S-matrix approach to two-pion production in $e^+e^-$ annihilation and $\tau$ decay

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Abstract

Based on the S-matrix approach, we introduce a modified formula for the $\pi^\pm$ electromagnetic form factor which describes very well the experimental data in the energy region $2m_\pi \leq \sqrt{s} \leq 1.1$ GeV. Using the CVC hypothesis we predict $B(\tau^- \to \pi^-\pi^0\nu_\tau) = (24.75 \pm 0.38)\%$, in excellent agreement with recent experiments.

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I. Introduction.

The processes $e^+e^- \to \pi^+\pi^-$ and $\tau^- \to \pi^-\pi^0\nu_\tau$ provide a clean environment for a consistency check of the Conserved Vector Current (CVC) hypothesis [1]. Actually, the measurement of the $\pi^\pm$ electromagnetic form factor in $e^+e^-$ annihilation is used to predict [2] the dominant hadronic decay of the tau lepton, namely $\tau^- \to \pi^-\pi^0\nu_\tau$. The weak pion form factor involved in $\tau$ decay is obtained by removing the (model-dependent) $I=0$ contribution (arising from isospin violation and included via $\rho - \omega$ mixing) from the measured pion electromagnetic form factor.

In a previous paper [3] we have applied the S-matrix approach to the $e^+e^- \to \pi^+\pi^-$ data of Ref. [4] and determined the pole parameters of the $\rho^0$ resonance. In particular, we have fitted the data of Ref. [4] by assuming a constant value for the strength of the $\rho - \omega$ mixing parameter and using different parametrizations to account for the non-resonant background. As a result, the pole position of the scattering amplitude was found [3] to be insensitive to the specific background chosen to fit the experimental data.

The purpose of this Brief Report is two-fold. We first argue that the pole position in $e^+e^- \to \pi^+\pi^-$ is not modified by taking the $\rho - \omega$ mixing parameter as a function of the center-of-mass energy, as already suggested in recent papers [5]. Then we propose a new parametrization for the scattering amplitude of $e^+e^- \to \pi^+\pi^-$, based on the S-matrix approach, which looks very similar to the Breit-Wigner parametrization with an energy-dependent width. This results into an improvement in the quality of the fits (respect to Ref. [3]) while the pole position and $\rho - \omega$ mixing parameters remain unchanged (as it should be). Finally, we make use of CVC to predict the $\tau^- \to \pi^-\pi^0\nu_\tau$ branching ratio, which is found to be in excellent agreement
II. Energy-dependent $\rho - \omega$ mixing.

We start by giving a simple argument to show that the pole position would not be changed if we choose the $\rho - \omega$ mixing parameter to be $m_{\rho\omega}^2(s) \propto s$ (namely $m_{\rho\omega}^2(0) = 0$), where $\sqrt{s}$ is the total center-of-mass energy in $e^+e^- \rightarrow \pi^+\pi^-$. 

Let us consider Eq.(7) of Ref. [3] and replace $y \rightarrow y's/s_\omega^\frac{5}{2}$, where $s_V = m_V^2 - i m_V \Gamma_V$. This yields the following expression for Eq. (7) of Ref. [3]:

$$F_{\pi}(s) = \frac{A}{s - s_\rho} \left(1 + \frac{y's}{s_\omega} \frac{m_{\omega}^2}{s - s_\omega}\right) + B(s)$$

$$= \frac{A'}{s - s_\rho} \left(1 + y'' \frac{m_{\omega}^2}{s - s_\omega}\right) + B(s),$$

where $A$ and $B(s)$ denote the residue at the pole and non-resonant background terms, respectively. The second equality above follows from the approximations:

$$A' \equiv A \left(1 + \frac{y' m_{\omega}^2}{s_\omega}\right) \approx A(1 + y'),$$

$$y'' \equiv \frac{y'}{1 + y'/m_{\omega}^2/s_\omega} \approx \frac{y'}{1 + y'}$$

i.e. by neglecting small imaginary parts of order $y'\Gamma_\omega/m_\omega \approx 10^{-5}$ [3]. Thus, since introducing $m_{\rho\omega}^2 \propto s$ is equivalent to a redefinition of the residue at the pole and of the $\rho - \omega$ mixing parameter, we conclude that the pole position

\footnote{In the Vector Meson Dominance model, $y$ is related to the usual $\rho - \omega$ mixing strength through $y = m_{\rho\omega}^2 f_\rho/(m_{\rho}^2 f_\omega) \approx -2 \times 10^{-3}$ [3].}
would not be changed if we take a constant or an energy-dependent $\rho - \omega$ mixing parameter.

### III. Electromagnetic pion form factor.

Next, we consider a new parametrization for the pion electromagnetic form factor. This parametrization is obtained by modifying the pole term in the following way:

$$s - m_\rho^2 + im_\rho \Gamma_\rho \theta(\bar{s}) \to D(s) \equiv [1 - ix(s)\theta(\bar{s})](s - m_\rho^2 + im_\rho \Gamma_\rho \theta(\bar{s})), \quad (2)$$

where $\theta(\bar{s})$ is the step function, with argument $\bar{s} = s - 4m_\pi^2$.

Observe that if we chose:

$$x(s) = -m_\rho \left( \frac{\Gamma_\rho(s) - \Gamma_\rho}{s - m_\rho^2} \right), \quad (3)$$

then Eq. (2) becomes:

$$D(s) = s - m_\rho^2 + m_\rho \Gamma_\rho x(s) \theta(s - 4m_\pi^2) + im_\rho \Gamma_\rho(s) \quad (4)$$

which, when inserted in (1), looks very similar to a Breit-Wigner with an energy-dependent width, which we will chose to be:

$$\Gamma_\rho(s) = \Gamma_\rho \left( \frac{s - 4m_\pi^2}{m_\rho^2 - 4m_\pi^2} \right)^{3/2} \frac{m_\rho}{\sqrt{s}} \theta(s - 4m_\pi^2) \quad (5)$$

with the obvious identification $\Gamma_\rho = \Gamma(m_\rho^2)$.

Using Eq. (2) we are lead to modified expressions for Eqs. (8), (9) and (15) of Ref. [3], namely:

$$F_\pi^{(1)}(s) = \left( -\frac{am_\rho^2}{D(s)} + b \right) \left( 1 + \frac{ym_\omega^2}{s - s_\omega} \right) \quad (6)$$
Using Eqs. (6-8), we have repeated the fits to the experimental data of Barkov et al. [4] in the energy region \(2m_\pi \leq \sqrt{s} \leq 1.1\) GeV. As in Ref. [3], the free parameters of the fit are \(m_\rho, \Gamma_\rho, a, b\) and \(y\). The results of the best fits are shown in Table 1.

From a straightforward comparison of Table 1 and the corresponding results in Ref. [3] (see particularly, Eqs. (10), (11), (16) and Table I of that reference), we observe that the quality of the fits are very similar. Furthermore, the pole position, namely the numerical values of \(m_\rho\) and \(\Gamma_\rho\), and of the \(\rho-\omega\) mixing parameter \(y\), are rather insensitive to the new parametrizations (as it should be). The major effect of the new parametrizations is observed in the numerical values of \(a\) (the residue at the pole) and \(b\) (which describes the background).

An interesting consequence of the results in Table 1 is an improvement in the value of \(F_\pi(0)\), which should equal 1 (the charge of \(\pi^+\)). Indeed, from Eqs. (6-8) and Table 1 we obtain:

\[
F_\pi^{(1)}(0) = a + b = 0.997 \pm 0.015 \ (0.962 \pm 0.020)
\]

\[
F_\pi^{(2)}(0) = a + b = 0.997 \pm 0.015 \ (0.960 \pm 0.017)
\]

\[
F_\pi^{(4)}(0) = \frac{a}{1 - b} = 1.011 \pm 0.010 \ (0.987 \pm 0.013)
\]
where the corresponding values obtained in Ref. [3] are shown in brackets. An evident improvement is observed.

Let us close the discussion on this new parametrization with a short comment: using $F_\pi^{(4)}(s)$ (with imaginary parts and $y$ set to zero) we are able to reproduce very well the data of Ref. [6] in the space-like region $-0.253 \text{ GeV}^2 \leq s \leq -0.015 \text{ GeV}^2$.

IV. Prediction for $\tau^- \rightarrow \pi^-\pi^0\nu_\tau$.

Finally, using the previous results on the pion electromagnetic form factor, we consider the decay rate for $\tau^- \rightarrow \pi^-\pi^0\nu_\tau$. As is well known [2], the CVC hypothesis allows to predict the decay rate for $\tau^- \rightarrow (2n\pi)^-\nu_\tau$ in terms of the measured cross section in $e^+e^- \rightarrow (2n\pi)^0$. Since for the $\tau^- \rightarrow \pi^-\pi^0\nu_\tau$ case the kinematical range extends up to $\sqrt{s} = m_\tau$, let us point out that we have verified that our parametrizations for $F_\pi(s)$ reproduce very well the data of $e^+e^- \rightarrow \pi^+\pi^-$ in the energy region from 1.1 GeV to $m_\tau$.

The decay rate for $\tau^- \rightarrow \pi^-\pi^0\nu_\tau$ at the lowest order is given by [2]:

$$
\Gamma^0(\tau^- \rightarrow \pi^-\pi^0\nu_\tau) = \frac{G_F|V_{ud}|^2m_\tau^3}{384\pi^3} \int_{4m_\pi^2}^{m_\tau^2} ds \left(1 + \frac{2s}{4m_\tau^2}\right) \left(1 - \frac{s}{m_\tau^2}\right)^2 \left(\frac{s - 4m_\pi^2}{s}\right)^{3/2} |F_\pi^{I=1}(s)|^2 \quad (10)
$$

where $V_{ud}$ is the relevant Cabibbo-Kobayashi-Maskawa mixing angle. In the above expression we have neglected isospin breaking in the pion masses. The form factor $F_\pi^{I=1}(s)$ in the Eq. (10) is obtained from Eqs. (6-8) by removing the I=0 contribution due to $\rho - \omega$ mixing (namely, $y = 0$).
According to Ref. [7], after including the dominant short-distance electroweak radiative corrections the expression for the decay rate becomes:
\[
\Gamma(\tau^- \rightarrow \pi^- \pi^0 \nu_\tau) = \left(1 + \frac{2\alpha}{\pi} \ln \frac{M_Z}{m_\tau}\right) \Gamma^0(\tau^- \rightarrow \pi^- \pi^0 \nu_\tau).
\] (11)

We have not included the effects of long-distance electromagnetic radiative corrections, but we expect that they would not exceed 2.0 %.

In order to predict the branching ratio, we use Eqs. (6)-(8) with \(y = 0\), the results of Table 1 and the following values of fundamental parameters (ref. [7, 8]):

\[
\begin{align*}
m_\tau & = 1777.1 \pm 0.5 \text{ MeV} \\
G_F & = 1.16639(2) \times 10^{-5} \text{ GeV}^{-2} \\
|V_{ud}| & = 0.9750 \pm 0.0007.
\end{align*}
\]

With the above inputs we obtain:

\[
B(\tau^- \rightarrow \pi^- \pi^0 \nu_\tau) = \left(\frac{\tau_{\text{true}}}{2.956 \times 10^{-13} \text{s}}\right) \cdot \begin{cases} 
(24.66 \pm 0.26)\% & \text{from Eq. (6)} \\
(24.62 \pm 0.26)\% & \text{from Eq. (7)} \\
(24.96 \pm 0.32)\% & \text{from Eq. (8)} 
\end{cases}
\]

(12)

or, the simple average

\[
B(\tau^- \rightarrow \pi^- \pi^0 \nu_\tau) = (24.75 \pm 0.38)\% 
\]

(13)

which is in excellent agreement with recent experimental measurements and other theoretical calculations (see Table 2). Eq. (13) includes the errors (added in quadrature) coming from the fit to \(e^+e^- \rightarrow \pi^+\pi^-\) and the 1 % error in the \(\tau\) lifetime [8]: \(\tau_\tau = (295.6 \pm 3.1) \times 10^{-15} \text{s}\).
In summary, based on the S-matrix approach we have considered a modified parametrization for the $\pi^\pm$ electromagnetic form factor, which describes very well the experimental data of $e^+e^- \rightarrow \pi^+\pi^-$ in the energy region from threshold to 1.1 GeV. The pole position of the S-matrix amplitude is not changed by this new parametrization. Using CVC, we have predicted the $\tau^- \rightarrow \pi^-\pi^0\nu_\tau$ branching ratio, which is found to be in excellent agreement with experiment.

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TABLE CAPTIONS

1. Best fits to the pion electromagnetic form factor of Ref. [4], using Eqs. (6-8).

2. Summary of recent experimental measurements (Exp.) and theoretical results (Th.) for the $\tau^{-} \rightarrow \pi^{-}\pi^{0}\nu_{\tau}$ branching ratio. The errors in the first entry arise from use of $e^{+}e^{-} \rightarrow \pi^{+}\pi^{-}$ data, the $\tau$ lifetime and radiative correction effects [9], respectively.
### Table 1

| $F^{(1)}_\pi$ | $m_\rho$ (MeV) | $\Gamma_\rho$ (MeV) | $a$ | $b$ | $y(10^{-3})$ | $\chi/d.o.f$ |
|----------------|-----------------|---------------------|-----|-----|---------------|--------------|
| 756.74±        | 143.78±         | 1.236±              | −0.239± | −1.91± | 0.998         |
| 0.82           | 1.16            | 0.008               | 0.013 | 0.15 |               |
| $F^{(2)}_\pi$  | 756.58±         | 144.05±             | 1.237± | −0.240± | −1.91±        | 1.008        |
| 0.82           | 1.17            | 0.008               | 0.013 | 0.15 |               |
| $F^{(4)}_\pi$  | 757.03±         | 141.15±             | 1.206± | −0.193± | −1.86±        | 0.899        |
| 0.76           | 1.18            | 0.008               | 0.009 | 0.15 |               |

### Table 2

| Reference     | $B(\tau^- \rightarrow \pi^-\pi^0\nu_\tau)$ (in %) |
|---------------|--------------------------------------------------|
| Th. [9]       | 24.58 ± 0.93 ± 0.27 ± 0.50                       |
| Th. [10]      | 24.60 ± 1.40                                     |
| Th./Exp. [11] | 24.01 ± 0.47                                     |
| Exp. [8]      | 25.20 ± 0.40                                     |
| Exp. [12]     | 25.36 ± 0.44                                     |
| Exp. [13]     | 25.78 ± 0.64                                     |