A Note on the Generalization of the GEMS Approach

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Abstract

This paper is a supplement of our earlier work \cite{1}. We map the vector potential of charged black holes into GEMS and find that its effect on the thermal spectrum is the same as that on the black hole side, i.e., it will induce a chemical potential in the thermal spectrum which is the same as that in the charged black holes. We also argue that the generalization of GEMS approach to non-stationary motions is not possible.

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1 Introduction

It is well known that horizon, radiation and entropy are closely related objects. The Hawking effect \cite{2} is related to the event horizon intrinsic to the spacetime geometry, while the Unruh effect \cite{3} is related to the horizon associated with a specific observer. However, these two effects are also closely related. Using a higher dimensional global embedding Minkowski spacetime (GEMS) of a curved spacetime, Deser and Levin \cite{4, 5, 6} find that the temperature detected by a static detector in a curved spacetime is equal to the temperature detected by the corresponding detector (a Rindler detector) in the GEMS. In a preceding paper \cite{1}, we generalize their work to detectors in general stationary motions in the spherically symmetric spacetimes and match the whole thermal spectra including the chemical potential in the case of uncharged black holes. However, we note in \cite{1} that the chemical potential can not be properly matched for charged black holes, since the vector potential does not map into the GEMS. In this paper, we will address this problem and give a natural mapping of the vector potential into the GEMS and show that the resulting thermal spectrum is the same as that of the charged black hole. We will also make some comments on the generalization to non-stationary motions and argue that the generalization to that cases is not possible.

2 Mapping the vector potential into the GEMS

We consider the simplest case of a RN black hole.\footnote{We only consider the non-extremal case.} The metric is

\[ ds^2 = F(r, M, Q)dt^2 - F^{-1}(r, M, Q)dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2), \]  

where \[ F(r, M, Q) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}. \]  

And the 1-form vector potential in a specific gauge which is natural for an outside observer is

\[ A = \frac{-Q}{r} dt. \]  

It has two horizons at \[ r_\pm = M \pm \sqrt{M^2 - Q^2}. \] The usual GEMS approach only embeds the metric into the higher dimensional space. However, this is not complete for the charged black hole. As discussed in \cite{1}, the static
detector outside the black hole will observe a thermal spectrum with chemical potential associated with the vector potential, while when viewed in the GEMS it is a standard Rindler detector and will detect a thermal spectrum without chemical potential. So we must take into account the vector potential and find a method to encode its information in the higher dimensional one. The most natural choice is to push-forward it into the GEMS. We will see that this is a correct choice, since it induces the desired chemical potential in the thermal spectrum detected by the Rindler detector in the higher dimensional space.

The RN black hole can be embedded in a flat $D = 7$ space with metric

$$ds^2 = (dz^0)^2 - \sum_{i=1}^{5} (dz^i)^2 + (dz^6)^2, \quad (4)$$

as follows:

$$
\begin{align*}
    z^0 &= \kappa^{-1} \sqrt{F(r, M, Q)} \sinh(\kappa t), \\
    z^1 &= \kappa^{-1} \sqrt{F(r, M, Q)} \cosh(\kappa t), \\
    z^2 &= \int \left( r^2 (r_+ + r_-) + r_+^2 (r + r_+) \right) \left( r^2 (r - r_-) \right)^{1/2} dr, \\
    z^3 &= r \sin \theta \cos \phi, \\
    z^4 &= r \sin \theta \sin \phi, \\
    z^5 &= r \cos \theta, \\
    z^6 &= \int \left( \frac{4r_+^5 r_-}{r^4 (r_+ - r_-)^2} \right)^{1/2} dr,
\end{align*}
$$

where

$$\kappa = \kappa(r_+) = \frac{r_+ - r_-}{2r_+}, \quad (6)$$
is the surface gravity of the outer horizon. For the cases we are interested in ($r, \theta = \text{const}$), the coordinates $z^2, z^5, z^6$ are constants, and the motions in the higher dimensional space are effectively four dimensional ones in the usual Minkowski spacetime (with coordinates $z^0, z^1, z^3, z^4$). In the following, we only consider this subspace. So in the GEMS, there are two coordinate systems. One uses $z^0$ as time coordinate and is the usual Minkowski coordinate system which can be extended to cover the entire spacetime (when including the other dimensions) and corresponds to the maximal Kruskal extension of RN black hole; the other one uses $t$ as time coordinate and is a
Rindler coordinate system which has a horizon at \( r = r_+ \) and can only cover the region \( r > r_+ \). It should be emphasized that the above embedding can be extended to cover \( r < r_+ \) [7] so that the observer restricted to \( r > r_+ \) will lose information and hence detect radiation. For a detector outside the RN black hole moving according to constant \( r, \theta \) and \( \phi \) with \( r > r_+ \), the corresponding detector in the GEMS can be read from eq. (5), and it is just a Rindler detector with a coordinate acceleration \( a = \kappa \). The vector potential [3], when push-forwarded to the GEMS, becomes

\[
\tilde{A} = -\frac{Q}{r} dt \\
= -\frac{1}{\kappa} \left( \frac{z^1}{(z^1)^2 - (z^0)^2} \right) r \frac{Q}{r} dz^0 + \frac{1}{\kappa} \left( \frac{z^0}{(z^1)^2 - (z^0)^2} \right) r \frac{Q}{r} dz^1.
\]  

Here eq. (7) is the form in the Rindler coordinates, while eq. (8) is the form in the Minkowski coordinates. Following the spirit of GEMS approach, we would like to calculate the thermal spectrum detected by the Rindler detector of the Rindler coordinates in the Minkowski vacuum. Because of the presence of the vector potential, it is simplest to use thermal Green function method [8, 9, 10].

We consider a charged scalar field \( \phi(x) \) with charge \( q \). The thermal Green function of a grand canonical ensemble with inverse temperature \( \beta \) is defined as

\[
G_{\beta}(x, x') = \langle T \phi(x) \phi(x')^\dagger \rangle_\beta,
\]

where \( T \) is the time-ordering operator and the ensemble expectation value is given by

\[
\langle \Phi \rangle_\beta = \frac{\text{Tr} \left[ e^{-\beta(H-\mu N)} \Phi \right]}{\text{Tr} \left[ e^{-\beta(H-\mu N)} \right]},
\]

where \( \mu \) is the chemical potential, \( H \) is the Hamiltonian, \( N \) is the number operator, and \( \text{Tr} \) means taking trace over a group of complete bases. Note, in the above definition, that the time evolution of the field is governed by the usual Heisenberg equation, i.e.,

\[
\phi(t) = e^{iHt} \phi(0) e^{-iHt}.
\]

It is well known that the thermal Green function defined above has the following quasi-periodicity in imaginary time [3]:

\[
G_{\beta}(t - t' + i\beta) = e^{\beta \mu} G_{\beta}(t - t').
\]
We will now consider the zero temperature Green function of Minkowski space and show that it just corresponds to the thermal Green function in Rindler coordinates.

First, we note that the the vector potential in the Minkowski coordinates is singular on the horizon of the Rindler space \( r = r_+ \) which corresponds to \((z^1)^2 - (z^0)^2 = 0\). In order to give a well-behaved vector potential on the horizon, we perform the following gauge transformation:

\[
\tilde{A}' = \tilde{A} + d\chi, \tag{13}
\]

\[
\phi'(x) = e^{iq\chi(x)}\phi(x), \tag{14}
\]

with

\[
\chi(x) = \frac{Q}{r_+}t. \tag{15}
\]

Now in this \( \tilde{A}' \) gauge the zero temperature Green function of Minkowski spacetime is suitably analytic in the Minkowski coordinates [10, 11], and when expressed in terms of the Rindler coordinates, it is easy to see from eq.(5) that

\[
G_{\tilde{A}'}(t-t' + i\beta) = G_{\tilde{A}'}(t-t'), \tag{16}
\]

with

\[
\beta = \frac{2\pi}{\kappa}. \tag{17}
\]

However, the usual Rindler observers (mapped from the usual observers outside the RN black hole in the usual gauge) will describe their observations using the Rindler coordinates and the \( \tilde{A} \) gauge, so we should change back to the \( \tilde{A} \) gauge and get

\[
G_{\tilde{A}}(t-t') = e^{-iq\chi(x)}e^{iq\chi(x')}G_{\tilde{A}'}(t-t')
\]

\[
= \exp[-\frac{iQ}{r_+}(t-t')]G_{\tilde{A}'}(t-t'). \tag{18}
\]

Now it is easy to see that

\[
G_{\tilde{A}}(t-t' + i\beta) = e^{\beta\mu}G_{\tilde{A}}(t-t'), \tag{19}
\]

with chemical potential

\[
\mu = \frac{qQ}{r_+}, \tag{20}
\]

and \( \beta \) still given by eq.(17). This is just the same thermal spectrum detected by a static detector outside the RN black hole.
Furthermore, if the detector outside the black hole follows the trajectory

$$r = r_0, \quad \theta = \theta_0, \quad \phi = \Omega t,$$  \hspace{1cm} (21)

where $r_0$, $\theta_0$ and $\Omega$ are constants, then as discussed in [1], the detector will detect an average number of particles with energy $\omega$ as follows

$$N_\omega = \frac{1}{e^{\beta(\omega - \mu')} - 1},$$  \hspace{1cm} (22)

with chemical potential

$$\mu' = \frac{qQ}{r_+} - m\Omega,$$  \hspace{1cm} (23)

and $\beta$ given by eq.(17), where $m$ is the orbital angular quantum number of the particles.

To see the corresponding spectrum in the GEMS, we simply use the same arguments as above. We see from eq.(5) that the trajectory (21) is mapped into the GEMS as a 4-dimensional Rindler motion superposed with a circular motion perpendicular to the acceleration of the Rindler motion. To go to the rest frame of the detector, we just perform the coordinate transformation

$$\tilde{\phi} = \phi - \Omega t.$$  \hspace{1cm} (24)

We now focus on a mode with angular quantum number $m$. Its zero temperature Green function in the Minkowski spacetime is suitably analytic in the original coordinates and in the $\tilde{A}'$ gauge, and takes the following form

$$G_{\tilde{A}'}(t - t', \phi - \phi') = F_{\tilde{A}'}(t - t') e^{im(\phi - \phi')}.$$  \hspace{1cm} (25)

It will be periodic in $t - t'$, namely we have

$$F_{\tilde{A}'}(t - t' + i\beta) = F_{\tilde{A}'}(t - t')$$  \hspace{1cm} (26)

with $\phi - \phi'$ invariant and $\beta$ given by eq.(17). The embedding observers will describe their observations using coordinate $\tilde{\phi}$ and the $\tilde{A}$ gauge, so after change back to $\tilde{A}$ gauge and perform the coordinate transformation (24), we get

$$G_{\tilde{A}}(t - t', \tilde{\phi} - \tilde{\phi}') = F_{\tilde{A}}(t - t') e^{im(\tilde{\phi} - \tilde{\phi}')} e^{i\Omega(t - t')} e^{-iqQ(t - t')/r_+}$$  \hspace{1cm} (27)

Taking $\tilde{\phi} - \tilde{\phi}'$ invariant, it is easy to see that

$$G_{\tilde{A}}(t - t' + i\beta, \tilde{\phi} - \tilde{\phi}') = e^{\beta\tilde{\mu}} G_{\tilde{A}}(t - t', \tilde{\phi} - \tilde{\phi}'),$$  \hspace{1cm} (28)
with $\beta$ again given by eq. (17) and chemical potential given by

$$\tilde{\mu} = \frac{qQ}{r_+} - m\Omega. \quad (29)$$

This is just the same thermal spectrum eq. (22, 23) detected by a rotating detector outside the RN black hole.

3 Comments on the Generalization to Non-Stationary motions

The above discussions together with results of [1] indicate that the GEMS approach can be generalized to general stationary motions in curved spacetimes. It is interesting to ask whether this approach can be generalized to even more general motions, i.e., non-stationary motions. One reason of this generalization is the hope of using the techniques available in the quantum field theory on flat Minkowski spacetime to study the more complicated issues in curved spacetimes. In fact, this is one of our motivations to study this approach. However, we will argue that the generalization to non-stationary motions is not possible.

First, we note that in some cases, the global embedding space of a curved spacetime will have more than one time dimension. Take RN black hole for example. Its embedding space (4) has an extra time dimension $z^6$ which only depends on $r$. For a non-stationary motion with varying $r$, we have to deal with a quantum field theory on a spacetime with two times which we even do not know how to define it. On the other hand, although the non-stationary motion in the curved spacetime is complicated, we can still deal with it using technique of particle detector. So in this case, the generalization is not possible.

For the Schwarzschild spacetime whose embedding space is an ordinary $D = 6$ Minkowski spacetime with only one time dimension, the situation is not better either. For example, suppose we want to consider a freely falling detector outside a Schwarzschild black hole. Since a static detector in the curved space maps to a static detector in the Rindler space, and a rotating detector maps to a rotating detector in the Rindler space, naively, we will expect a freely falling detector maps to a freely falling detector in the Rindler space. Furthermore, a freely falling detector in the Rindler space is just an inertial detector in the Minkowski space. For such a detector, it is well known that it will detect nothing in the Minkowski vacuum. So, following the spirit of the GEMS approach, it seems that we may conclude that a freely
falling detector in the Hartle-Hawking vacuum in a Schwarzschild spacetime will detect nothing. Indeed, it is known that a freely falling detector sees essentially no particles near the horizon \([14]\). It seems that we have matched the two sides. Unfortunately, the naive expectation is not correct. Generally speaking, the embedding does not preserve geodesics. The simplest example is that a geodesic (a large circle) on a 2-sphere \(S^2\) is not a geodesic (a straight line) in its embedding Euclidean space \(E^3\). For the case we are interested in, this can be seen from the global embedding of the Schwarzschild spacetime. It can be embedded in a \(D = 6\) Minkowski spacetime with metric

\[
ds^2 = (dz^0)^2 - (dz^1)^2 - (dz^2)^2 - (dz^3)^2 - (dz^4)^2 - (dz^5)^2,
\]

as follows \([12]\):

\[
z^0 = 4M \sqrt{1 - \frac{2M}{r} \sinh \left( \frac{t}{4M} \right)},
\]

\[
z^1 = 4M \sqrt{1 - \frac{2M}{r} \cosh \left( \frac{t}{4M} \right)},
\]

\[
z^2 = \int dr \sqrt{\frac{2Mr^2 + 4M^2r + 8M^3}{r^3}},
\]

\[
z^3 = r \sin \theta \sin \phi,
\]

\[
z^4 = r \sin \theta \cos \phi,
\]

\[
z^5 = r \cos \theta.
\]

On the other hand, for a freely falling observer outside the Schwarzschild black hole, its rest frame \((\tau, R)\) is related to the Schwarzschild coordinates \((t, r)\) through the Lemaitre transformation \([13]\) as follows:

\[
r = (2M)^{1/3} \left[ \frac{3}{2} (R - \tau) \right]^{2/3},
\]

\[
t = \tau - 2(2M)^{2/3} \left[ \frac{3}{2} (R - \tau) \right]^{1/3}
\]

\[
-2M \ln \left[ \frac{3}{2} (R - \tau)^{1/3} - (2M)^{1/3} \right]/\left[ \frac{3}{2} (R - \tau)^{1/3} + (2M)^{1/3} \right],
\]

with \(\theta, \phi\) invariant. A freely falling observer corresponds to \((R, \theta, \phi) = \text{const.}\) From eqs.\([31,32]\), it can be shown by straightforward calculations that the freely falling observer in the Schwarzschild spacetime does not map to the inertial observer in the embedding Minkowski spacetime. In fact, we can also
anticipate this by noting that the geodesic motion in the Minkowski space is stationary rather than non-stationary. On the contrary, the embedding motion is complicated and the corresponding detector will detect particles. Thus, we see that even for this relatively simple non-stationary motion, the results of the two sides cannot match each other. So we conclude that the generalization of the GEMS approach to non-stationary motions is not possible.

4 Conclusion

We see that after properly mapping the vector potential of the RN black hole into the GEMS, we can match the whole thermal spectrum completely, including the chemical potential, for both static and rotating detectors. The generalization to other spherically symmetric charged black holes, such as RN-dS and RN-AdS black holes, and the generalization to other dimensions are straightforward. Thus we have confirmed that the proposal of Deser and Levin in [4, 5, 6] is valid in these stationary motions. We also argue that the generalization to non-stationary motions is not possible. This means that the GEMS approach can be only applied to stationary motions in curved spacetime.

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