The scale dependence of the power spectrum can be parametrised in the usual way as \( P_A \propto k^{n_s-1} \), so that \( n_s = 1 \) corresponds to a flat spectrum. Comparing this with Eq. \( \text{(3)} \) we find that the spectral index is

\[
  n_s - 1 = 3 - 2\nu \quad \Rightarrow \quad n_s = 4 - \sqrt{1 + 48\alpha - 4 \left( \frac{m}{H} \right)^2}.
\]

To obtain a scale-invariant spectrum of vector field perturbations we need

\[
  \alpha \approx \frac{1}{6} \left[ 1 + \frac{1}{2} \left( m/H \right)^2 \right].
\]

Hence, we see that we need \( \alpha \gtrsim \frac{1}{6} \). If \( m \gtrsim H \) then scale invariance is attained only when \( \alpha \) is tuned according to Eq. \( \text{(6)} \). However, if \( m \ll H \) then scale-invariance simply requires \( \alpha \approx \frac{1}{6} \). In the latter case \( m \) and \( H \) do not have to balance each other through the condition in Eq. \( \text{(6)} \) and can be treated as free parameters. We feel that this is a more natural setup, so, in the following, we assume \( \alpha \approx \frac{1}{6} \) unless stated otherwise. Since the latest observations deviate from exact scale invariance, \( \alpha \) should not be exactly equal to 1/6. Indeed, according to the 5-year
WMAP results \( n_s = 0.960 \pm 0.014 \) at 1-\( \sigma \). This implies that, when \( m \ll H \), we need \( 6 \alpha = 1.03 \pm 0.01 \).

To study the evolution of the vector field we consider that, for a homogeneous massive Abelian vector field the temporal component \( A_i \) is zero \( \Box \), while the spatial components satisfy the following equation of motion

\[
\ddot{A} + H \dot{A} + \left( m^2 + \frac{\dot{V}}{6} R \right) A = 0, \tag{7}
\]

where we assume that the homogeneous vector field lies along the \( z \)-direction with \( A_z = (0, 0, 0, A(t)) \). During and after inflation, it is easy to show that

\[
R = 3(3w - 1)H^2, \tag{8}
\]

where \( w \) is the barotropic parameter of the Universe: \( w \approx -1 \) \( \{ w = \frac{1}{3} \} \) \{ \mbox{quasi} \}Sitter inflation \{ \mbox{matter domination} \} \{ \mbox{radiation domination} \} \{ \mbox{matter domination} \} \}. Using the above and considering \( m \ll H \) we can obtain the following solution for the zero-mode of the vector field

\[
A = W_0 a + C a^2 (3w-1), \tag{9}
\]

where \( W_0 \) and \( C \) are constants of integration. Thus, the growing mode for the vector field, in all cases, scales as \( A \propto a \). This can be understood as follows.

As discussed in Refs. \[3,4\] \( A_\mu \) is the \textit{comoving} vector field; with the Universe expansion factored-out. The spatial components of the physical vector field, in a FRW geometry are \( W_i \equiv A_i/a \) where \( i = 1, 2, 3 \). This can be understood just by considering the mass term in Eq. \( (1) \), which can be written as

\[
\frac{1}{2} m^2 A_\mu A^\mu = \frac{1}{2} m^2 (A_\perp^2 - A_\parallel A_\perp/a^2), \tag{10}
\]

where Einstein summation is assumed. Since the Lagrangian density is a physical quantity we see that the spatial components of the physical vector field are \( W_i \equiv A_i/a \). Writing the physical vector field as \( W_\mu = (0, 0, 0, W(t)) \) with \( W \equiv A/a \), we can obtain its equation of motion from Eq. \( (7) \) as

\[
\ddot{W} + 3H \dot{W} + m^2 W = 0, \tag{11}
\]

which is identical to the one of a massive scalar field and we used Eq. \( (8) \). When \( m \ll H \) Eq. \( (11) \) has the solution

\[
W = W_0 + C a^2 (w-1), \tag{12}
\]

where \( W_0 \) and \( C \) are constants of integration, consistent with Eq. \( (8) \). Thus, as long as \( m \ll H \), the physical vector field develops a condensate which remains constant \( W \approx W_0 \). This is the physical interpretation of \( A \propto a \).

We can follow the evolution of the vector field condensate by considering the energy momentum tensor, which can be written in the form

\[
T^\mu_\nu = \text{diag}(\rho_A, -p_\perp, -p_\perp, -p_\parallel), \tag{13}
\]

where

\[
\rho_A = \frac{1}{2} \dot{W}^2 + \frac{1}{2} m^2 W^2 \tag{14}
\]

and the transverse and longitudinal pressures are

\[
\begin{align}
p_\perp &= \frac{2}{3}(W^2 - m^2 W^2) + \frac{1}{3}(2HW + H\dot{W} + 3H^2 W)W \tag{15}\\
p_\parallel &= -\frac{1}{6}(W^2 - m^2 W^2) - \frac{2}{3}(2HW + H\dot{W} + 3H^2 W)W.
\end{align}
\]

Thus, the energy-momentum tensor for the homogeneous vector field is, in general, anisotropic because \( p_\parallel \neq p_\perp \). This is why the vector field cannot be taken to drive inflation, for if it did it would generate a substantial large-scale anisotropy, which would be in conflict with the isotropy in the CMB. Therefore, we have to investigate whether, \textit{after} inflation, there is a period in which the vector field becomes isotropic (i.e. \( p_\parallel \approx p_\perp \)) and can imprint its perturbation spectrum onto the Universe.

Considering the growing mode in Eqs. \( (9) \) and \( (12) \), from Eqs. \( (11) \) and \( (12) \) we see that, during and after inflation, when \( m \ll H \), we have

\[
\rho_A \approx \frac{1}{2} m^2 W^2, \quad \text{and} \quad p_\perp \approx -\frac{1}{2} p_\parallel \approx \frac{1}{2}(1-w)H^2 W^2. \tag{16}
\]

Hence, the density of the vector field remains roughly constant, while the vector field condensate remains anisotropic during the hot big bang.

The above are valid under the condition \( m \ll H \). However, after the end of inflation \( H(t) \propto t^{-1} \), so there will be a moment when \( m \sim H \). After this moment, due to Eq. \( (8) \), the curvature coupling becomes negligible and the vector field behaves as a massive minimally-coupled Abelian vector boson. As shown in Ref. \[3\], when \( m \gg H \) a massive vector field undergoes (quasi)harmonic oscillations of frequency \( \sim m \), because the friction term in Eqs. \( (7) \) and \( (11) \) becomes negligible. In this case, on average over many oscillations, it has been shown that

\[
\frac{W^2}{2} \approx m^2 W^2 \tag{17}
\]

Hence, Eqs. \( (11) \) and \( (12) \) become

\[
\rho_A \propto \frac{m^2 W^2}{2} \quad \text{and} \quad p_\perp \propto -\frac{1}{2} p_\parallel \propto \frac{1}{2} m H \left[ 1 + \frac{3}{2}(1-w)\frac{H}{m} \right] W^2.
\]

The effective barotropic parameters of the vector field are

\[
0 < \omega_\perp \simeq -\frac{1}{2} w_\parallel = \frac{3}{2} \left[ 1 + \frac{3}{2}(1-w) \left( \frac{H}{m} \right) \right] \left( \frac{H}{m} \right) \ll 1, \tag{18}
\]

where \( w_\perp = p_\perp/\rho_A \) and \( w_\parallel = p_\parallel/\rho_A \). By virtue of the condition \( m \gg H \), we see that, after the onset of the oscillations, \( w_\perp, w_\parallel \to 0 \). This means that the oscillating massive vector field behaves as pressureless \textit{isotropic} matter, which can dominate the Universe without generating a large-scale anisotropy. Moreover, its density can be shown to decrease as \( \rho_A \propto a^{-3} \) (like dust) as expected \[3\]. Thus, if the Universe is radiation dominated, \( \rho_A/\rho \propto a^{-3} \) while oscillations occur, so the field has a chance to dominate the Universe and imprint its curvature perturbation according to the curvaton scenario \[3\].

At the onset of the oscillations we have

\[
\Omega \equiv \frac{\rho_A}{\rho} \sim \left( \frac{W_0}{m_P} \right)^2, \tag{19}
\]

where \( \Omega \)
where we used the flat Friedman equation $\rho = 3m_P^2H^2$ with $m_P = 2.4 \times 10^{18}$ GeV being the reduced Planck mass. To avoid excessive anisotropy the density of the vector field must be subdominant before the onset of oscillations, which means that $W_0 < m_P$.

Let us assume that inflation is driven by some inflaton field, which after inflation ends, oscillates around its VEV until its decay into a thermal bath of relativistic particles at reheating. In this scenario the Universe is matter dominated (by inflaton particles) until reheating. Using the above findings we can estimate the Hubble scale when the vector field dominates the Universe as

$$H_{\text{dom}} \sim \min\{m, \Gamma\} \left(\frac{W_0}{m_P}\right)^4,$$  \hspace{1cm} (20)

where $\Gamma$ is the decay rate of the inflaton field. If inflation gives away directly to a thermal bath of particles then $\Gamma = \frac{1}{\tau}$, where $\tau$ is the time to decay the field into the thermal bath. Then $\Gamma$ is determined by the density ratio of the vector field at decay $\rho_{\text{dec}}$ and its perturbation $\delta W$. The condition in Eq. (6), $\nu \approx \frac{1}{2}$ and Eq. (3) gives $\rho_{\text{dec}} \geq \frac{1}{\tau}$. Hence, from Eqs. (20) and (21) we can write

$$\frac{H_{\text{dom}}}{\Omega_{\text{dec}}H_{\text{dec}}} \sim \frac{\zeta}{\Omega_{\text{dec}}\zeta_A},$$  \hspace{1cm} (22)

where $\zeta_A$ is the curvature perturbation attributed to the curvaton field. In a foliage of spacetime of spatially flat hypersurfaces $\zeta_A = -H \frac{\delta \rho_A}{\rho_A} = \frac{1}{3} \frac{\delta \rho_A}{\rho_A} \bigg|_{\text{dec}},$  \hspace{1cm} (23)

where we used that the vector field decays after the onset of oscillations in which case $\delta \rho_A \propto a^{-3}$. Note that, since $\zeta_A$ is determined by the fractional perturbation of the field’s density, which is a scalar quantity, the perturbation $\zeta_A$ is scalar and not vector in nature.

Now, since Eq. (24) is a linear differential equation, $W$ and its perturbation $\delta W$ satisfy the same equation of motion. Therefore, they evolve in the same way, which means that $\delta W/W$ remains constant, before and after the onset of oscillations. As shown in Ref. [2], during the (quasi)harmonic oscillations of the massive vector field, $\rho_A = m^2W^2/2$, where $W$ is the amplitude of the oscillating physical vector field. From the above we obtain

$$\zeta_A = \frac{2}{3} \frac{\delta W}{W} \bigg|_{\text{osc}} = \frac{2}{3} \frac{\delta W}{W} \bigg|_{d_{\text{dec}}},$$  \hspace{1cm} (24)

where ‘osc’ denotes the onset of oscillations and the star denotes the time when cosmological scales exit the horizon during inflation.

If $m \ll H$ during inflation the physical vector field (not being conformally invariant) undergoes particle production and obtains an approximately flat superhorizon spectrum of perturbations, as shown. Indeed, under the condition in Eq. (6), $\nu \approx \frac{1}{2}$ and Eq. (3) gives $\rho_{\text{dec}} \geq \frac{1}{\tau}$. Hence, from Eqs. (20) and (21) we can write

$$\frac{H_{\text{dom}}}{\Omega_{\text{dec}}H_{\text{dec}}} \sim \frac{\zeta}{\Omega_{\text{dec}}\zeta_A},$$  \hspace{1cm} (22)

Thus, from the above and Eq. (22) we obtain

$$\zeta \sim \Omega_{\text{dec}}H_{\text{dec}}/W_0,$$  \hspace{1cm} (27)

Using this, Eqs. (20) and (21), after some algebra, we get

$$\frac{H_*}{m_P} \sim \frac{\zeta}{\Omega_{\text{dec}}} \left(\frac{\max\{H_{\text{dom}}, \Gamma_A\}}{\min\{m, \Gamma\}}\right)^{1/4}.$$  \hspace{1cm} (28)

The Hot Big Bang has to begin before nucleosynthesis (which occurs at temperature $T_{\text{BBN}} \sim 1$ MeV). Hence, max{$H_{\text{dom}}, \Gamma_A$} $\geq T_{\text{BBN}}^2/m_P$. Using this and also $\min\{m, \Gamma\} \lesssim H_*$, we obtain the bound

$$H_* \gtrsim \zeta^{4/5} \Omega_{\text{dec}}^{-2/5} (T_{\text{BBN}}m_P)^{1/5} \Rightarrow V_*^{1/4} \gtrsim 10^{12}$ GeV, \hspace{1cm} (29)

where we used that $\Omega_{\text{dec}} \lesssim 1$ and $\zeta = 4.8 \times 10^{-5}$ from COBE observations. This is similar to the case of a scalar field curvaton [8].

Another bound on the inflation scale is obtained by considering that $\Gamma_A \sim g^2m$, where $g$ is the vector field coupling to its decay products, for which $g \gtrsim m/m_P$ due to gravitational decay. Thus, max{$H_{\text{dom}}, \Gamma_A$} $\gtrsim g^2m$. Combining with Eq. (28) we obtain the bound

$$H_* \gtrsim \zeta^{4/5} \Omega_{\text{dec}}^{-1/2}(m_Pm)^{1/2} \Rightarrow V_*^{1/4} \gtrsim 10^{13}$ GeV, \hspace{1cm} (30)

where we took $m \gtrsim 1$ TeV.

Finally, an upper bound on inflation scale can be obtained by combining Eq. (27) with the requirement $W_0 < m_P$, thereby finding

$$H_* < \zeta m_P/\Omega_{\text{dec}} \Rightarrow V_*^{1/4} < 10^{17}$$ GeV, \hspace{1cm} (31)

where we considered that $\Omega_{\text{dec}} \gtrsim 10^{-2}$, in order to avoid excessive non-Gaussianity in the CMB [5].

We also need to consider the hazardous possibility of the thermal evaporation of the vector field condensate. Were this to occur, all memory of the superhorizon spectrum of perturbations would be erased. Considering that the scattering rate of the massive vector bosons with the thermal bath is $\Gamma_{\text{sc}} \sim g^4T$ we can obtain a bound such that the condensate does not evaporate before the vector field decays. Since $\Gamma_{\text{sc}}/\Gamma_A \sim a^{-1}$, we need to enforce this bound at the onset of the oscillations, when $\Gamma_{\text{sc}} \sim g^4\sqrt{m_Pm}$. Hence, the range for $g$ is

$$\frac{m}{m_P} \lesssim g \lesssim \left(\frac{m}{m_P}\right)^{1/4},$$  \hspace{1cm} (32)
where the lower bound is due to gravitational decay. Note that, in the case when the vector curvaton dominates the Universe before its decay the condensate may not evaporate even if the above upper bound is violated. This is because, after domination, the density of the thermal bath is exponentially smaller than \( \rho_A \) by a factor of \( (H_{\text{dom}}/H)^{2/3} \). Moreover, even if it does evaporate the condensate has already imprinted \( \zeta_A \) onto the Universe at domination rendering the evaporation bound irrelevant.

The above lower bounds on \( H_* \) can be substantially relaxed by employing the so-called mass increment mechanism according to which, the vector field obtains its bare mass at a phase transition (denoted by ‘pt’) with \( m/H_{\text{pt}} \gg 1 \). The mechanism was firstly introduced for the scalar curvaton in Ref. [9] and has been already implemented in the vector curvaton case in Ref. [4].

To illustrate our findings let us consider a specific example. Let us choose \( m \sim 10^{-10} \) \( \text{GeV} \) and also \( \Gamma_A \sim 10^{-10} \) \( \text{GeV} \) such that the temperature at the vector field decay is \( T_{\text{dec}} \sim 10 \) \( \text{TeV} \). Such a particle may be potentially observable in the LHC. These values suggest \( g \sim 10^{-7} \), which lies comfortably within the range in Eq. [52]. For the decay rate of the inflaton let us choose \( \Gamma \sim 10^{-2} \) \( \text{GeV} \) so that the reheating temperature satisfies the gravitino overproduction constraint \( T_{\text{reh}} \sim \sqrt{m_{\text{pl}} \Gamma} \sim 10^8 \) \( \text{GeV} \). Assume at first that the vector curvaton decays before domination \( \Gamma_A > H_{\text{dom}} \). Then Eq. (28) reduces to \( H_* m_P \sim 10^{-2} \zeta/\sqrt{T_{\text{dec}}} \). Using this and Eq. (27) we get \( W_0 / m_P \sim 10^{-2} \sqrt{T_{\text{dec}}} \). Hence, the lowest value for the inflation Hubble scale is \( H_* \gtrsim 10^{12} \) \( \text{GeV} \). It can be readily checked that the bound in Eq. (30) is weaker by a factor \( 10^{-5} \). Suppose now that the vector curvaton dominates before its decay \( \Gamma_A < H_{\text{dom}} \). Using Eq. (20) we get \( W_0 > 10^{-2} m_P \), while Eq. (28) suggests \( H_* \sim \zeta W_0 \). Taking into account the bound \( W_0 > m_P \), we find that the maximum value for the Hubble scale is \( H_* < 10^{14} \) \( \text{GeV} \). The relation between \( H_* \) and \( W_0 \) in both cases is depicted in Fig. 1.

![FIG. 1: Parameter space for \( H_* \) and \( W_0 \) in our example.](image)

Let us consider now the case when \( \alpha \neq \frac{1}{2} \). If \( \alpha = O(1) \) then, according to Eq. (6), a scale invariant spectrum is possible only if \( m \sim H_* \). Hence, the oscillations begin immediately after the end of inflation. With this in mind the previous analysis remains valid. In particular, the bound in Eq. (29) remains the same. However, the bound in Eq. (30) becomes much more stringent:

\[
H_* \gtrsim \zeta^2 m_P \Rightarrow V_*^{1/4} \gtrsim 10^{14} \text{GeV} .
\] (33)

Hence, in view of Eq. (31), we see that the inflation energy scale is constrained near that of grand unification.

In summary, we have discussed a concrete model which generates the curvature perturbation in the Universe with a single massive Abelian vector boson field non-minimally coupled to gravity through an \( RA^2 \) coupling. The vector field can act as a curvaton, imposing its scalar perturbation spectrum well after the end of inflation without introducing a large-scale anisotropy. We have shown that there is ample parameter space for the model to work by considering all relevant constraints in the cosmology. The VEV of the vector curvaton is zero, which means that it does not violate Lorentz invariance in the vacuum. Our model does not need to rely on scalar fields at all since inflation might take place due to purely geometrical effects, such as in \( f(R) \)-gravity models [10] (e.g. \( R^2 \)-inflation [11]). The remaining challenge is to realise our mechanism in the context of a realistic setup beyond the standard model [12].

Recently, vector fields have been employed to drive inflation [12] (see also Ref. [13]). To avoid a large-scale anisotropy the authors of Ref. [7] introduce a large number of vector fields randomly orientated in space. However, they do not consider the generation of curvature perturbations, which could proceed along the lines of this work, albeit introduced during and not after inflation.

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