PHYSICAL BASIS FOR A CONSTANT LAG TIME

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ABSTRACT

We show that the constant time lag prescription for tidal dissipation follows directly from the equations of motion of a tidally-forced viscous fluid body, given some basic assumptions. They are (i) dissipation results from a viscous force that is proportional to the velocity of the tidal flow (ii) tidal forcing and dissipation are weak and non-resonant (iii) the equilibrium structure of the forced body is spherically-symmetric. The lag time is an intrinsic property of the tidally-forced body and is independent of the orbital configuration.

1. INTRODUCTION

The origin of tidal dissipation in gaseous planets and stars is, for the most part, an un-solved problem. Due to the extreme weakness of dissipation in these nearly perfect fluids, even identifying the correct theoretical framework to address this issue has proven to be difficult. Despite this, there has been a recent revival in tidal theory that has produced many promising results (Ogilvie & Lin 2004; Arras 2004; Wu 2005; Ivanov & Papaloizou 2007; Goodman & Lackner 2009; Weinberg et al. 2012).

In order to study the long-term evolution of orbits that are shaped by tidal dissipation, a parameterization of the strength of tidal dissipation is often performed. One of the most common parameterizations is to incorporate a tidal lag time, where the forced response lags behind the equilibrium tidal deformation by some fixed amount of time, in the event that the dissipation is weak. Furthermore, we show in 2 that this constant lag time is an intrinsic property of the tidally forced body, in that it only depends upon its internal structure. A brief discussion and summary are given in 3.

2. ASSUMPTIONS AND DERIVATION OF THE CONSTANT LAG TIME MODEL

2.1. the tidal interaction

Consider a spherical, self gravitating fluid object (star/planet) which is weakly perturbed by a slowly varying external gravitational field of the form

\[ U_T(r, \theta, \phi, t) = -\sum_{\ell m} r^\ell \Psi_{\ell m}(t). \]

If, for example, the perturbation arises from a point mass \( m_{\text{per}} \) orbiting the object at distance \( D(t) \) with angular coordinates \( \theta'(t), \phi'(t) \) we have

\[ \Psi_{\ell m}(t) = -G m_{\text{per}} \sum_{\ell m} \frac{4\pi}{2\ell + 1} \frac{1}{D(t)^{\ell+1}} Y^{*}_{\ell m}(\theta'(t), \phi'(t)). \]

To leading order in perturbation theory, the interaction potential is given by (cf. Newcomb 1962)

\[ H_I = \int d^3x \, \rho \, \xi \cdot \nabla U_T \]

where \( \int d^3x \) is taken over the volume of the forced body, \( \rho(x) \) is its unperturbed fluid density and \( \xi(x, t) \) is the Lagrangian displacement field.

We may write the interaction energy as

\[ H_I = -\sum_{\ell m} q_{\ell m} \Psi_{\ell m} \]

where \( q_{\ell m} \) is the multipole moment (cf. Press & Teukolsky 1977)

\[ q_{\ell m}(t) = \int d^3x \rho \, \xi \cdot \nabla r^{\ell} Y^{*}_{\ell m}(\theta, \phi). \]

Note that for \( \ell = 2 \), the \( \Psi_{\ell m} \)’s have dimensions of frequency squared.
The rate of energy transfer $\dot{E}$ between the orbit and and the forced body is given by

$$\dot{E} = -\int d^3 x \rho \cdot \nabla U_T = - \sum_{\ell m} \dot{q}_{\ell m} \Psi_{\ell m}. \tag{6}$$

Secular orbital evolution is entirely determined by the relationship between the tidal potential, represented by the $\Psi_{\ell m}$’s, and the corresponding induced multipolar moments, $q_{\ell m}$’s. That is, the gravitational potential induced by the perturbation of the body’s mass distribution is fully determined by the $q_{\ell m}$’s. While the fluid response $\xi$ may be complicated, only its multipolar moments affect secular orbital evolution.

2.2. equation of motion, the equilibrium tide and higher-order corrections

We neglect rotation when considering the equilibrium structure as well as the dynamics of the fluid perturbations. Furthermore, we ignore the effects of rotation in the limit of non-resonant forcing\footnote{Throughout, ‘non-resonant’ refers to the fluid oscillations of the forced body.}. The tidal problem may be expressed as

$$\ddot{\xi} + C \cdot \dot{\xi} + D \cdot \xi = -\nabla U_T = - \sum_{\ell m} \Psi_{\ell m}(t) \Xi_{\ell m}(x) \tag{7}$$

where $C$ is an Hermitian operator that is responsible for the restoring force (Lynden-Bell & Ostriker 1967) and $D$ is a time-independent differential operator that leads to dissipation. In the co-ordinate system of the forced body, the spatial dependance of the tidal forcing is given by the vector

$$\Xi_{\ell m}(x) = \nabla_x \cdot Y_{\ell m}(\theta, \phi), \tag{8}$$

which for $\ell = 2$ has dimensions of length.

If the time-dependence of the forcing is slow such that the inertia of the fluid is small, then

$$\ddot{\xi} \ll C \cdot \dot{\xi} \ll C \cdot \xi. \tag{9}$$

and in the weak friction approximation

$$D \cdot \dot{\xi} \ll C \cdot \xi. \tag{10}$$

Given this ordering, we may approximate the solution as

$$\xi(x, t) = \xi^{(0)} - C^{-1} \cdot \left[ D \cdot \dot{\xi}^{(0)} + \xi^{(0)} \right] \tag{11}$$

and the equilibrium tide solution $\xi^{(0)}$ is given by

$$\xi^{(0)}(x, t) = - \sum_{\ell m} \Psi_{\ell m}(t) \cdot \Xi_{\ell m}(x). \tag{12}$$

2.3. a single lag time for a given $\ell$

The interaction energy, given by eq. [4] and the energy transfer rate, given by eq. [5] only depend on the multipole moments $q_{\ell m}$. When computing the multipole moments $q_{\ell m}$’s by inserting eqs. [11] and [12] into eq. [5] coefficients of the following form are encountered

$$M_{\ell \ell' m m'} = \int \rho \cdot \Xi_{\ell m} \cdot M \cdot \Xi_{\ell' m'} \tag{13}$$

where $M = C^{-1}$ or $M = C^{-1} \cdot D \cdot C^{-1}$. By assuming that $C$ and $D$ are rotationally invariant, which is equivalent to assuming the equilibrium structure of the forced body is spherically symmetric, then

$$M_{\ell \ell' m m'} = M \delta_{\ell \ell'} \delta_{m m'}. \tag{14}$$

It follows that we can express a given multipole moment to leading order as

$$q_{\ell m}(t) = q_{\ell m}^{(0)}(t) - \tau_{\ell} \dot{q}_{\ell m}^{(0)}(t) - P_{\ell}^{-2} \ddot{q}_{\ell m}^{(0)}(t) \tag{15}$$

where a single lag time $\tau_{\ell}$ for a given $\ell$ is

$$\tau_{\ell} = \frac{\int d^3 x \rho \Xi_{\ell m}^* \cdot C^{-1} \cdot D \cdot C^{-1} \cdot \Xi_{\ell m}}{\int d^3 x \rho \Xi_{\ell m}^* \cdot C^{-1} \cdot \Xi_{\ell m}} \tag{16}$$

and $P_{\ell}^2$ is given by

$$P_{\ell}^2 = \frac{\int d^3 x \rho \left| C^{-1} \cdot \Xi_{\ell m} \right|^2}{\int d^3 x \rho \Xi_{\ell m}^* \cdot C^{-1} \cdot \Xi_{\ell m}}. \tag{17}$$

Though the integrals above contain the azimuthal quantum number $m$, spherical symmetry requires invariance under rotations and consequently, independence of $m$.

The term $\propto q_{\ell m}^{(0)}$ does not lead energy or angular momentum transfer since it contribution is a full derivative in time, which does not accumulate over long time-scales. The leading order terms of the multipole deformation $q_{\ell m}(t)$ that are responsible for apsidal precession and secular orbital evolution are thus given by

$$q_{\ell m}(t) = q_{\ell m}^{(0)}(t) - \tau_{\ell} \dot{q}_{\ell m}^{(0)}(t) \simeq q_{\ell m}^{(0)}(t - \tau_{\ell}) \tag{18}$$

where we assumed that $\tau_{\ell}$ is small in comparison to the characteristic time in which the tidal potential varies. Sole consideration of the quadrupolar $\ell = 2$ response of the expression above is equivalent to eqs. 2 & 3 of Hut (1981), which together serve as the starting point and underlying assumption of his analysis. Furthermore, by inserting the above expression for the $q_{\ell m}$ into eq. [6] we may write the secular energy transfer rate as

$$\dot{E} = \sum_{\ell m} \tau_{\ell} \dot{q}_{\ell m}^{(0)} \Psi_{\ell m}. \tag{19}$$

Again, by restricting to $\ell = 2$, the relation above is equivalent to eq. 40 of Eggelston et al. (1998), which serves as the starting point of their derivation for the secular equations for orbital evolution.

The lag time $\tau_{\ell}$ is completely determined by the equilibrium structure of the forced body. That is, $\tau_{\ell}$ is determined by $\rho(x), C$ and $D$ all of which, under the assumptions previously mentioned, only depend upon the equilibrium structure of the object in question. Similar conclusions can be deduced from the analysis of Willem et al. (2010). Those authors considered the case of a spherically symmetric radiative star where dissipation results from thermal diffusion and turbulent viscosity.

2.4. constant density equilibrium structure

As an illustrative example, consider a constant density equilibrium structure. Reiseneisserg (1994 and references therein) points out that for such an idealized system, $\Xi_{\ell m}$ is an eigenfunction of $C$, with eigenvalue $\omega_{\ell}$. Furthermore, and by construction, $\Xi_{\ell m}$ is responsible for
the entire multipolar response from the tidal acceleration \(-\nabla U_T \propto \Xi_{\ell m}\). In this case, the equilibrium tide becomes

\[
\xi_{\ell m}^{(0)} = -\frac{\Psi_{\ell m}(t)}{\omega_{\ell m}^2} \Xi_{\ell m}.
\]

(20)

The displacement field \(\Xi_{\ell m}\) is fundamental mode of the forced body, which is sometimes referred to as the \(f\) - mode or the Kelvin mode. \(\Xi_{\ell m}\) has no radial nodes and its period of oscillation is, essentially, the free-fall time at the surface.

In this limit, the lag time \(\tau_\ell\) becomes

\[
\tau_\ell = \frac{\gamma_\ell}{\omega_\ell^2} = \frac{\int d^3 x \rho \Xi_{\ell m} \cdot D \cdot \Xi_{\ell m}}{\int d^3 x \rho \Xi_{\ell m}^2 \Xi_{\ell m}}
\]

(21)

where \(\gamma_\ell\) is the damping rate of the \(\Xi_{\ell m}\)'s and

\[
P_\ell^2 = \omega_\ell^{-2}.
\]

(22)

Each \(\Xi_{\ell m}\) need not be an eigenvector of \(D\) in order for eq. 21 to be correct. As in the more general case, the only requirement on \(D\) for producing a single constant time lag \(\tau_\ell\) is for it to be invariant under rotations. Finally, for a constant density equilibrium structure, the tidal problem is equivalent to a set of decoupled forced damped harmonic oscillators with equal constant lag times.

3. DISCUSSION AND SUMMARY

The constant time lag model of Hut (1981) is perhaps the most widely used prescription for parameterizing the strength of tidal dissipation. Hut states that the most attractive feature of the constant \(\tau\) model for tidal dissipation is its simplicity. Perhaps this is true. However, as we have shown, the constant \(\tau\) model for tidal dissipation follows from some very basic physical assumptions. Namely, the tidal forcing non-resonant, the forced body is a spherically symmetric fluid and the dissipation as well as the tidal forcing, is weak. In fact, these assumptions are well-approximated in many astrophysical environments and can be, in principle, tested with observations of stars (cf. Dong et al. 2012) and extra-solar planets.

We demonstrated that the lag time \(\tau_\ell\) is an intrinsic property of the tidally-forced object. In other words, two identical objects placed in vastly different orbital configurations possess identical values for \(\tau_\ell\) as long as the underlying assumptions of a constant \(\tau_\ell\) remain valid.

For example, Socrates et al. (2012) show that in order for a Jupiter analogue to undergo high-\(\epsilon\) migration, the required lag time must be at least ten times stronger than that inferred from the Jupiter-Io interaction. Therefore, either the assumptions that led to a constant lag time are either incorrect or, for example, another element must be added to the theory of high-\(\epsilon\) migration.

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