General fractional integral inequalities for convex and $m$-convex functions via an extended generalized Mittag-Leffler function

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Abstract

In this paper some new general fractional integral inequalities for convex and $m$-convex functions by involving an extended Mittag-Leffler function are presented. These results produce inequalities for several kinds of fractional integral operators. Some interesting special cases of our main results are also pointed out.

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1 Introduction, definitions, and preliminaries

Convex functions are very important in the field of integral inequalities. A lot of fractional integral inequalities and novel results have been established due to convex functions (for more details, see [1, 8, 13, 14]).

Definition 1 A function $f : I \rightarrow \mathbb{R}$, where $I$ is an interval in $\mathbb{R}$, is said to be a convex function if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \quad (1)$$

holds for $t \in [0,1]$ and $x, y \in I$.

A convex function $f : I \rightarrow \mathbb{R}$ is also equivalently defined by the Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_{a}^{b} f(x) dx \leq \frac{f(a) + f(b)}{2},$$

where $a, b \in I$, $a < b$.

The concept of $m$-convexity was introduced in [17] and since then many properties, especially inequalities, have been obtained for this class of functions (see [3, 6, 7, 18]).

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**Definition 2** A function \( f : [0, b] \rightarrow \mathbb{R}, b > 0 \) is called \( m \)-convex, where \( m \in [0, 1] \), if for every \( x, y \in [0, b] \) and \( t \in [0, 1] \), we have
\[
 f(tx + m(1 - t)y) \leq tf(x) + m(1-t)f(y).
\]

For \( m = 1 \), we recapture the definition of convex functions, and for \( m = 0 \), the definition of star-shaped functions defined on \([0, b]\). We recall that a function \( f : [0, b] \rightarrow \mathbb{R} \) is called *star-shaped* if
\[
 f(tx) \leq tf(x) \quad \text{for all } t \in [0, 1] \text{ and } x \in [0, b].
\]

If we denote by \( K_m(b) \) the set of \( m \)-convex functions defined on \([0, b]\) for which \( f(0) < 0 \), then
\[
 K_1(b) \subset K_m(b) \subset K_0(b),
\]
whenever \( m \in (0, 1) \). Note that in the class \( K_1(b) \) there are only convex functions \( f : [0, b] \rightarrow \mathbb{R} \) for which \( f(0) \leq 0 \) (see [4]), while \( K_0(b) \) contains *star-shaped* functions.

**Example 1.1 ([6])** The function \( f : [0, \infty) \rightarrow \mathbb{R} \), given by
\[
 f(x) = \frac{1}{12} (x^4 - 5x^3 + 9x^2 - 5x),
\]
is a \( \frac{16}{17} \)-convex function but it is not \( m \)-convex for any \( m \in (\frac{16}{17}, 1] \).

For more results and inequalities related to \( m \)-convex functions, one can consult, for example, [3, 6, 7] along with the references therein.

Recently in [2] Andrić et al. defined an extended generalized Mittag-Leffler function \( E_{\mu,\alpha,l,c}^{\gamma,\lambda,k,e}(t;p) \) as follows.

**Definition 3** ([2]) Let \( \mu, \alpha, l, \gamma, c \in \mathbb{C}, \Re(\mu), \Re(\alpha) > 0, \Re(l) > 0, \Re(c) > \Re(\gamma) > 0 \) with \( p > 0, \delta > 0 \), and \( 0 < k \leq \delta + \Re(\mu) \). Then the extended generalized Mittag-Leffler function \( E_{\mu,\alpha,l,c}^{\gamma,\lambda,k,e}(t;p) \) is defined by
\[
 E_{\mu,\alpha,l,c}^{\gamma,\lambda,k,e}(t;p) = \sum_{n=0}^{\infty} \beta_p(\gamma + nk, c - \gamma) \frac{(c)_{nk}}{\Gamma(\mu n + \alpha)} \frac{t^n}{n!},
\]
where \( \beta_p \) is the generalized beta function defined by
\[
 \beta_p(x,y) = \int_0^1 t^{x-1} (1-t)^{y-1} e^{-pt} \, dt
\]
and \((c)_{nk}\) is the Pochhammer symbol defined as \((c)_{nk} = \frac{\Gamma(c+nk)}{\Gamma(c)}\).

In [2] properties of the generalized Mittag-Leffler function are discussed, and it is given that \( E_{\mu,\alpha,l,c}^{\gamma,\lambda,k,e}(t;p) \) is absolutely convergent for \( k < \delta + \Re(\mu) \). Let \( S \) be the sum of series of absolute terms of the Mittag-Leffler function \( E_{\mu,\alpha,l,c}^{\gamma,\lambda,k,e}(t;p) \), then we have \(|E_{\mu,\alpha,l,c}^{\gamma,\lambda,k,e}(t;p)| \leq S\).

We use this property of Mittag-Leffler function in our results where we need.
The corresponding left and right sided extended generalized fractional integral operators are defined as follows.

**Definition 4** ([2]) Let \( \omega, \mu, \alpha, l, \gamma, c \in \mathbb{C}, \mathfrak{R}((\mu)), \mathfrak{R}((\alpha)), \mathfrak{R}(l) > 0, \mathfrak{R}(c) > \mathfrak{R}(\gamma) > 0 \) with \( p \geq 0, \delta > 0 \) and \( 0 < k \leq \delta + \mathfrak{R}(\mu) \). Let \( f \in L_1[a,b] \) and \( x \in [a,b] \). Then the extended generalized fractional integral operators \( \epsilon_{\mu,a,l,\omega,a}^{\gamma,k,c}f \) and \( \epsilon_{\mu,a,l,\omega,b}^{\gamma,k,c}f \) are defined by

\[
(\epsilon_{\mu,a,l,\omega,a}^{\gamma,k,c}f)(x;p) = \int_{a}^{x} (x-t)^{\mu-1} E_{\mu,1}^{\gamma,k,c} (\omega(x-t)^{\mu};p) f(t) \, dt
\]

and

\[
(\epsilon_{\mu,a,l,\omega,b}^{\gamma,k,c}f)(x;p) = \int_{x}^{b} (t-x)^{\mu-1} E_{\mu,1}^{\gamma,k,c} (\omega(t-x)^{\mu};p) f(t) \, dt.
\]

From extended generalized fractional integral operators, we have

\[
(\epsilon_{\mu,a,l,\omega,a}^{\gamma,k,c} 1)(x;p)
= \int_{a}^{x} (x-t)^{\mu-1} E_{\mu,1}^{\gamma,k,c} (w(x-t)^{\mu};p) \, dt
= \int_{a}^{x} (x-t)^{\mu-1} \sum_{n=0}^{\infty} B_p(\gamma+nk,\gamma) \frac{(c)_{nk}}{\Gamma(\mu+n+\alpha)} \frac{w^n(x-t)^{\mu+n}}{(\mu+n+\alpha)} \, dt
= \sum_{n=0}^{\infty} B_p(\gamma+nk,\gamma) \frac{(c)_{nk}}{\Gamma(\mu+n+\alpha)} \frac{w^n}{(\mu+n+\alpha)} \int_{a}^{x} (x-t)^{\mu+n-1} \, dt
= (x-a)^{\mu} \sum_{n=0}^{\infty} B_p(\gamma+nk,\gamma) \frac{(c)_{nk}}{\Gamma(\mu+n+\alpha)} \frac{w^n}{(\mu+n+\alpha)} (x-a)^{\mu+n} \frac{1}{\mu+n+\alpha}.
\]

Hence

\[
(\epsilon_{\mu,a,l,\omega,a}^{\gamma,k,c} 1)(x;p) = (x-a)^{\mu} E_{\mu,1}^{\gamma,k,c} (w(x-a)^{\mu};p),
\]

and similarly

\[
(\epsilon_{\mu,a,l,\omega,b}^{\gamma,k,c} 1)(x;p) = (b-x)^{\mu} E_{\mu,1}^{\gamma,k,c} (w(b-x)^{\mu};p).
\]

We use the following notations in our results:

\[
C_{a,a^{+}}(x;p) = (\epsilon_{\mu,a,l,\omega,a}^{\gamma,k,c} 1)(x;p)
\]

and

\[
C_{a,b^{+}}(x;p) = (\epsilon_{\mu,a,l,\omega,a}^{\gamma,k,c} 1)(x;p).
\]

For more information related to Mittag-Leffler functions and corresponding fractional integral operators, the readers are referred to [9–12, 15, 16, 19].
In this paper we give general fractional integral inequalities for convex and m-convex functions by involving an extended Mittag-Leffler function and deduce some results already published in [1, 5, 6, 8, 13]. Also we give a Hadamard type inequality for convex and m-convex functions by involving an extended Mittag-Leffler function.

2 Main results

Here we give some fractional integral inequalities for convex and m-convex functions via an extended generalized Mittag-Leffler function and corresponding fractional integral operators given in (3) and (4). The following lemma is useful to establish the results.

**Lemma 2.1** Let $f : [a, mb] \rightarrow \mathbb{R}$ be a differentiable function such that $f' \in L_1[a, mb]$ with $0 \leq a < mb$. Also let $g : [a, mb] \rightarrow \mathbb{R}$ be a continuous function on $[a, mb]$, then the following identity for extended generalized fractional integral operators holds:

\[
\left( \int_a^{mb} g(s)E^{\gamma,\delta,k,c}_{\mu,a,l}(\omega s^\mu; \rho) \, ds \right)^{\alpha} \left[ f(a) + f(mb) \right] \\
- \alpha \int_a^{mb} \left( \int_s^t g(s)E^{\gamma,\delta,k,c}_{\mu,a,l}(\omega s^\mu; \rho) \, ds \right)^{\alpha-1} g(t)E^{\gamma,\delta,k,c}_{\mu,a,l}(\omega t^\mu; \rho) f(t) \, dt \\
- \alpha \int_a^{mb} \left( \int_t^{mb} g(s)E^{\gamma,\delta,k,c}_{\mu,a,l}(\omega s^\mu; \rho) \, ds \right)^{\alpha-1} g(t)E^{\gamma,\delta,k,c}_{\mu,a,l}(\omega t^\mu; \rho) f(t) \, dt \\
= \int_a^{mb} \left( \int_a^t g(s)E^{\gamma,\delta,k,c}_{\mu,a,l}(\omega s^\mu; \rho) \, ds \right)^{\alpha} f(t) \, dt \\
- \int_a^{mb} \left( \int_t^a g(s)E^{\gamma,\delta,k,c}_{\mu,a,l}(\omega s^\mu; \rho) \, ds \right)^{\alpha} f(t) \, dt.
\]

(7)

**Proof** On integrating by parts one can have

\[
\int_a^{mb} \left( \int_a^t g(s)E^{\gamma,\delta,k,c}_{\mu,a,l}(\omega s^\mu; \rho) \, ds \right)^{\alpha} f(t) \, dt \\
= \left( \int_a^{mb} g(s)E^{\gamma,\delta,k,c}_{\mu,a,l}(\omega s^\mu; \rho) \, ds \right)^{\alpha} f(mb) \\
- \alpha \int_a^{mb} \left( \int_a^t g(s)E^{\gamma,\delta,k,c}_{\mu,a,l}(\omega s^\mu; \rho) \, ds \right)^{\alpha-1} g(t)E^{\gamma,\delta,k,c}_{\mu,a,l}(\omega t^\mu; \rho) f(t) \, dt
\]

(8)

and

\[
\int_a^{mb} \left( \int_t^{mb} g(s)E^{\gamma,\delta,k,c}_{\mu,a,l}(\omega s^\mu; \rho) \, ds \right)^{\alpha} f(t) \, dt \\
= - \left( \int_a^{mb} g(s)E^{\gamma,\delta,k,c}_{\mu,a,l}(\omega s^\mu; \rho) \, ds \right)^{\alpha} f(a) \\
+ \alpha \int_a^{mb} \left( \int_t^a g(s)E^{\gamma,\delta,k,c}_{\mu,a,l}(\omega s^\mu; \rho) \, ds \right)^{\alpha-1} g(t)E^{\gamma,\delta,k,c}_{\mu,a,l}(\omega t^\mu; \rho) f(t) \, dt.
\]

(9)

 Subtracting (9) from (8), we get (7) which is the required identity. □

If we take $m = 1$ in (7), then we get the following identity for a convex function.
**Corollary 2.2** Let \( f : [a, b] \subseteq [0, \infty) \to \mathbb{R} \) be a differentiable function such that \( f' \in L_1[a, b] \) with \( a < b \). Also let \( g : [a, b] \to \mathbb{R} \) be continuous on \([a, b]\), then the following identity for extended generalized fractional integral operators holds:

\[
\left( \int_a^b g(s)E_{\mu,a,\delta}^{\gamma,k,c}(\cos s; p) \, ds \right)^{\alpha} \left[ f(a) + f(b) \right]
- \alpha \int_a^b \left( \int_a^t g(s)E_{\mu,a,\delta}^{\gamma,k,c}(\cos s; p) \, ds \right)^{a-1} g(t)E_{\mu,a,\delta}^{\gamma,k,c}(\cos t; p) f(t) \, dt
- \alpha \int_a^b \left( \int_t^b g(s)E_{\mu,a,\delta}^{\gamma,k,c}(\cos s; p) \, ds \right)^{a-1} g(t)E_{\mu,a,\delta}^{\gamma,k,c}(\cos t; p) f(t) \, dt
= \int_a^b \left( \int_a^b g(s)E_{\mu,a,\delta}^{\gamma,k,c}(\cos s; p) \, ds \right)^{a} f'(t) \, dt
- \int_a^b \left( \int_t^b g(s)E_{\mu,a,\delta}^{\gamma,k,c}(\cos s; p) \, ds \right)^{a} f'(t) \, dt.
\] (10)

We use identity (7) to establish the following fractional integral inequality.

**Theorem 2.3** Let \( f : [a, mb] \to \mathbb{R} \) be a differentiable function such that \( f' \in L_1[a, mb] \) with \( 0 \leq a < mb \). Also let \( g : [a, mb] \to \mathbb{R} \) be a continuous function on \([a, mb]\). If \( |f'| \) is an \( m \)-convex function on \([a, mb]\), then the following inequality for extended generalized fractional integral operators holds:

\[
\left| \left( \int_a^b g(s)E_{\mu,a,\delta}^{\gamma,k,c}(\cos s; p) \, ds \right)^{a} (f(a) + f(mb)) \right|
- \alpha \int_a^b \left( \int_a^t g(s)E_{\mu,a,\delta}^{\gamma,k,c}(\cos s; p) \, ds \right)^{a-1} g(t)E_{\mu,a,\delta}^{\gamma,k,c}(\cos t; p) f(t) \, dt
- \alpha \int_a^b \left( \int_t^b g(s)E_{\mu,a,\delta}^{\gamma,k,c}(\cos s; p) \, ds \right)^{a-1} g(t)E_{\mu,a,\delta}^{\gamma,k,c}(\cos t; p) f(t) \, dt
\leq \frac{(mb - a)\alpha^{|\alpha|} \|g\|_\infty \|f\|_\infty \alpha}{(\alpha + 1)} \right| (f'(a) + m|f'(b)|)
\] (11)

for \( k < \delta + \Re(\mu) \) and \( \|g\|_\infty = \sup_{t \in [a,mb]} |g(t)| \).

**Proof** From Lemma 2.1, we have

\[
\left| \left( \int_a^b g(s)E_{\mu,a,\delta}^{\gamma,k,c}(\cos s; p) \, ds \right)^{a} (f(a) + f(mb)) \right|
- \alpha \int_a^b \left( \int_a^t g(s)E_{\mu,a,\delta}^{\gamma,k,c}(\cos s; p) \, ds \right)^{a-1} g(t)E_{\mu,a,\delta}^{\gamma,k,c}(\cos t; p) f(t) \, dt
- \alpha \int_a^b \left( \int_t^b g(s)E_{\mu,a,\delta}^{\gamma,k,c}(\cos s; p) \, ds \right)^{a-1} g(t)E_{\mu,a,\delta}^{\gamma,k,c}(\cos t; p) f(t) \, dt
\leq \int_a^b \left( \int_a^t g(s)E_{\mu,a,\delta}^{\gamma,k,c}(\cos s; p) \, ds \right)^{a} |f'(t)| \, dt
+ \int_a^b \left( \int_t^b g(s)E_{\mu,a,\delta}^{\gamma,k,c}(\cos s; p) \, ds \right)^{a} |f'(t)| \, dt.
\] (12)
Using absolute convergence of the Mittag-Leffler function and \( \|g\|_\infty = \sup_{t \in [a,b]} |g(t)| \), we have

\[
\left| \left( \int_a^b g(s)E_{\mu,\alpha,l}^{\gamma,k,c} (\omega s^\nu; p) \, ds \right)^\alpha \right| (f(a) + f(mb))
- \alpha \int_a^b \left( \int_a^t g(s)E_{\mu,\alpha,l}^{\gamma,k,c} (\omega s^\nu; p) \, ds \right)^{a-1} g(t)E_{\mu,\alpha,l}^{\gamma,k,c} (\omega t^\nu; p)f(t) \, dt \\
- \alpha \int_a^b \left( \int_t^b g(s)E_{\mu,\alpha,l}^{\gamma,k,c} (\omega s^\nu; p) \, ds \right)^{a-1} g(t)E_{\mu,\alpha,l}^{\gamma,k,c} (\omega t^\nu; p)f(t) \, dt \\
\leq \|g\|_\infty S^{\alpha} \left( \int_a^b (t-a)^\alpha |f'(t)| \, dt + \int_a^b (mb-t)^\alpha |f'(t)| \, dt \right).
\]

(13)

Since \( |f'| \) is an \( m \)-convex function, we have

\[
|f'(t)| \leq \frac{mb-t}{mb-a} |f'(a)| + m \frac{t-a}{mb-a} |f'(b)|
\]

for \( t \in [a,mb] \).

Using (14) in (13), we have

\[
\left| \left( \int_a^b g(s)E_{\mu,\alpha,l}^{\gamma,k,c} (\omega s^\nu; p) \, ds \right)^\alpha \right| (f(a) + f(mb))
- \alpha \int_a^b \left( \int_a^t g(s)E_{\mu,\alpha,l}^{\gamma,k,c} (\omega s^\nu; p) \, ds \right)^{a-1} g(t)E_{\mu,\alpha,l}^{\gamma,k,c} (\omega t^\nu; p)f(t) \, dt \\
- \alpha \int_a^b \left( \int_t^b g(s)E_{\mu,\alpha,l}^{\gamma,k,c} (\omega s^\nu; p) \, ds \right)^{a-1} g(t)E_{\mu,\alpha,l}^{\gamma,k,c} (\omega t^\nu; p)f(t) \, dt \\
\leq \|g\|_\infty S^{\alpha} \left( \int_a^b (t-a)^\alpha \left( \frac{mb-t}{mb-a} |f'(a)| + m \frac{t-a}{mb-a} |f'(b)| \right) \, dt \\
+ \int_a^b (mb-t)^\alpha \left( \frac{mb-t}{mb-a} |f'(a)| + m \frac{t-a}{mb-a} |f'(b)| \right) \, dt \right).
\]

(15)

After simple calculation of the above inequality, we get (11) which is required. \( \Box \)

If we take \( m = 1 \) in (11), then we get the following result for a convex function.

**Corollary 2.4** Let \( f : [a, b] \subseteq [0, \infty) \to \mathbb{R} \) be a differentiable function such that \( f' \in L^1[a,b] \) with \( a < b \). Also let \( g : [a, b] \to \mathbb{R} \) be a continuous function on \([a,b]\). If \( |f'| \) is a convex function on \([a,b]\), then the following inequality for extended generalized fractional integral operators holds:

\[
\left| \left( \int_a^b g(s)E_{\mu,\alpha,l}^{\gamma,k,c} (\omega s^\nu; p) \, ds \right)^\alpha \right| f(a) + f(b) \\
- \alpha \int_a^b \left( \int_a^t g(s)E_{\mu,\alpha,l}^{\gamma,k,c} (\omega s^\nu; p) \, ds \right)^{a-1} g(t)E_{\mu,\alpha,l}^{\gamma,k,c} (\omega t^\nu; p)f(t) \, dt \\
- \alpha \int_a^b \left( \int_t^b g(s)E_{\mu,\alpha,l}^{\gamma,k,c} (\omega s^\nu; p) \, ds \right)^{a-1} g(t)E_{\mu,\alpha,l}^{\gamma,k,c} (\omega t^\nu; p)f(t) \, dt \\
\leq \|g\|_\infty S^{\alpha} \left( \int_a^b (t-a)^\alpha |f'(a)| \, dt + \int_a^b (mb-t)^\alpha |f'(b)| \, dt \right) \\
+ \int_a^b (mb-t)^\alpha \left( \frac{mb-t}{mb-a} |f'(a)| + m \frac{t-a}{mb-a} |f'(b)| \right) \, dt \right).
\]
\[
- \alpha \int_a^b \left( \int_a^b g(s) E^{\gamma, \lambda, \kappa, \varepsilon}_{\mu, \sigma, \theta} (\cos^\alpha; p) \, ds \right)^{\alpha-1} g(t) E^{\gamma, \lambda, \kappa, \varepsilon}_{\mu, \sigma, \theta} (\cos^\alpha; p) f(t) \, dt \\
\leq \frac{(b-a)^{\alpha-1} \|g\|_{\infty}^{\alpha} S^\alpha}{(\alpha + 1)} \left[ |f'(a)| + |f'(b)| \right] 
\]

(16)

for \( k < \delta + \Re(\mu) \) and \( \|g\|_{\infty} = \sup_{t \in [a,b]} |g(t)| \).

**Remark 2.5** In Theorem 2.3.

(i) If we put \( p = 0 \), then we get [6, Theorem 3.2].

(ii) If we put \( \omega = p = 0 \) and \( m = 1 \), then we get [13, Theorem 6].

(iii) If we take \( \omega = p = 0, m = 1, \alpha = \frac{1}{2} \), and \( g(s) = 1 \), then we get [8, Corollary 2.3].

(iv) For \( g(s) = 1 \) along with \( \omega = p = 0, m = 1, \) and \( \alpha = \mu \), we get [13, Corollary 2].

**Remark 2.6** In Corollary 2.4.

(i) If we put \( p = 0 \), then we get [1, Theorem 3.2].

(ii) If we put \( \omega = p = 0 \), then we get [13, Theorem 6].

(iii) For \( \omega = p = 0, \alpha = \frac{1}{2} \), and \( g(s) = 1 \), we get [8, Corollary 2.3].

(iv) For \( g(s) = 1 \) along with \( \omega = p = 0 \), we get [13, Corollary 2].

Next we give the following fractional integral inequality.

**Theorem 2.7** Let \( f : [a, mb] \to \mathbb{R} \) be a differentiable function such that \( f \in L_1[a, mb] \) with \( 0 \leq a < mb \). Also let \( g : [a, mb] \to \mathbb{R} \) be a continuous function on \([a, mb]\). If \( |f'|^q \) is a convex function on \([a, mb]\), then for \( q > 0 \) the following inequality for extended generalized fractional integral operators holds:

\[
\left| \int_a^{mb} g(s) E^{\gamma, \lambda, \kappa, \varepsilon}_{\mu, \sigma, \theta} (\cos^\alpha; p) \, ds \right|^\alpha \left( f(a) + f(mb) \right) \\
- \alpha \int_a^{mb} \left( \int_a^{t} g(s) E^{\gamma, \lambda, \kappa, \varepsilon}_{\mu, \sigma, \theta} (\cos^\alpha; p) \, ds \right)^{\alpha-1} g(t) E^{\gamma, \lambda, \kappa, \varepsilon}_{\mu, \sigma, \theta} (\cos^\alpha; p) f(t) \, dt \\
- \alpha \int_a^{mb} \left( \int_t^{mb} g(s) E^{\gamma, \lambda, \kappa, \varepsilon}_{\mu, \sigma, \theta} (\cos^\alpha; p) \, ds \right)^{\alpha-1} g(t) E^{\gamma, \lambda, \kappa, \varepsilon}_{\mu, \sigma, \theta} (\cos^\alpha; p) f(t) \, dt \\
\leq \frac{2(mb-a)^{\alpha-1} \|g\|_{\infty}^{\alpha} S^\alpha}{(\alpha + 1)^{\alpha}} \left( \frac{|f'(a)|^q + m|f'(b)|^q}{2} \right)^\frac{1}{q}
\]

(17)

for \( k < \delta + \Re(\mu) \) and \( \|g\|_{\infty} = \sup_{t \in [a,mb]} |g(t)| \) and \( \frac{1}{p} + \frac{1}{q} = 1 \).

**Proof** From Lemma 2.1 and by using Hölder’s inequality, we have

\[
\left| \int_a^{mb} g(s) E^{\gamma, \lambda, \kappa, \varepsilon}_{\mu, \sigma, \theta} (\cos^\alpha; p) \, ds \right|^\alpha \left( f(a) + f(mb) \right) \\
- \alpha \int_a^{mb} \left( \int_a^{t} g(s) E^{\gamma, \lambda, \kappa, \varepsilon}_{\mu, \sigma, \theta} (\cos^\alpha; p) \, ds \right)^{\alpha-1} g(t) E^{\gamma, \lambda, \kappa, \varepsilon}_{\mu, \sigma, \theta} (\cos^\alpha; p) f(t) \, dt \\
- \alpha \int_a^{mb} \left( \int_t^{mb} g(s) E^{\gamma, \lambda, \kappa, \varepsilon}_{\mu, \sigma, \theta} (\cos^\alpha; p) \, ds \right)^{\alpha-1} g(t) E^{\gamma, \lambda, \kappa, \varepsilon}_{\mu, \sigma, \theta} (\cos^\alpha; p) f(t) \, dt 
\]
Using absolute convergence of the Mittag-Leffler function and \( \|g\|_{\infty} = \sup_{t \in [a, b]} |g(t)| \), we have

\[
\left( \int_a^b g(s)E_{\mu,\lambda}^{\gamma,\delta,k,c}(\omega s^\mu;p)\,ds \right)^{\frac{1}{p}} \left( \int_a^b |f'(t)|^q\,dt \right)^{\frac{1}{q}}
\]

\[
+ \left( \int_a^b \left| \int_t^b g(s)E_{\mu,\lambda}^{\gamma,\delta,k,c}(\omega s^\mu;p)\,ds \right|^{\frac{1}{p}}\,dt \right)^{\frac{1}{p}} \left( \int_a^b |f'(t)|^q\,dt \right)^{\frac{1}{q}}.
\]  

(18)

Since \( |f'(t)|^q \) is an \( m \)-convex function, we have

\[
|f'(t)|^q \leq \frac{mb-t}{mb-a} |f'(a)|^q + \frac{t-a}{mb-a} |f'(b)|^q.
\]  

(20)

Using (20) in (19), we have

\[
\left( \int_a^b g(s)E_{\mu,\lambda}^{\gamma,\delta,k,c}(\omega s^\mu;p)\,ds \right)^{\frac{1}{p}} \left( \int_a^b |f'(t)|^q\,dt \right)^{\frac{1}{q}}
\]

\[
+ \left( \int_a^b \left| \int_t^b g(s)E_{\mu,\lambda}^{\gamma,\delta,k,c}(\omega s^\mu;p)\,ds \right|^{\frac{1}{p}}\,dt \right)^{\frac{1}{p}} \left( \int_a^b |f'(t)|^q\,dt \right)^{\frac{1}{q}}
\]

\[
\times \left( \int_a^b \frac{mb-t}{mb-a} |f'(a)|^q + \frac{t-a}{mb-a} |f'(b)|^q \right)^{\frac{1}{2}}.
\]  

(21)

After simple calculation of the above inequality, we get (17) which is required.

If we take \( m = 1 \) in (17), then we get the following result for a convex function.

**Corollary 2.8** Let \( f : [a, b] \subseteq [0, \infty) \to \mathbb{R} \) be a differentiable function such that \( f' \in L_1[a, b] \) with \( a < b \). Also let \( g : [a, b] \to \mathbb{R} \) be a continuous function on \([a, b]\). If \( |f'|^q \) is a convex function on \([a, b]\), then for \( q > 0 \) the following inequality for extended generalized fractional
integral operators holds:

\[
\left| \left( \int_{a}^{b} g(s) E_{\mu,\alpha,\lambda}^{\gamma,k,c} (\omega t^\alpha; p) \, ds \right) \right|^2 \left[ f(a) + f(b) \right] \\
- \alpha \int_{a}^{b} \left( \int_{a}^{t} g(s) E_{\mu,\alpha,\lambda}^{\gamma,k,c} (\omega t^\alpha; p) \, ds \right) \left( \int_{t}^{b} g(s) E_{\mu,\alpha,\lambda}^{\gamma,k,c} (\omega t^\alpha; p) \, ds \right) \, dt \\
\leq \frac{2(b-a)^{\alpha+1} \|g\|_\infty \mathcal{S}^q}{(ap+1)^{\frac{1}{q}}} \left[ \frac{\|f\|_\infty \mathcal{S}^q}{2} \right] \frac{1}{q} (22)
\]

for \( k < \delta + \Re(\mu) \) and \( \|g\|_\infty = \sup_{t \in [a,b]} |g(t)| \) and \( \frac{1}{p} + \frac{1}{q} = 1 \).

**Remark 2.9** In Theorem 2.7.

(i) If we put \( p = 0 \), then we get [6, Theorem 3.6].

(ii) If we put \( \omega = p = 0 \) and \( m = 1 \), then we get [13, Theorem 7].

(iii) If we take \( \omega = p = 0, m = 1 \) along with \( \alpha = \frac{\mu}{\lambda} \), then we get [8, Theorem 2.5].

(iv) If we take \( g(s) = 1, \omega = p = 0 \), then we get [5, Theorem 2.3].

(v) If we put \( \omega = p = 0, m = 1, \) and \( \alpha = 1 \), then we get [5, Corollary 3].

**Remark 2.10** In Corollary 2.8.

(i) If we put \( p = 0 \), then we get [1, Theorem 3.5].

(ii) If we put \( \omega = p = 0 \), then we get [13, Theorem 7].

(iii) If we put \( \omega = p = 0, \alpha = 1 \), then we get [13, Corollary 3].

(iv) If we take \( \omega = p = 0 \) along with \( \alpha = \frac{\mu}{\lambda} \), then we get [8, Theorem 2.5].

(v) If we take \( g(s) = 1 \) and \( \omega = p = 0 \), then we get [5, Theorem 2.3].

In the next result we give Hadamard type inequalities for \( m \)-convex functions via an extended Mittag-Leffler function.

**Theorem 2.11** Let \( f : [a, mb] \to \mathbb{R} \) be a function such that \( f \in L_1[a, mb] \) with \( 0 \leq a < mb \). If \( f \) is \( m \)-convex on \([a, mb]\), then the following inequalities for extended generalized fractional integral operators hold:

\[
2f\left( \frac{a + mb}{2} \right) C_{a+1} (\frac{a+mb}{2}; mb; p) \\
\leq \left( e_{\mu,\alpha,\lambda,\omega'} (\frac{a+mb}{2}; mb; p) \right) + m^{\alpha+1} \left( e_{\mu,\alpha,\lambda,\omega'} (\frac{a+mb}{2}; mb; p) \right) \left( \frac{a}{m}; p \right) \\
\leq \frac{1}{mb-a} \left( f(a) - m^2 f \left( \frac{a}{m^2} \right) C_{a+1} (\frac{a+mb}{2}; mb; p) \\
+ m^{\alpha+1} \left( f(b) + mf \left( \frac{a}{m} \right) \right) C_{a+1} (\frac{a+mb}{2}; mb; p) \right),
\]

where \( \omega' = \frac{2^\mu \omega}{(mb-a)^\mu} \).
Proof Since $f$ is an $m$-convex function, we have

\[ 2f\left(\frac{a + mb}{2}\right) \leq f\left(\frac{t}{2} a + 2 - \frac{t}{2} \right) + mf\left(\frac{2 - t}{2m} a + \frac{t}{2} b\right). \quad (24) \]

Also from $m$-convexity of $f$, we have

\[ f\left(\frac{t}{2} a + \frac{2 - t}{2} b\right) + mf\left(\frac{2 - t}{2m} a + \frac{t}{2} b\right) \leq \frac{t}{2} \left(f(a) - m^2 f\left(\frac{a}{m^2}\right) + m f\left(\frac{a}{m^2}\right)\right). \quad (25) \]

Multiplying (24) by $\mu^{-1} E_{\mu,a,d}^{\gamma,k,c} (\omega t^\mu; p)$ on both sides and then integrating over $[0,1]$, we have

\[ 2f\left(\frac{a + mb}{2}\right) \int_0^1 t^{\mu-1} E_{\mu,a,d}^{\gamma,k,c} (\omega t^\mu; p) \, dt \]
\[ \leq \int_0^1 t^{\mu-1} E_{\mu,a,d}^{\gamma,k,c} (\omega t^\mu; p) f\left(\frac{t}{2} a + 2 - \frac{t}{2} mb\right) \, dt \]
\[ + m \int_0^1 t^{\mu-1} E_{\mu,a,d}^{\gamma,k,c} (\omega t^\mu; p) f\left(\frac{2 - t}{2m} a + \frac{t}{2} b\right) \, dt. \quad (26) \]

Putting $u = \frac{t}{2} a + 2 - \frac{t}{2} mb$ and $v = \frac{2 - t}{2m} a + \frac{t}{2} b$ in (26), we have

\[ 2f\left(\frac{a + mb}{2}\right) \int_{\frac{a + mb}{2}}^{mb} (mb - u)^{\mu-1} E_{\mu,a,d}^{\gamma,k,c} (\omega (mb - u)^\mu; p) \, du \]
\[ \leq \int_{\frac{a + mb}{2}}^{mb} (mb - u)^{\mu-1} E_{\mu,a,d}^{\gamma,k,c} (\omega (mb - u)^\mu; p) f(u) \, du \]
\[ + m^{\mu+1} \int_{\frac{a + mb}{2m}}^{mb} \left(v - \frac{a}{m}\right)^{\mu-1} E_{\mu,a,d}^{\gamma,k,c} \left(m^\nu \omega \left(v - \frac{a}{m}\right)^\mu; p\right) f(v) \, dv. \]

By using (3), (4), and (5) we get the first inequality of (23).

Now multiplying (25) by $\mu^{-1} E_{\mu,a,d}^{\gamma,k,c} (\omega t^\mu; p)$ on both sides and then integrating over $[0,1]$, we have

\[ \int_0^1 t^{\mu-1} E_{\mu,a,d}^{\gamma,k,c} (\omega t^\mu; p) f\left(\frac{t}{2} a + 2 - \frac{t}{2} mb\right) \, dt \]
\[ + m \int_0^1 t^{\mu-1} E_{\mu,a,d}^{\gamma,k,c} (\omega t^\mu; p) f\left(\frac{2 - t}{2m} a + \frac{t}{2} b\right) \, dt \]
\[ \leq \frac{1}{2} \left(f(a) - m^2 f\left(\frac{a}{m^2}\right)\right) \int_0^1 t^{\mu-1} E_{\mu,a,d}^{\gamma,k,c} (\omega t^\mu; p) \, dt \]
\[ + m \left(f(b) + mf\left(\frac{a}{m^2}\right)\right) \int_0^1 t^{\mu-1} E_{\mu,a,d}^{\gamma,k,c} (\omega t^\mu; p) \, dt. \quad (27) \]
Putting \( u = \frac{t}{2} a + \frac{m}{2} \frac{t}{2} b \) and \( v = \frac{t}{2} a + \frac{t}{2} b \) in (27), we have

\[
\int_{\frac{a + mb}{2m}}^{mb} (mb - u)^{\alpha-1} E^{\nu,\delta,k,c}_{\mu,\alpha,l} (\omega' (mb - u)^{\mu}; p) f(u) \, du
\]
\[
+ \int_{\frac{a + mb}{2m}}^{mb} \left( v - \frac{a}{m} \right)^{\alpha-1} E^{\nu,\delta,k,c}_{\mu,\alpha,l} (m^{\mu} \omega' (v - \frac{a}{m})^{\mu}; p) f(v) \, dv
\]
\[
\leq \frac{1}{2} \left( f(a) - m^{\nu} f\left( \frac{a}{m^2} \right) \right) \int_{\frac{a + mb}{2m}}^{mb} (mb - u)^{\alpha} E^{\nu,\delta,k,c}_{\mu,\alpha,l} (\omega' (mb - u)^{\mu}; p) \, dt
\]
\[
+ m^{\nu+1} \left( f(b) + mf\left( \frac{a}{m^2} \right) \right)
\]
\[
\times \int_{\frac{a + mb}{2m}}^{mb} \left( v - \frac{a}{m} \right)^{\alpha-1} E^{\nu,\delta,k,c}_{\mu,\alpha,l} (m^{\mu} \omega' (v - \frac{a}{m})^{\mu}; p) \, dt.
\] (28)

By using (3), (4), and (6), we get the second inequality of (23). \(\square\)

If we take \( m = 1 \) in (23), then we get the following Hadamard type inequality for a convex function.

**Corollary 2.12** Let \( f : [a, b] \subseteq [0, \infty) \to \mathbb{R} \) be a function such that \( f \in L_{1}[a, b] \) with \( a < b \).
If \( f \) is convex on \([a, b]\), then the following inequalities for extended generalized fractional integral operators hold:

\[
f\left( \frac{a + b}{2} \right) C_{\omega, \left( \frac{a + b}{2} \right)}^{\nu,\alpha,\delta,\lambda}(b; p)
\]
\[
\leq \left[ \left( \nu^{\gamma,\delta,k,c}_{\mu,\alpha,l; \omega', \left( \frac{a + b}{2} \right)} f(b; p) \right) + \left( \nu^{\gamma,\delta,k,c}_{\mu,\alpha,l; \omega', \left( \frac{a + b}{2} \right)} f(a; p) \right) \right]
\]
\[
\leq \frac{f(a) + f(b)}{2} C_{\omega, \left( \frac{a + b}{2} \right)}^{\nu,\alpha,\delta,\lambda}(a; p),
\] (29)

where \( \omega' = \frac{2^{\nu}m}{(b-a)^{\nu}} \).

**Remark 2.13** In Theorem 2.11.
(i) If we put \( p = 0 \), then we get [6, Theorem 3.10].
(ii) If we put \( \omega = p = 0, m = 1, \) and \( \alpha = 1 \), then we get the classical Hadamard inequality.

**Remark 2.14** In Corollary 2.12.
(i) If we put \( p = 0 \), then we get [1, Theorem 3.9].
(ii) If we put \( \omega = p = 0 \) and \( \alpha = 1 \), then we get the classical Hadamard inequality.
(iii) If we take \( \omega = p = 0 \), then we get [14, Theorem 4].

### 3 Concluding remarks

We have investigated more general fractional integral inequalities. By selecting specific values of parameters quite interesting results can be obtained. For example selecting \( p = 0 \), fractional integral inequalities for fractional integral operators defined by Salim and Faraj in [12], selecting \( l = \delta = 1 \), fractional integral inequalities for fractional integral operators
defined by Rahman et al. in [11], selecting \( p = 0 \) and \( l = \delta = 1 \), fractional integral inequalities for fractional integral operators defined by Shukla and Prajapati in [15] (see also [16]), selecting \( p = 0 \) and \( l = \delta = k = 1 \), fractional integral inequalities for fractional integral operators defined by Prabhakar in [10], selecting \( p = \omega = 0 \), fractional integral inequalities for Riemann–Liouville fractional integral operators.

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