STABILIZERS AND ORBITS OF SMOOTH FUNCTIONS

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Abstract. Let $f : \mathbb{R}^m \to \mathbb{R}$ be a smooth function such that $f(0) = 0$. We give a condition on $J(id)$ for arbitrary preserving orientation diffeomorphism $\phi : \mathbb{R} \to \mathbb{R}$ such that $\phi(0) = 0$ for the function $\phi \circ f$ is right equivalent to $f$, i.e. there exists a diffeomorphism $h : \mathbb{R}^m \to \mathbb{R}^m$ such that $\phi \circ f = f \circ h$ at $0 \in \mathbb{R}^m$. The requirement is that $f$ belongs to its Jacobi ideal. This property is rather general: it is invariant with respect to the stable equivalence of singularities, and holds for non-degenerated, simple, and many other singularities.

We also globalize this result as follows. Let $M$ be a smooth compact manifold, $f : M \to [0, 1]$ a surjective smooth function, $\mathcal{D}_M$ the group of diffeomorphisms of $M$, and $\mathcal{D}_R^{[0,1]}$ the group of diffeomorphisms of $\mathbb{R}$ that have compact support and leave $[0, 1]$ invariant. There are two natural right and left-right actions of $\mathcal{D}_M$ and $\mathcal{D}_M \times \mathcal{D}_R^{[0,1]}$ on $C^\infty(M, \mathbb{R})$. Let $\mathcal{S}_M(f)$, $\mathcal{S}_{\mathcal{M}\mathcal{R}}(f)$, $\mathcal{O}_M(f)$, and $\mathcal{O}_{\mathcal{M}\mathcal{R}}(f)$ be the corresponding stabilizers and orbits of $f$ with respect to these actions. We prove that if $f$ satisfies $J(id)$ at each critical point and has additional mild properties, then the following homotopy equivalences hold: $\mathcal{S}_M(f) \approx \mathcal{S}_{\mathcal{M}\mathcal{R}}(f)$ and $\mathcal{O}_M(f) \approx \mathcal{O}_{\mathcal{M}\mathcal{R}}(f)$. Similar results are obtained for smooth mappings $M \to S^1$.

Résumé. Soit $f : \mathbb{R}^m \to \mathbb{R}$ une application différentiable telle que $f(0) = 0$. On introduit une condition $J(id)$ et prouve que si $f$ lui satisfait, alors pour chaque difféomorphisme $\phi : \mathbb{R} \to \mathbb{R}$ tel que $\phi(0) = 0$ et $\phi$ préserve l’orientation de $\mathbb{R}$ il existe un difféomorphisme $h : \mathbb{R}^m \to \mathbb{R}^m$ tel que $\phi \circ f = f \circ h$ en $0 \in \mathbb{R}^m$. La condition $J(id)$ requiert que $f$ appartienne à son idéal Jacobien en $0$. Cette propriété est invariante par rapport à l’équivalence stable des singularités. Aussi les singularités non-degénérées et simples satisfont $J(id)$.

Ce résultat local implique le théorème global suivant. Soient $M$ une variété différentiable, $f : M \to [0, 1]$ une application différentiable surjective, $\mathcal{D}_M$ le groupe des difféomorphismes de $M$ et $\mathcal{D}_R^{[0,1]}$ le groupe des difféomorphismes de $\mathbb{R}$ de support compact et préserve $[0, 1]$. Il y a deux actions droite et gauche-droite de $\mathcal{D}_M$ et $\mathcal{D}_M \times \mathcal{D}_R^{[0,1]}$ respectivement sur $C^\infty(M, \mathbb{R})$. Soient $\mathcal{S}_M(f)$, $\mathcal{S}_{\mathcal{M}\mathcal{R}}(f)$, $\mathcal{O}_M(f)$ et $\mathcal{O}_{\mathcal{M}\mathcal{R}}(f)$ les stabilisateurs et orbites correspondantes de $f$ par rapport ces actions. On prouve que si $f$ satisfait $J(id)$ en chaque point critique et aussi une autre condition naturelle, alors on a les équivalences homotopiques suivantes: $\mathcal{S}_M(f) \approx \mathcal{S}_{\mathcal{M}\mathcal{R}}(f)$ et $\mathcal{O}_M(f) \approx \mathcal{O}_{\mathcal{M}\mathcal{R}}(f)$. Les résultats ressemblants sont obtenus pour les applications différentiables $M \to S^1$.

Keywords: singularities, diffeomorphisms, right and left-right action, flow, homotopy type.

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1. Introduction

Let $M$ be a smooth ($C^\infty$) connected compact $m$-dimensional manifold, $P$ either the real line $\mathbb{R}$ or the circle $S^1$, and $\mathcal{D}_M$ and $\mathcal{D}_P$ the groups of diffeomorphisms of $M$ and $P$ respectively. There are natural left actions of the groups $\mathcal{D}_M$ and $\mathcal{D}_M \times \mathcal{D}_P$ on $C^\infty(M, P)$ defined by the following formulas: if $f \in C^\infty(M, P)$, $h \in \mathcal{D}_M$, and $\phi \in \mathcal{D}_P$, then

\begin{align}
(1.1) & \quad h \cdot f = f \circ h^{-1} \\
(1.2) & \quad (h, \phi) \cdot f = \phi \circ f \circ h^{-1}.
\end{align}

These actions are often called right and left-right respectively. They were studied by many authors, see e.g. [1, 3] for references.

For $f \in C^\infty(M, P)$ let $\mathcal{S}_M(f) = \{ h \in \mathcal{D}_M \mid f = f \circ h \}$ be the stabilizer and $\mathcal{O}_M(f) = \{ f \circ h^{-1} \mid h \in \mathcal{D}_M \}$ the orbit of $f$ under the right action (1.1).
If $\mathcal{D}_p'$ is a subgroup of $\mathcal{D}_p$, then the left-right action of the group $\mathcal{D}_M \times \mathcal{D}_p'$ on $C^\infty(M, P)$ is well-defined by (1.2).

Let also $S_{MP}^r(f) = \{(h, \phi) \in \mathcal{D}_M \times \mathcal{D}_p' \mid \phi \circ f = f \circ h\}$ be the corresponding stabilizer and $C_{MP}^r(f) = \{(\phi \circ f \circ h^{-1}) \mid (h, \phi) \in \mathcal{D}_M \times \mathcal{D}_p'\}$ the orbit of $f$.

Evidently, $S_M(f) \equiv S_M(f) \times \text{id}_P \subset S_{MP}^r(f)$ and $O_M(f) \subset C_{MP}^r(f)$.

The aim of this paper is to show that for almost all mappings $f \in C^\infty(M, P)$ and some natural subgroups $\mathcal{D}_p' \subset \mathcal{D}_p$ we have the following homotopy equivalences (in the corresponding $C^\infty$-topologies): $S_M(f) \approx S_{MP}^r(f)$, $O_M(f) \approx O_{MP}^r(f)$ for $P = \mathbb{R}$, and $O_M(f) \times S^1 \approx O_{MP}^r(f)$ for $P = S^1$. In fact, we obtain precise relationships between the topological (not just homotopy) types of these spaces.

1.1. Conditions on $f$. In order to formulate our main results (Theorems 1.3 and 1.4) we introduce the following conditions (V) and (J). Say that $f \in C^\infty(M, P)$ satisfies the condition (V) if

(V) $f$ is constant at every connected component of $\partial M$ and has only finitely many critical values.

For each point $z \in M$ let $C^\infty_z(M)$ be the algebra of germs of smooth functions at $z$. If $f \in C^\infty_z(M)$, then the Jacobi ideal $\Delta(f, z)$ of $f$ at $z$ is the ideal in $C^\infty_z(M)$ generated by partial derivatives of $f$ at $z$. Evidently, it does not depend on a particular choice of local coordinates at $z$.

We will say that a mapping $f \in C^\infty(M, P)$ satisfies the condition (J) if the following holds true.

(J) Let $z$ be a critical point of $f$ and $f : \mathbb{R}^m \to \mathbb{R}$ a local representation of $f$ at $z$ such that $f(z) = 0$. Then the germ of a function $f$ at $z$ belongs to the Jacobi ideal $\Delta(f, z)$.

This means that there are smooth functions $H_1, \ldots, H_m$ such that

\begin{equation}
(1.3) \quad f(x) = \sum_{i=1}^m f_i'(x)H_i(x), \quad x = (x_1, \ldots, x_m) \in \mathbb{R}^m.
\end{equation}

Moreover, if we define a vector field $H$ near $z$ by $H = (H_1, \ldots, H_m)$, then (1.3) can also be written in the following form:

\begin{equation}
(1.4) \quad f(x) = H.f(x), \quad x \in \mathbb{R}^m
\end{equation}

where $H.f$ is a derivative of $f$ along $H$.

Notice, that conditions (V) and (J) are invariant with respect to left-right action.

1.1.1. Suppose that a function $f \in C^\infty(M, P)$ satisfies (V). Then the set of critical points of $f$ may be infinite. Moreover, there may be critical points on $\partial M$.

The values of $f$ on the connected components of $\partial M$ will be called boundary values. All critical and boundary values of $f$ will be called exceptional and the inverse images of these exceptional values under $f$ will be called exceptional level-sets. Since $M$ is assumed compact, it follows that the set of exceptional values is finite.

Let $n$ be the total number of exceptional values of $f$.

If $n = 0$, then $M$ is closed, $P = S^1$, and $f : M \to S^1$ is a locally trivial fibration.

Otherwise, $n \geq 1$. If $P = S^1$, then we shall always regard $S^1$ as the group of reals modulo $n$:

\[ S^1 \equiv \mathbb{R}/n\mathbb{Z}, \]

not $\mathbb{R}/\mathbb{Z}$ as usual! Therefore in both cases of $P$ we can assume that $1, \ldots, n$ are all of the exceptional values of $f$. But for $P = S^1$, they are taken modulo $n$, and in particular we have that $n \equiv 0$.

Let us define the following groups. If $f \in C^\infty(M, \mathbb{R})$, then let
for every diffeomorphism \( \phi \) form \( \Theta \):

- \( D_R^{[1,n]} \) be the subgroup of \( D_R \) consisting of diffeomorphisms that preserve orientation of \( \mathbb{R} \), have compact support, and leave the image \([1,n]\) of \( f \) invariant;
- \( D_R \) be the subgroup of \( D_R^{[1,n]} \) consisting of diffeomorphisms that also fix every exceptional value \( 1, \ldots, n \) of \( f \);
- \( D_{MR} = D_M \times D_R^{[1,n]} \).

If \( f \in C^\infty(M,S^1) \), then let
- \( D_{s1}^+ \) be the group of preserving orientation diffeomorphisms of \( S^1 \);
- \( D_S^+ \) be the subgroup of \( D_{s1}^+ \) preserving the set \( \{1, \ldots, n\} \) of exceptional values of \( f \). If \( n = 0 \), then \( D_S^+ = D_{s1}^+ \);
- \( D_{s1}^- \) be the (normal) subgroup of \( D_S^+ \) fixing the set \( \{1, \ldots, n\} \) point-wise, thus \( D_{s1}^-/D_{s1}^+ \) is a cyclic group \( \mathbb{Z}_n \) of order \( n \);
- \( D_{MS1} = D_M \times D_{s1}^+ \).

Then \( D_M \) and \( D_{MP} \) act on \( C^\infty(M,P) \) by formulas (1.1) and (1.2). Let \( S_M(f), S_{MP}(f), O_M(f) \), and \( O_{MP}(f) \) be respectively the stabilizers and the orbits of \( f \) under these actions. Evidently,

\[
S_M(f) \times \text{id}_P \subset S_{MP}(f) \quad \text{and} \quad O_M(f) \subset O_{MP}(f).
\]

Finally, we endow the spaces \( D_M, D_P, \) and \( C^\infty(M,P) \) with the corresponding \( C^\infty \) Whitney topologies. These topologies yield certain topologies on \( D_{MP} \) and on the corresponding stabilizers and orbits of \( f \).

### 1.2. Stabilizers

Let us say that a diffeomorphism \( \phi \in D_P \) is left-trivial or \( L \)-trivial for \( f \) if there exists a diffeomorphism \( h \in D_M \) such that \( (h, \phi) \in S_{MP}(f) \), i.e. \( \phi \circ f = f \circ h \). Thus applying \( \phi \) to \( f \) (acting from the left) we remain in the right orbit \( O_M(f) \) of \( f \). This explains the term “left-trivial”.

Let \( p : D_M \times D_P \to D_P \) be the standard projection. Evidently, \( p \) is a homomorphism. Consider the restriction of \( p \) to \( S_{MP}(f) \), then its kernel coincides with \( S_M(f) \), and the image \( p(S_{MP}(f)) \subset D_P \) consists of all \( L \)-trivial for \( f \) diffeomorphisms.

**Theorem 1.3.** Suppose that \( f \in C^\infty(M,P) \) satisfies (V) and (J). Then

1. Case \( P = \mathbb{R} \):
   \[
p(S_{MR}(f)) = D_R^n.
   \]
2. Case \( P = S^1, n \geq 1 \):
   \[
   D_{s1}^+ \subseteq p(S_{MS1}(f)) \subseteq D_S^+.
   \]
3. Case \( P = S^1, n = 0 \):
   \[
p(S_{MS1}(f)) = D_{s1}^-.
   \]

In the cases (1) and (2) \( p \) admits a continuous section \( \Theta : D_P^+ \to S_{MP}(f) \) i.e. \( p \circ \Theta = \text{id}(D_P^+) \). Moreover, this section is a homomorphism.

In the case (3) \( p \) admits a continuous section \( \Theta : D_{s1}^+ \to S_{MS1}(f) \) iff the fibration \( f : M \to S^1 \) trivial. Such a section can also be chosen to be a homomorphism.

**Remark 1.3.1.** The section of the projection \( p \) in Theorem 1.3 must be of the form \( \Theta(\phi) = \theta(\phi, \phi) \), where \( \theta : D_P^+ \to D_M \) is a continuous mapping such that for every diffeomorphism \( \phi \in D_P^+ \) (fixing each exceptional value of \( f \)) we have \( \phi \circ f = f \circ \theta(\phi) \in O_M(f) \). Notice also that \( \Theta \) is a homomorphism iff \( \theta \) is.

**Remark 1.3.2.** In the case (2) denote \( \tilde{S}_{MS1}(f) = p^{-1}(D_{s1}^+) \). Since \( D_{s1}^+/D_{s1}^- \approx \mathbb{Z}_n \), it follows from Theorem 1.3 that

\[
S_{MS1}(f) / \tilde{S}_{MS1}(f) \approx p(S_{MS1}(f)) / D_{s1}^+ \approx \mathbb{Z}_c,
\]

for some \( c \) that divides \( n \).
Theorem 1.3 (Another formulation). The following sequences of group homomorphisms are exact

(1) Case $P = \mathbb{R}$:

$$1 \to S_M(f) \to S_{MB}(f) \overset{\varphi}{\to} D^e_R \to 1,$$

(2) Case $P = S^1$, $n \geq 1$:

$$1 \to S_M(f) \to \tilde{S}_{MS^1}(f) \overset{\varphi}{\to} D^e_{S^1} \to 1,$$

(3) Case $P = S^1$, $n = 0$:

$$1 \to S_M(f) \to S_{MS^1}(f) \overset{\varphi}{\to} D^e_{S^1} \to 1.$$  

They always split in the cases (1) and (2), and in the case (3) if $f : M \to S^1$ is a trivial fibration.

Notice that the existence of splittings in the cases (1) and (2) is rather natural, since $p$ is a principal $S_M(f)$-fibration and $D^e_R$ and $D^e_{S^1}$ for $n \geq 1$ are contractible, see Lemma 6.6.1.

Thus in the case (1) we obtain a homeomorphism $S_{MB}(f) \cong S_M(f) \times D^e_R$, whence the embedding $S_M(f) \subset S_{MB}(f)$ is a homotopy equivalence.

In the case (2) we have that $\tilde{S}_{MS^1}(f) \cong S_M(f) \times D^e_{S^1}$. Since $S_{MS^1}(f)/\tilde{S}_{MS^1}(f)$ is a cyclic group $\mathbb{Z}_c$, we get $S_{MS^1}(f) \cong S_M(f) \times D^e_{S^1} \times \mathbb{Z}_c$. Whence $S_{MS^1}(f)$ is homotopy equivalent to $S_M(f) \times \mathbb{Z}_c$.

Moreover, suppose that $\mathbb{Z}_c$ is non-trivial, i.e. $c > 1$. Then there exists $(h, \phi) \in S_{MS^1}(f)$, i.e., $\phi \circ f = f \circ h$, such that $\phi$ cyclically shifts exceptional values $1, \ldots, n$ of $f$ and $h$ cyclically shifts the corresponding exceptional level-sets of $f$:

$$\phi(k) = k + n/c, \quad h(L_k) = L_{k+n/c}, \quad (k = 1, \ldots, n),$$

where $L_k = f^{-1}(k)$ and each sum is taken modulo $n$. Thus the level-sets

$$L_k, \ L_{k+n/c}, \ldots, L_{k+n(c-1)/c}$$

are homeomorphic each with other. This situation is not typical, though it could be stable (e.g. for generic Morse function). Thus for most functions we should have $D^e_{S^1} = p(S_{MS^1}(f))$. In this case $S_{MS^1}(f)$ is homeomorphic with $S_M(f) \times D^e_{S^1}$ and the embedding $S_M(f) \subset S_{MS^1}(f)$ is a homotopy equivalence.

1.3.3. Interpretation: holonomy. Let $f : M \to B$ be a finite-dimensional vector bundle over a smooth manifold $B$. For each $b \in B$ let $M_b = f^{-1}(b)$ be the corresponding fiber. Choose some connection on $M$. Then for each smooth path $\omega : I \to B$ there exists a smooth isotopy (consisting even of linear isomorphisms) $h_t : M_{\omega(0)} \to M_{\omega(t)} \subset M$, called holonomy along $\omega$. It follows that, if $\phi_0 : B \to B$ is an isotopy and $\phi_0 = \text{id}_B$, then there is an isotopy $h_t : M \to M$ such that $h_0 = \text{id}_M$ and the following diagram is commutative:

$$
\begin{array}{ccc}
M & \overset{f}{\longrightarrow} & B \\
\downarrow h_t & & \downarrow \phi_t \\
M & \overset{f}{\longrightarrow} & B
\end{array}
$$

i.e., $\phi_t \circ f = f \circ h_t$.

In other words, $(h_t, \phi_t)$ belongs to the stabilizer of $f$ under the left-right action of the group $\mathcal{D}_M \times \mathcal{D}_B$ on $C^\infty(M, B)$. Moreover, if $G \subset \mathcal{D}_B$ is a simply-connected subgroup, then we obtain a continuous homomorphism $\theta : G \to \mathcal{D}_M$ such that $\phi \circ f = f \circ \theta(\phi)$ for all $\phi \in G$.

Notice that the vector-bundle projection $f$ is a smooth mapping without critical points. If we replace $f$ with an arbitrary smooth mapping between arbitrary smooth manifolds, and try to construct such a “holonomy homomorphism”, then we should put some restrictions on $f$ and $G$. In particular, $G$ must preserve the image of $f$ and its “exceptional” values. Theorem 1.3 gives rather general conditions when such a homomorphism exists for the case $\dim B = 1$.  

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1.4. Orbits. We will now describe the relationships between the orbits.

**Definition 1.4.1.** A critical point \( z \) of \( f \in C^\infty(M, P) \) is essential, if for every neighborhood \( U \) of \( z \) there is a neighborhood \( \mathcal{U} \) of \( f \) in \( C^\infty(M, P) \) with \( C^\infty \)-topology such that each \( g \in \mathcal{U} \) has a critical point in \( U \).

**Example 1.4.2.** Let \( f(x) = x^2 \) and \( g(x) = x^3 \). Then 0 \( \in \mathbb{R} \) is an essential critical point for \( f \) but not for \( g \).

**Theorem 1.5.** Suppose that \( f \) satisfies the conditions (V) and (J) and each critical level-set of \( f \) includes either an essential critical point or a connected component of \( \partial M \).

If \( P = \mathbb{R} \), then the embedding \( \mathcal{O}_M(f) \subset \mathcal{O}_{MR}(f) \) extends to a homeomorphism

\[
\mathcal{O}_M(f) \times \mathbb{R}^{n-2} \approx \mathcal{O}_{MR}(f).
\]

Let \( P = S^1 \) and \( c \) be the index of \( \mathcal{D}_{S^1} \) in \( p(\mathcal{S}_{MS^1}(f)) \), as in Theorem 1.3.

a) If \( n = 0 \), then \( \mathcal{O}_M(f) = \mathcal{O}_{MS^1}(f) \).

b) If \( n \) is even and \( n/c \) is odd, then

\[
\mathcal{O}_M(f) \times S^1 \times \mathbb{R}^{n-1} \approx \mathcal{O}_{MS^1}(f),
\]

where \( S^1 \times \mathbb{R}^{n-1} \) is the total space of a unique non-trivial \((n-1)\)-dimensional fibration over \( S^1 \).

c) Otherwise,

\[
\mathcal{O}_M(f) \times S^1 \times \mathbb{R}^{n-1} \approx \mathcal{O}_{MS^1}(f).
\]

1.6. Structure of the paper. In Section 2 for each germs of “admissible” smooth functions \( \alpha : \mathbb{R} \to \mathbb{R} \) at \( 0 \in \mathbb{R} \) (see Definition 2.0.1) we introduce and study a certain group \( L(\alpha) \) of germs of diffeomorphisms \( \mathbb{R} \to \mathbb{R} \) at 0.

In Section 3 we consider the local left-right action of the groups of germs of diffeomorphisms of \( \mathbb{R}^m \) and \( \mathbb{R} \) on the germs of smooth functions \( f : \mathbb{R}^m \to \mathbb{R} \) such that \( f(0) = 0 \). We give a sufficient condition \( J(\alpha) \) on \( f \) when \( L(\alpha) \) consists of L-trivial for \( f \) diffeomorphisms (Theorem 3.2). The most complete result (Theorem 3.3) which is also a local variant of Theorem 1.3 is obtained for \( f \) satisfying condition \( J(\text{id}) \) being a local analogue of \( J \).

Section 4 We show that the condition \( J \) holds for a very large class of singularities and is invariant with respect to a stable equivalence of singularities (Lemma 4.0.5). On the other hand, there are singularities that do not satisfy this condition (Claim 4.0.5).

In Section 5 we prove Theorem 1.3.

Section 6 The finite-dimensional spaces of adjacent classes of \( \mathcal{D}_{\mathbb{R}}^{[1,n]} / \mathcal{D}_{\mathbb{R}}^* \) and \( \mathcal{D}_{S^1}^* / \mathcal{D}_{S^1}^* \) (Theorem 6.1) are described.

Section 7 We give a sufficient condition for the continuity of the mapping \( k \) corresponding to each \( g \in \mathcal{O}_{MP}(f) \) the ordered set of exceptional values of \( g \) (Lemma 7.0.1). We also show that without this condition \( k \) may loose continuity.

Finally in Section 8 we prove Theorem 1.5.

2. Groups \( L(\alpha) \)

Let \( C^\infty_0(\mathbb{R}) \) be the algebra of germs of smooth functions at \( 0 \in \mathbb{R} \). For each \( \mu \in C^\infty_0(\mathbb{R}) \) we will denote by \( I_\mu \) the ideal \( \mu \cdot C^\infty_0(\mathbb{R}) \) in \( C^\infty_0(\mathbb{R}) \).

**Definition 2.0.1.** We will say that a function \( \alpha \in C^\infty_0(\mathbb{R}) \) is admissible, provided

1. \( \alpha(0) = 0 \), \( \alpha'(0) \) is equal either 0 or 1, and
2. there is a neighborhood \( U \) of \( 0 \in \mathbb{R} \) such that the intersection \( \alpha^{-1}(0) \cap U \) is nowhere dense in \( U \). thus \( \alpha \) in not constant on open intervals near 0.
For each admissible $\alpha \in C_0^\infty(\mathbb{R})$ we will now define a certain group $L(\alpha)$ of germs of diffeomorphisms of $\mathbb{R}$ at 0. They are analogous to the groups $G_d$ of [2 §5] of quasi-homogeneous diffeomorphisms $C^\infty \to C^\infty$ of order $d \geq 0$. Our situation is simpler since we consider diffeomorphisms of $\mathbb{R}$, on the other hand dimension 1 allows to prove more. The proximity to the identity $id_\mathbb{R}$ for diffeomorphisms of $L(\alpha)$ is defined not just up to order $d$ but up to the admissible function $\alpha$ which can be flat. Moreover, our approach to $L(\alpha)$ differs from [2 §5]. We study this group using its characterization as the set of smooth shifts along trajectories of the vector field $\alpha(s) \frac{d}{ds}$ on $\mathbb{R}$, see Theorem 2.3. The formulas for shift-functions will play a key role in the proof of Theorem 1.3 and its local variant Theorem 8.2.

2.1. Definition of $L(\alpha)$. Let $\alpha \in C_0^\infty(\mathbb{R})$ be an admissible function. Then we define $L(\alpha)$ to be the the subset of $D_0(\mathbb{R})$ consisting of preserving orientation diffeomorphisms $\phi$ of the following form:

$$\phi(s) = s + \alpha(s)\beta_\phi(s), \quad \beta_\phi \in C_0^\infty(\mathbb{R}).$$

In other words, $\phi = id_\mathbb{R} \in I_\alpha$. Consider two cases.

a) Suppose that $\alpha(0) = 0$ and $\alpha'(0) = 1$, i.e. $\alpha(s) = s\bar{\alpha}(s)$, where $\bar{\alpha} \in C_0^\infty(\mathbb{R})$ and $\bar{\alpha}(0) = \alpha'(0) = 1$. In this case $L(\alpha) = D_0(\mathbb{R})$. Indeed, if $\phi \in D_0(\mathbb{R})$, then $\phi(0) = 0$, whence $\phi(s) - s = s\omega(s) = \alpha(s)\frac{d\alpha}{ds}(s)$, where $\omega \in C_0^\infty(\mathbb{R})$. Thus $\phi \in L(\alpha)$.

b) Otherwise, $\alpha(0) = \alpha'(0) = 0$, i.e. $\alpha(s) = s^k\bar{\alpha}(s)$ for some $\bar{\alpha} \in C_0^\infty(\mathbb{R})$. Then each $\phi \in L(\alpha)$ has the following form $\phi(s) = s + s^k\bar{\alpha}(s)\beta_\phi(s)$. In particular, $\phi'(0) = 1$.

Notice that $L(s^k)$ consists of diffeomorphisms of the form $\phi(s) = s + s^k\beta_\phi(s)$. Equivalently, $\phi \in L(s^k)$ iff $\phi'(0) = 1$, and $\phi^{(p)}(0) = 0$ for $p = 2, 3, \ldots, k - 1$.

**Remark 2.1.1.** $L(\alpha\gamma) \subseteq L(\alpha)$ for each $\gamma \in C_0^\infty(\mathbb{R})$, and $L(\alpha\gamma) = L(\alpha)$ iff $\gamma(0) \neq 0$.

In particular, $L(\alpha) \subset L(id)$ for all admissible $\alpha$.

**Remark 2.1.2.** Condition (2) of Definition 2.0.1 implies that $\beta_\phi$ is uniquely determined near 0 by $\phi$. Indeed, if this condition is violated, then there is a sequence of mutually disjoint closed intervals $A_k$ converging to 0 and such that $|\alpha|_{A_k} \equiv 0$. Then varying $\beta_\phi$ on $A_k$ we do not change $\phi$.

2.2. Another description of $L(\alpha)$. We show that $L(\alpha)$ coincides with the set of smooth shifts along trajectories of the vector field $\alpha(s) \frac{d}{ds}$ on $\mathbb{R}$.

Let $\alpha : V \to \mathbb{R}$ be an admissible smooth function. Define a vector field $F$ on $\mathbb{R}$ by $F(s) = \alpha(s) \frac{d}{ds}$. Let also $F : V \times I \to \mathbb{R}$ be the local flow generated by $F$, where $V$ is a neighborhood of $0 \in \mathbb{R}$ and $I$ an open interval containing $0 \in \mathbb{R}$.

**Theorem 2.3.** Let $\phi \in C_0^\infty(\mathbb{R})$. Then $\phi \in L(\alpha)$ iff $\phi$ is a smooth shift along trajectories of $F$, i.e. $\phi(s) = F(s, \sigma_\phi(s))$ for some $\sigma_\phi \in C_0^\infty(\mathbb{R})$.

Moreover $\beta_\phi = \sigma_\phi \omega$, where $\omega \in C_0^\infty(\mathbb{R})$ and $\omega(0) = 1$. Hence $\beta_\phi(0) = \sigma_\phi(0)$.

We will call $\sigma_\phi$ a *shift-function* of $\phi$ with respect to $F$. Before proving this theorem let us deduce some corollaries.

**Lemma 2.3.1.** (c.f. [2 Prop. 5.2]). $L(\alpha)$ is a group.

**Proof.** Let $\phi, \psi \in L(\alpha)$. By Theorem 2.3 $\phi(s) = F(s, \sigma_\phi(s))$ and $\psi(s) = F(s, \sigma_\psi(s))$ for some $\sigma_\psi, \sigma_\phi \in C_0^\infty(\mathbb{R})$. It is easy to see that by Proposition 3] that

$$\psi \circ \phi(s) = F(s, \sigma_\psi \sigma_\phi(s)) = \psi^{-1}(s) = F(s, -\sigma_\phi \circ \psi^{-1}(s)).$$

Hence, again by Theorem 2.3 $\psi \circ \phi, \psi^{-1} \in L(\alpha)$. Thus $L(\alpha)$ is a group. 

Lemma 2.3.2. Let \( \phi \in L(\alpha) \) and \( \mu \in C^\infty_0(\mathbb{R}) \) be such that \( \alpha \mu \in C^\infty_0(\mathbb{R}) \) is admissible. Then the following conditions are equivalent:

1. \( \phi \in L(\alpha \mu) \)
2. \( \beta_\phi \in I_\mu \)
3. \( \sigma_\phi \in I_\mu \).

Thus the group \( L(\alpha \mu) \) consists of all smooth shifts along trajectories of \( \mathcal{F} \) whose shift-functions are proportional to \( \mu \), i.e. belong to the ideal \( I_\mu \).

Proof. The equivalence (2)⇔(3) holds by Theorem 2.3 since \( I_{\beta_\phi} = I_{\sigma_\phi} \).

(1)⇔(2) Notice that \( \phi \in L(\alpha \mu) \) means that \( \phi(s) = s + \alpha(s)\mu(s)\omega(s) \) for some \( \omega \in C^\infty_0(\mathbb{R}) \). By (2.1) and Remark 2.1.2 this condition is equivalent to the following \( \beta_\phi = \mu \omega \in I_\mu \).

Lemma 2.3.3. Let \( \psi, \phi \in L(\alpha) \) and \( \xi = \psi \circ \phi \circ \psi^{-1} \). Then \( \sigma_\xi = \sigma_\phi \circ \psi^{-1} \cdot \nu \), where \( \nu \in C^\infty_0(\mathbb{R}) \) and \( \nu(0) = 1 \). Hence \( \xi \in L(\alpha \mu) \) iff \( \sigma_\phi \circ \psi^{-1} \in I_\mu \).

Proof. From (2.2) we get

\[ \xi(s) = \psi \circ \phi \circ \psi^{-1}(s) = \mathcal{F}(s, -\sigma_\phi \circ \psi^{-1}(s) + \sigma_\phi \circ \psi^{-1}(s) + \sigma_\phi \circ \phi \circ \psi^{-1}(s)) \].

Thus \( \sigma_\xi = (\sigma_\phi + \sigma_\phi \circ \phi - \sigma_\phi) \circ \psi^{-1} \).

Then it follows from Hadamard lemma and Theorem 2.3 that

\[ \sigma_\phi \circ \phi(s) - \sigma_\phi(s) = (\phi(s) - s) \sigma_\psi = \alpha \beta_\phi \sigma_\phi = \alpha \sigma_\phi \omega \sigma_\psi \]

for some \( \sigma_\psi \in C^\infty_0(\mathbb{R}) \). Hence \( \sigma_\xi = (\sigma_\phi \cdot (1 + \alpha \omega \sigma_\psi)) \circ \psi^{-1} = \sigma_\phi \circ \psi^{-1} \cdot \nu \), where \( \nu = (\text{id}_\mathbb{R} + \alpha \omega \sigma_\psi) \circ \psi^{-1} \) and \( \nu(0) = 1 \).

Corollary 2.3.4. \( L(\alpha \mu) \) is a normal subgroup of \( L(\alpha) \) iff the ideal \( I_\mu \subset C^\infty_0(\mathbb{R}) \) is invariant with respect to the action of \( L(\alpha) \) on \( C^\infty_0(\mathbb{R}) \) by

\[ \psi \cdot \sigma = \sigma \circ \psi^{-1} \], \quad \psi \in L(\alpha), \quad \sigma \in C^\infty_0(\mathbb{R}). \]

Corollary 2.3.5. (c.f. [2, Prop. 5.3]). For each \( k \geq 1 \) the group \( L(s^k \alpha) \) is normal in \( L(\alpha) \).

Proof. It suffices to show that \( \sigma \circ \phi \in I_{s^k} \) for every \( \sigma \in I_{s^k} \) and \( \phi \in L(\text{id}) \). Indeed, we have that \( \sigma(s) = s^k \sigma \) and \( \phi(s) = s \omega(s) \) for some \( \sigma, \omega \in C^\infty_0(\mathbb{R}) \). Then

\[ \sigma \circ \phi(s) = \phi(s^k \sigma(\phi(s))) = s^k \omega(s^k \sigma(\phi(s))) \in I_{s^k}. \]

Define the following mapping \( \tau_\alpha : L(\alpha) \to \mathbb{R} \) by \( \tau_\alpha(\phi) = \beta_\phi(0) = \sigma_\phi(0) \). Then it follows from (2.2) that \( \tau_\alpha \) is a surjective homomorphism whose kernel is \( L(\alpha) \).

Thus for every admissible \( \alpha \) we obtain a sequence of normal subgroups

\[ L(\alpha) \supset L(s\alpha) \supset L(s^2 \alpha) \supset \cdots \]

such that each factor is isomorphic to \( \mathbb{R} \). Hence for each \( k \) the group \( L(\alpha)/L(s^k \alpha) \) is a Lie group diffeomorphic with \( \mathbb{R}^k \) (c.f. [2, Prop. 5.5]).

2.4. Proof of Theorem 2.3. It suffices to establish the following proposition that describes formulas for \( \mathcal{F} \).

Proposition 2.4.1. There is a smooth function \( \gamma \) on \( V \times I \) such that

\[ \mathcal{F}(s, t) = s + t\alpha(s)\gamma(s, t), \]

and \( \gamma(0, t) \equiv 1 \). In particular, \( \mathcal{F}_t \in L(\alpha) \) for every \( t \in I \).

First we deduce Theorem 2.3.

Sufficiency. It follows from (2.3) that if

\[ \phi(s) = \mathcal{F}(s, \sigma_\phi(s)) = s + \alpha(s)\sigma_\phi(s) \gamma(s, \sigma_\phi(s)), \]

then \( \phi \in L(\alpha) \) with \( \beta_\phi(s) = \sigma_\phi(s) \gamma(s, \sigma_\phi(s)) \) and \( \omega(s) = \gamma(s, \sigma_\phi(s)) \), where \( \omega(0) = \gamma(0, \sigma_\phi(0)) = 1 \).
Necessity. Suppose that $\phi \in L(\alpha)$. In order to show that $\phi(s) = F(s, \sigma_0(s))$ we have to show that $\beta_0(s) = \sigma_0(s)\gamma(s, \sigma_0(s))$ for some $\sigma_0 \in C_0^\infty(\mathbb{R})$.

Consider the function $\delta(s, t) = t \gamma(s, t)$. Notice that $\delta_t^2(s, 0) = \gamma(s, 0) = 1$, whence there exists a smooth function $q(s, t)$ such that $t = \delta(s, q(s, t))$. Therefore, we put $\sigma_0(s) = q(s, \beta_0(s))$. Then $\beta_0(s) = \delta(s, \sigma_0(s)) = \sigma_0(s) \cdot \gamma(s, \sigma_0(s))$. \hfill \Box

2.5. Proof of Proposition 2.4.1. For simplicity we will sometimes omit the dependence on $s$ and $(s, t)$. Recall that $F(s, 0) = \alpha(s)$, whence the Taylor expansion of $F(s, t)$ in $t$ at $(s, 0)$ has the following form:

\begin{equation}
F(s, t) = F(s, 0) + t F_1(s, 0) + \cdots = s + t \alpha(s) + \cdots
\end{equation}

Then

\[ \gamma(s, t) = \frac{F(s, t) - s}{t \alpha(s)}. \]

This function is defined only for those $(s, t)$ for which $t \alpha(s) \neq 0$. Nevertheless, since $F(s, 0) = s$, it follows that $(F(s, t) - s)/t$ is smooth. Moreover, for $t \neq 0$ we have that $\alpha(s) = 0$ iff $F(s, t) = s$.

Lemma 2.5.1. $\gamma(s, t)$ satisfies the following differential equation:

\begin{equation}
\gamma'(s, t) = \gamma^2(s, t) \cdot t \cdot \mu(s, t),
\end{equation}

where $\mu$ is a certain smooth function on $U \times I$. Whence

\[ \gamma(s, t) = \frac{1}{c(t) - t \int_0^1 \mu(z, t) dz}, \]

where $c(t)$ is a smooth function such that $c(0) = 1$. Thus $\gamma$ is smooth on $U \times (-\varepsilon, \varepsilon)$ for sufficiently small $\varepsilon > 0$ and $\gamma(0, t) = \frac{1}{c(t)} = 1$.

Proof. A simple calculation shows that if $\alpha(s) \neq 0$ then

\begin{equation}
\gamma'(s, t) = \frac{\alpha \cdot F' - \alpha - (F - s) \alpha'}{t \alpha^2}.
\end{equation}

Claim 2.5.2. The first term of the numerator in (2.6) is equal to $\alpha(s) \cdot F'(s, t) = \alpha \circ F(s, t)$.

Proof. Notice that $F$ defines the following differential equation on $\mathbb{R}$: $\frac{dF}{ds} = \alpha(s)$, whence $dt = \frac{ds}{\alpha(s)}$. Then for every $s \in V_1$ the time $t$ along the trajectory of $F$ between $s$ and $F(s, t)$ is equal to $t = \int_0^s \frac{dt}{\alpha(t)}$, (notice that if $\alpha(s) \neq 0$, then $\alpha \neq 0$ between $s$ and $F(s, t)$).

Differentiating both sides of this equality in $s$ we get

\[ 0 = \frac{F'}{\alpha(F)} \frac{1}{\alpha} = \frac{\alpha \cdot F' - \alpha \cdot F}{\alpha \cdot \alpha(F)}, \]

whence $\alpha(s) \cdot F'(s, t) = \alpha \circ F(s, t)$. \hfill \Box

On the other hand,

\[ \alpha \circ F(s, t) = \alpha + (F - s) \alpha_s' + (F - s)^2 \mu(s, t), \]

where $\mu(s, t)$ is a certain smooth function.

Hence (2.5) can be rewritten in the following form:

\[ \gamma' = \frac{\alpha \circ F - \alpha - (F - s) \alpha'}{t \alpha^2} = \frac{(F - s)^2}{t^2 \alpha^2} t \mu(s, t) = \gamma^2 t \mu(s, t), \]
which gives us (2.5). Notice that \( \gamma(s,0) = \frac{1}{c(0)} \), whence \( \gamma(s,t) = \frac{1}{c(0)} + t\beta(s,t) \) for some smooth function \( \beta(s,t) \). Therefore

\[
\mathcal{F}(s,t) = s + \frac{t\alpha(s)}{c(0)} + t^2\alpha(s)\beta(s,t).
\]

Comparing this with (2.4) we obtain that \( c(0) = 1 \).

Lemma 2.5.1 and Proposition 2.4.1 are proved. \( \square \)

3. Stabilizers of local left-right action

Let \( C_0^\infty(\mathbb{R}^m) \) be the algebra of germs of smooth functions \( \mathbb{R}^m \to \mathbb{R} \) at \( 0 \in \mathbb{R}^m \) and \( \mathfrak{m}(\mathbb{R}^m) \) a unique maximal ideal of \( C_0^\infty(\mathbb{R}^m) \) consisting of functions \( f \) such that \( f(0) = 0 \). Let also \( \mathcal{D}_0(\mathbb{R}^m) \) be the groups of germs of preserving orientation diffeomorphisms \( h : \mathbb{R}^m \to \mathbb{R}^m \) at \( 0 \in \mathbb{R}^m \) such that \( h(0) = 0 \).

Designate respectively by \( C_0^\infty(\mathbb{R}) \), \( \mathfrak{m}(\mathbb{R}) \), and \( \mathcal{D}_0(\mathbb{R}) \) the analogous objects for \( \mathbb{R} \). Then the groups \( \mathcal{D}_0(\mathbb{R}^m) \) and \( \mathcal{D}_0(\mathbb{R}^m) \times \mathcal{D}_0(\mathbb{R}) \) acts on \( \mathfrak{m}(\mathbb{R}^m) \) by formulas (1.1) and (1.2) respectively. We will call these actions \textit{local right} and \textit{local left-right} respectively.

We also say that \( f, g \in C_0^\infty(\mathbb{R}^m) \) are \textit{left-right} (right) equivalent iff they belong to the same orbit with respect to a local left-right (right) action.

For \( f \in \mathfrak{m}(\mathbb{R}^m) \) let

\[
\mathcal{S}_{\mathbb{R}^m, \mathbb{R}}(f) = \{(h, \phi) \in \mathcal{D}_0(\mathbb{R}^m) \times \mathcal{D}_0(\mathbb{R}) \mid \phi \circ f = f \circ h\}
\]

be the stabilizer of \( f \) with respect to this action.

3.1. Property \( J(\alpha) \). Notice that each \( f \in \mathfrak{m}(\mathbb{R}^m) \) yields a homomorphism of algebras:

\[
f^* : C_0^\infty(\mathbb{R}) \to C_0^\infty(\mathbb{R}^m), \quad f^*(\alpha) = \alpha \circ f.
\]

Denote by \( \Delta(f,0) \subset C_0^\infty(\mathbb{R}^m) \) the Jacobi ideal of \( f \) at \( 0 \in \mathbb{R}^m \), i.e. the ideal generated by partial derivatives \( f'_1, \ldots, f'_{m} \) of \( f \).

**Definition 3.1.1.** Let \( \alpha \in \mathfrak{m}(\mathbb{R}) \). We say that \( f \in \mathfrak{m}(\mathbb{R}^m) \) has property \( J(\alpha) \) at \( 0 \in \mathbb{R}^m \) if

\[
f^*(\alpha) = \alpha \circ f \in \Delta(f,0).
\]

Equivalently, there exists a vector field \( H \) at \( 0 \in \mathbb{R}^m \) such that

\[
H \cdot f = \alpha \circ f : \mathbb{R}^m \to \mathbb{R}^m.
\]

For instance, if \( \alpha = \text{id}_\mathbb{R} \), then \( J(\text{id}) \) means that \( f \in \Delta(f,0) \), which is precisely the condition (J). More generally, if \( \alpha(s) = s^k \), then \( J(s^k) \) means that \( f^k \in \Delta(f,0) \).

Notice also that for every \( \beta \in C_0^\infty(\mathbb{R}) \) the condition \( J(\alpha) \) implies \( J(\alpha\beta) \), i.e. if \( \alpha \circ f = H \cdot f \), then \( (\alpha\beta) \circ f = (\alpha \circ f)(\beta \circ f) = (\beta \circ f \cdot H) \circ f \in \Delta(f,0) \). Hence, if \( \beta(0) \neq 0 \), then \( J(\alpha) \) and \( J(\alpha\beta) \) are equivalent.

**Lemma 3.1.2.** (1) Property \( J(\alpha) \) is invariant with respect to local right equivalence.

(2) Property \( J(s^k) \) is invariant with respect to local left-right equivalence.

**Proof.** (1) Suppose that \( f \) has \( J(\alpha) \), i.e. \( \alpha \circ f = H \cdot f \) for some vector field \( H = (H_1, \ldots, H_m) \) at \( 0 \in \mathbb{R}^m \). We have to prove that for each \( h \in \mathcal{D}_0(\mathbb{R}^m) \) the function \( g = f \circ h \) has \( J(\alpha) \) as well. Indeed, since \( \nabla g = \nabla(f \circ h) = Th \cdot (Th)^{-1} \cdot g \) we get

\[
\alpha \circ g = \alpha \circ f \circ h = H \cdot f(h) = \sum_{i=1}^{m} (H_i \circ h) \cdot (f'_{x_i} \circ h) = [(H \circ h) \cdot (Th)^{-1}] \cdot g \in \Delta(g,0).
\]
(2) Suppose that \( f \) has \( J(s^k) \), i.e. \( f^k = H.f \) for some vector field \( H \) at \( 0 \in \mathbb{R}^m \).

We have to prove that for each \((\phi, h) \in D_0(\mathbb{R}^m) \times D_0(\mathbb{R})\) the function \( g = \phi \circ f \circ h^{-1} \) has \( J(s^k) \). Due to (1) we can assume that \( h = \text{id}_{\mathbb{R}^m} \), whence \( g = \phi \circ f \). Then

\[
H.g = \sum_{i=1}^{m} H_i \cdot (\phi \circ f)'_{x_i} = \phi'(f) \sum_{i=1}^{m} H_i f'_{x_i} = \phi'(f) \cdot H.f = \phi'(f) f^k.
\]

Notice that \( \phi(s) = \tilde{\phi}(s)s \), where \( \tilde{\phi} \in C_0^\infty(\mathbb{R}) \) and \( \tilde{\phi}(0) = \phi'(0) > 0 \). Therefore

\[
g^k = (\phi \circ f)^k = \tilde{\phi}(f)^k f^k = \frac{\tilde{\phi}(f)^k}{\phi'(f)} \phi'(f)^k = \left( \frac{\tilde{\phi}(f)}{\phi'(f)} H \right) \circ (\phi \circ f) \in \Delta(g, 0). \quad \square
\]

**Remark 3.1.3.** (c.f. Corollary 2.3.3). It seems that for arbitrary \( \alpha \) the property \( J(\alpha) \) is not a local left-right invariant. The crucial property of the function \( \alpha(s) = s^k \) that we have used for the proof of (2) is that \( \alpha(\phi \circ f) = \omega(f) \), i.e. \( (\phi \circ f)^k = \omega f^k \) for some \( \omega \in C_0^\infty(\mathbb{R}) \). This property may not hold in general. For example, let

\[
\alpha(s) = \begin{cases} 
eq 1/s, & s > 0, \\ 0, & s \leq 0, \end{cases} \quad \phi(s) = 2s, \quad f(s) = s.
\]

Then \( \alpha \circ \phi \circ f(s) = e^{-1/2s} = e^{1/2s} \cdot \alpha \circ f(s) \) for \( s > 0 \) and the function \( \omega(s) = e^{1/2s} \) does not extend to a smooth function near 0.

3.1.4. The following theorem establishes the relationships between the group \( L(\alpha) \), the condition \( J(\alpha) \) for \( f \), and the stabilizer \( S_{\mathbb{R}^m, \mathbb{R}}(f) \).

Let \( p : D_0(\mathbb{R}^m) \times D_0(\mathbb{R}) \to D_0(\mathbb{R}) \) be natural projection.

**Theorem 3.2.** Suppose that \( f \in \mathfrak{m}(\mathbb{R}^m) \) has property \( J(\alpha) \) for some admissible \( \alpha \in \mathfrak{m}(\mathbb{R}) \). Then \( L(\alpha) \subset p(S_{\mathbb{R}^m, \mathbb{R}}(f)) \) and \( p \) has a section \( \Theta : L(\alpha) \to S_{\mathbb{R}^m, \mathbb{R}}(f) \) being a homomorphism.

Equivalently, if \( H.f = \alpha \circ f \) for some vector field \( H \) at \( 0 \in \mathbb{R}^m \), then there exists a homomorphism \( \theta : L(\alpha) \to D_0(\mathbb{R}^m) \) such that

\[
\phi \circ f = f \circ \theta(\phi),
\]

i.e. \( (\theta(\phi), \phi) \in S_{\mathbb{R}^m, \mathbb{R}}(f) \). Thus each \( h \in L(\alpha) \) is \( L \)-trivial for \( f \).

Moreover, let \( H \) be a flow generated by \( H \). Then \( \theta \) is defined by the following formula: \( \theta(\phi)(x) = H(x, \sigma_0 \circ f(x)) \), where \( \phi \in L(\alpha) \) and \( \sigma_0 \in C_0^\infty(\mathbb{R}) \) is a shift-function of \( \phi \) with respect to the flow generated by the vector field \( F(s) = \alpha(s) \cdot \mathbb{R}^m \).

As a particular case of this theorem we obtain the following local variant of Theorem 1.3.

**Theorem 3.3.** Suppose that \( f \in \mathfrak{m}(\mathbb{R}^m) \) satisfies \( J(\text{id}) \), i.e. \( f \in \Delta(f, 0) \). Then \( p(S_{\mathbb{R}^m, \mathbb{R}}(f)) = D_0(\mathbb{R}) = L(\text{id}) \) and \( p \) admits a section \( \Theta : D_0(\mathbb{R}) \to S_{\mathbb{R}^m, \mathbb{R}}(f) \) being a homomorphism. Thus each \( \phi \in D_0(\mathbb{R}) \) is \( L \)-trivial for \( f \).

**Corollary 3.3.1.** Suppose that \( f, g \in \mathfrak{m}(\mathbb{R}^m) \) satisfy \( J(\text{id}) \). Then \( f \) and \( g \) are left-right equivalent iff they are right equivalent.

**Proof.** If \( f \) and \( g \) are left-right equivalent, i.e. \( g = \phi \circ f \circ h^{-1} \) for a certain \((h, \phi) \in D_0(\mathbb{R}^m) \times D_0(\mathbb{R})\), then \( g = f \circ \theta(\phi) \circ h^{-1} \). The converse statement is evident. \( \square \)

We shall prove Theorem 3.2 in Section 3.5.

3.4. **Characterization of property** \( J(\alpha) \). Let \( H \) be a vector field on \( \mathbb{R}^m \) and \( \mathcal{H} : U \times I \to \mathbb{R}^m \) a local flow generated by \( H \), where \( U \) is a neighborhood of \( 0 \in \mathbb{R}^m \) and \( I \) is an open interval containing \( 0 \in \mathbb{R} \).
Let $\alpha : \mathbb{R} \to \mathbb{R}$ be a smooth function, $F(s) = \alpha(s)\frac{d}{ds}$ a vector field on $\mathbb{R}$, and $\mathcal{F} : V \times I \to \mathbb{R}$ a local flow generated by $F$, where $V$ is a neighborhood of $0 \in \mathbb{R}$.

Notice that $F$ is a section of a (trivial) tangent bundle $T\mathbb{R}$ defined by

$$F : \mathbb{R} \to T\mathbb{R} \equiv \mathbb{R} \times \mathbb{R}, \quad F(s) = (s, \alpha(s)).$$

**Lemma 3.4.1.** Suppose that $f : \mathbb{R}^m \to \mathbb{R}$ is a smooth function such that $f(0) = 0$. Then the following conditions on $H$, $F$, $\mathcal{H}$, $\mathcal{F}$, and $\alpha$ are equivalent:

1. The following diagram is commutative:
   $$\begin{align*}
   TU & \xrightarrow{Tf} TV \\
   V & \xrightarrow{\mathcal{F}} I
   \end{align*}
   \quad \text{i.e.} \quad H.f = \alpha \circ f,
   $$
   which is precisely condition $J(\alpha)$.

2. For each $t \in I$ the following diagram is commutative:
   $$\begin{align*}
   U & \xrightarrow{f} V \\
   V & \xrightarrow{\mathcal{F}_t} I
   \end{align*}
   \quad \text{i.e.} \quad \mathcal{F}(f(x), t) = f \circ \mathcal{H}(x, t);
   $$

3. For each smooth function $\sigma : V \to I$ define two mappings
   $$\begin{align*}
   h_\sigma : U & \to \mathbb{R}^m \\
   h(x) & = \mathcal{H}(x, \sigma \circ f(x))
   \end{align*}$$

   $$\begin{align*}
   \phi_\sigma : V & \to \mathbb{R} \\
   \phi(s) & = \mathcal{F}(s, \sigma(s)).
   \end{align*}$$

   Then the following diagram is commutative:
   $$\begin{align*}
   U & \xrightarrow{f} V \\
   V & \xrightarrow{\phi_\sigma} I
   \end{align*}
   \quad \text{i.e.} \quad \phi_\sigma \circ f = f \circ h_\sigma.
   $$

In this case $h_\sigma$ is an embedding iff $\phi_\sigma$.

4. There exist smooth isotopies
   $$\begin{align*}
   \mathcal{H} : U \times [0, 1] & \to \mathbb{R}^m \\
   \mathcal{F} : V \times [0, 1] & \to \mathbb{R}
   \end{align*}
   $$

   such that $\mathcal{F}_0 = \text{id}_V$, $\mathcal{H}_0 = \text{id}_U$, $\mathcal{F}_t \circ f = f \circ \mathcal{H}_t$,

   $$\frac{\partial \mathcal{F}}{\partial t}(s, 0) = \alpha(s), \quad \text{and} \quad \frac{\partial \mathcal{H}}{\partial t}(s, 0) = H(s).$$

**Proof.** (1)⇒(2) Let $x \in U$. It suffices to prove that if $\omega(t) = \mathcal{F}(x, t)$ is a trajectory of $H$, i.e. $\omega(0) = H(\omega(t))$, then $f \circ \omega(t)$ is a trajectory of $F$, i.e. $(f \circ \omega(t))'_t = F(f \circ \omega(t))$. Indeed,

$$f \circ \omega(t) = \omega'(t), f = H.f(\omega(t)) \quad \overset{1}{=} \quad F(f \circ \omega(t)).$$

(2)⇔(3) Statement (3) can be obtained by substituting into (2) the function $\sigma \circ f(x)$ instead of $t$. Conversely, (2) is a particular case of (3) for the constant function $\sigma(s) = t$.

It remains to prove that $h_\sigma$ is an embedding iff $\phi_\sigma$ is. By [4] Theorem 19] $h_\sigma$ (resp. $\phi_\sigma$) is an embedding that preserves orientations of trajectories of $\mathcal{F}$ iff $H.(\sigma \circ f) > -1$ (resp. $\mathcal{F}.\sigma > -1$). But under assumption (1) these expressions coincide:

$$H.(\sigma \circ f) = \sigma'(f) \cdot H.f = \sigma'(f) \cdot F \circ f = F.\sigma(f).$$
Thus the group $p\mathcal{H}$ section we will prove that a very large class of functions withstands the pathology of singularities and in particular of flat functions. This yields the following problems whose solutions would probably give new invariance to the stable equivalence of singularities (Definition 4.0.6).

3.6. Problems.

By Theorem 3.2 we know that if $f$ satisfies $J(\alpha)$, then $L(\alpha)$ is included in the group $p(S_{\mathbb{R}^m,\mathbb{R}}(f))$ of all $L$-trivial diffeomorphisms for $f$. In the next section we will prove that a very large class of functions $f$ satisfies the “maximal” condition $J(id)$ which is the same as $(J)$. The property $(J)$ appears typical since it holds for non-degenerate critical points, all simple singularities $A_k$, $D_k$, $E_6$, $E_7$, $E_8$, and even for formal series. Thus for these functions by Theorem 3.2 we have that $p(S_{\mathbb{R}^m,\mathbb{R}}(f)) = D_0(\mathbb{R}) = L(id)$. Moreover, we also show that $(J)$ is invariant with respect to the stable equivalence of singularities (Definition 4.0.9).

On the other hand, there are singularities that do not satisfy $(J)$, see Claim 4.0.9. This yields the following problems whose solutions would probably give new invariants of pathological singularities and in particular of flat functions.

1) If $L(\alpha) \subset p(S_{\mathbb{R}^m,\mathbb{R}}(f))$, does the condition $(J)$ holds for $f$?

2) Suppose that $L(\sigma \alpha) \subseteq p(S_{\mathbb{R}^m,\mathbb{R}}(f)) \subseteq L(\alpha)$. Recall that $L(\alpha)/L(\sigma \alpha) \approx \mathbb{R}$, thus the group $G = p(S_{\mathbb{R}^m,\mathbb{R}}(f))/L(\sigma \alpha)$ is a subgroup in $\mathbb{R}$. Is $G$ closed or everywhere dense in $\mathbb{R}$? Does it coincide with either 0 or $\mathbb{R}$? In the last two cases $p(S_{\mathbb{R}^m,\mathbb{R}}(f))$ coincides with either $L(\sigma \alpha)$ or $L(\alpha)$.

3) Is it true in general, that for every $f \in m(\mathbb{R}^m)$ we have that $p(S_{\mathbb{R}^m,\mathbb{R}}(f))$ coincides with some group $L(\alpha)$?
4. Condition J(id)

In this section we give examples of singularities having property J(id). This property is invariant with respect to a stable equivalence (Corollary 3.1.2) and holds for non-degenerate, simple singularities (Corollary 4.0.4) and also for formal series. We also show that there are singularities for which J(id) fails.

Let \( C^\infty_0(R^m) \) be the algebra of germs of smooth functions \( R^m \to R \) at \( 0 \in R^m \), \( m(R^m) \) a unique maximal ideal of \( C^\infty_0(R^m) \) consisting of functions \( f \) such that \( f(0) = 0 \), and \( \Delta(f,0) \) the Jacobian ideal of \( f \) generated by partial derivatives of \( f \).

Recall that \( f \) has property J(id) at \( 0 \in R^m \) provided \( f \in \Delta(f,0) \). Moreover, by (2) of Lemma 3.1.2 J(id) is the property of the local left-right orbit of \( f \).

**Lemma 4.0.1.** Let \( m = 1 \). Then \( f \) has J(id) if and only if \( f(x) = \alpha(x)f'(x) \) for some \( \alpha \in C^\infty_0(R) \).

**Lemma 4.0.2.** Suppose that \( g \in m(R^m) \) has J(id), so \( H.g = g \) for some vector field \( H \) at 0. Let also \( f \in m(R^m) \) satisfies one of the following conditions:

1. \( f = g^a \) for some \( a \in R \setminus 0 \) (emphasize that \( f \) is assumed \( C^\infty \));
2. \( f(x) = \begin{cases} e^{-\frac{1}{x}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} \)

Then \( f \) also has property J(id), i.e. \( f \in \Delta(g,0) \).

**Proof.** We will show that in all the cases \( \beta H.f = f \) for some \( \beta \in C^\infty_0(R) \).

1. \( \frac{1}{2} H.f = \frac{1}{2} H.(g^a) = g^{a-1}.g = g^{a-1}g = f \).
2. It is well known that the following function \( \psi(x) = \begin{cases} e^{\frac{1}{x}}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} \) is smooth, whence so is \( f = \psi \circ g \). Then \( gH.f = gH.g e^{-\frac{1}{x}} = e^{-\frac{1}{x}} = f \).

**Lemma 4.0.3.** Let \( f \in m(R^m) \). Suppose that in some local coordinates at 0 the function \( f \) satisfies one of the following conditions:

1. \( f(x_1, \ldots, x_m) = x_1 \), which is equivalent to the assumption that 0 is a regular point of \( f \);
2. \( f(tx_1, \ldots, tx_m) = t^n f(x_1, \ldots, x_m) \) for all \( t > 0 \), i.e. \( f \) is homogeneous;
3. \( f(x_1, \ldots, x_m) = x_1^{a_1} \pm x_2^{a_2} \pm \cdots \pm x_k^{a_k} \);
4. \( f(x_1, \ldots, x_m) = x_1^{a_1} + x_2^{b_2}a_2 + x_3^{a_3}a_3 + \cdots + x_k^{b_k}a_k \);
5. \( m = 1 \) and \( f^{(s)}(0) \neq 0 \) for some \( s \geq 1 \), i.e. \( f \) is not flat at 0;

where \( a_i \geq 1, b_j \geq 0, \) and \( k \leq m \). Then \( f \) has property J(id).

**Proof.** (i) \( f = x_1 f'_x \).

(ii) This case follows from the well-known Euler identity for homogeneous functions: \( f(x) = \sum_{i=1}^{m} x_if'_x.i \). Let us recall its proof. Define a vector field by the formula

\[
H(x_1, \ldots, x_m) = \frac{1}{n}(x_1, \ldots, x_m).
\]

Then

\[
H.f(x) = \lim_{s \to 0} \frac{f(x + sH(x)) - f(x)}{s} = \lim_{s \to 0} \frac{f((1 + s/n)x) - f(x)}{s} = \lim_{s \to 0} \frac{(1 + s/n)^n - 1}{s} f(x) = f(x).
\]

(iii) \( f = \frac{a_1}{q_1} f^'_x x_1 \pm \frac{a_2}{q_2} f^'_x x_2 \pm \cdots \pm \frac{a_k}{q_k} f^'_x x_k \).

(iv) \( f = \frac{a_1}{q_1} f^'_x x_1 + \left(1 - \frac{b_1}{a_1}\right) \frac{a_2}{q_2} f^'_x x_2 + \left[1 - \frac{b_3}{a_3} \left(1 - \frac{b_1}{a_1}\right)\right] \frac{a_3}{q_3} f^'_x x_3 + \cdots \)
(v) Let \( s \geq 1 \) be the first number for which \( f^{(s)}(0) \neq 0 \). Then \( f \) is right equivalent to \( x^n \), and our statement follows from (ii).

**Corollary 4.0.4.** Non-degenerate singularities \( \sum \pm x_i^2 \) and simple singularities \( A_k(x) = x^k \) \((k \geq 1)\), \( B_k(x,y) = x^2y + y^{k-1} \) \((k \geq 4)\), \( E_6(x,y) = x^3 + y^4 \), 
\( E_7(x,y) = x^3 + xy^3 \), \( E_8(x,y) = x^3 + y^5 \), see e.g. [1], have \( J(\text{id}) \).

**Lemma 4.0.5.** Let \( f \in m(\mathbb{R}^m) \), \( g \in m(\mathbb{R}^n) \). Define \( h \in m(\mathbb{R}^{m+n}) \) by the formula \( h(x,y) = f(x) + g(y) \), where \( x = (x_1, \ldots, x_m) \in \mathbb{R}^m \) and \( y = (y_1, \ldots, y_n) \in \mathbb{R}^n \). Then \( f \) and \( g \) have property \( J(\text{id}) \) iff \( h \) has.

**Proof.** Suppose that \( f(x) = F.f(x) \) and \( g(y) = G.g(y) \) for certain vector fields \( F \) and \( G \) defined on \( \mathbb{R}^m \) and \( \mathbb{R}^n \) respectively. We can regard these vector fields as components of the following vector field \( H(x,y) = (F(x),G(y)) \) on \( \mathbb{R}^{m+n} \). Then \( G.f = F.g = 0 \), whence

\[
H.h(x,y) = F.f(x) + G.g(y) = f(x) + g(y) = h(x,y).
\]

Conversely, suppose that \( h(x,y) = H.h(x,y) \), where

\[
H(x,y) = (F(x,y),G(x,y))
\]

is a certain vector field on \( \mathbb{R}^{m+n} \). Notice that

\[
H.h(x,y) = (F,G).(f+g)(x,y) = \sum_{i=1}^{m} f'_{x_i}(x)F_i(x,y) + \sum_{j=1}^{n} g'_{y_j}(y)G_j(x,y),
\]

where \( F = (F_1, \ldots, F_m) \) and \( G = (G_1, \ldots, G_n) \).

Let \( F(x) = F(x,0) \) and \( G(y) = G(0,y) \) be vector fields on \( \mathbb{R}^m \) and \( \mathbb{R}^n \) respectively. By (i) of Lemma 4.0.3 we can assume that \( 0 \in \mathbb{R}^m \) and \( 0 \in \mathbb{R}^n \) are critical points for \( f \) and \( g \) respectively. Then \( f'_{x_i}(0) = 0 \) and \( g'_{y_j}(0) = 0 \), whence

\[
f(x) = h(x,0) = H.h(x,0) = \sum_{i=1}^{m} f'_{x_i}(x)F_i(x,0) = \tilde{F}.f(x)
\]

and similarly \( g(y) = \tilde{G}.g(y) \).

**Definition 4.0.6** (see e.g. [1]). Two functions \( f_k \in m(\mathbb{R}^{m_k}) \) \( k = 1,2 \), are stably equivalent if there are non-degenerated quadratic forms

\[
g_k(y_1, \ldots, y_{m_k}) = \sum_{i=1}^{n_k} \pm y_i^2, \quad k = 1,2
\]

such that \( m_1 + n_1 = m_2 + n_2 \) and functions \( f_1 + g_1 \) and \( f_2 + g_2 \) defined on \( \mathbb{R}^{m_1+n_1} \) are right equivalent (belong to the same orbit with respect to local right action).

By either (ii) or (iii) of Lemma 4.0.3 each quadratic form has property \( J(\text{id}) \), whence from Lemma 4.0.5 we get the following statement:

**Corollary 4.0.7.** Suppose that functions \( f_k \in m(\mathbb{R}^{m_k}) \) \( k = 1,2 \) are stably equivalent. Then \( f_1 \) has property \( J(\text{id}) \) iff \( f_2 \) has.

**Proposition 4.0.8.** Let

\[
f = \sum_{i_1, \ldots, i_m \geq 0} a_{i_1 \ldots i_m} x_1^{i_1} \cdots x_m^{i_m} \in \mathbb{R}[[x_1, \ldots, x_m]]
\]

be a formal series without first term, i.e. \( f(0) = 0 \). Then \( f \) has \( J(\text{id}) \) in \( \mathbb{R}[[x_1, \ldots, x_m]] \).
Proof. We should find formal series $H_1, \ldots, H_m \in \mathbb{R}[x_1, \ldots, x_m]$ such that

$$f = \sum_{i=1}^{m} f'_i H_i.$$  

This relation gives a system of linear equations on the coefficients of $H_i$ such that the solution can be found recurrently. The details are left to the reader. \qed

This statement gives a hope, that $J(\text{id})$ holds for analytical functions, and in particular for polynomials (as usual, we have to seek for converging series $H_i$), but I do not know is it true. On the other hand, the following statement shows that $J(\text{id})$ can fail in the non-analytical case.

**Claim 4.0.9.** Let $f \in \mathfrak{m}(\mathbb{R}^m)$. Suppose that there is a sequence $\{z_i\}$ of critical points of $f$ converging to 0 and such that $f(z_i) \neq 0$, in particular, the critical value 0 of $f$ is not isolated. Then $f$ does not satisfy $J(\text{id})$, i.e. $f \notin \Delta(f,0)$.

**Remark 4.0.10.** Such a function exists. For example let

$$g(x) = \begin{cases} e^{\frac{1}{x^1 - x^2}}, & x \neq \frac{1}{n}, \\ 0, & x = \frac{1}{n}, n \in \mathbb{Z}. \end{cases}$$

It is easy to verify that $g$ is $C^\infty$. Then the function $f(x) = \int_0^x g(t) dt$ satisfies the conditions of Claim 4.0.9.

**Proof.** Suppose that $f = H_x f$ for some smooth vector field $H$. Since $z_i$ is a critical point, we have $df(z_i) = 0$ for all $i$, whence $H_x f(z_i) = 0$. On the other hand $f(z_i) \neq 0$ by the assumption. Hence the equality $f(z_i) = H_x f(z_i)$ is impossible. \qed

5. Proof of Theorem 1.3

**Lemma 5.0.1.** Suppose that $f \in C^\infty(M,P)$ satisfies the condition (V). Let also $p : D_M \times D_P \to D_P$ be the projection onto the second multiple. Then

- **Case** $P = \mathbb{R}$: $p(S_{\text{MR}}(f)) \subset D^e_{\mathbb{R}}$
- **Case** $P = S^1$: $p(S_{\text{MS}}(f)) \subset D^e_{S^1}$.

**Proof.** Let $(h, \phi) \in S_{\text{MS}}(f)$, i.e., $\phi \circ f = f \circ h$. Then $\phi = p(h, \phi)$. We claim that $\phi$ preserves the set $E_f = \{1, \ldots, n\}$ of exceptional values of $f$. Indeed, $h$ interchanges level-sets of $f$ in same manner as $\phi$ interchanges values of $f$. Denote by $\Sigma_f$ the set of critical points of $f$. Since $h$ is a diffeomorphism, it preserves the sets $f^{-1}(f(\Sigma_f))$ and $f^{-1}(f(\partial M))$. Therefore, $\phi$ preserves $f(\Sigma_f) \cup f(\partial M) = E_f$.

This proves our lemma for the case $P = S^1$ as by the definition $D^e_{S^1}$ consists of preserving orientation diffeomorphisms that also preserves $E_f$. If $P = \mathbb{R}$, then $\phi$ also preserves the ordering of the finite set $E_f$ and therefore fixes it point-wise, i.e. $\phi \in D^e_{\mathbb{R}}$.

Now Theorem 1.3 follows from Proposition 5.0.2 and 5.0.3 below.

**Proposition 5.0.2.** Suppose that $n \geq 1$ and $f$ satisfies the conditions (V) and (J). Then $D_P \subset p(S_{\text{MP}}(f))$ and $p$ admits a continuous section

$$\Theta : D_P \to S_{\text{MP}}(f)$$

being a homomorphism.

**Proposition 5.0.3.** If $n = 0$, then $p(S_{\text{MR}}(f)) = D^e_{\mathbb{R}}$ but a global section of $p$ exists if and only if $f : M \to S^1$ is a trivial fibration.
5.1. **Proof of Proposition 5.0.2.** The section $\Theta : D^p \rightarrow S_{MP}(f)$ of the projection $p$ must have the following form $\Theta(\phi) = (\theta(\phi), \phi)$, where $\theta : D^p \rightarrow D_M$ is a continuous homomorphism such that $\phi \circ f = f \circ \theta(\phi)$. Thus we have to construct $\theta$.

For $i = 1, \ldots, n$ let

- $L_i = f^{-1}(i)$ be the $i$-th exceptional level of $f$,
- $V_i = f^{-1}(i, i + 1)$ be the part of $M$ between the levels $L_i$ and $L_{i+1}$,
- $U_i = f^{-1}(i - \frac{1}{2}, i + \frac{1}{2})$ be a neighborhood of $L_i$,
- $U_i^- = f^{-1}(i - \frac{1}{2}, i)$ and $U_i^+ = f^{-1}(i, i + \frac{1}{2})$ respectively lower and upper part of $U_i$.

Thus the restrictions of $f$ to $U_i$ and to $V_i$ can be regarded as functions

$$U_i \rightarrow (i - \frac{1}{2}, i + \frac{1}{2}) \quad \text{and} \quad V_i \rightarrow (i, i + 1).$$

**Lemma 5.1.1.** In some neighborhood of $L_i$ there is a vector field $G_i$ such that

$$G_i.f = f - i. \tag{5.1}$$

**Proof.** Notice that for each point $z \in L_i$ there is a neighborhood $U_z \subset U_i$ and a vector field $G_z$ defined on $U_z$ such that such that $G_z.f = f - i$. For critical points of $f$ this follows from the condition (J) and for the regular ones from (1) of Lemma 1.0.3. In the latter case we may put $G_i = (x_1, 0, \ldots, 0)$ in that coordinates for which $f(x_1, \ldots, x_m) = x_1 + i$.

Let $\{U'_i\}$ be a finite refinement of the covering $\{U_z\}_{z \in L_i}$ of $L_i$. Then on each $U'_i$ we have a vector field $G'_i$ such that $G'_i.f = f - i$. Let $\mu_j : U'_i \rightarrow [0, 1]$ be a partition of unity subordinated to $\{U'_i\}$, i.e. supp $\mu_j \subset U'_i$ and $\sum_j \mu_j \equiv 1$. Define the following vector field $G_i$ in a neighborhood of $L_i$ by $G_i = \sum_j \mu_j G'_j$. Then

$$G_i.f = \left(\sum_j \mu_j G'_j\right) . f = \sum_j \mu_j (G'_j.f) = \sum_j \mu_j (f - i) = f - i. \quad \Box$$

Decreasing $U_i$ if necessary, we can assume that $G_i$ is defined on some neighborhood of $U_i$. Then $G_i.f < 0$ on $U_i^-$ and $G_i.f > 0$ on $U_i^+$. Let $G_i : U_i \times (-\delta, \delta) \rightarrow M$ be a local flow generated by $G_i$.

Notice, that $f$ has no critical points in $V_i$, whence there is a vector field $H_i$ on $V_i$ such that $H_i.f > 0$. We can assume that $H_i$ generates a global flow $H_i : V_i \times \mathbb{R} \rightarrow V_i$.

Moreover, not violating condition (5.1) we can also suppose that $G_i(z) = H_i(z)$ for $z \in V_i$, and $G_i(z) = -H_{i-1}(z)$ for $z \in V_{i-1}$ provided $G_i(z)$ is defined. Then

$$G_i(z, t) = H_i(z, t) \quad \text{for } z \in U_i^+ \quad \text{and} \quad G_i(z, t) = H_{i-1}(z, -t) \quad \text{for } z \in U_i^-.$$  

**Remark 5.1.2.** Schematically vector fields $G_i$ and $H_i$ can be represented by the behavior of $f$ along their trajectories, see Figures (1) and (2). Thus $f$ increases along $H_i$ and along “upper” part of $G_i$ and decreases along “lower” part of $G_i$. A bold point on the arrow means that $G_i$ is tangent to the $i$-th level-set of $f$.

Suppose that either $P = \mathbb{R}$ or $P = S^1$ but $n$ is even. Then vector fields $H_i$ and $G_i$ give a global vector field $F$ on $M$. To construct $F$ we should just change signs of all $H_{2i}$ and $G_{2i}$ (having even indices), see Figures (1) and (2). In Figure (1) such a vector field is shown for a height function on 2-torus. The critical points of $f$ are in bold. Together with white points they constitute the set of singular points of $F$. Existence of $F$ would simplify the proof. But we shall not use this approach since for the case $P = S^1$ and $n$ is odd such a vector field $F$ does not exist, see Figure (2).
Lemma 5.1.3. For each $\phi \in \mathcal{D}_\mathbb{P}$ there exists a diffeomorphism $h \in \mathcal{D}_M$ such that $\phi \circ f = f \circ h$, whence $\phi \circ f \in \mathcal{O}_{MP}(f)$ in fact belongs to $\mathcal{O}_M(f)$.

Proof. Let $\phi \in \mathcal{D}_\mathbb{P}$. We shall find $C^\infty$ functions $\sigma_i : V_i \to \mathbb{R}$ and $\rho_i : U_i' \to (-\delta, \delta)$, where $U_i' \subset U_i$ is a neighborhood of $L_i$ such that the following mappings $h_i : V_i \to V_i$ and $g_i : U_i' \to U_i$ defined by:

\begin{equation}
(5.2) \quad h_i(z) = H_i(z, \sigma_i(z)) \quad \text{and} \quad g_i(z) = G_i(z, \rho_i(z)),
\end{equation}

are diffeomorphisms on their images, $\phi \circ f = f \circ h_i$ on $V_i$, and $\phi \circ f = f \circ g_i$ on $U_i'$.

Moreover, all these diffeomorphisms will coincide at common points of their ranges and therefore define a self-diffeomorphism $h$ of $M$ satisfying the statement of our lemma.

Step 1. Definition of $\sigma_i$ and $h_i$. For each $x \in V_i$ let $\omega_x \subset V_i$ be the trajectory of $x$ with respect to $\mathcal{H}_i$. Notice that $f$ maps $\omega_x$ diffeomorphically onto the open interval $(i,i+1)$ and $\omega_x$ transversely intersects level-sets of $f$. Let $y$ be a unique intersection point of $\omega_x$ with the level-set $f^{-1}(\phi \circ f(x))$ of $f$, see Figure 3. Set

\begin{equation}
h_i(x) = y,
\end{equation}

then $f \circ h_i(x) = \phi \circ f(x)$. Let also $\sigma_i(x)$ be the time along $\omega_x$ with respect to $\mathcal{H}_i$ from $x$ to $y$. Then $y = h_i(x) = H_i(x, \sigma_i(x))$.

Let us show that $\sigma_i$ is smooth. This will imply a smoothness of $h_i$. 

Figure 1.

Figure 2.
Let $z \in V_i$. Then $H_i(z) \neq 0$, whence we can assume that in some local coordinates $(x_1, \ldots, x_m)$ at $z$ we have $H_i(x) = (1, 0, \ldots, 0)$. For simplicity designate $\bar{x} = (x_2, \ldots, x_m)$ and $x = (x_1, \ldots, x_m) = (x_1, \bar{x})$.

It follows that $\mathcal{H}_i(x,t) = (x_1 + t, \bar{x})$. Suppose that a point $y$ belongs to the trajectory $\gamma_x$ of a point $x = (x_1, \bar{x})$. Then $y = (y_1, \bar{x})$. Moreover, the time between $x$ and $y$ on $\gamma_x$ is equal to $y_1 \cdot x_1$.

Since $H_i, f = f'_x \neq 0$, it follows that there exists a unique smooth function $q(x)$ such that $x_1 = q(f(x), \bar{x})$. Then from the definition of $\sigma_i$ we get that

$$
\sigma_i(x) = q(\phi \circ f(x), \bar{x}) - q(f(x), \bar{x}) = q(\phi \circ f(x), \bar{x}) - x_1,
$$

whence $\sigma_i$ and $h_i$ are smooth.

Let us verify that $h_i$ is a diffeomorphism at each point $z \in V_i$. By [4] Theorem 19], this is true if and only if the following relation holds true for all $z \in V_i$:

$$
H_i,\sigma_i(z) + 1 > 0.
$$

Notice that

$$
H_i,\sigma_i = \frac{\partial \sigma}{\partial x_1} = q'_x(\phi \circ f(x, \bar{x}) \cdot \phi'(f(x)) \cdot f'_x(x) - 1.
$$

Since $\phi$ is a preserving orientation diffeomorphism, we get $\phi' > 0$. Moreover, $f'_x$ and $q'_x$ have same signs. Therefore $H_i,\sigma_i + 1 > 0$.

**Step 2. Definition of $\rho_i$ and $g_i$.** Suppose that $z \in L_i$. Recall that we wish to define $g_i$ by (4.1). For simplicity assume that $f(z) = 0$. Then $\phi(0) = 0$ and by the Hadamard lemma $\phi(s) = s \cdot \phi(s)$ for some smooth function $\phi(s)$ such that $\phi(0) = \phi'(0) = 0$.

Then by Lemma 3.4.2 see [4], the relation $\phi \circ f = f \circ \mathcal{G}_i(x, \rho_i(x))$ can be rewritten as follows:

$$
f(x) \cdot \phi'(f(x)) = f(x) \cdot e^{\epsilon \rho_i(x)},
$$

whence we set

$$
\rho_i(x) = \epsilon \ln \phi(f(x)).
$$

It follows that $\rho_i$ and $g_i$ are smooth on some neighborhood $U_i^+$ of $L_i$.

Let us show that $g_i$ is a diffeomorphism, i.e. that $G_i,\rho_i(z) > -1$. Notice that

$$
G_i,\rho_i = G_i(\epsilon \ln \phi(f)) = \frac{\epsilon \phi'_i(f)}{\phi(f)} \cdot G_i f = \frac{\epsilon \phi'_i(f)}{\phi(f)} f.
$$

Since $f(z) = 0$, we get $G_i,\rho_i(z) = 0 > -1$. Hence $g_i$ is a diffeomorphism near $z$.

**Step 3. Coherency of $h_i$ and $g_i$.** We have to show that $g_i = h_i$ on $U_i^+$ and $g_i = h_{i-1}$ on $U_i^-$. Let $z \in U_i^+$. Then $\phi \circ f = f \circ h_i(z) = f \circ g_i(z)$. In other words:

$$
f \circ \mathcal{H}_i(z, \sigma_i(z)) = f \circ \mathcal{G}_i(z, \rho_i(z)).
$$
Notice that $f$ is monotone along trajectories of $H_i$ and those trajectories of $G_i$ that do not belong to $L_i$. Since $G_i(z, t) = H_i(z, t)$ on $U^+_i$, we get $\sigma_i(z) = \rho_i(z)$ and $g_i(z) = G_i(z, \rho_i(z)) = H_i(z, \rho_i(z)) = h_i(z)$.

Similarly, since $G_i(z, -t) = H_i(z, -t)$ on $U^-_i$, it follows that $\rho_i(z) = -\sigma_i^{-1}(z)$ and $g_i(z) = G_i(z, \rho_i(z)) = H_i(z, \rho_i(z)) = h_i(z)$. Lemma 5.1.3 is proved.

Thus the correspondence $\phi \mapsto h$ is a mapping $\theta : D^p \to D_M$ such that $(\theta(\phi), \phi) \in S_{MF}(f)$. It remains to prove the following two statements.

Claim 5.1.4. $\theta$ is a homomorphism.

Proof. Let $\phi_1, \phi_2 \in D^p$, $\phi_0 = \phi_1 \circ \phi_2$, $\theta_k = \theta(\phi_k), k = 0, 1, 2$. We have to show that $\hat{\theta} = \theta_1 \circ \theta_2 \circ \theta_0^{-1}$ coincides with id$_M$.

First notice that $\hat{\theta} \in S_M(f)$. Indeed, since $\phi_k \circ f = f \circ \theta_k$, we obtain that $f \circ \theta_1 \circ \theta_2 \circ \theta_0^{-1} = \phi_1 \circ \phi_2 \circ \phi_0^{-1} \circ f = f$.

Thus $\hat{\theta} \in S_M(f)$ and therefore it preserves each level-set of $f$. Let $z \in V_i$ and $\gamma_z$ be the trajectory of $z$ with respect to $H_i$. By the construction each $\theta_k$ preserves $\gamma_z$ for $k = 0, 1, 2$, whence so does $\hat{\theta}$. Therefore $\hat{\theta}$ also preserves the intersection $z = f^{-1}(f(z) \cap \gamma_z)$, i.e. $\theta(z) = z$. Thus $\hat{\theta}$ is the identity on $M \setminus L$.

If $z \in L_i$, then $\theta_k(z) = G_i(z, \rho_k^{-1}(z))$ for some smooth function $\rho_k^{-1}, k = 0, 1, 2$. But it follows from (5.5) that $\rho_k$ is constant on $L_i$ as well as $f$. This implies that $\hat{\theta}$ is identity on $L_i$ also.

Claim 5.1.5. $\theta : D^p \to D_M$ is continuous between strong $C^\infty$-topologies of these groups.

Proof. This follows from formulas (5.2), (5.3), and (5.5). We leave the details to the reader. 

5.2. Proof of Proposition 5.0.3. Recall that $M$ is closed and $f : M \to S^1$ is a locally trivial fibration. Let us show that $p(S_{MS}(f)) = D^S_1$.

Let $H$ be a gradient vector field for $f$, then $H.f > 0$ on all of $M$. Let also $H$ be the flow on $M$ generated by $H$. We claim that for every $\phi \in D^S_1$, there exists a smooth function $\sigma : M \to \mathbb{R}$ such that the mapping $h : M \to M$ defined by $h(x) = H(x, \sigma(x))$ is a diffeomorphism and $\phi \circ f = f \circ h$.

To construct $\sigma$, notice that there exists an isotopy $\phi_t : S^1 \to S^1$ such that $\phi_0 = \text{id}_{S^1}$ and $\phi_1 = \phi$. Let $x \in M$, and $\gamma : [0, 1] \to S^1$ be the path of the point $f(x)$ under $\phi_t$, i.e. $\gamma(t) = \phi_t(f(x))$. Since $f$ is a locally trivial fibration, we see that there exists a unique lifting $\omega : [0, 1] \to M$ of $\gamma$ to $M$ that starts at $x$ and lies in the trajectory of $x$ with respect to $H$. Thus $\omega(0) = x$, $f(\omega(x)) = \gamma(t) = \phi_t(f(x))$, and $\omega(0, 1)$ is a part of $H$-trajectory of $x$.

Let $\sigma(x)$ be the time along $\omega$ with respect to $H$ and $h(x) = H(x, \sigma(x))$. Then similarly to Step 2 of Lemma 5.1.3 it can be shown that $\sigma$ and $H$ are smooth.

Notice that the definition of $\sigma$ and $h$ depends on the isotopy $\phi_t$. Since $D^S_1$ is not contractible, it follows that in general we can not choose $h$ to continuously depend on $\phi$.

Suppose $f$ is a trivial fibration, i.e. $M = N \times S^1$ and $f : N \times S^1 \to S^1$ is given by $f(x, s) = s$. Then the homomorphism $\theta : D^S_1 \to D_M$ can be defined by $\theta(\phi)(x, s) = (x, \phi(s))$ and we also have that $\phi \circ f(x, s) = \phi(s) = f \circ \theta(\phi)(x, s)$.

Conversely, suppose that there exists a continuous mapping $\theta : D^S_1 \to D_M$ such that $\phi \circ f = f \circ \theta(\phi)$. We can always assume that $\theta(\text{id}_{S^1}) = \text{id}_M$. Let $\phi_t : S^1 \to S^1$ be the “rotation” isotopy of $S^1$: $\phi_t(s) = s + t \mod 1$. This isotopy yields an isotopy $h_t = \theta(\phi_t)$ of $M$ ($h_t$ is $\text{a priori}$ just continuous, but this is enough for the proof of triviality of $f$). Since $\phi_0 = \phi_1 = \text{id}_P$, we also have that $h_0 = \theta(\phi_0) = \theta(\phi_1) = h_1$. 


Denote \( N = f^{-1}(0) \). Then \( h_t \) yields a continuous map \( q : N \times S^1 \to M \) defined by \( q(x, s) = h_s(x) \). It is easy to verify that \( q \) is a homeomorphism. Moreover, the composition \( \left[ N \times S^1 \to M \to S^1 \right] \) is given by

\[
f \circ q(x, s) = f \circ h_s(x) = \phi_s \circ f(x) = f(x) + s = 0 + s = s,
\]

i.e. it coincides with the projection \( N \times S^1 \to S^1 \). Hence \( f \) is a trivial fibration. \( \square \)

6. Factor spaces \( \mathcal{D}^{[1,n]}_R / \mathcal{D}^c_R \) and \( \mathcal{D}^+_S / \mathcal{D}^c_S \).

Let us endow the spaces \( \mathcal{D}_R \) and \( \mathcal{D}^+_S \) with strong \( C^r \) Whitney topologies for some \( r = 0, 1, \ldots, \infty \). These topologies yields certain topologies on \( \mathcal{D}^{[1,n]}_R, \mathcal{D}^+_R, \mathcal{D}^c_S \), and on the spaces of left adjacent classes \( \mathcal{D}^{[1,n]}_R / \mathcal{D}^c_R \) and \( \mathcal{D}^+_S / \mathcal{D}^c_S \).

Lemma 6.0.1. The groups \( \mathcal{D}^{[1,n]}_R, \mathcal{D}^+_R, \) and \( \mathcal{D}^+_S \) for \( n \geq 1 \) are contractible.

Proof. For each of these groups its contracting formula: \( H(\phi, t)(x) = (1 - t)x + t\phi(x) \). \( \square \)

Theorem 6.1. There are homeomorphisms

\[
\mathcal{D}^{[1,n]}_R / \mathcal{D}^c_R \approx \mathbb{R}^{n-2} \quad \text{and} \quad \mathcal{D}^+_S / \mathcal{D}^c_S \approx S^1 \times \mathbb{R}^{n-1}.
\]

Proof. Case \( P = \mathbb{R} \). Let \( q : \mathcal{D}^{[1,n]}_R \to \mathcal{D}^{[1,n]}_R / \mathcal{D}^c_R \) be the natural projection. Consider the following subset of \( \mathbb{R}^{n-2} \):

\[
\Delta^{n-2} = \{(x_2, \ldots, x_{n-1}) \in \mathbb{R}^{n-2} \mid 1 < x_2 < x_3 < \cdots < x_{n-2} < n\}.
\]

Evidently, \( \Delta^{n-2} \) is open and convex subset of \( \mathbb{R}^{n-2} \), whence it is diffeomorphic with \( \mathbb{R}^{n-2} \). We shall show that \( \mathcal{D}^{[1,n]}_R / \mathcal{D}^c_R \) is canonically homeomorphic with \( \Delta^{n-2} \).

Consider the evaluation map \( e : \mathcal{D}^{[1,n]}_R \to \Delta^{n-2} \) defined by the formula \( e(\phi) = (\phi(2), \ldots, \phi(n - 1)) \) for \( \phi \in \mathcal{D}^{[1,n]}_R \). This mapping evidently factors to a unique bijection \( e : \mathcal{D}^{[1,n]}_R / \mathcal{D}^c_R \to \Delta^{n-2} \) such that

\[
e = c \circ q : \mathcal{D}^{[1,n]}_R \longrightarrow \mathcal{D}^{[1,n]}_R / \mathcal{D}^c_R \longrightarrow \Delta^{n-2}.
\]

Our aim is to prove that \( e \) is a homeomorphism for all \( C^r \)-topologies. It is easy to see that \( e \) is continuous in \( C^0 \)-topology (\( C^0 \)-continuous). Hence it is so in all \( C^r \)-topologies for \( r = 1, \ldots, \infty \). Then from the definition of the factor-topology on \( \mathcal{D}^{[1,n]}_R / \mathcal{D}^c_R \), we obtain that \( e \) is also \( C^r \)-continuous for \( r = 0, \ldots, \infty \).

We will now show that \( e \) admits a section \( s : \Delta^{n-2} \to \mathcal{D}^{[1,n]}_R \) which is \( C^r \)-continuous for every \( r = 0, \ldots, \infty \). It follows that so is \( e^{-1} = e \circ s \), whence \( e \) is a \( C^r \)-homeomorphism.

Consider another subset of \( \mathbb{R}^n \):

\[
\Delta^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid 0 < x_1 < x_2 < \cdots < x_n < n + 1\}.
\]

Then \( \Delta^{n-2} \) can be identified with the subset \( \{1\} \times \Delta^{n-2} \times \{n\} \) of \( \Delta^n \).

Lemma 6.1.1. There exists a smooth function \( u : \Delta^n \times \mathbb{R} \to \mathbb{R} \) with the following properties:

1. \( u(x_1, \ldots, x_n, t) > 0 \) for all \( (x_1, \ldots, x_n, t) \in \Delta^n \times \mathbb{R} \);
2. \( u(x_1, \ldots, x_n, t) = t \) for \( t \leq 0 \) and \( t \geq n + 1 \);
3. \( u(x_1, \ldots, x_n, k) = x_k \) for \( k = 1, 2, \ldots, n \);
4. \( u(1, 2, \ldots, n, t) = t \) for \( t \in \mathbb{R} \).
Finally, let us verify (3′) and (1)-(4) will be satisfied.

Proof of Lemma 6.1.1. Suppose that \( u \) is defined. Set \( v = u' \). Then

\[
\begin{align*}
(1') & \quad v > 0; \\
(2') & \quad v(x_1, \ldots, x_n, t) = 1, \text{ for } t \not\in (0, n + 1); \\
(3') & \quad \int \sum_{k=0}^{n} q_{k, k+1}^{'}(t, x_{k+1} - x_k) dt = 1 + q_{0,1}(t, x_1) + q_{1,2}(t, x_2 - x_1) + \cdots + q_{n,n+1}(t, x_{n+1} - x_n). \\
(4') & \quad v_{1,2, \ldots, n, t} = 1, \\
\end{align*}
\]

and \( u(x, t) = \int_0^t v(x, s) ds \).

Thus it suffices to construct a function \( v \) satisfying the conditions (1′)-(4′). Then (1)-(4) will be satisfied.

For every pair \( a < b \in \mathbb{R} \) we will now define a smooth function \( q_{a,b}(t, s) \geq 0 \) such that

\[
\begin{align*}
(2'') & \quad q_{a,b}(t, s) = 0 \text{ for } t \not\in (a, b) \text{ and } s \in \mathbb{R}, \text{ and} \\
(3'') & \quad \int_a^b q_{a,b}(t, s) dt = s - (b - a). \\
\end{align*}
\]

It follows from (3′′) and the condition \( q_{a,b}(t, s) \geq 0 \) that

\[
\begin{align*}
(4'') & \quad q_{a,b}(t, b - a) = 0 \text{ for } t \in \mathbb{R}. \\
\end{align*}
\]

Then \( v \) can be defined as follows:

\[
\begin{align*}
v(x_1, \ldots, x_n, t) = 1 + \sum_{k=0}^{n} q_{k, k+1}(t, x_{k+1} - x_k) = \\
= 1 + q_{0,1}(t, x_1) + q_{1,2}(t, x_2 - x_1) + \cdots + q_{n,n+1}(t, x_{n+1} - x_n). \end{align*}
\]

Indeed, \( v > 0 \). Further, (2′) and (4′) follows from (2′′) and (4′′) respectively. Finally, let us verify (3′). If \( k = 0, \ldots, n \), then we have

\[
\begin{align*}
\int_k^{k+1} v(x_1, \ldots, x_n, t) dt & \overset{(2'')}{=} \int_k^{k+1} (1 + q_{k, k+1}(t, x_{k+1} - x_k)) dt \overset{(3'')}{=} \\
&= (k+1 - k) + (x_{k+1} - x_k - 1) = x_{k+1} - x_k.
\end{align*}
\]
Thus it remains to construct $q_{a,b} \geq 0$ satisfying (2′) and (3″). Consider the following $C^\infty$-functions
\[
\alpha(t) = \begin{cases} 
\frac{1}{b-a} e^{-\frac{t-a}{b-a}}, & t \in (a, b) \\
0, & t \notin (a, b),
\end{cases}
\]
\[
\beta(t, c) = e^{\alpha(t)} - 1, \quad \text{and} \quad \gamma(c) = \int_a^b \beta(t, c) dt.
\]
Then $\gamma(c)$ is smooth, $\lim_{c \to -\infty} \gamma(c) = -(b-a)$, $\lim_{c \to +\infty} \gamma(c) = +\infty$, $\gamma(0) = 0$, and
\[
\gamma'(c) = \int_a^b \beta'(t, c) dt = \int_a^b \alpha(t) e^{\alpha(t)} dt > 0.
\]
Thus $\gamma$ is a diffeomorphism of $\mathbb{R}$ onto $(-(b-a), +\infty)$.

Let $\gamma^{-1} : (-(b-a), \infty) \to \mathbb{R}$ be a diffeomorphism inverse to $\gamma$. Then the following function satisfies (2′) and (3″):
\[
q_{a,b}(t, s) = \beta(t, \gamma^{-1}(s-(b-a))).
\]
Indeed, (2′) is obvious. Let us verify (3″):
\[
\int_a^b q_{a,b}(t, s) dt = \int_a^b \beta(t, \gamma^{-1}(s-(b-a))) dt = \gamma(\gamma^{-1}(s-(b-a))) = s-(b-a).
\]

**Proof.** Case $P = S^1$. Recall that the $n$-th configuration space of $S^1$ is the following subset
\[
\mathcal{F}_n(S^1) = \{ (x_1, \ldots, x_n) \in T^n \mid x_i \neq x_j \text{ for } i \neq j \}
\]
of $n$-dimensional torus $T^n = S^1 \times \cdots \times S^1$.

Let $F_n(S^1)$ be the connected component of $\mathcal{F}_n(S^1)$ containing the point $(1, \ldots, n)$. Denote also
\[
\Delta^{n-1} = \{ (x_2, \ldots, x_n) \in \mathbb{R}^{n-1} \mid 0 < x_2 < x_3 < \cdots < x_n < n \}.
\]
Then $\Delta^{n-1}$ is an open and convex subset of $\mathbb{R}^{n-1}$, whence it is diffeomorphic with $\mathbb{R}^{n-1}$.

**Lemma 6.1.2.** $F_n(S^1)$ is diffeomorphic with $\Delta^{n-1} \times S^1$.

**Proof.** Recall that we regard $S^1$ as $\mathbb{R}/n\mathbb{Z}$. Consider the following mapping $\xi : \mathcal{F}_n(S^1) \to \Delta^{n-1} \times S^1$ defined by
\[
\xi(x_1, \ldots, x_{n-1}, x_n) = (x_1 - x_n, \ldots, x_{n-1} - x_n, \lfloor x_n \rfloor),
\]
where the differences are taken modulo $n$, and $\lfloor x \rfloor$ means $x$ mod $n$. Since $x_n \neq x_{a'}$ for $a \neq a'$, it follows that $\xi$ is well defined. Notice also that $\xi(1, \ldots, n-1, n) = (1, \ldots, n-1, [n])$. Evidently, $\xi$ is smooth and admits a smooth right inverse $s : \Delta^{n-1} \times S^1 \to \mathcal{F}_n(S^1)$ defined by $s(t_1, \ldots, t_{n-1}, x) = (t_1 + x, \ldots, t_{n-1} + x, x)$. Since $\Delta^{n-1} \times S^1$ is connected, it follows that $\xi$ yields a diffeomorphism of the connected component of $\mathcal{F}_n(S^1)$ containing the point $(1, \ldots, n)$, i.e. $F_n(S^1)$, onto $\Delta^{n-1} \times S^1$. □

Let $e : D^+_S \to \mathcal{F}_n(S^1)$ be the evaluation map defined by
\[
e(\phi) = (\phi(1), \ldots, \phi(n)), \quad \phi \in D^+_S.
\]
Since $D^+_S$ is connected, it follows that the image $e(D^+_S)$ coincides with the connected component $F_n(S^1)$ of $\mathcal{F}_n(S^1)$ containing the point $e(\text{id}_S) = (1, \ldots, n)$.

\[
\int_a^b q_{a,b}(t, s) dt = \int_a^b \beta(t, \gamma^{-1}(s-(b-a))) dt = \gamma(\gamma^{-1}(s-(b-a))) = s-(b-a).
\]
Again $e$ is constant on the adjacent classes of $\mathcal{D}^+_S$ by $\mathcal{D}^c_S$ and yields a bijective continuous mapping $\tilde{e} : \mathcal{D}^+_S/\mathcal{D}^c_S \to F_n(S^1)$ such that

$$e = \tilde{e} \circ q : \mathcal{D}^+_S \xrightarrow{q} \mathcal{D}^+_S/\mathcal{D}^c_S \xrightarrow{\tilde{e}} F_n(S^1),$$

where $q$ is a factor-mapping. Thus in order to show that $\tilde{e}$ is a homeomorphism it suffices to prove that $\xi \circ \tilde{e} : \mathcal{D}^+_S/\mathcal{D}^c_S \to \Delta^{n-1} \times S^1$ is a homeomorphism in all Whitney topologies.

Consider the composition:

$$E = \xi \circ \tilde{e} \circ q : \mathcal{D}^+_S \xrightarrow{q} \mathcal{D}^+_S/\mathcal{D}^c_S \xrightarrow{\tilde{e}} F_n(S^1) \xrightarrow{\xi} \Delta^{n-1} \times S^1$$

As in the previous case it suffices to find a continuous section $s : \Delta^{n-1} \times S^1 \to \mathcal{D}^+_S$ of $E$.

Let $u : \Delta^{n-1} \times \mathbb{R} \to \mathbb{R}$ be the function constructed in Lemma [6.1] but for $n - 1$. Then $u$ maps the set $\Delta^{n-1} \times [0, n]$ onto $[0, n]$ so that $u(x, 0) = 0$ and $u(x, n) = n$ for all $x \in \Delta^{n-1}$. Hence $u$ yields a continuous mapping $\omega : \Delta^{n-1} \times S^1 \to S^1$ defined by factorization of $[0, n]$ onto $S^1$ by gluing the ends 0 and $n$.

Moreover, $u_1'(x, 0) = u_1'(x, n) = 1$ and $u_1(s)(x, 0) = u_1(s)(x, n) = 0$ for $s \geq 2$ and $x \in \Delta^{n-1}$. Hence, $\omega$ is in fact $C^\infty$. Then it is easy to verify that the following mapping $s : \Delta^{n-1} \times S^1 \to \mathcal{D}^+_S$ defined by:

$$(6.3) \quad s(x_1, \ldots, x_{n-1}, t) = \omega(x_1, \ldots, x_{n-1}, t) + t \mod n$$

is a continuous section of $E$. Theorem [6.1] is completed. □

6.2. Cyclic actions on $F_n(S^1)$. Consider the action of the group $\mathbb{Z}_n$ on $F_n(S^1)$ by cyclic shift $\nu : F_n(S^1) \to F_n(S^1)$ of coordinates:

$$\nu(x_1, \ldots, x_n) = (x_2, \ldots, x_n, x_1).$$

Evidently, this action is free and $\nu(F_n(S^1)) = F_n(S^1)$. Moreover, $\nu$ preserves orientation of $F_n(S^1)$ iff and only if $n$ is odd.

Suppose that $n = cd$ and let $\mathbb{Z}_c$ be a cyclic subgroup of order $c$ of $\mathbb{Z}_n$ generated by $\nu^d$.

Lemma 6.2.1. Suppose that $n$ is even and $d = n/c$ is odd, then $F_n(S^1)/\mathbb{Z}_c$ is diffeomorphic with a “twisted” product $\Delta^{n-1} \times S^1$. Otherwise, we have $F_n(S^1)/\mathbb{Z}_c \approx \Delta^{n-1} \times S^1$.

Proof. The proof is direct and based on the remark that $\nu^d$ reverses orientation if and only if $n$ is even and $d$ is odd. □

7. Exceptional values

Suppose that $f \in C^\infty(M, P)$ satisfies the condition (V). To each $g \in \mathcal{O}_{MP}(f)$ we will now correspond the set of its exceptional values and give the condition on $f$ when this correspondence is continuous.

If $P = \mathbb{R}$, then for every $g \in \mathcal{O}_{MR}(f)$ the ordered (by increasing) set of its exceptional values represents a point in

$$\Delta^{n-2} \equiv \{1\} \times \Delta^{n-2} \times \{n\} \subset \Delta^n,$$

since the minimal and maximal values 1 and $n$ are fixed with respect to $\mathcal{D}_{MR}$. Hence we have a well-defined mapping

$$k : \mathcal{O}_{MR}(f) \to \Delta^{n-2}.$$

Suppose that $P = S^1$. Then the exceptional values of $f$ are ordered only cyclically. Therefore the set of exceptional values of $g \in \mathcal{O}_{MS}(f)$ is a point $[g]$ in the factor space $F_n(S^1)/\mathbb{Z}_n$. Moreover, since we act by the connected group $\mathcal{D}^+_S$, we see that $[g]$ belongs to the connected component of $F_n(S^1)/\mathbb{Z}_n$ containing the class of
a set $1, \ldots, n$ of exceptional values of $f$. This connected component is $F_n(S^1)/\mathbb{Z}_n$, whence we get the following mapping:

$$k : \mathcal{O}_{MS^1}(f) \to F_n(S^1)/\mathbb{Z}_n.$$ 

By Lemma 6.2.1 $F_n(S^1)/\mathbb{Z}_n$ is $S^1 \times \Delta^{n-1}$ if $n$ is odd and $S^1 \tilde{\times} \Delta^{n-1}$ for even $n$.

We will now give a sufficient conditions for continuity of $k$ and show that without this condition $k$ can lose continuity even in $C^\infty$-topology of $\mathcal{O}_{MP}(f)$.

**Lemma 7.0.1.** Suppose that every critical level-set of $f$ includes either a connected component of $\partial M$ or an essential critical point of $f$. Then the mapping $k$ is continuous in $C^\infty$-topology of $\mathcal{O}_{MP}(f)$.

**Proof.** It suffices to prove continuity of $k$ at $f$. Choose some $\varepsilon \in (0, 1/3)$ and let $W = \bigcup_{i=1}^n (i-\varepsilon, i+\varepsilon)$ be a neighborhood of the set of exceptional values of $f$. We have to find a $C^\infty$-neighborhood $\mathcal{U}$ of $f$ in $C^\infty(M, P)$ such that for $k(\mathcal{U} \cap \mathcal{O}_{MP}(f)) \subset W$.

For every $i = 1, \ldots, n$ let $C_i = \Sigma_f \cap f^{-1}(i)$ be the set of critical points belonging to the $i$-th exceptional level-set of $f$. Evidently, $C_i \cap C_j = \emptyset$ for $i \neq j$. Also $C_i = \emptyset$, iff $i$ is a boundary but not a critical value of $f$.

We will assume that $M$ is given with some Riemannian metric. Then for every $g \in C^\infty(M, P)$ the norm $\|dg(x)\|$ of the differential of $g$ at every $x \in M$ is well defined. Notice that $\|df\| = 0$ on $C_i$. Hence there is a $\delta > 0$ and for every $i = 1, \ldots, n$ a compact neighborhood $V_i$ of $C_i$ such that

- $V_i = \emptyset$ provided $C_i = \emptyset$;
- $\|df\| > \delta$ on $M \setminus \bigcup_{i=1}^n V_i$, where $V = \bigcup_{i=1}^n V_i$, and
- $V_i \cap V_j = \emptyset$ for $i \neq j$.

Let $\Lambda$ be the set of critical values of $f$ that are not boundary ones. If $i \in \Lambda$, then by the assumptions there is an essential critical point $z_i \in C_i \subset V_i$. Hence for a neighborhood $V_i$ we can find a neighborhood $\mathcal{U}_i$ of $f$ in $C^\infty(M, P)$ such that every $g \in \mathcal{U}_i$ has a critical point in $V_i$.

Let also $\mathcal{U}_0$ be a $C^1$-neighborhood of $f$ in $C^\infty(M, P)$ consisting of smooth functions $g$ such that

1. $\|dg\| > \delta$ on $M \setminus \bigcup_{i=1}^n V_i$, and
2. $|f - g| < \varepsilon$.

Denote $\mathcal{U} = \mathcal{U}_0 \cap \left( \bigcap_{i \in \Lambda} \mathcal{U}_i \right)$. We claim that $k(\mathcal{U} \cap \mathcal{O}_{MP}(f)) \subset W$.

Indeed, suppose that $g \in \mathcal{U} \cap \mathcal{O}_{MP}(f)$. We have to show that the $i$-th exceptional value of $g$ differs from $i$ less than $\varepsilon$. Since $g$ has precisely $n$ exceptional values, it suffices to prove that every interval of the form $(i-\varepsilon, i+\varepsilon)$ for $i = 1, \ldots, n$ contains an exceptional value of $g$.

It follows from (i) that $g$ may have critical points only in $V_i$. If $i \in \Lambda$, then $g$ has a critical point in $V_i$ by the construction of $\mathcal{U}_i$.

Moreover, by (ii) the boundary values of $g$ on the connected components of $\partial M$ differ from the corresponding values of $f$ less than $\varepsilon$. Thus every interval $(i-\varepsilon, i+\varepsilon)$ for $i = 1, \ldots, n$ contains an exceptional value of $g$. Lemma is proved. \hfill \Box

### 7.1. Example of a function for which $k$ is not continuous

Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth non-decreasing function having precisely two flat critical points $a < b$, i.e. $f^{(r)}(a) = f^{(r)}(b) = 0$ for $r \geq 1$.

Let $U_a$ and $U_b$ be two disjoint $\varepsilon$-neighborhoods of $a$ and $b$ for some $\varepsilon > 0$, $U$ be a neighborhood of $[a, b]$, and $0 < \delta < \frac{1}{4}(f(b) - f(a))$.

Let $\mathcal{V}$ be a neighborhood of $f$ in $C^\infty(\mathbb{R})$ consisting of functions $g$ such that

$$\|g - f\|_\mathcal{V} = \sup_{x \in \mathcal{V}} \sum_{i=0}^n |g^{(r)}(x) - f^{(r)}(x)| < \delta.$$
It is easy to see that \( V \) contains a function \( g = \phi \circ f \circ h^{-1} \in \mathcal{O}_{MR}(f) \), where \( \phi \) and \( h \) are diffeomorphisms of \( \mathbb{R} \) such that \( h(a) \in U_b, h(b) = b, g \circ h(a) = \phi \circ f(a) \in (f(b) - \delta, f(b) + \delta) \), and \( g \circ h(b) = \phi \circ f(b) = f(b) \).

The construction of \( g \) is illustrated on the right diagram of Figure 4. The idea is that a critical point \( a \) can be eliminated by arbitrary small perturbation of \( f \) in any \( C^r \)-topology. On the other hand, we can “create” a critical point equivalent to \( a \) in arbitrary small neighborhood of \( b \). All this can be produced by conjugating \( f \) so that \( ||g - f||_U < \delta \).

Thus both critical values \( g \circ h(a) \) and \( g \circ h(b) \) of \( g \) belong to \( U_b \), i.e. the distance \( \mathbb{R}^2 \) between points \((g \circ h(a), g \circ h(b))\) and \((f(a), f(b))\) is greater that \( \delta \).

Since this holds for arbitrary small \( \delta \), we obtain that \( k \) is not continuous in \( C^r \)-topology for arbitrary \( r \geq 0 \), whence it is not continuous in \( C^\infty \)-topology.

8. PROOF OF THEOREM 1.5

Suppose that \( f \in C^\infty(M, P) \) satisfies the conditions (J) and (V), and every critical level-set of \( f \) either has an essential critical point or includes a connected component of \( \partial M \). By Lemma 7.0.1 the latter assumption implies that the mapping \( k \) corresponding to each \( g \in \mathcal{O}_{MP}(f) \) its ordered set of exceptional values is continuous.

8.1. CASE \( \mathbb{R} \). Notice that we have the following commutative diagram:

\[
\begin{array}{cccc}
S_M(f) & \longrightarrow & D_M & \longrightarrow & D_M/S_M(f) & \longrightarrow & \mathcal{O}_M(f) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
S_{MR}(f) & \longrightarrow & D_{MR} & \longrightarrow & D_{MR}/S_{MR}(f) & \longrightarrow & \mathcal{O}_{MR}(f) \\
\downarrow p & & \downarrow & & \downarrow & & \downarrow k \\
D_R & \longrightarrow & D_R^{[1,n]} & \longrightarrow & D_R^{[1,n]}/D_R^c & \longrightarrow & \Delta^{n-2} \\
\end{array}
\]

in which \( c \) is a homeomorphism and the other right horizontal arrows are continuous bijections. Moreover, every upper vertical arrow is an embedding and every lower vertical one is a mapping onto.

The following lemma implies Theorem 1.5 for the case \( P = \mathbb{R} \).

**Lemma 8.1.1.** Suppose \( f \in C^\infty(M, \mathbb{R}) \) satisfies (V). Suppose also that

1. the mapping \( k : \mathcal{O}_{MR}(f) \to \Delta^{n-2} \) is continuous and there exist
2. a section \( s : \Delta^{n-2} \to D_R^{[1,n]} \) of \( c \circ q \) (see Lemma 6.1.1), and
3. a section of \( p \), i.e. a homomorphism \( \theta : D_R \to D_M \) such that \( \phi \circ f = f \circ \theta(\phi) \) (see Lemma 5.0.1).

Then there is a homeomorphism \( \mathcal{O}_{MR}(f) \approx \mathcal{O}_M(f) \times \Delta^{n-2} \).
Proof. Consider the following composition:

\[ \text{sk} = s \circ k : \mathcal{O}_{\text{MR}}(f) \xrightarrow{k} \Delta^{n-2} \xrightarrow{s} \mathcal{D}^{[1,n]}_k. \]

Then each \( g \in \mathcal{O}_{\text{MR}}(f) \) can be represented in the form

\[ g = \text{sk}(g) \circ (\text{sk}(g)^{-1} \circ g). \]

Claim 8.1.2. \( \text{sk}(g)^{-1} \circ g \in \mathcal{O}_M(f) \).

Proof. Since \( g \in \mathcal{O}_{\text{MR}}(f) \), it can be represented in the form \( g = \phi \circ f \circ h^{-1} \), where \( \phi \in \mathcal{D}^{[1,n]}_g \) and \( h \in \mathcal{D}_M \).

Notice that \( \text{sk}(g)^{-1} \circ \phi \in \mathcal{D}^*_R \), whence by Proposition 5.0.2

\[ \text{sk}(g)^{-1} \circ g = (\text{sk}(g)^{-1} \circ \phi) \circ f \circ h^{-1} = f \circ \theta \circ [\text{sk}(g)^{-1} \circ \phi] \circ h^{-1} \in \mathcal{O}_M(f). \]

Now it is easy to see that the following two mappings are continuous and mutually inverse:

\[ \beta : \mathcal{O}_{\text{MR}}(f) \to \mathcal{O}_M(f) \times \Delta^{n-2}, \quad \beta(g) = (\text{sk}(g)^{-1} \circ g, k(g)), \]

\[ \alpha : \mathcal{O}_M(f) \times \Delta^{n-2} \to \mathcal{O}_{\text{MR}}(f), \quad \alpha(\psi, z) = s(z) \circ \psi, \]

where \( g \in \mathcal{O}_M(f) \), \( \psi \in \mathcal{O}_M(f) \) and \( z \in \Delta^{n-2} \).

\[ \square \]

8.2. Case \( P = S^1, \ n = 0 \). In this case \( f \) is a locally trivial fibration over \( S^1 \). Evidently, \( \mathcal{O}_M(f) \subset \mathcal{O}_{\text{MS}^1}(f) \). Conversely, suppose that \( g = \phi \circ f \circ h^{-1} \in \mathcal{O}_{\text{MS}^1}(f) \) for \( \phi \in \mathcal{D}^*_{S^1} \) and \( h \in \mathcal{D}_M \). Then by Proposition 5.0.3 for \( \phi \) there exists a diffeomorphism \( h_1 \) such that \( \phi \circ f = f \circ h_1 \). Whence \( g = f \circ h_1 \circ h^{-1} \in \mathcal{O}_M(f) \). Thus \( \mathcal{O}_M(f) = \mathcal{O}_{\text{MS}^1}(f) \).

8.3. Case \( P = S^1, \ n \geq 1 \). Define the mapping \( \tau : \mathcal{D}_{\text{MS}^1} \to \mathcal{O}_{\text{MS}^1}(f) \times F_n(S^1) \) by

\[ \tau(h, \phi) = (\phi \circ f \circ h^{-1}; \phi(1), \ldots, \phi(n)). \]

Evidently, \( \tau \) is continuous. Designate its image by \( \tilde{\mathcal{O}}_{\text{MS}^1}(f) \):

\[ \tilde{\mathcal{O}}_{\text{MS}^1}(f) := \{ \tau(D_{\text{MS}^1}) \subset \mathcal{O}_{\text{MS}^1}(f) \times F_n(S^1) \}. \]

Let also \( p_1 : \tilde{\mathcal{O}}_{\text{MS}^1}(f) \to \mathcal{O}_{\text{MS}^1}(f) \) be the restrictions of the standard projection \( \mathcal{O}_{\text{MS}^1}(f) \times F_n(S^1) \to \mathcal{O}_{\text{MS}^1}(f) \) to \( \mathcal{O}_{\text{MS}^1}(f) \).

Recall that \( \mathbb{Z}_n \) acts on \( F_n(S^1) \) by cyclic shift \( \nu \) of coordinates:

\[ \nu \cdot \{ x_a \} = \{ x_{a+1} \}, \]

where for simplicity a point \( \{ x_1, \ldots, x_n \} \in F_n(S^1) \) is designated by \( \{ x_a \} \). This action together with the trivial action on \( \mathcal{O}_{\text{MS}^1}(f) \) yields an action of \( \mathbb{Z}_n \) on \( \mathcal{O}_{\text{MS}^1}(f) \times F_n(S^1) \).

By Theorem 1.9 \((p(S_{\text{MS}^1}(f)))/\mathcal{D}^*_{S^1} \) is a cyclic group of some order \( c \) dividing \( n \). Denote \( d = n/c \) and define \( \tilde{\phi} \in \mathcal{D}^*_{S^1} \) by

\[ \tilde{\phi}(z) = z + d \mod n. \]

Let \( \mathbb{Z}_c \) be a subgroup of \( \mathbb{Z}_n \) generated by \( \nu^d \).

Lemma 8.3.1. \( \tilde{\mathcal{O}}_{\text{MS}^1}(f) \) is invariant under \( \nu^d \) and \( p_1 \) yields a continuous bijection \( \mu : \tilde{\mathcal{O}}_{\text{MS}^1}(f)/\mathbb{Z}_c \to \mathcal{O}_{\text{MS}^1}(f) \). If \( k \) is continuous, then \( p_1 \) is a \( k \)-sheet covering and \( \mu \) is a homeomorphism.

Proof. First we show that \( \tilde{\phi} \in p(S_{\text{MS}^1}(f)), \) i.e. \( (\tilde{h}, \tilde{\phi}) \in S_{\text{MS}^1}(f) \) for some \( \tilde{h} \in \mathcal{D}_M \).

Indeed, since \( p(S_{\text{MS}^1}(f))/\mathcal{D}^*_{S^1} \approx \mathbb{Z}_n \), there is \( (h_1, \phi_1) \in S_{\text{MS}^1}(f) \) such that \( \phi_1(a) = a + d \) for all \( a = 1, \ldots, n \). Then \( \tilde{\phi}^{-1} \circ \phi_1 \in \mathcal{D}^*_{S^1} \), whence

\[ f = \tilde{\phi}^{-1} \circ \phi_1 \circ f \circ h_1^{-1} = \tilde{\phi} \circ f \circ (\tilde{\phi}^{-1} \circ \phi_1) \circ h_1^{-1}. \]
Suppose that \((g, \{x_a\}) \in \tilde{O}_{MS}^1(f)\), i.e. \(g = \phi \circ f \circ h^{-1}\) and \(\phi(a) = x_a\) for \(a = 1, \ldots, n\). We have to show that \(\nu^2 \cdot (g, \{x_a\}) = (g, \{x_{a+d}\})\) also belongs to \(\tilde{O}_{MS}^1(f)\). Then
\[
g = \phi \circ f \circ h^{-1} = \phi \circ \tilde{\phi} \circ \tilde{h}^{-1} \circ h^{-1}.
\]
Moreover, \(\phi \circ \tilde{\phi}(a) = \phi(d + a) = x_{d+a}\). Thus putting \(\tilde{\phi} = \phi \circ \tilde{\phi}\) and \(\tilde{h} = h \circ h\) we see that \(\tau(\tilde{h}, \tilde{\phi}) = (g, \{x_{d+a}\})\), i.e. \((g, \{x_{d+a}\})\) belongs to the image \(\tilde{O}_{MS}^1(f)\) of \(\tau\). Thus \(\tilde{O}_{MS}^1(f)\) is invariant under \(\mathbb{Z}_c\).

Since \(\mathbb{Z}_c\) is finite and its action is free and \(p_1\)-equivariant, it follows that the factor mapping \(O_{MS}^1(f) \to \tilde{O}_{MS}^1(f)/\mathbb{Z}_c\) is a \(c\)-sheet covering. Moreover, since \(p_1\) is onto, we obtain that \(p_1\) yields a continuous bijection \(\tilde{\mu} : \tilde{O}_{MS}^1(f)/\mathbb{Z}_c \to O_{MS}^1(f)\).

Finally, suppose that \(k\) is continuous. We will show that \(p_1\) is a local homeomorphism. This will imply that so is \(\mu\), whence \(\mu\) is in fact a homeomorphism. It suffices to prove that \(p_1\) admits continuous local sections, i.e. for every \((g, x) \in \tilde{O}_{MS}^1(f)\) there exists a neighborhood \(U_g\) of \(g\) in \(O_{MS}^1(f)\) and a continuous mapping \(G : U_g \to F_n(S^1)\) such that \((g', G(g')) \in \tilde{O}_{MS}^1(f)\) and \(G(g) = x\).

Notice that we have the following maps:
\[
F_n(S^1) \xrightarrow{\nu} F_n(S^1)/\mathbb{Z}_n \xrightarrow{k} O_{MS}^1(f).
\]

Let \((g, x) \in \tilde{O}_{MS}^1(f)\) and \([x] = \nu(x)\) be the class of \(x\) in \(F_n(S^1)/\mathbb{Z}_n\). Then \(k(g) = [x]\). Since \(\nu\) is a covering, there is a neighborhood \(U_x\) of \(x\) in \(F_n(S^1)\) and a neighborhood \(V[x]\) of \([x]\) in \(F_n(S^1)/\mathbb{Z}_n\) such that \(\nu\) homeomorphically maps \(U_x\) onto \(V[x]\). Let \(U_g = k^{-1}(V[x])\). Since \(k\) is continuous, we obtain that \(U_g\) is an open neighborhood of \(g\). Then the mapping \(\nu^{-1} \circ k : U_g \to U_x\) is a local section of \(p_1\). \(\square\)

Consider now the projection \(k_2 : \tilde{O}_{MS}^1(f) \subset O_{MS}^1(f) \times F_n(S^1) \to F_n(S^1)\). Since the composition
\[
k_2 \circ \tau : D_{MS} \xrightarrow{\tau} O_{MS}^1(f) \times F_n(S^1) \xrightarrow{k_2} F_n(S^1),
\]
is given by \((h, \phi) \mapsto (\phi(1), \ldots, \phi(n-1))\), we get the following commutative diagram:
\[
\begin{array}{cccc}
S_M(f) & \longrightarrow & D_M & \longrightarrow & D_M/S_M(f) & \longrightarrow & O_M(f) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\tilde{S}_{MS}^1(f) & \longrightarrow & D_{MS} & \longrightarrow & D_{MS}/\tilde{S}_{MS}^1(f) & \longrightarrow & \tilde{O}_{MS}^1(f) \\
\downarrow p & & \downarrow & & \downarrow k_2 & & \downarrow c \\
D^+_{S^1} & \longrightarrow & D^+_{S^1} & \xrightarrow{q} & D^+_{S^1}/D^+_{S^1} & \xrightarrow{c} & F_n(S^1)
\end{array}
\]
in which \(c\) is a homeomorphism and the other right horizontal arrows are continuous bijections, upper vertical and left horizontal arrows are embeddings, and lower vertical ones are surjective mappings.

Notice that

1. \(k_2\) is continuous,
2. \(c \circ q\) admit a continuous section \(s\) (Lemma 6.0.1), and
3. \(p\) admit a continuous section \(\Theta\) being a homeomorphism (Theorem 1.3).

Then by the arguments similar to the proof of Lemma 8.1.1, it follows that the embedding \(O_M(f) \equiv O_M(f) \times (1, \ldots, n) \subset \tilde{O}_{MS}^1(f)\) extends to a homeomorphism
\[
\alpha : O_M(f) \times F_n(S^1) \approx \tilde{O}_{MS}^1(f), \quad \alpha(g, x) = (s(x) \circ g, x),
\]
where \(g \in O_M(f)\) and \(x \in F_n(S^1)\).
Finally, by Lemma 8.3.1 we have that $\mathcal{O}_{MS^1}(f) = \tilde{\mathcal{O}}_{MS^1}(f)/\mathbb{Z}_c$. Since $\mathbb{Z}_c$ trivially acts on $\mathcal{O}_M(f)$, we obtain a homeomorphism:

$$\mathcal{O}_{MS^1}(f) \approx \mathcal{O}_M(f) \times (F_n(S^1)/\mathbb{Z}_c).$$

Then it remains to apply Lemma 6.2.1. Theorem 1.5 is completed. □

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