A second addition formula for continuous $q$-ultraspherical polynomials *

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Dedicated to Mizan Rahman

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Abstract

This paper provides the details of Remark 5.4 in the author’s paper “Askey-Wilson polynomials as zonal spherical functions on the $SU(2)$ quantum group”, SIAM J. Math. Anal. 24 (1993), 795–813. In formula (5.9) of the 1993 paper a two-parameter class of Askey-Wilson polynomials was expanded as a finite Fourier series with a product of two $3\phi_2$’s as Fourier coefficients. The proof given there used the quantum group interpretation. Here this identity will be generalized to a 3-parameter class of Askey-Wilson polynomials being expanded in terms of continuous $q$-ultraspherical polynomials with a product of two $2\phi_2$’s as coefficients, and an analytic proof will be given for it. Then Gegenbauer’s addition formula for ultraspherical polynomials and Rahman’s addition formula for $q$-Bessel functions will be obtained as limit cases. This $q$-analogue of Gegenbauer’s addition formula is quite different from the addition formula for continuous $q$-ultraspherical polynomials obtained by Rahman and Verma in 1986. Furthermore, the functions occurring as factors in the expansion coefficients will be interpreted as a special case of a system of biorthogonal rational functions with respect to the Askey-Roy $q$-beta measure. A degenerate case of this biorthogonality are Pastro’s biorthogonal polynomials associated with the Stieltjes-Wigert polynomials.

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1 Introduction

Rahman and Verma [25] obtained the following addition formula for continuous q-ultraspherical polynomials:

\[ p_n(\cos \theta; a, aq^{\frac{1}{2}}, -a, -aq^{\frac{1}{2}} | q) = \sum_{k=0}^{n} \frac{(q; q)_n}{(q)_k} \frac{(a^4 q^n, a^4 q^{-1}, a^2 q^{\frac{1}{2}}, -a q^{\frac{1}{2}}, -a^2; q)_k}{(q)_k (q; q)_{n-k}} \frac{a^{n-k}}{(a^4 q^{-1}; q)_k (a^4 q^{\frac{1}{2}}, -a^2 q^{\frac{1}{2}}, -a^2; q)_n} \]

The formula is here written in the form given in [11, Exercise 8.11]. Use [11] also for notation of (q-)hypergeometric functions and (q-)shifted factorials. Throughout it is supposed that \(0 < q < 1\).

Formula (1.1) is given in terms of Askey-Wilson polynomials (see [4] or [11, §7.5]):

\[ p_n(\cos \theta; a, b, c, d | q) := a^{-n} (ab, ac, ad; q)_n r_n(\cos \theta; a, b, c, d | q) \quad (n \in \mathbb{Z}_{\geq 0}) \]  

(1.2)

(symmetric in a, b, c, d), where

\[ r_n(\cos \theta; a, b, c, d | q) := \phi_3(q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta}; ab, ac, ad | q). \]  

(1.3)

The continuous q-ultraspherical polynomials are the special case \(b = aq^{\frac{1}{2}}, c = -a, d = -aq^{\frac{1}{2}}\) of the Askey-Wilson polynomials, often notated as follows (see [11, (7.4.14)]):

\[ C_n(x; a^2 | q) = \frac{(a^4; q)_n}{(q; q)_n a^n} r_n(x; a, aq^{\frac{1}{2}}, -a, -aq^{\frac{1}{2}} | q) = \frac{(a^2; q)_n}{(q, a^4 q^n; q)_n} p_n(x; a, aq^{\frac{1}{2}}, -a, -aq^{\frac{1}{2}} | q). \]  

(1.4)

A further specialization to \(a = \frac{1}{2}\), i.e., \((a, b, c, d) = (\frac{1}{2}, \frac{3}{2}, -\frac{1}{2}, -\frac{3}{2})\), yields the continuous q-Legendre polynomials. For this case \(a = \frac{1}{2}\) Koelink was able to give two different proofs of the addition formula (1.1) from a quantum group interpretation on \(SU_q(2)\), see [18] and [19].

If \(a\) is replaced by \(q^{\frac{1}{2}\lambda}\) in (1.1) and the limit is taken for \(q \uparrow 1\), then a version of the addition formula for ultraspherical polynomials is obtained:

\[ C_n(\cos \theta) = \sum_{k=0}^{n} \frac{2^{2k} \left(2 \lambda + 2k - 1\right)}{(2 \lambda - 1)_{n+k+1}} \frac{(\lambda)^2}{(\lambda)_k} \times (\sin \phi)^k C_n^{\lambda+k}(\cos \phi) (\sin \psi)^k C_n^{\lambda+k}(\cos \psi) C_k^{\lambda-k} \left(\frac{\cos \theta - \cos \phi \cos \psi}{\sin \phi \sin \psi}\right). \]  

(1.5)

Here ultraspherical polynomials are defined by

\[ C_n(\cos \theta) := \frac{(2\lambda)^n}{n!} 2F_1\left(-n, n + 2\lambda; \lambda + \frac{1}{2}; 1 - \cos \theta\right). \]  

(1.6)
By elementary substitution the addition formula (1.5) transforms into the familiar addition formula for ultraspherical polynomials:

\[ C_n^\lambda (\cos \phi \cos \psi + \sin \phi \sin \psi \cos \theta) = \sum_{k=0}^{n} 2^{2k} \frac{(2\lambda + 2k - 1)(n-k)! (\lambda)^2_k}{(2\lambda - 1)_{n+k+1}} \times (\sin \phi)^k C_{n-k}^{\lambda+k}(\cos \phi) (\sin \psi)^k C_{n-k}^{\lambda+k}(\cos \psi) C_k^{\lambda-1/2}(\cos \theta), \quad (1.7) \]

see [310.9(34)], but watch out for the misprint \(2^m\) which should be \(2^{2m}\); see also the references given in [11, Lecture 4]. For the removable singularity at \(\lambda = 1/2\) in (1.7) observe that

\[ \lim_{\lambda \to 1/2} \frac{k}{2\lambda - 1} C_k^{\lambda - 1/2}(\cos \theta) = \cos(k\theta) \quad (k > 0); \quad C_0^{\lambda - 1/2}(\cos \theta) = 1. \quad (1.8) \]

An elementary transformation comparable to the passage from (1.5) to (1.7) cannot be performed on the \(q\)-level. It is a generally observed phenomenon in \(q\)-theory that, for a classical formula involving a parameter dependent function with an argument transformed by a parameter dependent transformation, a possible \(q\)-analogue has the transformation parameters occurring in the function parameters. Compare for instance the \(p_k\) factor in (1.6) with the \(C_k^{\lambda - 1/2}\) factor in (1.7).

In [21 (5.9)] I obtained the following formula as a spin-off of the interpretation of certain Askey-Wilson polynomials as zonal spherical functions on the quantum group \(SU_q(2)\):

\[ p_n(\cos \theta; -q^{1/2}(\sigma+\tau+1), -q^{1/2}(\sigma-\tau+1), q^{1/2}(\sigma-\tau+1), q^{1/2}(\sigma+\tau+1) | q) = \sum_{k=-n}^{n} \frac{(q^{n+1}; q)_n (q; q)_{2n}}{(q; q)_{n+k}(q; q)_{n-k}} q^{\frac{1}{2}(n-k)(n-k+\sigma+\tau)} \times {}_3\phi_2 \left( \begin{array}{ccc} q^{-n+k}, q^{-n}, q^{-n-\sigma} \\ q^{-2n}, 0 \end{array} ; q, q \right) {}_3\phi_2 \left( \begin{array}{ccc} q^{-n+k}, q^{-n}, q^{-n-\tau} \\ q^{-2n}, 0 \end{array} ; q, q \right) e^{ik\theta}. \quad (1.9) \]

Here the summand, with \(e^{ik\theta}\) omitted, is invariant under the transformation \(k \to -k\), as can be seen by twofold application of [11 (3.2.3)]. The \(\phi_2\)’s were viewed in [21] as dual \(q\)-Krawtchouk polynomials (see their definition in [11 §3.17]), but here we will prefer to consider them as certain (unusual) \(q\)-analogues of ultraspherical polynomials, for which an expression in terms of a \(2\phi_2\) is more suitable. For this purpose apply a \(2\phi_2 \to 2\phi_2\) transformation obtained from formulas (1.5.4) and (III.7) in [11]. Then, after the substitutions \(q^{1/2} \to is^{-1}, q^{1/2} \to it\) in (1.9), we can write (1.9) equivalently as

\[ p_n(\cos \theta; st^{-1}q^{1/2}, s^{-1}tq^{1/2}, -stq^{1/2}, -s^{-1}t^{-1}q^{1/2} | q) = (-1)^n q^{n^2} (q; q)_n \sum_{k=0}^{n} q^{\frac{1}{2}k} (q^{n+1}; q)_k (q^{-n}; q)_k (st)^{k-n} (-q^{-k+1}s^2, -q^{-k+1}t^{-2}; q)_{n-k} \times {}_2\phi_2 \left( \begin{array}{ccc} q^{-n+k}, q^{n+k+1} \\ q^{k+1}, -q^{k+1}s^2 \end{array} ; q, -qs^2 \right) {}_2\phi_2 \left( \begin{array}{ccc} q^{-n+k}, q^{n+k+1} \\ q^{k+1}, -q^{k+1}t^{-2} \end{array} ; q, -qt^{-2} \right) (1 + \delta_{k,0}) \cos(k\theta). \quad (1.10) \]
After the substitutions \( q' = \tan^2 \frac{1}{2} \phi, \quad q'' = \tan^2 \frac{1}{2} \psi \) in \((1.9)\), the limit for \( q \uparrow 1 \) becomes the case \( \lambda = \frac{1}{2} \) of the addition formula \((1.7)\) (combined with \((1.8)\)), i.e., the addition formula for Legendre polynomials \( P_n(x) = C_n^{1/2}(x) \).

The first main result of this paper is the following addition formula for continuous \( q \)-ultraspherical polynomials. Thus I will finally fulfill my promise of bringing out “reference [9]” of my paper [21], which reference was mentioned there as being in preparation.

**Theorem 1.1.** We have, in notation \((1.3)\),

\[
r_n^{(s,t)}(\cos \theta; a q^{-s}, a q^{-t} \mid q) = (-1)^n a^{2n} q^{n(n+1)} \left( 1 + a^2 \right) \sum_{k=0}^{n} \frac{(1 - a^2 q^k) (q^{-n}, a^4 q, a^4 q^{n+1}; q)_k (a^2 s t - 1 q^{1/2})^k}{(1 - a^4 q^k) (q, q)_k (a^2 q^2; q)_k^2 (a^2 s^2 q^1; q)_k (a^2 t^2 - 1 q^{1/2})^k} \cdot 2 \phi_2 \left( \frac{q^{-n+k}, a^4 q^{n+k+1}}{a^2 q^{k+1}, -a^2 s^2 q^{k+1}}; q, -s^2 q \right) \\
\times 2 \phi_2 \left( \frac{q^{-n+k}, a^4 q^{n+k+1}}{a^2 q^{k+1}, -a^2 t^2 q^{k+1}}; q, -t^2 q \right) r_n(\cos \theta; a, -a, a q^{-s}, -a q^{-t} \mid q), \quad (1.11)
\]

or, in notation \((1.2)\),

\[
p_n^{(s,t)}(\cos \theta; a q^{-s}, a q^{-t} \mid q) = (-1)^n a^{2n} \sum_{k=0}^{n} a^{n-k} (a^2 q^{-k+1}; q)_{n-k} \cdot 2 \phi_2 \left( \frac{q^{-n+k}, a^4 q^{n+k+1}}{a^2 q^{k+1}, -a^2 s^2 q^{k+1}}; q, -s^2 q \right) \\
\times 2 \phi_2 \left( \frac{q^{-n+k}, a^4 q^{n+k+1}}{a^2 q^{k+1}, -a^2 t^2 q^{k+1}}; q, -t^2 q \right) p_n(\cos \theta; a, -a, a q^{-s}, -a q^{-t} \mid q). \quad (1.12)
\]

For \( a = 1 \) formula \((1.12)\) specializes to formula \((1.10)\) (with usage of a Chebyshev case of the Askey-Wilson polynomials, see [31 (4.21)]). Its limit case for \( q \uparrow 1 \) (after the substitutions \( a = q^{1/2} \lambda - \frac{1}{2}, \quad s = \tan(\frac{1}{2} \phi), \quad t = \tan(\frac{1}{2} \psi) \)) is the addition formula \((1.7)\) for ultraspherical polynomials of general order \( \lambda \). It is also possible to obtain Rahman’s addition formula [21, (1.10)] for \( q \)-Bessel functions as a formal limit case of \((1.12)\), see Remark 2.2. For precise correspondence of \((1.11)\) with \((1.1)\) one should replace \( a \) by \( a q^{-\frac{1}{2}} \) in \((1.1)\).

The left-hand side of \((1.12)\) is invariant under the symmetries \( (s, t) \rightarrow (t, s), \quad (s, t) \rightarrow (-s^{-1}, t) \) and \( (s, t) \rightarrow (s, -t^{-1}) \). These symmetries are also visible on the right-hand side of \((1.12)\) if we take into account that

\[
\frac{(a^2 s^2 q^{k+1}; q)_{n-k}}{s^{n-k}} 2 \phi_2 \left( \frac{q^{-n+k}, a^4 q^{n+k+1}}{a^2 q^{k+1}, -a^2 s^2 q^{k+1}}; q, -s^2 q \right)
\]

is invariant under the transformation \( s \rightarrow s^{-1} \) up to the factor \((-1)^{n-k}\) (by \((3.14)\)).

The proof of Theorem 1.1 (see details in [21]) is quite similar to the proof of \((1.1)\) in [21]. We consider \((1.11)\) as a connection formula which connects Askey-Wilson polynomials of different
order. There are certain choices of the orders of the Askey-Wilson polynomials in a connection formula

$$r_n(\cos \theta; \alpha, \beta, \gamma, \delta \mid q) = \sum_{k=0}^{n} A_{n,k} r_k(\cos \theta; a, b, c, d \mid q)$$  \hspace{1cm} (1.13)$$

for which the connection coefficients $A_{n,k}$ factorize. Formula (1.11) is one example; formula (1.11) is another example.

At the end of §2 a degenerate addition formula for continuous $q$-ultraspherical polynomials will be given as a limit case of the addition formula (1.11). As a further limit case we obtain a degenerate addition formula for $q$-Bessel functions.

The $\psi_2$ factors on the right-hand side of (1.11) tend, after the mentioned substitutions and as $q \uparrow 1$, to ultraspherical polynomials $C_{n-k}^{\lambda+1}$ of argument $\cos \phi$ resp. $\cos \psi$, so we might expect that these $\psi_2$’s also satisfy some kind of (bi)orthogonality relations. This is indeed the case $\alpha = \beta$ of Theorem 1.2 below. See its proof in §3.

**Theorem 1.2.** Let $\alpha, \beta > -1$. Define a system of rational functions in $t$ by

$$p_n^{(\alpha, \beta)}(t; q, \text{rat}) := \psi_2^{(q^{-n}, q^{n+\alpha+\beta+1} \mid q^{\alpha+1}, -qtq^{\beta+1}; q)} (n \in \mathbb{Z}_{\geq 0}),$$  \hspace{1cm} (1.14)$$

and define, with additional parameter $c \in \mathbb{R}$, a measure on $[0, \infty)$ by

$$d\mu_{\alpha, \beta, c, q}(t) := \frac{q^{-c^2} \Gamma_q(c) \Gamma_q(1-c) \Gamma_q(\alpha + \beta + 2) t^{c-1} (-tq^{\beta+1}, -t^{-1}q^{\alpha+2}; q)_{\infty}}{\Gamma(c) \Gamma(1-c) \Gamma_q(\alpha + 1) \Gamma_q(\beta + 1) (-tq^{-c}, -t^{-1}q^{1+c}; q)_{\infty}} dt,$$  \hspace{1cm} (1.15)$$

where $\Gamma_q$ is defined in §2. Then

$$\int_0^{\infty} p_n^{(\alpha, \beta)}(t; q, \text{rat}) p_m^{(\beta, \alpha)}(t^{-1}; q, \text{rat}) d\mu_{\alpha, \beta, c, q}(t) = \frac{(-1)^n (1 - q^{n+\alpha+\beta+1})(q; q)_n \delta_{n,m}}{q^{c^2(n-1)} (1 - q^{2n+\alpha+\beta+1})(q^{\alpha+\beta+2}; q)_n}.$$  \hspace{1cm} (1.16)$$

The case $n = m = 0$ of (1.16), i.e., the integration formula $\int_0^{\infty} d\mu_{\alpha, \beta, c, q}(t) = 1$, is precisely the $q$-beta integral of Askey and Roy [3 (3.4)], which extends a $q$-beta integral of Ramanujan [26 (19)]. The Askey-Roy integral was independently obtained by Gasper (see [9] and also [10]) and by Thiruvenkatachar & Venkatachaliengar (see [2] p.93).

Note that the two $\psi_2$’s in (1.11) (with $a = q^{\frac{1}{2}c}$), can be rewritten in terms of the function (1.14): as $p_n^{(\alpha+k, \alpha+k)}(s^2; q, \text{rat})$ and $p_n^{(\alpha+k, \alpha+k)}(t^{-2}; q, \text{rat})$, respectively.

The biorthogonality measure in (1.16) is evidently not unique, because of the parameter $c$. Further illustration of the non-uniqueness of the measure for these biorthogonality relation is provided by a $q$-integral variant of (1.16). In order to state this, we need the following definition of $q$-integral on $(0, \infty)$ ($f$ arbitrary function on $(0, \infty)$ for which the sum absolutely converges):

$$\int_0^{s \infty} f(t) \, dq \, t := (1 - q) \sum_{k=-\infty}^{\infty} sq^k f(sq^k) \quad (s > 0).$$  \hspace{1cm} (1.17)$$
Theorem 1.3. The functions defined by (1.14) also satisfy the biorthogonality relations
\[
\frac{\Gamma_q(\alpha + \beta + 2)}{\Gamma_q(\alpha + 1) \Gamma_q(\beta + 1)} \int_0^{\infty} p_n^{(\alpha, \beta)}(t; q, \text{rat}) p_m^{(\beta, \alpha)}(t^{-1}; q, \text{rat}) \frac{t^{-1} (-t \beta + 1, -t^{-1} \alpha + 2; q)_{\infty}}{(-t, -t^{-1} q; q)_{\infty}} dt
\]
\[
= \frac{(-1)^n (1 - q^{n+\alpha+\beta+1}) (q; q)_n \delta_{n,m}}{q^{n(n-1)} (1 - q^{2n+\alpha+\beta+1}) (q^{\alpha+\beta+2}; q)_n} \quad (\alpha, \beta > -1, \ s > 0). \quad (1.18)
\]

The case \(n = m = 0\) of (1.18) is a \(q\)-beta integral first given by Gasper [10], but it is essentially Ramanujan’s \(1\psi_1\) sum; see some further discussion in §3 The proof of Theorem 1.3 is by a completely analogous argument as I will give in §3 for the proof of Theorem 1.2.

The paper concludes in §4 with some open questions and with some specializations of Theorem 1.2 Pastro’s [22] biorthogonal polynomials associated with the Stieltjes-Wigert polynomials occur as a special case.

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2 Proof of the new addition formula

In this section I prove the second addition formula for continuous \(q\)-ultraspherical polynomials, stated in Theorem 1.1. Let us first consider the general connection formula (1.13). We can split up this connection into three successive connections of more simple nature:
\[
r_n(\cos \theta; \alpha, \beta, \gamma, \delta \mid q) = \sum_{j=0}^{n} c_{n,j} (\alpha e^{i\theta}, \alpha e^{-i\theta}; q)_j,
\]
\[
(\alpha e^{i\theta}, \alpha e^{-i\theta}; q)_j = \sum_{l=0}^{j} d_{j,l} (\alpha e^{i\theta}, \alpha e^{-i\theta}; q)_l,
\]
\[
(\alpha e^{i\theta}, \alpha e^{-i\theta}; q)_l = \sum_{k=0}^{l} e_{l,k} r_k(\cos \theta; a, b, c, d \mid q).
\]

Then
\[
A_{n,k} = \sum_{j=0}^{n-k} \sum_{l=0}^{j} c_{n,j+k} d_{j+k,l+k} e_{l+k,k}
\]
and
\[
A_{n,k} = \sum_{m=0}^{n-k} \sum_{i=0}^{m} c_{n-i} d_{n-i,n-m} e_{n-m,k}.
\]
The coefficients $c_{n,j}$, $d_{j,l}$ and $e_{l,k}$ can be explicitly given as

\[
\begin{align*}
    c_{n,j} &= \frac{(q^{-n}, \alpha \beta \gamma \delta q^{n-1}; q)_j q^j}{(\alpha \beta, \alpha \gamma, \alpha \delta; q)_j}, \\
    d_{j,l} &= \frac{(q, a\alpha; q)_j (a^{-1} \alpha; q)_{j-l} \alpha^l}{(q, a\alpha; q)_l (q; q)_{j-l} \alpha^l}, \\
    e_{l,k} &= \frac{(abcd; q)_k (q^2; q)_k (abcd; q)_{k+l}}{(abcdq^{k-1}, q; q) (abcd; q)_{k+l}}.
\end{align*}
\]  

(2.6) Here (2.6) follows from (1.3), formula (2.7) can be obtained by rewriting the $q$-Saalschütz formula \[\text{[1]} (1.7.2)] and (2.8) follows from \[\text{[1]} (2.6), (2.5)].

It turns out that in the two double sums (2.4), (2.5) \[\text{[1]} \] of $A_{n,k}$ the inner sum can be written as a balanced $4\phi_3$ of argument $q$:

\[
A_{n,k} = (-1)^k q^\frac{k}{2} (q^{-n+k+1}; q) \frac{(q^{-n}, \alpha \beta \gamma \delta q^{n-1}, ab, ac, ad; q)_k \alpha^k}{(abcdq^{k-1}, \alpha \beta, \alpha \gamma, \alpha \delta; q)_k \alpha^k} \\
\times \sum_{j=0}^{n-k} \frac{(q^{-n+k}, \alpha \beta \gamma \delta q^{n+k-1}, ab^2, ac^2, ad^2; q)_j q^j}{(\alpha \beta q^k, \alpha \gamma q^k, \alpha \delta q^k; q)_j} 4\phi_3 \left( \frac{q^{-j}, ab^k, ac^k, ad^k}{(abq^k, \alpha \gamma q^k, \alpha \delta q^k; q)} \right)
\]

(2.9)

and

\[
A_{n,k} = (-1)^{n-k} q^{-n(n-2k-1)} \frac{(1 - abcdq^{2k-1})(q^{-n}; q)_k (\alpha \beta \gamma \delta q^{n-1}, ab, ac, ad; q)_n \alpha^n}{(1 - abcdq^{k-1})(q; q)_k (abcdq^k, \alpha \beta, \alpha \gamma, \alpha \delta; q)_n \alpha^n} \\
\times \sum_{m=0}^{n-k} \frac{(q^{-n+k}, abcd^{-1}q^{1-n-k}, ab^{-1}, (aa)^{-1}q^{1-n}; q)_m q^m}{(q, ab^{-1}q^{1-n}, (ac)^{-1}q^{1-n}, (ad)^{-1}q^{1-n}; q)_m} \\
\times 4\phi_3 \left( \frac{q^{-m}, (ab)^{-1}q^{-n+1}, (aa)^{-1}q^{-n+1}, (ad)^{-1}q^{-1-n+1}}{(\alpha \beta \delta^{-1}q^{-2n+2}, (aa)^{-1}q^{-1-n+1}, \alpha \alpha^{-1}q^{-n+1}; q)} \right).
\]

(2.10)

The sums in (2.9) and (2.10) can be compared with the Bateman type product formula \[\text{[1]} (8.4.7)]:

\[
\begin{align*}
    r_n(\cos \phi; a, aq^n, -a, -aq^n \mid q) r_n(\cos \psi; a, aq^n, -a, -aq^n \mid q) &= \frac{1 + a^2 q^n}{1 + a^2} (-1)^n q^{-\frac{n}{2}} \\
    \times \sum_{m=0}^{n} \frac{(q^{-n}, a^2 q^n, -e^{i\phi} q^\frac{1}{2}, -a^2 e^{i\phi} q^\frac{1}{2}; q)_m q^m}{(q, a^2 q^n, -a^2 q^n, -a^2 q^n; q)_m} \\
    \times 4\phi_3 \left( \frac{q^{-m}, a^2, a^2 e^{2i\phi}, a^2 e^{-2i\phi}}{a^4, -a^2 e^{i\phi - i\psi} q^\frac{1}{2}, -e^{i\phi - i\psi} q^{-m+\frac{1}{2}}; q} \right).
\end{align*}
\]

(2.11)

The sum in (2.9) can be matched with the right-hand side of (2.11) precisely for those values of the parameters $a, b, c, d, \alpha, \beta, \gamma, \delta$ in (1.13) which occur in the Rahman-Verma addition formula.
\( \text{[1.14]} \), i.e., for \( ab = cd \), \( a^2 = cdq \), \( \alpha = \beta q^{\frac{1}{2}} = -\gamma q^{\frac{1}{2}} = -\delta \). In fact, this will prove \( \text{[1.14]} \). The proof in \( \text{[2.5]} \) is essentially along these lines.

Next we see that the sum in \( \text{[2.10]} \) can be matched with the right-hand side of \( \text{[2.11]} \) precisely for those values of the parameters \( a, b, c, d, \alpha, \beta, \gamma, \delta \) in \( \text{[1.13]} \) when \( \alpha \beta = \gamma \delta = a^2q \), \( b = -c = aq^{\frac{1}{2}} = -dq^{\frac{1}{2}} \). Thus put \( b = aq^{\frac{1}{2}}, c = -aq^{\frac{1}{2}}, d = -a, \alpha = ast^{-1} q^{\frac{1}{2}}, \beta = ats^{-1} q^{\frac{1}{2}}, \gamma = -astq^{\frac{1}{2}}, \delta = -ast^{-1}t^{-1} q^{\frac{1}{2}} \) in \( \text{[1.13]} \) and \( \text{[2.10]} \). Then these two formulas specialize to:

\[
r_n(\cos \theta; ast^{-1} q^{\frac{1}{2}}, ats^{-1} q^{\frac{1}{2}}, -astq^{\frac{1}{2}}, -ast^{-1}t^{-1} q^{\frac{1}{2}} | q) = \sum_{k=0}^{n} A_{n,k} r_k(\cos \theta; a, -a, aq^{\frac{1}{2}}, -aq^{\frac{1}{2}} | q),
\]

\[
A_{n,k} = (-1)^n s^n t^{-n} q^{\frac{1}{2}n(n+2k+2)} \frac{(1 + a^2)(1 - a^4q^{2k}) (q^{-n}, a^4q; q)_k (a^4q^{n+1}; q)_n^2}{(1 + a^2q^n)(1 - a^4q^k) (q, a^4q^{n+1}; q)_k (a^2q; q)_n^2} \times \sum_{m=0}^{n-k} \frac{1}{(-s^2a^2q, -t^2a^2q; q)_n} \frac{(q^{-n+k}, a^{-4}q^{-n-k}, st^{-1}q^{\frac{1}{2}}, a^{-2}ts^{-1}q^{-n+\frac{1}{2}}, q_m q^m_m)}{\left( q, a^{-2}q^{\frac{1}{2}-n}, -a^{-2}q^{\frac{1}{2}-n}, -a^{-2}q^{1-n}, q_m \right)} \times 4\phi_3 \left( q^{-m}, a^{-2}q^{-n}, -a^{-2}s^2q^{-n}, -a^{-2}t^2q^{-n} \middle| a^{-4}q^{2n}, a^{-2}ts^{-1}q^{-n+\frac{1}{2}}, ts^{-1}q^{-m+\frac{1}{2}}, q, q \right).
\]

(2.13)

If we next make the successive substitutions \( n \to n - k, a \to a^{-1}q^{-\frac{1}{2}n}, e^{i\phi} \to is^{-1}, e^{i\psi} \to -it^{-1} \) in \( \text{[2.11]} \) and compare with \( \text{[2.13]} \) then we can write \( \text{[2.12]} \) as follows:

\[
r_n(\cos \theta; ast^{-1} q^{\frac{1}{2}}, ats^{-1} q^{\frac{1}{2}}, -astq^{\frac{1}{2}}, -as^{-1}t^{-1} q^{\frac{1}{2}} | q)
= \frac{s^n t^{-n} q^{-\frac{1}{2}n(n-1)}(a^4q^{n+1}; q)_n^2}{(a^2q; q)_n^2 (-s^2a^2q, -t^2a^2q; q)_n} \sum_{k=0}^{n} (-1)^k q^{k(n+\frac{1}{2})} \frac{1 - a^{-2}q^k}{1 - a^{-2}} \frac{(q^{-n}, a^4q; q)_k}{(q, a^4q^{n+1}; q)_k} \times r_{n-k}(\frac{s-s^{-1}}{2i}, a^{-1}q^{-\frac{1}{2}n}, a^{-1}q^{-\frac{1}{2}n+\frac{1}{2}}, -a^{-1}q^{-\frac{1}{2}n}, -a^{-1}q^{-\frac{1}{2}n+\frac{1}{2}} | q)
\times r_{n-k}(\frac{r-1}{2i}, a^{-1}q^{-\frac{1}{2}n}, a^{-1}q^{-\frac{1}{2}n+\frac{1}{2}}, -a^{-1}q^{-\frac{1}{2}n}, -a^{-1}q^{-\frac{1}{2}n+\frac{1}{2}} | q) \times r_k(\cos \theta; a, -a, aq^{\frac{1}{2}}, -aq^{\frac{1}{2}} | q).
\]

(2.14)

In order to make the two \( r_{n-k} \) factors on the right-hand side above into closer \( q \)-analogues of the two \( C_{n-k}^{\lambda+k} \) factors on the right-hand side of \( \text{[1.7]} \), we will use the following string of identities:

\[
r_n(\cos \theta; a, aq^{\frac{1}{2}}, -a, -aq^{\frac{1}{2}} | q) = 4\phi_3 \left( q^{-n}, a^4q^{n-1}, ae^{i\theta}, ae^{-i\theta} \middle| q^{-\frac{1}{2}}, -a^2, -a^2q^{\frac{1}{2}} \right)
= \frac{(a^2;q)_n a^n e^{in\theta}}{(a^4;q)_n} 2\phi_1 \left( q^{-n}, a^2 \middle| a^{-2}q^{1-n}, q, a^{-2}e^{-2i\theta}q \right)
= \frac{(a^2, a^{-2}e^{-2i\theta}q^{1-n}; q)_n a^n e^{in\theta}}{(a^4, q)_n} 2\phi_2 \left( q^{-n}, a^{-4}q^{1-n} \middle| q^{-1-n}, a^{-2}e^{-2i\theta}q^{1-n}, q, e^{-2i\theta}q \right).
\]

(2.15)
For the proof use successively (7.4.14), (7.4.2) and (1.5.4) in [1].

**Proof of Theorem 1.1**

This follows from (2.14) by twofold substitution of (2.15). Here replace $n$ by $n-k$ and $a$ by $a^{-1} q^{-\frac{1}{2}n}$ in (2.14), and replace $e^{i\theta}$ by $is^{-1}$ for the first substitution and by $it$ for the second substitution. □

**Remark 2.1.** Let the divided difference operator $D_q$ acting on a function $F$ of argument $e^{i\theta}$ be given by

$$D_q F(e^{i\theta}) := \frac{\delta_q F(e^{i\theta})}{\delta_q \cos \theta}, \quad \delta_q F(e^{i\theta}) := F(q^{\frac{1}{2}} e^{i\theta}) - F(q^{-\frac{1}{2}} e^{i\theta}).$$

Then $D_q$, acting on both sides of the addition formula (1.12), sends this to the same formula with $n$ replaced by $n-1$ and $a$ replaced by $q^{\frac{1}{2}a}$ (apply [10, (3.1.9)]). Thus, if formula (1.12) is already known for $a = 1$ (i.e., if formula (1.10) is known) then the procedure just sketched yields this formula for $a = q^{\frac{1}{2}j}$ for all $j \in \mathbb{Z}_{\geq 0}$, i.e., for infinitely many distinct values of $a$. Since, for fixed $n$, both sides of (1.12) are rational in $a$, formula (1.12) will then be valid for general $a$. Since formula (1.9), equivalent to (1.10), can be obtained by a quantum group interpretation, we can say that it is possible to prove formula (1.12) by arguments in a quantum group setting, followed by minor analytic, but not very computational reasoning.

**Proof that (1.11) has limit (1.7) as $q \uparrow 1$**

(after the substitutions $a = q^{\frac{1}{2}\lambda - \frac{1}{4}}, s = \tan(\frac{1}{2}\phi), t = \tan(\frac{1}{2}\psi)$).

The limits for $q \uparrow 1$ of the factors $r_n(\cos \theta)$, $r_k(\cos \theta)$, $2\phi_2(-s^2q)$ and $2\phi_2(-t^2q)$ in (1.11), after the above substitutions and after substitution of (1.3) for the $r_n$ and $r_k$ factors yields respectively:

\[
\begin{align*}
&2F_1(-n, n + 2\lambda; \lambda + \frac{1}{2}, \frac{1}{2}; -\frac{1}{2} \cos \phi \cos \psi + \sin \phi \sin \psi \cos \theta), \quad 2F_1(-k, k + 2\lambda - 1; \lambda; \frac{1}{2} - \frac{1}{2} \cos \theta), \\
&2F_1(-n + k, 2\lambda + n + k; \lambda + k + \frac{1}{2}, \frac{1}{2}; -\frac{1}{2} \cos \phi), \quad 2F_1(-n + k, 2\lambda + n + k; \lambda + k + \frac{1}{2}, \frac{1}{2} + \frac{1}{2} \cos \psi).
\end{align*}
\]

Express these $2F_1$’s as ultraspherical polynomials by (1.6). The limit of the coefficients on the right-hand side of (1.11) (after the above substitutions) is also easily computed. □

**Remark 2.2.** Replace $s$ by $sq^{\frac{1}{2}n}$ and $t$ by $tq^{-\frac{1}{2}n}$ in (1.12) and let $n \rightarrow \infty$. Then formally we obtain Rahman’s addition formula for $q$-Bessel functions, see [21] (1.10)] (but watch out for the misprint $\Gamma_q(\nu + 1)$ which should be $\Gamma_q^2(\nu + 1)$):

\[
\begin{align*}
&2\phi_1 \left( -astq^{\frac{1}{2}} e^{i\theta}, -astq^{\frac{1}{2}} e^{-i\theta} \middle| q, -a^{-2}t^{-2} \right) = \frac{1}{(q - a^{-2}t^{-2}; q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k q^{\frac{1}{2}k^2} a^{-k} s^k t^{-k}}{(q, a^2q, a^4q^2; q)_k} \sum_{k=0}^{\infty} \frac{(1)_k (q^{\frac{1}{2}} - q^{\frac{1}{2}} s^{k+1})}{(a^2q^{k+1}; q)_{k+1}} p_k(\cos \theta; a, -a, aq^{\frac{1}{2}}, -aq^{\frac{1}{2}} | q).
\end{align*}
\]
According to [24, (1.10)], the further conditions $0 < a < 1$, $0 < s < t^{-1}$, $\theta \in \mathbb{R}$ should be imposed here. If we replace in (2.19) $s$ by $(1 - q)s$, $t$ by $(1 - q)^{-1}t$, $a$ by $q^{-\alpha}$, and if we let $q \uparrow 1$ then we formally obtain the familiar Gegenbauer’s addition formula for Bessel functions $J_\alpha$, see [23, 7.15(32)].

In (2.16) the left-hand side is an Askey-Wilson $q$-Bessel function (see [20, §2.3]), earlier studied under the name $q$-Bessel function on a $q$-quadratic grid in [14] and [6]. The $\phi_1$’s on the right-hand side of (2.16) are Jackson’s second $q$-Bessel functions, usually written (see [11, Exercise 1.24]) as

$$J^{(2)}_\alpha(x; q) = \frac{(q^{\alpha+1}; q)_\infty}{(q; q)_\infty} (x/2)^\alpha \phi_1 \left( -q^{\alpha+1}; q, -\frac{1}{2} x^2 q^{\alpha+1} \right). \quad (2.17)$$

Koelink [17 (3.6.18)] gave an interpretation of the case $a = 1$ of (2.16) on the quantum group of plane motions. This interpretation is similar to the interpretation given in [21 (5.9)] for equation (1.19).

**Remark 2.3.** If we multiply both sides of the addition formula (1.11) with $(-a^2t^{-2}; q)_n$ and if we next let $t \to iaq^{\frac{1}{2}}n$ in (1.11) then we obtain a degenerate form of the addition formula (1.11):

$$\frac{(-ise^{i\theta}q^{\frac{1}{2}(1-n)}, -ise^{-i\theta}q^{\frac{1}{2}(1-n)}; a^2)}{(-a^2s^2q; q)_n} = \sum_{k=0}^{n} (-ia^{-1}q^{\frac{1}{2}(n+1)})^k \frac{(q^{-n}, a^4; q)_k}{(q, a^2; q)_k} \frac{s^k}{(-a^2s^2q; q)_k} \times 2^\phi_2 \left( q^{-n+k}, a^4q^{n+k+1}; a^2q^{k+1}, -a^2s^2q^{k+1}; q, -s^2q \right) r_k(\cos \theta; a, aq^{\frac{1}{2}}, -aq^{\frac{1}{2}}, -a | q). \quad (2.18)$$

Integrated forms of (1.11) and (2.18) can be obtained by integrating both sides of these formulas with respect to the measure $\frac{(e^{2i\theta}q^\infty)}{(a^2e^{2i\theta}q^\infty)} \, d\theta$ on $[0, \pi]$, i.e., with respect to the orthogonality measure for the continuous $q$-ultraspherical polynomials $r_k(\cos \theta; a, aq^{\frac{1}{2}}, -aq^{\frac{1}{2}}, a - q | q)$ (see [11, §7.4]). This will yield a product formula and an integral representation, respectively, for the functions $p^{(a)}_n(s^2; q, \text{rat})$ (with $a = q^{\frac{1}{2}}$).

**Remark 2.4.** In (2.18) replace $s$ by $sq^{\frac{1}{2}}n$ and let $n \to \infty$. Then we formally obtain a degenerate addition formula for $q$-Bessel functions:

$$\frac{(-ise^{i\theta}q^{\frac{1}{2}}, -ise^{-i\theta}q^{\frac{1}{2}}; q)_\infty}{(a^2s^2q^{\frac{1}{2}}; q)_\infty} = \sum_{k=0}^{\infty} i^k a^{-k} s^{k} q^{\frac{1}{2}k^2} \frac{(a^4; q)_k}{(q, a^2; q)_k} \times 0\phi_1 \left( -a^2q^{k+1}; q, -s^2q^{k+1} \right) r_k(\cos \theta; a, aq^{\frac{1}{2}}, -aq^{\frac{1}{2}}, -a | q) \quad (2.19)$$

If we replace in (2.19) $a$ by $q^{\frac{1}{2}}a$, and if we substitute (2.17) and (1.4) then we can rewrite (2.19)
as

\[ (-isq\frac{1}{2}e^{i\theta}, -isq\frac{1}{2}e^{-i\theta}; q)_\infty = q^{\frac{1}{2}a^2} \frac{(q; q)_{\infty}}{(q^{a+1}; q)_{\infty}} \sum_{k=0}^{\infty} \frac{q^k q^{\frac{1}{2}k^2 + \frac{1}{2}ka}}{1 - q^{a+k}} J^{(2)}_{a+k}(2sq^{-\frac{1}{2}}; q) C_k(\cos \theta; q^a | q). \]

(2.20)

It is interesting to compare formula (2.20) with Ismail & Zhang [15, (3.32)]. They expand there a generalized q-exponential function \( E_{q}(z; -i, b/2) \) in terms of the \( C_k(z; q^a | q) \) and they obtain almost the same expansion coefficients as in (2.20), including Jackson’s second q-Bessel functions, but they have a factor \( q^{\frac{1}{4}k^2} \), where (2.20) has a factor \( q^{\frac{1}{2}k^2 + \frac{1}{2}ka} \).

3 Rational biorthogonal functions for the Askey-Roy q-beta measure

The Askey-Roy q-beta integral (see [3, (3.4)], [10], [2, pp. 92,93] Exercise 6.17(ii)) is as follows:

\[
\int_0^{\infty} t^{\alpha-1} \frac{(-tq^{c+1}, -t^{-1}q^{a+2}; q)_\infty}{(-tq^{-c}, -t^{-1}q^{1+c}; q)_\infty} dt = \frac{q^{c^2} \Gamma(c) \Gamma(1-c) \Gamma(q(\alpha+1) \Gamma(q(\beta+1))}{\Gamma_q(c) \Gamma_q(1-c) \Gamma_q(\alpha+\beta+2)}, \]

\( \text{Re} \, \alpha, \text{Re} \, \beta > -1, \, c \in \mathbb{R}. \) (3.1)

Here the q-gamma function is defined by

\[ \Gamma_q(z) := \frac{(q; q)_{\infty}}{(q^z; q)_{\infty}} (1 - q)^{1-z}. \] (3.2)

The special case \( c = \alpha + 1 \) of (3.1) goes back (without proof) to Ramanujan in Chapter 16 of his second notebook (see [5, p.29, Entry 14]), and later in his paper [26, (19)], with subsequent proof by Hardy [12].

When we let \( q \uparrow 1 \) in (3.1) then we formally obtain the beta integral on \((0, \infty)\):

\[
\int_0^{\infty} t^{\alpha} (1 + t)^{-\alpha-\beta-2} dt = \frac{\Gamma(\alpha+1) \Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} \quad (\text{Re} \, \alpha, \text{Re} \, \beta > -1). \] (3.3)

When we move the orthogonality relations

\[
\int_{-1}^{1} P^{(\alpha,\beta)}(n)(x) P^{(\alpha,\beta)}(m)(x) (1-x)^\alpha (1+x)^\beta dx = \frac{2^{\alpha+\beta+1} \Gamma(n+\alpha+1) \Gamma(n+\beta+1)}{(2n+\alpha+\beta+1) n! \Gamma(n+\alpha+\beta+1) \delta_{n,m}} \delta_{\alpha, \beta} > -1 \]

(3.4)

of the Jacobi polynomials

\[ P^{(\alpha,\beta)}(n)(x) := \frac{(\alpha+1)_n}{n!} {}_2F_1 \left( \begin{array}{c} -n, n + \alpha + \beta + 1 \\ \alpha + 1 \end{array} ; \frac{1}{2}(1 - x) \right), \] (3.5)
(see [3, §10.8]) to $[0, \infty)$ by the substitution $x = (1 - t)/(1 + t)$, then we obtain orthogonality relations

$$
\int_0^\infty P_n^{(\alpha, \beta)} \left( \frac{1 - t}{1 + t} \right) P_m^{(\alpha, \beta)} \left( \frac{1 - t}{1 + t} \right) t^\alpha (1 + t)^{-\alpha - \beta - 2} \, dt
= \frac{\Gamma(n + \alpha + 1) \Gamma(n + \beta + 1)}{(2n + \alpha + \beta + 1) n! \Gamma(n + \alpha + \beta + 1)} \delta_{n,m} \quad (\alpha, \beta > -1). \quad (3.6)
$$

Thus the rational functions $t \mapsto P_n^{(\alpha, \beta)}((1 - t)/(1 + t))$, $n \in \mathbb{Z}_{\geq 0}$, are orthogonal with respect to the beta measure on $[0, \infty)$ of which the total mass is given in (3.3). We would like to find $q$-analogues of these orthogonal rational functions such that the orthogonality measure is the $q$-beta measure in (3.1).

From (1.13) we have

$$
d\mu_{\alpha, \beta, c, q}(t) = \frac{w_{\alpha, \beta, c, q}(t) \, dt}{\int_0^\infty w_{\alpha, \beta, c, q}(s) \, ds}, \quad \text{where} \quad w_{\alpha, \beta, c, q}(t) := t^{c-1} \frac{(-tq^{\beta+1}, -t^{-1}q^{\alpha+2}; q)_\infty}{(-tq^{-c}, -t^{-1}q^{1+c}; q)_\infty}. \quad (3.7)
$$

Observe from (3.1) that, for $k, l \in \mathbb{Z}_{\geq 0}$,

$$
\int_0^\infty \frac{t^k}{(-tq^{\beta+1}; q)_k} \frac{t^{-l}}{(-t^{-1}q^{\alpha+2}; q)_l} w_{\alpha, \beta, c, q}(t) \, dt = q^{-(k-l)(c+k-1)} \int_0^\infty w_{\alpha+k, \beta+l, c+k-l, q}(t) \, dt
= q^{c(k-1)} \frac{\Gamma(c + k - l) \Gamma(1 - c - k + l) \Gamma_q(\alpha + k + 1) \Gamma_q(\beta + l + 1)}{\Gamma_q(c + k - l) \Gamma_q(1 - c - k + l) \Gamma_q(\alpha + \beta + k + l + 2)}
= \frac{(q^{\alpha+1}; q)_k (q^{\beta+1}; q)_l}{q^{1/2(k-1)(k-l-1)} (q^{\alpha+2}; q)_{k+l}} \frac{q^{2l} \Gamma(c) \Gamma(1 - c) \Gamma_q(\alpha + 1) \Gamma_q(\beta + 1)}{\Gamma_q(c) \Gamma_q(1 - c) \Gamma_q(\alpha + \beta + 2)}.
$$

Thus

$$
\int_0^\infty q^{1/2(k-1)} \frac{t^k}{(q^{\alpha+1}, -tq^{\beta+1}; q)_k} \frac{t^{-l}}{(q^{\beta+1}, -t^{-1}q^{\alpha+2}; q)_l} \, d\mu_{\alpha, \beta, c, q}(t) = \frac{q^{kl}}{(q^{\alpha+\beta+2}; q)_{k+l}}. \quad (3.8)
$$

**Proof of Theorem 1.2**

Multiply both sides of (3.8) with $\frac{(q^{-n}, q^{n+\alpha+\beta+1}; q)_k q^k}{(q; q)_k}$ and sum from $k = 0$ to $n$. Then the right-hand side becomes

$$
\frac{1}{(q^{\alpha+\beta+2}; q)_l} 2\phi_1 \left( \frac{q^{-n}, q^{n+\alpha+\beta+1}}{q^{\alpha+\beta+l+2}; q^{l+1}} ; q, q^{l+1} \right) = \frac{(q^{-l+1}; q)_n}{(q^{\alpha+\beta+2}; q)_{l+n}} = \frac{(q; q)_n \delta_{n,l}}{(q^{\alpha+\beta+2}; q)_{2n}} \quad (l = 0, 1, \ldots, n)
$$
by \([11.5.2]\). Thus
\[
\int_0^\infty 2\phi_2\left(q^{-n}, q^{n+\alpha+\beta+1}; q, -tq\right) \frac{q^{l(l-1)/2}(tq^{-1})^l}{(q^{\beta+1}, -t^{-1}q^{\alpha+2}; q)_l} d\mu_{\alpha,\beta,c,q}(t) = \frac{(q; q)_n \delta_{n,l}}{(q^{\alpha+\beta+2}; q)_{2n}},
\]
\[l = 0, 1, \ldots, n, \quad (3.9)\]
\[
\int_0^\infty \frac{q^{k(k-1)/2}t^k}{(q^{\alpha+1}, -tq^{\beta+1}; q)_k} 2\phi_2\left(q^{-m}, q^{m+\alpha+\beta+1}; q, -t^{-1}q^2\right) d\mu_{\alpha,\beta,c,q}(t) = \frac{(q; q)_m \delta_{m,k}}{(q^{\alpha+\beta+2}; q)_{2m}},
\]
\[k = 0, 1, \ldots, m, \quad (3.10)\]

where \((3.10)\) is obtained by a similar argument as \((3.9)\). Then \((3.9)\) and \((3.10)\) together with \((1.14)\) imply the biorthogonality relations \((1.16)\). \(\square\)

The \(q\)-integral version of the Askey-Roy integral \((3.11)\) is
\[
\frac{(-sq^{-c}, -s^{-1}q^{1+c}; q)_\infty}{s^c(-s, -s^{-1}q; q)_\infty} \int_0^{s^{-\infty}} t^{\epsilon-1} \frac{(-tq^{\beta+1}, -t^{-1}q^{\alpha+2}; q)_\infty}{(-tq^{-c}, -t^{-1}q^{1+c}; q)_\infty} d_q t = \frac{\Gamma_q(\alpha + 1) \Gamma_q(\beta + 1)}{\Gamma_q(\alpha + \beta + 2)}. \quad (3.11)
\]

Here \(s > 0\), Re \(\alpha\), Re \(\beta > -1\), \(c \in \mathbb{R}\), and the \(q\)-integral is defined by \((1.17)\). The case \(s = q^c\) of \((3.11)\) (which is no real restriction) was given in \([10]\) (see also \([2. (2.27)]\)) and in \([11\text{ Exercise 6.17(i)]}\). Another approach to \((3.11)\) is presented in \([7]\).

For \(f\) any function on \((0, \infty)\) for which the sum below converges absolutely, we have:
\[
\frac{(-sq^{-c}, -s^{-1}q^{1+c}; q)_\infty}{s^c(-s, -s^{-1}q; q)_\infty} \int_0^{s^{-\infty}} f(t) t^{\epsilon-1} \frac{(-tq^{\beta+1}, -t^{-1}q^{\alpha+2}; q)_\infty}{(-tq^{-c}, -t^{-1}q^{1+c}; q)_\infty} d_q t = \frac{(1 - q)(-sq^{\beta+1}, -s^{-1}q^{\alpha+2}; q)_\infty}{(-s, -s^{-1}q; q)_\infty} \sum_{k = -\infty}^{\infty} f(sq^k) \frac{(-sq^{-\alpha-1}; q)_k}{(-sq^{\beta+1}; q)_k} q^{k(\alpha+1)}. \quad (3.12)
\]

The right-hand side, and thus the left-hand side of \((3.12)\) is independent of \(c\). Henceforth we will take \(c = 0\) without loss of information. Then \((3.11)\) together with \((3.12)\) takes the form
\[
\int_0^{s^{-\infty}} t^{\epsilon-1} \frac{(-tq^{\beta+1}, -t^{-1}q^{\alpha+2}; q)_\infty}{(-t, -t^{-1}q; q)_\infty} d_q t = (1 - q)(-sq^{\beta+1}, -s^{-1}q^{\alpha+2}; q)_\infty \frac{(1 - q)(-sq^{\beta+1}, -s^{-1}q^{\alpha+2}; q)_\infty}{(-s, -s^{-1}q; q)_\infty} \sum_{k = -\infty}^{\infty} \frac{(-sq^{-\alpha-1}; q)_k}{(-sq^{\beta+1}; q)_k} q^{k(\alpha+1)} = \frac{\Gamma_q(\alpha + 1) \Gamma_q(\beta + 1)}{\Gamma_q(\alpha + \beta + 2)}. \quad (3.13)
\]

The second equality in \((3.13)\) is Ramanujan’s \(1\psi_1\) sum \([11. (5.2.1)]\). This observation is the usual way to prove \((3.11)\). With a completely analogous argument as used for the proof of Theorem \(1.2\) we can next prove Theorem \(1.3\) Details are omitted.

In completion of this section, observe the following symmetry of the functions \(p_n^{(\alpha, \beta)}(t; q, \text{rat})\):
\[
p_n^{(\alpha, \beta)}(t; q, \text{rat}) = \frac{1}{(-tq^{\beta+1}; q)_n} 2\phi_2\left(q^{-n}, q^{n+\alpha+\beta+1}; q, -tq^{\beta+1}; q, -tq^{\beta+n+1}\right).
\]
Here the first and last equality are just (1.14). We use [11, (1.5.4)] for the second and fourth equality, while the third equality is obtained by reversion of summation order in a terminating $q$-hypergeometric series (see [11, Exercise 1.4.(ii)]).

4 Concluding remarks

The results of this paper lead to several interesting questions. I formulate two of these questions here. I also discuss specializations of Theorem 1.2.

As I already mentioned in §1, the new addition formula in the case of the continuous $q$-Legendre polynomials (formula (1.9)) was first obtained in a quantum group context, where a two-parameter family of Askey-Wilson polynomials, including the continuous $q$-Legendre polynomials, was interpreted as spherical functions on the $SU_q(2)$ quantum group. Here the left and right invariance of the spherical functions was no longer with respect to the diagonal quantum subgroup, but infinitesimally with respect to twisted primitive elements in the corresponding quantized universal enveloping algebra. The $\sigma$ and $\tau$ variables in the addition formula (1.9) are parameters for the twisted primitive elements occurring in the left respectively right invariance property. On the other hand, the $3\phi_2$ factors on the right-hand side of (1.9), involving $\sigma$ resp. $\tau$, can be rewritten as the functions $p_{n-k}(., q, \text{rat})$ of argument $-q^{-\sigma}$ resp. $-q^{-\tau}$. So I wonder whether an interpretation of these last functions and of their biorthogonality (discussed in §3) can be given in the context of $SU_q(2)$.

A second question is whether the biorthogonality relations for the functions $p_{n}(\alpha, \beta)(t; q, \text{rat})$ (Theorems 1.2 and 1.3) fit into a more general class of biorthogonal rational functions. In fact, several papers have appeared during the last 10 or 20 years which discuss explicit systems of biorthogonal rational functions depending on many parameters and expressed as $q$-hypergeometric functions, see Rahman [23], Wilson [28], Ismail & Masson [13] and Spiridonov & Zhedanov [27]. However, I did not see how the functions $p_{n}(\alpha, \beta)(t; q, \text{rat})$ and their biorthogonality relations can be obtained as special or limit case of families discussed in these references. If the functions $p_{n}(\alpha, \beta)(t; q, \text{rat})$ are indeed unrelated to the functions discussed in these references, then it is a natural question how to generalize the system of biorthogonal functions $p_{n}(\alpha, \beta)(t; q, \text{rat})$.

On the other hand, our functions $p_{n}(\alpha, \beta)(t; q, \text{rat})$ have some interesting limit cases, one of which has occurred earlier in literature. When we take limits for $\alpha \to \infty$ and/or $\beta \to \infty$ in
\[ \begin{align*}
&\phi_n^{(\alpha,\infty)}(t;q,\text{rat}) := \frac{1}{\phi_1(q^{-n}; q^{\alpha+1}; q, -qt)}, \\
&\phi_n^{(\infty,\beta)}(t;q,\text{rat}) := \frac{1}{\phi_1(q^{-n}; -qt^{\beta+1}; q, -qt)}, \\
&\phi_n^{(\infty,\infty)}(t;q,\text{rat}) := \frac{1}{\phi_1(q^{-n}; 0; q, -qt)}.
\end{align*} \]  \tag{4.1}

In (4.1) and (4.3) we have polynomials of degree \( n \) in \( t \), rather than rational functions in \( t \). The limit case \( \beta \rightarrow \infty \) of the biorthogonality relations (1.10) then becomes:

\[ \int_0^{\infty} \phi_n^{(\alpha,\infty)}(t;q,\text{rat}) \phi_m^{(\infty,\alpha)}(t^{-1}q;q,\text{rat}) \, d\mu_{\alpha,\infty,c}(t) = (-1)^n q^{-\frac{1}{2}n(n-1)} (q;q)_n \delta_{n,m}, \] \tag{4.4}

where

\[ d\mu_{\alpha,\infty,c}(t) := \frac{q^{-c^2} \Gamma_q(c) \Gamma_q(1-c) (q^{\alpha+1}; q)_\infty}{\Gamma(c) \Gamma(1-c) (1-q) (q;q)_\infty} \frac{t^{c-1} (-t^{-1}q^{\alpha+2}; q)_\infty}{(-tq^{-c}, -t^{-1}q^{1+c}; q)_\infty} \, dt \quad (\alpha > -1). \] \tag{4.5}

The further limit case \( \alpha \rightarrow \infty \) of the biorthogonality relations (4.4) then becomes

\[ \frac{q^{-c^2} \Gamma_q(c) \Gamma_q(1-c)}{\Gamma(c) \Gamma(1-c) (1-q) (q;q)_\infty} \int_0^{\infty} \phi_n^{(\infty,\infty)}(t;q,\text{rat}) \phi_m^{(\infty,\infty)}(t^{-1}q;q,\text{rat}) \, dt \frac{t^{c-1} \, dt}{(-tq^{-c}, -t^{-1}q^{1+c}; q)_\infty} = (-1)^n q^{-\frac{1}{2}n(n-1)} (q;q)_n \delta_{n,m}. \] \tag{4.6}

Similar limit cases can be considered for the biorthogonality relations (1.18). The biorthogonality relations (4.6) are essentially the ones observed by Pastro \[22, \text{pp. 532, 533}\]. He also points out that biorthogonality relations of the form \( \int P_n(t)Q_m(t^{-1}) \, d\mu(t) = h_n \delta_{n,m} \) with \( P_n \) and \( Q_n \) polynomials of degree \( n \) can be rewritten as orthogonality relations on the real line for Laurent polynomials. This is indeed the case in (4.6). Pastro also observes that the biorthogonality measure occurring in (4.6) is a (non-unique) orthogonality measure for the Stieltjes-Wigert polynomials (see \[16, \S3.27\]).

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