AN APPLICATION OF COLLAPSING LEVELS TO THE REPRESENTATION THEORY OF AFFINE VERTEX ALGEBRAS

DRAŽEN ADAMOVIĆ, VICTOR G. KAC, PIERLUIGI MÖSENEDER FRAJRIA, PAOLO PAPI, AND OZREN PERŠE

Abstract. We discover a large class of simple affine vertex algebras $V_k(g)$, associated to basic Lie superalgebras $g$ at non-admissible collapsing levels $k$, having exactly one irreducible $g$–locally finite module in the category $O$. In the case when $g$ is a Lie algebra, we prove a complete reducibility result for $V_k(g)$–modules at an arbitrary collapsing level. We also determine the generators of the maximal ideal in the universal affine vertex algebra $V^g(k)$ at certain negative integer levels. Considering some conformal embeddings in the simple affine vertex algebras $V_{-1/2}(C_n)$ and $V_{-4}(E_7)$, we surprisingly obtain the realization of non-simple affine vertex algebras of types $B$ and $D$ having exactly one non-trivial ideal.

1. Introduction

Affine vertex algebras are one of the most interesting and important classes of vertex algebras. Categories of modules for simple affine vertex algebra $V_k(g)$, associated to a simple Lie algebra $g$, have mostly been studied in the case of positive integer levels $k \in \mathbb{Z}_{\geq 0}$. These categories enjoy many nice properties such as: finitely many irreducibles, semisimplicity, modular invariance of characters (cf. [26], [31], [33], [41]).

In recent years, affine vertex algebras have attracted a lot of attention because of their connection with affine $W$–algebras $W_k(g,f)$, obtained by quantum Hamiltonian reduction (cf. [21], [23], [34], [35]). Since the quantum Hamiltonian reduction functor $H_f(\cdot)$ maps any integrable $g$–module to zero (cf. [14], [34]), in order to obtain interesting $W$–algebras, one has to consider affine vertex algebras $V_k(g)$, for $k \notin \mathbb{Z}_{\geq 0}$.

It turns out that for certain non-admissible levels $k$ (such as negative integer levels), the associated vertex algebras $V_k(g)$ have finitely many irreducibles in category $O$ (cf. [15], [17], [40]), and their characters satisfy certain modular-like properties (cf. [14]). These affine vertex algebras then give $C_2$–cofinite $W$–algebras $W_k(g,f)$, for properly chosen nilpotent element $f$ (cf. [36], [38]).

In this paper, we classify irreducible modules in the category $KL_k$ (i.e. the category of $g$–locally finite $V_k(g)$–modules in $O^k$ (see Subsection 2.2)) for a large family of collapsing levels $k$. Recall from [4] that a level $k$ is called collapsing if the simple $W$–algebra $W_k(g,\theta)$, associated to a minimal nilpotent element $e_{-\theta}$, is isomorphic to its affine vertex subalgebra $V_k(g^\theta)$ (see Definition 2.2 and (2.7)). In the present paper we keep the notation of [4]. In particular, the highest root is normalized by the condition $(\theta,\theta) = 2$. We discover a large family of vertex algebras having one irreducible module in the category $KL_k$, which in a way extends the results on Deligne series from [15]. Part (1) is proven there in the Lie algebra case.

Theorem 1.1. Assume that the level $k$ and the basic simple Lie superalgebra $g$ satisfy one of the following conditions:

1. $k = -\frac{h^\vee}{2} - 1$ and $g$ is one of the Lie algebras of exceptional Deligne’s series $A_2$, $G_2$, $D_4$, $F_4$, $E_6$, $E_7$, $E_8$, or $g = \text{psl}(m|m)$ ($m \geq 2$), $\text{osp}(n + 8|n)$ ($n \geq 2$), $\text{so}(2|1)$, $F(4)$, $G(3)$ (for both choices of $\theta$);
2. $k = -\frac{h^\vee}{2} + 1$ and $g = \text{osp}(n + 4m + 8|n)$, $n \geq 2, m \geq 0$.
3. $k = -\frac{h^\vee}{2} + 1$ and $g = D_{2m}$, $m \geq 2$.
4. $k = -10$ and $g = E_8$.
Then $V_k(g)$ is the unique irreducible $V_k(g)$–module in the category $KL_k$.

We also prove a complete reducibility result in $KL_k$ (cf. Theorem 5.9, Theorem 5.7):

**Theorem 1.2.** Assume that $g$ is a Lie algebra and $k \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$. Then $KL_k$ is a semi-simple category in the following cases:

- $k$ is a collapsing level.
- $W_k(g, \theta)$ is a rational vertex operator algebra.

It is interesting that in some cases we have that $KL_k$ is a semi-simple category, but there can exist indecomposable but not irreducible $V_k(g)$–modules in the category $O$. In order to prove Theorem 1.2 we modified methods from [28] and [20] in a vertex algebraic setting. In particular we prove that the contravariant functor $M \mapsto M^\sigma$ from [20] acts on the category $KL_k$ (cf. Lemma 5.6). Then for the proof of complete reducibility in $KL_k$ it is enough to check for each highest weight $V_k(g)$–module in $KL_k$ is irreducible (cf. Theorem 5.5).

Representation theory of a simple affine vertex algebra $V_k(g)$ is naturally connected with the structure of the maximal ideal in the universal affine vertex algebra $V^k(g)$. In the second part of paper we present explicit formulas for singular vectors which generate the maximal ideal in $V^{2-2\ell}(D_{2\ell})$ (which is case (3) of Theorem 1.1) and $V^{-2}(D_4)$. In the second case, we show that the Hamiltonian reduction functor $H_0(\cdot, \cdot)$ gives an equivalence of the category of $g$–locally finite $V^{-2}(D_4)$–modules $KL_{-2}$ and the category of modules for a rational vertex algebra $V_{-4}(A_1)$. Singular vectors in $V^k(g)$ for certain negative integer levels $k$ have also been constructed in [2].

We also apply our results to study the structure of conformally embedded subalgebras of some simple affine vertex algebras.

As in [3], for a subalgebra $\mathfrak{f}$ of a simple Lie algebra $g$, we denote by $\tilde{V}(k, \mathfrak{f})$ the vertex subalgebra of $V_k(g)$ generated by $x(-1)1$, $x \in \mathfrak{f}$. If $\mathfrak{f}$ is a reductive quadratic subalgebra of $g$, then we say that $\tilde{V}(k, \mathfrak{f})$ is conformally embedded in $V_k(g)$ if the Sugawara-Virasoro vectors of both algebras coincide.

We also say that $\mathfrak{f}$ is conformally embedded in $g$ at level $k$ if $\tilde{V}(k, \mathfrak{f})$ is conformally embedded in $V_k(g)$.

We are able to prove that in the cases listed in Theorem 1.3 below, $\tilde{V}(k, \mathfrak{f})$ is not simple. On the other hand, we show that $V_{-1/2}(C_5)$ contains a simple subalgebra $V_{-2}(B_2) \otimes V_{5/2}(A_1)$ (see Corollary 7.3). For the conformal embedding of $D_6 \times A_1$ into $E_7$ at level $k = -4$, we show that $\tilde{V}(-4, D_6 \times A_1) = V_{-4}(D_6) \otimes V_{-4}(A_1)$ where $V_{-4}(D_6)$ is a quotient of the universal affine vertex algebra $V^{-4}(D_6)$ by two singular vectors of conformal weights two and three (cf. [29]). Moreover, $V_{-4}(D_6)$ has infinitely many irreducible modules in the category of $g$–locally finite modules, which we explicitly describe. All of them appear in $V_{-4}(E_7)$ as submodules or subquotients.

**Theorem 1.3.** Let $V_k(D_\ell)$, $V_k(B_\ell)$, be the vertex algebras defined in [6.3], [21], [4.6]. Consider the following conformal embeddings:

1. $D_\ell \times A_1$ into $C_{2\ell}$ for $\ell \geq 4$ at level $k = -\frac{1}{2}$.
2. $B_\ell \times A_1$ into $C_{2\ell+1}$ for $\ell \geq 3$ at level $k = -\frac{1}{2}$.
3. $D_6 \times A_1$ into $E_7$ at level $k = -4$.

Then,

- $\tilde{V}(-\frac{1}{2}, D_\ell \times A_1) = V_{-2}(D_\ell) \otimes V_{-\ell}(A_1)$ in case (1),
- $\tilde{V}(-\frac{1}{2}, B_\ell \times A_1) = V_{-2}(B_\ell) \otimes V_{-\ell-1/2}(A_1)$ in case (2),
- $\tilde{V}(-4, D_6 \times A_1) = V_{-4}(D_6) \otimes V_{-4}(A_1)$ in case (3).

Moreover, the algebras $V_k(D_\ell)$, $V_k(B_\ell)$, are non-simple, with a unique non-trivial ideal.

The decompositions of the embeddings above is still an open problem, and will be a subject of our forthcoming papers.

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2. Preliminaries

We assume that the reader is familiar with the notion of vertex (super)algebra (cf. [18, 25, 32]) and of simple basic Lie superalgebras (see [30]) and their affinizations (see [31] for the Lie algebra case).

Let $V$ be a conformal vertex algebra. Denote by $A(V)$ the associative algebra introduced in [41], called the Zhu algebra of $V$.

2.1. Basic Lie superalgebras and minimal gradings. For the reader’s convenience we recall here the setting and notation of [4] regarding basic Lie superalgebras and their minimal gradings. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a simple finite dimensional basic Lie superalgebra. We choose a Cartan subalgebra $\mathfrak{h}$ for $\mathfrak{g}_0$ and let $\Delta$ be the set of roots. Assume $\mathfrak{g}$ is not $\mathfrak{osp}(3|n)$. A root $\theta$ is called minimal if it is even and there exists an additive function $\varphi : \Delta \to \mathbb{R}$ such that $\varphi(\Delta) \neq 0$ and $\varphi(\theta) > \varphi(\eta), \forall \eta \in \Delta \setminus \{\theta\}$. Fix a minimal root $-\theta$ of $\mathfrak{g}$. We may choose root vectors $e_\theta$ and $e_{-\theta}$ such that

$$[e_\theta, e_{-\theta}] = x \in \mathfrak{h}, \quad [x, e_{\pm \theta}] = \pm e_{\pm \theta}.$$  

Due to the minimality of $-\theta$, the eigenspace decomposition of $ad x$ defines a minimal $\frac{1}{2}\mathbb{Z}$-grading (4.1)):

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{1/2} \oplus \mathfrak{g}_1,$$

where $\mathfrak{g}_{\pm 1} = \mathbb{C}e_{\pm \theta}$. We thus have a bijective correspondence between minimal gradings (up to an automorphism of $\mathfrak{g}$) and minimal roots (up to the action of the Weyl group). Furthermore, one has

$$\mathfrak{g}_0 = \mathfrak{g}^0 \oplus \mathfrak{c} x, \quad \mathfrak{g}^0 = \{ a \in \mathfrak{g}_0 \mid (a|x) = 0 \}.$$  

Note that $\mathfrak{g}^0$ is the centralizer of the triple $\{ f_\theta, x, e_\theta \}$. We can choose $\mathfrak{h}^0 = \{ h \in \mathfrak{h} \mid (h|x) = 0 \}$, as a Cartan subalgebra of the Lie superalgebra $\mathfrak{g}^0$, so that $\mathfrak{h} = \mathfrak{h}^0 \oplus \mathfrak{c} x$.

For a given choice of a minimal root $-\theta$, we normalize the invariant bilinear form ($\cdot | \cdot$) on $\mathfrak{g}$ by the condition

$$\langle \theta | \theta \rangle = 2.$$  

The dual Coxeter number $h^\vee$ of the pair $(\mathfrak{g}, \theta)$ (equivalently, of the minimal gradation (2.1)) is defined to be half the eigenvalue of the Casimir operator of $\mathfrak{g}$ corresponding to ($\cdot | \cdot$), normalized by (2.3). Since $\theta$ is the highest root, we have that $2h^\vee = (\theta | \theta + 2\rho)$ hence

$$\langle \rho | \theta \rangle = h^\vee - 1.$$  

The complete list of the Lie superalgebras $\mathfrak{g}^0$, the $\mathfrak{g}^0$-modules $\mathfrak{g}_{\pm 1/2}$ (they are isomorphic and self-dual), and $h^\vee$ for all possible choices of $\mathfrak{g}$ and of $\theta$ (up to isomorphism) is given in Tables 1,2,3 of [35]. We reproduce them below. Note that in these tables $\mathfrak{g} = \mathfrak{osp}(m|n)$ (resp. $\mathfrak{g} = \mathfrak{spo}(m|n)$) means that $\theta$ is the highest root of the simple component $\mathfrak{so}(m)$ (resp. $\mathfrak{sp}(n)$) of $\mathfrak{g}_0$. Also, for $\mathfrak{g} = \mathfrak{sl}(m|n)$ or $\mathfrak{psl}(m|n)$ we always take $\theta$ to be the highest root of the simple component $\mathfrak{sl}(m)$ of $\mathfrak{g}_0$ (for $m = 4$ we take one of the simple roots). Note that the exceptional Lie superalgebras $\mathfrak{g} = F(4)$ and $\mathfrak{g} = G(3)$ appear in both Tables 2 and 3, which corresponds to the two inequivalent choices of $\theta$, the first one being a root of the simple component $\mathfrak{sl}(2)$ of $\mathfrak{g}_0$.

| $\mathfrak{g}$ is a simple Lie algebra. |
| $\theta | \theta | \theta | h^\vee | h^\vee | h^\vee | h^\vee |
| --- | --- | --- | --- | --- | --- | --- |
| $\mathfrak{g}_0$ | $\mathfrak{g}_{1/2}$ | $\mathfrak{g}_{-1/2}$ | $\mathfrak{h}$ | $\mathfrak{g}^0$ | $\mathfrak{c} x$ | $\mathfrak{h}^0$ | $\mathfrak{h}^0$ |
| $\mathfrak{sl}(n), n \geq 2$ | $\mathfrak{sl}(n-2)$ | $\mathfrak{C}^{(n-2)} \oplus (\mathfrak{C}^{(n-2)})^*$ | $\mathfrak{n}_p$ | $\mathfrak{F}_p$ | $\mathfrak{sp}(0)$ | $\mathfrak{sl}_2(\mathfrak{C})$ | $\mathfrak{so}(2)$ |
| $\mathfrak{so}(n), n \geq 3$ | $\mathfrak{so}(n-2) \oplus \mathfrak{so}(n-4)$ | $\mathfrak{C}^2 \otimes (\mathfrak{C}^2)^*$ | $\mathfrak{n}_p$ | $\mathfrak{F}_p$ | $\mathfrak{so}(10)$ | $\mathfrak{so}(4)$ | $\mathfrak{so}(12)$ |
| $\mathfrak{sp}(n), n \geq 2$ | $\mathfrak{sp}(n-2)$ | $\mathfrak{C}^{(n+2)} \oplus (\mathfrak{C}^{(n+2)})^*$ | $\mathfrak{n}_p$ | $\mathfrak{F}_p$ | $\mathfrak{so}(12)$ | $\mathfrak{so}(n)$ | $\mathfrak{so}(2n)$ |
| $\mathfrak{G}_2$ | $\mathfrak{sl}(2)$ | $\mathfrak{C}^2 \otimes (\mathfrak{C}^2)^*$ | 4 | $\mathfrak{E}_8$ | $\mathfrak{F}_7$ | dim = 56 | 30 |

| $\mathfrak{g}$ is not a Lie algebra but $\mathfrak{g}^0$ is and $\mathfrak{g}_{\pm 1/2}$ is purely odd $\lbrace m \geq 1 \rbrace$. |
Both $\mathfrak{g}$ and $\mathfrak{g}^\hat{\cdot}$ are not Lie algebras ($m, n \geq 1$).

\[
\mathfrak{g}^\hat{\cdot} = \bigoplus_{i \in I} \mathfrak{g}_i^\hat{\cdot},
\]

where each summand is either the (at most 1-dimensional) center of $\mathfrak{g}^\hat{\cdot}$ or is a basic simple Lie superalgebra different from $\mathfrak{psl}(n|m)$. Let $C_{\mathfrak{g}_i^\hat{\cdot}}$ be the Casimir operator of $\mathfrak{g}_i^\hat{\cdot}$ corresponding to $(\cdot, \cdot)_{\mathfrak{g}_i^\hat{\cdot} \times \mathfrak{g}_i^\hat{\cdot}}$.

We define the dual Coxeter number $h_{0,i}^\vee$ of $\mathfrak{g}_i^\hat{\cdot}$ as half of the eigenvalue of $C_{\mathfrak{g}_i^\hat{\cdot}}$ acting on $\mathfrak{g}_i^\hat{\cdot}$ (which is 0 if $\mathfrak{g}_i^\hat{\cdot}$ is abelian).

Denote by $V_q(\mu)$ (or $V(\mu)$) the irreducible finite-dimensional highest weight $\mathfrak{g}$-module with highest weight $\mu$. Denote by $P_+$ the set of highest weights of irreducible finite-dimensional representations of $\mathfrak{g}$.

Since $\mathfrak{h} = \mathfrak{h}^\hat{\cdot} \oplus \mathbb{C}x$, we have, in particular, that $\mu \in \mathfrak{h}^\hat{\cdot}$ can be uniquely written as

\[
(2.5) \quad \mu = \mu|_{\mathfrak{h}^\hat{\cdot}} + \ell \theta,
\]

with $\ell \in \mathbb{C}$. If $\mu \in P_+$, then, since $\theta(\mathfrak{h}^\hat{\cdot}) = 0$, $\mu(\mathfrak{h}^\hat{\cdot}) = 2\ell \in \mathbb{Z}$, so $\ell \in \frac{1}{2}\mathbb{Z}_{\geq 0}$.

### 2.2. Affine Lie algebras, vertex algebras, $W$-algebras

Let $\widehat{\mathfrak{g}}$ be the affinization of $\mathfrak{g}$:

\[
\widehat{\mathfrak{g}} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g} \oplus \mathbb{C}K \oplus \mathbb{C}d
\]

with the usual commutation relations. We let $\delta$ be the fundamental imaginary root. Let $\alpha_0 = \delta - \theta$ the affine simple root. Since $\theta$ is even, hence non-isotropic, so that $\alpha_0^\vee = K - \theta^\vee$ makes sense.

Denote by $\hat{L}(\lambda)$ (or $\hat{L}_q(\lambda)$) the irreducible highest weight $\widehat{\mathfrak{g}}$-module with highest weight $\lambda$.

Denote by $V^k(\mathfrak{g})$ the universal affine vertex algebra associated to $\mathfrak{g}$ of level $k \in \mathbb{C}$. We shall assume that $k \neq -h^\vee$. Then (see e.g. [32]) $V^k(\mathfrak{g})$ is a conformal vertex algebra with Segal-Sugawara conformal vector $\omega_q$. Let $Y(\omega_q, z) = \sum L_q(n)z^{-n-2}$ be the corresponding Virasoro field. Denote by $V_k(\mathfrak{g})$ the (unique) simple quotient of $V^k(\mathfrak{g})$. Clearly, $V_k(\mathfrak{g}) \cong L_q(k\lambda_0)$ as $\widehat{\mathfrak{g}}$-modules.

Denote by $W^k(\mathfrak{g}, \theta)$ the affine $W$-algebra obtained from $V^k(\mathfrak{g})$ by Hamiltonian reduction relative to a minimal nilpotent element $e_{-\theta}$. Denote by $W_k(\mathfrak{g}, \theta)$ the simple quotient of $W^k(\mathfrak{g}, \theta)$. Recall that the vertex algebra $W^k(\mathfrak{g}, \theta)$ is strongly and freely generated by elements $J^{(a)}$, where $a$ runs over a basis of $\mathfrak{g}^\hat{\cdot}$, $G^{(v)}$, where $v$ runs over a basis of $\mathfrak{g}_{-1/2}$, and the Virasoro vector $\omega$. The elements $J^{(a)}$, $G^{(v)}$ are primary of conformal weight 1 and 3/2, respectively, with respect to $\omega$.

Let $V^k(\mathfrak{g}^\hat{\cdot})$ be the subalgebra of the vertex algebra $W^k(\mathfrak{g}, \theta)$, generated by $\{J^{(a)} \mid a \in \mathfrak{g}^\hat{\cdot}\}$. The vertex algebra $V^k(\mathfrak{g}^\hat{\cdot})$ is isomorphic to a universal affine vertex algebra. More precisely, letting

\[
(2.6) \quad k_i = k + \frac{1}{2}(h^\vee - h_{0,i}^\vee), \quad i \in I,
\]

the map $a \mapsto J^{(a)}$ extends to an isomorphism $V^k(\mathfrak{g}^\hat{\cdot}) \cong \bigotimes_{i \in I} V_{k_i}(\mathfrak{g}_i^\hat{\cdot})$. 

We also set $\mathcal{V}_k(\mathfrak{g}^2)$ to be the image of $\mathcal{V}^k(\mathfrak{g}^2)$ in $W_k(\mathfrak{g}, \theta)$. Clearly we can write

\[(2.7) \quad \mathcal{V}_k(\mathfrak{g}^2) \cong \bigotimes_{i \in I} \mathcal{V}_k(\mathfrak{g}^i),\]

where $\mathcal{V}_k(\mathfrak{g}^i)$ is some quotient (not necessarily simple) of $V^k(\mathfrak{g}^i)$.

2.3. **Category $\mathcal{O}$ and Hamiltonian reduction functor.** Recall that $\mathfrak{g}$-module $M$ is in category $\mathcal{O}^k$ if it is $\mathfrak{h}$-diagonalizable with finite dimensional weight spaces, $K$ acts as $kId_M$ and $M$ has a finite number of maximal weights.

There is a remarkable functor $H_\theta$ from $\mathcal{O}^k$ to the category of $W^k(\mathfrak{g}, \theta)$-modules whose properties will be very important in the following. We recall them in a form suitable for our purposes (see [12] for details; there $H_\theta$ is denoted by $H^k$).

**Theorem 2.1.**

1. $H_\theta$ is exact.
2. If $L(\lambda)$ is a irreducible highest weight $\mathfrak{g}$-module, then $\lambda(\alpha_0^\vee) \in \mathbb{Z}_{>0}$ implies $H_\theta(L(\lambda)) = \{0\}$. Otherwise $H_\theta(L(\lambda))$ is isomorphic to the irreducible $W^k(\mathfrak{g}, \theta)$-module with highest weight $\phi_\lambda$ defined by formula (67) in [12].

2.4. **Collapsing levels.**

**Definition 2.2.** Assume $k \neq -h^\vee$. If $W_k(\mathfrak{g}, \theta) = \mathcal{V}_k(\mathfrak{g}^2)$, we say that $k$ is a collapsing level.

**Theorem 2.3.** [4] Theorem 3.3] Let $p(k)$ be the polynomial listed in Table 4 below. Then $k$ is a collapsing level if and only if $k \neq -h^\vee$ and $p(k) = 0$. In such cases,

\[(2.8) \quad W_k(\mathfrak{g}, \theta) = \bigotimes_{i \in I^*} \mathcal{V}_k(\mathfrak{g}^i),\]

where $I^* = \{i \in I \mid k_i \neq 0\}$. If $I^* = \emptyset$, then $W_k(\mathfrak{g}, \theta) = \mathbb{C}$.

Table 4

| $\mathfrak{g}$ | $p(k)$ |
|---------------|--------|
| $o(2m)$, $m \neq m$ | $(k + 4)(k + m - n)/2$ |
| $o(2m)$ | $k(k + 1)$ |
| $so(2m)$ | $(k + 2)(k + m - n)/2$ |
| $sp(2m)$ | $(k + 2)(k + m - n)/2$ |
| $D(2, 1; a)$ | $(k - a)(k + 1 + a)$ |
| $F(4)$, $g^2 = so(7)$ | $(k + 2/3)(k - 2/3)$ |
| $g^2 = D(2, 1; 2)$ | $(k + 2/3)(k + 1)$ |
| $G(3)$, $g^2 = G_2$ | $(k - 1/2)(k + 3/4)$ |
| $G(3)$, $g^2 = osp(3/2)$ | $(k + 2)/3(k + 3/4)$ |

2.5. **Weyl vertex algebra.** Let $M_\ell$ denote the Weyl vertex algebra (also called symplectic bosons) generated by even elements $a_i^\pm$, $i = 1, \ldots, k$ satisfying the following $\lambda$-brackets

\[\{ (a_i^+)\lambda(a_j^+) \} = 0, \quad \{ (a_i^-)\lambda(a_j^-) \} = \delta_{ij}.\]

Recall also that the symplectic affine vertex algebra $V_{-1/2}(C_k)$ is realized as a $\mathbb{Z}_2$-orbifold of $M_\ell$ (see [22]).

3. **The category $KL_k$**

Let $k$ be a noncritical level. Note that the Casimir element of $\widehat{\mathfrak{g}}$ can be expressed as $\Omega = d + L_\theta(0)$; it commutes with the $\widehat{\mathfrak{g}}$-action.

Consider the category $\mathcal{O}^k$ of modules for the universal affine vertex algebra $V^k(\mathfrak{g})$, i.e. the category of restricted $\widehat{\mathfrak{g}}$-modules of level $k$. Regard $M \in \mathcal{O}^k$ as a $\widehat{\mathfrak{g}}$-module by letting $d$ act as $-L_\theta(0)$. Let $KL_k$ be the category of modules $M \in \mathcal{O}^k$ such that, as $\widehat{\mathfrak{g}}$-modules, are in $\mathcal{O}^k$ and which admit the following weight space decomposition with respect to $L_\theta(0)$:

\[M = \bigoplus_{\alpha \in \mathbb{C}} M(\alpha), \quad L_\theta(0)|M(\alpha) \equiv \alpha \text{ Id}, \quad \dim M(\alpha) < \infty.\]
Our definition is related but different from the one introduced in [13]. Let $KL_k$ be the category of all modules in $KL^k$ which are $V_k(\mathfrak{g})$–modules.

**Remark 3.1.** If $V_k(\mathfrak{g})$ has finitely many irreducible modules in the category $KL^k$, one can show that every $V_k(\mathfrak{g})$–module $M$ in $KL_k$ is of finite length. This happens when $k$ is admissible (cf. [12]) and when $V_k(\mathfrak{g})$ is quasi-lisse (cf. [14]). But when $V_k(\mathfrak{g})$ has infinitely many irreducible modules in $KL^k$ (as in the cases considered in [39], [11]), then one can have modules in $KL_k$ of infinite length.

Recall that there is a one-to-one correspondence between irreducible $\mathbb{Z}_{\geq 0}$–graded modules for a conformal vertex algebra $V$ (with a conformal vector $\omega$, such that $Y(\omega, z) = \sum_{i \in \mathbb{Z}} L(i) z^{-i-2}$) and irreducible modules for the corresponding Zhu algebra $A(V)$ [11]. This implies, in particular, that there is a one-to-one correspondence between irreducible finite-dimensional $A(V)$–modules and irreducible $\mathbb{Z}_{\geq 0}$–graded $V$–modules whose graded components, which are eigenspaces for $L(0)$, are finite-dimensional. In the case of affine vertex algebras, we have the following simple interpretation.

**Proposition 3.2.** Let $\tilde{V}_k(\mathfrak{g})$ be a quotient of $V^k(\mathfrak{g})$ (not necessary simple). Consider $\tilde{V}_k(\mathfrak{g})$ as a conformal vertex algebra with conformal vector $\omega_{\mathfrak{g}}$. Then there is a one-to-one correspondence between irreducible $\tilde{V}_k(\mathfrak{g})$ in the category $KL^k$ and irreducible finite-dimensional $A(\tilde{V}_k(\mathfrak{g}))$–modules.

**Corollary 3.3.** Assume that $\mathfrak{g}$ is a simple basic Lie superalgebra and $\tilde{V}_k(\mathfrak{g})$ is a quotient of $V^k(\mathfrak{g})$ such that the trivial module $\mathbb{C}$ is the unique finite-dimensional irreducible $A(\tilde{V}_k(\mathfrak{g}))$–module. Then $\tilde{V}_k(\mathfrak{g}) = V_k(\mathfrak{g})$.

**Proof.** Assume that $\tilde{V}_k(\mathfrak{g})$ is not simple. Then it contains a non-zero graded ideal $I \neq \tilde{V}_k(\mathfrak{g})$ with respect to $L_0(0)$:

\[
I = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} I(n + n_0), \quad L_0(0)|I(r) = r \text{Id}, \quad I(n_0) \neq 0.
\]

Since $I \neq \tilde{V}_k(\mathfrak{g})$, we have that $n_0 > 0$, otherwise $1 \in I$.

We can consider $I(n_0)$ as a finite-dimensional module for $\mathfrak{g}$ and for the Zhu algebra $A(\tilde{V}_k(\mathfrak{g}))$.

Since the Casimir element $C_\mathfrak{g}$ of $\mathfrak{g}$ acts on $I(n_0)$ as the non-zero constant $2(k + h^\vee)n_0$, we conclude that $C_\mathfrak{g}$ acts by the same constant on any irreducible $\mathfrak{g}$–subquotient of $I(n_0)$. But any irreducible subquotient of $I(n_0)$ is an irreducible finite–dimensional $A(\tilde{V}_k(\mathfrak{g}))$–module, and therefore it is trivial. This implies that $C_\mathfrak{g}$ acts non-trivially on a trivial $\mathfrak{g}$–module, a contradiction. \hfill $\Box$

Take the Chevalley generators $e_i, f_i, h_i$, $i = 0, \ldots, \ell$, of the Kac–Moody Lie algebra $\hat{\mathfrak{g}}$ such that $e_i, f_i, h_i$, $i = 1, \ldots, \ell$, are the Chevalley generators of $\mathfrak{g}$. Let $\sigma$ be the Chevalley anti-automorphism of $\hat{\mathfrak{g}}$ defined by

\[
e_i \mapsto f_i, \quad f_i \mapsto e_i, \quad h_i \mapsto h_i, \quad d \mapsto d \quad (i = 0, \ldots, \ell).
\]

Assume that $M$ is from the category $\mathcal{O}$ of non-critical level $k$. Then $M$ admits the decomposition into weight spaces $M = \bigoplus_{\mu \in \Omega(M)} M_\mu$, where $\Omega(M)$ is the set of weights of $M$ and $\dim M_\mu < \infty$ for every $\mu \in \Omega(M)$. For a finite-dimensional vector spaces $U$, let $U^*$ denote its dual space. Then we have the contravariant functor $M \mapsto M^\sigma$ [20] acting on modules from the category $\mathcal{O}$. Here $M^\sigma = \bigoplus_{\mu \in \Omega(M)} M_\mu^*$ is the $\hat{\mathfrak{g}}$–module uniquely determined by

\[
\langle yw', w \rangle = \langle w', \sigma(y)w \rangle, \quad y \in \hat{\mathfrak{g}}, \quad w', w \in M^\sigma, \quad w \in M.
\]

It is easy to see that $M$ admits the decomposition

\[
M = \bigoplus_{\alpha \in \mathbb{C}} M(\alpha), \quad L_0(0)|M(\alpha) = \alpha \text{Id}
\]

such that :

- for any $\alpha \in \mathbb{C}$ we have $M(\alpha - n) = 0$ for $n \in \mathbb{Z}$ sufficiently large;
- for any $\mu \in \Omega(M)$ there exist $\alpha \in \mathbb{C}$ such that $M_\mu \subset M(\alpha)$.
Proposition 3.4. Assume that a module $M$ is in the category $O^k$. Then $M$ is in the category $KL^k$ if and only if $M$ is $g$-locally finite.

Proof. If $M$ is in $KL^k$ then it admits a decomposition as in (3.1). Since the spaces $M(\alpha)$ are $g$-stable and finite-dimensional, $M$ is $g$-locally finite.

Let us prove the converse. If $M$ is a highest weight module which is $g$-locally finite, then clearly all eigenspaces for $L_g(0)$ are finite-dimensional. Assume now that $M$ is an arbitrary $g$-locally finite module in the category $O^k$. Take $\alpha \in \mathbb{C}$ such that $M(\alpha) \neq \{0\}$. Then from [20] Proposition 3.1 we see that $M$ has an increasing filtration (possibly infinite)

$$\{0\} = M_0 \subset M_1 \subset \cdots \subset M$$

such that for every $j \in \mathbb{Z}_{>0}$, $M_j/M_{j-1} \cong \widetilde{L}(\lambda_j)$ is a highest weight $V^k(g)$-module with highest weight $\lambda_j$, which is $g$-locally finite. Let $h_{\lambda_j}$ denotes the lowest conformal weight of $\widetilde{L}(\lambda_j)$. Since the factors $M_j/M_{j-1}$ are highest weight modules, their $L_g(0)$-eigenspaces are finite-dimensional. This implies that the $L_g(0)$-eigenspaces of $M_j$ is finite-dimensional. By using the properties of the category $O$ one sees the following:

- There exists a finite subset $\{d_1, \cdots, d_s\} \subset \mathbb{C}$ such that $\alpha \in \bigcup_{i=1}^s (d_i + \mathbb{Z}_{>0})$.

- For $d \in \mathbb{C}$ there exist only finitely many subquotients $\widetilde{L}(\lambda_j)$ in (3.2) such that $h_{\lambda_j} = d$.

This implies that there is $j_0 \in \mathbb{Z}_{>0}$ such that $\alpha < h_{\lambda_j}$ for $j \geq j_0$. Therefore $M(\alpha) \subset M_{j_0}$. This proves that $M(\alpha)$ is finite-dimensional.

Remark 3.5. We will use several times the following fact, which is a consequence of the previous proposition: for any $k \not\in \mathbb{Z}_{>0}$ and any irreducible highest weight module $L(\lambda)$ in the category $KL^k$, one has $\lambda(\alpha_0^g) \not\in \mathbb{Z}_{>0}$.

Since $\sigma(L_g(0)) = L_g(0)$, if $M$ is in the category $KL^k$, then $M^\sigma$ is also in the category $KL^k$. The next result shows that this functor acts on the category $KL_k$. In the proof we find an explicit relation of $M^\sigma$ with the contragredient modules, defined for ordinary modules for vertex operator algebras [23].

Lemma 3.6.

1. Assume that $M$ is a $V_k(g)$-module in the category $O$. Then $M^\sigma$ is also a $V_k(g)$-module in the category $O$.

2. Assume that $M$ is a $V_k(g)$-module in the category $KL_k$. Then $M^\sigma$ is also in $KL_k$.

Proof. Assume that $M$ is a $V_k(g)$-module in the category $O$. Take the weight decomposition $M = \bigoplus_{\mu \in \text{OR}(M)} M_\mu$, and set $M^e = \bigoplus_{\mu \in \text{OR}(M)} M_\mu^e$. By applying the same approach as in the construction of the contragredient module from [23] Section 5, we get a $V_k(g)$-module $(M^e, Y_{M^e}(-, z))$, with vertex operator map

$$\langle Y_{M^e}(v, z)w, w' \rangle = \langle w', Y_{M^e}(e^{zL_1(-)}L_0(0)v, z)w \rangle,$$

where $w' \in M^e$, $w \in M$. The $\hat{g}$-action on $M^e$ is uniquely determined by

$$\langle x(n)w, w' \rangle = -\langle w', x(-n)w \rangle \quad (x \in \mathfrak{g}).$$

As a vector space $M^e = M^\sigma$, but we have different actions of $\hat{g}$. (Note that, in general, $M^e$ can be outside of the category $O$.)

Take the Lie algebra automorphism $h \in \text{Aut}(g)$ such that

$$e_i \mapsto -f_i, \quad f_i \mapsto -e_i, \quad h_i \mapsto -h_i \quad (i = 1, \ldots, \ell).$$

Then $h$ can be lifted to an automorphism of $V^k(g)$. Since the maximal ideal of $V^k(g)$ is unique, then it is $h$-invariant, thus $h$ is also an automorphism of $V_k(g)$. Then we can define a $V_k(g)$-module $(M^e_h, Y_{M^e_h}(-, z))$ where

$$M^e_h := M^e, \quad Y_{M^e_h}(v, z) = Y_{M^e}(hv, z).$$
On $M^c_k$ we have
\[(e_i(n)w', w) = \langle w', f_i(-n)w \rangle \]
\[(f_i(n)w', w) = \langle w', e_i(-n)w \rangle \]
\[\langle h_i(n)w', w \rangle = \langle w', h_i(-n)w \rangle \]
where $i = 1, \ldots, \ell$. This implies that $M^c_k = M^\sigma$. This proves the assertion (1).

Assume now that $M$ is in the category $KL_k$. Then all $L_\theta(0)$-eigenspaces are finite-dimensional, thus
\[M^c = \bigoplus_{\mu \in \Omega(M)} M^c_\mu = \bigoplus_{\alpha \in \mathbb{C}} M(\alpha)^*\]
This implies the $V_k(\mathfrak{g})$-module $(M^c, Y_{M^c}^{\ast}(\cdot, z))$ coincides with the contragredient module [23], realized on the restricted dual space $\bigoplus_{\alpha \in \mathbb{C}} M(\alpha)^*$, with the vertex operator map [5.3]. Since the $L_\theta(0)$-eigenspaces of $M^c$ are finite-dimensional, we conclude that $M^c$ and $M^\sigma = M^c_k$ are $V_k(\mathfrak{g})$-modules in $KL_k$. Claim (2) follows. $\square$

4. Constructions of vertex algebras with one irreducible module in $KL_k$ via collapsing levels

By [4], if $k$ is a collapsing level, then either $W_k(\mathfrak{g}, \theta) = \mathbb{C}$, $W_k(\mathfrak{g}, \theta) = M(1)$, or $W_k(\mathfrak{g}, \theta) = V_k(\mathfrak{a})$ for a unique simple component $\mathfrak{a}$ of $\mathfrak{g}^\mathbb{D}$. Here the level $k'$ is computed with respect to the invariant bilinear form of $\mathfrak{a}$ normalized so that the minimal root has squared length 2. For $\mathfrak{a} = sl(m|n)$, $m \geq 2$, the minimal root is always chosen to be the lowest root of $sl(m)$. For $\mathfrak{a} = osp(m|n)$ we write $so(m|n)$ vs. $osp(m|n)$ to specify the choice of the minimal root. In all other cases the minimal root of $\mathfrak{a}$ is unique.

To simplify notation define $V_k(\mathfrak{g}^\mathbb{D})$ to be as follows:
\[V_k(\mathfrak{g}^\mathbb{D}) = \begin{cases} \mathbb{C} & \text{if } W_k(\mathfrak{g}, \theta) = \mathbb{C}; \text{ in this case we set } k' = 0; \\ M(1) & \text{if } W_k(\mathfrak{g}, \theta) = M(1); \text{ in this case we set } k' = 1; \\ V_k(\mathfrak{a}) & \text{otherwise.} \end{cases} \]

In Table 5 we summarize all the relevant data.

Assume that $k \notin \mathbb{Z}_{\geq 0}$ and that:

1) $k$ is a collapsing level for $\mathfrak{g}$;
2) $V_k(\mathfrak{g}^\mathbb{D})$ is the unique irreducible $V_k(\mathfrak{g}^\mathbb{D})$-module in the category $KL_{k'}$.

Assume that $L(\hat{\Lambda})$ is an irreducible $V_k(\mathfrak{g})$-module in the category $KL_k$. Set $\mu = \hat{\Lambda}|_h$. By Proposition 3.4 we have $\mu \in P_+$, hence, by (2.5), the weight $\mu$ has the form $\mu = \mu^2 + \ell \theta$ with $\ell \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, where $\mu^2 = \mu|_{h^2}$.

Since $k \notin \mathbb{Z}_{\geq 0}$, by Theorem 2.4, $H_\theta(L(\hat{\Lambda}))$ is a non-trivial irreducible module for $W_k(\mathfrak{g}, \theta)$. Since $L(\hat{\Lambda})$ is a quotient of the Verma module $M(\hat{\Lambda})$, then, by exactness of $H_\theta$, $H_\theta(L(\hat{\Lambda}))$ is the quotient of a Verma module for $W_k(\mathfrak{g}, \theta) = V_k(\mathfrak{g}^\mathbb{D})$ hence it is an irreducible highest weight module. By (6.14) its highest weight $\Lambda^k(K)$ is $k'$ and $\Lambda^k|_{h^2} = \mu^2$. Therefore
\[H_\theta(L(\hat{\Lambda})) = L_{\mathfrak{g}^\mathbb{D}}(\Lambda^2)\]
In particular $H_\theta(L(\hat{\Lambda}))$ is in the category $KL_{k'}$.

Moreover, under the identification of the centralizer $\mathfrak{g}^f$ of $f$ in $\mathfrak{g}$ with $\mathfrak{g}_0 \oplus \mathfrak{g}_{1/2}$ via $ad(f)$ (see Example 6.2 of [35]), we get that $x$ acts on $H_\theta(L(\hat{\Lambda}))$ via $J_0^{(f)}$, and $J^{(f)}$ is the conformal vector of $W(k, \theta)$ (see the proof of Theorem 5.1 of [35]). Since the level is collapsing we know, by Proposition 4.1 of [4], that the conformal vector of $W_k(\mathfrak{g}, \theta)$ coincides with the Segal-Sugawara vector conformal
Table 5
Values of $k$ and $k'$.  

| $g$                        | $V_{k'}(g^2)$                           | $k$  | $k'$         |
|---------------------------|----------------------------------------|------|--------------|
| $sl(m|n)$, $m \neq n, m > 3, m - 2 \neq n$ | $V_{k'}(sl(m - 2|n))$                   | $\frac{n - m}{2}$ | $\frac{n - m + 2}{2}$ |
| $sl(3|n)$, $n \neq 3, n \neq 1, n \neq 0$ | $V_{k'}(sl(1|n))$                      | $\frac{n - 3}{2}$ | $\frac{1 - n}{2}$ |
| $sl(3)$                    | $\mathbb{C}$                           | $-\frac{3}{2}$ | $0$          |
| $sl(2|n)$, $n \neq 2, n \neq 1, n \neq 0$ | $V_{k'}(sl(n))$                       | $\frac{n - 2}{2}$ | $-\frac{n}{2}$ |
| $sl(2|1) = spo(2|2)$       | $\mathbb{C}$                           | $-\frac{1}{2}$ | $0$          |
| $sl(m|n)$, $m \neq n, n + 1, n + 2, m \geq 2$ | $M(1)$                               | $-1$ | $1$          |
| $psl(m|m)$, $m \geq 2$    | $\mathbb{C}$                           | $-1$ | $0$          |
| $spo(n|m)$, $m \neq n, n + 2, n \geq 4$ | $V_{k'}(spo(n - 2|m))$               | $\frac{m - n - 1}{4}$ | $\frac{m - n - 2}{4}$ |
| $spo(2|m)$, $m \geq 5$   | $V_{k'}(so(m))$                        | $\frac{m - 6}{4}$ | $\frac{4 - m}{2}$ |
| $spo(2|3)$                 | $V_{k'}(sl(2))$                       | $-\frac{4}{3}$ | $1$          |
| $spo(2|1)$                 | $\mathbb{C}$                           | $-\frac{4}{3}$ | $0$          |
| $spo(n|m)$, $m \neq n + 1, n \geq 2$ | $\mathbb{C}$                       | $-1/2$ | $0$          |
| $osp(m|n)$, $m \neq n, n \neq 0, m \geq 7$ | $V_{k'}(osp(m - 4|n))$            | $\frac{n - m - 4 + 2}{2}$ | $\frac{8 - m + n}{2}$ |
| $osp(m|n)$, $n \neq m, 0; 4 \leq m \leq 6$ | $V_{k'}(osp(m - 4|n))$            | $\frac{n - m - 4 + 2}{2}$ | $\frac{m - n - 8}{4}$ |
| $osp(n + 8|n)$, $n \geq 0$ | $\mathbb{C}$                           | $-2$ | $0$          |
| $D(2, 1; a)$              | $V_{k'}(sl(2))$                       | $a$  | $\frac{1 + 2a}{1 + a}$ |
| $D(2, 1; a)$              | $V_{k'}(sl(2))$                       | $-a - 1$ | $\frac{1 + 2a}{1 + a}$ |
| $F(4)$                    | $V_{k'}(D(2, 1; 2))$                   | $-1$ | $\frac{1}{2}$ |
| $F(4)$                    | $V_{k'}(so(7))$                       | $\frac{2}{3}$ | $-2$          |
| $F(4)$                    | $\mathbb{C}$                           | $-\frac{2}{3}$ | $0$          |
| $E_6$                     | $V_{k'}(sl(6))$                       | $-4$ | $-1$          |
| $E_6$                     | $\mathbb{C}$                           | $-3$ | $0$          |
| $E_7$                     | $V_{k'}(so(12))$                      | $-6$ | $-2$          |
| $E_7$                     | $\mathbb{C}$                           | $-4$ | $0$          |
| $E_8$                     | $V_{k'}(E_7)$                         | $-10$ | $-4$          |
| $E_8$                     | $\mathbb{C}$                           | $-6$ | $0$          |
| $F_4$                     | $V_{k'}(sp(6))$                       | $-3$ | $-\frac{1}{2}$ |
| $F_4$                     | $\mathbb{C}$                           | $-5/2$ | $0$          |
| $G_2$                     | $V_{k'}(sl(2))$                       | $-\frac{4}{3}$ | $1$          |
| $G_2$                     | $\mathbb{C}$                           | $-\frac{4}{3}$ | $0$          |
| $G(3)$                    | $V_{k'}(G_2)$                         | $\frac{1}{2}$ | $-\frac{5}{3}$ |
| $G(3)$                    | $\mathbb{C}$                           | $-\frac{2}{3}$ | $0$          |
| $G(3)$                    | $V_{k'}(osp(3|2))$                     | $-\frac{2}{3}$ | $1$          |
| $G(3)$                    | $\mathbb{C}$                           | $-\frac{1}{3}$ | $0$          |
Proof. Then irreducible \( W \) which extends a result of [15] for Lie algebras to the super case. By using the above analysis and properties of Hamiltonian reduction, we get the following lemma, Theorem 4.2.

Assume that the level

\[
(4.4)
\]

Proof. By Theorem 2.1, if \((4.4) \) imply that \( \bar{\theta} \) of \( H(\bar{\lambda}) \) by \( cI \) with

\[
(4.1)
\]

Now condition (2) implies that \( \mu \bar{\theta} = 0 \), so \( \mu = \ell \theta \) and

\[
\frac{(\mu + 2\rho, \mu)}{2(k + h^\vee)} - \mu(x) = \frac{(\ell \theta + 2\rho, \ell \theta)}{2(k + h^\vee)} - \ell \theta = 0.
\]

By using formula (2.4), we get

\[
(4.2)
\]

- Consider first the case \( k = -h^\vee/2 + 1 \) (this holds for \( g = D_{2n}, n \geq 2 \) and \( g = \mathfrak{osp}(n + 4m + 8n), n \geq 0 \)). Then (4.2) gives that

\[
2\ell^2 + (2h^\vee - 2)\ell
\]

\[
= \frac{(k + h^\vee)}{k + h^\vee} = 0.
\]

We get \( \ell = 0 \) or \( 2\ell + h^\vee - 4 = 0 \).

- Next we consider the case \( k = -h^\vee/6 - 1 \). We get

\[
(4.3)
\]

\[
6\ell^2 + h^\vee h
\]

\[
= \frac{(k + h^\vee)}{h^\vee - 6} = 0.
\]

We conclude that \( \ell = 0 \) or \( \ell = -\frac{h^\vee}{6} \).

By using the above analysis and properties of Hamiltonian reduction, we get the following lemma, which extends a result of [15] for Lie algebras to the super case.

**Lemma 4.1.** Assume that \( k = -\frac{h^\vee}{\theta} - 1 \) and \( g \) is one of the Lie algebras of exceptional Deligne’s series \( A_2, G_2, D_4, F_4, E_6, E_7, E_8 \), or \( g = \mathfrak{psl}(m|m) \) \( (m \geq 2) \), \( \mathfrak{osp}(n + 8n) \) \( (n \geq 2) \), \( \mathfrak{spo}(2|1) \), \( F(4) \), \( G(3) \) \( (\text{for both choices of } \theta) \).

Assume that \( L(\lambda) \) is a \( V_k(g) \)-module in the category \( \mathcal{O} \). Then one of the following condition holds:

1. \( \lambda(\alpha_i^\vee) \in \mathbb{Z}_{\geq 0} \);
2. \( \bar{\lambda} \) is either 0 or \( -\frac{h^\vee}{\theta} \), where \( \bar{\lambda} \) is the restriction of \( \lambda \) to \( \mathfrak{h} \).

Proof. By Theorem 2.1 if \( L(\lambda) \) is a \( V_k(g) \)-module for which \( \lambda(\alpha_i^\vee) \notin \mathbb{Z}_{\geq 0} \), then \( H_\theta(L(\lambda)) \) is an irreducible \( W_k(g, \theta) = H_\theta(V_k(g)) \)-module. The conditions on \( g \) exactly correspond to the cases when \( W_k(g, \theta) \) is one-dimensional (cf. [11], [13]), so the discussion that precedes the Lemma and relation (4.4) imply that \( \lambda \) is as in (2). \( \square \)

**Lemma 4.1** implies:

**Theorem 4.2.** Assume that the level \( k \) and the Lie superalgebra \( g \) satisfy one of the following conditions:

1. \( k = -\frac{h^\vee}{\theta} - 1 \) and \( g \) is one of the Lie algebras of exceptional Deligne’s series \( A_2, G_2, D_4, F_4, E_6, E_7, E_8 \), or \( g = \mathfrak{psl}(m|m) \) \( (m \geq 2) \), \( \mathfrak{osp}(n + 8n) \) \( (n \geq 2) \), \( \mathfrak{spo}(2|1) \), \( F(4) \), \( G(3) \) \( (\text{for both choices of } \theta) \);
2. \( k = -h^\vee/2 + 1 \) and \( g = \mathfrak{osp}(n + 4m + 8n), n \geq 2, m \geq 0. \)
3. \( k = -h^\vee/2 + 1 \) and \( g = D_{2m}, m \geq 2. \)
4. \( k = -10 \) and \( g = E_8. \)

Then \( V_k(g) \) is the unique irreducible \( V_k(g) \)-module in the category \( KL_k \).

Proof. If the Lie superalgebra \( g \) is as in (1), then Lemma 4.1 and Remark 3.5 imply that \( \bar{\lambda} \) is either 0 or \( -\frac{h^\vee}{\theta} \). Since in all cases in (1) we have that \( h^\vee \in \mathbb{Z}_{\geq 0} \), one obtains that the irreducible highest weight \( g \)-module with highest weight \( \bar{\lambda} = -\frac{h^\vee}{\theta} \) cannot be finite-dimensional. Therefore \( L(\lambda) \) can not be a
module in $KL_k$. This proves that $\lambda = 0$ and therefore $V_k(\mathfrak{g})$ is the unique irreducible $V_k(\mathfrak{g})$-module in the category $KL_k$.

Let us consider the case $\mathfrak{g} = osp(n + 4m + 8|n)$. Then for every $m \in \mathbb{Z}_{\geq 0}$ we have:

\begin{align*}
(4.5) \quad & h^\vee = 4m + 6, \\
(4.6) \quad & k = -h^\vee/2 + 1 = -2(m + 1), \\
(4.7) \quad & 2\ell + h^\vee - 4 \neq 0 \quad \forall \ell \in \frac{1}{2}\mathbb{Z}_{\geq 0}.
\end{align*}

We prove the claim by induction. In the case $m = 0$, the claim was proved in (1). Assume now that the claim holds for $\mathfrak{g}' = osp(n + 4(m - 1) + 8, n)$, and $k' = -2m$.

By Theorem 2.2 $k = -2(m + 1)$ is a collapsing level and $W_k(\mathfrak{g}, \theta) = V_k(\mathfrak{g'})$.

By inductive assumption $V_k(\mathfrak{g'})$ is the unique irreducible $V_k(\mathfrak{g'})$ in the category $KL_{k'}$. By applying (4.3) and (1.7) we get that $\ell = 0$ and therefore $V_k(\mathfrak{g})$ is the unique irreducible $V_k(\mathfrak{g})$-module in the category $KL_k$.

The assertion now follows by induction on $m$.

(3) is a special case of (2), by taking $n = 0$.

(4) follows from the fact that $H_\theta(V_{-10}(E_8)) = V_{-4}(E_7)$ and case (1) by applying formula (4.2). \hfill $\square$

**Remark 4.3.** Theorem 4.4 can be also proved by non-cohomological methods, using explicit formulas for singular vectors and Zhu algebra theory. As an illustration, we shall present in Theorem 8.6 a direct proof in the case of $D_{2n}$ at level $k = -h^\vee/2 + 1$.

In the following sections we shall study some other applications of collapsing levels. We shall restrict our analysis to the case of Lie algebras. In what follows we let $\omega_1, \ldots, \omega_n$ be the fundamental weights for $\mathfrak{g}$ and $\Lambda_0, \ldots, \Lambda_n$ the fundamental weights for $\widehat{\mathfrak{g}}$.

### 5. On Complete Reductibility in the Category $KL_k$

In this Section we prove complete reducibility results in the category $KL_k$ when $\mathfrak{g}$ is a Lie algebra. We start with a preliminary result, which also holds in the super setting.

**Lemma 5.1.** Assume that the Lie superalgebra $\mathfrak{g}$ and level $k$ satisfy the conditions of Theorem 4.2. Assume that $M$ is a highest weight $V_k(\mathfrak{g})$-module from the category $KL_k$. Then $M$ is irreducible.

**Proof.** By using the classification of irreducible modules from Theorem 1.2 we know that the highest weight of $M$ is necessary $k\Lambda_0$, and therefore $M$ is a $\mathbb{Z}_{\geq 0}$-graded with respect to $L_\mathfrak{g}(0)$. Denote a highest weight vector by $w_{k\Lambda_0}$. We have that

$$L_\mathfrak{g}(0)v = 0 \iff v = \nu w_{k\Lambda_0} \quad (\nu \in \mathbb{C}).$$

Assume that $M$ is not irreducible. Then it contains a non-zero graded submodule $N \neq M$ with respect to $L_\mathfrak{g}(0)$:

$$N = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} N(n + n_0), \quad L_\mathfrak{g}(0)|_{N(r)} = r \text{Id}, \quad N(n_0) \neq 0.$$

Since $N \neq M$, we have that $n_0 > 0$, otherwise $w_{k\Lambda_0} \in M$.

We can consider $N(n_0)$ as a finite-dimensional module for $\mathfrak{g}$ and for the Zhu algebra $A(V_k(\mathfrak{g}))$. Note that Theorem 1.2 and Proposition 5.2 imply that any irreducible finite-dimensional $A(V_k(\mathfrak{g}))$-module is trivial. Since the Casimir element $C_\mathfrak{g}$ acts on $N(n_0)$ as the non-zero constant $2(k + h^\vee) n_0$, we conclude that $C_\mathfrak{g}$ acts by the same constant on any irreducible $\mathfrak{g}$-subquotient of $N(n_0)$. But any irreducible subquotient of $N(n_0)$ is an irreducible finite-dimensional $A(V_k(\mathfrak{g}))$-module, and therefore it is trivial. This implies that $C_\mathfrak{g}$ acts non-trivially on a trivial $\mathfrak{g}$-module, a contradiction. \hfill $\square$

The following Lemma is a consequence of [28, Theorem 0.1].

**Lemma 5.2.** [28] Assume that $\mathfrak{g}$ is a simple Lie algebra and $k$ is a rational number, $k > -h^\vee$. Then, in the category of $V_k(\mathfrak{g})$-modules, we have: $\text{Ext}^1(V_k(\mathfrak{g}), V_k(\mathfrak{g})) = (0)$. 


Theorem 5.3. Assume that $g$ is a simple Lie algebra and that the level $k$ satisfies the conditions of Theorem 4.2. Then any $V_k(g)$-module $M$ from the category $KL_k$ is completely reducible.

Proof. Since $M$ is in $KL_k$ we have that any irreducible subquotient of $M$ is isomorphic to $V_k(g)$. $M$ has finite length. This implies that $M$ is $\mathbb{Z}_{\geq 0}$-graded:

$$M = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} M(n), \quad L_g(0)_{|M(r)} = r\text{Id}.$$ 

Assume that $M(0) = \text{span}_\mathbb{C}\{w_1, \ldots, w_s\}$. Then by Lemma 5.1 we have that $V_k(g)w_i \cong V_k(g)$ for every $i = 1, \ldots, s$. Now using Lemma 5.2 we get $M \cong \bigoplus V_k(g)w_i$ and therefore $M$ is completely reducible. 

\[\square\]

Remark 5.4. We expect that the previous theorem holds in the case when $g$ is the Lie superalgebra from Theorem 4.2. We shall study this case in [7].

We shall now prove much more general result on complete reducibility in $KL_k$.

Theorem 5.5. Assume that level $k \in \mathbb{Q}$, $k > -h^\vee$, and the simple Lie algebra $g$ satisfy the following property:

\[(5.1)\] Every highest weight $V_k(g)$-module in $KL_k$ is irreducible.

Then the category $KL_k$ is semi-simple.

Proof. We shall present a sketch of the proof and omit some standard representation theoretic arguments which can be found in [20] and [28].

- Since every irreducible $V_k(g)$-module in $KL_k$ is isomorphic to $L(\lambda)$ for certain rational, non-critical weight $\lambda$, then [28] Theorem 0.1 implies that $\text{Ext}^1(L(\lambda), L(\lambda)) = (0)$ in the category $KL_k$.

- We prove that in the category $KL_k$ we have

\[(5.2)\] $\text{Ext}^1(L_1, L_2) = (0)

for any two irreducible modules $L_1$ and $L_2$ from $KL_k$.

It remains to consider the case $L_1 \neq L_2$. Take an exact sequence in $KL_k$:

$$0 \to L(\lambda_1) \to M \to L(\lambda_2) \to 0,$$

where $\lambda_1 \neq \lambda_2$. Then $M$ contains a singular vector $w_{\lambda_1}$ of highest weight $\lambda_1$ and a subsingular vector $w_{\lambda_2}$ of weight $\lambda_2$ and $w_{\lambda_1}$ generates a submodule isomorphic to $L(\lambda_1)$. Consider the case $\lambda_1 - \lambda_2 \notin \mathbb{Z}_+$. Then $\lambda_2$ is a maximal element of the set $\Omega(M)$ of weights of $M$, and therefore the subsingular vector $w_{\lambda_2}$ in $M$ of weight $\lambda_2$ is a singular vector. By (5.1), it generates an irreducible module isomorphic to $L(\lambda_2)$ and we conclude that $M \cong L(\lambda_1) \oplus L(\lambda_2)$.

If $\lambda_1 - \lambda_2 \in \mathbb{Q}_+$ we can use the contravariant functor $M \mapsto M^\sigma$ and get an exact sequence

$$0 \to L(\lambda_2) \to M^\sigma \to L(\lambda_1) \to 0.$$ 

Since $M^\sigma$ is again a $V_k(g)$-module in $KL_k$ (cf. Lemma 3.1) by the first case we have that $M^\sigma = L(\lambda_1) \oplus L(\lambda_2)$. This implies that

$$M = L(\lambda_1)^\sigma \oplus L(\lambda_2)^\sigma = L(\lambda_1) \oplus L(\lambda_2).$$

- Assume now that $M$ is a finitely generated module from $KL_k$. Then from [20] Proposition 3.1 we see that $M$ has an increasing filtration

\[(5.3)\] $(0) = M_0 \subseteq M_1 \subseteq \cdots$

such that

1. for every $j \in \mathbb{Z}_{\geq 0}$, $M_j/M_{j-1}$ is an highest weight module in category $\mathcal{O}$;
2. for any weight $\lambda$ of $M$, there exists $r$ such that $(M/M_r)_{\lambda} = 0.$
Since $M$ is finitely generated as $\mathfrak{g}$-module, we can assume that its generators are weight vectors of weights say $\mu_1, ..., \mu_p$. Since they are a finite number there certainly exists $t$ such that $(M/M_t)\mu_i = 0$ for all $i = 1, ..., p$. Hence the filtration $L(\mu_i)$ is finite and stops at $M = M_t$.

Since $M$ is in category $KL_k$, we have that the factors of (5.3) are in category $KL_k$. Hence, by our assumption, they are irreducible. Therefore (5.3) is a composition series of finite length. Using assumption (5.1), relation (5.2) and induction on $t$ we get that

$$M \cong \bigoplus_{j=1}^t L(\lambda_j).$$

- Finally, we shall consider the case when $M$ is not finitely generated. Since $M$ is in $KL_k$, it is countably generated. So $M = \bigcup_{n=1}^{\infty} M^{(n)}$ such that each $M^{(n)}$ is finitely generated $V_k(\mathfrak{g})$-module. By previous case $M^{(n)}$ is completely reducible, so:

$$M^{(n)} = \bigoplus_{i=1}^{n} L(\lambda_{i,n}).$$

Therefore $M$ is a sum of irreducible modules from $KL_k$ and by using classical algebraic arguments one can see that $M$ is a direct sum of countably many irreducible modules from $KL_k$ appearing in decompositions (5.4).

The claim follows. \qed

In order to apply Theorem 5.5 the basic step is to check relation (5.1). We have the following method.

**Lemma 5.6.** Let $k \in \mathbb{Q} \setminus \mathbb{Z}_{\geq 0}$. Assume that $H_\theta(U)$ is an irreducible, non-zero $W_k(\mathfrak{g}, \theta) = H_\theta(V_k(\mathfrak{g}))$–module for every non-zero highest weight $V_k(\mathfrak{g})$–module $U$ from the category $KL_k$. Then every highest weight $V_k(\mathfrak{g})$–module in $KL_k$ is irreducible.

**Proof.** Assume that $M$ is a highest weight $V_k(\mathfrak{g})$–module in $KL_k$. Then $H_\theta(M)$ is an irreducible $H_\theta(V_k(\mathfrak{g}))$–module. If $M$ is not irreducible, then it contains a highest weight submodule $U$ such that $\{0\} \not\subseteq U \not\subseteq M$. Modules $U$ and $M/U$ are again highest weight modules in $KL_k$. By the assumption of the Lemma we have that $H_\theta(U)$ is a non-trivial submodule of $H_\theta(M)$. Irreducibility of $H_\theta(M)$ implies that $H_\theta(U) = H_\theta(M)$, and therefore $H_\theta(M/U) = \{0\}$, a contradiction. \qed

**Theorem 5.7.** Assume that $\mathfrak{g}$ is a simple Lie algebra and $k \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$ such that $W_k(\mathfrak{g}, \theta)$ is rational. Then $KL_k$ is a semi-simple category.

**Proof.** Assume that $\tilde{L}(\lambda)$ is a highest weight $V_k(\mathfrak{g})$–module in $KL_k$. Clearly $\lambda(\alpha_+^\vee) \notin \mathbb{Z}_{\geq 0}$ and by Theorem 2.1 $H_\theta(\tilde{L}(\lambda)) \neq (0)$. Since $H_\theta(\tilde{L}(\lambda))$ is non-zero highest weight module for the rational vertex algebra $W_k(\mathfrak{g}, \theta)$, we conclude that $H_\theta(\tilde{L}(\lambda))$ is irreducible. Now assertion follows from Theorem 5.5 and Lemma 5.6. \qed

**Remark 5.8.** The previous theorem proves that the category $KL_k$ is semisimple in the following (non-admissible) cases:

- $\mathfrak{g} = D_4, E_6, E_7, E_8$ and $k = -\frac{h^\vee}{\ell}$ using results from [38].

Moreover, using Theorem 5.5 and Lemma 5.6 we can prove the semi-simplicity of $KL_k$ for all collapsing levels not accounted by Theorem 1.1. We list here only non-admissible levels, since in admissible case $KL_k$ is semi-simple by [12].

**Theorem 5.9.** The category $KL_k$ is semisimple in the following cases:

1. $\mathfrak{g} = D_\ell$, $\ell \geq 3$ and $k = -2$;
2. $\mathfrak{g} = B_\ell$, $\ell \geq 2$ and $k = -2$;
3. $\mathfrak{g} = A_\ell$, $\ell \geq 2$ and $k = -1$;
4. $\mathfrak{g} = A_{2\ell-1}$, $\ell \geq 2$, $k = -\ell$;
(5) \( g = D_{2\ell - 1}, \ell \geq 3 \) and \( k = -2\ell + 3 \);
(6) \( g = C\ell, k = -1 - \ell/2 \);
(7) \( g = E_6, k = -4 \);
(8) \( g = E_7, k = -6 \);
(9) \( g = F_4, k = -3 \).

Proof. We will give a proof of relations (1) and (2) in Corollaries 6.8 and 7.7 respectively. Case (1) for \( \ell \neq 3 \) will follow from Theorem 5.7. Note also that case (1) for \( \ell = 3 \) is a special case of case (4), and that case (2) for \( \ell = 2 \) is a special case of (6). The proof in cases (3) – (6) is similar, and it uses the classification of irreducible modules from [10], [11], [16] and the results on collapsing levels [4]. Cases (7) – (9) are reduced to cases we have already treated. Here are some details.

Case (3):
- \([10], [4] \) \( H_\theta(V_{-1}(A_\ell)) \) is isomorphic to the Heisenberg vertex algebra \( M(1) \) of central charge \( c = 1 \).
- By using the fact that every highest weight \( M(1) \)-module is irreducible, we see that if \( U \) is a highest weight \( V_{-1}(A_\ell) \)-module in \( KL_{-1} \), then \( H_\theta(U) \) is a non-trivial irreducible \( M(1) \)-module.

Case (4):
- \([10], [4] \) \( H_\theta(V_{-\ell}(A_{2\ell-1})) = V_{-\ell+1}(A_{2\ell-3}) \).
- For \( \ell = 2 \), we have that every highest weight \( V_{-\ell+1}(A_{2\ell-3}) = V_{-1}(sl(2)) \)-module \( \tilde{L}(\lambda) \) in \( KL_{-1} \) with highest weight \( \lambda = -(1 + j)\Lambda_0 + jA_1, j \in \mathbb{Z}_{\geq 0} \), is irreducible.
- By induction, we see that for every highest weight \( V_{-\ell}(A_{2\ell-1}) \)-module \( U \) in \( KL_{-\ell} \), \( H_\theta(U) \) is a non-trivial irreducible \( V_{-\ell+1}(A_{2\ell-3}) \)-module.

Case (5)
- \( H_\theta(V_{-2\ell+3}(D_{2\ell-1})) \cong V_{-2\ell+5}(D_{2\ell-3}) \).
- By induction we see that for or every highest weight \( V_{-2\ell+3}(D_{2\ell-1}) \)-module \( U \) in \( KL_{-2\ell+3} \), \( H_\theta(U) \) is a non-trivial irreducible \( V_{-2\ell+5}(D_{2\ell-3}) \)-module.

Case (6)
- \( H_\theta(V_{-1-\ell/2}(C_\ell)) \cong V_{-1/2-\ell/2}(C_{\ell-1}) \).
- For \( \ell = 2 \), we have that every highest weight \( V_{-1/2-\ell/2}(C_{\ell-1}) = V_{-3/2}(sl(2)) \)-module in \( KL_{-3/2} \) is irreducible.
- By induction, we see that for every highest weight \( V_{-1-\ell/2}(C_\ell) \)-module \( U \) in \( KL_{-1-\ell/2} \), \( H_\theta(U) \) is a non-trivial irreducible \( V_{-1/2-\ell/2}(C_{\ell-1}) \)-module.

The proof follows by applying Theorem 5.5 and Lemma 5.6.

Cases (7) – (8)

We have
\[
H_\theta(V_{-4}(E_6)) = V_{-1}(A_3), \quad H_\theta(V_{-6}(E_7)) = V_{-2}(D_6),
\]
and these cases are settled in (3) and Theorem 11 (3) respectively. Case (9) follows from the fact that \( H_\theta(V_{-3}(F_4)) \) is isomorphic to the admissible affine vertex algebra \( V_{-1}(C_3) \) which is semisimple in \( KL_{-1/2} \) (cf. [11]).

Remark 5.10. The problem of complete-reducibility of modules in \( KL_k \) when \( g \) is a Lie superalgebra will be also studied in [2]. An important tool in the description of the category \( KL_k \) will be the conformal embedding of \( V_k(g_0) \) to \( V_k(g) \) where \( g_0 \) is the even part of \( g \).

Note that in the category \( O \) we can have indecomposable \( V_k(g) \)-modules in some cases listed in Theorem 5.9. See [10] Remark 5.8 for one example.
6. The vertex algebra $V^{-2}(D\ell)$ and its quotients

In this section we exploit Hamiltonian reduction and the results on conformal embeddings from [4] to investigate the quotients of the vertex algebra $V^{-2}(D\ell)$. In particular we are interested in a non-simple quotient $V_{-2}(D\ell)$ which appears in the analysis of certain dual pairs (see [13]) as well as in the simple quotient $V_{-2}(D\ell)$. We will show that the vertex algebra $V_{-2}(D\ell)$ has infinitely many irreducible modules in the category $KL^2$, while by [13], $V_{-2}(D\ell)$ has finitely many irreducible modules in $KL^2$. Recall that $-2$ is a collapsing level for $D\ell$ [3].

Consider the vector

$$w_1 := (e_{e_1+e_2}(-1) e_{e_3+e_4}(-1) - e_{e_1+e_3}(-1) + e_{e_2+e_4}(-1)) 1.$$  

It is a singular vector in $V^{-2}(D\ell)$ (cf. [13]). Note that this vector is contained in the subalgebra $V^{-2}(D_4)$ of $V^{-2}(D\ell)$.

By using the explicit expression for singular vectors $v_n$ in $V^{n-\ell+1}(D\ell)$ (see [41]), we have that

$$w_2 := v_{\ell-3} = \left( \sum_{i=2}^{\ell} e_{e_1-e_{1}}(-1) e_{e_1+e_{1}}(-1) \right)^{\ell-3} 1$$

is a singular vector in $V^{-2}(D\ell)$.

For $\ell = 4$ we also have a third singular vector (cf. [40])

$$w_3 := (e_{e_1+e_2}(-1) e_{e_3-e_4}(-1) - e_{e_1+e_3}(-1) + e_{e_2-e_4}(-1)) 1.$$  

6.1. The vertex algebra $V_{-2}(D\ell)$ for $\ell \geq 4$. Define the vertex algebra

$$V_{-2}(D\ell) = V^{-2}(D\ell)/J_{\ell},$$

where

$$J_{\ell} = \langle w_1, w_3 \rangle \quad (\ell = 4), \quad J_{\ell} = \langle w_1 \rangle \quad (\ell \geq 5).$$

The following proposition is essentially proven in [6].

**Proposition 6.1.**

1. There is a non-trivial vertex algebra homomorphism $\overline{\Phi} : V_{-2}(D\ell) \to M_{2\ell}$ where $M_{2\ell}$ the Weyl vertex algebra of rank $\ell$.

2. $V_{-2}(D\ell)$ is not simple, and $L((-2-t)\Lambda_0 + t\Lambda_1)$, $t \in \mathbb{Z}_{\geq 0}$ are $V_{-2}(D\ell)$-modules.

**Proof.** The homomorphism $\Phi : V^{-2}(D\ell) \to M_{2\ell}$ was constructed in [6] Section 7]. By direct calculation one proves that $\Phi(w_1) = 0$ for $\ell \geq 4$ and $\Phi(w_3) = 0$ for $\ell = 4$. Finally [6] Lemma 7.1] implies that $L((-2-t)\Lambda_0 + t\Lambda_1)$, $t \in \mathbb{Z}_{\geq 0}$ are $V_{-2}(D\ell)$-modules. Since the simple vertex algebra $V_{-2}(D\ell)$ has only finitely many irreducible modules in the category $\mathcal{O}$ [13], we have that $V_{-2}(D\ell)$ is not simple. \hfill $\square$

Next, we exploit the fact that in the case $g = D\ell$, $k = -2$ is a collapsing level, i.e., in the affine $W$-algebra $W^k(g, \theta)$, all generators $G^{(u)}$ at conformal weight $3/2$, $u \in g_{-1/2}$, belong to the maximal ideal (see [3] for details). This implies that there exists a non-trivial ideal $I$ in $V^{-2}(g)$ such that $G^{(u)} \in H_0(I)$ for all $u \in g_{-1/2}$.

Note also that $g^\delta = A_1 \oplus D\ell_{-2}$, so we have that $V^{\ell-4}(A_1) \otimes V^0(D\ell_{-2})$ is a subalgebra of $W^{-2}(D\ell, \theta)$. In the case $\ell = 4$ we identify $D_2$ with $A_1 \oplus A_1$.

**Lemma 6.2.** We have

- $x_{(-1)} 1 \in H_0(J_{\ell})$ for all $x \in D\ell_{-2} \subset g^\delta$,
- $G^{(u)} \in H_0(J_{\ell})$ for all $u \in g_{-1/2}$.

**Proof.** Assume that $\ell \geq 5$. Since $w_1$ is a singular vector in $V^{-2}(D\ell)$, the ideal $J_{\ell}$ is a highest weight module of highest weight $\lambda = -2\Lambda_0 + e_1 + e_2 + \Lambda_3 + \Lambda_4$. Now, the Main Theorem from [12] implies that $H_0(J_{\ell})$ is a non-trivial highest weight module. By formula [33] (6.14]) the highest weight is $(0, \omega_2)$ and, by [31], the conformal weight of its highest weight vector is 1. Up to a non-zero constant,
there is only one vector in \( W^{-2}(D_\ell, \theta) = V^{-t-4}(A_1) \otimes V^0(D_{\ell-2}) \) that has these properties, namely \( t^{e_3+e_4}_{(-1)} 1 \), and therefore \( H_\theta(J_\ell) \) contains all generators of \( V^0(D_{\ell-2}) \).

In the case \( \ell = 4, w_1 \) and \( w_3 \) generate submodules \( N_1 \) and \( N_3 \) of highest weights \( \lambda_1 = -2 \lambda_0 + \epsilon_1 + \epsilon_2 + e_3 + e_4 \), \( \lambda_3 = -2 \lambda_0 + \epsilon_1 + \epsilon_2 + e_3 - e_4 \), respectively. Applying the same arguments as above we get that \( t^{e_3+e_4}_{(-1)} 1 \in H_\theta(I) \), which implies that \( H_\theta(J_\ell) \) contains all generators of \( V^0(D_2) = V^0(A_1) \otimes V^0(A_1) \).

Now, claim follows by applying the action of generators of \( V^0(D_{\ell-2}) \) to \( G^{(u)} \) (see \[4\]). \( \square \)

**Proposition 6.3.** We have

(1) \( H_\theta(V^{-2}(D_\ell)) = V^{t-4}(A_1) \).

(2) \( H_\theta(L((-2 - t)\lambda_0 + t\lambda_1)) \cong L_{A_1}(\ell - 4 - t)\lambda_0 + t\lambda_1 \), \( t \in \mathbb{Z}_{\geq 0} \).

(3) The set \( \{ L((-2 - t)\lambda_0 + t\lambda_1) | t \in \mathbb{Z}_{\geq 0} \} \) provides a complete list of irreducible \( V^{-2}(D_\ell) \)-modules from the category \( KL^{-2} \).

**Proof.** By Lemma 6.2 we see that the vertex algebra \( H_\theta(V^{-2}(D_\ell)) \) is generated only by \( x_{(-1)} 1, x \in A_1 \subset D_\ell^\natural \). So there are only two possibilities: either \( H_\theta(V^{-2}(D_\ell)) = V^{t-4}(A_1) \) or \( H_\theta(V^{-2}(D_\ell)) = V_{\ell-4}(A_1) \). Moreover, for every \( t \in \mathbb{Z}_{\geq 0} \), \( H_\theta(L((-2 - t)\lambda_0 + t\lambda_1)) \) must be the irreducible \( H_\theta(V^{-2}(D_\ell)) \)-module with highest weight \( tw_1 \) with respect to \( A_1 \). So \( H_\theta(L((-2 - t)\lambda_0 + t\lambda_1)) \cong L_{A_1}(\ell - 4 - t)\lambda_0 + t\lambda_1 \), \( t \in \mathbb{Z}_{\geq 0} \). Therefore, \( H_\theta(V^{-2}(D_\ell)) \) contains infinitely many irreducible modules, which gives that \( H_\theta(V^{-2}(D_\ell)) = V^{t-4}(A_1) \). In this way we have proved claims (1) and (2).

Let us now prove claim (3).

Assume that \( L(k\lambda_0 + \mu) \) \( (\mu \in P_+ , k = -2) \) is an irreducible \( V_\ell(D_\ell) \)-module in the category \( KL^k \). Then \( H_\theta(L(k\lambda_0 + \mu)) \) is a non-trivial irreducible \( V^{t-4}(A_1) \)-module. The representation theory of \( V^{t-4}(A_1) \) implies that:

\[ H_\theta(L(k\lambda_0 + \mu)) \cong L_{A_1}(\ell - 4 - j)\lambda_0 + j\lambda_1 \quad \text{for} \quad j \in \mathbb{Z}_{\geq 0}. \]

Since \( D_\ell^\natural = A_1 \times D_{\ell-2} \), we conclude that \( \mu^s = j\omega_1 \) and therefore, by (2.5),

\[ \mu = j\omega_1 + s\omega_2 = (s + j)e_1 + se_2 \quad (s \in \mathbb{Z}_{\geq 0}). \]

By using the action of \( L(0) = \omega_0 \) on the lowest component of \( H_\theta(L(k\lambda_0 + \mu)) \) we get

\[ \frac{(\mu + 2\rho, \mu)}{2(k + h^\vee)} - \mu(x) = \frac{j(j + 2)}{4(\ell - 2)} \quad (x = \theta^\vee/2). \]

Since \( 2(k + h^\vee) = 2(-2 + 2\ell - 2) = 4(\ell - 2) \) and \( \mu(x) = (2s + j)/2 \) we get

\[ (\mu + 2\rho, \mu) - (h^\vee - 2)(2s + j) = j(j + 2). \]

By direct calculation we get

\[ (\mu + 2\rho, \mu) = (s + j)^2 + s^2 + h^\vee(s + j) + (h^\vee - 2)s, \]

which gives an equation:

\[ (s + j)^2 + s^2 + h^\vee(s + j) + (h^\vee - 2)s - (h^\vee - 2)(2s + j) = j(j + 2). \]

\( \iff (s + j)^2 + s^2 + h^\vee(s + j) - (h^\vee - 2)(s + j) = j(j + 2). \)

\( \iff (s + j)(s + j + 2) = j(j + 2) \)

\( \iff s = 0 \quad \text{or} \quad s = -2j - 2. \)

Since \( \mu \in P_+ \) we conclude that \( s = 0 \). Therefore \( \mu = j\omega_1 \) for certain \( j \in \mathbb{Z}_{\geq 0} \). The proof of claim (3) is now complete. \( \square \)

6.2. **The simple vertex algebra** \( V^{-2}(D_\ell) \). Next we use the fact that the simple affine \( W \)-algebra \( W^{-2}(D_\ell, \theta) \) is isomorphic to the simple affine vertex algebra \( V_{\ell-4}(A_1) \), for \( \ell \geq 4 \).

**Proposition 6.4.** The set \( \{ L((-2 - j)\lambda_0 + j\lambda_1) | j \in \mathbb{Z}_{\geq 0}, j \leq \ell - 4 \} \) provides a complete list of irreducible \( V^{-2}(D_\ell) \)-modules from the category \( KL^{-2} \).
Proof. Assume that $N$ is an irreducible $V_{-2}(D_4)$–module from the category $KL_{-2}$. Then $N$ is also irreducible as $V_{-2}(D_4)$–module, and therefore $N \cong L((-2 - j)\Lambda_0 + j\Lambda_1)$ for certain $j \in \mathbb{Z}_{\geq 0}$. Since $H_\theta(N)$ must be an irreducible $H_\theta(V_{-2}(D_4)) = V_{-2}(D_4, \theta) = V_{-4}(A_1)$–module, we get $j \leq \ell - 4$, as desired. \hfill\Box

Now we want to describe the maximal ideal in $V^{-2}(D_4)$. The next lemma states that any non-trivial ideal in $V_{-2}(D_4)$ is automatically maximal.

Lemma 6.5. Let \( \{0\} \neq I \subseteq V_{-2}(D_4) \) be any non-trivial ideal in $V_{-2}(D_4)$. Then we have

1. $H_\theta(I)$ is the maximal ideal in $V^{\ell-4}(A_1)$.
2. $I$ is a maximal ideal in $V_{-2}(D_4)$ and $I = L(-2(\ell - 2)\Lambda_0 + 2(\ell - 3)\Lambda_1)$.

Proof. Assume that $I$ is a non-trivial ideal in $V_{-2}(D_4)$. Then $I$ can be regarded as a $V_{-2}(D_4)$–module in the category $KL_{-2}$ and therefore, by Proposition 6.3 (3), it contains a non-trivial subquotient isomorphic to $L((-2 - j)\Lambda_0 + j\Lambda_1)$ for some $j \in \mathbb{Z}_{\geq 0}$. Since, by part (2) of the aforementioned Proposition, $H_\theta(L((-2 - j)\Lambda_0 + j\Lambda_1)) \neq 0$ for every $j \in \mathbb{Z}_{\geq 0}$, we conclude that $H_\theta(I)$ is a non-trivial ideal in $H_\theta(V_{-2}(D_4)) = V^{\ell-4}(A_1)$. But since $V^{\ell-4}(A_1)$, $\ell \geq 4$, contains a unique non-trivial ideal, which is automatically maximal, we have that $H_\theta(I)$ is a maximal ideal in $V^{\ell-4}(A_1)$. So

\[ H_\theta(V_{-2}(D_4)/I) \cong V_{-4}(A_1). \]

Assume now that $V_{-2}(D_4)/I$ is not simple. Then it contains a non-trivial singular vector $v'$ of weight $-2 + j\Lambda_0 + j\Lambda_1$ for $j \in \mathbb{Z}_{\geq 0}$. By [12], we have that $H_\theta(V^{-2}(D_4), v')$ is a non-trivial ideal in $V_{-4}(A_1)$ generated by a singular vector of $A_1$–weight $j\omega_1$. This is a contradiction. So $I$ is the maximal ideal.

Since the maximal ideal in $V^{\ell-4}(A_1)$ is generated by a singular vector of $A_1$–weight $2(\ell - 3)\omega_1$, and since the maximal ideal is simple, we conclude that $I = V_{-2}(D_4)v_{\text{sing}}$ for a certain singular vector $v_{\text{sing}}$ of weight $\lambda = -2(\ell - 2)\Lambda_0 + 2(\ell - 3)\Lambda_1$. It is also clear that this singular vector is unique, up to scalar factor. Therefore, $I = L(-2(\ell - 2)\Lambda_0 + 2(\ell - 3)\Lambda_1)$.

Note that in the previous lemma we proved the existence of a singular vector which generates the maximal ideal without presenting a formula for such a singular vector. Since the vector in [6.2] has the correct weight, we also have an explicit expression for this singular vector:

\[ \left( \sum_{i=2}^{\ell} e_{c_1 - c_i}(-1) e_{c_1 + c_i}(-1) \right)^{\ell-3} 1 \]

Corollary 6.6.

1. The maximal ideal in $V^{-2}(D_4)$ is generated by the vectors $w_1$ and $w_2$ for $\ell \geq 5$ and by the vectors $w_1, w_2, w_3$ for $\ell = 4$.

2. The homomorphism $\Phi : V_{-2}(D_4) \rightarrow M_{2\ell}$ is injective. In particular, the vertex algebra $V_{-2}(D_4) \otimes V_{-4}(A_1)$ is conformally embedded into $V_{-1/2}(C_{2\ell})$.

3. $\text{ch}(V_{-2}(D_4)) = \text{ch}(V_{-2}(D_4)) + \text{ch}(L(-2(\ell - 2)\Lambda_0 + 2(\ell - 3)\Lambda_1))$.

Remark 6.7. D. Gaiotto in [27] has started a study of the decomposition of $M_{2\ell}$ as a $V^{-2}(D_4) \otimes V_{-4}(A_1)$–module in the case $\ell = 4$. By combining results from [6] Section 8] and results from this Section we get that

\[ \text{Com}(V_{-4}(A_1), M_{2\ell}) \cong V_{-2}(D_4). \]

So the vertex algebra responsible for the decomposition of $M_{2\ell}$ is exactly $V_{-2}(D_4)$. Therefore in the decomposition of $M_{2\ell}$ only modules for $V_{-2}(D_4)$ can appear. In our forthcoming papers we plan to apply the representation theory of $V_{-2}(D_4)$ to the problem of finding branching rules.

Corollary 6.8. For $\ell \geq 3$ the category $KL_{-2}$ is semi-simple.

Proof. The assertion in the case $\ell \geq 4$ follows from Theorem 5.7 since then $W_{-2}(D_4, \theta) = V_{-4}(sl(2))$ is a rational vertex algebra.

In the case $\ell = 3$, we have that a highest weight $V_{-2}(D_4)$–module $M$ is isomorphic to $\tilde{L}((-2 - j)\Lambda_0 + j\Lambda_1)$ where $j \in \mathbb{Z}_{\geq 0}$. The irreducibility of $M$ follows easily from the fact that $H_\theta(M)$ is isomorphic
to an irreducible $V_{-1}(\mathfrak{sl}(2))$–module $L_{A_1}(-1-j)\Lambda_0 + jA_1)$. Now claim follows from Theorem 5.9 and Lemma 5.6.

7. THE VERTEX ALGEBRA $V^{-2}(B_\ell)$ AND ITS QUOTIENTS

In this section let $\ell \geq 2$. Note that $k = -2$ is a collapsing level for $B_\ell$ [6], and that the simple affine $W$–algebra $W_{-2}(B_\ell, \theta)$ is isomorphic to $V_{\ell-\frac{7}{2}}(A_1)$. This implies that $H_\theta(V_{-2}(B_\ell)) = V_{\ell-\frac{7}{2}}(A_1)$. But as in the case of the affine Lie algebra of type $D$, we can construct an intermediate vertex algebra $V$ so that $H_\theta(V) = V^{\ell-7/2}(A_1)$.

Remark 7.1. The formula for a singular vector of conformal weight two in $V^{-2}(B_\ell)$ was given in [15, Theorem 4.2] for $\ell \geq 3$, and in [15, Remark 4.3] for $\ell = 2$. Note that, for $\ell \geq 4$, the vector $\sigma(w_2)$ from [15] is equal to the vector $w_1$ from relation (7.1), i.e. it is contained in the subalgebra $V^{-2}(D_4)$. For $\ell = 3$, we have

$$w_1 = (e_{c_1+c_2}(-1)e_{c_2}(-1) - e_{c_1+c_3}(-1)e_{c_2}(-1) + e_{c_1}(-1)e_{c_2+c_3}(-1))\mathbf{1}.$$  

For $\ell = 2$, the singular vector of conformal weight two in $V^{-2}(B_2)$ is equal to

$$w_1 = (e_{c_1+c_2}(-1)e_{c_2}(-1) + \frac{1}{2}h_{c_2}(-1)e_{c_1}(-1) - e_{c_1-c_2}(-1)e_{c_2}(-1))\mathbf{1}.$$  

Consider the singular vector in $V^{-2}(B_\ell)$ denoted by $\sigma(w_2)$ in [15, Theorem 4.2] and [17, Section 7]. Let us denote that singular vector by $w_1$ in this paper (see Remark 7.1 for explanation).

Then we have the quotient vertex algebra

$$V_{-2}(B_\ell) = V^{-2}(B_\ell)/\langle w_1 \rangle.$$  

As in the case of the vertex algebra $V_{-2}(D_\ell)$, we have the non-trivial homomorphism $V_{-2}(B_\ell) \to M_{2\ell+1}$.

The proof of the following result is completely analogous to the proof of Proposition 6.3 and it is therefore omitted.

Proposition 7.2. We have

(1) There is a non-trivial homomorphism $\overline{\Phi} : V_{-2}(B_\ell) \to M_{2\ell+1}$.

(2) $H_\theta(V_{-2}(B_\ell)) = V^{\ell-7/2}(A_1)$.

(3) $H_\theta(L((-2-t)\Lambda_0 + t\Lambda_1)) \cong L_{A_1}((\ell - 2\ell - t)\Lambda_0 + t\Lambda_1)$, $t \in \mathbb{Z}_{\geq 0}$.

(4) The set

$$\{L((-2-t)\Lambda_0 + t\Lambda_1) \mid t \in \mathbb{Z}_{\geq 0}\}$$  

provides a complete list of irreducible $V_{-2}(B_\ell)$–modules from the category $KL^{-2}$.

We have the following result on classification of irreducible modules.

Proposition 7.3. Assume that $\ell \geq 3$. Then the set $\{L((-2-j)\Lambda_0 + j\Lambda_1) \mid j \in \mathbb{Z}_{\geq 0}, j \leq 2(\ell - 3) + 1\}$ provides a complete list of irreducible $V_{-2}(B_\ell)$–modules from the category $KL_{-2}$.

Proof. The proof is analogous to the proof of Proposition 6.4: it uses the exactness of the functor $H_\theta$ and the representation theory of affine vertex algebras. In particular, we use the result from [8] which gives that the set

$$\{L((-\ell - 7/2 - j)\Lambda_0 + j\Lambda_1) \mid j \in \mathbb{Z}_{\geq 0}, j \leq 2(\ell - 3) + 1\}$$  

provides a complete list of irreducible $V_{\ell-7/2}(A_1)$–modules from the category $KL_{\ell-7/2}$.  

An important consequence is the simplicity of the vertex algebra $V_{-2}(B_2)$.

Corollary 7.4. The vertex algebra $V_{-2}(B_2)$ is simple if and only if $\ell = 2$. In particular, the set (7.2) provides a complete list of irreducible modules for $V_{-2}(B_2)$ in $KL_{-2}$. 

Proof. Since by Proposition\textsuperscript{8.2} $V_{-2}(B_2)$ has infinitely many irreducible modules in the category $KL^{-2}$, and, by Proposition\textsuperscript{8.3} $V^{-2}(B_2)$ has finitely many irreducible modules in the category $KL^{-2}$ (if $\ell \geq 3$), we conclude that $V_{-2}(B_2)$ cannot be simple for $\ell \geq 3$.

Let us consider the case $\ell = 2$. Assume that $V_{-2}(B_2)$ is not simple. Then it must contain an ideal $I$ generated by a singular vector of weight $\lambda = -(2-j)\lambda_0 + j\Lambda_1$ for certain $j > 0$. By applying the functor $H_0$, we get a non-trivial ideal in $V^{-3/2}(A_1)$, against the simplicity of $V^{-3/2}(A_1)$. \hfill $\Box$

Next we notice that $V^{\ell-7/2}(A_1)$ has a unique non-trivial ideal $J$ which is generated by a singular vector of $A_1$–weight $2(\ell-2)\omega_1$. The ideal $J$ is maximal and simple (cf. \textsuperscript{[5]}). By combining this with properties of the functor $H_0$ from \textsuperscript{[12]}, one proves the existence of a unique maximal ideal $I$ (which is also simple) in $V_{-2}(B_2)$ such that $I \cong L(-2(\ell-1)\Lambda + 2(\ell-2)\Lambda_1))$.

Remark 7.5. The explicit expression for a singular vector which generates $I$ is more complicated than in the case $D$, and it won't be presented here.

In \textsuperscript{[6]} we constructed a homomorphism $V_{-2}(B_2)\otimes V_{-\ell-1/2}(A_1) \rightarrow M_{2t+1}$. The results of this section enable us to find the image of this homomorphism.

Corollary 7.6. We have:
\begin{enumerate}
\item The vertex algebra $V_{-2}(B_2)\otimes V_{-\ell-1/2}(A_1)$ is conformally embedded into $V_{-1/2}(C_{2t+1})$.
\item The vertex algebra $V_{-2}(B_2)$ for $\ell \geq 3$ contains a unique ideal $I \cong L(-2(\ell-1)\Lambda + 2(\ell-2)\Lambda_1)$ and $\text{ch}(V_{-2}(B_2)) = \text{ch}(V_{-2}(B_2)) + \text{ch}(L(-2(\ell-1)\Lambda + 2(\ell-2)\Lambda_1))$.
\end{enumerate}

Finally, we apply Theorem\textsuperscript{5.3} and prove that $KL^{-2}$ is a semi-simple category.

Corollary 7.7. If $\ell \geq 2$, then every $V_{-2}(B_2)$–module in $KL^{-2}$ is completely reducible.

Proof. It suffices to prove that every highest weight $V_{-2}(B_2)$–module in $KL^{-2}$ is irreducible. Assume that $\ell \geq 3$. If $M \cong L(\lambda)$ is a highest weight module in $KL^{-2}$ then the highest weight is $\lambda = -(2+j)\Lambda + j\Lambda_1$ where $0 \leq j \leq 2(\ell-3)j + 1$. Since $H_0(L(\lambda))$ is a non-zero highest weight $V_{-\ell+7/2}(s(l(2)))$–module, then the complete reducibility result from \textsuperscript{[8]} implies that $H_0(L(\lambda))$ is irreducible. The assertion now follows from Lemma\textsuperscript{5.4}. The proof in the case $\ell = 2$ is similar, and it uses the classification of irreducible $V_{-2}(B_2)$–modules from Corollary\textsuperscript{7.4} and the fact that every highest weight $V_{-3/2}(s(l(2))) = H_0(V_{-2}(B_2))$–module in $KL^{-3/2}$ is irreducible. \hfill $\Box$

8. ON THE REPRESENTATION THEORY OF $V_{2-\ell}(D_{\ell})$

8.1. The vertex algebra $\overline{V}_{2-\ell}(D_{\ell})$. Let $g$ be a simple Lie algebra of type $D_{\ell}$. Recall that $2 - \ell = -h'/2 + 1$ is a collapsing level \textsuperscript{[3]}. We have the singular vector
\begin{equation}
(8.1) v_n = \left( \sum_{i=2}^{\ell} e_{c_i - \epsilon_i, (1)} e_{c_i + \epsilon_i, (1)} \right)^n 1
\end{equation}
in $V^{|-\ell|}(D_{\ell})$, for any $n \in \mathbb{Z}_{>0}$. As in \textsuperscript{[10]}, we consider the vertex algebra
\begin{equation}
(8.2) \overline{V}_{2-\ell}(D_{\ell}) = V^{2-\ell}(D_{\ell})/(v_1),
\end{equation}
where $v_1$ denotes the ideal in $V^{2-\ell}(D_{\ell})$ generated by the singular vector $v_1$. We recall the following result on the classification of irreducible $\overline{V}_{2-\ell}(D_{\ell})$–modules in the category $KL^{2-\ell}$.

Proposition 8.1. \textsuperscript{[10]}
\begin{enumerate}
\item The set\textsuperscript{[10]}
\{ $V(t\omega), V(t\omega - 1) \mid t \in \mathbb{Z}_{>0}$ \}
provides a complete list of irreducible finite-dimensional modules for the Zhu algebra $A(\overline{V}_{2-\ell}(D_{\ell}))$.
\item The set\textsuperscript{[10]}
\{ $L((2 - t - \ell)\Lambda_0 + t\Lambda_0), L((2 - t - \ell)\Lambda_0 + t\Lambda_{t-1}) \mid t \in \mathbb{Z}_{>0}$ \}
provides a complete list of irreducible $\overline{V}_{2-\ell}(D_{\ell})$–modules from the category $KL^{2-\ell}$.
\end{enumerate}
Theorem 8.2. The vector

\[ \sum_{\ell=1}^{n} \prod_{i=1}^{2} \epsilon_{i} + \epsilon_{p(i)}(-1) \]  

is a singular vector in \( V^{n-2\ell+1}(D_{2\ell}) \), for any \( n \in \mathbb{Z}_{>0} \).

Proof. Direct verification of relations \( e_{\kappa - \kappa + 1}(0)w_{n} = 0 \), for \( k = 1, \ldots, 2\ell - 1 \), \( e_{\kappa + 1 + \kappa}(0)w_{n} = 0 \) and \( e_{-(\kappa + 1)}(1)w_{n} = 0 \).

Remark 8.3. The vector \( w_{n} \) has conformal weight \( n \ell \) and its \( g \)-highest weight equals \( 2n\omega_{2\ell} = n(\epsilon_{1} + \ldots + \epsilon_{2\ell}) \). In particular, for \( n = 1 \), the vector \( w_{1} \) has conformal weight \( \ell \) and highest weight \( 2\omega_{2\ell} = \epsilon_{1} + \ldots + \epsilon_{2\ell} \).

Example 8.4. Set \( n = 1 \) for simplicity. For \( \ell = 2 \) we recover the singular vector

\[ w_{1} = (e_{\kappa + e_{2}}(1)e_{\kappa + e_{2}}(-1) - e_{\kappa + e_{3}}(-1)e_{\kappa + e_{3}}(1) + e_{\kappa + e_{3}}(-1)e_{\kappa + e_{3}}(1))1 \]

in \( V^{-2}(D_{4}) \) of conformal weight 2 from [30]. For \( \ell = 3 \), the formula for the singular vector in \( V^{-4}(D_{6}) \) of conformal weight 3 is more complicated. It is a sum of 5!! = 15 monomials:

\[ w_{1} = (e_{\kappa + e_{3}}(-1)e_{\kappa + e_{3}}(1)e_{\kappa + e_{3}}(-1) - e_{\kappa + e_{2}}(-1)e_{\kappa + e_{3}}(-1)e_{\kappa + e_{3}}(1) - e_{\kappa + e_{2}}(-1)e_{\kappa + e_{3}}(-1)e_{\kappa + e_{3}}(1))1 \]
Denote by \( \bar{\vartheta} \) the automorphism of \( V^{n-2\ell+1}(D_{2\ell}) \) induced by the automorphism of the Dynkin diagram of \( D_{2\ell} \) of order two such that

\[
(8.4) \quad \bar{\vartheta}(\epsilon_k - \epsilon_{k+1}) = \epsilon_k - \epsilon_{k+1}, \quad k = 1, \ldots, 2\ell - 2,
\]

\[
(8.5) \quad \bar{\vartheta}(\epsilon_{2\ell-1} - \epsilon_{2\ell}) = \epsilon_{2\ell-1} + \epsilon_{2\ell}, \quad \bar{\vartheta}(\epsilon_{2\ell-1} + \epsilon_{2\ell}) = \epsilon_{2\ell-1} - \epsilon_{2\ell}.
\]

Theorem 8.2 now implies that \( \bar{\vartheta}(w_n) \) is a singular vector in \( V^{n-2\ell+1}(D_{2\ell}) \), for any \( n \in \mathbb{Z}_{>0} \), also. The vector \( \bar{\vartheta}(w_n) \) has conformal weight \( n\ell \) and its highest weight for \( g \) is \( 2n\omega_{2\ell-1} = n(\epsilon_1 + \ldots + \epsilon_{2\ell-1} - \epsilon_{2\ell}) \).

We consider the associated quotient vertex algebra

\[
\tilde{V}_{n-2\ell+1}(D_{2\ell}) := V^{n-2\ell+1}(D_{2\ell})/(v_n, w_n, \bar{\vartheta}(w_n)),
\]

where \( v_n \) is given by relation (8.1) (for \( D_{2\ell} \)):

\[
v_n = \left( \sum_{i=2}^{2\ell} e_{\epsilon_i - \epsilon_i}(1)e_{\epsilon_i + \epsilon_i}(1) \right)^n 1.
\]

In particular, for \( n = 1 \) we have the vertex algebra

\[
\tilde{V}_{2-2\ell}(D_{2\ell}) := V^{2-2\ell}(D_{2\ell})/(v_1, w_1, \bar{\vartheta}(w_1)).
\]

Clearly, \( \tilde{V}_{2-2\ell}(D_{2\ell}) \) is a quotient of vertex algebra \( \nabla_{2-2\ell}(D_{2\ell}) \) from Subsection 8.4. The associated Zhu algebra is

\[
A(\tilde{V}_{2-2\ell}(D_{2\ell})) = U(g)/(\bar{v}, \bar{w}, \bar{\vartheta}(\bar{w}))
\]

where

\[
\bar{v} = \sum_{i=2}^{2\ell} e_{\epsilon_i - \epsilon_i} e_{\epsilon_i + \epsilon_i}, \quad \bar{w} = \sum_{p \in \Pi_{2\ell}} s(p) \prod_{i \in \{1, \ldots, 2\ell\}} e_{\epsilon_i + \epsilon_{\pi(i)}}.
\]

**Lemma 8.5.** We have:

1. \( \bar{w}V(t\omega_{2\ell}) \neq 0 \), for \( t \in \mathbb{Z}_{>0} \).
2. \( \bar{\vartheta}(\bar{w})V(t\omega_{2\ell-1}) \neq 0 \), for \( t \in \mathbb{Z}_{>0} \).

**Proof.** (1) Let \( t = 1 \). Denote by \( v_{\omega_{2\ell}} \) the highest weight vector of \( V(\omega_{2\ell}) \), and by \( v_{-\omega_{2\ell}} \) the lowest weight vector of \( V(\omega_{2\ell}) \). One can easily check, using the spinor realization of \( V(\omega_{2\ell}) \), that there exists a constant \( C \neq 0 \) such that

\[
\bar{w}(v_{-\omega_{2\ell}}) = Cv_{\omega_{2\ell}}.
\]

For general \( t \in \mathbb{Z}_{>0} \), the claim follows using the embedding of \( V(t\omega_{2\ell}) \) into \( V(\omega_{2\ell})^{\otimes t} \). Claim (2) follows similarly.

**Theorem 8.6.** We have:

(i) The trivial module \( \mathbb{C} \) is the unique finite-dimensional irreducible module for \( A(\tilde{V}_{2-2\ell}(D_{2\ell})) \).

(ii) \( V_{2-2\ell}(D_{2\ell}) \) is the unique irreducible \( g \)-locally finite module for \( \tilde{V}_{2-2\ell}(D_{2\ell}) \).

(iii) The vertex operator algebra \( \tilde{V}_{2-2\ell}(D_{2\ell}) \) is simple, i.e.

\[
V_{2-2\ell}(D_{2\ell}) = V^{2-2\ell}(D_{2\ell})/(v_1, w_1, \bar{\vartheta}(w_1)).
\]

**Proof.** (i) Proposition 8.1 implies that the set

\[
\{ V(t\omega_{2\ell}), V(t\omega_{2\ell-1}) \mid t \in \mathbb{Z}_{>0} \}
\]

provides a complete list of finite-dimensional irreducible modules for the algebra \( U(g)/(\bar{v}) = A(\nabla_{2-2\ell}(D_{2\ell})) \).

Lemma 8.5 shows that \( V(t\omega_{2\ell}) \) and \( V(t\omega_{2\ell-1}) \) are not modules for \( A(\tilde{V}_{2-2\ell}(D_{2\ell})) \), for \( t \in \mathbb{Z}_{>0} \). Claim (i) follows. Claims (ii) and (iii) follow from (i) by applying Proposition 3.2 and Corollary 3.3.
Remark 8.7. A general character formula for certain simple affine vertex algebras at negative integer levels has been recently presented by V. G. Kac and M. Wakimoto in [37], (more precisely, \( \mathfrak{g} = A_n, C_n \) for \( k = -1 \) and \( \mathfrak{g} = D_4, E_6, E_7, E_8 \) for \( k = -2, -3, -4, 6 \)). Note that conditions (i)-(iii) of [37] Theorem 3.1 hold for vertex algebras \( V_{-b}(D_n) \), \( n > 4 \), \( b = 1, \ldots, n - 2 \), too. We conjecture that condition (iv) of this theorem holds as well; therefore formula (3.1) in [37] gives the character formula.

9. CONFORMAL EMBEDDING OF \( V(-4, D_6 \times A_1) \) INTO \( V_{-4}(E_7) \)

In this section, we apply the results on representation theory of \( V_{-4}(D_6) \) from previous sections to the conformal embedding of \( V(-4, D_6 \times A_1) \) into \( V_{-4}(E_7) \). This gives us an interesting example of a maximal semisimple equal rank subalgebra such that the associated conformally embedded subalgebra is not simple.

We use the construction of the root system of type \( E_7 \) from [19], [29], and the notation for root vectors similar to the notation for root vectors for \( E_6 \) from [4].

For a subset \( S = \{i_1, \ldots, i_k\} \subseteq \{1, 2, 3, 4, 5, 6\}, i_1 < \ldots < i_k, \) with odd number of elements (so that \( k = 1, 3 \) or 5), denote by \( e_{(i_1 \ldots i_k)} \) a suitably chosen root vector associated to the positive root

\[
\frac{1}{2} \left( \epsilon_s - \epsilon_7 + \sum_{i=1}^{6} (-1)^{p(i)} \epsilon_i \right)
\]

such that \( p(i) = 0 \) for \( i \in S \) and \( p(i) = 1 \) for \( i \notin S \). We will use the symbol \( f_{(i_1 \ldots i_k)} \) for the root vector associated to corresponding negative root.

Note now that the subalgebra of \( E_7 \) generated by positive root vectors

\[
e_{(1)}(1), e_{(12)} + e_{(34)}, e_{(13)}, e_{(14)}, e_{(23)}, e_{(24)}, e_{(156)}, e_{(123456)}
\]

and the associated negative root vectors is a simple Lie algebra of type \( D_6 \). There are 30 root vectors associated to positive roots for \( D_6 \):

\[
e_{(12)}, e_{(13)}, e_{(14)}, e_{(23)}, e_{(24)}, e_{(15)}, e_{(25)}, e_{(35)}, e_{(235)}, e_{(1235)}, e_{(125)}, e_{(135)}, e_{(145)}, e_{(1245)}, e_{(1345)}
\]

Furthermore, the subalgebra of \( E_7 \) generated by \( e_{(156)} \) and the associated negative root vector is a simple Lie algebra of type \( A_1 \). Thus, \( D_6 \oplus A_1 \) is a semisimple subalgebra of \( E_7 \).

It follows from [4], [9] that the affine vertex algebra \( V(-4, D_6 \times A_1) \) is conformally embedded in \( V_{-4}(E_7) \). Remark that \( V(-4, A_1) = V_{-4}(A_1) \) (since \( V^{-4}(A_1) = V_{-4}(A_1) \)). This implies that \( V(-4, D_6 \times A_1) \cong V(-4, D_6) \otimes V_{-4}(A_1) \).

It was shown in [15] that

\[
v_{E_7} = (e_{(156)}(-1) e_{(12356)}(-1) + e_{(256)}(-1) e_{(13456)}(-1) + e_{(356)}(-1) e_{(12456)}(-1) + e_{(456)}(-1) e_{(13256)}(-1)) 1
\]

is a singular vector in \( V^{-4}(E_7) \). Moreover,

\[
V^{-4}(E_7) \cong V^{-4}(E_7)/\langle v_{E_7} \rangle.
\]

Vectors \( (e_{(123456)}(-1))^{s} 1, \) for \( s \in \mathbb{Z}_{>0} \) are (non-trivial) singular vectors for the affinization of \( D_6 \oplus A_1 \) in \( V_{-4}(E_7) \) of highest weights \( -(s + 4) \Lambda_0 + s A_0 \) for \( D_6^{(1)} \) and \( -(s + 4) \Lambda_0 + s A_1^{(1)} \) for \( A_1^{(1)} \). Thus there exist highest weight modules \( \bar{L}_{D_6}(-(s + 4) \Lambda_0 + s A_0) \) and \( \bar{L}_{A_1^{(1)}}(-(s + 4) \Lambda_0 + s A_1^{(1)}) \), for \( D_6^{(1)} \) and \( A_1^{(1)} \),
respectively such that $(\widetilde{V}(-4, D_6) \otimes V_{-4}(A_1).)(e_{(12346)}(-1))^* \mathbf{1}$ is isomorphic to $\tilde{L}_{D_6}(-(s + 4)\Lambda_0 + s\Lambda_6) \otimes L_{A_1}(-(s + 4)\Lambda_0 + s\Lambda_1)$. This implies that
\[ L_{D_6}(-(s + 4)\Lambda_0 + s\Lambda_6) \otimes L_{A_1}(-(s + 4)\Lambda_0 + s\Lambda_1) \]
are irreducible $\widetilde{V}(-4, D_6 \times A_1)$–modules, for $s \in \mathbb{Z}_{>0}$.

In particular, $L_{D_6}(-(s + 4)\Lambda_0 + s\Lambda_6)$ are irreducible ($D_6$–locally finite) $\widetilde{V}(-4, D_6)$–modules, for $s \in \mathbb{Z}_{>0}$. In the next proposition, we use the notation from \([S2], [S3], [S4], [S5]\).

**Proposition 9.1.** We have:
(1) Assume that $L_{D_6}(-(6\Lambda_0 + 2\Lambda_6)$ and $L_{D_6}(-(6\Lambda_0 + 2\Lambda_5)$ are highest weight $\overline{V}_{-4}(D_6)$–modules from the category $KL^{−4}$, not necessarily irreducible. Then
\[
\overline{L}_{D_6}(-(6\Lambda_0 + 2\Lambda_6) \boxtimes \overline{L}_{D_6}(-(6\Lambda_0 + 2\Lambda_5) = 0,
\]
where $\boxtimes$ is the tensor functor for $KL^{−4}$–modules. In other words, we cannot have a non-zero $\overline{V}_{-4}(D_6)$–module $M$ from $KL^{−4}$ and a non-zero intertwining operator of type
\[
(9.4) \quad \left( \overline{L}_{D_6}(-(6\Lambda_0 + 2\Lambda_6) \boxtimes \overline{L}_{D_6}(-(6\Lambda_0 + 2\Lambda_5) \right).\]
(2) Relations $w_1 \neq 0$ and $\vartheta(w_1) = 0$ hold in $V_{-4}(E_7)$. In particular, $\widetilde{V}(-4, D_6)$ is not simple.

**Proof.** For the proof of assertion (1) we first notice that the following decomposition of $D_6$–modules holds:
\[
V_{D_6}(2\omega_6) \otimes V_{D_6}(2\omega_5) = V_{D_6}(2\omega_5 + 2\omega_6) \oplus V_{D_6}(\omega_3 + \omega_5 + \rho) \oplus V_{D_6}(2\omega_3)
\]
\[ \oplus V_{D_6}(\omega_1 + \omega_5 + \omega_6) \oplus V_{D_6}(\omega_1 + \omega_3) \oplus V_{D_6}(2\omega_1).\]
(9.5)
Assume that $M$ is a non-zero $\overline{V}_{-4}(D_6)$–module in the category $KL^{−4}$ such that there is a non-trivial intertwining operator of type \([M1]\). Then the Frenkel-Zhu formula for fusion rules implies that $M$ must contain a non-trivial subquotient whose lowest graded component appears in the decomposition of $V_{D_6}(2\omega_6) \otimes V_{D_6}(2\omega_5)$. But by Proposition \([S1]\) the $D_6$–modules appearing in \([9,5]\) cannot be lowest components of any $\overline{V}_{-4}(D_6)$–module. This proves assertion (1).

Assertion (1) implies that if $w_1 \neq 0$ and $\vartheta(w_1) \neq 0$ in $V_{-4}(E_7)$, then
\[
Y(w_1, z)\vartheta(w_1) = 0,
\]
a contradiction since $V_{-4}(E_7)$ is a simple vertex algebra. The same fusion rules argument shows that if $\vartheta(w_1) \neq 0$ in $V_{-4}(E_7)$, then
\[
Y(\vartheta(w_1), z)e_{(12346)}(-1)^2 \mathbf{1} = 0,
\]
which again contradicts the simplicity of $V_{-4}(E_7)$. So, $\vartheta(w_1) = 0$.

But if $w_1 = 0$, then, by Theorem \([S6]\) (iii), we have that $\widetilde{V}(-4, D_6) = V_{-4}(D_6)$. Theorem \([1.2]\) implies that $\widetilde{V}(-4, D_6)$ is not simple, since the simple vertex operator algebra $V_{-4}(D_6)$ has only one irreducible $D_6$–locally finite module, a contradiction. So $w_1 \neq 0$ and claim (2) follows.

\[
(9.6) \quad V_{-4}(D_6) = \frac{V^{-4}(D_6)}{< v_1, \vartheta(w_1) >}.
\]

**Theorem 9.2.** We have:
(1) $\widetilde{V}(-4, D_6) \cong V_{-4}(D_6)$.
(2) The set \(\{ L_{D_6}(-(s + 4)\Lambda_0 + s\Lambda_6) \mid s \in \mathbb{Z}_{\geq 0}\} \) provides a complete list of irreducible $V_{-4}(D_6)$–modules.

**Proof.** We first notice that $\widetilde{V}(-4, D_6)$ is a certain quotient of $\frac{V^{-4}(D_6)}{< v_1, \vartheta(w_1) >}$, and that
\[
H_{\theta}(\frac{V^{-4}(D_6)}{< v_1, \vartheta(w_1) >}) = V_{-2}(D_4).
\]
Since $\mathcal{V}_{-2}(D_4)$ contains a unique non-trivial ideal which is maximal and simple, we conclude that $V_{\mathfrak{a}_1,\theta(w_1)}^{\mathcal{V}_{-2}(D_4)}$ also contains a unique ideal, and it must be the ideal generated by $w_1$. Since in $\tilde{V}(-4,D_6)$ we have that $w_1 \neq 0$, we conclude that $\tilde{V}(-4,D_6) \cong V_{\mathfrak{a}_1,\theta(w_1)}^{\mathcal{V}_{-4}(D_6)}$.

The proof of assertion (2) follows from (1), the classification result of $\mathcal{V}_{-4}(D_6)$-modules from Proposition 8.1 and Lemma 8.5. □

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