The Extended Golden Section and Time Series Analysis

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Abstract The Golden ratio has played an important role in musical composition, architecture, visual art, science, and increasingly in signal processing [1,2,3]. Underlying many of these applications are several extensions of the golden proportions including the Golden \( p \)-Section by Stakhov, the generalized Golden section by Bradley, and others [4,5]. In this paper we review and introduce generalizations of the Golden ratio. We show that there exists a fundamental connection between the limit of two consecutive terms of recursive sequences, the generalized \((p,q)\)-Golden ratio and the Golden ratio generated by the characteristic equation. We apply these generalizations to forecasting financial time series to illustrate one of their applications in signal processing.

Keywords: Golden Ratio, Euclid, Fibonacci, Aesthetic Ratio, Time Series, Signal Processing

1 Introduction

Traditionally, the Golden ratio has arisen in fields ranging from mathematics to architecture and visual art [6,7,8,9,10,11,12,13,14,15,16,17]. The Golden ratio, also known as the Golden mean or proportion, has appeared in many geometrical constructions, even recently when Janusz Kapusta discovered a new world of geometrical relationships residing within the square and the circle. This marked the first visual construction connecting the Golden and Silver mean proportions in a single diagram [18]. Today the Golden ratio plays an increasing role in engineering and modern physical research [19,20,21,22,23,24,25,26,27,28,29]. Some of its applications in signal processing include:

- Face detection evaluation [1]
- Fashion and textile design [2,3,30]
- Analog-to-digital converter design [31,32]
- Traffic signal timing optimization [33]
- Heart and perception based biometrics [34,35,36]
- Audio and speech sampling [37,38]
- Barcode generation [39]

The Golden ratio was first referenced by Euclid (300 B.C.) in his book 'The Elements' [4,40,41]. In this book, Euclid outlined the geometrical problem named the "Division in Extreme and Mean Ratio" [42,43].

Euclid’s Theorem. Divide a line \( AB \) into two segments, a larger one \( CB \) and a smaller one \( AC \) such that:

\[ CB^2 = AB \cdot AC, \]

where \( CB > AC \) and \( AB = AC + CB \). Dividing both parts of the expression (1) by \( CB \) and then by \( AC \), we can rewrite expression (1) in the following form:

\[ \frac{CB}{AC} = \frac{AB}{CB}. \]

Equation (2) implies the following algebraic equation:

\[ x^2 - x - 1 = 0, \]

where \( x = \frac{CB}{AC} \). The positive root of (3) is \( \phi \approx 1.618 \) and is called the Golden ratio or proportion.
Kepler later discovered that the Golden ratio can be expressed as the ratio of two consecutive Fibonacci numbers [20]. Fibonacci numbers have the property that each term is the sum of the two preceding terms: \( f_k = f_{k-1} + f_{k-2}, \ k \geq 2 \), where \( f_0 = 0 \) and \( f_1 = 1 \).

Extensions of the Golden ratio have been considered by many authors. They can be broadly categorized as:

1. Generalizations of Euclid’s theorem [4,5,14,44,45,46].
2. Generalizations of the characteristic equation [47,48,49,4,5,42,50,51].
3. Generalizations of the limit of successive terms recursive sequences [4,52].

In this paper we review and introduce generalizations of the Golden ratio and investigate their properties and application. We examine the generalizations to Euclid’s theorem to formulate and present a solution of the extended Euclid problem. We categorize existing recursive sequences and new generalizations of the Golden section using their characteristic equations. We show that the new extension of the Golden sections is generated by the limits of the quotient of two consecutive terms of a recursive sequence. Lastly, we apply the generalizations to analyze financial stock prices.

The benefits and challenges of this work lie in identifying the relationship between Golden ratio, generalizations of Euclid’s theorem, the characteristic equation, and the limit of successive terms recursive sequences. The rest of the article is organized as follows: Section 2 introduces a new generalization of the Golden ratio by extending Euclid’s problem. Section 3 views this generation based on the characteristic equation generated by homogeneous recurrence relations. Section 4 defines the generalized Golden ratio as a limit of successive terms recursive sequences and investigates its properties. Section 5 summarizes our results and discusses its applications in financial time series analysis.

2 Extending Euclid’s Problem: a New Generalization of the Golden Section

In this section we present a review of the generalizations to Euclid’s theorem and introduce a new generalization of the extended Euclid problem.

*Golden p-Section* [4]: Let \( p \) be a non-negative integer. Divide line \( AB \) into 2 pieces such that:

\[
\frac{CB}{AC} = \left(\frac{AB}{CB}\right)^p,
\]

where \( CB > AC \) and \( AB = AC + CB \). From (4) we obtain:

\[
x^{p+1} - x^p - 1 = 0,
\]

where \( x = \frac{AB}{CB} \). We call the positive root of (5) the Golden \( p \)-section or \( p \)-ratio. In Table 1, we illustrate other known sequences with varying parameter \( p \) [4,50,53]. The ratio of two consecutive \( p \)-Fibonacci or \( p \)-Lucas sequences decays to the solution of (5).

| \( p \) | Fibonacci | Lucas |
|---|---|---|
| 0 | 0, 1, 2, 4, 8, 16, 32, 64, \ldots | 1, 1, 2, 4, 8, 16, 32, 64, \ldots |
| 1 | 1, 1, 2, 3, 5, 8, 13, \ldots | 2, 1, 3, 4, 7, 11, 18, 29, \ldots |
| 2 | 0, 1, 1, 2, 3, 4, 6, \ldots | 3, 1, 1, 4, 5, 6, 10, 15, \ldots |
| 3 | 0, 1, 1, 1, 2, 3, 4, \ldots | 4, 1, 1, 1, 5, 6, 7, 8, \ldots |
| 4 | 0, 1, 1, 1, 1, 2, 3, \ldots | 5, 1, 1, 1, 1, 6, 7, 8, \ldots |

*Golden \((p,q)\)-Section*: Let \( m \) be a real number. We generalize the definition of the Golden section by dividing a segment \( AB \) into \( s + e \) pieces, such that \( AB = sAC + eCB \) and:

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1. There are \( s \) pieces \( AC \) of equal length.
2. There are \( e \) pieces \( CB \) of equal length.
3. Each of the \( s \) pieces is shorter than each of the \( e \) pieces, or \( CB > AC \).
4. The sum of the \( s \) \( AC \) pieces and \( e \) \( CB \) pieces is the whole segment \( AB \), or \( AB = sAC + eCB \).
5. The ratio of the length of a single larger piece to the smaller piece multiplied by a constant is equal to the power \( m \) of the length of the whole segment to that of the larger piece multiplied by a constant:

\[
\frac{CB}{AC} = \left(\frac{AB}{AC}\right)^m,
\]

where \( m, \alpha, \beta \) are real (irrational or rational) numbers.

If \( m \) is a rational number, \( \frac{p}{q} \), where \( p \) and \( q \) are integers and \( q \neq 0 \), then (6) has the following form:

\[
\frac{CB}{AC} = \left(\frac{AB}{CB}\right)^p.
\]

Thus, the ratio between the longer piece \( CB \) and the shorter one \( AC \) raised to the \( q \)th power is equal to the ratio between the whole line segment \( AB \) and the longer part \( CB \) raised to the \( p \)th power.

**Proposition.** If \( m \) is a rational number, \( \frac{p}{q} \), where \( p \) and \( q \) are integers and \( q \neq 0 \), then (7) implies the two algebraic equations:

1. \( \alpha^p x^{p+1} - e\alpha^p x^p - \beta s = 0 \) for \( q = 1 \) and \( p = 0, 1, 2, 3 \ldots \)
2. \( \beta^q x^{q+1} - \alpha ex - \alpha s = 0 \) for \( p = 1 \) and \( q = 1, 2, 3 \ldots \)

**Proof.** Let’s assume \( q = 1 \) and \( p = 0, 1, 2, 3 \ldots \), then equation (7) can be written as:

\[
\frac{CB}{AC} = \left(\frac{AB}{CB}\right)^p.
\]

Denoting

\[
x = \frac{AB}{CB},
\]

and using the relationship \( AB = sAC + eCB \), we can write:

\[
x = \frac{sAC + eCB}{CB} = e + \frac{1}{\alpha CB} s.
\]

Using (9) we obtain:

\[
\frac{CB}{AC} = \frac{1}{\beta(\alpha x)^p}.
\]

Substituting (11) into (10), we have:

\[
x = e + \frac{s}{\beta(\alpha x)^p} \Rightarrow x = e + \frac{\beta s}{(\alpha x)^p} \Rightarrow \alpha^p x^{p+1} - e\alpha^p x^p - \beta s = 0.
\]

A similar argument can be made for the case \( p = 1, q = 1, 2, 3 \ldots \). Using (7) and \( AB = sAC + eCB \) we obtain:

\[
\left(\frac{CB}{AC}\right)^q = \frac{AB}{CB} = \alpha \frac{sAC + eCB}{CB} = \frac{sAC}{CB} + e = \alpha\left(\frac{1}{\alpha CB} + e\right).
\]

Denoting \( x = \frac{CB}{AC} \), we have:

\[
\left(\frac{CB}{AC}\right)^q = \alpha\left(\frac{1}{\alpha CB} + e\right) \Rightarrow (\beta x)^q = \alpha\left(\frac{1}{x} + e\right) \Rightarrow \beta^q x^{q+1} - \alpha ex - \alpha s = 0.
\]

\( \square \)
We call the positive root of these equations:

$$\alpha^p x^{p+1} - \alpha x^p - \beta s = 0, \quad (15)$$

$$\beta^q x^{q+1} - \alpha x^q - \alpha s = 0 \quad (16)$$

the generalized Golden \((p,q)\)-section or \(\phi_{p,q}\). Note that:

1. Every \(p,q,s\), and \(e\) in \((7)\) generates its own variant of the division.
2. The definition of the extended Golden \((p,q)\)-section contains the definition of classical Golden section, Bradley’s Golden section \([5]\), and Stakhov’s Golden \(p\)-section \([4]\). See Table 2.
3. Examples of the generalized Golden \((p,q)\)-section are given below.

| Author | Definition | Equation | \(p\) | \(q\) | \(s\) | \(e\) | Reference |
|--------|------------|----------|------|------|------|------|-----------|
| Generalized Golden \((p,q)\)-section | \(\beta \frac{CB}{AC} = \left(\frac{\alpha}{CB}\right)^p\) | \(\alpha^p x^{p+1} - \alpha x^p - \beta s = 0\) | 0, 1, 2, 3... | 1 | 1, 2, 3... | New |
| Euclid’s Section | \(\alpha \frac{AC}{AB} = \frac{CB}{AC}\) | \(x^2 - x = 1 = 0\) | 1 | 1 | 1 | 1 | [4] |
| Golden \(p\)-Section | \(\beta \frac{AB}{AC} = \left(\frac{\alpha}{CB}\right)^p\) | \(x^{p+1} - x^p - 1 = 0\) | 0, 1, 2, 3... | 1 | 1 | 1 | [4] |
| Bradley’s Section | \(\beta \frac{AC}{AB} = \frac{CB}{AC}\) | \(x^2 - ex - 1 = 0\) | 1 | 1 | 1 | 1, 2, 3... | [5] |
| Multi Parameters Section | \(\beta \frac{AC}{AB} = \left(\frac{\alpha}{CB}\right)^p\) | \(x^{p+1} - \alpha x - \beta = 0\) | 0, 1, 2, 3... | 1 | 1 | 1 | [54] |

Table 2. Extended sections with \(m \in \mathbb{R}\), \(\beta = \alpha = 1\)

3 Generalizing the Golden Section Based on the Characteristic Equation

In this section, we review some recursive sequences, most notably Fibonacci and Lucas numbers and their generalizations. We also categorize existing and new generalizations of the Golden section under characteristic equations and recursive relations.

**Definition** [53]. A sequence \(\{a_n\}\) is said to be defined recursively if its initial values or conditions are specified by \(a_0 = C_0\), \(a_1 = C_1\) \(\ldots\) \(a_{k-1} = C_{k-1}\) and the sequence terms \(a_n\) are defined by one or more of the previous terms \(a_n = \mathbb{R}(a_{n-1}, a_{n-2}, \ldots, a_{n-k})\) for \(n > k - 1\) in the sequence:

$$a_0, a_1, a_2, \ldots, a_{n-1}. \quad (17)$$

**Definition** [53]. A linear homogeneous recurrence relation of degree \(k\) with constant coefficients \(\{c_n\}\) is a relation where each element of a sequence \(\{a_n\}\) is a linear combination of previous terms:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}, \quad (18)$$

where \(a_0 = C_0\), \(a_1 = C_1\) \(\ldots\) \(a_{k-1} = C_{k-1}\) and \(c_1, c_2, \ldots, c_k \in \mathbb{R}\).

The degree \(k\) can be any number and allows us to make \(k\) independent choices (\(\AA\)degrees of freedom*). Note a sequence \(\{a_n\}\) is unique if it satisfies \((18)\).
Definition. A relation is a partially homogeneous recurrence relation if some of the constant coefficients \( c_k \) are zero (see illustrative examples in Table 4).

The partial recurrence relation (18) has degree \( m \) if \( a_n \) can be expressed in terms of some previous \( m < k \) terms. In this article, we shall restrict our attention to the second degree homogeneous linear recurrence relations. Table 3 and Table 4 present second degree \((k = 2)\) commonly used homogeneous and partially homogeneous recurrences.

### Table 3. Second degree \((k = 2)\) common homogeneous recurrences

| Numbers | Recurrence Relation \( k \geq 2 \) | Initial Conditions | Sequences | Limit of Quotient | Reference |
|---------|-----------------------------------|--------------------|-----------|-------------------|-----------|
| Fibonacci | \( f_k = f_{k-1} + f_{k-2} \) | \( f_0 = 0, f_1 = 1 \) | 0, 1, 1, 2, 3, 5, 8, 13... | \((1 + \sqrt{5})/2\) | [14,41,53,55] |
| Lucas | \( l_k = l_{k-1} + l_{k-2} \) | \( l_0 = 2, l_1 = 1 \) | 2, 1, 3, 4, 7, 11, 18, 29... | \((1 + \sqrt{5})/2\) | [14,41,53] |
| Weighted Fibonacci | \( f_k = \alpha f_{k-1} + \beta f_{k-2} \) | \( f_0 = 0, f_1 = 1 \) | \((a = 1, b = 1)\) | \((\alpha + \alpha^2 + 4\beta)/2\) | [50,53] |
| Weighted Lucas | \( l_k = \alpha l_{k-1} + \beta l_{k-2} \) | \( l_0 = 2, l_1 = 1 \) | \((a = 1, b = 1)\) | \((\alpha + \sqrt{\alpha^2 + 4\beta})/2\) | [50,53] |
| m-Fibonacci | \( f_{k-1,m} + f_{k-2,m} \) | \( f_0 = 0, f_1 = 1 \) | \((m = 1)\) | \((m + \sqrt{m^2 + 4})/2\) | [56] |
| Pell | \( p_k = 2p_{k-1} + p_{k-2} \) | \( p_0 = 0, p_1 = 1 \) | 0, 1, 2, 5, 12, 29, 70, 169... | \(1 + \sqrt{2}\) | [56,57] |
| Pell-Lucas | \( p_k = 2p_{k-1} + p_{k-2} \) | \( p_0 = 2, p_1 = 2 \) | 2, 2, 6, 14, 34, 82, 198, 478... | \(1 + \sqrt{2}\) | [56] |

### Table 4. Second degree \((k = 2)\) common and new partially homogeneous recurrences

| Numbers | Recurrence Relation \( k \geq 2 \) | Initial Conditions | Sequences | Limit of Quotient | Reference |
|---------|-----------------------------------|--------------------|-----------|-------------------|-----------|
| Padovan | \( d_k = d_{k-2} + d_{k-3} \) | \( d_0 = d_1 = d_2 = 1 \) | 1, 1, 1, 2, 2, 3, 5, 7... | \(\approx 1.324\) | [50] |
| Perrin | \( p_k = p_{k-2} + p_{k-3} \) | \( p_0 = 3, p_1 = 0, p_2 = 2 \) | 3, 0, 2, 3, 2, 5, 7... | \(\approx 1.324\) | [50] |
| p-Fibonacci | \( f_k = f_{k-1} + f_{k-p-1} \) | \( f_0 = 0, f_1 = ... = f_{p+1} = 1 \) | \((p = 2)0, 1, 1, 1, 2, 3, 4, 6...\) | \(\phi_p\) | [4,50,53] |
| p-Lucas | \( l_k = l_{k-1} + l_{k-p-1} \) | \( l_0 = p + 1, l_1 = ... = l_{p+1} = 1 \) | \((p = 2)3, 1, 1, 1, 4, 5, 6, 10...\) | \(\phi_p\) | [4,50,53] |

### Solving linear homogeneous recurrences \([53]\).

To solve the recurrence relation (18), we will guess that the solution has the form \( a_n = r^n \). We substitute into the recurrence (18) to get:

\[
r^n = c_1 r^{n-1} + c_2 r^{n-k},
\]

where \( c_1, c_2, \ldots, c_k \in \mathbb{R} \). Dividing both sides of (19) by \( r^{n-k} \) gives:

\[
r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c_k = 0.
\]

Equation (20) is known as the characteristic equation of the recurrence relation (18). It is a polynomial of degree \( k \) and thus by the Fundamental Theorem of Algebra has \( k \) complex roots \( r_1, r_2, \ldots, r_k \) counted by multiplicity. The general solution is therefore: \( a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \ldots + \alpha_k r_k^n \). More precisely, it can be shown that for \( c_1, c_2, \ldots, c_k \in \mathbb{R} \):

1. If the characteristic equation (20) has \( k \) distinct roots \( r_1, r_2, \ldots, r_k \), then the sequence \( \{a_n\} \) is a solution of the recurrence relation (18) if and only if:

\[
a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \ldots + \alpha_k r_k^n,
\]
where \( n = 0, 1, 2, \ldots \) and \( \alpha_1, \alpha_2, \ldots, \alpha_k \) are constants that are determined by the \( k \) initial conditions \( a_0 = C_0, a_1 = C_1, \ldots, a_{k-1} = C_{k-1} \).

2. If the characteristic equation (20) has \( t \) roots with multiplicities \( m_1, m_2, \ldots, m_t \). Then a sequence \( \{a_n\} \) is a solution of the recurrence relation (18) if and only if:

\[
a_n = (\alpha_{1,0} + \alpha_{1,1}n + \ldots + \alpha_{1,m_1-1})r_1^n + \ldots + (\alpha_{t,0} + \alpha_{t,1}n + \ldots + \alpha_{t,m_t-1})r_t^n,
\]

where \( n = 0, 1, 2, \ldots \) and \( \alpha_{i,j} \) are constants for \( 1 \leq i \leq t \) and \( 0 \leq j \leq m_i - 1 \) that depend on initial conditions. Note the roots of this polynomial are called the characteristic roots of the recurrence relation. The positive root is called the generalized Golden ratio.

**Example 1:** Consider a second order linear homogeneous recurrence of the form \( f_n = f_{n-1} + f_{n-2} \), \( f_0 = 0, f_1 = 1 \). The characteristic equation of \( f_n \) is \( x^2 - \alpha x - \beta = 0 \). The general form of the solution is \( f_n = \alpha^n + \beta^n \) where \( \alpha \) and \( \beta \) are unknowns. Using the initial conditions, \( \alpha + \beta = 0, \alpha \beta = 1 \), we get:

\[
f_k = \frac{r_1^k - r_2^k}{r_1 - r_2},
\]

where \( r_1 = \frac{\alpha + \sqrt{\alpha^2 + 4\beta}}{2} \) and \( r_2 = \frac{\alpha - \sqrt{\alpha^2 + 4\beta}}{2} \). For the Fibonacci sequence, \( \alpha = \beta = 1 \) and the characteristic equation is \( x^2 - x - 1 = 0 \). This gives us the roots \( \phi = \frac{1 + \sqrt{5}}{2} \approx 1.618 \) and \( \phi' = \frac{1 - \sqrt{5}}{2} \approx 0.618 \). We can also generate Fibonacci and Lucas numbers from Binet’s formula:

\[
f_n = \frac{\phi^n - (\frac{1}{\phi})^n}{\sqrt{5}}.
\]

**Example 2:** Consider a second order linear homogeneous recurrence of the form \( f_{k,m} = mf_{k-1,m} + f_{k-2,m} \), with \( f_0 = 0, f_1 = 1 \), and \( m > 0 \). This \( m \)-Fibonacci sequence may be expressed in the form:

\[
f_{k,m} = a r_1^k + b r_2^k,
\]

where \( r_1 = \frac{m + \sqrt{m^2 + 4 \cdot 2^k - 1}}{2} \), \( r_2 = \frac{m - \sqrt{m^2 + 4 \cdot 2^k - 1}}{2} \). \( \alpha, \beta \) are constants. For values \( k = 0, 1 \) we obtain:

\[
f_{0,m} = a + b = 0 \quad f_{1,m} = ar_1 + br_2,
\]

which implies \( a = \frac{1}{r_1 - r_2} \). We can use this result to derive Binet’s formula:

\[
f_{k,m} = ar_1^k + br_2^k = \frac{r_1^k - r_2^k}{r_1 - r_2}.
\]

By simplifying (27), we obtain:

\[
f_{k,m} = \frac{1}{\sqrt{m^2 + 4 \cdot 2^k - 1}} \sum_{i=0}^{k-1} \frac{k}{2i+1} m^{k-1-2i} (\sqrt{m^2 + 4})^{2i+1}.
\]

**Example 3:** Consider a second order linear homogeneous recurrence of the form \( g_{k,\phi} = mg_{k-1,\phi} + g_{k-2,\phi} \), where \( m = \phi = \frac{1 + \sqrt{5}}{2} \). Using a similar procedure to that in previous example, we can derive the general form of this Fibonacci-Golden sequence:

\[
g_{k,m} = \frac{1}{\sqrt{\phi^2 + 4}} (r_1^k - r_2^k),
\]

where \( r_1 = \frac{\phi + \sqrt{\phi^2 + 4}}{2} \) and \( r_2 = \frac{\phi - \sqrt{\phi^2 + 4}}{2} \). By simplifying (29), we obtain:

\[
g_{k,m} = \frac{1}{\sqrt{\phi^2 + 4}} \sum_{i=0}^{k-1} \frac{k}{2i+1} (\phi^{k-1-2i} (\sqrt{\phi^2 + 4})^{2i+1}).
\]

**Example 4:** Consider a second order partial linear homogeneous recurrence of the form \( f_k = f_{k-1} + f_{k-p-1} \) where \( f_0 = f_1 = \ldots = f_{p+1} = 1 \) (see Table 1. The characteristic equation of \( a_n \) is \( \lambda^{p+1} - \lambda^p - 1 = 0 \). The root of this equation is called the Golden \( p \)-section [4]. For \( p = 0, 1, 2, 3 \) the roots are \(.5, .618, .683, .725\), respectively. Table 5 summarizes the extensions.
Table 5. Extended Golden sections and their generating equations ($\alpha = \beta = 1$)

| Golden section          | $p$ | $q$ | $e$ | $s$ | Generating Equation                                                                 | Root            | Reference |
|-------------------------|-----|-----|-----|-----|-------------------------------------------------------------------------------------|-----------------|-----------|
| Euclid                  | 1   | 1   | 1   | 1   | $x^2 - x - 1 = 0$                                                                    | $\frac{1 + \sqrt{5}}{2}$ | [53]      |
| Family of Metallic Means| 1   | 1   | $\mathbb{Z}^+$ | $\mathbb{Z}^+$ | $x^2 - sx - e = 0$                                                                  | $\pm \sqrt{\frac{s^2 - 4e}{s}}$ | [47,48,49]|
| Silver Mean             | 1   | 1   | 2   | 1   | $x^2 - 2x - 1 = 0$                                                                    | $1 + \sqrt{2}$  | [47,48,49]|
| Bronze Mean             | 1   | 1   | 3   | 1   | $x^2 - 3x - 1 = 0$                                                                    | $\frac{3 \pm \sqrt{13}}{2}$ | [47,48,49]|
| Metallic Mean           | 1   | 1   | 4   | 1   | $x^2 - 4x - 1 = 0$                                                                    | $2 + \sqrt{13}$ | [47,48,49]|
| Bradley                 | 1   | 1   | $\mathbb{Z}^+$ | 1   | $x^2 - mx - 1 = 0$                                                                  | $\frac{m + \sqrt{m^2 + 4}}{2}$ | [5]       |
| Copper Mean             | 1   | 1   | 1   | 2   | $x^2 + x - 2 = 0$                                                                    | 2               | [47,48,49]|
| Nickel Mean             | 1   | 1   | 1   | 3   | $x^2 + x - 3 = 0$                                                                    | $\frac{1 + \sqrt{5}}{2}$ | [47,48,49]|
| Complex                 | 1   | 1   | 1   | $\frac{3}{2}$ | $x^n + x - \frac{3}{2} = 0$                                                        | $\frac{1 + \sqrt{5}}{2}$ | [58]      |
| Three Roots             | 2   | 1   | 1   | 1   | $x^3 - 3x^2 - 1 = 0$                                                                  | $\approx 1.466$  | [4]       |
| Four Roots              | 3   | 1   | 1   | 1   | $x^4 - 4x^3 - 1 = 0$                                                                  | $\approx 1.380$  | [4]       |
| Trigonometric Mean      | 1   | 1   | $2\cos x$ | 1   | $x^2 - 2\cos x - 1 = 0$                                                             | $1 + \sqrt{\cos^2 x + 1}$ | New       |
| Aesthetic               | 1   | 1   | $-\frac{7}{2}$ | 11  | $x^7 + 7x - 11 = 0$                                                                 | $\approx 1.322$  | New       |
| Generalized Golden      | $\mathbb{Z}^+$ | R   | R   | $\alpha^p x^{p+1} - \cos x - \cos \alpha x - \alpha s = 0$ | New               |           |
| Generalized Golden      | 1   | $\mathbb{Z}^+$ | R   | R   | $\beta^p x^{p+1} - \cos x - \cos \alpha x - \alpha s = 0$ | New               |           |

Properties [40]. The generalized Golden ratio, $\phi = e + \frac{s}{p}$, can be expressed by:

1. A series of continued fractions:

$$\phi = s + \frac{e}{s + e \frac{e}{s + e \frac{e}{s + e \ldots}}}.$$  \hspace{1cm} (31)

2. A series of continued square roots:

$$\phi = \sqrt{s + e \sqrt{s + e \sqrt{s + e \sqrt{s + e \ldots}}}}.$$  \hspace{1cm} (32)

4 Generalizing the Golden Section Based on Recursive Sequences

Johannes Kepler [20] observed that the ratio of consecutive Fibonacci numbers converges to the Golden ratio. In this section, we present a review of existing and new generalized Golden sections generated by the limits of the quotient of two consecutive terms of a recursive sequence [49].

Definition. A sequence is monotonic if it is either increasing or decreasing.

Proposition 1. If the sequence $\{a_n\}$ is generated by a linear monotonic homogeneous degree $k = 2$ recurrence: $a_n = ea_{n-1} + sa_{n-1}$, $n \geq 2$, then:

1. The ratio of consecutive $\{a_n\}$ numbers converges to the solution of the following equation: $x^2 - ex - s = 0$.
2. If $e = s = 1$, then $\lim_{n \to \infty} \frac{a_n}{a_{n-1}} = \frac{1 + \sqrt{5}}{2}$.
3. If $e = -7$, $s = 11$, then $\lim_{n \to \infty} \frac{a_n}{a_{n-1}} \approx 1.322$. We call this the Aesthetic ratio (see (34)).
4. The limit of the sequence is independent of the initial conditions.

Proposition 2. If the sequence $\{a_n\}$ is generated by a linear monotonic partial homogeneous degree $k = 2$ recurrence, i.e. $a_n = ea_{n-1} + sa_{n-p-1}$, $n \geq 2$, then:

1. The ratio of consecutive $\{a_n\}$ numbers converges to the solution of the following equation: $x^{p+1} - ex^p - s = 0$.  

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2. If \( e = s = 1 \), then \( \lim_{n \to \infty} a_n \approx \alpha \), where \( \alpha \) is a Fibonacci \( p \)-number illustrated in Table 1. The limit of the sequence is independent of the initial conditions. In [4] Stakhov showed that \( \tau_p \) is the solution to this equation.

Proposition 3. If the sequence \( \{a_n\} \) is generated by a linear monotonic homogeneous degree \( k = 2 \) recurrence: \( a_n = ea_{n-m} + sa_{n-k} \), \( n \geq 2 \), then:

1. The ratio of consecutive \( \{a_n\} \) numbers converges to the solution of the following equation:

\[
x^{m+k} - ex^k - sx^m = 0.
\]

We call the sequences that generate (33) with real or irrational initial conditions the family of \((k, m)\) Fibonacci sequences and the roots of (33) the family of generalized ratios.

2. The limit of the sequence is independent of the initial conditions.

Proof.

\[
x = \lim_{n \to \infty} \frac{a_n}{a_{n-1}} = \lim_{n \to \infty} \frac{ea_{n-m} + sa_{n-k}}{a_{n-1}} = e \lim_{n \to \infty} \frac{a_{n-1}}{a_{n-m}} + s \lim_{n \to \infty} \frac{a_{n-k}}{a_{n-k}} = e \lim_{n \to \infty} \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_{n-m+1}}{a_{n-m}} + s \lim_{n \to \infty} \frac{a_{n-k}}{a_{n-k}}
\]

\[
= e \frac{a_{m-1}}{a_{m-2}} \cdots \frac{a_{n-m+1}}{a_{n-m}} + s \frac{a_{n-k}}{a_{n-k}} \Rightarrow x^{m+k} - ex^k - sx^m = 0.
\]

Propositions 1 and 2 can be proved in a similar fashion.

The parameters \( m \) and \( k \) can generate both new and commonly used sequences and their corresponding ratios. For example,

- If \( m = k = 1 \), then \( a_n = a_{n-1}(e + s) \) and \( x^2 - ex - sx = 0 \). For \( a_1 = e = s = 1 \), \( x = \sqrt{2} \) and we have the base-2 sequence: \( 2^0, 2^1, 2^2, 2^3, \ldots \).
- If \( m = 1 \) and \( 2 \), then \( x^{k+1} = ex^k + sx^{k+1} \Rightarrow x^{k+1} = ex + sx \). For \( e = s = 1 \), we have the Fibonacci and Lucas sequences whose two consecutive terms converge to the Golden Ratio (see Table 2).
- If \( m = k + 1 \) and \( e = s = 1 \) and \( k = 2 \), we have Perrin and Padovan sequences (and their variations) whose two consecutive terms converge to \( \approx 1.324 \) (see Table 4).
- If \( m = k + 1 \) and \( e = s = 1 \) and \( k = 3 \), we have new sequences derived from our generalized equation (33) whose two consecutive terms converge to \( \lim_{n \to \infty} a_{n-1} \).

Aesthetic Ratio. For a sequence \( \{a_n\} \) that is generated by a linear monotonic homogeneous degree \( k = 2 \) recurrence, we designate the Aesthetic ratio as:

\[
\lim_{n \to \infty} \frac{a_n}{a_{n-1}} \approx 1.322,
\]

where \( a_n = ea_{n-1} + sa_{n-2} \), \( n \geq 2 \), \( e = -7 \), \( s = 11 \). The Aesthetic ratio derives its name from its use in art. It can be seen on a statistical study of 565 famous works of art by Bellini, Caravaggio, Cesanne, Goya, van Gogh, Delacroix, Rembrandt, and Toulouse-Lautrec [59]. It is the average ratio of the two dimensions of these paintings, as summarized in Table 6.

Definition. Let \( \{b_n\} \) be a monotonic homogeneous or partial homogeneous degree \( k \) recurrence. We call the quantity:

\[
\phi_n = \lim_{n \to \infty} \frac{b_n}{b_{n-1}},
\]

the generalized Golden section for the sequence \( \{b_n\} \).
Table 6. Ratio of dimensions of famous paintings

| Artist          | Number of Paintings | Average Error |
|-----------------|---------------------|---------------|
| Bellini         | 53                  | 1.46 ± 0.10   |
| Caravaggio      | 37                  | 1.32 ± 0.15   |
| Cezanne         | 100                 | 1.26 ± 0.27   |
| Delacroix       | 42                  | 1.32 ± 0.17   |
| Van Gogh        | 69                  | 1.32 ± 0.19   |
| Goya            | 34                  | 1.04 ± 0.04   |
| Rembrandt       | 39                  | 1.33 ± 0.14   |
| Toulouse-Lautrec| 64                  | 1.36 ± 0.12   |

Tables 3 and 4 summarizes existing and new generalized Golden sections generated by taking the limit of the quotient of two consecutive terms of a recursive sequence.

**Properties.**
1. Assume that the sequence \( \{f_n\} \) is generated by a second order linear homogeneous recurrence and satisfies (35). For the arbitrary integer constants \( m \) and \( k \) the following holds:

\[
\lim_{n \to \infty} \frac{f_{n+m}}{f_{n+k}} = \gamma^{m-k}
\]

where \( m > k \).

2. Assume that the sequence \( \{f_n\} \) is generated by a second order linear homogeneous recurrence with the solution (23). Then:

\[
\lim_{k \to \infty} \frac{f_k}{f_{k-1}} = r_1 \quad \lim_{k \to \infty} \frac{f_{k-1}}{f_k} = r_2.
\]

**Proof.** Since \( r_1 > r_2 \) then \( \lim_{k \to \infty} \frac{r_k}{r_{k-1}} = 0 \) and \( f_k = \frac{r_k - r_2}{r_1 - r_2} \) we have:

\[
\lim_{k \to \infty} \frac{f_k}{f_{k-1}} = \lim_{k \to \infty} \frac{r_k - r_2}{r_k - r_1} = \frac{1 - \lim_{k \to \infty} \frac{r_2}{r_k}}{1 - \lim_{k \to \infty} \frac{r_2}{r_1}} = r_1.
\]

Using a similar approach, we can show \( \lim_{k \to \infty} \frac{f_{k-1}}{f_k} = r_2 \). \( \square \)

**Definition.** We say two recursive sequences, \( f_k = f(t_{k-1}, t_{k-2}) \) and \( g_k = g(t_{k-1}, t_{k-2}) \), where \( k \geq 2 \), generate a Golden section power sequence \( \phi^1, \phi^2, \ldots, \phi^n \) if

\[
\phi^n = \alpha f_n + \beta g_n
\]

where \( n \in \mathbb{Z}^+ \).

**Example 5:** Let the roots of the characteristic equation be \( x^2 - x - 1 = 0 \). It can be shown that the classical Fibonacci and Lucas sequences generate a Golden section power sequence.

\[
\phi^n = \left( \frac{l_n + f_n \sqrt{5}}{2} \right) \quad \phi^{-n} = \left( \frac{l_n - f_n \sqrt{5}}{2} \right)
\]

**Proof.** Proof by mathematical induction.

\[
\phi = \frac{1 + \sqrt{5}}{2} \quad \phi^2 = \left( \frac{1 + \sqrt{5}}{2} \right)^2 = \frac{3 + \sqrt{5}}{2},
\]

\[
\phi^3 = \left( \frac{1 + \sqrt{5}}{2} \right)^3 = \left( \frac{3 + \sqrt{5}}{2} \right) \left( \frac{1 + \sqrt{5}}{2} \right) = \frac{4 + 2\sqrt{5}}{2}
\]

\[
\phi^4 = \left( \frac{1 + \sqrt{5}}{2} \right)^4 = \left( \frac{4 + 2\sqrt{5}}{2} \right) \left( \frac{1 + \sqrt{5}}{2} \right) = \frac{7 + 3\sqrt{5}}{2} \Rightarrow \phi^n = \frac{l_n + f_n \sqrt{5}}{2}.
\]

A similar argument can be used to show \( \phi^{-n} = \frac{l_n - f_n \sqrt{5}}{2} \). \( \square \)
Example 6: The 2-Fibonacci, $f_k = 2f_{k-1} + f_{k-2}, f_0 = 0, f_1 = 1$, and modified Pell sequences, $p_k = 2p_{k-1} + p_{k-2}, p_0 = 1, p_1 = 3$, generate a Silver mean power sequence

$$\gamma^n = 2 * p_n + p_{n-1} + (2f_n + f_{n-1})\sqrt{2}, \quad \gamma^{-n} = 2 * p_n + p_{n-1} - (2f_n + f_{n-1})\sqrt{2},$$

where $\gamma = 1 + \sqrt{2}$

Proof. Proof by mathematical induction.

$$\gamma = 1 + \sqrt{2}, \quad \gamma^2 = (1 + \sqrt{2})^2 = 3 + 2\sqrt{2},$$

$$\gamma^3 = (1 + \sqrt{2})^3 = (1 + \sqrt{2})(3 + 2\sqrt{2}) = (2 * 3 + 1) + (2 * 2 + 1)\sqrt{2},$$

$$\gamma^4 = (1 + \sqrt{2})^4 = (1 + \sqrt{2})(7 + 5\sqrt{2}) = (2 * 7 + 3) + (2 * 5 + 2)\sqrt{2}.$$

$$\Rightarrow \gamma^n = 2p_n + p_{n-1} + (2f_n + f_{n-1})\sqrt{2} \quad (40)$$

A similar argument can be made to show $\gamma^{-n} = 2 * p_n + p_{n-1} - (2f_n + f_{n-1})\sqrt{2}.$

5 Simulation

This section explores the application of the generalized golden ratio for forecasting financial time series. Forecasting methodologies include correlation analysis [60], moving averaging models [61,62,63], logistic regression [64], artificial neural networks [65,66,67], support vector machines [68], and decision tree analysis [69,70]. This section presents simulation results from applying the generalized golden ratio to one commonly used moving average model - the moving average crossover. This model is chosen because of its:

- Human interpretability with intuitive and transparent inputs and outputs
- Support by academic and industry practitioners
- Extensive use as a filter in DSP that is optimal for reducing random noise while retaining a sharp step response
- Simple and efficient low-pass filtering operation that smooths out price fluctuations
- Application to any universe of stocks

Golden Section Moving Average: The golden $(p,q)$ section $N$-day moving average is calculated by:

$$y_{p,q}[N] = \frac{1}{\phi_{p,q}} \cdot x[1] + \phi_{p,q} \cdot x[N] + \frac{1}{N} \sum_{k=2}^{N-1} x[k], \quad (42)$$

where $\phi_{p,q}$ is the positive root of (15) and $x[N]$ is the data value at day $N$.

Moving Average Crossover Model: The model generates a signal when a stock price’s short-term moving average crosses its long-term moving average. This cross may indicate that the stock is exhibiting upward (downward) momentum, and thus its price that is moving up (down) in the short term is likely to continue moving up (down) [71]. The signal from the generalized golden ratio moving average model is:

$$R = \begin{cases} 
BUY & \text{if } y_{p,q}[M] > y_{p,q}[N] \\
SELL & \text{if } y_{p,q}[M] < y_{p,q}[N] \\
\text{HOLD} & \text{otherwise}
\end{cases}, \quad (43)$$

where $M$ and $N$ are the short and long term moving average periods in days, respectively.

Table 7 provides performance results from using the signal described in [43] to trade SPY (SPDR S&P 500), the exchange traded fund that tracks the S&P 500 stock market index. The results are calculated over a ten year time horizon from January 2007 to July 2017, and short and long term moving averages of $M = 10$ and $N = 50$ are selected to be consistent with industry practice. $(p,q)$ parameters

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are taken from the examples of the Golden section provided earlier in this paper to generate separate strategies. The performance of each strategy is evaluated against the benchmark of buying and holding the index over the same time period using a risk-adjusted return measure. Two such often used risk measures are the Sharpe and Sortino ratios. The Sharpe ratio is defined as:

\[
\text{Sharpe Ratio} = \frac{R_p - R_f}{\sqrt{\sigma_p}},
\]

where \(R_p\) is the average portfolio return, \(R_f\) is the risk free rate, \(\sigma_p\) is the portfolio standard deviation.

The Sharpe ratio captures how well strategy returns compensate the investor for the amount of risk taken. The Sortino ratio is a variation of the Sharpe ratio that only factors downside risk rather than total risk of the portfolio [72]. The greater a strategy’s Sharpe or Sortino ratio, the better return for the same risk.

The Golden section strategies outperform the benchmark index robustly across \((p, q)\) parameters with an average Sharpe and Sortino ratios of .56 and 0.78, respectively. The highest Sharpe and Sortino ratios are reached with parameters \(p = 2, q = 3\), and Figure 1 shows the corresponding strategy’s full backtest performance and moving average series. Backtests are done on the Quantopian platform and take into account transaction costs. Results suggest that the Golden section can be used in moving average indicators to detect price momentum trends.

**Figure 1.** Golden \((2, 3)\)-Section MA Signals and Backtest Results

**Table 7.** Generalized Golden Section Crossover Strategy Performance

| \((p, q)\)-Section | \(\phi\) | Sharpe Ratio | Sortino Ratio | Cumulative Return |
|-------------------|---------|--------------|---------------|------------------|
| \((3, 4)\)        | 1.701   | 0.56         | 0.78          | 101%             |
| \((2, 3)\)        | 1.678   | 0.60         | 0.84          | 107%             |
| \((1, 1)\)        | 1.618   | 0.60         | 0.83          | 106%             |
| \((4, 3)\)        | 1.555   | 0.54         | 0.73          | 87%              |
| \((3, 2)\)        | 1.529   | 0.52         | 0.71          | 81%              |
| Benchmark         | SPY     | 0.43         | 0.60          | 112%             |
6 Conclusion

In this paper we review existing variations of the Golden section and introduce generalizations to Euclid’s problem. We extend the concepts of Golden ratio from different points of view: through the generalization of Euclid’s section definition, the generalization of the characteristic equation, and the definition of limit of successive terms in recursive sequences. Finally, we illustrate the application of generalized Golden sections in detecting stock price trends. Future work will apply higher order degree homogeneous linear recurrence relations to build upon the examined applications in financial time series forecasting.

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References

1. M. Hassaballah, K. Murakami, and S. Ido, “Face detection evaluation: a new approach based on the golden ratio,” Signal, Image and Video Processing, vol. 7, no. 2, pp. 307–316, 2013.
2. Y. Liang, C.-T. Li, Y. Guan, and Y. Hu, “Gait recognition based on the golden ratio,” EURASIP Journal on Image and Video Processing, vol. 2016, no. 1, p. 22, 2016.
3. Z. Kuzmaleva and J. Iliev, “The golden and fibonacci geometry in fashion and textile design,” Proc. of the eRA, vol. 10, pp. 15–64, 2016.
4. A. Stakhov, “The golden section and modern harmony mathematics,” Applications of Fibonacci Numbers, vol. 7, 1998.
5. S. Bradley, “A geometric connection between generalized fibonacci sequences and nearly golden sections,” The Fibonacci Quarterly, vol. 38, no. 2, pp. 174–80, 1999.
6. R. Herz-Fischler, The Shape of the Great Pyramid. Wilfrid Laurier University Press, 2000.
7. P. Engstrom, “Sections, golden and not so golden,” The Fibonacci Quarterly, vol. 26, no. 4, pp. 118–27, 1988.
8. A. van Zanten, “The golden ratio in the arts of painting, building and mathematic,” Nieuw Arch. Wisk., vol. 17, no. 2, pp. 229–45, 1999.
9. G. Runion, The Golden Section, and Related Curiosia. Scott Foresman and Company, 1972.
10. L. Marek-Cm, “The golden mean in the topology of four-manifolds in conformal field theory in the mathematical probability theory and in cantorian spacetime,” Chaos, Solitons and Fractals, vol. 17, no. 4-6, pp. 613–638, 1989.
11. A. Stakhov, “The golden section in measurement theory,” Computers and Mathematics with Applications, vol. 17, no. 4-6, pp. 613–638, 1989.
12. Y. Yazlik and N. Taskara, “A note on generalized-horadam sequence,” Computers and Mathematics with Applications, vol. 63, no. 1, pp. 36–41, 2012.
13. H. Kim and J. Neggers, “Fibonacci mean and golden section mean,” Computers and Mathematics with Applications, vol. 56, no. 1, pp. 228–232, 2008.
14. J. Kapur, “Some generalizations of the golden ratio,” International Journal of Mathematical Education in Science and Technology, vol. 19, no. 4, pp. 511–17, 1988.
15. H. Osborne. “Symmetry as an aesthetic factor,” Computers and Mathematics with Applications, vol. 12, no. 1-2, pp. 77–82, 1986.
16. J. Pelaez, “A new measure of consistency for positive reciprocal matrices,” Computers and Mathematics with Applications, vol. 46, pp. 1839–1845, 2003.
17. G. Doczi, “Seen and unseen symmetries: A picture essay,” Computers and Mathematics with Applications, vol. 12, no. 1-2, pp. 38–62, 1986.
18. J. Kapusta, “The square, the circle and the golden proportion: A new class of geometrical constructions,” Forma, vol. 19, pp. 293–313, 2004.
19. S. Sen and R. Agarwal, “Golden ratio in science, as random sequence source, it computation and beyond,” Computers and Mathematics with Applications, vol. 56, no. 2, pp. 469–498, 2008.
20. J. Kepler, A New Year Gift: On Hexagonal Snow. Oxford University Press, 1966.
21. E. Naschie, “The vak of vacuum fluctuation: Spontaneous self-organization and complexity theory interpretation of high energy particle physics and the mass spectrum,” Chaos, Solitons and Fractals, vol. 18, no. 2, pp. 401–20, 2003.
22. ———, “The concepts of e infinity: an elementary introduction to the cantorian-fractal theory of quantum physics,” Chaos, Solitons and Fractals, vol. 22, no. 2, pp. 495–511, 2004.
23. R. Heyrovská, “The golden ratio ionic and atomic radii and bond lengths,” Mol. Physics, vol. 103, pp. 877–82, 2005.
24. L. Marek-Crnjac, “On the mass spectrum of the elementary particles of the standard model using el naschie’s Zs golden field theory,” Chaos, Solitons and Fractals, vol. 15, no. 4, pp. 611–18, 2003.
25. Y. Abu-Mostafa and R. McEliece, “Maximal codeword lengths in huffman codes,” Computers and Mathematics with Applications, vol. 39, no. 11, pp. 129–134, 2000.
26. E. Cureg and A. Mukherjea, “Numerical results on some generalized random fibonacci sequences,” Computers and Mathematics with Applications, vol. 59, no. 1, pp. 233–246, 2010.
27. M. Esmaeili and M. Esmaeili, “A fibonacci-polynomial based coding method with error detection and correction,” Computers and Mathematics with Applications, vol. 60, no. 10, pp. 2738–2752, 2010.
28. L. Guillermo and M. Gomez, “Intertemporal issues associated with the control of macro-economic systems,” Computers and Mathematics with Applications, vol. 24, no. 8-9, pp. 77–98, 1992.
29. I. Tanackov, J. Tepić, and M. Kostelac, “The golden ratio in probabilistic and artificial intelligence,” Tehnicki Vjesnik/Technical Gazette, vol. 18, no. 4, 2011.
30. S. Lian-ying, “Analysis of the golden section in garment sculpt design effect,” Jiangzi Science, vol. 31, pp. 816–819, 2013.
31. I. Daubechies and O. Yilmaz, “Robust and practical analog-to-digital conversion with exponential precision,” IEEE Transactions on Information Theory, vol. 52, no. 8, pp. 3533–3545, 2006.
32. I. Daubechies, C. S. Gunturk, Y. Wang, and O. Yilmaz, “The golden ratio encoder,” IEEE Transactions on Information Theory, vol. 56, no. 10, pp. 5097–5110, 2010.
33. M. Esmaili and M. Esmaili, “A fibonacci-polynomial based coding method with error detection and correction,” Computers and Mathematics with Applications, vol. 32, no. 2, pp. 95–8, 2004.
34. V. Spinadel, “A generalization of the golden section,” The Metallic Means and Design. Nexus II: Architecture and Mathematics, 1998.
35. M. Sigalotti, “The golden ratio in special relativity,” Chaos, Solitons and Fractals, vol. 30, no. 3, pp. 521–524, 2006.
36. A. Ajluni, C. Martin, A. Yalamarthy, and J. Maleszewski, “A study on adult female human perception of the golden ratio in paintings using psychological survey,” Art and Science in Paris, 2010.
37. M. Kleider, B. Rafaely, B. Weiss, and E. Bachmat, “Golden-ratio sampling for scanning circular microphone arrays,” IEEE transactions on audio, speech, and language processing, vol. 18, no. 8, pp. 2091–2098, 2010.
38. C. Schettker, L. Kobelt, and P.-O. Dehaye, “Golden ratio sequences for low-discrepancy sampling,” Journal of Graphics Tools, vol. 16, no. 2, pp. 95–104, 2012.
39. S. Agaian and Y. Zhou, “Generalized phi number system and its applications for image decomposition and enhancement,” SPIE Electronic Images, vol. 7881, 2011.
40. M. Livio, The Golden Ratio. Broadway Books, 2002.
41. A. Stakhov, “The golden section in the measurement theory,” Computers and Mathematics with Applications, vol. 17, no. 4-6, pp. 613–38, 1989.
42. R. Herz-Fischler, A Mathematical History of Division in Extreme and Mean Ratio. Wilfrid Laurier University Press, 1987.
43. M. Sigalotti, “The golden ratio in special relativity,” Chaos, Solitons and Fractals, vol. 30, no. 3, pp. 521–524, 2006.
44. S. Lipovetsky and F. Lootsma, “Generalized golden sections, repeated bisections and aesthetic pleasure,” European Journal of Operational Research, vol. 121, no. 1, pp. 213–16, 2000.
45. T. Heath, Euclid’s Elements. Green Lion Press, 2002.
46. D. Fowler, “A generalization of the golden section,” The Fibonacci Quarterly, vol. 20, no. 2, pp. 146–58, 1982.
47. V. Spinadel, The Metallic Means and Design. Nexus II: Architecture and Mathematics, 1998.
48. ——, “The metallic means family and multifractal spectra,” Nonlinear Analysis, vol. 36, pp. 721–45, 1999.
49. ——, “A new family of irrational numbers with curious properties,” Computers and Mathematics with Applications, vol. 32, no. 2, pp. 95–8, 2004.
50. T. Koshy, Fibonacci and Lucas Numbers with Applications. Wiley, 2001.
51. V. Krcadinac, “A new generalization of the golden ratio,” The Fibonacci Quarterly, vol. 44, no. 4, pp. 335–40, 2006.
52. M. Rakoczevic, “Further generalization of golden mean in relation to el naschie’s Zs divine Zs equation,” FME Transactions, vol. 32, no. 2, pp. 95–8, 2004.
53. T. Koshy, Fibonacci and Lucas Numbers with Applications. Wiley, 2001.
54. S. Hashemiparast and O. Hashemiparast, “Multi parameters golden ratio and some applications,” Applied Mathematics, vol. 2, pp. 808–15, 2011.
55. M. de Villiers, “A fibonacci generalization and its dual,” International Journal of Mathematical Education, vol. 31, no. 3, pp. 447–77, 2000.
56. S. Falcon, “The k-fibonacci sequence and the pascal 2-triangle,” *Chaos, Solitons and Fractals*, vol. 33, pp. 38–49, 2006.
57. M. Bicknell, “A primer on the pell sequence and related sequences,” *The Fibonacci Quarterly*, vol. 13, no. 4, pp. 345–49, 1975.
58. C. Harman, “Complex fibonacci numbers,” *The Fibonacci Quarterly*, vol. 19, no. 1, pp. 82–86, 1981.
59. A. Olariu, “Golden section and the art of painting,” *National Institute for Physics and Nuclear Engineering*, pp. 1–4, 1999.
60. W. A. Woodward, H. L. Gray, and A. C. Elliott, *Applied Time Series Analysis with R*. CRC press, 2017.
61. B. R. Marshall, R. H. Cahan, and J. M. Cahan, “Does intraday technical analysis in the us equity market have value?” *Journal of Empirical Finance*, vol. 15, no. 2, pp. 199–210, 2008.
62. Y. Zhu and G. Zhou, “Technical analysis: An asset allocation perspective on the use of moving averages,” *Journal of Financial Economics*, vol. 92, no. 3, pp. 519–544, 2009.
63. Y. Han, K. Yang, and G. Zhou, “A new anomaly: The cross-sectional profitability of technical analysis,” *Journal of Financial and Quantitative Analysis*, vol. 48, no. 5, pp. 1433–1461, 2013.
64. A. Dutta, G. Bandopadhyay, and S. Sengupta, “Prediction of stock performance in indian stock market using logistic regression,” *International Journal of Business and Information*, vol. 7, no. 1, 2015.
65. S. Thawornwong and D. Enke, “The adaptive selection of financial and economic variables for use with artificial neural networks,” *Neurocomputing*, vol. 56, pp. 205–232, 2004.
66. D. Enke and S. Thawornwong, “The use of data mining and neural networks for forecasting stock market returns,” *Expert Systems with applications*, vol. 29, no. 4, pp. 927–940, 2005.
67. C.-F. Tsai, Y.-C. Lin, D. C. Yen, and Y.-M. Chen, “Predicting stock returns by classifier ensembles,” *Applied Soft Computing*, vol. 11, no. 2, pp. 2452–2459, 2011.
68. M.-C. Lee, “Using support vector machine with a hybrid feature selection method to the stock trend prediction,” *Expert Systems with Applications*, vol. 36, no. 8, pp. 10896–10904, 2009.
69. N. Ren, M. Zargham, and S. Rahimi, “A decision tree-based classification approach to rule extraction for security analysis,” *International Journal of Information Technology & Decision Making*, vol. 5, no. 01, pp. 227–240, 2006.
70. R. K. Lai, C.-Y. Fan, W.-H. Huang, and P.-C. Chang, “Evolving and clustering fuzzy decision tree for financial time series data forecasting,” *Expert Systems with Applications*, vol. 36, no. 2, pp. 3761–3773, 2009.
71. I. A. El-Khodary, “A decision support system for technical analysis of financial markets based on the moving average crossover,” *World Applied Sciences Journal*, vol. 6, no. 11, pp. 1457–1472, 2009.
72. T. N. Rollinger and S. T. Hoffman, “Sortino: A ‘sharper’ ratio,” *Chicago, IL: Red Rock Capital. http://www.redrockcapital.com/assets/RedRock_Sortino_white_paper.pdf*, 2013.