Convergence rate of mean-square error of kernel estimators of non-homogeneous Poisson process intensity function

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Abstract. The estimator has been constructed similar to the Parzen-Rosenblatt window method using a single realization of a Poisson process at a fixed time interval. The intensity function of a non-homogeneous Poisson process is estimated up to a multiplicative constant. The convergence rate of the mean square error was found in the series of schemes with an unlimited increase of the intensity.

1. Introduction and problem statement

Poisson processes are known to be useful to model different random phenomena; see, for example, the books [1-3]. The nuclear decay of atoms is a classic example of Poisson process [4].

A non-homogeneous Poisson process (NHPP) is an adequate tool for the analysis of a system whose rate (e.g. arrival rate in an inventory system) varies over time or space. The range of application of NHPP is very wide. In [5] NHPP is used to model the occurrences of petroleum reservoirs whose size is highly inhomogeneous. The paper [6] presents the use of NHPP for computing the rate of fast radio bursts. In [7] a non-homogeneous Poisson model is considered to study noise exposure. In [8] NHPP is used to study the reliability of service water pumps of a pressurised water reactor plant. NHPP model is widely used in software reliability engineering as a software reliability model [9]. In the last two papers the process is associated with the occurrence of failures. This is typical for engineering applications of NHPP.

Thus, the problem of estimation of Poisson intensity functions arises in many areas except inventory control problems.

In [10], we consider a perishable product with a fixed shelf life under fixed-order quantity policy and a Poisson demand with a highly price-sensitive intensity of customers’ flow. It is supposed that the product is sold for sure if the price is low enough. That is the case, for example, in relation to essential goods. A model of the intensity of demand control through the price is investigated allowing us to sell a lot without leftovers almost surely during the product’s lifetime. In [11], we consider a generalization of this dynamic pricing control model. Currently, dynamic price management is of a great practical interest due to advances in digital technologies. These papers are based on the ideas of A. F. Terpugov, who was one of the first in Russia to develop the perishable inventory theory; see, e.g. [12].
In [10, 11] the intensity of customers’ flow changes over time and we try to find the more suitable form of the dependence adding the adjustable parameters to the model. In [11] the task of the expected profit optimization with respect to the parameter is solved and the analogous optimization is possible with respect to the unknown function. But one question that remains open is matching of the models with real data. So, we are interested in estimating the unknown intensity function of the Poisson process that models demand.

In such a situation, it is natural to start with nonparametric methods. In [13, 14] we have considered classical and recursive kernel estimators of the intensity function of NHPP. Kernel methods have been thoroughly explored in the density estimation literature; see, e.g. [15, 16]. The estimators were proposed by Rosenblatt [17] for density estimation and studied by Parsen [18] and Nadaraya [19]. The extension of a kernel approach to the intensity estimation is quite natural. Rudemo [20] investigated a histogram and kernel estimators in the density and Poisson framework. Willett and Nowak [21] consider the nonparametric Poisson intensity estimation problem in a density estimation framework.

Thus, in this paper we do not assume any parametric form of the function except some regularity conditions, and suppose that only a single realization of the process is available at \([0,T]\), where \(T\) denotes the length of the fixed interval of observations (in our interpretation \(T\) is a product’s life time). The estimation a Poisson intensity from a single trajectory has been studied in [21-23] and for a cyclic Poisson process in different settings by Mangku; see, e.g. [24].

Let \(\{t_i, i = 1,N, 0 \leq t_i \leq T\}\) be a realization of a Poisson process with unknown intensity function \(\lambda(\cdot)\) at time interval \([0,T]\). Here \(N\) is the number of arrivals at \([0,T]\).

Probability mass function of random variable \(N\)

\[
p_n = P(N = n) = \frac{\Lambda(0,T)^n}{n!} \exp(-\Lambda(0,T)), \quad n \geq 0,
\]

where \(\Lambda(0,T) = \int_0^T \lambda(t)dt\).

Let us consider the following estimator of \(S(\cdot) = \lambda(\cdot) / \Lambda(0,T)\) at a point \(t \in (0,T)\) similar to the Parzen-Rosenblatt window method

\[
S_N = \frac{1}{N h_N} \sum_{i=1}^{N} K \left( \frac{t - t_i}{h_N} \right),
\]

(1)

where \((h_n)\) is a sequence of real numbers, \(h_n \downarrow 0\) and \(nh_n \to \infty\); \(K(\cdot)\) is a kernel function at \([-T, T]\) taking zero values outside the interval and \(\int_{-T}^{T} K(u)du = 1\). If \(N = 0\) then we assume that \(S_N = 0\).

Note that a sample size \(N\) is a random variable and we deal with the estimator based on a random number of observations.

We are interested in an asymptotic behavior of \(S_N\). Since the expectation \(E(N) = \Lambda(0,T)\), we can consider the two forms of the asymptotic: \(T \to +\infty\) [25] or \(\lambda \to +\infty\). In the last case a series of schemes may be used [26], which is a classical approach; see, for example, [27]. The last approach is quite natural in the framework of our task since in [10, 11] we consider the case when the lot is large and at the end of the period it should be sold almost surely. This means that the intensity of customers’ flow during the period should be high.

2. The convergence rate of the mean-square error

Here we present an extension of the results obtained in [13], where only mean-square convergence of the estimators was proved.

Let series of observations is done at \([0,T]\) with the intensity of the process in \(n\)-th trial \(\lambda_n(\cdot) = n\lambda(\cdot)\). Let us denote the value of statistics (1) in \(n\)-th trial

\[
E_n = \frac{1}{N h_N} \sum_{i=1}^{N} K \left( \frac{t - t_i}{h_N} \right),
\]

(2)

for a cyclic Poisson process with unknown intensity function \(\lambda(\cdot)\) at time interval \([0,T]\). Here \(N\) is the number of arrivals at \([0,T]\).

Probability mass function of random variable \(N\)

\[
p_n = P(N = n) = \frac{\Lambda(0,T)^n}{n!} \exp(-\Lambda(0,T)), \quad n \geq 0,
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where \(\Lambda(0,T) = \int_0^T \lambda(t)dt\).

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E_n = \frac{1}{N h_N} \sum_{i=1}^{N} K \left( \frac{t - t_i}{h_N} \right),
\]

(2)
\[ S_n = \frac{1}{N_n h_n} \sum_{i=1}^{N_n} K \left( \frac{t - t_i}{h_n} \right), \]

where \( N_n \) and \((t_i)\) are the numbers of observations and the realization of the process in \( n \)-th trial respectively.

Let \( h_n < t < T - h_n \) \( \forall n \geq n_0, n_0 \) is fixed.

Denote

\[ T_j = \int_{-\infty}^{\infty} u K(u) du, \quad b(S_n) = E \{ S_n \} - \lambda(t) / \Lambda(0, T). \]

Joint distribution of \( t_i \) and \( N \) [28]

\[ p_{ti}(x) = \lim_{\Delta x \to 0} \frac{P(t_i < x + \Delta x, N = n) - P(t_i < x, N = n)}{\Delta x} = \frac{\Lambda(0, x)^{i-1} \Lambda(x, T)^{n-i}}{(i-1)! (n-i)!} e^{-\Lambda(0,T)} \lambda(x), \]

\[ 0 < x < T, \quad n \geq i \geq 1. \]

Theorem 1 (the bias convergence rate). Let the kernel \( K(\cdot) \) and the intensity function \( \lambda(\cdot) \) satisfy the following conditions:

\[ \int_{-\infty}^{+\infty} |u| K(u) du < \infty, \quad T_j = 0, \quad j = 1, \ldots, v-1, \quad T_v \neq 0, \quad K(u) = K(-u); \]

\[ \sup_{x \in \mathbb{R}} \lambda(x) < \infty, \lambda(\cdot) \text{ is continuously differentiable at a point } t \text{ up to the } v \text{-th order inclusively.} \]

Then as \( n \to \infty \)

\[ \left| b(S_n) - \frac{T_v}{v! \Lambda(0, T)} \lambda^v(t) \right| = o(h_n^v). \]

Proof. Let \( p_i(x|n) \) be the conditional density function \( t_i \) given \( N = n \). Then the expected value

\[ E \{ S \} = \sum_{n=1}^{\infty} p_n \sum_{k=0}^{\infty} \int_{0}^{T} K \left( \frac{t-x}{h_n} \right) p_i(x|n) dx \]

\[ = \sum_{n=1}^{\infty} \frac{1}{nh_n} \sum_{i=1}^{T} \int_{0}^{T} K \left( \frac{t-x}{h_n} \right) \Lambda(0, x)^{i-1} \Lambda(x, T)^{n-i} e^{-\Lambda(0,T)} \lambda(x) dx. \]

Taking into account that

\[ \sum_{i=1}^{\infty} \frac{\Lambda(0, x)^{i-1} \Lambda(x, T)^{n-i}}{(i-1)! (n-i)!} = \frac{\Lambda(0,T)^{n-i}}{(n-i)!}, \]

we get

\[ E \{ S \} = \sum_{n=1}^{\infty} \frac{\Lambda(0,T)^{n-1}}{n!h_n} e^{-\Lambda(0,T)} \int_{0}^{T} K \left( \frac{t-x}{h_n} \right) \lambda(x) dx. \]

Denote

\[ I_n = \frac{1}{h_n} \int_{0}^{T} K \left( \frac{t-x}{h_n} \right) \lambda(x) dx = \int_{(T-t)h_n}^{\infty} K(x) \lambda(t-h_n x) dx \]

and expand \( \lambda(\cdot) \) by the Taylor series at the point \( t \) with the Lagrange form of the remainder.

For \( n >> 1 \), taking into account the kernel finiteness, we get for \( 0 < \theta < 1 \)

\[ I_n = \lambda(t) + \frac{T_v}{v!} \lambda^v(t) h_n^v + \frac{T_v}{v!} \int_{0}^{\infty} K(x) \left( \lambda^v(t-\theta h_n x) - \lambda^v(t) \right) dx. \]

Further, based on the convergence \( \lambda^v(t-\theta h_n x) \to 0 \) as \( n \to \infty \), we get
\[ \Delta_n = I_n - \lambda(t) = \frac{T_n}{\nu!} \lambda^{(\nu)}(t)h_n^\nu + o\left(h_n^\nu\right) \]

and
\[ b(S_n) = \frac{1}{\Lambda(0,T)} \sum_{k=0}^{\infty} \frac{\Lambda(0,T)^k n^k}{k!} \Delta_k \frac{e^{-n\Lambda(0,T)}}{1 - e^{-n\Lambda(0,T)}} = \frac{T_n}{\Lambda(0,T)\nu!} \lambda^{(\nu)}(t)h_n^\nu + o\left(h_n^\nu\right). \]

The theorem is proved.

Let us consider the rate of convergence of the variance of estimator \( S_n \).

Let \( h_i = i', -1 < r < 0 \).

Theorem 2 (the variance convergence rate). Let the kernel \( K(\cdot) \) and the intensity function \( \lambda(\cdot) \) satisfy the following conditions:

\[
\int_{\mathbb{R}} |K(u)| du < \infty \quad \text{and} \quad \int_{\mathbb{T}} K^2(x) dx < \infty ;
\]

\[
\sup_{x \in \mathbb{R}} \lambda(x) < \infty \quad \text{and} \quad \lambda(\cdot) \quad \text{is continuously differentiable at a point} \ t.
\]

Then as \( n \to \infty \)

\[
\left| \text{Var}\{ S_n \} - \frac{\mathbb{E}\{ K^2(x) dx \}}{n h_i} \right| = o\left( \frac{1}{n h_i} \right).
\]

Proof. Joint distribution of \( t_i, t_j, N \) [28]

\[
p_{ij}(x, y) = \lim_{\Delta x, \Delta y \to 0} \frac{P(t_i < x + \Delta x, t_j < y + \Delta y, N = n) - P(t_i < x, t_j < y, N = n)}{\Delta x \Delta y} = \frac{\Lambda(0,x)^{i-1} \Lambda(x,y)^{j-i-1} \Lambda(y,T)^{n-j}}{(i-1)! (j-i-1)! (n-j)!} e^{-\Lambda(x,T)} \Lambda(x,y) \lambda(y), \quad 0 < x < T, x < y < T, n \geq 2, i = 1, n-1, j = i+1, n.
\]

The variance as \( n \to \infty \)

\[
\text{Var}\{ S_n \} = \mathbb{E}\{ S_n^2 \} - \left( \mathbb{E}\{ S_n \} \right)^2
\]

\[
= \mathbb{E}\left\{ \frac{1}{N_n h_i} \left[ \sum_{i=1}^{N_n} K\left( \frac{t-t_{im}}{h_i} \right) + \sum_{i<j} K\left( \frac{t-t_{im}}{h_i} \right) K\left( \frac{t-t_{jm}}{h_j} \right) + \sum_{i<j} K\left( \frac{t-t_{im}}{h_i} \right) K\left( \frac{t-t_{jm}}{h_j} \right) \right] \right\}^2
\]

\[
- \mathbb{E}\left\{ \frac{1}{N_n h_i} \sum_{i=1}^{N_n} K\left( \frac{t-t_{im}}{h_i} \right) \right\}^2 - \frac{1}{\Lambda(0,T)} \sum_{k=1}^{\infty} 1/kh_k \left( \frac{T}{h_k} \right) \sum_{i=1}^{N_n} \frac{\Lambda(0,x)^{i-1} \Lambda(x,y)^{j-i-1} \Lambda(y,T)^{n-j}}{(i-1)! (j-i-1)! (k-j)!} e^{-\Lambda(x,T)} \Lambda(x,y) \lambda(y) \lambda(y) dy dx \sum_{i=1}^{N_n} \sum_{j=i+1}^{N_n} \frac{\Lambda(0,x)^{i-1} \Lambda(x,y)^{j-i-1} \Lambda(y,T)^{n-j}}{(i-1)! (j-i-1)! (k-j)!} \times n^k e^{-n\Lambda(0,T)}
\]

\[
+ 2 \sum_{k=1}^{\infty} 1/k^2 h_k^2 \frac{T}{h_k} \frac{T}{h_k} \lambda(x) \lambda(y) \lambda(y) dy dx \sum_{i=1}^{N_n} \sum_{j=i+1}^{N_n} \frac{\Lambda(0,x)^{i-1} \Lambda(x,y)^{j-i-1} \Lambda(y,T)^{n-j}}{(i-1)! (j-i-1)! (k-j)!} \times n^k e^{-n\Lambda(0,T)} \left[ \sum_{k=1}^{\infty} \frac{\Lambda(0,T)^k n^k}{k!} I_k e^{-n\Lambda(0,T)} \right]^2.
\]

Note that
\[
\sum_{j=1}^{k-1} \sum_{i=1}^{k} \frac{\Lambda(x, y)^{j-1} \Lambda(x, y)^{(i-1)}}{(j-i)! (k-i)!} = \frac{\Lambda(0, T)^{k-2}}{(k-2)!}.
\]

Denote
\[
J_k = \frac{1}{h_k} \int_0^T K\left(\frac{t-x}{h_k}\right) \hat{\lambda}(x) \int_0^T K\left(\frac{t-y}{h_k}\right) \hat{\lambda}(y) dy dx = \int_{(T)} \lambda(x) \lambda(t-h_k x) \int_{(T)} \lambda(y) \lambda(t-h_k y) dy dx.
\]

It follows that
\[
\lim_{h_k \to 0} J_k = \lambda^2(t) \int_{(-T)} K(x) \int_{(-T)} \lambda^2(t) dy dx = \frac{\lambda^2(t)}{2}.
\]

Thus, the second term on the right-hand side of (2) as \(n \to \infty\)
\[
\frac{2}{\Lambda(0, T)^2} \sum_{k=1}^{k-1} \Lambda(0, T)^n k! \lambda \rightarrow \lambda(0, T)^2
\]

Consider
\[
\int_{(T)} K^2(x) \lambda^2(t-h_k x) dx
\]
and expand \(\lambda(\cdot)\) by the Taylor series at the point \(t\) similarly as in the previous theorem.

To finish the proof, we need to prove only that for any \(\alpha > 0\)
\[
\lim_{n \to \infty} \sum_{k=1}^{n} \left(\frac{n}{k}\right)^a \frac{n^k}{k!} e^{-n} = 1.
\]

Let’s divide the series in (3) into three parts
\[
S_{1a}(n) = \sum_{i=1}^{n} \left[\frac{n!}{k!} \frac{n^k}{k!} \right]^\alpha, S_{2a}(n) = \sum_{k=1}^{n} \left[\frac{n!}{k!} \frac{n^k}{k!} \right]^\alpha, S_{3a}(n) = \sum_{k=1}^{n} \left[\frac{n!}{k!} \frac{n^k}{k!} \right]^\alpha, \text{ and } \left[\frac{n!}{k!} \frac{n^k}{k!} \right]^\alpha \text{ is the integer part of number } a.
\]

Consider the similar parts of the series
\[
\sum_{k=1}^{n} \frac{n^k}{k!} = S_1(n) + S_2(n) + S_3(n). \text{ The main part here as } n \to \infty \text{ is } S_2(n), \text{ that is } \lim_{n \to \infty} S_2(n) e^{-n} = 1.
\]

It is easy to see that
\[
S_2(n) \left(\frac{n-n^{3/4}}{n}\right)^\alpha \leq S_{2a}(n) \leq S_2(n) \left(\frac{n+n^{3/4}}{n}\right)^\alpha, \quad \lim_{n \to \infty} S_{2a}(n) e^{-n} = 0, \text{ and } \lim_{n \to \infty} S_{3a}(n) e^{-n} = 0.
\]

The theorem is proved.

The mean-square error of \(S_n\)
\[
\text{MSE}(S_n) = \text{Var}(S_n) + b(S_n) = \int K^2(x) dx + \frac{\lambda(0, T)^2}{\Lambda(0, T)^2} \left(\frac{\lambda(0, T)^2}{\Lambda(0, T)^2}\right) + o\left(n h_n^2 + o\left(n h_n^2 + h_n^{2y}\right)\right),
\]
which coincides with the results obtained for kernel density estimator.

3. Simulation
We have simulated a non-homogeneous Poisson process at interval [0, 130] with intensity function \(\lambda(x) = x^2 - 15x + 60 + 5\sin(4x)\).

The Epanechnikov kernel with support [−130, 130] has been used.
The results of simulation are shown in Fig. 1-4. The bold curve is the true curve; the gray curves are the estimates. Mean relative absolute errors (MRAE) are given.

**Figure 1.** $N = 500$, MRAE ≈ 10.3%.

**Figure 2.** $N = 1000$, MRAE ≈ 8.24%.

**Figure 3.** $N = 6000$, MRAE ≈ 4.31%.

**Figure 4.** $N = 20000$, MRAE ≈ 2.8%.

4. Conclusion
The coincidence the convergence rates of the estimator under consideration with the kernel density estimator is not surprising since given the event “the number of points $N$ falling into $[0, T]$ is fixed” $(t_i)$ has the same probability density function $\lambda(t)/\Lambda(0, T)$. The structure of the estimator is similar to the kernel estimator of a density function. Distinctive feature of the statistic under consideration is due to a random sample size.

Here we estimate the intensity function up to a multiplicative constant and this is enough for our purpose. We are primarily interested in the form of the dependence. In [29] we consider a general form of intensity-of-time dependence containing an unknown function.

Although the discussion here is oriented towards customers’ arrivals to vendor who uses dynamic pricing strategy, the estimators can be applied in many various areas such as communications, meteorology, insurance, medical sciences, seismology.

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