Cluster-Cyclic Quivers with three Vertices and the Markov Equation,

with an appendix by Otto Kerner

Andre Beineke    Thomas Brüstle    Lutz Hille

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Abstract

Acyclic cluster algebras have an interpretation in terms of tilting objects in a Calabi-Yau category defined by some hereditary algebra. For a given quiver $Q$ it is thus desirable to decide if the cluster algebra defined by $Q$ is acyclic. We call $Q$ cluster-acyclic in this case, otherwise cluster-cyclic. In this note we classify the cluster-cyclic quivers with three vertices using a Diophantine equation studied by Markov.

1 Introduction

Cluster algebras have been introduced and studied by Fomin and Zelevinsky in [5,6]. In [2] it is shown that each acyclic cluster algebra admits an interpretation in terms of tilting objects in a so-called cluster category. This category is a Calabi-Yau category defined as a quotient of the derived category of modules over some hereditary algebra. It is thus desirable to study which cluster algebras are acyclic and which are not. While it has been shown in [6] that all finite cluster algebras are acyclic, the general case is not known.

The first non-trivial situation is that of cluster algebras of rank three. This case has recently been studied in [8] (which analyzes properties of the acyclic case by representation-theoretic methods) and in [1] (which describes the mutation classes of non-acyclic cluster algebras). The cluster algebras considered here are defined by a quiver $Q$; the quiver $Q$ corresponds to a skew-symmetric matrix $B$. In general, a cluster algebra is given by a skew-symmetrizable matrix. In this paper we give precise criteria for deciding which quivers with three vertices yield an acyclic cluster algebra (Theorem 1.1) and which do not (Theorem 1.2).

We consider a quiver $Q$ with three vertices 1, 2 and 3 and cyclic orientation. We assume $Q$ has $x$ arrows from 1 to 2, $y$ arrows from 2 to 3 and $z$ arrows from 3 to 1:
An essential tool in the definition of a cluster algebra is the mutation of a quiver. We describe the effect of these cluster mutations on the cyclic quiver $Q$. A cluster mutation in the vertex 2 defines a new quiver $\mu_2 Q$ which is obtained from $Q$ by reversing all arrows from 1 to 2 and from 2 to 3. Moreover, the quiver $\mu_2 Q$ has $z - xy$ arrows from 3 to 1, provided this number is non-negative. Then the quiver $\mu_2 Q$ is no longer cyclic, it is acyclic. If $z - xy$ is negative, then $\mu_2 Q$ has $xy - z$ arrows from 1 to 3 and it is cyclic:

The cluster mutations $\mu_1 Q$ and $\mu_3 Q$ in the vertices 1 and 3 are defined analogously. We say that $Q$ is cluster-acyclic if there exists a finite sequence of cluster mutations from $Q$ to an acyclic quiver $Q'$. Otherwise, if all sequences of cluster mutations yield cyclic quivers, we say that $Q$ is cluster-cyclic. Of course, these cluster mutations are compatible with per mutations of the triple $(x, y, z)$ defining the cyclic quiver $Q$. So we say that a triple of non-negative integers $(x, y, z)$ is cluster-acyclic (respectively, cluster-cyclic) if the corresponding quiver $Q$ has this property.

In this note, we characterize cluster-cyclic quivers in terms of a Diophantine equation first studied by Markov in [10]. For each triple $(x, y, z)$, we define its Markov constant as

$$C(x, y, z) := x^2 + y^2 + z^2 - xyz.$$  

It turns out that, with some exceptions, the value of the Markov constant characterizes cluster-acyclic quivers. We first note that $C(x, y, z)$ is invariant under cluster mutations (Lemma 3.1). We consider the action of the group $\Gamma$ on $\mathbb{R}^3$, where $\Gamma$ is generated by the three cluster mutations $\mu_1, \mu_2$ and $\mu_3$ and all permutations in $S_3$. For each $C \in \mathbb{R}$, this group action reduces to an action on the affine algebraic variety

$$V(C) := \{(x, y, z) \in \mathbb{A}^3 \mid C(x, y, z) = C\}.$$  

We are mainly interested in the integral points of $V(C)$, but will also consider it over $\mathbb{R}$ and over $\mathbb{C}$. Moreover, for $x, y \geq 2$, we define two functions

$$m^-(x, y) := \frac{1}{2}(xy - \sqrt{(x^2 - 4)(y^2 - 4)}) \quad \text{and} \quad m^+(x, y) := \frac{1}{2}(xy + \sqrt{(x^2 - 4)(y^2 - 4)}),$$

where $m^+(x, y) \geq m^-(x, y) \geq 2$ for all $x, y \geq 2$. If we consider a point in $V(4)$ and express $z$ as a function of $x$ and $y$, then we obtain the two solutions $z = m^+(x, y)$ and $z = m^-(x, y)$. Moreover, it is easy to check (using Lemma 3.1) that

$$m^+(x, m^-(x, y)) = m^+(x, xy - m^+(x, y)) = y = m^-(x, m^+(x, y)) = m^-(x, xy - m^-(x, y)).$$
Using these two functions and the Markov constant, we can characterize cluster-acyclic triples as follows. If the Markov constant is larger than 4, we have a cluster-acyclic triple; if the Markov constant is less then 0, we have an cluster-cyclic triple. If the Markov constant lies in the interval $[0, 4]$ then we have a finite number of cluster-acyclic triples for the Markov constants 0, 1, 2, 4 corresponding to four finite orbits (see Theorem 1.1 (3) below) and an infinite number of cluster-cyclic orbits (one contains one element, all others are infinite orbits) for the Markov constants 0 and 4 (see Theorem 1.2). Also note that 4 is the only value of the Markov constant with an infinite number of integral orbits; for all other values the number of orbits is finite (see Corollary 1.3).

**Theorem 1.1.** Let $Q$ be a cyclic quiver with numbers of arrows given by $x, y, z$ in $\mathbb{Z}^3_{\geq 0}$. Then the following conditions are equivalent.

1. The triple $(x, y, z)$ is cluster-acyclic.
2. The Markov constant satisfies $C(x, y, z) > 4$ or $\min\{x, y, z\} < 2$.
3. The Markov constant satisfies $C(x, y, z) > 4$ or the triple $(x, y, z)$ is in the following list (where we assume $x \geq y \geq z$):
   a) $C(x, y, z) = 0 : (x, y, z) = (0, 0, 0)$,
   b) $C(x, y, z) = 1 : (x, y, z) = (1, 0, 0)$,
   c) $C(x, y, z) = 2 : (x, y, z) = (1, 1, 0)$ or $(1, 1, 1)$,
   d) $C(x, y, z) = 4 : (x, y, z) = (2, 0, 0)$ or $(2, 1, 1)$.
4. $\min\{x, y, z\} < 2$, or $\min\{x, y\} \geq 2$ and precisely one of the following inequalities is satisfied: $z > m^+(x, y)$ or $z < m^-(x, y)$.

**Theorem 1.2.** Let $Q$ be a cyclic quiver with numbers of arrows given by $x, y, z$ in $\mathbb{Z}^3_{\geq 0}$. Then the following conditions are equivalent.

1. The triple $(x, y, z)$ is cluster-cyclic.
2. $x, y, z \geq 2$ and $C(x, y, z) \leq 4$.
3. $C(x, y, z) < 0$ or the triple is in one of the following orbits:
   a) $C(x, y, z) = 0 : (x, y, z)$ is in $\Gamma(3, 3, 3)$ or
   b) $C(x, y, z) = 4 : (x, y, z)$ is in $\Gamma(2, x, x)$ for some $x \geq 2$.
4. $x, y, z \geq 2$ and $m^+(x, y) \geq z \geq m^-(x, y)$.
5. $x, y, z \geq 2$ and in the orbit $\Gamma(x, y, z)$ there exists a unique element $f(x, y, z) := (x', y', z')$ satisfying $x' \geq y' \geq z'$ and $y'z' - x' \geq x'$.

**Corollary 1.3.** (1) The list in Theorem 1.1 (3) together with the orbit of $(x, 0, 0)$ (for $x > 2$) is the list of all finite cluster-acyclic orbits containing a triple with $x \geq y \geq z \geq 0$.

(2) The only finite cluster-cyclic orbit is $(2, 2, 2)$.

The existence of two functions $m^+$ and $m^-$ satisfying condition (4) in Theorem 1.2 has already been shown in [1] without knowing the Markov equation. For more information on the classical Markov equation we refer to [3] and to the original article of Markov [10]. This equation also appears in the work of Rudakov on vector bundles on $\mathbb{P}^2$ ([12]) and has known applications for mutations of exceptional sequences (see e.g. [7]). The Markov equation we study in this note differs slightly from the classical one: Markov constructed in [10] all integral solutions of the equation

$$x^2 + y^2 + z^2 - 3xyz = 0$$

whereas we consider a rescaled version (dividing all variables by 3) and allow arbitrary right hand side $C$ (see section 6).
This paper is organized as follows. In Section 2 we define the group action we are interested in and prove some preliminary lemmata. This closely follows [3]. In Section 3 we explain how to obtain the Markov equation and show its invariance under cluster mutations. This is well known for mutations of exceptional sequences (see [12]) and the classical Markov equation. We consider the real points of the algebraic variety defined by the Markov equation (it is a differentiable manifold for \( C \neq 4 \)) and compute its connected components in Section 4. It turns out that the group action respects the components. Moreover, we need to compute the singularities. In Section 5 we construct a fundamental domain for the action of the group \( \Gamma \) on the set of all cluster-cyclic triples with real coefficients. Moreover, we show that the number of orbits for any given Markov constant \( C \neq 4 \) is finite. Finally, in Section 6 we prove our main theorems and the corollary.

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## 2 The group action and mutations

Let \( \Gamma \) be the group which is freely generated by three generators \( \mu_1, \mu_2, \) and \( \mu_3 \) of order 2

\[
\mu_1 : (x, y, z) \mapsto (yz - x, y, z), \quad \mu_2 : (x, y, z) \mapsto (x, xz - y, z), \quad \mu_3 : (x, y, z) \mapsto (x, y, xy - z)
\]

and let \( \Gamma \) the semi-direct product of \( \Gamma \) with the symmetric group \( S_3 \) (see also below). It follows from Lemma 2.1 that \( \Gamma \) is the subgroup of \( \Gamma \) which is generated by all cluster mutations. Note that \( \Gamma \) describes the cluster mutation as defined in [5] only when the quiver is cyclic. In case of an acyclic quiver, the cluster mutation is given by a different formula. Since we only want to decide whether a cyclic quiver is cluster-cyclic or not, it is sufficient to study the action of \( \Gamma \) (or \( \Gamma \)) on \( \mathbb{Z}^3 \) (or on \( \mathbb{R}^3 \)). A triple \((x, y, z)\) is then cluster-acyclic precisely when there exists a triple in the \( \Gamma \)-orbit \( \Gamma(x, y, z) \) with one non-positive entry.

In case there are more than 3 vertices, one could also derive a version of the Markov equation in a manner similar to that described in Section 3. However, one then has to consider also acyclic quivers, and we see no way to make the group action and the Markov equation compatible with cluster mutations on acyclic quivers.

The group \( \Gamma \) acts via cluster mutations on the three-dimensional affine space \( \mathbb{A}^3 \) defined over any commutative ring \( k \). Moreover, the symmetric group \( S_3 \) acts on \( \Gamma \) via permutation of the generators:

\[
\sigma \cdot \mu_{a(l)} \mu_{a(l-1)} \cdots \mu_{a(2)} \mu_{a(1)} := \mu_{\sigma(a(l))} \mu_{\sigma(a(l-1))} \cdots \mu_{\sigma(a(2))} \mu_{\sigma(a(1))} \quad \text{for } \sigma \in S_3.
\]

Thus \( \Gamma \) is the semi-direct product of \( S_3 \) with \( \Gamma \). In Section 5, we construct a fundamental domain for the action of the group \( \Gamma \) on the set of all cluster-cyclic triples with Markov constant less than 4. For the cluster-cyclic triples with \( C = 4 \), however, we cannot construct a fundamental domain. This is explained in detail in Section 4 when we compute the connected components and the singularities of the sets \( V(C) \) over \( \mathbb{R} \).
In this section, we only prove some first elementary properties of the group action. Note that the cluster mutation $\mu_i(x, y, z)$ changes only one of the three components $x, y$ and $z$.

So we can define $(x, y, z) \leq \mu_i(x, y, z)$ if all three components satisfy this inequality. It is a partial order on the set of all real triples $(x, y, z)$. For us it is important that we can compare $(x, y, z)$ with $\mu_i(x, y, z)$ for all $i = 1, 2, 3$.

We distinguish the following three cases:

(M1) $(x, y, z) \leq \mu_i(x, y, z)$ for all $i = 1, 2, 3$,
(M2) $(x, y, z) \leq \mu_i(x, y, z)$ for precisely two indices $i$,
(M3) $(x, y, z) \leq \mu_i(x, y, z)$ for at most one index $i$.

Moreover, for any triple with $x \geq y \geq z$ the inequality $xy - z < z$ already implies $xz - y < y$ and $xz - y < y$ implies $yz - x < x$.

**Lemma 2.1.** Consider three integers $x, y, z \geq 0$.

a) If $(x, y, z) \neq (0, 0, 0)$ satisfies (M1), then $x, y, z \geq 2$ and the triple is cluster-cyclic. Moreover, in this case $(x, y, z)$ is the unique triple in its $\Gamma$–orbit satisfying condition (M1) whereas all other triples in its $\Gamma$–orbit satisfy condition (M2).

b) In the $\Gamma$–orbit of $(x, y, z)$ there is either a unique triple satisfying (M1) or a triple satisfying (M3).

c) If $(x, y, z)$ satisfies (M3), then $\min\{x, y, z\} < 2$, and the triple is cluster-acyclic.

d) If $(x, y, z)$ is a triple where one entry equals 2, then the triple is either cluster-acyclic or it is of the form $(x, x, 2)$ with $x \geq 2$.

e) If $(x, y, z)$ is a fixed point under the $\Gamma$–action, then either $C(x, y, z) = 0$ and $(x, y, z) = (0, 0, 0)$ or $C(x, y, z) = 4$ and $(x, y, z) = (2, 2, 2)$.

f) The only element $\gamma \in \Gamma$ acting trivially on $\mathbb{A}^3(\mathbb{Z})$ is the unit element.

**Proof.** Statement a) summarizes Markov’s result from [10], see also [3]. A proof adapted to the situation we consider here is given in [1] from which also most of the other statements can be derived. We give here the main arguments, starting with part c) of the lemma. Assume that the triple satisfies (M3) and suppose, without loss of generality, that $x \geq y \geq z$. Then $yz - x < x$ and $xz - y < y$ (this holds up to the action of $S_3$, then we use the assumption $x \geq y \geq z$) and thus $z^2y < 2xz < 4y$ and $z^2 < 4$. Consequently, $z = 0$ (the quiver is acyclic) or $z = 1$. In the second case, we apply the mutation $(x, y, 1) \mapsto (y - x, y, 1)$ to obtain an acyclic quiver.

To prove a), assume now we start with a triple $(x, y, z)$ satisfying (M1). Thus $x, y, z \geq 2$, otherwise we can mutate the triple to a smaller one. We start to mutate the triple as long we get a triple satisfying (M2); the triples we obtain in each step away from $(x, y, z)$ are then larger or equal to their predecessors. Assume $(x', y', z')$ is the first triple not satisfying (M2). Then it does not satisfy (M1), otherwise the previous one does not satisfy (M2). It also cannot satisfy (M3), since otherwise (using part c)) we got a triple with one entry smaller than 2, but $(x', y', z') \geq (x, y, z)$. One can think about all triples as the vertices of a graph, where each triples has at most three neighbours, the three mutated triples. This graph is also ordered, that is, it defines a poset. Using this poset it is not difficult to see the properties we claimed.

The uniqueness can also be shown as follows: Assume there are two triples in the $\Gamma$–orbit
satisfying (M1). Then there exists a sequence of mutations from the first to the second. Along this sequence there must be at least one triple which admits two smaller neighbours (use the poset as described above), thus it satisfies (M3). But this a contradiction to c).

To prove the first part of b) we note that we can always apply a mutation to obtain a smaller triple unless the triple satisfies (M1). So, by applying a finite number of mutations, one arrives either at a triple satisfying (M1) or one obtains a triple with some entry smaller than 2, then some element in the orbit satisfies (M3).

To prove d) we consider the following sequence of mutations (where we assume $x > y$):

$$(x, y, 2) \mapsto (2y - x, y, 2) \mapsto (2y - x, 3y - 2x, 2) \ldots$$

After a finite number of steps $ay - (a - 1)x < 0$, so the triple is cluster-acyclic.

If we have a fix point, then we are in case (M1), so $z^2x = 4x, y^2z = 4z$, and $x^2y = 4y$. So all values must be 0 or of absolute value 2. Then one checks all possible values explicitly.

Finally we prove f). There exist triples with $x > y > z$ satisfying (M1) (take for example $(5, 4, 3)$). Then for all elements $\gamma \in \Gamma$, we find $\gamma(x, y, z) \neq (x, y, z)$; otherwise we find an element $\gamma'$ (that is a subsequence of the mutations defining $\gamma$) with $\gamma'(x, y, z)$ satisfying (M3); we use the same argument as in part a). This yields a contradiction to the uniqueness in part b).

\[\square\]

3 The Markov equation

In this section, we explain why the Markov constant is invariant under mutations. First, we recall from \[12\] and \[7\] the appearance of the Markov equation from the study of exceptional sequences of vector bundles or modules, where one obtains a similar formula (for vector bundles on $\mathbb{P}^2$ the classical Markov equation appears). Note that exceptional sequences admit a similar notion of mutation together with an action of the braid group. It turns out that for exceptional sequences of length 3 the semidirect product of those mutations with the symmetric group is isomorphic to $\Gamma$.

For exceptional mutations (of exceptional sequences of length 3), one obtains the Markov constant as follows. Consider the Cartan matrix $D$ of the endomorphism ring of an exceptional sequence $(E_1, E_2, E_3)$. It has the form

$$D := \begin{pmatrix}
1 & x & y \\
0 & 1 & z \\
0 & 0 & 1
\end{pmatrix}$$

with integral entries $x, y, z$. Now compute its corresponding Coxeter matrix $\Phi$, then the Markov constant is the trace of $\Phi$ up to a constant:

$$\Phi = -D^t D^{-1} = -\begin{pmatrix}
1 & 0 & 0 \\
x & 1 & 0 \\
y & z & 1
\end{pmatrix}\begin{pmatrix}
1 & -x & xz - y \\
0 & 1 & -z \\
0 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
-1 & * & * \\
* & x^2 - 1 & * \\
* & * & y^2 + z^2 - xyz - 1
\end{pmatrix}$$

$$\text{Tr}(\Phi) = x^2 + y^2 + z^2 - xyz - 3.$$ 

It follows from tilting theory (see e.g. \[11\]) that the Coxeter matrices of two derived equivalent algebras are conjugate, so, in particular, their traces are equal. Since the mutation of a full
strongly exceptional sequence (it defines a tilting module as the direct sum of the elements of the sequence) yields an endomorphism ring which is derived equivalent, the Markov constant must be invariant under such mutations. This argument gives an explanation for why the Markov constant occurs; the invariance itself can be checked directly:

**Lemma 3.1.** The Markov constant $C(x,y,z) = x^2 + y^2 + z^2 - xyz$ is invariant under all cluster mutations $\mu_i$ for $i = 1, 2,$ and $3$ and invariant under the action of $\Gamma$.

**Proof.** Obviously, the Markov constant is invariant under any permutation of the variables $x, y, z$. Then it is sufficient to check invariance under the transformation $x \mapsto yz - x$:

$$(yz - x)^2 + y^2 + z^2 - (yz - x)yz = x^2 + y^2 + z^2 - xyz.$$ 

\[\square\]

From the Markov equation we get also the eigen values of the Coxter transformation. We first note that there exists always a largest real eigen value in the wild case (see [4]) or all eigen values are roots of unity in the tame case. Moreover, the characteristic polynomial is symmetric of degree three: $T^3 + (C(x,y,z) - 3)T^2 + (C(x,y,z) - 3)T + 1$ (see [2] and note that the trace of $\Phi$ is $C(x,y,z) - 3$). All together we obtain the following result. Note that the Markov constant already determines all eigen values. Moreover, all the exceptions in the main theorems satisfy $\lambda$ is not real ($\lambda$ is not real precisely when $0 < C(x,y,z) < 4$).

**Lemma 3.2.** The eigen values of the Coxter transformation $\Phi$ are $-1, \lambda, 1/\lambda$ for some element $\lambda$ satisfying either

1) $\lambda$ is real with $|\lambda| \geq 1$ and then $C(x,y,z) \geq 4$ or $C(x,y,z) \leq 0$ or

2) $\lambda$ is a complex number with $|\lambda| = 1$ and then $0 \leq C(x,y,z) \leq 4$.

Finally $-1 + \lambda + 1/\lambda = C(x,y,z) - 3$.

**Lemma 3.3.** a) Assume a triple $(x,y,z)$, with $x, y, z \geq 0$, is cluster-acyclic, then $C(x,y,z) > 0$ or $C(x,y,z) = 0$ and $x, y, z = 0$.

b) Assume a triple $(x,y,z)$ is cluster-cyclic, then $C(x,y,z) < 4$ or $C(x,y,z) = 4$ and the triple is in the $\Gamma$-orbit of $(u,u,2)$ for some $u \geq 2$.

**Proof.** Assume a triple is cluster-acyclic, then there exists an element $(x,y,z)$ in its orbit with $x, y \geq 0$ and $z \leq 0$. Then, $x^2 \geq 0$, $y^2 \geq 0$, $z^2 \geq 0$, and $-xyz \geq 0$ and we get equality only for $(0,0,0)$. Assume the triple is cluster-cyclic, then we can assume $x \geq y \geq z \geq 2$ and $yz - x \geq x$ (so that it is the unique element satisfying (M1), it is also the unique minimal element in the orbit, see Lemma 2.1). We consider $z = 2$. Then we obtain $2y \geq 2x$ so $x = y$ with $C(x,x,2) = 4$. If $z \geq 3$, then $zy \geq 2x$. Now we assume $x = y$, then $C(x,y,z) \leq 2x^2 + 9 - zx^2 \leq 0$ with equality only for $x, y, z = 3$. If $z \geq 3$ and $zy = 2x$, then $C(x,y,z) = -(z^2/4 - 1)y^2 + z^2) < 0$. Finally note that $V(C)_{x}$ (here we fix the value of $z$ and consider $V(C)_{z}$ as an affine algebraic variety with coordinates $x,y$) is a hyperboloid, thus convex and $C(x,y,z)$ is at most the value for $x = y$ or for $zy = 2x$ we computed above (all this is elementary computation, to see more details compare with the results in Section 4 and the computation of the fundamental domain in Section 5). \[\square\]

Using Lemma 3.3 we have already proven Theorem 1.1 and Theorem 1.2 for all values $C < 0$ and $C > 4$ (the details will be given later in Section 6). The remaining cases need some further
investigation, in particular, we need to determine the possible values of the Markov constant between 0 and 4. Finally, note that 4 is the maximal possible value of the Markov constant for a cluster-cyclic triple. This and Lemma 2.1 a) explains why we obtain the functions \( m^+ \) and \( m^- \) as solutions of the Markov equation for \( C = 4 \).

## 4 Connected components

In this section we consider the Markov equation over the real numbers and compute the slices

\[
V(C)_z := \{ (x, y) \in \mathbb{R}^2 \mid C(x, y, z) = C \}
\]

for a fixed value \( z \). Then the Markov equation is quadratic in \( x \) and \( y \) and we consider the quadrics \( V(C)_z \) for the various values of \( C \) and \( z \). It turns out that \( V(C)_z \) is an ellipsoid for \( |z| < 2 \), a pair of lines for \( |z| = 2 \) and a hyperbola for \( |z| > 2 \) (it might be empty for \( |z| \leq 2 \)). Using the geometry of \( V(C)_z \) we can easily determine the connected components and the singularities of \( V(C) \). It turns out that \( V(C) \) is smooth except for \( C = 4 \) (over \( \mathbb{C} \) it also has a singularity for \( C = 0 \) in \( (0,0,0) \)). If \( C = 4 \) then it has the 4 singularities \( (2,2,2), (2,-2,-2), (-2,2,-2), \) and \((-2,-2,2)\). Moreover, the \( \Gamma \)-action respects the connected components. Since we have for \( 0 \leq C \leq 4 \) five connected components in the smooth part \( V(C) \setminus V(C)_{\text{sing}} \), one bounded and four unbounded, we obtain both cluster-cyclic and cluster-acyclic orbits over \( \mathbb{R} \). We finally classify these orbits over the integers, and obtain all orbits for \( 0 \leq C \leq 4 \). The remaining values of \( C \) are easier to handle (Lemma 3.3). We also note that it is convenient to work with the \( \Gamma \)-action here since \( S_3 \) permutes some of the components, however it preserves all components contained in \( \mathbb{R}^3_{\geq 0} \).

**Lemma 4.1.** We fix \( z \) and consider the restricted action induced by the mutations \((x, y, z) \mapsto (yz - x, y, z)\) and \((x, y, z) \mapsto (x, xz - y, z)\).

a) If \(|z| < 2\) then \( V(C)_z \) is an ellipsoid (hence bounded) and for generic \( x \), \( y \) and \( z \) the \( \Gamma \)-orbit is dense in \( V(C)_z \). In particular, there is no fundamental domain for the restricted action. Moreover, the set \( V(C)_z \) is empty precisely when \( C < z^2 \).

b) If \(|z| = 2\) then \( V(C)_z \) consists of two lines for \( C > 4 \), of one line for \( C = 4 \) and is empty for \( C < 4 \). If \( C = 4 \) then the restricted action is trivial, otherwise each generator interchanges the two components and the composition of the two generators acts as a translation along the lines. A fundamental domain for the restricted action on the set \( \{(x, y) \in \mathbb{R}^2 \mid x \neq y\} \) is \( \{(x, y) \in \mathbb{R}^2 \setminus (0,0) \mid y \leq 2x, y \leq 0\} \).

c) If \(|z| > 2\) then \( V(C)_z \) is a hyperbola for \( C \neq z^2 \). It is the union of two lines \( x = \lambda y \) and \( x = y/\lambda \) for \( C = z^2 \), where \( \lambda + 1/\lambda = z \). For \( C < z^2 \) one component of \( V(C)_z \) is contained in \( \mathbb{R}^2_{\geq 0} \), the other one in \( \mathbb{R}^2_{< 0} \). It turns out that each orbit is unbounded in this case. For the restricted action we have a fundamental domain.

**Proof.** Assume \(|z| < 2\), thus \( x^2 + y^2 - xyz \geq 0 \) and \( C(x, y, z) = x^2 + y^2 + z^2 - xyz \geq z^2 \). To see that the \( \Gamma \)-orbit is dense compute the eigenvalues of the linear transformation defined by the generators. If they are not rational the orbit must be dense. And since the coefficient of \( xy \) is smaller than 2, the equation defines an ellipsoid. This shows a). To prove b) write the Markov equation as follows: \( C(x, y, z) = (x-y)^2 + 4 = C \). Thus \( V(C) \) is empty for \( C < 4 \) and for \( C = 4 \) we get one line, for \( C > 4 \) we get two lines. The action is generated by \((x, y) \mapsto (2y - x, y)\)
and \((x, y) \mapsto (x, 2x - y)\), and their composition is \((x, y) \mapsto (2y - x, 3y - 2x)\). This is a linear action preserving the orientation, so we only need to compute the image \(\mathbb{R}_{\geq 0}(-1, -2)\) of the line \(\mathbb{R}_{\geq 0}(1, 0)\) and a fundamental domain is the cone generated by these two rays. To prove c) we first compute the hyperboloids and the asymptotic lines. Finally, we can also use the linearity of the action to get a fundamental domain.

Now we use the previous result to get information on the action of \(\Gamma\) and the components.

**Theorem 4.2.** Consider the action of \(\Gamma\) on \(V(C)\).

a) The set \(V(C)\) is a smooth manifold for all \(C \neq 4\) and for \(C = 4\) it has four cone singularities. Denote by \(V(C)^{\text{smooth}}\) the open smooth part.

b) The group \(\Gamma\) respects the connected components of \(V(C)^{\text{smooth}}\).

c) The number of connected components depends on the value of \(C\) as follows:

| \(C\)   | \(\text{conn. comp. of } V(C)\) | \(\text{conn. comp. of } V(C)^{\text{smooth}}\) | \(\text{compact conn. comp.}\) |
|--------|-------------------------------|---------------------------------|-------------------------------|
| \(< 0\) | 4                             | 4                               | 0                             |
| \(0\)  | 5                             | 5                               | 1                             |
| \(0 < C < 4\) | 5                                 | 5                               | 1                             |
| \(C = 4\) | 1                                 | 5                               | 0                             |
| \(C > 4\) | 1                                 | 1                               | 0                             |

d) For \(C = 0\) the compact connected component is the point \((0, 0, 0)\). All compact connected components are contained in \([-2, 2]^3\). All orbits in a non-compact connected component have a sequence of elements that converges to infinity.

e) All finite orbits are contained in the compact components or in the singular part.

**Proof.** We compute the singularities over \(C\): consider the differential

\[
\frac{\partial C(x, y, z)}{\partial x} = 2x - yz.
\]

It must vanish for any choice of the coordinate, thus

\[
2x = yz, \quad 2y = xz, \quad \text{and} \quad 2z = xy \quad \text{yields} \quad 4x = y^2, \quad 4y = z^2 \quad \text{and} \quad 4z = x^2.
\]

Then either \(x = 0\) and \(z, y = 0\) too, or \(|y| = 2\) and \(|x| = |z| = 2\), which yields the triples \((2, 2, 2), (2, -2, -2), (-2, 2, -2), \) and \((-2, -2, 2)\). Now we consider \(V(C)\) over \(\mathbb{R}\) again. Since \((0, 0, 0)\) is an isolated point in \(V(0)\) (consider the slices \(V(C)_z\) we have already computed in the previous lemma) it is not a singular point (over \(\mathbb{R}\)). This proves a) and the first part of d). To show b) we note that the group acts via differentiable (in fact even algebraic) transformations, thus preserves the singularities. To finish the proof of b) we need to show c) first: Assume first \(C \geq 4\). Then \(V(C)_z\) is never empty and connected for \(|z| < 2\). Now vary \(z\), then we obtain a path from a point in \(V(C)_z\) to each other slice \(V(C)_{z'}\). Thus \(V(C)\) is connected. Assume now \(0 < C < 4\). Consider \(2 > |z| > 0\). For \(z = 2 - \varepsilon\) and \(\varepsilon > 0\) sufficiently small the variety \(V(C)_z\) is empty. On the other hand \(V(C)_{z'}\) is non-empty for \(z = 0\) and \(|z| > 2\). Playing this with all three coordinates we get 5 connected components, one is contained in \([-2, 2]^3\) and is compact, the four others are contained in \([2, \infty]^3\), \([-\infty, -2] \times [-\infty, -2] \times [2, \infty]\),
$[-\infty, -2] \times [2, \infty] \times [-\infty, -2]$, respectively $[2, \infty] \times [-\infty, -2] \times [-\infty, -2]$. Note that no component can be contained in $[\infty, -2] \times [2, \infty] \times [2, \infty]$, since the Markov constant is at least 4. If $C < 0$ the compact component vanishes and for $C = 0$ it is just the origin. If $0 \leq C \leq 4$ then we have both, a compact component and four unbounded components. The arguments are similar to the ones above. One considers the slices $V(C)_z$ for the various $z$ and applies the action of the symmetric group and the possible sign changes.

Now we can finish the proof of b): since the five components for $C < 4$ (respectively the components in the smooth part for $C = 4$) respect the inequality $|z|, |y|, |z| \leq 2$ in two variables, they must respect it also in the third one by part c). Note that $\Gamma$ (or any permutation) can change signs and permutes three of the non-compact components, it only preserves the compact component and the one in $[2, \infty]^3$.

5 The fundamental domain

The uniqueness in Lemma 2.1 a) suggests the existence of a fundamental domain for the $\Gamma$-action on the set of cluster-cyclic triples in $\mathbb{R}^3$. We note that the uniqueness was proven only for integral triples. Indeed, there exist real triples (they cannot be integral by Lemma 2.1) that are cluster-cyclic and only have triples of type (M2) in their orbit. We will show that those triples can only occur for the Markov constant 4. Thus for all real triples with fixed Markov constant $C$ there exists a unique triple satisfying (M1) only if the Markov constant is less than 4. Consequently, there should exist a fundamental domain for all cluster-acyclic triples with Markov constant strictly less than 4. We define

$$F := \{(x, y, z) \in \mathbb{R}^3 \mid x \geq y \geq z \geq 2, yz \geq 2x\} \text{ and}$$

$$F^o := F \setminus \{(x, x, 2) \mid x \geq 2\} = \{(x, y, z) \in F \mid C(x, y, z) < 4\}.$$

Using Lemma 3.3 we obtain $C(F) \subseteq (-\infty, 4]$ and $C(F^o) \subseteq (-\infty, 4)$, since the elements in $F$ and $F^o$ are cluster-acyclic and satisfy (M1).

**Theorem 5.1.** a) Let $(x, y, z)$ be a real cluster-cyclic triple. Then there exists a unique triple $f(x, y, z)$ in $F$ that is in the closure (with respect to the ordinary topology) of the orbit $\Gamma(x, y, z)$. If the triple $(x, y, z)$ is integral or of Markov constant strictly less than 4 then $f(x, y, z) \in F \cap \Gamma(x, y, z)$.

b) Consider the action of $\Gamma$ on the set $\{(x, y, z) \mid x, y, z > 2; C(x, y, z) < 4\}$. Then $F^o$ is a fundamental domain for this action and the set $S_3 \cdot F^o$ is a fundamental domain for the $\Gamma$-action.

**Proof.** We first note that Lemma 2.1 a), c), d), e), and f) are also true for real triples. Only b) might be not. We obtain, by the uniqueness, a fundamental domain for the action on the set of all triples which do have a triple satisfying (M1) in their orbit. We show, it is already the fundamental domain as claimed above. Obviously, each element in $F$ satisfies (M1). We need to show that $\Gamma \cdot F^o = \{(x, y, z) \mid x, y, z > 2; C(x, y, z) < 4\}$. Take any triple in $\{(x, y, z) \mid x, y, z > 2; C(x, y, z) \leq 4\}$. Now we start to mutate in the unique way, so that in each step we get a strictly smaller triple (using the partial order defined in Section 2). Assume we got a smallest triple $f(x, y, z)$, then it is an element in $F$ (since all elements satisfying
Since the Markov constant $C$ is in $F$ by definition. Assume there is not a minimal triple with Markov constant $C < 4$. Then the sequence of triples converges (it is decreasing and bounded below) and the limit $f(x, y, z)$ must be in $F$. If the triple $f(x, y, z)$ is in the interior of $F$ (this is $F$ without its boundary) then a small ball around $f(x, y, z)$ is also in $F$ contradicting the fact that no element of the convergent sequence is in $F$. A similar argument works if $f(x, y, z)$ is in the boundary of $F$ for $C(x, y, z) = C(f(x, y, z)) < 4$, take a small ball around the point $f(x, y, z)$ with all Markov constants strictly less than 4. Then we find that the ball around $f(x, y, z)$ is contained in $S_3F \cup \mu_1S_3F \cup \mu_2S_3F \cup \mu_3S_3F$ (one can see, that each of the elements $\mu_i$ fixes one of the boundaries.

**Lemma 5.2.** a) Assume $(x, y, z)$ is in $F_2$. Then $C(x, y, z) = 4$, $C(x, y, z) = 0$ or $C(x, y, z) < 0$.

b) $F_2 \cap V(0) = \{(3, 3, 3)\}$
c) $F_2 \cap V(4) = \{(x, x, 2) \mid x \in \mathbb{Z}_{\geq 2}\}$.

**Proof.** We need to compute the maximal value of the Markov constant on the slices of the fundamental region $F_2 := \{(x, y, z) \in F\}$ where we fix $z$ (we show that only 4, 0 and negative values may occur). Since $F_2$ is obviously bounded by lines, it is an intersection of affine half spaces, in particular it is convex. It is a little explicit computation to see that the function $C(x, y, z)$ takes its minimal value on the vertices of $F_2$, that is on $(z, z, z)$ or on $(z^2/2, z, z)$. The Markov constants are $3z^2 - z^3 = z^2(3 - z)$ and $-z^4/4 + 2z^2 = z^2(2 - z^2/4)$. Considering the values for $z = 2, 3, 4, \ldots$ we obtain that the maximum is $z^2(3 - z)$ and it is in the critical part $[0, 4]$ only for $z = 2$ or 3. Finally, we need to classify the orbits for $C = 4$, that is already done using Lemma 5.3 and for $C = 0$. For $C = 0$ it is the classical Markov equation and there are precisely two orbits: $(0, 0, 0)$ (it is acyclic) and $\Gamma \cdot (3, 3, 3) = \Gamma \cdot (3, 3, 3)$ (it is cyclic). For convenience we repeat the arguments: assume $C(x, y, z) = 0$, considering the equation modulo 3 shows that 3 divides each entry in the triple. Now use the fundamental region. If $z \geq 4$ then there is no integral point in $F$ with Markov constant 0. We can assume $z = 3$, since $z = 2$ can not appear. Since $C = 0$ is the maximum on $F_3$ all other elements in $F_{3\geq 3}$ must have Markov constant strictly smaller than 0.

**Corollary 5.3.** For each integer $C \neq 4$ the number of integral $\Gamma$-orbits with Markov constant $C$ is finite.

**Proof.** The same arguments (convexity and shape of $F_2$) show that the set $F_2 \cap \{(x, y, z) \mid C(x, y, z) \geq C\}$ is bounded for any $z$ and any $C$. Consequently, there are only finitely many lattice points in this set and there are only finitely many orbits. If the triple is cluster-acyclic we consider the set of all acyclic triples with Markov constant $C$. This set is also finite, since the equation

$$x^2 + y^2 + z^2 - xyz = C$$

has only finitely many integral solutions for $x, y \geq 0$ and $z \leq 0$.

## 6 Proof of the Main Theorems

In this section we finally collect all the arguments for the proof of the two main theorems. Since the Markov constant $C(x, y, z)$ is invariant under cluster mutation and permutation we

(M1) are in $F$ by definition). Assume there is not a minimal triple with Markov constant

$C < 4$. Then the sequence of triples converges (it is decreasing and bounded below) and the limit $f(x, y, z)$ must be in $F$. If the triple $f(x, y, z)$ is in the interior of $F$ (this is $F$ without its boundary) then a small ball around $f(x, y, z)$ is also in $F$ contradicting the fact that no element of the convergent sequence is in $F$. A similar argument works if $f(x, y, z)$ is in the boundary of $F$ for $C(x, y, z) = C(f(x, y, z)) < 4$, take a small ball around the point $f(x, y, z)$ with all Markov constants strictly less than 4. Then we find that the ball around $f(x, y, z)$ is contained in $S_3F \cup \mu_1S_3F \cup \mu_2S_3F \cup \mu_3S_3F$ (one can see, that each of the elements $\mu_i$ fixes one of the boundaries.

**Lemma 5.2.** a) Assume $(x, y, z)$ is in $F_2$. Then $C(x, y, z) = 4$, $C(x, y, z) = 0$ or $C(x, y, z) < 0$.

b) $F_2 \cap V(0) = \{(3, 3, 3)\}$
c) $F_2 \cap V(4) = \{(x, x, 2) \mid x \in \mathbb{Z}_{\geq 2}\}$.

**Proof.** We need to compute the maximal value of the Markov constant on the slices of the fundamental region $F_2 := \{(x, y, z) \in F\}$ where we fix $z$ (we show that only 4, 0 and negative values may occur). Since $F_2$ is obviously bounded by lines, it is an intersection of affine half spaces, in particular it is convex. It is a little explicit computation to see that the function $C(x, y, z)$ takes its minimal value on the vertices of $F_2$, that is on $(z, z, z)$ or on $(z^2/2, z, z)$. The Markov constants are $3z^2 - z^3 = z^2(3 - z)$ and $-z^4/4 + 2z^2 = z^2(2 - z^2/4)$. Considering the values for $z = 2, 3, 4, \ldots$ we obtain that the maximum is $z^2(3 - z)$ and it is in the critical part $[0, 4]$ only for $z = 2$ or 3. Finally, we need to classify the orbits for $C = 4$, that is already done using Lemma 5.3 and for $C = 0$. For $C = 0$ it is the classical Markov equation and there are precisely two orbits: $(0, 0, 0)$ (it is acyclic) and $\Gamma \cdot (3, 3, 3) = \Gamma \cdot (3, 3, 3)$ (it is cyclic). For convenience we repeat the arguments: assume $C(x, y, z) = 0$, considering the equation modulo 3 shows that 3 divides each entry in the triple. Now use the fundamental region. If $z \geq 4$ then there is no integral point in $F$ with Markov constant 0. We can assume $z = 3$, since $z = 2$ can not appear. Since $C = 0$ is the maximum on $F_3$ all other elements in $F_{3\geq 3}$ must have Markov constant strictly smaller than 0.

**Corollary 5.3.** For each integer $C \neq 4$ the number of integral $\Gamma$-orbits with Markov constant $C$ is finite.

**Proof.** The same arguments (convexity and shape of $F_2$) show that the set $F_2 \cap \{(x, y, z) \mid C(x, y, z) \geq C\}$ is bounded for any $z$ and any $C$. Consequently, there are only finitely many lattice points in this set and there are only finitely many orbits. If the triple is cluster-acyclic we consider the set of all acyclic triples with Markov constant $C$. This set is also finite, since the equation

$$x^2 + y^2 + z^2 - xyz = C$$

has only finitely many integral solutions for $x, y \geq 0$ and $z \leq 0$.

## 6 Proof of the Main Theorems

In this section we finally collect all the arguments for the proof of the two main theorems. Since the Markov constant $C(x, y, z)$ is invariant under cluster mutation and permutation we
can consider the various values of $C$ case by case. Assume first $C > 4$ or $C < 0$. Then parts of the theorems follow from Lemma 3.3 in Theorem 1.1 (1), (2) and (3) are equivalent. Since $m^+(x, y) = z$ or $m^-(x, y) = z$ precisely when $C(x, y, z) = 4$ we obtain $m^+(x, y) < z$ or $m^-(x, y) > z$ precisely when $C(x, y, z) > 4$ for all $x, y \geq 2$. Thus, under the assumption $C(x, y, z) > 4$ condition (1) is also equivalent to (4). Similar arguments work for Theorem 1.2 under the assumption $C(x, y, z) < 0$: we get (1) if and only if (2) and (3) by Lemma 3.3. The same argument as above and $x, y, z \geq 2$ (otherwise $C(x, y, z) \geq 0$ by Lemma 3.1 a) and its proof) shows the equivalence with (4). Finally (5) is equivalent by Theorem 1.1 (a).

Now we consider $C(x, y, z) = 0$. Then we are in the case of the classical Markov equation

$$X^2 + Y^2 + Z^2 - 3XYZ = 0,$$

where $3X = x, 3Y = y, 3Z = z$.

We have already shown that each Markov triple is divisible by 3, so we can define new variables $X, Y, Z$ and get a new equation as above. The solutions are well-known and the only orbits are $(0, 0, 0)$ (it is cluster-acyclic) and $\Gamma(3, 3, 3) = \Gamma(3, 3, 3)$ (it is cluster-cyclic and corresponds to the orbit of $(1,1,1)$ for the classical Markov equation). Comparing with the statement in the theorems finishes this case.

Now we consider $0 < C(x, y, z) < 4$. In this case there is no integral triple in the fundamental domain $F$, thus all triples must be cluster-acyclic. Using the connected components we can, case by case, classify all triples with $x, y, z \leq 2$ explicitly (gives the list in Theorem 1.1 (3) and nothing in Theorem 1.2).

Finally, we need to consider $C(x, y, z) = 4$. Again we classify all triples with $x, y, z \leq 2$ (we obtain the two orbits in Theorem 1.1 (3) and one orbit in Theorem 1.2 (3)).

References

[1] I. Assem, M. Blais, T. Brüstle, A. Samson, *Mutation classes of skew-symmetric $3 \times 3$-matrices*, arXiv:math.RT/0610627, to appear in Comm. Alg.

[2] A. Buan, R. Marsh, M. Reineke, I. Reiten and G. Todorov, *Tilting theory and cluster combinatorics*, Adv. Math. 204 (2006), 572-612.

[3] J. W. S. Cassels, *An Introduction to Diophantine Approximation*. Facsimile reprint of the 1957 edition. Cambridge Tracts in Mathematics and Mathematical Physics, No. 45. Hafner Publishing Co., New York, 1972.

[4] J. A. de la Pea, M. Takane, *Spectral properties of Coxeter transformations and applications*. Arch. Math. (Basel) 55 (1990), no. 2, 120–134.

[5] S. Fomin and A. Zelevinsky, *Cluster algebras I. Foundations*, J. Amer. Math. Soc. 15(2), (2002), 497-529 (electronic)

[6] S. Fomin and A. Zelevinsky, *Cluster algebras II. Finite type classification*, Inventiones Mathematicae 154(1), (2003), 63-121.

[7] A. L. Gorodentsev, A. N. Rudakov, *Exceptional vector bundles on projective spaces*. Duke Math. J. 54 (1987), no. 1, 115–130.
[8] O. Kerner, *Wild cluster tilted algebras of rank three*, preprint, July 2006.

[9] H. Lenzing, *Coxeter transformations associated with finite-dimensional algebras*. Computational methods for representations of groups and algebras (Essen, 1997), 287–308, Progr. Math. 173, Birkhäuser, Basel, 1999.

[10] A. Markoff, *Sur les formes quadratiques binaires indéfinies*, Mathematische Annalen, Band XVII (1882), p. 379-399.

[11] C. M. Ringel, *Tame algebras and integral quadratic forms*. Lecture Notes in Mathematics, 1099. Springer-Verlag, Berlin, 1984.

[12] A. N. Rudakov *Markov numbers and exceptional bundles on* $P^2$. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 52 (1988), no. 1, 100–112, 240; translation in Math. USSR-Izv. 32 (1989), no. 1, 99–112.
Appendix

The aim of the appendix is to present an interpretation of the Markov constant \( C(x, y, z) = x^2 + y^2 + z^2 - xyz \) in the mutation acyclic case in terms of a first Hochschild cohomology group.

Let \( H = KQ \) be a basic hereditary algebra, where \( Q \) is a finite quiver without oriented cycles, and \( K \) is an algebraically closed field. The \( n \) vertices of \( Q \) are \( \{1, \ldots, n\} \). The set of arrows is denoted by \( Q_1 \). If \( \alpha \) is an arrow, then \( s(\alpha) \) denotes its starting point and \( t(\alpha) \) its terminating point. By \( e_i \) we denote the primitive idempotent of \( H \), corresponding to the vertex \( i \).

Denote by \( H^1(H) \cong \text{Ext}^1_{H^e}(H, H) \) the first Hochschild cohomology group of \( H \) with coefficients in \( H \), where \( H^e \) is the enveloping algebra of \( H \). It is shown in [2, 1.6] that \( \dim H^1(H) \) can be expressed as

\[
\dim H^1(H) = d - n + \sum_{\alpha \in Q_1} \nu(\alpha),
\]

where \( d \) is the number of connected components of the quiver \( Q \), and \( \nu(\alpha) \) is the number of paths from \( s(\alpha) \) to \( t(\alpha) \) which means \( \nu(\alpha) = \dim e_{t(\alpha)} H e_{s(\alpha)} \).

It should be mentioned that, similar to the Markov constant, also \( \dim H^1(H) \) is related to the trace of the Coxeter transformation of the hereditary algebra \( H \), see for example [3, 3.2.1].

Let \( C(x, y, z) \), with \( x, y, z > 0 \) be a cyclic quiver with three vertices

\[
\begin{array}{c}
1 \\
\downarrow v \\
2 \\
\downarrow u \\
3 \\
\end{array}
\]

where \( i \xrightarrow{a} j \) means that there are \( a \) arrows from \( i \) to \( j \).

If the cyclic quiver \( C(x, y, z) \) is mutation acyclic, then, after a finite sequence of mutations one gets an acyclic quiver \( Q(r, s, t) \) of the form

\[
\begin{array}{c}
1 \\
\downarrow v \\
2 \\
\downarrow u \\
3 \\
\end{array}
\]

with \( r, s > 0 \) and \( t \geq 0 \). Hence \( C(x, y, z) \) is the quiver of a cluster tilted algebra \( \Gamma \) of type \( H = KQ(r, s, t) \), see for example [1]. In this case one has:

**Theorem** If \( C(x, y, z) \) is the quiver of a cluster tilted algebra \( \Gamma \) of type \( H \), where \( H \) is connected hereditary of rank three, then

\[
C(x, y, z) - 2 = \dim H^1(H).
\]
Proof. Starting with the path algebra $H$ of the quiver $Q(r, s, t)$ one gets from Happel’s result the formula
\[
\dim H^1(H) = r^2 + s^2 + t^2 + rst - 2.
\]
Mutation at the vertex 2 of the quiver $Q(r, s, t)$ yields a cyclic quiver $\mu_2Q(r, s, t)$ of the form

But the Markov constant of this cyclic quiver is
\[
C(t + rs, s, r) = (t + rs)^2 + s^2 + r^2 - (t + rs)rs = r^2 + s^2 + t^2 + rst,
\]
which proves the theorem, since the Markov constant is invariant under mutations, as long as the quiver is cyclic.

Remark. Let $C(x, y, z)$ be a fixed cyclic quiver, which is mutation acyclic, hence the quiver of a cluster tilted algebra $\Gamma$. Using its Markov constant $C(x, y, z)$ one can determine the finite list of connected hereditary path-algebras $H$, with three vertices such that $\dim H^1(H) + 2 = C(x, y, z)$. $\Gamma$ then is cluster tilted of type $H$, where $H$ belongs to this list. The list can be quite big, as the following example shows: Let $H$ be the path algebra of a quiver of type $Q(2, s, t)$. Then $\dim H^1(H) = 2 + (s + t)^2$.

References

[1] A. Buan, R. Marsh and I. Reiten, Cluster mutations via quiver representations, arXiv:math.RT/0412077
[2] D. Happel, Hochschild cohomology of finite-dimensional algebras, in Sem. M.P. Malliavin, Springer Lect. Notes Math. 1404 (1989), 108–126.
[3] F. Lukas, Elementare Moduln über wilden erblichen Algebren, Thesis, Düsseldorf 1992.