TENSOR FIELDS AND CONNECTIONS ON HOLOMORPHIC ORBIT SPACES OF FINITE GROUPS

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ABSTRACT. For a representation of a finite group \( G \) on a complex vector space \( V \) we determine when a holomorphic \( (p,q) \)-tensor field on the principal stratum of the orbit space \( V/G \) can be lifted to a holomorphic \( G \)-invariant tensor field on \( V \). This extends also to connections. As a consequence we determine those holomorphic diffeomorphisms on \( V/G \) which can be lifted to orbit preserving holomorphic diffeomorphisms on \( V \). This in turn is applied to characterize complex orbifolds.

1. Introduction

Locally, an orbifold \( Z \) can be identified with the orbit space \( B/G \), where \( B \) is a \( G \)-invariant neighborhood of the origin in a vector space \( V \) with a finite group \( G \subset GL(V) \) and, using this identification, one can easily define local (and then global) tensor fields and other differential geometrical objects in \( Z \) as appropriate \( G \)-invariant tensor fields and objects on \( B \subset V \). In particular, one can naturally define Riemannian orbifolds, Einstein orbifolds, symplectic orbifolds, Kähler-Einstein orbifolds etc.

We study complex orbifolds, that is, orbifolds modeled on orbit spaces \( V/G \), where \( G \) is a finite subgroup of \( GL(V) \) for a complex vector space \( V \). In particular, the orbit spaces \( Z = M/G \) of a discrete proper group \( G \) of holomorphic transformations of a complex manifold \( M \) are complex orbifolds.

An orbifold \( X \) has a structure defined by the sheaf \( \mathcal{S}_X \) of local invariant holomorphic functions in a local uniformizing system. \( X \) has also a stratification by strata \( S \) which are glued from local isotropy type strata of local uniformizing systems. In particular, the regular stratum \( X_0 \) is an open dense complex manifold in \( X \).

Holomorphic geometric objects on \( X \) (e.g. tensor fields and connections) are locally defined as invariant objects on the uniformizing system. Their restrictions to the regular stratum \( X_0 \) are usual holomorphic geometric objects on the complex manifold \( X_0 \).

A natural question is to characterize these restrictions, i.e. to describe tensor fields and connections on \( X_0 \) which are extendible to \( X \). We look at the lifting problem for connections because this allows a very elegant approach to the lifting problem for holomorphic diffeomorphisms. And the last problem has immediate consequences for characterizing complex orbifolds, i.e., for answering the following

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question: Which data does one need besides $\mathfrak{F}_X$ and $X_0$ to characterize a complex orbifold $X$? The main goal of the paper is to answer these questions.

We have first to investigate the local situation, thus we consider a finite subgroup $G \subset GL(V)$ and the orbit space $Z = V/G$ with the structure given by the sheaf $\mathfrak{F}_{V/G}$ of invariant holomorphic functions on $V$, and the orbit type stratification. The prime role is played by strata of codimension 1 with the order $s$ of the reflection divisor corresponding stabilizer groups, which are arranged in the reflection divisor $D_{V/G}$ which keeps track of all complex reflections in $G$. It turns out that the union $Z_1$ of $Z_0$ and of all codimension 1 strata is a complex manifold, see 3.5. We characterize all $G$-invariant holomorphic tensor fields and connections on $V$ in terms of the reflection divisor of the corresponding meromorphic tensor field and connection on $Z_1$, see 3.7 and 4.2. Our result gives a generalization 3.9 of Solomon’s theorem [10], see 3.10. Using the lifting property of connections we are able to prove that a holomorphic diffeomorphism $Z = V/G \to V/G' = Z'$ between two orbit spaces has a holomorphic lift to $V$ which is equivariant over an isomorphism $G \to G'$ if and only if $f$ respects the regular strata and the reflection divisors, i.e. $f(Z_0) \subset Z'_0$ and $f_* (D_Z) \subset D_Z'$. In fact we give two proofs of this result, which in [4] is carried over to the algebraic geometry setting for algebraically closed ground fields of characteristic 0. The related problem of lifting (smooth) homotopies from (general) orbit spaces has been treated in [1] and [9].

Applying the local results we prove that a complex orbifold $X$ is uniquely determined by the sheaf $\mathfrak{F}_X$, the regular stratum $X_0$, and the reflection divisor $D_X$ alone, see 6.6.

2. Preliminaries

2.1. The orbit type stratification. Let $V$ be an $n$-dimensional complex vector space, $G$ a finite subgroup of $GL(V)$, and $\pi : V \to V/G$ the quotient projection. The ring $\mathbb{C}[V]^G$ has a minimal system of homogeneous generators $\sigma^1, \ldots, \sigma^m$. We will use the map $\sigma = (\sigma^1, \ldots, \sigma^m) : V \to \mathbb{C}^m$. Denote by $Z$ the affine algebraic variety in $\mathbb{C}^m$ defined by the relations between $\sigma^1, \ldots, \sigma^m$. It is known that $\sigma(V) = Z$.

We consider the orbit space $V/G$ endowed with the quotient topology as a local ringed space defined by the following sheaf of rings $\mathfrak{F}_{V/G}$: if $U$ is an open subset of $V/G$, $\mathfrak{F}_{V/G}(U)$ is equal to the space of $G$-invariant holomorphic functions on $\pi^{-1}(U)$. Clearly one may consider sections of $\mathfrak{F}_{V/G}$ on $U$ as functions on $U$. We call these functions holomorphic functions on $U$. It is known that the map of the orbit space $V/G$ to $Z$ induced by the map $\sigma$ is a homeomorphism. Moreover, this homeomorphism induces an isomorphism of the sheaf $\mathfrak{F}_{V/G}(U)$ and the structure sheaf of the complex algebraic variety $Z$ (see [7]). Via the above isomorphism we identify the local ringed spaces $V/G$ and $Z$. Under this identification the projection $\pi$ is identified with the map $\sigma$. Let $G$ and $G'$ be finite subgroups of $GL(V)$ and let $Z = V/G$ and $Z' = V/G'$ be the corresponding orbit spaces. By definition a holomorphic diffeomorphism of the orbit space $Z$ to the orbit space $Z'$ is an isomorphism of $Z$ to $Z'$ as local ringed spaces.

Let $K$ be a subgroup of $G$, $(K)$ the conjugacy class of $K$. Denote by $V_{(K)}$ the set of points of $V$ whose isotropy groups belong to $(K)$ and put $Z_{(K)} = \pi(V_{(K)})$. It is known that $\{ Z_{(K)} \}$ is a finite stratification of $Z$, called the isotropy type stratification, into locally closed irreducible smooth algebraic subvarieties (see [1]). Denote by $Z_i$ the union of the strata of codimension greater than $i$ and put $Z_i =$
Z \setminus Z^i$. Then $Z_0$ is the principal stratum of $Z$, i.e. $Z_0 = Z_{(K)}$ for $K = \{\text{id}\}$. It is known that $Z_0$ is a Zariski open subset of $Z$ and a complex manifold. It is clear that the restriction of the map $\sigma$ to the set $V_{reg}$ of regular points of $V$ is an tale map onto $Z_0$.

In this paper we consider the orbit space $Z = V/G$ with the above structure of local ringed space and the stratification $\{Z_{(K)}\}$.

**2.2. The divisor of a tensor field.** We shall use divisors of meromorphic functions on a complex manifold $X$. For technical reasons (see e.g. the last formula of this section) we define $\text{div}(0) = \sum S \propto S$, where the sum runs over all complex subspaces of $X$ of codimension 1.

Let $f$ and $g$ be two meromorphic functions on $X$. Then we have $\text{div}(f + g) \geq \min\{\text{div}(f), \text{div}(g)\}$, where $\text{div}(f)$ denote the divisor of $f$. Taking the minimum means: For each irreducible complex subspace $S$ of $X$ of codimension 1 belonging to the support of $f$ or $g$ take the minimum of the coefficients in $Z$ of $S$ in $\text{div}(f)$ and $\text{div}(g)$.

Let $P$ be a meromorphic tensor field (i.e., with meromorphic coefficient functions in local coordinates) on $X$. In local holomorphic coordinates $y^1, \ldots, y^n$ on an open subset $U \subset X$ the tensor field $P$ can be written as

$$P|_U = \sum_{i_1, \ldots, i_p, j_1, \ldots, j_q} P_{i_1, \ldots, i_p}^{j_1, \ldots, j_q} \frac{\partial}{\partial y^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial y^{i_p}} \otimes dy^{j_1} \otimes dy^{j_q},$$

and we define the divisor of $P$ on $U$ as the minimum of all divisors $\text{div}(P_{i_1, \ldots, i_p}^{j_1, \ldots, j_q}) \in \text{Div}(U)$ for all coefficient functions of $P$. The resulting coefficient of the complex subspace $S$ of codimension 1 in $\text{div}(P) \in \text{Div}(U)$ does not depend on the choice of the holomorphic coordinate system; e.g., for a vector field $\sum_i X^i \frac{\partial}{\partial y^i}$ we have

$$\text{div}\left(\sum_i X^i \frac{\partial u^k}{\partial y^i}\right) \geq \min_i \text{div}\left(X^i \frac{\partial u^k}{\partial y^i}\right) = \min_i \left(\text{div}(X^i) + \text{div}\left(\frac{\partial u^k}{\partial y^i}\right)\right) \geq \min_i \text{div}(X^i).$$

Finally we define the divisor of $P$ on $X$ by gluing the local divisors for any holomorphic atlas of $X$. Note that a tensor field $P$ is holomorphic if and only if $\text{div}(P) \geq 0$.

**3. Invariant tensor fields**

**3.1.** Let $P$ be a $G$-invariant holomorphic tensor field of type $(p, q)$ on $V$. Since $\sigma$ is an tale map on $V_{reg}$, there is a unique holomorphic tensor field $Q$ on $Z_0$ of type $(p, q)$ such that the pullback $\sigma^*(Q)$ coincides with the restriction of $P$ to $V_{reg}$. It is clear that the tensor field $P$ is uniquely defined by $Q$.

Consider a holomorphic tensor field $Q$ of type $(p, q)$ on $Z_0$ and its pullback $\sigma^*(Q)$ which is a $G$-invariant holomorphic tensor field on $V_{reg}$. Then by the Hartogs extension theorem, $\sigma^*(Q)$ has a $G$-invariant holomorphic extension to $V$ iff it has a holomorphic extension to $\sigma^{-1}(Z_1)$.

Denote by $\mathcal{H}$ the set of all reflection hyperplanes corresponding to all complex reflections in $G$ and, for each $H \in \mathcal{H}$, by $e_H$ the order of the cyclic subgroup of $G$ fixing $H$. It is clear that $\sigma(\cup_{H \in \mathcal{H}} H)$ contains all strata of codimension 1. This implies immediately the following
3.2. Proposition. If $\mathfrak{H} = \emptyset$, for each holomorphic tensor field $P_0$ on $Z_0$ the pullback $\sigma^*(P_0)$ has a $G$-invariant holomorphic extension to $V$. □

3.3. The reflection divisor of the orbit space. Consider the set $R_Z$ of all hyper surfaces $\sigma(H)$ in $Z$, where $H$ runs through all reflection hyperplanes in $V$. Note that $\sigma(H)$ is a complex subspace of $Z_1$ of codimension 1. We endow each $S = \sigma(H) \in R_Z$ with the label $e_{\mathcal{H}}$ of the hyperplane $H$. It is easily seen that this label does not depend on the choice of $H$, we denote it by $e_S$ and we consider $e_S \cdot S$ as an effective divisor on $Z$ and we consider the effective divisor in $Z_1$

$$D = D_{V/G} = D = \sum_{S \in R_Z} e_S \cdot S,$$

which we call the reflection divisor.

3.4. Basic example. Let the cyclic group $\mathbb{Z}_r = \mathbb{Z}/r\mathbb{Z}$ with generator $\zeta_r = e^{2\pi i/r}$ act on $\mathbb{C}$ by $z \mapsto e^{2\pi ik/r}z$ for $r \geq 2$. The generating invariant is $\tau(z) = z^r$.

We consider first a holomorphic tensor field $P = f(z)(dz)^{\sigma q} \otimes (\frac{\partial}{\partial y})^{\sigma p}$ on $\mathbb{C}$. It is invariant, $\zeta_r^* P = P$, if and only if $f(\zeta_r z) = \zeta_r^{-q} f(z)$, so that in the expansion $f(z) = \sum_{k \geq 0} f_k z^k$ at 0 of $f$ the coefficient $f_k \neq 0$ at most when $k \equiv p - q \mod r$. Writing $p - q = rs + t$ with $s \in \mathbb{Z}$ and $0 \leq t < r$ we see that $P$ is invariant if and only if $f(z) = z^t g(z^r)$ for holomorphic $g$.

We use the coordinate $y = \tau(z) = z^r$ on $\mathbb{C}/\mathbb{Z}_r = \mathbb{C}$, $\tau^* dy = rz^{r-1}dz$ and $\tau^*(\frac{\partial}{\partial y}|_{\mathbb{C}\setminus 0}) = \frac{dz}{rz^{r-1}} \frac{\partial}{\partial |_{\mathbb{C}\setminus 0}}$, and we write

$$P|_{\mathbb{C}\setminus 0} = g(z^r) z^t (dz)^{\sigma q} \otimes \left(\frac{\partial}{\partial y}\right)^{\sigma p}$$

$$= g(y) z^{r-t s} (r z^{r-1})^{p-q} (dy)^{\sigma q} \otimes \left(\frac{\partial}{\partial y}\right)^{\sigma p}$$

$$= g(y) y^{s-t p} (r y)^{-s} (dy)^{\sigma q} \otimes \left(\frac{\partial}{\partial y}\right)^{\sigma p}$$

(we omitted $\tau$). Thus a holomorphic tensor field $P$ of type $({\sigma q \over p q})$ on $\mathbb{C}$ is $\mathbb{Z}_r$-invariant if and only if $P|_{\mathbb{C}\setminus 0} = \tau^* Q$ for a meromorphic tensor field

$$Q = g(y) y^m (dy)^{\sigma q} \otimes \left(\frac{\partial}{\partial y}\right)^{\sigma p}$$

on $\mathbb{C}$ with $g$ holomorphic with $g(0) \neq 0$ and with

$$m \geq p - q - s.$$ 

It is easily checked that the above inequality is equivalent to the following one

$$mr + (p - q)(r - 1) \geq 0.$$ 

3.5. Suppose $\mathfrak{H} \neq \emptyset$. Let $z \in Z_1 \setminus Z_0$ and $v \in \sigma^{-1}(z)$. Then there is a unique hyperplane $H \in \mathfrak{H}$ such that $v \in H$ and the isotropy group $G_v$ is isomorphic to a cyclic group. It is evident that the order $r_z = e_H$ of $G_v$ depends only on $z = \sigma(v)$ and is locally constant on $Z_1 \setminus Z_0$.

By the holomorphic slice theorem (see [3], [6]) there is a $G_v$-invariant open neighborhood $U_v$ of $v$ in $V$ such that the induced map $U_v/G_v \to V/G$ is a local biholomorphic map at $v$.

Choose orthonormal coordinates $z^1, \ldots, z^n$ in $V$ with respect to a $G$-invariant Hermitian inner product on $V$, so that $H = \{z^n = 0\}$. Then the ring $\mathbb{C}[V]^G$ is generated by $z^1, \ldots, z^{n-1}, (z^n)^r$, where $r = r_z$. 
Put \( \tau^1 = z^1, \ldots, \tau^{n-1} = z^{n-1}, \tau^n = (z^n)^r \), and \( \tau = (\tau^1, \ldots, \tau^n) : U_0 \to \mathbb{C}^n \). Then there are holomorphic functions \( f^i \) \((i = 1, \ldots, n)\) in an open neighborhood \( W_z \) of \( z \in \mathbb{C}^n \) such that \( \tau^a = f^a \sigma |_{U_0} \). On the other hand, we know that in an open neighborhood of \( v \) all \( \sigma^a \) for \( a = 1, \ldots, m \) are holomorphic functions of the \( \tau^i \). We denote by \( y^i \) the holomorphic function on \( Z \) such that \( \tau^i = y^i \sigma \). Then we can use \( y^i \) as coordinates of \( Z \) defined in the open neighborhood \( W_z \subseteq \mathbb{C}^n \) of \( z \). Note that we found holomorphic coordinates near each point of \( Z_1 \), so we have:

**Corollary.** The union \( Z_1 \) of all codimension \( \leq 1 \) strata, with the restriction of the sheaf \( \mathfrak{F}_{V/G} \), is a complex manifold.

### 3.6. The reflection divisor of a meromorphic tensor field on \( Z_1 \)

Let \( \Gamma_M(T^\mu_1(Z_1)) \) be the space of meromorphic tensor fields (i.e. with meromorphic coefficient functions in local holomorphic coordinates on the complex manifold \( Z_1 \)), and let \( P \in \Gamma_M(T^\mu_1(Z_1)) \).

Let \( S \) be an irreducible component of \( Z_1 \setminus Z_0 \) and let \( z \in S \). Local coordinates \( y^1, \ldots, y^n \) on \( U \subseteq Z_1 \), centered at \( z \), are called adapted to the stratification of \( Z_1 \) if \( S = \{ y^n = 0 \} \) near \( z \). By definition the coordinates \( y^1, \ldots, y^n \) from 3.5 have this property. Denote by \( O_Z \) the ring of germs of holomorphic functions and by \( M_z \) the field of germs of meromorphic functions, both at \( z \in Z_1 \).

Let \( y^1, \ldots, y^n \) be local coordinates on \( U \subseteq Z_1 \), centered at \( z \), adapted to the stratification of \( Z_1 \). Then on \( U \) the meromorphic tensor field \( P \) is given by:

\[
P|_U = \sum_{i_1, \ldots, i_p, j_1, \ldots, j_q} P_{j_1 \ldots j_q}^{i_1 \ldots i_p} \frac{\partial}{\partial y^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial y^{i_p}} \otimes dy^{j_1} \otimes dy^{j_q},
\]

where the \( P_{j_1 \ldots j_q}^{i_1 \ldots i_p} \) are meromorphic on \( U \). Let us fix one nonzero summand of the right hand side: for the coefficient function we have \( P_{j_1 \ldots j_q}^{i_1 \ldots i_p} = (y^n)^m f \) for some integer \( m \) such that the germs at \( z \) of \( y^n \), \( g \), and \( h \) are pairwise relatively prime in \( O_z \) where \( f = g/h \in M_z \). Suppose that the factor \( \frac{\partial}{\partial y^n} \) appears exactly \( q' \) times and the factor \( dy^n \) appears exactly \( q' \) times in this summand. The integer

\[
\mu = mr + (q'-p')(r-1),
\]

a priori depending on \( z \), is constant along an open dense subset of \( S \) and it is called the reflection residua of the summand at \( S \). Finally let \( \mu_S(P) \) be the minimum of the reflection residua at \( S \) of all summands of \( P \) in the representation of \( P \).

Let \( \tilde{y}^1, \ldots, \tilde{y}^n \) be arbitrary local coordinates on \( U \subseteq Z_1 \), centered at \( z \), adapted to the stratification of \( Z_1 \). In a neighborhood of \( z \) we have \( \tilde{y}^n = f \tilde{y}^n \), where \( f \) is a holomorphic function such that \( f(z) \neq 0 \). Remark that \( \tilde{y}^n \) divides \( \frac{\partial}{\partial y^n} \) and \( \frac{\partial}{\partial y^i} \) \((i = 1, \ldots, n)\) in \( O_z \). A straightforward calculation using the above remark shows that the values of \( \mu_S(P) \) calculated in the coordinates \( \tilde{y}^i \) and in the coordinates \( y^i \) are the same. Then \( \mu_S(P) \) does not depend on the choice of the system of local coordinates adapted to the stratification of \( Z_1 \). For details see [R]: there we checked this in the algebraic geometry setting where the use of tensor fields is less familiar.

We now can define the reflection divisor

\[
\text{div}_P(P) = \text{div}_{U/G}(P) \in \text{Div}(U)
\]
as follows: take the divisor \( \text{div}(P) \), and for each irreducible component \( S \) of \( Z_1 \setminus Z_0 \) do the following: if \( S \) appears in the support of \( \text{div}(P) \in \text{Div}(U) \), replace its
coefficient by $\mu_S(P)$; if it does not appear, add $\mu_S(P)S$ to it. If $S$ is not contained in $Z_1 \setminus Z_0$, we keep its coefficient in $\text{div}(P)$.

Finally we glue the global reflection divisor $\text{div}_D(P) \in \text{Div}(Z_1)$ from the local ones, using a holomorphic atlas for $Z_1$.

3.7. Theorem. Let $G \subset GL(V)$ be a finite group, with reflection divisor $D = D_{V/G} = D_Z$. Then we have:

- Let $P$ be a holomorphic $G$-invariant tensor field on $V$. Then the reflection divisor $\text{div}_D(\pi_* P) \geq 0$.
- Let $Q \in \Gamma_M(T^p_q(Z_1))$ be a meromorphic tensor field on $Z_1$. Then the $G$-invariant meromorphic tensor field $\pi^* Q$ extends to a holomorphic $G$-

The above remains true for $G$-invariant holomorphic tensor fields defined in a $G$-stable open subset of $V$.

Proof. This follows directly from Hartogs’ extension theorem, the basic example 3.4 using $y^1, \ldots, y^{n-1}$ as dummy variables, and the definition of the reflection divisor $\text{div}_D(P)$ as explained in 3.6.

3.9. Corollary. The mapping $\sigma$ establishes an injective correspondence between the space of holomorphic $G$-invariant tensor fields of type $(p^q)$ on $V$ which are skew-symmetric with respect to the covariant entries, and the space of holomorphic tensor fields on $Z_1$ of the same type and the same skew-symmetry condition. If $p = 0$ the correspondence is bijective.

The above remains true for $G$-invariant holomorphic tensor fields defined in a $G$-stable open subset of $V$.

Proof. Let $P$ be a holomorphic $G$-invariant tensor field on $V$ satisfying the conditions of the corollary. For each nonzero decomposable summand of $\pi_* P$ take the integers $m, p', q'$ defined in 3.6. By skew symmetry of $P$ we have $q' \leq 1$. By Theorem 3.7 we get $\text{div}_D(\pi_* P) \geq 0$ and thus $mr \geq (p' - q')(r - 1) > -r$. So $m \geq 0$ and the summand is holomorphic on $Z_1$.

If $Q$ is a holomorphic differential form on $Z_1$ its pullback $\sigma^* Q$ is a $G$-invariant holomorphic form on $\sigma^{-1}(Z_1)$ and then has a holomorphic extension to the whole of $V$.

3.10. Remarks. Note that Corollary 3.9 is a generalization of Solomon’s theorem (see [10]): If $G \subset GL(V)$ is a finite complex reflection group then every $G$-invariant polynomial exterior $q$-form $\omega$ on $V$ can be written as $\omega = \sigma^* \varphi$ for a polynomial $q$-form $\varphi$ on $\mathbb{C}^n$, where $\sigma = (\sigma^1, \ldots, \sigma^m) : V \to \mathbb{C}^n$ is the mapping consisting of $G$-

Actually, in the case of a reflection group $Z = \mathbb{C}^n$ and each holomorphic $(p^q)$-

tensor field $Q$ on $Z_1$ has a holomorphic extension to $Z$ by Hartogs’ extension theorem.

4. INVARIANT COMPLEX CONNECTIONS

4.1. Let $\Gamma$ be a holomorphic $G$-invariant complex connection on $V$. Then the image $\sigma_\Gamma$ of $\Gamma$ under the map $\sigma$ defines a holomorphic complex connection on $Z_0$.

Let $z \in Z_1 \setminus Z_0$, $v \in \sigma^{-1}(z)$, and $r$ the order of $G_v$. Consider the coordinates $z^i$ in $V$ defined in 3.5. Denote by $\Gamma_{ijk}$ the components of the connection $\Gamma$ with respect to
these coordinates. By assumption, the $\Gamma_{jk}^i$ are holomorphic functions on $V$. Recall the standard formula for the image $\gamma$ of $\Gamma$ under a holomorphic diffeomorphism $f = (y^a(x^i))$

$$\gamma_a^i \circ f = \frac{\partial y^a}{\partial x^j} \frac{\partial x^j}{\partial y^i} \Gamma_{jk}^i(x^i) - \frac{\partial^2 y^a}{\partial x^j \partial x^k} \frac{\partial x^j}{\partial y^i} \frac{\partial x^k}{\partial y^i}.$$

Remark that the similar formula is true for the transformation of the components of connection under the change of coordinates.

Consider the generator $g$ of the cyclic group $G_v$ given by 3.5. Since $g$ acts linearly, the connection reacts to it like a $\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$-tensor field. Thus by the considerations of 3.4 we get in the notation of 3.5, where $i,j,k = 1, \ldots, n-1$:

$$\Gamma_{jk}^i = \tilde{\Gamma}_{jk}^i \circ \sigma, \quad \Gamma_{nk}^i = \frac{1}{r} z^n \tilde{\Gamma}_{jk}^i \circ \sigma, \quad \Gamma_{jn}^i = r(z^n)^{r-1} \tilde{\Gamma}_{jn}^i \circ \sigma, \quad \Gamma_{nk}^n = \tilde{\Gamma}_{nk}^n \circ \sigma,$$

$$\Gamma_{nn}^n = r(z^n)^{r-2} \tilde{\Gamma}_{nn}^n \circ \sigma, \quad \Gamma_{nn}^n = r(z^n)^{r-1} \tilde{\Gamma}_{nn}^n \circ \sigma,$$

where the $\tilde{\Gamma}_{bc}^a$ are holomorphic functions of the coordinates $y^a (a = 1, \ldots, n)$ introduced in 3.5.

Using the transformation formula for connections, we get the following formulas for the components $\gamma_{bc}^a$ of the meromorphic connection $\sigma_1 \Gamma$ with respect to the coordinates $y^a$.

$$\gamma_{jk}^i = \tilde{\Gamma}_{jk}^i, \quad \gamma_{nk}^i = y^n \tilde{\Gamma}_{jk}^i, \quad \gamma_{jn}^i = \tilde{\Gamma}_{jn}^i, \quad \gamma_{nk}^n = \tilde{\Gamma}_{nk}^n,$$

$$\gamma_{nn}^n = \tilde{\Gamma}_{nn}^n, \quad \gamma_{nk}^n = \tilde{\Gamma}_{nk}^n, \quad \gamma_{nn}^i = \frac{1}{y^n} \tilde{\Gamma}_{nn}^i, \quad \gamma_{nn}^n = \tilde{\Gamma}_{nn}^n - \frac{r-1}{ry^a}.$$

Let $\tilde{y}^a$ for $a = 1, \ldots, n$ be other local coordinates centered at $z$ and adapted to the stratification of $Z_1$. Then in a neighborhood of $z$ we have

$$y^a = f \tilde{y}^a, \quad \tilde{y}^a = \tilde{f} y^a,$$

where $f$ and $\tilde{f}$ are holomorphic functions in a neighborhood of $z$ and $\tilde{f} f = 1$. Then we have

$$\frac{\partial y^a}{\partial \tilde{y}^i} = \frac{\partial f}{\partial \tilde{y}^i} \tilde{y}^a, \quad \frac{\partial \tilde{y}^a}{\partial y^i} = \frac{\partial f}{\partial y^i} y^a \quad (i = 1, \ldots, n-1)$$

and on $S = \{y^n = 0\}$

$$\frac{\partial y^n}{\partial \tilde{y}^n} = f, \quad \frac{\partial \tilde{y}^n}{\partial y^n} = \tilde{f}.$$

Using these formulas one can check that in the coordinates $\tilde{y}^a$ the formulas 4.1.1 have the same form as in the coordinates $y^a$. For example, for the new component $\tilde{\gamma}_{nn}^a$ we have

$$\tilde{\gamma}_{nn}^a + \frac{r-1}{r \tilde{y}^n} \left( 1 - \tilde{f} \frac{\partial \tilde{y}^n}{\partial \tilde{y}^n} \left( \frac{\partial y^n}{\partial \tilde{y}^n} \right)^2 \right) \frac{\partial \tilde{y}^n}{\partial y^n} f = h,$$

where $h$ is a holomorphic function near $z$. Since on $S = \{y^n = 0\}$ we have

$$1 - \tilde{f} \frac{\partial \tilde{y}^n}{\partial \tilde{y}^n} \left( \frac{\partial y^n}{\partial \tilde{y}^n} \right)^2 = 1 - \tilde{f}^2 f^2 = 0,$$
4.1.1 where \( \tilde{\Gamma} \) for each \( z \).

4.2. Theorem. Let \( \gamma \) be a holomorphic complex linear connection on \( Z_0 \) such that for each \( z \in Z_1 \setminus Z_0 \) it has an extension to a neighborhood of \( z \) whose components in the coordinates adapted to the stratification of \( Z_1 \) are defined by the formulas 4.1.1 where \( \Gamma_{bc}^a \) are holomorphic. Then there is a unique \( G \)-invariant holomorphic complex linear connection \( \Gamma \) on \( V \) such that \( \sigma_* \Gamma \) coincides with \( \gamma \) on \( Z_0 \). This remains true if we replace \( V \) by a \( G \)-open subset of \( G \).

Proof. Since \( \sigma \) is tale on the principal stratum, there is a unique \( G \)-invariant complex linear connection \( \Gamma_0 \) on \( \sigma^{-1}(Z_0) \) such that \( \sigma_* \Gamma_0 = \gamma \). The condition of the theorem implies that the connection \( \Gamma_0 \) has a holomorphic extension to \( \sigma^{-1}(Z_1) \). Then by Hartogs’ extension theorem the connection \( \Gamma_0 \) has a unique holomorphic extension \( \Gamma \) to the whole of \( V \).

5. Lifts of diffeomorphisms of orbit spaces

5.1. Let \( G \) and \( G' \) be finite subgroups of \( GL(V) \) and \( GL(V') \) and let \( F \) be a holomorphic diffeomorphism \( V \to V' \) which maps \( G \)-orbits to \( G' \)-orbits bijectively.

Then the map \( F \) induces an isomorphism \( f \) of the sheaves \( \mathfrak{F}_{V/G} \to \mathfrak{F}_{V'/G'} \), i.e. a holomorphic diffeomorphism of orbit spaces \( V/G \) and \( V'/G' \).

Lemma. There is a unique isomorphism \( a : G \to G' \) such that \( F \circ a = a \circ F \) for every \( g \in G \).

Note that \( a \) and its inverse \( a^{-1} \) map complex reflections to complex reflections.

Proof. The cardinalities of the two groups are the same since \( F \) maps a generic regular orbit to a regular orbit. Consequently, it maps regular points to regular points and we have \( \sigma' \circ F = f \circ \sigma : V \to V'/G' \) for a holomorphic diffeomorphism \( f : V/G \to V'/G' \), where \( \sigma : V \to V/G \) and \( \sigma' : V' \to V'/G' \) are the quotient projections.

Fix some \( G \)-regular \( v \in V \). Then \( F(v) \) and \( F(gv) \) for \( g \in G \) are regular points of \( V' \) of the same orbit. Therefore, there is a unique \( a(g) \in G \) such that \( F(gv) = a(g)(F(v)) \). We have \( \sigma' \circ F \circ g = f \circ \sigma \circ g = f \circ \sigma = \sigma' \circ \sigma a(g) \circ F \). Since \( \sigma' \) is tale on \( V'_{\text{reg}} \) we see that \( F \circ g = a(g) \circ F \) locally near \( v \) and thus globally. By uniqueness, the map \( g \to a(g) \) is an isomorphism of \( G \) onto \( G' \).

In this section we study when a diffeomorphism \( f \) of the orbit spaces \( Z \to Z' \) has a holomorphic lift \( F \).

5.2. Corollary. Let \( F : V \to V \) be a holomorphic diffeomorphism which maps \( G \)-orbits onto \( G' \)-orbits, and \( f : Z \to Z' \) the corresponding holomorphic diffeomorphism of the orbit spaces. Then \( f \) maps the isotropy type stratification of \( Z \) onto that of \( Z' \) and, moreover, it maps \( D_Z \) to \( D_{Z'} \).

Proof. This follows from Lemma 5.1 and the definition 3.3 of the reflection divisor.
5.3. Theorem. Let $G$ and $G'$ be two finite subgroups of $GL(V)$ and let $f : Z \rightarrow Z'$ be a holomorphic diffeomorphism of the corresponding orbit spaces such that $f(Z_0) = Z'_0$ and $f_*(D_Z) = D_{Z'}$. If $Q$ is a holomorphic tensor field of type $(^p_q)$ on $Z_0$ which satisfies the conditions of Theorem 3.7, then $f_*(Q)$ also satisfies these conditions on $Z'_0$ and thus there exists a unique $G'$-invariant holomorphic tensor field $Q'$ of type $(^p_q)$ such that $\sigma'_* Q'$ coincides with $f_* Q$ on $Z'_0$.

This is also true for holomorphic connections if we replace Theorem 3.7 by Theorem 4.2. The theorem remains true if we replace $V$ by invariant open subsets of $V$.

Proof. Since $f(Z_0) = Z'_0$ the tensor field $f_* Q$ is also holomorphic on $Z'_0$. Let $z \in Z_1 \setminus Z_0$. Then there is a complex space $S \in R_Z$ of codimension 1 such that $z \in S$. By assumption $f(z) \in Z'_1 \setminus Z'_0$ and $f(z) \in f(S) \in R_{Z'}$ and $r_z = e_S = e_{f(S)} = r_{f(z)}$.

Now, obviously $f_*(Q)$ satisfies the conditions of Theorem 3.7 at $f(x)$. Thus there exists a $G'$-invariant holomorphic tensor field $Q'$ on $V$ with $\sigma'_* Q' = f_* Q$.

A similar argument applies to connections. 

5.4 Theorem. Let $G$ and $G'$ be two finite subgroups of $GL(V)$. Let $f : Z \rightarrow Z'$ be a holomorphic diffeomorphism of the orbit spaces such that $f(Z_0) = Z'_0$ and $f_*(D_Z) = D_{Z'}$.

Then $f$ lifts to a holomorphic diffeomorphism $F : V \rightarrow V$, i.e. $\sigma' \circ F = f \circ \sigma$.

The local version is also true. Namely, if $B$ is a ball in the vector space $V$ centered at 0 (for an invariant Hermitian metric), $U = \sigma(B)$, and $f : U \rightarrow Z'$ is a local holomorphic diffeomorphism of $U$ onto a neighborhood $U'$ of $\sigma'(0)$ such that $f(U \cap Z_0) = U' \cap Z'_0$ and $f$ maps $D_Z \cap U$ to $D_{Z'} \cap U'$, then there is a holomorphic lift $F : B \rightarrow V$.

Proof. Let $\Gamma$ be the natural flat connection on $V$. Then $\Gamma$ is uniquely defined by the holomorphic connection $\sigma_* \Gamma$ on $Z_0$ which satisfies the conditions of Theorem 4.2. By Theorem 5.3 there is a unique $G$-invariant holomorphic complex linear connection $\Gamma'$ on $V$ such that $\sigma'_* \Gamma'$ coincides with $f_*(\sigma_* \Gamma)$ on $Z'_0$. It is evident that $\Gamma'$ is a torsion free flat connection, since $\Gamma$ is it and $\Gamma'$ is locally isomorphic to $\Gamma$ on an open dense subset.

Let $v \in V$ be $G$-regular and let $v' \in V$ be $G'$-regular, such that $(f \circ \sigma)(v) = \sigma'(v')$. Then there is a biholomorphic map $F$ of a neighborhood $W$ of $v$ onto a neighborhood of $v'$ such that $\sigma' \circ F = f \circ \sigma$ on $W$ and $F(v) = v'$. Moreover by construction $F$ is a locally affine map of the affine space $(V, \Gamma)$ into $(V, \Gamma')$ equipped with the above structures of locally affine spaces, thus we have

$$ (1) \quad F = \exp_{v'} T_v F \circ \exp_v^\Gamma $$

where $\exp_v^\Gamma : T_v V \rightarrow V$ is the holomorphic geodesic exponential mapping centered at $v$ given by the connection $\Gamma$ and its induced spray. It is globally defined, thus complete and a holomorphic diffeomorphism since $\Gamma$ is the standard flat connection. Likewise $\exp_{v'}^{\Gamma'}$ is the holomorphic exponential mapping of the flat connection $\Gamma'$.

The formula above extends $F$ to a globally defined holomorphic mapping if $\exp_{v'}^{\Gamma'} : T_v V \rightarrow V$ is also globally defined (complete). Assume for contradiction that this is not the case. Let $F$ be maximally extended by equation (1); it still projects to $f : Z \rightarrow Z'$. We consider $\exp_v^\Gamma$ as a real exponential mapping, and then there is a real geodesic which reaches infinity in finite time and this is the image under $F$ of a finite part $\exp_v^\Gamma((0, t_0)w)$ of a real geodesic of $\Gamma$ emanating at $v$. The sequence
The groups \( \sigma \) complex reflections. Put

Next we prove Theorem 5.4 in the case when the group 5.6.

\[
\text{Proof.} \quad \text{Consider the restriction} \quad f \to G \quad \text{there is a holomorphic lift} \quad F \quad \text{of} \quad f^{-1}. \quad \text{Evidently the map} \quad F \circ F' \quad \text{preserves each} \quad G\text{-orbit.} \quad \text{Then, for a} \quad G\text{-regular point} \quad v \in V, \quad \text{there is a} \quad g \in G \quad \text{such that} \quad F' \circ F = g \quad \text{in a neighborhood of} \quad v \quad \text{and, then, on the whole of} \quad V. \quad \text{Similarly} \quad F \circ F' = g' \in G'. \quad \text{This implies that} \quad F \quad \text{is a holomorphic diffeomorphism of} \quad V. \quad \text{By definition the lift} \quad F \quad \text{respects the partitions of} \quad V \quad \text{into orbits.} \quad \Box
\]

We give a second proof of Theorem 5.4 based on the known results about the fundamental groups of \( V_{\text{reg}} \) and \( Z_0 \) for finite complex reflection groups. It is an extension of the proof of 5.3, using results of 5.2.

5.5. Lemma. Let \( G \) and \( G' \) be two finite subgroups of \( GL(V) \) and let \( f : Z \to Z' \) be a holomorphic diffeomorphism of the corresponding orbit spaces. Suppose \( v_0 \in V_{\text{reg}}, \quad v'_0 \in V'_{\text{reg}}, \) and \( f \circ \sigma(v_0) = \sigma'(v'_0). \) If the image of the fundamental group \( \pi_1(V_{\text{reg}}, v_0) \) under \( f \circ \sigma \) is contained in the subgroup \( \sigma'_* \circ \pi_1(V_{\text{reg}}, v'_0) \) of \( \pi_1(Z_0, \sigma'(v'_0)) \), the holomorphic lift of \( f \circ \sigma \) mapping \( v_0 \) to \( v'_0 \) exists.

\textbf{Proof.} Consider the restriction \( \varphi \) of the map \( f \circ \sigma \) to \( V_{\text{reg}} \). Since the restriction of \( \sigma \) to \( V_{\text{reg}} \) is a covering map onto \( Z_0 \), the condition of the lemma implies that there is a holomorphic lift \( F_0 \) of the map \( \varphi \) to \( V_{\text{reg}} \). The map \( F_0 \) is bounded on \( B \cap V_{\text{reg}} \) for each compact ball \( B \) in \( V \) since its image is contained in the compact set \( (\sigma'_{-1}) (f(\sigma(B))) \). Then by the Riemann extension theorem \( F_0 \) has a holomorphic extension \( F \) to \( V \) which is the required holomorphic lift of \( f \). \Box

5.6. Next we prove Theorem 5.4 in the case when the group \( G \) is generated by complex reflections. Put

\[
B := \pi_1(Z_0) \quad \text{and} \quad P := \pi_1(V_{\text{reg}}).
\]

The groups \( B \) and \( P \) are called the braid group and the pure braid group associated to \( G \), respectively. It is clear that the map \( \sigma \) induces an isomorphism of \( P \) onto a subgroup of \( B \).

The following results about the groups \( B \) and \( P \) are well known (see, for example, 2). The braid group \( B \) is generated by those elements which are represented by loops around the hypersurfaces \( \sigma(H) \) for \( H \in \mathcal{H} \). The pure braid group \( P \) is generated by the elements of \( B \) of the type \( s^e H \), where \( s \) is any of the above generators of \( B \) represented by a loop around the hypersurface \( \sigma(H) \). This implies the following

\textbf{Proposition.} Suppose the group \( G \) is generated by complex reflections. Let \( f \) be a holomorphic diffeomorphism of the orbit space \( Z = \mathbb{C}^n \) with \( f(Z_0) = Z_0 \) which also preserves \( D_2 \). Then \( f|_{Z_0} \) preserves the subgroup \( P \) of \( B \). \Box

The following proposition is an immediate consequence of Lemma 5.5 and Proposition 5.6.

5.7. Proposition. Suppose the groups \( G \) and \( G' \) are generated by complex reflections. Let \( f : Z \to Z' \) be a holomorphic diffeomorphism between the corresponding orbit spaces, such that \( f(Z_0) = Z_0' \) and \( f_*(D_2) = D_2' \).
Then $f$ has a holomorphic lift $F$ to $V$. □

**Second proof of 5.4.** Now let $G \subset GL(V)$ be a finite group and let $G_1$ be the subgroup generated by all complex reflections in $G$. Clearly $G_1$ is a normal subgroup of $G$. Let $G_2 = G/G_1$. Let $\sigma_1^1, \ldots, \sigma_1^n$ be a system of homogeneous generators of $C[V]^{G_1}$ and $\sigma_1 : V \to C^n$ the corresponding orbit map. Then the action of $G$ on $V$ induces the action of the group $G_2$ on $V_1 := C^n = \sigma_1(V)$. Since each representation of the group $G_2$ is completely reducible, by standard arguments of invariant theory, we may assume that the generators $\sigma_i^j$’s are chosen in such a way that the above action of $G_2$ on $V_1 = C^n$ is linear. Then the representation of $G_2$ on $V_1$ contains no complex reflections. Let $\sigma_2^1, \ldots, \sigma_2^m$ be a system of homogeneous generators of $C[V_1]^{G_2}$ and $\sigma_2 : V_1 \to C^m$ the corresponding orbit map. Then $\sigma_i^j = \sigma_2^i \sigma_1^j$ $(i = 1, \ldots, m)$ is a system of generators of $C[V]^{G}$ with orbit map $\sigma = \sigma_2 \sigma_1$. Similarly for $G'$.

Let $f : Z \to Z'$ be a holomorphic diffeomorphism, such that $f(Z_0) = Z'_0$ and $f_*(DZ) = DZ'$. Since the group $G_2$ contains no complex reflections the set $V_{1,reg}$ of regular points of the action of $G_2$ on $V_1$ is obtained from $V_1$ by removing some subsets of codimension $\geq 2$. And similarly for $G'$. Then the fundamental group $\pi_1(V_{1,reg}) = \pi_1(V_1) = 0$ is trivial and by lemma 5.5 the diffeomorphism $f$ has a holomorphic lift $F_1 : V_1 \to V'_1$ which is a holomorphic diffeomorphism mapping the principal stratum to the principal stratum, and the reflection divisor to the reflection divisor, since $G_2$ contains no complex reflections on $V_1$. Thus the diffeomorphism $F_1$ has a holomorphic lift to $V$ by Proposition 5.7, which is a holomorphic lift of $f$. □

### 6. An intrinsic characterization of a complex orbifold

We recall the definition of orbifold.

**6.1. Definition.** Let $X$ be a Hausdorff space. An atlas of a smooth $n$-dimensional orbifold on $X$ is a family $\{U_i\}_{i \in I}$ of open sets that satisfy:

1. $\{U_i\}_{i \in I}$ is an open cover of $X$.
2. For each $i \in I$ we have a local uniformizing system consisting of a triple $(\tilde{U}_i, G_i, \varphi_i)$, where $\tilde{U}_i$ is a connected open subset of $\mathbb{R}^n$ containing the origin, $G_i$ is a finite group of diffeomorphisms acting effectively and properly on $\tilde{U}_i$, and $\varphi_i : \tilde{U}_i \to U_i$ is a continuous map of $\tilde{U}_i$ onto $U_i$ such that $\varphi_i \circ g = \varphi_i$ for all $g \in G_i$ and the induced map of $\tilde{U}_i/G_i$ onto $U_i$ is a homeomorphism. The finite group $G_i$ is called a local uniformizing group.
3. Given $\tilde{x}_i \in \tilde{U}_i$ and $\tilde{x}_j \in \tilde{U}_j$ such that $\varphi_i(\tilde{x}_i) = \varphi_j(\tilde{x}_j)$, there is a diffeomorphism $g_{ij} : V_j \to V_i$ from a neighborhood $V_j \subseteq U_j$ of $\tilde{x}_j$ onto a neighborhood $V_i \subseteq U_i$ of $\tilde{x}_i$ such that $\varphi_j = \varphi_i \circ g_{ij}$.

Two atlases are equivalent if their union is again an atlas of a smooth orbifold on $X$. An orbifold is the space $X$ with an equivalence class of atlases of smooth orbifolds on $X$.

If we take in the definition of orbifold $\mathbb{C}^n$ instead of $\mathbb{R}^n$ and require that $G_i$ is a finite group of holomorphic diffeomorphisms acting effectively and properly on $\tilde{U}_i$ and the maps $g_{ij}$ are biholomorphic, we get the definition of complex analytic $n$-dimensional orbifold.
6.2. Theorem. [1] Let $M$ be a smooth manifold and $G$ a proper discontinuous group of diffeomorphisms of $M$. Then the orbit space $M/G$ has a natural structure of smooth $n$-dimensional orbifold. If $M$ is a complex $n$-dimensional manifold and $G$ is a group of holomorphic diffeomorphisms of $M$, the orbit space $M/G$ is a complex $n$-dimensional orbifold.

6.3 Definitions. In the definition of atlas of a complex orbifold on $X$ we can always take $\tilde{U}_i$ to be balls of the space $\mathbb{C}^n$ (with respect to some Hermitian metric) centered at the origin and the finite subgroups $G_i$ to be subgroups of the $GL(n)$ acting naturally on $\mathbb{C}^n$. In the sequel we consider atlases of complex orbifolds satisfying these conditions.

Let $X$ be a complex orbifold with an atlas $(\tilde{U}_i, G_i, \varphi_i)$. A function $f : U_i \to \mathbb{C}$ is called holomorphic if $f \circ \varphi_i$ is a holomorphic function on $\tilde{U}_i$. The germs of holomorphic functions on $X$ define a sheaf $\mathcal{F}_X$ on $X$. It is evident that the sheaf $\mathcal{F}_X$ depends only on the structure of complex orbifold on $X$.

Consider a uniformizing system $(\tilde{U}_i, G_i, \varphi_i)$ of the above atlas and the corresponding action of $G_i$ on $\mathbb{C}^n$. Then we have the isotropy type stratification of the orbit space $\mathbb{C}^n/G_i$, the induced stratification of $U_i$, and the divisor $D_i$.

By corollary 5.2 we get the stratification on $X$ by gluing the strata on the $U_i$'s. Denote by $X_0$ the principal stratum of this stratification. By definition, for each $x \in X_0$, for each uniformizing system $(U_i, G_i, \varphi_i)$, and for each $y \in \tilde{U}_i$ such that $\varphi_i(y) = x$, the isotropy group $G_y$ of $y$ is trivial. Note that $X_0$ is a complex manifold. Note that $X_1$ is also a complex manifold since this holds locally as noted in 3.5.

Denote by $R_X$ the set of all strata of codimension 1 of $X$. Since the pullbacks of the reflection divisors $D_{U_i}$ to $U_i \cap U_j$ agree by 5.2 we may glue them into the reflection divisor $D_X$ on $X_1$.

6.4. Definition. Let $X$ and $\tilde{X}$ be two smooth orbifolds. The orbifold $\tilde{X}$ is called a covering orbifold for $X$ with a projection $p : \tilde{X} \to X$ if $p$ is a continuous map of underlying topological spaces and each point $x \in X$ has a neighborhood $U = \tilde{U}/G$ (where $U$ is an open subset of $\mathbb{R}^n$) for which each component $V_i$ of $p^{-1}(U)$ is isomorphic to $\tilde{U}/G_i$, where $G_i \subseteq G$ is some subgroup. The above isomorphisms $U = \tilde{U}/G$ and $V_i = U/G_i$ must respect the projections.

Note that the projection $p$ in the above definition is not necessarily a covering of the underlying topological spaces. It is clear that a covering orbifold for a complex orbifold is a complex orbifold. Hereafter we suppose that all orbifolds and their covering orbifolds are connected.

6.5. Theorem. [1] An orbifold $X$ has a universal covering orbifold $p : \tilde{X} \to X$. More precisely, if $x \in X_0$, $\tilde{x} \in \tilde{X}_0$ and $p(\tilde{x}) = x$, for any other covering orbifold $p' : \tilde{X}' \to X$ and $\tilde{x}' \in \tilde{X}'$ such that $p'(\tilde{x}') = x$ there is a cover $q : \tilde{X} \to \tilde{X}'$ such that $p = p' \circ q$ and $q(\tilde{x}) = \tilde{x}'$. For any points $\tilde{x}, \tilde{x}' \in p^{-1}(x)$ there is a deck transformation of $\tilde{X}$ taking $\tilde{x}$ to $\tilde{x}'$.

Now we prove the main theorem of this section.

6.6. Theorem. An $n$-dimensional complex orbifold $X$ is uniquely determined by the sheaf of holomorphic functions $\mathcal{F}_X$, the principal stratum $X_0$, and the reflection divisor $D_X$. 

Proof. For each $x \in X$, there exists $V = \mathbb{C}^m$, a finite group $G \subset GL(m)$, a ball $B$ in $V$ centered at $0$, an open subset $U$ of $X$ containing $x$, and an isomorphism $\psi : \pi(B) \to U$ between the sheaves $\mathcal{F}|_{\pi(B)}$ and $\mathcal{F}|_U$. Consider the map $\pi : V \to Z = V/G$, the stratum $Z_0$ and the reflection divisor $D_Z$. We suppose also that $\psi(Z_0 \cap B/G) \subseteq X_0$ and $\psi_\pi(D_{\pi(B)}) = D_U$. It suffices to prove that the germ of the uniformizing system $\{B, G, \psi \circ \pi|B\}$ at $x$ is the germ of some uniformizing system of the orbifold $X$.

Let $y \in V_{\text{reg}} \cap B$. Then the ring $\mathcal{F}_Z(\pi(y))$ of germs of $\mathcal{F}_Z$ at $\pi(y)$ is isomorphic to the ring of germs of holomorphic functions on $\mathbb{C}^n$ at $0$ and thus we have $m = n$.

Consider the uniformizing system $(\tilde{U}_i, G_i, \varphi_i)$ of the orbifold $X$, where $\tilde{U}_i$ is a ball in $\mathbb{C}^n$ centered at the origin, $G_i$ is a finite subgroup of the group $GL(n)$ acting naturally on $V = \mathbb{C}^n$, and where $\varphi_i(0) = x$. Consider the map $\pi_i : V \to V/G_i$ given by some system of generators of $\mathbb{C}[V]^{G_i}$. We may assume that $\varphi_i = \psi_i \circ \pi_i|_{\tilde{U}_i}$, where $\psi_i : \mathcal{F}_{\tilde{U}_i}/G_i \to \mathcal{F}_{\tilde{U}_i}$ is an isomorphism of sheaves.

Then the maps $\psi$ and $\psi_i$ define a map (germ) $f$ of a holomorphic diffeomorphism $B/G$ to $U_i/G_i$ at $0 := \pi(0)$ such that $f(0) = 0 := \pi_i(0)$. Then $f$ induces an isomorphism $\mathcal{F}_{V/G}(0) \to \mathcal{F}_{V/G}(0)$, it maps $(B/G)_0$ to $(U_i/G_i)_0$ and $f_* (D_{B/G}) = D_{U_i/G_i}$. Thus by theorem 5.4 there is a germ of a holomorphic diffeomorphism $F : B \to \tilde{U}_i$ which is equivariant for a suitable isomorphism $G \to G_i$. □

6.7. Corollary. Let $M$ be a complex simply connected manifold, $G$ a proper discontinuous group of holomorphic diffeomorphisms of $M$, and $\mathcal{F}_X$ the corresponding sheaf on the orbifold $X = M/G$. The $G$-manifold $M$ is a universal covering orbifold for the orbifold $X$ and it is defined uniquely up to a natural isomorphism of universal coverings by the sheaf $\mathcal{F}_X$, the principal stratum $X_0$, and by the reflection divisor $D_X$.

Proof. Evidently the manifold $M$ is a covering orbifold for $X$. If $\tilde{X}$ is a universal covering orbifold for $X$, by definition 6.4 there is a cover $\tilde{q} : \tilde{X} \to M$. By definition $X$ should be a manifold and $q$ a cover of manifolds. Therefore, $q$ is a diffeomorphism. Then the statement of the corollary follows from theorem 6.6. □

An automorphism of the sheaf $\mathcal{F}_X$ is called a holomorphic diffeomorphism of the orbit space $X$. Theorem 6.5 and corollary 6.7 imply the following analogue of Theorem 5.4.

6.8. Theorem. Let $M$ be a complex simply connected manifold, $G$ a proper discontinuous group of holomorphic diffeomorphisms of $M$, and $\mathcal{F}_X$ the corresponding sheaf on the orbifold $X = M/G$. Each holomorphic diffeomorphism $f$ of the orbit space $X$ preserving $X_0$ and $D_X$ has a holomorphic lift $F$ to $M$, which is $G$-equivariant with respect to an automorphism of $G$. The lift $F$ is unique up to composition by an element of $G$. 


Proof. By theorem 6.6 and corollary 6.7 the manifold $M$ with the map $f_{op} : M \to X$, where $p : M \to X$ is the projection, is a universal covering orbifold for $X$. Then there is a holomorphic diffeomorphism $F : M \to M$ such that $pF = f_{op}$. The equivariance property holds locally by 5.1, thus globally. The lift is uniquely given by choosing $F(x)$ for a regular point $x$ in the orbit $f(p(x))$. \qed

6.9. Let $V$ be a complex vector space with a linear action of a finite group $G$. The group $C^*$ acts on $V$ by homotheties and induces an action on $Z = V/G$.

Corollary. In this situation, the $G$-module $V$ is uniquely defined up to a linear isomorphism by the sheaf $\mathfrak{F}_{V/G}$ with the action of $C^*$, by $Z_0$, and the reflection divisor $D_Z$. \qed

Proof. Consider the orbit space $Z = V/G$ of a $G$-module $V$ with the sheaf $\mathfrak{F}_{V/G}$, regular stratum $Z_0$, reflection divisor $D_Z$, and the action of $C^*$ induced by the action of $C^*$ on $V$ by homotheties. Suppose that we have another $G'$-module $V'$ with the same data on $Z' = V'/G'$ such that there is a biholomorphic map $f : Z \to Z'$ preserving these data. By Theorem 4.5 there is a biholomorphic lift $F : V \to V'$, and by lemma 5.1 there is an isomorphism $a : G \to G'$ such that $F \circ g = a(g) \circ F$. Thus we may assume that $G = G'$, $V = V'$, $Z = Z'$, and $a$ is the identity map. By definition the pullback $A$ of the vector field on the orbit space $V/G$ defined by the action of the group $C^*$ on $V/G$ coincides with the vector field on $V$ defined by the above action of the group $C^*$ on $V$. By construction $F^* A = A$ and then the map $F$ commutes with the action of $C^*$ on $V$, i.e. for each $t \in C^*$ and $v \in V$ we have $F(tv) = tF(v)$. Since $F$ is biholomorphic it is a linear automorphism of the vector space $V$. By definition $F$ is then an automorphism of the $G$-module $V$. \qed

6.10. Tensor fields and connections on orbifolds. The local results in section 3 show that the correct definition of a $\left(\frac{p}{q}\right)$-tensor field $Q$ on an orbifold $X$ is as follows: $Q$ is a meromorphic $\left(\frac{p}{q}\right)$-tensor field on $X_1$ such that $\text{div}_{D_X}(Q) \geq 0$.

Likewise, we can define connections on orbifolds by requiring the local conditions of section 4.

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