Abstract

We consider the situation that two players have cardinal preferences over a finite set of alternatives. These preferences are common knowledge to the players, and they engage in bargaining to choose an alternative. In this they are assisted by an arbitrator (a mechanism) who does not know the preferences.

Our main positive result suggests a satisfactory-alternatives mechanism wherein each player reports a set of alternatives. If the sets intersect, then the mechanism chooses an alternative from the intersection uniformly at random. If the sets are disjoint, then the mechanism chooses an alternative from the union uniformly at random. We show that a close variant of this mechanism succeeds in selecting Pareto efficient alternatives only, as pure Nash equilibria outcomes.

Then we characterize the possible and the impossible with respect to the classical bargaining axioms. Namely, we characterize the subsets of axioms can be satisfied simultaneously by the set of pure Nash equilibria outcomes of a mechanism. We provide a complete answer to this question for all subsets of axioms. In all cases that the answer is positive, we present a simple and intuitive mechanism which achieves this goal. The satisfactory-alternatives mechanism constitutes a positive answer to one of these possibility cases (arguably the most interesting case). Our negative results exclude the
possibility of an efficient mechanism with unique equilibrium outcome, and exclude the possibility of an efficient symmetric mechanism which is invariant with respect to repetition of alternatives.

1 Introduction

As a motivating example, consider a situation where an arbitrator has to decide how to split the assets between two partners who have decided to dissolve the partnership. Each partner has a utility for every subset of the assets and these utilities are common knowledge to the partners. The arbitrator does not know the utilities of the partners. The arbitrator may ask the partners for information about their utilities, but their answers might be strategic. Does the arbitrator have a simple mechanism for splitting the assets in an “efficient” and “fair” way? This is the question that we address in this paper.

More formally, we consider a situation where an arbitrator has to choose among a finite set of alternatives. Two players have cardinal preferences over the alternatives. These preferences are common knowledge to the players but the arbitrator does not know the preferences. We consider a classical mechanism-design framework. Both players simultaneously submit a signal to the arbitrator. Then, the arbitrator chooses a distribution over the alternatives according to the pair of signals. We assume that players are risk neutral. We ask an implementation theory question: Does there exist a mechanism that induces a good choice of alternative? In this question there are two terms that need further explanation:

1. In order to specify the behavior that is induced by a mechanism we should specify the solution concept that is assumed to be used by the players. In this paper we focus on Nash equilibrium.

2. What qualifies a choice of alternative as good? In Section 3 we focus on two properties: Pareto efficiency and anonymity (which reflects fairness). In Section 4 we take a more general approach, and consider a wider set of properties: the classical bargaining axioms.

Our main positive result (Section 3) suggests a simple and intuitive mechanism which
is called *satisfactory-alternatives mechanism*. The mechanism is based on the following idea: Each player reports the set of alternatives that are satisfactory for him. If the sets are disjoint (i.e., there is no alternative that will satisfy both players), then the arbitrator chooses uniformly at random an alternative from the union of the sets. If the sets intersect, then the arbitrator chooses uniformly at random an alternative from the intersection of the sets. Our main result (Theorem 1 and Corollary 2) states that a close variant of this mechanism enjoys the following properties:

1. A pure Nash equilibrium always exists.
2. All pure Nash equilibria are (approximately) Pareto efficient.
3. The mechanism treats equally both players (this is obvious).

Moreover, we provide an exact characterization of pure Nash equilibria outcomes of the satisfactory-alternatives mechanism. The set of outcomes is closely related to the notion of *average-fixed point* (see Section 3.1). Roughly speaking, the set of outcomes is the set of Pareto efficient alternatives that Pareto dominates an average fixed point, or an average fixed point itself if it is Pareto efficient.

The simple and realistic mechanism satisfactory-alternatives is used in practice by the commercial coordination program Doodle, see [22], where the organizer of an event suggests a finite number of possible times for the event, and asks all the potential participants to report a yes/no answer about each possible time. Note that in Doodle the organizer does not commit to choose uniformly at random from the intersection or from the union. However, in the absence of contrary guidance, the potential meeting participants may like generations of earlier thinkers [8, 6], follow the principle of indifference and use this assumption in their strategy. At the very least it is reasonable for them to assume that the organizer will assign a positive probability to every alternative in the intersection (if the intersection is non-empty) and he will assign a positive probability to every alternative in the union (if the intersection is empty). Our positive result can be extended to cases where the mechanism chooses the alternative in a non-uniform manner (see Section 3.6).

Pareto-efficiency and anonymity are not the only possible properties that we may desire of a “good” mechanism. In section 4 we consider the classical bargaining axioms, along with
several additional axioms which raises naturally in our settings. Our results draw a complete picture of the possible and the impossible with respect to these axioms when the analysis is done on the pure Nash equilibria outcomes of the mechanism. Our negative results exclude the possibility of an efficient mechanism with unique equilibrium outcome (Theorem 2) and exclude the possibility of an efficient symmetric mechanism which is invariant with respect to repetition of alternatives (Theorem 3). Our positive results suggest simple mechanisms for each one of the possibility cases. The most interesting possibility case is the case of a mechanism that satisfies symmetry and Pareto-efficiency in all equilibria, which is satisfied by the satisfactory-alternatives mechanism.

1.1 Related Literature

Starting with Nash [16] in 1950, the bargaining problem has been widely studied from different perspectives. The axiomatic approach aims to understand what should be the solution of a bargaining problem by introducing basic properties that it should satisfy. However, this approach provides no strategic reasoning for how this solution arises. The so-called “Nash program” aims to support solutions in a non-cooperative framework. The Nash program has two thrusts. One thrust is implementation theory, where the aim is to understand what can and what cannot be supported in a non-cooperative framework. The other thrust puts a premium on simplicity and realism of the bargaining mechanism, and derives insights on the bargaining problem from these constraints. All these aspects are related to the present paper. We introduce here the most relevant literature.

Simple Bargaining Models. The goal is to come up with a simple bargaining model that captures (in some sense) a realistic bargaining scenario. A classical example of such a model is Rubinstein [18]. He suggested the model where players make alternating offers one to the other until one of them accepts. Binmore et al. [3] showed that the subgame-perfect equilibrium in Rubinstein’s model converges to the Nash solution when players’ patience tends to infinity. Another example of a simple bargaining model is Anbarci [1], who also considered an alternating-offers procedure but with a finite number of alternatives, where the same alternative cannot be offered twice. Anbarci showed that the subgame perfect equilibrium outcome converges to the monotonic area solution, when the alternatives are
distributed uniformly along the bargaining set, and the number of alternatives tends to infinity.

The problem is that in these models, every outcome (including the inefficient ones) can be obtained as a Nash equilibrium; the advantages of the models pertain only to subgame-perfect equilibria, which require indefinite iteration. The purpose of the present paper is to ask: what happen if we consider a one-shot bargaining process? We believe that the mechanism suggested in this paper is a simple and a realistic one. In the example presented at the beginning of the paper, it is reasonable that the arbitrator will simply ask each player which alternatives are satisfactory, and then will try to satisfy both players by treating equally both players in the case agreement is impossible. Surprisingly, such a simple idea succeeds in selecting Pareto efficient alternatives only.

**Implementation Theory in Bargaining.** In implementation theory (see [5] for a survey), the goal is to design a mechanism such that all equilibria outcomes will have a desirable property.

Ideally, we would like to have a mechanism for which the unique equilibrium is the desired bargaining solution according to some framework (e.g., the Nash bargaining solution, or the Kalai-Smorodinsky bargaining solution, or any other reasonable solution). This goal can be achieved if we are interested in a subgame-perfect implementation. [4, 12, 13] present several versions of finite-stage mechanisms with a unique subgame-perfect equilibrium that is the desired solution (Nash, Kalai-Smorodinsky, or other). Unfortunately, such strong positive results are impossible if we consider Nash implementation. This has been established in the literature as described below.

The classical approach in implementation theory is very general: it considers a social choice set-valued function $f$, and then asks whether $f$ can be implemented by a mechanism. Namely, whether there exists a mechanism where the set of Nash equilibria outcomes is exactly $f$. The advantage of this approach is its generality, and the surprising fact that this question has a clean if-and-only-if answer, see Maskin [10] and Moore and Repullo [14].

An application of the negative results of [10, 14] in our setting implies impossibility of a mechanism where the unique Nash equilibrium outcome is the Nash bargaining solution or the Kalai-Smorodinsky bargaining solution. This follows from the fact that neither of these
solutions satisfies Maskin monotonicity, see e.g., [12]. Our negative results (Theorems 2 and 3) provide alternative evidence for the non-existence of a mechanism with unique Nash equilibrium outcome that is some bargaining solution that satisfies efficiency and symmetry. Theorem 2 excludes the possibility of uniqueness of efficient equilibrium outcome, even without assuming symmetry. This theorem does use that one of the players may be indifferent between two alternatives; this is necessary to exclude examples such as dictatorship. In some bargaining models, indifference is not allowed by definition; our next negative result, Theorem 3, does not rely on the possibility of indifference. It shows (in particular) that if we have a mechanism with unique Nash equilibrium outcome that is a bargaining solution that satisfies symmetry, then this bargaining solution must violate IRA (see definition in Section 4.1); Namely, the solution must depend not only on the set of possible alternatives, but also on the number of times each alternative appears in the multi-set of alternatives. Standard bargaining models usually satisfy IRA.

An application of the positive result in [14] to our settings implies (for instance) the existence of a mechanism where the set of equilibria outcomes is the set of all Pareto optimal outcomes. A disadvantage of this approach is that the constructed mechanism that implements this social choice set-valued function is very complicated. For instance, in the mechanisms of [10, 14] it is crucial that player will be able to submit an (unbounded) natural number, which is unrealistic. A number of papers have simplified the mechanisms of [10, 14], see e.g., [11, 19, 20, 17]. However, these simplifications hold only for the case of more than two players. Moreover, the constructed mechanisms are far more complicated than the satisfactory-alternatives mechanism that we suggest (this is not surprising because their purpose was to design a mechanism that fits every implementable social choice function, whereas we restrict our attention to the bargaining problem). Another advantage of the suggested satisfactory-alternatives mechanism, is the fact that it selects a strict subset of the Pareto optimal outcomes. Namely, it provides some prediction about the possible outcomes. For instance, in the “splitting-the-pie” problem the prediction is that a player will receive between 39% – 61% of the pie, see Proposition 1.

A more recent paper by Vartiainen [21] studies the Nash bargaining problem in the cake sharing game, in the case where the arbitrator is aware of the cake sharing structure. [21]
obtain a negative result on the non-existence of a symmetric Pareto efficient mechanism with unique Nash equilibrium outcome in the cake sharing game.

**Axiomatic Bargaining.** In the seminal work [16] Nash introduced the bargaining problem and presented four axioms: Pareto efficiency (PE), Symmetry (SYM), Invariance with respect to positive affine transformations (ITA), and Independence of irrelevant alternatives (IIA). He showed that there exists a unique solution that satisfies these four axioms. Among these four axioms the most doubtful is the last one (see [9] page 128 and [7]). This criticism led to other solutions for the bargaining problems. Kalai and Smorodinsky [7] replaced the IIA axiom with monotonicity, and derived a different solution. Anbarci and Bigelow [2] replaced the IIA axiom with area monotonicity, and derived another solution.

Following the terminology of Kalai-Smorodinsky [7], who defined a solution for a bargaining problem to be a mapping that satisfies PE, SYM, and ITA, our main focus will be on these three (more widely accepted) axioms.

### 2 Settings

Two players are bargaining over a finite collection of $n$ alternatives $A = (a^k)_{k \in [n]}$ where $a^k = (a^k_1, a^k_2) \in [0,1]^2$, with $a^k_i$ being the utility of player $i$ for the $k$'th alternative. (We write $[n] = \{1, \ldots, n\}$.) Both players know $A$.

A *mechanism for $n$ alternatives* $M_n$ is specified by a pair of signal sets $(\Sigma_1(n), \Sigma_2(n))$ and by a mapping $f_n : \Sigma_1(n) \times \Sigma_2(n) \rightarrow \Delta([n])$. Namely, for every pair of signals of the players the mechanism produces a distribution over indices (i.e., over $[n]$), which will be called a *randomized allocation*. The distribution over indices induces a distribution over alternatives, which we be called an *allocation*. A *mechanism* $M$ is a sequence of mechanisms $M = (M_n)_{n=1}^{\infty}$. We assume that players are risk neutral, and then, every mechanism induces a two player game $\Gamma_M(A)$ for every collection of alternatives $A$.

The main focus of this paper will be on the pure Nash equilibria outcomes of the mechanism. We denote by $NEO_M(A) \subset [0,1]^2$ the set of pure Nash equilibria outcomes of the game $\Gamma_M(A)$. Obviously, the focus on pure Nash equilibria is plausible only in case of existence of such an equilibrium. Hence, the existence of pure Nash equilibrium have to be proved for
every suggested mechanism. The negative results (Section 4.3) are more general, and they hold for Nash equilibria outcomes (not necessarily pure) as well.

3 Satisfactory Alternatives Mechanism

This paper is concerned with whether, in the bargaining setting introduced in the previous section, there exist “good” mechanisms—where we have three specific goals in mind for a good mechanism:

1. We would like the mechanism to lead to efficient outcomes.

2. We would like the mechanism to be fair.

3. We would like the mechanism to be simple and realistic.

For two vectors $x = (x_1, x_2), y = (y_1, y_2)$ we denote $x \gg y$ if $x_1 > y_1$ and $x_2 > y_2$.

Given a collection $A$, an allocation $x = (x_1, x_2)$ is $\varepsilon$-Pareto efficient if there is no alternative $a \in A$ such that $a \gg x + (\varepsilon, \varepsilon)$. For $\varepsilon = 0$ we will say that $x$ is Pareto efficient.

The first goal can be defined formally as follows:

**Definition 1.** A mechanism $M$ is $\varepsilon$-Pareto efficient in all equilibria if every $x \in NEO(A)$ is $\varepsilon$-Pareto efficient.

We can be formalize the second goal as follows, where a mechanism is fair if it refers to players as anonymous:

**Definition 2.** A mechanism $M$ is anonymous if $\Sigma_1(n) = \Sigma_2(n) = \Sigma(n)$ and $f_n(\sigma_1, \sigma_2) = f_n(\sigma_2, \sigma_1)$ for every $n \in \mathbb{N}$ and every $\sigma_1, \sigma_2 \in \Sigma(n)$.

Namely, the mechanism ignores the identity of the player who sent the signal.

Regarding the third goal, we will not define explicitly the notion of simplicity and realism of a mechanism. But, as we will see, the presented mechanism will be simple because the amount of information that each player will have to report is small, and it is realistic because it is used in practice, for instance, by Doodle 22.
In the mechanism satisfactory alternatives ($\mathcal{SA}$) each player $i = 1, 2$ submits a set of satisfactory alternatives $L^i \subset [n]$, i.e., $\Sigma_i(n) = 2^n$.

If the sets are disjoint (i.e., $L^1 \cap L^2 = \emptyset$) then we say that players disagree, and the randomized allocation is a uniform distribution over $L^1 \cup L^2$ (or $[n]$ if $L^1 \cup L^2 = \emptyset$). Otherwise, when $L^1 \cap L^2 \neq \emptyset$, we say that players reach an agreement, and the randomized allocation is the uniform distribution over $L^1 \cap L^2$. Formally the mapping $f_n$ is defined by

$$f_n(L_1, L_2) = \begin{cases} UN([n]) & \text{if } L_1 = L_2 = \emptyset, \\ UN(L_1 \cup L_2) & \text{if } L_1 \cap L_2 = \emptyset \text{ and } L_1 \cup L_2 \neq \emptyset, \\ UN(L_1 \cap L_2) & \text{if } L_1 \cap L_2 \neq \emptyset, \end{cases}$$

where the first condition is needed only in order that $f_n$ will be well defined for all pairs of sets.

The presented mechanism might contain inefficient equilibria as demonstrated in the following example.

**Example 1.** Let the collection of alternatives be

$$A = \{(1,0), (0,1), (0.99, 0.99), \left(\frac{2}{3}, \frac{2}{3}\right)\}.$$ 

It is easy to check that the pure action profile $(L_1, L_2) = (\{1, 4\}, \{2, 4\})$ is a pure Nash equilibrium of the game $\Gamma_{\mathcal{SA}}(A)$ with the outcome $\left(\frac{2}{3}, \frac{2}{3}\right)$ which is inefficient.

By considering the above example in more detail, we can see that both players do not lose by adding the efficient alternative $(0.99, 0.99)$ into their set. However, since the opponent does not include this alternative in his set neither player gains from adding it either.

In order to resolve this problematic issue, we provide to each player an incentive to add this efficient alternative irrespective of whether the opponent includes it in his set or not. We consider the mechanism $\mathcal{SA}_\delta$ which is identical to the $\mathcal{SA}$ mechanism except in one aspect. In the case $L^1 \cap L^2 \neq \emptyset$ the randomized allocation is the uniform distribution over $L^1 \cap L^2$ with probability $1 - \delta$ (not with probability 1), and the uniform distribution over $L^1 \cup L^2$
with probability $\delta$. Formally,

$$
 f_n(L_1, L_2) = \begin{cases} 
    UN([n]) & \text{if } L_1 = L_2 = \emptyset, \\
    UN(L_1 \cup L_2) & \text{if } L_1 \cap L_2 = \emptyset \text{ and } L_1 \cup L_2 \neq \emptyset, \\
    (1 - \delta)UN(L_1 \cap L_2) + \delta UN(L_1 \cup L_2) & \text{if } L_1 \cap L_2 \neq \emptyset,
\end{cases}
$$

Before we state our main positive result, which is an exact characterization of the pure Nash equilibria outcomes of the mechanism $SA_\delta$, we introduce a fixed-point notion which (as we will see in the results) is closely related to the mechanism $SA_\delta$.

### 3.1 Average fixed Point

We start with several notations. For a finite collection of vectors $B \subset [0,1]^2$ we denote $\text{avg}(B) = \frac{1}{\|B\|} \sum_{b \in B} b$, where $\text{avg}(B) \in [0,1]^2$.

For the collection of alternatives $A$ and a point $x \in [0,1]^2$ we denote by $A_{<, <}(x) = \{ a \in A : a_1 < x_1, a_2 < x_2 \}$ the set of alternatives in the third quadrant of the axis starting at $x$. Similarly, we denote the other quadrants (e.g., $A_{<, >}(x)$ is the second quadrant). Similarly, we denote $A_{\leq, \leq}(x) = \{ a \in A : a_1 \leq x_1, a_2 \leq x_2 \}$ the third quadrant that includes the axis.

**Definition 3.** A vector $x = (x_1, x_2)$ is a *boundaries-included average fixed point* of the collection $A$ if $\text{avg}(A \setminus A_{<, <}(x)) = x$.

We call the average fixed point boundaries-included because the alternatives on the lower boundary $A_{\leq, \leq}(x) \setminus A_{<, <}(x)$ are included in the computation of the average.

A slightly less restrictive notion of average fixed point, allows a situation where part of the points on the lower boundary belong to the averaging set and part do not.

**Definition 4.** A vector $x = (x_1, x_2)$ is an *average fixed point* of the collection $A$ if there exists a subset $B \subset A_{\leq, \leq}(x) \setminus A_{<, <}(x)$ such that $\text{avg}((A \setminus A_{\leq, \leq}(x)) \cup B) = x$.

Figure 1 demonstrates the definition of an average fixed point.

The following lemma demonstrates that even the more restrictive notion of boundaries-included average fixed point always exists, which obviously guarantees the existence of an average fixed point.
Lemma 1. Every collection $A$ admits at least one boundaries-included average fixed point.

Proof. We set $x^1 = \text{avg}(A)$, and for $t \geq 2$ we set $x^t = \text{avg}(A \setminus A_{<,<(x^{t-1}))}$.

If $A_{<,<(x^1)} = \emptyset$ then $x^1$ is a boundaries-included average fixed point. Otherwise, we know that $x^2 >> x^1$ because only strictly-below-average alternatives were eliminated from the set $A$. Similarly, if $A_{<,<(x^2)} \setminus A_{<,<(x^1)} = \emptyset$ then $x^2$ is a boundaries-included average fixed point. Otherwise, $x^3 >> x^2$ because only strictly-below-average alternatives were eliminated from the set $A \setminus A_{<,<(x^1)}$. There are at most $n$ different outcomes, therefore for some $t \leq n + 1$ we will have $A_{<,<(x^t)} \setminus A_{<,<(x^{t-1})} = \emptyset$ and $x^t$ is a boundaries-included average fixed point. 

A natural question arises: Is a boundaries-included average fixed point necessarily unique? The following example demonstrates that the answer is no.

Example 2. Let $A = ((1,1), (0.98,0), (0,0.98))$, then both $(1,1)$ and $(0.66,0.66)$ are boundaries-included average fixed points.

Actually, we can construct examples with an arbitrary large number of boundaries-included average fixed points.

Example 3. For every $k$ let $A$ be a collection of size $2 \sum_{j=0}^{k} 3^j = 3^{k+1} - 1$. For $0 \leq j \leq k$ the collection contains the outcome $(2^{-j},0)$ exactly $3^j$ times, and the outcome $(0,2^{-j})$ exactly $3^j$ times.
For every $1 \leq j \leq k$ the point $(y_j, y_j)$ where
\[
y_j = \frac{\sum_{i=0}^{j} \frac{3^i}{2^i}}{2 \sum_{i=0}^{j} 3^i} = \frac{2^{-j} - 2 \cdot 3^{-j-1}}{1 - 3^{-j-1}}
\] (1)
is a boundaries-included average fixed point of $A$. This is because $2^{-j-1} < y_j < 2^{-j}$, therefore the set $A \setminus A_{\leq, <}(y_j, y_j)$ includes exactly the points that were averaged in equation (1).

The set of average fixed points is not necessarily a singleton. However, the following lemma shows that the set of average fixed points has the following structure: it must be a sequence of (weakly) Pareto dominating outcomes.

**Lemma 2.** For every two average fixed points $x, y$, either $x \leq y$ or $y \leq x$.

**Proof.** Assume by way of contradiction that $x, y$ satisfies $x_1 > y_1$ and $x_2 < y_2$. Let $B_x \subseteq A_{\leq, <}(x) \setminus A_{\leq, <}(x)$ be such that $x = \text{avg}((A \setminus A_{\leq, <}(x)) \cup B_x)$. We denote $A_x = (A \setminus A_{\leq, <}(x)) \cup B_x$ and we denote $B_x^C = (A_{\leq, <}(x) \setminus A_{\leq, <}(x)) \setminus B_x$ the complementary lower boundary points (which are not included in the averaging). Similarly we denote $A_y, B_y$ and $B_y^C$.

![Figure 2: The sets $A_x, A_y, B_x, B_y, F$ and $G$.](image)

Note that
\[
A_y = (A_x \cup F) \setminus G
\] (2)

where (see Figure 2)
\[
F = \{ a \in A : y_1 \leq a_1 \leq x_1, a_2 \leq x_2 \} \setminus B_y^C
\] and
\[
G = \{ a \in A : a_1 \leq y_1, x_2 \leq a_2 \leq y_2 \} \setminus B_y.
\]
With respect to the average \(x\) (and specifically the average \(x_2\) of player 2), when we switch from \(A_x\) to \(A_y\), we add points that are weakly below \(x_2\) (the set \(F\)), and we remove points that are weakly above \(x_2\) (the set \(G\)). Therefore, \(y_2 \leq x_2\), which is a contradiction. \(\Box\)

A straightforward corollary from Lemma 2 is that for symmetric sets all average fixed points are located on the diagonal.

**Corollary 1.** Let \(A\) be a symmetric collection of alternatives, and let \(x = (x_1, x_2)\) be an average fixed point of \(A\), then \(x_1 = x_2\).

**Proof.** By symmetry of the collection, \((x_2, x_1)\) is also an average fixed point. By Lemma 2 it must be the case that \((x_1, x_2) \leq (x_2, x_1)\) or the opposite \((x_2, x_1) \leq (x_1, x_2)\). In both cases it follows that \(x_1 = x_2\). \(\Box\)

### 3.2 Characterization of the pure Nash equilibria

Before the statement of our main positive result, we introduce several notions.

For a collection \(A\), we denote by \(PE(A) = \{a \in A : \text{ there is no } b \in A \text{ such that } b >> a\}\) the set of Pareto efficient points of \(A\). We denote by \(AFP(A)\) the set of average fixed points of \(A\). By Lemma 1 we know that \(AFP(A) \neq \emptyset\).

Our main positive result is an exact characterization of pure Nash equilibria outcomes of the mechanism \(SA_\delta\).

**Theorem 1.** For every collection \(A\) and for every \(0 < \delta \leq 1\), the set of pure Nash equilibria outcomes of the game \(\Gamma_{SA_\delta}(A)\) is exactly the union of the following two types of equilibria outcomes:

1. The set of agreement equilibria outcomes

\[
AG(A) = \{(1 - \delta) a + \delta x : a \in PE(A), x \in AFP(A), \text{ and } a \geq x\}.
\]

2. The set of disagreement equilibria outcomes

\[
DIS(A) = \{x \in AFP(A) : \text{ There is no } a \in A \text{ such that } a >> x\}.
\]
The set of equilibrium outcomes is demonstrated in Figure 3.

The proof is presented in Section 3.5. A straightforward Corollary, shows that $SA_\delta$ is indeed an anonymous mechanism all of whose equilibrium outcomes are (approximately) efficient.

**Corollary 2.** The mechanism $SA_\delta$ admits a pure Nash equilibrium for every collection $A$, and $SA_\delta$ is an anonymous mechanism which is $\delta$-Pareto efficient in all equilibria.

**Proof of Corollary 2.** It is easy to check that $SA_\delta$ is anonymous. It is also easy to check that all the elements in $AG(A) \cup DIS(A)$ are $\delta$-Pareto efficient. The remaining part is to show that $AG(A) \cup DIS(A) \neq \emptyset$. By Lemma 1 $AFP(A) \neq \emptyset$. For some average fixed point $x \in AFP(A)$, if $x$ is not Pareto dominated by any alternative, then $x \in DIS(A)$. Otherwise, there exists an $a \in PE(A)$ that Pareto dominates $x$, and then $(1-\delta)a + \delta x \in AG(A)$. □

Actually, in the proof of Theorem 1 we show a stronger result that characterizes the set of Nash equilibria *action-profiles* (not only outcomes).

For a collection $A$, the set of Nash equilibria action profiles of the game $\Gamma_{SA_\delta}(A)$ is the union of the following two types of equilibria:

**Agreement equilibria**, which exist if there exists a Pareto dominated average fixed point $x \leq a$ for $a \in PE(A)$. The equilibrium action profile is demonstrated in Figure 4.

**Disagreement equilibium**, which exist if there exists a Pareto efficient average fixed point $x$. The equilibrium action profile is demonstrated in Figure 5.
3.3 Splitting the pie example

One of the central problems that has been studied in the context of bargaining is the problem of “splitting the pie”, where there is a unit of good that should be split among the bargainers.

There are several modelings of this problem in settings with a finite number of alternatives.

1. The collection is \( A = ((0, 1), (1, 0)) \).

2. The collection is an \( \frac{1}{k} \)-grid of the line \( \text{conv}((0, 1), (1, 0)) \). Namely, \( A = ((\frac{c}{k}, 1 - \frac{c}{k}))^{k}_{c=0} \).

There is also a modelling which does not assume efficiency in the definition of the problem.
3. The collection is an $\frac{1}{k}$-grid of the triangle $\text{conv}((1,0),(0,1),(0,0))$. Namely,

$$A = \left\{ \left( \frac{c}{k}, \frac{d}{k} \right) : c, d \in \mathbb{N} \text{ and } \frac{c}{k} + \frac{d}{k} \leq 1 \right\}.$$  \hspace{1cm} (3)

For both modellings 1 and 2 the unique outcome of both mechanisms $SA$ and $SA_\delta$ is the point $(\frac{1}{2}, \frac{1}{2})$. This is not very interesting. Actually $(\frac{1}{2}, \frac{1}{2})$ is the unique outcome of every anonymous mechanism $M$ for both modellings 1 and 2. This follows from the fact that the game $\Gamma_M(A)$ is a symmetric 1-sum game, and therefore has a unique Nash equilibrium outcome $(\frac{1}{2}, \frac{1}{2})$.

It is interesting to analyse the outcomes of the mechanism $SA_\delta$ for small values of $\delta$ and $k$ in modelling (3). The following proposition states that the outcomes get close to the Pareto efficient segment that connects the points $(0.39,0.61)$ and $(0.61,0.39)$.

**Proposition 1.** Let $A = A(k)$ be the collection in equation (3). For every $k$ and every $\delta > 0$ all the pure Nash equilibria outcomes of the mechanism $SA_\delta$ are $(\delta + \frac{1}{k})$-close to the segment $\text{conv}((x,1-x),(1-x,x))$, where $x \approx 0.39$ is the solution of the equation $x^3 - x + \frac{1}{3} = 0$ in the segment $x \in [0, \frac{1}{2}]$.

![Figure 6: Equilibria of the splitting-the-pie example.](image)

**Proof.** First we approximate the average fixed points of the set $A = A(k)$ up to an error of $\frac{1}{k}$. By Corollary all average fixed points of $A$ are of the form $(x,x)$ for $x \in [0, \frac{1}{2}]$. 

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We consider the continuous version where we replace the sets \( A \setminus A_{\zeta,\varepsilon}(x,x) \) and \( A \setminus A_{\varepsilon,x}(x,x) \) by the set
\[
B = \text{conv}((0,0),(0,1),(1,0)) \setminus \text{conv}((0,0),(0,x),(x,0),(x,x))
\]
with the uniform density. The center of mass of \( B \) approximates both the average of \( A \setminus A_{\zeta,\varepsilon}(x,x) \) and the average of \( A \setminus A_{\varepsilon,x}(x,x) \) up to an error of \( \frac{1}{k} \) because the difference between these expressions depends only the boundary points that are close to \( A_{\varepsilon,x}(x,x) \) which are at most \( \frac{1}{k} \) fraction of all points in \( A \setminus A_{\varepsilon,x}(x,x) \).

The center of mass of \( B \) is given by
\[
\frac{1}{2 - x^2} \left[ \frac{1}{6} - x^2 \left( \frac{x}{2} , \frac{x}{2} \right) \right]
\]
where \( \left( \frac{1}{3} , \frac{1}{3} \right) \) is the center of mass of \( \text{conv}((0,0),(0,1),(1,0)) \), \( \left( \frac{x}{2} , \frac{x}{2} \right) \) is the center of mass of \( \text{conv}((0,0),(0,x),(x,0),(x,x)) \), and \( \frac{1}{2} \) and \( x^2 \) are the corresponding areas of these sets. By the average fixed point assumption we deduce from formula (4) that
\[
\frac{1}{2 - x^2} \left[ \frac{1}{6} - x^3 \left( \frac{x}{2} \right) \right] = x \Rightarrow x^3 - x + \frac{1}{3} = 0
\]
This equation has a unique solution for \( x \in [0, \frac{1}{2}] \). Therefore all average fixed points \((y,y)\) of \( A(k) \) are located \( \frac{1}{k} \) close to \((x,x)\). Finally, by the characterization of equilibria outcome in Theorem 1, we get that all equilibria are agreement equilibria, where the agreement point is \((a, 1-a)\) for \( y \leq a \leq 1 - y \) and the outcome is \( \delta \) close to \((a, 1-a)\). See Figure 6.

It is worth explaining why allowing sub-efficient alternatives can create a wider list of efficient bargaining solutions. The sub-efficient alternatives increase the level of punishment; i.e., player \( i \) can reduce the payoff of player \( 3 - i \) below \( \frac{1}{2} \) (Figure 6). Therefore, new equilibria arise where player \( i \) plays a “clever” punishment strategy in which the opponent’s best option is to agree on a division where he gets less than \( \frac{1}{2} \). An example of such a “clever” punishment strategy in Figure 6 for player 1 is the strategy that includes the 22 alternatives in the bottom-right trapezoid and one additional alternative \((0.6,0.4)\). This strategy is “clever” in the above sense because it balances between two opposite goals of player 1: On the one hand, to punish player 2 in order to force player 2 to agree to an unfair division; and on the other hand, to exclude alternatives that are bad for himself, because with a positive probability \( \delta \) these bad alternatives are taken into account (even in the case of agreement).
3.4 Pareto frontier

The mechanism is allowed to return randomized allocations, whereas we measured the efficiency of a mechanism with respect to the pure alternatives. Consider, for instance, the following collection of alternatives:

\[ A = \{(1,0), (0.4,0), (0.4,0), (0,1), (0,0.4), (0,0.4)\} \]

that is similar to the one in Example 3. Both points \((0.5,0.5)\) and \((0.3,0.3)\) are average fixed points of \(A\) that are Pareto efficient (with respect to \(A\)). Therefore, by Theorem 1, both points are equilibria outcomes. It is reasonable to argue that the equilibrium outcome \((0.3,0.3)\) is not Pareto-optimal, because the mechanism can choose a randomized allocation with expected utilities \((0.5,0.5)\).

A stronger (and arguably more suitable in our settings) notion of Pareto optimality is the Pareto frontier. Given a collection \(A\), an allocation \(x = (x_1, x_2)\) is \(\varepsilon\)-close to the Pareto frontier if there is no alternative \(y \in \text{conv}(A)\) such that \(y \gg x + (\varepsilon, \varepsilon)\).

**Definition 5.** A mechanism \(M\) is \(\varepsilon\)-close to the Pareto frontier in all equilibria if every \(x \in \text{NEO}(A)\) is \(\varepsilon\)-close to the Pareto frontier.

We argue that the mechanism \(SA_{\delta}\) can be modified to a similar mechanism that is arbitrarily close to the Pareto frontier in all equilibria.

A \(k\)-uniform distribution over the indexes \([n]\) is a uniform distribution over a multiset of size \(k\) of indexes in \([n]\). We denote by \(kUN([n])\) the set of all \(k\)-uniform distributions over \([n]\).

In the modified mechanism \(SA^k_{\delta}\), each player submits a set of satisfactory \(k\)-uniform distributions over alternatives. Namely, each player \(i = 1, 2\) submits a list \(L_i \subset kUN([n])\). The mechanism \(SA^k_{\delta}\) chooses the randomized allocation exactly in the same way as \(SA_{\delta}\) does. The only difference, is that here we have a uniform distribution over \(k\)-uniform distributions, which induces a distribution over indexes.

\(^1\text{Note that the number of }k\text{-uniform different distributions is finite, and is equal to }\binom{n+k-1}{k-1}.\)
Proposition 2. The mechanism $\mathcal{SA}_k^\delta$ admits a pure Nash equilibrium for every collection $A$, and $\mathcal{SA}_k^\delta$ is an anonymous mechanism which is $(\delta + \frac{1}{k})$-close to the Pareto frontier in all equilibria.

Proof. The mechanism $\mathcal{SA}_k^\delta$ over the collection $A$ is identical to the mechanism $\mathcal{SA}_\delta$ over the collection $k\cdot UN(A)$, where $k\cdot UN(A) = \{\mathbb{E}_{i\sim \mu}(a_i) : \mu$ is a $k$-uniform distribution over $[n]\}$ is the set of expected outcomes under $k$-uniform distributions over $A$. By Corollary 2 this proves existence of pure Nash equilibrium.

![Figure 7: Points that are Pareto dominated by $k$-uniform outcomes.](image)

For every line $conv(a, b)$ on the Pareto frontier, where $a, b \in PE(A)$, the outcomes $\{\frac{m}{k}a + \frac{k-m}{k}b\}_{m=1}^k$ are $k$-uniform distribution outcomes on the Pareto frontier. Figure 7 demonstrates that every point that is $\frac{1}{k}$-far from the Pareto frontier is Pareto dominated by one of such outcomes $\frac{m}{k}a + \frac{k-m}{k}b$. Therefore $\delta$-Pareto efficiency with respect to $k\cdot UN(A)$ implies $(\delta + \frac{1}{k})$-closeness to the Pareto frontier of $A$. By Corollary 2 all equilibria of the mechanism $\mathcal{SA}_\delta$ over the collection $k\cdot UN(A)$ are $\delta$-Pareto efficient (with respect to $k\cdot UN(A)$), which implies that all equilibria of the mechanism $\mathcal{SA}_k^\delta$ over the collection $A$ are $(\delta + \frac{1}{k})$-close to the Pareto frontier.

3.5 Proof of Theorem [1]

We start with introducing several additional notations. For a set of alternatives $S \subset A$ we denote by $I_S = \{i \in [n] : a_i \in S\}$ the corresponding set of indexes. In the opposite direction,
for a list of indexes $L$, we denote by $A_L \subset A = \{a_l \in A : l \in L\}$ the corresponding set of alternatives.

For a fixed-point $x$ which includes the boundary point $B \subset A_{\leq}(x) \setminus A_{<}(x)$ (i.e., $\text{avg}(A \setminus A_{\leq}(x)) \cup B = x$) we partition the boundary points in $B$ into two sets $B_i = \{a \in B : a_i = x_i\}$, where $B_1 \cup B_2 = B$.

We start with showing that every outcome $x = (x_1, x_2) \in DIS(A)$ is a disagreement equilibrium outcome.

We split the alternatives in $A \setminus A_{\leq}(x)$ into two groups:

$$D_i = \{a \in A : a_i > x_i\} \text{ for } i = 1, 2.$$  

The sets $B_i$ and $D_i$ are demonstrated in Figure 8.

![Figure 8: A disagreement equilibrium.](image)

The fixed point $x$ belongs to $DIS(A)$, therefore, there is no $a \in A$ such that $a >> x$. So the sets $B_1 \cup D_1$ and $B_2 \cup D_2$ are disjoint. Therefore the payoff profile for the profile $(I_{B_1 \cup D_1}, I_{B_2 \cup D_2})$ is $x$ (because $x$ is a fixed point). We argue that the action profile $(I_{B_1 \cup D_1}, I_{B_2 \cup D_2})$ is a Nash equilibrium. Player 1 includes all the above-average alternatives ($D_1$) in his list and excludes all the below-average alternatives from his list. Therefore, player 1 cannot increase his payoff by remaining in a disagreement. Note that all the alternatives in $B_2 \cup D_2$ are below-average alternatives for player 1. Therefore, every agreement will reduce the payoff of player 1. Symmetric arguments prove that player 2 has no profitable deviation.
Before we show that every outcome in $AG(A)$ is a disagreement equilibrium outcome, we introduce a Lemma that will be useful in its proof.

**Lemma 3.** Let $x$ be an average fixed point and let $S_2 \subset A$ be a list (of player 2) that includes all the alternatives in $A_{<}(x) \cup B_2$ and excludes all the alternatives in $A_{\leq}(x) \setminus B_2$. Then

$$\max_{S_1 \subset A} \text{avg}_1(S_1 \cup S_2) = x.$$  

**Proof.** For $S_1 = \{a \in A : a_1 \geq x_1 \}$ we have $\text{avg}_1(S_1 \cup S_2) = x_1$, this is because $x$ is an average fixed point and every choice of the boundary points $\{a \in A : a_1 = x_1 \}$ does not effect $\text{avg}_1$. This is also the maximal value of $\text{avg}_1(S_1 \cup S_2) = x_1$, because every elimination of above-average or addition of below-average alternative will reduce the average.

Now we show that every outcome $(1 - \delta)a + \delta x \in AG(A)$ is an agreement equilibrium outcome. We denote by $R = A_{\leq}(a) \cap A_{\geq}(x)$ the alternatives in the rectangle that is formed by the two points $a$ and $x$. We also denote $C_1 = A_{>}(x_2, a_2) \setminus R$, and $C_2 = A_{>}(a_1, x_2) \setminus R$. Let $R = R_1 \cup R_2$ be an arbitrary partition of the alternatives in $R$. The set $B_i, R$ and $C_i$ are demonstrated in Figure 9.

![Figure 9: An agreement equilibrium.](image)

We argue that the action profile $(L_1, L_2) = (I_{B_1 \cup C_1 \cup R_1 \cup (a)}, I_{B_2 \cup C_2 \cup R_2 \cup (a)})$ is an agreement equilibrium. First it is easy to check that $L_1 \cap L_2 = \{a\}$ and $L_1 \cup L_2 = I_{(A \setminus A_{\leq}(x)) \cup B}$, therefore the outcome is indeed $o = (1 - \delta)a + \delta x$. Player 1 cannot improve the “disagreement” payoff.
$x_1$ in $o_1$, because all the above-average alternatives are played in $L_1 \cup L_2$ (not necessarily by player 1). Moreover, all the below-average alternatives are not played in $L_1$. Now we show that the “agreement” payoff $a_1$ cannot be improved by a unilateral deviation of player 1. Player 1 can break the agreement (i.e., to remove the action $a$ from his set), by Lemma 3 this will reduce his payoff to $x_1$ or less. Finally, player 1 cannot improve the agreement payoff $a_1$ by switching to another (or adding an additional) agreement alternative because for all $b \in L_2$, $b_1 \leq a_1$. Symmetric arguments prove that player 2 has no profitable deviation.

Now we turn to the second part of the proof where we show that the constructed above equilibria are all the pure Nash equilibria of the game.

We start with showing that in every agreement equilibrium the agreement is unique:

**Lemma 4.** Let $(L_1, L_2)$ be a pure Nash equilibrium such that $L_1 \cap L_2 \neq \emptyset$, then for every $a, b \in L_1 \cap L_2$, $a = b$.

**Proof.** Assume that $L_1 \cap L_2 \neq \emptyset$ and the intersection contains indexes with different outcomes. Without loss of generality, we assume that not all outcomes of player 1 are identical at the indexes $L_1 \cap L_2$, and that $i \in L_1 \cap L_2$ obtains the minimal payoff for player 1. Player 1 has a profitable deviation to $L'_1 = L_1 \setminus \{i\}$ because $\text{avg}_1(A_{L'_1 \cup L_2}) = \text{avg}_1(A_{L_1 \cup L_2})$ (because the union remains unchanged) and $\text{avg}_1(A_{L'_1 \cap L_2}) > \text{avg}_1(A_{L_1 \cap L_2})$.

Now we claim that for every equilibrium the union of the sets forms an average fixed point.

**Lemma 5.** Let $(L_1, L_2)$ be a pure Nash equilibrium and let $x = \text{avg}(A_{L_1 \cup L_2})$, then $A \setminus A_{\leq x}(x) \subset A_{L_1 \cup L_2} \subset A \setminus A_{\leq x}(x)$.

**Proof.** If there exists an index $l$ such that $l \notin L_1 \cup L_2$ such that $a^i_l > x$, then player $i$ has a profitable deviation to $L_i \cup \{l\}$. Therefore, $A \setminus A_{\leq x}(x) \subset A_{L_1 \cup L_2}$.

If there exists an index $l \in L_1 \cup L_2$ such that $a_l \ll x$, then we consider two cases.

Case 1: $l \in L_i$ but $l \notin L_{3-i}$. Then player $i$ has a profitable deviation to $L_i \setminus \{l\}$.

Case 2: $l \in L_1 \cap L_2$. Then by Lemma 4 $a_l$ is the unique agreement outcome. Player 1 has a profitable deviation to a disagreement by excluding the alternative $a_l$ (and all the identical alternatives $a_k$ such that $a_k = a_l$) from his set.
Therefore $A_{L_1 \cup L_2} \subset A \setminus A_{c,c}(x)$. 

The following lemma shows that the agreement outcome (if exists) is better than the disagreement outcome, and it is efficient:

**Lemma 6.** Let $(L_1, L_2)$ be a pure Nash equilibrium such that $L_1 \cap L_2 \neq \emptyset$ and let $a \in A_{L_1 \cap L_2}$, then

1. $a \geq \text{avg}(A_{L_1 \cup L_2})$.
2. $a \in PE(A)$.

**Proof.**
1. Assume to the contrary that $a_1 < \text{avg}_1(A_{L_1 \cup L_2})$. By Lemma 4, $a$ is the unique agreement outcome. Player 1 has a profitable deviation to a disagreement by excluding the alternative $a$ (and all the identical alternatives $b$ that are played by player 2) from his set. A similar argument excludes the possibility of $a_2 < \text{avg}_2(A_{L_1 \cup L_2})$.

2. Assume to the contrary that there exists $a' >> a$. By (1) we know that $a' >> a \geq \text{avg}(A_{L_1 \cup L_2})$. By Lemma 5 we get that $a' \in A_{L_1 \cup L_2}$. Without loss of generality we assume $a' \in L_2$. Then player 1 can increase his payoff by including $a'$ into the set of agreements.

The following lemma shows that a disagreement equilibrium has to be efficient:

**Lemma 7.** Let $(L_1, L_2)$ be a pure Nash equilibrium such that $L_1 \cap L_2 \neq \emptyset$, then there is no $a \in A$ such that $a >> \text{avg}(A_{L_1 \cup L_2})$.

**Proof.** Assume to the contrary that there exists $a >> \text{avg}(A_{L_1 \cup L_2})$. By Lemma 5 we get that $a \in A_{L_1 \cup L_2}$. Without loss of generality we assume $a \in L_2$. Then player 1 can increase his payoff by adding $a$ into his set and turning the disagreement into agreement on $a$.

Summarizing, every disagreement equilibrium outcome is an average fixed point (Lemma 5) and is not dominated by any alternative (Lemma 7) which restricts the set of outcomes to be in $DIS(A)$. In every agreement equilibrium the agreement is on a unique outcome (Lemma 4) which is Pareto efficient (Lemma 6). In addition, the set $L_1 \cup L_2$ forms an average fixed point (Lemma 5), and $\text{avg}(L_1 \cup L_2)$ is Pareto dominated by the agreement outcome (Lemma 6). This restricts the set of outcomes to be in $AG(A)$. 

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3.6 More general class of satisfactory-alternatives mechanisms

The satisfactory alternatives mechanism chooses an alternative uniformly at random from the intersection, or from the union. For the proofs of the results, it is sufficient to assume that the the distribution by which the mechanism chooses the alternative (from the intersection or from the union) assign positive probability to each alternative.

For instance we can consider a mechanism where each index $i$ has a weight $w_i > 0$. In case the lists are disjoint ($L_1 \cap L_2 = \emptyset$), the mechanism chooses the index $j \in L_1 \cup L_2$ with probability $w_j / \sum_{i \in L_1 \cup L_2} w_i$. In case the lists intersect ($L_1 \cap L_2 \neq \emptyset$), the mechanism chooses an index $j \in L_1 \cup L_2$ with probability $\delta w_j / \sum_{i \in L_1 \cup L_2} w_i$, and it chooses an index $j \in L_1 \cap L_2$ with probability $(1 - \delta) w_j / \sum_{i \in L_1 \cup L_2} w_i$. We denote this mechanism by $\mathcal{SA}_\delta^w$ where $w = (w_1, \ldots, w_n)$.

The following proposition states that the mechanism $\mathcal{SA}_\delta^w$ enjoys the same properties as $\mathcal{SA}_\delta$ does.

**Proposition 3.** The mechanism $\mathcal{SA}_\delta^w$ admits a pure Nash equilibrium for every collection $A$, and $\mathcal{SA}_\delta$ is an anonymous mechanism which is $\delta$-Pareto efficient in all equilibria.

**Sketch of the proof.** We argue that all the arguments that hold for $\mathcal{SA}_\delta$ can be translated to arguments on $\mathcal{SA}_\delta^w$.

First, the average fixed point notion is replaced by the corresponding notion of $w$-weighted average fixed point. Existence of an average fixed point is proved similar to Lemma 1.

The proof of Theorem also can be translated (almost without changes) to a proof for the case $\mathcal{SA}_\delta^w$. This follows from the fact that all the arguments in the proof rely on the facts that adding an above average alternative increases the average, and adding a below-average alternative decreases the average. This two basic facts hold for weighted averages as well.

4 The Possible and the Impossible for One-shot Bargaining Mechanisms

In this section we focus on the question: what can and what cannot be achieved by Nash equilibria in our settings (Section 2). Our approach is axiomatic. We start with an introduction of the axioms.
4.1 Axioms

At the core of most bargaining solutions (e.g., Nash [16], Kalai-Smorodinsky [7], or area monotonic solution [2]) are the following axioms: Invariance with respect to positive affine transformations (IAT), Symmetry (SYM), and Pareto optimality (PO).

One additional axiom appears in each one of the above mentioned solutions. In Nash [16] it is independence of irrelevant alternatives; in Kalai-Smorodinsky [7] it is monotonicity; in [2] it is area monotonicity. The focus of this section will be on the core axioms IAT, SYM, and PO.

Our setting differs from the standard bargaining settings in the following respects.

(a) We do not have a disagreement outcome.
(b) In the standard axiomatic bargaining approach the solution is unique by definition. In our settings it is plausible that the set of Nash equilibria outcomes is not a singleton. Therefore, we consider the more general case where the solution to a bargaining problem is a set of outcomes. Uniqueness is an additional axiom and we allow a situation where uniqueness is violated.
(c) The standard definition of a bargaining problem, assumes that the possible outcomes is a set. Therefore a repetition of an outcome multiple times does not effect the set of possible outcomes, hence it does not effect the solution. In our model the collection of alternatives is a multiset. An alternative may appear once or multiple times in the collection. The requirement that this replication of an alternative will not effect the solution requires an additional axiom.

Below we present natural analogs of the standard axioms IAT, SYM, and PE (Axioms 1,2,3) in our settings. As we will see in some cases there are several natural analogs. We also present the additional axioms of uniqueness and Invariance with respect to repetition of alternatives (Axioms 4,5) which address items (b) and (c) above.

1. Various definitions for Pareto optimallity.

Definition [1] defines the notion of Pareto efficiency of a mechanism. The discussion in Section [5.4] demonstrates that, in our settings, more suitable notion of Pareto optimality
might be the Pareto frontier (see Definition 5). In Proposition 2 we show that we can modify the concrete mechanism satisfactory-alternatives, which satisfies Pareto efficiency, into a mechanism that satisfies closeness to the Pareto frontier. The arguments in the proof of Proposition 2 are general. Actually, the same idea can modify every mechanism that satisfies Pareto efficiency into to a mechanism that satisfy closeness to the Pareto frontier.

**Proposition 4.** Given a mechanism $M$ that satisfies $\delta$-Pareto efficiency, let $M^k$ be a mechanism which is identical to the mechanism $M$ over the collection of alternatives is $k$-$UN(A)$ (rather than $A$). Then the mechanism $M^k$ satisfies $(\delta + \frac{1}{k})$-closeness to the Pareto frontier.

The proof is similar to the proof of Proposition 2. This proposition allows us to focus on Pareto efficiency only.

Definition 1 requires that all equilibria will be Pareto efficient. A weaker notion of Pareto efficiency requires only the existence of Pareto efficient equilibrium.

2. Symmetry.

Anonymity of a mechanism (see Definition 2) is a strong version of symmetry. A weaker notion of symmetry requires only that the equilibrium outcomes of the mechanism will be symmetric, while the mechanism may treat differently to the two players.

A collection $A$ is called symmetric if for every $x_1, x_2$ it holds that $|\{k : a^k = (x_1, x_2)\}| = |\{k : a^k = (x_2, x_1)\}|$. A set $S \subset [0, 1]^2$ is symmetric if $(x_1, x_2) \in S \Rightarrow (x_2, x_1) \in S$.

A mechanism $M$ is symmetric if for every symmetric collection $A$ the set of outcomes $NEO_M(A)$ is symmetric.

Note that anonymity is indeed a stronger requirement, because an anonymous mechanism forms a symmetric game, and therefore the set of Nash equilibrium outcomes is symmetric. All our positive results will satisfy this stronger requirement of anonymity. Whereas, for the negative results it is sufficient to assume the weaker axiom of symmetry.

3. Invariance with respect to positive affine transformations (IAT).

Let $T_i(x) = \alpha_i x + \beta_i$ be a linear mapping for $i = 1, 2$. For a set $S \subset \mathbb{R}^2$ we denote $(T_1, T_2) S := \{(T_1(s_1), T_2(s_2)) : (s_1, s_2) \in S\}$.

A mechanism $M$ is invariant with respect to positive affine transformations if for every pair of linear mappings $T_1, T_2$ such that $\alpha_i > 0$ and every collection $A$ such that $(T_1, T_2) A \subset$
[0, 1]^2$, it holds that $NEO_M((T_1, T_2)A) = (T_1, T_2)NEO_M(A)$.

**Observation 1.** Every mechanism satisfies IAT, simply because the set of Nash equilibrium outcomes is invariant with respect to affine transformations.

4. **Uniqueness (UNI)**, which requires that $|NEO_M(A)| = 1$ for every collection $A$.

5. **Invariance with respect to repetition of alternatives (IRA).**

Let $A = (a_k)_{k \in [n]}$ be a collection of alternatives. For $j \in [n]$ we denote by $(A, j) := (a_1, a_2, ..., a_n, a_j)$ the collection of size $n + 1$ that contains the alternative $a_j$ twice.

A mechanism $M$ is invariant with respect to repetition of alternatives if for every $A = (a_k)_{k \in [n]}$ and every $j \in [n]$ holds $NEO_M(A) = NEO_M((A, j))$.

### 4.2 Our Goal

Our goal is to understand whether there exists an efficient mechanism that satisfies additional desirable properties 2-5, where “efficient” can mean that all equilibria are efficient, or that there exists an efficient equilibrium. For this question we provide a complete answer.

4.3 **Negative results**

The paper considers mechanisms that admit a pure Nash equilibrium and proves positive results on the set of pure Nash equilibria outcomes. In the current section our negative results will be more general. We will prove that the results cannot hold for (possibly mixed) Nash equilibria outcomes. These result are more general, because if we focus on mechanisms that admit a pure Nash equilibrium, and we focus only on the subset of pure Nash equilibria outcomes, then the results hold with exactly the same proofs.

**Theorem 2.** There is no mechanism that satisfies uniqueness and existence of $\varepsilon$-Pareto efficient equilibrium for $\varepsilon < \frac{1}{2}$.

**Proof.** Informally, the idea is to consider the collection $((0, 1), (1, 1))$ where player 2 is indifferent between the two alternatives. Player 2 may act as if the collection is $((0, 0), (1, 1))$.

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2Note that this does not require uniqueness of Nash equilibrium, only uniqueness of the outcome, that is, of the utilities of the players.
which in an approximately efficient mechanism should result in a high weight for the second action. Or, Player 2 may act as if the collection is \(((0,1),(1,0))\), which should result in a lower weight to the second action.

Formally, let \(M\) be a mechanism that satisfies existence of \(\varepsilon\)-Pareto efficient equilibrium for \(\varepsilon < \frac{1}{2}\).

For the collection \(A = ((0,0),(1,1))\) there exists an \(\varepsilon\)-Pareto efficient Nash equilibrium \((z_1, z_2)\) of the game \(\Gamma_M(A)\), where \(z_i \in \Delta(\Sigma_i(2))\). In terms of randomized allocations, it means that alternative 2 is chosen with probability \(p > \frac{1}{2}\) and no player can increase this probability by a unilateral deviation.

For the collection \(A' = ((0,1),(1,0))\), let \((z'_1, z'_2)\) be a Nash equilibrium of the game \(\Gamma_M(A')\). Let \((q, 1-q)\) be the outcome at the equilibrium \((z'_1, z'_2)\). Without loss of generality we assume that \(q \leq \frac{1}{2}\). In terms of randomized allocations, it means that alternative 2 is chosen with probability \(q \leq \frac{1}{2}\) and player 1 cannot increase this probability by deviation.

Now consider the game \(\Gamma_M(A'')\) where \(A'' = ((0,1),(1,1))\). For the action profile \((z_1, z_2)\), the outcome is \((p, 1)\), and player 1 cannot increase the probability of the second alternative to be chosen; i.e., player 1 has no profitable deviation. Obviously, player 2 has no profitable deviation either. For the action profile \((z'_1, z'_2)\), the outcome is \((q, 1)\), and player 1 cannot increase the probability of the second alternative to be chosen. Therefore we have two different equilibria outcomes \((p, 1)\) and \((q, 1)\), which implies that \(M\) does not satisfy uniqueness.

In our settings, we allow a situation where a player can be indifferent between two alternatives. The proof of Theorem 2 relies on it. Some of the bargaining models do not allow such situation. The following impossibility results holds even in the case where an indifference is not allowed.

**Theorem 3.** There is no mechanism that satisfies SYM, IRA and \(\varepsilon\)-Pareto efficiency in all equilibria for \(\varepsilon < \frac{1}{2}\).

**Proof.** Let \(M\) be a mechanism that satisfies SYM and IRA.

For the collection \(A = ((1,0),(0,1))\) the game \(\Gamma_M(A)\) is a constant sum game. By the Minmax theorem, every constant sum game has a unique Nash equilibrium outcome (the value of the game). By symmetry this outcome is \((\frac{1}{2}, \frac{1}{2})\).
For the collection $A^\prime = ((1,0),(0,1),(0,1))$ the game $\Gamma_M(A)$ is again a constant sum game. By IRA, the unique equilibrium outcome of the game $\Gamma_M(A)$ is $(\frac{1}{2},\frac{1}{2})$. By Minmax theorem there exists a mixed strategy $z_1 \in \Delta(\Sigma_1(3))$ that guarantees (against any strategy of player 2) a payoff of $\frac{1}{2}$. In terms of randomized allocations, the mixed strategy $z_1$ guarantees that the first index will be chosen with probability of at least $\frac{1}{2}$.

We repeat the same arguments for player 2 with the collection $A'' = ((1,0),(1,0),(0,1))$. We get that there exists a strategy $z_2 \in \Delta(\Sigma_2(3))$ that guarantees that the third index will be chosen with probability of at least $\frac{1}{2}$.

Now consider the collection $A''' = ((1,0),(\frac{3}{4} + \frac{\varepsilon}{2},\frac{3}{4} + \frac{\varepsilon}{2}),(0,1))$. We claim that $(z_1,z_2)$ is a Nash equilibrium in the game $\Gamma_M(A''')$ with the outcome $(\frac{1}{2},\frac{1}{2})$. This will complete the proof because the outcome $(\frac{1}{2},\frac{1}{2})$ is not $\varepsilon$-Pareto efficient, for $\varepsilon < \frac{1}{2}$ (because $\frac{1}{2} + \varepsilon < \frac{3}{4} + \frac{\varepsilon}{2}$).

We first show that the outcome is $(\frac{1}{2},\frac{1}{2})$. Since $z_1$ guarantees that the first index is chosen with probability of at least $\frac{1}{2}$, and $z_2$ guarantees that the third index is chosen with probability of at least $\frac{1}{2}$, the randomized allocation at the profile $(z_1,z_2)$ must be 1 and 3 with equal probability $\frac{1}{2}$, which leads to the payoff $(\frac{1}{2},\frac{1}{2})$.

Now we show that no player has a profitable deviation. Since player 2 is playing $z_2$, index 3 will be chosen with probability of at least $\frac{1}{2}$ for all actions $\sigma_1 \in \Sigma_1(3)$. Therefore, with probability of at least $\frac{1}{2}$ the payoff of player 1 will be 0. All the payoffs are bounded by 1, therefore player 1 cannot get a payoff above $\frac{1}{2}$ by any deviation. Symmetric arguments prove that player 2 has no profitable deviation either.

4.4 Positive results

Theorem 2 excludes the possibility of an efficient mechanism with unique equilibrium. Therefore, we will ignore uniqueness.

Observation 1 implies that IAT automatically holds for all mechanisms. Therefore, we will ignore IAT as well.

The remaining three axioms are Pareto efficiecy (with a stronger and a weaker notion of efficiency), symmetry, and IRA.

Theorem 3 excludes the possibility of a mechanism that is symmetric, IRA, and efficient in all equilibria. This leaves us with the following three possibilities:
1. To relinquish symmetry. Then the question is whether there exists a mechanism that satisfies IRA and is efficient in all equilibria.

2. To weaken the requirement on efficiency. Then the question is whether there exists a mechanism that satisfies symmetry, IRA, and with at least one efficient equilibrium.

3. To relinquish IRA. Then the question is whether there exists a mechanism that is symmetric and efficient in all equilibria.

The answer to all these three questions is: yes such a mechanism exists (see Observation 2 and Corollaries 3 and 2). Moreover the mechanisms that constitute positive answers to all these three questions are simple.

We start with the first question on the existence of IRA and efficient mechanism. We consider the \textit{dictator mechanism} ($\mathcal{D}$), where player 1 decides on the allocation. Formally $\Sigma_1(n) = [n], \Sigma_2(n) = \emptyset$ and $f(\sigma_1, \sigma_2) = \sigma_1$.

\textbf{Observation 2.} The dictator mechanism satisfies IRA and Pareto efficiency in all equilibria.

Now we turn to the second question on the existence of an IRA symmetric mechanism that admits a Pareto efficient equilibrium. It is well known that existence of efficient equilibrium can be obtained by a coordination mechanism that incentivizes both players to agree on an outcome. We present here a simple version of this idea that holds in our settings. The mechanism is called \textit{coordination with uniform dictatorial disagreement} (CUDD). Each player submits a pair of indexes, an agreement index and an index for the case of disagreement. If the agreement index is identical then we say that players reach an \textit{agreement} and the allocation is this index. Otherwise, if the agreement index is not identical, then we say that players \textit{disagree} and the allocation is the uniform distribution over the two disagreement indexes of both players. Formally $\Sigma_1(n) = \Sigma_2(n) = [n]^2$, an element in $\Sigma_i(n)$ is denoted by $(g_i, d_i)$ where $g_i$ is the agreement index, and $d_i$ is the disagreement index. The mapping $f_n$ is defined as follows

$$f_n((g_1, d_1), (g_2, d_2)) = \begin{cases} g_1 & \text{if } g_1 = g_2, \\ UN(\{d_1, d_2\}) & \text{otherwise.} \end{cases}$$
For a collection $A$ we denote by $\overline{m}_i = \max_{a_k \in A} a^i_k$ and $\underline{m}_i = \min_{a_k \in A} a^i_k$ the maximal and the minimal utility of player $i$. We also denote by $A^i_{\text{max}} = \{ a_k \in A : a^i_k = \overline{m}_i \}$ the set of alternatives that maximizes player’s $i$ utility.

The exact characterization of pure Nash equilibria outcomes of the mechanism $CUDD$ is as follows.

**Proposition 5.** For every collection $A$ the set of pure Nash equilibria outcomes of the game $\Gamma_{CUDD}(A)$ is exactly the union of the following two types of equilibria outcomes:

1. The set of agreement equilibria outcomes

\[ AG(A) = \{ a \in A : a_i \geq \frac{\overline{m}_i + \underline{m}_i}{2} \text{ for } i = 1, 2 \}. \]

2. The set of disagreement equilibria outcomes

\[ DIS(A) = \{ \frac{b + c}{2} : b \in A^1_{\text{max}}, c \in A^2_{\text{max}} \}. \]

![Figure 10: Equilibrium Outcomes of $CUDD$.](image)

The set of equilibrium outcomes is demonstrated in Figure [10].
Proof of Proposition 5. First we show that all outcomes in the set $AG(A) \cup DIS(A)$ are pure Nash equilibrium outcomes.

For $a_j \in A_{1\text{max}}$ and $a_k \in A_{2\text{max}}$ the action profile $((j,j), (k,k))$ is an equilibrium.

The alternative $a_k \in A_{2\text{max}}(\frac{m_1+m_1}{2}, \frac{m_2+m_2}{2})$ can be obtained by the following agreement equilibrium. Let $p(i)$ be a punishment strategy of player $i$ that minimizes the payoff of the opponent, namely $a_{p(i)} = \frac{m_{3-i}}{2}$. Then the action profile $((k,p(1)), (k,p(2)))$ is an equilibrium, because by a deviation to a disagreement player $i$ can get a payoff of at most $\frac{m_{1}+m_2}{2}$.

Now we show that no other outcome can be obtained in a pure Nash equilibrium.

An agreement on an alternative $k$ such that $a_k < \frac{m_1+m_1}{2}$ is not stable because player 1 can deviate to $(j,j)$ where $a_j \in A_{1\text{max}}$ and get a payoff of at least $\frac{m_1+m_1}{2}$. The same argument for player 2 excludes an agreement on an alternative $k$ such that $a_k < \frac{m_2+m_2}{2}$.

In case of a disagreement, in equilibrium, each player maximizes his own payoff which leads to an outcome of the form $a_1 + a_2$ where $a_j \in A_{1\text{max}}$ and $a_k \in A_{2\text{max}}$.

A straightforward Corollary shows that $CUDD$ indeed constitutes a positive answer to Question 2 posed at the beginning of this section.

Corollary 3. The mechanism $CUDD$ satisfies symmetry, IRA, and existence of Pareto efficient equilibrium with respect to the set of pure Nash equilibria outcomes.

Proof of Corollary 3. It is easy to see that the set $AG(A) \cup DIS(A)$ is symmetric for symmetric collections, and that it is invariant with respect to repetition of alternatives.

For existence of Pareto efficient outcome, we consider some outcome $\frac{b+c}{2}$ such that $b \in A_{1\text{max}}$ and $c \in A_{2\text{max}}$. If there is no $a \in A$ such that $a \geq \frac{b+c}{2}$ then $\frac{b+c}{2} \in DIS(A)$ Pareto efficient equilibrium outcome. Otherwise, a Pareto efficient alternative $a \in PE(A)$ such that $a \geq \frac{b+c}{2}$ satisfies $a \in AG(A)$, because

$$a_1 \geq \frac{b_1 + c_1}{2} = \frac{m_1 + c_1}{2} \geq \frac{m_1 + m_1}{2}$$

and similarly for player 2

$$a_2 \geq \frac{b_2 + c_2}{2} = \frac{b_2 + m_2}{2} \geq \frac{m_2 + m_2}{2}.$$
Finally, the third question on the existence of a symmetric mechanism that is efficient in all equilibria was addressed in Section 3. We saw that the satisfactory-alternatives mechanism $SA_d$ enjoys these two properties.

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