Plane 5D worlds and simple compactification

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Abstract

We obtain a new kind of exact solution to vacuum Einstein field equations that contain both Minkowsian world and a special 5D curved spacetime with particularly free structure. This special world is defined by an arbitrary function and a space of three parameters. We suggest that this solution could correct (in principle) certain aspects of the physics in flat spacetime.

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1 Introduction

The use of a fifth dimension in Theoretical Physics has been a common way to accomplish the unification of the gravitation and the other interactions since the seminal work of Nordström [1] and Kaluza [2]. The problem of recover the standard 4-D spacetime can be solve by the Klein’s mechanism [3], which explains how the additional coordinates are length-like and could be detected. The idea of extra-dimension compactification is necessary, there are also certain research oriented to the idea extra-coordinates are not physically real (Projective theories) [4]. There are also the Non-compactified theories, in which extra-dimensions are not necessarily length-like or compact (see [5] and references therein).

In recent years, higher dimensional theories of extended objects, like String Theory [6], provide a promising scheme to construct renormalizable quantum fields that unify all the interactions of Nature.

Since the exact solution found by Davidson and Owen in the mid-eighties [7], the dynamics of 5D-physics has been widely study since the late eighties by Wesson, and his collaborators in their Theory of Induced Matter [8]. Latter, Randall and Sundrum [9] proused a Brane-theory in 5D that explain the large hierarchy between the quantum gravity effect and the standard model. In [10] there an extensive review full of reference of Randall-Sundum Brane worlds principally applied to cosmology.

In the present work, the existence of solutions to the Einstein’s field equations when the metric tensor dependents only of a new extra-coordinate in plane symmetry are discussed. The anzatz followed in the present article is mainly based on [11], who showed that the compactification (vanishing) of one coordinate could happen for vacuum gravity for a static cylindrical world. The philosophy in the work is to discard a wide range of solutions because they do not represent asimptotically flat spacetimes. But we show in [12] that a large family of spacetimes can be constructed by using an internal freedom that Einstein’s field equation give for this specific symmetry and the Israel’s junction conditions.

2 Field equation and its solutions

Let be the following line element:

\[ ds^2 = e^{\gamma(\eta)} dt^2 - e^{\tau(\eta)} dx^2 - e^{\mu(\eta)} dy^2 - e^{\nu(\eta)} dz^2 - e^{\rho(\eta)} d\eta^2 \]  

where the new coordinate ” \eta ” is introduced. Here, all the functions
\( \gamma, \tau, \mu, \nu, \rho \) are only \( \eta \)-dependent. Notice that our procedure is similar in spirit to that of [14], only that we do not include a thin matter wall in energy-stress tensor proportional to a Dirac \( \delta \)-function.

Let us define the symbol \( ()' \) as \( d/d\eta \). Then, the vacuum Einstein field equations \( R_{AB} = 0 \) are

\[
\begin{align*}
\gamma'' + \gamma'^2 - \gamma' \rho' + \gamma' (\tau' + \mu' + \nu') &= (2) \\
\tau'' + \tau'^2 - \tau' \rho' + \tau' (\gamma' + \mu' + \nu') &= (3) \\
\mu'' + \mu'^2 - \mu' \rho' + \mu' (\gamma' + \tau' + \nu') &= (4) \\
\nu'' + \nu'^2 - \nu' \rho' + \nu' (\gamma' + \tau' + \mu') &= (5) \\
2(\gamma'' + \tau'' + \mu'' + \nu'') - \rho'(\gamma' + \tau' + \mu' + \nu') + \gamma'^2 + \tau'^2 + \mu'^2 + \nu'^2 &= (6)
\end{align*}
\]

Let \( \chi = \gamma + \tau + \mu + \nu \), an arbitrary function where its square is naturally

\[
\chi^2 = \gamma^2 + \tau^2 + \mu^2 + \nu^2 + 2\gamma'(\tau' + \mu' + \nu') + 2\tau'(\mu' + \nu') + 2\mu' \nu' \quad (7)
\]

And now by summing \( (2), (3), (4) \) and \( (5) \) we get

\[
2\chi'' + \chi'^2 - \rho' \chi' = 0 \quad (8)
\]

which gives the integral

\[
\rho = \xi + 2 \ln(\chi') + \chi \quad (9)
\]

where \( \xi \) is a arbitrary constant. The value of \( \xi \) can be perfectly chosen as zero, because in \( g_{\eta \eta} = e^\rho = \chi^2 e^{\xi} \) the constant \( e^\xi \) is just an amplification parameter of the length element that only depends of the chosen unit system. According to \( (8) \) and \( (2) \) it is easy to see that:

\[
\gamma = A_0 \chi + B_0 \quad (10)
\]

Applying this procedure to equations \( (3), (4) \) and \( (5) \), we get

\[
\begin{align*}
\tau &= A_1 \chi + B_1 \\
\mu &= A_2 \chi + B_2 \\
\nu &= A_3 \chi + B_3
\end{align*} \quad (11, 12, 13)
\]

Without loss of generally, let us choose \( B_\alpha = 0, (\forall \alpha = 0, 1, 2, 3) \). Therefore, every metric function (except that one related to the extra-coordinate)
maintain a linear relation with $\chi$. Finally, by using the relation in (8) in (6) we obtain the important relation:

$$\chi' (F - 1) = 0$$

(14)

where $F = A_0^2 + A_1^2 + A_2^2 + A_3^2$ is a real non-negative constant. Therefore, from (14) we must analyze the two separate cases:

**Case: $F \neq 1$** This implies that necessary $\chi$ = constant. Hence, $\chi$ is only a scaling parameter, the 4-D Minkowskian space-time is recovered as we expected if $\chi = 0$:

$$ds^2 = dt^2 - (dx^2 + dy^2 + dz^2)$$

(15)

**Case: $F = 1$** According to equation (14), the $F$ parameter must be 1, or

$$A_0^2 + A_1^2 + A_2^2 + A_3^2 = 1$$

(16)

Since $\chi = \gamma + \tau + \mu + \nu$ by definition, it is clearly seen from equations (10) to (13) that

$$A_0 + A_1 + A_2 + A_3 = 1$$

(17)

Then, one can find a map $Q : \mathbb{R}^2 \to \mathbb{R}^4$ that describes the solution to the algebraic equations (16) and (17). For simplicity, let define $A_0 = u$ and $A_1 = v$ and then, the $Q$ map that relates $(u, v)$ with $(A_0, A_1, A_2, A_3)$ is given by the parametrization

$$A = \begin{pmatrix}
A_0 \\
A_1 \\
A_2 \\
A_3
\end{pmatrix} = \begin{pmatrix}
u \\
u \\
-u - 1 + \sqrt{\Delta} \\
u + 1 + \sqrt{\Delta}
\end{pmatrix}$$

(18)

where $\Delta = -3(u^2 + v^2) + 2(u + v - uv) + 1$. The necessity of real integration constants restricts the domain of $Q$ to the region of the $uv$-plane described by the ellipse: $\Delta (u, v) = -3(u^2 + v^2) + 2(u + v - uv) + 1 = 0$ and all its interior points (Figure 1).

One particularly simple solution is, for example: $A_0 = -\frac{1}{2}$, $A_1 = A_2 = A_3 = \frac{1}{2}$. Then, the line element under this conditions would be
Figure 1: The domain of $Q$: All the $(u, v)$ points on the ellipse and inside it define the parametrization of the integration constants

\[ ds^2 = e^{-\frac{1}{2}\chi}dt^2 - e^{\frac{1}{2}\chi}dx^2 - e^{\frac{1}{2}\chi}dy^2 - e^{\frac{1}{2}\chi}dz^2 - (\chi')^2e^{\chi}d\eta^2 \]  

(19)

A formal classification of the spacetimes based on the behavior of the $\chi$-function will be treated with exhaustive details in [12].

3 Conclusions

Let summarize these results here. We found a large class of exact solution to the field equation of Einstein gravity and discovered a new kind of freedom (maybe of mathematical nature) that they permit. A spacetime with the form [19] can be used to study quantum fields theories (QFT’s) is a curved spacetime that corrects the minkowkian plane world. Depending on the form of the arbitrary $\chi$, we can construct a new special classically fluctuating spacetime that can correct (in principle) classical and quantum trajectories of particles [12]. In a more traditional stand, when we applied the conventional Kaluza-Klein unification of electromagnetism and gravitation, EM energy “fusses” with the gravitational one (that is locally undefined), in such way that Kaluza-Klein world could contain zero total energy [15].

We can call this non-trivial solution to Einstein equations R0X. The $R\theta$ is for Ricci flatness that every vaccum solution has. The “X” is for a specific
kind of the $\chi$-function we choose for a spacetime. As we will discuss with extensive details in [15], if $\chi$ is a map $\chi : \mathbb{V} \to \mathbb{W}$, where $\mathbb{V}, \mathbb{W} \subset \mathbb{R}$ and the target set $\mathbb{W}$ is finite ($\mathbb{W} = [0, a]$, $a \in \mathbb{R}$) and $\lim_{\eta \to \pm \infty} \chi = 0$, then it is easy to see from equation (19) that traditional flat world in 4D is recovered. Thus, when $\chi$ is a well-tempered function the extra-dimension disappears. We may suggest that this simple compactification procedure (far more easier that the Calabi-Yau one) can be use to formulate a consistent Brane theory that explain the large difference between the electro-weak and Planck scale.

Now, if the domain $\mathbb{V} = \mathbb{R} - \{S_n\}$, where $\{S_n\}$ is a finite set of point where $\chi$ is ill-defined, the R0X manifold would have a finite number of point where the Riemann tensor does not exist and thus the gravitational tidal forces would be infinite. As in [15] we show that under certain conditions R0X has exact solutions to the geodesic equations, R0X could be a ideal “laboratory” to explore certain proposal made in the past, such as the Penrose’s cosmic censorship conjecture [16]. Also, the study of new singularities could generated a rich arena for the recently developed Loop Quantum Gravity theory [17].

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