REMARK ON THE GLOBAL NON-EXISTENCE OF
SEMIRELATIVISTIC EQUATIONS WITH NON-GAUGE
INVARIANT POWER TYPE NONLINEARITY WITH MASS

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Abstract. The non-existence of global solutions for semirelativistic equations
with non-gauge invariant power type nonlinearity with mass is studied in the
framework of weighted $L^1$. In particular, a priori control of weighted integral
of solutions is obtained by introducing a pointwise estimate of fractional
derivative of some weight functions. Especially, small data blowup with small
mass is obtained.

1. Introduction

We consider the Cauchy problem for the following semirelativistic equations
with non-gauge invariant power type nonlinearity:

\begin{equation}
\begin{cases}
i \partial_t u + (m^2 - \Delta)^{1/2} u = \lambda |u|^p, & t \in [0, T), \ x \in \mathbb{R}^n, \\
u(0) = u_0, & x \in \mathbb{R}^n,
\end{cases}
\end{equation}

with $m \geq 0$, $\lambda \in \mathbb{C} \setminus \{0\}$, where $\partial_t = \partial / \partial t$ and $\Delta$ is the Laplacian in $\mathbb{R}^n$. Here
$(m^2 - \Delta)^{1/2}$ is realized as a Fourier multiplier with symbol $(m^2 + |\xi|^2)^{1/2}$: $(m^2 - \Delta)^{1/2} = \mathfrak{F}^{-1} (m^2 + |\xi|^2)^{1/2} \mathfrak{F}$, where $\mathfrak{F}$ is the Fourier transform defined by

$$(\mathfrak{F}u)(\xi) = \hat{u}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} u(x) e^{-ix \cdot \xi} dx.$$ 

We remark that the Cauchy problem such as (1.1) arises in various physical set-
tings and accordingly, semirelativistic equations are also called half-wave equations,
fractional Schrödinger equations, and so on, see [3, 15, 16] and reference therein.

The local existence for (1.1) in the $H^s(\mathbb{R}^n)$ framework is easily seen if $s > n/2$, where $H^s(\mathbb{R}^n)$ is the usual Sobolev space defined by $(1 - \Delta)^{-s} L^2(\mathbb{R}^n)$. Here the
local existence in the $H^s(\mathbb{R}^n)$ framework means that for any $H^s(\mathbb{R}^n)$ initial data,
there is a positive time $T$ such that there is a solution for the corresponding integral
equation,

\begin{equation}
u(t) = e^{it(m^2 - \Delta)^{1/2}} u_0 - i \lambda \int_0^t e^{i(t-t')(m^2 - \Delta)^{1/2}} |u(t')|^p dt',
\end{equation}
in $C([0, T); H^s(\mathbb{R}^n))$. We remark that for $s > n/2$, local solution for (1.2) may
be constructed by a standard contraction argument with the Sobolev embedding
$H^s(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ which holds if and only if $s > n/2$. We also remark that in
the one dimensional case, $s > 1/2$ is also the necessary condition for the local existence
in the $H^s(\mathbb{R})$ framework because the non-existence of local weak solutions to (1.1)
with some $H^{1/2}(\mathbb{R})$ data is shown in [8]. In general setting, the necessary condition
is still open and partial results are discussed in [2, 9, 13]. We also remark that in
massless case, (1.1) is scaling invariant. Namely, when \( u \) is a solution to (1.1) with initial data \( u_0 \), then for any \( \rho > 0 \), the pair,
\[
u_\rho(t, x) = \rho^{1/(p-1)}u(\rho t, \rho x), \quad u_0, \rho = \rho^{1/(p-1)}u_0(\rho x)
\]
also satisfies (1.1). Then the case where \( (s, q) \) satisfies that for \( u_0 \in H^s_q(\mathbb{R}^n) \),
\[
\|(-\Delta)^{s/2}u_0, \rho\|_{L^q_s(\mathbb{R}^n)} \to \infty \quad \text{as} \quad \rho \to \infty \iff s - \frac{n}{q} + \frac{1}{p-1} > 0
\]
is called \( H^s_q(\mathbb{R}^n) \) scaling subcritical case, where \( H^s_q(\mathbb{R}^n) = (1 - \Delta)^{-s/2}L^q(\mathbb{R}^n) \). Moreover, if \( s = n/q + 1/(p-1) \), we call the case as \( H^s_q(\mathbb{R}^n) \) scaling critical case. In the \( H^s_q(\mathbb{R}^n) \) scaling subcritical case, in general, the local existence in \( H^s(\mathbb{R}^n) \) framework is expected but this is not our case because the case where \( n = 1 \) and \( s = 1/2 \) is \( H^s(\mathbb{R}) \) scaling subcritical with any \( p > 1 \).

In the present paper, we revisit the global non-existence of (1.1). In order to go back to prior works, we define weak solutions for (1.1) and its lifespan.

**Definition 1.1.** Let \( u_0 \in L^2(\mathbb{R}^n) \). We say that \( u \) is a weak solution to (1.1) on \([0, T]\), if \( u \) belongs to \( L^1_{\text{loc}}(0, T; L^2(\mathbb{R}^n)) \cap L^1_{\text{loc}}(0, T; L^p(\mathbb{R}^n)) \) and the following identity
\[
\int_0^\infty (u(t)|i\partial_t \psi(t) + (m^2 - \Delta)^{1/2}\psi(t))dt = i(u_0)\psi(0) + \lambda \int_0^\infty (|u(t)|^2|\psi(t))dt
\]
holds for any \( \psi \in C([0, \infty); H^1(\mathbb{R}^n)) \cap C^1([0, \infty); L^2(\mathbb{R}^n)) \) satisfying
\[
sup \psi < [0, T] \times \mathbb{R},
\]
where \( (\cdot | \cdot) \) is the usual \( L^2(\mathbb{R}^n) \) inner product defined by
\[
(f | g) = \int_{\mathbb{R}^n} f(x)g(x)dx.
\]
Moreover we define \( T_w \) as
\[
T_w = \inf\{ T > 0 ; \text{ There is no weak solutions for (1.1) on } [0, T]. \}.
\]

At first, in \( L^1(\mathbb{R}) \) scaling critical and subcritical massless cases, the global non-existence is shown in [10].

**Proposition 1.2** ([10] Theorem 1.3]). If \( n = 1, m = 0, 1 < p \leq 2, \) and \( u_0 \in (L^1 \cap L^2)(\mathbb{R}) \) satisfying that
\[
(1.3) \quad \text{Re}(m_0) = 0, \quad -\text{Im}\left( \int_{\mathbb{R}} \mathcal{N}_0(x)dx \right) > 0,
\]
then there is no global weak solution, namely, if \( T \) is big enough, there is no weak solution on \([0, T]\).

Here we remark that the case when \( p = 2 \) is \( L^1(\mathbb{R}) \) scaling critical.

Later, Inui [15] obtained the following global non-existence in \( H^s(\mathbb{R}^n) \) scaling critical and subcritical cases for large data with \( 0 < s < n/2 \) and in \( L^2(\mathbb{R}^n) \) scaling subcritical massless case for small data:

**Proposition 1.3** ([15] Theorem 1.2]). Let \( s \geq 0 \) and \( m \geq 0 \). We assume that \( 1 < p \leq 1 + 2/(n - 2s) \). Let \( f \in H^s(\mathbb{R}^n) \) satisfy
\[
(1.4) \quad \text{Re}(\mathcal{N}f) = 0, \quad -\text{Im}(\mathcal{N}f) \geq \begin{cases} |x|^{-k}, & \text{if } |x| \leq 1, \\ 0, & \text{if } |x| > 1, \end{cases}
\]
with $k < n/2 - s(\leq 1/(p - 1))$. If initial value $u_0$ is given by $\mu f$ with positive constant $\mu$, then there exists $\mu_0$ such that there is no global weak solution for $\mu > \mu_0$. Moreover, for any $\mu \in [\mu_0, \infty)$, $T_w$ is estimate by

$$T_w \leq C\mu^{-\frac{1}{2^* - k}}.$$

with positive constant $C$ which is independent of $\mu$.

**Proposition 1.4** ([15 Theorem 1.4]). We assume that $1 < p < 1 + 2/n$, $m = 0$. Let $f \in L^2(\mathbb{R}^n)$ satisfy

$$\text{Re}(\lambda f) = 0, \quad -\text{Im}(\lambda f) \geq \begin{cases} 0, & \text{if } |x| \leq 1, \\ |x|^{-k}, & \text{if } |x| > 1, \end{cases}$$

with $n/2 < k < 1/(p - 1)$. If initial value $u_0(x)$ is given by $\mu f(x)$ with $\mu > 0$, then there is no global weak solution. Moreover, there exist $\varepsilon > 0$ and a positive constant $C > 0$ such that

$$T_w \leq \begin{cases} C\mu^{-\frac{1}{2^* - k}}, & \text{if } 0 < \mu < \varepsilon, \\ 2, & \text{if } \mu > \varepsilon. \end{cases}$$

We remark that for $0 < s < n/2$, there are $H^s(\mathbb{R}^n)$ functions satisfying (1.4). For details, see [14 Example 5.1].

In [10][15], the non-existence of weak solutions are shown by a test function method introduced by Baras-Pierre [1] and Zhang [17, 18]. In the classical test function argument, the classical Leibniz rule plays a critical role. On the other hand, the fractional derivative $(m^2 - \Delta)^{1/2}$ of compact supported functions is not controlled pointwisely like classical derivative. Indeed, since $(m^2 - \Delta)^{1/2}$ is nonlocal, $\text{supp} \ (m^2 - \Delta)^{1/2}\phi$ is bigger than $\text{supp} \ \phi$ for $\phi \in C_c^\infty(\mathbb{R}^n)$ in general, where $C_c^\infty(\mathbb{R}^n)$ denotes the collection of smooth compactly supported functions. Therefore, it is impossible to have the following pointwise estimate: There exists a positive constant $C$ such that for any $\phi \in C_c^\infty(\mathbb{R}^n)$,

$$|(m^2 - \Delta)^{1/2}\phi^\ell(x)| \leq C|\phi^{\ell-1}(x)(m^2 - \Delta)^{1/2}\phi)(x)|, \quad \forall x \in \mathbb{R}^n$$

with $\ell > 1$. In order to avoid from the difficulty of nonlocality, in [10][15], (1.1) is transformed into

$$\partial_t^2 v + m^2 v - \Delta v = -|\lambda|^2 \partial_t |u|^p,$$

where $v = \text{Im}(\lambda u)$. (1.7) may be obtained by applying $-\text{Im}(\lambda(i\partial_t - (m^2 - \Delta)^{1/2}))$ to both sides of (1.4). Propositions above were obtained by applying test function method to (1.7) with some special test functions. Here we remark that test function method is relatively indirect method. Especially, it is impossible to see the behavior of blowup solution with test function method because the lifespan is obtained by comparison between initial data and scaling parameter.

On the other hand, in [7], the global nonexistence of (1.1) was studied in a more direct manner.

**Proposition 1.5** ([17 Proposition 4]). Let $m = 0$. Let

$$X(T) = C([0, T]; L^2(\mathbb{R}^n)) \cap C^1([0, T], H^{-1}(\mathbb{R}^n)) \cap L^\infty(0, T; L^p(\mathbb{R}^n)).$$

Let $u_0 \in L^2(\mathbb{R}^n)$ satisfy

$$M_R(0) > C_{n,p,\alpha} R^{n-1/(p-1)},$$

(1.8)
with some $R > 0$ and $\alpha \in \mathbb{C}$ satisfying that

\[
\text{Re}(\alpha \lambda) > 0.
\]

Here $M_R(0)$ and $C_{n,p,\alpha}$ is given by

\[
M_R(0) = -\text{Im} \left( \alpha \int_{\mathbb{R}^n} u_0(x) \langle x/R \rangle^{-n-1} dx \right),
\]

\[
C_{n,p,\alpha}^p = 2^{1+p'/p} p^{-p'/p} \text{Re}(\alpha \lambda)^{-p'} |\alpha|^{p+p'} A_{n,n+1}^p \left( \int_{\mathbb{R}^n} \langle x \rangle^{-n-1} dx \right)^p
\]

and constant $A_{n,n+1}$ is determined below. Then there is no solution for (1.10) in $X(T)$ with $u(0) = u_0$ and $T > T_{n,p,\lambda,\alpha,R}$, where

\[
T_{n,p,\lambda,\alpha,R} = (p-1)^{-1} D_{n,p,\lambda,\alpha}^{-1} R^{n(p-1)} (M_R(0) - C_{n,p,\alpha} R^{n-1/(p-1)-p+1}),
\]

\[
D_{n,p,\lambda,\alpha} = 2^{-1} \text{Re}(\alpha \lambda) |\alpha|^{-p} \left( \int_{\mathbb{R}^n} \langle x \rangle^{-n-1} dx \right)^{-p+1}.
\]

We remark that in the subcritical massless case, Propositions 1.2, 1.3, and 1.4 may be relaxed. For details, see Corollaries 1, 2, and 3 in [7] and also Corollaries 1.10, 1.11, and 1.12 below.

Proposition 1.5 may be obtained by a modification of test function method of [11]. Particularly, one can show that, for solution $u$ to (1.1),

\[
M_R(t) = -\text{Im} \left( \alpha \int_{\mathbb{R}^n} u(t,x) \langle x/R \rangle^{-n-1} dx \right)
\]

satisfies the ordinary differential inequality,

\[
\frac{d}{dt} (M_R(t) - C_1) \geq C_2 (M_R(t) - C_1)^p
\]

with some positive constants $C_1$ and $C_2$. Since a priori weight $L^1$ control of blowup solutions (1.10) is given, the approach of [11] may be regarded as relatively direct comparing to test function methods of [10, 15]. In order to show (1.10), again, pointwise control of weight functions like (1.6) is required. Since (1.6) fails for general compactly supported functions, we consider the estimate of weight functions decaying polynomially and obtain the following:

**Lemma 1.6.** Let $\langle x \rangle = (1 + |x|^2)^{1/2}$. For $q > 0$, there exists a positive constant $A_{n,q}$ depending only on $n$ and $q$ such that for any $x \in \mathbb{R}^n$,

\[
|((-\Delta)^{1/2} \langle \cdot \rangle^{-q}) (x)| \leq \begin{cases} 
A_{n,q} (x)^{-q-1}, & \text{if } 0 < q < n, \\
A_{n,q} (x)^{-n-1} (1 + \log(1 + |x|)), & \text{if } q = n, \\
A_{n,q} (x)^{-n-1}, & \text{if } q > n.
\end{cases}
\]

Lemma 1.6 may be shown by a direct computation with the following representation:

\[
((-\Delta)^{1/2} f)(x) = B_{n,s} \lim_{\varepsilon \searrow 0} \int_{|y| \geq \varepsilon} \frac{f(x) - f(x + y)}{|y|^{n+1}} dy,
\]

where

\[
B_{n,s} = \left( \int_{\mathbb{R}^n} \frac{1 - \cos(\xi_1)}{|\xi|^{n+1}} d\xi \right)^{-1}.
\]
For details of this representation, for example, we refer the reader [6]. If one regards \((-\Delta)^{s/2}\) as \(\nabla\), Lemma 1.6 seems natural at least for \(0 < q < n\). When \(q \geq n\), the decay rate of fractional derivative is worse than the expectation form the classical first derivative but it is sufficient to prove Proposition 1.5 and actually sharp. For details, see Remarks 1 and 2 in Section 2 of [7].

We also remark that Córdoba and Córdoba [4] showed that

\[
(-\Delta)^{s/2} (\phi^2)(x) \leq 2\phi(x)((-\Delta)^{s/2} \phi)(x)
\]

for any \(0 \leq s \leq 2\), \(\phi \in \mathcal{S}(\mathbb{R}^2)\), and \(x \in \mathbb{R}^2\), where \(\mathcal{S}\) denotes the collection of rapidly decreasing functions. In general, \(\phi \geq 0\) does not imply \((-\Delta)^{s/2} \phi \geq 0\), and therefore (1.12) does not imply (1.6) even with positive \(\phi\). We also remark that they also used the integral representation of \((-\Delta)^{s/2}\), which is (1.11) when \(s = 1\).

The aim of this paper is to generalize Proposition 1.5 by introducing the following pointwise estimate:

\[
|(m^2 - \Delta)^{1/2}(\cdot)^{-n-1}(x)| \leq C(\cdot)^{-n-1}
\]

for any \(x \in \mathbb{R}^n\) with some positive constant \(C\).

The difficulty to study (1.13) is the non-existence of integral representation of \((m^2 - \Delta)^{1/2}\) like (1.11). Therefore, we divide our operator into two parts as follows:

\[(m^2 - \Delta)^{1/2} = (-\Delta)^{1/2} + \mathcal{R},\]

where \(\mathcal{R}\) is a Fourier multiplier with the following symbol:

\[
(m^2 + |\xi|^2)^{1/2} - |\xi| = \int_0^m (\theta^2 + |\xi|^2)^{-1/2} \theta d\theta.
\]

Thanks to Lemma 1.6, it is sufficient to show the pointwise control of \(\mathcal{R}\). Fortunately, \(\mathcal{R}\) consists of Bessel potential and the Bessel potential \((1 - \Delta)^{-1/2}\) has an integral kernel \(K\). In particular, we have the following:

**Proposition 1.7** ([12, Proposition 1.2.5]). Let \(K\) be a measurable function satisfying

\[(1 - \Delta)^{-1/2} \phi = K * \phi,\]

for \(\phi \in \mathcal{S}\), where \(*\) denotes the convolution. Then \(K\) is strictly positive and \(\|K\|_{L^1(\mathbb{R}^n)} = 1\). Moreover there is a positive constant \(\tilde{B}_n\) depending only on \(n\) and satisfying that

\[
K(x) \leq \tilde{B}_n e^{-|x|/2}, \quad \text{if } |x| > 2,
\]

\[
K(x) \leq \tilde{B}_n \begin{cases} 
\log(\frac{2}{|x|}) + 1 + O(|x|^2), & \text{if } n = 1, \\
1 + |x|^{1-n}, & \text{if } n > 1,
\end{cases}
\]

if \(|x| < 2\).

Since \(K\) has only integrable singularity at the origin and decays exponentially, nonlinear estimate \(\mathcal{R}\) may be obtained by a direct computation. The next estimate is essential in this paper.
Corollary 1.10. Let \( \text{of (1.1)} \) are shown to be estimated similarly to solutions of (1.1) without mass. However, if mass is sufficiently small, solutions (1.1) is not scaling invariant essentially, Propositions 1.2 and 1.4 seem difficult to corollaries of Proposition 1.9. Here, we remark that since the Cauchy problem

Corollary 1.11. Let \( T \)

Especially,

Then there is no solution for \( (1.1) \) in \( X(T) \) with \( u(0) = u_0 \) and \( T > \tilde{T}_{n,p,m,\lambda,\alpha,R} \), where

\[
\tilde{T}_{n,p,m,\lambda,\alpha,R} = (p - 1)^{-1} D_{n,p,\lambda,\alpha} R^{(n+2)/(p-1)} (M_R(0) - \langle R m \rangle^{(n+2)/(p-1)}) \tilde{C}_{n,p,\alpha} R^{n-1/(p-1)} - \langle x \rangle^{-n+1} dx \]

Now, in the subcritical case, Propositions 1.2, 1.3 and 1.4 may be obtained as corollaries of Proposition 1.9. Here, we remark that since the Cauchy problem \( (1.1) \) is not scaling invariant essentially, Propositions 1.2 and 1.4 seem difficult to be extended in case of general mass. However, if mass is sufficiently small, solutions of \( (1.1) \) are shown to be estimated similarly to solutions of \( (1.1) \) without mass.

Corollary 1.10. Let \( 1 < p < 1 + 1/n \). Let \( \alpha \in \mathbb{C} \) and \( u_0 \in (L^1 \cap L^2)(\mathbb{R}^n) \) satisfy \( (1.9) \) and

\[
- \text{Im} \left( \alpha \int_{\mathbb{R}^n} u_0(x) dx \right) > 0.
\]

Then, for sufficiently small \( m \), there exists no solution in \( X(T) \) for sufficiently large \( T \).

Corollary 1.11. Let \( m \in \mathbb{R} \). Let \( u_0(x) = \mu f(x) \) where \( \mu \gg 1 \) and \( f \) satisfies

\[
- \text{Im}(\alpha f(x)) \geq \begin{cases} 
|x|^{-k}, & \text{if } |x| \leq 1, \\
0, & \text{if } |x| > 1,
\end{cases}
\]

with some \( k < \min(n/2, 1/(p-1)) \) and \( \alpha \) satisfying \( (1.9) \). Then there exists some \( R_1 > 0 \) satisfying \( (1.10) \) and

\[
\tilde{T}_{n,p,m,\lambda,\alpha,R_1} \leq C \mu^{-1/(p-1)}.
\]
Corollary 1.12. Let \( u_0(x) = \mu f(x) \) where \( 0 < \mu \ll 1 \) and \( f \) satisfies

\[
-\text{Im}(\alpha f(x)) \geq \begin{cases} 
0, & \text{if } |x| \leq 1, \\
|x|^{-k}, & \text{if } |x| > 1,
\end{cases}
\]

with some \( n/2 < k < 1/(p-1) \) and \( \alpha \) satisfying (1.19). Then, for sufficiently small \( m \), there exists some \( R_2 > 0 \) satisfying (1.16) and

\[
\tilde{T}_{n,p,m,\lambda,\alpha,R_2} \leq C\mu^{-\frac{1}{r(p-1)-\min(n,k)}}.
\]

We remark that Corollaries 1.10, 1.11, and 1.12 correspond to Propositions 1.2, 1.3, and 1.4, respectively.

In the next section, we show Proposition 1.8. In Section 3, we show the proof of Proposition 1.9 and Corollaries 1.10, 1.11, and 1.12.

2. Proof of Proposition 1.8

In order to show (1.14), it is sufficient to show for any \( q > n/2 \),

\[
|\mathcal{R}\langle \cdot \rangle^{-n-1}(x)| \leq 2^{q/2} |\langle \cdot \rangle^q K|_{L^1(\mathbb{R}^n)} (m)^{q+1} \langle x \rangle^{-q}.
\]

For \( \theta > 0 \),

\[
(\theta^2 - \Delta)^{-1/2} \theta f = \theta^3^{-1} ((\theta^2 + |\cdot|^2)^{-1/2} \hat{f}) = \theta^n \theta^{-1} ((1 + |\cdot|^2)^{-1/2} (\hat{f})_1/\theta)
\]

\[
= (1 - \Delta)^{-1/2} f_\theta.
\]

Therefore,

\[
\theta(\theta^2 - \Delta)^{-1/2} \langle \cdot \rangle^{-q} = (1 - \Delta)^{-1/2} \langle \cdot /\theta \rangle^{-q} \leq \langle \theta \rangle^q K * \langle \cdot \rangle^{-q},
\]

where we have used the fact that \( K \) is positive and for any \( x \in \mathbb{R}^n \),

\[
\langle x \rangle \leq \langle x/\theta \rangle \langle \theta \rangle.
\]

Then by (2.2),

\[
\mathcal{R}\langle \cdot \rangle^{-q}(x) \leq \int_0^m \langle \theta \rangle^q d\theta \cdot K * \langle \cdot \rangle^{-q}(x)
\]

\[
\leq 2^{q/2} (m)^{q+1} \int_{\mathbb{R}^n} K(y) \langle y \rangle^q dy \cdot \langle x \rangle^{-q},
\]

where we have used the fact that, for any real numbers \( x \) and \( y \),

\[
\langle x \rangle \leq \sqrt{2}(x - y) \langle y \rangle.
\]

This implies (2.1). (1.15) is shown by the following direct computation:

\[
|(m^2 - \Delta)^{1/2} \langle \cdot /R \rangle^{-n-1}| = R^{-1} \|(R^2 m^2 - \Delta)^{1/2} \langle \cdot \rangle^{-n-1} R\|
\]

3. Proof of nonexistence results

3.1. Proof of Proposition 1.9

Proposition 1.9 is shown by the proof of Proposition 1.5 with replacing \( A_{n,n+1} \) by \( \tilde{A}_n \langle Rm \rangle^{n+2} \). So we omit the detail.
3.2. Proof of Corollary 1.10

Let $R_0$ be a positive number satisfying that for any $R > R_0$,

$$M_R(0) > \frac{1}{2} \text{Im} \left( \alpha \int_{\mathbb{R}^n} u_0(x) dx \right).$$

We remark that such $R_0$ exists because of (1.17) and the Lebesgue dominant theorem. Moreover, let $R \geq R_0$ be a positive number satisfying that

$$(3.1) \quad \frac{1}{2} \text{Im} \left( \alpha \int_{\mathbb{R}^n} u_0(x) dx \right) > \tilde{C}_{n,p,\alpha} 2^{(n+2)/(p-1)} R^{n-1/(p-1)}.$$

If $m < R^{-1}$, then (3.1) implies (1.16) and therefore Proposition 1.9 implies Corollary 1.10.

Proof of Corollary 1.11. For $0 < R < 1$, by (1.18),

$$M_R(0) \geq \mu \int_{|x| \leq 1} |x|^{-k} |x/R|^{-n-1} dx \geq 2^{-n-1} \mu \int_{|x| \leq R} |x|^{-k} dx = (n-k)^{-1} 2^{-n-1} \omega_n \mu R^{n-k},$$

where $\omega_n$ is the volume of $S_{n-1}$. Let $I_1 = (n-k)^{-1} 2^{-n-1} \omega_n$ and

$$R_1 = \left( \frac{\mu I_1}{2^{(n+p+1)/(p-1)} \tilde{C}_{n,p,\alpha}} \right)^{-1/(p-1)}.$$

We put $\mu \gg 1$ so that $R_1 < 1/\max(1, m)$. Then

$$M_{R_1}(0) - \tilde{C}_{n,p,\alpha}(R_1 m)^{(n+2)/(p-1)} R_1^{n-1/(p-1)} \geq R_1^{n-k}(\mu I_1 - 2^{(n+2)/(p-1)} \tilde{C}_{n,p,\alpha} R_1^{k-1/(p-1)}) \geq 2^{-1} R_1^{n-k} \mu I_1 > 0$$

and therefore (1.16) is satisfied. Moreover,

$$\tilde{T}_{n,p,m,\lambda,\alpha,R_1} \leq (p-1)^{-1} 2^{-1} D^{-1} \mu I_1 \left( \frac{\mu I_1}{2^{(n+p+1)/(p-1)} \tilde{C}_{n,p,\alpha}} \right)^{k/(p-1)} (\mu I_1)^{-p+1} = (p-1)^{-1} 2^{-1} D^{-1} \mu I_1 \tilde{C}_{n,p,\alpha}^{k/(p-1)} (\mu I_1)^{1/(p-1)} - \frac{1}{p-1} \tilde{C}_{n,p,\alpha}^{k/(p-1)}.$$
Proof of Corollary 1.12. For $R \gg 1$, by \((1.19)\),
\[
M_R(0) \geq \mu \int_{|x| \geq 1} |x|^{-k} (x/R)^{-n-1} \, dx \\
\geq 2^{-n-1} \mu \int_{1 \leq |x| \leq R} |x|^{-k} \, dx \\
\geq 2^{-n-1} \omega_n \mu \int_1^R r^{-n-k-1} \, dr, \\
\geq 2^{-n-1} \omega_n \mu \left\{ \begin{array}{ll}
(n-k)^{-1} (R^{n-k} - 1), & \text{if } k < n, \\
2^{n-k} r^{n-k-1} \, dr, & \text{if } k \geq n,
\end{array} \right.
\geq I_2 \mu R^{(n-k)+},
\]
where $(n-k)_+ = \max(n-k,0)$ and
\[
I_2 = \begin{cases} 
2^{-n-2} \omega_n (n-k)^{-1}, & \text{if } k < n, \\
2^{-n+1} \omega_n \int_1^2 r^{n-k-1} \, dr, & \text{if } k \geq n.
\end{cases}
\]
Let
\[
R_2 = \left( \frac{\mu I_2}{2^{(n+1)(p-1)/(p-1)} C_{n,p,\alpha}} \right)^{\min(n,k)-1/(p-1)},
\]
where $R_2 \gg 1$ if $\mu \ll 1$. Then, by choosing $m$ so that $m \leq 1/R_2$,
\[
M_{R_2}(0) - \tilde{C}_{n,p,\alpha} (R_2 m)^{(n+2)/(p-1) - 1/(p-1)} R_2^{n-1/(p-1)} \\
\geq R_2^{(n-k)_+} (\mu I_2 - 2^{(n+2)/(p-1)} \tilde{C}_{n,p,\alpha} R_2^{\min(n,k)-1/(p-1)}) \\
\geq 2^{-1} R_2^{(n-k)_+} \mu I_2 > 0
\]
and therefore \((1.16)\) is satisfied. Moreover,
\[
\tilde{T}_{n,p,m,\lambda,\alpha,R_2} \\
\leq (p-1)^{-1} D_{n,p,\lambda,\alpha} R_2^{(p-1)-(n-k)_+} (\mu I_2 - \tilde{C}_{n,p,\alpha} R_2^{\min(n,k)-1/(p-1)})^{-p+1} \\
\leq (p-1)^{-1} 2^{p-1} D_{n,p,\lambda,\alpha} (2^{(p+1)/(p-1)} \tilde{C}_{n,p,\alpha})^{-\min(n,k)-1/(p-1)} (\mu I_2)^{-1/(p-1)} {\min(n,k)}^{-1/(p-1)}.
\]

\begin{flushright}
$\square$
\end{flushright}

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