Round-Hashing for Data Storage: Distributed Servers and External-Memory Tables

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Abstract

This paper proposes round-hashing, which is suitable for data storage on distributed servers and for implementing external-memory tables in which each lookup retrieves at most a single block of external memory, using a stash. For data storage, round-hashing is like consistent hashing as it avoids a full rehashing of the keys when new servers are added. Experiments show that the speed to serve requests is tenfold or more than the state of the art. In distributed data storage, this guarantees better throughput for serving requests and, moreover, greatly reduces decision times for which data should move to new servers as rescanning data is much faster.

1 Introduction

Consistent hashing was invented by Karger et al. [13] for shared web caching and highest random weight hashing (also known as rendezvous hashing) was invented by Thaler and Ravishankar [24] for web proxy servers. Both hashing methods were conceived independently around the mid 90s, and shared similar goals (with different implementations): cached web pages are assigned to servers, so that when a server goes down, its cached web pages are reassigned to the other servers so as to preserve their load balancing; similarly, when a new server is added, some cached web pages are moved to it from the others.

Consistent hashing, in its basic version, maps both web pages and servers to the circular universe \([0 \ldots 2^w - 1]\), where each hash value requires \(w\) bits: each web page starts from its hash value in the circular universe and is assigned to the server whose hash value is clockwise met first; this can be done in \(O(\log m)\) time using a search data structure of size \(O(m)\) for \(m\) servers. Rendezvous hashing, for a given web page \(p\), applies hashing to the pairs \((p, i)\) for each server \(i\), and then assigns \(p\) to the server \(i = i_0\) that gives the maximum hash value among these pairs; this is computed in \(O(\log m)\) time using a tree of size \(O(m)\) as discussed by Wang and Ravishankar [26]. Table 1 reports a summary of these bounds. Both methods apply their rule above when a server is deleted or added. They have been successfully exploited in the industry, e.g. Akamai for consistent hashing, and Microsoft’s cache array routing protocol (CARP) for rendezvous hashing. Among the notable applications, it is worth mentioning Chord [23] for building distributed hash tables in peer-to-peer networks (such as BitTorrent), and Amazon’s Dynamo [7] for distributed and cloud computing, with data stored in main memory for speed on a wide set of machines.
Table 1: Performance of the hashing methods for $m$ buckets (servers). Here $s_0 \ll m$ is a constant slack parameter (typically $s = 64$ or 128), and $\alpha = \frac{\text{number of stored keys}}{m}$ is the load factor. Although creating a new bucket moves $O(\alpha)$ keys on the average, each hashing method can take different time to decide which are the keys to move: “local” means that few other buckets scan their keys, while “distributed” means that all buckets scan their keys in parallel to decide which ones have to move to the new bucket. The $\tilde{O}()$ notation indicates an expected cost.

Recently, Lamping and Veach presented jump consistent hashing \[14\] at Google, observing that consistent hashing can be tailored for data centers and data storage applications in general. In this scenario, servers cannot disappear, as this would mean loss of valuable data; rather, they can be added to increase storage capacity. As a result, the hash values “jump” to higher values for the keys moved to a new bucket; moreover, the hash values are a contiguous range $[0 \ldots m - 1]$ for $m$ servers, rather than a subset of $m$ integers from $[0 \ldots 2^w - 1]$. This has a dramatic impact on the performance of the jump consistent hash, as illustrated in \[14\], observing that only balance and monotonicity should be guaranteed from the original proposal in \[13\]. The auxiliary storage is just $O(1)$, as shown in Table 1; average query cost is the $m$-th harmonic number, so $O(\log m)$, with no worst case guarantee.

In this paper, we study the problem of consistent hashing for data storage in the above scenario depicted by Lamping and Veach. For the presentation’s sake, the keys are the hash values of the web pages, and the servers are the buckets, numbered from 0 to $m - 1$, where the keys have to be placed. At any time, we want to support the following operations, besides the initialization:

- Return the current number $m$ of buckets.
- Given a key $u$, find its corresponding bucket number in $[0 \ldots m - 1]$.
- Add a new bucket having number $m$, thus the range becomes $[0 \ldots m]$.
- Release the last bucket $m - 1$, thus the range becomes $[0 \ldots m - 2]$ .

We observe that linear hashing, introduced by Litwin \[15\] and Larson \[15\] at the beginning of the 80s, can be successfully employed in this scenario, thus taking $O(\log m)$ time and $O(1)$ space, as reported in Table 1. We use a constant slack parameter $s_0 \ll m$ (typically $s_0 = 64$ or 128) to guarantee that the number of keys in the most populated bucket is at most $1 + 1/s_0$ times the number of keys in the least populated bucket.

**Our first contribution** In this paper we present a new hashing scheme, called *round-hashing*, which applies *round-mapping* to the hashed value, allowing us to achieve $O(1)$ time and space in the worst case. This is a desirable feature, as otherwise hashing with no worst-case guarantee can pose...

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\[1\] Data is split into shards, where each shard is handled by a cluster of machines with replication, thus it is not acceptable for shards to disappear \[13\].

\[2\] This operation is not actually mentioned in the original setting, but it comes for free in our case.
security issues, such as algorithmic complexity attacks [3,4] for low-bandwidth denial of service exploiting its worst-case behavior.

Round-hashing computes the hash value of the given key and invokes round-mapping for this value, as described in Section 2. Here we give a glimpse using the example with the bucket numbers shown in Figure 1, where we set $s_0 = 3$. Round-mapping does not materialize these numbers, and proceeds in rounds $q = 0, 1, \ldots$ when adding new bucket numbers, where at the end of round $q$ there are $m = 2^q s_0$ bucket numbers. For instance, consider when round $q - 1 = 3$ ends ($m = 8 s_0$) and a new bucket number is requested, so round $q = 4$ begins.

We observe that each round is divided into steps, where each step $s = s_0, s_0 + 1, \ldots, 2 s_0 - 1$ handles the next $8 = 2^{q-1}$ requests of new bucket numbers. At the end of these steps, $2^{q-1} s_0$ bucket numbers have been added to the $2^{q-1} s_0$ ones inherited from round $q - 1$, and thus we find the claimed $m = 2^q s_0$ bucket numbers at the end of round $q$.

Let us go back to the first request for $q = 4$. It is for block number $m = 24$, which is implicitly inserted after the first $s = s_0 = 3$ elements, between 2 and 12. Next comes $m = 25$, so we skip other $s = 3$ elements, inserting it between 20 and 6. In general, for each step $s$, we insert a new bucket number by skipping the next $s$ elements from the current position. When step $s$ completes its virtual scan, it means that $2^{q-1}$ new block numbers have been served (8 in our example), so we can set $s := s + 1$ and repeat the scan from the beginning if $s < 2 s_0$.

In summary, assigning a new block number is simple: skip the next $s$ elements from the current position and insert the new bucket number; if the end of the scan is reached, increase $s$ and restart from the beginning if $s < 2 s_0$. Any time we have a permutation of $[0 \ldots m - 1]$: what is non-trivial is that, for any given (hash value) $u \in [0 \ldots m - 1]$, round-mapping computes the block number at position $u$ in this evolving permutation, in $O(1)$ time using $O(1)$ additional memory, without general division and modulo operations. We refer the reader to Section 2 for a formal description.

Finally, looking back at Table 1 when a new bucket is created, we observe that $O(\alpha)$ keys on average are moved from the other buckets, where $\alpha$ is the load factor, namely, the number of stored keys divided by the number $m$ of buckets. However, the hashing methods take different time to decide which are the keys to move. Specifically, consistent hashing has to examine the $O(\alpha)$ keys

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3We are assuming, wlog, that the buckets have all the same size.
in the two neighbor servers in the worst case, and update the data structure in $O(\log m)$ time.\footnote{For the sake of discussion, we consider the basic version of consistent hashing, and refer the reader to \cite{13,14} for the version with multiple hash values per server.} Rendezvous hashing requires that each bucket scans its keys and test whether the new bucket is now the maximum for some of them. Hence all the keys are scanned, $O(m\alpha)$, but only $O(\alpha)$ of them are moved in total. Jump consistent hashing needs to perform a similar task, to see which keys “jump” to the new bucket. Linear hashing and our round-hashing require to scan the keys in $s_0 = O(1)$ buckets to find the $O(\alpha)$ ones to move. Note that the methods in the last three rows of Table 1 need little data structure bookkeeping as space usage is $O(1)$.

When a new bucket is created, there are pros and cons to involve “all” vs “few” buckets to decide which keys move. At one end, when involving all buckets (rendezvous, jump consistent), Lamping and Veach observe that it is better to take few keys from each of them to relieve a hot spot, but this requires many servers scanning and sending data. At the other end (consistent, linear, round-), involving few buckets may not relieve a hot spot soon, but it makes sense if the data storage is distributed geographically among many data centers, and the most efficient way to move data is to make a copy on physical devices, moving them with a truck to the new data center. Note that suitably increasing $s_0$ can combine the best of these two behaviors, so the choice depends on the application domain.

**Our second contribution** We performed an experimental study of the above hashing methods, since Table 1 does not give the full picture from an algorithm engineering point of view. The code is publicly available at \url{https://github.com/veluca93/round_hashing} to replicate the experiments.

Our first observation addresses how balanced are the buckets filled with the hashing methods in Table 1. By uniformly sampling all the possible keys, their hash values can be used to estimate how far the number of keys in buckets are from the ideal load factor $\alpha$, reporting the least and the most populated buckets after the experiments. We observed that jump consistent hashing is very close to $\alpha$, ranging from 0.988 $\alpha$ to 1.012 $\alpha$; the experimental study in \cite{14} shows that it compares favorably with consistent hashing (rendezvous hashing is not directly compared). We can match this performance by setting $s_0 = 128$ for linear hashing and $s_0 = 64$ for round-hashing.

Our second observation relates to the real cost of instructions on a commodity processor. Concretely to illustrate our points, we refer to Intel processors \cite{11}. Here Euclidean division is not our friend: integer division and modulo operations on 64-bit integers take 85–100 cycles, whereas addition takes 1 cycle (and can be easily pipelined). Interestingly, this goes in the direction of the so-called AC$^0$-RAM dictionaries (e.g. see Andersson et al. \cite{2}) and Practical RAM (e.g. see Brodnik et al. \cite{4} and Miltersen \cite{18}), where integer division and multiplication are not permitted, among others.\footnote{We thank Rasmus Pagh for pointing us the reference on AC$^0$-RAM dictionaries.} However, multiplication should be taken with a grain of salt as, surprisingly, it takes 3–4 cycles (which becomes 1 cycle when it can be pipelined). Also, the modulo operation for powers of two or for small constants proportional to $s_0$, can be replaced with a few shift and multiplication operations \cite{10} as available, for instance, in the gcc compiler from version 2.6. Our implementation of round-hashing avoids general integer division and modulo operations because they are almost two orders of magnitude slower than the other operations: using them could nullify the advantage of the $O(1)$ time complexity. We applied this tuning also to linear hashing whenever possible.

As a result, to find the bucket number for a key, round-hashing is almost an order of magnitude faster than jump consistent hashing, and even much faster than the other hashing methods in Table 1. This is crucial for the system throughput: first, round-hashing can serve tenfold or more requests; second, when a new bucket number is added, it improves the performance of rescanning
the keys to decide which ones move to the new bucket. We refer the reader to Section 3 for further details on our experimental study.

Our third contribution We apply round-hashing to obtain a variant of dynamic hash tables, called round-table, that addresses the issues of a high-throughput server with many lookup requests, relatively few updates, and where some keys can be kept in a stash in main memory. We follow the classical two-level external-memory model [1] to evaluate the complexity. Let \( n \) be the number of keys currently stored in the table, and \( B \) be the maximum number of keys that fit inside one block transfer from main memory to external memory, or vice versa. A stash of size \( k \) keys can be kept in main memory. We measure space occupancy using the space utilization \( 1 - \epsilon \), where \( 0 \leq \epsilon < 1 \), defined as the ratio of the \( n \) of keys divided by the number of external-memory blocks times \( B \), hence the number of blocks is \( \lceil \frac{n}{B(1-\epsilon)} \rceil \). In other words, \( \epsilon \) represents the “waste” of space in external memory, so the lower \( \epsilon \), the better.

Round-table achieves the following bounds. Each lookup reads just 1 block from external memory in the worst case, taking \( O(1) \) CPU time and thus requiring only \( O(1) \) words from main memory. Each update (insertion or deletion) requires to access at most \( 4s_0 \) blocks in external memory, in the worst case, taking \( O(s_0(B + \log n / \log \log n)) \) CPU time w.h.p. (expected time is \( O(s_0B) \)) and using \( O(B) \) memory cells. The number of keys in the stash is \( k \approx n / \exp(B) \). For example, setting \( s_0 = 2/\epsilon \) when \( \epsilon > 0 \), we obtain \( k \approx n \exp\left(\frac{\sqrt{\frac{\pi}{2} \cdot \frac{1}{\epsilon^2}}}{\sqrt{\frac{\pi}{2} B (2 - \epsilon)}}\right) \), noting that the update cost becomes \( O(\epsilon^{-1}) \) in this case. Thus \( k \) is exponentially smaller than \( O(n/B) \) in main memory, obtained by storing at least one word per external block (as B-trees do). Experiments in Section 4 confirm our estimation.

Looking at the vast literature on hashing, apart from the previously mentioned methods, Mirrokni et al. [19] provide a version of consistent hashing that keeps bucket load within a factor of \( 1 + \epsilon \), but it cannot guarantee at most one memory access. The expected optimal bound of \( 1 + O(1/\sqrt{B}) \) memory accesses in Jensen and Pagh [12] does not require the stash, with no guarantee of at most one memory access. As for the work on tables with one external-memory access, some results [17, 8] rely on perfect hashing, but are either not dynamic or cannot reach arbitrarily high utilization. A recent cuckoo hashing based approach [21], combined with in-memory Bloom filters to ensure that lookups access the correct position, is not simple to dynamize. A general scheme [22] relies on perfect hashing to store the stash on external memory, thus having higher worst-case cost for insertions. The result in [5] achieves single-access lookups, but at the cost of \( O\left(\frac{n}{B(1-\epsilon)}\right) \) internal memory. A solution based on predecessor search needs \( O\left(\frac{n}{B}\right) \) internal memory, as discussed in [20].

2 Round-Hashing and Round-Mapping

Round-hashing maps the hash value of the input key into a position \( u \) along the unitary circumference \( C \), and invokes round-mapping on \( u \) to find the corresponding bucket number (note that \( u \) can be seen as a fraction of the unit). To this end, given the integer slack parameter \( s_0 > 0 \), round-mapping maintains a partition of \( C \) into arcs of length proportional to either \( 1/s \) or \( 1/(s+1) \) for some integer \( s \) \( (s_0 \leq s \leq 2s_0 - 1) \): if there are \( m \) arcs in \( C \), they are consecutively numbered from 0 to \( m - 1 \) in clockwise order and in one-to-one correspondence with a permutation of \( \{0, 1, \ldots, m - 1\} \), called the bucket numbers. Also we refer to arcs as short when their length is proportional to \( 1/(s+1) \), and long when their length is proportional to \( 1/s \). Round-mapping supports the following operations, which are better visualized using \( C \) and its arcs in clockwise order.

- \( \text{init()} \): divide the circumference \( C \) into \( s = s_0 \) long arcs, and set \( m = s_0 \).
We exploit the invariant property that short arcs are numbered from 0 to \( p^j \) such that \( 0 \leq p^j \leq \lfloor \log_2 (m + 1) \rfloor \), where divisions and modulo operations involve just powers of two or constants in the range \( b = \{ 0, 1 \} \). Schema, we give a closed formula for \( \text{len} \) that can be computed in \( O(1) \) time in the word RAM model, where \( \text{len} \) is the length of the bucket numbers that can be computed in \( O(1) \) time in the worst case, which is \( O(1) \) when \( s_0 = O(1) \).

We say that a round starts when \( s = s_0 \). Let the rounds be numbered as \( q = 0, 1, 2, \ldots \), and let the length \( \text{len}(q) = s_0 2^q \) of a round \( q \) represent how many bucket numbers have been allocated by the last step \( s \). For example, choosing \( s_0 = 3 \), the first rounds are shown in Figure 1. At step \( s = 4 \) of round \( q = 4 \), each call to \( \text{newBucket}() \) takes \( s \) consecutive bucket numbers and inserts a new bucket number: after 0 1 2 24 it inserts 32, after 12 16 20 25 it inserts 33, after 6 8 10 26 it inserts 34, and so on. Note that 32, 33, 34, etc., are native of round \( q = 4 \). Figure 1 shows of which round the bucket numbers are native, using framed chunks. After the last step \( s = 2s_0 - 1 \), each round contains twice the bucket numbers than after the last step of the previous round. Also, the concatenation of every other chunk of \( s \) non-native bucket numbers, produces exactly the outcome of the previous round. We will exploit this regular pattern.

As it can be noted, a mapping between the arcs and a permutation is maintained: for example, in round \( q = 4 \), arc \( j \) for \( j = 0, 1, 2 \) corresponds to bucket number \( b(j) = j \); arc \( j = 4 \) has \( b(j) = 24 \), arc \( j = 5 \) has \( b(j) = 32 \), \ldots, and arc \( j = 47 \) has \( b(j) = 47 \). Note that we do not maintain this mapping explicitly; still, \( \text{findBucket}(u) \) is able to identify arc \( j \) and its bucket number \( b(j) \) in constant time.

### 2.1 Implementation of \( \text{findBucket}(u) \)

We exploit the invariant property that short arcs are numbered from 0 to \( p \), and thus \( p + 1 \) is a multiple of \( s + 1 \), where \( p \) is maintained as the last added short arc. We also use \( \text{pow}(a) \), where \( a > 0 \), to denote the largest integer exponent \( e \geq 0 \) such that \( 2^e \) divides \( a \) (a.k.a. 2-adic order). Equivalently, \( \text{pow}(a) \) is the position of the least significant bit 1 in the binary representation of the unsigned integer \( a > 0 \).

First, consider the ideal situation: after the last step \( s = 2s_0 - 1 \) of round \( q \), we have \( \text{len}(q) \) buckets, numbered consecutively from 0 to \( \text{len}(q) - 1 \). We also have \( \text{len}(q) \) arcs on the circle, numbered consecutively from 0 to \( \text{len}(q) - 1 \). As arc \( j \) is mapped to bucket number \( b(j) \) using our scheme, we give a closed formula for \( b(j) \) that can be computed in \( O(1) \) time in the word RAM model, where divisions and modulo operations involve just powers of two or constants in the range \( [s_0 \ldots 2s_0] \).

Let \( j = s_0 i + x \) where \( x \in \{ 0, 1, \ldots, s_0 - 1 \} \). If \( i = 0 \), then \( b(j) = b(x) = x \). Thus \( b(j) = j \) for \( 0 \leq j < s_0 \). Hence, let assume \( i > 0 \) in the rest of the section, and thus we need to compute \( b(j) \) for \( j \geq s_0 \).
Algorithm 1: Mapping from arcs to buckets

1 Function findBucket(u)
2 \[ j \leftarrow \text{arc hit by } u \]
3 if \( j < s_0 \) then return \( j \)
4 if \( j > p \) then \( j' \leftarrow j - \frac{p+1}{s+1}, \ s' = s \)
5 else \( j' \leftarrow j, \ s' = s + 1 \)
6 \( x \leftarrow (j' \mod s') \mod s_0 \)
7 \( q' \leftarrow q + \frac{s'-1}{s_0} \)
8 \( i = (1 + \left\lfloor \frac{x-1}{s_0} \right\rfloor) \cdot \left\lfloor \frac{j'}{s} \right\rfloor + \left\lfloor \frac{j's'}{s_0} \right\rfloor \)
9 return \( pos(i, x, q') \)

10 Function pos(i, x, q)
11 \( e \leftarrow \text{position of the least significant bit 1 in } i \)
12 return \( \left\lfloor \frac{(s_0 + x)2^q + i}{2^{\text{pow}(i) + 1}} \right\rfloor \)

We say that the bucket number in position \( j \) belongs to chunk \( i \) (hence, a chunk is of length \( s_0 \)). For odd values of \( i \), the bucket number is native for round \( q \). For even values of \( i \), the bucket number is native for round \( q - \text{pow}(i) \), as it can be checked in Figure 1. For example, in round \( q \) after the last step, bucket number 9 is in position \( j = 37 = 3 \cdot 12 + 1 \), so \( i = 12 \) and 9 is native for round \( q - \text{pow}(i) = 4 - 2 = 2 \). In general, as \( \text{pow}(i) = 0 \) when \( i \) is odd, we can always say that the bucket number is native for round \( q - \text{pow}(i) \) for \( i > 0 \). Another useful observation is that the smallest native number in round \( q \) is \( \text{len}(q - 1) \) by construction (e.g. 24 in round \( q = 4 \)).

In the ideal situation, we find the native round for the bucket number at position \( j \): as its chunk is preserved in the native round, we can use its offset \( x \) inside the chunk to recover the value of that bucket number. In the native round \( q \), each chunk \( i \) starts with bucket number \( \text{len}(q - 1) \) as previously observed, increased by one for each such chunk, thus the first bucket number in chunk \( i \) is \( \text{len}(q - 1) + \lfloor i/2 \rfloor \). Also, any two adjacent numbers in the chunk, differ by \( 2^{i-1} \) by construction. Summing up, there are two cases for the bucket number for \( j \):

- \( i \) odd and thus native for round \( q \): the bucket number is \( \left\lfloor \frac{(s_0 + x)2^q + i}{2^{\text{pow}(i) + 1}} \right\rfloor \)
- \( i \) even and thus native for round \( q - \text{pow}(i) \): the bucket number is \( \left\lfloor \frac{(s_0 + x)2^q + i}{2^{\text{pow}(i) + 1}} \right\rfloor \)

As \( \text{pow}(a) = 0 \) when \( a \) is odd, we can compactly write these positions in the ideal situation as

\[
\text{pos}(i, x, q) = \left\lfloor \frac{(s_0 + x)2^q + i}{2^{\text{pow}(i) + 1}} \right\rfloor
\]

Second, consider the general situation, with an intermediate step \( s_0 \leq s \leq 2s_0 - 1 \) in round \( q \). Recall that we know the position \( p \) of the last created arc. This gives the following picture. The first \( p + 1 \) short arcs in clockwise order can be seen as \( \frac{p+1}{s+1} \) consecutive groups, each of \( s + 1 \) arcs, and the remaining arcs are long and form groups of \( s \) arcs each. Let us set \( s' = s + 1 \) in the former groups, and \( s' = s \) in the latter groups. In the following, we equally say that each group contains \( s' \) arcs or that each group contains \( s' \) bucket numbers. In general, we say \( s' \) entries (arcs or bucket numbers) when it is clear from the context.

A common feature is that the first \( s_0 \) entries of each group are inherited from the previous round, and the last \( s' - s_0 \) entries in each group are those added in the current round: each new entry is appended at the end of each group, so the entry in position \( p \) is the last in its group.
Now, given a position $j$, we want to compute $b(j)$, the corresponding bucket number. The idea is to reduce this computation to the ideal situation analyzed before.

If $j > p$, we conceptually remove one entry for each group such that $s' = s + 1$. This is equivalent to set $j := j - \frac{p+1}{s+1}$ and, consequently, $p := p - \frac{p+1}{s+1}$. Now, we conceptually have all the groups of the same size $s'$, which are sequentially numbered starting form 0.

Let $i' = \lfloor j/s' \rfloor$ be the number of the group that contains the entry corresponding to $j$. We now decide whether $j$ is one of the first $s_0$ entries of its group or not. We have two cases, according to the value of $r = j \% s'$.

If $r < s_0$, the wanted entry is one of the first $s_0$ entries of its group. If we concatenate those entries over all groups, we obtain the ideal situation of the previous round $q - 1$. There, the wanted entry occupies position $j' = s_0i' + r$. Hence, $b(j) = pos(i', r, q - 1)$ in the ideal situation.

If $r \geq s_0$, the wanted entry is one of the last $s'$ entries of its group. Analogously, if we concatenate those entries over all groups, the position of the wanted entry becomes $j'' = (s' - s_0)i' + r - s_0$, where $x = r - s_0$ is the internal offset. However, we cannot solve this directly. We use instead the observation that the futures entries that will contribute to get the ideal situation for round $q$, will be appended at the end of each group. In this ideal situation, the wanted entry correspond to arc $2i' + 1$ and is at position $j'' = s_0(2i' + 1) + r - s_0$ for round $q$. Thus, $b(j) = pos(2i' + 1, r - s_0, q)$ in the ideal situation.

We can summarize the entire computation of $b(j)$ in an equivalent formula computed by Algorithm 1 that can be computed in $O(1)$ time.

**Lemma 1** findBucket() can be implemented in $O(1)$ time using bitwise operations.

Interestingly, findBucket() is much faster then other approaches known in the literature for consistent hashing, as we will see in Section 3.

**Theorem 1** Round-mapping with integer parameter $s_0 > 1$ can be implemented using $O(1)$ words, so that init(), numBuckets() and findBucket() take $O(1)$ time, and newBucket() and freeBucket() take $O(s_0)$ time.

## 3 Distributed Servers

We experimentally evaluated round-hashing and our C implementation of Algorithm 1 on a commodity hardware based on Intel Xeon E3-1545M v5 CPU and 32Gb RAM, running Linux 4.14.34, and using gcc 7.3.1 compiler. We give some implementation details on the experimented algorithms, observing that we decided not to run consistent hashing [13] and rendezvous hashing [24] as they are outperformed by jump consistent hashing as discussed in detail in [14]. Specifically, we ran the following code.

- Jump consistent hashing [14]: we employed the implementation provided by the authors’ optimized code.
- Linear hashing [15, 16]: the pseudocode is provided but not the code, which we wrote in C.
- As for the $O(\log m)$ hash functions, we followed the approach suggested in [15]: we employed the fast and high-quality pseudo-random number generator in [25] using the key to hash as a seed and the $j$th output as the outcome of the $j$th hash function. This takes constant time per hash function. Moreover, we replaced all modulo operations with the equivalent faster operations, as we did for round-mapping.
Round-hashing (this paper): we employed the first output from the pseudo-random number generator in \cite{25} as hash value. We chose the size of our hash range to be a power of two, so that mapping a hash value to an arc number can be done without divisions: we computed the product between the number of buckets and the hash value, divided by the maximum possible hash value. Note that some care is required to compute the product correctly as it may overflow.

It is worth noting that replacing the expensive division was very effective in our measurements. In particular, we replaced the division by $s'$ in Algorithm 1 with the precomputed equivalent combination of multiplication and shift: as $s_0 \leq s' \leq 2s_0$, this can be done at initialization time with a constant amount of work. This reduced the time per round-hashing call from 14.02ns to 8.71ns, a 60% decrease, which is an interesting lesson that we learned.

Figure 2 shows the running times for the above implementations, when computing ten million hash values, as the number of buckets varies on the x-axis. On the y-axis, the running times are reported for jump consistent hashing, linear hashing, and three versions of our round-hashing: the full round-hashing cost (i.e. given a key, return its bucket number); the cost of round-mapping alone (i.e. given a position $u$ in the circumference, return its bucket number); and the cost of computing just the hash value using \cite{25}. As it can be seen, as the overhead of the latter is negligible, the costs of round-hashing and round-mapping are very close and constant along the x-axis, outperforming the non-constant costs of jump consistent hashing and linear hashing, which behave similarly when the number of buckets is large. Note that round-hashing has at least an order of magnitude improvement at around $2^{16}$ buckets and on, which indicates that it scales well.

All the running times in Figure 2 were normalized by the time needed to compute the sum of all the values. Looking at the absolute figures, the running time for the sum is about 0.4ns per element, and that of round mapping is 8–10ns per element (and the pseudo-random number generator in \cite{25} takes twice the cost of the sum).

Speed is not the whole story as it is important also how the hash values in the range are
Given a universe $U$ of keys, and a random hashing function $h : U \rightarrow I$, where $I = \{0, 1, \ldots, |I| - 1\}$, we build a hash table that keeps a stash of keys in main memory. Armed with the round-hashing, we obtain a hash table called round-table that uses $O(k + 1)$ words in main memory, where $k$ denotes the number of stash keys. We consider the stash to be a set of $k$ keys, where notation $\text{stash}[b]$ indicates the set $\{x \in \text{stash} : \text{findBucket}(h(x)/|I|) = b\}$ (e.g. a hash table in main memory with maximum size $O(B + \log n/\log \log n)$ w.h.p. via a classical load balancing argument). To check if $x \in \text{stash}$, we check if $x \in \text{stash}[b]$ where $b = \text{findBucket}(x)$. Also, for a user given parameter $\epsilon$, the guaranteed space utilization in external memory is $1 - \epsilon$.

### Table 2: Statistics on how much hash space is assigned to a given bucket, with a total of 10000 buckets. Note that the actual bucket sizes are obtained by multiplying the numbers in columns min, max, 1%, 99% by the load factor $\alpha$. Extremal values and percentiles are a ratio from the ideal value.

| $s_0$ | $\frac{\sigma}{\mu}$ | min | max | 1%   | 99%   | percentile ratio |
|-------|----------------|-----|-----|------|------|-----------------|
| jump consistent h. | 0.316 | 0.988 | 1.012 | 0.993 | 1.007 | 1.014 |
| round-hashing | 1 | 29.325 | 0.610 | 1.221 | 0.610 | 1.221 | 2.001 |
| | 2 | 20.272 | 0.814 | 1.221 | 0.814 | 1.221 | 1.500 |
| | 4 | 7.192 | 0.977 | 1.221 | 0.977 | 1.221 | 1.250 |
| | 8 | 4.465 | 0.976 | 1.085 | 0.976 | 1.085 | 1.112 |
| | 16 | 2.560 | 0.976 | 1.028 | 0.976 | 1.028 | 1.053 |
| | 32 | 0.613 | 0.976 | 1.002 | 0.976 | 1.002 | 1.027 |
| | 64 | 0.421 | 0.989 | 1.002 | 0.989 | 1.002 | 1.013 |
| | 128 | 0.277 | 0.995 | 1.002 | 0.995 | 1.002 | 1.007 |
| linear hashing | 1 | 29.329 | 0.602 | 1.232 | 0.605 | 1.228 | 2.030 |
| | 2 | 20.274 | 0.803 | 1.234 | 0.808 | 1.228 | 1.520 |
| | 4 | 7.203 | 0.964 | 1.232 | 0.969 | 1.225 | 1.264 |
| | 8 | 4.476 | 0.965 | 1.095 | 0.970 | 1.090 | 1.124 |
| | 16 | 2.583 | 0.965 | 1.041 | 0.970 | 1.034 | 1.066 |
| | 32 | 0.685 | 0.968 | 1.014 | 0.973 | 1.009 | 1.037 |
| | 64 | 0.527 | 0.980 | 1.014 | 0.984 | 1.009 | 1.025 |
| | 128 | 0.417 | 0.985 | 1.014 | 0.990 | 1.009 | 1.019 |

4 External-Memory Tables

We have distribution properties that are similar to jump consistent hashing, as long as we choose suitable values: $s_0 = 64$ for round-hashing and $s_0 = 128$ for linear hashing. Figure 2 has been plotted using these values of $s_0$. 

To check if $x \in \text{stash}$, we check if $x \in \text{stash}[b]$ where $b = \text{findBucket}(x)$. Also, for a user given parameter $\epsilon$, the guaranteed space utilization in external memory is $1 - \epsilon$. 

The columns in the table report the parameters for $10^4$ buckets, where the actual bucket sizes are obtained by multiplying parameters in $\{\min, \max, 1\%, 99\%\}$ by the load factor $\alpha = 10^9/10^4$. Specifically, $s_0$ useful for linear hashing and round-hashing, the standard error $\sigma/\mu$ where $\sigma$ is the variance and $\mu$ is the average of the bucket sizes, the minimum and maximum bucket size, the 1% and 99% percentiles of the size, and the ratio between the latter two. This ratio is the most important parameter in the table as it shows how well-balanced are buckets. It can be easily seen that both round-hashing and linear-hashing can match almost perfectly, with round-hashing having a slightly better distribution. Based on this table, we can see that round-hashing and linear-hashing have distribution properties that are similar to jump consistent hashing, as long as we choose suitable values: $s_0 = 64$ for round-hashing and $s_0 = 128$ for linear hashing. Figure 2 has been plotted using these values of $s_0$. 

The columns in the table report the parameters for $10^4$ buckets, where the actual bucket sizes are obtained by multiplying parameters in $\{\min, \max, 1\%, 99\%\}$ by the load factor $\alpha = 10^9/10^4$. Specifically, $s_0$ useful for linear hashing and round-hashing, the standard error $\sigma/\mu$ where $\sigma$ is the variance and $\mu$ is the average of the bucket sizes, the minimum and maximum bucket size, the 1% and 99% percentiles of the size, and the ratio between the latter two. This ratio is the most important parameter in the table as it shows how well-balanced are buckets. It can be easily seen that both round-hashing and linear-hashing can match almost perfectly, with round-hashing having a slightly better distribution. Based on this table, we can see that round-hashing and linear-hashing have distribution properties that are similar to jump consistent hashing, as long as we choose suitable values: $s_0 = 64$ for round-hashing and $s_0 = 128$ for linear hashing. Figure 2 has been plotted using these values of $s_0$.
### Table 3: Percentage of elements on the stash as \( s_0 \) and \( \epsilon \) change, with \( B = 1024 \).

The lookup algorithm is straightforward while the insertion algorithm is a bit more complex (see the appendix for the pseudocode). After checking that the key is not in the table, it proceeds with the insertion. For this, we need to maintain the claimed space utilization of \((1 - \epsilon)\). That is, if \( \lceil \frac{n}{B(1-\epsilon)} \rceil > \text{numBuckets()} \), we need one more block. We invoke \texttt{newBucket()}\(^\dagger\), and receive a list of \( z < 2s_0 \) block numbers. We have to distribute the keys stored in these \( z \) blocks over \( z + 1 \) blocks, where the extra block has number \texttt{numBuckets()}\(^\dagger\) as it is the latest allocated block number by round-mapping. In the distribution, the keys from the stash are also involved, as described below in the function \texttt{distribute}. After that, \texttt{findBucket()}\(^\dagger\) finds the external-memory block \texttt{block()}\(^\dagger\) that should contain the key: if it is full, the key is added to the stash.

Function \texttt{distribute(b_0, b_1, \ldots, b_{z-1})} takes these \( z \) block numbers from \texttt{newBucket()}, knowing that \( b_z = \text{numBuckets()} \) is the new allocated block number, and thus allocates \texttt{block(b_z)}\(^\dagger\). Then it loads \texttt{block(b_{z-1})} and moves to \texttt{block(b_z)} all keys \( x \in \text{block(b_{z-1})} \) such that \texttt{findBucket(x) = b_z}\(^\dagger\). Also, for each \( x \in \text{stash}[b_{z-1}] \) such that \texttt{findBucket(x) = b_z}, it moves \( x \) to \texttt{block(b_z)}\(^\dagger\), if there is room, or to \texttt{stash}[b_z]\(^\dagger\) otherwise. Next, we repeat this task for \( b_{z-2} \) and \( b_{z-1} \) while also taking care of moving keys from \texttt{stash[b_{z-1}]}\(^\dagger\) to \texttt{block(b_{z-1})} if there is room, and so on. In this way, the cost of \texttt{distribute} is \( 2z + 1 \) block transfers, using \( O(B) \) space in main memory, taking \( O(s_0(B + \log n/\log \log n)) \) CPU time w.h.p., and \( O(s_0B) \) expected time.

The deletion algorithm is similar to the insertion one (see the appendix for the pseudocode), and its performance can be bound in the same way as above. We check the condition \( \lceil \frac{n}{B(1-\epsilon)} \rceil < \text{numBuckets()} - 1 \) for \( n > 0 \) to run \texttt{freeBucket()}\(^\dagger\) using a slightly different \texttt{distribute} that proceeds in reverse. Note that the rhs of the condition is \texttt{numBuckets()}\(^\dagger\) - 1 to avoid \texttt{newBucket()}\(^\dagger\) being called too soon.

In Appendix \[B\]\(^\dagger\), we show that as long as we choose \( s_0 > \frac{2}{\epsilon} \) we have that the stash size of a hash table implemented with round-hashing is similar to the behaviour we would get with an uniform hash function (that would require rehashing). Thus, we recommend choosing \( s_0 \epsilon > 2 \), as confirmed by the experiments below. Moreover, in Appendix \[C\]\(^\dagger\), we show how to keep a copy of the stash in external memory, without increasing space usage but increasing the number of block operations per update to \( O(1 + \epsilon s_0) \).

To evaluate our approach, we consider the worst-case stash size (over the number of keys) across multiple values of \( n \) (going from \( 2^{10}B \) to \( 2^{13}B \)) for \( B = 512, 1024, 2048 \) as \( \epsilon \) and \( s_0 \) vary. The results are reported in Tables 3, 4 and 5 (the last two can be found in the appendix), where the left side of every column reports the ratio predicted by the analysis of Appendix \[B\]\(^\dagger\), and the right side shows the effective maximum ratio reported during the experiment. As our analysis is substantially different when \( \epsilon s_0 > 1 \), we reported those values in bold to highlight them. Finally, the last row reports the
best values one can hope to achieve for that value of $\epsilon$, that is, the values that our analysis predicts for a uniform hash function.

Looking at these results, we can make some observations. First, the values predicted by the analysis match the results fairly well, especially when $s_0 \epsilon \gg 1$ or $s_0 \epsilon \ll 1$. In particular, it almost never happens that the analysis is wrong by more than a factor of 3. Second, when $s_0$ is small, stash size is fairly high, even for low space utilization. This is to be expected, as in this case different buckets may have very different assignment probabilities. Third, as $s_0$ grows, stash size quickly approaches the one that we would expect from the ideal case. Nonetheless, the improvement is fairly small when $s_0$ goes over 32, even at low utilization. We thus recommend $s_0$ to be chosen near 32 for practical usage.

We also considered how stash size varies over time, as more elements are inserted. To study that, we fixed $s_0 = \frac{2}{\epsilon}$, as recommended in the analysis section, and plotted the size of the stash against the number of elements in the table. The plots can be found in Figure 3. These plots clearly show the “cyclic” behavior of round-table: when a new round begins, the distribution of keys in buckets is further away from being uniform and, as a result, the stash size increases. As more steps of the round are completed, the spikes in stash size get progressively smaller as round-table balances keys in a better way, until a new round starts again and the table reverts to its previous behavior.

Figure 3: Stash size (on the y-axis) as $n$ grows (on the x-axis) for $s_0 = \frac{2}{\epsilon}$ and different values of $\epsilon$. 
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In the following, we consider the arcs along the circle handled by round-mapping as buckets. To give bounds on the expected stash size for a hash function that does not map a value into a bucket equiprobably [9], we will first study the expected number of elements that overflow from a bucket of size $B$ that is expected to reach a load factor of $1 - \delta$, i.e. when any of the $n$ values are mapped to that bucket with probability $O\left(\frac{B(1-\delta)}{n}\right)$. We will consider the cases in which $\delta = 0, \delta < 0$ and $0 < \delta < 1$ separately.

The number of values that are assigned to the given bucket is a random variable with binomial distribution $B\left(n, \frac{B(1-\delta)}{n}\right)$, which, as $n$ grows, can be approximated with a normal variable with distribution $N(B(1-\delta), B(1-\delta))$. The number of values that overflow from that bucket is given by

\[ \text{Overflow} = n - B(1-\delta). \]
1. 0 if $x < B$ values end up in that bucket.
2. $x - B$ if $x \geq B$ values end up in that bucket.

From this follows that the expected number of overflown values from the bucket is given by

$$
\frac{1}{\sqrt{2\pi B(1-\delta)}} \int_B^\infty (x-B)e^{\frac{(x-B(1-\delta))^2}{2B(1-\delta)}} dx
$$

With some calculations, we find out that the value of this integral is given by

$$
\frac{\sqrt{B(1-\delta)}e^{-\frac{\delta}{2(1-\delta)}}}{\sqrt{2\pi}} - \frac{\delta B}{2} \text{erfc}\left(\sqrt{\frac{B}{2(1-\delta)}}\right)
$$

where $\text{erfc}$ is the complementary error function.

**A.1 Case $\delta = 0$**

In this case, we easily get that the expected number of values that overflow from a single bucket is

$$\sqrt{\frac{B}{2\pi}}.$$

We can use this result to bound the stash size in the case $\epsilon = 0$ with a uniform hash function: indeed, we get that the expected size of the stash grows as

$$\frac{n}{\sqrt{2\pi B}}.$$

**A.2 Case $\delta > 0$**

As $0 < \text{erfc}(x) < 2$, we remark that the first term of the above expression is an upper bound. This, as before, proves that the size of the stash with an uniform hash function grows at most as

$$n\frac{e^{-\frac{\delta}{2(1-\epsilon)}}}{\sqrt{2\pi B(1-\epsilon)}}$$

which decreases exponentially in $B$. Under the assumption that $\epsilon\sqrt{B}$ is big enough, we can replace $\text{erfc}$ in the above expression with its Taylor series to get a bound of

$$n\frac{\sqrt{1-\epsilon}}{\epsilon^2 B\sqrt{2\pi B}}e^{-\frac{\delta}{2\frac{1}{1-\epsilon}}}$$

that improves the previous one by a factor of $\frac{1}{B}$. Of course, this second bound only holds when $B$ and $\epsilon$ are large enough.

**A.3 Case $\delta < 0$**

In this case, we can replace $\text{erfc}$ with its upper bound 2 and ignore the first term, as it is quite smaller than the second one. By multiplying by the number of buckets as in the previous cases, we get the bound of $n\frac{-\delta}{1-\delta}$ for a hash table that tries to fit $n$ values in buckets that only have space for $\frac{n}{1-\delta}$, as one would expect. This corresponds to an expected stash size per bucket of $-\delta B$. 
B Analysis of stash size with round-hashing

Our version of consistent hashing guarantees that, at any time, the expected load on the most-loaded bucket will be, at most, $1 + \frac{1}{s} \leq 1 + \frac{1}{s_0}$ times the expected load on the least-loaded one. We can give an upper bound for the load factor of those buckets as $(1 - \epsilon)(1 + \frac{1}{s}) \leq 1 - \epsilon + \frac{1}{s}$. If $s_0 > \frac{2}{\epsilon}$, our hash table implemented with round-mapping this version of consistent hashing will behave at least as well as a hash table implemented with uniform hashing and a load factor of $1 - \frac{2}{\epsilon}$, in terms of stash size.

We will now consider what happens when $s_0 \leq \frac{1}{\epsilon}$. For simplicity, we will first study the behavior of the stash in the first step $s = s_0$ of every round. Let $c$ be the number of buckets that was present the last time the buckets were all equally sized, and let $c + q$ be the current number of buckets. We know that there are $c - s_0q$ buckets with size proportional to $\frac{1}{c}$ and $(s_0 + 1)q$ buckets with size proportional to $\frac{s_0}{c(1+s_0)} = \frac{1}{c + \frac{1}{s_0}}$. As the second kind of buckets is less loaded than they should be with a uniform hash function, we will ignore them as they will not contribute to the stash more than the uniform case.

We expect each of the bigger buckets to be assigned $\frac{n}{c}$ keys, while only having space for $B$. Thus, the $c - s_0q$ bigger buckets will behave as buckets that have a load factor of $1 - \delta$ with $\delta$ given by

$$-\delta = \frac{n}{c} - \frac{B}{c - s_0q} = \left(1 + \frac{q}{c}\right)(1 - \epsilon) - 1 \approx \frac{q}{c} - \epsilon$$

where we used the fact that $c + q = \frac{n}{(1-\epsilon)}$. The analysis in Subsection A gives us an expected stash size of

$$(c - s_0q)\left(\frac{q}{c} - \epsilon\right)B = n(1 - \epsilon)\frac{(1 - s_0\frac{q}{c})(\frac{q}{c} - \epsilon)}{1 + \frac{q}{c}}$$

Since $c > s_0q$, we can use standard tools from analysis to find that the maximum value of this function is realized when

$$\frac{q}{c} = \sqrt{\left(1 + \frac{1}{s_0}\right)(1 + \epsilon)} - 1$$

with a value of

$$n(1 - \epsilon)\left[1 + \epsilon s_0 - 2s_0 \sqrt{\left(1 + \frac{1}{s_0}\right)(1 + \epsilon)} - 1\right]$$

Note that this expression is decreasing in $s_0$: this proves that the worst-case behavior for stash occurs in the first step of a round, and the above formula actually gives an upper bound on the amount of keys in the stash.

Ignoring smaller terms as $s_0$ grows and $\epsilon$ goes to 0, we can rewrite the expression as $n\frac{1-\epsilon}{4s_0}(1 - s_0\epsilon)^2$.

In particular, when $\epsilon = 0$, we expect the additional stack size (wrt. a uniform hash function) to grow as $n\frac{1}{4s_0}$.

C Keeping an external-memory copy of the stash

In practical applications, it may be useful to keep a copy of the stash on external memory (for example, to have a copy in case of application crashes). We now show a variation on our hash table that achieves this at the cost of increasing the number of block transfer for the updates by an expected constant factor. To achieve this, we treat the “leftover” space in underflown buckets as an array (because of our allocation rule, we know that there is always enough “leftover” space to store
the full contents of the stash). We start by filling it from the lowest-indexed bucket and proceed on towards buckets with bigger indexes.

To insert a key in the stash on external memory, we try to fit it in the last underflown block used. If the block is full, we find a new block by scanning all the buckets on the left until we find a non-full one: this process will terminate in \(O(1)\) block transfers w.h.p.\(^6\) as a bucket is not full with probability at least \(\frac{1}{2}\).

Deletion works in a similar way: if the key to be deleted is not in the last position of our virtual array, we swap it with the last one and go back one position.

If we want to insert a "legitimate" key in a block that has stash keys in it, we can identify a stash key since it has the incorrect hash for its bucket. We can then move it to the front of the virtual array and proceed as usual. We proceed in the opposite way if space is freed up in a block that should be used in the virtual array.

It remains to see how we reassign the keys after a newBucket() or freeBucket() operation. In the worst case, it may cause us to perform \(O(s_0 B)\) stash operations, which could require up to \(O(s_0 B)\) block transfers. However, we may decide to delay those stash operations, doing \(O(s_0)\) of them on each of the subsequent update operations, without causing significant changes in how the hash table performs.

We can compute the expected number of stash operations and prove that this method performs better in expectation, without requiring \(O(s_0)\) operations per update. We consider the case of newBucket(), as freeBucket() behaves in the same way. We use the results of Subsection A. Since, before looking into the the stash, we move values within the buckets themselves, the number of stash operations may be bounded by the number of empty cells in the buckets after internal re-arranging. As the load factor on those buckets will be approximately \(1 - \epsilon - \frac{1}{s_0}\), we expect \(s_0\) of those buckets to still have space for \(B(\epsilon s_0 + 1)\) elements, plus the elements that were put into the stash. We can give an upper bound for the elements that are put into the stash during this procedure by increasing the load factor to \(1 - \frac{1}{s_0}\) and using the formulas we obtain from the analysis:

\[
(s_0 + 1) \sqrt{B \left(1 - \frac{1}{s_0}\right) e^{-\frac{B}{2s_0^2(1 - \frac{1}{s_0})}}} \frac{B^{\frac{s_0}{2}}}{\sqrt{2\pi}}
\]

This can be bounded as \(\sqrt{B} + \frac{B}{2\sqrt{\pi}} e^{-\tau^2}\) for some \(\tau\). Since \(\tau e^{-\tau^2} < 1\), we can bound the total number of stash insertions by \(O(B)\).

This gives a total of \(O(B(1 + \epsilon s_0))\) stash operations. By delaying some of them to the next \(O(B)\) updates, we get an expected number of block transfers per operation of \(O(1 + \epsilon s_0)\), which is constant as long as \(s_0\) is not chosen too big (in particular, it is constant for \(s_0 = \frac{2}{\epsilon}\)).

**D Stash sizes for \(B = 512\) and \(B = 2048\)**

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\(^6\)If we want to reduce the number of block transfers, we can keep an in-memory dictionary that holds the indexes of the buckets that are full, because of keys not in the stash, and perform our scans on that data structure: the expected number of full buckets is smaller than the expected stash size.
Table 4: Percentage of elements on the stash as $s_0$ and $\epsilon$ change, with $B = 512$.

| $s_0$ | 0  | 0.001 | 0.01 | 0.03 | 0.05 | 0.1  |
|-------|----|-------|------|------|------|------|
|       | real est. | real est. | real est. | real est. | real est. | real est. |
| 1     | 17.2% 18.9% | 17.1% 18.8% | 16.7% 17.9% | 15.9% 16.1% | 15.1% 14.7% | 13% 12% |
| 4     | 5.6% 7.3% | 5.5% 7.2% | 5.1% 6.4% | 4.2% 4.8% | 3.4% 3.6% | 1.7% 1.7% |
| 16    | 2.2% 3.3% | 2.1% 3.2% | 1.7% 2.4% | 1% 1% | 0.5% 0.3% | 0.06% 0.4% |
| 32    | 1.9% 2.5% | 1.8% 2.4% | 1.4% 1.7% | 0.8% 0.7% | 0.4% 0.9% | 0.03% 0.09% |
| 64    | 1.9% 2.2% | 1.7% 2.1% | 1.4% 1.4% | 0.7% 1.1% | 0.4% 0.5% | 0.03% 0.04% |
| 256   | 1.9% 1.9% | 1.7% 1.8% | 1.3% 1.5% | 0.7% 0.8% | 0.4% 0.3% | 0.03% 0.02% |
| ideal | 1.8% - | 1.7% - | 1.3% - | 0.7% - | 0.3% - | 0.01% - |

Table 5: Percentage of elements on the stash as $s_0$ and $\epsilon$ change, with $B = 2048$.

| $s_0$ | 0  | 0.001 | 0.01 | 0.03 | 0.05 | 0.1  |
|-------|----|-------|------|------|------|------|
|       | real est. | real est. | real est. | real est. | real est. | real est. |
| 1     | 17.2% 18% | 17.1% 17.9% | 16.7% 17.1% | 15.9% 15.6% | 15.1% 14.4% | 13% 12% |
| 4     | 5.6% 6.5% | 5.5% 6.4% | 5.1% 5.5% | 4.2% 4.2% | 3.4% 3.3% | 1.6% 1.7% |
| 16    | 1.6% 2.4% | 1.5% 2.3% | 1.1% 1.5% | 0.5% 0.5% | 0.2% 0.06% | 0.0009% 0.02% |
| 32    | 1.1% 1.7% | 1% 1.6% | 0.6% 0.8% | 0.2% 0.09% | 0.03% 0.2% | 0.0002% 0.0002% |
| 64    | 0.9% 1.3% | 0.8% 1.2% | 0.5% 0.5% | 0.1% 0.3% | 0.02% 0.05% | 0.0001% 1e-05% |
| 256   | 0.9% 1% | 0.8% 0.9% | 0.5% 0.6% | 0.1% 0.1% | 0.01% 0.01% | 0.0002% 1e-06% |
| ideal | 0.9% - | 0.8% - | 0.5% - | 0.09% - | 0.008% - | 4e-07% - |

E  Pseudocode

Algorithm 2: Lookup algorithm

```plaintext
Function lookup(x)
  if x \in stash then return true
  b \leftarrow findBucket(h(x)/|I|)
  if x \in block(b) then return true
  return false
```

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Algorithm 3: Insertion algorithm

1 Function insert(x)
2     if lookup(x) then return false
3     n \leftarrow n + 1  // initially n = 0
4     if \left\lceil \frac{n}{B(1-\epsilon)} \right\rceil > numBuckets() then
5          distribute(newBucket())
6     b \leftarrow findBucket(h(x)/|I|)
7     if |block(b)| < B then
8          block(b) = block(b) \cup \{x\}
9     else
10        stash = stash \cup \{x\}
11     return true

Algorithm 4: Deletion algorithm

1 Function delete(x)
2     found = false
3     if x \in stash then
4          stash = stash \setminus \{x\}
5          found = true
6     else
7          b \leftarrow findBucket(h(x)/|I|)
8          if x \in block(b) then
9          block(b) = block(b) \setminus \{x\}
10         found = true
11     if found then
12         n \leftarrow n - 1
13         if n > 0 and \left\lceil \frac{n}{B(1-\epsilon)} \right\rceil < numBuckets() - 1 then
14            distribute(freeBucket())
15     return found