Conclusive Discrimination by $N$ Sequential Receivers between $r \geq 2$ Arbitrary Quantum States

E. R. Loubenets*,**,*** and M. Namkung***

*National Research University Higher School of Economics, Moscow 101000, Russia
**Steklov Mathematical Institute of Russian Academy of Sciences, Moscow 119991, Russia
***Center for Quantum Information, Korea Institute of Science and Technology (KIST), Seoul 02792, South Korea

Received March 15, 2023; Revised April 27, 2023; Accepted May 5, 2023

Abstract. In the present paper, we develop a general mathematical framework for discrimination between $r \geq 2$ quantum states by $N \geq 1$ sequential receivers for the case in which every receiver obtains a conclusive result. This type of discrimination constitutes an $N$-sequential extension of the minimum-error discrimination by one receiver. The developed general framework, which is valid for a conclusive discrimination between any number $r \geq 2$ of quantum states, pure or mixed, of an arbitrary dimension and any number $N \geq 1$ of sequential receivers, is based on the notion of a quantum state instrument, and this allows us to derive new important general results. In particular, we find a general condition on $r \geq 2$ quantum states under which, within the strategy in which all types of receivers’ quantum measurements are allowed, the optimal success probability of the $N$-sequential conclusive discrimination between these $r \geq 2$ states is equal to that of the first receiver for any number $N \geq 2$ of further sequential receivers and specify the corresponding optimal protocol. Furthermore, we extend our general framework to include an $N$-sequential conclusive discrimination between $r \geq 2$ arbitrary quantum states under a noisy communication. As an example, we analyze analytically and numerically a two-sequential conclusive discrimination between two qubit states via depolarizing quantum channels. The derived new general results are important both from the theoretical point of view and for the development of a successful multipartite quantum communication via noisy quantum channels.

DOI: 10.1134/S1061920823020085

1. INTRODUCTION

Constructing quantum measurements for distinguishing between quantum states with an optimal value of a figure of merit is one of the key problems of quantum information processing, which is referred to as a quantum state discrimination [1, 2]. For a quantum state discrimination, various strategies based on the constraints imposed by the quantum measurement theory can be proposed.

In 2013, J.A. Bergou [3] firstly devised the sequential unambiguous state discrimination for which a receiver discriminates a sender’s quantum state and a further receiver discriminates posterior states from the previous receiver. However, in [3], every receiver performs the so called unambiguous state discrimination for which every receiver’s conclusive result is always confident, even though this receiver can obtain the inconclusive result with a nonzero probability, and, until now, there have been numerous theoretical and experimental developments [4–11] on only this type of sequential state discrimination.

Unfortunately, the unambiguous state discrimination can be performed if only (i) in the case of pure initial states, they are linearly independent; and (ii) in the case of mixed initial states, a support space spanned by eigenvectors of nonzero eigenvalue of one mixed state is not identical to that of any other mixed state [12]. Therefore, another strategy for a sequential state discrimination should be devised.

Recently, another scenario for a sequential state discrimination has been proposed [13], where every receiver’s quantum measurement is performed on a quantum system in the conditional posterior state after a measurement of the previous receiver and outputs only a conclusive result. The latter means that receivers’ quantum measurements are designed to remove a possibility for every receiver to obtain a result “I don’t know which quantum state was prepared.” In view of this, we call this discrimination scenario a sequential conclusive quantum state discrimination. However, the analytical and numerical results derived in [13] refer only to the $N$-sequential conclusive discrimination between two pure qubit states under receivers’ generalized quantum measurements with some specific conditional posterior states.1

1For the constraint used in [13] on receivers’ quantum measurements, see Section 2 of that paper.
In the present paper, we develop a general mathematical framework for the description of an \( N \)-sequential conclusive discrimination between \( r \times 2 \) arbitrary quantum states, pure or mixed, in the case where both beforehand and during a sequential discrimination a communication between receivers via a separate classical channel and via an extra quantum channel (by encoding the information on their outcomes into orthogonal quantum states) is prohibited [3]. This type of quantum state discrimination constitutes an \( N \)-sequential extension of the minimum-error discrimination by one receiver. The developed general framework incorporates the description of \( N \)-sequential discrimination considered in [13] only as a particular case.

Mathematically, the new general framework for an \( N \)-sequential conclusive state discrimination is based on the notion of a quantum state instrument describing a consecutive measurement of \( N \) receivers, and this allows us to derive three mutually equivalent representations for the success probability via correspondingly: (i) the quantum state instrument under a consecutive measurement by \( N \) receivers; (ii) the POV (positive operator-valued) measures of all \( N \) receivers, and conditional posterior states after every sequential measurement; (iii) the product of the success probabilities of all \( N \) receivers and, also, to present a general upper bound for the optimal success probability expressed explicitly in terms of \( r \times 2 \) quantum states and their a priori probabilities.

Based on this, we find a new general condition on \( r \times 2 \) quantum states sufficient for the optimal success probability of an \( N \)-sequential conclusive discrimination between these \( r \times 2 \) states to be equal to the optimal success probability of the first receiver and specify the corresponding optimal protocol for this case.

We show that, for an \( N \)-sequential conclusive discrimination between two \( (r = 2) \) arbitrary quantum states, this general condition is always fulfilled; therefore, in this case, the optimal success probability is given by the Helstrom bound for any number \( N \) of sequential receivers, and specify for this case the optimal protocol which is general in the sense that it is true for any two quantum states, pure or mixed, of an arbitrary dimension, and also present the specific optimal protocols in case of discrimination between two states satisfying some extra condition. For the general optimal protocol, we construct its implementation via indirect measurements.

Furthermore, we extend our general framework to include an \( N \)-sequential conclusive state discrimination under communication via noisy quantum channels. As an example, we apply the developed formalism for the analysis of the two-sequential conclusive discrimination between two qubit states in the presence of a depolarizing noise.

The developed general mathematical framework is true for any number \( N \) of sequential receivers, any number \( r \times 2 \) of arbitrary quantum states, pure or mixed, any type of receivers’ quantum measurements and for arbitrary noisy quantum communication channels.

The new general results derived in the present paper are important both from the theoretical point of view and for the development of a successful multipartite quantum communication via noisy channels.

The present paper is organized as follows.

In Section 2, we present the main issues of the quantum measurement theory, specify the notions of a quantum state instrument and its statistical realizations.

In Section 3, we develop a new general framework for the description of an \( N \)-sequential conclusive discrimination between \( r \times 2 \) arbitrary quantum states prepared with any a priori probabilities.

In Section 4, we specify the optimal success probability under a definite strategy for an \( N \)-sequential conclusive state discrimination and establish a new upper bound on this probability. For the strategy for which all possible receivers’ quantum measurements are allowed, we find a new general condition on \( r \times 2 \) quantum states sufficient for the optimal success probability of the \( N \)-sequential conclusive state discrimination to be equal to the optimal success probability of the first receiver for any number \( N \times 2 \) of sequential receivers and specify for this case the corresponding optimal protocol. We show that, in case \( r = 2 \), this general condition is fulfilled for any two quantum states, possibly infinite dimensional, therefore, within the strategy where all possible receivers’ quantum measurements are allowed, the optimal success probability for the \( N \)-sequential conclusive state discrimination between two arbitrary quantum states is given by the Helstrom bound for any number \( N \times 2 \) of sequential receivers, and we specify the corresponding optimal protocols for this case.

In Section 5, we develop a general framework for an \( N \)-sequential conclusive discrimination between \( r \times 2 \) arbitrary quantum states via noisy communication channels.

In Section 6, we analyze analytically and numerically a two-sequential conclusive discrimination between two qubit states via depolarizing quantum channels.

In Section 7, we summarize the main results.
2. PRELIMINARIES

For the description of a general scenario for an $N$-sequential conclusive state discrimination in Sections 3 and 4, we briefly recall the main notions of the quantum measurement theory [2, 14, 15]. Let a measurement with outcomes $\omega$ in a finite set $\Omega$ be performed on a quantum system described in terms of a complex Hilbert space $\mathcal{H}$. Denote by $\mathcal{L}(\mathcal{H})$ the vector space of all bounded linear operators on $\mathcal{H}$ and by $\mathcal{T}(\mathcal{H}) \subseteq \mathcal{L}(\mathcal{H})$ the vector space of all trace class operators on $\mathcal{H}$.

The complete description of every quantum measurement, projective or generalized, includes the knowledge of both the statistics of observed outcomes and a family of posterior states, each conditioned by an outcome observed under a single trial of this measurement. In mathematical terms, the complete description of a quantum measurement is specified [2, 14–16] by the notion of a quantum state instrument $\mathcal{M}(\cdot)$ with values $\mathcal{M}(B) = \sum_{\omega \in B} \mathcal{M}(\omega)$, $B \subseteq \Omega$, that are completely positive bounded linear mappings $\mathcal{M}(\omega)[\cdot] : \mathcal{T}(\mathcal{H}) \to \mathcal{T}(\mathcal{H})$, satisfying the relation

$$\sum_{\omega \in \Omega} \text{tr}\{\mathcal{M}(\omega)[T]\} = \text{tr}\{\mathcal{M}(\Omega)[T]\} = \text{tr}\{T\}, \quad T \in \mathcal{T}(\mathcal{H}).$$

(1)

To every quantum state instrument $\mathcal{M}$, there corresponds a unique observable instrument $\mathcal{N}(\cdot)$ with values $\mathcal{N}(B) = \sum_{\omega \in B} \mathcal{N}(\omega)$, $B \subseteq \Omega$, that are normal completely positive bounded linear mappings $\mathcal{N}(\omega)[\cdot] : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$, $\mathcal{N}(\omega)[I_\mathcal{H}] = I_\mathcal{H}$, defined for a state instrument $\mathcal{M}$ via the duality relation

$$\text{tr}\{\mathcal{M}(B)[T]Y\} = \text{tr}\{T\mathcal{N}(B)[Y]\}, \quad B \subseteq \Omega, \quad T \in \mathcal{T}(\mathcal{H}), \ Y \in \mathcal{L}(\mathcal{H}),$$

(2)

and the values $\mathcal{M}(B) = \sum_{\omega \in B} \mathcal{M}(\omega)$, $B \subseteq \Omega$, of the POV measure $\mathcal{M}$, satisfying the relation $\mathcal{M}(\Omega) = I_\mathcal{H}$ and describing the statistics of this quantum measurement, are given by

$$\mathcal{M}(B) = \mathcal{N}(B)[I_\mathcal{H}], \quad B \subseteq \Omega.$$  

(3)

Given a state instrument $\mathcal{M}$ describing a quantum measurement and a quantum system state $\rho$ (density operator) on $\mathcal{H}$ before this measurement, the probability to observe under this measurement an outcome $\omega$ in a subset $B \subseteq \Omega$ has the form

$$\mu(B \mid \rho) = \text{tr}\{\mathcal{M}(B)[\rho]\} = \text{tr}\{\rho\mathcal{N}(B)[I_\mathcal{H}]\} = \text{tr}\{\rho\mathcal{M}(B)\}. $$

(4)

After a single measurement trial in which an outcome $\omega \in \Omega$ is observed, the state of a quantum system is given by the relation [16–18]

$$\rho_{\text{out}}(\omega \mid \rho) = \frac{\mathcal{M}(\omega)[\rho]}{\mu(\omega \mid \rho)},$$

(5)

and is referred to as the posterior state conditioned by an observed outcome $\omega \in \Omega$, or the conditional posterior state. The unconditional posterior state is defined by $\rho_{\text{out}}(\rho) := \rho_{\text{out}}(\Omega \mid \rho) = \mathcal{M}(\Omega)[\rho]$.

Every state instrument $\mathcal{M}(\cdot)[\cdot]$ admits the Stinespring–Kraus representation (for details, see [16–20])

$$\mathcal{M}(\omega)[T] = \sum_{l} K_l(\omega)TK_l^\dagger(\omega), \quad T \in \mathcal{T}(\mathcal{H}), \ \omega \in \Omega,$$

(6)

which may be not unique. Here $K_l(\omega) \in \mathcal{L}(\mathcal{H})$, $\omega \in \Omega$, $l \in \{1, \ldots, L\}$, are bounded linear operators, with the operator norms $\|K_l(\omega)\| \leq 1$, the so-called Kraus operators [21], that satisfy the relation

$$\sum_{\omega, l} K_l^\dagger(\omega)K_l(\omega) = I_\mathcal{H}.$$  

(7)

It follows from (2) and (7) that, in terms of Kraus operators, for every $\omega \in \Omega$,

$$\mathcal{M}(\omega) = \sum_{l} K_l(\omega)^\dagger K_l(\omega),$$

(8)

$$\mu(\omega \mid \rho) = \sum_{l} \text{tr}\left\{\rho K_l^\dagger(\omega)K_l(\omega)\right\},$$

(9)

$$\rho_{\text{out}}(\omega \mid \rho) = \frac{\sum_{l} K_l(\omega)\rho K_l^\dagger(\omega)}{\mu(\omega \mid \rho)}.$$  

(10)
If, in representation (6), a set \(\{1, \ldots, L\}\) contains only one element, i.e., the Kraus operators are labelled only by outcomes \(\omega \in \Omega\), then the state instrument \(\mathcal{M}\) is said to be pure since, in this case, the mapping \(\mathcal{M}(\omega)[\cdot]\) “transforms” a pure initial state \(|\psi\rangle\langle\psi|\) to the pure conditional posterior state

\[
\rho_{\text{out}}(\omega | |\psi\rangle\langle\psi|) = \frac{K(\omega)|\psi\rangle\langle\psi|K^\dagger(\omega)}{\langle\psi|K^\dagger(\omega)K(\omega)|\psi\rangle}, \quad \omega \in \Omega.
\]  

As proved by Ozawa [22], for every observable instrument \(\mathcal{N}\) describing a generalized quantum measurement, there exists a statistical realization

\[
\Xi := \{\tilde{\mathcal{H}}, \sigma, P, U\},
\]  

possibly not unique, consisting of four elements: a complex Hilbert space \(\tilde{\mathcal{H}}\), a density operator \(\sigma\) on \(\tilde{\mathcal{H}}\), a projection-valued measure \(P\) on \(\Omega\) with values \(P(\omega), \omega \in \Omega\), that are projections on \(\mathcal{H}\), and a unitary operator \(U\) on \(\mathcal{H} \otimes \tilde{\mathcal{H}}\) such that, for all \(Y \in \mathcal{L}(\tilde{\mathcal{H}})\),

\[
\mathcal{N}(\omega)[Y] = \text{tr}_{\tilde{\mathcal{H}}} \left\{ (I_\mathcal{H} \otimes \sigma) U^\dagger(Y \otimes P(\omega))U \right\}, \quad \omega \in \Omega,
\]  

where the record \(\text{tr}_{\tilde{\mathcal{H}}}[\cdot]\) means the partial trace over the Hilbert space \(\tilde{\mathcal{H}}\). It follows from relations (2) and (13) that, in terms of the elements of a statistical realization (12), the values \(\mathcal{M}(\omega)[\cdot]\) of a quantum state instrument are given by

\[
\mathcal{M}(\omega)[\rho] = \text{tr}_{\tilde{\mathcal{H}}} \left\{ (I_\mathcal{H} \otimes P(\omega)) U(\rho \otimes \sigma)U^\dagger (I_\mathcal{H} \otimes P(\omega)) \right\}, \quad \omega \in \Omega,
\]  

for all density operators \(\rho\) on \(\mathcal{H}\).

**Remark 1.** The existence, for every generalized quantum measurement, of a statistical realization \(\Xi = \{\tilde{\mathcal{H}}, \sigma, P, U\}\) means that every generalized quantum measurement on a state \(\rho\) on \(\mathcal{H}\) can be realized via the indirect measurement specified by the elements of this statistical realization \(\Xi\). Namely, via a direct measurement \(P(\cdot)\) on some auxiliary quantum system, being initially in a state \(\sigma\) on a Hilbert space \(\tilde{\mathcal{H}}\), after its interaction with the original system which results in the composite system state \(U(\rho \otimes \sigma)U^\dagger\) on \(\mathcal{H} \otimes \tilde{\mathcal{H}}\).

The representations (6) and (14) imply that, if there exists a statistical realization \(\Xi\) for a state instrument \(\mathcal{M}\), where a state \(\sigma = |b\rangle\langle b|\) is pure, while the values of a projection-valued measure \(P\) have the form \(P(\omega) = |\xi_\omega\rangle\langle\xi_\omega|\), where \(\{|\xi_\omega\rangle, \omega \in \Omega\}\) is an orthonormal basis of \(\tilde{\mathcal{H}}\), then, for this state instrument \(\mathcal{M}\), there is a representation (6) for which the Kraus operators are labelled only by an outcome \(\omega \in \Omega\) and are defined by the relations (see Lemma 1 in [17])

\[
U(|f\rangle\langle g|) = (f, K(\omega)g), \quad \text{for all } f, g \in \mathcal{H}.
\]  

In the physical notation,

\[
K(\omega) = |\xi_\omega, U\rangle b\rangle.
\]  

3. **N-SEQUENTIAL CONCLUSIVE DISCRIMINATION BETWEEN \(r \geq 2\) QUANTUM STATES**

For an \(N\)-sequential conclusive quantum state discrimination, introduce a general scenario where a sender, say, Alice, prepares a quantum system, described in terms of a complex Hilbert space \(\mathcal{H}\), possibly infinite-dimensional, in one of states \(\rho_1, \ldots, \rho_r\), \(r \geq 2\), pure or mixed, with a priori probabilities \(q_1, \ldots, q_r\), and sends her quantum system in the initial state

\[
\rho_m = \sum_{j=1}^{r} q_j \rho_j, \quad \sum_{j} q_j = 1, \quad q_j > 0,
\]  

through a quantum channel to a chain of \(N\) receivers. On receiving a quantum system from Alice, the first receiver is allowed to perform a conclusive measurement on this system for the discrimination between states \(\{\rho_1, \ldots, \rho_r\}\) while a quantum system of Alice in the posterior state, conditioned by an outcome \(j_1 \in \{1, \ldots, r\}\)
observed by the first receiver, is further transmitted via a quantum channel to a sequential receiver. This procedure is repeated until a conclusive measurement by an N-th receiver occurs.

We assume that, both beforehand and during a sequential discrimination, a communication between receivers via a separate classical channel is prohibited [3], the same concerns also the communication between receivers via an extra quantum channel (by encoding the information on their outcomes into orthogonal quantum states) since the permission would allow one the trivial strategy where the first receiver performs an optimal measurement and communicates the information on his outcome to sequential receivers.

Under the above constraints, the N-sequential conclusive state discrimination scenario described below can provide a practical multipartite quantum communication protocol.

Denote by $M^{(r)}_n(\cdot), n = 1, \ldots, N$, a quantum state instrument describing a conclusive quantum measurement of an n-th sequential receiver with outcomes $j_n$ in the set $\{1, \ldots, r\}$. The quantum observable instrument $N^{(r)}_n(\cdot)$ and the POV measure $M^{(r)}_n(\cdot)$ corresponding to $M^{(r)}_n(\cdot)$ are specified by (1) and (3) in Section 2.

Let the quantum channels between sequential receivers be ideal. The sequence $A|\rightarrow 1 \rightarrow \cdots \rightarrow k$ of quantum measurements performed by $k \in \{1, \ldots, N\}$ sequential receivers, each with an outcome $j_k \in \{1, \ldots, r\}$, constitutes a consecutive measurement with an outcome $\omega = (j_1, \ldots, j_k) \in \{1, \ldots, r\}^k$ on the Alice quantum system in an initial state (17) and is described by the quantum state instrument $M^{(r)}_{A|\rightarrow 1 \rightarrow \cdots \rightarrow k}$ with the values [16]

$$M^{(r)}_{A|\rightarrow 1 \rightarrow \cdots \rightarrow k}(j_1, \ldots, j_k | \cdot) := M^{(r)}_k(j_k) \left[ M^{(r)}_{k-1}(j_{k-1}) \left[ \cdots M^{(r)}_1(j_1) [ \cdot \cdots ] \right] \right], \quad (j_1, \ldots, j_k) \in \{1, \ldots, r\}^k. \quad (18)$$

By (1) and (2), the corresponding quantum observable instrument $N^{(r)}_{A|\rightarrow 1 \rightarrow \cdots \rightarrow k}$ and the POV measure $M^{(r)}_{A|\rightarrow 1 \rightarrow \cdots \rightarrow k}$ of $A|\rightarrow 1 \rightarrow \cdots \rightarrow k$ consecutive measurement, described by the state instrument (18), have the forms

$$N^{(r)}_{A|\rightarrow 1 \rightarrow \cdots \rightarrow k}(j_1, \ldots, j_k | \cdot) = N^{(r)}_k(j_k) \left[ N^{(r)}_{k-1}(j_{k-1}) \left[ \cdots N^{(r)}_1(j_1) [ \cdot \cdots ] \right] \right], \quad (j_1, \ldots, j_k) \in \{1, \ldots, r\}^k,$$

$$M^{(r)}_{A|\rightarrow 1 \rightarrow \cdots \rightarrow k}(j_1, \ldots, j_k | \rho_{in}) = N^{(r)}_k(j_k) \left[ N^{(r)}_{k-1}(j_{k-1}) \left[ \cdots N^{(r)}_1(j_1) [ \rho_j ] \right] \right], \quad (j_1, \ldots, j_k) \in \{1, \ldots, r\}^k. \quad (19)$$

It follows from relations (4), (18), and (20) that, in the case of an input state (17) before the consecutive measurement $A|\rightarrow 1 \rightarrow \cdots \rightarrow k$, the probability to receive an outcome $\omega = (j_1, \ldots, j_k) \in \{1, \ldots, r\}^k$ under this measurement is equal to

$$\mu_{M^{(r)}_{A|\rightarrow 1 \rightarrow \cdots \rightarrow k}}(j_1, \ldots, j_k | \rho_{in}) = \sum_j q_j \text{tr} \left\{ M^{(r)}_{A|\rightarrow 1 \rightarrow \cdots \rightarrow k}(j_1, \ldots, j_k | \rho_j) \right\} = \sum_j q_j \mu_{M^{(r)}_{A|\rightarrow 1 \rightarrow \cdots \rightarrow k}}(j_1, \ldots, j_k | \rho_j), \quad (21)$$

where

$$\mu_{M^{(r)}_{A|\rightarrow 1 \rightarrow \cdots \rightarrow k}}(j_1, \ldots, j_k | \rho_j) = \text{tr} \left\{ M^{(r)}_{A|\rightarrow 1 \rightarrow \cdots \rightarrow k}(j_1, \ldots, j_k | \rho_j) \right\} = \text{tr} \left\{ \rho_j M^{(r)}_{A|\rightarrow 1 \rightarrow \cdots \rightarrow k}(j_1, \ldots, j_k) \right\}. \quad (22)$$

Therefore, within an N-sequential conclusive quantum state discrimination, the probability for N receivers to take the proper decisions on discriminating between states $\rho_1, \ldots, \rho_r$, given with a priori probabilities $q_1, \ldots, q_r$, i.e., the success probability under a consecutive measurement $A|\rightarrow 1 \rightarrow \cdots \rightarrow N$, has the form

$$P^{\text{success}}_{M^{(r)}_{A|\rightarrow 1 \rightarrow \cdots \rightarrow N}}(\rho_1, \ldots, \rho_r | q_1, \ldots, q_r) = \sum_{j=1, \ldots, r} q_j \text{tr} \left\{ M^{(r)}_{A|\rightarrow 1 \rightarrow \cdots \rightarrow N}(j_1, \ldots, j_N | \rho_j) \right\} = \sum_{j=1, \ldots, r} q_j \text{tr} \left\{ \rho_j M^{(r)}_{A|\rightarrow 1 \rightarrow \cdots \rightarrow N}(j_1, \ldots, j_N) \right\}, \quad (23)$$

where $M^{(r)}_{A|\rightarrow 1 \rightarrow \cdots \rightarrow N}$ is a POV measure (20).

Along with the representation (23), let us also introduce two other equivalent representations for the success probability $P^{\text{success}}_{M^{(r)}_{A|\rightarrow 1 \rightarrow \cdots \rightarrow N}}$. Denote by

$$\tau^{(k)}_{out}(j_1, \ldots, j_k | \rho_j), \quad k \geq 1, \quad (24)$$

RUSSIAN JOURNAL OF MATHEMATICAL PHYSICS Vol. 30 No. 2 2023
the posterior state on $\mathcal{H}$ conditioned by the outcome $(j, \ldots, j) \in \{1, \ldots, r\}^k$ observed under the consecutive measurement $\mathcal{M}_{A|1 \rightarrow \cdots \rightarrow k}$ on the state $\rho_j$. In view of relations (5), (18), and (21), we have

$$
\tau_{\text{out}}^{(k)}(j, \ldots, j) |\rho_j = \frac{\mathcal{M}_{\mathcal{M}_{A|1 \rightarrow \cdots \rightarrow (k-1)}(j, \ldots, j) |\rho_j}}{\mu_{\mathcal{M}_{A|1 \rightarrow \cdots \rightarrow (k-1)}(j, \ldots, j) |\rho_j}} \quad (25)
$$

and, therefore,

$$
\mathcal{M}_{A|1 \rightarrow \cdots \rightarrow k}(j, \ldots, j) |\rho_j = \mathcal{M}_{k}(j) \left[ \mathcal{M}_{A|1 \rightarrow \cdots \rightarrow (k-1)}(j, \ldots, j) |\rho_j \right] \quad (26)
$$

Since the length of a tuple $(j, \ldots, j) \in \{1, \ldots, r\}^k$ is equal to a number $k$ standing in indices at the probability $\mu$ and at the posterior state $\tau_{\text{out}}^{(k)}$, for brevity, we omit below the upper decoration at $j, \ldots, j$.

By (26), for all $k \geq 2$, we come to the following representation of the success probability $\mathbb{P}_{\mathcal{M}_{A|1 \rightarrow \cdots \rightarrow k}}$ under a consecutive measurement $A| \rightarrow 1 \rightarrow \cdots \rightarrow k$:

$$
\mathbb{P}_{\mathcal{M}_{A|1 \rightarrow \cdots \rightarrow k}}(\rho_1, \ldots, \rho_r | q_1, \ldots, q_r) = \sum_{j=1}^{r} q_j \mu_{\mathcal{M}_{A|1 \rightarrow \cdots \rightarrow (k-1)}(j, \ldots, j) |\rho_j} \times \text{tr} \left\{ \mathcal{M}_{k}(j) \left[ \tau_{\text{out}}^{(k-1)}(j, \ldots, j) |\rho_j \right] \right\}. \quad (27)
$$

Relations (22) and (27) imply the following general statement.

**Proposition 1.** Under an $N$-sequential conclusive discrimination between $r \geq 2$ arbitrary states $\rho_1, \ldots, \rho_r$ described by a quantum state instrument $\mathcal{M}_{A|1 \rightarrow \cdots \rightarrow N}$, $N \geq 2$, the success probability $\mathbb{P}_{\mathcal{M}_{A|1 \rightarrow \cdots \rightarrow N}}$ admits the representation (23) and, also, the following representations equivalent to (23):

$$
\mathbb{P}_{\mathcal{M}_{A|1 \rightarrow \cdots \rightarrow N}}(\rho_1, \ldots, \rho_r | q_1, \ldots, q_r) = \sum_{j=1}^{r} q_j \text{tr} \left\{ \rho_j \mathcal{M}_{1}(j) \right\} \\
\times \text{tr} \left\{ \tau_{\text{out}}^{(1)}(j |\rho_j) \mathcal{M}_{2}(j) \right\} \times \cdots \times \text{tr} \left\{ \tau_{\text{out}}^{(N-1)}(j, \ldots, j |\rho_j) \mathcal{M}_{N}(j) \right\}, \quad (28)
$$

$$
\mathbb{P}_{\mathcal{M}_{A|1 \rightarrow \cdots \rightarrow N}}(\rho_1, \ldots, \rho_r | q_1, \ldots, q_r) = \mathbb{P}_{\mathcal{M}_{A|1 \rightarrow \cdots \rightarrow N}}(\rho_1, \ldots, \rho_r | q_1, \ldots, q_r) \\
\times \prod_{n=2, \ldots, N} \mathbb{P}_{\mathcal{M}_{A|1 \rightarrow \cdots \rightarrow n}} \left( \tau_{\text{out}}^{(n-1)}(1, \ldots, 1 |\rho_1), \ldots, \tau_{\text{out}}^{(n-1)}(r, \ldots, r |\rho_r) | Q_{n-1}^{(1)}, \ldots, Q_{n-1}^{(r)} \right). \quad (29)
$$

Here: (i) $\mathcal{M}_{n}(j) := \mathcal{N}(j)[\mathbb{E}_H]$ is the value of the POVM measure describing the measurement of an $n$-th sequential receiver; (ii) $\tau_{\text{out}}^{(n-1)}(j, \ldots, j |\rho_j)$, $n \geq 2$, is the posterior state after $(n-1)$-th measurement, each originated from a state $\rho_j$ and conditioned by the outcome “$i$” under the conclusive measurements of all previous receivers and

$$
Q_{n-1}^{(j)} := \frac{q_{n-1}\mu_{\mathcal{M}_{A|1 \rightarrow \cdots \rightarrow (n-1)}(j, \ldots, j) |\rho_j}}{\mathbb{P}_{\mathcal{M}_{A|1 \rightarrow \cdots \rightarrow (n-1)}}(\rho_1, \ldots, \rho_r | q_1, \ldots, q_r)}, \quad \sum_{j=1}^{r} Q_{n-1}^{(j)} = 1, \quad n \geq 2, \quad (30)
$$

is the a priori probability of the posterior state $\tau_{\text{out}}^{(n-1)}(j, \ldots, j |\rho_j)$, originated from the initial state $\rho_j$, before the measurement of every sequential receiver $n \geq 2$. 

RUSSIAN JOURNAL OF MATHEMATICAL PHYSICS  Vol. 30  No. 2  2023
It follows from representations (28), (29) and the general upper bound derived in [23] on the success probability in the case of a single receiver

\[ \mathbb{P}^\text{success}_{\mathcal{A}_{i=1}^r} (\rho_1, \ldots, \rho_r \mid q_1, \ldots, q_r) \leq \frac{1}{r} \left( 1 + \sum_{1 \leq i < j \leq r} \| (q_i \rho_i - q_j \rho_j) \|_1 \right), \tag{31} \]

it follows that, for every protocol of an \( N \)-sequential conclusive discrimination between \( r \geq 2 \) quantum states \( \rho_1, \ldots, \rho_r \), the success probability admits the upper bound

\[ \mathbb{P}^\text{success}_{\mathcal{A}_{i=1}^N} (\rho_1, \ldots, \rho_r \mid q_1, \ldots, q_r) \leq \frac{1}{r} \left( 1 + \sum_{1 \leq i < j \leq r} \| (q_i \rho_i - q_j \rho_j) \|_1 \right). \tag{32} \]

Here, the recoed \( \| \|_1 \) means the trace norm of the Hermitian operator \( (q_i \rho_i - q_j \rho_j) \). Recall that, for any bounded Hermitian operator,

\[ X = X^{(+)} - X^{(-)}, \quad X^{(+)}, X^{(-)} \geq 0, \quad \| X \|_1 = \| X^{(+)} \|_1 + \| X^{(-)} \|_1, \]

\[ \| X^{(\pm)} \|_1 = \text{tr}\{ X^{(\pm)} \}. \]

We stress that, in the general scenario for an \( N \)-sequential conclusive quantum state discrimination which we have introduced above, no constraints are imposed on the discriminated states \( \rho_1, \ldots, \rho_r \) and a priori probabilities \( q_1, \ldots, q_r \).

4. OPTIMAL PROTOCOLS

Let \( \mathcal{M}_{r,N}^{(\text{cond})} \) denote the set of quantum state instruments \( \mathcal{M}_{r,N}^{(r)} \mathcal{A}_{i=1}^N \) describing an \( N \)-sequential conclusive measurement with outcomes in \( \{1, \ldots, r\}_N \), under some extra condition on receivers’ quantum measurements which is determined by some scenario strategy.\(^2\) The optimal success probability within this strategy for an \( N \)-sequential conclusive state discrimination is defined by

\[ \mathbb{P}^\text{opt.success}_{\mathcal{A}_{i=1}^N} (\rho_1, \ldots, \rho_r \mid q_1, \ldots, q_r) \mid_{\mathcal{M}_{r,N}^{(\text{cond})}} = \max_{\mathcal{M}_{r,N}^{(\text{cond})}} \mathbb{P}^\text{success}_{\mathcal{A}_{i=1}^N} (\rho_1, \ldots, \rho_r \mid q_1, \ldots, q_r), \tag{34} \]

where the success probabilities \( \mathbb{P}^\text{success}_{\mathcal{A}_{i=1}^N} \) are given by any of the equivalent representations (23), (28), (29). If no extra condition on receivers’ quantum measurements is put, then the optimal success probability is given by

\[ \mathbb{P}^\text{opt.success}_{\mathcal{A}_{i=1}^N} (\rho_1, \ldots, \rho_r \mid q_1, \ldots, q_r) = \max_{\mathcal{M}_{r,N}} \mathbb{P}^\text{success}_{\mathcal{A}_{i=1}^N} (\rho_1, \ldots, \rho_r \mid q_1, \ldots, q_r), \tag{35} \]

where \( \mathcal{M}_{r,N} = \{ \mathcal{M}_{r,N}^{(\text{cond})} \} \) is the convex set of all possible quantum state instruments \( \mathcal{M}_{i=1}^N \mathcal{A}_{i=1}^N \) describing a consecutive measurement of \( N \) receivers. Clearly,

\[ \mathbb{P}^\text{opt.success}_{\mathcal{A}_{i=1}^N} (\rho_1, \ldots, \rho_r \mid \mathcal{M}_{r,N}^{(\text{cond})}) \leq \mathbb{P}^\text{opt.success}_{\mathcal{A}_{i=1}^N} (\rho_1, \ldots, \rho_r \mid \mathcal{M}_{r,N}). \tag{36} \]

Since, in the product of the success probabilities standing in (29), the success probability of every receiver depends on measurement parameters of the previous receivers, it follows that, in the general case, the optimal success probability in (35) need not be equal to the product of the optimal success probabilities of \( N \) receivers.

\(^2\)This is, for example, the case in [13], where the receivers’ quantum measurements are described by specific quantum instruments.
4.1. Optimal Success Probability

It follows from relations (29), (31), and (35) that, for an arbitrary number \( r \geq 2 \) of quantum states \( \rho_1, \ldots, \rho_r \), pure and mixed, and any a priori probabilities \( q_1, \ldots, q_r \), the optimal success probability is

\[
\mathbb{P}^{\text{opt.success}}_{A|1 \rightarrow \cdots \rightarrow N}(\rho_1, \ldots, \rho_r | q_1, \ldots, q_r) \leq \mathbb{P}^{\text{opt.success}}_{A|1}(\rho_1, \ldots, \rho_r | q_1, \ldots, q_r) \leq \frac{1}{r} \left( 1 + \sum_{1 \leq i < j \leq r} \| q_i \rho_i - q_j \rho_j \|_1 \right). \tag{37}
\]

If \( N \) receivers discriminate between two states \( (r = 2) \), then the upper bound in the second line of (37) reduces to the Helstrom upper bound [1]:

\[
\mathbb{P}^{\text{opt.success}}_{A|1 \rightarrow \cdots \rightarrow N}(\rho_1, \rho_2 | q_1, q_2) \leq \mathbb{P}^{\text{opt.success}}_{A|1}(\rho_1, \rho_2 | q_1, q_2) = \frac{1}{2} \left( 1 + \| q_1 \rho_1 - q_2 \rho_2 \|_1 \right), \tag{38}
\]

which is attained on *any quantum state instrument* \( \mathcal{M}_1^{(2)}(\cdot) \) of the first receiver with\(^3\) the POV measure

\[
M^{(2)}_{\text{opt}}(j) = P_0(j), \quad j = 1, 2, \tag{39}
\]

which is projection-valued and defined by

\[
P_0(1) = \sum_{\lambda_k > 0} E(\lambda_k), \quad P_0(2) = \mathbb{I}_\mathcal{H} - P_0(1), \tag{40}
\]

\[
(P_0(j))^2 = P_0^{(2)}(j), \quad j = 1, 2, \quad P_0(1)P_0(2) = P_0(2)P_0(1) = 0.
\]

Here \( E(\lambda_k) \) are the spectral projections of the Hermitian operator \( q_1 \rho_1 - q_2 \rho_2 = \sum \lambda_k \lambda_k E(\lambda_k) \), \( \sum \lambda_k E(\lambda_k) = \mathbb{I}_\mathcal{H} \), and \( P_0(1) \) is the orthogonal projection onto the invariant subspace of the operator \( (q_1 \rho_1 - q_2 \rho_2) \), corresponding to its positive eigenvalues.

**Remark 2.** We stress that, in the case of discrimination between \( r > 2 \) arbitrary quantum states \( \rho_1, \ldots, \rho_r \), the precise expression for the optimal POV measure in terms of \( r > 2 \) states to be discriminated and their a priori probabilities is not known even for a single receiver, and, after the optimal measurement of the first receiver, the conditional posterior states corresponding to different outcomes need not be mutually orthogonal.

The following statement introduces the special condition on \( r \geq 2 \) quantum states where, under an \( N \)-sequential conclusive quantum state discrimination scenario specified in Section 3, the upper bound in the first line of (37) is attained.

**Theorem 1.** Let \( \rho_1, \ldots, \rho_r \), \( r \geq 2 \), be arbitrary quantum states, pure or mixed, on a Hilbert space \( \mathcal{H} \) of dimension \( d \geq r \), given with any a priori probabilities \( q_1, \ldots, q_r \), and, under an \( N \)-sequential conclusive discrimination between these states, let \( M_{\text{opt}}^{(r)} \) be an optimal POV measure of the first receiver measurement:

\[
\mathbb{P}^{\text{opt.success}}_{A|1 \rightarrow \cdots \rightarrow N}(\rho_1, \ldots, \rho_r | q_1, \ldots, q_r) = \max_{\mathcal{M}_{\text{opt}}^{(r)}} \mathbb{P}^{\text{success}}_{A|1}(\rho_1, \ldots, \rho_r | q_1, \ldots, q_r) = \sum_{j=1, \ldots, r} q_j \text{tr} \left\{ \rho_j M_{\text{opt}}^{(r)}(j) \right\}. \tag{41}
\]

If the optimal measurement of the first receiver on states \( \rho_1, \ldots, \rho_r \) can be realized via a quantum state instrument\(^4\)

\[
\mathcal{M}_{1}^{(r)}(j)[\cdot] = K_r(j)[\cdot]K_r^\dagger(j), \quad K_r^\dagger(j)K_r(j) = M_{\text{opt}}^{(r)}(j), \tag{42}
\]

where the Kraus operators \( K_r(j), j = 1, \ldots, r \), satisfy the relation

\[
K_r^\dagger(j)P^{(r)}(j)K_r(j) = M_{\text{opt}}^{(r)}(j), \quad j = 1, \ldots, r, \tag{43}
\]

for some mutually orthogonal projections \( P^{(r)}(j) \) on \( \mathcal{H} : \)

\[
\sum_{j=1, \ldots, r} P^{(r)}(j) = \mathbb{I}_\mathcal{H}, \quad P^{(r)}(j_{n_1})P^{(r)}(j_{n_2}) = \delta_{n_1n_2} P^{(r)}(j_{n_1}), \tag{44}
\]

\(^3\)One and the same POV measure may correspond to different quantum state instruments, see Section 2.

\(^4\)See the representation (6).
then, under this conclusive state discrimination for any number \( N \geq 2 \) of sequential receivers, the optimal success probability (35) is equal to the optimal success probability (41) of the first receiver:

\[
P^{\text{opt, success}}_{A|\rightarrow 1\rightarrow \ldots \rightarrow N}(\rho_1, \ldots, \rho_r | q_1, \ldots, q_r) = P^{\text{opt, success}}_{A|\rightarrow 1}(\rho_1, \ldots, \rho_r | q_1, \ldots, q_r),
\]

and is attained under the \( N \)-sequential conclusive state discrimination protocol described by the quantum state instrument

\[
\mathcal{J}^{(r)}_{A|\rightarrow 1\rightarrow \ldots \rightarrow N}(j_1, \ldots, j_N)[.] = P^{(r)}(j_N) \ldots P^{(r)}(j_2) K_r(j_1)[.] K_r^\dagger(j_1) P^{(r)}(j_2) \ldots P^{(r)}(j_N), \quad j_n = 1, \ldots, r.
\]

**Proof.** In view of (44), for the quantum state instrument (46), we have

\[
\mathcal{J}^{(r)}_{A|\rightarrow 1\rightarrow \ldots \rightarrow N}(j_1, \ldots, j)[.] = P^{(r)}(j number, r | q_1, \ldots, q_r) = \sum_{j=1, \ldots, r} q_j \text{tr} \left\{ \mathcal{S}_{A|\rightarrow 1\rightarrow \ldots \rightarrow N}^{(r)}(j_1, \ldots, j)[\rho_j] \right\}
\]

\[
= \sum_{j=1, \ldots, r} q_j \text{tr} \left\{ \rho_j K_r(j)[.] K_r(j)[.] P^{(r)}(j) \right\}
\]

\[
= \sum_{j=1, \ldots, r} q_j \text{tr} \left\{ \rho_j M_{\text{opt}}^{(r)}(j) \right\}
\]

\[
= P^{\text{opt, success}}_{A|\rightarrow 1}(\rho_1, \ldots, \rho_r | q_1, \ldots, q_r).
\]

In view of (37), this proves the assertion of Theorem 1.

**Remark 3.** Under the optimal protocol (46), after the optimal measurement (42) of the first receiver, the conditional posterior states corresponding to different outcomes need not be mutually orthogonal; however, due to condition (43) and updated in view of (30) a priori probabilities for these posterior states, the success probability of the second receiver is equal to 1.

For \( r = 2 \), for any two quantum states, the optimal POV measure of the first receiver measurement is given by the orthogonal projections in (40), so that relations (42), (43) are true if the quantum state instrument of the first receiver in (42) is given by

\[
\mathcal{M}^{(2)}_1(j)[.] = P_0(j)[.] P_0(j),
\]

and \( \{ P_0(1), P_0(2) \} \) is the set of the orthogonal projections given in (40). If, for some two states, each of orthogonal projections \( P_0(j) \) is nonzero, \( j = 1, 2 \), then, for these states, relations (42) and (43) are also true if the quantum state instrument of the first receiver has the form

\[
\tilde{\mathcal{M}}^{(2)}_1[j.] = \sum_{j=1, 2} K_2(j)[.] K_2^\dagger(j),
\]

where the Kraus operators \( K_2(j) \) are defined by the relations

\[
K_2(j) = \sum_{i=1, \ldots, k(j)} |\phi_i(j)\rangle \langle v_i(j)|, \quad \sum_{i=1, \ldots, k(j)} \langle v_i(j)| \langle v_i(j)| = P_0(j), \quad j = 1, 2,
\]

\[
\langle v_i(j_1) | v_i(j_2) \rangle = \langle \phi_i(j_1) | \phi_i(j_2) \rangle = \delta_{i_1 i_2} \delta_{j_1 j_2},
\]

while the orthogonal projections in (44) are given by

\[
\tilde{P}^{(2)}(j) = \sum_{i=1, \ldots, k(j)} |\phi_i(j)\rangle \langle \phi_i(j)|, \quad j = 1, 2.
\]
Here: (i) \( \{|v_i(j)\}, i = 1, \ldots, k(j)\}, j = 1, 2, \) are any two orthonormal bases spanning the invariant subspaces of the Hermitian operator \( (q_1 \rho_1 - q_2 \rho_2) \), corresponding to its positive and nonpositive eigenvalues, respectively, and having dimensions \( k(j), j = 1, 2, \sum_{j=1,2} k(j) = d; \) (ii) \( \{|\phi_i(j)\}, i = 1, \ldots, k(j), j = 1, 2\} \) is any orthonormal basis of the Hilbert space \( \mathcal{H} \). If for the bases we have

\[
\{|\phi_i(j)\}, i = 1, \ldots, k(j), j = 1, 2\} = \{|v_i(j)\}, i = 1, \ldots, k(j), j = 1, 2\},
\]

then \( \tilde{K}_2(j) = \tilde{P}^{(2)}(j) = P_0(j) \), and the quantum state instrument (50) reduces to the state instrument (49).

Therefore, for an \( N \)-sequential conclusive discrimination between two quantum states \( \rho_1, \rho_2 \), the statement of Theorem 1 holds.

**Theorem 2.** Let \( \rho_1, \rho_2 \), given with any a priori probabilities \( q_1, q_2 \), be two arbitrary quantum states, pure or mixed, on a Hilbert space \( \mathcal{H} \) of an arbitrary dimension \( d \geq 2 \). For an \( N \)-sequential conclusive discrimination between arbitrary two states \( \rho_1, \rho_2 \), the optimal success probability (35) is equal to the Helstrom bound

\[
P_{\text{opt,success}}^{(A_1 \rightarrow \ldots \rightarrow A_N)} (\rho_1, \rho_2 | q_1, q_2) = \frac{1}{2} (1 + \|q_1 \rho_1 - q_2 \rho_2\|_1),
\]

for every number \( N \geq 1 \) of sequential receivers, and is attained under the general optimal protocol, described by the quantum state instrument

\[
\mathcal{M}^{(2)}_{A_1 \rightarrow \ldots \rightarrow A_N} (j_1, \ldots, j_N) [\cdot] := P_0(j_N) \cdots P_0(j_1)[\cdot] P_0(j_1) \cdots P_0(j_N),
\]

where \( P_0(j), j = 1, 2, \) are orthogonal projections in (40). If, for two states to be discriminated, each of the orthogonal projections \( P_0(j) \) is nonzero, \( j = 1, 2, \) then, for these two states, the optimal success probability (54) is also attained under either of the optimal protocols described by the quantum state instrument of the form:

\[
\tilde{\mathcal{M}}^{(2)}_{A_1 \rightarrow \ldots \rightarrow A_N} (j_1, \ldots, j_N) [\cdot] = \tilde{P}^{(2)}(j_N) \cdots \tilde{P}^{(2)}(j_2) \tilde{K}_2(j_1)[\cdot] \tilde{K}^{(2)}_2(j_1) \tilde{P}^{(2)}(j_2) \cdots \tilde{P}^{(2)}(j_N),
\]

where the Kraus operators \( \tilde{K}_2(j), j = 1, 2, \) are defined in (51) and the orthogonal projections \( \tilde{P}^{(2)}(j) \) are given by (52).

Note that, under either of the optimal protocols, described by the quantum state instruments (55) and (56), the measurement of the first receiver for the discrimination between two quantum states \( \rho_1, \rho_2 \) is the optimal one with the POV measure given by projections in (40). If, in (40), each of the orthogonal projections \( P_0(j) \) is nonzero, \( j = 1, 2, \) then, after the optimal measurement of the first receiver, the conditional posterior states, corresponding to different outcomes \( j = 1, 2, \) are mutually orthogonal (though not pure in general) and are defined due to (10) by the corresponding Kraus operators in (49) and (50); otherwise, they are given by \( \rho_1 \) (or \( \rho_2 \)).

### 4.2. Implementation via Indirect Measurements

Let, under an \( N \)-sequential conclusive discrimination between two arbitrary quantum states, pure or mixed, all receivers perform indirect measurements.

**Proposition 2.** For any two quantum states \( \rho_1, \rho_2 \) on a Hilbert space \( \mathcal{H} \), possibly infinite-dimensional, the general optimal \( N \)-sequential conclusive discrimination protocol (55) is implemented if every \( n \)-th receiver performs the indirect measurement described by the statistical realization\(^5\)

\[
\Xi_n = \left\{ \mathbb{C}^2, |b_n\rangle\langle b_n|, \tilde{P}_n, U_n \right\},
\]

\(^5\)See Section 2.
where \(|b_n⟩⟨b_n|\) is a pure state on \(\mathbb{C}^2\), the projection-valued measure \(\overline{P}_n\) on \(\mathbb{C}^2\) has the elements \(\overline{P}_n(1) = |b_n⟩⟨b_n|\) and \(\overline{P}_n(2) = |b_n⟩⟨b_n| = I_{\mathbb{C}^2} - |b_n⟩⟨b_n|\), and \(U_n\) is the unitary operator on \(\mathcal{H} \otimes \mathbb{C}^2\) having the CNOT-like form:

\[
U_n = P_0(1) \otimes I_{\mathbb{C}^2} + P_0(2) \otimes (|b_n⟩⟨b_n| + |b_n⟩⟨b_n|).
\]

Here \(P_0(j), j = 1, 2, \) are projections (40) on the Hilbert space \(\mathcal{H}\).

**Proof.** Let the indirect measurement of every \(n\)-th receiver be described by the statistical realization (57). Then, by (14), the quantum state instrument \(\mathcal{M}^{(2)}_n(\cdot)\), describing the \(n\)-th receiver’s indirect measurement (57), is given by

\[
\mathcal{M}^{(2)}_n(j)[σ] = \text{tr}_\mathcal{H} \left\{ \mathbb{E}_\mathcal{H} \otimes \overline{P}_n(j) \left( U(σ \otimes |b_n⟩⟨b_n|)U^\dagger \right) \mathbb{E}_\mathcal{H} \otimes \overline{P}_n(j) \right\}
\]

\[
= P_0(j)[σ] P_0(j), \quad j = 1, 2,
\]

for any state \(σ\) on a Hilbert space \(\mathcal{H}\). This implies the optimal \(N\)-sequential conclusive state discrimination protocol (55) and proves the assertion of Proposition 2.

5. \(N\)-SEQUENTIAL CONCLUSIVE DISCRIMINATION UNDER A NOISE

In Sections 3 and 4, we have introduced the general framework for the description of an \(N\)-sequential conclusive state discrimination between \(r \geq 2\) quantum states, pure or mixed, in the case of an ideal multipartite quantum communication among all participants. However, in a realistic situation, a multipartite quantum communication among participants is noisy and, in this section, we proceed to specify a general framework for the description of an \(N\)-sequential conclusive state discrimination via noisy quantum communication channels.

Let \(\Lambda_1\) be a noisy quantum channel between a sender and a first receiver and \(\Lambda_n\) is that between an \((n-1)\)-th receiver and an \(n\)-th receiver. In this case, similarly to (18), a quantum state instrument describing a consecutive measurement with an outcome \((j_1, \ldots, j_k) \in \{1, \ldots, r\}^k\), performed by \(k \in \{1, \ldots, N\}\) sequential receivers communicating via noisy quantum channels, takes the following form:

\[
\mathcal{M}^{(r)}_{\Lambda_1|\cdots|\Lambda_n}(j_1,\ldots,j_k)[\cdot] = \mathcal{M}^{(r)}_1(j_k) \left[ \Lambda_k \left[ \mathcal{M}^{(r)}_{\Lambda_{k-1}|\cdots|\Lambda_n}(j_{k-1}) \left[ \cdots \Lambda_2 \left[ \mathcal{M}^{(r)}_1(j_1) \left[ \Lambda_1[\cdot] \right] \right] \right] \right] \right],
\]

\((j_1, \ldots, j_k) \in \{1, \ldots, r\}^k\).

Here the symbol \(\rightsquigarrow\) means a quantum state discrimination under a noisy communication between participants.

It follows from (60) that, under an \(N\)-sequential conclusive state discrimination via noisy quantum communication channels, the representation (23) for the success probability is to be replaced by

\[
P^{\text{success}}_{\mathcal{M}^{(r)}_{\Lambda_1|\cdots|\Lambda_n}}(\rho_1, \ldots, \rho_r | q_1, \ldots, q_r) \bigg|_{\Lambda_1, \ldots, \Lambda_N}
\]

\[
= \sum_{j=1, \ldots, r} q_j \text{tr} \left\{ \mathcal{M}^{(r)}_N(j) \left[ \Lambda_N \left[ \mathcal{M}^{(r)}_{\Lambda_{N-1}|\cdots|\Lambda_n}(j) \left[ \cdots \Lambda_2 \left[ \mathcal{M}^{(r)}_1(j_1) \left[ \Lambda_1[\rho_j] \right] \right] \right] \right] \right] \right\}
\]

\[
= \sum_{j=1, \ldots, r} q_j \mu_{\mathcal{M}^{(r)}_{\Lambda_1|\cdots|\Lambda_n}}(j_1, \ldots, j_k | \rho_j),
\]

where

\[
\mu_{\mathcal{M}^{(r)}_{\Lambda_1|\cdots|\Lambda_n}}(j_1, \ldots, j_k | \rho_j) = \text{tr} \left\{ \mathcal{M}^{(r)}_{\Lambda_1|\cdots|\Lambda_n}(j_1, \ldots, j_k) | \rho_j \right\}.
\]

The posterior state \(\bar{\sigma}^{(k)}_{\text{out}}(j_1, \ldots, j_k | \rho_j)\) conditioned by an outcome \((j, \ldots, j) \in \{1, \ldots, r\}^k\) observed under a consecutive measurement (60) on a state \(\rho_j\) is given by

\[
\bar{\sigma}^{(k)}_{\text{out}}(j_1, \ldots, j_k | \rho_j) = \frac{\mu_{\mathcal{M}^{(r)}_{\Lambda_1|\cdots|\Lambda_n}}(j_1, \ldots, j_k | \rho_j)}{\mu_{\mathcal{M}^{(r)}_{\Lambda_1|\cdots|\Lambda_n}}(j_1, \ldots, j_k | \rho_j)}.
\]
As we noted after (26), for simplicity, we omit below the upper decoration at the outcome $j, \ldots, j$.

In view of relation (4) between values $M_k^{(r)}(\cdot)$ of a state instrument describing a measurement of a $k$-th receiver and the values $M_k^{(r)}(\cdot)$ of the POV measure corresponding to this instrument, relation (63) implies that, similarly to (28) and (29), under an $N$-sequential conclusive state discrimination via a noisy communication, the success probability (61) admits also two other equivalent representations:

$$
\mathbb{P}_{\mathcal{M}_A^{(r)}}^{\text{success}} ( \rho_1, \ldots, \rho_r | q_1, \ldots, q_r | \Lambda_1, \ldots, \Lambda_N ) = \sum_{j=1}^{r} q_j \text{tr} \left\{ \Lambda_1 [\rho_j] M_1^{(r)}(j) \right\} \text{tr} \left\{ \Lambda_2 \left[ \sigma_{\text{out}}^{(1)}(j) | \rho_j \right] M_2^{(r)}(j) \right\}
$$

$$
\times \cdots \text{tr} \left\{ \Lambda_N \left[ \sigma_{\text{out}}^{(N-1)}(j, \ldots, j | \rho_j) \right] M_N^{(r)}(j) \right\},
$$

(64)

and

$$
\mathbb{P}_{\mathcal{M}_A^{(r)}}^{\text{success}} ( \rho_1, \ldots, \rho_r | q_1, \ldots, q_r | \Lambda_1, \ldots, \Lambda_N ) = \mathbb{P}_{\mathcal{M}_1^{(r)}}^{\text{success}} ( \Lambda_1 [\rho_1], \ldots, \Lambda_1 [\rho_r] | q_1, \ldots, q_r )
$$

$$
\times \prod_{n=2}^{N} \mathbb{P}_{\mathcal{M}_n^{(r)}}^{\text{success}} \left( \Lambda_n \left[ \sigma_{\text{out}}^{(n-1)}(1, \ldots, 1 | \rho_1) \right], \ldots, \Lambda_n \left[ \sigma_{\text{out}}^{(n-1)}(r, \ldots, r | \rho_r) \right] Q_n^{(1)}, \ldots, Q_n^{(r)} \right),
$$

(65)

where

$$
Q_{n-1}^{(j)} := \frac{q_j \mu_{\mathcal{M}_A^{(r)}}^{(r)} (j, \ldots, j | \rho_j)}{\sum_j q_j \mu_{\mathcal{M}_A^{(r)}}^{(r)} (j, \ldots, j | \rho_j)}
$$

(66)

is the a priori probability for the conditional posterior state $\sigma_{\text{out}}^{(n-1)}(r, \ldots, r | \rho_r)$ coming to $n$-th receiver and originated from the initial state $\rho_j$.

Also, in case of a noisy communication, the upper bound similar to (37) takes the form

$$
\mathbb{P}_{\mathcal{M}_A^{(r)}}^{\text{success}} ( \rho_1, \ldots, \rho_r | q_1, \ldots, q_r | \Lambda_1, \ldots, \Lambda_N ) \leq \frac{1}{r} \left( 1 + \sum_{1 \leq i < j \leq r} \| \Lambda_1 [q_i \rho_i - q_j \rho_j] \|_1 \right).
$$

(67)

Specified for a two-sequential discrimination between two arbitrary quantum states ($N = 2$), the representation (65) for the success probability reduces to

$$
\mathbb{P}_{\mathcal{M}_1^{(r)}}^{\text{success}} ( \rho_1, \ldots, \rho_r | q_1, \ldots, q_r | \Lambda_1, \Lambda_2 ) = \mathbb{P}_{\mathcal{M}_1^{(r)}}^{\text{success}} ( \Lambda_1 [\rho_1], \ldots, \Lambda_1 [\rho_r] | q_1, \ldots, q_r )
$$

$$
\times \mathbb{P}_{\mathcal{M}_2^{(r)}}^{\text{success}} \left( \Lambda_2 \left[ \sigma_{\text{out}}^{(1)}(1 | \rho_1) \right], \ldots, \Lambda_2 \left[ \sigma_{\text{out}}^{(1)}(r | \rho_r) \right] Q_1^{(1)}, \ldots, Q_1^{(r)} \right),
$$

(68)

where the conditional posterior states $\sigma_{\text{out}}^{(1)}(j | \rho_j)$ and their a priori probabilities $Q_1^{(j)}$, $j = 1, \ldots, r$, before a measurement of the second receiver, are given by

$$
\sigma_{\text{out}}^{(1)}(j | \rho_j) = \frac{M_1^{(2)}(j) [\Lambda_1 [\rho_j]]}{\mu_{\mathcal{M}_1^{(2)}}^{(2)}(j | \Lambda_1 [\rho_j])},
$$

(69)

$$
Q_1^{(j)} = \frac{q_j \mu_{\mathcal{M}_1^{(2)}}^{(2)}(j | \Lambda_1 [\rho_j])}{\mathbb{P}_{\mathcal{M}_1^{(2)}}^{\text{success}} ( \Lambda_1 [\rho_1], \ldots, \Lambda_1 [\rho_r] | q_1, \ldots, q_r )},
$$

and

$$
\mathbb{P}_{\mathcal{M}_1^{(2)}}^{\text{success}} ( \Lambda_2 \left[ \sigma_{\text{out}}^{(1)}(1 | \rho_1) \right], \ldots, \Lambda_2 \left[ \sigma_{\text{out}}^{(1)}(r | \rho_r) \right] Q_1^{(1)}, \ldots, Q_1^{(r)} ) = \sum_{j=1}^{r} Q_1^{(j)} \text{tr} \left\{ \Lambda_2 \left[ \sigma_{\text{out}}^{(1)}(j | \rho_j) \right] M_2^{(2)}(j) \right\}
$$

(70)
It follows from (68) that, for a two-sequential discrimination between arbitrary $r \geq 2$ quantum states, the optimal success probability has the form
\[
\mathbb{P}_{\text{opt.success}}^{A_1 \rightarrow A_2} (\rho_1, \ldots, \rho_r \mid q_1, \ldots, q_r) \big| \Lambda_1, \Lambda_2 = \max_{M_1^{(r)}} \left( \mathbb{P}_{\text{success}}^{A_1 \rightarrow A_2} (\Lambda_1 \rho_1, \ldots, \Lambda_1 \rho_r \mid q_1, \ldots, q_r) \right)
\times \max_{M_2^{(r)}} \left( \mathbb{P}_{\text{success}}^{A_1 \rightarrow A_2} (\Lambda_2 [\sigma_{\text{out}}(1) | \rho_1], \ldots, \Lambda_2 [\sigma_{\text{out}}(r) | \rho_r]) \mid q_1^{(1)}, \ldots, q_r^{(r)} \right)
\] (71)
and, similarly to our note in Remark 2, is not in general equal to the product of the optimal success probabilities of the first and the second receiver.

Taking further into account the upper bound (37) found in [23], and also relations (69), we derive the following general statement.

**Theorem 3.** The optimal success probability, for a two-sequential discrimination between $r \geq 2$ arbitrary quantum states under a noise, admits the upper bound
\[
\mathbb{P}_{\text{opt.success}}^{A_1 \rightarrow A_2} (\rho_1, \ldots, \rho_r \mid q_1, \ldots, q_r) \big| \Lambda_1, \Lambda_2 = \frac{1}{r} \max_{M_1^{(r)}} \left( \mathbb{P}_{\text{success}}^{A_1 \rightarrow A_2} (\Lambda_1 \rho_1, \ldots, \Lambda_1 \rho_r \mid q_1, \ldots, q_r) \right)
+ \sum_{1 \leq i < j \leq r} \| \mathbb{A}_2 \left[ q_i \mathcal{M}_1^{(r)}(i) [\Lambda_1 \rho_i] - q_j \mathcal{M}_1^{(r)}(j) [\Lambda_1 \rho_j] \right] \|_1
\]
(72)
where
\[
\mathbb{P}_{\text{success}}^{A_1 \rightarrow A_2} (\Lambda_1 \rho_1, \ldots, \Lambda_1 \rho_r \mid q_1, \ldots, q_r) \leq \frac{1}{r} \left( 1 + \sum_{1 \leq i < j \leq r} \| \Lambda_1 [q_i \rho_i - q_j \rho_j] \|_1 \right).
\] (73)

The equality in the first line of (72) holds for $r = 2$ and reads:
\[
\mathbb{P}_{\text{opt.success}}^{A_1 \rightarrow A_2} (\rho_1, \rho_2 \mid q_1, q_2) \big| \Lambda_1, \Lambda_2 = \max_{M_1^{(2)}} \frac{1}{2} \left( \mathbb{P}_{\text{success}}^{A_1 \rightarrow A_2} (\Lambda_1 \rho_1, \Lambda_1 \rho_2 \mid q_1, q_2) + \| \mathbb{A}_2 \left[ q_1 \mathcal{M}_1^{(2)}(1) [\Lambda_1 \rho_1] - q_2 \mathcal{M}_1^{(2)}(2) [\Lambda_1 \rho_2] \right] \|_1 \right).
\] (74)

Since (74) constitutes the maximization of a convex function on a convex set $\{\mathcal{M}_1^{(2)}\}$ of all quantum instruments of the first receiver, the extremum in (74) is attained at the extreme points of this set, i.e., on the subset of quantum instruments with elements $P(j) : P(j)$, $j = 1, 2$, where $P(j)$ are orthogonal projections on $\mathcal{H}$ and $\sum_{j=1,2} P(j) = I_\mathcal{H}$.

Taking into account that, for any positive trace class operators $T, \tilde{T}$,
\[
\|T - \tilde{T}\|_1 \leq ||T||_1 + ||\tilde{T}||_1 = \text{tr}[T] + \text{tr} [\tilde{T}]
\] (75)
and that $\text{tr} \{A[T]\} = \text{tr} \{T\}$ for any trace class operator $T$, we have in (72)
\[
\sum_{1 \leq i < j \leq r} \| \mathbb{A}_2 \left[ q_i \mathcal{M}_1^{(r)}(i) [\Lambda_1 \rho_i] - q_j \mathcal{M}_1^{(r)}(j) [\Lambda_1 \rho_j] \right] \|_1
\leq \sum_{1 \leq i < j \leq r} \left( q_i \text{tr} \left\{ \mathbb{A}_2 \left[ \mathcal{M}_1^{(r)}(i) [\Lambda_1 \rho_i] \right] \right\} + q_j \text{tr} \left\{ \mathbb{A}_2 \left[ \mathcal{M}_1^{(r)}(i) [\Lambda_1 \rho_j] \right] \right\} \right)
= \sum_{1 \leq i < j \leq r} \left( q_i \text{tr} \left\{ \Lambda_1 \rho_i \mathcal{M}_1^{(r)}(i) \right\} + q_j \text{tr} \left\{ \Lambda_1 \rho_j \mathcal{M}_1^{(r)}(j) \right\} \right)
= (r - 1) \mathbb{P}_{\text{success}}^{A_1 \rightarrow A_2} (\Lambda_1 \rho_1, \ldots, \Lambda_1 \rho_r \mid q_1, \ldots, q_r),
\] (76)
so that, for $N = 2$, the upper bound in (72) is tighter than the general upper bound in (67).
6. TWO-SEQUENTIAL CONCLUSIVE DISCRIMINATION BETWEEN TWO STATES UNDER A DEPOLARIZING NOISE

Consider the optimal success probability (74) for a two-sequential conclusive discrimination for the case in which the sender prepares two qubit states $\rho_j, \ j = 1, 2$, with Bloch representations

$$\rho_j = \frac{1}{2} (I_2 + \vec{r}_j \cdot \vec{\sigma}), \quad \vec{r}_j = \text{tr}(\rho_j \vec{\sigma}) \in \mathbb{R}^3, \quad (77)$$

and the a priori probabilities $q_j, \ j = 1, 2$, and quantum communication channels between a sender and a receiver and between two receives are depolarizing,

$$\Lambda^{(\text{pol})}_n[T] := (1 - \gamma_n)T + \gamma_n \text{tr}\{T\} \frac{I_2}{2}, \quad \gamma_n \in [0, 1], \quad n = 1, 2. \quad (78)$$

Here $\sigma_k, \ k = 1, 2, 3$, stand for the Pauli operators, $I_2$ is the identity operator, and $T$ is an arbitrary linear operator on $\mathbb{C}^2$. For a qubit state $\rho_j$ with representation (77), relation (78) reads

$$\Lambda^{(\text{pol})}_1[\rho_j] = \frac{1}{2} (I_2 + (1 - \gamma_1) \vec{r}_j \cdot \vec{\sigma}). \quad (79)$$

For a single receiver ($N = 1$), the optimal success probability under a depolarizing noise is given by the Helstrom bound (38) for the discrimination between noisy states $\Lambda^{(\text{pol})}_1[\rho_1]$ and $\Lambda^{(\text{pol})}_1[\rho_2]$, i.e.,

$$p^\text{opt.success}_1 (\rho_1, \rho_2 | q_1, q_2) = \frac{1}{2} \left(1 + \sqrt{1 - q_1 q_2} \frac{1}{2} \gamma_1^2 \right). \quad (80)$$

Taking into account that, for a Hermitian operator on a finite-dimensional Hilbert space, the trace norm is given by the sum of the absolute values of its eigenvalues and that the eigenvalues of the qubit operator $(q_1 \Lambda^{(\text{pol})}_1[\rho_1] - q_2 \Lambda^{(\text{pol})}_1[\rho_2])$ are equal to $\frac{1}{2} (q_1 - q_2 \pm (1 - \gamma_1) \| q_1 \vec{r}_1 - q_2 \vec{r}_2 \|_{\mathbb{R}^3})$, we see that

$$p^\text{opt.success}_1 (\rho_1, \rho_2 | q_1, q_2) = \frac{1}{2} \left(1 + \frac{1}{2} (q_1 - q_2 + (1 - \gamma_1) \| q_1 \vec{r}_1 - q_2 \vec{r}_2 \|_{\mathbb{R}^3}) \right) \quad (81)$$

It follows from (80) and (81) that

$$p^\text{opt.success}_1 (\rho_1, \rho_2 | q_1, q_2) = \frac{1}{2} \left(1 + \frac{1}{2} (q_1 - q_2 \pm (1 - \gamma_1) \| q_1 \vec{r}_1 - q_2 \vec{r}_2 \|_{\mathbb{R}^3}) \right) \quad (82)$$

if

$$(1 - \gamma_1) \| q_1 \vec{r}_1 - q_2 \vec{r}_2 \|_{\mathbb{R}^3} > |q_1 - q_2| \quad (83)$$

and

$$p^\text{opt.success}_1 (\rho_1, \rho_2 | q_1, q_2) = \frac{1}{2} \left(1 + \frac{1}{2} (q_1 - q_2 \pm (1 - \gamma_1) \| q_1 \vec{r}_1 - q_2 \vec{r}_2 \|_{\mathbb{R}^3}) \right) \quad (84)$$

otherwise. Relations (80) - (84) imply

$$p^\text{opt.success}_1 (\rho_1, \rho_2 | q_1, q_2) = \max \left\{ \frac{1}{2} + \frac{1}{2} (q_1 - q_2 \pm (1 - \gamma_1) \| q_1 \vec{r}_1 - q_2 \vec{r}_2 \|_{\mathbb{R}^3}), \max\{q_1, q_2\} \right\}. \quad (85)$$

**Remark 4.** If the initial qubit states are pure: $\rho_1 = |\psi_1\rangle\langle\psi_1|$, $\rho_2 = |\psi_2\rangle\langle\psi_2|$, then $\| \vec{r}_1 \|_{\mathbb{R}^3} = \| \vec{r}_2 \|_{\mathbb{R}^3} = 1$ and, as is well-known,

$$\| q_1 \vec{r}_1 - q_2 \vec{r}_2 \|_{\mathbb{R}^3} = \sqrt{q_1^2 + q_2^2 - 2q_1q_2 \vec{r}_1 \cdot \vec{r}_2} = \sqrt{1 - 4q_1q_2 |\langle\psi_2|\psi_1\rangle|^2}. \quad (86)$$

Therefore, for pure states, expression (85) reduces to

$$p^\text{opt.success}_1 (\rho_1, \rho_2 | q_1, q_2) = \max \left\{ \frac{1}{2} + \frac{1}{2} (1 - \gamma_1) \sqrt{1 - 4q_1q_2 |\langle\psi_2|\psi_1\rangle|^2}, \max\{q_1, q_2\} \right\}. \quad (87)$$
Note that, since
\[ \|q_1\vec{r}_1 - q_2\vec{r}_2\|_{\mathbb{R}^3} = \sqrt{1 - 4q_1q_2|\langle \psi_2 | \psi_1 \rangle|^2} \geq |q_1 - q_2|, \]
for \( \gamma_1 = 0 \), it follows that the maximum (87) reduces the well-known expression
\[ \frac{1}{2} + \frac{1}{2}\sqrt{1 - 4q_1q_2|\langle \psi_2 | \psi_1 \rangle|^2} \]
for the Helstrom bound in case of pure initial states.

For two receivers \( (N = 2) \), we derive the following new result proved rigorously in the appendix.

**Theorem 4.** Let \( \rho_1 \) and \( \rho_2 \) be arbitrary qubit states given with any a priori probabilities \( q_1, q_2 \). In the case of depolarizing channels (78), the optimal success probability (74) of a two-sequential discrimination between qubit states \( \rho_1 \) and \( \rho_2 \) is given by
\begin{align}
\mathcal{P}_{\text{opt.success}}(\rho_1, \rho_2 \mid q_1, q_2) |_{\gamma_1, \gamma_2} = \max \left\{ \left(1 - \frac{\gamma_2}{2}\right) \left(\frac{1}{2} + \frac{1}{2}(1 - \gamma_1)\|q_1\vec{r}_1 - q_2\vec{r}_2\|_{\mathbb{R}^3}\right), \max\{q_1, q_2\} \right\},
\end{align}
where \( \vec{r}_j, j = 1, 2 \), are the Bloch vectors of states \( \rho_j \) in the representation (77).

For an arbitrary \( \gamma_1 \in [0, 1] \) and \( \gamma_2 = 0 \), expression (90) coincides with expression (85), and this agrees with Theorem 2.

In Figs. 1–4, we present the numerical results related to (85) and (90). For simplicity, we assume that \( \gamma_1 = \gamma_2 = \gamma \).

In Figs. 1 and 2, we consider qubit states \( \rho_1 \) and \( \rho_2 \) with the Bloch vectors
\[ \vec{r}_1 = (0.3, 0.3, 0.3), \quad \vec{r}_2 = (0.3, 0.3, -0.3), \]
and a priori probabilities \( q_1 = q_2 = 0.5 \) in Fig. 1 and a priori probabilities \( q_1 = 0.55, q_2 = 0.45 \) in Fig. 2.

In Figs. 3 and 4, we take qubit states \( \rho_1 \) and \( \rho_2 \) with the Bloch vectors
\[ \vec{r}_1 = (0.2, 0.3, -0.4), \quad \vec{r}_2 = (-0.2, -0.3, 0.35), \]
and a priori probabilities \( q_1 = q_2 = 0.5 \) in Fig. 3 and \( q_1 = 0.55, q_2 = 0.45 \) in Fig. 4.

In each of Figs. 1–4: (i) a dotted line corresponds to the Helstrom bound (38) for qubit states \( \rho_1 \) and \( \rho_2 \) with the corresponding a priori probabilities \( q_1, q_2 \); (ii) a thin line describes the dependence on \( \gamma \) of the optimal success probability (85) of the first receiver \( (N = 1) \) under a depolarizing quantum channel between a sender and the first receiver, and (iii) a bold line describes the dependence on \( \gamma \) of the two-sequential \( (N = 2) \) optimal success probability (90) in case of depolarizing quantum channels between a sender and the first receiver and between two sequential receivers.
According to the presented numerical results, if \( \gamma \) is not so large, then the dependence of the optimal success probability \( (90) \) on \( \gamma \) is presented by a nearly straight line. This is since, for the considered states and a priori probabilities, in the corresponding domains of \( \gamma \), the impact in \( (90) \) of the term quadratic in \( \gamma \) is much less than that of the term linear in \( \gamma \).

As clearly seen in Figs. 1–4, for every case “i” of qubit states and a priori probabilities, under values \( \gamma \leq \gamma_i \), the optimal success probability \( (90) \) of the two-sequential \( (N = 2) \) discrimination between qubit states \( \rho_1 \) and \( \rho_2 \) is equal to the product of the optimal success probability \( (85) \) of discrimination under a noise between states \( \rho_1 \) and \( \rho_2 \) by the first receiver and the expression \( (1 - \frac{\gamma^2}{2}) \), which, in view of \( (85) \), constitutes the optimal success probability for the discrimination by the second receiver under a noise if, in \( (68) \) and \( (69) \), the conditional posterior states are pure and mutually orthogonal and \( \max\{Q_1, Q_2\} \leq 1 - \frac{\gamma}{2} \) for updated a priori probabilities.

7. CONCLUSION

In the present paper, we have developed a general mathematical framework for the description of an \( N \)-sequential conclusive discrimination between \( r \geq 2 \) quantum states, pure or mixed, for the case in which, both beforehand and during a sequential discrimination, a communication between receivers via a separate classical channel and via an extra quantum channel (by encoding the information on their outcomes into orthogonal quantum states) is prohibited [3]. This type of quantum state discrimination constitutes an \( N \)-sequential extension of the minimum-error discrimination by a single receiver. The developed general framework incorporates the description of the \( N \)-sequential discrimination considered in [13] as a particular case only.

For this general \( N \)-sequential conclusive quantum state discrimination scenario, we have

- derived (Proposition 1) three mutually equivalent general representations \( (23), (28), (29) \) for the success probability;
- found (Theorem 1) a new general condition on \( r \geq 2 \) quantum states sufficient for the optimal success probability to be equal to the optimal success probability of the first receiver for any number \( N \geq 2 \) of further sequential receivers and specified the corresponding optimal protocol \( (46) \);
- shown (Theorem 2) that, in the case of discrimination between two arbitrary quantum states, this sufficient condition is always fulfilled, so that the optimal success probability of an \( N \)-sequential conclusive discrimination between two arbitrary quantum states is given by the Helstrom bound for any number \( N \geq 2 \) of sequential receivers and is attained under the optimal protocol \( (55) \), which is general in the sense that it is true for the \( N \)-sequential conclusive discrimination between any two quantum states, pure or mixed, of an arbitrary dimension, and also presented the specific optimal protocols \( (56) \) valid for two states satisfying some extra condition;
- explicitly constructed (Proposition 2) receivers’ indirect measurements implementing the general optimal protocol \( (55) \) for the \( N \)-sequential conclusive discrimination between any two quantum states.
Furthermore, we have extended our general framework to include by (64) and (65) the case of an $N$-sequential conclusive discrimination between $r \geq 2$ arbitrary quantum states under arbitrary noisy quantum communication channels. For the optimal success probability in the case of a two-sequential discrimination under a noise between $r \geq 2$ arbitrary quantum states, we have specified (Theorem 3) a new general upper bound (72), which is attained for $r = 2$.

As an example, we have analyzed analytically (Theorem 4) and further numerically a 2-sequential conclusive discrimination between arbitrary two qubit states via depolarizing quantum channels.

The developed general framework is true for any number $N \geq 1$ of sequential receivers, any number $r \geq 2$ of arbitrary quantum states, pure or mixed, to be discriminated, all types of receivers’ quantum measurements, and arbitrary noisy quantum communication channels.

The new general results derived within the developed framework are important both from the theoretical point of view and for a successful multipartite quantum communication via noisy channels.

FUNDING

The study by E.R. Loubenets in Section 2, Section 3, and Section 4.1 of this work was supported by the Russian Science Foundation under grant No 19-11-00086 and performed at the Steklov Mathematical Institute of Russian Academy of Sciences. The study by E.R. Loubenets in Section 5 and Section 6 was performed at the National Research University Higher School of Economics. The study by Min Namkung in Section 4.2, Section 5 and Section 6 was performed until August 2021 at the National Research University Higher School of Economics and, from September 2021, at the Kyung Hee University under the support from the National Research Foundation of Korea (NRF) grant (NRF2020M3E4A1080088) funded by the Korea government (Ministry of Science and ICT).

A. APPENDIX

In this section, we present the proof of Theorem 4. In the qubit case, the extreme points of the set $\{M_1(2)\}$ in (74) of quantum state instruments of the first receiver are given by:

(i) the state instruments $\{\Pi_{\vec{n}}^{(\pm)}(j), \vec{n} \in \mathbb{R}^3, \|\vec{n}\|_{\mathbb{R}^3} = 1\}$ with elements

$$\Pi_{\vec{n}}^{(\pm)}(1)[\cdot] = E_{\sigma_{\vec{n}}}^{(\pm)}[\cdot] E_{\sigma_{\vec{n}}}^{(\pm)}, \quad \Pi_{\vec{n}}^{(\pm)}(2)[\cdot] = E_{\sigma_{\vec{n}}}^{(\mp)}[\cdot] E_{\sigma_{\vec{n}}}^{(\mp)},$$

where $E_{\sigma_{\vec{n}}}^{(\pm)}$ are the spectral projections corresponding to eigenvalues $\pm 1$ of the Hermitian operator $\sigma_{\vec{n}} := \vec{n} \cdot \vec{\sigma}$ that describe the projection of qubit spin $\vec{\sigma}$ on a unit direction $\vec{n}$;

(ii) the state instruments $\Pi_j(\cdot), j = 1, 2$, with elements

$$\Pi_1(1)[\cdot] = \mathbb{I}_{\mathbb{C}^2}[\cdot] \mathbb{I}_{\mathbb{C}^2}, \quad \Pi_1(2)[\cdot] = 0,$$

$$\Pi_2(1)[\cdot] = 0, \quad \Pi_2(2) = \mathbb{I}_{\mathbb{C}^2}[\cdot] \mathbb{I}_{\mathbb{C}^2}.$$ 

Therefore,

$$\mathbb{P}^{\text{opt.success}}_{A_1 \rightarrow A_2} (\rho_1, \rho_2|q_1, q_2) |_{\gamma_1, \gamma_2} = \max_{E_{\Pi_{\vec{n}}^{(A)}}(\rho_1, \rho_2, q_1, q_2)} \mathcal{E}_{\Pi}(\rho_1, \rho_2, q_1, q_2) |_{\gamma_1, \gamma_2},$$

$$= \max \left\{ \max_{\vec{n}} \mathcal{E}_{\Pi_{\vec{n}}^{(\pm)}}(\rho_1, \rho_2, q_1, q_2) |_{\gamma_1, \gamma_2}, \max_{j = 1, 2} \max \mathcal{E}_{\Pi_j}(\rho_1, \rho_2, q_1, q_2) \right\},$$

where $\mathcal{E}_{\Pi_{\vec{n}}^{(\pm)}}(\rho_1, \rho_2, q_1, q_2)$ is the expression to be maximized in (74).

It follows from (78), (79), and (A1) that

$$q_j \text{tr} \left\{ \Lambda_1[\rho_j] E_{\sigma_{\vec{n}}}^{(\pm)} \right\} = \frac{1}{2} q_j + \frac{1}{2} q_j (1 - \gamma_1)(\vec{r}_j \cdot \vec{n}),$$

and thus, in the expression $\mathcal{E}_{\Pi_{\vec{n}}^{(\pm)}}$ standing under the maximum sign in (74), for every extreme state instru-
$\Pi_\alpha^{(\pm)}$, the first term equals to
\[
\mathbb{P}_{\Pi_\alpha^{(\pm)}}^{\text{success}}(\rho_1, \rho_2 \mid q_1, q_2) |_{\gamma_1} := \mathbb{P}_{\Pi_\alpha^{(\pm)}}^{\text{success}}(A_1[\rho_1], A_1[\rho_2] \mid q_1, q_2)
\]
\[
= q_1 \text{tr} \left\{ A_1[\rho_1] E_{\sigma_\alpha}^{(\pm)} \right\} + q_2 \text{tr} \left\{ A_1[\rho_2] E_{\sigma_\alpha}^{(\mp)} \right\}
\]
\[
= \frac{1}{2} + \frac{1}{2} (1 - \gamma_1) (q_1 \vec{n}_1 - q_2 \vec{n}_2) \cdot \vec{n},
\]
and the second term is the trace norm of the Hermitian operator
\[
A_2 \left[ q_1 \Pi_\alpha^{(\pm)}(1) [A_1[\rho_1]] - q_2 \Pi_\alpha^{(\pm)}(2) [A_1[\rho_2]] \right] = \gamma_2 \left[ q_1 \text{tr} \left\{ A_1[\rho_1] E_{\sigma_\alpha}^{(\pm)} \right\} - q_2 \text{tr} \left\{ A_1[\rho_2] E_{\sigma_\alpha}^{(\mp)} \right\} \right]
\]
\[
+ (1 - \gamma_2) \left[ q_1 E_{\sigma_\alpha}^{(\pm)} \text{tr} \left\{ A_1[\rho_1] E_{\sigma_\alpha}^{(\pm)} \right\} - q_2 E_{\sigma_\alpha}^{(\mp)} \text{tr} \left\{ A_1[\rho_2] E_{\sigma_\alpha}^{(\mp)} \right\} \right]
\]
\[
= E_{\sigma_\alpha}^{(\pm)} \left\{ (1 - \gamma_2) \mathbb{P}_{\Pi_\alpha^{(\pm)}}^{\text{success}} |_{\gamma_1} - q_2 \text{tr} \left\{ A_1[\rho_2] E_{\sigma_\alpha}^{(\mp)} \right\} \right\}
\]
\[
- E_{\sigma_\alpha}^{(\mp)} \left\{ (1 - \gamma_2) \mathbb{P}_{\Pi_\alpha^{(\pm)}}^{\text{success}} |_{\gamma_1} - q_1 \text{tr} \left\{ A_1[\rho_1] E_{\sigma_\alpha}^{(\pm)} \right\} \right\}
\]
and is equal to
\[
\left| (1 - \frac{\gamma_2}{2}) \mathbb{P}_{\Pi_\alpha^{(\pm)}}^{\text{success}} |_{\gamma_1} - q_2 \text{tr} \left\{ A_1[\rho_2] E_{\sigma_\alpha}^{(\mp)} \right\} \right| + \left| (1 - \frac{\gamma_2}{2}) \mathbb{P}_{\Pi_\alpha^{(\pm)}}^{\text{success}} |_{\gamma_1} - q_1 \text{tr} \left\{ A_1[\rho_1] E_{\sigma_\alpha}^{(\pm)} \right\} \right|.
\]
Here, the terms $q_1 \text{tr} \left\{ A_1[\rho_1] E_{\sigma_\alpha}^{(\pm)} \right\}$ and $\mathbb{P}_{\Pi_\alpha^{(\pm)}}^{\text{success}}(\rho_1, \rho_2 | q_1, q_2) |_{\gamma_1}$ are given by (A4) and (A5), respectively, and, in (A6), we write $\mathbb{P}_{\Pi_\alpha^{(\pm)}}^{\text{success}} |_{\gamma_1} := \mathbb{P}_{\Pi_\alpha^{(\pm)}}^{\text{success}}(\rho_1, \rho_2 | q_1, q_2) |_{\gamma_1}$, for short.

For the extreme state instruments (A2), we have
\[
\mathbb{P}_{\Pi_\alpha}^{\text{success}}(\rho_1, \rho_2 \mid q_1, q_2) := \mathbb{P}_{\Pi_\alpha}^{\text{success}}(A_1[\rho_1], A_1[\rho_2] \mid q_1, q_2) = q_j,
\]
also, the trace norm standing in (74) is equal to $q_j$.

Equations (A5)–(A8) imply that, for the state instruments (A1) and (A2), the expression $\mathcal{E}_{\Pi(\cdot)} |_{\gamma_1, \gamma_2}$ in (A3), takes the following values:
\[
\mathcal{E}_{\Pi_\alpha^{(\pm)}}(\rho_1, \rho_2 \mid q_1, q_2) |_{\gamma_1, \gamma_2} = \frac{1}{2} \mathbb{P}_{\Pi_\alpha^{(\pm)}}^{\text{success}}(\rho_1, \rho_2 \mid q_1, q_2) |_{\gamma_1}
\]
\[
+ \frac{1}{2} \left| (1 - \frac{\gamma_2}{2}) \mathbb{P}_{\Pi_\alpha^{(\pm)}}^{\text{success}}(\rho_1, \rho_2 \mid q_1, q_2) |_{\gamma_1} - q_2 \text{tr} \left\{ A_1[\rho_2] E_{\sigma_\alpha}^{(\mp)} \right\} \right|
\]
\[
+ \frac{1}{2} \left| (1 - \frac{\gamma_2}{2}) \mathbb{P}_{\Pi_\alpha^{(\pm)}}^{\text{success}}(\rho_1, \rho_2 \mid q_1, q_2) |_{\gamma_1} - q_1 \text{tr} \left\{ A_1[\rho_1] E_{\sigma_\alpha}^{(\pm)} \right\} \right|,
\]
\[
\mathcal{E}_{\Pi_\alpha^{(\pm)}}(\rho_1, \rho_2 \mid q_1, q_2) |_{\gamma_1, \gamma_2} = q_j, \quad j = 1, 2.
\]
Substituting (A9) and (A10) into (A3), we obtain
\[
\mathbb{P}_{A_1 \uparrow \downarrow 1 \downarrow 2}^{\text{opt,success}}(\rho_1, \rho_2 | q_1, q_2) |_{\gamma_1, \gamma_2} = \max \left\{ \max_{\Pi_\alpha^{(\pm)}} \mathcal{E}_{\Pi_\alpha^{(\pm)}}(\rho_1, \rho_2, q_1, q_2) |_{\gamma_1, \gamma_2}, \max \{q_1, q_2\} \right\}.
\]
Further, relation (A9) yields
\[
\mathcal{E}_{\Pi_\alpha^{(\pm)}}(\rho_1, \rho_2 | q_1, q_2) |_{\gamma_1, \gamma_2} = \left(1 - \frac{\gamma_2}{2}\right) \mathbb{P}_{\Pi_\alpha^{(\pm)}}^{\text{success}}(\rho_1, \rho_2 | q_1, q_2) |_{\gamma_1}
\]
if
\[
\left(1 - \frac{\gamma_2}{2}\right) \mathbb{P}_{\Pi_\alpha^{(\pm)}}^{\text{success}}(\rho_1, \rho_2 | q_1, q_2) |_{\gamma_1} \geq \max \left\{ q_1 \text{tr} \left\{ A_1[\rho_1] E_{\sigma_\alpha}^{(\pm)} \right\}, q_2 \text{tr} \left\{ A_1[\rho_2] E_{\sigma_\alpha}^{(\mp)} \right\} \right\},
\]
and
\[
\mathcal{E}_{\Pi_\alpha^{(\pm)}}(\rho_1, \rho_2 | q_1, q_2) |_{\gamma_1, \gamma_2} = \max \left\{ q_1 \text{tr} \left\{ A_1[\rho_1] E_{\sigma_\alpha}^{(\pm)} \right\}, q_2 \text{tr} \left\{ A_1[\rho_2] E_{\sigma_\alpha}^{(\mp)} \right\} \right\} \leq \max \{q_1, q_2\},
\]
only otherwise.
Taking relations (A12) and (A14) into account in expression (A11), we derive

\[ P_{\text{opt.success}}^{A|\sim 1\rightarrow 2} (\rho_1, \rho_2 | q_1, q_2) |_{\gamma_1, \gamma_2} = \max \left\{ \max_{\vec{n} \in J_1} \left\{ \left( 1 - \frac{\gamma_2}{2} \right) P_{\text{success}}^{\text{opt}} (\rho_1, \rho_2 | q_1, q_2) |_{\gamma_1} \right\}, \max \{q_1, q_2\} \right\}, \]  

(A15)

where \( J_1 \) is the set of vectors \( \vec{n} \in \mathbb{R}^3 \), satisfying relation (A13). Note that, since, by (A5),

\[ P_{\text{success}}^{\text{opt}} (\rho_1, \rho_2 | q_1, q_2) |_{\gamma_1} = \frac{1}{2} \pm \frac{1}{2} (1 - \gamma_1) |q_1 \vec{r}_1 - q_2 \vec{r}_2| \cdot \vec{n}, \]

(A16)

the maximum of this expression over \( \vec{n} \in J_1 \) is attained at the unit vectors

\[ \vec{n}_{\pm} = \frac{\pm (q_1 \vec{r}_1 - q_2 \vec{r}_2)}{\|q_1 \vec{r}_1 - q_2 \vec{r}_2\|_{\mathbb{R}^3}}, \]

(A17)

respectively, and is equal to

\[ \max_{\vec{n} \in J_1} \left\{ P_{\text{success}}^{\text{opt}} (\rho_1, \rho_2 | q_1, q_2) |_{\gamma_1} \right\} = \max_{\vec{n} \in J_1} \left\{ \frac{1}{2} + \frac{1}{2} (1 - \gamma_1) \|q_1 \vec{r}_1 - q_2 \vec{r}_2\|_{\mathbb{R}^3} \right\} \]

(A18)

It follows from (A15) and (A18) that, under a depolarizing noise, the optimal two-sequential success probability \( P_{\text{opt.success}}^{A|\sim 1\rightarrow 2} (\rho_1, \rho_2 | q_1, q_2) |_{\gamma_1, \gamma_2} \)

\begin{itemize}
  \item equals to
    \[ \max \left\{ \left( 1 - \frac{\gamma_2}{2} \right) P_{\text{success}}^{\text{opt}} (\rho_1, \rho_2 | q_1, q_2) |_{\gamma_1} + \max \{q_1, q_2\} \right\} \]
    (A19)

    \[ = \max \left\{ \left( 1 - \frac{\gamma_2}{2} \right) \left( \frac{1}{2} + \frac{1}{2} (1 - \gamma_1) \|q_1 \vec{r}_1 - q_2 \vec{r}_2\|_{\mathbb{R}^3} \right), \max \{q_1, q_2\} \right\} \]

    for all \( \gamma_1, \gamma_2 \) satisfying relation (A13) with \( \vec{n} = \vec{n}_{+} \);
  \item equals to
    \[ \max \{q_1, q_2\} \]
    (A20)

for all other \( \gamma_1, \gamma_2 \).
\end{itemize}

Note that

\[ P_{\text{opt.success}}^{A|\sim 1\rightarrow 2} (\rho_1, \rho_2 | q_1, q_2) = \max \left\{ P_{\text{success}}^{\text{opt}} (\rho_1, \rho_2 | q_1, q_2) |_{\gamma_1}, \max \{q_1, q_2\} \right\}, \]

(A21)

where \( P_{\text{success}}^{\text{opt}} (\rho_1, \rho_2 | q_1, q_2) |_{\gamma_1} \) is given by (A5) with \( \vec{n} = \vec{n}_{+} \).

For a noisy parameter \( \gamma_2 \), let us now specify the range where the optimal success probability \( P_{\text{opt.success}}^{A|\sim 1\rightarrow 2} |_{\gamma_1, \gamma_2} \)

is given by expression (A19).

It follows from relation (A13), specified for \( \vec{n} = \vec{n}_{+} \), that, for a given \( \gamma_1 \), expression (A19) holds for all \( \gamma_2 \leq \gamma_2^{(1)} \), where

\[ 1 - \frac{\gamma_2^{(1)}}{2} = \max \left\{ q_1 \text{tr} \left( A_1 [\rho_1] B_{\vec{n}_{+}}^{(\pm)} \right), q_2 \text{tr} \left( A_1 [\rho_2] B_{\vec{n}_{+}}^{(\pm)} \right) \right\} \leq \max \{q_1, q_2\}, \]

(A22)

while the expression (A20) is true for all \( \gamma_2 > \gamma_2^{(1)} \).

On the other hand, it follows from (A19) that \( P_{\text{opt.success}}^{A|\sim 1\rightarrow 2} (\rho_1, \rho_2 | q_1, q_2) |_{\gamma_1, \gamma_2} \) is equal to

\[ \left( 1 - \frac{\gamma_2}{2} \right) P_{\text{success}}^{\text{opt}} (\rho_1, \rho_2 | q_1, q_2) |_{\gamma_1} \]

(A23)

for all \( \gamma_2 \leq \gamma_2^{(2)} \), where

\[ 1 - \frac{\gamma_2^{(2)}}{2} = \max \{q_1, q_2\}, \]

(A24)

\[ P_{\text{success}}^{\text{opt}} (\rho_1, \rho_2 | q_1, q_2) |_{\gamma_1} \]
and is equal to \(\max\{q_1, q_2\}\) for all \(\gamma_2 > \gamma_2^{(2)}\).

Comparing (A22) and (A24), we conclude that

\[
\left(1 - \frac{\gamma_2^{(1)}}{2}\right) \leq \left(1 - \frac{\gamma_2^{(2)}}{2}\right) \Leftrightarrow \gamma_2^{(1)} \geq \gamma_2^{(2)},
\]

(A25)

and hence, along with \(\gamma_2 > \gamma_2^{(1)}\), equality (A20) is also true for all \(\gamma_2^{(2)} \leq \gamma_2 \leq \gamma_2^{(1)}\).

Taking this into account when analyzing the validity of expressions (A19) and (A20), we come to the assertion of Theorem 4.

REFERENCES

[1] C. W. Helstrom, *Quantum Detection and Estimation Theory*, Academic Press, 1976.
[2] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory*, North-Holland, 1979.
[3] J. A. Bergou, E. Feldman, and M. Hillery, “Extracting Information from a Qubit by Multiple Observers: Toward a Theory of Sequential State Discrimination”, *Phys. Rev. Lett.*, **111** (2013), 100501.
[4] C.-Q. Pang, F.-L. Zhang, L.-F. Xu, M.-L. Liang, and J.-L. Chen, “Sequential State Discrimination and Requirement of Quantum Dissonance”, *Phys. Rev. A*, **88** (2013), 052331.
[5] J.-H. Zhang, F.-L. Zhang, and M.-L. Liang, “Sequential State Discrimination with Quantum Correlation”, *Quantum. Inf. Process.*, **17** (2018), 260.
[6] M. Namkung and Y. Kwon, “Optimal Sequential State Discrimination between Two Mixed Quantum States”, *Phys. Rev. A*, **96** (2017), 022318.
[7] M. Namkung and Y. Kwon, “Analysis of Optimal Sequential State Discrimination for Linearly Independent Pure Quantum States”, *Sci. Rep.*, **8** (2018), 6515.
[8] M. Hillery and J. Mihm, “Sequential Discrimination of Qudits by Multiple Receivers”, *J. Phys. A: Math. Theor.*, **50** (2017), 435301.
[9] M. Namkung and Y. Kwon, “Generalized Sequential State Discrimination for Multiparty QKD and Its Optical Implementation”, *Sci. Rep.*, **10** (2020), 8247.
[10] M. A. Solis-Prosser, P. Gonzalez, J. Fuenzalida, S. Gomez, G. B. Xavier, A. Delgado, and G. Lima, “Experimental Multiparty Sequential State Discrimination”, *Phys. Rev. A*, **94** (2016), 042309.
[11] M. Namkung and Y. Kwon, “Sequential State Discrimination of Coherent States”, *Sci. Rep.*, **8** (2018), 16915.
[12] T. Rudolph, R. W. Spekkens, and P. S. Turner, “Unambiguous Discrimination of Mixed States”, *Phys. Rev. A*, **68** (2003), 010301(R).
[13] D. Fields, A. Varga, and J. A. Bergou, *Sequential Measurements on Qubits by Multiple Observers: Joint Best Guess Strategy*, IEEE International Conference on Quantum Computing and Engineering (QCE), 2020.
[14] E. B. Davies, *Quantum Theory of Open Systems*, Academic Press, 1976.
[15] P. Busch, M. Grabowski, and P. J. Lahti, *Operational Quantum Physics*, Springer, 1995.
[16] A. S. Holevo, *Statistical Structure of Quantum Theory*, Springer, Berlin, 2001.
[17] E. R. Loubenets, “Quantum Stochastic Approach to the Description of Quantum Measurements”, *J. Phys. A: Math. Gen.*, **34** (2001), 7639–7675.
[18] O. E. Barndof-Nielsen and E. R. Loubenets, “General Framework for the Behaviour of Continuously Observed Open Quantum Systems”, *J. Phys. A*, **35** (2002), 565–588.
[19] M. E. Shirokov, “Entropy Reduction of Quantum Measurements”, *J. Math. Phys.*, **52** (2011), 052202.
[20] M. E. Shirokov, “On Properties of the Space of Quantum States and Their Application to the Construction of Entanglement Monotones”, *Izv. Math.*, **74**:4 (2010), 849–882.
[21] K. Kraus, *States, Effects, and Operations: Fundamental Notions of Quantum Theory*, Springer, 1983.
[22] M. Ozawa, “Quantum Measuring Processes of Continuous Observables”, *J. Math. Phys.*, **25** (1984), 79.
[23] E. R. Loubenets, “General Lower and Upper Bounds under Minimum-Error Quantum State Discrimination”, *Phys. Rev. A.*, **105** (2022), 032410.