals. J. Combin. Theory, 5:244–257, 1968.
[7] Thomas H. Brylawski. Some properties of basic families of subsets. Discrete Math., 6:333–341, 1973.
[8] J. de Sousa and D.J.A. Welsh. A characterisation of binary transversal structures. J. Math. Anal. Appl., 40:55–59, 1972.
[9] Curtis Greene. A multiple exchange property for bases. Proc. Amer. Math. Soc., 39:45–50, 1973.
[10] Philip Hall. On representatives of subsets. J. London Math. Soc., 10(1):26–30, 1935.
[11] A.W. Ingleton. Non-base-orderable matroids. In Proceedings of the Fifth British Combinatorial Conference (Univ. Aberdeen, Aberdeen, 1975), pages 355–359. Congressus Numerantium, No. XV, Utilitas Math., Winnipeg, Man., 1976.
[12] A.W. Ingleton. Transversal matroids and related structures. In Higher combinatorics (Proc. NATO Advanced Study Inst., Berlin, 1976), volume 31 of NATO Adv. Study Inst. Ser., Ser. C: Math. Phys. Sci., pages 117–131. Reidel, Dordrecht-Boston, Mass., 1977.

[13] A.W. Ingleton and M.J. Piff. Gammoids and transversal matroids. J. Combin. Theory Ser. B, 15:51–68, 1973.

[14] Joseph P.S. Kung. Basis-exchange properties. In Neil White, editor, Theory of Matroids, volume 26 of Encyclopedia of Mathematics and its Applications, pages 62–75. Cambridge University Press, Cambridge, 1986.

[15] J.H. Mason. Matroids as the study of geometrical configurations. In Higher combinatorics (Proc. NATO Advanced Study Inst., Berlin, 1976), volume 31 of NATO Adv. Study Inst. Ser., Ser. C: Math. Phys. Sci., pages 133–176. Reidel, Dordrecht-Boston, Mass., 1977.

[16] Brendan D. McKay and Adolfo Piperno. nauty and Traces User’s Guide (Version 2.5), 2013, http://cs.anu.edu.au/~bdm/nauty/nug25.pdf.

[17] James Oxley. Matroid Theory, volume 21 of Oxford Graduate Texts in Mathematics. Oxford University Press, Oxford, second edition, 2011.

[18] Rudi A. Pendavingh and J.G. van der Pol. Counting matroids in minor-closed classes. J. Combin. Theory Ser. B, 111:126–147, 2015.

[19] Rudi A. Pendavingh, Stefan van Zwam, et al. Sage Matroid Package, included in Sage Mathematics Software 6.3, 2014.

[20] M.J. Piff and D.J.A. Welsh. On the vector representation of matroids. J. London Math. Soc. (2), 2:284–288, 1970.

[21] Alexander Schrijver. Combinatorial optimization. Polyhedra and efficiency. Vol. B, volume 24 of Algorithms and Combinatorics. Springer-Verlag, Berlin, 2003. Matroids, trees, stable sets, Chapters 39–69.

[22] Julie A. Sims. A complete class of matroids. Quart. J. Math. Oxford Ser. (2), 28(112):449–451, 1977.

[23] Julie A. Sims. Some Problems in Matroid Theory. PhD thesis, Linacre College, Oxford University, 1980.

[24] D.R. Woodall. An exchange theorem for bases of matroids. J. Combin. Theory Ser. B, 16:227–228, 1974.