ZETA FUNCTIONS AND TOPOLOGY OF HEISENBERG CYCLES FOR LINEAR ERGODIC FLOWS

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ABSTRACT. Placing a Dirac-Schrödinger operator along the orbit of a flow on a compact manifold \( M \) defines an \( \mathbb{R} \)-equivariant spectral triple over the algebra of smooth functions on \( M \). We study some of the properties of these triples, especially their zeta functions, which have the form \( \text{Trace}(fH^{-s}) \) with \( f \) the restriction to \( \mathbb{R} \) of a function on \( M \) and \( H = \frac{-\partial^2}{\partial x^2} + x^2 \) the harmonic oscillator. The meromorphic continuation property and pole structure of these zeta functions is related to ergodic time averages in dynamics. The construction reproduces the ‘Heisenberg cycles’ of Lesch and Moscovici, in the case of the periodic flow on the circle, where it produces a spectral triple over the smooth irrational torus in the irrational rotation algebra \( A_\hbar \). We strengthen a result of these authors, showing that the zeta function \( \text{Trace}(aH^{-s}) \) extend meromorphically for any element \( a \) of the \( \mathbb{C}^* \)-algebra \( A_\hbar \). Another variant of the construction produces a spectral cycle for \( A_\hbar \otimes A_1/\hbar \) and a spectral triple over a suitable subalgebra with the meromorphic continuation property if \( \hbar \) satisfies a Diophantine condition. The class of this cycle defines a fundamental class in the sense that it determines a KK-duality. We employ the Local Index Theorem of Connes and Moscovici in order to elaborate an index theorem of Connes for certain classes of differential operators on the line and compute the intersection form on K-theory induced by the fundamental class.

1. INTRODUCTION

The irrational rotation algebra \( A_\hbar := C(\mathbb{T}) \rtimes_\hbar \mathbb{Z} \), the crossed product of \( C(\mathbb{T}) = C(\mathbb{R}/\mathbb{Z}) \) by the action of \( \mathbb{Z} \) by translation by \( \hbar \in \mathbb{R} \setminus \mathbb{Q} \) mod \( \mathbb{Z} \) on \( \mathbb{T} \), is one of the key motivating examples in Noncommutative Geometry. Early results of Connes and Rieffel classified finitely generated projective modules over \( A_\hbar \) or over its natural Schwartz subalgebra \( A^\infty_\hbar \), by an analogue of the first Chern number of a line bundle over \( \mathbb{T}^2 \), defined for \( e \in A^\infty_\hbar \subset A_\hbar \), by

\[
c_1(e) := \frac{1}{2\pi i} \cdot \tau(e[\delta_1(e), \delta_2(e)]),
\]

where \( \delta_1, \delta_2 \) are the derivations of \( A_\hbar \) generating the natural \( \mathbb{R}^2 \)-action, and \( \tau \) is the trace. In fact these numbers are integers, a fact related to the Quantum Hall effect in solid state physics.

The reason for the integrality lies in the following. The densely defined operators \( \partial_t, \partial_t \) on \( L^2(\mathbb{T}^2) \) assemble to the operator

\[
\tilde{\mathcal{D}} := \begin{bmatrix}
0 & \frac{\partial}{\partial x} - i\frac{\partial}{\partial y} \\
\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} & 0
\end{bmatrix}.
\]

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on $L^2(T^2) \oplus L^2(T^2)$, and the representation of $C(T^2)$ on $L^2(T^2)$ by multiplication operators can be adjusted by introducing phase factors to give a representation $\lambda_\mathbb{H}: A_\mathbb{H} \rightarrow \mathbb{B}(L^2(T^2))$ which makes the triple $(L^2(T^2) \oplus L^2(T^2), \lambda_\mathbb{H}, \bar{\partial})$ a $2$-summable spectral triple over $A_\mathbb{R}^\mathbb{H}$ whose Chern character may be computed using the Local Index Formula of Connes and Moscovici to be the class of the cyclic cocycle

\begin{equation}
(1.1) \quad \tau_2(a^0, a^1, a^2) = \tau(a^0\delta_1(a^1)\delta_2(a^2) - a^0\delta_2(a^1)\delta_1(a^2)), \quad a^0, a^1, a^2 \in A_\mathbb{R}.
\end{equation}

The integrality of the Chern numbers $\tau_2(e, e, e)$ follows from the Connes-Moscovici Index Theorem which implies that for any idempotent $e \in A_\mathbb{R}^\mathbb{H}$,

$$c_1(e) = \tau_2(e, e, e) = \langle [e], [\bar{\partial}] \rangle \in \mathbb{Z},$$

where the right hand side is the pairing between K-theory and K-homology. But it is a result going back to early direct computations of Connes \cite{3} involving in particular a calculation of the cyclic cohomology of $A_\mathbb{R}^\mathbb{H}$.

In this article, we study a slightly different method of constructing spectral triples, using the operators $x \pm d/dx$, the annihilation and creation operators of quantum mechanics. They assemble to form a spectral triple over a suitable smooth subalgebra of $C_\alpha(\mathbb{R}) \rtimes \mathbb{R}_d$, where $C_\alpha(\mathbb{R})$ is the C*-algebra of uniformly continuous, bounded functions on $\mathbb{R}$ and $\mathbb{R}_d$ is the group of real numbers with the discrete topology. The operator of the triple is $D = \begin{bmatrix} 0 & x - d/dx \\ x + d/dx & 0 \end{bmatrix}$, whose closure is self-adjoint. The operator $D$ commutes mod bounded operators with group translations and smooth bounded functions on $\mathbb{R}$ with bounded derivatives, and $D^2$ is essentially the direct sum of two copies of the harmonic oscillator

$$H = -\frac{d^2}{dx^2} + x^2$$

on $\mathbb{R}$, which has discrete spectrum consisting of the odd positive integers. The representation $\pi$ of $C_\alpha(\mathbb{R}) \rtimes \mathbb{R}_d$ on $L^2(\mathbb{R})$ lets $f \in C_\alpha(\mathbb{R})$ act by the corresponding multiplication operator $(f\xi)(x) = f(x)\xi(x)$, and a group element $t \in \mathbb{R}_d$ by the group translation unitary operator $(u_t\xi)(x) = \xi(x-t)$.

The triple just describes gives a spectral (unbounded) cycle for $KK_0(C_\alpha(\mathbb{R}) \rtimes \mathbb{R}_d, \mathbb{C})$. We call it the Heisenberg cycle.

The Heisenberg cycle pulls back to any C*-subalgebra of $C_\alpha(\mathbb{R}) \rtimes \mathbb{R}_d$, and in this article we are most interested in subalgebras arising from ergodic flows. If $\alpha$ is a smooth flow on a compact manifold $M$. If $p \in M$ the function $f_p(t) := f(\alpha_t(p))$ is uniformly continuous on $\mathbb{R}$ if $f$ is continuous. It follows that restriction to an orbit defines an embedding $B_\alpha := C(M) \rtimes_\alpha \mathbb{R}_d \subset C_\alpha(\mathbb{R}) \rtimes \mathbb{R}_d$. So one can associate a Heisenberg cycle to any smooth flow and corresponding class $[B_\alpha] \in KK_0(C(M) \rtimes \mathbb{R}_d, \mathbb{C})$ or, in $KK_0(C(M) \rtimes \Lambda, \mathbb{C})$, if one has a subgroup $\Lambda \subset \mathbb{R}_d$ of particular interest for the context.

If one takes the trivial subgroup, then the class in $KK_0(C(M), \mathbb{C})$ is equal to the class in K-homology of the point $p \in M$, and so contains no interesting topological information (this follows from constructing a certain homotopy in KK, see \cite{5}). However, simple examples show that for certain natural (nontrivial) choices of subgroup, one obtains a great deal of topological information about the crossed-products.

In the case of the periodic flow on $\mathbb{T}$ and $\Lambda = \mathbb{Z} \mathbb{H}$, where $\mathbb{H}$ is irrational, the C*-algebra $B_\alpha$ is $C(\mathbb{T}) \rtimes \mathbb{R}_d$ which contains the irrational rotation algebra $A_\mathbb{H} = C(\mathbb{T}) \rtimes \mathbb{H}$ by restricting to the subgroup $\mathbb{H} \mathbb{Z} \subset \mathbb{R}_d$. This is the irrational rotation algebra $A_\mathbb{H}$. The Heisenberg cycle for this algebra, has been studied by Connes \cite{2}, \cite{3}, and Moscovici and Lesch \cite{13}. 


The latter authors refer to Heisenberg modules. One can build a Heisenberg module by twisting the Dirac-Dolbeault cycle of Connes by a Morita bimodule; such bimodules come from compact transversals to the Krönecker flow. One obtains thus a family of such cycles (for $A_h$) all having a somewhat similar form, and involving the Dirac-Schrödinger operators $x \pm d/dx$ on $L^2(\mathbb{R})$, or a finite sum of copies of $L^2(\mathbb{R})$.

Our Heisenberg cycles are defined over a much larger algebra $C_u(\mathbb{R}) \rtimes \mathbb{R}_d$ than $A_h$. One gets cycles for $C(M) \rtimes \Lambda$ for flows on manifolds $M$, and so in principal can be used to find topological invariants of flows. We restrict ourselves in this note to looking at Krönecker flow. For the subgroup $\Lambda \subset \mathbb{R}_d$ generated by $1, \hbar$, the crossed-product $B_\hbar := C(\mathbb{T}^2) \rtimes \Lambda$ is isomorphic to $A_h \otimes A_{1/\hbar}$. In the second part of the paper we compute some topological invariants of these Heisenberg cycles, using the Local Index Theorem of Connes and Moscovici and N. Higson’s exposition of it in [10]. The Heisenberg cycle for $A_h \otimes A_{1/\hbar} = B_\hbar$ induces a KK-duality between $A_h$ and $A_{1/\hbar}$, and we compute the index pairing $K_0(A_h) \times K_0(A_{1/\hbar}) \to \mathbb{Z}$ and show that it has matrix

$$\begin{bmatrix} 1 & -[1/\hbar] \\ -[\hbar] & 1 \end{bmatrix}$$

with respect to the bases consisting of the unit and the Rieffel projections. This strengthens an index calculation of Connes in [13] for classes of differential operators on the real line. This is based on our computation of the Chern character of the Heisenberg cycle over $A_h$, which we show is given by the mixed degree cyclic cochain

$$\tau - \hbar \tau_2,$$

where $\tau_2$ is as in [1.1].

The main technical contribution of this note concerns the meromorphic extension problem of the zeta functions

$$\zeta(a, s) := \text{Trace}(a H^{-s})$$

for $a \in C_u(\mathbb{R}) \rtimes \mathbb{R}_d$. Establishing such meromorphic extensions is necessary to apply the Local Index Theorem, at least in the presentation [10], as the cyclic cocycles involved in the local Chern character formula are obtained as poles of such zeta functions.

The meromorphic extension property in the classical situation, asserts that if $\Delta_M$ is the Laplacian on a compact manifold, and $f \in C^\infty(M)$, then $\text{Trace}(f \Delta_M^{-s})$ extends meromorphically to $\mathbb{C}$, with certain poles; it is proved by the theory of asymptotic expansions, specifically of the kernel of $f e^{-t \Delta_M}$, because the Mellin transform transforms the meromorphic extension problem into a problem about the asymptotics of the heat kernel as $t \to 0$. Such asymptotic expansions are also available for the situation of the Schwartz algebra of the irrational rotation algebra $A_h$, as noted by [13], who used them to deduce the meromorphic extendibility of $\zeta(a H^{-s})$ for $a \in A_h^*$ in the smooth irrational torus. In fact we show that $\zeta(a, s)$ meromorphically extends for $a$ in the $C^\ast$-algebra $A_h$, and that more generally, the zeta functions for flows appear to be related to ergodic time averages in dynamics. If $\alpha$ is a smooth ergodic flow on $M$ then we show that

$$\lim_{s \to 1^+} (s - 1) \cdot \text{Trace}(f \mu H^{-s}) = \int_M f \, d\mu$$

for a.e. $p \in M$, $f \in C(M)$, and $\mu$ any $\alpha$-invariant measure. Therefore, the residue trace, defined spectrally, recovers the invariant measure $\mu$. The proof is based on an integral formula for $\text{Trace}(f H^{-s})$; in fact, as we show more generally that if $f \in C_u(\mathbb{R})$ and if $\lim_{r \to +\infty} \frac{1}{r} \int_0^r f(t) \, dt$ exists, then the limit equals $\lim_{s \to 1^+} (s - 1) \cdot \text{Trace}(f \mu H^{-s})$ of $f \mu$. The result then follows from the Birkhoff Ergodic Theorem.
The meromorphic extension property from this point of view, for a given smooth flow, requires a strengthening of the Birkhoff Ergodic theorem for that situation which gives a finer estimate for the deviation \( \int_0^T f(t)dt\mu - T \int_M f d\mu \). From our integral formula for the zeta function, it is apparent that the meromorphic extension property of \( \text{Trace}(f_\rho H^{-s}) \) would follow, for example, for any \( f \in C^\infty(M) \), if one was guaranteed smooth solvability of the cohomological equation \( Xu = f \) for a smooth flow with generating vector field \( X \).

The condition \( \int_M f d\mu = 0 \) of \( f \) is an obvious obstruction to \( Xu = f \) being continuously solvable for any \( \alpha \)-invariant \( \mu \). For the standard periodic flow on the circle, this is the only obstruction. This is because if \( f \) is continuous and \( p \)-periodic and \( \int_0^T f d\mu = 0 \) then the antiderivative \( F(T):=\int_0^T f(t)dt \) is also \( p \)-periodic, so \( F \) solves the equation continuously. As we show, then the zeta function can be meromorphically extended to \( \text{Re}(s) > 1 - \frac{2}{d} \) by solving the equation \( n \) times, and so meromorphically extended to \( \mathbb{C} \). For the Kröner flow on \( \mathbb{T}^2 \), the cohomological equation \( Xu = f \) is smoothly solvable for smooth \( f \) of zero Lebesgue mean if \( \alpha \) satisfies a Diophantine condition, and it follows that \( \text{Trace}(f_\rho H^{-s}) \) extends meromorphically to \( \mathbb{C} \) with a simple pole at \( s = 1 \) in this case as well, if \( f \) is smooth. As observed in [8] a more refined statement is possible. The passage from \( f \) to \( u \) in solving the cohomological equation involves a specific loss of Sobolev regularity related to the Diophantine constant. Hence if \( f \) lies in a sufficiently high Sobolev space for \( \mathbb{T}^2 \), then \( \text{Trace}(f_\rho H^{-s}) \) can be meromorphically extended a certain finite distance.

The issue of estimating deviations from ergodic averages is a research topic of significant activity (see [9], [10]). It would be interesting to see if the meromorphic extension property for the Heisenberg cycles holds for more general flows.

2. Spectral cycles from the canonical anti-commutation relations

The Heisenberg group \( H = \{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \mid x,y,z \in \mathbb{R} \} \) has Lie algebra \( \mathfrak{h} \) the 3-by-3 strictly upper triangular matrices under matrix commutator. Let \( X, Y \) be the elements

\[
X = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},
\]

of \( \mathfrak{h} \). Then

\[
[X,Y] = Z := \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},
\]

while \( Z \) is central in \( \mathfrak{h} \). It follows that if \( \pi \) is any irreducible representation of \( H \), \( \pi(Z) = \pi([X,Y]) = [\pi(X),\pi(Y)] \) is a multiple of the identity operator:

\[
[\pi(X),\pi(Y)] = \hbar,
\]

for some \( \hbar \in \mathbb{R} \), a ‘Planck constant.’

The name Heisenberg group originates in these relations, which have the same form as the canonical commutation relations in quantum mechanics, where \( x \) and \( \frac{\hbar}{i\hbar} \) model position and momentum operators.

From the above remarks, we obtain a classification of irreducible representations of \( H \). Either \( \hbar = 0 \), in which case \( \pi(Z) = 0 \) and hence \( \pi(X) \) and \( \pi(Y) \) commute, which implies the representation is 1-dimensional, and is completely determined by the pair of real numbers...
(\pi(X), \pi(Y))$, or $\hbar \neq 0$, in which case one can show that the representation is isomorphic to the following interesting representation $\pi_\hbar$ of $\mathfrak{h}$ by unbounded operators on $L^2(\mathbb{R})$. Let

$$\pi_\hbar(x) = x, \quad \pi_\hbar(\hbar \frac{d}{dx}) = \hbar \frac{d}{dx}.$$ 

Then $[\hbar, \hbar \frac{d}{dx}] = \hbar$, so the required identity is satisfied to give a representation.

Application of functional calculus to the operators $x$ and $\hbar \frac{d}{dx}$ produces the operators

$$u = e^{2\pi i x}, \quad v_\hbar := e^{-\hbar \frac{d}{dx}},$$

where $u$ is multiplication by the periodic function $e^{2\pi i x}$ and

$$\left(v_\hbar \xi \right)(x) = \xi(x - \hbar).$$

We have

$$uv_\hbar = e^{-2\pi i \hbar} v_\hbar u.$$ 

If $\hbar \in \mathbb{R} \setminus \mathbb{Q}$ then the rational rotation algebra is the C*-algebra

$$\mathcal{A}_\hbar := C(\mathbb{T}) \rtimes_\hbar \mathbb{Z},$$

where $\mathbb{Z}$ acts on the circle $\mathbb{T} := \mathbb{R} / \mathbb{Z}$ with generator the automorphism induced by translation by $\hbar \mod \mathbb{Z}$. If $U \in C(\mathbb{T}) \rtimes_\hbar \mathbb{Z}$ is the generator $U(t) = e^{2\pi i t}$ of $C(\mathbb{T})$ and $V$ the generator of the $\mathbb{Z}$ action in the crossed-product, then a quick computation shows that

$$UV = e^{-2\pi i \hbar} VU \in \mathcal{A}_\hbar,$$

and it follows that we obtain, for each $\hbar$, rational or not, a representation

$$\pi_\hbar : \mathcal{A}_\hbar \to \mathbb{B}(L^2(\mathbb{R}))$$

of $\mathcal{A}_\hbar$ on $L^2(\mathbb{R})$. Note that $\pi_\hbar$ depends on $\hbar$ as a real number, while $\mathcal{A}_\hbar$ only depends on the class of $\hbar \mod \mathbb{Z}$.

We are going to fit these representations into a spectral cycle for $\text{KK}_0(\mathcal{A}_\hbar, C)$, using the properties of the harmonic oscillator

$$(2.1) \quad H := -\frac{d^2}{dx^2} + x^2,$$

a second-order elliptic operator on $\mathbb{R}$, whose domain we will take initially to be the Schwartz space $\mathcal{S}(\mathbb{R})$. Actually, the construction is more general, and produces a spectral cycle for $C_0(\mathbb{R}) \rtimes \mathbb{R}_d$, with $\mathbb{R}_d$ denoting $\mathbb{R}$ with the discrete topology.

Let $A = x + \frac{d}{dx}$, initial domain the Schwartz space $\mathcal{S}(\mathbb{R})$, and $A^* = x - \frac{d}{dx}$. The relations

$$(2.2) \quad AA^* = H + 1, \quad A^*A = H - 1, \quad [A,A^*] = 2, \quad [H,A] = -2A, \quad [H,A^*] = 2A^*.$$

hold as operators on $\mathcal{S}$. (See [15].)

Now set $\psi_0 := \pi^{-\frac{1}{2}} e^{-\frac{x^2}{2}} \in L^2(\mathbb{R})$. In quantum mechanics, $\psi_0$ is called the ground state, and the states inductively defined by $\psi_k := (2k)^{-\frac{1}{2}} \cdot A^* \psi_{k-1}$ the ‘excited states’. Observe that due to $HA^* = A^*H + 2A^*$, from (2.2), we see by induction that $\psi_k$ is a unit-length eigenvector of $H$ with eigenvalue $2k + 1$:

$$(2.3) \quad H\psi_k = (2k)^{-\frac{1}{2}} \cdot HA^* \psi_{k-1} = (2k)^{-\frac{1}{2}} \cdot (A^*H + 2A^*) \psi_{k-1} = (2k)^{-\frac{1}{2}} \cdot ((2k - 1) \cdot A^* \psi_{k-1} + 2A^* \psi_{k-1}) = (2k + 1) \cdot \psi_k.$$

It follows from $[H,A] = -2A$ that

$$A \psi_k = \sqrt{2k} \cdot \psi_{k-1}, \quad A^* \psi_k = \sqrt{2k + 2} \cdot \psi_{k-1}.$$
The eigenvectors of $H$ are given by $\hat{\xi}_k = H_k(x)e^{-x^2/2}$ where $H_k$ is the $k$th Hermite polynomial. This follows from induction using the recurrence

$$H_k(x) = (2k)^{-\frac{1}{2}} (2xH_{k-1}(x) - H'_{k-1}(x))$$

to define the polynomials.

The vectors $\{\psi_k\}$ form an orthonormal basis for $L^2(\mathbb{R})$ by the Stone-Weierstrass Theorem, and each $\psi_k$ is in the Schwartz class $S(\mathbb{R})$.

With respect to this basis, $H$ is diagonal with eigenvalues the odd integers $1, 3, 5, \ldots:$

$$H = \begin{bmatrix} 1 & 0 & 0 & \ldots \\ 0 & 3 & 0 & \ldots \\ 0 & 0 & 5 & \ldots \\ \vdots & \vdots & \vdots & \cdots \end{bmatrix}.$$

In particular, $H$ has a canonical extension to a self-adjoint operator on $L^2(\mathbb{R})$, and $f(H)$ is a compact operator for all $f \in C_0(\mathbb{R})$, and a bounded operator for all $f \in C_b(\mathbb{R})$.

If $f \in L^2(\mathbb{R})$, let $(\hat{f}(n))$ denote the sequence of its Fourier coefficients with respect to the spectral decomposition of $L^2(\mathbb{R})$ into eigenspaces of $H$ discussed above.

**Lemma 2.1.** If $f \in L^2(\mathbb{R})$, then $f \in S$ if and only if $(\hat{f}(n))$ is a rapidly decreasing sequence of integers:

$$|\hat{f}(n)| = O(n^{-k})$$

for any $k$.

The proof is routine, see [15].

Let $D$ be the unbounded operator

$$D = \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix}$$

on $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$, defined initially on Schwartz functions; it admits a canonical extension to a densely defined self-adjoint operator on $L^2(\mathbb{R})$. Since $D^2 = \begin{bmatrix} H - 1 & 0 \\ 0 & H + 1 \end{bmatrix}$, $1 + D^2 = \begin{bmatrix} H & 0 \\ 0 & H + 2 \end{bmatrix}$, which is now diagonal with respect to the basis described above, and invertible as an unbounded operator.

If $f \in C_c^0(\mathbb{R})$ is a smooth bounded function with bounded first derivative, acting by a multiplication operator on $L^2(\mathbb{R})$, then the commutator $[f, D] = \begin{bmatrix} 0 & -f' \\ f' & 0 \end{bmatrix}$ is a bounded operator. Let

$$C_u(\mathbb{R}) := \{ f \in C_b(\mathbb{R}) \mid f \text{ is uniformly continuous} \}$$

be the $C^*$-algebra of bounded uniformly continuous functions on $\mathbb{R}$. The group $\mathbb{R}_d$ of real numbers with the discrete topology, acts on $C_u(\mathbb{R})$. Let $\pi : C_u(\mathbb{R}) \times \mathbb{R}_d \rightarrow \mathbb{B}(L^2(\mathbb{R}))$ the representation of $C_u(\mathbb{R})$ by multiplication operators and $\mathbb{R}$ by translations.

**Proposition 2.2.** The triple

$$\left( L^2(\mathbb{R}) \oplus L^2(\mathbb{R}), \pi \oplus \pi, \ D = \begin{bmatrix} 0 & A^* \\ A & 0 \end{bmatrix} \right)$$

is a spectral triple over $C_u(\mathbb{R})[\mathbb{R}^d] \subset C_u(\mathbb{R}) \times \mathbb{R}_d$; it is 2-dimensional in the sense that $|D|^{-2} \in L^{1,\infty}$. 
We refer to the cycle above as the *Heisenberg cycle.*

As \( C^\infty_u(\mathbb{R})[\mathbb{R}_d] \) is dense in \( C_u(\mathbb{R}) \times \mathbb{R}_d \), the Proposition implies that the associated Fredholm module

\[
L^2(\mathbb{R}) \oplus L^2(\mathbb{R}), \pi \oplus \pi, \quad F := \chi(D) = \begin{bmatrix} 0 & A^*(H+2)^{-\frac{1}{2}} \\ A H^{-\frac{1}{2}} & 0 \end{bmatrix}
\]

obtained by applying a normalizing function \( \chi \), here chosen to be \( \chi(x) = x(1+x^2)^{-\frac{1}{2}} \), defines a cycle for \( \text{KK}_0(C_u(\mathbb{R}) \times \mathbb{R}_d, \mathbb{C}) \), because \( [\pi(a),AH^{-\frac{1}{2}}] \) is a compact operator for \( a \in C_u(\mathbb{R}) \times \mathbb{R}_d \) by standard functional calculus arguments (see [11], [12]) due to \( [\pi(a),A] \) being bounded for dense \( a \).

The corresponding class in \( \text{KK}_0(C_u(\mathbb{R}) \times \mathbb{R}_d, \mathbb{C}) \) is non-zero: it has index +1.

Since the spectrum of \( H \) grows linearly, \( [\pi(a)H^{-t}] \) is trace-class for \( \text{Re}(s) > 1 \) and the zeta function \( \text{Trace}(\pi(a)H^{-s}) \) is holomorphic for \( \text{Re}(s) > 1 \) and \( a \in C_u(\mathbb{R}) \times \mathbb{R}_d \). One of our main interests is in the possible meromorphic continuation properties of such zeta functions.

The irrational rotation algebra \( A_h \) is a subalgebra of \( C_u(\mathbb{R}) \times \mathbb{R}_d \) and the restriction of the representation \( \pi \) above to \( A_h \) lets \( f \in C(\mathbb{T}) = C(\mathbb{R}/\mathbb{Z}) \) act by multiplication on \( L^2(\mathbb{R}) \) by the corresponding periodic function, and the group \( \mathbb{Z} \) by \( n \mapsto u_n h \), with, recall, \( u_n \xi(x) = \xi(x-t) \). However \( C_u(\mathbb{R}) \times \mathbb{R}_d \) contains numerous other subalgebras of related interest. We first point out a generic example of such a subalgebra, arising from dynamics.

**Lemma 2.3.** If \( M \) is a compact manifold and \( \{\alpha_t\}_{t \in \mathbb{R}} \) is a smooth flow on \( M \), then if \( p \in M \), then mapping \( f \in C(M) \) to the uniformly continuous function \( f_p(t) := f(\alpha_p) \) on \( \mathbb{R} \), and mapping \( t \in \mathbb{R}_d \) to \( u_t \), determine a \(*\)-algebra homomorphism

\[
\mu: C(M) \times_{\alpha} \mathbb{R}_d \to C_u(\mathbb{R}) \times \mathbb{R}_d
\]

where \( \mathbb{R}_d \) is the group of real numbers with the discrete topology. It is injective if the flow is minimal, and restricts to a \(*\)-algebra homomorphism \( C^\infty(M)[\mathbb{R}_d] \to C^\infty_u(\mathbb{R})[\mathbb{R}_d] \).

The proof is the observation that if the vector field \( X \) generates the flow, then \( f \in C^\infty(M) \) implies \( \frac{d}{dt} f_p(t) = X(f)(p(t)) \) which is bounded in \( t \) so \( f_p \) is uniformly continuous on \( \mathbb{R} \). In particular the Heisenberg cycle pulls back to a cycle for \( C(M) \times \mathbb{R}_d \), and a spectral triple over \( C^\infty(M)[\mathbb{R}_d] \).

Returning to irrational rotation, fix \( h \in \mathbb{R} \), so that we have the representation \( \pi_h \) of \( A_h \) determined by \( \pi_h(\pi) = C(\mathbb{T}) = C(\mathbb{R}/\mathbb{Z}) \) acting by multiplication operators by \( \mathbb{Z}-\)periodic functions, \( n \in \mathbb{Z} \) by \( u_n h \).

**Lemma 2.4.** Let \( \pi^h: A_{1/h} \to C(\mathbb{R}/\mathbb{Z}) \) be the representation obtained by letting \( f \in C(\mathbb{T}) = C(\mathbb{R}/\mathbb{Z}) \) act by multiplication by \( f(\frac{x}{h}) \) and \( n \in \mathbb{Z} \) by translation by \( n \).

Then \( \pi_h(A_h) \) and \( \pi^h(A_{1/h}) \) commute. The tensor product \( \rho_h(a \otimes b) := \pi_h(a) \pi^h(b) \) of the representations gives an representation of \( B_h \) on \( L^2(\mathbb{R}) \) factoring through

\[
\mu: B_h = A_h \otimes A_{1/h} \to C_u(\mathbb{R}) \times \mathbb{R}_d.
\]

The representation \( \rho \) is injective if \( h \in \mathbb{R} \setminus \mathbb{Q} \).

The \( C^\ast \)-algebra \( A_h \otimes A_{1/h} =: B_h \) is the crossed product of \( C(\mathbb{T}^2) \) by the group \( \mathbb{Z}^2 \) with action

\[
(n,m) \cdot (x,y) = (x+nh,y+\frac{m}{h}).
\]
and the homomorphism $\rho$ embeds $B_0$ into $C_\kappa(\mathbb{R}) \times \mathbb{R}_d$ by letting $f \in C(\mathbb{T}^2)$ map to $f_\kappa(t) := f(t, \frac{1}{\kappa})$, and embedding $\mathbb{Z}^2$ isomorphically to the dense subgroup
\[ \Lambda := \{ nh + m \mid n, m \in \mathbb{Z} \} \subset \mathbb{R}. \]
We can consider the $\mathbb{Z}^2$ action as factoring through the translation action of the subgroup $\Lambda$ acting through the Kronecker flow
\[ \alpha_r(x, y) = (x + r, y + \frac{r}{\kappa}) \]
along lines of slope $1/\kappa$, because
\[ \alpha_{nh+m}(x, y) = (x + nh, y + \frac{m}{\kappa}). \]

The restriction of $\rho$ to $C(\mathbb{T}^2)$ is thus a special case of Lemma \ref{lemma2.3} with

**Proposition 2.5.** Let $\gamma : \mathbb{R} \to \mathbb{T}^2$ be the group homomorphism $\gamma(t) = (ht, t)$, let $U$ be its image.

a) $U$ is dense if $\kappa \notin \mathbb{Q}$.

b) An element $(x, y) \in \mathbb{R}^2$ projects to an element of $U$ if and only if $hy = x + n + m\kappa$ for some integers $n, m$ if and only if $hy = x \mod \Lambda \subset \mathbb{R}$.

c) The subgroup $\Lambda$ is contained in $U$ for all integers $n, m$.

d) If $U' := \gamma(\mathbb{R})$ with $\gamma(t) = (t, \kappa t)$ then $\Lambda = U \cap U'$.

**Proof.** a)-c) are routine. The (dense) subgroup $\Lambda \subset \mathbb{T}^2$ is obtained as follows. The line of slope $1/\kappa$ in $\mathbb{R}^2$ through the origin intersects the vertical lines $x = n$, for $n \in \mathbb{Z}$, in the points $\mathbb{Z}^2$-congruent to $(nh, 0)$, and through the horizontal lines $y = m$, in the points congruent to $(0, \frac{m}{\kappa})$. Summing all of these points in $\mathbb{T}^2$ gives $\Lambda$, which is contained in $U$ by c). The flip $\mathbb{R}^2 \to \mathbb{R}^2$ interchanges lines of slope $\kappa$ and of $1/\kappa$ and leaves $\mathbb{Z}^2$ invariant interchanging vertical and horizontal lines. The last statement follows from symmetry. \hfill $\Box$

The subgroup $\Lambda$ consists therefore of all points of $\mathbb{T}^2$ in the intersection of the two dense subgroups $U$ and $U'$, projections of lines of slope $\kappa$ and $1/\kappa$. We call $\Lambda$ the **homoclinic subgroup**.

**Definition 2.6.** The **Heisenberg bi-cycle** is the spectral triple over $C^\infty(\mathbb{T}^2)[A] \subset A_\kappa \otimes A_{1/\kappa}$ obtained by pulling back the Heisenberg cycle of Proposition \ref{proposition2.2} by the $*$-homomorphism $\mu : C(\mathbb{T}^2) \times \Lambda \to C_\kappa(\mathbb{R}) \times \mathbb{R}_d$ of Lemma \ref{lemma2.4} (restricted to $C^\infty(\mathbb{T}^2)[A]$).

The class of the Heisenberg cycle is denoted $\Delta_\kappa \in KK_0(A_\kappa \otimes A_{1/\kappa}, \mathbb{C})$ (see \ref{proposition2.4} with $\pi$ replaced by $\rho := \pi \circ \mu$).

In the next section we will show that the zeta functions $\text{Trace}(\rho(a)H^{-s})$ extend meromorphically to $\mathbb{C}$ for $a \in C^\infty(\mathbb{T}^2)[A] \subset C(\mathbb{T}^2) \times \mathbb{R}_d$, provided that $\kappa$ satisfies a Diophantine condition.

The inclusion $A_\kappa \to A_\kappa \otimes A_{1/\kappa}$ pulls the Heisenberg bi-cycle back to a spectral cycle for $KK_0(A_\kappa, \mathbb{C})$, which is a spectral triple over the smooth subalgebra $A^\infty_\kappa$. As it is of special interest to us, we single it out in a definition.

**Definition 2.7.** The **Heisenberg cycle** is the even, 2-dimensional spectral cycle
\[ \left( L^2(\mathbb{R}) \oplus L^2(\mathbb{R}), \pi_\kappa \oplus \pi_\kappa, D = \begin{bmatrix} 0 & A \nabla \\ A & 0 \end{bmatrix} \right), \]
for $KK_0(A_\kappa, \mathbb{C})$, defining a spectral triple over the Schwartz subalgebra $A^\infty_\kappa$ of the rotation algebra $A_\kappa := C(\mathbb{T}) \rtimes_\kappa \mathbb{Z}$. 

The class in $\text{KK}_0(C(\mathbb{T}) \rtimes \mathbb{Z}, \mathbb{C})$ of the Heisenberg cycle is denoted $[D_\hbar]$.

**Remark 2.8.** The injection $A_{1/\hbar}$ into $A_{\hbar} \otimes A_{1/\hbar}$ pulls the Heisenberg bi-cycle back to a cycle and class $[D^0]$ for $\text{KK}_0(A_{1/\hbar}, \mathbb{C})$. But the unitary $U : L^2(\mathbb{R}) \to L^2(\mathbb{R})$, $U \xi(x) = \sqrt{\hbar} \xi(\hbar x)$ conjugates the representation $\pi_{1/\hbar}$ to the representation $\pi^0$ (in notation of Lemma 2.4). This effects the operator by a homotopically trivial re-scaling, and hence $[D^0] = [D_{1/\hbar}] \in \text{KK}_0(A_{1/\hbar}, \mathbb{C})$.

If $0 < h < 1$ then the spectral triple describing $[D_\hbar]$ is studied in [13]. It is related in an exact way to Connes’ Dolbeault class (for any $\hbar$) as we now establish, although the result is already proved in [13] (for $0 < h < 1$, and in slightly different language). Let $L^2(A_{\hbar})$ denote the GNS Hilbert space associated to the trace $\tau : A_{\hbar} \to \mathbb{C}$. On $L^2(A_{\hbar})$ the derivations $\delta_1, \delta_2$ are defined $\delta_1(U) = 2\pi i U$, $\delta_1(V) = 0$, $\delta_2(U) = 0$, $\delta_2(V) = 2\pi i V$. Then the derivations assemble to give

$$\delta := \begin{bmatrix} 0 & \delta_1 - i \delta_2 \\ \delta_1 + i \delta_2 & 0 \end{bmatrix}.$$  

The GNS representation $\lambda : A_{\hbar} \to B( L^2(A_{\hbar}))$ then fits into a spectral triple over $A_{\hbar}^\tau$ with Hilbert space $L^2(A_{\hbar}) \oplus L^2(A_{\hbar})$ and representation $\lambda \oplus \lambda$.

We let $[\delta] \in \text{KK}_0(A_{\hbar}, \mathbb{C})$ be its class. Since $\tau$ is not just a state, but a trace, the right multiplication operation of $A_{\hbar}$ on itself determines another, commuting representation $\lambda^{op} : A_{\hbar}^{op} \to B( L^2(A_{\hbar}))$. As $A_{\hbar}$ is naturally isomorphic to its opposite algebra we obtain a pair of commuting representations of $A_{\hbar}$ on $L^2(A_{\hbar})$ and on $L^2(A_{\hbar}) \oplus L^2(A_{\hbar})$. These observations determines a cycle and class

$$\Delta_\delta \in \text{KK}_0(A_{\hbar} \otimes A_{\hbar}, \mathbb{C}).$$

Note that all of this only depends on the class of $\hbar$ mod $\mathbb{Z}$.

Connes proves that cup-cap product

$$PD_\delta : \text{KK}_* (D_1, A_{\hbar} \otimes D_2) \to \text{KK}_* (A_{\hbar} \otimes D_1, D_2)$$

(for any $D_i$) with $\Delta_\delta$ induces an isomorphism, that is, yields a self $\text{KK}$-duality for $A_{\hbar}$. The result is refined in [6].

**Lemma 2.9.** Define on $C_c(\mathbb{R})$ the inner products

$$\langle \xi, \eta \rangle_{C(\mathbb{R}/h\mathbb{Z}) \rtimes \mathbb{Z}}(x, m) = \sum_{n \in \mathbb{Z}} \xi(x - nh) \cdot \eta(x - nh - m), \ x \in \mathbb{R}/h\mathbb{Z}, \ m \in \mathbb{Z},$$

and

$$\langle \xi, \eta \rangle_{C(\mathbb{R}/h\mathbb{Z}) \rtimes \mathbb{Z}}(x, m) = \sum_{n \in \mathbb{Z}} \xi(x - n) \eta(x - n - mh), x \in \mathbb{R}/\mathbb{Z}, \ m \in \mathbb{Z}.$$  

Give $C_c(\mathbb{R})$ the $C(\mathbb{R}/h\mathbb{Z}) \rtimes \mathbb{Z} \cdot C(\mathbb{R}/\mathbb{Z}) \rtimes h\mathbb{Z}$ bimodule structure with

$$\langle n \xi \rangle(x) = \xi(x - n), \quad (f \xi)(x) = f(x) \xi(x), \quad \langle \xi n \rangle(x) = \xi(x + n h), \quad (\xi f)(x) = f(x) \xi(x).$$

Then $C_c(\mathbb{R})$ completes to a Morita equivalence $C(\mathbb{R}/h\mathbb{Z}) \rtimes \mathbb{Z} \cdot C(\mathbb{R}/\mathbb{Z}) \rtimes h\mathbb{Z}$ bimodule $\mathcal{E}_h$, that is, to a Morita equivalence $A_{1/\hbar} \rtimes h\mathbb{Z}$-bimodule.

The relation between Connes’ Dolbeault class $[\delta]$ and the Heisenberg $[D_\hbar]$ is based on the following simple relationship between modules.

**Lemma 2.10.** The tensor product of Hilbert modules $\mathcal{E}_h \otimes_{A_{\hbar}} L^2(A_{\hbar})$ over the representation $\lambda : A_{\hbar} \to B(L^2(A_{\hbar}))$, is naturally isomorphic to $L^2(\mathbb{R})$ as a Hilbert space.

Under this identification:
a) The representation $\lambda$ of $A_{1/\hbar}$ on $\mathcal{E}_h \otimes_{A_h} L^2(A_h)$ induced by its representation on $\mathcal{E}_h$ corresponds to the representation $\pi^h$ on $L^2(\mathbb{R})$ of Lemma 2.2.

b) The representation $\lambda^{op}$ of $A_h$ on $L^2(A_h)$ commutes with the representation $\lambda$ involved in the tensor product. Hence $A_h$ is also represented on $\mathcal{E}_h \otimes_{A_h} L^2(A_h)$ by $1 \otimes \lambda^{op}$. This representation identifies with $\pi_h$ on $L^2(\mathbb{R})$ of Lemma 2.2.

**Proof.** If $f_1, f_2 \in C_c(\mathbb{R})$, then their $A_h = C(\mathbb{R}/\mathbb{Z}) \rtimes_{\mathbb{Z}}$-valued inner product is given in the above Lemma. Let $\delta_0 \in L^2(A_h)$ the vector corresponding to $1 \in A_h$ and consider the elements $f_1 \otimes \delta_0 \in \mathcal{E}_h \otimes_{A_h} L^2(A_h)$. Their inner product is given by

$$
\langle f_1 \otimes \delta_0, f_2 \otimes \delta_0 \rangle = \langle \delta_0, (f_1, f_2)_{A_h} \delta_0 \rangle = \tau((f_1, f_2)) = \int_0^1 \langle f_1, f_2 \rangle_{A_h}(x, 0)dx
$$

where $\tau : A_h \to \mathbb{C}$ is the trace. It follows that $f \mapsto f \otimes \delta_0$ induces a Hilbert space isometry $L^2(\mathbb{R}) \to \mathcal{E}_h \otimes_{A_h} L^2(A_h)$. Since elements of the form $\xi \otimes \delta_0, \xi \in \mathcal{E}_h$, are dense in the tensor product (because the GNS representation is cyclic), this isometry is actually a unitary. The other statements are easy to check.

$$
\Box
$$

**Corollary 2.11.** Let $[\mathcal{E}_h] \in \text{KK}_0(A_{1/\hbar}, A_h)$ be the class of the Morita equivalence bimodule $\mathcal{E}_h, \Delta_h$ of the Heisenberg bi-cycle (Definition 2.6) and $\text{PD}_\bar{\theta}$ be Connes’ Poincaré duality (2.5). Then

a) $\text{PD}_\bar{\theta}([\mathcal{E}_h]) = [\Delta_h] \in \text{KK}_0(A_h \otimes A_{1/\hbar}, \mathbb{C})$.

b) The class $\Delta_h \in \text{KK}_0(A_h \otimes A_{1/\hbar}, \mathbb{C})$ determines a KK-duality between $A_h$ and $A_{1/\hbar}$.

c) If $[p_\hbar] \in K_0(A_h)$ denotes the class of the Rieffel projection then $\text{PD}_\bar{\theta}([p_\hbar]) = [\Delta_h]$.

**Proof.** $\text{PD}_\bar{\theta}([\mathcal{E}_h]) = (1_{A_h} \otimes [\mathcal{E}_h]) \otimes_{A_h \otimes A_h} \Delta_\bar{\theta} \in \text{KK}_0(A_h \otimes A_{1/\hbar}, \mathbb{C})$. By definition. The module composition involved in the Kasparov product results in (two copies of) $L^2(\mathbb{R})$ with (two copies of) the Heisenberg representation $p_\hbar$ of Theorem 2.4 by Lemma 2.10. The operator $D$ satisfies the connection condition for the axiomatic approach to the product by [11].

Let $\text{PD}_h$ denote the analogue of (2.5) using $\Delta_h$ in place of $\Delta_\bar{\theta}$. Then for $x \in K_* (A_{1/\hbar})$, $y \in K_* (A_h)$,

$$
\langle \text{PD}_h(x), y \rangle = \langle y \otimes_{\mathbb{C}} x, \Delta_h \rangle = \langle y \otimes_{\mathbb{C}} x, (1_{A_h} \otimes [\mathcal{E}_h]) \otimes_{A_h \otimes A_h} \Delta_\bar{\theta} \rangle = \langle y \otimes_{\mathbb{C}} \mathcal{E}_h^h(x), \Delta_\bar{\theta} \rangle.
$$

Since $[\mathcal{E}_h]$ is an equivalence in KK, the intersection form for $\Delta_h$ is obtained by twisting the form for $\Delta_\bar{\theta}$ by an isomorphism, and hence is non-degenerate, since Connes’ is.

By definition

$$
\langle [p_\hbar], [\Delta_h] \rangle = \langle [p_\hbar] \otimes 1_{A_h} \rangle \otimes_{A_h \otimes A_h} \Delta_\bar{\theta} = (u \otimes 1_{A_h})^* (\mathcal{E}_h \otimes 1_{A_h}) \otimes_{A_h \otimes A_h} \Delta_\bar{\theta}
$$

where $u : \mathbb{C} \to A_{1/\hbar}$ is the unital inclusion, where the non-trivial step was the penultimate one, which used c).

We are going to show using cyclic cohomology calculations that $\langle [p_\hbar], [\Delta_h] \rangle = - [\hbar]$.

This is enough to describe the intersection form induced by $\Delta_h$. We note the result for the record here.
Proposition 2.12. For any $h$ give $K_0(A_h)$ the ordered free abelian group basis $\{ [1], [p_h] \}$. Then the matrix of the intersection form induced by $\Delta_h$ is

$$
\begin{bmatrix}
1 & -[1/h] \\
-[1/h] & 1
\end{bmatrix}
$$

Proof. By the definitions $\langle \text{PD}_h([p_h]), [p_{1/h}] \rangle = \langle [p_h] \otimes \mathbb{C} [p_{1/h}], \Delta_h \rangle$. As noted above $\Delta_h = \text{PD}_h([\mathcal{E}_h]) := (1_{A_h} \otimes [\mathcal{E}_h]) \otimes_{A_h \otimes A_h} \Delta_\mathfrak{d}$ so this may be written

$$
\langle [p_h] \otimes \mathbb{C} [p_{1/h}], (1_{A_h} \otimes \mathbb{C} [\mathcal{E}_h]))^*(\Delta_\mathfrak{d})
$$

Moving $[\mathcal{E}_h]$ to the other side and noting that $[p_{1/h}] \otimes_{A_{1/h}} [\mathcal{E}_h] = [1] \in K_0(A_h)$ gives that

$$
(2.8) \quad \langle \text{PD}_h([p_h]), [p_{1/h}] \rangle = \langle [p_h] \otimes \mathbb{C} [p_{1/h}], \Delta_\mathfrak{d} \rangle = \langle \text{PD}_h([p_{1/h}]), [1] \rangle = \langle [D_h], [1] \rangle = 1.
$$

We end by the remark that Heisenberg cycles, over $C_u(\mathbb{R}) \rtimes \Gamma_d$, are only topologically interesting if $G$ is non-trivial. Lück and Rosenberg construct a homotopy in KK-theory by considering the operators $\lambda x + d/dx$ for $\lambda \in [1, \infty)$. This field can be continuously extended to $[1, \infty]$ by adding a copy of $\mathbb{C}$ to $L^2(\mathbb{R})$ at infinity, and extending the operator by the direct sum of the multiplication operator $x/|x|$ on $L^2(\mathbb{R})$, and 0 on the 1-dimensional summand. Their argument implies the following.

Proposition 2.13. The class in $KK_0(C_u(\mathbb{R}), \mathbb{C})$ of the Heisenberg cycle over $C_u(\mathbb{R})$, is equal to the class $[\text{ev}_0] \in KK_0(C_u(\mathbb{R}), \mathbb{C})$, of the point-evaluation homomorphism $C_u(\mathbb{R}) \to \mathbb{C}$, $f \mapsto f(0)$.

In particular, if $[D_\alpha]$ is the Heisenberg cycle for an ergodic flow on $M$, then $[D_\alpha]) = [\text{ev}_p] \in K_0(C(M), \mathbb{C})$.

This shows that it is essential to consider the crossed products $C_u(\mathbb{R}) \rtimes \Gamma$, for suitable non-trivial groups $G \subset \mathbb{R}$, in order to see interesting topological phenomena.

However, the geometry of the Heisenberg cycles is by contrast interesting, even without taking into account a group action, as we discuss in the next section.

## 3. Zeta functions and ergodic flows

Let $f \in C_u(\mathbb{R})$ be a bounded, uniformly continuous function.

We consider the zeta function $\text{Trace}(fH^{-s})$ where $H$ is the harmonic oscillator. (2.1), which is analytic for $\text{Re}(s) > 1$.

Theorem 3.1. If $f \in C_u(\mathbb{R})$ then the difference of analytic functions on $\text{Re}(s) > 1$

$$
(3.1) \quad \Gamma(s) \cdot \text{Tr}(fH^{-s}) - \frac{1}{2\sqrt{\pi}} \cdot \int_0^1 \int_{\mathbb{R}} t^{s-1} \text{csch} t \cdot f(x\sqrt{\coth t}) \cdot e^{-x^2} dx dt
$$
extends analytically to $\mathbb{C}$.

Remark 3.2. If $f = 1$ is constant (3.1) gives that

$$
\Gamma(s) \cdot \text{Trace}(H^{-s}) = \int_0^1 \int_{\mathbb{R}} t^{s-1} \text{csch} t dt
$$

up to an entire function. The Mellin transform of $\text{csch} t$ is $2(1 - 2^{-s})\Gamma(s)\zeta(s)$, where $\zeta$ is the Riemann zeta function. This is meromorphic on the whole complex plane; dividing by $\Gamma(s)$, we get a single simple pole at $s = 1$. The residue there is equal to 1.
Before proving the Theorem we discuss applications. If \( f \in C_u(\mathbb{R}) \), let \( F(T) = \int_0^T f(t) dt \). F is uniformly continuous, not necessarily bounded, but \(|F(T)| = O(T)| as \( T \to \infty \).

**Lemma 3.3.** If \( f \in C_u(\mathbb{R}) \) admits \( n \) successive bounded anti-derivatives, \( 1, f, f^2, \ldots, f^n \), then \( \text{Trace}(fH^{-s}) \) extends analytically to \( \text{Re}(s) > 1 - \frac{n}{2} \).

**Proof.** By Theorem 3.1

\[
(3.2) \quad \Gamma(s) \cdot \text{Tr}(fH^{-s}) \sim \frac{1}{2\sqrt{\pi}} \int_0^1 \int_{\mathbb{R}} t^{s-2} \text{csch} t \cdot \int_{\mathbb{R}} f(x\sqrt{\coth t}) \cdot e^{-x^2} dx dt
\]

where \( \sim \) means up to an entire function. Let \( F = f^1 \), then integration by parts gives

\[
(3.3) \quad = \frac{1}{\sqrt{\pi}} \int_0^1 \int_{\mathbb{R}} t^{s-2} \text{csch} t \sqrt{\tanh t} \cdot F(x\sqrt{\coth t}) xe^{-x^2} dx dt = \frac{1}{\sqrt{\pi}} \int_0^1 t^{s-2} \text{csch} t \sqrt{\tanh t} \cdot \phi(t) dt,
\]

with \( \phi(t) = \int_{\mathbb{R}} F(x\sqrt{\coth t}) xe^{-x^2} dx \). The function \( t^{s-2} \text{csch} t \sqrt{\tanh t} \cdot \phi(t) \) is \( \sim t^{s-3/2} \cdot \phi(t) \) as \( t \to 0 \), and is integrable over \([0, 1]\) for \( \text{Re}(s) > \frac{1}{2} \) if \( \phi \) is continuous and bounded as \( t \to 0 \). In particular this holds if \( F \) is bounded on \( \mathbb{R} \). So we have verified analyticity for \( \text{Re}(s) > \frac{3}{4} \). Repeating the argument, if \( 2f = F \) is the second anti-derivative then the previous expression can be written

\[
(3.4) \quad \frac{1}{\sqrt{\pi}} \int_0^1 t^{s-2} \text{csch} t \sqrt{\tanh t} \cdot \int_{\mathbb{R}} 2f(x\sqrt{\coth t}) (1 - 2x)e^{-x^2} dx dt
\]

which is analytic now for \( \text{Re}(s) > 0 \) if \( 2f \) is also bounded. One repeats this argument \( n \) times and the statement follows.

The **cohomological equation** in dynamics refers to the differential equation

\[ Xu = f \]

where \( X \) is a generating vector field for a smooth flow \( \alpha \) on a compact manifold \( M \). Let \( p \in M, f \in C(M) \) and \( f_p(t) = f(\alpha_t(p)) \).

Then \( f_p \in C_u(\mathbb{R}) \). If \( f \in C^\infty(M) \) then \( f_p \in C^\infty_u(\mathbb{R}) \).

An obstruction to solving the cohomological equation for given \( f \) with respect to any \( \alpha \)-invariant probability measure \( \mu \). This follows from differentiating the equation \( \int_M u \circ \alpha_t \ d\mu = \int_M u \ d\mu \), which gives that \( \int_M Xu \ d\mu = 0 \), that is, \( \int_M f \ d\mu = 0 \) if \( Xu = f \) has a solution.

Conversely, if one can solve \( Xu = f \) for given \( f \), then \( u_p(t) := u(\alpha_t(p)) \) supplies a bounded anti-derivative of \( f_p \). Hence Lemma 3.3 gives the following.

**Proposition 3.4.** Let \( \alpha \) be a smooth flow on \( M \) with generator \( X \) and \( \mu \) any \( \alpha \)-invariant measure. If \( f \in C^\infty(M) \) and \( \int_M f \ d\mu = 0 \) implies that \( Xu = f \) for some \( u \in C^\infty(M) \), then \( \text{Trace}(f_pH^{-s}) \) extends meromorphically to \( \mathbb{C} \) and

\[ \text{Res}_{s=1} \text{Trace}(f_pH^{-s}) = \int_M f \ d\mu \]

for any \( p \in M \).
In some simple situations of elliptic dynamics, e.g. the periodic flow on the circle, having mean zero is the only obstruction to solving the cohomological equation for \( f \): indeed, in this case one can make a extremely strong statement not even requiring smoothness.

**Lemma 3.5.** Let \( f \) be continuous and \( \rho \)-periodic on \( \mathbb{R} \) with zero mean: \( \int_0^\rho f(t)dt = 0 \). Then \( F(T) \) is also continuous, \( \rho \)-periodic, with zero mean.

**Corollary 3.6.** If \( f \) is continuous and \( \rho \)-periodic then \( \text{Trace}(fH^{-s}) \) meromorphically extends to \( \mathbb{C} \) with a simple pole at \( s = 1 \) and

\[
\text{Res}_{s=1} \text{Trace}(fH^{-s}) = \frac{1}{\rho} \int_0^\rho f(t)dt =: \mu(f).
\]

*Proof.* \( \tilde{f} := f - \mu(f) \) has zero mean. Applying the previous lemma gives that \( f \) has bounded anti-derivatives of all orders; the result follows from Lemma 3.3 and Remark 3.2.

**Definition 3.7.** Let \( \alpha \) be a smooth, ergodic Riemannian flow on a compact Riemannian manifold \( M \) (\( \alpha_t : M \to M \) is a Riemannian isometry for all \( t \)). Let \( \Delta \) be the Laplacian on \( M \), \( 0 = \lambda_0 < \lambda_1 < \cdots \), \( \mu \) normalized volume measure on \( M \) and \( L^2(M) = \oplus_{n=0}^\infty H_n \) the \( \Delta \)-spectral decomposition of \( L^2(M) \), \( H_n = \ker(\lambda_n - \Delta) \).

The vector field \( X \) commutes with \( \Delta \) as an operator on \( C^\infty(M) \) and so leaves each \( H_n \) invariant. For \( n > 0 \), \( \epsilon_n = \|X|_{H_n}\| \). Since \( \alpha \) is ergodic, the kernel of \( X \) consists of constant functions, and, moreover, \( H_0 \) is the space of constant functions.

A Riemannian flow \( \alpha \) satisfies a Diophantine condition if there exists \( C \geq 0 \) and \( \gamma > 0 \) such that \( \epsilon_n \geq Cn^{-\gamma} \).

**Corollary 3.8.** If \( \alpha \) is a smooth, Riemannian, ergodic flow on \( M \) satisfying a Diophantine condition, and \( f \in C^\infty(M) \) then \( \text{Trace}(f_\rho H^{-s}) \) extends meromorphically to \( \mathbb{C} \) with a simple pole at \( s = 1 \) and

\[
\text{Res}_{s=1} \text{Trace}(f_\rho H^{-s}) = \int_M f d\mu
\]

for any \( \alpha \)-invariant measure \( \mu \) and any \( \rho \in M \).

*Proof.* Proceeding as in the discussion above, let \( f \) be smooth on \( M \), then \( f \in L^2(M) \) and \( f = \sum_{n=0}^\infty f_n \) where \( s_n \) are \( \lambda_n \)-eigenvectors for \( \Delta \). The linear operators \( e_n = X|_{H_n} \) have no kernel for \( n > 0 \) because the flow is ergodic, and \( e_0 = 0 \). Note that \( f_0 = \int_M f d\mu \). Assuming that this is zero, we can set

\[
u := \sum_{n=1}^\infty e_n^{-1} f_n s_n.
\]

If \( f \) is smooth, the sequence \( \{\|f_n\|\} \) has rapid decay. The Diophantine assumption implies that \( \{e_n^{-1} f_n\} \) also has rapid decay, and hence defines a smooth function on \( M \).

This shows that the only obstruction to solving the cohomological equation \( Xu = f \) for \( f \) smooth, is \( \int_M f d\mu = 0 \). The result follows from Lemma 3.4.

The hypothesis holds if \( h \in \mathbb{R} \setminus \mathbb{Q} \) is an irrational number satisfying a Diophantine condition, and \( f \in C^\infty(T^2) \), \( f_\rho(t) = f(\alpha(t)) \) with \( \alpha(x,y) = (x + t, y + ht) \) Kronecker flow. Then \( X = \frac{\partial}{\partial x} + \frac{h}{m} \frac{\partial}{\partial y} \) acts on the eigenfunctions \( e^{inx+my} \) for \( \Delta \) on \( T^2 \) by the constant \( n + hm \). The usual Diophantine condition on an irrational number gives \( \gamma \) such that \( |n + hm| \geq C(n^2 + m^2)^{-1/2} \), and this implies the flow is Diophantine in the broader sense above.
Corollary 3.9. Let \( \omega = (\omega_1, \ldots, \omega_n) \in \mathbb{R}^n \) where \( \omega_1, \ldots, \omega_n \) are rationally independent. Assume the following Diophantine condition. There exists \( C > 0 \) and \( \gamma > 0 \) such that 
\[
|\sum_{i=1}^n k_i \omega_i| \geq C|k|^{-\gamma},
\]
with \( |k| \) the word length of \( k = (k_1, \ldots, k_n) \) in \( \mathbb{Z}^n \). Then if \( \omega_n(x) = x + 100 \) is the corresponding linear flow on \( \mathbb{T}^n \), and \( f \in C^\infty(\mathbb{T}^n) \), then for any \( p \in \mathbb{T}^n \), 

\[
\text{Trace}(f \cdot H^1) \text{ extends meromorphically to } \mathbb{C} \text{ with a simple pole at } s = 1 \text{ and}
\]

\[
\text{Res}_{s=1} \text{Trace}(f H^{-s}) = \int_{\mathbb{T}^n} f \, d\mu.
\]

\( \mu \) Lebesgue measure on \( \mathbb{T}^n \).

We now proceed to the proof of Theorem 3.1. A computation of the heat kernel of \( e^{-tH} \) follows from solving a differential equation: the heat kernel \( k_t \) satisfies \( (\frac{\partial}{\partial t} + H) \cdot k_t = 0 \), where \( \phi_t(x) = \int_{\mathbb{R}} k_t(x, y) \phi(y) \, dy \), for \( \phi \in \mathcal{S}(\mathbb{R}) \), and \( t \geq 0 \), together with the initial condition \( \lim_{t \to 0} \phi_t = \phi \). Consider the ansatz

\[
k_t(x, y) = \exp \left( \frac{a_1}{2} x^2 + b_1 x y + \frac{a_2}{2} y^2 + c_t \right).
\]

Setting this equal to 0 and solving for coefficients gives the ordinary differential equations

\[
\frac{a_1}{2} = a_t^2 - 1 = b_t^2, \quad c_t = a_t.
\]

Solving these gives

\[
a_t = -\coth(2t + C), \quad b_t = \csch(2t + C), \quad c_t = -\frac{1}{2} \log \sinh(2t + C) + D.
\]

Using the initial conditions we get \( C = 0 \) and \( D = \log(2\pi) - \frac{1}{4} \). See [1].

We obtain the following, called Mehler’s formula [14].

Lemma 3.10.

\[
(3.5) \quad k_t(x, y) = \frac{1}{\sqrt{2\pi \sinh 2t}} \exp \left( -\tanh \cdot \frac{(x + y)^2}{4} - \coth \cdot \frac{(x - y)^2}{4} \right)
\]

Proof. (Of Theorem 3.1) The operator \( H^{-s} \) is trace-class for \( \text{Re}(s) > 1 \), and the operator-valued integral \( \int_0^t t^{-1} e^{-tH} dt \) converges in norm to \( \Gamma(s) \cdot H^{-s} \). Hence if \( a \in \mathbb{B}(L^2) \),

\[
(3.6) \quad \Gamma(s) \cdot a H^{-s} = \int_0^t t^{-1} a e^{-tH} dt.
\]

and taking traces gives

\[
(3.7) \quad \Gamma(s) \cdot \text{Trace}(a H^{-s}) = \int_0^t t^{-1} \text{Trace}(a e^{-tH}) dt.
\]

Furthermore, if \( a \) is any bounded operator then

\[
\int_1^\infty t^{-1} \text{Trace}(a e^{-tH}) dt
\]

extends to an analytic function on \( \mathbb{C} \). Hence

\[
(3.8) \quad \Gamma(s) \cdot \text{Trace}(a H^{-s}) = \int_0^1 t^{-1} \text{Trace}(a e^{-tH}) dt.
\]

extends analytically to \( \mathbb{C} \).

Now let \( f \in C_0(\mathbb{R}) \), set \( a = f \). Then \( f e^{-tH} \) is an integral operator with kernel \( f(x) k_t(x, y) \), and hence 

\[
\text{Trace}(f e^{-tH}) = \int_{\mathbb{R}} f(x) k_t(x, x) \, dx.
\]

Applying Mehler’s formula Lemma 3.10 gives
Making the change of variables $x \mapsto \frac{x}{\sqrt{\tanh(t)}}$ gives

$$
\Gamma(s) \cdot \text{Trace}(aH^{-s}) = \int_0^1 t^{s-1} \frac{1}{\sqrt{2\pi \sinh 2t}} \int f(x)e^{-x^2\tanh t} \, dx \, dt.
$$

The result follows from the identity $\coth|t| = \frac{\sinh^2 t}{\cosh^2 t} = \csch^2 t$.

If $\text{Trace}(fH^{-s})$ meromorphically extends past $\Re(s) = 1$, then the residue of the pole at $s = 1$ defines kind of ‘mean’ of $f \in C_c(\mathbb{R})$. In certain examples of flows where $f = g_p$ for $p \in M$. $f(t) := g(\alpha(p))$, we have noted (Proposition 3.4) that this spectrally defined mean agrees with the geometric mean $\int_M f \, d\mu$ over the manifold.

The spectrally defined mean does not require meromorphic continuation, but only existence of the limit $\lim_{s \to 1^+} (s - 1) \text{Trace}(fH^{-s})$, which is a weaker condition which we now discuss.

**Definition 3.11.** Let $\mathcal{D} \subset C_c(\mathbb{R})$ be the closed linear subspace of all $f$ such that

$$\text{Res Tr}(f) := \lim_{s \to 1^+} (s - 1) \text{Trace}(fH^{-s})$$

exists.

The residue trace $\text{Res Tr}$ defines a positive linear functional of norm 1 on $\mathcal{D}$. This is immediate from the following geometric description of $\text{Res Tr}$ given below.

**Theorem 3.12.** If $f \in C_c(\mathbb{R})$ then $f \in \mathcal{D}$ if and only if

$$\mu_s(f) := \lim_{\lambda \to \infty} \frac{1}{2\sqrt{\pi}} \int_0^1 \int_\mathbb{R} f(xt^{-\lambda})e^{-x^2} \, dx \, dt$$

exists, and if this holds, then $\text{Res Tr}(f) = \mu_s(f)$.

**Proof.** Choose $\varepsilon > 0$. Since $\Gamma(1) = 1$ by Theorem 3.1 we have for $\Re(s) > 1$,

$$\lim_{s \to 1^+} \text{Trace}(fH^{-s}) = \lim_{s \to 1^+} \frac{s - 1}{2\sqrt{\pi}} \int_0^1 \int_\mathbb{R} t^{s-1} \csch f(x\sqrt{\coth t})e^{-x^2} \, dx \, dt$$

In the proof, we noted that the part of the integral corresponding to $t \geq \delta$ extends analytically to $\mathbb{C}$. Hence it contributes zero to the limit, and we may choose $\delta > 0$ small enough that $|\csch t - 1| < \varepsilon$ for $0 < t < \delta$, so that $|\csch t - \frac{1}{t}| < \frac{\varepsilon}{t}$ for $t < \delta$. Let $\phi(t) = \int_\mathbb{R} f(x\sqrt{\coth t}) \cdot e^{-x^2} \, dx$, then

$$\left| \int_0^\delta t^{s-1} \csch \left(1 - \frac{1}{t}\right) \phi(t) \, dt \right| < \frac{\varepsilon}{s - 1} \cdot \|f\|$$

by a brief computation. Letting $\varepsilon \to 0$ we see that the limit on the right hand side of (3.13), if it exists, equals the limit

$$\lim_{s \to 1^+} \frac{s - 1}{2\sqrt{\pi}} \int_0^1 \int_\mathbb{R} t^{s-2} f(x\sqrt{\coth t}) \cdot e^{-x^2} \, dx \, dt.$$
Let $\lambda = \frac{1}{s-1}$ and substitute $t \to t^\lambda$ in the above expression, and, noting $\delta \to 1$ as $\lambda \to \infty$, we deduce that

\begin{equation}
(3.16) \quad \mu_u(f) = \lim_{s \to 1+} (s-1) \text{Trace}(f H^{-t}) = \lim_{\lambda \to \infty} \frac{1}{2\sqrt{\pi}} \int_0^1 \int_{\mathbb{R}} f(x \sqrt{\coth t^\lambda}) \cdot e^{-x^2} \, dx \, dt.
\end{equation}

Since Lipschitz functions are dense in $C^\infty_u(\mathbb{R})$ and $\mathcal{D}$ is closed, we may assume $f$ is Lipschitz, and it follows that

\begin{equation}
(3.17) \quad |\int_0^1 \int_{\mathbb{R}} (f((xt)^{-\lambda}) - f(x^t^{-\lambda})) \cdot e^{-x^2} \, dx \, dt| \\
\leq \text{const.} \lim_{\lambda \to \infty} \int_0^1 |\sqrt{\coth t^\lambda - r^{-\frac{2}{\alpha}}}| \, dt
\end{equation}

which $\to 0$ as $\lambda \to \infty$. This proves the result.

The theorem can be expressed this way:

**Theorem 3.13.** Let $\mu_0 = \frac{1}{2\sqrt{\pi}} e^{-x^2} \, dx$, the Gaussian probability measure on $\mathbb{R}$. For $t \in \mathbb{R}_+$ let $\rho_t : \mathbb{R} \to \mathbb{R}$, $\rho_t(x) = tx$, and $\mu_t : = (\rho_t)_* \mu_0$. Then

\begin{equation}
\lim_{\lambda \to \infty} \int_0^1 \mu_t \, dt = \text{Res Tr} \in \mathcal{D}'
\end{equation}

in the weak topology on $\mathcal{D}'$.

We deduce the following.

**Corollary 3.14.** If $f \in C_u(\mathbb{R})$ and $\lim_{T \to \pm \infty} \frac{1}{T} \int_0^T f(t) \, dt$ exists, then $f \in \mathcal{D}$ and

\begin{equation}
(3.18) \quad \text{Res Tr}(f) = \lim_{T \to \pm \infty} \frac{1}{T} \int_0^T f(t) \, dt.
\end{equation}

In particular, if $\alpha$ is an ergodic flow on a compact smooth manifold $M$, $\mu$ an $\alpha$-invariant probability measure, $f \in C(M)$, $f_p(t) := f(\alpha^t p)$, then $f_p \in \mathcal{D}$ and

\begin{equation}
\text{Res Tr}(f_p) = \int_M f \, d\mu
\end{equation}

for a.e. $p \in M$.

**Proof.** Integration by parts, the change of variables $u \to ut^\lambda$, and a slight re-arrangement gives

\begin{equation}
(3.19) \quad \int_0^1 \int_{\mathbb{R}} f((xt)^{-\lambda})e^{-x^2} \, dx \, dt = 2 \int_0^1 \int_{\mathbb{R}} f((ut)^{-\lambda})xe^{-x^2} \, dudxdt
\end{equation}

\begin{equation}
= 2 \int_0^1 \int_{\mathbb{R}} \int_0^{u^{-\lambda}} f(u)x e^{-x^2} \, dudxdt = 2 \int_0^1 \int_{\mathbb{R}} \int_0^{u^{-\lambda}} f(u)x^2 e^{-x^2} \, dudxdt
\end{equation}

Now letting $\lambda \to \infty$ and using the hypothesis that $L := \lim_{T \to \pm \infty} \frac{1}{T} \int_0^T f(t) \, dt$ exists gives

\begin{equation}
(3.20) \quad \lim_{\lambda \to \infty} \int_0^1 \int_{\mathbb{R}} f((xt)^{-\lambda})e^{-x^2} \, dx \, dt = L \sqrt{\pi}
\end{equation}

By Theorem 3.12

\begin{equation}
(3.21) \quad \text{Res Tr}(f) = \lim_{\lambda \to \infty} \frac{1}{2\sqrt{\pi}} \int_0^1 \int_{\mathbb{R}} f((xt)^{-\lambda})e^{-x^2} \, dx \, dt
\end{equation}

giving (3.18).
The second statement follows from combining the first with the Birkhoff Ergodic Theorem.

**Remark 3.15.** By a slightly more elaborate argument the assumption of Corollary \[3.14\] can be weakened to: there exists \(0 \leq \beta < 1\) such that

\[
L := \lim_{x \to \pm \infty} \frac{1}{x^{1-\beta}} \int_0^x f(u)u^{-\beta} \, du
\]

exists. Then \(f \in \mathcal{D}\) and \(\text{Res} \text{ Tr}(f) = L\) remains true.

We next produce some estimates related to group translation operators on \(L^2(\mathbb{R})\).

**Lemma 3.16.** Let \(U_\alpha\) be the unitary induced by translation on the real line by \(\alpha \neq 0\). Then if \(f \in C_u(\mathbb{R})\) and \(a = fU_\alpha \in C_u(\mathbb{R}) \times \mathbb{R}_d\) then

\[
\Gamma(s) \cdot \text{Tr}(fU_\alpha^{-s}) = \frac{1}{2\sqrt{\pi}} \int_0^\infty t^{s-1} \text{csch} t \exp(-\frac{\alpha^2}{4} \coth t) \mu_{\alpha t}(f) \, dt
\]

where \(\mu_{\alpha t}(f) = \int_\mathbb{R} f(x\sqrt{\coth t} + \alpha)e^{-x^2} \, dx\).

**Proof.** The argument proceeds as in the proof of Lemma \[3.10\]. The operator \(fU_\alpha e^{-tH}\) is a compact integral operator with kernel

\[
k'(x,y) = f(x)k_t(x-\alpha, y).
\]

Hence for \(\text{Re}(s) > 1\),

\[
\Gamma(s) \cdot \text{Tr}(fU_\alpha^{-s}) = \int_0^\infty \int_\mathbb{R} t^{s-1} k_t(x-\alpha, x) \, dx \, dt
\]

\[
= \int_0^\infty \int_\mathbb{R} t^{s-1}(2\pi \sinh 2t)^{-\frac{1}{2}} f(x) \exp \left( -\frac{(2x-\alpha)^2}{4} \tanh t - \frac{\alpha^2}{4} \coth t \right) \, dx \, dt
\]

Basic manipulations yield \[3.22\].

**Lemma 3.17.** If \(f \in C_u(\mathbb{R})\), \(\alpha \in \mathbb{R}\) nonzero, then the function \(\phi_{f,\alpha}(s) := \Gamma(s) \cdot \text{Trace}(fU_\alpha^{-s})\), \(\text{Re}(s) > 1\), extends to an analytic function on \(\mathbb{C}\). There are constants \(C_s'\) and \(C_s''\) depending holomorphically on \(s\) such that

\[
|\phi_{f,\alpha}(s)| \leq (C_s' \alpha^{-2 \text{Re}(s)} + C_s'') e^{-\frac{\alpha^2}{4} s} \|f\|
\]

for all \(s \in \mathbb{C}\).

**Proof.** The point is that not only does \(\text{Trace}(fU_\alpha^{-s})\) meromorphically extend to \(\mathbb{C}\), the formula \[3.22\] defines \(\phi_{f,\alpha}(s)\) by a direct integral formula valid for all \(s \in \mathbb{C}\).

As shown above, a suitable family of states \(\mu_{\alpha t}\) on \(C_u(\mathbb{R})\), and a constant which we omit,

\[
\Gamma(s) \cdot \text{Tr}(fU_\alpha^{-s}) \sim \int_0^\infty t^{s-1} \text{csch} t \exp(-\frac{\alpha^2}{4} \coth t) \mu_{\alpha t}(f) \, dt
\]

\[
= \int_0^1 t^{s-1} \text{csch} t \exp(-\frac{\alpha^2}{4} \coth t) \mu_{\alpha t}(f) \, dt
\]

\[
+ \int_1^\infty t^{s-1} \text{csch} t \exp(-\frac{\alpha^2}{4} \coth t) \mu_{\alpha t}(f) \, dt = \zeta_1(s) + \zeta_2(s).
\]
Consider first $\zeta_1(s)$. Since $\frac{\tanh t}{t} = \frac{\sinh t}{t}$ are bounded on $[0,1]$, we can bound the integrand of $\zeta_1(s)$ by

$$t^{s-2}e^{-\beta t}||f||,$$

A change of variables gives

$$\int_0^1 t^{s-2}e^{-\beta t}dt = \int_1^{\infty} t^{-s}e^{-\beta t}dt.$$

If $A_s = \int_0^\infty t^{-s}e^{-\beta t}dt$ then $A_s = \frac{e^{-\beta}}{s} + \frac{1}{\beta}A_{s+1}$. by integration by parts, and it follows that

$$|\int_0^1 t^{s-2}e^{-\beta t}dt| \leq \text{const.} \beta^{-\text{Re}(s)}e^{-\beta}$$

where the constant does not depend on $\beta$ or $s$, and hence that

$$|\zeta_1(s)| \leq C_s \cdot ||f|| \cdot \alpha^{-2\text{Re}(s)}e^{-\frac{\alpha^2}{4}}.$$

We can bound $\zeta_2(s)$ as follows

$$(3.26) \quad |\int_1^\infty t^s \cosh \exp\left(-\frac{\alpha^2}{4} \coth t\right) \mu_{\alpha}\phi dt| \leq ||f||e^{-\frac{\alpha^2}{4}} \cdot \int_1^\infty t^s \cosh dt = C_s e^{-\frac{\alpha^2}{4}}||f||$$

The significance of Lemma 3.17 is that it sheds light on when $\text{Trace}(fH^{-s})$ meromorphically extends, when $a = \sum_{a \in \Gamma} f_a U_a$ is an element of $C_m(\mathbb{R}) \rtimes \mathbb{R}_d$, and $\Gamma \subset \mathbb{R}_d$ is a finitely generated subgroup.

The easiest case is that of a cyclic subgroup, and this produces a very strong result. If $f \in C_m(\mathbb{R}) \rtimes \mathbb{R}$ for $\Gamma \subset \mathbb{R}$ a subgroup, let $f_0$ the coefficient of $f$ at the identity 0 $\in \Gamma$. Let $D^\infty$ denote the subspace of $D \subset C_m(\mathbb{R})$ of $f$ such that $\text{Trace}(fH^{-s})$ meromorphically extends to $\mathbb{C}$ with a simple pole at $s = 1$.

**Theorem 3.18.** Let $f \in C_m(\mathbb{R}) \rtimes h \mathbb{Z}$, where $h \in \mathbb{R}$ is nonzero.

Then, if $f_0 \in D^\infty$, then $\text{Trace}(fH^{-s})$, $\text{Re}(s) > 1$, meromorphically extends to $\mathbb{C}$, with a simple pole at $s = 1$, and

$$\text{Res}_{s=1} \text{Trace}(fH^{-s}) = \mu_a(f),$$

where $\mu_a$ is the uniform mean of $f$.

**Proof.** Suppose first that $f$ has expansion $f = \sum f_a U_n h$ with $f_0 = 0$. Then

$$\Gamma(s) \cdot \text{Trace}(fH^{-s}) = \sum_n \text{Trace}(f_n U_n H^{-s}) = \sum_n \phi_n(s)$$

where $\phi_n(s)$ abbreviates $\phi_{f_n, nh}(s)$ of Lemma 3.17. The series converges absolutely and uniformly on compact subsets of $\mathbb{C}$ because of the bound

$$|\phi_n(s)| \leq \left(C_n h^{-2\text{Re}(s)} n^{-2\text{Re}(s)} + C_n h^2\right) e^{-\frac{\alpha^2}{4}n^2}.$$ 

due to the Lemma, shows that $\phi_n \to 0$ exponentially fast as $n \to \pm \infty$.

In the general case, $f = f - f_0$, $\text{Trace}((f - f_0)H^{-s})$ extends to an entire function for arbitrary $f \in C_m(\mathbb{R})$, and $\text{Trace}(f_0H^{-s})$ to a meromorphic function with the stated pole structure if $f_0 \in D^\infty$ by definition, see Theorem 3.12 for the equivalent condition to being in $D$. 

□
**Corollary 3.19.** If $a \in A_h := C(\mathbb{T}) \times_h \mathbb{Z}$, then $\text{Trace}(aH^{-s})$ meromorphically extends to $\mathbb{C}$ with a simple pole at $s = 1$ and residue there $\tau(a)$, $\tau$ the standard trace.

Hence the Heisenberg cycle Definition 3.17 over the irrational rotation algebra $A_h$ defines a spectral triple over $A_h^\infty$ with the meromorphic continuation property over the whole $C^*$-algebra $A_h$.

**Definition 3.20.** A finitely generated subgroup $\Lambda \subset \mathbb{R}$ with word length function $|\cdot|_\Gamma$ satisfies a Diophantine property if

$$|\alpha| \geq C|\alpha|_\Gamma^{-\gamma}$$

for some $\gamma > 0$ and $C > 0$.

**Definition 3.21.** Let $B_\Gamma := C_u(\mathbb{R}) \rtimes \Gamma$, where $\Gamma \subset \mathbb{R}$ is a countable subgroup. Then $B_\Gamma^\infty$ denotes the completion of the (twisted) group algebra $C_u^\infty(\mathbb{R})[\Gamma]$ with respect to the family of semi-norms $p_{s,m}(f) = \sum_{\gamma \in \Gamma} |f_\gamma^{(m)}| \cdot |\gamma|_\Gamma$. If a sequence $\phi$ satisfies a Diophantine property, let $f \in B_\Gamma^\infty$ have rapid decay in the sense that $B$ has a Diophantine property, let $f_0 \in D$. Then $\text{Trace}(fH^{-s})$ meromorphically extends to $\mathbb{C}$ with a simple pole at $s = 1$ and $f_0 = 0$. It suffices to prove that $B_\Gamma^\infty$ is closed under holomorphic functional calculus.

**Theorem 3.23.** Suppose $\Gamma \subset \mathbb{R}$ has a Diophantine property, let $f \in B_\Gamma^\infty$ and assume $f_0 \in \mathcal{D}$. Then $\text{Trace}(fH^{-s})$ meromorphically extends to $\mathbb{C}$ with a simple pole at $s = 1$ and $\text{Res}_{s=1}(f) = \mu_0(f)$.

**Proof.** Write $f = \sum_{\alpha \in \Gamma} f_\alpha U_\alpha \in B_\Gamma^\infty \subset \mathbb{B}(L^2(\mathbb{R}))$ and assume $f_0 = 0$. It suffices to prove that $\text{Trace}(fH^{-s})$ extends to an analytic function on $\mathbb{C}$. This equals

$$\sum_{\alpha \in \Gamma} \text{Trace}(f_\alpha U_\alpha^{-s}) = \sum_{\alpha \in \Gamma} \phi_{f_\alpha,\alpha}(s)$$

where $\phi_{f_\alpha,\alpha}$ is notation as in Lemma 3.17 and it suffices to show that this is an absolutely summable sequence of analytic functions, uniformly on compact subsets of $\mathbb{C}$.) Shorten notation $\phi_\alpha := \text{Trace}(f_\alpha U_\alpha^{-s})$. By the same Lemma

$$|\phi_\alpha(s)| \leq \left( C' |\alpha|^{\gamma} + C'' e^{-\frac{|a|^2}{4}} \right) \cdot \|f_\alpha\|$$

for all $s \in \mathbb{C}$. Since there are potentially infinitely many $\alpha$ with small absolute value, the exponential term is no longer of any use and we discard it, obtaining a polynomial bound for $\phi_\alpha(s)$ of order $|\alpha|^\mu$ for $\mu = -2\text{Re}(s) \in \mathbb{R}$. Since $\Gamma$ is finitely generated, there exists a constant $C_\Gamma$ such that $|\alpha| \leq C_\Gamma |\alpha|_\Gamma$ for all $\alpha \in \Gamma$. Combining with the Diophantine assumption gives that

$$|\alpha|_\Gamma^{-\gamma} \leq |\alpha| \leq C'|\alpha|_\Gamma^{-\gamma}$$

If $\mu \geq 0$ we get

$$C^\mu |\alpha|_\Gamma^{-\mu\gamma} \leq |\alpha|^\mu \leq C'' |\alpha|^\mu$$

Hence

$$\sum_{\alpha \in \Gamma} \|f_\alpha\| \cdot |\alpha|^\mu \leq C \sum_{\alpha \in \Gamma} \|f_\alpha\| \cdot |\alpha|_\Gamma^{-\mu\gamma}$$

and the last term is finite by (3.27).

If $\mu < 0$ then we use the bound

$$|\alpha|^\mu \leq \text{const.}|\alpha|^{-\mu\gamma}$$
Again, $\sum_{\alpha \in \Gamma} \|f_{\alpha}\| \cdot |\alpha|^2_{\Gamma}^{\mu T}$ is finite by assumption on $f$. \hfill $\square$

**Remark 3.24.** We make two comments about the proof.

a) The Diophantine condition on $\Gamma$ only seems relevant for the zone $0 < \Re(s) < 1$.

b) A weaker condition on $f$ than (3.27) still seems to ensures the result. It suffices to assume that

$$\sum_{\alpha \in \Gamma} \|f_{\alpha}\| \cdot |\alpha|^2 \cdot e^{-\frac{|\mu|^2}{\hbar}} < \infty$$

for all real $s > 0$. I do not know if such $f$ form a holomorphically closed subalgebra of $C_u(\mathbb{R}) \rtimes \Gamma$. However, it does make clear that if $\Gamma \subset \text{discrete}$, then there is no condition at all on $f$, except, of course, that $f$ defines a bounded operator in $C_u(\mathbb{R}) \rtimes \Gamma$, and that $f_0 \in \mathcal{D}^\infty$.

Let $\alpha$ be a smooth flow on $M$ compact. Let $p \in M$ and $\pi_p: C(M) \times \mathbb{R}_d \to C_u(\mathbb{R}) \rtimes \mathbb{R}_d \subset \mathbb{B} \left( L^2(\mathbb{R}) \right)$ be the *-homomorphism induced by restriction of functions to the orbit of $p$.

Pulling back the Heisenberg cycle for $C_u(\mathbb{R}) \rtimes \mathbb{R}_d$ we obtain a cycle for $C(M) \times \mathbb{R}_d$, and for $C(M) \times \Gamma$ for any $\Gamma \subset \mathbb{R}$ a subgroup.

From the results above, if both the flow and the group satisfy Diophantine conditions (Definitions 3.7 and 3.20 respectively) then the pulled-back cycle determines a spectral triple over a suitable smooth subalgebra of $C(M) \rtimes \Gamma$. However we are mainly interested in applying this to the situation of Kronecker flow $\alpha^h$ on $\mathbb{T}^2$, where both Diophantine properties are implied by a Diophantine condition on the number $h$.

**Theorem 3.25.** Let $B_h := A_h \otimes A_1 / h \cong C(\mathbb{T}^2) \rtimes \Gamma$, with $\Gamma \subset \mathbb{R}$ the group generated by 1, $h$. Let $B_h^\infty$ be the Schwartz subalgebra of $B_h$ of all $\sum_{\alpha \in \Gamma} f_{\alpha} U_{\alpha}$ with

$$\sum_{\alpha \in \Gamma} \|X^n f\| \cdot |\alpha|^2 < \infty,$$

where $X$ generates the flow.

Then the Heisenberg bi-cycle of Definition 2.6 determines a spectral triple over $B_h^\infty$ with the meromorphic extension property.

4. **TOPOLOGY OF HEISENBERG CYCLES**

We now use cyclic cohomology to perform some K-theory and index-pairing calculations with the Heisenberg cycles over $A_{\hbar}$ and $B_{\hbar} = A_{\hbar} \otimes A_1 / h$.

We will focus on $A_{\hbar}$. Let $[D_{\hbar}] \in \text{KK}_0(A_{\hbar}, \mathbb{C})$ be the Heisenberg class of Definition 2.7 involving

a) The unitary action of $\mathbb{Z}$ on $L^2(\mathbb{R})$ with $n$ acting by $U_n^\hbar$.

b) The action of $C(\mathbb{T}) = C(\mathbb{R}/\mathbb{Z}) = C(\mathbb{R})^\mathbb{Z}$ of $\mathbb{Z}$-periodic functions by multiplication operators.

c) The operator $D = \begin{bmatrix} 0 & x - i dx \\ x + i dx & 0 \end{bmatrix}$

From a) and b) we get the representation $\pi_{\hbar}: A_{\hbar} \to \mathbb{B} \left( L^2(\mathbb{R}) \right)$

It is important to note that $A_{\hbar}$ only depends, of course, on the class of $\hbar$ mod $\mathbb{Z}$, but $\pi_{\hbar}$ depends on $\hbar$, not just its equivalence class. (This is a difference between our set-up and that of [13], as they only allow $0 < \hbar < 1$.)

In particular, fixing $0 < h < 1$, then the $[D_{h+b}]$, for $b \in \mathbb{Z}$, define a $\mathbb{Z}$-family of spectral cycles over the single $A_{h}$. 
In fact, due to the identity $[\mathcal{E}_h] \otimes_{A_h} [\partial] = [D_h]$, this is accounted for by a similar fact about the Morita equivalences $\mathcal{E}_h$ of Theorem 2.9. Namely, that for $h$ varying within a $\mathbb{Z}$-coset of $\mathbb{R}$, we obtain a $\mathbb{Z}$-parameterized family of Morita $A_{1/h}$-$A_h$-bimodules: note that $A_{1/h}$ changes as an integer is added to $h$, and so does the bimodule, but the ring of scalars $A_h$ in the right multiplication remains the same. Twisting $\partial$ by these bimodules and forgetting the left action gives a $\mathbb{Z}$-parameterized family of spectral triples these are exactly the $[D_{h+b}]$.

Actually, there is a certain internal symmetry of $A_h$, which we call the ‘Heisenberg twist’ (which appeared already in [6]) whose iterates act transitively on these sets of data in both the K-theory and K-homology picture. The Heisenberg twist is a KK-morphism determined by the bundle of cycles $D_h$, over $\mathbb{R}$, as we explain below.

We are going to use the Local Index Theorem of [5] in dimension 2 to compute the pairing of the class $[D_h]$ with $K_0(A_h)$; the result, and its proof, effectively computes the index theory of the class $\Delta_h$ as well. We will use the development of the Local Index Formula by N. Higson in [10] and our results on the harmonic oscillator residue trace of the previous section.

**Remark 4.1.** The index formula Corollary 4.8 we arrive at is very similar to one due to Connes (see [3], [2]), who discovered it using his computation of the cyclic theory of $A_n^\pi$. There are some minor differences, as Connes fixes $0 < h < 1$ and computes the index map $K_0(A_{1/h}) \to \mathbb{Z}$ induced by $D_h$, and we are allowing arbitrary $h$ and computing the index pairing with $K_0(A_h)$. Connes’ used some basic results about the cyclic cohomology of $A_h$ to deduce his theorem. Our method uses the Local Index Theorem and is more general.

In the paper [2], Alain Connes described an invariant of a finitely generated projective module over $A_h$, generalizing the first Chern number of a complex vector bundle over $\mathbb{T}^2$.

Connes’ construction was the following. Let $A$ be any C*-algebra endowed with an action of $\mathbb{R}^2$ by automorphisms with $(s,t)$ acting by $\alpha_s \circ \beta_t$.

Let $\delta_t: A \to A$ be the densely defined derivations
\[ \delta_t(a) := \lim_{t \to 0} \frac{\alpha_t(a) - a}{t}, \quad \delta_2(a) := \lim_{t \to 0} \frac{\beta_t(a) - a}{t}, \quad a \in A^\pi, \]
where $A^\pi = \cap_{n \in \mathbb{N}, m \in \mathbb{Z}} \text{dom}(\delta^n) \cap \text{dom}(\delta^m)$, the *-subalgebra of elements such that $(s,t) \mapsto \alpha_s(\beta_t(a))$ is smooth.

In addition, let $\tau: A \to \mathbb{C}$ be an $\mathbb{R}^2$-invariant tracial state.

Then Connes’ invariant of a f.g.p. module $eA^\pi$, where $e$ is a projection in $A^\pi$, is given by
\[ c_1(e) := \frac{1}{2\pi i} \tau(e[\delta_1(e), \delta_2(e)]). \]

We call $c_1(e)$ the *first Chern number of $e$.*

The number
\[ c_1(e) := \frac{1}{2\pi i} \tau(e[\delta_1(e), \delta_2(e)]) \]
only depends on the equivalence class of $e$ in $K_0(A^\pi)$ (see [4]).

Moreover, $c_1(\mathcal{E} \oplus \mathcal{E}') = c_1(\mathcal{E}) + c_1(\mathcal{E}')$ and $c_1$ thus determines a group homomorphism $K_0(A) \to \mathbb{R}$.

Let $\bar{h} \in \mathbb{R}$ and $A_\bar{h} = C(\mathbb{T}) \rtimes_\mathbb{Z} \mathbb{Z}$ the corresponding rotation algebra, with $u \in A_\bar{h}$ the generator of the $\mathbb{Z}$-action. Then the $\mathbb{R}^2$-action with $\alpha_u(f) = f(x-t)$, $\alpha_u(\bar{u}) = u \bar{u}$, $\beta_u(a) = e^{2\pi i u_n} \bar{u}^n$, $\beta_u(f) = f$, gives rise to the derivations
\[ \delta_1(\sum_n f_n[n]) = \sum_n f'_n[n], \quad \delta_2(\sum_n f_n[n]) = \sum_n 2\pi i n \cdot f_n[n]. \]
Recall the Rieffel modules described in Lemma 2.9. For any \( h \), \( E_h \) is the completion of \( C_c(\mathbb{R}) \) with respect to the \( C(\mathbb{R}) \)-valued inner product

\[
(\xi, \eta)_{A_h}(x, m) = \sum_{n \in \mathbb{Z}} \xi(x-n)\eta(x-n-m), x \in \mathbb{R} / \mathbb{Z}, \ m \in \mathbb{Z}
\]

Give \( C_c(\mathbb{R}) \) the right \( C(\mathbb{R} / \mathbb{Z}) \times_h \mathbb{Z} = A_h \) module structure with

\[
(\xi \rho)(x) = \xi(x+nh), \ (\xi f)(x) = f(x)\xi(x).
\]

This extends to a multiplication \( E_h \times A_h \to E_h \) giving a finitely generated and projective Hilbert \( A_h \)-module. To find a projection \( p_h \) such that \( p_h A_h \cong E_h \), recall that the \( A_1/h \)-valued inner product is given by

\[
\langle \xi, \eta \rangle_{A_h}(x, m) = \sum_{n \in \mathbb{Z}} \xi(x-nh)\eta(x-nm), x \in \mathbb{R} / h\mathbb{Z}, \ m \in \mathbb{Z},
\]

If we can find \( \xi \in E_h \) such that \( C(\mathbb{R} / h\mathbb{Z}) \times (\xi, \eta)_{A_h} = 1 \), then the map \( \eta \mapsto (\eta, \xi)_{A_h} \) embeds \( E_h \) in the trivial rank-one \( A_h \)-module, and the required projection is \( (\xi, \xi)_{A_h} \). We thus seek \( \xi \) such that

\[
\sum_{n \in \mathbb{Z}} \xi(x-nh)\eta(x-nm) = \delta_{hm}
\]

in Kronecker notation. In particular, \( \xi \) should have support contained within an interval, e.g. \([0,1]\), of length 1, for the outcome to be zero if \( m \neq 0 \). But this means that if \( h > 1 \) the function \( \sum_{n \in \mathbb{Z}} \xi(x-nh)^2 \) will have zeros. A necessary condition therefore to find such \( p_h \) is that \( 0 < h < 1 \). For larger \( h \), one must find a finite set \( \xi_1, \ldots, \xi_m \) of vectors in \( E_h \), and use them to embed \( E_h \) in \( A_h^m \). One obtains projections \( p_h \) in matrix algebras over \( A_h \). This makes the computation of curvature more complicated.

If \( 0 < h < 1 \) then the problem can be solved: the ensuing projection is of the form

\[
p_h = f + gu + g^{-h}u^*
\]

where \( f \) and \( g \) are suitably chosen functions. For \( a \in (0,1) \), and \( \varepsilon > 0 \) small, \( f \) equals zero on \([0,a]\) and on \([a+h+\varepsilon, 2\pi]\), and \( f = 1 \) on \([a+\varepsilon, a+h]\). We choose \( f \) so that \( f(x) + f(x+h) = 1 \). We set \( g = \sqrt{f-f^2} \) on \([a+h, a+h+\varepsilon]\) and is zero otherwise.

The following calculation from (4) is reproduced below for the benefit of the reader.

**Lemma 4.2.** Let \( p_h \in A_h = C(\mathbb{T}) \times_h \mathbb{Z} \) be the Rieffel projection Then \( c_1(p_h) = +1 \).

**Proof.** For brevity, for \( f \in C(\mathbb{T}) \) understood as a \( \mathbb{Z} \)-periodic function on \( \mathbb{R} \), let \( f^h(x) := f(x-h) \) denote the action.

The Rieffel projection is given by \( p_h = f + gu + g^{-h}u^* \) as above. We then have to compute \( c_1(e) = \frac{1}{2\pi i} \tau([\delta_1(p_h), \delta_2(p_h)]) \). We first compute

\[
(4.1) \quad \frac{1}{2\pi i} [\delta_1(p_h), \delta_2(p_h)] = u^*(gg' - g^hf^h) + 2g'(g^h - gg') + 2g^2f'h)u
\]

Multiplying this on the left by \( p_h \) produces a terrific mess, but we are only interested in its trace, so the only part which is relevant is

\[
(4.2) \quad g^2(f' - f^h) + 2f(g^h - gg') + (g^2f' - g^hf')u + (wv')^{-h}
\]

which we want to integrate over \( \mathbb{T} \).

Set \( w = g^2, v = f - f^h \). The integral is given by

\[
(4.3) \quad \int w^2 + f(w - w^h) + (wv')^{-h}
\]
The middle term is
\[ \int f w' - \int f w = \int f^h w' - \int f w' = - \int v w'. \]

Hence we are reduced to computing
\[ \int w v' - v w' + (w v')' = 3 \int w v', \]

by integration by parts. Next, since \( f^h = 1 - f \) on \( \text{supp}(g) \), we can replace \( f' - f^h \) by \( -2 f' \) and get
\[ (4.4) \quad -6 \int f'(f - f^2) = -6 \int f' f + 6 \int f' f^2 = -3 \int (f^2)' + 2 \int (f^3)' = 3 - 2 = 1, \]

where the integration is understood to be restricted to the support of \( g \). This completes the calculation.

\[ \square \]

The first Chern class of any projection in \( \mathbb{A}_R^\infty \) is an integer, a fact related to the Quantum Hall Effect (see [3]). This is due to agreement of the first Chern number with the index pairing (an integer) of the projection and the Dirac-Dolbeault spectral cycle. This a consequence of the Local Index Formula of Connes and Moscovici, which we are going to work out in the case of the Heisenberg cycles, but first state in low dimensions.

**Theorem 4.3.** (Connes-Moscovici, [3]) Let \((H, \pi, D)\) be an even, 2-dimensional spectral triple over \( \mathbb{A}_R^\infty \subset \mathbb{A} \), for a C*-algebra \( \mathbb{A} \), regular and with the meromorphic continuation property over \( \mathbb{A}_R^\infty \).

Let \([D] \in \text{KK}_0(\mathbb{A}, \mathbb{C})\) be the class of the triple. Let \( \Delta := D^2 \), and let \( \varepsilon \) be the grading operator on \( H \).

Define functionals
a) \( \psi_0 : \mathbb{A}_R^\infty \rightarrow \mathbb{C} \),

\[ \psi_0(a) := \text{Res}_{s=0} \Gamma(s) \cdot \text{Tr}(\varepsilon a(\Delta + \text{proj}_{\text{ker} D})^{-s}), \]

and
b) \( \psi_2 : \mathbb{A}_R^\infty \otimes \mathbb{A}_R^\infty \otimes \mathbb{A}_R^\infty \rightarrow \mathbb{C} \),

\[ \psi_2(a^0, a^1, a^2) := \frac{1}{2} \text{Res}_{s=1} (\varepsilon a^0 [D, a^1] [D, a^2] \Delta^{-s}). \]

Then, if \( e \in \mathbb{A}_R^\infty \) is a projection, then
\[ \langle [e], [D] \rangle = \psi_0(e) - \psi_2(e - \frac{1}{2}, e, e), \]

where \( \langle [e], [D] \rangle \in \mathbb{Z} \) is the pairing between the \( K_0(\mathbb{A}) \)-class \([e] \) and the \( \text{KK}_0(\mathbb{A}, \mathbb{C}) \) class \([D] \).

The following Lemma shows that zero-dimensional part of the Chern character of a Heisenberg cycle comes from taking the pole at \( s = 1 \) of the zeta function \( \text{Trace}(a H^{-s}) \) discussed in the previous section.

**Lemma 4.4.** If \( a \in B(L^2(\mathbb{R})) \) then
\[ \Gamma(s) \text{Tr}(\varepsilon a(\Delta + \text{proj}_{\text{ker} D})^{-s}) - \text{Trace}(a H^{-s-1}) \]
extends analytically to $\Re(s) > -1$. In particular, if $\Psi_0$ is the functional b) in Theorem 4.3 and if $\text{Tr}(aH^{-s})$ meromorphically extends to $\mathbb{C}$ then
\[ \Psi_0(a) = \text{Res} \text{Tr}(a) := \text{Res}_{s=1} \text{Tr}(aH^{-s}). \]

**Proof.** We refer to Theorem 4.3. The Hilbert space for the Heisenberg triple is the direct sum of two copies of $L^2(\mathbb{R})$, and $D = \begin{bmatrix} 0 & x - d/dx & 0 \\ x + d/dx & 0 & 0 \end{bmatrix}$. The kernel of $D$ is the same as the kernel of $x + d/dx$, and is spanned by the ground state $\psi(x) = \pi^{-\frac{1}{4}} e^{\frac{x^2}{2}}$, and $\Delta = \begin{bmatrix} H - 1 & 0 & 0 \\ 0 & H + 1 & 0 \\ 0 & 0 & H + 1 \end{bmatrix}$ where $H$ is the harmonic oscillator. If $p = \text{proj}_{\ker D}$ then $\Delta + p = \begin{bmatrix} H - 1 + p & 0 & 0 \\ 0 & H + 1 & 0 \\ 0 & 0 & H + 1 \end{bmatrix}$. The first copy of the Hilbert space $L^2(\mathbb{R})$ is even in the grading, the second is odd and so the meromorphic function whose pole at $s = 0$ gives $\Psi_0(a)$.

(4.5) $\Gamma(s) \text{Tr}(e^{2s}(\Delta + p)^{-1}) = \Gamma(s) \cdot \text{Tr}(a(H - 1 + p)^{-1}) - \Gamma(s) \cdot \text{Tr}(a(H + 1)^{-1})$.

for $a \in C_c(\mathbb{R})[\mathbb{R}_d]$. Applying Mellin transform and small calculation gives
\[ = \int_0^\infty t^{s-1} \sinh t \cdot \text{Tr}(ae^{-tH})dt + E(s) \]
\[ \text{where} \]
\[ E(s) = \int_0^\infty t^{s-1} e^{t} \cdot \text{Tr}(e^{tH} - e^{-t(H+p)})dt = \text{Tr}(ap) \cdot \int_0^\infty t^{s-1} (1 - e^{-t})dt, \]
giving that $E(s)$ extends meromorphically with a simple pole at $s = -1$, and in particular, $E(s)$ is analytic for $\Re(s) > -1$. Hence
\[ \Psi_0(a) = \text{Res}_{s=0} \int_0^\infty t^{s-1} \sinh t \text{Tr}(ae^{-tH})dt. \]

Since $\sinh t \sim t$ as $t \to 0$,

(4.6) $\Psi_0(a) = \text{Res}_{s=0} \int_0^\infty t^s \text{Tr}(ae^{-tH})dt = \text{Res}_{s=1} \int_0^\infty t^{s-1} \text{Tr}(ae^{-tH})dt$
\[ = \text{Res}_{s=1} \text{Tr}(aH^{-s}). \]

$\square$

Let $B = C(M) \rtimes \Lambda$, for a flow $\alpha$ on $M$ and $\Lambda \subset \mathbb{R}$ a Diophantine subgroup. Let $\mu$ be an $\alpha$-invariant measure, $X$ generate the flow, and assume that $Xf = f$ is smoothly solvable for any smooth $f$ such that $\int_M f d\mu = 0$. Fix $p \in M$ and let $\pi: C(M) \rtimes \Lambda \rightarrow \mathbb{B}(L^2(\mathbb{R}))$ the corresponding representation, with $\pi(f) = f_p, f_p(t) = f(\alpha_\tau(p))$.

We have shown that $B^w \subset B$ has the property that $\text{Tr}(\pi(b)H^{-s})$ has the meromorphic extension property and that
\[ \text{Res}_{s=1}(\pi(b)H^{-s}) = \tau_\mu(b), \]
where $\tau_\mu: B \rightarrow \mathbb{C}$ is the trace induce by $\mu$, and $b \in B^w$. Let $\delta_1^\mu, \delta_2^\mu$ be the derivations of $B^w$ defined
\[ \delta_1^\mu(f) = X(f), \delta_1^\mu(U_\alpha) = 0, \quad \delta_2^\mu(f) = 0, \delta_2^\mu(U_\alpha) = \alpha. \]

**Lemma 4.5.** In the above notation, the functional $\Psi_2$ of Theorem 4.3(b) is given on $B^w$ by

(4.7) $\Psi_2(b^0, b^1, b^2) = \tau_\mu(\delta_1^\mu(a^0) \delta_1^\mu(a^1) \delta_2^\mu(a^2) - \delta_2^\mu(a^0) \delta_1^\mu(a^1) \delta_1^\mu(a^2))$

for all $b_0, b_1, b_2 \in B^w$. 

Proof. Note that
\[ \delta_1(b) = [\pi(b), \frac{d}{dx}], \quad \delta_2(b) = [\pi(b), x]. \]
Expand \([D, a^1][D, a^2]\) as a block matrix
\[ \begin{bmatrix} x - \frac{d}{dx}b^1 & x + \frac{d}{dx}b^2 \\ x - \frac{d}{dx}b^1 & x + \frac{d}{dx}b^2 \end{bmatrix} \]
We deduce
\[(4.8) \quad \frac{1}{2} \text{Res} \text{ Tr} (e^b [D, b^1] [D, b^2] \Delta^{-1}) = \text{Res} \text{ Tr} (b^0 [x, b^1] [d/\,dx, a^2]) - \text{Res} \text{ Tr} (b^0 [d\,dx, b^1] [x, b^2]) = \tau_\mu (b^0 \delta_1(b^1)\delta_2(b^2) - b^0 \delta_2(b^1)\delta_1(b^2)) \]

We have thus proved the following.

**Theorem 4.6.** Let \(B = C(M) \rtimes \Lambda\), for a smooth flow \(\alpha\) on a compact manifold \(M\). Let \(\Lambda \subset \mathbb{R}\) be a Diophantine subgroup. Let \(\mu\) be an \(\alpha\)-invariant measure, \(X\) generate the flow, and assume that \(X_u = f\) is smoothly solvable for any smooth \(f\) such that \(\int f\,d\mu = 0\).

Then the Chern character of the Heisenberg cycle determined by a point \(p \in M\) is given by \(\tau_\mu = \tau_\Delta\), where \(\tau_\mu\) is the trace on \(B\) determined by \(\mu\), and \(\tau_\Delta\) is the cyclic 2-cocycle
\[ \tau_\Delta^a(b^0, b^1, b^2) = \tau_\mu (b^0 \delta_1(a^1)\delta_2(b^2) - b^0 \delta_2(a^1)\delta_1(b^2)) \]
on \(B^a\).

**Corollary 4.7.** Let \(\mathfrak{h} \in \mathbb{R}\) and \(A_\mathfrak{h} := C(\mathbb{T}) \rtimes \mathbb{Z}\) the corresponding rotation algebra.

Let \([D_\mathfrak{h}]\) be the class of the Heisenberg cycle (Definition 2.27)
\[ \bigg( L^2(\mathbb{R}) \oplus L^2(\mathbb{R}).\pi_\mathfrak{h}, D := \begin{bmatrix} 0 & x - \frac{d}{dx} \\ x + \frac{d}{dx} & 0 \end{bmatrix} \bigg). \]

Then the Chern character of \([D_\mathfrak{h}]\) is given by \(\tau = \mathfrak{h} \tau_2\), where
\[ \tau_2(a^0, a^1, a^2) = \tau (a^0 \delta_1(a^1)\delta_2(a^2) - a^0 \delta_2(a^1)\delta_1(a^2)) \],
the curvature cocycle of Connes.

**Corollary 4.8.** Let \(e \in A_\mathfrak{h}\) be a projection, \([e] \in K_0(A_\mathfrak{h})\) its class. Then
\[ \langle [e], [D_\mathfrak{h}] \rangle = \tau(e) - \mathfrak{h}c_1(e), \]
where \(c_1(e)\) is its first Chern number.

In particular, if \(p_\mathfrak{h}\) is the Rieffel projection for \(\mathfrak{h}\) then
\[ \langle [p_\mathfrak{h}], [D_\mathfrak{h}] \rangle = - |\mathfrak{h}| \]
where \(|\mathfrak{h}|\) is the greatest integer \(< \mathfrak{h}\).

The proof is immediate from the formula (4.7) because \(c_1(p_\mathfrak{h}) = 1\) and \(\tau(p_\mathfrak{h}) = \mathfrak{h} - |\mathfrak{h}|\).

This yields the proof of Proposition 2.12.

We close with a remark. The integrality of Connes’ first Chern number is due to the index result that
\[ \langle [\delta], [e] \rangle = c_1(e), \]
where \(c_1(e)\) is the first Chern number of \(e\). If one combines this with our Index computation for \([D_\mathfrak{h}]\) then we obtain the following ‘gap-labelling’ result. It is of course well-known; but the argument below does not depend on computation of \(K_0(A_\mathfrak{h})\).
Corollary 4.9. Suppose \( h \in \mathbb{R} \) is nonzero. Then if \( \tau : A_h \to \mathbb{C} \) is the trace,

\[
\tau_* : K_0(A_h) \to \mathbb{R}
\]

the induced group homomorphism, then then range of \( \tau_* (K_0(A_h)) \) is the subgroup \( \mathbb{Z} + h\mathbb{Z} \subset \mathbb{R} \).

Proof. If \( e \in A_h^\infty \) is a projection, then application of our results above gives that \( \tau(e) + h \cdot c_1(e) \) is an integer. On the other hand, \( c_1(e) \) is an integer. This implies \( \tau(e) = m + nh \) for a pair of integers \( m,n \). Finally, \( A_h^\infty \) is dense and holomorphically closed in \( A_h \), so any projection in \( A_h \) is represented by a projection in \( A_h^\infty \).

The interest in this argument is that it computes the range of the trace using a combination of two spectral cycles, but does not rely on computation of \( K_* (A_h) \). It is conceivable that such a method could be applied in connection with other ‘Gap-Labelling’ problems.

5. Transverse foliations

There is a well-known procedure for producing finitely generated projective (f.g.p.) modules over \( A_h \), using Morita equivalence. Fixing one of the standard linear loops in \( T^2 \) determines a Morita equivalence of \( A_h \) with the C*-algebra \( B_h = C(T^2) \rtimes_h \mathbb{R} \) of the Kronecker foliation \( f_h \) into lines of slope \( h \). On the other hand, any linear loop in \( T^2 \) is transverse to \( f_h \). These linear loops, parameterized by pairs of relatively prime integers, determine therefore Morita equivalences between \( A_h \) and what turn out to be other rotation algebras, and in particular, unital algebras. Hence they determine f.g.p. modules over \( A_h \); these parameterize (the positive part of) the K-theory.

In [6] we showed that there is an analogue of this procedure using non-compact transversals: if \( h' \neq h \) then the Kronecker foliations \( f_h \) and \( f_{h'} \) are transverse. Their product foliates \( T^2 \times T^2 \) and its restriction to the diagonal gives an equivalent étale groupoid. This reasoning produces for every \( h' \neq h \) a f.g.p. module \( L_{h,h'} \) over \( A_h \otimes A_{h'} \) and corresponding K-theory class \( [L_{h,h'}] \in KK_0(\mathbb{C} A_h \otimes A_{h'}) \). In particular, fixing \( h' = h + b \) for any integer \( b \neq 0 \) gives a f.g.p. module over \( A_h \otimes A_h \). We denote it \( L_b \).

Let

\[
\text{PD} : KK_0(\mathbb{C} A_h \otimes A_h) \to KK_0(A_h \otimes A_h),
\]

be Connes’ Poincaré duality map [3]. In [6] it is proved that

\[
(5.1) \quad \text{PD}([L_b]) = \tau_b,
\]

where \( \tau_b \) is the Kasparov morphism defined in terms of Dirac-Schrödinger operators as follows. We take the standard right Hilbert \( A_h \)-module \( L^2(\mathbb{R}) \otimes A_h \). We let \( \mathbb{Z} \) act on the left by the following formula, where we designate a dense set of elements of our Hilbert module in the form \( \sum_{n \in \mathbb{Z}} \xi_n \cdot [n] \), with \( \xi_n \in L^2(\mathbb{R}) \otimes C(T) \):

\[
k \cdot \left( \sum_{n \in \mathbb{Z}} \xi_n \cdot [n] \right) := \sum_{n \in \mathbb{Z}} k(\xi_n) \cdot [k + n],
\]

where \( k(\xi)([x], t) = \xi([x - kh], t - k), \xi \in L^2(\mathbb{R}) \otimes C(T) \).

Let \( f \in C(T) \) act by

\[
f \cdot \left( \sum_{n \in \mathbb{Z}} \xi_n \cdot [n] \right) := \sum_{n \in \mathbb{Z}} f^b \cdot \xi_n \cdot [n].
\]

where \( f^b(t, [x]) = f([x + tb]) \).
These two assignments determine a covariant pair and representation
\[ \pi_h : A_H \rightarrow B(L^2(\mathbb{R}) \otimes A_H). \]

**Definition 5.1.** The Heisenberg twist \( \tau_b \in KK_0(A_H, A_H) \) is the class of the spectral cycle
\[ (L^2(\mathbb{R}) \otimes A_H \oplus L^2(\mathbb{R}) \otimes A_H, \pi_b \oplus \pi_b, D \oplus 1_{A_H}), \]
with \( D := \begin{bmatrix} 0 & x - d/dx \\ x + d/dx & 0 \end{bmatrix} \).

The equality \( (5.1) \) gives rise to an explicit geometric cycle representing the unit in Connes’ duality \( [6] \). The relationship between the Heisenberg twist and the Heisenberg cycles is implied by the following Lemma.

**Lemma 5.2.** Let \( h, \mu \in \mathbb{R} \).

a) The group multiplication \( m : \mathbb{T} \times \mathbb{T} \rightarrow \mathbb{T} \) intertwines the diagonal \( \mathbb{Z} \)-action on \( \mathbb{T} \times \mathbb{T} \) by group addition of \( (h, \mu) \) and group addition on \( \mathbb{T} \) of \( h + \mu \), and so determines a *-homomorphism
\[ \chi : A_{h+\mu} \rightarrow A_h \otimes A_\mu. \]

In this notation:
\[ \chi^\ast([D_h] \otimes C [D_\mu]) = [D_{h+\mu}], \]

b) If \( \mu = b \in \mathbb{Z} \), then
\[ \tau^b = \chi^\ast([D_h] \otimes 1_{A_h}) \]
where \( \tau^b \) is the Heisenberg twist.

c) \( \tau^b \otimes_{A_h} [D_h] = [D_{h+b}] \) for any \( h \in \mathbb{R}, b \in \mathbb{Z} \).

**Proof.** For c) we have
\[ \tau^b \otimes_{A_h} [D_h] = \chi^\ast([D_h] \otimes 1_{A_h}) \otimes_{A_h} [D_h] = \chi^\ast([D_h] \otimes C [D_h]) = [D_{b+h}] \]

using first part b) and then part a).

b) follows from an inspection at the level of cycles: they differ only in the representations, which are clearly homotopic.

We now prove a).

Consider \( \chi^\ast([D_h] \otimes C [D_\mu]) \), a class in \( KK(A_{\theta+\eta}, \mathbb{C}) \). By the standard method of computing external products, it is represented by the following spectral cycle. The Hilbert space is module is \( L^2(\mathbb{R}, \mathbb{C}^2) \otimes L^2(\mathbb{R}, \mathbb{C}^2) \) and operator \( D \otimes 1 + 1 \otimes D \). With \( u, v \in A_H \) the standard unitary generators, \( u = \zeta, v = [1], \) the representation is given by:
\[ (u \cdot \phi)(x, y) = e^{2\pi i(x+y)} \phi(x, y), \quad (v \cdot \phi)(x, y) = \phi(x - \theta, y - \eta) \]

We apply a homotopy to the representation, with \( t \in [0, 1/2] \):
\[ (\pi_t(v) \cdot \phi)(x, y) = \phi(x - (1-t)\theta - t\eta, y - t\theta - (1-t)\eta) \]

The resulting cycle is \( (L^2(\mathbb{R}, \mathbb{C}^2) \otimes L^2(\mathbb{R}, \mathbb{C}^2), D \otimes 1 + 1 \otimes D), \) where \( \Delta \) is the “diagonal” representation of \( A_{\theta+\eta} \):
\[ (u\phi)(x, y) = e^{2\pi i(x+y)} \phi(x, y), \quad (v\phi)(x, y) = \phi \left( x - \frac{\theta + \eta}{2}, y - \frac{\theta + \eta}{2} \right) \]

Next consider the class \( [D_{h+b}] \). The the unit in \( 1_C \in KK(\mathbb{C}, \mathbb{C}) \) can be represented by the cycle \( (L^2(\mathbb{R}, \mathbb{C}^2), 1_1), \). Taking the intersection product of this with \( [D_{b+\eta}] \) yields a cycle which is equivalent to \( [D_{b+\eta}] \), but more closely resembles the cycle described in the
previous paragraph: the Hilbert space is \( L^2(\mathbb{R}, \mathbb{C}^2) \otimes L^2(\mathbb{R}, \mathbb{C}^2) \), the operator is \( D \otimes 1 + 1 \otimes D \), and the representation is given by:

\[
(u \cdot \phi)(x, y) = e^{2\pi i x} \phi(x, y), \quad (v \cdot \phi)(x, y) = \phi(x - \theta - \eta, y)
\]

From here we take a homotopy by rotating this representation around \( \mathbb{R}^2 \) to lie along the diagonal (i.e. so that \((a \cdot \phi)(x, y)\) depends only on \(x + y\)), and the result follows.

By b) the equation \( \tau_b \otimes A_b \cdot [D_h] = [D_{h+b}] \) follows immediately. \( \square \)

**Corollary 5.3.** The Heisenberg twist acts by the identity on \( K_1(A_h) \).

With respect to the ordered basis \( \{ [1], [p_h] \} \) for \( K_0(A_h) \), where \( p_h \) is the Rieffel projection, the morphism \( \tau_b \) acts by matrix multiplication by \( \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \).

**Proof.** The first statement follows from [6].

Consider \( (\tau^b)_*([p_h]) \in K_0(A_h) \). Write

\[
(\tau^b)_*([p_h]) = x[1] + y[p_h].
\]

Pairing both sides with \([D_h]\) and using the Index Theorem Corollary [4, 8] twice gives

\[
x = (\tau^b)_*([p_h]) \otimes A_b \cdot [D_h] = [p_h] \otimes A_b \cdot (\tau^b)^*([D_h]) = [p_h] \otimes A_b \cdot [D_{h+b}] = b.
\]

Pairing the same equation with \([D_{h+b}]\) and computing give that \( y = 1 \).

\( \square \)

It follows from similar simple arguments that in the classical case \( \hbar = b \in \mathbb{Z} \), the Heisenberg classes \([D_b] \in KK_0(C(\mathbb{T}^2), \mathbb{C}) = K_0(\mathbb{T}^2)\) are given by

\[ [D_b] = [pt] + b \cdot [\bar{\partial}] \in K_0(\mathbb{T}^2). \]

For \( b = 1 \), we have noted that \([D_1] = [\bar{\partial} \cdot \mathcal{P}] \). This corresponds to \([\bar{\partial} \cdot \mathcal{P}] = [pt] + [\bar{\partial}] \), which of course follows from the Riemann-Roch formula. We have

\[
\langle [D_n], [E] \rangle = \dim E + n \cdot c_1(E),
\]

for any complex vector bundle \( E \) over \( \mathbb{T}^2 \). Therefore the classes \([D_n]\) taken together determine both the the dimension and first Chern number, the two basic invariants of a complex vector bundle over \( \mathbb{T}^2 \).

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