Phase space trajectories in quantum mechanics

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Abstract

An adapted representation of quantum mechanics sheds new light on the relationship between quantum states and classical states. In this approach the space of quantum states splits into a product of the state space of classical mechanics and a Hilbert space, and expectation values of observables decompose into their classical value plus a quantum correction. The splitting is preserved under time evolution of the Schrödinger equation under certain assumptions, and the time evolution of the classical part of a quantum state is governed by Hamilton’s equation. The new representation is obtained from the usual Hilbert space representation of quantum mechanics by introducing a gauge degree of freedom in a time-dependent unitary transformation, followed by a non-conventional gauge fixing condition.
1 Introduction

A challenge in the interpretation of quantum mechanics consists in the difficulty to explain the emergence of classical physics in the so-called classical limit. The latter is often associated with a hypothetical limit $\hbar \to 0$ of Planck's constant $\hbar$. However, unlike for instance special relativity, where taking the limit $c \to \infty$ of the velocity of light in the equations of motion more or less straightforwardly leads to the Newtonian formulation of classical mechanics, the limit $\hbar \to 0$ is usually understood in a more symbolic way, since the mathematical model underlying quantum mechanics differs fundamentally from the model(s) of classical mechanics. In particular, the space of states for a point-particle in $\mathbb{R}^n$ looks very different in quantum mechanics ($\hbar \neq 0$) than in classical mechanics ($\hbar = 0$); pure quantum states are modeled by projective rays in an infinite-dimensional Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n)$, whereas classical pure states are points in phase space $\mathbb{R}^{2n}$, interpreted as position and momentum vectors of the particle.

This apparent discrepancy has found a mathematical explanation in terms of $C^*$-algebras for observables, whose associated states are defined as normalized bounded linear functionals on the algebra. The latter is commutative in the classical case and acquires a non-commutative deformation in quantum mechanics. By means of the Gelfand representation, pure states on a commutative $C^*$-algebras can be identified with points in a classical phase space, and by means of the Gelfand–Naimark–Segal representation, pure states on a non-commutative $C^*$-algebra can be identified with (projective equivalence classes of) unit vectors in a Hilbert space. This formal relationship does not yet explain the emergence of classical physics within a quantum world, however.

Several approaches to the classical limit have been considered in the literature, among them the WKB approximation, the correspondence principle or large $N$ limit, the Wigner density, coherent states, stationary path integrals, deformation quantization and decoherence. While each of these
offers some relevant insights into the classical limit, they all fall short of providing a simple and comprehensive derivation. See [1] for a concise review or [2] for a more extensive review with historical and philosophical explanations. A recurrent theme is that the results apply to a limited set of special states (in particular for WKB, large $N$ and coherent states), often based on some ad-hoc constructions, or states are ignored completely, like in deformation quantization.

In this note it will be shown that the traditional representation of quantum mechanics can be adapted by means of a time-dependent unitary transformation in such a way that it allows for a natural splitting of states and observables into a classical and a quantum part, and that the classical Hamiltonian equations of motions arise in this setting as a gauge fixing condition for the Schrödinger equation. The classical part of a quantum state or observable is exactly the state or observable of the corresponding classical mechanics model in the Hamiltonian formalism, i.e. the (pure) state consists of a trajectory in phase space and the observable is modeled by a function on phase space.

The second part of the paper, starting with section 5, deals with the classical limit. In a first step we will define a framework in which the limit $\hbar \to 0$ can be formulated. This involves the promotion of $\hbar$ to an operator on state space, which acts upon families of states parametrized by $\hbar > 0$ as a multiplication operator. Based on ideas from deformation quantization we will introduce a filtration on the extended operator algebra, which will play an important role in the derivation of the classical limit. In section 6 we will clarify the conditions under which the previously found splitting into classical and quantum contributions of states and observables is preserved under time evolution. When the conditions are satisfied, the time evolution of the classical part of a quantum state decouples completely from the purely quantum part.

The results presented in the paper shed new light on the relation between the Schrödinger equation in quantum mechanics and Hamilton’s equations in classical mechanics, with the latter appearing as a gauge condition on the quantum state in our formalism. In addition, they seem to elucidate the classical limit $\hbar \to 0$, allowing us to formulate the conditions under which the transition from quantum to classical takes place. Nevertheless, the interpretation of the results is not straightforward, since they require us to consider families of states $\{\psi_\hbar\}$ for $\hbar > 0$, whereas our physical reality is bound to a fixed value $\hbar_0$. We will discuss some aspects of this in section 7 without coming to a conclusive result. In particular, it remains open whether the “naive limit” approach $\hbar \to 0$ pursued in this note can in itself explain the emergence of classical physics from quantum mechanics. One possible approach to apply the results presented here would be to consider the large $N$-limit, where $\hbar$ can be identified with $1/N$ and hence it may be possible to use the results for a rigorous derivation of the classical limit. These applications are left for future work, however.

Appendix A gives a geometric interpretation of the representation of quantum mechanics provided here. It elucidates the connection to Fedosov’s approach to deformation quantization, as well as to the geometric quantization framework. Based on this interpretation, Appendix B discusses the generalization to curved phase spaces.

Throughout the paper we have to deal with unbounded operators on a Hilbert space. In order not to distract from the core topic by technical details we will pretend that these operators were defined on the whole Hilbert space and mapped it into itself. Furthermore, we will ignore convergence questions of infinite series. Hence, the mathematical presentation is not rigorous. A popular approach to avoid the issues with unbounded operators is to only consider their image under the exponential map. This strategy is not directly applicable here, however, since the quantum filtration on the
algebra of observables introduced in section 5, a core concept of this paper, is not preserved under the exponential map.

2 Textbook quantum mechanics

This paragraph serves to introduce our notation. We start with quantum mechanics on $\mathbb{R}^n$ in its textbook formulation, working in units where $\hbar$ is dimensionless ($\hbar \in \mathbb{R}_{>0}$). Let $q^1, \ldots, q^n, p_1, \ldots, p_n$ be coordinates on the classical phase space $\mathbb{R}^{2n}$. We use the symbol $y^\alpha$ ($\alpha = 1, \ldots, 2n$) to collectively refer to the $q^j, p_k$ coordinates, i.e. $y^j = q^j$ and $y^{n+j} = p_j$ for $j = 1, \ldots, n$. Furthermore, in the context of the Hilbert space $\mathcal{H} \simeq L^2(\mathbb{R}^n)$, we use a second set of coordinates $x^1, \ldots, x^n$ on $\mathbb{R}^n$. On $\mathcal{H}$ we have an action of the Weyl algebra $W$ generated by $\hat{q}^j$ and $\hat{p}_k$, defined as follows (where $j, k = 1, \ldots, n$ and $\psi \in \mathcal{H}$):

$$\hat{q}^j \psi(x) = x^j \psi(x), \quad \hat{p}_k \psi(x) = \frac{\hbar}{i} \frac{\partial}{\partial x^k} \psi(x).$$

(1)

We will denote these generators collectively by $\hat{y}^\alpha$, ($\alpha = 1, \ldots, 2n$), i.e. $\hat{y}^j = \hat{q}^j$ and $\hat{y}^{n+j} = \hat{p}_j$ for $j = 1, \ldots, n$. Then the canonical commutation relations read

$$[\hat{y}^\alpha, \hat{y}^\beta] = i\hbar \omega^{\alpha\beta},$$

(2)

where $\omega$ is the symplectic form, concretely $\omega^{\alpha\beta}_{\alpha,\beta=1,\ldots,2n} = \begin{pmatrix} 0_{n \times n} & \mathbb{1}_{n \times n} \\ -\mathbb{1}_{n \times n} & 0_{n \times n} \end{pmatrix}$. To each smooth function $f(y)$ on phase space $M = \mathbb{R}^{2n}$ we associate an operator $\hat{f}$ on $\mathcal{H}$ by means of its Taylor expansion (summation over $\alpha_1, \ldots, \alpha_k$ from 1 to 2$n$ is understood):

$$\hat{f} = \sum_{k=0}^{\infty} \frac{1}{k!} (\partial_{\alpha_1} \cdots \partial_{\alpha_k} f)(0) \hat{y}^{\alpha_1} \cdots \hat{y}^{\alpha_k}$$

(3)

$$= f(0) + (\partial_{\alpha} f)(0) \hat{y}^\alpha + \frac{1}{2} (\partial_{\alpha} \partial_{\beta} f)(0) \hat{y}^\alpha \hat{y}^\beta + \cdots,$$

Here 0 denotes the origin in $\mathbb{R}^{2n}$. This is the quantum operator in Weyl-ordering, or symmetric ordering, for $f$. The expectation value of $\hat{f}$ in the state $\psi$ is given by the $L^2$-inner product $\langle \hat{f} \rangle_\psi := \langle \psi | \hat{f} \psi \rangle$, and the time evolution of a quantum state $\psi$ is governed by the Schrödinger equation

$$i\hbar \partial_t \psi(t, x) = \hat{H} \psi(t, x),$$

(4)

where $H \in C^\infty(M)$ is the Hamiltonian function of the system.

Alternatively to the presented dynamics, it is possible to assign the dynamic behaviour of the system entirely to the observable and consider the state as time-independent. For this purpose, we define the time-dependent observable $\hat{f}_H(t) = e^{-\frac{it}{\hbar} \hat{H}} \hat{f} e^{\frac{it}{\hbar} \hat{H}}$, then the Schrödinger equation implies

$$i\hbar \partial_t \hat{f}_H(t) = [\hat{H}, \hat{f}_H(t)],$$

(5)

and the expectation value of the observable $f$ at time $t$ is

$$\langle \hat{f} \rangle_\psi(t) = \langle \psi(t) | \hat{f} \psi(t) \rangle = \langle \psi(0) | \hat{f}_H(t) \psi(0) \rangle$$

(6)
3 The trajectory gauge

For an arbitrary point \( y = (q,p) \) in phase space we define a unitary operator on \( \mathcal{H} \) (summation over \( j = 1, \ldots, n \), respectively \( \alpha, \beta = 1, \ldots, 2n \), is understood), the so-called Weyl operator:

\[
U_y = U_{(q,p)} = \exp \left[ \frac{i}{\hbar} (q^j \hat{p}_j - p_j \hat{q}^j) \right] = \exp \left[ - \frac{i}{\hbar} \omega_{\alpha \beta} y^\alpha \hat{y}^\beta \right].
\]

(7)

For an observable \( f \in C^\infty(M) \) we define the transformed operator \( \tilde{f}y \):

\[
\tilde{f}_y = U_y \tilde{f} U_y^{-1} = \sum_{k=0}^{\infty} \frac{1}{k!} (\partial_{\alpha_1} \cdots \partial_{\alpha_k} f)(y) \hat{y}^{\alpha_1} \cdots \hat{y}^{\alpha_k}
\]

(8)

Symbolically, we can write this as \( \tilde{f}_y = f(y + \hat{y}) \). In particular, for the coordinate functions \( y^\alpha \) we have \( \hat{y}^\alpha = q^\alpha + \hat{q}^\alpha \). In the next step, we take this ansatz further and allow the unitary transformation to be time dependent. Let \( I \subset \mathbb{R} \) be a (time) interval, and \( c : I \to M \) be a differentiable trajectory in phase space. For \( t \in I \) define the operator \( U(t) \) as

\[
U(t) := U_{c(t)} = \exp \left[ - \frac{i}{\hbar} \omega_{\alpha \beta} c^\alpha(t) \hat{y}^\beta \right].
\]

(9)

as well as \( \tilde{\psi}(t) = U(t) \psi(t) \) for \( \psi \in C^\infty(I, \mathcal{H}) \), and \( \tilde{f}(t) = U(t) \tilde{f} U(t)^{-1} \) for an observable \( f \in C^\infty(M) \). Due to the additional time dependence in \( \tilde{\psi} \), the Schrödinger equation (4) expressed in terms of \( \tilde{\psi}(t) \) and \( \tilde{H}(t) = U(t) \tilde{H} U(t)^{-1} \) acquires an additional term. It reads:

\[
\begin{align*}
\frac{i}{\hbar} \partial_t \tilde{\psi}(t, x) &= -\frac{i}{\hbar}(\partial_t U(t)) U(t)^{-1} \tilde{\psi}(t, x) + \tilde{H}(t) \tilde{\psi}(t, x) \\
&= \omega_{\alpha \beta} (\partial_t c^\alpha(t))(\hat{y}^\beta + \frac{1}{2} c^\beta(t)) \tilde{\psi}(t, x) + \tilde{H}(t) \tilde{\psi}(t, x),
\end{align*}
\]

(10)

where the result for \( \partial_t U(t) \) can be obtained from the explicit form \( U(t) = \exp \left[ \frac{i}{\hbar} (q^j(t) \hat{p}_j - p_j(t) \hat{q}^j) \right] \) by means of the Baker-Campbell-Hausdorff formula. Due to (8) we can write (10) as

\[
\begin{align*}
\frac{i}{\hbar} \partial_t \tilde{\psi}(t, x) &= \omega_{\alpha \beta} (\partial_t c^\alpha(t))(\hat{y}^\beta + \frac{1}{2} c^\beta(t)) + \sum_{k=0}^{\infty} \frac{1}{k!} \left( \partial_{\alpha_1} \cdots \partial_{\alpha_k} H \right)(c(t)) (\hat{y}^{\alpha_1} \cdots \hat{y}^{\alpha_k}) \tilde{\psi}(t, x) \\
&= \left[ H(c(t)) + \frac{1}{2} \omega_{\alpha \beta} (\partial_t c^\alpha(t)) c^\beta(t) + \left( \partial_{\alpha} H \right)(c(t)) - \omega_{\alpha \beta} (\partial_t c^\beta(t)) \right] \hat{y}^\alpha \\
&\quad + \sum_{k=2}^{\infty} \frac{1}{k!} \left( \partial_{\alpha_1} \cdots \partial_{\alpha_k} H \right)(c(t)) (\hat{y}^{\alpha_1} \cdots \hat{y}^{\alpha_k}) \tilde{\psi}(t, x)
\end{align*}
\]

(11)

This equation looks very similar to the original Schrödinger equation (4), except that the Hamiltonian function is evaluated in the point \( c(t) \in \mathbb{R}^{2n} \) instead of 0, and the zeroth and first order terms in the Taylor expansion of \( H \) are modified.

4 Gauge fixing

Equation (12) is still fully equivalent to our original Schrödinger equation, for all choices of trajectory \( c \). Hence, \( c \) can be considered a gauge parameter. Equation (12) guides us at a particular
choice for this trajectory, however. If we impose on \(c\) the differential equation
\[
\partial_t c^\alpha(t) = \omega^{\alpha\beta} \partial_\beta H(c(t)),
\]
then the first order term in \(\hat{y}^\alpha\) in the new Schrödinger equation \([12]\) vanishes, and the latter simplifies to
\[
\frac{i\hbar}{\hbar} \partial_t \tilde{\psi}(t, x) = \left[ H(c(t)) - \frac{1}{2} (\partial_\alpha H(c(t))) c^\alpha(t) + \sum_{k=2}^{\infty} \frac{1}{k!} (\partial_{\alpha_1} \ldots \partial_{\alpha_k} H)(c(t)) \hat{y}^{\alpha_1} \ldots \hat{y}^{\alpha_k} \right] \tilde{\psi}(t, x) \tag{14}
\]
Note that condition \([13]\) is nothing but Hamilton’s equation of motion. If we write \(c(t) = (q(t), p(t))\) and \(\dot{q} = \partial_t q(t)\), etc., then it becomes
\[
\dot{q}^j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q^j} \tag{15}
\]
This appearance of Hamilton’s equation(s) as a gauge fixing condition in quantum mechanics is quite astonishing, and we will see below how to interpret it.

The expectation value of an observable \(f\) becomes
\[
\langle \hat{f} \rangle_{\tilde{\psi}(t)} = \langle \hat{f} \rangle_{\tilde{\psi}(t)} = f(c(t)) + (\partial_\alpha f)(c(t)) \langle \hat{y}^\alpha \rangle_{\tilde{\psi}(t)} + \ldots, \tag{16}
\]
where the first term on the right-hand side is exactly the classical expectation value of \(f\) along the trajectory \(c\). By imposing appropriate initial conditions on \(c\) we should be able to choose it in such a way that it represents the corresponding classical state of our quantum system. In this case, all the higher order terms in \([16]\) should vanish in the classical limit. We will try to verify this observation in the remaining sections of the paper.

5 The quantum filtration

In order to study the limiting behaviour for \(\hbar \to 0\) of the Schrödinger equation \([14]\) we will extend the algebra of observables and the state space in a way that allows us to treat \(\hbar\) as a variable. Let us focus on the observables first. Consider the complex algebra \(\mathcal{W}_\hbar\) generated by purely formal symbols
\[
1, \ h^{1/2}, \ \hat{y}^\alpha, \ h^{-1/2} \hat{y}^\alpha \quad (\alpha = 1, \ldots, 2n) \tag{17}
\]
on which we impose that 1 acts as identity, \(h^{1/2}\) commutes with everything, and the equivalence relations
\[
h^{1/2} \cdot (h^{-1/2} \hat{y}^\alpha) \sim \hat{y}^\alpha \tag{18}
\]
\[
\hat{y}^\alpha \cdot (h^{-1/2} \hat{y}^\beta) \sim (h^{-1/2} \hat{y}^\alpha) \cdot \hat{y}^\beta \tag{19}
\]
\[
[h^{-1/2} \hat{y}^\alpha, h^{-1/2} \hat{y}^\beta] \sim i\hbar^{\alpha\beta} 1 \tag{20}
\]
hold. Note that this also implies the canonical commutator \([\hat{y}^\alpha, \hat{y}^\beta] = i\hbar \omega^{\alpha\beta}\), if we write \(h\) for \((h^{1/2})^2\). We can define the \(h\)-degree on this algebra by assigning degree \(\frac{1}{2}\) to both \(h^{1/2}\) and \(\hat{y}^\alpha\), and degree 0 to 1 and \(h^{-1/2} \hat{y}^\alpha\). Furthermore, let us introduce a filtration by defining the subspace \(W_d \subset \mathcal{W}_\hbar\) to be generated by all monomials in the generators \([17]\) of \(h\)-degree at least \(d/2\), for \(d \in \mathbb{N}\).
Note that the equivalence relations are compatible with the $\hbar$-degree and hence the filtration. We have $W_h = W_0 \supset W_1 \supset W_2 \supset \ldots$, and the relation

$$W_d \cdot W_e \subseteq W_{e+d}$$

holds true for all $d, e \in \mathbb{N}$. Now consider a time evolution of the form $i\hbar \partial_t \hat{f}(t) = [\hat{A}, \hat{f}(t)]$ on this algebra, for some fixed operator $\hat{A} \in W_h$ and a time-dependent $\hat{f}(t)$ (i.e. equation (5)). We might be tempted to write this as

$$\partial_t \hat{f}(t) = -\frac{i}{\hbar} \hat{A}, \hat{f}(t)].$$

However, the operator $-\frac{i}{\hbar} \hat{A}, \cdot ]$ is not well-defined on $W_h$ since the latter does not contain $\hbar^{-1}$. This would be incompatible with our filtration. In the special case that

$$\hat{A} = \sum_{k=0}^{\infty} \frac{1}{k!} A_{k; \alpha_1 \ldots \alpha_k} \hat{g}^{\alpha_1} \ldots \hat{g}^{\alpha_k}$$

(with $A_{k; \alpha_1 \ldots \alpha_k}$ totally symmetric in $\alpha_1, \ldots, \alpha_k$, and possibly dependent on $h^{1/2}$) does not contain a linear term:

$$A_{1; \alpha} = 0, \quad \text{for all } \alpha,$$

we can nevertheless make sense of (22), since then

$$-\frac{i}{\hbar} [\hat{A}, \cdot ] := -\sum_{k=2}^{\infty} \frac{i}{k!} A_{k; \alpha_1 \ldots \alpha_k} \left( (h^{-1/2} \hat{g}^{\alpha_1}) (h^{-1/2} \hat{g}^{\alpha_2}) \hat{g}^{\alpha_3} \ldots \hat{g}^{\alpha_k}, \cdot \right]$$

Furthermore, this operator respects the filtration of the operator algebra. Note that for the original Hamiltonian operator $\hat{H} = \sum_k \frac{1}{\hbar} \partial_{\alpha_1} \ldots \partial_{\alpha_k} H(0) \hat{g}^{\alpha_1} \ldots \hat{g}^{\alpha_k}$ of equation (4), condition (24) is violated (unless $H$ is constant), whereas for the modified Hamiltonian of equation (14) it is satisfied. This gives a first hint why the gauge condition (13) may be useful.

The quantum filtration on the operator algebra has been introduced in the so-called deformation quantization context, and has proved extremely valuable there [3]. We will denote $W_h$ as the extended Weyl algebra.

6 The classical limit

In the previous section we have promoted $\hbar$ to an operator in the quantum algebra. Next, we would like to introduce an $\hbar$-dependency in the the state space as well. So let’s assume for now that we are given a smooth family $\psi_h = \psi_h(t_0)$ at fixed time $t_0$ of states in Hilbert space, parametrized by $\hbar > 0$. It is an element of $C^\infty(\mathbb{R}_{>0}) \otimes \mathcal{H}$. On this space we have an action of the operator algebra $W_h$, defined in the previous section, which is generated by the canonical operators $\hat{g}^{\alpha}$ plus $\hbar$, the latter acting by multiplication. Let $H$ be a Hamiltonian function, $c$ a trajectory satisfying Hamilton’s equation (13) with $c(t_0) = 0$, and $\tilde{\psi}(t) := U(t)\psi$ with $U(t)$ defined in terms of $c$ as in (24) (we drop the $\hbar$ index on $\psi$ to avoid notational overload, but still consider $\psi$ to be parameterized by $\hbar$). These conditions imply that $\tilde{\psi}(t_0) = \psi$. We denote the operator from (14) by $\hat{H}$:

$$\hat{H}(t) := H(c(t)) - \frac{1}{2} (\partial_{\alpha_1} H(c(t))) c^{\alpha_1}(t) + \sum_{k=2}^{\infty} \frac{1}{k!} (\partial_{\alpha_1} \ldots \partial_{\alpha_k} H)(c(t)) \hat{g}^{\alpha_1} \ldots \hat{g}^{\alpha_k}. \quad (26)$$
Then the time evolution of the expectation values of the canonical operators on $\hat{\psi}(t)$ are
\[
\langle \hat{g}^{\alpha_1} \ldots \hat{g}^{\alpha_k} \rangle_{\hat{\psi}(t)} = \langle \psi(t_0) | e^{-\frac{i}{\hbar} \hat{H} t} \hat{g}^{\alpha_1} \ldots \hat{g}^{\alpha_k} e^{\frac{i}{\hbar} \hat{H} t} | \psi(t_0) \rangle
\]
\[
= \langle \psi(t_0) | (e^{-\frac{i}{\hbar} [\hat{H}, \cdot]} \cdot (\hat{g}^{\alpha_1} \ldots \hat{g}^{\alpha_k})) | \psi(t_0) \rangle
\]
according to Baker-Campbell-Hausdorff. Now assume that the initial family $\psi(t_0) = \psi_0(t_0)$ obeys the following regularity condition:
\[
\langle \psi(t_0) | \hat{g}^{\alpha_1} \ldots \hat{g}^{\alpha_k} \psi(t_0) \rangle = \kappa_{\alpha_1, \ldots, \alpha_k} \hbar^{k/2} + O(\hbar^{(k+1)/2}) \quad \forall k \in \mathbb{N}, \ \alpha_1, \ldots, \alpha_k = 1, \ldots, 2n,
\]
where $O(\hbar^{(k+1)/2})$ denotes higher order monomials in $\hbar^{1/2}$. This is equivalent to demanding the map
\[
W_\hbar \rightarrow C^\infty(\mathbb{R}_{>0}), \quad A \mapsto \langle \psi(t_0) | \hat{A} \psi(t_0) \rangle
\]
to be filtration-preserving, where $W_\hbar$ carries its $\hbar$-filtration and the image of the map in $C^\infty(\mathbb{R}_{>0})$ is filtered by monomial degree (monomials in $\hbar^{1/2}$). We already know that the operator $-\frac{i}{\hbar} [\hat{H}, \cdot]$ preserves the filtration (but not the $\hbar$-degree, in general) on the operator algebra, hence this is also true for its exponential, and we can conclude that (28) holds true for all times:
\[
\langle \hat{\psi}(t) | \hat{g}^{\alpha_1} \ldots \hat{g}^{\alpha_k} \hat{\psi}(t) \rangle = \tilde{\kappa}_{\alpha_1, \ldots, \alpha_k} \hbar^{k/2} + O(\hbar^{(k+1)/2}) \quad \forall k \in \mathbb{N}, \ \alpha_1, \ldots, \alpha_k = 1, \ldots, 2n
\]
for some $\hbar$-independent $\tilde{\kappa}_{\alpha_1, \ldots, \alpha_k}(t)$. The expectation value of an observable $f$ in the state $\psi$, governed by the Hamiltonian $H$, is
\[
\langle \hat{f} \rangle_{\psi(t)} = \langle \hat{f}(t) \rangle_{\hat{\psi}(t)} = \frac{\langle f(c(t)) + \partial_\alpha f(c(t)) \langle \hat{\psi}(t) | \hat{g}^\alpha \hat{\psi}(t) \rangle + \frac{1}{2} \partial_\alpha \partial_\beta f(c(t)) \langle \hat{\psi}(t) | \hat{g}^\alpha \hat{g}^\beta \hat{\psi}(t) \rangle + \ldots \rangle_{O(h^0)}}{R(H)}
\]
which implies that in the classical limit $\hbar \rightarrow 0$ the quantum mechanical expectation value $\langle \hat{f} \rangle_{\psi(t)}$ becomes equal to the classical expectation value $f(c(t))$:
\[
\lim_{\hbar \rightarrow 0} \langle \hat{f} \rangle_{\psi(t)} = f(c(t)).
\]
The derivation of this result has been made possible by our use of the representation in terms of $c$, $\hat{\psi}$, $\hat{f}$ and $\hat{H}$, making use of the fact that $-\frac{i}{\hbar} [\hat{H}, \cdot]$ preserves the $\hbar$-filtration of the operator algebra. In principle, the textbook representation of quantum mechanics:
\[
\langle \hat{f} \rangle_{\psi(t)} = f(0) + \partial_\alpha f(0) \langle \psi(t) | \hat{g}^\alpha \psi(t) \rangle + \frac{1}{2} \partial_\alpha \partial_\beta f(0) \langle \psi(t) | \hat{g}^\alpha \hat{g}^\beta \psi(t) \rangle + \ldots
\]
must lead to the same result. However, in this case, terms are not sorted by $\hbar$-degrees, so all of the infinite number of terms can contribute to the classical result at order $O(h^0)$, which makes it impossible to calculate the classical limit directly from (33). It was the Hamilton equation (13), as gauge condition on $c$ that enabled us to sort the terms by $\hbar$-degree. Nevertheless, any other choice for $c$ is possible, and for a constant trajectory $c(t) = 0 \forall t$ we get back the textbook formulation of quantum mechanics. Only if $c$ satisfies Hamilton’s equation (13), then the filtration on the operator algebra is preserved under time evolution and we get a clean split of the expectation value into a classical and a quantum contribution, as in (31).

Examples of families of wave functions that satisfy (28) are the eigenfunctions of the $n$-dimensional harmonic oscillator, see the example in section 8 below. Particularly for these oscillator eigenfunctions and a specific class of Hamiltonians, the result (32) has been obtained previously by Hepp [4], who also used the Weyl operators 7 in his derivation.
7 Interpretation

In the derivation of the limiting behaviour of observable expectation values we had to make two assumptions:

1. We are given a family of initial wave functions \( \{ \psi_\hbar(t_0) \}_{\hbar > 0} \).

2. The family of initial wave functions satisfies \( (28) \).

When modeling a physical system, we would normally demand the prescription of a single wave function \( \psi_\hbar(t_0) \) as initial condition for the dynamical system. Here \( \hbar \) denotes the physical value of the variable \( \hbar \). This initial state then has to be obtained through measurements. Since there is no differential equation in \( \hbar \) that would allow us to interpolate the value of \( \psi \) beyond \( \hbar = \hbar_0 \), it appears that for a generic physical system condition 1. is already violated.

Assume, however, that \( \psi_n \) is an eigenstate of an observable operator \( \hat{f} \):

\[
\hat{f} \psi_n = \alpha_n \psi_n
\]

for some \( \alpha_n \in \mathbb{R} \). For a given physical observable \( f \in C^\infty(\mathbb{R}^{2n}) \) equation \( (34) \) actually defines a family \( \{ \psi_{n,\hbar} \}_{\hbar > 0} \) parametrized by \( \hbar \):

\[
\hat{f} \psi_{n,\hbar} = \alpha_n(\hbar) \psi_{n,\hbar},
\]

where now \( \hat{f} \) is considered an element of the extended Weyl algebra \( W_\hbar \). If the initial state \( \psi_{\hbar_0}(t_0) \) is determined by means of an expansion into eigenvectors of \( \hat{f} \), i.e.

\[
\psi_{\hbar_0}(t_0) = \sum_n k_n \psi_{n,\hbar_0},
\]

with \( k_n \in \mathbb{C} \) for all \( n \in \mathbb{N} \) (assuming a discrete spectrum of \( \hat{f} \) for simplicity), then this representation naturally generalizes to arbitrary values of \( \hbar \):

\[
\psi_\hbar(t_0) := \sum_n k_n \psi_{n,\hbar}.
\]

In order to determine the initial state of a physical system some kind of measurement will have to be performed, which according to the standard interpretation involves the projection onto an eigenstate of the corresponding observable. Hence, one could argue, that the initial state after a measurement will always consist of an expansion \( (36) \) with \( k_n = \delta_{nm} \) for some \( m \). This would imply that a natural generalization of the initial state to arbitrary \( \hbar \)-values does indeed exist. In order to apply our result on the classical limit, it then remains to verify condition \( (28) \).

Depending on the context, other interpretations of the conditions may be more helpful. When considering a large \( N \)-limit, for instance, one can often identify \( \hbar \) with \( 1/N \), and hence make sense of the family of initial states \( \{ \psi_\hbar(t_0) \}_{\hbar \in 1/N} \).

8 Example: 1D harmonic oscillator

Let \( q, p \) be coordinates on phase space \( \mathbb{R}^2 \) of a one-dimensional point particle. Consider the Hamiltonian

\[
H(q, p) = \frac{1}{2}(p^2 + q^2).
\]
Its corresponding quantum operator is
\[ \hat{H}_{(q,p)} = \frac{1}{2}(q^2 + p^2) + q\hat{q} + p\hat{p} + \frac{1}{2}(\hat{q}^2 + \hat{p}^2). \] (39)

If we were to set \( q = p = 0 \) we would obtain the common representation as \( \frac{1}{2}(\hat{p}^2 + \hat{q}^2) \) here. The modified operator \( \hat{H} \) is
\[ \hat{H}_{(q,p)} = \frac{1}{2}(\hat{q}^2 + \hat{p}^2). \] (40)

Note how the modified zeroth order term \( H(c(t)) - \frac{1}{2}(\partial_\alpha H(c(t)))c^\alpha(t) \) vanishes completely in this example, and we end up with the ordinary quantum harmonic oscillator Hamiltonian. In general, the correction to the zeroth order term cancels out the quadratic term in \( H(c(t)) \). Classical solutions \( c : \mathbb{R} \to \mathbb{R}^2 \) of Hamilton’s equation
\[ \partial_t c(t) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} c(t) \]
are of the form
\[ c(t) = \exp \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} t \right\} c(0) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} c(0). \] (41)

The Schrödinger equation (14) becomes
\[ i\hbar \partial_t \tilde{\psi} = \frac{1}{2}(\hat{p}^2 + \hat{q}^2)\tilde{\psi}. \] (42)

As is well known, stationary solutions to this equation are labelled by \( n = 0, 1, 2, \ldots \), so let \( |n\rangle \) be the \( n \)-th oscillator eigenfunction, and \( \tilde{\psi}(t) = |n(t)\rangle \) its time-dependent counterpart. Since we have \( \langle n|\hat{q}^\alpha|n\rangle = 0 \), according to (3) the expectation value of the position observable is
\[ \langle \hat{q} \rangle(t) = \langle \hat{q}(t) \rangle |n(t)\rangle = c^\alpha(t) = c^\alpha(0) + \sin(t)c^\alpha(0), \] (43)
and there are no quantum corrections at all to the center of mass motion. In general, the wave function \( \tilde{\psi} \) encapsulates the quantum fluctuations around the classical trajectory. For example, the energy is
\[ \langle H \rangle = \langle \hat{H} \rangle |n(t)\rangle = \frac{1}{2}(c^\alpha(0)^2 + c^\beta(0)^2) + \hbar\omega \left(n + \frac{1}{2}\right). \] (44)

In order to validate the limiting behaviour (28) let us introduce creation and annihilation operators:
\[ \hat{a} = \sqrt{\frac{1}{2\hbar}}(\hat{q} + i\hat{p}), \quad \hat{a}^\dagger = \sqrt{\frac{1}{2\hbar}}(\hat{q} - i\hat{p}) \] (45)
They act on the eigenstates \( |n\rangle \) as follows:
\[ \hat{a}^\dagger |n\rangle = \sqrt{n + 1}|n + 1\rangle \]
\[ \hat{a} |n\rangle = \sqrt{n}|n - 1\rangle \] (46) (47)
This implies that condition (28) is satisfied for the states \( |n\rangle \).

For completeness, let us check what the solutions we just constructed in our new quantum mechanics formalism look like in the textbook formalism. We have
\[ \psi(t) = U(t)^{-1}\tilde{\psi}(t) = \exp \left[ \frac{i}{\hbar} \left( c^\alpha(t)\hat{q} - c^\beta(t)\hat{p} \right) \right] |n(t)\rangle. \] (48)
In terms of the creation and annihilation operators (45) this reads
\[
\psi(t) = \exp \left( -ie^{it}a^\dagger \right) |n(t)\rangle,
\]
with \( z = c^q(0) + ie^q(0) \). For \( n = 0 \) this type of wave function is called a coherent state. Coherent states are known to resemble classical solutions as closely as possible, which is compatible with the fact that in our adapted formalism their quantum contribution is the ground state.

The example of the harmonic oscillator is quite special, because the Hamiltonian contains only quadratic terms and hence the modified Schrödinger equation looks exactly like the ordinary Schrödinger equation. This is related to the fact that the quantum operator \( \hat{H} \) in this case preserves not only the \( \hbar \)-filtration on the operator algebra, but even the \( \hbar \)-degree itself. For more general, non-quadratic systems this will not be the case. It is left as an exercise to the reader to verify that condition (28) is violated for the eigenstates of the hydrogen atom Hamiltonian.

9 Summary

We have defined a representation of quantum mechanics where every state consists of a pair \((c, \tilde{\psi})\), with \( c : I \to \mathbb{R}^{2n} \) a trajectory in phase space and \( \tilde{\psi} \) a wave function. An equivalence relation is defined on the set of all such pairs, which identifies pairs that can be transformed into each other by means of the transformations (9). The choice of trajectory \( c \) to represent a certain quantum state can be thought of as a kind of gauge fixing, and the simplest choice \( c(t) = 0 \ \forall t \) gives us back the textbook representation of quantum mechanics. The general equation of motion is the modified Schrödinger equation (12) for \( \tilde{\psi} \) and there is no restriction on \( c \), but we found out that the Schrödinger equation simplifies if we select \( c \) such that it satisfies Hamilton’s equation (13). In this case the quantum equation of motion is governed by the adapted Hamilton operator
\[
\hat{H}(t) = H(c(t)) - \frac{1}{2}(\partial_\alpha H(c(t)))c^\alpha(t) + \sum_{k=2}^{\infty} \frac{1}{k!} \left( \partial_{\alpha_1} \ldots \partial_{\alpha_k} H \right)(c(t))\hat{y}^{\alpha_1} \ldots \hat{y}^{\alpha_k}.
\]
which is missing the linear term in the canonical operators \( \hat{y}^\alpha \). This in turn implies that it preserves the quantum filtration on the operator algebra, which allowed us to deduce that the expectation value of an observable is given by the classical value of the observable plus quantum corrections, under certain assumptions on the initial state.

The table below summarizes the three equivalent representations of quantum mechanics that were discussed above.

|                  | QM textbook | QM gauged | QM gauge fixed |
|------------------|-------------|-----------|---------------|
| State            | \( \psi \)  | [\( (c, \psi) \)] | \( (c, \tilde{\psi}) \) |
| Observable       | \( \hat{f} \) | \( \tilde{\hat{f}} \) | \( \tilde{\hat{f}} \) |
| Equation of motion | \( i\hbar \delta_t \psi = \hat{H} \psi \) | see (12) | \( i\hbar \delta_t \tilde{\psi} = \hat{H}(t) \tilde{\psi} \) |
|                  | \( \langle \psi | \hat{f} | \psi \rangle \) | \( \langle \tilde{\psi} | \tilde{\hat{f}} | \tilde{\psi} \rangle \) | \( \langle \tilde{\psi} | \tilde{\hat{f}} | \tilde{\psi} \rangle = f(c(t)) + \mathcal{O}(\hbar^{1/2}) \) |

The relation between the quantities with and without tilde is given by \( \tilde{\psi}(t) = U_{c(t)} \psi(t) \) and \( \tilde{\hat{f}} = U_{c(t)} \hat{f} U_{c(t)}^{-1} \), with the Weyl operator \( U_{c(t)} \) defined in (9). The expression \( [c, \psi] \) denotes the equivalence class of pairs \((c, \psi)\), where \( c \) is a phase space trajectory and \( \psi \) a wave function, and the equivalence relation is defined as \((c, \psi) \sim (d, U_d U_c^{-1} \psi)\).
Appendix A: Geometric interpretation

Consider a trivial vector bundle $\mathcal{F} = \mathbb{R}^{2n} \times L^2(\mathbb{R}^n)$ over the phase space $M = \mathbb{R}^{2n}$, with fibre equal to the Hilbert space $L^2(\mathbb{R}^n)$. Sections of this vector bundle are functions $\psi \in \Gamma(\mathcal{F}) = C^\infty(M) \otimes L^2(\mathbb{R}^n)$, i.e. functions $\psi_{(q,p)}(x)$, with $q, p, x \in \mathbb{R}^n$. As above, we use coordinates $y^\alpha$ with $\alpha = 1, \ldots, 2n$ on the phase space that include both $q$ and $p$s. We can define a covariant derivative $D$ on the space of sections as follows:

$$D = d - \frac{i}{\hbar} \left[ \theta + \omega_{\alpha\beta} \dot{y}^\alpha dy^\beta \right],$$

(52)

where $d = dq^\alpha \frac{\partial}{\partial q^\alpha} + dp^\beta \frac{\partial}{\partial p^\beta}$ is the exterior derivative, and $\theta$ is any 1-form on $\mathbb{R}^{2n}$ satisfying $d\theta = \omega = \frac{1}{2} \omega_{\alpha\beta} dy^\alpha \wedge dy^\beta$. A convenient choice is $\theta = \frac{1}{2} \omega_{\alpha\beta} y^\alpha dy^\beta$, which we adopt. It can be shown that the covariant derivative is flat, i.e. its curvature form vanishes. This implies that we can find global solutions to the equation $D\phi = 0$. Explicitly, these solutions have the form

$$\phi_{(q,p)}(x) = \chi(q + x)e^{-\frac{i}{\hbar}p(x + \frac{x}{2})},$$

(53)

where $\chi$ is any differentiable function. The operator corresponding to a function $f$, acting on the fibre $\mathcal{H}_{(q,p)}$, is

$$\hat{f}_{(q,p)} = \sum_{k=0}^{\infty} \frac{1}{k!} \partial_{\alpha_1} \cdots \partial_{\alpha_k} f(q,p) \dot{y}^{\alpha_1} \cdots \dot{y}^{\alpha_k}$$

(54)

$$= f(q,p) + \partial_\alpha f(q,p) \dot{y}^\alpha + \frac{1}{2} \partial_\alpha \partial_\beta f(q,p) \dot{y}^\alpha \dot{y}^\beta + \ldots .$$

Finally, the parallel transport operator $U := U\left((q_0, p_0), (q, p)\right)$ for $D$, defined by $\phi_{(q,p)} = U\phi_{(q_0,p_0)}$ for the solutions (53), is the Weyl operator

$$U = \exp \left[ \frac{i}{\hbar} \left( (p_0 - p) \dot{q} + (q - q_0) \dot{p} + \frac{1}{2} (qp_0 - pq_0) \right) \right].$$

(55)

Here summation over indices is understood, i.e. $pq$ means $p_j q^j$, etc. The important property we need is that $U$ satisfies the parallel transport equation

$$\partial_t U(y, c(t)) = -A_{c(t)}(\dot{c}(t))U(y, c(t)),$$

(56)

where $y \in \mathbb{R}^{2n}$ is a point in phase space, $c$ is any curve starting in $y$, and $A$ is the connection form of $D$, i.e.

$$A = -\frac{i}{\hbar} \omega_{\alpha\beta} \left[ \frac{1}{2} y^\alpha + \dot{y}^\alpha \right] dy^\beta.$$  

(57)

Consider the Schrödinger equation on the single fibre $\mathcal{F}_y$. It reads $i\hbar \partial_t \psi(t) = \tilde{H}_y \psi(t)$. If we define

$$\phi(t) := U(y,c(t))\psi(t) \in \mathcal{F}_{c(t)}$$  

(58)

then the Schrödinger equation can be formulated for the flat section $\phi$:

$$i\hbar \partial_t \phi = i\hbar (\partial_t U) \psi + i\hbar U \partial_t \psi$$

$$= -i\hbar A(\dot{c}) \psi U + U \tilde{H}_y \psi$$

$$= (\tilde{H}_{c(t)} - i\hbar A_{c(t)}(\dot{c}(t))) \phi(t),$$

(59)
where we used that $\tilde{H}_{c(t)} = U \tilde{H} y U^{-1}$. We could now insert the explicit expressions for $\tilde{H}$ and $A$, but this will not be very enlightening in general. Instead, we consider special curves $c$, those who satisfy Hamilton’s equation
\[
\partial_t c^\alpha(t) = \omega^{\alpha\beta} \partial_\beta H(c(t)).
\] (60)
Then we get
\[
- i\hbar A(\dot{c}) = - \partial_\alpha H \dot{y}^\alpha - \frac{1}{2} \partial_\alpha H y^\alpha,
\] (61)
and the first term on the rhs. cancels the terms linear in $\dot{y}^\alpha$ of $\tilde{H}$ in (59). Thus
\[
\hbar \partial_t \phi = \left( H - \frac{1}{2} \partial_\alpha H \dot{y}^\alpha + \sum_{k=2}^{\infty} \frac{1}{k!} \partial_{\alpha_1} \ldots \partial_{\alpha_k} H \dot{y}^{\alpha_1} \ldots \dot{y}^{\alpha_k} \right)|_{c(t)} \phi,
\] (62)
This is again our equation (13).

The phase space $\mathbb{R}^{2n}$ with its symplectic 2-form $\omega$ forms a so-called symplectic manifold, and the vector bundle $\mathfrak{H}$ can be viewed as the bundle of symplectic spinors over $\mathbb{R}^{2n}$. There is an action of the symplectic group on $\mathbb{R}^{2n}$ (the group of linear transformations preserving the symplectic form). The symplectic group has a universal covering group, the so-called metaplectic group. The latter does not possess any finite-dimensional representation, but it can be represented on the Hilbert space $L^2(\mathbb{R}^n)$. Its Lie algebra is generated by symmetrized operators $\{\dot{y}^\alpha, \dot{y}^\beta\} = \frac{1}{2}(\dot{y}^\alpha \dot{y}^\beta + \dot{y}^\beta \dot{y}^\alpha)$. Compare to the Spin group, whose Lie algebra is generated by the commutators of Dirac matrices $[\gamma^\alpha, \gamma^\beta]$.

**Appendix B: Generalization to curved phase space**

The construction presented in **Appendix A** generalizes to curved phase spaces. It is well-known that the Hamiltonian formulation of classical mechanics extends to symplectic manifolds $(M, \omega)$ of dimension $2n$, where $\omega$ is a non-degenerate, closed 2-form on $M$. In local coordinates $y^1, \ldots, y^{2n}$ on $M$ we have $\omega = \frac{i}{2} \omega_{\alpha\beta}(y) dy^\alpha \wedge dy^\beta$. And since $\omega$ is closed, $d\omega = 0$, locally we can find a 1-form $\theta = \theta_\alpha dy^\alpha$ which satisfies $d\theta = \omega$. Let $c : I \subset \mathbb{R} \rightarrow M$ be a trajectory in phase space, then Hamilton’s equation for $c$ reads:
\[
\partial_t c^\alpha(t) = \omega^{\alpha\beta}(c(t)) \partial_\beta H(c(t)),
\] (63)
where $H \in C^\infty(M)$ is the Hamilton function. The existence of the 2-form $\omega$ implies that there is an action of the symplectic group on the fibers of the tangent and cotangent bundles over $M$. The symplectic group $Sp(2n)$ is defined as the subgroup of all linear transformations on a symplectic vector space which leave the form $\omega$ invariant, i.e. transformations $U$ which satisfy $\omega_U(X,Y) = \omega_U(U \cdot X, U \cdot Y)$ for all $X,Y \in T_pM$. Now assume that the first Chern class $c_1(M)$ is even, and that the quantization condition
\[
\frac{[\omega]}{2\pi \hbar} \in H^2(M, \mathbb{Z})
\] (64)
for the cohomology class $[\omega]$ is satisfied. Then the $Sp(n)$-structure on the tangent space $TM$ can be lifted to an action of the metaplectic group on a Hilbert bundle $\mathfrak{H}$ on $M$, i.e. a vector bundle whose fibers are all isomorphic to the Hilbert space $L^2(\mathbb{R}^n)$. This is completely analogous to the lift of $SO(n)$-actions on the tangent bundle of an oriented Riemannian manifold to $Spin(n)$-actions on the associated spinor bundle, except that the spin representation is finite-dimensional and the metaplectic action is not.
It can be shown that on every symplectic manifold there is a torsion-free connection \( \nabla \) on the tangent bundle (and associated bundles) which preserves the symplectic form \( \omega \), i.e. \( \nabla \omega = 0 \). Contrary to the Riemannian case, this symplectic connection is not unique, but we simply choose an arbitrary one. The connection lifts to the Hilbert bundle \( \mathcal{H} \), just like the spin connection lifts to the spinor bundle. In local coordinates we can write

\[
\nabla_{\alpha} = \partial_{\alpha} + \Gamma^\gamma_{\alpha\beta} dy^\beta \otimes \partial_\gamma
\]

(65)
on the tangent space, and on \( \mathcal{H} \):

\[
\nabla_{\alpha} = \partial_{\alpha} - \frac{i}{2\hbar} \Gamma_{\beta\gamma\alpha} \hat{y}^\beta \hat{y}^\gamma,
\]

(66)
where \( \hat{y}^\alpha \) are the canonical operators acting on a fibre \( \mathcal{H}_y \), satisfying \( [\hat{y}^\alpha, \hat{y}^\beta] = i\hbar \omega^{\alpha\beta}(y) \). Furthermore, \( \Gamma^\alpha_{\beta\gamma} := \omega^\alpha_{\delta\gamma} \Gamma^\delta_{\beta\gamma} \). In the previous section we chose our quantum states \( \phi \) as parallel sections of the vector bundle \( \mathcal{H} \) over \( \mathbb{R}^{2n} \), and it would be obvious to demand \( \nabla \phi = 0 \) in the general case as well, for sections \( \phi \in \Gamma(\mathcal{H}) \). However, the connection \( \nabla \) in general has a non-vanishing curvature, which implies that the equation \( \nabla \phi = 0 \) does not possess any solutions. Therefore, we first need to tweak the connection a little. Fedosov has shown that by adding higher order terms in the operators \( \hat{y}^\alpha \) it is possible to make the curvature form vanish projectively [3]. Fedosov’s connection \( D \) assumes the form

\[
D^\mathcal{H}_{\alpha} = \partial_{\alpha} + \frac{i}{\hbar} \omega_{\alpha\beta} \hat{y}^\beta - \frac{i}{2\hbar} \Gamma_{\beta\gamma\alpha} \hat{y}^\beta \hat{y}^\gamma - \frac{i}{8\hbar} R^\mathcal{H}_{\beta\gamma\delta\alpha} \hat{y}^{[\beta} \hat{y}^\gamma \hat{y}^\delta]} + O(\hbar^1)
\]

(67)
where \( R_{\alpha\beta\gamma\delta} = \omega_{\alpha\kappa} R^\mathcal{H}_{\beta\gamma\delta} \) are the components of the curvature form of \( \nabla \), and \( \hat{y}^{[\alpha} \hat{y}^\beta \hat{y}^{\gamma]} \) denotes the totally symmetrized product of the three operators. Higher order terms in the connection are determined by a recursive formula, which can be found in [3]. In the derivation of this result Fedosov makes heavy use of the quantum filtration introduced in section 5 which treats operators \( \hat{y}^\alpha \) as having quantum level \( 1/2 \), like \( \hbar^{1/2} \). It should be noted that in general nothing can be said about the convergence of the series (67), which is why Fedosov is very careful to define it only on some operator space of formal series in \( \hbar^{1/2} \) and the \( \hat{y}^\alpha \), similarly to the one defined in section 5. We’ll pretend instead that (67) was well-defined, just to see where this leads us, but should be aware that the remainder of this section is mathematically ill-founded for generic symplectic manifolds.

The curvature form \( F^\mathcal{H} \in \Gamma(\Lambda^2 T^* M \otimes L(\mathcal{H})) \) of \( D^\mathcal{H} \) actually does not vanish completely, but is equal to

\[
F^\mathcal{H} = \frac{i}{\hbar} \omega \otimes 1^\mathcal{H},
\]

(68)
where \( 1^\mathcal{H} \) denotes the fibre-wise identity operator on \( \mathcal{H} \). So we are not quite there yet. Enter geometric quantization: similarly to Fedosov’s deformation quantization, geometric quantization was born out of an attempt to explicitly construct the quantum theory associated to the Hamiltonian mechanics on \( (M, \omega) \). An important ingredient in this construction is the so-called pre-quantum bundle \( B \). This is a line-bundle, i.e. a complex 1-dimensional vector bundle on \( M \), which carries a connection \( \nabla_B \). In a local trivialization of \( B \) this pre-quantum connection can be written as

\[
\nabla_B = d - \frac{i}{\hbar} \theta,
\]

(69)
and its curvature form \( F_B \) is

\[
F_B = -\frac{i}{\hbar} \omega \otimes 1_B.
\]

(70)
Equations (68) and (70) let us deduce that the product bundle $\mathcal{H} \otimes B$ carries a flat connection

$$ D := D^B \otimes 1_B + 1_H \otimes \nabla_B. \quad (71) $$

Explicitly,

$$ D = d - \frac{i}{\hbar} (\partial_\alpha \phi \hat{y}^\alpha dy^\beta) - \frac{i}{2\hbar} \Gamma_{\alpha\beta\gamma} \hat{y}^\alpha y^\beta d\gamma - \frac{i}{8\hbar} R_{\alpha\beta\gamma\delta} \hat{y}^{(\alpha} y^{\beta} y^\gamma y^{\delta}) + \mathcal{O}(\hbar^1) \quad (72) $$

Now it makes sense to consider the equation $D\phi = 0$ for sections $\phi \in \Gamma(\mathcal{H} \otimes B)$. The connection $D$ also induces a covariant derivative on the sections of the bundle of linear operators on $\mathcal{H}$. For an observable $f \in C^\infty(M)$ we define a quantum operator $\hat{f} \in \Gamma(L(\mathcal{H}))$ by the constraints

$$ D\hat{f} = 0, \quad \text{and} \quad [\hat{f}_y, \hat{g}_y] = \{f, g\}(y)\hat{1}_y + \mathcal{O}(\hbar^{1/2}) \quad \forall y \in M, \quad (73) $$

where $\{f, g\} := \omega^{\alpha\beta} \partial_\alpha f \partial_\beta g$ is the Poisson bracket. Fedosov’s recursive formula for $D^\phi$ allows us to also determine the form of $\hat{f}$ recursively:

$$ \hat{f} = f + \partial_\alpha f \hat{y}^\alpha + \frac{1}{2} (\partial_\alpha \partial_\beta - \Gamma_{\alpha\beta\gamma} \partial_\gamma) f \hat{y}^\alpha \hat{y}^\beta + \mathcal{O}(\hbar^{3/2}). \quad (74) $$

This is Fedosov’s generalization of the Weyl quantization rule (5). Note that the expression $(\partial_\alpha \partial_\beta - \Gamma_{\alpha\beta\gamma} \partial_\gamma) f \hat{y}^\alpha \hat{y}^\beta$ is the image of $\nabla df \in \Gamma(Sym^2(T^*M))$ under the Weyl representation $dy^\alpha \rightarrow \hat{y}^\alpha$, hence is independent of the selected coordinates. Choose an arbitrary point $y \in M$, then the Schrödinger equation for a wave function $\psi_y \in \mathcal{H}_y \otimes B_y$ reads

$$ i\hbar \partial_t \psi_y(t) = \hat{H}_y \psi_y(t). \quad (75) $$

We can extend the wave function $\psi_y$ from the single fibre $\mathcal{H}_y$ to a parallel section $\psi \in \Gamma(\mathcal{H} \otimes B)$ by defining $\psi_z = U(y, z)\psi_y$, where $z \in M$ and $U(y, z)$ is the parallel transport operator associated to the connection $D$. Let $c : I \rightarrow M$ be a solution to the Hamilton equation, like in Appendix A we can consider the time-dependent parallel transport $\phi(t) = U(y, c(t))\psi_y(t)$. The Schrödinger equation formulated in terms of $\phi$ is:

$$ i\hbar \partial_t \phi = (\hat{H}_c(t) - i\hbar A_{c(t)}(\dot{c}(t)))\phi(t), \quad (76) $$

where $A$ is the connection form of our connection $D$, see (72). Explicitly evaluating $A$ on $\dot{c}$ gives

$$ - i\hbar A(\dot{c}) = -\theta_\alpha \omega^{\alpha\beta} \partial_\beta H - \partial_\alpha H \hat{y}^\alpha + \frac{1}{2} \Gamma_{\alpha\beta\gamma} \partial_\gamma H \hat{y}^\alpha \hat{y}^\beta - \frac{1}{8} R_{\alpha\beta\gamma\delta} \omega^{\delta\kappa} \partial_\kappa H \hat{y}^\alpha \hat{y}^\beta \hat{y}^\gamma + \mathcal{O}(\hbar^2). \quad (77) $$

and inserting this into the Schrödinger equation:

$$ i\hbar \partial_t \phi = \left( H - \theta_\alpha \omega^{\alpha\beta} \partial_\beta H + \frac{1}{2} \partial_\alpha \partial_\beta H \hat{y}^\alpha \hat{y}^\beta + \mathcal{O}(\hbar^{3/2}) \right) \phi. \quad (78) $$

Up to this order in $\hbar$, the equation looks exactly like in the flat case, but this will not be true for higher orders. Since the right hand side of (78) does not contain any terms linear in the $\hat{y}^\alpha$, we can deduce again that the $\hbar$-filtration is preserved under the time evolution, and for the expectation value of $\hat{f}$ we get

$$ \langle \hat{f} \rangle_{\phi}(t) = f(c(t)) + \underbrace{\partial_\alpha f(c(t)) \langle \hat{y}^\alpha \rangle_{\phi(t)}}_{\mathcal{O}(\hbar^{1/2})} + \frac{1}{2} (\underbrace{\partial_\alpha \partial_\beta - \Gamma_{\alpha\beta\gamma} \partial_\gamma}_{\mathcal{O}(\hbar^1)}) f \langle \hat{y}^\alpha \hat{y}^\beta \rangle_{\phi(t)} + \mathcal{O}(\hbar^{3/2}), \quad (79) $$
which in the classical limit converges to its classical value: \( \lim_{\hbar \to 0} \langle \tilde{f} \rangle_{\phi}(t) = f(c(t)) \).

This approach of representing quantum mechanical states by means of parallel sections of a metaplectic spinor bundle was proposed in [5], although the pre-quantum line bundle is still missing from the construction there. The results of the present paper have been presented first in [6], where they are derived in a top-down approach starting from the geometrical construction of this section. This paper is an attempt to present the results in a bottom-up approach instead, in order to make them more accessible.

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