Accurately modelling the complete gravitational-wave signal from precessing binary black holes through the late inspiral, merger and ringdown remains a challenging problem. The lack of analytic solutions for the precession dynamics of generic double-spin systems, and the high dimensionality of the problem, obfuscate the incorporation of strong-field spin-precession information into semi-analytic waveform models used in gravitational-wave data analysis. Previously, an effective precession spin, \( \chi_p \), was introduced to reduce the number of spin degrees of freedom. Here, we show that \( \chi_p \) alone does not accurately reproduce higher-order multipolar modes, in particular the ones that carry strong imprints due to precession such as the \((2,1)\)-mode. To improve the higher-mode content, and in particular to facilitate an accurate incorporation of precession effects in the strong-field regime into waveform models, we introduce a new dimensional reduction through an \textit{effective precession spin vector}, \( \bar{\chi} \), which takes into account precessing spin information from both black holes. We show that this adapted effective precession spin (i) mimics the precession dynamics of the fully precessing configuration remarkably well, (ii) captures the signature features of precession in higher-order modes, and (iii) reproduces the final state of the remnant black hole with high accuracy for the overwhelming majority of configurations. We demonstrate the efficacy of this two-dimensional precession spin in the strong-field regime, paving the path for meaningful calibration of the precessing sector of semi-analytic waveform models with a faithful representation of higher-order modes through merger and the remnant black hole spin.

I. INTRODUCTION

Since the first detection of gravitational waves (GWs) from a binary black hole (BBH) merger in 2015 [1], the GW observatories Advanced LIGO [2, 3] and Virgo [4, 5] have reported detections of GWs from tens of compact binary mergers [6, 7], including the first multi-messenger observation of a binary neutron star inspiral, GW170817 [8, 9], the first intermediate mass black hole GW190521 [10, 11], the first unequal mass BBH GW190412 [12] and the first neutron star – black hole candidates GW190814 [13] and GW190426 [7]. Additional GW candidates have been reported from analyses of the the publicly available data in Refs. [14, 15]. In order to fully exploit the scientific potential of these observations and infer the source properties, highly accurate and computationally efficient models of the emitted GW signal are required.

Recent years have seen significant improvements in the modellng of the complete inspiral-merger-ringdown (IMR) signal of compact binaries with the inclusion of spin-induced precession effects [16, 17] as well as higher-order harmonics [18–20]. While the state-of-the-art waveform models are sufficiently accurate for current observations, where the uncertainty in the measurement of the BH properties is dominated by the statistical uncertainty due to detector noise, future upgrades to the current interferometer network [21] and third-generation ground-based detectors such as the Einstein Telescope [22] and Cosmic Explorer [23], will operate at unprecedented sensitivities, shifting focus onto systematic modelling errors as the dominant source of error [24]. The development of ever more accurate models by increasing their physics content is of paramount importance.

The current generation of waveform models can broadly be split into three families: effective-one-body (EOB) models [25], phenomenological (Phenom) models [26, 27], and numerical relativity (NR) surrogate models [28, 29]. While the EOB and Phenom models describe the complete GW signal throughout inspiral, merger and ringdown (IMR), NR surrogates are mainly restricted to the strong-field regime. Phenom models are currently the most widely used due to their considerable computational efficiency relative to other models, which is particularly crucial for parameter estimation due to the large number of required model evaluations, and their wide parameter space validity. Computational efficiency, however, often comes at the cost of simplified physics content, which can severely impact the accuracy of parameter measurements [30]. In this paper, we focus on one particularly urgent open problem in waveform modelling: devising a feasible strategy to accurately incorporate precession effects into BBH waveform models in the strong-field regime.

Spin-induced orbital precession occurs when the spins of one or both compact objects are misaligned with the orbital angular momentum [31, 32]. This introduces phase and amplitude modulations into the GW signal, and a richer mode structure, amplifying high-order modes relative to the \((2,2)\)-mode. Due to their...
complexity, precessing waveforms encode vast amounts of information which can be used to break parameter degeneracies [33–39]. This facilitates better measurements and more stringent tests of general relativity, but this also makes them difficult to model across the binary parameter space.

Current precessing IMR waveform models are built in an approximate way by applying a time-dependent rotation encoding the orbital precession dynamics to waveform modes obtained in a frame that co-precesses with the orbit [40, 41]. The four spin components instantaneously orthogonal to the orbital angular momentum, i.e. within the instantaneous orbit plane, source the orbital precession [31]. This relatively large number of spin degrees of freedoms complicates the inclusion of precession into semi-analytic waveform models. Therefore, developing an efficient dimensional reduction strategy is crucial for modelling strong-field precession across the parameter space.

Previously, the effective precession spin parameter $\chi_P$ [42] was introduced to reduce the four orthogonal spin components to one whilst capturing the dominant precession effects in the waveforms. However, a simple $\chi_P$-parameterisation fails to accurately reproduce the phenomenology of higher-order modes (HMs), including precession-induced mode-mixing and the asymmetry between positive and negative $m$-modes [30]. In precessing systems where the relative power in HMs can be comparable to the dominant quadrupolar mode, this can lead to significant systematic errors [19, 20], as recent observations are starting to indicate [11].

Here, we introduce a new two-dimensional effective precession spin vector, $\vec{\chi}_{\perp}$, which incorporates two-spin effects. Focusing on the strong-field regime, We show that a $\vec{\chi}_{\perp}$-parameterisation (i) matches the opening angle of the precession cone at a given reference time; (ii) significantly better reproduces higher-order modes than a $\chi_P$-parameterisation; (iii) more accurately mimics the precession dynamics and (iv) matches the final state. This vectorial effective spin mapping could facilitate more accurate waveform modelling of precession in the strong-field regime.

The paper is organised as follows: In Sec. II we briefly summarise the phenomenology of precessing binaries and current waveform modelling efforts, before introducing the new effective precession spin vector $\vec{\chi}_{\perp}$ in Sec. II B. We describe the methodology used in this work in Sec. III. In Sec. IV we present our results and subsequently discuss the accuracy and caveats of this spin mapping in Sec. V. Throughout this paper, we will use $G = c = 1$.

II. PRECESSING BINARIES

A. Modelling Precession

Binary black holes on quasi-spherical orbits are intrinsically characterised by seven parameters: the mass ratio $q = m_1/m_2 \geq 1$, where $m_i$ with $i \in [1, 2]$ denotes the component mass, and the spin angular momentum $\vec{S}_i$ of each black hole, or its dimensionless counterpart $\chi_i = \vec{S}_i/m_i^2$. If the black holes’ spins are misaligned with the direction of the orbital angular momentum $\vec{L}$ of the binary motion, spin-induced precession of the orbital plane occurs [31, 32]. This introduces characteristic amplitude and phase modulations in the GW signal, excites higher-order modes, and modifies the final state of the merger remnant. The precession of the orbital plane is driven by the spin components $\vec{S}_{1\perp}$ and $\vec{S}_{2\perp}$ instantaneously perpendicular to $\vec{L}$, defined as $\vec{S}_{1\perp} = \vec{S}_1 \times \vec{L}$. In a precessing binary, $\vec{L}(t)$ and the orientation of the two spins $\vec{S}_i$ become time-dependent. In the case of simple precession, throughout the inspiral, $\vec{L}(t)$ traces a cone centred around the direction of the total angular momentum $\vec{J}$, which remains approximately spatially fixed [31, 32], i.e., $\vec{J}(t) \simeq \vec{J}_I \rightarrow -\infty \forall t$, where $\vec{J} = \vec{L} + \vec{S}_1 + \vec{S}_2$. The opening angle of this precession cone, $\lambda_L(t)$, is defined as [31]

$$\cos(\lambda_L(t)) \equiv \vec{L}(t) \cdot \vec{J}(t) = \frac{L(t) + S_{||}(t)}{\sqrt{(L(t) + S_{||}(t))^2 + S_{\perp}^2(t)}}.$$  

(1)

where $\vec{S}(t) = \vec{S}_1(t) + \vec{S}_2(t)$ is the total spin of the binary with $S_{\perp}(t) = ||\vec{S}(t) \times \vec{L}(t)||$ and $S_{||}(t) = \vec{S}(t) \cdot \vec{L}(t)$. The precession cone opening angle grows on the precession timescale, which lies between the shorter orbital timescale and the longer radiation reaction timescale, i.e., the time it takes for the binary to merge. This separation of timescales allows us to define approximate closed-form solutions of the post-Newtonian (PN) precession equations in the inspiral [38, 43, 44]. Due to the high dimensionality of the problem, however, precessing waveforms that include inspiral, merger and ringdown are commonly modelled by applying a time-dependent rotation $\vec{R}(t)$ to the waveform modes obtained in a co-precessing frame that tracks the precession of the orbital plane [40, 41], i.e.,

$$h_{\ell m}^P(t) \simeq \sum_{m' = -\ell}^{\ell} \vec{R}_{\ell mm'}(t) h_{\ell mm'}^{\text{co-prec}}(t).$$  

(2)

Such a decomposition is possible due to the approximate decoupling between the precession and inspiral dynam-

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1 We note the difference between the orbital angular momentum $\vec{L}$ and its Newtonian approximation $\vec{L}_N$, however, in this work we will not distinguish between them.
ics [41]. In addition, to simplify the problem, the coprecessing frame modes may be identified with aligned-spin modes [41, 45] as is done for the current generation of phenomenological waveform models [16, 18, 19]. We note, however, that this approximation introduces significant systematic errors due to the neglect of spin-induced mode asymmetries in precessing systems [30]. Additionally, no semi-analytic precessing IMR waveform model currently incorporates NR information in modelling the precession dynamics through merger, again due to the high dimensionality of the problem. Efficient dimensional reduction strategies such as the use of effective parameterisations to reduce the number of spin degrees of freedom may be a way forward to calibrate the precession dynamics in the strong-field regime. Insights from PN theory have previously led to the construction of an effective precession spin \( \chi_p \) defined as [42]

\[
S_p := \max(A_1 S_{1\perp}, A_2 S_{2\perp}),
\]

\[
\chi_p := \frac{S_p}{A_1 m_1^2},
\]

where \( A_1 = 2 + 3/2q \), \( A_2 = 2 + 3q/2 \) and \( S_{\perp} = ||\vec{S}_{\perp}|| \) such that the Kerr limit \( \chi_1 \leq 1 \) is obeyed. It is constructed such that it captures the average amount of precession exhibited by a generically precessing system over many precession cycles defined at some reference time \( t_{\text{ref}} \) during the inspiral. We note that \( \chi_p \) will assume a (slightly) different value depending on \( t_{\text{ref}} \) but this time(frequency)-dependence can be mitigated through the inclusion of additional precession averaged spin effects [46]. An alternative effective parameterisation based on the total spin can be found in [47].

The effective precession spin \( \chi_p \) is regularly used to make statements about the measurement of precession at a certain reference frequency (time) in GW inference, see e.g. [6, 39, 48], and may also present a natural way for making statements about the measurement of precession based on the total spin can be found in [47].

\[
\vec{x}_1(t_{\text{ref}}) \rightarrow \vec{S}_1(t_{\text{ref}})/m_1^2, \quad \vec{x}_2(t_{\text{ref}}) \rightarrow \vec{0}, \quad (7)
\]

where \( \vec{S}_1(t_{\text{ref}}) = m_1^2 \vec{x}_1(t_{\text{ref}}) + m_2^2 \vec{x}_2(t_{\text{ref}}) \). (8)

We find, however, that this mapping can be further improved by assigning it conditionally to either the primary or secondary BH, depending on which BH has the largest in-plane spin magnitude \( S_{1\perp}(t_{\text{ref}}) \) at the reference time. This conditional placement ensures that a binary with an in-plane spin on only one BH is correctly reproduced. Furthermore, we impose the Kerr limit on the BH spin by including appropriate normalisation factors into the definition of \( \vec{x}_\perp \). With these constraints, we obtain the following effective precession spin vector \( \vec{x\perp}(t_{\text{ref}}) \):

\[
\vec{x}_\perp(t_{\text{ref}}) \equiv \begin{cases} 
\frac{\vec{S}_1(t_{\text{ref}})}{m_1^2(1 + q^2 \chi_{1\perp}(t_{\text{ref}})/q^2)^{1/2}}, & \text{if } S_{1\perp}(t_{\text{ref}}) \geq S_{2\perp}(t_{\text{ref}}), \\
\frac{\vec{S}_1(t_{\text{ref}})}{m_2^2(1 + q^2 \chi_{1\perp}(t_{\text{ref}})/q^2)^{1/2}}, & \text{if } S_{1\perp}(t_{\text{ref}}) < S_{2\perp}(t_{\text{ref}}).
\end{cases}
\] (9)

We stress that the mass ratio \( q \) and the spin components along \( \hat{L}(t_{\text{ref}}) \) remain unaltered in this particular mapping. Explicitly, in a Cartesian binary source frame with mode, which is predominantly emitted along \( \hat{L} \), in a precessing system power is transferred from the \( (2, 2) \)-mode to higher-order modes. These higher-order modes can become comparable in strength to the quadrupolar mode in the later inspiral and merger, and some modes, especially the \( (2, \pm 1) \)-modes, can be particularly strong [40]. Therefore, the accurate modelling of higher-order modes is particularly important in precessing systems. We will show in Sec. IV A that the simple \( \chi_p \)-parameterisation of Eqs. (6) fails to accurately reproduce the behaviour of higher-order precessing modes, motivating the introduction for a new effective precession spin vector \( \vec{x}_{\perp} \) to address this issue.

**B. A new effective precession spin**

To aid the calibration of the precessing sector of semi-analytic IMR waveform models, we seek to capture the dominant behaviour through dimensional reduction by reducing the number of in-plane spin components through an effective map. To this end, we introduce a new dimensionless effective precession spin vector, \( \vec{\chi}_{\perp}(t) \in \mathbb{R}^2 \).

Our starting point for the construction of \( \vec{\chi}_{\perp} \) is the opening angle of the precession cone at a reference time \( t = t_{\text{ref}} \), \( \lambda_L(t_{\text{ref}}) \), given by Eq.(1), which captures the amount of precession in the system. We recall that the opening angle depends explicitly on the in-plane spin components through \( S_L(t) \), we therefore seek a mapping such that this quantity is approximately preserved at the reference time at which the mapping is applied. To do so, we first place the in-plane spin projection of the total spin of the system onto the larger black hole, such that

\[
\vec{\chi}_1(t_{\text{ref}}) \rightarrow \vec{S}_1(t_{\text{ref}})/m_1^2, \quad \vec{\chi}_2(t_{\text{ref}}) \rightarrow \vec{0}, \quad (7)
\]

where

\[
\vec{S}_1(t_{\text{ref}}) = m_1^2 \vec{\chi}_1(t_{\text{ref}}) + m_2^2 \vec{\chi}_2(t_{\text{ref}}). \quad (8)
\]
\( \hat{L}(t_{\text{ref}}) \equiv \hat{z} \), we have:

\[
\hat{X}_1 = (\chi_{1x}, \chi_{1y}, \chi_{1z}) \rightarrow \hat{X}_1^\perp = (\chi_{\perp x}, \chi_{\perp y}, \chi_{1z})
\]

(10)

\[
\hat{X}_2 = (\chi_{2x}, \chi_{2y}, \chi_{2z}) \rightarrow \hat{X}_2 = (0, 0, \chi_{2z}),
\]

(11)

for \( S_1 \perp (t_{\text{ref}}) \geq S_2 \perp (t_{\text{ref}}) \) and \( 1 \leftrightarrow 2 \) else. We demonstrate the efficacy of this vectorial parameterisation, in particular for higher-order modes, in Sec. IV.

### III. METHODOLOGY

#### A. Waveforms

To assess the efficacy of the new effective parameterisation Eq. (4), we compare the waveforms obtained from the seven-dimensional system characterised by \((q, \chi_1, \chi_2)\) to the five-dimensional effective system described by \((q, \chi_1, \chi_2, \chi_3)\) as well as to the four-dimensional system given by \((q, \chi_1, \chi_2, \chi_3)\). As we are particularly interested in testing the efficacy of such mappings in the strong-field regime, we use the NR surrogate model NRSur7dq4 [49] as provided through the public gwsurrogate Python package [50] to generate late inspiral-merger-ringdown waveforms for all our analyses. The computational efficiency of this model allows us to assess the mappings over a dense sampling of the intrinsic parameter space. However, due to the limited parameter range of the NR simulations it is built upon, the surrogate is limited to dimensionless spin magnitudes \(|\chi_i| \leq 0.8\) and mass ratios \(q \leq 4\). While precession effects are even more pronounced for higher mass ratios, the importance of the in-plane spin on the smaller BH decreases and therefore we expect any dimensional reduction that is built to capture the dominant precession spin to perform even better in this limit.

The surrogate model represents an interpolant across a discrete set of NR simulations [28, 51, 52]. The precessing waveform modes up to \( \ell \leq 4 \) are obtained by following the strategy outlined in Sec. II, where the co-precessing modes are further decomposed into co-orbital modes to further simplify their structure,

\[
h_{\ell m}^{\text{co-prec}}(t) = e^{i\Omega(t) t} h_{\ell m}^{\text{coorb}}(t),
\]

(12)

where \( \Omega(t) \) is the relative angular velocity relating the two frames [29].

Unlike most other waveform models, the surrogate makes use of four unit quaternions \( \{\hat{q}_0(t), \hat{q}_1(t), \hat{q}_2(t), \hat{q}_3(t)\} \) instead of three Euler angles to describe the precession dynamics of the orbital plane [53]. Importantly, the precessing modes are obtained in an inertial frame corresponding to \( \hat{L}(t_0) = \hat{z} \) at the initial time \( t_0 \), as opposed to the more commonly used \( \hat{J} \)-aligned frame. In this coordinate frame, the \( xy \)-plane coincides with the initial orbital plane of the binary with the \( x \)-axis parallel to the separation vector pointing from the smaller black hole to the larger one.

Due to this binary source frame choice, caution must be taken when interpreting the physical meaning of the quaternions.

The resulting waveforms are of a fixed length, from \( t_0 = -4300M \), the negative sign indicating pre-merger, up to \( t = +100M \) after the merger. This relatively short length makes them unsuitable for describing the waveforms of low mass binaries \( M \lesssim 70M_\odot \), assuming a starting frequency of 20Hz. The surrogate determines the coalescence time \( t_c \) as the peak of the quadrature sum of the mode amplitudes,

\[
t_c = \max_\ell \sqrt{\sum_{\ell m} |h_{\ell m}(t)|^2},
\]

(13)

and shifts the time arrays such that the peak amplitude occurs at \( t_c = 0 \).

#### B. Faithfulness for precessing waveforms

For our quantitative comparisons, we define \( h \) to represent the fully precessing waveform with all six spin degrees of freedom, and \( h_\sigma \) the corresponding waveform produced by an effective mapping, where \( \sigma \in \{p, q, \chi_\perp\} \). Hereafter, we refer to \( h \) as the signal waveform, and to \( h_\sigma \) as the template waveform. The effective mappings are applied at the surrogate initial time \( t_0 \), such that the full and mapped spins are used as initial data to produce the signal and template waveforms respectively.

To quantify how well either mapping reproduces the full waveform, we compute the match (faithfulness) between \( h \) and \( h_\sigma \), which is defined as the noise-weighted inner product between the two waveforms maximised over a time and phase shift of the template waveform:

\[
\mathcal{M}(h, h_\sigma) = \max_{\ell, \varpi, \vartheta, \varphi} \frac{\langle h, h_\sigma \rangle}{\sqrt{\langle h, h \rangle \langle h_\sigma, h_\sigma \rangle}},
\]

(14)

where the inner product is defined as

\[
\langle h, h_\sigma \rangle = 4R \int_{f_{\min}}^{f_{\max}} \tilde{h}(f) \tilde{h}_\sigma^*(f) \frac{S_n(f)}{S_n(f)},
\]

(15)

with \( S_n(f) \) the one-sided power spectral density (PSD) of the detector noise, \( \tilde{h} \) indicates the Fourier transform of \( h \) and \( \ast \) complex conjugation. In what follows, \( h \) and \( h_\sigma \) either denote individual waveform modes \( h_{\ell m} \), or the complex strain, defined as

\[
h(t, \theta, \phi) = \sum_{\ell = -\ell}^{\ell} \sum_{m=-\ell}^{\ell} h_{\ell m}(t) -2 Y_{\ell m}(\theta, \phi),
\]

(16)

where \(-2 Y_{\ell m}(\theta, \phi)\) are the spin-weighted spherical harmonics of spin weight \( s = -2 \) and \((\theta, \phi)\) are the polar and azimuthal angles on the unit sphere in the binary source frame.
| (ℓ,m)-mode | $E_{\ell m}$ threshold |
|-------------|-------------------------|
| (2, ±1)     | $1.0 \times 10^{-6}$    |
| (3, ±3)     | $5.5 \times 10^{-7}$    |

TABLE I. Mode energy thresholds for odd $m$-modes. If the energy of a particular mode is below its threshold, the mode is considered to be numerical noise and excluded from the mode-by-mode match calculation.

To assess how accurately individual modes, in particular higher-order modes, are reproduced under the effective mapping, we compute individual mode-by-mode matches between each spin mapping and the full-spin waveform, i.e. for each pair $(\ell, m)$, $h$, $h_\sigma$ in Eq. (14) are replaced by individual modes $h \rightarrow h_{\ell m}$ and $h_\sigma \rightarrow h_{\sigma, \ell m}$.

As the odd $m$-modes are sourced by mass and spin asymmetries, they are often contaminated by numerical noise for systems with small asymmetries. We therefore employ an additional cut on the energy $E_{\ell m}$ contained in the inertial-frame $(2, \pm 1)$ and $(3, \pm 3)$-modes of the fully spinning mode prior to calculating the match, where the mode energy is given by

$$E_{\ell m} = \frac{1}{16\pi} \int_{t_c}^{t_f} |\dot{h}_{\ell m}(\tau)|d\tau. \quad (17)$$

Based on calculations of the energy contained in those modes for binaries without mass or spin asymmetries, we find the energy thresholds for these modes given by the values listed in Tab. I. Modes with $E_{\ell m}$ less than these values are discarded in the mode-by-mode match calculations performed in Sec. IV A.

We perform mode-by-mode match calculations using both white noise, i.e. $S_n(f) = 1$ and the projected aLIGO PSD for the fourth observing run [54], denoted $\mathcal{M}_{\text{White}}$ and $\mathcal{M}_{\text{O4}}$ respectively. The white noise matches are to assess the systematic errors induced by the mappings in the absence of detector-specific frequency sensitivities, whilst the PSD-weighted matches demonstrate the effect for a given detector. For the detector PSD matches we choose a starting frequency of $f_{\text{min}} = 20$Hz, and truncate the waveforms at $t = 50M$ after the peak as determined by Eq. (13) to remove post-merger numerical noise.

Whilst the individual mode matches allow us to assess how well higher-order modes in particular are captured by the lower dimensional spin parameterisation, GW detectors measure the strain, which also depends on extrinsic parameters of the source such as the luminosity distance, the effective polarisation angle $\kappa$ [55] and the binary inclination $i$ relative to the line-of-sight of an observer.

Following Refs. [20, 39, 56], we compute the strain match by analytically optimising over the template polarisation angle $\kappa_{\sigma}$ and numerically optimising over the template reference phase $\phi_{0\sigma}$ and template coalescence time $t_{\text{co}}$

$$\mathcal{M}_{\text{strain}}(M, i, \phi_0, \kappa) = \max_{t_{\text{co}}, \phi_{0\sigma}, \kappa_{\sigma}} \frac{\langle h, h_\sigma \rangle}{\sqrt{\langle h, h \rangle \langle h_\sigma, h_\sigma \rangle}} \bigg|_{t_{\text{co}} = t_{\text{ref}}}. \quad (18)$$

We do not optimise over any intrinsic parameters. We note that Eq. (18) still depends on the signal polarisation $\kappa$ and reference phase $\phi_0$. By averaging over these two angles, we obtain the sky-and-polarisation-averaged strain match,

$$\overline{\mathcal{M}}_{\text{strain}}(M, i) = \frac{1}{8\pi^2} \int_0^{2\pi} d\kappa \int_0^{2\pi} d\phi_0 \mathcal{M}_{\text{strain}}(M, i, \phi_0, \kappa). \quad (19)$$

Additionally, to account for the correlation between low matches and low signal-to-noise ratio (SNR), we also compute the SNR-weighted strain match [20, 39] given by

$$\overline{\mathcal{M}}_{\text{SNR}}(M, i) = \left( \sum_i \frac{(\mathcal{M}(h(h_\sigma)))^3 \langle h_i, h_\sigma \rangle^{3/2}}{\sum_i \langle h_i, h_i \rangle^{3/2}} \right)^{1/3}, \quad (20)$$

where the sum is over a discrete range of source polarisations $\kappa$ and initial phases $\phi_0$ as detailed in Sec. III C.

We note that we do not apply the post-merger truncation at $t = 50M$, nor do we impose the mode energy thresholds of Tab. I when computing the sky-and-polarisation-averaged and the SNR-weighted strain matches. For strain matches we take into account all modes up to $\ell = 4$ as provided by the NR surrogate.

Lastly, rather than showing the agreement between two waveforms, it can be advantageous to quantify the disagreement through the mismatch $\mathcal{M}\mathcal{M}$ instead:

$$\overline{\mathcal{M}}\overline{\mathcal{M}}_{\text{strain}} = 1 - \overline{\mathcal{M}}_{\text{strain}}, \quad (21)$$
$$\overline{\mathcal{M}}\overline{\mathcal{M}}_{\text{SNR}} = 1 - \overline{\mathcal{M}}_{\text{SNR}}. \quad (22)$$

C. Binary Configurations

The mode-by-mode matches are computed for a large number of mass ratios and spins that systematically sample the validity range of the surrogate model with the details provided in the second column of Tab. II. We choose the initial time as the reference time, i.e. $t_{\text{ref}} \equiv t_0$, and sample the initial spins in a spherical coordinate systems using the spin magnitudes $|\vec{\chi}_i|$, the azimuthal orientations $\phi_i = \arccos(\hat{S}_i \cdot \hat{x})$, and the cosine of the tilt angles $\cos(\theta_i) = \hat{S}_i \cdot \hat{L}$. Specifically, we keep the initial azimuthal orientation of the spin of the larger BH $\phi_1$ of $\vec{\chi}_1 \parallel$ fixed, whilst rotating $\vec{\chi}_2 \perp$ to achieve a range of angular azimuthal separations, and vary the initial tilt angles $\theta_i$ systematically. Further, we only choose configurations with at least one spinning BH, demanding that at least one BH has a non-zero in-plane spin, thereby excluding aligned-spin or nonspinning binaries. This amounts to a total of 47,136 unique binary configurations in terms of
their intrinsic parameters \( \{ q, \chi_1, \chi_2 \} \). When considering a detector PSD, we additionally consider three values of the total mass, 75, 150 and 250 \( M_\odot \).

For the strain matches as given in Eqn. (19) and (20), additional extrinsic parameters, namely binary inclination \( \iota \), initial phase \( \phi_0 \), and polarisation \( \kappa \), need to be taken into account. In such high dimensions, systematic sampling becomes unfeasible. Therefore, for the sky-and-polarisation-averaged matches we draw the intrinsic and extrinsic binary parameters from random uniform distributions as detailed in the third column of Tab. II, considering a total of 20,833 unique binary configurations.

For the SNR-weighted strain matches, we first draw 100 binary configurations randomly from the 20,833 used to compute the sky-and-polarisation-averaged strain matches, only considering their intrinsic parameters, \( \{ q, |\chi_1|, |\chi_2|, \cos(\theta_1), \cos(\theta_2), \phi_1, \phi_2 \} \). We fix the source inclination at a moderate inclination of \( \iota = \pi/3 \). As detailed in the last column of Tab. II, for each binary configuration we choose eight initial phase and four polarisation values and compute 32 matches \( \mathcal{M}(h, h_\sigma) \), one for each pair \( \{ \phi_0, \kappa \} \), which are then summed into a single SNR-weighted match for each binary configuration as per Eq. (20). We repeat this calculation for each of the total masses detailed in Tab. II, noting that we use the same 100 intrinsic binary configurations for each \( M \).

This yields 800 SNR-weighted strain matches for each mapping \( \sigma \in \{ \chi_p, \chi_\perp \} \).

### IV. RESULTS

#### A. Mode Analysis

We first assess how well the vectorial effective spin \( \chi_\perp \) reproduces individual modes, in particular higher-order modes, for different mass ratios. In Fig. 1, we show the \( (2, \pm 1) \)-modes for a fiducial precessing binary with \( q = 1.4 \) and initial spins \( \chi_1 = (0.075, 0.043, 0.05), \chi_2 = (-0.346, 0.6, -0.4) \). We find that \( \chi_\perp \) (orange) captures the fully precessing modes (blue) significantly better than a simple \( \chi_p \) parameterisation (purple). In particular, we see that unlike \( \chi_p \), \( \chi_\perp \) reproduces the amplitude and phasing of the mode oscillations on the orbital timescale and the amplitude modulations on the precession timescale much more faithfully. Additionally, amplitude modulations in the ringdown signal, which are completely missed in the \( \chi_p \)-parameterisation, are much better captured. We note that the precession of this fiducial binary is dominated by the secondary BH spin – a region in the spin parameter space where \( \chi_p \) knowingly performs poorly. Additional
examples are presented in Figs. 14, 15 and 16 in App. B, including a mass ratio \( q = 3 \) binary with the precession dominated by the primary BH spin in Fig. 15, where we still observe noticeable improvements with \( \chi_\perp \) over \( \chi_p \).

To quantify the efficacy of \( \chi_\perp \) across the parameter space, we compute white noise mode-by-mode matches for the (2, \( \pm 2 \))-modes and a selection of higher-order modes for all binaries listed in the second column of Tab. II. Fig. 2 shows the cumulative match results for the (2, 2) and (2, 1)-modes for mass ratio \( q = 1 \) and \( q = 3 \). Results for additional modes and mass ratios are shown in Fig. 13 in App. A.

We expect the \( \chi_p \)-parameterisation to perform well at replicating the dominant (2, \( \pm 2 \))-mode behaviour, and indeed we see similar results in this mode for both parameterisations, if slightly improved with the new effective spin, except for the equal mass case, where we find a more marked improvement. We attribute this to the fact that \( \chi_p \) is designed to replicate the average precession rate, but in equal mass configurations the in-plane spin vectors precess at the same rate and become orientationally locked, which is not captured correctly by \( \chi_p \). Additionally, \( \chi_\perp \) takes into account the in-plane spins on both black holes, while \( \chi_p \) selects only the larger spin component leading to a systematic underestimation of the total in-plane spin for equal mass cases.

We observe the most dramatic improvements in the (2, \( \pm 1 \)) modes. For example, for \( q = 3 \) shown in Fig. 2, the percentage of matches below 0.99 decreases dramatically from 94.73% with \( \chi_p \) to 8.2% with \( \chi_\perp \). Note that the long tails toward very low matches for the \( \chi_p \)-parameterisation, and the comparatively short ones of \( \chi_\perp \), are a generic feature across all higher-order modes we analyzed, suggesting that \( \chi_\perp \) better replicates the higher mode behaviour even when it performs at its worst.

Intriguingly, for the (2, \( \pm 1 \)) and (4, \( \pm 4 \))-modes, both parameterisations perform worst at \( q \sim 1.5 \), after which their performance improves with increasing mass ratio. To further investigate this intermediate region between the equal mass regime and higher mass ratios where the secondary spin becomes less important, we performed additional white noise matches at mass ratios \( q \in \{1.2, 1.4, 1.6, 1.8\} \). We find that the performance of both spin mappings improves with increasing mass ratio for the (2, \( \pm 2 \)) and (3, \( \pm 3 \))-modes, with \( \chi_\perp \) consistently outperforming \( \chi_p \). We also find that \( \chi_\perp \) performs worst around \( q \sim 1.4 \) for the (2, \( \pm 1 \)) and (4, \( \pm 4 \))-modes, but that the distributions for both mappings is fairly flat between \( q = 1.2 \) and \( q = 2 \), and even at its worst \( \chi_\perp \) still vastly outperforms \( \chi_p \). For example, in the (2, 1)-mode, at \( q = 1.4 \), the percentage of matches below 0.99 is 100% with \( \chi_p \), and only 26.35% with \( \chi_\perp \). We also note that we find only minor differences between the positive and negative \( m \)-modes for both mappings, and neither performs consistently worse at replicating either positive or negative \( m \)-modes.

Additionally, our new mapping shows moderate improvements for the (3, \( \pm 3 \)) and striking improvements for the (4, \( \pm 4 \))-modes, with the improvements particularly marked at equal mass and at our highest mass ratio \( q = 4 \). In summary, we find that the \( \chi_\perp \)-parameterisation performs consistently better than \( \chi_p \) for every mass ratio and across all modes, and in particular for odd \( m \)-modes.

In addition to the white noise matches, we repeat the analysis using the projected O4 aLIGO PSD [54] with \( f_{\text{min}} = 20 \text{Hz} \) for three total masses \( M \in \{75, 150, 250\} M_\odot \) compatible with the fixed length of the NR surrogate. For the PSD mode-by-mode matches, we obtain qualitatively similar results to the white noise matches as shown in Fig. 13 in App. A. All of the matches improve slightly compared to the white noise matches across both mappings due to the frequency weighting of the PSD, but the features of our results and conclusions remain the same: The \( \chi_\perp \) mapping significantly improves upon \( \chi_p \) for the (2, \( \pm 1 \))- and (4, \( \pm 4 \))-modes, with moderate improvements for the (3, \( \pm 3 \)), and comparable if slightly better performance for the (2, \( \pm 2 \))-modes.

### B. Strain Analysis

In the previous section we have demonstrated the improvement of \( \chi_\perp \) over \( \chi_p \) at the level of individual \( h_{\text{inr}} \)-modes. We now assess the degree to which the improvement in the higher-order modes impacts the strain. Fig. 3 shows the strain for the fiducial binary at an inclination of \( \iota = \pi/3 \). The excellent agreement between the fully precessing waveform (blue) and the one parameterised by \( \chi_\perp \) (orange) throughout the late inspiral as well as the merger-ringdown is clearly visible. To quantify this agreement, we first compute the sky-and-polarisation-averaged strain mismatches for 20,833 binary configurations as detailed in Tab. II using the O4 PSD and \( f_{\text{min}} = 20 \text{Hz} \). Our results for both effective parameterisations are shown in Fig. 4. Using \( \chi_\perp \) rather than \( \chi_p \), we find a median improvement of more than one order of magnitude from \( 4 \times 10^{-3} \) to \( 2 \times 10^{-4} \). Furthermore, we note the non-negligible tail of extremely low mismatches below \( 10^{-6} \) for \( \chi_\perp \).

As low matches are often correlated with low SNRs and, therefore, with a lower detection probability, we also compute the SNR-weighted mismatch Eq. (22) for 100 randomly drawn intrinsic binary configurations as given in the fourth column of Tab. II for a moderate inclination of \( \iota = \pi/3 \) at \( t_0 \). Similarly to the sky-and-polarisation-averaged strain mismatches, we see an improvement of around one order of magnitude when using the \( \chi_\perp \)-parameterisation instead of \( \chi_p \), as shown in Fig. 5. The worst two cases for each parameterisation are highlighted in both panels. We see that the worst cases for \( \chi_\perp \) (red and orange) perform similarly under both mappings, if slightly better with the new \( \chi_\perp \)-mapping. These cases both have a mass ratio of \( q \sim 1.5 \), which as noted previously in Sec. IV A, is a mass ratio where both parameterisations perform worst. The worst cases for \( \chi_p \) (purple and blue) on the other hand, show significant im-
FIG. 1. Amplitude of the $(2,1)$-mode (left) and the $(2,1)$-mode (right) for a fiducial precessing binary with $q = 1.4$, $\vec{\chi}_1(t_0) = (0.075, 0.043, 0.05)$ and $\vec{\chi}_2(t_0) = (-0.346, 0.6, -0.4)$. The fully precessing signal waveform is shown in blue, and the template waveforms parameterised by $\chi_p$ and $\chi_\perp$ are shown in purple and orange, respectively. The $\chi_\perp$ reproduces the phenomenology of this mode markedly better than the $\chi_p$ mapping, especially in the merger-ringdown portion of the waveform.

FIG. 2. Cumulative histograms of white noise matches for the $(2,2)$-mode (top row) and the $(2,1)$-mode (bottom row) for 9,120 binaries with ratios $q = 1$ (left column) and 9,504 binaries with $q = 3$ (right column). Details of how these binaries are systematically sampled can be found in Tab. II. Orange histograms show the results using the $\chi_\perp$-parameterisation, purple ones $\chi_p$. The dashed horizontal lines indicate the percentage of matches below 0.99. Using $\chi_\perp$, we see a clear improvement over the $\chi_p$-mapping for the $(2,1)$-mode, and comparable if slightly better performance for the $(2,2)$-mode. Results for additional modes and mass ratios are presented in Fig. 12 in App. A.
with both the $\chi$ cannot be seen, it is indistinguishable from the orange line. The $\chi$-mapping reproduces the strain amplitude much more faithfully than $\chi_p$, especially in the late inspiral portion of the waveform. Note that in the late inspiral, where the blue line cannot be seen, it is indistinguishable from the orange line.

The better agreement between the initial opening angles suggests that the spins themselves are captured more faithfully. To show this, as a second diagnostic we compare the spin evolutions of a fully precessing binary with its effective counterparts. Figure 7 shows the time evolution of the total in-plane spin $S_\perp$ (blue) for the fully precessing fiducial binary and those of the $\chi_\perp$ (orange) and $\chi_p$ (purple) parameterisations for the fiducial binary. We obtain these by transforming the spin evolutions in the inertial frame to the co-precessing frame using the quaternions. It is evident that the two-dimensional $\chi_\perp$-mapping represents the full spin dynamics much more faithfully than $\chi_p$.

As a third diagnostic, we investigate how faithfully both mappings reproduce the fully spinning precession dynamics, which is represented by the unit quaternions $\hat{q}_i$. In Fig. 8 we show the time evolution of the four unit quaternion components of the fiducial binary. The new effective spin mapping $\chi_\perp$ clearly replicates the time evolution of each quaternion component much more accurately than $\chi_p$, with the most dramatic improvement observed for $\hat{q}_1$ and $\hat{q}_2$.

To quantify the improvement in replicating the precession dynamics, we perform a match calculation for each of the four quaternion components $\hat{q}_i$, $i \in \{0, 1, 2, 3\}$, similar to the white match calculation in Eq. (14) but replacing the waveforms $h$ and $h_s$ with the quaternion components, $M_{\hat{q}_i} = M(\hat{q}_i, \hat{q}_{\sigma,i})$, where $\sigma$ is the quaternion component produced by the effective mapped system, with $\sigma \in \{\chi_p, \chi_\perp\}$.

We compute the quaternion matches for the same 20,833 binaries used in the sky-and-polarisation-averaged strain match calculations. Figure 9 shows the results for $\hat{q}_1$ (left) and $\hat{q}_2$ (right), which show the largest improvements: the percentage of matches below 0.99 improves

\[
\Delta \lambda_L(t_0) \equiv \lambda_L(t_0) - \lambda_{\sigma L}(t_0),
\]

where $\lambda_L$ is given by Eq. (1). The definition of $\lambda_{\sigma L}$ is the same as for $\lambda_L$, but replaces $S_\perp$ with $S_{\sigma \perp}$, where $S_{\perp}$ is the initial total in-plane spin magnitude before the mapping and $S_{\sigma \perp}$ is the total in-plane spin magnitude after the mapping. All quantities are evaluated at the initial time $t_0$, and we approximate $L$ by its Newtonian value $L = \mu \sqrt{M r}$, where $\mu = m_1 m_2 / M$ is the reduced mass and $r = M^{1/3} \omega_{\text{orb}}^{2/3}$ with $\omega_{\text{orb}}$ the orbital angular frequency.

Figure 6 shows $\Delta \lambda_L(t_0)$ against the strain mismatch for each of the 20,833 binaries, calculated with both the $\chi_\perp$ (orange) and $\chi_p$ (purple) effective spin mappings. For the new mapping, we see a clear correlation between lower values of $\Delta \lambda_L(t_0)$ and lower strain mismatches. Overall, the $\chi_\perp$-parameterisation yields a more accurate initial cone opening angle resulting in a more faithful representation of the fully precessing waveform.

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from 98.65% with \( \chi_p \) to 46.33% with \( \hat{\chi}_\perp \) for \( \hat{q}_1 \), and for \( \hat{q}_2 \) it improves from 95.71% to 46.37%. We see a negligible difference in the results for \( \hat{q}_0 \), which is well reproduced by both spin mappings: none of cases have a match value below 0.99. We see a small improvement in the results for \( \hat{q}_3 \), with the percentage of quaternion matches below 0.99 dropping from 40.5% for \( \chi_p \) to 34.21% for \( \hat{\chi}_\perp \). These results indicate that the observed improvements when using \( \hat{\chi}_\perp \) can indeed be attributed to a more faithful representation of precession dynamics itself.

**C. Accuracy of the Final Spin and Recoil**

Finally, we quantify how well the \( \hat{\chi}_\perp \)-parameterisation is able to reproduce the final spin and recoil of the rem-
nant black hole. The final mass and spin of the remnant determine the quasinormal modes of the ringdown [57–60], so it is therefore crucial to understand the accuracy with which the final state can be replicated by the reduced set of spin parameters. We will focus on the final spin estimates as previous comparisons against NR simulations have shown that the final mass estimate is only very weakly dependent on precession [61].

To evaluate the final spin using the surrogate model we first evolve the BH spins in the inertial frame from \( \hat{t}_0 \) to a time \( t = -100M \) before merger, which are then used to evaluate the remnant fits of Ref. [62] via the public PYTHON package surfinBH [63]. The same procedure is followed to obtain the results under the two spin parameterisations, where either effective spin map is applied at the initial time \( t_0 \). We evaluate the remnant spin for the 20,833 binary configuration of column two in Tab. II.

We assess the accuracy of the final state under the two mappings by calculating the differences in the remnant spin magnitude \( \Delta \chi_f \), the final spin tilt angle \( \Delta \theta_f \), the azimuthal spin angle \( \Delta \phi_f \), the recoil velocity \( \Delta v_f \) and its tilt angle \( \Delta \theta_{v_f} \), defined as

\[
\Delta \chi_f = ||\hat{\chi}_f|| - ||\hat{\chi}_{f\sigma}||, \\
\Delta \theta_f = ||\arccos(\hat{z} \cdot \hat{\chi}_f) - \arccos(\hat{z} \cdot \hat{\chi}_{f\sigma})||, \\
\Delta \phi_f = \arccos(\hat{\chi}_{f\perp} \cdot \hat{\chi}_{f\sigma\perp}), \\
\Delta v_f = ||\hat{\sigma}_f|| - ||\hat{\sigma}_{f\sigma}||, \\
\Delta \theta_{v_f} = ||\arccos(\hat{z} \cdot \hat{v}_f) - \arccos(\hat{z} \cdot \hat{v}_{f\sigma})||,
\]

where \( \sigma \in [\chi_p, \chi_{\perp}] \), and \( \hat{\chi}_{f\perp} \) indicates the \( xy \)-components of the final spin vector in the inertial frame. We note that the remnant spin and recoil velocities are also returned in the inertial coordinate frame of the NR surrogate, which has no particular physical mean-

FIG. 8. Time evolution of the quaternion components \( \hat{q}_0 \) and \( \hat{q}_3 \) (left), and \( \hat{q}_1 \) and \( \hat{q}_2 \) (right) for the fiducial precessing binary. The system parameterised by \( \chi_{\perp} \) (orange) reproduces the precession dynamics of the fully spinning system (blue) much more faithfully than the \( \chi_p \) mapping (purple).

FIG. 9. Cumulative distribution of matches for two of the four quaternion elements \( \hat{q}_1 \) (left) and \( \hat{q}_2 \) (right), between the fully precessing dynamics and each of the \( \chi_{\perp} \) (orange) and \( \chi_p \)-mapped (purple) systems. The dashed horizontal lines indicate the percentages of matches which are below 0.99 for each effective mapping. We see significant improvements for the \( \chi_{\perp} \)-parameterisation over \( \chi_p \).
ing post-merger. However, as we are computing relative differences in magnitudes and angles, this gauge choice has no effect on the results presented here.

We find marginal improvements in the accuracy of the final spin magnitude and tilt angle using the \( \chi_\perp \) mapping as opposed to \( \chi_p \). The median tilt angle difference \( \Delta \theta_f \) improves slightly from \( 2.30 \times 10^{-3} \) rad with \( \chi_p \) to \( 1.43 \times 10^{-3} \) rad with \( \chi_\perp \); the absolute value \( \Delta \chi_f \) also improves slightly from \( 1.75 \times 10^{-3} \) for \( \chi_p \) to \( 9.96 \times 10^{-4} \) for \( \chi_\perp \). However, the largest improvement is found for the azimuthal angle \( \Delta \phi_f \), which encapsulates the difference in the relative angle in the \( xy \)-plane of the inertial frame as shown in Fig. 10. We see a dramatic difference between the two mappings, with \( \chi_\perp \) effectively reproducing the azimuthal orientation with a median error of less than 0.1 rad, whereas the \( \chi_p \) mapping poorly replicates the azimuthal orientation with a median difference of more than 1 rad, and a significant proportion of differences around \( \Delta \phi = \pi \). We also note the significantly long tail of the \( \chi_\perp \) histogram towards angle differences of zero.

![Image](image_url)

**FIG. 10.** Error in the azimuthal angle of the final spin state, \( \Delta \phi_f \), in radians, between the final spin state produced by the fully precessing waveform, \( \chi_f \), and the resulting final spin state of the waveform produced by the \( \chi_\perp \) (orange) and \( \chi_p \) (purple) mappings. The new effective spin \( \chi_\perp \) reproduces the azimuthal angle of the the remnant spin much more accurately, reducing the median error to less than 0.1 rad.

We now analyze the effect of the two mappings on the recoil velocity \( v_f \) of the final black hole. For the recoil velocity tilt angle \( \Delta \theta_{v_f} \), i.e. the polar direction of the recoil, we find a large improvement from a median of error of 0.67 rad for \( \chi_p \) to 0.08 rad for \( \chi_\perp \) as shown in the right panel of Fig. 11. For the recoil velocity itself, we only find a modest improvement in \( \Delta v_f \) from a median error of \( 3.81 \times 10^{-4} \) c for \( \chi_p \) to \( 1.47 \times 10^{-4} \) c for \( \chi_\perp \), corresponding to an improvement in accuracy of \( \sim 70 \) km/s on average.

To summarise, overall the \( \chi_\perp \)-parameterisations reproduces the final state, in particular the orientation of the final spin and the direction of the recoil, much more accurately. Both effective spin parameterisations perform similarly in determining the final spin magnitude.

We note that the comparison using \( \chi_p \) is not directly comparable to the definition of the final spin used in semi-analytical waveform models, which use (a variety of) in-plane spin corrections to modify the final spin of an aligned-spin binary [19]. The current generation of precessing Phenom models [16, 18, 39] apply a correction of the form \( S_p/M_f^2 \), where \( M_f \) is the remnant mass and \( S_p \) is an effective in-plane spin contribution. In [16, 18], \( S_p \) is taken to be defined as in Eq. (3), which is similar to the results presented here obtained by applying the \( \chi_p \)-mapping [61, 64]. For the more recent model presented in [19], a range of different final spin mappings have been implemented including the \( \chi_p \) mapping as well as a precession-averaged mapping that attempts to account for the change in the aligned-spin components due to nutation effects. The EOB models [20] employ the final spin fits of [65] which introduce corrections to the aligned-spin final state fits that depend on the angle between the two in-plane spin vectors and the projection of the spins along the orbital angular momentum. As discussed in [20, 58] and above, a crucial choice is the separation at which the spins are used to evaluate the final state, taken to be \( r = 10M \) in [20]. This approach enables the effective one body models to account for the evolution of the spin vectors ensuring that the same waveform is produced irrespective of the initial separation.

**V. DISCUSSION**

The inclusion of fully relativistic precession effects in semi-analytical IMR waveform models in the strong-field regime remains a challenging problem, with none of the current waveform models from either the Phenom or the EOB waveform family including calibration to NR in the precessing sector. The high dimensionality of the precessing BBH parameter space obfuscates a clear path for calibration. Effective spin parameterisations to reduce the number of spin degrees of freedom are a promising way forward to including fully relativistic precession in the strong-field regime. Previously, a scalar quantity, \( \chi_p \), was introduced to this effect but its efficacy was only demonstrated for the inspiral regime [42]. Here, we have assessed its applicability in the strong-field regime. Crucially, we find that whilst \( \chi_p \) does faithfully represent the \( (2, 2) \)-mode of the majority of fully precessing systems, it does not accurately reproduce higher-order modes. Since higher-order modes are excited by mass and spin asymmetries, which can be very pronounced for precessing binaries, they carry crucial parameter degeneracy breaking power [66–71] making the accurate modelling of higher-order modes critical. Therefore, NR calibration through a simple \( \chi_p \)-parameterisation is unlikely to be sufficient to satisfy the accuracy requirements for future GW observations.

To improve upon the shortcomings of \( \chi_p \), we have in-
introduced a new two-dimensional effective precession spin vector, $\vec{\chi}_\perp$, and have performed extensive studies comparing the efficacy of $\vec{\chi}_\perp$ to that of $\chi_p$ in the strong-field regime using the NR surrogate waveform model NR85x7dq4 [49]. When analysing individual $h_{tm}$-modes, in particular the (2, ±1)-modes, we find that $\vec{\chi}_\perp$ performs significantly better than $\chi_p$, but both effective parameterisations yield comparable results for the quadrupolar (2, ±2)-modes. Correspondingly, we also find a significant improvement in the precessing strain matches with the new mapping, from which we conclude that the improved efficacy of $\vec{\chi}_\perp$ over $\chi_p$ for the higher-order modes has a significant effect on the accuracy of the overall strain, demonstrating the importance of accurately modelling higher-order modes in precessing systems. Furthermore, we find that $\vec{\chi}_\perp$ performs better compared to $\chi_p$ in the equal mass limit (see Fig. 2). In this limit, the BH spins precess at the same rate, locked in orientation relative to each other. The parameter $\chi_p$, which is defined to mimic the average rate of precession, performs knowingly poorly in this limit [42]: $\vec{\chi}_\perp$, on the other hand, is constructed such that it approximates the total in-plane spin of the fully precessing system a some reference time, leading to a significantly improved behaviour in the equal-mass limit as anticipated from PN theory [31, 72]. As expected, we find that $\vec{\chi}_\perp$ performs increasingly better for larger mass ratios $q \gtrsim 2$, where the spin on the smaller BH becomes negligible and hence the approximation with a single in-plane spin becomes more accurate. However, in the intermediate region between these two regimes, whilst still a considerable improvement upon $\chi_p$, we find a small drop in accuracy at a mass ratio of $q \sim 1.4$, where two-spin effects are important but are not fully captured in $\vec{\chi}_\perp$.

We have further demonstrated that the overall improvement relative to a $\chi_p$-parameterisation can be attributed to a more accurate replication of the precession dynamics itself when using the $\vec{\chi}_\perp$-parameterisation. Indeed, in the case where only one of the two objects has non-zero in-plane spin components, the full dynamics are returned exactly, which is not the case for $\chi_p$. The accurate capture of the precession dynamics of particular interest as a natural way for incorporating strong-field precession information into waveform models is through calibrating the precession dynamics itself, i.e. the rotation operator $\mathbf{R}$ of Eq. (2) or, equivalently, the quaternions.

Lastly, we have also quantified how well the $\vec{\chi}_\perp$-mapping is able to replicate the final spin and recoil velocity of the remnant black hole. We find a considerable improvement in the accuracy with which we replicate the azimuthal direction of the remnant spin, and moderate improvements in the accuracy of the magnitude and direction of the recoil velocity. This suggests our $\vec{\chi}_\perp$ mapping is better able to replicate the final direction of GW emission, compared to $\chi_p$. Previous work has demonstrated that the relative orientation of the in-plane spins at merger plays a crucial role in determining the final state properties [73–76]. We attribute the observed improvements to the (partial) incorporation of two-spin effects, which are crucial for determining the recoil direction and velocity of the final BH.

Despite its significantly better performance in all areas, there are also caveats associated with $\vec{\chi}_\perp$: (i) For spin configurations with similar in-plane spin magnitudes, i.e. $S_{1\perp} \simeq S_{2\perp}$, we expect larger mismatches due to, by construction, the neglect of larger in-plane spin-spin couplings. (ii) We have normalised $\vec{\chi}_\perp$ such that the Kerr limit is not violated. Consequently, for binaries with large spin magnitudes, $\vec{\chi}_\perp$ will underestimate the magnitude of the in-plane spin in the system. Due to the limited spin parameter range of the surrogate, we have not been able to fully quantify the effect of this on the performance of the mapping. (iii) The conditional placement of $\vec{\chi}_\perp$ on either of the two black holes introduces a discontinuity, in the sense that waveforms with $\vec{\chi}_\perp$ placed on the...
primary BH show slightly different features than those with $\vec{\chi}_\perp$ on the secondary BH. We note that all of our $\vec{\chi}_\perp$-mapped individual waveforms are physical and continuous, but that a shift in phenomenological features can occur between binary configurations with $S_{1\perp} = S_{2\perp} + \epsilon$, where $\vec{\chi}_\perp$ is placed on the primary BH, against the same binary configuration with slightly smaller $S_{1\perp} = S_{2\perp} - \epsilon$, where $\vec{\chi}_\perp$ will be placed on the secondary BH. With this in mind, we tested the performance of $\vec{\chi}_\perp$ without the conditional placement. We recalculated the sky-and-polarisation-averaged strain matches shown in Fig. 4 with $\vec{\chi}_\perp$ always placed upon the primary BH irrespective of whether the precession is dominated by the primary or secondary BH, and indeed find little difference from the original $\vec{\chi}_\perp$ strain match distribution, with the median mismatch increasing minimally from $2 \times 10^{-4}$ to $2.08 \times 10^{-4}$. Additionally, we recalculated the white noise mode-by-mode matches of equation (14), again with $\vec{\chi}_\perp$ always placed on the primary BH. Whilst we find little difference between the $(2, \pm 2)$-mode results for $\vec{\chi}_\perp$ with and without conditional placement, we find that it has a marked effect on the results for higher-order modes. For example, in the $(2, 1)$-mode at mass ratio $q = 3$, the percentage of mismatches below 0.99 using $\vec{\chi}_\perp$ without conditional placement rises to 41.9%, compared to just 9.1% if we include the conditional placement (under the $\chi_p$-mapping the value is 98.8%). We therefore conclude that for higher-order modes, it is crucial to accurately capture spin asymmetries by placing the effective spin appropriately, to achieve an accurate mapped waveform mode.

Lastly, we have also tested whether the improvements found by using $\vec{\chi}_\perp$ over $\chi_p$ are entirely due to the conditional placement, and whether a conditional placement of $\chi_p$ would have similar effects. An example of imposing this condition also on $\chi_p$ is shown in Fig. 16 in App. B for the fiducial binary, and we do indeed see that the phenomenology is captured better. To quantify the improvement in the performance of $\chi_p$ when imposing conditional placement, we recalculated the sky-and-polarisation-averaged strain matches shown in Fig. 4 with an analogous conditional placement for $\chi_p$. We find only a small improvement compared to the $\chi_p$-mapping without conditional placement, with the median strain mismatch improving from $4 \times 10^{-3}$ to $3.4 \times 10^{-3}$, compared to a median of $2 \times 10^{-4}$ with $\vec{\chi}_\perp$. We therefore surmise that the improvements we see in the performance of $\vec{\chi}_\perp$ over $\chi_p$ are due to a combination of both the new parameterisation itself, and the conditional placement, and that both are a necessary requirement to reproduce accurate precessing higher-order waveform modes.

Finally, we also note that the efficacy of $\vec{\chi}_\perp$ has not been investigated for the special case of transitional precession, which leads to the tumbling of the total angular momentum $\mathbf{J}$ when $L \simeq 3$ and $\dot{L} = -\mathbf{S}$. As with all effective mapping that neglect some spin contributions, however, we expect that the fine-tuned conditions needed for the occurrence of the transitional precession phase are not preserved under the mapping.

In conclusion, our results demonstrate that by introducing the two-dimensional vector quantity $\vec{\chi}_\perp$, which partially accounts for two-spin effects, we can accurately reproduce the waveforms of fully precessing binaries, and in particular their higher-order modes, in the strong-field regime across a wide range of the BBH parameter space. The effective reduction of four in-plane spin components to two provides a clear and tractable path forward to meaningfully incorporating precession effects in the strong-field regime into semi-analytic waveform models with higher-order modes, which we leave to future work.

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Appendix A: Complete Mode-by-Mode Match Results

Here we present the complete results of the white noise and O4 aLIGO PSD matches, for which a selection was presented in Fig. 2. As described in Sec. III C, we systematically sampled a total of 47,136 intrinsic binary configurations $\{q, \vec{\chi}_1, \vec{\chi}_2\}$, across the parameter range of the NR surrogate. For the PSD matches, which also require a total mass scale, we choose three masses $M \in \{75, 150, 250\} M_\odot$ compatible with the fixed length of the surrogate waveforms, bringing the total sampled binaries to 141, 408. Full details of the sampling are given in Tab. II.

The white match results are shown in Fig. 12. We split the results by mass ratio $q \in [1, 1.5, 2, 3, 4]$ and mode $(\ell, m) \in [(2 \pm 2), (2 \pm 1), (3 \pm 3), (4, \pm 4)]$. We show the percentages of matches below 0.99 between the fully spinning waveform, and the waveforms produced by each of the two effective spins, $\vec{\chi}_\perp$ in orange, and $\chi_p$ in purple. We first note that at the match threshold of 0.99, we see an improvement by using $\vec{\chi}_\perp$ over $\chi_p$, across all mass ratios and modes. These improvements are particularly dramatic for higher modes, particularly the $(2, \pm 1)$-modes. For example, at mass ratio $q = 4$, the percentage of $(2, 1)$-mode matches below 0.99 using the $\chi_p$ parameterisation is 96.3%, which improves dramatically with $\vec{\chi}_\perp$ to just 4.3%. The parameterisations perform more similarly for the $(2, \pm 2)$-modes, but even for the quadrupolar modes we see small improvements. For example, at
mass ratio $q = 1.5$ we see the percentage of $(2, 2)$-mode matches below 0.99 improving from 8.5% with $\chi_p$ to 4.7% with $\chi_p^\perp$.

Using the O4 aLIGO PSD, we similarly split the results by mass ratio and mode, and additionally by the total mass of the system. The results are shown in Fig. 13. The PSD-weighted match results are qualitatively very similar to the white match results, albeit with small improvements across all matches due to the frequency weighting of the PSD, with only minor differences between each total mass.

**Appendix B: Additional Examples**

Here we provide additional examples of waveform modes and corresponding precession dynamics produced using the effective spin mappings, both $\chi_p$ and $\chi_p^\perp$. For each of the three binaries considered here, we show the $(2, 1)$-mode, as well as two of the four quaternion component evolutions. In all of the figures, the fully precessing system's results are shown in blue, the results parameterized by $\chi_p^\perp$ in orange, and by $\chi_p$ in purple.

Our first example is an equal mass binary, i.e. $q = 1$, with initial spins $\chi_1(t_0) = (0.225, 0.13, -0.15)$, $\chi_2(t_0) = (0.09, 0.15, 0.1)$. In this equal mass limit, we expect $\chi_p$ to perform poorly, and $\chi_p^\perp$ to perform much better, as discussed more thoroughly in Sec. V. Indeed, in the left panel of Fig. 14, we see that $\chi_p^\perp$ better replicates the $(2, 1)$-mode for this particular binary, with an amplitude closer to that of the fully precessing waveform and slightly improved phasing. We see the improvement by using $\chi_p^\perp$ as opposed to $\chi_p$ more clearly in the dynamics as shown in the right panel. The good agreement between the time evolution of fully precessing quaternion components in blue and those of the $\chi_p^\perp$-mapped system in orange, is in stark contrast to the $\chi_p$-mapped components in purple, which matches the dynamics poorly.

The second example is a binary with $q = 3$, and initial spins $\chi_1(t_0) = (0.45, 0.26, -0.3)$, $\chi_2(t_0) = (0.15, 0.08, 0.1)$. We note that in this example, unlike the fiducial binary shown in Fig. 1, $\chi_p^\perp$ is mapped onto the primary BH. The left panel of Fig. 15 shows the $(2, 1)$-mode, where we see that unlike in the previous example, the amplitudes of the two mapped waveform modes, are very similar to that of the fully precessing mode (blue). However, $\chi_p^\perp$ clearly much better matches the phasing of the fully precessing mode, with the orange and blue lines being indistinguishable for much of the inspiral, in contrast to $\chi_p$ which shows a clear dephasing, especially in the merger-ringdown. We also see that, like for the other fiducial binaries, the quaternion components mapped by $\chi_p^\perp$, much more faithfully represent the fully precessing quaternions, compared to the $\chi_p$-mapped components, demonstrating that $\chi_p^\perp$ better replicates the precession dynamics of the fully precessing system.

Lastly, we illustrate the impact of the conditional placement using the original fiducial binary of Fig. 1, but this time showing the effect of an analogous conditional placement with the $\chi_p$-parameterisation. In Fig. 16, we show the fully precessing fiducial binary in blue, $\chi_p^\perp$-mapped system in orange, and the $\chi_p$-mapped system...
with conditional placement in purple. We note that now both \( \chi_L \) and \( \chi_P \) are placed on the secondary BH for this binary. We can see in the left panel that the conditional placement does improve the accuracy with which \( \chi_P \) reproduces the (2,1)-mode, however the \( \chi_perp \)-mapping still reproduces the phasing of the mode significantly better. Additionally, we see in the right panel that the conditional placement of \( \chi_P \) does not produce the precession dynamics more accurately, with \( \chi_perp \) still much more closely matching the time evolution of the fully precessing quaternion components.

[1] B.P. Abbott et al. (LIGO Scientific, Virgo), “Observation of Gravitational Waves from a Binary Black Hole Merger,” Phys. Rev. Lett. 116, 061102 (2016), arXiv:1602.03837 [gr-qc].
[2] B.P. Abbott et al. (KAGRA, LIGO Scientific, VIRGO), “Prospects for Observing and Localizing Gravitational-Wave Transients with Advanced LIGO, Advanced Virgo and KAGRA,” Living Rev. Rel. 21, 3 (2018), arXiv:1304.0670 [gr-qc].
[3] M. Tse et al., “Quantum-Enhanced Advanced LIGO Detectors in the Era of Gravitational-Wave Astronomy,” Phys. Rev. Lett. 123, 231107 (2019).
[4] F. Acernese et al. (VIRGO), “Advanced Virgo: a second-generation interferometric gravitational wave detector,” Class. Quant. Grav. 32, 024001 (2015), arXiv:1408.3978 [gr-qc].
[5] F. Acernese et al. (Virgo), “Increasing the Astrophysical Reach of the Advanced Virgo Detector via the Application of Squeezed Vacuum States of Light,” Phys. Rev. Lett. 123, 231108 (2019).
[6] B.P. Abbott et al. (LIGO Scientific, Virgo), “GWTC-1: A Gravitational-Wave Transient Catalog of Compact Binary Mergers Observed by LIGO and Virgo during the First and Second Observing Runs,” Phys. Rev. X 9, 031040 (2019), arXiv:1811.12907 [astro-ph.HE].
[7] R. Abbott et al. (LIGO Scientific, Virgo), “GWTC-2: Compact Binary Coalescences Observed by LIGO and Virgo During the First Half of the Third Observing Run,” (2020), arXiv:2010.14527 [gr-qc].
[8] B.P. Abbott et al. (LIGO Scientific, Virgo), “GW170817: Observation of Gravitational Waves from a Binary Neutron Star Inspiral,” Phys. Rev. Lett. 119, 161101 (2017), arXiv:1710.05832 [gr-qc].
[9] B.P. Abbott et al. (LIGO Scientific, Virgo, Fermi GBM, INTEGRAL, IceCube, AstroSat Cadmium Zinc Telluride Imager Team, IPN, Insight-Hxmt, ANTARES, Swift, AGILE Team, 1M2H Team, Dark Energy Camera GW-EM, DES, DLT40, GRAWITA, Fermi-LAT, ATCA, ASKAP, Las Cumbres Observatory Group, OzGrav, DWF (Deeper Wider Faster Program), AST3, CA竞技, VINROUGE, MASTER, J-GEM, GROWTH, JAGWAR, CaltechNRAO, TTV-NRAO, NuSTAR, Pan-STARRS, MAXI Team, TZAC Consortium, KU, Nordic Optical Telescope, ePESTO, GROND, Texas Tech University, SALT Group, TOROS, BOOTES, MWA, CALET, IKI-GW Follow-up, H.E.S.S., LOFAR, IWA, HAWC, Pierre Auger, ALMA, Euro VLBI Team, Pi of Sky, Chandra Team at McGill University, DFN, ATLAS Telescopes, High Time Resolution Universe Survey, RIMAS, RATIR, SKA South Africa/MeerKAT), “Multi-messenger Observations of a Binary Neutron Star Merger,” Astrophys. J. Lett. 848, L12 (2017), arXiv:1710.05833 [astro-ph.HE].
[10] R. Abbott et al. (LIGO Scientific, Virgo), “GW190521: A Binary Black Hole Merger with a Total Mass of 150 M⊙,” Phys. Rev. Lett. 125, 101102 (2020), arXiv:2009.01075 [gr-qc].
[11] R. Abbott et al. (LIGO Scientific, Virgo), “Properties and Astrophysical Implications of the 150 M⊙ Binary Black Hole Merger GW190521,” Astrophys. J. Lett. 900, L13 (2020), arXiv:2009.01190 [astro-ph.HE].
[12] R. Abbott et al. (LIGO Scientific, Virgo), “GW190412: Observation of a Binary-Black-Hole Coalescence with Asymmetric Masses,” Phys. Rev. D 102, 043015 (2020), arXiv:2004.08342 [astro-ph.HE].
[13] R. Abbott et al. (LIGO Scientific, Virgo), “GW190814: Gravitational Waves from the Coalescence of a 23 Solar Mass Black Hole with a 2.6 Solar Mass Compact Object,” Astrophys. J. Lett. 896, L44 (2020), arXiv:2006.12611 [astro-ph.HE].
[14] Alexander H. Nitz, Thomas Dent, Gareth S. Davies, Sumit Kumar, Collin D. Capano, Ian Harry, Simone Mozzon, Laura Nuttall, Andrew Lundgren, and Mártón Tápai, “2-OGC: Open Gravitational-wave Catalog of binary mergers from analysis of public Advanced LIGO and Virgo data,” Astrophys. J. 891, 123 (2019), arXiv:1910.05331 [astro-ph.HE].
[15] Tejaswi Venumadhav, Barak Zackay, Javier Roulet, Liang Dai, and Matias Zaldarriaga, “New binary black hole mergers in the second observing run of Advanced LIGO and Advanced Virgo,” Phys. Rev. D 101, 083030 (2020), arXiv:1904.07214 [astro-ph.HE].
[16] Mark Hannam, Patricia Schmidt, Alejandro Bohé, Leïla Hagem, Sascha Hua, Frank Ohme, Geraint Pratten, and Michael Purrrer, “Simple Model of Complete Precessing Black-Hole-Binary Gravitational Waveforms,” Phys. Rev. Lett. 113, 151101 (2014), arXiv:1308.3271 [gr-qc].
[17] Yi Pan, Alessandra Buonanno, Andrea Taracchini, Lawrence E. Kidder, Abdul H. Mroué, Harald P. Pfeiffer, Mark A. Scheel, and Béla Szilágyi, “Inspiral-merger-ringdown waveforms of spinning, precessing black-hole binaries in the effective-one-body formalism,” Phys. Rev. D 89, 084006 (2014), arXiv:1307.6232 [gr-qc].
[18] Sebastian Khan, Frank Ohme, Katerina Chatziioannou, and Mark Hannam, “Including higher order multipoles in gravitational-wave models for precessing binary black holes,” Phys. Rev. D 101, 024056 (2020), arXiv:1911.06050 [gr-qc].
[19] Geraint Pratten et al., “Let’s twist again: computationally efficient models for the dominant and sub-dominant harmonic modes of precessing binary black holes,” (2020), arXiv:2004.06503 [gr-qc].
[20] Serguei Ossokine et al., “Multipolar Effective-One-Body Waveforms for Precessing Binary Black Holes: Construction and Validation,” Phys. Rev. D 102, 044055 (2020), arXiv:2004.09442 [gr-qc].
[21] David Reitze et al., “Cosmic Explorer: The U.S. Contribution to Gravitational-Wave Astronomy beyond LIGO,” Bull. Am. Astron. Soc. 51, 035 (2019), arXiv:1907.04833 [astro-ph.IM].

[22] Michele Maggiore et al., “Science Case for the Einstein Telescope,” JCAP 03, 050 (2020), arXiv:1912.02622 [astro-ph.CO].

[23] Benjamín P. Abbott et al. (LIGO Scientific), “Exploring the Sensitivity of Next Generation Gravitational Wave Detectors,” Class. Quant. Grav. 34, 044001 (2017), arXiv:1607.08697 [astro-ph.IM].

[24] Michael Pürrer and Carl-Johan Haster, “Gravitational waveforms: The Impact of spin-induced precession,” Phys. Rev. D 59, 084006 (1999), arXiv:gr-qc/9811091 [gr-qc].

[25] L. Santamaría et al., “Matching post-Newtonian and numerical relativistic waveforms: systematic errors and a new phenomenological model for non-precessing black hole binaries,” Phys. Rev. D 82, 064016 (2010), arXiv:1005.3306 [gr-qc].

[26] Jonathan Blackman, Scott E. Field, Mark A. Scheel, Chad R. Galley, Daniel A. Hemberger, Patricia Schmidt, and Rory Smith, “A Surrogate Model of Gravitational Waveforms from Numerical Relativity Simulations of Precessing Black Hole Mergers,” Phys. Rev. D 95, 104023 (2017), arXiv:1701.00550 [gr-qc].

[27] Antoni Ramos-Buades, Patricia Schmidt, Geraint Pratten, and Sascha Husa, “Validity of common modeling approximations for precessing binary black holes with higher-order modes,” Phys. Rev. D 101, 103014 (2020), arXiv:2001.10936 [gr-qc].

[28] Thocharis A. Apostolatos, Curt Cutler, Gerald J. Sussman, and Kip S. Thorne, “Spin induced orbital precession and its modulation of the gravitational wave forms from merging binaries,” Phys. Rev. D 49, 6274–6297 (1994).

[29] Lawrence E. Kidder, “Coalescing binary systems of compact objects to post-Newtonian 5/2 order. 5. Spin effects,” Phys. Rev. D 52, 821–847 (1995), arXiv:gr-qc/9506022.

[30] Alberto Vecchio, “LISA observations of rapidly spinning massive black hole binary systems,” Phys. Rev. D 70, 042001 (2004), arXiv:astro-ph/0304051.

[31] M.V. van der Sluys, C. Röver, A. Strooer, V. Raymond, I. Mandel, N. Christensen, V. Kalogera, R. Meyer, and A. Vecchio, “Gravitational-Wave Astronomy with Inspiral Signals of Spinning Compact-Object Binaries,” Astrophys. J. Lett. 688, L61 (2008), arXiv:0710.1897 [astro-ph].

[32] Ryan N. Lang and Scott A. Hughes, “Measuring coalescing massive binary black holes with gravitational waves: The Impact of spin-induced precession,” Phys. Rev. D 74, 122001 (2006), [Erratum: Phys.Rev.D 75, 089902 (2007), Erratum: Phys.Rev.D 77, 109901 (2008)], arXiv:gr-qc/0608062.

[33] Hee-Suk Cho, Evan Ochsner, Richard O’Shaughnessy, Chunglee Kim, and Chang-Hwan Lee, “Gravitational waves from black hole-neutron star binaries: Effective Fisher matrices and parameter estimation using higher harmonics,” Phys. Rev. D 87, 024004 (2013), arXiv:1209.4949 [gr-qc].

[34] R. O’Shaughnessy, Benjamin Farr, E. Ochsner, Hee-Suk Cho, V. Raymond, Chunglee Kim, and Chang-Hwan Lee, “Parameter estimation of gravitational waves from precessing black hole-neutron star inspirals with higher harmonics,” Phys. Rev. D 89, 102005 (2014), arXiv:1403.0544 [gr-qc].

[35] Katerina Chatziioannou, Neil Cornish, Antoine Klein, and Nicolas Yunes, “Spin-Precession: Breaking the Black Hole–Neutron Star Degeneracy,” Astrophys. J. Lett. 798, L17 (2015), arXiv:1402.3581 [gr-qc].

[36] Geraint Pratten, Patricia Schmidt, Riccardo Buscicchio, and Lucy M. Thomas, “Measuring precession in asymmetric compact binaries,” Phys. Rev. Res. 2, 043096 (2020), arXiv:2006.16153 [gr-qc].

[37] Patricia Schmidt, Mark Hannam, Sascha Husa, and P. Ajith, “Tracking the precession of compact binaries from their gravitational-wave signal,” Phys. Rev. D 84, 024046 (2011), arXiv:1012.2879 [gr-qc].

[38] Patricia Schmidt, Mark Hannam, and Sascha Husa, “Towards models of gravitational waveforms from generic binaries: A simple approximate mapping between precessing and non-precessing inspiral signals,” Phys. Rev. D 86, 104063 (2012), arXiv:1207.3088 [gr-qc].

[39] Patricia Schmidt, Frank Ohme, and Mark Hannam, “Towards models of gravitational waveforms from generic binaries II: Modelling precession effects with a single effective precession parameter,” Phys. Rev. D 91, 024043 (2015), arXiv:1408.1810 [gr-qc].

[40] Michael Kesden, Davide Gerosa, Richard O’Shaughnessy, Emanuele Berti, and Ulrich Sperhake, “Effective potentials and morphological transitions for binary black-hole spin precession,” Phys. Rev. Lett. 114, 081103 (2015), arXiv:1411.0674 [gr-qc].

[41] Davide Gerosa, Michael Kesden, Ulrich Sperhake, Emanuele Berti, and Richard O’Shaughnessy, “Multiscale analysis of phase transitions in precessing black-hole binaries,” Phys. Rev. D 92, 064016 (2015), arXiv:1506.03492 [gr-qc].

[42] L. Pekowsky, R. O’Shaughnessy, J. Healy, and D. Shoemaker, “Comparing gravitational waves from nonprecessing and precessing black hole binaries in the corotating frame,” Phys. Rev. D 88, 024040 (2013), arXiv:1304.3176 [gr-qc].

[43] Davide Gerosa, Matthew Mould, Daria Gangardt, Patricia Schmidt, Geraint Pratten, and Lucy M. Thomas, “A generalized precession parameter χp to interpret gravitational-wave data,” (2020), arXiv:2011.11948 [gr-qc].

[44] Sarp Akcay, Rossella Gamba, and Sebastiano Bernuzzi, “A hybrid post-Newtonian—effective-one-body scheme for spin-precessing compact-binary waveforms,” (2020), arXiv:2005.05338 [gr-qc].

[45] B.P. Abbott et al. (LIGO Scientific, Virgo), “Properties of the Binary Black Hole Merger GW150914,” Phys. Rev. Lett. 116, 241102 (2016), arXiv:1602.03840 [gr-qc].
[49] Vijay Varma, Scott E. Field, Mark A. Scheel, Jonathan Blackman, Davide Gerosa, Leo C. Stein, Lawrence E. Kidder, and Harald P. Pfeiffer, “Surrogate models for precessing binary black hole simulations with unequal masses,” Phys. Rev. Research. 1, 033015 (2019), arXiv:1905.09300 [gr-qc].

[50] Jonathan Blackman, Scott E. Field, Chad R. Galley, and Vijay Varma, “gwsurrogate,” https://pypi.python.org/pypi/gwsurrogate/.

[51] Scott E. Field, Chad R. Galley, Jan S. Hesthaven, Jason Kaye, and Manuel Tiglio, “Fast prediction and evaluation of gravitational waveforms using surrogate models,” Phys. Rev. X 4, 031006 (2014), arXiv:1308.3566 [gr-qc].

[52] Jonathan Blackman, Scott E. Field, Chad R. Galley, Béla Szilágyi, Mark A. Scheel, Manuel Tiglio, and Daniel A. Hemberger, “Fast and Accurate Prediction of Numerical Relativity Waveforms from Binary Black Hole Coalescences Using Surrogate Models,” Phys. Rev. Lett. 115, 121102 (2015), arXiv:1502.07758 [gr-qc].

[53] Michael Boyle, “Angular velocity of gravitational radiation from precessing binaries and the corotating frame,” Phys. Rev. D 87, 104006 (2013), arXiv:1302.2919 [gr-qc].

[54] LIGO Scientific Collaboration and The Virgo Collaboration, “Noise curves used for Simulations in the update of the Observing Scenarios Paper,” https://dcc.ligo.org/LIGO-T1500602/public (2019).

[55] Collin Capano, Yi Pan, and Alessandra Buonanno, “Impact of higher harmonics in searching for gravitational waves from nonspinning binary black holes,” Phys. Rev. D 89, 102003 (2014), arXiv:1311.1286 [gr-qc].

[56] Stanislav Babak, Andrea Taracchini, and Alessandra Buonanno, “Validating the effective-one-body model of spinning, precessing binary black holes against numerical relativity,” Phys. Rev. D 95, 024010 (2017), arXiv:1607.05661 [gr-qc].

[57] William H. Press, “Long Wave Trains of Gravitational Waves from a Vibrating Black Hole,” Astrophys. J. Lett. 170, L105–L108 (1971).

[58] S. Chandrasekhar and Steven L. Detweiler, “The quasi-normal modes of the Schwarzschild black hole,” Proc. Roy. Soc. Lond. A 344, 441–452 (1975).

[59] Steven L. Detweiler, “Black holes and gravitational waves. III. The resonant frequencies of rotating holes,” Astrophys. J. 330, 292–295 (1980).

[60] Kostas D. Kokkotas and Bernd G. Schmidt, “Quasinormal modes of stars and black holes,” Living Rev. Rel. 2, 2 (1999), arXiv:gr-qc/9900303.

[61] Alejandro Bohé, Mark Hannam, Sascha Husa, Frank Ohme, Michael Pürrer, and Patricia Schmidt, “PhenomPv2 – technical notes for the LAL implementation,” https://dcc.ligo.org/LIGO-T1500062/public (2016).

[62] Vijay Varma, Davide Gerosa, Leo C. Stein, François Hébert, and Hao Zhang, “High-accuracy mass, spin, and recoil predictions of generic black-hole merger remnants,” Phys. Rev. Lett. 122, 011101 (2019), arXiv:1809.00215 [gr-qc].

[63] Vijay Varma and Leo C. Stein, “vijayvarma392/surfinBH: Surrogate Final BH properties,” (2018).

[64] Xisco Jiménez-Forteza, David Keitel, Sascha Husa, Mark Hannam, Sebastian Khan, and Michael Pürrer, “Hierarchical data-driven approach to fitting numerical relativity data for nonprecessing binary black holes with an application to final spin and radiated energy,” Phys. Rev. D 95, 064024 (2017), arXiv:1611.00332 [gr-qc].

[65] Fabian Hofmann, Enrico Barausse, and Luciano Rezzolla, “The final spin from binary black holes in quasicircular orbits,” Astrophys. J. Lett. 825, L19 (2016), arXiv:1605.01938 [gr-qc].

[66] Vijay Varma, Parameswaran Ajith, Sascha Husa, Juan Calderón Bustillo, Mark Hannam, and Michael Pürrer, “Gravitational-wave observations of binary black holes: Effect of nonquadrupole modes,” Phys. Rev. D 90, 124004 (2014), arXiv:1409.2349 [gr-qc].

[67] Philip B. Graff, Alessandra Buonanno, and B. S. Sathyaprakash, “Missing Link: Bayesian detection and measurement of intermediate-mass black-hole binaries,” Phys. Rev. D 92, 022002 (2015), arXiv:1504.04766 [gr-qc].

[68] Juan Calderón Bustillo, Sascha Husa, Alicia M. Sintes, and Michael Pürrer, “Impact of gravitational radiation higher order modes on single aligned-spin gravitational wave searches for binary black holes,” Phys. Rev. D 93, 084019 (2016), arXiv:1511.02060 [gr-qc].

[69] Juan Calderón Bustillo, Pablo Laguna, and Deirdre Shoemaker, “Detectability of gravitational waves from binary black holes: Impact of precession and higher modes,” Phys. Rev. D 95, 104035 (2017), arXiv:1612.02340 [gr-qc].

[70] Ian Harry, Juan Calderón Bustillo, and Alex Nitz, “Searching for the full symphony of black hole binary mergers,” Phys. Rev. D 97, 023004 (2018), arXiv:1709.09181 [gr-qc].

[71] Marta Colleoni, Maite Mateu-Lucena, Héctor Estelé, Cecilia García-Quirós, David Keitel, Geraint Pratten, Antoni Ramos-Buades, and Sascha Husa, “Towards the routine use of subdominant harmonics in gravitational-wave inference: re-analysis of GW190412 with generation X waveform models,” (2020), arXiv:2010.05830 [gr-qc].

[72] Alessandra Buonanno, Yan-bei Chen, Yi Pan, and Michele Vallisneri, “A Quasi-physical family of gravity-wave templates for precessing binaries of spinning compact objects. 2. Application to double-spin precessing binaries,” Phys. Rev. D 70, 104003 (2004), [Erratum: Phys.Rev.D 74, 029902 (2006)], arXiv:gr-qc/0405090.

[73] Jose A. Gonzalez, Ulrich Sperhake, Bernd Bruegmann, Mark Hannam, and Sascha Husa, “Total recoil: The Maximum kick from nonspinning black-hole binary inspiral,” Phys. Rev. Lett. 98, 091101 (2007), arXiv:gr-qc/0610154.

[74] Bernd Bruegmann, Jose A. Gonzalez, Mark Hannam, Sascha Husa, and Ulrich Sperhake, “Exploring black hole superkicks,” Phys. Rev. D 77, 124047 (2008), arXiv:0707.0135 [gr-qc].

[75] Manuela Campanelli, Carlos O. Lousto, Yosef Zlochower, and David Merritt, “Maximum gravitational recoil,” Phys. Rev. Lett. 98, 231102 (2007), arXiv:gr-qc/0702133.

[76] John G. Baker, William D. Boggs, Joan Centrella, Bernard J. Kelly, Sean T. McWilliams, M.Coleman Miller, and James R. van Meter, “Modeling kicks from the merger of generic black-hole binaries,” Astrophys. J. Lett. 682, L29–L32 (2008), arXiv:0802.0416 [astro-ph].
FIG. 13. Full results for the O4 PSD-weighted matches between the fully spinning waveform and each of the effective spin parameterisations, $\vec{\chi}_\parallel$ (orange) and $\chi_p$ (purple). We show percentages of matches, split by total mass, mass ratio and mode, which have a match less than 0.99. The dashed horizontal lines indicate 1% and 10%. Similarly to the white noise matches of Fig. 12, we see improvements by using $\vec{\chi}_\parallel$ over $\chi_p$ across all masses, mass ratios and modes, but we note the dramatic improvements in performance for higher-order modes, particularly the $(2, \pm 1)$-modes. We also note the high degree of similarity between each of the three total masses.
FIG. 14. Amplitude of the $(2,1)$-mode (left), and time evolution of the two quaternion components, $q_1(t)$ and $q_2(t)$ (right), for an equal mass binary with initial spins $\chi_1(t_0) = (0.225, 0.13, -0.15)$, and $\chi_2(t_0) = (0.09, 0.15, 0.1)$. We show the fully spinning waveform mode and quaternion components in blue. The mode and quaternions parameterized by $\chi_p$ are shown in purple, and $\vec{\chi}_\perp$ in orange. We see that $\vec{\chi}_\perp$ more faithfully reproduces the fully precessing $(2,1)$-mode, and much more accurately reproduces the precession dynamics, than $\chi_p$.

FIG. 15. Amplitude of the $(2,1)$-mode (left), and time evolution of the two quaternion components, $q_1(t)$ and $q_2(t)$ (right), for a $q = 3$ fiducial binary, with initial spins $\vec{\chi}_1(t_0) = (0.45, 0.26, -0.3)$ and $\vec{\chi}_2(t_0) = (0.15, 0.087, 0.1)$. The fully precessing waveform mode and quaternions are shown in blue, with the system parameterized by $\chi_p$ shown in purple, and $\vec{\chi}_\perp$ in orange. We see that $\vec{\chi}_\perp$ more faithfully reproduces the fully precessing $(2,1)$-mode, capturing the correct phasing of the mode, and more accurately reproduces the precession dynamics represented by the quaternion components, than $\chi_p$. 
FIG. 16. Amplitude of the (2, 1)-mode (left), and time evolution of the two quaternion components, $q_1(t)$ and $q_2(t)$ (right), for the same fiducial binary as in Fig.s 1 and 8, but with $\chi_p$ conditionally placed on the secondary BH. The fully precessing waveform mode and quaternions are shown in blue, with the system parameterized by a conditionally-placed $\chi_p$ shown in purple, and $\vec{\chi}_\perp$ in orange. We see that although the conditional placement of $\chi_p$ does lead to an improvement in the accuracy with which it reproduces the (2, 1)-mode, $\vec{\chi}_\perp$ still outperforms $\chi_p$. Additionally, conditionally placing $\chi_p$ does not improve the accuracy with which it reproduces the precession dynamics, with $\vec{\chi}_\perp$ still much more closely matching the time evolution of the fully precessing $\dot{q}_1$ and $\dot{q}_2$ quaternion components.