Experimental Purification of Single Qubits.

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Abstract

We report the experimental realization of the purification protocol for single qubits sent through a depolarization channel. The qubits are associated with polarization encoded photon particles and the protocol is achieved by means of passive linear optical elements. The present approach may represent a convenient alternative to the distillation and error correction protocols of quantum information.
Modern quantum data processing using realistic (imperfect) quantum gates and long-distance quantum communication in the presence of a noisy environment requires a large supply of qubits with a high degree of purity. Indeed the fidelity of most quantum information (QI) protocols critically depends on the preservation of the purity of the QI carriers. It is therefore crucial to develop techniques that protect quantum states from the unavoidable losses and decoherence processes accompanying the transmission. One of these techniques is the quantum error correction [1,2] which works by encoding the quantum state into a higher-dimensional Hilbert space. Alternatively, one can distribute several copies of an entangled state and extract fewer highly-entangled states by means of entanglement distillation [3–7] in order to be able to subsequently transmit an arbitrary state with high fidelity by quantum teleportation [8–11]. Yet another option is to transmit several copies of the state over the noisy channel and then purify the resulting mixed states at the receiver’s station [12–14].

The present work realizes the purification procedure that was theoretically proposed by Cirac et al. in 1999 [12]. It addresses the issue of the purification of $N$ equally prepared qubits in the mixed state $\rho = \xi |\phi\rangle \langle \phi| + \frac{1}{2}(1 - \xi)\mathbb{I}$, where $0 \leq \xi \leq 1$. This procedure allows to distill from a set of mixed states a subset of states with a higher degree of purity, i.e. it probabilistically increases the purity by filtering out some of the noise. The procedure is based on a set of projections onto the symmetric subspace of the $N$ qubits (i.e. the subspace spanned by all the states that are invariant under any permutation of the $N$ qubits) and onto orthogonal subspaces that contain symmetric subspaces for subsets of the initial $N$ qubits. The procedure is designed to be optimal and universal, i.e., it acts with the same fidelity for all input states. Since it is optimal, the purity cannot be further increased by any means. In this paper we consider the case of two qubits, i.e. $N = 2$. The purification procedure for $N = 2$ reduces to a projection of the two-qubit state onto the symmetric subspace, and it is equivalent to the symmetrization procedure proposed as a theoretical method to stabilize quantum computation in the presence of noise [15].

For $N = 2$ the purification procedure works as follows. Consider two independent qubits, $a$ and $b$, both originally in the state $|\phi\rangle$, that are transmitted over a noisy channel from which
they emerge in a mixed state represented by the density matrix 
\[ \rho_a = \xi |\phi\rangle \langle \phi| + \frac{1}{2} (1 - \xi) I = \frac{1 + \xi}{2} |\phi\rangle \langle \phi| + \frac{1 - \xi}{2} |\phi^\perp\rangle \langle \phi^\perp| , \]
where \( |\phi^\perp\rangle \) is a state orthogonal to \( |\phi\rangle \). Our goal is then to
purify the transmitted qubits in order to obtain two qubits that are as close as possible to
the original state \( |\phi\rangle \). The overall 2-qubit input state \( \rho^\text{in}_{ab} = \rho^\text{in}_a \otimes \rho^\text{in}_b \) is expressed in the
basis \( \{ |\phi\rangle_a, |\phi^\perp\rangle_a, |\phi\rangle_b, |\phi^\perp\rangle_b \} \) by the matrix
\[
\rho^\text{in}_{ab} = \frac{1}{4} \begin{pmatrix}
(1 + \xi)^2 & 0 & 0 & 0 \\
0 & 1 - \xi^2 & 0 & 0 \\
0 & 0 & 1 - \xi^2 & 0 \\
0 & 0 & 0 & (1 - \xi)^2 \\
\end{pmatrix}
\] (1)

As mentioned above, the purification protocol consists of the projection of the 2-qubit state
onto the symmetric subspace: if the projection is successful we obtain two equal output
qubits that are the optimal ”purified” ones, otherwise we discard the output states. We
note that this protocol can be implemented, for every qubit encoding, by a quantum circuit
requiring an ancilla qubit and a Toffoli gate [16]. After a successful projection the output
qubits are in the state
\[
\rho^\text{out}_{ab} = \frac{\Pi \rho^\text{in}_{ab} \Pi^\dagger}{\text{Tr}[\Pi \rho^\text{in}_{ab} \Pi^\dagger]} = \frac{1}{3 + \xi^2} \begin{pmatrix}
(1 + \xi)^2 & 0 & 0 & 0 \\
0 & \frac{1 - \xi^2}{2} & \frac{1 - \xi^2}{2} & 0 \\
0 & \frac{1 - \xi^2}{2} & \frac{1 - \xi^2}{2} & 0 \\
0 & 0 & 0 & (1 - \xi)^2 \\
\end{pmatrix}
\] (2)

where \( \Pi = \mathbb{I}_{ab} - |\Psi^-_{ab}\rangle \langle \Psi^-_{ab}| \) is the projector onto the symmetric subspace and \( |\Psi^-_{ab}\rangle = (|01\rangle - |10\rangle)/\sqrt{2} \) is the singlet state of two qubits. The success probability of the procedure
is \( p = \text{Tr}[\Pi \rho^\text{in}_{ab} \Pi^\dagger] = \frac{3 + \xi^2}{4} \). Since \( \rho^\text{out}_{ab} \) belongs to the symmetric subspace, the reduced density
matrices of the resulting single qubits, expressed in the basis \( \{ |\phi\rangle, |\phi^\perp\rangle \} \), are found to be
identical,
\[
\rho^\text{out}_a = \rho^\text{out}_b = \frac{1}{2} \begin{pmatrix}
1 + \xi_P & 0 \\
0 & 1 - \xi_P \\
\end{pmatrix},
\] (3)
where $\xi_p = \frac{4}{3} \xi^2 \geq \xi$ and the purification gain factor is $\eta = \frac{4}{3} \xi^2$. Note that $p$ and $\eta$ are related by the equation $\eta p = 1$ so a higher purification gain factor is necessarily accompanied by a lower probability of success.

We report the implementation of the above protocol for qubits encoded in the polarization of single photons (see Figure 1). The qubit to be purified is $\frac{1+\xi}{2} |\phi\rangle \langle \phi| + \frac{1-\xi}{2} |\phi^+\rangle \langle \phi^+|$ where $|\phi\rangle = \alpha |H\rangle + \beta |V\rangle$ and $|H\rangle, |V\rangle$ respectively correspond to the horizontal and vertical linear polarizations. In the present experiment, pairs of photons with wavelength $\lambda = 532$ nm and coherence time $\tau_{coh} = 80$ fs, were generated in a Type I, BBO crystal slab in the product state $|H\rangle_a |H\rangle_b$ by spontaneous parametric down conversion (SPDC) process excited by a CW fourth-harmonic-generation laser (Coherent Verdi +MBD-266). The output state was first encoded in the state $|\phi\rangle_a |\phi\rangle_b$ by means of two equal waveplates (wp) $WP(|\phi\rangle)$ and then each photon, injected into a noisy channel $P$, emerged in the mixed state: $\rho_a = \rho_b = \xi |\phi\rangle \langle \phi| + (1-\xi)\frac{I}{2}$. The two mixed qubits, associated with the two modes $k_a$ and $k_b$, were linearly superimposed at beam-splitter $BS$ with a mutual time delay $\Delta t$ micrometrically adjustable by a translation stage with position settings $Z = 2\Delta tc$, with $c$ denoting the velocity of light. The value $Z = 0$ was assumed to correspond to the full overlapping of the photon pulses injected into $BS$, i.e. to the maximum photon interference leading to the simultaneous detection of two photons on either output modes $k_1$ or $k_2$ of $BS$ [17][18]. Recently it has been shown that the projection of the overall state in the symmetric subspace, precisely the one implying the present purification procedure, is unambiguously identified by the maximum interference condition: $Z = 0$ [18][19].

Let us give more details about the realization of the two equal depolarizing channels $P$ and $P'$ operating on the $BS$ input modes $k_a, k_b$, respectively. Each channel implemented the quantum map $\mathcal{E}(\rho) = \xi \rho + (1-\xi)\mathcal{E}_{DEP}(\rho)$ where $\mathcal{E}_{DEP}(\rho)$ maps any unknown input state $\rho$ into a fully mixed one. This transformation can be achieved by stochastically applying the full set of Pauli operators $\{I, \sigma_x, \sigma_y, \sigma_z\}$ with the same statistical weight, that is, $\mathcal{E}_{DEP}(\rho) = \frac{1}{4}(I \rho I + \sigma_x \rho \sigma_x + \sigma_y \rho \sigma_y + \sigma_z \rho \sigma_z)$. Let us consider here only one of the noisy channels, say $P$, the one operating on the mode $k_a$. The $\mathcal{E}(\rho)$ map was realized by means of a pair of
equal electro-optic (EO) LiNbO$_3$ Pockels cells (P-cell), $P_X$ and $P_Z$, carefully aligned with a $45^\circ$ mutual spatial orientation of the optical axes of the EO crystals (see Fig. 1). All P-cells were Shanghai Institute of Ceramics devices with a $\lambda/2$-voltage: $V_{\lambda/2} = 390$ V. Each P-cell of the pair was driven by a CW periodic square-wave electric field with maximum $V = V_{\lambda/2}$, fixed frequency $f = T^{-1}$ and variable pulse duration $\tau$ corresponding to a duty-cycle $\nu = \tau/T$ adjustable in the range: $0 < \nu < 1/2$. The excitation pulses feeding the two P-cells were mutually delayed by a time equal to $\tau/2$ (see inset of Fig. 1). Consider a single excitation cycle. In the time intervals $\Delta \tau = \tau/2$ in which only one P-cell was active, either the $\sigma_x$ or the $\sigma_z$ transformation was implemented depending on the corresponding crystal orientation. In the interval $\Delta \tau = \tau/2$ in which both P-cells were simultaneously active, the $\sigma_y$ transformation was realized. In summary, each operators $I, \sigma_x, \sigma_y, \sigma_z$ was applied to the input state over a time $\Delta \tau = \tau/2$ and the total depolarizing process lasted a time $2\tau$ over each period $T$, thus achieving an average depolarizing fraction $(1 - \xi) = 2\nu$. In order to avoid any correlation between the two qubits to be purified, $\rho_a$ and $\rho_b$, the two channels $P$ and $P'$ were driven by different frequencies: $f = 6$ KHz and $f' = 1.7 \times f$. Correspondingly, different values of $\tau$ were adopted for the two channels in order to realize, within each experimental run, equal values of $\xi$ for the two input qubits.

In the analysis we have assumed an identical preparation of the two input qubits, while the output ones are described by the same density matrix $\rho = \rho_a = \rho_b$. With this assumption, carefully checked over each channel, the verification of the purification procedure lies on the tomography of the density matrix of one of the input and one of the output qubits. For the sake of simplicity, we only analyzed the measurements performed on the $BS$ output mode $k_1$ (see Fig. 1), selecting counts in coincidence between the detectors $[D_1, D_2]$ to trigger the realization of the projection of $\rho$ onto the symmetric sub-space. The detectors $D_1, D_2$ were coupled to mode $k_1$ by a 50:50 beam-splitter $BS_1$. $D_1$ provided the measured outcomes of a simple tomographic setup consisting of a $\lambda/2$-wp, a $\lambda/4$-wp and a polarizing beam splitter ($PBS$).

Consider first the projector switched off, by setting $Z \gg c_{\text{coh}}$, i.e., by spoiling any
interference on the photons impinging on \( BS \). A tomographic reconstruction of the qubit in the mode \( k'_1 \) based on the measurement of the corresponding Stokes parameters by 4 different settings of the wp’s \( \frac{1}{2}, \frac{3}{4} \) was undertaken. It is easy to see that this qubit, corresponding to the qubit to be purified, is expressed by the density matrix \( \rho'_1 = \frac{\rho'^{in} + \rho'^{out}}{2} \) [20]. By turning on the projector, i.e., by restoring the \( BS \) interference setting \( Z = 0 \), the mode \( k_1 \) contains the two photons described by the density matrix \( \rho'^{out} \). In this case, we measured on the mode \( k'_1 \) the purified qubit \( \rho'_1 = \rho'^{out} \). From the density matrices reconstructed in absence and in presence of interference, we obtain \( \xi, \xi_p \) and thus the purification factor \( \eta = \xi_p / \xi \). In addition, from the coincidence rates determined for \( Z = 0 \) and for \( Z \gg c_{\text{coh}} \) we inferred the success probability \( p \) of the purification protocol. We may check that an increase of the purification gain factor for any qubit pair, i.e., a larger \( \eta \), corresponds to a lower success probability of the overall protocol, as expected. In Fig. 2, we plotted the experimental values of \( \eta \) and \( p \) obtained for different \( \xi \)'s, i.e., for different experimental values of \( \nu = (1 - \xi)/2 \), for three input states: \( |H\rangle, |L\rangle = 2^{-\frac{1}{2}}(|H\rangle + |V\rangle) \), and \( |E\rangle = [\cos(3\pi/16)|H\rangle + i\sin(3\pi/16)|V\rangle \) corresponding, respectively, to horizontal, 45°-diagonal, and a very general elliptical polarizations of the input qubits. The mutual agreement of the data for different input states demonstrates the universality of the purification procedure. The deviations of the experimental data from the theoretical values were mainly due to the imperfections of the optical components. In particular, the non-ideal properties of the main \( BS \) were found to be highly critical. In order to achieve the projection onto the symmetric subspace, the \( BS \) transmittances \( T_H \) and \( T_V \) for the \( H \) and \( V \) polarization modes should be equal, with a high level of precision, and any difference between \( T_H \) and \( T_V \) partially spoils the purification. Notice, however, that deviations of \( T_H = T_V \) from 50% only decrease the success probability \( p \) but do not alter the purification gain factor \( \eta \).

We may generalize the above method by accounting for any possible asymmetry of the preparation of the input qubits. Allowing the input qubits to have different degree of mixedness, i.e. \( \rho'^{in}_a = \zeta |\phi\rangle \langle \phi| + (1 - \zeta)\frac{1}{2}, \rho'^{in}_b = \kappa |\phi\rangle \langle \phi| + (1 - \kappa)\frac{1}{2} \), the output qubits are still in the state given by Eq.3 with \( \xi_p = 2(\zeta + \kappa)/(3 + \kappa\zeta) \). The purification
factor is \( \eta = \frac{\zeta + \kappa}{1 + \zeta + \kappa} = 1/p \) where \( \xi \equiv \frac{1}{2}(\zeta + \kappa) \) is the average input mixedness factor. This process may be investigated by recourse to the quantum ”relative entropy” that measures the closeness of any output state \( \sigma \) with respect to a corresponding input pure state \( \rho \), e.g. after propagation through a noisy communication channel:
\[
S(\rho \parallel \sigma) \equiv Tr(\rho \log \rho) - Tr(\rho \log \sigma) \quad [21].
\]
Suppose that two qubits are equally prepared in the pure state \( \rho = |\psi\rangle_a \langle \psi|_a \otimes |\psi\rangle_b \langle \psi|_b \), \( S(\rho) = 0 \). After corruption by noise the entropy is:
\[
S(\rho \parallel \rho^{\text{in}}) = \log \left[ \frac{1}{2}(1 + \zeta) \right] + \log \left[ \frac{1}{2}(1 + \kappa) \right].
\]
If the qubits are further purified by symmetrization the following result is obtained:
\[
S(\rho \parallel \rho^{\text{out}}) = \log \left[ \frac{1}{2}(1 + \zeta) \right] + \log \left[ \frac{1}{2}(1 + \kappa) \right] - \log \eta.
\]
Then the symmetrization leads to the positive information gain \( \Delta S = S(\rho \parallel \rho^{\text{in}}) - S(\rho \parallel \rho^{\text{out}}) = \log \eta \) at the expense of a reduced rate \( p \) of success: \( \Delta S = -\log p, \frac{3}{4} \leq p \leq 1. \)

An interesting case is represented by the purification of a fully mixed state by a pure state, e.g. by the initial conditions \( \zeta = 1 \) and \( \kappa = 0 \). This precisely corresponds to the probabilistic quantum cloning process recently realized by our Laboratory in Rome by a symmetrization procedure [16], [18]. There \( \xi = \frac{1}{2} \), and a purification gain factor \( \eta = 4/3 \) was attained with a success probability: \( p = 3/4. \)

In conclusion, we have experimentally demonstrated the optimal purification of two depolarized qubits using the interference of two photons at a beam splitter, conditionally effecting symmetrization. The experimentally observed purification gain factors are in very good agreement with the theoretical estimates. We therefore envision that single qubit purification may become a viable procedure for protecting quantum states against noise. Note also that the projection on the symmetric subspace of more than two qubits can be carried out with a sequence of beam splitters, which is in the reach of present optical technology.

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REFERENCES

[1] P. W. Shor, *Phys. Rev. A* **52**, R2493 (1995).

[2] A.M. Steane, *Phys. Rev. Lett.* **77**, 793 (1996).

[3] C.H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J.A. Smolin, and W.K. Wootters, *Phys. Rev. Lett.* **76**, 722 (1996).

[4] D. Deutsch, A. Ekert, R. Jozsa, C. Macchiavello, S. Popescu, and A. Sanpera, *Phys. Rev. Lett.* **77**, 2818 (1996).

[5] T. Yamamoto, M. Koashi, S.K. Ozdemir, and N. Imoto, *Nature* (London) **421**, 343 (2003).

[6] J.W. Pan, S. Gasparoni, R. Ursin, G. Weihs, and A. Zeilinger, *Nature* (London) **423**, 417 (2003).

[7] Z. Zhao, T. Yang, Y.A. Chen, A.N. Zhang, and J.W. Pan, *Phys. Rev. Lett.* **90**, 207901 (2003).

[8] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, *Phys. Rev. Lett.* **70**, 1895 (1993).

[9] D. Boschi, S. Branca, F. De Martini, L. Hardy, and S. Popescu, *Phys. Rev. Lett.* **80**, 1121 (1998).

[10] D. Bouwmeester, J.W. Pan, K. Mattle, M. Eibl, H. Weinfurter, and A. Zeilinger, *Nature* (London) **390**, 575 (1997).

[11] I. Marcikic, H. de Riedmatten, W. Tittel, H. Zbinden H, and N. Gisin, *Nature* (London) **421**, 509 (2003).

[12] J.I. Cirac, A.K. Ekert, and C. Macchiavello, *Phys. Rev. Lett.* **82**, 4344 (1999).

[13] M. Keyl, and R.F. Werner, The rate of optimal purification procedures. *Ann. Henri*
\[ \rho_1' = \xi \left| \phi \right> \left< \phi \right| + (1 - \xi) \frac{1}{2} I \]

both for symmetric, \( \xi = \kappa = \zeta \), and asymmetric preparation, 
\( \xi = (\kappa + \zeta)/2 \)

\[ \rho_1 = \xi |\phi\rangle \langle \phi| + (1 - \xi) \frac{1}{2} \]

\[ = (\cos (\frac{\theta}{2}) |H\rangle + i \sin (\frac{\theta}{2}) |V\rangle) \]

with \( \theta = \frac{3}{8} \pi \). Filled markers denote the experimental purification factor \( \eta \) data while open markers denote the experimental data of the procedure probability \( p \). For simplicity we report only one error-bar for the probability values obtained. The statistical error is the same for all data reported.
