Fair Division via Social Comparison

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Abstract

In the classical cake cutting problem, a resource must be divided among agents with different utilities so that each agent believes they have received a fair share of the resource relative to the other agents. We introduce a variant of the problem in which we model an underlying social network on the agents with a graph, and agents only evaluate their shares relative to their neighbors’ in the network. This formulation captures many situations in which it is unrealistic to assume a global view, and also exposes interesting phenomena in the original problem.

Specifically, we say an allocation is locally envy-free if no agent envies a neighbor’s allocation and locally proportional if each agent values her own allocation as much as the average value of her neighbor’s allocations, with the former implying the latter. While global envy-freeness implies local envy-freeness, global proportionality does not imply local proportionality, or vice versa. A general result is that for any two distinct graphs on the same set of nodes and an allocation, there exists a set of valuation functions such that the allocation is locally proportional on one but not the other.

We fully characterize the set of graphs for which an oblivious single-cutter protocol—a protocol that uses a single agent to cut the cake into pieces—admits a bounded protocol with $O(n^2)$ query complexity for locally envy-free allocations in the Robertson-Webb model. We also consider the price of envy-freeness, which compares the total utility of an optimal allocation to the best utility of an allocation that is envy-free. We show that a lower bound of $\Omega(\sqrt{n})$ on the price of envy-freeness for global allocations in fact holds for local envy-freeness in any connected undirected graph. Thus, sparse graphs surprisingly do not provide more flexibility with respect to the quality of envy-free allocations.

1 Introduction

The fair allocation of resources is a fundamental problem for interacting collections of agents. A central issue in fair allocation is the process by which each agent compares her allotment to those of others’. While theoretical models have tended to focus on global comparisons—in which an agent makes comparisons to the full population—a rich line of empirical work with its origins in the social sciences has suggested that in practice, individuals often focus their comparisons on their social network neighbors. This literature, known as social comparison theory, dates back to work of Festinger [15], and has been explored extensively by economists and sociologists since; for example, see Akerlof [2] and Burt [9]. The primary argument is that in many contexts, an individual’s view of their subjective well-being is based on a comparison with peers, defined through an underlying social network structure, rather than through comparison with the overall population [20].

In this work, we find that the perspective of social comparison theory motivates a rich set of novel theoretical questions in classical resource allocation problems. In particular, we consider the cake cutting problem, which refers to the challenge of allocating a single divisible, continuous, good in a fair and efficient manner. The “cake” is intended to stand for a good over which
different agents have different preferences for difference pieces. This problem has a wide range of applications including international border settlements, divorce and inheritance settlements, and allocating shared computational resources.

Agent preferences are modeled through functions that map subintervals of the $[0, 1]$ interval, which represents the entire cake, to real numbers according to the value the agent assigns to that piece. We normalize these valuations so that each agent’s value for the whole cake is 1. The entire cake is to be allocated, but agents need not receive a single, contiguous interval (and valuations are additive across pieces).

Following our goal of understanding the properties of local comparisons to network neighbors, we study cake cutting in a setting where there is an underlying network on the agents, and fairness considerations are defined locally to an agent’s neighbors’ in the network. Given a graph $G$ and a cake $[0, 1]$ to be allocated, we define a locally proportional allocation to be one where each agent values her allocation at least as much as the average value from the allocations given to her neighbors in $G$. We define a locally envy-free allocation to be one where no agent envies the allocation of any neighbor in $G$. Analogous to graphical games $[16]$, it seems quite plausible that agents may care about fairness within a local part of the population rather than with respect to the population as a whole.

As in the global case, it is straightforward to see that a locally envy-free allocation for $G$ is also locally proportional for $G$. It is also clear that if $H$ is a subgraph of $G$ on the same node set (i.e. if it contains a subset of the edges of $G$), then a locally envy-free allocation for $G$ must also be locally envy-free for $H$, since the constraints defining local envy-freeness for $G$ are a superset of the constraints defining local envy-freeness for $H$. For local proportionality, however, the constraints for different graphs $G$ and $H$ (even when one is a subgraph of the other) can operate quite differently, and so as a first question we ask:

**Problem 1** For graphs $G$ and $H$, what is the relationship between the set of locally proportional allocations for $G$ and the set of locally proportional allocations for $H$?

Network topology plays a crucial role in our results. Note that if the network under consideration is the complete graph $K_n$, then the local definitions coincide with their global analogues; in this sense, the local formulations contain the standard definitions as special cases. At the other extreme, if the network is the empty graph $I_n$, then any allocation satisfies local envy-freeness and proportionality. In light of this, we can pose the following problem:

**Problem 2** Are there non-trivial classes of graphs for which we can give efficient protocols for locally envy-free or at least locally proportional allocations?

We believe that, in addition to capturing real-world contexts, posing the cake cutting problem on a network will give further insight into the structure of the original problem.

We also consider the effect of fairness on welfare as measured through the price of fairness $[10]$. The price of fairness is defined as the worst case ratio over all inputs between the social welfare of the optimal allocation (the allocation maximizing the sum of agent valuations) and the social welfare of the optimal fair allocation—the envy-free or proportional allocation that maximizes the sum of agent valuations. We will refer to this ratio in the case of envy-free allocations as the price of envy-freeness, and the ratio in the case of proportional allocations as the price of proportionality. When there is an underlying graph $G$ governing the comparisons, these ratios become the price of local envy-freeness and the price of local proportionality for $G$. We pose the following question:

**Problem 3** How do the achievable lower and upper bounds on the price of local envy-freeness and local proportionality depend on the structure of the graph $G$?
Caragiannis et al. [10] give an \( \Omega(\sqrt{n}) \) lower bound for the price of global envy-freeness and proportionality, a matching upper bound of \( O(\sqrt{n}) \) for the price of global proportionality, and a loose upper bound of \( n - 1/2 \) for the price of global envy-freeness.

**Overview of Results.** With respect to Problem 1, we show that in fact the set of locally proportional allocations do not satisfy any natural containment relations. In particular, for any two distinct connected graphs \( G \) and \( H \) on the same set of nodes, there exists a set of valuations for the agents and an allocation that is locally proportional for \( G \) but not for \( H \). Note that this includes the case where \( G \) is the complete graph, and so global proportionality does not imply local proportionality on any other connected graph \( H \).

For Problem 2, we start from the structure of the classical Cut-and-Choose solution for two agents: one agent divides the cake and the other selects a piece. This solution does not provide any guarantees for global allocations with more than two agents, but with other underlying graphs \( G \) it turns out to have a natural and non-trivial generalization. Specifically, we fully characterize the family of graphs \( G \) for which a locally envy-free allocation can be produced by a protocol in which a single designated node performs all the cuts at the outset, based only on its valuation function.

Finally, for Problem 3, we start from the \( \Omega(\sqrt{n}) \) lower bound on the price of global envy-freeness, which provides a lower bound for the price of local envy-freeness in complete graphs. We show that this \( \Omega(\sqrt{n}) \) lower bound on the price of local envy-freeness holds in any connected, undirected graph \( G \). We consider this to be surprising, because one might think that sparse graphs provide more flexibility with respect to the quality of envy-free allocations. The known upper bound for the price of global envy-freeness serves as a loose upper bound for the price of local proportionality and the price of local envy-freeness.

In outline, Section 2 defines local envy-freeness and local proportionality, and shows that the former implies the latter, and that both are implied by global envy-freeness. This section also establishes a lack of equivalence between global proportionality and either of the two local fairness concepts. Section 3 considers the notion of protocols with a single cutter as described above, and it characterizes the family of graphs (formally, cones of directed acyclic graphs and their subgraphs) for which such a protocol can yield a locally envy-free allocation. Section 4 turns to the price of fairness. We conclude by giving a list of open directions that we hope Problems 1, 2, and 3 will inspire.

### 1.1 Background on the Cake Cutting Problem

Cake cutting algorithms can be traced back to the Hebrew Bible. In the Book of Genesis, when Abraham and Lot decided to separate, they were presented with the challenge of dividing up the land. Abraham suggested to mark the place on which they stand as the cutting point and asks Lot to pick a side. Lot chose the side that is well-watered and Abraham went in the opposite direction. This is precisely the Cut-and-Choose Protocol from above, which has been shown to give envy-free allocations for two agents.

The cake cutting problem has a wide range of applications including international border settlements, divorce and inheritance settlements, and allocating shared computational resources. One historically prominent instance took place at the Potsdam conference, held after the defeat of Germany in 1945. One of the goals of the conference was to divide up Berlin and Austria into four occupation zones [7]. These regions are heterogeneous since they have different geographies, infrastructure and populations, and the four occupying powers had different preferences over the different subregions.
The formal study of the cake cutting problem was initiated in the 1940s due to Banach et al. [7, 17, 26]. Later, Steinhaus observed that the Cut-and-Choose protocol could be extended to three players, and asked whether it can be generalized for any number of agents [7]. This was resolved affirmatively for proportional allocations by Banach et al. [26] and initiated an interesting line of research for envy-free allocations; see Brams and Taylor [7]. See Procaccia [22, 23] for a recent survey from a computer science perspective.

A central problem has been finding envy-free protocols. In 1988, Garfunkel [5] even called this among the most important problems in 20th century mathematics. An envy-free protocol for any number of agents was proposed by Brams and Taylor [6], but may need an unbounded number of queries even for four agents. They thus posed the question whether there are bounded, envy-free protocols for \( n \geq 4 \). Aziz and Mackenzie [3] recently provided a bounded protocol for \( n = 4 \). The case for \( n > 4 \) remained open until recently when Aziz and Mackenzie announced a discrete and bounded protocol [4]. Although bounded, their solution has a very high multiple-exponential query complexity and it remains an open question whether there are more efficient protocols.

The query model was formalized by Roberton and Webb [24]. The focus in this model has been to minimize the number of cut and eval queries. The cut query, formally \( \text{cut}(x, y, \alpha) \), asks an agent for a cut point \( x \) such that the subinterval \( [x, y] \) has value \( \alpha \) to that agent and the eval query, formally \( \text{eval}(x, y) \), asks the agent for her value valuation of the subinterval \( [x, y] \). It is possible to encode all (decentralized) cake cutting protocols using this model. Note that these queries do not result in an allocation themselves but rather provide information about the agents’ valuations in order to compute an allocation. The query complexity of a cake cutting problem is thus the worst case number of queries required in order to output an allocation satisfying the desired fairness criteria.

Procaccia [21] gives a lower bound of \( \Omega(n^2) \) for envy-free cake cutting, which shows a gap between this and the \( \Theta(n \log n) \) bound known for the proportional cake cutting problem [19]. Special cases of the problem have also received attention. Kurokawa et al. [18] for example establish that an algorithm that computes an envy-free allocation for \( n \) agents with piecewise uniform functions using \( f(n) \) queries would also be able to compute an envy-free allocation with general valuation functions using at most \( f(n) \) queries. Considering only contiguous allocations, Stromquist [27] shows that for any \( n \geq 3 \), there are no finite envy-free cake cutting algorithms, even with an unbounded number of queries. A positive result has been achieved by considering approximately envy-free allocations [22]. An allocation is said to be \( \epsilon \)-envy-free if each agent values the allocation of any other agent to be at most \( \epsilon \) more than their own. This relaxation provides a simple algorithm for any \( n \) using \( n \lceil 1/\epsilon \rceil \) cut queries, as shown in [23]. Alon [1] and Dubins and Spanier [13] also study allocations under alternate notions of fairness.

Due to the difficult nature of the envy-free cake cutting problem, researchers have imposed restrictions on different aspects in order to give useful protocols and gain insight into the problem. A few examples are: restricting valuation functions to be only piecewise constant or uniform [18], relaxing envy-freeness to approximate envy-freeness [23], considering partial allocations that simultaneously satisfy envy-freeness and proportionality [11], and limiting allocations to be contiguous pieces [27].

Incentive compatibility is not a standard consideration in the cake cutting literature. One important exception is the work of Chen et al. [11], who give a polynomial time algorithm outside of the Robertson-Webb model for finding proportional and envy-free allocations for piecewise uniform valuation functions and any \( n \), while also achieving strategyproofness.

To the best of our knowledge, there is no preexisting work on the cake cutting problem in which fairness is determined via comparisons defined by an underlying graph. In perhaps the most closely related paper, Chevaleyre et al. [12] analyze how the network topology of negotiation
affects the convergence to an envy-free allocation for multiple indivisible goods. In their setting, agents are only able to negotiate with, and also envy agents that are in their neighborhood. They ask under what conditions a sequence of negotiations can lead to a state where there is no envy. There are a number of differences between this work and ours: first, they consider indivisible goods, which leads to a different set of questions; and second, in their setting the network constrains not just the comparisons that are made in determining fairness, but also the allowable interactions in the division protocol.

2 Relating Global and Local Properties

Let \( N = \{1, 2, \ldots, n\} \) denote the set of agents. The cake is represented using the interval \([0, 1]\) and a piece of cake is a finite union of non-overlapping (interior disjoint) subintervals of \([0, 1]\). Allocated pieces are a finite union of subintervals. Each agent \( i \) has a valuation function \( V_i \) that maps subintervals to values in \( \mathbb{R} \). Given subinterval \([x, y] \subseteq [0, 1]\), we write \( V_i(x, y) \) instead of \( V_i([x, y]) \) for simplicity. We assume that valuation functions are additive, non-atomic, and non-negative. Non-atomicity gives us \( V_i(x, x) = 0 \) for all \( x \in [0, 1] \), so we can ignore boundaries when defining cut-points. We normalize valuations so that \( V_i(0, 1) = 1 \) for each agent \( i \).

**Definition 1 (Allocation)** An allocation is a partition of the unit interval \([0, 1]\) into pieces (subintervals) \( \{A_1, A_2, \ldots, A_n\} \) such that \( \bigcup_i A_i = [0, 1] \) and the pieces are pairwise disjoint. Each agent \( i \) is assigned the corresponding piece \( A_i \).

As is standard, this ensures that the entire cake is allocated. If we remove this constraint, then we can have trivial solutions that satisfy fairness, such as assigning each agent nothing in the case of envy-freeness. This assumption is a natural one to make since the valuation functions are assumed to be non-negative and additive and thus satisfy free-disposal. A feasible allocation is one where no subinterval is assigned to more than one agent.

In order to avoid direct revelation of valuation functions, which may be cumbersome, cake cutting procedures are typically given as protocols that interact with the agents in order to output a feasible allocation. The Robertson-Webb query model, which is typically used for cake cutting protocols is defined with the following two types of queries:

- \( \text{eval}_i(x, y) \): this asks agent \( i \) for the valuation \( V_i(x, y) \).
- \( \text{cut}_i(x, y, \alpha) \): given \( y, \alpha \in [0, 1] \), this asks agent \( i \) to pick \( x \in [0, 1] \) such that \( V_i(x, y) = \alpha \).

These queries are used to gather information regarding the valuations of the agents and need not directly determine an allocation. Rather, a cake cutting protocol can use other steps for determining allocations.

**Definition 2 (Query complexity)** The query complexity of a cake cutting protocol is the worst case number of queries that the protocol requires to output an allocation over all possible valuation functions.

The query complexity of a cake cutting problem is the minimum query complexity over all known protocols for computing the desired allocation.

2.1 Global and Local Fairness

Given a set of agents and an allocation \( \mathcal{A} = (A_1, A_2, \cdots, A_n) \), we formally define two global fairness criteria:
This follows from the fact that $A$ is the former of which violates intuition. We provide a counter example. 

In particular, \[ V \] to be edge (\(i, j\)), agent \(i\) can view agent \(j\)'s allocation. Agent \(i\)'s neighborhood is the set of all nodes to which it has directed edges (\(i, j\)), and we denote this set of nodes by \(N_i\). We define \(i\)'s degree to be \(d_i = |N_i|\). We define local analogues for fairness concepts:

**Definition 3 (Proportional, Envy-free)** An allocation \(A\) is proportional if \(V_i(A_i) \geq 1/n\), for all \(i \in N\), and is envy-free if \(V_i(A_i) \geq V_j(A_j)\), for all \(i, j \in N\).

Suppose we are given a directed graph \(G = (V, E)\), where the nodes correspond to agents and edges signify relations between the agents. In particular, we assume that given a directed edge (\(i, j\)), agent \(i\) can view agent \(j\)'s allocation. Agent \(i\)'s neighborhood is the set of all nodes to which it has directed edges (\(i, j\)), and we denote this set of nodes by \(N_i\). We define \(i\)'s degree to be \(d_i = |N_i|\). We define local analogues for fairness concepts:

**Definition 4 (Local proportional, local envy free)** Given a graph \(G\), an allocation \(A\) is locally proportional if \(V_i(A_i) \geq \frac{\sum_{j \in N_i} V_j(A_j)}{|N_i|}\) for all \(i, j \in N_i\) and locally envy-free if \(V_i(A_i) \geq V_j(A_j)\).

In a locally proportional allocation, each agent assigns as much value to her allocation as the average value she has for her neighbors’ allocation. In a locally envy-free allocation, each agent values her allocation at least as much as her neighbors’ allocation.

When \(G = K_n\), the complete graph on \(n\) vertices, these local fairness definitions coincide with their global analogues. Whereas, if \(G = I_n\), the empty graph on \(n\) nodes, any allocation is trivially locally envy-free. So, the graph topology plays a significant role in computing locally fair allocations.

**Lemma 2.1** A locally envy-free allocation \(A\) on some graph \(G\) is also locally envy-free on all subgraphs \(G' \subseteq G\).

**Proof** We want to show that given a node \(u\) and \(v \in N_u\), \(u\) does not envy \(v\)'s allocation in \(G'\). This follows from the fact that \(A\) is a locally envy-free allocation on \(G\), and if \((u, v)\) is an edge in \(G'\), then it is also an edge in \(G\).

One consequence of this lemma is that local envy-freeness is implied by global envy-freeness. Since globally envy-free allocations exist for all sets of agent valuations \([1]\), a locally envy-free allocation exists for every graph \(G\) and every set of agent valuations.

**Lemma 2.2** If an allocation \(A\) is locally envy-free on a graph \(G\), then it is also locally proportional on the same graph.

**Proof** If an allocation \(A = (A_1, A_2, \ldots, A_n)\) is locally envy-free, then for any \(i \in V\), \(V_i(A_i) \geq V_j(A_j)\), for all \(j \in N_i\). Therefore, \(V_i(A_i) \geq \frac{\sum_{j \in N_i} V_j(A_j)}{|N_i|}\).

Therefore, locally proportional allocations also exist. By considering \(G = K_n\), we also recover that global envy-freeness implies global proportionality. While global envy-freeness implies local envy-freeness, global proportionality does not necessarily imply local proportionality, or vice versa, the former of which violates intuition. We provide a counter example.

**Example 1** Let \(n = 4\) and \(G = C_4\), the cycle graph on 4 nodes, where the nodes are labeled clockwise. Assume agents 2, 3, and 4 have the uniform valuation function \(V_1(x, y) = |y - x|\) for any subinterval \([x, y]\) \(\subseteq [0, 1]\). Let agent 1 have the piecewise uniform valuation function where \(V_1(0, 1/4) = 1/2\) and \(V_1(3/4, 1) = 1/2\), and no value for the remaining subinterval. It is easy to verify that the following allocation is locally proportional on \(K_4\),

\[ A = ([0, 1/8], [1/8, 3/8], [3/8, 5/8], [5/8, 1]). \]

In particular, \(V_i(A_i) = 1/4\) for \(i \in \{1, 2, 3\}\) and \(V_4(A_4) = 3/8\). This allocation is however not locally proportional on \(C_4\) since \(V_1(A_1 \cup A_2 \cup A_4) = 1\), but \(V_1(0, 1/8) = 1/4 < 1/3\). It is also not locally envy-free since \(V_1(A_4) > V_1(A_1)\).
Then, the allocation well as $V_i$. Assume that all other nodes besides $i$ have a uniform valuation function over the entire cake. Then, the allocation $A_j = (j - 1)/n, j/n$ is locally proportional from the perspective of every other agent $j \in N$ on both $H$ and $G$. Now, define $i$'s valuation function to be the piecewise uniform valuation function where, $V_i((i - 1)/n, i/n) = 1/|N_i(G)| + 1$ and $V_i(A_i \cup A_{i_2} \cdots \cup A_{i_{\ell}}) = 1 - 1/|N_i(G)|$. Agent $i$'s valuation for the allocation of nodes $\{i_{\ell+1}, i_{\ell+2}, \cdots, i_k\}$ as well as $V_i(G)\setminus N_i(G)$ is 0. This allocation $A$ is therefore locally proportional on $G$. For $A$ to be locally proportional on $H$, we need $V_i(A_i) \geq 1/|N_i(H)|$. However, since $|N_i(H)| < |N_i(G)|$, and $V_i(A_i \cup A_{i_1} \cup A_{i_2} \cdots \cup A_{i_{\ell}}) = 1$, we only have that $V_i(A_i) < 1/|N_i(H)|$.

Now, suppose $H \not\subseteq G$. Then, there exists an edge $(i, j)$ in the edge-set of $H$ that is not in the edge-set of $G$. Assume that all nodes $k \neq i$ have a uniform valuation over the entire cake. Suppose further that we have the allocation where each $k$ is assigned the piece $A_k = ((k - 1)/n, k/n)$. As above, this allocation is locally proportional from the perspective of each agent $k$ on both $G$ and $H$. Define $i$'s valuation function to be $V_i((j - 1)/n, j/n) = 1$ and 0 on the remainder of the cake. Then, $V_i(A_i) = 0$ and $V_i(A_k) = 0$ for all $k \neq j$. Since $j \notin N_i(G)$, this allocation is locally proportional on $G$. However, it is not locally proportional on $H$ since $V_i(\cup_{k \in N_i}A_k) = 1$, but $V_i(A_i) = 0$.

3 Envy-Free Network Allocations

In this section, we consider the question of finding efficient protocols for computing locally envy-free allocations. We assume that graphs $G$ are directed, unless specified otherwise. When we mean the component of a directed graph, we will instead take the graph obtained by replacing each directed edge with an undirected one, and a component in the directed graph is the corresponding subgraph to the connected component in the undirected analogue. We will use strongly connected component when we mean to take directed reachability into account.

Lemma 3.1 Suppose we have a bounded protocol for computing locally envy-free allocations on $G$. The same protocol can be used to compute locally envy-free allocations on the following two
classes of graphs: (i) \( H = G \cup G' \) where \( G \) and \( G' \) are disjoint components, and (ii) when \( H \) is a graph with a directed cut such that every edge across the cut goes from a node in \( G \) to a node in \( H \setminus G = G' \).

**Proof** In both instances, we simply apply the protocol on \( G \) and allocate agents in \( G' \) the empty allocation. This is a locally envy-free allocation on \( H \), since no two agents in \( G \) envy one another by the assumption on the protocol, and no agent envies the allocation of another agent in \( G' \).

A few consequences of this lemma are that: given a graph \( G \) with more than one connected component, we can reduce the search for a bounded protocol on \( G \) to any one of the connected components. Furthermore, if \( G \) is a directed acyclic graph (DAG), then there exists at least one node with no incoming edges. Therefore, the allocation where such a node gets the entire cake—or where it is divided among a set of such nodes—is locally envy-free.

### 3.1 Directed Acyclic Graphs and Their Cones

We consider a conceptually useful class of graphs for which we can give a protocol with query complexity of \( O(n^2) \).

**Definition** Given a graph \( G = (V,E) \), we say that \( G' \) is a cone of \( G \) if it is the join of \( G \) and a single node \( c \), which we call the apex. That is, \( G' \) has node set \( V \cup \{c\} \), and edge set consisting of the edges of \( G \), together with undirected edges \((u,c)\) for all \( u \in V \). We denote the cone \( G' \) of \( G \) by \( G \star c \).

We consider cones of DAGs. These are the class of graphs where there is a single node \( c \) that lies on all cycles. We now show how to compute a locally envy-free allocation on any graph that is the cone of a DAG.

![Figure 2: Cone of a Directed Acyclic Graph](image)

**Protocol 1**: Cone of DAGs

1: Agent \( c \) cuts the cake into \( n \) pieces that she values equally.
2: Topologically sort and label the nodes \( N \setminus \{c\} \) such that for every edge \((i,j), i \leq j\).
3: Nodes \( N \setminus \{c\} \) pick a piece they prefer most in increasing order of their index.
4: Agent \( c \) takes the remaining piece.

**Theorem 3.2** Given a graph \( G \) that is a cone of a DAG, Protocol 1 computes a locally envy-free allocation on \( G \) using a bounded number of queries.

**Proof** We first show that the allocation is locally envy-free. First, there is no envy between agent \( c \) and any other agent since agent \( c \) cuts the cake into \( n \) pieces she values equally. Therefore, her
valuation for all the allocated pieces is $1/n$. Each of the other agents picks a piece before $c$, and so are able to pick a piece that they value at least as much as the remaining piece that agent $c$ is assigned. Finally, given any directed edge $(i, j)$ such that $i, j \neq c$, note that $V_i(A_i) \geq V_i(A_j)$, since if such an edge exists, then $i < j$ and thus $i$ selects a piece before $j$.

To count the number of queries, the first step requires $n - 1$ cut queries by agent $c$. Then, each agent $i$ must perform $n - i + 2$ eval queries to determine the piece for which they have the highest value. Therefore, the protocol above uses $(n^2 + 3n - 4)/2$ queries.

The importance of cones of DAGs can be seen in the following result, which shows that they emerge naturally as the characterization of graphs on which a particular fundamental kind of protocol succeeds.

Definition 5: An oblivious single-cutter protocol is one in which a single agent $i$ first divides up the cake into a set of pieces $P_1, P_2, \ldots, P_t$ (potentially $t > n$), and then all remaining operations consist of agents choosing from among these pieces.

The classical Cut-and-Choose protocol is an oblivious single-cutter protocol that works for all sets of valuation functions on the complete, two-node graph $K_2$. Protocol 1 is an oblivious single-cutter protocol which works for any graph that is the cone of a DAG. We show that subgraphs of cones of DAGs are in fact precisely the graphs on which oblivious single-cutter protocols are guaranteed to produce a locally envy-free allocation.

Theorem 3.3: If $G$ is a graph for which an oblivious single-cutter protocol produces a locally envy-free allocation for all sets of valuation functions, then $G$ is a subgraph of the cone of a DAG.

Proof: Suppose, by way of contradiction, that $G$ is not a subgraph of the cone of a DAG, but that there is an oblivious single-cutter protocol on $G$, in which a node $i$ starts by dividing the cake into pieces using only knowledge of her own valuation function. Since $G$ is not a subgraph of the cone of a DAG, the graph $G_i = G \setminus \{i\}$ is not acyclic, so there is a cycle $C = (c_1, c_2, \ldots, c_m, c_1)$ in $G_i$.

Let $i$ have a valuation function such that she produces a partition of the cake into pieces $P_1, P_2, \ldots, P_t$. Because agent $i$ produces these pieces without knowledge of the valuation functions of the other agents, we can imagine that we adversarially choose the valuations of the other agents after these pieces have been produced. In particular, consider the valuation functions in which each node $c_j$ on the cycle $C$ values piece $P_r$ (for $r < t$) at $2 \cdot 3^{-r}$, and the last piece $P_t$ with the remaining value. These valuations have the property that for each $r$, the piece $P_r$ is more valuable than the union of all pieces $P_{r+1} \cup P_{r+2} \cup \cdots \cup P_t$.

After the protocol is run, each agent $c_j$ on $C$ will get a subset of the pieces produced by $i$. Let $s$ be the minimum index of any piece allocated to an agent $c_j$ on $C$. Then, the agent $c_{j-1}$ who has a directed edge to $c_j$ will have a union of pieces that she values less than she would value $P_s$, and hence envies $c_j$. This contradicts the assumption that the protocol produces a locally envy-free allocation on $G$.

We highlight an important connection between computing locally envy-free allocations on graphs and what is known in the literature as irrevocable advantage. Given a partial allocation, an agent $i$ is said to have irrevocable advantage over agent $j$ (or dominate $j$) if agent $i$ remains unenvious of agent $j$’s allocation even if the entire remaining piece of the cake (the residue) is added to agent $j$’s allocation.

With the additional guarantee that each agent dominates some number of other agents, this concept is often used to extend partial globally envy-free allocations to complete ones. For
instance, it is a key concept in the Aziz-Mackenzie protocol for $K_n$. Their protocol can be decomposed to three subprotocols: Core, Permutation, and Post-Double Domination Protocols, in order. The Core Protocol computes partial envy-free allocation where each agent dominates at least two other agents, while the Post Double Domination Protocol extends this to a complete allocation. We will use Protocol 1 to show that given a partial envy-free allocation on $K_n$ where each agent dominates at least $n - 2$ other agents, we can extend the allocation to a complete one, thereby generalizing the Post Double Domination Protocol [3] for any $n$. This is presented in Appendix A.

4 Price of Fairness

Finally, we consider the efficiency of allocation from the perspective of local fairness. We follow the approach introduced by Caragiannis et al. [10] of studying the price of envy-freeness, and for this we begin with the following definitions. Recall that for an allocation $A$ into pieces $\{A_1, A_2, \ldots, A_n\}$ for the $n$ agents, we use $V_i(A_i)$ to denote agent $i$’s valuation for its piece.

**Definition 6 (Optimality)** An allocation $A$, is said to be optimal if $\sum_i V_i(A_i) \geq \sum_i V_i(B_i)$ for any allocation $B$. We denote this optimal allocation by $A^\ast$.

We define the optimal locally envy-free (resp. optimal locally proportional) allocations, denoted by $A^{\text{LEF}}\ast$ (resp. $A^{\text{LP}}\ast$), analogously by imposing the constraint that $A$ and $B$ be locally envy-free (resp. locally proportional) and maximizing sum of the values across all agents.

**Definition 7 (Price of Local Envy-Freeness, Proportionality)** Given an instance of a cake cutting problem on a graph $G$, the price of local envy-freeness is the ratio,

$$\frac{\sum_i V_i(A_i^\ast)}{\sum_i V_i(A_{\text{LEF}}^\ast)},$$

where the sum is over all agents $i \in N$. We likewise define the price of proportionality by taking the denominator to be $\sum_i V_i(A_{\text{LP}}^\ast)$.

We are measuring the degradation in efficiency when considering allocations that maximize the welfare in both instances under the given constraints.

To quantify the loss of efficiency, we are interested in giving a tight lower and upper bound. More specifically, given a graph $G$ and a fairness concept in consideration, say local envy-freeness, we seek to find an input (i.e., a valuation profile) for which the price of local envy-freeness is high. This corresponds to a lower bound on the price of fairness. On the other hand, the upper bound will be given via an argument that shows, for any valuation profile, the price of fairness cannot exceed that stated.

The main result on global envy-freeness, due to Caragiannis et al. [10] is an $\Omega(\sqrt{n})$ lower bound on the price of (global) fairness: there exist valuation functions for which the ratio is $\Omega(\sqrt{n})$. (Very little is know about the upper bound for the price of envy-freeness: an upper bound of $n$ is immediate, and the best known upper bound is $n - 1/2$ [10].)

These existing results are for the standard model in which each agent can envy every other agent. Using our graph-theoretic formulation, we can study the price of local fairness. As defined in Section 2, this is the ratio of the total welfare of the optimal allocation to the maximum total welfare of any allocation that is locally envy-free.

The numerator of this ratio— based on the optimal allocation —is independent of $G$, while the denominator is a maximum over a set of allocations that is constrained by $G$. Now, if we
Thus, each values of $k$ if a connected undirected graph $G$ has size at least $\varepsilon n/k - 1$, such that for each $j \in S_i$, there is an $i$-$j$ path in $S \cup \{i\}$. That is, each $j \in S_i$ can reach $i \in L$ without passing through any other nodes of $L$.

We next show that if a connected undirected graph $G$ has such a structure, with appropriate values of $k$ and $\varepsilon$, then we obtain a lower bound on the price of local envy-freeness on $G$.

**Theorem 4.1** For any connected undirected graph $G$, there exist valuation functions for which the price of local envy-freeness on $G$ is $\Omega(\sqrt{n})$.

To prove this theorem, we start by adapting a set of valuation functions that Caragiannis et al [10] used in their lower bound for global envy-freeness. To argue about their effect on allocations in an arbitrary graph $G$, we need to reason about the paths connecting agents in $G$ to others with different valuation functions. This, in turn, requires a delicate graph-theoretic definition and argument: we introduce a structure that we term a $(k, \varepsilon)$-linked partition; we show that if $G$ contains this structure, then we can carry out the lower bound argument in $G$; and finally we show that every connected undirected graph contains a $(k, \varepsilon)$-linked partition.

**Definition** For a connected graph $G = (V, E)$, a natural number $k \geq 1$, and a real number $0 < \varepsilon \leq 1$, we define a $(k, \varepsilon)$-linked partition as follows. It consists of a set $L \subseteq V$ of size $k$, and a partition of $S = V - L$ into sets $\{S_i : i \in L\}$ each of size at least $(\varepsilon n/k) - 1$, such that for each $j \in S_i$, there is an $i$-$j$ path in $S \cup \{i\}$. That is, each $j \in S_i$ can reach $i \in L$ without passing through any other nodes of $L$.

We now use a valuation function adapted from the construction of Caragiannis et al [10], who considered the price of global envy-freeness. We partition the full resource to be allocated, the interval $[0,1]$, into $\sqrt{n}$ disjoint intervals $I_1, \ldots, I_{\sqrt{n}}$. We will give each $i \in L$ a valuation $v_i$ that places all value on a distinct interval $I_i$, and each $j \in S$ a valuation that is uniform on $[0,1]$. The optimal allocation for this set of valuations has total welfare of $\sqrt{n}$, which is achieved by giving each $i \in L$ the entire interval where it places value.

Now let us consider any envy-free allocation $A = \{A_i : i \in V\}$. Let $\mu_i$ be a real number denoting the Lebesgue measure of the set $A_i$ assigned to node $i$. If $j \in S$, then $j$’s valuation for its set, $v_j(A_j)$ is equal to $\mu_j$. If $i \in L$, then $i$’s valuation $v_i(A_i)$ is $\sqrt{n}$ times the measure of $A_i \cap I_i$; hence $v_i(A_i) \leq \mu_i \sqrt{n}$.

For each $i \in L$, each $j \in S_i$ has a path $P_j$ to $i$ entirely through nodes of $S$; let the nodes on this path, beginning at $i$, be $P_j = i, j_1, j_2, \ldots, j_d = j$. The immediate neighbor $j_1$ of $i$ on $P_j$ must satisfy $\mu_{j_1} \geq \mu_i$, since $j_1$ is in $S$ and hence has a uniform valuation on intervals. For each successive $j_t$ on $P$, we must have $\mu_{j_t} = \mu_{j_{t-1}}$, since $j_t$ and $j_{t-1}$ have the same valuation on all sets, and the allocation is locally envy-free. Thus, by induction we have $\mu_{j_t} \geq \mu_i$ for all $t$, and hence $\mu_j \geq \mu_i$.

We can now derive a set of inequalities that establishes the lower bound. First we have,

$$
\sum_{j \in S_i} v_j(A_j) = \sum_{j \in S_i} \mu_j \geq \sum_{j \in S_i} \mu_i \geq \mu_i (\varepsilon \sqrt{n} - 1).
$$
Let us assume $n$ is large enough that $\varepsilon \sqrt{n} - 1 \geq \varepsilon \sqrt{n}/2$, so we have,

$$\sum_{j \in S_i} v_j(A_j) \geq \mu_i(\varepsilon \sqrt{n}/2).$$

Since $v_i(A_i) \leq \mu_i \sqrt{n}$ for $i \in L$, we have,

$$\sum_{j \in S_i} v_j(A_j) \geq \varepsilon v_i(A_i)/2. \quad (1)$$

Thus, the total welfare of the allocation is,

$$\sum_{h \in V} v_h(A_h) = \sum_{i \in L} v_i(A_i) + \sum_{j \in S} v_j(A_j)$$

$$= \sum_{i \in L} [v_i(A_i) + \sum_{j \in S_i} v_j(A_j)]$$

$$\leq \sum_{i \in L} [(2\varepsilon^{-1} + 1) \sum_{j \in S_i} v_j(A_j)]$$

$$= (2\varepsilon^{-1} + 1) \sum_{i \in L} \sum_{j \in S_i} v_j(A_j)$$

$$= (2\varepsilon^{-1} + 1) \sum_{j \in S} v_j(A_j) \leq (2\varepsilon^{-1} + 1),$$

where the first inequality is by $(1)$ and the second since

$$\sum_{j \in S} v_j(A_j) = \sum_{j \in S} \mu_j \leq 1,$$

because all agents in $S$ get disjoint intervals. Since $(2\varepsilon^{-1} + 1)$ is a constant, while the optimal allocation has total welfare $\sqrt{n}$, this implies an $\Omega(\sqrt{n})$ lower bound for the price of envy-freeness on $G$.

Finally, we establish that every connected undirected graph $G$ has a $(k, \varepsilon)$-linked partition for appropriate values of $k$ and $\varepsilon$. We begin by showing that it is enough to find a structure satisfying a slightly more relaxed definition, in which the set $L$ can have more than $k$ elements, and we do not need to include all the nodes of $G$. Specifically, we have the following definition:

**Definition** For a connected graph $G = (V, E)$, a natural number $k \geq 1$, and a real number $0 < \varepsilon \leq 1$, we define a $(k, \varepsilon)$-linked subpartition as follows. It consists of a set $L \subseteq V$ of size $\ell \geq k$, together with disjoint subsets $S_1, S_2, \ldots, S_\ell \subseteq S = V - L$, each of size of size at least $(\varepsilon n/k) - 1$, such that for each $j \in S_i$, there is an $i$-$j$ path in $S \cup \{i\}$.

The following lemma says that it is sufficient to find a $(k, \varepsilon)$-linked subpartition.

**Lemma 4.3** If a connected undirected graph $G$ contains a $(k, \varepsilon)$-linked subpartition, then it contains a $(k, \varepsilon)$-linked partition.

**Proof** We start with a $(k, \varepsilon)$-linked subpartition of $G$, with disjoint sets $L$ of size $\ell \geq k$, and $S_1, S_2, \ldots, S_\ell \subseteq S = V - L$. First, for every node $v \not\in L \cup S$, we assign it to a subset $S_i$ as follows: we find the shortest path from $v$ to any node in $L$; suppose it is to $i \in L$. We add $v$ to $S_i$. Note
that this preserves the property that all $S_i$ are disjoint, and $v$ has a path to $i$ that does not meet any other node of $L$, since if $h \in L$ were to lie on this path, it would be closer to $v$ than $i$ is.

At this point, every node of $G$ belongs to $L \cup S$. We now must remove nodes from $L$ to reduce its size to exactly $k$ while preserving the properties of a $(k, \varepsilon)$-linked partition. To do this, we choose a node $i \in L$ arbitrary, remove $i$ from $L$, and remove the set $S_i$ from the collection of subsets. We then assign each node in $S_i \cup \{i\}$ to an existing subset $S_h$ exactly as in the previous paragraph. After this process, the size of $L$ has been reduced by $1$, and we still have a partition of $V - L$ into subsets $S_i$ with the desired properties. Continuing in this way, we can reduce the size of $L$ to exactly $k$, at which point we have a $(k, \varepsilon)$-linked partition.

Finally, we prove the following graph-theoretic result, which together with Lemma 4.2 establishes Theorem 4.1.

**Theorem 4.4** For every $k \geq 2$ and with $\varepsilon = 1/2$, every connected undirected graph has a $(k, \varepsilon)$-linked subpartition.

**Proof** It is enough to find the required structure on a spanning tree $T$ of $G$, since if the paths required by the definition exist in $T$, then they also exist in $G$. Thus, it is sufficient to prove the result for an arbitrary tree $T$.

We root $T$ at an arbitrary node, and let $X$ be the set of leaves of $T$. If $|X| \geq k$, then we can choose any $k$ leaves of $T$ and partition the remaining nodes of $T$ arbitrarily into sets of size $(n - k)/k$ to satisfy the definition. Otherwise, $|X| < k$. In this case, we begin by including all nodes of $X$ in $L$.

Now, we process the nodes of $T$, working upward from the leaves, so that when we get to a node $v$ in $T$, we have already processed all descendents of $v$. Each node is processed once, and at that point we decide whether to add it to $L$, and if not which set $S_i$ to place it in, given the current set $L$. For a node $v$, we say that $w$ is downward-reachable from $v$ if $w$ is a descendent of $v$, and if the $v$-$w$ path in $T$ does not contain any internal nodes belonging to $L$.

When we process a node $v$, we do one of two things:

(i) We label $v$ with the name of a node in $L$ that is downward-reachable from $v$; or

(ii) We place $v$ in $L$.

Let $b = (\varepsilon n/k) - 1$. We perform action (i) if there is any $w \in L$ that is downward-reachable from $v$, such that there are not yet $b$ nodes labeled with $w$. In this case, we label $v$ arbitrarily with one such $w$. Since $v$ and all its descendents are now processed, $v$ will continue to have a path to $w$ that does not pass through any other nodes of $L$. Otherwise suppose there is no such $w$; that is, all $w \in L$ that are downward-reachable from $v$ have $b$ nodes labeled with $w$. In this case, we perform action (ii). Note that at this point, every $w \in L$ that is a descendent of $v$ has a set $S_w$ of exactly $b$ nodes, and these nodes can all reach $w$ without passing through any other node of $L$.

Our procedure comes to an end when we process the root node $v^*$. There are three cases to consider, the first two of which are straightforward.

First, if we place $v^*$ into $L$, then $T - v^*$ is partitioned into $L$ and sets $\{S_w : w \in L\}$ such that $|S_w| = b$ for each $w$. Thus, if we remove $v^*$ from $L$, we have a $(k, \varepsilon)$-linked subpartition, since the sets $S_w$ are disjoint and of size at least $b$, and

$$|L| = (n - 1)/(\varepsilon n/k) - 1 \geq (n/(\varepsilon n/k)) = k/\varepsilon > k.$$

Otherwise, $v^*$ is labeled with some downward-reachable $u \in L$. Our second case, which is also straightforward, is that after this labeling of the root, all sets $S_w$ for $w \in L$ have size exactly $b$, then we have a $(k, \varepsilon)$-linked subpartition.

13
We know that we have introduced a new line of inquiry for the envy-free and proportional cake cutting problems with a leaf that is a descendent of \( x \) same leaf are \( < k \) downward-reachable from the root, then there is a \( v \) that any active \( w \) that any active \( v \) in the step when we placed \( v \) has enough elements; if that holds, then we have a \((k, \varepsilon)\)-linked subpartition.

To show this, we proceed as follows. We say that a node \( w \in L \) is active if \( |S_w| < b \). We first observe that any active \( w \) must be downward-reachable from the root \( v^* \). Indeed, if \( w \) is active and not downward-reachable from the root, then there is a \( v \in L \) such that \( w \) is a descendant of \( v \). But in the step when we placed \( v \) in \( L \), it was not possible to label \( v \) with \( w \), and hence we must have had \( |S_w| = b \) at that point.

Next we claim that there are \( < k \) active \( w \) in \( L \). To prove this, for each active \( w \), we associate \( w \) with a leaf that is a descendant of \( w \). (This can be \( v \) itself if \( w \) is a leaf.) Observe that the same leaf \( x \) cannot be associated with two distinct active \( w, w' \), for then on the path from \( v^* \) to \( x \), one of \( w \) or \( w' \) would be closer to \( v^* \), and the other would not be downward-reachable from \( v^* \). Given that we can associate a distinct leaf to each active \( w \), and there are \( < k \) leaves, there are \( < k \) active \( w \in L \).

We say that a node \( w \in L \) is inactive if it is not active; that is, if \( |S_w| = b \). Let \( L_0 \) be the inactive nodes of \( L \) and \( L_1 \) be the active nodes of \( L \). We have

\[
|L_0| + \sum_{w \in L_0} |S_w| + |L_1| + \sum_{w \in L_1} |S_w| = n.
\]

We know that \( |L_1| < k \) and \( |S_w| < b \) for each \( w \in L_1 \); hence, using the fact that \( \varepsilon = 1/2 \), we have

\[
|L_1| + \sum_{w \in L_1} |S_w| < kb = n/(2k) = n/2.
\]

It follows that \( |L_0| + \sum_{w \in L_0} |S_w| > n/2 \). But since \( |S_w| = b \) for each \( w \in L_0 \), we have

\[
n/2 < |L_0| + \sum_{w \in L_0} |S_w| = |L_0|(1 + b) = |L_0|n/(2k),
\]

from which it follows that \( |L_0| > k \). We now conclude the construction by declaring \( L \) to be \( L_0 \); since the sets \( S_w \) for \( w \in L_0 \) are all pairwise disjoint and each has size at least \( b \), we have the desired \((k, \varepsilon)\)-linked subpartition.

## 5 Conclusion and Future Work

We have introduced a new line of inquiry for the envy-free and proportional cake cutting problems by considering local notions of fairness. We show interesting relations between these local fairness concepts and their global analogues. Besides introducing this new model, our main contribution has been to fully classify the class of graphs for which there is an oblivious single-cutter protocol for computing locally envy-free allocations. Furthermore, we quantify the degradation in welfare resulting from adding the local envy-freeness constraint on the allocations; in particular, we show that the known \( \Omega(\sqrt{n}) \) lower-bound for the (global) price of envy-freeness continues to hold even for sparse graphs.

It is of interest to give efficient protocols for computing locally envy-free allocations on rich classes of graphs without the single-cutter constraint. Since local envy-freeness is a stronger condition than local proportionality, the same problem can also be considered for locally proportional allocations. Finally, whether there is a similar lower bound of \( \Omega(\sqrt{n}) \) for the price of local proportionality is an open question. Currently, the upper bound for both local fairness concepts is the loose \( n - 1/2 \) bound, and giving tighter bounds is another direction.
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\section{Taking Advantage of Irrevocable Advantage}

Given a partial envy-free allocation, an agent $i$ is said to dominate an agent $j$, if $i$ remains envy-free of $j$ even if the entire residue (the remaining subinterval of the cake) is allocated to $j$. We can thus define:

\textbf{Definition 8} A domination graph on $n$ is a graph where $V$ is the set of agents and there is a directed edge $(i,j)$ if $i$ has irrevocable advantage over $j$.

Constraints on the number of agents each agent must dominate at a certain stage in the protocol can be used to extend partial envy-free allocations to complete ones. One salient example is the Aziz-Mackenzie protocol for $K_4$, where they obtain a partial envy-free allocation using what they call the \textit{Core Protocol} and the \textit{Permutation Protocol} such that each agent is guaranteed to dominate at least two other agents. They then use the Post-Double Domination Protocol to extend this to a complete envy-free allocation. We generalize this protocol to any $n$. That is, given a partial envy-free allocation such that each agent dominates $n-2$ other agents, we can apply Protocol 1 to extend the allocation to a complete allocation. This provides an alternate proof to a recent paper by Segal-Halevi et al. \cite{25}. We first define a special class of graphs.

\textbf{Definition 9} A pseudoforest is a graph where each vertex has at most one outgoing edge.

Each component of a pseudoforest is a subgraph of a cone of a DAG; since each node has at most one outgoing edge, there are at most $n$ edges. If there are fewer than $n$, then it is a DAG. If there are exactly $n$, then there exists a cycle. Find this cycle and remove an edge $e = (u,v)$ from the cycle. The resulting graph will be a DAG, and we can therefore Protocol 1 by setting $u = c$. This makes the protocol key to extending partial envy-free allocations to complete ones under particular domination criteria.

\textbf{Lemma A.1} We can extend a partial globally envy-free allocation in which each agent dominates at least $n-2$ other agents to a complete, envy-free allocation.

\textbf{Proof} Suppose we have a partial globally envy-free allocation $(P_1, P_2, \cdots, P_n)$, with residue $R$. If each agent dominates at least $n-2$ other agents, then the complement of the domination graph, denoted by $G^c$, is a pseudoforest. We apply Protocol 1 on $G^c$ using residue $R$ and denote this allocation by $(R_1, R_2, \cdots, R_n)$. We want to show that $(P_1 \cup R_1, P_2 \cup R_2, \cdots, P_n \cup R_n)$ is a globally envy-free allocation. Suppose it is not. Then, there exists $i,j$ such that $V_i(P_i \cup R_i) < V_j(P_j \cup R_j)$, but this is only possible if either $V_i(P_i) < V_j(P_j)$ or $V_i(R_i) < V_j(R_j)$.

The assumption that each agent dominates at least $n-2$ agents is necessary. In particular, suppose that there exists one agent that dominates only $n-3$ other agents, such as in Example 2.

Figure 3: Counterexample to the extension lemma.
**Example 2** Suppose that each agent $i \neq 2$ dominates every other agent but agents $i + 1 \mod n$ and that agent 2 dominates agents $\{4, 5, \ldots, n\}$. The domination graph is given in Figure 3. The complement of the domination graph is the cycle graph $(1, 2, 3, \ldots, n, 1)$ plus the edge $(2, 1)$. It therefore consists of more than one simple cycle, and hence a direct application of Protocol 1 to the complement of the domination graph will not extend a partial allocation to a complete allocation.