ON TWO THEOREMS OF DARBOUX

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Abstract. We provide precise formulations and proofs of two theorems from Darboux's lectures on orthogonal systems [2]. These results provide local existence and uniqueness of solutions to certain types of first order PDE systems where each equation contains a single derivative for which it is solved:

\[ \frac{\partial u_i}{\partial x_j}(x) = f_{ij}(x, u(x)). \]

The data prescribe values for the unknowns \( u_i \) along certain hyperplanes through a given point \( \bar{x} \).

The first theorem applies to determined systems (the number of equations equals the number unknowns), and a unique, local solution is obtained via Picard iteration. While Darboux's statement of the theorem leaves unspecified "certaines conditions de continuité," it is clear from his proof that he assumes Lipschitz continuity of the maps \( f_{ij} \). On the other hand, he did not address the regularity of the data. We provide a precise formulation and proof of his first theorem.

The second theorem is more involved and applies to overdetermined systems of the same general form. Under the appropriate integrability conditions, Darboux used his first theorem to treat the cases with two and three independent variables. We provide a proof for any number of independent variables.

While the systems are rather special, they do appear in applications; e.g., the second theorem contains the classical Frobenius theorem on overdetermined systems as a special case. The key aspect of the proofs is that they apply to non-analytic situations. In an analytic setup the results are covered by the general Cartan-Kähler theorem.

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1. Introduction

Darboux, in Chapter I of Livre III in his monograph “Systèmes Orthogonaux” [2], stated three integrability theorems for certain types of first order systems of PDEs. The second one of these is the classical Frobenius theorem and will not be considered in detail in what follows (see Remark 4.3); the remaining ones are his “Théorème I” and “Théorème III” and these are the “two theorems” in our title.

The theorems give local existence for certain types of first order systems of PDEs of the general form

\[ u_{i,x_j}(x) = f_{ij}(x, u(x)), \]

where \( u \) denotes the vector of unknown functions \( u_1, \ldots, u_N \), the independent variables \( x = (x_1, \ldots, x_n) \) ranges over an open set about a fixed point \( \bar{x} \in \mathbb{R}^n \), and the \( f_{ij} \) are given scalar functions. The data that the systems under consideration are to take on consist of given functions prescribed along (typically several) hyperplanes through the point \( \bar{x} \in \mathbb{R}^n \). Darboux makes repeated use of Théorème I in his proof of Théorème III.

Concerning regularity, we assume throughout that the functions \( f_{ij} \) as well as the assigned data functions are (at least) continuous. By a “solution” we always mean a classical, \( C^1 \) function \( u \) which satisfy the PDEs and the data at every point of some neighborhood of \( \bar{x} \).

We note the special character of the equations: each equation contains a single derivative with respect to which it is solved. As commented by Spivak in [7] in connection with the PDE formulation of the Frobenius theorem, it is rather laughable to call these PDEs at all as they do not relate different partial derivatives to each other. Yet, these are basic equations and they are useful. Our motivation for revisiting Darboux’s results has been their application to PDE systems that appear when asking for the existence of
maps \( g : \mathbb{R}^n \to \mathbb{R}^n \) whose Jacobian matrix has a set of prescribed eigenvector fields [5].

1.1. Determined systems. The first theorem concerns determined systems, i.e. the number \( N \) of dependent variables \( u_i \) equals the number of equations. The data in this case consists of \( N \) prescribed functions along certain hyperplanes through \( \bar{x} \) (see Section 2 for details).

Darboux employed Picard iteration to establish local existence of a solution, and the proof is quite similar to that of the classical Picard-Lindelöf theorem for local existence to Cauchy problems for ODEs [4]. Nevertheless, some care is required to obtain a precise result which is explicit about the regularity of the data and the functions \( f_{ij} \). In Darboux’s proof the first step is to apply a change of coordinates \( x \mapsto y \) and \( u \mapsto v \) so as to obtain an equivalent PDE system

\[
(v_i)_{y_j}(y) = F_{ij}(y, v(y)) \tag{1}
\]

The change of variables is made so that the point \( \bar{x} \) corresponds to the origin and the data for the \( v \)-variables vanish identically. As is made explicit in his proof, Darboux assumes Lipschitz continuity of the maps \( F_{ij} \) (with respect to the dependent variables \( v \)). Next, Darboux sets up the appropriate Picard iteration and establishes existence of a local solution \( v \). Finally, he establishes uniqueness by exploiting (in a now-standard fashion) the assumed Lipschitz continuity of the \( F_{ij} \). With that his proof is finished.

However, the following “detail” was not addressed by Darboux: the change of variables depends on the originally prescribed data, and no regularity assumptions are imposed on the data. Our first task is to address this issue and provide local existence under precise regularity conditions on the original right hand sides \( f_{ij} \), as well as on the original \( u \)-data. Different setups are possible; we opt for Lipschitz continuity of the \( f_{ij} \) (with respect to \( u \)) and just continuity of the \( u \)-data. The precise statement is given in Theorem 2.3 below.

We note two points about this result. First, it provides an easy, “ODE method” proof for local solutions to a certain type of PDE systems under mild regularity conditions. Even in an analytic setting, the standard Cauchy-Kowalevskaya theorem does not apply as the data are prescribed along several hyperplanes. (The only exception to this last statement is the case when the given system (1) degenerates to an ODE system, i.e. the independent variable \( x_i \) in (1) is the same in all equations.)

Concerning the regularity setup we use, other choices are possible. E.g., it would be of interest to formulate an existence result for Carathéodory-type solutions for systems (1). However, we will not pursue this in the present paper.

1.2. Overdetermined systems. The second theorem (Darboux’s Théorème III) concerns overdetermined systems of the same general form (1). Under
the appropriate integrability conditions, which are rather restrictive, it guarantees existence of a unique solution near a point $\bar{x}$, again for data given along hyperplanes through $\bar{x}$. This result is more challenging to prove, and Darboux limited himself to the cases with two or three independent variables. It is not immediate to generalize his argument to cases with more independent variables. We shall provide a proof by induction on the number of independent variables.

Concerning regularity assumptions, we note that a key ingredient in both Darboux’s and our proof for the second theorem, is the application of the first theorem. In particular, it will be applied to the 1st order system satisfied by quantities of the form “LHS(1) − RHS(1).” We will therefore need to require more regularity than for the first theorem. Another reason for increased regularity is to make the integrability conditions valid.

We note that the second theorem is a consequence of the much more general Cartan-Kähler theorem. However, that theorem depends on the Cauchy-Kowalevskaya theorem and only applies to the much more restrictive setting of analytic systems with analytic data. For the precious few results on overdetermined systems which do not require analyticity, see [8] and references therein. The work [8] is formulated in a smooth ($C^\infty$) setting; the proofs are not easy. In contrast, the present work employs entirely elementary tools.

2. DARBOUX’S THEOREM ON DETERMINED SYSTEMS

2.1. Setup and notation. We consider determined systems of PDEs with the following structure:

(i) there are $N$ first order PDEs for $N$ unknown functions,
(ii) each equation contains exactly one derivative and is assumed solved for this derivative, and
(iii) each unknown appears differentiated exactly once.

There is no constraint regarding which first derivatives appear in the equations. E.g., they could all be with respect to the same independent variable (giving an ODE system with parameters) or with respect to some or all of the independent variables.

We denote the independent variables by $x = (x_1, \ldots, x_n)$ and the dependent variables by $u = (u_1, \ldots, u_N) \in \mathbb{R}^N$. For each $i$, $1 \leq i \leq n$, let $u_{i1}, \ldots, u_{iN_i}$ ($0 \leq N_i \leq N$) denote the dependent variables that appear differentiated with respect to $x_i$ in the system. Note that we allow for the possibility that $N_i = 0$, i.e. none of the equations involve differentiation with respect to $x_i$, in which case $x_i$ plays the role of a parameter. According to assumption (iii) we have that

$$N = \sum_{i=1}^{n} N_i,$$  \hspace{1cm} (2)
and we relabel coordinates so as to write $u$ in the form
\[ u = (u_1, \ldots, u_N) = (u_{11}, \ldots, u_{1N_1}, u_{21}, \ldots, u_{2N_2}, \ldots, u_{n1}, \ldots, u_{nN_n}). \] (3)

The PDEs involving differentiation with respect to $x_i$ then appear consecutively, and we write them as
\[ \frac{\partial u_{ih}}{\partial x_i} = f_{ih}(x_1, \ldots, x_n, u_1, \ldots, u_N). \] (4)

Here and below it is understood that the index $i$ ranges between 1 and $n$, and that for each such $i$ the index $h$ ranges between 1 and $N_i$ (unless $N_i = 0$).

We next describe the data for the system. For a given point $\bar{x} \in \mathbb{R}^n$ and for each $i$, we prescribe $N_i$ functions $\varphi_{ih}$ of $n-1$ arguments, to which the unknown function $u$ should reduce when restricted to the hyperplane $\{x_i = \bar{x}_i\}$ through $\bar{x}$. We use the following notation: for any $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}$ set
\[ x_{i'} := (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in \mathbb{R}^{n-1} \] (5)
and
\[ x_{i'}^\xi := (x_1, \ldots, x_{i-1}, \xi, x_{i+1}, \ldots, x_n) \in \mathbb{R}^n. \] (6)

The data requirement is then that each component $u_{ih}$ satisfy
\[ u_{ih}(x_{i'}^\xi) = \varphi_{ih}(x_{i'}). \] (7)

We define the point $\bar{\varphi} \in \mathbb{R}^N$ by giving its components according to (3) as
\[ \bar{\varphi}_{ih} := \varphi_{ih}(\bar{x}_{i'}). \] (8)

Of course, it is assumed that each function $\varphi_{ih}$ is defined on a neighborhood of $\bar{x}_{i'}$, and that each function $f_{ih}$ is defined on a neighborhood of $(\bar{x}, \bar{\varphi})$. For convenience we shall equip any Euclidean space $\mathbb{R}^k$ under consideration with the 1-norm
\[ |y| := \sum_{j=1}^{k} |y_j|, \quad y \in \mathbb{R}^k, \]
and let $B_r(y)$ denote the corresponding closed ball
\[ B_r(y) := \{ z \in \mathbb{R}^k : |z| \leq r \}. \]

**Remark 2.1.** A special case is that all dependent variables appear differentiated with respect to the same independent variable, $x_1$ say. In this case the system reduces to a system of ODEs with parameters $x_2, \ldots, x_n$, and the data consist of $N$ functions of $x_2, \ldots, x_n$ which the solution should reduce to when $x_1 = \bar{x}_1$. If in addition $x_1$ is the only independent variable then we would prescribe $N$ constants as data at $x_1 = \bar{x}_1$ (pure ODE case).

Before formulating and proving Darboux’s first theorem we include a representative example.
Example 2.2. Consider the system

\begin{align}
    u_x &= f(x, y, z, u, v, w, \xi) \tag{9} \\
    v_x &= g(x, y, z, u, v, w, \xi) \tag{10} \\
    w_y &= h(x, y, z, u, v, w, \xi) \tag{11} \\
    \xi_y &= k(x, y, z, u, v, w, \xi), \tag{12}
\end{align}

with independent variables \( x_1 = x, \ x_2 = y, \ x_3 = z \), and dependent variables \( u_{11} = u, \ u_{12} = v, \ u_{21} = w, \ u_{22} = \xi \). \( (\text{Here} \ n = 3, \ N = 4, \ N_1 = 2, \ N_2 = 2, \ N_3 = 0, \text{and} \ f_{11} = f, \ f_{12} = g, \ f_{21} = h, \ f_{22} = k.) \) The type of data considered in Darboux’s first theorem are given as follows: near a given point \((\bar{x}, \bar{y}) \in \mathbb{R}^2\) we prescribe four functions \( \varphi_{11}(y, z), \varphi_{12}(y, z), \varphi_{21}(x, z), \text{and} \varphi_{22}(x, z) \). Darboux’s first theorem then guarantees the existence and uniqueness of a solution \((u(x, y), v(x, y), w(x, y))\) to the system (9)-(12) near \((\bar{x}, \bar{y})\) which satisfies

\begin{align}
    u(\bar{x}, y, z) &= \varphi_{11}(y, z) \tag{13} \\
    v(\bar{x}, y, z) &= \varphi_{12}(y, z) \tag{14} \\
    w(x, \bar{y}, z) &= \varphi_{21}(x, z) \tag{15} \\
    \xi(x, \bar{y}, z) &= \varphi_{22}(x, z). \tag{16}
\end{align}

In this case the \( z \)-variable plays the role of a parameter.

2.2. Statement and proof of Darboux’s first theorem. Recall that by a “solution” we mean a classical, \( C^1 \) solution satisfying the PDEs in (4), as well as the data requirements (7), in a pointwise manner.

Theorem 2.3. With the notations introduced above, assume there exist numbers \( a, b, L > 0 \) such that the functions \( f_{ih} \) and \( \varphi_{ih} \) \( (1 \leq i \leq n, \ 1 \leq h \leq N_i) \) satisfy:

(A1) Each \( f_{ih} \) maps \( B_{a,b} := B_a(\bar{x}) \times B_b(\bar{\varphi}) \) continuously into \( \mathbb{R} \), with

\[ |f_{ih}(x, u) - f_{ih}(x, v)| \leq L|u - v| \quad \text{for} \ (x, u), (x, v) \in B_{a,b}. \tag{17} \]

(A2) Each \( \varphi_{ih} \) maps \( B_a(\bar{x}^i) \) continuously into \( \mathbb{R} \), with

\[ |\varphi_{ih}(x^i) - \bar{\varphi}_{ih}| \leq \frac{b}{2N} \quad \text{for} \ x^i \in B_a(\bar{x}^i). \tag{18} \]

Then, with

\[ M := \max_{i,h} \sup \{|f_{ih}(x, u)| : (x, u) \in B_{a,b}\} \quad \text{and} \quad \sigma := \min \left(a, \frac{b}{2MN}\right), \]

the PDE system

\[ \frac{\partial u_{ih}}{\partial x_i} = f_{ih}(x_1, \ldots, x_n, u_1, \ldots, u_N), \tag{19} \]

with data

\[ u_{ih}(x_{\bar{x}^i}) = \varphi_{ih}(x^i), \tag{20} \]

has a unique solution \( u(x) \) defined on \( B_{\sigma}(\bar{x}) \).
Remark 2.4. Given \( f_{ih} \) satisfying (A1) and continuous \( \varphi_{ih} \), we can satisfy (18) by reducing \( a \) if necessary.

Proof. We follow the standard procedure of realizing the solution as the fixed point of a map defined on a suitable space of continuous functions. For this, define the function \( \varphi : B_a(\bar{x}) \to \mathbb{R}^N \) by giving its components according to (3) as

\[
\varphi(x)_{ih} := \varphi_{ih}(x'^i),
\]

and then set

\[
X := \{ u \in C(B_\sigma(\bar{x}); \mathbb{R}^N) : |u(x) - \varphi(x)| \leq \frac{b}{2} \quad \forall x \in B_\sigma(\bar{x}) \}. \tag{21}
\]

Equipped with sup-metric the set \( X \) is complete; for convenience we equip \( X \) with the following equivalent metric \( d \):}

\[
d(u, v) := \sup_{x \in B_\sigma(\bar{x})} e^{-K|x - \bar{x}|} |u(x) - v(x)|, \quad \text{for } u, v \in X, \tag{22}
\]

where

\[
K := 2LN. \tag{23}
\]

On the complete metric space \((X, d)\) we define the functional map \( u \mapsto \Phi[u] \) by giving its components according to (3): for \( u \in X \) and \( x \in B_\sigma(\bar{x}) \), set

\[
\Phi[u]_{ih}(x) := \varphi_{ih}(x'^i) + \int_{\bar{x}_i}^{x_i} f_{ih}(x'_\xi, u(x'_\xi)) \, d\xi. \tag{24}
\]

The first thing to verify is that \( \Phi[u](x) \) is in fact well-defined whenever \( u \in X \) and \( x \in B_\sigma(\bar{x}) \). The first term on RHS(24) is defined since \( \sigma \leq a \). For the integral on RHS(24) we note that whenever \( \xi \) is between \( \bar{x}_i \) and \( x_i \), then

\[
|x'_\xi - \bar{x}| \leq |x - \bar{x}| \leq \sigma \leq a, \tag{25}
\]

and

\[
|u(x'_\xi) - \varphi| \leq |u(x'_\xi) - \varphi(x'_\xi)| + |\varphi(x'_\xi) - \bar{\varphi}|
\leq \frac{b}{2} + \sum_{i,h} |\varphi_{ih}(x'^i) - \bar{\varphi}_{ih}|
\leq \frac{b}{2} + N \cdot \frac{b}{2N} = b. \quad \text{(by (A2))}
\]

Thus, whenever \( \xi \) is between \( \bar{x}_i \) and \( x_i \), and \( u \in X \), then

\[
(x'_\xi, u(x'_\xi)) \in B_{a,b},
\]

such that \( f_{ih}(x'_\xi, u(x'_\xi)) \) is defined according to (A1). This shows that each \( \Phi[u]_{ih}(x) \) is defined whenever \( x \in B_\sigma(\bar{x}) \) and \( u \in X \).
Next, we show that \( \Phi \) maps \( X \) into itself. For \( x \) and \( u \) as above, clearly \( \Phi[u] \) is a continuous map. Furthermore,

\[
|\Phi[u](x) - \varphi(x)| = \sum_{i,h} |\Phi[u]_{ih}(x) - \varphi_{ih}(x^{t_i})| = \sum_{i,h} \left| \int_{\tilde{x}_i}^{x_i} f_{ih}(x^{t_i}, u(x^{t_i})) \, d\xi \right|
\leq M \sum_{i,h} |x_i - \tilde{x}_i| \leq MN|x - \tilde{x}| \leq MN\sigma \leq \frac{b}{2},
\]

which shows that \( \Phi[u] \) belongs to \( X \) whenever \( u \in X \).

Finally, we argue that \( \Phi : X \to X \) is a strict contraction. For this assume \( u, v \in X \) and \( x \in B_\sigma(\tilde{x}), \) and compute:

\[
|\Phi[u](x) - \Phi[v](x)| = \sum_{i,h} |\Phi[u]_{ih}(x) - \Phi[v]_{ih}(x)|

\leq \sum_{i=1}^{N_i} N_i L \cdot \int_{\tilde{x}_i}^{x_i} \left| u(x^{t_i}) - v(x^{t_i}) \right| d\xi

\leq L \cdot d(u, v) \cdot \sum_{i=1}^{N_i} e^{K|x^{t_i} - \tilde{x}_i|} d\xi.
\]

(Here \( \land \) and \( \lor \) denote “minimum” and “maximum,” respectively.) From this and the choice \( \lbrack 23 \rbrack \) for \( K \) we obtain that (recall \( | \cdot | \) denotes 1-norm and that \( N \) is given by \( \lbrack 2 \rbrack \))

\[
e^{-K|x-x|} |\Phi[u](x) - \Phi[v](x)| \leq L \cdot d(u, v) \cdot \sum_{i=1}^{N_i} e^{-K|x_i - \tilde{x}_i|} \int_{\tilde{x}_i}^{x_i} e^{K|x^{t_i} - \tilde{x}_i|} d\xi

= L \cdot d(u, v) \cdot \sum_{i=1}^{N_i} \frac{1}{K} \left( e^{K|x_i - \tilde{x}_i|} - 1 \right)

< L \cdot d(u, v) \cdot \frac{N}{K} = \frac{1}{2} d(u, v).
\]

It follows from \( \lbrack 22 \rbrack \) that

\[
d(\Phi[u], \Phi[v]) \leq \frac{1}{2} d(u, v),
\]

i.e. \( \Phi : X \to X \) is a strict contraction. According to the contraction mapping theorem \( \Phi \) has a unique fixed point \( u \in X \), i.e.

\[
u_{ih}(x) = \varphi_{ih}(x^{t_i}) + \int_{\tilde{x}_i}^{x_i} f_{ih}(x^{t_i}, u(x^{t_i})) \, d\xi
\]

for each \( 1 \leq i \leq n, 1 \leq h \leq N_i \). It is immediate that the PDEs in \( \lbrack 19 \rbrack \) are satisfied, and that \( u \) satisfies the data requirements in \( \lbrack 20 \rbrack \).
Finally, uniqueness follows from the fact that any solution (in the classical, pointwise sense that we consider) is a fixed point of $\Phi$, together with the uniqueness of such fixed points.

3. Darboux’s theorem on overdetermined systems

3.1. Preliminaries. By “Darboux’s theorem on overdetermined systems” we mean Théorème III of Chapter I of Livre III in [2]. From here on we refer to this simply as “Darboux’s theorem,” while Theorem 2.3 above will be called “Darboux’s first theorem.”

Darboux’s theorem requires more work to state and prove. In this section we first describe the class of systems under consideration, and then give two examples. Finally, we comment on Darboux’s original treatment which provided a proof for the cases of two and three independent variables. Then, in Section 4 we introduce the class of “Darboux systems” and formulate Darboux’s theorem for these. The proof of the existence part of the theorem proceeds by induction on the number of independent variables. To highlight its structure we outline the argument for the case $n = 3$ in Section 5. The detailed proof for an arbitrary number of independent variables is carried out in Section 6.

Convention 1. In what follows “an unknown” refers to a dependent variable that is to be solved for. We allow for an unknown to be a vector of unknown, scalar functions. The independent variables are denoted by $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

We now change notation slightly from earlier and let $u$ denote an unknown, while $U$ will denote a list of all unknowns. The system we consider prescribes certain (possibly more than one) partials $u_{x_i}$ in terms of $x$ and some of the other unknowns (possibly all or none of them). In particular, the systems we consider are assumed to be closed in the sense that each unknown appears differentiated with respect to at least one $x_i$. Thus, each equation has the general form

$$u_{x_i}(x) = F_i^u(x; U(x)), \quad (26)$$

where $F_i^u$ is a given function. Whenever (26) appears in the given system, we say that the system prescribes $u_{x_i}$. At this stage, before we consider integrability conditions, each $F_i^u$ may depend on any collection of unknowns. Without loss of generality we make the following:

Convention 2. In writing down the equations (26), it is assumed that all scalar unknowns for which the system (26) prescribes exactly the same partials, are already grouped together in a single vector $u$ of unknowns.

According to this convention, for a given system of the form (26), there is a one-to-one correspondence between the set of unknowns $U$ and a certain set $\mathcal{I}$ of strictly increasing multi-indices over $\{1, \ldots, n\}$. Namely, for each unknown $u$ in (26) we can associate a unique, strictly increasing multi-index
\( I_u := (i_1, \ldots, i_m) \ (1 \leq i_1 < \cdots < i_m \leq n) \) with the property that the given system (26) prescribes exactly the partials \( u_{x_{i_1}}, \ldots, u_{x_{i_m}} \). If \( I_u \) only contains one index \( i \), we write \( I_u = (i) \).

The set of multi-indices \( I \) is available once the system (26) is given, and it is convenient to use \( I \) to index the unknowns. Thus, for each \( I \in I \) we let \( u^I \) denote the (according to Convention 2, unique) unknown for which (26) prescribes exactly the partials \( u_{x_{i}}, \ i \in I \). With this notation, and upon renaming the right-hand sides in (26), we obtain a system of the following form: for each \( I \in I \) the unknown \( u^I \) satisfies the equations
\[
 u^I_{x_i}(x) = F^I_i(x; U(x)) \quad \text{for each } i \in I. \tag{27}
\]

To describe the data for the system (27), let \( \bar{x} \in \mathbb{R}^n \) be a given point and assume \( u^I \) is an unknown. With \( I = (i_1, \ldots, i_m) \), we let \( I^c \) denote the strictly increasing multi-index of indices \( i \) not belonging to \( I \), we let \( x_I := (x_{i_1}, \ldots, x_{i_m}) \), and similarly define \( x_{I^c} \). We then give data that prescribe the unknown \( u^I \) along the hyperplane through \( \bar{x} \) spanned by those direction for which the system does not prescribe its partials. That is, we require
\[
 u^I(x)|_{x_I = \bar{x}} = \bar{u}^I(x_{I^c}), \tag{28}
\]
where \( \bar{u}^I \) is a given function.

For convenience we introduce the following terminology.

**Definition 3.1.** The system (27) is said to be overdetermined provided that there is at least one unknown for which more than one partial derivative is prescribed by the system.

From now on it is assumed that the given system (27) under consideration is overdetermined. (If it is not then we are in a situation covered by Darboux’s 1st theorem.) For an overdetermined system we need to impose integrability conditions, and these will put constraints on which unknowns the functions \( F^I_i \) on the right hand side of (27) can depend on. This will be detailed in Section 4 below. We first consider the simplest situations where Darboux’s theorem applies.

### 3.2. Two examples

We consider two examples with \( n = 2 \) independent variables. For brevity we do not employ the index notation introduced above.

**Example 3.2.** The simplest situation where Darboux’s theorem applies is the following: 3 equations for 2 (scalar or vector) unknowns in 2 independent variables.\(^1\)

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\(^1\)Here and below we use the shorthand notation \( i \in I \) to mean that \( i \) is one of the indices in \( I \).
variables. Let the unknowns be $u$ and $w$, let the independent variables be $x$ and $y$, and assume that the equations are

$$u_x = F(x, y, u, w)$$

$$w_x = f(x, y, w)$$

$$w_y = \varphi(x, y, u, w).$$

The data take the form

$$u(\bar{x}, y) = \bar{u}(y)$$

$$w(\bar{x}, \bar{y}) = \bar{w},$$

where $\bar{u}$ is a given function and $\bar{w}$ is a given constant. Note that $f$ does not depend explicitly on $u$. This is a consequence of the single integrability condition in this case: 

$$\left(\frac{w_x}{w_y}\right)_y = \left(\frac{w_y}{w_x}\right)_x = \frac{\partial_y[f(x, y, u, w)]}{\partial_x[\varphi(x, y, u, w)]}.$$ 

The requirement that this last equation, after applying the chain rule, should involve only partials of $u$ or $w$ that are prescribed by (29)-(31), implies that $f$ must be independent of $u$. Indeed, the only $u_y$-term is $f_u u_y$, and as $u_y$ is not prescribed by the system, we need to assume that $f_u \equiv 0$. Substituting from (30)-(31), we obtain the requirement that

$$f_y + f_w \varphi = \varphi_x + \varphi_u F + \varphi_w f.$$

(If $w$, say, is a vector of unknowns, then $f_w$ denotes the Jacobian matrix of $f$ with respect to $w$.) This integrability condition need to be imposed as an identity in $(x, y, u, w)$, i.e. we require that

$$f_y(x, y, w) + f_w(x, y, w)\varphi(x, y, u, w) = \varphi(x, y, u, w) + \varphi_u(x, y, u, w) F(x, y, u, w) + \varphi_w(x, y, u, w) f(x, y, w),$$

for all $(x, y, u, w) \approx (\bar{x}, \bar{y}, \bar{u}(\bar{y}), \bar{w})$. Under this condition, and some mild regularity assumptions, Darboux’s theorem will guarantee the existence of a unique local solution $(u(x, y), w(x, y))$ to (29)-(31) near $(\bar{x}, \bar{y})$ that takes on the data (32)-(33).

Note that the system (29)-(31) is not “maximal” in the sense that, while $y$ is an independent variable, there is no unknown that appears differentiated with respect to only $y$. The next example considers the simplest case of a maximal system.

**Example 3.3.** Consider a system with 4 equations for 3 (scalar or vector) unknowns in 2 independent variables: let the unknowns be $u$, $v$, $w$, let the independent variables be $x$ and $y$, and assume that the equations are

$$u_x = F(x, y, u, v, w)$$

$$v_y = \Phi(x, y, u, v, w)$$

$$w_x = f(x, y, v, w)$$

$$w_y = \varphi(x, y, u, w).$$

The data are prescribed near a point \((\bar{x}, \bar{y})\) and take the following form:

\[
\begin{align*}
u(\bar{x}, y) &= \bar{u}(y) \quad (38) \\
v(x, \bar{y}) &= \bar{v}(x) \quad (39) \\
w(\bar{x}, \bar{y}) &= \bar{w}, \quad (40)
\end{align*}
\]

where \(\bar{u}(y)\) is defined near \(\bar{y}\), \(\bar{v}(x)\) is defined near \(\bar{x}\), and \(\bar{w}\) is a constant.

As in Example 3.2, the requirement that the condition \("(w_x)_y = (w_y)_x\"") should not involve the unspecified partials \(u_y\) and \(v_x\), implies that \(f\) is independent of \(u\) and \(\phi\) is independent of \(v\). Substituting from the system we obtain that

\[
f_y + f_v \Phi + f_w \varphi = \varphi_x + \varphi_u F + \varphi_w f
\]

should hold as an identity for all \((x, y, u, v, w) \approx (\bar{x}, \bar{y}, \varphi_1(\bar{y}), \varphi_2(\bar{x}), \varphi_3)\).

Then, under suitable regularity assumptions, Darboux’s theorem guarantees the existence of a unique, local solution \((u(x, y), v(x, y), w(x, y))\) to (34)-(37) near the point \((\bar{x}, \bar{y})\) taking on the data (38)-(40).

**3.3. Comments on Darboux’s formulation and proofs.** While Darboux [2] stated his theorem (Theorem 4.2 below) for any number \(n\) of independent variables, he provided proofs only for the cases \(n = 2\) and \(n = 3\).

In fact, by letting the unknowns \(u, v, w\) in Example 3.3 denote vectors, we obtain precisely the setting that Darboux considered for \(n = 2\). (Darboux worked at the level of individual, scalar unknowns, which required an extra layer of indices.)

Concerning Darboux’s proofs, we note that his proof for the \(n = 2\) case makes use of his first theorem, i.e. Theorem 2.3 above. His proof for the \(n = 3\) case then makes use of both the result for \(n = 2\), as well as his first theorem. This indicates that one can obtain a proof for any number of independent variables via an induction argument. This is what we do below. In particular, we have not been able to provide a “direct” proof along the same lines as in the proof of his first theorem (Theorem 2.3 above).

Let’s elaborate. For an overdetermined system satisfying the assumptions of Darboux’s theorem, it is easy enough to define a suitable functional map (corresponding to \(\Phi\) in the proof of Theorem 2.3), with the property that the sought-for solution is a fixed point of the functional map. However, we have not been able to give a proof for the required, opposite implication: if \(U\) is fixed point of the functional map, then \(U\) is the sought-for solution.

Instead, for the existence part, we shall extend Darboux’s strategy for \(n = 2\) and \(n = 3\): induction on the number of independent variables, and repeated use of his first theorem. Finally, uniqueness will follow from the inductive construction we employ.

**4. Formulation of Darboux’s theorem**

We now return to the general system (27), which is assumed to be overdetermined. Below we describe the relevant integrability conditions and introduce the resulting class of “Darboux systems.” We then state Darboux’s
theorem which amounts to the unique, local solvability of such systems under certain mild regularity conditions.

4.1. Integrability conditions. Let the system (27) be given and let \( \mathcal{I} \) denote the corresponding set of multi-indices \( I \) as detailed in Section 3.1. If \( u^I(x) \) is part of a solution \( U(x) \) to (27), and \( i, j \in I \) with \( i \neq j \), then we must have that \( (u^I_{x_i})_{x_j} = (u^I_{x_j})_{x_i} \). With the notation above this amounts to having

\[
F^I_{i,x_j}(x; U(x)) + \sum_{J \in \mathcal{I}} F^I_{i,u^J}(x; U(x))u^J_{x_j}(x)
= F^I_{j,x_i}(x; U(x)) + \sum_{J \in \mathcal{I}} F^I_{j,u^J}(x; U(x))u^J_{x_i}(x),
\]

(42)

Here \( F^I_{i,x_j} \) denotes the partial derivative of \( F^I_i \) with respect to \( x_j \), while \( F^I_{i,u^J} \) denotes the Jacobian matrix of \( F^I_i \) with respect to \( u^J \). For (42) to hold we must require that:

(i) all partials of unknowns \( u^J \) appearing in (42) are also prescribed by the original system (27), and that

(ii) upon substitution from the original system for these, an identity in \((x, U)\) is obtained.

As in the examples above, the condition in (i) places constraints on which unknowns each \( F^I_i \) can depend on. Namely, we need to require that \( F^I_i \) depends explicitly on (at most) those unknowns \( u^J \) with the property that each index in \( I \setminus i \) also belongs to \( J \). (Here we employ the following shorthand notation: if \( I = (i_1, \ldots, i_m) \) and \( i = i_j \) for some \( 1 \leq j \leq m \), then \( I \setminus i \) denotes the multi-index \((i_1, \ldots, i_{j-1}, i_{j+1}, \ldots, i_m)\).) We express this by writing \( J \supset I \setminus i \). Next, whenever \( i \in I \in \mathcal{I} \) we define

\[
\mathcal{I}^I_i := \{ J \in \mathcal{I} \mid J \supset I \setminus i \}
\]

(43)

together with the corresponding set of unknowns

\[
U^I_i := \{ u^J \mid J \in \mathcal{I}^I_i \}.
\]

(44)

With this notation, condition (i) requires that each \( F^I_i \) appearing in the system depends explicitly on (at most) the unknowns in \( U^I_i \). Thus (i) amounts to requiring that the equations for the unknown \( u^I \) take the form

\[
u^I_{x_j}(x) = F^I_i(x; U^I_i(x)) \quad \text{whenever} \quad i \in I \in \mathcal{I},
\]

(45)

Condition (ii) then becomes the requirement that: whenever \( I \in \mathcal{I}, i, j \in I, \) and \( i \neq j \), then

\[
F^I_{i,x_j}(x; U^I_i) + \sum_{J \in \mathcal{I}^I_i} F^I_{i,u^J}(x; U^I_i)F^J_i(x; U^I_j)
= F^I_{j,x_i}(x; U^I_j) + \sum_{J \in \mathcal{I}^I_j} F^I_{j,u^J}(x; U^I_j)F^J_i(x; U^I_i)
\]

(46)
holds as an identity in \((x, U)\) near \((\bar{x}, \bar{U})\). Note that, in the sum on the lefthand side of (46), each function \(F^J_i\) is given by the original system (45). This is because \(J \in \mathcal{I}\), so that according our notation, \(u^J_i\) is an unknown in the system, and \(J \supset I \setminus i\) together with \(j \neq i\) imply that \(j \in J\); thus \(u^J_{i,j}\) is prescribed by the system, i.e. \(F^J_i\) appears as one of the righthand sides in (45). The same remarks apply to the functions \(F^J_i\) in the sum on righthand side of (46).

**Definition 4.1.** Let \(\mathcal{I}\) be a set of strictly increasing multi-indices over \(\{1, \ldots, n\}\), and let \(u^I, I \in \mathcal{I}\), be the corresponding unknowns. Then, with the notation introduced above, a system of equations of the form (45), where the maps \(F^I_i(x; U^I_i)\) are defined on a neighborhood of a given point \((\bar{x}, \bar{U})\), is called a Darboux system near \((\bar{x}, \bar{U})\) provided the integrability conditions (46) is satisfied.

We can now formulate Darboux’s theorem as follows.

**Theorem 4.2 (Darboux’s theorem).** Assume that (45) is a Darboux system near \((\bar{x}, \bar{U})\) according to Definition 4.1, and let the data (28) be assigned on a neighborhood of \((\bar{x}, \bar{U})\) for each unknown \(u^I_i\). Assume that all functions \(F^I_i\) in (45) are \(C^2\)-smooth near \((\bar{x}, \bar{U})\), and that all functions \(\bar{u}^I_i\) in (28) are \(C^2\)-smooth near \(\bar{x}\). Then there is a neighborhood of \(\bar{x}\) on which the system (45) has a unique \(C^2\)-smooth solution taking on the assigned data (28).

**Remark 4.3.** Note that we allow for the possibility that the integrability conditions (46) are vacuously met. However, in that case each unknown appears differentiated with respect to only one independent variable. Since we assume that the given system is closed, this case is covered by Darboux’s first theorem (Theorem 2.3 above). In what follows it therefore suffices to restrict attention to overdetermined Darboux systems.

We also remark that Darboux’s theorem also covers the opposite extreme case where all first partials of all unknowns are prescribed, and the corresponding integrability conditions are met. This result is the standard Frobenius theorem for overdetermined systems, cf. Theorem 1, Chapter 6 in [6].

5. Outline of proof for the case \(n = 3\)

To motivate the structure of the proof for general \(n\), we provide an outline of Darboux’s proof for the case \(n = 3\). Although our general proof, when specialized to \(n = 3\), will differ slightly from Darboux’s proof, it will help to explain the structure of the proof for arbitrary \(n\). We recall that [2] establishes the existence part of Darboux’s theorem for maximal systems when the number of independent variables is \(n = 2\). This provides the base-step for the inductive proof.

We consider a Darboux system in the three independent variables. For brevity we do not apply the index notation introduced above and instead follow (partially) the notation of Darboux [2]. The independent variables
are $X := (x, y, z)$ and the independent variables are the seven (scalar, say) unknowns $U := (u, v, w, p, q, r, s)$. For concreteness we assume that the PDE system under consideration is maximal in the sense that for every choice of one, two, or three independent variables, there is an unknown which appears differentiated with respect to exactly the chosen variables. Without loss of generality we assume the data are prescribed on hyperplanes through $X = 0$.

The system thus consists of 12 equations of the following form:

\[
\begin{align*}
    u_x &= F(X; U) \quad (47) \\
    v_y &= \Phi(X; U) \quad (48) \\
    w_z &= \Psi(X; U) \quad (49) \\
    p_y &= f_1(X; U) \quad (50) \\
    p_z &= f_2(X; U) \quad (51) \\
    q_x &= \varphi_0(X; U) \quad (52) \\
    q_z &= \varphi_2(X; U) \quad (53) \\
    r_x &= \psi_0(X; U) \quad (54) \\
    r_y &= \psi_1(X; U) \quad (55) \\
    s_x &= \theta_0(X; U) \quad (56) \\
    s_y &= \theta_1(X; U) \quad (57) \\
    s_z &= \theta_2(X; U) \quad (58)
\end{align*}
\]

with prescribed data of the form

\[
\begin{align*}
    u(0, y, z) &= \bar{u}(y, z) \quad (59) \\
    v(x, 0, z) &= \bar{v}(x, z) \quad (60) \\
    w(x, y, 0) &= \bar{w}(x, y) \quad (61) \\
    p(x, 0, 0) &= \bar{p}(x) \quad (62) \\
    q(0, y, 0) &= \bar{q}(y) \quad (63) \\
    r(0, 0, z) &= \bar{r}(z) \quad (64) \\
    s(0, 0, 0) &= \bar{s}. \quad (65)
\end{align*}
\]

The system is clearly overdetermined; to be a Darboux system the two constraints (i) and (ii) above must be satisfied. It is straightforward to verify that the first constraint implies the following dependencies:

\[
\begin{align*}
    f_1 &= f_1(X; w, p, q, s), \quad f_2 = f_2(X; v, p, r, s), \quad (66) \\
    \varphi_0 &= \varphi_0(X; w, p, q, s), \quad \varphi_2 = \varphi_2(X; u, q, r, s), \quad (67) \\
    \psi_0 &= \psi_0(X; v, p, q, s), \quad \psi_1 = \psi_1(X; u, q, r, s), \quad (68) \\
    \theta_0 &= \theta_0(X; p, s), \quad \theta_1 = \theta_1(X; q, s), \quad \theta_2 = \theta_2(X; r, s). \quad (69)
\end{align*}
\]

The second constraint then requires that the following integrability conditions are satisfied as identities on a full $\mathbb{R}^3_X \times \mathbb{R}_U^7$-neighborhood of the point
(0,  \bar{u}(0),  \bar{v}(0),  \bar{w}(0),  \bar{\rho}(0),  \bar{q}(0),  \bar{r}(0),  \bar{s}):

\begin{align*}
\varphi_{0,z} + \varphi_{0,y}I + \varphi_{0,p}f_2 + \varphi_{0,q} \varphi_2 + \varphi_{0,s} \theta_2 &= \varphi_{2,x} + \varphi_{2,u} F + \varphi_{2,q} \varphi_0 + \varphi_{2,r} \psi_0 + \varphi_{2,s} \theta_0, \\
\psi_{0,y} + \psi_{0,v} I + \psi_{0,p} f_1 + \psi_{0,r} \psi_1 + \psi_{0,s} \theta_1 &= \psi_{1,x} + \psi_{1,u} F + \psi_{1,q} \varphi_0 + \psi_{1,r} \psi_0 + \psi_{1,s} \theta_0, \\
\theta_{0,y} + \theta_{0,p} f_1 + \theta_{0,s} \theta_1 &= \theta_{1,x} + \theta_{1,q} \varphi_0 + \theta_{1,s} \theta_0, \\
\theta_{0,z} + \theta_{0,p} f_2 + \theta_{0,s} \theta_2 &= \theta_{2,x} + \theta_{2,r} \psi_0 + \theta_{2,s} \theta_0, \\
\theta_{1,z} + \theta_{1,q} \varphi_2 + \theta_{1,s} \theta_2 &= \theta_{2,y} + \theta_{2,r} \psi_1 + \theta_{2,s} \theta_1.
\end{align*}

The overall approach for establishing existence of a solution taking on the assigned data is to build the solution by solving several smaller, auxiliary systems. Each auxiliary system is either a Darboux system in two independent variables (and thus solvable according to the base-step, n = 2), or a determined system (solvable according to Darboux's first theorem). All equations occurring in these smaller systems will be copies of equations forming the original system (17)-(58).

For the case n = 3 there will be three auxiliary systems, which we refer to as system 1 through 3, respectively. System 3 will be the last system to be solved, and this will be a determined system which is solved via an application of Darboux's first theorem. What we want to arrange is that the resulting solution of system 3 is in fact the sought-for solution of the original system (17)-(58) with the original data (59)-(65). The main issue is how to provide data for system 3 so that this works out, and for this we make use of the solutions to systems 1 and 2.

System 3 consists of copies of the u_x, v_y, w_z, p_y, q_x, r_x, and s_x equations from the original, given system (17)-(58). (In the general case, system n will consist of copies of all equations u_{2,i} = F_i(x; U_i) from (15) with I ∈ \mathcal{I} and i = \text{min } I.) Note that these equations form a determined system. For clarity we denote the unknowns of system 3 by  \hat{U} = (\hat{u}, \hat{v}, \hat{w}, \hat{p}, \hat{q}, \hat{r}, \hat{s}). The challenge is to assign appropriate \hat{U}-data so that the resulting solution solves (17)-(58) with data (59)-(65). As system 3 is solved using Darboux's first theorem, the relevant \hat{U}-data must prescribe

\hat{u}(0, y, z), \hat{v}(x, 0, z), \hat{w}(x, y, 0), \hat{p}(x, 0, z), \hat{q}(0, y, z), \hat{r}(0, y, z), \hat{s}(0, y, z). (76)

For the three first we simply use the original data in (59)-(61). However, the remaining four pieces of data need to be generated, and for this we proceed to identify and solve systems 1 and 2.

\footnote{This is not precisely what Darboux does in [2]; his argument appears to proceed in the opposite order from what we do.}
For system 1, we make copies of the $q_z$, $r_y$, $s_y$, and $s_z$ equations from the original system and restrict to the hyperplane $\{x = 0\}$. (In the general case, system 1 will consist of copies of all equations $u^I_{x_1} = F^I_i(x; U^I_i)$ from (85) where $I \in \mathcal{I}$, $1 \in I \neq (1)$ and $1 < i \in I$, which are then restricted to $\{x_1 = 0\}$.) For clarity we introduce new labels $q, \underline{r}, \underline{s}$ for the resulting unknowns, which are functions of $(y, z)$. System 1 thus takes the form

\begin{align*}
\dot{q}_x &= \varphi_2(0, y, z; \bar{u}(y, z), (q, \underline{r}, \underline{s})(y, z)) \quad (77) \\
\dot{r}_y &= \psi_1(0, y, z; \bar{u}(y, z), (q, \underline{r}, \underline{s})(y, z)) \quad (78) \\
\dot{s}_y &= \theta_1(0, y, z; \bar{u}(y, z), (q, \underline{r}, \underline{s})(y, z)) \quad (79) \\
\dot{s}_z &= \theta_2(0, y, z; \bar{u}(y, z), (r, \underline{s})(y, z)). \quad (80)
\end{align*}

Observe that this is a closed system: each unknown appearing on one of the right-hand sides in (77)-(80) also appears at least once in one of the left-hand sides. For this system we use the original data to assign the data

\begin{align*}
\bar{q}(y, 0) &= \bar{q}(y), \quad \varphi(0, z) = \bar{r}(z), \quad \underline{s}(0, 0) = \bar{s}.
\end{align*}

As the equations (77)-(80) are obtained by duplication and restriction to $\{x = 0\}$ of equations from the original system, it turns out that system 1 is again a Darboux system, but now in the two independent variables $(y, z)$. According to the base-step of the proof it has a local solution $(q, r, s)(y, z)$ defined for $(y, z) \approx (0, 0)$ and taking on the assigned data.

At this point we can assign three more of the data in (76) by setting

\begin{align*}
\hat{q}(0, y, z) &= q(y, z), \quad \hat{r}(0, y, z) = \underline{r}(y, z), \quad \hat{s}(0, y, z) = \underline{s}(y, z).
\end{align*}

However, it still remains to assign appropriate data for $p(x, 0, z)$. To do so we exploit the $p_z$ equation from the original system, which we have not yet used. Proceeding as for system 1 we make a copy of this equation, and then restrict it to the hyperplane $\{y = 0\}$. However, differently from what occurred for system 1, we do not obtain a closed system in this way: we get a single equation for $p(x, 0, z)$ which contains both $r(x, 0, z)$ and $s(x, 0, z)$ on its right-hand side. To obtain a closed system we add copies of the $r_x$ and $s_x$ equations (note that these also appear in system 3), and restrict them to $\{y = 0\}$. We denote the resulting unknowns by $\hat{p}, \hat{r}, \hat{s}$; these are functions of $(x, z)$ and are required to satisfy the following system 2:

\begin{align*}
\hat{p}_z &= f_2(x, 0, z; \bar{v}(x, z), (\hat{p}, \hat{r}, \hat{s})(x, z)) \quad (81) \\
\hat{r}_x &= \psi_0(0, x, z; \bar{v}(x, z), (\hat{p}, \hat{r}, \hat{s})(x, z)) \quad (82) \\
\hat{s}_x &= \theta_0(x, 0, z; (\hat{p}, \hat{s})(x, z)). \quad (83)
\end{align*}

(This system, which is also considered in Darboux’s argument, is not exactly what our general approach below uses as system 2; see Remark 5.1.) The equations (81)-(83) form a determined system and to provide data we use both the original data as well as the solution to system 1:

\begin{align*}
\hat{p}(x, 0) &= \bar{p}(x), \quad \hat{r}(0, z) = \bar{r}(z), \quad \hat{s}(0, z) = \bar{s}(0, z).
\end{align*}
According to Darboux’s first theorem it possesses a local solution \((\tilde{p}, \tilde{r}, \tilde{s})(x, z)\) near \((\bar{x}, \bar{z}) = (0, 0)\).

We can now assign appropriate data for system 3:
\[
\hat{u}(0, y, z) = \bar{u}(y, z), \quad \hat{v}(x, 0, z) = \bar{v}(x, z), \quad \hat{w}(x, y, 0) = \bar{w}(x, y),
\]
\[
\hat{p}(x, 0, z) = \tilde{p}(x, z), \quad \hat{q}(0, y, z) = \tilde{q}(y, z), \quad \hat{r}(0, y, z) = \tilde{r}(y, z),
\]
and
\[
\hat{s}(0, y, z) = \tilde{s}(y, z).
\]
Note that this makes use of the original data, as well as the solutions of both system 1 and system 2. Darboux’s first theorem guarantees the existence of a solution \(\hat{U}(x, y, z)\) of system 3 on a full neighborhood of \(0 \in \mathbb{R}^3\).

The claim now is that this \(\hat{U}\) is in fact a solution of the original system (47)-(58) and takes on the original data (59)-(65). Thanks to the construction of the solutions to system 1 and 2, it is straightforward to verify that \(\hat{U}\) attains the data (59)-(65). Next, as system 3 is a subsystem of the original system, all that remains to be verified is that \(\hat{U}\) solves the remaining \(p_z, q_z, r_y, s_y,\) and \(s_z\) equations in (47)-(58). For this we follow Darboux and define the quantities
\[
A(X) := \hat{p}_z(X) - f_2(X; (\hat{v}, \hat{p}, \hat{r}, \hat{s})(X)) \quad (84)
\]
\[
B(X) := \hat{q}_z(X) - \varphi_2(X; (\hat{u}, \hat{q}, \hat{r}, \hat{s})(X)) \quad (85)
\]
\[
C(X) := \hat{r}_y(X) - \psi_1(X; (\hat{u}, \hat{q}, \hat{r}, \hat{s})(X)) \quad (86)
\]
\[
D(X) := \hat{s}_y(X) - \theta_1(X; (\hat{q}, \hat{s})(X)) \quad (87)
\]
\[
E(X) := \hat{s}_z(X) - \theta_2(X; (\hat{r}, \hat{s})(X)). \quad (88)
\]
Without going into the details (carried out for the general case below), it turns out that these quantities solve a linear, homogeneous, and determined system with vanishing data near \(0 \in \mathbb{R}^3\). (Here the integrability conditions (70)-(75) are used.) The uniqueness part of Darboux’s first theorem then implies that they must vanish identically near \(0 \in \mathbb{R}^3\). This means precisely that \(\hat{U}\) satisfies also the \(p_z, q_z, r_y, s_y,\) and \(s_z\) equations in the original system, on a full neighborhood of \(0 \in \mathbb{R}^3\). This establishes the existence part of Darboux’s theorem in the case \(n = 3\).

**Remark 5.1.** The outline above essentially follows Darboux’s original proof for the case \(n = 3\). As already noted, our argument for the general case with any number \(n\) of independent variables will differ slightly from that of Darboux when specialized to \(n = 3\). Specifically, in our setup for \(n = 3\), system 2 will be an overdetermined Darboux system, rather than a determined system.

### 6. Proof of Darboux’s theorem

We now return to the general case and consider a given Darboux system (15) in \(n\) independent variables and with data (28). To simplify the notation,
and without loss of generality, we assume from now on that \( \bar{x} = 0 \). Thus the assigned data are:

\[
u^I(x)|_{\{x_I=0\}} = \bar{u}^I(x_I), \tag{89}\]

Our proof proceeds by induction on the number \( n \) of independent variables.

As noted above, [2] provides a proof of the existence part of Darboux’s theorem when the number of independent variables is 2 and 3. This takes care of the two base-steps for the inductive proof of the general case. We now assume \( n > 3 \), and the induction hypothesis is that Theorem 4.2 holds for the cases with \( n - 2 \) and \( n - 1 \) independent variables. Before providing a detailed proof, we provide an outline of the argument.

6.1. Outline of proof for general \( n \). As in the case \( n = 3 \), we begin by defining \( n \) auxiliary systems which are referred to as system 1 through \( n \). System \( n \), whose solution will turn out to be the sought-for solution, consists of a copy of each equation \( u^I_{x_i} = F^I_i(x; U^I_i) \) from (15) with \( I \in \mathcal{I} \) and \( i = \min I \). This is a determined system, and the main part of the proof is about how to generate appropriate data for this system. As above this is accomplished by solving the auxiliary systems 1 through \( n - 1 \), and then using their solutions to provide data for system \( n \).

Once such data are available, Darboux’s first theorem guarantees the existence of a solution \( \hat{U} \) of system \( n \). The fact that \( \hat{U} \) satisfies the original data (89) will be a direct consequence of the construction. On the other hand, to show that \( \hat{U} \) also satisfies the remaining equations in the original system (15) (i.e. the equations not in system \( n \)), requires an argument. For this we follow Darboux’s approach in the case \( n = 3 \) and show that the functions \( \hat{u}^I_{x_i} - F^I_i(x; \hat{U}^I_i) \) where \( I \in \mathcal{I}, |I| > 1 \), and \( i \in I \) is such that \( i \neq \min I \), solve a linear, homogeneous, and determined system with vanishing data. For this, the integrability conditions (16) are utilized. The uniqueness part of Darboux’s first theorem then implies that these quantities vanish identically near \( \bar{x} = 0 \), completing the existence part of Darboux’s theorem for the case of \( n \) independent variables.

The remaining issue is to define and solve system 1 through \( n - 1 \). For each \( \ell = 1, \ldots, n - 1 \), we copy certain equations from the original system (details below), and restrict them to the hyperplane \( \{x_\ell = 0\} \). The selection is made so as to guarantee that the chosen set of equations form a Darboux system in \( n - 1 \) independent variables. We refer to the resulting system as “system \( \ell \).” The data for system \( \ell \) are provided by the original data (89). (This is a technical point where our argument differs from that of Darboux [2], cf. Remark 5.1.) According to the induction hypothesis, it has a local solution near \( 0 \in \mathbb{R}^{n-1} \).

Having solved system 1 through \( (n - 1) \), we use their solutions to provide the appropriate data for system \( n \), and the argument for the existence part of Darboux’s theorem in \( n \) independent variables is concluded as indicated above.
6.2. System $\ell$ for $\ell = 1, \ldots, n - 1$. To define these systems we introduce the following notation: for $\ell = 1, \ldots, n - 1$, we let
\[
I_\ell := \{I \in I \mid \ell \in I \neq (\ell)\},
\]
and denote the unknowns of system $I_\ell$ by $u^{\ell}_{I}$ for $I \in I_\ell$. These will be functions of $x^{\ell} = (x_1, \ldots, x_{\ell-1}, x_{\ell+1}, \ldots, x_n)$. The equations of system $\ell$ are then obtained as follows: for each $I \in I_\ell$ and for each $i \in I \setminus \ell$ we make a copy of the equation occurring in equation (45), restrict it to the hyperplane $\{x_\ell = 0\}$, and rename any unknown $u^K$ appearing in it as $u^{\ell,K}$.

At this point we need to address a notational issue. Namely, assume $i \in I \in I_\ell$ and $i \neq \ell$. According to the procedure above, system $\ell$ will then contain a copy of the equation
\[
u^{I}_{x_i} = F^{I}_{i}(x; U^{I}_{i})
\]
from the original system (45); the corresponding equation in system $\ell$ is obtained by restricting this equation to the hyperplane $\{x_\ell = 0\}$. Now, in the particular case that $I$ is the 2-index $(\ell, i)$ or $(i, \ell)$, then $I \setminus i = (\ell)$ and, according to (43)-(44), $F^{I}_{i}$ may depend explicitly on the unknown $u^{(\ell)}$. In the corresponding equation in system $\ell$, any such occurrence of $u^{(\ell)}$ is to be replaced by the originally given data $\bar{u}^{(\ell)}(x^{\ell})$ (because we are restricting to the hyperplane $\{x_\ell = 0\}$). It is awkward to treat the 2-indices $(\ell, i)$ and $(i, \ell)$ separately, and we shall simply write the corresponding equations in system $\ell$ as
\[
u^{\ell,I}_{x_i}(x^{\ell}) = F^{I}_{i}(x^{\ell}_{0}; [\bar{u}^{(\ell)}(x^{\ell})], U^{\ell,I}_{i}(x^{\ell}))
\]
where we have used the notation in (9) for $x^{\ell}_{0}$, and the square brackets indicate that the bracketed term is present only when $I = (\ell, i)$ or $I = (i, \ell)$. Finally, $U^{\ell,I}_{i}$ denotes the collection of the remaining dependent-variable arguments of $F^{I}_{i}$, i.e., whenever $i \in I \in I_\ell$, we set
\[
I^{\ell,I}_{i} := \{K \in I \mid K \supset I \setminus i, K \neq (\ell)\}
\]
and
\[
U^{\ell,I}_{i} := \{u^{\ell,K} \mid K \in I^{\ell,I}_{i}\}.
\]
Thus, system $\ell$ consists of the following equations: for each $I \in I_\ell$ we have the $|I| - 1$ equations in (91) with $i \in I \setminus \ell$. As data for $u^{\ell,I}$ we make use of the original data (89) and require
\[
u^{\ell,I}(x^{\ell})|_{\{x_{\ell'} = 0\}} = \bar{u}^{I}(x^{\ell'}),
\]
where $I^{\ell}$ denotes the multi-index $I \setminus \ell$. We note that the right hand side in (91) is independent of $\ell$.

For later reference we record the following: whenever $\ell \in I$ and $|I| \geq 3$, then
\[
\begin{cases}
\text{the square-bracketed term in (91) is absent,} \\
I^{\ell,I}_{i} = I^{I}_{i} \text{ by (93), and} \\
u^{\ell,K} \in U^{\ell,I}_{i} \text{ if and only if } u^{K} \in U^{I}_{i}.
\end{cases}
\]
We now have that:

**Claim 6.1.** For each \( \ell = 1, \ldots, n - 1 \), system \( \ell \) is a closed Darboux system in \( n - 1 \) independent variables.

**Proof.** To show that system \( \ell \) is closed we need to argue that whenever an unknown \( u_{\ell,K}^{\ell} \) appears on the righthand side of one of the equations in (91), then system \( \ell \) also contains an equation where \( u_{\ell,K}^{\ell} \) appears on the lefthand side. If \( u_{\ell,K}^{\ell} \) appears on the righthand side of the equation (91), then \( K \supset I \setminus i, \) and \( K \neq (\ell) \). As \( \ell \in I \) and \( \ell \neq i \), we have that \( \ell \in K \neq (\ell) \).

It follows that \( K \in I_\ell \), and that for each \( k \in K \setminus \ell \) the equation

\[
u_{\ell,K} = F_k^K (x'_0; [\bar{u}(x')_\ell], U_{\ell,K}^K (x')_\ell)
\]

is prescribed by system \( \ell \).

Next, we need to verify the integrability conditions corresponding to equality of 2nd mixed partial derivatives for system \( \ell \). I.e., we must argue that the two properties corresponding to (i) and (ii) in Section 4.1 hold in the present situation. For this, assume that system \( \ell \) prescribes both \( u_{x_i}^{\ell,I} \) and \( u_{x_j}^{\ell,J} \), where \( i \neq j \). In particular, this means that \( i, j, \) and \( \ell \) all belong to \( I \), and that they are all distinct. It follows that \(|I| \geq 3\), such that (95) applies and the square-bracketed argument on the righthand side of (91) is absent. Thus, the equations in question read:

\[
u_{x_i}^{\ell,I} = F_i^I (x'_0; U_{x_i}^{\ell,J} (x')_\ell), \tag{96}
\]

and

\[
u_{x_j}^{\ell,J} = F_j^I (x'_0; U_{x_j}^{\ell,J} (x')_\ell), \tag{97}
\]

In particular, as both \( i \) and \( j \) are different from \( \ell \), both \( x_i \) and \( x_j \) appear as independent variables in (96) and (97). Applying \( \partial_{x_j} \) to (96), \( \partial_{x_i} \) to (97), and equating the results we obtain (dropping arguments for now)

\[
F_{i,x_j}^{I,i} + \sum_{J \in I_{\ell,j}} F_{i,a}^{I,i} u_{x_j}^{\ell,J} = F_{j,x_i}^{I,j} + \sum_{J \in I_{\ell,j}} F_{j,a}^{I,j} u_{x_i}^{\ell,J}. \tag{98}
\]

The first issue is to argue that all the partials derivatives \( u_{x_i}^{\ell,J} \) and \( u_{x_j}^{\ell,J} \) in the two sums in (98) are all prescribed by system \( \ell \). Consider \( u_{x_i}^{\ell,J} \) in the lefthand sum. This partial derivative is prescribed by system \( \ell \) provided \( j \in I \setminus \ell \) and \( J \in I_\ell \). The first condition is satisfied since \( j \) and \( \ell \) are distinct and both belong to \( I \). The second condition is met because \( J \in I_{\ell,j} \), so that \( J \supset I \setminus i \supset j, \ell; \) in particular, \( \ell \in J \neq (\ell) \), i.e. \( J \in I_\ell \). A similar argument shows that each partial \( u_{x_i}^{\ell,J} \) in the righthand sum of (98) is prescribed by system \( \ell \).

Again, as \( i, j, \ell \) are distinct and belong to \( I \), we have that (95) applies. Therefore, upon substituting from system \( \ell \), we obtain from (98) the
requirement that
\[
F_{i,x_j}^I(x_0^{\ell}; U_i^{\ell,J}) + \sum_{J \in \mathcal{I}_I^J} F_{i,u,J}^I(x_0^{\ell}; U_i^{\ell,J}) F_{j}^J(x_0^{\ell}; U_j^{\ell,J})
= F_{j,x_i}^I(x_0^{\ell}; U_j^{\ell,J}) + \sum_{J \in \mathcal{I}_J^I} F_{j,u,I}^J(x_0^{\ell}; U_j^{\ell,J}) F_{i}^I(x_0^{\ell}; U_i^{\ell,J})
\]
(99)

should hold as an identity with respect to \(x^{\ell}\) and the dependent variables occurring. We finish the proof by arguing that this is a consequence of the integrability condition (10). Indeed, since (10) by assumption holds as an identity in \((x, U)\), we may restrict (10) to the hyperplane \(\{x_\ell = 0\}\). The result is that the following identity holds
\[
F_{i,x_j}^I(x_0^{\ell}; U_i^{\ell,J}) + \sum_{J \in \mathcal{I}_I^J} F_{i,u,J}^I(x_0^{\ell}; U_i^{\ell,J}) F_{j}^J(x_0^{\ell}; U_j^{\ell,J})
= F_{j,x_i}^I(x_0^{\ell}; U_j^{\ell,J}) + \sum_{J \in \mathcal{I}_J^I} F_{j,u,I}^J(x_0^{\ell}; U_j^{\ell,J}) F_{i}^I(x_0^{\ell}; U_i^{\ell,J}).
\]
(100)

Finally, it follows from (95) that (99) results from (100) upon renaming any dependent variable \(u^K\) in (100) by \(u^{1,K}\).

From the inductive hypothesis we conclude that, for each \(\ell = 1, \ldots, n - 1\), system \(\ell\) with data (94) possess a unique solution \(u^{\ell,1}(x^{\ell})\) defined on a full \((n - 1)\)-dimensional neighborhood of \(0 \in \mathbb{R}_{x^{\ell}}^{n-1}\).

Before we continue we make the following observation that will be important later. It is a consequence of the fact that the data assigned for system \(\ell\) are independent of \(\ell\), cf. (94).

Claim 6.2. Assume that \(I \in \mathcal{I}\) contains two distinct indices \(k\) and \(\ell\). Then the solutions \(u^{k,1}\) and \(u^{\ell,1}\) of systems \(k\) and \(\ell\), respectively, satisfy
\[
u^{k,1}(x^{t^k})|_{\{x_\ell = 0\}} = u^{\ell,1}(x^{t^\ell})|_{\{x_\ell = 0\}}.
\]
(101)

Proof. Assume \(k < \ell\). If \(I\) is the 2-index \((k, \ell)\), then \(I^k = (\ell)\), \(I^\ell = (k)\), and the data assignment (94) gives
\[
u^{k,1}(x^{t^k})|_{\{x_\ell = 0\}} = u^{k,1}(x^{t^k})|_{\{x_\ell = 0\}}
= \tilde{u}^{k}(x^{t^\ell})
= u^{\ell,1}(x^{t^\ell})|_{\{x_\ell = 0\}} = u^{\ell,1}(x^{t^\ell})|_{\{x_\ell = 0\}},
\]
which verifies (101) in this case. Next assume \(|I| \geq 3\), \(k < \ell < n\), and set
\[
\mathcal{I}_{k,\ell} := \{I \in \mathcal{I} | k, \ell \in I \neq (k, \ell)\}.
\]

Then, for each \(I \in \mathcal{I}_{k,\ell}\), system \(k\) contains the equations
\[
u^{k,1}_{x_i}(x^{t^k}) = F_{i}^{1}(x_0^{t^k}; U_i^{k,1}(x^{t^k})) \quad i \in I \setminus \{k, \ell\},
\]
(102)

and system \(\ell\) contains the equations
\[
u^{\ell,1}_{x_i}(x^{t^\ell}) = F_{i}^{1}(x_0^{t^\ell}; U_i^{\ell,1}(x^{t^\ell})) \quad i \in I \setminus \{k, \ell\}.
\]
(103)
We now let 
\[ \tilde{x} := x'^{k,l} = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{\ell-1}, x_{\ell+1}, \ldots, x_n), \]
and define the functions 
\[ v^I(\tilde{x}) := u^{k,I}(x'^{k})|_{x_k=0} \quad \text{and} \quad w^I(\tilde{x}) := u^{\ell,I}(x'^{\ell})|_{x_{k}=0}. \]
Letting 
\[ V^I_i := \{ v^J | J \supset I \setminus i, J \neq (k,\ell) \}, \]
and 
\[ W^I_i := \{ w^J | J \supset I \setminus i, J \neq (k,\ell) \}, \]
and restricting (102) and (103) to \{x_\ell = 0\} and \{x_k = 0\}, respectively, we deduce that the sets of functions 
\[ \{v^I(\tilde{x}) | I \in \mathcal{I}_{k,\ell}\} \quad \text{and} \quad \{w^I(\tilde{x}) | I \in \mathcal{I}_{k,\ell}\} \]
solve the following system-data pairs: 
\[ v^I_i(x) = F^I_i(x'^{k}; \hat{u}^{(k,\ell)}(\tilde{x}), V^I_i(\tilde{x})) \quad \text{for} \ i \in I \setminus \{k,\ell\} \]
and 
\[ w^I_i(x) = F^I_i(x'^{\ell}; \hat{u}^{(k,\ell)}(\tilde{x}), W^I_i(\tilde{x})) \quad \text{for} \ i \in I \setminus \{k,\ell\} \]
respectively. Here we are using the same notational convention as earlier: the square-bracketed terms are present only if \( I \) is one of the 3-indices \((i,k,\ell),(k,i,\ell),(k,\ell,i)\).

We note that the two systems above are identical, and we claim that they are closed Darboux systems in \( n-2 \) independent variables. The argument for this claim is entirely similar to that for Claim 6.1 above; we omit the details. Since the systems above for the \( v^I \) and the \( w^I \) have the same data, it follows from the uniqueness part of Darboux’s theorem (for \( n-2 \) independent variables) that \( v^I \equiv w^I \), i.e. \( \square \) holds.

6.3. System \( n \). To generate our candidate for the solution of the original system we proceed to define system \( n \), which will be a determined system in all \( n \) independent variables \( x = (x_1, \ldots, x_n) \). We denote its unknowns by \( \hat{u}^I \) where \( I \) now ranges over all of \( \mathcal{I} \). For each \( I \in \mathcal{I} \), and with \( i := \min I \), we copy the \( u_{x_i} \)-equation from the original system (15) and then rename any unknown \( u^K \) appearing in it as \( \hat{u}^K \): 
\[ \hat{u}^I_{x_i}(x) = F^I_i(x; \hat{U}^I_i(x)) \quad \text{with} \ i = \min I, \]
where 
\[ \hat{U}^I_i := \{ \hat{u}^K | K \in \mathcal{I}, K \supset I \setminus i \}. \]
The data for system \( n \) are prescribed as follows:
(1) if \( I = (i) \) for some \( i \in \{1, \ldots, n\} \), then 
\[ \hat{u}^{(i)}(x)|_{x_i=0} := \hat{u}^{(i)}(x'^{i}); \]
\[ \begin{align*}
\end{align*} \]
(2) if $|I| > 1$ and $\min I = i$, then
\[ \hat{u}^I(x)|_{\{x_i=0\}} := u^{i,I}(x^{i'}). \]  
(106)

Note that we use the solutions for systems 1 through $n-1$ to assign the data for system $n$. Since (104) contains exactly one equation for each of the unknowns $\hat{u}^I$, where $I$ ranges over $\mathcal{I}$, it follows that system $n$ is a closed and determined system. Also, the data are of the form required by Darboux’s first theorem. Hence, according to Theorem 2.3, system $n$ with the data in (105)-(106) has a unique, local solution $\hat{U}$ defined in a full $n$-dimensional neighborhood of $0 \in \mathbb{R}^n$.

It remains to verify that the solution $\hat{U}$ of system $n$ in fact solves the original problem (45) & (89). For this, consider first the data requirements, which are straightforward to verify. Indeed, if $|I| = 1$, say $I = (i)$, then according to (105) we have
\[ \hat{u}^I(x)|_{\{x_i=0\}} \equiv \hat{u}^{(i)}(x)|_{\{x_i=0\}} \equiv \hat{u}^{i,I}(x_{I'}). \]
and if $|I| > 1$, with $\min I = i$, then according to (106) and (94) we have
\[ \hat{u}^I(x)|_{\{x_i=0\}} \equiv (\hat{u}^I(x)|_{\{x_i=0\}})|_{\{x_i=0\}} = u^{i,I}(x^{i'}|_{\{x_i=0\}}). \]

This shows that $\hat{U}$ takes on the original data (89).

To show that $\hat{U}$ solves all equations in (45), we start by noting that, by construction, the solutions $\hat{u}^I$ ($I \in \mathcal{I}$) of the $n$th system solve all equations in (45) with $i = \min I$. The remaining equations in the original system are those equations in (45) where an unknown $u^I$ is differentiated with respect to an $x_i$ where $i \in I$ and $i > \min I$.

To verify these remaining equations we shall employ the same strategy used by Darboux. For this we define, for each $I \in \mathcal{I}$, the functions
\[ \Delta^I_i(x) := \hat{u}^I_{x_i}(x) - F^I(x; \hat{U}^I(x)) \quad \text{for } i \in I, \]  
(107)
and set
\[ \mathcal{D} := \{ \Delta^I_i \mid i \in I \in \mathcal{I}, i > \min I \}. \]  
(108)

Note that $\Delta^I_i \equiv 0$ whenever $i = \min I$. We will show, through an application of Theorem 2.3, that also all $\Delta^I_i$ in $\mathcal{D}$ vanish identically on a full neighborhood of $0 \in \mathbb{R}^n$. By definition of $\Delta^I_i$ this establishes that $\hat{U}$ solves all the equations in the original system (45). Verifying that all $\Delta^I_i$ in $\mathcal{D}$ vanish identically near $x = 0$, will be accomplished in two steps by showing that:

(A) whenever $\Delta^I_i \in \mathcal{D}$, then the partial derivative
\[ \partial_{x_\ell} \Delta^I_i \quad \text{where } \ell = \min I, \]
is given as a linear combination (with variable coefficients) of other functions $\Delta^I_k$ in $\mathcal{D}$;

(B) each $\Delta^I_i \in \mathcal{D}$ vanishes identically along $\{x_\ell = 0\}$ for $\ell = \min I$. 

It follows from (A) and (B) that the functions $\Delta^I_i$ in $\mathcal{D}$ satisfy a determined system of linear PDEs of the type covered by Darboux's first theorem, and with vanishing data. According to the uniqueness part of that theorem, it follows that all $\Delta^I_i$ vanish identically near $0 \in \mathbb{R}^n$, which is the desired conclusion.

6.3.1. Proof of (A). We now fix $I \in \mathcal{I}$ and $i \in I$ such that $\ell := \min I < i$. Then, as $\ell = \min I$ and $\hat{u}^I$ is part of the solution to the $n$th system, we obtain:

$$\partial_{x_i} \Delta^I_i(x) = [\hat{u}^I_{x_i}(x)]_{x_i} - [F^I_i(x; \hat{U}^I_i(x))]_{x_i}$$

$$= [F^I_i(x; \hat{U}^I_i(x))]_{x_i} - [F^I_i(x; \hat{U}^I_i(x))]_{x_i}$$

$$= F^I_{\ell,x_i}(x; \hat{U}^I_{\ell}(x)) + \sum_{K \supset I \setminus \ell} F^I_{\ell,uK}(x; \hat{U}^I_{\ell}(x))\hat{u}_{x_i}^K(x)$$

$$- F^I_{i,x_i}(x; \hat{U}^I_i(x)) - \sum_{K \supset I \setminus i} F^I_{i,uK}(x; \hat{U}^I_i(x))\hat{u}_{x_i}^K(x)$$

$$= \sum_{K \supset I \setminus \ell} A^I_{\ell,K}(x)\Delta^K(x) - \sum_{K \supset I \setminus i} A^I_{i,K}(x)\Delta^K(x), \quad (109)$$

where in the last step we have used the integrability conditions [16], and also introduced the coefficient functions

$$A^I_{\ell,K}(x) := F^I_{\ell,uK}(x; \hat{U}^I_{\ell}(x)).$$

Note that, in the first sum in (109), $K \supset I \setminus \ell \supset i$, such that $\Delta^K$ is defined by (107); ditto for the second sum in (109), with $i$ and $\ell$ interchanged. Finally, since the $\hat{u}^K$ solve the $n$th system, we have that:

- those $\Delta^K$ in the first sum in (109) for which $\min K = i$ vanish identically, and
- those $\Delta^K$ in the second sum in (109) for which $\min K = \ell$ vanish identically.

This shows that whenever $\Delta^I_i \in \mathcal{D}$ and $\ell = \min I$, then $\partial_{x_i} \Delta^I_i$ is a linear combination of functions from $\mathcal{D}$.

q.e.d.(A)

6.3.2. Proof of (B). Fix $I$ and $i$ as above, i.e. $\ell := \min I < i \in I$, and calculate:

$$\Delta^I_i(x)_{x_i=0} = \hat{u}^I_{x_i}(x'_{0}) - F^I_i(x'_{0}; \hat{U}^I_i(x'_{0}))$$

$$= \partial_{x_i} \left[ \hat{u}^I(x'_{0}) \right] - F^I_i(x'_{0}; \hat{u}^I(x'_{0}); \hat{U}^I_i(x'_{0})),$$

where we have set

$$\hat{U}^I_i := \{ \hat{u}^K | K \in \mathcal{I}^I_i \},$$

with $\mathcal{I}^I_i$ given in [92] (recall that $i \in I \in \mathcal{I}_\ell$). We then split $\mathcal{I}^I_i$ into two parts as follows:

$$\mathcal{I}^I_{i,a} := \{ K \in \mathcal{I}^I_i | \min K = \ell \}$$
and
\[ T^{\ell,I}_{i,a} := \{ K \in T^{\ell,I}_{i} \mid \min K < \ell \}. \]
According to the data assignment (106) for system \( n \), we have
\[ \hat{u}^K(x'_0) = u^{\ell,K}(x'\ell) \quad \text{whenever } K \in T^{\ell,I}_{i,a}. \tag{111} \]
We now have the following claim:

**Claim 6.3.** With \( I, \ell, \) and \( i \) as above, we have
\[ \hat{u}^K(x'_0) = u^{\ell,K}(x'\ell) \quad \text{whenever } K \in T^{\ell,I}_{i,b} \text{ as well.} \tag{112} \]
Assuming this result for now (proved below), we get from (110) that
\[ \Delta^i_I(x)_{\{x_i=0\}} = \partial_{x_i} \left[ u^{I,I}(x'\ell) - F^i_I(x'_0; \hat{u}^l(x'(\ell)), U^{\ell,I}_{i}(x'\ell)) \right]. \]
This last expression vanishes identically since \( u^{\ell,I} \) is part of the solution to system \( \ell \), verifying the claim in (B).

It only remains to argue for Claim 6.3. With \( I, \ell, \) and \( i \) as above, we introduce the following notation: whenever \( K \in T^{\ell,I}_{i,b} \), we set
\[ v^K(x'\ell) := u^{\ell,K}(x'\ell) \quad \text{and} \quad w^K(x'\ell) := \hat{u}^K(x'_0). \]
Claim 6.3 amounts to the statement that \( v^K(x'\ell) = w^K(x'\ell) \). We shall establish that the functions \( v^K \) and \( w^K \) satisfy the same determined system with the same data. The uniqueness part of Darboux’s first theorem will then yield the conclusion.

We have, with \( K \in T^{\ell,I}_{i,b} \) and \( k := \min K \), that
\[ v^K(x'\ell) = u^{\ell,K}(x'\ell) \]
\[ = F^k_I(x'_0; \hat{u}^l(x'(\ell)), U^{\ell,K}_{k}(x'\ell)), \]
\[ = G^k_I(x'_0; V^{\ell,K}_{k,b}(x'\ell)), \tag{113} \]
where we regard \( U^{\ell,K}_{k,b}(x'\ell) \) as a set of known functions, and we have set \( V^{\ell,K}_{k,b} = U^{\ell,K}_{k,b} \). We note that the resulting system of equations contains exactly one equation for each \( K \in T^{\ell,I}_{i,b} \). The data for this system is given as
\[ v^K(x'\ell)_{\{x_k=0\}} = u^{\ell,K}(x'\ell)_{\{x_k=0\}}. \]
Similarly,
\[ w^K(x'_0) = \hat{u}^K(x'_0) \]
\[ = F^k_I(x'_0; \hat{u}^l(x'(\ell)), \hat{U}^{\ell,K}_{k,b}(x'_0)), \]
\[ = F^k_I(x'_0; \hat{u}^l(x'(\ell)), \hat{U}^{\ell,K}_{k,b}(x')) \]
where we have set
\[ \hat{U}^{\ell,K}_{k,a} := \{ \hat{u}^K \mid K \in T^{\ell,I}_{i,a} \} \quad \text{and} \quad \hat{U}^{\ell,K}_{k,b} := \{ \hat{u}^K \mid K \in T^{\ell,I}_{i,b} \} \]
According to (111) we have
\[ \hat{U}^{\ell,K}_{k,a}(x'\ell) \equiv U^{\ell,K}_{k,a}(x'\ell), \]
so that
\[
    w_k^K(x^*^\ell) = \hat{u}_k^K(x_0^*^\ell) = F_k^K(x_0^*^\ell, [\hat{w}^K(x^*^\ell)], U_{k,a}^\ell K(x^*^\ell), \hat{U}_{k,b}^\ell K(x^*^\ell))
\]
\[
    \equiv G_k^K(x_0^*^\ell, W_{k,b}^\ell K(x^*^\ell)),
\]
where \(G_k^K\) was defined in (113) and we have set \(W_{k,b}^\ell K = \hat{U}_{k,b}^\ell K\). Again, these equations for the \(w^K, K \in T_{i,b}^\ell I\) form a determined system. Finally, the data for \(w^K\) are given as follows:
\[
    w_k^K(x^*^\ell)|_{x_k=0} = \hat{u}_k^K(x_0^*^\ell)|_{x_k=0} \equiv \hat{u}_k^K(x_0^*^k)|_{x_k=0} = u_{k,K}^K(x^*^k)|_{x_k=0},
\]
where we have used that \(k = \min K\), together with the data assignment (106) for \(\hat{u}_k^K\). Finally, we apply Claim 6.2 to conclude that the data for \(w^K\) are given by
\[
    w_k^K(x^*^\ell)|_{x_k=0} = u_{k,K}^\ell(x^*^\ell)|_{x_k=0}.
\]
Thus, the \(v^K\) and the \(w^K\) solve the same determined system with the same data, and the uniqueness part of Darboux’s first theorem implies that \(v^K \equiv w^K\) for each \(K \in T_{i,b}^\ell I\).

The uniqueness part of Darboux’s theorem follows from the inductive construction above. This concludes the proof of Theorem 4.2.

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