How Not to Compute a Fourier Transform

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May 02, 2024
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September 6, 2023
Abstract

We revisit the Fourier transform of a Hankel function, of considerable importance in the theory of knife edge diffraction. Our approach is based directly upon the underlying Bessel equation, which admits manipulation into an alternate second order differential equation, one of whose solutions is precisely the desired transform, apart from an \textit{a priori} unknown constant and a second, undesired solution of logarithmic type. A modest amount of analysis is then required to exhibit that constant as having its proper value, and to purge the logarithmic accompaniment. The intervention of this analysis, which relies upon an interplay of asymptotic and close-in functional behaviors, prompts our somewhat ironic, mildly puckish caveat, our negation \textit{“not”} in the title. In a concluding section we show that this same transform is still more readily exhibited as an easy byproduct of the inhomogeneous wave equation in two dimensions satisfied by the Green’s function $G$, itself proportional to a Hankel function. This latter discussion lapses of course into the argot of physicists and, in its rôle of a mere afterthought, makes no claim whatsoever to any originality.

\textbf{key words} — knife edge diffraction; Fourier cosine transform; Bessel’s differential equation; asymptotic Hankel function estimate; two-dimensional Laplace equation; Green’s function for two-dimensional Helmholtz equation
1 Introduction

Sommerfeld’s celebrated, period $4\pi$ treatment of knife edge diffraction [1,2] was quick to elicit alternate viewpoints [3,4] of comparable ingenuity if, regrettably, far less renown. These achievements rested on their deserved laurels for half a century, only to be roused from their slumbers during the early forties by a minor renaissance anchored around the Wiener-Hopf [W-H] technique, first reported in [5,6], and then belatedly in [7]. Although initiated in a framework of integral equations, the W-H program soon abandoned this mooring in favor of Fourier transforms applied directly to the underlying differential (wave) equation [8,9,10], and this latter approach, too, was ultimately complemented by the method of dual integral equations cum plane wave spectrum [11,12,13].

The integral equation springboard in [5,6,7] pivots around the Green’s function for the two-dimensional wave equation, a quantity proportional to the Hankel function $H^{(2)}_0(k\rho)$ of the second kind and order zero. There quickly ensues a need for the Fourier transform thereof having the known outcome

$$L(k, w) = \int_0^\infty H^{(2)}_0(kx) \cos(wx) dx = \frac{1}{\sqrt{k^2 - w^2}}. \quad (1)$$

Our aim here is to arrive at (1) by an alternate route, one which exploits the underlying Bessel equation satisfied by $H^{(2)}_0(kx)$, with an emphasis upon wave number $k$ as variable and with $x$ temporarily regarded as an incidental multiplier. We note of course that the quadrature in (1) is valid throughout the horizontal strip $|\Im w| < |\Im k|$, which allows us to restrict initial attention to the situation with $\Im w = 0$, $\Re w > 0$, and then to extend its outcome throughout that strip on the basis of analytic continuation. And then, since incidental multiplier $x$ in Bessel’s equation is readily produced via differentiation with respect to $w$, the ordinary differential equation (ODE) is promoted at once to partial differential equation (PDE) status in variable pair $(k, w)$.

On exploiting next a variable scaling $\zeta = k/w$,

$$L(k, w) = \frac{1}{w} \int_0^\infty H^{(2)}_0(kx/w) \cos(x) dx = \frac{1}{w} F(\zeta), \quad (2)$$

our PDE regresses to second-order ODE status in $\zeta$, an equation among whose two independent, easily ascertained solutions one indeed finds the second line from (1), but with an initially undetermined constant multiplier, and accompanied by a somewhat more complicated, unwelcome logarithmic structure, similarly prefaced with its own constant factor.

And so, after this initial burst of easy progress, the remainder of the calculation devolves into fixing these constants in a way to dismiss that logarithmic companion and indeed to match (1). This latter activity, which marries the asymptotic behavior of $L$ to the close-in behavior of $H^{(2)}$ vis-à-vis their respective arguments, requires a modicum of spade work, diligence which underwrites our tongue-in-cheek qualifier “how not to compute” embedded in paper title. On the other hand, diligent or not, this latter effort,
too, is not without some minor flecks of mathematical allure. Such, at any rate, is our own modest perception thereof.

We conclude the note with yet another derivation of (1), couched in the physicist’s language of Green’s functions and Dirac deltas. Hopefully the tone of this grace note will not ruffle too many mathematical sensibilities. This material is essentially common folklore among physicists and is included, bereft of any claim to originality, mainly for the sake of completeness. We beg editorial indulgence for allowing us thus to gild the lily.

2 From Bessel’s ODE to PDE and then back to an ODE once more

Bessel’s ODE equation for $H^{(2)}_0(kx)$ yields in the first instance

$$
\left[ k \frac{\partial^2}{\partial k^2} + \frac{\partial}{\partial k} + x^2 k \right] H^{(2)}_0(kx) = 0,
$$

whence it readily follows that

$$
\left[ k \frac{\partial^2}{\partial k^2} + \frac{\partial}{\partial k} - k \frac{\partial^2}{\partial w^2} \right] L(k, w) = 0,
$$

Passage from (3) to (4) is premised on the assumption that the differential operator

$$
k \frac{\partial^2}{\partial k^2} + \frac{\partial}{\partial k} - k \frac{\partial^2}{\partial w^2}
$$

can be transferred with analytic impunity, en masse, to the interior of the defining quadrature from (1). This should certainly be legitimate since, even though its terms destroy convergence when permitted to operate on $H^{(2)}_0(kx) \cos(wx)$ individually, in unison they simply annihilate $H^{(2)}_0(kx) \cos(wx)$, yielding a constant, null outcome about whose integrability there can assuredly be no doubt.

Introduction of (2) has the effect of converting PDE (4) into the homogeneous ODE

$$
(\zeta - \zeta^3)F_{\zeta\zeta}(\zeta) + (1 - 4\zeta^2)F_\zeta(\zeta) - 2\zeta F(\zeta) = 0
$$

which further condenses into

$$
\left( (\zeta - \zeta^3)F(\zeta) \right)_{\zeta\zeta} + \left( (2\zeta^2 - 1)F(\zeta) \right)_\zeta = 0
$$

and hence allows an immediate integration into the form

$$
F(\zeta) = \frac{A}{\sqrt{1 - \zeta^2}} + \frac{B}{\sqrt{1 - \zeta^2}} \log \left( \frac{\zeta}{1 + \sqrt{1 - \zeta^2}} \right),
$$

with constants $A$ and $B$ still to be determined. And, from what has been said before, the desired form (1) now literally drops into our lap, provided only that values $A = i$ and $B = 0$ can somehow be assured.
Fixing integration constants $A$ and $B$

Constants $A$ and $B$ are found next by utilizing the dominant terms of $H_0^{(2)}(kx/w)$ as $w \to \infty$. Thus, from (2) and (7) it follows on the one hand that

$$L(k, w) \approx \frac{A}{w} + \frac{B}{w} \log \left( \frac{k}{2w} \right).$$

At the same time, from the first line of (2),

$$L(k, w) = \lim_{\beta \downarrow 0^+} \frac{1}{w} \int_0^\infty e^{-\beta x} \cos(x) H_0^{(2)}(kx/w) dx \approx \lim_{\beta \downarrow 0^+} \frac{1}{w} \int_0^\infty e^{-\beta x} \cos(x) \left[ 1 - \frac{2i}{\pi} \left\{ \log \left( \frac{k}{2w} \right) + \gamma \right\} - \frac{2i}{\pi} \log(x) \right] dx,$$

with $\gamma = 0.5772156649\ldots$ being the Euler-Mascheroni constant. Here

$$\int_0^\infty e^{-\beta x} \cos(x) dx = \frac{\beta}{1 + \beta^2}$$

and thus vanishes together with $\beta$, something which allows us to say at once that $B = 0$.

It remains hence to contend with

$$\int_0^\infty e^{-\beta x} \cos(x) \log(x) dx = \beta \int_0^\infty e^{-\beta x} \sin(x) \log(x) dx - \int_0^\infty e^{-\beta x} \sin(x) \frac{x}{x} dx,$$

its right-hand side gotten under a routine integration by parts. And, while the second integral on the right has of course the known limit $\pi/2$ when $\beta \downarrow 0^+$, a short excursus intrudes now in order to ascertain the corresponding limit for the first.

We begin by writing

$$\beta \int_0^\infty e^{-\beta x} \sin(x) \log(x) dx = \beta M(\beta, 1),$$

with function $M(\beta, \xi)$ defined as

$$M(\beta, \xi) = \int_0^\infty e^{-\beta x} \sin(\xi x) \log(x) dx.$$

It is evident by inspection that $M(\beta, \xi)$ satisfies the Laplace equation

$$M(\beta, \xi)_{\beta \beta} + M(\beta, \xi)_{\xi \xi} = 0.$$

On the other hand, at least in the first quadrant with $\beta > 0$, $\xi > 0$,

$$M(\beta, \xi) = \xi^{-1} M(\beta/\xi, 1) - \xi^{-1} \log(\xi) \int_0^\infty e^{-\beta x/\xi} \sin(x) dx$$

$$= \xi^{-1} M(\beta/\xi, 1) - \xi \log(\xi)/(\beta^2 + \xi^2).$$
By direct calculation\(^6\) Laplace’s equation (14) is then reduced to the following inhomogeneous ODE
\[
\left((1 + \eta^2)N(\eta)\right)_{\eta\eta} = (1 + \eta^2)^{-1} - 4(1 + \eta^2)^{-2}
\]  
for the function \(N(\eta) = M(\eta, 1)\). Integration gives
\[
N(\eta) = \frac{1}{1 + \eta^2} \left\{ C + D\eta - \frac{1}{2} \log(1 + \eta^2) - \eta \arctan(\eta) \right\}
\]  
with still other constants \(C\) and \(D\) whose precise values are not of any immediate interest.\(^7\) What is of interest is our newly acquired ability to state with some confidence that
\[
\lim_{\beta \downarrow 0^+} \beta M(\beta, 1) = \lim_{\beta \downarrow 0^+} \beta N(\beta) = 0.
\]  
And finally, on putting together the remnant of (9) with that of (11), we duly find
\[
L(k, w) \approx \frac{i}{w},
\]  
so that indeed \(A = i\) and we are done.

### 4 Green’s function byproduct

We would be highly remiss were we not to admit that the simplest, if perhaps not the most mathematically palatable access route to transform (1) originates with the PDE for the Green’s function of the two-dimensional wave equation, having a product of Dirac deltas as its source. The Green’s function \(G(x, y)\) per se is \(G(x, y) = i H_0^{(2)}(k \rho)/4\), and it satisfies\(^8\)
\[
\left(\nabla^2 + k^2\right) G(x, y) = \delta(x) \delta(y)
\]  
or else, following Fourier transformation
\[
\tilde{G}(w, y) = \int_{-\infty}^{\infty} e^{iwx} G(x, y) dx
\]  
the reduced equation
\[
\left[ \frac{\partial^2}{\partial y^2} + k^2 - w^2 \right] \tilde{G}(w, y) = \delta(y),
\]  
the solution of which latter is easily found to read
\[
\tilde{G}(w, y) = \frac{i}{2\sqrt{k^2 - w^2}} e^{-i\sqrt{k^2 - w^2}|y|}.
\]
Then
\[
\int_0^\infty H_0^{(2)}(kx) \cos(wx) \, dx = -2i \tilde{G}(w, 0)
\]
\[
= \frac{1}{\sqrt{k^2 - w^2}}
\]
and so we are done once again. At the same time this latest intrusion by Dirac deltas compels us to acknowledge that all three derivations of (1), neither Basset’s nor ours in any way excluded, skate, in one mode or another, upon a thin ice sheet of analytic delicacy.

5 Remarks

It goes without saying that use of the alternate time dependence, \( \exp(-i\omega t) \), \( \omega > 0 \), induces complex conjugation \( \text{en masse} \), \( H_0^{(2)}(kp) \to H_0^{(1)}(kp) \), \( \Im k \geq 0 \), \( |\Im w| < \Im k \) as an initial strip of analyticity, extendable thereafter to the whole plane when properly cut.

On a somewhat different tack, one must remember that the limit \( \beta \downarrow 0^+ \) cannot be recklessly enforced, beginning with evaluation (10). For example, joining (10) is the further elementary integral
\[
\int_0^\infty e^{-\beta x} \sin(x) \, dx = \frac{1}{1 + \beta^2}
\]
already implied in (15). But, in connection with that limit, there is of course no intent to set
\[
\int_0^\infty \sin(x) \, dx = 1
\]
\[
\int_0^\infty \cos(x) \, dx = 0,
\]
both proposed quadratures being clearly nonsensical. Meaningless likewise would be the tentative assignments
\[
\int_0^\infty \sin(x) \log(x) \, dx = C
\]
\[
\int_0^\infty \cos(x) \log(x) \, dx = -\pi/2.
\]

A nagging anxiety that retaining in (9) only the lowest-order close-in Hankel terms may not be adequate is alleviated somewhat by considering, and then at once dismissing yet another potential contribution proportional to \( \log(k/2w) \) [16], \( \text{viz.,} \)
\[
\Delta = -\frac{2i}{\pi w} \log \left( \frac{k}{2w} \right) \sum_{r=1}^\infty (-1)^r \left( \frac{k}{2w^r r!} \right)^2 \lim_{\beta \downarrow 0^+} \int_0^\infty e^{-\beta x} x^r \cos(x) \, dx.
\]
However, with a view to (10),
\[
\lim_{\beta \downarrow 0^+} \int_0^\infty e^{-\beta x} x^{2r} \cos(x) dx = \lim_{\beta \downarrow 0^+} \left( -\frac{d}{d\beta} \right)^{2r} \sum_{n=0}^\infty (-1)^n \beta^{2n+1} = 0,
\]
and therefore the asymptotic development of \( L(k, w) \) is entirely free of logarithmic dependence at every power of multiplier \( 1/w \). All other analytic fragments contributing to \( H_0^{(2)}(kx/w) \) in the first line of (9) can be treated in similar fashion, with similarly null outcomes.

And as for the constants \( C \) and \( D \) from (17), we can fix those via outright numerical quadrature of
\[
N(\eta) = \int_0^\infty e^{-\eta x} \sin(x) \log(x) dx
\]
at, say, \( \eta = \mu \) and \( \eta = \nu, \mu \neq \nu \). Thus
\[
\begin{pmatrix} 1 & \mu \\ 1 & \nu \end{pmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} (1 + \mu^2) N(\mu) + \frac{1}{2} \log(1 + \mu^2) + \mu \arctan(\mu) \\ (1 + \nu^2) N(\nu) + \frac{1}{2} \log(1 + \nu^2) + \nu \arctan(\nu) \end{pmatrix},
\]
the matrix on the left having \( \nu - \mu \neq 0 \) as determinant and hence being open to inversion as
\[
\begin{pmatrix} 1 & \mu \\ 1 & \nu \end{pmatrix}^{-1} = \frac{1}{\nu - \mu} \begin{pmatrix} \nu & -\mu \\ -1 & 1 \end{pmatrix}.
\]
While system (31) can thus be solved in the usual way, one naturally imposes the self-consistency demand that any other pair \( \mu' \neq \nu' \) yield an identical outcome, apart from any inaccuracy attributable to the numerical treatment of (30). But, even though it be a sine qua non desideratum, we are in no position to offer any sort of an \textit{ab initio} proof of its fulfillment.

6 Acknowledgement

An arXiv preprint of the material herein contained has previously been published [17].
Notes

1 A Hankel function of the second kind is paired with the adoption of a simple harmonic dependence upon time $t$ evolving in accordance with $\exp(i\omega t)$, $\omega = 2\pi/T > 0$, with $T$ being the period. It is likewise assumed that $\Re k > 0$, $\Im k \leq 0$, the latter intended to account for the physically welcome possibility of wave dissipation. In the absence thereof, $k = 2\pi/\lambda$, $\lambda = cT$ being the wavelength dictated by light speed $c$.

2 While our notation aims to be as close as possible to that of [5,6], we have dispensed with an extraneous factor of $\sqrt{2/\pi}$, and, with a view to its imminent, analytically more robust rôle, have included wave number $k$ as an explicit argument. Functional form $1/\sqrt{k^2 - w^2}$ in line two, crucial for the succeeding W-H developments, is gotten under analytic continuation from a formula due to Alfred Barnard Basset [14, p. 32], but stated only as an exercise no less, without any indication of an overt proof (!!), and subsequently quoted in [15, p. 388, Eq. (10)]. Details of the requisite analytic continuation are provided in [5, pp. 25-26]. The branch of $\sqrt{k^2 - w^2}$ is understood to give $k$ when $w = 0$.

3 Indeed, such extension can sweep beyond this strip so as to include the entire $w$ plane, suitably slit with branch cuts emanating from $w_{\pm} = \pm k$.

4 A subscript, such as $\zeta$, denotes a derivative with respect to the variable indicated.

5 Our abrupt, deus ex machina recourse, in the second line of (9), to the lowest order only of the close-in Hankel function behavior is triggered by the circumstance that all of its remaining terms are burdened by still higher powers of $1/w$ [16]. We indicate in the Remarks section at paper’s end how some of these terms can be disposed of.

6 In both (16) and (17) hand manipulations have been successfully collated against their MATHEMATICA counterparts.

7 Additional comments regarding $C$ and $D$, but short of any immediate evaluation, are likewise found in a concluding Remarks section.

8 We hasten to repeat that there is no claim whatsoever to originality at this point, the material being in its entirety a part of common knowledge among physicists.

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