Estimating a Structural Distribution Function by Grouping

Bert van Es and Stamatis Kolios

Korteweg-de Vries Institute for Mathematics
University of Amsterdam
The Netherlands

February 1, 2022

Abstract

By the method of Poissonization we confirm some existing results concerning consistent estimation of the structural distribution function in the situation of a large number of rare events. Inconsistency of the so called natural estimator is proved. The method of grouping in cells of equal size is investigated and its consistency derived. A bound on the mean squared error is derived.

AMS classification: 62G05; secondary 62G20
Keywords: multinomial distribution, large number of rare events, Poissonization, mean squared error, linguistics.

1 Introduction and results

The concept of a structural distribution function originates from linguistics. Let \( M \) denote the size of the vocabulary of an author and consider a text of this author that contains \( n \) words. Every choice of a word in the text from the vocabulary can be seen as the realization of a multinomial random vector. The whole text consists of a sequence of such choices \( X^{(i)} = (X_{1,M}^{(i)}, \ldots, X_{M,M}^{(i)}), \quad i = 1, 2, \ldots, n \), which are assumed to be independent. So each \( X^{(i)} \) is Multinomial\((1, p_{1,M}, p_{2,M}, \ldots, p_{M,M})\) distributed, where \( p_{1,M}, p_{2,M}, \ldots, p_{M,M} \) denote the cell probabilities. In linguistics the vector of those word probabilities is viewed as a characteristic of the author. More specifically one is interested in estimating the so called structural distribution function.
Definition 1.1 The Structural Distribution Function $F_M$ is the empirical distribution function based on $M$ times the cell probabilities. Hence

$$F_M(x) = \frac{1}{M} \sum_{j=1}^{M} I_{[M p_{j,M} \leq x]}.$$  (1.1)

We will investigate the estimation problem for the case of a large number of rare events, i.e. we assume

$$n, M \to \infty \quad \text{and} \quad n/M \to \lambda, \text{ where } 0 < \lambda < \infty. \quad (1.2)$$

So in the linguistic context both sizes of the text and the vocabulary are large, and the text size is proportional to the size of the vocabulary. Assuming that, under (1.2), $F_M$ converges weakly to a distribution function $F$ we want to estimate $F$ at a fixed positive point $x$. The problem of estimation of $p_{1,M}, p_{2,M}, \ldots, p_{M,M}$ is thus asymptotically replaced by estimation of $F$.

The estimators we consider are based on the cell counts of the $n$ observations of $X$, i.e.

$$\nu_{j,M} = \sum_{i=1}^{n} X_{j,M}^{(i)}, \ j = 1, 2, \ldots, M. \quad (1.3)$$

Since the cell probabilities can be estimated by the cell frequencies an obvious estimator of $F$ seems to be the natural estimator $\hat{F}_M$ which is defined as the empirical distribution function based on $M$ times the cell frequencies $\nu_{j,M}/n$. Hence

$$\hat{F}_M(x) = \frac{1}{M} \sum_{j=1}^{M} I_{[\frac{M}{n} \nu_{j,M} \leq x]}. \quad (1.4)$$

The method of Poissonization is based on the following idea. Instead of considering the cell counts based on $n$ observations of $X$, we introduce the cell counts $\rho_{j,M}$ based on $N$ observations of $X$, where $N$ is a Poisson($n$) distributed random variable independent of the $X$’s. So

$$\rho_{j,M} = \sum_{i=1}^{N} X_{j,M}^{(i)}, \ j = 1, 2, \ldots, M. \quad (1.5)$$

The advantage of Poissonization is that the $\rho_{j,M}$ are independent Poisson($np_{j,M}$) random variables, while $(\nu_{1,M}, \ldots, \nu_{M,M})$ are Multinomial($n, p_{1,M}, p_{2,M}, \ldots, p_{M,M}$) distributed.

The natural estimator based on $\rho_{1,M}, \rho_{2,M}, \ldots, \rho_{M,M}$, denoted by $\tilde{F}_M(x)$, is then equal to

$$\tilde{F}_M(x) = \frac{1}{M} \sum_{j=1}^{M} I_{[\frac{M}{N} \rho_{j,M} \leq x]}.$$

(1.6)
Let $Z_M$ denote a random variable with distribution function $F_M$ and $Z$ a random variable with distribution function $F$. The following theorem establishes the inconsistency of the natural estimator. This has already been proved by Klaassen and Mnatsakanov (2000) without using Poissonization.

**Theorem 1.1** Let (1.2) hold and let $F_M \xrightarrow{w} F$ (or equivalently $Z_M \xrightarrow{w} Z$). Then
\[
\hat{F}_M(x) \xrightarrow{P} F_{Y/\lambda}(x),
\] (1.7)
where the conditional distribution of $Y$ given $Z = z$ is Poisson($\lambda z$), for positive $z$, and of $Y$ given $Z = 0$ is degenerate at zero.

Inconsistency of $\hat{F}_M$ also follows from the fact that it is a distribution function with jumps only at multiples of $M/n$. Hence, in the limit, it can only have mass at multiples of $1/\lambda$. However, knowledge of the limit is useful since based on the exact limit given by Theorem [1.1], Klaassen and Mnatsakanov (2000) have constructed a consistent estimator of $F$ by Laplace inversion.

The inconsistency of the natural estimator seems to occur since $n$ increases too slowly with regard to the number of cells $M$. We can reduce that number by replacing the $M$ cells by $m$ groups and assuming $n/m \to \infty$. We define the **grouped cell probabilities** $q_{j,M}$ by
\[
q_{j,M} = \sum_{i=k_{j-1}+1}^{k_j} p_{i,M}, \ j = 1, 2, \ldots, m
\] (1.8)
and the **grouped cell frequencies** $\bar{\nu}_{j,M}$ as
\[
\bar{\nu}_{j,M} = \sum_{i=k_{j-1}+1}^{k_j} \nu_{i,M}, \ j = 1, 2, \ldots, m
\] (1.9)
where the **cell limits** $k_j, \ j = 0, 1, \ldots, m$, are integers such that $0 = k_0 < k_1 < \ldots < k_m = M$. We restrict ourselves to the situation where the $m$ groups are of equal size $k$, so $M = km$ and $k_j = jk$.

Let $F_m$ denote the empirical distribution function based on $m$ times the grouped cell probabilities. So
\[
F_m(x) = \frac{1}{m} \sum_{j=1}^{m} I_{[mq_j,M \leq x]}.
\] (1.10)
Define the estimator $\hat{F}_m(x)$ based on the grouped cell counts by
\[
\hat{F}_m(x) = \frac{1}{m} \sum_{j=1}^{m} I_{[\frac{m}{n}\bar{\nu}_j,M \leq x]}.
\] (1.11)
The Poissonized version $\tilde{F}_m(x)$, based on the grouped Poisson counts

$$\tilde{\rho}_{j,M} = \sum_{i=k_{j-1}+1}^{k_j} \bar{\nu}_{i,M}, \ j = 1, \ldots, m, \quad (1.12)$$

is obtained by replacing the $\bar{\nu}$’s by $\tilde{\rho}$’s in (1.11). Note that $\tilde{\rho}_{j,M}$ has a Poisson($nq_{j,M}$) distribution and that the $\tilde{\rho}$’s are independent. Note also that for $m = M$ and hence $k = 1$, a situation excluded by condition (1.13) below, we regain the natural estimator $\hat{F}_M(x)$.

The following theorem establishes the weak consistency of the estimator based on the grouped counts.

**Theorem 1.2** Let (1.2) hold. Assume further that

$$\frac{n}{m \log m} \to \infty. \quad (1.13)$$

If $F_m \Rightarrow F$ and the distributions induced by the $F_m$ are concentrated on a fixed bounded set, then

$$\hat{F}_m(x) \Rightarrow F(x), \quad (1.14)$$

for every continuity point $x$ of $F$.

Let us sketch the proofs of the two theorems. The proofs consist of three parts. We have to derive the limit of the expectation of the Poissonized estimator, we have to show that the variance of the Poissonized estimator vanishes asymptotically, and we have to prove that Poissonization is allowed, i.e. that the difference between the original estimator and its Poissonized version asymptotically vanishes in probability. Here we only derive the limits of the expectation. The complete proofs are given in Section 2.

We can rewrite the expectation of $\tilde{F}_m(x)$ as follows

$$E \tilde{F}_m(x) = E \frac{1}{m} \sum_{j=1}^{m} I[m \tilde{\rho}_{j,M} \leq x] = \frac{1}{m} \sum_{j=1}^{m} P\left(\frac{m}{n} \tilde{\rho}_{j,M} \leq x\right). \quad (1.15)$$

Recall that for $m = M$ this gives the expectation of the Poissonized natural estimator $\tilde{F}_M(x)$.

Now consider a two stage procedure. We draw a value $z$ from the sequence of points $mq_{1,M}, mq_{2,M}, \ldots, mq_{m,M}$ with equal probability $1/m$. The corresponding random variable is denoted by $Z_m$. Note that it has distribution function $F_m$. Given $Z_m = z$ the random variable $Y_m$ is equal to $m/n$ times a Poisson($\frac{m}{n}z$) distributed random variable. Then we have by conditioning on $Z_m$

$$E \tilde{F}_m(x) = \frac{1}{m} \sum_{j=1}^{m} P\left(\frac{m}{n} \tilde{\rho}_{j,M} \leq x\right) = E\left(P(Y_m \leq x|Z_m)\right) = P(Y_m \leq x). \quad (1.16)$$
Hence \( E \hat{F}_m(x) \) equals the distribution function of \( Y_m \) at \( x \). We derive weak convergence of this distribution function by the continuity theorem for characteristic functions. The characteristic function of \( Y_m \), denoted by \( \phi_m \), is given by

\[
\phi_m(t) = E(e^{itY_m}) = E(E(e^{itY_m | Z_m})) = \int e^{\frac{m}{m+i}(e^{it} - 1)} dF_m(z), \tag{1.17}
\]

since the characteristic function of a Poisson(\( \mu \)) distribution is equal to \( e^{\mu(e^{it} - 1)} \). In the case of the natural estimator we have \( m = M \) and hence by (1.12)

\[
\phi_m(t) \to \int e^{\lambda z(e^{it/\lambda} - 1)} dF(z), \tag{1.18}
\]

the characteristic function of the limit distribution function in (1.7). For the estimator based on the grouped counts we have \( m/n \to 0 \) by (1.13) and hence

\[
\phi_m(t) \to \int e^{itz} dF(z), \tag{1.19}
\]

the characteristic function of \( F \). By the continuity theorem (1.18) and (1.19) imply the conclusions of the two theorems.

**Remark 1.1** In Theorem 1.2 we can replace the condition \( F_m \xrightarrow{w} F \) by \( F_M \xrightarrow{w} F \) if the \( p_{j,M} \)'s, \( j = 1, 2, \ldots, M \) are ordered. A proof can be found in Section 3.

**Remark 1.2** The condition of the weak convergence of \( F_m \) to \( F \) is implied by a stronger condition in Klaassen and Mnatsakanov (2000). Define \( f_M \) by

\[
f_M(t) = \sum_{j=1}^{M} Mp_{j,M} I_{\left[\frac{j-1}{M}, \frac{j}{M}\right]}(t), \quad 0 < t \leq 1. \tag{1.20}
\]

Note that the structural distribution function \( F_M \) is the distribution function of \( f_M(U) \), where \( U \) is uniformly distributed on the interval \((0, 1] \). Assume that \( f_M \) converges uniformly on \((0, 1] \) to a density function \( f \), i.e.

\[
\sup_{0 < t \leq 1} |f_M(t) - f(t)| \to 0. \tag{1.21}
\]

Klaassen and Mnatsakanov proved, without requiring equal cell sizes, that this condition implies weak consistency. Moreover, the condition (1.13) is slightly stronger then the corresponding one required by Klaassen and Mnatsakanov.
Let us consider the rate of convergence and the choice of the number of groups \( m \). Define the Mean Squared Error (MSE) of \( \hat{F}_m(x) \) as

\[
\text{MSE}(\hat{F}_m(x)) = E(\hat{F}_m(x) - F(x))^2. \tag{1.22}
\]

A standard computation shows that the mean squared error is equal to the sum of the squared bias and the variance.

Consider the situation where the \( p_{j,M} \)'s are generated by a distribution function \( G \), via

\[
p_{j,M} = G\left(\frac{j}{M}\right) - G\left(\frac{j-1}{M}\right), \quad j = 1, \ldots, M. \tag{1.23}
\]

Then we also have \( q_{j,M} = G\left(\frac{j}{m}\right) - G\left(\frac{j-1}{m}\right), j = 1, \ldots, m \). If \( G \) has a density \( g \) that is continuous and bounded then we have

\[
mg_{j,M} = m\left(G\left(\frac{j}{m}\right) - G\left(\frac{j-1}{m}\right)\right) = mg(\xi_{j,M}) \frac{1}{m} = g(\xi_{j,M}), \tag{1.24}
\]

where \( \xi_{j,M} \) is a point in the interval \( ((j-1)/m, j/m) \]. Assuming that \( g \) is also uniformly continuous on \((0, 1]\) this implies \( f_m(t) \to g(t) \), uniformly on \([0, 1]\). So in this situation the limit density \( f \) in (1.21) is equal to \( g \).

Let us first present some simulation results. Figures 1, 2 and 3 show estimates of \( F \) based on a simulated sample where \( G(x) = 2x - x^2 \) and \( g(x) = 2(1 - x) \) for \( 0 \leq x \leq 1 \). We have chosen \( M = 1000 \) and \( n = 3000 \). So \( \lambda \) equals three. Since it equals the distribution function of \( g(U) \), with \( U \) uniformly distributed on \([0, 1]\), the limit structural function \( F \) is given by

\[
F(x) = \begin{cases} 
0 & \text{if } x < 0, \\
\frac{1}{2}x & \text{if } 0 \leq x \leq 2, \\
1 & \text{if } x > 2. 
\end{cases} \tag{1.25}
\]

Figure 1 shows the result of the natural estimator. Next we show two figures of estimates based on grouping. In Figure 2 we have \( k = 25 \) and thus \( m = 40 \) while for Figure 3 we have chosen \( k = 100 \) and thus \( m = 10 \). Figure 1 shows that the natural estimator is inconsistent, having jumps only at multiples of \( 1/3 \). Figures 2 and 3 show that by grouping we achieve consistency, and that the choice of \( m \) is important. All in all the figures suggest that \( k \) too small or too large is not wise and that there might be an optimal cell size.

The next theorem gives some insight in the choice of \( m \). It gives bounds on the mean squared error of \( \hat{F}_m(x) \). These bounds depend on \( m \).
Figure 1: \( \hat{F}_M(x) \) for \( M = 1000, n = 3000 \) (\( m = M = 1000, k = 1 \))

**Theorem 1.3** Let (1.2) hold. Assume that the cell probabilities \( p_{j,M}, j = 1, \ldots, M \) are generated by a distribution function \( G \) as in (1.23) and that \( G \) has a density that is uniformly continuous on \((0, 1]\). Assume further that \( G \) has a bounded second derivative \( g \) that bounded away from zero on \((0, 1]\), and that, for some \( 0 < \alpha < 1/6 \),

\[
\frac{n}{m(\log m)^{1/2\alpha}} \to \infty. \tag{1.26}
\]

Then we have, if \( m \gg n^{1/3} \),

\[
\text{MSE}(\hat{F}_m(x)) \leq \frac{9}{4\pi^2}(24\tau)^{4/3} \left( \frac{m}{n} \right)^{2/3} + \frac{1}{4m} + o \left( \left( \frac{m}{n} \right)^{2/3} \right) + o \left( \frac{1}{m} \right), \tag{1.27}
\]

and if \( m \ll n^{1/3} \)

\[
\text{MSE}(\hat{F}_m(x)) \leq \frac{1}{4m} + o \left( \frac{1}{m} \right). \tag{1.28}
\]

The key idea of the proof is to exploit the fact that we have derived the convergence of \( E \hat{F}_m(x) \), which is in fact equal to the distribution function of \( Y_m \), to \( F(x) \) from the convergence of its characteristic function \( \phi_m \), cf. (1.17), to the characteristic function of \( F \). By Esseen’s smoothing lemma we get a bound on the distance of distribution functions from the distance of their characteristic functions. By expanding (1.17) we obtain a rate of convergence for the bias \( E \hat{F}_m(x) - F(x) \) of the Poissonized estimator. The bound on the variance of the Poissonized estimator is the same as in the proof
Figure 2: $\hat{F}_m(x)$ based on grouping with $m = 40, k = 25, M = 1000, n = 3000$

of Theorem 1.2. The remainder of the proof consists of showing that Poissonization is allowed in this context too.

Straightforward calculations show that the right hand side of (1.27) is asymptotically minimized by $m_n$ if

$$ m_n \sim \left( \frac{\pi^6}{6^3(24\tau)^4} \right)^{1/5} n^{2/5}. $$

(1.29)

This gives a mean squared error

$$ \text{MSE}(\hat{F}_m(x)) \leq \frac{33}{4} \left( \frac{(24\tau)^2}{6\pi^3} \right)^{2/5} n^{-2/5} + o(n^{-2/5}). $$

(1.30)

The bound (1.28) of Theorem 1.3 gets smaller as $m$ increases. However, the order of $m$ is bounded by $n^{1/3}$. Hence, for $m \ll n^{1/3}$ we get

$$ \text{MSE}(\hat{F}_m(x)) \gg \frac{1}{4} n^{-1/3} + o(\frac{1}{4} n^{-1/3}). $$

(1.31)

Note that the bound in (1.30) is smaller than the one given in (1.31). Therefore (1.30) gives the minimal upper bound.

Remark 1.3 The assumption that there exists a known ordering of words in a vocabulary, necessary for grouping, for which (1.23) holds is not realistic. Consistent estimators as the one in Klaassen and Mnatsakanov (2000), which do not require such an ordering, seem to have a logarithmic rate of convergence, as opposed to the algebraic rate in Theorem 1.2.
2 Proofs

2.1 Proof of Theorem 1.1

The limit of $E\hat{F}_M(x)$ is derived in the previous section. It remains to check (1.18) reformulated in the following lemma.

**Lemma 2.1** Under the conditions of Theorem 1.1, we have

$$\phi_M(t) = \int e^{\frac{n}{M}z(e^{it\frac{M}{n}}-1)}dF_M(z) \rightarrow \int e^{\lambda z(e^{it\lambda}-1)}dF(z).$$  \hspace{1cm} (2.1)

The proof is given in Section 3.

A bound on the variance of $\hat{F}_M(x)$ is given by

$$\text{Var}(\hat{F}_M(x)) = \text{Var} \left( \frac{1}{M} \sum_{j=1}^{M} I_{[\frac{M}{n}\rho_j,M \leq x]} \right) = \frac{1}{M^2} \sum_{j=1}^{M} \text{Var} \left( I_{[\frac{M}{n}\rho_j,M \leq x]} \right) \leq \frac{1}{M^2} \sum_{j=1}^{M} \frac{1}{4} = \frac{1}{4M} \rightarrow 0.$$

All this implies that $\hat{F}_M$ is weakly consistent for $F_{Y/\lambda}$.

Finally we show that Poissonization is allowed. We have

$$|\hat{F}_M(x) - \hat{F}_M(x)| = \left| \frac{1}{M} \sum_{j=1}^{M} I_{[\frac{M}{n}\nu_j,M \leq x]} - \frac{1}{M} \sum_{j=1}^{M} I_{[\frac{M}{n}\rho_j,M \leq x]} \right|$$
\[
\leq \frac{1}{M} |N - n| = \frac{n}{M} \left| \frac{N}{n} - 1 \right| \to 0,
\]
almost surely and in probability. This implies that \( \hat{F}_M \) is weakly consistent for \( F_{Y/\lambda} \) too as stated in the theorem.

### 2.2 Proof of Theorem 1.2

The limit of \( \mathbb{E} \tilde{F}_m(x) \) is derived in the previous section. It remains to check (1.19) reformulated in the following lemma.

**Lemma 2.2** Under the conditions of Theorem 1.2 we have

\[
\phi_m(t) = \int e^{\frac{m}{n} z (e^{it\bar{\rho}} - 1)} dF_m(z) \to \int e^{itz} dF(z). \tag{2.2}
\]

The proof can be found in Section 3.

Here we bound the variance of \( \tilde{F}_m(x) \) as follows

\[
\text{Var} \tilde{F}_m(x) = \text{Var} \frac{1}{m} \sum_{j=1}^{m} I_{\left[ \frac{m}{n} \bar{\rho}_{j,M} \leq x \right]} = \frac{1}{m^2} \sum_{j=1}^{m} \text{Var} I_{\left[ \frac{m}{n} \bar{\rho}_{j,M} \leq x \right]} \tag{2.3}
\]

\[
\leq \frac{1}{m^2} \sum_{j=1}^{m} \frac{1}{4} = \frac{1}{4m} \to 0.
\]

This implies that \( \tilde{F}_m(x) \) is a weakly consistent for \( F(x) \).

In order to transfer the weak consistency result to the original estimator we must show that we may indeed Poissonize, i.e. we must show that \( \hat{F}_m(x) - \tilde{F}_m(x) \) vanishes in probability.

We need the Bernstein inequality for Poisson random variables. If \( X \) has a Poisson distribution then

\[
P \left( \frac{|X - \mathbb{E} X|}{(\mathbb{E} X)^{1/2}} \geq \epsilon \right) \leq 2 \exp \left( - \frac{\epsilon^2}{2 + \epsilon (\mathbb{E} X)^{-1/2}} \right). \tag{2.4}
\]

cf. Lemma 8.3.4 in Reiss (1993). It also follows from Inequality 1 on page 485 of Shorack and Wellner (1986).

Write \( z_{j,n} = m q_{j,M} \). Note that, since the distributions induced by the \( F_m \) are concentrated on a bounded set, we have \( \max_{1 \leq j \leq m} z_{j,n} \leq c \) for some constant \( c > 0 \). Hence, for all \( \delta > 0 \), we have

\[
\sum_{j=1}^{m} P \left( \left| \frac{m}{n} \bar{\rho}_{j,M} - z_{j,n} \right| \geq \delta \right) =
\]
\[ \sum_{j=1}^{m} P\left( \frac{|\bar{\nu}_{j,M} - \frac{n}{m} z_{j,n}|}{nq_{j,M}} \geq \left( \frac{n}{m} \right)^{1/2} \frac{1}{\sqrt{z_{j,n}}} \delta \right) = \sum_{j=1}^{m} 2 \exp \left( -\delta^2 \frac{n}{m} \frac{1}{2 + \delta \frac{1}{z_{j,M}}} \right) \leq 2m \exp \left( -\delta^2 \frac{n}{m} \frac{1}{2c + \delta} \right) = 2 \exp \left( \log m \left( -\frac{n}{m} \log m \frac{\delta^2}{2c + \delta} + 1 \right) \right) \to 0, \]

by (1.13).

By max \[1 \leq j \leq m q_{j,M} \to 0\] we have \[\text{Var} (\bar{\nu}_{j,M}) = nq_{j,M} (1 - q_{j,M}) \sim nq_{j,M} = \text{Var} (\bar{\rho}_{j,M}).\]

By the Bernstein inequality for binomial random variables, cf. Shorack and Wellner (1986), p 440, it now follows that for \(\delta > 0\)

\[ \sum_{j=1}^{m} P\left( \left| \frac{m}{n} \bar{\nu}_{j,M} - z_{j,n} \right| \geq \delta \right) \to 0. \]

This implies that with probability approaching one we have

\[ \left| \frac{m}{n} \bar{\nu}_{j,M} - z_{j,n} \right| < \delta \quad \text{and} \quad \left| \frac{m}{n} \bar{\rho}_{j,M} - z_{j,n} \right| < \delta, \quad j = 1, \ldots, m. \]

Consequently, (2.7) implies

\[ \frac{1}{m} \sum_{j=1}^{m} \left( \mathbb{I}_{[z_{j,n} \leq x - \delta]} - \mathbb{I}_{[z_{j,n} \leq x + \delta]} \right) \leq \hat{F}_m(x) - \tilde{F}_m(x) \leq \frac{1}{m} \sum_{j=1}^{m} \left( \mathbb{I}_{[z_{j,n} \leq x + \delta]} - \mathbb{I}_{[z_{j,n} \leq x - \delta]} \right). \]

By the weak convergence of \(F_m\) to \(F\), if \(x - \delta\) and \(x + \delta\) are continuity points of \(F\), the left and right hand side converge to \(F(x - \delta) - F(x + \delta)\) and \(F(x + \delta) - F(x - \delta)\) respectively. Now, for given \(\epsilon > 0\), choose \(\delta\) such that \(F(x + \delta) - F(x - \delta)\) is smaller than \(\epsilon\) and we have shown

\[ P(|\hat{F}_m(x) - \tilde{F}_m(x)| \geq \epsilon) \to 0. \]

Hence \(\hat{F}_m(x) - \tilde{F}_m(x)\) vanishes in probability, proving that Poissonization is allowed.

### 2.3 Proof of Theorem 1.3

First we consider the mean squared error of the Poissonized estimator. By a standard calculation we have

\[ \text{MSE}(\tilde{F}_m(x)) = (\mathbb{E} \tilde{F}_m(x) - F(x))^2 + \text{Var}(\tilde{F}_m(x)) \] (2.10)
A bound on the variance is already given by (2.3). It is harder to obtain a bound on the bias. We shall use the convergence of the characteristic function of \( Y_m \) to the characteristic function of \( Y \) in the proof of Theorem 1.1 and Esseen’s smoothing lemma, see Feller (1966), Section XIV 3, Lemma 2 on page 538.

**Lemma 2.3 (Esseen’s smoothing lemma)** Let \( F \) be a probability distribution function with vanishing expectation and characteristic function \( \varphi \). Suppose \( F - G \) vanishes at \( \pm \infty \) and that \( G \) has a derivative \( g \) such that \( |g| \leq \tau \). Finally, suppose that \( g \) has a continuously differentiable Fourier transform \( \gamma \) such that \( \gamma(0) = 1 \) and \( \gamma'(0) = 0 \). Then, for all \( x \) and \( T > 0 \)

\[
|F(x) - G(x)| \leq \frac{1}{\pi} \int_{-T}^{T} \left| \frac{\varphi(t) - \gamma(t)}{t} \right| dt + \frac{24\tau}{\pi T}. \tag{2.11}
\]

Now apply this lemma with \( F \) equal to the distribution function of \( Y_m \) and \( G \) equal to the limit structural distribution function \( F \). Note that both distribution functions have expectation one and that the induced distributions are concentrated on \([0, \infty)\). Then

\[
|E \tilde{F}_m(x) - F(x)| \leq \frac{1}{\pi} \int_{-T}^{T} \left| \frac{E e^{itY_m} - E e^{itZ}}{t} \right| dt + \frac{24\tau}{\pi T}. \tag{2.12}
\]

Let us first consider the integrand. Write

\[
|E e^{itY_m} - E e^{itZ}| = | \int e^{\frac{m}{m^*}z(e^{it\frac{m}{m^*}} - 1)} dF_m(z) - \int e^{itz} dF(z) |
\]

\[
\leq | \int e^{\frac{m}{m^*}z(e^{it\frac{m}{m^*}} - 1)} dF_m(z) - \int e^{itz} dF_m(z) | + | \int e^{itz} dF_m(z) - \int e^{itz} dF(z) | \tag{2.13}
\]

For \( n \) large we have

\[
e^{\frac{m}{m^*}z(e^{it\frac{m}{m^*}} - 1)} = e^{\frac{m}{m^*}z(1-it\frac{m}{m^*} - \frac{1}{2}t^2(\frac{m}{m^*})^2 + R_n(t) - 1)} = e^{itz - \frac{1}{2}t^2\frac{m}{m^*} + R_n(t)\frac{m}{m^*}}, \tag{2.15}
\]

where \( R_n(t) = e^{it\frac{m}{m^*}} - 1 - it\frac{m}{m^*} + \frac{1}{2}t^2\frac{m^2}{m^*} \). Note that

\[
|R_n(t)| \leq \frac{1}{6} t^3 \frac{m^3}{n^3} \tag{2.16}
\]

and that for \( w \in \mathbb{C} \), and \( |w| \) small enough, we have

\[
|e^{w} - 1| \leq 4|w|. \tag{2.17}
\]

Hence

\[
|e^{\frac{m}{m^*}zR_n(t)} - 1| \leq \frac{2}{3} |z||t^3\frac{m^2}{n^2}. \tag{2.18}
\]
So we can bound the term $(2.13)$ as follows:

\[
| \int e^{\frac{nt}{m}z} (e^{it\frac{n}{m}z} - 1) dF_m(z) - \int e^{itz} dF_m(z) | \\
\leq | \int (e^{it\frac{n}{m}z} - e^{itz}) dF_m(z) | + | \int (e^{\frac{nt}{m}z} (e^{it\frac{n}{m}z} - 1) - e^{it\frac{n}{m}z} - e^{itz}) dF_m(z) | \\
\leq \int | e^{\frac{nt}{m}z} - 1 | dF_m(z) + \int | e^{\frac{nt}{m}z} - 1 | dF_m(z) \\
\leq \frac{1}{2} \int t^2 dz dF_m(z) + \frac{2 m^2}{3 n^2} |t|^3 \int dF_m(z). \\
= \frac{1}{2} \int t^2 + \frac{2}{3 n^2} m^2 |t|^3.
\]

For $(2.18)$ to hold we have tacitly assumed that $(n/m)zR_n(t)$ vanishes for $-T \leq t \leq T$. By $(2.10)$ and the fact that $Z_m$ is almost surely bounded by the same constant for all $m$, it suffices to check that $(m^2/n^2)j^3 \to 0$ for $-T \leq t \leq T$. Further on in the proof $T$ will depend on $n$. The condition is satisfied for our two choices of $T_n$ in $(2.27)$ and $(2.30)$.

For the first term in $(2.12)$ we get

\[
\frac{1}{\pi} \int_{-T}^{T} \frac{1}{|t|} \left| \int e^{\frac{nt}{m}z} (e^{it\frac{n}{m}z} - 1) dF_m(z) - \int e^{itz} dF_m(z) \right| dt \\
\leq \frac{1}{2 \pi} \frac{m}{n} \int_{-T}^{T} \frac{1}{|t|} dt + \frac{2}{3 \pi} \frac{m^2}{n^2} \int_{-T}^{T} t^2 dt \\
= \frac{1}{2 \pi} \frac{m}{n} T^2 + \frac{4}{9 \pi} \frac{m^2}{n^2} T^3.
\]

Let the function $f_m$ be defined by

\[
f_m(t) = \sum_{j=1}^{m} mq_{j,m} I_{[\frac{(j-1)}{m}, \frac{j}{m})}, \quad 0 < t \leq 1.
\]

Then $F_m$ is the distribution function of $f_m(U)$ where $U$ is uniformly distributed on $(0, 1]$. Since $f_m$ converges uniformly to $g$ the limit distribution function $F$ is the distribution function of $g(U)$. Hence

\[
\int e^{itz} dF_m(z) - \int e^{itz} dF(z) = \int_{0}^{1} (e^{itf_m(u)} - e^{itg(u)}) du.
\]

Integrated over the intervals $((j-1)/m, j/m]$, the constant $mq_{j,M}$ yields the same value as $g$ integrated over these intervals. So we can write

\[
e^{itf_m(u)} - e^{itg(u)} = e^{itf_m(u)} \left( 1 - e^{it(g(u) - f_m(u))} \right) \\
= e^{itf_m(u)} (it(f_m(u) - g(u)) + R_n(t, u)),
\]
where
\[ |R_n(t, \mathbf{u})| \leq \frac{1}{2} t^2 (g(\mathbf{u}) - f_m(\mathbf{u}))^2 \] (2.23)

And hence, if \( g \) has a bounded derivative on \((0, 1]\),
\[
\left| \int (e^{itf_m(\mathbf{u})} - e^{itg(\mathbf{u})}) \, d\mathbf{u} \right| = \left| \int e^{itf_m(\mathbf{u})} R_n(t, \mathbf{u}) \, d\mathbf{u} \right| 
\leq \frac{1}{2} t^2 \int (f_m(\mathbf{u}) - g(\mathbf{u}))^2 \, d\mathbf{u} \leq \frac{c}{2 m^2},
\]
where \( c \) is a positive constant. This implies
\[
\frac{1}{\pi} \int_{-T}^{T} \frac{1}{|t|} \left| \int e^{itz} dF_m(z) - \int e^{itz} dF(z) \right| dt \leq \frac{1}{\pi} \int_{-T}^{T} \frac{1}{|t|} \frac{c}{2 m^2} dt = \frac{c T^2}{2 \pi m^2}.
\] (2.24)

Hence, for all \( x \) and \( T > 0 \)
\[
|E \tilde{F}_m(x) - F(x)| \leq \frac{4}{9\pi} \frac{m^2}{n^2} T^3 + \frac{1}{2\pi} \frac{m}{n} T^2 + \frac{c}{2\pi} \frac{1}{m^2} T^2 + \frac{24\tau}{\pi T}. \] (2.25)

First assume that \( m \gg n^{1/3} \). Then equation (2.25) becomes asymptotically
\[
|E \tilde{F}_m(x) - F(x)| \leq \frac{1}{2\pi} \frac{m}{n} T^2 + \frac{24\tau}{\pi T}. \] (2.26)

The value \( T_n \) that minimizes the right hand side of (2.26) is given by
\[
T_n = (24\tau)^{1/3} \left( \frac{n}{m} \right)^{1/3}. \] (2.27)

Hence the bias can be asymptotically bounded by
\[
|E \tilde{F}_m(x) - F(x)| \leq \frac{3}{2\pi} (24\tau)^{2/3} \left( \frac{m}{n} \right)^{1/3} + o \left( \left( \frac{m}{n} \right)^{1/3} \right) \] (2.28)

and the mean squared error by
\[
\text{MSE}(\tilde{F}_m(x)) \leq \frac{9}{4\pi^2} (24\tau)^{4/3} \left( \frac{m}{n} \right)^{2/3} + \frac{1}{4m} + o \left( \left( \frac{m}{n} \right)^{2/3} \right) + o \left( \frac{1}{m} \right). \] (2.29)

If \( m \ll n^{1/3} \), by minimizing the third and fourth term in (2.23), we get, by choosing
\[
T_n = c^{-1/3} (24\tau)^{1/3} m^{2/3}, \] (2.30)
that asymptotically
\[
\text{MSE}(\tilde{F}_m(x)) \leq \frac{9}{4\pi^2} c^{2/3} (24\tau)^{4/3} m^{-4/3} + \frac{1}{4m} + o(m^{-4/3}) + o \left( \frac{1}{m} \right) = \frac{1}{4m} + o \left( \frac{1}{m} \right). \] (2.31)
We have now derived the asymptotic bounds on the mean squared error of the Poissonized estimator. We will show that Poissonization is allowed. By the triangle inequality we have
\[ \text{MSE}(\hat{F}_m(x))^{1/2} \leq \text{MSE}(\tilde{F}_m(x))^{1/2} + (\mathbb{E}(\hat{F}_m(x) - \tilde{F}_m(x))^2)^{1/2}. \] (2.32)

The second term on the right hand side can be dealt with using the following lemma. Its proof is given in Section 3.

**Lemma 2.4** Under the conditions of Theorem 1.3 and we have for any \( 0 < \alpha < \frac{1}{6} \)
\[ \mathbb{E}(\tilde{F}_m(x) - \hat{F}_m(x))^2 = O\left((\frac{m}{n})^{1-2\alpha}\right) + O\left(\frac{1}{m^2}\right). \] (2.33)

By this order bound and (2.29) and (2.31) it follows that \( (\mathbb{E}(\hat{F}_m(x) - \tilde{F}_m(x))^2)^{1/2} \) is asymptotically negligible compared to \( \text{MSE}(\tilde{F}_m(x))^{1/2} \). Hence Poissonization is allowed.

### 3 Technical proofs

#### 3.1 Proof of Lemma 2.1

Recall that \( \frac{(n/M)Y_M}{Z_M}, \) given \( Z_M = z \), has a Poisson\( (\frac{n}{M}z) \) distribution. We have \( Z_M \xrightarrow{w} Z \), so \( F_M(w) \rightarrow F(w) \) at all continuity points \( w \) of \( F \). Let \( \psi_M \) denote the characteristic function of \( \frac{(n/M)Y_M}{Z_M} \). Then
\[ \psi_M(t) = \mathbb{E}\left(e^{it(n/M)Y_M}\right) = \mathbb{E}\left(\mathbb{E}\left(e^{it(n/M)Y_M}|Z_M\right)\right) = \int_{-\infty}^{\infty} e^{\frac{\lambda}{M}z}(e^{it}-1) dF_M(z). \] (3.1)

Consider \( t \) fixed. For \( z \in [0, w] \) we have
\[ |e^{\frac{\lambda}{M}z}(e^{it}-1) - e^{\lambda z}(e^{it}-1)| = \left|e^{\frac{\lambda}{M}z}(e^{it}-1)\right| \left|1 - e^{\lambda z}(e^{it}-1)\right| \leq |1 - e^{\lambda (\frac{z}{M})z}(e^{it}-1)| \rightarrow 0 \]
or equivalently
\[ e^{\frac{\lambda}{M}z}(e^{it}-1) \rightarrow e^{\lambda z}(e^{it}-1). \]

This also holds for \( z \) replaced by \( z_n \), for every sequence \( \{z_n\} \) with values in \( [0, w] \), showing that the convergence is uniform in \( z \). Hence for \( \epsilon > 0 \) and \( n \) large enough
\[ |e^{\frac{\lambda}{M}z}(e^{it}-1) - e^{\lambda z}(e^{it}-1)| \leq \epsilon/2 \] (3.2)
for all \( z \in [0, w] \).
Let \( w \) be a continuity point of \( F \). Note that, because \( F \) and \( F_M \) vanish on the negative half line, the point \(-1\) is also a continuity point. Then, according to the Helly-Bray theorem and because characteristic functions are continuous, we can conclude that
\[
\int_{-1}^{w} e^{\lambda z (e^{it} - 1)} dF_M(z) \to \int_{-1}^{w} e^{\lambda z (e^{it} - 1)} dF(z). \tag{3.3}
\]
So for \( n \) large enough
\[
\left| \int_{-1}^{w} e^{\lambda z (e^{it} - 1)} dF_M(z) - \int_{-1}^{w} e^{\lambda z (e^{it} - 1)} dF(z) \right| \leq \epsilon/2. \tag{3.4}
\]
Because of (3.2) and (3.4) we now have
\[
\left| \int_{-1}^{w} e^{\lambda z (e^{it} - 1)} dF_M(z) - \int_{-1}^{w} e^{\lambda z (e^{it} - 1)} dF(z) \right| \leq \epsilon. \tag{3.5}
\]
Next choose the continuity point \( w \) such that \( 1 - F_M(w) < \epsilon/2 \). Since \( F_M(w) \to F(w) \) we also have \( 0 \leq 1 - F_M(w) < \epsilon/2 \), for \( n \) large enough. This implies
\[
\left| \int_{-1}^{w} e^{\lambda z (e^{it} - 1)} dF_M(z) - \int_{-1}^{w} e^{\lambda z (e^{it} - 1)} dF(z) \right| \leq \int_{-1}^{w} |e^{\lambda z (e^{it} - 1)}| dF_M(z) + \int_{-1}^{w} |e^{\lambda z (e^{it} - 1)}| dF(z) \tag{3.6}
\]
\[
= 1 - F_M(w) + 1 - F(w) < \epsilon.
\]
The inequalities (3.3) and (3.6) show that
\[
\left| \int_{-\infty}^{w} e^{\lambda z (e^{it} - 1)} dF_M(z) - \int_{-\infty}^{w} e^{\lambda z (e^{it} - 1)} dF(z) \right| < 2\epsilon \tag{3.7}
\]
for \( n \) large enough. Hence
\[
\int_{-\infty}^{w} e^{\lambda z (e^{it} - 1)} dF_M(z) \to \int_{-\infty}^{w} e^{\lambda z (e^{it} - 1)} dF(z). \tag{3.8}
\]
Since convergence of characteristic functions is uniform on bounded intervals we also have (2.1).
3.2 Proof of Lemma 2.2

The proof is similar to the proof in the previous section. Note that
\[
\lim_{n \to \infty} n \frac{mz}{m} \left( e^{it \frac{m}{n}} - 1 \right) = itz,
\] (3.9)
uniformly for \( z \in [-1, w] \). Let \( \epsilon > 0 \) and \( w \) be a continuity point of \( F \). By the Helly-Bray theorem and (3.3), we have, for \( n \) large enough,
\[
\left| \int_{-1}^{w} e^{\frac{mz}{m}(e^{it \frac{m}{n}} - 1)} dF_m(z) - \int_{-1}^{w} e^{itz} dF(z) \right|
\leq \left| \int_{-1}^{w} e^{\frac{mz}{m}(e^{it \frac{m}{n}} - 1)} dF_m(z) - \int_{-1}^{w} e^{itz} dF_m(z) \right| + \left| \int_{-1}^{w} e^{itz} dF_m(z) - \int_{-1}^{w} e^{itz} dF(z) \right| < \epsilon.
\] (3.10)

Now choose \( w \) such that \( 1 - F(w) < \epsilon/2 \). Then, for \( n \) large enough,
\[
\left| \int_{-1}^{w} e^{\frac{mz}{m}(e^{it \frac{m}{n}} - 1)} dF_m(z) - \int_{-1}^{w} e^{itz} dF(z) \right|
\leq \int_{-1}^{w} |e^{\frac{mz}{m}(e^{it \frac{m}{n}} - 1)}| dF_m(z) + \int_{w}^{\infty} |e^{itz}| dF(z)
\leq 1 - F_m(w) + 1 - F(w) < \epsilon.
\] (3.11)

As in the previous section the inequalities (3.11) and (3.10) prove the lemma.

3.3 Proof of Remark 1.1

We assume that the set of the \( p_{j,M} \)'s is ordered. So \( p_{1,M} \leq p_{2,M} \leq \cdots \leq p_{M,M} \). Let \( x \) be a continuity point of \( F \). We want to show that
\[
|F_M(x) - F_m(x)| = \left| \frac{1}{M} \sum_{j=1}^{M} I_{[M p_{j,M} \leq x]} - \frac{1}{m} \sum_{i=1}^{m} I_{[m q_{i,M} \leq x]} \right|
\] (3.12)
vanishes since this implies that \( F_m(x) \to F(x) \) follows from \( F_M(x) \to F(x) \).

Assume that in the \( \beta \) first groups of the \( m, \beta = 0, \ldots, m \), we have \( M p_{j,M} \leq x \) and that in the \((\beta+1)\)th group for the first \( \alpha, \alpha = 1, \ldots, k \), of the \( p_{j,M} \)'s we have \( M p_{j,M} \leq x \) and that for the others \( M p_{j,M} > x \). Then in total exactly \( k \beta + \alpha \) of the \( p_{j,M} \)'s satisfy \( M p_{j,M} \leq x \). Note that both \( \beta \) and \( \alpha \) depend on \( M \) and \( x \).
Let us focus on the \( i \)-th group, where \( i = 0, \ldots, \beta \). Then we have \( M_{p_{j,M}} \leq x \) for all \( j = k_{i-1} + 1, \ldots, k_{i-1} + k = k_i \) and hence for all \( i = 1, \ldots, \beta \)

\[
M \sum_{j=k_{i-1}+1}^{k_i} p_{j,M} \leq kx. \tag{3.13}
\]

This implies \( m_{q_{i,M}} \leq x \).

We can now bound the difference (3.12). We get

\[
\left| \frac{1}{M} \sum_{j=1}^{M} I_{[M_{p_{j,M}} \leq x]} - \frac{1}{m} \sum_{i=1}^{m} I_{[m_{q_{i,M}} \leq x]} \right|
\]

\[
= \left| \frac{k\beta + \alpha}{M} - \frac{\beta}{m} - \frac{1}{m} I_{[m_{q_{\beta+1,M}} \leq x]} \right| = \left| \frac{\alpha}{M} - \frac{c}{m} \right| \to 0,
\]

since \( \alpha/M \leq k/M \to 0 \).

### 3.4 Proof of Lemma 2.4

Let \( \delta_n = (m/n)^{1/2-\alpha} \) and let \( A_n \) denote the event

\[
\left| \frac{m}{n} \bar{v}_{j,M} - z_{j,n} \right| < \delta_n \quad \text{and} \quad \left| \frac{m}{n} \bar{\rho}_{j,M} - z_{j,n} \right| < \delta_n, \quad j = 1, \ldots, m. \tag{3.14}
\]

Then, as in (2.5) we have, for \( n \) large enough

\[
P(A_n^c) \leq \sum_{j=1}^{n} \left\{ P\left( \left| \frac{m}{n} \bar{v}_{j,M} - z_{j,n} \right| \geq \delta_n \right) + P\left( \left| \frac{m}{n} \bar{\rho}_{j,M} - z_{j,n} \right| \geq \delta_n \right) \right\}
\]

\[
\leq 4m \exp\left( -\delta_n^2 \frac{n}{m} \frac{1}{2c + \delta_n} \right)
\]

\[
= 4 \exp\left( -\left( \frac{n}{m} \right)^{2\alpha} \frac{1}{c} - \log m \right)
\]

\[
\leq 4 \exp\left( -\log m \left( \frac{1}{c m^{2\alpha}} \log m - 1 \right) \right)
\]

\[
\leq \frac{1}{m^2}.
\]

Using (2.8) we write

\[
E \left( \bar{F}_m(x) - \hat{F}_m(x) \right)^2
\]

\[
= E \left( \bar{F}_m(x) - \hat{F}_m(x) \right)^2 I_{A_n} + E \left( \bar{F}_m(x) - \hat{F}_m(x) \right)^2 I_{A_n^c}
\]

\[
\leq \left( \frac{1}{m} \sum_{j=1}^{m} \left( I_{[z_{j,n} \leq x + \delta_n]} - I_{[z_{j,n} \leq x - \delta_n]} \right) \right)^2 + P(A_n^c)
\]

\[
= (F_m(x + \delta_n) - F_m(x - \delta_n))^2 + P(A_n^c).
\]
Now recall that $F_m$ is the empirical distribution function based on the values $mq_{j,M}, j = 1, \ldots, m$. If $g'(x) > 0$ then each of these values are order $1/m$ apart. Hence there are order $\delta_n/(1/m) = m\delta_n$ values in the interval $(x - \delta_n, x + \delta_n]$, each contributing $1/m$ to the probability. So

$$F_m(x + \delta_n) - F_m(x - \delta_n) = O(\delta_n).$$  
(3.15)

Hence

$$E(\tilde{F}_m(x) - \hat{F}_m(x))^2 = O(\delta_n^2) + O\left(\frac{1}{m^2}\right),$$  
(3.16)

which completes the proof of the lemma.

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