COMPACT SPACES AS QUOTIENTS OF PROJECTIVE FRA"ISSE LIMITS

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Abstract. We develop a theory of projective Fra"isse limits in the spirit of Irwin-Solecki. The structures here will additionally support dual semantics as in [SI10, SI12]. Let $Y$ be a compact metrizable space and let $G$ be a closed subgroup of $\text{Homeo}(Y)$. We show that there is always a projective Fra"isse limit $K$ and a closed equivalence relation $r$ on its domain $K$ that is definable in $K$, so that the quotient of $K$ under $r$ is homeomorphic to $Y$ and the projection $K \twoheadrightarrow Y$ induces a continuous group embedding $\text{Aut}(K) \hookrightarrow G$ with dense image.

Introduction

The interplay between Fra"isse theory and dynamics of non-Archimedean Polish groups established by the correspondence between closed subgroups of $S_\infty$ and countable Fra"isse structures has given rise to a reach theory with various interesting results; see for example the survey [Kec12]. Taking this fact as a departure point it is natural to seek what would the natural correspondence be for projective Fra"isse structures.

Projective Fra"isse structures were introduced by T. Irwin and S. Solecki in [ISS06] and since then they have been used by several authors in the study of the dynamics of compact spaces such as the pseudo-arc, the Lelek fan and the Cantor space; see [ISS06, BK13, Kwi12, Kwi13]. There is a standard process that is implemented in all these papers. One starts with a compact metricizable space $Y$ where $Y$ or $\text{Homeo}(Y)$ is under investigation. Then one defines an appropriate class $\mathcal{K}$ of finite model theoretic $L$-structures which “approximate” $Y$. If this class $\mathcal{K}$ satisfies the projective Fra"isse axioms then the projective Fra"isse limit $K$ of $\mathcal{K}$ is uniquely defined and it is a compact, zero-dimensional metrizable space $K$ together with closed interpretations for the model-theoretic content $L$; see [ISS06], or Section 2. In all cases above, whenever $Y$ is not totally disconnected [ISS06, BK13], the language $L$ contained—implicitly or explicitly—a special binary predicate $r$ whose interpretation in all finite structures of $\mathcal{K}$ is a reflexive, symmetric graph, and whose interpretation in $K$ is a closed equivalence relation. Moreover, the space $K/r$ is homeomorphic to $Y$ and the quotient projection map $K \twoheadrightarrow Y$ induces a continuous group embedding $\text{Aut}(K) \hookrightarrow \text{Homeo}(Y)$ whose image is dense in $\text{Homeo}(Y)$. This correspondence between $K$ and $Y$ allows one to study $Y$ and $\text{Homeo}(Y)$ through the combinatorial properties of the finite structures in $\mathcal{K}$. 
The purpose of this note is to turn these heuristics into a general theorem. The price we have to pay is that we have to include dual predicates in our language $L$ and use them to further endow our structures with dual semantics. Dual predicates and structures were first introduced by S. Solecki in [SH10, SH12], where they were used to institute a uniform structural approach to various Ramsey theoretic results. Dual predicates in $L$ quantify over dual tuples (finite clopen partitions). We will call a $L$-structure purely dual if $L$ contains only dual predicates.

In Section 2, we rewrite the projective Fraïssé theory developed in [ISS06] so that it includes the dual semantics found in [SH10, SH12]. In Section 3, we use a standard orbit completion argument and equipping $\text{Homeo}(K)$ with the compact-open topology we get the following proposition.

**Proposition 3.1.** Let $G$ be a closed subgroup of $\text{Homeo}(K)$ where $K$ is zero-dimensional, compact, metrizable space. Then there is a purely dual projective Fraïssé structure $K$ on domain $K$ such that $\text{Aut}(K) = G$.

Proposition 3.1 is essentially the dual of the statement that every closed subgroup of $S_\infty$ is the automorphism group of a Fraïssé limit on domain $\mathbb{N}$. In Section 4, we show how to turn any topological $L$-structure into a purely dual one without losing any information. We also show that the above theorem is false if we do not allow our structures to support a dual structure. Therefore the context of dual structures is strictly more general than the context of direct structures.

In Section 5, we fix a special binary relation symbol $r$ whose interpretation will always be a reflexive and symmetric closed relation. This should be paralleled with the metric Fraïssé theory where the symbol $d$ is reserved as a signifier for the metric. A formal relational language $L$ will be decorated with the subscript $r$ whenever $r$ belongs to $L$. Therefore, we always have $r \in L_r$ and $L_r$-structures are, in particular, reflexive $r$-graphs. We say that an $L_r$-structure $K$ is a pre-space if $K$ is moreover transitive and therefore an equivalence relation. We apply Proposition 3.1 to get the following result.

**Theorem 5.2.** Let $G$ be a closed subgroup of $\text{Homeo}(Y)$, for some compact metrizable space $Y$. Then there is a projective Fraïssé pre-space $K$ such that $K/r^K$ is homeomorphic to $Y$, and the quotient projection $K \mapsto Y$ induces a continuous group embedding $\text{Aut}(K) \hookrightarrow G$, with dense image in $G$.

We shall note here that R. Camerlo characterized all possible quotients $M/r^M$ of Fraïssé structures in the language $\{r\}$ to be certain combinations of singletons, Cantor spaces and pseudo-arcs [RC10].

**Acknowledgements.** This set of notes includes –I think– the first written account of Fraïssé theory with dual predicates. This is not however the original idea of this paper. The idea of using dual predicates was first introduced in a systematic way by S. Solecki [SH10, SH12] in Ramsey theory to formalize dual Ramsey statements.
such as the Graham and Rothschild theorem [GR71]. S. Solecki had known for a fact that projective Fraïssé theory can be developed including dual predicates and he was the one who pointed out that dual structure might be needed for the proof of Proposition 3.1. An example in Section 4 shows that for this theorem to be true dual predicates are actually necessary. I would like to thank S. since many of the ideas included here grew out of the discussions we had. I would also like to thank Ola Kwiatkowska for sharing with me her insight on this problem, as well as Alex Kruckman and Gianluca Basso for their useful comments.

1. Preliminaries

In what follows, $K$ will always denote a zero-dimensional, compact, metrizable space. Our main objects of study will be spaces $K$ as above, which support both usual model-theoretic structure as well as dual structure. To make this precise, following S. Solecki [SH0, SH12], we consider the following two types of tuples.

- In the classical model-theoretic context, a tuple of size $n > 0$ in $K$ corresponds to an injection $i : \{0, \ldots, n-1\} \to K$. We will call this kind of tuple a direct tuple. We denote the set of all direct tuples in $K$ of size $n > 0$ by $K^{[n]}$.

- In the dual context, a tuple of size $n$ in $K$ corresponds to a surjection $e : K \to \{0, \ldots, n-1\}$. Since our intention is to work with topological structures, we endow $\{0, \ldots, n-1\}$ with the discrete topology and we impose a further regularity condition, that $e$ is also continuous. We will call this kind of tuple a dual tuple. We denote the set of all dual tuples in $K$ of size $n$ by $[n]^K$.

Notice that for $K$ zero-dimensional, compact, metrizable space, the set $[n]^K$ is at most countably infinite and moreover these functions suffice to separate points of $K$, i.e., for every $x_1, x_2 \in K$ there is an $e \in [n]^K$ such that $e(x_1) \neq e(x_2)$. Notice also that the set $[n]^K$ of all dual tuples naturally corresponds to the set $\text{CP}_n(K)$ of all clopen, ordered, $n$-partitions of $K$, i.e.,

$$\text{CP}_n(K) = \{ (\Delta_0, \ldots, \Delta_{n-1}) : \Delta_i \subset K \text{ clopen, } \Delta_i \cap \Delta_j = \emptyset \ \forall i \neq j, \cap \Delta_i = K \}.$$ 

Whenever it is convenient for notational purposes we will not distinguish between the set $\{\Delta_0, \ldots, \Delta_{n-1}\}$ and the tuple $(\Delta_0, \ldots, \Delta_{n-1})$. If for example $P \in \text{CP}_n(K)$ and $\Delta$ is a clopen set appearing some of the $n$ entries of $P$ then we will write $\Delta \in P$.

1.1. Topological $\mathcal{L}$-structures. We will work with relational only languages $\mathcal{L}$. The structures we describe here, with additional dual function symbols, were introduced in [SH0, SH12]. To each relational symbol $R$ in $\mathcal{L}$ corresponds some natural number $\text{arity}(R) > 0$ which is the arity of the symbol $R$. Moreover, for every symbol in $\mathcal{L}$ we have predetermined our intention to use it in the direct or in the dual
context. We will make the convention here of using lower case letters \( r, p, q, \ldots \) for direct relational symbols, and capital letters \( R, P, Q, \ldots \) for dual relational symbols. We will call the language \( \mathcal{L} \) purely direct if it contains only direct symbols and purely dual if it contains only dual symbols.

By a topological \( \mathcal{L} \)-structure \( K \) we mean a zero-dimensional, metrizable, compact space \( K \) together with appropriate interpretations for every symbol in \( \mathcal{L} \).

- If \( r \in \mathcal{L} \) is a direct relation symbol of arity \( n \), an appropriate interpretation for \( r \) is any closed subset \( r^K \) of \( K^n \).
- If \( R \in \mathcal{L} \) is a dual relation symbol of arity \( n \), an appropriate interpretation for \( R \) is any subset \( R^K \) of \( [n]^K \), or equivalently, any subset of \( \text{CP}_n(K) \).

We call a topological \( \mathcal{L} \)-structure purely direct \( \mathcal{L} \)-structure whenever \( \mathcal{L} \) is purely direct and purely dual \( \mathcal{L} \)-structure whenever \( \mathcal{L} \) is purely dual.

1.2. Morphisms. Following [ISS06], we will be working with epimorphisms. The epimorphisms here will additionally preserve the dual structure. Such epimorphisms were introduced in [S10, S12].

Let \( A, B \) be two dual topological \( \mathcal{L} \)-structure. By an epimorphism \( f \) from \( A \) to \( B \) we mean a continuous surjection \( f : A \to B \) such that:

- for every \( r \in \mathcal{L} \) of arity say \( m \) and every \( \beta \in B^m \) we have
  \[ \beta \in r^B \iff \exists \alpha \in r^A \quad \beta = f \circ \alpha \]
- for every \( R \in \mathcal{L} \) of arity say \( m \) and for every \( \beta \in [m]^B \) we have
  \[ \beta \in R^B \iff \beta \circ f \in R^A \]

An isomorphism between \( A \) and \( B \) is a bijective epimorphism and an automorphism of \( A \) is an isomorphism from \( A \) to \( A \).

Let \( K \) be a topological \( \mathcal{L} \)-structure, let \( A \) be a zero-dimensional, metrizable, compact space and let \( f : K \to A \) be a continuous surjection. Notice that there is a unique topological \( \mathcal{L} \)-structure \( A \) on domain \( A \) that renders \( f \) an epimorphism. We call this structure \( A \), the structure induced by the map \( f \).

Let \( f : K \to A, h : K \to B \) be epimorphisms. We say that \( f \) factors through \( h \) if there is an epimorphism \( f_h : B \to A \) such that \( f_h \circ h = f \).

**Lemma 1.** Let \( A, B \) and \( K \) be topological \( \mathcal{L} \)-structures with \( A, B \) finite. Let also \( f : K \to A \) and \( g : K \to B \) be epimorphisms. Then there is a finite topological \( \mathcal{L} \)-structure \( C \) and an epimorphism \( h : K \to C \) such that both \( f \) and \( g \) factor through \( h \).

**Proof.** Let \( C = (\Delta_0, \ldots, \Delta_{n-1}) \) a clopen partition of \( K \) whose every entry \( \Delta_i \) is a (nonempty) set of the form \( f^{-1}(a) \cap g^{-1}(b) \), where \( a \in A \) and \( b \in B \). Let \( h : K \to C \) be the inclusion map, i.e, \( h(x) = \Delta_i \) if and only if \( x \in \Delta_i \). This map is a continuous surjection, so, it induces a structure \( C \) on domain \( C \). It is immediate now that both \( f \) and \( g \) factor through \( h \). \( \square \)
Given a sequence $A_1, A_2, \ldots, A_i, \ldots$ of finite topological $L$-structures together with epimorphisms $\pi_i : A_{i+1} \to A_i$, we can define a new structure $M$ and epimorphisms $\pi_i^\infty : M \to A_i$ through an inverse limit construction. Let

$$M = \{(a_1, a_2, \ldots) \in \prod_{i \in \mathbb{N}} A_i : \forall i \geq 1 \, \pi_i(a_{i+1}) = a_i\}. $$

$M$ is a closed subset of the compact space $\prod_{i \in \mathbb{N}} A_i$ and it will serve as the domain of $M$. We define $\pi_i^\infty$ to be the projection map from $M$ to $A_i$.

For $r \in L$ of arity say $m$, and $\beta \in M^m$ we let $\beta \in r^M$ if and only if $\pi_i^\infty \circ \beta \in r^{A_i}$ for all $i \in \mathbb{N}$. For $R \in L$ of arity say $m$, and $\gamma \in [m]^M$, notice that there is an $i_0 \in \mathbb{N}$ such that $\gamma$ factors through $\pi_i^\infty$. Let $\alpha \in [m]^A$ be such that $\gamma = \alpha \circ \pi_i^\infty$. We let $\gamma \in R^M$ if and only if $\alpha \in R^{A_{i_0}}$, which happens if and only if for every $i > i_0$ we have $(\alpha \circ \pi_{i_0} \circ \ldots \circ \pi_{i-1}) \in R^A$.

This turns $M$ into a topological $L$-structure and every $\pi_i^\infty$ to an epimorphism. We call $M$ the inverse limit of the inverse system $\{(A_i, \pi_i) : i \in \mathbb{N}\}$ and we write $M = \varprojlim (A_i, \pi_i)$.

2. Projective Fraïssé structures

In Chapter 7 of [Hod93], Hodges reviews the theory of Fraïssé limits of direct structures via direct morphisms (embeddings). Following Hodges and [ISS06] we present here the theory of Fraïssé limits of topological $L$-structures via dual morphisms. To avoid confusion, we should emphasize two things. First, what in [ISS06] is called topological $L$-structure, here it falls under the name purely direct topological $L$-structure. Secondly, in contrast with the definition that we will be using here, in [ISS06] a projective Fraïssé class is not bound to satisfy the hereditary property (HP).

We say that a topological $L$-structure $M$ is projectively Fraïssé or projectively ultra-homogeneous if for every two epimorphisms $f_1, f_2$ of $M$ on some finite topological $L$-structure $A$ there is an automorphism $g$ of $M$ such that $f_1 \circ g = f_2$.

For every topological $L$-structure $M$ we denote by $\text{Age}(M)$ the class of all the finite topological $L$-structures $A$ such that $M$ epimorphs on $A$. We call a class $K$ of topological $L$-structures an age if $K = \text{Age}(M)$ for some topological $L$-structure $M$. It is immediate that if $K$ is an age, then $K$ is not empty, any subclass of $K$ of pairwise non-isomorphic structures is at most countable, and the following two properties hold for $K$.

- **Hereditary Property (HP):** if $A \in K$ and $A$ epimorphs onto a structure $B$, then $B \in K$.
- **Joint Surjecting Property (JSP):** if $A, B \in K$ then there is $C \in K$ that epimorphs onto both $A$ and $B$.

The converse is also true i.e. if $K$ is a non empty class of finite topological $L$-structures such that any subclass of $K$ of pairwise non-isomorphic structures is at most countable and the above two properties hold for $K$ then $K$ is an age.
To see this, let $A_1, A_2, \ldots$ be a list of structures in $\mathcal{K}$ that up to isomorphism exhaust $\mathcal{K}$. Using the JSP we can find a new list $B_1, B_2, \ldots$ of structures in $\mathcal{K}$ such that $B_1 = A_1$ and for $i > 1$, $B_{i+1}$ epimorphs on both $A_i$ and $B_i$. Let $\pi_i : B_{i+1} \to B_i$ be such epimorphisms and let $M = \varprojlim (B_i, \pi_i)$. Then $\mathcal{K} = \text{Age}(M)$ because by construction $M$ epimorphs to every $A_i$ and moreover, every epimorphism of $M$ to some finite dual topological $\mathcal{L}$-structure $A$ factors through an epimorphism from some $B_i \in \mathcal{K}$ that was used in the inverse system so, by HP we have that $A \in \mathcal{K}$.

Given now that the structure $M$ is projectively ultrahomogeneous, it is easy to see that its age $\mathcal{K}$ satisfies moreover the following property.

- Projective Amalgamation Property (PAP): if $A, B, C \in \mathcal{K}$ and $f_A : A \to C$, $f_B : B \to C$ are epimorphisms, then there is $D \in \mathcal{K}$ and epimorphisms $g_A : D \to A$, $g_B : D \to B$ such that $f_A \circ g_A = f_B \circ g_B$.

To check that this is true, notice first that since $A, B \in \mathcal{K}$, there are epimorphisms $h_A : M \to A$ and $h_B : M \to B$. But then $f_A \circ h_A$ and $f_B \circ h_B$ are both epimorphisms from $M$ to $C$. So, by projective ultra-homogeneity of $M$ there is $\phi \in \text{Aut}(M)$ such that $f_A \circ h_A \circ \phi = f_B \circ h_B$. Using Lemma 1 we can find $D \in \mathcal{K}$ and an epimorphism $h_D : M \to D$ such that $h_A$ and $h_B$ are both maps that close these diagrams, i.e., $g_A \circ h_D = h_A$ and $g_B \circ h_D = h_B \circ \phi^{-1}$. The functions $g_A$ and $g_B$ are the required epimorphisms from $D$ to $A$ and $B$ in respect.

In Theorem 2.1 we will see that the converse is also true, i.e., if an age $\mathcal{K}$ has PAP then we can built from it a projective Fraïssé structure $M$ with $\text{Age}(M) = \mathcal{K}$. An age $\mathcal{K}$ that satisfies PAP is called projective Fraïssé class.

Let $M$ be a topological $\mathcal{L}$-structure with $\text{Age}(M) = \mathcal{K}$. We say that $M$ has the finite extension property if for every $A, B \in \mathcal{K}$ and $f : B \to A$, $g : M \to A$ epimorphisms, there is an epimorphism $h : M \to B$ such that $f \circ h = g$. We say that $M$ has the one point extension property if the above holds when the size of $B$ is one more than the size of $A$. Notice that for any topological $\mathcal{L}$-structure $M$, $M$ has the one point extension property if and only if $M$ has the finite extension property.

**Lemma 2.** Let $M$ and $N$ be two topological $\mathcal{L}$-structure of the same age $\mathcal{K}$. Let $A \in \mathcal{K}$ and let $f : M \to A$ and $g : N \to A$ be two epimorphisms. If $M$ and $N$ have the finite extension property then there is an isomorphism $h : M \to N$ such that $g \circ h = f$.

**Proof.** We will use a back and forth type of argument. For every $n \in \mathbb{N}$ we will construct $A_n \in \mathcal{K}$ and epimorphisms $f_n : M \to A_n$, $g_n : N \to A_n$, and for every $n > 0$ we will also construct an epimorphism $\pi_{n-1} : A_n \to A_{n-1}$. At the end of the construction $M$ and $N$ will be proven to be isomorphic to $\varprojlim (A_n, \pi_n)$. By using these indirect isomorphisms we will get the desired isomorphism $h$. Let $\{e_n : n \in \mathbb{N}\}$
be an enumeration of dual tuples $[m]^M$ of $M$ for every $m > 0$ and let $\{e'_n : n \in \mathbb{N}\}$ be an enumeration of dual tuples $[m]^N$ of $N$ for every $m > 0$.

$n = 0$. Let $A_0 = A$, $f_0 = f$ and $g_0 = g$.

odd $n > 0$. Using Lemma 1 we can find a structure $A_n$ and an epimorphism $f_n : M \to A_n$ such that both $f_{n-1}$ and $e_{n-1}$ factor through $f_n$. Let $\pi_{n-1} : A_n \to A_{n-1}$ be the epimorphism that closes the one diagram, i.e., $\pi_{n-1} \circ f_n = f_{n-1}$. Finally define $g_n : N \to A_n$ to be any map such that $\pi_{n-1} \circ g_n = g_{n-1}$. A map like this exists, since $N$ satisfies the finite extension property.

even $n > 0$. Again, using Lemma 1 we can find a structure $A_n$ and an epimorphism $g_n : N \to A_n$ such that both $g_{n-1}$ and $e'_{n-1}$ factor through $g_n$. Let $\pi_{n-1} : A_n \to A_{n-1}$ be the epimorphism that closes the one diagram, i.e., $\pi_{n-1} \circ g_n = g_{n-1}$. Finally define $f_n : M \to A_n$ to be any map such that $\pi_{n-1} \circ f_n = f_{n-1}$. A map like this exists, since $M$ satisfies the finite extension property.

Let now $B = \lim_{\to_{\subseteq}}(A_n, \pi_n)$. The maps $\mu : M \to B$ with $\mu(x) = (f_0(x), f_1(x), \ldots)$ and $\nu : N \to B$ with $\nu(x) = (g_0(x), g_1(x), \ldots)$ are bijections since the families $\{e_n\}$ and $\{e'_n\}$ separate points of $M$ and $N$ in respect. It is moreover easy to see that $\mu$ and $\nu$ are actually isomorphisms. So, the map $h : M \to N$ with $h = \nu^{-1} \circ \mu$ is also an isomorphism which by construction satisfies the desired property $g \circ h = f$.

Lemma 3. Let $M$ be a topological $\mathcal{L}$-structure with $\text{Age}(M) = \mathcal{K}$ then the following are equivalent:

1. $M$ is projectively ultrahomogeneous;
2. $M$ has the finite extension property;
3. $M$ has the one point extension property.

Proof. It is immediate that (2) and (3) are equivalent. We prove that (1) is also equivalent to (2).

(1) $\to$ (2) Let $A, B \in \mathcal{K}$ and $f : B \to A$, $g : M \to A$ epimorphisms. Since $B \in \mathcal{K} = \text{Age}(M)$, there is an epimorphism $j : M \to B$. So, $f \circ j : M \to A$ is an epimorphism, and by the projective ultra-homogeneity of $M$ there is $\phi \in \text{Aut}(M)$ with $g \circ \phi = f \circ j$. Let $h = j \circ \phi^{-1}$. Then $h : M \to B$ is an epimorphism such that $f \circ h = g$.

(2) $\to$ (1) Let $f_1, f_2 : M \to A$ be epimorphisms for some $A \in \mathcal{K}$. Then by Lemma 2 there is $g \in \text{Aut}(M)$ such that $f_1 \circ g = f_2$.

Theorem 2.1. For every projective Fraïssé class $\mathcal{K}$ there is a unique, up to isomorphism, projectively ultra-homogeneous topological $\mathcal{L}$-structure $M$ such that $\text{Age}(M) = \mathcal{K}$.

Proof. First notice that is $M_1, M_2$ share the same age and are both projectively ultra-homogeneous, by Lemma 3 they have finite extension property. Let $A$ any structure in $\mathcal{K}$. Since $\mathcal{K}$ is the age of both $M_1, M_2$, there are epimorphisms $f_1 : M_1 \to A$ and $f_2 : M_2 \to A$. Lemma 2 gives as then an isomorphism $h$ between $M_1$ and $M_2$.
3. Closed subgroups of Homeo($K$)

By definition and since $K$ is compact, every automorphism of a topological $\mathcal{L}$-structure $K$ is also a homeomorphism, therefore, Aut($K$) can be seen as a subgroup of Homeo($K$). We will view Homeo($K$) as a topological group equipped with the compact-open topology $\tau_{co}$. The collection of the sets

$$V(F, U) = \{ g \in \text{Homeo}(K) : g(F) \subset U \},$$

where $F$ is a compact subset of $K$ and $U$ is an open subset of $K$, provide a subbase for $\tau_{co}$. In this topology the group Aut($K$) of automorphisms of a dual topological $\mathcal{L}$-structure $K$ is a closed subgroup of Homeo($K$). To check this, let $g \not\in \text{Aut}(K)$. We will find an open neighborhood $V_g$ of $g$ in Homeo($K$) which does not intersect Aut($K$). Since $g \not\in \text{Aut}(K)$, one of the following holds:

1. there is $R \in \mathcal{L}$ of arity say $m$ and a dual tuple $e \in [m]^K$ such that $K \models R(e)$ if and only if $K \not\models R(e \circ g^{-1})$, or
2. there is $r \in \mathcal{L}$ of arity say $m$ and a tuple $i \in K^{[m]}$ such that $K \models r(i)$ but $M \not\models r(g \circ i)$, or
3. there is $r \in \mathcal{L}$ of arity say $m$ and a tuple $i \in K^{[m]}$ such that $K \not\models r(i)$ but $M \models r(g \circ i)$.

In the first case notice that if $g \not\in \text{Aut}(K)$ then there is $R \in \mathcal{L}$ of arity say $m$ and But then $V(e^{-1}(0), g \circ e^{-1}(0)) \cap \ldots \cap V(e^{-1}(m - 1), g \circ e^{-1}(m - 1))$ is an open subset of Homeo($K$) containing $g$ and lying entirely out of Aut($K$).

In the second case, because $r^M$ is closed, we can find an open rectangle $U_0 \times \ldots \times U_{m-1}$ around $((g \circ i)(0), \ldots, (g \circ i)(m - 1))$ which does not intersect $r^M$. Therefore, let $V_g = V(\{i(0)\}, U_0) \cap \ldots \cap V(\{i(m - 1)\}, U_{m-1})$.

For the last case, notice that if we let $(b_0, \ldots, b_{m-1}) = ((g \circ i)(0), \ldots, (g \circ i)(m - 1))$, then, as in the previous case we can find open neighborhood $V_{g^{-1}}$ of $g^{-1}$ such that for every $f \in V_{g^{-1}}$, $f(b_0, \ldots, b_{m-1}) \not\in r^M$. Let then $V_g = V_{g^{-1}} = \{f^{-1} : f \in V_{g^{-1}}\}$.

Using the continuity of the inversion operator $f \rightarrow f^{-1}$ in $\tau_{co}$ we have that $V_g$ is open and moreover $g \in V_g \subset \text{Aut}(K)^c$.

The following proposition says that the inverse of the above observation is true, i.e., for every closed subgroup $G$ of Homeo($K$) there is topological $\mathcal{L}$-structure $K$ on $K$ such that $G = \text{Aut}(K)$. Moreover, $K$ can be taken to be purely dual and projectively ultra-homogeneous.

**Proposition 3.1.** Let $G$ be a closed subgroup of Homeo($K$). Then there is a purely dual projective Fraïssé structure $K$ on domain $K$ such that Aut($K$) = $G$.

**Proof.** For every $n > 0$, the group $G$ acts on $[n]^K$ in a natural way: for $g \in G$ and $e \in [n]^K$ let

$$g \cdot e := e \circ g^{-1}.$$

We denote this action by $G \actson [n]^K$. Notice that this action corresponds to the following action $G \actson \text{CP}_n$ of $G$ on $\text{CP}_n$: for $g \in G$ and $P = (\Delta_0, \ldots, \Delta_{n-1}) \in \text{CP}_n$.
For each $n > 0$ let $(C^n_i : i \in I_n)$ be the collection of all orbits of $G \acts [n]^K$.

Consider now the language $\mathcal{L} = \bigcup_{n=1}^\infty L^n$, where $L^n$ in the language that consists of $n$-ary relational symbols $\{O^n_i : i \in I_n\}$, one for every orbit $O^n_i$. We turn $K$ into a topological $\mathcal{L}$-structure $K$. For $e \in [m]^K$ we let

$$K \models O^n_i(e) \text{ if and only if } e \in O^n_i.$$ 

It is immediate that $G \subseteq \text{Aut}(K)$. We work now towards the converse inclusion.

Let $g \in \text{Aut}(K)$ and let $V(F, U)$ be an open neighborhood of $g$ in $\text{Homeo}(K)$. We can assume that $U \neq K$. We will find $h \in G \cap V(F, U)$ which will prove that $G \supseteq \text{Aut}(K)$. Because $g(F)$ is compact and $U$ is a union of clopen sets, $g(F)$ can be covered with finitely many of them, so we can assume without loss of generality that $U$ is clopen and $U \neq K$. Notice that $g \in V(g^{-1}(U), U) \subset V(F, U)$. Consider the following two dual tuples $e_1, e_2 \in 2^K$, with $e_1^{-1}(\{0\}) = g^{-1}(U)$, $e_1^{-1}(\{1\}) = K \setminus g^{-1}(U)$ and $e_2^{-1}(\{0\}) = U$, $e_2^{-1}(\{1\}) = K \setminus U$. Since $g$ is an automorphism of $K$ and since $e_1 = g \cdot e_2$, we have that $e_1$ and $e_2$ lie in the same orbit $O^n_i$ for some $i \in I_2$. Therefore, there is an $h \in G$ that sends $g^{-1}(U)$ into $U$ and therefore $h \in G \cap V(F, U)$, which proves that $G = \text{Aut}(K)$.

We prove now that $K$ is projectively Fraïssé. First notice that for every dual tuple $e \in [m]^K$, there is a unique $i \in I_m$ such that $K \models O^n_i(e)$. Let $C \in K$ and let $f_1, f_2$ be two epimorphisms of $K$ onto $C$. We can assume without the loss of generality that $C = \{0, \ldots, m - 1\}$ for some $m > 0$ and therefore $f_1, f_2 \in [m]^K$. Because $f_1$ and $f_2$ induce the same structure $C$, there is a unique $i \in I_m$ such that $K \models O^n_i(f_1)$ and $K \models O^n_i(f_2)$. Therefore, $f_1$ and $f_2$ lie in the same orbit $G \acts [m]^K$, so there is $g \in \text{Aut}(K)$ such that $f_1 \circ g = f_2$, showing that $K$ is projectively ultra-homogeneous. 

4. TURNING A STRUCTURE TO A PURELY DUAL ONE

Here we show that it is always possible to translate the direct structure into a dual one without losing any information. We provide a counterexample to show that the converse is not always possible. Although purely dual structures are sufficient for the development of the general theory, in Section 5 it will be convenient to make use of direct relations. Moreover, there are many examples of structures whose most natural presentation would involve both direct and dual structure.

Let $\mathcal{L}$ be a language and $M$ a topological $\mathcal{L}$-structure. Let also $s \in \mathcal{L}$ be a direct relation of arity $n$. For every $k$ with $0 < k \leq n$ and for every $f \in [k]^n$ (if $f$ is therefore a surjection), we introduce a dual relational symbol $R^f_s$ of arity $k + 1$. Let

$$\mathcal{L}_s = \mathcal{L} \cup \{R^f_s : f \in [k]^n \text{ for some } 0 < k \leq n\} \setminus \{s\}.$$ 

We turn $M$ into an $\mathcal{L}_s$-structure $M_s$ on the same domain $M$. We encode $s^M$ using the new dual symbols as follows: for $f \in [k]^n$ we let $M_s \models R^f_s(\Delta_0, \ldots, \Delta_k)$, if
and only if there are $a_0, \ldots, a_{n-1} \in M$ such that

$$M \models s(a_0, \ldots, a_{n-1}) \text{ and } a_i \in \Delta f(i) \text{ for every } i \in n.$$ 

It can easily be checked that Aut($M$) can be fully recovered from Aut($M_\ell$), that $M$ is projectively Fraïssé if and only if $M_\ell$ is, and that Aut($M$) and Aut($M_\ell$) are equal as permutation groups on $M$.

There are cases of topological $\mathcal{L}$-structures which can be turned into purely direct structures. However, this is not the case always. The main observation is that if $r$ is direct relation of arity $k$ which belongs to $\mathcal{L}$ and $M$ is a topological $\mathcal{L}$-structure then $r^M$ is a set-wise invariant closed subset of $M^k$. Let now $K = 2^\mathbb{N}$ and let $\mu$ be the uniform probability measure on $2^\mathbb{N}$. The group Aut($K, \mu$) of all continuous measure preserving bijections can be easily seen to be a closed proper subgroup of Homeo($K$) which for every $n > 0$ leaves no proper subset of $K^{[n]}$ invariant. Therefore, the canonical Fraïssé structure given by an application of Proposition 3.1 on Aut($K, \mu$) cannot be turned into a purely direct one.

5. Compact Polish spaces as quotients of dual Fraïssé structures

We fix a special binary relation symbol $r$ whose interpretation will always be a reflexive and symmetric closed relation. A formal relational language $\mathcal{L}$ will be decorated with the subscript $r$ whenever $r \in \mathcal{L}$. Therefore, an $\mathcal{L}_r$-structure is always going to be a reflexive $r$-graph perhaps with some extra structure. We say that an $\mathcal{L}_r$-structure $K$ is a pre-space if $r^K$ is moreover transitive and therefore an equivalence relation.

As we noted in the introduction, T. Irwin and S. Solecki used purely direct Fraïssé theory to express the pseudo-arc $P$ as a quotient of a projective Fraïssé $\{r\}$-structure $\mathbb{P}$ via $r^\mathbb{P}$. Moreover, through their construction, the group Aut($\mathbb{P}$) naturally embedded in Homeo($P$) as a dense subgroup. In [RC10], R. Camerlo characterized all different projective Fraïssé classes of $\{r\}$-structures. Their limits are pre-spaces with quotients $M/r^M$ which vary between certain combinations of singletons, Cantor spaces and pseudo-arcs [RC10]. In [BK13], D. Bartošová and A. Kwiatkowska express the Lelek fan $L$ as the quotient of the projective Fraïssé limit $\mathbb{L}$ of a certain class of directed graphs. Their limit $\mathbb{L}$ can be seen again as pre-space in some $\mathcal{L}_r$. Here again the group Aut($\mathbb{L}$) naturally embedded in Homeo($L$) as a dense subgroup.

In this section we show that under the notion of projective Fraïssé limit we developed here the same representation applies to every second-countable compact space $Y$. Since this is trivial for finite spaces, we will restrict ourselves to the case where $Y$ is infinite.

The following proposition will be used in the proof of Theorem 5.2.

**Proposition 5.1.** Let $G, H$ be Polish groups and let $S$ be a dense subgroup of $G$. Then any continuous homomorphism $f : S \to H$ extends to a continuous homomorphism $\tilde{f} : G \to H$.

---

1He allows Fraïssé classes to lack hereditary property.
The proof of Proposition 5.1 is an easy exercise given that every Polish admits a compatible left-invariant metric $d$ and given this metric we can define a new compatible complete metric $D$ by $D(x, y) = d(x, y) + d(x^{-1}, y^{-1})$. For more details, see page 6 of [BK96].

**Theorem 5.2.** Let $G$ be a closed subgroup of $\text{Homeo}(Y)$, for some compact metrizable space $Y$. Then there is a projective Fraïssé pre-space $K$ such that $K/r^K$ is homeomorphic to $Y$, and the quotient projection $K \mapsto Y$ induces a continuous group embedding $\text{Aut}(K) \rightarrow G$, with dense image in $G$.

**Proof.** Let $Y$ be an infinite compact Polish space and let $H$ be a countable, dense subgroup of $G$. In what follows, we define a countable Boolean algebra $(F, 0_F, 1_F, \wedge, \vee, ')$ of closed subsets of $Y$ as well as an action of $H$ on $F$ via Boolean algebra automorphisms. Every set $F \in F$ will be regular closed. Recall that an open set $U$ is called regular open if $\text{int}(U) = U$ and a closed set $F$ is called regular closed if $\text{int}(F) = F$.

We define $0_F$, $1_F$ and the operations $\wedge$, $\vee$, $'$ as follows:

- $0_F = \emptyset$;
- $1_F = Y$;
- $F_1 \wedge F_2 = \text{int}(F_1 \cap F_2)$;
- $F_1 \vee F_2 = F_1 \cup F_2$;
- $F' = F^c$.

The boolean algebra axioms are satisfied by the above configuration since $F$ consists of regular closed sets (see also [Hal63] for the boolean algebra of regular open sets).

To construct the boolean algebra fix first a compatible complete metric $d$ on $Y$. For every $n$ chose a finite open cover $\{V_0^n, \ldots, V_n^n\}$ of $Y$ such $\text{diam}(V_i^n) < 1/n$ for every $i \in \{0, \ldots, k_n\}$. Since $\overline{V}$ is regular closed for every open $V$ we have that the collection $J = \{F_i^n : F_i^n = \overline{V_i^n}, n \in \mathbb{N}, 0 \leq i \leq k_n\}$ consists of regular closed sets. We define $F$ to be the least family of closed subsets of $Y$ such that:

1. $J \subset F$;
2. $F$ is closed under the boolean operators $\wedge, \vee, ' $ and
3. $F$ is closed under translation by elements of $H$, i.e., if $h \in H$ and $F \in F$ then $h(F) \in F$.

Notice that all these operations preserve regularity and since $J$ and $H$ are countable $F$ is a countable family of regular closed sets. Notice that this implies that the only $F \in F$ that has empty interior is the empty set. The group $H$ is acting on $F$ with Boolean algebra automorphisms: for every $h \in H$ and $F \in F$ let

$$h \cdot F = h(F).$$

Let $K = S(F)$ be the Stone space of all ultrafilters $x$ on $F$. This space comes with a topology whose basic clopen sets can be taken to be the sets of the form
Let \( \bar{F} = \{ x : F \in x \} \) for \( F \in \mathcal{F} \). The space \( K \) is a compact, second-countable, and zero-dimensional. Let \( p : K \to Y \) be the natural projection defined by:

\[
\{ p(x) \} = \bigcap_{F \in x} F.
\]

The map \( p \) is continuous surjection with \( p(\bar{F}) = F \) for every \( F \in \mathcal{F} \). We can turn now \( K \) to a \( \{ \tau \} \)-structure \( K_\tau \) by setting \( K_\tau \models \tau(x_0, x_1) \) if and only if \( p(x_0) = p(x_1) \).

Notice that \( \hat{h} \) is a continuous surjection from \( K_\tau \) to a \( \{ \tau \} \)-structure \( K_\tau \) by \( p \) since \( K_\tau \models \tau(x_0, x_1) \) if and only if \( p(x_0) = p(x_1) \).

The map \( \hat{h} \) is also continuous. To see that, let \( h \in H_K \) and let \( V(L, U) \) be an open neighborhood of \( T_0(h)(L) \) in \( \text{Homeo}(Y) \), i.e., \( T_0(h)(L) \subset U \). Since the family \( \{ \text{int}(F) : F \in J \} \) constitutes a basis of \( Y \) and since \( T_0(h)(L) \) is compact, we can find \( F_1, \ldots, F_k \in \mathcal{F} \) such that \( T_0(h)(L) \subseteq F_1 \cup \ldots \cup F_k \subseteq U \). Let \( F_0 = F_1 \cup \ldots \cup F_k \), then both \( F_0 \) and \( h^{-1}(F_0) \) belong to \( \mathcal{F} \). Moreover, \( V(h^{-1}(F_0), F_0) \) is an open neighborhood of \( h \) in \( H_K \) that is mapped via \( T_0 \) completely inside \( V(L, U) \), proving that \( T_0 \) is continuous at \( h \).

By applying the Proposition 3.1, we can endow \( K \) with a topological Fraïssé structure \( K_0 \) in a purely dual language \( \mathcal{L}_\tau \) such that \( \bar{H}_K = \text{Aut}(K_0) \) (the closure here is taken in \( \text{Homeo}(K) \)). By Proposition 5.1 the map \( T_0 \) extends to a continuous homomorphism \( T : \text{Aut}(K_0) \to G \). We denote the image of \( \text{Aut}(K_0) \) under \( T \) by \( \hat{H} \). Notice that \( \hat{H} \) lies densely in \( \text{Homeo}(K) \) since \( H < \hat{H} \leq \text{Homeo}(K) \), and since the same is true for \( H \). Moreover, by the density of \( H_K \) in \( \bar{H}_K \) the continuity of \( T \) and the fact that every \( F \in \mathcal{F} \) has non-empty interior we get that for every \( h \in \bar{H}_K \) and for every \( F \in \mathcal{F} \) the following equality holds

\[
T(h)(F) = h(\hat{F}). \tag{1}
\]

We combine now the structures \( K_0 \) and \( K_\tau \) into one \( \mathcal{L}_\tau \)-structure \( K \) on domain \( K \), where \( \mathcal{L}_\tau = \mathcal{L} \cup \{ \tau \} \). Notice that \( \tau \) is invariant under \( \text{Aut}(K_0) \) since \( (x_0, x_1) \in \tau \) if and only if for all \( F_0, F_1 \in \mathcal{F} \) with \( x_0 \in \bar{F}_0 \) and \( x_1 \in \bar{F}_1 \) we have that \( F_0 \cap F_1 \neq \emptyset \). Thus \( \bar{H}_K = \text{Aut}(K_0) \) if and only if every \( A_0 \in \text{Age}(K_0) \) uniquely extends to an \( A \in \text{Age}(K) \) and \( K \) is a also a projective Fraïssé structure. The fact that \( p(\bar{F}) = F \) for every \( F \in \mathcal{F} \) and the relation \( \circledast \) above let us view \( T : \text{Aut}(K) \to G \) as the homomorphism induced by the quotient \( p : K \to Y \).
We are left to show that $T$ is injective. Let $h \in \text{Aut}(M)$ so that $h \neq \text{id}_{\text{Aut}(M)}$. By the continuity of $h$ we can find a non-empty $F$ in $\mathcal{F}$ so that $F \cap h(F) = \emptyset$. Therefore, the interiors in $Y$ of $p(F)$ and $p(h(F))$ do not intersect and because the interior in $Y$ of every non-empty $F$ in $\mathcal{F}$ is non-empty we have that $T(h) \neq \text{id}_{\text{Homeo}(Y)}$. □

We should remark here that the image $\hat{H}$ of $\text{Aut}(K)$ under $T$ is in general a meager subset of $G$. This can be seen as follows: first notice that as a corollary of Pettis theorem we have that if $f : B \to D$ is a Baire-measurable homomorphism between Polish groups and $f(B)$ is not meager, then $f$ is open (see for example Theorem 1.2.6 [BK96]). Now notice that for $F \in \mathcal{F}$ the set $V(\hat{F}, \hat{F})$ is open in $\text{Homeo}(K)$ but the set $V(F, F)$ is rarely open in $G$ (except if $Y$ is zero-dimensional or if $G$ contains very few homeomorphisms). Therefore $T$ will fail in general to be an open map.

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