Composite fermions for fractionally filled Chern bands

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We address the question of whether fractionally filled bands with a nontrivial Chern index in zero external field could also exhibit a Fractional Quantum Hall Effect (FQHE). Numerical works suggest this is possible. Analytic treatments are complicated by a non-vanishing band dispersion and a non-constant Berry flux. We propose embedding the Chern band in an auxiliary lowest Landau level (LLL) and then using composite fermions. We find some states which have no analogue in the continuum, and dependent on the interplay between interactions and the lattice. The approach extends to two-dimensional time-reversal invariant topological insulators.

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Models with no net magnetic flux but with a quantized Hall conductance $\sigma_{xy}$ have been known for some time.\textsuperscript{11} The breaking of time-reversal symmetry necessary for $\sigma_{xy} \neq 0$ manifests itself as a nontrivial Berry flux for the band in the Brillouin zone (BZ), which integrates to a nonzero integer Chern index, leading to a quantized Hall conductance when the band is full.\textsuperscript{12}

Could these “Chern bands” (CBs) exhibit the FQHE at partial filling in the presence of suitable interactions? A necessary condition is a hierarchy of scales where the bandgaps $\Delta$, the interaction strength $U$, and the bandwidth of the CB $W$ satisfy $\Delta \gg U \gg W$. Efforts have concentrated on “flattening” the CB\textsuperscript{2} with numerical work suggests that FQH-like states are realized.\textsuperscript{13} On the analytical front, Qi\textsuperscript{14} has constructed a basis in which known FQHE wavefunctions can be transcribed into the CB.

Recently Parameswaran et al.\textsuperscript{15} examined the algebra of $\tilde{v}CB(q)$, the density operators projected into the CB. Unlike in the LLL, where the projected density operators obey the magnetic translation algebra,\textsuperscript{15} in a CB the algebra does not close because the Berry curvature is not a constant. Equivalently, consider $R^CB$ the projected position vector of the electron with matrix elements

\begin{equation}
(R^CB)_{kk'} = \left( i \frac{\partial}{\partial k} + A_\mu(k) \right) \delta^2(k-k')
\end{equation}

where $A_\mu = -i(u(k)\partial_\mu|u(k))$ and $[R^CB, R^CB']_{kk'} = iB_{\mu\nu}(k)\delta^2(k-k')$ and the curvature $B(k)$ is related to the Chern index by $C = \frac{1}{2\pi} \int_{BZ} B$.

In an LLL problem in the continuum $B(k)$ is a constant and the projected electronic guiding center coordinates $R^*$ have $c$-number commutation relations $[R^*_\mu, R^*_\nu] = -il^2$, where $l$ is the magnetic length. This $c$-number commutator implies the closure of the magnetic translation algebra,\textsuperscript{16} the analyticity of wavefunctions, and the flux attachment that underlies the Laughlin wavefunction,\textsuperscript{17} Composite Bosons,\textsuperscript{18} and Composite Fermions (CFs).\textsuperscript{19}

Our main result is that one can use all the beautiful properties of the LLL, including flux-attachment, CFs, etc., by embedding the CB in an auxiliary LLL with a periodic potential. The price to be paid is either a more complicated density operator or the presence of extra states, but the profit is in the fact that one can carry out dynamical calculations for an arbitrarily varying $B(k)$. The applicability of CFs extends to analogues of the $\nu = \frac{1}{3}$ CF-Fermi Liquid\textsuperscript{12} or the $\nu = \frac{2}{5}$ state\textsuperscript{20} in CBs. New states for fractionally filled CBs, with no counterparts in the continuum, are clearly exhibited in this embedding, which comes in two variants (i) and (ii).

**Embedding (i):** This maps the CB into a modified LLL and seems to display the phenomenology of the gapped states seen in numerics\textsuperscript{6,10} on fractionally filled CBs. It also makes an explicit connection to work on fractional quantum Hall states on a lattice in an external field.\textsuperscript{21} Consider a set of noninteracting Landau levels (LLs) with a periodic potential (with the lattice symmetries of the target CB), with one quantum of electronic flux per unit cell. For a periodic potential $V$ on the same scale as the cyclotron energy $\omega_c$, LL-mixing will induce a set of dispersing bands with nonuniform Chern density in the BZ. We assume that by manipulating the harmonics of $V$ we can flatten the energy band and approximate the Chern density of the CB in the lowest band of our mixed LLs, which is the modified LLL, or MLLL with bandwidth $W$.

Now turn on repulsive interactions $U \gg W$, and $U \ll \Delta = \max(V,\omega_c)$ and fractionally fill the MLLL, and attach two units of flux a la Jain\textsuperscript{22}. In general the CFs will see $\nu' = \frac{n}{q}$ units of effective flux per unit cell, and each CF-Landau level (CFLL) will break up into $p'$ subbands. Filling an integer number $n'$ of these subbands will yield a gapped state. If the $n' = \nu'p'$ is an integer multiple of $p'$, these gapped states are adiabatically connected to the principal fractions $\nu' = \frac{p}{2p' + 1}$ seen in the LLL\textsuperscript{16}. For such states, the Hall conductance is identical to the filling factor, and the lattice seems to be an unimportant detail. On the other hand, if $n'$ is not an integer multiple of $p'$, the resulting gapped states rely crucially on the interplay of interactions and the lattice potential. Such states will not have any analogues in the translationally invariant LLL, and can display a Hall conductance which is different from the filling factor\textsuperscript{19}.

**Embedding (ii):** This maps the CB into a subband of the LLL with a periodic potential $V$ with a rational flux $p/q$ per unit cell, with $p > 1$. No other LLs are present. The LLL breaks up into $p$ subbands each with...
degeneracy $1/p$ of the LLL, each characterized by a Chern index $\chi$. Identify a subband with Chern index 1, which will play the role of the CB in the larger space. We assume that by manipulating $V$ one can flatten the target subband so that $W \ll V$, as well as achieve the required flux density $B(k)$ in the BZ.

Now turn on repulsive interactions $U$ which satisfy $W \ll U \ll V$. We argue by adiabatic continuity that the CFs (or composite bosons$^{15}$), which are created in the larger LLL Hilbert space will survive for an extensive set of fractions, i.e., that one can follow the gapped ground state from weak $V \ll U$ to strong $V \gg U$ without band crossings. If for $V \gg U$ the CF-bands break up into groups with degeneracy $1/p$ of an LLL, with distinct groups separated by energies of order $V$, (with the correct total Chern index for each group) one can safely use $\hat{\rho}_{LLL}$ in place of $\hat{\rho}_T$ because inter-subband matrix elements of the former will be suppressed by the large $V$.

For both (i) and (ii), for a given number of flux attached to transform from the electron in a CF, many more fractions are possible than in the liquid states in the LLL$^{14,16}$, because the periodic potential makes it possible to fill a CFLL partially by creating subbands. The generic problem of the FQHE in a periodic potential was analyzed by Kohl and Read$^{22}$ (KR). We use their analysis to identify the fractional quasiparticle charge and the Hall conductance for each gapped state. Without Galilean invariance, $\sigma_{xy}$ need not be the same as the filling fraction$^{23}$. There is numerical evidence for such states for lattice hard-core bosons in an external field$^{23}$.

It will become evident that this approach is easily extended to two-dimensional time-reversal invariant TIs$^{22,23}$.

To illustrate Embedding (i) we consider a periodic square lattice potential with one flux quantum per unit cell ($a^2 = 2\pi l^2$). In order to get nonconstant Chern density $B(p)$, we need to mix higher LLs. We will use the basis of states which have the translation properties appropriate to the potential: $\psi_{p,n}(x) = \frac{1}{\sqrt{a}} \sum_j e^{ijp_xa+i(y+2\pi j)a} \Phi_n(x-ja-p_y l^2)$

Here $n$ is a LL index and $\Phi_n$ are oscillator eigenfunctions. Now the one-body Hamiltonian is

$$H = \sum_{G} V(-G) \hat{\rho}_e(G) = \sum_{\mathbf{p}} c_{\mathbf{p}}^\dagger h(\mathbf{p}) c_{\mathbf{p}}$$

where $G$ are the reciprocal lattice vectors and $\hat{\rho}_e$ is the electronic density operator in the full Hilbert space including all LLs, and $h(\mathbf{p})$ is a matrix labeled by LL indices. In computing the Chern density of the lowest band of $H$, it is important to remember that the basis functions of Eq. (2) cannot be localized$^{23}$. Their $k$-dependence contributes a uniform background Chern density of $B_{\text{Bloch}} = a^2/2\pi$, to be added to that from the gauge connection of the lowest eigenstate. One manipulates the Chern density of the lowest band by tuning the harmonics of $V$ to approximate the Chern density of the given CB by the Chern density of this target band, as well as make it flat in energy. Given this, it is clear that the algebra of $\hat{\rho}_T(q)$ will be identical to that of $\hat{\rho}_{\text{CB}}(q)$.

To fully exploit the mapping of the CB to the MLLL we express $\hat{\rho}_T(q)$ in terms of the LLLL projected density $\rho_{\text{LLL}}(q)$, which satisfies the magnetic translation algebra$^{13}$. By standard techniques$^{24}$ one can construct a unitary transformation which decouples the target band from the others. Since the mixing terms between different LLs can only be superpositions of $\hat{G} \cdot \mathbf{R}^T$ it is evident that

$$\hat{\rho}_T(q) = \sum_{G} \hat{r}(G,q) \rho_{\text{LLL}}(q + G)$$

where the function $\hat{r}(G,q)$ vanishes as $q \rightarrow 0$ for $G \neq 0$. Given the unitary transformation $U_{n_1 n_2}(\mathbf{p})$ diagonalizing $h(\mathbf{p})$, the coefficients $r$ can be extracted as

$$\hat{r}(G,q) = \sum_{\mathbf{p},\mathbf{n}_1 \mathbf{n}_2} e^{in_1 \mathbf{G} \cdot \mathbf{R}} \sum_{l} e^{-i\mathbf{p} \cdot \mathbf{x}} \rho_{\mathbf{n}_1 \mathbf{n}_2}^{l}(\mathbf{p}) \rho_{\mathbf{n}_1 \mathbf{n}_2}^{l}(\mathbf{p}) U_{n_1 n_2}(\mathbf{p})$$

where $\mathbf{p} = (\mathbf{p} + q) \text{mod} \mathbf{G}$, $\rho_{\mathbf{n}_1 \mathbf{n}_2}(\mathbf{q})$ are the usual matrix elements of the density. Generically this $\hat{\rho}_T$ does not close under commutation$^{12}$.

To be more specific, consider $\nu^{\text{CB}} = \nu_f = \frac{1}{3}$. Attach two units of flux to the electron to form CFs, which see $\frac{1}{3} \nu_f = \frac{1}{3} - 2 = \frac{1}{3}$ quanta of effective flux per CF, thereby filling 1 CFLL. Since the target band we have constructed is adiabatically connected to the LLL, we can expect the Laughlin liquid state of the LLL to be adiabatically connected to the $\frac{1}{3}$ state in the CB. In particular, the qualitative properties, such as the ground state degeneracy and Hall conductance, will be the same as in the corresponding liquid state. Similar logic holds for $\frac{1}{5}$. These appear to be the states seen in recent numerics in fractionally filled CBs$^{26,27}$.

However, this does not exhaust the possibilities. Consider $\nu^{\text{CB}} = \nu_f = \frac{2}{3}$. Upon attaching two units of flux, the CFs see $\frac{2}{3} \nu_f = \frac{2}{3} - 2 = \frac{2}{3}$ quanta of effective flux per particle, implying $\nu^{\text{CF}} = \frac{2}{3}$. In a translationally invariant system one would expect this state to be gapless (unless it is paired). However, in a periodic potential the CFs see $\frac{1}{3}$ quanta of effective flux per unit cell, implying that each CFLL breaks up into three subbands$^{23}$. One can fill two of subbands arising from the $n = 0$ CFLL and obtain a gapped state, as shown schematically in Fig. (1). A surprising property of this state is that its Hall conductance cannot be $\frac{2}{3}$. To see this, we use the analysis of KR$^{22}$, who find that given the total CF-Chern index of all the filled CF-subbands (dubbed $\sigma_{xy}^{CF}$ in dimensionless units), the quasihole charge and electronic Hall conductance $\sigma_{xy}$ (again dimensionless) are

$$e_{qh} = \frac{1}{1 + 2\sigma_{xy}^{CF}}; \quad \sigma_{xy} = e_{qh} \sigma_{xy}^{CF}$$
and the statistics parameter for the quasiholes is \( \theta = \pi (1 - 2e_{ph}^*). \)

The CF-Chern index of each subband is an integer. For \( \frac{\theta}{2} \) the possibilities for \( \sigma_{xy} \) are \( \frac{1}{2}, \frac{2}{2}, \frac{3}{2} \) etc, depending on precisely what the CF-Chern indices of the occupied subbands are. While we do not know what \( \sigma_{xy} \) is we do know that it is not \( \frac{\theta}{2} \). This state, not adiabatically connected to a liquid-like state in the LLL, exists thanks to the interplay of interactions and periodic potential.

By attaching one quantum of flux, it is possible to map a hard-core boson problem into a fermion problem (the reverse of the usual transformation from electrons to Composite Bosons\(^{20-21}\)). If the hard-core bosons live in a CB, then one can use the techniques presented above to analyze their incompressible states with a strong interaction. In the related problem of hard-core bosons on a lattice in the presence of external flux, in addition to the noninteracting problem with the periodic potential producing LL-mixing. The CF’s have a filling of \( \frac{2}{3} \) and see \( \frac{2}{3} \) quanta of effective flux per unit cell. Each CFLL breaks up into three subbands, and two of the lowest subbands are occupied. The grey boxes indicate band-crossing regions where the state may not be gapped.

![Diagram](image)

FIG. 1: A schematic picture of a filling of \( \frac{2}{3} \) for Embedding (i). The MLLL on the right is the modified lowest band of the noninteracting problem with the periodic potential producing LL-mixing. The CF’s have a filling of \( \frac{2}{3} \) and see \( \frac{2}{3} \) quanta of effective flux per unit cell. Each CFLL breaks up into three subbands, and two of the lowest subbands are occupied. The grey boxes indicate band-crossing regions where the state may not be gapped.

Besides having the right hierarchy of energies, in Embedding (ii) the subbands merging into the lowest (insert) electronic subband must match its Chern number, namely 0, by adiabatic continuity. This implies that the single occupied CF-subband going into the target electronic subband carries a CF-Chern index of 2, which in turn implies\(^{19}\) that the Hall conductance of this state is \( \frac{2}{5} \), once again different from the filling of the CB.

Flux attachment offers a way to not only construct excellent wavefunctions in the LLL\(^{14-16}\), but also, via an extended Hamiltonian approach\(^{20-21}\) developed by us, a way to compute dynamical response functions, temperature-dependent properties, and much more. This approach, which starts from an extended Hilbert space of CFs supplemented by constraints to “project” to the physical space, has special advantages when applied to our embedding of the CB into an LLL. In the lattice FQHE problem, while trial wavefunctions can be constructed, they seem to have a much weaker overlap with the exact ground state even for Laughlin fractions\(^{20-21}\). The likely reason is that the density operator (Eq. (1)), its algebra\(^{12}\), and thus the entire dynamics, are different. Our approach does not rely on wavefunctions and is immune to this problem. Secondly, the entire formalism of...
“projecting” response functions to the physical subspace (the conserving approximation) carries over unchanged, and there are no significant finite-size limitations.

In summary, we propose two ways to analyze fractionalized Chern bands. Embedding (i) uses LL-mixing induced by a periodic potential $V$ to approximate the Chern density of the CB by that of the lowest band (the MLLL) in the LL problem by a suitable manipulation $V$. The density operator of the MLLL (needed for computations in the interacting theory) can be expressed as a superposition of LLL-projected density operators (Eq. 4). Upon introducing interactions $U$ much larger than the bandwidth $W$, we argue for the existence of two types of CF-states with fractional filling in this case: (a) States which are adiabatically connected to gapped liquid states in the LLL, which are the gapped states seen in numerics on CBs and for FQHE on a lattice, and (b) More exotic states whose very existence depends on an interplay of interactions and the lattice potential. The latter have no analogue in the LLL and a Hall conductance not equal to the filling factor, and may have been seen in a problem of lattice hard-core bosons in an external field.

Embedding (ii) maps the CB into a subband of an auxiliary LLL split into electronic subbands by a periodic potential. The density operator does not have to be projected into the target subband in order to carry out computations. If the fractional filling of the CB is such that the chemical potential lies in a CF-subband gap, and the final set of CF-subbands break up into groups according to the electronic hierarchy with the total Chern indices in each group consistent with the electronic Chern indices, then the dynamics of the fractionally filled CB is well-approximated by the CFs in a periodic potential.

Now for 2D time-reversal invariant TIs. We label the pair of Chern bands making up the noninteracting TI (with equal and opposite Chern index) by a pseudospin index which need not be conserved and merely determines the sign of the uniform magnetic fields in the auxiliary LLL problem. Interactions between CFs with different pseudospins can be included as before.

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