A STRENGTHENED ALEXANDROV MAXIMUM PRINCIPLE OR UNIFORM HÖLDER CONTINUITY For solutions of the Monge–Ampère equation with bounded right-hand side

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Abstract. This article is about the convex solution $u$ of the Monge–Ampère equation on an at least 2-dimensional open bounded convex domain with Dirichlet boundary data and nonnegative bounded right-hand side.

For convex functions with zero boundary data, an Alexandrov maximum principle $|u(x)| \leq C \text{dist}(x, \partial \Omega)^\alpha$ is equivalent to (uniform) Hölder continuity with the same constant and exponent. Convex $\alpha$-Hölder continuous functions are $W^{1,p}$ for $p < 1/(1-\alpha)$. We prove Hölder continuity with the exponent $\alpha = 2/n$ for $n \geq 3$ and any $\alpha \in (0, 1)$ for $n = 2$, provided that the boundary data satisfy this Hölder continuity, and show that these bounds for the exponent are sharp. The only means is to bound the Hessian determinant of a certain explicit function on an $n$-dimensional cylinder and to use the comparison principle.

1. Introduction

The subject of this article is the convex solution of the Monge–Ampère equation

\[
\begin{cases}
\det D^2 u = f \quad \text{in } \Omega, \\
u = g \quad \text{on } \partial \Omega,
\end{cases}
\]

for an open, bounded, and convex set $\Omega \subset \mathbb{R}^n$, $n \geq 2$, $0 \leq f \in L^1(\Omega)$ and a convex function $g \in C(\Omega)$. The suitable notion of a weak formulation for this equation, the Alexandrov formulation [Ale58], uses the subdifferential

$$\partial u(x) := \{\xi \in \mathbb{R}^n \mid u \geq u(x) + \langle \xi, \bullet - x \rangle \text{ in } \Omega\},$$

the Lebesgue measure $|\bullet|$, and the Monge–Ampère measure

$$\mu_u : A \mapsto \left| \bigcup_{x \in A} \partial u(x) \right|.$$
and seeks a convex function \( u \in C(\Omega) \) with

\[
\begin{align*}
\mu_u &= fdx \quad \text{in } \Omega, \\
u &= g \quad \text{on } \partial \Omega.
\end{align*}
\]

1.1. Related prior work. When this article was already completed, the author got to know that the main result (1.2) was already proven by Caffarelli in [Caf90] for \( n > 3 \). The Alexandrov formulation provides uniqueness of the convex solution for (1.2) and existence if \( g = 0 \) or \( \Omega \) is strictly convex (even for finite Borel measures \( \nu \) instead of \( fdx \) [Fig17, Theorem 2.13, p. 20, Theorem 2.14, p. 24]. A brief history of the Monge–Ampère equation can be found in the introduction of [Fig17]. The regularity of solutions has been subject of intensive research. First of all, Alexandrov showed his maximum principle [Ale58] for \( u \) vanishing on the boundary, which implies that \( u \in C^{0,1/n} \) (see below). Trudinger–Urbas [TU83] showed that a uniformly convex \( C^{1,1} \) domain, a right-hand side \( f(x, u, \nabla u) \) bounded from above by \( \mu(|u|) \text{dist}(x, \partial \Omega)^\beta(1+|\nabla u|^2)^\alpha \) with a positive and nodcreasing \( \mu, \alpha \geq 0, \beta \geq 0, \beta \geq \alpha - n - 1 \) and further conditions lead to \( u \in C^2 \cap C^{0,1} \). Urbas [Urb88] proved a global Hölder estimate for \( C^{1,1} \) domains and a right-hand side \( f(x, u, \nabla u) \) bounded from below by \( \mu \text{dist}(x, \partial \Omega)^\beta(1+|\nabla u|^2)^\alpha \). In a later work [Urb91], he dealt with the equation of prescribed Gauss curvature. Furthermore, there are estimates at the boundary, if \( u \) behaves on the boundary like \( |x|^2 \) [Sav12, Sav13b, Sav13a]. And there are global \( W^{1,p} \) and \( W^{2,p} \) estimates for solutions of uniformly elliptic equations [Win99] (the Monge–Ampère equations are elliptic but not uniformly) and the linearized Monge–Ampère equations [LN14, LN17]. A global \( W^{1,p} \) estimate for the Monge–Ampère equation was not known to the author yet.

1.2. Basic properties. The Monge–Ampère measure has the following scaling properties: If \( \det D^2u = f \), more generally if \( \mu_u = fdx \), then

\[
\mu_{cu, A} = c^n (\det L_A)^2 f \circ A dx
\]

for \( c \geq 0 \), an affine map \( A : \mathbb{R}^n \to \mathbb{R}^n \), and its linear map \( L_A \). As an elliptic equation, the Monge–Ampère equation satisfies a comparison principle: If for continuous convex functions \( u \) and \( v \) on the closure of a bounded open set \( U \subset \Omega \),

\[
\begin{align*}
u &\leq u & \text{on } \partial U \\
\mu_u &\geq \mu_v & \text{in } U,
\end{align*}
\]

then also

\[
\begin{align*}
u &\leq u & \text{in } U.
\end{align*}
\]

The Monge–Ampère measure is superadditive [Fig17, Lemma 2.9, p. 17], i.e.

\[
\mu_{u+v} \geq \mu_u + \mu_v.
\]

The Alexandrov maximum principle (AMP) says that the solution \( u \) of (1.2) with \( g = 0 \) satisfies \( |u(x)| \leq C \text{dist}(x, \partial \Omega)^\alpha \) for \( \alpha = 1/n \) [Ale58, Fig17, Theorem 2.8].

1.3. Outline. For nonvanishing boundary values, the AMP is generalized by Hölder continuity of a convex function (see (2.6)) in the preliminary Section 2 as mentioned in [TW08, Lemma 3.3]. Additionally, an \( \alpha \)-Hölder continuous convex function \( u \) satisfies \( |\nabla u(x)| \lesssim \text{dist}(x, \partial \Omega)^{n-1} \) for every choice of \( \nabla u(x) \in \partial u(x) \) (see Lemma 3) which leads to \( u \in W^{1,p} \) for \( p < 1/(1-\alpha) \) (see Lemma 8) aside from other preliminaries. In Section 3 the Hessian determinants of certain explicit functions of the form \( w(x) = a(x_1) b(x') \) on a cylinder \( [0, h] \times B^{n-1}_p \) are bounded from...
below. This function has the aimed growth behavior $|w(x)| \leq x_n^\beta$ (for $n = 2$ $|w(x)| \lesssim x_1(1-\ln x_1)$, respectively) near the boundary. Then, in Section 4 the comparison principle enables us to infer from that the main result Theorem 18 that the solution of (1.2) with right-hand side bounded from above has the aimed growth behavior

$$|w(x)| = \frac{1}{\lambda} u(x) - u(x + \lambda z)$$

and for $p = \infty$, it denotes the usual essential supremum.

**Definition 1.** For $h, \rho > 0$, we will use cylinders

$$K_{h,\rho} := (0, h) \times B^n_{\rho^{-1}}.$$

**2.2. Uniformly continuous convex functions.**

**Definition 2.** For a function $u : \overline{\Omega} \to \mathbb{R}^n$ let

$$\omega_u(\delta) := \sup_{x,y \in \overline{\Omega}, |x-y| \leq \delta} |u(x) - u(y)|$$

be the modulus of continuity of $u$.

Obviously, $\omega_u$ is nonnegative, monotonically increasing, sublinear in $u$: For $\lambda \geq 0$

$$\omega_{u + v} \leq \lambda \omega_u + \omega_v,$$

and we have the following composition rule: For $u : \overline{\Omega_1} \to \overline{\Omega_2}$ and $v : \overline{\Omega_2} \to \mathbb{R}^n$ it holds that $\omega_{v \circ u} \leq \omega_v \circ \omega_u$. For affine $u$, this reads

$$\omega_{v \circ u}(\delta) \leq \omega_v(\delta)$$

Now let $\overline{\Omega}$ be compact and $u : \overline{\Omega} \to \mathbb{R}$ be convex. Since on a line segment $x + (0, \infty)z \cap \overline{\Omega}$, the function $0 < \lambda \mapsto \frac{u(x) - u(x+\lambda z)}{\lambda}$ is monotonically decreasing,

$$\omega_u(\bullet)/\bullet$$

is monotonically decreasing.
The setting is illustrated in Figure 2.1. For any \( x, y \in \overline{\Omega} \), the quantity \( u(x) - u(y) \) will not decrease if the line segment \( xy \) is translated in the direction \( x-y \) to the boundary of \( \partial \Omega \), i.e., if \( x \) and \( y \) are replaced by \( x' = x + \lambda(x-y) \in \partial \Omega \) and \( y' = y + \lambda(x-y) \), respectively, with \( \lambda \geq 0 \). This shows

\[
\omega_u(\delta) = \sup_{x \in \partial \Omega, y \in \overline{\Omega}, |x-y| \leq \delta} u(x) - u(y)
\]

for convex functions \( u \) on compact domains \( \overline{\Omega} \). If \( u \) vanishes on the boundary additionally, the latter quantity equals

\[
\omega_u(\delta) = \sup_{y \in \overline{\Omega}, \text{dist}(y, \partial \Omega) \leq \delta} -u(y)
\]

which is closely related to the AMP. Another direct consequence of (2.5) is the following comparison principle of the modulus of continuity: For a convex function \( u \) on \( \overline{\Omega} \) with a (nonnecessarily convex) minorant \( v \) with \( u \geq v \) in \( \Omega \),

\[
u = v \quad \text{on} \quad \partial \Omega,
\]

it holds that \( \omega_u \leq \omega_v \).

In particular, this implies that for fixed boundary conditions \( g|_{\partial \Omega} \) (with a convex \( g \in C(\overline{\Omega}) \)), the modulus \( \omega_g \) of any convex extension \( \tilde{g} \) of these is minimized by their convex envelope \( (g|_{\partial \Omega})_* \). The superadditivity, the subadditivity and the comparision principles of the Monge–Ampère measure and of the modulus compose to the following subadditivity between the Monge–Ampère measure and the modulus:

**Lemma 3.** For \( i \in \{1, 2\} \), let \( u_i \) be convex functions on \( \overline{\Omega} \). Let \( u_{12} \) be convex and continuous with

\[
\mu_{u_{12}} \leq \mu_{u_1} + \mu_{u_2} \quad \text{in} \quad \Omega,
\]

\[
u_{12} = u_1 + u_2 \quad \text{on} \quad \partial \Omega.
\]

Then

\[
\omega_{u_{12}} \leq \omega_{u_1} + \omega_{u_2}.
\]

**Proof.** The superadditivity (1.3) says that \( \mu_{u_{12}} \leq \mu_{u_1 + u_2} \), the comparision principle (1.4) that \( u_{12} \geq u_1 + u_2 \), and the comparision principle (2.7) and the subadditivity (2.2) of the modulus of continuity show that

\[
\omega_{u_{12}} \leq \omega_{u_1 + u_2} \leq \omega_{u_1} + \omega_{u_2}.
\]
Lemma 4 (Bound for the gradient with \( \omega_u \)). Let \( u : \overline{\Omega} \to \mathbb{R} \) be convex and \( x \in \Omega \). Then for any vector \( p \in \partial u(x) \)
\[
|p| \leq \sup_{y \in \partial \Omega} \frac{|u(y) - u(x)|}{|y - x|} \leq \frac{\omega_u(\text{dist}(x, \partial \Omega))}{\text{dist}(x, \partial \Omega)}.
\]

Proof. Let \( y \) be a point on \( \partial \Omega \) with \( \lambda (y - x) = p \) for some \( \lambda \geq 0 \). (\( y \) is unique if \( p \neq 0 \), because \( \Omega \) is bounded and convex.) Then convexity of \( u \) in \( x \) yields
\[
u(y) - u(x) \geq \langle p, y - x \rangle = |p| |y - x|.
\]
This shows \( |p| \leq \sup_{y \in \partial \Omega} |u(y) - u(x)| / |y - x| \), which is, since \( \omega_u(\cdot) / \cdot \) decreases monotonically \((2.4)\), bounded by \( \omega_u(\text{dist}(x, \partial \Omega)) / \text{dist}(x, \partial \Omega) \).

As a composition of the scaling properties \((1.3)\), the composition rule \((2.3)\) of the modulus and the chain rule of calculus, the next lemma sums up how to reduce solutions for arbitrary bounded right-hand sides and domains to the normalized case \( \Lambda = 1 \) and \( \Omega \) included in the unit ball \( B_1^d \).

Lemma 5 (Reduction to a normalized problem). If \( u \in C(\overline{\Omega}) \) is convex, \( \mu_u = f \, dx \), and \( L \in \text{Aff}(\mathbb{R}^n) \), then
\[
v := \Lambda^{1/n} |\det L|^{-2/n} u \circ L \in C(L \overline{\Omega})
\]
is convex, satisfies
\[
\mu_v = \Lambda f \circ L \, dx,
\]
and
\[
\omega_u(\delta) \leq \Lambda^{1/n} |\det L|^{-2/n} \omega_u(|L| \delta),
\]
\[
|\nabla v| \leq \Lambda^{1/n} |L| |\det L|^{-2/n} |\nabla u|.
\]

2.3. Hölder continuous convex functions are Sobolev. The next two propositions will be used to estimate the integral \( \int_{\Omega} \text{dist}(x, \partial \Omega)^2 \, dx \).

Lemma 6. Recall that \( \Omega_h := \{ x \in \Omega \mid \text{dist}(x, \partial \Omega) \geq h \} \), let
\[
F := \{ (x, |x-y|, y) \mid x \in \partial \Omega, y \in \Omega, |x-y| = \text{dist}(y, \partial \Omega) \},
\]
and
\[
\tilde{\Omega} := \pi_{12}(F) := \{ (x, h) \mid \exists y \in \Omega : (x, h, y) \in F \}.
\]

Then
(1) \( \Omega_h \) is convex.
(2) \( F \) is univalent with respect to \( y \), i.e. given \( x \in \partial \Omega \) and \( h \in (0, \infty) \), there is at most one \( y \) such that \( (x, h, y) \in F \). The corresponding function \( f : \tilde{\Omega} \to \Omega \)
\[
(x, h) \mapsto y,
\]
where \( (x, h, y) \in F \)
equals the projection \( f(x, h) = \pi(x, \Omega_h) := \arg \min_{y \in \Omega} |x-y| \). (Nevertheless we want to retain the domain \( \tilde{\Omega} \) of \( f \).)
(3) \( f \) is surjective.
(4) If \( (x, |x-y|, y) \in F \), then also \( (x, |x-z|, z) \in F \) for all \( z \in xy \).
(5) Equip \( \partial \Omega \times \mathbb{R} \) with the metric induced by the Euclidean norm on \( \mathbb{R}^{n+1} \):
\[
d(\langle x, a \rangle, \langle y, b \rangle) := |\langle x, a \rangle - \langle y, b \rangle| = \sqrt{|x-y|^2 + |a-b|^2}.
\]
Then \( f \) is 1-Lipschitz.
Proof. If $\Omega$ is the empty set, then also $\Omega_h$, $F$ and $\partial \Omega$ are empty, $f$ is the function $\emptyset \to \emptyset$ and the statements are trivial. Let us investigate the case where $\Omega \neq \emptyset$.

(1) $x \in \Omega_h$, if and only if $x + B^n_h \subseteq \Omega$. Let $x, y \in \Omega_h$. Since $\Omega$ is convex, it contains the convex hull $\overline{xy} + B_h$ of $(x + B^n_h) \cup (y + B^n_h)$, hence $\lambda x + (1-\lambda)y + B^n_h \subseteq \Omega$ and $\lambda x + (1-\lambda)y \in \Omega_h$.

(2) If $(x, h, y) \in F$, then $y \in \Omega_h$ and there cannot be any point in $\Omega_h$ closer to $x$ than $y$. This means that $y$ is the projection of $x$ onto $\Omega_h$, which is unique [Bré83 Theorem V.2, p. 79].

(3) Since $\partial \Omega$ is compact, every $y \in \Omega$ has a proximum $x = \arg \min_{x \in \partial \Omega} |x - y|$ on $\partial \Omega$ and $f(x, |x - y|) = y$.

(4) We have to show that $\operatorname{dist}(z, \partial \Omega) = |x - z|$. The triangle inequality, the infimizing property of $\operatorname{dist}$ and the fact that $z$ lies on the line segment $\overline{xy}$ show that

$$\operatorname{dist}(y, \partial \Omega) \leq \operatorname{dist}(z, \partial \Omega) + |z - y| \leq |x - z| + |z - y| = |x - y|.$$  

Since $\operatorname{dist}(y, \partial \Omega) = |x - y|$, all terms of this line must be equal and $\operatorname{dist}(z, \partial \Omega) = |x - z|$.

(5) Let $(x, a, z), (y, b, w) \in F$ and without loss of generality let $a \leq b$. According to (4), there is $v \in \overline{yw}$ with $(y, a, v) \in F$. We decompose

$$f(x, a) - f(y, b) = (f(x, a) - f(y, a)) + (f(y, a) - f(y, b))$$

$$= (\pi(x, \Omega_a) - \pi(y, \Omega_a)) + (f(y, a) - f(y, b)).$$

Since $v \in \overline{yw}$, the length of the second term $|f(y, a) - f(y, b)| = b - a$. Since $\Omega_h$ is closed and convex, the projection $\pi(\bullet, \Omega_a)$ is 1-Lipschitz according to [Bré83 Proposition V.3, p. 80], so the length of the first term $|f(x, a) - f(y, a)| \leq |x - y|$. Because of the well-known fact that for $\Omega_a$ as for any convex set,

$$\langle y - \pi(y, \Omega_a), z - \pi(y, \Omega_a) \rangle \leq 0 \quad \text{for all } z \in \Omega_a$$

[Bré83 Theorem V.2, p. 79], and since $f(y, a) - f(y, b)$ points in the same direction as $y - f(y, a)$, we know that the angle between the vectors is not acute:

$$\langle f(y, a) - f(y, b), f(x, a) - f(y, a) \rangle \leq 0.$$

This implies that

$$|f(x, a) - f(y, b)| \leq \sqrt{|x - y|^2 + |a - b|^2}.\qedhere$$

Corollary 7 (Volume of the layers). Let $\Omega \subset B^n_R$ be a bounded convex set. Then for all $0 < a < b < \infty$

$$(2.8) \quad |\Omega_a \setminus \Omega_b| \leq \mathcal{H}^{n-1}(\partial \Omega)(b - a) \leq \mathcal{H}^{n-1}(\partial B^n_R) R^{n-1}(b - a),$$

where $\mathcal{H}^{n-1}(\partial \Omega)$ denotes the $(n - 1)$-dimensional Hausdorff measure.

Proof. In terms of Lemma 6 the set of interest equals

$$\Omega_a \setminus \Omega_b = f \left( \tilde{\Omega} \cap (\partial \Omega \times [a, b]) \right).$$

Since $f$ is 1-Lipschitz, $|\Omega_a \setminus \Omega_b|$ can be estimated from above by the $n$-dimensional Hausdorff measure of the preimage $\tilde{\Omega} \cap (\partial \Omega \times [a, b])$ (cf. [EG15 Theorem 2.8, p. 97]), which is at most

$$(b - a)\mathcal{H}^{n-1}(\partial \Omega).$$
Lemma 8 (Hölder continuous convex functions are Sobolev). Let \( \Omega \subset B^n_R \), \( r := \max \text{dist}(\cdot, \partial \Omega) \), and let \( u \in C(\Omega) \) be convex and satisfy
\[
|u(x) - u(y)| \leq C_H |x - y|^\alpha
\]
for all \( x, y \in \Omega \) and some \( \alpha \in (0, 1] \). Further, let \( p \in [0, \infty) \), \( \beta \in \mathbb{R} \) with \((1 - \alpha)p - \beta = q < 1\). Then
\[
\int_\Omega |\nabla u(x)|^p \text{dist}(x, \partial \Omega)^\beta dx \leq H^{n-1}(\partial B^1_1) R^{n-1} C_H^p \frac{r^{1-q}}{1-q}.
\]
Especially, \( u \in W^{1,p}(\Omega) \) for all \( p < 1/(1-\alpha) \).

Proof. As a locally Lipschitz function, \( u \) is differentiable a.e. according to Rademacher’s theorem, and \( \nabla u(x) \) coincides with the classical derivative of \( u \) in \( x \) a.e. The following holds for these points and the other null set can be ignored when integrating. Lemma 4 and the Hölder continuity of \( u \) show that
\[
|\nabla u(x)| \leq \frac{\omega_u(\text{dist}(x, \partial \Omega))}{\text{dist}(x, \partial \Omega)} \leq C_H \text{dist}(x, \partial \Omega)^{\alpha-1},
\]
and thus
\[
|\nabla u(x)|^p \text{dist}(x, \partial \Omega)^\beta \leq C_H^p \text{dist}(x, \partial \Omega)^{-q}.
\]
Using the measure on \([0, \infty)\) defined by
\[
\sigma([a, b)) := |\Omega_a \setminus \Omega_b|,
\]
the integral of interest is bounded by
\[
\int_\Omega |\nabla u(x)|^p \text{dist}(x, \partial \Omega)^\beta dx \leq C_H^p \int_{t=0}^r t^{-q} dt \leq H^{n-1}(\partial B^1_1) R^{n-1} C_H^p \frac{r^{1-q}}{1-q}
\]
by means of Corollary 7. This can be further estimated by
\[
C_H^p \int_{t=0}^r t^{-q} dt \leq H^{n-1}(\partial \Omega) C_H^p \int_0^r t^{-q} dt \leq H^{n-1}(\partial B^1_1) R^{n-1} C_H^p \frac{r^{1-q}}{1-q}
\]
since \( q < 1 \). □

The following last preliminary lemma prepares the Theorem about converse bounds.
Lemma 9. Recall that $K_{h,r} = (0,h) \times B^m_r$. Assume that $\Omega$ is contained in the upper half space $(0,\infty) \times \mathbb{R}^{n-1}$, that the boundary of $\Omega$ contains the flat hyperdisc

$$\bar{F} := \{0\} \times B^{n-1}_r \subset \partial \Omega$$

for some $r > 0$ and let $u \in C(\bar{\Omega})$ be convex and $u|_{\bar{F}}$ be affine. Then there is a map $L \in \text{Aff}(\mathbb{R}^n)$ such that (the configuration is depicted in Figure 2.2)

$$F := \{0\} \times B^{n-1}_r \subset \bar{L}\bar{F} \cup \bar{K}_{2,2} \subset \bar{L}\bar{\Omega} \subset [0,\infty) \times \mathbb{R}^{n-1}.$$

Further, there is an affine function $\ell_g: \mathbb{R}^n \to \mathbb{R}$ which depends only on $L$ and the boundary data $g := u|_{\partial \Omega}$ such that

$$u \circ L^{-1} - \ell_g \leq 0 \quad \text{in } K_{2,2},$$

$$u \circ L^{-1} - \ell_g = 0 \quad \text{on } F.$$

Proof. Since $\Omega$ is open and convex, there is a small cylinder $K$ with a base included in $\bar{F}$. Let $L$ be a map which maps $K$ to $K_{2,2}$. At first, subtract the affine function $u|_{\bar{F}} \circ L^{-1}$:

$$v(x) := u(L^{-1}x) - u|_{\bar{F}}(0,(L^{-1}x)') .$$

Then $v|_F = 0$. Let $D$ be the “uppermost boundary points above $F$”:

$$D := \{(\max \{ x_1 \mid (x_1,x') \in L\bar{\Omega} \}, x') \mid x' \in B^{n-1}_r \}.$$

Since $K_{2,2} \subset \Omega$, the first component $x_1$ of a point $x \in D$ is at least 2. Since $\bar{D} \subset L\bar{\Omega}$ is compact, $v|_{\bar{D}}$ is bounded: $v|_{\bar{D}} \leq M \in \mathbb{R}$. Then

$$u_0(x) := v(x) - \frac{M}{2} x_1$$

vanishes on $F$ and is, because of its convexity, nonpositive everywhere between $F$ and $D$, as well as on $D$. Thus, let $\ell_g$ be the sum of the two subtracted functions. \qed

3. Explicit functions on cylinders with Hessian determinants bounded from below or above

Definition 10. Let $\nu_h \in C(\bar{K}_{h,1})$ be the convex solution of

$$\begin{cases} 
\mu_v = 1dx & \text{in } K_{h,1} \\
 v = 0 & \text{on } \partial K_{h,1} .
\end{cases}$$

For $n = 2$ and $0 < \varepsilon \leq 1/2$, let $\alpha(x) := x_1^{1-\varepsilon}$. For $n \geq 3$ let $\alpha(x) := x_1^{2/n}$ and, for brevity, $\alpha := \alpha$, so we can summarize all these cases by

$$\alpha(x) := x_1^{2/n - \delta_n \varepsilon} .$$

For $n = 2$ let

$$\mathfrak{a}(x) := x_1 (1 - \ln x_1)$$

and

$$\mathfrak{m}(x) := x_1 \left( \frac{1}{2} - \ln x_1 \right)^{1/2} ,$$

both continuously extended by $\mathfrak{a}(0) := \mathfrak{m}(0) := 0$. Again for brevity, we set for $n \geq 3$

$$\mathfrak{a} := \mathfrak{m} := a .$$

Further, let

$$b(x') := \frac{1}{2} |x'|^2 - 1.$$
and

\[ w_\varepsilon: K_{1,\varepsilon} \to (-\infty, 0] \]

\[(x_1, x') \mapsto a_\varepsilon(x_1)b(x'), \]

\[ \bar{w}: K_{1,\varepsilon} \to (-\infty, 0] \]

\[(x_1, x') \mapsto \bar{w}(x_1)b(x'). \]

**Remark 11.** The functions \( w_\varepsilon, w, \) and \( \bar{w} \) vanish on the bottom base of the cylinder \( K_{1,\varepsilon} \), where \( x_1 = 0 \), and on the side, where \( |x'| = \sqrt{2} \).

**Lemma 12.** For every \( 0 < \varepsilon \leq 1/2 \), there exists a small positive radius \( \rho > 0 \) such that \( w_\varepsilon \) restricted to \( \overline{K_{1,\rho}} \) is convex and its Hessian determinant is bounded from below by a positive number \( \lambda \). For \( n = 2 \), valid values are

\[ \lambda := \lambda_\varepsilon := \frac{\varepsilon}{4} \text{ for } \rho := \rho_\varepsilon := \sqrt{\frac{\varepsilon}{2}}. \]

**Proof.** In \( \overline{K_{1,\rho}} \), we have \( b(x') \leq \rho^2/2 - 1 \), \( \partial_i b(x') = x_i \) and \( \partial_{ij} b(x') = \delta_{ij} \) for \( i, j \in \{2, \ldots, n\} \), so

\[
D^2 w_\varepsilon(x_1, x') = 
\begin{pmatrix}
  a''_\varepsilon(x_1)b(x') & a'_\varepsilon(x_1)x_2 & \cdots & a'_\varepsilon(x_1)x_n \\
  a'_\varepsilon(x_1)x_2 & a_\varepsilon(x_1) & 0 & 0 \\
  \vdots & 0 & \ddots & 0 \\
  a'_\varepsilon(x_1)x_n & 0 & 0 & a_\varepsilon(x_1)
\end{pmatrix}.
\]

Sylvester’s criterion means that \( w_\varepsilon \) is convex, if all the principal minors

\[ M_{k, \ldots, n} := \det \begin{pmatrix}
  \partial_{kk} w_\varepsilon & \cdots & \partial_{kn} w_\varepsilon \\
  \vdots & \ddots & \vdots \\
  \partial_{nk} w_\varepsilon & \cdots & \partial_{nn} w_\varepsilon
\end{pmatrix} \]

are positive for all \( x \) in the interior \( K_{1,\rho} \). For \( k \geq 2 \) and \( \rho < \sqrt{2} \), obviously \( M_{k, \ldots, n} = a_\varepsilon(x_1)^{n-k+1} > 0 \). For \( k = 1 \), we have to calculate the determinant of the whole matrix, e.g. by the Leibniz formula, and to bound it:

\[ \det D^2 w_\varepsilon(x_1, x') = a''_\varepsilon(x_1)b(x')a_\varepsilon(x_1)^{n-1} - \sum_{i=2}^{n} a'_i(x_1)^2 x_i^2 a_\varepsilon(x_1)^{n-2} \]

\[ = \left( \frac{a''_\varepsilon(x_1)}{a_\varepsilon(x_1)}b(x') - \left( \frac{a'_\varepsilon(x_1)}{a_\varepsilon(x_1)} |x'| \right)^2 \right) a_\varepsilon(x_1)^n. \]

(3.1)

Note that

\[ \frac{a'_\varepsilon(x_1)}{a_\varepsilon(x_1)} = \left( \frac{2}{n} - \delta_{n2}\varepsilon \right) x_1^{-1}, \]

\[ \frac{a''_\varepsilon(x_1)}{a_\varepsilon(x_1)} = \left( \frac{2}{n} - \delta_{n2}\varepsilon \right) \left( \frac{2}{n} - 1 - \delta_{n2}\varepsilon \right) x_1^{-2}, \]

which yields

\[ \det D^2 w_\varepsilon(x_1, x') \geq \left( \frac{2}{n} - \delta_{n2}\varepsilon \right) \left( 1 - \frac{2}{n} + \delta_{n2}\varepsilon \right) \left( 1 - \frac{1}{2}\rho^2 \right) - \left( \frac{2}{n} - \delta_{n2}\varepsilon \right) \rho^2 \right) x_1^{-2\delta_{n2}\varepsilon}. \]
which is bounded away from 0 for \( \varepsilon \leq 1/2 \) and sufficiently small \( \rho \). For \( n = 2 \) this is
\[
\det D^2 w_{\varepsilon}(x_1, x') \geq (1-\varepsilon) \left( \varepsilon - \left(1 - \frac{\varepsilon}{2} \right) \rho^2 \right) x_1^{-2\varepsilon}.
\]
Since \( \varepsilon \leq 1/2 \), this is bounded by
\[
\det D^2 w_{\varepsilon}(x_1, x') \geq \frac{(1-\varepsilon)\varepsilon}{2} \geq \frac{\varepsilon}{4} = \lambda_{\varepsilon},
\]
if
\[
\rho_{\varepsilon} := \sqrt{\frac{\varepsilon}{2}} \leq \sqrt{\frac{\varepsilon}{2-\varepsilon}}.
\]

\[\square\]

**Lemma 13 (Strengthened AMP on a square).** There exists a dimensional constant \( C_n > 0 \) such that the solution \( v_2 : K_{2,1} \rightarrow (-\infty, 0] \) and the function \( w \) from Definition 11 satisfy for every \( (x_1, x') \in K_{1,1} \)
\[
|v_2(x_1, x')| \leq C_n a(x_1).
\]

**Proof.** At first, we prove it only for \( v_1 \) on \( K_{1,1} \). Take the function \( w_{\varepsilon} \), the lower bound \( \lambda \leq \det D^2 w_{\varepsilon} \) and the radius \( \rho \) from Lemma 12, stretch the domain by \( 1/\rho \) in the radial \( x' \) directions, and multiply the function by \( \lambda^{-1/\rho} \rho^{-2(n-1)/n} \cdot w_{\varepsilon}(x_1, \rho x') \).

The factor is chosen such that according to the reduction to a normalized problem (Lemma 5),
\[
\det D^2 w_{1,\varepsilon} \geq 1.
\]

The function \( w_{1,\varepsilon} \leq 0 \) on \( \partial K_{1,1} \). Therefore, the comparison principle yields \( 0 \geq v \geq w_{1,\varepsilon} \). For \( n \geq 3 \), note that \( b \leq 1 \) and we are done. For \( n = 2 \), we continue as follows:
\[
|v(x)| \leq |w_{1,\varepsilon}(x)| \leq \lambda_{\varepsilon}^{-1/\rho} \rho^{-1} x_1^{-1-\varepsilon} = \sqrt{\varepsilon} x_1^{1-\varepsilon}/\varepsilon.
\]

In order to minimize this for fixed \( x_1 \) with respect to \( \varepsilon \), we differentiate:
\[
\frac{\partial}{\partial \varepsilon} x_1^{1-\varepsilon} = x_1^{1-\varepsilon} \left( -\ln x_1 - \frac{1}{\varepsilon} \right).
\]

Thus, the minimum is expected at \( \varepsilon = -1/\ln x_1 \). Therefore
\[
|v(x_1, x_2)| \leq \sqrt{\varepsilon} x_1^{1+1/\ln x_1} (-\ln x_1) = \sqrt{\varepsilon} x_1 (\ln x_1),
\]
if \( \varepsilon = -1/\ln x_1 \leq 1/2 \), i.e. for \( x_1 \leq e^{-2} \). Since for all remaining \( x_1 \in [e^{-2}, 1] \),
\[
|v(x)| \leq |w_{1,1/2}(x)| \leq \sqrt{\varepsilon} x_1^{-1/2} / (1/2) \leq C_2 x_1 (1 - \ln x_1),
\]
for a sufficiently large \( C_2 \geq \sqrt{\varepsilon} \), altogether we get the global bound
\[
|v(x)| \leq C_2 x_1 (1 - \ln x_1) = C_2 g(x_1).
\]

To extend this estimate to \( v_2 \) on \( K_{2,1} \), note that on each line \( [0,1] \times \{x'\} \), \( w_{\varepsilon} \) attains its minimum at \( x_1 = 1 \), so it can be convexly extended by reflection:
\[
w_{\varepsilon}(x_1, x') := w_{\varepsilon}(2-x_1, x') \quad \text{for } 1 < x_1 \leq 2,
\]
preserving $\mu_w \geq \lambda \varepsilon$. Thus, the estimates can be extended to the doubled domain.

\[ \square \]

\textbf{Remark 14.} Sharper estimates can be achieved, if $a_{\varepsilon}$ and $b_{\varepsilon}(x') := b_{\varepsilon}(|x'|)$ are defined as the solutions of the differential equations

\[
a_{\varepsilon}'' a_{\varepsilon}^{n-1} = -x^{-2\delta_n \varepsilon}
\]

\[
a_{\varepsilon}(0) = a_{\varepsilon}(2) = 0
\]

\[
\left(\frac{2}{n} - \delta_n \varepsilon\right) \left(1 - \frac{2}{n} + \delta_n \varepsilon\right) b_{\varepsilon}'' + \left(\frac{2}{n} - \delta_n \varepsilon\right)^2 b_{\varepsilon}''' b_{\varepsilon}^2 = -1
\]

\[
b_{\varepsilon}(-1) = b_{\varepsilon}(1) = 0.
\]

This improves the constant $C_n$, but not the quality of the estimate.

\textbf{Lemma 15 (Converse AMP on a rectangle).} The Monge–Ampere measure $\mu_{w_{\varepsilon}}$ of the convex envelope of $\mathcal{F}(x) = \mathcal{N}(x_1) b(x')$ from Definition 14 has a density which is bounded from above.

\textbf{Proof.} According to [Fig17, Proposition A.35, p. 184], the Monge–Ampere measure of the convex envelope can be bounded by

\[
\mu_{w_{\varepsilon}}(E) \leq \int_E \chi_{\{x \mid \mathcal{F}(x) = \mathcal{N}(x_1) b(x')\}} \det D^2 \mathcal{F} dx
\]

for all Borel sets $E$, so it suffices to bound $\det D^2 \mathcal{F}$ from above. Start with (3.1) for $w_{\varepsilon}$ instead of $w_{\varepsilon}$ and recall that $\mathcal{F} > 0 > \mathcal{F}'$, $b \geq -1$ to estimate

\[
\det D^2 \mathcal{F}(x_1, x') = \left(\frac{\mathcal{F}'(x_1)}{\mathcal{F}(x_1)} b(x') - \left(\frac{\mathcal{F}(x_1)}{\mathcal{F}(x_1)} |x'|\right)^2\right) \mathcal{F}(x_1)^{n-1}
\]

\[
\leq -\mathcal{F}'(x_1) \mathcal{F}(x_1)^{n-1}
\]

\[
= \left\{\begin{array}{ll}
\frac{1 - \ln x_1}{1 - \frac{2}{n}} & \text{for } n = 2,
\frac{2}{n} (1 - \frac{2}{n}) & \text{otherwise}.
\end{array}\right.
\]

\[ \square \]

4. Improved Hölder continuity and $W^{1,p}$ estimates for functions with bounded Hessian determinant.

\textbf{Lemma 16 (Strengthened Alexandrov Maximum Principle).} Let $L \in \text{Aff} (\mathbb{R}^n)$ with $L \Omega \subset B^1_1$, $u \in C(\Omega)$ be a convex function with Monge–Ampere measure $\mu_u \leq A dx$ and $u|_{\partial \Omega} = 0$. For the same constant $C_n > 0$ as in Lemma 13 and $a$ from Definition 10 it holds that

\[
|u(x)| \leq C_n |\det L|^{-2/n} \Lambda_1^{1/n} \mathcal{A}(\|L\| \text{dist}(x, \partial \Omega))
\]

for $a$ as in Definition 10 and for all $x \in \Omega$.

\textbf{Remark 17.} For $n = 2$ and any $\alpha < 1$, $|u(x)| \leq \mathcal{A}(\|L\| \text{dist}(x, \partial \Omega))$ implies $|u(x)| \leq \text{dist}(x, \partial \Omega)^\alpha$.

\textbf{Proof.} Lemma 15 reduces the statement to the normalized case where $L = \text{id}$, $\Lambda = 1$. Let $\nu_2$ be as in Definition 10. The following configuration is depicted in Figure 1.1. Let $x_0$ be a proximality of $x$ on $\partial \Omega$ and we can assume that after a motion, $\Omega$ is included in $K_{2,1}$ and touches the bottom base $B_2 := \{0\} \times B_{1}^{n-1}$ of $K_{2,1}$ at $x_0$. The
comparison principle yields that $|u| \leq |v|$. Since $\Omega$ is included in a ball of radius 1, it holds that $x_1 = |x-x_0| = \text{dist}(x, \partial \Omega) \leq 1$. Thus, Lemma 13 implies

$$|u(x)| \leq |v(x)| \leq C_n a(1) = C_n a(\text{dist}(x, \partial \Omega)).$$

With the equivalence of the AMP and Hölder continuity (2.6) and the subadditivity between the Monge–Ampère measure and the modulus of continuity (Lemma 3), this implies our main result:

**Theorem 18** (Hölder continuity). Let $\Omega \subset \mathbb{R}^n$ be open, bounded and convex, $L \in \text{Aff}(\mathbb{R}^n)$ which maps $\Omega$ into $B^n_1$, $f : \Omega \to \mathbb{R}$ with $f \leq \Lambda < \infty$, and $g$ be a convex function on $\Omega$. Let $u$ be the continuous convex solution of (1.2). For the same dimensional constant $C_n > 0$ as in Lemma 13, it holds for the modulus of continuity of $u$ that

$$\omega_u(\delta) \leq \omega_g(\delta) + C_n |\det L|^{-2/n} \Lambda^{1/n} \begin{cases} |L|^{\delta} (1 - \ln(|L|^{\delta})) & \text{for } n = 2, \\ \left(|L|^{\delta}\right)^{2/n} & \text{otherwise.} \end{cases}$$

Particularly, if $g \in C^\alpha(\bar{\Omega})$ for some $\alpha < 1$, then $u \in C^{\min\{\alpha, 2/n\}}(\bar{\Omega})$.

**Proof.** Apply (2.6) and Lemma 3 where $u_1$ is the solution for zero boundary conditions from the last lemma, $u_2 = g$, and $u_{12}$ is the seeked solution $u$. □

Together with Lemma 8 this implies the following immediately.

**Corollary 19** (dist-weighted $W^{1,p}$ regularity). Let $\Omega \subset B^n_R$, and $f$ and $u$ be as in Theorem 18, where $g \in C^\alpha(\bar{\Omega})$ is convex. Then the following holds true for the weak derivative $\nabla u$ of $u$. Let $p, \beta \in [0, \infty)$ with $(1 - \min\{2/n, \alpha\})p - \beta =: q < 1$. Then

$$\int_{\Omega} |\nabla u(x)|^p \text{dist}(x, \partial \Omega)^\beta dx \leq C(n, R, \Lambda, p, \beta, \omega_g) < \infty.$$

In particular, $u \in W^{1,p}(\Omega)$ for all $p < (1 - \min\{2/n, \alpha\})^{-1}$.

**Remark 20.** (The dependence of $\|\nabla u\|_p$ on a normalizing map and $\Lambda$.) Let $g := 0$ and let $L \in \text{Aff}(\mathbb{R}^n)$ such that

$$B^n_1 \subset L\Omega \subset B^n_1.$$
Then there is a constant $C(n, p) > 0$ such that

$$\left( \frac{\int_{\Omega} |\nabla u(x)|^p \, dx}{\mu(\Omega)} \right)^{1/p} \leq \frac{|L|}{(\det L)^{2/n}} \Lambda^{1/n} C(n, p).$$

**Proof.** Write $u$ as $\Lambda^{1/n} \det L^{-2/n} v \circ L$, such that $v$ maps from the normalized domain $L \Omega \subset B_n^\alpha$ and has the Monge–Ampère measure $\det D^2 v(Lx) = \det D^2 u(x)/\Lambda \leq 1$. Corollary [13] bounds the left-hand side of (4.1) for $v$ instead of $u$ dependently only on $n$ and $p$. Meanwhile, Lemma [5] says that $\int_{\partial K} |\nabla u(x)| \leq \det L^{-2/n} |L| |\nabla v(Lx)| \Lambda^{1/n}$, so the same factor arises if we estimate the $p$-mean of the pointwise norms of the gradients:

$$\left( \frac{\int_{\Omega} |\nabla u(x)|^p \, dx}{\mu(\Omega)} \right)^{1/p} \leq \frac{|L|}{(\det L)^{2/n}} \left( \frac{\int_{L \Omega} |\nabla v(x)|^p \, dx}{\mu(\Omega)} \right)^{1/p} \leq \frac{|L|}{(\det L)^{2/n}} \Lambda^{1/n} C(n, p).$$

\[\square\]

### 5. Converse bounds for the exponents

Now we show that the exponent of $\delta$ in Theorem [13] cannot be larger than $2/n$ and that $\int_{\Omega} |\nabla u(x)|^{n/(n-2)} \, dx = \infty$, if the boundary of $\Omega$ contains a flat piece where $u$ is affine and $\det D^2 u \geq \lambda > 0$.

**Theorem 21** (Converse Hölder and Sobolev bounds). Assume that the boundary of $\Omega$ contains some flat boundary piece, namely that $\hat{F} = \{0\} \times B_{\delta}^{n-1} \subset \overline{\Omega} \subset [0, \infty) \times \mathbb{R}^{n-1}$.

Let $u \in C(\overline{\Omega})$ be a convex function with Monge–Ampère measure $\mu_\Omega \geq \lambda dx > 0$ and let $u|_{\hat{F}}$ be affine. Let $L \in \text{Aff}(\mathbb{R}^n)$ be the map from Lemma [7] such that $F := \{0\} \times B_\delta^{n-1} \subset L \hat{F} \cup K_{2, 2} \subset L \overline{\Omega} \subset [0, \infty) \times \mathbb{R}^{n-1}$.

1. Then there exists a constant $C_{L, g}$, depending only on $L$ and $u|_{\partial K_1}$, and a constant $C_n > 0$, depending only on $n$, such that for every $x \in L^{-1} K_{1, 1}$ and its projection $x_0 := L^{-1}(0, (Lx)')$ it holds that

$$u(x_0) - u(x) \geq -C_{L, g} |x - x_0| + \frac{C_n \lambda^{1/n}}{|\det L|^{2/n}} \left( \frac{|L| |x - x_0|}{(\det L)^{2/n}} \right)^{1/2} \quad \text{if } n = 2,$$

$$\text{otherwise.}$$

In particular, $u$ is not Lipschitz continuous and not Hölder continuous with exponent $\alpha > 2/n$.

2. For $n \geq 2$, $\|\nabla u\|_\infty = \infty$. For $n \geq 3$, $\|\nabla u\|_{n/(n-2)} = \infty$.

**Proof.** (1) Lemma [5] reduces the statement to the case where $L = \text{id}$. Let $\ell_g$ be the affine map given by Lemma [9]. The function $u_0 := u - \ell_g$ is non-positive on $\partial K_{2, 2}$ and vanishes on $F$. Thus, for $x \in K_{2, 2}$ we have $u(0, x') - u(x) = -u_0(x) + \ell_g(0, x') - \ell_g(x)$.

Hence, letting $C_{L, g}$ be the Lipschitz constant of $\ell_g$, it remains to find a constant $C_n > 0$ such that for every $x \in K_{1, 1}$

$$|u_0(x)| \geq \lambda^{1/n} C_n u(x_1).$$
for π from Definition 19. The scaling property (1.3) reduces the statement to the normalized case where λ = 1. Let \( B_T := \{1\} \times \mathbb{B}^{n-1}_r \) be the top base of \( K_{1,\sqrt{2}} \).

If \( u_0 \) vanished anywhere in \( \Omega \), the convexity would constrain it to vanish at all, contradicting \( \mu_{u_0} \geq dx \). So \( u_0 < 0 \) in \( \Omega \) and attains a negative maximum on the compactum \( B_T \subset K_{2,2} \). Therefore, there exists a small constant \( C > 0 \) such that for the convex envelope \( \bar{\pi} \), of \( \pi(x) = \pi(x_1)b(x') \) from Definition 19 it holds that

- \( C\bar{\pi} \geq u_0 \) on \( B_T \) and anyhow \( C\bar{\pi} = 0 \geq u_0 \) everywhere else on \( \partial K_{1,\sqrt{2}} \),
- \( \mu_{C\bar{\pi}} \leq 1dx \leq \mu_{u_0} \).

(The latter condition is enabled by Lemma 15 and the scaling properties.) Then the comparison principle (1.4) says that \( C\bar{\pi} \geq u_0|_{K_{1,\sqrt{2}}} \) and we get for \( x \in K_{1,1} \), where \( -b(x') = 1 - x^2/2 \geq 1/2 \),

\[-u_0(x) \geq -C\bar{\pi}(x) \geq -C\bar{\pi}(x) \geq \frac{1}{2}C\bar{\pi}(x_1).\]

(2) The statements are independent of addition or subtraction of an affine function, so it suffices to show it for \( u_0 \). For \( n = 2 \) we have just seen that \( u_0 \) is not Lipschitz at the flat piece. As a continuous function on the closure of an open domain, it is not possible to make it Lipschitz by removing a null set from the domain, so \( \|\nabla u_0\|_\infty = \infty \). For \( n \geq 3 \), assume that the affine map \( L \) from the first step is the identity without loss of generality. Start with the estimate

\[
\int_\Omega |\nabla u_0(x)|^p \, dx \geq \int_{B_1^{|n-1}|} \left| \frac{\partial u_0}{\partial x_1}(x_1, x') \right|^p \, dx_1 \, dx'.
\]

We claim that \( \int_0^1 |\partial_1 u_0(x_1, x')|^p \, dx_1 = \infty \) for each \( x' \in B_1^{n-1} \), such that the whole integral is infinite. If it were not infinite, for every \( \delta > 0 \) there would have to be an \( \varepsilon > 0 \) such that

\[
\int_0^\varepsilon |\partial_1 u_0(x_1, x')|^p \, dx_1 < \delta
\]

by the monotone convergence theorem. Contradicting that, the Jensen inequality, the fundamental theorem of calculus (which is applicable because a continuous convex function on a compactum is absolutely continuous), and (4.1) yield

\[
\int_0^\varepsilon \left| \frac{\partial u_0}{\partial x_1}(x_1, x') \right|^p \, dx_1 \geq \varepsilon \left( \frac{u_0(\varepsilon, x') - u_0(0, x')}{\varepsilon} \right)^p \geq C\varepsilon^{1+p(2/n-1)} = C
\]

for \( p = n/(n-2) \).

\( \square \)

6. Conclusion and a remaining question

The solution \( u \) of a problem with right-hand side 1 and affine boundary data on a flat boundary piece grows like \( x^{2/n} \) near the flat part of the boundary for \( n \geq 3 \). For \( n = 2 \), it has been established that the modulus of continuity of \( u \) lies between

\[ \delta(1 - \log \delta)^{1/2} \lesssim \omega_u(\delta) \lesssim \delta(1 - \log \delta). \]

But what is an explicit function \( a \) with

\[ a \lesssim \omega_u \lesssim a. \]
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