ALGEBRAIC PROOFS OF SOME FUNDAMENTAL THEOREMS IN ALGEBRAIC $K$-THEORY

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Abstract. We present new proofs of the additivity, resolution and cofinality theorems for the algebraic $K$-theory of exact categories. These proofs are entirely algebraic, based on Grayson’s presentation of higher algebraic $K$-groups via binary complexes.

Introduction

The beautiful and relatively young discipline of algebraic $K$-theory has seen tremendous development and far-reaching applications in many other mathematical disciplines over the last decades. This paper makes a contribution to a project (begun in [Gra12] and [Gra13]) reformulating its foundations.

The algebraic $K$-theory of an exact category was first described by Segal and Waldhausen, obtained by modifying Segal’s construction of the $K$-theory of a symmetric monoidal category. Quillen’s alternative $Q$-construction gives a very powerful tool for proving fundamental theorems in algebraic $K$-theory, which he exploited to prove the additivity, resolution, dévissage and localisation theorems [Qui73]. Waldhausen’s later work on the $S$-construction, in particular his version of the additivity theorem, made simpler proofs of the theorems cited above and the cofinality theorem possible [Sta89]. Common in all of these approaches is the use of some non-trivial content from homotopy theory.

Grayson [Gra12] recently gave the first presentation of the higher algebraic $K$-groups of an an exact category by generators and relations; we take this presentation as our definition of $K_n\mathcal{N}$. The object of this paper is to present new completely algebraic proofs of the additivity, resolution and cofinality theorems in higher algebraic $K$-theory of exact categories.

We assume throughout that the reader is familiar with exact categories. They are first systematically defined in [Qui73], a very nice exposition is [Buh10]. In section 1 we review the necessary details of Grayson’s presentation and present a new proof of the additivity theorem, in its form concerning so-called extension categories.

Date: November 26th, 2013.
2010 Mathematics Subject Classification. 19D99.
Key words and phrases. higher algebraic $K$-groups, acyclic binary complexes, additivity theorem, resolution theorem, cofinality theorem.
**Theorem** (Additivity). Let $B$ be an exact category, with exact subcategories $A$ and $C$ closed under extensions in $B$. Let $E(A, B, C)$ denote the associated extension category. Then $K_nE(A, B, C) \cong K_nA \times K_nC$, for every $n \geq 0$.

Using Grayson’s presentation, the proof of the additivity theorem is rather simple. In sections 2 and 3 we present more involved proofs of the resolution and cofinality theorems.

**Theorem** (Resolution). Let $M$ be an exact category and let $P$ be a full, additive subcategory that is closed under extensions. Suppose also that:

1. If $M' \to M \to M''$ is an exact sequence in $M$ with $M$ and $M''$ in $P$ then $M'$ is in $P$ as well.
2. Given $j : M \to P$ in $M$ with $P$ in $P$ there exists $j' : P' \to P$ and $f : P' \to M$ in $M$ with $P'$ in $P$ such that $jf = j'$.
3. Every object of $M$ has a finite resolution by objects of $P$.

Then the inclusion functor $P \hookrightarrow M$ induces an isomorphism $K_n(P) \cong K_n(M)$ for every $n \geq 0$.

**Theorem** (Cofinality). Let $M$ be a cofinal exact subcategory of an exact category $N$. Then the inclusion functor $M \hookrightarrow N$ induces an injection $K_0M \hookrightarrow K_0N$ and isomorphisms $K_nM \cong K_nN$ for $n > 0$.

Grayson defines the $n$th algebraic $K$-group of an exact category, denoted $K_nN$, as a quotient group of the Grothendieck group of a certain related exact category $(B^n)^\alpha N$, whose objects are so-called *acyclic binary multicomplexes* (see Definition 1.5). Each of the theorems above makes some comparison between the $K$-groups of a pair of exact categories. These theorems all have well-known algebraic folk proofs for the Grothendieck group $K_0$ so the general schema for our proofs is then as follows. First we verify that the hypotheses on our exact categories of interest also hold for their associated categories of acyclic binary multicomplexes. Then we apply the algebraic $K_0$ proof to obtain a comparison between their Grothendieck groups. Finally we verify that the required comparison still holds when we pass to the quotients defining the higher algebraic $K$-groups.

The remaining theorems regarded as fundamental in the algebraic $K$-theory of exact categories are the dévissage and localisation theorems. These theorems concern abelian categories, say $A$ and $B$. While the associated categories $(B^n)^\alpha A$ and $(B^n)^\alpha B$ are still exact, they will no longer be abelian, so a strategy more sophisticated than the approach of this paper will be necessary to prove these theorems in the context of Grayson’s new definition of the higher algebraic $K$-groups.

**Acknowledgements.** The author thanks Daniel Grayson for an essential insight into the proof of the resolution theorem, and for the results of his paper [Gra12], which we draw upon extensively. Thanks are also due for his helpful comments on a late draft of this paper.

The author also thanks his supervisor Bernhard Köck for his careful readings of successive drafts of this work, for suggesting numerous improvements to its content, and for his encouragement and enthusiasm.
1. Grayson’s binary complex algebraic $K$-theory

In this section we recall the definitions and main result of [Gra12]. As a first application we give a simple new proof of the additivity theorem, as previously proven in [Qui73], as well as [McC93] and [Gra11], and whose version for the $S$-construction is commonly considered to be the fundamental theorem in the algebraic $K$-theory of spaces. We shall work throughout with exact categories in the sense of Quillen [Qui73], that is, additive categories with a distinguished collection of short exact sequences that satisfies a certain set of axioms.

**Definition 1.1.** A bounded acyclic complex, or long exact sequence, in an exact category $N$ is a bounded chain complex $N$ whose differentials factor through short exact sequences of $N$. That is, the differentials $d_k : N_k \to N_{k-1}$ factor as $N_k \to Z_{k-1} \to N_{k-1}$ such that each $Z_{k-1} \to N_{k-1}$ is a short exact sequence of $N$.

In an abelian category the long exact sequences defined above agree with the usual long exact sequences. Care must be taken in the case of a general exact category, as the following example shows.

**Example 1.2.** Let $R$ be a ring with a finitely-generated, stably-free, non-free projective module $P$ (so $P \oplus R^m \cong R^n$ for some $m$ and $n$). We have short exact sequences of $R$-modules $0 \to P \xrightarrow{i} R \xrightarrow{p} R^{m} \xrightarrow{j} R^{n} \xrightarrow{q} 0$ and $0 \to R^{m} \xrightarrow{j} R^n \xrightarrow{q} P \to 0$, where $i,j,p$ and $q$ are the obvious inclusions and projections. The sequence $0 \to R^{m} \xrightarrow{j} R^n \xrightarrow{q} R^n \xrightarrow{p} R^{m} \to 0$ is a chain complex in the exact category of finitely-generated free modules, $\text{Free}(R)$, and it is exact as a sequence of $R$ modules, but, in the sense of the definition above, it is not long exact in $\text{Free}(R)$.

The category $C^aN$ of bounded acyclic complexes in an exact category $N$ is itself an exact category ([Gra12], §6); a composable pair of chain maps is declared to be short exact if and only if each composable pair of term-wise morphisms is short exact in $N$. That is, a composition of chain maps $N' \xrightarrow{\phi} N \xrightarrow{\psi} N''$ is short exact if and only if every $N'_k \xrightarrow{\phi_k} N_k \xrightarrow{\psi_k} N''_k$ is short exact. A word of warning here: a morphism in $C^aN$ that has admissible epimorphisms of $N$ as its term-wise morphisms is not necessarily an admissible epimorphism in $C^aN$.

**Example 1.3.** Let $i, j, p$ and $q$ be the morphisms in Example 1.2, and note that $iq + jp = 1$. The diagram below has exact rows, and is in fact a morphism of $C^a\text{Free}(R)$.

\[
\begin{array}{c}
R^n \\
\downarrow \scriptstyle{[0 \ 1]} \\
0 \\
\end{array} & \xrightarrow{[j \ 1]} & \begin{array}{c}
R^n \oplus R^m \\
\downarrow \scriptstyle{[-p \ 1]} \\
R^m \\
\end{array} & \xrightarrow{[1 \ -p \ -1]} & \begin{array}{c}
R^n \oplus R^m \\
\downarrow \scriptstyle{[0 \ 1]} \\
R^m \\
\end{array} & \xrightarrow{[p \ 1]} & \begin{array}{c}
R^n \\
\downarrow \scriptstyle{[0 \ 1]} \\
0 \\
\end{array}
\end{array}
\]

Each vertical arrow is an admissible epimorphism of $\text{Free}(R)$, but the diagram is not an admissible epimorphism of $C^a\text{Free}(R)$—its kernel is the complex discussed in Example 1.2 which is not acyclic in $\text{Free}(R)$.

\footnote{The q stands for “quasi-isomorphic to the zero complex”. We will not make use of the notion of a quasi-isomorphism—the q is a reminder that we are dealing only with the acyclic complexes in $N$.}
**Definition 1.4.** A *binary complex* in $\mathcal{N}$ is a chain complex with two independent differentials. More precisely, a binary complex is a triple $(N, d, \tilde{d})$ such that $(N, d)$ and $(N, \tilde{d})$ are chain complexes in $\mathcal{N}$. We call a binary chain complex *acyclic* if each of the complexes $(N, d)$ and $(N, \tilde{d})$ is acyclic in $\mathcal{N}$. A morphism between binary complexes is a morphism between the underlying graded objects that commutes with both differentials. A short exact sequence is a composable pair of such morphisms that is short exact term-wise.

Since $C^q\mathcal{N}$ is an exact category, the reader may easily check that the category $B^q\mathcal{N}$ of bounded acyclic binary complexes in $\mathcal{N}$ is also an exact category. There is a diagonal functor $\Delta : C^q\mathcal{N} \to B^q\mathcal{N}$, sending $(N, d)$ to $(N, d, \tilde{d})$. A binary complex that is in the image of $\Delta$ is also called diagonal. The diagonal functor is split by the *top* and *bottom* functors $\top, \bot : B^q\mathcal{N} \to C^q\mathcal{N}$; it is clear that $\Delta, \top$ and $\bot$ are all exact.

Taking the category of acyclic binary complexes behaves well with respect to subcategories closed under extensions. If $\mathcal{M}$ is a full subcategory closed under extension in $\mathcal{N}$ (later just called an *exact subcategory*), then $B^q\mathcal{M}$ is a subcategory closed under extensions in $B^q\mathcal{N}$. It is important here that the binary complexes in $B^q\mathcal{N}$ are bounded. Starting at the final non-zero term, one argues by induction that the objects that the extension factors through are actually in $\mathcal{M}$, using the hypothesis on $\mathcal{M}$ and $\mathcal{N}$, and the 3 × 3 Lemma ([Buh10], Corollary 3.6).

Since $B^q\mathcal{N}$ is an exact category, we can iteratively define an exact category $(B^q)^{\dot{n}}\mathcal{N} = B^qB^q\cdots B^q\mathcal{N}$ for each $n \geq 0$. The objects of this category are bounded acyclic binary complexes of bounded acyclic binary complexes ... of objects of $\mathcal{N}$. Happily, this may be neatly unwrapped: it is obvious that the following is an equivalent definition of $(B^q)^{\dot{n}}\mathcal{N}$.

**Definition 1.5.** The exact category $B^q\mathcal{N}$ of bounded acyclic binary multicomplexes of dimension $n$ in $\mathcal{N}$ is defined as follows. A *bounded acyclic binary multicomplex of dimension* $n$ is a bounded (i.e. only finitely many non-zero), $\mathbb{Z}^n$-graded collection of objects of $\mathcal{N}$ together with a pair of acyclic differentials $d^i$ and $\tilde{d}^j$ in each direction $1 \leq i \leq n$ such that, for $i \neq j$,

\[
\begin{align*}
d^i d^j &= d^i d^j \\
d^i \tilde{d}^j &= \tilde{d}^i d^j \\
\tilde{d}^i d^j &= \tilde{d}^i \tilde{d}^j \\
\tilde{d}^i \tilde{d}^j &= \tilde{d}^i \tilde{d}^j.
\end{align*}
\]

In other words, any pair of differentials in different directions commute. A morphism $\phi : N \to N'$ between such binary multicomplexes is a $\mathbb{Z}^n$-graded collection of morphisms of $\mathcal{N}$ that commutes with all of the differentials of $N$ and $N'$. A short exact sequence in $(B^q)^{\dot{n}}\mathcal{N}$ is a composable pair of such morphisms that is short exact term-wise.

In addition to $(B^q)^{\dot{n}}\mathcal{N}$, for each $n \geq 1$ we have an exact category $C^q((B^q)^{\dot{n}-1})\mathcal{N}$ of bounded acyclic complexes of objects of $(B^q)^{\dot{n}-1}\mathcal{N}$. For each $i$ with $1 \leq i \leq n$ there is a diagonal functor $\Delta_i : C^q((B^q)^{\dot{n}-1})\mathcal{N} \to (B^q)^{\dot{n}}\mathcal{N}$ that consists of ‘doubling up’ the differential of the (non-binary) acyclic complex and regarding it as direction $j$ in the resulting acyclic binary multicomplex. Any object of $(B^q)^{\dot{n}}\mathcal{N}$ that is in the
image of one of these $\Delta_i$ is called diagonal. The diagonal binary multicomplexes are those that have $d^i = \bar{d}^i$ for at least one $i$.

We can now formulate Grayson's presentation of the algebraic $K$-theory groups of $\mathcal{N}$, which we shall take to be their definition for the remainder of this paper.

**Theorem / Definition 1.6** ([Gra12, Corollary 7.4]). For $\mathcal{N}$ an exact category and $n \geq 0$, the abelian group $K_n\mathcal{N}$ is presented as follows. There is one generator for each bounded acyclic binary multicomplex of dimension $n$, and there are relations $[N'] + [N''] = [N]$ if there is a short exact sequence $N' \rightarrow N \rightarrow N''$ in $(B^n)\mathcal{N}$, and $[T] = 0$ if $T$ is a diagonal acyclic binary multicomplex.

We now present a new, elementary proof of the additivity theorem. Let $\mathcal{A}$ and $\mathcal{C}$ be exact subcategories of an exact category $\mathcal{B}$.

**Definition 1.7.** The extension category $\mathcal{E}(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is the exact category whose objects are short exact sequences $\mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C}$ in $\mathcal{B}$, with $\mathcal{A}$ in $\mathcal{A}$ and $\mathcal{C}$ in $\mathcal{C}$, and whose morphisms are commuting rectangles.

**Theorem 1.8** (Additivity). $K_n\mathcal{E}(\mathcal{A}, \mathcal{B}, \mathcal{C}) \cong K_n\mathcal{A} \times K_n\mathcal{C}$, for every $n \geq 0$.

**Proof.** The exact functors

$$\mathcal{E}(\mathcal{A}, \mathcal{B}, \mathcal{C}) \rightarrow \mathcal{A} \times \mathcal{C}$$

$$(\mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C}) \mapsto (\mathcal{A} \rightarrow \mathcal{A} \oplus \mathcal{C} \rightarrow \mathcal{C})$$

induce mutually inverse isomorphisms between $K_0\mathcal{E}(\mathcal{A}, \mathcal{B}, \mathcal{C})$ and $K_0\mathcal{A} \times K_0\mathcal{C}$, (see, e.g., [Wei13, II.9.3]). For each $n > 0$, the categories $(B^n)\mathcal{A}$ and $(B^n)\mathcal{C}$ are exact subcategories of $(B^n)\mathcal{B}$, so we can define an $n^{th}$ extension category

$$\mathcal{E}^n(\mathcal{A}, \mathcal{B}, \mathcal{C}) := \mathcal{E}((B^n)\mathcal{A}, (B^n)\mathcal{B}, (B^n)\mathcal{C}).$$

From the above, for each $n$ the induced map

$$K_0\mathcal{E}^n(\mathcal{A}, \mathcal{B}, \mathcal{C}) \rightarrow K_0((B^n)\mathcal{A} \times (B^n)\mathcal{C})$$

is an isomorphism. But a short exact sequence of binary multicomplexes is exactly the same thing as a binary multicomplex of short exact sequences, so the categories $\mathcal{E}^n(\mathcal{A}, \mathcal{B}, \mathcal{C})$ and $(B^n)\mathcal{E}(\mathcal{A}, \mathcal{B}, \mathcal{C})$ are isomorphic, so we therefore have an isomorphism

$$K_0((B^n)\mathcal{A} \times (B^n)\mathcal{C}) \cong K_0((B^n)\mathcal{A} \times (B^n)\mathcal{C})$$

for each $n$. Identifying the categories $\mathcal{E}^n(\mathcal{A}, \mathcal{B}, \mathcal{C})$ and $(B^n)\mathcal{E}(\mathcal{A}, \mathcal{B}, \mathcal{C})$, a binary multicomplex in $(B^n)\mathcal{E}(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is diagonal in direction $i$ if and only if its constituent binary multicomplexes in $(B^n)\mathcal{A}$, $(B^n)\mathcal{B}$ and $(B^n)\mathcal{C}$ are also diagonal in direction $i$. Similarly, if $\mathcal{A}$ in $(B^n)\mathcal{A}$ and $\mathcal{C}$ in $(B^n)\mathcal{C}$ are diagonal then the
binary multicomplexes corresponding to \((A \twoheadrightarrow A \rightarrow 0)\) and \((0 \rightarrow C \rightarrow C)\) are diagonal as well, so the isomorphism \(K_0(B^3)^n E(A, B, C) \cong K_0(B^3)^n A \times K_0(B^3)^n C\) restricts to an isomorphism \(T^n_{E(A, B, C)} \cong T^n_A \times T^n_B\). Passing to the quotients yields the result.

2. THE RESOLUTION THEOREM

The resolution theorem relates the \(K\)-theory of an exact category to that of a larger exact category all of whose objects have a finite resolution by objects of the first category. Its most well-known application states that the \(K\)-theory of a regular ring is isomorphic to its so-called \(G\)-theory (the \(K\)-theory of the exact category of all finitely-generated \(R\)-modules). As in the proof of the additivity theorem, we adapt a simple proof for \(K_0\) to work for all \(K_n\). The main difficulty in this proof is verifying that the hypotheses of the theorem pass to exact categories of acyclic binary multicomplexes.

The general resolution theorem for exact categories ([Qui73, §4 Corollary 2]) is a formal consequence of the following theorem, which is Theorem 3 of [Qui73].

**Theorem 2.1.** Let \(\mathcal{P}\) be a full, additive subcategory of an exact category \(\mathcal{M}\) that is closed under extensions and satisfies:

1. For any exact sequence \(P' \rightarrow P \rightarrow M\) in \(\mathcal{M}\), if \(P\) is in \(\mathcal{P}\) then \(P'\) is in \(\mathcal{P}\).
2. For any \(M\) in \(\mathcal{M}\) there exists a \(P\) in \(\mathcal{P}\) and an admissible epimorphism \(P \twoheadrightarrow M\).

Then the inclusion functor \(\mathcal{P} \hookrightarrow \mathcal{M}\) induces an isomorphism \(K_n \mathcal{P} \cong K_n \mathcal{M}\) for all \(n \geq 0\).

**Proof.** For \(K_0\) the inverse to the induced homomorphism \(K_0 \mathcal{P} \rightarrow K_0 \mathcal{M}\) is given by the map

\[
\phi: \quad K_0 \mathcal{M} \rightarrow K_0 \mathcal{P} \\
[M] \mapsto [P] - [P'],
\]

where \(P' \twoheadrightarrow P \rightarrow M\) is a short exact sequence of \(\mathcal{M}\). The proof of Theorem 2.1 for \(n = 0\) is the simple exercise of checking that \(\phi\) is well-defined. We noted earlier that if \(\mathcal{P}\) is closed under extensions in \(\mathcal{M}\) then \((B^3)^n \mathcal{P}\) is closed under extensions in \((B^3)^n \mathcal{M}\) for each \(n\), and by the same reasoning, one easily sees that if \(\mathcal{P}\) and \(\mathcal{M}\) satisfy hypothesis (1) of the theorem, then so do \((B^3)^n \mathcal{P}\) and \((B^3)^n \mathcal{M}\) for each \(n\). The following proposition is about hypothesis (2).

**Proposition 2.2.** Let \(\mathcal{P}\) and \(\mathcal{M}\) satisfy the hypotheses of Theorem 2.1. For every object \(M\) of \((B^3)^n \mathcal{M}\) there exists a short exact sequence \(P' \twoheadrightarrow P \rightarrow M\) of \((B^3)^n \mathcal{M}\) with \(P'\) and \(P\) in \((B^3)^n \mathcal{P}\). Furthermore, if \(M\) is a diagonal binary multicomplex then we may choose \(P\) and \(P'\) to be diagonal as well.

We shall prove Proposition 2.2 shortly. We now continue with the proof of Theorem 2.1. Together with the known isomorphism for \(K_0\), the first part of the proposition implies that the induced map \(K_0(B^3)^n \mathcal{P} \rightarrow K_0(B^3)^n \mathcal{M}\) is an isomorphism for each \(n\). Clearly this isomorphism sends elements of \(T^n_{\mathcal{P}}\) to elements of \(T^n_{\mathcal{M}}\). Since the value of \(\phi\) is independent of the choice of resolution, the second part of the proposition implies that \(\phi\) maps elements of \(T^n_{\mathcal{M}}\) to elements of \(T^n_{\mathcal{P}}\). The isomorphism therefore descends to an isomorphism \(K_n \mathcal{P} \rightarrow K_n \mathcal{M}\). \(\square\)
It remains then to prove Proposition [2.2] so for the rest of this section we assume the hypotheses of Theorem [2.1]. The idea of the proof is to construct, for each $M$ in $(B^n)^a M$, a morphism of acyclic binary chain complexes $P \to M$ that is term-wise and admissible epimorphism, i.e. $P_j \to M_j$. By the assumption on $P$ and $M$ each of these admissible epimorphisms is part of a short exact sequence $P_j \to P_j \to M_j$ with the $P_j$ in $P$. The $P_j$ form a binary complex with the induced maps, and we show that this binary complex is in fact acyclic. The result will then follow from an induction on the dimension. We shall rely on the following fact.

**Lemma 2.3.** Let $f_i : Q_i \to N$, $i = 1, \ldots, m$ be a family of morphisms in an exact category, at least one of which is an admissible epimorphism. Then the induced morphism

$$[f_1 \ldots f_m] : \bigoplus_{i=1}^m Q_i \to N$$

is an admissible epimorphism as well. □

**Proof.** The general case follows from the case $m = 2$. In this case, the morphism $[f_1, f_2]$ factors as the composition

$$Q_1 \oplus Q_2 \xrightarrow{[f_1 \ 0 \ 1]} N \oplus Q_2 \xrightarrow{[\ 0 \ f_2 \ 1]} N \oplus Q_2 \xrightarrow{[\ 1 \ 0]} N,$$

all of which are admissible epimorphisms. □

We begin resolving binary complexes in the less involved case, in which we assume $M$ to be diagonal.

**Lemma 2.4.** Given a diagonal bounded acyclic binary complex $M$ in $B^n M$ there exists a short exact sequence $P' \to P \to M$ where $P'$ and $P$ are diagonal objects of $B^n P$.

**Proof.** We may consider $M$ as an object of $C^n N$, as $\Delta : C^n N \to B^n N$ is a full embedding for any exact category $N$. Represent $M$ in $C^n N$ as below. Without loss of generality we assume that $M$ ends at place 0.

$$
\begin{array}{cccccccc}
0 & \to & M_n & \xrightarrow{d} & \cdots & \xrightarrow{d} & M_k & \xrightarrow{d} & M_{k-1} & \xrightarrow{d} & \cdots & \xrightarrow{d} & M_0 & \to & 0,
\end{array}
$$

Since $P$ and $M$ satisfy the hypotheses of Theorem [2.1] there exists an object $Q_k$ of $P$ and an admissible epimorphism $\epsilon_k : Q_k \to M_k$ in $M$ for each $0 \leq k \leq n$. The diagram below is a morphism in $B^n M$ with target $M$, and its upper row, the source of the morphism, is an object of $C^n P$.

$$
\begin{array}{cccccccc}
0 & \to & 0 & \xrightarrow{\epsilon_k} & Q_k & \xrightarrow{d_{\epsilon_k}} & \cdots & \xrightarrow{d_{\epsilon_k}} & Q & \to & Q
\end{array}
$$

We denote the top row by $P^k$ and the morphism of complexes by $\zeta^k : P^k \to M$. We do this for each $k \in \{0, \ldots, n\}$ and form the sum

$$\zeta := [\zeta^n \ldots \zeta^0] : \bigoplus P^k \to M.$$
of $M$. By construction and Lemma 2.3 it is an admissible epimorphism of $\mathcal{M}$. We now have a morphism $\zeta : P \rightarrow M$ that has term-wise morphisms all admissible epimorphisms. The kernels of the term-wise admissible epimorphisms are all in $\mathcal{P}$ by the hypotheses on $\mathcal{P}$ and $\mathcal{M}$. These kernels form a (as we have seen in §1, not \textit{a priori} acyclic) bounded chain complex $P'$ in $\mathcal{P}$ under the induced maps between them. Moreover, $\mathcal{P}'$ must be diagonal as $P$ and $M$ are.

It remains to show that this chain complex is acyclic (i.e., that $P$ is in $B^a\mathcal{P}$), then $P' \rightarrow P \rightarrow M$ will be a short exact sequence of acyclic complexes. The objects of $P'$ are in $\mathcal{P}$ and $P'$ is acyclic in $\mathcal{M}$ by standard homological algebra, but this does not guarantee the acyclicity of $P'$ in $\mathcal{P}$, as evidenced by Example 1.3. Suppose that $M$ factors through objects $Z_j$ of $\mathcal{M}$, and that each $P^k$ factors through objects $Y_j^k$ of $\mathcal{P}$. Then $P'$ factors through the kernels of the corresponding morphisms

$$
\bigoplus Y_j^k \rightarrow Z_j.
$$

Since $\mathcal{P}$ is closed under kernels of admissible epimorphisms to objects of $\mathcal{M}$, it is enough to show that each of these morphisms is an admissible epimorphism, and by Lemma 2.3 it is enough in turn to show that, for each $j$, one of the morphisms $Y_j^k \rightarrow Z_j$ is an admissible epimorphism. The diagram below shows that this is true for $k = j$, as $Y_j^j = Q_j$.

Finally we consider $P'$, $P$ and $M$ now as diagonal binary complexes (by applying $\Delta$). Then $P' \rightarrow P \rightarrow M$ is the required short exact sequence of acyclic diagonal binary complexes. \hfill \Box

A little more work is required if the binary complex $M$ is not diagonal. The idea in this case is due to Grayson, and relies on the acyclicity of the chain complexes

$$0 \rightarrow Q \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow Q \oplus Q \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow Q \oplus Q \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \cdots \rightarrow Q \rightarrow 0,$$

and

$$0 \rightarrow Q \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow Q \oplus Q \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \rightarrow Q \oplus Q \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \cdots \rightarrow Q \rightarrow 0,$$

of arbitrary length, where $Q$ is an object of any exact category.

**Lemma 2.5.** Given an arbitrary bounded acyclic binary complex $M$ in $B^a\mathcal{M}$ there exists a short exact sequence $P' \rightarrow P \rightarrow M$, where $P'$ and $P$ are objects of $B^a\mathcal{P}$.

**Proof.** Let $M$ denote the element of $B^a\mathcal{M}$ given by the binary complex below

$$
0 \rightarrow M_n \xrightarrow{d} M_{n-1} \xrightarrow{d} \cdots \xrightarrow{d} M_k \xrightarrow{d} M_{k-1} \xrightarrow{d} \cdots \xrightarrow{d} M_0 \xrightarrow{d} 0
$$

and as before let $\epsilon_k : Q_k \rightarrow M_k$ be admissible epimorphisms in $\mathcal{M}$ with $Q_k$ in $\mathcal{P}$. Inductively define two collections of morphisms $\delta_{k,i}, \delta'_{k,i} : Q_k \rightarrow M_{k-i}$ by

$$
\begin{cases}
\delta_{k,1} = d \circ \epsilon_k \\
\delta'_{k,1} = d' \circ \epsilon_k
\end{cases}
$$
and
\[
\begin{align*}
\delta_{k,l+1} &= d \circ \delta'_{k,l}, \\
\delta'_{k,l+1} &= d' \circ \delta_{k,l}.
\end{align*}
\]
Since each of its differentials is acyclic, the top row of the diagram below is an object of \(B^n\mathcal{P}\) for each \(k \in \{0, \ldots, n\}\).

\[
\begin{array}{ccccccc}
0 & \longrightarrow & Q_k & \longrightarrow & Q_k \oplus Q_k & \longrightarrow & \cdots & \longrightarrow & Q_k \oplus Q_k & \longrightarrow & Q_k \\
\downarrow{\varepsilon_k} & & \downarrow{\begin{bmatrix} \delta_{k,1} & \delta'_{k,1} \end{bmatrix}} & & \downarrow{\begin{bmatrix} \delta_{k,2} & \delta'_{k,2} \end{bmatrix}} & & \cdots & & \downarrow{\begin{bmatrix} \delta_{k,n} & \delta'_{k,n} \end{bmatrix}} & & \downarrow{\delta_k} & \longrightarrow & Q_k \\
\cdots & \longrightarrow & M_k & \longrightarrow & M_{k-1} & \longrightarrow & \cdots & \longrightarrow & M_{k-2} & \longrightarrow & M_0 & \longrightarrow & 0.
\end{array}
\]

The morphisms \(\delta_{k,l}\) and \(\delta'_{k,l}\) have been constructed so that the vertical morphisms commute with the top and bottom differentials, so the diagram represents a morphism in \(B^n\mathcal{M}\), which we shall again denote by \(\zeta^k : P^k \rightarrow M\). Following the same method of proof as of the previous lemma, each \(P^k\) is acyclic so their direct sum is acyclic as well and so
\[
\zeta := [\zeta^0 \cdots \zeta^n] : \bigoplus P^k \rightarrow M.
\]
is a morphism in \(B^n\mathcal{M}\). By construction and Lemma \ref{lem:2.3} again, each term-wise morphism \(\zeta^i : P_j \rightarrow M_j\) is an admissible epimorphism in \(\mathcal{M}\). Each of these morphisms therefore has a kernel in \(\mathcal{P}\) and these kernels form a binary complex with the induced maps.

We wish to show that both differentials of this binary complex is acyclic in \(\mathcal{P}\). Consider the top differential first. Denoting the objects that the top differentials of \(M\) and each \(P^k\) factor through by \(Z_j\) and \(Y^k_j\), it is enough to show for each \(j\) that one of the morphisms \(Y^k_j \rightarrow Z_j\) is an admissible epimorphism, exactly as in the proof of Lemma \ref{lem:2.4}. Taking \(k = j\) again yields the result, as shown by the diagram below.

\[
\begin{array}{ccc}
Q_j & \longrightarrow & Q_j \oplus \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
\downarrow & & \downarrow \\
M_j & \longrightarrow & Z_j \\
\downarrow & & \\
& \longrightarrow & M_{j-1}.
\end{array}
\]

The bottom differential is dealt with entirely analogously. \(\square\)

**Proof of Proposition \ref{prop:2.2}** We proceed by induction on \(n\). In the base case \(n = 0\), there is nothing to show. For the inductive step, we view an acyclic binary multicomplex \(M\) in \((B^n)^{n+1}\mathcal{M}\), as an acyclic binary multicomplex of objects of \((B^n)^n\mathcal{M}\), i.e. as an object of \(B^n(B^n)^n\mathcal{M}\). By the inductive hypothesis, the inclusion of \((B^n)^n\mathcal{P}\) into \((B^n)^n\mathcal{M}\) satisfies the hypotheses of Theorem \ref{thm:2.1} so by Lemma \ref{lem:2.5} there exists a short exact sequence \(P' \rightarrow P \rightarrow M\) in \(B^n(B^n)^n\mathcal{M}\) with \(P'\) and \(P\) in \(B^n(B^n)^n\mathcal{P}\) = \((B^n)^{n+1}\mathcal{P}\), and so the first part follows. For the second part, suppose that \(M\) is diagonal in some direction \(i\). We consider \(M\) as a diagonal acyclic binary complex of (not necessarily diagonal) objects of \((B^n)^n\mathcal{M}\), that is, we “expand” \(M\) along the \(i\) direction. Then by Lemma \ref{lem:2.4} there exist diagonal acyclic binary
complexes \( P' \) and \( P \) in \((B^n)^{n+1} \mathcal{P}\) that are diagonal in direction \( i \), and an exact sequence \( P' \to P \to M \), so the proof is complete.

3. The cofinality theorem

Unlike the additivity and resolution theorems, the cofinality theorem was not proved by Quillen in [Qui73]. A proof for exact categories based on work by Gersten was given in [Gra79]. More general versions can be found in [Sta89] and [TT90]. It is proven in [Gra12] that the hypotheses of the cofinality theorem are satisfied by the appropriate exact categories of acyclic binary complexes, the main work in our proof is in ensuring that the results pass to the quotients defining \( K_n \).

Definition 3.1. An exact subcategory \( \mathcal{M} \) of an exact category \( \mathcal{N} \) is said to be cofinal in \( \mathcal{N} \) if for every object \( N_1 \) of \( \mathcal{N} \) there exists another object \( N_2 \) of \( \mathcal{N} \) such that \( N_1 \oplus N_2 \) is isomorphic to an object of \( \mathcal{M} \).

An obvious example of a cofinal exact subcategory is the category of free \( R \)-modules inside the category of projective \( R \)-modules, for any ring \( R \). More generally, every exact category is cofinal in its Karoubification ([TT90, Appendix A]). The cofinality theorem relates the \( K \)-theory of the cofinal subcategory to the \( K \)-theory of the exact category containing it. Throughout this section \( \mathcal{M} \to \mathcal{N} \) is the inclusion of a cofinal exact subcategory of an exact category \( \mathcal{N} \).

Define an equivalence relation on the objects of \( \mathcal{N} \) by declaring \( N_1 \sim N_2 \) if there exist objects \( M_1 \) and \( M_2 \) of \( \mathcal{M} \) such that

\[
N_1 \oplus M_1 \cong N_2 \oplus M_2.
\]

Since \( \langle M \rangle = 0 \) for every \( M \) in \( \mathcal{M} \), the cofinality of \( \mathcal{M} \) in \( \mathcal{N} \) ensures that equivalence classes of \( \sim \) form a group under the natural operation \( \langle N_1 \rangle + \langle N_2 \rangle = \langle N_1 + N_2 \rangle \); we denote this group by \( K_0(\mathcal{N} \text{ rel. } \mathcal{M}) \). The following lemma and its corollary were first observed in the proof of Theorem 1.1 in [Gra79].

Lemma 3.2. The sequence:

\[
0 \to K_0 \mathcal{M} \to K_0 \mathcal{N} \to K_0(\mathcal{N} \text{ rel. } \mathcal{M}) \to 0
\]

is well-defined and exact.

□

Corollary 3.3. For any pair of objects \( N_1, N_2 \) of \( \mathcal{N} \) with the same class in \( K_0 \mathcal{N}/K_0 \mathcal{M} \) there exists a (single) object \( N' \) in \( \mathcal{N} \) such that each \( N_i \oplus N' \) is in \( \mathcal{M} \).

Proof. By the lemma, if \( N_1 \) and \( N_2 \) have the same class in \( K_0 \mathcal{N}/K_0 \mathcal{M} \) then \( \langle N_1 \rangle = \langle N_2 \rangle \). From cofinality there exists a \( P \) in \( \mathcal{N} \) such that \( N_1 \oplus P \) is in \( \mathcal{M} \), so \( 0 = \langle N_1 + P \rangle = \langle N_2 + P \rangle \). Hence there exist objects \( P_1 \) and \( P_2 \) of \( \mathcal{M} \) such that each \( (N_i + P) \oplus P_i \) is an object of \( \mathcal{M} \). Setting \( N' = P \oplus P_1 \oplus P_2 \), each \( N_i \oplus N' \) is an object of \( \mathcal{M} \). □

We show now that cofinality of \( \mathcal{M} \) in \( \mathcal{N} \) passes to the associated categories of acyclic binary multicomplexes. We can say much more in general however.

Lemma 3.4. For all \( n \geq 0 \), the exact subcategory \((B^n)^n \mathcal{M}\) is cofinal in \((B^n)^n \mathcal{N}\). More precisely, if \( N \) is in \((B^n)^n \mathcal{N}\) and \( i \in \{1, \ldots, n\}\) is any direction then there exists an object \( T \) in \((B^n)^n \mathcal{N}\) that is diagonal in direction \( i \) such that \( N \oplus T \) is in...
(B^3)^n \mathcal{M}. Moreover, if N is diagonal in direction \( j \in \{1, \ldots, n\}, j \neq i \), then T may be shown to be diagonal in directions i and j.

Proof. (Part of the following proof is adapted from the proof of Lemma 6.2 in [Gra12] ) We proceed by induction on n. The statements for the base case \( n = 0 \) mean only that \( \mathcal{M} \) is cofinal in \( \mathcal{N} \), which is assumed throughout this section.

For the inductive step we fix \( i \in \{1, \ldots, n + 1\} \) and \( N \in (B^3)^{n+1}\mathcal{N} \). We first assume that \( N \) is diagonal in direction \( j \neq i \), and “expand along \( j \)” to consider \( N \) as a diagonal acyclic binary complex of objects of \((B^3)^n \mathcal{N}\). Let \( C_k \) in \((B^3)^n \mathcal{N}\) be the image of \( d_k^j = \tilde{d}_k^i \) (between the terms \( k \) and \( k-1 \)). By the inductive hypothesis, for each \( 'k \) there exists an object \( T_k \) in \((B^3)^n \mathcal{N}\) that is diagonal in direction \( i \) such that \( C_k \otimes T_k \) is an object of \((B^3)^n \mathcal{M}\). Let \( (T', e) \) be the acyclic chain complex of objects of \((B^3)^n \mathcal{N}\) given by taking the direct sum of the identity maps \( T_k \to T_k \) concentrated in degrees \( k \) and \( k - 1 \). That is, \((T', e)\) is the complex

\[
0 \to \cdots \to T_{k+1} \oplus T_{k+1} \to T_{k+1} \oplus T_k \to T_k \oplus T_{k-1} \to \cdots \to 0
\]

with differential \( e = (\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) \). Since each \( T_k \) is diagonal in direction \( i \), the complex \( T' \) is diagonal in direction \( i \), when regarded as an object of \( C^n(B^3)^n \mathcal{N}\). The image of the differential \( d_k^j \oplus c_k = d_k^j \oplus c_k \) on the acyclic complex \( \bigoplus j_{\mathcal{N}}(N) \oplus T' \) is equal to \( C_k \oplus T_k \), and hence belongs to \((B^3)^n \mathcal{M}\). Since \((B^3)^n \mathcal{M}\) is closed under extensions in \((B^3)^{n+1} \mathcal{M}\), we obtain that the complex \( \bigoplus j_{\mathcal{N}}(N) \oplus T' \) belongs to \( C^n(B^3)^n \mathcal{M}\). We define a binary complex \( T := \Delta_j(T') \), which is an object of \((B^3)^n \mathcal{N} = (B^3)^{n+1} \mathcal{N}\) and is diagonal in directions \( i \) and \( j \). Then \( N \oplus T = \Delta_j(\bigoplus j_{\mathcal{N}}(N) \oplus T') \) belongs to \((B^3)^n \mathcal{M} = (B^3)^{n+1} \mathcal{M}\).

If \( N \) is not diagonal in any direction different from \( i \), we again consider \( N \) as an acyclic binary complex

\[
\cdots \xrightarrow{\delta} N_{k+1} \xrightarrow{\delta} N_k \xrightarrow{\delta} N_{k-1} \xrightarrow{\delta} \cdots
\]

of objects \( N_k \) in \((B^3)^n \mathcal{N}\), but now “expanded along” direction \( i \). Let \( C_k \) and \( \tilde{C}_k \) in \((B^3)^n \mathcal{N}\) denote the images of the (normally different) differentials \( d_k^j \) and \( \tilde{d}_k^i \) (between \( k \) and \( k-1 \)). The classes of both \( C_k \) and \( \tilde{C}_k \) are equal to the finite sum \( \sum_{i=-\infty}^{k} (-1)^{k-i} [N_{k-i}] \) in \( K_0(B^3)^{n} \mathcal{N}\), by a standard argument, so they have the same class in

\[
\text{coker}(K_0(B^3)^n \mathcal{M} \to K_0(B^3)^n \mathcal{N})
\]

By the inductive hypothesis and Corollary[3.3] there exists a single object \( T_k \) in \((B^3)^n \mathcal{N}\) such that \( C_k \oplus T_k \) and \( \tilde{C}_k \oplus T_k \) are both objects of \((B^3)^n \mathcal{N}\). As above, from the objects \( T_k \) we form the acyclic binary complex \( T \) in \( B^n(B^3)^n \mathcal{N} = (B^3)^{n+1} \mathcal{N}\), which is diagonal in direction \( i \). Then \( N \oplus T \) is an object of \((B^3)^{n+1} \mathcal{M}\), as was to be shown. \( \square \)

Following Lemmas[3.2] and[3.4], we now regard \( K_0(B^3)^n \mathcal{M} \) as a subgroup of \( K_0(B^3)^n \mathcal{N}\) for each \( n \). It is clear moreover that this inclusion respects the subgroups generated by the classes of diagonal binary multicomplexes, i.e. \( T^a_{M} \subseteq T^a_{N} \). The following proposition concerning representations of elements of \( T^a_{N}/T^a_{M} \) is key to our proof of the cofinality theorem.

**Proposition 3.5.** Let \( x + T^a_{M} \) be a class in \( T^a_{N}/T^a_{M} \), for \( n \geq 1 \). Then \( x + T^a_{M} = [t] + T^a_{M} \), where \([t]\) is the class in \( K_0(B^3)^n \mathcal{N}\) of a diagonal acyclic binary multicomplex \( t \) in \((B^3)^n \mathcal{N}\).
Proof. The idea is to take the class of a general element \( x \in T^n_M \) and transform it into a direct sum of diagonal complexes that are all diagonal in the same direction, without altering the class of \( x \) modulo \( T^n_M \). To begin, we write

\[
x + T^n_M = \sum_{j=1}^{n} ([t'_j] - [t''_j]) + T^n_M,
\]

where each \( t'_j \) and \( t''_j \) is an actual acyclic binary multicomplex diagonal in direction \( j \), and pick a distinguished direction \( i \). By Lemma 3.3, for each \( j \neq i \) there exist acyclic binary multicomplexes \( s'_j \) and \( s''_j \) that are both diagonal in directions \( i \) and \( j \) such that \( t'_j \oplus s'_j \) and \( t''_j \oplus s''_j \) are objects of \((B^n) M\). The binary complexes \( t'_j, s'_j, t''_j \) and \( s''_j \) are all diagonal in direction \( j \), so their direct sums are also diagonal in direction \( j \), and we have \([t'_j \oplus s'_j] \in T^n_M \) and \([t''_j \oplus s''_j] \in T^n_M \). Therefore \([s'_j] = -[t'_j]\) and \([s''_j] = -[t''_j]\) in \( T^n_M \). But the \( s'_j \) and \( s''_j \), along with \( t'_j \) and \( t''_j \) are all diagonal in direction \( i \), so taking \( u_1 \) to be the sum of the positive classes in our new expansion of \( x + T^n_M \), and \( u_2 \) to be the sum of the negative classes, we have

\[
x + T^n_M = [u_1] - [u_2] + T^n_M,
\]

where \( u_1 \) and \( u_2 \) are acyclic binary multicomplexes that are diagonal in direction \( i \). Finally, we use Lemma 3.3 again to find an acyclic binary multicomplex \( u'_2 \) that is also diagonal in direction \( i \), such that \([u_2 \oplus u'_2] \in T^n_M \). Then

\[
x + T^n_M = [u_1] + [u'_2] + T^n_M,
\]

and setting \( t = u_1 \oplus u'_2 \) yields the result. \( \square \)

Theorem 3.6 (Cofinality). The inclusion functor \( M \rightarrow N \) induces an injection \( K_0 M \rightarrow K_0 N \) and isomorphisms \( K_n M \cong K_n N \) for \( n > 0 \).

Proof. The case \( n = 0 \) is part of Lemma 3.2, so we proceed directly to \( n > 0 \). We have the following diagram of abelian groups, whose rows are exact.

\[
\begin{array}{ccccccccc}
0 & \rightarrow & T^n_M & \rightarrow & K_0(B^n) M & \rightarrow & K_0 N & \rightarrow & 0 \\
0 & \rightarrow & T^n_N & \rightarrow & K_0(B^n) N & \rightarrow & K_0 N & \rightarrow & 0 \\
& \downarrow & & \downarrow & & \downarrow & & & \\
0 & \rightarrow & T^n_M / T^n_N & \rightarrow & K_0(B^n) N / K_0(B^n) M & \rightarrow & 0 \\
\end{array}
\]

The snake lemma implies that the homomorphism \( K_0 M \rightarrow K_0 N \) is an isomorphism if and only if the induced homomorphism

\[
T^n_M / T^n_N \rightarrow K_0(B^n) N / K_0(B^n) M
\]

is an isomorphism. Denote this homomorphism by \( \psi \). We first show that \( \psi \) is surjective. Let \( b \) be a generic element of \( K_0(B^n) N \), so \( b = [b_1] - [b_2] \), where \( b_1 \) and \( b_2 \) are acyclic binary multicomplexes of dimension \( n \). By Lemma 3.3, there exist diagonal acyclic binary multicomplexes \( s_1 \) and \( s_2 \) such that \( b_1 \oplus s_1 \) is an object of \( (B^n) M \) for \( i = 1, 2 \). Then \([b_1 \oplus s_1] - [b_2 \oplus s_2] \in K_0(B^n) M \), and is therefore zero in \( K_0(B^n) N / K_0(B^n) M \). Set \( s = [s_1] - [s_2] \in T^n_N \). Then \( b + K_0(B^n) N \) is the image of \(-s + T^n_M \) under \( \psi \), so \( \psi \) is surjective.
For the injectivity of \( \psi \), suppose that \( x \in T^n_M \) such that \( x + T^n_M \) is in \( \ker(\psi) \). By Proposition 3.5 we may write \( x + T^n_M = [t] + T^n_M \) for an actual acyclic binary multicomplex \( t \) diagonal in some direction \( i \). Since \( [t] + T^n_M \) is in the kernel of \( \psi \), we must have \([t] \in K_0(B^n)^nM \) (considered as a subgroup of \( K_0(B^n)^nN \)). In the notation of Lemma 3.2 we have \([t] = 0 \) in 
\[
K_0(B^n)^nN/K_0(B^n)^nM \cong K_0((B^n)^nN \text{ rel. } (B^n)^nM),
\]
so there exist acyclic binary multicomplexes \( a_1 \) and \( a_2 \) in \((B^n)^nM\) such that \( t \oplus a_1 \cong a_2 \). Consider the composite exact functor \( \Delta_i T_i : (B^n)^nN \to (B^n)^nN \), that replaces the bottom differential in direction \( i \) of an acyclic binary multicomplex with a second copy of the top differential. The binary multicomplexes \( \delta_i T_i(a_1) \) and \( \delta_i T_i(a_2) \) are diagonal in direction \( i \), and \( \delta_i T_i(t) = t \), since \( t \) is already diagonal in direction \( i \).

Applying the induced homomorphism on \( K_0 \) we have
\[
[t] = K_0(\Delta_i T_i)([t]) = K_0(\delta_i T_i)([a_1] - [a_2]) = [\delta_i T_i(a_1)] - [\delta_i T_i(a_2)] \in T^n_M.
\]
Hence \( x \in T^n_M \) for any \( x \) such that \( x + T^n_M \) is in the kernel of \( \psi \), so the kernel of \( \psi \) is trivial and \( \psi \) is injective.

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