Lambda number of the power graph of a finite group

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Abstract

The power graph $\Gamma_G$ of a finite group $G$ is the graph with the vertex set $G$, where two distinct elements are adjacent if one is a power of the other. An $L(2,1)$-labeling of a graph $\Gamma$ is an assignment of labels from nonnegative integers to all vertices of $\Gamma$ such that vertices at distance two get different labels and adjacent vertices get labels that are at least 2 apart. The lambda number of $\Gamma$, denoted by $\lambda(\Gamma)$, is the minimum span over all $L(2,1)$-labelings of $\Gamma$. In this paper, we obtain bounds for $\lambda(\Gamma_G)$, and give necessary and sufficient conditions when the bounds are attained. As applications, we compute the exact value of $\lambda(\Gamma_G)$ if $G$ is a dihedral group, a generalized quaternion group, a $P$-group or a cyclic group of order $pq^n$, where $p$ and $q$ are distinct primes and $n$ is a positive integer.

Keywords: Power graph, $L(2,1)$-labeling, $\lambda$-number, finite group

MSC 2010: 05C25, 05C78

1 Introduction

All graphs considered in this paper are finite, simple and undirected. Let $\Gamma$ be a graph with the vertex set $V(\Gamma)$ and the edge set $E(\Gamma)$. The distance between vertices $x$ and $y$ is the length of a shortest path from $x$ to $y$ in $\Gamma$. For nonnegative integers $j$ and $k$, an $L(j, k)$-labeling of $\Gamma$ is a nonnegative integer valued function $f$ on $V(\Gamma)$ such that

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\( |f(u) - f(v)| \geq k \) whenever \( u \) and \( v \) are vertices of distance two and \( |f(u) - f(v)| \geq j \) whenever \( u \) and \( v \) are adjacent. The span of \( f \) is the difference between the maximum and minimum values of \( f \). The \( L(j,k) \)-labeling number \( \lambda_{j,k}(\Gamma) \) of \( \Gamma \) is the minimum span over all \( L(j,k) \)-labelings of \( \Gamma \). The classical work of the \( L(j,k) \)-labeling problem is when \( j = 2 \) and \( k = 1 \). The \( L(2,1) \)-labeling number of a graph \( \Gamma \) is also called the \( \lambda \)-number of \( \Gamma \) and denoted by \( \lambda(\Gamma) \).

The problem of studying \( L(j,k) \)-labelings of a graph is motivated by the radio channel assignment problem [15] and by the study of the scalability of optical networks [26]. In 1992, Griggs and Yeh [14] formally introduced the notion of the \( L(j,k) \)-labeling of a graph, and showed that the \( L(2,1) \)-labeling problem is NP-complete for general graphs. The \( L(j,k) \)-labelling problem, in particular in the \( L(2,1) \) case, has been studied extensively; see [11, 12, 22, 27] for examples. Surveys of results and open questions on the \( L(j,k) \)-labeling problem can be found in [28].

Graphs associated with groups and other algebraic structures have been actively investigated, since they have valuable applications (cf. [21]) and are related to automata theory (cf. [16, 17]). Zhou [29] studied \( L(j,k) \)-labelings of Cayley graphs of abelian groups. Kelarev, Ras and Zhou [20] established connections between the structure of a semigroup and the minimum spans of distance labellings of its Cayley graphs. In this paper we study \( L(2,1) \)-labelings of the power graph of a finite group.

The undirected power graph \( \Gamma_G \) of a finite group \( G \) has the vertex set \( G \) and two distinct elements are adjacent if one is a power of the other. The concepts of a power graph and an undirected power graph were first introduced by Kelarev and Quinn [18] and by Chakrabarty et al. [5], respectively. Since this paper deals only with undirected graphs, we use the term “power graph” to refer to an undirected power graph. Many interesting results on power graphs have been obtained in [3, 4, 8, 9, 19, 23–25]. A detailed list of results and open questions on power graphs can be found in [1].

Section 2 gives some preliminary results. In Section 3, we obtain a sharp lower bound for the \( \lambda \)-number of the power graph of a finite group \( G \); as applications, we compute \( \lambda(\Gamma_G) \) if \( G \) is a dihedral group, a generalized quaternion group or a \( P \)-group. In Section 4, we construct an upper bound for \( \lambda(\Gamma_G) \), and classify all groups such that the upper bound is attained.

2 Preliminaries

A path covering of a graph \( \Gamma \), denoted by \( C(\Gamma) \), is a collection of vertex-disjoint paths in \( \Gamma \) such that each vertex in \( V(\Gamma) \) is contained in a path in \( C(\Gamma) \). The path covering number \( c(\Gamma) \) of \( \Gamma \) is the minimum cardinality of a path covering of \( \Gamma \). Let \( \Gamma^c \) denote
Proposition 2.1. ([12, Theorem 1.1]) Let $\Gamma$ be a graph of order $n$.

(i) Then $\lambda(\Gamma) \leq n - 1$ if and only if $c(\Gamma^c) = 1$.

(ii) Let $r$ be an integer at least 2. Then $\lambda(\Gamma) = n + r - 2$ if and only if $c(\Gamma^c) = r$.

A vertex $x$ is a cut vertex in a graph $\Gamma$ if $\Gamma - x$ contains more connected components than $\Gamma$ does, where $\Gamma - x$ is the graph obtained by deleting the vertex $x$ from $\Gamma$.

Proposition 2.2. Let $\Gamma$ be a graph of order $n$ with a cut vertex $x$. Suppose that all connected components of $\Gamma - x$ are $\Gamma_1, \ldots, \Gamma_t$ and $|V(\Gamma_i)| = n_i$ for $i \in \{1, \ldots, t\}$, where $n_t \leq n_{t-1} \leq \cdots \leq n_1$. If $n_1 \leq \sum_{i=2}^{t} n_i$, then $\lambda(\Gamma) \leq n$.

Proof. Note that $n \geq 3$. If $n = 3$, then $\Gamma$ is a path, and so $\lambda(\Gamma) = 3$. In the following, suppose $n \geq 4$. Write $\Delta = (\Gamma - x)^c$, and pick $v \in V(\Delta)$. Assume $v \in V(\Gamma_k)$ for some $k \in \{1, \ldots, t\}$. Then

$$\deg_{\Delta}(v) \geq n - 1 - n_k,$$

where $\deg_{\Delta}(v)$ is the degree of $v$ in $\Delta$. Since $\sum_{i=1}^{t} n_i = n - 1$, we have $n_k \leq n_1 \leq \frac{n-1}{2}$, which implies that $\deg_{\Delta}(v) \geq \frac{n-1}{2}$. It follows from Dirac’s theorem ([2, Theorem 4.3]) that $\Delta$ has a Hamilton cycle, and so $c(\Delta) = 1$. Note that $\Delta = \Gamma^c - x$. Then $c(\Gamma^c) \leq 2$. By Proposition 2.1, we get the desired result.

Let $\Gamma$ be a graph. A subset of $V(\Gamma)$ is a clique if any two distinct vertices in this subset are adjacent in $\Gamma$. The clique number $\omega(\Gamma)$ is the maximum cardinality of a clique in $\Gamma$. It is easy to see that

$$\lambda(\Gamma) \geq 2\omega(\Gamma) - 2. \quad (1)$$

We give a sufficient condition for reaching the lower bound in (1).

Proposition 2.3. Let $C$ be a clique of a graph $\Gamma$ such that $|C| = \omega(\Gamma)$. Then $\lambda(\Gamma) = 2\omega(\Gamma) - 2$ if there exist partitions

$$\{A_1, \ldots, A_s\} \quad \text{and} \quad \{C_1, \ldots, C_s, C_{s+1}\}$$

of $V(\Gamma) \setminus C$ and $C$, respectively, satisfying the follows for each index $i \in \{1, \ldots, s\}$.

(i) $|A_i| \leq |C_i| - 1$.

(ii) Every vertex in $A_i$ and every vertex in $C_i$ are nonadjacent in $\Gamma$.

Proof. For $1 \leq i \leq s$ and $1 \leq j \leq s + 1$, write

$$A_i = \{u_{i1}, \ldots, u_{im_i}\}, \quad C_j = \{v_{j1}, \ldots, v_{jn_j}\}.$$
Let \( f \) be an integer valued function on \( V(\Gamma) \) such that
\[
v_{1t} = 2t, \quad f(v_{jt}) = 2(\sum_{k=1}^{j-1} n_k + t) \quad \text{for} \quad j \geq 2, \quad f(u_{it}) = f(v_{it}) + 1.
\]

By (i), we have \( m_i \leq n_i - 1 \), which implies that \( f \) is well-defined, and furthermore, the minimum and maximum values of \( f \) are \( 2 \) and \( 2 \sum_{k=1}^{s+1} n_k \), respectively. It follows from (ii) that \( f \) is an \( L(2,1) \)-labeling of \( \Gamma \), and so
\[
\lambda(\Gamma) \leq 2 \sum_{k=1}^{s+1} n_k - 2 = 2\omega(\Gamma) - 2.
\]

By (1), the desired result follows. \( \square \)

For a graph \( \Gamma \), a subset of \( V(\Gamma) \) is an independent set if no two of which are adjacent, and the independence number \( \alpha(\Gamma) \) is the maximum cardinality of an independent set in \( \Gamma \).

**Proposition 2.4.** Let \( \Gamma \) be a graph of order \( n \). Then \( \lambda(\Gamma) \leq 2n - \alpha(\Gamma) - 1 \).

**Proof.** Write \( r = c(\Gamma^c) \). If \( r = 1 \), by Proposition 2.1 (i), one has
\[
\lambda(\Gamma) \leq n - 1 \leq 2n - \alpha(\Gamma) - 1.
\]
Suppose \( r \geq 2 \). Let \( A \) be an independent set of \( \Gamma \) with \( |A| = \alpha(\Gamma) \). Then the subgraph induced by \( A \) of \( \Gamma^c \) has a Hamilton path, and so \( c(\Gamma^c) \leq n - |A| + 1 \). From Proposition 2.1 (ii), the desired inequality holds. \( \square \)

The independence number of the power graph of a finite abelian group has been studied. Denote by \( \mathbb{Z}_n \) the cyclic group of order \( n \).

**Lemma 2.5.** ([6, Theorem 10]) Let \( G \) be a finite abelian group. Then \( \alpha(\Gamma_G) = 2 \) if and only if \( G \cong \mathbb{Z}_{pq^n} \), where \( p \) and \( q \) are distinct primes and \( n \) is a positive integer.

**Lemma 2.6.** ([13, Theorem 5.4.10 (ii)]) A \( p \)-group having a unique subgroup of order \( p \) is either cyclic or generalized quaternion.

We extend Lemma 2.5 to the following result.

**Proposition 2.7.** Let \( G \) be a finite group. Then \( \alpha(\Gamma_G) = 2 \) if and only if \( G \cong \mathbb{Z}_{pq^n} \), where \( p \) and \( q \) are distinct primes and \( n \) is a positive integer.

**Proof.** The sufficiency follows from Lemma 2.5. Now suppose that \( \alpha(\Gamma_G) = 2 \). Then \( G \) has at most two distinct subgroups of prime order, and furthermore the order of \( G \) has at most two distinct prime divisors.
Suppose that $G$ is a $p$-group. If $G$ has a unique subgroup of order $p$, by Lemma 2.6 we conclude that $\Gamma_G$ is complete or has independence number at least 3, a contradiction. It follows that $G$ has two distinct subgroups of order $p$. Note that the center $Z$ of $G$ is nontrivial. Choose a subgroup $A$ of order $p$ in $Z$, and another subgroup $B$ of order $p$ in $G$. Then $AB \cong \mathbb{Z}_p \times \mathbb{Z}_p$, which implies that $G$ has at least 3 subgroups of order $p$, a contradiction.

The above contradiction implies that the order of $G$ has exactly two distinct prime divisors, say, $p$ and $q$. By Lemma 2.6, all Sylow subgroups of $G$ are cyclic. Let $P$ be a Sylow $p$-subgroup of $G$ and $Q$ a Sylow $q$-subgroup of $G$. If there exists a Sylow $p$-subgroup $P_1$ of $G$ such that $P \neq P_1$, then $\{a, b, c\}$ is an independent set of $\Gamma_G$, where $\langle a \rangle = P$, $\langle b \rangle = P_1$ and $\langle c \rangle = Q$, a contradiction. As a result, $G$ has a unique Sylow $p$-subgroup, and so $P$ is normal in $G$. Similarly, we conclude that $Q$ is also normal in $G$. Consequently, $G$ is abelian. By Lemma 2.5, we get the desired result.

\[\square\]

### 3 Lower bound

In this section, we give a lower bound for the $\lambda$-number of the power graph of a finite group, and compute the exact value of $\lambda(\Gamma_G)$ if $G$ is a dihedral group, a generalized quaternion group or a $P$-group.

**Theorem 3.1.** Let $G$ be a group of order $n$. Then $\lambda(\Gamma_G) \geq n$, with equality if and only if $(\Gamma_G - e)^c$ contains a Hamilton path, where $e$ is the identity of $G$.

**Proof.** Suppose that $f$ is an $L(2,1)$-labeling of $\Gamma_G$ with labels in $\{0, 1, 2, \ldots, \lambda(\Gamma_G)\}$. Since $\Gamma_G$ has diameter at most 2, the labels of all vertices under $f$ are pairwise distinct, which implies that $f$ is injective, and so $\lambda(\Gamma_G) \geq n - 1$. If $\lambda(\Gamma_G) = n - 1$, then $f$ is a bijection, and hence there exists an element $x \in G \setminus \{e\}$ such that $|f(x) - f(e)| = 1$, which contradicts that $x$ and $e$ are adjacent in $\Gamma_G$. Therefore, we have $\lambda(\Gamma_G) \geq n$.

Note that $e$ is adjacent to every other vertex in $\Gamma_G$. Then $e$ is an isolated vertex in $(\Gamma_G)^c$, which implies that $c((\Gamma_G)^c) = 2$ if and only if $(\Gamma_G - e)^c$ contains a Hamilton path. Hence, we get the desired result from Proposition 2.1 (ii).

It is hard to get a further characterization of the groups $G$ satisfying $\lambda(\Gamma_G) = n$, where $n$ is the order of $G$. In the remaining of this section, we give some groups such that the $\lambda$-numbers of their power graphs reach the lower bound in Theorem 3.1.

**Example 3.2.** For $n \geq 3$, we have $\lambda(\Gamma_{D_{2n}}) = 2n$, where $D_{2n}$ is the dihedral group of order $2n$.
Proof. Suppose

\[ D_{2n} = \langle a, b : a^n = b^2 = e, bab = a^{-1} \rangle. \]

Write \( B = \{b, ab, a^2b, \ldots, a^{n-1}b\} \). Then \( D_{2n} = \langle a \rangle \cup B \). Define an integer valued function \( f \) on \( D_{2n} \) as

\[
\begin{align*}
f(e) &= 0, \quad f(a) = 3, \quad f(a^2) = 5, \quad \ldots, \quad f(a^{n-1}) = 2(n - 1) + 1, \\
f(b) &= 2, \quad f(ab) = 4, \quad f(a^2b) = 6, \quad \ldots, \quad f(a^{n-1}b) = 2(n - 1) + 2.
\end{align*}
\]

Note that each element of \( B \) is an involution. Then

\[ E(\Gamma_{D_{2n}}) = E(\Gamma_{\langle a \rangle}) \cup \{\{e, x\} : x \in B\}, \]

which implies that \( f \) is an \( L(2, 1) \)-labeling of \( \Gamma_{D_{2n}} \), and so \( \lambda(\Gamma_{D_{2n}}) \leq 2n \). Thus, by Theorem 3.1 we get \( \lambda(\Gamma_{D_{2n}}) = 2n \).

Example 3.3. For \( n \geq 2 \), we have

\[ \lambda(\Gamma_{Q_{4n}}) = \begin{cases} 
4n + 1, & \text{if } n \text{ is a power of 2}, \\
4n, & \text{otherwise},
\end{cases} \]

where \( Q_{4n} \) is the generalized quaternion group of order \( 4n \).

Proof. Suppose

\[ Q_{4n} = \langle x, y : x^n = y^2, x^{2n} = 1, y^{-1}xy = x^{-1} \rangle. \]

Then \( y^{-1} = x^n, |x^i| = 4 \) and \( (x^i)^{-1} = x^{2n-i}y \) for \( i \in \{1, \ldots, n-1\} \). Therefore

\[
\begin{align*}
V(\Gamma_{Q_{4n}}) &= \{e, x, \ldots, x^{2n-1}\} \cup \left( \bigcup_{i=0}^{n-1} \{x^iy, (x^i)^{-1}\} \right), \\
E(\Gamma_{Q_{4n}}) &= E(\Gamma_{\langle x \rangle}) \cup \bigcup_{i=0}^{n-1} E(\Gamma_{\langle x^iy \rangle}). \quad (2)
\end{align*}
\]

The power graph \( \Gamma_{Q_{4n}} \) is shown in Figure 1.

First suppose that \( n \) is a power of 2. Then \( x^n \) is adjacent to every other vertex of \( \Gamma_{Q_{4n}} \), which implies that \( x^n \) is an isolated vertex in \( (\Gamma_{Q_{4n}} - e)^c \), and so \( (\Gamma_{Q_{4n}} - e)^c \) does not have a Hamilton path. According to Theorem 3.1, one gets \( \lambda(\Gamma_{Q_{4n}}) \geq 4n + 1 \).

Define an integer valued function \( f \) on \( V(\Gamma_{Q_{4n}}) \) such that

\[
\begin{align*}
f(e) &= 0, \quad f(x^n) = 2, \quad f(x) = 2n, \quad f(x^i) = 2i \text{ for } i \in \{2, \ldots, 2n-1\} \setminus \{n\}, \\
f(x^jy) &= 2(j + 1) + 3 \text{ and } f(x^{2n-j}y) = 2n + 2(j + 1) + 1 \text{ for } j \in \{1, \ldots, n-1\}, \\
f(y) &= 5, \quad f(x^ny) = 4n.
\end{align*}
\]
Figure 1: Γ_{Q_4^n}

It follows from (2) that $f$ is an $L(2, 1)$-labeling of $Γ_{Q_4^n}$ with span $4n + 1$, and so $\lambda(Γ_{Q_4^n}) \leq 4n + 1$. Thus, we have $\lambda(Γ_{Q_4^n}) = 4n + 1$.

In the following, suppose that $n$ is not a power of 2. In view of Theorem 3.1, it suffices to prove that $\lambda(Γ_{Q_4^n}) \leq 4n$. Note that there exists an element $x_0$ of odd order in $⟨x⟩$. Write $⟨x⟩ \setminus \{e, x^n, x_0\} = \{z_1, z_2, \ldots, z_{2n-3}\}$.

Let $f$ be an integer valued function on $V(Γ_{Q_4^n})$ such that

$\begin{align*}
  f(e) &= 0, \\
  f(x^n) &= 2, \\
  f(x_0) &= 3, \\
  f(z_i) &= 2(i + 1) + 1 \text{ for } i \in \{1, \ldots, 2n - 3\}, \\
  f(x^i y) &= 2(j + 2) \text{ and } f(x^{2n-i} y) = 2n + 2(j + 1) \text{ for } j \in \{1, \ldots, n - 1\}, \\
  f(y) &= 4, \\
  f(x^n y) &= 4n - 1.
\end{align*}$

Since $x_0$ and $x^n$ are nonadjacent in $Γ_{Q_4^n}$, it follows from (2) that $f$ is an $L(2, 1)$-labeling of $Γ_{Q_4^n}$ with span $4n$, as desired. \hfill $\blacksquare$

Now we give a sufficient condition for reaching the lower bound in Theorem 3.1.

**Lemma 3.4.** Let $G$ be a noncyclic group of order $n$. Suppose that all maximal cyclic subgroups of $G$ are $M_1, \ldots, M_t$, and $|M_i| = n_i$ for $1 \leq i \leq t$, where $n_t \leq n_{t-1} \leq \cdots \leq n_1$. If

$\begin{align*}
  |M_i \cap M_j| = 1 \text{ for } 1 \leq i < j \leq t \quad (3)
\end{align*}$

and

$\begin{align*}
  n_1 + t - 2 \leq \sum_{i=2}^{t} n_i, \quad (4)
\end{align*}$

then $\lambda(Γ_G) = n$. 

7
Proof. Note that $e$ is a cut vertex of $\Gamma_G$. By (3), the connected components of $\Gamma_G - e$ are $\Gamma_{M_1} - e, \ldots, \Gamma_{M_t} - e$. Hence, the desired result follows from Theorem 3.1 and Proposition 2.2.

A group is a $\mathcal{P}$-group [7] if every nonidentity element of the group has prime order.

Corollary 3.5. Let $G$ be a $\mathcal{P}$-group of order $n$. Then

$$\lambda(\Gamma_G) = \begin{cases} 
2(n - 1), & \text{if } n \text{ is a prime}, \\
n, & \text{otherwise}.
\end{cases}$$

Proof. If $n$ is a prime, then $G$ is cyclic, and so $\Gamma_G$ is complete, which implies that $\lambda(\Gamma_G) = 2(n - 1)$. In the following, assume that $n$ is not a prime. Note that $G$ is noncyclic and each maximal cyclic subgroup of $G$ has prime order. With reference to Lemma 3.4, the equation (3) is valid. In order to get the desired result, it suffice to show that (4) holds. Now by [7, Main Theorem], one of the following cases occurs:

(a) $G$ is a $p$-group of exponent $p$, where $p$ is a prime.
(b) $G \cong A_5$, the alternating group on 5 letters.
(c) $G$ is isomorphic to a Frobenius group $[P]Q$, where $P$ is a Sylow $p$-subgroup of $G$ and of exponent $p$, $Q$ is a Sylow $q$-subgroup of $G$ and of order $q$.

We use the notation in Lemma 3.4 to verify (4). Note that if $n_1 = n_2$ then (4) holds. If (a) or (b) occurs, then $n_1 = n_2$. Suppose (c) occurs. Then $G$ has a unique Sylow $p$-subgroup $P$, and the number of Sylow $q$-subgroups is at least $p$. If $p < q$, then $n_1 = n_2 = q$. If $p > q$ and $|P| > p$, then $n_1 = n_2 = p$. If $p > q$ and $|P| = p$, then $G$ has $p$ Sylow $q$-subgroups, which implies that

$$t = p + 1, \quad n_1 = p, \quad n_2 = \cdots = n_t = q,$$

and so (4) holds.

By Corollary 3.5, we get the following example.

Example 3.6. Let $\mathbb{Z}_p^n$ denote the elementary abelian $p$-group of order $p^n$. Then $\lambda(\Gamma_{\mathbb{Z}_p^n}) = p^n$ for $n \geq 2$.

4 Upper bound

In this section, we shall prove the following result.

Theorem 4.1. Let $G$ be a group of order $n$. If $G$ is not cyclic of prime power order, then $\lambda(\Gamma_G) \leq 2n - 4$, with equality if and only if $G$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_{2q}$, where $q$ is an odd prime.
We remark that if $G$ is a cyclic group of prime power order then $\Gamma_G$ is complete and so its $\lambda$-number is equal to $2n - 2$, where $n$ is the order of $G$. In order to prove Theorem 4.1, we first give some useful lemmas.

**Lemma 4.2.** Let $p$ and $q$ be distinct primes. For a positive integer $n$, we have

$$\lambda(\Gamma_{\mathbb{Z}_{pq^n}}) = \begin{cases} 2q^{n-1}(pq - p + 1) - 2, & \text{if } p < q; \\ 2q^n(p - 1), & \text{if } q < p. \end{cases}$$

*Proof.* For $1 \leq i \leq n$, write

$$X_i = \{ g \in \mathbb{Z}_{pq^n} : |g| = pq^{n-i} \} = \{ x_{i1}, \ldots, x_{ir_i} \},$$

$$Y_i = \{ g \in \mathbb{Z}_{pq^n} : |g| = q^{n+1-i} \} = \{ y_{i1}, \ldots, y_{is_i} \}.$$

Let $Z = G \setminus (\bigcup_{i=1}^{n} (X_i \cup Y_i))$. Then

$$Z = \{ g \in \mathbb{Z}_{pq^n} : |g| = pq^n \text{ or } |g| = 1 \},$$

$$r_i = \varphi(pq^{i-1}), s_i = \varphi(q^{n-i}) \text{ and } |Z| = \varphi(pq^n) + 1,$$

where $\varphi$ is the Euler’s totient function. Note that every vertex in $X_i$ and every vertex in $Y_i$ are nonadjacent in $\Gamma_{\mathbb{Z}_{pq^n}}$.

Suppose $p < q$. Then $(\bigcup_{i=1}^{n} Y_i) \cup Z$ is a clique of order $\omega(\Gamma_{\mathbb{Z}_{pq^n}})$ in $\Gamma_{\mathbb{Z}_{pq^n}}$ by [25, Theorem 2]. Note that $r_i \leq s_i - 1$. By Proposition 2.3, we have

$$\lambda(\Gamma_{\mathbb{Z}_{pq^n}}) = 2\omega(\Gamma_{\mathbb{Z}_{pq^n}}) - 2 = 2q^{n-1}(pq - p + 1) - 2.$$  

Suppose $q < p$. Then $(\bigcup_{i=1}^{n} X_i) \cup Z$ is a clique of order $\omega(\Gamma_{\mathbb{Z}_{pq^n}})$ in $\Gamma_{\mathbb{Z}_{pq^n}}$. If $(p, q) \neq (3, 2)$, then $r_i \leq s_i - 1$. It follows from Proposition 2.3 that

$$\lambda(\Gamma_{\mathbb{Z}_{pq^n}}) = 2\omega(\Gamma_{\mathbb{Z}_{pq^n}}) - 2 = 2q^n(p - 1).$$

In the following, assume $(p, q) = (3, 2)$. Then $r_i = s_i = 2^{n-i}$ for $1 \leq i \leq n - 1$ and $r_n = s_n + 1 = 2$. Let $r_{n+1} = 2^n + 1$, and write

$$Z = \{ x_{n+1,1}, \ldots, x_{n+1,r_{n+1}} \}.$$  

Define an integer valued function $f$ on $V(\Gamma_{\mathbb{Z}_{3^{2n}}})$ such that

$$x_{it} = 2t, \quad f(x_{jt}) = 2(\sum_{k=1}^{n} r_k + t) \text{ for } j \geq 2, \quad f(y_{it}) = f(x_{it}) + 1.$$  

For $1 \leq i \leq n - 1$, any vertex of $Y_i$ is not adjacent to any vertex of $X_i \cup X_{i+1}$ in $\Gamma_{\mathbb{Z}_{3^{2n}}}$. Hence $f$ is an $L(2, 1)$-labeling of $\Gamma_{\mathbb{Z}_{3^{2n}}}$ with minimum value 2 and maximum value $2\sum_{k=1}^{n+1} r_k$. Since

$$2\omega(\Gamma_{\mathbb{Z}_{3^{2n}}}) - 2 = 2^{n+2} = 2\sum_{k=1}^{n+1} r_k - 2,$$

we get the desired result by (1). \qed
Lemma 4.3. Let $G$ be a group of order $n$. If $G$ has pairwise distinct elements $u_1, u_2, u_3$ and $u_4$ such that $u_i$ and $u_{i+1}$ are nonadjacent in $\Gamma_G$ for $1 \leq i \leq 3$, then $\lambda(\Gamma) \leq 2n - 5$.

Proof. Note that $n \geq 4$. If $c((\Gamma_G)^c) = 1$, then $\lambda(\Gamma) \leq n - 1 \leq 2n - 5$ by Proposition 2.1 (i). Now suppose $c((\Gamma_G)^c) \geq 2$. Since $(\Gamma_G)^c$ contains a path $(u_1, u_2, u_3, u_4)$, we have $c((\Gamma_G)^c) \leq n - 3$. It follows from Proposition 2.1 (ii) that $\lambda(\Gamma) \leq 2n - 5$. \hfill \Box

Now we prove Theorem 4.1.

Proof of Theorem 4.1. Since $\Gamma_G$ is not complete, we have $\alpha(\Gamma_G) \geq 2$. If $\alpha(\Gamma_G) = 2$, by Proposition 2.7 and Lemma 4.2, we conclude that $\lambda(\Gamma_G) \leq 2n - 4$, with equality if and only if $G$ is isomorphic to $\mathbb{Z}_{2q}$ for some odd prime $q$. In the following, suppose $\alpha(\Gamma_G) \geq 3$. From Proposition 2.4, one has $\lambda(\Gamma_G) \leq 2n - 4$. If $G$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, then $\lambda(\Gamma_G) = 2n - 4$ by Example 3.6. Now assume $\lambda(\Gamma_G) = 2n - 4$. Then $\alpha(\Gamma_G) = 3$ by Proposition 2.4. It suffice to show that $G$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Suppose that $n$ is divisible by distinct primes $p$ and $q$, where $p < q$. It follows from [10, Section 4, I] that the number of subgroups of order $p$ and $q$ are $k_1 p + 1$ and $k_2 q + 1$, respectively, where $k_1$ and $k_2$ are nonnegative integers. If $k_1$ or $k_2$ is positive, then $\Gamma_G$ has an independent set containing 4 pairwise distinct elements of prime order, contrary to $\alpha(\Gamma_G) = 3$. So $k_1 = k_2 = 0$. Let $P$ and $Q$ be a Sylow $p$-subgroup and a Sylow $q$-subgroup of $G$, respectively. It follows from Lemma 2.6 that $P$ is either cyclic or generalized quaternion, and $Q$ is cyclic. Pick an element $y$ of order $q$ in $Q$. If $p > 2$, choose an element $z$ of order $p$ in $P$, then $y, z, y^{-1}$ and $z^{-1}$ are pairwise distinct elements in $G$ such that

$$\{\{y, z\}, \{z, y^{-1}\}, \{y^{-1}, z^{-1}\}\} \cap E(\Gamma_G) = \emptyset, \tag{5}$$

and by Lemma 4.3 we get $\lambda(\Gamma_G) \leq 2n - 5$, a contradiction. Hence $p = 2$. If there exists an element $z$ of order 4 in $P$, then (5) holds, a contradiction. Therefore, $P$ is isomorphic to $\mathbb{Z}_2$, and so the generator $x$ of $P$ is the unique element of order 2 in $G$, which implies that $x$ belongs to the center of $G$. If $Q$ has an element $z$ of order $q^2$, replace $y$ with $x y$ in (5), then we get a contradiction. Therefore $Q = \langle y \rangle$. If $n$ has a prime divisor $r$ with $r \notin \{p, q\}$, pick an element $z$ of order $r$ in $G$, then (5) holds, a contradiction. It follows that $G$ is isomorphic to $\mathbb{Z}_{2q}$, and so Proposition 2.7 implies that $\alpha(\Gamma_G) = 2$, a contradiction.

The above contradiction implies that $G$ is a $p$-group. If $p \geq 3$, then by Lemma 2.6, $G$ has at least $p+1$ subgroups of order $p$, contrary to $\alpha(\Gamma_G) = 3$. Hence $p = 2$. If $G$ has a unique subgroup of order 2, then Lemma 2.6 and Example 3.3 give $\lambda(\Gamma_G) = n + 1$, a contradiction. Thus, the number of subgroups of order 2 in $G$ is at least 3. If $G$ has an element of order 4, we obtain a contradiction by Lemma 4.3. It follows that
$G$ is elementary abelian, and from Example 3.6 we deduce that $G$ is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$, as desired.

\begin{acknowledgement}

This work was completed during Ma and Feng’s visit to the Beijing Normal University. Wang’s research was supported by National Natural Science Foundation of China (11371204, 11671043) and the Fundamental Research Funds for the Central University of China.

\end{acknowledgement}

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