Abstract. For two-dimensional manifold $M$ with locally symmetric connection $\nabla$ and with $\nabla$-parallel volume element $\text{vol}$ one can construct a flat connection on the vector bundle $TM \oplus E$, where $E$ is a trivial bundle. The metrizable case, when $M$ is a Riemannian manifold of constant curvature, together with its higher dimension generalizations, was studied by A.V. Shchepetilov [J. Phys. A: 36 (2003), 3893-3898]. This paper deals with the case of non-metrizable locally symmetric connection. Two flat connections on $TM \oplus (\mathbb{R} \times M)$ and two on $TM \oplus (\mathbb{R}^2 \times M)$ are constructed. It is shown that two of those connections – one from each pair – may be identified with the standard flat connection in $\mathbb{R}^N$, after suitable local affine embedding of $(M, \nabla)$ into $\mathbb{R}^N$.

1. Introduction

In the article [9] R. Sasaki proposed to add the property of describing pseudospherical surfaces to other remarkable properties – such as applicability of the inverse scattering method, infinite number of conservation laws and Bäcklund transformations – which characterize soliton equations in $1 + 1$ dimensions. He expressed the $\mathfrak{sl}(2, \mathbb{R})$-valued 1-form $\Omega$, which arises in the corresponding linear scattering problem $dv = \Omega v$, $v = (\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix})$, by 1-forms $\omega^1$, $\omega^2$ and $\omega^2_1$

$$\Omega = \begin{pmatrix} -\frac{1}{2} \omega^2 & \frac{1}{2} (\omega^2_1 + \omega^1) \\ \frac{1}{2} (-\omega^2 + \omega^1) & \frac{1}{2} \omega^2 \end{pmatrix}$$

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In such a way, that the integrability condition \( d\Omega - \Omega \wedge \Omega = 0 \) is equivalent to the structural equations \( dw^1 = \omega^2_1 \wedge \omega^2, dw^2 = -\omega^2_1 \wedge \omega^1 \) and \( dw^2_1 = \omega^1 \wedge \omega^2 \) of a pseudospherical surface \( (K = -1) \). This \( \mathfrak{sl}(2,\mathbb{R}) \)-valued 1-form \( \Omega \) itself can be interpreted as the connection form of a connection on some principal \( SL(2,\mathbb{R}) \)-bundle. The condition \( d\Omega - \Omega \wedge \Omega = 0 \) means that the curvature of this connection vanishes. In this respect the connection \( \Omega \) differs from the Levi-Civita connection of the considered pseudospherical metric. On the other hand, \( \Omega \) appeared to be somehow related to the Levi-Civita connection, because the Levi-Civita connection form \( \left( \begin{array}{cc} 0 & -\omega^2_1 \\ \omega^2_1 & 0 \end{array} \right) \) is contained in \( \Omega \). As might be expected, the question of finding the geometric interpretation of \( \Omega \) occurred.

In the paper [11] A.V. Shchepetilov explained the geometric meaning of the Sasaki connection. Using an equivalent representation of \( \Omega \), \( \mathfrak{so}(2,1) \)-valued, he constructed a flat connection \( \nabla \) on the vector bundle \( TM \oplus E \), where \( TM \) is the tangent bundle and \( E = \mathbb{R} \times M \) is a trivial one-dimensional vector bundle (our notation is slightly different from that in [10])

\[
\hat{\nabla}_X (Y \oplus f) = (\nabla_X Y + fX) \oplus (X(f) + g(X,Y)).
\]

Here \( g \) is a metric on \( M \), \( \nabla \) is its Levi-Civita connection, \( f \in C^\infty(M) \) is a section of \( E \) and \( X, Y \) are vector fields on \( M \).

Shchepetilov considered also manifolds with metric of constant positive curvature \( K = +1 \). The corresponding flat connection \( \hat{\nabla} \) on \( TM \oplus E \) is

\[
\hat{\nabla}_X (Y \oplus f) = (\nabla_X Y + fX) \oplus (X(f) - g(X,Y)).
\]

The aim of this paper is to construct a similar flat connection \( \hat{\nabla} \) for a two-dimensional manifold with non-metrizable locally symmetric connection \( \nabla \) and with \( \nabla \)-parallel volume element. Our main motivation for research is as follows. Firstly, manifold with locally symmetric linear connection can be thought of as a generalization of a constant sectional curvature Riemannian manifold. Secondly, sometimes more important than \( (M,g) \) or \( (M,\nabla) \) alone is an embedding of \( M \) into \( \mathbb{R}^3 \). For example, every isometric embedding of a pseudospherical surface \( (M,g) \) into \( \mathbb{R}^3 \) corresponds to some particular solution of the sine-Gordon equation. Therefore restriction to those non-flat locally symmetric connections which are induced on hypersurfaces in \( \mathbb{R}^{n+1} \) is legitimated. If such hypersurface \( f \) is degenerate and its type number \( r \) is greater than 1, then around each generic point of \( M \) there exists a local cylinder decomposition which contains as a part a non-degenerate hypersurface in \( \mathbb{R}^{r+1} \) with some locally symmetric connection (see [4]). On the other hand, if \( f \) is non-degenerate and \( n > 2 \), then \( \nabla \) is the Blaschke connection, \( \nabla h = 0 \), \( S = \rho \text{id} \), \( \rho = \text{const} \), \( \rho \neq 0 \) and \( f(M) \) is an open part of a quadric with center [4]. Similarly as in the second proof of Berwald theorem in [3] one can then define a pseudo-scalar product \( G \) in \( \mathbb{R}^{n+1} \) such that \( G(f_*X,f_*Y) = h(X,Y) \), \( G(f_*X,\xi) = 0 \) and \( G(\xi,\xi) = \rho \), where \( \xi \) is the affine normal. It is easy to check that relative to this pseudo-scalar product \( f \) is a hypersurface of constant sectional curvature \( \rho \). If \( f \) is non-degenerate, \( n = 2 \) and the induced locally symmetric connection satisfies the condition \( \dim \text{im} \hat{\nabla} = 2 \), then there also exists a pseudo-scalar product on \( \mathbb{R}^{n+1} = \mathbb{R}^3 \) relative to which \( f \) has constant Gaussian curvature and \( \xi \) is perpendicular to \( f \) [3].
Affine analogues of the Sasaki-Shchepetilov connection

On the contrary, if \( f : M \to \mathbb{R}^{n+1} \) is of type number 1 or if \( f : M \to \mathbb{R}^3 \) is nondegenerate and \( \text{dim} \ker R = 1 \), then the connection as a connection of 1-codimensional nullity (\( \text{dim} \ker R = n - 1 \)) is not metrizable \([7]\), therefore we have reason for generalizing Shchepetilov’s construction. The present paper deals with the case \( n = 2 \).

2. Preliminaries

Let \( M \) be a connected two-dimensional real manifold and let \( \nabla \) be a locally symmetric connection on \( M \), satisfying the condition \( \text{dim} \ker R = 1 \), where for \( p \in M \)

\[ \text{im} R|_p := \text{span}\{R(X, Y)Z : X, Y, Z \in T_p M\} \]

and \( R \) is the curvature tensor of \( \nabla \). Such connections were studied by B. Opozda in \([5]\). Opozda proved that for every \( p \in M \) there is a coordinate system \((u, v)\) around \( p \) such that

\[ \nabla_{\partial_u} \partial_u = \nabla_{\partial_u} \partial_v = 0 \quad \text{and} \quad \nabla_{\partial_v} \partial_v = \varepsilon u \partial_u, \]  

where \( \varepsilon \in \{1, -1\} \). A local coordinate system in which a locally symmetric connection \( \nabla \) is expressed by \([3]\) will be called a canonical coordinate system for \( \nabla \) \([3]\). It is not unique. It is easy to check that if \( u, v \) and \( \pi, \tau \) are canonical coordinate systems then on each connected component of the intersection of their domains we have \( \pi = Au + \chi(v), \tau = \delta v + B \), where \( A, B, \delta \) are constants, \( \delta^2 = 1 \), and \( \chi \) satisfies the differential equation \( \chi'' + \varepsilon \chi = 0 \).

The Ricci tensor \( \text{Ric}(X, Y) := \text{trace}[V \mapsto R(V, X)Y] \) of such a connection is symmetric and for every \( p \in M \) there exists a \( \nabla \)-parallel volume element around \( p \). Here we assume that a \( \nabla \)-parallel volume element \( \text{vol} \) exists on the whole \( M \).

It follows, that for every \( p \in M \) we can find around \( p \) a local basis \((X_1, X_2)\) of \( TM \), satisfying the conditions:

\[ X_1 \in \ker \text{Ric}, \quad \text{Ric}(X_2, X_2) = \varepsilon \quad \text{and} \quad \text{vol}(X_1, X_2) = 1. \]  

For example, on the domain of canonical coordinates \((u, v)\) as in \([3]\) we may take \( X_1 = \frac{1}{c} \partial_u \) and \( X_2 = \partial_v \), where \( c \) is the non-zero constant such that \( \text{vol} = c \, du \wedge dv \). Let \( \omega^1, \omega^2 \) be the dual basis for \((X_1, X_2)\). The local connection form is \( (\omega^i_j) = \begin{pmatrix} 0 & \omega^1_2 \\ 0 & 0 \end{pmatrix} \) and the structural equations are \( d\omega^1 = -\omega^1_2 \wedge \omega^2, d\omega^2 = 0 \) and \( d\omega^1_2 = \varepsilon \omega^1 \wedge \omega^2 \).

The following proposition is easy to check.

**Proposition 2.1**

Let \( M \) be a two-dimensional manifold with locally symmetric connection \( \nabla \) satisfying condition \( \text{dim} \ker R = 1 \). Let \( \omega^1, \omega^2 \) and \( \omega^1_j \) be the dual basis and the local connection forms for some local basis of \( TM \) satisfying the condition \([1]\). Then each of the following four 1-forms \( \Omega_i \)

\[ \Omega_1 = \begin{pmatrix} 0 & -\omega^1_2 & \omega^1 \\ 0 & 0 & \omega^2 \\ 0 & -\varepsilon \omega^2 & 0 \end{pmatrix}, \quad \Omega_2 = \begin{pmatrix} 0 & -\omega^1_2 & \varepsilon \omega^2 \\ 0 & 0 & 0 \\ -\omega^2 & \omega^1 & 0 \end{pmatrix}, \]
those \( \mathfrak{gl}(N,\mathbb{R}) \)-valued \((N = 3 \text{ or } N = 4)\) 1-forms were obtained in [8] as the local connection forms of connections on some principal \( GL(N,\mathbb{R}) \)-bundle \( P \) and seem to be analogous to the Sasaki connection form. The bundle \( P(M,G) \), \( G = GL(N,\mathbb{R}) \), is an extension of the bundle \( Q(M,H) \) consisting of all linear frames on \( M \) which satisfy (4). The structure group is \( H := \{ (t,1) : t \in \mathbb{R} \} \cup \{ (0,t^{-1}) : t \in \mathbb{R} \} \). Here we need not explain what the bundle \( P(M,G) \) is. It suffices to know that there exists \( f : Q \to P \) such that the triple \( (f, \text{id}_M, \iota) \) is a homomorphism of principal fibre bundles \( Q(M,H) \) and \( P(M,G) \). The homomorphism \( \iota : H \to G \) of structure groups is given by \( \iota(a) := (0, t^{-1}) \), where \( I_{N-2} \) is the identity \((N - 2) \times (N - 2)\) matrix. Each of the forms \( \Omega \) is a local connection form associated with a local section \( f \circ \sigma \) of \( P \), where \( \sigma \) is some local section of \( Q \).

In the construction of \( P \) and \( \Omega \) in [8] and in the present paper we consider the left action of \( H \) on \( Q \) : \( a * q := qa^{-1} \), where \( (v_1, v_2)h := (h_1v_1 + h_2v_2, h_3v_1 + h_4v_2) \) for \( h = \begin{pmatrix} h_1 & h_2 \\ h_3 & h_4 \end{pmatrix} \in H \), and some left action of \( G \) on \( P \). Another possible way is to consider traditionally a right action, but we have then \(-\Omega\) instead of \( \Omega \).

### 3. The connections on the vector bundle \( TM \oplus E \)

We will use the definition of the covariant derivative of a section of an associated bundle which comes from [1], and is described for example in [2]. Since we consider here the left action of \( G \) on \( P \) and the right action of \( G \) on \( \mathbb{R}^N \), \( z * c := c^{-1}z \), some details may be different from that of [1] and [2].

Let \( TM \) be the tangent bundle of \( M \) and let \( E \) be the trivial bundle, \( E = \mathbb{R}^{N-2} \times M \).

**Proposition 3.1**

The bundle \( TM \oplus E \) is a vector bundle associated to \( P \) with fibre \( \mathbb{R}^N \)

\[
P \times_G \mathbb{R}^N = (P \times \mathbb{R}^N)/\sim,
\]

with the equivalence relation \( \sim \) given by \( (cp, z * c^{-1}) \sim (p, z) \).

**Proof.** For \( x \in M \) we take a basis \( q = (v_1, v_2) \in Q \) of \( T_xM \) and identify \((z^1v_1 + z^2v_2) \oplus (z^3, \ldots, z^N) \) from \( (TM \oplus E)_x \) with \( [(f(q), z)] \in (P \times \mathbb{R}^N)/\sim \). This identification is correct, because if we take another basis \( q' = (v'_1, v'_2) \in Q_x \), then \( q' = a * q = qa^{-1} \) for some \( a \in H \) and \( z^1v_1 + z^2v_2 = z'^1v'_1 + z'^2v'_2 \) with \( z'^1 = a^1z^1 + a^2z^2 \), \( z'^2 = a^3z^1 + a^4z^2 \). It follows that \( (z'^1v'_1 + z'^2v'_2) \oplus (z'^3, \ldots, z'^N) = (z^1v_1 + z^2v_2) \oplus (z^3, \ldots, z^N) \) for \( z' = \iota(a)z = z * (\iota(a))^{-1} \). We obtain \( [(f(q'), z')] = [(f(a * q), z * (\iota(a))^{-1})] = [(\iota(a)f(q), z * (\iota(a))^{-1})] = [(f(q), z)] \).

Let \( [(p, z)] \in P \times_G \mathbb{R}^N \) and let \( \pi(p) = x \), where \( \pi : P \to M \). Let \( q = (v_1, v_2) \in Q_x \), then \( f(q) \in P_x \). Since \( G \) acts transitively on fibres of \( P \), there exists \( b \in G \)
such that \( p = bf(q) \). It follows that \([[(p, z)] = [(bf(q), z)] = [[bf(q), (z * b) * b^{-1}]] = [[[f(q), z * b]] = [[[f(q), b^{-1}z]], \text{ therefore we have to identify } [[(p, z)] \text{ with } (y_{1}v_{1} + y_{2}v_{2}) \oplus (y^{3}, \ldots, y^{N})\), where \( y = b^{-1}z \).

To each local section \( \eta \) of an associated vector bundle \( P \times_{G} \mathbb{R}^{N} \) corresponds some mapping \( \overline{\eta} : |P| \to \mathbb{R}^{N} \) - called the Crittenden mapping - which satisfies the condition \( \overline{\eta}(bp) = \overline{\eta}(p) * b^{-1} \). Since we have actually defined the right action of \( G \) on \( \mathbb{R}^{N} \) using the left action, \( x * c := c^{-1}x \), we can write this condition simply as \( \overline{\eta}(bp) = b\overline{\eta}(p) \). By definition of the Crittenden mapping, \( [(p, \overline{\eta}(p))] = \eta(\pi(p)) \). Conversely, to each mapping \( \overline{\eta} : |P| \to \mathbb{R}^{N} \) satisfying the condition \( \overline{\eta}(b * p) = \overline{\eta}(p) * b^{-1} \) corresponds a local section of the associated bundle. Let \( X \) be a vector field on \( M \). For every connection form \( \Omega_{i} \) from Proposition 2.1 we will find the covariant derivative \( \overline{\nabla}_{X} \eta \) of a local section \( \eta \) of \( TM \oplus E \).

**Theorem 3.2**
Let \( \eta = \Psi \oplus \Psi \), with a vector field \( Y \) on \( U \subset M \) and \( \Psi : U \to \mathbb{R}^{N(i)} \), be a local section of \( TM \oplus E \). Here \( N(1) = N(2) = 1 \) and \( N(3) = N(4) = 2 \). Let \( \overline{\nabla}_{X} \eta \) denote the covariant derivative of \( \eta \) with respect to the connection corresponding to local connection form \( \Omega_{i} \) from Proposition 2.1. Then

\[
\begin{align*}
\overline{\nabla}_{X}^{1}(Y \oplus \Psi) &= (\nabla_{X}Y - \Psi X) \oplus (X(\Psi) + \text{Ric}(X, Y)), \\
\overline{\nabla}_{X}^{2}(Y \oplus \Psi) &= (\nabla_{X}Y - \Psi LX) \oplus (X(\Psi) - \text{vol}(X, Y)), \\
\overline{\nabla}_{X}^{3}(Y \oplus (\Psi^{1}, \Psi^{2})) &= (\nabla_{X}Y - \Psi^{1}X - \varepsilon\Psi^{2}LX) \oplus (X(\Psi^{1}) + \text{Ric}(X, Y), X(\Psi^{2}))
\end{align*}
\]

and

\[
\begin{align*}
\overline{\nabla}_{X}^{4}(Y \oplus (\Psi^{1}, \Psi^{2})) &= (\nabla_{X}Y - \Psi^{1}LX) \oplus (X(\Psi^{1}) - \text{vol}(X, Y), X(\Psi^{2}) - \varepsilon\text{Ric}(X, Y)),
\end{align*}
\]

with the \((1, 1)\) tensor field \( L \) such that \( \text{vol}(LX, Y) = \text{Ric}(X, Y) \) for every \( X, Y \).

**Proof.** By definition of the covariant derivative, the Crittenden mapping corresponding to \( \overline{\nabla}_{X} \eta \) is equal to \( X^{H}(\overline{\eta}) \), where \( X^{H} \) is the horizontal lift of \( X \) to \( P|_{U} \).

We use a local section \( \tau = f \circ \sigma \) of \( P \), where \( \sigma = (V_{1}, V_{2}) \) is a local section of \( Q \). Let \( Y = Y^{1}V_{1} + Y^{2}V_{2} \), then \( \overline{\tau} = (Y^{1}, Y^{2}, \Psi) \).

Let \( \overline{\Omega} \) be the connection form on \( P \). The local connection form is \( \tau^{\ast}\overline{\Omega} = \Omega_{\sigma} \). We have

\[
\overline{\nabla}_{X}\eta(\tau(x)) = X^{H}_{\tau(x)}(\overline{\eta}), \quad X^{H}_{\tau(x)} = d_{x}\tau(\tau_{x}) + B_{\tau(x)}^{*},
\]

where the right-invariant vector field \( B = -\Omega_{\sigma}(X_{x}) \), which we easily obtain from the condition \( \overline{\Omega}(X^{H}_{\tau(x)}) = 0 \):

\[
0 = \overline{\Omega}(d_{x}\tau(X_{x})) + \overline{\Omega}(B_{\tau(x)}^{*}) = (\tau^{\ast}\overline{\Omega})_{x}(X_{x}) + B = \Omega_{\sigma}(X_{x}) + B.
\]

The first part of \( X^{H}_{\tau(x)}(\overline{\eta}) \) is equal to

\[
(d_{x}\tau(X_{x}))(\overline{\eta}) = X_{x}(\overline{\eta} \circ \tau) = (X_{x}(Y^{1}), X_{x}(Y^{2}), X_{x}(\Psi)).
\]
The second part is

\[
B^*_{τ(x)}(\tilde{η}) = \left. \frac{d}{dt} \tilde{η}(b_t τ(x)) \right|_{t=0} = \left. \frac{d}{dt} b_t \tilde{η}(x) \right|_{t=0} = \left. \frac{d}{dt} b_t \right|_{t=0} \tilde{η}(τ(x)) = B\tilde{η}(τ(x)).
\]

Here \((b_t)\) is 1-parameter subgroup of \(G\) generated by \(B\). It follows that

\[
\tilde{∇}_X η(τ(x)) = \begin{pmatrix} X_1(Y_1) \\ X_2(Y_2) \\ X_1(Ψ) \end{pmatrix} - Ω_σ(X_1) \begin{pmatrix} Y_1(x) \\ Y_2(x) \\ Ψ(x) \end{pmatrix}.
\] (5)

For \(Ω_σ = Ω_1\) we obtain

\[
\tilde{∇}_X η ∘ τ = \begin{pmatrix} X_1^1 \\ X_1^2 \\ X_1(Ψ) \end{pmatrix} - \begin{pmatrix} 0 & -ω_2^1(X) & ω_1^2(X) \\ 0 & 0 & ω_2^1(X) \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ Ψ \end{pmatrix}
\]

and

\[
\tilde{∇}_X η = ((X(Y_1) + ω_2^1(X)Y^2 - ω_1^2(X)Ψ)V_1 + (X(Y^2) - ω_2^2(X)Ψ)V_2) + (X(Ψ) + εω_2^1(X)Y^2).
\]

Since \(∇_X V_1 = 0\), we have

\[
∇_X Y = ∇_X (Y_1V_1 + Y_2V_2) = X(Y_1)V_1 + X(Y_2)V_2 + Y_1V_2 + Y_2V_1 = X(Y_1)V_1 + X(Y_2)V_2 + Y^2ω_2^1(X)V_1.
\]

We have also

\[
Ric(X, Y) = Ric(ω_1^1(X)V_1 + ω_2^2(X)V_2, Y_1V_1 + Y_2V_2) = \omega_2^2(X)Y^2 Ric(V_2, V_2) = \omega_2^2(X)Y^2 ε,
\]

because \(V_1\) is a local section of \(ker Ric\).

We obtain finally

\[
\tilde{∇}_X(Y ⊕ Ψ) = (∇_X Y - ΨX) ⊕ (X(Ψ) + Ric(X, Y)).
\] (6)

If we take \(Ω_σ = Ω_2\), then we obtain from (5)

\[
\tilde{∇}_X η ∘ τ = \begin{pmatrix} X_1(Y_1) \\ X_1(Y_2) \\ X_1(Ψ) \end{pmatrix} - \begin{pmatrix} 0 & -ω_2^1(X) & εω_2^1(X) \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ Ψ \end{pmatrix},
\]

which gives

\[
\tilde{∇}_X(Y ⊕ Ψ) = ((X(Y_1) + ω_2^1(X)Y^2 - εω_2^1(X)Ψ)V_1 + X(Y^2)V_2) ⊕ (X(Ψ) + ω_2^2(X)Y^1 - ω_1^1(X)Y^2).
\]
Affine analogues of the Sasaki-Shchepetilov connection \[ [43] \]

because \( \text{vol}(V_1, V_2) = 1 \).

Let \( (\tilde{V}_1, \tilde{V}_2) \) be another local basis of \( TM \) satisfying (4). Then in the intersection of the corresponding domains we have \( \tilde{V}_3 = \delta V_1, \tilde{V}_2 = tV_1 + \delta V_2 \) with \( \delta \in \{1,-1\} \). For the new dual basis we obtain \( \omega^1 = t\omega^1 - \delta \omega^2, \omega^2 = \delta \omega^2 \). It follows that \( \tilde{\omega}^2 V_1 = \omega^2 V_1 \), therefore the vector field \( LX := \varepsilon \omega^2(X)V_1 \) is defined on the whole \( M \) and \( L \) is a \((1,1)\) tensor field.

Note that for every \( Z \) we have

\[
\text{vol}(LX, Z) = \text{vol}(\varepsilon \omega^2(X)V_1, Z) = \varepsilon \omega^2(X)\omega^2(Z) \text{vol}(V_1, V_2) = \varepsilon \omega^2(X)\omega^2(Z) = \text{Ric}(X, Z).
\]

For the second connection we finally obtain the global formula

\[
\tilde{\nabla}_X (Y \oplus \Psi) = (\nabla_X Y - \Psi LX) \oplus (X(\Psi) - \text{vol}(X, Y)). \tag{8}
\]

For \( \Omega_2 = \Omega_3 \) we have

\[
\tilde{\nabla}_X \eta \circ \tau = \begin{pmatrix} X(Y^1) \\ X(Y^2) \\ X(\Psi^1) \\ X(\Psi^2) \end{pmatrix} - \begin{pmatrix} 0 & -\omega^1_2(X) & \omega^1(X) & \omega^2(X) \\ 0 & 0 & \omega^2(X) & 0 \\ 0 & -\varepsilon \omega^2(X) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} Y^1 \\ Y^2 \\ \Psi^1 \\ \Psi^2 \end{pmatrix},
\]

hence

\[
\tilde{\nabla}_X (Y \oplus (\Psi^1, \Psi^2)) = (X(Y^1) + \omega^1_2(X)Y^2 - \omega^1(X)\Psi^1 - \omega^2(X)\Psi^2)V_1 + (X(Y^2) - \omega^2(X)\Psi^1)V_2 \\
\oplus (X(\Psi^1) + \varepsilon \omega^2(X)Y^2, X(\Psi^2)),
\]

which gives

\[
\tilde{\nabla}_X (Y \oplus (\Psi^1, \Psi^2)) = (\nabla_X Y - \Psi^1 X - \varepsilon \Psi^2 LX) \oplus (X(\Psi^1) + \text{Ric}(X, Y), X(\Psi^2)). \tag{9}
\]

For \( \Omega_2 = \Omega_4 \) we obtain

\[
\tilde{\nabla}_X \eta \circ \tau = \begin{pmatrix} X(Y^1) \\ X(Y^2) \\ X(\Psi^1) \\ X(\Psi^2) \end{pmatrix} - \begin{pmatrix} 0 & -\omega^1_2(X) & \varepsilon \omega^2(X) & 0 \\ 0 & 0 & 0 & 0 \\ -\omega^2(X) & \omega^1(X) & 0 & 0 \\ 0 & \omega^2(X) & 0 & 0 \end{pmatrix} \begin{pmatrix} Y^1 \\ Y^2 \\ \Psi^1 \\ \Psi^2 \end{pmatrix}
\]

and

\[
\tilde{\nabla}_X (Y \oplus (\Psi^1, \Psi^2)) = (\nabla_X Y - \Psi^1 LX) \oplus (X(\Psi^1) - \text{vol}(X, Y), X(\Psi^2) - \varepsilon \text{Ric}(X, Y)). \tag{10}
\]
4. Flatness of $\hat{\nabla}$

**Theorem 4.1**
Each of four connections $\hat{\nabla}^i$ in Theorem 3.2 is flat.

**Proof.** We will compute

$$\hat{R}(X, Y)(Z \oplus \Psi) = (\hat{\nabla}_X \hat{\nabla}_Y - \hat{\nabla}_Y \hat{\nabla}_X - \hat{\nabla}_{[X,Y]})(Z \oplus \Psi)$$

for each of four connections (6), (8), (9) and (10).

If we use $\nabla X - \nabla Y - [X,Y] = T(X,Y) = 0$, then for the connection (6) we obtain

$$\hat{R}(X, Y)(Z \oplus \Psi) = (R(X, Y)Z - (Ric(Y, Z)X - Ric(X, Z)Y))$$

$$\oplus (\nabla Y \text{Ric}(Y, Z) - (\nabla X \text{Ric})(X, Z) - \Psi(Ric(X, Y) - Ric(Y, X)))$$

But Ric is symmetric, $\nabla R = 0$ implies $\nabla \text{Ric} = 0$, and for each two-dimensional manifold

$$R(X, Y)Z = Ric(Y, Z)X - Ric(X, Z)Y. \quad (11)$$

Therefore $\hat{R}(X, Y)(Z \oplus \Psi) = 0 \oplus 0$.

For the connection (8) we obtain

$$\hat{R}(X, Y)(Z \oplus \Psi)$$

$$= (R(X, Y)Z + \text{vol}(Y, Z)LX - \text{vol}(X, Z)LY - \Psi((\nabla X \text{L}Y - (\nabla Y \text{L})X))$$

$$\oplus (\nabla Y \text{vol})(X, Z) - (\nabla X \text{vol})(Y, Z) + \Psi(\text{vol}(X, LY) - \text{vol}(Y, LX)).$$

From $\nabla \text{vol} = 0$ it follows that $R \cdot \text{vol} = 0$, therefore

$$0 = (R(X, Y) \cdot \text{vol})(Z, W) = - \text{vol}(R(X, Y)Z, W) - \text{vol}(Z, R(X, Y)W)$$

$$= - \text{vol}(R(X, Y)Z, W) + \text{vol}(R(X, Y)W, Z),$$

hence

$$\text{vol}(R(X, Y)Z, W) = \text{vol}(R(X, Y)W, Z). \quad (12)$$

For an arbitrary vector field $W$ using (12), (7) and (11) we obtain

$$\text{vol}(R(X, Y)Z + \text{vol}(Y, Z)LX - \text{vol}(X, Z)LY, W)$$

$$= \text{vol}(R(X, Y)W, Z) + \text{vol}(Y, Z)\text{Ric}(X, W) - \text{vol}(X, Z)\text{Ric}(Y, W)$$

$$= \text{vol}(R(X, Y)W + \text{Ric}(X, W)Y - \text{Ric}(Y, W)X, Z)$$

$$= 0.$$

From the non-degeneracy of vol it follows that

$$R(X, Y)Z + \text{vol}(Y, Z)LX - \text{vol}(X, Z)LY = 0. \quad (13)$$

Moreover, $\nabla \text{Ric} = 0$, $\nabla \text{vol} = 0$ and (7) imply $\nabla L = 0$. We have also $\text{vol}(X, LY) - \text{vol}(Y, LX) = - \text{vol}(LY, X) + \text{vol}(LX, Y) = - \text{Ric}(Y, X) + \text{Ric}(X, Y) = 0$. Hence $\hat{R}(X, Y)(Z \oplus \Psi) = 0 \oplus 0$ for $\hat{\nabla}$ given by (8).
For the connection (9) we obtain
\[
\begin{align*}
R(X,Y)(Z \oplus (\Psi^1, \Psi^2)) &= (R(X,Y) - \text{vol}(Y,Z)LX - \text{vol}(X,Z)LY - \Psi^1((\nabla_X L)(Y) - (\nabla_Y L)(X))) \\
&\quad \oplus ((\nabla_X \text{vol})(X,Z) - (\nabla_Y \text{vol})(Y,Z) + \Psi^1(\text{vol}(X,LY) - \text{vol}(Y,LX)), 0) \\
&= 0 \oplus (0,0).
\end{align*}
\]

Note that im $L \subset \ker \text{Ric}$.

For (10) we have
\[
\begin{align*}
R(X,Y)(Z \oplus (\Psi^1, \Psi^2)) &= (R(X,Y) + \text{vol}(Y,Z)LX - \text{vol}(X,Z)LY - \Psi^1((\nabla_X L)(Y) - (\nabla_Y L)(X))) \\
&\quad \oplus ((\nabla_Y \text{vol})(X,Z) - (\nabla_X \text{vol})(Y,Z) + \Psi^1(\text{vol}(X,LY) - \text{vol}(Y,LX)), 0) \\
&= 0 \oplus (0,0).
\end{align*}
\]

5. Some remarks about interpretation of $\nabla$

As is shown in [10], in the metric case using (at least local) embedding of $(M,g)$ with $K = \pm 1$ into Euclidean or pseudoeuclidean space $E$ we may identify $\nabla$ with the restriction of the flat connection on $T_E = E \times E$ to $E \times M$ and identify the trivial one-dimensional summand $E$ with the normal bundle of the surface.

We consider now the case of non-metrizable locally symmetric connection on $M$, dim $M = 2$. Let $f : M \to \mathbb{R}^3$ be an immersion and let $\nabla$ be the connection induced on $M$ by $f$ and the transversal vector field $\xi$. If we identify the bundle $f^*(TM) \oplus \mathbb{R}\xi$ with $TM \oplus E$, then to the vector field $f^*(Y) + \Psi\xi$ corresponds the section $Y \oplus \Psi$ of $TM \oplus E$. The Gauss and Weingarten formulae yield that to $D_X(f^*Y + \Psi\xi)$ corresponds
\[
D_X(Y \oplus \Psi) = (\nabla_X Y - \Psi SX) \oplus (X(\Psi) + h(X,Y) + \Psi \tau(X)),
\]
where $h$ is the affine fundamental form, $S$ is the shape operator and $\tau$ is the transversal connection form (see [3] for the definitions). We look for $f$ and $\xi$ such that $\tilde{D} = \nabla$. Comparing the right-hand side of (14) with that of (6) and (8) for the section $0 \oplus 1$ gives $\tau = 0$, which means that we may restrict ourselves to equiaffine transversal vector fields.

Furthermore, since $h$ is always symmetric and vol is anti-symmetric, we see that there are no $f$ and $\xi$ which allow to identify in the above described way the connection (8) with the standard connection $D$ on the bundle $\mathbb{R}^3 \times M$.

As concerns (6), it should be $h = \text{Ric}$, which implies that we should consider some realization of $\nabla$ on a degenerate surface $f$ with the type number $tf$ equal to 1. Such realizations were described by B. Opozda in [7]. Using a general description
given in Proposition 6.2 of [7] and claiming that $\xi = -f$, we easily obtain the
following particular local realizations of $\nabla$
\[
 f(u,v) = (u, \cos v, \sin v) \in \mathbb{R}^3 \quad \text{for } \varepsilon = 1
\]
and
\[
 f(u,v) = \left( u, \frac{\sqrt{2}}{2} e^{-v}, \frac{\sqrt{2}}{2} e^v \right) \in \mathbb{R}^3 \quad \text{for } \varepsilon = -1.
\]
Here $u, v$ is some fixed local canonical coordinate system for $\nabla$. The volume element $\text{vol} = du \wedge dv$ is the element induced by $(f, \xi)$ from $\mathbb{R}^3$.

For a centro-affine immersion $(f, \xi = -f)$ and $n = 2$ we have $SX = X$ and $\text{Ric}(X,Y) = h(X,Y)\text{tr} S - h(SX,Y) = (n - 1)h(X,Y) = h(X,Y)$. It follows that using the immersion (15) or (16) we may identify (6) with the standard $\mathbb{S}$ from (9) from (16) we identify (16) with the standard connection $D$.

To obtain $\nabla = \tilde{D}$ for $\nabla$ given by (6) we choose and fix some local canonical coordinate system $u, v$ for $\nabla$ and use for example the immersion $f: M \to \mathbb{R}^4$, $f(u,v) = (u, \cos v, \sin v, 0)$ if $\varepsilon = 1$ and $f(u,v) = (u, \sqrt{2} e^{-v}, \sqrt{2} e^v, 0)$ if $\varepsilon = -1$, and the two-dimensional transversal bundle spanned by $\xi_1(u,v) = -f(u,v)$ and $\xi_2(u,v) = (-v, 0, 0, 1)$. The induced connection (which is equal to $\nabla$), the affine fundamental forms $h^1$, $h^2$, the shape operators $S_1$, $S_2$, and the normal connection forms $\tau^i_j$ are defined by the following decompositions (cf [3])
\[
 D_X f_* Y = f_* \nabla_X Y + h^1(X,Y)\xi_1 + h^2(X,Y)\xi_2,
\]
\[
 D_X \xi_1 = -f_* S_1 X + \tau^1_1(X)\xi_1 + \tau^1_2(X)\xi_2,
\]
\[
 D_X \xi_2 = -f_* S_2 X + \tau^2_1(X)\xi_1 + \tau^2_2(X)\xi_2.
\]
We obtain $\tau^i_j = 0$, $S_1 X = X$, $S_2 = dv(\cdot)\partial_u = \varepsilon L$, $h^2 = 0$ and $h^1(\partial_u, \partial_u) = h^1(\partial_v, \partial_u) = \varepsilon$. The volume element $\text{vol} = du \wedge dv$ is induced from $\mathbb{R}^4$, $\text{vol}(X,Y) = \det(f_* X, f_* Y, \xi_1, \xi_2)$. Identifying the vector field $f_* (Y) + \Psi^1 \xi_1 + \Psi^2 \xi_2$ with the section $Y \oplus (\Psi^1, \Psi^2)$ of $TM \oplus E$ we obtain $\nabla_X (Y \oplus (\Psi^1, \Psi^2))$ as in (9) from $D_X (f_* Y + \Psi^1 \xi_1 + \Psi^2 \xi_2)$.

Similarly as it was for (6), the above immersion $f$ is degenerate. By definition (see [3]), an immersion $f: M \to \mathbb{R}^4$ is non-degenerate if the symmetric bilinear function $G_{\sigma}$ is non-degenerate. For a local frame field $\sigma = (X_1, X_2)$ the function $G_{\sigma}$ is defined by the formula (cf [3])
\[
 G_{\sigma}(Y, Z) = \frac{1}{2} \left( \det(f_* (X_1), f_* (X_2), D_Y f_* (X_1), D_Z f_* (X_2))
 + \det(f_* (X_1), f_* (X_2), D_Z f_* (X_1), D_Y f_* (X_2)) \right).
\]
For $\sigma = (\partial_u, \partial_v)$ we obtain $G_{\sigma} = 0$.

It is impossible to obtain in a similar way the connection [10], because $\text{vol}$ is anti-symmetric.
6. Some further remarks

In general, to each immersion \((f, \xi)\) and to each local basis \(\sigma = (X_1, X_2)\) of \(TM\) corresponds some \(GL(3, \mathbb{R})\)-valued 1-form \(\Omega_\sigma\)

\[
\Omega_\sigma = \begin{pmatrix} -\omega^1_1 & -\omega^2_1 & S^1(\cdot) \\ -\omega^1_2 & -\omega^2_2 & S^2(\cdot) \\ -h(\cdot, X_1) & -h(\cdot, X_2) & -\tau \end{pmatrix}.
\]

Here \(\omega^i_j\) are local connection forms of the induced connection and \(S = S^1(\cdot)X_1 + S^2(\cdot)X_2\) is the shape operator. The condition \(d\Omega_\sigma - \Omega_\sigma \wedge \Omega_\sigma = 0\) is equivalent to the fundamental Gauss, Codazzi and Ricci equations. The formula (5) gives on \(TM \oplus E\) a flat connection \(\bar{D}\) described by formula (14).

The considered in the present paper 1-forms \(\Omega_i\) were constructed as satisfying additional condition \(\Omega_i = A\omega^1 + B\omega^2 + C\omega^i\) with constant \(A, B\) and \(C\). For given \(\Omega_\sigma\), such constant \(A, B\) and \(C\) may not exist, in such a case the connection \(\bar{D}\) is always different from \(\bar{\nabla}\). For example, \((M, \nabla)\) can be affinely immersed also as a non-degenerate surface in \(\mathbb{R}^3\). Such immersions and transversal fields are described in [5]. If we use one of them, then we obtain \(\bar{D}\) different from [6] and [8].

For each given connection \(\nabla\) on \(M\), for each \((1, 1)\) tensor field \(A\) and \((0, 2)\) tensor field \(\alpha\) we can define some connection \(\bar{\nabla}^{A,\alpha}\) on \(TM \oplus E\) by the formula

\[
\bar{\nabla}^{A,\alpha}(Y \oplus \Psi) = (\nabla_X Y + \Psi AX) \oplus (X(\Psi) + \alpha(X, Y)).
\]

We may look for such connections \(\nabla\) for which there exist \(A\) and \(\alpha\) such that \(\bar{\nabla}^{A,\alpha}\) is flat.

It is easy to compute

\[
\bar{R}_{A,\alpha}(X, Y)(Y \oplus \Psi)
= \left(\nabla X Y \alpha + (\nabla_X Y)(\alpha)\right) \oplus \left(\nabla Y \alpha \right)
= \left(\nabla X Y \alpha + (\nabla_X Y)(\alpha)\right) \oplus \left(\nabla Y \alpha \right)
= \left(\nabla X Y \alpha - \nabla Y \alpha \right)
\]

7. The case of indefinite metric

To complete the description we consider now a two-dimensional manifold with indefinite metric \(g\) of constant curvature \(\kappa\). We can assume, by replacing \(g\) by \(-g\) if necessary, that \(\kappa > 0\). Let \(\kappa = \frac{1}{\rho^2}\). We take a local basis \(X_1, X_2\) such that \(g(X_1, X_1) = 1 = -g(X_2, X_2), g(X_1, X_2) = 0\). The local connection forms are \(\omega^1_1 = \omega^2_2 = 0, \omega^1_2 = \omega^2_1 = \omega\). The structural equations are \(d\omega^1 = -\omega \wedge \omega^2\), \(d\omega^2 = -\omega \wedge \omega^1\), \(d\omega = -\kappa \omega^1 \wedge \omega^2\) and the 1-form

\[
\Omega_\sigma = \begin{pmatrix} 0 & -\omega & -\frac{1}{\rho} \omega^1 \\ -\omega & 0 & -\frac{1}{\rho} \omega^2 \\ \frac{1}{\rho} \omega^1 & \frac{1}{\rho} \omega^2 & 0 \end{pmatrix}.
\]
satisfies the condition \( d\Omega_{\sigma} - \Omega_{\sigma} \wedge \Omega_{\sigma} = 0 \). Using (5) we obtain

\[
\nabla_{X} (Y \oplus \Psi) = \left( (X(Y^1) + \omega(X)Y^2 + \frac{1}{\rho} \omega^1(X)\Psi)X_1 + (X(Y^2) + \omega(X)Y^1 + \frac{1}{\rho} \omega^2(X)\Psi)X_2 \right) \oplus \left( X(\Psi) - \frac{1}{\rho} (\omega^1(X)Y^1 - \omega^2(X)Y^2) \right) \tag{17}
\]

Let \( \mathbb{R}^{2,1} = \mathbb{R}^3 \) with the scalar product \( \langle (v^1, v^2, v^3), (w^1, w^2, w^3) \rangle = v^1w^1 + v^2w^2 - v^3w^3 \). Let \( Q = \{ x \in \mathbb{R}^3 : \langle x, x \rangle = \rho^2 \} \). Let \( f: M \to Q \subset \mathbb{R}^{2,1} \) be a local isometric immersion. Then \( g(X,Y) = (f_*(X), f_*(Y)) \) and the connection induced by \( f \) and the normal vector field \( \xi = \frac{1}{\rho} f \) is the Levi-Civita connection of \( g \). We have \( h(X,Y) = g(SX,Y) \) and \( SX = -\frac{1}{\rho} X \). From (14) we obtain

\[
\nabla_{X} (Y \oplus \Psi) = \left( \nabla_{X} Y + \frac{1}{\rho} \Psi(X) \right) \oplus \left( X(\Psi) - \frac{1}{\rho} g(X,Y) \right) \tag{18}
\]

and we see that \( \nabla = \nabla \).

If \( \kappa = -\frac{1}{\rho^2} \), then to \( -g \) corresponds the positive curvature \( -\kappa = \frac{1}{\rho^2} \) and the formula (17) gives the flat connection

\[
\nabla_{X} (Y \oplus \Psi) = \left( \nabla_{X} Y + \frac{1}{\rho} \Psi(X) \right) \oplus \left( X(\Psi) - \frac{1}{\rho} (-g)(X,Y) \right) \tag{18}
\]

If \( \rho = 1 \), then from (18) we obtain (1) and from (17) we obtain (2). It follows that Shchepetilov’s formulae hold also for indefinite metric \( g \).

8. Summary

For a locally symmetric connection \( \nabla \) with one-dimensional \( \text{im}R \) we have constructed two flat connections on the vector bundle \( TM \oplus (\mathbb{R} \times M) \) and two flat connections on \( TM \oplus (\mathbb{R}^2 \times M) \). From each pair only one connection may be identified with the standard connection in \( \mathbb{R}^N \), \( N = 3 \) or \( N = 4 \), after suitable local embedding of \( (M, \nabla) \) into \( \mathbb{R}^N \). Those embeddings are degenerate.

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