Exact Results for Tunneling Problems of Bogoliubov
Excitations in the Critical Supercurrent State

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Abstract We show the exact solution of Bogoliubov equations at zero-energy in the
critical supercurrent state for arbitrary shape of potential barrier. With use of this
solution, we prove the absence of perfect transmission of excitations in the low-energy
limit by giving the explicit expression of transmission coefficient. The origin of disap-
pearance of perfect transmission is the emergence of zero-energy density fluctuation
near the potential barrier.

Keywords Bose-Einstein condensate · Gross-Pitaevskii equation · Bogoliubov
equations · anomalous tunneling · critical supercurrent state

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1 Introduction

In 2001-2003, Kovrizhin and his collaborators\textsuperscript{1} have shown an interesting theoretical
prediction on the tunneling properties of Bogoliubov excitations\textsuperscript{2}; the Bogoliubov ex-
citations show perfect transmission across a potential barrier in the low-energy limit.
It is called anomalous tunneling. It is quite different from an ordinary particle obeying
Schrödinger equation, which shows perfect reflection in the low-energy limit. Later,
Danshita \textit{et al.}\textsuperscript{3} have extended the problem in the presence of the supercurrent. Solv-
ing the delta-functional barrier problem, they have shown that (a) perfect transmission
occurs even when the condensate supercurrent exists, except for the critical supercur-
rent state; (b) under the critical supercurrent, the partial transmission occurs. Thus, a
consistent explanation for both perfect transmission in the non-critical states and the
absence of perfect transmission in the critical state had been highly desired. As for the
perfect transmission in the non-critical states, its physical mechanism has been investi-
gated in many works\textsuperscript{4,5} and importance of the similarity between the condensate and
low-energy excitations has been pointed out. However, few results have been known on
the critical supercurrent state. It is crucial to understand the reason why the critical

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supercurrent state shows an exceptional behavior, for the purpose of constructing a
unified picture for non-critical and critical states. In our study, we prove the absence
of perfect transmission in the critical supercurrent state for arbitrary shape of barrier,
and clarify its physical mechanism.

2 Formulation

We begin with time-dependent Gross-Pitaevskii (GP) equation

$$i \frac{\partial}{\partial t} \psi(x, t) = \left( -\frac{1}{2} \frac{\partial^2}{\partial x^2} + U(x) \right) \psi(x, t) + |\psi(x, t)|^2 \psi(x, t). \quad (1)$$

Here we use a dimensionless description. Assuming the solution in the form of

$$\psi(x, t) = e^{-i \mu t} n_\Psi(x) + h u(x) e^{-i \epsilon t} - v^* \left( x \right) e^{i \epsilon t} i o, \quad (2)$$

and ignoring the higher-order terms of $u$ and $v$, we obtain stationary GP equation

$$\hat{L} \Psi(x) = 0, \quad \hat{L} = -\frac{1}{2} \frac{d^2}{dx^2} + U(x) - \mu + |\Psi(x)|^2 \quad (3)$$

for the condensate wavefunction, and Bogoliubov equations

$$\begin{pmatrix} \hat{L} + |\Psi(x)|^2 & -(\Psi(x))^2 \\ -(\Psi(x))^* & \hat{L} + |\Psi(x)|^2 \end{pmatrix} \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} = \epsilon \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} \quad (4)$$

for wavefunctions of Bogoliubov excitations.

Henceforth, we would like to consider the problem depicted in Fig. 1. Therefore,
we assume that $U(x)$ is a short-ranged potential barrier, and that the condensate
wavefunction has the following asymptotic form:

$$\Psi(x \to \pm \infty) = \exp \left[ i \left( q x \pm \frac{\varphi}{2} + \text{const.} \right) \right]. \quad (5)$$

This asymptotic behavior determines chemical potential as $\mu = 1 + q^2/2$. Setting
$\Psi(x) = A(x) \exp[i \Theta(x)]$, one obtains

$$\hat{H} A = 0, \quad \hat{H} = -\frac{1}{2} \frac{d^2}{dx^2} + U + \frac{q^2}{2} \left( \frac{1}{A^2} - 1 \right) - 1 + A^2, \quad (6)$$

$$A^2 \frac{d \Theta}{dx} = q. \quad (7)$$

Here the second equation is already integrated once, and a constant of integration
becomes $q$ from Eq. (5). Figure 2 shows the Josephson relation. From this figure, the
condition for the critical supercurrent state is given by

$$\frac{\partial q}{\partial \varphi} = 0. \quad (8)$$

By means of the condensate phase $\Theta$, we introduce the following quantities:

$$S = u e^{-i \Theta} + v e^{i \Theta}, \quad G = u e^{-i \Theta} - v e^{i \Theta}. \quad (9)$$
Bogoliubov equations are then rewritten as

$$\hat{H} S - i q \frac{d}{dx} \left( \frac{G}{A} \right) = \epsilon G,$$  \hspace{1cm} (10)

$$\left( \hat{H} + 2 A^2 \right) G - i q \frac{d}{dx} \left( \frac{S}{A} \right) = \epsilon S.$$  \hspace{1cm} (11)

$S$ and $G$ can be interpreted as phase and density fluctuations, because one can show the following expressions from Eq. (2):

$$|\psi|^2 = A^2 \left[ 1 + \frac{2}{A} \text{Re}(Ge^{-i\Theta}) \right] + O(S,G)^2,$$  \hspace{1cm} (12)

$$\frac{\psi}{|\psi|} = e^{-i\Theta + i\epsilon} \left[ 1 + \frac{i}{A} \text{Im}(Se^{-i\Theta}) \right] + O(S,G)^2.$$  \hspace{1cm} (13)

In order to obtain the transmission amplitude, we construct the tunneling solution with the following asymptotic form:

$$\begin{pmatrix} S \\ G \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{k_1}{\sqrt{2(\epsilon - qk_1)}} \end{pmatrix} e^{ik_1x} + \begin{pmatrix} 1 \\ \frac{k_2}{\sqrt{2(\epsilon - qk_2)}} \end{pmatrix} r e^{ik_2x} \quad (x \to -\infty)$$

$$\begin{pmatrix} S \\ G \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{k_1}{\sqrt{2(\epsilon - qk_1)}} \end{pmatrix} t e^{ik_1x} \quad (x \to +\infty).$$  \hspace{1cm} (14)

Here $k_1$ and $k_2$ are real positive and negative roots of the dispersion relation $\epsilon = qk + \frac{1}{2} \sqrt{k^2 + 4}$.  

3 Exact Solution of Bogoliubov Equations at $\epsilon = 0$ in the Critical Supercurrent State for Arbitrary Shape of Potential Barrier

Since Bogoliubov equations are two-component and second-order linear differential equations, the general solution can be expressed in terms of four linearly-independent
solutions. We have obtained the exact solutions of Bogoliubov equations at $\epsilon = 0$ valid only for the critical current state, in other words, only when the condition (8) holds. They are given by

$$
(S_1, G_1) = \begin{pmatrix} A \\ 0 \end{pmatrix}, \quad (S_{II}, G_{II}) = \begin{pmatrix} A \int_0^x \frac{dx}{A^2} - 2i q A A_3 \int_0^x \frac{G_{IV} dx}{A^3} \\ -2i q A A_3 \int_0^x \frac{dx}{A^3} \end{pmatrix},
$$

$$
(S_{III}, G_{III}) = \begin{pmatrix} -2i q A A_3 \\ A_3 \end{pmatrix}, \quad (S_{IV}, G_{IV}) = \begin{pmatrix} -2i q A A_3 \int_0^x \frac{dx}{A^3} \\ A_3 \int_0^x \frac{dx}{A^3} \end{pmatrix}.
$$

Here we have introduced the following notations:

$$
A_\phi := \frac{\partial A}{\partial \phi}, \quad A_3(x) := \int_0^x \frac{A_\phi(x') dx'}{A(x')^3}.
$$

$(S_1, G_1)$, which can be expressed as $(u, v) = (\Psi, \Psi^*)$, is a well-known solution. The solution $(S_{III}, G_{III})$, which can be expressed as $(u, v) = (\partial \Psi/\partial \phi, -\partial \Psi^*/\partial \phi)$, is specific to the critical current state, and represents the localized density fluctuation near the potential barrier. See Fig. 3. $(S_{II}, G_{II})$ and $(S_{IV}, G_{IV})$ are exponentially divergent solutions. However, as we will see in the next section, all four solutions are necessary to prove the absence of perfect transmission.

4 Absence of Perfect Transmission: Sketch of Proof

Using the exact zero-energy solution shown in the previous section, we can prove the absence of perfect transmission. In the following, we show the sketch of proof. The detailed calculation is given in Ref. [8]. Henceforth, we assume $U(x) = U(-x)$ for simplicity.

i) Our goal is to obtain the tunneling solution (14) up to first-order in $\epsilon$, that is,

$$
S(x) \longrightarrow \begin{cases} 1 + i \epsilon \bar{t}(0) + \epsilon \left( \bar{t}(1) + \frac{t_i(0)}{1+q} + \frac{t_i(0)}{1+q} x \right) + O(\epsilon^2) & (x \rightarrow -\infty) \\
\bar{t}(0) + \epsilon \left( \bar{t}(1) + \frac{t_i(0)}{1+q} x \right) + O(\epsilon^2) & (x \rightarrow +\infty). \end{cases}
$$

Here $t = t(0) + \epsilon t(1) + \cdots$, and $\bar{t} = \bar{t}(0) + \epsilon \bar{t}(1) + \cdots$.

ii) In order to achieve the above purpose, we construct the solution of Bogoliubov
equations in the form of power series with respect to $\epsilon$: $(S,G) = \sum_{n=0}^{\infty} \epsilon^n (S^{(n)}, G^{(n)})$. We then obtain the following inhomogeneous differential equations:

\begin{align*}
\hat{H} S^{(n)} - \frac{i q}{A} \frac{d}{dx} \left( \frac{G^{(n)}}{A} \right) &= G^{(n-1)}, \\
(\hat{H} + 2A^2) G^{(n)} - \frac{i q}{A} \frac{d}{dx} \left( \frac{S^{(n)}}{A} \right) &= S^{(n-1)},
\end{align*}

where the right-hand sides should read as zero if $n = 0$. Since four homogeneous solutions are already given in Eq. (15), a particular solution can be found by the method of variation of parameters. Thus, $(S^{(n)}, G^{(n)})$ can be determined from $(S^{(n-1)}, G^{(n-1)})$ recursively.

iii) Bogoliubov equations with finite energy $\epsilon$ have four linearly-independent solutions. Generally, two of the four behave as plane waves far from the potential barrier, and the other two are unphysical solutions which diverge exponentially. In solving the tunneling problem, we are particularly interested in the former two solutions.

iv) Therefore, we extend the non-divergent zero-energy solutions, i.e., $(S_I, G_I)$ and $(S_{III}, G_{III})$, to first-order in $\epsilon$. The particular solution of Eqs. (18) and (19) obtained by the method of variation of parameters diverges exponentially, so we must cancel the divergent term by adding the divergent homogeneous solutions $(S_{II}, G_{II})$ and $(S_{IV}, G_{IV})$. This manipulation is most important. After the calculation, we can obtain the asymptotic forms of the solutions extended up to first-order as follows:

\begin{align*}
S_{\text{total}}^{\text{I}}(x) &\to 1 + \epsilon \left( \frac{q^2 - \eta}{i q (1 - q^2)} x + \tilde{\gamma} \text{sgn} x \right) + O(\epsilon^2), \\
\text{const.} \times S_{\text{total}}^{\text{III}}(x) &\to \text{sgn} x + \epsilon \left( \frac{q^2 + \eta}{i q (1 - q^2)} |x| + \tilde{\lambda} \right) + O(\epsilon^2).
\end{align*}

Here $\tilde{\gamma}$ and $\tilde{\lambda}$ are constants, and $\eta$ is defined by the following integral:

$$\eta := \left[ \int_{0}^{\infty} dx \frac{\partial A}{\partial \varphi} \right] / \left[ \int_{0}^{\infty} dx \frac{\partial A}{A^3} \frac{\partial \varphi}{\partial \varphi} \right].$$

v) By making a linear combination of $S_{\text{total}}^{\text{I}}$ and $S_{\text{total}}^{\text{III}}$, we can construct the solution of the form (17). The transmission amplitude is obtained explicitly as

$$t^{(0)} = t(\epsilon \to 0) = \frac{2 q \eta}{q^2 + \eta^2}. \quad (23)$$

Unless $\eta \neq \pm q$, $0 < |t^{(0)}|^2 < 1$ holds. Thus, the absence of perfect transmission is proved. We have confirmed in the delta-functional barrier model that $\eta = q$ occurs only when there is no potential barrier.

5 Discussion

Let us consider the physical origin of the absence of perfect transmission. The tunneling solution in the low-energy limit can be written as [8]

\begin{align*}
\lim_{\epsilon \to 0} \left( \begin{array}{c} u \\ v \end{array} \right) &\propto \left( \begin{array}{c} \Psi \\ \psi^* \end{array} \right) - 2 i \frac{q - \eta}{q + \eta} \frac{\partial}{\partial \varphi} \left( \frac{\psi}{\psi^*} \right) \neq \left( \begin{array}{c} \psi \\ \psi^* \end{array} \right).
\end{align*}
This is drastically different from the non-critical states, in which the wavefunctions of excitations coincide with the condensate wavefunction\([4, 9]\). In the non-critical states, there is only one non-divergent solution \((u, v) = (\Psi, \Psi^\ast) \leftrightarrow (S, G) = (A, 0)\) at \(\epsilon = 0\), so \(G\) cannot contribute to the wavefunction of excitations in the low-energy limit. In the critical supercurrent state, on the other hand, another non-divergent solution \((u, v) = (\partial \Psi / \partial \varphi, -\partial \Psi^\ast / \partial \varphi) \leftrightarrow (S, G) = (-2i q A A_3, A_\varphi)\) arises, and contributes to the low-energy wavefunction of excitations. As shown in Fig. 3 this solution represents the density fluctuation localized near the potential barrier. Thus, the presence of local density fluctuation near the barrier in the low-energy limit is the origin of the absence of perfect transmission.

We further note that the critical supercurrent states are at the “phase boundary” which separates steady flow states and nonstationary flow states. (See, e.g., Fig. 1 in Ref. \([10]\).) Therefore, our result suggests that the emergence of low-energy density fluctuation can characterize the destabilization of the superflow. In Ref. \([11]\), it has been shown by numerical simulation of time-dependent GP equation that if \(q \geq q_c\), soliton-phonon creation occurs and stationary superflow no longer exists. We expect that the local density fluctuation \(G = \partial A / \partial \varphi\) possesses the information on the “manner of collapse of stationary superflow”. That is, the profile of the density fluctuation could describe a part of collapse phenomena, such as the soliton-phonon creation. Thus, the investigation of the role of low-energy density fluctuation near the critical supercurrent state is left as a future work.

6 Conclusion

In conclusion, with use of the exact solution of Bogoliubov equations at zero-energy, we have exactly proved the absence of perfect transmission of Bogoliubov excitations in the critical supercurrent state for arbitrary shape of potential barrier. The origin of the absence of perfect transmission is the emergence of low-energy density fluctuation near the potential barrier. Because of this density fluctuation, wavefunctions of excitations do not coincide with the condensate wavefunction in the low-energy limit.

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