Coexistence of magnetic and charge order in a two-component order parameter description of the layered superconductors

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Abstract
We consider an effective theory of superconductivity for layered superconductors using a two-component order parameter, and show that it allows the formation of a condensate with magnetic and charge degrees of freedom. This condensate is an inhomogeneous state, topologically stable, that exists without the presence of an applied magnetic field. In particular, it is associated to a charge density in the superconducting layers. We show that well defined angular momentum states have for their lowest moment an hexadecapole charge distribution, i.e. quartic in the momenta. Our approach is based on solving first order equations (FOE) that generalize the Abrikosov–Bogomolny equations of the Ginzburg–Landau theory with one order parameter. The FOE solve the variational equations of the theory in the limit of a small order parameter, which is achieved for the special temperature that corresponds to the crossing of the superconducting dome and the pseudogap transition line. This topologically stable state is a condensate of skyrmions that breaks time reversal symmetry and produces a weak local magnetic field below the threshold of experimental observation.

Keywords: order parameter, charge density wave, skyrmions

(Some figures may appear in colour only in the online journal)

1. Introduction

The concept of an order parameter was introduced in 1937 by Lev Landau to describe the second order phase transition in the specific heat of tin that takes place at the passage to the superconducting state and had been observed a few years before. In 1950 the celebrated Ginzburg–Landau (GL) theory was proposed to provide a gauge invariant macroscopic description of superconductivity and incorporated the second order phase transition and also London’s theory which accounted for the Meissner effect. Interestingly this macroscopic description was developed without any knowledge of the microscopic mechanism of superconductivity, such as the existence of pairing, which was only proposed in 1956 by Leon Cooper. According to the BCS theory of superconductivity paired electrons condense into a single state whose description is attainable through the order parameter approach near to the critical temperature, $T_c$. The discovery of the high-$T_c$ superconductors [1] brought a renewed interest in the order parameter approach because of the lack of understanding of the pairing mechanism. Hereafter high-$T_c$ superconductors are those that display a layered structure such that superconductivity originates in the layers. This definition comprises several families of superconductors, such as the cuprates [1], the pnictides [2], the borocarbides [3], the superconductors Sr$_2$RuO$_4$ [4], MgBi$_2$ [5], CeCoIn$_5$ [6] and others. We argue in this paper that the description of the high-$T_c$ superconductors demands two complex order parameters.
Multiple order parameter theories provide the optimal framework for the description of multiple phases and also of inhomogeneous condensates [7].

Soon after the discovery of the high-$T_c$ superconductors by Bednorz and Müller [1] in 1987 two types of order parameter approaches were applied to them [8], namely, the anisotropic GL and the Lawrence–Doniach theories. The former is just the traditional GL theory with a mass anisotropy tensor to cope with the inertia acquired by Cooper pairs to move perpendicularly to the layers. This theory does not have layers and therefore cannot recognize them as the sources of the superconducting state. In the latter theory superconductivity only exists within the layers and the space between them is a perfect void. There is coupling between nearest neighbor layers through the Josephson effect. As successful applications of these two theories to the high-$T_c$ superconductors we quote the description of the torque [9] and of the THz spectroscopy [10], respectively. Here we consider a third kind of order parameter approach that takes the layers as the source of superconductivity and yet has the condensate outside them existing in an evanescent way. The high-$T_c$ superconductor is a stack of layers embedded in a metallic media since the condensate still exists in the inter-layer space, although it decays exponentially away from the layers [11]. The layers contain supercurrent circulation which demands distinct order parameter behavior just above and below them. In this paper we show that these distinct properties lead to an inhomogeneous condensate with intrinsic magnetic and charge orders. The intertwining of pairing, charge, and spin degrees of freedom has been the subject of intense research lately [12, 13]. The existence of spontaneously circulating currents in the layers has been proposed before as microscopic orbital currents [14–17] in order to explain for the breaking of time reversal symmetry [18, 19]. We find the present approach advantageous because it provides a topological explanation for the stability of this state.

In the past decade it became clear that superconductivity exists above $T_c$, fact that can only be handled by the anisotropic GL and the Lawrence–Doniach theories from the point of view of thermal fluctuations of the order parameter. However recent understanding of the so-called temperature versus doping phase diagram shows that superconductivity above $T_c$ cannot be simply explained by thermal fluctuations. The high-$T_c$ superconductor acquires new properties according to the doping, namely, the number of carriers available for conduction in the layers. The $T_c$ versus doping line of this diagram defines a dome shaped curve. The superconducting state is called underdoped, optimally doped, and overdoped, respectively, according to the doping level relative to the maximum $T_c$. Thus besides the superconducting state there are other states [20], such as the so-called pseudogap state [21], that are not caused by thermal fluctuations. The pseudogap emerges at a temperature $T^*$, claimed to be a phase transition line by some [22]. In the underdoped regime this temperature is above $T_c$ and decreases with increasing doping level. At some doping $T^* = T_c$, and beyond, one expects that the pseudogap line enters the superconducting dome to finally reach a quantum critical point at $T = 0$ [23]. The microscopic nature of the pseudogap remains controversial. In this paper we assume that the pseudogap is also a condensate, and so can be described by the order parameter approach. The presence of two transition lines, namely, $T_c$ and $T^*$ is suggestive of a two-component order parameter $\Psi$, while the original anisotropic GL and Lawrence–Doniach theories have only one, $\psi$. Multi-component order parameter theories have been proposed for the high-$T_c$ superconductors since long ago [24].

A two-component order parameter theory admits associated first order equations (FOE) that generalize the Abrikosov–Bogomolny equations of the GL theory, and solve the full variational equations in the limit of a small order parameter, in particular close to the special temperature $T^* = T_c$. We show here that the FOE admit solutions that describe a skyrmion condensate, that has energy above the ground state energy but is topologically protected from decaying. Such condensate does not require an applied external magnetic field. Skyrmions are topologically non-trivial solutions of classical and quantum field theories, first proposed by Tony Skyrme in 1962 to describe mesons and baryons [25]. They are still studied today to describe for example atomic nuclei, see [26] for an example and related references. A new wave of interest of skyrmions in solid state physics has arisen recently due to their possible application in the field of spintronics: skyrmions can form small domains in magnetic materials [27] and their presence or absence can be used to represent bit states that can be used for example for data storage; the possibility of writing and deleting single skyrmions using scanning tunneling microscopy has been shown in [28]. In all these examples as well as in our work the condition that the skyrmion states are topologically protected from decay is of fundamental importance. One can think of the non-trivial topology as being associated to the particular configuration of the spin for skyrmions in magnetic materials, and of the local magnetic field in our work, that assumes a somehow ‘knotted’ form, mathematically encoded by the topological charge of equation (8), that cannot be continuously untied and reduced to the ground state in the absence of abrupt changes.

We briefly summarize the experimental predictions of this paper taken under the assumption that the pseudogap indeed corresponds to a skyrmionic condensate in the neighborhood of $T_c$ and $T^*$. Then the pseudogap is a condensate that displays both magnetic and charge properties whose patterns are obtained here for states of definite angular momentum. There is a supercurrent circulating in and out of the layers, and so, there is supercurrent within a layer and in between the layers as well. This intricate supercurrent pattern produces a very low nearly undetectable magnetic field. Assuming a given value for this local magnetic field we predict the pseudogap energy density and find a natural oscillatory frequency of charges in and out of the layers that falls in the THz regime.
2. The topological equations

Interestingly, the prediction of a crystalline ordered state made of topological excitations, i.e., the vortex lattice, was done based on the so-called FOE, and not on the second order variational GL equations. This important remarks stems directly from Abrikosov’s original work [29], where the GL free energy only enters to determine which vortex lattices, among the possible ones, has the lowest energy. The FOE were rediscovered by Bogomolny [30] in the context of string theory and shown to solve exactly the GL second order equations for a particular value of the coupling constant (κ = 1/√2). The Abrikosov–Bogomolny equations are given by, 

\[ D_\mu \Psi = 0 \quad \text{and} \quad h_3 = C_3 - \frac{\hbar q}{mc}|\psi|^2, \]

where \( C_3 \) is a constant and \( D_\mu \equiv D_\mu + iD_\mu \). The covariant derivative, \( D_\mu \), \( i = 1, 2, 3 \), is described below. These equations demand uniaxial symmetry, chosen along the external magnetic field direction, \( H = C_3 \hat{e}_3 \). Then the single component order parameter and the local magnetic field must be given by \( \psi(x_1, x_3) \) and \( h = h_3(x_1, x_3) \hat{e}_3 \), respectively. In case of no external field these equations give the trivial solution of a spatially homogeneous state. The first Abrikosov–Bogomolny equation becomes \( \nabla \psi = 0 \), and can be expressed as \( \partial \psi / \partial z = 0, z = x_1 + i x_2 \). The only possible solution, assuming periodicity in the plane, is of a spatially constant order parameter, according to Liouville’s theorem. By selecting the constant \( C_3 = \frac{\hbar q}{mc}|\psi|^2 \) in the second Abrikosov–Bogomolny equation then \( h_3 = 0 \). Interestingly there is another set of FOE, the Seiberg–Witten equations, which describe four dimensional massless magnetic monopoles [31]. Thus the FOE form a family that render topological solutions and for this simple reason we call them the topological equations. We claim that another pair of such equations is required to describe the topological excitations of the high-\( T_c \) superconductors [32]:

\[
\vec{\sigma} \cdot \vec{D} \Psi = 0, \quad \Psi := \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}, \quad \text{and} \quad h = h_3 = \hat{h} + \frac{\hbar q}{mc} |\psi|^2 \hat{\sigma} \Psi.
\]

To proof this relation just check that equation (1) can be expressed as \( \vec{\sigma} \cdot \vec{D} \Psi = 2\pi i \vec{\sigma} \cdot \vec{A} \Psi / \Phi_0 \), where \( \Phi_0 = \hbar c / q \) is the flux unit. Using that \( \nabla_i (\Psi^\dagger \sigma_3 \Psi) = (\sigma_3 \nabla_i \Psi)^\dagger \Psi + \Psi^\dagger (\sigma_3 \nabla_i \Psi) \), it follows that, \( \nabla_i (\Psi^\dagger \sigma_3 \Psi) = \nabla_i (\Psi^\dagger \Psi) + \Psi^\dagger (\sigma_3 \nabla_i \Psi) / \Phi_0 = 0 \).

We seek the solution of equations (1) and (2) for a stack of layers separated by distance \( d \) and without the presence of an applied field \( \vec{H} = 0 \). We shall show that \( \Psi \) arises in the layers and evanescence away from them such as in a metallic medium able to sustain a three-dimensional state. For the moment assume an inhomogeneous solution of the FOE, \( \Psi \equiv \Psi(\vec{x}) \) and \( \vec{h} \equiv \hat{h}(\vec{x}) \). Then the local inhomogeneous field implies on a spatially circulating supercurrent, both volumetric and superluminal, \( \vec{J}(\vec{x}) \), or, equally, a superluminal magnetization, \( \vec{M}(\vec{x}) = -c \vec{\varepsilon}_3 \times \vec{J}(\vec{x}) \), where axis 3 is perpendicular to the layers. Thus it results from equations (1) and (2) that,

\[
\vec{J} = -c \mu_0 \vec{\nabla} \times (\Psi^\dagger \vec{\sigma} \Psi),
\]

\[
\vec{J} = -2c \mu_0 \vec{\varepsilon}_3 \times \Psi^\dagger (\Psi^* \vec{\sigma} \Psi). \tag{4}
\]

Hence the above equation is just a consequence of Ampère’s law applied to a stack of layers with metallic medium in between them. To prove the above relation just apply the divergence operator to the parallel boundary condition, which gives that, \( \vec{\nabla} \cdot [\vec{\varepsilon}_3 \times \vec{h}(\vec{x})] = -4\pi \vec{\nabla} \cdot \vec{J}. \) Then from Ampère’s law it follows that \( \vec{\nabla} \cdot (\vec{\varepsilon}_3 \times \vec{h}) = -4\pi \vec{\varepsilon}_3 \cdot \vec{J} / c \), which leads to equation (6).

Therefore according to the present model the volumetric supercurrent is constantly entering and exiting each given layer and transforming itself into the superficial supercurrent. This means that there is charge entering and exiting the layer at a constant rate. Equation (6) describes the net volumetric supercurrent between the layers that transforms itself into the superficial supercurrent at each spatial point where \( \vec{x}_3 \cdot (\vec{J}(\vec{x}) - \vec{J}(\vec{x})) \neq 0 \). We interpret this as a surface charge density within the layer, \( \sigma \), defined by,

\[
\vec{\nabla} \cdot \vec{J} + \frac{\sigma}{d \tau} = 0, \quad \text{where} \quad \frac{\sigma}{d \tau} \equiv \vec{\varepsilon}_3 \cdot (\vec{J}(\vec{x}) - \vec{J}(\vec{x})). \tag{7}
\]

In summary the present model determines the rate of charge density, \( \sigma / d \tau \), that enters and exits at each point of a given layer. We find remarkable that a magnetostatic description of a stack of two-dimensional layers embedded in a metallic...
medium leads to an inhomogeneous charge rate density within the layers [33].

3. Time reversal symmetry and the topological charge

Interestingly the FOE, given by equations (1) and (2), automatically break time reversal symmetry as they admit two solutions, one associated to a local magnetic field and the other to the reversed field. For simplicity consider the no applied field case \( \vec{H} = 0 \) in equations (1) and (2). Assume \( \Psi \) to be a known solution and consider another state \( \Psi' = U \Psi^* \), where \( U = e^{i \sigma_2} \), \( U U^\dagger = 1 \), where \( \sigma \) can be any angle. Then one obtains that \( U \bar{\sigma} \Psi \bar{U}^\dagger = -\bar{\sigma} \). The time reversal operation flips the magnetic field and also the spin and it is well known that this can be achieved by a unitary rotation proportional to \( \sigma_2 \), the only imaginary Pauli matrix. The conjugation operation over the real expectation value of the spin gives that, \( (\Psi')^* \bar{\sigma} \Psi^* = (\Psi')^* \bar{\sigma} \Psi^* = (\Psi)^* U \bar{\sigma} \Psi U^\dagger = -(U \bar{\sigma} U^\dagger) \Psi \). Therefore we have shown that \( \Psi'^* \bar{\sigma} \Psi = -\Psi^* \bar{\sigma} \Psi \), which according to equation (2), also implies that \( \bar{h}(\Psi') = -\bar{h}(\Psi) \). Concerning equation (1) take its complex conjugate and rotate it, \( U(\vec{\sigma} \cdot \vec{D} \Psi)^* = 0 \). It follows from this global transformation that \( U \bar{\sigma} \Psi U^\dagger \cdot U \bar{\sigma} \Psi U^\dagger = \bar{\sigma} \cdot \vec{D}(-\bar{X}) \Psi \). We reach the conclusion that the topological equations naturally break the time-reversal symmetry since they present two independent sets of solutions, namely, \( (\Psi, \bar{h}) \) and \( (\Psi', -\bar{h}) \).

Under the present scenario it is easy to conclude for the existence of topological solutions according to the following argument. Consider the case of no applied magnetic field and yet the presence of circulating supercurrents, volumetric between the layers and superficial within the layers, that establish a spatial local magnetic field \( \bar{h} \). Assume the presence of closed magnetic field stream lines that pierce twice a given layer such that the magnetic field component parallel to the layer flips direction from one side to the other of this layer. Looking from just one side of this layer one sees that a given stream line has a fountain and a sinkhole in this layer. The spatial arrangement of such closed loops is such that the sinkhole of all magnetic field stream lines are concentrated into a few points, the skyrmion cores, whereas the fountains are not necessarily concentrated and in fact are scattered within the unit cell. Recall that this intricate magnetic field arrangement due to the volumetric and superficial supercurrents also results in constant charge rate passing through the layers and creating positive and negative spots within a layer. This results in a highly inhomogeneous state with free energy higher than that of the homogeneous state and so expected to decay. However this does not happen, the state remains stable thanks to its topological properties. Integration over a single layer, chosen at \( x_3 = 0 \),

\[
Q = \frac{1}{4\pi} \int_{x_1 = 0^+} \left( \frac{\partial \bar{h}}{\partial x_1} - \frac{\partial \bar{h}}{\partial x_2} \right) \cdot \vec{h} \, dx^2,
\]

where \( \bar{h} = \bar{h}/\bar{h}^2 \), reveals that this inhomogeneous solution has \( Q \neq 0 \), whereas the homogeneous solution has \( Q = 0 \). This is the skyrmion state and each \( Q \) state belongs to a different topological class. Clearly the time reversal symmetry, \( \vec{h} \rightarrow -\vec{h} \), is broken by the skyrmions. We find that the topological number \( Q \) counts the number of skyrmion cores within the unit cell. Thus the skyrmions are magnetic excitations [34] with a core that establishes a well defined sense of rotation in the cell, and for this reason they are also chiral solutions [35]. The superficial current \( J_0 \) is very strong within the core as compared to the rest of the cell, where it is weak. At the center of the skyrmion core the rotation ceases. The unique sense of flow set by the core makes the skyrmion state break the time-reversal symmetry. This preferred chirality of the skyrmions should rotate circularly polarized light passing through the layers and lead to the dichroism observed below the pseudogap line [18, 19].

4. The layered solution

The solution of equations (1) and (2) for a stack of layers under the simplifying assumption that all layers are identical in the very special limit of a weak \( \vec{h} \) field since this is the interesting physical limit to be treated. As it is well known NMR/NQR [36, 37] and μSR [38, 39] experiments set a very restrictive limit to the maximum magnetic field inside the cuprates, which cannot be larger than \( \sim 7-0.7 \text{G} \). In such a case the solution can be found recursively, namely, firstly \( \Psi \) is obtained from equation (1) in the absence of \( \bar{h} \), and, next, \( \bar{h} \) is determined from equation (2) using the previously obtained \( \Psi \). The smallness of \( \bar{h} \) dismisses the requirement of further iterations of the topological equations, such that it becomes enough to solve equation (1) as \( \vec{\sigma} \cdot \vec{V} \Psi = 0 \). This solution has been obtained elsewhere [11] and for a single layer at \( x_3 = 0 \) is given by,

\[
\Psi = \sum_{k \neq 0} e^{i \vec{k} \cdot \vec{r}} e^{-i \vec{k} \cdot \vec{r}} \left[ \begin{array}{c} 1 \\ -\frac{k_3}{k} x_3 \end{array} \right],
\]

where \( k_z = k_1 \pm ik_2 \) and \( k \equiv |\vec{k}| \), \( \vec{k} = k_1 \vec{e}_1 + k_2 \vec{e}_2 \). Notice that the up and down components satisfy \( \psi_{u}(0^+) = \psi_{d}(0^+) \) and \( \psi_{d}(0^-) = -\psi_{u}(0^+) \), meaning that they correspond to symmetric and antisymmetric modes across the layer of zero thickness. Thus the present phenomenological approach is indicative of microscopic bond and anti-bond states along the direction perpendicular to the layer. A finite thickness single layer would produce a double-well potential perpendicularly to the layers such that symmetric and anti-symmetric states are similar in energy. This scenario could be a direct consequence of a two-band, two-superconducting-gap behavior perpendicularly to the layers. The discontinuous behavior across the model zero thickness layer caused by the anti-symmetric state is necessary because what takes place immediately above and below the layer is very different. From this solution the multi-layer solution valid for
0 < x_1 < d is straightforwardly obtained,
\[ \Psi = \sum_{k \neq 0} c_k^\dagger \frac{e^{i \vec{k} \cdot \vec{r}}}{\sinh (k d/2)} \left( \cosh \left[ k (x_1 - d/2) \right] \right), \] (10)

In both cases the superficial current \( \vec{J}_s \) in the layers is immediately determined from,
\[ \frac{\vec{J}}{2e \mu_B} = \Psi^\dagger (0^+) \sigma_2 \Psi (0^+) \vec{J}_1 - \Psi^\dagger (0^+) \sigma_1 \Psi (0^+) \vec{J}_2, \] (11)
where
\[ \Psi^\dagger (0^+) \sigma_2 \Psi (0^+) = -\sum_{k, \neq 0} c_k^\dagger c_k e^{i \vec{k} \cdot \vec{r}}, \] (12)
and
\[ \Psi^\dagger (0^+) \sigma_1 \Psi (0^+) = \sum_{k, \neq 0} c_k^\dagger c_k e^{i \vec{k} \cdot \vec{r}}. \] (13)

The order parameter \( \Psi \) of equation (10) is intrinsically inhomogeneous since \( \vec{k} \neq 0 \). We choose to study here a periodic structure characterized by a unit cell with sides \( L_1 \) and \( L_2 \). Therefore \( k_i = 2\pi n_i / L_i, i = 1, 2 \), where \( n_1 \) and \( n_2 \) are integers. Thus the volumetric cell has volume \( V = Ad \), \( A = L_1 L_2 \) being the rectangular area within the layer where we find \( Q \) skyrmions. Albeit its complexity, the superficial supercurrent in the unit cell, which has area \( A \) and a perimeter \( P \), satisfies some simple properties:

(i) Null average supercurrent within the unit cell:
\[ \int_A \vec{J} \cdot d\vec{x} = 0; \]
(ii) No net supercurrent circulation at the edge of the unit cell: \( \vec{f}_b \cdot \vec{J} \cdot d\vec{l} = 0 \); and
(iii) No in and out of the unit cell supercurrent flow:
\[ \vec{f}_b \cdot \vec{J} \cdot d\vec{l} = 0, \] where \( \vec{n} \cdot d\vec{l} = 0. \)

It is straightforward to check that
\[ \int_A \vec{J} \cdot d\vec{x} = -4e \mu_B \sum_{k \neq 0} c_k^\dagger \frac{1}{\sinh (k d/2)} \] using that
\[ \int_A \exp \left[ i \vec{k} \cdot \vec{r} \right] d\vec{x} = \delta_{\vec{k}, \vec{0}} A. \] This summation vanishes provide that the coefficients satisfy \( k_{\perp} k_{\perp} = k_{\perp} k_{\perp} = k_{\perp} k_{\perp} = k_{\perp} k_{\perp} \) and so we find that (i) is valid under this condition. We use Stoke’s theorem, \( \vec{f}_b \cdot \vec{J} \cdot d\vec{l} = \int_{\partial A} \vec{V} \times \vec{J} \cdot d\vec{x} \), and Gauss’ theorem, \( \vec{f}_b \cdot \vec{J} \cdot d\vec{l} = \int_A \vec{V} \cdot \vec{J} \cdot d\vec{x} \) to find that assertions (ii) and (iii) are true, respectively, because \( \int_{\partial A} \vec{V} \times \vec{J} \cdot d\vec{x} = 0 \) for any \( j = 1, 2 \) since \( (k_i - k_i) \int \exp \left[ i (k_i - k_i) \cdot \vec{r} \right] d\vec{x} = 0 \) for \( i = 1, 2 \). An important and direct consequence of (ii) is that there is no net charge rate entering or exit the cell,
\[ \int_A \frac{\partial \sigma}{\partial t} d^2 x = 0, \] (14)
according to equation (7).

5. The GL theory

We show here that the topological equations do solve the GL variational equations for the special choice of temperature \( T = T_c \) under the approximation of a very weak local magnetic field. There is no applied magnetic field but there is a circulating supercurrent that creates this very small magnetic field. This solution corresponds to a lattice of skyrmions that we claim to be solution for the GL theory in this particular temperature. Recall that the GL theory is an order parameter expansion valid for temperatures near to the critical one where the order parameter is supposed to be small since the superconducting state is at the brink of disappearance. Obviously at the transition temperature itself the order parameter should be very small indeed. Landau’s argument is that in the \( T_c \) neighborhood powers of the order parameter higher than four can be safely neglected. Firstly, consider the case of the traditional GL theory, without the presence of an external field, whose Gibbs free energy is the sum of three terms, namely, the kinetic, the condensate and the field density energies,

\[ F = F_k + F_c + F_l, \] (15)
\[ F_k = \left\langle \frac{\partial \psi^2}{2m} \right\rangle, \] (16)
\[ F_c = \left\langle -\alpha_0 |\psi|^2 + \frac{1}{2} \beta |\psi|^4 \right\rangle, \] (17)
\[ F_l = \left\langle \frac{c^2}{8\pi} \right\rangle, \] (18)

where \( \langle \cdot \rangle \equiv \int (...) d^3 x / V \) and \( V \) is the bulk volume, \( m \) is the Cooper pair mass, \( \alpha_0 \equiv c_0 (T_c - T) \), \( c_0 > 0 \), and \( \beta > 0 \). For \( T = T_c \) the condensate becomes positive, \( F_c > 0 \), and so all of the contributions to the free energy are positive, since \( F_k > 0 \) and \( F_l > 0 \) hold for any temperature. Thus the lowest energy state has \( F = 0 \), and corresponds to the homogeneous state \( \psi = 0 \) without any local magnetic field present, \( \vec{h} = 0 \). However the situation becomes far more complex in case of the two-component order parameter GL theory given by,

\[ F = F_k + F_c + F_l, \] (19)
\[ F_k = \left\langle \frac{\partial \psi^2}{2m} \right\rangle, \] (20)
\[ F = \left\{ -\alpha_0 |\Psi|^2 - \vec{a} \cdot \Psi^\dagger \partial \Psi + \frac{1}{2} \Psi \ast \Psi \ast \cdot \beta \cdot \Psi \ast \cdot \Psi \right\}. \]

The condensate energy density is assumed to be the most general one with no extra assumptions other than its own stability. The second order term is the most general one and contains four independent parameters, \( \alpha_0 \), \( \alpha_3 \), and two other ones, \( \vec{a}_0 \), where parallel means to the layers as we shall see here. The fourth order term must be real and positive to warrant stability of the condensate energy: \( \Psi \ast \cdot \Psi \ast \cdot \beta \cdot \Psi \ast \cdot \Psi \). \( \Psi \equiv \beta_{abcd} \Psi_a \Psi_b \Psi_c \Psi_d > 0 \), where the indices \( a, b, c, d \) run over \( \ast \) and \( \ast \) in the most general tensor \( \beta_{abcd} \) which contains the required symmetry to render the fourth order term also positive. However the presence of two critical temperatures introduces new features into the theory. The presence of two critical temperatures restricts, according to the above argument, the validity of this GL free energy expansion to the temperature range \( T \approx T^* \), and \( T \approx T_c \) where the order parameter is expected to be small. Consequently the validity of this GL free energy expansion is also limited to situations such that \( T^* \approx T_c \). Assume that the pseudogap and the superconducting transition temperatures can be associated to \( T^* \) and \( T_c \), respectively [40]. Thus from the point of view of the temperature versus doping diagram, the present arguments restrict a GL free energy expansion to the top of the superconducting dome where these two lines cross each other. The temperature \( T = T_c = T^* \) corresponds to \( \alpha_0 + \alpha_3 = 0 \) and \( \alpha_0 - \alpha_3 = 0 \), since \( \alpha_0 |\Psi|^2 + \vec{a} \cdot \Psi^\dagger \partial \Psi = (\alpha_0 + \alpha_3) |\Psi|^2 + (\alpha_0 - \alpha_3) |\Psi|^2 + \vec{a} \cdot \Psi^\dagger \partial \Psi \). We also assume that at this crossing temperature that \( \vec{a}_0 = 0 \), where parallel is associated to the direction along the layers by choice of coordinate system. We shall see that the topological solution automatically satisfies that \( \langle \Psi \ast \vec{a} \Psi \rangle = 0 \). Similarly to the one-component GL theory the free energy of the two-component case also becomes a sum of three positive terms in this special temperature. Thus one naturally expects that its fundamental state has \( \Psi \neq 0 \) and \( \vec{h} = 0 \). Indeed this is the case, but we shall show here that there is an excited inhomogeneous state above this homogeneous state such that \( \Psi \neq 0 \) and \( \vec{h} \neq 0 \). This is the skyrmion state, made stable because of its topological properties. The previous solution of the two-component GL theory, given by equation (10), obtained under the only assumption that the order parameter is small, fact that defines an expansion parameter \( \epsilon \), namely, \( \Psi = O(\epsilon) \). However the order parameter is not dimensionless since \( |\Psi|^2 \Psi \) is a density and has the dimension of \( 1/V \), where \( V \) is the volume. Therefore we seek to determine \( \epsilon \propto 1/\sqrt{V} \) and leave to show elsewhere a more careful analysis that treats the present expansion in terms of a dimensionless order parameter. We show that the topological equations, provide a solution of the variational equations to order \( O(\epsilon^3) \) for the temperature \( T = T_c = T^* \).

The variational second order equations of the two-component order parameter theory are given by,

\[
\frac{\mathcal{D}^2 \Psi}{2m} = \alpha_0 \Psi + \vec{a} \cdot \partial \Psi - \left( \Psi \ast \cdot \beta \cdot \Psi \right) \cdot \Psi,
\]

\[
\vec{V} \times \vec{h} = \frac{4 \pi}{c} \vec{j}, \quad \vec{j} = \frac{q}{2m} \left\{ \Psi^\dagger \mathcal{D} \Psi + \text{c.c.} \right\}.
\]

The cubic order term in the order parameter means, for instance, that the ‘d’ component of \( \langle \Psi^\ast \cdot \beta \cdot \Psi \rangle \cdot \Psi \) is \( \beta_{abcd} \Psi_a \Psi_b \Psi_c \Psi_d \).

The keystone of the present approach is the existence of a dual formulation of the kinetic energy density [11, 41] given by,

\[
F_k = \left\{ \frac{1}{2m} \left| \vec{a} \cdot \mathcal{D} \Psi \right|^2 + \mu_0 \vec{h} \cdot \Psi^\dagger \partial \Psi - \frac{\hbar}{4m} \vec{V} \left\{ \Psi^\dagger \left( \vec{a} \times \vec{D} \right) \Psi + \text{c.c.} \right\} \right\}.
\]

While the original formulation of the kinetic energy leads to the above standard formulation of the variational equation, the dual one leads to an equivalent, but distinct formulation, given by,

\[
\frac{1}{2m} \left( \vec{a} \cdot \vec{D} \right)^2 \Psi = \Psi - \mu_0 \vec{h} \cdot \partial \Psi + \alpha_0 \Psi + \vec{a} \cdot \partial \Psi - \left( \Psi \ast \cdot \beta \cdot \Psi \right) \cdot \Psi,
\]

\[
\vec{V} \times \left( \vec{h} + 4 \pi \mu_0 \Psi^\dagger \partial \Psi \right)
= 2\pi \mu_0 \left\{ \Psi^\dagger \partial \left( \vec{a} \cdot \vec{D} \Psi \right) + \text{c.c.} \right\}.
\]

For instance, to show that the GL equation admits this two-fold formulation, namely, that equations (25) and (22) are equivalent, just use that \( \mathcal{D}^2 \Psi = \delta_{ij} D_i D_j \Psi \), and that \( \delta_{ij} = \sigma_i \sigma_j - i \epsilon_{ijk} \sigma_k \), where \( I \) is the two by two identity matrix, \( \epsilon_{ijk} \) is the totally anti-symmetric Levi–Civita tensor, and so, the local magnetic field is \( \vec{h} = -i \epsilon_{ijk} D_i D_j \).

Next we show that the second order variational equations are solved by the FOE until order lower than \( O(\epsilon^3) \). The right side of the GL equation, given by equation (25), is of order \( O(\epsilon^3) \), since \( \Psi = O(\epsilon) \) and \( \vec{h} = O(\epsilon^2) \). The terms of order \( O(\epsilon) \) in the right side are considered to vanish at \( T = T_c = T^* \), and the ones of order \( O(\epsilon^3) \) are negligibly small and can be approximated to zero, while the left side vanishes by virtue of the topological equation given by equation (1). Remarkably Ampère’s law, given by equation (26), is exactly solved by the topological equations.

The free energy is of order \( O(\epsilon^3) \) for \( T = T_c = T^* \). To show this firstly write equation (1) as,

\[
\mathcal{D} \Psi = i \vec{a} \times \mathcal{D} \Psi.
\]
Then the kinetic energy density of equation (24) becomes,
\[ F_k = \left( \frac{\hbar^2}{4m} V^2 \Psi^2 - \frac{1}{4\pi} \frac{\epsilon^2}{\hbar} \right) \tag{28} \]
by use of equation (2). The condensate energy density, equation (21), for \( T = T_c = T^* \) features \( O(\epsilon^3) \) and the field energy, equation (18), is also \( F_2 = O(\epsilon^4) \). Therefore the total energy has only one term of order \( O(\epsilon^3) \), and becomes,
\[ F = \left( \frac{\hbar^2}{4m} V^2 \Psi^2 \right) + O(\epsilon^4). \tag{29} \]

Interestingly the above \( O(\epsilon^3) \) term is a surface one that does not vanish thanks to the distinct behavior of the skyrmion solution infinitesimally above and below a layer. In fact this term is responsible for the gap of the inhomogeneous state above the homogeneous one because it reaches a constant value in its infrared limit (small \( k \)) \cite{11}. The free energy follows from equations (10) and (29), which give that,
\[ \epsilon = \frac{2}{\sinh \left( \frac{gd}{2} \right)} \left[ e^{i\pi\epsilon} \cos \left( \frac{g(x_1 + x_2) - m\pi}{2} \right) \right. \\
+ e^{i\pi\epsilon} \cos \left( \frac{g(x_2 - x_1) - m\pi}{2} \right) \cosh (g\tau_3) \]
\[ \times \left. \left[ e^{i\pi\epsilon} \cos \left( \frac{g(x_1 - x_2) - m\pi}{2} \right) \right. \\
- i e^{i\pi\epsilon} \cos \left( \frac{g(x_2 - x_1) - m\pi}{2} \right) \sinh (g\tau_3) \right] \right] \tag{30} \]
\[ + \frac{\epsilon}{\sinh \left( \frac{\sqrt{\pi}gd}{2} \right)} \left[ e^{i\pi\epsilon} \cos \left( \frac{g(x_1 + x_2) - m\pi}{2} \right) \right. \\
+ e^{i\pi\epsilon} \cos \left( \frac{g(x_2 - x_1) - m\pi}{2} \right) \cosh (\sqrt{2}g\tau_3) \]
\[ \times \left. \left[ e^{i\pi\epsilon} \cos \left( \frac{g(x_1 - x_2) - m\pi}{2} \right) \right. \\
- i e^{i\pi\epsilon} \cos \left( \frac{g(x_2 - x_1) - m\pi}{2} \right) \sinh (\sqrt{2}g\tau_3) \right] \right] \tag{31} \]

Thus we have proven that at least for \( T = T_c = T^* \) the topological equations solve the variational ones which means that the free energy can be determined from the above expression.

Nevertheless our criterion for abandoning terms of order \( O(\epsilon^3) \) and higher in the variational equations has introduced a handicap into the problem. The normalization parameter \( \epsilon \), so far just assumed to be small, cannot be determined by keeping just terms below order \( O(\epsilon^3) \). The local magnetic field, according to equation (2), and the free energy, equation (29) are both of order \( O(\epsilon^3) \), but there is no scheme to determine this parameter. Indeed the \( \epsilon \) parameter should be determined by the condensate energy, which here was completely abandoned because of its smallness. Therefore we introduce an external phenomenological criterion to define \( \epsilon \), which corresponds to the knowledge of the local magnetic field inside the superconductor. In some sense such knowledge is a way to phenomenologically include the residual higher order terms \( O(\epsilon^4) \) present in the free energy that were abandoned. Therefore we shall use this to argue through equation (2) that \( \epsilon \sim \sqrt{\hbar_{\text{exp}}} \).

6. States of angular momentum

The topological equations have a degenerate set of solutions, which means that these equations do leave room in parameter space for further minimization of the free energy minimization. They present an undetermined number of solutions as seen in the coefficients \( c_k \) of the order parameter which are not determined by the topological equations. This freedom also shows that for \( T = T_c = T^* \) there is a large degeneracy in the problem in case terms of order \( O(\epsilon^3) \) in the variational equations are neglected. To explicitly calculate the \( \partial \sigma / \partial t \) of the skyrmion state we need to know the coefficients \( c_k \) in equation (10). Here we find solutions that represent an order parameter with a definite angular momentum perpendicular to the layers:
\[ J_j \Psi_m = h \left( m + \frac{1}{2} \right) \Psi_m, \text{ where } J_j = J_3 + \frac{\hbar}{2} \pi \tau. \tag{31} \]

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\[ J_j \Psi_m = h \left( m + \frac{1}{2} \right) \Psi_m, \text{ where } J_j = J_3 + \frac{\hbar}{2} \pi \tau. \tag{31} \]
of $\epsilon$ is determined from the assumption that the calculated mean value of the local magnetic field corresponds to the experimental threshold, namely, $h_{\text{exp}} = |\langle \hat{h} \rangle|$. Therefore assuming the angular momentum states previously discussed we obtain that \[ h_{\text{exp}} = 16\pi\mu_B\epsilon^2 \left( \frac{1}{\sinh^2 \left( \frac{\pi^2 \mu_B}{L} \right)} + \frac{1}{\sinh^2 \left( \frac{\pi \mu_B}{L} \right)} \right) \] (35).

The present predictions for the inhomogeneous state gap are based on the selection of parameters, namely, the experimental threshold for the local field, $h_{\text{exp}} = 0.01$ Gauss, and the ratio $d/L = 0.75$. For the latter we have in mind the compound $YBa_2Cu_3O_{7-0.08}$ as this material presents the checkerboard pattern with $L = 4a = 1.6$ nm, where the crystallographic cell has size $a = 0.4$ nm and $d = 1.2$ nm. Notice that in the present model there is no commensurability with the underlying crystallographic structure such that the ratio $L/a$ can be any. The only relevant concern of the present model is whether the existence of skyrmions sets some limits on the ratio $L/d$. Indeed we have found an upper bound for $L/d$ much above the above taken value \cite{42} of $d/L = 0.75$. In convenient units the Bohr magneton is $\mu_B = 9.2 \text{Gauss} \cdot \text{nm}^2$. 

**Table 1.** The coefficients $a_m$ for $m = -4, \ldots, +4$ are listed in this table.

| $m$ | $p_m$ | $q_m$ |
|-----|-------|-------|
| $-4$ | $\sqrt{2}$ | 2 |
| $-3$ | $-4$ | $-3\sqrt{2}$ |
| $-2$ | $3\sqrt{2}$ | 4 |
| $-1$ | $-2$ | $-\sqrt{2}$ |
| 0 | $-\sqrt{2}$ | $-2$ |
| 1 | 4 | $3\sqrt{2}$ |
| 2 | $-3\sqrt{2}$ | $-4$ |
| 3 | 2 | $\sqrt{2}$ |
| 4 | $\sqrt{2}$ | 2 |

**Figure 1.** The superficial supercurrent, $J_s$, given by equation (4), is shown within the unit cell area, defined by $0 \leq x_i/L \leq 1$, $e = 1, 2$, for each of the $m = 0, 1, 2$, and 3 states. These states are eigenvectors of $J_3$ defined by equation (31). Notice that the superficial charge only exists in the layers and not in the interlayer space.
and then we obtain that [42],

$$e^2 = 5.3 \times 10^{-4} \text{ nm}^{-3}. \quad (36)$$

As previously discussed [11] the inhomogeneous state gap of equation (30) becomes $F = 0.5 \text{ meV nm}^{-3}$. A dimensional analysis shows that the charge density rate is controlled by the mean magnetic field and the size of the tetragonal lattice, $L$:

$$\frac{\partial \sigma}{\partial t} \sim \frac{h_{\text{exp}} c}{L}. \quad (37)$$

This follows from the following argument. According to equation (4) $J \sim c_{\mu B} e^2$ since $\Psi \sim e$. Thus equation (7) sets that $\partial \sigma/\partial t \sim c_{\mu B} e^2/L$ since $V \sim 1/L$, as it becomes evident in equation (33). From the other side equation (35) sets that $h_{\text{exp}} \sim \mu_B e^2$ which leads to the above result. Under these values one obtains that,

$$\frac{\partial \sigma}{\partial t} \sim 10^{-7} \text{ A nm}^{-2}. \quad (38)$$

This is the estimated charge density rate which achieves positive and negative values within the cell such that its average value vanishes as shown in equation (14).

From this value we also estimate the time rate that pairs enter the layers, based on an extra assumption beyond the scope of the present model. We assume that pairs cross a layer at a known rate and we interpret this as the origin of the charge density wave observed by Ghiringhelli et al [43]. They have proposed the presence of a charge density wave in the CuO$_2$ layers of the cuprates [10] which also falls in the THz regime.

7. The charge density wave

In the present magnetostatic model charge is constantly crossing a layer at a known rate and we interpret this as the origin of the charge density wave observed by Ghiringhelli et al [43]. They have proposed the presence of a charge density wave in the CuO$_2$ layers of the cuprates using resonant soft x-ray scattering. This two-dimensional charge density wave in the underdoped compound YBa$_2$Cu$_3$O$_{6.68}$ with an incommensurate periodicity that sets a tetragonal lattice because it is found to exist in orthogonal directions, namely, along and perpendicular to the so-called CuO chains. Interestingly they find that this structure holds both above and below $T_c$. The present model of two-dimensional layers embedded in a kind of metallic medium displays an inhomogeneous charge distribution in the layers. However from a three-dimensional perspective there is no charge accumulation in any point since both the volumetric and the superficial supercurrent render the total supercurrent divergenceless. The interpretation of the model is that non-static charges cross the layers at constant rate and, in so doing, produce positive and negative spots associated to their entrance and exit in the layers. This follows from the previously calculated $\partial \sigma/\partial t$ in the layers. We address here the question of the multipole moment of this definitely inhomogeneous charge distribution interpreted as a charge density in the layer. To achieve this goal we briefly review a few aspects of multipole expansion suitable for our analysis. The electrostatic energy density $U$ associated to a charge density $\sigma(\vec{x})$ in presence of an applied electrostatic potential $V(\vec{x})$ is given by,

$$U = \int_A \frac{d^2x}{A} \sigma(\vec{x}) V(\vec{x}). \quad (40)$$

Obviously here we are considering a fixed time window $\Delta t < 1/f$, $f$ given equation (39):

$$\sigma \approx \frac{\partial \sigma}{\partial t} \Delta t. \quad (41)$$

Expanding the electrostatic potential around a point $\vec{x} = 0$ within the area $A$ gives that,

$$U = QV(0) + \sum_{N=1}^{\infty} \frac{1}{N!} Q_{n_1, n_2 \cdots n_N} V_{n_1, n_2 \cdots n_N}(0), \quad (42)$$

$$Q_{n_1, n_2 \cdots n_N} \equiv \int_A d^2x \sigma(\vec{x}) x_{n_1} x_{n_2} \cdots x_{n_N}. \quad (43)$$

$$f \sim 0.3 \times 10^{12} \text{ Hz}. \quad (39)$$

Interestingly this number falls in the same order of magnitude of the Josephson plasma frequency between layers in the cuprates [10] which also falls in the THz regime.
\[ V_{i_1i_2 \ldots i_N}(0) \equiv \frac{\partial^2 V(\vec{x})}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_N}} |_{\vec{x}=0}, \]  

(44)

where \( Q = \int d^2x \sigma(\vec{x}) \) is the total charge and \( V(0) \) the potential at this selected origin. The tensors \( Q_{i_1i_2 \ldots i_N} \) are the multipole moments of this charge distribution. A way to calculate these tensors is simply to obtain the Fourier transform of the charge density and then expand it in powers of the wave number:

\[ \int_A d^2x e^{i \vec{k} \cdot \vec{x}} \sigma(\vec{x}) = Q + \sum_{N=1}^{\infty} \frac{i^N}{N!} Q_{i_1i_2 \ldots i_N} k_{i_1} k_{i_2} \cdots k_{i_N}. \]  

(45)

Next we apply the above ideas to the deconfined momentum states and obtain the Fourier transform of the charge density rate given by equation (33)

\[ \int_A e^{i \vec{k} \cdot \vec{x}} \sigma_m d^2x = -c\mu_B \alpha_m \]

\[ \times \frac{48 g^2}{(4 g^2 - 5 g^2 k_1^2 + k_3^2)(4 g^2 - 5 g^2 k_2^2 + k_3^2)} \Delta t. \]  

(46)

Expanding in powers of wave number gives that,

\[ \int_A e^{i \vec{k} \cdot \vec{x}} \sigma_m d^2x = -c\mu_B 12 \pi^2 \alpha_m \frac{1}{g^2} \]

\[ \times (k_2 k_1^3 - k_1 k_2^3) \Delta t + O(k^5), \]  

(47)

fact that configures an hexadecapole charge distribution since the first non-vanishing moments are,

\[ Q_{1112} = Q_{2221} = -c\mu_B 12 \pi^2 \alpha_m \frac{1}{g^2} \Delta t. \]  

(48)

Figure 1 displays the superficial current density within the unit cell for \( m = 0, 1, 2 \), and 3 associated to \( J_3 \) given by equation (31). The skyrmion cores can be seen in these plots and correspond to \( Q = +1, +2, -2, -1 \), respectively. Interestingly the \( m = 0 \) state has a core sitting at the corner and the figure display one fourth of it sitting at each corner. This also holds for the \( m = 1 \) state, whose core is at the corner but has double charge. The remaining two states, \( m = 2, 3 \) can be interpreted from the two previous cases, \( m = 1, 0 \) such that the circulation in the middle is strengthened in the center and weakened at the corner, which reverts the skyrmion number.

Figure 2 shows the charge density rate crossing the unit cell and generating a hexadecapole moment. Positive and negative charged spots represent the entrance and exit of the volumetric supercurrent in the unit cell at constant rate. Interestingly all the \( m \) states have the same charge density rate spatial distribution that only differs by the amplitude, according to the coefficients of table 1. Notice that the \( m = 2 \) state has opposite signal in comparison with the other \( m > 0 \) states. Interestingly the position of the skyrmion cores, shown in figure 1, and the positive and negative spots of the charge density rate, shown in figure 2, are not obviously correlated.

8. Conclusion

We have shown that the two-component order parameter theory describes layers constantly crossed by the supercurrent forming positive and negative charged spots in the condensate. Therefore this condensate is an inhomogeneous state with a gap above the ground state that produces a local magnetic field below the threshold of experimental observation and without the presence of an applied external field. This state is topologically stable and given by a lattice of skyrmions that display an elaborate pattern of volumetric and superficial currents circulating through the stack of layers that breaks time reversal symmetry.

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