Spin light of a neutron in classical and quantum theories

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Abstract. In this paper, we try to give a systematic exposition of the classical and quantum description of relativistic neutrons radiation. Here we show that the correspondence principle in the spin precession entails the complete agreement of the classical and quantum properties of radiation (total power, angular distribution, linear and circular polarization in a full accordance with the results of well-known quantum theory of A. Ternov, V. Bagrov, A. Khapaev. Finally, we give estimates for the time and length of radiation self-polarization of the neutrons.

1 Introduction
Spin light is a radiation associated with the spin precession of the proper magnetic moment of elementary particles [1].

In the classical theory the spin precession of the neutron with a magnetic moment \( \mu_N = -|\mu| = -1,93|\mu_{nucl}| < 0 \) in the uniform magnetic field \( \vec{H} = (0,0,H) \) is described by solution of Bargmann-Michel-Telegdi (BMT) equation [2]

\[
\Pi^{\mu\nu} = \Pi_1^{\mu\nu} - \frac{1}{\Omega} q^{[\mu\alpha}\Pi_1^{\nu\alpha]} \sin \Omega \tau - \left( 2\Pi_1^{\mu\nu} + \frac{1}{\Omega^2} q^{[\mu\alpha} q_{\alpha\beta}\Pi_1^{\beta\nu]} \right) (1 - \cos \Omega \tau) \alpha\alpha.
\]

Here \( \Pi^{\mu\nu} \) is a dimensionless spin tensor with the invariant

\[
\frac{1}{2} \Pi_{\mu\nu} \Pi^{\mu\nu} = \frac{1}{2} \Pi_{\mu\nu} \Pi_1^{\mu\nu} = 1.
\]

Index "1" corresponds to the initial conditions, a brackets denote an antisymmetrization \( a^{[\mu\nu]} = a^{\mu\nu} - a^{\nu\mu} \), \( \Omega \) is a neutron spin precession frequency in the units of the proper time \( \tau \):

\[
\Omega = \frac{2|\mu|H}{\hbar}\sqrt{1 - \beta^2 \cos^2 \alpha} = \omega_\gamma,
\]

when it moves in accordance with \( \vec{\beta} = \beta(sina,0,cosa) \), the tensor \( q^{\mu\nu} \) is space-like part of the tensor \( h^{\mu\nu} = -2(|\mu|/\hbar)H^{\mu\nu} \).

Description of the spin precession in quantum theory is given by a nonstationary wave function \( \Psi(\vec{r},t) \) of Dirac-Pauli equation, and the initial conditions are given by the formula (see [3])

\[
\hat{\Pi} \psi(\vec{r},0) = \lambda \psi(\vec{r},0).
\]

In the following it is showed that for the same initial conditions the identical solutions for the spin precession as in classical theory can be obtained.
2 Classical theory of neutron radiation

The classical theory of neutron spin light was built by V.A. Bordovitsyn et al [2]. Here we give a comparison of this theory with the quantum theory of radiation of the neutron.

2.1 Total power of radiation

At the heart of the classical theory of radiation is the field stress tensor in the wave zone [4]

$$\tilde{H}_{\mu\nu} = -|\mu|c \varepsilon \tilde{\Pi}_{[\mu} \tilde{\Pi}_{\nu]} \tilde{r}^3 \sim \frac{1}{\tilde{r}^3}. $$

Here $\tilde{r}^\mu = (\tilde{r}, \tilde{r})$ is the light-like vector drawn from the world point of neutron to the observer at time $\tilde{t}$, the dot denotes the derivative with respect to the proper time $\tau$. In addition, there are $\tilde{H}$ and $\tilde{\Pi}$ tensors of the mixed and convective fields correspondingly.

Knowing the stress tensor fields, we can construct a tensor density of energy in the wave zone

$$\tilde{\mathcal{E}}_{\mu\nu} = -\frac{1}{4\pi} (\tilde{H}_{\mu\rho} \tilde{H}_{\nu} - \frac{1}{4} g_{\mu\nu} \tilde{H}_{\alpha\beta} \tilde{H}^{\alpha\beta})$$

or

$$\tilde{\mathcal{E}}_{\mu\nu} = -\frac{c^2 \tilde{r}_\alpha \tilde{\Pi}^{\alpha\beta}}{4\pi} \tilde{r}_\beta \tilde{r}^\nu. $$

To obtain the radiation power in the most general form, we will proceed from the expression

$$\tilde{P}_\mu = \frac{1}{c} \oint \tilde{\mathcal{E}}_{\mu \nu} d\sigma_\nu. $$

Here $d\sigma_\nu$ is the element of the spatial hypersurface: $d\sigma_\nu = \varepsilon^2 c_{\nu} d\Omega_0 d\tilde{r} \tau$, $\varepsilon = \tilde{r} \tau [1 - (\tilde{r}^3 \tilde{\beta})] = -\tilde{r}_\nu \tilde{r}^\nu / c$; $\tilde{n}$ is the unit vector along the direction of wave motion; $c_{\nu}$ is a unit space-like vector of Rohrlich; $c_{\nu} \varepsilon^\nu = 1$; $c_{\nu} \nu^\nu = 0$; $\tilde{r}^\mu = \varepsilon (\tilde{r}^\mu + \nu^\mu / c)$; $d\Omega_0$ is the element of solid angle in a rest frame; $d\Omega_0 = c^2 / (n_{\nu} \nu^\nu)^2 d\Omega$; in addition $\Pi^{\nu\rho} \nu_\alpha = 0$ and $\Pi^{\nu\rho} \nu_\alpha = 0$ due to the property of $\Pi^{\rho\sigma}$ tensor.

Further we provide the calculation using the technique of covariant integration [2] and guiding by the subsidiary equations above. Thus we obtain a general expression for the power of radiation

$$\frac{d\tilde{P}_\mu}{d\tau} = \frac{\mu^2 v^\mu}{3c^3} \tilde{\Pi}_{\alpha\beta} \tilde{\Pi}^{\alpha\beta}. $$

(3)

Next we consider the particular type of spin precession. Expression of $\tilde{\Pi}_{\alpha\beta}$ is obtained from the classical solution of the BMT

$$\begin{cases} 
\Pi_x = -|\zeta| \sin \omega t / \sqrt{1 - \beta^2 \cos^2 \alpha}, \\
\Pi_y = |\zeta| \gamma \cos \omega t, \\
\Pi_z = 0. 
\end{cases}$$

Here the initial condition for a spin vector $\zeta_{\mu=0} = (0, \zeta, 0)$ It should be noted, that this case is equal to the initial spin direction parallel to the $Y$ axis as in the quantum theory (see the condition (2)). After averaging over the time from (3) the following formula for the total power of radiation is obtained

$$\tilde{W} = \frac{2^5 \mu^6 H^4 (1 - \beta^2 \cos^2 \alpha)^2}{3c^7 h^4 (1 - \beta^2)^2}, $$

(4)

which is in a full accordance with the result of quantum theory of Ternov-Bagrov-Khapaev [5].
2.2 Angular distribution and polarization

To study the polarization properties, we use the following considerations. The angular distribution \(dW/d\Omega\) can be obtained with respect to formula [2]

\[
dW = \frac{r^2 c}{4\pi} \int E^2 (1 - (\vec{n} \vec{\beta}))d\Omega,
\]

here \(\vec{E} = \sqrt{E_\theta^2 + E_\varphi^2}\) is the electromagnetic field strength tensor, obtained from the tensor of the radiation field by the rule \(E_s = (\vec{E} \vec{n}_s)\), where index \(s = \theta, \varphi\) and \(\vec{n}_s\) are the vectors perpendicular to the normal vector \(\vec{n} = [\vec{n}_\theta \vec{n}_\varphi]\) (these vectors define the polarization). In the explicit form

\[
\vec{n}_\theta = (\cos \theta \cos \varphi, \cos \theta \sin \varphi, -\sin \theta), \quad \vec{n}_\varphi = (-\sin \varphi, \cos \theta, 0), \quad \vec{n} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta).
\]

The relation between \(\vec{n}\) and \(E_s\)

\[
E_s = -|\mu| \frac{\vec{n}[\vec{n}[(\vec{n} - \vec{\beta})\vec{H}]]}{c^2 \gamma^3 r (1 - \vec{n} \vec{\beta})^3}.
\]

Finally, to obtain the angular distribution in a more convenient form the transformation of rotation of the coordinate axes should be done to let the \(\vec{\beta}\) axis coincides with the \(X'\) axis (right-hand turn around the \(Y\) axis)

\[
\vec{n}' = (\sin \theta' \cos \varphi', \sin \theta' \sin \varphi', \cos \theta'),
\]

\[
\vec{n} = (\cos \alpha \sin \theta' \cos \varphi' + \sin \alpha \cos \theta' \cos \varphi', \sin \theta' \sin \varphi', -\sin \alpha \sin \theta' \cos \varphi' + \cos \alpha \cos \theta').
\]

Thus, one can obtain the angular distribution of the radiation with respect to the certain polarization

\[
\frac{dW_s}{d\Omega} = \frac{8\mu^6 H^4 (1 - \beta^2 \cos^2 \alpha)^2}{\pi c^2 \gamma^4 (1 - \beta \cos \theta')^3} E_s^2.
\]

For the components of linear polarization in the case of an axially symmetric spin-light \((\alpha = 0)\), we have

\[
E_\theta = -\frac{\mu \zeta \Omega^2 (\cos \theta - \beta) \sin(\omega t - \varphi)}{r \gamma^2 c^2 (1 - \beta \cos \theta)^3},
\]

\[
E_\varphi = -\frac{\mu \zeta \Omega^2 \cos(\omega t - \varphi)}{r \gamma^2 c^2 (1 - \beta \cos \theta)^2},
\]

where \(\Omega\) is determined in (1); \(\zeta = \pm 1\). In the case of an axially nonsymmetric spin-light \((\alpha = \pi/2)\) easy to obtain

\[
E_\theta = -\frac{\mu \zeta \Omega^2 \cos \theta (\cos \varphi \sin \omega t - \gamma^2 \sin \varphi \cos \omega t)}{r \gamma^3 c^2 (1 - \beta \sin \theta \cos \varphi)^3},
\]

\[
E_\varphi = -\frac{\mu \zeta \Omega^2 \sin \varphi \sin \omega t + \gamma \cos \omega t (\cos \varphi - \beta \sin \varphi)}{r \gamma^3 c^2 (1 - \beta \cos \theta)^3}.
\]

Then, for the components of the right \((s = 1)\) and the left \((s = -1)\) circular polarization we have correspondingly

\[
E_s = \frac{1}{\sqrt{2}} \frac{\mu \zeta \Omega^2 (1 + s \cos \theta)(1 + s \beta)}{r \gamma^3 c^2 (1 - \beta \cos \theta)^3}.
\]
It should be noted that the components of circular polarization can be obtained from the equations of motion automatically. One can see that the angular distribution ($\alpha = 0$) has an azimuthal symmetry in contrast to the angular distribution ($\alpha = \pi/2$).

After integrating over angles and after averaging over time we obtain correspondingly the following expressions for the total radiation power

$$W = \frac{2^5 \mu^6 H^4}{3 c^3 h^4},$$

$$W_{\theta} = \frac{4 + 2\beta^2 - 2\beta^4}{16} W,$$

$$W_{\varphi} = \frac{12 - 2\beta^2 + \beta^4}{16} W.$$

For the circular polarization we have

$$W_{\pm 1} = \frac{1}{2} \left( 1 \pm \frac{1}{2} \beta \right) W.$$

From this in the nonrelativistic case one can obtain

$$W_{\theta} = \frac{1}{4} W, \quad W_{\varphi} = \frac{3}{4} W,$$

and correspondingly in the ultrarelativistic case:

$$W_{\theta} = \frac{5}{16} W, \quad W_{\varphi} = \frac{11}{16} W.$$

3 Comparison with the quantum theory of Ternov-Bagrov-Khapaev

Now we show that absolutely all the properties of neutron radiation in classical and quantum theory are the same.

In the classical theory of radiation frequency can be found from the condition of invariance of 4-dimensional scalar product

$$k_{\mu} v^{\mu}|_{\beta \to 0} = k_{\mu 0} v^{\mu 0} = inv.$$

Here $k^{\mu}$ is the 4-dimensional light-like wave vector, $k^{\mu} = \tilde{\omega}/c(1, \tilde{n})$, $k^{\mu} k^{\mu} = 0$, $v^{\mu} = (v^0, \tilde{v})$ is a 4-dimensional velocity; $\tilde{\omega}$ is the radiation frequency,

$$\tilde{\omega} = \frac{\tilde{\omega}_0}{\gamma(1 - (\tilde{n} \tilde{\beta}))}.$$  \hspace{1cm} (5)

Thus, in the rest frame radiates the fundamental frequency is radiated. It coincides with the mechanical precession frequency. The relation between radiation frequency in the rest frame $\tilde{\omega}_0$ and precession frequency $\omega$ is the following

$$\tilde{\omega}_0 = \omega \gamma = \Omega.$$

Let us turn to the laboratory frame. On the basis of (5) we get

$$\tilde{\omega} = \frac{\omega}{1 - (\tilde{n} \tilde{\beta})} = \frac{2|\mu|H}{\hbar} \sqrt{\frac{1 - \gamma^2 \cos^2 \alpha}{1 - \beta \cos \theta' \cos \theta}}.$$  \hspace{1cm} (6)
Now we show that the same formula can be obtained in the relativistic quantum theory. Thereby, in the quantum theory radiation frequency can be obtained on the basis of conservation laws of energy and momentum (in the dimensionless form), taking into account the spin states

\[
\begin{align*}
\gamma_\zeta &- \gamma_\zeta' = \hbar \omega/m_0 c^2, \\
\vec{b} - \vec{b}' &= \hbar \vec{n}/m_0 c^2,
\end{align*}
\]

here \(\gamma_\zeta\) is obtained from the solution of Dirac-Pauli equation (with respect to spin). It is easy to see that

\[\vec{b} = 2\gamma_\zeta \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \vec{b}'; \quad \gamma_\zeta = \hbar \omega/m_0 c^2;
\]

which is exactly the same with (6), when \(\zeta = 1\). This corresponds to the established in [5] selection rule: only a neutron spin of which was oriented in the direction of the magnetic field emits. In this case the radiation necessarily accompanied by a change of spin direction.

All the essential features of the neutron radiation follows the intermediate relation, derived from the relation

\[
\frac{|\mu| H}{m_0 c^2} (\zeta \sqrt{1 - \beta^2 \cos^2 \alpha} - \zeta' \sqrt{1 + \beta^2 \cos^2 \alpha}) = \frac{\hbar \omega \gamma_\zeta}{m_0 c^2} (1 - \beta \cos \theta') > 0. \tag{7}
\]

Let's turn to the rest frame \((\beta \rightarrow 0)\). Formula (7) becomes the following (this condition is satisfied only if \(\zeta = -\zeta' = 1\))

\[
\frac{|\mu| H}{m_0 c^2} (\zeta - \zeta') = \frac{\hbar \omega \gamma_\zeta}{m_0 c^2} > 0; \quad \zeta = -\zeta' = 1.
\]

In the rest frame the relation between the power of radiation \(W_0\) and the total probability of the radiation \(w_0\) is

\[W_0 = \hbar \omega_0 w_0.
\]

where

\[w_0 = \frac{32}{3} \frac{|\mu|^5 H_0^3}{c^2 \hbar^4} |< \zeta'| \bar{\sigma} |\zeta >|^2,
\]

and the matrix element built on Pauli matrices \(\bar{\sigma}\) is easily to obtained by suitable spinors

\[< \zeta'| \bar{\sigma} |\zeta > = \frac{1 - \zeta \zeta'}{2} (\vec{r} + i \vec{s}) + \frac{1 + \zeta \zeta'}{2} (\vec{k} \bar{\zeta}). \tag{8}
\]

The first member of the right part of (8) corresponds to the spin-flip transitions \((\zeta = -\zeta')\), and the second one corresponds to the transitions without flip \((\zeta = \zeta')\). Then we take into account the known fact that here all the transitions are spin-flip transitions [5]. It leads to

\[|< \zeta'| \bar{\sigma} |\zeta >|^2 = 2 \frac{1 - \zeta \zeta'}{2} + \frac{1 + \zeta \zeta'}{2} |_{\zeta = -\zeta'} = 2,
\]

and

\[W_0 = \frac{2^7}{3} \frac{|\mu|^5 H_0^3}{c^2 \hbar^4}.
\]

Such result we can see in works [5] and [6]. However, comparing with (4) one can find a two orders difference in the coefficient. This is due to the fact that in classical theory, we performed a time averaging. While the authors of articles below didn’t performed it. Accordingly there is a need to average over initial states and summation over final

\[
\sum_{\zeta \zeta'} \frac{1 - \zeta \zeta'}{2} = \frac{1}{2} \sum_{\zeta \zeta'} \frac{1 + \zeta \zeta'}{2} = \frac{1}{4}.
\]

It is easy to see that after this procedure all of the differences in the coefficients are vanished. Other features are exactly the same.
4 Radiative self-polarization of neutrons

This effect was first discovered for relativistic electrons in the quantum theory of synchrotron radiation, and later confirmed experimentally (see, for example, [7]). In the case of neutron spin-light radiation self-polarization time in quantum theory is determined by the reciprocal of the total probability of radiation [6].

\[ T_0 = \frac{1}{w_0} = \frac{3}{32} \frac{c^3 \hbar^4}{|\mu|^3 H_0^3} \frac{1}{\langle \zeta' | \vec{\sigma} | \zeta \rangle^2} = \frac{3}{64} \frac{c^3 \hbar^4}{|\mu|^3 H_3^3}. \] (9)

To get the same effect in the classical theory let us deal with the rest frame where the spin precession leads to the minimum potential energy:

\[ U = -\langle \vec{\mu} \vec{H} \rangle_{\beta = 0} = \langle \mu_0 \vec{H}_0 \rangle = \mu_0 H_0 \cos \lambda |_{\lambda \to \pi} \to \min, \]

when the neutron spin is oriented against the direction of the magnetic field; \( \lambda \) is the angle between the direction of magnetic field and magnetic moment. Thus, the magnetic moment and also spin are oriented against the direction of magnetic field.

Considering the process of radiation as a non stationary in the rest frame where \( \vec{\mu}|_{\beta = 0} = -\mu_0 \zeta = -\mu_0 (\zeta || + \zeta \perp) \) we obtained the following equation for the radiative self-polarization [8]

\[ \frac{d\zeta ||}{d\tau} = \frac{1}{2T_0} (1 - \zeta ||^2), \]

The solutions of this equation for an initially unpolarized neutron leads to

\[ \zeta ||(\tau) = \frac{\exp(-\tau/T_0)}{\exp(-\tau/T_0) + 1} \bigg|_{\tau \to \infty} \to 1, \]

where \( T_0 \) is given by (9). Taking into account the time of radiative self-polarization for the neutron with the velocity \( \vec{\beta} = \beta (\sin \alpha, 0, \cos \alpha) \) we find the expression

\[ T = \frac{3}{64} \frac{c^3 \hbar^4}{|\mu|^3 H_3^3} \frac{1}{2 \gamma^2 (1 - \beta^2 \cos^2 \alpha)^{3/2}}. \]

To sum up, for a relativistic \( 10^{13} \text{GeV} \) neutron and \( H = 10^6 \text{Gs} \), \( \alpha = \pi/2 \) we find the time \( T \approx 1.3 \cdot 10^{-7} \text{sec} \) and length of the self-polarization \( L \approx 4 \cdot 10^4 \text{sm} \). However, as shown by this calculation, for the neutron to compared with the electron this effect is small and becomes noticeable only at high magnetic fields and at very high energies. The classical and the quantum theories of the spin light and radiative self-polarization of the neutron fully adequate to each other.

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