Relativistic Bound States in 2+1 and 1+1 Dimensions in the Null-Plane \(^a\)

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The Faddeev-like equation for the component of the three-boson vertex for a relativistic contact interaction is

\[
v(q^\mu) = 2\tau(M^2) \int \frac{d^n k}{(2\pi)^n} \frac{i}{k^2 - M^2 + i\varepsilon} \frac{i}{(P_3 - q - k)^2 - M^2 + i\varepsilon} v(k^\mu), \tag{1}\]

where \(P_3^\mu = (M_3^2, 0, 0, 0)\) is the three-boson four-momentum in the center of mass system, \(n\) is the dimension of the space-time, the mass of the two-boson subsystem is given by \(M_2^2 = (P_3 - q)^2\) and the single boson mass is \(M\). The factor 2 comes from symmetrization of the total vertex.

The total three-boson vertex is the sum of three Faddeev components in which each boson is spectator once.

The two-boson scattering amplitude, \(\tau(M_2)\), enters in the kernel of the integral equation for the vertex in Eq.(1). It is easily obtained as:

\[
\tau^{(n)}(M_2) = \left\{i\lambda^{-1} - B^{(n)}(M_2)\right\}^{-1}, \tag{2}\]

where \(\lambda\) is the coupling constant of the zero-range interaction, \(M_2\) is the mass of the two boson system and \(B(M_2)\) is the kernel of the integral equation for the scattering amplitude

\[
B^{(n)}(M_2) = -\int \frac{d^n k}{(2\pi)^n} \left\{ (k^2 - M^2 + i\varepsilon) \left( (P - k)^2 - M^2 + i\varepsilon \right) \right\}^{-1}, \tag{3}\]

where \(M\) is the boson mass and \(P_\mu = (M_2, 0, 0, 0)\).

The value of \(\lambda\) is chosen such that the two-boson system has one bound-state. The scattering amplitude, Eq.(2), has a pole at the bound-state mass

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\( \lambda^{-1} = B^{(n)}(M_{2B}). \) \( \text{(4)} \)

In two dimensions the two-boson scattering amplitude for \( M_{2B} < 2M \) is
\[
\tau^{(2)}(M_2) = -2\pi i \left\{ \frac{\text{atan} \left( \frac{2\beta(M_{2B})}{M_{2B}^2\beta(M_{2B})} \right)}{M_{2B}^2\beta(M_{2B})} - \frac{\text{atan} \left( \frac{2\beta(M_2)}{M_2^2\beta(M_2)} \right)}{M_2^2\beta(M_2)} \right\}^{-1}, \quad \text{(5)}
\]
where \( \beta(M_2) = \sqrt{\frac{M_2^2}{M_{2B}} - 1} \).

In three dimensions the scattering amplitude is
\[
\tau^{(3)}(M_2) = -8\pi i \left\{ M_{2B}^{-1} \ln \left( \frac{2M + M_{2B}}{2M - M_{2B}} \right) - M_2^{-1} \ln \left( \frac{2M + M_2}{2M - M_2} \right) \right\}^{-1}. \quad \text{(6)}
\]
For our purpose of the bound-state calculation is enough to know \( \tau^{(n)}(M_2) \) for \( M_2 < 2M \).

The momentum variables in the integral equation are the momenta in the null-plane for an on-mass-shell particle, \( q^+ \) and \( q_\perp \). The transversal momentum is needed in three space-time dimensions.

Let us discuss the limits of the variables \( y = \frac{q^+}{M_{3B}} \) and \( q_\perp \). In 1+1 space-time dimensions only the momentum fraction is enough to describe the spectator boson. The mass of the two-boson subsystem must be real and in 1+1 dimensions it implies
\[
(M_2)^2 = (M_{3B} - q^+) \left( M_{3B} - \frac{M_2^2}{q^+} \right) > 0. \quad \text{(7)}
\]
From the above inequality follows \( 1 > y > \frac{M_2^2}{M_{2B}^2} \).

In 2+1 dimensions, we deduce the range of values of the perpendicular momentum allowed by the reality of the mass of the two-boson subsystem. Then
\[
(M_2)^2 = (M_{3B} - q^+) \left( M_{3B} - \frac{q_\perp^2 + M^2}{q^+} \right) - q_\perp^2 > 0. \quad \text{(8)}
\]
Solving the inequality for \( q_\perp^2 \), we obtain \( q_\perp^2 < (1 - y)(M_{2B}^2y - M_2^2) \). The limits for \( y \) are \( 1 > y > \frac{M^2}{M_{3B}^2} \), and the lower bound comes from \( q_\perp^2 > 0 \).

The equation for the Faddeev component of the vertex in 1+1 space-time dimensions is obtained as the result of the \( k^- \) integration in the momentum
loop of Eq. (1). We also use Eq. (5) and the limit in the internal momentum fraction $x$

$$v(y) = \frac{i}{2\pi} \tau^{(2)}(M_2) \int_{M_2^2 \pi^2}^{1-y} \frac{dx}{x(1-y-x)} \frac{v(x)}{M_2^2 - M_0^2}.$$  \hspace{1cm} (9)

where $(M_2)^2$ is given by Eq. (8) and the free mass of the virtual three-boson state in 1+1 dimensions is:

$$M_{03}^2 = \frac{M^2}{x} + \frac{M^2}{y} + \frac{M^2}{1-y-x}.$$ \hspace{1cm} (10)

The equation for the Faddeev component of the vertex in 2+1 space-time dimensions is found after the $k^-$ integration of in the momentum loop of Eq. (1),

$$v(y, \vec{q}_\perp) = \pi^{-2} \tau^{(3)}(M_2) \int_{M_2^2 \pi^2}^{1-y} \frac{dx}{x(1-y-x)} \int_{-k_{\max}^\perp}^{k_{\max}^\perp} d^2k_\perp \frac{v(x, k_\perp)}{M_{2B}^2 - M_{03}^2},$$ \hspace{1cm} (11)

where $M_2$ is given by Eq. (8), as well as $k_{\max}^\perp = \sqrt{(1-x)(M_{2B}^2 - M^2)}$. The mass of the virtual three-boson state is:

$$M_{03}^2 = \frac{k_\perp^2 + M^2}{x} + \frac{q_\perp^2 + M^2}{y} + \frac{(q + k)^2}{1-y-x}.$$ \hspace{1cm} (12)

The dependence of $v$ on $q^-$ is not specified because the spectator boson is on mass-shell. $q^+$ and $q_\perp$ describe the spectator boson propagation. The relativistic equations in 1+1 and 2+1 dimensions, have a lower bound for the mass of the three-boson system which comes from the limits on the $x$ integration and the condition $y > \frac{M_{2B}}{M_{2B}^2}$ which implies $M_{3B} > \sqrt{2M}$. The same limit was obtained in 3+1 dimensions in [1].

The Faddeev component of the ground state vertex, in 2+1 dimensions is rotationally symmetric in the x-y plane in Eq. (11). The mass of the boson gives the scale of the system. Here, the solution is presented for $M = 1$. In Fig. (1), the numerical results for the ground-state binding energies ($E_{3B} = M_{2B} + M - M_{3B}$) of the three-boson system are shown in two and three-dimensions. In the nonrelativistic limit, for $E_{2B} = 0$, the results approach the well-known values in [3].

In summary, we give an example of how null-plane dynamics can be elaborated. We develop a zero-range model of the three-boson bound state in the null-plane and solve numerically the dynamical equation for the ground-state in 1+1 and 2+1 space-time dimensions.
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Figure 1: The ratio between the three-boson and two-boson binding energies as a function of the two-boson binding energy in units of the mass of the single boson. Results for 2+1 (solid line) and 1+1 (dashed line).