Higher order corrections to the equation-of-state of QED at high temperature

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Abstract
We elaborate on the computation of the pressure of thermal quantum electrodynamics, with massless electrons, to the fifth ($e^5$) order. The calculation is performed within the Feynman gauge and the imaginary-time formalism is employed. For the $e^4$ calculation, the method of Sudakov decomposition is used to evaluate some ultraviolet finite integrals which have a collinear singularity. For the $e^5$ contribution, we give an alternative derivation and extend the discussion to massive electrons and nonzero chemical potential. Comments are made on expected similarities and differences for prospective three-loop calculations in QCD.

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1 Introduction

Recently we reported on the order $e^4$ \cite{1} and $e^5$ \cite{2} contributions to the pressure of a massless QED plasma at temperature $T$ and zero chemical potential $\mu_e$. Here we would like to fill in the discussion, particularly for the more difficult $e^4$ calculation, as some of the techniques might be of wider interest. In addition, an alternative derivation of the $e^5$ contribution will be given.

For motivation we note that the equation-of-state of a relativistic plasma is of relevance in astrophysics \cite{3, 4, 5, 6}. It was obtained at nonzero $T$ and $\mu_e$ by Akhiezer and Peletminskii \cite{3} to the third order ($e^3$) and later extended to order $e^4$ at $T = 0$, but nonzero $\mu_e$, in both QED and quantum chromodynamics (QCD)\cite{3, 4}. Thus the calculation of the $T \neq 0$, $\mu_e = 0$ contribution at order $e^4$ fills a gap in our knowledge. It is also a step towards same order calculations for QCD which is believed to exist in a perturbative quark-gluon phase at high temperature. Currently the pressure of the high-temperature phase of QCD is known to order $g^4 \ln g$, excluding the normalisation of the logarithm \cite{7, 8, 9}.

The calculations in this paper are performed within the framework of the imaginary-time formalism (see \cite{10, 11} and references therein) whereby the energies take on discrete Matsubara values, $q_0 = i n \pi T$, $n$ being an even (odd) integer for bosons (fermions). The usual zero-temperature ultraviolet (UV) singularities are regularised by dimensional continuation \cite{12}. For simplicity we restrict ourselves to the Feynman gauge but renormalization via minimal subtraction \cite{13} ensures that the coupling constant is gauge-fixing independent \cite{14}, and hence so will then be our final answer for the pressure \cite{5}.

At intermediate stages of the calculation we will encounter various types of infrared (IR) singularities. The first kind is due to many-body effects and gives rise to power-like singularities in some diagrams. As is well known, these are removed when the static electric propagator is dressed to take into account the screening of electric fields in a plasma \cite{13, 3, 11}. As a result of this resummation the expansion for the pressure is in powers of $\sqrt{e^2}$ rather than $e^2$, the famous $e^3$ “plasmon” of Gell-Mann and Brueckner \cite{15} illustrating this at the lowest order. In this paper we will obtain also the order $e^5$ plasmon contribution.

We remind the reader that the identification of bubble (i.e. no external legs) diagrams which require the use of screened propagators so as to produce a consistent perturbative expansion is most easily done in the imaginary-time formalism where
only the zero mode of the photon propagator lacks an infrared cutoff of order $T$. More generally one can perform a consistent resummation in imaginary-time for any Greens function whose external legs are bosonic and static (at zero energy). For Greens functions which have external fermionic lines, or nonstatic external bosonic lines, one must first analytically continue [16, 11] to real-time to obtain the physical Greens function before the resummation can be discussed. In the latter case it is in general necessary to use nontrivial propagators and vertices to restore the perturbative expansion, as discussed by Braaten and Pisarski [17].

Though individual bubble diagrams in imaginary time may be IR finite (after using dressed propagators if need be), once the frequency sums are performed the diagram in general splits into several pieces (integrals) each of which individually may contain mass-shell and/or collinear singularities. Since it is convenient to evaluate the different integrals separately, a regularisation has to be used for these “spurious” IR divergences. We will use dimensional regularisation (DR) for this also [18, 4]. Actually, in this paper we will make no attempt to distinguish between IR and UV divergences at each stage but will only verify that sums of diagrams which a priori should be finite, are indeed so. Such finite sums form gauge-invariant subsets, contributing to the pressure amounts which are conveniently labelled by different powers of $N$, the number of fermion flavours.

We mention here also two other difficulties, related to IR divergences, in evaluating some integrals. As in the case of the 3-loop pressure in $\phi^4$ theory [19], we find some integrals containing singularities along the path of integration which are intrepreted in the principal value sense. In some cases we will transform, by a change of variables in the integral split at the point of singularity, these principal value singularities into infrared singularities and then evaluate them by dimensional regularisation. Also some difficult 3-loop integrals which are UV finite because of statistical distribution factors, but contain a collinear singularity, are in this paper handled by using Sudakov variables [20] in dimensional regularisation (see [21]).

Our notation is as follows: the wave-vector, $Q_\mu = (q_0, \vec{q})$ is contracted with a Minkowski metric, $Q^2 = q_0^2 - \vec{q}^2$, and the measure of loop integrals is denoted compactly by

$$\int[dQ] \equiv T \sum_{q_0, \text{even}} \int \frac{d^{D-1}q}{(2\pi)^{D-1}} ,$$

2
\[ \int \{dQ\} \equiv T \sum_{q_{\alpha, \text{odd}}} \int \frac{d^{D-1}q}{(2\pi)^{D-1}}. \]

The fermions are kept as four-component objects, \( Tr(\gamma_{\mu} \gamma_{\nu}) = 4g_{\mu\nu} \), and the gauge propagator is \( D_{\mu\nu}(K) = g_{\mu\nu}/K^2 \).

In the next section we review the calculation of the pressure in massless QED to third \((e^3)\) order. In Sect.3 the diagrams and integrals which contribute at fourth order are discussed. Some frequency-sums are evaluated in Sect.4 while in Sect.5 we explain our use of Sudakov variables to evaluate two difficult integrals. The final pressure to order \(e^4\) is summarised in Sect.6 and the renormalisation group briefly discussed. In Sect.7 we rederive the \(e^5\) contribution to the pressure. The results of this paper are summarised and discussed in Sect.8 while the appendices contain some useful identities and technical derivations.

## 2 Lower Orders

Before discussing the 3-loop calculation, let us briefly review the lower order results for the pressure of QED with \(N\) massless Dirac fermions. The ideal gas pressure \(P_0\) due to electrons, positrons and photons is given by,

\[
P_0 = \frac{T}{V} \ln \left\{ \left[ \text{Det}_+ (\partial^2 g_{\mu\nu}) \right]^{-\frac{1}{2}} \text{Det}_+(\partial^2) \text{Det}_-(i \partial) \right\} \]

\[= \left[ (D-2) + \frac{7}{8}(4N) \right] \frac{T^4 \xi(4)}{\pi^2} \]

\[= \frac{\pi^2}{45} T^4 \left( 1 + \frac{7}{4}N \right). \]

In (2.1) the \(\pm\) subscripts refer respectively to periodic and antiperiodic boundary conditions, and the second determinant in is the ghost contribution which is required for proper counting of physical degrees of freedom in the ideal gas pressure [10]. The first correction \(P_2\) is given by the two-loop diagram of Fig.1,

\[
P_2 = -\frac{\mu^{4-D}e^2N}{2} \int \frac{\{dK\} [dQ] \cdot \text{Tr}(\gamma_{\mu}(K-P)\gamma_{\nu}KD^{\nu\mu}(P))}{K^2(K-P)^2}. \]
\[ = (D - 2) \epsilon^2 N f_1(2b_1 - f_1) \mu^{4-D} T^{2D-4} \]  
(2.2)

\[ = - \frac{5e^2T^4N}{288}. \]

The last line above follows from the previous one as \( D \to 4 \). In the rest of this paper we will often take this limit, where consistently possible, without comment. In (2.2), \( e \) is the renormalised coupling, \( \mu \) the mass-scale of dimensional regularisation and the integrals \( b_1 \) and \( f_1 \) are defined through

\[ b_n \equiv \int \frac{[dQ]}{(Q^2)^n}, \]

\[ f_n \equiv \int \frac{[dQ]}{(Q^2)^n}. \]

We have scaled out the temperature in the above integrals so that \( T = 1 \) there. The simplest way to evaluate \( b_n \) is to first perform the momentum integrals and then the frequency sums [22]. This gives

\[ b_n = \frac{2}{(2\pi)^{2n}} \int \frac{[dQ]}{(Q^2)^n}, \]

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\[ f_n \equiv \int \frac{[dQ]}{(Q^2)^n}. \]
\[ (2.5) \]

Though (2.4) is finite, we have evaluated it \( \Pi \) using dimensional continuation \( 3 \rightarrow D - 1 \). Then the second term in (2.4) vanishes and the first term gives the result (2.5) as \( D \rightarrow 4 \). The point of this excursion is to verify, in this example, that scaleless integrals like those above may indeed be consistently dropped in DR. This fact will be useful later.

It is of interest to note that one may also calculate the plasmon contribution in real-time but then the analysis is much more intricate. In a real-time analysis, the transverse photons are relevant at intermediate stages but their net contribution to the \( e^3 \) plasmon term vanishes \([24, 25]\) and one recovers the result of the imaginary-time analysis where it is clear from the outset (by power counting) that only the longitudinal photons contribute. In the language of Braaten and Pisarski \([17]\), for the calculation of the pressure of QED in imaginary time (recall the discussion in Sect. 1), the only soft line is the static photon line. Thus resummation (which gives the \( e^3, e^5 \) etc. terms) involves dressing this soft line with the relevant static “hard thermal loop” which is just the electric mass \( \Pi_{00}(0,0) \) \([17]\). For more discussion comparing the real and imaginary-time approaches to the calculation of the pressure see \([24, 25]\), in particular the conclusion of the second reference of \([24]\).

### 3 Three Loop Diagrammatics

The order \( e^4N \) diagrams are shown in Fig. 2 and 3. For massless fermions, the Ward identity \( Z_1 = Z_2 \) implies the mutual cancellation of the counterterm diagrams (Fig.3), so the sum \( G_1 + G_2 \) (Fig. 2) is UV finite. After performing the spinor traces and some algebra,

\[ G_1 = \left( -\frac{e^4N}{2} \right) T^{3D-8} \mu^{8-2D} 2 (D - 2)^2 \left( f_2(f_1 - b_1)^2 + I_2 - 2I_3 \right), \quad (3.1) \]

\[ G_2 = \left( -\frac{e^4N}{4} \right) T^{3D-8} \mu^{8-2D} 2 (2-D) \left( (8b_1 - 16f_1)I_1 + 2(D - 4)I_2 + 8(D - 3)I_3 + 4I_4 \right). \quad (3.2) \]
As before, we have scaled all the momenta by $1/T$ so that the integrals are dimensionless (i.e. $T = 1$ there). The integrals $b_n$ and $f_n$ were defined in the last section, while the rest are

\[
\begin{align*}
I_1 &= \int \frac{dKdR}{K^2R^2(K+R)^2}, \\
I_2 &= \int \frac{dK[dQdP]}{K^2Q^2P^2(K+Q+P)^2}, \\
I_3 &= \int \frac{dKdRdS}{K^2R^2S^2(K+R+S)^2}, \\
I_4 &= \int \frac{dK[dPdQ]P^2}{K^2Q^2(K+Q)^2(K+P)^2(K+P+Q)^2}, \\
I_5 &= \int \frac{dK[dQdP](P \cdot Q)}{K^2P^2Q^2(K+Q)^2(K+P)^2}.
\end{align*}
\]

Some simplification is possible. Firstly, within DR, $I_1 = 0$ as shown by Arnold and Espinoza \[23\] using scaling arguments. We will deduce the same result by direct evaluation in the next section. Using scaling arguments as in \[23\] one may also show

\[
I_2 = \frac{1}{6} \left(2^{11-3D} - 1\right) H_1 - \frac{1}{6} I_3 \tag{3.3}
\]

where

\[
H_1 = \int \frac{[dQdPdK]}{K^2Q^2P^2(K+Q+P)^2}
\]

is the integral analysed by Frenkel, Saa and Taylor \[19\] in their 3-loop pressure calculation in hot $\phi^4$ theory. Next, some algebraic rewriting gives for $I_4$,

\[
\begin{align*}
I_4 &= \int \frac{dKdRdS}{K^2R^2S^2(K+R)^2(K+R+S)^2} (K+S)^2 \\
&= 2 f_1 I_1 - \frac{1}{2} I_3 \\
&= -\frac{1}{2} I_3.
\end{align*}
\]

The last line is valid because $f_1$ is finite near $D = 4$ but $I_1 = 0$ as mentioned above. Relabelling $I_3 \to H_2$ and $I_5 \to H_3$, we obtain
\[
\frac{G_1 + G_2}{e^4 N T^{3D-8} \mu^{8-2D}} = \left( \frac{D-2}{6} \right) \left( 2 \left( 1 - 2^{(11-3D)} \right) H_1 + (20 - 3D) H_2 \right) + (D-2)^2 \left( 2 H_3 - f_2 (f_1 - b_1)^2 \right). \quad (3.4)
\]

Consider now the order \( e^4 N^2 \) diagrams. The photon wave-function renormalization required for \( G_3 \) (Fig. 4a) is provided by \( X_1 \) (Fig. 4b). Diagram \( G_3 \) also has an IR singularity which contributes to the \( e^3 \) plasmon as the first term of the series in eq. (2.3). In principle this term should be subtracted from \( G_3 \) to avoid overcounting but since, as discussed earlier, it vanishes in DR, double-counting is automatically avoided. We have

\[
G_3 = \frac{e^4 N^2}{4} T^{3D-8} \mu^{8-2D} 16 \left( (D-4) b_2 f_1^2 + \frac{(D-4)}{4} H_2 + 4 H_4 \right), \quad (3.5)
\]

and

\[
X_1 = -(Z_3 - 1) e^2 N (D-2) T^{3D-8} \mu^{8-2D} f_1 (2b_1 - f_1) \left( \frac{T}{\mu} \right)^{4-D}, \quad (3.6)
\]

where

\[
Z_3 - 1 = \frac{e^2 N}{6\pi^2(D-4)} + O(e^4).
\]

In (3.5) we have dropped a vanishing contribution proportional to \( I_1 \). The new integral in (3.5) is

\[
H_4 = \int \frac{[dQ][dK dR]}{Q^4 K^2 R^2 (Q + K)^2 (Q + R)^2}.
\]

In summary, our task for the calculation of the order \( e^4 \) contribution to the pressure as given by eqns. (3.4, 3.5, 3.6) has been reduced to the evaluation of the following four integrals:

\[
H_1 = \int \frac{[dQ dP dK]}{K^2 Q^2 P^2 (K + Q + P)^2}, \quad (3.7)
\]

\[
H_2 = \int \frac{[dK dR dS]}{K^2 R^2 S^2 (K + R + S)^2}; \quad (3.8)
\]
\[ H_3 = \int \frac{\{dK\} [dQ dP] (P \cdot Q)}{K^2 P^2 Q^2 (K + Q)^2 (K + P)^2}, \quad (3.9) \]

\[ H_4 = \int \frac{[dQ] \{dK dR\} (K \cdot R)^2}{Q^4 K^2 R^2 (Q + K)^2 (Q + R)^2}. \quad (3.10) \]

In the next section we will do the frequency sum in \( H_1 \) and re-obtain the expression of [19]. The analysis of \( H_2 \) is completely analogous to \( H_1 \). The new integrals we have to analyse are \( H_3 \) and \( H_4 \). These are the main concern of the next two sections.

4 The Frequency Sums

There are several variations in the literature [11, 19] for performing the frequency sums. Here we sketch one way (cf. [20]).

\( \bullet I_1 \)

Consider the two-loop integral

\[ I_1 = \int \{dK dQ\} \frac{1}{K^2 Q^2 (K + Q)^2}. \]

The first step is standard and involves writing the \( k_0 \) sum as a contour integral,

\[ \int \{dK\} \frac{1}{K^2 (K + Q)^2} = \int \frac{d^{D-1} K}{(2\pi)^{D-1}} \times \frac{1}{2\pi i} \oint \frac{-N_{k_0}}{K^2 (K + Q)^2} \]

(4.1)

where

\[ N_{k_0} = \frac{1}{e^{k_0} + 1}, \]

and the anticlockwise contour circles the poles of \( N_{k_0} \) only. Next the contour is deformed to enclose only the poles of the propagator. Upon using the identity

\[ \frac{1}{2x} (f(x) - f(-x)) = \int_{-\infty}^{\infty} dx_0 \delta(x_0^2 - x^2) \epsilon(x_0) f(x_0), \]

(4.2)

and noting that \( q_0 \) is an odd multiple of \( i \), the result can be written compactly as

\[ I_1 = \int \frac{d^D K}{(2\pi)^{D-1}} (N_{k_0} - n_{k_0}) \epsilon(k_0) \delta(K^2) \int \{dQ\} \frac{1}{K^2 (K + Q)^2}, \]

where

\[ n_{k_0} = \frac{1}{e^{k_0} - 1}. \]

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The above procedure is repeated on the \( q_0 \) sum but note that \( k_0 \) is now real and continuous,

\[
I_1 = \int \frac{d^D K d^D Q}{(2\pi)^{2D-2}} \frac{\epsilon(k_0) \delta(K^2) \epsilon(q_0) \delta(Q^2)}{(K+Q)^2} (N_{k_0} - n_{k_0}) (N_{q_0} - N_{-k_0 - q_0}).
\]

Using the identities in the appendix and symmetrizing, this becomes

\[
I_1 = \int \frac{d^D K d^D Q}{(2\pi)^{2D-2}} \frac{\epsilon(k_0) \delta(K^2) \epsilon(q_0) \delta(Q^2)}{(K+Q)^2} (N_{k_0} N_{q_0} - n_{k_0} N_{q_0} - n_{-k_0} N_{-q_0} - 1/2).
\]

Now use the identities in the appendix again to write the expression above in terms of \( n_k \) and \( N_q \) where \( k = |k_0| \) and \( q = |q_0| \). Then terms which are independent of statistical factors contribute zero in DR, terms with only one statistical factor also vanish by Lorentz covariance and DR, while terms with two statistical factors vanish in DR after the energy integrals are done. Thus \( I_1 = 0 \) in agreement with \([23]\).

\[\bullet H_1 \text{ and } H_2\]

When applied to \( H_1 \), the frequency sum algorithm discussed above yields

\[
H_1 = \int \frac{[dK \, dQ \, dP]}{K^2 Q^2 P^2 (K + Q + P)^2}
\]

\[
= \int \frac{d^D K \, d^D Q \, d^D P \, \epsilon(k_0) \delta(K^2) \epsilon(q_0) \delta(Q^2) \epsilon(p_0) \delta(P^2)}{(2\pi)^{3(D-1)} (K + P + Q)^2} \times (4n_{k_0} n_{q_0} n_{p_0} + 2n_{k_0} n_{p_0})
\]

\[
= \int \frac{d^D K \, d^D Q \, d^D P \, \delta(K^2) \delta(Q^2) \delta(P^2)(4n_{k_0} n_{q_0} n_{p_0} + 6n_{k_0} n_{p_0})}{(2\pi)^{3(D-1)} (K + P + Q)^2} \tag{4.3}
\]

where \( k = |k_0| \) and we have skipped a few steps which make use of the identities in the appendix. We have also dropped terms which vanish in DR. To write \([13]\) in the form of Ref. \([19]\), use

\[
\int d^{D-1}q \int_{-\infty}^{\infty} dk_0 \, dq_0 \, dp_0 \frac{\delta(K^2) \delta(P^2) \delta(Q^2)}{(K + Q + P)^2} \equiv \frac{1}{2\pi i} \int d^{D-1}q \int_{-\infty}^{\infty} dk_0 \, dq_0 \, dp_0
\]

\[
\times \frac{\delta(K^2) \delta(P^2)}{(Q^2 - i0^+)((K + Q + P)^2 - i0^+)} \tag{4.4}
\]

\[\text{to get}\]

\[
H_1 = \frac{-6}{2\pi i} \int \frac{d^D K \, d^D Q \, d^D P}{(2\pi)^{3(D-1)}} \frac{\delta(K^2) \delta(P^2) n_k \, n_p}{(Q^2 - i0^+)((K + Q + P)^2 - i0^+)}
\]

\[
-4 \int \frac{d^D K \, d^D Q \, d^D P \, \delta(K^2) \delta(Q^2) \delta(P^2) n_k \, n_p \, n_q}{(K + Q + P)^2} \tag{4.5}
\]
This is the expression of [19]. Keeping only the nonvanishing terms as $D \to 4$ gives

\[ H_1 = \frac{1}{3(2^6\pi^2)} \left( 5.6658 - \frac{1}{(D - 4)} \right). \] (4.6)

We leave it as an exercise for the interested reader to show that $H_2$ (3.8) is obtained from (4.5) by replacing $n_i$ with $-N_i$, and then

\[ H_2 = \frac{1}{3(2^8\pi^2)} \left( 2.6045 - \frac{1}{(D - 4)} \right). \] (4.7)

**$H_3$**

For $H_3$ we obtain

\[ H_3 = \int \frac{d^DQ \, d^DK \, d^DP}{(2\pi)^{3(D-1)}} \epsilon(q_0) \, \delta(Q^2)\epsilon(k_0) \, \delta(K^2)\epsilon(p_0) \, \delta(P^2) \, \{\mathcal{R}\}, \] (4.8)

where

\[ \{\mathcal{R}\} \equiv \frac{T_K}{(K + Q)^2(K + P)^2} + \frac{(T_K|K \to K - Q)}{(K - Q)^2(K - Q + P)^2} + \frac{(T_K|K \to K - P)}{(K - P)^2(K - P + Q)^2}, \]

\[ T_K \equiv N_{k_0} \left( n_{q_0} [n_{q_0} P \cdot Q - N_{-q_0} P \cdot (Q + K)] - N_{-p_0} [n_{q_0} Q \cdot (P + K) - N_{-q_0} (Q + K) \cdot (P + K)] \right), \]

and $(T_K|K \to K - Q)$ means replace $K_\mu$ by $K_\mu - Q_\mu$ in the expression for $T_K$. After the usual simplification,

\[ H_3 = J_1 + K_1 + L_1. \] (4.9)

The piece $L_1$ contains integrals that can be performed exactly using relations (A.17)-(A.28) of Appendix A.

\[ L_1 = \frac{\omega(D)\omega^2(D - 1)}{(2\pi)^{3(D-1)}} 2^{(D-5)} \, B(1/2, \, D/2 - 1) \, B(D/2, \, D/2 - 2) \]
\[ \times \mathcal{M}_2(D - 5) \left[ \mathcal{M}_1(D - 3) + \mathcal{M}_2(D - 3) \right]^2 \]
\[ + \frac{\omega(D)\omega(D - 1)}{(2\pi)^{3(D-1)}} \, B(D/2 - 1, \, D - 3) \, 2^{D-6} \, N_1, \]
where
\[
\mathcal{N}_1 = 3\mathcal{M}_1 \left( \frac{3D-10}{2} \right) \mathcal{M}_2 \left( \frac{3D-10}{2} \right) \sum_{\sigma, \gamma = \pm 1} (\sigma \gamma) C_0(\sigma, \gamma) \\
+ 2 \left[ \mathcal{M}_1 \left( \frac{3D-10}{2} \right) + \mathcal{M}_2 \left( \frac{3D-10}{2} \right) \right]^2 \sum_{\sigma, \gamma = \pm 1} C_1(\sigma, \gamma).
\]

The functions \( \omega, \mathcal{M}_i \) and \( C_i \) are defined in the appendix and \( B(a, b) \) is the usual beta-function. Expanding about \( D = 4 \),
\[
L_1 = \frac{-1}{3(2^9\pi^4)} \left( \frac{2\pi^2 + 7\pi^2 \ln 2 + 6\pi^2 \zeta'(0) + 18\zeta'(2)}{(D-4)} + 35.478 \right).
\]

In (4.9) \( K_1 \) is an integral similar to that appearing in \( H_1 \) and \( H_2 \),
\[
K_1 = -\int \frac{d^4K \; d^4Q \; d^4P}{(2\pi)^9} \frac{\delta(P^2)\delta(Q^2)\delta(K^2)N_p N_q n_k}{(K + P + Q)^2} \\
= \frac{1.1439}{2(2\pi)^6}.
\]

Finally, \( J_1 \) is defined by
\[
J_1 = \int \frac{d^D K \; d^D Q \; d^D P}{(2\pi)^{3(D-1)}} \delta_+(K^2)\delta_+(Q^2)\delta_+(P^2)N_k n_q (N_p n_n + n_p) \sum_{\sigma, \gamma = \pm 1} (-\sigma) S_1(\sigma, \gamma),
\]
with
\[
S_1(\sigma, \gamma) = \frac{P \cdot Q}{K \cdot Q K \cdot P + P \cdot (\sigma K + \gamma Q)},
\]
and \( \delta_+(Q^2) = \delta(Q^2) \theta(q^0) \). The evaluation of \( J_1 \) is described in Sect.5.

Collecting the pieces,
\[
H_3 = J_1 + K_1 + L_1 \\
= \frac{1}{2^7\pi^4} \left( 1.095 - \frac{0.4112335165}{(D-4)} \right).
\]

\( \bullet H_4 \)

The only extra point here is the doubled propagator \( 1/(Q^2)^2 \). Thus the contribution to
the contour integrals coming from this propagator must be extracted correctly using
the appropriate formula of complex analysis. Other than this technical point, the rest
of the derivation proceeds as discussed for \( H_3 \) but the expressions are lengthier. So
we will only state the results.

We obtain

\[
H_4 = J_2 + K_2 + L_2, \quad (4.15)
\]

where \( J_2 \) is an integral similar to \( J_1 \) and is discussed in the next section.

\[
K_2 = \frac{1}{2} \int \frac{d^4K \, d^4Q \, d^4P}{(2\pi)^9} \frac{\delta(K^2) \delta(Q^2) \delta(P^2) \, N_k \, N_q \, N_p}{(K + P + Q)^2} \delta(K_2) \delta(Q_2) \delta(P_2) \, N_{k_0} \, N_{q_0} \, N_{p_0} \cdot (\sigma K + \gamma Q) . \quad (4.16)
\]

\[
L_2 = \frac{1}{3^3 \cdot 2^{10} \cdot \pi^4} \left( \frac{17\pi^2 + 12\pi^2 \ln 2 + 24\pi^2 \zeta'(0) + 72\zeta'(2)}{(D - 4)} - 0.5098 \right) \cdot (D - 4) . \quad (4.17)
\]

Finally

\[
H_4 = J_2 + K_2 + L_2
\]

\[
= \frac{1}{2^7 \pi^6} \left( 1.407 - \frac{2.254840072}{(D - 4)} \right) . \quad (4.18)
\]

\section{Sudakov Variables}

In this section we describe the evaluation of the integrals \( J_1 \) and \( J_2 \) appearing in the
last section.

\begin{itemize}
\item \( J_1 \)
\end{itemize}

The integral \( J_1 \) was defined as \( (1.12) \),

\[
J_1 = \sum_{\sigma, \gamma = \pm 1} (-\sigma) \int \frac{d^D Q \, d^D P}{(2\pi)^{3(D - 1)}} \frac{\delta_+(Q^2) \, \delta_+(P^2) \, N_{q_0} \, (N_{p_0} + N_{p_0})}{N_{k_0}} \ \times \int d^D K \, \delta_+(K^2) \frac{P \cdot Q}{K \cdot Q} \frac{N_{k_0}}{K \cdot Q + P \cdot (\sigma K + \gamma Q)} . \quad (5.1)
\]
This integral is quite complicated and we have not succeeded in evaluating it in closed form. However for our purposes we only require the order \((D - 4)^{-1}\) and \(O(D - 4)^0\) terms from (5.1) in the limit \(D \to 4\). Although (5.1) is UV finite, a \(1/(D - 4)\) pole is expected because of the collinear singularity as \(K.Q \to 0\).

Let us concentrate on the \(K\)-subintegral appearing in (5.1),

\[
\int d^D K \delta_+(K^2) \frac{P \cdot Q}{K \cdot Q} \frac{N_{k_0}}{K \cdot Q + P \cdot (\sigma K + \gamma Q)}.
\]

(5.2)

For the rest of this subsection we will mostly discuss this subintegral with the implicit understanding that it occurs inside (5.1) so that the constraints \(P^2 = Q^2 = 0\) and \(p_0 = p, \ q_0 = q\) hold. It is sufficient to extract only the \(O(D - 4)^{-1}\) and \(O(D - 4)^0\) pieces from (5.2) because the remaining \((P, Q)\) integrals in (5.1) are UV and IR finite (so that we will only need their \(O(D - 4)^0\) and \(O(D - 4)^1\) pieces to get the full \(O(D - 4)^{-1}\) and \(O(D - 4)^0\) terms for \(J_1\) when our result for (5.2) is substituted back into (5.1)).

Since the integral (5.2) involves three independent scalar products in \(D\) dimensions, it poses a formidable problem. We begin our task by decomposing the loop momentum \(K\) in a Sudakov base [20] constructed on \(P\) and \(Q\) in such a way that the \(D\) dimensional angular variables will disappear from the denominator in (5.2). Thus we write

\[K \equiv \alpha P + xQ + K_\perp\]

(5.3)

with \(K_\perp\) denoting a transversal vector \((P \cdot K_\perp = Q \cdot K_\perp = 0)\) of \(D - 2\) nonzero components. Note that the delta and step-functions in (4.12) imply \(K^2_\perp \leq 0\). Since the Jacobian in going from the \(K\)-basis to the \((\alpha, x, K_\perp)\) basis is \(s \equiv P.Q\), we obtain for (5.2) (upon using (5.3) to eliminate \(P.K\) and \(Q.K\)),

\[
\int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} dx \int d^{D-2} K_\perp \frac{\delta(2sx\alpha + K^2_\perp)}{\alpha} \frac{N_{k_0}}{\alpha + (\sigma x + \gamma)} \theta(k_0).
\]

(5.4)

\[
= \int_{-\infty}^{\infty} dx \int d^{D-2} K_\perp \frac{1}{(-K^2_\perp)} \frac{N_{k_0}}{\frac{(-K^2_\perp)}{2sx} + (\sigma x + \gamma)},
\]

(5.5)

where now

\[k_0 = \frac{(-K^2_\perp)}{2sx}p + xq + (K_\perp)_0\]

(5.6)

which follows from (5.3) and the fact that \(p_0 = p\) and \(q_0 = q\). The original collinear singularity of the \(K\)-integral now appears as an endpoint IR singularity as \(D \to 4\) in the radial part of the \(K_\perp\)-integral. This is not surprising since we note from (5.3)
that the collinear singularity of (5.2) is encountered on the plane spanned by the two massless momenta \( P \) and \( Q \). The region of integration which contributes to this singularity is therefore identified by the limit \( K_\perp \to 0 \).

The combination of Sudakov methods and dimensional regularisation to study discontinuities in zero temperature type integrals is discussed in [21]. In the present case though we have obtained a simplified structure for (5.2) in terms of (5.5), the dependence of the statistical factor on \( k_0 \) is still a complication. We therefore isolate the IR singularity in (5.5) into a simpler integral by rewriting

\[
N_{k_0} \theta(k_0) = [N_{k_0} \theta(k_0) - N_{xq} \theta(xq)] + N_{xq} \theta(xq). \tag{5.7}
\]

The reason for this rearrangement is because now the term in square-brackets on the right-hand-side above gives an UV and IR finite \( \dagger \) contribution to (5.5) (and hence to \( J_1 \)) as \( D \to 4 \), while the simpler last term in (5.7) will give the pole (and some finite parts). Calling the net contribution of the square bracket in (5.7) to \( J_1 \) as \( J_1A \), we obtain (see Appendix B)

\[
J_1A = \frac{6.101}{2^7\pi^6}. \tag{5.8}
\]

The last term on the right-hand-side of (5.7) contributes to (5.5) the amount

\[
\int_{-\infty}^{\infty} dx \int d^{D-2}K_\perp \frac{1}{(-K_\perp^2)} \frac{N_{xq} \theta(xq)}{2sx + (\sigma x + \gamma)}. \tag{5.9}
\]

Since \( K_\perp \) is a space-like vector with \( D - 2 \) independent components we may write \( K_\perp = \sum_{i=1}^{D-2} \sigma_i e_i \), where the \( e_i \) are orthonormal space-like vectors: \( e_i \cdot e_j = -\delta_{ij} \). Then (5.9) becomes

\[
\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{d\sigma_1 \ldots d\sigma_{D-2}}{(\sigma_1^2 + \ldots + \sigma_{D-2}^2)} \frac{N_{xq} \theta(xq)}{(\sum \sigma_i^2 / 2sx) + (\sigma x + \gamma)}. \tag{5.10}
\]

\[
= \omega(D-1) \int_{-\infty}^{\infty} dx \int_{0}^{\infty} dr \int_{0}^{D-4} d\sigma_{D-5} r^2 \frac{N_{xq} \theta(xq)}{2sx + (\sigma x + \gamma)}. \tag{5.11}
\]

\[
= \frac{\omega(D-1)}{2} \int_{0}^{\infty} dx \int_{0}^{\infty} dy y^{D-3} \frac{N_{xq}}{2sx + (\sigma x + \gamma)}. \tag{5.12}
\]

\[
= \frac{\omega(D-1)}{2} (2s)^{D/2-2} \int_{0}^{\infty} dx \frac{x^{D/2-2}}{e^{xq} + 1} \int_{0}^{\infty} dz \frac{z^{D/2-3}}{z + (\sigma x + \gamma)}. \tag{5.13}
\]

\( \dagger \)The reader who does not find this too apparent may find a posteriori satisfaction by examining the simplified form of the relevant equation in Appendix B.
In going from (5.10) to (5.11) we switched from the Cartesian coordinates $\sigma_i$ to spherical coordinates and the function $\omega(D - 1)$ as defined by (A.25) is the result of doing the angular integrals while $\int dr$ is the remaining radial integral. When (5.13) is substituted back into $J_1$, we will need the sum

$$\sum_{\sigma, \gamma = \pm 1} (-\sigma) \frac{1}{z + (\sigma x + \gamma)} = \sum_{\sigma = \pm 1} (-\sigma) \left( \frac{1}{z + \sigma(x + 1)} + \frac{1}{z + \sigma(x - 1)} \right).$$  \hspace{1cm} (5.14)$$

The change of variables $z = y(x + 1)$ for the first term in (5.14) and $z = y|x - 1|$ for the second term decouples the $x - z$ integral in (5.13) into a product of an $x$-integral and a simple $y$-integral. The resulting $y$-integrals can be evaluated explicitly using eqns. (A.26-A.28). The $x$-integrals are still too complicated to be evaluated in closed form but their integrands may be expanded about $D = 4$ directly or after some integration by parts. In this way we obtain the net contribution, $J_{1B}$, of (5.13) to $J_1$ as

$$J_{1B} = \int_0^{\infty} dq \, q \, n_q \left\{ \frac{4g_0}{(D - 4)}[\text{Box1}] + g_0 \left( 2[\text{Box2}] + 6 \ln(q)[\text{Box1}] \right) + 4g_1[\text{Box1}] \right\} + O(D - 4),$$ \hspace{1cm} (5.15)

where

$$[\text{Box1}] = -\int_0^{\infty} dx \, \frac{N_{xq}}{x + 1} + \int_0^1 dx \, \frac{N_{xq} - N_q}{(1 - x)} + \int_1^{\infty} dx \, q \, N_{xq} (N_{xq} - 1) \ln(x - 1),$$ \hspace{1cm} (5.16)

$$[\text{Box2}] = \frac{-\pi^2}{6} N_q - \int_0^{\infty} dx \, \frac{N_{xq}}{x + 1} \ln x(x + 1) + \int_0^1 dx \frac{N_{xq} - N_q}{(1 - x)} \ln x(1 - x)$$

$$+ \int_1^{\infty} dx \left( \frac{N_{xq}}{x} \ln x(x - 1) + q N_{xq} (N_{xq} - 1) \frac{[\ln x(x - 1)]^2}{2} \right),$$ \hspace{1cm} (5.17)

and

$$g_0 = \frac{1}{512 \pi^4},$$ \hspace{1cm} (5.18)

$$g_1 = \frac{-2 \pi^2 + \pi^2 \ln 2 - 3 \pi^2 \ln \pi + 18 \zeta'(2)}{256 \pi^6}.$$ \hspace{1cm} (5.19)

Evaluating the integrals above numerically we obtain

$$J_{1B} = \frac{1}{2^7 \pi^4} \left( \frac{r_1}{(D - 4)} + 3.3175 \right)$$ \hspace{1cm} (5.20)
with

\[ r_1 = \int_0^\infty dq \, q \, n_q \{ \text{Box1} \} \]  \hspace{1cm} (5.21)

\[ = -0.7167667897 \ldots. \]  \hspace{1cm} (5.22)

Thus finally

\[ J_1 = J_{1A} + J_{1B} \]

\[ = \frac{1}{2^7 \pi^4} \left( 3.936 - \frac{0.7167667897}{(D - 4)} \right). \]  \hspace{1cm} (5.23)

Before going on to \( J_2 \), let us recapitulate the story of \( J_1 \). Our objective was to obtain the order \( 1/(D - 4) \) and \( O(D - 4)^0 \) pieces of \( J_1 \) as \( D \to 4 \). We first used a technique popular and effective at zero temperature, the Sudakov decomposition, which exposed the singularity structure of \( J_1 \) in a more manageable form. However, unlike \( T = 0 \) cases, further progress was hampered by the presence of the statistical factor which in the Sudakov basis obtains a complicated energy dependence. Our next step was to isolate the singularity of \( J_1 \) into a simpler integral by adding and subtracting terms in the original integral (see (5.7)). The simpler part of \( J_1 \), which we called \( J_{1B} \), had a statistical factor which did not depend on the transversal momentum \( K_\perp \) of the Sudakov decomposition and so we were able in this case to proceed further in extracting the singular and finite pieces of \( J_{1B} \) as \( D \to 4 \). The other part of \( J_1 \), which was created by the shift (5.7) and which we called \( J_{1A} \), was a complicated but finite object at \( D = 4 \) so we could evaluate it numerically.

\section*{J_2}

In the evaluation of \( H_2 \) in Sect. 4 we required,

\[ J_2 \equiv J_{21} + J_{22}, \]  \hspace{1cm} (5.24)

where

\[ J_{21} = - \int \frac{d^D K \, d^D Q \, d^D P}{(2\pi)^{3(D-1)}} N_k N_q N_p \delta_+(K^2) \delta_+(P^2) \delta_+(Q^2) \sum_{\sigma \gamma = \pm 1} (-\sigma) S_1(\sigma, \gamma), \]  \hspace{1cm} (5.25)
and

\[ J_{22} = \int \frac{d^D K d^D Q d^D P}{(2\pi)^{3(D-1)}} N_k N_q N_p \delta_+(K^2) \delta_+(Q^2) \delta_+(P^2) \sum_{\sigma \gamma = \pm 1} (\sigma \gamma) S_2(\sigma, \gamma), \quad (5.26) \]

and where we have defined

\[ S_N(\sigma, \gamma) = \left( \frac{P \cdot Q}{K \cdot Q} \right)^N \frac{1}{K \cdot Q + P \cdot (\sigma K + \gamma Q)}. \quad (5.27) \]

The evaluation of \( J_{21} \) is similar to that of \( J_1 \) discussed earlier. We find

\[ J_{21} = \frac{0.221}{2^7 \pi^6}. \quad (5.28) \]

Notice however that unlike \( J_1 \) there is no singular \( 1/(D - 4) \) contribution from \( J_{21} \).

Consider next the \( K \)-integral in \( J_{22} \) \((5.26)\) :

\[ \int d^D K \delta_+(K^2) \left( \frac{P \cdot Q}{K \cdot Q} \right)^2 \frac{N_{k_0}}{K \cdot Q + P \cdot (\sigma K + \gamma Q)}. \quad (5.29) \]

In the Sudakov basis this becomes

\[ \int_{-\infty}^\infty dx \int d^{D-2} K_\perp \frac{2sx}{(-K_\perp^2)^2} \frac{N_{k_0} \theta(k_0)}{\frac{(-K_\perp^2)}{2sx} + (\sigma x + \gamma)}, \quad (5.30) \]

with \( k_0 \) again given by \((5.6)\). As \( D \to 4 \), the IR singularity in this \( K_\perp \) integral is now more severe than the case of \((5.5)\) so that the rearrangement \((5.7)\) by itself is not sufficient to simplify the integral. However what is eventually required in \((5.26)\) is

\[ \sum_{\sigma \gamma = \pm 1} (\sigma \gamma) [\text{eqn.}(5.30)], \quad (5.31) \]

and we see that the IR behaviour of \((5.31)\) as \( K_\perp \to 0 \) is similar to that of \((5.5)\). Hence the split \((5.7)\) can be used as before to obtain the pole and finite pieces for \( J_{22} \). Since the analysis is very much the same as before (see however some comments at the end of Appendix B), we simply state the result :

\[ J_{22} = \frac{1}{2^7 \pi^6} \left( 1.430 - \frac{0.6420474257}{D - 4} \right). \quad (5.32) \]

Thus

\[ J_2 = J_{21} + J_{22} \]

\[ = \frac{1}{2^7 \pi^6} \left( 1.651 - \frac{0.6420474257}{D - 4} \right). \quad (5.33) \]
6 Results to Fourth Order

The pressure up to and including order $e^4$ then follows from Sect. 2 and eqns. (3.4)-(3.6), (4.6), (4.7), (4.14), (4.18):

$$
\frac{P}{T^4} = \frac{\pi^2}{45} \left( 1 + \frac{7}{4}N \right) - \frac{5e^2N}{288} + \frac{e^3}{12\pi} \left( \frac{N}{3} \right)^{3/2}
$$

$$
+ \frac{e^4}{\pi^6} \left( 0.4056 \right) - e^4N^2 \left( \frac{0.4667}{\pi^6} + \frac{5}{6\pi^2 \times 288 \ln \frac{T}{\mu}} \right). \quad (6.1)
$$

The coupling above is an implicit function of the renormalisation scale $\mu$. One may choose $\mu = T$ so as to eliminate the logarithm at this and higher orders. Then the pressure is a function of $e(T)$. In principle the value of $e(T)$ may be determined by comparing the perturbative calculation of some other observable at super-high temperature $T$ (where the electron mass is negligible) with its experimentally measured value. Alternatively one can use the renormalisation group to relate $e(T)$ to the coupling at some other scale $\Lambda$. Perhaps the most instructive thing to do is to write (6.1) in terms of the perturbative renormalization-group-invariant coupling, at the energy scale $T$, given by

$$
e^2(T) = e^2 \left( 1 + \frac{e^2N}{6\pi^2 \ln \frac{T}{\mu}} \right) + O(e^6). \quad (6.2)
$$

Define $\alpha(T) = e^2(T)/4\pi$ we arrive at

$$
\frac{P}{T^4} = \frac{\pi^2}{45} \left( 1 + \frac{7}{4}N \right) - \frac{5\pi^2}{72} \frac{\alpha(T)N}{\pi} + \frac{2\pi^2}{9\sqrt{3}} \left( \frac{\alpha(T)N}{\pi} \right)^{3/2}
$$

$$
+ \left( \frac{0.658 \pm 0.006}{N} - 0.757 \pm 0.004 \right) \left( \frac{\alpha(T)N}{\pi} \right)^2 + O \left( \alpha(T)^{5/2} \right). \quad (6.3)
$$

Notice the disappearance of the logarithm. We have also indicated in (6.3) estimates of numerical uncertainties due to the evaluation of some integrals by quadratures.

7 Fifth Order

The next correction to the pressure is of order $e^5$ and comes about by dressing the photon lines of the 3-loop diagrams. Its calculation is completely analogous to that of
the $e^3$ term reviewed in Sect.2 and the reader is encouraged to re-read the discussion there and in the references quoted. Since it has already been discussed at length in [2], here we only sketch another derivation using the "ring-summation" formula.

Consider the static, renormalised, one-loop photon polarisation tensor $\Pi_{\mu\nu}(q_0 = 0, q)$. From gauge invariance, $Q_{\mu}\Pi^{\mu\nu}(Q) = 0$, one obtains $\Pi_0(0, q) = 0$ while explicit calculations yield $\Pi_{ij}(0, q \to 0) = O(e^2q^2)$ and $\Pi_{00}(0, q \to 0) = m^2 + O(e^2q^2)$. Thus for diagrams $G_1$ and $G_2$ (Fig.2), one deduces from the usual power counting that it is only necessary to dress one of the photon lines with the static one-loop electric polarisation tensor to get the $e^5$ contribution. The resulting dressed diagrams are of the form of Fig.5 and are contained in the full ring sum which is summarised by the formula (see the third reference of [11]),

$$ P_{\text{ring}} = -\frac{1}{2} \int [dQ] Tr \{ \ln(1 - D(Q) \hat{\Pi}(Q)) + D(Q) \hat{\Pi}(Q) \}, \quad (7.1) $$

where the trace, $Tr$, is over Lorentz indices, $\hat{\Pi}$ is the full self-energy and $D(Q)$ the bare propagator. Define $\hat{F}(q) \equiv \hat{\Pi}_{00}(0, q)$. Then the restriction of (7.1) to the static electric sector (as mentioned earlier this is the only sector which will give the $e^5$ contribution) gives

$$ T \int \frac{d^{D-1}q}{(2\pi)^{D-1}} \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \frac{n F^2(q)}{(q^2)^n} (F^1(q))^{n-1}. \quad (7.2) $$

Now write $\hat{F}(q) = F^1(q) + F^2(q) + \ldots$, where the superscripts refer to the loop order. Then diagrams like those of Fig.5 are obtained by truncating (7.2) to the appropriate sector,

$$ [\text{eq.(7.2)}] \rightarrow T \int \frac{d^{D-1}q}{(2\pi)^{D-1}} \sum_{n=2}^{\infty} \frac{(-1)^n}{n} \frac{n F^2(q)}{(q^2)^n} (F^1(q))^{n-1}. \quad (7.3) $$

By scaling $q = m\vec{x}$ one deduces that for the $e^5$ contribution it is sufficient to set $q = 0$ in the $F^i$ of eq.(7.4). Thus the order $e^5N^{3/2}$ contribution to the pressure is given by

$$ [\text{eq.(7.4)}] \rightarrow \frac{T \sqrt{F^1(0)}}{8\pi} F^2(0) = \left. \frac{e^2T \sqrt{F^1(0)}}{8\pi} \frac{\partial^2 P_2}{\partial \mu_e^2} \right|_{\mu_e=0} \quad (7.5) $$

$$ = -\frac{e^5T^4 N^{3/2}}{64\pi^2 \sqrt{3}}, \quad (7.6) $$
where \( F^1(0) = m^2 = e^2 T^2 N/3 \) and \( P_2 \) denotes the pressure at two-loop order at chemical potential \( \mu_e \) \([1,1]\). In \((7.5)\) we used the relation \( \Pi_{00}(0,0) = e^2 \partial^2 P / \partial \mu_e^2 \) \([1,1]\). The result \((7.6)\) may also be obtained by direct calculation of the left-hand-side of \((7.5)\) \([2]\).

At order \( e^5 \) we still have the \( e^5 N^{5/2} \) contribution obtained by dressing \( G_3 \) (Fig. 4a). Now the ring summation formula \((7.2)\) has to be truncated in the sector where only iterations of \( F^1(q) \) occur, but for one of them the subleading momentum dependence is taken while the rest are at zero momentum:

\[
[\text{eq.(7.2)}] \to \quad \frac{T}{2} \int \frac{q^{D-1} q}{(2\pi)^{D-1}} \sum_{n=3}^\infty (-1)^n \left( \frac{\partial F^1}{\partial q^2} \right)_{q^2=0} \left( \frac{F^1(0)}{q^2} \right)^n = -\frac{m_3^2 T}{8\pi} \frac{\partial F^1}{\partial q^2} \bigg|_{q^2=0} \quad (7.7)
\]

\[
= -\frac{e^5 T^4 N^{5/2}}{8\pi \sqrt{27}} \frac{\gamma - 1 + \ln(4/\pi)}{12\pi^2} + \frac{e^5 T^4 N^{5/2}}{8\pi \sqrt{27}} \frac{\ln(T/\mu)}{6\pi^2}. \quad (7.8)
\]

As in the last section, the logarithm eventually disappears when the pressure is written in terms of the temperature-dependent coupling \((6.3)\).

The derivation of \((7.5)\) may be extended naturally to re-obtain the identity of \([2]\) linking the contribution of the pressure at order \(2n+3\) \( (n \geq 1) \) from diagrams with one-fermion loop to the pressure at order \(2n\). The relation \((7.5)\) and its higher order extensions were stated in \([2]\) for the case of massless fermions at zero chemical potential. Clearly one can relax these restrictions since only the dressing of the photon is involved. Thus in general

\[
P_{2n+3}^{1F} = \frac{e^2 T \sqrt{F^1(0)}}{8\pi} \frac{\partial^2 P_{2n}^{1F}}{\partial \mu_e^2} \quad (7.10)
\]

relates the gauge-invariant pressure at order \(2n+3\) \( (n \geq 1) \), from diagrams with one fermion loop, of QED with massive electrons at non-zero temperature but arbitrary chemical potential, to the pressure at order \(2n\) (the nonzero \(T\) is required so as to isolate the zero mode to be dressed). In this general case, \( F^1(0) \) is the lowest order \( (e^2) \) electric screening mass at nonzero \( T \), \( \mu_e \) and electron mass.
8 Conclusion

With $\alpha(T) = e^2(T)/4\pi$, and defining $g^2 = \alpha(T)N/\pi$, the pressure of QED with $N$ massless Dirac fermions at nonzero temperature, $T$, is given to fifth order by

$$\frac{P}{T^4} = a_0 + g^2a_2 + g^3a_3 + g^4(a_4 + b_4/N) + g^5(a_5 + b_5/N) + O(g^6), \quad (8.1)$$

with

$$a_0 = \frac{\pi^2}{45} (1 + \frac{7}{4}N), \quad (8.2)$$

$$a_2 = \frac{5\pi^2}{72}, \quad (8.3)$$

$$a_3 = \frac{2\pi^2}{9\sqrt{3}}, \quad (8.4)$$

$$a_4 = -0.757 \pm 0.004, \quad (8.5)$$

$$b_4 = 0.658 \pm 0.006, \quad (8.6)$$

$$a_5 = \frac{\pi^2[1 - \gamma - \ln(4/\pi)]}{9\sqrt{3}} = 0.11473..., \quad (8.7)$$

$$b_5 = \frac{-\pi^2}{2\sqrt{3}} = -2.849... \quad (8.8)$$

Real world QED corresponds to $N = 1$, but since we have ignored the electron mass the results are applicable only at extremely high temperatures [9]. Numerically, the fourth and fifth order terms we have found are small corrections in the regime where the coupling itself is small. However, since perturbative QED is not asymptotically free, the effective coupling $\alpha(T)$ increases slowly with temperature so the results might be of use for physics of the very early universe or, more speculatively, for studies of strongly coupled QED (some references are in [2]).

It is an amusing fact that the order $e^5$ contribution to the pressure of QED is much easier to calculate than the $e^4$ contribution. Indeed, as discussed in the last section, the fifth order calculation may even be extended to massive electrons and nonzero chemical potential (but nonzero $T$). However we have not bothered to give the explicit expressions in those cases because the fourth order calculation at nonzero $T$ is itself only known for massless electrons at zero chemical potential [3].

A 3-loop calculation in QCD will differ from the QED case in two respects. Firstly there is an increase in the number of diagrams. This however is not a problem (except for tedium) as we feel that our approach using the frequency-sum algorithm discussed...
in Sect.5 and the Sudakov method of Sect.6 is general enough to handle any new integrals that might arise. The second difference is that the static electric polarisation tensor in QCD behaves as $\Pi_{00}(0, q \to 0) = M^2 + qT$ and this gives rise to the $g^4 \ln g$ term \cite{7} from the sum of ring diagrams. Thus in this case one has to be more careful in using dimensional regularisation (as in this paper) to extract this term and also the constant under the logarithm.

Note added in proof: The three-loop free energy of hot Yang-Mills theory has been obtained in Ref.\cite{28}.

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Appendix A

A1. Identities for statistical factors
Let \( n_x = 1/(e^x - 1) \) and \( N_x = 1/(e^x + 1) \). Also, denote the step function by \( \theta(x) \) and the sign function by \( \epsilon(x) \). We have

\[
\begin{align*}
n_{-x} &= -(1 + n_x), \quad (A.1) \\
N_{-x} &= 1 - N_x, \quad (A.2) \\
n_x &= -\theta(-x) + \epsilon(x)n_{|x|}, \quad (A.3) \\
N_x &= \theta(-x) + \epsilon(x)N_{|x|}, \quad (A.4) \\
n_{x+y}(n_x - n_y) &= n_x n_y, \quad (A.5) \\
n_{x+y}(N_x - N_y) &= N_x n_y, \quad (A.6) \\
N_{x+y}(N_x + n_y) &= N_x n_y, \quad (A.7) \\
n_{x+y+z}(n_x n_z + n_y n_{-z} - n_x n_{-y}) &= n_x n_y n_z, \quad (A.8) \\
N_{x+y+z}(N_x N_{-y} - N_x N_z + N_y N_z) &= N_x N_y N_z. \quad (A.9)
\end{align*}
\]

The equations \((A.1)-(A.7)\) follow from the definitions of \( n_x \) and \( N_x \) while the last two are obtained by iterating \((A.3)-(A.7)\).

A2. Standard Results \([27]\)

\[
\begin{align*}
\sum_{n=1}^{\infty} \frac{1}{n^\alpha} &= \zeta(\alpha), \quad (A.10) \\
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^\alpha} &= \left(1 - 2^{1-\alpha}\right) \zeta(\alpha), \quad (A.11) \\
\mathcal{M}_1(\alpha) \equiv \int_0^{\infty} dx \frac{x^\alpha}{e^x - 1} &= \Gamma(\alpha + 1)\zeta(\alpha + 1), \quad (A.12) \\
\mathcal{M}_2(\alpha) \equiv \int_0^{\infty} dx \frac{x^\alpha}{e^x + 1} &= (1 - 2^{-\alpha})\Gamma(\alpha + 1)\zeta(\alpha + 1), \quad (A.13) \\
\int_0^{\infty} dx \frac{x^{p-1}}{e^{rx} - y} &= \frac{1}{y r^p} \Gamma(p) \sum_{n=1}^{\infty} \frac{y^n}{n^p}, \quad (p > 0, \ r > 0, \ -1 < y < 1), \quad (A.14)
\end{align*}
\]
\[
\int_0^\infty dx \ x^{\nu-1} e^{-\mu x} = \frac{1}{\mu^\nu} \Gamma(\nu), \quad \text{Re}(\mu, \nu) > 0. \tag{A.15}
\]

\[
\int_0^1 dx \ x^{\mu-1} (1-x)^{\nu-1} (ax + b (1-x) + c)^{-(\mu+\nu)} = \frac{(a+c)^{-\mu}(b+c)^{-\nu}B(\mu, \nu)}{a \geq 0, \ b \geq 0 \ c > 0, \ \text{Re} \ \mu > 0, \ \text{Re} \ \nu > 0.} \tag{A.16}
\]

A3. Derived Relations

\[M_3(\alpha, \beta) \equiv \int_0^\infty \int_0^\infty dx \ dy \ x^\alpha y^\beta n_{x+y} = \Gamma(\alpha+1) \Gamma(\beta+1) \zeta(\alpha+\beta+2), \tag{A.17}\]

\[M_4(\alpha, \beta) \equiv \int_0^\infty \int_0^\infty dx \ dy \ x^\alpha y^\beta n_{x+y} = \left(1-2^{-(1+\alpha+\beta)}\right) \Gamma(\alpha+1) \Gamma(\beta+1) \zeta(\alpha+\beta+2) \tag{A.18}\]

The result (A.17) is obtained by using in sequence the relations (A.14), (A.15) and (A.10), and similarly for (A.18).

• Some Covariant Integrals.

Define

\[F_N(+, \pm) \equiv \int d^D K \frac{\delta_+(K^2)\delta_+(Q^2)\delta_+(P^2)}{K \cdot (P + Q) \pm P \cdot Q \ (K \cdot Q)^N}, \tag{A.19}\]

\[F_N(-, \pm) \equiv \int d^D K \frac{\delta_+(K^2)\delta_+(Q^2)\delta_+(P^2)}{K \cdot (P - Q) \pm P \cdot Q \ (K \cdot Q)^N}. \tag{A.20}\]

Then

\[F_N(+, \pm) = \delta_+(Q^2)\delta_+(P^2) \left(\frac{P \cdot Q}{2}\right)^{D/2-2-N} C_N(+, \pm), \tag{A.21}\]

\[F_N(-, \pm) = \delta_+(Q^2)\delta_+(P^2) \left(\frac{P \cdot Q}{2}\right)^{D/2-2-N} C_N(-, \pm), \tag{A.22}\]

where

\[\frac{C_N(+, \pm)}{\omega(D-1) 2^{D-5-N}} = B(D/2-1, \ D/2-1-N) \ P_N(\pm), \tag{A.23}\]

\[\frac{C_N(-, \pm)}{\omega(D-1) 2^{D-5-N}} = B(3+N-D, \ D/2-1-N) \ P_N(\pm) - B(3+N-D, \ D/2-1) \ P_N(\mp), \tag{A.24}\]
and

\[ \omega(D) = \frac{2\pi^{D-1}}{\Gamma\left(\frac{D-1}{2}\right)}, \quad (A.25) \]

\[ P_N(+) = \pi \csc \left( (N + 3 - D)\pi \right), \quad (A.26) \]

\[ P_N(-) = \pi \cot \left( (N + 3 - D)\pi \right). \quad (A.27) \]

The result (A.21) is obtained as follows: Since the integral is covariant, it may be evaluated in any convenient frame. Choose \( \vec{p} = -\vec{q} \). The only nontrivial integrals are the radial integral \( \int_0^\infty dk \) and the angular integral \( \int_{-1}^1 d\cos \theta \), where \( \theta \) is the angle between \( \vec{k} \) and \( \vec{q} \). These two integrals may be decoupled by a simple change of variables and the angular integral then evaluated using (A.16) while the radial integrals are

\[ P_N(\pm) \equiv \int_0^\infty dz \frac{z^{D-3-N}}{z \pm 1}. \quad (A.28) \]

For the + case the integral is standard \([27]\) while the – case is interpreted in the principal value sense and the result is indicated in (A.27). Consider instead the integral

\[ \hat{P}_N(-) \equiv \int_0^\infty dz \frac{z^{D-3-N}}{z - 1 \mp i0^+} \quad (A.29) \]

\[ = P_N(-) \pm i\pi. \quad (A.30) \]

Compared to \( P_N(-) \), the integral \( \hat{P}_N(-) \) has the original pole at \( z = 1 \) shifted above or below the real-axis and this shift may be viewed as a "regularisation" of \( P_N(-) \) which is then given by the real part of \( \hat{P}_N(-) \). Combining eqns. (A.28-A.27) and (A.30), we can relate \( \hat{P}_N(-) \) to \( P_N(+) \) which is regular,

\[ \hat{P}_N(-) = e^{\mp i\pi(D-3-N)} P_N(+). \quad (A.31) \]

A direct derivation of this result can be obtained by an analytic continuation of \( P_N(\pm) \) for \( z < 0 \). We omit this presentation since it requires more involved considerations of the integrals for \( z < 0 \).
A4. Relations between Gamma and Zeta functions
One has the standard formulae \[27\]
\[
\Gamma(-n + \epsilon) = \frac{(-1)^n}{n!} \left( \frac{1}{\epsilon} + (1 + \frac{1}{2} + \ldots + \frac{1}{n} - \gamma) + O(\epsilon) \right) \tag{A.32}
\]
\[
\gamma = \lim_{z \to 1} \left( \zeta(z) - \frac{1}{z - 1} \right) = 0.5772157 \ldots \tag{A.33}
\]
\[
\pi^{1-z} \zeta(z) = 2^z \Gamma(1 - z) \zeta(1 - z) \sin \frac{\pi z}{2}, \tag{A.34}
\]
and one may deduce
\[
\zeta(0) = -\frac{1}{2}, \quad \zeta'(0) = -\frac{1}{2} \ln 2\pi, \quad \zeta''(0) = -2.00635645 \ldots,
\]
\[
\zeta(-1) = -\frac{1}{12}, \quad \zeta'(-1) = -0.16542114369 \ldots,
\]
\[
\zeta''(-1) = -0.2502044 \ldots,
\]
\[
\zeta(2) = \frac{\pi^2}{6}, \quad \zeta'(2) = -0.937548254 \ldots,
\]
\[
\zeta''(2) = 1.9892802342 \ldots
\]
The specific values above may be obtained by a Taylor expansion of both sides of (A.12) and (A.13) with respect to an appropriate value of \( \alpha \), and using (A.32)-(A.34).

Appendix B
Here we describe how to obtain (5.8). The contribution of the square bracket in (5.7) to (5.5) is finite as \( D \to 4 \) and is given by
\[
V \equiv \int_{-\infty}^{\infty} dx \int d^2 K_\perp \frac{1}{(-K_\perp^2)} \left[ N_{k_0} \theta(k_0) - N_{xq} \theta(xq) \right] \tag{B.1}
\]
where \( s = P \cdot Q = pq - \vec{p} \cdot \vec{q} \) and \( k_0 = -\frac{K_\perp^2}{2s} p + xq + (K_\perp)_0 \). As in the discussion of (5.3) we first decompose \( K_\perp \) along two orthonormal space-like vectors. Thus we write
\[
K_\perp = \sigma_1 e^{(1)} + \sigma_2 e^{(2)} \tag{B.2}
\]
where the \( e^i \) are two orthonormal space-like vectors : \( e^{(1)} \cdot e^{(2)} = 0 \) and \( (e^{(1)})^2 = (e^{(2)})^2 = -1 \). However unlike the case of (5.3), the integral in (B.1) depends explicitly on the complicated energy \( k_0 \) and so we now require an explicit parametrisation of
our basis in order to proceed. Though the Lorentz symmetry of the integrals in $J_1$, of which (B.1) is a part of, is broken by the heat bath, we still have three dimensional rotational invariance in the $(\vec{p}, \vec{q})$ integrals. Therefore we can choose the following explicit basis (we are grateful to Cosmas Zachos for discussions on this point) for the evaluation of (B.1) and its contribution to $J_1$:

$$
P_\mu = p(1, 0, 0, 1)\\
Q_\mu = q(1, 0, \sin \phi, \cos \phi)\\
e^{(1)} = (0, 1, 0, 0)\\
e^{(2)} = \left( \frac{\sin \phi}{1 - \cos \phi}, 0, 1, \frac{\sin \phi}{1 - \cos \phi} \right).
$$  \hfill (B.3)

where $\phi$ is the angle between $\vec{p}$ and $\vec{q}$.

The basis (B.3) satisfies the requirements

$$
P^2 = Q^2 = e^{(1)} \cdot P = e^{(1)} \cdot Q = e^{(2)} \cdot P = e^{(2)} \cdot Q = 0,
$$

and $p_0 = p$, $q_0 = q$, $s = P \cdot Q = pq(1 - \cos \phi) \geq 0 \ (-\pi \leq \phi \leq \pi)$. Thus (B.1) can be written as

$$
V = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} d\sigma_1 \int_{-\infty}^{\infty} d\sigma_2 \ \frac{1}{\sigma_1^2 + \sigma_2^2} \ \frac{[N_{k_0} \theta(k_0) - N_{xq} \theta(xq)]}{(\sigma_1^2 + \sigma_2^2) + (\sigma x + \gamma)}
$$

with $k_0 = \frac{(\sigma_1^2 + \sigma_2^2)}{2xs} p + xq + \sigma_2 e^{(2)}_0$.

Now it is convenient to transform from the Cartesian $(\sigma_1, \sigma_2)$ coordinates to the polar coordinates defined by

$$
\sigma_1 = y \sin \theta, \quad \sigma_2 = y \cos \theta
$$

and then change variables $y^2 \to y$ to obtain

$$
V = \int_{-\infty}^{\infty} dx \int_{0}^{\infty} \frac{dy}{y} \int_{0}^{\pi} d\theta \ \frac{[N_{k_0} \theta(k_0) - N_{xq} \theta(xq)]}{\frac{y}{2xs} + (\sigma x + \gamma)}
$$

with $k'_0 = \frac{wp}{2xs} + xq + \sqrt{y} \cos \theta e^{(2)}_0$.

Next split the $x-$integral in (B.6) as $\int_{0}^{\infty} dx + \int_{-\infty}^{0} dx$ and in the second part do $x \to -x$, $\theta \to \pi - \theta$, and note that eventually we need $\sum_{\gamma=\pm1} V$, to get (after some simplification)

$$
\sum_{\gamma=\pm1} V = \sum_{\gamma=\pm1} \int_{0}^{\infty} dx \int_{0}^{\infty} \frac{dz}{z} \int_{0}^{\pi} d\theta \ \frac{[\epsilon(k_0) N_{|k_0|} - N_{xq}]}{z + (\sigma x + \gamma)}
$$

(B.7)
where
\[
\hat{k}_0 = pz + xq + \sqrt{2sxz} \cos \theta e_0^{(2)}
\]
\[
= pz + xq + \sqrt{2pqxz (1 + \cos \phi)} \cos \theta.
\] (B.8)

Furthermore it follows from (B.8) that \(\hat{k}_0 \geq 0\), and thus we can write (B.7) as
\[
\sum_{\gamma = \pm 1} V = \sum_{\gamma = \pm 1} \int_0^\infty dx \int_0^\infty dz \int_0^\pi d\theta \frac{(N_{k_0} - N_{xq})}{z + (\sigma x + \gamma)}.
\] (B.9)

Finally \(J_{1A}\) is given by
\[
J_{1A} = \int \frac{d^3p \, d^3q}{4pq} \frac{(N_p + n_p)n_q}{(2\pi)^9} \sum_{\gamma = \pm 1} V
\]
\[
= \frac{(4\pi)(2\pi)}{4(2\pi)^9} \int_0^\infty dp \, p \int_0^\infty dq \, q \int_{-1}^1 d\cos \phi \sum_{\gamma = \pm 1} V, \tag{B.10}
\]
with \(V\) defined through equations (B.9) and (B.8). The integrals in (B.10) can now be performed numerically. One technical point that should be noted is the principal value singularity in the factor \(\frac{1}{z + (\sigma x + \gamma)}\) occurring in (B.9) and (B.10). Consider, for example, the case \(P \frac{1}{y-1}\) in (B.9). Doing the change of variables \(z = y|x - 1|\) gives us the factor \(P \frac{1}{y-1}\) to deal with in the new \(\int_0^\infty dy\) integral. We use the definition of the principal value to write
\[
\int_0^\infty dy \, P \frac{1}{y-1} \times (\text{rest})
\]
\[
= \lim_{\delta \to 0} \left( \int_0^{1-\delta} \frac{dy}{y-1} \times (\text{rest}) + \int_{1+\delta}^\infty \frac{dy}{y-1} \times (\text{rest}) \right), \tag{B.11}
\]
where “(rest)” denotes the rest of the integrand in (B.9) after the change \(z = y|x - 1|\).

In the second term in (B.11) one can do the change of variables \(y \to \frac{1}{y}\) and then combine the result with the first term to get an integral of the form
\[
\int_0^1 \frac{dy}{y-1} \times (\text{new rest}), \tag{B.12}
\]
where the limit \(\delta \to 0\) can be taken because now the integral is finite as \(y \to 1\) since (new rest) \(\to (y - 1)\) as \(y \to 1\). Once the principal value singularity has been removed by transforming (B.11) into (B.12), the numerical integration of (B.10) can proceed and we obtain
\[
J_{1A} = \frac{6.101}{2^7\pi^6}. \tag{B.13}
\]
Some remarks are in order. The change of variables $z = y|x - 1|$ that yielded (B.11) from (B.10) results in an $x$-integral of the form $\int_0^\infty dx(\cdot)/|x - 1|$. In this case one appears to have created a new singularity at $x = 1$. However this is not true because the rest of the integrand actually compensates it as $x \to 1$. In other places in this paper, for example in the evaluation of $J_{1B}$ (5.14) and also $F_N(−,±)$ (A.20) near $D = 4$ dimensions, the change $z = y|x - 1|$ actually creates a singularity at $x = 1$ when $D = 4$. Noting that $\int_0^\infty dx(\cdot)/|x - 1|$ actually means $\int_0^1 dx(\cdot)/(1 - x) + \int_1^\infty dx(\cdot)/(x - 1)$, we have used dimensional continuation separately for each part to regulate the singularity at $x = 1$ in those cases.

For another example, consider an integral of the form

$$\int_0^\infty dz \int_0^\infty dx \left[ \frac{1}{z - (x - 1)} + \frac{1}{z + (x - 1)} \right] (\star), \quad (B.14)$$

which appears in the evaluation of $J_2$ (5.31). If one now naively does the change $z = y|x - 1|$ to simplify the integrals, the $x$-integral becomes $\int_0^\infty \frac{dx(\cdot)}{|x - 1|^2}$ (new* and the singularity at $x = 1$ is not compensated. The safe way to proceed in this case is to first do $x \to xz$ in (B.14) to get

$$\int_0^\infty dx \int_0^\infty dz \left[ \frac{1}{z(1 - x) + 1} + \frac{1}{z(x + 1) - 1} \right] (\star : x \to xz). \quad (B.15)$$

Then do $z \to z|x - 1|$ for the first term, $z \to z(x + 1)$ for the second, and finally proceed as per (B.11)-(B.12) to remove the principal value prescription so that the integrals may be handled numerically.
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Figure Captions

Fig.1:
Contribution to order $e^2$. The wavy line represents the photon propagator.

Fig.2:
The order $e^4N$ contributions $G_1$ and $G_2$.

Fig.3:
Ultraviolet counterterm diagrams for Fig.2.

Fig.4:
Fig.4a is the $e^4N^2$ contribution ($G_3$) while Fig.4b is the corresponding counterterm diagram $X_1$.

Fig.5:
Diagrams obtained by self-energy insertions along one photon propagator in the diagrams of Fig.2.
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-ph/9409269v1
This figure "fig2-1.png" is available in "png" format from:

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