Holomorphic Integer Graded Vertex Superalgebras

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A Lie superalgebra is a $\mathbb{Z}_2$-graded vector space $g = g_0 \oplus g_1$ with a bilinear bracket $[\cdot, \cdot] : g \times g \to g$ such that

\[
[a, b] \in g_{p(a)+p(b)},
\]

\[
[a, b] = -p(a, b) [b, a],
\]

\[
[[a, b], c] = [a, [b, c]] - p(a, b) [b, [a, c]],
\]

where $a, b$ have degrees $p(a), p(b)$ respectively and $p(a, b) = (-1)^{p(a)p(b)}$. 
Lie Superalgebras

Definition (Killing Form)

For $\mathfrak{g}$, a finite dimensional Lie superalgebra, the Killing form is

$$\kappa : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$$

$$\kappa(a, b) = \text{STr}_{\mathfrak{g}} \text{ad}(a) \text{ad}(b)$$
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- Invariant: $\kappa([a, b], c) = \kappa(a, [b, c])$
- Supersymmetric: $\kappa(a, b) = p(a, b)\kappa(b, a)$
- Consistent: $\kappa(g_0, g_1) = 0$
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The Killing form of a simple Lie superalgebra might be degenerated.
Lie Superalgebras

Definition
Killing Form
Classification

Holomorphic $\mathbb{Z}$-graded vertex superalgebras

Proposition
If $\mathfrak{g}$ be a finite dimensional Lie superalgebra whose Killing form is non-degenerated, then it decomposes into a direct sum of simple Lie superalgebras with non-degenerated Killing form.
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**Definition**

A simple Lie superalgebra $\mathfrak{g}$ is said to be of classical type if $\mathfrak{g}_1$ is a completely reducible $\mathfrak{g}_0$-module.
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**Proposition**

If $\mathfrak{g}$ is a finite dimensional simple Lie superalgebra with non-degenerated Killing form then $\mathfrak{g}$ is of classical type.
The finite dimensional simple Lie superalgebras fall into the following classes:

**Classical type with non degenerated Killing form:** $A(m,n)$ with $m \neq n$, $B(m,n)$, $C(n)$, $D(m,n)$ with $m \neq n - 1$, $F(4)$, $G(3)$.
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| Non classical type with vanishing Killing form: | $W(n)$, $S(n)$, $H(n)$, $\tilde{S}(n)$. |
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Where $B(m, n) := \mathfrak{osp}(2m + 1, 2n)$, $m \geq 0$, $n \geq 1$ and

- $\mathfrak{osp}_0(2m + 1, 2n) := \{X \in \mathfrak{gl}_0(2m + 1, 2n) : (Xu, v) = -(u, Xv)\}$

- $\mathfrak{osp}_1(2m + 1, 2n) := \{X \in \mathfrak{gl}_1(2m + 1, 2n) : (Xu, v) = -(-1)^{p(u)}(u, Xv)\}$
A vertex superalgebra is a super vector space $V$, a distinguished vector $1 \in V_0$ and linear map

$$Y_z : V \rightarrow \text{Field}(V), \quad v \mapsto Y(v, z),$$

such that the following axioms are satisfied:

(Vacuum Axiom) $Y(1, z) = \text{id}$, $Y(v, z)1 \in V[[z]]$, $Y(v, z)1|_{z=0} = v$;

(Translation Invariance) $[T, Y(v, z)] = \partial_z Y(v, z)$;

(Locality Axiom) $[Y_{z_1}(v_1), Y_{z_2}(v_2)](z_1 - z_2)^N = 0$ for $N >> 0$.

Where the translation endomorphism $T \in \text{End}(V)$ is defined as $Tv = \partial_z Y(v, z)1|_{z=0}$.
**Definition**

A conformal structure on $V$ of central charge $c$ consist of a vector $\omega \in V$ whose modes $L_n = \omega_{(n+1)}$ satisfy the Virasoro relations,

$$[L_m, L_n] = (m - n)L_{m+n} + \delta_{m,-n} \frac{m^3 - m}{12} c,$$

the action of $L_0$ on $V$ is semisimple with finite dimensional eigenspaces (write $V_n$ for the eigenspace with eigenvalue $n$) and $L_{-1} = T$. 
**Conformal Field Theory Type Algebras**

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**Definition**

A conformal vertex $V$ algebra is said to be of CFT type (conformal field theory type) if

- $V_n = 0$ for $n < 0$,
- $V_0 = \mathbb{C}1$, 

Conformal Field Theory Type Algebras

**Proposition**

Let $V = \bigoplus_{n \in \mathbb{Z}_+} V_n$ be a conformal vertex super algebra of CFT type then the product

$$\cdot_{(0)} \cdot : V_1 \otimes V_1 \rightarrow V_1$$

$$a \otimes b \mapsto a_{(0)}b$$

equips $V_1$ with a Lie superalgebra structure.
Definition

A vertex algebra is $C_2$-cofinite if the subspace $V_{(-2)}V$ has finite codimension.
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**Proposition**

Let $V$ be a $\mathbb{Z}$-graded conformal vertex superalgebra. If $V$ is simple, of CFT type and $L_1 V_1 = 0$ and

$$\langle u, v \rangle = p(u, v)u_{(1)} v$$

defines a nondegenerated invariant bilinear form.

In this context invariant means that for all $a, u, v \in V$ we have

$$(Y(a, z)u, v) = p(a, u)(u, Y(e^{z L_1}(-z^{-2})^{L_0})v).$$
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**Definition**

Holomorphic $\mathbb{Z}$-graded vertex superalgebras Let $V$ be a $\mathbb{Z}$-graded conformal vertex superalgebra. We call $V$ *holomorphic* if it is *self-contragredient*, $C_2$-*cofinite* and *rational* and if, moreover the unique irreducible ordinary $V$-module is the adjoint module $V$ itself (in particular $V$ is simple).
Problem

Classify holomorphic integer graded vertex superalgebras at small central charge.
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Compute the possible Lie superalgebras that might appear as the $V_1$ part.
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Proposition

The central charge must be a multiple of 8.
Main Result. Case $c = 24$

Theorem

Let $V$ be a holomorphic $\mathbb{Z}$-graded vertex superalgebra of central charge 24 for which $sdim(V_1) \neq 24$. Then $V_1$ is either zero or is isomorphic to a finite sum

$$g = g^{(1)} \oplus \cdots \oplus g^{(r)}$$

of simple Lie superalgebras, each even or of type $B(0,n) = osp(1,2n)$. Moreover there are at most 1332 possible Lie superalgebras for $V_1$. 
Sketch of the proof

1. Prove that if \( \text{sdim}(V_1) \geq 24 \) then the Killing form on \( V_1 \) is nondegenerated and therefore \( V_1 \) decomposes into a sum of simple Lie superalgebras.

2. Prove the formula 
   \[
   h_i \Lambda_i = \text{sdim}(V_1) - 24
   \]
   for all \( i = 1, \ldots, r \).

3. Find bounds for the superdimension of \( V_1 \), for instance, we found 
   \[
   0 \leq \text{sdim}(V_1) \leq 1344
   \]

4. Develop a script using a computer (for instance using Python) to find a list of all possible decompositions for \( V_1 \) as a direct sum of simple Lie superalgebras.
Sketch of the proof

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Lie superalgebras
Vertex superalgebras
Holomorphic \( \mathbb{Z} \)-graded vertex superalgebras

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  \[
  \frac{h_i^\vee}{k_i} = \frac{sdim(V_1)}{24}
  \]
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- Develop a script using a computer (for instance using Python) to find a list of all possible decompositions for $V_1$ as a direct sum of simple Lie superalgebras.
Cases $c = 8, c = 16$

**Theorem**

Let $V$ be a holomorphic $\mathbb{Z}$-graded vertex superalgebra of central charge $c = 8$ or $c = 16$ then $V$ is purely even.
Reduce the list of those 1332 Lie superalgebras for V1.
Next steps...

Reduce the list of those 1332 Lie superalgebras for V1.

Analyze the case of super dimension 24.
Next steps...

- Reduce the list of those 1332 Lie superalgebras for $V_1$.
- Analyze the case of super dimension 24.
- Produce non-trivial examples (non purely even) of holomorphic integer graded vertex superalgebras.
Next steps...

Reduce the list of those 1332 Lie superalgebras for $V_1$.

Analyze the case of super dimension 24.

Produce non trivial examples (non purely even) of holomorphic integer graded vertex superalgebras.

Very ambitious goal

Classify holomorphic integer graded vertex superalgebras at a certain central charge.
End of the presentation

Thank you

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