STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY PURELY
SPATIAL NOISE

S. V. LOTOTSKY AND B. L. ROZOVSKII

Abstract. We study stochastic parabolic and elliptic PDEs driven by purely spatial white noise. Even the simplest equations driven by this noise often do not have a square-integrable solution and must be solved in special weighted spaces. We demonstrate that the Cameron-Martin version of the Wiener chaos decomposition is an effective tool to study both stationary and evolution equations driven by space-only noise. The paper presents results about solvability of such equations in weighted Wiener chaos spaces and studies the long-time behavior of the solutions of evolution equations with space-only noise.

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1. Introduction

Stochastic PDEs of the form

$$\dot{u}(t, x) = A u(t, x) + M u(t, x) \cdot \dot{W}(t, x),$$

where $A$ and $M$ are (linear) partial differential operators and $\dot{W}(t, x)$ is the space-time white noise are usually referred to as bilinear evolution SPDEs. The theory and the applications of these equations have been actively investigated for a few decades now (see for example [2, 9, 15, 18, 20, 21]). In contrast, little is known about the evolution equations or stationary (elliptic) equations driven by purely spatial white noise $\dot{W}(x)$. An important exception to this statement is the paper by Y. Hu [6]. This paper deals with heat equation with random potential:

$$\dot{u}(t, x) = \Delta u(t, x) + u(t, x) \diamond \dot{W}(x),$$

where $\phi \diamond \dot{W}$ denotes the Skorohod integral. A surprising discovery made in [6] was that the spatial regularity of the square integrable solution is better than in the case of similar equation driven by the space-time white noise.

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Even less attention was paid to stochastic elliptic equations such as

$$A u(x) + M u(x) \circ \dot{W}(x) = f(x).$$

(1.3)

A number of interesting examples of stochastic elliptic PDEs, including stochastic Poisson equation and stationary Schrödinger equation with a stochastic potential, are discussed in the book [4], see also the references therein. These examples demonstrate that elliptic SPDEs are of substantial interest in many areas of science.

Another reason to study elliptic SPDEs is that they generate stationary solutions of the related parabolic SPDEs. In particular, it is natural to expect that, under certain conditions, the solution of the elliptic equation (1.3) will describe the limiting behavior, as $t \to +\infty$, of the stochastic evolution equation

$$\frac{\partial v(t, x)}{\partial t} = A v(t, x) + M v(t, x) \circ \dot{W}(x) - f(x).$$

(1.4)

The objective of this paper is to develop a systematic approach to dealing with bilinear SPDEs driven by purely spatial white noise. In the case of spatial white noise, there is no natural and convenient filtration, therefore we consider only anticipating solutions. We cover the same classes of operators $A$ and $M$ that were investigated previously for $\mathcal{F}_t^W$-adapted (nonanticipating) solutions of the equation (1.1) driven by space-time white noise. In fact, we go a bit further.

Traditionally, $\mathcal{F}_t^W$-adapted solutions of equation (1.1) were studied under the following assumptions:

(1) The operator $A - \frac{1}{2} M M^*$ is elliptic (possibly degenerate) operator.

It is well known (see e.g. KR, [14]) that unless assumption (1) holds, equation (1.1) has no solutions in $L_2(\Omega; X)$ for any reasonable choice of the state space $X$.

It was shown recently in [14] (see also [17]) that if (i) the operator $A$ is elliptic and (ii) the order of $M$ is smaller than the order of $A$, then there exists a unique nonanticipating generalized solution of equation (1.1). This solution is often referred to as Wiener Chaos solution and is given by the Wiener chaos expansion

$$u(t) = \sum_{|\alpha|<\infty} u_\alpha(t) \xi_\alpha,$$

where $\{\xi_\alpha\}_{|\alpha|<\infty}$ is the Cameron-Martin orthonormal basis in the space $L_2(\Omega, \mathcal{F}_t^W; X)$ of square integrable random elements in $X$ adapted to the filtration $\mathcal{F}_t^W$. The Cameron-Martin basis $\{\xi_\alpha\}$ is indexed by multiindices $\alpha = (\alpha_1, \alpha_2, \ldots)$ with non-negative integer entries, and $|\alpha| = \sum_k \alpha_k$. It was shown that for certain positive weights $Q = \{q(\alpha)\}_{|\alpha|<\infty}$, the weighted norm

$$\|u\|_{Q, X}^2 := \sum_{|\alpha|<\infty} q^2(\alpha) \left\| u_\alpha \right\|_{L_2((0,T);X)}^2 < \infty,$$

where $X$ is the appropriate Hilbert space characterizing the “regularity” of the solution. Note that by the reasons explained above,

$$\mathbb{E} \|u\|_{L_2((0,T);X)}^2 = \sum_{|\alpha|<\infty} \|u_\alpha(t)\|_{L_2((0,T);X)}^2 = \infty.$$
In this paper we abandon assumption (ii) and allow the the operators $A$ and $M$ to be of the same order. Examples of equations that require the assumption $\text{ord}(A) = \text{ord}(M)$ include equation

$$
\dot{v}(t,x) = \Delta \left( v(t,x) + \varepsilon v(t,x) \diamond \dot{W} \right) + f(x),
$$

and its elliptic counterpart

$$
\Delta \left( u(x) + \varepsilon u(x) \diamond \dot{W} \right) = f(x)
$$

that could be interpreted as the heat and Poisson equations, respectively, in random media. These two equations are nontrivial perturbations of the deterministic Heat and Poisson equations: the expectations $\bar{v}(t,x) = \mathbb{E} v(t,x)$, $\bar{u}(x) := \mathbb{E} u(x)$ solve the deterministic heat and Poisson equations, respectively, while $\text{Var}(v(t,x)) = \text{Var}(u(x)) = \infty$ for every $\varepsilon \neq 0$. Only the weights $Q = \{q(\alpha)\}_{|\alpha|<\infty}$ can make the variances finite.

Equations of types (1.3) and (1.4) can be also addressed in the framework of the White noise analysis; see [4, 19], and references therein. While being reasonably general, this approach has substantial limitations:

1. The white noise solutions “live” in the Hida space $\mathcal{S}^*$ of generalized distributions or even larger Kondratiev space $(\mathcal{S})_{-1}$. In contrast, the stochastic support of Wiener chaos solutions is much smaller and can be characterized precisely by weights $Q = \{q(\alpha)\}_{|\alpha|<\infty}$ suitable for the equation in question.

2. There seems to be little or no connection between the white noise solution and the traditional (square integrable) solutions. On the other hand, the Wiener chaos approach is a bona fide extension of the classical solution. In particular, both definitions coincide when all the weights are equal to 1.

3. The white noise solution, being constructed on a special white noise probability space, is weak in the probabilistic sense. Path-wise uniqueness does not apply to such solutions because of the ”averaging” nature of the solution spaces. On the contrary, the Wiener chaos solutions are strong in the probabilistic sense and quite helpful in numerical simulations (see e.g. [5]).

In this paper, we establish unique solvability in weighted Wiener Chaos spaces for both stationary equations of the type (1.3) and evolution equations of the type (1.4), and establish the convergence, as $t \to +\infty$, of the solution of the evolution equation to the solution of the stationary equation. Section 2 reviews the definition of the Skorokhod integral in the framework of the Malliavin calculus and shows how to compute the integral using Wiener chaos. Sections 3 and 4 are about solvability of abstract evolution and stationary equations, respectively, driven by a general (not necessarily white) space noise; Section 4 describes also the limiting behavior of the solution of the evolution equation; Section 5 illustrates the general results by applying them to bilinear SPDEs driven by purely spatial white noise.
2. Weighted Wiener Chaos and Malliavin Calculus

Let $\mathbb{F} = (\Omega, \mathcal{F}, P)$ be a complete probability space, and $\mathcal{U}$, a real separable Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathcal{U}}$. On $\mathbb{F}$, consider a zero-mean Gaussian family

$$\tilde{W} = \left\{ \tilde{W}(h), \ h \in \mathcal{U} \right\}$$

so that

$$\mathbb{E} \left( \tilde{W}(h_1) \tilde{W}(h_2) \right) = \langle h_1, h_2 \rangle_{\mathcal{U}}.$$

It suffice, for our purposes, to assume that $\mathcal{F}$ is the $\sigma$-algebra generated by $\tilde{W}$. Given a real separable Hilbert space $X$, we denote by $L^2(\mathcal{F}; X)$ the Hilbert space of square-integrable $\mathcal{F}$-measurable $X$-valued random elements $f$. In particular,

$$(f, g)^2_{L^2(\mathcal{F}; X)} := \mathbb{E}(f, g)^2_X.$$

When $X = \mathbb{R}$, we write $L^2(\mathcal{F})$ instead of $L^2(\mathcal{F}; \mathbb{R})$.

**Definition 2.1.** A formal series

$$\tilde{W} = \sum_k \tilde{W}(u_k) u_k,$$  \hspace{1cm} (2.1)

where $\{u_k, k \geq 1\}$ is a complete orthonormal basis in $\mathcal{U}$, is called (Gaussian) white noise on $\mathcal{U}$.

The white noise on $\mathcal{U} = L^2(G)$, where $G$ is a domain in $\mathbb{R}^d$, is usually referred to as a spatial or space white noise (on $L^2(G)$). The space white noise is of central importance for this paper.

Below, we will introduce a class of spaces that are convenient for treating nonlinear functionals of white noise, in particular, solutions of SPDEs driven by white noise.

Given an orthonormal basis $\mathcal{U} = \{u_k, k \geq 1\}$ in $\mathcal{U}$, define a collection $\{\xi_k, k \geq 1\}$ of independent standard Gaussian random variables so that $\xi_k = \tilde{W}(u_k)$. Denote by $\mathcal{J}$ the collection of multi-indices $\alpha$ with $\alpha = (\alpha_1, \alpha_2, \ldots)$ so that each $\alpha_k$ is a non-negative integer and $|\alpha| := \sum_{k \geq 1} \alpha_k < \infty$. For $\alpha, \beta \in \mathcal{J}$, we define

$$\alpha + \beta = (\alpha_1 + \beta_1, \alpha_2 + \beta_2, \ldots), \quad \alpha! = \prod_{k \geq 1} \alpha_k!.$$

By (0) we denote the multi-index with all zeroes. By $\varepsilon_i$ we denote the multi-index $\alpha$ with $\alpha_i = 1$ and $\alpha_j = 0$ for $j \neq i$. With this notation, $n \varepsilon_i$ is the multi-index $\alpha$ with $\alpha_i = n$ and $\alpha_j = 0$ for $j \neq i$.

Define the collection of random variables $\Xi = \{\xi_\alpha, \alpha \in \mathcal{J}\}$ as follows:

$$\xi_\alpha = \prod_k \left( \frac{H_{\alpha_k}(\xi_k)}{\sqrt{\alpha_k!}} \right),$$  \hspace{1cm} (2.2)

where

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2}$$  \hspace{1cm} (2.3)

is Hermite polynomial of order $n$. 
Theorem 2.2. (Cameron and Martin [1]) The collection \( \Xi = \{ \xi_\alpha, \ \alpha \in \mathcal{J} \} \) is an orthonormal basis in \( L_2(\mathbb{F}) \): if \( \eta \in L_2(\mathbb{F}) \) and \( \eta_\alpha = \mathbb{E}(\eta \xi_\alpha) \), then \( \eta = \sum_{\alpha \in \mathcal{J}} \eta_\alpha \xi_\alpha \) and \( \mathbb{E}|\eta|^2 = \sum_{\alpha \in \mathcal{J}} \eta_\alpha^2 \).

Expansions with respect to the Cameron-Martin basis \( \Xi \) is usually referred to as Wiener Chaos. Next, we introduce a modification of the Wiener Chaos expansion which will be called weighted Wiener Chaos.

Let \( \mathcal{R} \) be a bounded linear operator on \( L_2(\mathbb{F}) \) defined by \( \mathcal{R}\xi_\alpha = r_\alpha \xi_\alpha \) for every \( \alpha \in \mathcal{J} \), where the weights \( \{r_\alpha, \ \alpha \in \mathcal{J} \} \) are positive numbers. By Theorem 2.2, \( \mathcal{R} \) is bounded if and only if the weights \( r_\alpha \) are uniformly bounded from above: \( r_\alpha < C \) for all \( \alpha \in \mathcal{J} \), with \( C \) independent of \( \alpha \). The inverse operator \( \mathcal{R}^{-1} \) is defined by \( \mathcal{R}^{-1} \xi_\alpha = r_\alpha^{-1} \xi_\alpha \).

We extend \( \mathcal{R} \) to an operator on \( L_2(\mathbb{F}; X) \) by defining \( \mathcal{R}f \) as the unique element of \( L_2(\mathbb{F}; X) \) so that, for all \( g \in L_2(\mathbb{F}; X) \),

\[
\mathbb{E}(\mathcal{R}f, g)_X = \sum_{\alpha \in \mathcal{J}} r_\alpha \mathbb{E}((f, g)X\xi_\alpha).
\]

Denote by \( \mathcal{R}L_2(\mathbb{F}; X) \) the closure of \( L_2(\mathbb{F}; X) \) with respect to the norm

\[
\|f\|_{\mathcal{R}L_2(\mathbb{F}, X)} := \|\mathcal{R}f\|_{L_2(\mathbb{F}; X)}^2.
\]

Then the elements of \( \mathcal{R}L_2(\mathbb{F}; X) \) can be identified with a formal series \( \sum_{\alpha \in \mathcal{J}} f_\alpha \xi_\alpha \), where \( f_\alpha \in X \) and \( \sum_{\alpha \in \mathcal{J}} \|f_\alpha\|_X^2 r_\alpha^2 < \infty \).

We define the space \( \mathcal{R}^{-1}L_2(\mathbb{F}; X) \) as the dual of \( \mathcal{R}L_2(\mathbb{F}; X) \) relative to the inner product in the space \( L_2(\mathbb{R}; X) \):

\[
\mathcal{R}^{-1}L_2(\mathbb{F}; X) = \{ g \in L_2(\mathbb{F}; X) : \mathcal{R}^{-1}g \in L_2(\mathbb{F}; X) \}.
\]

For \( f \in \mathcal{R}L_2(\mathbb{F}; X) \) and \( g \in \mathcal{R}^{-1}L_2(\mathbb{F}) \) we define the scalar product

\[
\langle \langle f, g \rangle \rangle := \mathbb{E}((\mathcal{R}f)(\mathcal{R}^{-1}g)) \in X.
\]

In what follows, we will identify the operator \( \mathcal{R} \) with the corresponding collection \( \{r_\alpha, \ \alpha \in \mathcal{J} \} \). Note that if \( u \in \mathcal{R}_1L_2(\mathbb{F}; X) \) and \( v \in \mathcal{R}_2L_2(\mathbb{F}; X) \), then both \( u \) and \( v \) belong to \( \mathcal{R}L_2(\mathbb{F}; X) \), where \( r_\alpha = \min(r_{1,\alpha}, r_{2,\alpha}) \). As usual, the argument \( X \) will be omitted if \( X = \mathbb{R} \).

The spaces \( \mathcal{R}L_2(\mathbb{F}; X) \) and \( \mathcal{R}^{-1}L_2(\mathbb{F}; X) \) are weighted versions of \( L_2(\mathbb{F}; X) \). General properties of such weighted, or sequence, spaces were studied by Köthe ([5]).

Important particular cases of the space \( \mathcal{R}L_2(\mathbb{F}; X) \) correspond to the following weights:

(1)

\[
r_\alpha^2 = \prod_{k=1}^{\infty} q_k^{\alpha_k},
\]

where \( \{q_k, k \geq 1\} \) is a non-increasing sequence of positive numbers with \( q_1 \leq 1 \) (see [14, 17]).
(2) \[ \rho^2 = (\alpha!)^2(2N)^{\ell_\alpha}, \quad \rho \leq 0, \quad \ell \leq 0, \quad \text{where} \quad (2N)^{\ell_\alpha} = \prod_{k \geq 1} (2k)^{\ell_{\alpha k}}. \] (2.5)

This set of weights defines Kondratiev’s spaces \( (S)_{\rho,\ell}(X) \) (cf. \([3, 4]\)).

Now we will sketch the basics of Malliavin calculus on \( RL_2(F; X) \).

Denote by \( D \) the Malliavin derivative on \( L_2(F) \) (see e.g. \([16]\)). In particular, if \( F : \mathbb{R}^N \rightarrow \mathbb{R} \) is a smooth function and \( h_i \in U, \ i = 1, \ldots, N \), then

\[
D F(\dot{W}(h_1), \ldots, \dot{W}(h_N)) = \sum_{i=1}^{N} \frac{\partial F}{\partial x_i}(\dot{W}(h_1), \ldots, \dot{W}(h_N)) h_i \in L_2(F; U). \quad (2.6)
\]

It is known \([16]\) that the domain \( D_{1,2}(F) \) of the operator \( D \) is a dense linear subspace of \( L_2(F) \).

The adjoint of the Malliavin derivative on \( L_2(F) \) is the Itô-Skorokhod integral and is traditionally denoted by \( \delta \) \([16]\). We will keep this notation for the extension of this operator to \( RL_2(F; X \otimes U) \).

For \( f \in RL_2(F; X \otimes U) \), we define \( \delta(f) \) as the unique element of \( RL_2(F; X) \) with the property

\[
\langle \langle \delta(f), \varphi \rangle \rangle = \mathbb{E}(R f, R^{-1} D \varphi)_U \quad (2.7)
\]

for every \( \varphi \) satisfying \( \varphi \in R^{-1} L_2(F) \) and \( D \varphi \in R^{-1} L_2(F; U) \).

Next, we derive the expressions for the Malliavin derivative \( D \) and its adjoint \( \delta \) in the basis \( \Xi \).

**Proposition 2.3.** For each \( \alpha \in J \), we have

\[
D(\xi_\alpha) = \sum_{k \geq 1} \sqrt{\alpha_k} \xi_{\alpha - \varepsilon_k} u_k. \quad (2.8)
\]

**Proof.** The result follows by direct computation using the property \((2.6)\) of the Malliavin derivative and the relation \( H'_n(x) = nH_{n-1}(x) \) for the Hermite polynomials (cf. \([16]\)).

Obviously, the set \( J \) is not invariant with respect to substraction. In particular, the expression \( \alpha - \varepsilon_k \) is undefined if \( \alpha_k = 0 \). In \((2.8)\) and everywhere below in this paper where undefined expressions of this type appear, we use the following convention: if \( \alpha_k = 0 \), then \( \sqrt{\alpha_k} \xi_{\alpha - \varepsilon_k} = 0 \).

**Proposition 2.4.** For \( \xi_\alpha \in \Xi \), \( h \in X \), and \( u_k \in U \), we have

\[
\delta(\xi_\alpha h \otimes u_k) = h \sqrt{\alpha_k + 1} \xi_{\alpha + \varepsilon_k}. \quad (2.9)
\]

**Proof.** It is enough to verify \((2.7)\) with \( f = h \otimes u_k \xi_\alpha \) and \( \varphi = \xi_\beta \), where \( h \in X \). By \((2.8)\),

\[
\mathbb{E}(f, D \varphi)_U = \sqrt{\beta_k} h \mathbb{E}(\xi_{\alpha} \xi_{\beta - \varepsilon_k}) = \begin{cases} 
\sqrt{\alpha_k + 1} h, & \text{if } \alpha = \beta - \varepsilon_k, \\
0, & \text{if } \alpha \neq \beta - \varepsilon_k.
\end{cases}
\]
In other words,
\[ \mathbb{E}(\xi_\alpha h \otimes u_k, \mathbf{D}\xi_\beta)_t = h \mathbb{E}(\sqrt{\alpha_k + 1} \xi_{\alpha + \epsilon_k} \xi_\beta) \]
for all \( \beta \in \mathcal{J}. \) \( \Box \)

**Remark 2.5.** The operator \( \delta \mathbf{D} \) is linear and unbounded on \( L_2(\mathbb{F}) \); it follows from Propositions 2.3 and 2.4 that the random variables \( \xi_\alpha \) are eigenfunctions of this operator:
\[ \delta(\mathbf{D}(\xi_\alpha)) = |\alpha| \xi_\alpha. \quad (2.10) \]

To give an alternative characterization of the operator \( \delta \), we define a new operation on the elements of \( \Xi. \)

**Definition 2.6.** For \( \xi_\alpha, \xi_\beta \) from \( \Xi, \) define the Wick product
\[ \xi_\alpha \diamond \xi_\beta := \sqrt{\left[ \frac{(\alpha + \beta)!}{\alpha! \beta!} \right]} \xi_{\alpha + \beta}. \quad (2.11) \]

In particular, taking in (2.6) \( \alpha = k\varepsilon_i \) and \( \beta = n\varepsilon_i, \) and using (2.2), we get
\[ H_k(\xi_i) \diamond H_n(\xi_i) = H_{k+n}(\xi_i). \quad (2.12) \]

**Proposition 2.7.** If \( f \in \mathcal{R}L_2(\mathbb{F}; X) \) and \( \eta \in \mathcal{R}L_2(\mathbb{F}), \) then \( f \diamond \eta \) is an element of \( \bar{R}L_2(\mathbb{F}; X) \) for a suitable operator \( \bar{R}. \)

**Proof.** It follows from (2.6) that \( f \diamond \eta = \sum_{\alpha \in \mathcal{J}} F_\alpha \xi_\alpha \) and
\[ F_\alpha = \sum_{\beta, \gamma \in \mathcal{J}: \beta + \gamma = \alpha} \sqrt{\left[ \frac{(\beta + \gamma)!}{\beta! \gamma!} \right]} f_\beta \eta_\gamma. \]

Therefore, each \( F_\alpha X \) is an element of \( X, \) because, for every \( \alpha \in \mathcal{J}, \) there are only finitely many multi-indices \( \beta, \gamma \) satisfying \( \beta + \gamma = \alpha. \) It is known [4, Proposition 2.3.3] that
\[ \sum_{\alpha \in \mathcal{J}} (2N)^{q\alpha} \quad \text{if and only if} \quad q < -1. \quad (2.13) \]

Therefore, \( f \diamond \eta \in \bar{R}L_2(\mathbb{F}; X), \) where the operator \( \bar{R} \) can be defined using the weights \( \bar{s}_\alpha^2 = (2N)^{-2\alpha}/(1 + \|F_\alpha\|^2_X). \)

Below we summarize the properties of the operator \( \delta. \)
Theorem 2.8. If $f$ is an element of $\mathcal{RL}_2(\mathbb{F}; X \otimes U)$ so that $f = \sum_{k \geq 1} f_k \otimes u_k$, with $f_k = \sum_{\alpha \in J} f_{k,\alpha} \xi_{\alpha} \in \mathcal{RL}_2(\mathbb{F}; X)$, then

$$\delta(f) = \sum_{k \geq 1} f_k \diamond \xi_k,$$  \hspace{1cm} (2.15)

and

$$(\delta(f))_{\alpha} = \sum_{k \geq 1} \sqrt{\alpha_k} f_{k,\alpha - \xi_k}.$$  \hspace{1cm} (2.16)

Proof. By linearity and (2.14),

$$\delta(f) = \sum_{k \geq 1} \sum_{\alpha \in J} \delta(\xi_{\alpha} f_{k,\alpha} \otimes u_k) = \sum_{k \geq 1} \sum_{\alpha \in J} f_{k,\alpha} \xi_{\alpha} \otimes \xi_k = \sum_{k \geq 1} f_k \otimes \xi_k,$$

which is (2.15). On the other hand, by (2.9),

$$\delta(f) = \sum_{k \geq 1} \sum_{\alpha \in J} f_{k,\alpha} \sqrt{\alpha_k + 1} \xi_{\alpha + \xi_k} = \sum_{k \geq 1} \sum_{\alpha \in J} f_{k,\alpha - \xi_k} \sqrt{\alpha_k} \xi_{\alpha},$$

and (2.16) follows. \qed

Remark 2.9. Together with (2.14), the arguments in [3, Section 2.5] show that the operator $\delta$ can be considered an extension of the Skorokhod integral to the weighted spaces $\mathcal{RL}_2(\mathbb{F}; X \otimes U)$.

One way to describe a multi-index $\alpha$ with $|\alpha| = n > 0$ is by its characteristic set $K_\alpha$, that is, an ordered $n$-tuple $K_\alpha = \{k_1, \ldots, k_n\}$, where $k_1 \leq k_2 \leq \ldots \leq k_n$ characterize the locations and the values of the non-zero elements of $\alpha$. More precisely, $k_1$ is the index of the first non-zero element of $\alpha$, followed by $\max(0, \alpha_{k_1} - 1)$ of entries with the same value. The next entry after that is the index of the second non-zero element of $\alpha$, followed by $\max(0, \alpha_{k_2} - 1)$ of entries with the same value, and so on. For example, if $n = 7$ and $\alpha = (1, 0, 2, 0, 0, 1, 0, 3, 0, \ldots)$, then the non-zero elements of $\alpha$ are $\alpha_1 = 1$, $\alpha_3 = 2$, $\alpha_6 = 1$, $\alpha_8 = 3$. As a result, $K_\alpha = \{1, 3, 3, 6, 8, 8, 8\}$, that is, $k_1 = 1$, $k_2 = k_3 = 3$, $k_4 = 6$, $k_5 = k_6 = k_7 = 8$.

Using the notion of the characteristic set, we now state the following analog of the well-known result of Itô [7] connecting multiple Wiener integrals and Hermite polynomials.

Proposition 2.10. Let $\alpha \in J$ be a multi-index with $|\alpha| = n \geq 1$ and characteristic set $K_\alpha = \{k_1, \ldots, k_n\}$. Then

$$\xi_{\alpha} = \frac{\xi_{k_1} \diamond \xi_{k_2} \diamond \cdots \diamond \xi_{k_n}}{\sqrt{\alpha!}}.$$  \hspace{1cm} (2.17)

Proof. This follows from (2.2) and (2.12), because by (2.12), for every $i$ and $k$,

$$H_k(\xi_i) = \underbrace{\xi_i \diamond \cdots \diamond \xi_i}_{k \text{ times}}.$$  \hspace{1cm} \qed
3. Evolution Equations Driven by White Noise

3.1. The setting. In this section we study anticipating solutions of stochastic evolution equations driven by Gaussian white noise on a Hilbert space \( U \).

**Definition 3.1.** The triple \((V, H, V')\) of Hilbert spaces is called **normal** if and only if

1. \( V \hookrightarrow H \hookrightarrow V' \) and both embeddings \( V \hookrightarrow H \) and \( H \hookrightarrow V' \) are dense and continuous;
2. The space \( V' \) is the dual of \( V \) relative to the inner product in \( H \);
3. There exists a constant \( C > 0 \) so that \(|\langle h, v \rangle_H| \leq C\|v\|_V\|h\|_V\) for all \( v \in V \) and \( h \in H \).

For example, the Sobolev spaces \((H^{\ell+\gamma}(\mathbb{R}^d), H^{\ell}(\mathbb{R}^d), H^{\ell-\gamma}(\mathbb{R}^d))\), \( \gamma > 0, \ell \in \mathbb{R} \), form a normal triple.

Denote by \( \langle v', v \rangle \), \( v' \in V', v \in V \), the duality between \( V \) and \( V' \) relative to the inner product in \( H \). The properties of the normal triple imply that \(|\langle v', v \rangle| \leq C\|v\|_V\|v'\|_{V'}\), and, if \( v' \in H \) and \( v \in V \), then \( \langle v', v \rangle = (v', v)_H \).

We will also use the following notation:

\[
V = L_2((0, T); X), \quad H = L_2((0, T); H), \quad V' = L_2((0, T); V').
\]  

(3.1)

Given a normal triple \((V, H, V')\), let \( A : V \rightarrow V' \) and \( M : V \rightarrow V' \otimes U \) be bounded linear operators.

**Definition 3.2.** The solution of the stochastic evolution equation

\[
\dot{u} = Au + f + \delta(Mu), \quad 0 < t \leq T,
\]  

(3.2)

with \( f \in \mathcal{R}L_2(\mathbb{F}; V) \) and \( u|_{t=0} = u_0 \in \mathcal{R}L_2(\mathbb{F}; H) \), is a process \( u \in \mathcal{R}L_2(\mathbb{F}; V) \) so that, for every \( \varphi \) satisfying \( \varphi \in \mathcal{R}^{-1}L_2(\mathbb{F}) \) and \( D\varphi \in \mathcal{R}^{-1}L_2(\mathbb{F}; U) \), the equality

\[
\langle u(t), \varphi \rangle = \langle u_0, \varphi \rangle + \int_0^t \langle Au(s) + f(s) + \delta(Mu)(s), \varphi \rangle ds
\]  

(3.3)

holds in \( V' \); see (2.4) for the definition of \( \langle \cdot, \cdot \rangle \).

**Remark 3.3.**

(a) The solutions described by Definitions 3.2 and 4.1 belong to the class of “variational solutions”, which is quite typical for partial differential equations (see \[10, 11, 12, 20\], etc.)

(b) Since \( \langle u(t), \varphi \rangle \in V \) and \( \langle u(t), \varphi \rangle_t \in V' \), by the standard embedding theorem (see e.g. \[12\] Section 1.2.2) there exists a version of \( \langle u(t), \varphi \rangle \in C([0, T]; H) \). Clearly, one could also select a version of \( u(t) \) such that \( \langle u(t), \varphi \rangle \in C([0, T]; H) \). In the future, we will consider only this version of the solution. By doing this we ensure that formula (3.3) which is understood as an equality in \( V' \) yields \( u|_{t=0} = u_0 \in \mathcal{R}L_2(\mathbb{F}; H) \).

**Remark 3.4.** To simplify the notations and the overall presentation, we assume that \( A \) and \( M \) do not depend on time, even though many of the results in this paper can easily be extended to time-dependent operators.
Fix an orthonormal basis \( \mathcal{U} \) in \( \mathcal{U} \). Then, for every \( v \in V \), there exists a collection \( v_k \in V', k \geq 1 \), so that
\[
Mv = \sum_{k \geq 1} v_k \otimes u_k.
\]
We therefore define the operators \( M_k : V \to V' \) by setting \( M_kv = v_k \) and write
\[
Mv = \sum_{k \geq 1} (M_kv) \otimes u_k.
\]
By (2.15), equation (3.2) becomes
\[
\dot{u}(t) = Au(t) + f(t) + M_{u}(t) \diamond \dot{W},
\]
where
\[
M_{u} \diamond \dot{W} := \sum_{k \geq 1} (M_kv) \diamond \xi_k.
\]

3.2. Equivalence Theorem. In this section we investigate stochastic Fourier representation of equation (3.4).

Recall that every process \( u = u(t) \) from \( \mathcal{R}L_2(\mathbb{F}; \mathcal{V}) \) is represented by a formal series
\[
u(t) = \sum_{\alpha \in \mathcal{J}} u_{\alpha}(t) \xi_{\alpha},
\]
with \( u_{\alpha} \in \mathcal{V} \) and
\[
\sum_{\alpha} r_{\alpha}^2 \|u_{\alpha}\|^2_{\mathcal{V}} < \infty.
\]

Theorem 3.5. Let \( u = \sum_{\alpha \in \mathcal{J}} u_{\alpha} \xi_{\alpha} \) be an element of \( \mathcal{R}L_2(\mathbb{F}; \mathcal{V}) \). The process \( u \) is a solution of equation (3.2) if and only if the functions \( u_{\alpha} \) have the following properties:

1. every \( u_{\alpha} \) is an element of \( C([0, T]; H) \)
2. the system of equalities
\[
u_{\alpha}(t) = u_{0,\alpha} + \int_0^t \left( Au_{\alpha}(s) + f_{\alpha}(s) + \sum_{k \geq 1} \sqrt{\alpha_k} M_k u_{\alpha-\xi_k}(s) \right) ds
\]
holds in \( V' \) for all \( t \in [0, T] \) and \( \alpha \in \mathcal{J} \).

Proof. Let \( u \) be a solution of (3.2) in \( \mathcal{R}L_2(\mathbb{F}; \mathcal{V}) \). Taking \( \varphi = \xi_{\alpha} \) in (3.3) and using relation (2.16), we obtain equation (3.7). By Remark 3.3 \( u_{\alpha} \in \mathcal{V} \cap C([0, T]; H) \).

Conversely, let \( \{u_{\alpha, \alpha} \in \mathcal{J} \) be a collection of functions from \( \mathcal{V} \cap C([0, T]; H) \) satisfying (3.6) and (3.7). Set \( u(t) := \sum_{\alpha \in \mathcal{J}} u_{\alpha}(t) \xi_{\alpha} \). Then, by Theorem 2.8 equation (3.7) yields that, for every \( \alpha \in \mathcal{J} \),
\[
\langle \langle u(t), \xi_{\alpha} \rangle \rangle = \langle \langle u_0, \xi_{\alpha} \rangle \rangle + \int_0^t \langle \langle Au(s) + f(s) + \delta(Mu)(s), \xi_{\alpha} \rangle \rangle ds.
\]
By continuity, we conclude that for any \( \varphi \in \mathcal{R}^{-1}L_2(\mathbb{F}) \) such that \( D\varphi \in \mathcal{R}^{-1}L_2(\mathbb{F}; \mathcal{U}) \), equality
\[
\langle \langle u(t), \varphi \rangle \rangle = \langle \langle u_0, \varphi \rangle \rangle + \int_0^t \langle \langle Au(s) + f(s) + \delta(Mu)(s), \varphi \rangle \rangle ds
\]
holds in \( V' \). By Remark 3.3 \( \langle \langle u(t), \varphi \rangle \rangle \in C([0, T]; H) \).
This simple but very helpful result establishes the equivalence of the “physical” (3.4) and the (stochastic) Fourier (3.7) forms of equation (3.2). System of equations (3.7) is often referred in the literature as the propagator of equation (3.4). Note that the propagator is lower-triangular and can be solved by induction on $|\alpha|$.

3.3. Existence and uniqueness. Below, we will present several results on existence and uniqueness of evolution equations driven by Gaussian white noise.

Before proceeding with general existence-uniqueness problems, we will introduce two simple examples that indicate the limits of the “quality” of solutions of bi-linear SPDEs driven by general Gaussian white noise.

Example 3.6. Consider equation

$$u(t) = \phi + \int_0^t (b u(s) \circ \xi - \lambda u(s)) ds,$$

(3.8)

where $\phi, \lambda$ are real numbers, $b$ is a complex number, and $\xi$ is a standard Gaussian random variable. In other words $\xi$ is Gaussian white noise on $\mathcal{U} = \mathbb{R}$. With only one Gaussian random variable $\xi$, the set $\mathcal{J}$ becomes $\{0, 1, 2, \ldots\}$ so that $u(t) = \sum_{n \geq 0} u(n)(t) H_n(\xi) / \sqrt{n!}$, where $H_n$ is Hermite polynomial of order $n$ (2.3). According to (3.7),

$$u(n)(t) = \phi I_{(n=0)} - \int_0^t \lambda u(n)(s) ds + \int_0^t b \sqrt{n} u(n-1)(s) ds.$$

It follows that $u(0)(t) = \phi e^{-\lambda t}$ and then, by induction, $u(n)(t) = \phi \frac{(bt)^n}{n!} e^{-\lambda t}$. As a result,

$$u(t) = e^{-\lambda t} \left( \phi + \sum_{n \geq 1} \frac{(bt)^n}{n!} H_n(\xi) \right) = \phi e^{-\lambda t + (bt(\xi - |b|t^2/2)}.$$

Obviously, the solution of the equation is a square integrable on any fixed time interval. However, as the next example indicates, the solutions of SPDEs driven by stationary noise are much more intricate then the non-anticipating, or adapted, solutions of SPDEs driven by general Gaussian white noise.

Example 3.7. With $\xi$ as in the previous examples, consider a partial differential equation

$$u_t(t, x) = a u_{xx}(t, x) + (\beta u(t, x) + \sigma u_x(t, x)) \circ \xi, \ t > 0, \ x \in \mathbb{R},$$

(3.9)

with some initial condition $u_0 \in L_2(\mathbb{R})$. By taking the Fourier transform and using the results of Example 3.6 with $\phi = \hat{u}_0(y), \lambda = -ay^2, \ b = \beta + \sqrt{-1}y\sigma$, we find

$$\hat{u}_t(t, y) = -y^2 a \hat{u}_y(t) + (\beta + \sqrt{-1}y\sigma) \hat{u}_y(t, y) \circ \xi;$$

$$\hat{u}_t(t, y) = \hat{u}_0(y) \exp \left( -tay^2 + (\sigma^2 y^2 - \beta^2) t^2/2 + \sqrt{-1}\beta y t^2 + (\sqrt{-1}\sigma y + \beta) t \xi \right).$$

If $\sigma = 0$, i.e. the “diffusion” operator in equation (3.9) is of order zero, then the solution belongs to $L_2(\mathbb{F}; L_2(\mathbb{R}))$ for all $t$. However, if $\sigma > 0$, then the solution $u(t, \cdot)$
will, in general, belong to \( L_2(\mathbb{F}; L_2(\mathbb{R})) \) only for \( t \leq 2a/\sigma^2 \). This blow-up in finite time is in sharp contrast with the solution of the equation

\[
u_t = a\nu_{xx} + \sigma \nu_x \circ \dot{w},
\]

(3.10)
driven by the standard one-dimensional white noise \( \dot{w}(t) = \partial_t W(t) \), where \( W(t) \) is the one-dimensional Brownian motion; a more familiar way of writing (3.10) is in the Itô form

\[
du = a\nu_{xx}dt + \sigma \nu_x dW(t).
\]

(3.11)

It is well known (see, for example, [20]) that the solution of (3.11) belongs to \( L_2(\mathbb{F}; L_2(\mathbb{R})) \) for every \( t > 0 \) as long as \( u_0 \in L_2(\mathbb{R}) \) and

\[
a - \sigma^2/2 \geq 0.
\]

(3.12)

The existence of a square integrable (global) solution of an Itô’s SPDE with square integrable initial condition hinges on the parabolic condition which in the case of equation (3.10) is given by (3.12). Example 3.7 shows that this condition is not in any way sufficient for SPDEs involving a Skorokhod-type integral. The next theorem provides sufficient conditions for the existence and uniqueness of a solution to equation (3.4) in the space \( \mathcal{R}L_2(\mathbb{F}; \mathcal{V}) \), which appears to be a reasonable extension of the class of square integrable solutions.

Firstly, we introduce an additional assumption on the operator \( A \) that will be used throughout this section:

(A): For every \( U_0 \in H \) and \( F \in \mathcal{V}' := L_2((0, T); \mathcal{V}') \), there exists a function \( U \in \mathcal{V} \) that solve the deterministic equation

\[
\partial_t U(t) = AU(t) + F(t), \quad U(0) = U_0,
\]

(3.13)

and there exists a constant \( C = C(A, T) \) so that

\[
\|U\|_{\mathcal{V}} \leq C(A, T)(\|U_0\|_H + \|F\|_{\mathcal{V}}).
\]

(3.14)

Remark 3.8. Assumption (A) implies that a solution of equation (3.13) is unique and belongs to \( C \((0, T); H\) \) (cf. Remark 3.3). The assumption also implies that the operator \( A \) generates a semi-group \( \Phi = \Phi_t \), \( t \geq 0 \), and, for every \( v \in \mathcal{V} \),

\[
\int_0^T \left\| \int_0^t \Phi_{t-s}M_k v(s) \, ds \right\|_{\mathcal{V}}^2 \, dt \leq C^2_k \|v\|_{\mathcal{V}}^2,
\]

(3.15)

with numbers \( C_k \) independent of \( v \).

Remark 3.9. There are various types of assumptions on the operator \( A \) that yield the statement of the assumption (A). In particular, (A) holds if the operator \( A \) is coercive in \((\mathcal{V}, H, \mathcal{V}')\):

\[
\langle Av, v \rangle + \gamma \|v\|_{\mathcal{V}}^2 \leq C\|v\|_H^2
\]

for every \( v \in \mathcal{V} \), where \( \gamma > 0 \) and \( C \in \mathbb{R} \) are both independent of \( v \).

Theorem 3.10. Assume (A). Consider equation (3.4) in which \( u_0 \in \mathcal{R}L_2(\mathbb{F}; H) \), \( f \in \mathcal{R}L_2(\mathbb{F}; \mathcal{V}') \) for some operator \( \mathcal{R} \), and each \( M_k \) is a bounded linear operator from \( \mathcal{V} \) to \( \mathcal{V}' \).

Then there exist an operator \( \mathcal{R} \) and a unique solution \( u \in \mathcal{R}L_2(\mathbb{F}; \mathcal{V}) \) of (3.4).
Proof. By Theorem 3.5, it suffices to prove that the propagator \( h \) has a unique solution \( u_\alpha(t) \) such that for each \( \alpha, u_\alpha \in \mathcal{V} \cap C([0,T]; H) \) and \( u := \sum_{\alpha \in \mathcal{J}} u_\alpha \xi_\alpha \in \mathcal{R}L_2(\mathbb{F}; \mathcal{V}) \).

For \( \alpha = (0) \), that is, when \(|\alpha| = 0\), equation \( (3.7) \) reduces to
\[
    u_0(t) = u_{0,0} + \int_0^t (Au_0 + f_0)(s)ds.
\]

By (A), this equation has a unique solution and
\[
    \|u_0\|_V \leq C(A, T) \left( \|u_{0,0}\|_H + \|f_0\|_V \right).
\]

Using assumption (A), it follows by induction on \(|\alpha|\) that, for every \( \alpha \in \mathcal{J} \), equation
\[
    \partial_t u_\alpha(t) = Au_\alpha(t) + f_\alpha(t) + \sum_{k \geq 1} \sqrt{\alpha_k} M_k u_{\alpha - \varepsilon_k}(t), \quad u_\alpha(0) = u_{0,\alpha}
\]
has a unique solution in \( \mathcal{V} \cap C([0,T]; H) \). Moreover, by \( (3.14) \),
\[
    \|u_\alpha\|_V \leq C(A, M, T) \left( \|u_{0,\alpha}\|_H + \|f_\alpha\|_V + \sum_{k \geq 1} \sqrt{\alpha_k} \|u_{\alpha - \varepsilon_k}\|_V \right).
\]

Since only finitely many of \( \alpha_k \) are different from 0, we conclude that \( \|u_\alpha\|_V < \infty \) for all \( \alpha \in \mathcal{J} \).

Define the operator \( \mathcal{R} \) on \( L_2(\mathbb{F}) \) using the weights
\[
    r_\alpha = \min \left( r_\alpha, \frac{(2N)^{-\kappa\alpha}}{1 + \|u_\alpha\|_V} \right),
\]
where \( \kappa > 1/2 \) (cf. (2.5)). Then \( u(t) := \sum_{\alpha \in \mathcal{J}} u_\alpha(t) \xi_\alpha \) is a solution of \( (3.4) \) and, by \( (2.13) \), belongs to \( \mathcal{R}L_2(\mathbb{F}; \mathcal{V}) \). □

While Theorem 3.10 establishes that under very broad assumptions one can find an operator \( \mathcal{R} \) such that equation \( (3.4) \) has a unique solution in \( \mathcal{R}L_2(\mathbb{F}; \mathcal{V}) \), the choice of the operator \( \mathcal{R} \) is not sufficiently explicit (because of the presence of \( \|u_\alpha\|_V \)) and is not necessarily optimal.

Consider equation \( (3.4) \) with non-random \( f \) and \( u_0 \). In this situation, it is possible to find more constructive expression for \( r_\alpha \) and to derive explicit formulas, both for \( \mathcal{R}u \) and for each individual \( u_\alpha \).

Theorem 3.11. If \( u_0 \) and \( f \) are non-random, then the following holds:

1. the coefficient \( u_\alpha \), corresponding to the multi-index \( \alpha \) with \(|\alpha| = n \geq 1 \) and characteristic set \( K_\alpha = \{k_{1}, \ldots, k_{n}\} \), is given by
\[
    u_\alpha(t) = \frac{1}{\sqrt{|\alpha|!}} \sum_{\sigma \in \mathcal{P}_n} \int_0^t \int_{0}^{s_1} \cdots \int_{0}^{s_n} \Phi_{t - s_n} M_{k_{\sigma(n)}} \cdots \Phi_{s_2 - s_1} M_{k_{\sigma(1)}} u_0(s_1) ds_1 \cdots ds_n,
\]
where
- \( \mathcal{P}_n \) is the permutation group of the set \( \{1, \ldots, n\} \);
- \( \Phi_t \) is the semigroup generated by \( A \);
(2) the weights \( r_\alpha \) can be taken in the form

\[
r_\alpha = \frac{q^\alpha}{2^{|\alpha|}\sqrt{|\alpha|!}}, \quad \text{where } q^\alpha = \prod_{k=1}^{\infty} q_k^{a_k},
\]

(3.18)

and the numbers \( q_k, k \geq 1 \), are chosen so that \( \sum_{k \geq 1} q_k^2 k^2 C_k^2 < 1 \), with \( C_k \) from \( (3.17) \).

(3) With \( q_k \) and \( r_\alpha \) from \( (3.18) \), we have

\[
\sum_{|\alpha| = n} q^\alpha u_\alpha(t) \xi_\alpha = \int_0^t \int_{0}^{s_n} \cdots \int_{0}^{s_2} \Phi_{t-s_n} \delta(M\Phi_{s_{n-1}} \cdots \delta(Mu_{(0)}) \cdots) ds_1 \cdots ds_{n-1} ds_n,
\]

where \( M = (q_1 M_1, q_2 M_2, \ldots) \), and

\[
\mathcal{R} u(t) = u_{(0)}(t)
\]

\[
+ \sum_{n=1}^{\infty} \frac{1}{2^n n!} \int_0^t \int_{0}^{s_n} \cdots \int_{0}^{s_2} \Phi_{t-s_n} \delta(M\Phi_{s_{n-1}} \cdots \delta(Mu_{0}(s_1))) \cdots) ds_1 \cdots ds_{n-1} ds_n.
\]

(3.20)

**Proof.** If \( u_0 \) and \( f \) are deterministic, then equation \( (3.7) \) becomes

\[
u_{(0)}(t) = u_0 + \int_0^t A u_{(0)}(s) ds + \int_0^t f(s) ds, \quad |\alpha| = 0;
\]

(3.21)

\[
u_\alpha(t) = \int_0^t A u_\alpha(s) ds + \sum_{k \geq 1} \sqrt{\alpha_k} \int_0^t M_k u_{\alpha-\epsilon_k}(s) ds, \quad |\alpha| > 0.
\]

(3.22)

Define \( \tilde{u}_\alpha = \sqrt{\alpha!} u_\alpha \). Then \( \tilde{u}_{(0)} = u_{(0)} \) and, for \( |\alpha| > 0 \), \( (3.22) \) implies

\[
\tilde{u}_\alpha(t) = \int_0^t A \tilde{u}_\alpha(s) ds + \sum_{k \geq 1} \alpha_k M_k \tilde{u}_{\alpha-\epsilon_k}(s) ds
\]

or

\[
\tilde{u}_\alpha(t) = \sum_{k \geq 1} \alpha_k \int_0^t \Phi_{t-s} M_k \tilde{u}_{\alpha-\epsilon_k}(s) ds = \sum_{k \in K_\alpha} \int_0^t \Phi_{t-s} M_k \tilde{u}_{\alpha-\epsilon_k}(s) ds.
\]

By induction on \( n \),

\[
\tilde{u}_\alpha(t) = \sum_{\sigma \in \mathcal{P}_\alpha} \int_0^t \cdots \int_{0}^{s_n} \cdots \int_{0}^{s_2} \Phi_{t-s_n} M_{k_{\sigma(n)}} \cdots \Phi_{s_2-s_1} M_{k_{\sigma(1)}} u_{(0)} ds_1 \cdots ds_n,
\]

and \( (3.17) \) follows.

Since \( (3.20) \) follows directly from \( (3.19) \), it remains to establish \( (3.19) \). To this end, define

\[
U_n(t) = \sum_{|\alpha| = n} q^\alpha u_\alpha(t) \xi_\alpha, \quad n \geq 0.
\]
Let us first show that, for each \( n \geq 1 \), \( U_n \in L_2(\mathbb{F}; V) \). Indeed, for \( \alpha = (0) \), \( u_\alpha(0) = u_0 \), \( f_\alpha = f \) and
\[
 u(0)(t) = \Phi_t u_0 + \int_0^t \Phi_{t-s} f(s) ds.
\]
By (3.14), we have
\[
\| u(0) \|_V \leq C(\mathbf{A}, T) (\| u_0 \|_H + \| f \|_{V'}) \| u_0 \|_H + \| f \|_{V'} \sqrt{n}. \tag{3.23}
\]
When \( |\alpha| \geq 1 \), \( f_\alpha = 0 \) and the solution of (3.22) is given by
\[
 u_\alpha(t) = \sum_{k \geq 1} \sqrt{\alpha_k} \int_0^t \Phi_{t-s} M_k u_{\alpha-\varepsilon_k}(s) ds. \tag{3.24}
\]
By (3.17), together with (3.14), (3.23), and (3.15), we have
\[
\| u_\alpha \|_V^2 \leq C^2(\mathbf{A}, T) \frac{|\alpha|!^2}{\alpha!} (\| u_0 \|_H^2 + \| f \|_{V'}^2) \prod_{k \geq 1} C_{2\alpha_k}. \tag{3.25}
\]
It is known (see, for example, [4, page 45]) that
\[
|\alpha|! \leq \alpha! (2N)^{2\alpha}; \tag{3.26}
\]
recall that
\[
(2N)^{2\alpha} = 2^{2|\alpha|} \prod_{k \geq 1} k^{2\alpha_k}.
\]
Then
\[
\sum_{|\alpha|=n} q^{2\alpha} \| u_\alpha \|_V^2 \leq C^2(\mathbf{A}, T) \frac{|\alpha|!^2}{\alpha!} (\| u_0 \|_H^2 + \| f \|_{V'}^2) 2^{2n} n! \prod_{|\alpha|=n} \prod_{k \geq 1} (kC_k q_k)^{2\alpha_k}
\]
\[
= C_A 2^{2n} n! \left( \sum_{k \geq 1} k^2 C_k^2 q_k^2 \right)^n < \infty,
\]
because of the selection of \( q_k \), and so \( U_n \in L_2(\mathbb{F}; V) \). Moreover, if the weights \( r_\alpha \) are defined by (3.18), then
\[
\sum_{\alpha \in \mathcal{F}} r_\alpha^2 \| u_\alpha \|_V^2 = \sum_{n \geq 0} \sum_{|\alpha|=n} r_\alpha^2 \| u_\alpha \|_V^2 \leq C^2_A \sum_{n \geq 1} \left( \sum_{k \geq 1} k^2 C_k^2 q_k^2 \right)^n < \infty
\]
because of the assumption \( \sum_{k \geq 1} k^2 C_k^2 q_k^2 < 1 \).
Next, the definition of \( U_n(t) \) and (3.24) imply that (3.19) is equivalent to
\[
U_n(t) = \int_0^t \Phi_{t-s} \delta(M U_{n-1}(s)) ds, \ n \geq 1. \tag{3.27}
\]
Accordingly, we will prove (3.27). For \( n = 1 \), we have
\[
U_1(s) = \sum_{k \geq 1} q_k u_{\varepsilon_k}(t) \xi_k = \sum_{k \geq 1} \int_0^t q_k \Phi_{t-s} M_k u(0) \xi_k dt = \int_0^t \Phi_{t-s} \delta(M U_0(s)) ds,
\]
where the last equality follows from (2.14). More generally, for \( n > 1 \) we have by definition of \( U_n \) that
\[
(U_n)_\alpha(t) = \begin{cases} 
q^\alpha u_\alpha(t), & \text{if } |\alpha| = n, \\
0, & \text{otherwise}.
\end{cases}
\]
From the equation
\[
q^\alpha u_\alpha(t) = \int_0^t A q^\alpha u_\alpha(s) ds + \sum_{k \geq 1} \int_0^t q_k \sqrt{\alpha_k} M_k q^{\alpha - \varepsilon_k} u_{\alpha - \varepsilon_k}(s) ds
\]
we find
\[
(U_n(t))_\alpha = \begin{cases}
\sum_{k \geq 1} \sqrt{\alpha_k} q_k \int_0^t \Phi_{t-s} M_k q^{\alpha - \varepsilon_k} u_{\alpha - \varepsilon_k}(s) ds, & \text{if } |\alpha| = n, \\
0, & \text{otherwise},
\end{cases}
\]
and then (3.27) follows from (2.16). Theorem 4.5 is proved.

Formula (3.19) is similar to the multiple Wiener integral representation of the solution of a stochastic parabolic equation driven by the Wiener process; see [14, Theorem 3.8].

**Example 3.12.** Consider the equation
\[
u(t, x) = u_0(x) + \int_0^t u_{xx}(s, x) ds + \sum_{k \geq 1} \int_0^t \sigma_k u_{xx}(s, x) \phi \xi_k ds.
\]

With no loss of generality assume that \( \sigma_k \neq 0 \) for all \( k \). Standard properties of the heat kernel imply assumption (A) and inequality (3.15) with \( C_k = \sigma_k^2 \). Then the conclusions of Theorem 3.11 hold, and we can take \( q_k^2 = k^{-2} 4^{-k} (1 + \sigma_k^2)^{-k} \). Note that Theorem 3.11 covers equation (3.28) with no restrictions on the numbers \( \sigma_k \).

In the existing literature on the subject, equations of the type (3.4) are considered only under the assumption

\((H)\): each \( M_k \) is a bounded linear operators from \( V \) to \( H \).

Obviously this assumption rules out equation (3.28) but still covers equation (3.9).

Of course, Theorem 3.11 does not rule out a possibility of a better-behaving solution under additional assumptions on the operators \( M_k \). Indeed, it was shown in [13] that if (H) is assumed and the space-only Gaussian noise in equation (3.4) is replaced by the space-time white noise, then a more delicate analysis of equation (3.4) is possible. In particular, the solution can belong to a much smaller Wiener chaos space even if \( u_0 \) and \( f \) are not deterministic.

If the operators \( M_k \) are bounded in \( H \) (see e.g. equation (3.9) with \( \sigma = 0 \)), then, as the following theorem shows, the solutions can be square integrable (cf. [10]).
**Theorem 3.13.** Assume that the operator $A$ satisfies
\[
\langle Av, v \rangle + \kappa \|v\|_V^2 \leq C_A \|v\|_H^2
\]
for every $v \in V$, with $\kappa > 0$, $C_A \in \mathbb{R}$ independent of $v$, and assume that each $M_k$ is a bounded operator on $H$ so that $\|M_k\|_{H^{-\infty}} \leq c_k$ and
\[
C_M := \sum_{k \geq 1} c_k^2 < \infty.
\]
If $f \in V'$ and $u_0 \in H$ are non-random, then there exists a unique solution $u$ of (3.4) so that $u(t) \in L_2(\mathbb{F}; H)$ for every $t$ and
\[
\mathbb{E}\|u(t)\|_H^2 \leq C(C_A, C_M, \kappa, t) \left( \int_0^t \mathbb{E}\|f(s)\|_V^2 \, ds + \|u_0\|_H^2 \right).
\]

**Proof.** Existence and uniqueness of the solution follow from Theorem 3.10 and Remark 3.9, and it remains to establish (3.31).

It follows from (3.7) that
\[
u_\alpha = \frac{1}{\sqrt{\alpha!}} \sum_{k \in \mathbb{N}_0} \int_0^t \Phi_{t-s} M_k u_{\alpha-k}(s) \, ds,
\]
where $\Phi$ is the semi-group generated by $A$ and $K_\alpha$ is the characteristic set of $\alpha$. Assumption (3.29) implies that $\|\Phi_t\|_{H^{-\infty}} \leq e^{pt}$ for some $p \in \mathbb{R}$. A straightforward calculation using relation (3.32) and induction on $|\alpha|$ shows that
\[
\|u_\alpha(t)\|_H \leq e^{pt} \frac{t|\alpha|c^\alpha}{\sqrt{\alpha!}} \|u_0\|_H,
\]
where $c^\alpha = \prod_k \epsilon_k^\alpha$ and $u_0(t) = \Phi_t u_0 + \int_0^t \Phi_{t-s} f(s) \, ds$. Assumption (3.29) implies that $\|u_0\|_H^2 \leq C(C_A, \kappa, t) \left( \int_0^t \mathbb{E}\|f(s)\|_V^2 \, ds + \|u_0\|_H^2 \right)$. To establish (3.31), it remains to observe that
\[
\sum_{\alpha \in \mathcal{J}} \frac{c^{2\alpha} t^{2|\alpha|}}{\alpha!} = e^{C_M t^2}.
\]

Theorem 3.13 is proved. \hfill \Box

**Remark 3.14.** Taking $M_k u = c_k u$ shows that, in general, bound (3.33) cannot be improved. When condition (3.30) does not hold, a bound similar to (3.31) can be established in a weighted space $RL_2(\mathbb{F}; H)$, for example with $r_\alpha = q^\alpha$, where $q_k = 1/(2^k (1 + c_k))$. For special operators $M_k$, a more delicate analysis might be possible; see, for example, [6].

If $f$ and $u_0$ are not deterministic, then the solution of (3.4) might not satisfy
\[
\mathbb{E}\|u(t)\|_H^2 \leq C(C_A, C_M, \kappa, t) \left( \int_0^t \mathbb{E}\|f(s)\|_V^2 \, ds + \mathbb{E}\|u_0\|_H^2 \right)
\]
even if all other conditions of Theorem 3.13 are fulfilled. An example can be constructed similar to Example 9.7 in [13]: an interested reader can verify that the solution of the equation $u(t) = u_0 + \int_0^t u(s) \circ \xi \, ds$, where $\xi$ is a standard Gaussian random
variable and \( u_0 = \sum_{n \geq 0} a_n \frac{H_n(\xi)}{\sqrt{n!}} \), satisfies \( \mathbb{E} u^2(1) \geq \frac{1}{10} \sum_{n \geq 1} a_n^2 n^{\sqrt{n}} \). For equations with random input, one possibility is to use the spaces \((S)_{-1,q}\); see (2.5). Examples of the corresponding results are Theorems 4.6 and 5.1 below and Theorem 9.8 in [13].

### 4. Stationary equations

#### 4.1. Definitions and Analysis.

The objective of this section is to study stationary stochastic equation

\[ Au + \delta(Mu) = f. \tag{4.1} \]

**Definition 4.1.** The solution of equation (4.1) with \( f \in \mathcal{R}L_2(\mathbb{F}; V') \), is a random element \( u \in \mathcal{R}L_2(\mathbb{F}; V) \) so that, for every \( \varphi \) satisfying \( \varphi \in \mathcal{R}^{-1}L_2(\mathbb{F}) \) and \( D\varphi \in \mathcal{R}^{-1}L_2(\mathbb{F}; U) \), the equality

\[ \langle \langle Au, \varphi \rangle \rangle + \langle \langle \delta(Mu), \varphi \rangle \rangle = \langle \langle f, \varphi \rangle \rangle \tag{4.2} \]

holds in \( V' \).

As with evolution equations, we fix an orthonormal basis \( \mathcal{U} \) in \( U \) and use (2.15) to rewrite (4.1) as

\[ Au + (Mu) \diamond \dot{W} = f, \tag{4.3} \]

where

\[ Mu \diamond \dot{W} := \sum_{k \geq 1} M_k u \diamond \xi_k. \tag{4.4} \]

Taking \( \varphi = \xi_\alpha \) in (4.2) and using relation (2.16) we conclude, as in Theorem 3.5, that \( u = \sum_{\alpha \in J} u_\alpha \xi_\alpha \) is a solution of equation (4.1) if and only if \( u_\alpha \) satisfies

\[ Au_\alpha + \sum_{k \geq 1} \sqrt{\alpha_k} M_k u_{\alpha - \varepsilon_k} = f_\alpha \tag{4.5} \]

in the normal triple \((V, H, V')\). This system of equation is lower-triangular and can be solved by induction on \(|\alpha|\).

The following example illucidates the limitations on the "quality" of the solution of equation (4.1).

**Example 4.2.** Consider equation

\[ u = 1 + u \diamond \xi. \tag{4.6} \]

Similar to Example 3.6, we write \( u = \sum_{n \geq 0} u_n H_n(\xi)/\sqrt{n!} \), where \( H_n \) is Hermite polynomial of order \( n \) (2.3). Then (4.3) implies \( u_n = I_{n=0} + \sqrt{n} u_{n-1} \) or \( u_0 = 1 \), \( u_n = \sqrt{n!} \), \( n \geq 1 \), or \( u = 1 + \sum_{n \geq 1} H_n(\xi) \). Clearly, the series does not converge in \( L_2(\mathbb{F}) \), but does converge in \((S)_{-1,q}\) for every \( q < 0 \) (see (2.5)). As a result, even a simple stationary equation (4.6) can be solved only in weighted spaces.

**Theorem 4.3.** Consider equation (4.3) in which \( f \in \overline{\mathcal{R}}L_2(\mathbb{F}; V') \) for some \( \overline{\mathcal{R}} \).

Assume that the deterministic equation \( AU = F \) is uniquely solvable in the normal triple \((V, H, V')\), that is, for every \( F \in V' \), there exists a unique solution \( U = A^{-1}F \in \mathcal{R}L_2(\mathbb{F}; V) \).
V so that $\|U\|_V \leq C_A \|F\|_V$. Assume also that each $M_k$ is a bounded linear operator from $V$ to $V'$ so that, for all $v \in V$

$$\|A^{-1}M_kv\|_V \leq C_k \|v\|_V,$$

with $C_k$ independent of $v$.

Then there exists an operator $R$ and a unique solution $u \in RL_2(\mathbb{F}; V)$ of (3.4).

**Proof.** The argument is identical to the proof of Theorem 3.10. □

**Remark 4.4.** The assumption of the theorem about solvability of the deterministic equation holds if the operator $A$ satisfies $\langle Av, v \rangle \geq \kappa \|v\|^2_V$ for every $v \in V$, with $\kappa > 0$ independent of $v$.

An analog of Theorem 3.11 exists if $f$ is non-random. With no time variable, we introduce the following notation to write multiple integrals in the time-independent setting:

$$\delta_B(0)(\eta) = \eta, \delta_B^{(n)}(\eta) = \delta_B(\delta_B^{(n-1)}(\eta)), \eta \in RL_2(\mathbb{F}; V),$$

where $B$ is a bounded linear operator from $V$ to $V \otimes U$.

**Theorem 4.5.** Under the assumptions of Theorem 4.3, if $f$ is non-random, then the following holds:

1. the coefficient $u_\alpha$, corresponding to the multi-index $\alpha$ with $|\alpha| = n \geq 1$ and the characteristic set $K_\alpha = \{k_1, \ldots, k_n\}$, is given by

$$u_\alpha = \frac{1}{\sqrt{\alpha!}} \sum_{\sigma \in P_n} B_{k_\sigma(n)} \cdots B_{k_\sigma(1)} u(0),$$

where

- $P_n$ is the permutation group of the set $(1, \ldots, n)$;
- $B_k = -A^{-1}M_k$;
- $u(0) = A^{-1}f$.

2. the operator $R$ can be defined by the weights $r_\alpha$ in the form

$$r_\alpha = \frac{q_\alpha}{2|\alpha| \sqrt{|\alpha|!}}, \text{ where } q_\alpha = \prod_{k=1}^\infty q_k^{|\alpha|},$$

where the numbers $q_k$, $k \geq 1$ are chosen so that $\sum_{k=1}^\infty q_k^2k^2C_k^2 < 1$, and $C_k$ are defined in (4.7).

3. With $r_\alpha$ and $q_k$ defined by (4.9),

$$\sum_{|\alpha|=n} q_\alpha u_\alpha \xi_\alpha = c^{(n)}_\mathbb{B}(A^{-1}f),$$

where $\mathbb{B} = -(q_1A^{-1}M_1, q_2A^{-1}M_2, \ldots)$, and

$$Ru = A^{-1}f + \sum_{n \geq 1} \frac{1}{2^n \sqrt{n!}} \delta^{(n)}_\mathbb{B}(A^{-1}f),$$
Proof. While the proofs of Theorems 3.11 and 4.5 are similar, the complete absence of time makes equation (4.3) different from either (3.4) or anything considered in [14]. Accordingly, we present a complete proof.

Define \( \widetilde{u}_\alpha = \sqrt{\alpha!} u_\alpha \). If \( f \) is deterministic, then \( \tilde{u}(0) = A^{-1} f \) and, for \( |\alpha| \geq 1 \),

\[
A \tilde{u}_\alpha + \sum_{k \geq 1} \alpha_k M_k \tilde{u}_{\alpha - \varepsilon_k} = 0,
\]

or

\[
\tilde{u}_\alpha = \sum_{k \geq 1} \alpha_k B_k \tilde{u}_{\alpha - \varepsilon_k} = \sum_{k \in K_\alpha} B_k \tilde{u}_{\alpha - \varepsilon_k},
\]

where \( K_\alpha = \{k_1, \ldots, k_n\} \) is the characteristic set of \( \alpha \) and \( n = |\alpha| \). By induction on \( n \),

\[
\tilde{u}_\alpha = \sum_{\sigma \in \mathcal{P}_n} B_{k_{\sigma(n)}} \cdots B_{k_{\sigma(1)}} u_{(0)},
\]

and (4.8) follows.

Next, define

\[
U_n = \sum_{|\alpha| = n} q^\alpha u_\alpha \xi_\alpha, \quad n \geq 0.
\]

Let us first show that, for each \( n \geq 1 \), \( U_n \in L_2(\mathbb{F}; V) \). By (4.8) we have

\[
\|u_\alpha\|_V^2 \leq C_A^2 \frac{(|\alpha|!)^2}{\alpha!} \|f\|_V^2 \prod_{k \geq 1} C_k^2 q_k^2.
\]

(4.12)

By (3.26),

\[
\sum_{|\alpha| = n} q^{2\alpha} \|u_\alpha\|_V^2 \leq C_A^2 2^{2n} n! \prod_{k \geq 1} (kC_k q_k)^{2\alpha_k} = C_A^2 2^{2n} n! \left( \sum_{k \geq 1} k^2 C_k^2 q_k^2 \right)^n < \infty,
\]

because of the selection of \( q_k \), and so \( U_n \in L_2(\mathbb{F}; V) \). If the weights \( r_\alpha \) are defined by (4.9), then

\[
\sum_{\alpha \in \mathcal{J}} r_\alpha^2 \|u\|_V^2 = \sum_{n \geq 0} \sum_{|\alpha| = n} r_\alpha^2 \|u_\alpha\|_V^2 \leq C_A^2 \sum_{n \geq 0} \left( \sum_{k \geq 1} k^2 C_k^2 q_k^2 \right)^n < \infty,
\]

because of the assumption \( \sum_{k \geq 1} k^2 C_k^2 q_k^2 < 1 \).

Since (4.11) follows directly from (4.10), it remains to establish (4.10), that is,

\[
U_n = \delta_B(U_{n-1}), \quad n \geq 1.
\]

(4.13)

For \( n = 1 \) we have

\[
U_1 = \sum_{k \geq 1} q_k u_{\varepsilon_k} \xi_k = \sum_{k \geq 1} B_k u_{(0)} \xi_k = \delta_B(U_0),
\]

where the last equality follows from (2.15). More generally, for \( n > 1 \) we have by definition of \( U_n \) that

\[
(U_n)_\alpha = \begin{cases} q^\alpha u_\alpha, & \text{if } |\alpha| = n, \\ 0, & \text{otherwise}. \end{cases}
\]
From the equation
\[ q^\alpha A u_\alpha + \sum_{k \geq 1} q_k \sqrt{\alpha_k} M_k q^{\alpha - \varepsilon_k} u_{\alpha - \varepsilon_k} = 0 \]
we find
\[
(U_n)_\alpha = \begin{cases} 
\sum_{k \geq 1} \sqrt{\alpha_k} q_k B_k q^{\alpha - \varepsilon_k} u_{\alpha - \varepsilon_k}, & \text{if } |\alpha| = n, \\
0, & \text{otherwise.}
\end{cases}
\]
and then (4.13) follows from (2.16). Theorem 4.5 is proved. \hfill \Box

Here is another result about solvability of (4.3), this time with random \( f \). We use the space \((S)_{\rho, q}, \) defined by the weights (2.5).

**Theorem 4.6.** In addition to the assumptions of Theorem 4.3, let \( C_A \leq 1 \) and \( C_k \leq 1 \) for all \( k \). If \( f \in (S)_{-1, -\ell}(V') \) for some \( \ell > 1 \), then there exists a unique solution \( u \in (S)_{-1, -\ell-4}(V) \) of (4.3) and
\[
\|u\|(S)_{-1, -\ell-4}(V) \leq C(\ell)\|f\|(S)_{-1, -\ell}(V').
\]  
(4.14)

**Proof.** Denote by \( u(g; \gamma) \), \( g \in J \), \( g \in V' \), the solution of (4.3) with \( f_\alpha = gI_{(\alpha, \gamma)} \), and define \( \tilde{u}_\alpha = (\alpha!)^{-1/2} u_\alpha \). Clearly, \( u_\alpha(g; \gamma) = 0 \) if \( |\alpha| < |\gamma| \) and so
\[
\sum_{\alpha \in J} \|u_\alpha(f_\gamma; \gamma)\|_V^2 r_\alpha^2 = \sum_{\alpha \in J} \|u_{\alpha + \gamma}(f_\gamma; \gamma)\|_V^2 r_{\alpha + \gamma}^2.
\]  
(4.15)

It follows from (4.5) that
\[
\tilde{u}_{\alpha + \gamma}(f_\gamma; \gamma) = \bar{u}_\alpha(f_\gamma(\alpha!^{-1/2}; 0)).
\]  
(4.16)

Now we use (4.12) to conclude that
\[
\|\bar{u}_{\alpha + \gamma}(f_\gamma; \gamma)\|_V \leq \frac{|\alpha|!}{\sqrt{\alpha! \gamma!}} \|f\|_{V'}. 
\]  
(4.17)

Coming back to (4.15) with \( r_\alpha^2 = (\alpha!)^{-1}(2N)^{(-\ell-4)\alpha} \) and using inequality (3.26) we find:
\[
\|u(f_\gamma; \gamma)\|(S)_{-1, -\ell-4}(V) \leq C(\ell)(2N)^{-2\gamma} \frac{\|f_\gamma\|_{V'}}{\sqrt{\gamma!}}.
\]
where
\[
C(\ell) = \left( \sum_{\alpha \in J} \left( \frac{|\alpha|!}{\alpha!} \right)^2 (2N)^{(-\ell-4)\alpha} \right)^{1/2}.
\]

(2.13) and (3.26) imply \( C(\ell) < \infty \). Then (4.14) follows by the triangle inequality after summing over all \( \gamma \) and using the Cauchy-Schwartz inequality. \hfill \Box

**Remark 4.7.** Example 4.2 in which \( f \in (S)_{0, 0} \) and \( u \in (S)_{-1, q}, q < 0 \), shows that, while the results of Theorem 4.6 are not sharp, a bound of the type \( \|u\|(S)_{\rho, q}(V) \leq C\|f\|(S)_{\rho, q}(V') \) is, in general, impossible if \( \rho > -1 \) or \( q \geq \ell \).
4.2. Convergence to Stationary Solution. Let \((V, H, V')\) be a normal triple of Hilbert spaces. Consider equation
\[
\dot{u}(t) = (Au(t) + f(t)) + M_k u(t) \odot \xi_k, \tag{4.18}
\]
where the operators \(A\) and \(M_k\) do not depend on time, and assume that there exists an \(f^* \in \mathcal{R}L_2(\mathcal{F}; H)\) such that \(\lim_{t \to -\infty} \|f(t) - f^*\|_{\mathcal{R}L_2(\mathcal{F}; H)} = 0\). The objective of this section is to study convergence, as \(t \to +\infty\), of the solution of (4.18) to the solution \(u^*\) of the stationary equation
\[
-Au^* = f^* + M_k u^* \odot \xi_k. \tag{4.19}
\]

**Theorem 4.8.** Assume that

(C1) Each \(M_k\) is a bounded linear operator from \(H\) to \(H\), and \(A\) is a bounded linear operator from \(V\) to \(V'\) with the property
\[
\langle Av, v \rangle + \kappa \|v\|_V^2 \leq -c \|v\|_H^2 \tag{4.20}
\]
for every \(v \in V\), with \(\kappa > 0\) and \(c > 0\) both independent of \(v\).

(C2) \(f \in \mathcal{R}L_2(\mathcal{F}; H)\) and there exists an \(f^* \in \mathcal{R}L_2(\mathcal{F}; H)\) such that \(\lim_{t \to +\infty} \|f(t) - f^*\|_{\mathcal{R}L_2(\mathcal{F}; H)} = 0\).

Then, for every \(u_0 \in \mathcal{R}L_2(\mathcal{F}; H)\), there exists an operator \(\mathcal{R}\) so that

1. There exists a unique solution \(u \in \mathcal{R}L_2(\mathcal{F}; V)\) of (4.18),
2. There exists a unique solution \(u^* \in \mathcal{R}L_2(\mathcal{F}; V)\) of (4.19), and
3. The following convergence holds:
\[
\lim_{t \to +\infty} \|u(t) - u^*\|_{\mathcal{R}L_2(\mathcal{F}; H)} = 0. \tag{4.21}
\]

**Proof.**

1. Existence and uniqueness of the solution of (4.18) follow from Theorem 3.10 and Remark 3.9.

2. Existence and uniqueness of the solution of (4.19) follow from Theorem 4.3 and Remark 4.4.

3. The proof of (4.21) is based on the following result.

**Lemma 4.9.** Assume that the operator \(A\) satisfies (4.20) and \(F = F(t)\) is a deterministic function such that \(\lim_{t \to +\infty} \|F(t)\|_H = 0\). Then, for every \(U_0 \in H\), the solution \(U = U(t)\) of the equation \(U(t) = U_0 + \int_0^t A U(s) ds + \int_0^t F(s) ds\) satisfies \(\lim_{t \to +\infty} \|U(t)\|_H = 0\).

**Proof.** If \(\Phi = \Phi_t\) is the semi-group generated by the operator \(A\) (which exists because of (4.20)), then
\[
U(t) = \Phi_t U_0 + \int_0^t \Phi_{t-s} F(s) ds.
\]
Condition (4.20) implies \(\|\Phi_t U_0\|_H \leq e^{-ct}\|U_0\|_H\), and then
\[
\|U(t)\|_H \leq e^{-ct}\|U_0\|_H + \int_0^t e^{-c(t-s)}\|F(s)\|_H ds.
\]
The convergence of \( \|U(t)\|_H \) to zero now follows from the Toeplitz lemma (see Appendix). Lemma 4.9 is proved.

To complete the proof of Theorem 4.8, we define \( v_\alpha(t) = u_\alpha(t) - u_\alpha^* \) and note that

\[
\dot{v}_\alpha(t) = A v_\alpha(t) + (f_\alpha(t) - f_\alpha^*) + \sum_k \sqrt{\alpha_k} M_k v_{\alpha-\varepsilon_k}.
\]

By Theorem 4.3, \( u_\alpha^* \in V \) and so \( v_\alpha(0) \in H \) for every \( \alpha \in \mathcal{J} \). By Lemma 4.9, \( \lim_{t \to +\infty} \|v_\alpha(0)\|_H = 0 \). Using induction on \( |\alpha| \) and the inequality \( \|M_k v_{\alpha-\varepsilon_k}(t)\|_H \leq c_k \|v_{\alpha-\varepsilon_k}(t)\|_H \), we conclude that \( \lim_{t \to +\infty} \|v_\alpha(t)\|_H = 0 \) for every \( \alpha \in \mathcal{J} \). Since \( v_\alpha \in C((0, T); H) \) for every \( T \), it follows that \( \sup_{t \geq 0} \|v_\alpha(t)\|_H < \infty \). Define the operator \( \mathcal{R} \) on \( L_2(\mathbb{F}) \) so that \( \mathcal{R}_\xi_\alpha = r_\alpha \xi_\alpha \), where

\[
r_\alpha = \frac{(2N)^{-\alpha}}{1 + \sup_{t \geq 0} \|v_\alpha(t)\|_H}.
\]

Then (1.21) follows by the dominated convergence theorem.

Theorem 4.8 is proved.

5. BILINEAR PARABOLIC AND ELLIPTIC SPDEs

Let \( G \) be a smooth bounded domain in \( \mathbb{R}^d \) and \( \{h_k, k \geq 1\} \), an orthonormal basis in \( L_2(G) \). We assume that

\[
\sup_{x \in G} |h_k(x)| \leq c_k, \quad k \geq 1. \tag{5.1}
\]

A space white noise on \( L_2(G) \) is a formal series

\[
\tilde{W}(x) = \sum_{k \geq 1} h_k(x) \xi_k, \tag{5.2}
\]

where \( \xi_k, k \geq 1 \), are independent standard Gaussian random variables.

5.1. Dirichlet Problem for parabolic SPDE of the Second Order. Consider the following equation:

\[
u_t(\xi_t, x) = a_{ij}(x) D_i D_j u(t, x) + b_i(x) D_i u(t, x) + c(x) u(t, x) + f(t, x) + (\sigma_i(x) D_i u(t, x) + \nu(x) u(t, x) + g(t, x)) \circ \tilde{W}(x), \quad 0 < t \leq T, \quad x \in G, \tag{5.3}
\]

with zero boundary conditions and some initial condition \( u(0, x) = u_0(x) \); the functions \( a_{ij}, b_i, c, f, \sigma_i, \nu, g \), and \( u_0 \) are non-random. In [5.3] and in similar expressions below we assume summation over the repeated indices. Let \( (V, H, V') \) be the normal triple with \( V = H^1_0(G) \), \( H = L_2(G) \), \( V' = H^{-1}_2(G) \). In view of (5.2), equation (5.3) is a particular case of equation (5.4) so that

\[
A u = a_{ij}(x) D_i D_j u + b_i(x) D_i u + c(x) u, \quad M_k u = (\sigma_i(x) D_i u + \nu(x) u) h_k(x), \tag{5.4}
\]

and \( f(t, x) + g(t, x) \circ \tilde{W}(x) \) is the free term.

We make the following assumptions about the coefficients:
D1 The functions $a_{ij}$ are Lipschitz continuous in the closure $\bar{G}$ of $G$, and the functions $b_i, c, \sigma_i, \nu$ are bounded and measurable in $\bar{G}$.

D2 There exist positive numbers $A_1, A_2$ so that $A_1|y|^2 \leq a_{ij}(x)y_iy_j \leq A_2|y|^2$ for all $x \in \bar{G}$ and $y \in \mathbb{R}^d$.

Given a $T > 0$, recall the notation $\mathcal{V} = L_2((0, T); V)$ and similarly for $\mathcal{H}$ and $\mathcal{V}'$ (see (3.11)).

**Theorem 5.1.** Under the assumptions D1 and D2, if $f \in \mathcal{V}'$, $g \in \mathcal{H}$, $u_0 \in \mathcal{H}$, then there exists an $\ell > 1$ and a number $C > 0$, both independent of $u_0, f, g$, so that $u \in \mathcal{R}L_2(\mathbb{F}; \mathcal{V})$ and

$$
\|u\|_{\mathcal{R}L_2(\mathbb{F}; \mathcal{V})} \leq C \cdot (\|u_0\|_H + \|f\|_{\mathcal{V}'} + \|g\|_H),
$$

where the operator $\mathcal{R}$ is defined by the weights

$$
\gamma^2_\alpha = c^{-2\alpha}(|\alpha|!)^{-1}(2\mathbb{N})^{-2\alpha}
$$

and $c^\alpha = \prod_k c_k^{\alpha_k}$, with $c_k$ from (5.1); the number $\ell$ in general depends on $T$.

**Proof.** We derive the result from Theorem 3.11. Consider the deterministic equation $\dot{U}(t) = AU(t) + F$. Assumptions D1 and D2 imply that there exists a unique solution of this equation in the normal triple $(V, H, V')$, and the solution satisfies

$$
\sup_{0 < t < T} \|U(t)\|_H + \|U\|_{\mathcal{V}} \leq C \cdot (\|U(0)\|_H + \|F\|_{\mathcal{V}'})
$$

where the number $C$ depends on $T$ and the operator $A$. Moreover, (5.1) implies that (5.15) holds with $C_k = C_0c_k$ for some positive number $C_0$ independent of $k$, but possibly depending on $T$.

To proceed, let us assume first that $g = 0$. Then the statement of the theorem follows directly from Theorem 3.11 if we take in (3.18) $q_k = c_k^{-1}(2k)^{-\ell}$ with sufficiently large $\ell$.

It now remains to consider the case $g \neq 0$ and $f = u_0 = 0$. Even though $g$ is non-random, $g\xi_k$ is, and therefore a direct application of Theorem 3.11 is not possible. Instead, let us look more closely at the corresponding equations for $u_\alpha$. For $\alpha = (0)$,

$$
u(0)(t) = \int_0^t A\nu(0)(s)ds,
$$

which implies $\nu(0)(t) = 0$ for all $t$. For $\alpha = \varepsilon_k$,

$$
u_{\varepsilon_k}(t) = \int_0^t A\nu_{\varepsilon_k}(s)ds + h_k \int_0^t g(s)ds,
$$

or

$$
u_{\varepsilon_k}(t) = \int_0^t \Phi_{t-s}h_k g(s)ds,
$$

so that

$$
\|\nu_{\varepsilon_k}\|_{\mathcal{V}} \leq C_0c_k\|g\|_H.
$$

If $|\alpha| > 1$, then

$$
u_{\alpha}(t) = \int_0^t A\nu_{\alpha}(s)ds + \sum_{k \geq 1} \sqrt{\alpha_k} M_k u_{\alpha - \varepsilon_k},
$$
which is the same as (3.22). In particular, if $|\alpha| = 2$ and $\{i, j\}$ is the characteristic set of $\alpha$, then

$$u_\alpha(t) = \frac{1}{\sqrt{\alpha!}} \int_0^t \Phi_{t-s} \left( M_i u_{\varepsilon_j}(s) + M_j u_{\varepsilon_i}(s) \right) ds.$$

More generally, by analogy with (3.25), if $|\alpha| = n > 2$ and $\{k_1, \ldots, k_n\}$ is the characteristic set of $\alpha$, then

$$u_\alpha(t) = \frac{1}{\sqrt{\alpha!}} \sum_{\sigma \in P_n} \int_0^t \cdots \int_0^{s_n} \Phi_{t-s_n} M_{k_\sigma(n)} \cdots \Phi_{t-s_2} M_{k_\sigma(2)} u_{\varepsilon_{\sigma(1)}}(s_2) ds_2 \cdots ds_n.$$

By the triangle inequality and (5.8),

$$\|u_\alpha\|_V \leq \frac{|\alpha| C_0 |\varepsilon^\alpha|}{\sqrt{\alpha!}} \|g\|_H,$$

and then (5.9) follows from (3.26) if $\ell$ is sufficiently large.

This completes the proof of Theorem 5.1. \hfill \square

**Theorem 5.2.** In addition to $D_1$ and $D_2$, assume that

1. $\sigma_i = 0$ for all $i$;
2. the operator $A$ in $G$ with zero boundary conditions satisfies (4.20).

If there exist functions $f^*$ and $g^*$ from $H$ so that

$$\lim_{t \to +\infty} (\|f(t) - f^*\|_H + \|g(t) - g^*\|_H) = 0,$$

then the solution $u$ of equation (5.3) satisfies

$$\lim_{t \to +\infty} \|u(t) - u^*\|_{RL^2(\mathbb{F};H)} = 0,$$

where the operator $R$ is defined by the weights (5.6) and $u^*$ is the solution of the stationary equation

$$a_{ij}(x) D_i D_j u^*(x) + b_i(x) D_i u^*(x) + c(x) u^*(x) + f^*(x)$$

$$\quad + (\nu(x) u^*(x) + g^*(x)) \cdot \mathbf{W}(x) = 0, \quad x \in G; \quad u|_{\partial G} = 0.$$

**Proof.** This follows from Theorem 4.8. \hfill \square

**Remark 5.3.** The operator $A$ satisfies (4.20) if, for example, each $a_{ij}$ is twice continuously differentiable in $\bar{G}$, each $b_i$ continuously differentiable in $\bar{G}$, and

$$\inf_{x \in G} c(x) - \sup_{x \in G} (|D_i D_j a_{ij}(x)| + |D_i b_i(x)|) \geq \varepsilon > 0;$$

this is verified directly using integration by parts.
5.2. Elliptic SPDEs of the full second order. Consider the following Dirichlet problem:

\[-D_1 \left( a_{ij} (x) D_j u (x) \right) + D_i \left( \sigma_{ij} (x) D_j (u (x)) \right) \circ \dot{W} (x) = f (x), \quad x \in G, \tag{5.13} \]
\[u_{|\partial G} = 0,\]

where \( \dot{W} \) is the space white noise (5.2). Assume that the functions \( a_{ij}, \sigma_{ij}, f, \) and \( g \) are non-random. Recall that according to our summation convention, in (5.13) and in similar expressions below we assume summation over the repeated indices.

We make the following assumptions:

**E1:** The functions \( a_{ij} = a_{ij}(x) \) and \( \sigma_{ij} = \sigma_{ij}(x) \) are measurable and bounded in the closure \( \tilde{G} \) of \( G \).

**E2:** There exist positive numbers \( A_1, A_2 \) so that \( A_1 |y|^2 \leq a_{ij}(x)y_i y_j \leq A_2 |y|^2 \) for all \( x \in \tilde{G} \) and \( y \in \mathbb{R}^d \).

**E3:** The functions \( h_k \) in (5.2) are bounded and Lipschitz continuous.

Clearly, equation (5.13) is a particular case of equation (4.3) with

\[ A u (x) := -D_1 \left( a_{ij} (x) D_j u (x) \right) \tag{5.14} \]

and

\[ M_k u (x) := h_k (x) D_i \left( \sigma_{ij} (x) D_j u (x) \right). \tag{5.15} \]

Assumptions **E1** and **E3** imply that each \( M_k \) is a bounded linear operator from \( H_2^{-1}(G) \) to \( H_2^{-1}(G) \). Moreover, it is a standard fact that under the assumptions **E1** and **E2** the operator \( A \) is an isomorphism from \( V \) onto \( V' \) (see e.g. [12]). Therefore, for every \( k \) there exists a positive number \( C_k \) such that

\[ \| A^{-1} M_k v \|_V \leq C_k \| v \|_V, \quad v \in V. \tag{5.16} \]

**Theorem 5.4.** Under the assumptions **E1** and **E2**, if \( f \in H_2^{-1}(G) \), then there exists a unique solution of the Dirichlet problem (5.13) \( u \in RL_2 (\mathbb{F}; \tilde{H}_2^1(G)) \) such that

\[ \| u \|_{RL_2 (\mathbb{F}; \tilde{H}_2^1(G))} \leq C \cdot \| f \|_{H_2^{-1}(G)}. \tag{5.17} \]

The weights \( r_{\alpha} \) can be taken in the form

\[ r_{\alpha} = \frac{q^\alpha}{2^{|\alpha|} \sqrt{|\alpha|!}}, \quad \text{where} \quad q^\alpha = \prod_{k=1}^{\infty} q_k^{a_k}, \tag{5.18} \]

and the numbers \( q_k, k \geq 1 \) are chosen so that \( \sum_{k \geq 1} C_k^2 q_k^2 k^2 < 1, \) with \( C_k \) from (5.16).

**Proof.** This follows from Theorem 4.5.

**Remark 5.5.** With an appropriate change of the boundary conditions, and with extra regularity of the basis functions \( h_k \), the results of Theorem 5.4 can be extended to stochastic elliptic equations of order \( 2m \). The corresponding operators are

\[ A u = (-1)^m D_{i_1} \cdots D_{i_m} \left( a_{i_1 \cdots i_m j_1 \cdots j_m} (x) D_{j_1} \cdots D_{j_m} u (x) \right) \tag{5.19} \]
and
\[ M_k u = h_k(x) D_{i_1} \cdots D_{i_m} \left( \sigma_{i_1 \ldots i_m j_1 \ldots j_m}(x) D_{j_1} \cdots D_{j_m} u(x) \right). \] (5.20)

Since \( G \) is a smooth bounded domain, regularity of \( h_k \) is not a problem: we can take \( h_k \) as the eigenfunctions of the Dirichlet Laplacian in \( G \).

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Appendix.

A version of the Toeplitz lemma.
Lemma. Assume that $f = f(t)$ is an integrable function and $\lim_{t \to +\infty} |f(t)| = 0$. Then, for every $c > 0$, $\lim_{t \to +\infty} \int_0^t e^{-ct} f(s) ds = 0$.

Proof. Given $\varepsilon > 0$, choose $T$ so that $|f(t)| < \varepsilon$ for all $t > T$. Then $\left| \int_0^t e^{-ct} f(s) ds \right| \leq e^{-ct} \int_0^T e^{cs} |f(s)| ds + \varepsilon \int_T^t e^{-ct} ds$. Passing to the limit as $t \to +\infty$, we find $\lim_{t \to +\infty} \left| \int_0^t e^{-ct} f(s) ds \right| \leq \varepsilon/c$, which completes the proof. \(\square\)

Current address, S. V. Lototsky: Department of Mathematics, USC, Los Angeles, CA 90089

E-mail address, S. V. Lototsky: lototsky@math.usc.edu

URL: http://math.usc.edu/~lototsky

Current address, B. L. Rozovskii: Division of Applied Mathematics, Brown University, Providence, RI 02912

E-mail address, B. L. Rozovskii: roszovsky@dam.brown.edu

URL: http://www.dam.brown.edu/people/rozovsky.html