We discuss fluctuation-induced forces in a system described by a continuous Landau-Ginzburg model with a quenched disorder field, defined in a \( d \)-dimensional slab geometry \( \mathbb{R}^{d-1} \times [0, L] \). A series representation for the quenched free energy in terms of the moments of the partition function is presented. In each moment an order parameter-like quantity can be defined, with a particular correlation length of the fluctuations. For some specific strength of the non-thermal control parameter, it appears a moment of the partition function where the fluctuations associated to the order parameter-like quantity becomes long-ranged. In this situation, these fluctuations become sensitive to the boundaries. In the Gaussian approximation, using the spectral zeta-function method, we evaluate a functional determinant for each moment of the partition function. The analytic structure of each spectral zeta-function depending on the dimension of the space for the case of Dirichlet, Neumann Laplacian and also periodic boundary conditions is discussed in a unified way. Considering the moment of the partition function with the largest correlation length of the fluctuations, we evaluate the induced force between the boundaries, for Dirichlet boundary conditions. We prove that the sign of the fluctuation-induced force for this case depend in a non-trivial way on the strength of the non-thermal control parameter.

I. INTRODUCTION

In this paper we relate microscopic/mesoscopic physics with macroscopic phenomena discussing fluctuation-induced forces in systems described by a continuous Landau-Ginzburg model, confined between boundaries, in the presence of a quenched disorder \[1–4\]. The random frozen variable is a non-thermal control parameter, that drives the system to the critical regime. In a system where a characteristic length is defined by the distance between the boundaries, near the second-order phase transition the fluctuations associated to some order parameters-like become long-ranged. For the moment of the partition function with the largest correlation length of the fluctuations, the fluctuation spectrum becomes highly sensitive to the geometry of the system. Our main result is that for the case of Dirichlet boundary conditions we found a connection between the sign of the fluctuation-induced force and the strength of the non-thermal control parameter. Varying continuously the intensity of this control parameter, the induced force between the boundaries can be repulsive or attractive. Although the study of fluctuation-induced forces produced in disordered systems defined in film geometries presents great technical difficulties, our objective here is to discuss a quite simplified situation where the effects of the Gaussian fluctuations associated to the order parameters-like quantity driven by the disorder field can be calculated. This global approach to obtain the thermodynamic quantities as entropy, pressure of radiation associated to the electromagnetic or a scalar field confined in a box or the Casimir force between boundaries, for clean systems has been widely discussed in the literature. See for example the Ref. \[5\].

To discuss quenched disordered systems, two main approaches has been developed. In the first one the spatial heterogeneities of the disorder is taken into account. Using the strong disorder renormalization group, the effects of local fluctuations of the disorder field can be discussed \[6\]. In the second one, an averaging over the disorder is performed, where the quenched free energy must be calculated. This quenched free energy is obtained performing the average of the free energy over the ensemble of all realizations of the disorder \[7–10\]. A fundamental characteristic of many disordered systems is the appearance of a complex structure, i.e., a multivalley free energy landscape. A very powerful tool that is able to exhibit this multivalley landscape due to the influences of the disorder is the distributional zeta-function method \[11–17\]. This formalism is quite simple and remarkably efficient. Using the machinery of the distributional zeta-
function one is able to derive the free energy landscape after a coarse grained procedure, where the quenched free energy is written as a series of the moments of the partition function. Therefore, one has to discuss the structure of the series representation of the average free energy as a function of the strength of the disorder, where the ground state of the fields in each moment of the partition function may change. In the series representation may appear specific moments of the partition function with a critical regime-like situation. In this case the fluctuations associated to order parameters-like quantities becomes sensitive to the boundaries. The strength of the disorder is a control parameter that play the same role as the temperature plays in clean thermal systems. We are using the terminology, order parameter-like, for the non-zero average of the fields for each moment of the partition function.

Quantum fields are fundamental building blocks in our description of nature. In the canonical formalism of field quantum theory, observables are represented by self-adjoint operators and one gives a physical meaning to vacuum expectation values for products of field operators. One can show that the renormalized vacuum expectation value of the stress-energy tensor of a scalar quantum field may have a local negative energy density. This result is based in the fact that only smeared fields are well defined operators in a field quantum theory, i.e., an operator value tempered distribution. In other words, although the energy operator associated to any quantum field is self-adjoint and positive, the (0-0) component of the stress-energy tensor can be negative if we work with operators smeared with functions with continuous derivatives of all orders and with bounded support [18]. As a consequence of the above discussion, the original Casimir effect can be interpreted as the effect of changes in the vacuum modes associated to the quantized electromagnetic field by the presence of boundaries [19–24]. Experimental support for the Casimir effect, can be found in Refs [25–27]. Although the original Casimir effect was discussed for hard-wall boundary conditions, it is worth to point out that one can use a smoothly increasing potential that represents some distribution of matter which interact with the quantum field [28]. Also, this effect is not restricted only to bosonic fields, any constrained quantum fields, as for example a massless fermionic field is also a source of the effect as a consequence of the interaction of quantum field vacuum modes with classical surfaces [29–30]. Even in the quasi-particle Landau scenario, the Casimir effect can be studied. We take as an example the phononic analog of the Casimir effect where the speed of light is substituted by the speed of sound in the medium [31].

In the canonical formalism, to obtain a finite result from the divergent vacuum energy, different methods have been established. One approach analyses the local energy density of the quantized field [32–37]. Another one, known as the global approach, investigates the total energy of the quantized field in the presence of the boundaries [38–39]. Although, for the case of a neutral scalar field, at principle there is an ambiguity to define the renormalized energy of minimal and conformally coupled scalar fields in the presence of boundaries, this conceptual difficulty was solved by Ford and Svaiter in Ref. [40], where it was discussed the situation of a fluctuating boundary. In the global approach, there are two natural ways to regularize and renormalize the divergent vacuum energy. The first one is a cut-off method, where an ultraviolet regulator function is introduced in the divergent sum. In principle, the definition of the renormalized vacuum energy of the electromagnetic field is given by the difference between the sum of the discrete spectrum eigenvalues of the Laplacian operator given by the difference between the sum of the discrete spectrum eigenvalues of the Laplacian operator associated to the constrained and the free field configuration. The second one is an analytic regularization procedure, where the spectral zeta-function regularization can be included [41–45]. Although these two methods, the cut-off method and analytic regularization procedures, are quite different in its grounds, it is possible to compare them and prove the analytic equivalence between them in some specific situations [46–49].

After the original formulation of the Casimir effect, the question if the attraction or repulsion between classical surfaces needs a field quantum theory scenario arises. In a confined system in the critical regime, where a length scale of the fluctuations are very large, the influences of the boundaries may appear. Therefore order parameter fluctuations play the role of the quantum vacuum modes mimetizing the original Casimir conceptual scenario. In the Landau-Ginzburg model in the presence of boundaries, for the case where the dimension of the order parameter is greater than one, confined massless Goldstone modes may lead to the critical Casimir effect [50–60]. One instructive example is the case of a binary liquid, where near to the critical point, fluctuations become long range inducing force between boundaries. On the other hand, has been demonstrated that quenched disorder fields are able to generate a critical regime in fluids and magnets systems [61–62]. Based in the above discussions, we perform a systematic approach to evaluate finite-size effects in disordered critical systems.

The theory of the statistical mechanics Casimir effect is based in the concept of finite size scaling, using the covariant operator for the constrained field Gaussian measure, constructed by the eigenfunctions of the Laplace operator in the presence of boundaries. For disordered systems, the finite-size scaling correlation was defined in Ref. [63]. Inspired in the statistical Casimir effect, we discuss fluctuation-induced forces between boundaries. Using the Gaussian approximation, here we will develop a global approach, discussing the series of the eigenvalues of Laplace operators, using generalized zeta-functions and an analytic regularization procedure. The key point of this global approach is that there are moments of the partition function that contributes to the force between the boundaries, induced by geometric restrictions, i.e., the constraints in the fluctuation spectra of each specific
moment. Although this global approach does not show the connection between the structure of the divergences and the geometry of the boundaries, its simplicity reveals the relation between the intensity of the effect and the correlation lengths of the fluctuations associated to the order parameters-like in some moments of the partition function. Also, it shows the link among the dimension of the space, the boundary conditions and the structure of the divergences in the associated spectral zeta-functions. Note that although we are in the statistical field theory framework, we are not using an ultraviotet cut-off in the model. Using the argument of universality in the critical behaviour, where the results of macroscopic measurements must be independent of the cut-off parameter, we can remove a natural physical cut-off and use an analytic regularization procedure to obtain finite results. In the perturbative expansion of interacting field theories, the analytic regularization procedure was introduced by Bollini et al. in Ref. [64].

Inspired in the thermal statistical Casimir effect, in this work we study the fluctuation-induced forces generated by a quenched disorder field (a non-thermal control parameter) linearly coupled with a scalar field defined by the Landau-Ginzburg model with $Z_2$ symmetry, in a $d$-dimensional slab geometry $\mathbb{R}^{d-1} \times [0, L]$. This Landau-Ginzburg model in a film geometry may represents a confined one-component fluid. The cases of the Dirichlet, Neumann Laplacian and also periodic boundary conditions are discussed. We present the force between the boundaries, for the case of Dirichlet boundary conditions that depends on the contributions of the term of the series, with the largest correlation length of the fluctuations, the nearly critical scenario. We prove that the sign of the fluctuation-induced force depends on the strength of the control parameter. One comment is in order. To implement the renormalization program in systems where translational invariance is broken it is required the introduction of counterterms which are surface interactions [65–69]. Since in this work we adopt a global approach, we are not introducing these boundary contributions in the model.

This paper is organized as follows. In section II we discuss the Landau-Ginzburg model defined in the continuum, in the presence of a quenched disorder. In section III the disordered average generating functional of correlations functions of the model is presented. In section IV in this scenario of confined fluctuations near the critical regime, the spectral zeta-function method and an analytic regularization procedure is discussed. Conclusions are given in section V. In the appendix A the distributional zeta-function method is defined and a series representation for the quenched free energy measured in units of temperature is obtained. Here we are using that $\hbar = k_B = 1$.

II. A LANDAU-GINZBURG MODEL WITH DISORDERED FIELDS

Let us start from the Euclidean field theories that are defined on a lattice [70, 71] to obtain the Landau-Ginzburg model in some limit. The Euclidean free and interacting field theories are defined on a cubic lattice $\mathbb{Z}^d$. In each point of the lattice, $x \in \mathbb{Z}^d$ one defines a random variable $\varphi(x)$ distributed according a probability measure and the continuum limit is defined. This is exactly the classical statistical field theory scenario, where the degrees of freedom that defines the model are random variables. To calculate equilibrium properties of the system one perform ensemble averages of these random variables.

In the classical statistical field theory scenario, the Landau-Ginzburg functional i.e., the action functional $S(\varphi)$ for the scalar field is given by

$$S = \int d^d x \left[ \frac{1}{2} \varphi(x) \left( -\Delta + m_0^2 \right) \varphi(x) + \lambda_0 \frac{\varphi^4(x)}{4!} \right].$$

The symbol $\Delta$ denotes the Laplacian in $\mathbb{R}^d$ and $\lambda_0$ and $m_0^2$ are respectively the coupling constant and a parameter that gives the distance of the model from the critical point. We call it the square mass of the model. Note that we are using the action $S(\varphi) = \beta H(\varphi)$, where $H(\varphi)$ is the Hamiltonian of the model. The action is the energy measured in units of temperature. The literature has been emphasized that the critical behaviour of the model is described by a disordered zero-temperature fixed point, i.e., for large distances the effects of the disorder field is much stronger than the thermal effects. Therefore we are interested to discuss the statistical field theory, where the thermal and disorder fluctuations dominate over the quantum fluctuations. Let us define a probability measure

$$dP(\varphi) = \frac{1}{Z} [d\varphi] \exp(-S(\varphi)), \quad (2)$$

where $[d\varphi]$ is a formal Lebesgue measure, given by $[d\varphi] = \prod_x d\varphi(x)$. The normalization is $(1) = 1$, and therefore the functional integral that defines the partition function is given by

$$Z = \int [d\varphi] \exp(-S(\varphi)). \quad (3)$$

The average value for any polynomial of the field $f(\varphi)$ can be obtained using that

$$\langle f(\varphi) \rangle = \frac{1}{Z} \int [d\varphi] f(\varphi) \exp(-S(\varphi)). \quad (4)$$

For instance, the $n$-point correlation functions of the model are given by

$$\langle \varphi(x_1) \cdots \varphi(x_n) \rangle = \frac{1}{Z} \int [d\varphi] \prod_{i=1}^n \varphi(x_i) \exp(-S(\varphi)). \quad (5)$$
Introducing an external source \( j(x) \), the generating functional of all \( n \)-point correlation functions \( Z(j) \) can be defined as \( [72, 73] \)

\[
Z(j) = \int [d\varphi] \exp \left( -S(\varphi) + \int d^d x \, j(x) \varphi(x) \right). \tag{6}
\]

For the case where \( \lambda_0 = 0 \) we can write the above expression as \( Z(j) = \int \exp(\varphi(j)) d\mu(\varphi) \). Therefore there is a probability measure on distributional-value fields. The generating function of correlation functions can be represented by a functional Taylor series. One has

\[
Z(j) = \sum_{m=0}^{\infty} \frac{1}{m!} \int \prod_{i=1}^{m} d^d x_i \prod_{k=1}^{m} j(x_k) \frac{\delta^k Z(j)}{\delta j(x_1) \ldots \delta j(x_m)} \bigg|_{j=0}. \tag{7}
\]

Taking functional derivatives with respect to the external source and setting to zero in the end, we obtain the \( n \)-point correlations functions of the model

\[
\langle \varphi(x_1) \ldots \varphi(x_k) \rangle = Z^{-1}(0) \frac{\delta^k Z(j)}{\delta j(x_1) \ldots \delta j(x_k)} \bigg|_{j=0}. \tag{8}
\]

Notice that these \( n \)-point correlation functions are given by the sum of all diagrams with \( n \)-external legs, including the disconnected ones. Next, using the linked cluster theorem, it is possible to define the generating functional of connected correlation functions, given by \( W(j) = \ln Z(j) \).

Using a functional Taylor expansion, the generating functional of connected correlation functions for the pure system can be written as

\[
W(j) = \sum_{m=0}^{\infty} \frac{1}{m!} \int \prod_{i=1}^{m} d^d x_i \prod_{k=1}^{m} j(x_k) \frac{\delta^k W(j)}{\delta j(x_1) \ldots \delta j(x_m)} \bigg|_{j=0}. \tag{9}
\]

For the sake of completeness, let us discuss the behaviour of the two-point correlation function in the Gaussian approximation in a generic \( d \)-dimensional space. In an interacting field theory a perturbative series in the coupling constant is able to construct all the \( n \)-point correlation functions where the building block of this perturbative expansion is the two-point correlation function. For a Gaussian random field with mean zero, the free covariance satisfies the Laplace equation

\[
(-\Delta + m_0^2) G_0(x, y) = \delta^d(x - y). \tag{10}
\]

Using a Fourier representation for \( G_0(x, y) \) one can show that

\[
G_0(x - y, m_0^2) = \frac{1}{(2\pi)^{d/2}} \left( \frac{m_0}{|x - y|} \right)^{d-2} K_{d-2}(m_0|x - y|), \tag{11}
\]

where \( K_{d/2}(z) \) is a modified Bessel function \( [74] \). For \( d = 3 \) we have that

\[
G_0(x - y, m_0^2) = \frac{1}{4\pi|x - y|} \exp(-m_0|x - y|). \tag{12}
\]

Finally, for \( d = 2 \)

\[
G_0(x - y, m_0^2) = -\frac{1}{2\pi} \ln(m_0|x - y|). \tag{13}
\]

In the general \( d \)-dimensional case, for large \( m_0|x - y| \) one can write

\[
G_0(x - y, m_0^2) = \frac{1}{2^{d/2}\pi^{d/2}} m_0^{d-3} |x - y|^{-d+3} \exp(-m_0|x - y|). \tag{14}
\]

From the above equation, the concept of correlation length of the fluctuation can be defined. The correlation length of the fluctuations is defined as \( \xi = m_0^{-1} \). For finite \( \xi \) the correlation of the fluctuations decay exponentially at large distances, showing the the interaction of the fluctuations is only significant in the volume of the order of \( \xi^d \).

To deal with systems in the presence of disorder, let us introduce the functional \( Z(j, h) \), the generating functional of correlation functions for one disorder realization, where we again use an external source \( j(x) \). This functional integral is defined

\[
Z(j, h) = \int [d\varphi] \exp \left( -S(\varphi, h) + \int d^d x \, j(x) \varphi(x) \right), \tag{15}
\]

where the action functional in the presence of disorder is defined as

\[
S(\varphi, h) = S(\varphi) + \int d^d x \, h(x) \varphi(x), \tag{16}
\]

for \( h(x) \in L^2(\mathbb{R}^n) \). In the above equation \( S(\varphi) \) is the pure Landau-Ginzburg action functional, defined in Eq. \( [4] \) and \( h(x) \) is a quenched random source. This is the simplest scalar model with a disorder field linearly coupled to the scalar field of the theory. We would like to point out that one can discuss also the quenched random mass model given by

\[
S(\varphi, \eta) = S(\varphi) + \frac{\eta}{4} \int d^d x \, \varphi^2(x). \tag{17}
\]

This model is known as the random-temperature disorder, where small density of impurities lead to randomness in the local transition temperature. In this work we will discuss only the quenched random source model. There are two different ways to perform disordered averages. The first one is the annealed average. In this case we define the average of the generating functional of correlation functions first and then define the annealed average free energy. Using that \( d\lambda = \prod_x dh(x) \) is a functional
measure and \([dh]P(h)\) is the probability distribution of the disorder, we have

\[
\mathbb{E}[Z(j, h)] = \int [dh]P(h)Z(j, h). \tag{18}
\]

For the case of annealed disorder, we use that the annealed free energy is defined by \(F_a(j) = -W_a(j)\) where

\[
W_a(j) = \ln \mathbb{E}[Z(j, h)], \tag{19}
\]

i.e., the free energy measured in units of temperature. Taking functional derivatives with respect to the external source and setting to zero in the end, we obtain the \(n\)-point correlations functions of the model. For example

\[
\left. \frac{\delta W_a(j)}{\delta j(x)} \right|_{j=0} = \left. \frac{1}{\mathbb{E}[Z(0, h)]} \frac{\delta \mathbb{E}[Z(j, h)]}{\delta j(x)} \right|_{j=0}, \tag{20}
\]

and so on. The second way to perform disorder average is the quenched average. One defines first \(W(j, h) = -\ln Z(j, h)\) i.e., the disordered free energy, for one disorder realization, also measured in units of temperature. Next one defines the average free energy performing the average over all realizations of the disorder. We have then a new generating functional, written as

\[
\mathbb{E}[W(j, h)] = \int [dh]P(h) \ln Z(j, h). \tag{21}
\]

To obtain the correlation functions we take the functional derivative of \(\mathbb{E}[W(j, h)]\) with respect to the external source \(j(x)\) directly. In this case, to obtain all the correlation functions of the model, one has to deal with the contribution \(\langle Z(j, h) \rangle^{-1}\), i.e., the disordered dependent generating functional in the denominator. In the next section, we show how it is possible to compute \(\mathbb{E}[W(j, h)]\), and the average free energy using the distributional zeta-function method.

### III. THE DISTRIBUTIONAL ZETA-FUNCTION METHOD

Recall that a measure space \((\Omega, \mathcal{W}, \eta)\) consists in a set \(\Omega\), a \(\sigma\)-algebra \(\mathcal{W}\) in \(\Omega\), and a measure \(\eta\) on this \(\sigma\)-algebra. Given a measure space \((\Omega, \mathcal{W}, \eta)\) and a measurable \(f : \Omega \to (0, \infty)\), we define the associated generalized \(\zeta\)-function as

\[
\zeta_{\eta, f}(s) = \int_{\Omega} f(\omega)^{-s} \, d\eta(\omega)
\]

for those \(s \in \mathbb{C}\) such that \(f^{-s} \in L^1(\eta)\), where in the above integral \(f^{-s} = \exp(-s \log(f))\) is obtained using the principal branch of the logarithm.

Definition: For any real number \(\kappa\), let \(|\kappa|\) denote the largest integer \(\leq \kappa\), that is, the integer \(r\) for which \(r \leq \kappa < r + 1\). Consider that: 1 - if \(\Omega = \mathbb{R}^+\), \(\mathcal{W}\) is the Lebesgue \(\sigma\)-algebra, \(\eta\) is the Lebesgue measure, and \(f(\omega) = |\omega|\) we obtain the classical Riemann zeta-function [22]; where \([x]\) means the integer part of \(x\), using the definition above. Next, 2 - if \(\Omega\) and \(\mathcal{W}\) are as in item 1, \(\eta(E)\) counts the prime numbers in \(E\) and \(f(\omega) = \omega\) we retrieve the prime zeta-function [76] [79], 3 - if \(\Omega\), \(\mathcal{W}\), and \(f\) are as in item 2 and \(\eta(E)\) counts the eigenvalues of an elliptic operator, with their respective multiplicity, we obtain the families of superzeta-functions [80], and finally 4 - if \(\Omega\), \(\mathcal{W}\), and \(f\) are as in item 2 and \(\eta(E)\) counts the eigenvalues of an elliptic operator, with their respective multiplicity, we obtain the spectral zeta-functions. This spectral zeta-function will be used to find the Casimir energy of the critical system.

For a given probability distribution of the disorder, one is mainly interested in obtaining the average free energy defined by Eq. (21). For a general disorder probability distribution, using the disordered functional integral \(Z(j, h)\) given by Eq. (15), the distributional zeta-function, \(\Phi(s, j)\), is defined as

\[
\Phi(s, j) = \int [dh]P(h) \frac{1}{Z(j, h)^s}, \tag{22}
\]

for \(s \in \mathbb{C}\), this function being defined in the region where the above integral converges. The average generating functional defined by Eq. (21) can be written as

\[
\mathbb{E}[W(j, h)] = -(d/ds)\Phi(s, j)|_{s=0^+}, \quad \text{Re}(s) \geq 0, \tag{23}
\]

where one defines the complex exponential \(n^{-s} = \exp(-s \log(n))\), with \(\log n \in \mathbb{R}\). Using analytic tools, the quenched free energy of a system in the presence of an external field is \(F_q(j) = -\mathbb{E}[W(j, h)]\) where

\[
\mathbb{E}[W(j, h)] = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} a^k}{kk!} \mathbb{E} \left[ \left( Z(j, h) \right)^k \right] - \ln(a) + \gamma + R(a, j). \tag{24}
\]

Again, we are defining the free energy measured in units of temperature. The quantity \(a\) is a dimensionless arbitrary constant, \(\gamma\) is the Euler-Mascheroni constant, and \(R(a)\) is given by

\[
R(a, j) = -\int [dh]P(h) \int_0^\infty \frac{dt}{t} \exp(-Z(j, h)t). \tag{25}
\]

One can show that the contribution of \(|R(a)|\) in the critical regime is quite small, since the dominant contribution to the quenched free energy is given by a singular moment in the set of the moments of the disordered generating functional of correlation functions that for simplicity will be called by the moments of partition functions of the model. In this singular moment of the partition function, the fluctuations of the order parameter-like will become
highly sensitive to the boundaries. To proceed we assume that the probability distribution of the disorder is written as \( dh P(h) \), where

\[
P(h) = p_0 \exp \left( -\frac{1}{2 \sigma^2} \int d^d x (h(x))^2 \right) .
\] (26)

The quantity \( \sigma \) is a positive parameter associated with the disorder and \( p_0 \) is a normalization constant. This defines a delta correlated processes, i.e., \( \mathbb{E}[h(x)h(y)] = \sigma^2 \delta^d(x - y) \). To discuss the ordered phase in the model, i.e., the infrared regime, the microscopic details of the disorder must be irrelevant. After integrating over the disorder we get that each moment of the partition function \( \mathbb{E} [(Z(j, h))^k] \) can be written as

\[
\mathbb{E} [Z^k(j, h)] = \int \prod_{i=1}^k [d\varphi_i] \exp \left( -S_{\text{eff}}(\varphi_i, j_i) \right) ,
\] (27)

where the effective action \( S_{\text{eff}}(\varphi_i) \) describing the field theory with \( k \)-field components is given by

\[
S_{\text{eff}}(\varphi_i^{(k)}, j_i^{(k)}) = \int d^d x \left[ \sum_{i=1}^k \left( \frac{1}{2} \varphi_i^{(k)}(x) (-\Delta + m_0^2) \varphi_i^{(k)}(x) \right) + \frac{\lambda_0}{4!} \varphi_i^{(k)}(x)^4 \right] - \frac{\sigma^2}{2} \sum_{i,j=1}^k \varphi_i^{(k)}(x) \varphi_j^{(k)}(x) - \sum_{i=1}^k \varphi_i^{(k)}(x) j_i^{(k)}(x) .
\] (28)

In order to avoid unnecessary complications, and for practical purposes, we assume the following configuration of the scalar fields \( \varphi_i^{(k)}(x) = \varphi_j^{(k)}(x) = \varphi^{(k)}(x) \) in the function space and also \( j_i^{(k)}(x) = j_j^{(k)}(x) = j^{(k)}(x) \). All the terms of the series have the same structure and one minimizes each term of the series one by one. For instance, in many systems with quenched disorder the free energy has a multivale structure. Let us discuss the consequences of our choice for the free energy landscape. We have that the \( k \)-th moment of the partition function is written as

\[
\mathbb{E} [(Z(j, h))^k ] = \left[ \int [d\varphi] \exp \left( -S^{(k)}(\varphi^{(k)}, j^{(k)}) \right) \right]^k ,
\] (29)

In this case, the new effective action, obtained from Eq. (28), is written as:

\[
S^{(k)}(\varphi^{(k)}, j^{(k)}) = \int d^d x \left( \frac{1}{2} \varphi^{(k)}(x) (-\Delta + m_0^2 - k \sigma^2) \varphi^{(k)}(x) \right) + \frac{\lambda_0}{4!} \varphi^{(k)}(x)^4 - \varphi^{(k)}(x) j^{(k)}(x) .
\] (30)

We can write \( \mathbb{E}[W(j, h)] \) as

\[
\mathbb{E}[W(j, h)] = \lim_{N \to \infty} \sum_{k=1}^N c_k \mathbb{E} [(Z(j, h))^k] ,
\] (31)

where \( c_k(a) = (-a)^{k+1} / k! \). For simplicity we will adopt the convention that \( c_k(a) = c_k \). To describe a second-order phase transition, one moment of the partition function will dominate the series, to have the critical regime. In what follows let us define

\[
W^{(k)}(j) = c_k \mathbb{E} [(Z(j, h))^k] .
\] (32)

Let us discuss the situation similar to a second-order phase transitions for each moment of the partition function. We are assuming that \( m_0^2 > \sigma^2 \). We have three different contribution of the terms of the series depending on the sign of \( m_0^2 - k \sigma^2 \): (i) the case where \( m_0^2 - k \sigma^2 > 0 \), (ii) the case where \( m_0^2 \equiv k \sigma^2 \), (iii) when this quantity is negative, one has to shift field to a new minimum, i.e. \( \phi^{(k)}(x) = \varphi^{(k)}(x) \pm \eta^{(k)} \), where

\[
\eta^{(k)} = \left( \frac{6(k \sigma^2 - m_0^2)}{\lambda_0} \right)^{1/2} .
\] (33)

This situation is similar to the spontaneous symmetry breaking in statistical field theory and the behaviour of each term of the series is described by the (+) or (−) cases. From now on, we choose the minus sign above. In this case, we find a positive squared mass with self-interactions terms \( (\phi^{(k)}(x))^3 \) and \( (\phi^{(k)}(x))^4 \), where the potential \( V(\phi^{(k)}) \) reads

\[
V(\phi^{(k)}) = (k \sigma^2 - m_0^2) (\phi^{(k)}(x))^2 + \sqrt{\frac{\lambda_0}{3!} (k \sigma^2 - m_0^2)} (\phi^{(k)}(x))^3 + \frac{\lambda_0}{4!} (\phi^{(k)}(x))^4 .
\] (34)

In the usual picture of second-order phase transition there is one thermodynamic quantity, zero in the disordered phase and non-zero in the ordered one. Nevertheless, it is tempting to interpret each term of the series as describing a domain with its own order parameter. In other words, one can interpret the series as describing a system that consists of distinct subsystems where the local average works. In this situation, in each moment of the partition function we may define order parameters-like, i.e., \( \phi_0^{(k)}(x) \). The critical scenario is dominated by one or few terms of the series.

In field theory, the effective action can be written in terms of three contributions. The first term coming from the stationarity of the action, (the tree-level approximation if field quantum theory), the second one gives the contribution from the Gaussian fluctuations, (the oneloop level in field quantum theory) and finally the radiative corrections. Here we have the same situation for each moment of the partition function. We obtain that for any moment of the partition function the first contribution is

\[
-\Delta \varphi_0^{(k)}(x) + (m_0^2 - k \sigma^2) \varphi_0^{(k)}(x) + \frac{\lambda_0}{3!} (\varphi_0^{(k)}(x))^3 = j^{(k)}(x)
\] (35)
The Fourier transform of the susceptibility-like quantity \( \chi^{(k)}(x-y) \) is obtained as

\[
\chi^{(k)}(q) = \frac{1}{q^2 + m_0^2 - k\sigma^2 + \frac{1}{2}\lambda_0(\varphi_0^{(k)})^2}.
\]

(36)

Let us consider first the terms of the series that defines the average free energy where \( k < k_c \). Then \( \varphi_0^{(k)}(x) = 0 \) and

\[
\chi^{(k)}(q) = \frac{1}{q^2 + m_0^2 - k\sigma^2}.
\]

(37)

The correlation length \( \xi^{(k)}(\sigma, m_0) = (m_0^2 - k\sigma^2)^{-\frac{1}{2}} \). The situation can be discussed also for \( k > k_c \), for a given strength of the disorder. In this case we get

\[
(\varphi_0^{(k)})^2 = \frac{6(k\sigma^2 - m_0^2)}{\lambda_0^2}.
\]

(38)

The Fourier transform of the susceptibility-like quantity \( \chi^{(k)}(x-y) \) reads

\[
\chi^{(k)}(q) = \frac{1}{q^2 + 2(k\sigma^2 - m_0^2)}.
\]

(39)

The correlation length of the fluctuations in each moment of the partition function is given by \( \xi^{(k)}(\sigma, m_0) = (m_0^2 - k\sigma^2)^{-\frac{1}{2}} \). Again, for simplicity we will write \( \xi^{(k)}(\sigma, m_0) = \xi^{(k)} \). As in the critical regime, there are fluctuations whose correlation lengths are of the order of the characteristic length scale of the system, defined by the distance between the boundaries. In these moment of the partition function where \( (k\sigma^2 - m_0^2) \approx 0 \), the fluctuations of the order parameter-like becomes highly sensitive to the boundaries.

The next step is to present the contribution of the moments of the partition function of the quenched free energy, for non-zero order parameters-like, where the sensitivity to the boundary can not be neglected. We shall now discuss this situation in more details. The contributions to the functional \( E[W(j, h)] \) is written

\[
E[W(j, h)] = \sum_{k=1}^{N} c_k \left[ \int [d\varphi^{(k)}] \exp(-S^{(k)}(\varphi^{(k)}, j^{(k)})) \right]^k,
\]

(40)

where \( S^{(k)}(\varphi^{(k)}, j^{(k)}) \) is given by Eq. (30). For \( k\sigma^2 > m_0^2 \), we have to expand each functional integral around the minimum (Eq. (43)) up to the lowest-order quadratic term and integrate out the fluctuations. Starting from the elliptic operator that we are interested defined as \( D = -\Delta + 2(k\sigma^2 - m_0^2) \), we define \( D(z, y; k) \) such that

\[
D(x, y; k) = \left( -\Delta + 2(k\sigma^2 - m_0^2) \right) \delta^d(x - y)
\]

(41)

and the inverse kernel \( K(x, z; k) \) that satisfies

\[
\int d^d z K(x, z; k) D(z, y; k) = \delta^d(x - y).
\]

(42)

Therefore up to the Gaussian approximation we can write \( E[W(j, h)] \) as

\[
E[W(j, h)] = \sum_{k=1}^{N} \frac{c_k}{(\det D(k))^{k/2}} \exp \left( -\int d^d x \int d^d y j^{(k)}(x) K(x, y; k) j^{(k)}(y) \right)^k.
\]

(43)

As we discussed, we consider the theory in finite-size geometry in one dimension with the spatial coordinate \( x_d = z \) compactified. We are considering a slab defined as \( \Omega = [x \equiv (x_1, x_2, \ldots, x_{d-1}, z) : 0 \leq z \leq L] \subset \mathbb{R}^d \). We will discuss the Dirichlet, Neumann, and periodic boundary conditions cases in a unified way. In systems where the translational invariance is broken, in a local approach one can use a Fourier representation for the fields. Since the system processes translational invariance along the direction parallel to the plates, one has to adopt a mixed representation, to implement the renormalization program. The Fourier transform of the susceptibility-like quantity \( \chi^{(k)}((x-y)|_{(y, z, z') \rightarrow (x, z, z')}) \) reads

\[
\chi^{(k)}(q_{||}, n) = \frac{1}{(q_{||})^2 + (2n\pi/L)^2 + 2(k\sigma^2 - m_0^2)}.
\]

(44)

One would expect that the sensitivity to the boundaries is given by the correlation length of the fluctuations \( \xi^{(k)} = (2(k\sigma^2 - m_0^2))^{-\frac{1}{2}} \), where for \( \xi^{(k)} \ll L \), the induced force between the boundaries is negligible. In the next section we discuss how to evaluate the functional determinants using the spectral zeta-function method and its connection with the fluctuation-induced force on the boundaries.

**IV. FLUCTUATION-INDUCED FORCE IN SYSTEMS WITH DISORDER**

In the previous section we interpret the series representation for the quenched free energy where each moment defines its own order parameter-like quantity with a particular correlation length of the fluctuations. For some specific strength of the non-thermal control parameter, it appears moments with long-range correlation lengths. Each of these moments contribute to the quenched free energy by mean of a functional determinant. To evaluate each of these functional determinants, the formalism of spectral zeta-function is a standard procedure. Since, the results that we will obtained will play a fundamental role in our approach, we will discuss the spectral zeta-function method with some details.

Now suppose a infinite sequence of non-zero real or complex numbers \( \lambda_n \). If the sequence of numbers is zeta regularizable we define the regularized product \( \prod_n \lambda_n \). The zeta regularized product of these numbers is defined
as \( \exp(-\zeta'(0)) \) where this generalised zeta-function is given by

\[
\zeta(s) = \sum_n \lambda_n^{-s}, \quad \text{Re}(s) > s_0 \tag{45}
\]

for \( s \in \mathbb{C} \), this function being defined in the region of the complex plane where the sum converges and \( \zeta'(0) = \frac{d}{ds} \zeta(s)|_{s=0+} \). Since the series defined above is absolutely convergent, therefore the sum is analytic in the half-plane \( \text{Re}(s) > s_0 \). The generalised zeta-function has a meromorphic continuation with at most simple poles, to the half-plane containing the origin. We must show that this continuation is analytic at the origin. Performing an analytic extension one can find \( \zeta'(0) = \frac{d}{ds} \zeta(s)|_{s=0+} \).

Suppose a bounded domain in \( \mathbb{R}^d \), i.e., \( \Omega \subset \mathbb{R}^d \). In this case a positive definite self-adjoint elliptic operator has a complete set of orthonormal eigenfunctions with eigenvalues \( \lambda_n \). If \( \lambda_n \) is the sequence of positive eigenvalues of \( -\Delta \) on a bounded domain, then the zeta regularized product is the determinant of the Laplacian and the zeta-function is called the spectral zeta-function of the Laplacian \([31]-[34]\). One can write that

\[
\left[ \det D(k) \right]^{-\frac{1}{2}} = \exp \left[ \frac{k}{2} \left( \zeta'(0, k) \right) \right] \tag{46}
\]

The standard way to discuss the analytic structure of the spectral zeta-function is to introduce a diffusion kernel, the integrated diffusion kernel, and discuss an asymptotic expansion \([35]-[37]\). In the standard spectral zeta-function approach, to define the effective action one introduces a scale \( \mu \) with mass dimension to keep the generalized zeta-function dimensionless. In this case, the elliptic operator \( D \) is transformed to \( \mu^{-2} D \). The scaling properties of this formalism is given by

\[
- \frac{d}{ds} \zeta_{\mu^{-2} D}(s, \mu, k)|_{s=0+} = \ln \mu^2 \zeta(0, \mu, k) - \frac{d}{ds} \zeta_D(s, \mu, k)|_{s=0}. \tag{47}
\]

In the absence of sources, one can write the quenched free energy as \( F_q = -\mathbb{E}[W(j, h)]|_{j=0} \) where

\[
\mathbb{E}[W(j, h)]|_{j=0} = \sum_{k=1}^{N} c_k \exp \left[ \frac{k}{2} \left( \zeta'(0, k) \right) \right]. \tag{48}
\]

We would like to point out that in systems without disorder there is a connection between the effective action defined by \( \zeta'(0) = \frac{d}{ds} \zeta(s)|_{s=0+} \) and the Casimir energy \([38]\). A standard calculation to obtain the Casimir energy can be found in Ref. \([39]\). These authors discussed the vacuum to vacuum amplitude for a massless scalar theory in Minkowski space-time. After a Wick rotation and using the spectral zeta-function they obtained that \( \langle 0_+|0_- \rangle = \exp(-T E(L)) \), where \( T \) is the total Euclidean time and the Casimir energy is \( E(L) = -\frac{\pi^2 L^2}{2d^2} \), where \( A \) is the area of the boundaries. Since, for the case of flat boundaries, these quantities coincides, here we will discuss each contribution for the free energy using an analytic regularization procedure, calculation \( \zeta(-\frac{1}{2}, k) \) instead of \( \zeta'(0, k) \). It is clear that using this procedure we do not need to introduce the scale \( \mu \) with mass dimension that has been used to perform the calculation of \( \zeta'(0) = \frac{d}{ds} \zeta(s)|_{s=0+} \).

Let us assume a "thermodynamic limit" with respect to the surface area, i.e., \( L_1, L_2, ..., L_d \gg L_d \), and \( 2(\kappa^2 - m_0^2) > 0 \). To proceed one define the spectral zeta-function \( \zeta_d(s, k) \) such that

\[
\zeta_d(s, k) = \frac{1}{(2\pi)^{d-1}} \left( \prod_{i=1}^{d-1} L_i \right) \int \frac{d^{d-1} q_i}{(2\pi)^{d-1}} \left[ \sum_{n \in \mathbb{Z}} (q_1^2 + ... + q_{d-1}^2 + \frac{2\pi n}{L_d})^2 + 2(\kappa^2 - m_0^2) \right]^{-s} \tag{49}
\]

for \( s \in \mathbb{C} \). Performing a continuation beyond the domain of convergence, the Casimir-like energy associated to any specific moment is found after evaluate the spectral zeta function for \( s = -\frac{1}{2} \). Using that the finite length \( L_d = L \) and performing the angular part of the integral over the continuous mode spectrum of the \((d-1)\) non-compact dimensions we get that

\[
\int d\Omega_{d-1} = \frac{2(\pi)^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}. \tag{50}
\]

Note that we are assuming \( d \geq 2 \). Remember that we are interested in discuss the fluctuation-induced force generated by specific moments of the partition function, where the fluctuations of the order parameters-like quantities are sensitive to the boundaries. This occurs when the correlation lengths of the fluctuations of the order parameters-like quantities satisfies \( \xi^{(d)} > L \).

As we discussed, for any real number \( \kappa \), let \( \lfloor \kappa \rfloor \) denote the largest integer \( \leq \kappa \), that is, the integer \( r \) for which \( r \leq \kappa < r + 1 \). Therefore \( \kappa - 1 < \lfloor \kappa \rfloor \leq \kappa \). \([40]\). We have a family of moments such that \( \frac{m_0^2}{\pi^2} \leq k \leq \frac{1}{\pi^2} \left( \frac{\pi^2}{\pi^2} + m_0^2 \right) \). We are interested to discuss the contribution of the moments of the partition function where \( k \geq \frac{m_0^2}{\pi^2} \), i.e., where each of the order parameters-like quantities does not vanish.

All the calculations will be performed for one specific moment of the partition function. At this point, let us define \( A(d) \), such as

\[
A(d) = \frac{1}{2^{d-2} \pi^{\frac{d-1}{2}} \Gamma(\frac{d-1}{2})}, \tag{51}
\]

and the modified spectral zeta-function, i.e., the spectral zeta-function per unit area \( Z_d(s, k) \) given by

\[
Z_d(s, k) = \frac{\zeta_d(s, k)}{A(d) \left( \prod_{i=1}^{d-1} L_i \right)}. \tag{52}
\]
where we included the factor $A(d)$ in the definition. The expression for $Z_d(s, k)$ is written as

$$Z_d(s, k) = \left( \frac{L}{\sqrt{4\pi}} \right)^{2s} \int_0^\infty dp \, p^{d-2} \sum_{n \in \mathbb{Z}} \left( \pi n^2 + \frac{L^2}{4\pi} \left( p^2 + 2(k\sigma^2 - m_0^2) \right) \right)^{-s}. \quad (53)$$

At this point, we would like to point out that the functional determinant $\prod_k (\lambda_k + \lambda)$ was also discussed by Voros [91]. One can show that if the Eq. (45) is zeta regularizable, then so is

$$\zeta(s, \lambda) = \sum_{n=0}^\infty (\lambda n + \lambda)^{-s}, \quad \text{Re} \, (s) > s_0, \quad (54)$$

for $(\lambda \notin (-\infty, -\lambda_0))$. For $\lambda \to \infty$ an asymptotic expansion for $\sum_{n=0}^\infty \exp(-\lambda n t)$ allow one to show that the above expression is zeta regularizable. Since we are discussing periodic and Neumann boundary conditions, the zero mode problem appears. There are three different ways to deal with this problem. The first one is to assume that we are working in compact domains. The second one is the following. Remember that in order to obtain the analytic extension of the zeta-function, Riemann discussed a special case of the Jacobi theta-function, a function with modular symmetry that satisfies a functional equation [92]. Recently it was discussed the consequences of introducing, in the integral representation of the Riemann zeta-function, a cut-off $f(\lambda, t) = \exp \left[ -\lambda (t + \frac{1}{t}) \right]$ which is invariant under the transformation $t \to 1/t$. It was proved that this modified zeta-function generalize the Riemann functional equation with the same symmetry $s \to (1 - s)$ in the critical strip. [93]. One way to proceed is to introduce a modification in the integral representation of the zeta-function introducing a function that exhibit the symmetry $t \to 1/t$, using that $\lambda \in \mathbb{R}_+$. The problem of this approach is the introduction of an undetermined cut-off parameter. Therefore, in this work we are using a third approach. We will present in the end of the calculations the contribution of the zero mode. To proceed let us use a Mellin transform. We can show that the modified zeta-function can be written as

$$Z_d(s, k) = \frac{1}{\Gamma(s)} \left( \frac{L}{\sqrt{4\pi}} \right)^{2s} \int_0^\infty dp \, p^{d-2} \int_0^\infty dt \, t^{s-1} \sum_{n \in \mathbb{Z}} \exp \left[ -\left( \pi n^2 + \frac{L^2}{4\pi} \left( p^2 + 2(k\sigma^2 - m_0^2) \right) \right) t \right] \quad (55)$$

Let us define the dimensionless quantities $m^2(k) = \frac{L^2}{4\pi} (k\sigma^2 - m_0^2)$ and $\sigma^2 = \frac{L^2}{4\pi} p^2$. The quantity $m^2(k)$ defines the finite size scaling, i.e., close to the critical point, finite size effects is controlled by the ratio $\frac{L}{\xi(\sigma)}$. Is usually defined the universal scaling function $v\left(\frac{L}{\xi(\sigma)}\right)$. We also define $B(s, d)$ such that

$$B(s, d) = 2(\sqrt{4\pi})^{d-2s-1} \frac{\Gamma\left(\frac{d-1}{2}\right)}{\Gamma(s)}. \quad (56)$$

Therefore we can write

$$Z_d(s, k) = \frac{B(s, d)}{2\Gamma\left(\frac{d+1}{2}\right)} \frac{1}{L^{d-2s-1}} \int_0^\infty dr \, r^{d-2} \int_0^\infty dt \, t^{s-1} \exp \left[ -\left( m^2(k) + r^2 \right) t \right] \Theta(t). \quad (57)$$

where $\Theta(v)$, the theta-function, an example of a modular form, is defined by

$$\Theta(v) = \sum_{n \in \mathbb{Z}} \exp(-\pi n^2 v). \quad (58)$$

Using the Poisson summation formula, for $v > 0$ one can show the transformation formula $\Theta(v) = \frac{1}{\sqrt{\pi}} \Theta\left(\frac{1}{\sqrt{v}}\right)$. Splitting the $t$ integral in two contributions and performing the $r$ integral we can write

$$Z_d(s, k) = Z_d^{(1)}(s, k) + Z_d^{(2)}(s, k) \quad (59)$$

where

$$Z_d^{(1)}(s, k) = \frac{1}{L^{d-2s-1}} B(s, d) \int_0^1 dt \, t^{s-\frac{d}{2}} \exp\left(-m^2(k)t\right) \Theta(t) \quad (60)$$

and

$$Z_d^{(2)}(s, k) = \frac{1}{L^{d-2s-1}} B(s, d) \int_1^\infty dt \, t^{s-\frac{d}{2}} \exp\left(-m^2(k)t\right) \Theta(t). \quad (61)$$

Changing variables in the integral given by Eq. (60) and using the symmetry of the theta-function we can write $Z_d^{(1)}(s, k)$ as

$$Z_d^{(1)}(s, k) = \frac{1}{L^{d-2s-1}} B(s, d) \int_1^\infty dt \, t^{-s+\frac{d}{2}-1} \exp\left(-\frac{m^2(k)}{t}\right) \Theta(t) \quad (62)$$

Let us define the psi-function $\psi(v)$ such that

$$\psi(v) = \sum_{n=1}^\infty \exp(-\pi n^2 v) \quad (63)$$

so that $\psi(v) = \frac{1}{2} (\Theta(v) - 1)$. We can write that $Z_d(s, k)$ has four contributions given by $I_d^{(i)}(s, k), i = 1, ..., 4$. We have

$$Z_d(s, k) = \frac{1}{L^{d-2s-1}} B(s, d) \left( 2I_d^{(1)}(s, k) + 2I_d^{(2)}(s, k) + I_d^{(3)}(s, k) + I_d^{(4)}(s, k) \right), \quad (64)$$

where $I_d^{(i)}(s, k), i = 1, ..., 4$, are integrals that can be expressed in terms of modular forms. These integrals can be evaluated using the properties of the theta-function and the modular symmetry.
where each of these integrals are given by

\[ I_d^{(1)}(s, k) = \int_1^\infty dt \, t^{s - \frac{d}{2} - \frac{1}{2}} \exp \left( -m^2(k)t \right) \psi(t), \quad (65) \]

\[ I_d^{(2)}(s, k) = \int_1^\infty dt \, t^{-s + \frac{d}{2} - 1} \exp \left( -\frac{m^2(k)}{t} \right) \psi(t), \quad (66) \]

\[ I_d^{(3)}(s, k) = \int_1^\infty dt \, t^{s - \frac{d}{2} - \frac{1}{2}} \exp \left( -m^2(k)t \right) \quad (67) \]

and finally

\[ I_d^{(4)}(s, k) = \int_1^\infty dt \, t^{-s + \frac{d}{2} - 1} \exp \left( -\frac{m^2(k)}{t} \right). \quad (68) \]

Now we can use the standard result that a function which is analytic on a domain \( \Omega \subset \mathbb{C} \), has a unique extension to a function defined in \( \mathbb{C} \) except to a discrete set of points. Let us discuss this procedure for the case of the Dirichlet Laplacian where only the integrals \( I_d^{(1)}(s, k) \) and \( I_d^{(2)}(s, k) \) appears. Using the fact that \( \psi(t) = O(e^{-\pi t}) \) as \( t \to \infty \), the integrals \( I_d^{(1)}(s, k) \) and \( I_d^{(2)}(s, k) \) represent an everywhere regular functions of \( s \) for \( m^2(k) \in \mathbb{R}^+ \). The upper bound insure uniform convergence of the integrals on every bounded domain in \( \mathbb{C} \). As in the standard quantum field theory scenario, the contribution to the average free energy from each moment of the partition function of the system can be evaluated for \( s = -\frac{1}{2} \).

In the Fig. 1 we depict the behaviour of the integral given by Eq. (65) for \( s = -1/2 \), for arbitrary dimensionality of the space and also dimensionless quantity \( m(k) = m \).

![FIG. 1. Behavior of \( I_d^{(1)}(s, k) \) given by Eq. (65) for \( s = -1/2 \), for arbitrary dimensionality of the space and also dimensionless quantity \( m(k) = m \).](image1)

For the Neumann Laplacian and also the periodic boundary conditions, not only the integrals \( I_d^{(1)}(s, k) \) and \( I_d^{(2)}(s, k) \) but also the integrals \( I_d^{(3)}(s, k) \) and \( I_d^{(4)}(s, k) \) also appears. In the absence of the exponential decay of the \( \psi(v) \) function and for \( m^2(k) \in \mathbb{R}^+ \) we have to discuss the polar structure of the integral \( I_d^{(3)}(s, k) \). Note that we are assuming that \( m^2(k) \) is small, but different from zero. One can write that the integral \( I_d^{(3)}(s, k) \) can be written as

\[ I_d^{(3)}(s, k) = (m^2(k))^{\frac{d}{2} - s - \frac{1}{2}} \Gamma \left( s - \frac{d}{2} + 1, m^2(k) \right), \quad (69) \]

where \( \Gamma(\alpha, x) \) is the incomplete gamma function defined by

\[ \Gamma(\alpha, x) = \int_x^\infty dt \, t^{\alpha - 1} e^{-t}. \quad (70) \]

For noninteger \( \alpha \), \( \Gamma(\alpha, x) \) is a multivalue function of \( x \), with a branch point at \( x = 0 \). Using the series representation for the \( \Gamma(\alpha, x) \) incomplete gamma function we get

\[ I_d^{(3)}(s, k) = (m^2(k))^{\frac{d}{2} - s - \frac{1}{2}} \Gamma \left( s - \frac{d}{2} + 1, m^2(k) \right) + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n!} \left( \frac{(m^2(k))^n}{s - \frac{d}{2} + 1, n} + \right) \quad (71) \]

Note that in principle, that the contribution of \( I_d^{(3)}(s, k) \) is finite for odd dimensional space. Let us discuss the
The polar structure of $I^{(4)}_{d}(s, k)$. We can write that

$$I^{(4)}_{d}(s, k) = \left( m^2(k) \right)^{\frac{d}{2} - s} \gamma \left( s - \frac{d}{2}, m^2(k) \right),$$

(72)

where $\gamma(\alpha, x)$ is the incomplete gamma function defined by

$$\gamma(\alpha, x) = \int_{0}^{x} dt \, t^{\alpha - 1} e^{-t},$$

(73)

for $\text{Re}(\alpha) > 0$. Using again a series representation for the $\gamma(\alpha, x)$ incomplete gamma function we can write that

$$I^{(4)}_{d}(s, k) = \sum_{n=0}^{\infty} \frac{(-1)^n \left( m^2(k) \right)^n}{n! \left( s - \frac{d}{2} + n \right)}.$$  

(74)

The contribution of $I^{(4)}_{d}(s, k)$ is finite for even dimensional space. Therefore it is not possible to define the Casimir-like energy per unit area associated to the Neumann Laplacian using an analytic regularization procedure in the Gaussian approximation [94]. This obstruction is related to the presence of the zero mode [95]. Going beyond the Gaussian approximation, the infrared problem can be discussed [96]. For the case of Dirichlet boundary conditions we can write that the terms of the series representation of $E[V(j, h)]|_{j=0}$, i.e., moments of the partition function where the fluctuations of the order parameters-like quantities become sensitive to the boundaries defined by $F_d(L)$ as

$$F_d(L) = \sum_{k=k_1}^{k_2} c_k(a) \exp \left[ \frac{k}{2} \zeta_d \left( -\frac{1}{2}, k \right) \right],$$

(75)

where $k_1 = \left[ \frac{m^2}{\pi^2} \right]$ and $k_2 = \left[ \frac{1}{2} \left( \frac{m^2}{\pi^2} + m_0^2 \right) \right]$. As, in the critical phenomena, the fluctuations occur over a wide range of scale. To proceed, let us investigate the leading contribution to the series representation for the free energy, where the correlation length of the fluctuations attains its maximum value. Next we can use a renormalization procedure, i.e., or a suitable choice of $a$, i.e., $a = \exp \left( [\zeta_d(\frac{1}{2}, k_1)] \right)$, we can write that the force per unit area is given by

$$f_d(L) = \frac{(-1)^{k_1+1}}{2k_1!} \frac{1}{\prod_{i=1}^{d-1} L_i} \frac{\partial}{\partial L} \zeta_d \left( -\frac{1}{2}, k_1 \right).$$

(76)

Two comments are in order. The first one is that any physical observable must be independent of the dimensionless quantity $a$. For instance, the average free given by Eq. [24] is $a$ independent. The renormalization procedure also eliminates the dependence of the force per unit area on the dimensionless parameter. The second one is that using the cut-off method, the definition of the force between the boundaries, is given in terms of subtraction of two configurations free energies, the free energy per-unit area and the free energy density. The structure of the free energy is related to the Weyl’s theorem that links the asymptotic distribution of eigenvalues of a generic elliptic differential operator with geometric parameters of the boundary that defines the domain where the field is defined [97, 98]. In an analytic regularization procedure, this procedure of subtraction of configurations is not necessary. Using the definition of $Z_d(s, k)$ we can write

$$f_d(L) = \frac{(-1)^{k_1+1}}{2k_1!} A(d) \frac{\partial}{\partial L} \frac{1}{Z_d} \left( -\frac{1}{2}, k_1 \right).$$

(77)

Therefore the induced-force per unit area in the case of Dirichlet boundary condition depends on the contribution coming from the leading term, with the largest correlation length of the fluctuations. The sign of the induced-force depends on the sign of $\zeta_d(\frac{1}{2}, k_1)$, that is negative using that $\Gamma(\frac{1}{2}) = -2\sqrt{\pi}$ and also from the value of $\left( \frac{m^2}{\pi^2} \right)$, being odd or even. We can write the induced-force per unit area as

$$f_d(L) = \frac{2}{\sqrt{\pi}} \frac{(-1)^{k_1}}{k_1!} \left( 1 \right)^{d+1} \left( I_d^{(1)} \left( -\frac{1}{2}, k_1 \right) + I_d^{(2)} \left( -\frac{1}{2}, k_1 \right) \right).$$

(78)

V. CONCLUSIONS

In the canonical approach of field quantum theory, the original Casimir effect, may be viewed as the effect of changes in the vacuum modes associated to the quantized electromagnetic field by the presence of boundaries. The literature have been discussed a scenario where classical fluctuations play the role of the quantum vacuum modes mimetizing the original Casimir conceptual framework.
In a confined system with quenched disorder, a sensitivity to the boundaries may arise, where the distance to the critical situation is given by some non-thermal control parameter.

Here we discussed the application of the spectral and distributional zeta-function methods to describe fluctuation-induced forces arising from a quenched disorder field in a continuous Landau-Ginzburg model, inspired in the statistical Casimir effect. We discuss a series representation of the quenched free energy. Each moment defines an order parameter-like quantity with a specific correlation length of the fluctuations. Next we discussed the contribution to the fluctuation-induced force generated by specific moments of the partition function, where the fluctuations of the order parameter-like quantity are sensitive to the boundaries. This occurs when the correlation length of the fluctuations of some order parameters-like quantity are sensitive to the boundaries. This happens in the statistical Casimir effect. We discuss a series representation of the average free energy, it is possible to obtain the force between the boundaries, due to the interaction of the critical fluctuations generated by the moment of the partition function, with the largest correlation length of the fluctuations.

Assuming a \(d\)-dimensional slab geometry \(\mathbb{R}^{d-1} \times [0, L]\) we study Dirichlet, Neumann and periodic boundary conditions in an unified way. Using a Gaussian approximation, and the result that for the case of flat boundaries, the effective action defined by the spectral zeta function method and the Casimir energy coincides, we discuss the quenched free energy of the system using an analytic regularization procedure. Using the symmetry of a modular form, we study the cases of all four boundary conditions. The structure of the divergences of the spectral zeta-function as a function of the dimension of the space is presented. We show that it is not possible to define the fluctuation-induced force to the Neumann Laplacian using an analytic regularization procedure. The sign of the induced-force for the case of Dirichlet boundary conditions depends on the sign of \(|\zeta_{d}(−\frac{1}{2}, k_1)|\) and also from \(|\frac{m^2}{\pi^2}|\), being odd or even. We can therefore conclude that the sign of the fluctuation-induced force depends on the strength of the control parameter. In other words, varying continuously the intensity of the non-thermal control parameter, i.e., \(\sigma^2\), the induced force can be repulsive or attractive between the boundaries.

With respect to the results obtained in this work, a remark is appropriate. The situation similar to the ratio \(|\frac{m^2}{\pi^2}|\) appears in the Casimir effect, associated to electromagnetic fields. For the case of two boundaries, one a dielectric surface and the second one a permeable surface with large dielectric constant \(\epsilon\) and large permeability \(\mu\) respectively, the transition between the attractive or repulsive behaviour depends on the ratio \(\sqrt{\frac{\epsilon}{\mu}}\). A natural continuation of this work is to discuss the generalized Heisenberg ferromagnet with a \(N\)-dimensional order parameter \([100]\), defined on a slab \(\mathbb{R}^{d-1} \times [0, L]\), invariant under the \(O(N)\) symmetry group in the presence of quenched disorder with strength \(\sigma\). For \(d > 2\), for instance, where large correlation lengths may appear, one can discuss the fluctuation-induced force between boundaries in a Bose-Einstein condensate in the presence of disorder \([107]\). This subject is under investigation by the authors.

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**Appendix A: The distributional zeta-function method**

The usual approach is to define zeta functions in terms of countable collection of numbers, as for example prime numbers, length of closed paths, etc. The starting point of our derivation is to use the definition of the distributional zeta-function \(\Phi(s, j)\), inspired in the spectral zeta-function, as

\[
\Phi(s, j) = \int [dh] P(h) \frac{1}{Z(j, h)^s}, \quad (A1)
\]

for \(s \in \mathbb{C}\), this function being defined in the region where the above integral converges. Note that \(\Phi(s, j)\) is a function of \(s\) and a functional of the external source \(j(x)\) introduced by convenience. The disordered-averaged generating functional \(\mathbb{E}[W(j, h)]\) can be written as

\[
\mathbb{E}[W(j, h)] = -(d/ds)\Phi(s, j)|_{s=0^+}, \quad \text{Re}(s) \geq 0, \quad (A2)
\]

where \(\Phi(s, j)\) is well defined. To proceed, we use Euler’s integral representation for the Gamma function given by

\[
\frac{1}{Z(j, h)^s} = \frac{1}{\Gamma(s)} \int_0^\infty dt t^{s-1}e^{-Z(j, h)t}, \quad \text{for Re}(s) > 0.
\]

Although the above Mellin integral converges only for \(\text{Re}(s) > 0\), as \(Z(j, h) > 0\), we will show how to obtain from the above expression a formula for the generating functional valid for \(\text{Re}(s) \geq 0\). Substituting Eq. (A3) in Eq. (A1) we get

\[
\Phi(s, j) = \frac{1}{\Gamma(s)} \int [dh] P(h) \int_0^\infty dt t^{s-1}e^{-Z(j, h)t}. \quad (A4)
\]

We already know that the distributional zeta-function \(\Phi(s, j)\) is defined for \(\text{Re}(s) \geq 0\). Now we will use the above expression for computing its derivative at \(s = 0\) by analytic tools. We assume at principle the commutativity
of the following operations, disorder average, differentiation, integration if necessary. To continue, take \( a > 0 \) and write \( \Phi = \Phi_1 + \Phi_2 \) where

\[
\Phi_1(s, j) = \frac{1}{\Gamma(s)} \int [dh] P(h) \int_0^a dt \, t^{s-1} e^{-Z(j, h)^t} \tag{A5}
\]

and

\[
\Phi_2(s, j) = \frac{1}{\Gamma(s)} \int [dh] P(h) \int_a^\infty dt \, t^{s-1} e^{-Z(j, h)^t}, \tag{A6}
\]

where \( a \) is a dimensionless parameter. The disorder average generating functional can be written as

\[
\mathbb{E}[W(j, h)] = \frac{d}{ds} \Phi_1(s, j) \bigg|_{s=0^+} - \frac{d}{ds} \Phi_2(s, j) \bigg|_{s=0^+}. \tag{A7}
\]

Let us define the integer moment of the generating functional of correlation functions, \( \mathbb{E} \left[ (Z(j, h))^k \right] \), where

\[
\mathbb{E} \left[ (Z(j, h))^k \right] = \int [dh] P(h) (Z(j, h))^k. \tag{A8}
\]

The integral \( \Phi_2(s, j) \) defines an analytic function defined in the whole complex plane. The contribution of \( \Phi_1(s, j) \) reads

\[
\Phi_1(s, j) = \frac{a^s}{\Gamma(s + 1)} + \frac{1}{\Gamma(s)} \sum_{k=1}^\infty \frac{(-1)^k a^{k+s}}{k!(k+s)} \mathbb{E} \left[ (Z(j, h))^k \right]. \tag{A9}
\]

an expression valid for \( \text{Re}(s) \geq 0 \). The function \( \Gamma(s) \) has a pole at \( s = 0 \) with residue 1, therefore

\[
\left. \frac{d}{ds} \Phi_1(s; j) \right|_{s=0^+} = \sum_{k=1}^\infty \frac{(-1)^k a^k}{k!} \mathbb{E} \left[ (Z(j, h))^k \right] + f(a), \tag{A10}
\]

where

\[
f(a) = \left. \frac{d}{ds} \left( \frac{a^s}{\Gamma(s + 1)} \right) \right|_{s=0^+} = (\log a + \gamma) \tag{A11}
\]

and \( \gamma \) is Euler’s constant \( 0.577 \ldots \) The derivative of \( \Phi_2 \) in Eq. (A6) is given by

\[
\left. \frac{d}{ds} \Phi_2(s; j) \right|_{s=0^+} = \int [dh] P(h) \int_a^\infty dt \, e^{-Z(j, h)^t} = -R(a, j). \tag{A12}
\]

Hence, using analytic tools, and integrating over the disorder, we get a new representation for the disordered-averaged generating functional \( W(j, h) \) given by

\[
\mathbb{E}[W(j, h)] = \sum_{k=1}^\infty \frac{(-1)^k a^k}{k!} \mathbb{E} \left[ (Z(j, h))^k \right] - \log a - \gamma + R(a, j). \tag{A13}
\]

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