EXISTENCE OF SOLUTION FOR KIRCHHOFF TYPE PROBLEM
IN ORLICZ-SOBOLEV SPACES VIA LERAY-SCHAUDER’S
NONLINEAR ALTERNATIVE

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Abstract. In this paper, we establish the existence of weak solution in Orlicz-
Sobolev space for the following Kirchhoff type problem

\[
\begin{align*}
- M \left( \int_\Omega \Phi(\nabla u) \, dx \right) \text{div}(a(\nabla u) \nabla u) &= f(x, u) \quad \text{in} \quad \Omega, \\
u &= 0 \quad \text{on} \quad \partial \Omega,
\end{align*}
\]

where \( \Omega \) is a bounded subset in \( \mathbb{R}^N \), \( N \geq 1 \) with Lipschitz boundary \( \partial \Omega \).

The used technical approach is mainly based on Leray-Schauder’s non linear
alternative.

1. Introduction. The study of nonlinear elliptic problems involving quasilinear
homogeneous type operators is based on the theory of Sobolev spaces in order to find
weak solutions. In certain equations, in particular in the case of nonhomogeneous
differential operators, when trying to impose certain conditions on these operators
(like growth conditions), the problem cannot be studied with the classical Lebesgue
and Sobolev spaces. Then, the adequate functional spaces are what we call Orlicz
spaces. As an example of application in the field of image processing, let us consider
the restoration model with the following additive noise:

\[
f = u + \text{bruit}.
\]

The energy function associated with the equation (1.1) is then:

\[
J(u) := \int_\Omega (f - u)^2 \, dx,
\]

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with \( \Omega \) a bounded open set of \( \mathbb{R}^2 \). As this is a poorly conditioned problem, then we need to add an energy regularization term, which is usually written in the form:

\[
\int_{\Omega} \Phi(|\nabla u|) \, dx,
\]

this term represents diffusion. The energy function then takes the following form:

\[
J(u) := \frac{\lambda}{2} \int_{\Omega} (f - u)^2 \, dx + \int_{\Omega} \Phi(|\nabla u|) \, dx.
\]

The problems of minimization with sublinear regularization terms of this kind are in general badly posed in spaces with bounded variation (BV). On the other hand, thanks to the Orlicz-Sobolev spaces, this model of restoration, is not only well-posed but also the existence and the uniqueness of a minimum are guaranteed (see [29, Theorem 3.1.1]).

These spaces consist of functions satisfying certain integrability conditions and having weak derivatives. Many properties of Orlicz-Sobolev spaces come from [2, 17, 24] and references therein. The interest in studying these kinds of problems is motivated by some recent advances in the study of eigenvalue problems involving non-homogeneous operators in the divergence form. This type of problem arises in many different applications, such as, elasticity (see [31]), fluid dynamics (see [22, 28]), calculus of variations and differential equations with nonstandard growth (see [27, 28]).

In this paper, we are interested to study the existence of a weak solutions of the following problem

\[
\begin{cases}
-M \left( \int_{\Omega} \Phi(|\nabla u|) \, dx \right) \text{div}(a(|\nabla u|) \nabla u) = f(x, u) \text{ in } \Omega, \\
u = 0 \text{ on } \partial \Omega,
\end{cases}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^N \) \((N \geq 1)\) with smooth boundary \( \partial \Omega \), \( f : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \) is a Carathéodory function satisfying some appropriate conditions, \( M : \mathbb{R}^+ \rightarrow \mathbb{R} \) is a nondecreasing continuous function satisfying the following condition:

\[
\text{there exists } m_0 > 0 \text{ such that } M(t) \geq m_0 \text{ for all } t \geq 0, \quad (M_0)
\]

and \( a, \Phi : \mathbb{R} \rightarrow \mathbb{R} \) which will be specified later.

The problem \((P)\) is related to the stationary version of the Kirchhoff equation

\[
\rho \frac{\partial^2 u}{\partial t^2} = \left( \frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 \, dx \right) \frac{\partial^2 u}{\partial x^2},
\]

presented by Kirchhoff [23] in 1883, is an extension of the classical d’Alembert’s wave equation by considering the changes in the length of the string during vibrations. In \((1.3)\), \( L \) is the length of string, \( h \) is the area of the cross section, \( E \) is the Young modulus of the material, \( \rho \) is the mass density, and \( P_0 \) is the initial tension. The Kirchhoff’s model takes into account the length changes of the string produced by transverse vibrations. Some interesting results can be found, for example in [10]. On the other hand, Kirchhoff-type boundary value problems model several physical and biological systems where \( u \) describes a process which depends on the average of itself, as for example, the population density. We refer the reader to [3, 21] for some related works. In recent years, some problems involving \( p \)-Kirchhoff type operators
have been studied in many papers, we refer to [7, 5, 6, 4, 14, 26], in which the authors use different methods to establish the existence of solutions.

On the other hand, many authors have studied some non-linear problems in modular spaces, by using the Leray-Shauder’s non linear alternative. Recently, for $M = 1$ and $a(t) = t^{p-2}$, the problem (P) reduces to the p-Laplacian problem

$$
\begin{cases}
-\text{div}(|\nabla u|^{p-2}\nabla u) = f(x, u) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
$$

(1.4)

In [16], Dinca et al have assumed that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function that satisfies the growth condition

$$|f(x, t)| \leq c|t|^q - 1 + a(x) \text{ a.e in } \Omega \text{ all } t \in \mathbb{R},$$

where $c$ is a positive constant, $1 < q < p^*$ and $a \in L^{q'}(\Omega)$. By using Schauders fixed point theorem, they studied the existence of solution to the problem (1.4).

In [16], Dinca proposed a generalization of (1.4) by adapting the fundamental properties of the Leary-Schauder degree used in proving the existence of a week solution in $W^{1, p}(\cdot)$ to Dirichlet problem:

$$
\begin{cases}
-\text{div}(|\nabla u|^{p(x)-2}\nabla u) = f(x, u) & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega
\end{cases}
$$

(1.5)

such that, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Caratheodory function that satisfies the growth condition

$$|f(x, t)| \leq c|t|^q - 1 + a(x) \text{ a.e in } \Omega \text{ all } t \in \mathbb{R},$$

where $c$ is a positive constant, $a \in L^{q'}(\Omega)$, $a(x) \geq 0$ for a.e. $x \in \Omega$ and $p, q : \overline{\Omega} \to (0, +\infty)$ are continuous functions such that

$$1 < p^- := \inf_{x \in \Omega} p(x) \leq p^+ := \sup_{x \in \overline{\Omega}} p(x) < \infty,$$

$$1 < q^- := \inf_{x \in \Omega} q(x) \leq q^+ := \sup_{x \in \overline{\Omega}} q(x) < \infty,$$

and

$$\frac{1}{q(x)} + \frac{1}{q'(x)} = 1 \forall x \in \Omega.$$

In [8], Srati et al have proposed to study the fractional version of the Problem (P) with $M = 1$, that is, they have studied the following nonlocal problem

$$
\begin{cases}
(-\Delta)^s_{a(.)} u = f(x, u) & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
$$

(1.6)

where $\Omega$ is a bounded open subset of $\mathbb{R}^N$, $N \geq 1$, with Lipschitz boundary $\partial \Omega$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying some growth conditions, and $(-\Delta)^s_{a(.)}$ is the fractional $a(.)$-Laplacian operator of elliptic type which is defined as

$$
(-\Delta)^s_{a(.)} u(x) = 2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_{\varepsilon}(x)} a \left( \frac{|u(x) - u(y)|}{|x-y|^s} \right) \frac{u(x) - u(y)}{|x-y|^s} \frac{dy}{|x-y|^{N+s}},
$$

for all $x \in \mathbb{R}^N$. By Leray-Schauder’s nonlinear alternative, the authors obtained the existence of a weak solution in a fractional Orlicz-Sobolev space for the problem (1.6).
Motivated by the above papers, the aim of this paper is to prove the existence of a weak solution for the problem \((P)\) in Orlicz-Sobolev space, by using Leray-Schauder’s Theorem. To our knowledge, this is the first contribution to studying a Kirchhoff problem in this class of Orlicz-Sobolev spaces by Leray-Schauder’s nonlinear alternative. Moreover, the presence of the functions \(M\), which implies that the first equation in \((P)\) is no longer a pointwise equation, therefore it is often called nonlocal problem.

This paper is divided into four sections. In the second section, we recall some properties of Orlicz Sobolev spaces. In the third section, applying several important properties of \(\text{div}(a(|\nabla u|)\nabla u)\), we obtain the existence of weak solution of problem \((P)\). Finally, in the fourth section, we present some examples which illustrate our results.

2. Some preliminary results. To deal with this situation, we will give just a brief review of some basic concepts and facts of the theory of Orlicz-Sobolev spaces. We refer the reader to [1, 25], for further reference and for some of the proofs of the results in this section.

Let \(\Omega\) be an open subset of \(\mathbb{R}^N\), \(N \geq 1\). We assume that \(a : \mathbb{R} \to \mathbb{R}\) in \((P)\) is such that :

\[
\phi : \mathbb{R} \to \mathbb{R} \text{ defined by:} \quad \phi(t) = \begin{cases} 
\frac{a(|t|)t}{|t|} & \text{for } t \neq 0, \\
0 & \text{for } t = 0,
\end{cases}
\]

is an increasing homeomorphism from \(\mathbb{R}\) onto itself. Let

\[
\Phi(t) = \int_0^t \phi(\tau)d\tau.
\]

Then, \(\Phi\) is an \(N\)-function, see [1], that is, \(\Phi : \mathbb{R}^+ \to \mathbb{R}^+\) is a continuous, convex, increasing function, with \(\frac{\Phi(t)}{t} \to 0\) as \(t \to 0\) and \(\frac{\Phi(t)}{t} \to \infty\) as \(t \to \infty\).

For the function \(\Phi\) introduced above we define the Orlicz space :

\[
L_\Phi(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable : } \int_{\Omega} \Phi(\lambda|u(x)|)dx < \infty \text{ for some } \lambda > 0 \right\}.
\]

The space \(L_\Phi(\Omega)\) is a Banach space endowed with the Luxemburg norm

\[
||u||_\Phi = \inf \left\{ \lambda > 0 : \int_{\Omega} \Phi\left(\frac{|u(x)|}{\lambda}\right)dx \leq 1 \right\}.
\]

The conjugate \(N\)-function of \(\Phi\) is defined by

\[
\overline{\varphi}(t) = \int_0^t \varphi(\tau)d\tau,
\]

where \(\varphi : \mathbb{R} \to \mathbb{R}\) is given by

\[
\varphi(t) = \sup \{ s : \phi(s) \leq t \}.
\]

Furthermore, it is possible to prove a Hölder type inequality, that is

\[
\left| \int_{\Omega} uv dx \right| \leq 2||u||_\Phi||v||_{\overline{\varphi}} \quad \forall u \in L_\Phi(\Omega) \text{ and } v \in L_{\overline{\varphi}}(\Omega).
\] (2.1)

Throughout this paper, we assume that

\[
1 < \varphi^- := \inf_{t \geq 0} \frac{t\phi(t)}{\Phi(t)} \leq \varphi^+ := \sup_{t \geq 0} \frac{t\phi(t)}{\Phi(t)} < +\infty.
\] (2.2)
The above relation implies that \( \Phi \in \Delta_2 \), that is, \( \Phi \) satisfies the global \( \Delta_2 \)-condition (see \[27\]):

\[
\Phi(2t) \leq K\Phi(t) \quad \text{for all } t \geq 0,
\]

where \( K \) is a positive constant.

Furthermore, we assume that \( \Phi \) satisfies the following condition

the function \([0, \infty) \ni t \mapsto \Phi(\sqrt{t})\) is convex. (2.3)

The above relation assures that \( L\Phi(\Omega) \) is an uniformly convex space (see \[27\]).

**Definition 2.1.** Let \( A, B \) be two \( N \)-functions. \( A \) is stronger (resp essentially stronger) than \( B \), \( A \succ B \) (resp \( A \succ\succ B \)) in symbols, if

\[
B(x) \leq A(ax), \quad x \geq x_0 \geq 0,
\]

for each (resp for some) \( x_0 \) (depending on \( a \)) and \( a > 0 \).

**Remark 2.1.** (see [1, Section 8.5]). \( A \succ\succ B \) is equivalent to the condition

\[
\lim_{x \to \infty} \frac{B(\lambda x)}{A(x)} = 0,
\]

for all \( \lambda > 0 \).

Now, we define the Orlicz-Sobolev space \( W^{1,\Phi}_1(\Omega) \) as follows

\[
W^{1,\Phi}_1(\Omega) =: \left\{ u \in L\Phi(\Omega) : \frac{\partial u}{\partial x_i} \in L\Phi(\Omega), \quad i = 1,...,N \right\}
\]

endowed with the norm,

\[
|||u|||_{1,\Phi} := ||u||_\Phi + ||\nabla u||_\Phi,
\]

where

\[
||\nabla u||_\Phi = \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_\Phi.
\]

By [1], \( W^{1,\Phi}_1(\Omega) \), unlike the Sobolev spaces, are in general neither separable nor reflexive. A key tool to guarantee these properties is represented by the \( \Delta_2 \) condition. Moreover \( L\Phi(\Omega) \) (resp. \( W^{1,\Phi}_1(\Omega) \)) is a separable Banach space (resp. reflexive) space if and only if \( \Phi \in \Delta_2 \) (resp. \( \Phi \in \Delta_2 \) and \( \Phi \in \Delta_2 \)). Furthermore if \( \Phi \in \Delta_2 \) and \( \Phi(\sqrt{t}) \) is convex, then the spaces \( L\Phi(\Omega) \) and \( W^{1,\Phi}_1(\Omega) \) are uniformly convex.

Let \( W^{1,\Phi}_0(\Omega) \) denote the closure of \( C^\infty_0(\Omega) \) in \( W^{1,\Phi}_1(\Omega) \). By [20, Lemma 5.7] we obtain that on \( W^{1,\Phi}_0(\Omega) \) we may consider an equivalent norm

\[
||u|| := ||\nabla u||_\Phi.
\]

Given two Banach spaces \( X \) and \( Y \), the symbol \( X \hookrightarrow Y \) means that \( X \) is continuously embedded in \( Y \) and also the symbol \( X \hookrightarrow\hookrightarrow Y \) means that there is a compact embedding of \( X \) in \( Y \). Moreover, we will use the symbols \( \hookrightarrow \) to denote the strong convergence and \( \rightharpoonup \) to denote the weak convergence. For simplicity, we use \( c_i \), to denote the general non-negative or positive constant.

Let us now introduce the Orlicz-Sobolev conjugate \( \Phi_* \) of \( \Phi \), which is given by

\[
\Phi_*^{-1}(t) = \int_0^t \frac{\Phi^{-1}(\tau)}{\sqrt[N]{\tau}} d\tau, \quad (2.4)
\]
and we assume the following conditions:
\[
\lim_{t \to 0} \int_t^1 \frac{\Phi^{-1}(\tau)}{\tau^{\frac{N}{N-1}}} d\tau < +\infty, \tag{2.5}
\]
and
\[
\lim_{t \to +\infty} \int_1^t \frac{\Phi^{-1}(\tau)}{\tau^{\frac{N}{N-1}}} d\tau = \infty. \tag{2.6}
\]

Then we have the following theorems.

**Theorem 2.1.** (cf. [1]) Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^N \) with \( C^{0,1} \)-regularity and bounded boundary. If (2.5) – (2.6) hold, then
\[
W^1L_\Phi(\Omega) \hookrightarrow L_\Phi^*(\Omega). \tag{2.7}
\]

**Theorem 2.2.** (cf. [1]) Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^N \) with \( C^{0,1} \)-regularity and bounded boundary. If Relations (2.5) – (2.6) yield, then
\[
W^1L_\Phi(\Omega) \hookrightarrow L_B(\Omega), \tag{2.8}
\]
is compact for all \( B \prec\prec \Phi_\star \).

A very important role in manipulating the Orlicz space is played by the modular of the \( L_\Phi(\Omega) \), which is defined by
\[
L_\Phi(\Omega) \rightarrow \mathbb{R} \quad u \mapsto \int_\Omega \Phi(|\nabla u(x)|) dx.
\]

**Proposition 2.1.** (see [27]) Assume condition (2.2) is satisfied. Then the following relations hold true,
\[
||u||^{\varphi^-} \leq \int_\Omega \Phi(|\nabla u(x)|) dx \leq ||u||^{\varphi^+} \quad \text{if} \quad ||u|| > 1,
\]
\[
||u||^{\varphi^+} \leq \int_\Omega \Phi(|\nabla u(x)|) dx \leq ||u||^{\varphi^-} \quad \text{if} \quad ||u|| < 1.
\]

**Theorem 2.3.** (Schauder fixed point [18]) Let \( T \) be a compact and continuous mapping from a Banach space \( X \) into itself, such that the set
\[
\{ x \in X : x = \lambda Tx \text{ for some } 0 \leq \lambda \leq 1 \}
\]
is bounded. Then \( T \) has a fixed point.

3. **Main results.** In this section, we are going to prove the existence of weak solution for Problem \((P)\).

Let \( g : \mathbb{R} \rightarrow \mathbb{R} \) be an odd increasing homeomorphism. By setting
\[
G(t) = \int_0^t g(\tau) d\tau, \quad \overline{G}(t) = \int_0^t \overline{g}(\tau) d\tau, \tag{3.1}
\]
where \( \overline{g}(t) = \sup \{ s : g(s) \leq t \} \), we obtain complementary \( N \)-functions which define corresponding Orlicz spaces \( L_G \) and \( L_{\overline{G}} \). Then, we assume that \( f : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \) is a carathéodory function and there exists a non negative constant \( c_1 \), such that:
\[
|f(x,s)| \leq c_1 (b(x) + g(s)) \quad \text{for all} \quad (x,s) \in \Omega \times \mathbb{R}, \tag{f_1}
\]
where \( b \in L_{\overline{G}}(\Omega) \). Also we suppose that
\[
1 < \liminf_{t \to +\infty} \frac{tg(t)}{G(t)} \leq \limsup_{t \to +\infty} \frac{tg(t)}{G(t)} < +\infty, \tag{f_2}
\]
\[ \lim_{t \to +\infty} \frac{G(t)}{\Phi_*(kt)} = 0 \quad \text{for all } k > 0. \quad (f_3) \]

We start our analysis with the following Remarks.

**Remark 3.1.** Condition \((f_3)\) has the important consequence that the embedding

\[ W_0^1 L_\Phi(\Omega) \hookrightarrow L_G(\Omega) \quad (3.2) \]

is compact, a fact that is crucial in our development and it corresponds to subcritical type of condition.

**Remark 3.2.** We note that in Definition 3.1 the functional

\[ L(u) = \int_\Omega F(x,u) dx \]

where \(F(x,t) = \int_0^t f(x,\tau) d\tau\), is well defined. Indeed, if \(u \in W_0^1 L_\Phi(\Omega)\), then by condition \((f_3)\), we have that \(u \in L_G\) and \(b \in L_G\) and by using \((f_1)\), there exists \(c_2 > 0\) such that

\[ |F(x,u)| \leq \left| \int_0^u f(x,t) dt \right| \leq c_2 (b(x) + G(|u|)), \quad (3.3) \]

hence,

\[ \int_\Omega |F(x,u(x))| dx < +\infty. \]

**Definition 3.1.** We say that \(u \in W_0^1 L_\Phi(\Omega)\) is a weak solution of Problem \((P)\) if

\[ M \left( \int_\Omega \Phi(|\nabla u(x)|) dx \right) \int_\Omega a(|\nabla u(x)|) \nabla u(x) \nabla v(x) dx = \int_\Omega f(x,u) v dx \quad (3.4) \]

for all \(u, v \in W_0^1 L_\Phi(\Omega)\).

The main result of this paper is given by the following theorem.

**Theorem 3.1.** Assume that \(f\) satisfies \((f_1)-(f_3)\) then there exists \(R\) such that problem \((P)\) has a solution \(u \in W_0^1 L_\Phi(\Omega)\), with \(||u|| \leq R\).

In order to prove our main results, we introduce the following energy functionals \(\Psi, J : W_0^1 L_\Phi(\Omega) \rightarrow \mathbb{R}\)

\[ \Psi(u) = \int_\Omega \Phi(|\nabla u|) dx, \quad J(u) = \widehat{M} \left( \int_\Omega \Phi(|\nabla u|) dx \right) \quad (3.5) \]

where \(\widehat{M}(t) = \int_0^t M(\tau) d\tau, \forall t \geq 0\) and we pose

\[ T u := -M \left( \int_\Omega \Phi(|\nabla u|) dx \right) \text{div}(a(|\nabla u|) \nabla u). \]

The dual space of \((W_0^1 L_\Phi(\Omega), ||||)\) is denoted by \((W_0^1 L_\Phi(\Omega))^*, ||||^*\).

Now, we are going to prove some Lemmas we will use in the sequel.

**Lemma 3.1.** \(T : W_0^1 L_\Phi(\Omega) \rightarrow (W_0^1 L_\Phi(\Omega))^*\) is a continuous, bounded and strictly monotone operator.
Proof. By a standard argument [19, Lemma 3.4], we have $\Psi : W_0^1 L_\Phi(\Omega) \to \mathbb{R}$ is $C^1$ with derivative given by

$$\langle \Psi'(u), v \rangle = \int_{\Omega} a(|\nabla u(x)|) \nabla u(x) \nabla v(x) dx,$$

for all $u, v \in W_0^1 L_\Phi(\Omega)$.

Now, we prove that $J' : W_0^1 L_\Phi(\Omega) \to (W_0^1 L_\Phi(\Omega))^*$ is $C^1$ on $W_0^1 L_\Phi(\Omega)$ it is easy to see that

$$\langle J'(u), v \rangle = M(\Psi(u)) \int_{\Omega} a(|\nabla u(x)|) \nabla u(x) \nabla v(x) dx,$$

for all $u, v \in W_0^1 L_\Phi(\Omega)$. It follows that for each $u \in W_0^1 L_\Phi(\Omega)$, $J'(u) \in (W_0^1 L_\Phi(\Omega))^*$. Let $u_n \to u$ strongly in $u \in W_0^1 L_\Phi(\Omega)$, we consider

$$\langle J'(u_n), v \rangle = M(\Psi(u_n)) \langle \Psi'(u_n), v \rangle = M(\Psi(u_n)) \int_{\Omega} a(|\nabla u_n(x)|) \nabla u_n(x) \nabla v(x) dx.$$

We have $\Psi'(u_n) \to \Psi'(u)$ in $(W_0^1 L_\Phi(\Omega))^*$ (see [19]). Moreover by continuity of $\Psi$, we have

$$\Psi(u_n) \to \Psi(u) \text{ in } (W_0^1 L_\Phi(\Omega))^*.$$

Now, taking into account the continuity of the function $M$ we obtain

$$M(\Psi(u_n)) \to M(\Psi(u)).$$

So we have

$$J'(u_n) \to J'(u) \text{ in } (W_0^1 L_\Phi(\Omega))^*,$$

then $J' \in C^1(W_0^1 L_\Phi(\Omega), \mathbb{R})$.

We denote $T = J' : W_0^1 L_\Phi(\Omega) \to (W_0^1 L_\Phi(\Omega))^*$ then

$$\langle Tu, u \rangle = M(\Psi(u)) \int_{\Omega} a(|\nabla u(x)|) \nabla u(x) \nabla u(x) dx.$$

Obviously $T$ is continuous and bounded.

Let $u, v \in W_0^1 L_\Phi(\Omega)$ with $u \neq v$ and let $\lambda, \mu \in [0, 1]$ with $\lambda + \mu = 1$. Since $a$ is increasing, then $a(|t|)t$ is also increasing and

$$\langle \Psi'(u) - \Psi'(v), u - v \rangle = \int_{\Omega} (a(|\nabla u(x)|) \nabla u(x) - a(|\nabla v(x)|) \nabla v(x))(\nabla u(x) - \nabla v(x)) dx.$$

From the above information, the operator $\Psi'$ is strictly monotone, so by [30, Proposition 25.10], $\Psi$ is strictly convex. Moreover, since $M$ is non decreasing the function $\widehat{M}$ is convex in $[0, +\infty)$. Thus, we have

$$J(\lambda u + \mu v) = \widehat{M}(\lambda \Psi(u) + \mu \Psi(v)) \leq \lambda \widehat{M}(\Psi(u)) + \mu \widehat{M}(\Psi(v)).$$

This shows that $J$ is strictly convex as already, which is strictly monotone. \qed

Lemma 3.2. $T$ is mapping of type $(S_+)$ that is for a sequence $u_n$ that converges weakly to $u$ in $W_0^1 L_\Phi(\Omega)$ and

$$\limsup_{n \to +\infty} \langle Tu_n, u_n - u \rangle \leq 0 \quad (3.7)$$

we have $u_n \to u$ in $W_0^1 L_\Phi(\Omega)$. 
Proof. Since \( u_n \) converges weakly to \( u \) in \( W^1_0L_\Phi(\Omega) \), then \( ||u_n|| \) is a bounded sequence of real numbers, this and Proposition 2.1, imply that \( J(u_n) \) is bounded, then we have for a subsequence noted also \( u_n \),
\[
J(u_n) \to c_2.
\]
Since \( J \) is weakly lower semi continuous, we get
\[
J(u) \leq \liminf_{n \to +\infty} J(u_n) = c_2.
\]
On the other hand, by the convexity of \( J \), we have
\[
J(u) \geq J(u_n) + \langle J'(u_n), u_n - u \rangle. \tag{3.8}
\]
Making use of (3.7), we obtain
\[
J(u) = c_2. \tag{3.9}
\]
Assume by contradiction that \( u_n \) does not converge to \( u \) in \( W^1_0L_\Phi(\Omega) \). Hence there exists a subsequence of \( u_n \), still denoted by \( u_n \) and there exists \( \varepsilon_0 > 0 \) such that
\[
\left| \left| \frac{u_n + u}{2} \right| \right| \geq \varepsilon_0,
\]
by Proposition 2.1, we have
\[
J\left( \frac{u_n + u}{2} \right) \geq m_0 \max\{\varepsilon_0^-, \varepsilon_0^+\}. \tag{3.10}
\]
Moreover by (3.10), one has
\[
J(u) - m_0 \max\{\varepsilon_0^-, \varepsilon_0^+\} \geq \limsup_{n \to +\infty} J\left( \frac{u_n + u}{2} \right). \tag{3.11}
\]
Thus, by (3.9) and (3.11) we obtain a contradiction and hence \( u_n \) converges strongly to \( u \).

Lemma 3.3. \( \mathcal{T} : W^1_0L_\Phi(\Omega) \to (W^1_0L_\Phi(\Omega))^* \) is a homomorphism.

Proof. First, we assume that the operator
\[
\mathcal{T} : W^1_0L_\Phi(\Omega) \to (W^1_0L_\Phi(\Omega))^*
\]
is invertible by Minty-Browder theorem (see [30, Theorem 26]), it suffices to prove that \( \mathcal{T} \) is strictly monotone, hemicontinuous and coercive in the sense of monotone operators. From Lemma 3.1, the operator
\[
\mathcal{T} : W^1_0L_\Phi(\Omega) \to (W^1_0L_\Phi(\Omega))^*
\]
is strictly monotone. For any \( u \in W^1_0L_\Phi(\Omega) \) with \( ||u|| > 1 \), by \((M_0)\), (2.2) and proposition 2.1, we have
\[
\varphi^- \Phi(t) \leq \varphi(t)t,
\]
for all $t > 0$ and

$$\langle Tu, u \rangle = \frac{M(\Psi(u)) \int_{\Omega} a(|\nabla u|) \nabla u \nabla u dx}{||u||}$$

$$\geq \frac{C_{1}^{\infty} \psi^{\cdot}}{||u||}$$

$$\geq \frac{C_{1}^{\infty} \psi^{\cdot}}{||u||} = C_{1}^{\infty} \psi^{\cdot} - 1.$$  

From which we have the coercivity of $T$.

From Lemma 3.1, $T$ is hemicontinuous. Thus, in view of the Minty-Browder theorem, there exists $T^{-1} : (W_{0}^{1}L_{\Phi}(\Omega))^{*} \to W_{0}^{1}L_{\Phi}(\Omega)$ and it is bounded. Let us prove that $T^{-1}$ is continuous by showing that it is sequentially continuous. Let $u_{n} \subset (W_{0}^{1}L_{\Phi}(\Omega))^{*}$ be a sequence converging to $u \in (W_{0}^{1}L_{\Phi}(\Omega))^{*}$ and let $v_{n} = T^{-1}(u_{n})$ and $v = T^{-1}(u)$. Then, $v_{n}$ is bounded in $W_{0}^{1}L_{\Phi}(\Omega)$ and without loss of generality, we can assume that it converges weakly to a certain $v_{0} \in W_{0}^{1}L_{\Phi}(\Omega)$. Since $u_{n}$ converges strongly to $u$, it is easy to see that

$$\lim_{n \to +\infty} T(v_{n})(v_{n} - v_{0}) = \lim_{n \to +\infty} u_{n}(v_{n} - v_{0}) = 0,$$

or

$$\lim_{n \to +\infty} M(\Psi(v_{n})) \int_{\Omega} a(|\nabla v_{n}|) \nabla v_{n} \nabla v_{n} - \nabla v_{0} dx = 0. \quad (3.12)$$

By combining Lemma 3.2 with (3.12) and the fact that $v_{n}$ converges weakly to $v_{0} \in W_{0}^{1}L_{\Phi}(\Omega)$. Then we have $v_{n}$ converges strongly to $v_{0}$ in $W_{0}^{1}L_{\Phi}(\Omega)$. Thus, this completes the proof.

**Proof of Theorem 3.1.** From Remark 3.1, $W_{0}^{1}L_{\Phi}(\Omega)$ is compactly embedded in $L_{G}(\Omega)$. Denote by $j$ the injection of $W_{0}^{1}L_{\Phi}(\Omega)$ in $L_{G}(\Omega)$ and $j^{*} : L_{G}(\Omega) \to (W_{0}^{1}L_{\Phi}(\Omega))^{*}$ with $j^{*} v = \delta v j$ for all $v \in L_{G}(\Omega)$ its adjoint. It follows from the assumption (f) that the Nemytskii operator $N_{f}$ generated by $f$, $(N_{f} u)(x) = f(x, u)$ is well defined from $L_{G}(\Omega)$ into $L_{G}(\Omega)$, continuous and bounded. In order to prove that the problem $(P)$ has a weak solution $u \in W_{0}^{1}L_{\Phi}(\Omega)$, it is sufficient to prove that the equation

$$Tu = (j^{*}N_{f})u, \quad (3.13)$$

has a solution in $W_{0}^{1}L_{\Phi}(\Omega)$. Indeed, if $u \in W_{0}^{1}L_{\Phi}(\Omega)$ is a solution of (3.13), then for all $v \in W_{0}^{1}L_{\Phi}(\Omega)$, one has

$$\langle Tu, v \rangle = \langle (j^{*}N_{f})u, v \rangle = \langle N_{f}(j u), j v \rangle,$$

which can be rewritten

$$M(\Psi(u)) \int_{\Omega} a(|\nabla u(x)|) \nabla u(x) \nabla v(x) dx = \int_{\Omega} f(x, u)v dx,$$

so, $u \in W_{0}^{1}L_{\Phi}(\Omega)$ is a weak solution of problem $(P)$. By Lemma 3.3, $T$ is a homeomorphism from $W_{0}^{1}L_{\Phi}(\Omega)$ into $(W_{0}^{1}L_{\Phi}(\Omega))^{*}$. The equation (3.13) is equivalent to

$$u = T^{-1}(j^{*}N_{f})u. \quad (3.14)$$

Therefore, to prove that the problem $(P)$ has a weak solution in $W_{0}^{1}L_{\Phi}(\Omega)$ it remains to show that the compact operator

$$T := T^{-1}(j^{*}N_{f}) : W_{0}^{1}L_{\Phi}(\Omega) \to (W_{0}^{1}L_{\Phi}(\Omega))^{*},$$
has a fixed point.

By Theorem 2.3, to prove that $T$ admits a fixed point it sufficient to verify the following property. There exists a constant $R > 0$ such that

$$H = \{ u \in W_0^1 L_\Phi(\Omega) : u = \lambda Tu \text{ for some } \lambda \in (0, 1) \} \subset B(0, R).$$

For $\lambda = 0$, the only solution of equation $u = \lambda Tu$ is $u = 0$.

For $\lambda \in [0, 1)$, let $u \in W_0^1 L_\Phi(\Omega)$ which satisfies $u = T^{-1}((j^* N_{f_j})u)$. Then, we have

$$\langle Tu, u \rangle = \lambda \langle (j^* N_{f_j})u, u \rangle.$$

If $||u|| < 1$, by (2.2) and Proposition 2.1, we have

$$\varphi_\Phi(t) \leq \varphi(t) t \text{ for all } t > 0,$$

and

$$m_0 \varphi^- ||u||^\gamma = \leq m_0 \varphi^- \int_\Omega \Phi(|\nabla u|) dx$$

$$\leq m_0 \int_\Omega a(|\nabla u|)|\nabla u| u dx$$

$$\leq M(\Psi(u)) \int_\Omega a(|\nabla u|)|\nabla u| u dx$$

$$\leq \langle Tu, u \rangle$$

$$\leq \langle (j^* N_{f_j})u, u \rangle$$

$$\leq ||j^*||_* ||N_{f_j}(ju)||_{L_\Psi} ||u||.$$

In order to estimate $||N_{f_j}(ju)||_{L_\Psi}$, we deduce from assumption ($f_1$) that

$$|(N_{f_j}(x))| = |f(x, v)| \leq c_1(b(x) + g(v)).$$

From this we can see that

$$||(N_{f_j})||_{L_\Psi} = \inf \left\{ \lambda > 0; \int_\Omega \mathcal{G} \left( \frac{N_{f_j}}{\lambda} \right) \leq 1 \right\},$$

using the assumption ($f_1$), we get

$$\mathcal{G}(N_{f_j}) \leq \mathcal{G}(c_1 b(x) + c_1 g(v)) \leq \mathcal{G} \left( \frac{2c_1 b(x)}{2} + \frac{2c_1 g(v)}{2} \right),$$

using again the fact that $\mathcal{G}$ is convex and for some positive constants $d_1$ and $d_2$, one has

$$\mathcal{G}(N_{f_j}) = \leq \frac{1}{2} \mathcal{G}(2c_1 b(x)) + \frac{1}{2} \mathcal{G}(2c_1 g(v))$$

$$\leq d_1 \mathcal{G}(b(x)) + d_2 \mathcal{G}(g(v)).$$

From [27], it follows that, for a suitable positive constant $c \mathcal{G}(g(t)) \leq c \mathcal{G}(t)$. Then we have

$$\mathcal{G}(N_{f_j}) \leq c(\mathcal{G}(b(x)) + \mathcal{G}(v)).$$

So,

$$||N_{f_j}(v)||_{L_\Psi} \leq c(||b||_{L_\Psi} + ||v||_{L_\Psi}).$$

By taking (3.16) for $v = ju$, $u \in W_0^1 L_\Phi(\Omega)$, we have

$$||N_{f_j}(ju)||_{L_\Psi} \leq c(||b||_{L_\Psi} + ||j||_* ||u||).$$

(3.17)
In particular, if \( u \in W^{1,0}_0(\Omega) \), for some \( \lambda \in (0, 1] \), we conclude from (3.14), (3.15) and (3.17) that

\[
m_0 \varphi^- ||u||^\gamma = \leq ||j||_\gamma (c||b||_L + ||j||_\delta ||u||)||u||
\leq c||j||_\gamma ||b||_L ||u|| + c||j||_\delta ||u||^2
= c_2 ||u|| + c_3 ||u||^2,
\]
where \( c_2 = ||j||_\gamma c||b||_L, \) \( c_3 = ||j||_\delta^2 c \) and \( \gamma = \varphi^- \). Using the young’s inequality, we have

\[
c_3 ||u||^2 \leq \frac{1}{4} ||u||^\gamma + 4 \frac{\gamma}{\gamma - 2} c_3^\frac{\gamma}{\gamma - 2},
\]
and

\[
c_2 ||u|| \leq \frac{1}{4} ||u||^\gamma + 4 \frac{\gamma}{\gamma - 2} c_2^\frac{\gamma}{\gamma - 2}.
\]
Inserting these two inequalities into (3.18), one has

\[
\frac{1}{2} ||u||^\gamma \leq 4 \frac{\gamma}{\gamma - 2} c_3^\frac{\gamma}{\gamma - 2} + 4 \frac{\gamma}{\gamma - 2} c_2^\frac{\gamma}{\gamma - 2}.
\]
Set

\[
R_1 = (2 \frac{\gamma - 2}{\gamma} c_3^\frac{\gamma}{\gamma - 2} + 2 \frac{\gamma - 1}{\gamma} c_2^\frac{\gamma}{\gamma - 2})^\frac{1}{\gamma},
\]
then \( ||u|| \leq R_1 \). Similarly, if \( ||u|| \geq 1 \) we have \( ||u|| \leq R_2 \) where

\[
R_2 = (2 \frac{\gamma - 2}{\gamma} c_3^\frac{\gamma}{\gamma - 2} + 2 \frac{\gamma - 1}{\gamma} c_2^\frac{\gamma}{\gamma - 2})^\frac{2}{\gamma},
\]
with \( \gamma = \varphi^+ \), this implies that the set \( H \) is bounded.

4. Examples. We present in this section some examples of functions that satisfy the conditions of Theorem 3.1.

Example 1. As a first example, we can take

\[
M(t) = 1 (4.1)
\]
\[
\varphi(t) = \log(1 + |t|)|t|^{p-2}t (4.2)
\]
\[
f(x, s) = f(s) = |s|^{\delta - 1}s (4.3)
\]
where \( p \in (1, N-1) \) and \( p < \delta \leq \frac{N(p-1) + p}{N-p} \).

We consider the problem

\[
(P_1) \quad \left\{ \begin{array}{ll}
- \text{div} (\log(1 + |\nabla u|)|\nabla u|^{p-2} \nabla u) = |u|^{\delta - 1} u & \text{in } \Omega \\
u = 0 & \text{on } \partial \Omega,
\end{array} \right.
\]

So, from (4.1)-(4.2), we have

\[
\Phi(t) = \frac{1}{p} \log(1 + |t|)|t|^p - \frac{1}{p} \int_0^{|t|} \frac{\tau^{p}}{1 + \tau} d\tau (4.4)
\]
\[
\hat{M}(t) = t \text{ and } F(x, s) = F(s) = |s|^{\delta + 1} \frac{\delta}{\delta + 1}. (4.5)
\]
We will next show that all the hypotheses of Theorem 3.1 are satisfied.

- First, it easy to see that \( \Phi \) is an \( N \)-function and by Example 2 [12, p. 243] it follows that

\[
\varphi^+ = p + 1 \quad \text{and} \quad \varphi^- = p. \quad (4.6)
\]
Then (2.2) holds true.
• Next, we have
\[ M(t) = 1 > 0 \text{ for all } t \geq 0 \]
thus, for \( m_0 = 1 \) \((M_0)\) holds true.

• On the other hand, by [13], it follows that \((f_1)-(f_3)\) hold true, and
\[
\int_1^\infty \frac{d\tau}{\tau^{(N+1)/N}} = \infty
\]
\[
\int_0^1 \frac{d\tau}{\tau^{(N+1)/N}} < \infty.
\]

Hence from Theorem 3.1, problem \((P_2)\) has a nontrivial nonnegative weak solution \( u \in W^{1,L}(\Omega) \).

**Example 2.** As a second example, we can take
\[ M(t) = a + bt^{\alpha - 1} \]
\[ \varphi(t) = p\frac{|t|^{p-2}t}{\log(1 + |t|)} \text{ for all } t \geq 0, \]
where \( a, b > 0, \alpha > 1 \) and \( 1 < p < \infty \).

Then, in this case problem \((P)\) becomes
\[
(P_2) \begin{cases}
(a + b\Psi(u))^{\alpha - 1} \text{ div } \left( \frac{p|u|^{p-2}u}{\log(1 + |u|)} \right) = f(x,u) & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\]
with
\[ \Psi(u) = \int_\Omega p\frac{|u|^p}{\log(1 + |u|)} + \int_0^{|u|} \frac{\tau^p}{(1 + \tau)(\log(1 + \tau))^2} d\tau dx \]
It is easy to see that \( \Phi \) is an \( N \)-function. Moreover, by [13][Example 3], we have
\[ p - 1 \leq t\varphi(t) \leq p \forall t \geq 0. \]
Thus, \((2.2)\) holds true with \( \varphi^- = p - 1 \) and \( \varphi^+ = p \). Next, we have
\[ M(t) = a + bt^{\alpha - 1} \geq a > 0 \text{ for all } t \geq 0 \]
thus, for \( m_0 = a \) \((M_0)\) holds true.

Hence, we derive an existence result for problem \((P_2)\) which is given by the following corollary.

**Corollary 4.1.** Assume that \( f \) satisfies \((f_1)-(f_3)\). Then, problem \((P_2)\) has a nontrivial weak solution.

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