ABSTRACT. In this paper, we first evaluate topological distributions of the sets of all doubling spaces, all uniformly disconnected spaces, and all uniformly perfect spaces in the space of all isometry classes of compact metric spaces equipped with the Gromov–Hausdorff distance. We then construct branching geodesics of the Gromov–Hausdorff distance continuously parameterized by the Hilbert cube, passing through or avoiding sets of all spaces satisfying some of the three properties shown above, and passing through the sets of all infinite-dimensional spaces and the set of all Cantor metric spaces. Our construction implies that for every pair of compact metric spaces, there exists a topological embedding of the Hilbert cube into the Gromov–Hausdorff space whose image contains the pair. From our results, we observe that the sets explained above are geodesic spaces and infinite-dimensional.

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1. Introduction

In this paper, we denote by $\mathcal{M}$ the set of all isometry classes of non-empty compact metric spaces, and denote by $\mathcal{GH}$ the Gromov–Hausdorff distance. We refer to $(\mathcal{M}, \mathcal{GH})$ as the Gromov–Hausdorff space. We denote by $\mathcal{Q}$ the product space $[0,1]^\mathbb{N}$ of the countable copies of the unit interval. The space $\mathcal{Q}$ is called the Hilbert cube.

In this paper, we first evaluate topological distributions of the sets of all doubling spaces, all uniformly disconnected spaces, and all uniformly perfect spaces in $(\mathcal{M}, \mathcal{GH})$, respectively. We then show that the existence of continuum many branching geodesics passing through or avoiding sets of all spaces satisfying some of the three properties shown above, or passing through the sets of all infinite-dimensional spaces and the set of all Cantor metric spaces, by constructing a family of geodesics continuously parametrized by the Hilbert cube. This construction implies that for a given pair of compact metric spaces, there exists a topological embedding from the Hilbert cube into $\mathcal{M}$ whose image contains a pair of compact metric spaces. From our results, we observe that the sets explained above are geodesic spaces and infinite-dimensional.

Before precisely stating our results, we introduce basic concepts. Let $N \in \mathbb{N}$. A metric space $(X,d)$ is said to be $N$-doubling if for all $x \in X$ and $r \in (0,\infty)$ there exists a subset $F$ of $X$ satisfying $B(x,r) \subset \bigcup_{y \in F} B(y,r/2)$ and $\text{Card}(F) \leq N$, where $B(x,r)$ is the closed ball centered at $x$ with radius $r$, and the symbol “Card” stands for the cardinality. A metric space is said to be doubling if it is $N$-doubling for some $N$.

Let $\delta \in (0,\infty)$. A metric space $(X,d)$ is said to be $\delta$-uniformly disconnected if for every non-constant finite sequence $\{z_i\}_{i=1}^N$ in $X$ we have $\delta d(z_1,z_N) \leq \max_{1 \leq i \leq N} d(z_i,z_{i+1})$. A metric space is said to be uniformly disconnected if it is $\delta$-uniformly disconnected for some $\delta \in (0,\infty)$. Note that a metric space is uniformly disconnected if and only if it is bi-Lipschitz embeddable into an ultrametric space (see [3, Proposition 15.7]).

Let $c \in (0,1)$. A metric space $(X,d)$ is said to be $c$-uniformly perfect if for every $x \in X$, and for every $r \in (0,\delta_d(X))$, there exists $y \in X$ with $c \cdot r \leq d(x,y) \leq r$, where $\delta_d(X)$ stands for the diameter. A metric space is said to be uniformly perfect if it is $c$-uniformly perfect for some $c \in (0,1)$.

Let $(X,d)$ and $(Y,e)$ be metric spaces. A homeomorphism $f : X \to Y$ is said to be quasi-symmetric if there exists a homeomorphism $\eta : [0,\infty) \to [0,\infty)$ such that for all $x,y,z \in X$ and for every $t \in [0,\infty)$ the inequality $d(x,y) \leq td(x,z)$ implies the inequality $e(f(x),f(y)) \leq \eta(t)e(f(x),f(z))$. For example, all bi-Lipschitz homeomorphisms are quasi-symmetric. Note that the doubling property, the
uniform disconnectedness, and the uniform perfectness are invariant under quasi-symmetric maps. In this paper, we denote by $\Gamma$ the Cantor set. David and Semmes [3] proved that if a compact metric space is doubling, uniformly disconnected, and uniformly perfect, then it is quasi-symmetrically equivalent to the Cantor set $\Gamma$ equipped with the Euclidean metric ([3, Proposition 15.11]).

Let $X$ be a topological space. A subset $S$ of $X$ is said to be nowhere dense if the complement of the closure of $S$ is dense in $X$. A subset of $X$ is said to be meager if it is the union of countable nowhere dense subsets of $X$. A subset of $X$ is said to be comeager if its complement is meager. A subset of $X$ is said to be $F_\sigma$ (resp. $G_\delta$) if it is the union of countably many closed subsets of $X$ (resp. the intersection of countably many open subsets of $X$). A subset of $X$ is said to be $F_{\sigma\delta}$ (resp. $G_{\delta\sigma}$) if it is the intersection of countably many $F_\sigma$ subsets of $X$ (resp. the union of countably many $G_\delta$ subsets of $X$).

There are some results on topological distributions in the Gromov–Hausdorff space and spaces of metrics. Rouyer [14] proved that several properties on metric spaces are generic. For example, it was proven that the set of all compact metric spaces homeomorphic to the Cantor set, the set of all compact metric spaces with zero Hausdorff dimension and lower box dimension, and the set of all compact metric spaces with infinite upper box dimension are comeager in $(M, GH)$.

For a metrizable space $X$, we denote by $\text{Met}(X)$ the space of all metrics generating the same topology of $X$. We consider that $\text{Met}(X)$ is equipped with the supremum metric. In [7] and [8], the author determined the topological distributions of the doubling property, the uniform disconnectedness, the uniform perfectness, and their negations in $\text{Met}(X)$ for a suitable space $X$. For example, in [7], it was proven that the set of all non-doubling metrics and the set of all non-uniformly doubling metrics are dense $G_\delta$ in $\text{Met}(X)$ for a non-discrete space $X$.

Let $\mathcal{D}_0$, $\mathcal{U}_D$, and $\mathcal{U}_P$ be the sets of all doubling metric spaces, all uniformly disconnected metric spaces, and all uniformly perfect metric spaces in $M$, respectively.

**Theorem 1.1.** The sets $\mathcal{D}_0$, $\mathcal{U}_D$, and $\mathcal{U}_P$ are dense $F_\sigma$ and meager in the Gromov–Hausdorff space $(M, GH)$.

To simplify our description, the symbols $\mathcal{P}_1$, $\mathcal{P}_2$, and $\mathcal{P}_3$ stand for the doubling property, the uniform disconnectedness, and the uniform perfectness, respectively. Let $\mathcal{P}$ be a property of metric spaces. If a metric space $(X, d)$ satisfies the property $\mathcal{P}$, then we write $T_{\mathcal{P}}(X, d) = 1$; otherwise, $T_{\mathcal{P}}(X, d) = 0$. For a triple $(u_1, u_2, u_3) \in \{0, 1\}^3$, we say that a metric space $(X, d)$ is of type $(u_1, u_2, u_3)$ if we have $T_{\mathcal{P}_k}(X, d) = u_k$ for all $k \in \{1, 2, 3\}$.

A topological space is said to be a **Cantor space** if it is homeomorphic to the Cantor set $\Gamma$. The author [6] proved that for every $(u, v, w) \in \ldots$
\{0,1\}^3$ except $(1,1,1)$, the set of all quasi-symmetric equivalence classes of Cantor metric spaces of type $(u,v,w)$ has exactly continuum many elements. In [8], the author determined the topological distribution of the set of all metrics of type $(u,v,w)$ in $\text{Met}(\Gamma)$. In this paper, we develop these results in the context of the Gromov–Hausdorff space.

Let $Q_1 = \mathcal{D}\emptyset$, $Q_2 = \mathcal{U}\emptyset$, and $Q_3 = \mathcal{U}\mathcal{P}$. For $k \in \{1,2,3\}$, and for $u \in \{0,1,2\}$, we define

$$E_k(u) = \begin{cases} M \setminus Q_k & \text{if } u = 0; \\ Q_k & \text{if } u = 1; \\ M & \text{if } u = 2. \end{cases}$$

For $(u,v,w) \in \{0,1,2\}^3$, we also define

$$X(u,v,w) = E_1(u) \cap E_2(v) \cap E_3(w).$$

The next theorem is an analogue of [8, Theorem 1.6].

**Theorem 1.2.** Let $(u,v,w) \in \{0,1,2\}^3$. Then the following statements hold true.

1. If $\{u,v,w\} \setminus \{2\} = \{1\}$, then the set $X(u,v,w)$ is dense $F_\sigma$.
2. If $\{u,v,w\} \setminus \{2\} = \{0\}$, then the set $X(u,v,w)$ is dense $G_\delta$.
3. If $\{u,v,w\} \setminus \{2\} = \{0,1\}$, then the set $X(u,v,w)$ is dense $F_{\sigma\delta}$ and $G_{\delta\sigma}$.

Let $(X,d)$ be a metric space, and $a,b \in \mathbb{R}$. A continuous map $\gamma : [a,b] \to X$ is said to be a curve. For $x,y \in X$, a curve $\gamma : [0,1] \to X$ is said to be a geodesic from $x$ to $y$ if $\gamma(0) = x$ and $\gamma(1) = y$, and for all $s,t \in [0,1]$ we have

$$d(\gamma(s), \gamma(t)) = |s-t| \cdot d(x,y).$$

Note that if there exists a curve whose length is $d(x,y)$, then there exists a geodesic from $x$ to $y$ (see [1, Chapter 2]). A metric space is said to be a geodesic space if for all two points, there exists a geodesic connecting them.

Ivanov, Nikolaeva, and Tuzhilin [10] proved that $(M, \mathcal{G}\mathcal{H})$ is a geodesic space by showing the existence of the mid-point of all two points of $M$. Klibus [12] proved that the closed ball in the Gromov–Hausdorff space centered at the one-point metric space is a geodesic space. Chowdhury and Mémoli [2] constructed an explicit geodesic in $(M, \mathcal{G}\mathcal{H})$ using an optimal closed correspondence (see also [9]). They also showed that $(M, \mathcal{G}\mathcal{H})$ permits branching geodesics by constructing branching geodesics from the one-point metric space. Mémoli and Wan [13] showed that every Gromov–Hausdorff geodesic is realizable as a geodesic in the Hausdorff hyperspace of some metric space. For every pair of two distinct compact metric spaces, they also constructed countably many geodesics connecting that metric spaces. As a development of their results on branching geodesics, in this paper, for all
pairs of compact metric spaces in $\mathcal{M}$, we construct branching geodesics connecting them continuously parametrized by $Q$.

**Definition 1.1.** Let $A$ be a closed subset of $[0, 1]$ with $\{0, 1\} \subset A$. Let $(X, d), (Y, e) \in \mathcal{M}$. We say that a continuous map $F : [0, 1] \times Q \to \mathcal{M}$ is an $A$-branching bunch of geodesics from $(X, d)$ to $(Y, e)$ if the following are satisfied:

1. for every $q \in Q$, we have $F(0, q) = (X, d)$ and $F(1, q) = (Y, e)$;
2. for every $s \in A$ and for all $q, r \in Q$ we have $F(s, q) = F(s, r)$;
3. for each $q \in Q$, the map $F_q : [0, 1] \to \mathcal{M}$ defined by $F_q(s) = F(s, q)$ is a geodesic from $(X, d)$ to $(Y, e)$;
4. for all $(s, q), (t, r) \in ([0, 1] \setminus A) \times Q$ with $(s, q) \neq (t, r)$, we have $F(s, q) \neq F(t, r)$.

![Figure 1. A-branching bunch of geodesics](image)

**Theorem 1.3.** Let $(u, v, w) \in \{0, 1, 2\}^3$. Let $(X, d), (Y, e) \in \mathcal{X}(u, v, w)$. Let $A$ be a closed subset of $[0, 1]$ with $\{0, 1\} \subset A$. Then there exists an $A$-branching bunch of geodesics $F : [0, 1] \times Q \to \mathcal{X}(u, v, w)$ from $(X, d)$ to $(Y, e)$.

We say that a map $D : \mathcal{M} \to [0, \infty]$ is a *dimensional function* if

1. for every $(X, d) \in \mathcal{M}$, and for every closed subset $A$ of $X$, we have $D(A, d) \leq D(X, d)$;
2. for all $\epsilon \in (0, \infty)$, there exists a compact metric space $(Y, e)$ with $D(Y, e) = \infty$ and $d_\epsilon(Y) \leq \epsilon$.

For example, the covering dimension (topological dimension) $\dim$, the Hausdorff dimension $\dim_H$, the packing dimension $\dim_P$, the lower box dimension $\dim_B$, the upper box dimension $\dim_U$, and the Assouad dimension $\dim_A$ are dimensional functions.
For a dimensional function $D$, we denote by $I(D)$ the set of all compact metric spaces $(X,d) \in M$ with $D(X,d) = \infty$. Note that $I(D) \neq \emptyset$.

**Theorem 1.4.** Let $D$ be a dimensional function. Let $(X,d), (Y,e) \in I(D)$, and $A$ a closed subset of $[0,1]$ with $\{0,1\} \subset A$. Then there exists an $A$-branching bunch of geodesics $F : [0,1] \times Q \to I(D)$ from $(X,d)$ to $(Y,e)$.

Let $CA$ denote the set of all Cantor metric space in $M$. Note that $CA$ is comeger in $M$ (see [14]). By the same method of the proof of Theorem 1.3, we obtain:

**Theorem 1.5.** Let $(X,d), (Y,e) \in CA$, and $A$ a closed subset of $[0,1]$ with $\{0,1\} \subset A$. Then there exists an $A$-branching bunch of geodesics $F : [0,1] \times Q \to CA$ from $(X,d)$ to $(Y,e)$.

From Theorems 1.3, 1.4, and 1.5 we obtain the following four statements. Theorem 1.6 is an improvement of [13, Theorem 5.13].

**Theorem 1.6.** Let $S$ be any one of $X(u,v,w)$ for some $(u,v,w) \in \{0,1,2\}^3$ or $I(D)$ for some dimensional function $D$ or $CA$. Then for all $(X,d), (Y,e) \in S$ satisfying $\mathcal{GH}((X,d),(Y,e)) > 0$, there are exact continuum many geodesics from $(X,d)$ to $(Y,e)$ passing through $S$.

In this paper, a metric space is said to be infinite-dimensional if its covering dimension is infinite.

**Theorem 1.7.** Let $S$ be any one of $X(u,v,w)$ for some $(u,v,w) \in \{0,1,2\}^3$ or $I(D)$ for some dimensional function $D$ or $CA$. Then for every non-empty open subset $O$ of $M$, the set $S \cap O$ is infinite-dimensional.

The following theorem states that the sets $I(D)$ and $X(u,v,w)$ and $CA$ are everywhere infinite-dimensional.

**Theorem 1.8.** Let $S$ be any one of $X(u,v,w)$ for some $(u,v,w) \in \{0,1,2\}^3$ or $I(D)$ for some dimensional function $D$ or $CA$. Then for every non-empty open subset $O$ of $M$, the set $S \cap O$ is infinite-dimensional.

Since all separable metrizable spaces are topologically embeddable into $Q$, we have:

**Corollary 1.9.** If $S$ is any one of $X(u,v,w)$ for some $(u,v,w) \in \{0,1,2\}^3$, or $I(D)$ for some dimensional function $D$ or $CA$, then every separable metrizable space $X$ can be topologically embeddable into $S$.

**Remark 1.1.** It is open whether all separable (or compact) metric spaces are isometrically embeddable into $M$. On the other hand, Wan [16] proved that all separable ultrametric spaces are isometrically embeddable into the Gromov–Hausdorff ultrametric space.
The organization of this paper is as follows: In Section 2, we prepare and explain basic concepts and statements on metric spaces. In Section 3, we prove Theorems 1.1 and 1.2. In Section 4, we introduce specific versions of telescope spaces and sequentially metrized Cantor spaces introduced in [6]. Using telescope spaces, we also construct a family of compact metric spaces continuously parameterized by $\mathbb{Q}$, which are not isometric to each other. In Section 5, we first prove Theorems 1.3, 1.4, and 1.5. As its applications, we next prove Theorems 1.6, 1.7, and 1.8. In Section 6, for the convenience for the readers, we exhibit a table of symbols.

2. Preliminaries

In this section, we prepare and explain the basic concepts and statements on metric spaces.

2.1. Generalities. In this paper, we denote by $\mathbb{N}$ the set of all positive integers. The symbol $\vee$ stands for the maximal operator of $\mathbb{R}$. Let $X$ be a set. A metric $d$ on $X$ is said to be an ultrametric if for all $x, y, z \in X$ the metric $d$ satisfies $d(x, y) \leq d(x, z) \vee d(z, y)$. In this paper, for a metric space $(X, d)$, and for a subset $A$ of $X$, we represent the restricted metric $d |_A$ as the same symbol $d$ as the ambient metric $d$. For a subset $A$ of $X$, we denote by $\delta_d(A)$ the diameter of $A$, and we define $\alpha_d(A) = \inf \{d(x, y) \mid x \neq y, x, y \in A\}$.

The following two lemmas are used to prove our results.

Lemma 2.1. Let $L \in (0, \infty)$, and $A$ be a closed subset of $[0, 1]$. Then there exists an $L$-Lipschitz function $\zeta : [0, 1] \rightarrow [0, \infty)$ with $\zeta^{-1}(0) = A$.

Proof. For $x \in [0, 1]$, let $\xi(x)$ be the distance between $x$ and $A$. Then the function $L \cdot \xi : [0, 1] \rightarrow [0, \infty)$ is a desired one. □

Lemma 2.2. Let $x, y, u, v \in \mathbb{R}$. Then, we have

$$|x \vee y - u \vee v| \leq |x - u| \vee |y - v|.$$

Proof. We only need to consider the case of $x \vee y = x$ and $u \vee v = v$. By $u \leq v$, we obtain $x - v \leq x - u \leq |x - u|$. By $y \leq x$, we obtain $v - x \leq v - y \leq |y - v|$. These imply the inequality. □

2.2. Quasi-symmetrically invariant properties. For a metric space $(X, d)$, the Assouad dimension $\dim_A(X, d)$ of $(X, d)$ is defined by the infimum of all $\beta \in (0, \infty)$ for which there exists $C \in (0, \infty)$ such that for every finite subset $A$ of $X$ we have $\text{Card}(A) \leq C \cdot (\delta_d(A)/\alpha_d(A))^\beta$. By the definition of the doubling property, we obtain the following two lemmas.

Lemma 2.3. A metric space $(X, d)$ is doubling if and only if there exist $\beta \in (0, \infty)$ and $C \in [1, \infty)$ such that for every finite subset $A$ of $X$ we have $\text{Card}(A) \leq C \cdot (\delta_d(A)/\alpha_d(A))^\beta$. 

Lemma 2.4. Let \((X,d)\) be a metric space, and let \(A\) be a subset of \(X\). Then, we have 
\[ \dim_A(A,d) \leq \dim_A(X,d). \]

Note that the doubling property is equivalent to the finiteness of the Assouad dimension.

By definitions of ultrametric spaces, we obtain:

Lemma 2.5. Every ultrametric space is \(\delta\)-uniformly disconnected for all \(\delta \in (0,1)\).

For two metric spaces \((X,d)\) and \((Y,e)\), we denote by \(d \times e\) the \(\ell^\infty\)-product metric defined by 
\[ (d \times e)((x,y),(u,v)) = d(x,u) \lor e(y,v). \]

Note that \(d \times e\) generates the product topology of \(X \times Y\).

In this paper, we sometimes use the disjoint union \(\bigsqcup_{i \in I} X_i\) of a non-disjoint family \(\{X_i\}_{i \in I}\). Whenever we consider the disjoint union \(\bigsqcup_{i \in I} X_i\) of a family \(\{X_i\}_{i \in I}\) of sets (this family is not necessarily disjoint), we identify the family \(\{X_i\}_{i \in I}\) with its disjoint copy unless otherwise stated. If each \(X_i\) is a topological space, we consider that \(\bigsqcup_{i \in I} X_i\) is equipped with the direct sum topology.

Next we review the basic statements of the doubling property, the uniform disconnectedness, and the uniform perfectness.

The following is presented in [4, Corollary 10.1.2].

Lemma 2.6. Let \((X,d)\) and \((Y,e)\) be metric spaces. Then, 
\[ \dim_A(X \times Y,d \times e) \leq \dim_A(X,d) + \dim_A(Y,e). \]

By Lemmas 2.13 and 2.14 in [6], we obtain the following lemma (see also [6, Remark 2.16]):

Lemma 2.7. Let \((X,d)\) and \((Y,e)\) be metric spaces. Then for all \(k \in \{1,2\}\), we obtain 
\[ T_p^k(X \times Y,d \times e) = T_p^k(X,d) \land T_p^k(Y,e), \] 
where \(\land\) is the minimum operator.

By [6, Lemma 5.14], we obtain:

Lemma 2.8. Let \((X,d)\) and \((Y,e)\) be metric spaces. Let \(h \in \text{Met}(X \sqcup Y)\) with \(h|_{X^2} = d\) and \(h|_{Y^2} = e\). Then for all \(k \in \{1,2,3\}\), we obtain 
\[ T_p^k(X \sqcup Y,h) = T_p^k(X,d) \land T_p^k(Y,e). \]

The following is identical with [6, Lemma 2.15].

Lemma 2.9. Let \((X,d)\) and \((Y,e)\) be metric spaces. If either of the two is uniformly perfect, then so is \((X \times Y,d \times e)\).

By the definition of the uniform perfectness, we obtain:

Lemma 2.10. Let \((X,d)\) be a metric space, and \(A, B\) be subsets of \(X\). If \((A,d)\) and \((B,d)\) are uniformly perfect, then so is \((A \cup B,d)\).

We refer the readers to [5] for the details of the following:
Proposition 2.11. The doubling property, the uniform disconnectedness, and the uniform perfectness are invariant under quasi-symmetric maps.

By the definitions of the doubling property, and the uniformly disconnectedness, we obtain:

Proposition 2.12. Let \((X, d)\) be a metric space, and \(A\) be a subset of \(X\). If \((X, d)\) is doubling (resp. uniformly disconnected), then so is \((A, d)\).

For a property \(\mathcal{P}\) of metric spaces, and for a metric space \((X, d)\) we define \(\mathcal{S}_\mathcal{P}(X,d)\) as the set of all points in \(X\) of which no neighborhoods satisfy \(\mathcal{P}\) (see [6, Definition 1.3]). By Proposition 2.11, we obtain:

Lemma 2.13. Let \(k \in \{1, 2, 3\}\). If metric spaces \((X, d)\) and \((Y, e)\) are quasi-symmetric equivalent to each other, then so are \(\mathcal{S}_{\mathcal{P}_k}(X, d)\) and \(\mathcal{S}_{\mathcal{P}_k}(Y, e)\). In particular, \(\mathcal{S}_{\mathcal{P}_k}(X, d)\) is an isometric invariant.

For a triple \((u, v, w) \in \{0, 1\}^3\) except \((1,1,1)\), we say that a metric space \((X, d)\) is of totally exotic type \((u, v, w)\) if \((X, d)\) is of type \((u, v, w)\), and if \(\mathcal{S}_{\mathcal{P}_k}(X, d) = X\) holds for all \(k \in \{1, 2, 3\}\) with \(T_{\mathcal{P}_k}(X, d) = 0\). This concept was introduced in the author’s paper [6], and the existence of such spaces was proven in [6, Theorem 1.7].

Theorem 2.14. For every \((u, v, w) \in \{0, 1\}^3\) except \((1,1,1)\), there exists a Cantor metric space of totally exotic type \((u, v, w)\).

The definition of totally exotic types and Proposition 2.12 imply:

Lemma 2.15. Let \((u, v, w) \in \{0, 1\}^3\) except \((1,1,1)\). Let \((X, d)\) be a metric space of totally exotic type \((u, v, w)\). Let \((Y, e)\) be a metric space. Then for all \(k \in \{1, 2\}\) with \(T_{\mathcal{P}_k}(X, d) = 0\), we obtain \(\mathcal{S}_{\mathcal{P}_k}(X \times Y, d \times \infty, e) = X \times Y\).

2.3. The Gromov–Hausdorff distance. For a metric space \((Z, h)\), and for subsets \(A, B\) of \(Z\), we denote by \(\mathcal{H}(A, B; Z, h)\) the Hausdorff distance of \(A\) and \(B\) in \(Z\). For metric spaces \(X\) and \(Y\), the Gromov–Hausdorff distance \(\mathcal{GH}((X, d), (Y, e))\) between \(X\) and \(Y\) is defined as the infimum of all values \(\mathcal{H}(i(X), j(Y); Z, h)\), where \((Z, h)\) is a metric space, and \(i : X \to Z\) and \(j : Y \to Z\) are isometric embeddings. For \(\epsilon \in (0, \infty)\), and for metric spaces \(X\) and \(Y\), a pair \((f, g)\) with \(f : X \to Y\) and \(g : Y \to X\) is said to be an \(\epsilon\)-approximation between \((X, d)\) and \((Y, e)\) if the following conditions hold:

1. For all \(x, y \in X\), we have \(|d(x, y) - e(f(x), f(y))| < \epsilon\);
2. For all \(x, y \in Y\), we have \(|e(x, y) - d(g(x), g(y))| < \epsilon\);
3. For each \(x \in X\) and for each \(y \in Y\), we have \(d(g \circ f(x), x) < \epsilon\) and \(e(f \circ g(x), x) < \epsilon\).
Remark that if there exists a map \( f : (X, d) \to (Y, e) \) satisfying (1) and \( \bigcup_{y \in f(X)} B(y, e) = Y \), then there exists \( g : (Y, e) \to (X, d) \) such that \( (f, g) \) is a 3\( \varepsilon \)-approximation between \((X, d)\) and \((Y, e)\) (see [11]).

The proof of the next lemma is presented in [1] and [11].

**Lemma 2.16.** Let \( \{(X_i, d_i)\}_{i \in \mathbb{N}} \) be a sequence of compact metric spaces, and \((X, d)\) be a compact metric space. Then \( \widetilde{GH}((X_i, d_i), (X, d)) \to 0 \) as \( i \to \infty \) if and only if there exists a sequence \( \{\varepsilon_i\}_{i \in \mathbb{N}} \) in \((0, \infty)\) converging to 0, and for each \( i \in \mathbb{N} \) there exists an \( \varepsilon_i \)-approximation between \((X_i, d_i)\) and \((X, d)\).

Let \((X, d)\) and \((Y, e)\) be compact metric spaces. We say that a subset \( R \) of \( X \times Y \) is a correspondence if \( \pi_X(R) = X \) and \( \pi_Y(R) = Y \), where \( \pi_X \) and \( \pi_Y \) are projections into \( X \) and \( Y \), respectively. We denote by \( \mathcal{R}(X, Y) \) (resp. \( \mathcal{CR}(X, d, Y, e) \)) the set of all correspondences (resp. closed correspondences in \( X \times Y \)) of \( X \) and \( Y \). For \( R \in \mathcal{R}(X, Y) \), we define the distortion \( \text{dis}(R) \) of \( R \) by

\[
\text{dis}(R) = \sup_{(x,y), (u,v) \in R} |d(x, u) - e(y, v)|.
\]

The proof of the next lemma is presented in [1].

**Lemma 2.17.** For all compact metric spaces \((X, d)\) and \((Y, e)\), we obtain

\[
\widetilde{GH}((X, d), (Y, e)) = \frac{1}{2} \inf_{R \in \mathcal{R}(X, Y)} \text{dis}(R).
\]

Let \((X, d)\) and \((Y, e)\) be compact metric spaces. We denote by \( \mathcal{CR}_{opt}(X, d, Y, e) \) the set of all \( G \in \mathcal{CR}(X, d, Y, e) \) satisfying that \( \text{dis}(G) = \inf_{R \in \mathcal{R}(X, Y)} \text{dis}(R) \). An element of \( \mathcal{CR}_{opt}(X, d, Y, e) \) is said to be optimal. The proof of the next lemma is presented in [2] and [9].

**Lemma 2.18.** If \((X, d)\) and \((Y, e)\) are compact metric spaces, then \( \mathcal{CR}_{opt}(X, d, Y, e) \neq \emptyset \).

For a set \( X \), we define \( \Delta_X \in \mathcal{R}(X, X) \) by \( \Delta_X = \{(x, x) \mid x \in X \} \), and we call it the trivial correspondence of \( X \).

### 2.4. Amalgamation of metrics

Since the following lemma seems to be classical, we omit the proof. For \( n \in \mathbb{N} \), we put \( \hat{n} = \{1, 2, \ldots, n\} \), and we consider that \( \hat{n} \) is always equipped with the discrete topology.

**Lemma 2.19.** Let \((X_i, d_i)\) be a sequence of metric spaces. Let \( e \in \text{Met}(\hat{n}) \). Assuming that \( \delta_{\hat{n}}(X_i) \leq \alpha_{\varepsilon}(\hat{n}) \), we define a symmetric function \( D : (\prod_{i=1}^n X_i)^2 \to [0, \infty) \) by

\[
D(x, y) = \begin{cases} 
  d_i(x, y) & \text{if } x, y \in X_i \\
  e(i, j) & \text{if } x \in X_i \text{ and } y \in X_j \text{ with } i \neq j
\end{cases}
\]

Then \( D \in \text{Met}(\prod_{i=1}^n X_i) \).
Lemma 3.1. The set \( \text{definition of dimensional functions.} \)

\( \square \) statement (1). For the statement (2), we use the condition (2) in the DO is that for every finite subset of \( X \) Since \( \text{Proof.} \)

Thus, \( (e_i) \) and \( Z = \{e_i\} \) by \( e(i, j) = d(a_i, a_j) \). Put \( \eta = \min \{\epsilon, \alpha e(\hat{n})\} \). Take a Cantor metric space \( (M, h) \) of type \( (u, v, w) \) (see \([6, \text{Theorem 1.2}]\)). By replacing with \( (M, (\eta/\delta(M)h)) \) if necessary, we may assume that \( \delta(M) \leq \eta \). For each \( i \in \{1, \ldots, n\} \), let \( (X_i, d_i) \) be an isometric copy of \( (M, h) \). Let \( Z = \prod_{i=1}^{n} X_i \). Let \( D \) be a metric stated in Lemma 2.19 induced from \( e \) and \( \{(X_i, d_i)\}_{i=1}^{n} \). Then, by Lemma 2.8 the space \( (Z, D) \) is of type \( (u, v, w) \), and we obtain

\[ \mathcal{GH}((X, d), (Z, D)) \leq \mathcal{GH}((X, d), (A, d)) + \mathcal{GH}((A, d), (Z, D)) \leq 2\epsilon. \]

Thus, \( \mathcal{X}(u, v, w) \) is dense in \( \mathcal{M} \).

The statements (2) and (3) can be proven in the same way as the statement (1). For the statement (2), we use the condition (2) in the definition of dimensional functions. \( \square \)

3. Topological distributions

In this section, we prove Theorems 1.1 and 1.2

Lemma 3.1. The set \( \mathcal{D} \mathcal{O} \) is \( F_{\sigma} \) and meager in \( \mathcal{M} \).

\( \text{Proof.} \) Since \( \mathcal{M} \setminus \mathcal{D} \mathcal{O} \) is dense (see Lemma 2.20), it suffices to show that \( \mathcal{D} \mathcal{O} \) is \( F_{\sigma} \). For \( C, \beta \in (0, \infty) \), let \( S(C, \beta) \) be the set of all \( (X, d) \) such that for every finite subset of \( X \) we have

\[ \text{Card}(A) \leq C \cdot \left( \frac{\delta_d(A)}{\alpha_d(A)} \right)^{\beta}. \]

We now prove that each \( S(C, \beta) \) is closed in \( \mathcal{M} \). Take a convergent sequence \( \{(X_i, d_i)\}_{i \in \mathbb{N}} \) in \( S(C, \beta) \), and let \( (X, d) \) be its limit space. Then, by Lemma 2.16 there exist a positive sequence \( \{\epsilon_i\}_{i \in \mathbb{N}} \) converging to 0, and sequences \( \{f_i : (X_i, d_i) \to (X, d)\}_{i \in \mathbb{N}} \) and \( \{g_i : (X, d) \to (X_i, d_i)\}_{i \in \mathbb{N}} \) such that for each \( i \in \mathbb{N} \) the pair \( (f_i, g_i) \) is an \( \epsilon_i \)-approximation. Take an arbitrary finite subset \( A \) of \( X \). Take a sufficiently large \( i \in \mathbb{N} \), then \( g_i : A \to g_i(A) \) is bijective. Thus,

\[ \text{Card}(A) = \text{Card}(g_i(A)) \leq C \cdot \left( \frac{\delta_d(g_i(A))}{\alpha_d(g_i(A))} \right)^{\beta} \leq C \cdot \left( \frac{\delta_d(A) + \epsilon_i}{\alpha_d(A) - \epsilon_i} \right)^{\beta}. \]
By letting $i \to \infty$, we obtain

$$\text{Card}(A) \leq C \cdot \left( \frac{\delta_d(A)}{\alpha_d(A)} \right)^{\beta}. $$

Then $(X, d) \in S(C, \beta)$, and hence $S(C, \beta)$ is closed in $\mathcal{M}$.

By Lemma 3.2, we obtain

$$\mathcal{D} \sigma = \bigcup_{C, \beta \in \mathbb{Q}_{>0}} S(C, \beta).$$

Therefore, we conclude that $\mathcal{D} \sigma$ is $F_\sigma$ in $\mathcal{M}$. 

\begin{lemma}
The set $\mathcal{U} \mathcal{D}$ is $F_\sigma$ and meager in $\mathcal{M}$.
\end{lemma}

\textbf{Proof.} Since $\mathcal{M} \setminus \mathcal{U} \mathcal{D}$ is dense (see Lemma 2.20), it suffices to show that $\mathcal{U} \mathcal{D}$ is $F_\sigma$. For $\delta \in (0, 1)$, we denote by $S(\delta)$ the set of all $\delta$-uniformly disconnected compact metric spaces. We now prove that each $S(\delta)$ is closed in $\mathcal{M}$. Take a convergent sequence $\{(X_i, d_i)\}_{i \in \mathbb{N}}$ in $S(\delta)$, and let $(X, d)$ be its limit compact metric space. Then, by Lemma 2.16, there exist a positive sequence $\{\epsilon_i\}_{i \in \mathbb{N}}$ converging to 0, and sequences $\{f_i : (X_i, d_i) \to (X, d)\}_{i \in \mathbb{N}}$ and $\{g_i : (X, d) \to (X_i, d_i)\}_{i \in \mathbb{N}}$ such that for each $i \in \mathbb{N}$ the pair $(f_i, g_i)$ is an $\epsilon_i$-approximation. Take a finite non-constant sequence $\{z_i\}_{i=1}^N$ in $(X, d)$. For a sufficiently large $i \in \mathbb{N}$, the sequence $\{g_i(z_k)\}_{k=1}^N$ is non-constant. Since $(X_i, d_i)$ is $\delta$-uniformly disconnected, we obtain

$$\delta d(g_i(z_1), g_i(z_N)) \leq \max_{1 \leq k \leq N} d(g_i(z_k), g_i(z_{k+1})).$$

By letting $i \to \infty$, we obtain $\delta d(z_1, z_N) \leq \max_{1 \leq i \leq N} d(z_i, z_{i+1})$. This implies that $(X, d) \in S(\delta)$. Since

$$\mathcal{U} \mathcal{D} = \bigcup_{\delta \in \mathbb{Q} \cap (0, 1)} S(\delta),$$

we conclude that $\mathcal{U} \mathcal{D}$ is $F_\sigma$ in $\mathcal{M}$. 

\begin{lemma}
The set $\mathcal{U} \mathcal{P}$ is $F_\sigma$ and meager in $\mathcal{M}$.
\end{lemma}

\textbf{Proof.} Since $\mathcal{M} \setminus \mathcal{U} \mathcal{P}$ is dense (see Lemma 2.20), it suffices to show that $\mathcal{U} \mathcal{P}$ is $F_\sigma$. For $c \in (0, 1)$ and $t \in (0, \infty)$, let $S(c, t)$ be the set of all metric spaces $(X, d)$ such that for all $x \in X$ there exists $y \in X$ with $cr \leq d(x, y) \leq r$. We prove that $S(c, t)$ is closed in $\mathcal{M}$. Take a convergent sequence $\{(X_i, d_i)\}_{i \in \mathbb{N}}$ in $S(c, t)$, and let $(X, d)$ be its limit compact metric space. Then, by Lemma 2.16, there exist a positive sequence $\{\epsilon_i\}_{i \in \mathbb{N}}$ converging to 0, and sequences $\{f_i : (X_i, d_i) \to (X, d)\}_{i \in \mathbb{N}}$ and $\{g_i : (X, d) \to (X_i, d_i)\}_{i \in \mathbb{N}}$ such that for each $i \in \mathbb{N}$ the pair $(f_i, g_i)$ is an $\epsilon_i$-approximation. Take arbitrary $z \in X$, and for each $n \in \mathbb{N}$ put $x_n = g_i(z)$. Then $x_i \in X_i$ and $d(f_i(x_i), z) \leq \epsilon_i$. By
(X_i, d_i) ∈ S(c, t), there exists y_i with cr ≤ d_n(x_i, y_i) ≤ r. Combining these inequalities and |d_n(x_i, y_i) − d(f_i(x_i), f_i(y_i))| ≤ ε_i, we obtain

\[ cr - 2ε_i ≤ d(z, f_i(y)) ≤ r + 2ε_i. \]

(3.1)

By extracting a subsequence if necessary, we may assume that the sequence \( \{f_i(y_i)\}_{i ∈ \mathbb{N}} \) is convergent. Let w be its limit point. By letting \( n → ∞ \), by (3.1), we obtain \( cr ≤ d(z, w) ≤ r \). This implies that \( (X, d) ∈ S(c, t) \), and hence \( S(c, t) \) is closed in \( M \). Since

\[ \mathcal{UP} = \bigcup_{c ∈ \mathbb{Q} \cap [0, 1), t ∈ \mathbb{Q} > 0} S(c, t), \]

we conclude that \( \mathcal{UP} \) is \( F_σ \).

We now prove Theorems 1.1 and 1.2.

**Proof of Theorem 1.1.** By combining Lemmas 3.1, 3.2, and 3.3, we obtain Theorem 1.1.

**Proof of Theorem 1.2.** From Theorem 1.1 and Lemma 2.20, and the fact that the intersection of an \( F_σ \) set and a \( Gδ \) set of a metric space is \( F_σδ \) and \( Gδσ \), Theorem 1.2 follows.

- \( \square \)

4. CONSTRUCTIONS OF METRIC SPACES

In this section, we prepare constructions of metrics spaces to prove Theorem 1.3.

Let \( X = \{(X_i, d_i)\}_{i ∈ \mathbb{N}} \) be a family of metric spaces satisfying the inequality \( δ_d(X_i) ≤ 2^{−i−1} \) for all \( i ∈ \mathbb{N} \). We put

\[ T(X) = \{∞} \sqcup \coprod_{i ∈ \mathbb{N}} X_i, \]

and define a symmetric function \( d_X : (T(X))^2 → [0, ∞) \) by

\[ d_X(x, y) = \begin{cases} d_i(x, y) & \text{if } x, y ∈ X_i \text{ for some } i, \\ |2^{−i} - 2^{−j}| & \text{if } x ∈ X_i, y ∈ X_j \text{ for some } i \neq j, \\ 2^{−i} & \text{if } x = ∞, y ∈ X_i \text{ for some } i. \end{cases} \]

Then \( d_X \) is a metric on \( T(X) \). This construction is a specific version of the telescope space defined in \( \mathbb{[}6\mathbb{]} \). The space \( (T(X), d_X) \) is the same as the telescope space \( (T(X, \mathcal{R}), d_{(X, \mathcal{R})}) \), where \( \mathcal{R} \) is the telescope base defined in \( \mathbb{[}6\mathbb{]} \) Definition 3.2]. The symbol \( \mathcal{R} \) is a pair of the space \( \{0\} \sqcup \{2^{−n} \mid n ∈ \mathbb{N} \} \) with the Euclidean metric and its numbering map. We review the basic properties of this construction.

**Proposition 4.1.** Let \( X = \{(X_i, d_i)\}_{i ∈ \mathbb{N}} \) be a sequence of metric spaces with \( δ_d(X_i) ≤ 2^{−i−1} \). Then, the following hold true.

1. If there exists \( N ∈ \mathbb{N} \) such that each \( (X_i, d_i) \) is \( N \)-doubling, then \( (T(X), d_X) \) is doubling.
Proof. The statements (1), (2), and (3) follow from Proposition 3.8, Proposition 3.9, Proposition 3.10 in [6], respectively.

We denote by $2^\omega$ the set of all maps from $\mathbb{Z}_{\geq 0}$ into $\{0,1\}$. We define $v : 2^\omega \times 2^\omega \to [0,\infty]$ by $v(x,y) = \min \{ n \in \mathbb{Z}_{\geq 0} \mid x(n) \neq y(n) \}$ if $x \neq y$; otherwise $v(x,y) = \infty$. For each $c \in (0,1)$, we define $g_c : \mathbb{Z}_{\geq 0} \cup \{\infty\} \to [0,\infty)$ by $g_c(n) = c^n$ if $n \in \mathbb{Z}_{\geq 0}$ and $g_c(\infty) = 0$. We define a metric $\beta_c$ on $2^\omega$ by $\beta_c(x,y) = g_c(v(x,y))$. For every $c \in (0,1)$, the metric space $(2^\omega, \beta_c)$ is a Cantor space. This metric space is a specific version of a sequentially metrized Cantor space defined in the author’s paper [6]. The following is implicitly shown in the proof of [6, Lemma 6.3].

Lemma 4.2. Let $c \in (0,1)$. If $N \in (0,\infty)$ satisfies that for all $k \in \mathbb{N}$ we have $\text{Card}(\{ n \in \mathbb{N} \mid c^k/2 \leq c^n \leq c^k \}) \leq N$, then $(2^\omega, \beta_c)$ is $(2^{N+1})$-doubling. In particular, $(2^\omega, \beta_c)$ is $(2^{\log 2/\log c^{-1}+2})$-doubling.

Lemma 4.3. Let $c \in (0,1)$. Then

1. The space $(2^\omega, \beta_c)$ is $(2^{\log 2/\log c^{-1}+2})$-doubling.
2. The space $(2^\omega, \beta_c)$ is an ultrametric space.
3. The space $(2^\omega, \beta_c)$ is $c$-uniformly perfect.
4. For all $x \in 2^\omega$, and for all $r \in (0,\infty)$, we have $\dim_A(B(x,r), \beta_c) = \log 2/\log c^{-1}$.

Proof. Lemma 4.2 implies the statement (1). The statements (2) and (3) follow from the definition of $\beta_c$. The statement (4) follows from the fact that $(B(x,r), \beta_c)$ is isometric to $(2^\omega, \tilde{e}^m \beta_c)$ for some $m \in \mathbb{N}$.

Definition 4.1. Let $\mathcal{P} = \{ (P_{i,j}, p_{i,j}) \}_{i \in \{1, 2, 3\}, j \in \mathbb{N}}$ be a sequence of compact metric spaces indexed by $\{1, 2, 3\} \times \mathbb{N}$. For $q \in \mathbb{Q}$, we put $\mathcal{P}_q = (\mathcal{P}, q)$. We now define a metric space $(U(\mathcal{P}_q), e_{\mathcal{P}_q})$ induced from $\mathcal{P}_q$ by using the telescope construction. We define $l : \mathbb{R} \to [0,\infty)$ by $l(x) = \sqrt{2 - 2\cos(x)}$, and define a homeomorphism $\theta : [0,1] \to [\pi/6, \pi/3]$ by $\theta(t) = (\pi/6)(t+1)$. Assume that $\delta_{p_{i,j}}(P_{i,j}) \leq 2^{-j-1}(\pi/6)$ for all $i \in \{1, 2, 3\}$ and $j \in \mathbb{N}$. For $q = \{q_i\}_{i \in \mathbb{N}} \in \mathbb{Q}$ and $j \in \mathbb{N}$, define a metric $e_{j,q} \in \text{Met}(\check{3})$ by $e_{j,q}(a,b) = \begin{cases} 2^{-j-1} & \text{if } \{a,b\} = \{1, 2\} \text{ or } \{2, 3\}; \\ 2^{-j-1} \cdot l(\theta(q_j)) & \text{if } \{a,b\} = \{1, 3\}. \end{cases}$ The metric space $(\check{3}, e_{j,q})$ is the set of vertices of the isosceles triangle whose apex angle is $\theta(q_j)$ and whose length of the legs is $2^{-j-1}$. For
each $j \in \mathbb{N}$, we put $T_j = P_{1,j} \sqcup P_{2,j} \sqcup P_{3,j}$ and let $k_j$ be a metric stated in Lemma 2.19 induced from $\{(P_{i,j}, p_{i,j})\}_{i \in \{1,2,3\}}$ and $e_{j,q} \in \text{Met}\left(\hat{3}\right)$. Put $\tilde{P}_q = \{(T_j, k_j)\}_{j \in \mathbb{N}}$. Then we have $\delta_{k_j}(T_j) = 2^{-j-1}$. We define $$(U(P_q), e_{P_q}) = \left(\tilde{T}(\tilde{P}_q), d_{\tilde{P}_q}\right).$$

Figure 2. $\left(\hat{3}, e_{j,q}\right)$
Figure 3. $(T_j, k_j)$

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{$(U(P_q), e_{P_q})$ in Definition 4.1}
\end{figure}

By Proposition 4.1, we obtain:

**Proposition 4.4.** Let $\mathcal{P} = \{(P_{i,j}, p_{i,j})\}_{i \in \{1,2,3\}, j \in \mathbb{N}}$ be a sequence of metric spaces with $\delta_{p_{i,j}}(P_{i,j}) \leq 2^{-j-1}(\pi/6)$. Let $q \in \mathbb{Q}$. Then we have:

(1) *If there exists $N \in \mathbb{N}$ such that each $(P_{i,j}, p_{i,j})$ is $N$-doubling, then $(U(P_q), e_{P_q})$ is doubling.*
(2) If there exists $\delta \in (0, 1)$ such that each $(P_{i,j}, p_{i,j})$ is $\delta$-uniformly disconnected, then $(U(\mathcal{P}_q), e_{P_q})$ is uniformly disconnected.

(3) If there exists $c \in (0, 1)$ such that each $(P_{i,j}, p_{i,j})$ is $c$-uniformly perfect, then $(U(\mathcal{P}_q), e_{P_q})$ is uniformly perfect.

The following proposition states the continuity of $\{(U(\mathcal{P}_q), e_{P_q})\}_{q \in \mathbb{Q}}$.

**Proposition 4.5.** Let $\mathcal{P} = \{(P_{i,j}, p_{i,j})\}_{i \in \{1,2,3\}, j \in \mathbb{N}}$ be a sequence with $\delta_{P_{i,j}}(P_{i,j}) \leq 2^{-i-1}l(\pi/6)$. Then the map $G : \mathbb{Q} \to \mathcal{M}$ defined by $G(q) = (U(\mathcal{P}_q), e_{P_q})$ is continuous.

**Proof.** For $q \in \mathbb{Q}$, put $(X_q, d_q) = (U(\mathcal{P}_q), e_{P_q})$. Note that $X_q = X_r$ for all $q, r \in \mathbb{Q}$. We fix $q \in \mathbb{Q}$, and take arbitrary $\epsilon \in (0, \infty)$. Since $l \circ \theta : [0, 1] \to [0, 1]$ is uniformly continuous, there exists $\delta \in (0, \infty)$ such that for all $s, t \in [0, 1]$ with $|s - t| \leq \delta$, we obtain $|l \circ \theta(s) - l \circ \theta(t)| \leq \epsilon$. Take $n \in \mathbb{N}$ such that $2^{-n} \leq \min\{\delta, \epsilon\}$. We denote by $V$ the set of all $r \in \mathbb{Q}$ satisfying that for every $i \in \{1, 2, \ldots, n\}$ we obtain $|q_i - r_i| \leq 2^{-i}$. Note that $V$ is a neighborhood of $q$ in $\mathbb{Q}$. We then estimate $\text{dis}(\Delta_X_q)$. By the definition of the metric space $(U(\mathcal{P}_q), e_{P_q})$, for every $r \in V$, the quantity $|d_q(x, y) - d_r(x, y)|$ only takes values in the set $\{1 \cdot 2^{-i-1} \mid i \in \mathbb{N}\}$. If $i \in \{1, 2, \ldots, n\}$, by $|q_i - r_i| \leq \delta$, we obtain $|l(\theta(q_i)) - l(\theta(r_i))| \cdot 2^{-i-1} \leq \epsilon$. If $i > n$, using $l \circ \theta(x) \leq 1$ for all $x \in [0, 1]$, we obtain $|l(\theta(q_i)) - l(\theta(r_i))| \cdot 2^{-i-1} \leq 2 \cdot 2^{-n} \leq 2\epsilon$. This implies that $\text{dis}(\Delta_X_q) \leq 2\epsilon$. Thus, we conclude that for every $r \in V$, we obtain $\mathcal{G}(G(q), G(r)) \leq \epsilon$, and hence $G$ is continuous. $\square$

A topological space is said to be **perfect** if it has no isolated points. For a metric space $(X, d)$, and $v \in [0, \infty)$ we denote by $A(v, X, d)$ the set of all $x \in X$ for which there exists $r \in (0, \infty)$ such that for every $\epsilon \in (0, r)$ we have $\dim_A(B(x, \epsilon), d) = v$. Note that if $(X, d)$ and $(Y, e)$ are isometric to each other, then so are $A(v, X, d)$ and $A(v, Y, e)$ for all $v \in [0, \infty)$.

Under certain assumptions on a family $\mathcal{P}$, we can prove that the family $\{(U(\mathcal{P}_q), e_{P_q})\}_{q \in \mathbb{Q}}$ are not isometric to each other.

**Proposition 4.6.** Let $\mathcal{P} = \{(P_{i,j}, p_{i,j})\}_{i \in \{1,2,3\}, j \in \mathbb{N}}$ be a sequence of compact metric spaces with $\delta_{P_{i,j}}(P_{i,j}) \leq 2^{-j-1}l(\pi/6)$. We assume that either of the following conditions holds true:

1. Each $(P_{i,j}, p_{i,j})$ is the one-point metric space.
2. For all $i \in \{1,2,3\}$ and $j \in \mathbb{N}$, we have:
   1. the space $(P_{i,j}, p_{i,j})$ is perfect;
   2. $\mathcal{A}(\dim_A(P_{i,j}, p_{i,j}), P_{i,j}, p_{i,j}) = (P_{i,j}, p_{i,j})$;
   3. the values $\{\dim_A(P_{i,j}, p_{i,j})\}_{i \in \{1,2,3\}, j \in \mathbb{N}}$ are different from each other.

Then for all $q, r \in \mathbb{Q}$ with $q \neq r$, and for all $K, L \in (0, \infty)$, the spaces $(U(\mathcal{P}_q), K \cdot e_{P_q})$ and $(U(\mathcal{P}_r), L \cdot e_{P_r})$ are not isometric to each other.
Proof. Put \((X, d) = (U(P_q), K \cdot e_{P_q})\) and \((Y, e) = (U(P_r), L \cdot e_{P_r})\). For the sake of contradiction, suppose that there exists an isometry \(I : (X, d) \to (Y, e)\).

We first assume that the condition [1] is true. By the definition, the spaces \((X, d)\) and \((Y, e)\) have unique accumulation points, say \(\omega, \omega'\), respectively. Then \(I(\omega) = \omega'\). Thus, we obtain \(\{d(\omega, x) \mid x \in X\} = \{d(\omega', y) \mid y \in Y\}\). Since \(\{d(\omega, x) \mid x \in X\} = \{0\} \cup \{K \cdot 2^{-i} \mid i \in \mathbb{N}\}\), and \(\{e(\omega', y) \mid y \in Y\} = \{0\} \cup \{L \cdot 2^{-i} \mid i \in \mathbb{N}\}\), we obtain \(K = L\). For each \(i \in \mathbb{N}\), the set \(\{x \in X \mid d(\omega, x) = K2^{-i}\}\) consists of three points, and they form an isosceles triangle. The apex angle of this isosceles triangle is equal to \(\theta(q_i)\), and \(I(\{x \in X \mid d(\omega, x) = K2^{-i}\}) = \{y \in Y \mid e(\omega', y) = L2^{-i}\}\). Then we obtain \(\theta(q_i) = \theta(r_i)\), and hence \(q = r\). This is a contradiction.

Second, we assume that the condition [2] holds true. Note that both \((X, d)\) and \((Y, e)\) contain \(P_{i,j}\) for all \((i, j) \in \{1, 2, 3\} \times \mathbb{N}\) and the element represented as \(\infty\). For each \((i, j) \in \{1, 2, 3\} \times \mathbb{N}\), we put \(W_k(i, j) = A(\dim_A(P_{i,j}, Y, d))\) and \(W_k(i, j) = A(\dim_A(P_{i,j}, Y, e))\). By the assumption [2-b] and [2-c], and by the fact that each \(P_{i,j}\) is open in \(X\) and \(Y\), respectively, for all \(k \in \{1, 2\}\) we have

\[
W_k(i, j) = \begin{cases} P_{i,j} & \text{if } \infty \notin W_k(i, j) \\ \{\infty\} \cup P_{i,j} & \text{if } \infty \in W_k(i, j). \end{cases}
\]

By Definition [4.1] for all \(k \in \{1, 2\}\), if \(\infty \in W_k(i, j)\), then \(\infty\) is an isolated point of \(W_k(i, j)\). Then, since each \(P_{i,j}\) is perfect (the assumption [2-a]), for each \((i, j) \in \{1, 2, 3\} \times \mathbb{N}\), we can take a non-isolated point \(a_{i,j} \in W_1(i, j)\). Note that \(a_{i,j} \neq \infty\) and \(a_{i,j} \in P_{i,j}\). Since \(I\) is an isometry and \(A\) is invariant under isometries, the point \(I(a_{i,j})\) is a non-isolated point and \(I(a_{i,j}) \in W_1(i, j)\). Since \(\infty \notin W_2(i, j)\) or \(\infty\) is an isolated point of \(W_2(i, j)\), we have \(I(a_{i,j}) \in P_{i,j} \subset Y\). Put \(Q = \{(a_{i,j}), d_{i,j}\}\) \(i \in \{1, 2, 3\}, j \in \mathbb{N}\), where \(d_{i,j}\) is the trivial metric. Then \((U(Q), e_Q)\) and \((U(Q), e_Q)\) are isometric to each other. Thus by the first case, we obtain \(q = r\). This leads to the proposition. \(\Box\)

5. Geodesics

In this section, we prove Theorem [1.3] and [1.4]. As its applications, we next prove Theorem [1.6] [1.7] and [1.8].

The following construction of geodesics using optimal closed correspondences is presented in [2] and [3].

**Proposition 5.1.** Let \((X, d)\) and \((Y, e)\) be compact metric spaces with \(GH((X, d), (Y, e)) > 0\). Let \(R \in C_{opt}(X, Y, e)\). For each \(s \in (0, 1)\), define a metric \(D_s\) on \(R\) by \(D_s((x, y), (u, v)) = (1 - s)d(x, u) + se(y, v)\).
We define $\gamma : [0, 1] \to \mathcal{M}$ by

$$
\gamma(s) = \begin{cases} 
(X, d) & \text{if } s = 0; \\
(Y, e) & \text{if } s = 1; \\
(R, D_s) & \text{if } s \in (0, 1).
\end{cases}
$$

Then $\gamma$ is a geodesic in $(\mathcal{M}, \mathcal{GH})$ from $(X, d)$ to $(Y, e)$. Moreover,

1. For every $s \in (0, 1)$, the set $\{(x, z) \in X \times R \mid \pi_X(z) = x\}$ is in $\mathcal{CR}_{\text{opt}}(\gamma(0), \gamma(s))$.
2. For every $s \in (0, 1)$, the set $\{(z, x) \in R \times Y \mid \pi_Y(z) = y\}$ is in $\mathcal{CR}_{\text{opt}}(\gamma(s), \gamma(1))$.
3. For all $s, t \in (0, 1)$, we have $\Delta_R \in \mathcal{CR}_{\text{opt}}(\gamma(s), \gamma(t))$.

The following is a useful criterion of geodesics. The proof is presented in [2, Lemma 1.3].

**Lemma 5.2.** Let $\gamma : [0, 1] \to X$ be a curve. If for every $s, t \in [0, 1]$, we have $d(\gamma(s), \gamma(t)) \leq |s - t| \cdot d(\gamma(0), \gamma(1))$, then $\gamma$ is a geodesic.

In Propositions 5.3 and 5.4, we construct modifications of the geodesic stated in Proposition 5.1. In the proof of Theorem 1.3, Proposition 5.3 is used to obtain a geodesic consisting of perfect metric spaces. Proposition 5.4 is used to attach $\{(U_q, e_q)\}_{q \in Q}$ in Definition 4.1 to a geodesic constructed in Proposition 5.3.

**Proposition 5.3.** Let $(X_0, d_0)$ and $(X_1, d_1)$ be compact metric spaces with $\mathcal{GH}((X_0, d_0), (X_1, d_1)) > 0$. Let $f : [0, 1] \to \mathcal{M}$ be a geodesic from $(X_0, d_0)$ to $(X_1, d_1)$. Let $(Y, e)$ be in $\mathcal{M}$, and $L \in (0, \infty)$ satisfy $L\delta(Y) \leq 2 \mathcal{GH}((X_0, d_0), (X_1, d_1))$. Let $A$ be a closed subset of $[0, 1]$ with $\{0, 1\} \subset A$, and let $\zeta : [0, 1] \to [0, \infty)$ be an $L$-Lipschitz map with $\zeta^{-1}(0) = A$. Put $f(s) = (X_s, d_s)$. We define a map $F : [0, 1] \to \mathcal{M}$ by

$$
F(s) = \begin{cases} 
 f(s) & \text{if } s \in A; \\
(X_s \times Y, d_s \times \zeta(s) \cdot e) & \text{if } s \in [0, 1] \setminus A.
\end{cases}
$$

Then $F$ is a geodesic from $(X_0, d_0)$ to $(X_1, d_1)$. Put $F(s) = (Z_s, D_s)$. Moreover,

1. for all $s, t \in [0, 1] \setminus A$, and for all $R \in \mathcal{CR}_{\text{opt}}(f(s), f(t))$, the set

$$
\{ ((x, a), (x', a)) \in Z_s \times Z_t \mid (x, x') \in R, a \in Y \}
$$

is in $\mathcal{CR}_{\text{opt}}(F(s), F(t))$;
2. for all $s \in A$ and $t \in [0, 1] \setminus A$, and for all $R \in \mathcal{CR}_{\text{opt}}(f(s), f(t))$, the set

$$
\{ (x, (x', a)) \in Z_s \times Z_t \mid (x, x') \in R, a \in Y \}
$$

is in $\mathcal{CR}_{\text{opt}}(F(s), F(t))$.
Proof. By Lemma [5.2] it suffices to show that
\begin{equation}
\mathcal{G}(F(s), F(t)) \leq |s - t| \cdot \mathcal{G}((X_0, d_0), (X_1, d_1)).
\end{equation}
We assume that \(s, t \in [0, 1] \setminus A\), and take an optimal correspondence \(R \in \mathcal{C}(X_s, d_s, X_t, d_t)\). Put
\[ U = \{ ((x, a), (x', a)) \in Z_s \times Z_t \mid (x, x') \in R, a \in Y \} \]
Then, \(U \in \mathcal{R}(Z_s, Z_t)\). For all \(((x, a), (x', a)), ((y, b), (y', b)) \in U\), by Lemma [2.2] we obtain
\[ |D_s((x, a), (y, b)) - D_t((x', a), (y', b))| \leq |d_s(x, y) - d_t(x', y')| + |\zeta(s)e(a, b) - \zeta(t)e(a, b)|. \]
Since \(R\) is optimal, we obtain
\[ |d_s(x, y) - d_t(x', y')| \leq 2|s - t|\mathcal{G}(X_0, d_0, X_1, d_1). \]
By the assumption,
\[ |\zeta(s)e(a, b) - \zeta(t)e(a, b)| = |\zeta(s) - \zeta(t)|e(a, b) \leq |s - t|L\delta_e(Y) \]
\[ \leq 2|s - t|\mathcal{G}((X_0, d_0), (X_1, d_1)). \]
Thus, \(\text{dis}(U) \leq 2|s - t|\mathcal{G}((X_0, d_0), (X_1, d_1))\). This implies the inequality [5.1]. The remaining cases can be solved in a similar way. \(\square\)

**Proposition 5.4.** Let \((X_0, d_0)\) and \((X_1, d_1)\) be compact metric spaces with \(\mathcal{G}((X_0, d_0), (X_1, d_1)) > 0\). Let \(f : [0, 1] \to \mathcal{M}\) be a geodesic from \((X_0, d_0)\) to \((X_1, d_1)\). Let \((Y, e) \in \mathcal{M}\), and let \(L \in (0, \infty)\) satisfy \(L\delta_e(Y) \leq 2\mathcal{G}((X_0, d_0), (X_1, d_1))\). Let \(A\) be a closed subset of \([0, 1]\) with \([0, 1] \setminus A\) \(\subseteq A\), and let \(\zeta : [0, 1] \to [0, \infty)\) be an \(L\)-Lipschitz map with \(\zeta^{-1}(0) = A\). We assume that there exists a set \(R\) such that for all \(s \in (0, 1)\) we can represent \(f(s) = (R, d_s)\). Let \(o \in R\), and \(\omega \in Y\). We define
\[ Z = R \times \{\omega\} \cup \{o\} \times Y, \]
and define a map \(F : [0, 1] \to \mathcal{M}\) by
\[ F(s) = \begin{cases} f(s) & \text{if } s \in A; \\ (Z, d_s \times_\infty (\zeta(s) \cdot e)) & \text{if } s \in [0, 1] \setminus A. \end{cases} \]
If for all \(s, t \in (0, 1)\), there exists \(K \in \mathcal{C}(R, d_s, R, d_t)\) with \((o, o) \in K\), then \(F\) is a geodesic from \((X_0, d_0)\) to \((X_1, d_1)\).

**Proof.** Since \(F\) is continuous, by Lemma [5.2] it suffices to show the equality \(\mathcal{G}(F(s), F(t)) \leq |s - t| \cdot \mathcal{G}((X_0, d_0), (X_1, d_1))\) for all \(s, t \in (0, 1)\). Take \(K \in \mathcal{C}(R, d_s, R, d_t)\) with \((o, o) \in K\). We define a correspondence \(U \in \mathcal{R}(Z, Z)\) by
\[ U = \{ ((x, \omega), (x', \omega)) \mid (x, x') \in K \} \cup \{ ((o, y), (o, y)) \mid y \in Y \}. \]
The rest of the proof can be presented in the same way as the proof of Proposition 5.3. Remark that to obtain the inequality

$$|d_s(x, y) - d_t(x', y')| \leq 2|s - t|G\mathcal{H}((X_0, d_0), (X_1, d_1)),$$

we need to use the assumption that \((o, o) \in K\). □

Proof of Theorem 1.3. Let \(A\) be a closed subset of \([0, 1]\). Let \((u, v, w) \in \{0, 1, 2\}^3\). Let \((X, d), (Y, e) \in \mathcal{X}(u, v, w)\) with \(G\mathcal{H}((X, d), (Y, e)) > 0\). We only need to consider the case of \((u, v, w) \in \{0, 1\}^3\).

Let \(R \in \mathcal{C}\mathcal{R}_{opt}(X, d, Y, e)\), and for each \(s \in (0, 1)\), put

$$d_s = (1 - s) \cdot d + s \cdot e.$$

Then by Proposition 5.1, the map \(\gamma : [0, 1] \to M\) defined by

$$\gamma(s) = \begin{cases} (X, d) & \text{if } s = 0; \\
(Y, e) & \text{if } s = 1; \\
(R, d_s) & \text{if } s \in (0, 1) \end{cases}$$

is a geodesic from \((X, d)\) to \((Y, e)\).

To construct a branching bunch of geodesics, in the following, we first construct an “expansion” \(\{(W, E_s)\}_{s \in (0, 1)}\) of the geodesics \(\gamma\) such that each \((W, E_s)\) is the product space of \(\gamma(s)\) and a Cantor metric space \((C, h)\) in \(\mathcal{X}(u, v, w)\). Such a choice of \((C, h)\) is needed to prove \(F(s, q) \in \mathcal{X}(u, v, w)\). By the construction, each \((W, E_s)\) is perfect and this property is convenient for our aim (note that each \(\gamma(s)\) is not always perfect). If \((u, v, w) = (1, 1, 1)\), let \((C, h)\) be a Cantor metric space of type \((1, 1, 1)\); otherwise, let \((C, h)\) be a Cantor metric space of totally exotic type \((u, v, w)\) (see Theorem 2.14). We may assume that \(\delta_h(C) \leq 2\). Let \(\zeta_1 : [0, 1] \to [0, \infty)\) be a \(G\mathcal{H}((X, d), (Y, e))\)-Lipschitz map with \(\zeta_1^{-1}(0) = \{0, 1\}\) (see Lemma 2.1). For \(s \in (0, 1)\), put

\[(W, E_s) = (R \times C, d_s \times \infty (\zeta_1(s) \cdot h)).\]

We define a map \(f : [0, 1] \to M\) by

$$f(s) = \begin{cases} (X, d) & \text{if } s = 0; \\
(Y, e) & \text{if } s = 1; \\
(W, E_s) & \text{if } s \in (0, 1) \end{cases}$$

By Proposition 5.3, the map \(f\) is a geodesic from \((X, d)\) to \((Y, e)\). Note that by the latter parts of Propositions 5.1 and 5.3 for all \(s, t \in (0, 1)\) the trivial correspondence \(\Delta_W\) is in \(\mathcal{C}\mathcal{R}_{opt}(f(s), f(t))\).

To construct geodesics continuously parametrized by the Hilbert cube, we use the spaces \((U(P_q), e_{P_q})\) for some \(P\) as “identifiers” corresponding to \(q \in Q\) (see Proposition 4.6). In the following, we define \(P = \{(P_{i,j}, p_{i,j})\}_{i \in \{1, 2, 3\}, j \in \mathbb{N}}\), depending on \((u, v, w)\), to construct
(\(U(P_q), e_{P_q}\)). In the case of \(w = 1\), we define

\[
M = \begin{cases} 
\sup_{s \in [0,1]} \dim_A(W, E_s) & \text{if } (u, v, w) = (1, 1, 1); \\
1 & \text{otherwise.}
\end{cases}
\]

We now show that \(M\) is always finite. In the case of \((u, v, w) = (1, 1, 1)\), put \((K, k_s) = (X \times Y \times C, d_s \times \infty (\zeta_1(s) \cdot h))\). Then, the space \((W, E_s)\) is a subspace of \((K, k_s)\) for all \(s \in [0,1]\). Thus, Lemma 2.4 implies \(\dim_A(W, E_s) \leq \dim_A(K, k_s)\). Since the Assouad dimension is a bi-Lipschitz invariant, and since \(d_s\) is bi-Lipschitz equivalent to \(d \times e\), or \(d \circ e\) in any case, put \((\mathcal{P}, \varepsilon)\) if \(\gamma\) is a geodesic. Then, for all \(s \in [0,1]\), then, we have \(\sup_{s \in [0,1]} \dim_A(W, E_s) \leq \dim_A(X, d) + \dim_A(Y, e) + \dim_A(C, h)\), and hence \(\sup_{s \in [0,1]} \dim_A(W_s, E_s)\) is finite. Namely, the number \(M\) is always finite.

In the case of \(w = 1\), for \(i \in \{1, 2, 3\}\), and \(j \in \mathbb{N}\), let \(c(i, j) \in (0, 1)\) satisfy

\[
\dim_A(2^i, \beta_{c(i,j)}) = M + i + 2^{-j}.
\]

Put \((P_{i,j}, p_{i,j}) = (2^i, 2^{j-1}(\pi/6) \cdot \beta_{c(i,j)})\). Then, by Lemma 4.3

1. each \((P_{i,j}, p_{i,j})\) is \((2^M+6)\)-doubling;
2. each \((P_{i,j}, p_{i,j})\) is an ultrametric space, and hence it is \(\delta\)-uniformly disconnected for all \(\delta \in (0,1)\);
3. each \((P_{i,j}, p_{i,j})\) is \((2^{-1/M})\)-uniformly perfect.

In the case of \(w = 0\), for \(i \in \{1, 2, 3\}\), and \(j \in \mathbb{N}\), let \((P_{i,j}, p_{i,j})\) be the one-point metric space.

In any case, put \(\mathcal{P} = \{(P_{i,j}, p_{i,j})\}_{i \in \{1,2,3\}, j \in \mathbb{N}}\). For each \(q \in \mathbb{Q}\), put \((U_q, e_q) = (U(P_q), e_{P_q})\). By Proposition 4.4 we obtain \((U_q, e_q) \in \mathcal{X}(1, 1, 1)\) if \(w = 1\); otherwise, \((U_q, e_q) \in \mathcal{X}(1, 1, 0)\). Recall that \(U_q\) contains the element represented as \(\infty\). Take \(o \in W\). Let \(z_2 : [0,1] \to [0,\infty)\) be a \(\mathcal{G}(\mathcal{X}, d), (Y, e)\)-Lipschitz map with \(z_2^{-1}(0) = A\) (see Lemma 2.1). For \(s \in [0,1]\) and \(q \in \mathbb{Q}\), we put

\[
Z_q = W \times \{\infty\} \cup \{o\} \times U_q,
\]

and define a metric \(H_{s,q}\) on \(Z_q\) by

\[
H_{s,q} = E_s \times \infty (\zeta_2(s) \cdot e_q).
\]

Note that if \(s \in A \cap (0,1)\), then \((Z_q, H_{s,q})\) is a pseudo-metric space, and there exists an isometry between \((W, E_s)\) and the quotient metric space of \((Z_q, H_{s,q})\) by the relation \(x \sim y\) defined by \(H_{s,q}(x, y) = 0\). The family \(\{\{Z_q, H_{s,q}\}\}_{s,q \in [0,1] \times \mathbb{Q}}\) is considered as an “expansion of a geodesic \(\gamma\) equipped with the identifiers \(\{(U_q, e_q)\}_{q \in \mathbb{Q}}\)”.

We define a map \(F : [0,1] \times \mathbb{Q} \to \mathcal{M}\) by

\[
F(s, q) = \begin{cases} 
f(s) & \text{if } s \in A; \\
(Z_q, H_{s,q}) & \text{if } s \in [0,1] \setminus A.
\end{cases}
\]
We now prove that the map \( F : [0, 1] \times Q \to M \) is an \( A \)-branching bunch of geodesics from \((X, d)\) to \((Y, e)\).

By Proposition 4.5, we first observe that the map \( F \) is continuous.

By the definition of \( F \), the conditions [1] and [2] in Definition 1.1 are satisfied.

Since for each \( q \in Q \) we have \( \delta_{e_q}(U_q) \leq 1 \), and since the trivial correspondence \( \Delta_W \) is in \( C_{reg}(W, E_u, W, E_t) \), by Proposition 5.4 the map \( F_q : [0, 1] \to M \) is a geodesic. Thus the condition [3] in Definition 1.1 is satisfied.

We now prove the condition [4] in Definition 1.1. Before doing that, depending on \((u, v, w)\), we define isometrically invariant operations picking out the space \((U_q, e_q)\) from \( F(s, q) \). Let \((S, m)\) be an arbitrary compact metric space. We denote by \( t \) and \( t \) picking out the space \((U_q, e_q)\) from \( F(s, q) \). Let \((S, m)\) be an arbitrary compact metric space. We denote by \( C_M(S, m) \) the set of all \( x \in S \) for which there exists \( r \in (0, \infty) \) such that for all \( \epsilon \in (0, r) \) we have \( \text{dim}_A(B(x, \epsilon), m) \leq M \). We denote by \( I(S, m) \) the set of all isolated points of \((S, m)\). Let \((u, v, w) \in \{0, 1\} \), we define

\[
B_{(u,v,w)}(S, m) = \begin{cases} 
\text{CL}_S(S \setminus C_M(S, m)) & \text{if } (u, v, w) = (1, 1, 1); \\
\text{CL}_S(I(S, m)) & \text{if } w = 0; \\
\text{CL}_S(S \setminus S_{\partial_1}(S, m)) & \text{if } u = 0 \text{ and } w = 1; \\
\text{CL}_S(S \setminus S_{\partial_2}(S, m)) & \text{if } (u, v, w) = (1, 0, 1),
\end{cases}
\]

where \( \text{CL}_S \) is the closure operator of \( S \). Then \( B_{(u,v,w)}(S, m) \) is an isometric invariant, i.e., if \((S, m)\) and \((S', m')\) are isometric, then so are \( B_{(u,v,w)}(S, m) \) and \( B_{(u,v,w)}(S', m') \). By the definitions of \( M, P, \) and \( Q \), and by Lemma 2.15, we see that for every \((s, q) \in (0, 1) \times A \times Q \) the space \( B_{(s,q)}(F(s,q)) \) is isometric to \((U_q, \zeta_2(s) \cdot e_q)\).

Take \((s, q), (t, r) \in (0, 1) \times A \times Q \). We now prove that if \( F(s, q) \) and \( F(t, r) \) are isometric to each other, then \((s, q) = (t, r)\). Under this assumption, since \( B_{(s,q)}(F(s,q)) \) is invariant under isometric maps, \((U_q, \zeta_2(s) \cdot e_q)\) and \((U_r, \zeta_2(t) \cdot e_r)\) are isometric. Thus, by Proposition 4.6, we obtain \( q = r \). Since \( F_q \) is a geodesic, we observe that \( s = t \). This implies the condition [4] in Definition 1.1.

Subsequently, we prove that for all \((s, q) \in [0, 1) \times Q \) we have \( F(s, q) \in X(u, v, w) \). By Lemma 2.7, we first observe that \( T_{p_1}(W, E_u) = u \) and \( T_{p_2}(W, E_s) = v \). Then we also observe that \( T_{p_1}(W \times U_q, E_t \times \zeta_2(s) e_q)) = u \) and \( T_{p_2}(W \times U_q, E_t \times \zeta_2(s) e_q)) = v \). Since \( F(s, q) \) is a subspace of \((W \times U_q, E_t \times \zeta_2(s) e_q)\), and since \( F(s, q) \) contains \((W, E_s)\), by Proposition 2.12, we obtain \( T_{p_1}(F(s, q)) = u \) and \( T_{p_2}(F(s, q)) = v \). By the definitions of \((U_q, e_q)\) and \( F(s, q) \), and by Lemmas 2.9 and 2.10, we infer that \( T_{p_1}(F(s, q)) = w \). Thus, we obtain \( F(s, q) \in X(u, v, w) \).

Therefore we conclude that Theorem 1.3 holds true.

**Corollary 5.5.** For every \((u, v, w) \in \{0, 1, 2\}^3\), the set \( X(u, v, w) \) is a geodesic space.

Next we prove Theorems 1.4, 1.5, and 1.6.
Lemma 5.6. For a dimensional function $\mathcal{D}$, there exists a perfect compact metric space $(X, d)$ such that $\mathcal{D}(X, d) = \infty$.

Proof. Take a compact metric space $(Y, e)$ with $\mathcal{D}(Y, e) = \infty$, and take a perfect metric space $(P, h)$. Then $(X \times P, e \times_\infty h)$ is a desired one. □

Proof of Theorem 1.4. Let $(C, h) \in \mathcal{CA}$. We take a correspondence $R \in \mathcal{CR}_{opt}(X, d, Y, e)$. For each $s \in (0, 1)$, put $d_s = (1 - s) \cdot d + s \cdot e$. Depending on the doubling property of the metric spaces $(S, d)$, we can construct the map $F : [0, 1] \times Q \to M$ as in the proof of Theorem 1.3 in the case of $(u, v, w) = (1, 1, 1)$ or $(0, 1, 1)$, respectively. Since a perfect totally disconnected compact metric space is in $\mathcal{CA}$, we have $F(s, q) \in \mathcal{CA}$ for all $(s, q) \in [0, 1] \times Q$. This completes the proof. □

Proof of Theorem 5.8. Let $E$ be a metrizable linear space. A convex set of $E$ is homeomorphic to the Hilbert cube $Q$ if and only if it is an infinite-dimensional compact AR.

Corollary 5.9. Define $m : [0, 1] \to [0, 1]$ by $m(s) = \min\{s, 1 - s\}$, and define $K = \{(s, m(s) \cdot q) \mid s \in [0, 1], q \in Q\}$. Then $K$ is homeomorphic to the Hilbert cube $Q$.

Proof. The set $K$ is a compact convex infinite-dimensional subset of the locally convex topological linear space $\mathbb{R}^N$. By the Dugundji extension theorem (see [L5] Theorem 1.13.1), $K$ is an AR. Thus, by Theorem 5.8, the set $K$ is homeomorphic to $Q$. □

Proof of Theorem 1.7. Let $(X, d), (Y, e) \in \mathcal{S}$. Let $F : [0, 1] \times Q \to X(u, v, w)$ be a $\{0, 1\}$-branching bunch of geodesics stated in Theorem
We define a map $G : K \to \mathcal{X}(u, v, w)$ by

$$G(a, b) = \begin{cases} (X, d) & \text{if } a = 0; \\ (Y, e) & \text{if } a = 1; \\ F(a, b/m(a)) & \text{otherwise.} \end{cases}$$

Subsequently, $G$ is continuous and injective. Since $K$ is compact, the map $G$ is a topological embedding. This completes the proof. □

**Proof of Theorem 1.8.** Let $S$ be any one of $\mathcal{X}(u, v, w)$ for some triple $(u, v, w) \in \{0, 1, 2\}^3$ or $\mathcal{J}(\mathcal{D})$ for some dimensional function $\mathcal{D}$ or $\mathcal{C}_A$. Let $O$ be a non-empty open subset of $M$. By Lemma 2.20, the set $S$ is dense in $M$. Thus, $S \cap O$ is non-empty. Take $(X, d) \in S \cap O$ and take $r \in (0, \infty)$ such that the closed ball centered at $(X, d)$ with radius $r$ is contained in $O$. Take $(Y, e) \in S$ with $\mathcal{GH}((X, d), (Y, e)) = r$. The existence of such $(Y, e)$ is guaranteed by the existence of geodesics in $S$. Applying Theorems 1.3 and 1.7 to $(X, d)$, $(Y, e)$ and $A = \{0, 1\}$, we obtain a $\{0, 1\}$-branching bunch $F : [0, 1] \times Q \to S$ of geodesics from $(X, d)$ to $(Y, e)$. By the choice of $r$, we obtain $F([0, 1] \times Q) \subset S \cap O$. Since $F([0, 1] \times Q)$ contains a homeomorphic copy of $Q$, the set $S \cap O$ is infinite-dimensional. □

### 6. Table of Symbols

| Symbol | Description |
|--------|-------------|
| $M$    | The set of all isometry classes of compact metric spaces. |
| $\mathcal{GH}$ | The Gromov–Hausdorff distance. |
| $Q$    | The Hilbert cube. |
| $B(x, r)$ | The closed ball centered at $x$ with radius $r$. |
| $\mathbb{N}$ | The set of all positive integers. |
| Met$(X)$ | The set of all topologically compatible metrics of a metrizable space $X$. |
| $\mathcal{P}_k$ | $\mathcal{P}_1$: the doubling property, $\mathcal{P}_2$: the uniform disconnectedness, $\mathcal{P}_3$: the uniform perfectness. |
| $\mathcal{DO}$ | The set of all doubling compact metric spaces in $M$. |
| $\mathcal{UD}$ | The set of all uniformly disconnected compact spaces in $M$. |
| $\mathcal{UP}$ | The set of all uniformly perfect compact spaces in $M$. |
| $T_\mathcal{P}(X, d)$ | The truth value of the statement that $(X, d)$ satisfies a property $\mathcal{P}$. |
| $Q_k$ | $Q_1 = \mathcal{DO}$, $Q_2 = \mathcal{UD}$, and $Q_3 = \mathcal{UP}$. |
| Symbol | Description |
|--------|-------------|
| $E_k(u)$ | $E_k(0) = M \setminus Q_k$, $E_k(1) = Q_k$, and $E_k(2) = \mathcal{M}$. |
| $X(u, v, w)$ | $X(u, v, w) = E_1(u) \cap E_2(v) \cap E_3(w)$. |
| $\mathcal{D}$ | The set of all $(X, d)$ in $\mathcal{M}$ with $\mathcal{D}(X, d) = \infty$ for a dimensional function $\mathcal{D}$. |
| $\mathcal{C} \mathcal{A}$ | The set of all compact metric spaces homeomorphic to the (middle-third) Cantor set. |
| $\lor$ | The maximum operator on $\mathbb{R}$. |
| $\land$ | The minimum operator on $\mathbb{R}$. |
| Card($A$) | The cardinality of a set $A$. |
| $\delta_d(A)$ | The diameter of $A$ by a metric $d$. |
| $\alpha_d(A)$ | $\alpha_d(A) = \inf \{d(x, y) \mid x \neq y, x, y \in A\}$. |
| dim$_A(X, d)$ | The Assouad dimension of $(X, d)$. |
| $d \times \infty$ | The product metric defined by $(d \times \infty)(x, y) = d(x, u) \lor e(y, v)$. |
| $X \sqcup Y$ | The direct sum of $X$ and $Y$. |
| $\prod_{i \in I} X_i$ | The direct sum of a family $\{X_i\}_{i \in I}$. |
| $\mathcal{S}_P(X, d)$ | The set of all points in $(X, d)$ of which no neighborhoods satisfy a property $P$. |
| $\mathcal{R}(X, Y)$ | The set of all correspondences of $X$ and $Y$ (see Subsection 2.3). |
| $\mathcal{C} \mathcal{R}(X, d, Y, e)$ | The set of all closed correspondences in $X \times Y$ in $\mathcal{R}(X, d, Y, e)$. |
| $\mathcal{C} \mathcal{R}_{opt}(X, d, Y, e)$ | The set of all optimal $R \in \mathcal{C} \mathcal{R}(X, d, Y, e)$. |
| dis($R$) | $\text{dis}(R) = \sup_{(x, y), (u, v) \in R} |d(x, u) - e(y, v)|$. |
| $\Delta_X$ | $\Delta_X = \{(x, x) \mid x \in X\} \in \mathcal{R}(X, X)$. |
| $\hat{n}$ | $\hat{n} = \{1, \ldots, n\}$. |
| $(T(X, d), d_X)$ | The telescope space for $X$ defined in Section 4. |
| $P_q$ | $P_q = (P, q)$, where $q \in \mathbb{Q}$ and $P = \{(P_{i,j}, p_{i,j})\}_{i \in \{1,2,3\}, j \in \mathbb{N}}$ be a sequence of compact metric spaces indexed by $\{1, 2, 3\} \times \mathbb{N}$ (see Definition 4.1). |
| $(U(P_q), e_{P_q})$ | The telescope space constructed in Definition 4.1. |
| $2^\omega$ | The set of all maps from $\mathbb{Z}_{\geq 0}$ into $\{0, 1\}$. |
| $v(x, y)$ | $v(x, y) = \min\{n \in \mathbb{Z}_{\geq 0} \mid x(n) \neq y(n)\}$ if $x \neq y$; otherwise $v(x, y) = \infty$. |
| $g_c : \mathbb{Z}_{\geq 0} \cup \{\infty\} \to [0, \infty)$ | $g_c(n) = c^n$ if $n \in \mathbb{Z}_{\geq 0}$ and $g_c(\infty) = 0$ for $c \in (0, 1)$. |
| $\beta_c$ | The metric $\beta_c$ on $2^\omega$ defined by $\beta_c(x, y) = g_c(v(x, y))$ for $c \in (0, 1)$. |
\[ A(v, X, d) \]

The set of all \( x \in X \) for which there exists \( r \in (0, \infty) \) such that for every \( \epsilon \in (0, r) \) we have \( \dim A(B(x, \epsilon), d) \). The auxiliary invariant under isometries for the proof of Proposition 4.6.

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