ON THE DENSITY ARISING FROM THE DOMAIN OF ATTRACTION BETWEEN SUM AND SUPREMUM: THE $\alpha$-SUN OPERATOR

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Abstract. We explore the analytic properties of the density function $h(x; \gamma, \alpha)$, $x \in (0, \infty)$, $\gamma > 0$, $0 < \alpha < 1$ which arises from the domain of attraction problem for a statistic interpolating between the supremum and sum of random variables. The parameter $\alpha$ controls the interpolation between these two cases, while $\gamma$ parametrises the type of extreme value distribution from which the underlying random variables are drawn from. For $\alpha = 0$ the Fréchet density applies, whereas for $\alpha = 1$ we identify a particular Fox H-function, which are a natural extension of hypergeometric functions into the realm of fractional calculus. In contrast for intermediate $\alpha$ an entirely new function appears, which is not one of the extensions to the hypergeometric function considered to date. We derive series, integral and continued fraction representations of this latter function.

1. Our Motivation

Consider a sequence of independent random variables $X_1, X_2, \ldots$ with a distribution $F \in D(\Phi_\gamma)$, where $D(.)$ is the domain of attraction of the distribution $\Phi_\gamma$ and this latter distribution is the extreme value distribution $\Phi_\gamma(x) = \exp(-x^{-\gamma})I_{[0, \infty)}(x)$. Now define the following sequence $Y_0, Y_1, Y_2, \ldots$

(1.1) $Y_n = \max\{Y_{n-1}, \alpha Y_{n-1} + X_n\}, \quad n \geq 1, \quad Y_0 = X_0 \in \mathbb{R}.$

For $\alpha = 0$ we have the supremum, $Y_n = \max\{X_0, X_1, X_2, \ldots, X_n\}$, while for $\alpha = 1$ we have the sum, $Y_n = X_0 + \sum_{j=1}^{n} X_jI_{[0, \infty)}(X_j)$. In this way one can interpolate between the sum and the supremum of a sequence of i.i.d. random variables via the parameter $\alpha$.

There is a compelling motivation for our model (1.1) and the associated operator interpolating between sum and supremum. Many decisions which we take are of the following form: Accept the status quo, in which case $Y_n = Y_{n-1}$, or risk losing some portion of what we have, in the process of taking some action, so that the value of our current state is reduced to a portion $\alpha Y_{n-1}$, in which case we can increase our value by a stochastic $X_n$ to obtain $Y_n = \alpha Y_{n-1} + X_n$. Suppose we are so lucky.

2010 Mathematics Subject Classification. 60E07, 60G70, 30A80, 33.20, 33A35.
or wise that we can evaluate $X_n$ before taking the decision whether to act or not. Then we can maximize the outcome by formulating our decision as per (1.1).

In their study of this model - the $\alpha$-sun process - Greenwood and Hooghiemstra [18, Theorem 2, Eq. (2.4)] derived a linear, homogeneous integral equation for the density

$$h(x) = \frac{\gamma}{x} \int_0^x du \frac{h(u)}{(x - \alpha u)^\gamma},$$

subject to the following conditions: $\alpha \in (0, 1)$, $\gamma \in (0, \infty)$ and $h(x)$ is a real, normalised density on $x \in (0, \infty)$. No further conditions can apply. We investigate the solutions $h(x) = h(x; \gamma, \alpha)$ to this integral equation here.

In a subsequent work Hooghiemstra and Greenwood [25] took up the related problem of finding the domain of attraction for the other extreme value distributions, namely $F \in D(\Psi_\gamma) \cup D(\Lambda)$ where $\Psi_\gamma(x) = \exp(-(-x)^\gamma)I_{(-\infty,0)}(x) + I_{(0,\infty)}(x)$ and $\Lambda(x) = \exp(-e^{-x})$. For the latter case a solution for the density was given there by Eq. (10) in Cor. 2 of §2,

$$h(x; \alpha) = \frac{1 - \alpha}{\Gamma(1 - (1 - \alpha)^{-1})} \exp \left(-x - e^{-(1 - \alpha)x}\right).$$

In the former case one seeks solutions to an integral equation similar to (1.2), as given by Eq. (9) in Cor. 1,

$$h(x) = \frac{\gamma}{|x|} \int_{x/\alpha}^x du |x - \alpha u|^\gamma h(u), \quad x \in (-\infty, 0).$$

However no explicit solutions to either (1.2) nor (1.4) have been reported in any studies following this line of enquiry.

Using different methods Schlather [48] has treated a related problem. The Schlather paper says, at the very beginning, that the 1991 and 1997 Greenwood and Hooghiemstra papers combine the CLT with extreme value theory. However this is not the case. We take up this distinction in the discussion §5. See the more recent thesis by Anja Janßen [27] continuing this other line of enquiry.

The study of density solutions to a variant of (1.2), specialised to $\alpha = 1$ but possessing an additional parameter $m \in \mathbb{N}$, was initiated in [49], then pursued in [39] and most recently by [28]. This distinct integral equation takes the form

$$h(x) = \frac{\gamma}{x^m} \int_0^x du \frac{h(u)}{(x - u)^\gamma},$$

and the corresponding random variables are called "generalized stable" and our case includes the classical stable case $m = 1$. Such random variables are of interest in the case $m = 2$, $\gamma \in (0, 1)$ which is especially investigated in Section 3 of [49] and Section 7 of [39], because of its relevance to particle transport on a one-dimensional lattice. The first work [49] identifies the solutions in terms of Fox $H$-functions and shows that for all $m \in \mathbb{N}, \gamma \in (1 - m, 1)$ there exists a density solution to (1.5) having a convergent power series representation at infinity and a Fréchet-like behaviour at
zero. The second work [39] considers invariance under length-biasing and shows that for all \( m \in \mathbb{N} \), \( \gamma \in (1 - m, 1) \) there exists a unique density solution to (1.5), whose corresponding random variable can be represented in the case \( \gamma \in (1 - m, 2 - m) \) as a finite independent product involving the Gamma and positive stable random variables. We have several common results with [28], which are given later, and this latter work shows that there exists a density solution, which is then unique and can be expressed in terms of the Beta distribution, if and only if \( m > 1 - \gamma \).

These density solutions extend the class of generalised one-sided stable distributions introduced in [49] and subsequently investigated in [39]. This work studies various analytical aspects of these densities, and solves some open problems about infinite divisibility formulated in [49].

From the analytical viewpoint there is another thread that our study connects with, and this concerns the solutions of integro-differential equations or more specifically differential-delay Volterra integral equations. For this to become apparent let us consider the cumulative distribution functions

\[
\begin{align*}
(1.6) & \quad f(x) := \int_0^x du h(u), \quad f(0) = 0, f(\infty) = 1, \quad \text{from the solution of (1.2)}, \\
(1.7) & \quad f(x) := \int_x^{\infty} h(-u), \quad f(0) = 1, f(\infty) = 0, \quad \text{from the solution of (1.4)}.
\end{align*}
\]

An important point to note is that henceforth we will frame our discussion of (1.4) for \( x \in \mathbb{R}_+ \) by defining another density \( h_<(x) := h(-x) \). Where there is no chance of confusion we will often drop the subscript \(<\). Thus our defining integral equation will become

\[
(1.8) \quad h_<(x) = \frac{\gamma}{x} \int_x^{x/\alpha} du (x - \alpha u)^\gamma h_<(u), \quad x \in (0, \infty).
\]

A simple exercise then recasts (1.2) into the retarded version of such equations

\[
(1.9) \quad f'(x) - (1 - \alpha)^{-\gamma} \gamma x^{-\gamma-1} f(x) = -\alpha \gamma^2 x^{-\gamma-1} \int_0^1 dq (1 - \alpha q)^{-\gamma-1} f(qx),
\]

whereas (1.4) becomes the advanced version

\[
(1.10) \quad f'(x) + (1 - \alpha)^{\gamma} \gamma x^{\gamma-1} f(x) = \alpha \gamma^2 x^{\gamma-1} \int_1^\alpha dq (1 - \alpha q)^\gamma f(qx),
\]

where these are to solved as boundary value problems, as defined above. This pair of equations is a specialised form of a general class of integro-differential equations formulated by Iserles and Liu [26], although they did not derive solutions of the general system nor our particular case. This class also covers the Pantograph type equations [11], [3] and cell division and growth models [21], [22]. An early study presaging these developments is one on a stochastic absorption problem [16].

We will only treat the solutions of (1.2) or (1.9) in this work, and defer the solutions of (1.8) or (1.10) to a subsequent study. Following this introduction we find explicit solutions to the two special cases \( \alpha = 0 \) and \( \alpha = 1 \) in §2. The density
functions we find here are either well-known or fairly well studied extensions to
the hypergeometric functions, and we include these cases because they serve as
"bookends" to the novel aspects of our results. After that we treat the general
case $0 < \alpha < 1$ in §3, which can’t be handled in the same way as the preceding
section. Our main results are given in Prop. 3.9. The density function arising here
is somewhat more exotic, one of a class of functions lying beyond the generalised
hypergeometric, Meijer-G or Fox H-functions. We give explicit series, integral and
continued fraction representations of the specific function arising in our study. In
§4 we evaluate and plot the densities for a range of $\gamma \in \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ and $\alpha \in \{\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$. A number of implications and conclusions are taken up in the Discussion §5.

2. Special Cases $\alpha = 0$ and $\alpha = 1$

2.1. Case $\alpha = 0$: the Fréchet and Weibull distributions. In this case (1.2)
reduces to

\begin{equation}
(2.1) \quad h_0(x) = \frac{\gamma}{x^{\gamma+1}} \int_0^x du \, h_0(u).
\end{equation}

A simple differentiation of this yields the ordinary differential equation

\[
\frac{d}{dx} h_0 + \left[ \frac{\gamma + 1}{x} - \frac{\gamma}{x^{\gamma+1}} \right] h_0 = 0,
\]

with the solution

\[
h_0 = C x^{-\gamma-1} \exp(-x^{-\gamma}),
\]

and the normalisation gives $C = \gamma$. Thus

\begin{equation}
(2.2) \quad h_0(x) = \gamma x^{-\gamma-1} \exp[-x^{-\gamma}] = \frac{d}{dx} \exp[-x^{-\gamma}].
\end{equation}

See [15], [14], [20], [47].

It will be of interest to compute the Mellin transform of $h_0(x)$ as given by (2.2),
which we denote by $H_0(s)$ for $\Re(s) < 1 + \gamma$

\begin{equation}
(2.3) \quad H_0(s) = \Gamma\left(\frac{1 + \gamma - s}{\gamma}\right).
\end{equation}

Thus we can construct $h_0(x)$ from the inverse Mellin transform

\[
h_0(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds \, x^{-s} \Gamma\left(\frac{1 + \gamma - s}{\gamma}\right), \quad c < 1 + \gamma.
\]

2.2. Case $\alpha = 1$. In the retarded case (1.2) reduces to a linear, homogeneous and
singular integral equation

\begin{equation}
(2.4) \quad h_1(x) = \frac{\gamma}{x} \int_0^x du \, \frac{h_1(u)}{(x-u)^\gamma}.
\end{equation}

**Proposition 2.1.** Let $0 < \Re(\gamma) < 1$. Then the solution $h_1(x)$ has the integral
representation

\begin{equation}
(2.5) \quad h_1(x) =\frac{1}{\pi} \int_0^\infty dr \, \exp[-xr - \cos(\pi \gamma)\Gamma(1-\gamma)r^\gamma] \sin\left(\frac{\pi}{\Gamma(\gamma)} r^\gamma\right).
\end{equation}
Proof. Being a singular integral equation with difference kernel, (2.4) can be treated with a Laplace transform. For $\Re(p) > 0$ let

$$L_1(p) := \int_0^\infty dx \, e^{-px} h_1(x).$$

Then the Laplace transform of (2.4) gives us

$$-\gamma^{-1} \frac{d}{dp} L_1(p) = \int_0^\infty dx \int_0^x du \, e^{-px} h_1(u)(x-u)^{-\gamma},$$

$$= \int_0^\infty du \, h_1(u) \int_u^\infty dx \, e^{-px}(x-u)^{-\gamma},$$

$$= \Gamma(1-\gamma)p^{-\gamma} L_1(p).$$

This latter equation is easily solved for $L_1(p)$ and upon the inversion of the transform yields

$$h_1(x) = C \frac{\Gamma(1-\gamma)}{2\pi} \int_{c-i\infty}^{c+i\infty} dp \, e^{px-\Gamma(1-\gamma)p^\gamma}, \quad c > 0,$$

where $C$ is a normalisation constant. The integrand of the inversion formula has no non-analytic features other than a branch cut in the $p$-plane on $[0, -\infty)$ and it is possible to deform the vertical contour to a Hankel loop $(-\infty, 0, -\infty)$ either side of the cut. Subsequently, in the second proof of Prop. 3.4, we will see that $C = 1$. This then gives the stated result, (2.5).

Remark 2.1. When $\alpha = 1$ we are inevitably lead to the additional constraint of $\Re(\gamma) < 1$. This is clear from the strength of the singularity at the end-point of the integration range in (2.4) and Gamma function $\Gamma(1-\gamma)$ in (2.6).

**Proposition 2.2.** For $\Re(\gamma) < 1$ the density $h_1(x)$ has a convergent large-$x$ expansion

$$h_1(x) = \frac{1}{\pi x} \sum_{n=1}^\infty (-1)^{n-1} \sin(\pi n\gamma) \frac{\Gamma(1+n\gamma)}{n!} \left( \frac{\Gamma(1-\gamma)}{x^\gamma} \right)^n.$$

Proof. Rewriting the integrand of (2.5) in terms of exponentials instead of trigonometric functions and then performing the Taylor expansion of $\exp(-e^{\pm i\pi\gamma}\Gamma(1-\gamma)p^\gamma)$ about $r^\gamma = 0$ the resulting integrals can be done, after recognising the sum and integral are uniformly and absolutely convergent.

Remark 2.2. Formula (2.7) can be simplified when $\gamma \in \mathbb{Q}$. If $\gamma$ is rational then Gauss’s multiplication formula for the Gamma function [45, Eq. (5.5.6)] can be
employed to write the sum in a hypergeometric form. For example one has

\begin{align}
\tag{2.8}
h_1(x; \frac{1}{2}) &= \frac{1}{2x^{3/2}} e^{-\pi/4x}, \\
\tag{2.9}
h_1(x; \frac{1}{3}) &= \frac{\Gamma\left(\frac{2}{3}\right)}{\sqrt[3]{x^{4/3}}} \text{Ai}\left(3^{-1/3}\Gamma\left(\frac{2}{3}\right)x^{-1/3}\right), \\
\tag{2.10}
h_1(x; \frac{2}{3}) &= \frac{4\pi e^{-2\Gamma\left(\frac{4}{3}\right)^3/x^2}}{3^{3/6}\Gamma\left(-\frac{2}{3}\right)x^{7/3}} \\
&\quad \times \left[3^{2/3}x^{2/3}\text{Ai}'\left(3^{2/3}\Gamma\left(\frac{4}{3}\right)^2 x^{-4/3}\right) - 3\Gamma\left(\frac{4}{3}\right)\text{Ai}\left(3^{2/3}\Gamma\left(\frac{4}{3}\right)^2 x^{-4/3}\right)\right],
\end{align}

using the standard definitions of Airy function [45, Chapter 9].

The following result for general \( \gamma \) appears to be simplest identification amongst the extensions of hypergeometric function families.

**Proposition 2.3** (§3.1.3, [28]). The density \( h_1(x; \gamma) \) is the Fox \( H \)-function

\begin{equation}
\tag{2.11} h_1(x; \gamma) = \gamma^{-1} (\Gamma(1 - \gamma))^{-1/\gamma} H^{0, 1}_{1, 1} \left( \frac{x}{(\Gamma(1 - \gamma))^{1/\gamma}} ; \begin{pmatrix} 1 - \gamma^{-1}, \gamma^{-1} \\ (0, 1) \end{pmatrix} \right),
\end{equation}

as per the definition of [30][Eq. (51), pg. 316, Chap. 7].

**Proof.** This follows from the Mellin-Barnes integral representation of the Fox \( H \)-function, see (3.11). This also agrees with the formulae in §3.1.3 of [28], with \( m = n = 1 \) and the scale factor \( c = (\gamma \Gamma(1 - \gamma))^{-1/\gamma}. \)

\[\square\]

![Figure 1. Densities \( h_1(x; \gamma) \) versus \( x \) for \( \gamma = \{\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}\} \).](image)
3. General $\alpha$ Case

The kernel in (1.2) is not separable nor of convolution type unless $\alpha = 1$ so we need to employ more powerful methods in the generic $\alpha$ case, than those utilised in §2. In the general case the Mellin transform of the density $h(x)$ is defined by

$$H(s) := \int_0^\infty dx \, x^{s-1}h(x),$$

for a suitable vertical strip in the $s$-plane, to be clarified below, but containing the line $\Re(s) = 1$. This constitutes our normalisation $H(1) = 1$.

**Proposition 3.1.** For $0 < \alpha < 1$, $\gamma > 0$ and $\Re(s) < 1 + \gamma$ the retarded Mellin transform $H(s; \gamma, \alpha)$ satisfies the linear, homogeneous functional equation

$$H(s) = \frac{\gamma}{1 - \gamma - s} \, 2F_1(\gamma, 1 + \gamma - s; 2 + \gamma - s; \alpha) H(s - \gamma),$$

where $2F_1(a, b; c; z)$ is the Gauss hypergeometric function [45, §15].

**Proof.** Taking the Mellin transform of both sides of (1.2) we compute

$$\int_0^\infty dx \, x^{s-1}h(x) = \gamma \int_0^\infty dx \, x^{s-2} \int_0^x du \, \frac{h(u)}{(x - \alpha u)^\gamma},$$

$$= \gamma \int_0^\infty du \, h(u) \int_u^\infty dx \, \frac{x^{s-2}}{(x - \alpha u)^\gamma},$$

$$= \gamma \int_0^\infty du \, h(u) u^{s-1-\gamma} \int_1^\infty dx \, \frac{x^{s-2}}{(x - \alpha)^\gamma}. $$

The nested integral exists only for $\Re(s) < 1 + \gamma$ and the integrand permits an expansion about $\alpha = 0$, which can be integrated term-by-term to yield

$$\int_1^\infty dx \, \frac{x^{s-2}}{(x - \alpha)^\gamma} = \frac{1}{1 + \gamma - s} \sum_{k=0}^\infty \frac{(\gamma)_k(1 + \gamma - s)_k}{k!(2 + \gamma - s)_k} \alpha^k. $$

This allows us to make the identification with the stated Gauss hypergeometric function and (3.2).

From its definition (3.1) $H(s)$ exists for $s$ in the strip $a < \Re(s) < b$, for some $a < b$. Due to the existence of the norm we have $a < 1 < b$. In addition $H(s - \gamma)$ exists for $a < \Re(s) - \gamma < b$, or $a + \gamma < \Re(s) < b + \gamma$. There must be a common overlap of these two intervals so we conclude $a + \gamma < b$. Exploiting all of the above inequalities we find that $b < 1 + \gamma$ and $1 - \gamma < a$ and so in summary we deduce $1 - \gamma < a < 1 < b < 1 + \gamma$. □

3.1. **On the nature of the** $2F_1(\gamma, 1 + \gamma - s; 2 + \gamma - s; \alpha)$. Firstly with regard to the analytical character of this hypergeometric function with respect to $s$ we can make the following immediate observations:

(a) For $0 < |\alpha| < 1$ the $2F_1$ is convergent for all $\Re(s) < 1 + \gamma$. The $2F_1$ has simple poles on the right at $s = 1 + \gamma + k$ for $k \in \mathbb{Z}_{\geq 0}$. 


(b) For $|\alpha| = 1$ we require in addition that $\Re(\gamma) < 1$. The poles are the same as for $|\alpha| < 1$,

(c) Our $2F_1$ function is one of a pair of solutions to the hypergeometric differential equation about $\alpha = 0$ that is bounded as $s \to -\infty$ (by $(1 - \alpha)^{-\gamma}$).

This distinguishes itself from the other member of the pair which has the simple algebraic form $\alpha^{s-\gamma-1}$, and which diverges as $s \to -\infty$.

This hypergeometric function can linearly mapped into other forms via the Kummer or linear fractional transformations of the $\alpha$-plane, and we record three versions [45, 15.8.1] of these

$$2F_1(\gamma, 1 + \gamma - s; 2 + \gamma - s; \alpha) = (1 - \alpha)^{-\gamma}2F_1(\gamma, 1; 2 + \gamma - s; -\frac{\alpha}{1-\alpha}),$$

$$= (1 - \alpha)^{s-\gamma-1}2F_1(2 - s, 1 + \gamma - s; 2 + \gamma - s; -\frac{\alpha}{1-\alpha}),$$

$$= (1 - \alpha)^{1-\gamma}2F_1(2 - s, 1; 2 + \gamma - s; \alpha).$$

The hypergeometric function is a special case of the most general, because $c = b+1$ and only contains one general argument and two general parameters. Thus it can be identified in a number of ways. In the first instance it is an incomplete Beta function, see [45, 8.17.7],

$$2F_1(\gamma, 1 + \gamma - s; 2 + \gamma - s; \alpha) = (1 + \gamma - s)\alpha^{s-\gamma-1}B_\alpha(1 + \gamma - s, 1 - \gamma).$$

In the second instance it is a generalisation of the Hurwitz-Lerch zeta function, namely the higher-order Apostol-Lerch function defined by the integral [33], [34], [24]

$$\zeta(z, \sigma, \nu, \lambda) = \frac{1}{\Gamma(\sigma)} \int_0^\infty dx \frac{x^{\sigma-1}e^{-x\nu}}{(1-ze^{-x})^\lambda},$$

for $|z| \leq 1, \Re(\nu) > 0, \Re(\sigma) > 1$ and $\Re(\lambda) > 0$. Thus

$$2F_1(\gamma, 1 + \gamma - s; 2 + \gamma - s; \alpha) = (1 + \gamma - s)\zeta(\alpha, 1 + \gamma - s, \gamma).$$

A useful alternative integral representation of the $2F_1$ is

$$2F_1(\gamma, 1; 2 + \gamma - s; \frac{\alpha}{1-\gamma}) = 1 - \alpha\gamma(1 - \alpha) \int_0^1 dt t^{1+\gamma-s}(1 - \alpha t)^{-\gamma-1}. \quad (3.3)$$

### 3.2. Inequalities

We will require bounds on the density in our subsequent analysis, which can be simply derived.

**Proposition 3.2.** For $0 < x < \infty$, $0 < \alpha < 1$ and $\gamma > 0$ then retarded density $h(x; \gamma, \alpha)$ is a real, positive density and satisfies the bounds

$$\gamma x^{-\gamma-1} \exp \left[ -(1 - \alpha)^{-\gamma} x^{-\gamma} \right] \leq h(x) \leq \gamma (1 - \alpha)^{-\gamma} x^{-\gamma-1} \exp \left[ -x^{-\gamma} \right]. \quad (3.4)$$

**Proof.** We start by noting the simple inequality $x^{-\gamma} \leq |x - \alpha u|^{-\gamma} \leq (1 - \alpha)^{-\gamma} x^{-\gamma}$ for $0 \leq u \leq x$. Furthermore define the cumulative distribution function $f(x) := \int_0^x du h(u) < 1$ in the standard way. Employing this inequality in (1.2) we have

$$\gamma x^{-\gamma-1} f(x) \leq h(x) = f'(x) \leq \gamma (1 - \alpha)^{-\gamma} x^{-\gamma-1} f(x).$$
Integrating this from the reference point \( x_0 = \infty \) \( (f(\infty) = 1) \) to \( x < \infty \) without crossing any standard placement of the branch cut, e.g. \([0, -\infty)\), we deduce a reversed inequality

\[
\exp[-(1 - \alpha)\gamma x^{-\gamma}] \leq f(x) \leq \exp[-x^{-\gamma}].
\]

Combining these two inequalities we conclude (3.4).

\[\square\]

**Corollary 3.1.** Let \( \Re(s) = \sigma \). For \( \sigma < 1 + \gamma \) the Mellin transform \( H(s; \gamma, \alpha) \) satisfies the bound

\[
|H(s)| \leq (1 - \alpha)^{-\gamma} \Gamma \left( \frac{1 + \gamma - \sigma}{\gamma} \right).
\]

**Proof.** This follows immediately from (3.4). It should be noted that the stronger result \( |H(s)| \leq 1 \) holds when \( \Re(s) = 1 \). \[\square\]

If one were to define a new transform \( K(s) \) by

\[
K(s) := \frac{(1 - \alpha)^s}{\Gamma(1 + \gamma - 1 - \gamma^{-1}s)} H(s),
\]

then the functional equation becomes

\[
K(s) = 2 F_1(\gamma, 1; 2 + \gamma - s; \frac{\alpha}{\alpha - 1}) K(s - \gamma).
\]

This, or slight variants of it, will turn out to have better analytic properties with respect to \( s \).

3.3. **Check with** \( \alpha = 0 \). At \( \alpha = 0 \) the retarded functional equation (3.2) reduces to

\[
H_0(s) = \frac{\gamma}{1 + \gamma - s} H_0(s - \gamma).
\]

**Proposition 3.3.** The solution of (3.8) is

\[
H_0(s) = \Gamma \left( \frac{1 + \gamma - s}{\gamma} \right).
\]

**Proof.** Equation (3.8) is a linear, homogeneous first order recurrence relation with the particular solution (2.3) or (3.9) which also satisfies the boundary condition \( H(1) = 1 \). It is also the only solution because the only permissible \( \gamma \)-periodic function \( P(s) \), satisfying \( P(s) = P(s - \gamma) \), and making \( H_0(s)P(s) \) another solution, is the constant function. This crucial fact follows because \( |H(s)| \) must be bounded (by unity) as \( \Im(s) \to \pm \infty \) with \( \Re(s) = 1 \). \[\square\]

We can also verify (3.8) is satisfied by (2.3) from the direct computation of its Mellin transform as given by (2.2).
3.4. Check with $\alpha = 1$. For $\alpha = 1$ the Gauss hypergeometric function evaluates to
\[ _2F_1(\gamma, 1 + \gamma - s; 2 + \gamma - s; 1) = \frac{\Gamma(1 - \gamma)\Gamma(2 + \gamma - s)}{\Gamma(2 - s)}, \]
and the functional equation becomes
\[ (3.10) \quad H_1(s) = \frac{\gamma\Gamma(1 - \gamma)\Gamma(1 + \gamma - s)}{\Gamma(2 - s)} H_1(s - \gamma). \]

**Proposition 3.4** (see Eq. (11), §2.1 of [28]). The solution of (3.10) is
\[ (3.11) \quad H_1(s) = (\Gamma(1 - \gamma))^{(s-1)/\gamma} \frac{\Gamma\left(1 + \frac{\gamma - s}{\gamma}\right)}{\Gamma(2 - s)}. \]

**First Proof.** Define $t = s/\gamma$ and $P_1(t)$ by
\[ H_1(s) = (\Gamma(1 - \gamma))^t \sin\left(\pi\gamma t\right)\Gamma(\gamma t)\Gamma(\gamma^{-1} - t)P_1(t). \]
Then (3.10) translates as $P_1(t + 1) = P_1(t)$ and so $P_1$ is a 1-periodic function. However $P_1(t)$ is, in fact a constant, due to identical arguments given in the proof of Proposition 3.3. This constant value is
\[ P_1 = \frac{1}{\pi\gamma (\Gamma(1 - \gamma))^{1/\gamma}}, \]
and the stated result follows. This result can also be found from a specialisation of Eq. (11), §2.1 of [28].

**Second Proof.** We can directly compute the Mellin transform of (2.5) and verify (3.11). Taking this transform we compute
\[ H_1(s) = \frac{C \Gamma(s)}{\pi} \int_0^\infty dr \ r^{-s} \sin\left(\frac{\pi}{\Gamma(\gamma)} r\right) e^{-\cos(\pi\gamma)\Gamma(1-\gamma)r^{\gamma}}. \]
under the condition $\Re(s) < 1 + \gamma$ and the more restrictive $\Re(\gamma) < \frac{1}{2}$, $(\cos(\pi\gamma) > 0)$. Under an internal change of variable this integral becomes a known one, [17][pg. 490, 3.944, # 5], with an evaluation leading to (3.11). In addition we see that $C = 1$. \qed

3.5. **The general solution.** Treating the functional equation (3.2) as a recurrence relation and recurring to the left $k$ times we deduce
\[ (3.12) \quad H(s - k\gamma) = \prod_{l=1}^k \left(1 + \frac{l\gamma - s}{\gamma}\right) \frac{1}{\Gamma(1 + \gamma - s; 2 + l\gamma - s; \alpha)} H(s), \]
and thus setting $s = 1$
\[ (3.13) \quad H_k := H(1 - k\gamma) = k! \prod_{l=1}^k \frac{1}{\Gamma(1 + l\gamma; 1 + l\gamma; \alpha)}. \]
This is meaningful because the denominators never vanish and $H(s)$ is analytic for $\Re(s) < 1 + \gamma$ as we demonstrate in the following Lemma.
Definition 3.1. Let $F_j := _2F_1(\gamma, j\gamma; 1 + j\gamma; \alpha)$ for $j \in \mathbb{Z}_{\geq 0}$.

Lemma 3.1. Let $0 < \alpha < 1$ and $\gamma > 0$. The hypergeometric factors $F_j$ satisfy

1. $F_0 = 1$,
2. are monotonically increasing, $F_{j+1} > F_j$, and
3. are bounded above, $F_j < \alpha^{-\gamma}$.

Proof. The factors have the integral representation

\[
F_j = (1 - \alpha)^{-\gamma} - \alpha \gamma \int_0^1 du u^{\gamma - 1} (1 - \alpha u)^{-\gamma - 1}.
\]

From this we observe that

\[
F_{j+1} - F_j = \alpha \gamma \int_0^1 du u^{\gamma - 1} (1 - \alpha u)(1 - \alpha u)^{-\gamma - 1} > 0.
\]

In addition $\alpha \gamma \int_0^1 du u^{\gamma - 1} (1 - \alpha u)^{-\gamma - 1} > 0$ so that $F_j < \alpha^{-\gamma}$. Note that $F_\infty = _2F_1(\gamma, -; -; \alpha) = (1 - \alpha)^{-\gamma}$. □

In contrast recurring to the right encounters the simple poles at $s = 1 + k\gamma$, as exemplified by the formula

\[
H(s + k\gamma) = \gamma^k \prod_{l=0}^{k-1} \frac{1}{1 - \gamma - s} _2F_1(\gamma, 1 - l\gamma - s; 2 - l\gamma - s; \alpha) \times H(s).
\]

Proposition 3.5. The consecutive product of hypergeometric functions has the leading order behaviour as $N \to \infty$

\[
\prod_{k=1}^N _2F_1(\gamma, 1; 2 + k\gamma - s; \frac{\alpha}{\alpha - 1}) \sim _{N \to \infty} C \left[ N + 1 + \frac{2 - s}{\gamma} \right]^{-\alpha/(1 - \alpha)},
\]

where $C$ is a constant independent of $N$. An alternative form of this is

\[
\prod_{k=1}^N _2F_1(\gamma, k\gamma; 1 + k\gamma; \alpha) \sim _{N \to \infty} C(1 - \alpha)^{-N\gamma} \left[ N + 1 + \frac{1}{\gamma} \right]^{-\alpha/(1 - \alpha)}.
\]

Proof. Firstly we write

\[
(1 - \alpha)^\gamma _2F_1(\gamma, 1 - s + k\gamma; 2 - s + k\gamma; \alpha) = _2F_1(\gamma, 1; 2 + k\gamma - s; \frac{\alpha}{\alpha - 1}) = 1 + A_k + B_k,
\]

where

\[
A_k = -\frac{\alpha}{1 - \alpha} \frac{\gamma}{2 - s + k\gamma}, \quad B_k = \frac{\alpha^2 \gamma (\gamma + 1)(1 - \alpha)^\gamma}{2 - s + k\gamma} \int_0^1 du u^{2-s+k\gamma}(1 - \alpha u)^{-\gamma - 2}.
\]

This is valid because $\Re(s) < 1 + \gamma$ assures that $2 - \Re(s) + k\gamma > 0$ for $k \geq 1$. We will also require $1 + A_k > 0$ and this requires the slightly stronger condition $2 - \Re(s) + k\gamma > \gamma \frac{\alpha}{1 - \alpha}$. Splitting the foregoing sum in the following way

\[
\log (1 + A_k + B_k) = \log (1 + A_k) + \log \left( 1 + \frac{B_k}{1 + A_k} \right),
\]
and employing the elementary inequality \[ |\log \left( 1 + \frac{B_k}{1 + A_k} \right) | \leq \frac{|B_k|}{|1 + A_k|} \] along with the bound
\[
|B_k| \leq \frac{\alpha^2}{(1 - \alpha)^2} \frac{\gamma (\gamma + 1)}{(2 - s + k\gamma)(3 - s + k\gamma)},
\]
we see that the second term in (3.18) is of \(O(k^{-2})\) as \(k \to \infty\). Thus in the sum over \(k\) it contributes a term of only \(O(1)\) as \(N \to \infty\).

Now focusing on the first term of (3.18) we see that this is bracketed by another elementary inequality
\[
A_k - \frac{1 - 2 A_k^2}{1 + A_k} \leq \log (1 + A_k) \leq A_k - \frac{1 - 2 A_k^2}{1 + A_k} (\text{as } A_k < 0).
\]
Again the second terms of the lower and upper bounds are of \(O(k^{-2})\) as \(k \to \infty\) and contribute a term of only \(O(1)\) as \(N \to \infty\). Collecting all of these contributions in the \(k\)-sum we have, as \(N \to \infty\),
\[
\sum_{k=1}^{N} \log \,_{2} F_{1}(\gamma, 1; 2 + k\gamma - s; \frac{\alpha}{\alpha - r}) = -\frac{\alpha}{1 - \alpha} \sum_{k=1}^{N} \frac{\gamma}{2 - s + k\gamma} + O(1),
\]
\[
= -\frac{\alpha}{1 - \alpha} \psi(N + 1 + (2 - s)/\gamma) + O(1),
\]
\[
= -\frac{\alpha}{1 - \alpha} \log \left[ N + 1 + (2 - s)/\gamma \right] + O(1).
\]
Thus (3.16) follows. \(\square\)

For \(k \in \mathbb{Z}_{\geq 0}\) we have a denumerable sequence of \(K(s_k)\) values and we know that \(K(s)\) is bounded in the finite \(s\)-plane. Clearly \(K(s)\) is entire, and the question we pose is that of interpolation of an entire function, i.e. reconstructing it from this sequence of values on \(\mathbb{Z}_{\geq 0}\). This question was addressed in 1884 with the Mittag-Leffler theorem [37], [10], [7], [46], by the Pringsheim interpolation formula [44], and by Guichard’s theorem [19]. However use of these involve restrictions which are not satisfied by \(K(s)\).

A more useful function turns out to be
\[
(3.19) \quad G(t) := \frac{1}{\Gamma \left( 1 + \frac{1 - s}{\gamma} \right)} H(s), \quad t := \frac{1 - s}{\gamma},
\]
which eliminates the simple poles at \(s = 1 + k\gamma, k \in \mathbb{N}\), so that in the finite-\(t\) plane
\[
(3.20) \quad |G(t)| \leq (1 - \alpha)^{-\gamma} [\cosh(\pi \Im(t))]^{1/2}.
\]
However, this is at the expense of a growth \(\exp(\pi |\Im(t)|/2)\) as \(\Im(t) \to \pm \infty\). At the prescribed values \(s = 1 - k\gamma, k \in \mathbb{Z}_{\geq 0}\), we deduce
\[
(3.21) \quad G_k := G(t = k) = \frac{1}{\prod_{l=1}^{k} \,_{2} F_{1}(\gamma, l\gamma; 1 + l\gamma; \alpha)},
\]
which have the growth properties
\[
(3.22) \quad G_k \sim (1 - \alpha)^{k\gamma} \left[ k + 1 + \frac{1}{\gamma} \right]^{\alpha/(1 - \alpha)}.\]
Definition 3.2. Recall $F_j := z F_1(\gamma, j \gamma; 1 + j \gamma; \alpha)$ for $j \in \mathbb{N}$, and let $t = (1-s)/\gamma$.

Define the analytic generating function $g(x; \gamma, \alpha)$

\begin{equation}
(3.23) \quad g(x) := \sum_{l=0}^{\infty} \frac{(-x)^l}{l!} G_l = \sum_{l=0}^{\infty} \frac{(-x)^l}{l!} \prod_{j=0}^{l} F_j.
\end{equation}

Proposition 3.6. $g(x)$ is an entire function of $x$ for $\gamma > 0$ and $0 < \alpha < 1$ due to (3.22).

Because of the bounded growth (3.20) and consequent analyticity of $G(t)$ we can use the Ramanujan interpolation formula, also known as the Ramanujan master theorem, given in the first of his quarterly reports of 1913, and published in [2, §1.2, pg. 298, Ramanujan Notebooks, Part I]

\begin{equation}
(3.24) \quad \Gamma(t) G(-t) = \int_{0}^{\infty} dx \ x^{t-1} g(x) = \mathcal{M}[g(x); t].
\end{equation}

See [1, §10.12, pg. 550 and Ex. 34(b)] and [23, Chap. XI] for proofs of this relation under the same hypotheses of Carlson’s theorem, and [4, App. B] for recent developments with these methods. Here we recall some details of Hardy’s theorem and proof, which is framed in terms of the quantity

\begin{equation}
(3.25) \quad \phi(u) := \frac{G(u)}{\Gamma(1+u)}.
\end{equation}

and the half-plane $\mathcal{H}(\delta) := \{ \Re(u) > -\delta, 0 < \delta < 1 \}$.

Proposition 3.7 (Hardy 1940, Chap. XI). Let

1. $\phi(u)$ be regular in $\mathcal{H}(\delta)$,
2. $|\phi(u)| < C \exp [P \Re(u) + A|\Im(u)|]$ in $\mathcal{H}(\delta)$, and
3. $A < \pi$.

Then (3.24) is valid.

In our case we have (3.20) and

\begin{equation}
(3.26) \quad |\phi(u)| \leq (1 - \alpha)^{-\gamma} c \cosh(\pi |\Im(u)|) \frac{1}{\Gamma(1 + \Re(u))},
\end{equation}

so $A = \pi$, and we need to extend the result of Hardy.

Corollary 3.2 ([5], Lemma 3.1). Prop. 3.7 extends to $A = \pi$, in particular the integrand of the inverse Mellin transform to (3.24) has algebraic decay

\begin{equation}
(3.27) \quad |\Gamma(t) G(-t)| \leq (1 - \alpha)^{-\gamma} \Gamma(1 - \Re(t)) |\Im(t)|^{2\Re(t)-1},
\end{equation}

as $|\Im(t)| \to \infty$. This means that $G(-t) \in Q_{\infty}(2, \frac{1}{2})$, one of the classes defined in [5]. Consequently the inverse Mellin transform converges absolutely and uniformly for $\Re(t) < 0$.

Proof. From (3.5) we have

\begin{equation}
(3.28) \quad |G(-t)| \leq (1 - \alpha)^{-\gamma} \frac{\Gamma(1 - \Re(t))}{|\Gamma(1 - t)|},
\end{equation}
and from Eq. 2.4.4 of §2.4 of [41] we have
\[
\log \left| \Gamma(t) \right| \rightarrow -\frac{1}{2} \pi |\Re(t)| - \frac{1}{2} \log |\Im(t)|,
\]
\[
\log \left| \Gamma(1-t) \right| \rightarrow -\frac{1}{2} \pi |\Re(t)| + (\frac{1}{2} - \Re(t)) \log |\Im(t)|.
\]
Thus the exponential decay of the integrand vanishes and one is left with the algebraic decay of (3.27).

As a consequence of (3.19) we have
\[
(3.29) \quad \mathcal{M}[h(x); s] = \frac{\Gamma\left(\frac{1 + \gamma - s}{\gamma}\right)}{\Gamma\left(\frac{s - 1}{\gamma}\right)} \mathcal{M}\left[\frac{s - 1}{\gamma}, g(x); s - 1\right].
\]
This relationship can be translated uniquely into one connecting \(h(x)\) and \(g(x)\), given that the inverse Mellin transform of the ratio on the right-hand side of the above equation is available.

**Lemma 3.2.** Let \(g(x)\) be defined by (3.23). Then the density \(h(x; \gamma, \alpha)\) is given by the Hankel transform of order zero of \(g(\cdot)\)
\[
(3.30) \quad h(x) = \gamma^2 x^{\gamma-1} \int_0^\infty dt \ t^{-\gamma-1} J_0(2[xt]^{-\gamma/2}) g(t^{-\gamma}).
\]
Here \(J_0(z)\) is the standard Bessel function of order zero. The inversion of this Hankel transform is
\[
(3.31) \quad t^{-\gamma/4} g(t^{-\gamma}) = \int_0^\infty dx \ J_0(2[tx]^{-\gamma/2}) h(x).
\]
**Remark 3.1.** A check can be made in the special case of \(\alpha = 0\). Here \(F_j = 1\) and \(g(x) = e^{-x}\). One finds that (3.30) reduces to
\[
\frac{1}{2} \exp(-x^{-\gamma}).
\]
which is a known integral, see [17, pg. 716, 6.631, #(1)], and that this evaluates to
\[
\frac{1}{2} \exp(-x^{-\gamma}).
\]
Thus we recover \(h_0(x)\) as given by (2.2).

**Remark 3.2.** In the case \(\alpha = 1\) we find
\[
F_j = \Gamma(1 - \gamma) \frac{\Gamma(1 + j\gamma)}{\Gamma(1 + (j - 1)\gamma)}, \quad \prod_{j=0}^l F_j = (\Gamma(1 - \gamma))^j \Gamma(1 + l\gamma).
\]
Consequently we find that the following result.

**Proposition 3.8.** The generating function \(g\) for \(\alpha = 1\) is a Wright Bessel function of the form
\[
(3.32) \quad g(x) = \phi(\gamma, 1, -x/\Gamma(1 - \gamma)),
\]
in the notation of [30, pg. 241]. Other notations express this as, see [51],
\[
\frac{1}{2} \exp(-x^{-\gamma}).
\]
or
\[ g(x) = W(\gamma, 1; -x/\Gamma(1 - \gamma)). \]

Using the known Mellin transform of the Wright Bessel function we can compute that
\[ G(t) = \frac{1}{\Gamma\left(\frac{s-1}{\gamma}\right)} \mathcal{M}\left[ g(x); \frac{s-1}{\gamma} \right] = \frac{(\Gamma(1 - \gamma))^{(s-1)/\gamma}}{\Gamma(2 - s)}. \]

This agrees with the evaluation of \( G(t) \) using \( H_1(s) \) from (3.11).

Combining (3.32) and (3.30) with (2.11) we are lead to the following identity

\[
\gamma^2 x^{-\gamma-1} \int_0^{\infty} dt \, t^{-\gamma-1} J_0(2[xt]^{-\gamma/2}) \phi(\gamma, 1, -t^{-\gamma}/\Gamma(1 - \gamma))
= \gamma^{-1} (\Gamma(1 - \gamma))^{-1/\gamma} H_{0,1}^{1,1} \left( \frac{x}{(\Gamma(1 - \gamma))^{1/\gamma}}; (1 - \gamma^{-1}, 1) \right). \tag{3.33}
\]

It remains to directly verify this.

**Lemma 3.3.** Let \( 0 < \alpha < 1 \). The finite product \( G_t \), defined by (3.21), can be written as an analytic function of complex \( \Re(l) > 1 + \gamma^{-1} \) in an infinite product form

\[
G_t = (1 - \alpha)^\gamma t \prod_{j=1}^{\infty} \frac{F_{j+l}}{F_j}. \tag{3.34}
\]

and its interpolating function is

\[
G(-t) = (1 - \alpha)^{-\gamma t} \prod_{j=1}^{\infty} \frac{F_{j-t}}{F_j}. \tag{3.35}
\]

A useful relation, to be used later is

\[
F_{-t} \prod_{j=1}^{\infty} \frac{F_{j-t}}{F_j} = F_{\infty} \prod_{j=0}^{\infty} \frac{F_{j-t}}{F_j}. \tag{3.36}
\]

**Proof.** To show convergence of the product, [29, §29, Satz 5, s. 228], we compute bounds for \( \frac{F_{j+l}}{F_j} - 1 \). The factors can be written (see (3.3))

\[
F_j = (1 - \alpha)^{-\gamma} - \alpha \gamma \int_0^1 du \, u^\gamma (1 - \alpha u)^{-\gamma-1}. \]
Then we note

\[
\left| \frac{F_{j+1}}{F_j} - 1 \right| = \frac{\alpha \gamma (1 - \alpha)^\gamma \int_0^1 du \,(1 - \alpha u)^{-\gamma - 1} [u^\gamma - u^{(j+t)\gamma}]}{1 - \alpha \gamma (1 - \alpha)^\gamma \int_0^1 du \,(1 - \alpha)^{-\gamma - 1} u^\gamma},
\]

\[
\leq \frac{\alpha \gamma (1 - \alpha)^{-1} \int_0^1 du \,[u^\gamma - u^{(j+t)\gamma}]}{1 - \alpha \gamma (1 - \alpha)^{-1} \int_0^1 du \,u^\gamma},
\]

\[
= \frac{\alpha \gamma^2 l}{[1 + (j + l)\gamma] [1 - \alpha(1 + \gamma) + j\gamma(1 - \alpha)]}
\]

\[
= O(j^{-2}) \text{ as } j \to \infty,
\]

uniformly for \(l, \alpha, \gamma\) under the conditions stated.

We recall \(\lim_{j \to \infty} F_j = F_\infty = (1 - \alpha)^{-\gamma}\) is well defined. The situation we have is that we take \(N \to \infty\) for fixed \(l\) so

\[
\prod_{j=1}^N \frac{F_{j+1}}{F_j} = \frac{F_{N+1} \ldots F_{N+l}}{F_1 \ldots F_l}, \quad \text{as } N \geq l + 1,
\]

\[
\lim_{N \to \infty} \frac{1}{F_1 \ldots F_l} \prod_{j=l}^{N} F_{N+1} \ldots F_{N+l},
\]

\[
= F_\infty^l G_l,
\]

and (3.34) follows. Now set \(l \mapsto l - 1\) or \(t \mapsto t + 1\) in (3.35)

\[
\prod_{j=1}^N \frac{F_{j-1+l}}{F_j} = \frac{F_{N+1} \ldots F_{N+l-1}}{F_1 \ldots F_{l-1}}, \quad \text{as } N \geq l,
\]

\[
\lim_{N \to \infty} \frac{1}{F_1 \ldots F_{l-1}} \prod_{j=l-1}^{N} F_{N+1} \ldots F_{N+l-1},
\]

\[
= F_\infty^{l-1} G_{l-1},
\]

\[
\prod_{j=1}^\infty \frac{F_{j-1+l}}{F_j} = \frac{F_l}{F_\infty} \prod_{j=1}^\infty \frac{F_{j+l}}{F_j}.
\]

Thus (3.36) also follows. \(\square\)

In summary we have our penultimate result where the generating function and density are shown to possess integral representations.

**Proposition 3.9.** Let \(0 \leq \alpha < 1 - \delta, \delta > 0\). The generating function \(g(x; \gamma, \alpha)\) has the Mellin-Barnes integral representation \(0 < c < 1 + \gamma^{-1}\)

\[
(3.37)
\]

\[
g(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dt \, x^{-t} \Gamma(t) (1 - \alpha)^{-\gamma t} \prod_{j=1}^\infty \frac{2F_1(\gamma, (j-t)\gamma; 1 + (j-t)\gamma; \alpha)}{2F_1(\gamma, j\gamma; 1 + j\gamma; \alpha)},
\]
and consequently the density \( h(x; \gamma, \alpha) \) has a similar representation \( c < 1 \)

\[
(3.38) \quad h(x) = \frac{\gamma}{2\pi i x} \int_{c-i\infty}^{c+i\infty} dt \, x^{-\gamma t} \Gamma(1-t)(1-\alpha)^{-\gamma t} \prod_{j=1}^{\infty} \frac{\mathbf{2}F_{1}(\gamma, (j-t)\gamma; 1 + (j-t)\gamma; \alpha)}{\mathbf{2}F_{1}(\gamma, j\gamma; 1 + j\gamma; \alpha)}.
\]

The Mellin transform of the density \( H(s; \gamma, \alpha) \) is

\[
(3.39) \quad H(s) = (1 - \alpha)^{1-s} \Gamma \left( \frac{1 + \gamma - s}{\gamma} \right) \prod_{j=1}^{\infty} \frac{\mathbf{2}F_{1}(\gamma, 1-s + j\gamma; 2 - s + j\gamma; \alpha)}{\mathbf{2}F_{1}(\gamma, j\gamma; 1 + j\gamma; \alpha)}.
\]

**Proof.** The exponential generating function sum can be written as a Mellin-Barnes inversion integral of (3.24)

\[
(4.0) \quad g(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} dt \, x^{-t} \Gamma(t) G(-t),
\]

given convergence of the integrand as \( \Im(t) \to \pm \infty \) on the vertical contour \( \Re(t) = c \).

The second formula follows from the first under the absolute convergence of both integrals and the validity of the integral

\[
\int_{0}^{\infty} du \, u^{1-2t} J_{0}(zu) = 2^{1-2t} \pi^{-2} \frac{\Gamma(1-t)}{\Gamma(t)},
\]

when \( \frac{3}{4} < \Re(t) < 1 \). \( \square \)

**Second Proof.** There is a simple and direct way to verify (3.38). For generic \( \gamma \notin \mathbb{Q} \) the full set of singularities of the integrand in (3.38) are: a sequence of simple poles due to \( \Gamma(1-t) \) at \( t_{k} = k, \, k \in \mathbb{N} \), and a double sequence of simple poles due to the denominator factor in \( \mathbf{2}F_{1}(\gamma, (j-t)\gamma; 1 + (j-t)\gamma; \alpha) \) at \( t_{j,k} = j + \gamma^{-1}k \) for \( j, k \in \mathbb{N} \). Thus we are able to displace the contour to the left so that \( c < 0 \), which will be necessary for our derivation. Now we substitute (3.38) into the right-hand side of (1.2)

\[
\text{RHS} = \frac{\gamma^{2}}{2\pi i x} \int_{c-i\infty}^{c+i\infty} dt \, \Gamma(1-t)(1-\alpha)^{-\gamma t} \prod_{j=1}^{\infty} \frac{F_{j-t}}{F_{j}} \int_{0}^{x} du \, (x - \alpha u)^{-\gamma} u^{-1-\gamma t},
\]

where we have abbreviated the parameter dependence of the hypergeometric functions and employed the absolute and uniform convergence of the integrals for \( \Re(t) < -1 \) to exchange the order. The \( u \)-integration evaluates to

\[
\frac{x^{-\gamma-\gamma t}}{-\gamma t} 2F_{1}(\gamma, -t\gamma; 1 - t\gamma; \alpha) = \frac{x^{-\gamma-\gamma t}}{-\gamma t} F_{-t}.
\]

Simplifying the right-hand side using (3.36) we find

\[
\text{RHS} = \frac{\gamma}{2\pi i x} \int_{c-i\infty}^{c+i\infty} dt \, x^{-\gamma(t+1)} \Gamma(-t)(1-\alpha)^{-\gamma-\gamma t} \prod_{j=0}^{\infty} \frac{F_{j-t}}{F_{j}},
\]

and making a translation of the integration variable \( v = t + 1 \) we see

\[
\text{RHS} = \frac{\gamma}{2\pi i x} \int_{c+1-i\infty}^{c+1+i\infty} dv \, x^{-\gamma v} \Gamma(1-v)(1-\alpha)^{-\gamma v} \prod_{j=0}^{\infty} \frac{F_{j+1-v}}{F_{j+1}}.
\]
Because $c + 1 < 1$ we can shift the contour one unit to the left thus restoring the original one, and the product index can be relabelled, we conclude that the resulting integral is now identical to (3.38).

Remark 3.3. The observations on the large $N$ behaviour of the products in Prop. 3.5 and the integral representations given above show that the previous Proposition does not hold in the neighbourhood of $\alpha = 1$. When $\alpha = 1$ the finite product has the following evaluation

$$L \prod_{j=1}^{L} \frac{\Gamma(L\gamma + 1 - t\gamma)}{\Gamma(L\gamma + 1)} \frac{\Gamma(1 - t\gamma)}{\Gamma(1 - t\gamma)},$$

and so the limit vanishes. Furthermore we know from (3.11) that the inverse Mellin transform shows, for $c < 1 + \gamma$,

$$h_1(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds x^{-s} \frac{\Gamma(1 - \gamma)}{\Gamma(s - 1)} \frac{\Gamma(1 + \gamma - s)}{\Gamma(2 - s)}.$$

Clearly there is a transition regime from $\alpha < 1$ to $\alpha = 1$ where the limiting behaviour as $L \to \infty$ crosses over from one to the other - a so-called "double scaling limit". We can explicitly demonstrate this with the following matching procedure:

using (3.42) the relevant part of the integrand of (3.38) is

$$L \to \infty \quad \alpha \to 1$$

$$\left(1 - \alpha\right)^{-\gamma t} \prod_{j=1}^{L} \frac{F_j}{F_j} \sim \frac{\left(\gamma(1 - \alpha)L\right)^{-\gamma t}}{\Gamma(1 - t\gamma)}.$$

Now if we change variables $t = (s - 1)/\gamma$ the integrand of (3.38) will match that of (3.43) if we simultaneously take $L \to \infty$ with $\alpha \to 1^+$ such that the product

$$(1 - \alpha)L = \frac{1}{\gamma(\Gamma(1 - \gamma))^{1/\gamma}},$$

is fixed.

3.6. The large $\gamma$ case. A particularly instructive case of our foregoing analysis is when $\gamma$ is real and large whilst $0 < \alpha < 1$. In this way we have a simple example where $\alpha$ is explicitly exhibited. It is apparent that in the opposite case of $\gamma \to 0$ only the trivial solution of $h(x)$ with Dirac-atomic measure at the origin is admissible. In this regime the leading order behaviour of the hypergeometric function is [40], [9] is

$$2F_1\left(a + \epsilon\lambda, b; c + \lambda; z\right) \sim (1 - \epsilon z)^{-b}, \quad |\lambda| \to \infty,$$

excluding the region around the singular point $z_S = 1/\epsilon$. In our application we require the connection formula

$$2F_1\left(\gamma_j, (k - t)\gamma; 1 + (k - t)\gamma; \alpha\right) = (1 - \alpha)^{-\gamma} 2F_1\left(\gamma, 1 + (k - t)\gamma; -\frac{\alpha}{1 - \alpha}\right),$$
with $\lambda = (k-t)\gamma$ and $0 < \epsilon = 1/(k-t) < 1$ so our singular point is

$$\alpha_S = 1 + 1/(k-1-t) > 1,$$

which is excluded from our consideration. Using the definition 3.1 we have

$$F_j \sim (1-\alpha)^{-\gamma} \left[1 + \frac{1}{j} \frac{\alpha}{1-\alpha}\right]^{-1}, \quad j \geq 1.$$

With this knowledge one can easy verify the following results.

**Proposition 3.10.** For $0 \leq \alpha < 1$ and for $\gamma \to \infty$ the leading order hypergeometric product $G_l$, $l \geq 0$ is

$$G_l = (1-\alpha)^\gamma \frac{\Gamma(l + \frac{1}{1-\alpha})}{\Gamma(\frac{1}{1-\alpha})}.$$

Consequently the generating function is given by the confluent hypergeometric function

$$g(x) = {}_1F_1(\frac{1}{1-\alpha}; 1; -(1-\alpha)^\gamma x).$$

This is an entire function of $x$ and $\frac{1}{1-\alpha}$ with exponential decay as $x \to \infty$ and its Mellin transform is

$$\mathcal{M}[g(x); t] = (1-\alpha)^{-\gamma t} \frac{\Gamma(t)\Gamma(\frac{1}{1-\alpha} - t)}{\Gamma(1-t)\Gamma(\frac{1}{1-\alpha})},$$

for $0 < \Re(t) < \frac{1}{1-\alpha}$, using Oberhettinger’s tables [38, 13.49, pg. 145]. Using the Master Theorem (3.24) we obtain the interpolating function

$$G(t) = (1-\alpha)^{\gamma t} \frac{\Gamma(\frac{1}{1-\alpha} + t)}{\Gamma(1+t)\Gamma(\frac{1}{1-\alpha})}.$$

**Proof.** The steps indicated above are self-explanatory. One can also directly solve the functional equation (3.2) directly. Expanding to sufficient order for large $\gamma$

$$\frac{\gamma}{1 + \gamma - s} {}_2F_1(\gamma, 1 + \gamma - s; 2 + \gamma - s; \alpha)$$

$$= \frac{\gamma}{1 + \gamma - s} (1-\alpha)^{1-\gamma} {}_2F_1(2 - s, 1; 2 + \gamma - s; \alpha),$$

$$= (1-\alpha)^{1-\gamma} \left[1 + \frac{\alpha}{\gamma} + \frac{(\alpha-1)(1-s)}{\gamma} + O(\gamma^{-2})\right],$$

$$= (1-\alpha)^{-\gamma} \left[\frac{1-\alpha/\gamma}{1-\alpha} + \frac{1-s}{\gamma}\right]^{-1} \{1 + O(\gamma^{-2})\}.$$

Solving the functional equation

$$\frac{H(s)}{H(s-\gamma)} = \frac{(1-\alpha)^{-\gamma}}{\frac{1-\alpha/\gamma}{1-\alpha} + \frac{1-s}{\gamma}}.$$
in the same manner as in the proof of (3.4) and appealing to the bound (3.5) for \(\Re(s) \to \pm \infty\) we deduce

\[(3.55)\quad H(s) = (1 - \alpha)^{1-s} \frac{\Gamma(\frac{1-\alpha}{1-\alpha} + \frac{1-\gamma}{\gamma})}{\Gamma(\frac{1-\alpha}{1-\alpha})}.\]

Recalling (3.29) this gives us (3.52) after discarding the small term \(\alpha/\gamma\). \[\square\]

Remark 3.4. Inverting the Mellin transform (3.55) gives an elementary function form for the density

\[(3.56)\quad h_\gamma(x) = (1 - \alpha)^{\frac{\gamma - \alpha}{1 - \alpha}} \frac{1}{\Gamma(\frac{1-\alpha}{1-\alpha})} x^{1-\gamma} \exp \left[-(1 - \alpha)^{-\gamma} x^{-\gamma}\right].\]

3.7. Continued Fraction Forms. Our final representation of \(g(x)\) is as a continued fraction, however this is best expressed in terms of a related function rather than \(g\) itself.

Definition 3.3. Let \(L(z; \gamma, \alpha)\) be the Laplace transform of the generating function \(g(x)\), for \(\Re(z) > 0\).

\[(3.57)\quad L(z) := \mathcal{L}[g(x); z] = \int_0^\infty dx \, e^{-zx} g(x).\]

Thus we have the convergent expansion due to (3.22) for \(|z| > (1 - \alpha)^\gamma\)

\[(3.58)\quad L(z) = \sum_{l=0}^{\infty} \frac{(-1)^l}{\prod_{j=0}^{l} F_j} z^{-l-1}.\]

Proposition 3.11. \(L(z)\) is a special \(M\)-continued fraction, see [8, pg. 38, Chap. 1], given by

\[(3.59)\quad \frac{1}{L(z)} = z + \sum_{m=1}^{\infty} \frac{F_{m-1}z}{F_m z - 1},\]

where our notation is equivalent to the alternative forms

\[b_0 + \sum_{j=1}^{\infty} \frac{a_j}{b_j} = b_0 + \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3 + \ldots}}}.\]

Proof. Eq. (3.59) is found by adapting a result known to Euler, see [8, (1.7.2)], for the recasting of a series

\[(3.60)\quad e_n = 1 + \sum_{k=1}^{n} \prod_{j=1}^{k} (-E_j) z^k,\]
\[ e_n = \frac{1}{1 + \sum_{j=1}^{n} \frac{E_j z}{1 - E_j z}}. \]

It is straightforward to show the partial numerators and denominators \( A_n \) and \( B_n \) of \( e_n = \frac{A_n}{B_n} \) satisfy the recurrence relations
\[
A_n = b_n A_{n-1} + a_n A_{n-2},
\]
\[
B_n = b_n B_{n-1} + a_n B_{n-2},
\]
with \( b_0 = 0, a_1 = 1, b_1 = 1 \) and for \( m \geq 1, a_{m+1} = E_m z \) and \( b_{m+1} = 1 - E_m z \). The solutions to this recurrence give \( B_n = 1 \) and \( A_n \) the right-hand side of (3.60).

**Remark 3.5.** As a check for \( \alpha = 0 \) we recall \( F_j = 1, j \geq 0 \) so that
\[
L_0(z) = z + \frac{z}{z - 1 + R} = z + R,
\]
however we observe \( R = \frac{z}{z - 1 + R} \) so that \((R - 1)(R + z) = 0\). Choosing \( R = 1 \) as the appropriate root we verify that
\[
L_0(z) = \frac{1}{z + 1},
\]
which should be compared with the Laplace transform of \( g_0(x) = e^{-x} \).

**Remark 3.6.** For \( \alpha = 1 \) we can record a result which may be known
\[
\frac{1}{L_1(z)} = z + \sum_{m=1}^{\infty} \frac{\Gamma((1+m-1)\gamma)\Gamma(1-\gamma) z}{\Gamma((1+m)\gamma)\Gamma(1+(m-1)\gamma)\Gamma(1-\gamma) z - 1}
\]
On the other hand the Laplace transform of \( g_1 \) is the Mittag-Leffler function \( E_\gamma(\cdot) \), see [30, pg. 269]
\[
L_1(z) = \frac{1}{z} E_\gamma \left( -\frac{1}{(1-\gamma)z} \right).
\]
This is an entire function of \( z \) of order \( \frac{1}{\Re(\gamma)} \) and type unity.

**Remark 3.7.** From the results of Prop. 3.10 we can show that for \( \gamma \to \infty \) the Laplace transform has the algebraic form
\[
L_\gamma(z) = z^{-1} \left( 1 + (1 - \alpha) \gamma z \right)^{-\frac{1}{(1-\gamma)z}}.
\]
This can be established from the convergent expansion (3.58) about \( z = \infty \) using (3.49) or computing the Laplace transform of (3.50) with the assistance of [12, pg.
215, §2.2 (11)] assuming $\Re(z) > 0$. Therefore we note the M-continued fraction form

\[
\frac{1}{L_\gamma(z)} = z + \sum_{m=1}^{\infty} \frac{(m-1)}{(m+\frac{1}{2}+\alpha)(1-\alpha)\gamma} z - \frac{1}{1-\alpha}
\]

4. Numerical Evaluations

The most robust numerical strategy for computing the density is to use the Mellin-Barnes integral representation (3.38) because the integrand is smooth and decays exponentially as $\Im(t) \to \pm \infty$. There are oscillations but they have a period of the same order as the foregoing decay length and therefore are rapidly damped. Our methods employed 64-bit or real (kind=8) precision with the gfortran F90 compiler GNU Fortran (Ubuntu 7.5.0-3ubuntu1 18.04) 7.5.0 and software from the following sources:

- We have employed DQAGI for the Mellin-Barnes integrals and DQAGS for the normalisation and integral equations from the QUADPACK package[43]. The absolute and relative error goals were $10^{-6}$ and $10^{-4}$ respectively. A convenient choice for $c$ is half.
- The complex gamma function was computed with the error control ACM algorithm 421 [31] with the error value set at $10^{-8}$.
- The complex Gauss hypergeometric function was computed using algorithm AEA\_v1.0 from the CPC program library [36]. An error criteria is computed from certain contiguous relations.
- Extrapolation of the Gaussian hypergeometric function products up to $2^{13}$ factors was performed by ACM algorithm 602 HURRY, an acceleration algorithm for scalar sequences and series [13].

Two criteria were used to quantify the all-up global errors - the normalisation of the density and the residual of the solution in satisfying the integral equation (1.2) at $x = 1$. These are tabulated in Tables 1 and 2 respectively over a range of $\gamma \in (0, 1]$ and $\alpha \in (0, 1)$ values. Our error settings are relatively weak and no attempt has been made in pushing the methods to their accuracy limits.

**Table 1. Absolute Normalisation Errors**

| $\gamma$ | $\alpha$ | 0.10 | 0.20 | 0.30 | 0.40 | 0.50 | 0.60 |
|---------|---------|------|------|------|------|------|------|
| 0.25    |         | 0.318E-05 | 0.254E-04 | 0.633E-04 | 0.333E-06 | 0.888E-05 | 0.111E-05 |
| 0.50    |         | 0.331E-05 | 0.126E-05 | 0.201E-06 | 0.122E-04 | 0.733E-05 | 0.138E-05 |
| 0.75    |         | 0.162E-04 | 0.516E-05 | 0.301E-05 | 0.128E-05 | 0.106E-06 | 0.150E-05 |
| 1.00    |         | 0.160E-04 | 0.389E-04 | 0.664E-05 | 0.380E-06 | 0.260E-05 | 0.157E-04 |
Table 2. Absolute Residual Errors

| γ   | 0.10  | 0.20  | 0.30  | 0.40  | 0.50  | 0.60  |
|-----|-------|-------|-------|-------|-------|-------|
| 0.25| 0.237E-06 | 0.143E-07 | 0.653E-07 | 0.209E-06 | 0.689E-09 | 0.746E-07 |
| 0.50| 0.126E-05 | 0.517E-08 | 0.122E-06 | 0.176E-06 | 0.677E-07 | 0.341E-07 |
| 0.75| 0.622E-05 | 0.347E-05 | 0.284E-05 | 0.383E-06 | 0.253E-06 | 0.129E-06 |
| 1.00| 0.596E-06 | 0.369E-05 | 0.131E-06 | 0.485E-06 | 0.301E-05 | 0.876E-05 |

We present our results in a sequence of Figures 2, 3, 4, 5, 5 and 6 plotting the density $h(x; \gamma, \alpha)$ versus $x$ with increasing values of $\gamma$ and within each plot $\alpha$ increases through the range $[0, 1)$.

![Figure 2](image)

Figure 2. Densities $h(x; \frac{1}{4}, \alpha)$ versus $x$ for $\alpha = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ computed using (3.38) and (2.5).
Figure 3. Densities $h(x; \frac{1}{2}, \alpha)$ versus $x$ for $\alpha = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ computed using (3.38) and (2.5).

Figure 4. Densities $h(x; \frac{3}{4}, \alpha)$ versus $x$ for $\alpha = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\}$ computed using (3.38) and (2.5).
Figure 5. Densities $h(x; 1, \alpha)$ versus $x$ for $\alpha = \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}\}$ computed using (3.38).

Figure 6. Large-$\gamma$ densities as computed using (3.56). In this case $\gamma = 4$ and $\alpha \in \{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}\}$. 
The densities are unimodal and exhibit a tail as \( x \to \infty \) with algebraic decay, having the leading form \( x^{-1-\gamma} \) for all \( \gamma > 0 \) and \( 0 < \alpha \leq 1 \). For non-rational \( \gamma \) there is a cascade of lower order terms with algebraic decay \( x^{-1-j\gamma} \) for \( j \in \mathbb{N} \). This is illustrated by formula (2.7) in the case \( \alpha = 1 \) and from the general formula (3.38) for \( 0 < \alpha < 1 \). This latter formula can be evaluated as a sum over the algebraically decaying terms by enclosing the poles of the Gamma function at \( t \in \mathbb{N} \) to the right of the contour, similar to (2.7)

\[
(4.1) \quad h(x) = \gamma x^{-1} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} [(1-\alpha)x]^{-\gamma(j+1)} \prod_{k=0}^{j} \binom{2F1}{\gamma, -k\gamma; 1-k\gamma; \alpha}.
\]

For fixed \( \gamma \) and \( \alpha \) increasing from 0 to 1 the maxima occurs at increasing \( x \)-values and the maximal density itself decreases. The locations and densities of the maxima at \( \alpha = 0 \) are easily seen to be

\[
(4.2) \quad x_{\text{max}} = \left(\frac{\gamma+1}{\gamma}\right)^{-1/\gamma}, \quad h_{\text{max}} = e^{-\frac{1}{\gamma}-1} \left(\frac{1}{\gamma} + 1\right)^{1/\gamma} (\gamma + 1).
\]

This latter situation is where the bounds (3.4) are sharp.

**Table 3.** Maxima for \( \alpha = 1 \) and some rational \( \gamma \)

| \( \gamma \) | \( x_{\text{max}} \) | \( h_{\text{max}} \) |
|-------------|-------------|-------------|
| 1/4 | 0.00539575 | 2.61146 |
| 1/3 | 0.0579035 | 0.744108 |
| 1/2 | 0.523599 | 0.294463 |
| 2/3 | 1.70192 | 0.217484 |
| 3/4 | 2.82353 | 0.202068 |

The behaviour as \( x \to 0^+ \), \( \arg(x) = 0 \) is quite different, however. The density has an algebraic divergence here, but it is totally overwhelmed by an exponential suppression. This is simply illustrated by the large-\( \gamma \) formula (3.56) and the \( \alpha = 1 \) special cases (2.8). The asymptotic behaviour of the Fox H-function as \( x \to 0 \) with \( \arg(x) = 0 \) is known, see [35] pg. 2 Definition 1.1 and Theorem 1.1, Case 4, Eq. (1.109) in (ii) of Theorem 1.3, so for \( 0 < \gamma < 1 \) and \( \alpha = 1 \) we have

\[
(4.3) \quad h_1(x) \sim \frac{1}{\gamma} \frac{\Gamma(1-\gamma)}{\Gamma(1-\gamma)\left(1-\gamma\right)^{1-\gamma}/(1-\gamma)} x^{-\gamma/(1-\gamma)} \exp \left[-(1-\gamma)\gamma^{1/(1-\gamma)} (\Gamma(1-\gamma))^{1/(1-\gamma)} x^{-\gamma/(1-\gamma)} \right].
\]

An outstanding task is to find the analogous leading order asymptotics for \( x \to 0 \) and general \( \alpha \).
5. Discussion

We pick up here the thread in Section §1 concerning related works in the probability literature. Schlather [48] treats a different operation from the one considered in [18], [25] or here in that he takes a $1/p$ root of a sum of $p^{th}$ powers. In addition to being an altogether different problem producing a very different solution, he employs a different method. For example he finds a different family of limit distributions parametrised by $c$ with $0 \leq c \leq \infty$. And he discusses domains of attraction, but with norming depending on $c$ (unlike in the foregoing works) and of course different limits.

One property of $h(x; \gamma, \alpha)$ for fixed $\alpha$ and $\gamma$ is that because it is obtained as a normed limit it is infinitely divisible, and so has a Lévy measure, in the Kolmogorov or Lévy canonical representations of the characteristic function. Can we see this transparently from our product representation for $H(s)$, as given by (3.39)? The distribution $h(x; \gamma, \alpha)$ is also associated with a Lévy process, i.e. a process with stationary, independent increments. For $\alpha = 1$ we have recovered a known identification of $h(x; \gamma, \alpha)$ as a Fox $H$-function, and we have mentioned some previous works [49], [39], [28] where there is a nice statistical interpretation of this fact. However the sequence (1.1) is in the domain of attraction of the normal distribution for $\alpha = 1$, so there is a "discontinuity": (1.1) interpolates between Fréchet and normal, whereas (1.2) interpolates between Fréchet and this Fox $H$-function.

Regarding the nature of the generic solution $h(x; \gamma, \alpha)$ we have found here, it appears that this is a novel extension of the hypergeometric function to the best of our knowledge. Standard extensions through the Mittag-Leffler, Wright-Bessel and Meijer-G functions to the Fox $H$-functions clearly appear when $\alpha = 1$, but not for $0 < \alpha < 1$. The clearest way to distinguish the generic functions found here is to think of the standard hypergeometric or Fox-$H$ function as series

$$H \left( \begin{array}{c}
(A_1, a_1) \ldots (A_P, a_P) \\
(B_1, b_1) \ldots (B_Q, b_Q) 
\end{array} ; x \right) = \sum_{n \geq 0} C_n x^n,$$

and that the coefficients $C_n$ satisfy a first order difference equation with respect to $n$, in unit steps in the forward direction $n \rightarrow n + 1,$

$$C_{n+1} = \frac{(A_1 + a_1n)\ldots(A_P + a_Pn)}{(B_1 + b_1n)\ldots(B_Q + b_Qn)} C_n, \{A_1, a_1, \ldots, A_P, a_P, B_1, b_1, \ldots, B_Q, b_Q\} \in \mathbb{C},$$

i.e. rational coefficients in $n$. Such equations can be solved in terms of products of gamma functions or Pochhammer symbols given suitable boundary conditions. However in our generic case the coefficient of the first order difference equation is itself a hypergeometric function of $n$, which is the solution of a second-order difference equation with respect to $n$, but as $n \rightarrow n + 1/\gamma$. Alternatively the original coefficients $C_n$, or effectively $H(s = 1 - n\gamma)$, would satisfy a non-linear difference equation with rational coefficients in $n$ but with incommensurate steps
of $1$ and $\gamma^{-1}$

\begin{equation}
(5.4) \quad 0 = [1 + \gamma - s + \alpha(1 - s)] \frac{H(s)}{H(s - \gamma)} + \alpha(s - 2) \frac{H(s - 1)}{H(s - 1 - \gamma)} + (s - \gamma) \frac{H(s + 1)}{H(s + 1 - \gamma)}.
\end{equation}

This is an entirely higher level of special function.

It may be of interest to note that another family of extensions to the Gauss hypergeometric functions has been explored recently, under the monikers of extended hypergeometric functions [6], extended $\tau$-Gauss hypergeometric functions [50], [42], extended, generalised $\tau$-Gauss hypergeometric functions [32]. The common theme here is that a pair of Pochhammer symbol ratios in $C_n$, which are equivalent to the beta function $B(a, b)$ [45, §5.12], are replaced by

\begin{equation}
(5.5) \quad B(a, b; \sigma) = \int_0^1 dt \, t^{a-1}(1-t)^{b-1} \exp \left[ -\frac{\sigma}{t(1-t)} \right], \Re(\sigma) > 0,
\end{equation}

along with

$$F_\sigma(a, b; c; z) = \sum_{n=0}^\infty (a)_n \frac{B(b+n, c-b; \sigma)}{B(b, c-b)} \frac{z^n}{n!},$$

or ultimately

$$B(a, b; \sigma; \alpha, \beta, \rho, \lambda) = \int_0^1 dt \, t^{a-1}(1-t)^{b-1} F_1 \left( \alpha; \beta; -\frac{\sigma}{t^\rho(1-t)^\lambda} \right), \Re(\sigma) > 0.$$

There is no evidence that the extended beta function (5.5) satisfies a difference equation like that of (5.4).

6. ACKNOWLEDGEMENTS

The authors would like to acknowledge Jesse Goodman for organising the 2020 Taupo Probability Workshop and the funding support of the University of Auckland, which facilitated this collaboration.

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