REGULARITY OF ANALYTIC TORSION FORM ON FAMILIES OF NORMAL COVERINGS

BING KWAN SO AND GUANGXIANG SU

ABSTRACT. We prove the smoothness of the $L^2$-analytic torsion form on some fiber bundles with non-compact fibers of positive Novikov-Shubin invariant. We do so by generalizing the arguments of Azzali-Goette-Schick to an appropriate Sobolev space, and proving that the Novikov-Shubin invariant remains positive in the Sobolev settings, using an argument of Alvarez Lopez-Kordyukov.

Bismut-Lott [3] introduced a torsion form on a compact fiber bundle $M \to B$ and a flat vector bundle $E \to M$:

$$\int_0^\infty -F^2(t) + \frac{\text{str}_\Psi(\mathbb{N}I_0)}{2} + \frac{(\dim(Z) \text{rk}(E) \text{str}_\Psi(I_0))}{4} - \frac{\text{str}_\Psi(\mathbb{N}I_0)}{2}(1-2t) e^{-t} \frac{dt}{t}.$$ (1)

One important characteristic of the torsion form is that it satisfies the transgression formula. There has been several attempts to generalize the notion to $L^2$-torsion forms on fiber bundles with the non-compact fibers. The main technical obstacle is that in the non-compact case the spectrum of the Laplacian operator may be $[0, \infty)$, and the integral in Equation (1) may not converge.

To proceed, one considers the special case where the Novikov-Shubin invariant is (sufficiently) positive. In particular, Gong-Rothenberg [5] defined the analytic torsion form as in (1), and they proved that the torsion is smooth, provided the Novikov-Shubin invariant is at least half of the the base dimension. Heitsch-Lazarov [7] generalized essentially the same arguments to foliations. Subsequently, Azzali-Goette-Schick [1] proved, by direct computation, that the integrands defining the torsion form, as well as several other invariants related to the signature operator, converge provided the Novikov-Shubin invariant is positive (or of acyclic determinant class). However, they did not prove the smoothness of the $L^2$-analytic torsion form, but left the problem open [1, Section 6.3]. To consider transgression formulas, they had to use weak derivatives.

The objective of this paper is to establish the regularity of the analytic torsion form, in the case when the Novikov-Shubin invariant is positive.

We achieve this result by generalizing Azzali-Goette-Schick’s arguments to some Sobolev spaces. In Section 1, we define such Sobolev norms on the spaces of kernels on the fibered product groupoid. Unlike [1], we consider Hilbert-Schmit type norms on the space of smoothing operators. Given a kernel, the Hilbert-Schmit norm can be explicitly written down. As a result, we are able to take into account derivatives in both the fiber-wise and transverse directions, with the help of a splitting similar to [6]. The main result in this section is Corollary 1.14 where we check that for any

Chern Institute of Mathematics, Nankai University, Tianjin, 300071, China.
e-mail: bkso@graduate.hku.hk (B.K. So), guangxiangsu@gmail.com (G. Su).
smooth bounded operator \( A, m = 0, 1, \cdots \), one has the estimate
\[
\|\hat{A}\psi\|_{HS m} \leq \left( \sum_{0 \leq l \leq m} C^l_{m,l} \|A\|_{op l}\right) \|\psi\|_{HS m},
\]
which is an analogue of the key ingredient used in the proof of [1] Theorem 4.2.

Then in section 2, we turn to prove that having positive Novikov-Shubin invariant implies positivity of the Novikov-Shubin invariant in the Sobolev settings. We adapt an argument of Alvarez Lopez-Kordyukov [8]. We remark that [8] requires extra conditions (which are always satisfied for signature operators on fiber bundles), therefore such result is non-trivial.

With these preliminaries in place, we can simply apply [1] in Section 3, and conclude the integral (1) converges in all Sobolev norms, and hence the regularity of the torsion form.

1. Preliminaries

Let \( Z \to M \xrightarrow{\pi} B \) be a fiber bundle, \( E \xrightarrow{\pi} M \) be a vector bundle. We assume \( B \) is compact. For each \( x \in B \), write \( Z_x := \pi^{-1}(x) \subset M \).

Recall that the ‘infinite dimensional bundle’ over \( B \) in the sense of Bismut is a vector bundle with typical fiber \( \Gamma_{\infty}(E|_{Z_x}) \) (or other function spaces) over each \( x \in B \). We denote by \( E_b \) such Bismut bundle. The space of smooth sections on \( E_b \) is, as a vector space, \( \Gamma_{\infty}(E) \). Each element \( s \in \Gamma_{\infty}(E) \) is regarded as a map
\[
x \mapsto s|_{Z_x} \in \Gamma_{\infty}(E|_{Z_x}), \quad \forall x \in B.
\]
In other words, one defines a section on \( E_b \) to be smooth, if the images of all \( x \in B \) fit together to form an element in \( \Gamma_{\infty}(E) \). In the language of non-commutative geometry, one considers \( \Gamma_{\infty}(E_b) \) as a \( C_{\infty}(B) \)-module with action
\[
(f, s) \mapsto (\pi^* f)s, \quad \forall f \in C_{\infty}(B), s \in \Gamma_{\infty}(E).
\]
For any \( x \in B \), denote \( E_x := E|_{Z_x} \).

Let \( V := \ker(dx) \) and fix a splitting \( TM = H \oplus V \). Denote by \( P^V, P^H \) respectively the projections to \( V \) and \( H \). Given any vector field \( X \in \Gamma_{\infty}(TB) \), denote the horizontal lift of \( X \) by \( X^H \in \Gamma_{\infty}(H) \subset \Gamma_{\infty}(TM) \). Note that \( [X^H, Y] \in \Gamma_{\infty}(V) \) for any vertical vector field \( Y \in \Gamma_{\infty}(V) \).

Let \( \nabla^E \) be a connection on \( E \). Fix Riemannian metrics on \( M \) and \( B \) and denote respectively by \( \nabla^TM, \nabla^TB \) the Levi-Civita connections. One naturally defines the connections
\[
\nabla^V_X Y := [X^H, Y], \quad \forall Y \in \Gamma_{\infty}(V_b) \cong \Gamma_{\infty}(V)
\]
\[
\nabla^E_X s := \nabla^E_{X^H} s, \quad \forall s \in \Gamma_{\infty}(E_b) \cong \Gamma_{\infty}(E).
\]

**Definition 1.1.** The covariant derivative on \( E_b \) is the map \( \nabla^E : \Gamma_{\infty}(\otimes^* T^* B \otimes \otimes^* V'_b \otimes E_b) \to \Gamma_{\infty}(\otimes^* T^* B \otimes \otimes^* V'_b \otimes E_b) \),
\[
\nabla^E s(X_0, X_1, \cdots X_k; Y_1, \cdots Y_l) := \nabla^E_{X_0} s(X_1, \cdots X_k; Y_1, \cdots Y_l)
\]
\[
- \sum_{j=1}^l s(X_1, \cdots \hat{X}_j \cdots X_k; Y_1, \cdots, \nabla^V_{X_j} Y_j, \cdots, Y_l)
\]
\[ - \sum_{i=1}^{k} s(X_1, \ldots, \nabla^{TB}_{X_i} X_l, \ldots, X_k; Y_1, \ldots Y_l), \]
for any \( k, l \in \mathbb{N}, X_0, \ldots, X_k \in \Gamma^\infty(TB), Y_1, \ldots Y_l \in \Gamma^\infty(V). \)

Clearly, taking covariant derivative can be iterated, which we denote by \((\nabla^{E_s})^m, m = 1, 2, \ldots.\) Note that \((\nabla^{E_s})^m\) is a differential operator of order \( m.\)

Also, we define \(\partial^V : \Gamma^\infty(\otimes^s T^*B \otimes \otimes^l V_0' \otimes E_b) \to \Gamma^\infty(\otimes^{s-1} T^*B \otimes \otimes^{l+1} V_0' \otimes E_b),\)
\[\partial^V s(X_1, \ldots X_k; Y_0, Y_1, \ldots Y_l) := \nabla_{Y_0}^{E_s} s(X_1, \ldots X_k; Y_1, \ldots Y_l) \]
\[ - \sum_{j=1}^{l} s(X_1, \ldots X_k; Y_1, \ldots, P^V (\nabla_{Y_0}^{TM} Y_j), \ldots, Y_l). \]

\textbf{Remark 1.2.} The operators \((\nabla^{E_s})\) and \(\partial^V\) are just respectively the \((0, 1)\) and \((1, 0)\) parts of the usual covariant derivative operator.

Equip \(E\) with a metric \(\langle \cdot, \cdot \rangle_{E}\). One naturally extends the point-wise inner product to \(E', \otimes T^*M \otimes E',\) etc. Recall that, in particular on \(E \otimes E',\) one has
\[\langle \psi_1, \psi_2 \rangle_{E \otimes E'} = \text{tr}(\psi_1 \psi_2^*),\]
where \(\text{tr}\) denotes the point-wise matrix trace. We shall denote by \(|\cdot|\) the point-wise norm.

We assume further \(M\) is a Riemannian manifold with bounded geometry (i.e. there exist a cover by foliated charts such that the metric and all its derivatives are uniformly bounded), and \(E\) is a vector bundle with bounded geometry.

Given any \(s \in \Gamma^\infty_c(E_b)\), regard \((\nabla^{E_s})^i(\partial^E)^j s\) as a \(C^\infty(B)\) linear map from \(\otimes^s TB\) to \(\otimes^j V_0' \otimes E_b.\)

\textbf{Definition 1.3.} Define the \(m\)-th Sobolev norm by
\[\|s\|^2_m := \sum_{i+j \leq m} \sup_{X_0 \in \otimes^s TB, y \in Z_x} \int_{|X_0| = 1} |(\nabla^{E_s})^i(\partial^E)^j s)(X_1)|^2(y) \mu_x(y).\]

Denote by \(W^m(E)\) be the Sobolev completion of \(\Gamma^\infty_c(E)\) with respect to \(|\cdot|_m.\)

\textbf{Remark 1.4.} Because \(|\cdot|\) is a norm and by the Cauchy-Schwarz inequality, it follows \(|\cdot|_m\) is also a norm.

Let \(G\) be a finitely generated discrete group acting on \(M\) freely, properly discontinuously, and co-compact. Let \(M_0 := M/G.\) Suppose \(G\) also acts on \(B\) such that for any \(p \in M, \pi(pg) = \pi(p)g.\) For each \(x \in B,\) denote by \(G_x\) the subgroup fixing \(x, Z_x := \pi^{-1}(x).\) Since the projection \(\pi\) is \(G\)-invariant, \(M_0\) is also foliated, denote such foliation by \(V_0.\) The leaf through each \(x \in B\) is given by \(Z_x/G_x.\) Fix a distribution \(H_0 \subset TM_0\) complementary to \(V_0,\) and a Riemannian metric on \(M_0.\)

Since the projection form \(M\) to \(M_0\) is a local diffeomorphism, one gets a \(G\)-invariant splitting \(TM = V \oplus H,\) where \(V = \text{Ker}(d\pi),\) and a \(G\)-invariant metric on \(M.\) For each \(x \in B,\) denote by \(\mu_x\) the Riemannian volume on \(Z_x.\)

\textbf{Definition 1.5.} Let \(E \xrightarrow{\phi} M\) be a complex vector bundle. We say that \(E\) is a contravariant \(G\)-bundle if \(G\) also acts on \(E\) from the right, such that for any \(v \in\)
For any vectors $X \in \mathbb{R}^n$, $\phi(vg) = \phi(v)g \in M$, and moreover $G$ acts as a linear map between the fibers.

The group $G$ then acts on sections of $E$ from the left by

$$s \mapsto g^* s, \quad (g^* s)(p) := s(pg)g^{-1} \in \phi^{-1}(p), \quad \forall \ p \in M.$$  

1.1. **The fibered product groupoid.** Recall that the fibered product groupoid is, as a manifold, $M \times_B M := \{(p,q) \in M \times M : \pi(p) = \pi(q)\}$ and is equipped with groupoid operations:

$$s(p,q) := q, \quad t(p,q) := p, \quad (p,q)^{-1} := (q,p),$$

$$(p_1,q_1)(p_2,q_2) := (p_1,q_2), \quad \text{whenever } q_1 = p_2.$$

Note that $G$ acts on $M \times_B M$ by the diagonal action

$$(p,q)g := (pg, qg).$$

The manifold $M \times_B M$ is a fiber bundle over $B$, with typical fiber $Z \times Z$.

**Notation 1.6.** With some abuse in notations, we shall often write elements in $M \times_B M$ as a triple $(x,y,z)$, where $x \in B, y,z \in Z_x$. We let $s(x,y,z) = (x,z), t(x,y,z) = (x,y) \in M$.

One naturally has the splitting [6, Section 2]

$$T(M \times_B M) = \mathcal{H} \oplus V_t \oplus V_s,$$

where $V_s := \text{Ker}(dt), V_t := \text{Ker}(ds)$. Note that $V_s \cong s^{-1}V, V_t \cong t^{-1}(V)$, hence the notation.

Let $E \to M$ be a contravariant $G$-vector bundle. We denote

$$\hat{E} \to M \times_B M := t^{-1}E \otimes s^{-1}E',$$

where $E'$ is the dual of $E$. Given a $G$-invariant connection $\nabla^E$ on $E$, let

$$\nabla^{\hat{E}} := \nabla^{t^{-1}E} + \nabla^{s^{-1}E'}$$

be the tensor sum of the pullback connections. Here, recall that, fixing any local base $\{e_1, \cdots, e_r\}$ of $E'$ on some $U \subset M$, any section can be written as

$$s = \sum_{i=1}^r u_i \otimes s^* e_i$$

on $s^{-1}(U)$, where $u_i \in \Gamma^\infty(t^{-1}E)$, and by definition

$$\nabla^{\hat{E}}_X \sum_{i=1}^r u_i \otimes s^* e_i = \sum_{i=1}^r (\nabla^{t^{-1}E}_X u_i) \otimes s^* e_i + u_i \otimes s^*(\nabla^{E'}_{ds(X)} e_i).$$

For any vectors $X$ on $M$.

Similar to Equation (2), define the operators on $\Gamma^\infty(\otimes^*B \otimes (V'_t)_y \otimes (V'_s)_z \otimes \hat{E})$

$$\hat{\nabla}^E \psi(X_0, X_1, \cdots X_k; Y_1, \cdots Y_l, Z_1, \cdots Z_l')$$

$$:= \nabla^{\hat{E}}_{X_0} \psi(X_1, \cdots X_k; Y_1, \cdots Y_l, Z_1, \cdots Z_l')$$

$$- \sum_{1 \leq j \leq l} \psi(X_1, \cdots X_k; Y_1, \cdots, \nabla^{V'_j}_{X_0} Y_j, \cdots, Y_l, Z_1, \cdots Z_l').$$
\[ - \sum_{1 \leq j \leq \ell'} \psi(X_1, \ldots, X_k; Y_1, \ldots, Y_i, Z_1, \ldots, Z_{2 \ell'}) \]

\[ - \sum_{1 \leq i \leq k} \psi(X_1, \ldots, \nabla_{X_0}^{TB} X_i, \ldots, X_k; Y_1, \ldots, Y_i, Z_1, \ldots, Z_{2 \ell'}) \]

\[ \hat{\partial}^s \psi(X_1, \ldots, X_k; Y_0, Y_1, \ldots, Y_i, Z_1, \ldots, Z_{2 \ell'}) \]

\[ \hat{\partial}^t \psi(X_1, \ldots, X_k; Y_1, \ldots, Y_i, Z_0, Z_1, \ldots, Z_{2 \ell'}) \]

\[ := \nabla_{Y_0}^E \psi(X_1, \ldots, X_k; Y_1, \ldots, Y_i, Z_1, \ldots, Z_{2 \ell'}) \]

\[ - \sum_{1 \leq j \leq \ell} \psi(X_1, \ldots, X_k; Y_1, \ldots, P^V \nabla_{Y_0}^{TM} Y_j, \ldots, Y_i, Z_1, \ldots, Z_{2 \ell'}) \]

\[ - \sum_{1 \leq j \leq \ell'} \psi(X_1, \ldots, X_k; Y_1, \ldots, Y_i, Z_1, \ldots, P^V [Y_0, Z_j], \ldots, Z_{2 \ell'}) \]

\[ \hat{\partial}^t \psi(X_1, \ldots, X_k; Y_1, \ldots, Y_i, Z_0, Z_1, \ldots, Z_{2 \ell'}) \]

\[ := \nabla_{Y_0}^E \psi(X_1, \ldots, X_k; Y_1, \ldots, Y_i, Z_0, Z_1, \ldots, Z_{2 \ell'}) \]

\[ - \sum_{1 \leq j \leq \ell} \psi(X_1, \ldots, X_k; Y_1, \ldots, P^V [Z_0, Y_j], \ldots, Y_i, Z_1, \ldots, Z_{2 \ell'}) \]

\[ - \sum_{1 \leq j \leq \ell'} \psi(X_1, \ldots, X_k; Y_1, \ldots, Y_i, Z_1, \ldots, P^V (\nabla_{Z_0}^{TM} Z_j), \ldots, Z_{2 \ell'}) \]

Given any vector fields \( Y, Z \in V \). Let \( Y^s, Z^t \) be respectively the lifts of \( Y \) and \( Z \) to \( V^s \) and \( V^t \). Then \([Y^s, Z^t] = 0\). It follows that as differential operators,

\[ [\hat{\partial}^s, \hat{\partial}^t] = 0. \]

Also, it is straightforward to verify that

\[ [\nabla_{Y_0}^E, \hat{\partial}^s] \text{ and } [\nabla_{Y_0}^E, \hat{\partial}^t] \]

are both zeroth order differential operators (i.e. smooth bundle maps).

Fix a local trivialization

\[ x_\alpha : \pi^{-1}(B_\alpha) \to B_\alpha \times Z, \quad p \mapsto (\pi(p), \varphi \alpha(p)) \]

where \( B = \bigcup \alpha B_\alpha \) is a finite open cover, and \( \varphi \alpha|_{\pi^{-1}(z)} : Z_\alpha \to Z \) is a diffeomorphism.

Such a trivialization induces a local trivialization of the fiber bundle \( M \times_B M \xrightarrow{t} M \) by \( M = \bigcup M_\alpha, M_\alpha := \pi^{-1}(B_\alpha), \)

\[ x_\alpha : t^{-1}(M_\alpha) \to M_\alpha \times Z, \quad (p, q) \mapsto (p, \varphi \alpha(q)). \]

On \( M_\alpha \times Z \) the source and target maps are explicitly given by

\[ s \circ (x_\alpha)^{-1}(p, z) = (x_\alpha)^{-1}(\pi(p), z), \quad \text{and} \quad t \circ (x_\alpha)^{-1}(p, z) = p. \]

To such trivialization, one has the natural splitting

\[ T(M_\alpha \times Z) = H^\alpha \oplus V^\alpha \oplus TZ, \]

where \( H^\alpha \) and \( V^\alpha \) are respectively \( H \) and \( V \) restricted to \( M_\alpha \times \{z\}, \ z \in Z \). It follows form \([7]\) that

\[ V^\alpha = d\hat{x}_\alpha(V^s), \quad TZ = d\hat{x}_\alpha(V^t). \]
As for vector fields along $\hat{H}$, given any vector field $X$ on $B$, let $X^H, X^\hat{H}$ be respectively the lifts of $X$ to $H$ and $\hat{H}$. Since $dt(X^\hat{H}) = ds(X^H) = X^H$, it follows that
\[
d\xi_\alpha(X^H) = X^{H\alpha} + d\varphi^\alpha(X^H).
\]
Note that $d\varphi^\alpha(X^H) \in TZ \subseteq T(M_\alpha \times Z)$.

Corresponding to the splitting $T(M_\alpha \times Z) = H^\alpha \oplus V^\alpha \oplus TZ$, one defines the covariant derivative operators. Let $\nabla^{TM_\alpha}$ be the Levi-Civita connection on $M$ pulled back to $M_\alpha$, and fix a Levi-Civita connection $\nabla^{TZ}$ on $Z$. Define for any smooth section $\phi \in \Gamma(M \times M_\alpha \times Z)$
\[
\hat{\nabla}^\alpha \phi(X_0, X_1, \ldots, X_k; Y_1, \ldots, Y_l, Z_1, \ldots, Z_m)
\]
and for any smooth section $\psi \in \Gamma(\nabla^\alpha \phi(X_0, X_1, \ldots, X_k; Y_1, \ldots, Y_l, Z_1, \ldots, Z_m)$
\[
\hat{\nabla}^\alpha \psi(X_0, X_1, \ldots, X_k; Y_1, \ldots, Y_l, Z_1, \ldots, Z_m)
\]
\[
\frac{\partial}{\partial X_i} \phi(X_1, \ldots, X_k; Y_1, \ldots, Y_l, Z_1, \ldots, Z_m)
\]
\[
\frac{\partial}{\partial Y_j} \phi(X_1, \ldots, X_k; Y_1, \ldots, Y_l, Z_1, \ldots, Z_m)
\]
\[
\frac{\partial}{\partial Z_l} \phi(X_1, \ldots, X_k; Y_1, \ldots, Y_l, Z_1, \ldots, Z_m)
\]

Remark 1.7. If $Z_1, \ldots, Z_m$ are vector fields on $Z$, lifted to $M_\alpha \times Z$, then
\[
\hat{\nabla}^\alpha \phi(X_1, \ldots, X_k; Y_0, Y_1, \ldots, Y_l, Z_1, \ldots, Z_m) = \hat{\nabla}^\alpha \phi(X_1, \ldots, X_k; Y_0, Y_1, \ldots, Y_l, Z_1, \ldots, Z_m).
\]

We express the (pullback of) the covariant derivatives $\hat{\nabla}^E_\alpha \psi, \hat{\nabla}^y_\alpha \psi, \hat{\nabla}^z_\alpha \psi$ in terms of $\hat{\nabla}^\alpha \psi^\alpha, \hat{\nabla}^\alpha \psi^\alpha, \hat{\nabla}^\alpha \psi^\alpha$, and $\hat{\nabla}^\alpha \psi^\alpha$, where $\psi^\alpha := (x^{-1}_\alpha)^* \psi$. One directly verifies
\[
\hat{\nabla}^E_\alpha \psi(X_0, X_1, \ldots, X_k; Y_1, \ldots, Y_l, Z_1, \ldots, Z_m)
\]
\[
= (x^{-1}_\alpha)^* \left( \hat{\nabla}^\alpha_0 (X^{H\alpha} + d\varphi^\alpha(X^H)) \psi^\alpha \right)(X_1, \ldots, X_k; d\xi_\alpha(Y_1, \ldots, Y_l, Z_1, \ldots, Z_m))
\]
\[
- \sum_{1 \leq j \leq l} \psi^\alpha(X_1, \ldots, X_k; d\xi_\alpha Y_1, \ldots, [X^{H\alpha}, d\xi_\alpha Y_j], \ldots, d\xi_\alpha Y_l; d\xi_\alpha (Z_1, \ldots, Z_m))
\]
For any $M, E$ mention function on $M$. As a vector space, the Riemannian distance between $\infty$ 

We introduce a Sobolev type generalization of the Hilbert-Schmidt norm on $\Psi = (\infty, \infty)$. For any $(\infty, \infty)$

\[ \sum_{1 \leq j \leq \ell} \psi^\alpha(X_1, \cdots, X_k; dx_\alpha(Y_1, \cdots, Y_l), dx_\alpha Z_l, \cdots, [X_0^H, \cdots, \nabla^T \nabla X_0^H, X_1, \cdots, X_k; Y_1, \cdots, Y_l, Z_1, \cdots, Z_l]) \]

\[ = (x_\alpha^{-1})^* (\nabla^\alpha \psi^\alpha(X_0, X_1, \cdots, X_k; Y_0, Y_1, \cdots, Y_l, Z_1, \cdots, Z_l)) \]

\[ + \hat{\partial}^Z \psi^\alpha(X_0, X_1, \cdots, X_k; Y_0, Y_1, \cdots, Y_l, Z_1, \cdots, Z_l) \]

\[ + \sum_{1 \leq j \leq \ell} \psi^\alpha(X_1, \cdots, X_k; dx_\alpha(Y_1, \cdots, Y_l), dx_\alpha Z_l, \cdots, (\nabla^T \nabla^T d \psi^\alpha(X_0^H))(dx_\alpha Z_j), \cdots, dx_\alpha Z_l)) \]

Applying similar computations to $\hat{\partial}^k$ and $\hat{\partial}^t$, one gets:

\[ \hat{\partial}^k \psi^\alpha(X_1, \cdots, X_k; Y_0, Y_1, \cdots, Y_l, Z_1, \cdots, Z_l) \]

\[ = (x_\alpha^{-1})^* (\hat{\partial}^k \psi^\alpha(X_1, \cdots, X_k; dx_\alpha(Y_0, Y_1, \cdots, Y_l, Z_1, \cdots, Z_l)) \]

\[ + \sum_{1 \leq j \leq \ell} \psi^\alpha(X_1, \cdots, X_k; dx_\alpha(Y_1, \cdots, Y_l), dx_\alpha Z_l, \cdots, (\nabla^T \nabla^T d \psi^\alpha(X_0^H))(dx_\alpha Z_j), \cdots, dx_\alpha Z_l)) \]

\[ \Psi_\infty(M \times_B M, E) := \left\{ \psi \in \Gamma(\hat{\partial}) : \begin{array}{l}
\text{For any } m \in \mathbb{N}, \epsilon > 0, \exists C_m > 0 \\
\text{such that } \forall i + j + k \leq m, X_i \in \odot^j TB, \\
|(|\nabla^E\hat{\partial}^k|^j(\hat{\partial}^k)^k \psi(X_l)| \leq C_m e^{-\epsilon d} \right\} \]

The convolution product structure on $\Psi_\infty(M \times_B M, E)$ is defined by

\[ \psi_1 \circ \psi_2(x, y, z) := \int_{Z_x} \psi_1(x, y, w) \psi_2(x, w, z) \mu_z(w). \]

We introduce a Sobolev type generalization of the Hilbert-Schmidt norm on $\Psi_\infty(M \times_B M, E)^G$, the space of $G$-invariant kernels. Fix a smooth compactly supported function $\chi \in C_c^\infty(M)$, such that

\[ \sum_{g \in G} g^* \chi = 1. \]

For any $G$-invariant $\psi \in \Psi_\infty(M \times_B M, E)^G$, recall that the standard trace of $\psi$ is

\[ \text{tr}_\psi(\psi)(x) := \int_{x \in Z_x} \chi(x, z) \text{tr}(\psi(x, z, z)) \mu_z(z) \in C^\infty(B). \]
Theorem 1.12. Given any fixed co-compact subset $U \subseteq Z$, one has the estimates for any $(x, z) \in B_\alpha \times U$:

$$
\int_{y \in Z_x} |\nabla^\alpha A\psi| (x, y, z) \mu_x(y) 
$$
\[
\begin{align*}
&\leq (C_{1,1} ||A||^2_{op,1} + C_{1,0} ||A||^2_{op,0}) \int_{y \in Z_x} |\tilde{\nabla}^\alpha \psi^\alpha|^2 + |\hat{\partial}^\alpha \psi^\alpha|^2 + |\psi^\alpha|^2 \mu_x(y) \\
&\int_{y \in Z_x} |\hat{\partial}^\alpha \hat{A} \psi^\alpha|^2(x, y, z) \mu_x(y) \\
&\leq (C_{1,1} ||A||^2_{op,1} + C_{1,0} ||A||^2_{op,0}) \int_{y \in Z_x} |\tilde{\nabla}^\alpha \psi^\alpha|^2 + |\hat{\partial}^\alpha \psi^\alpha|^2 + |\psi^\alpha|^2 \mu_x(y) \\
&\int_{y \in Z_x} |\hat{\partial}^\alpha \hat{A} \psi^\alpha|^2(x, y, z) \mu_x(y) \leq ||A||^2_{op,0} \int_{y \in Z_x} |\tilde{\nabla}^\alpha \psi^\alpha|^2 \mu_x(y),
\end{align*}
\]
for some constants $C_{1,1}, C_{0,0} > 0$. Note that here, $Z_x \subset M_\alpha \times \{z\}$.

**Proof.** Let $Z = \bigcup \lambda Z_\lambda$ be a locally finite cover. Then $U \subseteq \bigcup \lambda Z_\lambda$ for some finite sub-cover. Without loss of generality we may assume $E|_{Z_\lambda}$ are all trivial. For each $\lambda$ fix an orthonormal basis $\{(e^\alpha_r)^\lambda\}$ of $E|_{B_\alpha \times Z_\lambda}$.

Consider $\tilde{\nabla}^\alpha(\hat{A} \psi^\alpha)(X_0, \cdots, X_k; Y_1, \cdots, Y_l, Z_1, \cdots, Z_r)$. It suffices to restrict to the case when $Y_j, Z_j$ are respectively vector fields on $M_\alpha$ and $Z$, lifted to $M_\alpha \times Z$.

Then for any $|X_0| = \cdots = |X_k| = 1$,

\[
\tilde{\nabla}^\alpha(\hat{A} \psi^\alpha)(X_0, \cdots, X_k; Y_1, \cdots, Y_l, Z_1, \cdots, Z_r)|_{M_\alpha \times Z_\lambda} = \sum_{r} (\tilde{\nabla}^\alpha_{X_0} A(u^\lambda_r|_{M_\alpha \times Z_\lambda})(X_1, \cdots, X_k)(x, y)) \otimes s^* e^\lambda_r + (A u^\lambda_r) \otimes s^*(\tilde{\nabla}^\alpha_{X_0} e^\lambda_r).
\]

Taking point-wise norm and integrating, we get for any $(x, y, z) \in M_\alpha \times Z_\lambda$:

\[
\begin{align*}
&\int |\tilde{\nabla}^\alpha(\hat{A} \psi^\alpha)(X_0, \cdots, X_k; Y_1, \cdots, Y_l, Z_1, \cdots, Z_r)|_{M_\alpha \times Z_\lambda}^2(x, y, z) \mu_x(y) \\
&\leq 2 \text{rk}(E) \sum_{r} \int |(\tilde{\nabla}^\alpha_{X_0} A(u^\lambda_r|_{M_\alpha \times Z_\lambda})(X_1, \cdots, X_k)(x, y)) \otimes s^* e^\lambda_r|^2 \\
&\quad + \|A u^\lambda_r\| \otimes s^*(\tilde{\nabla}^\alpha_{X_0} e^\lambda_r)|^2 \mu_x(y) \\
&\leq 2 \text{rk}(E) \sum_{r} \left( ||A||^2_{op,1} \int |\tilde{\nabla}^\alpha_{X_0} u^\lambda_r|_{M_\alpha \times Z_\lambda}^2(X_0, \cdots, X_k)(x, y) \\
&\quad + ||\hat{\partial}^\alpha u^\lambda_r|_{M_\alpha \times Z_\lambda}^2 + |u^\lambda_r|^2 \right)(X_1, \cdots, X_k)(x, y) \mu_x(y) \\
&\quad + (\sup |\nabla^\alpha_{X_0} e^\lambda_r|^2) ||A||^2_{op,0} \int |u^\lambda_r|_{M_\alpha \times Z_\lambda}^2 \mu_x(y) \right) \\
&\leq 2 \text{rk}(E) ||A||^2_{op,1} \int \sum_{r} \left( ||\tilde{\nabla}^\alpha_{X_0} (u^\lambda_r|_{M_\alpha \times Z_\lambda}) \otimes s^* e^\lambda_r|^2(X_0, \cdots, X_k) \\
&\quad + ||\hat{\partial}^\alpha u^\lambda_r|_{M_\alpha \times Z_\lambda} \otimes s^* e^\lambda_r|^2 + |u^\lambda_r|^2 \right)(X_1, \cdots, X_k) \mu_x(y) \\
&\quad + 2 \text{rk}(E)(\sum \sup |\nabla^\alpha_{X_0} e^\lambda_r|^2) ||A||^2_{op,0} \int \sum_{r} |u^\lambda_r|_{M_\alpha \times Z_\lambda} \otimes s^* e^\lambda_r|^2 \mu_x(y) \right) \\
&\leq 4 \text{rk}(E) ||A||^2_{op,1} \int |\tilde{\nabla}^\alpha \psi^\alpha|^2(X_0, \cdots, X_k) + ||\hat{\partial}^\alpha \psi^\alpha|^2 + |\psi^\alpha|^2 \right)(X_1, \cdots, X_k) 
\end{align*}
\]
\[ + \sum_r |u_r|^2 (\sup \|\nabla^E e_r^\alpha\|^2) \mu_x(y) \]
\[ + 2 \text{rk}(E)(\sum_r \sup \|\nabla^E e_r^\alpha\|^2)\|A\|_{op}^2 \int |\psi^\alpha|\mu_x(y), \]
where in the last line we used
\[ \nabla^E_x \otimes \text{id}(\psi^\alpha) = \sum_r (\nabla^E_x u_r^\alpha) \otimes s^r e_r^\alpha, \]
and \( \{s^r e_r^\alpha\} \) is an orthonormal basis of \( s^r E' \). The first inequality follows after obvious rearrangements and taking supremum over all patches. Using the same arguments with \( \tilde{\nabla}^\alpha \) in place of \( \nabla^\alpha \), one gets the second inequality.

As for the last inequality, since \( t^{-1}E |_{M_\alpha \times \{z\}} \) and the connection \( (x_\alpha^{-1})^* \nabla^s t^{-1}E \) is trivial along \( \exp tZ_0 \), one can write
\[ \nabla_{Z_0}\tilde{A}u \otimes s^r e = \tilde{A}\left( \int_{t=0}^{t} u_{M_\alpha \times \{exp tZ\}} \otimes s^r e + u \otimes \nabla_{Z_0} s^r E' s^r e \right). \]
It follows that
\[ \tilde{\nabla}^Z \tilde{A} \psi^\alpha = \tilde{A}(\tilde{\nabla}^Z \psi^\alpha), \]
and from which the third inequality follows. \( \square \)

Using Equations (11), (12), (13), Theorem 1.12 and the observation that the tensors are bounded on compact subsets, it follows that estimates similar to Theorem 1.12 holds for the operators \( \nabla^E, \tilde{\nabla}, \hat{\nabla} \) for any smooth bounded operator \( A \). Integrating with respect to \( z \in Z \) (note that the domain of integration is compact), one arrives at

**Corollary 1.13.** For any smooth bounded \( G \)-invariant operators \( A, \psi \in \Psi^{-\infty}(M \times_B M)^G \), \( A \psi \in \Psi^{-\infty}(M \times_B M)^G \) and one has
\[ \|\tilde{A} \psi\|_{\text{HS}1} \leq (C_{1,1} \|A\|_{op} + C_{1,0} \|A\|_{op,0}) \|\psi\|_{\text{HS}1}, \]
for some constants \( C_{1,1}, C_{1,0} > 0 \).

Clearly, the arguments leading to Corollary 1.13 can be repeated, and we obtain:

**Corollary 1.14.** For any smooth bounded operator \( A, m = 0, 1, \cdots \),
\[ \|\tilde{A} \psi\|_{\text{HS}m} \leq \left( \sum_{0 \leq l \leq m} C_{m,l} \|A\|_{op,l} \right) \|\psi\|_{\text{HS}m}, \]
for some constants \( C_{m,l} > 0 \).

2. **Large Time Behavior of the Heat Kernel**

2.1. **Example: The Bismut super-connection.** Let \( M \rightarrow B \) be a fiber bundle with a \( G \) action, and \( TM = H \otimes V \) be a \( G \)-invariant splitting, as in the last section. We shall further assume the metric on \( H \cong \pi^{-1}TB \) is given by the pull back some Riemannian metric on \( B \). In other words, \( V \) is a Riemannian foliation.

Let \( E \rightarrow M \) be a flat, contravariant \( G \)-vector bundle, and \( \nabla \) be a flat connection on \( E \).
Definition 2.1. Let \( \Theta \) be the \( V \)-valued horizontal 2-form defined by
\[
\Theta(X_1^H, X_2^H) := -P_V[X_1^H, X_2^H], \quad \forall X_1, X_2 \in \Gamma^\infty(TB),
\]
where \( P_V \) denotes the projection onto \( V \). Define \( \iota_\Theta \) to be the contraction with \( \Theta \).

Since the vertical distribution \( V \) is integrable, the deRham differential \( d_V \) along \( V \) is well defined.

**Definition 2.2.** A standard flat Bismut super-connection is an operator of the form
\[
d := d_V + \nabla^\wedge V'_\otimes E_\circ + \iota_\Theta.
\]

**Remark 2.3.** The operator \( d \) is just the deRham differential. However, the grading, the identification \( \wedge^\bullet H' \otimes \wedge^\bullet V' \otimes E \cong \wedge^\bullet T^* M \otimes E \), and the splitting into Bismut super-connection is not canonical.

Let \( \zeta_H, \zeta_V \) be respectively the Hodge star operators on \( \wedge^\bullet H' \) and \( \wedge^\bullet V' \). Define the zeroth order operator
\[
R(\omega \otimes \eta \otimes s) := \sum_i \omega \wedge \zeta_i \otimes \left[ \nabla^V_{X_i} \otimes E_\circ, \zeta_V \otimes \text{id}_E \right](\zeta_V \eta) \otimes s,
\]
for any sections \( \omega, \eta, s \) of \( \wedge^\bullet H', \wedge^\bullet V', E \) respectively, where \( \{X_i\} \) is any local orthonormal frame of \( H \) and \( \{\zeta_i\} \) is dual. It is straightforward to verify:

**Lemma 2.4.** [8 Proposition 3.1] One has
\[
\nabla^\wedge V'_\otimes E_\circ d_V + d_V \nabla^\wedge V'_\otimes E_\circ = 0
\]
\[
\nabla^\wedge V'_\otimes E_\circ d_V^* + d_V^* \nabla^\wedge V'_\otimes E_\circ = Rd_V + d_V^* R.
\]

### 2.2. The regularity result of Alvarez Lopez and Kordyukov.

As in [8], we make the more general assumption that there exists a transversally elliptic uniformly bounded first order differential operator \( Q \), and zero degree operators \( R_1, R_2, R_3, R_4 \), all \( G \)-invariant, such that
\[
(15) \quad Qd_V + d_V Q = R_1 d_V + d_V R_2
\]
\[
Qd_V^* + d_V^* Q = R_3 d_V^* + d_V^* R_4.
\]

Clearly, in our example, the operators considered in Lemma 2.4 satisfy Equation (15).

Write \( D_0 := d_V + d_V^* \), \( \Delta := D_0^2 \), and denote by \( \Pi_0, \Pi_{d_V}, \Pi_{d_V^*} \) respectively the orthogonal projection onto \( \text{Ker}(\Delta) \), \( \text{Rg}(d_V) \), \( \text{Rg}(d_V^*) \).

In this section, we shall consider the operators
\[
B_1 := R_1 \Pi_{d_V} + R_3 \Pi_{d_V^*}
\]
\[
B_2 := \Pi_{d_V} R_2 + \Pi_{d_V} R_4
\]
\[
B := B_2 \Pi_0 + B_1 (\text{id} - \Pi_0).
\]

We recall some elementary formulas regarding these operators from [8]:

Lemma 2.5. [S] Lemma 2.2] One has

\[ Qd_V + d_V Q = B_1d_V + d_V B_2 \]
\[ Qd^*_V + d^*_V Q = B_1^*d^*_V + d^*_V B_2 \]
\[ [Q, \Delta] = B_1\Delta - \Delta B_2 - D_0(B_1 - B_2)D_0. \]

Here, recall [S Theorem 2.2] that, the projection operator \( \Pi_0 \) is represented by a smooth kernel. Moreover, since \( \Pi_0 = \Pi_0^2 \), one has

\[ \|\Pi_0\|^2_{HS, 0} = \int \chi(x, z) \text{tr}(\Pi_0(x, z)) \mu_x(y) < \infty, \]

therefore \( \Pi_0 \in \tilde{\Psi}^{-\infty}_0(M \times B M, \wedge^* V' \otimes E). \) One can furthermore estimate the derivatives of \( \Pi_0 \). First, recall that

Lemma 2.6. One has [S Corollary 2.8]

\[ [Q + B, \Pi_0] = 0. \]

Proof. Here we give a different proof. From definition we have

\[ B = (\Pi_{d^*} R_2 + \Pi_d R_4)\Pi_0 + R_1\Pi_d + R_3\Pi_{d^*}, \]

where we used \( \Pi_d \Pi_0 = \Pi_{d^*} \Pi_0 = 0 \). Hence

\[ B\Pi_0 = \Pi_0 B = (\Pi_{d^*} R_2 + \Pi_d R_4)\Pi_0 - \Pi_0 R_1\Pi_d - \Pi_0 R_3\Pi_{d^*}. \]

For any \( s \) one has

\[ \Pi_{ds} \rightarrow \lim_{n \rightarrow \infty} d\tilde{s}_n, \]

for some sequence \( \tilde{s}_n \) (in some suitable function spaces). It follows that

\[ \Pi_0 R_1 \Pi_{ds} = \lim_{n \rightarrow \infty} \Pi_0 R_1 d\tilde{s}_1 \]
\[ = \lim_{n \rightarrow \infty} \Pi_0 (Qd + dQ - dR_2)\tilde{s}_1 = \Pi_0 Q \Pi_{ds}. \]

Similarly, one has \( \Pi_0 R_3 \Pi_{d^*} = \Pi_0 Q \Pi_{d^*} \) and by considering the adjoint \( \Pi_{d^*} R_2 \Pi_0 = \Pi_{d^*} Q \Pi_0 \), and \( \Pi_{d^*} R_4 \Pi_0 = \Pi_{d^*} Q \Pi_0 \). It follows that

\[ [Q + B, \Pi_0] = (\text{id} - \Pi_{d^*} - \Pi_d)Q \Pi_0 - \Pi_0 Q (\text{id} - \Pi_{d^*} - \Pi_d) = 0. \]

In other words, \( \|[Q, \Pi_0]\|_{HS_m} = \|[B, \Pi_0]\|_{HS_m}, \) provided the right hand side is finite. Hence, using elliptic regularity, one can prove inductively that

\[ \Pi_0 \in \tilde{\Psi}^{-\infty}_m(M \times B M, \wedge^* V' \otimes E), \quad \forall m. \]

Next, we recall the main result of [S]

Lemma 2.7. For any \( m = 0, 1, \cdots \),

1. The heat kernel \( e^{-t\Delta} \), and the operators \( D_0 e^{-t\Delta}, \Delta e^{-t\Delta} \) map \( W^m(E) \) to itself as a bounded operator, moreover, there exist constants \( C^0_m, C^1_m, C^2_m > 0 \) such that

\[ \|e^{-t\Delta}\|_{op, m} \leq C^0_m \]
\[ \|D_0 e^{-t\Delta}\|_{op, m} \leq t^{-\frac{1}{2}} C^1_m \]
\[ \|\Delta e^{-t\Delta}\|_{op, m} \leq t^{-1} C^2_m, \]

for all \( t > 0. \)
(2) As \( t \to \infty \), \( e^{-t\Delta} \) strongly converges as an operator on \( \mathcal{W}^m(E) \), moreover \( (t,s) \mapsto e^{-t\Delta}s \) is a continuous map form \([0, \infty) \times \mathcal{W}^m(E)\) to \( \mathcal{W}^m(E) \);

(3) One has the Hodge decomposition

\[
\mathcal{W}^m(E) = \text{Ker}(\Delta) + \text{Rg}(\Delta) = \text{Ker}(D_0) + \text{Rg}(D_0),
\]

where the kernel, image and closure are in \( \mathcal{W}^m(E) \).

Note that the operator norm \( \| \cdot \|_{op,m} \) is different from that of \([8]\), but the same arguments apply.

We recall more results in \([8, \text{Section 2}]\).

**Lemma 2.8.** For any first order differential operator \( A \), one has the Duhamel type formula

\[
[A, e^{-t\Delta}] = -\int_0^t e^{-(t-t')\Delta}[A, \Delta] e^{-t'\Delta} dt'.
\]

In particular if \( A \) is uniformly bounded, then \( [A, e^{-t\Delta}] \in \Psi^{-\infty}_\infty(M \times_B M, E) \).

For the proof of Equation (16), see [6] (and observe that \([A, \Delta] \) is a fiber-wise differential operator, which is uniformly bounded provided \( A \) is uniformly bounded. Therefore the off-diagonal estimate for the heat kernel implies \( [A, e^{-t\Delta}] \in \Psi^{-\infty}_\infty(M \times_B M, E) \).

**Lemma 2.9.** \([8, \text{Lemma 2.4}]\) For all \( m \)

\[
\| [Q, e^{-t\Delta}] \|_{op,m} \leq C_m^3,
\]

for some constants \( C_m^3 > 0 \).

**Proof.** Using the third equation of Lemma 2.5, Equation (16) becomes

\[
[Q, e^{-t\Delta}] = \int_0^t e^{-(t-t')\Delta} D_0(B_1 - B_2) D_0 e^{-t'\Delta} dt' - \int_0^t e^{-(t-t')\Delta}(B_1 \Delta - \Delta B_2) e^{-t'\Delta} dt'.
\]

Using Lemma 2.7, we estimate the first integral

\[
\left\| \int_0^t e^{-(t-t')\Delta} D_0(B_1 - B_2) D_0 e^{-t'\Delta} dt' \right\|_{op,m} \leq \| B_1 - B_2 \|_{op,m}(C_m^3)^2 \int_0^t \frac{dt'}{(t-t')^{\frac{1}{2}}}
\]

\[
= \| B_1 - B_2 \|_{op,m}(C_m^3)^2 \pi.
\]

As for the second integral, we split the domain of integration into \([0, \frac{t}{2}]\) and \([\frac{t}{2}, t]\), and then integrate by part to get

\[
\int_0^t e^{-(t-t')\Delta}(B_1 \Delta - \Delta B_2) e^{-t'\Delta} dt' = \int_0^{\frac{t}{2}} e^{-(t-t')\Delta} \Delta(-B_1 - B_2) e^{-t'\Delta} dt'
\]

\[
- \int_{\frac{t}{2}}^t e^{-(t-t')\Delta}(B_1 - B_2) \Delta e^{-t'\Delta} dt'
\]

\[
+ e^{-(t-t')\Delta} B_1 e^{-t'\Delta} \bigg|_{t'=\frac{t}{2}} - e^{-(t-t')\Delta} B_2 e^{-t'\Delta} \bigg|_{t'=\frac{t}{2}}.
\]
Again using Lemma 2.5 its $\| \cdot \|_{op_m}$-norm is bounded by

$$C_m^0 C_m^1 (\| B_1 \|_{op_m} + \| B_2 \|_{op_m}) \left( \int_{t_0}^{t} \frac{dt'}{t - t'} + \int_{t_0}^{t} \frac{dt'}{t'} \right) + C_m^0 (C_m^{\alpha} + 1) (\| B_1 \|_{op_m} + \| B_2 \|_{op_m}),$$

which is uniformly bounded because $\int_{0}^{1} \frac{dt}{t} = \int_{\frac{t}{2}}^{1} \frac{dt}{t} = \log 2$.

**Definition 2.10.** We say that $M \rightarrow B$ has positive Novikov-Shubin invariant if there exists $\gamma > 0$ such that for sufficiently large $t$,

$$\|e^{-t\Delta} - \Pi_0\|_{HS 0} \leq C_0 t^{-\gamma}$$

for some constant $C_0 > 0$.

**Remark 2.11.** Since $e^{-\frac{t}{2}\Delta} - \Pi_0$ is non-negative, self adjoint and $(e^{-\frac{t}{2}\Delta} - \Pi_0)^2 = e^{-t\Delta} - \Pi_0$, one has

$$\|e^{-\frac{t}{2}\Delta} - \Pi_0\|_{HS 0} = \|e^{-t\Delta} - \Pi_0\|_{\tau},$$

where $\| \cdot \|_{\tau}$ is defined in [1]. Hence our definition of having positive Novikov-Shubin is equivalent to that of [1]. Our arguments here is similar to the proof of [4, Theorem 7.7].

Lemma 2.8 suggests $[Q + B, e^{-t\Delta}]$ converges to zero. Indeed, we shall prove a stronger result, namely, $[Q + B, e^{-t\Delta}]$ decay polynomially in the $\| \cdot \|_{HS_m}$-norm for all $m$.

**Lemma 2.12.** Suppose there exists $\gamma > 0$ such that $\|e^{-t\Delta}\|_{HS_m} \leq C_m (t^{-\gamma})$, then there exists $C'_m > 0$ such that

$$\|[Q + B, e^{-t\Delta}]\|_{HS_m} = \|[Q + B, e^{-t\Delta} - \Pi_0]\|_{HS_m} \leq C'_m t^{-\gamma m}.$$

**Proof.** We follow the proof of [8, Lemma 2.6]. By Lemma 2.5 we get

$$[Q + B, \Delta] = (\Delta (B_1 + B_2) + D_0(B_1 - B_2) D_0)(id - \Pi_0),$$

it follows that

$$\Pi_0 [Q + B, e^{-\frac{t}{2}\Delta}] = [Q + B, e^{-\frac{t}{2}\Delta}] \Pi_0 = 0.$$

Write

$$[Q + B, e^{-t\Delta}] = [Q + B, e^{-\frac{t}{2}\Delta}] e^{-\frac{t}{2}\Delta} + [Q + B, e^{-\frac{t}{2}\Delta}] [Q + B, e^{-\frac{t}{2}\Delta}]$$

$$= [Q + B, e^{-\frac{t}{2}\Delta}] (e^{-\frac{t}{2}\Delta} - \Pi_0) + (e^{-\frac{t}{2}\Delta} - \Pi_0) [Q + B, e^{-\frac{t}{2}\Delta}].$$

Taking $\| \cdot \|_{HS_m}$ and using Corollary 1.14, Lemma 2.9 the claim follows.

**Remark 2.13.** We regard $\Delta$ as an operator on $\wedge^* H_0 \otimes \wedge^* V_0 \otimes E_0$ that is $\Gamma^\infty (\wedge^* H)$-linear. One may also regard $\Delta$ as an operator on $\wedge^* V_0 \otimes E_0$, that preserves the grading. Clearly, the trace norms, and hence of notion of Novikov-Shubin invariant are equivalent.

**Theorem 2.14.** Suppose $\|e^{-t\Delta} - \Pi_0\|_{HS 0} \leq C_0 t^{-\gamma}$ for some $\gamma > 0, C_0 > 0$. Then for any $m$, there exists $C_m'' > 0$ such that

$$\|e^{-t\Delta} - \Pi_0\|_{HS_m} \leq C_m'' t^{-\gamma}, \quad \forall t > 1.$$
Proof. We prove the theorem by induction. The case \( m = 0 \) is given. Suppose that for some \( m \), \( \|e^{-t\Delta} - \Pi_0\|_{HS^m} \leq C_m t^{-\gamma} \). Consider \( \|e^{-t\Delta} - \Pi_0\|_{HS^{m+1}} \).

Since \( Q \) is a first order differential operator, for any kernel \( \psi \in \Psi^\infty(M \times B M, E)^G \), \( [Q, \psi] \) is also a kernel lying in \( \Psi^\infty(M \times B M, E)^G \), that is in particular given by a composition of the covariant derivatives \( \hat{\nabla}^E, \hat{\partial}, \hat{\partial}^t \) and some tensors acting on \( \psi \). Since \( \|\psi\|_{HS^m} \) is by definition the \( \| \cdot \|_{HS^0} \) norm of the \( m \)-th derivatives of \( \psi \), elliptic regularity implies

\[
\|\psi\|_{HS^{m+1}} \leq \tilde{C}_m (\|\psi\|_{HS^m} + \|D_0\psi\|_{HS^m} + \|\psi D_0\|_{HS^m} + \|[Q, \psi]\|_{HS^m}),
\]

for some constants \( \tilde{C}_m > 0 \). Put \( \psi = e^{-t\Delta} - \Pi_0 \). The theorem then follows from the estimates

\[
\|D_0(e^{-t\Delta} - \Pi_0)\|_{HS^m} \leq \sum_{0 \leq l \leq m} C_{m,l}' \|D_0(e^{-\frac{t}{2}\Delta} - \Pi_0)\|_{op,l} \|e^{-\frac{t}{2}\Delta} - \Pi_0\|_{HS^m} \\
\leq \sum_{0 \leq l \leq m} C_m C_l (t^2)^{-\frac{1}{2}} C_m t^{-\gamma}
\]

\[
\|[Q, e^{-t\Delta} - \Pi_0]\|_{HS^m} \leq \|[Q + B, e^{-t\Delta} - \Pi_0]\|_{HS^m} + \|[B, e^{-t\Delta} - \Pi_0]\|_{HS^m} \\
\leq C_m t^{-\gamma} + 2 \sum_{0 \leq l \leq m} C_m t^{l \gamma}.
\]

Note that we used Lemma \[2.12\] for the last inequality. \( \square \)

3. Sobolev Convergence

Let \( \nabla^E \) be a flat connection on \( E \). Define the grading operators on \( \wedge^* H' \otimes \wedge^t V' \otimes E \)

\[
\begin{align*}
\mathcal{N}\Omega_{|\wedge^* H' \otimes \wedge^t V' \otimes E} & := q, \\
\mathcal{N}_{|\wedge^* H' \otimes \wedge^t V' \otimes E} & := q'.
\end{align*}
\]

In this section, we consider the rescaled Bismut super-connection \[2, \text{Chapter 9.1}\]

\[
\hat{\partial}(t) := \frac{t^2}{2} t^{-\frac{\gamma p}{2}} (d + d^*) t^{-\frac{\gamma p}{2}} = \frac{1}{2} \left( t^\frac{1}{2} (dv + d^v) + (\nabla^E_v + (\nabla^E_v)^*) + t^{\frac{1}{2}} (\Lambda \Theta - \iota \Theta) \right).
\]

Denote

\[
D_0 := -\frac{1}{2} (dv - d^v), \quad \Omega_t := -\frac{1}{2} ((\nabla^E_v - (\nabla^E_v)^*) - t^{\frac{1}{2}} (\Lambda \Theta - \iota \Theta)) \quad D(t) := t^\frac{1}{2} D_0 + \Omega_t.
\]

The curvature of \( \hat{\partial}(t) \) can be expanded in the form:

\[
\hat{\partial}(t)^2 = -D(t)^2 = t \Delta + t^\frac{1}{2} \Omega_t D_0 + t^\frac{1}{2} D_0 \Omega_t + \Omega_t^2.
\]

**Definition 3.1.** The heat kernel \( e^{-D(t)^2} \) is defined by Duhamel’s expansion:

\[
e^{-\hat{\partial}(t)^2} = e^{D(t)^2} := e^{-t\Delta} + \sum_{n=1}^{\dim B} \int_{(r_0, \ldots, r_k) \in \Sigma^n} e^{-r_0 t \Delta} (t^\frac{1}{2} \Omega_t D_0 + t^\frac{1}{2} D_0 \Omega_t + \Omega_t^2) e^{-r_1 t \Delta} \\
\cdots (t^\frac{1}{2} \Omega_t D_0 + t^\frac{1}{2} D_0 \Omega_t + \Omega_t^2) e^{-r_n t \Delta} d\Sigma^n
\]

where \( \Sigma^n := \{(r_0, r_1 \cdots, r_n) \in [0,1]^{n+1} : r_0 + \cdots + r_n = 1\} \).
We then follow [1] Section 4] to estimate the Hilbert-Schmit norms of $e^{-\vartheta(t)^2}$. Using Corollary [1,14] and Theorem [2,14] we observe that the arguments still hold if one replaces the operator and $\| \cdot \|_r$ norm respectively by $\sum_{0 \leq l < m} C''_{m,k} \| \cdot \|_{opt}$ and $\| \cdot \|_{HS}$ for any $m$. We conclude with the following analogue of [1, Theorem 4.1]:

**Theorem 3.2.** For $k = 0, 1, 2$ and any $m \in \mathbb{N}$,

$$\lim_{t \to \infty} D(t)^k e^{-\vartheta(t)^2} = \Pi_0(\Omega \Pi_0)^k e^{(\Omega \Pi_0)^2}$$

in the $\| \cdot \|_{HS,m}$-norm, where $\Omega := -\frac{\nabla E_0 - (\nabla E_0)^*}{2}$. Moreover, as $t \to \infty$,

$$\| D(t)^k e^{-\vartheta(t)^2} - \Pi_0(\Omega \Pi_0)^k e^{(\Omega \Pi_0)^2} \|_{HS,m} = O(t^{-\gamma})$$

for some $\gamma' > 0$.

**Outline of proof.** Let $\gamma' := 1 - (1 + \frac{2\gamma}{\dim B + 2})^{-1}$, $\bar{r}(t) := t^{-\gamma'}$. Fix $\bar{t}$ such that $\bar{r}(\bar{t}) < (\dim B + 1)^{-1}$. Split the domain of integration $\Sigma_n = \bigcup_{I \neq \{0, \ldots, n\}} \Sigma^n_{\bar{r}, I}$, where

$$\Sigma^n_{\bar{r}, I} := \{(r_0, \ldots, r_n) : r_i \geq \bar{r}, \forall i \in I\}.$$ 

Define

$$K(t, n, I, c_0, \ldots, c_n; a_1, \ldots, a_n) := \int_{\Sigma^n_{\bar{r}, I}} (t^2 D_0)^c_0 e^{-r_0 t^2} \prod_{i=1}^n (\Theta^a_i (t^2 D_0)^{a_i} e^{-r_i t^2}) d\Sigma^n,$$

for $c = 0, 1, 2, a_j = 1, 2$. Then one has

$$e^{-\vartheta(t)^2} = e^{D(t)^2} = \sum K(t, n, I, c_0, \ldots, c_n; a_1, \ldots, a_n)$$

by grouping terms involving $D_0$ together.

By some elementary computations, one has the analogue of [1, Proposition 4.6, 4.7]:

$$\lim_{t \to \infty} K(t, n, c_0, \ldots, c_n; a_1, \ldots, a_n) = \begin{cases} \frac{1}{n!} \Pi_0 \Omega^{a_1} \Pi_0 \cdots \Pi_0 & \text{if } I = \emptyset, c_0 = \cdots = c_n = 0 \\ 0 & \text{otherwise} \end{cases}$$

in the $\| \cdot \|_{HS,m}$-norm. Moreover, for $t$ sufficiently large

$$\| K(t, n, c_0, \ldots, c_n; a_1, \ldots, a_n) - \lim_{t \to \infty} K(t, n, c_0, \ldots, c_n; a_1, \ldots, a_n) \|_{HS,m} \leq C \bar{s}(t)^{-\frac{\dim B}{2}} \left( \frac{\bar{s}(t) t}{2} \right)^{-\gamma} + \| \Pi_0 \|_{HS,m} \bar{s}(t)^{\frac{1}{2}}. \quad \square$$

3.1. Application: the $L^2$ analytic torsion form. Our main application of Theorem 3.2 is in establishing the smoothness and transgression formula of the $L^2$ analytic torsion form. Here, we briefly recall the definitions.

On $\wedge^* T^* M \otimes E \cong \wedge^* H' \otimes \wedge^* V' \otimes E$, define $\Theta_1, \Theta$ to be the degree operators of $\wedge^* H' \cong \pi^{-1}(\wedge^* T^* B)$ and $\wedge^* V'$ respectively.

Define

$$F^\wedge(t) := (2\pi \sqrt{-1})^{-\frac{\dim B}{2}} \text{str}_\psi(2^{-1} \bar{s}(1 + 2D(t)^2)e^{-\vartheta(t)^2}).$$

**Lemma 3.3.** One has the limits [3, Theorem 3.21]

$$F^\wedge(t) = \begin{cases} \frac{1}{2} \dim(Z) \text{rk}(E) \text{str}_\psi(\Pi_0) + O(t) & \text{if } \dim(Z) \text{ odd} \\ O(t^\frac{1}{2}) & \text{if } \dim(Z) \text{ even} \end{cases} \quad \text{as } t \to 0$$

$$F^\wedge(t) = \frac{1}{2} \text{str}_\psi(\bar{s} \Pi_0) \quad \text{as } t \to \infty.$$
Finally, define

**Definition 3.4.** The $\mathcal{L}^2$ analytic torsion form is defined to be

$$
\tau := \int_0^{\infty} -F^\wedge(t) + \frac{\str \Psi(\tau \Pi_0)}{2} + \left( \frac{\dim(Z) \rk(E)}{4} \str \Psi(\Pi_0) - \frac{\str \Psi(\tau \Pi_0)}{2} \right) (1-2t)e^{-t} dt.
$$

The integral is well defined since by Theorem 3.2 the integrand is $O(t^{-\gamma-1})$.

Moreover,

**Theorem 3.5.** The form $\tau$ is smooth, i.e. $\tau \in \Gamma^\infty(\wedge^\bullet T^*B)$.

**Proof.** Using [2, Proposition 9.24], the derivatives of the $t$-integrand are bounded as $t \to 0$. It follows its integral over $[0, 1]$ is smooth.

We turn to study the large time behavior. Consider $\str(2^{-1}\tau(e^{-\bar{\theta}(t)^2} - \Pi_0))$. Using the semi-group property, we can write

$$
e^{-\bar{\theta}(t)^2} = 2^{-\frac{\dim(Z)}{2}} e^{-\bar{\theta}(\frac{t}{2})^2} e^{-\bar{\theta}(\frac{t}{2})^2} 2^{-\frac{\dim(Z)}{2}}.
$$

Also, since $\str(\tau \Pi_0(\tau \Pi_0)^{2j}) = \str((\tau \Pi_0(\tau \Pi_0), \Pi_0(\tau \Pi_0)^{2j-1})) = 0$ for any $j \geq 1$ one has

$$
\str(\tau \Pi_0) = \str(\tau \Pi_0(e^{(\tau \Pi_0)^2} \Pi_0 e^{(\tau \Pi_0)^2}).
$$

Therefore

$$
\str(2^{-1}\tau(e^{-\bar{\theta}(t)^2} - \Pi_0)) = \str(2^{-1}\tau(e^{-\bar{\theta}(\frac{t}{2})^2} - \Pi_0 e^{(\tau \Pi_0)^2}))
= \str \left( 2^{-1}\tau e^{-\bar{\theta}(\frac{t}{2})^2} (e^{-\bar{\theta}(\frac{t}{2})^2} - \Pi_0 e^{(\tau \Pi_0)^2}) \right)
+ \str \left( 2^{-1}\tau e^{-\bar{\theta}(\frac{t}{2})^2} - \Pi_0 e^{(\tau \Pi_0)^2} \Pi_0 e^{(\tau \Pi_0)^2} \right).
$$

Now consider $\str \Psi \left( 2^{-1}\tau \Pi_0 e^{-\bar{\theta}(\frac{t}{2})^2} (e^{-\bar{\theta}(\frac{t}{2})^2} - \Pi_0 e^{(\tau \Pi_0)^2}) \right)$. To shorten notations, denote $G := 2^{-1}\tau \Pi_0 e^{-\bar{\theta}(\frac{t}{2})^2} (e^{-\bar{\theta}(\frac{t}{2})^2} - \Pi_0 e^{(\tau \Pi_0)^2})$. Writing $G$ as a convolution product, it then follows that

$$
\left| \int_{\mathcal{Z}} \chi(x, z) \str(G(x, z, z)) \mu_x(z) \right|
= \int_{\mathcal{Z}} \chi \left( \frac{\dim(Z)}{2} \int_{\mathcal{Z}} e^{-\bar{\theta}(\frac{t}{2})^2} (x, y, z) (e^{-\bar{\theta}(\frac{t}{2})^2} - \Pi_0 e^{(\tau \Pi_0)^2}) \mu_x(y) \right) \mu_x(z)
\leq \frac{\dim(Z)}{2} \| e^{-\bar{\theta}(\frac{t}{2})^2} \|_{H^0} \| e^{-\bar{\theta}(\frac{t}{2})^2} - \Pi_0 e^{(\tau \Pi_0)^2} \|_{H^0},
$$

where we used the Cauchy-Schwarz inequality twice. Since $\| e^{-\bar{\theta}(\frac{t}{2})^2} \|_{H^0}$ is bounded for $t$ large (by triangle inequality), the expression above is $O(t^{-\gamma})$.

We turn to estimate its derivatives. For any vector field $X$ on $B$,

$$
\nabla_X^TB \str \Psi(G) = \int (L_X^H \chi(x, z)) \str(G(x, z, z)) \mu_x(z)
+ \int \chi(x, z) (\nabla_X^{-1}L^TB \str \Psi(G(x, z, z))) \mu_x(z)
+ \int \chi(x, z) \str(G(x, z, z)) (L_X^H \mu_x(z)).
$$
Differentiating under the integral sign is valid because we knew a-priori the integrands are all $\mathcal{L}^1$. Since $L_{X^H}\mu_x(z)$ equals $\mu_x(z)$ multiplied by some bounded functions, it follows that the last term $\int \chi(x,z)\str(G(x,z,z))(L_{X^H}\mu_x(z))$ is $O(t^{-\gamma})$.

For the first term, we write $L_{X^H}\chi(x,z) = \sum_{g \in G}(g^*\chi)(x,z)(L_{X^H}\chi)(x,z)$. The sum is finite because $L_{X^H}$ is compactly supported. By $G$-invariance,

$$\int (g^*\chi)(x,z)\str(G(x,z,z))\mu_x(z) = \int \chi(x,z)\str(G(x,z,z))\mu_x(z).$$

Since $(L_{X^H}\chi)(x,z)$ is bounded, it follows that $\int (L_{X^H}\chi)(x,z)\str(G(x,z,z))\mu_x(z)$ is also $O(t^{-\gamma})$.

As for the second term, we differentiate under the integral sign, then use the Leibniz rule to get

$$|\nabla_{X^H}^{\pi^{-1}}\str(G(x,z,z))|\leq \dim Z\left(\int_{Z_x} |\nabla_{X^H}^{\pi^{-1}}\str(G(x,z,z))|\right)_{\mathcal{L}^1} e^{-\tilde{\alpha}(\frac{t}{2})^2}(x,z,y),$$

$$+ \int_{Z_x} e^{-\tilde{\alpha}(\frac{t}{2})^2}(x,z,y)\nabla_{X^H}^{\pi^{-1}}\str(G(x,z,z))\mu_x(y)$$

$$+ \int_{Z_x} e^{-\tilde{\alpha}(\frac{t}{2})^2}(x,z,y)\str(G(x,z,z))\mu_x(y)$$

$$\leq \dim Z\left(\int_{Z_x} \left|\nabla_{X^H}^{\pi^{-1}}\str(G(x,z,z))\right|\right)_{\mathcal{L}^1} e^{-\tilde{\alpha}(\frac{t}{2})^2}(x,z,y),$$

$$+ \int_{Z_x} e^{-\tilde{\alpha}(\frac{t}{2})^2}(x,z,y)\str(G(x,z,z))\mu_x(y)$$

$$= O(t^{-\gamma}).$$

Clearly the above arguments can be repeated and one concludes that all derivatives of $\str_{\Psi}(G)$ are $O(t^{-\gamma})$.

By exactly the same arguments, we have as $t \to \infty$,

$$\str_{\Psi}\left(2^{-1}\mathcal{N}(e^{-\tilde{\alpha}(\frac{t}{2})^2} - \Pi_0 e^{(\Omega\Pi_0)^2})\right) = O(t^{-\gamma}).$$

As for $\str_{\Psi}(2^{-1}\mathcal{N}(D(t)^2 e^{-\tilde{\alpha}(t)^2}))$, one has $D(t)^2 = 2(2^{-\frac{\mathcal{N}}{2}} D(t)^2 2^{-\frac{\mathcal{N}}{2}})$. Therefore

$$\str_{\Psi}\left(\frac{\mathcal{N}}{2}(D(t)^2 e^{-\tilde{\alpha}(t)^2})\right) = \str_{\Psi}\left(\mathcal{N}(D(t)^2 e^{-\tilde{\alpha}(t)^2} - \Pi_0 (\Omega\Pi_0)^2) e^{(\Omega\Pi_0)^2} e^{(\Omega\Pi_0)^2})\right)$$

$$= \str_{\Psi}\left(\mathcal{N}(D(t)^2 e^{-\tilde{\alpha}(t)^2} - \Pi_0 (\Omega\Pi_0)^2) e^{(\Omega\Pi_0)^2} e^{-\tilde{\alpha}(t)^2})\right)$$

$$- \str_{\Psi}\left(\mathcal{N}(\Pi_0 (\Omega\Pi_0)^2) e^{(\Omega\Pi_0)^2} e^{-\tilde{\alpha}(t)^2})\right),$$

which is also $O(t^{-\gamma})$ as $t \to \infty$ by similar arguments.

Finally, since all derivatives of the $t$-integrand in Definition 4.4 is $\mathcal{L}^1$, derivatives of $\tau$ exist and equals differentiation under the $t$-integration sign. Hence we conclude that the torsion $\tau$ is smooth.
Remark 3.6. In the acyclic determinant class case, the analogue of Remark \[2.11\] reads
\[
\int_0^\infty ||e^{-t\Delta}\parallel_{HS_0}^2 \frac{dt}{t} = \int_0^\infty \parallel e^{-t\Delta}\parallel_{\pi} \frac{dt}{t} < \infty
\] (note that \(H_0 = 0\) by hypothesis). Unlike having positive Novikov-Shubin invariant, the heat kernel is not of determinant class in \(\parallel \cdot \parallel_{HS_0}\).

Given a power series \(f(x) = \sum a_j x^j\). For clarity, let \(h\) be the metric on \(\wedge^* V \otimes E\) we denote
\[
f(\nabla^{\wedge^* V'} \otimes E, h) := \text{str} \left( \sum_j a_j \left( \frac{1}{2} (\nabla^{\wedge^* V'} \otimes E - (\nabla^{\wedge^* V'} \otimes E^{*}))^j \right) \right) \in \Gamma^\infty(\wedge^* T^* M)
\]
\[
f(\nabla^{\wedge^* V'} \otimes E, h)_{H^* (Z, E)} := \text{str} \left( \sum_j a_j \left( \frac{1}{2} H_0 (\nabla^{\wedge^* V'} \otimes E - (\nabla^{\wedge^* V'} \otimes E^{*}))^j \right) \right) \in \Gamma^\infty(\wedge^* T^* B).
\]
Note that the summations are only up to \(\dim M\). Also, if \(\dim Z = 2n\) is even, define the Euler form
\[
e(TZ) := \text{tr} \left( \frac{1}{n!} \left( -\frac{RT_{Z_z}}{2\pi} \right)^n \right),
\]
where \(R^T_{Z_z}\) is the Riemannian curvature.

A classical argument \[3 \, 9, 1\] then gives:

Corollary 3.7. If \(\dim Z = 2n\) is even one has the transgression formula
\[
d\tau(x) = \int_{Z_x} \chi(x, z) e(TZ) f(\nabla^{\wedge^* V'} \otimes E) - f(\nabla^{\wedge^* V'} \otimes E)_{H^* (Z, E)},
\]
with \(f(x) = xe^{x^2}\).

Now let \(h_l\) be a family of \(G\)-invariant metrics on \(\wedge^* V \otimes E, l \in [0, 1]\). Define
\[
\tilde{f}(\nabla^{\wedge^* V'} \otimes E, h_l) := \int_0^1 (2\pi \sqrt{-1})^{-\frac{n_0}{2}} \text{str} \left( (h_l)^{-1} \frac{dh_l}{dl} f'(\nabla^{\wedge^* V'} \otimes E, h_l) \right) dl,
\]
and similarly for \(\tilde{f}(\nabla^{\wedge^* V'} \otimes E, h_l)_{H^* (Z, E)}\). Note that \(f'(\nabla^{\wedge^* V'} \otimes E, h_l)\) uses the adjoint connection with respect to \(h_l\).

One has an anomaly formula \[3 \, \text{Theorem 3.24}\].

Lemma 3.8. Modulo exact forms
\[
\tau_1 - \tau_0 = \int_{Z_x} \chi(x, z) \tilde{e}(TZ, h_l) f(\nabla^{\wedge^* V'} \otimes E, h_l) + \int_{Z_x} \chi(x, z) e(TZ, h_1) \tilde{f}(\nabla^{\wedge^* V'} \otimes E, h_l) - \tilde{f}(\nabla^{\wedge^* V'} \otimes E, h_l)_{H^*(Z, E)},\tag{17}
\]
for any \(\dim Z - 1\) form \(\tilde{e}(TZ, h_l)\) such that \(d\tilde{e}(TZ, h_l) = e(TZ, h_1) - e(TZ, h_0)\).

In particular, the degree-0 part of Equation (17) is the anomaly formula for the \(L^2\)-Ray-Singer analytic torsion, which is a special case of \[11 \, \text{Theorem 3.4}\].

Remark 3.9. Let \(Z_0 \to M_0 \to B\) be a fiber bundle with compact fiber \(Z_0, Z \to M \to B\) be the normal covering of the fiber bundle \(Z_0 \to M_0 \to B\). i.e., \(G = \pi_1(Z)\) acts fiber-wisely and \(Z_0 = Z/G\). Then one can define the Bismut-Lott and \(L^2\)-analytic...
torsion form $\tau_{M_0 \to B}, \tau_{M_0 \to B} \in \Gamma^\infty(\wedge^\bullet T^*B)$, and one has the respective transgression formulas

$$d\tau_{M_0 \to B} = \int_{z_0} \chi(x) e(TZ_0) f(\nabla^\wedge V' \otimes E_0) - f(\nabla^\wedge V' \otimes E_0) \text{H}^\bullet(Z_0, E_0)$$

$$d\tau_{M \to B} = \int_{Z_0} \chi(x, z) e(TZ) f(\nabla^\wedge V' \otimes E) - f(\nabla^\wedge V' \otimes E) \text{H}^\bullet(Z, E).$$

Suppose further that the DeRham cohomologies are trivial:

$$H^\bullet(Z_0, E|Z_0) = H^\bullet_{L^2}(Z, E|Z) = \{0\}.$$  

Then $d(\tau_{M \to B} - \tau_{M_0 \to B}) = 0$. Hence $\tau_{M \to B} - \tau_{M_0 \to B}$ defines some class in the DeRham cohomology of $B$. We also remark that this form was also mentioned in [1, Remark 7.5], as a weakly closed form.

**References**

[1] S. Azzali, S. Goette, and T. Schick. Large time limit and $L^2$ local index for families. preprint arXiv:1306.5659v1, 2013.

[2] N. Berline, E. Getzler, and M. Vergne. *Heat kernels and Dirac operators*. Springer-Verlag, 1992.

[3] J.M. Bismut and J. Lott. Flat bundles, direct images and higher real analytic torsion. *J. Amer. Math. Soc.*, 8(2):291–363, 1995.

[4] J.M. Bismut, X. Ma, and W. Zhang. Asymptotic torsion and toeplitz operators. preprint http://www.math.u-psud.fr/~bismut/liste-prepub.html, 2011.

[5] D. Gong and M. Rothenberg. Analytic torsion forms on non-compact fiber bundles. preprint http://webdoc.sub.gwdg.de/ebook/serien/mpi_mathematik/1997/105.ps, 1996.

[6] J.L. Heitsch. Bismut super-connections and the Chern character for Dirac operators on foliated manifolds. *K-Theory*, 9:507–528, 1995.

[7] J.L. Heitsch and C. Lazarov. Riemann-Roch-Grothendieck and torsion for foliations. *J. Geom. Anal.*, 12(3):437–468, 2002.

[8] J.A. Alvarez Lopez and Y.A. Kordyukov. Long time behavior of leafwise heat flow for Riemannian foliations. *Compositio Math.*, 125(2):129–153, 2001.

[9] X. Ma and W. Zhang. Eta-invariants, torsion forms and flat vector bundles. *Math. Ann.*, 340:569–624, 2008.

[10] Y. Nistor, A. Weinstein, and P. Xu. Pseudodifferential operators on differential groupoids. *Pac. J. Math.*, 189(1):117–152, 1999.

[11] W. Zhang. An extended Cheeger-Müller theorem for covering spaces. *Topology*, 44:1093–1131, 2005.