On SIC-POVMs and MUBs in Dimension 6

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We provide a partial solution to the problem of constructing mutually unbiased bases (MUBs) and symmetric informationally complete POVMs (SIC-POVMs) in non-prime-power dimensions. An algebraic description of a SIC-POVM in dimension six is given. Furthermore it is shown that several sets of three mutually unbiased bases in dimension six are maximal, i.e., cannot be extended.

Keywords: Mutually unbiased bases, SIC-POVMs, finite geometry

1 Introduction
In a recent paper, Wootters discusses relations between quantum measurements and finite geometries [11]. Both projective quantum measurements corresponding to so-called mutually unbiased bases (MUBs) and generalized quantum measurements described by symmetric informationally complete positive operator valued measures (SIC-POVMs) can be linked to finite affine planes. While some constructions of MUBs are directly related to affine planes over finite fields (see, e.g., [4]), the link between SIC-POVMs and finite affine planes is on a more abstract level. Both finite affine planes and maximal sets of MUBs can be constructed if the dimension is a prime power (see, e.g., [1, 5]). In neither case constructions for non-prime-powers are known, but there is numerical evidence that SIC-POVMs exist in all dimensions [6].

In this note we provide an explicit algebraic expression for a SIC-POVM in dimension six. This shows that the problem of constructing SIC-POVMs is not equivalent to the existence of a finite affine plane, as it is well-known that there is no finite affine plane of order six [8]. For MUBs in dimension six, however, the question remains open whether a maximal set of seven MUBs exists. It is known that three MUBs in dimension six exist, and it is widely believed that there are no four MUBs. Here we show that particular sets of three MUBs are maximal in the sense that they cannot be extended to four MUBs.

2 The Weyl-Heisenberg Group and the Jacobi Group
Before we discuss the construction of SIC-POVMs and MUBs, we recall properties of the well-known Weyl-Heisenberg group and investigate its automorphism group.

The Heisenberg group $H_d$ is a finite subgroup of $GL(d, \mathbb{C})$ generated by the cyclic shift operator $X$ and the phase operator $Z$ given by

$$X := \sum_{j=0}^{n-1} |j+1\rangle\langle j| \quad \text{and} \quad Z := \sum_{j=0}^{n-1} \omega^{j}_{d} |j\rangle\langle j|,$$  

(1)
where $\omega_d := \exp(2\pi i / d)$ is a primitive $d$-th root of unity and the cyclic shift is modulo $d$. The group $H_d$ is also called Weyl-Heisenberg group as the operators $X$ and $Z$ are the discrete Weyl operators (see \cite{[9], Kap. IV.D} and \cite{[10], Chapt. IV.D}). Each element of $H_d$ can be uniquely written as $\omega_d^a X^a Z^b$ with $a, b, c \in \{0, \ldots, d-1\}$. Two elements $\omega_d^a X^a Z^b$ and $\omega_d^c X^c Z^c$ commute iff $ab' - a'b = 0 \mod d$.

The center $\zeta(H_d)$ of the group $H_d$ is generated by $\omega_d I$, where $I$ denotes the identity matrix. Ignoring the global phase factor, the group $H_d$ is isomorphic to the direct product of two cyclic groups of order $d$, i.e., $H_d/\zeta(H_d) \cong \mathbb{Z}_d \times \mathbb{Z}_d$ where $\mathbb{Z}_d := \mathbb{Z} / d\mathbb{Z}$ denotes the ring of integers modulo $d$. The matrices $X^a Z^b$ are mutually orthogonal with respect to the trace inner product and form a vector space basis of all $d \times d$ matrices.

In our constructions we also consider the normalizer of the Heisenberg group $H_d$ in the group $U(d, \mathbb{C})$ of unitary matrices. The Fourier matrix

$$\text{DFT}_d := \frac{1}{\sqrt{d}} \left( \omega_d^{jk} \right)_{j,k=0}^{d-1}$$

acts via conjugation on $H_d$. It interchanges the operators $X$ and $Z$ (up to a phase factor). Another matrix that acts on $H_d$ modulo the center is the matrix $P_d$ which is defined as

$$P_d^{\text{even}} := \sum_{j=0}^{d-1} \omega_d^{j^2 / 2} |j\rangle \langle j|, \quad \text{if } d \text{ is even},$$

and

$$P_d^{\text{odd}} := \sum_{j=0}^{d-1} \omega_d^{j(j-1)/2} |j\rangle \langle j|, \quad \text{if } d \text{ is odd}.$$  

If $d$ is odd, $P_d^{\text{odd}}$ acts on $H_d$ via conjugation. If $d$ is even, conjugation with $P_d^{\text{even}}$ might introduce an additional overall phase factor $\omega_d^{1/2} = \omega_{2d}$. Therefore we enlarge the center of $H_d$ by the scalar multiple of identity $\omega_{2d} I$. Then $P_d$ acts on $\langle H_d, \omega_{2d} I \rangle$ via conjugation. What is more, the group generated by $\text{DFT}_d$ and $P_d$ acts transitively on $H_d$ modulo the center. We define the Jacobi group as the group which is generated by $X$, $Z$, $\text{DFT}_d$, and $P_d$, i.e.,

$$J_d := \langle X, Z, \text{DFT}_d, P_d \rangle.$$  

The Jacobi group is widely studied in the context of number theory \cite{[2]}. In quantum computing, the group is also known as the Clifford group. We essentially use the fact that the Jacobi group is a semi-direct product of the Heisenberg group $H_d$ and $SL(2, \mathbb{Z}_d)$. The Heisenberg group is a normal subgroup of the Jacobi group, and the quotient group $J_d/\langle \zeta(J_d), H_d \rangle$ is isomorphic to $SL(2, \mathbb{Z}_d)$.

## 3 Symmetric Informationally Complete POVMs

The most general quantum measurement is represented by positive operator valued measures (POVMs). A POVM is called informationally complete if the statistics of the measurement allows the reconstruction of the quantum state on which the measurement is carried out. As a quantum state in dimension $d$ is described by $d^2 - 1$ real parameters, an informationally complete POVM must have at least $d^2$ elements. In order to achieve maximal statistical
independence of the outcomes, the inner product between the elements should be constant. Such a POVM is called a symmetric, informationally complete POVM, or SIC-POVM for short. It consists of \(d^2\) operators of the form \(E_j = \Pi_j/d\), where the rank-one projectors \(\Pi_j = |\phi_j\rangle \langle \phi_j|\) satisfy the condition
\[
\text{Tr} \Pi_j \Pi_k = \frac{1}{d+1} \quad \text{for } j \neq k,
\]
or, equivalently,
\[
|\langle \phi_j| \phi_k \rangle|^2 = \frac{1}{d+1} \quad \text{for } j \neq k.
\]
It has been conjectured that SIC-POVMs exist in all dimensions \([12, 6]\).

**Conjecture 1 (Zauner [12])** For every dimension \(d \geq 2\) there exists a SIC-POVM whose elements are the orbit of a positive rank-one operator \(E_0\) under the Heisenberg group. What is more, \(E_0\) commutes with an element \(T\) of the Jacobi group \(J_d\). The action of \(T\) on \(H_d\) modulo the center has order three.

Zauner provides algebraic solutions for dimensions \(d = 2, 3, 4, 5\), and numerical solutions for \(d = 6, 7\). A weaker form of his conjecture—the existence of SIC-POVMs that are the orbit under the Heisenberg group—has been numerically verified up to dimension \(d = 45\) \([6]\) (as of May 25, 2009, Andrew Scott has found numerical solutions up to \(d = 67\) \([7]\)).

A SIC-POVM that is invariant under the Heisenberg group is given by
\[
\{hE_0h^{-1}: h \in H_d\},
\]
where \(E_0 = |\phi_0\rangle \langle \phi_0|/d\) for some normalized quantum state \(|\phi_0\rangle \in \mathbb{C}^d\). The image of the corresponding projectors \(\Pi_i\) is the set
\[
\{X^aZ^b|\phi_0\rangle: a, b = 0, \ldots, d-1\}.
\]
Note that a global change of basis \(U\) yields another SIC-POVM which is invariant under the conjugated group \(UH_dU^{-1}\). If \(U\) is an element of the Jacobi group, we get an equivalent SIC-POVM that is invariant under the very same representation of the Heisenberg group.

In the following we briefly sketch how we found an algebraic solution for a SIC-POVM in dimension \(d = 6\). Starting from Zauner’s conjecture, we search for an initial vector \(|\phi_0\rangle\) that is an eigenvector of an element of the Jacobi group. The Jacobi group \(J_6\) has 124416 elements, but it is sufficient to consider only one element of each of the 767 non-trivial conjugacy classes. Let \(|b_0\rangle, \ldots, |b_{d-1}\rangle\) be a basis of a possibly degenerate eigenspace of an element of \(J_d\). A generic vector in this eigenspace has the form
\[
|\phi_0\rangle = \sum_{j=0}^{d-1} (x_{2j} + ix_{2j+1})|b_j\rangle,
\]
where \(x_0, \ldots, x_{2d-1}\) are real variables and \(i^2 = -1\). Combining Eqs. (4), (6), and (7) yields a system of polynomial equations for the variables \(x_j\). Using the computer algebra system MAGMA \([3]\), we check whether this system has a solution. While the polynomial equations are stated over some cyclotomic field \(\mathbb{Q}(\omega_m)\), where \(\omega_m := \exp(2\pi i/m)\) is a primitive \(m\)-th root of unity, we find a solution in the field \(\mathbb{K} := \mathbb{Q}(\sqrt{3}, \sqrt{7}, \theta_1, \theta_2, \theta_3, i)\) which is an algebraic...
extension of degree 96 over the rational numbers \( \mathbb{Q} \). The field \( K \) is generated by \( \sqrt{3}, \sqrt{7}, \) the complex number \( i \) with \( i^2 = -1 \), and the real algebraic numbers \( \theta_j \) given by

\[
\theta_1 := \sqrt[3]{\frac{3\sqrt{21} + 9}{224}}, \quad \theta_3 := \sqrt[3]{\frac{3\sqrt{21} + 7\sqrt{7} + 21}{148384}},
\]

and \( \theta_2 \) is the real root of the polynomial

\[
f(x) = x^3 - \frac{\sqrt{21} - 3}{28} x + \frac{21 - 5\sqrt{21}}{126} \theta_1.
\]

One solution for the initial state is \( |\phi_0\rangle = \theta_3(v_1,v_2,v_3,v_4,v_5,v_6)^t \), where the values of \( v_j \) are given in Fig. 1. As mentioned before, a global change of basis by an element of the Jacobi group yields a SIC-POVM that is invariant under the Heisenberg group, too. The action of the Jacobi group \( J_6 \) on the Heisenberg group \( H_6 \) modulo the center corresponds to the group \( SL(2,\mathbb{Z}_6) \) of order 144. But as the initial state \( |\phi_0\rangle \) is fixed by an element of order three, we get only \( 144/3 = 48 \) different SIC-POVMs that are invariant under the Heisenberg group. Starting with the complex conjugated state \( |\phi_0\rangle \) (or replacing \( \Pi_0 \) by its transpose \( \Pi_0^t \), respectively) we get 48 different SIC-POVMs. In total we obtain 96 SIC-POVMs that are invariant under the Heisenberg group, all related by complex conjugation or by a global basis change corresponding to conjugation by an element of \( J_6 \). Note that this agrees with the number of SIC-POVMs listed in [6, Table 1].

### 4 Mutually Unbiased Bases

In the previous section we have considered generalized quantum measurements with \( d^2 \) elements that allow the reconstruction of the \( d^2 - 1 \) real parameters of a quantum state from the measurement statistics. If one allows only projective measurements, each measurement provides only \( d-1 \) independent real parameters. Hence we need at least \( d+1 \) different orthonormal bases \( B_j := \{ |\psi_k^j\rangle : k = 0, \ldots, d-1 \} \) for the projective measurements. Again, the measurement results should be maximally independent which leads to the following condition for the basis states \( |\psi_k^j\rangle \):

\[
|\langle \psi_k^j | \psi_m^l \rangle|^2 = \begin{cases} 1/d & \text{for } j \neq l, \\ \delta_{k,m} & \text{for } j = l. \end{cases}
\]

We call a single vector \( |\psi\rangle \) unbiased with respect to a set \( B \) of vectors iff

\[
|\langle \psi | \psi_k \rangle|^2 = 1/d \quad \text{for all } |\psi_k\rangle \in B.
\]

The maximal size of a set of mutually unbiased bases in dimension \( d \) is \( d+1 \). Constructions for such maximal sets of MUBs are known when the dimension \( d \) is the power of a prime number (see [1, 5] and references therein). For non-prime-powers, a lower bound is given by:

**Lemma 1 (see [5])** Let \( N(d) \) denote the maximal size of a set of mutually unbiased bases in dimension \( d \). Furthermore, let \( d = p_1^{e_1} \cdots p_r^{e_r} \) be the factorization of \( d \) into powers of distinct primes \( p_j \). Then

\[
N(d) \geq \min\{N(p_1^{e_1}), N(p_2^{e_2}), \ldots, N(p_r^{e_r})\}.
\]

In particular, \( N(d) \geq 3 \) for all \( d \geq 2 \).
Proof. The statement follows directly from the proof of Theorem 3.2 in [1].

In order to obtain three MUBs in any dimension, we use the following construction of MUBs which is based on partitioning a set of unitary matrices that are mutually orthogonal into subsets of commuting matrices.

**Theorem 1** Let $\mathcal{B} = C_1 \cup \ldots \cup C_\mu$ with $C_j \cap C_l = \{ I \}$ for $j \neq l$ be a set of $\mu(d-1)+1$ unitary matrices which are mutually orthogonal with respect to the inner product $\langle A, B \rangle := \text{Tr}(AB^\dagger)$. Furthermore, let each class $C_j$ of the partition of $\mathcal{B}$ contain $d$ commuting matrices $U_{j,t} \in U(d, \mathbb{C})$, $0 \leq t \leq d-1$, where $U_{j,0} := I$. For fixed $j$, let the basis $B_j$ contain the joint eigenvectors $| \psi_k^j \rangle$ of the matrices $U_{j,t}$. Then the bases $B_j$ form a set of $\mu$ mutually unbiased bases, i.e.,

$$|\langle \psi_k^j | \psi_m^j \rangle|^2 = 1/d \quad \text{for} \ j \neq l.$$

**Proof.** The statement follows directly from the proof of Theorem 3.2 in [1].

This theorem allows the construction of at least three MUBs in any dimension.
Lemma 2 For fixed $k$, $1 \leq k \leq d - 1$, $\gcd(k, d) = 1$, the eigenvectors of the operators \( \{X, Z, XZ^k \} \) form three mutually unbiased bases. Using the Fourier transformed basis, it follows that the eigenvectors of the operators \( \{X, Z, X^kZ \} \) form three mutually unbiased bases, too.

**Proof.** Let \( C_A := \{A^0, A, A^2, \ldots, A^{d-1}\} \) for \( A = X, Z, XZ^k \). Clearly, the matrices in each set \( C_A \) commute. Modulo the center of \( H_d \), each set \( C_A \) can be written as \( \{X^{\lambda \alpha}Z^{\lambda \beta} : \lambda = 0, \ldots, d - 1\} \) for fixed \( (a, b) \in \mathbb{Z}_d \times \mathbb{Z}_d \). Hence the sets \( C_A \) correspond to the \( x \)-axes, the \( z \)-axes, and a line with slope \( k \). For \( \gcd(k, d) = 1 \), the intersection of any two sets contains only the identity matrix. The second part of the statement follows from the fact that the Fourier transform interchanges the operators \( X \) and \( Z \), i.e., the Fourier transform corresponds to the interchange of the coordinates of \( \mathbb{Z}_d \times \mathbb{Z}_d \).

As far as we know, it is still open what the maximal size \( N(6) \) of a set of mutually unbiased bases in dimension six is. Zauner conjectured that \( N(6) = 3 \) [12]. This conjecture is supported by the —so far mere numerical— similarity between the number of MUBs and the number of mutually orthogonal latin squares [5].

From Lemma 2 we know how to construct three MUBs in dimension six. In order to find four MUBs, one can start with these three bases and search for another basis that is mutually unbiased with respect to the three initial bases. A more general approach is to start with four MUBs, one can start with these three bases and search for another basis that is mutually unbiased with respect to the six vectors in \( B \). For \( \gcd(k, d) = 1 \), the intersection of any two sets contains only the identity matrix. The second part of the statement follows from the fact that the Fourier transform interchanges the operators \( X \) and \( Z \), i.e., the Fourier transform corresponds to the interchange of the coordinates of \( \mathbb{Z}_d \times \mathbb{Z}_d \).

**Theorem 2** Let \( B_X \) and \( B_Z \) denote the eigenbasis of the operators \( X \) and \( Z \), respectively. Then there are exactly 48 normalized vectors \( |w\rangle \in \mathbb{C}^6 \) with \( v_1 = 1/\sqrt{6} \) such that

\[
|\langle w | v \rangle|^2 = 1/6 \quad \text{for all } |w\rangle \in B_X \cup B_Z.
\]

From the 48 vectors one can construct 16 different orthonormal bases \( B_i \) such that the three bases \( MUB_i := \{B_X, B_Z, B_i\} \) are mutually unbiased. All \( MUB_i \) are maximal in the sense that there is no vector \( |\psi\rangle \) that is unbiased with respect to all three bases in \( MUB_i \).

**Proof.** Let \( |v\rangle \in \mathbb{C}^6 \) be an arbitrary vector that is unbiased with respect to \( B_X \) and \( B_Z \). The squared norm of the inner product of \( |v\rangle \) with an eigenvector of \( Z \), i.e., an element of the standard basis, must be equal to \( 1/6 \). Hence the squared norm of each component of \( |v\rangle \) is \( 1/6 \). Multiplying \( |v\rangle \) by a phase factor, one can, without loss of generality, assume that the first component of \( |v\rangle \) is \( 1/\sqrt{6} \). Hence the vector \( |v\rangle \) has the form

\[
|v\rangle = \frac{1}{\sqrt{6}} (1, x_1 + ix_6, x_2 + ix_7, x_3 + ix_8, x_4 + ix_9, x_5 + ix_{10})^t,
\]

where the variables \( x_k \) are real and obey the equations \( x_k^2 + x_{k+5}^2 = 1 \). From the inner product of \( |v\rangle \) with the eigenvectors of \( X \) we get additional polynomial equations for the variables \( x_k \). Using the computer algebra system MAGMA [14] one can show that this system of polynomial equations has 48 real solutions for the variables \( x_k \). The resulting set of vectors \( V = \{|v^\ell\rangle : \ell = 1, \ldots, 48\} \) given in the Appendix contains 16 orthonormal bases \( B_i \) each of which is unbiased with respect to \( B_X \) and \( B_Z \) (see Fig. 2 for the index sets of these bases).

Assume that we could find a vector that is unbiased with respect to \( B_X \), \( B_Z \), and some \( B_i \). Hence \( V \) contains a vector that is unbiased with respect to the six vectors in \( B_i \). Each vector in \( V \) is unbiased with respect to either 4 or 12 other vectors in \( V \). In Fig. 3 we list
those vectors that are unbiased with respect to 12 other vectors. None of these 12-element sets contains one of the bases $B_i$, thus all MUBs are maximal.

\[ B_1 := \{1, 5, 9, 14, 18, 22\} \quad B_7 := \{4, 7, 11, 15, 19, 24\} \quad B_{12} := \{12, 15, 34, 36, 45, 47\} \]

\[ B_2 := \{1, 6, 10, 14, 18, 21\} \quad B_8 := \{4, 23, 37, 40, 41, 44\} \quad B_{13} := \{25, 31, 33, 40, 42, 47\} \]

\[ B_3 := \{2, 5, 9, 13, 17, 22\} \quad B_9 := \{7, 20, 26, 27, 30, 31\} \quad B_{14} := \{26, 32, 34, 39, 41, 48\} \]

\[ B_4 := \{2, 6, 10, 13, 17, 21\} \quad B_{10} := \{8, 19, 25, 28, 29, 32\} \quad B_{15} := \{27, 29, 36, 38, 44, 46\} \]

\[ B_5 := \{3, 8, 12, 16, 20, 23\} \quad B_{11} := \{11, 16, 33, 35, 46, 48\} \quad B_{16} := \{28, 30, 35, 37, 43, 45\} \]

\[ B_6 := \{3, 24, 38, 39, 42, 43\} \]

**Fig. 2.** The 16 bases that are unbiased with respect to the eigenbases $B_X$ and $B_Z$ of $X$ and $Z$.

| unbiased vectors | unbiased vectors |
|------------------|------------------|
| 1 \{15, 16, 19, 20, 26, 29, 35, 39, 40, 43, 44, 47\} | 13 \{3, 4, 19, 20, 26, 29, 35, 36, 40, 43, 47, 48\} |
| 2 \{15, 16, 19, 20, 25, 30, 36, 40, 43, 44, 48\} | 14 \{3, 4, 19, 20, 25, 30, 35, 36, 39, 44, 47, 48\} |
| 5 \{11, 12, 23, 24, 27, 31, 32, 33, 38, 41, 45\} | 17 \{3, 4, 15, 16, 25, 26, 29, 30, 35, 39, 44, 47\} |
| 6 \{11, 12, 23, 24, 27, 31, 32, 34, 37, 42, 46\} | 18 \{3, 4, 15, 16, 25, 26, 29, 30, 36, 40, 43, 48\} |
| 9 \{7, 8, 23, 24, 27, 32, 33, 34, 37, 42, 45, 46\} | 21 \{7, 8, 11, 12, 27, 32, 33, 37, 41, 42, 45\} |
| 10 \{7, 8, 23, 24, 28, 31, 33, 34, 38, 41, 45, 46\} | 22 \{7, 8, 11, 12, 28, 31, 34, 37, 38, 41, 42, 46\} |

**Fig. 3.** Vectors in $V$ that are unbiased with respect to at least 6 other vectors.

Using the automorphism group of the Heisenberg group, we can extend this result as follows:

**Corollary 1** Theorem 2 still holds when replacing the eigenbases $B_X$ and $B_Z$ of $X$ and $Z$, respectively, by the eigenbases of the operators $X^a Z^b$ and $X^{a'} Z^{b'}$, where $ab' - a'b = 1 \mod 6$.

**Proof.** The action of the Jacobi group $J_d$ on the Heisenberg group $H_d$ modulo the center corresponds to the group $SL(2, \mathbb{Z}_d)$. As $ab' - a'b = 1 \mod 6$, there exists a matrix $S \in J_6$ whose action on $H_6$ corresponds to

\[ \begin{pmatrix} a & a' \\ b & b' \end{pmatrix}, \]

i.e., $X \equiv (1, 0)$ is mapped to $X^a Z^b \equiv (a, b)$ and $Z \equiv (0, 1)$ is mapped to $X^{a'} Z^{b'} \equiv (a', b')$.

Hence the new bases can be obtained from $B_X$ and $B_Z$ via the global change of basis $S$. □

In particular, this implies that in dimension six the MUBs of Lemma 3 with three bases are maximal, i.e., there is no vector $|v\rangle \in \mathbb{C}^6$ such that $|\langle w|v\rangle|^2 = 1/6$ for all $|w\rangle \in B_X \cup B_X Z^k \cup B_Z$, $k = 1, 5$.

Finally, assume that we start with the eigenbases of two arbitrary operators $A, B \in H_6$ that have non-degenerate eigenspaces. Furthermore, let the intersection of the cyclic subgroups $H_A$ and $H_B$ of $H_6$ generated by $A$ and $B$, respectively, be contained in the center of $H_6$, i.e., $H_A \cap H_B \subseteq \zeta(H_6)$. The two subgroups correspond to lines in $\mathbb{Z}_6 \times \mathbb{Z}_6$ that intersect only in the origin. From Theorem 1 it follows that the two eigenbases bases are mutually unbiased. As the group $SL(2, \mathbb{Z}_6)$ acts transitively on the pairs of lines with six points that intersect only in the origin $(0, 0)$, from Theorem 2 it follows that the two bases cannot be contained in a set of four MUBs in dimension six.
Therefore, if a set of four or more MUBs in dimension six exists, the bases cannot be related to the Heisenberg group.

5 MUBs in Dimension 4

As 4 is a prime power, a maximal set with 5 MUBs exists. It can, e.g., be constructed using the common eigenvectors of the following sets of two-qubit Pauli matrices

\[
\{id \otimes \sigma_x, \sigma_x \otimes id, \sigma_x \otimes \sigma_x\}, \{\sigma_x \otimes \sigma_y, \sigma_y \otimes \sigma_x, \sigma_z \otimes \sigma_x\}, \\
\{\sigma_x \otimes \sigma_z, \sigma_y \otimes \sigma_x, \sigma_z \otimes \sigma_y\}, \{id \otimes \sigma_z, \sigma_z \otimes id, \sigma_z \otimes \sigma_z\}.
\]

However, if we use the shift operator \(X\) and the phase operator \(Z\) as defined in (1), we get again only three MUBs. Direct computation using MAGMA yields

**Proposition 1** Starting with the eigenbases \(B_Z\) and \(B_X\) of the operators \(Z\) and \(X\), respectively, given as the row-vectors of the following matrices

\[
B_Z := \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \quad \text{and} \quad B_X := \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & i & -1 & -i \\
1 & -1 & 1 & -1 \\
1 & -i & -1 & i
\end{pmatrix},
\]

the vectors of any third unbiased basis are given by the rows of the matrix

\[
B_3 := \frac{1}{2} \begin{pmatrix}
1 & e^{ia} & 1 & -e^{ia} \\
1 & -e^{ia} & 1 & e^{ia} \\
1 & e^{ib} & -1 & e^{ib} \\
1 & -e^{ib} & -1 & -e^{ib}
\end{pmatrix} \quad \text{where} \quad a, b \in [0, \pi).
\]

For all choices of the parameters \(a\) and \(b\), the three MUBs are maximal in the sense that there is no further vector that is unbiased with respect to all three bases.

6 Conclusions

We have explicitly constructed a SIC-POVM in dimension six, the smallest dimension that is not a prime power. Unfortunately, we do not know whether this implies that also a maximal set of MUBs—or just more than three MUBs—in dimension six exist. As there is no affine plane of order six, we cannot make use of the geometric analogies between SIC-POVMs and MUBs pointed out by Wootters. Our results rather support the conjecture that there are at most three MUBs in dimension six. This is true when starting with the eigenbasis of some operators of the Heisenberg group with its geometric structure over \(\mathbb{Z}_d \times \mathbb{Z}_d\).

It seems as if the relation between MUBs and finite geometries was stronger than the relation between SIC-POVMs and finite geometries. Yet, it is still an open problem to determine the maximal number of mutually unbiased bases in non-prime-power dimensions, even for the smallest case \(d = 6\).

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Appendix

In the following we list the 48 vectors that are unbiased with respect to the eigenbases $B_X$ and $B_Z$ of $X$ and $Z$. Note that the vectors are not normalized, the first coordinate is set to one.

Here $\omega = \exp(2\pi i/12)$ denotes a primitive 12-th root of unity and $\theta = \sqrt{-2\omega^3 + 4\omega + 3}/48$.

1. $(1, \omega^2, 1, -\omega^2, -\omega^3)$
2. $(1, \omega^5, -\omega^2, -\omega^3, -\omega^2, \omega^5)$
3. $(1, \omega^3, (2\omega^2 - 6\omega + 2)\theta + (2\omega^3 - \omega^2 + \omega + 1)/2, (-8\omega^3 + 16\omega - 12)\theta - \omega^3 + 2\omega^2 - 1, (12\omega^3 - 8\omega^2 - 12\omega + 16)\theta + \omega^5 - \omega^2 + \omega, (-4\omega^3 + 6\omega^2 + 2\omega - 6)\theta + (\omega^2 - \omega + 1)/2)$
4. $(1, \omega^3, (2\omega^2 - \omega + 2)\theta + (2\omega^3 - \omega^2 - \omega + 1)/2, (8\omega^3 - 16\omega + 12)\theta - \omega^3 + 2\omega^2 - 1, (-12\omega^3 + 8\omega^2 + 12\omega - 16)\theta + \omega^3 - \omega^2 + \omega, (4\omega^3 - 6\omega^2 - 2\omega + 6)\theta + (\omega^2 - \omega + 1)/2)$
5. $(1, -\omega, 1, \omega^2, \omega^3, \omega^2)$
6. $(1, -\omega, \omega^2, \omega^3, \omega^4, -\omega)$
7. \((1 - \omega, (6\omega^3 - 2\omega^2 - 6\omega + 4)\theta + (-\omega^3 + \omega^2 - \omega)/2, (-8\omega^3 + 16\omega - 12)\theta + \omega^3 - 2\omega^2 + 1, (8\omega^3 - 12\omega + 8)\theta - 2\omega^3 + \omega^2 + \omega - 1, (2\omega^3 - 6\omega^2 + 2\omega)\theta + (\omega^3 - \omega^2 - \omega + 2)/2)\)

8. \((1 - \omega, (-6\omega^3 + 2\omega^2 - 6\omega - 4)\theta + (-\omega^3 + \omega^2 - \omega)/2, (8\omega^3 - 16\omega + 12)\theta + \omega^3 - 2\omega^2 + 1, (-8\omega^3 + 12\omega + 8)\theta - 2\omega^3 + \omega^2 + \omega - 1, (-2\omega^3 + 6\omega^2 - 2\omega)\theta + (\omega^3 - \omega^2 - \omega + 2)/2)\)

9. \((1, \omega^3, \omega^4, \omega^5, 1, -\omega)\)

10. \((1, \omega^3, -\omega^2, \omega^3, 1, -\omega^3)\)

11. \((1, \omega^3, (-6\omega^3 + 4\omega^2 - 2)\theta + (-\omega^3 + 2\omega - 1)/2, (-8\omega^3 + 16\omega - 12)\theta + \omega^3 - 2\omega^2 + 1, (12\omega^3 - 16\omega^2 + 8)\theta + \omega^3 - 2\omega + 1, (2\omega^3 - 6\omega^2 + 2\omega)\theta + (\omega^3 - \omega^2 + 2\omega)/2)\)

12. \((1, \omega^3, (6\omega^3 - 4\omega^2 + 2)\theta + (-\omega^3 + 2\omega - 1)/2, (8\omega^3 - 16\omega + 12)\theta + \omega^3 - 2\omega^2 + 1, (-12\omega^3 + 16\omega^2 - 8)\theta + \omega^3 - 2\omega + 1, (-2\omega^3 + 4\omega - 6)\theta + (\omega^3 - 2\omega^2 - 1)/2)\)

13. \((1, -\omega^3, \omega^4, -\omega^3, 1, \omega)\)

14. \((1, -\omega^3, -\omega^2, -\omega^3, 1, \omega^3)\)

15. \((1, -\omega^3, (-6\omega^3 + 4\omega^2 - 2)\theta + (-\omega^3 + 2\omega - 1)/2, (8\omega^3 - 16\omega + 12)\theta + \omega^3 - 2\omega^2 + 1, (12\omega^3 - 16\omega^2 + 8)\theta + \omega^3 - 2\omega + 1, (2\omega^3 - 6\omega^2 + 2\omega)\theta + (\omega^3 - \omega^2 + 2\omega)/2)\)

16. \((1, -\omega^3, (6\omega^3 - 4\omega^2 + 2)\theta + (-\omega^3 + 2\omega - 1)/2, (8\omega^3 - 16\omega + 12)\theta + \omega^3 - 2\omega^2 + 1, (-12\omega^3 + 16\omega^2 - 8)\theta + \omega^3 - 2\omega + 1, (2\omega^3 - 6\omega^2 + 2\omega)\theta + (\omega^3 - \omega^2 - 1)/2)\)

17. \((1, \omega, 1, -\omega^3, \omega^4, -\omega^3)\)

18. \((1, \omega, \omega^4, -\omega^3, \omega^4, \omega)\)

19. \((1, \omega, (6\omega^3 - 2\omega^2 - 6\omega + 4)\theta + (-\omega^3 + \omega^2 - \omega)/2, (8\omega^3 - 16\omega + 12)\theta - \omega^3 - 2\omega^2 + 1, (8\omega^3 - 12\omega + 8)\theta - 2\omega^3 + \omega^2 + \omega - 1, (-2\omega^3 + 6\omega^2 - 2\omega)\theta + (\omega^3 - \omega^2 + \omega - 2)/2)\)

20. \((1, \omega, (-6\omega^3 + 2\omega^2 + 6\omega - 4)\theta + (-\omega^3 + \omega^2 - \omega)/2, (-8\omega^3 + 16\omega - 12)\theta - \omega^3 - 2\omega^2 + 1, (-8\omega^3 + 12\omega - 8)\theta - 2\omega^3 + \omega^2 + \omega - 1, (2\omega^3 - 6\omega^2 + 2\omega)\theta + (\omega^3 - \omega^2 + \omega - 2)/2)\)

21. \((1, -\omega^3, 1, \omega^3, -\omega^2, \omega^3)\)

22. \((1, -\omega^3, -\omega^2, 1, -\omega^3, -\omega^3)\)

23. \((1, -\omega^3, (2\omega^3 - 6\omega^2 + 2)\theta + (-\omega^3 - 2\omega - \omega + 1)/2, (8\omega^3 - 16\omega + 12)\theta + \omega^3 - 2\omega^2 + 1, (12\omega^3 - 8\omega^2 - 12\omega + 16)\theta + \omega^3 - 2\omega + 1, (2\omega^3 - 6\omega^2 + 2\omega)\theta + (\omega^3 - \omega^2 + \omega - 2)/2)\)

24. \((1, -\omega^3, (-2\omega^3 + 6\omega - 2)\theta + (2\omega^3 - \omega^2 - \omega + 1)/2, (-8\omega^3 + 16\omega - 12)\theta + \omega^3 - 2\omega^2 + 1, (-12\omega^3 + 8\omega^2 + 12\omega - 16)\theta + \omega^3 + \omega^2 + \omega, (4\omega^3 - 6\omega^2 - 2\omega + 6)\theta + (\omega^3 - \omega + 2)/2)\)

25. \((1, (4\omega^3 - 6\omega^2 - 2\omega + 6)\theta + (\omega^3 - \omega - 1)/2, (6\omega^3 - 2\omega^2 - 6\omega + 4)\theta + (-\omega^3 + \omega^2 - \omega)/2, -\omega^3, (2\omega^3 - 6\omega + 2)\theta + (2\omega^3 - \omega^2 - \omega + 1)/2, (2\omega^3 - 6\omega^2 + 2\omega)\theta + (\omega^3 - \omega^2 + \omega - 2)/2)\)

26. \((1, (4\omega^3 - 6\omega^2 - 2\omega + 6)\theta + (\omega^3 - \omega - 1)/2, (12\omega^3 - 8\omega^2 - 12\omega + 16)\theta + \omega^3 - \omega^2 + \omega, (8\omega^3 - 16\omega + 12)\theta - \omega^3 + 2\omega^2 - 1, (-2\omega^3 + 6\omega - 2)\theta + (2\omega^3 - \omega^2 - \omega + 1)/2, \omega^3\)
34. \((1, (2\omega^3 - 4\omega + 6)\theta + (-\omega^3 + 2\omega^2 - 1)/2, (12\omega^3 - 16\omega^2 + 8)\theta + \omega^3 - 2\omega + 1,
(8\omega^3 + 16\omega - 12)\theta + \omega^3 - 2\omega^2 + 1, (-6\omega^3 + 4\omega^2 - 2)\theta + (-\omega^3 + 2\omega - 1)/2, \omega^3)\)

35. \((1, (2\omega^3 - 4\omega + 6)\theta + (\omega^3 - 2\omega^2 + 1)/2, (6\omega^3 - 4\omega^2 + 2)\theta + (-\omega^3 + 2\omega - 1)/2, -\omega^3,
(-6\omega^3 + 4\omega^2 - 2)\theta + (-\omega^3 + 2\omega - 1)/2, -(2\omega^3 + 4\omega - 6)\theta + (\omega^3 - 2\omega^2 + 1)/2)\)

36. \((1, (2\omega^3 - 4\omega + 6)\theta + (\omega^3 - 2\omega^2 + 1)/2, (-12\omega^3 + 16\omega^2 - 8)\theta + \omega^3 - 2\omega + 1,
(-8\omega^3 + 16\omega - 12)\theta - \omega^3 + 2\omega^2 - 1, (6\omega^3 - 4\omega^2 + 2)\theta + (-\omega^3 + 2\omega - 1)/2, -\omega^3)\)

37. \((1, (-2\omega^3 + 6\omega^2 - 2\omega)\theta + (\omega^3 - \omega^2 - \omega + 2)/2, (2\omega^3 - 6\omega + 2)\theta + (2\omega^3 - \omega^2 - \omega + 1)/2, \omega^3,
(-6\omega^3 - 2\omega^2 - 6\omega + 4)\theta + (-\omega^3 + \omega^2 - \omega)/2, (-4\omega^3 + 6\omega^2 + 2\omega - 6)\theta + (-\omega^3 + \omega - 1)/2)\)

38. \((1, (-2\omega^3 + 6\omega^2 - 2\omega)\theta + (\omega^3 - \omega^2 - \omega + 2)/2, (-8\omega^3 + 12\omega - 8)\theta - 2\omega^3 + \omega^2 + \omega - 1,
(8\omega^3 - 16\omega + 12)\theta + \omega^3 - 2\omega^2 + 1, (-6\omega^3 + 2\omega^2 + 6\omega - 4)\theta + (-\omega^3 + \omega^2 - \omega)/2, -\omega)\)

39. \((1, (-2\omega^3 + 6\omega^2 - 2\omega)\theta + (\omega^3 - \omega^2 - \omega - 2)/2, (8\omega^3 - 12\omega + 8)\theta - 2\omega^3 + \omega^2 + \omega - 1,
(8\omega^3 - 16\omega + 12)\theta - \omega^3 + 2\omega^2 - 1, (6\omega^3 - 2\omega^2 - 6\omega + 4)\theta + (-\omega^3 + \omega^2 - \omega)/2, \omega)\)

40. \((1, (2\omega^3 - 6\omega^2 + 2\omega)\theta + (\omega^3 - \omega^2 - \omega + 2)/2, (8\omega^3 - 12\omega + 8)\theta - 2\omega^3 + \omega^2 + \omega - 1,
(8\omega^3 - 16\omega + 12)\theta - \omega^3 + 2\omega^2 + 1, (6\omega^3 - 2\omega^2 - 6\omega + 4)\theta + (-\omega^3 + \omega^2 - \omega - 2)/2, -\omega)\)

41. \((1, (2\omega^3 - 6\omega^2 + 2\omega)\theta + (\omega^3 - \omega^2 - \omega + 2)/2, (-2\omega^3 + 6\omega - 2)\theta + (2\omega^3 - \omega^2 - \omega + 1)/2, \omega^3,
(-6\omega^3 + 2\omega^2 + 6\omega - 4)\theta + (-\omega^3 + \omega^2 - \omega)/2, (-4\omega^3 + 6\omega^2 + 2\omega - 6)\theta + (\omega^2 - \omega - 1)/2)\)

42. \((1, (2\omega^3 - 6\omega^2 + 2\omega)\theta + (\omega^3 - \omega^2 - \omega + 2)/2, (2\omega^3 - 6\omega + 2)\theta + (2\omega^3 - \omega^2 - \omega + 1)/2, -\omega^3,
(6\omega^3 - 2\omega^2 - 6\omega + 4)\theta + (-\omega^3 + \omega^2 - \omega)/2, (4\omega^3 + 6\omega^2 - 2\omega + 6)\theta + (\omega^2 - \omega + 1)/2)\)

43. \((1, (2\omega^3 - 6\omega^2 + 2\omega)\theta + (\omega^3 - \omega^2 - \omega + 2)/2, (2\omega^3 - 6\omega + 2)\theta + (2\omega^3 - \omega^2 - \omega + 1)/2, -\omega^3,
(6\omega^3 - 2\omega^2 - 6\omega + 4)\theta + (-\omega^3 + \omega^2 - \omega)/2, (4\omega^3 - 6\omega^2 - 2\omega + 6)\theta + (\omega^2 - \omega - 1)/2)\)

44. \((1, (2\omega^3 - 6\omega^2 + 2\omega)\theta + (\omega^3 - \omega^2 + \omega - 2)/2, (-8\omega^3 + 12\omega - 8)\theta - 2\omega^3 + \omega^2 + \omega - 1,
(-8\omega^3 + 16\omega - 12)\theta - \omega^3 + 2\omega^2 - 1, (-6\omega^3 + 2\omega^2 + 6\omega - 4)\theta + (-\omega^3 + \omega^2 - \omega)/2, \omega)\)

45. \((1, (-2\omega^3 + 4\omega - 6)\theta + (\omega^3 + 2\omega^2 - 1)/2, (6\omega^3 - 4\omega^2 + 2)\theta + (-\omega^3 + 2\omega - 1)/2, \omega^3,
(-6\omega^3 + 4\omega^2 - 2)\theta + (-\omega^3 + 2\omega - 1)/2, (2\omega^3 - 4\omega + 6)\theta + (-\omega^3 + 2\omega^2 - 1)/2)\)

46. \((1, (-2\omega^3 + 4\omega - 6)\theta + (\omega^3 + 2\omega^2 - 1)/2, (-12\omega^3 + 16\omega^2 - 8)\theta + \omega^3 - 2\omega + 1,
(8\omega^3 - 16\omega + 12)\theta + \omega^3 - 2\omega^2 + 1, (6\omega^3 - 4\omega^2 + 2)\theta + (-\omega^3 + 2\omega - 1)/2, \omega^3)\)

47. \((1, (-2\omega^3 + 4\omega - 6)\theta + (\omega^3 - 2\omega^2 + 1)/2, (-6\omega^3 + 4\omega^2 - 2)\theta + (-\omega^3 + 2\omega - 1)/2, -\omega^3,
(6\omega^3 - 4\omega^2 + 2)\theta + (-\omega^3 + 2\omega - 1)/2, (2\omega^3 - 4\omega + 6)\theta + (\omega^3 - 2\omega^2 + 1)/2)\)

48. \((1, (-2\omega^3 + 4\omega - 6)\theta + (\omega^3 - 2\omega^2 + 1)/2, (12\omega^3 - 16\omega^2 + 8)\theta + \omega^3 - 2\omega + 1,
(8\omega^3 - 16\omega + 12)\theta - \omega^3 + 2\omega^2 - 1, (-6\omega^3 + 4\omega^2 - 2)\theta + (-\omega^3 + 2\omega - 1)/2, -\omega^3)\)