Nucleon structure functions in the truncated moments approach

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Abstract. We demonstrate advantages of the truncated Mellin moments (TMM) approach in the analysis of DIS data. We present a novel method for determination of the Bjorken sum rule (BSR) from restricted in \(x\) variable experimental data. We show how to incorporate different uncertainties for each kinematic bin. We apply our analysis to recent COMPASS data.

1. Introduction

Our understanding of the matter structure and fundamental particle interactions at high energies is mostly provided by deep inelastic scattering (DIS) of leptons on hadrons and hadron-hadron collisions. In virtue of QCD factorization in these processes hadron properties can be described in terms of the parton distribution functions (PDFs) \(f_p(x, \mu^2)\). They are universal process-independent densities explaining how the whole hadron momentum \(P\) is partitioned in \(x \cdot P\) between partons of type \(p\). Here hard momentum transfer \(q^2 = Q^2 \gg P^2 = m_h^2\), and the Bjorken variable \(x\) satisfies \(0 < x = Q^2/(2Pq) < 1\). These distributions \(f_p(x, \mu^2)\) are formed by nonperturbative strong interaction at hadronic scale \(m_h^2\), while the dependence on the normalization scale \(\mu^2\) is governed within perturbative QCD by the well-known Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution equations \([1]\). Alternatively, one can study how to evolve with this scale \(\mu^2\) the Mellin moments of the parton densities \(f(n, \mu^2)\), which are integrals of PDFs weighted with \(x^n\) over the whole range \((0,1)\) of \(x\). These moments provide a natural framework of QCD analysis as they originate from the basic formalism of operator product expansion and are essential in testing sum rules. However, these standard moments, in principle, cannot be extracted from any experiment due to kinematic constraints inevitably appearing in real DIS of lepton-hadron and hadron-hadron collisions. Therefore, it is demanded to invent new “real observables” with a goal to overcome these kinematic constraints. Generalized moments of the parton distribution \(f(x,\mu^2)\), \(f(z; n, \mu^2) = \int^1_z f(x, \mu^2)x^{n-1} dx\), contains the unavoidable lower limit of integration \(z \equiv x_{\text{min}} = Q^2/(2(Pq)_{\text{max}}) > 0\), and in this way the kinematic constraint can be taken into account. The idea of “truncated” Mellin moments of the parton densities in QCD analysis was introduced and developed in the late 1990s \([2]\). Later on, diagonal integro-differential DGLAP-type evolution equations for the single and double truncated
moments of the parton densities were derived in [3] and [4, 5]. The main finding of the truncated Mellin moments (TMM) approach is that the \( n \)th moment of the parton density also obeys the DGLAP equation, but with a rescaled evolution kernel \( P'(z) = z^n P(z) \) [3]. The TMM approach has already been successfully applied, e.g., in spin physics to derive a generalization of the Wandzura-Wilczek relation in terms of the truncated moments and to obtain the evolution equation for the structure function \( g_2 \) [5]. The evolution equations for double cut moments and their application to study the quark-hadron duality were also discussed in [6]. The advantages of the TMM approach to QCD factorization for DIS structure functions were presented in [7]. A valuable generalization of the TMM approach incorporating multiple integrations as well as multiple differentiations of the original parton distribution has been obtained in [8]. Recently, based on the TMM, we constructed a device which allows one to improve an experimental determination of the Bjorken sum rule (BSR) [9]. Here, we present the main results of our recent approach [10] - the generalized BSR that allows an effective determination of the BSR value from the experimental data in a restricted kinematic range of \( x \). We also show how to incorporate different uncertainties for each kinematic bin. We apply our analysis to recent COMPASS data on the spin structure function \( g_1 \).

2. Generalized Bjorken sum rule
The generalized truncated moment \( f(z, n, \omega) \), obtained as a Mellin convolution of the PDF \( f \) with any normalized function \( \omega(x) \),

\[
\begin{align}
f(x, n; \omega) &= (\omega * x^n f(x)) = \int_x^1 \omega(x/z) f(z, \mu^2) z^n \frac{dz}{z}, \\
\int_0^1 \omega(t) dt &= 1,
\end{align}
\]

obeys the DGLAP evolution equation with the rescaled kernel [9]:

\[
P(y) = P(y) \cdot y^n.
\]

In a case of the non-singlet polarized structure function \( g_1 \) and for \( n = 0 \), one obtains the generalized structure function \( g_1; \omega \),

\[
g_1;\omega(x) = (\omega * g_1)(x) = \int_x^1 \omega(x/z) g_1(z, Q^2) \frac{dz}{z}
\]

with the same DGLAP evolution kernel as \( g_1 \), [10], namely \( P(y) \). In this way, we define the truncated BSR, \( \Gamma_1(x_0) \), and simultaneously, the generalized cut Bjorken sum rules (gBSR), \( \Gamma_{1;\omega}(x_0) \),

\[
\begin{align}
\Gamma_1(x_0) &= \int_{x_0}^1 g_1(x) dx, \\
\Gamma_{1;\omega}(x_0) &= \int_{x_0}^1 g_{1;\omega}(x) dx,
\end{align}
\]

which are equal to the ordinary BSR as \( x_0 \to 0 \):

\[
\Gamma_{1;\omega}(0) = \int_0^1 g_{1;\omega}(x) dx = \int_0^1 g_1(x) dx \equiv \Gamma_1(0).
\]
Now, one can estimate the value of $\Gamma_1(0)$ from the smooth extrapolation of the truncated moments $\Gamma_1(\omega(x_0))$ in $x_0$. To this aim, one constructs a bunch of different $\Gamma_1(\omega(x_0))$ based on the simple sign-changing normalized function $\omega(x)$ depending on three parameters $z_1$, $z_2$, $A$, 

$$\omega(z) = -A \delta(z - z_1) + (1 + A) \delta(z - z_2).$$

Here the $\omega$-model parameters are $z_2 > z_1 > x_0 > 0$ and $A > 0$ for the sign change. This leads to a “shuffle” of the initial function $g_1$ in $x$ variable and the generalized truncated BSR $\Gamma_1(\omega(x_0))$,

$$\Gamma_1(\omega(x_0)) = \int_{x_0/z_2}^{1} g_1(x) \, dx + A \int_{x_0/z_2}^{\infty} g_1(x) \, dx,$$

approaches the limit $\Gamma_1(0)$ more quickly than the ordinary $\Gamma_1(x_0)$ in Eq. (4). In this way, the total BSR limit $\Gamma_1(0)$ can be determined very effectively with use of a few first orders of Taylor expansion:

$$\Gamma_1(0) = \Gamma_1(\omega(x_0) - x_0) = \Gamma_1(\omega(x_0)) - x_0 \Gamma_1'(\omega(x_0)) + x_0^2 \frac{1}{2} \Gamma_1''(\omega(x_0)) + \cdots.$$  

Figs. 1 and 2 illustrate the features of $\Gamma_1(\omega)$, where we plot the bunch $\Gamma_1(\omega(x_0))$, Eq. (8) for different values of $A$, including: “constant behavior” value $A = A_{00}$, “quasi-linear behavior” value $A = A_{02}(\bar{x})$ and the standard truncated BSR $\Gamma_1(x_0)$, Eq. (4) (thick black curve). We use the popular form of parametrization of $g_1$ at $Q_0^2 = 1$ GeV$^2$,

$$g_1(x, Q_0^2) = N \cdot x^a(1-x)^b(1 + \gamma x),$$

where $a = 0$ in Fig. 1 and $a = -0.4$ in Fig. 2, respectively, $b = 3$, $\gamma = 5$ and the coefficient $N$ is the norm. In our tests, in order to obtain a smooth approach of the bunch in the experimentally available $x$ region, we fixed $z_1 = 0.7$ and $z_2 = 0.9$. One can see significantly rapid saturation of $\Gamma_1(\omega(x_0))$ to $\Gamma_1(0)$ in comparison with $\Gamma_1(x_0)$. The quasi-linear regime of gBSR near 0 visibly starts at rather large values of $x_0 > 0.1$ for the different parametrization in Eq. (10). This should ensue the applicability of the first order approximation (IAPX),

$$\Gamma_1(0) \approx \Gamma_1\text{IAPX}(x_0) = \Gamma_1(\omega(x_0)) - x_0 \Gamma_1'(\omega(x_0)).$$
even for JLab experimental conditions, where the admissible $x$ bunches are rather far from 0. In practice, one can use fit to the data instead of the ready input parametrization. It is worthy to notice that the analysis based on the bunch behavior allows one to shift the available region of $x$ to smaller values, $x_0 = x \cdot z_2$. In this manner, using data from large $x$ and choosing suitable values of $z_1$ and $z_2$, one is able to get an answer in a much smaller $x$ region.

Presented above idea of the generalized BSR can be used to analysis of the experimental data incorporating different uncertainties for each measurement. Usually, the results for $g_1$ are extracted from the data for kinematic bins \{$Q^2_i, x_i\}$ with uncertainties $\Delta(g_1(x_i))$. In order to compute moments of $g_1$ and verify the BSR, bins must be evolved to a common scale $Q^2$. Then, the experimental bins \{$x_i\}_1^n$ can be used to construct multi-point weight function $\omega$ in Eq. (1),

$$\omega(z) = - \sum_{i=1}^{n-1} A_i \delta(z - z_i) + (1 + A) \delta(z - z_n),$$

where

$$A = \sum_{i=1}^{n-1} A_i > 0.$$  

This multi-point \{$\{z_i, A_i\}_1^n$\} ansatz is a generalization of the previous two-point \{$\{z_1, z_2, A\}$\} one, Eq. (7), and leads to a new presentation for $\Gamma_1; \omega$,

$$\Gamma_1; \omega(x_0) = \int_{x_0/z_n}^1 g_1(x) dx + A \int_{x_0/z_n}^{x_0/z_n-1} g_1(x) dx + A \sum_{i=1}^{n-2} w_i \int_{x_0/z_i}^{x_0/z_{i+1}} g_1(x) dx.$$  

In order to use the generalized BSR to the determination of $\Gamma_1(0)$, one only needs to construct the set of $\{A_i\}$. Choosing the most appropriate weights $w_i$ and hence $A_i$,

$$w_i = \frac{1}{A} \sum_{k=1}^{i} A_k; \quad A_i = A \cdot (w_i - w_{i-1}),$$

one is able to tune the analysis to the real experimental constraints. The most natural way of implementing the experimental uncertainties of $g_1$ into the ansatz of $\Gamma_1; \omega$ is to use $w_i$ inversely proportional to the relative uncertainties $\Delta(g_1)/g_1$ at each $x_i$:

$$w_i \sim \frac{g_1(x_i)}{\Delta(g_1(x_i))},$$


to increase the weights of the data with smaller uncertainties.

3. Analysis of data
In this section we present practical estimation of $\Gamma_1(0)$ from the recent COMPASS data [11] using two-point and multi-point versions of the gBSR approach described in the previous section. We apply the simplified equations of our approach, written in terms of experimental parameters.
3.1. Two-point estimation of $\Gamma_1(0)$

For practical purposes, we write here the essential formulas for the generalized BSR in terms of experimental data and demonstrate the effective method for the estimation of $\Gamma_1(0)$. Thus, the gBSR, Eq. (8), where the lower limit of integrations has to be strictly related to the minimal $x$ accessible experimentally, $x_{\text{min}}$, takes the form

$$
\Gamma_{1;\omega}(x_{\text{min}}, r) = \int_{x_{\text{min}}}^{1} g_1(x)\, dx + A \int_{x_{\text{min}}}^{x_{\text{min}}/r} g_1(x)\, dx.
$$

(17)

The experimental lower value $x_{\text{min}}$ in the above equation is related to $x_0$ from Eq. (8) via $x_0 = x_{\text{min}} z_2$. The ratio parameter $r \equiv z_1/z_2$ and $x_0 < x_{\text{min}} < r < 1$. We have tested all methods of estimation of $\Gamma_1(0)$, described in [10] and have found that a very effective method, universal for the different small-$x$ behavior of $g_1$ and for $x_{\text{min}} \lesssim 0.1$, is the first order approximation, Eq. (11). With use of the parameters $x_{\text{min}}$ and $r$, it reads

$$
\Gamma_1(0) \approx \Gamma_{1;\omega}^{\text{APX}}(x_{\text{min}}, r) = \Gamma_{1;\omega}(x_{\text{min}}, r) + (A + 1) x_{\text{min}} g_1(x_{\text{min}}) - A \frac{x_{\text{min}}}{r} g_1(x_{\text{min}}/r)
$$

(18)

with $A$ obtained requiring the second derivative of $\Gamma_{1;\omega}(x_0)$ to vanish (first order approximation),

$$
A = \left[r^2 \frac{g_1'(x_{\text{min}}/r)}{g_1'(x_{\text{min}})} - 1\right]^{-1}.
$$

(19)

Below we present our results on determination of the BSR based on the recent COMPASS data, where $x_{\text{min}} = 0.0036$. We follow the method described above using Eqs. (17)–(19). Our fit to the data at $Q^2 = 3$ GeV$^2$ is $g_1(x) \sim x^{-0.36}(1 - x)^3(1 + 3.9 x)$. One can see from Fig. 3 that $\Gamma_{1;\omega}(x)$ approaches $\Gamma_1(0)$ more quickly than the original BSR $\Gamma_1(x)$.

![Figure 3. $\Gamma_{1;\omega}(x, r, Q^2)$, Eq. (17), for $A(x_{\text{min}} = 0.0036, r)$, Eq. (19) for three values of $r : 0.9, 0.5, 0.3$, together with the truncated BSR $\Gamma_1(x, Q^2)$, Eq. (4), as a function of $x$. The results are based on our fit to the COMPASS data.](image)

We find the estimated total BSR $\Gamma_1(0)$ from Eqs. (17)–(19) for $x_{\text{min}} = 0.0036$ and different $r$ (0.9, 0.5, 0.3, 0.1) $\Gamma_{1;\omega}^{\text{APX}}(x_{\text{min}}, r) = 0.190$. This should be compared to the final estimation provided by COMPASS collaboration, $\Gamma_1(0) = 0.192 \pm 0.007_{\text{stat}} \pm 0.015_{\text{syst}}$. 

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3.2. Multi-point estimation of $\Gamma_1(0)$

Here we develop the idea of the generalized BSR to analysis of the COMPASS data on $g_1(x)$ incorporating uncertainties for each measurement. We consider the results for $n$ experimental values of $x$: $x_{\text{min}} \equiv x_1 < x_2 < x_3 < \cdots < x_n \equiv x_{\text{max}} < 1$, where $x_i = x_0/z_{n-i+1}$ and corresponding $g_1(x_i)$ together with their relative statistical uncertainties $U_i = \Delta g_1(x_i)/g_1(x_i)$. In terms of the experimental parameters the multi-point gBSR, Eq. (14), has the form

$$\Gamma_1(\omega) = \int_{x_1}^1 g_1(x) \, dx + A \sum_{i=1}^{n-1} \tilde{w}_i \int_{x_i}^{x_{i+1}} g_1(x) \, dx,$$

where we choose the weights $\tilde{w}_i = U_i/U_i$ reflecting the increase of the experimental uncertainties of $g_1$ with decreasing $x$. The first order approximation for $\Gamma_1(0)$ in the multi-point case is a generalization of Eqs. (18) and (19):

$$\Gamma_1(0) \approx \Gamma_1^\text{APX}(\{x_i\}^n_1) = \Gamma_1(\omega(\{x_i\}^n_1)) + x_1 g_1(x_1) + A \sum_{i=1}^{n-1} \tilde{w}_i [x_i g_1(x_i) - x_{i+1} g_1(x_{i+1})],$$

$$A = \frac{x_1^2 g_1'(x_1)}{\sum_{i=1}^{n-1} \tilde{w}_i [x_{i+1}^2 g_1'(x_{i+1}) - x_i^2 g_1'(x_i)].}$$

In our analysis we use fit function to evolve $g_1(x, Q^2)$ to the common value of $Q^2 = 3$ GeV$^2$ and also to find $g_1'(x_i)$. We obtain from Eqs. (20)–(22) $\Gamma_1(0) \approx \Gamma_1^\text{APX}(\{x_i\}^n_1) = 0.190$, the same value as in the previous case of 2-point estimation. This result is in a good agreement with the value $0.192 \pm 0.007_{\text{stat}} \pm 0.015_{\text{syst}}$ provided by COMPASS [11] from reanalyzed data.

4. Conclusions

The generalized BSR, based on the truncated Mellin moment approach provides a powerful tool to test QCD at experimental constraints.

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