REFUTING SPECTRAL COMPATIBILITY
OF QUANTUM MARGINALS

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Abstract. The spectral variant of the quantum marginal problem asks: Given prescribed spectra for a set of quantum marginals, does there exist a compatible joint state? The main idea of this work is a symmetry-reduced semidefinite programming hierarchy for detecting incompatible spectra. The hierarchy is complete, in the sense that it detects every incompatible set of spectra. The refutations it provides are dimension-free, certifying incompatibility in all local dimensions. The hierarchy equally applies to the sums of Hermitian matrices problem, to optimize trace polynomials on the positive cone, to the compatibility of invariants, and to certify vanishing Kronecker coefficients.

1. Introduction

The compatibility of quantum marginals (also known as reduced density matrices) is central to quantum phenomena such as entanglement and non-locality. It also plays a key role in quantum algorithms like quantum error correction and adiabatic quantum computation. At the heart of this quantum marginal problem lies a constraint satisfaction problem with prohibitive computational complexity: it is QMA-complete, with QMA being the quantum analogue of NP [Liu06]. This renders molecular-structure and ground state calculations in chemistry1 and physics challenging. Consequently, a large literature focuses on conditions for quantum marginals to be compatible [Sch14, Wal14, Hub17, Kla17].

A more fundamental problem is to decide compatibility of the spectra instead of the reduced density matrices. The simplest formulation of the spectral variant of the quantum marginal problem is perhaps the following: Given a set of prescribed eigenvalues \( \vec{\lambda}_{AB} \) and \( \vec{\lambda}_{BC} \) associated to subsystems \( AB \) and \( AC \), does there exist a joint state \( \varrho_{ABC} \) such that its reduced density matrices \( \varrho_{AB} = \text{tr}_C(\varrho_{ABC}) \) and \( \varrho_{BC} = \text{tr}_A(\varrho_{ABC}) \) have spectra \( \lambda_{AB} \) and \( \lambda_{BC} \)? If such a joint state exists, the spectra are said to be compatible; they are incompatible otherwise. This spectral formulation also maintains an intimate connection to fundamental questions in representation and matrix theory [Kly98].

The works by Klyachko [Kly06] and Christandl, Şahinoğlu and Walter [CSW18] allow one to establish compatibility of prescribed spectra through representation theoretic

1For fermionic systems, the quantum marginal problem is also known as the \( N \)-representability problem, the full set of conditions for two-electron reduced states were given by Mazziotti in [Maz12].
methods: compatible spectra correspond to families of non-vanishing Kronecker and recoupling coefficients. The positivity of these coefficients can be established with algorithms from algebraic combinatorics and geometric complexity theory [BI08, BVW18]. It is much harder, however, for these methods to determine the incompatibility of marginal spectra.

The aim of this manuscript is to provide such converse method: a semidefinite programming hierarchy for certifying spectral incompatibility, where the marginals are allowed to overlap (Section 5). It is complete, in the sense that it detects every set of incompatible spectra. Our formulation in terms of a symmetric extension hierarchy is furthermore symmetry-reduced, drastically reducing the size of the optimization problem (Section 6). This approach can produce spectral incompatibility certificates for both finite fixed local dimensions and for arbitrary local dimensions (Section 7). In particular, it allows a modern desktop computer to work with up to four copies of four-partite states, and five copies of tri-partite states of arbitrary local dimensions (Section 9).

2. Contribution

Let $\rho \in L((C^d)^\otimes n)$ be an $n$-partite quantum state of local dimension $d$ and $A$ a collection of subsystems of $\{1, \ldots, n\}$. Given a subsystem $A \in A$, denote by $\rho_A = \text{tr}_{A^c}(\rho)$ the reduced density matrix on $A$ and by $\mu_A$ the eigenvalues of $\rho_A$, i.e., the spectrum on $A$. We want to answer the following:

**Problem.** Let $A$ be a collection of subsets of $\{1, \ldots, n\}$. Given prescribed spectra $\{\mu_A | A \in A\}$, does there exist a joint state $\rho$ for which the spectrum of $\rho_A = \text{tr}_{A^c}(\rho)$ equals $\mu_A$ for all $A \in A$?

We provide the following symmetry-reduced semidefinite programming hierarchy for determining spectral incompatibility for overlapping marginals.

**Theorem A.** Let $A$ be a collection of subsets of $\{1, \ldots, n\}$ with associated marginal spectra $\{\mu_A | A \in A\}$. The spectra are compatible with a joint quantum state on $(C^d)^\otimes n$ if and only if every level in the hierarchy (SDP-SC) is feasible. If a level of the hierarchy returns a negative value, then the spectra are incompatible.

For the proof see Theorem 7 and Theorem 10. The symmetry-reduction allows to work with up to four copies (i.e., the fourth step of the hierarchy) of four-partite systems and five copies of three-partite systems on a modern desktop computer (see Figure 1 and Table 2).

**Theorem B.** When the number of copies is less or equal than the local dimension ($k \leq d$), the incompatibility witnesses produced by the hierarchy (SDP-SC) are dimension-free and the spectra are incompatible in all local dimensions.

For the proof see Theorem 9. As a consequence, the SDP refutations stabilize when the number of copies equals the local dimension, certifying incompatibility for all local dimensions.

2.1. Further applications.

(1) **Kronecker and recoupling coefficients.** Klyachko has shown that $\text{spect}(\rho_A)$, $\text{spect}(\rho_B)$, and $\text{spect}(\rho_{AB})$ are compatible if and only if dilations of associated Young tableaux $\lambda, \mu, \nu$ allow for a non-vanishing Kronecker coefficient, $g(m\lambda, m\mu, m\nu) \neq 0$ for some $m > 0$ [Kly04],\(^2\) A similar statement holds for marginals of tripartite systems [CŠW18]. Deciding positivity of Kronecker coefficients is an NP-hard

\(^2\)See also the work by Christandl and Mitchison [CM06] that showed one direction of this statement.
task [IMW17]. The algorithm by Baldoni, Vergne, and Walter allows to compute
dilated Kronecker coefficients [BVW18]. Our hierarchy provides a converse
method: to show the vanishing of Kronecker coefficient \( g(m\lambda, m\mu, m\nu) \) for any
dilation \( m \in \mathbb{N} \), by certifying the corresponding spectral incompatibility.

(2) Sums of Hermitian Matrices. The Sums of Hermitian Matrices problem
(solved by Klyachko [Kly98] and related to honeycombs by Knutsen and Tao [KT01])
asks: given Hermitian matrices \( A \) and \( B \) with spectra \( \text{spec}(A) \) and \( \text{spec}(B) \), what
are the constraints on the spectrum of \( A + B \)? It can be shown that this problem
is denoted by \( U \). A \( n \)-partite state

is denoted by \( \psi \). The marginal or
\( \text{spec}(\pi \sigma \rho ) \). It is
\( \rho \). In what follows, \( A \) is a collection of subsets \( A \subseteq \{1, \ldots, n\} \). In what follows, \( A \) is a collection of subsets \( A \subseteq \{1, \ldots, n\} \). The coordinate-free definition of the partial trace states that

\[
\text{tr} \left( (M \otimes 1)N \right) = \text{tr} \left( M \text{tr}_2(N) \right)
\]
holds for all \( M \in L(\mathcal{H}_1) \) and \( N \in L(\mathcal{H}_1 \otimes \mathcal{H}_2) \). Finally, the set of unitary \( d \times d \) matrices
is denoted by \( \mathcal{U}(d) \).

3.1. Quantum systems. Denote by \( L(\mathcal{H}) \) the space of linear maps acting on a Hilbert
space \( \mathcal{H} \). Quantum states on \( n \) systems with \( d \) levels each are represented by positive
operators of trace one acting on \( (C^d)^\otimes n \), i.e., satisfying \( \rho \geq 0 \), \( \text{tr}(\rho) = 1 \). The marginal or
reduced state of an \( n \)-partite state \( \rho \) on subsystem \( A \) is denoted by \( \rho_A = \text{tr}_{A^c}(\rho) \), where \( A^c \)
is the complement of \( A \) in \( \{1, \ldots, n\} \). In what follows, \( A \) is a collection of subsets \( A \subseteq \{1, \ldots, n\} \). The coordinate-free definition of the partial trace states that

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is denoted by \( \mathcal{U}(d) \).

3.2. Symmetric group. Our work makes use of \( k \) copies of \( n \)-particle states, with the
symmetric group acting on both copies and their subsystems. The symmetric group
permuting \( k \) elements is \( S_k \). The group ring \( \mathbb{C}S_k \) is formed by formal sums \( \mathbb{C}S_k = \{ \sum_{\sigma \in S_k} a_\sigma \sigma : a_\sigma \in \mathbb{C} \} \). Linearly extending the multiplication of \( S_k \) gives the multipli-
cation on \( \mathbb{C}S_k \). An element \( a = \sum_{\sigma \in S_k} a_\sigma \sigma \) has the adjoint \( a^* = \sum_{\sigma \in S_k} a_\sigma \sigma^{-1} \); it is
Hermitian if \( a = a^* \).

Denote \( S_k^n = S_k \times \cdots \times S_k \) the \( n \)-fold cartesian product of \( S_k \). Let \( S_k \) act on \( \sigma = (\sigma_1, \ldots, \sigma_n) \in S_k^n \) via

\[
\pi \sigma \pi^{-1} := (\pi \sigma_1 \pi^{-1}, \ldots, \pi \sigma_n \pi^{-1})
\]
where $\pi \in S_k$, and by linear extension also on $CS^n_k$. Finally, $(CS^n_k)^{S_k}$ is the subspace of $CS^n_k$ invariant under the diagonal action of $S_k$.

\[(CS^n_k)^{S_k} = \{ a \in CS^n_k : a = \pi a \pi^{-1}, \pi \in S_k \}.\]

### 3.3. Representations

Let $\sigma \in S_k$ act on $(\mathbb{C}^d)^{\otimes k}$ by its representation $\eta_d(\sigma)$, that permutes the tensor factors,

\[\eta_d(\sigma) |v_1 \rangle \otimes \cdots \otimes |v_k \rangle = |v_{\sigma^{-1}(1)} \rangle \otimes \cdots \otimes |v_{\sigma^{-1}(k)} \rangle.\]

Now consider $\sigma = (\sigma_1, \ldots, \sigma_n) \in S^n_k$. It acts on $((C^d)^{\otimes k})^{\otimes n}$ as

\[\eta(\sigma) := \eta_d(\sigma_1) \otimes \cdots \otimes \eta_d(\sigma_n),\]

with $\eta_d(\sigma_i)$ acting on the collection of the $k$ copies of the $i$'th tensor factor.\(^3\)

Another action we need is that of permuting the $k$ copies of $n$-partite states. For $\pi \in S_k$, the representation $\tau$ acts diagonally conjugate on $((\mathbb{C}^d)^{\otimes k})^{\otimes n}$,

\[\tau(\pi) \eta(\sigma) \tau(\pi^{-1}) := \eta_d(\pi \sigma_1 \pi^{-1}) \otimes \cdots \otimes \eta_d(\pi \sigma_n \pi^{-1}) = \eta(\pi \sigma \pi^{-1}),\]

making it compatible with Eq. (2).

Finally, a representation $R$ is orthogonal if $R(g^{-1}) = R(g)^T$. For the symmetric group, Young’s orthogonal representation is orthogonal [JK84]. In the software Sagemath, it can be obtained with the command `SymmetricGroupRepresentation` [The22].

### 4. Spectra are polynomial in $\varrho$

#### 4.1. Spectrum from $\varrho^{\otimes k}$

We first show how the spectral quantum marginal problem can be formulated as a constraint that is polynomial in $\varrho$. This is done with a generalization of the swap trick.

Consider a single quantum system $\varrho \in L(C^d)$. It is clear that $\text{tr}(\varrho^\ell) = \sum \mu_i^\ell$, where the $\mu_i$ are the eigenvalues of $\varrho$. A complex $d \times d$ matrix has $d$ eigenvalues, such that the set $\{\text{tr}(\varrho^\ell) : \ell = 1, \ldots, d\}$ determines the spectrum of $\varrho$.

Recall that $\sigma \in S_k$ acts on $(\mathbb{C}^d)^{\otimes k}$ via its representation $\eta_d(\sigma)$ that permutes the tensor factors,

\[\eta_d(\sigma) |v_1 \rangle \otimes \cdots \otimes |v_k \rangle = |v_{\sigma^{-1}(1)} \rangle \otimes \cdots \otimes |v_{\sigma^{-1}(k)} \rangle.\]

For a cycle $(\alpha_1 \ldots \alpha_\ell) \in S_k$ of length $\ell \leq k$ and a Hermitian matrix $B \in L(C^d)$, it is known that $\text{tr}(\eta_d((\alpha_1 \ldots \alpha_\ell))B^{\otimes k}) = \text{tr}(B^\ell) \text{tr}(B)^{k-\ell}$ [Hub21]. For a density matrix this simplifies further to

\[\text{tr}(\eta_d((\alpha_1 \ldots \alpha_\ell))\varrho^{\otimes k}) = \text{tr}(\varrho^\ell).\]

Consequently, under a global trace, the permutation operators acting on copies of a state $\varrho$ can recover its spectrum.

#### 4.2. Permuting subsystems of copies

A similar strategy works with multipartite states. Then, we additionally need to consider the action of permutations on subsystems.

Recall that the element $\sigma \in S^n_k$ acts on $((\mathbb{C}^d)^{\otimes k})^{\otimes n}$ via

\[\eta(\sigma) := \eta_d(\sigma_1) \otimes \cdots \otimes \eta_d(\sigma_n),\]

with $\eta_d(\sigma_i)$ acting on the $k$ copies of the $i$'th tensor factor. Now, for a subset $A \subseteq \{1, \ldots, n\}$, define $\sigma^A = (\sigma_1^A, \ldots, \sigma_n^A) \in S^n_k$ through

\[\sigma_i^A = \begin{cases} \sigma & \text{if } i \in A \\ \text{id} & \text{if } i \not\in A \end{cases}.\]

\(^3\)This is the same setting as found in Ref. [Rai00].
By Eq. (9), the operator $\eta(\sigma^A)$ acts on the collection of subsystems contained in $A$ with $\sigma$, and with the identity matrix on the remaining subsystems.

With some abuse of notation, $\eta(\sigma^A)$ can be thought of acting on $((C^d)^{\otimes n})^\otimes k$ as well as on any tensor space containing the subsystem $A$. For $\ell \leq k$, Eq. (8) generalizes to

$$\text{tr}\left( \eta((\alpha_1 \ldots \alpha_\ell)^A)\varrho^\otimes k \right) = \text{tr}\left( \eta((\alpha_1 \ldots \alpha_\ell)^A)\varrho^\otimes k \right) = \text{tr}\left( \varrho^\ell_A \right).$$

(11)

where we have used the coordinate-free definition of the partial trace in Eq. (1).

Let a prescribed spectrum $\mu_A$ on subsystem $A$ be given. Define

$$q_{A,\ell} = \sum_{\mu_i \in \mu_A} \mu_i^\ell.$$

(12)

If a $\varrho$ realizing $\mu_A$ exists, then for any $(\alpha_1 \ldots \alpha_\ell) \in S_k$,

$$q_{A,\ell} = \text{tr}(\eta(\sigma^A)\varrho^\otimes k) = \text{tr}(\varrho^\ell_A).$$

(13)

More generally, for any local unitary invariant polynomial function of reductions, one can define for a suitable $\sigma \in S^2_k$ the quantity

$$q_{A,\sigma} = \text{tr}(\eta(\sigma^A)\varrho^\otimes k).$$

(14)

4.3. Compatibility conditions. Denote by $H = (C^d)^\otimes n$ the space of a $n$-qudit system. Our discussion makes the following immediate.

**Proposition 1.** Let $A$ be a collection of subsystems of $\{1, \ldots, n\}$ and $\mu = \{\mu_A | A \in A\}$ be spectra of prescribed reductions. Let $m$ be the size of the largest spectrum and $q_{A,\ell}$ be given in terms of $\mu_A$ by Eq. (12). Then $\mu$ is compatible with a joint state, if and only if there exists $\varrho \in L(H)$, such that for all $\ell$-cycles $\sigma = (\alpha_1, \ldots, \alpha_\ell)$ with $\ell = 1, \ldots, m$, and $A \in A$,

$$\text{tr}(\eta(\sigma^A)\varrho^\otimes m) = q_{A,\ell}.$$ 

(15)

**Proof.** If a compatible $\varrho$ exists, then $\text{tr}(\eta(\sigma^A)\varrho^\otimes m)$ evaluates through Eq. (13) to $q_{A,\ell}$. Conversely, if there exists a $\varrho$ satisfying Eq. (15) for all $\ell$-cycles and $A \in A$, then its spectrum on $A$ is completely determined and equal to $\mu_A$ for all $A \in A$. \qed

4.4. Symmetric extension relaxation. We relax Proposition 1 to:

**Proposition 2.** Let $A$ be a collection of subsystems of $\{1, \ldots, n\}$ and $\mu = \{\mu_A | A \in A\}$ be prescribed spectra of reductions. If the spectra $\mu$ are compatible with a joint state, then for every $k \in \mathbb{N}$ there exists a state $\varrho_k \in L(H^\otimes k)$ such that for all $\ell$-cycles $\sigma = (\alpha_1, \ldots, \alpha_\ell)$ with $\ell = 1, \ldots, k$, and $A \in A$,

$$\text{tr}(\eta(\sigma^A)\varrho_k) = q_{A,\ell}.$$ 

(16)

It is clear that the constraints in Proposition 2 are weaker than those in Proposition 1.

**Remark 3.** One could add the constraint of a positive partial transpose $\varrho_k^{TR} \geq 0, \forall R \subseteq \{1, \ldots, n\}$ to Proposition 2. However, this approach is not directly suitable to the symmetry reduction method employed in this manuscript.
4.5. **Invariance.** One can see that if \( \varrho_k \) satisfies Eq. (16), then so do the states in the set
\[
\{ (U_1 \otimes \ldots \otimes U_n)^{\otimes k} \varrho_k ( (U_1 \otimes \ldots \otimes U_n)^{\dagger} )^{\otimes k} : U_1, \ldots, U_n \in \mathcal{U}(d) \}.
\]
This can be understood from the fact that the eigenvalues of a matrix are unitary invariants. As a second invariance, also the states in
\[
\{ \tau(\pi) \varrho_k \tau(\pi)^{-1} : \pi \in S_k \},
\]
where \( \tau(\pi) \) acts diagonally on \( ((\mathcal{C}^d)^{\otimes k})^\otimes \), will satisfy Eq. (16).

These are the symmetries of local unitary invariants (including local spectra), expressed as polynomials in the density matrix. We will use both symmetries in the next section to formulate an invariant hierarchy of semidefinite programs.

5. **SDP refutation**

5.1. **Primal and dual programs.** We follow Watrous [Wat12] and Doherty, Parrilo, and Spedalieri [DPS04] to recall: a semidefinite program (SDP) is specified by a hermiticity preserving linear map \( \Xi : L(X) \to L(Y) \) and Hermitian operators \( C \) and \( D \). Define the inner product \( \langle A, B \rangle = \text{tr}(A^\dagger B) \), and denote the set of positive and hermitian operators on a Hilbert space \( \mathcal{H} \) by \( \text{Herm}(\mathcal{H}) \) and \( \text{Pos}(\mathcal{H}) \) respectively. Then the primal and dual problem of the semidefinite program read

\[
\begin{align*}
\text{Primal:} & \quad \max_X \langle C, X \rangle \quad \text{such that} \quad \Xi(X) = D \quad X \in \text{Pos}(X) \\
\text{Dual:} & \quad \min_Y \langle D, Y \rangle \quad \text{such that} \quad \Xi^*(Y) \geq C \quad Y \in \text{Herm}(Y)
\end{align*}
\]

Operators \( X \) and \( Y \) satisfying (19) and (20) are said to be primal and dual feasible, respectively. Denote the set of primal and dual feasible operators by \( \mathcal{P} \) and \( \mathcal{D} \). Every semidefinite program satisfies weak duality, that is, for all \( X \in \mathcal{P} \) and \( Y \in \mathcal{D} \),

\[
\langle D, Y \rangle - \langle C, X \rangle = \langle \Xi(X), Y \rangle - \langle C, X \rangle = \langle \Xi^*(Y) - C, X \rangle \geq 0.
\]

Interestingly, weak duality (21) can be used to give an SDP refutation for the feasibility \( (C = 0) \) of a primal problem: if there exists a feasible \( Y \in \mathcal{D} \) with \( \langle D, Y \rangle < 0 \), weak duality (21) is violated. This implies that the primal problem is infeasible. The operator \( Y \) provides then a certificate of infeasibility.

5.2. **Primal hierarchy.** Incorporating the symmetries (17) and (18), Proposition 2 can be formulated as a hierarchy of semidefinite programs for feasibility \( (C = 0) \), indexed by \( k \in \mathbb{N} \).

\[
\begin{align*}
\text{Primal:} & \quad \max_X \langle 0, X \rangle \quad \text{such that} \quad \text{tr}(X) = 1 \\
& \quad \text{tr} \left( \eta(\sigma^A)X \right) = q_{A,\sigma} \quad \forall A \in \mathcal{A}, \quad \sigma \in S_k \\
& \quad X = \tau(\pi)X\tau(\pi)^{-1} \quad \forall \pi \in S_k \\
& \quad X = \mathcal{U}^{-1}X\mathcal{U} \quad \forall \mathcal{U} = (U_1 \otimes \ldots \otimes U_n)^{\otimes k} : U_1, \ldots, U_n \in \mathcal{U}(d) \\
& \quad X \in \text{Pos}(\mathcal{H}^{\otimes k})
\end{align*}
\]

Note that some elements in \( S_k \), for example (12)(34), are of the form \( \sigma \times \sigma^{-1} \). For these the corresponding constraints are “quadratic”: \( \text{tr} \left( \eta(\sigma^A) \otimes \eta(\sigma^A)^\dagger X \right) = q_{A,\sigma \times \sigma^{-1}} = q_{A,\sigma}^2 \).
This will be relevant for completeness of the hierarchy, which we show in Theorem 10. For now we return to the question of feasibility of this program.

In (22) and using the notation of (19), we write

\[
\Xi(X) = \bigoplus_{A \in A, \sigma \in S_k} \Xi_{A,\sigma}(X)
\]

where the hermitian maps \( \Xi_{A,\sigma} \) and their duals are given by

\[
\Xi_{A,\sigma}(X) = \frac{1}{2} \text{tr} \left( (\eta(\sigma^A) + \eta(\sigma^A)\dagger)X \right),
\]

\[
(\Xi_{A,\sigma})^* (y^{A,\sigma}) = \frac{1}{2} y^{A,\sigma} (\eta(\sigma^A) + \eta(\sigma^A)\dagger),
\]

with associated constants \( D^{A,\sigma} = q_{A,\sigma} \).

5.3. Dual hierarchy. Consider now the dual of the hierarchy in Eq. (22). We first start by identifying the symmetries present in the dual. The objective function of the dual program is

\[
\langle D, Y \rangle = \langle \Xi(X), Y \rangle = \langle X, \Xi^*(Y) \rangle.
\]

We can now apply the symmetries of \( X \) to see that

\[
\langle X, \Xi^*(Y) \rangle = \langle \eta(\pi)X\eta(\pi)^{-1}, \Xi^*(Y) \rangle = \langle X, \eta(\pi)^{-1}\Xi^*(Y)\eta(\pi) \rangle,
\]

\[
\langle X, \Xi^*(Y) \rangle = \langle \Upsilon X\Upsilon^{-1}, \Xi^*(Y) \rangle = \langle X, \Upsilon^{-1}\Xi^*(Y)\Upsilon \rangle,
\]

holds for all \( \pi \in S_k \) and unitaries of the form \( \Upsilon = (U_1 \otimes \ldots \otimes U_n)^{\otimes k} \).

Thus, we can write dual of the hierarchy in Eq. (22), indexed by \( k \in \mathbb{N} \), as

\[
\text{Dual :}
\]

\[
\begin{align*}
\text{minimize } & \sum_{A \in A, \sigma \in S_k} y^{A,\sigma} q_{A,\sigma} \\
\text{such that } & \Xi^* = \tau(\pi)\Xi^*\tau(\pi)^{-1} \quad \forall \pi \in S_k \\
& \Xi^* = \Upsilon \Xi^*\Upsilon^{-1} \quad \forall \Upsilon = (U_1 \otimes \ldots \otimes U_n)^{\otimes k} : U_1, \ldots, U_n \in \mathcal{U}(d) \\
& \Xi^* \in \text{Pos}(\mathcal{H}^{\otimes k})
\end{align*}
\]

where

\[
\Xi^* = \Xi^*(Y) = \sum_{A \in A, \sigma \in S_k} y^{A,\sigma} \eta(\sigma^A)
\]

Remark 4. We say that an element \( \sigma = (\sigma_1, \ldots, \sigma_n) \in \mathbb{C}S^n_k \) factorizes if the operator \( \eta(\sigma) \) factorizes along the copies where its cycles act. Thus, factorizing permutations can be evaluated by polynomials in \( q_{A,\ell} \). For example, \((12), (12)(34), (34)\) \( \in S_3^3 \) yields

\[
\text{tr}(\eta(\sigma)\varrho^4) = \text{tr}(\varrho^2_{AB}) \text{tr}(\varrho^2_{BC}) = q_{AB,2} q_{BC,2}.
\]

The dual program (27) can be strengthened by replacing the sum over \( \ell \)-cycles by a sum over factorizing permutations. This becomes only relevant when \( k \geq 4 \), as one readily sees that all factorizing permutations are \( \ell \)-cycles for \( k \leq 3 \).
5.4. SDP refutation. If for some $k \in \mathbb{N}$ the dual program (27) is feasible with $(D, Y) < 0$, then by violation of weak duality in Eq. (21), the primal problem must be infeasible. Consequently, by Proposition 2 the spectra corresponding to $q_{A,\ell}$ are incompatible. For moderate sizes, such semidefinite programs can be solved by a computer.\footnote{In order for the dual program to be numerically bounded, one can change the dual to a feasibility problem with the constraint $\langle D, Y \rangle = -1$.}

The SDP refutation for detecting incompatibility of prescribed spectra can now be understood in simple terms: suppose there exists a density matrix $\rho$ with $\text{tr}(\eta((\alpha_1 \ldots \alpha_\ell)^A)\rho^\otimes k) = q_{A,\ell}$. If one finds a positive semidefinite operator $F = \Xi'(Y)$ satisfying the conditions in (27) and for which

$$\text{tr}(F\rho^\otimes k) = \sum_{A \in A} \sum_{\sigma \in S_k} y^{A,\sigma} \text{tr}(\eta(\sigma^A)\rho^\otimes k) = \sum_{A \in A} \sum_{\sigma \in S_k} y^{A,\sigma} q_{A,\sigma} < 0$$

holds, then one has arrived at a contradiction, because the trace inner product of two semidefinite operators must be non-negative.

6. Symmetry-reduction

Consider the symmetries appearing in Eq. (27),

$$\tau(\pi)\Xi'(Y)\tau(\pi)^{-1} = \Xi'(Y) \quad \forall \pi \in S_k$$

$$U \Xi'(Y)U^{-1} = \Xi'(Y) \quad \forall U = (U_1 \otimes \ldots \otimes U_n)^\otimes k : U_1, \ldots, U_n \in U(d).$$

From the Schur-Weyl duality it follows that the actions commute, $[\pi, U] = 0$.

Let us now decompose $((C^d)^\otimes k)^\otimes n$ under these symmetries. Consider the collection of the $k$ first subsystems. By the Schur-Weyl duality, the space $(C^d)^\otimes k$ decomposes as

$$(C^d)^\otimes k \simeq \bigoplus_{\text{height}(\lambda) \leq d} \mathcal{U}_\lambda \otimes S_\lambda,$$

where the unitary group acts on $\mathcal{U}_\lambda$ and the symmetric group on $S_\lambda$. Consequently,

$$( (C^d)^\otimes k )^\otimes n \simeq \bigotimes_{i=1}^{n} \left( \bigoplus_{\text{height}(\lambda_i) \leq d} \mathcal{U}_{\lambda_i} \otimes S_{\lambda_i} \right).$$

An operator $X$ on $((C^d)^\otimes k)^\otimes n$ that is invariant under the symmetries (30) will have the form

$$X = \bigotimes_{i=1}^{n} \left( \bigoplus_{\text{height}(\lambda_i) \leq d} \mathbb{1}_{\lambda_i} \otimes X_{\lambda_i} \right).$$

Then $X \geq 0$ if and only if $X_{\lambda_1} \otimes \ldots \otimes X_{\lambda_n} \geq 0$ for all $\lambda_1, \ldots, \lambda_n \vdash k$ with height$(\lambda_i) \leq d$.

Denote by $R_{\lambda}(\sigma)$ an irreducible orthogonal representation of $\sigma$ corresponding to the partition $\lambda \vdash n$. For $\sigma = (\sigma_1, \ldots, \sigma_n) \in S_k^n$, denote similarly

$$(34) \quad R_{\lambda_1, \ldots, \lambda_n}(\sigma) := R_{\lambda_1}(\sigma_1) \otimes \ldots \otimes R_{\lambda_n}(\sigma_n).$$

\underline{Lemma 5.} Let $G$ be a finite group and $R, T$ isomorphic orthogonal representations, i.e., there exists a bijective intertwiner $\phi$ such that $R \xrightarrow{\phi} T$. Then

$$(35) \quad R(a) = \sum_{g \in G} a_g R(g) \geq 0 \quad \iff \quad T(a) = \sum_{g \in G} a_g T(g) \geq 0.$$
Proof. The sets \( \{ \sum_{g \in G} a_g R(g) \mid a \in C \} \) and \( \{ \sum_{g \in G} a_g T(g) \mid a \in C \} \) form \( C^* \)-algebras. Then for any \( R(a) \geq 0 \), there exists \( b \in CG \) such that \( R(a) = \eta_d(bb^*) = \eta_d(b)\eta_d(b)^\dagger \) [BGSV12]. Then
\[
T(a) = T(bb^*) = T(b)T(b^*) = T(\sum_{g \in G} b_g g)T(\sum_{h \in G} b_h h^{-1}) = T(b)T(b)^\dagger \geq 0.
\]
The reverse direction is analogous. □

Proposition 6. In the dual program (27), it holds that
\[
\Xi^*(Y) = \sum_{A \in \mathcal{A}} \sum_{\sigma \in S_k} y^{A,\sigma} q_{A,\sigma} \geq 0
\]
if and only if
\[
F_{\lambda_1,\ldots,\lambda_n} = \sum_{A \in \mathcal{A}} \sum_{\sigma \in S_k} y^{A,\sigma} R_{\lambda_1,\ldots,\lambda_n}(\sigma^A) \geq 0
\]
for all \( \lambda_1,\ldots,\lambda_n \vdash k \) with \( \text{height}(\lambda_i) \leq d \).

Proof. The variable \( \Xi^*(Y) \) is positive semidefinite if and only if it is positive semidefinite in each of its irreducible components. By Lemma 5, this is the case if and only if is positive in an isomorphic orthogonal representation. □

Proposition 6 allows to find SDP refutations with a fewer number of variables but of equal strength than the naive approach of Eq. (27). This symmetry-reduced hierarchy is the following.

Theorem 7. Let \( \mathcal{A} \) be a collection of subsets of \( \{1,\ldots,n\} \) with associated marginal spectra \( \{ \mu_A \mid A \in \mathcal{A} \} \). If a term in the hierarchy (SDP-SC) returns a negative value, then the spectra are incompatible with a joint quantum state on \( (\mathbb{C}^d)^\otimes n \).

Proof. Proposition 2 states a necessary condition for spectral compatibility. Proposition 6 allows a symmetry-reduction of the corresponding SDP formulation (22). A negative value in (SDP-SC) violates weak duality (21). Consequently, the putative marginal spectra \( \{ \mu_A \mid A \in \mathcal{A} \} \) are then incompatible on \( (\mathbb{C}^d)^\otimes n \). □

Note that the symmetry-reduced hierarchy (SDP-SC) is equivalent to the program (27) while having a smaller number of variables.

6.1. Scaling. Given a combination of partitions \( (\lambda_1,\ldots,\lambda_n) \), the associated irreducible representation has dimension \( \prod_{i=1}^n \chi_{\lambda_i}(id) \). A Hermitian matrix of size \( N \times N \) has \( N(N - 1)/2 + N \) complex variables. Accordingly, the symmetry-reduced SDP contains roughly
\[
\frac{1}{2} \sum_{\lambda_1,\ldots,\lambda_n \vdash n} \prod_{i=1}^n \chi_{\lambda_i}^2(id)
\]
complex variables. Table 1 shows the relative growth of the naive unsymmetrized SDP versus that of the symmetrized SDP.
Example 8. Consider three copies of a three-qubit state with associated space \((\mathbb{C}^2)^{\otimes 3}\). Under the action of \(U(2)\), the space \((\mathbb{C}^2)^{\otimes 3}\) decomposes into irreducible representations (irreps) associated to the partitions \(3 = 3\) and \(2 + 1 = 3\), whose dimensions are 1 and 2, respectively. Thus the full space carries the irreps

\[
\begin{array}{ccc}
\text{Irreducible Representation} & \text{dimension} \\
\otimes & 1 & 1 \\
\otimes & 1 & 2 \\
\otimes & 2 & 2 \\
\otimes & 2 & 2 \\
\end{array}
\]

as well as permutations thereof. The total number of complex variables in above symmetry-reduced space is 76, fewer than the 136 complex variables required for an SDP of four qubits.

7. Dimension-free incompatibility

We now show that when \(k \leq d\), the incompatibility witnesses found by the hierarchy (SDP-SC) are dimension-free. This means they certify incompatibility of spectra of joint states having arbitrary local dimensions.

Theorem 9. When the number of copies is less or equal than the local dimension \((k \leq d)\), the incompatibility witnesses produced by the hierarchy (SDP-SC) are dimension-free and the detected spectra are incompatible in all local dimensions.

Proof. Let an incompatibility witness for dimension \(d\) using \(k\) copies be given.

First, consider the case of local dimension \(d' > d\): Recall that the program (SDP-SC) is equivalent to the program (27). Now consider a feasible \(F = \Xi^*(Y)\) from (27). It can be written as \(F = \eta_d(f)\) with \(f \in \mathbb{C}S^n_k\). Because of \(k \leq d\) and the Schur-Weyl decomposition (32), \(f \in \ker(\eta_d)\). This implies two things: First, because \(F \geq 0\), there is an element \(a \in \mathbb{C}S^n_k\) such that \(f = aa^*\). Consequently, if \(F \geq 0\) then also \(F' = \eta_d(f) = \eta_d(aa^*) \geq 0\) for all \(d'\). Second, the decomposition of \(F\) and \(F'\) into permutations is identical. Thus, the expectation values \(\text{tr}(F \rho \otimes k)\) and \(\text{tr}(F' \rho' \otimes k)\) coincide for \(\rho\) and \(\rho'\) with spectra \(\mu\). Thus, if \(F\) is an infeasibility certificate for spectra \(\mu\) in dimension \(d\) then \(F'\) is an infeasibility certificate for spectra \(\mu\) in dimension \(d'\).

Now, we consider the case \(d' \leq d\): Through the direct sum \(\mathbb{C}^d = \mathbb{C}^d' \oplus \mathbb{C}^{(d-d')}\), the space \((\mathbb{C}^d')^\otimes n\) embeds into \((\mathbb{C}^d)^\otimes n\). Clearly, spectral compatibility in the smaller space \((\mathbb{C}^d')^\otimes n\) implies compatibility in the larger space \((\mathbb{C}^d)^\otimes n\). Consequently, incompatibility in \((\mathbb{C}^d)^\otimes n\) implies incompatibility in \((\mathbb{C}^d')^\otimes n\).

Thus, if \(k \leq d\) and \(F = \eta_d(f)\) certifies for spectra to be incompatible with a joint state on \((\mathbb{C}^d)^\otimes n\), then the same spectra are also incompatible on \((\mathbb{C}^d')^\otimes n\) with \(d' \in \mathbb{N}_+\). \(\square\)

For numerical calculations, this dimension-free property can be helpful: any incompatibility witness found, as long as \(k \leq d\), will certify the spectra to be incompatible with a joint state with any local Hilbert space dimensions.

8. Completeness and convergence

8.1. Completeness. We now show that the hierarchy (22) is complete, that is, feasible at every level of the hierarchy if and only if the spectra are compatible. For this we use a
strategy similar to that in a recent work by Ligthart and Gross [LG22] \(^5\) where de Finetti together with “quadratic constraints” yields completeness. The quantum de Finetti theorem states [CKMR07]: Suppose \(g_t \in L((C^D)^{\otimes \ell})\) is permutation-invariant and infinitely symmetrically extendable, that is, there exists \(g_k \in L((C^D)^{\otimes k})\) for every \(k > t\), such that

\[
\text{tr}_{k-\ell}(g_k) = g_t, \quad \tau(\pi)g_k\tau(\pi)^{-1} = g_k, \forall \pi \in S_k.
\]

Then

\[
g_t = \int g^\otimes \ell \, dm(\varrho).
\]

for a measure \(m\) on the set of states in \(L(C^D)\).

**Theorem 10.** The marginal spectra \(\{\mu_A | A \in A\}\) are compatible with a joint quantum state on \((C^d)^{\otimes n}\), if and only if every term in the hierarchy (22) is feasible.

**Proof.** Consider the permutation \(\sigma = (1...\ell)\) for \(\ell \leq \lfloor \frac{k}{2} \rfloor\). A primal feasible \(X\) fulfills the constraints appearing in the primal hierarchy (22) of the form

\[
\text{tr}(\eta(\sigma^A)X) = q_{A,\sigma} = q_{A,\ell},
\]

(41)

\[
\text{tr}(\eta(\sigma^A) \otimes \eta(\sigma^A)\dagger X) = q_{A,\sigma}^2 = q_{A,\ell}^2,
\]

where we understand \(\eta(\sigma^A)\dagger\) appearing above to act on a disjoint set of \(\ell\) tensor factors (e.g., on tensor factors \(\ell + 1\) to \(2\ell\)). As a consequence,

\[
\text{tr} \left( (\eta(\sigma^A) - q_{A,\sigma} 1) \otimes (\eta(\sigma^A) - q_{A,\sigma} 1)X\dagger \right) = 0.
\]

(42)

Now assume that the primal is feasible for each level \(k\). By the quantum de Finetti theorem, the reduction of \(X\) onto size \(2\ell\) is separable as

\[
\text{tr}_{k-2\ell}(X) = \int g^\otimes 2\ell \, dm(\varrho).
\]

Then the constraint of Eq. (42) factorizes as

\[
\begin{align*}
\int \text{tr} \left( (\eta(\sigma^A) - q_{A,\sigma} 1) \otimes (\eta(\sigma^A) - q_{A,\sigma} 1)X\dagger \right) \, dm(\varrho) \\
= \int \left| \text{tr} \left( (\eta(\sigma^A) - q_{A,\sigma} 1) \otimes (\eta(\sigma^A) - q_{A,\sigma} 1)X\dagger \right) \right|^2 \, dm(\varrho) = 0
\end{align*}
\]

(44)

For this to be satisfied, each term \(g^\otimes \ell\) in \(\int g^\otimes \ell \, dm(\varrho)\) of non-zero measure must satisfy \(\text{tr}(\eta(\sigma^A)g^\otimes \ell) = q_{A,\sigma}\). The same reasoning holds for all \(\sigma^A\) that constrain on the spectrum. This makes \(\int g^\otimes \ell \, dm(\varrho)\) a convex combination (up to measure zero) of compatible states \(\varrho\). Consequently, the primal hierarchy (22) is feasible for every \(k \in \mathbb{N}\) if and only if a state compatible with the marginal spectra \(\{\mu_A | A \in A\}\) exists. \(\square\)

\[8.2.\] **Convergence.** Suppose the primal SDP is feasible up to level \(k\) in the hierarchy. What guarantee can be given for a state \(\varrho\) to exist whose moments \(\text{tr}(\varrho_A)\dagger\) are close to the desired ones \(q_{A,\ell}\)? A finite version of the quantum de Finetti theorem states that, if the primal problem is feasible up to some level \(k\) of the hierarchy, then the state \(\text{tr}_{k-\ell}(g_k)\) is close to a separable state [CKMR07]: Suppose \(g_t \in L((C^d)^{\otimes \ell})\) is permutation-invariant and symmetrically extendable for some \(k > t\). Then there exists a measure \(m\) on the set of states in \(L(C^d)\), such that

\[
\|g_t - \int g^\otimes \ell \, dm(\varrho)\|_1 \leq \frac{2d^2\ell}{k},
\]

\[5\]We thank Laurens T. Ligthart for explaining to us their proof.
where \( \|X\|_1 = \frac{1}{2} \text{tr} |X| = \frac{1}{2} \text{tr} \sqrt{XX^\dagger} \) is the trace norm of \( X \). This allows us to show the following:

**Corollary 11.** Let \( \mathcal{A} \) be a collection of subsets of \( \{1, \ldots, n\} \) with associated marginal spectra \( \{\mu_A | A \in \mathcal{A}\} \). If level \( k \) in the hierarchy (22) is feasible, then there exists a state \( \varrho \) on \( (\mathbb{C}^d)^\otimes n \) such that for all \( 2 \leq \ell \leq \lfloor \frac{k}{2} \rfloor \),

\[
|\text{tr}(\varrho_A^\ell) - q_{A,\ell}|^2 \leq \frac{12|\mathcal{A}|d^{2n}}{k}(\ell - 1)(\ell + 2).
\]

**Proof.** We follow the strategy that if \( X \) is close in trace distance to some \( Y = \int \varrho^{\otimes k} dm(\varrho) \), then the difference in their expectation values \( |\langle \sigma^A \rangle_X - \langle \sigma^A \rangle_Y| \) for any \( \ell \)-cycle \( \sigma^A \) is small. By using quadratic constraints, this can be further strengthened, such that \( |\langle \sigma^A \rangle_X - \langle \sigma^A \rangle_{\varrho^{\otimes k}}| \) is small for some \( \varrho^{\otimes k} \) in the decomposition of \( \int \varrho^{\otimes k} dm(\varrho) \). Finally, we consider the sum of squares over all \( A \) and \( j \leq \ell \) to show that there is a state close w.r.t. all \( A \) and powers \( \ell \).

Let \( \sigma = (\alpha_1 \ldots \alpha_\ell) \) be some \( \ell \)-cycle. Then the primal feasible variable \( X \) at level \( 2\ell \leq k \) of the hierarchy satisfies both,

\[
\text{tr}(\sigma^A X) = q_{A,\ell},
\]

\[
\text{tr}(\sigma^A \otimes \sigma^A \dagger X) = q_{A,\ell}^2,
\]

Due to the finite quantum de Finetti theorem, there exists a measure \( m \) such that

\[
\|X - \int \varrho^{\otimes 2\ell} dm(\varrho)\|_1 \leq \frac{4d^{2n}\ell}{k}.
\]

Let \( Y = \int \varrho^{\otimes 2\ell} dm(\varrho) \) and consider the expression

\[
\int |\text{tr} \left( (\sigma^A - q_{A,\ell}1) \varrho^{\otimes \ell} \right) |^2 dm(\varrho) = \text{tr} \left( (\sigma^A - q_{A,\ell}1) \otimes (\sigma^A - q_{A,\ell}1) \dagger Y \right),
\]

which is non-negative.

We now derive an upper bound for this expression. For this, observe that

\[
\text{tr} \left( (\sigma^A - q_{A,\ell}1) \otimes (\sigma^A - q_{A,\ell}1) \dagger Y \right) = \text{tr} \left( (\sigma^A - q_{A,\ell}1) \otimes (\sigma^A - q_{A,\ell}1) \dagger (Y - X) \right) - q_{A,\ell} \text{tr} \left( (1 \otimes 1) (Y - X) \right)
\]

\[
= \text{tr} \left( (\sigma^A \otimes \sigma^A) \dagger (Y - X) \right) - q_{A,\ell} \text{tr} \left( 1 \otimes 1 (Y - X) \right) - q_{A,\ell} \text{tr} \left( (1 \otimes 1) (Y - X) \right)
\]

\[
+ 2 \sum_i s_i (X - Y) s_i (U)
\]

where we used the fact that \( \text{tr} \left( 1 \otimes 1 (Y - X) \right) = 0 \) and that the whole expression vanishes on \( X \). Each term in (50) is further bounded with von Neumann’s trace inequality: for any unitary \( U \) it holds that

\[
|\text{tr} \left( XU \right) - \text{tr} \left( YU \right)| \leq \sum_i s_i (X - Y) s_i (U)
\]

\[
= \sum_i s_i (X - Y)
\]

\[
= 2 \|X - Y\|_1,
\]

\[
(51)
\]
where \( s_i(U) \) denote the \( i \)-th largest singular value of \( U \) (which equal one for unitary matrices), together with the identity \( \|A\|_1 = \frac{1}{2} \sum_i s_i(A) \). With this, (50) is bounded by
\[
\text{tr} \left( (\eta(\sigma^A) - q_{A,\ell}) \otimes (\eta(\sigma^A) - q_{A,\ell})^* Y \right) \leq (2 + 4q_{A,\ell}) \|Y - X\|_1.
\]
At this point, we use the finite quantum de Finetti theorem in Eq. (48). Together with the fact that \( q_{A,\ell} \leq 1 \), we get from Eqs. (49) and (52) that
\[
\int \left| \text{tr} \left( (\eta(\sigma^A) - q_{A,\sigma}) \otimes (\eta(\sigma^A) - q_{A,\sigma})^*\right) \right|^2 \, dm(\varrho) \leq \frac{24d^2n\ell}{k}.
\]
Note that the left-hand side of Eq. (53) can be interpreted as the average of \( \left| \text{tr} \left( (\eta(\sigma^A) - q_{A,\ell}) \otimes (\eta(\sigma^A) - q_{A,\ell})^*\right) \right|^2 \) over all \( \varrho \) in the decomposition of \( Y \). Thus, there must exist a \( \varrho \) of non-zero measure such that
\[
\left| \text{tr}(\varrho^A) - q_{A,\ell}\right|^2 = \left| \text{tr} \left( (\eta(\sigma^A) - q_{A,\ell}) \otimes (\eta(\sigma^A) - q_{A,\ell})^*\right) \right|^2 \leq \frac{24d^2n\ell}{k}.
\]
A similar argument can be made for all spectra. Consider the sum of the left-hand sides of Eqs. (50) over all \( A \) and \( j \leq \ell \). Thus,
\[
\int \sum_{A \in A} \sum_{j=2}^\ell \left| \text{tr} \left( (\eta((1\ldots j)^A) - q_{A,j}) \otimes (\eta((1\ldots j)^A) - q_{A,j})^*\right) \right|^2 \, dm(\varrho)
\]
\[
= \sum_{A \in A} \sum_{j=2}^\ell \text{tr} \left( (\eta((1\ldots j)^A) - q_{A,j}) \otimes (\eta((1\ldots j)^A) - q_{A,j})^*\right) \leq \frac{24|A|d^2n}{k} \sum_{j=2}^\ell j
\]
\[
= \frac{12|A|d^2n}{k} (\ell - 1)(\ell + 2).
\]
Thus, there is again a state \( \varrho \) in the decomposition of \( Y \) with
\[
\sum_{A \in A} \sum_{j=2}^\ell \left| \text{tr} \left( (\eta((1\ldots j)^A) - q_{A,j}) \otimes (\eta((1\ldots j)^A) - q_{A,j})^*\right) \right|^2 \leq \frac{12|A|d^2n}{k} (\ell - 1)(\ell + 2).
\]
As the left-hand side of this inequality is a sum of positive terms, each of them must be bounded individually, yielding the claim.

9. Numerical results

9.1. Spectra of three-partite states. As an example, consider a three-partite state \( \varrho_{ABC} \) with two-body marginals \( \varrho_{AB}, \varrho_{AC}, \) and \( \varrho_{BC} \) of rank two. Their spectra are thus of the form
\[
\text{spect}(\varrho_{AB}) = (\lambda_{AB}, 1 - \lambda_{AB})
\]
\[
\text{spect}(\varrho_{AC}) = (\lambda_{AC}, 1 - \lambda_{AC})
\]
\[
\text{spect}(\varrho_{BC}) = (\lambda_{BC}, 1 - \lambda_{BC}).
\]
Evaluating the symmetry-reduced semidefinite programming hierarchy in Theorem 7, we obtain the incompatibility regions shown in Figure 1. One sees that the use of four copies \((k = 4, \text{dotted line})\) in the hierarchy excludes a larger region of spectra than two only \((k = 2, \text{dashed line})\). The use of factorizing permutations [see Remark 4] is even stronger \((k = 4, \text{solid line})\).
Figure 1. **Regions of spectral incompatibility.** Consider prescribed eigenvalues $\lambda_{AB}, \lambda_{AC}, \lambda_{BC}$ of rank-2 two-body marginals of three-partite states. We plot the regions of certified incompatibility for values in the interval $[0, \frac{1}{2}]$, as the problem is symmetric under the exchange of $\lambda_{ij} \leftrightarrow 1 - \lambda_{ij}$. The infeasible regions are below (for $\lambda_{AC} < \lambda_{AB}$) and to the left (for $\lambda_{AC} > \lambda_{AB}$) of the lines. Shown are the boundaries of infeasibility for $k = 2$ (dashed lines), $k = 4$ (dotted lines), and $k = 4$ with factorizing permutations (solid lines), with the height of the Young tableaux $d$ equal to the number of copies $k$. Due Theorem 9, the infeasibility regions are valid for tripartite states of arbitrary local dimensions.

Recall that $d$ controls the height of Young tableaux used and that $k$ is the number of copies. In the symmetry-reduced formulation, the number of variables saturates when $d = k$ and contains fewer variables when $d < k$. We choose the saturated parameters $k = d = 2$ and $k = d = 4$. Due to Theorem 9, our spectral incompatibility regions are then valid for tripartite systems of arbitrary local dimensions.

A precise boundary of the region can be obtained through a divide and conquer algorithm with a precision of $10^{-3}$, implemented with the Python interface PICOS [SS22] and
the solver MOSEK [ApS21]. The infeasibility boundaries are described by

\begin{align}
  k = 2 : & \quad (\lambda_{AB} - \frac{1}{2})^2 + (\lambda_{AC} - \frac{1}{2})^2 - (\lambda_{BC} - \frac{1}{2})^2 \leq \frac{1}{4}, \\
  k = 4 : & \quad (\lambda_{AB} - \frac{1}{2})^2 + (\lambda_{AC} - \frac{1}{2})^2 - (\lambda_{BC} - \frac{1}{2})^2 \\
 & \quad - b \left( (\lambda_{AB} - \frac{1}{2})^4 + (\lambda_{AC} - \frac{1}{2})^4 - (\lambda_{BC} - \frac{1}{2})^4 \right) \leq c,
\end{align}

(58)

and two inequalities with exchanged roles of the parties. The constants in Eq. (58) are

\begin{align}
  b &= 0.393931, \\
  c &= 0.225380.
\end{align}

(59)

9.2. Purity inequalities. The semidefinite programming hierarchy of Theorem 7 yields the infeasibility regions of Fig. 1. Note that the SDP does not fix any individual eigenvalues but their power sums. This can also be seen from the resulting incompatibility witnesses, which have the form \( \sum_{\sigma \in S_k} y_{\sigma} \eta(\sigma) \). We transform the eigenvalue inequalities into moments through

\begin{align}
  (\lambda_{AB} - \frac{1}{2})^2 & \quad \rightarrow \quad \frac{1}{2} \text{tr}(\varrho_{AB}^2) - \frac{1}{4}, \\
  (\lambda_{AB} - \frac{1}{2})^4 & \quad \rightarrow \quad \frac{1}{2} \text{tr}(\varrho_{AB}^4) - \frac{3}{4} \text{tr}(\varrho_{AB}^2) + \frac{5}{16}.
\end{align}

(60)

The inequalities in Eqs. (58) then correspond to the following the two constraints

\begin{align}
  k = 2 : & \quad 1 + \text{tr}(\varrho_{BC}^2) - \text{tr}(\varrho_{AB}^2) - \text{tr}(\varrho_{AC}^2) \geq 0, \\
  k = 4 : & \quad 1 + (1 + \alpha) \left( \text{tr}(\varrho_{BC}^2) - \text{tr}(\varrho_{AB}^2) - \text{tr}(\varrho_{AC}^2) \right) \\
 & \quad - \alpha \left( \text{tr}(\varrho_{BC}^4) - \text{tr}(\varrho_{AB}^4) - \text{tr}(\varrho_{AC}^4) \right) \geq 0,
\end{align}

(61)

where \( \alpha = \frac{8b}{16c+5b+4} = 0.329107 \). These inequalities are valid for all tripartite states of arbitrary local dimension, and correspond to the incompatibility witnesses

\begin{align}
  k = 2 : & \quad A \otimes P \otimes P + P \otimes A \otimes A, \\
  k = 4 : & \quad 1 + (1 + \alpha) \left[ \eta((12)^{BC}) - \eta((12)^{AC}) - \eta((12)^{AB}) \right] - \\
 & \quad - \alpha \left[ \eta((1234)^{BC}) - \eta((1234)^{AC}) - \eta((1234)^{AB}) \right],
\end{align}

(62)

where \( P \) and \( A \) are the projectors onto the symmetric and antisymmetric subspaces of \((C^d)^{\otimes 2}\). We note that the inequality for \( k = 2 \) is a linear combination of Rains’ shadow inequalities [Rai00], while the \( k = 4 \) relation seems to be new. Because of Theorem 9, these inequalities hold for tripartite systems of arbitrary dimensions.

9.3. Flat marginal spectra. As a final example, we consider the two-body marginal spectra of pure three- and four-partite states. To simplify the discussion, we assume the two-body marginals to be flat,

\begin{equation}
  \text{spect}(\varrho_S) = \left( \frac{1}{r_S}, \ldots, \frac{1}{r_S}, 0, \ldots, 0 \right),
\end{equation}

(63)

so that \( r_S \) is the rank of the marginal on \( S = \{i, j\} \) with \( i \neq j \). Some constraints on ranks are known: from the Schmidt decomposition, it follows that tracing out a subsystem of dimension \( d \) from a pure state yields a state of rank at most \( d \). Additionally, Cadney et al. [CHLW14] have conjectured the inequality \( r_{AB} r_{AC} \geq r_{BC} \).\(^6\)

\(^6\)The conjecture is claimed proven in the preprint [SCSH21].
Let us apply the symmetry-reduced SDP hierarchy. Consider the case of three-partite systems and fix the ranks $r_{AB}$, $r_{AC}$ and $r_{BC}$ and the local dimension $d$. We ask whether the spectra are compatible with a pure joint state and apply the SDP hierarchy of Theorem 7 with $k = 4$ and factorizing permutations. The nonexistence of pure states with flat marginal spectra is shown in Fig. 2 (top). These numerical results agree with the known and conjectured rank inequalities. In the case of four-partite states, we fix ranks $r_{AB}$, $r_{AC}$ and $r_{AD}$ instead. Here the hierarchy yields stronger results, shown in Fig. 2 (bottom). In particular, depending on the local dimension we can exclude states with flat marginals and ranks $[r_{AB}, r_{AC}, r_{AD}]$ equal to $[3, 2, 2]$, $[4, 2, 2]$, $[4, 3, 2]$, and $[4, 3, 3]$.

It is interesting to see that there exist states that are excluded in dimension $d$, but which can be shown to exist in dimension $d' > d$. This shows that our hierarchy can obtain meaningful constraints also on spectra whose compatibility is dimension-dependent.

10. Related work

In the past years, extensive work approached the quantum marginal problem [Kly06, Sch14], developing constraints on operators [BSS06], von Neumann entropies [Osb08, CLL13], purities [EHGS18] and ranks [CHLW14] of subsystems. The perhaps most systematic approach to date uses representation theory of the symmetric group [Kly04, CM06] and generalizes the polygon inequalities [Hig03, HSS03]. Here, we highlight the existence of critical [BRVR18, BLRR19] and absolutely maximally entangled states [HGS17, YSW+21, RBB+22] as guiding problems, achieving extreme values in spectra and entropies. This has also led to the development of methods to reconstruct the joint state from partial information [CG22, AFT21], to tackle the question of uniqueness in the reconstruction [KJTV19, WHG17, Kla17], to detect entanglement from partial information [PMMG18, NBA20], and to investigate marginals of random states [CDKW14, CM21]. Fermionic settings are treated in [Maz12, CLL+21].

The key systematic approach for overlapping marginals of spin systems is that of symmetric extensions [CJK+14, YSW+21]. Our work is inspired by Hall [Hal07], Yu et al. [YSW+21], and Huber et al. [HKMV22]. However, these are neither applicable to more fundamental spectral formulation of the problem nor can they give results that are valid for all local dimensions.

11. Conclusions

Our main result, Theorem 7, combines the techniques of symmetric extension and symmetry reduction to certify the incompatibility of marginal spectra. This simple idea turns out to be quite powerful, allowing for a complete hierarchy for spectral compatibility in arbitrary local dimensions (Theorem 9). At the same time, it can be used to differentiate different dimensions with respect to spectral compatibility: There exist spectra which are incompatible in dimension $d$, but compatible in $d' > d$.

A natural question is how to include constraints arising from a positive partial transpose in the hierarchy (27). A symmetry-reduction similar to the one employed here would require a decomposition of the Brauer algebra. For the case of $k = 3$ copies, the Brauer Algebra can be expressed as a linear combination of elements from $C_{S_3}$ [EW01] and it is possible to formulate a semidefinite program analogous to Theorem 7 [HHM+].

\footnote{This is not in contradiction to Thm. 9, as the incompatibility of these states was shown by using $k > d$.}
| system   | copies | $N_{\text{naive}}$ | $N_{\text{sym}}$ | # blocks | max. size |
|----------|--------|---------------------|-------------------|----------|-----------|
| 2 qubits | 2      | 136                 | 4                 | 4        | 1         |
|          | 3      | 2080                | 17                | 4        |           |
|          | 4      | 32896               | 116               | 9        | 9         |
|          | 5      | 524800              | 932               | 9        | 25        |
|          | 6      | $\approx 8.3 \cdot 10^6$ | 8912             | 16       | 81        |
|          | 7      | $\approx 1.3 \cdot 10^8$ | 92633            | 16       | 196       |
| 3 qubits | 2      | 2080                | 8                 | 8        | 1         |
|          | 3      | 131328              | 76                | 8        |           |
|          | 4      | $\approx 8.3 \cdot 10^6$ | 1480             | 27       | 27        |
|          | 5      | $\approx 5.3 \cdot 10^8$ | 37544            | 27       | 125       |
| 4 qubits | 2      | 32896               | 16                | 16       | 1         |
|          | 3      | $\approx 8.3 \cdot 10^6$ | 353              | 16       |           |
|          | 4      | $\approx 2.1 \cdot 10^9$ | 19856            | 81       | 81        |
| 5 qubits | 2      | 524800              | 32                | 32       | 1         |
|          | 3      | $\approx 5.3 \cdot 10^8$ | 1684             | 32       |           |
|          | 4      | $\approx 5.4 \cdot 10^{11}$ | 272800          | 243      | 243       |
| 2 ququarts | 3  | 266085              | 26                | 9        | 4         |
|          | 4      | $\approx 2.1 \cdot 10^7$ | 305              | 16       | 9         |
|          | 5      | $\approx 1.7 \cdot 10^9$ | 5525             | 25       | 36        |
|          | 6      | $\approx 1.4 \cdot 10^{11}$ | 132885          | 49       | 256       |
| 3 ququarts | 3 | $\approx 1.9 \cdot 10^8$ | 140              | 27       | 8         |
|          | 4      | $\approx 1.4 \cdot 10^{11}$ | 6448            | 64       | 27        |
|          | 5      | $\approx 1.0 \cdot 10^{14}$ | 550994          | 125      | 216       |
| 4 ququarts | 3 | $\approx 1.4 \cdot 10^{11}$ | 776              | 81       | 16        |
|          | 4      | $\approx 9.2 \cdot 10^{14}$ | 143201          | 256      | 81        |
| 2 ququarts | 4 | $\approx 2.1 \cdot 10^9$ | 338              | 25       | 9         |
|          | 5      | $\approx 5.5 \cdot 10^{11}$ | 7393             | 36       | 36        |
| 3 ququarts | 4 | $\approx 1.4 \cdot 10^{14}$ | 7412             | 125      | 27        |
|          | 5      | $\approx 5.7 \cdot 10^{17}$ | 850392          | 216      | 216       |
| 4 ququarts | 4 | $\approx 9.2 \cdot 10^{18}$ | 170888          | 625      | 81        |

Table 1. Number of complex variables in the naive and symmetry-reduced SDP. For the symmetry-reduced SDP the number of blocks and the size of the largest block is shown. In comparison, an SDP size commonly solvable on modern laptops is that of a seven-qubit density matrix with 8255 complex variables. Note that $d$ controls the height of Young tableaux used and $k$ is the number of copies. In the symmetry-reduced formulation, the number of variables saturates when $d = k$; containing fewer variables when $d < k$. 
Table 2. Compatibility of two-body marginals with flat spectra.
We apply the hierarchy of Thm. 7 for $k \in \{2, 3, 4\}$ to two-body marginals with flat spectra. Cases are excluded by incompatibility certificates (blue), by known rank constraints (black triangle), and by conjectured rank constraints (c). Also shown are compatible pure states (green). Left: Fix the ranks $[r_{AB}, r_{AC}, r_{BC}]$ and assume flat spectra for the two-body marginals of three-partite pure states of different local dimensions. Our hierarchy coincides with known and conjectured rank inequalities. Right: fix the ranks $[r_{AB}, r_{AC}, r_{AD}]$ and assume flat spectra of the two-body marginals of four-partite pure states. Additional cases $[3, 2, 2], [4, 2, 2], [4, 3, 2], [4, 3, 3]$ are excluded depending on the local dimension.
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