Cherrier-Escobar problem for the elliptic Schrödinger-to-Neumann map.

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Abstract

In this paper, we study a Cherrier-Escobar problem for the extended problem corresponding to the elliptic Schrödinger-to-Neumann map on a compact 3-dimensional Riemannian manifold with boundary. Using the algebraic topological argument of Bahri-Coron\textsuperscript{[6]}, we show solvability under the assumption that the extended problem corresponding to the elliptic Schrödinger-to-Neumann map has a positive first eigenvalue, a positive Green’s function, and also verifies the strong maximum principle.

Key Words: Barycenter technique, PS-sequences, elliptic Schrödinger-to-Neumann map, Self-action estimates, Inter-action estimates.

AMS subject classification: 53C21, 35C60, 58J60, 55N10.

1 Introduction and statement of the results

Boundary Value problems (BVP) with critical nonlinearity on the boundary of the form

\[
\begin{aligned}
-\Delta_g u + qu &= 0 \quad \text{in } M, \\
-\frac{\partial u}{\partial n_g} + \frac{n-2}{2(n-1)} H_g &= u^{\frac{n}{n-2}} \quad \text{on } \partial M, \\
u > 0 &\quad \text{on } \overline{M},
\end{aligned}
\]

have received a lot of attention in the last decades. In (1), \((M,g)\) is a \(n\)-dimensional compact Riemannian manifold with boundary \(\partial M\) and interior \(M\), with \(n \geq 3\). Furthermore, \(\Delta_g\) denotes the Laplace–Beltrami with respect to \(g\), \(\frac{\partial}{\partial n_g}\) denotes the inner Neumann derivative with respect to \(g\), \(H_g\) denotes the mean curvature of \(\partial M\) with respect to \(g\) in the normal direction, and the potential \(q\) is a bounded smooth function defined on \(M\). In this paper, we study the particular case \(\text{dim } \overline{M} = 3\) and \(H_g = 0\). Hence, the BVP of interest becomes

\[
\begin{aligned}
-\Delta_g u + qu &= 0 \quad \text{in } M, \\
-\frac{\partial u}{\partial n_g} &= u^3 \quad \text{on } \partial M, \\
u > 0 &\quad \text{on } \overline{M}.
\end{aligned}
\]

Looking at the Cherrier-Escobar problem studied in \textsuperscript{[17]} as the \(\frac{1}{2}\)-Yamabe problem, we have BVP (2) is related to a Cherrier-Escobar problem for the elliptic Schrödinger-to-Neumann map \(P_q\) defined by \(P_q : \mathcal{C}^\infty(\partial M) \to \mathcal{C}^\infty(\partial M)\), \(u \rightarrow \frac{\partial U_q}{\partial n_g}\) where \(U_q\) is the unique solution of

\[
\begin{aligned}
-\Delta_g U_q + q U_q &= 0 \quad \text{in } M, \\
U_q &= u \quad \text{on } \partial M.
\end{aligned}
\]

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Indeed, \( u \) is solution of BVP (2) implies
\[
\begin{cases}
P_qu = u^3 & \text{on } \partial M, \\
u > 0 & \text{on } \partial M,
\end{cases}
\]
and \( u \) is solution of (4) implies that \( U_q \) defined by (3) is solution of BVP (2) thanks to fact that the extended problem associated to \( P_q \) verifies the strong maximum principle as we will assume (see (7)-(8) below). We used the terminology the elliptic Schrödinger-to-Neumann map, since \( P_q \) is clearly the Dirichlet to Neumann associated to Schrödinger operator \(-\Delta_g + q\).

As in [1], it is easy to see that a necessary condition of the existence of solution to (2) is that the first eigenvalue \( \lambda_1(P_q) \) of the extended problem corresponding to \( P_q \) i.e. of the eigenvalue problem
\[
\begin{cases}
-\Delta_g u + qu = 0 & \text{in } M, \\
-\frac{\partial u}{\partial n_g} = \lambda u & \text{on } \partial M,
\end{cases}
\]
is strictly positive.

We use the symbol \( \lambda_1(P_q) \), since it can be easily checked that it corresponds also to the first eigenvalue of \( P_q \). Hence, from now on we will assume that
\[
\lambda_1(P_q) > 0.
\]

As already said, we will assume that the extended problem corresponding to \( P_q \) satisfies the strong maximum principle, in the sense that
\[
\begin{cases}
P_qu \geq 0 & \text{on } \partial M, \\
u \geq 0 & \text{on } \partial M, \\
u \neq 0 & \text{on } \partial M,
\end{cases}
\]
implies
\[
U_q > 0 \quad \text{on } \overline{M}
\]
with \( U_q \) defined by (3). Thus thanks to elliptic regularity theory (see [20]), a smooth solution of (2) can be found by looking at critical points of the functional \( J_q \) defined by
\[
J_q(u) := \frac{\langle u, u \rangle_q}{\langle \int_{\partial M} u^4 dS_g \rangle^2}, \quad u \in H^1(M) := \{ u \in H^1(M) : u \geq 0 \text{ and } u \neq 0 \},
\]
with
\[
\langle u, u \rangle_q = \int_M (|\nabla_g u|^2 + qu^2) \, dV_g,
\]
where \( \nabla_g \) denotes the covariant derivative with respect to \( g \), \( dV_g \) denotes the volume form with respect to \( g \), and \( dS_g \) denotes the volume form with respect to the Riemannian metric induced by \( g \) on \( \partial M \). Moreover, in formula (10), \( H^1(M) \) denotes the usual Sobolev space of functions which are \( L^2 \)-integrable together with their first derivatives.

The existence of solutions for (2) are known under the assumption \( q = R_g \) up to a positive constant, where \( R_g \) denotes the scalar curvature of \( (M, g) \). There are a lot of works related to equation (11), see [2], [3], [11], [12], [14], [15], and [18]. In this paper, we use the Barycenter technique of Bahri-Coron [6] to study (2) for general potentials \( q \). We prove the following result:

**Theorem 1.1.** Let \((M, g)\) be 3-dimensional compact Riemannian manifold with boundary \( \partial M \) and interior \( M \) such that \( H_g = 0 \). Let also \( q \) be a bounded smooth potential defined on \( M \). Assuming that \( \lambda_1(P_q) > 0 \) (see (5) for the definition), the extended problem corresponding to \( P_q \) satisfies the strong maximum principle (in the sense of (7)-(8)), and the Green’s function \( G \) of the extended problem corresponding to \( P_q \) defined by (16) is strictly positive, then the BVP (2) has a least one solution.
As in [1], to prove Theorem 1.1 we will use the Barycenter technique of Bahri-Coron [6] which is possible since $\dim M = 3$ and $H_g = 0$ imply the problem under study is a Global one (for the definition of "Gobal" for Yamabe and Cherrier-Escobar type problems, see [19]). Indeed, as in [1] and [19], we will follow the scheme of the Algebraic topological argument of Bahri-Coron [6] as performed in the work [17] of the second author and Mayer. As in [1], one of the main difficulty with respect to the works [17] and [19] is the presence of the linear term "qua" and the lack of conformal invariance. As in [1], to deal with such a difficulty, we use the fact that $\dim M = 3$, $H_g = 0$, and the Brendle [9]-Schoen [21]’s bubble construction to run the scheme of the Algebraic topological argument of Bahri-Coron [6] for the existence.

The plan of the paper is as follows. In Section 2, we fix some notations and discuss some preliminaries. We start by introducing some technical estimates. In Section 3, we recall the profile decomposition of Palais-Smale (PS)-sequences for $J_g$. We also state a Deformation Lemma for $J_g$ taking into account the possible bubbling phenomena involved in the study (2). In Section 4, we derive some sharp self-action estimates needed for the application of the Barycenter technique of Bahri-Coron [6] for existence. In Section 5, we derive some sharp inter-action estimates needed for the application of the Barycenter technique of Bahri-Coron [6] for existence. In Section 6, we present the algebraic topological argument for existence. In Section 7, we collect some technical estimates.

2 Notations and preliminaries

In this Section, we fix some notations and discuss some preliminaries. We start by introducing some notations.

For $x = (x_1, x_2, x_3) \in \mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}_+$, with $\mathbb{R}_+ = [0, +\infty)$, we set $\bar{x} = (x_1, x_2) \in \mathbb{R}^2$, so that $x = (\bar{x}, x_3)$, and we set also $r = |x| = \sqrt{x_1^2 + x_2^2 + x_3^2}$. We define $\mathbb{R}_+^3 = \mathbb{R}^2 \times \mathbb{R}_+$ with $\mathbb{R}_+ = (0, \infty)$. We identify the boundary of $\mathbb{R}_+^3$ denoted $\partial \mathbb{R}_+^3$ with $\mathbb{R}^2$. For $r > 0$, we define the half ball $B_r^+(0)$ with radius $r$ centered at 0, by

$$B_r^+(0) = B_r(0) \cap \mathbb{R}_+^3,$$

with $B_r(0)$ denoting the Euclidean ball centered at 0 with radius $r$ in $\mathbb{R}^3$, namely

$$B_r(0) = \{ x \in \mathbb{R}^3 : |x| < r \}.$$

Furthermore, we define

$$\partial B_r^+(0) = B_r(0) \cap \mathbb{R}^2.$$

Large positive constants are usually denoted by $C$ and the value of $C$ is allowed to verify from formula to formula and also within the same line.

For $r > 0$ and $a \in \partial M$, we define

$$B(a, r) = \{ x \in M : d_g(a, x) < r \},$$
$$\hat{B}(a, r) = \{ x \in \partial M : d_g(a, x) < r \},$$
$$\hat{B}_r(0) = \{ x \in \mathbb{R}^2 : |x| < r \}.$$

We set

$$\|u\|_q = \sqrt{\langle u, u \rangle_q}, \quad u \in H^1(M), \quad (11)$$

and $\langle u, u \rangle_q$ is as in (10).

For $\lambda > 0$, we set

$$\delta_{0, \lambda}(x) = c_0 \left[ \frac{\lambda}{(1 + \lambda x_3)^2 + \lambda^2 |x|^2} \right]^\frac{1}{2}, \quad x = (\bar{x}, x_3) \in \mathbb{R}_+^3 \quad (12)$$

with $c_0 > 0$ such that
We define $g$, with $C$ where $\Delta = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_{i}^2}$

is the Euclidean Laplacian on $\mathbb{R}^{3}$.

It is a well-know fact that (and easy to check)

$$\int_{\mathbb{R}^{3}} |\nabla \delta_{0,\lambda}|^2 = \int_{\mathbb{R}^{3}} |\nabla \delta_{0,1}|^2 = \int_{\mathbb{R}^{3}} \delta_{0,\lambda}^4 = \int_{\mathbb{R}^{3}} \delta_{0,1}^4 \quad (13)$$

Moreover, we set

$$S = \frac{\int_{\mathbb{R}^{3}} |\nabla \delta_{0,1}|^2}{(\int_{\mathbb{R}^{3}} \delta_{0,1}^4)^{\frac{1}{2}}} \quad (14)$$

We define

$$c_1 = \int_{\mathbb{R}^{3}} \left( \frac{1}{1 + |y|^2} \right)^{\frac{1}{2}} dy \quad (15)$$

For $a \in \partial M$, we let $G(a,x)$ be the unique solution of

$$\begin{cases}
\Delta_{g} G(a,x) + qG(a,x) = 0, & x \in M, \\
- \frac{\partial G(a,x)}{\partial n_{a}} = 2\pi \delta_{a}(x), & x \in \partial M. 
\end{cases} \quad (16)$$

Since $q$ is a bounded smooth function defined on $M$, $H_{g} = 0$, and $\dim \overline{M} = 3$, then Green’s function $G(a,x)$ satisfies the following estimates

\begin{align*}
|G(a,x) - \frac{1}{d_{g}(a,x)}| & \leq C, \quad \text{for} \quad x \neq a \in \overline{M}, \\
|\nabla \left( G(a,x) - \frac{1}{d_{g}(a,x)} \right)| & \leq \frac{C}{d_{g}(a,x)}, \quad \text{for} \quad x \neq a \in \overline{M},
\end{align*} \quad (17) (18)

with $C$ a positive constant.

Moreover, we have $\overline{M}$ compact implies that there exists

$$\delta_{0} > 0 \quad (19)$$

such that $\forall \ a \in \partial M$ and $\forall \ 0 < 2\delta \leq \delta_{0}$, the Fermi coordinates centered at $a$ defines a smooth map

$$\psi_{a} : B_{2\delta}^{+}(0) \rightarrow \overline{M}, \quad (20)$$

identifying a neighborhood $O(a)$ of $a$ in $\overline{M}$. We will identify a point $y = \psi_{a}(x) \in O(a)$ with $x \in B_{2\delta}^{+}(0)$. With this agreement and recalling that $H_{g} = 0$, we have that an expansion of the Riemannian metric $g$ and $\sqrt{|g|}$ (where $|g|$ denotes the modulus of the determinant of $g$) on $B_{2\delta}^{+}(0)$ is given by the following formulas

$$g^{ij}(x) = \delta_{ij} + 2(L_{g})_{ij}x_{i}(x) + \frac{1}{3}R_{ikjl}g^{ij}x_{k}x_{l}(x) + g_{nk}^{ij}x_{n}x_{k}(x) + \{3(L_{g})_{ij}(L_{g})_{kj} + R_{ijnk}[g]\} x_{n}^{2}(x) + o(|x|^{3}), \quad x \in B_{2\delta}^{+}(0)$$

$$\sqrt{|g(x)|} = 1 - \frac{1}{6} R_{ic}[g]_{ij}x_{i}x_{j}(x) - \left[ \frac{1}{2}||L_{g}||^{2} + R_{ij}[g]_{nn} \right] x_{n}^{2}(x) + o(|x|^{3}), \quad x \in B_{2\delta}^{+}(0). \quad (21)$$

In the formulas in (21), $n = 3$, $(L_{g})_{ij}$ denotes the component of the second fundamental form of $\partial M$ with respect to $g$, $\ (R_{ic}[g])_{ab}, \ a,b = 1,..,n$ denotes the component of the Ricci tensor of $\overline{M}$ with
respect to \( g, \hat{g} := g|_{\partial M} \) is the Riemannian metric induced by \( g \) on \( \partial M \), \( (R_{\alpha\beta\gamma\delta}[\hat{g}])_{ij}, i, j = 1, \ldots, n-1 \) are the components of the Ricci tensor of \( \partial M \) with respect to \( \hat{g} \), \( R_{abcd}[\hat{g}] \), \( a, b, c, d = 1, \ldots, n \) denotes the components of Riemann tensor of \( M \) with respect to \( \hat{g} \), and \( R_{ij\alpha\beta}[\hat{g}], i, k = 1, \ldots, n-1 \) denotes the components of Riemann curvature tensor of \( \partial M \) with respect to \( \hat{g} \). All the tensors in the right of (21) are evaluated at \( 0 \), and we also use Einstein summation convention for repeated indexes.

Let \( \chi : \mathbb{R} \to \mathbb{R} \) be a smooth cut-off function satisfying

\[
\chi(t) = \begin{cases} 
1, & \text{if } t \leq 1, \\
0, & \text{if } t \geq 2.
\end{cases}
\]  

(22)

For \( 0 < 2\delta < \delta_0 \), we define

\[
\chi_\delta(x) = \chi\left(\frac{|x|}{\delta}\right), \quad x \in \mathbb{R}^3.
\]  

(23)

For \( 0 < 2\delta < \delta_0 \), \( a \in \partial M \), and \( \lambda > 0 \), we define the Brendle\[9\]-Schoen\[21\]'s bubble

\[
u_{a,\lambda}(x) = u_{a,\lambda,\delta}(x):
\]

\[
\hat{\delta}_{a,\lambda}(x) = \delta_{a,\lambda}(\psi_a^{-1}(x)),
\]  

(25)

and

\[
\chi_{\delta}^a = \chi_{\delta}(\psi_a^{-1}(x)),
\]  

(26)

with

\[
\psi_a : B^+(0) \to \hat{M}.
\]  

(27)

Thus, recalling that we are under the assumption \( G > 0 \) (for the definition of \( G \) see (16)), then \( \forall a \in \partial M \) and \( \forall 0 < 2\delta < \delta_0 \), we have

\[
u_{a,\lambda} \in H^1(M), \quad \forall \nu_{a,\lambda} > 0 \quad \text{in } \hat{M}.
\]  

(28)

For \( a_i, a_j \in \partial M \), and \( \lambda_i, \lambda_j > 0 \), we define

\[
\epsilon_{ij} = \left[ \frac{1}{\lambda_i} + \frac{1}{\lambda_j} + \lambda_i \lambda_j g^{-2}(a_i, a_j) \right]^{\frac{1}{2}}.
\]  

(29)

Moreover, for \( 0 < 2\delta < \delta_0 \), \( a_i, a_j \in \partial M \), and \( \lambda_i, \lambda_j > 0 \), we define

\[
\epsilon_{ij} = \int_{\partial M} u_{a_i,\lambda_i} u_{a_j,\lambda_j} dS_g
\]  

(30)

and

\[
\epsilon_{ij} = \int_M \nabla_g u_{a_i,\lambda_i} \nabla_g u_{a_j,\lambda_j} dV_g + \int_M q u_{a_i,\lambda_i} u_{a_j,\lambda_j} dV_g.
\]  

(31)

To end the section, we derive the following \( C^0 \)-estimate needed for the energy and the interaction estimates required for the application of the Barycenter technique of Bahri-Coron\[6\] for existence.

**Lemma 2.1.** Assuming that \( \theta > 0 \) is small, then there exists \( C > 0 \) such that \( \forall a \in \partial M \), \( \forall 0 < 2\delta < \delta_0 \) and \( \forall \epsilon \in [\frac{1}{10} \leq \delta_0, \epsilon \), we have

\[
|(-\Delta_g + q) u_{a,\lambda}(x)| \leq C \left[ \frac{1}{\delta^2 \sqrt{\lambda}} \right] \left[ \int_{\{y \in M: d_g(a, y) \leq 2\delta\}} (x) + \hat{\delta}_{a,\lambda}(x) \right] \left[ \int_{\{y \in M: d_g(a, y) \leq 2\delta\}} (x) \right], \forall x \in M,
\]  

(32)

and

\[
\frac{\partial u_{a,\lambda}(x)}{\partial n_g} - u_{a,\lambda}^3 \leq C \left[ \frac{1}{1 + \lambda^2 d_g^2(a, x)} \right] \left[ \int_{\{y \in \partial M: d_g(a, y) \geq 4\}} (x) \right], \forall x \in \partial M.
\]  

(33)
We will estimate each of the $G_a(\cdot) := G(a, \cdot)$ and $\tilde{G}_a(\cdot) = c_0 G_a(\cdot)$. Then, we have

$$u_{a,\lambda}(x) = \chi_a^\delta(x) \delta_{a,\lambda}(x) + (1 - \chi_a^\delta(x)) \frac{\tilde{G}_a(x)}{\sqrt{\lambda}}, \quad x \in M.$$  \hfill (34)

To deal with $\delta_2$, first we observe

$$(-\Delta_g + q) u_{a,\lambda}(x) = (-\Delta_g + q) \left( \chi_a^\delta(x) \delta_{a,\lambda}(x) + (1 - \chi_a^\delta(x)) \frac{\tilde{G}_a(x)}{\sqrt{\lambda}} \right)$$

$$= (-\Delta_g + q) \left( \chi_a^\delta(x) \left( \delta_{a,\lambda}(x) - \frac{\tilde{G}_a(x)}{\sqrt{\lambda}} \right) \right) + (-\Delta_g + q) \left[ \frac{\tilde{G}_a(x)}{\sqrt{\lambda}} \right], \quad x \in M.$$  \hfill (35)

Now, since $x \in M$ and $a \in \partial M$, then $(-\Delta_g + q) \left[ \frac{\tilde{G}_a(x)}{\sqrt{\lambda}} \right] = 0$. This implies

$$(-\Delta_g + q) u_{a,\lambda}(x) = -\Delta_g \chi_a^\delta(x) \left[ \delta_{a,\lambda}(x) - \frac{\tilde{G}_a(x)}{\sqrt{\lambda}} \right] - 2\nabla_g \chi_a^\delta(x) \nabla_g \left[ \delta_{a,\lambda}(x) - \frac{\tilde{G}_a(x)}{\sqrt{\lambda}} \right]$$

$$- \chi_a^\delta(x) \Delta_g \left[ \delta_{a,\lambda}(x) - \frac{\tilde{G}_a(x)}{\sqrt{\lambda}} \right] + q \chi_a^\delta(x) \left( \delta_{a,\lambda}(x) - \frac{\tilde{G}_a(x)}{\sqrt{\lambda}} \right), \quad x \in M.$$  \hfill (36)

Thus,

$$(-\Delta_g + q) u_{a,\lambda}(x) = \sum_{m=1}^4 I_m$$

with

$$I_1 = -\Delta_g \chi_a^\delta(x) \left[ \delta_{a,\lambda}(x) - \frac{\tilde{G}_a(x)}{\sqrt{\lambda}} \right], \quad x \in M,$$

$$I_2 = -2 \langle \nabla_g \chi_a^\delta(x), \nabla_g \left[ \delta_{a,\lambda}(x) - \frac{\tilde{G}_a(x)}{\sqrt{\lambda}} \right] \rangle, \quad x \in M,$$

$$I_3 = q \chi_a^\delta(x) \delta_{a,\lambda}(x), \quad x \in M,$$

$$I_4 = -\chi_a^\delta(x) \Delta_g \delta_{a,\lambda}(x), \quad x \in M.$$  \hfill (37)

We will estimate each of the $I_m$'s one at a time. For $I_1$, we have

$$I_1 = -\Delta_g \chi_a^\delta(x) \left[ \delta_{a,\lambda}(x) - \frac{c_0}{\sqrt{\lambda} d_g(a,x)} + \frac{c_0}{\sqrt{\lambda} d_g(a,x)} - \frac{\tilde{G}_a(x)}{\sqrt{\lambda}} \right], \quad x \in M.$$  \hfill (38)

Applying (12) and (17), we derive

$$\left| \delta_{a,\lambda}(x) - \frac{c_0}{\sqrt{\lambda} d_g(a,x)} \right| \leq \frac{C}{\sqrt{\lambda}}, \quad x \in M,$$  \hfill (39)

and

$$\left| \frac{c_0}{\sqrt{\lambda} d_g(a,x)} - \frac{\tilde{G}_a(x)}{\sqrt{\lambda}} \right| \leq \frac{C}{\sqrt{\lambda}}, \quad x \in M.$$  \hfill (40)

Using (22) and (23), we get

$$\left| \frac{c_0}{\sqrt{\lambda} d_g(a,x)} - \frac{\tilde{G}_a(x)}{\sqrt{\lambda}} \right| \leq \frac{C}{\sqrt{\lambda}}, \quad x \in M.$$  \hfill (41)

Hence, combining (35) and (36), we obtain

$$|I_1| \leq \frac{C}{\delta^2} \delta^2 \delta_\lambda \left( \frac{c_0}{\sqrt{\lambda} d_g(a,x)} - \frac{\tilde{G}_a(x)}{\sqrt{\lambda}} \right), \quad x \in M.$$  \hfill (42)

In order to estimate $I_2$, we first write

$$I_2 = 2 \langle \nabla_g \chi_a^\delta(x), \nabla_g \left[ \delta_{a,\lambda}(x) - \frac{c_0}{\sqrt{\lambda} d_g(a,x)} + \frac{c_0}{\sqrt{\lambda} d_g(a,x)} - \frac{\tilde{G}_a(x)}{\sqrt{\lambda}} \right] \rangle, \quad x \in M.$$  \hfill (43)
In the next step, using (12) and (18), we have the following:
\[
\left| \nabla_g \left[ \hat{\delta}_{a,\lambda}(x) - \frac{e_0}{\sqrt{\lambda}d_g(a,x)} \right] \right| \leq \frac{C}{\sqrt{\lambda}d_g(a,x)}, \quad x \in M, 
\] (41)
and
\[
\left| \nabla_g \left[ \frac{e_0}{\sqrt{\lambda}d_g(a,x)} - \frac{\hat{G}_a(x)}{\sqrt{\lambda}} \right] \right| \leq \frac{C}{\sqrt{\lambda}d_g(a,x)}, \quad x \in M. 
\] (42)

On the other hand, using (22), we derive
\[
\left| \nabla_g x^2 \right| \leq \frac{C}{\delta^2 \sqrt{\lambda}} 1_{\{y \in M: \delta \leq d_g(a,y) \leq 2\delta\}}(x), \quad x \in M. 
\] (43)

Hence, combining (41) and (42), we get
\[
|I_2| \leq \frac{C}{\delta^2 \sqrt{\lambda}} 1_{\{y \in M: d_g(a,y) \leq 2\delta\}}(x), \quad x \in M. 
\] (44)

For $I_3$, using (22) and (23), we obtain
\[
|I_3| \leq C \hat{\delta}_{a,\lambda}(x) 1_{\{y \in M: d_g(a,y) \leq 2\delta\}}(x), \quad x \in M. 
\] (45)

To deal with $I_4$, we first estimate $\Delta_g \hat{\delta}_{a,\lambda}$ on $\psi(B_2^+(0))$ (see (20)). For this, identifying $\psi(a,x)$ with $x \in B_2^+(0)$, we have
\[
\Delta_g \hat{\delta}_{a,\lambda}(x) = \frac{1}{\sqrt{|g(x)|}} \partial_t \left[ g^{ij}(x) \sqrt{|g(x)|} \delta_{ij} \hat{\delta}_{a,\lambda}(x) \right] + \frac{1}{\sqrt{|g(x)|}} \partial_n \left[ \sqrt{|g(x)|} \partial_n \hat{\delta}_{a,\lambda}(x) \right] 
\]
\[
= \partial_t \left[ g^{ij}(x) \frac{x^i}{r} \right] \partial_r \hat{\delta}_{a,\lambda}(x) + g^{ij}(x) \frac{x^i}{r} \partial_r \hat{\delta}_{a,\lambda}(x) + \frac{1}{\sqrt{|g(x)|}} \partial_t \left[ g^{ij}(x) \frac{x^i}{r} \right] \partial_r \hat{\delta}_{a,\lambda}(x) 
\]
\[
+ \frac{1}{\sqrt{|g(x)|}} \partial_n (\sqrt{|g(x)|}) \partial_n \hat{\delta}_{a,\lambda}(x) + \partial_n \hat{\delta}_{a,\lambda}(x) 
\] (46)

where
\[
A = \left( g^{ij}(x) \frac{x^i x^j}{r^2} - 1 \right) \partial_r \delta_{a,\lambda}(x) + \left( g^{ij}(x) \frac{x^i}{r} \partial_r (\sqrt{|g(x)|}) + \partial_t (g^{ij}(x) \frac{x^i}{r}) - \frac{n-1}{r} \right) \partial_r \delta_{a,\lambda}(x) 
\]
\[
+ \partial_n (\sqrt{|g(x)|}) \partial_r \delta_{a,\lambda}(x), 
\] (47)
and
\[
\Delta \delta_{a,\lambda}(x) = \partial_r \delta_{a,\lambda}(x) + \left( \frac{n-1}{r} \right) \partial_r \delta_{a,\lambda}(x) + \partial_n \delta_{a,\lambda}(x). 
\] (48)

In (46), $n = 3$,
\[
\partial_i = \frac{\partial}{\partial x_i}, \quad i = 1, \ldots, n, \text{ and } \partial_r = \frac{\partial}{\partial r}. 
\]

Since
\[
\Delta \delta_{a,\lambda}(x) = 0, \quad \text{in } B_{2\delta}^+(0) \cap \mathbb{R}^3_+, 
\]
then (46) implies
\[
\Delta_g \hat{\delta}_{a,\lambda}(x) = \left( g^{ij}(x) \frac{x^i x^j}{r^2} - 1 \right) \partial_r^2 \hat{\delta}_{a,\lambda}(x) + \left( g^{ij}(x) \frac{x^i}{r} \partial_r (\sqrt{|g(x)|}) + \partial_t (g^{ij}(x) \frac{x^i}{r}) - \frac{n-1}{r} \right) \partial_r \hat{\delta}_{a,\lambda}(x) 
\]
\[
+ \partial_n (\sqrt{|g(x)|}) \partial_r \hat{\delta}_{a,\lambda}(x), \quad x \in B_{2\delta}^+(0). 
\] (49)

Combining (21) and (49), we get
\[
|\Delta_g \hat{\delta}_{a,\lambda}(x)| \leq C \left[ |x|^2 \partial_r^2 \hat{\delta}_{a,\lambda}(x) + |x| (\partial_r \delta_{a,\lambda}(x) + \partial_n \delta_{a,\lambda}(x)) \right], \quad x \in B_{2\delta}^+(0). 
\] (50)
Using Lemma 7.1 in (50), we obtain the following estimate for $I_4$

$$|I_4| \leq C \left( \frac{\lambda}{1 + \lambda^2 \delta_g^2(a, x)} \right)^{1/2} 1_{\{y \in M: d_y(a, y) \leq 2\delta\}}(x), \quad x \in M. \quad (51)$$

Hence, the result for (52) follows from (50), (44), (15), and (51).

For the formula (53), we first use (54) to derive

$$-\frac{\partial u_{a, \lambda}(x)}{\partial n_g} - u_{a, \lambda}^3(x) = \left( -\frac{\partial}{\partial n_g} \right) \left[ \chi_{a, \lambda}^2(x) \delta_{a, \lambda}(x) + (1 - \chi_{a, \lambda}^2(x)) \frac{G_a(x)}{\sqrt{\lambda}} \right] - u_{a, \lambda}^3(x)$$

$$= -\frac{\partial}{\partial n_g} \left[ \chi_{a, \lambda}^2(x) \delta_{a, \lambda}(x) \right] - \frac{\partial}{\partial n_g} \left[ (1 - \chi_{a, \lambda}^2(x)) \frac{G_a(x)}{\sqrt{\lambda}} \right] - u_{a, \lambda}^3(x), \quad x \in \partial M.$$

To continue, we write

$$-\frac{\partial}{\partial n_g} \left[ (1 - \chi_{a, \lambda}^2(x)) \frac{G_a(x)}{\sqrt{\lambda}} \right] = -\frac{\partial}{\partial n_g} \frac{G_a(x)}{\sqrt{\lambda}} (1 - \chi_{a, \lambda}^2(x)) + \frac{\partial}{\partial n_g} \chi_{a, \lambda}^2(x) \frac{G_a(x)}{\sqrt{\lambda}}, \quad x \in \partial M.$$

Next, using the definition $\chi_{a, \lambda}$ (see (26)), the symmetry of $\chi_{a, \lambda}$ in $\psi_{a}(B_{2a}^+(0))$ after passing to Euclidean coordinates, and $\frac{\partial \delta_{a, \lambda}(x)}{\partial n_g} = 0$ for $x \in \partial M$, $x \neq a$, we have

$$-\frac{\partial}{\partial n_g} \left[ (1 - \chi_{a, \lambda}^2(x)) \frac{G_a(x)}{\sqrt{\lambda}} \right] = 0, \quad \text{for} \quad x \in \partial M, \quad \text{and} \quad x \neq a.$$

Hence for $x \in \partial M$ and $x \neq a$, we have

$$-\frac{\partial u_{a, \lambda}(x)}{\partial n_g} - u_{a, \lambda}^3(x) = \frac{\partial}{\partial n_g} \left[ \chi_{a, \lambda}^2(x) \delta_{a, \lambda}(x) \right] - u_{a, \lambda}^3(x).$$

Using again the definition of $\chi_{a, \lambda}$ and the symmetry of $\chi_{a, \lambda}$ in $\psi_{a}(B_{2a}^+(0))$ as before, we obtain

$$-\frac{\partial u_{a, \lambda}(x)}{\partial n_g} - u_{a, \lambda}^3(x) = -\chi_{a, \lambda}^2(x) \frac{\partial \delta_{a, \lambda}(x)}{\partial n_g} - u_{a, \lambda}^3(x), \quad x \in \partial M, \quad x \neq a. \quad (52)$$

Clearly, (52) is true for $x = a$. On other hand, since identifying $x \in B_{2a}^+(0)$ with $\psi_a(x)$, we have

$$-\frac{\partial \delta_{a, \lambda}(x)}{\partial n_g} = \frac{\partial \delta_{a, \lambda}(x)}{\partial x_3} = \delta_{a, \lambda}^3(x), \quad x \in \partial B_{2a}^+(0),$$

then

$$-\frac{\partial u_{a, \lambda}(x)}{\partial n_g} - u_{a, \lambda}^3(x) = 0, \quad \forall \quad x \in \partial B_{a}^+(0).$$

Therefore, (52) implies

$$\left| -\frac{\partial u_{a, \lambda}(x)}{\partial n_g} - u_{a, \lambda}^3(x) \right| \leq C \left( \frac{\lambda}{1 + \lambda^2 \delta_g^2(a, x)} \right)^{1/2} 1_{\{y \in \partial M: d_y(a, y) \leq 2\delta\}}(x), \quad x \in \partial M.$$

\section{PS-sequences and Deformation Lemma}

In this Section, we discuss the asymptotic behavior of Palais-Smale (PS)-sequences for $J_q$. We also introduce the neighborhoods of potential critical points at infinity of $J_q$ and their associated selection maps. As in other applications of the Barycenter technique of Bahri-Coron (see [1], [15], and [18]), we also recall the associated Deformation Lemma.

By some arguments which are classical by now see for example (3), [1], [12], and [29]), we have the following the profile decomposition for (PS)-sequences of $J_q$.

\textbf{Lemma 3.1.} Suppose that $(u_k) \subset H^1_0(M)$ is a (PS)-sequences for $J_q$, that is $\nabla J_q(u_k) \to 0$ and $J_q(u_k) \to c$ up to a subsequence, and $\int_{\partial M} u_k^4 \, dS_g = c^2$ for $k \in \mathbb{N}^*$, then up to a subsequence, we have there exists $u_\infty \geq 0$, an integer $p \geq 0$, a sequence of points $a_{i, k} \in \partial M$, $i = 1, \ldots, p$, and a
sequence of positive numbers \( \lambda_{i,k} \), \( i = 1, \cdots p \), such that
1) \[
\begin{cases}
-\Delta_g u_\infty + qu_\infty = 0 & \text{in } M, \\
\frac{\partial u_\infty}{\partial n_\infty} = u_\infty^3 & \text{on } \partial M.
\end{cases}
\]

2) \[
\|u_k - u_\infty - \sum_{i=1}^p u_{a_i,k} \|_q \to 0, \text{ as } k \to \infty.
\]

3) \[
J_q(u_k)^2 \to J_q(u_\infty)^2 + pS^2, \text{ as } k \to \infty.
\]

4) For \( i \neq j \), \[
\frac{\lambda_{i,k}}{\lambda_{j,k}} + \frac{\lambda_{j,k}}{\lambda_{i,k}} + \lambda_{i,k} \lambda_{j,k} G^{-2}(a_{i,k}, a_{j,k}) \to +\infty, \text{ as } k \to \infty,
\]
where \( G \) is as in \([13]\), and \( \| \|_q \) is as in \([11]\).

To introduce the neighborhoods of potential critical points at infinity of \( J_q \), we first fix \( \varepsilon_0 > 0 \) and \( \varepsilon_0 \simeq 0 \). (53)

Furthermore, we choose \( \nu_0 > 1 \) and \( \nu_0 \simeq 1 \). (54)

Then for \( p \in \mathbb{N}^* \), and \( 0 < \varepsilon \leq \varepsilon_0 \), we define \( V(p, \varepsilon) \) the \((p, \varepsilon)\)-neighborhood of potential critical points at infinity of \( J_q \) by
\[
V(p, \varepsilon) := \{ u \in H^1_+(M) : \exists a_1, \cdots, a_p \in \partial M, \ a_1, \cdots, a_p > 0, \ \lambda_1, \cdots, \lambda_p > 0, \ \lambda_i \geq \frac{1}{\varepsilon} \text{ for } i = 1, \cdots, p, \|
\]
\[
\|u - \sum_{i=1}^p \alpha_i u_{a_i, \lambda_i} \|_q \leq \varepsilon, \quad \frac{\alpha_i}{\alpha_j} \leq \nu_0 \text{ and } \varepsilon_{i,j} \leq \varepsilon \text{ for } i \neq j = 1, \cdots, p, \}
\]

Concerning the sets \( V(p, \varepsilon) \), for every \( p \in \mathbb{N}^* \) there exists \( 0 < \varepsilon_p \leq \varepsilon_0 \) such that for every \( 0 < \varepsilon \leq \varepsilon_p \), we have
\[
\forall u \in V(p, \varepsilon) \text{ the minimization problem } \min_{B^p} \|u - \sum_{i=1}^p \alpha_i u_{a_i, \lambda_i} \|_q
\]
has a solution \((\bar{\alpha}, A, \bar{\lambda}) \in B^p \), which is unique up to permutations, (55)

where \( B^p \) is defined as
\[
B^p := \{ (\bar{\alpha} = (\alpha_1, \cdots, \alpha_p), \ A = (a_1, \cdots, a_p), \ \bar{\lambda} = \lambda_1, \cdots, \lambda_p) \in \mathbb{R}^p \times (\partial M)^p \times (0, +\infty)^p \}
\]
\[
\frac{\alpha_i}{\alpha_j} \leq \nu_0 \text{ and } \varepsilon_{i,j} \leq \varepsilon, \ i \neq j = 1, \cdots, p, \}
\]

We define the selection map \( s_p \) via
\[
s_p : V(p, \varepsilon) \to (\partial M)^p / \sigma_p : u \to s_p(u) = A \text{ and } A \text{ is given by (55)}.
\]

To state the Deformation Lemma needed for the application of the algebraic topological argument of Bahri-Coron\([6]\) for existence, we first set
\[
W_p := \{ u \in H^1_+(M) : J_q(u) \leq (p + 1)\frac{\varepsilon}{2} S \},
\]
for \( p \in \mathbb{N} \), where \( S \) is as in \([14]\).

As in \([1], [17], \text{ and } [19]\), we have Lemma 3.1 implies the following Deformation Lemma.

**Lemma 3.2.** Assuming that \( J_q \) has no critical points, then for every \( p \in \mathbb{N}^* \), up to taking \( \varepsilon_p \) given by (55) smaller, we have that for every \( 0 < \varepsilon \leq \varepsilon_p \), the topological pair \((W_p, W_{p-1})\) retracts by deformation onto \((W_{p-1} \cup A_p, W_{p-1})\) with \( V(p, \varepsilon) \subset A_p \subset V(p, \varepsilon) \) where \( 0 < \varepsilon < \frac{\varepsilon}{p+1} \) is a very small positive real number and depends on \( \varepsilon \).
4 Self-action estimates

In this Section, we establish some sharp self-action estimates that are essential for the use of the Bahri-
Coron’s Barycenter technique for existence. We start with the numerator of the functional $J_q$.

**Lemma 4.1.** Assuming that $\theta > 0$ is small, then there exists $C > 0$ such that $\forall \, a \in \partial M$, $\forall \, 0 < 2\delta < \delta_0$ and $\forall \, 0 < \frac{1}{\lambda} < \theta\delta$, we have

$$
\int_M \left( |\nabla u_{a,\lambda}|^2 + qu_{a,\lambda}^2 \right) dV_g \leq \int_{\partial M} u_{a,\lambda}^4 dS_g + \frac{C}{\lambda} \left[ 1 + \delta + \frac{1}{\lambda^2} \right],
$$

where $\delta_0$ is as in (19).

**Proof.** Setting $I = \int_M \left( |\nabla u_{a,\lambda}|^2 + qu_{a,\lambda}^2 \right) dV_g$

we have by Green’s first identity

$$
I = \int_{\partial M} u_{a,\lambda}^4 dS_g + \int_M \left( \Delta_g u_{a,\lambda} + qu_{a,\lambda} \right) u_{a,\lambda} dV_g + \int_{\partial M} \left[ \frac{\partial u_{a,\lambda}}{\partial n} - u_{a,\lambda} \right] u_{a,\lambda} dS_g.
$$

For the second term, we get

$$
\int_M \left( \Delta_g u_{a,\lambda} + qu_{a,\lambda} \right) u_{a,\lambda} dV_g 
\leq \frac{C}{\delta^2 \sqrt{\lambda}} \int_M u_{a,\lambda} \mathbf{1}_{\{x \in M: \delta \leq d_g(a,x) \leq 2\delta\}} dV_g
$$

and

$$
\int_{\partial M} \left[ \frac{\partial u_{a,\lambda}}{\partial n} - u_{a,\lambda} \right] u_{a,\lambda} dS_g 
\leq \frac{C}{\sqrt{\lambda}} \int_{B_d^+(0)} \frac{1}{|x|} dx
$$

For the first term on the right-hand side of the formula above, we observe

$$
\int_M u_{a,\lambda} \mathbf{1}_{\{x \in M: \delta \leq d_g(a,x) \leq 2\delta\}} dV_g 
\leq \frac{C}{\lambda} \int_{\partial M} \frac{1}{\sqrt{\lambda d_g(a,x)}} dV_g
$$

and

For the second term, we get

$$
\frac{C}{\lambda} \int_{B_d^+(0)} \frac{1}{|x|^2} dx
$$

For the first term on the right-hand side of the formula above, we observe

$$
\int_M u_{a,\lambda} \mathbf{1}_{\{x \in M: \delta \leq d_g(a,x) \leq 2\delta\}} dV_g 
\leq \frac{C}{\lambda} \int_{\partial M} \frac{1}{\lambda d_g^2(a,x)} dV_g
$$

and

$$
\int_{\partial M} \frac{1}{|x|^2} dx
$$

For the second term, we get

$$
\frac{C}{\lambda} \int_{B_d^+(0)} \frac{1}{|x|^2} dx
$$

Finally, we have

$$
\int_{\partial M} \frac{1}{|x|^2} dx
$$

and

$$
\frac{C}{\lambda} \int_{B_d^+(0)} \frac{1}{|x|^2} dx
$$
Now, for the final term, we have

\[
\int_M \left( \frac{\lambda}{1 + \lambda^2 d_g^2(a, x)} \right)^2 u_{a, \lambda} \mathbb{1}_{\{x \in M: d_g(a, x) \leq 2\delta\}} \, dV_g \leq C \int_{\{x \in M: d_g(a, x) \leq 2\delta\}} \left( \frac{\lambda}{1 + \lambda^2 d_g^2(a, x)} \right) \, dV_g
\]

\[
\leq C \int_{\{x \in M: d_g(a, x) \leq 2\delta\}} \frac{1}{\lambda^2 d_g^2(a, x)} \, dV_g
\]

\[
\leq C \lambda \int_{B_{2\delta}(0)} \frac{1}{|x|^2} \, dx
\]

\[
\leq C \lambda \int_0^{2\delta} 1 \, dr
\]

\[
\leq C \frac{\delta}{\lambda}.
\]

Thus, combining (57)-(59), we get

\[
|I_1| \leq C \lambda (1 + \delta).
\]

Next, in the case of $I_2$, we have

\[
|I_2| \leq \oint_{\partial M} \left[ -\frac{\partial u_{a, \lambda}}{\partial n} - u_{a, \lambda} \right] u_{a, \lambda} \, dS_g
\]

\[
\leq C \oint_{\partial M} \left( \frac{\lambda}{1 + \lambda^2 d_g^2(a, x)} \right)^2 u_{a, \lambda} \mathbb{1}_{\{x \in \partial M: d_g(a, x) \geq \delta\}} \, dS_g.
\]

On the right-hand side of the above formula, we observe

\[
\oint_{\partial M} \left( \frac{\lambda}{1 + \lambda^2 d_g^2(a, x)} \right)^2 u_{a, \lambda} \mathbb{1}_{\{x \in \partial M: d_g(a, x) \geq \delta\}} \, dS_g \leq C \oint_{\{x \in \partial M: d_g(a, x) \geq \delta\}} \left( \frac{\lambda}{1 + \lambda^2 d_g^2(a, x)} \right)^2 \, dS_g
\]

\[
\leq C \oint_{\{x \in \partial M: d_g(a, x) \geq \delta\}} \frac{1}{\lambda^2 d_g^2(a, x)} \, dS_g
\]

\[
\leq C \lambda^2 \left[ \int_{B_{\delta}(0) \setminus B_{\delta}(0)} \frac{1}{|x|^2} \, dx + 1 \right]
\]

\[
\leq C \lambda^2 \left[ \int_{\mathbb{R}^d \setminus B_{\delta}(0)} r^{-3} \, dr + 1 \right]
\]

\[
\leq C \lambda^2 \left[ \int_{\delta}^{+\infty} r^{-3} \, dr + 1 \right]
\]

\[
\leq C \lambda^2 \frac{1}{\lambda^2 \delta^2}.
\]

Thus, we have

\[
|I_2| \leq C \lambda \left( \frac{1}{\lambda^2 \delta^2} \right).
\]

As a result, by combining (60) and (62), we obtain

\[
\int_M \left( |\nabla u_{a, \lambda}|^2 + qu_{a, \lambda}^2 \right) \, dV_g \leq \oint_{\partial M} u_{a, \lambda}^4 \, dS_g + C \lambda \left[ 1 + \frac{1}{\lambda^2 \delta^2} \right].
\]

For the denominator of $J_q$, we have.

**Lemma 4.2.** Assuming that $\theta > 0$ is small, then there exists $C > 0$ such that $\forall \, a \in \partial M$, $\forall \, 0 < 2\delta < \delta_0$ and $\forall \, 0 < \frac{1}{\lambda} \leq \theta \delta$, we have

\[
\oint_{\partial M} u_{a, \lambda}^4 \, dS_g = \int_{\mathbb{R}^d} \delta_{0, \lambda}^4 \, dx + O \left( \frac{1}{\lambda^2 \delta^2} \right),
\]

where $\delta_0$ is as in (19).

**Proof.** We have

\[
\oint_{\partial M} u_{a, \lambda}^4 \, dS_g = \oint_{\{x \in \partial M: d_g(a, x) \leq \delta\}} u_{a, \lambda}^4 \, dS_g + \oint_{\{x \in \partial M: \delta < d_g(a, x) \leq 2\delta\}} u_{a, \lambda}^4 \, dS_g
\]

\[
+ \oint_{\{x \in \partial M: d_g(a, x) > 2\delta\}} u_{a, \lambda}^4 \, dS_g.
\]

\[
(63)
\]
We are going to estimate each terms of the right-hand side of the formula of (63). To begin, we start with the first term, we have

\[
\oint_{\{x \in \partial M: d_q(a, x) \leq \delta\}} u_{a, \lambda}^4 \, dS_g = \oint_{\{x \in \partial M: d_q(a, x) \leq \delta\}} \delta_{a, \lambda}^4 \, dS_g
= \oint_{B_\delta(0)} \delta_{0, \lambda}^4 \, dx
= \oint_{\mathbb{R}^2} \delta_{0, \lambda}^4 \, dx - \int_{\{x \in \mathbb{R}^2: |x| \geq \delta\}} \delta_{0, \lambda}^4 \, dx
= \oint_{\mathbb{R}^2} \delta_{0, \lambda}^4 \, dx + O \left( \frac{1}{\lambda^2 \delta^2} \right).
\]  

(64)

To get to the second term, we have

\[
\oint_{\{x \in \partial M: \delta < d_q(a, x) \leq 2\delta\}} u_{a, \lambda}^4 \, dS_g \leq C \oint_{\{x \in \partial M: \delta < d_q(a, x) \leq 2\delta\}} \left( \frac{\lambda}{1 + \lambda^2 d_q(a, x)^2} \right)^2 \, dS_g
\leq C \oint_{\{x \in \partial M: \delta < d_q(a, x) \leq 2\delta\}} \frac{1}{\lambda^2 d_q^2(a, x)} \, dS_g
\leq C \frac{1}{\lambda^2} \oint_{B_\delta(0) \setminus B_{\delta}(0)} \frac{1}{|x|^4} \, dx
\leq C \frac{1}{\lambda^2} \int_{\delta}^{2\delta} r^{-3} \, dr
\leq C \frac{1}{\lambda^2 \delta^2}.
\]

(65)

Next, using (17), we estimate the final term as follows

\[
\oint_{\{x \in \partial M: d_q(a, x) > 2\delta\}} u_{a, \lambda}^4 \, dS_g = \oint_{\{x \in \partial M: d_q(a, x) > 2\delta\}} \left( \frac{c_0}{\sqrt{\lambda}} G_a(x) \right)^4 \, dS_g
= \frac{C}{\lambda^2} \oint_{\{x \in \partial M: d_q(a, x) > 2\delta\}} G_a^4(x) \, dS_g
\leq C \oint_{\{x \in \partial M: d_q(a, x) > 2\delta\}} \frac{1}{\lambda^2 d_q^2(a, x)} \, dS_g
\leq C \oint_{\{x \in \partial M: 2\delta < d_q(a, x) \leq \delta_0\}} \frac{1}{\lambda^2 d_q^2(a, x)} \, dS_g
+ C \oint_{\{x \in \partial M: d_q(a, x) > 2\delta_0\}} \frac{1}{\lambda^2 d_q^2(a, x)} \, dS_g
\leq C \frac{1}{\lambda^2} \oint_{\{x \in \partial M: 2\delta < d_q(a, x) \leq \delta_0\}} \frac{1}{d_q^2(a, x)} \, dS_g + C
\leq C \frac{1}{\lambda^2} \int_{B_{\delta}(0) \setminus B_{2\delta}(0)} \frac{1}{|x|^4} \, dx + C
\leq C \frac{1}{\lambda^2} \int_{2\delta}^{\infty} r^{-3} \, dr + C
\leq C \frac{1}{\lambda^2 \delta^2}.
\]

(66)

Finally, combining, (63)-(66), we have

\[
\oint_{\partial M} u_{a, \lambda}^4 \, dS_g = \oint_{\mathbb{R}^2} \delta_{0, \lambda}^4 \, dx + O \left( \frac{1}{\lambda^2 \delta^2} \right).
\]

We establish now the \( J_q \)-energy estimate of \( u_{a, \lambda} \) needed for the application of the Barycenter technique of Bahri-Coron[6] for existence.

**Corollary 4.3.** Assuming that \( \theta > 0 \) is small, then there exists \( C > 0 \) such that \( \forall \ a \in \partial M, \ \forall \ 0 < 2\delta < \delta_0 \) and \( \forall \ 0 < \frac{\lambda}{\theta} \leq \theta \delta \), we have

\[
J_q(u_{a, \lambda}) \leq C \left[ 1 + C \left( \frac{1}{\lambda} + \frac{\delta}{\lambda} + \frac{1}{\delta^2 \lambda^2} \right) \right],
\]
where $\delta_0$ is as in (19).

**Proof.** It follows from the properties of $\delta_{0,\lambda}$ (see (13), Lemma 4.1, and Lemma 4.2).

## 5 Interaction estimates

Throughout this Section, we derive some sharp inter-action estimates required for the algebraic topological argument of Bahri-Coron[6] for existence. We start with the following technical inter-action estimates. We anticipate that it will provide the needed inter-action estimates between $\epsilon_{ij}$ and $\epsilon_{ji}$, see (30) and (31) for the definitions of $\epsilon_{ij}$ and $\epsilon_{ji}$.

**Lemma 5.1.** Assuming that $\theta > 0$ is small, then there exists $C > 0$ such that $\forall \ a_i, a_j \in \partial M$,

$\forall 0 < 2\delta < \delta_0$, and $\forall 0 < \frac{1}{\lambda_i} \frac{1}{\lambda_j} \leq \theta$, we have

$$
\int_M \left[ (\Delta_g + q) u_{a_i, a_j} \right] \leq C \left[ \frac{1}{\delta^2} \sqrt{\lambda_j} \mathbb{1}_{\{y \in M : \delta \leq d_g(a_j, y) \leq 2\delta\}}(x) + \hat{\delta}_{a_i, a_j}(x) \mathbb{1}_{\{y \in M : d_g(a_j, y) \leq 2\delta\}}(x) \right. \right.
+ \left. \left. \left( \frac{\lambda_i}{1 + \lambda_j^2 d_g^2(a_j, x)} \right)^{\frac{3}{2}} \mathbb{1}_{\{y \in M : d_g(a_j, y) \leq 2\delta\}}(x) \right], \ x \in M,
$$

(67)

and

$$
\left| \frac{\partial u_{a_i, a_j}(x)}{\partial n_g} - u_{a_i, a_j}^3(x) \right| \leq C \left[ \left( \frac{\lambda_i}{1 + \lambda_j^2 d_g^2(a_j, x)} \right)^{\frac{3}{2}} 1_{\{y \in \partial M : d_g(a_j, y) \geq \delta\}}(x) \right], \ x \in \partial M.
$$

(68)

For $x \in \{y \in M : d_g(a_j, y) \leq 2\delta\}$, we have

$$
\left( \frac{\lambda_i}{1 + \lambda_j^2 d_g^2(a_j, x)} \right) \geq \left( \frac{\lambda_j}{1 + 4\lambda_j^2 \delta^2} \right) \geq \frac{1}{\lambda_j \delta^2} \left[ 1 + O\left( \frac{1}{\lambda_j^2 \delta^2} \right) \right] \geq \frac{1}{2\lambda_j \delta^2}.
$$

This implies

$$
\frac{1}{\sqrt{\lambda_j \delta}} \leq C \left( \frac{\lambda_i}{1 + \lambda_j^2 d_g^2(a_j, x)} \right)^{\frac{3}{2}}.
$$

Thus, using (67) and Lemma 7.1, we get

$$
A_j \leq C \left( 1 + \frac{1}{\delta} \right) \left( \frac{\lambda_j}{1 + \lambda_j^2 d_g^2(a_j, x)} \right)^{\frac{3}{2}} \mathbb{1}_{\{y \in M : d_g(a_j, y) \leq 4\delta\}}(x), \ x \in M.
$$

(69)

For $B_j$ given by (68), we have

$$
B_j \leq C \left( \frac{\lambda_j}{1 + \lambda_j^2 d_g^2(a_j, x)} \right)^{\frac{3}{2}} 1_{\{y \in \partial M : d_g(a_j, y) \geq \delta\}}(x), \ x \in \partial M.
$$

(70)

Now, using (69), we obtain

$$
\int_M A_j u_{a_i, a_j} \ dV_g \leq C \left( 1 + \frac{1}{\delta} \right) \int_{\{x \in M : d_g(a_i, x) \leq 4\delta\}} \left( \frac{\lambda_j}{1 + \lambda_j^2 d_g^2(a_j, x)} \right)^{\frac{3}{2}} \left( \frac{\lambda_j}{1 + \lambda_j^2 d_g^2(a_i, x)} \right)^{\frac{3}{2}} dV_g.
$$

(71)
Also, using (70), we have

\[
\oint_{\partial M} B_j \ u_{a_i, \lambda_i} \ dS_g \leq C \oint_{\{x \in \partial M : d_g(a_j, x) \geq \delta\}} \left( \frac{\lambda_j}{1 + \lambda_j^2 d_g^2(a_j, x)} \right)^{\frac{1}{2}} \left( \frac{\lambda_i}{1 + \lambda_i^2 d_g^2(a_i, x)} \right)^{\frac{1}{2}} \ dS_g. \tag{72}
\]

In the next step, we are going to estimate \( I_1 \) as follows

\[
I_1 = \int_{\{x \in M : d_g(a_j, x) \leq 4\delta\}} \left( \frac{\lambda_j}{1 + \lambda_j^2 d_g^2(a_j, x)} \right)^{\frac{1}{2}} \left( \frac{\lambda_i}{1 + \lambda_i^2 d_g^2(a_i, x)} \right)^{\frac{1}{2}} \ dV_g
\]

\[
= \int_{\{x \in M : 2d_g(a_j, x) \leq \frac{\delta}{2} + d_g(a_i, x)\}} \left( \frac{\lambda_j}{1 + \lambda_j^2 d_g^2(a_j, x)} \right)^{\frac{1}{2}} \left( \frac{\lambda_i}{1 + \lambda_i^2 d_g^2(a_i, x)} \right)^{\frac{1}{2}} \ dV_g
\]

\[
+ \int_{\{x \in M : 2d_g(a_j, x) \geq \frac{\delta}{2} + d_g(a_i, x)\}} \left( \frac{\lambda_j}{1 + \lambda_j^2 d_g^2(a_j, x)} \right)^{\frac{1}{2}} \left( \frac{\lambda_i}{1 + \lambda_i^2 d_g^2(a_i, x)} \right)^{\frac{1}{2}} \ dV_g.
\]

Using the triangle inequality, we estimate \( I_1^1 \) as follows

\[
I_1^1 \leq C \sqrt{\frac{\lambda_j}{\lambda_i}} \int_{\{x \in M : d_g(a_j, x) \leq 8\delta\}} \left( \frac{\lambda_j}{1 + \lambda_j^2 d_g^2(a_j, x)} \right)^{\frac{1}{2}} \left( \frac{\lambda_i}{1 + \lambda_i^2 d_g^2(a_i, x)} \right)^{\frac{1}{2}} \ dV_g
\]

\[
\leq C \sqrt{\frac{\lambda_j}{\lambda_i}} \int_{\{x \in M : d_g(a_j, x) \leq 8\delta\}} \frac{1}{d_g(a_j, x)} \ dV_g
\]

\[
\leq C \sqrt{\frac{\lambda_j}{\lambda_i}} \int_{B_{8\delta}^{a_j}(0)} \frac{1}{|x|} \ dx.
\]

So, for \( I_1^1 \), we have

\[
I_1^1 = O \left( \delta^2 \left( \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j d_g^2(a_i, a_j) \right)^{-\frac{1}{2}} \right). \tag{73}
\]

For \( I_1^2 \), we observe

\[
I_1^2 \leq C \sqrt{\frac{\lambda_j}{\lambda_i}} \int_{\{x \in M : d_g(a_j, x) \geq 4\delta\}} \left( \frac{\lambda_j}{1 + \lambda_j^2 d_g^2(a_j, x)} \right)^{\frac{1}{2}} \left( \frac{\lambda_i}{1 + \lambda_i^2 d_g^2(a_i, x)} \right)^{\frac{1}{2}} \ dV_g
\]

\[
\leq C \sqrt{\frac{\lambda_j}{\lambda_i}} \int_{\{x \in M : d_g(a_j, x) \leq 4\delta\}} \frac{1}{d_g(a_j, x)} \ dV_g
\]

\[
\leq C \sqrt{\frac{\lambda_j}{\lambda_i}} \int_{B_{4\delta}^{a_j}(0)} \frac{1}{|x|} \ dx.
\]

Thus, we get for \( I_1^2 \)

\[
I_1^2 = O \left( \delta^2 \left( \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j d_g^2(a_i, a_j) \right)^{-\frac{1}{2}} \right). \tag{74}
\]

Hence, using (73) and (74), we obtain

\[
I_1 = O \left( \delta^2 \left( \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j d_g^2(a_i, a_j) \right)^{-\frac{1}{2}} \right). \tag{75}
\]
For the last step, we are going to estimate $I_2$. We first write the following for $I_2$

\[
I_2 = \oint_{\{x \in \partial M: d_g(a_j, x) \geq \delta\}} \left( \frac{\lambda_j}{1 + \lambda_j^2 d_g^2(a_j, x)} \right)^{\frac{1}{2}} \left( \frac{\lambda_i}{1 + \lambda_i^2 d_g^2(a_i, x)} \right)^{\frac{1}{2}} dS_g
\]

\[
I_2 = \oint_{\{x \in \partial M: d_g(a_i, x) \leq \frac{1}{\lambda_j} + d_g(a_j, a_i)\} \cap \{x \in \partial M: d_g(a_j, x) \geq \delta\}} \left( \frac{\lambda_j}{1 + \lambda_j^2 d_g^2(a_j, x)} \right)^{\frac{1}{2}} \left( \frac{\lambda_i}{1 + \lambda_i^2 d_g^2(a_i, x)} \right)^{\frac{1}{2}} dS_g
\]

\[
I_2 = \oint_{\{x \in \partial M: d_g(a_i, x) > \frac{1}{\lambda_j} + d_g(a_j, a_i)\} \cap \{x \in \partial M: d_g(a_j, x) \geq \delta\}} \left( \frac{\lambda_j}{1 + \lambda_j^2 d_g^2(a_j, x)} \right)^{\frac{1}{2}} \left( \frac{\lambda_i}{1 + \lambda_i^2 d_g^2(a_i, x)} \right)^{\frac{1}{2}} dS_g.
\]

In order to estimate $I_2^1$, we set

\[
A = \{ x \in \partial M : 2d_g(a_i, x) \leq \frac{1}{\lambda_j} + d_g(a_j, a_i) \} \cap \{ x \in \partial M : d_g(a_j, x) \geq \delta \},
\]

\[
r_{ij} = \frac{1}{2} \left( \frac{1}{\lambda_j} + d_g(a_j, a_i) \right),
\]

and observe

\[
I_2^1 \leq C \frac{1}{\lambda_j} \oint_A \left( \frac{1}{1 + \lambda_j^2 d_g^2(a_j, a_i)} \right)^{\frac{1}{2}} \left( \frac{\lambda_i}{1 + \lambda_i^2 d_g^2(a_i, x)} \right)^{\frac{1}{2}} \left( \frac{1}{1 + \lambda_i^2 d_g^2(a_i, x)} \right)^{\frac{1}{2}} dS_g
\]

\[
\leq C \left( \frac{1}{\sqrt{\lambda_i}} \right) \left( \frac{1}{\lambda_j^2 \delta^2} \right) \left( \frac{\lambda_i}{1 + \lambda_i^2 d_g^2(a_i, x)} \right)^{\frac{1}{2}} \oint_{\{x \in \partial M: 2d_g(a_i, x) \leq r_{ij}\}} \frac{1}{d_g(a_i, x)} dS_g
\]

\[
\leq C \left( \frac{1}{\sqrt{\lambda_i}} \right) \left( \frac{1}{\lambda_j^2 \delta^2} \right) \left( \frac{\lambda_i}{1 + \lambda_i^2 d_g^2(a_i, x)} \right)^{\frac{1}{2}} \int_{B_{r_{ij}}(0)} \frac{1}{|x|} dx
\]

\[
\leq C \left( \frac{1}{\sqrt{\lambda_i}} \right) \left( \frac{1}{\lambda_j^2 \delta^2} \right) \left( \frac{1}{1 + \lambda_i^2 d_g^2(a_i, x)} \right)^{\frac{1}{2}} \left( \frac{1}{\lambda_j} + d_g(a_j, a_i) \right)
\]

\[
\leq C \left( \frac{1}{\sqrt{\lambda_i}} \right) \left( \frac{1}{\lambda_j^2 \delta^2} \right) \left( \frac{1}{1 + \lambda_j^2 d_g^2(a_j, a_i)} \right)^{\frac{1}{2}}.
\]

This implies

\[
I_2 = O \left( \frac{1}{\lambda_j \delta^2} \left( \frac{\lambda_i + \lambda_j d_g^2(a_i, a_j)}{\lambda_j} \right)^{\frac{1}{2}} \right).
\]

For $I_2^2$, we derive

\[
I_2^2 \leq C \left( \frac{\lambda_j}{\sqrt{\lambda_i}} \right) \oint_{\{x \in \partial M: d_g(a_j, x) \geq \delta\}} \left( \frac{1}{1 + \lambda_j^2 d_g^2(a_j, a_i)} \right)^{\frac{1}{2}} \left( \frac{\lambda_j}{1 + \lambda_i^2 d_g^2(a_i, x)} \right)^{\frac{1}{2}} dS_g
\]

\[
\leq C \left( \frac{\lambda_j}{\sqrt{\lambda_i}} \right) \left( \frac{1}{1 + \lambda_j^2 d_g^2(a_j, a_i)} \right)^{\frac{1}{2}} \frac{1}{\lambda_j^2} \oint_{\{x \in \partial M: d_g(a_j, x) \geq \delta\}} \frac{1}{d_g^2(a_j, x)} dS_g
\]

\[
\leq C \left( \frac{\lambda_j}{\sqrt{\lambda_i}} \right) \left( \frac{1}{1 + \lambda_j^2 d_g^2(a_j, a_i)} \right)^{\frac{1}{2}} \frac{1}{\lambda_j} \oint_{\{x \in \partial M: d_g(a_j, x) \geq \delta\}} \frac{1}{d_g^2(a_j, x)} dS_g
\]

\[
+ C \left( \frac{\lambda_j}{\sqrt{\lambda_i}} \right) \left( \frac{1}{1 + \lambda_j^2 d_g^2(a_j, a_i)} \right)^{\frac{1}{2}} \frac{1}{\lambda_j} \oint_{\{x \in \partial M: d_g(a_j, x) \leq \delta\}} \frac{1}{d_g^2(a_j, x)} dS_g
\]

\[
\leq C \left( \frac{\lambda_j}{\sqrt{\lambda_i}} \right) \left( \frac{1}{1 + \lambda_j^2 d_g^2(a_j, a_i)} \right)^{\frac{1}{2}} \frac{1}{\lambda_j} \left[ \int_{B_{\delta_0}(0) \setminus B_\delta(0)} \frac{1}{|x|^\frac{1}{2}} dx + C \right]
\]

\[
\leq C \left( \frac{\lambda_j}{\sqrt{\lambda_i}} \right) \left( \frac{1}{1 + \lambda_j^2 d_g^2(a_j, a_i)} \right)^{\frac{1}{2}}.
\]
Thus, we get for $I_2$
\[ I_2 = O \left( \frac{1}{\lambda_j \delta^2} \left( \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j d_g^2(a_i, a_j) \right)^{-\frac{1}{2}} \right). \quad (77) \]

Hence, combining (70) and (77), we have
\[ I_2 = O \left( \frac{1}{\lambda_j \delta^2} \left( \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j d_g^2(a_i, a_j) \right)^{-\frac{1}{2}} \right). \quad (78) \]

Therefore, using (69), (70), (75), and (78), we obtain
\[ \int_M \left( -\Delta_g + q \right) u_{a_j, \lambda_j} u_{a_i, \lambda_i} dV_g + \oint_{\partial M} \left( -\partial u_{a_j, \lambda_j} \delta_{a_j} - u_{a_j, \lambda_j} \right) u_{a_i, \lambda_i} dS_g \]
\[ \leq C \left[ \delta + \frac{1}{\lambda_j \delta^2} \right] \left( \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j d_g^2(a_i, a_j) \right)^{-\frac{1}{2}}. \]

Hence, the proof of Lemma 5.1 is complete. \[ \blacksquare \]

Clearly Lemma 5.1 implies the following sharp interaction-estimate relating $\epsilon_{ij}$, $\epsilon_{ij}$, and $\epsilon_{ij}$ (for their definitions, see (20)-(31)).

**Corollary 5.2.** Assuming that $\theta > 0$ is small and $\mu_0 > 0$ is small, then $\forall a_i, a_j \in \partial M$, $\forall 0 < 2\delta < \delta_0$, and $\forall 0 < \frac{1}{\lambda_i} \leq \frac{1}{\lambda_j} \leq \theta \delta$ such that $\epsilon_{ij} \leq \mu_0$, we have
\[ \epsilon_{ij} = \epsilon_{ij} + O \left( \delta + \frac{1}{\lambda_i \delta^2} \right) \epsilon_{ij}, \]
where $\delta_0$ is as in (19).

The following lemma gives a refined inter-action estimate relating $\epsilon_{ij}$ and $\epsilon_{ij}$.

**Lemma 5.3.** Assuming that $\theta > 0$ is small and $\mu_0 > 0$ is small, then $\forall a_i, a_j \in \partial M$, $\forall 0 < 2\delta < \delta_0$, and $\forall 0 < \frac{1}{\lambda_i} \leq \frac{1}{\lambda_j} \leq \theta \delta$ such that $\epsilon_{ij} \leq \mu_0$, we have
\[ \epsilon_{ij} = c_0 \epsilon_{ij} \left[ \left( 1 + O \left( \delta + \frac{1}{\lambda_i \delta^2} \right) \right) \left( 1 + o_{\delta_{ij}}(1) + O(\epsilon_{ij} \delta^{-1}) \right) + O \left( \epsilon_{ij} \frac{1}{\delta^3} \right) \right], \]
where $c_0$ is as in (20) and $c_1$ is as in (15).

**Proof.** By using (21), we have
\[ u_{a_i, \lambda_i}(x) = \chi_{\delta}^a(x) \hat{\delta}_{a_i, \lambda_i}(x) + (1 - \chi_{\delta}^a(x)) \frac{c_0}{\sqrt{\lambda}} G_{a_i}(x), \quad x \in \partial M, \]
with $G_{a_i}(x) = G(a_i, x)$. On the other hand, by using the definition of $\hat{\delta}_{a, \lambda}$ (see (25)), we have
\[ \chi_{\delta}^a(x) \hat{\delta}_{a_i, \lambda_i}(x) = c_0 \chi_{\delta}^a(x) \left[ \frac{\lambda_i}{1 + \lambda_i^2 G_{a_i}^{-2}(x) \frac{d_g^2(x, a_i)}{G_{a_i}(x)}} \right]^{\frac{1}{2}}, \quad x \in \partial M. \]

For $x \in \{ y \in \partial M : d_g(a_i, y) \leq 2\delta \}$, the quantity $\left[ 1 + \lambda_i^2 G_{a_i}^{-2}(x) \frac{d_g^2(x, a_i)}{G_{a_i}(x)} \right]$ can be estimated by
\[ 1 + \lambda_i^2 G_{a_i}^{-2}(x) \frac{d_g^2(x, a_i)}{G_{a_i}^{-2}(x)} = 1 + \lambda_i^2 G_{a_i}^{-2}(x) (1 + O(\delta)) \]
\[ = 1 + \lambda_i^2 G_{a_i}^{-2}(x) + O \left( \lambda_i^2 \delta G_{a_i}^{-2}(x) \right) \]
\[ = (1 + \lambda_i^2 G_{a_i}^{-2}(x)) \left[ 1 + O \left( \frac{\lambda_i^2 \delta G_{a_i}^{-2}(x)}{1 + \lambda_i^2 G_{a_i}^{-2}(x)} \right) \right] \]
\[ = (1 + \lambda_i^2 G_{a_i}^{-2}(x)) [1 + O(\delta)]. \]
Hence, we have
\[
\chi_0^a(x)\delta_{a_1,\lambda_i}(x) = c_0\chi_0^a(x) \left[ \frac{\lambda_i}{(1 + \lambda_i^2 G_{a_i}^2(x)) [1 + O(\delta)]} \right]^{\frac{1}{2}} = c_0\chi_0^a(x) [1 + O(\delta)] \left[ \frac{\lambda_i}{1 + \lambda_i^2 G_{a_i}^2(x)} \right]^{\frac{1}{2}}, \quad x \in \partial M. \tag{79}
\]

Furthermore, we have
\[
c_0(1 - \chi_0^a(x)) \left[ \frac{\lambda_i}{1 + \lambda_i^2 G_{a_i}^2(x)} \right]^{\frac{1}{2}} = (1 - \chi_0^a(x)) \frac{c_0}{\sqrt{\lambda_i}} G_{a_i}(x) \left[ \frac{1}{1 + \lambda_i^2 G_{a_i}^2(x)} \right]^{\frac{1}{2}}, \quad x \in \partial M.
\]

Since on \( \{ x \in \partial M : d_g(x, a_i) \geq \delta \} \), we have
\[
\frac{1}{1 + \lambda_i^2 G_{a_i}^2(x)} = 1 + O \left( \frac{G_{a_i}^2(x)}{\lambda_i^2} \right) = 1 + O \left( \frac{1}{\lambda_i^2 \delta^2} \right),
\]
then we get
\[
c_0(1 - \chi_0^a(x)) \left[ \frac{\lambda_i}{1 + \lambda_i^2 G_{a_i}^2(x)} \right]^{\frac{1}{2}} = (1 - \chi_0^a(x)) \frac{c_0}{\sqrt{\lambda_i}} G_{a_i}(x) \left( 1 + O \left( \frac{1}{\lambda_i^2 \delta^2} \right) \right), \quad x \in \partial M.
\]

This implies
\[
(1 - \chi_0^a(x)) \frac{c_0}{\sqrt{\lambda_i}} G_{a_i}(x) = c_0 \left( 1 + O(\delta) \right) \chi_0^a(x) \left[ \frac{\lambda_i}{1 + \lambda_i^2 G_{a_i}^2(x)} \right]^{\frac{1}{2}} \left( 1 + O \left( \frac{1}{\lambda_i^2 \delta^2} \right) \right), \quad x \in \partial M. \tag{80}
\]

Thus, combining (79) and (80), we get
\[
u_{a_1,\lambda_i}(x) = c_0 \left( 1 + O(\delta) \right) \chi_0^a(x) \left[ \frac{\lambda_i}{1 + \lambda_i^2 G_{a_i}^2(x)} \right]^{\frac{1}{2}} \left( 1 + O \left( \frac{1}{\lambda_i^2 \delta^2} \right) \right), \quad x \in \partial M.
\]

Hence, we obtain
\[
u_{a_1,\lambda_i}(x) = c_0 \left( 1 + O(\delta) \right) \left[ \frac{\lambda_i}{1 + \lambda_i^2 G_{a_i}^2(x)} \right]^{\frac{1}{2}}, \quad x \in \partial M. \tag{81}
\]

Now, we are going to complete our task by using (81). We begin by writing the estimate in the following form:
\[
\epsilon_j = \iint_{\partial M} u_{a_j,\lambda_j}^3 u_{a_1,\lambda_1} dS_g = \int_{B(a_j,\delta)} u_{a_j,\lambda_j}^3 u_{a_1,\lambda_1} dS_g + \int_{\partial M - B(a_j,\delta)} u_{a_j,\lambda_j}^3 u_{a_1,\lambda_1} dS_g. \tag{82}
\]

We are going to estimate \( J_1 \) and \( J_2 \) separately. In the case of \( J_2 \), we have
\[
\int_{\partial M - B(a_j,\delta)} u_{a_j,\lambda_j}^3 u_{a_1,\lambda_1} dS_g \leq C \int_{\partial M - B(a_j,\delta)} \left( \frac{1}{\lambda_j} \right)^{\frac{3}{2}} \left( \frac{1}{\delta} \right)^3 u_{a_1,\lambda_1} dS_g
\]
\[
\leq C \left( \frac{1}{\lambda_j} \right)^{\frac{3}{2}} \frac{1}{\delta^3} \int_{\partial M - B(a_j,\delta)} u_{a_1,\lambda_1} dS_g
\]
\[
\leq C \left( \frac{1}{\lambda_j \delta^2} \right)^{\frac{3}{2}} \int_{\partial M - (B(a_j,\delta) \cup B(a_i,\delta))} u_{a_1,\lambda_1} dS_g
\]
\[
+ C \left( \frac{1}{\lambda_j \delta^2} \right)^{\frac{3}{2}} \int_{\partial M - (B(a_j,\delta) \cup B(a_i,\delta))} u_{a_1,\lambda_1} dS_g
\]
\[
\leq C \left( \frac{1}{\lambda_j} \right)^{\frac{3}{2}} \left( \frac{1}{\delta} \right)^3 \frac{1}{\sqrt{\lambda_i}} + C \left( \frac{1}{\lambda_j} \right)^{\frac{3}{2}} \left( \frac{1}{\delta} \right)^3 \frac{1}{\sqrt{\lambda_i}} dS_g
\]
\[
\leq C \left( \frac{1}{\lambda_j} \right)^{\frac{3}{2}} \left( \frac{1}{\delta} \right)^3 \frac{1}{\sqrt{\lambda_i}} (1 + \delta^2)
\]
\[
\leq \frac{C}{\lambda_j^3 \sqrt{\lambda_i} \delta^4}.
\]
Thus, we get for $J_2$

$$J_2 = O \left( \varepsilon_{ij}^2 \frac{1}{\delta^2} \right). \quad (83)$$

In the next step, using (81) for $J_1$, we have

$$
\int_{B(a_j,\delta)} u_{a_j,\lambda_j}^3 \, da_j = c_0^4 \int_{B(a_j,\delta)} \left( \frac{\lambda_j}{\lambda_i} \right)^{\frac{4}{2}} \left[ 1 + O(\delta) + O \left( \frac{1}{\lambda_i^2 \delta^2} \right) \right] \frac{xy}{1 + \lambda_i^2 \delta^2(x)} \, ds_y

= c_0^4 \left[ 1 + O(\delta) + O \left( \frac{1}{\lambda_i^2 \delta^2} \right) \right] \int_{B_{\lambda_j}(0)} \left( \frac{1}{1 + |y|^2} \right)^{\frac{1}{2}} \left[ \frac{\lambda_j}{\lambda_i} \right]^2 \frac{1}{1 + \lambda_i^2 G_{\lambda_j}^{-2}(\psi_{a_j}(\frac{y}{\lambda_j}))} \, ds_y

= c_0^4 \left[ 1 + O(\delta) + O \left( \frac{1}{\lambda_i^2 \delta^2} \right) \right] \int_{B_{\lambda_j}(0)} \left( \frac{1}{1 + |y|^2} \right)^{\frac{1}{2}} \left[ \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j G_{\lambda_j}^{-2} \left( \psi_{a_j}(\frac{y}{\lambda_j}) \right) \right].
$$

Recalling that $\lambda_i \leq \lambda_j$, then for $\varepsilon_{ij} \sim 0$, we have

1) Either $\varepsilon_{ij}^2 \sim \lambda_i \lambda_j G_{\lambda_j}^{-2}(a_j).$

2) or $\varepsilon_{ij}^{-2} \sim \lambda_i^{-1}.$

In order to estimate $J_1$, we first define the following sets

$$A_1 = \left\{ y \in \mathbb{R}^2 : \left( \frac{y}{\lambda_j} \right) \leq \epsilon G_{a_j}^{-1}(a_j) \cap \bar{B}_{\lambda_j}(0) \right\},$$

$$A_2 = \left\{ y \in \mathbb{R}^2 : \left( \frac{y}{\lambda_j} \right) \leq \epsilon \frac{1}{\lambda_i} \cap \bar{B}_{\lambda_j}(0) \right\},$$

and

$$\mathcal{A} = A_1 \cup A_2,$$

with $\epsilon > 0$ very small. Then by Taylor expansion on $\mathcal{A}$, we have

$$\left[ \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j G_{\lambda_j}^{-2} \left( \psi_{a_j} \left( \frac{y}{\lambda_j} \right) \right) \right]^{\frac{1}{2}} = \left[ \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j G_{\lambda_j}^{-2}(a_j) \right]^{\frac{1}{2}}$$

$$+ \left[ \left( -\frac{1}{2} \nabla G_{\lambda_j}^{-2} \circ \psi_{a_j}(a_j) \lambda_j y \right) \left[ \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j G_{\lambda_j}^{-2} \left( \psi_{a_j} \left( \frac{y}{\lambda_j} \right) \right) \right]^{\frac{1}{2}}

+ O \left[ \left( \frac{\lambda_j}{\lambda_i} \right) |y|^2 \left[ \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j G_{\lambda_j}^{-2} \left( \psi_{a_j} \left( \frac{y}{\lambda_j} \right) \right) \right]^{\frac{1}{2}} \right].
$$

So, we write $J_1$ such as

$$J_1 = c_0^4 \left[ 1 + O(\delta) + O \left( \frac{1}{\lambda_i^2 \delta^2} \right) \right] \left( \sum_{m=1}^4 I_m \right), \quad (84)$$

with

$$I_1 = \left[ \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j G_{\lambda_j}^{-2}(a_j) \right]^{\frac{1}{2}} \int_{\mathcal{A}} \left( \frac{1}{1 + |y|^2} \right)^{\frac{1}{2}}$$

$$I_2 = \left[ \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j G_{\lambda_j}^{-2}(a_j) \right]^{\frac{1}{2}} \int_{\mathcal{A}} \left( \frac{1}{1 + |y|^2} \right)^{\frac{1}{2}} \left[ \nabla G_{\lambda_j}^{-2} \circ \psi_{a_j}(a_j) \lambda_j y \right],

I_3 = \left[ \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j G_{\lambda_j}^{-2}(a_j) \right]^{\frac{1}{2}} \int_{\mathcal{A}} \left( \frac{1}{1 + |y|^2} \right)^{\frac{1}{2}} \left[ \nabla G_{\lambda_j}^{-2} \circ \psi_{a_j}(a_j) \lambda_j \right],

I_4 = \int_{B_{\lambda_j}(0) - \mathcal{A}} \left( \frac{1}{1 + |y|^2} \right)^{\frac{1}{2}} \left[ \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j G_{\lambda_j}^{-2} \left( \psi_{a_j} \left( \frac{y}{\lambda_j} \right) \right) \right]^{\frac{1}{2}}.
$$

Now, let us estimate $I_1$. We have

$$I_1 = \left[ \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j G_{\lambda_j}^{-2}(a_j) \right]^{\frac{1}{2}} \left[ c_1 + \int_{\mathbb{R}^2 - \mathcal{A}} \left( \frac{1}{1 + |y|^2} \right)^{\frac{1}{2}} \right],$$
where $c_1$ is as in (16). On the other hand, we set

$$T_{ij} = \lambda_j \epsilon G_{a_i}^{-1}(a_i, a_j),$$

$$L_{ij} = \epsilon \frac{\lambda_j}{\lambda_i},$$

and have

$$\int_{\mathbb{R}^2 - A} \left( \frac{1}{1 + |y|^2} \right) \frac{2}{2} \leq \int_{\mathbb{R}^2 - B_{\delta ij}(0)} \left( \frac{1}{1 + |y|^2} \right) \frac{2}{2} + \int_{\mathbb{R}^2 - B_{\epsilon ij}(0)} \left( \frac{1}{1 + |y|^2} \right) \frac{2}{2},$$

if $\epsilon_{ij}^{-2} \sim \lambda_i \lambda_j G_{a_i}^{-2}(a_j)$, and

$$\int_{\mathbb{R}^2 - A} \left( \frac{1}{1 + |y|^2} \right) \frac{2}{2} \leq \int_{\mathbb{R}^2 - B_{\delta ij}(0)} \left( \frac{1}{1 + |y|^2} \right) \frac{2}{2} + \int_{\mathbb{R}^2 - B_{\epsilon ij}(0)} \left( \frac{1}{1 + |y|^2} \right) \frac{2}{2},$$

if $\epsilon_{ij}^{-2} \sim \frac{\lambda_j}{\lambda_i}$. We have

$$\int_{\mathbb{R}^2 - B_{\delta ij}(0)} \left( \frac{1}{1 + |y|^2} \right) \frac{2}{2} = O \left( \frac{1}{\lambda_i \delta} \right).$$

Moreover, if $\epsilon_{ij}^{-2} \sim \lambda_i \lambda_j G_{a_i}^{-2}(a_j)$, then

$$\int_{\mathbb{R}^2 - B_{\epsilon ij}(0)} \left( \frac{1}{1 + |y|^2} \right) \frac{2}{2} = O \left( \frac{1}{\lambda_j G_{a_i}^{-1}(a_j)} \right)$$

$$= O \left( \frac{1}{\sqrt{\lambda_i \lambda_j G_{a_i}^{-1}(a_j)}} \right)$$

$$= O \left( \epsilon_{ij} \right).$$

Furthermore if $\epsilon_{ij}^{-2} \sim \frac{\lambda_j}{\lambda_i}$, then

$$\int_{\mathbb{R}^2 - B_{\epsilon ij}(0)} \left( \frac{1}{1 + |y|^2} \right) \frac{2}{2} = O \left( \epsilon_{ij} \right).$$

This implies

$$\int_{\mathbb{R}^2 - A} \left( \frac{1}{1 + |y|^2} \right) \frac{2}{2} = O \left( \epsilon_{ij} + \frac{1}{\lambda_i \delta} \right)$$

$$= O \left( \epsilon_{ij} + \epsilon_{ij} \frac{1}{\delta} \right)$$

$$= O \left( \epsilon_{ij} \frac{1}{\delta} \right).$$

Thus, we get

$$I_1 = \left[ \frac{\lambda_j}{\lambda_i} + \lambda_i \lambda_j G_{a_i}^{-2}(a_j) \right] \frac{1}{2} \left[ c_1 + O \left( \epsilon_{ij} \frac{1}{\delta} \right) \right]$$

$$= \epsilon_{ij} \left( 1 + o_{\epsilon_{ij}}(1) \right) \left[ c_1 + O \left( \epsilon_{ij} \frac{1}{\delta} \right) \right]$$

Hence, we obtain

$$I_1 = c_1 \epsilon_{ij} \left[ 1 + o_{\epsilon_{ij}}(1) + O \left( \epsilon_{ij} \frac{1}{\delta} \right) \right].$$

By symmetry, we have

$$I_2 = 0.$$  \hspace{1cm} (85)

Next, for $I_3$, we derive

$$\int_{A} \frac{|y|^2}{(1 + |y|^2)^2} \leq \int_{B_{\epsilon ij}(0)} \frac{|y|^2}{(1 + |y|^2)^2} + \int_{B_{\epsilon ij}(0)} \frac{|y|^2}{(1 + |y|^2)^2}$$

$$= O \left( \epsilon \lambda_j G_{a_i}^{-1}(a_j) + \epsilon \frac{\lambda_j}{\lambda_i} \right).$$

Thus, we have

$$I_3 = \epsilon_{ij}^3 \left( \frac{\lambda_j}{\lambda_i} \right) \left( 1 + o_{\epsilon_{ij}}(1) \right) \left[ O \left( \epsilon \lambda_j G_{a_i}^{-1}(a_j) + \epsilon \frac{\lambda_j}{\lambda_i} \right) \right]$$

$$= \epsilon_{ij}^3 \left( 1 + o_{\epsilon_{ij}}(1) \right) \left[ O \left( \sqrt{\lambda_i \lambda_j} G_{a_i}^{-1}(a_j) + \frac{\lambda_j}{\lambda_i} \right) \right].$$

By symmetry, we have

$$I_2 = 0.$$  \hspace{1cm} (86)
Hence, we obtain
\[ I_3 = O\left( \varepsilon_i^2 \right). \]  
(87)

Finally, we estimate \( I_4 \) as follows.

If \( \varepsilon_{ij}^{-2} \sim \lambda_i \), then
\[ I_4 \leq C \varepsilon_{ij} \int_{B_{\lambda_i}(0) \setminus A} \left( \frac{1}{1 + |y|^2} \right)^{\frac{1}{2}} \leq C \varepsilon_{ij} \left( \frac{\lambda_i}{\lambda_j} \right)^{-1} \leq C \varepsilon_{ij}^3. \]  
(88)

If \( \varepsilon_{ij}^{-2} \sim \lambda_i \lambda_j G^{-2}_{a_i}(a_j) \), then we argue as follows. In case \( d_y(a_i, a_j) \geq 2\delta \), since
\[ G_{a_i} \left( \psi_{a_j} \left( \frac{y}{\lambda_j} \right) \right) \leq C \delta^{-1} \]
for \( y \in \hat{B}(0, \lambda_j \delta) \), then we have
\[ I_4 \leq C \int_{B_{\lambda_j \delta}(0) \setminus A} \left( \frac{1}{1 + |y|^2} \right)^{\frac{1}{2}} \frac{1}{\sqrt{\lambda_i \lambda_j} \delta} \]
\[ \leq C \left( \frac{1}{\lambda_j G^{-a_i^{-1}}_{a_i}(a_j)} \right) \]
\[ \leq C \left( \frac{G^{-a_i^{-1}}_{a_i}(a_j)}{\lambda_i \lambda_j G^{-a_i^{-1}}_{a_i}(a_j) \delta} \right) \]
\[ \leq C \varepsilon_{ij} \varepsilon_{ij} \frac{1}{\delta}. \]

Thus, when \( d_y(a_i, a_j) \geq 2\delta \), we have
\[ I_4 = O\left( \varepsilon_i^2 \frac{1}{\delta} \right). \]  
(89)

In case \( d_y(a_i, a_j) < 2\delta \), we first observe that
\[ \hat{B}_{\lambda_j \delta}(0) \setminus A \subset A_1 \cup A_2 \]
with
\[ A_1 = \{ y \in \mathbb{R}^2 : \epsilon \lambda_j G^{-a_i^{-1}}_{a_i}(a_j) \leq |y| \leq E \lambda_j G^{-a_i^{-1}}_{a_i}(a_j) \} \]
and
\[ A_2 = \{ y \in \mathbb{R}^2 : E \lambda_j G^{-a_i^{-1}}_{a_i}(a_j) \leq |y| \leq \lambda_j \delta \}, \]
where \( 0 < \epsilon < E \).

Thus, we have
\[ I_4 \leq I^1_4 + I^2_4, \]  
(90)

with
\[ I^1_4 = \int_{A_1} \left( \frac{1}{1 + |y|^2} \right)^{\frac{1}{2}} \left[ \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j G^{-a_i} G^{-a_i} \left( \psi_{a_j} \left( \frac{y}{\lambda_j} \right) \right) \right]^{-\frac{1}{2}}, \]
and
\[ I^2_4 = \int_{A_2} \left( \frac{1}{1 + |y|^2} \right)^{\frac{1}{2}} \left[ \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j G^{-a_i} G^{-a_i} \left( \psi_{a_j} \left( \frac{y}{\lambda_j} \right) \right) \right]^{-\frac{1}{2}}. \]
We estimate $I_1^4$ as follows:

$$I_1^4 \leq C \left[ 1 + \lambda_i^2 G_{a_i^2}(a_j) \right]^{-2} \int_{\{y \in \mathbb{R}^2 : |y| \leq E \lambda_i G_{a_i^{-1}}(a_j)\}} \left[ \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j G_{a_i^{-2}} \left( \frac{\psi_{a_j} \left( \frac{y}{\lambda_j} \right)}{\lambda_j} \right) \right]^{-\frac{1}{2}}$$

$$\leq C \left[ 1 + \lambda_i^2 G_{a_i^2}(a_j) \right]^{-2} \left( \frac{\lambda_i}{\lambda_j} \right)^{\frac{1}{2}} \int_{\{y \in \mathbb{R}^2 : |y| \leq E \lambda_i G_{a_i^{-1}}(a_j)\}} \left[ 1 + \lambda_i^2 \lambda_i^{-1} \psi_{a_i} \left( \frac{y}{\lambda_j} \right) + \frac{1}{1 + |y|^2} \right]^{-\frac{1}{2}}$$

$$\leq C \left[ \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j G_{a_i^{-2}}(a_j) \right]^{-2} \frac{1}{\lambda_j G_{a_i^{-1}}(a_j)}.$$ 

Thus, using (90) and (92), we have that if $d_y(a_i, a_j) < 2\delta$, then

$$I_4 = O \left( \varepsilon_{ij}^2 \right). \quad (91)$$

For $I_2^4$, we have

$$I_2^4 = \int_{A_2} \left( \frac{1}{1 + |y|^2} \right)^{\frac{1}{2}} \left[ \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j G_{a_i^{-2}} \left( \psi_{a_j} \left( \frac{y}{\lambda_j} \right) \right) \right]^{-\frac{1}{2}}$$

$$\leq C \left[ \frac{\lambda_i}{\lambda_j} + \lambda_i \lambda_j G_{a_i^{-2}}(a_j) \right]^{-2} \frac{1}{\lambda_j G_{a_i^{-1}}(a_j)}.$$ 

This implies

$$I_2^4 = O \left( \varepsilon_{ij}^2 \right). \quad (92)$$

Thus, combining (92) and (94), we have that if $d_y(a_i, a_j) < 2\delta$, then

$$I_4 = O \left( \varepsilon_{ij}^2 \right). \quad (93)$$

Now, using (93) and (94), we infer that in case $\varepsilon_{ij}^{-2} \geq \lambda_i \lambda_j G_{a_i^{-2}}(a_j)$,

$$I_4 = O \left( \varepsilon_{ij}^2 \right). \quad (95)$$

Finally combining (93), (94), we get

$$I_4 = O \left( \varepsilon_{ij}^2 \right). \quad (96)$$

Using (94)–(97), and (95), we obtain the following for $J_1$ (see (82))

$$J_1 = c_0^4 \left[ 1 + O \left( \delta + \frac{1}{\lambda_i^2 \delta^2} \right) \right] \left[ c_1 \varepsilon_{ij} \left( 1 + o_{\varepsilon_{ij}}(1) + O(\varepsilon_{ij} \delta^{-1}) \right) \right]. \quad (98)$$

Thus, using (82), (83), and (98), we arrive to

$$\oint_{\delta \mathcal{M}} u_{a_i, a_j}^3 dS_g = c_0^4 \left[ 1 + O \left( \delta + \frac{1}{\lambda_i^2 \delta^2} \right) \right] \left[ c_1 \varepsilon_{ij} \left( 1 + o_{\varepsilon_{ij}}(1) + O(\varepsilon_{ij} \delta^{-1}) \right) \right]$$

$$+ O \left( \varepsilon_{ij}^2 \frac{1}{\delta^4} \right).$$

Therefore, we obtain

$$\oint_{\delta \mathcal{M}} u_{a_i, a_j}^3 dS_g = c_0^4 c_1 \varepsilon_{ij} \left[ \left( 1 + O \left( \delta + \frac{1}{\lambda_i^2 \delta^2} \right) \right) \left( 1 + o_{\varepsilon_{ij}}(1) + O(\varepsilon_{ij} \delta^{-1}) \right) \right]$$

$$+ O \left( \varepsilon_{ij}^2 \frac{1}{\delta^4} \right). \quad (99)$$
Hence, recalling \( \epsilon_{ij} = \int_{\partial M} u_{a_i, \lambda_j} u_{a_j, \lambda_i} dS_g \), then the result follows from (17).

Clearly switching the index \( i \) and \( j \) in Lemma 5.3 we have the following corollary which is equivalent to Lemma 5.3. We decide to present the following corollary, because its form suits more our presentation of the Barycenter technique of Bahri-Coron [6] which follows the work [17] as done in [19].

**Corollary 5.4.** Assuming that \( \theta > 0 \) is small and \( \mu_0 > 0 \) is small then \( \forall \ a_i, a_j \in \partial M, \forall 0 < 2\delta < \delta_0 \), and \( \forall 0 < \frac{1}{\lambda_i}, \frac{1}{\lambda_j} \leq \theta \delta \) such that \( \epsilon_{ij} \leq \mu_0 \), we have

\[
\epsilon_{ij} = c_i^0 c_1 \epsilon_{ij} \left[ 1 + O \left( \delta + \frac{1}{\lambda_j^2} \right) \right] (1 + o_{\epsilon_{ij}} (1) + O(\epsilon_{ij} \delta^{-1})) + O \left( \frac{\epsilon_{ij}^2}{\delta^2} \right),
\]

where \( \delta_0 \) is as in (19).

We now show some sharp high-order inter-action estimates that are required for the application of the algebraic topological argument of Bahri-Coron [6] for existence. We begin with the balanced high-order inter-action estimate shown below.

**Lemma 5.5.** Assuming that \( \theta > 0 \) is small and \( \mu_0 > 0 \) is small then \( \forall \ a_i, a_j \in \partial M, \forall 0 < 2\delta < \delta_0 \), and \( \forall 0 < \frac{1}{\lambda_i}, \frac{1}{\lambda_j} \leq \theta \delta \) such that \( \epsilon_{ij} \leq \mu_0 \), we have

\[
\int_{\partial M} u_{a_i, \lambda_j} u_{a_j, \lambda_i} dS_g = O \left( \frac{\epsilon_{ij}^2}{\delta^2} \log (\epsilon_{ij}^{-1}) \right).
\]

**Proof.** By symmetry, we can assume without loss of generality (w.l.o.g) that \( \lambda_j = \lambda_i \). Thus we have

1) Either \( \epsilon_{ij}^{-2} \sim \lambda_i \lambda_j G_{a_i}^{-2}(a_j) \).

2) Or \( \epsilon_{ij}^{-2} \sim \frac{1}{\lambda_i} \).

Now, if \( d_g(a_i, a_j) \geq 2\delta \), then we have

\[
I := \int_{\partial M} u_{a_i, \lambda_i} u_{a_j, \lambda_j} dS_g \\
\leq C \int_{B(a_i, \delta)} \frac{\lambda_j}{1 + \lambda_i^2 d_g^2(a_i, x)} \left( \frac{\lambda_i}{1 + \lambda_i^2 G_{a_i}^{-2}(x)} \right) dS_g \\
+ \frac{C}{\lambda_i \lambda_j} \delta^2 \int_{B(a_j, \delta)} \left( \frac{1}{1 + \lambda_i^2 d_g^2(a_j, x)} \right) dS_g + \frac{C}{\lambda_i \lambda_j \delta^4} \\
\leq C \int_{B_{a_i}(0)} \frac{1}{1 + |y|^2} \left( \frac{1}{\lambda_i} + \lambda_i \lambda_j G_{a_i}^{-2} \left( \frac{\psi_{a_j}}{\lambda_j} \right) \right) \\
+ \frac{C}{\lambda_i \lambda_j \delta^2} \int_{B_{a_j}(0)} \left( \frac{1}{1 + |y|^2} \right) + \frac{C}{\lambda_i \lambda_j \delta^4} \\
\leq CI_1 + \frac{C}{\lambda_i \lambda_j \delta^2} \log (\lambda_j) + C] + \frac{C}{\lambda_i \lambda_j \delta^4} \\
\leq CI_1 + \frac{C}{\lambda_i \lambda_j \delta^4} \log (\lambda_j).
\]

Now, we estimate \( I_1 \) as follows

If \( \epsilon_{ij}^{-2} \sim \frac{1}{\lambda_i} \), then we get

\[
I_1 \leq C \epsilon_{ij}^2 \left[ \log (\lambda_i \delta) + C \right].
\]

So, for \( I \) we have

\[
I \leq C \epsilon_{ij}^2 \left[ \log (\lambda_i \delta) + C \right] + \frac{C}{\lambda_i \lambda_j \delta^4} \log (\lambda_i \lambda_j) \\
\leq \frac{C}{\delta^4} \epsilon_{ij}^2 \log (\epsilon_{ij}^{-2} G_{a_i}^{-2}(a_j)) \\
= O \left( \frac{\epsilon_{ij}^2}{\delta^4} \log (\epsilon_{ij}^{-1}) \right).
\]
If $\varepsilon_{ij}^{-2} \sim \lambda_i \lambda_j G_{a_i}^{-2}(a_j)$, then we get

$$I_1 \leq \frac{C}{\lambda_i \lambda_j \delta^2} \left[ \log (\lambda_i \delta) + C \right].$$

So, for $I$ we have

$$I \leq \frac{C}{\lambda_i \lambda_j \delta^2} \left[ \log (\lambda_i \lambda_j) \right].$$

This implies

$$I \leq \frac{C}{\delta^4 \varepsilon_{ij}^2} \log (\varepsilon_{ij}^{-2} G_{a_i}^2(a_j)).$$

Hence, for $d(g(a_i, a_j) \geq 2\delta$, we obtain

$$I = O \left( \frac{\varepsilon_{ij}^2 \log (\varepsilon_{ij}^{-1})}{\delta^4} \right). \quad (99)$$

On the other hand, arguing as above, if $d(g(a_i, a_j) < 2\delta$, then we have also

$$I \leq I_1 + \frac{C}{\lambda_i \lambda_j \delta^2} \left[ \log (\lambda_i \delta) \right] + \frac{C}{\lambda_i \lambda_j \delta^4} \log (\lambda_i \lambda_j),$$

where $I_1$ is as in 98. Thus, if $\varepsilon_{ij}^{-2} \simeq \frac{\lambda_i \lambda_j}{\lambda_j^2}$, then

$$I \leq I_1 + \frac{C}{\delta^4 \varepsilon_{ij}^2} \log (\varepsilon_{ij}^{-1}).$$

This implies

$$I \leq I_1 + \frac{C}{\delta^4 \varepsilon_{ij}^2} \log (\varepsilon_{ij}^{-1}).$$

Next, if $\varepsilon_{ij}^{-2} \simeq \lambda_i \lambda_j G_{a_i}^{-2}(a_j)$, then we get

$$I \leq I_1 + \frac{1}{\lambda_i \lambda_j \delta^4 G_{a_i}^{-2}(a_j)} \left[ \log (\lambda_i \lambda_j G_{a_i}^{-2}(a_j)) + \log (G_{a_i}^2(a_j)) \right] \log (\lambda_i \lambda_j),$$

$$\leq I_1 + \frac{C}{\delta^4 \varepsilon_{ij}^2} \log (\varepsilon_{ij}^{-1}).$$

Now, in order to proceed, we will estimate $I_1$. To do so, we begin by defining the following sets:

$$A_1 = \left\{ y \in \mathbb{R}^2 : |y| \leq \epsilon \lambda_i \sqrt{G_{a_i}^2(a_i) + \frac{1}{\lambda_j^2}} \right\},$$

$$A_2 = \left\{ y \in \mathbb{R}^2 : \epsilon \lambda_i \sqrt{G_{a_i}^2(a_i) + \frac{1}{\lambda_j^2}} \leq |y| \leq E \lambda_i \sqrt{G_{a_i}^2(a_i) + \frac{1}{\lambda_j^2}} \right\},$$

$$A_3 = \left\{ y \in \mathbb{R}^2 : E \lambda_i \sqrt{G_{a_i}^2(a_i) + \frac{1}{\lambda_j^2}} \leq |y| \leq 4 \lambda_i \delta \right\},$$

with $0 < \epsilon < E < \infty$. Clearly by the definition of $I_1$ (see 98), we have

$$I_1 \leq \int_{A_1} L_{ij} + \int_{A_2} L_{ij} + \int_{A_3} L_{ij},$$

where

$$L_{ij} = \left( \frac{1}{1 + |y|^2} \right) \left( \frac{1}{\lambda_i + \lambda_i \lambda_j G_{a_j}^{-2}(a_j)} \right).$$

For $\int_{A_1} L_{ij}$, we have

$$\int_{A_1} L_{ij} \leq C \varepsilon_{ij}^2 \int_{A_1} \left( \frac{1}{1 + |y|^2} \right) \leq C \varepsilon_{ij}^2 \log \left( \frac{\lambda_i}{\lambda_j} \right) \left( \lambda_i \lambda_j G_{a_j}^{-2}(a_j) + \frac{\lambda_i}{\lambda_j} \right) \leq C \varepsilon_{ij}^2 \log (\varepsilon_{ij}^{-1}).$$
For $\int_{A_2} L_{i,j}$, we have

$$\int_{A_2} L_{i,j} \leq C \left( \frac{1}{\lambda_j^2} + \lambda_i^2 G_{a_i}^{-2}(a_i) \right) \int_{A_2} \left( \frac{1}{\lambda_j} + \lambda_i \lambda_j G_{a_j}^{-2} \left( \psi_{a_i} \left( \frac{\lambda_j}{\lambda_i} \right) \right) \right)$$

$$\leq C \left( \frac{\lambda_j}{\lambda_i} \right)^2 \varepsilon_{ij}^2 \int_{\{y \in \mathbb{R}^2 : |y| \leq E \lambda_i \}} \frac{1}{G_{a_i}^{-2}(a_i) + \frac{1}{\lambda_j}} \left( \frac{1}{\lambda_j} + \frac{\lambda_i}{\lambda_j} |\psi_{a_i}^{-1} \circ \psi_{a_j} \left( \frac{\lambda_j}{\lambda_i} \right) |^2 \right)$$

$$\leq C \left( \frac{\lambda_j}{\lambda_i} \right) \varepsilon_{ij}^2 \int_{\{y \in \mathbb{R}^2 : |y| \leq E \lambda_i \}} \frac{1}{G_{a_i}^{-2}(a_i) + \frac{1}{\lambda_j} \lambda_i^{-2}} \left( \frac{1}{1 + |y|^2} \right)$$

$$\leq C \varepsilon_{ij}^2 \log \left( \varepsilon_{ij}^{-1} \right).$$

For $\int_{A_3} L_{i,j}$, we have

$$\int_{A_3} L_{i,j} \leq \int_{A_3} \left( \frac{1}{1 + |y|^2} \right) \left( \frac{1}{\lambda_j^2} + \frac{\lambda_i^2}{\lambda_j} |y|^2 \right)$$

$$\leq C \left( \frac{\lambda_i}{\lambda_j} \right) \int_{A_3} \frac{1}{|y|^4}$$

$$\leq C \left( \frac{\lambda_i}{\lambda_j} \right) \frac{1}{\lambda_j^2 G_{a_j}^{-2}(a_j) + \left( \frac{\lambda_i}{\lambda_j} \right)^2}$$

$$\leq C \varepsilon_{ij}^2.$$  

Therefore, we have

$$I_1 \leq C \varepsilon_{ij}^2 \log \varepsilon_{ij}^{-1}.$$  

This implies for $d_\theta(a_i, a_j) < 2\delta$, we have

$$I = O \left( \frac{\varepsilon_{ij}^2}{\delta^4} \log \left( \varepsilon_{ij}^{-1} \right) \right).$$  

Hence, combining with the estimate for $d_\theta(a_i, a_j) \geq 2\delta$ (see (99)), we have

$$\oint_{\partial M} u_{a_i, a_j}^2 dS_g = O \left( \frac{\varepsilon_{ij}^2}{\delta^4} \log \left( \varepsilon_{ij}^{-1} \delta^{-1} \right) \right).$$

Finally, we establish a sharp unbalanced high-order interaction estimate that is required for the application of the Barycenter technique of Bahri-Coron\cite{6} for existence.

**Lemma 5.6.** Assuming that $\theta > 0$ is small and $\mu_0 > 0$ is small, then $\forall a_i, a_j \in \partial M$, $\forall 0 < 2\delta < \delta_0$, and $\forall 0 < \frac{1}{\lambda_i} \leq \frac{1}{\lambda_j} \leq \theta \delta$ such that $\varepsilon_{ij} \leq \mu_0$, we have

$$\oint_{\partial M} u_{a_i, a_j}^\alpha dS_g = O \left( \frac{\varepsilon_{ij}^\beta}{\delta^4} \right).$$

where $\delta_0$ is as in (19), $\alpha + \beta = 4$, and $\alpha > 2 > \beta > 1$.

**Proof.** Let $a = \frac{1}{\lambda_i}$ and $\beta = \frac{1}{\lambda_j}$. Then we have $a + \beta = 2$. Now, since $\lambda_j \leq \lambda_i$, then for $\varepsilon_{ij} \sim 0$ we have

1) Either $\varepsilon_{ij}^{-2} \sim \lambda_i \lambda_j G_{a_i}^{-2}(a_j)$.  
2) Or $\varepsilon_{ij}^{-2} \sim \frac{\lambda_i}{\lambda_j}$.
To continue, we write
\[
\oint_{\partial M} u_{\alpha,\lambda}^* u_{\alpha,\lambda}^\beta \, dS_g = \oint_{B(a,\delta)} u_{\alpha,\lambda}^* u_{\alpha,\lambda}^\beta \, dS_g + \oint_{\partial M - B(a,\delta)} u_{\alpha,\lambda}^* u_{\alpha,\lambda}^\beta \, dS_g
\]
and estimate $I_1$ and $I_2$. For $I_2$, we have
\[
I_2 = \oint_{\partial M - B(a,\delta) \cap B(a,\delta)} u_{\alpha,\lambda}^* u_{\alpha,\lambda}^\beta \, dS_g + \oint_{\partial M - (B(a,\delta) \cup B(a,\delta))} u_{\alpha,\lambda}^* u_{\alpha,\lambda}^\beta \, dS_g \\
\leq C \oint_{(\partial M - B(a,\delta)) \cap B(a,\delta)} \left( \frac{\lambda_i}{1 + \lambda_i^2 G_{a_i}^{-2}(x)} \right)^\alpha \left( \frac{\lambda_j}{1 + \lambda_j^2 G_{a_j}^{-2}(x)} \right)^\beta \, dS_g \\
+ C \oint_{\partial M - (B(a,\delta) \cup B(a,\delta))} \left( \frac{\lambda_i}{1 + \lambda_i^2 G_{a_i}^{-2}(x)} \right)^\alpha \left( \frac{\lambda_j}{1 + \lambda_j^2 G_{a_j}^{-2}(x)} \right)^\beta \, dS_g \\
\leq \frac{C}{\lambda_i^\alpha \lambda_j^\alpha} \oint_{B_{a,\lambda}(0)} \left( \frac{1}{\lambda^\beta} \right)^\beta + \frac{C}{\lambda_i^\alpha \lambda_j^\alpha} \lambda_i^\beta \lambda_j^\beta.
\]
Thus, we have for $I_2$
\[
I_2 \leq \frac{C}{\lambda_i^\alpha \lambda_j^\alpha} \lambda_i^\beta \lambda_j^\beta. \tag{100}
\]
Next, for $I_1$ we have
\[
I_1 = \oint_{B(a,\delta)} \left( \frac{\lambda_i}{1 + \lambda_i^2 d_g^2(a_i, x)} \right)^\alpha \left( \frac{\lambda_j}{1 + \lambda_j^2 G_{a_j}^{-2}(x)} \right)^\beta \, dS_g \\
= \oint_{B_{a,\lambda}(0)} \left( \frac{1}{\lambda^\beta} \right)^\beta + \left( \frac{1}{\lambda_i^\alpha \lambda_j^\alpha + \lambda_i \lambda_j G_{a_i}^{-2}(a_i)} \right)^\beta.
\]
Thus, if $\varepsilon_{ij}^{-2} \sim \frac{\lambda_i}{\lambda_j}$, then
\[
I_1 \leq C \varepsilon_{ij}^{2\beta} \left( \frac{1}{\lambda_i^\alpha} \right)^{2\beta - 2} + C \\
\leq C \varepsilon_{ij}^\beta.
\]
If $\varepsilon_{ij}^{-2} \sim \lambda_i \lambda_j G_{a_i}^{-2}(a_i)$ and $d_g(a_i, a_j) \geq 2\delta$, then we have
\[
I_1 \leq C \left( \frac{1}{\lambda_i^\alpha \lambda_j^\alpha} \right)^\beta \left( \frac{1}{\lambda_i^\beta} \right)^{2\beta - 2} + C \\
\leq \frac{1}{\delta^2} \left( \frac{1}{\lambda_i^\alpha \lambda_j} \right)^\beta \leq C \frac{1}{\delta^2} \left( \frac{1}{\lambda_i \lambda_j G_{a_i}^{-2}(a_i)} \right)^\beta \\
\leq C \frac{1}{\delta^2} \varepsilon_{ij}^{-\beta}.
\]
Now, if $\varepsilon_{ij}^{-2} \sim \lambda_i \lambda_j G_{a_i}^{-2}(a_j)$ and $d_g(a_i, a_j) < 2\delta$, then we get
\[
I_1 \leq C \oint_{B_{a,\lambda}(0)} \left( \frac{1}{1 + |y|^2} \right)^\alpha \left( \frac{1}{\lambda_i^\alpha \lambda_j^\alpha + \lambda_i \lambda_j G_{a_j}^{-2}(a_j)} \right)^\beta.
\]
Next, we define
\[
B = \left\{ y \in \mathbb{R}^2 : \frac{1}{2} d_g(a_i, a_j) \leq \frac{|y|}{\lambda_i} \leq 2 d_g(a_i, a_j) \right\}
\]
and have

\[ I_1 \leq C \int_B \left( \frac{1}{1 + |y|^2} \right)^{\hat{\alpha}} \left[ \frac{1}{\lambda_j^\alpha + \lambda_j^\alpha |\psi_{a_j}^{-1} \circ \psi_{a_i} (\frac{y}{\lambda_j})|^2} \right] \hat{\beta} \]

\[ + C \int_{B \setminus (0) - B} \left( \frac{1}{1 + |y|^2} \right)^{\hat{\alpha}} \left[ \frac{1}{\lambda_j^\alpha + \lambda_j^\alpha |\psi_{a_j}^{-1} \circ \psi_{a_i} (\frac{y}{\lambda_j})|^2} \right] \hat{\beta}. \]

For the second term, we have

\[ \int_{B \setminus (0) - B} \left( \frac{1}{1 + |y|^2} \right)^{\hat{\alpha}} \left[ \frac{1}{\lambda_j^\alpha + \lambda_j^\alpha |\psi_{a_j}^{-1} \circ \psi_{a_i} (\frac{y}{\lambda_j})|^2} \right] \hat{\beta} \leq C \varepsilon_{ij}^{\hat{\beta}} \left[ \left( \frac{1}{\lambda_j^\alpha} \right)^{\alpha - 2} + C \right] \leq C \varepsilon_{ij}^{\hat{\beta}}. \]

For the first term, we have

\[ \int_B \left( \frac{1}{1 + |y|^2} \right)^{\hat{\alpha}} \left[ \frac{1}{\lambda_j^\alpha + \lambda_j^\alpha |\psi_{a_j}^{-1} \circ \psi_{a_i} (\frac{y}{\lambda_j})|^2} \right] \hat{\beta} \leq C \left( \frac{1}{\lambda_j^\alpha + \lambda_j^\alpha |\psi_{a_j}^{-1} \circ \psi_{a_i} (\frac{y}{\lambda_j})|^2} \right) \]

\[ \leq C \left( \frac{1}{1 + \lambda_j^\alpha d_g^2(a_i, a_j)} \right)^{\hat{\alpha}} \int_{\{ y \in \mathbb{R}^2 : |y| \leq 2\lambda_j d_g(a_i, a_j) \}} \left[ \frac{1}{\lambda_j^\alpha + \lambda_j^\alpha |\psi_{a_j}^{-1} \circ \psi_{a_i} (\frac{y}{\lambda_j})|^2} \right] \hat{\beta} \]

\[ \leq C \left( \frac{1}{1 + \lambda_j^\alpha d_g^2(a_i, a_j)} \right)^{\hat{\alpha}} \int_{\{ z \in \mathbb{R}^2 : |z| \leq 4\lambda_j d_g(a_i, a_j) \}} \left[ \frac{1}{\lambda_j^\alpha + \lambda_j^\alpha |z|^2} \right] \hat{\beta} \]

\[ \leq C \left( \frac{1}{\lambda_j^\alpha + \lambda_j^\alpha d_g^2(a_i, a_j)} \right)^{\hat{\alpha} + \frac{\beta}{2}} \int_{\{ z \in \mathbb{R}^2 : |z| \leq 4\lambda_j d_g(a_i, a_j) \}} \left[ \frac{1}{1 + |z|^2} \right] \hat{\beta}. \]

If \( \lambda_j d_g(a_i, a_j) \) is bounded, then we get

\[ I_1 \leq C \left( \frac{1}{\lambda_j^\alpha + \lambda_j^\alpha d_g^2(a_i, a_j)} \right)^{\hat{\alpha}} \]

\[ \leq C \varepsilon_{ij}^{\hat{\beta}}. \]

If \( \lambda_j d_g(a_i, a_j) \) is unbounded, then we get

\[ I_1 \leq C \left( \frac{1}{\lambda_j^\alpha + \lambda_j^\alpha d_g^2(a_i, a_j)} \right)^{\hat{\alpha}} \left( \lambda_j d_g^2(a_i, a_j) \right)^{2 - \hat{\beta}} \]

\[ \leq C \left( \frac{1}{1 + \lambda_j^\alpha d_g^2(a_i, a_j)} \right)^{\hat{\alpha} + \beta - 1} \left( \lambda_j \right)^{\hat{\beta}} \]

\[ \leq C \left( \frac{1}{\lambda_j^\alpha + \lambda_j^\alpha d_g^2(a_i, a_j)} \right)^{\hat{\beta}} \left( \frac{1}{1 + \lambda_j^\alpha d_g^2(a_i, a_j)} \right)^{\hat{\alpha} - 1} \]

\[ \leq C \varepsilon_{ij}^{\hat{\beta}}. \]

Thus, we have for \( I_1 \)

\[ I_1 \leq C \frac{\varepsilon_{ij}^{\hat{\beta}}}{\delta^4}. \]

(101)

On the other hand, using the estimate for \( I_2 \) (see (100)), we have

\[ I_2 = O \left( \frac{\varepsilon_{ij}^{\hat{\beta}}}{\delta^4} \right). \]

(102)

Hence, combining (101) and (102), we have

\[ \oint_{\delta M} u_{a_i, \lambda_i} \partial_{a_j, \lambda_j} dS_g = O \left( \frac{\varepsilon_{ij}^{\hat{\beta}}}{\delta^4} \right). \]
6 Algebraic topological argument

In this Section, we present the algebraic topological argument for existence. We start by fixing some notations from algebraic topology. For a topological space $Z$ and $Y$ a subspace of $Z$, $H_*(Z, Y)$ stands for the relative homology with $\mathbb{Z}_2$ coefficients of the topological pair $(Z, Y)$. For $f : (Z, Y) \to (W, X)$ a map with $(Z, Y)$ and $(W, X)$ topological pairs, $f_*$ denotes the induced map in relative homology.

Furthermore, we discuss some algebraic topological tools needed for our application of the Barycenter technique of Bahri-Coron [6] for existence. We start with recalling the space of formal the barycenter of $\partial M$. We have the selection map

$$ B_p(\partial M) = \left\{ \sum_{i=1}^{p} \alpha_i \delta_{a_i} : a_i \in \partial M, \alpha_i \geq 0, \quad i = 1, \ldots, p, \quad \sum_{i=1}^{p} \alpha_i = 1 \right\}, \quad \text{and} \quad B_0(\partial M) = \emptyset, \quad (103) $$

where $\delta_a$ for $a \in \partial M$ is the Dirac measure at $a$. Since $\dim(\partial M) = 2$, then we have the existence of $\mathbb{Z}_2$ orientation classes (see [6] and [16])

$$ w_p \in H_{2p-1}(B_p(\partial M), B_{p-1}(\partial M)), \quad p \in \mathbb{N}^+. \quad (104) $$

Now to continue, we fix $\delta$ small such that $0 < 2\delta < \delta_0$ where $\delta_0$ is as in (19). Moreover, we choose $\theta_0 > 0$ and small. After this, we let $\lambda$ varies such that $0 < \frac{1}{\lambda} \leq \theta_0 \delta$ and associate for every $p \in \mathbb{N}^*$ the map

$$ f_p(\lambda) : B_p(\partial M) \to H^1_+ (M) $$

defined by the formula

$$ f_p(\lambda)(\sigma) = \sum_{i=1}^{p} \alpha_i u_{a_i, \lambda}, \quad \sigma = \sum_{i=1}^{p} \alpha_i \delta_{a_i} \in B_p(\partial M), $$

where $u_{a_i, \lambda}$ is as in (28) with $a$ replaced by $a_i$.

As in Proposition 3.1 in [17] and Proposition 6.3 in [19], using Corollary 5.3 Corollary 5.2 Corollary 5.4 Lemma 5.5 and Lemma 5.6, we have the following multiple-bubble estimate.

**Proposition 6.1.** There exist $\tilde{C}_0 > 0$ and $\tilde{c}_0 > 0$ such that for every $p \in \mathbb{N}^*$, $p \geq 2$ and every $0 < \varepsilon \leq \varepsilon_0$, there exists $\lambda_p := \lambda_p(\varepsilon)$ such that for every $\lambda \geq \lambda_p$ and for every $\sigma = \sum_{i=1}^{p} \alpha_i \delta_{a_i} \in B_p(\partial M)$, we have

1. If $\sum_{i \neq j} \varepsilon_{i, j} > \varepsilon$ or there exist $i_0 \neq j_0$ such that $\frac{a_{i_0}}{a_{j_0}} > \nu_0$, then

$$ J_q(f_p(\lambda)(\sigma)) \leq p^{\tilde{S}}. $$

2. If $\sum_{i \neq j} \varepsilon_{i, j} \leq \varepsilon$ and for every $i \neq j$ we have $\frac{a_i}{a_j} \leq \nu_0$, then

$$ J_q(f_p(\lambda)(\sigma)) \leq p^{\tilde{S}} \left( 1 + \frac{\tilde{C}_0}{\lambda} - \tilde{c}_0 \frac{(p-1)}{\lambda} \right), $$

where $\tilde{S}$ is as in (14), $\varepsilon_{i, j}$ is as in (29), $\lambda_i = \lambda_j = \lambda$, $\varepsilon_0$ is as in (53), and $\nu_0$ is as in (54).

As in Lemma 4.2 in [17] and Lemma 6.4 in [19], we have the selection map $s_1$ (see (55)), Lemma 3.2 and Corollary 3.3 imply the following topological result.

**Lemma 6.2.** Assuming that $J_q$ has no critical points, then there exists $\tilde{\lambda}_1 > 0$ such that for every $\lambda \geq \tilde{\lambda}_1$, we have

$$ f_1(\lambda) : (B_1(\partial M), B_0(\partial M)) \to (W_1, W_0) $$

is well defined and satisfies

$$ (f_1(\lambda))_*(w_1) \neq 0 \text{ in } H_2(W_1, W_0). $$

As in Lemma 4.3 in [17] and Lemma 6.5 in [19], we have the selection map $s_p$ (see (55)), Lemma 3.2 and Proposition 6.1 imply the following recursive topological result.
Lemma 6.3. Assuming that \( J_q \) has no critical points, then there exists \( \bar{\lambda}_p > 0 \) such that for every \( \lambda \geq \bar{\lambda}_p \), we have
\[
f_{p+1}(\lambda) : (B_{p+1}(\partial M), B_p(\partial M)) \to (W_{p+1}, W_p)
\]
and
\[
f_p(\lambda) : (B_p(\partial M), B_{p-1}(\partial M)) \to (W_p, W_{p-1})
\]
are well defined and satisfy
\[
(f_p(\lambda))_*(w_p) \neq 0 \text{ in } H_{2p-1}(W_p, W_{p-1})
\]
implies
\[
(f_{p+1}(\lambda))_*(w_{p+1}) \neq 0 \text{ in } H_{2(p+1)-1}(W_{p+1}, W_p).
\]
Finally, as in Corollary 3.3 in [17] and Lemma 6.6 in [19], we clearly have that Proposition 6.1 implies the following result.

Lemma 6.4. Setting
\[
\bar{p}_0 := [1 + \bar{C}_0/\bar{c}_0] + 2
\]
with \( \bar{C}_0 \) and \( \bar{c}_0 \) as in Proposition 6.3 and recalling (56), we have there exists \( \bar{\lambda}_{\bar{p}_0} > 0 \) such that \( \forall \lambda \geq \bar{\lambda}_{\bar{p}_0},\)
\[
f_{\bar{p}_0}(\lambda)(B_{\bar{p}_0}(\partial M)) \subset W_{\bar{p}_0-1}.
\]

Proof of Theorem 1.1
As in [17] and [19], the theorem follows by a contradiction argument from Lemma 6.2 - Lemma 6.4.

7 Appendix

In this Section, using the explicit expression of \( \delta_{0,\lambda} \) (see (12)) or Lemma A-1 in [19], we have the following technical estimates.

Lemma 7.1. Recalling the definition of \( \delta_{0,\lambda} \) see (12), and setting \( x = (\bar{x}, x_3) \) with \( \bar{x} = (x_1, x_2) \), we have on \( \mathbb{R}^3_+ \).
\[
\delta_{0,\lambda}(x) = O \left( \frac{\lambda}{1 + \lambda^2 |x|^2} \right)^\frac{1}{2},
\]
\[
\partial_{\bar{x}} \delta_{0,\lambda}(x) = O \left( \frac{\lambda^2}{1 + \lambda^2 |x|^2} \right)^\frac{1}{2} \left( \frac{\lambda}{1 + \lambda^2 |x|^2} \right)^\frac{1}{2},
\]
\[
\nabla_{\bar{x}} \delta_{0,\lambda}(x) = O \left( \frac{\lambda}{1 + \lambda^2 |x|^2} \right)^\frac{1}{2} \left( \frac{\lambda^2}{1 + \lambda^2 |x|^2} \right)^\frac{1}{2},
\]
\[
\nabla_{\bar{x}}^2 \delta_{0,\lambda}(x) = O \left( \frac{\lambda}{1 + \lambda^2 |x|^2} \right)^\frac{1}{2} \left( \frac{\lambda^2}{1 + \lambda^2 |x|^2} \right)^\frac{1}{2}.
\]

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