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ABSTRACT

At the critical point, one finds thoroughly mixed droplets of water and bubbles of gas. Indeed, any large part of the system which the density difference between the liquid and vapor just vanishes as does the latent heat of evaporation. At a pressure

\[ P \sim P_c \]

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ABSTRACT

Networks play a vital role in the development of predictive models of physical, biological, and social collective phenomena.\(^1\)\(^\text{-}^3\) A quite remarkable feature of many of these networks is that they are believed to be approximately scale free: the fraction of nodes with \(k\) incident links (the degree) follows a power law \(p(k) \propto k^{-\lambda}\) for sufficiently large degree \(k\).\(^4\)\(^,\)\(^5\) The value of the exponent \(\lambda\) as well as deviations from power law scaling provide invaluable information on the mechanisms underlying the formation of the network such as small degree saturation, variations in the local fitness to compete for links, and high degree cutoffs owing to the finite size of the network.\(^1\) Indeed real networks are not infinitely large and the largest degree of any network cannot be larger than the number of nodes. Finite size scaling\(^6\)\(^\text{-}^10\) is a useful tool for analyzing deviations from power law behavior in the vicinity of a critical point in a physical system arising due to a finite correlation length. Here we show that despite the essential differences between networks and critical phenomena, finite size scaling provides a powerful framework for analyzing self-similarity and the scale free nature of empirical networks. We analyze about two hundred naturally occurring networks with distinct dynamical origins, and find that a large number of these follow the finite size scaling hypothesis without any self-tuning. Notably this is the case of biological protein interaction networks, technological computer and hyperlink networks and informational citation and lexical networks. Marked deviations appear in other examples, especially infrastructure and transportation networks, but also social, affiliation and annotation networks. Strikingly, the values of the scaling exponents are not independent but satisfy an approximate exponential relationship.

Scale-free networks possess important emergent attributes such as long range correlations in their topology, robustness to failures, while yet being fragile to the right types of attacks. Recently, Broido and Clauset\(^11\) fitted a power law model to the degree distribution of a variety of empirical networks and suggested that scale free networks are rare. Voitalov \textit{et al.}\(^12\) rebutted that scale free networks are not as rare if deviations from pure power law behavior are permitted in the small degree regime. Here we use the machinery of finite size scaling in critical phenomena\(^6\)\(^,\)\(^7\) to analyze around two hundred large networks (those considered in\(^11\) and\(^12\)). Our results show that scale invariance is indeed a feature of many of these networks, with finite size effects accounting for quantifiable deviations. We benchmark our results against the quality measure of a well-known self-similar graph introduced by Barabási and Albert\(^4\) and contrast it with that of a Poisson random graph\(^13\)\(^,\)\(^14\) (which is not scale free), for both of which the exact behaviors are known.

A system at criticality exhibits scale invariance and power law behavior. In contrast to networks, criticality entails fine-tuning of parameters. Let us consider the familiar case of water, which boils at 100C under 1 atmosphere pressure: there is a density difference between the liquid and the vapor and there is a latent heat of evaporation that needs to be supplied in order to effect the phase change. This is why we can cook pasta or rice in water at the fixed temperature of 100C even though we continue to heat it. As one raises the pressure, the temperature of boiling goes up, the density difference goes down, as does the latent heat. At a special critical pressure \(P_c \sim 217.7\) atmospheres and critical temperature \(T_c \sim 373.9\)C, one observes a critical point at which the density difference between the liquid and vapor just vanishes as does the latent heat of evaporation. At a pressure higher than the critical value, the distinction between the liquid and vapor phases is lost.

The critical point exhibits scale invariance and power law singularities of physical quantities such as the compressibility, the specific heat, the density difference between the liquid and vapor, as well as the latent heat. Water at its critical point exhibits fluctuations at all scales between the molecular length scale and the size of the container, which could be macroscopically large. At the critical point, one finds thoroughly mixed droplets of water and bubbles of gas. Indeed, any large part of the system looks like the whole – the system is self-similar. The length scale of these droplets and bubbles extends from the molecular
scale up to the correlation length, which is a measure of the size of the largest droplet or bubble.

Unlike in physical systems, the networks we consider are not obtained as a result of any overt fine-tuning of parameters. Furthermore, we do not have representations of networks of different sizes to allow for a straightforward test of the finite size scaling hypothesis. In order to test whether these networks spontaneously self-organize to a critical-like state, we take a large network and construct by hand smaller sized representations of it in an unbiased manner. We then use the characteristics of the large original network as well as the derived sub-networks to test the finite size scaling hypothesis. Remarkably, many, but not all, networks satisfy such a venerable hypothesis and, even more surprisingly, the two exponents for all such networks satisfy a non-trivial scaling relationship. Box A provides a brief summary of finite size scaling applied to network topology. Box B presents an independent method of determining whether networks are scale free based on analyses of the size dependences of the ratio of moments of the degree distributions.

**Box A: Finite Size Scaling of networks**

A scale free network is postulated to have a degree distribution \( p(k) \propto k^{-\lambda} \) in the limit of large degree \( k \). For an infinitely sized network, the exponent \( \lambda > 1 \) in order for \( p(k) \) to be normalizable in the interval \([k_{\text{min}}, \infty)\). In what follows, we will consider the cumulative distribution \( P(k) = \int_k^\infty p(q)dq \propto k^{-\gamma} \) where \( \gamma = \lambda - 1 > 0 \).

Networks are of course not infinitely large. Consider a network comprising \( N \) nodes. There is an intrinsic limit on \( k \) given by the network size: the largest value of \( k \) is \( N - 1 \), when a node is linked to all other nodes in the system. Thus it is plausible that, below some \( k_{\text{max}} \), the degree distribution follows a power-law behavior as would be expected for an infinite network but falls more rapidly beyond \( k_{\text{max}} \). The finite size scaling hypothesis states that

\[
P(k, N) = k^{-\gamma}f(kN^d).
\]

The remarkable simplifying feature of the scaling hypothesis is that \( P \) is not any arbitrary function of the two variables \( k \) and \( N \) but rather \( k \) and \( N \) combine in a non-trivial manner to create a composite variable. The behavior of the system is fully defined by the two exponents, \( \gamma \) and \( d \), and the scaling function \( f \). The exponent \( d < 0 \) so that, for an infinite size network \((N \rightarrow \infty)\), the argument of \( f \) approaches zero. A pure power law decay of \( P(k, N) \) with \( k \) for very large \( N \) requires that \( f(x) \rightarrow \text{constant} \) as \( x \rightarrow 0 \). For a network with a finite number of nodes, the degree distribution does not follow a pure power law but is modified by the cutoff function \( f \).

A powerful way of assessing whether a network is self-similar and scale free is to confirm the validity of the scaling hypothesis and determine the two exponents and the scaling function \( f \). One may recast Eq. (1) as

\[
P(k, N)k^\gamma = f(kN^d).
\]

The path forward is then simple. For networks belonging to the same class but with different \( N \), one optimally selects two fitting parameters (our familiar exponents) \( \gamma \) and \( d \) by seeking to collapse plots of \( P(k, N)k^\gamma \) versus \( kN^d \) for different \( N \) on top of each other.\(^{15}\) The fidelity of the collapse plot provides a measure of self-similarity and scale free behavior, the optimal parameters are the desired exponents, and the collapsed curve is a plot of the scaling function.

We start out with a single representation of an empirical network with \( N \) nodes. For purposes of the scaling collapse plot, we seek additional representative networks of smaller sizes. In order to accomplish this, we obtained the mean degree distributions of multiple sub-networks of sizes \( nN/4 \), \( nN/2 \) and \( 3nN/4 \), which were then collapsed on to each other and the mother network to create a master curve. The quality, \( S \), of the collapse plot is then measured as the distance of the data from the master curve in units of standard errors (as a rule of thumb, the collapse is considered satisfactory if \( S \leq 2 \)).

Note that as a measure of the size of a network (or subnetwork), one may use the number of nodes \( N \) or alternatively the number of links \( E \). The scaling function in this case reads as follows:

\[
P(k, E)k^\gamma = f_E(kE^{d_E}).
\]

where the exponent \( \gamma \) is the same as before and the exponent \( d_E \) ought to be equal to the previously introduced exponent \( d \) for networks satisfying the finite size scaling hypothesis (see Box B). Thus the difference between the \( d \) and \( d_E \) values provides an independent quality measure of the scale-free attributes of the empirical network.

**Box B: Ratio of moments test**

A simple alternative and independent test of the scale free hypothesis is to study the size dependence of the ratio between the \( i \)-th and the \((i-1)\)-th moments of \( k \), for large enough \( i \). The \( i \)-th moment \( \langle k^i \rangle \) is defined to be

\[
\langle k^i \rangle = \int dk k^{i-1}k^{-\gamma}f(kN^d) \propto N^{-d(i-\gamma)}
\]
provided \( i > \gamma \). Instead if \( i \leq \gamma \), \( \langle k^i \rangle \) converges to \( (1 - i/\gamma)^{-1/k_{\text{min}}} \) for \( N \to \infty \). Therefore when \( i - 1 > \gamma \),

\[
\frac{\langle k^i \rangle}{\langle k^{i-1} \rangle} \propto N^{-d},
\]

independently of \( i \). Thus, for a scale free network, a log-log plot of the ratio of consecutive moments versus \( N \) is a straight line with slope \(-d\). Likewise

\[
\langle k^i \rangle = \int dk k^i e^{-kE} \propto E^{-d(i-\gamma)}
\]

when \( i > \gamma \), otherwise \( \langle k^i \rangle \sim \text{constant} \) for \( E \to \infty \). Therefore when \( i - 1 > \gamma \),

\[
\frac{\langle k^i \rangle}{\langle k^{i-1} \rangle} \propto E^{-dE}.
\]

The exponents \( d \) and \( d_E \) are not independent for scale free networks. On the one hand, equations (4) and (6) imply \( E \propto N^{d/d_E} \). On the other, in general \( \langle k \rangle \propto E/N \propto N^{d/d_E - 1} \). Because \( \langle k \rangle \) is constant for scale free networks with \( \gamma > 1 \), we obtain \( d = d_E \).

**Results**

We start analyzing the reference cases of Barabási-Albert\(^4\) and Erdős-Rényi\(^1\)\(^3\) models whose behavior is known. Figure 1 shows that for a Barabási-Albert graph the degree distributions of the (sub)networks result in a collapse of very high quality. The power law exponent \( \gamma \) yielding the best collapse is consistent with the value \( \Gamma \) obtained by maximum-likelihood fitting the

\[
P(k) \propto k^{-\gamma}
\]

**Barabási-Albert model**

![Figure 1](image)

**Figure 1.** Scaling analysis on a numerical realization of the Barabási-Albert model. The network has \( N = 10^4 \) nodes and the minimum node degree is \( k_{\text{min}} = 14 \). The best power-law fit on this network yields \( \Gamma = 1.89 \pm 0.03 \) with a \( p \)-value of 0.19. Panels (a), (b), (c) show results of the scaling analysis using the number of nodes as for Eqs. (2) and (5). Inset (a) reports the dependence of various moment ratios on \( N \); fitting these slopes yields \( d = -0.351 \pm 0.042 \). The main panel (a) shows the collapse of the cumulative degree distributions when scaled with \( N \). The best collapse is obtained with \( \gamma = 1.95 \pm 0.05 \) and yields \( S = 0.26 \). Panel (c) shows how the quality of the collapse reported in (a) varies on moving away from the optimal value of \( \gamma \). Panels (d), (e) further show results of the scaling analysis using the number of links as for Eqs. (3) and (7). In this case, the moment ratio test of inset (d) returns \( d_E = -0.343 \pm 0.055 \) while the best collapse of the cumulative degree distributions reported in the main panel (e) is obtained with \( \gamma = 1.95 \pm 0.05 \) and yields \( S = 0.26 \).
degree distribution of the mother network with a power law. Additionaly, the moments ratio are indeed parallel lines, with slopes satisfying \( d = d_E \). Figure 2 instead shows that for an Erdős-Rényi graph both the collapse and the power-law fitting are poor as expected, and consistently the moment ratio test does not yield parallel lines hence \( d \neq d_E \).

We then move to real network data. We consider a large set of empirical networks taken from the Index of Complex Networks (ICON) as well as from the Koblenz Network Collection (KONECT). These are the dataset used by Broido & Clauset and Voitalov et al. See the Methods section for a discussion on how we built the dataset. Overall, we have networks belonging to ten different categories: biological (PPI), social (i.e., friendship and communication), affiliation, authorship (including co-authorship), citation, text (i.e., lexical), annotation (i.e., feature, folksonomy, rating), hyperlink, computer, infrastructure. Figure 3 shows results of the finite size scaling analysis for selected network instances, while Figure 4 summarizes results of the scaling analysis for all the networks considered (see the Methods section for details). For each network, first we obtain \( d \) and \( d_E \) from the moment ratio test and check that they are compatible in most of the cases – see Figure 4(a). Then, using these values, we determine the power law exponent \( \gamma \) yielding the collapse of highest quality (lowest \( S \)). Figure 4(b) shows that each network is characterized by different values of \( \gamma \) and \( S \). Note this result is obtained when the different network categories are well balanced in the dataset, because networks that are very similar tend instead to cluster together in the \( \gamma - S \) plane. The two small side plots show examples of this behavior for protein interaction networks of various species and Facebook social networks of colleges in the US. In order to remove this artificial clustering effect, we have not considered in the dataset these (and other) cases of very similar networks nor repetitions of the same network (see Methods). This is the main reason why our dataset is smaller than that used by Broido & Clauset.

A striking observation worthy of further exploration emerging from Figure 4(c) is that the exponents \( \gamma \) and \( d \) of the scaling function are not independent but satisfy an approximate and perhaps universal exponential relation. Finally, Figure 4(d) shows that the value of \( \gamma \) computed from finite size scaling is in good agreement with \( \Gamma \) obtained from the maximum likelihood power law fit of the degree distribution (see Methods).
Figure 3. Scaling analysis (with $N$) on four real network instances. Top left panels: the 2005 version of the proteome-scale map for Human binary protein-protein interactions ($N = 3133, E = 6726$). Top right panels: the word adjacency graph extracted from the English text “The Origin of Species” by C. Darwin ($N = 7724, E = 46281$). Bottom left panels: symmetrized snapshot of the Internet structure at the level of Autonomous Systems in 2006 ($N = 22963, E = 48436$). Bottom right panels: the collaboration graph of authors of scientific papers from DBLP computer science bibliography ($N = 1314050, E = 10724828$).

Given that the quality of the collapse $S$ is a normal variable (see Methods) and defining the z-score between $d$ and $d_E$ as $Z_{dd_E} = |d - d_E|\left(\sigma_d^2 + \sigma_{d_E}^2\right)^{-1/2}$, we have strong evidence of a scale-free network when $S \leq 1$ and $Z_{dd_E} \leq 1$, weak evidence when
Figure 4. Visual summary of results from the finite size scaling analysis, in which each network dataset is represented as a point in a specific plane. Panel (a) shows the relation between $d$ and $d_E$ resulting from the moment ratio test, with the solid black line representing the identity. The three other panels focus on the scaling analysis with $N$. Panel (b) shows the relation between the power law exponent $\gamma$ and the quality of the collapse $S$. The two insets report additional network instances of similar nature (metabolic networks of various species from KEGG$^{21}$ and social networks of colleges and universities admitted to facebook.com$^{22}$) and show how they cluster together in this plane. Panel (c) reports the relation between the exponents $\gamma$ and $d$ of the scaling function. Finally panel (d) show the relation between $\gamma$ computed from finite size scaling and $\Gamma$ from the maximum likelihood power law fit of the degree distribution (see Methods).

$S \leq 3$ and $Z_{dd_E} \leq 3$, and no evidence otherwise. Overall, as shown in Table 1, the 185 networks of our dataset are classified as strong in the 35% of cases, weak in 32% and non-scale free in 33%. These figures do however vary substantially among the different network categories. On the one hand, biological networks are very often classified at least as weakly scale free, and the outliers have an $S$ very close to 3. The same happens for computer and hyperlink networks, with outliers respectively given by the Gnutella peer-to-peer file sharing network (that has the same character of a social networks$^{23}$) and by some hyperlink networks restricted to specific domains. Citation and text networks in our analysis are few, but are always scale free. On the other hand, infrastructure networks (i.e., road and flights network) are rarely scale free (with the notable exception of water distribution systems), possibly because of the heavy cost of establishing a connection. Between these two extremes, there are the social and other kinds of networks (see for instance the well-known discussion of the Facebook case presented in$^{24,25}$ and that of other information sharing social network presented in$^{26}$).

Discussion

Since the birth of network science, the scale invariance of complex networks has been regarded as a universal feature present in real data$^{16,27–31}$ as well as reproduced in models.$^{4,32–36}$ Thus the recent claim by Broido & Clauset$^{11}$ that scale free networks are rare created a stir, strengthening previous claims along the same direction.$^{16,37,38}$ The most surprising finding was that even
We now describe the procedure for deriving the master curve of the scaling function from the cumulative degree distributions with $d$-values of scale free networks. Voitalov et al. replied to these arguments fitting data to generalized power laws, that is, regularly varying distributions $p(k) = l(k)k^{-\lambda}$ (where $l(k)$ is a function that varies slowly at infinity and thus does not affect the power law tail). By allowing deviations from the pure power law distribution at low $k$, they argued that scale free networks are definitely not rare.

Gerlach & Altmann very recently touched on this issue, showing that correlations present in the data can lead to false rejections of statistical laws when using standard maximum-likelihood recipes (in the case of networks, this can be important in the presence of degree-degree correlations).

In this work we go beyond statistical arguments and apply powerful ideas from the study of critical phenomena in physics to an analysis of a wide range of networks. The concept of scale invariance is richer than the mere existence of a power law: complex networks are self-similar if any part exhibits the same properties as the whole. We have shown that a wide range of networks spontaneously, without fine-tuning, satisfy the finite size scaling hypothesis, which, in turn, supports the claim that complex networks are indeed scale-free. The conclusion that networks consist of self-repeating patterns on all length scales can also be tested and confirmed using a renormalization procedure that coarse-grains the system into boxes containing nodes within a given size. This fractal scaling is however not the same as the self-similarity of the degree distribution. Understanding the dynamical origins of the scaling behavior as well as understanding the relationship between the two exponents introduced in our study remains an intriguing unresolved issue for future research.

**Methods**

Here we outline the steps to test the finite size scaling hypothesis of Eq. (2). In order to test Eq. (3), one uses the number of edges $E$ (e) associated with each (sub-)network of size $N$ ($n$), and replaces $d$ obtained from the moments ratio test of Eq. (5) with $d_E$ from Eq. (7).

**Finite Size Scaling analysis**

Given an undirected network of size $N$, our analysis is based on the following steps.

1. We compute the cumulative degree distribution $P(k,N)$ and use the method of Clauset, Shalizi and Newman to estimate the best fitting power law parameters $\Gamma$ and $k_{min}$. Concerning the goodness-of-fit between the data and the power law, as a rule of thumb, if the resulting $p$-value is greater than 0.1, the power law is a plausible hypothesis for the data, otherwise it is rejected.

2. We generate an ensemble of 100 random sub-networks for each sub-size $n \in \{N/4, N/2, 3N/4\}$. It is known that sub-networks of scale free networks are not scale free because of deviations for low $k$ values (independent of the sampling scheme adopted). We avoid this issue by removing from each sub-network all the nodes with $k < k_{min}$ and all the link stubs associated with them. After this pruning process, we compute the mean cumulative degree distribution $P(k,n)$ over each sub-network ensemble.

3. Keeping the value of $d$ fixed, we determine the exponent $\gamma$ (and the associated error) that maximizes the quality of the collapse plot for the four cumulative degree distribution $P(k,n)$ (see below).

4. The exponent $d$ (and the associated error) is determined a priori using the moments ratio test. That is, we use least-squares to fit $\log(\langle k^d \rangle / \langle k^{d-1} \rangle)$ (computed on a sub-network of size $n$) versus $\log(n)$, and average over $i$. We use 20 equally spaced values of $n \in [N/4, N]$, for each of which we compute the moments ratio (and associated error used as fit weight) over an ensemble of 50 $n$-sized sub-network built as described above.

**Quality of collapse**

We now describe the procedure for deriving the master curve of the scaling function from the cumulative degree distributions of the various sub-networks, following the steps described in. The key premise is that when these distributions are properly

| number | TOTAL | 8   | 39  | 16  | 30  | 5   | 9   | 14  | 12  | 42  | 10  |
|--------|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| Strong | 35%   | 50% | 26% | 25% | 43% | 40% | 67% | 43% | 33% | 21% | 60% |
| Weak   | 32%   | 13% | 31% | 25% | 43% | 60% | 22% | 43% | 8%  | 33% | 40% |
| NSF    | 33%   | 38% | 44% | 50% | 13% | 0%  | 11% | 14% | 58% | 45% | 0%  |

Table 1. Summary of network classification split into categories. For each category we report the total number of networks and the percentage of strong, weak and non-scale free instances.
rescaled they can be fitted by a single (master) curve. The quality of the collapse plot is then measured as the distance of the data from the master curve, and the collapse is good if all the rescaled distributions overlap onto each other.

In practice for each (sub)network size \( n \in \{N/4, N/2, 3N/4, N\} \) we have the set \( \{j\} \) of ordered points for the cumulative degree distribution of the form \( \{(k_j, P(k_j, n))\} \). We apply to them the scaling laws and obtain:

\[
\begin{align*}
x_{nj} &= k_j n^d \\
y_{nj} &= P(k_j, n) k_j^\gamma
\end{align*}
\]

so that \( x_{nj} \) is the rescaled \( j^{th} \) degree in the distribution of the \( n \)-sized subnetwork, and \( y_{nj} \) is the rescaled value of such distribution relative to the \( j^{th} \) degree. We also assign an error on the latter quantity as \( dy_{nj} = dP(k_j, n) k_j^\gamma \), where \( dP(k_j, n) \) is the Poisson error on the count \( P(k_j, n) \).

The master curve is the function \( Y(x) \) best fitting all these points. To compute \( Y(x) \) we first need to define a set \( M \) of points for which the curves for the various \( n \) overlap. Specifically, we go through all pairs \( (nj) \); for each of them, we put in \( M \) the two points in the other sets \( n' \neq n \) that best approximate \( x_{nj} \) from below and above, \( i.e., \) the two points \( (n'j') \) and \( (n'(j'+1)) \) satisfying \( x_{n'j'} \leq x_{nj} \leq x_{n'(j'+1)} \). Least square fitting is then performed over this set of points and yields

\[
Y(x) = \frac{W_x W_y - W_x W_y}{\eta} + x \frac{W W_x - W_x W_y}{\eta}
\]

where \( w_{nj} = 1/dy_{nj}^2 \) for the fit weights and \( W = \sum_{(nj) \in M} w_{nj} \). \( W_x = \sum_{(nj) \in M} w_{nj} x_{nj} \), \( W_y = \sum_{(nj) \in M} w_{nj} y_{nj} \), \( W_{xx} = \sum_{(nj) \in M} w_{nj} x_{nj}^2 \), \( W_{xy} = \sum_{(nj) \in M} w_{nj} x_{nj} y_{nj} \). \( \eta = W_{xx} W_y - W_x W_{xy} \) for the fit parameters. The error associated with \( Y \) is

\[
dY^2(x) = \frac{1}{\eta}(W_{xx} - 2x W_x + x^2 W)
\]

Finally the quality of the collapse is evaluated as the mean square distance of the sets to the master curve in units of standard errors:

\[
S = \frac{1}{|M|} \sum_{(nj) \in M} \frac{(y_{nj} - Y_{nj})^2}{dy_{nj}^2 + dY^2_{nj}}
\]

where \( Y_{nj} \) and \( dY_{nj} \) are the position and standard error of the master curve at \( x_{nj} \).

Note that in principle it is possible to optimize the quality \( S \) of the collapse by varying the scaling parameters \( \gamma \) and \( d \) simultaneously. However, the numerical optimization procedure often gets stuck in a local minima of the fitness landscape. To avoid this trapping, we fix \( d \) to the value computed from the moments ratio test, and optimize \( S \) only with respect to \( \gamma \). By doing so, the error associated with \( \gamma \) is estimated with a \( S + 1 \) analysis: \( \Delta \gamma \) is such that \( S(\gamma + \Delta \gamma) = S(\gamma) + 1 \).

**Dataset**

We extract a collection of real network data from the Index of Complex Networks (ICON) at https://icon.colorado.edu/ as well as the Koblenz Network Collection (KONECT) at http://konect.uni-koblenz.de/. The full list of networks we consider together with detailed results of the finite size scaling analysis are reported in the Supplementary Table. To define the dataset we select networks (removing duplicates appearing in both ICON and KONECT) according to the following criteria.

First, to allow for a reliable scaling analysis, we only use networks with \( N > 1000 \) and \( E > 1000 \) (for computational reasons, we did not consider networks with more than 50 million links). We then include undirected networks, as well as the undirected version of both directed and bipartite networks. Similarly, we consider binary networks as well as the binarized version of weighted and multi-edge networks. We however ignore networks that are marked as incomplete in the database. Importantly, among database entries that possibly represent the same real-world network we select only one (or at most a few) entry, and consistently we do the same for temporal networks (when there is only one snapshot, we ignore time stamps of links).

In practice, in KONECT we select only the Wikipedia-related networks in English language. For ICON the implications are more profound. We ignore interactomes of the same species extracted from different experiments, the (almost 100) fungal growth networks, the (more than 100) Norwegian boards of directors graphs, the (more than 100) CAIDA snapshots denoting autonomous system relationships on the Internet, networks of software function for Callgraphs and digital circuits ITC99 and ISCAS89. We consider only one instance of Gnutella peer-to-peer file sharing network, as well as a few instances of the (more than 50) within-college Facebook social networks and of the (about 50) US States road networks. Among the (more than 100) KEGG metabolic networks, we select 17 species trying to balance the different taxonomies.
Thus, in our analysis, we do employ the same data source used by Broido & Clauset, but we avoid over-represented network instances. As explained in the main text, this procedure removes the clustering of similar networks shown in Figure 3, and leads to less biased conclusions on the scale-free nature of networks belonging to different categories.

**Additional information**

GCa conceived the experiment. MS and GCi performed the analysis of the dataset. All the authors contributed to the interpretation of results, both from the point of view of network theory as well as in terms of finite size scaling analysis. They also contributed to the writing and reviewing of the manuscript.

**Competing information**  The authors declare no competing interests.

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