The effect of the domain topology on the number of positive solutions of an elliptic Kirchhoff problem

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Abstract

Using minimax methods and Lusternik-Schnirelmann theory, we study multiple positive solutions for the Schrödinger - Kirchhoff equation

\[ M \left( \int_{\Omega_\lambda} |\nabla u|^2 \, dx + \int_{\Omega_\lambda} u^2 \, dx \right) [-\Delta u + u] = f(u) \]

in \( \Omega_\lambda = \lambda \Omega \). The set \( \Omega \subset \mathbb{R}^3 \) is a smooth bounded domain, \( \lambda > 0 \) is a parameter, \( M \) is a general continuous function and \( f \) is a superlinear continuous function with subcritical growth. Our main result relates, for large values of \( \lambda \), the number of solutions with the least number of closed and contractible in \( \overline{\Omega} \) which cover \( \overline{\Omega} \).

Keywords: Schrödinger - Kirchhoff type problem; Lusternik-Schnirelmann Theory; expanding domain.

1991 Mathematics Subject Classification. Primary 35J65, 34B15.

1 Introduction

In this paper we study multiple positive solutions for the following problem

\[ (P_\lambda) \]

\[ \begin{aligned}
\mathcal{L} u &= f(u), \quad \Omega_\lambda \\
\quad & \quad u > 0, \quad \Omega_\lambda \\
\quad & \quad u = 0, \quad \partial \Omega_\lambda
\end{aligned} \]

where \( \Omega \subset \mathbb{R}^3 \) is a smooth bounded domain, \( \lambda > 0 \) is a parameter, \( \Omega_\lambda := \lambda \Omega \) is an expanding domain and \( \mathcal{L} \) is the nonlocal operator given by

\[ \mathcal{L} u = M \left( \int_{\Omega_\lambda} |\nabla u|^2 \, dx + \int_{\Omega_\lambda} u^2 \, dx \right) [-\Delta u + u]. \]

In 1883, Kirchhoff [14] established the equation

\[ (K) \]

\[ \frac{\partial^2 u}{\partial t^2} - \left( \frac{P_0}{h} \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 \, dx \right) \frac{\partial^2 u}{\partial x^2} = 0 \]

*Supported by CAPES - Brazil - 7155123/2012-9
where $L$ is the length of the string, $h$ is the area of cross-section, $E$ is the Young modulus of the material, $\rho$ is the mass density and $P_0$ is the initial tension. This model was proposed to modify the classical d’Alembert’s wave equation, assuming a nonlinear dependence of the axial strain on the deformation of the gradient.

Owing to its importance in engineering, physics and material mechanics, a considerable effort has been devoted during the last years to the study the generalization of the stationary equation associated with problem (K). With no hope of being thorough, we mention some papers regarding the study of this class of problems: [2], [12], [13], [15], [16], [19], [20], [22] and reference therein. For an excellent didactic about this class of problems we cite [6] and for an overview of non-local problems we cite [10].

Problem $(P_\lambda)$ is a generalization of the stationary problem associated with problem (K). Before stating our main result, we need the following hypotheses on the functions $M$ and $f$.

The continuous function $M : \mathbb{R}_+ \to \mathbb{R}_+$ and the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:

$(M_1)$ There is $m_0 > 0$ such that $M(t) \geq m_0$, $\forall t \geq 0$.

$(M_2)$ The function $t \mapsto M(t)$ is increasing.

$(M_3)$ The function $t \mapsto \frac{M(t)}{t}$ is decreasing.

A typical example of function verifying the assumptions $(M_1) - (M_3)$ is given by $M(t) = m_0 + bt$, with $m_0 > 0$ and $b > 0$. More generally, each function of the form $M(t) = m_0 + bt + \sum_{i=1}^{k} b_i t^{\gamma_i}$ with $b_i \geq 0$ and $\gamma_i \in (0,1)$ for all $i \in \{1, 2, \ldots, k\}$ verifies the hypotheses $(M_1) - (M_3)$.

Now we give an example of a continuous but non-differentiable function that satisfies such hypotheses. Let $m_0, b_0, b_1$ and $t_0$ be positive constants such that $b_0 \neq b_1$ and $t_0 < \frac{m_0}{b_1 - b_0}$ if $b_0 < b_1$. We define the continuous function

$$M(t) = \begin{cases} 
    m_0 + b_0 t, & \text{if } 0 \leq t \leq t_0 \\
    m_0 + (b_0 - b_1)t_0 + b_1 t, & \text{if } t_0 \leq t 
\end{cases}$$

Since that $b_0 \neq b_1$, we have that $M$ is non-differentiable in $t_0$. Using the same reasoning, we can build continuous functions that are not differentiable in a finite number of points.

We assume that the locally Lipschitz continuous function $f$ vanishes in $(-\infty, 0)$ and verifies

$(f_1)$ \[ \lim_{t \to 0^+} \frac{f(t)}{t^3} = 0. \]

$(f_2)$ There is $q \in (4, 6)$ such that

$$\lim_{t \to \infty} \frac{f(t)}{t^{q-1}} = 0.$$
(f_3) There is \( \theta \in (4, 6) \) such that
\[
0 < \theta F(t) \leq f(t)t, \; \forall t > 0,
\]
where \( F(s) = \int_0^s f(t)dt. \)

(f_4) The application
\[
t \mapsto \frac{f(t)}{t^3}
\]
is nondecreasing in \( (0, \infty) \).

A typical example of locally Lipschitz continuous function verifying the assumptions \((f_1)-(f_4)\) is given by
\[
f(t) = \sum_{i=1}^{n} c_i (t^+)^{q_i - 1}
\]
with \( c_i \geq 0 \) not all zero and \( q_i \in \left[\theta, 6\right) \) for all \( i \in \{1, 2, \ldots, n\} \). Moreover, a very simple example of non-differentiable function verifying these hypotheses is given by
\[
f(t) = \begin{cases} 
c(t^+)^{q_2}, & \text{if } 0 \leq t \leq 1 \\
c(t^+)^{q_1}, & \text{if } t \geq 1,
\end{cases}
\]
where \( c > 0 \) and \( 4 < \theta \leq q_1 < q_2 < 6 \).

The main result of this paper is:

**Theorem 1.1** Suppose that the function \( M \) satisfies \( (M_1)-(M_3) \) and the function \( f \) satisfies \( (f_1)-(f_4) \). Then there exists \( \lambda^* > 0 \) such that, for each \( \lambda \in (\lambda^*, \infty) \), the problem \((P_\lambda)\) has at least \( \text{cat}\Omega \) positive weak solutions. Moreover, if \( \text{cat}\Omega > 1 \) then \((P_\lambda)\) has at least \( \text{cat}\Omega + 1 \) weak solutions.

For more informations about the Lusternik - Schnirelmann category, we refer to [7] and [8].

When \( M = 1 \) we have the following problem
\[
(BC) \quad \begin{cases} 
-\Delta u + u = f(u), & \Omega_\lambda \\
u > 0, & \Omega_\lambda \\
u = 0, & \partial \Omega_\lambda
\end{cases}
\]
that was studied first by Benci and Cerami in [7]. In order to obtain multiple solutions for this problem, these authors made comparisons between the category of some sublevel sets of the functional associated to the problem and the category of the domain \( \Omega_\lambda \). After this excellent paper, appeared several generalizations. A version of this problem for a class of quasilinear equation can be seen in [4]. In [5] there is a version considering the Schrödinger operator in the presence of magnetic potential. The case with \( p \)-Laplacian operator is in [1]. In all these works the nonlinearity \( f \) is \( C^1 \) class, because in the arguments used was important the regularity of the Nehari manifold associated to \((P_\lambda)\).
Problem \((P_\lambda)\) is a nonlocal version of the \((BC)\) considering the Kirchhoff operator. But the presence of Kirchhoff operator with \(M\) and \(f\) only continuous imply that several estimates used in \([7], [1], [4]\) and \([5]\) cannot be repeated for the functional energy associated to \((P_\lambda)\). To overcome this difficult we use an argument that can be found in \([18]\) and \([19]\), and we introduce some Lemmas, as for example, Lemmas \(2, 3\) and \(2, 4\). However, due to the presence of the function \(M\), some estimates more refined are need, such as in the study of the limit problem and in the Lemma \(3.2\).

An important point in this type of arguments is the existence of solution of a limit problem. In our case, the limit problem is given by

\[
(P_\infty) \quad \begin{cases} 
\mathcal{L}_\infty u = f(u), & \mathbb{R}^3 \\
u > 0, & \mathbb{R}^3.
\end{cases}
\]

This result was proved in \([3]\). In our paper we show this result of existence considering a less restrictive set of assumptions about \(f\) and \(M\).

The paper is organized as follows. In the section 2 we study the existence of solution of the limit problem and we prove a compactness result of the Nehari manifold associated to the functional of the limit problem. This study was not necessary in \([3]\). In the section 3 we study the behavior of minimax levels from the functional associated to the problem \((P_\lambda)\). The main result is proved in the section 4.

2 The limit problem

An important result that we shall use in this work is related to the existence of a positive ground-state solution for the problem

\[
(P_\infty) \quad \begin{cases} 
\mathcal{L}_\infty u = f(u), & \mathbb{R}^3 \\
u > 0, & \mathbb{R}^3.
\end{cases}
\]

where

\[
\mathcal{L}_\infty u = M \left( \int_{\mathbb{R}^3} |\nabla u|^2 \, dx + \int_{\mathbb{R}^3} u^2 \, dx \right) [-\Delta u + u].
\]

More precisely, we are concerned with the existence of a positive function \(u \in H^1(\mathbb{R}^3)\) verifying

\[
I_\infty(u) = c_\infty \quad \text{and} \quad I_\infty'(u) = 0,
\]

where

\[
I_\infty(u) = \frac{1}{2} \hat{M}(||u||^2) - \int_{\mathbb{R}^3} F(u) \, dx,
\]

\[
\hat{M}(t) = \int_0^t M(s) \, ds \quad \text{and} \quad c_\infty \text{ denotes the minimax level of the mountain pass theorem associated to the functional } I_\infty \text{ and given by}
\]

\[
c_\infty = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} I_\infty(\gamma(t)),
\]

with \(\Gamma = \{ \gamma \in C([0,1], H^1(\mathbb{R}^3) : \gamma(0) = 0 \text{ and } I_\infty(\gamma(1)) < 0}\}.

4
It is not difficult to check that $I_{\infty}$ is a $C^1$ functional,

$$I_{\infty}'(u)v = M(\|u\|^2)(u, v) - \int_{\mathbb{R}^3} f(u)v \, dx, \quad \forall u, v \in H^1(\mathbb{R}^3)$$

and its Nehari manifold is given by

$$N_{\infty} = \{ u \in H^1(\mathbb{R}^3) \setminus \{0\} : I_{\infty}'(u)u = 0 \}.$$

Here, $(,)$ and $\| \|$ denote, respectively, the standard inner product of $H^1(\mathbb{R}^3)$ and its induced norm.

In [3, Theorem 2.5] was proved that problem $(P_{\infty})$ has a positive solution. Since that in our paper we have a smaller number of the hypotheses, we give some details about this result.

2.1 Existence of positive ground-state solution for the limit problem

Notice from $(M_3)$, there is a positive constant $K$ such that

$$M(t) \leq K + M(1)t,$$  \hspace{1cm} (2.1)

for all $t \geq 0$. Thus, the growth condition $(M_3)$ and $(M_1)$ allow us to use [3, Lemma 2.1] and to conclude that functional $I_{\infty}$ has the Mountain Pass geometry and by a version of Mountain Pass Theorem (see [21, Theorem 1.15]), there is an $(PS)_{c_{\infty}}$ sequence for the functional $I_{\infty}$, that is, there is a sequence $(u_n) \subset H^1(\mathbb{R}^3)$ such that

$$I_{\infty}(u_n) \to c_{\infty} \text{ and } I_{\infty}'(u_n) \to 0.$$

Using [3, Lemma 2.4] we can assume that the weak limit of a $(PS)_{c_{\infty}}$ is nontrivial. Indeed, suppose that $u_n \not\to 0$. Since $u_n \not\to 0$, there are $(y_n) \subset \mathbb{R}^3$ and $R, \beta > 0$ such that

$$\liminf_{n \to \infty} \int_{B_R(y_n)} u_n^2 \, dx \geq \beta > 0.$$

Considering $v_n(x) = u_n(x + y_n)$, we can prove that $(v_n)$ is a $(PS)_{c_{\infty}}$ sequence for the functional $I_{\infty}$, $(v_n)$ is bounded in $H^1(\mathbb{R}^3)$ and there is $v \in H^1(\mathbb{R}^3)$ non-trivial, with $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^3)$.

In the next Proposition we obtain a positive ground-state solution for the autonomous problem $(P_{\infty})$.

**Theorem 2.1** Let $(u_n) \subset H^1(\mathbb{R}^3)$ be a $(PS)_{c_{\infty}}$ sequence for $I_{\infty}$. Then there is $u \in H^1(\mathbb{R}^3) \setminus \{0\}$ with $u \geq 0$ such that, passing a subsequence, we have $u_n \to u$ in $H^1(\mathbb{R}^3)$. Moreover, $u$ is a positive ground-state solution for the problem $(P_{\infty})$.

**Proof.** Firstly we prove that $(u_n)$ is bounded in $H^1(\mathbb{R}^3)$. Since $(u_n) \subset H^1(\mathbb{R}^3)$ is a $(PS)_{c_{\infty}}$ sequence for $I_{\infty}$, there exists $C > 0$ such that

$$C + \|u_n\| \geq I_{\infty}(u_n) - \frac{1}{\theta} I_{\infty}'(u_n)u_n, \quad \forall n \in \mathbb{N}$$
From (f3), we get
\[ C + \|u_n\| \geq \frac{1}{2} \hat{M}(\|u_n\|^2) - \frac{1}{\theta} M(\|u_n\|^2)\|u_n\|^2, \quad \forall n \in \mathbb{N} \]

Notice that by (M3) we have
\[ \hat{M}(t) \geq \frac{1}{2} M(t) t, \quad (2.2) \]
for all \( t \geq 0 \). This inequality allows us to conclude that,
\[ \frac{1}{t} \left[ \frac{1}{2} \hat{M}(t) - \frac{1}{\theta} M(t) t \right] \geq \left( \frac{\theta - 4}{4\theta} \right) m_0, \quad (2.3) \]
for all \( t > 0 \). Now, suppose by contradiction that, up to a subsequence, \( \|u_n\| \to \infty \). Thus,
\[ \frac{C}{\|u_n\|^2} + \frac{1}{\|u_n\|^2} \left[ \frac{1}{2} \hat{M}(\|u_n\|^2) - \frac{1}{\theta} M(\|u_n\|^2)\|u_n\|^2 \right], \quad \forall n \in \mathbb{N}. \]
Passing to limit we get
\[ 0 \geq \left( \frac{\theta - 4}{4\theta} \right) m_0 > 0, \]
which is an absurd. Hence, there exist \( u \in H^1(\mathbb{R}^3) \) and \( t_0 > 0 \) such that
\[ u_n \rightharpoonup u \text{ in } H^1(\mathbb{R}^3), \quad (2.4) \]
and
\[ \|u_n\| \to t_0. \quad (2.5) \]

Since \( M \) is continuous function, we get
\[ M(\|u_n\|^2) \to M(t_0^2). \quad (2.6) \]

From [3, Theorem 2.5] we conclude that
\[ M(t_0^2) = M(\|u\|^2). \quad (2.7) \]

Using (M2) we obtain \( \|u\| = t_0 \) and the lemma is proved. ■

An important property that we can derive from (2.4), (2.5) and (2.7) is that, up to a subsequence,
\[ u_n \to u \text{ in } H^1(\mathbb{R}^3) \quad (2.8) \]

### 2.2 A compactness result on the Nehari manifold associated to limit problem

We denote by \( H^{1,+}(\mathbb{R}^3) \) the open subset of \( H^1(\mathbb{R}^3) \) given by
\[ H^{1,+}(\mathbb{R}^3) = \{ u \in H^1(\mathbb{R}^3) : u^+ = \max\{0, u\} \neq 0 \}. \]
and $S^+_{\infty} = S_{\infty} \cap H^{1,+}(\mathbb{R}^3)$, where $S_{\infty}$ is unit sphere of $H^1(\mathbb{R}^3)$.

Note that $S^+_{\infty}$ is a non-complete $C^{1,1}$-manifold of codimension 1, modeled on $H^1(\mathbb{R}^3)$ and contained in the open $H^{1,+}(\mathbb{R}^3)$. Thus, $H^1(\mathbb{R}^3) = T_u S^+_{\infty} \oplus \mathbb{R} u$ for each $u \in S^+_{\infty}$, where $T_u S^+_{\infty} = \{ v \in H^1(\mathbb{R}^3) : (u, v) = 0 \}$.

As can be seen in [18, Theorem 3.1], to prove the main result, it is very important to obtain a result of compactness on of the Nehari manifold associated to limit problem. Since $f$ and $M$ are only continuous, we cannot claim that $\mathcal{N}_{\infty}$ is a continuous manifold. Here was necessary a new argument. In the Lemmas 2.1 and 2.2 we adapt arguments from the excellent book [18, Chapter 3], see also [17]. In the Lemma 2.3 we prove a result of the type Ekeland’s Principle on set $S^+_{\infty}$. The behavior of the functional $I_{\infty}$ near the boundary of $S^+_{\infty}$ (see Lemma 2.1(A4)) overcome the fact that this set is a non-complete $C^{1,1}$-manifold of $H^1(\mathbb{R}^3)$. In the Lemma 2.4 we prove the main result of the section.

**Lemma 2.1** Suppose that the function $M$ satisfies $(M_1)-(M_3)$ and the function $f$ satisfies $(f_1)-(f_4)$. Then:

(A1) For each $u \in H^{1,+}(\mathbb{R}^3)$, let $h : \mathbb{R}_u \to \mathbb{R}$ be defined by $h_u(t) = I_{\infty}(tu)$. Then, there is a unique $t_u > 0$ such that $h_u'(t) > 0$ in $(0, t_u)$ and $h_u'(t) < 0$ in $(t_u, \infty)$.

(A2) There is $\tau > 0$ independent on $u$ such that $t_u \geq \tau$ for all $u \in S^+_{\infty}$. Moreover, for each compact set $W \subset S^+_{\infty}$ there is $C_W > 0$ such that $t_u \leq C_W$, for all $u \in W$.

(A3) The map $\tilde{m}_{\infty} : H^{1,+}(\mathbb{R}^3) \to \mathcal{N}_{\infty}$ given by $\tilde{m}_{\infty}(u) = t_u u$ is continuous and $m_{\infty} := \tilde{m}_{\infty}|_{S^+_{\infty}}$ is a homeomorphism between $S^+_{\infty}$ and $\mathcal{N}_{\infty}$. Moreover, $m_{\infty}^{-1}(u) = \| u \|_\infty$.

(A4) If there is a sequence $(u_n) \subset S^+_{\infty}$ such that $\text{dist}(u_n, \partial H^{1,+}(\mathbb{R}^3)) \to 0$, then $\| m_{\infty}(u_n) \|_{\infty} \to \infty$ and $I_{\infty}(m_{\infty}(u_n)) \to \infty$.

**Proof.** Since that $(M_1)$ and (2.1) occurs, the items (A1), (A2) and (A3) follows of the simple adaptation of the [18, Proposition 8]. The item (A4) follows by [18, Lemma 26].

We use (A3) and (A4) to overcome the lack of differentiability of $\mathcal{N}_{\infty}$ (see Lemmas 2.3 and 2.4).

Now we define

$$\tilde{\Psi}_{\infty} : H^{1,+}(\mathbb{R}^3) \to \mathbb{R} \ e \ \Psi_{\infty} : S^+_{\infty} \to \mathbb{R},$$

by $\tilde{\Psi}_{\infty}(u) = I_{\infty}(\tilde{m}_{\infty}(u))$ and $\Psi_{\infty} := (\tilde{\Psi}_{\infty})|_{S^+_{\infty}}$.

Using the same type of arguments explored in ([18, Lemma 26]) we can prove the next lemma. Thus, we omit the proof.

**Lemma 2.2** Suppose that the function $M$ satisfies $(M_1)-(M_3)$ and the function $f$ satisfies $(f_1)-(f_4)$. Then:

(a) $\tilde{\Psi}_{\infty} \in C^1(H^{1,+}(\mathbb{R}^3), \mathbb{R})$ e

$$\tilde{\Psi}_{\infty}'(u)v = \frac{\| \tilde{m}_{\infty}(u) \|}{\| u \|} I_{\infty}(\tilde{m}_{\infty}(u))v, \ \forall u \in H^{1,+}(\mathbb{R}^3) \ e \ \forall v \in H^1(\mathbb{R}^3).$$
(b) \(\Psi_{\infty} \in C^1(S^+_\infty, \mathbb{R})\) e
\[
\Psi'(u)v = \|m(u)\|I'_{\infty}(m_{\infty}(u))v, \forall v \in T_uS^+_\infty.
\]
(c) If \((u_n)\) is a \((PS)_c\) sequence for \(\Psi_{\infty}\), then \((m_{\infty}(u_n))\) is a \((PS)_c\) sequence for the functional \(I_{\infty}\). If \((u_n) \subset N_{\infty}\) is a bounded \((PS)_c\) sequence for \(I_{\infty}\), then \((m_{\infty}^{-1}(u_n))\) is a \((PS)_c\) sequence for \(\Psi_{\infty}\).

(d) \(u\) is critical point of \(\Psi_{\infty}\) if, and only if, \(m_{\infty}(u)\) is a nontrivial critical point of \(I_{\infty}\). Moreover, the critical values are the same and
\[
\inf_{S^+_\infty} \Psi_{\infty} = \inf_{N_{\infty}} I_{\infty}.
\]

By using \((M_1) - (M_3)\) we have, as in [18, Remark 11, Remark 34], the following variational characterization of the infimum of \(I_{\infty}\) over \(N_{\infty}\):

\[
c_{\infty} = \inf_{u \in N_{\infty}} I_{\infty}(u) = \inf_{u \in H^{1,+}(\mathbb{R}^3)} \max_{t > 0} I_{\infty}(tu) = \inf_{u \in S^+_\infty} \max_{t > 0} I_{\infty}(tu). \tag{2.9}
\]

**Lemma 2.3** Let \((v_n) \subset S^+_\infty\) be a sequence such that \(\Psi_{\infty}(v_n) \to c_{\infty}\). Then, there is a sequence \((\tilde{v}_n) \subset S^+_\infty\) such that \((\tilde{v}_n)\) is a \((PS)_{c_{\infty}}\) sequence for \(\Psi_{\infty}\) in \(S^+_\infty\) and \(\|\tilde{v}_n - v_n\| = o_n(1)\).

**Proof.** Let \((V, d)\) be a complete metric space, where \(V = \overline{H^{1,+}(\mathbb{R}^3)}\) and \(d(u, v) = \|u - v\|\), and a map \(\zeta : V \to \mathbb{R} \cup \{\infty\}\) given by \(\zeta(u) = \Psi_{\infty}(u)\) if \(u \in H^{1,+}(\mathbb{R}^3)\) and \(\zeta(u) = \infty\) if \(u \not\in \partial H^{1,+}(\mathbb{R}^3)\). From Lemma 2.1 \((A_4)\), we have that \(\zeta\) is continuous and by \((M_3)\) and \((f_5)\) we conclude that \(\zeta\) is bounded below. Using Eland’s Variational Principle [11, Theorem 1.1], it follows that for each given \(\varepsilon, \lambda > 0\) and each \(u \in V\), with \(c_{\infty} < \zeta(u) < c_{\infty} + \varepsilon\), there is \(v \in V\) such that
\[
\zeta(v) \leq \zeta(u), \|u - v\| \leq \lambda e \zeta(u) > \zeta(v) - (\varepsilon \lambda)\|v - w\|, \forall w \neq v. \tag{2.10}
\]

Once that \((v_n) \subset S^+_\infty\) and \(\zeta(v_n) = \Psi_{\infty}(v_n) \to c_{\infty}\), from Lemma 2.1 \((A_4)\), there exists \(R > 0\) which independent on \(n \in \mathbb{N}\) such that
\[
dist(v_n, \partial H^{1,+}(\mathbb{R}^3)) > R, \forall n \in \mathbb{N}.
\]

Thus, for each \(z \in \overline{B_1(0)} \subset H^1(\mathbb{R}^3)\) and for each \(n \in \mathbb{N}\), we get
\[
v_n + tz \in B_\frac{R}{2}(v_n) \subset H^{1,+}(\mathbb{R}^3), \forall t \in (0, \frac{R}{2}). \tag{2.11}
\]

We can consider without loss of generality that \(\Psi_{\infty}(v_n) < c_{\infty} + \frac{1}{n}\). Hence, choosing \(u = v_n\) and \(\varepsilon = \lambda = \frac{1}{n}\) in (2.10), we obtain \(\tilde{v}_n \in V\) such that
\[
\zeta_{\infty}(\tilde{v}_n) \leq \Psi_{\infty}(v_n). \tag{2.12}
\]
\[ \| \tilde{w}_n - v_n \| \leq \frac{1}{n} \quad (2.13) \]

and
\[ \zeta_\infty (\tilde{w}_n + tz) > \Psi_\infty (\tilde{w}_n) - \left( \frac{1}{n^2} \right) \|tz\|, \quad \forall \ t \in (0, \frac{R}{2}), \quad (2.14) \]

where in \( (2.10) \) was chosen \( w = \tilde{w}_n + tz \). From \( (2.12), (2.13) \) and \( (2.11) \) we derive \( (\tilde{w}_n) \subset H^{1,+}(\mathbb{R}^3) \) and

\[ \| \tilde{w}_n + tz - v_n \| < \frac{1}{n} + R \frac{2}{n}, \quad \forall \ n \in \mathbb{N} \text{ e } \forall \ t \in (0, \frac{R}{2}). \]

Thus, for \( n \) large and \( t \in (0, \frac{R}{2}) \), we have \( \tilde{w}_n + tz \in B_{\frac{R}{2}}(v_n) \subset H^{1,+}(\mathbb{R}^3) \) and consequently \( \zeta_\infty (\tilde{w}_n) = \hat{\Psi}_\infty (\tilde{w}_n) \) and \( \zeta_\infty (\tilde{w}_n + tz) = \hat{\Psi}_\infty (\tilde{w}_n + tz) \).

By \( (2.14) \) we obtain
\[ \hat{\Psi}_\infty (\tilde{w}_n + tz) = \hat{\Psi}_\infty (\tilde{w}_n) > -\left( \frac{1}{n^2} \right) \|z\|, \quad (2.15) \]

for all \( t \in (0, \frac{R}{2}), n \geq n_0 \) and \( z \in \overline{B_1(0)} \). Using the definition of \( \hat{\Psi}_\infty \) follow that, for each \( u \in H^{1,+}(\mathbb{R}^3) \), we get \( \hat{\Psi}_\infty (tu) = \hat{\Psi}_\infty (u) \), for all \( t > 0 \). Now defining \( \hat{\nu}_n = \frac{\hat{\Psi}_\infty (\tilde{w}_n)}{\|\tilde{w}_n\|} \), we conclude that \( \{\hat{\nu}_n\} \subset S^+_\infty \). Moreover, from \( (2.12) \) we derive
\[ c_\infty \leq \hat{\Psi}_\infty (\hat{\nu}_n) \leq \Psi_\infty (v_n). \quad (2.16) \]

By \( (2.13) \), we obtain
\[ 1 - \frac{1}{n} < \| \tilde{w}_n \| < 1 + \frac{1}{n}, \quad \forall \ n \in \mathbb{N} \quad (2.17) \]

From \( (2.13) \) and a straightforward computation, we have
\[ \| \hat{\nu}_n - v_n \| \leq \frac{2}{n - 1} \quad (2.18) \]

Finally, from \( (2.15) \) we conclude
\[ \frac{\hat{\Psi}_\infty (\tilde{w}_n + tz) - \hat{\Psi}_\infty (\tilde{v}_n)}{t} > -\left( \frac{1}{n^2} \right) \|z\|, \quad (2.19) \]

for all \( t \in (0, \frac{R}{2}), n \geq n_0 \) and \( z \in \overline{B_1(0)} \), that implies
\[ \frac{\hat{\Psi}_\infty (\tilde{v}_n + \frac{t}{\|\tilde{w}_n\|} z) - \hat{\Psi}_\infty (\tilde{v}_n)}{\frac{1}{\|\tilde{w}_n\|}} > -\left( \frac{1}{n^2} \right) \|z\| \|\tilde{w}_n\|, \quad (2.20) \]

for all \( t \in (0, \frac{R}{2}), n \geq n_0 \) and \( z \in \overline{B_1(0)} \). Passing to the limit in \( t \to 0 \) and using \( (2.17) \) we have
\[ \hat{\Psi}'_\infty (\tilde{v}_n) z \geq -\left( \frac{1}{n^2} \right) (1 + \frac{1}{n}) \|z\| \quad (2.21) \]

for all \( n \geq n_0 \) and \( z \in \overline{B_1(0)} \). Considering now \( z \in \overline{B_1(0)} \cap T_{\hat{\nu}_n} S_\infty \), we get
\[ \| \hat{\Psi}'_\infty (\tilde{v}_n) \| \leq \left( \frac{1}{n^2} \right) (1 + \frac{1}{n}), \quad (2.19) \]
for all $n \geq n_0$. Passing to the limit in $n \to \infty$ in (2.16), (2.18) and (2.19) we prove the Lemma. □.

The following compactness property will be crucial in our arguments and is necessary because the Nehari manifold is not a regular manifold. If $M$ and $f$ are $C^1$ functions, we can to argue as in [1, Proposition 3.1].

**Lemma 2.4** Let $(u_n) \subset \mathcal{N}_\infty$ be a sequence such that $I_\infty(u_n) \to c_\infty$. Then,

$$ u_n(x) = w_n(x - y_n) + \Psi(x - y_n), $$

where $\{w_n\} \subset H^1(\mathbb{R}^3)$ with

$$ w_n \to 0, \quad \text{em} H^1(\mathbb{R}^3), $$

$\{y_n\} \subset \mathbb{R}^3$ is such that $|y_n| \to \infty$ and $\Psi \in H^1(\mathbb{R}^3)$ is a positive continuous function that satisfies

$$ I_\infty(\Psi) = c_\infty \quad \text{e} \quad I'_\infty(\Psi)\Psi = 0. $$

**Proof.** Follow from Lemmas 2.1 ((A3)), 2.2 and 2.3 that

$$ v_n = m^{-1}_\infty(u_n) = \frac{u_n}{\|u_n\|} \in S^+_\infty, \quad \forall n \in \mathbb{N} \quad (2.20) $$

and

$$ \Psi_\infty(v_n) = I_\infty(u_n) \to c_\infty = \inf_{S^+_\infty} \Psi_\infty. $$

By Lemma 2.7 there is a sequence $\{\tilde{v}_n\} \subset S^+_\infty$ such that $\{\tilde{v}_n\}$ is a $(PS)_{c_\infty}$ sequence for $\Psi_\infty$ on $S^+_\infty$ and

$$ \|\tilde{v}_n - v_n\|_\infty = o_n(1). \quad (2.21) $$

Thus,

$$ \tilde{\Psi}'_\infty(\tilde{v}_n)v = 2\lambda_n(\tilde{\bar{v}}_n, v) + o_n(1), \quad \forall v \in H^1(\mathbb{R}^3). $$

From Lemma 2.2 ((a)), we have

$$ \|m_\infty(\tilde{v}_n)\|I'_\infty(m_\infty(\tilde{v}_n))v = 2\lambda_n(\tilde{\bar{v}}_n, v) + o_n(1), \quad \forall v \in H^1(\mathbb{R}^3), $$

that implies

$$ I'_\infty(m_\infty(\tilde{v}_n))v = 2\lambda_n \left( \tilde{\bar{v}}_n, \frac{v}{m_\infty(\tilde{v}_n)} \right) + o_n(1), \quad \forall v \in H^1(\mathbb{R}^3). \quad (2.22) $$

Hence, for $v = m_\infty(\tilde{v}_n)$, we get

$$ 0 = I'_\infty(m_\infty(\tilde{v}_n))m_\infty(\tilde{v}_n) = 2\lambda_n + o_n(1). $$

Thus, we conclude that $\lambda_n \to 0$ as $n \to \infty$ and from (2.22), we obtain

$$ \|I'_\infty(m_\infty(\tilde{v}_n))\| \leq C|\lambda_n| + o_n(1) = o_n(1). $$

Thus, $\{m_\infty(\tilde{v}_n)\}$ is a $(PS)_{c_\infty}$ sequence for $I_\infty$ in $H^1(\mathbb{R}^3)$. From (2.8), we obtain $\{\tilde{w}_n\} \subset H^1(\mathbb{R}^3)$ and $\{y_n\} \subset \mathbb{R}^3$ such that

$$ \tilde{w}_n \to 0, \quad \text{in} \quad H^1(\mathbb{R}^3), $$

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\[ |y_n| \to \infty \] and
\[ m_\infty(\hat{v}_n)(x) = \hat{w}_n(x - y_n) + \hat{\Psi}(x - y_n), \]
where \( \hat{\Psi} \in H^1(\mathbb{R}^3) \) is a positive function satisfying
\[ I_\infty(\hat{\Psi}) = c_\infty \text{ and } I'_\infty(\hat{\Psi})\hat{\Psi} = 0. \] (2.23)

So \( \|\hat{\Psi}\| \geq \tau > 0 \), and defining
\[ \hat{v}_n(x) = \frac{m_\infty(\hat{v}_n)(x)}{\|m_\infty(\hat{v}_n)\|} = \frac{\hat{w}_n(x - y_n)}{\|m_\infty(\hat{v}_n)\|} + \frac{\hat{\Psi}(x - y_n)}{\|m_\infty(\hat{v}_n)\|}, \]
we conclude that
\[ \hat{v}_n(x) = w_n(x - y_n) + \frac{\hat{\Psi}(x - y_n)}{\|m_\infty(\hat{v}_n)\|}, \]
where \( w_n(x - y_n) := \frac{\hat{w}_n(x - y_n)}{\|m_\infty(\hat{v}_n)\|} \) is such that \( w_n \to 0 \) in \( H^1(\mathbb{R}^3) \). Follow from (2.21) that
\[ v_n(x) = w_n(x - y_n) + \frac{\hat{\Psi}(x - y_n)}{\|m_\infty(\hat{v}_n)\|} \]
From (2.20) and from the continuity of \( m_\infty \), we have
\[ u_n(x) = w_n(x - y_n) + m_\infty \left( \frac{\hat{\Psi}(x - y_n)}{\|m_\infty(\hat{v}_n)\|} \right). \]
Defining, \( \Psi := m_\infty \left( \frac{\hat{\Psi}}{\|m_\infty(\hat{v}_n)\|} \right) = m_\infty(\hat{\Psi}) \), it follows that
\[ u_n(x) = w_n(x - y_n) + \Psi(x - y_n), \]
with \( w_n \to 0 \) in \( H^1(\mathbb{R}^3) \). From (2.23), we derive
\[ I_\infty(\Psi) = c_\infty \text{ and } I'_\infty(\Psi)\Psi = 0. \]

3 Variational framework and behavior of minimax levels

From now on we will assume, without loss of generality, that \( 0 \in \Omega \). Let us fix real numbers \( R > r > 0 \) such that \( B_r(0) \subset \Omega \subset B_R(0) \) and the sets
\[ \Omega^+ := \{ x \in \mathbb{R}^N : \text{dist}(x, \Omega) \leq r \}, \quad \Omega^- := \{ x \in \Omega : \text{dist}(x, \partial \Omega) \geq r \} \]
are homotopically equivalent to \( \Omega \).
For each \( \lambda > 0 \), we shall denote by \( H^1_0(\Omega_\lambda) \) the Hilbert space obtained by the closure of \( C_0^\infty(\Omega_\lambda) \) under the scalar product
\[
\langle u, v \rangle_\lambda := \int_{\Omega_\lambda} \nabla u \nabla v \, dx + \int_{\Omega_\lambda} uv \, dx.
\]
The norm induced by this inner product is given by
\[
\|u\|_\lambda := \left( \int_{\Omega_\lambda} |\nabla u|^2 \, dx + \int_{\Omega_\lambda} |u|^2 \, dx \right)^{1/2}.
\]
In view of (f1) – (f2), we have that the functional \( J_\lambda : H^1_0(\Omega_\lambda) \to \mathbb{R} \) given by
\[
J_\lambda(u) := \frac{1}{2} M(\|u\|_\lambda^2) - \int_{\Omega_\lambda} F(u) \, dx
\]
is well defined. Moreover, \( J_\lambda \in C^1(H^1_0(\Omega_\lambda)) \) with the following derivative
\[
J'_\lambda(u)v = M(\|u\|_\lambda^2) \left[ \int_{\Omega_\lambda} \nabla u \nabla v \, dx + \int_{\Omega_\lambda} uv \, dx \right] - \int_{\Omega_\lambda} f(u)v \, dx.
\]
Thus the weak solutions of (P_\lambda) are precisely the critical points of \( J_\lambda \).

As in the previous section, we denote by \( H^{1,+}(\Omega_\lambda) \) the open subset of \( H^1_0(\Omega_\lambda) \) given by
\[
H^{1,+}(\Omega_\lambda) = \{ u \in H^1_0(\Omega_\lambda) : u^+ = \max\{0, u\} \neq 0 \},
\]
and \( S^+_\lambda = S_\lambda \cap H^{1,+}(\Omega_\lambda) \), where \( S_\lambda \) is unit sphere of \( H^1_0(\Omega_\lambda) \). Recalling that \( H^1_0(\Omega_\lambda) = T_u S^+_\lambda \oplus \mathbb{R} u \) for each \( u \in S^+_\lambda \), where \( T_u S^+_\lambda = \{ v \in H^1_0(\Omega_\lambda) : \langle u, v \rangle_\lambda = 0 \} \).

In view of the subcritical growth of \( f, (f_3), (M_1) \) and (2.1), it is standard to check that \( J_\lambda \) satisfies the Palais-Smale condition. Moreover, these hypotheses imply that \( J_\lambda \) has the mountain pass geometry. Hence, for each \( \lambda > 0 \), there exists \( u_\lambda \in H^1_0(\Omega_\lambda) \) such that \( J_\lambda(u_\lambda) = b_\lambda \) and \( J'_\lambda(u_\lambda) = 0 \), where \( b_\lambda \) denotes the mountain pass level of the functional \( J_\lambda \).

**Remark 3.1** We point out that analogous results to the Lemmas 2.1 and 2.2 are still true for the functional \( J_\lambda \). Moreover, in the present case, we denote the functions of those Lemmas by \( \tilde{m}_\lambda, m_\lambda, \tilde{\Psi}_\lambda \) and \( \Psi_\lambda \).

By using (M1) – (M3) and as [15, Remark 11, Remark 34], we can prove that \( b_\lambda \) can also be characterized as
\[
b_\lambda = \inf_{u \in \mathcal{M}_\lambda} J_\lambda(u) = \inf_{u \in H^{1,+}(\Omega_\lambda)} \max_{t > 0} J_\lambda(tu) = \inf_{u \in S^+_\lambda} \max_{t > 0} J_\lambda(tu), \tag{3.2}
\]
where \( \mathcal{M}_\lambda \) is the Nehari manifold associated to \( J_\lambda \), namely
\[
\mathcal{M}_\lambda := \{ u \in H^1_0(\Omega_\lambda) \setminus \{0\} : J'_\lambda(u)u = 0 \}. \tag{3.3}
\]
From (f1) and (f2), there exists \( r = r(\lambda) > 0 \) such that
\[
\|u\|_\lambda \geq r > 0, \tag{3.4}
\]
for all \( u \in M_\lambda \).

We recall that \( B_{\lambda r}(0) \subset \Omega_\lambda \) and define the triple \((J_{\lambda r}, b_{\lambda r}, M_{\lambda r})\) in a similar way, just replacing \( \Omega_\lambda \) by \( B_{\lambda r}(0) \).

For each \( x \in \mathbb{R}^3 \), let us denote by \( A_{\lambda,x} \) the following set

\[
A_{\lambda,x} := B_{\lambda R}(x) \setminus B_{\lambda r}(x)
\]

and define the functional \( \tilde{J}_{\lambda,x} : H^1_0(A_{\lambda,x}) \to \mathbb{R} \) by

\[
\tilde{J}_{\lambda,x}(v) := \frac{1}{2} \tilde{M} \left( \int_{A_{\lambda,x}} |\nabla v|^2 \, dx + \frac{1}{2} \int_{A_{\lambda,x}} |v|^2 \, dx \right) - \int_{A_{\lambda,x}} F(v) \, dx.
\]  \hspace{1cm} (3.5)

and the set

\[
\hat{A}_{\lambda,x} := \{ v \in H^1_0(A_{\lambda,x}, \mathbb{R}) \setminus \{0\} : \tilde{J}^\prime_{\lambda,x}(v)v = 0 \}.
\]

For \( v \in H^1(\mathbb{R}^N) \) with compact support, we consider the barycenter map

\[
\beta(v) := \frac{\int_{\mathbb{R}^3} x|\nabla v|^2 \, dx}{\int_{\mathbb{R}^3} |\nabla v|^2 \, dx}
\]

and we introduce the following quantity

\[
a_{\lambda,x} := \inf \left\{ \tilde{J}_{\lambda,x}(v) : v \in \hat{A}_{\lambda,x} \text{ and } \beta(v) = x \right\}.
\]

We present below an important property of the asymptotic behavior of the numbers \( a_{\lambda,0} \).

**Lemma 3.1** The following holds

\[
c_\infty < \liminf_{\lambda \to \infty} a_{\lambda,0}.
\]

**Proof.** Since \( c_\infty \leq a_{\lambda,0} \) for any \( \lambda > 0 \), we have that \( c_\infty \leq \liminf_{\lambda \to \infty} a_{\lambda,0} \). Suppose, by contradiction, that for some sequence \( \lambda_n \not\to \infty \) we have that \( a_{\lambda_n,0} \to c_\infty \). Then, we can obtain \( v_n \in \hat{A}_{\lambda_n,0} \subset N_\infty \) satisfying \( \tilde{J}_{\lambda_n,0}(v_n) = J_\infty(v_n) \to c_\infty \) and \( \beta(v_n) = 0 \), where we are understanding that the function \( v_n \) is extended to the whole space by setting \( v_n(x) := 0 \) for a.e. \( x \in \mathbb{R}^N \setminus A_{\lambda_n,0} \).

Thus, it follows from Lemma [24] that

\[
v_n(x) = w_n(x - y_n) + \tilde{v}(x - y_n)
\]  \hspace{1cm} (3.6)

with \( (w_n) \subset H^1(\mathbb{R}^3) \) satisfying \( w_n \to 0 \) strongly in \( H^1(\mathbb{R}^3) \), \( (y_n) \subset \mathbb{R}^3 \) being such that \( |y_n| \to \infty \), and \( \tilde{v} \in H^1(\mathbb{R}^3) \) verifying

\[
J_\infty(\tilde{v}) = c_\infty, \quad J_\infty^\prime(\tilde{v}) = 0.
\]  \hspace{1cm} (3.7)

The rest of the proof follows as in [1, Proposition 4.1]. ■

In the next result, we present the asymptotic behavior of the minimax \( b_\lambda \) as \( \lambda \to \infty \).
Lemma 3.2 We have that
\[\lim_{\lambda \to \infty} b_\lambda = c_\infty.\]

Proof. Let \( \varphi \in C_0^\infty(\mathbb{R}^3) \) be a function such that \( \varphi = 1 \) in \( B_1(0) \), \( \varphi = 0 \) in \( \mathbb{R}^3 \setminus B_2(0) \) and \( 0 \leq \varphi \leq 1 \). For each \( R > 0 \), we define

\[ \varphi_R(x) = \varphi \left( \frac{x}{R} \right) \quad \text{and} \quad w_R(x) = \varphi_R(x)w(x), \]

where \( w \) is a ground-state solution of \( (P_\infty) \). Arguing as [1, Proposition 4.2] we conclude that

\[ \limsup_{\lambda \to \infty} c_\lambda \leq I_\infty(t_RW_R). \tag{3.8} \]

Now we show that

\[ \lim_{R \to \infty} t_R = 1. \]

Indeed, since \( \|w_R\|_\lambda^2 = \|w_R\|^2 \), we have

\[ M(t_R^2\|w_R\|^2) = \frac{1}{\|w_R\|^4} \int_{\mathbb{R}^3} \left[ f(t_R w_R) \right] w_R^4. \tag{3.9} \]

From (f3) e (f4) and \( R > 1 \), we get

\[ \frac{M(t_R^2\|w_R\|^2)}{t_R^2\|w_R\|^2} \geq \frac{1}{\|w_R\|^4} \int_{B_1(0)} \left[ f(t_R a) \right] a^4 \geq \frac{1}{\|w_R\|^4} \int_{B_1(0)} \left[ f(t_R a) \right] a^4, \tag{3.10} \]

where \( a = \min_{|x| \leq 1} w(x) \).

Suppose that there is \( (R_n) \) a sequence such that \( t_{R_n} \to \infty \) as \( R_n \to \infty \). Thus, we have

\[ \frac{M(t_{R_n}^2\|w_{R_n}\|^2)}{t_{R_n}^2\|w_{R_n}\|^2} \geq \frac{1}{\|w_{R_n}\|^4} \int_{B_1(0)} \left[ f(t_{R_n} a) \right] a^4. \]

It follows from (f3) and Fatou’s Lemma that

\[ \frac{M(t_{R_n}^2\|w_{R_n}\|^2)}{t_{R_n}^2\|w_{R_n}\|^2} \to \infty, \]

which is a contradiction with (M3).

Suppose now \( t_{R_n} \to 0 \) as \( R_n \to \infty \). Using the growth of \( f \), given by (f1) – (f2), we have

\[ \frac{M(t_{R_n}^2\|w_{R_n}\|^2)}{t_{R_n}^2\|w_{R_n}\|^2} \to 0, \]

which is a contradiction with (M1). Thus, there exists \( t_0 > 0 \) such that, up to a subsequence, \( t_{R_n} \to t_0 \) and from (3.9), we get

\[ \frac{M(t_0^2\|w\|^2)}{t_0^2\|w\|^2} = \frac{1}{\|w\|^4} \int_{\mathbb{R}^3} \left[ f(t_0 w) \right] w^4. \]
Since that $w$ is a solution of $(P_\infty)$ we conclude that $t_0 = 1$ and $I_\infty(t_RW_R) \to I_\infty(w) = c_\infty$ as $R \to \infty$. Thus, by (3.8) we obtain
\[
\limsup_{\lambda \to \infty} c_\lambda \leq c_\infty.
\]
The reverse inequality follows from the definition of $c_\infty$ and $b_\lambda$.  

The following result is the key point in the comparison of the category of $\Omega$ with that of the sublevel sets of the functional $J_\lambda$ given by $J^{b,\lambda}_\lambda = \{ u \in \mathcal{N}_\lambda : J_\lambda(u) \leq b_\lambda \}$.

**Lemma 3.3** There exists $\lambda_* > 0$ such that $\beta(u) \in \Omega^{+}_\lambda$, whenever $u \in J^{b,\lambda}_\lambda$ and $\lambda \geq \lambda_*$.  

**Proof.** The result follows from Lemma 3.1 and arguments that were used in [1, Proposition 4.3].  

Replacing $\Omega_\lambda$ by $B_{\lambda r}(0)$ we can prove that there exists $u_{\lambda,r}$ a solution of problem $\lambda$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-$ $\Omega_-

Lemma 3.4 For each $\lambda \in [\lambda_*, \infty)$, we have
\[
cat_{\Psi^{b,\lambda}_\lambda} (m^{-1}_{\lambda}(\Phi_\lambda(\lambda \Omega_-))) = cat_{I^{b,\lambda}_\lambda} (\Phi_\lambda(\lambda \Omega_-)) \geq cat\Omega,
\]
where $\Psi^{b,\lambda}_\lambda = \{ u \in S^+_\lambda : \Psi_\lambda(u) \leq b_\lambda \}$.  

**Proof.** Suppose that
\[
\Phi_\lambda(\lambda \Omega_-) = \bigcup_{j=1}^k A_j,
\]
where $A_j \subset I^{b,\lambda}_\lambda \subset \mathcal{N}_\lambda$ is closed and contractible in $I^{b,\lambda}_\lambda$. Since that $m_\lambda : S^+_\lambda \to \mathcal{N}_\lambda$ is a homeomorphism we get
\[
cat_{\Psi^{b,\lambda}_\lambda} (m^{-1}_{\lambda}(\Phi_\lambda(\lambda \Omega_-))) = cat_{I^{b,\lambda}_\lambda} (\Phi_\lambda(\lambda \Omega_-)) = k.
\]
Now the rest of the proof follows from the Lemma 3.3 and from [1, Proposition 4.5].
4 Proof of Theorem 1.1

Firstly, we define the compact set $K := m_\lambda^{-1}(\Phi_r(\lambda\Omega_-))$ and we observe that $K \subset \Psi^{b_{\lambda,r}}_\lambda \subset S^+_\lambda$. Moreover, follow from the Lemma 3.4 that

$$\text{cat}\Omega \leq \text{cat}_{\Psi^{b_{\lambda,r}}_\lambda} K.$$  

Follow from [18], Theorem 27, with $c = b_\lambda < b_{\lambda,r} = d$, that $\Psi^{b_{\lambda,r}}_\lambda$ contains cat$\Omega$ critical points of $\Psi_\lambda$. From the Remark 3.1 we conclude that $I_\lambda$ has at least cat$\Omega$ critical points, with energy in $[b_\lambda, b_{\lambda,r}]$.

On the other hand, if cat$\Omega > 1$ we argue similarly to [8, Theorem 1.1]. Choosing $u^* \in S^+_\lambda$ such that $b_{\lambda,r} < \Psi_\lambda(u^*)$, we define

$$\Theta = \{tu^* + (1-t)u : t \in [0,1] \text{ e } u \in K\}.$$  

We observe that $\Theta$ is compact and $0 \notin \Theta$.

We also define the set

$$\Gamma = \left\{ \frac{w}{\|w\|_\lambda} : w \in \Theta \right\} \subset S^+_\lambda.$$  

Once $K \subset \Gamma$, with $K$ contractible in $\Gamma$ and

$$b_{\lambda,r} < \Psi_\lambda(u^*) \leq \max_{v \in \Gamma} \Psi_\lambda =: c,$$

it follows that $K$ is contractible in $\Psi^+_\lambda$. Over again, from [18], Theorem 27, with $2 \leq k = \text{cat}\Omega$ e $e = c$, we conclude that $\Psi^+_\lambda$ has another critical point in $\Psi^+_\lambda \setminus \Psi^{b_{\lambda,r}}_\lambda$. From Remark 3.1 it follows that $I_\lambda$ has another critical point with energy in $(b_{\lambda,r}, c]$. $\blacksquare$

Acknowledgement. This work was done while the author was visiting the “Departamento de ecuaciones diferenciales y análisis numérico” of the Universidad de Sevilla. They would like to express his gratitude to the Prof. Antonio Suarez for his warm hospitality.

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