Abstract

This paper studies how differentiable representations of certain subsemigroups of the Weyl-Heisenberg group may be obtained in suitably constructed rigged Hilbert spaces. These semigroup representations are induced from a continuous unitary representation of the Weyl-Heisenberg group in a Hilbert space. Aspects of the rigged Hilbert space formulation of time asymmetric quantum mechanics are also investigated within the context of the results developed here.

1 Introduction

Rigged Hilbert spaces have been used in quantum physics since the mid 1960’s. Although the motivation of the first contributions [1,2,3] was to provide a rigorous mathematical context for Dirac’s (already well-established) bra-and-ket formulation of quantum mechanics, subsequent investigations [4,5,6,7] have led to some interesting new physical results. Among these are a formulation of scattering theory which accommodates an asymmetric time evolution given by a semigroup of operators, and a related vector description.
for an (isolated) resonance state. The more recent of these works [7] extend the earlier results to relativistic resonances where it is shown that they can be characterized by irreducible representations of the causal Poincaré semigroup.

Many of the results of these theories can be subsumed under a general study of the representations of Lie groups and their subsemigroups in rigged Hilbert spaces. We call a subset $S$ of a Lie group $G$ a subsemigroup of the group if $S$ contains the identity element and remains invariant under the group multiplication of $G$. Notice that $S$ need not be closed under the inverse operation $x \to x^{-1}$. If $S$ is such a subsemigroup of a Lie group $G$, the problem in its broadest generality can be stated as follows: If $U$ is a continuous (often unitary) representation of $G$ in a Hilbert space $\mathcal{H}$, does there exist a rigged Hilbert space $\Phi \subset \mathcal{H} \subset \Phi^\times$ such that $\Phi$ reduces $U$ to a continuous representation of $S$?

It is clear that if $U|_\Phi$ is such a representation of $S$, then there also exists a dual representation $U|_{\Phi^\times}$ of $S$ in $\Phi^\times$. The semigroup time evolution of Gamow vectors [4,5,6,7], which describe the (isolated) resonance states, is given by such a representation in $\Phi^\times$ dual to a semigroup representation in $\Phi$. In particular, in the non-relativistic scattering theory developed in [4,5,6], time, i.e., the Lie group of real numbers under addition, is unitarily represented, by way of the mapping $(U(t)f)(E) = e^{iEt}f(E)$, in the Hilbert space of square integrable functions defined on the spectrum of the Hamiltonian $H$. The rigged Hilbert spaces $\Phi_- \subset \mathcal{H} \subset \Phi_-^\times$ and $\Phi_+ \subset \mathcal{H} \subset \Phi_+^\times$ of Hardy class functions, introduced to represent the in- and out-states, have the property that $\Phi_+$ and $\Phi_-$ reduce the unitary group representation $U(t)$ in $\mathcal{H}$ to representations of the semigroups (under addition) of negative and positive real numbers, respectively. In the relativistic theory developed in [7], there exist two rigged Hilbert spaces which reduce a unitary representation of the Poincaré group in $\mathcal{H}$ to continuous representations of the forward and backward causal Poincaré semigroups.

These cases provide examples to the general question posed above. At present, the complete answer to this question is not known to us. It is perhaps the case that the problem is unanswerable, at least in the affirmative, as stated above; it may be that such rigged Hilbert spaces are possible for only certain classes of subsemigroups of $G$, and this then means that it is necessary to develop a classification criterion for the subsemigroups $S$ of a Lie group $G$. (Recall that even in the Lie group theory, all subgroups are not considered to be of much interest. It is Lie subgroups that are generally investigated).
In this paper, we shall mainly restrict ourselves to semigroups which are the unions of the products of continuous one parameter subsemigroups.

The purpose of this paper is to study the representations of some sub-semigroups of the Weyl-Heisenberg group as illustrative of the above general question. Our treatment will also reveal the general group theoretical content underlying the rigged Hilbert space formulation of the time asymmetric quantum mechanics developed in [4, 5, 6].

It is convenient to introduce here some preliminary concepts which we shall make use of in the following sections.

**Definition 1.1.** A rigged Hilbert space consists of a triad of vector spaces

\[ \Phi \subset \mathcal{H} \subset \Phi^\times \]  

where:

1. \( \mathcal{H} \) is a Hilbert space
2. \( \Phi \) is a dense subspace of \( \mathcal{H} \) and it is endowed with a complete, locally convex, nuclear topology \( \tau_\Phi \) that is stronger than the \( \mathcal{H} \)-topology
3. \( \Phi^\times \) is the space of continuous antilinear functionals on \( \Phi \). It is complete in its weak* topology \( \tau^\times \) and it contains \( \mathcal{H} \) as a dense subspace.

**Definition 1.2.** A continuous representation of a Lie group \( G \) on a topological vector space \( \Psi \) is a continuous mapping \( T : G \times \Psi \to \Psi \) such that

1. for every \( g \in G \), \( T(g) \) is a linear operator in \( \Psi \)
2. for every \( \psi \in \Psi \) and \( g_1, g_2 \in G \), \( T(g_1g_2)\psi = T(g_1)T(g_2)\psi \)
3. \( T(e) = I \), the identity operator in \( \Psi \)

**Definition 1.3.** A differentiable representation of a Lie group \( G \) on a complete topological vector space \( \Psi \) is a mapping \( T : G \times \Psi \to \Psi \) which fulfills all the requirements of Definition 1.2 and has the additional property that for every one parameter subgroup \( \{ g(t) \} \) of \( G \), \( \lim_{t \to 0} \frac{T(g(t))\phi - \phi}{t} \) exists for all \( \phi \in \Psi \) (and, a fortiori, defines a continuous linear operator on \( \Psi \)).

The semigroup analogues of these definitions are obvious. For instance, in Definition 1.2 we simply replace the one parameter subgroups \( g(t) \) of \( G \) by one parameter subsemigroups \( g(t) \) of \( S \).
2 Weyl-Heisenberg Group and its Subsemigroups

The three dimensional Euclidean space $\mathbb{R}^3$ is a Lie group under the associative multiplication rule defined by

$$(a, b, c)(\alpha, \beta, \gamma) = (a + \alpha, b + \beta, c + \gamma + a\beta)$$ (2.1)

It is easily verified that the origin $(0, 0, 0)$ of $\mathbb{R}^3$ is the identity element and that each element $(a, b, c)$ has an inverse given by $(a, b, c)^{-1} = (-a, -b, -c + ab)$. Thus, under (2.1) $\mathbb{R}^3$ is a group, the well known Weyl-Heisenberg group. Throughout the rest of this paper we shall refer to this group by $G$. We shall denote an element of $G$ by $(a, b, c)$, or by $\xi$, where $\xi = (\xi_1, \xi_2, \xi_3)$.

The Lie algebra $\mathcal{G}$ of the group $G$ is also isomorphic to $\mathbb{R}^3$, and the elements $\chi_1 = (1, 0, 0), \chi_2 = (0, 1, 0)$ and $\chi_3 = (0, 0, 1)$ can be chosen as a basis for $\mathcal{G}$. In fact, $\mathcal{G}$ can be made into an associative algebra (of operators acting on $\mathbb{R}^3$ itself) by way of the multiplication rule $\mathcal{G} \otimes \mathcal{G} \to \mathcal{G}$ defined by

$$(a, b, c)(\alpha, \beta, \gamma) = (0, 0, a\beta)$$ (2.2)

Under (2.2), the basis elements $\chi_i$ fulfill the relations $\chi_i \chi_j = \delta_i^1 \delta_j^2 \chi_3$, and thereupon we have the very well-known Heisenberg commutation relations: $[\chi_1, \chi_2] = \chi_3$, $[\chi_1, \chi_3] = [\chi_2, \chi_3] = 0$.

Among the subsemigroups of $G$ are the following:

$$S_1(0) = \{\xi : \xi_1 \geq 0, \xi_2 = 0, \xi_3 \in \mathbb{R}\}$$

$$S_1 = \{\xi : \xi_1 \geq 0, \xi_2, \xi_3 \in \mathbb{R}\}$$

$$S_2(0) = \{\xi : \xi_1 = 0, \xi_2 \geq 0, \xi_3 \in \mathbb{R}\}$$

$$S_2 = \{\xi : \xi_2 \geq 0, \xi_1, \xi_3 \in \mathbb{R}\}$$

$$S_3 = \{\xi : \xi_1, \xi_2 \geq 0, \xi_3 \in \mathbb{R}\}$$

$$S_4 = \{\xi : \xi_1, \xi_2 \geq 0, \xi_1 \xi_2 \geq \xi_3 \geq 0\}$$ (2.3)

It is readily seen that each set in (2.3) is a topological semigroup. More specifically, (2.1) reduces to a continuous, associative multiplication on every $S_i$, and none is closed under the inverse operation $\xi \to \xi^{-1}$. Thus each $S_i$ is truly a topological subsemigroup of $G$. Furthermore, it is straightforward to verify that the set consisting of the inverses of the elements in each $S_i$ of (2.3) is also a subsemigroup of $G$. We shall denote this complementary semigroup to $S_i$ by $S_i^{-1}$. 

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Next, let $L^2$ be the Hilbert space of square integrable (with respect to Lebesgue measure) functions on the real line $\mathbb{R}$. The mapping $U : G \otimes L^2 \to L^2$, defined by

\[(U(\xi)f)(x) = e^{i\xi_1}e^{ix\xi_2}f(x + \xi_1)\]  

(2.4)

furnishes a continuous unitary representation of $G$ in $L^2$. The differential of $U$ at the identity $0$, $dU|_0$, yields a representation of the Lie algebra $\mathcal{G}$, a well known result from the classical representation theory. In particular, the basis elements $\chi_i$ acquire representation as the linear operators

\[(dU|_0(\chi_1)f)(x) = (Mf)(x) = ixf(x)\]

\[(dU|_0(\chi_2)f)(x) = (Df)(x) = \left(\frac{df}{dx}\right)(x)\]

\[(dU|_0(\chi_3)f)(x) = if(x)\]  

(2.5)

It is clear that the first two equalities may be defined not on the whole of $L^2$ but on a dense subspace thereof.

In the remainder of this paper we shall discuss how a rigged Hilbert space may be constructed so that the restriction $U|_{\Phi}$ of $U$ to $\Phi$ yields therein a non-trivial (i.e., one that does not extend to a representation of a subgroup of $G$) differentiable representation of two of the subsemigroups in (2.3). We shall also remark on how rigged Hilbert spaces maybe constructed for the other subsemigroups in (2.3).

### 3 A Differentiable Representation of $S_1(0)$ in a Rigged Hilbert Space

In this section we shall construct a rigged Hilbert space $\Psi \subset L^2 \subset \Psi^\ast$ such that the restriction of $U$ to $\Psi$ yields a representation of the subsemigroup $S_1(0)$ of $G$ defined in (2.3). The main technical result is the construction of the rigged Hilbert space.

#### 3.1 Construction of the Rigged Hilbert Space

**Definitions** Let $L^2_+$ and $L^2_-$ be the Hilbert spaces of square integrable (with respect to Lebesgue measure) functions supported in $(0, \infty)$ and $(-\infty,0)$, respectively. Let us denote the norms in these spaces by $\|\cdot\|_+$ and $\|\cdot\|_-$. 
The restriction of $L^2$-functions to $(0, \infty)$ and $(-\infty, 0)$ define, respectively, two projection operators $Q_+$ and $Q_-$ onto $L^2_+$ and $L^2_-$. Thus, $L^2_+ = Q_+L^2$, $L^2_- = Q_-L^2$, and $L^2 = L^2_+ \oplus L^2_-$. 

The mapping \( \mathbb{H}: L^2 \to L^2 \) defined by

\[
(\mathbb{H}f)(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} dx \frac{f(x)}{x-t}
\]

where the integral is defined as the Cauchy principal value, is called the Hilbert transform. It is well known that the operators $P_+ = \frac{1}{2}(I + i\mathbb{H})$ and $P_- = \frac{1}{2}(I - i\mathbb{H})$ are projections from $L^2$ onto $\mathcal{H}^2_+$ and $\mathcal{H}^2_-$, the Hilbert spaces of Hardy class functions from above and below, respectively [8]. Thus, \( \mathcal{H}^2_+ = P_+L^2 \), \( \mathcal{H}^2_- = P_-L^2 \), and \( L^2 = \mathcal{H}^2_- \oplus \mathcal{H}^2_+ \).

For any \( f \in L^1(\mathbb{R}) \), the function \( \hat{f} \) defined by the integral

\[
\hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-ixt}dx
\]  

(3.1)
is said to be the Fourier transform of \( f \). It is well known that the mapping \( \mathcal{F}: f \to \hat{f} \) defined by (3.1) extends to a unitary transformation on \( L^2 \).

Fourier transform \( \mathcal{F} \) provides a unitary equivalence between the two sets of projection operators introduced above: A Paley-Wiener theorem asserts that \( \mathcal{F}(Q_+(L^2)) = P_+(L^2) \), \( Q_+(L^2) = \mathcal{F}^{-1}(P_+(L^2)) \), \( \mathcal{F}^{-1}(Q_-(L^2)) = P_-(L^2) \), and \( Q_+(L^2) = \mathcal{F}(P_-(L^2)) \).

A remarkable theorem of C. van Winter [9] states that a function in \( \mathcal{H}^2_\pm \) is completely determined by its values on \( (0, \infty) \) (or on \( (-\infty, 0) \)). Further, the restriction of \( \mathcal{H}^2_\pm \)-functions to \( (0, \infty) \) form a dense subspace of \( L^2_+ \). Similarly, their restrictions to \( (-\infty, 0) \) are dense in \( L^2_- \). Transcribed to our notation, the theorem states that the \( L^2 \)-inclusions \( Q_+P_+ \subset Q_+ \) and \( Q_-P_- \subset Q_- \) are dense.

**Proposition 3.1.** The functions \( -iP_+(L^2) \) form a dense subspace of \( \mathcal{H}^2_+ \). Similarly, the functions \( iP_-(L^2_-) \) are dense in \( \mathcal{H}^2_- \).

**PROOF** Suppose \( f_0 \in \mathcal{H}^2_+ \), and \( \epsilon \), any positive number. For any \( h \in \mathcal{H}^2_-, \) we have \( P_+(f_0 + h) = P_+f_0 = f_0 \). Now, by the above mentioned theorem of van Winter, we can choose \( h \in \mathcal{H}^2_- \) such that

\[
|i f_0 + h|_+ < \frac{\epsilon}{2}
\]
For such an \( h \), let \( \tilde{g} = Q_- (i f_0 - h) \). Then,

\[
\| -i P_+ \tilde{g} - f_0 \| = \| P_+ (\tilde{g} - i f_0) \| \\
= \| P_+ (\tilde{g} - i f_0 + h) \| \\
\leq \| \tilde{g} - i f_0 + h \| \\
= (\| \tilde{g} - i f_0 + h \|^2 + \| -i f_0 + h \|_+^2)^{1/2} \\
= \| -i f_0 + h \|_+ < \frac{\epsilon}{2}
\]

(3.2)

Thus, \(-i P_+ (L^2_+)\) is dense in \( H^2_- \). The same argument shows that \( i P_- (L^2_-) \) is dense in \( H^2_+ \). □

This proposition shows that the denseness of the inclusions \( Q_+ P_\pm \subset Q_+ \) and \( Q_- P_\pm \subset Q_- \), i.e., van Winter’s theorem, implies the denseness of the complementary inclusions \( P_+ Q_\pm \subset P_+ \) and \( P_- Q_\pm \subset P_- \).

**Definitions** Let \( S \) be the space of Schwartz functions on \( \mathbb{R} \). That is, if \( f \in S \), then we have \( f \in C^\infty (\mathbb{R}) \) and \( \lim_{x \to \pm \infty} x^n f(x) = 0 \) for \( n = 0, 1, 2, \cdots \). It is well known that \( S \) is dense in \( L^2 \). There exists a locally convex topology\(^1\) under which \( S \) becomes a Fréchet space, and the Fourier transform \( F \) defined by (3.1) is a homeomorphism on this Fréchet space. Further, \( S_\pm \), the space of \( S \)-functions with the support in \((0, \pm \infty)\) is dense in \( L^2_\pm \). The above mentioned Paley-Wiener theorem implies that \( F(S_\mp) = S \cap H^2_\pm \) and that \( S \cap H^2_\pm \) is dense in \( H^2_\pm \).

Let \( \mathcal{N} \) be the subspace of Schwartz functions with vanishing moments of all orders. That is, if \( f \in \mathcal{N} \), then \( f \in S \) and \( \int_{-\infty}^{\infty} x^n f(x) dx = 0 \) for \( n = 0, 1, 2, \cdots \). Let \( \mathcal{N}_\pm \) be the space of \( \mathcal{N} \)-functions supported in \((0, \pm \infty)\). It is shown in Appendix A that \( \mathcal{N}_\pm \) is dense in \( L^2_\pm \). Since \( \mathcal{N}_- \oplus \mathcal{N}_+ \subset \mathcal{N} \), it then follows that \( \mathcal{N} \) is dense in \( L^2 \).

The unitarity of the Fourier transform \( F \) implies that the image of \( \mathcal{N} \) under \( F \) is dense in \( L^2 \). Let this space be denoted by \( \mathcal{M} \). A function \( f \) in \( \mathcal{M} \), being the Fourier transform of a function in \( \mathcal{N} \), is smooth, rapidly decaying and has vanishing derivatives of all orders at the origin. Further, from Appendix A, it is clear that \( \mathcal{N}_- \oplus \mathcal{N}_+ \subset \mathcal{N} \cap \mathcal{M} \), and so, the space \( \mathcal{N} \cap \mathcal{M} \) is dense in \( L^2 \). It is also straightforward to verify that \( \mathcal{N} \cap \mathcal{M} \) is a closed subspace of \( \mathcal{S} \). Its invariance under the Fourier transform is an interesting property.

\(^1\)This topology is defined by the countable family of norms \( |f|_{mn} = \sup_{x \in \mathbb{R}} |x^m \frac{d^n}{dx^n} f(x)| \), where \( m \) and \( n \) are positive integers. Equivalently, the norms \( \|f\|_n = \|(M^2 + D^2 + I)^n f\| \) can be used. See also (3.5).
Proposition 3.2. $\mp iP_\pm(N_-)$ is dense in $S \cap \mathcal{H}^2_\pm$.

PROOF Let $f_0$, $\tilde{g}$ and $\epsilon$ be as in the proof of Proposition 3.1. The denseness of $N_-$ in $L^2_-$ (see Appendix) allows us to choose $g \in N_-$ such that
\[ \|g - \tilde{g}\| < \frac{\epsilon}{2}. \]

Thus,
\[ \| -iP_+g - f_0 \| \leq \| -iP_+(g - \tilde{g}) \| + \| -iP_+\tilde{g} - f_0 \| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \]

It only remains to show that $-iP_+g \in S$. To that end, recall that the Fourier transform of the Hilbert transform of a function satisfies the equality $(\mathbb{H}f)(y) = -\frac{i}{2y}f(y)$. Therefore,
\begin{equation}
(-iP_+g)(y) = \left(\frac{1}{2}(\mathbb{H} - iI)g\right)(y) = -\frac{i}{2} \frac{y}{|y|} \tilde{g}(y) - \frac{i}{2} \tilde{g}(y)
\end{equation}

Since $g \in N$, $\tilde{g}$ has vanishing derivatives of all orders at $y = 0$. Thus, $(-iP_+g)$, and therewith also $(-iP_+g)$, belongs to $S$.

The same argument proves the denseness of $iP_-(N_-)$ in $S \cap \mathcal{H}^2_\pm$. □

Remark: Notice that $(-iP_+g)$ is in fact an element of $\mathcal{M}$. This means that $(-iP_+g) \in N \cap \mathcal{H}^2_\pm$, whenever $g \in N$. That is, $N$ has the interesting property that it is invariant under the Hilbert transform $\mathbb{H}$. Furthermore, it follows that $N \cap \mathcal{H}^2_\pm$ is dense in $\mathcal{H}^2_\pm$, and therefore also in $S \cap \mathcal{H}^2_\pm$.

Definitions: From Proposition 3.2 it follows that $-iP_+(N_-) \oplus iP_-(N_-)$ is a dense subspace of $L^2$. If $f$ is an element of this subspace, then for unique functions $g, h \in N_-$,
\begin{equation}
f = -iP_+g + iP_-h \tag{3.4}\end{equation}

We may introduce a locally convex topology on $-iP_+(N_-) \oplus iP_-(N_-)$ by defining a family of norms $\|f\|_n$ for $f$:
\begin{equation}
\|f\|_n^2 = \|-iP_+g\|_n^2 + \|P_-h\|_n^2 + \|-iP_-g\|_n^2 + \|-P_+h\|_n^2 \tag{3.5}\end{equation}

The norms on the right hand side of (3.5) refer to the topology that $-iP_+(N_-) \oplus iP_-(N_-)$ inherits as a subspace of $S$. For instance, $\|-iP_+g\|_n$ can be iteratively defined by
\begin{equation}
\|-iP_+g\|_{n+1}^2 = \|M(-iP_+g)\|_n^2 + \|D(-iP_+g)\|_n^2 + \|-iP_+g\|_n^2, n = 0, 1, 2, \cdots \tag{3.6}\end{equation}
where \( M \) and \( D \) are the multiplication and differentiation operators defined in (2.5) and \( \| \cdot \|_0 \) is the \( L^2 \)-norm. The topology induced on \( \mathcal{S} \) by the norms of (3.6) is equivalent to the one given by the more customary norms \( \| f \|_{mn} = \sup_{x \in \mathbb{R}} |x^m \frac{d^n}{dx^n} f(x)| \).

Let \( \Psi \) be the direct sum space \(-iP_+(\mathcal{N}_-) \oplus iP_-(\mathcal{N}_-)\) endowed with the topology given by the norms (3.5).

**Proposition 3.3.** \( \Psi \) is a nuclear Fréchet space.

**Proof.** Local convexity and metrizability of \( \Psi \) are obvious from (3.5). To see that \( \Psi \) is complete, suppose that \( \{ f_i \} \) is a Cauchy sequence in \( \Psi \). Since each \( f_i \) has the decomposition \( f_i = -iP_+g_i + iP_-h_i \) (3.7) for some \( g_i, h_i \in \mathcal{N}_- \), we obtain four Cauchy sequences: \( \{ -iP_+g_i \} \) and \( \{ -iP_-h_i \} \) in \( \mathcal{S} \cap \mathcal{H}^2_+ \); \( \{ iP_+g_i \} \) and \( \{ iP_-h_i \} \) in \( \mathcal{S} \cap \mathcal{H}^2_- \). Since these two spaces are complete, we conclude that there exist functions \( g, h \in \mathcal{S} \cap \mathcal{H}^2_+ \) and \( \tilde{g}, \tilde{h} \in \mathcal{S} \cap \mathcal{H}^2_- \) such that

\[
-iP_+g_i \rightarrow g \quad -iP_-h_i \rightarrow h \quad iP_+g_i \rightarrow \tilde{g} \quad iP_-h_i \rightarrow \tilde{h}
\]

The convergences in (3.8) are of course with respect the \( \mathcal{S} \)-topology (3.6). Therefore, the functions \(-iP_+g_i\) converge to \( g \) point-wise (and similarly for the other three sequences). Next, notice that the two functions \(-iP_+f\) and \( iP_-f \) obtained from any \( f \in \mathcal{N}_- \) coincide on \( (0, \infty) \): \(( -iP_+f)(x) = (\overline{f}(x) = ( iP_-f)(x) \) for \( x \in (0, \infty) \). Thus, \( g(x) = \tilde{g}(x) \) and \( h(x) = \tilde{h}(x) \) for \( x \in (0, \infty) \). Now let \( g_0 = i(g - \tilde{g}) \), \( h_0 = i(h - \tilde{h}) \), and

\[
f_0 = -iP_+g_0 + iP_-h_0
\]

The function \( f_0 \) is an element of \( \Psi \). The convergences (3.8) imply that \( f_i \rightarrow f_0 \). Hence, \( \Psi \) is a Fréchet space.

It is well-known that \( \mathcal{S} \) is a nuclear space. Since every subspace of a nuclear space is nuclear, \(-iP_+(\mathcal{N}_-) \oplus iP_-(\mathcal{N}_-)\) is nuclear. It then follows that \( \Psi \) is a nuclear space as its topology (3.5) is derived from the nuclear topology (3.6) of \( \mathcal{S} \).

Let \( \Psi^\times \) be the space of continuous antilinear functionals on \( \Psi \), endowed with the weak*-topology. Then, the triplet of spaces

\[
\Psi \subset L^2 \subset \Psi^\times
\]

(3.10)
constitutes a rigged Hilbert space.

**Remark:** From the proof of Proposition 3.3 in particular from the coincidence of the functions \(-iP_+f\) and \(iP_-f\) on \((0, \infty)\) for any \(f \in \mathcal{N}_-\), it follows that the elements of \(\Psi\) do not vanish on any subset of \((0, \infty)\) with non-zero (finite) measure. In fact, this property could have been used to define the space \(\Psi\). Furthermore, if \(f_+\) is the \(L^2_+\)-function obtained from some \(f \in \Psi\) by \(f_+ = Q_+f\), then it follows that \(f_+\) extends to both a unique function in \(N \cap H^2_+ \subset S \cap H^2_+\) and a unique function in \(N \cap H^2_- \subset S \cap H^2_-\). That is, \(Q_+(\Psi) \subset \Phi_+ \cap \Phi_-\), where \(\Phi_+ = Q_+(S \cap H^2_+)\) and \(\Phi_- = Q_+(S \cap H^2_-)\), the spaces defined in [4] and [5] in their study of scattering and time asymmetric quantum mechanics. The denseness of \(\Psi\) in \(L^2\) shows that \((Q_+(\Psi)\) and thus also) the intersection \(\Phi_+ \cap \Phi_-\) is dense in \(L^2_+\), extending the result in [5] that it is non-trivial.

### 3.2 Representation of \(S_1(0)\) in \(\Psi\)

**Proposition 3.4.** The restriction of \(U\) to \(S\), where \(U\) is the continuous unitary representation of \(G\) given in (2.4), yields a differentiable representation of \(G\) in \(S\).

**Proof:** From the definition (2.4), it follows directly that \(S\) remains invariant under all \(U(\xi), \xi \in G\). Then, direct computations show [10], for all \(f \in S\) and \(n = 0, 1, 2, \ldots\),

\[
\|U(\xi)f\|_n \leq (1 + \xi_1^2 + \xi_2^2)^{n/2} \|f\|_n
\]

and

\[
\lim_{\xi_1 \to 0} \left\| \left( \frac{U((\xi_1, 0, 0)) - I}{\xi_1} - D \right) f \right\|_n = 0
\]

\[
\lim_{\xi_2 \to 0} \left\| \left( \frac{U((0, \xi_2, 0)) - I}{\xi_2} - M \right) f \right\|_n = 0
\]

\[
\lim_{\xi_3 \to 0} \left\| \left( \frac{U((0, 0, \xi_3)) - I}{\xi_3} - iI \right) f \right\|_n = 0
\]

where the norms \(\|\cdot\|_n\) are those defined in (3.6). This proves that \(U|_S\) is a differentiable representation of \(G\). \(\square\)

**Remark:** Equations (3.12) show that the \(S\)-generators of the representation \(U|_S\) coincide on \(S\) with the \(L^2\)-generators of \(U\). This has an interesting implication for the rigged Hilbert formulation of quantum physics in that, just as in
the conventional Hilbert space theory, the concept of an observable has interpretation as the infinitesimal form of a symmetry (or asymmetry/semigroup) transformation.

Let us next consider the action of $U$ on the elements of $Ψ$.

**Proposition 3.5.** The restriction of $U$ to $Ψ$ yields a non-trivial differentiable representation of $S_1(0)$ in $Ψ$.

**Proof** If $Ψ$ is invariant under $U(ξ), ξ ∈ S_1(0)$, then it follows from the topology (3.3) and Proposition 3.4 that $U|Ψ$ furnishes a differentiable representation of $S_1(0)$. Therefore, we must simply show that $U(ξ), ξ ∈ S_1(0)$, leaves $Ψ$ invariant and that the resulting semigroup representation does not extend in $Ψ$ to a representation of $G$ or a subgroup thereof. To that end, let $f ∈ Ψ$. Then, there exist unique functions $g$ and $h$ in $N_−$ such that $f = −iP_+g + iP_−h$, and

$$U((ξ_1, 0, ξ_3))f = U((ξ_1, 0, ξ_3))(-iP_+g + U((ξ_1, 0, ξ_3))iP_−h$$

$$= −iP_+U((ξ_1, 0, ξ_3))g + iP_−U((ξ_1, 0, ξ_3))h$$

(3.13)

where the second equality follows from the commutativity of translations with the Hilbert transform: $U((ξ_1, 0, ξ_3))H = HU((ξ_1, 0, ξ_3))$.

From the construction of $N_−$ (Appendix A), it is clear that $U((ξ_1, 0, ξ_3))g ∈ N_−$ and $U((ξ_1, 0, ξ_3))h ∈ N_−$ if $ξ_1 ≥ 0$, or equivalently, if $(ξ_1, 0, ξ_3) ∈ S_1(0)$. That is, $Ψ$ is invariant under the operator semigroup $U((ξ_1, 0, ξ_3)), ξ_1 ≥ 0$. Furthermore, from Appendix A it is also clear that for any $ξ_1 < 0$, there exist functions $f$ in $N_−$ such that $U((ξ_1, 0, ξ_3))f ∉ N_−$. Thus, the semigroup representation $U(ξ), ξ ∈ S_1(0)$, in $Ψ$ does not extend to a representation of the whole of $G$ or even the subgroup $\{(ξ_1, 0, ξ_3) : ξ_1, ξ_3 ∈ R\}$. □

### 4 A Differentiable Representation of $S_2(0)$

As a corollary to the construction carried out in Section 3, we can obtain a rigged Hilbert space $Ψ ⊂ L^2 ⊂ Ψ^\times$ such that $Ψ$ reduces the continuous unitary representation of $G$ given by (2.4) to a differentiable representation of subsemigroup $S_2(0)$ defined in (2.3). This can be easily achieved by letting $Ψ$ be the Fourier transform $F(Ψ)$ of the nuclear Fréchet space $Ψ$ constructed in the above section. Since $F$ is a unitary mapping on $L^2$, it follows that the triad

$$Ψ ⊂ L^2 ⊂ Ψ^\times \quad (4.1)$$
is a rigged Hilbert space. The topology on $\tilde{\Psi}$ can be induced from the topology of $\Psi$ via the Fourier transform. That is, if $\varphi \in \tilde{\Psi}$, then $\varphi = \hat{f}$ for a unique $f \in \Psi$, and a locally convex nuclear topology can be defined on $\tilde{\Psi}$ by way of the norms

$$||\varphi||_n = ||f||_n$$

(4.2)

where $||f||_n$ are the norms in $\Psi$ defined by (3.5).

It is well known that $F$ (and $F^{-1}$) establishes a unitary equivalence between the operators $U((\xi_1, 0, \xi_3))$ and $U((0, \xi_1, \xi_3))$, i.e., $F \circ U((\xi_1, 0, \xi_3)) = U((0, \xi_1, \xi_3))$ and $U((\xi_1, 0, \xi_3)) = F^{-1} \circ U((0, \xi_1, \xi_3))$. It thus follows that $\tilde{\Psi}$ reduces the unitary representation $U$ of $G$ in $L^2$ given by (2.4) to a non-trivial differentiable (with respect to the topology given by (4.2)) representation of $S_2(0)$.

It is perhaps worthwhile to take a closer look at the properties of the functions in $\tilde{\Psi}$. Each such function $\varphi$ is the Fourier transform of a function in $\Psi$, i.e.,

$$\varphi = (-iP_+g) + (iP_-h)$$

(4.3)

for unique functions $g$ and $h$ in $\mathcal{N}_-$. From (3.3), we then have

$$\varphi(x) = -\frac{i}{2} \left( 1 + \frac{x}{|x|} \right) \hat{g}(x) + \frac{i}{2} \left( 1 - \frac{x}{|x|} \right) \hat{h}(x)$$

(4.4)

This means, the space $\tilde{\Psi}$ is the direct sum of the restrictions of $F(\mathcal{N}_-)$ to $(0, \infty)$ and to $(-\infty, 0)$. Since $F(\mathcal{N}_-) = \mathcal{M} \cap \mathcal{H}^2_+$, we have

$$\tilde{\Psi} = Q_+ (\mathcal{M} \cap \mathcal{H}^2_+) \oplus Q_- (\mathcal{M} \cap \mathcal{H}^2_+)$$

(4.5)

The topology of $\tilde{\Psi}$ can also be defined by way of the norms

$$||\varphi||^2_n = ||\hat{g}||^2_n + ||\hat{h}||^2_n$$

(4.6)

where $\hat{g}$ and $\hat{h}$ are as in (4.4) and the norms on the right hand side of (4.6) are as in (3.6). This topology is clearly equivalent to one given by the norms (4.2). In this light, the differentiable representation of $S_2(0)$ in $\tilde{\Psi}$ is just that which is induced from its differentiable representation in $\mathcal{M} \cap \mathcal{H}^2_+$. 

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5 Subsemigroups $S_1$, $S_2$ and $S_3$

Notice that the centrally significant feature of the preceding constructions of two rigged Hilbert spaces is the existence of dense subspaces of the Hilbert space which remain invariant under the differential $dU|_0$ but not the representation $U$. Once a subspace invariant under $dU|_0$ and $U(\xi)$, $\xi \in S_1(0)$, or $dU|_0$ and $U(\xi)$, $\xi \in S_2(0)$, was identified, it was possible to construct the rigged Hilbert space (3.10) or (4.1).

Such dense subspaces can be constructed also for the operator semigroups $\{U(\xi) : \xi \in S_1\}$, $\{U(\xi) : \xi \in S_2\}$ and $\{U(\xi) : \xi \in S_3\}$. It is interesting to notice, however, that any subspace which remains invariant under the operator semigroup $\{U(\xi) : \xi \in S_4\}$ will be invariant also under $\{U(\xi) : \xi \in S_3\}$. This means that the general method implied by the preceding two constructions (by way of dense subspaces invariant under the relevant operator semigroup) does not lead to a rigged Hilbert space for a non-trivial differentiable representation of the subsemigroup $S_4$, i.e., such a representation naturally extends to a representation of $S_3$.

A dense subspace of $L^2$ which is invariant under the operator Lie algebra $dU|_0$ and the semigroup $U(S_1)$ can be easily obtained from the $\Psi$ of Section 3. It was shown that $\Psi$ remains invariant under the operator semigroup $\{U(\xi) : \xi \in S_1(0)\}$. However, $\Psi$ is not invariant under any operator such as $U(\xi)$ where $\xi = (0, \xi_2, \xi_3)$. This is easily seen when either the Hardy class property or the vanishing moment (in $N$) property of the functions in $\Psi$ is considered. For instance, for an arbitrary element $f \in \Psi$, the integrals $\int_{-\infty}^{\infty} x^n f(x) dx = 0$ for $n = 0, 1, 2, \ldots$, but the integrals of the transformed element $U(\xi)f$, $\int_{-\infty}^{\infty} x^n (U((0, \xi_2, \xi_3))f)(x)dx = e^{i\xi_3} \int_{-\infty}^{\infty} x^n e^{i\xi_2 x} f(x)dx$ do not vanish for $n = 0, 1, 2, \ldots$ when $\xi_2 \neq 0$, i.e., $U((0, \xi_2, \xi_3))f \notin \Psi$. Therefore, let $\Psi_\xi = U((0, \xi_2, \xi_3))(\Psi)$. Unitarity of the operators $U((0, \xi_2, \xi_3))$ implies that $\Psi_\xi$ is dense in $L^2$. Thus, a dense subspace of $L^2$ which remains invariant under the operator semigroup $U(S_1)$ can be obtained by setting

$$\Psi_1 = \bigcup_{\xi \in S_1} \Psi_\xi$$

(5.1)

and, starting from the dense subspace (5.1), a rigged Hilbert space may be built as in Section 3 for a differentiable representation of the semigroup $S_1$.

In complete analogy to (5.1), we can construct a dense subspace invariant under the operator semigroup $\{U(\xi) : \xi \in S_2\}$, starting from the space $\tilde{\Psi}$ of Section 4.
As evident from the vector space $\mathcal{N} \cap \mathcal{M}$ introduced in Section 3, there also exist dense subspaces of $L^2$ which are invariant under the operator Lie algebra $dU|_0$ but not under any non-trivial $U(\xi)$, i.e., under any $U(\xi)$ where $\xi$ is a non-central element of $G$. Now consider the vector space

$$\Psi_3 = \bigcup_{\xi \in S_3} U(\xi) (\mathcal{N} \cap \mathcal{M})$$

(5.2)

The unitarity of $U(\xi)$ and the denseness of $\mathcal{N} \cap \mathcal{M}$ imply that $\Psi_3$ is dense in $L^2$. By construction, $\Psi_3$ is invariant under both $dU|_0$ and the operator semigroup $\{U(\xi) : \xi \in S_3\}$, but not under any $U(\xi)$ with $\xi \notin S_3$. Thus, a rigged Hilbert space furnishing a non-trivial differentiable representation of $S_3$ may be built from the dense subspace $\Psi_3$.

6 Concluding Remarks – Juxtaposition with Time Asymmetric Quantum Theory

In this paper we have investigated how differentiable representations of certain subsemigroups of the Weyl-Heisenberg group may be obtained in rigged Hilbert spaces. These representations were induced from a given continuous unitary representation of the Weyl-Heisenberg group $G$ in the Hilbert space of $L^2$-functions on $\mathbb{R}$. As stated earlier, the construction of the particular rigged Hilbert space, which we denote here as in Definition generically by $\Phi \subset \mathcal{H} \subset \Phi^\times$, begins with the identification of a dense subspace of $L^2$ which stays invariant under the action of the $L^2$-differential $dU|_0$ and the relevant operator subsemigroup $\{U(\xi) : \xi \in S_i\}$. In order to make certain that the ensuing differentiable representation is non-trivial, i.e., that it does not extend to a representation of the group $G$ or a subgroup thereof, it was necessary to verify that the dense subspace invariant for the subsemigroup $\{U(\xi) : \xi \in S_i\}$ does not remain invariant under certain $U(\xi)$ with $\xi \notin S_i$. Once such a dense subspace was identified, it was possible to introduce a topology on it, by means of the enveloping operator algebra of $dU|_0$, so as to obtain a differentiable representation of $S_i$ in the inner space $\Phi$ of the rigged Hilbert space. With respect to this topology, elements of the enveloping algebra of $dU|_0$ become continuous as operators in $\Phi$. Moreover, as seen from (3.12), the elements of the $\Phi$-differential of the semigroup representation $U(S_i)$ in $\Phi$ coincide with the corresponding elements of the $\mathcal{H}$-differential $dU|_0$ of the group representation $U(G)$ in the Hilbert space $\mathcal{H}$.
In a definite technical sense, the semigroup time evolution of the rigged Hilbert space formulation of quantum mechanics developed in [4,5,6] has at its heart the Weyl-Heisenberg subsemigroups \( S_1 \) and \( S_2 \) of (2.3). Recall first that the rigged Hilbert spaces of Hardy class functions constructed in [4,5,6] are

\[
S \cap \mathcal{H}_+^2 \subset L_+^2 \subset (S \cap \mathcal{H}_+^2)^\times, \quad S \cap \mathcal{H}_-^2 \subset L_+^2 \subset (S \cap \mathcal{H}_-^2)^\times
\] (6.1)

where \( \mid_+ \) indicates the restrictions of the \( S \cap \mathcal{H}_\pm^2 \)-functions to the half line \((0, \infty)\). From the above mentioned van Winter’s theorem [9], a function \( f_\pm \in S \cap \mathcal{H}_\pm^2 \mid_+ \) extends to a unique function \( f_\pm \in S \cap \mathcal{H}_\pm^2 \). This property is used in [4,5] to define a nuclear Fréchet topology on \( S \cap \mathcal{H}_\pm^2 \mid_+ \):

\[
\|f_\pm\|_n = \|f_\pm\|_n
\] (6.2)

where the norms on the right hand side refer to the Schwartz space topology (6.5) the space \( S \cap \mathcal{H}_\pm^2 \) inherits from \( S^2 \). The continuous unitary representation of \( \mathbb{R} \) given in \( L_+^2 \) by \((U(t)f)(E) = e^{iEt}f(E)\) reduces to a differentiable representation of \((0, \pm \infty)\) in \( S \cap \mathcal{H}_\pm^2 \mid_+ \). These semigroup representations can be related to the representations of \( S_1^{-1} \) and \( S_2 \) of (2.3) in the following way.

Observe that the mapping (2.4) yields a continuous representation of \( S_1^{-1} \) in \( L_+^2 \) by contractions:

\[
(U(\xi)f)(x) = e^{ix_3}e^{ix_2}f(x + x_1), \quad \xi \in S_1^{-1}, \ f \in L_+^2
\] (6.3)

and \( \|U(\xi)f\|_+ \leq \|f\|_+ \).

Further, the multiplication subgroup \( \{(0, \xi_2, \xi_3) : \xi_2, \xi_3 \in \mathbb{R}\} \) of \( S_1^{-1} \) is unitarily represented by \( U \) in \( L_+^2 \):

\[
(U((0, \xi_2, \xi_3))f)(x) = e^{ix_3}e^{ix_2}f(x)
\] (6.4)

The dense subspace \( S_+ \) remains invariant under both the operator semigroup \( \{U(\xi) : \xi \in S_1^{-1}\} \) and the basis elements \( M, D, I \) of the differential \( dU|_0 \). Moreover, \( S_+ \), a closed subspace of \( S \), is a nuclear Fréchet space, and therewith the triplet

\[
S_+ \subset L_+^2 \subset S_+^\times
\] (6.5)

\[\text{The countable family of norms used to characterize } S \text{ in [4,5] is not that of [6.5] but the more customary } \|f\|_n = \|(M^2 + D^2 + I)^n f\|_0.\]
constitutes a rigged Hilbert space. The continuous representation (6.3) of $S^{-1}_1$ in $L^2_+$ yields a differentiable representation of the semigroup in $S_+$.

The Fourier transform (3.1) establishes a unitary equivalence between (6.5) and the rigged Hilbert space

$$S \cap \mathcal{H}_-^2 \subset \mathcal{H}_-^2 \subset (S \cap \mathcal{H}_-^2)^\times \quad (6.6)$$

while its inverse $F^{-1}$ maps (6.5) unitarily onto

$$S \cap \mathcal{H}_+^2 \subset \mathcal{H}_+^2 \subset (S \cap \mathcal{H}_+^2)^\times \quad (6.7)$$

Since $F$ is a homeomorphism on $S$, $S \cap \mathcal{H}_\pm^2$ are closed subspaces of $S$. They are thus nuclear Fréchet spaces with respect to the Schwartz space topology (3.5).

The mappings from (6.5) onto (6.6) and (6.7) given by $F$ and $F^{-1}$ also transform the representation of $S^{-1}_1$ in (6.5) to a representation of $S^{-1}_2$ and $S_2$ in (6.6) and (6.7), respectively. In particular,

$$FU(\xi)F^{-1} = U((-\xi_2, \xi_1, \xi_3 - \xi_1 \xi_2))$$

$$F^{-1}U(\xi)F = U((\xi_2, -\xi_1, \xi_3 - \xi_1 \xi_2)), \quad (6.8)$$

and when $\xi \in S^{-1}_1$, the contractions $U((-\xi_2, \xi_1, \xi_3 - \xi_1 \xi_2))$ and $U((\xi_2, -\xi_1, \xi_3 - \xi_1 \xi_2))$ provide continuous representations of $S^{-1}_2$ and $S_2$ in $\mathcal{H}_-^2$ and $\mathcal{H}_+^2$, respectively. Further, in the nuclear Fréchet spaces $S \cap \mathcal{H}_-^2$ and $S \cap \mathcal{H}_+^2$ the mappings (6.8) furnish differentiable representations of the semigroups $S_2$ and $S^{-1}_2$, respectively. In particular, the differentiation operator $D$ generates the one parameter group of translations in both $S \cap \mathcal{H}_-^2$ and $S \cap \mathcal{H}_+^2$, whereas the multiplication operator $M$ generates only a one parameter semigroup, one in $S \cap \mathcal{H}_2$ for negative $\xi_1$ and another in $S \cap \mathcal{H}_2$ for positive $\xi_1$.

Consider the subsemigroup $S_2(0)$ of $S_2$. As just seen, it is represented differentiably by the mapping (2.4), $\xi \to U(\xi)$, in $S \cap \mathcal{H}_2^2$:

$$(U(\xi)f)(x) = e^{i\xi_3}e^{ix\xi_2}f(x), \quad \xi \in S_2(0), \ f \in S \cap \mathcal{H}_2^2 \quad (6.9)$$

The relation (6.8) and the description following it shows that this representation of $S_2(0)$ in $S \cap \mathcal{H}_2^2$ is non-trivial. Now, since the mapping $S \cap \mathcal{H}_+^2 \to S \cap \mathcal{H}_+^2$ is one-to-one and onto (6.9), it follows that (6.9) induces a non-trivial differentiable representation of $S_2(0)$ in $S \cap \mathcal{H}_+^2$.

Observe next that the differentiable representation (6.9) of the subsemigroup $S_2(0)$ in $S \cap \mathcal{H}_+^2$ can be identified with the unitary representation
of the subgroup \( \{ \xi = (0, \xi_2, \xi_3) : \xi_2, \xi_3 \in \mathbb{R} \} \) in \( L^2_+ \). Once this identification is made, we conclude that the continuous unitary representation of \( \mathbb{R} \) given by (6.4),

\[
(U((0, \xi_2, 0))f)(x) = e^{ix\xi_2}f(x)
\]

in \( L^2_+ \) is reduced by the subspace \( S \cap H^2_+ \) to a representation of the half line \((0, \infty)\). This is the semigroup that is interpreted in [4,5,6] as governing the asymmetric time evolution of out-states and the decaying Gamow vectors. We see here that it is induced from a continuous representation of the Weyl-Heisenberg semigroup \( S_2 \) in \( H^2_+ \). The latter representation, in turn, is equivalent to the continuous representation of the semigroup \( S_1^{-1} \) in \( L^2_+ \), given by (6.3). In this sense, it can be said that the semigroup time evolution of the quantum theory developed in [4,5,6], where the Hilbert space \( L^2_+ \) consists of the square integrable functions defined on the energy spectrum \((0, \infty)\) and \( \xi_2 \) is interpreted as time, \( t \), is obtained from the semigroup of translations in \( L^2_+ \) given by (6.3), \( (U(\xi)f)(E) = f(E - \xi), \xi \in (0, \infty) \).

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Appendix

A Denseness of \( \mathcal{N}_\pm \) in \( L^2_\pm \)

**Proposition A.1.** \( \mathcal{N}_+ \) is dense in \( L^2_+ \).

**Proof.** We prove this assertion by showing that any compactly supported \( C^\infty \)-function in \( L^2_+ \) can be approximated by functions in \( \mathcal{N}_+ \). Since the class of \( C^\infty \)-functions with compact support is dense in \( L^2_+ \), the proposition follows.

Let \( g \) be a compactly supported smooth function in \( L^2_+ \) and \( \epsilon \), any positive number. Without loss of generality, let us assume that \( \int_0^\infty g(x)dx \neq 0 \), for otherwise we may choose a compactly supported smooth function arbitrarily close to \( g \) in the \( L^2 \)-metric with this property. Now, suppose that there exists a family of compactly supported smooth functions \( f_k \), with supports contained in, say \((a_k, a_{k+1})\), such that
1.) The support of $f_k$ is to the right of the support of $f_{k-1}$ and disjoint from it; the support of $f_0$ is to the right of that of $g$.

2.) $\int_0^\infty x^i f_k(x)dx = 0$ for $i = 0, 1, 2, \ldots, k - 1$.

3.) $\int_0^\infty x^k f_k(x)dx = -\int_0^\infty x^k (g + f_0 + f_1 + \cdots + f_{k-1})(x)dx$

4.) $\|f_k\| < \frac{\epsilon}{2k+1 a_{k+1}^k}$

If such a family $\{f_k\}$ can be found, then set

$$f = \sum_{k=0}^\infty f_k + g \quad (A.1)$$

The $f$ is well defined since for each $0 < x < \infty$ all but one $f_k$ are zero.

Since the $f_k$ have disjoint supports

$$\int_{a_n}^{a_{n+1}} x^m \sum_{k=0}^\infty f_k(x)dx = \int_{a_n}^{a_{n+1}} \sum_{k=0}^\infty x^m f_k(x)dx$$

$$= \int_{a_n}^{a_{n+1}} x^m f_n(x)dx$$

$$= 0 \text{ for all } n > m \quad (A.2)$$

and so,

$$\sum_{n=0}^\infty \left( \int_{a_n}^{a_{n+1}} \sum_{k=0}^\infty x^m f_k(x)dx \right) = \sum_{n=0}^m \left( \int_{a_n}^{a_{n+1}} \sum_{k=0}^\infty x^m f_k(x)dx \right)$$

$$= \sum_{n=0}^m \int_{a_n}^{a_{n+1}} x^m f_n(x)dx \quad (A.3)$$
Therefore,
\[
\int_0^\infty x^m f(x)dx = \int_0^\infty \left( \sum_{k=0}^m x^m f_k(x) + x^m g(x) \right) dx
\]
\[
= \sum_{k=0}^m \int_0^\infty x^m f_k(x)dx + \int_0^\infty x^m g(x)dx
\]
\[
= 0, \quad \text{for } m = 0, 1, 2, \ldots
\]  
(A.4)
where the above property 2.) of the $f_k$ is used in the last equality of (A.2) and the property 3.), in the last equality of (A.4). Further, from the inequality 4.), it is clear that $x^m f \in L^1((0, \infty))$. Since the functions $f_k$ have increasing supports, it then follows that $f \in \mathcal{N}_+$. Furthermore, from the disjointness of the supports and the property 4.) above of the $f_k$, it follows readily
\[
\|g - f\| = \left\| \sum_{k=0}^\infty f_k \right\| = \left( \sum_{k=0}^\infty \|f_k\|^2 \right)^{\frac{1}{2}} \leq \left( \sum_{k=0}^\infty \frac{\epsilon^2}{2^{2k+2}a_{k+1}^{2k+1}} \right)^{\frac{1}{2}} < \epsilon
\]  
(A.5)
i.e., $\mathcal{N}_+$ is dense in $L^2_+$. 

It remains to show that the $C^\infty$ functions $f_k$ can be chosen subject to the conditions 1.)–4.) above. This can be done by induction. To that end, suppose the smooth functions $f_0, \ldots, f_{k-1}$ with their supports in $(a_0, a_1), \ldots, (a_{k-1}, a_k)$, respectively, have been aptly chosen. Assume further that the given smooth function is supported in $(0, a_0)$ with $a_0 > 1$. Define now a function $f_k$ by setting
\[
f_k(x) = \gamma_k \frac{d^k g}{dx^k} \left( \frac{a_0(x - a_k)}{a_{k+1} - a_k} \right)
\]  
(A.6)
The function $f_k$ is supported in $(a_k, a_{k+1})$, where $a_{k+1}$ and the constant $\gamma_k$ are to be chosen subject to the conditions (A.10) and (A.12) below. 

From the definition (A.6) it is clear
\[
\int_0^\infty x^n f_k(x)dx = 0 \quad \text{for } n = 0, 1, 2, \ldots, k - 1
\]  
(A.7)
and,

\[ \int_0^\infty x^k f_k(x) dx = (-1)^k k! I \gamma_k \left( \frac{a_{k+1} - a_k}{a_0} \right)^{k+1}, \tag{A.8} \]

where \( I = \int_0^\infty g(x) dx \). Next, in accordance with the condition 3.) above, we require

\[ \int_0^\infty x^k f_k(x) dx = \lambda \tag{A.9} \]

where \( \lambda = - \int_0^\infty x^k (g(x) + f_0(x) + \ldots + f_{k-1}(x)) dx \). Equalities (A.8) and (A.9) yield

\[ \gamma_k = \frac{(-1)^k \lambda}{k!} \left( \frac{a_0}{a_{k+1} - a_k} \right)^{k+1} \tag{A.10} \]

It remains only to choose \( a_{k+1} \). By the definition (A.6) of \( f_k \) and the inequality 4.) on its \( L^2 \) norm, we have

\[ \| f_k \| = |\gamma_k| \left( \frac{a_0}{a_{k+1} - a_k} \right)^{1/2} \left\| \frac{d^k g}{dx^k} \right\| < \frac{\epsilon}{2^{k+1} a_{k+1}^{k+3/2}} \tag{A.11} \]

Along with the relation (A.10) above, we then observe that \( a_{k+1} \) is to be chosen subject to the inequality

\[ \frac{(a_{k+1} - a_k)^{k+3/2}}{a_{k+1}^{k+1}} > \frac{|\lambda| 2^{k+1}}{\epsilon |I| k!} a_0^{k+3/2} \left\| \frac{d^k g}{dx^k} \right\| \tag{A.12} \]

By construction, \( a_k > a_0 > 1 \), and so it is clear that this inequality can be fulfilled by choosing \( a_{k+1} \) large enough. Once the \( a_{k+1} \) is picked out, the equation (A.10) determines \( \gamma_k \), and therewith the expression (A.6) completely determines the function \( f_k \). Thus, the functions \( f_k \) are simply the derivatives of the given smooth function \( g \) with their supports appropriately dilated and translated on the real axis, followed by a suitable overall scaling. \( \square \)

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