EXISTENCE OF SOLUTION TO A LOCAL FRACTIONAL NONLINEAR DIFFERENTIAL EQUATION

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Abstract. We prove existence of solution to a local fractional nonlinear differential equation with initial condition. For that we introduce the notion of tube solution.

1. Introduction

Fractional calculus is a branch of mathematical analysis that studies the possibility of taking noninteger order powers of the differentiation and/or integration operators. Even though the term “fractional” is a misnomer, it has been widely accepted for a long time: the term was coined by the famous mathematician Leibniz in 1695 in a letter to L'Hopital [26]. In the paper What is a fractional derivative? [23], Ortigueira and Machado distinguish between local and nonlocal fractional derivatives. Here we are concerned with local operators only. Such local approach to the fractional calculus dates back at least to 1974, to the use of the fractional incremental ratio in [9]. For an overview and recent developments of the local approach to fractional calculus we refer the reader to [22, 24, 27, 28] and references therein.

Recently, Khalil et al. introduced a new well-behaved definition of local fractional (noninteger order) derivative, called the conformable fractional derivative [21]. The new calculus is very interesting and is getting an increasing of interest – see [7, 11] and references therein. In [1], Abdeljawad proves chain rules, exponential functions, Gronwall’s inequality, fractional integration by parts, Taylor power series expansions and Laplace transforms for the conformable fractional calculus. Furthermore, linear differential systems are discussed [1]. In [5], Batarfi et al. obtain the Green function for a conformable fractional linear problem and then introduce the study of nonlinear conformable fractional differential equations. See also [3] where, using the conformable fractional derivative, a second-order conjugate boundary value problem is investigated and utilizing the corresponding positive fractional Green’s function and an appropriate fixed point theorem, existence of a positive solution is proved. For abstract Cauchy problems of conformable fractional systems see [2]. Here we are concerned with the following problem:

\[
\begin{aligned}
  x^{(\alpha)}(t) &= f(t, x(t)), & t \in [a, b], & a > 0, \\
  x(a) &= x_0,
\end{aligned}
\]

where \( f : [a, b] \times \mathbb{R} \to \mathbb{R} \) is a continuous function, \( x^{(\alpha)}(t) \) denotes the conformable fractional derivative of \( x \) at \( t \) of order \( \alpha \), \( \alpha \in (0, 1) \). For the first time in the literature of conformable fractional calculus, we introduce the notion of tube solution. Such idea of tube solution has been used with success to investigate existence of solutions for ordinary differentiable equations [12, 13], delta and nabla differential equations on time scales [6, 14, 16], and dynamic inclusions [15]. Roughly speaking, the tube solution method generalizes the method of lower and upper solution [8, 10, 17, 25].

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The paper is organized as follows. In Section 2, we present the main concepts of the local conformable fractional calculus and we give some useful preliminary results. In Section 3, we prove existence of solution to problem 11 by using the notion of tube solution and Schauder’s fixed-point theorem (see Theorem 19). We end with Section 4, where an illustrative example is given.

2. Preliminaries

We consider fractional derivatives in the conformable sense [21].

Definition 1 (Conformable fractional derivative [21]). Let \( \alpha \in (0, 1) \) and \( f : [0, \infty) \to \mathbb{R} \). The conformable fractional derivative of \( f \) of order \( \alpha \) is defined by

\[
T_\alpha(f)(t) := \lim_{\epsilon \to 0} \frac{f(t+\epsilon t^{1-\alpha})-f(t)}{\epsilon}.
\]

for all \( t > 0 \). Often, we write \( f^{(\alpha)} \) instead of \( T_\alpha(f) \) to denote the conformable fractional derivative of \( f \) of order \( \alpha \). In addition, if the conformable fractional derivative of \( f \) of order \( \alpha \) exists, then we simply say that \( f \) is \( \alpha \)-differentiable. If \( f \) is \( \alpha \)-differentiable in some \( t \in (0, a) \), \( a > 0 \), and \( \lim_{t \to 0^+} f^{(\alpha)}(t) \) exists, then we define \( f^{(\alpha)}(0) := \lim_{t \to 0^+} f^{(\alpha)}(t) \).

Theorem 2 [21]. Let \( \alpha \in (0, 1] \) and assume \( f, g \) to be \( \alpha \)-differentiable. Then,

1. \( T_\alpha(a f + b g) = a T_\alpha(f) + b T_\alpha(g) \) for all \( a, b \in \mathbb{R} \);
2. \( T_\alpha(f g) = f T_\alpha(g) + g T_\alpha(f) \);
3. \( T_\alpha(f/g) = (g T_\alpha(f) - f T_\alpha(g))/g^2 \).

If, in addition, \( f \) is differentiable at a point \( t > 0 \), then \( T_\alpha(f)(t) = t^{1-\alpha} f'(t) \).

Remark 3. From Theorem 2 it follows that if \( f \in C^1 \), then one has

\[
\lim_{\alpha \to 1} T_\alpha(f)(t) = f'(t)
\]

and

\[
\lim_{\alpha \to 0} T_\alpha(f)(t) = tf'(t).
\]

So \( T_\alpha(f) \) is “conformable” in the sense it coincides with \( f' \) in the case \( \alpha \to 1 \) and satisfies similar properties to the integer-order calculus. Note that the property \( \lim_{\alpha \to 0} T_\alpha(f) \neq f \) is not uncommon in fractional calculus, both for local and nonlocal operators: see, e.g., the local fractional derivative of [19, 20], for which property (2) also holds [4]; and the classical nonlocal Marchaud fractional derivative, which is zero when \( \alpha \to 0 \) [26]. Note, however, that we only have \( T_\alpha(f)(t) = t^{1-\alpha} f'(t) \) in case \( f \) is differentiable. If one considers a function that is not differentiable at a point \( t \), then the conformable derivative is not \( t^{1-\alpha} f'(t) \). For applications we refer the reader to [11].

Example 4. Let \( \alpha \in (0, 1] \). Functions \( f(t) = t^p \), \( p \in \mathbb{R} \), \( g(t) \equiv \lambda \), \( \lambda \in \mathbb{R} \), \( h(t) = e^{ct} \), \( c \in \mathbb{R} \), and \( \beta(t) = e^{\frac{1}{2}t^{\alpha}} \), are \( \alpha \)-differentiable with conformable fractional derivatives of order \( \alpha \) given by

1. \( T_\alpha(f)(t) = pt^{p-\alpha} \);
2. \( T_\alpha(g)(t) = 0 \);
3. \( T_\alpha(h)(t) = ct^{1-\alpha} e^{ct} \);
4. \( T_\alpha(\beta)(t) = e^{\frac{1}{2}t^{\alpha}} \).

Remark 5. Differentiability implies \( \alpha \)-differentiability but the contrary is not true: a nondifferentiable function can be \( \alpha \)-differentiable. For a discussion of this issue see [21].

Definition 6 (Conformable fractional integral [21]). Let \( \alpha \in (0, 1) \) and \( f : [a, \infty) \to \mathbb{R} \). The conformable fractional integral of \( f \) of order \( \alpha \) from \( a \) to \( t \), denoted by \( I^\alpha_a(f)(t) \), is defined by

\[
I^\alpha_a(f)(t) := \int_a^t \frac{f(\tau)}{\tau^{1-\alpha}} d\tau,
\]

where the above integral is the usual improper Riemann integral.

Theorem 7 [21]. If \( f \) is a continuous function in the domain of \( I^\alpha_a \), then

\[
T_\alpha(I^\alpha_a(f))(t) = f(t)
\]

for all \( t \geq a \).
**Notation 8.** Let $0 < a < b$. We denote by $\alpha \mathcal{J}_a^b [f]$ the value of the integral $\int_a^b \frac{f(t)}{t^{1-a}} \, dt$, that is, $\alpha \mathcal{J}_a^b [f] := I_a^b (f)(b)$.

**Proposition 9.** Assume $f \in L^1([a, b], \mathbb{R})$, $0 < a < b$. Then $|\alpha \mathcal{J}_a^b [f]| \leq \alpha \mathcal{J}_a^b [||f||]$. 

**Proof.** Let $f \in L^1([a, b], \mathbb{R})$. Then,

$$\left| \alpha \mathcal{J}_a^b [f] \right| = \left| \int_a^b \frac{f(t)}{t^{1-a}} \, dt \right| \leq \int_a^b \left| \frac{f(t)}{t^{1-a}} \right| \, dt = \int_a^b \frac{|f(t)|}{t^{1-a}} \, dt.$$

Therefore, $|\alpha \mathcal{J}_a^b [f]| \leq \alpha \mathcal{J}_a^b [||f||]$ and the proposition is proved. \qed

**Notation 10.** We denote by $C^{(\alpha)}([a, b], \mathbb{R})$, $0 < a < b$, $\alpha > 0$, the set of all real-valued functions $f : [a, b] \to \mathbb{R}$ that are $\alpha$-differentiable and for which the $\alpha$-derivative is continuous. We often abbreviate $C^{(\alpha)}([a, b], \mathbb{R})$ by $C^{(\alpha)}([a, b])$.

The next lemma is a consequence of the conformable mean value theorem proved in [21] by noting the discussion under Definition 2.1 in [1]. Note that $r(b) - r(a) = I_a^b (r^{(\alpha)})(b)$ follows from Lemma 2.8 in [1].

**Lemma 11.** Let $r \in C^{(\alpha)}([a, b])$, $0 < a < b$, such that $r^{(\alpha)}(t) < 0$ on $\{t \in [a, b] : r(t) > 0\}$. If $r(a) \leq 0$, then $r(t) \leq 0$ for every $t \in [a, b]$.

**Proof.** Suppose the contrary. If there exists $t \in [a, b]$ such that $r(t) > 0$, then there exists $t_o \in [a, b]$ such that $r(t_o) = \max_{a \leq t \leq b} (r(t)) > 0$ because $r \in C^{(\alpha)}([a, b])$ and $r(t) > 0$. There are two cases. (i) If $t_o > a$, then there exists an interval $[t_1, t_o]$ included in $[a, t_o]$ such that $r(t) > 0$ for all $t \in [t_1, t_o]$. It follows from the assumption $r^{(\alpha)}(t) < 0$ for all $t \in [t_1, t_o]$ and Lemma 2.8 of [1] that $I_a^{t_o} (r^{(\alpha)})(t_o) = r(t_o) - r(t_1) < 0$, which contradicts the fact that $r(t_o)$ is a maximum. (ii) If $t_o = a$, then $r(t_o) > 0$ is impossible from hypothesis. \qed

**Theorem 12.** If $g \in L^1([a, b])$, then function $x : [a, b] \to \mathbb{R}$ defined by

$$x(t) := e^{-\frac{1}{a}(\frac{t}{a})^\alpha} \left( e^{\frac{1}{a}(\frac{t}{a})^\alpha} x_0 + \alpha \mathcal{J}_a^b \left[ \frac{g(s)}{e^{\frac{1}{a}(\frac{s}{a})^\alpha}} \right] \right)$$

is solution to problem

$$\begin{cases} x^{(\alpha)}(t) + \frac{1}{a^\alpha} x(t) = g(t), & t \in [a, b], \ a > 0, \\ x(a) = x_0. \end{cases}$$

**Proof.** Let $x : [a, b] \to \mathbb{R}$ be the function defined by (3). We know from Theorems 2 and 5 that

$$x^{(\alpha)}(t) = t^{1-\alpha} \left( -\frac{1}{a} \left( \frac{1}{a} \right)^\alpha \right) ^{\alpha} e^{-\frac{1}{a}(\frac{t}{a})^\alpha} \left( e^{\frac{1}{a}(\frac{t}{a})^\alpha} x_0 + \alpha \mathcal{J}_a^b \left[ \frac{g(s)}{e^{\frac{1}{a}(\frac{s}{a})^\alpha}} \right] \right) + e^{-\frac{1}{a}(\frac{t}{a})^\alpha} \left( g(t) \right)$$

$$= \left( -\frac{1}{a} \right)^\alpha e^{-\frac{1}{a}(\frac{t}{a})^\alpha} \left( e^{\frac{1}{a}(\frac{t}{a})^\alpha} x_0 + \alpha \mathcal{J}_a^b \left[ \frac{g(s)}{e^{\frac{1}{a}(\frac{s}{a})^\alpha}} \right] \right) + g(t)$$

$$= -\left( \frac{1}{a} \right)^\alpha x(t) + g(t).$$

We just obtained that $x^{(\alpha)}(t) + \left( \frac{1}{a} \right)^\alpha x(t) = g(t)$. On the other hand,

$$x(a) = e^{-\frac{1}{a}(\frac{a}{a})^\alpha} \left( e^{\frac{1}{a}(\frac{a}{a})^\alpha} x_0 + \alpha \mathcal{J}_a^b \left[ \frac{g(s)}{e^{\frac{1}{a}(\frac{s}{a})^\alpha}} \right] \right) = e^{-\frac{1}{a}} \left( e^{\frac{1}{a}} x_0 + 0 \right) = x_0$$

and the proof is complete. \qed

Theorem 12 is enough for our purposes. It should be mentioned, however, that it can be generalized by benefiting from Lemma 2.8 in [1] with its higher-order version [1] Proposition 2.9.
Theorem 13. If \( g \in L^1([a, b]) \) and \( p(t) \) is continuous on \([a, b]\), then the function \( x : [a, b] \to \mathbb{R} \) defined by
\[
x(t) = \frac{1}{\mu(t)} \left( x(a) \mu(a) + I^\alpha_a(\mu g)(t) \right)
\] (4)
is a solution to the linear conformable equation
\[
x^{(\alpha)}(t) + p(t)x(t) = g(t), \quad x(a) = x_0, \quad a > 0.
\] (5)

Proof. Consider the integrating factor function \( \mu(t) = e^{\int_a^t p(s)ds} \). Then, by means of item (3) of Example 4 and the Chain Rule [1, Theorem 2.11], one can see that \( \mu \) is a solution to the linear conformable equation
\[
(x(t)\mu(t))^{(\alpha)} = \mu(t)g(t).
\] (6)

Apply \( I^\alpha_a \) to (4) and use Lemma 2.8 in [1] to conclude that (4) holds.

Theorem 12 follows as a corollary from Theorem 13 by putting \( \mu(t) = e^{\frac{1}{\alpha}(\frac{t}{\alpha})} \) and \( p(t) = e^{\frac{1}{\alpha}} \).

Proposition 14. If \( x : (0, \infty) \to \mathbb{R} \) is \( \alpha \)-differentiable at \( t \in [a, b] \), then
\[
|x(t)|^{(\alpha)} = \frac{x(t)x^{(\alpha)}(t)}{|x(t)|^2}.
\]

Proof. From Definition 1 we have
\[
|x(t)|^{(\alpha)} = \lim_{\epsilon \to 0} \frac{x(t + \epsilon\tau^{1-\alpha}) - x(t)}{\epsilon}
\]
\[
= \lim_{\epsilon \to 0} \frac{x(t + \epsilon\tau^{1-\alpha})^2 - x(t)^2}{\epsilon (|x(t + \epsilon\tau^{1-\alpha})| + |x(t)|)}
\]
\[
= \lim_{\epsilon \to 0} \left[ \frac{x(t + \epsilon\tau^{1-\alpha})^2 - x(t)^2}{\epsilon} \cdot \frac{1}{|x(t + \epsilon\tau^{1-\alpha})| + |x(t)|} \right]
\]
\[
= \left[ x(t)^2 \right]^{(\alpha)} \frac{1}{2|x(t)|}
\]
\[
= 2x(t)x^{(\alpha)}(t) \frac{1}{2|x(t)|},
\]
which proves the intended relation. \( \square \)

3. Main Result

We begin by introducing the notion of tube solution to problem (1).

Definition 15. Let \((v, M) \in C^{(\alpha)}([a, b], \mathbb{R}) \times C^{(\alpha)}([a, b], [0, \infty))\). We say that \((v, M)\) is a tube solution to problem (1) if
(i) \((y - v(t)) (f(t, y) - v^{(\alpha)}) \leq M(t)M^{(\alpha)}(t)\) for every \( t \in [a, b] \) and every \( y \in \mathbb{R} \) such that \(|y - v(t)| = M(t);\)
(ii) \(v^{(\alpha)}(t) = f(t, v(t))\) and \(M^{(\alpha)}(t) = 0\) for all \( t \in [a, b] \) such that \(M(t) = 0;\)
(iii) \(|x_0 - v(a)| \leq M(a)\).

Notation 16. We introduce the following notation:
\[
\mathcal{T}(v, M) := \left\{ x \in C^{(\alpha)}([a, b], \mathbb{R}) : |x(t) - v(t)| \leq M(t), \; t \in [a, b] \right\}.
\]

Consider the following problem:
\[
\begin{aligned}
x^{(\alpha)} + \frac{1}{\alpha}x(t) &= f(t, \bar{x}(t)) + \frac{1}{\alpha}\bar{x}(t), \quad t \in [a, b], \quad a > 0, \\
x(a) &= x_0,
\end{aligned}
\] (7)

where
\[
\bar{x}(t) := \begin{cases} 
\frac{M(t)}{x - v(t)} (x(t) - v(t)) + v(t) & \text{if } |x - v(t)| > M(t), \\
x(t) & \text{otherwise}.
\end{cases}
\] (8)
Let us define the operator $N : C([a, b]) \to C([a, b])$ by

$$N(x)(t) := e^{-\frac{1}{2}(\frac{t}{\alpha})^\alpha} \left( e^{\frac{1}{2}a^\alpha}x_0 + \alpha \gamma_a^t \frac{f(s, \bar{x}(s)) + \frac{1}{\alpha a} \bar{x}(s)}{e^{-\frac{1}{2}(\frac{t}{\alpha})^\alpha}} \right).$$

In the proof of Proposition 18 we use the concept of compact function.

**Definition 17** (See p. 112 of [18]). Let $X, Y$ be topological spaces. A map $f : X \to Y$ is called compact if $f(X)$ is contained in a compact subset of $Y$.

Compact operators occur in many problems of classical analysis. Note that operator $N$ is nonlinear because $f$ is nonlinear. In the nonlinear case, the first comprehensive research on compact operators was due to Schauder [18, p. 137]. In this context, the Arzelà–Ascoli theorem asserts that a subset is relatively compact if and only if it is bounded and equicontinuous [18, p. 607].

**Proposition 18.** If $(v, \tilde{M}) \in C(\alpha)([a, b], \mathbb{R}) \times C(\alpha)([a, b], [0, \infty))$ is a tube solution to (1), then $N : C([a, b]) \to C([a, b])$ is compact.

**Proof.** Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of $C([a, b], \mathbb{R})$ converging to $x \in C([a, b], \mathbb{R})$. By Proposition 9

$$|N(x_n(t)) - N(x(t))| = \left| e^{-\frac{1}{2}(\frac{t}{\alpha})^\alpha} \left( e^{\frac{1}{2}a^\alpha}x_0 + \alpha \gamma_a^t \frac{f(s, \bar{x}_n(s)) + \frac{1}{\alpha a} \bar{x}_n(s)}{e^{-\frac{1}{2}(\frac{t}{\alpha})^\alpha}} \right) - e^{-\frac{1}{2}(\frac{t}{\alpha})^\alpha} \left( e^{\frac{1}{2}a^\alpha}x_0 + \alpha \gamma_a^t \frac{f(s, \bar{x}(s)) + \frac{1}{\alpha a} \bar{x}(s)}{e^{-\frac{1}{2}(\frac{t}{\alpha})^\alpha}} \right) \right|$$

$$\leq \frac{K}{C} \alpha \gamma_a^t \left\{ \left| f(s, \bar{x}_n(s)) - f(s, \bar{x}(s)) \right| + \frac{1}{\alpha a} \gamma_a^t \left| \bar{x}_n(s) - \bar{x}(s) \right| \right\},$$

where $K := \max_{a \leq s \leq b} \{e^{-\frac{1}{2}(\frac{t}{\alpha})^\alpha}\}$ and $C := \min_{a \leq s \leq b} \{e^{-\frac{1}{2}(\frac{t}{\alpha})^\alpha}\}$. We need to show that the sequence $(g_n)_{n \in \mathbb{N}}$ defined by $g_n(s) := f(s, \bar{x}_n(s)) + \frac{1}{\alpha a} \bar{x}_n(s)$ converges in $C(\alpha)([a, b])$ to function $g(s) = f(s, \bar{x}(s)) + \frac{1}{\alpha a} \bar{x}(s)$. Since there is a constant $R > 0$ such that $\|\bar{x}\|_{C([a, b], \mathbb{R})} < R$ for all $n > N$. Thus, $f$ is uniformly continuous on $[a, b] \times B_R(0)$. Therefore, for $\epsilon > 0$ given, there is a $\delta > 0$ such that

$$|y - x| < \delta < \frac{C\epsilon a^\alpha}{2k(b^\alpha - a^\alpha)}$$

for all $x, y \in \mathbb{R}$;

$$|f(s, y) - f(s, x)| < \frac{C\epsilon a^\alpha}{2k(b^\alpha - a^\alpha)}$$

for all $s \in [a, b]$. By assumption, one can find an index $\hat{N} > N$ such that $\|\bar{x}_n - \bar{x}\|_{C([a, b], \mathbb{R})} < \delta$ for $n > \hat{N}$. In this case,

$$|N(x_n)(t) - N(x)(t)| \leq \frac{K}{C} \left( \alpha \gamma_a^b \left[ \frac{C\epsilon a^\alpha}{2k(b^\alpha - a^\alpha)} \right] + \frac{1}{\alpha a} \gamma_a^b \left[ \frac{C\epsilon a^\alpha}{2k(b^\alpha - a^\alpha)} \right] \right)$$

$$= \frac{2KC\epsilon a^\alpha}{2kC(b^\alpha - a^\alpha)\alpha \gamma_a^b[1]}$$

$$= \frac{\epsilon a^\alpha}{b^\alpha - a^\alpha} = \epsilon.$$

This proves the continuity of $N$. We now show that the set $N(C([a, b]))$ is relatively compact. Consider a sequence $(y_n)_{n \in \mathbb{N}}$ of $N(C([a, b]))$ for all $n \in \mathbb{N}$. It exists $x_n \in C([a, b])$ such that
\[ y_n = N(x_n). \] Observe that from Proposition 14 we have
\[
|N(x_n)(t)| = \left| e^{-\frac{\alpha}{\beta}(t)^\alpha} \left( e^{\frac{\alpha}{\beta}t_0} + \alpha \frac{\partial}{\partial t} \left[ \frac{f(s, \bar{x}_n(s)) + \frac{1}{\alpha} \bar{x}_n(s)}{e^{-\frac{\alpha}{\beta}(t)^\alpha}} \right] \right) \right| \\
\leq K \left( e^{\frac{\alpha}{\beta}t_0} + \alpha \frac{\partial}{\partial t} \left[ \left| f(t, \bar{x}_n(s)) + \frac{1}{\alpha} \bar{x}_n(s) \right| \right] \right) \\
\leq K \left( e^{\frac{\alpha}{\beta}t_0} + \alpha \frac{\partial}{\partial t} \left[ \left| f(t, \bar{x}_n(s)) \right| \right] + \alpha \frac{\partial}{\partial t} \left[ \left| \bar{x}_n(s) \right| \right] \right).
\]

By definition, there is an \( R > 0 \) such that \( |\bar{x}_n(s)| \leq R \) for all \( s \in [a, b] \) and all \( n \in \mathbb{N} \). The function \( f \) is compact on \( [a, b] \times B_R(0) \) and we can deduce the existence of a constant \( A > 0 \) such that \( |f(s, \bar{x}_n(s))| \leq A \) for all \( s \in [a, b] \). The sequence \( \{y_n\}_{n \in \mathbb{N}} \) is uniformly bounded for all \( n \in \mathbb{N} \). Observe also that for \( t_1, t_2 \in [a, b] \), we have
\[
\left| N(x_n)(t_2) - N(x_n)(t_1) \right| \\
\leq B \left( e^{-\frac{\alpha}{\beta}(t_2)^\alpha} - e^{-\frac{\alpha}{\beta}(t_1)^\alpha} \right) + \frac{K(A + \hat{R})}{C} |\alpha \beta t_2[1]| \\
\leq B \left( e^{-\frac{\alpha}{\beta}(t_2)^\alpha} - e^{-\frac{\alpha}{\beta}(t_1)^\alpha} \right) + \frac{K(A + \hat{R})}{C} \frac{1}{\alpha} |t_1 - t_2|,
\]
where \( B := e^{\alpha}t_0, \hat{R} := \frac{R}{e^{\alpha}t_0}, K := \max_{s \in [a, b]} e^{-\frac{\alpha}{\beta}(t)^\alpha} \), and \( C := \max_{s \in [a, b]} e^{-\frac{\alpha}{\beta}(t)^\alpha} \). This proves that the sequence \( \{y_n\}_{n \in \mathbb{N}} \) is equicontinuous. By the Arzelà–Ascoli theorem, \( N(C([a, b])) \) is relatively compact and hence \( N \) is compact. \( \square \)

**Theorem 19.** If \((v, M) \in C(\alpha)([a, b], \mathbb{R}) \times C(\alpha)([a, b], [0, \infty))\) is a tube solution to \( (1) \), then problem \( (1) \) has a solution \( x \in C(\alpha)([a, b], \mathbb{R}) \cap T(v, M) \).

**Proof.** By Proposition 18 the operator \( N \) is compact. It has a fixed point by the Schauder fixed point theorem (see p. 137 of \[18\]). Therefore, Theorem 12 implies that such fixed point is a solution to problem \( (7) - (8) \) and it suffices to show that for every solution \( x \) to problem \( (7) - (8), \ x \in T(v, M) \). Consider the set \( A := \{ t \in [a, b] : |x(t) - v(t)| > M(t) \} \). If \( t \in A \), then by virtue of Proposition 14 we have
\[
(x(t) - v(t) - M(t))| = \left( \frac{x(t) - v(t) - (x^{(\alpha)}(t) - v^{(\alpha)}(t))}{|x(t) - v(t)|} \right).
\]
Therefore, since \((v, M)\) is a tube solution to problem \( (1) \), we have on \( \{ t \in A : M(t) > 0 \} \) that
\[
\left( \frac{x(t) - v(t) - M(t)}{|x(t) - v(t)|} \right)^{\alpha} \\
= \left( \frac{x(t) - v(t) - (x^{(\alpha)}(t) - v^{(\alpha)}(t))}{|x(t) - v(t)|} \right)^{\alpha} - M^{(\alpha)}(t) \\
= (x(t) - v(t)) \left( f(t, \bar{x}(t)) + \frac{1}{\alpha} \bar{x}(t) - \frac{1}{\alpha} x(t) - v^{(\alpha)}(t) \right) - M^{(\alpha)}(t) \\
= \frac{(\bar{x}(t) - v(t)) \left( f(t, \bar{x}(t)) - v^{(\alpha)}(t) \right)}{M(t)} + \frac{|\bar{x}(t) - v(t)| (\bar{x}(t) - x(t))}{a^{\alpha}M(t)} - M^{(\alpha)}(t) \\
\leq \frac{M(t) M^{(\alpha)}(t)}{M(t)} + \frac{1}{a^{\alpha} |x(t) - v(t)|} - M^{(\alpha)}(t) < 0.
\]
On the other hand, we have on \( t \in \{ r \in A : M(\tau) = 0 \} \) that
\[
(|x(t) - v(t)| - M(t))^{(\alpha)} = \frac{\left( x(t) - v(t) \right) \left( f(t, x(t)) \right) + \left( \frac{1}{\alpha} x(t) - \frac{1}{\alpha} v(t) \right) - v^{(\alpha)}(t)}{|x(t) - v(t)|} - M^{(\alpha)}(t)
\]
\[
= \frac{\left( x(t) - v(t) \right) \left( f(t, x(t)) \right) - v^{(\alpha)}(t)}{|x(t) - v(t)|} - \frac{1}{\alpha} |x(t) - v(t)| - M^{(\alpha)}(t)
\]
\[
< -M^{(\alpha)}(t)
\]
\[
= 0.
\]
The last equality follows from Definition \[15\]. If we set \( r(t) := |x(t) - v(t)| - M(t) \), then \( r^{(\alpha)} < 0 \) on \( A := \{ t \in [a, b] : r(t) > 0 \} \). Moreover, since \((v, M)\) is a tube solution to problem \[11\] and \( x \) satisfies \( |x_0 - v(a)| \leq M(a) \), we know that \( r(a) \leq 0 \) and Lemma \[11\] implies that \( A = \emptyset \). Therefore, \( x \in T(v, M) \) and the theorem is proved. \( \square \)

4. An Example

Consider the conformable noninteger order system
\[
\begin{aligned}
x(t) &= a \sqrt{t} x^{3}(t) + bx(t)e^{cx(t)}, \quad t \in [1, 2], \\
x(1) &= 0,
\end{aligned}
\]
\]
where \( a, b \in (\infty, 0) \) and \( c \) is a real constant. According to Definition \[15\], \((v, M) \equiv (0, 1)\) is a tube solution. It follows from our Theorem \[13\] that problem \[9\] has a solution \( x \) such that \( |x(t)| \leq 1 \) for every \( t \in [1, 2] \).

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