CONFLUENCE IN QUANTUM $K$-THEORY OF WEAK FANO MANIFOLDS
AND $q$-OSCILLATORY INTEGRALS FOR TORIC MANIFOLDS

TODOR MILANOV AND ALEXIS ROQUEFEUIL

Abstract. For a smooth projective variety whose anti-canonical bundle is nef, we prove confluence of the small $K$-theoretic $J$-function, i.e., after rescaling appropriately the Novikov variables, the small $K$-theoretic $J$-function has a limit when $q \to 1$, which coincides with the small cohomological $J$-function. Furthermore, in the case of a Fano toric manifold $X$ of Picard rank 2, we prove the $K$-theoretic version of an identity due to Iritani that compares the $I$-function of the toric manifold and the oscillatory integral of the toric mirror. In particular, our confluence result yields a new proof of Iritani’s identity in the case of a toric manifold of Picard rank 2.

Contents

1. Introduction 2
  1.1. Foreword 2
  1.2. Plan of the paper 6
  1.3. Acknowledgements 7
2. The ABC of twisted Gromov–Witten theory 7
  2.1. Definition of ABC twists 7
  2.2. Removing the twists 9
  2.3. Fake $K$-theoretic GW invariants 12
  2.4. Stem invariants 13
3. Confluence of the small $K$-theoretic $J$-function 14
  3.1. The Kawasaki–Riemann–Roch formula 14
  3.2. Stems as Kawasaki strata 15
  3.3. Reconstruction of the small $K$-theoretic $J$-function 17
  3.4. Confluence of the small $K$-theoretic $J$-function 18
4. Oscillatory integrals in Gromov–Witten theories 21
  4.1. Symplectic toric manifolds 21
  4.2. Oscillatory integral and gamma class in quantum cohomology 25
  4.3. $q$-oscillatory integral in quantum $K$-theory 28
  4.4. Comparison theorem 36

Date: August 20, 2021.
2000 Math. Subj. Class. 14N35, 35Q53.
Key words and phrases: Frobenius structures, K-theoretic Gromov–Witten invariants, $q$-difference equations, $q$-gamma function.
1. Introduction

1.1. Foreword. Let $X$ be a smooth complex projective variety and let $K(X) = K^0(X; \mathbb{C})$ be the Grothendieck group of topological vector bundles on $X$. For simplicity, we will assume that $H^{\text{odd}}(X; \mathbb{C}) = 0$. In particular, the Chern character map $\text{ch} : K^0(X) \rightarrow H(X; \mathbb{C})$ is a ring isomorphism. Let us denote by $X_{0,n,d}$ the proper moduli stack of genus-0 stable maps of degree $d \in H_2(X; \mathbb{Z})$ with $n$ marked points. The operation that assigns to a point in the moduli space the cotangent line at the $i$-th marked point is functorial and it gives rise to a line bundle $L_i \rightarrow X_{0,n,d}$, while evaluation at the $i$-th marked point gives rise to a map of Deligne–Mumford stacks $ev_i : X_{0,n,d} \rightarrow X$ known as the evaluation map. Let $E_1, \ldots, E_n \in K(X)$, then following Givental and Y. P. Lee (see [Giv Lee04]) we introduce the $K$-theoretic Gromov–Witten (GW) invariants.

**Definition 1.1.1.** The $K$-theoretic Gromov–Witten invariants of $X$ are given by

$$(E_1L_1^{k_1}, \ldots, E_nL_n^{k_n})_{g,n,d} = \chi(O_{\text{virt}} \otimes ev_1^*(E_1)L_1^{k_1} \cdots ev_n^*(E_n)L_n^{k_n}) \in \mathbb{Z},$$

where $\chi(F)$ denotes the holomorphic Euler characteristic of $F$ and $O_{\text{virt}}$ is the so called virtual structure sheaf (see [Lee04]).

Let us fix a set $P_1, \ldots, P_r$ of ample line bundles, s.t., $p_i = c_1(P_i)$ form a $\mathbb{Z}$-basis of $H^2(X; \mathbb{Z}) \cap H^{1,1}(X; \mathbb{C}) \cong NS^1(X)$. If $d \in H_2(X; \mathbb{Z})$ then we define

$$Q^d := Q_1^{(p_1,d)} \cdots Q_r^{(p_r,d)},$$
where $Q_1, \ldots, Q_r$ are formal variables known as the *Novikov variables*. Let us assign to each $Q_i$ degree $m_i \in \mathbb{Z}$ defined by the identity $c_1(T_X) = \sum_{i=1}^r m_i p_i$. For $t \in K(X)$ put

$$
(1.1.2) \quad \langle E_1 L_1^{k_1}, \ldots, E_n L_n^{k_n} \rangle_{g,n}(t) = \sum_{d} \sum_{\ell=0}^{\infty} \frac{Q^d}{\ell!} \langle E_1 L_1^{k_1}, \ldots, L_n^{k_n}, t, \ldots, t \rangle_{g,n+t,d}.
$$

Let us fix a basis $\{\Phi_i\}_{i=1}^N \subset K(X)$ and denote by $\{\Phi^i\}_{i=1}^N$ the dual basis with respect to the Euler pairing

$$
g_{ij} := \chi(\Phi_i \otimes \Phi_j) = \int_X \text{ch}(\Phi_i) \text{ch}(\Phi_j) \text{Td}(X).
$$

Following Givental, we introduce

**Definition 1.1.3** ([IMT15], Definition 2.4). The small $K$-theoretic $J$-function of the variety $X$ is the formal power series

$$
J(q, Q) = 1 - q + \sum_{d \in \text{Eff}(X)} \sum_{i=1}^N \left( \frac{\Phi_i}{1 - q L_1} \right)_{0,1,d} Q^d \Phi^i \in K(X)(q)[[Q]].
$$

Let us recall also the definition of the cohomological GW invariants. Let us fix bases $\{\phi_i\}_{i=1}^N$ and $\{\phi^i\}_{i=1}^N$ of $H(X; \mathbb{C})$ dual to each other with respect to the Poincaré pairing, that is,

$$
(\phi_i, \phi^j) := \int_X \phi_i \cup \phi^j = \delta_{i,j}, \quad 1 \leq i, j \leq N.
$$

In fact, we choose $\phi_i := \text{ch}(\Phi_i)$ and $\phi^i := \text{ch}(\Phi^i) \text{Td}(X)$. The GW invariants are defined by

$$
\langle \phi_i \psi_1^{k_1}, \ldots, \phi_n \psi_n^{k_n} \rangle_{g,n,d} = \int_{[X_{g,n,d}]} \prod_{s=1}^N \text{ev}_s^* (\phi_{i_s}) \psi_s^{k_s},
$$

where $\text{ev}_s : X_{g,n,d} \to X$ is the evaluation map at the $s$-th marked point, $\psi_s = c_1(L_s)$, and $[X_{g,n,d}]^\text{virt}$ is the virtual fundamental cycle constructed in [BF07]. The cohomological $J$-function is defined by

$$
J^\text{coh}(z, Q) = -z + \sum_{d \in \text{Eff}(X)} \sum_{i=1}^N \left( \frac{\phi_i}{-z \psi} \right)_{0,1,d} Q^d \phi^i.
$$

The parameter $z$ in $J^\text{coh}(z, Q)$ is in fact redundant due to the homogeneity properties of the $J$-function. Namely, let us define the degree operator

$$
\text{deg} : H(X; \mathbb{C}) \to H(X; \mathbb{C}),
$$

which for a homogeneous element $\phi \in H^{2k}(X; \mathbb{C})$ is defined by $\text{deg}(\phi) = k\phi$. Using the formula for the dimension of the virtual fundamental cycle we get

$$
(1.1.4) \quad J^\text{coh}(z, Q_1, \ldots, Q_r) = z^{1 - \text{deg}} J^\text{coh}(1, z^{-m_1} Q_1, \ldots, z^{-m_r} Q_r).
$$

One way to compare cohomological and $K$-theoretical Gromov–Witten invariants is to compare their $J$-functions as solutions of their respective functional equations. In more details, one can use confluence of $q$-difference equations to obtain the cohomological $J$-function as a limit of the $K$-theoretical one. This has been studied for projective spaces in [Roq19] and for the quintic threefold in [Wen20]. Recall that a line bundle $L$ is said to be *nef* if it has a non-negative degree on all
complex curves in \( X \), that is, if \( f : C \to X \) is a holomorphic map from a complex curve \( C \) to \( X \), then \( \int_C c_1(f^*L) \geq 0 \). Our first main goal will be to prove the following confluence result.

**Theorem (Theorem 3.4.1).** If \( X \) is a smooth projective variety, such that, the anti-canonical bundle \( K_X^\vee \) is nef, then the limit

\[
\lim_{q \to 1} (q-1)^{\deg -1} \deg \left( J(q,(q-1)^{m_1}Q_1,\ldots,(q-1)^{m_r}Q_r) \right)
\]

exists and it coincides with \( J^{\text{coh}}(1,Q) \).

Let us point out that the title of our paper is a bit misleading, because weak Fano manifold \( X \) means that \( K_X^\vee \) is nef and big, while our theorem does not require the condition that \( K_X^\vee \) is big. In particular, our theorem applies to all Fano and Calabi–Yau manifolds. The proof of Theorem 3.4.1 is based on the reconstruction result of Givental–Tonita for the \( K \)-theoretic \( J \)-function in terms of cohomological GW invariants. More precisely, in the case of the small \( J \)-function, the recursion procedure outlined in the proof of Proposition 4 in [GT14] takes a very elegant form. Using the Fano condition and the formula for the dimension of the virtual fundamental cycle, the statement of Theorem 3.4.1 follows quite easily.

Our next goal is motivated by the problem of comparing the \( K \)-theoretic \( I \)-function of a toric manifold \( X \) with the corresponding oscillatory integral defined through the toric mirror of \( X \). The \( K \)-theoretic \( I \)-function was introduced by Givental in [Giv15a, Giv15b]. More precisely, using fixed-point localization techniques, Givental was able to prove that a certain \( q \)-hypergeometric series \( I_X^K \), called the small \( K \)-theoretic \( I \)-function (see Definition 4.3.14), belongs to the permutation-equivariant \( K \)-theoretic Lagrangian cone of \( X \). Let us point out that in the case of a Fano toric manifold, up to some simple normalization factor, \( I_X^K \) coincides with the small \( J \)-function of \( X \). Moreover, the small \( J \)-function in permutation equivariant \( K \)-theoretical GW theory coincides with the non-equivariant one, i.e., with the small \( J \)-function used in the current paper. On the other hand, from the toric data of \( X \) one can construct a \( K \)-theoretic version of the toric mirror model (see [Giv15c]). Namely, following Givental, let us consider the following family of functions:

\[
Y := (\mathbb{C}^*)^n \xrightarrow{W^K} \mathbb{C},
\]

\[
B := (\mathbb{C}^*)^r,
\]

where we denote by \( x_1,\ldots,x_n \) the standard coordinates on \( Y \), \( Q_1,\ldots,Q_r \) the standard coordinates on \( B \), and the maps \( W^K \) and \( \pi \) are given by

\[
W^K(x_1,\ldots,x_n) := \sum_{j=1}^n \sum_{l>0} \frac{x_j}{(1-q)^{l-1}} \in \mathbb{C},
\]

\[
\pi(x_1,\ldots,x_n) := \left( \cdots, \prod_{j=1}^n x_i^{m_{ij}}, \cdots \right) \in B.
\]
Let us recall first the case of quantum cohomology, which was investigated in details in [Iri09]. Put
\[ \mathcal{I}^{\text{coh}}(z, Q) := \int_{\mathbb{R}} \exp \left( W_{\pi^{-1}(Q)}^{\text{coh}}(x_1, \ldots, x_n) \right) \omega_{\pi^{-1}(Q)}, \]
where
\[ \omega_{\pi^{-1}(Q)} := \frac{d \log x_1 \wedge \cdots \wedge d \log x_n}{d \log Q_1 \wedge \cdots \wedge d \log Q_r}, \]
is interpreted naturally as a holomorphic volume form on the fiber \( \pi^{-1}(Q) \) and \( \Gamma_{\mathbb{R}} \) is the real Lefschetz thimble (see Remark 4.2.4). The comparison result goes as follows.

**Theorem** ([Iri09], see Theorem 4.2.8). If \( X \) is a Fano toric manifold, then the cohomological oscillatory integral \( \mathcal{I}^{\text{coh}} \) and the cohomological I-function (see Definition 4.2.5) are related by the identity
\[ \mathcal{I}^{\text{coh}}(z, Q) = \int_{[X]} \widehat{\Gamma}(TX) \cup z^\rho \cdot z^{\deg I^{\text{coh}}}(z, Q), \]
where \( \rho = c_1(TX) \cup \) is the operator of cup product multiplication by \( c_1(TX) \), \( \int_{[X]} \) denotes the intersection product by \( [X] \in H_*(X; \mathbb{C}) \), and \( \widehat{\Gamma}(TX) \) is a cohomological characteristic class given by
\[ \widehat{\Gamma}(TX) = \prod_{\delta_j/\text{Chern root of } TX} \Gamma(1 + \delta_j) \in H^*(X; \mathbb{C}). \]

Let us go back to the \( K \)-theoretical setting. The difficulty of the problem depends on the Picard rank of \( X \). The Picard rank 1 case does not contain non-Fano manifolds. Therefore, since we would like to see the role of the Fano condition, we will focus on the case when \( X \) is a Fano toric manifold of Picard rank 2. Let us point out that the Picard rank 2 case is the 1st one to consider if one is interested in extending Iritani’s result to non-Fano manifolds. Using the \( K \)-theoretic mirror family of functions, we construct a \( q \)-analogue of the oscillatory integral, by using the Jackson integral
\[ \left[ \int_0^{\infty} \right]_q f(x)d_qx := \sum_{d \in \mathbb{Z}} q^d f(q^d). \]
In Definition 4.3.4, we introduce the \( q \)-oscillatory integral
\[ \mathcal{I}^{K^{-\text{th}}}(q, Q_1, Q_2) := \left[ \int_{\mathbb{R}} \right]_q \exp \left( W_{\pi^{-1}(Q)}^{K^{-\text{th}}}(x_1, \ldots, x_n) \right) \omega_{\pi^{-1}(Q), q}. \]
In Proposition 4.3.15, we show that the functions \( I^{K^{-\text{th}}} \) and \( \mathcal{I}^{K^{-\text{th}}} \) are solutions to the same set of \( q \)-difference equations. In order to compare these two functions, we introduce a multiplicative characteristic class, which can be viewed as a \( q \)-analogue of Iritani’s gamma class \( \widehat{\Gamma}(TX) \). Denote by \( (z; q)_\infty := \prod_{l \geq 0} (1 - q^l z) \) the \( q \)-Pochhammer symbol (for \( |q| < 1 \)), and let \( \Gamma_q(x) := (1 - q)^{1 - x} \frac{(q^x)_\infty}{(q^{-x})_\infty} \) be the \( q \)-gamma function.

**Definition** (Definition 4.4.1). The \( q \)-gamma class of a symplectic toric manifold is defined to be, for \( |q| > 1 \),
\[ \gamma_q(TX) := \prod_{\delta_j/\text{Chern root of } TX} \delta_j (1 - q^{-1})^{\delta_j} \Gamma_q^{-1}(\delta_j) \in H^*(X; \mathbb{C}). \]
The 2nd main goal of this paper is to prove the following $K$-theoretical analogue of Iritani’s theorem:

**Theorem** (Theorem 4.4.4). Suppose that $q > 1$ is a real number and that the Novikov variables satisfy $Q_1, Q_2 \in q\mathbb{Z}$. Then the $K$-theoretic $I$-function $I_X^{K-th}$ and the oscillatory integral $\mathcal{I}^{K-th}$ are related by the following identity:

$$
\mathcal{I}^{K-th}(q, Q_1, Q_2) = \int_{[X]} \gamma_q(TX) \cup \text{ch}_q(I_X^{K-th}(q, Q_1, Q_2)),
$$

where $\int_{[X]}$ denotes the cap product with the fundamental class $[X] \in H_*(X_{\mu,K}; \mathbb{C})$, $\gamma_q(TX)$ is the $q$-gamma class of Definition 4.4.1 and

$$
\text{ch}_q(E) := \sum_{\delta, \text{Chern root of } E} q^\delta.
$$

Note that in quantum cohomology, the scope of [Iri09] goes much beyond the comparison theorem as we stated it here - in particular, the gamma class $\tilde{\Gamma}(TX)$ is used to define the A-side integral structure and the quantum cohomology central charge. The search for $K$-theoretical analogues of these constructions should motivate further investigations related to the $q$-gamma class $\gamma_q(TX)$.

1.2. Plan of the paper. This paper is essentially structured in two independent parts. The first part consists of Sections 2 and 3, whose goal is to prove the confluence of the small $K$-theoretical $J$-function to the small cohomological $J$-function. Section 2 deals with prerequisites to understand the reconstruction result of [GT14], which we will explain in Subsections 3.2 and 3.3. Finally, in Subsection 3.4, we state and prove the 1st main result of this paper - confluence of the small $K$-theoretic $J$-function (Theorem 3.4.1).

The second part consists of Section 4, in which we study two mirrors to the $J$-function: the $I$-function and the ($q$-)oscillatory integral. Subsection 4.1 fixes the notations for symplectic toric manifolds. Subsection 4.2 deals with the results already known in quantum cohomology. We will also give a new strategy to prove the comparison result between the small cohomological $I$-function and the oscillatory integral (see Theorem 4.2.8). In subsection 4.3, motivated by Givental’s proposal for mirror symmetry for the small $K$-theoretical $J$-function, we define a $q$-oscillatory integral (defined through the Jackson integral, see Definition 4.3.4). There is a subtlety in the case when the Novikov variables do not belong to the $q$-spiral $q\mathbb{Z} := \{ q^m \mid m \in \mathbb{Z} \}$. Following De Sole and Kac [DSK05] we were able to find a natural construction of the $q$-oscillatory integral for arbitrary values of the Novikov variables modulo a certain conjecture about the regularity of the $K$-theoretic quantum $q$-difference equations. Furthermore, in Section 4.4 we prove the 2nd main result of this paper, that is, the comparison between the $K$-theoretical $I$-function and the $q$-oscillatory integral (Theorem 4.4.4). Finally, in Section 4.5 we give a new proof of Iritani’s theorem by taking the limit $q \to 1$ in Theorem 4.4.4 and using the confluence result in Theorem 3.4.1.

For the sake of completeness, we have added three appendices. Appendix A contains an outline of the proof of the reconstruction result of [GT14]. Appendix B contains the proof of Iritani’s theorem 4.2.8 for the case of Picard rank 2, based on techniques similar to our proof of Theorem
4.4.4. Appendix C attempts to reproduce the results of Section 4.3 and Section 4.4 for a different $q$-analogue of the oscillatory integral, defined using continuous integration.

1.3. **Acknowledgements.** Both authors are supported by World Premier International Research Center Initiative (WPI Initiative), MEXT, Japan. The work of T. M. is partially supported by Grant-in-aid JP19F19802 and Grant-in-aid (Kiban C) 17K05193. The work of A. R. is partially supported by Grant-in-aid JP19F19802. The second author wishes to acknowledge the Japan Society for Promotion of Science for providing funding for their research projects at the University of Tokyo, Kavli IPMU.

2. **The ABC of twisted Gromov–Witten theory**

Our goal here is to give the definitions of Gromov–Witten invariants required for the reconstruction theorem of quantum $K$-theory from quantum cohomology of [GT14]. In Subsection 2.1 we define ABC-twisted (cohomological) GW invariants. In Subsection 2.2 we explain how to reconstruct ABC-twisted GW invariants in terms of the usual cohomological GW invariants. In the remaining subsections 2.3 and 2.4, we define the two particular kinds of ABC-twisted GW invariants that appear in the reconstruction theorem.

Suppose that $Y$ is an orbifold whose coarse moduli space $|Y|$ is a projective variety. Let $IY = \bigsqcup_{v=1}^{m} Y_v$ and $\overline{IY} = \bigsqcup_{v=1}^{m} \overline{Y_v}$ be respectively the inertia and the rigidified inertia orbifolds of $Y$, where the index $v$ ($1 \leq v \leq m$) enumerates the connected components of the coarse moduli space $|Y| = |\overline{Y}|$. Let us assume that $Y_1 = Y$ and so $Y_v$ (resp. $\overline{Y_v}$) with $v \neq 1$ are the so-called twisted (resp. rigidified twisted) sectors. We would like to recall Givental–Tonita’s twisted orbifold Gromov–Witten theory of $Y$, which plays a key role in the study of $K$-theoretic GW theory (see [GT14, Ton14]).

In fact, the only orbifolds that we will be interested in will be the global quotients $Y = [X/\mu_m]$, where $\mu_m$ is the multiplicative group of order $m$, $X$ is a smooth projective variety, and the action of $\mu_m$ is the trivial one. In this case the twisted sectors are parametrized by the elements $g$ of the group $\mu_m$ and we have

$$Y_g = [X/\mu_m], \quad \overline{Y_g} = [X/(\mu_m/(g))],$$

where $(g)$ denotes the cyclic subgroup generated by $g$.

2.1. **Definition of ABC twists.** Some standard references for orbifold GW theory is [AGV08, CR02] (see also [Ton14] for an overview). Let $Y_{g,n,d}$ be the moduli space of orbifold stable maps $f : (C, s_1, \ldots, s_n) \to Y$. The moduli space has connected components $Y^{v_1,\ldots,v_n}_{g,n,d}$ parametrized by $n$-tuples $(v_1, \ldots, v_n)$, $v_i \in [1,m] := \{1,2,\ldots,m\}$ and consisting of stable maps, such that, $f|_{s_i} \in \overline{Y}_{v_i}$. Let $\mathcal{C}^{(v_1,\ldots,v_n)}_{g,n,d} := Y^{(v_1,\ldots,v_n)}_{g,n+1,d}$ be the connected components of the universal curve $\mathcal{C}_{g,n,d}$ and $\pi : \mathcal{C}_{g,n,d} \to Y_{g,n,d}$ and $ev : \mathcal{C}_{g,n,d} \to Y$ be respectively the map forgetting the last marked point and the evaluation map at the last marked point. Finally, let $Z_v$ ($v \in [1,m]$) be the closed substack of $\mathcal{C}_{g,n,d}$ parametrizing stable maps $(C, s_1, \ldots, s_{n+1}, f)$, such that, if $C'$ denotes the irreducible component of $C$ that carries the $(n+1)$-st marked point $s_{n+1}$, then

(i) $C'$ carries exactly two nodes of $C$, say $z_+$ and $z_-$, and no other marked points.
(ii) The map $f$ maps $C'$ to $Y$ with degree 0, that is, $C'$ is contracted to a point.

(iii) The evaluation map at $z_+$ or $z_-$ lands in $Y_v$.

Let us recall that the forgetful map $\pi$ is characterized by the property that it forgets the $(n+1)$st marked point and it contracts the resulting unstable components. Therefore, the fibers of $\pi|_Z : Z := \bigsqcup_v Z_v \to Y_{g,n,d}$ are non-empty only if the domain curve $C$ of the corresponding stable map in $Y_{g,n,d}$ is singular. In this case, the points in the fiber of $\pi|_Z$ correspond to the singular points of $C$. The above moduli spaces and maps between them form the following diagram:

\[
\begin{array}{ccc}
Z_v & \xrightarrow{\iota_w} & C_{g,n,d}(v_1,\ldots,v_n) = Y_{g,n+1,d}(v_1,\ldots,v_n,1) \\
\downarrow & & \downarrow \\
\pi & & \pi \\
Y_{g,n,d}(v_1,\ldots,v_n) & \xrightarrow{(\ev_1,\ldots,\ev_n)} & Y_{v_1} \times \cdots \times Y_{v_n},
\end{array}
\]

where $\ev_i$ is the evaluation map at the $i$-th marked point and $\iota_w$ is the natural inclusion map.

The twisted GW invariants depend on the choice of the following 3 types of data:

(A) A finite number of orbifold vector bundles $E_\alpha \to Y$ (where $1 \leq \alpha \leq k_A$) and identically indexed multiplicative characteristic classes

\[A_\alpha(\ell) := \exp \left( \sum_{i=0}^{\infty} s_{\alpha,i}^A \chi_i(\ell) \right), \quad s_{\alpha,i}^A \in \mathbb{C}.\]

(B) A finite number of polynomials $f_\beta \in K^0(X)[\ell]$ where $(1 \leq \beta \leq k_B)$ and identically indexed multiplicative characteristic classes

\[B_\beta(\ell) := \exp \left( \sum_{i=0}^{\infty} s_{\beta,i}^B \chi_i(\ell) \right), \quad s_{\beta,i}^B \in \mathbb{C}.\]

(C) A finite number of orbifold vector bundles $E_{v,\gamma} \to Y$ (where $v \in [1,m], 1 \leq \gamma \leq k_v$) and identically indexed multiplicative characteristic classes

\[C_{v,\gamma}(\ell) := \exp \left( \sum_{i=0}^{\infty} s_{v,\gamma,i}^C \chi_i(\ell) \right), \quad s_{v,\gamma,i}^C \in \mathbb{C}.\]

Using the above data we define the following three types of cohomology classes:

**Definition 2.1.1.** The $ABC$-twists are the three cohomological classes on $Y_{g,n,d}$ defined by:

\[\Theta^A_{g,n,d} := \prod_{\alpha=1}^{k_A} A_\alpha(\pi_*(\ev_{n+1}^* E_\alpha)),\]

\[\Theta^B_{g,n,d} := \prod_{\beta=1}^{k_B} B_\beta(\pi_*(\ev_{n+1}^* f_\beta(L_{n+1}^{-1}) - \ev_{n+1}^* f_\beta(1))),\]

where $\ev_{n+1}^* f_\beta(\ell) \in \ev_{n+1}^* K^0(X)[\ell]$ and

\[\Theta^C_{g,n,d} := \prod_{v=1}^m \prod_{\gamma=1}^{k_v} C_{v,\gamma}(\pi_*(\ev_{n+1}^* (E_{v,\gamma}) \otimes \iota_v \mathcal{O}_{Z_v})),\]

where $\ev_{n+1}^* f_\beta(\ell) \in \ev_{n+1}^* K^0(X)[\ell]$ and $\iota_v : Z_v \to Y_v$. 


where \( \pi_s = R^0 \pi_s - R^1 \pi_s \) is the \( K \)-theoretic pushforward and \( L_{n+1} \to C_{g,n,d} \) is the orbifold line bundle corresponding to the cotangent line at the \((n+1)\)st marked point.

Put \( \Theta^{ABC}_{g,n,d} = \Theta^A_{g,n,d} \Theta^B_{g,n,d} \Theta^C_{g,n,d} \). Then the ABC-twisted GW-invariants are defined by the following formula:

**Definition 2.1.2.** Let \( k_1, \ldots, k_n \in \mathbb{Z}_{\geq 0} \). The ABC-twisted Gromov–Witten invariants of \( Y \) are defined by

\[
\langle \phi_{i_1} \psi^{k_1}, \ldots, \phi_{i_n} \psi^{k_n} \rangle_{g,n,d}^{ABC} = \int_{[Y_{g,n,d}]^{\text{virt}}} \Theta^{ABC}_{g,n,d} \prod_{s=1}^{n} \text{ev}_s^* (\phi_{i_s}) \psi_s^{k_s},
\]

where \( \{ \phi_i \}_{i=1}^{N} \) is a basis of \( H^* (TY; \mathbb{C}) = H^* (IY; \mathbb{C}) \), \( \psi_s \) is the 1st Chern class of the orbifold line bundle \( L_s \) corresponding to the cotangent line at the \( s \)-th marked point, and \([Y_{g,n,d}]^\text{virt}\) is the virtual fundamental cycle of \([Y]_{g,n,d}^{vir}\).

Let us also recall some properties of the grading in orbifold GW theory, which we will need later on to prove confluence of the \( K \)-theoretical \( J \)-function. Let \( Y = (Y_1 \to Y_0) \) be an orbifold groupoid representing the Morita equivalence class of \( Y \). The inertia orbifold \( IY \) is represented by the orbifold groupoid whose objects are pairs \((y, g), y \in Y_0 \) and \( g \in \text{Aut}(y) := \{ \xi \in Y_1 | s(\xi) = t(\xi) = y \} \) and whose morphisms \( \text{Mor}_{IY}((y, g), (y', g')) \) are morphisms \( f \in \text{Mor}_Y(y, y') \) such that \( g' \circ f = f \circ g \). Furthermore, the orbifold tangent bundle \( TY \) is represented by the orbifold groupoid \( T\text{Aut}_{Y_0} \), whose objects are pairs \((y, \xi), y \in Y_0 \) and \( \xi \in T_y Y_0 \) and whose morphisms \( \text{Mor}_{T Y}((y, \xi), (y', \xi')) \) consists of morphisms \( f \in \text{Mor}_Y(y, y') \) such that, \( df(\xi) = \xi' \). If \((y, g) \in IY \), then \( g \) acts naturally on \( T_y Y_0 \) and since \( g \) has finite order, it is diagonalizable. Let \( \lambda_i(y, g) = e^{2\pi i \alpha_i(y, g)} \), \( 0 \leq \alpha_i(y, g) < 1 \), \( 1 \leq i \leq \dim(Y) \) be the eigenvalues of \( g \). Then \( \nu(y, g) := \sum \alpha_i(y, g) \) is a rational number depending only on the connected component \( Y_v \) to which the point \((y, g)\) belongs to, so we put \( \nu(v) := \nu(y, g) \). If \( \phi \in H^{2k}(Y_v, \mathbb{C}) \), then the Chen–Ruan degree of \( \phi \) is defined to be \( \text{deg}_{CR}(\phi) := k + \nu(v) \).

**Proposition 2.1.3.** The complex dimension of the virtual fundamental cycle of \( \sum \nu(\nu_{v_1, \ldots, v_n}) \) is

\[
\dim \left[ \sum_{g,n,d} \nu(\nu_{v_1, \ldots, v_n}) \right] = 3g - 3 + n + D(1 - g) + \int_d c_1(TY) - \sum_{i=1}^{n} \nu(v_i).
\]

**Corollary 2.1.4.** If the twisted GW invariant of Definition (2.1.2) is non-zero, then the following inequality holds:

\[
\sum_{s=1}^{n} \text{deg}_{CR}(\phi_{i_s}) + k_s \leq 3g - 3 + n + D(1 - g) + \int_d c_1(TY).
\]

2.2. **Removing the twists.** The results of this section are not really needed in this paper. We include them just for completeness of the reconstruction theorem. To begin with let us organize the twisted GW invariants into a generating function. Let \( t = (t_{k,v,a}) \) be a sequence of formal variables.
where \( k \geq 0, v \in [1,m], \) and \( 1 \leq a \leq N_v := \dim H(Y_v; \mathbb{C}) \). Let \( \{ \phi_{v,a} \} (v \in [1,m], 1 \leq a \leq N_v) \) be a basis of \( H(Y_v; \mathbb{C}) \). Put

\[
t(z) := \sum_{k=0}^{\infty} \sum_{v=1}^{m} \sum_{a=1}^{N_v} t_{k,v,a} \phi_{v,a} z^k.
\]

The total descendent potential of the ABC-twisted GW invariants is defined by

\[
D^{ABC}(\mathbf{h}, \mathbf{t}) := \exp \left( \sum_{n,g,d} h^{g-1} Q^d n! \left( t_1(\psi_1), \ldots, t_n(\psi_1) \right) \right).
\]

Let us define also AB-twisted, A-twisted, and non-twisted (i.e. cohomological) GW invariants by formula (2.1.2) except that we replace \( \Theta^{ABC} \) with respectively \( \Theta^{AB} := \Theta^{A} \Theta^{B}, \Theta^{A}, \) and 1. The corresponding total descendent potentials \( D^{AB}, D^{A}, \) and \( D \) are defined in the same fashion. Our goal is to express \( D^{ABC} \) in terms of \( D \).

Let us begin with the C-twist. Let \( I : IY \to IY \) be the involution induced by the map \( (y,g) \mapsto (y,g^{-1}) \). The involution maps a connected component \( Y_v \) isomorphically to a connected component which we denote by \( Y_{I(v)} \). The C-twist is removed by the following formula:

**Proposition 2.2.1** ([Ton14]). The C-twist is removed from the total descendant potential \( D^{ABC} \) according to the following identity:

\[
D^{ABC}(\mathbf{h}, \mathbf{t}) = \exp \left( \frac{h}{2} \sum_{v=1}^{m} \sum_{k,a,b=1}^{\infty} A^v_{ka,lb} \frac{\partial^2}{\partial t_{v,k,a} \partial t_{I(v),l,b}} \right) D^{AB}(\mathbf{h}, \mathbf{t}),
\]

where the coefficients \( A^v_{ka,lb} \in \mathbb{C} \) are defined in terms of the twisting data of type C. \( \square \)

Let us recall the definition of the coefficients \( A^v_{ka,lb} \). First, we need to define a map

\[
\Delta_v : H(Y_v; \mathbb{C})[z] \to H(Y_v; \mathbb{C}) \otimes H(Y_{I(v)}; \mathbb{C})[z_1, z_2],
\]

which is a \( H(Y_v; \mathbb{C})[z] \)-modules morphism, where the ring structure on \( H(Y_v; \mathbb{C})[z] \) is the obvious one induced by the topological cup product, \( z \) acts on the tensor product via multiplication by \( z_1 + z_2 \) and \( \phi \in H(Y_v; \mathbb{C}) \) acts by cup-product multiplication on the first tensor factor, i.e., via the operator \( \phi \cup \otimes \mathbb{1} \mathbb{1} \). In order to define \( \Delta_v \) then we need only to specify the image of 1: we put

\[
\Delta_v(1) := \sum_{a,b=1}^{N_v} \eta^{ab} \phi_{v,a} \otimes \phi_{I(v),b},
\]

where

\[
\eta_{ab} := \frac{1}{r(v)} \int_{[Y_v]} \phi_{v,a} \cup I^*(\phi_{I(v),b})
\]

are the entries of the matrix of the orbifold Poincaré pairing and \( \eta^{ab} \) are the entries of the corresponding inverse matrix. Here \( r(v) \) denotes the order of the local automorphism \( g \) from a point \((y,g) \in Y_v\) and \([Y_v]\) is the fundamental cycle of the coarse moduli space \([Y_v]\). Let us identify
$H(Y_v; \mathbb{C})[z] = H(Y_v \times \mathbb{C}P^\infty; \mathbb{C})$, so that $z$ is the 1st Chern class of the universal line bundle $\ell := \mathcal{O}(1) \to \mathbb{C}P^\infty$. Then the coefficients of the above differential operator are defined by

$$
\sum_{k,l=0}^N \sum_{a,b=1} A^v_{ka,lb} \phi_{v,a} \otimes \phi_{I(v),b} z_1^{k_1} z_2^{l_2} := \Delta_v \left( \prod_{\gamma=1}^{k_v} C_{v,\gamma} (q^+ (E_{v,\gamma}) |_{Y_v} \otimes (1 - \ell)) - 1 \right),
$$

where $q : Y \to Y$ is the forgetful map $(y,g) \mapsto y$.

Let us continue with the B-twist. The relation in this case is easier.

**Proposition 2.2.2 ([Ton14]).** The B-twist is removed from the total descendant potential $D^{AB}$ according to the following identity:

$$D^{AB} (hc_B^2, tc_B) = \text{const}_B \ D^A (h, t(z) + z - \delta_B(z) z),$$

where the constants $c_B, \text{const}_B \in \mathbb{C}$ and the vector $\delta_B(z) \in H(I(Y; \mathbb{C})[z] \text{ depend only on the twisting data of type B.} \quad \Box$

Let us recall the definition of $c_B$ and $\delta_B(z) \in H(I(Y; \mathbb{C})[z]$. Let us identify again $H(I(Y; \mathbb{C})[z] = H(I(Y \times \mathbb{C}P^\infty; \mathbb{C})$ so that $z = c_1(\ell)$. Then

$$\prod_{\beta=1}^{k_B} B_\beta \left( - \frac{f_\beta(\ell) - f_\beta(1)}{\ell - 1} \right) =: c_B \delta_B(z),$$

where $c_B \in \mathbb{C}$ and $\delta_B(z) = 1 + \cdots$, where the dots stand for cohomology classes in $H(I(Y; \mathbb{C})[z]$ of degree > 0.

**Remark 2.2.4.** Formula (2.2.3) is slightly different from [Ton14], Theorem 1.2. Tonita did not say explicitly, but from the proof it becomes clear that he works with multiplicative characteristic classes $B_\beta$ for which $s_B^{B,0} = 0$. If $s_B^{B,0} = 0$, then $c_B = 1$ and formula (2.2.3) coincides with Tonita’s. \quad \Box

Finally, let us recall how to remove the A-twist. Let $E_{\alpha,v} := q^+ E_{\alpha} |_{Y_v}$ and $E_{\alpha,v,f} 0 \leq f < 1$ be the orbibundle on $Y_v$ whose fiber at a point $(y,g) \in Y_v$ consists of all $\xi \in (E_{\alpha,v})_{(y,g)}$, such that, $g(\xi) = e^{2\pi i f} \xi$. We have $E_{\alpha,v} = \bigoplus_f E_{\alpha,v,f}$. Let us recall Tseng’s operator:

**Definition 2.2.5 ([Tse10]).** Tseng’s operator $\Delta_A(z)$ is the cohomological operator defined by

$$\Delta_A(z) := \prod_{\alpha=1}^{k_A} \sqrt{A_{\alpha}(E^\text{inv}_{\alpha})} \exp \left( \sum_{i=0}^{\infty} \sum_{j=1}^{v,f} s_{\alpha,i+j} A_{\alpha,\gamma} \chi_i (E_{\alpha,v,f}) \frac{B_{j+1} (f) z^j}{(j+1)!} \right),$$

where $E^\text{inv}_{\alpha} = \sum_{v=1}^m E_{\alpha,v,0} \in K^0 (I(Y))$ and $B_j(t)$ are the Bernoulli polynomials defined by the following identify:

$$\frac{xe^{tx}}{e^x - 1} = \sum_{j=0}^{\infty} B_j(t) \frac{x^j}{j!}.$$

It turns out that $\Delta_A(z)$ is a symplectic transformation with respect to Givental’s symplectic loop space formalism (more precisely its orbifold version). The quantization $\hat{\Delta}_A$ yields a differential operator acting on the Fock space

$$\mathbb{C}(\hat{h})[[s^A, q_0, q_1 + 1, q_2, \ldots]],$$
where $s^A = (s^A_{\alpha,i})$ and $q_k = (q_{k,v,a})$ ($v \in \{1, m\}, 1 \leq a \leq N_v$) are formal vector variables. The components of $q_k$ should be thought of as linear coordinates on $H(IY; \mathbb{C})$ with respect to the basis $\{\phi_{v,a}\}$. The shift in $q_1 + 1$, also known as the dilaton shift, is only in the variable $q_{1,1,1}$ corresponding to the unit $\phi_{1,1,1} = 1 \in H([Y]; \mathbb{C})$. Both $D^A(h, t)$ and $D(h, t)$ are identified with elements in the Fock space via respectively the substitutions

$$q(z) = (t(z) - z) \prod_{\alpha=1}^{k_A} A_{\alpha}(E^{inv}_{\alpha})$$

and $q(z) = t(z) - z$. Then,

**Proposition 2.2.6 ([Ton14]).** The $A$-twist is removed from the total descendant potential $D^A$ according to the following identity:

$$D^A(h, q) = \text{const}_A \tilde{\Delta}_A D(h, q),$$

where $\text{const}_A$ is a constant depending only on the twisting data of type $A$.

### 2.3. Fake $K$-theoretic GW invariants

Suppose now that $X$ is a smooth projective variety. By definition, the tangent bundle $T_{g,n,d}$ of the moduli space $X_{g,n,d}$ is a $K$-theoretic vector bundle defined by

$$T_{g,n,d} := \pi_* \text{ev}^*_{n+1}(T_X - 1) - \pi_*(L_{n+1} - 1) - (\pi_*\iota_* O_Z)^\vee,$$

where the notation is the same as in Section 2.1 for the case $Y = X$. The fake $K$-theoretic GW invariants are defined as if applying Hirzebruch–Riemann–Roch (HRR) formula to $X_{g,n,d}$ (being a stack, the usual HRR formula does not hold).

**Definition 2.3.1.** The fake Gromov–Witten invariants of $X$ are defined by the formula

$$\langle \Phi_{i_1} L^{k_1}, \ldots, \Phi_{i_n} L^{k_n} \rangle_{\text{fake}}^{g,n,d} := \int_{[X_{g,n,d}]_{\text{virt}}} \text{td}(T_{g,n,d}) \prod_{s=1}^n \text{ch}(\text{ev}^*_{s}(\Phi_{i_s}) \otimes L^{k_s}_s) \in \mathbb{Q},$$

where the Todd class of a vector bundle is defined by

$$\text{td}(E) = \prod_{x \text{Chern root of } E} \frac{x}{1 - e^{-x}}.$$

Note that the fake $K$-theoretic invariants are ABC-twisted invariants with the following twisting data:

(A) Vector bundle $E := T_X - 1$ and corresponding multiplicative characteristic class

$$A(\ell) := \text{td}(\ell) = \frac{z}{1 - e^{-z}}.$$

(B) Polynomial $f(\ell) = \ell$ and corresponding multiplicative characteristic class

$$B(\ell) := \text{td}(-\ell) = \frac{1}{\text{td}(\ell)} = \frac{1 - e^{-z}}{z}.$$

(C) Trivial vector bundle and corresponding characteristic class

$$C(\ell) := \text{td}(-\ell)^\vee = \frac{1}{\text{td}(\ell)^\vee} = \frac{e^z - 1}{z}. $$
Note that in $H^*(X_{g,n,d})$, we can write twisting class as $\Theta_{g,n,d}^{ABC} = 1 + \cdots$, where the dots contain only cohomology classes on $X_{g,n,d}$ of degree $> 0$. In particular, if we have an $ABC$-twisted correlator as in Definition 2.1.2, such that, the cohomological degree of the correlator insertions add up to the dimension of the virtual fundamental cycle $[X_{g,n,d}]^\text{virt}$, then the higher degree terms of $\Theta_{g,n,d}^{ABC}$ do not contribute and hence the $ABC$-twisted invariant coincides with the usual GW invariant.

**Definition 2.3.2.** The fake $K$-theoretic $J$-function is by definition

$$J\text{fake}(q,Q,\tau) = 1 - q + \tau + \sum_{d,a,i} \frac{1}{\ell!} \left( \frac{\Phi_a}{1 - qL}, \tau, \ldots, \tau \right) \text{fake} Q^d \Phi^a \in K(X)(q)[[Q]],$$

where $\tau \in K^0(X)$ and the first insertion in the correlator should be expanded using $\frac{1}{1 - qL} = \sum_{l \geq 0} q^k L^k$.

### 2.4. Stem invariants.

Let $X$ be a smooth projective variety and $m > 1$ an integer. Let us fix a primitive $m$-th root of unity $\zeta$ and denote by $\eta = e^{2\pi i/m}$ the standard generator of the multiplicative group $\mu_m$. Following Givental and Tonita (see [GT14]) we will refer to the moduli space $[X/\mu_m]_{(\eta,1,\ldots,1,\eta^{-1})}$ as the moduli space of stems. The inertia orbifold of $Y := [X/\mu_m]$ consists of $m$ copies of $Y$, which correspond naturally to the elements $g$ of the group $\mu_m$. Let $1_g$ be the unit in $H([Y_g];\mathbb{C})$. Let us fix a basis $\{\phi_i\} (1 \leq i \leq N)$ of $H(X;\mathbb{C})$ and denote by $\phi_i 1_g$ the cohomology class $\phi_i$ but viewed as an element in $H([Y_g];\mathbb{C}) = H(X;\mathbb{C})$. Clearly, $\{\phi_i 1_g\} (1 \leq i \leq N, g \in \mu_m)$ is a basis of the orbifold cohomology $H([Y];\mathbb{C})$. Note that if $\phi_i \in H^{2d_i}(X;\mathbb{C})$ and $g = e^{2\pi i k/m}$, then the Chen–Ruan degree of $\phi_i 1_g$ is $d_i$, because the action of $g$ on the tangent bundle is trivial. We will be interested in the $ABC$-twisted GW invariants of $[X/\mu_m]$ with the following twisting data:

**Type A:** Vector bundles $E_{\alpha} = T_X \otimes \mathbb{C}_{\zeta^{-\alpha}}$ $(1 \leq \alpha \leq m)$, where $\mathbb{C}_{\zeta^k}$ is the orbibundle $[(X \times \mathbb{C})/\mu_m]$, where the action of $\mu_m$ on $\mathbb{C}$ is defined by requiring that the standard generator $\eta = e^{2\pi i/m}$ of $\mu_m$ acts by $\zeta^k$. The corresponding multiplicative classes are

$$A_1(\ell) := \text{td}(\ell) = \frac{z}{1 - e^{-z}},$$

$$A_{k+1}(\ell) := \text{td}_{\zeta^k}(\ell) = \frac{1}{1 - \zeta^k e^{-z}} \quad (1 \leq k \leq m - 1).$$

**Type B:** Polynomials $f_\beta(\ell) = \mathbb{C}_{\zeta^{1-\beta}} \ell \in K^0(Y)[\ell] (1 \leq \beta \leq m)$, where $\mathbb{C}_{\zeta^k}$ are the same as in the type A data above. The corresponding characteristic classes are

$$B_1(\ell) := \text{td}(-\ell) = \frac{1 - e^{-z}}{z},$$

$$B_{k+1}(\ell) := \text{td}_{\zeta^k}(-\ell) = 1 - \zeta^k e^{-z} \quad (1 \leq k \leq m - 1).$$

**Type C:** If $g \neq 1$, then we have only one orbibundle, that is, $k_g := 1$ and $E_{g,1}$ is the trivial line bundle on $Y$. If $g = 1$, then we have $m$ orbifold line bundles, that is, $k_1 = m$ and $E_{1,\gamma} := \mathbb{C}_{\zeta^{-1}}$
(1 ≤ γ ≤ m). The corresponding characteristic classes are

\[ C_{g,1}(\ell) := \text{td}(-\ell)^v = \frac{e^z - 1}{z}, \quad g \neq 1 \]
\[ C_{1,1}(\ell) := \text{td}(-\ell)^v = \frac{e^z - 1}{z}, \]
\[ C_{1,k+1}(\ell) := \text{td}_{\zeta k}(-\ell)^v = 1 - \zeta^k e^z \quad (1 \leq k \leq m - 1). \]

In other words,

\[ (2.4.1) \quad \Theta^A = \text{td}(\pi_* \ev^* (T_X - 1)) \prod_{k=1}^{m-1} \text{td}_{\zeta k}(\pi_* \ev^* (T_X \otimes \mathbb{C}_{\zeta k})), \]
\[ (2.4.2) \quad \Theta^B = \text{td}(-\pi_* (L - 1)) \prod_{k=1}^{m-1} \text{td}_{\zeta k}(-\pi_* ((L - 1) \otimes \ev^* \mathbb{C}_{\zeta k})), \]
\[ (2.4.3) \quad \Theta^C = \text{td}(-\pi_* \iota_* O_{Z_y})^v \text{td}(-\pi_* \iota_* O_{Z_1})^v \prod_{k=1}^{m-1} \text{td}_{\zeta k}(-\pi_* (\iota_* O_{Z_1} \otimes \ev^* \mathbb{C}_{\zeta k}))^v. \]

Following Givental–Tonita we define the stem invariants of \( X \) as an ABC-twisted GW invariant of \([X/\mu_m]\):

**Definition 2.4.4 ([GT14])**. The stems invariants of \( X \) are defined by the following:

\[ (\phi_{i_1} \psi^{k_1}_{1,1}, \ldots, \phi_{i_{n+2}} \psi^{k_{n+2}}_{n+2})|_{0,n+2,d} := (\phi_{i_1} \psi^{k_1}_{1,1}, \phi_{i_2} \psi^{k_2}_{2,2}, \ldots, \phi_{i_{n+1}} \psi^{k_{n+1}}_{n+1,1}, \phi_{i_{n+2}} \psi^{k_{n+2}}_{n+2,1})_{0,n+2,d}. \]

### 3. Confluence of the small K-theoretic J-function

The main goal in this section is to prove confluence of the K-theoretic J-function to its cohomological analogue (Theorem 3.4.1). As we already explained in the introduction, we need to recall the reconstruction of Givental–Tonita outlined in [GT14], Proposition 4. For the sake of completeness the proof of the recursion will be outlined in Appendix A.

#### 3.1. The Kawasaki–Riemann–Roch formula

To begin with, let us recall Kawasaki’s formula generalising the Hirzebruch–Riemann–Roch formula to orbifolds (see [Kaw79]). Let \( Y \) be as in Section 2, \( Y = (\mathcal{Y}_1 \rightrightarrows \mathcal{Y}_0) \) be a corresponding orbifold groupoid, and \( q : IY \to Y \) be the forgetful map \((y, g) \mapsto y\). If \( E \to Y \) is an orbifold complex vector bundle, we use the decompositions \( q^* E|_{\mathcal{Y}_0} =: \bigoplus_{0 \leq f < 1} E_{v,f} \) same as in the definition of Tseng’s operator, c.f. Definition 2.2.5.

**Definition 3.1.1.**

(i) Let \( E \to Y \) be an orbifold complex vector bundle. The **trace** of \( E \) is the K-theoretic orbifold vector bundle on \( IY \) defined by

\[ \text{Tr}(E) := \sum_{v=1}^{m} \sum_{0 \leq f < 1} e^{2\pi i f} E_{v,f} \]

(ii) The **inertia tangent** and **inertia normal** vector bundles are orbifold K-theoretic vector bundles on \( IY \) defined by respectively

\[ T_{IY} := \sum_{v=1}^{m} TY_{v,0} \quad \text{and} \quad N_{IY} := \sum_{v=1}^{m} \sum_{0 \leq f < 1} TY_{v,f} \].
(iii) The holomorphic Euler characteristic of the bundle $E$ is given by

$$
\chi(Y, E) := \sum_{i=0}^{D} (-1)^i \dim H^i(Y, E)
$$

Kawasaki’s formula can be stated as follows:

**Theorem 3.1.2.** [Kaw79] Let $E$ be a holomorphic orbifold vector bundle on $Y$. Its holomorphic Euler characteristic can be computed by the following formula:

$$
\chi(Y, E) = \int_{[Y]} \text{td}(T_Y) \frac{\text{ch} \circ \text{Tr}(E)}{\text{ch} \circ \text{Tr}(\varpi^*(N_{Y/X}^*))},
$$

where the denominator on the RHS is by definition

$$
\text{ch} \circ \text{Tr}(\varpi^*(N_{Y/X}^*)) = \sum_{v=1}^{m} \prod_{i=1}^{f} \prod_{x \in \text{chern roots}} (1 - e^{-2\pi i f e^{-x}}).
$$

\[\square\]

### 3.2. Stems as Kawasaki strata.

Recall that in Definition 1.1.3 of the $K$-theoretic $J$-function, we used $K$-theoretic GW invariants of the form $\left\{\frac{\Phi}{1-\eta L}\right\}_{0,1,d}$, which are understood as holomorphic Euler characteristic of bundles on the moduli space $X_{0,1,d}$. Let us apply Kawasaki’s formula (3.1.3) to this moduli space. It is convenient to think of the inertia orbifold $IX_{0,n,d}$ as the moduli space of $(C, s_1, \ldots, s_n, f, g)$, where $(C, s_1, \ldots, s_n, f)$ is a stable map in $X_{0,n,d}$ and $g \in \text{Aut}(C, s_1, \ldots, s_n, f)$. Suppose that $\zeta$ is a primitive $m$-th root of 1. Let us define

$$
I_\zeta X_{0,n,d} := \{(C, s_1, \ldots, s_n, f, g) \mid dg \text{ acts on } T^*_\zeta C \text{ via multipl. by } \zeta\}.
$$

We are going to construct an explicit isomorphism $\varphi$ of Deligne–Mumford stacks

$$
\bigcup_{d_0, \ldots, d_k} \left[\left[\frac{X}{\mu_m}\overline{[0,1]}_{d_0} \times \cdots \times \left[\frac{X}{\mu_m}\overline{[0,1]}_{d_k}\right]\right]_{0,k+2,d_0} \times_{\cdots} \right] \xrightarrow{\varphi} I_\zeta X_{0,1,d},
$$

where the disjoint union is over all sequences $d_0, \ldots, d_k \in \text{Eff}(X)$ of effective curve classes and all sequences $\eta_1, \ldots, \eta_k$ of primitive roots of 1, such that, $m(d_0 + \cdots + d_k) = d$ and $\eta_i \neq 1$ for all $i$. The fiber product is defined in terms of the evaluation maps in such a way that the $(i+1)$-st marked point in $\left[\frac{X}{\mu_m}\overline{[0,1]}_{d_0} \times \cdots \right]$ is identified with the marked point of $I_{\eta_i}X_{0,1,d_i}$.  

The isomorphism $\varphi$ is defined as follows (see Figure 1). Suppose that $C_0 = (C_0, s_1^0, \ldots, s_k^0, f_0)$ is an orbifold stable map in $\left[\frac{X}{\mu_m}\overline{[0,1]}_{d_0}\right]$ and that $C_i = (C_i, s_i^1, f_i, g_i) \in I_{\eta_i} X_{0,1,d_i}$, $(1 \leq i \leq k)$ are such that $f_0(s_i^0) = f_i(s_i^1)$ for all $1 \leq i \leq k$. The orbifold $C_0$ is a global quotient, that is, $C_0 = [\overline{C_0}/\mu_m]$ and an orbifold stable map $f_0: C_0 \to X$ is equivalent to the choice of a $\mu_m$-equivariant map $\overline{f_0}: \overline{C_0} \to X$. The marked points $s_i^0$ can be represented by the gerbes

$$
\begin{align*}
& s_i^0 = \left[\{s_i\}/\mu_m\right], \\
& s_{i+1}^0 = \left[\{s_i, \ldots, s_i, m\}/\mu_m\right] \quad (1 \leq i \leq k), \\
& s_{k+2}^0 = \left[\{s_k+2\}/\mu_m\right].
\end{align*}
$$
where $s_1$ and $s_{k+2}$ are the $\mu_m$-fixed non-singular points of $\overline{C}_0$ and \{s_{i,1},\ldots,s_{i,m}\} is a regular $\mu_m$-orbit of non-singular points. Note that there is a freedom in choosing the $\mu_m$-action on $\overline{C}_0$: if we pick an isomorphism $\phi: \mu_m \to \mu_m$ and define a new action by $(g,z) \mapsto \phi(g) \cdot z$, then the quotient orbifold $[\overline{C}_0/\mu_m]$ is equivalent to the original one. We eliminate the freedom by requiring that the standard generator $\eta$ of $\mu_m$ acts on the cotangent line $T^*_s \overline{C}_0$ by multiplication by $\zeta$, that is, by the given primitive $m$-th root of 1. Let us define a nodal curve

\begin{equation}
C = \overline{C}_0 \bigsqcup \bigcup_{i=1}^{k} C_i \times \{1, m\} / \sim,
\end{equation}

where the equivalence relation is such that the marked point $s_{i,a} \in \overline{C}_0$ is identified with $(s_1^i, a) \in C_i \times \{a\}$. Let us define an automorphism $g$ of $C$. To begin with, let us denote by $g_0 \in \text{Aut}(\overline{C}_0)$ the automorphism corresponding to the action of the generator $\eta \in \mu_m$. Put

\begin{equation}
g(z) := g_0(z) \quad \text{for } z \in \overline{C}_0
\end{equation}

\begin{equation}
g(z,a) := \begin{cases} (z, a + 1) & \text{if } z \in C_i \text{ and } 1 \leq a \leq m - 1, \\ (g_i(z), 1) & \text{if } z \in C_i \text{ and } a = m. \end{cases}
\end{equation}

and

\begin{align*}
f(z) &:= f_0(z) \quad \text{for } z \in \overline{C}_0, \\
f(z,a) &:= f_i(z) \quad \text{if } z \in C_i \text{ and } 1 \leq a \leq m.
\end{align*}

Clearly, $\mathcal{C} := (C, s_1, f, g)$ is a stable map representing a point in $I_{\zeta}X_{0,1,d}$.
Proposition 3.2.5 ([GT14]). The map
\[
\varphi: \prod_{d_0, \ldots, d_k} ([X/\mu_m]_{d_0, k+2, d_0}^{1, \ldots, 1, \eta_i^{-1}} \times \mathbb{X} \left( I_{g_1} X_{0,1,d_1} \times \cdots \times I_{g_k} X_{0,1,d_k} \right)) \to I_\zeta X_{0,1,d}
\]
we constructed (see Equation (3.2.1)) is an isomorphism of Deligne–Mumford stacks. \qed

Constructing the map in the inverse direction is in some sense the starting point in Givental and Tonita’s work (see [GT14], Section 7). Let us recall their idea.

Proof. Given a point \( C := (C, s_1, f, g) \) in \( I_\zeta X_{0,1,d} \), the marked point \( s_1 \) on \( C \) is called the horn. Note that \( g^m \) is an automorphism of \( C \) acting trivially on the irreducible component of \( C \) that carries the horn. Let \( \bar{C}_0 \) be the maximal connected subcurve of \( C \) that carries the horn and on which \( g^m \) acts trivially. The automorphism \( g \) acts on \( \bar{C}_0 \) with exactly two fixed non-singular points: the horn \( s_1 \) and one more point called the butt which we denote by \( s_{k+2} \). The number \( k \) here is defined as follows. Removing \( \bar{C}_0 \) from \( C \) we obtain a curve consisting of several connected components called legs. The action of the automorphism \( g \) splits the set of points on \( \bar{C}_0 \) at which the legs are attached, into \( k \) groups each consisting of \( m \) points \( s_{i,1}, \ldots, s_{i,m} \) forming an orbit of the cyclic group \( \langle g \rangle \). Clearly \( g \) defines an isomorphism between the legs attached to the points in the same orbit. Therefore, we may assume that these legs are copies of the same curve \( C_i \), that is, we denote the leg attached to \( s_{i,a} \) by \( C_i \times \{ a \} \). The point on the leg identified with \( s_{i,a} \) has the form \( (s_i^1, a) \), for some point \( s_i^1 \in C_i \). Furthermore, \( g^m \) induces an automorphism \( g_i \) of \( C_i \) fixing the point \( s_i^1 \in C_i \). The differential \( d g_i \) acts on \( T_{s_i^1} C_i \) with some eigenvalue \( \eta_i \neq 1 \), otherwise if \( \eta_i = 1 \), then \( g_i \) will act trivially on the irreducible component of \( C_i \) that carries \( s_i^1 \), so the domain of \( \bar{C}_0 \) can be extended. Clearly, \( (C_i, s_i^1, f|_{C_i}, g_i) \) is a point in \( I_{g_i} X_{0,1,d_i} \) for some \( d_i \) and \( f|_{\bar{C}_0} \) induces an orbifold stable map on the orbifold curve \( C_0 := \bar{C}_0/\langle g \rangle \). Therefore, this data defines a point on the fiber product. We get a map in the inverse direction of (3.2.1) which is the inverse that we were looking for. The above constructions are functorial and they can be done in families, so (3.2.1) is an isomorphism of Deligne–Mumford stacks. Let us point out that the marked point \( s_{k+2} \in \bar{C}_0 \) is downgraded to a regular point on \( C \). \qed

3.3. Reconstruction of the small $K$-theoretic $J$-function. Let us use Kawasaki’s formula (3.1.3) to express the small $K$-theoretic $J$-function as an integral over the inertia stacks \( I X_{0,1,d} = \bigsqcup_{m \geq 1} \bigsqcup_\zeta I_\zeta X_{0,1,d} \), where the 2nd disjoint union is over all primitive $m$-th roots of 1. We get
\[
J(q,Q) = 1 - q + \tau(q,Q) + \sum_{d,a} Q^d \Phi_a \int_{I_1 X_{0,1,d}} \text{td}(T_{I_1 X_{0,1,d}}) \frac{1}{\text{ch} \circ \text{Tr}(\wedge^* (N_{I_1 X_{0,1,d}}^\vee))} \frac{\text{ev}_1^* (\text{ch}(\Phi^a))}{1 - q e^\psi_1},
\]
where \( \tau(q,Q) = \sum_{m \geq 1} \sum_\zeta \tau_\zeta(q,Q) \) and
\[
(3.3.1) \quad \tau_\zeta(q,Q) := \sum_{d,a} Q^d \Phi_a \int_{I_\zeta X_{0,1,d}} \text{td}(T_{I_\zeta X_{0,1,d}}) \frac{1}{\text{ch} \circ \text{Tr}(\wedge^* (N_{I_\zeta X_{0,1,d}}^\vee))} \text{ch} \circ \text{Tr} \left( \frac{\text{ev}_1^* (\Phi^a)}{1 - q L_1} \right).
\]
According to Givental–Tonita (see [GT14], Proposition 1), the integral over $[I_1X_{0,1,d}]$ can be expressed in terms of fake $K$-theoretic invariants of $X$, that is,

**Theorem 3.3.2** ([GT14], Proposition 1). The $K$-theoretic small $J$-function can be expressed in terms of the fake $K$-theoretic $J$-function as follows:

\[
J(q, Q) = 1 - q + \tau + \sum_{d,a=0}^{\infty} \frac{Q^d \Phi_a}{d!} \left( \frac{\Phi_a}{1 - q L_1}, \tau, \ldots, \tau \right)_{0,1+d,0},
\]

where $\tau$ is defined by equation (3.3.1).

This follows from the isomorphism that we have constructed in Proposition 3.2.5 for the case $m = 1$ and $\zeta = 1$ after analyzing how the inertia tangent and the inertia normal bundles decompose under the isomorphism.

The remaining problem is to compute $\tau(q, Q)$. The answer is given by the following relation.

**Theorem 3.3.4** ([GT14], Propositions 3 and 4). The contribution from the other integrals $\tau(q, Q)$, defined in Equation (3.3.1), is computed by the formula below.

\[
\tau_q(q, Q) = \sum_{d_0,a} \sum_{k=0}^{\infty} \frac{Q^{md_0} \Phi_a}{k!} \times \\
\left[ \frac{\text{ch}(\Phi_a)}{1 - q \zeta e^{\psi_1/m}}, \tau^{(m)}(\psi_2, Q), \ldots, \tau^{(m)}(\psi_{k+1}, Q), 1 - \zeta^{-1} e^{\psi_{k+2}/m} \right]_{0, k+2, d_0},
\]

where $[\cdots]_{0, k+2, d_0}$ is the stem correlator of Definition 2.4.4 and $\tau^{(m)}(z, Q) \in H(X; \mathbb{C})[[z, Q]]$ is obtained from $\tau(q, Q)$ via the Chern character map and the Adam’s operations. To define the action Adam’s operations on $\tau$, put

\[
\tau(q, Q) := \sum_{d,a} \tau_{d,a}(q)Q^d \Phi_a,
\]

where $\tau_{d,a}(q)$ is a rational function in $q$. Then

\[
\tau^{(m)}(z, Q) := \sum_{d,a} \tau_{d,a}(q)Q^{md} \text{ch}(\Psi^m(\Phi_a)),
\]

where $\Psi^m : K^0(X) \to K^0(X)$ are the Adam’s operation, that is, ring homomorphisms which on line bundles $\ell$ are defined to be $\Psi^m(\ell) = \ell^m$.

The reason why the above relation is a recursion follows immediately from the observation that if we put a lexicographical order on $d := (d_1, \ldots, d_r) := (p_1, d), \ldots, (p_r, d)$ and compare the coefficients in front of $Q^d = Q_1^{d_1} \cdots Q_r^{d_r}$, then the RHS will involve only the components $\tau_{d',a'}(q)$ of $\tau(q, Q)$ for which $d' < d$. A sketch of proof of this theorem will be done in Appendix A.

### 3.4. Confluence of the small $K$-theoretic $J$-function.

We now have the tools to tackle the confluence of the small $K$-theoretic $J$-function to its cohomological analogue. Our goal is to prove the following theorem.
Theorem 3.4.1. If $X$ is a smooth projective variety, such that, the anti-canonical bundle $K_X$ is nef, then the limit
\[
\lim_{q \to 1} (q - 1)^{\deg - 1} \text{ch} \left( J(q, (q - 1)^{m_1} Q_1, \ldots, (q - 1)^{m_r} Q_r) \right)
\]
exists and it coincides with $J^{\text{coh}}(1, Q)$.

It is natural to ask whether the $q$-difference system satisfied by the $K$-theoretic $J$-function has also a similar limit when $q \to 1$. We expect that the techniques from our proof of Theorem 3.4.1 can be used to prove that if we pullback the variables $Q_1, \ldots, Q_r$ to $(q - 1)^{m_1} Q_1, \ldots, (q - 1)^{m_r} Q_r$, then the formal limit as $q \to 1$ of the resulting system of $q$-difference equations exists and it coincides with the system of differential equations satisfied by the small cohomological $J$-function, evaluated at $z = 1$.

Note that using Equation (1.1.4), we can recover the cohomological $J$-function for all $z$. We may assume that $\phi_i = \text{ch}(\Phi_i)$ is a homogeneous cohomology class. Let us denote the complex degree (i.e., half of the standard cohomology degree) of $\phi_i$ by $|\phi_i|$. First, we will prove the following lemma.

Lemma 3.4.2. The limit
\[
\lim_{q \to 1} (q - 1)^{\deg - 1} \text{ch} \left( \tau(q, (q - 1)^{m_1} Q_1, \ldots, (q - 1)^{m_r} Q_r) \right)
\]
exists and it is 0. In other words, using the decomposition $\tau(q, Q) = \sum_{d, a} \tau_{d, a}(q) Q^d \Phi_a$ of Equation (3.3.6), we claim that
\[
\tau_{d, a}(q) = 0 \quad \forall d \text{ and } a, \text{ s.t., } |\phi_a| - 1 + \int_d c_1(T_X) \leq 0.
\]

Proof. We argue by induction on the lexicographical order of $d$. Note that for $d = 0$, $\tau_{0, a} = 0$ for all $a$, because $\tau$ would be a sum of integrals over the virtual fundamental cycle of $I_\zeta X_{0,1,0}$, but due to the stability conditions the moduli spaces $X_{0,1,0}$ are empty. Let us compare the coefficients in front of $Q^d \phi_a$ in (3.3.5). Let us expand the $i$-th insertion involving $\tau^{(m)}$ on the RHS as
\[
\sum_{d_i, a_i} \tau_{d_i, a_i}(e^{m \psi_{i+1}}) Q^{m d_i} \phi_{a_i} m^{|\phi_{a_i}|}.
\]
We must have $d = m(d_0 + d_1 + \cdots + d_k)$. In particular, since $m > 1$, we have $d_i < d$ for all $1 \leq i \leq k$ and hence we may recall the inductive assumption, that is,
\[
|\phi_{a_i}| - 1 + \int_{d_i} c_1(T_X) > 0, \quad 1 \leq i \leq k.
\]
On the other hand, the stem correlator is defined through integration along the virtual fundamental cycle of $[X/\mu_m]^{(\eta_1, \ldots, \eta_k)}$ which has complex dimension
\[
3 \cdot 0 - 3 + k + 2 + D + \int_{d_0} c_1(X).
\]
While the total degree of the cohomology classes inside the correlator is at least
\[
|\phi^a| + \sum_{i=1}^k |\phi_{a_i}| > D - |\phi_a| + k - \sum_{i=1}^k \int_{d_i} c_1(X).
\]
Comparing with the dimension of the virtual fundamental cycle we get

\[ |\phi_a| - 1 + \frac{1}{m} \int_d c_1(X) > 0. \]

Since \(m > 1\) and \(K_X^\vee\) is nef we have \(\int_d c_1(X) \geq \frac{1}{m} \int_d c_1(X)\). Therefore, the above inequality implies that if \(\tau_{d,a} \neq 0\), then \(|\phi_a| - 1 + \int_d c_1(X) > 0\). This is exactly what we had to prove. \(\square\)

**Proof of Theorem 3.4.1.** Let us first prove that the limit in Theorem 3.4.1 exists. In order to do this, let us recall formula (3.3.3). The first three terms, that is, \(1 - q + \tau\) will contribute to the limit \(-1\). Let us expand the \(i\)-th insertion of \(\tau\) as \(\tau = \sum_{d_i, a_i} \tau_{d_i, a_i}(L_i)Q^{d_i} \Phi_{a_i}\) and express the fake \(K\)-theoretic correlator as a twisted GW invariant. We get a sum of terms of the form

\[
\frac{1}{l!} \Phi_a Q^{d+d_1+\ldots+d_l} \left[ \frac{ev_1^+(\phi^a/td(X))}{1-qe^{\psi_1}}, \tau_{d_1, a_1}(e^{\psi_2})\phi_{a_1}, \ldots, \tau_{d_l, a_l}(e^{\psi_l})\phi_{a_l} \right]_{ABC}^{1+1+l,d}.
\]

Note also that the first insertion in the above correlator has an expansion at \(q = 1\) of the form:

\[
\sum_{k=0}^{\infty} (1-q)^{-k-1}(\phi^a + \ldots)(q^k\psi^k + \ldots),
\]

where the dots stand for cohomology classes of higher degrees. Therefore, we have to prove that if the above correlator is not 0, then \(|\phi_a| - 2 - k + \int_d c_1(X) + \sum_{i=1}^{l} \int_{d_i} c_1(X) \geq 0\). According to the claim that we already proved, if \(\tau_{d_i, a_i} \neq 0\), then \(|\phi_{a_i}| - 1 + \int_{d_i} c_1(X) > 0\), that is,

\[
2 - \int_{d_i} c_1(X) \leq |\phi_{a_i}|.
\]

Note that the above twisted correlator is defined by integrating along the virtual fundamental cycle of \(X_{0,1+l,d}\) which has dimension \(l - 2 + D + \int_d c_1(X)\). Therefore, by comparing the degree of the cohomology classes in the correlator, we get the inequality

\[
D - |\phi_a| + k + \sum_{i=1}^{l} |\phi_{a_i}| \leq l - 2 + D + \int_d c_1(X).
\]

Recalling our estimate (3.4.4), we get

\[
l \leq |\phi_a| - 2 - k + \int_d c_1(X) + \sum_{i=1}^{l} \int_{d_i} c_1(X).
\]

This proves that the limit exists. In order to compute the limit, we need to single out only the terms for which the RHS of the above equality is 0. But this would be the case only if \(l = 0\), the dots (i.e. the higher degree terms) in (3.4.3) and the higher degree terms of the ABC-twisting class \(\Theta_{ABC}^{0,1,d} = 1 + \ldots\) are disregarded. Clearly the limit becomes

\[
-1 + \sum_{d,a} \sum_{k=0}^{\infty} Q^d \phi_a \{\phi^a \psi^k\}_{0,1,d} (-1)^{k+1} = J^{\text{coh}}(1, Q).
\]

\(\square\)
4. Oscillatory integrals in Gromov–Witten theories

In this Section, we will revisit Givental’s proposal for mirror symmetry in toric geometry. More precisely, if $X$ is a Fano toric manifold of Picard rank 2, then we would like to compare the following two solutions of the cohomological quantum differential equations and their $K$-theoretic analogues:

(i) The $I$-function, which is a certain hypergeometric-type series associated to the toric manifold (see e.g. [Giv96] in quantum cohomology and [Giv15a] in quantum $K$-theory)

(ii) Givental’s oscillatory integral model in quantum cohomology [Giv98] (see also Section 3 of [Iri09]) and its $q$-analogue in quantum $K$-theory (which will be introduced in Definition 4.3.4).

The main goal in this Section is to provide a comparison of the $K$-theoretic $I$-function and the $q$-oscillatory integral, which will be done in Theorem 4.4.4. We believe that our results can be extended to all compact toric orbifolds. The Picard rank 2 case is a natural case to investigate, because it includes both Fano and non-Fano manifolds. However, the non-Fano case will be pursued elsewhere.

4.1. Symplectic toric manifolds. In this Subsection, we recall some standard definitions from toric geometry, explain some of their geometrical properties and give an explicit construction of all symplectic toric Fano manifolds whose Picard rank is 2. For more details on the subject, we refer to Chapter 7 of [Aud04].

We consider the manifold $X$ to be a symplectic toric manifold, that is a smooth symplectic (or GIT) quotient $X = \mathbb{C}^n/\Gamma^k$ of the $n$-dimensional symplectic vector space by a linear action of a $k$-dimensional torus. We will describe such a toric manifold through its moment map.

**Definition 4.1.1 (Symplectic toric orbifolds).**

(i) Suppose that $\mu : \mathbb{Z}^n \to \mathbb{Z}^r$ is a linear map and let us denote by $\alpha_1, \ldots, \alpha_n \in \mathbb{Z}^r$ the images of the canonical basis of $\mathbb{Z}^n$ by the moment map $\mu$. Let $\text{Mat}(\mu) := (m_{ij})$ be the matrix of $\mu$, that is, the entries in the $i$th column are the coordinates of $\alpha_i$. Slightly abusing the terminology, we will refer to $\mu$ as the moment map. The cone associated to the moment map $\text{Cone}(\mu)$ in $\mathbb{R}^r$ spanned by the elements $\alpha_1, \ldots, \alpha_n \in \mathbb{R}^r$ is defined to be the cone $\text{Cone}(\alpha_1, \ldots, \alpha_n) \subseteq \mathbb{R}^r$.

(ii) We denote by $\text{BInd}(\mu)$ the set

$$\text{BInd}(\mu) := \{ \sigma = (\sigma_1, \ldots, \sigma_r) \in \{1, \ldots, n\} \mid \text{Vect}_\mathbb{Q} (\alpha_{\sigma_1}, \ldots, \alpha_{\sigma_r}) \},$$

where $\text{Vect}_\mathbb{Q} (v_1, \ldots, v_r)$ denotes the vector space spanned over $\mathbb{Q}$ by vectors $v_1, \ldots, v_r$. The singular cone associated to a moment map $\text{Cone}_{\text{sing}}(\mu) \subseteq \mathbb{R}^r$ is the union of boundaries given by

$$\text{Cone}_{\text{sing}}(\mu) = \bigcup_{\sigma \in \text{BInd}(\mu)} \partial \text{Cone}(\alpha_{\sigma_1}, \ldots, \alpha_{\sigma_r}),$$

where $\text{Cone}(v_1, \ldots, v_r) := \mathbb{R}_{\geq 0} v_1 + \cdots + \mathbb{R}_{\geq 0} v_r$ denotes the real cone spanned by $v_1, \ldots, v_r$. A chamber $K \subseteq \mathbb{R}^r$ is a connected component of $\text{Cone}(\mu) - \text{Cone}_{\text{sing}}(\mu)$. 


(iii) Consider the composition of maps

\[ \begin{array}{ccc}
\mathbb{C}^n & \xrightarrow{f} & \mathbb{R}^n \\
& \xrightarrow{\mu} & \mathbb{R}^r \\
\end{array} \]

\[ z_j \mapsto |z_j|^2 \]

Given a moment map $\mu$ and a chamber $K$, we define a symplectic manifold $X_{\mu,K}$ to be the quotient

\[ X_{\mu,K} := f^{-1}(K)/\mathbb{T}^r, \]

where the action of the $r$-dimensional torus $\mathbb{T}^r = (\mathbb{C}^*)^r$ on $\mathbb{C}^n \times f^{-1}(K)$ is given by the matrix of the moment map $\mu$: $(t \cdot z)_j = t_1^{m_{1j}} \cdots t_r^{m_{rj}} z_j$ for $t = (t_1, \ldots, t_r) \in \mathbb{T}^r$ and $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$.

Note that $f^{-1}(K)$ is not $\mathbb{T}^r$-invariant. The smallest $\mathbb{T}^r$-invariant subset of $\mathbb{C}^n$ containing $f^{-1}(K)$ is the following open subset of $\mathbb{C}^n$:

\[ U_K = \bigcup_{I: K \subset \text{Cone}((\alpha_I))} \mathbb{C}^I \times (\mathbb{C}^*)^I, \]

where the union is over all subsets $I = \{i_1, \ldots, i_m\} \subset \{1, 2, \ldots, n\}$, such that, the cone $\text{Cone}(\alpha_I) := \text{Cone}(\alpha_{i_1}, \ldots, \alpha_{i_m})$ contains the chamber $K$, $I$ is the complement of $I$ in $\{1, 2, \ldots, n\}$, and

\[ \mathbb{C}^I := \{ z \in \mathbb{C}^n \mid j \notin I \Rightarrow z_j = 0 \}, \]

\[ (\mathbb{C}^*)^I := \{ z \in \mathbb{C}^n \mid j \notin I \Leftrightarrow z_j = 0 \}. \]

Strictly speaking we should define $X_{\mu,K} = U_K/\mathbb{T}^r$. By definition, $f^{-1}(\eta) \subset U_K$ for all $\eta \in K$. It is a non-trivial result (see [Aud04, Theorem VII.2.1]) that every $\mathbb{T}^r$-orbit in $U_K$ intersects $f^{-1}(\eta)$ along a $(\mathbb{S}^1)^r$-orbit, that is, $X_{\mu,K} = f^{-1}(\eta)/(\mathbb{S}^1)^r$ is a symplectic reduction, where $\mathbb{S}^1 \subset \mathbb{C}$ is the unit circle.

**Proposition 4.1.2.** Consider a moment map $\mu$ and a chamber $K$, and denote the resulting symplectic toric variety by $X_{\mu,K}$.

(i) The quotient $X_{\mu,K}$ is compact if and only if the cone associated to its moment map $\text{Cone}(\mu)$ is contained in a half space of $\mathbb{R}^r$.

(ii) The quotient $X_{\mu,K}$ is smooth if and only if for any $\underline{\sigma} \in \text{Blind}(\mu)$ such that $K \cap \text{Cone}(\alpha_{\sigma_1}, \ldots, \alpha_{\sigma_r}) \neq \emptyset$,

the linear map $\mu$ restricted to $\text{Vect}(\alpha_{\sigma_1}, \ldots, \alpha_{\sigma_r})$ has determinant $\pm 1$.

(iii) The quotient $X_{\mu,K}$ is Fano if and only if for the vector $c_1(TX_{\mu,K}) := \alpha_1 + \cdots + \alpha_n \in \mathbb{R}^r$ is an element of the chamber $K$.

\[ \square \]

**Remark 4.1.3.** A cone $C$ of $\mathbb{R}^r$ is contained in a half space if and only if its dual cone has maximal dimension, i.e. $\dim C^\vee = r$.

Let $P_j = (U_K \times \mathbb{C})/\mathbb{T}^r$ ($1 \leq j \leq r$) be the line bundle on $X_{\mu,K}$, where the action of $\mathbb{T}^r$ on $\mathbb{C}$ is given by the character $\mathbb{T}^r \to \mathbb{C}^*$, $(t_1, \ldots, t_r) \mapsto t_j^{-1}$. Let $p_j := -c_1(P_j)$. 

**Proposition 4.1.4.** Let \( X_{\mu,K} \) be a symplectic toric manifold. Then, the cohomology ring and the topological \( K \)-ring of \( X_{\mu,K} \) are given by

\[
H^*(X_{\mu,K}; \mathbb{Q}) \simeq \mathbb{Q}[p_1, \ldots, p_r]/\left(\prod_{j \in J_\nu} \alpha_j(p)\right),
\]

\[
K(X_{\mu,K}) \simeq \mathbb{Z}[P_1^{\pm 1}, \ldots, P_r^{\pm 1}]/\left(\prod_{j \in J_\nu} (1 - U_j(P))\right),
\]

where \( \alpha_j(p) := m_{1j}p_1 + \cdots + m_{rj}p_r \), \( U_j(P) := P_1^{m_{1j}} \cdots P_r^{m_{rj}} \), and \( J_\nu = (j_{\nu,1}, \ldots, j_{\nu,l}) \subseteq \{1, \ldots, N\} \) are the maximal subsets with respect to inclusion, such that, the cone spanned by \( \alpha_{j_{\nu,1}}, \ldots, \alpha_{j_{\nu,l}} \) does not intersect the chamber \( K \).

**Example 4.1.5.** Let us consider for the manifold \( X = \text{Bl}_p \mathbb{P}^3 \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)) \) the following toric data: we use the moment map given by the matrix

\[
M = \begin{pmatrix}
1 & 1 & 1 & 0 & -1 \\
0 & 0 & 0 & 1 & 1
\end{pmatrix},
\]

and the chamber \( K = (\mathbb{R}_{>0})^2 \). We give below a figure of the toric data.

![Toric Data Diagram](example)

Then, the sets \( J_\nu \) of Proposition 4.1.4 are \{1, 2, 3\} and \{4, 5\}. The cohomology ring is given by

\[
H^*(X_{\mu,K}; \mathbb{Q}) \simeq \mathbb{Q}[p_1, p_2]/\left(p_1^3 = 0, p_2(p_2 - p_1) = 0\right),
\]

and the topological \( K \)-ring by

\[
K(X_{\mu,K}) \simeq \mathbb{Z}[P_1^{\pm 1}, P_2^{\pm 1}]/\left((1 - P_1)^3 = 0, (1 - P_2)(1 - P_1^{-1}P_2) = 0\right).
\]

**Proposition 4.1.6.** Suppose that \( X \) is a compact toric manifold with Picard rank 2. The isomorphism class of \( X \) can be represented by a toric manifold \( X_{\mu,K} \), such that, the chamber \( K = (\mathbb{R}_{>0})^2 \) and the matrix of the moment map \( \mu \) has the following form:

\[
\text{Mat}(\mu) = \begin{pmatrix}
1 & \cdots & 1 & 0 & -a_1 & \cdots & -a_k \\
0 & \cdots & 0 & 1 & 1 & \cdots & 1
\end{pmatrix},
\]

where the first column \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) is being repeated \( N > 0 \) times and \( a_j \in \mathbb{Z}_{\geq 0} \) \((1 \leq j \leq k)\). Furthermore, \( X_{\mu,K} \) is naturally isomorphic to a projectivised vector bundle, that is,

\[
X_{\mu,K} \simeq \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-a_1) \oplus \cdots \oplus \mathcal{O}(-a_k)) \to \mathbb{P}^{N-1}.
\]

Finally, \( X_{\mu,K} \) is Fano if and only if the inequality \( N > a_1 + \cdots + a_k \) holds.
4.1.2

**Proof.** Let \( X_{\mu,K} = f^{-1}(K)/\mathbb{T}^r \) be a symplectic toric manifold of Picard rank \( r \). Denote by \( M := \text{Mat}(\mu) \in M_{r,n}(\mathbb{Z}) \) the matrix of its moment map \( \mu \), where \( M_{r,n}(\mathbb{Z}) \) denotes the space of matrices of size \( r \times n \) with integer entries. The action of \( GL_r(\mathbb{Z}) \) on \( M_{r,n}(\mathbb{Z}) \) by left multiplication \( M \mapsto A \cdot M \) corresponds to changing the coordinates of the torus \( \mathbb{T}^r \). Identifying the moment map \( \mu \) with its matrix \( M \), we have \( X_{\mu,K} = X_{AM,A(K)} \). Moreover, permuting the columns of the matrix \( A \) amounts to a relabeling of the canonical basis of \( \mathbb{C}^n \supset f^{-1}(K) \), therefore the quotient \( X_{\mu,K} \) is still the same manifold.

Now, let us assume that the toric manifold \( X_{\mu,K} \) has Picard rank 2. Note that the chamber \( K \) is the interior of a cone \( \text{Cone}(C_1, C_2) \), where \( C_1 \) and \( C_2 \) are columns of \( M \). By permuting the columns of the matrix \( M \) we may assume that \( C_1 \) and \( C_2 \) are the 1st two columns of \( M \). By the smoothness condition of Proposition 4.1.2, the matrix \( A_K = (C_1|C_2) \in M_{2,2}(\mathbb{Z}) \) formed from the columns \( C_1 \) and \( C_2 \) has determinant \( \pm 1 \). Therefore, by multiplying \( M \) from the left by a matrix \( A \in GL_2(\mathbb{Z}) \), we can assume without loss of generality that the matrix \( A_K \) is the identity, and that the Kähler cone is the first quadrant: \( K = (\mathbb{R}_{>0})^2 \). Then, the matrix of the moment map will have the form

\[
M = \begin{pmatrix}
1 & 0 & a & \ldots \\
0 & 1 & b & \ldots \\
\end{pmatrix},
\]

where \( a, b \in \mathbb{Z} \). Let us investigate the sign of the integers \( a \) and \( b \). Since the Kähler cone \( K \) is the first quadrant \( (\mathbb{R}_{>0})^2 \), the case where \( a, b > 0 \) would contradict \( K = (\mathbb{R}_{>0})^2 \), so it is impossible. Furthermore, the case where \( a, b < 0 \) is also impossible, as the resulting toric manifold would fail the compactness condition of Proposition 4.1.2: the cone associated to the moment map would contain the line spanned by the vector \((a, b)\), its negative being in the Kähler cone \( K \). Therefore, the integers \( a \) and \( b \) must be of opposite signs.

If \( a > 0 \) and \( b < 0 \), then applying the smoothness condition of Proposition 4.1.2, we obtain that the matrix \( \begin{pmatrix} 0 & a \\ 1 & b \end{pmatrix} \) must have determinant \( \pm 1 \), therefore \( a = 1 \). Using a similar argument, if \( a < 0 \) and \( b > 0 \) then \( b = 1 \). Therefore, the columns of \( M \) following the first two columns are either of the type \( \begin{pmatrix} -k \\ 1 \end{pmatrix} \) or of the type \( \begin{pmatrix} 1 \\ -k \end{pmatrix} \), with \( k \in \mathbb{Z}_{\geq 0} \).

Finally, we show that \( M \) can not contain simultaneously columns of these two types. Let us assume that

\[
M = \begin{pmatrix}
1 & 0 & -b & \ldots \\
0 & 1 & -a & 1 & \ldots \\
\end{pmatrix}
\]

Then, the smoothness condition gives that

\[
\det \begin{pmatrix}
1 & -b \\
-a & 1 \\
\end{pmatrix} = 1 - ab = \pm 1
\]

Therefore \( a = 1, b = 2 \) or \( a = 2, b = 1 \). In both cases, the cone associated to the moment map \( M \) contains the line \( \text{Vect}_{\mathbb{R}}(\begin{pmatrix} 1 \\ -1 \end{pmatrix}) \), thus failing the compactness condition. Consequently, the remaining columns of \( M \) are always of the same type \( \begin{pmatrix} -k \\ 1 \end{pmatrix} \) or \( \begin{pmatrix} 1 \\ -k \end{pmatrix} \). Finally, multiplying \( M \) from the left by
the matrix \[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\] and permuting the 1st two columns of \(M\) if necessary, we can arrange that the matrix \(M\) contains only vectors of the type \[
\begin{pmatrix}
-k \\
1
\end{pmatrix}
\]. \(\square\)

Finally, we give a result for computing intersection products for such symplectic toric manifolds. Our formula will rely on the computation of Jeffrey–Kirwan residues for toric manifolds done in Section 2 of [SV04]. We will also refer to the same article for more details on these residues.

**Theorem 4.1.7** ([SV04], Theorem 2.6). Let \(X_{\mu,K}\) be a symplectic toric manifold, whose toric data is given as in the statement of Proposition 4.1.6: its chamber is \(K = (\mathbb{R}_{>0})^2\) and the matrix of its moment map is given by

\[
\begin{pmatrix}
1 & \cdots & 1 & 0 & -a_1 & \cdots & -a_k \\
0 & \cdots & 1 & 1 & \cdots & 1
\end{pmatrix}
\]

Denote by \(p_1, p_2\) the toric divisors of \(X_{\mu,K}\) (see Proposition 4.1.4), and let \(f(x,y) \in \mathbb{C}[x,y]\) be some polynomial. Then, the intersection product given by the polynomial \(f\) can be computed as an iterated residue as follows:

\[
\int_{[X_{\mu,K}]} f(p_1, p_2) = \text{Res}_{y=0} \text{Res}_{x=0} f(x,y) \frac{dx dy}{x^Ny(y-a_1x)\cdots(y-a_kx)}
\]

**Proof.** Recall that \(\alpha_i \in \mathbb{Z}^2\) denotes the vector given by the \(i\)-th column of the moment map. Starting from the right hand side of the identity we want to prove, we use Theorem 2.6 of [SV04] for the projective sequence \(\mathcal{A} = (\alpha_1, \ldots, \alpha_{N+k+1})\) and a sum-regular vector \(\xi \in K = (\mathbb{R}_{>0})^2\) located below the line \(\text{Vect}(c_1(TX_{\mu,K}))\). We obtain

\[
\text{Res}_{y=0} \text{Res}_{x=0} f(x,y) = \text{JK}_{(\mathbb{R}_{>0})^2} \left( \frac{f(x,y)}{x^Ny(y-a_1x)\cdots(y-a_kx)} \right),
\]

where \(\text{JK}_{(\mathbb{R}_{>0})^2}\) denotes the Jeffrey–Kirwan residue, see e.g. Equation (2.1) of [SV04]. Using Proposition 2.3 of [SV04], we get

\[
\text{JK}_{(\mathbb{R}_{>0})^2} \left( \frac{f(x,y)}{x^Ny(y-a_1x)\cdots(y-a_kx)} \right) = \int_{[X_{\mu,K}]} f(p_1, p_2)
\]

\(\square\)

4.2. **Oscillatory integral and gamma class in quantum cohomology.** In this Subsection, we recall the definition of the oscillatory integral and the \(I\)-function of a toric manifold. Then we would like to recall the results of Iritani from [Iri09, Iri20], which will be our guiding principle for what we would like to do in the \(K\)-theoretic settings.

4.2.i. **Oscillatory integral in quantum cohomology.** From now on, when considering a symplectic toric manifold \(X_{\mu,K}\), we will always assume that it satisfies the three conditions of Proposition 4.1.2, i.e. it is compact, smooth, and Fano.
Definition 4.2.1 (Landau–Ginzburg potential). Let $X_{\mu,K}$ be a symplectic toric manifold and denote by $m_{ij} \in \mathbb{Z}$ the coefficients of the matrix of its moment map $\mu$. The Landau–Ginzburg potential associated to the toric manifold $X_{\mu,K}$ is the following family of holomorphic functions:

$$Y := (\mathbb{C}^*)^n \xrightarrow{W} B := (\mathbb{C}^*)^r,$$

where we denote by $x_1, \ldots, x_n$ the standard global coordinates on the complex torus $Y$, $Q_1, \ldots, Q_r$ the standard global coordinates on the complex torus $B$, the maps $W$ and $\pi$ are given by

$$W(x_1, \ldots, x_n) := x_1 + \cdots + x_n \in \mathbb{C},$$

$$\pi(x_1, \ldots, x_n) := \left(\cdots, \prod_{j=1}^n x_i^{m_{ij}}, \cdots\right) \in B.$$ 

We will refer to the relations (after identification) $Q_i = \prod_{j=1}^n x_i^{m_{ij}}$ as the Batyrev constraints.

Definition 4.2.2 (Oscillatory integral). Consider the Landau–Ginzburg potential associated to a toric manifold $X_{\mu,K}$. Fix $(Q_1, \ldots, Q_r) \in B$. The formula

$$\omega_{\pi^{-1}(Q)} := \frac{d\log x_1 \wedge \cdots \wedge d\log x_n}{d\log Q_1 \wedge \cdots \wedge d\log Q_r}$$

defines a holomorphic form on $\pi^{-1}(Q_1, \ldots, Q_r)$. The oscillatory integral $I^{\mathrm{coh}}(z, Q)$ is the function defined by

$$I^{\mathrm{coh}}(z, Q) := \int_{\Gamma} e^{-W_Q/z} \omega_{\pi^{-1}(Q)},$$

where $W_Q := W|_{\pi^{-1}(Q_1, \ldots, Q_r)}$ and $\Gamma \subset \pi^{-1}(Q_1, \ldots, Q_r)$ is a semi-infinite cycle representing a homology class in

$$(4.2.3) \quad \lim_{l \to \infty} H_{n-r}(\pi^{-1}(Q_1, \ldots, Q_r), \mathrm{Re}(W_Q/z) > l; \mathbb{Z}).$$

The triple $(Y, W, \omega_{\pi^{-1}(Q)})$ was proposed by Givental (see [Giv98]) as a mirror model of the toric manifold $X_{\mu,K}$. On the other hand, there is a quantum field theory model known as the Landau–Ginzburg model, whose partition function is closely related to Givental’s oscillatory integral. We will refer to the triple $(Y, W, \omega_{\pi^{-1}(Q)})$ as the Givental’s mirror or the Landau–Ginzburg model of $X_{\mu,K}$.

Remark 4.2.4. We will be interested mostly in a specific integration cycle $\Gamma_{\mathbb{R}}$, which we will refer to as the real Lefschetz thimble. Taking $(Q_1, \ldots, Q_r) \in (\mathbb{R}_{>0})^r$, the corresponding real Lefschetz thimble $\Gamma_{\mathbb{R}}$ is defined by

$$\Gamma_{\mathbb{R}} := \{(x_1, \ldots, x_n) \in \pi^{-1}(Q_1, \ldots, Q_r) | \forall j, x_j \in \mathbb{R}_{>0}\}.$$ 

Note that if $Q \in (\mathbb{R}_{>0})^r$ and $z > 0$, then $\Gamma_{\mathbb{R}}$ represents a homology class in $(4.2.3)$. Therefore, we can define the corresponding oscillatory integral. We refer to Section 3.3.1 in [Iri09], for further details on the group $(4.2.3)$ of semi-infinite cycles.
4.2.ii. Small I-function. We introduce a second function, called Givental’s small I-function, that is related by mirror symmetry to cohomological Gromov–Witten invariants. While we immediately define it as a formal power series, this formula should be understood through fixed point localisation in cohomology (see [Giv96]) or through the theory of GKZ D-modules (see Lemma 4.6 of [Iri09]).

**Definition 4.2.5** (Cohomological small I-function). Let $X_{\mu,K}$ be a symplectic toric manifold. Write $\alpha_1, \ldots, \alpha_n \in \mathbb{R}^r$ for the image of the canonical basis of $\mathbb{Z}^n$ by the moment map $\mu$. Denote by $p_1, \ldots, p_r \in H^2(X_{\mu,K};\mathbb{Z})$ the toric divisors of Proposition 4.1.4. The cohomological small I-function $I_{\text{coh}}^\mu(z,Q)$ of the toric manifold $X_{\mu,K}$ is the cohomologically valued power series defined by

$$I_{\text{coh}}^\mu(z,Q) := e^{-\sum_{i=1}^n p_i \log(Q_i)/z} \sum_{d \text{ effective}} \prod_{r=1}^n \prod_{j=1}^n \alpha_j(p) - r z \in H^* (X_{\mu,K};\mathbb{Q}) \otimes \mathbb{C}[z^{\pm 1}][[Q]],$$

where $\alpha_j(p) := (\alpha_j, p) = m_{1j} p_1 + \cdots + m_{rj} p_r \in H^*(X_{\mu,K};\mathbb{Q})$ and $Q^d := Q_1^{d_1} \cdots Q_r^{d_r}$, $d_i := \int_I p_i$.

**Proposition 4.2.6** ([Iri09], Lemma 4.6). The oscillatory integral $I_{\text{coh}}^\mu$ of Definition 4.2.2 and the I-function $I_{\text{coh}}^\mu(z,Q)$ of Definition 4.2.5 satisfy the same system of differential equations

$$\Delta I_{\text{coh}}^\mu(z,Q) = \Delta I_{\text{coh}}^\mu(z,Q) = 0,$$

where $\Delta$ is the differential operator associated to the system of $r$ differential equations

$$\prod_{j: m_{ij} > 0} \prod_{r=0}^{m_{ij} - 1} (r - \alpha_j(Q \partial_i Q)) I(z,Q) = z^{-m_i} \prod_{j: m_{ij} < 0} \prod_{r=0}^{-m_{ij} - 1} (r - \alpha_j(Q \partial_i Q)) I(z,Q),$$

where $\alpha_j(Q \partial_i Q) := \sum_{a=1}^r m_{aj} Q_a \partial_i Q_a$ and $m_i = \sum_{j=1}^n m_{ij}$. □

4.2.iii. Comparison theorem. Since we have two solutions $I_{\text{coh}}^\mu$ and $I_{\text{coh}}^\mu$ of the same differential system, we would like to be able to compare these two solutions. Following Iritani, let us introduce a multiplicative characteristic class which plays an important role in quantum cohomology.

**Definition 4.2.7** (Cohomological Gamma class). Let $E \to X$ be a vector bundle, and denote by $\delta_1, \ldots, \delta_n$ its Chern roots. The cohomological Gamma class $\Gamma(E) \in H^*(X;\mathbb{Q})$ is defined by

$$\Gamma(E) := \prod_{j=1}^n \Gamma(1 + \delta_j) \in H^*(X_{\mu,K};\mathbb{Q}),$$

where $\Gamma(1 + \delta_j)$ is defined by substituting $x = \delta_j$ in the Taylor series expansion of $\Gamma(1 + x)$ at $x = 0$.

**Theorem 4.2.8** ([Iri09], Theorem 4.14, Equation (70)). Let $\Gamma_\mathbb{R}$ be the real Lefschetz thimble of Remark 4.2.4. Then, the oscillatory integral $I_{\text{coh}}^\mu$ and the I-function $I_{\text{coh}}^\mu$ are related by the identity

$$I_{\text{coh}}^\mu(z,Q) = \int_{[X_{\mu,K}]} \Gamma(X_{\mu,K}) \cup z^\mu z^{\deg} I_{\text{coh}}^\mu(z,Q),$$

where $\int_{[X_{\mu,K}]}$ denotes the intersection product by $[X_{\mu,K}] \in H_*(X_{\mu,K};\mathbb{C})$, $\Gamma(X_{\mu,K})$ is the Gamma class of the holomorphic tangent bundle $TX_{\mu,K}$, and $\rho$ is the operator of cup product multiplication by $c_1(TX_{\mu,K})$. □
We refer to the paper of Iritani [Iri09] for the proof of this statement in general. For later comparison with the $K$-theoretic case, we give a proof of Theorem 4.2.8 in Appendix B when $X_{\mu,K}$ has Picard rank 2.

4.3. $q$-oscillatory integral in quantum $K$-theory. The goal of this subsection is to introduce a $K$-theoretic analogue of Theorem 4.2.8 comparing the oscillatory integral with the $I$-function in cohomology. In quantum $K$-theory, we will consider Givental’s permutation equivariant $I$-function defined in Theorem p.8 of [Giv15b], and a $q$-analogue of the oscillatory integral (see Definition 4.3.4).

4.3.i. $q$-oscillatory integral in quantum $K$-theory. The $K$-theoretic analogue of the Landau–Ginzburg potential defined in Definition 4.2.1 was proposed by Givental in [Giv15c].

**Definition 4.3.1** ($K$-theoretic mirror family; [Giv15c], Theorem 2). Let $X_{\mu,K}$ be a symplectic toric manifold and write $m_{ij} \in \mathbb{Z}$ for the entries of the matrix $\text{Mat}(\mu)$ of the moment map $\mu$, and suppose that the length of $q \in \mathbb{C}^*$ is not 1. The $K$-theoretic mirror family associated to the toric manifold $X_{\mu,K}$ is the following family of holomorphic functions:

$$Y := (\mathbb{C}^*)^n \xrightarrow{W_q} \mathbb{C}$$

$$B := (\mathbb{C}^*)^r$$

where we denote by $x_1, \ldots, x_n$ the standard coordinates on $Y$, $Q_1, \ldots, Q_r$ the standard coordinates on $B$, and $W_q$ and $\pi$ are defined by

$$W_q(x_1, \ldots, x_n) := \sum_{j=1}^{n} \sum_{l>0} \frac{x_j}{l(1-q^l)} \in \mathbb{C},$$

$$\pi(x_1, \ldots, x_n) := \left(\ldots, \prod_{j=1}^{n} x_1^{m_{ij}}, \ldots\right) \in B.$$  

We will refer to the relations (after identification) $Q_i = \prod_{j=1}^{n} x_1^{m_{ij}}$ as the Batyrev constraints.

There are two ways to define an oscillatory integral that solves the system of $K$-theoretic quantum difference equations of $X_{\mu,K}$. One of them, as proposed by Givental in [Giv15c], is by using a Riemann (or Lebesgue) integral. In fact, we made an attempt to achieve our goals with such a definition, but we got into a problem which is somewhat tricky to resolve. The interested reader is referred to Appendix C, where we have tried to describe the difficulty which we encountered. The 2nd way is to use an appropriate multi-dimensional version of the Jackson integral, that is, to define a $q$-analogue of the oscillatory integral for quantum $K$-theory. This is the approach which we take in this paper. Such $q$-integrals appeared first for Grassmannians in Section 8 of [GY21]. According to Givental–Yan, the $q$-oscillatory integrals should resemble the formula

$$\left[ \int_{\Gamma} \right]_q \exp\left( W_{\pi^{-1}(Q)}(x_1, \ldots, x_n) \right) \omega_{\pi^{-1}(Q),q},$$
where the symbol \([f_q]_{\Gamma}\) means we should consider a sum for which the inputs \(x_j\) take values in some lattice in the semi-infinite cycle \(\Gamma \subset \pi^{-1}(Q)\) stable by multiplication by \(q\). In this paper, we will focus only on the \(q\)-oscillatory integral corresponding to the real cycle \(\Gamma_{\mathbb{R}}\) (see Remark 4.2.4).

The case of an arbitrary semi-infinite cycle, requires choosing a representative in the homology class that admits an appropriate \(q\)-discrete structure. Proving the existence of such a choice requires a separate investigation, so we do not pursue it in this paper.

For the sake of simplicity, let us consider the case of Picard rank 2 symplectic toric manifolds. The case of an arbitrary Picard rank is similar. To begin with, let us examine the analytic properties of the function \(W_q\). Assume that the length of \(q \in \mathbb{C}^*\) is not 1. The power series (in the definition of \(W_q\)) \(\sum_{l>0} \frac{x^l}{l(1-q^l)}\) has finite convergence radius. More precisely, using the ratio test, we get that if \(|q| < 1\) (resp. \(|q| > 1\)), then the series is convergent for \(|x_j| < 1\) (resp. \(|x_j| < q^{-1}\)). The function defined by \(f(x) = \exp\left(\sum_{l>0} \frac{x^l}{l(1-q^l)}\right)\) has the following analytical continuations:

\[
\exp\left(\sum_{l>0} \frac{x^l}{l(1-q^l)}\right) = \begin{cases} \\
\frac{1}{(x;q)_{\infty}} & \text{if } |q| < 1, \\
(q^{-1}x;q^{-1})_{\infty} & \text{if } |q| > 1,
\end{cases}
\]

where we denote by \((z;q)_{\infty} := \prod_{r \geq 0} (1-q^rz)\) the \(q\)-Pochhammer symbol, defined for \(|q| < 1\). To prove the first analytical continuation, we use the following Taylor series:

\[
\sum_{l>0} \frac{x^l}{l(1-q^l)} = \sum_{l>0} \sum_{k \geq 0} \frac{x^l}{l} q^{kl} = -\sum_{k \geq 0} \log(1-q^k x).
\]

To obtain the second one, we start from the same Taylor series and multiply both sides of the fraction by \(q^{-1}\), and use the first analytical continuation.

Note that these two analytical continuations are closely related to the \(q\)-exponential functions:

\[
e_q(x) := \sum_{d=0}^{\infty} x^d \prod_{l=1}^{d} \frac{1-q}{1-q^l} = \frac{1}{((1-q)x;q)_{\infty}}
\]

and

\[
E_q(x) := \sum_{d=0}^{\infty} q^{d(d-1)/2} x^d \prod_{l=1}^{d} \frac{1-q}{1-q^l} = ((q-1)x;q)_{\infty}.
\]

Using the \(q\)-binomial theorem (see Equation (1.3.2) p.8 of [GR]), one can show that if \(|q| < 1\), then

\[
\frac{1}{(x;q)_{\infty}} = e_q\left(\frac{x}{1-q}\right)
\]

and if \(|q| > 1\), then

\[
(q^{-1}x;q^{-1})_{\infty} = E_{q^{-1}}\left(\frac{x}{1-q}\right).
\]

From now on we will assume that \(|q| > 1\). Our motivation for choosing \(|q| > 1\) comes from the formula for the \(K\)-theoretic \(I\)-function of a Fano toric manifold, i.e., if \(|q| > 1\), then the \(I\)-function has an infinite radius of convergence with respect to the Novikov variables. Furthermore, for simplicity, let us assume that \(q\) is a real number.
Definition 4.3.3 (Jackson integral; see e.g. Appendix A of [DVZ09]). The Jackson integral is a $q$-analogue of the classical (Riemann) integral. We introduce the following formal definitions for a complex function:

$$
\left[ \int_0^\infty \right]_q f(x) dq x := \sum_{d \in \mathbb{Z}} q^d f \left( \frac{q^d}{A} \right),
$$

$$
\left[ \int_0^\infty/A \right]_q f(x) dq x := \sum_{d \in \mathbb{Z}} q^d A f \left( \frac{q^d}{A} \right).
$$

Let us point out that sometimes the Jackson integral is defined to be $(1 - q) \left[ \int_0^\infty \right]_q f(x) dq x$ for $q < 1$, so that in the limit $q \to 1$ it coincides with the Riemann integral. For our purposes, Definition 4.3.3 seems to be more convenient.

In order to write this $q$-oscillatory integral, we will fix coordinates $x_1, \ldots, x_{N+k+1}$ on the fibre $\pi^{-1}(Q_1, Q_2)$. For fixed $Q_1, Q_2 \in \mathbb{C}^*$, the the fibre $\pi^{-1}(Q_1, Q_2)$ is defined by the following equations:

$$
Q_1 = x_1 \cdots x_N x_{N+1}^{-a_1} \cdots x_{N+k+1}^{-a_k},
$$

$$
Q_2 = x_{N+1} \cdots x_{N+k+1}.
$$

We will consider an isomorphism $\varphi : (\mathbb{C}^*)^{N+k-1} \simeq \pi^{-1}(Q)$ by assuming all coordinates on the fibre except for $x_N$ and $x_{N+1}$ to be free, i.e., we define

$$
\varphi(x_1, \ldots, x_{N-1}, x_{N+2}, \ldots x_{N+k+1}) := (x_1, \ldots, x_{N-1}, Q_1 x_1^{-1} x_{N-1}^{-1} x_{N+1}^{-a_1} \cdots x_{N+k+1}^{-a_k} Q_2 x_{N+2}^{-1} \cdots x_{N+k+1}^{-1}, x_{N+2}, \ldots, x_{N+k+1}) \in \pi^{-1}(Q).
$$

The isomorphism $\varphi$ gives a parametrization of the real cycle $\Gamma_R$, that is, we have an isomorphism $(\mathbb{R}_{>0})^{N+k-1} \simeq \Gamma_R$. Moreover, the holomorphic volume form takes the form

$$
\omega_{\pi^{-1}(Q)} = \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_{N-1}}{x_{N-1}} \wedge \frac{dx_{N+1}}{x_{N+1}} \wedge \cdots \wedge \frac{dx_{N+k+1}}{x_{N+k+1}}.
$$

Using the parametrization $\varphi$, it is natural to define the $q$-oscillatory integral along the real Lefschetz thimble $\Gamma_R$ by using the multivariable version of Definition 4.3.3.

Definition 4.3.4 ($q$-oscillatory integral). Suppose that $Q_1, Q_2 \in q^{\mathbb{Z}} := \{q^n \mid n \in \mathbb{Z}\}$. The $q$-oscillatory integral along the real Lefschetz thimble is the function defined by the following Jackson integral along the coordinates $x_1, \ldots, x_{N-1}, x_{N+2}, \ldots, x_{N+k+1}$:

$$
\mathcal{I}^{K-\text{th}}(q, Q_1, Q_2) := \left[ \int_{\Gamma_R} \right]_q \prod_{j=1}^n E_q^{-1} \left( \frac{x_j}{1 - q} \right)_{\pi^{-1}(Q)} \omega_{\pi^{-1}(Q)} q
$$

$$
:= \sum_{d_1, \ldots, d_{N-1} \in \mathbb{Z}} \prod_{j=1}^n E_q^{-1} \left( \frac{x_j (d_j Q_1, Q_2)}{1 - q} \right),
$$

where $E_q^{-1}$ is the $q$-analogue of the classical exponential function.
where

\[ x_j(d, Q_1, Q_2) := q^{d_j}, \]
\[ x_N(d, Q_1, Q_2) := Q_1 q^{-\sum_{j=1}^{N-1} d_j + \sum_{j'=1}^k a_j d_{N+1+j'}}, \]
\[ x_{N+1}(d, Q_1, Q_2) := Q_2 q^{-\sum_{j'=1}^k d_{N+1+j'}} \]

and

\[ \omega_{\pi^{-1}}(Q)_q = \frac{d_q x_1}{x_1} \wedge \cdots \wedge \frac{d_q x_{N-1}}{x_{N-1}} \wedge \frac{d_q x_{N+1}}{x_{N+1}} \wedge \cdots \wedge \frac{d_q x_{N+1+k}}{x_{N+1+k}}. \]

4.3.ii. Non-discrete Novikov variables. Definition 4.3.4 does not work for \( Q_1, Q_2 \notin q^\mathbb{Z} \), because the corresponding Jackson integral is divergent. In this section we would like to outline a construction which should allow us to define the \( q \)-oscillatory integral for arbitrary values of \( Q_1 \) and \( Q_2 \). Our definition relies on a conjecture about the regularity of a certain system of \( q \)-difference equations. For toric manifolds of Picard rank 1, the regularity is known, but for Picard rank > 1 our conjecture seems to be a separate project. The results of this subsection would not be used in what follows. The reader not interested in our speculations could skip it.

We follow the construction of De Sole and Kac from \cite{DSK05}. Let us recall the Jacobi theta function

\[ \theta(x) := \theta_q^{-1}(x) = \sum_{n \in \mathbb{Z}} q^{-n(n+1)/2} x^n. \]

It is a holomorphic function in \( x \in \mathbb{C}^* \) with essential singularities at \( x = 0 \) and \( \infty \). Moreover, we have the so-called Jacobi triple product identity

\[ \theta(x) = \prod_{j=0}^{\infty} (1 - q^{-j-1}) \prod_{j=0}^{\infty} (1 + q^{-j} x) \prod_{j=0}^{\infty} (1 + q^{-j-1} x^{-1}), \]

which shows that \( \theta \) has simple zeroes at the points \( x = -q^m \) (\( m \in \mathbb{Z} \)). Finally, let us recall also the following property:

\[ \theta(q^m x) = q^{m(m+1)/2} x^m \theta(x), \quad m \in \mathbb{Z}, \]

which follows from the definition of \( \theta(x) \). Namely, suppose that \( A > 0 \) is a real number. Put

\[ k(A, t) := \frac{\theta(A^{-1})}{\theta(q^{-t} A^{-1})}. \]

Using (4.3.5), we get \( k(A, m) = A^{-m} q^{-m(m-1)/2} \) for all \( m \in \mathbb{Z} \). We will make use of the infinite-order difference operator \( k(A, y \partial_y) \) acting on the space of formal power series \( \mathbb{C}[[y]] \) via

\[ k(A, y \partial_y) \sum_{m=0}^{\infty} c_m y^m = \sum_{m=0}^{\infty} c_m k(A, m) y^m = \sum_{m=0}^{\infty} c_m q^{-m(m-1)/2} (y A^{-1})^m. \]

Using the ratio test, we get that if the series \( \sum_{m=0}^{\infty} c_m y^m \) has a non-zero radius of convergence, then the operator \( k(A, y \partial_y) \) will produce an entire function. Finally, comparing the Taylor series
expansions of the two exponential functions $e_{q^{-1}}(x)$ and $E_{q^{-1}}(x)$ at $x = 0$, we get

$$k(A, y\partial_y) e_{q^{-1}} \left( \frac{y A}{1 - q} \right) = E_{q^{-1}} \left( \frac{y}{1 - q} \right).$$

Motivated by the above formula, we would like to modify the definition of the $q$-oscillatory integral as follows. Suppose that $A_1, \ldots, A_n$ and $Q_1, Q_2$ are positive real numbers. Put

$$k(A, Q\partial_Q) := \prod_{j=1}^{n} k(A_j, \alpha_j(Q\partial_Q)), \quad \alpha_j(Q\partial_Q) := m_{1j}Q_1\partial_{Q_1} + m_{2j}Q_2\partial_{Q_2},$$

where $A = (A_1, \ldots, A_n)$. We would like to define

$$\mathcal{I}^{K^{-th}}(q, Q_1, Q_2) := k(A, Q\partial_Q) \left[ \int_{\mathbb{R}} \sum_{d_{N+1} \in \mathbb{Z}} e_{q^{-1}} \left( \frac{x_1 A_1}{1 - q} \right) \cdots e_{q^{-1}} \left( \frac{x_n A_n}{1 - q} \right) \omega_{\pi^{-1}}(Q), q \right],$$

where the action of the operator $k(A, Q\partial_Q)$ on the oscillatory integral will be defined next. Let us first focus on the analytic properties of the oscillatory integral

$$\mathcal{I}^{K^{-th}}(q, Q_1, Q_2) := \left[ \int_{\mathbb{R}} \sum_{d_{N+1} \in \mathbb{Z}} e_{q^{-1}} \left( \frac{x_1 A_1}{1 - q} \right) \cdots e_{q^{-1}} \left( \frac{x_n A_n}{1 - q} \right) \omega_{\pi^{-1}}(Q), q \right].$$

Using the ratio test, it is straightforward to check that the Jackson integral is convergent, that is, $\mathcal{I}^{K^{-th}}(q, Q_1, Q_2)$ is analytic for $|q| > 1$ and for all $Q_1, Q_2 \in \mathbb{C}$, such that, $\text{Re}(Q_1) > 0$, $\text{Re}(Q_2) > 0$. On the other hand, in order to define the action of the operator $k(A, Q\partial_Q)$ we need to expand $\mathcal{I}^{K^{-th}}(q, Q_1, Q_2)$ in a neighborhood of $(Q_1, Q_2) = 0$. This is exactly the place where we need to make a conjecture about the structure of such an expansion.

**Proposition 4.3.9.** The oscillatory integral $\mathcal{I}^{K^{-th}}(q, Q_1, Q_2)$ is a solution to the following system of $q$-difference equations

$$\prod_{j=m_{ij} > 0}^{m_{ij} - 1} \prod_{r=0}^{q^{-\alpha_j(Q\partial_Q)} r + 1} \left( q^{-\alpha_j(Q\partial_Q)} r - 1 \right) - a_i q^{-m_i} Q_i \prod_{j=m_{ij} < 0}^{-m_{ij} - 1} \prod_{r=0}^{q^{-\alpha_j(Q\partial_Q)} r + 1} \left( q^{-\alpha_j(Q\partial_Q)} r - 1 \right) f_i(Q) = 0,$$

where $\alpha_j(Q\partial_Q) = m_{1j}Q_1\partial_{Q_1} + m_{2j}Q_2\partial_{Q_2}$, $a_i := n_{j=1} A_j^{m_{ij}}$, and $m_i = \sum_{j=1}^{n} m_{ij}$.

The proof of the above proposition is similar to the proof of Proposition 4.3.15 below, so we omit it. To simplify the notation in our discussion let us explain how to define the action of $k(A, Q\partial_Q)$ on (4.3.8) when all constants $A_i = 1$. We would like to conjecture that the above system of $q$-difference equations (4.3.10) has a basis of solutions of the following form

$$f(Q) = \sum_{a,b} f_{a,b}(Q_1, Q_2) \ell(Q_1)^a \ell(Q_2)^b,$$

where $L \in \mathbb{Z}_{\geq 0}$ and the coefficients $f_{a,b}(Q_1, Q_2)$ are analytic functions at $(Q_1, Q_2) = 0$, where

$$\ell(x) = \frac{1}{2} - x\partial_x \log \theta(x) = \frac{1}{2} \frac{x\theta'(x)}{\theta(x)}.$$
is the so-called q-logarithm. Here the constant \( \frac{1}{2} = \frac{\theta'(1)}{\theta(1)} \) is chosen so that \( \ell(1) = 0 \). Let us mention some evidence for this conjecture: in the case of a toric manifold of Picard rank 1, the system of q-difference equations in Proposition 4.3.9 has a non resonant regular singularity at \( Q = 0 \) and the existence of a basis of solutions of the form (4.3.11) is known (see [HSS], Theorem 3.1.7 p. 127). Furthermore, in the Picard rank 2 case, we are able to prove the conjecture in the following cases:

**Proposition 4.3.13.** We assume that the manifold \( X_{\mu,K} \) satisfies the three following conditions: that all coefficients \( m_{ij} \) of the moment map are odd; that the numbers \( m_i := \sum_j m_{ij} \) are all equal to some positive integer \( M \) (i.e. \( c_1(TX_{\mu,K}) = MP_1 + MP_2 \)); and that the coefficients \( A_i \) are chosen so that the numbers \( a_i := \prod_{j=1}^n A_{ij}^{-1} \) are all equal to some number \( a \). Then, the q-difference system (4.3.10) has a basis of solutions of the form (4.3.11).

**Proof.** Under these assumptions, the q-difference system we have to solve is

\[
\left[ \prod_{j|m_{ij}>0} m_{ij}^{-1} \prod_{r=0}^{m_{ij}-1} \left( 1 - q^{-\alpha_j(c\partial q)_{ij} + r} \right) - a q^{-M} \right] f_q(Q) = 0,
\]

Using Theorem p.8 of [Giv15b], one obtains a K-theoretically-valued solution given by

\[
I_{X_{\mu,K}}^{K,-\ell}(q^{-1}, a q^{-M} Q),
\]

where \( I_{X_{\mu,K}}^{K,-\ell} \) the small K-theoretic I-function

\[
I_{X_{\mu,K}}^{K,-\ell}(q,Q) = P^{-\ell(Q)} \sum_{d=\left(d_1,d_2\right)} Q^d \prod_{j=1}^{\alpha_2(d)} \frac{\prod_{r=0}^{m_{ij}-1} (1 - U_j(P) q^r)}{\prod_{r=0}^{n} (1 - U_j(P) q^r)} \in K^0(X_{\mu,K})(q)[[Q]],
\]

where \( \ell(Q) \) is the q-logarithm (4.3.12), \( P^{-\ell(Q)} := \prod_{i=1}^n P_i^{-\ell(Q_i)} \in K^0(X_{\mu,K}), \) and \( P_i^{-\ell(Q_i)} \) should be understood as the expansion of the binomial \( (1 - (1 - P_i^{-1}))^{\ell(Q_i)} \). The function \( \ell(a q^{-M} Q) \) is another q-logarithm, therefore there exists a q-constant function \( f \) such that \( \ell(a q^{-M} Q) = f(Q) \ell(Q) \).

Let us consider the decomposition of the solution \( IX_{X_{\mu,K}}^{\ell(Q)}, a q^{-M} Q \) in the basis of \( K^0(X_{\mu,K}) \). In front of a vector of the form \( (1 - P_1^{-1})^\alpha (1 - P_2^{-1})^\beta \), we will find a function of the form (up to a q-constant)

\[
\sum_{\alpha=0}^\infty \sum_{\beta=0}^\infty f_{u,v}(Q_1, Q_2) \ell(Q_1)^u \ell(Q_2)^v,
\]

with \( f_{u,v} \) analytic at \( (Q_1, Q_2) = 0 \). \( \square \)

Our claim is that for any \( B_j \in \mathbb{R}_{>0} \) (1 \( \leq j \leq n \)), there is a natural way to define the action \( k(B_j, D_j)f(Q) \), where \( D_j = m_{ij}Q_i \partial Q_i + m_{ij}Q_j \partial Q_j \) and \( f \) has the form (4.3.11). Indeed, note that the difference operator \( q^{-D_j}B_j^{-1} \) commutes with the differential operators \( \ell(Q_i) - m_{ij}B_j \partial B_j \) (i \( = 1, 2 \)). Indeed, we have

\[
q^{-D_j}B_j^{-1} \left( \ell(Q_i) - m_{ij}B_j \partial B_j \right) = \ell(q^{-m_{ij}}Q_i)q^{-D_j}B_j^{-1} - m_{ij}(B_j \partial B_j + 1)q^{-D_j}B_j^{-1}.
\]
Using that $\ell(q^{-m}x) = \ell(x) + m$ for all $m \in \mathbb{Z}$, we get that the above expression coincides with $(\ell(Q_1) - m_{ij}B_j\partial_{B_j}) q^{-D_j}B_j^{-1}$. Let us write $f$ in the form

$$f(Q) = \sum_{a,b} f_{a,b}(Q_1, Q_2) \left( \ell(Q_1) - m_{1j}B_j\partial_{B_j} \right)^a \left( \ell(Q_2) - m_{2j}B_j\partial_{B_j} \right)^b \cdot 1,$$

that is, a differential operator in $B_j$ acting on 1. Since $k(B_j, D_j) = \theta(B_j^{-1}) \theta(q^{-D_j}B_j^{-1})$, using the commutativity of $q^{-D_j}B_j^{-1}$ and $\ell(Q_1) - m_{ij}B_j$, we get

$$k(B_j, D_j) f(Q) = \sum_{a,b} \theta(B_j^{-1}) \left( \ell(Q_1) - m_{1j}B_j\partial_{B_j} \right)^a \left( \ell(Q_2) - m_{2j}B_j\partial_{B_j} \right)^b \cdot \theta(q^{-D_j}B_j^{-1}) f_{a,b}(Q_1, Q_2).$$

On the other hand,

$$\theta(q^{-D_j}B_j^{-1}) Q_1^{d_1} Q_2^{d_2} = \theta(q^{-\alpha_j(d)}B_j^{-1}) Q_1^{d_1} Q_2^{d_2} = \theta(B_j^{-1})^{-1} q^{-\alpha_j(d)(\alpha_j(d)+1)/2} B_j^{-\alpha_j(d)} Q_1^{d_1} Q_2^{d_2},$$

where $\alpha_j(d) := d_1m_{1j} + d_2m_{2j}$. Finally,

$$\theta(B_j^{-1}) B_j\partial_{B_j} \theta(B_j^{-1}) = B_j\partial_{B_j} - \ell(B_j^{-1}) + \frac{1}{2}.$$

Our definition takes the form

$$k(B_j, D_j) f(Q) := \sum_{a,b=0}^{L} \sum_{d_1, d_2=0}^{\infty} f_{a,b,d_1,d_2} q^{-\alpha_j(d)(\alpha_j(d)+1)/2} Q_1^{d_1} Q_2^{d_2} \left( \ell(Q_1) - m_{1j}(B_j\partial_{B_j} - \ell(B_j^{-1}) + 1/2) \right)^a \left( \ell(Q_2) - m_{2j}(B_j\partial_{B_j} - \ell(B_j^{-1}) + 1/2) \right)^b \cdot B_j^{-\alpha_j(d)}.$$

The action of the composition $k(B, Q\partial Q) = k(B_1, D_1) \cdots k(B_n, D_n)$ is defined by

$$k(B, Q\partial Q) f(Q) := \sum_{a,b,d_1,d_2=0}^{L} f_{a,b,d_1,d_2} q^{-\frac{1}{2} \sum_{a=1}^{n} \alpha_j(d)(\alpha_j(d)+1)} Q_1^{d_1} Q_2^{d_2} \prod_{i=1}^{2} \left( \ell(Q_i) - \sum_{j=1}^{n} m_{ij}(B_j\partial_{B_j} - \ell(B_j^{-1}) + 1/2) \right)^{l_i} \cdot \prod_{j=1}^{n} B_j^{-\alpha_j(d)},$$

where $\cdot$ on the 2nd line denotes the action of a differential operator on a function. Note that under the Fano condition the number $\sum_{j=1}^{n} \alpha_j(d) \to \infty$ as $d_1 \to \infty$ or $d_2 \to \infty$. It follows that $k(B, Q\partial Q) f(Q)$ has the form (4.3.11) and that the coefficient in front of $\ell(Q_1)^a \ell(Q_2)^b$ is a convergent power series in $Q_1$ and $Q_2$ whose radius of convergence is $\infty$, that is, the coefficients are holomorphic for all $(Q_1, Q_2) \in \mathbb{C}^2$. Furthermore, if we expand $k(B, Q\partial Q) f(Q)$ as a Laurent series in $q^{-1}$, then the coefficients will be polynomials in $Q_i$ ($i = 1,2$) and $\ell(Q_i)$ ($i = 1,2$).

4.3.iii. Small I-function in quantum K-theory.

**Definition 4.3.14** (K-theoretic I-function; [Giv15b], Theorem p.8). Let $X_{\mu,K}$ be a symplectic toric manifold, and denote by $m_{ij} \in \mathbb{Z}$ ($1 \leq i \leq r$, $1 \leq j \leq n$) the entries of the matrix Mat($\mu$), and by $P_i$ the ring generators of $K^0(X_{\mu,K})$ – same as in Proposition 4.1.4. For a multi-index $d = (d_1, \ldots, d_r)$, we also use the notations $\alpha_j(d) := m_{1j}d_1 + \cdots + m_{rj}d_r$, $Q^d := Q_1^{d_1} \cdots Q_r^{d_r}$ and $U_j(P) = \prod_{i=1}^{r} P_{ij}^{m_{ij}}$. The
small \( K \)-theoretic \( I \)-function \( I^K_{X_{\mu,K}} \) of the toric manifold \( X_{\mu,K} \) is given by the \( K \)-theoretic formal series

\[
I^K_{X_{\mu,K}}(q,Q) = P^{-\ell(Q)} \sum_{d=(d_1,d_2)} Q^d \prod_{j=1}^n \frac{1 - U_j(P)q^r_j}{1 - U_j(P)q^r_j} \in K^0(X_{\mu,K})(q)[[Q]],
\]

where \( \ell(Q) \) is the \( q \)-logarithm \((4.3.12)\), \( P^{-\ell(Q)} := \prod_{i=1}^2 P_i^{-\ell(Q_i)} \in K^0(X_{\mu,K}) \), and \( P_i^{-\ell(Q_i)} \) should be understood as the expansion of the binomial \((1 - (1 - P_i^{-1}))^{\ell(Q_i)}\).

Using the ratio test we get that the \( K \)-theoretic \( I \)-function has the same analytic properties as the \( q \)-oscillatory integral, that is, if \( X_{\mu,K} \) is a Fano toric manifold and \( |q| > 1 \), then \( I^K_{X_{\mu,K}}(q,Q) \) can be expanded into a convergent power series in \( q^{-1} \), whose coefficients are polynomials in \( Q_i \) \((1 \leq i \leq 2)\) and \( \ell(Q_i) \) \((1 \leq i \leq 2)\).

**Proposition 4.3.15** (see also [GY21] p.21; [IMT15], Proposition 2.12). The \( K \)-theoretic oscillatory integral \( \mathcal{I}^{K-\text{th}} \) of Definition 4.3.4 and the small \( I \)-function \( I^K_{X_{\mu,K}} \) satisfy the same set of \( q \)-difference equations below (indexed by \( i \in \{1, \ldots, r\} \)):

\[
\left[ \prod_{j:m_{ij}>0} \prod_{r=0}^{m_{ij}-1} \left( 1 - q^{-r+\alpha_j(Q_0\partial Q)} \right) - Q_i \prod_{j:m_{ij}<0} \prod_{r=0}^{m_{ij}-1} \left( 1 - q^{-r+\alpha_j(Q_0\partial Q)} \right) \right] f_q(Q) = 0,
\]

where \( \alpha_j(Q_0\partial Q) = m_{ij} Q_i \partial Q_i + m_{2j} Q_0 \partial Q_2 \).

**Proof.** Denote by \( (m_{ij}) \) the coefficients of the matrix of the moment map \( \mu \). Note that for all \( j \in \{1, \ldots, n = N + 1 + k\} \) the following identity holds:

\[
q^{\alpha_j(Q_0\partial Q)} \mathcal{I}^{K-\text{th}}(q,Q_1,Q_2) = \left[ \int_{\Gamma_q} \prod_{j=1}^{m_1} q^{x_j \partial x_j} E_q^{-1} \left( \frac{x_1}{1-q} \right) \cdots E_q^{-1} \left( \frac{x_n}{1-q} \right) \omega_{\pi^{-1}}(Q) \right] q^r.
\]

Indeed, the action of the difference operator \( q^{x_j \partial x_j} \) inside the integrand amounts to rescaling \( x_j \rightarrow qx_j \). On the other hand, the integration is by definition an infinite sum over all \( x_a \in q^Z \) \((a \neq N, N+1)\) satisfying the relations \( x_1^{m_1} \cdots x_n^{m_n} = Q_i \) \((i = 1, 2)\). Changing the integration variables via \( y_a = x_a \) for \( a \neq j \) and \( y_j = qx_j \), we get that the sum defining the Jackson integral is a sum over all \( y_a \in q^Z \) \((a \neq N, N+1)\), satisfying the relations \( y_1^{m_1} \cdots y_n^{m_n} = q^{m_{ij} Q_i} \) \((i = 1, 2)\), while the integrand takes the form \( E_{q^{-1}}(y_1/(1-q)) \cdots E_{q^{-1}}(y_n/(1-q)) \). Clearly the resulting Jackson integral coincides with the LHS of the identity that we wanted to prove.

Next, we use that \( q^{x_{j \partial x_j}} E_q^{-1}(x/(1-q)) = (1-x)E_q^{-1}(x/(1-q)) \) to obtain that for all \( r \geq 0 \),

\[
\prod_{r=0}^{m-1} \left( 1 - q^{-r} q^{x_{j \partial x_j}} \right) E_q^{-1} \left( \frac{x}{1-q} \right) = x^m E_q^{-1} \left( \frac{x}{1-q} \right).
\]

Combining these two results we get

\[
(4.3.16) \quad \prod_{j:m_{ij}>0} \prod_{r=0}^{m_{ij}-1} \left( 1 - q^{-r} q^{\alpha_j(Q_0\partial Q)} \right) \mathcal{I}^{K-\text{th}}(q,Q_1,Q_2) = \\
\left[ \int_{\Gamma_q} \prod_{j:j'=m_{ij}>0} \prod_{r=0}^{m_{ij}-1} x_j^{m_{ij}'} E_q^{-1} \left( \frac{x_{j'}}{1-q} \right) \left( \prod_{j':m_{ij'}<0} E_q^{-1} \left( \frac{x_{j'}}{1-q} \right) \right) \omega_{\pi^{-1}}(Q) \right] q^r.
\]
Using the Batyrev relation we replace $\prod_{j':m_{ij'}>0} x_{j'}^{m_{ij'}}$ with $Q \prod_{j':m_{ij'}<0} x_{j'}^{-m_{ij'}}$. The RHS of (4.3.16) transforms into

\[(4.3.17)\quad Q_i\left[\int_{\Gamma_k} \left( \prod_{j':m_{ij'}\geq 0} E_{q^{-1}} \left( \frac{x_{j'}}{1-q} \right) \right) \left( \prod_{j':m_{ij'}<0} x_{j'}^{-m_{ij'}} E_{q^{-1}} \left( \frac{x_{j'}}{1-q} \right) \right) \omega_{q^{-1}}(Q) \right].\]

On the other hand, note that

\[(4.3.18)\quad \prod_{j':m_{ij'}<0} \prod_{r=0}^{m_{ij'}-1} \left( 1 - q^{-r} q^{\alpha_j(Q \partial q)} \right) I_{K^{-}\text{th}}(q, Q_1, Q_2) = \left[ \int_{\Gamma_k} \left( \prod_{j':m_{ij'}\geq 0} E_{q^{-1}} \left( \frac{x_{j'}}{1-q} \right) \right) \times \left( \prod_{j':m_{ij'}<0} x_{j'}^{-m_{ij'}} E_{q^{-1}} \left( \frac{x_{j'}}{1-q} \right) \right) \omega_{q^{-1}}(Q) \right].\]

We get that up to a factor of $Q_i$ the RHS of formula (4.3.18) coincides with (4.3.17). This completes the proof of the fact that the oscillatory integral is a solution to the $q$-difference system. For the $I$-function $I_{K^{-}\text{th}}^{X_{\mu,K}}$, we refer to Theorem p.8 of [Giv15b].

**Remark 4.3.19.** If our conjecture that the integral (4.3.8) has an expansion of the form (4.3.11) is true, then the $q$-oscillatory integral (4.3.7) makes sense and it has the following properties:

(a) The integral (4.3.7) is a solution to the system of $q$-difference equations in Proposition 4.3.15 for any choice of the positive real numbers $A_1, \ldots, A_n$.

(b) The integral (4.3.7) is a $q$-constant with respect to $A_i$ for all $1 \leq i \leq n$.

(c) If $Q_1, Q_2 \in q^Z$, then the two definitions of the $q$-oscillatory integral, that is, Definition 4.3.4 and (4.3.7) agree.

**Remark 4.3.20.** After rescaling the variables $Q_i$ by $(1-q)^{\deg(Q_i)} Q_i$, one can notice that the $q$-difference equation satisfied by our functions in the proposition above has a formal limit when $q \to 1$, using the formal limit

$$\lim_{q \to 1} \frac{1 - q^{Q_i \partial Q_i}}{1 - q} = Q_i \partial Q_i.$$ 

Moreover, this formal limit corresponds to the differential equation of Proposition 4.2.6, evaluated at $z = 1$. Confluence of the $q$-oscillatory integral will be investigated in the Subsection 4.5.

**4.4. Comparison theorem.** Let us recall the $q$-gamma function, defined by

$$\Gamma_{q^{-1}}(t) := (1-q)^{-t} (q^{-1}; q^{-1})_{\infty}^t (q^{-t}; q^{-1})_{\infty},$$

where $q > 1$. This function satisfies $\lim_{q \to 1} \Gamma_{q^{-1}}(t) = \Gamma(t)$, see e.g. Equation (1.10.3) and its proof p.21 in [GR]. We will make use of a multiplicative characteristic class defined via the following modification of the $q$-gamma function:

$$\gamma_q(t) := (1-q^{-1})^{1-t} \Gamma_{q^{-1}}(t) = \frac{(q^{-1}; q^{-1})_{\infty}}{(q^{-t}; q^{-1})_{\infty}}.$$
Definition 4.4.1 (**q-gamma class**). Suppose that \( q > 1 \) is a real number and that \( E \to X_{\mu,K} \) is a vector bundle. The **q-gamma class** \( \overline{\gamma}_q(E) \in H^*(X_{\mu,K};\mathbb{Q}) \) is defined by

\[
\overline{\gamma}_q(E) := \prod_{j=1}^m \delta_j \gamma_q(\delta_j) \in H^*(X_{\mu,K};\mathbb{Q}),
\]

where \( \delta_1, \ldots, \delta_m \) are the Chern roots of \( E \).

Let us compute the **q-gamma class** of the tangent bundle of a toric manifold \( X_{\mu,K} \). It is well known that \( TX_{\mu,k} = \sum_{j=1}^n U_j (P^{-1}) - 1 \) in \( K^0(X_{\mu,k}) \), where \( U_j (P^{-1}) = P^{-m_{ij}} \) and \( 1 \) is the trivial rank-1 bundle. On the other hand, we have \( \gamma_0(t+1) = (1-q^{-t}) \gamma_q(t) \) and

\[
\lim_{t \to 0} t \gamma_q(t) = \lim_{t \to 0} \frac{t}{1-q^{-t}} = \frac{1}{\log q}.
\]

In particular, if \( \epsilon \) is the trivial bundle of rank \( r \), then \( \gamma_q(\epsilon) = (\log q)^{-r} \) and the **q-gamma class** of a toric manifold takes the form

\[
\overline{\gamma}_q(TX_{\mu,K}) = (\log q)^r \prod_{j=1}^m \alpha_j(p) \gamma_q(\alpha_j(p)),
\]

where \( \alpha_j(p) := \sum_{i=1}^r p_i m_{ij} \) is the same as in Proposition 4.1.4.

Remark 4.4.2. Note also that in Equation (2.13) of [JMNT21], another **q-gamma class** is introduced for Grassmannians through a different motivation.

**Definition 4.4.3 (**q-Chern character**).** Let \( E \to X \) be a vector bundle, and denote by \( \delta_1, \ldots, \delta_m \) its Chern roots. The **q-Chern character** \( \text{ch}_q(E) \in H^*(X;\mathbb{Q}) \) is defined by

\[
\text{ch}_q(E) := (\log q)^{\deg} \circ \text{ch}(E) = \sum_{j=1}^m q^\delta_j \in H^*(X;\mathbb{C}).
\]

**Theorem 4.4.4.** Let \( X = X_{\mu,(\mathbb{R},\alpha)}^2 \) be a symplectic toric Fano manifold of Picard rank 2. Consider the associated **q-oscillatory integral** \( I^K_{-\text{th}} \) of Definition 4.3.4 and the **K-theoretic I-function** \( I^K_{X_{\mu,K}} \) of Definition 4.3.14. If \( q > 1 \) and \( Q_1, Q_2 \in q^\mathbb{Z} \), then the two functions are related by the following identity:

\[
I^K_{X_{\mu,K}}(q,Q_1,Q_2) = \int_{[X]} \overline{\gamma}_q(TX) \cup \text{ch}_q(I^K_{-\text{th}}(q,Q_1,Q_2)),
\]

where \( \int_{[X]} \) denotes the cap product with the fundamental class \([X] \in H_*(X_{\mu,K};\mathbb{C})\), \( \overline{\gamma}_q(TX_{\mu,K}) \) is the **q-gamma class** of Definition 4.4.1 and \( \text{ch}_q \) is the **q-Chern character** of Definition 4.4.3.

Our strategy to prove this identity goes as follows: we use the **q-Mellin transform** and its inversion formula to write the **q-oscillatory integral** \( I^K_{-\text{th}} \) as a Jeffrey–Kirwan residue. Then, we identify this Jeffrey–Kirwan residue as an intersection product using the results contained in Section 2 of [SV04], which will match with the right hand side of the identity we are trying to prove. Let us begin by defining the **q-Mellin transform** and its inverse. Then, we will state two computational lemmas, then give a proof of the theorem.
Definition 4.4.5 (q-Mellin transform). The \( q \)-Mellin transform of a function \( f \) is the formal Jackson integral

\[
\mathcal{M}_q(f)(p) := \left[ \int_0^\infty \right] q f(z) z^{p-1} d_q z = \sum_{n=-\infty}^\infty f(q^n) q^{np}.
\]

Notice that, at the level of functional operators, we have

\[
\mathcal{M}_q(q \partial_z) = q^{-p}, \quad \mathcal{M}_q(z) = \tau_1,
\]

where \( \tau_1 \) is the difference operator. Just like the Mellin transform changes a differential equation into a difference equation, the \( q \)-Mellin transform changes a \( q \)-difference equation into a difference equation. In general, the classical Mellin transform of a \( q \)-difference equation is not a difference equation.

Proposition 4.4.6 ([FBB06], Proposition 3). Let \( f \) be a function defined over the \( q \)-spiral \( q^\mathbb{Z} \), and assume there exist real numbers \( u > v \in \mathbb{R} \), such that,

\[
f(x) = O_{x \to 0^+}(x^u) \quad \text{and} \quad f(x) = O_{x \to 0^+}(x^v).
\]

Then, the \( q \)-Mellin transform \( \mathcal{M}_q f(p) \) is well defined in the complex strip \( \{ p \in \mathbb{C} : -u < \text{Re}(p) < -v \} \). \( \square \)

Theorem 4.4.7 (q-Mellin inversion formula; [FBB06], Theorem 2 p.315). Consider a function \( f \) defined on the \( q \)-spiral \( q^\mathbb{Z} \), and assume its Mellin transform is well defined on a complex strip \( \{ p \in \mathbb{C} : u < \text{Re}(p) < v \} \), for some \( u, v \in \mathbb{R} \), \( u < v \). Let \( \varepsilon \in \{ p \in \mathbb{C} : u < \text{Re}(p) < v \} \). Then, for any \( x \in q^\mathbb{Z} \),

\[
f(x) = \frac{\log(q)}{2\pi i} \int_{\varepsilon-i\pi/\log(q)}^{\varepsilon+i\pi/\log(q)} \mathcal{M}_q f(p) x^{-p} dp.
\]

\( \square \)

We now state two lemmas needed in the proof of Theorem 4.4.4.

Lemma 4.4.8. Let \( X_{\mu,K} \) be a Fano symplectic toric manifold of Picard rank 2 as in Proposition 4.1.6 and let \( q > 1 \). Denote by \( \mathcal{I}^{K-\text{th}} \) the \( q \)-oscillatory integral along the real Lefschetz thimble of Definition 4.3.4. The \( q \)-Mellin transform of the \( q \)-oscillatory integral is given by

\[
\mathcal{M}_q \mathcal{I}^{K-\text{th}}(q,p_1,p_2) = \gamma_q(p_1)^N \prod_{j=0}^k \gamma_q(p_2 - a_j p_1),
\]

where \( \gamma_q(p) = \left( \frac{q^{1/q}; q^{-1}}{q^p q^{-1}} \right)_\infty \).

The proof of this lemma will be done in Section 4.4.i.

Lemma 4.4.9. Let \( g(p_1,p_2) \) be the \( q \)-Mellin transform of the \( q \)-oscillatory integral, computed in the previous Lemma 4.4.8, that is,

\[
g(p_1,p_2) := \mathcal{M}_q \mathcal{I}^{K-\text{th}}(q,p_1,p_2) = \gamma_q(p_1)^N \prod_{j=0}^k \gamma_q(p_2 - a_j p_1).
\]
Then, the $q$-inverse Mellin transform of $g$ evaluated at $Q_1, Q_2 \in q^\mathbb{Z}$ can be computed by the following iterated residues:

$$M_q^{-1} g(Q_1, Q_2) = (\log q)^2 \text{Res}_{x_2=0} \text{Res}_{x_1=0} \omega^{K-\text{th}}(x_1, x_2) dx_1 dx_2,$$

where

$$\omega^{K-\text{th}}(x_1, x_2) = \sum_{d_1, d_2 \geq 0} Q_1^{-x_1+d_1} Q_2^{-x_2+d_2} \left( \gamma_q(x_1) \prod_{r=-\infty}^0 (1 - q^{-x_1+r}) \right)^N \prod_{j=0}^k \gamma_q(x_2 - a_j x_1) \prod_{r=-\infty}^0 \left(1 - q^{-x_2+a_j x_1+r}\right).$$

The proof of this lemma will be done in Section 4.4.ii.

**Proof of Theorem 4.4.4.** Using Lemma 4.4.8, we obtain that

$$M_q T^{K-\text{th}}(q, p_1, p_2) = \gamma_q(p_1)^N \prod_{j=0}^k \gamma_q(p_2 - a_j p_1).$$

Using the $q$-Mellin inversion formula of Theorem 4.4.7 and the computation of the inverse $q$-Mellin transform of Lemma 4.4.9, we obtain that

$$T^{K-\text{th}}(q, Q_1, Q_2) = (\log q)^2 \text{Res}_{x_2=0} \text{Res}_{x_1=0} \omega^{K-\text{th}}(x_1, x_2) dx_1 dx_2,$$

where $\omega^{K-\text{th}}(x_1, x_2)$ is the form defined in Lemma 4.4.9. Using Theorem 4.1.7, we identify the iterated residues above with the intersection product

$$T^{K-\text{th}}(q, Q_1, Q_2) = (\log q)^2 \int_{[ \mu, K]} x_1^N \prod_{j=0}^k (x_2 - a_j x_1) \omega^{K-\text{th}}(x_1, x_2),$$

where $x_1 = c_1(P_i^{-1}) = -c_1(P_i)$. Finally,

$$(\log q)^2 x_1^N \prod_{j=0}^k (x_2 - a_j x_1) \omega^{K-\text{th}}(x_1, x_2) = (\log q)^2 \left[ x_1^N \gamma_q(x_1)^N \prod_{j=0}^k (x_2 - a_j x_1) \gamma_q(x_2 - a_j x_1) \right] \times$$

$$\times \left[ Q_1^{-x_1} Q_2^{-x_2} \sum_{d_1, d_2 \geq 0} Q_1^{d_1} Q_2^{d_2} \left( \prod_{r=-\infty}^0 (1 - q^{-x_1+r}) \right)^N \prod_{j=0}^k \left(1 - q^{-x_2+a_j x_1+r}\right) \right].$$

The factor on the first line on the RHS of the above identity is the $q$-gamma class $\gamma_q(TX)$. Note that if $Q_i \in q^\mathbb{Z}$, then $\ell(Q_i) = -\log Q_i / \log q$. Therefore, the $q$-Chern character of $P^{\ell(Q)}$ is $Q_1^{-x_1} Q_2^{-x_2}$ and the factor on the second line on the RHS of the above identity is the $q$-Chern character of the $K$-theoretic $I$-function $\text{ch}_q \left( I_X^{K-\text{th}}(q, Q_1, Q_2) \right)$. \hfill $\square$

4.4.i. **Proof of Lemma 4.4.8.** This lemma is a corollary of the following $q$-integral representation of the $q$-gamma function due to Koelink–Koornwinder [KK92] (see also Theorem 3.2 in [DSK05]).

**Proposition 4.4.10.** Suppose that $q > 1$. The $q$-gamma function admits the following $q$-integral representation:

$$\Gamma_{q^{-1}}(p) = (1 - q^{-1})^{-p} \left[ \int_0^\infty \right] q \left( x^p \right) \frac{d_q x}{x}.$$
Proof of Lemma 4.4.8. By definition, the $q$-Mellin transform $\mathcal{M}_q \mathcal{I}^{K-\text{th}}(q, p_1, p_2)$ is given by the expression
\[
\left[ \int_{(\mathbb{R}, 0)^2} \right] q \left( \left[ \int_{\mathbb{R}_{\mathbb{Q}}} \right] E_{q^{-1}} \left( \frac{x_1}{1-q} \right) \cdots E_{q^{-1}} \left( \frac{x_n}{1-q} \right) \omega_{\pi^{-1}}(Q, q) \right) Q_1^{p_1} Q_2^{p_2} \frac{d_q Q_1}{Q_1} \frac{d_q Q_2}{Q_2}.
\]
By definition, the above integral is a sum over all $x_j \in q^{\mathbb{Z}}$ ($j \neq N, N+1$) and all $Q_1, Q_2 \in q^{\mathbb{Z}}$ satisfying the relations $Q_i = \prod_{j=1}^n x_j^{m_{ij}}$. These relations determine $x_N$ and $x_{N+1}$ in terms of $x_j$ ($j \neq N, N+1$) and $Q_1, Q_2$. Note that our sum can be viewed equivalently as a sum over all $x_j \in q^{\mathbb{Z}}$ ($1 \leq j \leq n$), where now we use the relations to solve for $Q_1$ and $Q_2$ in terms of $x_j$ ($1 \leq j \leq n$). Clearly, the above integral splits into a product of 1-dimensional integrals
\[
\mathcal{M}_q \mathcal{I}^{K-\text{th}}(q, p_1, p_2) = \prod_{j=1}^n \left[ \int_0^{\infty} \right] q E_{q^{-1}} \left( \frac{x_j}{1-q} \right) x_j^{\alpha_j(p)} \frac{d_q x_j}{x_j},
\]
where $\alpha_j(p) = p_1 m_{1j} + p_2 m_{2j}$. Recalling the explicit formulas for the moment matrix, we get
\[
\mathcal{M}_q \mathcal{I}^{K-\text{th}}(q, p_1, p_2) = \left( \left[ \int_0^{\infty} \right] q E_{q^{-1}} \left( \frac{x}{1-q} \right) x p \frac{d_q x}{x} \right)^N \prod_{j=0}^{k-1} \left[ \int_0^{\infty} \right] q E_{q^{-1}} \left( \frac{x}{1-q} \right) x^{p_{2j-1} - p_{1j}} \frac{d_q x}{x}.
\]
Finally, we use Proposition 4.4.10 to get
\[
\left[ \int_0^{\infty} \right] q E_{q^{-1}} \left( \frac{x}{1-q} \right) x p \frac{d_q x}{x} = (1 - q^{-1})^{p-1} \Gamma_{q^{-1}}(p) = \gamma_q(p).
\]
□

4.4.ii. Proof of Lemma 4.4.9. The proof of this lemma relies on proving a contour deformation result to compute the integral in the $q$-Mellin inversion formula of Theorem 4.4.7 using the residue theorem. Just like in the cohomological case (c.f. Subsection B.3), we are going to need a $q$-analogue of the Stirling formula.

Proposition 4.4.11 ([Moa84], Equation 2.13). Let $q < 1$ in this proposition only. For the usual $q$-gamma function $\Gamma_q$, the following $q$-analogue of the Stirling’s formula holds for any $z$ such that $\text{Re}(z) > 0$:
\[
\log \Gamma_q(z) \sim \left( z - \frac{1}{2} \right) \log \left( \frac{1-q^z}{1-q} \right) + \frac{1}{\log(q)} \text{Li}_2(1-q^z) + C_q + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \left( \frac{\log(q)}{q^z - 1} \right)^{2k-1} q^z p_{2k-3}(q^z),
\]
where $C_q \in \mathbb{C}$ is some constant depending on $q$, and $p_k$ is a degree $k$ polynomial satisfying the recursion relation
\[
p_0(z) = 1, \quad \forall k > 0, \quad p_k(z) = (z - z^2)p'_{k-1}(z) + (kz + 1)p_{k-1}(z).
\]
□

Proof of Lemma 4.4.9. To apply Theorem 4.4.7, we choose real numbers $\varepsilon_1, \varepsilon_2 > 0$ such that $0 < \max(1, a_j) \varepsilon_1 < \varepsilon_2$. We also choose to integrate first with respect to the coordinate $p_1$, then with respect to $p_2$. Thus, we will compute
\[
\mathcal{M}_q^{-1} g(Q_1, Q_2) := (\frac{\log(q)}{2\pi i})^2 \int_{p_2 = \varepsilon_2 - i\pi / \log(q)}^{\varepsilon_2 + i\pi / \log(q)} \left( \int_{p_1 = \varepsilon_1 - i\pi / \log(q)}^{\varepsilon_1 + i\pi / \log(q)} g(p_1, p_2) Q_1^{-p_1} Q_2^{-p_2} \frac{dp_1}{2\pi i} \right) \frac{dp_2}{2\pi i}.
\]
where
\[
g(p_1, p_2) = \gamma_q(p_1)^N \prod_{j=0}^{k} \gamma_q(p_2 - a_j p_1).
\]

We consider the following closed curve \( C(M) \), with the usual orientation:

\[
\begin{align*}
&\text{Im}(p_1) \\
&\text{Re}(p_1) \\
&-M + i\pi / \log(q) \quad \varepsilon_1 + i\pi / \log(q) \\
&\varepsilon_1 - i\pi / \log(q) \quad -M - i\pi / \log(q)
\end{align*}
\]

where we picked some large number \( M > 0 \) such that this curve does not encounter any of the poles of the function \( g \). Note also that \( \varepsilon_1 \) should be small enough so that the poles of the form \( p_1 = \frac{k + p_2}{a_j}, k \in \mathbb{Z}_{\geq 0} \) sit outside of the curve. Let us prove that for the inverse \( q \)-Mellin transform, when \( M \rightarrow +\infty \), integrating along the closed curve \( C(M) \) is the same as integrating along the cycle in the formula of Theorem 4.4.7, \( \int_{p_1 = \varepsilon_1 - i\pi / \log(q)}^{\varepsilon_1 + i\pi / \log(q)} g(p_1, p_2) Q_1^{p_1} Q_2^{-p_1} dp_1 \).

For the horizontal lines of \( C(M) \), let \( t \in (-M, \varepsilon_1) \) and \( Q_1 = q^{k_1}, k_1 \in \mathbb{Z}. \) Then, notice that the integrand satisfies
\[
g(t + i\pi / \log(q), p_2) Q_1^{t + i\pi / \log(q)} = g(t - i\pi / \log(q), p_2) Q_1^{t - i\pi / \log(q)}
\]

Therefore, when computing \( \int_{p_1 \in C(M)} g(p_1, p_2) Q_1^{-p_1} Q_2^{-p_1} dp_1 \), the integrals along the horizontal lines cancel each other.

For the vertical line \((-M + i\pi / \log(q), -M - i\pi / \log(q))\) of \( C(M) \), write \( p_1 = a + ib, a < 0. \) For such choice (recall \( q > 1 \)),
\[
|\gamma_q(p_1)| = \left| \left(1 - q^{-1}\right) \frac{(q^{-1}; q^{-1})_\infty}{(q^{-p_1}; q^{-p_1})_\infty} \right| \leq |1 - q| \left(1 - q^{-1}\right) \frac{(q^{-1}; q^{-1})_\infty}{(q^{-p_1}; q^{-p_1})_\infty} q^a.
\]

Therefore \(|\gamma_q|\) has exponential decay as \( a \rightarrow -\infty \). Let us explain that the function given by \(|\gamma_q(p_1)\gamma_q(-p_1)|\) will also have exponential decay when \( \text{Re}(p_1) \rightarrow \pm \infty \). Using Moak’s \( q \)-analogue of the Stirling formula (see Proposition 4.4.11, recall \( \gamma_q(p) = (1 - q^{-1})^{p-1} \Gamma_{q^{-1}}(p) \)), we have for \( \text{Re}(p) > 0 \),
\[
\log |\Gamma_{q^{-1}}(p)| \sim_{p \rightarrow \infty} -\text{Re}(p) \log(1 - q^{-1}),
\]
while, for $\text{Re}(p) < 0, p \notin \mathbb{Z}_{\leq 0}$, we have
\[
\left| (q^{-p}; q^{-1}) \right| \sim \left| \prod_{k = -\text{Re}(p) - k \geq 0} (1 - q^{p-k}) \right| \sim q^{-\text{Re}(p)(-\text{Re}(p)+1)/2}.
\]
Thus, the function given by $|\gamma_q(p_1) \gamma_q(-p_1)|$ has exponential decay when $\text{Re}(p_1) \to \pm \infty$. Finally, we can write the integrand as
\[
\left( \frac{\gamma_q(p_1)^N}{\prod_{j=0}^k \gamma_q(a_j p_1 - p_2)} \right) \left( \prod_{j=0}^k \gamma_q(p_2 - a_j p_1) \gamma_q(-p_2 + a_j p_1) \right).
\]
Our previous observation gives that the second factor in the big parentheses has exponential decay. Using the Fano condition $N - \sum a_j > 0$, the first factor $\left| \frac{\gamma_q(p_1)^N}{\prod_{j=0}^k \gamma_q(a_j p_1 - p_2)} \right|$ also has exponential decay. Therefore, we have proved for $|Q_1| < 1$,
\[
\lim_{M \to +\infty} \int_{p_1 \in \mathcal{C}(M)} g(p_1, p_2) Q_1^{p_1} Q_2^{p_2} dp_1 = \int_{p_1=\epsilon_1-i\pi/\log(q)}^{\epsilon_1+i\pi/\log(q)} g(p_1, p_2) Q_1^{p_1} Q_2^{p_2} dp_1.
\]
Now, we apply the residue theorem to the left hand side, using that the poles of the integrand inside the contour are exactly at $p_1 \in \mathbb{Z}_{\leq 0}$. We obtain
\[
\frac{1}{2\pi i} \int_{p_1=\epsilon_1-i\pi/\log(q)}^{\epsilon_1+i\pi/\log(q)} g(p_1, p_2) Q_1^{p_1} Q_2^{p_2} dp_1 = \sum_{d_1 \geq 0} \text{Res}_{p_1=-d_1} g(p_1, p_2) Q_1^{p_1} Q_2^{p_2} dp_1.
\]
We will do a similar contour deformation for the coordinate $p_2$, using the same contour as for the previous coordinate $p_1$. Using the same reasoning, we obtain that
\[
\lim_{M \to +\infty} \int_{p_2 \in \mathcal{C}(M)} \sum_{d_1 \geq 0} \text{Res}_{p_1=-d_1} g(p_1, p_2) Q_1^{p_1} Q_2^{p_2} dp_1 dp_2
\]
\[
= \int_{p_2=\epsilon_2-i\pi/\log(q)}^{\epsilon_2+i\pi/\log(q)} \sum_{d_1 \geq 0} \text{Res}_{p_1=-d_1} g(p_1, p_2) Q_1^{p_1} Q_2^{p_2} dp_1 dp_2.
\]
Recall that $g(p_1, p_2) := \gamma_q(p_1)^N \prod_{j=0}^k \gamma_q(p_2 - a_j p_1)$ and $a_0 := 0$, therefore when applying the residue theorem, the poles that are inside the contour are exactly given by $p_2 \in \mathbb{Z}_{\leq 0}$. Next, we do a change of variable $p_1 = x_1 - d_1, p_2 = x_2 - d_2$ and use Fubini theorem to permute the sums and residues. To obtain the identity announced in the statement of the lemma, it remains to use the difference equation for the function $\gamma_q$, that is,
\[
\gamma_q(p-1) = \frac{(q^{-1}; q^{-1})_\infty}{(q^{-p+1}; q^{-1})_\infty} = \frac{1}{1 - q^{-p+1}} \gamma_q(p).
\]
\[\square\]

4.5. Confluence of the comparison theorem. We would like to use Theorems 3.4.1 and 4.4.4 to give a 2nd proof of Theorem 4.2.8 in the case of Fano toric manifolds of Picard rank 2. If we compare the proofs of Theorem 4.4.4 and Theorem 4.2.8 for Picard rank 2, then we see that the argument in the $K$-theoretic case is somewhat easier. Therefore, it looks promising that our proof of Theorem 4.4.4 generalizes to all weak Fano toric manifolds and hence by using the confluence result from Theorem 3.4.1, we would be able to obtain a new proof of Iritani’s theorem.
Proposition 4.5.1. a) We have $0 < a_k < \frac{1}{\sqrt{k}}$ for all sufficiently large $k$.

b) If $Q > 0$ is a real number, then there exists a sequence of integers $b_k$, such that, $q_k^{b_k} \to Q$ when $k \to \infty$.

Proof. a) Put $g(q) = q^{k+1} - q^k - 1$. We have to prove that $q_k = 1 + a_k < 1 + 1/\sqrt{k}$ for $k \gg 0$. Since $g'(q) = q^{k-1}((k+1)q - k) > 0$ for $q > 1$. The inequality is equivalent to $g(1 + 1/\sqrt{k}) > g(q_k) = 0$. On the other hand,

$$g(1 + 1/\sqrt{k}) = \left(1 + \frac{1}{\sqrt{k}}\right)^k \frac{1}{\sqrt{k}} - 1 > \frac{2\sqrt{k}}{\sqrt{k}} - 1$$

and the above expression tends to $+\infty$ when $k \to +\infty$. In particular, the inequality that we need holds for $k \gg 0$.

b) It is sufficient to consider the case when $Q > 1$, that is, $c = \log Q > 0$. We claim that the interval $[c/a_k, (c + 1/\sqrt{k})/a_k]$ has length $> 1$ for $k \gg 0$ and hence it contains at least one integer $b_k$. Indeed,

$$\frac{1}{a_k} \left(c + \frac{1}{\sqrt{k}}\right) - \frac{c}{a_k} = \frac{1}{\sqrt{k}a_k} > 1,$$

where the last inequality holds for $k \gg 0$ thanks to part a). Let $b_k$ be an integer in the above interval, then

$$0 < a_kb_k - c < \frac{1}{\sqrt{k}}.$$ 

In particular, $a_kb_k \to c$ when $k \to \infty$. Finally,

$$q_k^{b_k} = (1 + a_k)^{b_k} = \left((1 + a_k)^{1/a_k}\right)^{a_kb_k} \to q = e^c = Q,$$

where we used that according to part a), $a_k \to 0$, so $(1 + a_k)^{1/a_k} \to e$. □

Proof of Theorem 4.2.8. We are going to apply the identity in Theorem 4.4.4 in the following settings. Let $Q_i^0$ ($i = 1, 2$) be positive real numbers. Let us choose integer sequences $b'_k$ ($k \geq 1$), such that, $q_k^{b'_k} \to Q_i^0$ when $k \to \infty$ (see Proposition 4.5.1, b)). Let us fix an integer $k > 0$ and set $q = q_k$, $Q_i = (q_k - 1)^{m_i}q_k^{b'_k}$, where $m_i = \sum_{j=1}^n m_{ij}$. Let us rewrite the $q$-oscillatory integral

$$\int_{\Gamma_k} \prod_{j=1}^n E_q^{-1} \left(\frac{x_j}{1 - q}\right) \omega_{q^{-1}}(Q_i)q$$

in two different ways. First, by definition the integral is a sum over all $x_j \in q\mathbb{Z}$ satisfying the relations $\prod_{j=1}^n x_j^{m_j} = Q_k$. Let us change the variables by $x_j = (q - 1)y_j$. Then, since $q - 1 = q_k - 1 = q_k^{-k} \in q\mathbb{Z}$, the
integral becomes a sum over all \( y_j \in q^Z \) satisfying the relations \( \prod_j y_j^{m_{ij}} = (q - 1)^{-m_i} Q_1 \). Therefore, the integral turns into
\[
(1 - q^{-1})^{N-1+k} \int_{\Gamma_{\mathbb{R}}} \prod_{j=1}^{n} E_{q^{-1}} (-y_j) \omega_{\pi^{-1}}((q-1)^{-m_1} Q_1, (q-1)^{-m_2} Q_2), q.
\]

In the limit when \( k \to \infty \), we have \( q = q_k \to 1 \), the Jackson integral \( (1 - q^{-1})^{N-1+k} \int_{\Gamma_{\mathbb{R}}} \) tends to the Riemannian integral \( \int_{\Gamma_{\mathbb{R}}} \) while the integrand tends to \( \prod_j e^{-y_j} \omega_{\pi^{-1}}(Q^\circ) \), where \( Q^\circ = (Q_1^\circ, Q_2^\circ) \) and we used that \( (q-1)^{-m_i} Q_i = q_k^{b_k^i} \to Q_i^\circ \). In other words, in the limit \( k \to \infty \), the Jackson integral (4.5.2) tends to \( I_{\coh}(1, Q^\circ) \).

The integral (4.5.2) coincides with \( (1 - q^{-1})^{N-1+k} T^{K,\text{th}}(q, Q_1, Q_2) \). Using Theorem 4.4.4, we rewrite (4.5.2) as
\[
(4.5.3) \quad (1 - q^{-1})^{N-1+k} \int_{[X]} \tilde{\gamma}_q(TX) \text{ch}_q \left( I_{\text{th}}^K(q, Q) \right),
\]
where \( X = X_{\mu, K} \). Under the Fano condition the \( I \)-function essentially coincides with the \( J \)-function (see [Giv15b], Theorem p.8), that is,
\[
(4.5.4) \quad I_{\text{th}}^K(q, Q) = (1 - q)^{-1} P^{-\ell}(Q) J_{\text{th}}^K(q, Q).
\]

Since \( Q_i \in q^Z \), we have \( P^{-\ell}(Q) = \prod_i P_i^{\log q_i} \). The \( q \)-Chern character of the \( I \)-function takes the form
\[
\text{ch}_q \left( I_{\text{th}}^K(q, Q) \right) = (\log q)^{\deg} \text{ch} \left( \prod_i P_i^{\log q_i} \right) = -\frac{\log q}{q-1} e^{-\sum_i p_i \log Q_i} (\log q)^{\deg-1} \text{ch} \left( J_{\text{th}}^K(q, Q) \right).
\]

Let us introduce also the \( \Gamma_{q^{-1}} \)-class of a vector bundle \( E \) by
\[
\tilde{\Gamma}_{q^{-1}}(E) := \prod_{\delta \text{: Chern roots of } E} \delta \Gamma_{q^{-1}}(\delta).
\]

Note that
\[
\tilde{\gamma}_q(TX) = \left( \frac{\log q}{1 - q^{-1}} \right)^2 (1 - q^{-1})^{-N+1-k} (1 - q^{-1})^\rho \Gamma_{q^{-1}}(TX)
\]
and that, since \( \rho = m_1 p_1 + m_2 p_2 \),
\[
(1 - q^{-1})^\rho e^{-\sum_i p_i \log Q_i} = q^\rho e^{-\sum_i p_i \log ((q-1)^{-m_i} Q_i)}.
\]

We get that (4.5.3) can be written as
\[
-\left( \frac{\log q}{1 - q^{-1}} \right)^3 \int_{[X]} \tilde{\Gamma}_{q^{-1}}(TX) q^{-\rho-1} e^{-\sum_i p_i \log ((q-1)^{-m_i} Q_i)} (\log q)^{\deg-1} \text{ch} \left( J_{\text{th}}^K(q, Q) \right).
\]

Let us compute the limit of the above expression when \( k \to \infty \). Since \( Q_i = (q_k - 1)^{m_i} q_k^{b_k^i} \) and \( q_k^{b_k^i} \to Q_i^\circ \), Theorem 3.4.1 implies that \( (\log q)^{\deg-1} \text{ch} \left( J_{\text{th}}^K(q, Q) \right) \to J_{\text{th}}^{\coh}(1, Q^\circ) \). Therefore, the limit of (4.5.3) is
\[
- \int_{[X]} \tilde{\Gamma}(TX) e^{-\sum_i p_i \log Q_i} J_{\text{th}}^{\coh}(1, Q^\circ).
\]
Since $J_X^{\text{coh}}(1, Q^o) = -J_X^{\text{coh}}(1, Q^o)$, we get precisely the identity in Theorem 4.2.8 for the case when $z = 1$. The general case follows from the homogeneity property (1.1.4).

\section*{Appendix A. Proof of the Givental–Tonita recursion}

We would like to outline the proof of formula (3.3.5). The idea is to express the integral in (3.3.1) as an integral over the fiber product in (3.2.1). The main difficulty is to find the image of the inertia tangent $T_{IX_0,1,d}$ and the inertia normal $N_{IX_0,1,d}$ bundles on $I_\xi X_{0,1,d}$ via the isomorphism (3.2.1). Strictly speaking, we have to solve this problem for any $B$-point of $I_\xi X_{0,1,d}$, where $B$ is an arbitrary scheme $B$. However, we will do this only in the case when $B = \text{Spec}\, \mathbb{C}$.

\subsection*{A.1. Virtual tangent space}

Suppose that $C = (C, s_1, \ldots, s_n, f)$ is a point in the moduli space $X_{g,n,d}$. The restriction of the virtual tangent bundle $T_{g,n,d}$ on $X_{g,n,d}$ to $C$ is the virtual vector space $T(C) = -T^0(C) + T^1(C) - T^2(C)$, where

\begin{align*}
T^0(C) &:= H^0(C, \mathcal{T}_C(-s_1 - \cdots - s_n)), \\
T^1(C) &:= H^1(C, \mathcal{T}_C(-s_1 - \cdots - s_n)) + H^0(C, f^* T_X) + \bigoplus_{z \in \text{Sing}(C)} T'_z \otimes T''_z, \\
T^2(C) &:= H^1(C, f^* T_X).
\end{align*}

Here the notation is as follows. $\mathcal{T}_C$ is the sheaf of holomorphic vector fields on $C$ and $\mathcal{T}_C(-s_1 - \cdots - s_n)$ is the sheaf of holomorphic vector fields vanishing at the marked points $s_1, \ldots, s_n$. $T'_z$ and $T''_z$ are the tangent spaces at $z$ to the two irreducible components of $C$ that meet at $z$. The groups $T^i(C)$ come from the deformation theory of the stable map $C = (C, s_1, \ldots, s_n, f)$. Namely, $T^0(C)$ is the Lie algebra of the group of automorphisms $\text{Aut}(C, s_1, \ldots, s_n)$, $T^1(C)$ is the vector space of infinitesimal deformations, and $T^2(C)$ is the obstruction space. We refer to [Pal76] for more details on deformation theory.

Suppose now that $(C, g) = (C, s_1, \ldots, s_n, f, g)$ is a point in the inertia moduli space $IX_{0,n,d}$. The automorphism $g$ acts on the virtual tangent space $T(C)$. Let us denote by

\[ T(C, g)_\lambda := \{ v \in T(C) \mid gv = \lambda v \} \]

the eigensubspace with eigenvalue $\lambda$ and let $I_\lambda T_{0,n,d}$ be the virtual vector bundle on $IX_{0,n,d}$ whose fiber at $(C, g)$ is $T(C, g)_\lambda$. Note that the inertia tangent bundle is $T_{IX_{0,n,d}} := I_1 T_{0,n,d}$ and the inertia normal bundle is $N_{IX_{0,n,d}} = \oplus_{\lambda} I_\lambda T_{0,n,d}$. It is convenient to introduce the notation

\[ \tilde{\text{td}}(T_{0,n,d}) = \prod_{\lambda \in \mathbb{C}^*} \text{td}_{-1}(I_\lambda T_{0,n,d}), \]

where $\text{td}_1$ is the usual Todd class and $\text{td}_\lambda$, $\lambda \neq 1$, is the \textit{moving Todd class}

\[ t_{\lambda}(E) = \prod_{\text{Chern roots } x} \frac{1}{1 - e^{-\lambda e^{-x}}}. \]

Then the integral in formula (3.3.1) can be written as

\begin{equation}
(A.1.1) \quad \tau_{\xi,d,a}(q) := \int_{[I_\xi X_{0,1,d}]} \tilde{\text{td}}(T_{0,1,d}) \frac{\ev_1^* \text{ch}(\Phi^a)}{1 - q \text{ch}(\text{Tr} L_1)}. \end{equation}
Let us recall the notation from Section 3.2. Suppose that \((C, g) = (C, s_1, f, g)\) is a point in \(I_\zeta X_{0,1,d}\) obtained via the isomorphism (3.2.1) from an orbifold stable map \(\mathcal{C}_0 := (C_0, s_0^1, \ldots, s_0^{k+2}, f_0) \in [X/\mu_m]_{0,k+2,d_0}^{1,1} \) and a collection of \(k\) stable maps \((C_i, g_i) = (C_i, s_i^1, f_i, g_i) \in I_n X_{0,1,d_i}\) \((1 \leq i \leq k)\). In other words, the curve \(\mathcal{C}\) decomposes as in (3.2.2) and the automorphism \(g\) has the form (3.2.3)–(3.2.4). Let \(\mathcal{C}_0 := (\mathcal{C}_0, s_1, s_{i,a}) (1 \leq i \leq k, 1 \leq a \leq m), s_{k+2}, f_0\) be the stable map, such that, \(\mathcal{C}_0 = [\mathcal{C}_0/\mu_m]\). Our first goal, is to express the eigensubspaces \(T(C, g)\lambda_i\) in terms of the eigensubspaces \(T(\mathcal{C}_0, g_0)\lambda_i\) and \(T(C_1, g_1)\lambda_i\) \((1 \leq i \leq k)\).

**Lemma A.1.2.** Let \(\mathcal{C} = (C, s_1, \ldots, s_n, f)\) be a stable map obtained from gluing two stable maps

\[
\mathcal{C}' = (C', s_1, \ldots, s_{n_1}, z', f_1) \quad \text{and} \quad \mathcal{C}'' = (C'', s_{n_1+1}, \ldots, s_{n+n_2}, z'', f_2),
\]

that is, we have \(n_1 + n_2 = n\), \(f_1(z') = f_2(z'')\), and the points \(z'\) and \(z''\) are identified yielding a node \(z\) of \(C\). Then the virtual tangent space decomposes as follows:

\[
T(C) = T(C') + T(C'') + T_zC' \otimes T_zC'' - T_{f(z)}X.
\]

**Proof.** Locally near the node \(z\) we have \(\mathcal{O}_{C,z} = \mathcal{C}\{x, y\}/\langle xy \rangle\), where \(\mathcal{O}_C\) is the structure sheaf of \(C\) and \(x\) (resp. \(y\)) is a holomorphic coordinate on \(C'\) (resp. \(C''\)) near \(z'\) (resp. \(z''\)). By definition, the stalk \(T_{C,z}\) of the tangent sheaf at \(z\) is given by the \(\mathcal{O}_{C,z}\)-module of derivations \(\text{Der}(\mathcal{O}_{C,z}, \mathcal{O}_{C,z})\). If \(u : \mathcal{O}_{C,z} \to \mathcal{O}_{C,z}\) is a derivation, then put \(u_1 := u(x)\), \(u_2 = u(y)\). We have \(0 = u(xy) = yu_1 + xu_2\) in \(\mathcal{O}_{C,z} \Rightarrow u_1 \in x\mathcal{O}_{C,z} = x\mathcal{C}\{x\}\) and \(u_2 \in y\mathcal{O}_{C,z} = y\mathcal{C}\{y\} \Rightarrow u \in x\mathcal{C}\{x\} \partial_x + y\mathcal{C}\{y\} \partial_y\). Therefore,

\[
T_{C,z} = x\mathcal{C}\{x\} \partial_x + y\mathcal{C}\{y\} \partial_y.
\]

Let \(\iota' : C' \to C\) and \(\iota'' : C'' \to C\) be the natural inclusions. Then the above formula shows that

\[
T_C(-s_1 - \cdots - s_n) = \iota'_* T_{C'}(-s_1 - \cdots - s_{n_1} - z') + \iota''_* T_{C''}(-s_{n_1+1} - \cdots - s_{n+n_2} - z'')
\]

By comparing stalks, we can prove that the following short exact sequence of sheaves on \(C\) is exact

\[
0 \longrightarrow f^* T_X \longrightarrow \iota'_* (f_1^* T_X) \oplus \iota''_* (f_2^* T_X) \longrightarrow \mathcal{O}_z \otimes T_{f(z)}X \longrightarrow 0,
\]

where \(\mathcal{O}_z\) is the structure sheaf of the point \(z\). Using the long exact cohomology sequence of (A.1.4) and (A.1.3) we get the formula stated in the lemma.

**A.2.** **Splitting into stems and legs.** Using Lemma A.1.2 we get the following formula for the virtual tangent space:

\[
T(C) = T(\mathcal{C}_0) - T_{s_{k+2}} \mathcal{C}_0 + \bigoplus_{i=1}^{k} \bigoplus_{a=1}^{m} T(C_i \times \{a\}) + \bigoplus_{i=2}^{k+2} \bigoplus_{a=1}^{m} (T_{s_{i,a}} \mathcal{C}_0 \otimes T_{s_{i,a}} C_i - T_{f_0(s_{i,a})} X),
\]

where the 2nd term on the RHS corresponds to forgetting that the but \(s_{k+2}\) is a marked point. The RHS of (A.2.1) splits naturally into \(k\) types of subspaces. Each type corresponds to the fiber of a certain virtual vector bundle on the fiber product in (3.2.1). We would like to work out the contributions of each of these \(k\) type of virtual vector bundles to the total Todd class \(\tilde{t}(T_{0,1,d})\). Our computation splits naturally into \(k\) cases.
Case 1: Contribution from $T(\widetilde{C}_0)$. Note that $\widetilde{C}_0$ is a $g$-invariant curve and that the restriction of $g$ to $\widetilde{C}_0$ is $g_0$. Let us denote by $T(\widetilde{C}_0)_\lambda$ the eigensubspace of $g_0$ with eigenvalue $\lambda$. Clearly, $\lambda = \zeta^{-k}$ ($0 \leq k \leq m - 1$) must be a $m$th root of 1.

Let $Y = [X/\mu_m]$ and let us consider the commutative diagram involving the universal curve

$$
\begin{array}{ccc}
C_0 & \xrightarrow{j} & Y_{0,k+3,d_0} \\
\pi \downarrow & & \downarrow \pi \\
\text{Spec}(\mathbb{C}) & \xrightarrow{j} & Y_{0,k+2,d_0}^{\eta,1,\ldots,1,\eta^{-1}}
\end{array}
$$

where $j$ is the inclusion corresponding to the orbifold stable map $C_0$. Let us denote by

$$
T_{0,k+2,d_0} := \pi_* \ev^*(TX) - \pi_* L^{-1} - (\pi_* \mathcal{O}_Z)^\vee
$$

the virtual tangent bundle on $Y_{0,k+2,d_0}^{\eta,1,\ldots,1,\eta^{-1}}$, where $\pi_* = R^0\pi_* - R^1\pi_*$ is the K-theoretic pushforward. Slightly abusing the notation we view $T_{0,k+2,d_0}^0$ as an element in the topological K-ring of $Y_{0,k+2,d_0}^{\eta,1,\ldots,1,\eta^{-1}}$. One can check that the eigensubspace $T(\widetilde{C}_0)_1$ coincides with $j^* T_{0,k+2,d_0}^0$ and that the eigensubspace $T(\widetilde{C}_0)_{\zeta^{-k}}$ coincides with

$$
j^* \left( \pi_* \ev^*(TX \otimes \mathbb{C}_{\zeta^k}) + \pi_* ((1 - L^{-1}) \otimes \ev^* \mathbb{C}_{\zeta^k}) - \left( \pi_* (\mathcal{O}_Z \otimes \ev^* \mathbb{C}_{\zeta^k})^\vee \right) \right).
$$

Comparing with formulas (2.4.1)–(2.4.2) we get that the contribution to $\widehat{\text{td}}(T_{0,1,d})$ is given by the cohomology class $\Theta^{ABC} \in H^*(Y_{0,k+2,d_0}^{\eta,1,\ldots,1,\eta^{-1}}, \mathbb{C})$ used in the definition of the stem invariants (see Section 2.4).

Case 2: Contribution from $T_{s_k+2}\widetilde{C}_0$. Note that this is a one dimensional vector space on which $g$ acts with eigenvalue $\zeta$. The cotangent line $T^*_s\widetilde{C}_0$ is by definition the fiber of the orbifold line bundle $L_{k+2}$ on $Y_{0,k+2,d_0}^{\eta,1,\ldots,1,\eta^{-1}}$. Therefore, we can identify $T_{s_k+2}\widetilde{C}_0$ with the fiber of an orbifold line bundle $L^{-1/m}_{k+2}$, that is, an $m$-th root of $L^{-1}_{k+2}$. The contribution to $\widehat{\text{td}}(T_{0,1,d})$ is given by

$$(A.2.2) \quad \text{td}_{\zeta^{-1}}(-L^{-1/m}_{k+2}) = 1 - \zeta^{-1}e^{\psi_{k+2}/m} \in H^*(Y_{0,k+2,d_0}^{\eta,1,\ldots,1,\eta^{-1}}, \mathbb{C}),$$

where $\psi_{k+2} = c_1(L_{k+2})$.

Case 3: Contribution from $\bigoplus_{i=1}^m T(C_i \times \{a\})$, $1 \leq i \leq k$. To begin with, note that we have the following identification of eigensubspaces:

$$
\left( \bigoplus_{i=1}^m T(C_i \times \{a\}) , g \right)_\lambda \cong T(C_i, g_i)^{\lambda^m}.
$$

Indeed, suppose that $(v_1, \ldots, v_m)$ is an element of the vector space on the LHS of the above isomorphism. Recalling the definition of $g_i$, we get

$$
\lambda(v_1, \ldots, v_m) = g(v_1, \ldots, v_m) = (g_i(v_m), v_1, \ldots, v_{m-1}).
$$

Comparing the components, we get

$$
v_{m-i} = \lambda^i v_m \quad (1 \leq i \leq m-1)
$$
and \(g_{i}(v_{m}) = \lambda^{m}v_{m}\). Therefore, the isomorphism is given by \((v_{1}, \ldots, v_{m}) \mapsto v_{m}\).

The vector space \(T(C_{i}, g_{i})\) is the fiber of the virtual vector bundle \(I_{\xi}T_{0,1,d_{i}}\) on \(I_{\eta}X_{0,1,d_{i}}\) introduced in Section A.1. The contribution to \(\tilde{td}(T_{0,1,d})\) is given by

\[
\prod_{x \in \text{Chern roots of } I_{T_{0,1,d_{i}},l}} \frac{x}{1 - e^{-x}} \prod_{\lambda \in \mathbb{C}^{*}} x \in \text{Chern roots of } I_{\lambda^{m}T_{0,1,d_{i}},l} \frac{1}{1 - \lambda^{-1}e^{-x}}.
\]

Let us rewrite the product over \(\lambda\) in the following way. Let us take the terms for which \(\lambda^{m} = 1\), that is, \(\lambda = \zeta^{-k}\) for \(1 \leq k \leq m - 1\) and combine them with the terms of the first product. Then the terms in the first product will change to \(\frac{x}{1 - e^{-mx}}\). The remaining part of the product over \(\lambda\) can be parametrized by \(\lambda = \xi^{1/m} \zeta^{-k}\), where \(\xi \neq 1\) and \(1 \leq k \leq m\). The above formula takes the form

\[
\prod_{x \in \text{Chern roots of } I_{T_{0,1,d_{i}},l}} \frac{x}{1 - e^{-mx}} \prod_{\xi^{1/m} \zeta^{-k} \in \mathbb{C}^{*}} x \in \text{Chern roots of } I_{\xi^{m}T_{0,1,d_{i}},l} \frac{1}{1 - \xi^{-1}e^{-mx}}.
\]

The above formula coincides with

(A.2.3) \(m^{-\text{rk}(I_{T_{0,1,d_{i}},l})} \tilde{td}(\Psi_{m}^{T_{0,1,d_{i}},l}) \in H^{*}(I_{\eta}X_{0,1,d_{i}}, \mathbb{C})\),

where \(\Psi_{m}\) is the \(m\)th Adam’s operation.

**Case 4:** Contribution from \(\bigoplus_{a=1}^{m} T_{s_{i,a}}C_{0} \otimes T_{s_{1}^{i},l}C_{i}, 1 \leq i \leq k\). To begin with, note that we have the following identification of eigensubspaces:

\[
\left( \bigoplus_{a=1}^{m} T_{s_{i,a}}C_{0} \otimes T_{s_{1}^{i},l}C_{i}, g \right)_{\lambda} \cong T_{s_{i+1},l}C_{0} \otimes T_{s_{1}^{i},l}C_{i}.
\]

Indeed, suppose that \((v_{1} \otimes w_{1}, \ldots, v_{m} \otimes w_{m})\) is an eigenvector, i.e., an element of the LHS. Recalling again the definition of \(g_{i}\), we get

\[
\lambda(v_{1} \otimes w_{1}, \ldots, v_{m} \otimes w_{m}) = g(v_{1} \otimes w_{1}, \ldots, v_{m} \otimes w_{m}) = (v_{m} \otimes g_{i}(w_{m}), v_{1} \otimes w_{1}, \ldots, v_{m-1} \otimes w_{m-1}).
\]

By definition, \(g_{i}(w_{m}) = \eta_{i}^{-1}w_{m}\). Comparing the components in the above equality, we get

\[
v_{i} \otimes w_{i} = \lambda^{m-i}v_{m} \otimes w_{m} \quad (1 \leq i \leq m - 1)
\]

and \(\lambda^{m} = \eta_{i}^{-1}\). The desired isomorphism is given by \((v_{1}, \ldots, v_{m} \otimes w_{m}) \mapsto v_{m} \otimes w_{m}\). Moreover, we proved that all possible values for the eigenvalue are given by \(\lambda = \eta_{i}^{-1/m} \zeta^{k} \quad (1 \leq k \leq m)\).

The vector spaces \(T_{s_{i+1},l}C_{0}\) and \(T_{s_{1}^{i},l}C_{i}\) are respectively the fibers of the orbifold vector bundles \(L_{i+1}^{(0)} := L_{i+1} \otimes \left[X/\mu_{m_{1}}\eta_{i},\ldots,\tilde{\eta}_{1}\left[1,\ldots,1,\eta^{-1}\right]\right]\) and \(L_{1}^{(i)} := L_{1} \otimes \left[I_{\eta}X_{0,1,d_{i}}\right].\) The contribution to \(\tilde{td}(T_{0,1,d})\) is given by

(A.2.4) \[
\frac{1}{\prod_{k=1}^{m} 1 - \eta_{i}^{1/m} \zeta^{-k} e^{\psi_{i+1}^{(0)} + \psi_{1}^{(i)}}} = \frac{1}{1 - \eta_{i} e^{m(\psi_{i+1}^{(0)} + \psi_{1}^{(i)})}} = \Psi_{m}^{T_{0,1,d}}\left(\frac{1}{1 - \eta_{i} \operatorname{ch}(L_{i+1}^{(0)} \otimes L_{1}^{(i)})}\right),
\]

where \(\psi_{i+1}^{(0)} = c_{1}(L_{i+1}^{(0)})\) and \(\psi_{1}^{(i)} = c_{1}(L_{1}^{(i)})\). Here, using the Chern character map, we extend the Adam’s operations to cohomology, that is, if \(\phi\) is a homogeneous cohomology class in \(H^{2k}(M; \mathbb{C})\) for some topological space \(M\), then \(\Psi_{m}^{T_{0,1,d}}(\phi) := m^{k}\phi\).
Case 5: Contributions from $\bigoplus_{a=1}^m T_{f_0(s_{i,a})}X$ ($1 \leq i \leq k$). Note that $g$ acts on the direct sum by cyclically permuting the summands. Therefore, the eigenspaces

$$\left(\bigoplus_{a=1}^m T_{f_0(s_{i,a})}X, g\right)_X \cong T_{f_0(s_{i+1})}X = T_{f_i(s^1_i)}X$$

and all possible values for $\lambda$ are $\lambda = \zeta^k$ ($1 \leq k \leq m$). The vector space $T_{f_i(s^1_i)}X$ is the fiber of the vector bundle $(\text{ev}_1^i)^*T_X$, where $\text{ev}_1^i : I_{\eta_i}X_{0,1,d_i} \to X$ is the evaluation map. The contribution to $\widetilde{td}(T_{0,1,d})$ is given by

$$(A.2.5) \prod_x \frac{1 - e^{-x}}{x} \prod_{k=1}^{m-1} \prod_{x}(1 - \zeta^{-k} e^{-x}) = \prod_x \frac{1 - e^{-mx}}{x} = m^{\dim(X)} (\text{ev}_1^i)^* \Psi^m \frac{1}{\text{td}(X)},$$

where in the products over $x$, the variable $x$ varies over the set of all Chern roots of $(\text{ev}_1^i)^*(T_X)$.

Combining the results from the above 5 cases, that is formulas (A.2.2)–(A.2.5), we get the following formula:

$$(A.2.6) \quad \text{ev}^* \left( m^{k^{\dim(X)} - \sum_{i=1}^k \text{rk}(I_{0,i}T_{0,1,d_i})} \left( \Theta_{0,k+2,d_0}^{ABC} \cup \left( 1 - \zeta^{-1} e^{\psi^{(0)}_{k+2}/m} \right) \otimes I^\otimes k \right) \cup \right. $$

$$\left. \left( 1 \otimes \Psi^m (\text{ev}_1^1)^* \frac{1}{\text{td}(X)} \right) \otimes \cdots \otimes \Psi^m (\text{ev}_1^k)^* \frac{1}{\text{td}(X)} \right) \cup \bigcup_{i=1}^{k} \frac{1}{1 - \eta_i \text{ch} \otimes \Psi^m \left( L_{i+1}^{(0)} \otimes L_{i}^{(1)} \right)} \right),$$

where $\iota$ is the natural inclusion of the fiber product into the direct product. More precisely, we have the following pullback diagram

$$(A.2.7) \quad [X/\mu_m]_{0,k+2,d_0}^{[1]} \times_{X^k} I_{\eta_1}X_{0,1,d_1} \times \cdots \times I_{\eta_k}X_{0,1,d_k} \xrightarrow{\text{ev}} X^k \quad \xrightarrow{\iota} \quad X^k \times X^k$$

where the right vertical arrow is the diagonal embedding, the upper horizontal arrow is the evaluation map at $s^1_i$ ($1 \leq i \leq k$), and the lower horizontal map is the evaluation map at $s^0_{i+1}$ ($1 \leq i \leq k$) and $s^1_i$ ($1 \leq i \leq k$).

A.3. Integration over the Kawasaki strata. Using formula (A.2.6), let us compute the integral (A.1.1). The integral over the fiber product can be rewritten as an integral over the corresponding direct product by using the Thom isomorphism, i.e., if $\iota : X \to Y$ is a regular embedding, then

$$(A.3.1) \quad \int_X \iota^* \Phi = \int_Y \Phi \cup \theta(N_i),$$

where $\theta(N_i)$ is the Thom class of the normal bundle $N_i$ to $X$ in $Y$. In our case, having in mind the pullback diagram (A.2.7), the normal bundle $N_i$ is the pullback of the normal bundle to the diagonal embedding $X^k \to X^k \times X^k$. Therefore, $\theta(N_i)$ is the pullback of the Thom class of the
normal bundle to the diagonal, which is well known to be the Poincare dual of the diagonal, that is,

\[(A.3.2) \quad \theta(N_i) = \sum_{a_1, \ldots, a_k=1}^{N} \left( (ev_2^0)^* \phi_{a_1} \cup \cdots \cup (ev_{k+1}^0)^* \phi_{a_k} \right) \otimes (ev_1^1)^* \phi^a_1 \otimes \cdots \otimes (ev_1^1)^* \phi^a_k,
\]

where \(\{\phi_a\}\) and \(\{\phi^a\}\) are dual bases of \(H^*(X; \mathbb{C})\) with respect to the Poincare pairing. For simplicity we will choose \(\phi_a\) to be homogeneous and denote its degree by \(2|\phi_a|\). Note that the corresponding dual vector \(\phi^a\) must be homogeneous too and if we denote its degree by \(2|\phi^a|\), then \(|\phi_a| + |\phi^a| = \dim(X)\). The integral \((A.1.1)\) takes the form

\[
\sum_{\eta_1, \ldots, \eta_k} \sum_{(d_0 + d_1 + \cdots + d_k) = m = d} \int_{[X/\mu_m]^{\eta_1, \ldots, \eta_k}} \cdots \int_{[T_{0,1,d}]} \theta(N_i) \frac{(ev_1^0)^* \phi^a}{1 - \eta_k \chi o \Psi^m} (I_{i+1}^{(1)} \otimes L_1^{(1)}),
\]

where the 1st sum is over all primitive roots of unity \(\eta_1, \ldots, \eta_k\), such that, \(\eta_k \neq 1\) for all \(i\), and the 2nd sum is over all degree classes \(d_0, \ldots, d_k\), such that, \((d_0 + \cdots + d_k) = d\). Here we used that under the isomorphism \((3.2.1)\) the line bundle \(L_1\) on \(L_{0,1,d}\) is a \(m\)th root of \(L_1^{(0)}\) – the line bundle \(L_1\) on \([X/\mu_m]^{\eta_1, \ldots, \eta_k} \). Let us substitute formulas \((A.2.6)\) and \((A.3.2)\) in the above formula and single out the terms in the integrand that involve cohomology classes on \(I_{\eta_i} X_{0,1,d_i}\) \((1 \leq i \leq k)\)

\[(A.3.3) \quad m^{-rk(I_{i}T_{0,1,d_i})-|\phi^a_i|} \Psi^m (I_{i+1}^{(1)} \otimes L_1^{(1)}) \frac{1}{1 - \eta_k \chi o \Psi^m}.
\]

The remaining terms of the integrand are given by

\[(A.3.4) \quad m^{k \dim(X)} \Theta_{ABC}^{0,k+2,d_0} \frac{(ev_1^0)^* \phi^a}{1 - \eta_k \chi o \Psi^m} (I_{i+1}^{(1)} \otimes L_1^{(1)}) \frac{1}{1 - \eta_k \chi o \Psi^m}.
\]

Let us integrate \((A.3.3)\) along the virtual fundamental cycle of \(I_{\eta_i} X_{0,1,d_i}\). First, note that the cohomology classes \(\phi^a_{td(X)} = \chi(\Phi^a)\) form a basis dual to \(\phi_a = \chi(\Phi_a)\) with respect to the K-theoretic Euler pairing. Furthermore, note that the rank of the inertia tangent bundle \(I_{i}T_{0,1,d_i}\) coincides with the dimension of the virtual fundamental cycle of \(I_{\eta_i} X_{0,1,d_i}\). On the other hand, the Adam’s operation \(\Psi^m\) in \((A.3.3)\) rescales each cohomology class by a power of \(m\) equal to its degree. Since, the cohomology classes on \(I_{\eta_i} X_{0,1,d_i}\) that contribute non-trivially to the integral should have total degree matching the degree of the virtual fundamental cycle, we get that the Adam’s operation cancels out with \(m^{-rk(I_{i}T_{0,1,d_i})}\), except for the part of the Adam’s operation acting on cohomology classes not supported on \(I_{\eta_i} X_{0,1,d_i}\), such as, \(\chi(L_{i+1}^{(1)})\). After these remarks, it is clear that the integral of \((A.3.3)\) along the virtual fundamental cycle of \(I_{\eta_i} X_{0,1,d_i}\) is precisely \(m^{-|\phi^a_i|} \tau_{\eta_i,d_i,a_i}(e^{m \psi_{i+1}})\), that is, we get an integral of the type \((A.1.1)\). Therefore, in order to compute the integral \((A.1.1)\) we have to multiply \(\prod_{i=1}^k m^{-|\phi^a_i|} \tau_{\eta_i,d_i,a_i}(e^{m \psi_{i+1}})\) with \((A.3.4)\) and integrate over the virtual fundamental cycle of \([X/\mu_m]^{\eta_1, \ldots, \eta_k} \). Using also that \(m^{\dim(X)}-|\phi^a| = m^{\dim(X)}\phi_a = \Psi^m \phi_a\), we get

\[
\tau_{\zeta,d,a}(q) = \sum_{\tau_{\eta_1,d_1,a_1}(e^{m \psi_2}) \cdots, \tau_{\eta_k,d_k,a_k}(e^{m \psi_{k+1}}) \Psi^m \phi_a, 1 - \zeta^{-1} e^{m \psi_{k+2}/m}}.
\]
where the sum is over all non-trivial primitive roots of unity $\eta_1, \ldots, \eta_k$, all effective degree classes $d_0, d_1, \ldots, d_k$, such that, $m(d_0 + \cdots + d_k) = d$ (just like above), and over all $1 \leq a_1, \ldots, a_k \leq N$. Finally, in order to complete the proof of (3.3.5), it remains only to note that

$$\tau_\zeta(q, Q) = \sum_{a=1}^{N} \sum_{d \in \text{Eff}(X)} \tau_{\zeta, d, a}(q) Q^d \Phi_a$$

and that

$$\sum_{\eta_i} \sum_{d_i} \sum_{a_i=1}^{N} \tau_{\eta_i, d_i, a_i}(e^{m\psi_{i+1}}) Q^{m\psi_{i}} \Phi_{a_i} = \tau(m)(\psi_{i+1}, Q),$$

where $\tau(m)(z, Q)$ is the same as in (3.3.7).

**APPENDIX B. A proof of Theorem 4.2.8 for Picard rank 2**

The strategy of our proof is different from the one originally used in [Iri09]. It relies on the inversion formula for the Mellin transform and the formula for the Poincaré pairing for toric cohomology in terms of Jeffrey–Kirwan residues (c.f. Theorem 4.1.7). Similar use of the inversion formula for the Mellin transform appears in the master’s thesis [Xia21] to study the Gamma integral structure of the blowup of $\mathbb{P}^N$ at a point.

**B.1. Strategy of the proof.** Let $X_{\mu,K}$ be a Fano symplectic toric manifold of Picard rank 2. According to Proposition 4.1.6, we may assume that the matrix of the moment map is

$$\text{Mat}(\mu) = \begin{pmatrix} 1 & \cdots & 1 & 0 & \cdots & -a_1 & \cdots & -a_k \\ 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \end{pmatrix},$$

where $a_1, \ldots, a_k \in \mathbb{Z}_{\geq 0}$. It is convenient to define $a_0 := 0$. We have $c_1(TX_{\mu,K}) = m_1p_1 + m_2p_2$, where $m_i = \sum_{j=1}^{k+1} m_{ij}$ is the sum of the entries in the $i$th row of $\text{Mat}(\mu)$, that is, $m_1 = N - a_1 - \cdots - a_k$ and $m_2 = k + 1$.

We will make use of the inversion theorem for *Mellin transform* of a smooth function $f : \mathbb{R}_{>0}^2 \to \mathbb{C}$. Recall that the Mellin transform of $f$ is defined by

$$\mathcal{M} f(p_1, p_2) := \int_{\mathbb{R}_{>0}^2} f(q_1, q_2) q_1^{p_1} q_2^{p_2} \frac{dq_1}{q_1} \wedge \frac{dq_2}{q_2}.$$

The Mellin transform $\mathcal{M} f$ is a holomorphic function for all inputs $(p_1, p_2)$ for which the integral is absolutely convergent. Let $E = \{(x, y) \in \mathbb{C}^2 | x_{\min} < \text{Re}(x) < x_{\max} \text{ and } y_{\min} < \text{Re}(y) < y_{\max}\} \subseteq \mathbb{C}^2$ be the product of two strips. Assume that the Mellin transform $\mathcal{M} f$ is holomorphic on the strip $E$. Let $F(p_1, p_2) := \mathcal{M} f(p_1, p_2)$, then the inverse Mellin transform of $F$ is given by the formula below ([SP78], Lemma 2 p.125; see also for the one dimensional case [ML86], Lemma 11.10.1 p.246):

$$f(q_1, q_2) = \mathcal{M}^{-1} F(q_1, q_2) := \lim_{T_1 \to +\infty} \lim_{T_2 \to +\infty} \frac{1}{(2\pi i)^2} \int_{\epsilon_1-iT_1}^{\epsilon_1+iT_1} dp_1 \int_{\epsilon_2-iT_2}^{\epsilon_2+iT_2} dp_2 F(p_1, p_2) q_1^{-p_1} q_2^{-p_2},$$

where $(\epsilon_1, \epsilon_2)$ is a real point of $E$.

We begin by announcing the results of two computations – Lemmas B.1.1 and B.1.4, then give our proof of the theorem. The proofs of both lemmas will follow after.
Lemma B.1.1. Let $X_{\mu, K}$ be a symplectic toric manifold of Picard rank 2 as in Proposition 4.1.6, let $\Gamma_\mathbb{R} = \{(x_1, \ldots, x_n) \in \pi^{-1}(Q_1, \ldots, Q_r) | x_j > 0\}$ be the real Lefschetz thimble and $\mathcal{I}^{\text{coh}}(z, Q) \equiv \int_{\Gamma_\mathbb{R}} e^{-W_\mathbb{R}/z} \omega_{x^{-1}(Q)}$ be the oscillatory integral. Then, the Mellin transform $\mathcal{M}\mathcal{I}^{\text{coh}}$ of the oscillatory integral exists and it is given by

$$\mathcal{M}\mathcal{I}^{\text{coh}}(z, p_1, p_2) = z^\rho \Gamma(p_1)^N \prod_{j=0}^k \Gamma(p_2 - a_j p_1).$$

For a proof of this lemma, see Section B.2. Note that the oscillatory integral has the following symmetry:

(B.1.2) \[ \mathcal{I}(z, Q_1, Q_2) = \mathcal{I}(1, z^{-m_1} Q_1, z^{-m_2} Q_2). \]

Indeed, if we change the variables in the oscillatory integral via $x_i = z y_i$, then the Batyrev constraints take the form $z^{-m_i} Q_i = \prod_{j=1}^{N+1+k} y_j^{m_{ij}}$ and the above identity follows. Furthermore, the I-function has a similar symmetry

(B.1.3) \[ I^{\text{coh}}(z, Q_1, Q_2) = z^{-\deg z} z^{-\rho} I^{\text{coh}}(1, z^{-m_1} Q_1, z^{-m_2} Q_2) \]

which can be checked easily. Let us point out that in the Fano case $-z e^{\sum P_i \log Q_i / z} I^{\text{coh}}(z, Q_1, Q_2)$ coincides with the J-function, so the above symmetry follows also from (1.1.4). Using the symmetries (B.1.2) and (B.1.3) we get that it is sufficient to prove Theorem 4.2.8 for $z = 1$.

Lemma B.1.4. Consider the Mellin transform

$$g(p_1, p_2) := \mathcal{M}\mathcal{I}^{\text{coh}}(1, p_1, p_2) = \Gamma(p_1)^N \prod_{j=0}^k \Gamma(p_2 - a_j p_1).$$

Then, its inverse Mellin transform $\mathcal{M}^{-1} g(Q_1, Q_2)$ can be computed by the following residue formula:

$$\mathcal{M}^{-1} g(Q_1, Q_2) = \text{Res}_{x_2=0} \text{Res}_{x_1=0} \omega(x_1, x_2) dx_1 dx_2,$$

where

$$\omega(x_1, x_2) := \sum_{d_1, d_2 \geq 0} Q_1^{-x_1 + d_1} Q_2^{-x_2 + d_2} \left( \Gamma(x_1) \prod_{r=-\infty}^{0} \frac{\prod_{i=0}^{d_1} (x_1 - r)}{\prod_{r=-\infty}^{d_2} (x_2 - a_j x_1 - r)} \prod_{j=0}^k \Gamma(x_2 - a_j x_1) \prod_{r=-\infty}^{d_2 - a_j d_1} \frac{(x_2 - a_j x_1 - r)}{(x_2 - a_j x_1 - r)} \right)^N.$$

The proof of Lemma B.1.4 will be given in Section B.3.

Proof of Theorem 4.2.8. Consider the oscillatory integral for the real Lefschetz thimble $\Gamma_\mathbb{R}$ evaluated at $z = 1$, $\mathcal{I}^{\text{coh}}(1, Q_1, Q_2)$. The Mellin transform of this oscillatory integral is computed in Lemma B.1.1, in which we obtain

$$\mathcal{M}\mathcal{I}^{\text{coh}}(1, p_1, p_2) = \Gamma(p_1)^N \prod_{j=0}^k \Gamma(p_2 - a_j p_1).$$

Using Lemma B.1.4, we compute the inverse Mellin transform of this expression to be

$$\mathcal{M}^{-1} \mathcal{M}\mathcal{I}^{\text{coh}}(1, Q_1, Q_2) = \text{Res}_{x_2=0} \text{Res}_{x_1=0} \omega(x_1, x_2) dx_1 dx_2,$$
Proof of Lemma B.2.

Using Theorem 4.1.7, we identify the iterated residues above with the intersection product

\[ M^{-1} M^{\text{coh}}(1, Q_1, Q_2) = \int_{[X_{\mu,K}]} x_1^N \prod_{j=0}^k (x_2 - a_j x_1) \omega(x_1, x_2) \]

Finally,

\[ x_1^N \prod_{j=0}^k (x_2 - a_j x_1) \omega(x_1, x_2) = \left[ x_1^N \Gamma(x_1)^N \prod_{j=0}^k (x_2 - a_j x_1) \Gamma(x_2 - a_j x_1) \right] \times \]

\[ \times \left[ Q_1^{-x_1} Q_2^{-x_2} \sum_{d_1,d_2 \geq 0} Q_1^{d_1} Q_2^{d_2} \left( \prod_{r=\infty}^{d_2} (x_1 - r) \prod_{r=-\infty}^{d_1} (x_1 - r) \right) \prod_{j=0}^k \prod_{r=\infty}^{d_2-a_j d_1} (x_2 - a_j x_1 - r) \right]. \]

In the right hand side, the factor in the first line is the gamma class \( \Gamma(TX_{\mu,K}) \), and the factor in the second line is the small \( I \)-function of the toric manifold \( T^{\text{coh}}(1, Q_1, Q_2) \). Using Equation 1.1.4, we recover the result for all \( z \).

Remark B.1.5. In the language of [SV04], consider the projective sequence \( \mathcal{A} = (\alpha_1, \ldots, \alpha_n) \). The choice of integrating first with respect to the input \( p_1 \) amounts to choosing in Theorem 2.6 a vector \( \xi \in K \) regular with respect to \( \Sigma \mathcal{A} \) that is located below the line \( \text{Vect}(c_1(TX_{\mu,K})) \). The other choice replaces the residue \( \text{Res}_{p_2=0} \text{Res}_{p_1=0} \) with the sum \( \sum_{j} \text{Res}_{p_1=0} \text{Res}_{p_2=a_j p_1} \).

B.2. Proof of Lemma B.1.1. The computation relies on applying the Fubini theorem to see the Mellin transform \( M^{\text{coh}} \) as an integral on the space \( Y_R := (\mathbb{R}_{>0})^{N+k+1} \).

Proof. We have

\[ M^{\text{coh}}(z, p_1, p_2) = \int_{(\mathbb{R}_{>0})^2} T^{\text{coh}}(z, Q_1, Q_2) Q_1^{p_1} Q_2^{p_2} \frac{dQ_1}{Q_1} \frac{dQ_2}{Q_2} \]

We recall the diagram of the Landau–Ginzburg model below.

\[ Y := (\mathbb{C}^*)^{n+N+k+1} \xrightarrow{W} \mathbb{C} \]

\[ B := (\mathbb{C}^*)^r \]

Notice that in the oscillatory integral \( T^{\text{coh}} \), we have a first integral along a fibre \( \pi^{-1}(Q_1, Q_2) \), while the Mellin transform introduces an integral over the base, i.e. all \( (Q_1, Q_2) \in B_R := (\mathbb{R}_{>0})^2 \), for which \( \frac{dQ_1}{Q_1} \frac{dQ_2}{Q_2} \) is a volume form. Using Fubini theorem, we can write

\[ M^{\text{coh}}(z, p_1, p_2) = \int_{Y_R} e^{-W(x_1, \ldots, x_{N+k+1})/z} Q_1^{p_1} Q_2^{p_2} \frac{dx_1}{x_1} \cdots \frac{dx_{N+k+1}}{x_{N+k+1}} \]
Using the Batyrev constraints $Q_i = \prod_{j=1}^n x_i^{m_{ij}}$, we can write the Mellin transform as

$$\mathcal{M}^{\text{coh}}(z, p_1, p_2) = \int_{(\mathbb{R}_0)^n} \prod_{j=1}^{N_k} \frac{dx_j}{x_j} \prod_{j=1}^{N_k+1} \frac{dx_j}{x_j} e^{-x_j/z} \prod_{j=1}^{N_k} x_j^{\alpha_j(p)} \prod_{j=1}^{N_k+1} x_j^{\alpha_j(p)} = z^{\varepsilon_1} (TX_{\mu,K})(p) \Gamma(p_1)^N \prod_{j=0}^{k} \Gamma(p_2 - a_j p_1)$$

B.3. Proof of Lemma B.1.4. To prove this lemma, the goal will be to obtain a contour deformation result to express the integrals along $p_l \in \varepsilon_l + i\mathbb{R}$ (where $l = 1, 2$) in terms of integrals along closed curves, for which we can then apply the residue theorem. The main ingredients to prove our contour deformation result will be Stirling’s formula and the Fano condition $N - \sum_1^k a_j > 0$.

Remark B.3.1. In general, the formula for the inverse Mellin transform relies on the choice of a base point $\varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2$, as in the formula

$$\mathcal{M}^{-1}(Q_1, Q_2) := \int_{p_2 \in \varepsilon_2 + i\mathbb{R}} \int_{p_1 \in \varepsilon_1 + i\mathbb{R}} \Gamma(p_1)^N \prod_{j=0}^{k} \Gamma(p_2 - a_j p_1) Q_1^{-p_1} Q_2^{-p_2} dp_1 dp_2$$

This point $\varepsilon$ could be chosen such that the input inside every gamma function is positive, i.e. for all $j \in \{0, \ldots, n\}$, $m_{1j}\varepsilon_1 + \cdots + m_{rj}\varepsilon_r > 0$. If $\sigma \subset \mathbb{R}^2$ denotes the cone spanned by the columns of the moment map of $X_{\mu,K}$, this condition is equivalent to choosing a point $\varepsilon$ in the interior of the dual cone $(\sigma^*)^0$. This interior is not empty as $X_{\mu,K}$ is compact, c.f. Remark 4.1.3.

Proof of Lemma B.1.4. We consider the inverse Mellin transform

$$\mathcal{M}^{-1}(Q_1, Q_2) := \int_{p_2 \in \varepsilon_2 + i\mathbb{R}} \int_{p_1 \in \varepsilon_1 + i\mathbb{R}} \Gamma(p_1)^N \prod_{j=0}^{k} \Gamma(p_2 - a_j p_1) Q_1^{-p_1} Q_2^{-p_2} dp_1 dp_2,$$

where positive numbers $\varepsilon_1, \varepsilon_2$ are chosen such that $0 < \max_1 \{1, \alpha_j\} \varepsilon_1 < \varepsilon_2 < 1$.

We begin by showing that

$$\mathcal{M}^{-1}(Q_1, Q_2) = \sum_{d_2 \in \mathbb{Z}_{\geq 0}} \text{Res}_{p_2 = -d_2} \sum_{d_1 \in \mathbb{Z}_{\geq 0}} \text{Res}_{p_1 = -d_1} \Gamma(p_1)^N \prod_{j=0}^{k} \Gamma(p_2 - a_j p_1) Q_1^{-p_1} Q_2^{-p_2}$$

Let us begin by treating the first integral, with respect to the coordinate $p_1$. We will deform the integration contour $\varepsilon_1 + i\mathbb{R}$ by the following contour: pick $R_M > 0$ some large number and define the closed contour $\mathcal{C}(R_M)$ as the union of the following pieces: a vertical line segment $L_1(R_M)$ from $(\varepsilon_1, -\sqrt{R_M^2 - 1})$ to $(\varepsilon_1, +\sqrt{R_M^2 - 1})$, a horizontal line $L_2(R_M)$ from $(\varepsilon_1, +\sqrt{R_M^2 - 1})$ to $(-1, +\sqrt{R_M^2 - 1})$, a circular arc $C(R_M)$ from $(-1, +\sqrt{R_M^2 - 1})$ to $(-1, -\sqrt{R_M^2 - 1})$ along the circle of radius $R_M$ and origin 0 in the half space $\{\text{Re}(p_1) < 0\}$, and a horizontal line $L_3(R_M)$ from $(-1, -\sqrt{R_M^2 - 1})$ to $(\varepsilon_1, -\sqrt{R_M^2 - 1})$.

Our goal is to show that the contributions to the integral of all parts except the vertical line segment $L_1(R_M)$ vanish when $R_M \to \infty$. We recall that for $a, b \in \mathbb{R}$ we have

$$|\Gamma(a + ib)|^2 = |\Gamma(a)|^2 \prod_{k=0}^{\infty} \frac{1}{1 + \frac{b^2}{(a+k)^2}}$$
Therefore, for a fixed, the function $b \mapsto |\Gamma(a + ib)|$ has exponential decay as $b \to \pm \infty$. Thus, for the integrals along the horizontal lines,

$$\lim_{R_M \to \infty} \int_{L_2(R_M)} \Gamma(p_1)^N \prod_{j=0}^k \Gamma(p_2 - a_j p_1) Q_1^{-p_1} dp_1 = \lim_{R_M \to \infty} \int_{L_3(R_M)} \Gamma(p_1)^N \prod_{j=0}^k \Gamma(p_2 - a_j p_1) Q_1^{-p_1} dp_1 = 0$$

Next, on the circular arc $C(R_M)$, we have $\text{Re}(p_1), \varepsilon_2 < 1$, thus

$$\left| \frac{\Gamma(p_1)^N}{\prod_{j=0}^k \Gamma(a_j p_1)} \right| \sim_{R_M \to \infty} (\text{Constant}) R_M^{1-N} e^{R_M \cos(\theta)(-N + \sum_j a_j - \log(a_j))} e^{N - \sum_j a_j (R_M \log(R_M) \cos(\theta) - R_M \theta \sin(\theta))}$$

The leading term in this expression is $e^{\cos(\theta)(N - \sum_j a_j) (R_M \log(R_M))}$, and the coefficient $\cos(\theta)(N - \sum_j a_j)$ is negative on the circular arc $C(R_M)$ due to the Fano condition $N - \sum_j a_j > 0$. Combining with $|\sin(\pi a_j z)| \sim e^{\pi a_j R_M |\sin(\theta)|}/2$, we get that the function $\left| \Gamma(p_1)^N \prod_{j=0}^k \Gamma(p_2 - a_j p_1) \right|$ has exponential decay as $R_M \to \infty$ on the circular arc $C(R_M)$. Finally, we get

$$\lim_{R_M \to \infty} \int_{C(R_M)} \Gamma(p_1)^N \prod_{j=0}^k \Gamma(p_2 - a_j p_1) Q_1^{-p_1} dp_1 = \lim_{R_M \to \infty} \int_{L_2(R_M)} \Gamma(p_1)^N \prod_{j=0}^k \Gamma(p_2 - a_j p_1) Q_1^{-p_1} dp_1$$

$$= \int_{\varepsilon_2 + i\mathbb{R}} \Gamma(p_1)^N \prod_{j=0}^k \Gamma(p_2 - a_j p_1) Q_1^{-p_1} dp_1$$

It remains to apply the residue theorem to the closed contour $C(R_M)$. As a function of $p_1$, the integrand $\Gamma(p_1)^N \prod_{j=0}^k \Gamma(p_2 - a_j p_1) Q_1^{-p_1}$ has poles at $p_1 \in \mathbb{Z}_{\leq 0}$ and for $p_2 - a_j p_1 \in \mathbb{Z}_{\leq 0}$. Since $p_2$ takes values on $\varepsilon_2 + i\mathbb{R}$, and since $0 < \max_j \{1, a_j\} \varepsilon_1 < \varepsilon_2 < 1$, the poles obtained from the condition $p_2 - a_j p_1 \in \mathbb{Z}_{\leq 0}$ lay outside of the integration contour $C(R_M)$. Therefore, we have computed the first integral with respect to the input $p_1$:

$$g(p_1, p_2) := \mathcal{M}^{-1} \mathcal{M}^{\text{coh}}(1, p_1, p_2) = \int_{\varepsilon_2 + i\mathbb{R}} \sum_{d_1 \geq 0} \text{Res}_{p_2 = -d_1} \Gamma(p_1)^N \prod_{j=0}^k \Gamma(p_2 - a_j p_1) Q_1^{-p_1} Q_2^{-Q_2} dp_2$$

The second integral with respect to the input $p_2$ can be computed using a similar contour deformation, proven by using Stirling’s formula once more (the Fano condition will not appear there).

To obtain the identity in the statement of this lemma, we do the change of variables (for $l = 1, 2$) $x_l := p_l - d_l$ and use Fubini’s theorem to permute the discrete sums and the residues to obtain (B.3.2)

$$\mathcal{M}^{-1} g(Q_1, Q_2) = \text{Res}_{x_2 = 0} \text{Res}_{x_1 = 0} \sum_{d_1, d_2 \geq 0} \Gamma(x_1 - d_1) \prod_{j=0}^k \Gamma(x_2 - a_j x_1 - (d_2 - a_j d_1)) Q_1^{-x_1 + d_1} Q_2^{-x_2 + d_2}$$
Then, using the difference equation satisfied by the gamma function, we obtain
\[ \Gamma(x_2 - a_j x_1 - (d_2 - a_j d_1)) = \Gamma(x_2 - a_j x_1) \prod_{r=-\infty}^{0}(x_2 - a_j x_1 - r) \prod_{r=-\infty}^{d_2-a_j d_1}(x_2 - a_j x_1 - r) \]

Applying this formula to every factor in the Equation B.3.2 above gives the formula given in the statement of Lemma B.1.4.

**Appendix C. Continuous oscillatory integral in quantum \( K \)-theory**

In this appendix we study another model of an oscillatory integral in quantum \( K \)-theory, by using the usual (continuous) integral instead of the Jackson integral. This model was introduced first in [Giv15c]. For such an oscillatory integral, we are only able to prove the analogue of our theorem for projective spaces using our strategy with Mellin transforms. When the toric manifold has Picard rank above 1, the contour deformation does not seem to work. Combined with the fact that in general, the Mellin transform of a \( q \)-difference equation is not a difference equation, we decided to move these results to an appendix.

### C.1. Oscillatory integral in quantum \( K \)-theory.

In this subsection, we will always consider \( |q| < 1 \).

**Definition C.1.1** (Oscillatory integral in quantum \( K \)-theory; [Giv15c], Theorem 2). Let \( q \in (0,1) \). Consider the \( K \)-theoretic Laudau–Ginzburg model associated to a toric manifold \( X_{\mu,K} \). Fix \( (Q_1,\ldots,Q_r) \in B \) and let \( \omega_{\pi^{-1}(Q)} \in \Lambda^*(T^*\pi^{-1}(Q_1,\ldots,Q_r)) \) be a volume form on \( \pi^{-1}(Q_1,\ldots,Q_r) \).

The (continuous) oscillatory integral \( I_{c}^{K-\text{th}} \) is the function defined by
\[ I_{c}^{K-\text{th}}(q,Q) := \int_{\Gamma \subset \pi^{-1}(Q_1,\ldots,Q_r)} e^{W_Q} \omega_{\pi^{-1}(Q)} \]
where \( W_Q := W|_{\pi^{-1}(Q_1,\ldots,Q_r)} \) and \( \Gamma \subset \pi^{-1}(Q_1,\ldots,Q_r) \) is a Lefschetz thimble.

Note that in this definition and unlike Subsection 4.3, we will understand the function \( e^{W_Q} \) as its analytical continuation given by the infinite product \((|q| < 1)\)
\[ \prod_{j=1}^{n} \frac{1}{(x_j;q)_{\infty}} \]
To choose which Lefchetz thimble we will consider, we have to compare our oscillatory integrals in \( K \)-theory and in cohomology. Recall that in cohomology, we were considering (for \( z,Q_i,x_j > 0 \))
\[ I_{c}^{\text{coh}}(z,Q) = \int_{\pi^{-1}(Q)\cap(\mathbb{R}_{>0})^n} \exp(-x_1/z - \cdots - x_n/z)\omega_{\pi^{-1}(Q)} \]
One can define a \( q \)-analogue of the exponential by the following formula
\[ e_q(x) := \sum_{d \geq 0} x^d \prod_{i=0}^{d} \frac{1 - q}{1 - q^i} \]
This function inherits its name from the observation that
\[ \lim_{q \to 1} e_q(x) = \exp(x) \]
Furthermore, we will pick a Lefschetz thimble $\Gamma_\mathbb{R}$ so that $e^{iW(x_1,\ldots,x_n)}$ is a $q$-analogue of $e^{-x_1/z\cdots-x_n/z}$, in the sense of the limit in Equation C.1.2. Therefore, we will need to take a real Lefschetz thimble $\Gamma_\mathbb{R}$ for which the coordinates $(x_j)$ are negative.

**Definition C.1.3.** Let $(Q_1,\ldots,Q_r) \in (\mathbb{R}_{>0})^r$, the corresponding real Lefschetz thimble $\Gamma_\mathbb{R} \subseteq \pi^{-1}(Q)$ is given by the negative points

$$\Gamma_\mathbb{R} := \{(x_1,\ldots,x_n) \in \pi^{-1}(Q_1,\ldots,Q_r) \mid x_j < 0\}$$

To match the signs in the Batyrev relations, we will replace $Q_i$ by $(-1)^{\deg Q_i} := (-1)^{\Sigma_j m_{ij}} Q_i$, and finally consider the oscillatory integral (defined for $|q| < 1, Q_i > 0$):

(C.1.4) $$\mathcal{I}_c^{K_{q}}(q,Q) = \int_{\Gamma_\mathbb{C} \cap \pi^{-1}((-1)^{\deg Q_1},\ldots,-(1)^{\deg Q_r})} e^{W_{(-1)^{\deg Q}} \frac{dQ_1}{Q_1} \cdots \frac{dQ_n}{Q_n}}$$

Note that if we were to consider $q > 1$, we would have to replace the analytical continuation of $W$ (given by $\frac{1}{(x;q)_\infty}$) in the integral by the expression $(q^{-1}x; q^{-1})_\infty$. Then, the integral would be immediately divergent.

**Proposition C.1.5 (Giv15c, Theorem 2; see also IMT15, Proposition 2.12).** The $K$-theoretical oscillatory integral $\mathcal{I}_c^{K_{q}}$ and the small $I$-function $I^{K_{q}}_{X,K}$ satisfy the same set of $q$-difference equations below (indexed by $i \in \{1,\ldots,r\}$):

$$\left[ \prod_{j:m_{ij} > 0} \prod_{r=0}^{m_{ij}-1} \left( 1 - q^{-r} q^{\Sigma_{r'} m_{ij} Q_{ij} \delta_{ij}} \right) - Q_i \prod_{j:m_{ij} < 0} \prod_{r=0}^{m_{ij}-1} \left( 1 - q^{-r} q^{\Sigma_{r'} m_{ij} Q_{ij} \delta_{ij}} \right) \right] f_i(Q) = 0$$

□

### C.2. Corresponding $q$-gamma class and comparison theorem

Since we changed the definition of the oscillatory integral, it turns out we will be using another $q$-analogue of the gamma function.

**Definition C.2.1 (Continuous $q$-gamma class).** Let $E \to X_{\mu,K}$ be a vector bundle, and denote by $\delta_1,\ldots,\delta_m \in H^2(X_{\mu,K};\mathbb{Q})$ its Chern roots. The continuous $q$-gamma class $\gamma_q^c(E) \in H^*(X_{\mu,K};\mathbb{Q})$ is defined by

$$\gamma_q^c(E) := \prod_{j=1}^{m} \delta_j \gamma_q^c(\delta_j) \in H^*(X_{\mu,K};\mathbb{Q}),$$

where

(C.2.2) $$\gamma_q^c(z) := \int_0^{\infty} \frac{x^z}{(x;q)_\infty} \frac{dx}{x}$$

In the definition of $\gamma_q^c(E)$, the right hand side should be understood as its power series expansion, using the Ramanujan identity below.

**Proposition C.2.3 (Ask80).** For $a \in \mathbb{C}, z > 0$ and $q \in (0, 1)$, the following Ramanujan formula holds:

$$\int_0^{\infty} t^z \frac{(-at; q)_\infty}{(-t; q)_\infty} \frac{dt}{t} = \frac{(a; q)_\infty}{(q; q)_\infty} \frac{\pi}{\sin(\pi z)} \frac{(q^{1-z}; q)_\infty}{(aq^{1-z}; q)_\infty}$$
Setting $a = 0$ in this proposition, we get
\[ \gamma_c^q(z) = \frac{\pi}{\sin(\pi z)} \frac{(q^{1-z}; q)_\infty}{(q; q)_\infty} \]
Note that this function satisfies the difference equation
\[ \gamma_c^q(z + 1) = \frac{1}{q^{-z} - 1} \gamma_c^q(z) \]

**Remark C.2.4.** The function $\gamma_q$ defined in Equation (C.2.2) is another $q$-analogue of the gamma function $\Gamma$, as we replacing the exponential $e^{-z}$ by the $q$-analogue $e_q(-x/(1-q)) = (x; q)_\infty^{-1}$. Recall that "the" $q$-gamma function $\Gamma_q$ introduced by Jackson (see e.g. Equation (1.10.1) in [GR]) is defined by
\[ \Gamma_q(z) := (1-q)^{1-z} \frac{(q; q)_\infty}{(q^z; q)_\infty} \]
Our two $q$-analogues are related by the formula
\[ \gamma_c^q(z) = \frac{\pi}{\sin(\pi z)} \frac{(1-q)^z}{\Gamma_q(1-z)} \]
Using Euler’s reflection formula for the classical gamma function, we obtain that also
\[ \lim_{q \to 1} (1-q)^{-z} \gamma_q^q(z) = \Gamma(z) \]
Note that we introduced the factor $(1-q)^{-z}$ in the left hand side as otherwise the difference equation satisfied by $\gamma_c^q(z)$ would have no formal limit when $q \to 1$.

We are now ready to state our theorem comparing the $q$-oscillatory integral and the $I$-function.

**Theorem C.2.6.** Let $X = \mathbb{P}^N$ be a projective space. Then, the oscillatory integral $\mathcal{I}^K_{c}$ defined in Equation (C.1.4) and the $I$-function $I^K_{c}$ of Definition 4.3.14 are related by the identity
\[ \mathcal{I}^K_{c}(q, Q) = \int_{[X]} \text{ch}_q(I^K_{X}(q, Q)) \cup \gamma^c_{q}(TX), \]
where $\int_{[X_{\mu,K}]}$ denotes the intersection product by $[X_{\mu,K}] \in H_*(X_{\mu,K}; \mathbb{C})$, $\gamma^c_{q}(TX_{\mu,K})$ is the continuous $q$-Gamma class of Definition C.2.1 and $\text{ch}_q$ is the $q$-Chern character of Definition 4.4.3.

**C.3. Proof of the comparison theorem for projective planes.** In this case, we will begin by writing our proof as if the target manifold was a symplectic toric manifold of Picard rank 2. We are able to compute the Mellin transform of the oscillatory integral, however trouble will appear when considering its inverse.

**Lemma C.3.1.** Let $X_{\mu,K}$ be a Fano symplectic toric manifold of Picard rank 2, whose moment map is given by (cf Proposition 4.1.6)
\[ \begin{pmatrix} 1 & \cdots & 1 & 0 & -a_1 & \cdots & -a_k \\ 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \end{pmatrix}, \]
where \( a_0 := 0, a_1, \ldots, a_k \in \mathbb{Z}_{\geq 0} \). Let \( \Gamma := \{(x_1, \ldots, x_n) \in \pi^{-1}(Q_1, \ldots, Q_r) | x_j < 0\} \) be the real Lefschetz thimble and \( I_{-}^{K^{-}\text{th}}(q, Q) = \int_{\Gamma} e^{W_{(-1)^{\deg Q}Q_1 \ldots Q_n}} \) be the oscillatory integral. Then, the Mellin transform \( \mathcal{M}_c^{K^{-}\text{th}} \) of the oscillatory integral exists and is given by

\[
\mathcal{M}_c^{K^{-}\text{th}}(q, p_1, p_2) = \gamma_q^c(p_1)^N \prod_{j=0}^{k} \gamma_q^c(p_2 - a_j p_1),
\]

where we recall that \( \gamma_q^c(z) := \frac{\pi}{\sin(\pi z)} \frac{(q^{1-z}; q)_\infty}{(q;q)_\infty} \).

The proof will be similar to its analogue in cohomology, see Lemma B.1.1: our goal is to use Fubini’s theorem to compute the Mellin transform as an integral on \( Y_\Gamma = (\mathbb{R}_{>0})^n \).

**Proof.** We recall the diagram of the \( K \)-theoretic mirror family of Definition 4.3.1:

\[
\begin{array}{ccc}
Y := (\mathbb{C}^*)^n & \xrightarrow{W} & \mathbb{C} \\
\downarrow & & \downarrow \pi \\
B := (\mathbb{C}^*)^r & & \\
\end{array}
\]

We also recall that in the expression of the oscillatory integral, \( W_Q \) (for a point \( Q \in B \)) designates the restriction of the map \( W \) along the fibre \( \pi^{-1}(Q) \), and that \( \omega_{\pi^{-1}(Q)} \) designates a volume form on the same fibre \( \pi^{-1}(Q) \). Using Fubini’s theorem and the Batyrev constraints \( (-1)^{\deg(Q)} Q_i = \prod_j x_j^{m_{ij}} \), we get

\[
\mathcal{M}_c^{K^{-}\text{th}}(q, p_1, p_2) = \int_{Q_1, Q_2 > 0} \int_{\Gamma} \exp(W_{(-1)^{\deg Q}(x_1, \ldots, x_n)}) Q_1^{p_1} Q_2^{p_2} \omega_{\pi^{-1}(Q)}^q \frac{dQ_1}{Q_1} \frac{dQ_2}{Q_2}
\]

\[
= \int_{(x_1, \ldots, x_n < 0)} \prod_{j=1}^{n} \frac{1}{x_j q_\infty} \left((-1)^{\deg(Q)} x_1^{m_{11}} \cdots x_n^{m_{nn}}\right)^{p_1} \left((-1)^{\deg(Q)} x_1^{m_{12}} \cdots x_n^{m_{nn}}\right)^{p_2} \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}
\]

We now do a change of variable and set \( x'_j = -x_j \). After this change of variable, we obtain

\[
\mathcal{M}_c^{K^{-}\text{th}}(q, p_1, p_2) = \prod_{j=1}^{n} \int_{x'_j > 0} \frac{1}{(-x'_j q_\infty)} x'_j^{\alpha_j(d)} \frac{dx'_j}{x'_j}
\]

Using the Ramanujan formula of Proposition C.2.3 and the description of Picard rank 2 symplectic toric manifolds, we obtain a product of functions \( \gamma_q \) as stated in the lemma.

\[\square\]

Now, our goal is to compute the inverse Mellin transform using a contour deformation result to apply the residue theorem.

**Conjecture C.3.2.** The inverse Mellin transform of \( \mathcal{M}_c^{K^{-}\text{th}}(q, p_1, p_2) \) can be computed by the following sum of iterated residues:

\[
\mathcal{M}^{-1}_c^{K^{-}\text{th}}(q, Q_1, Q_2) = \sum_{d_2 > 0} \operatorname{Res}_{p_2 = -d_2} \sum_{d_1 > 0} \operatorname{Res}_{p_1 = -d_1} \gamma_q^c(p_1)^N \prod_{j=0}^{k} \gamma_q^c(p_2 - a_j p_1) Q_1^{-d_1} Q_2^{-d_2} dQ_1 dQ_2
\]
Proof when $X = \mathbb{P}^N$. We recall that the comparison between the function $\gamma_q$ and the usual $q$-gamma function is given by (cf. Remark C.2.4)

$$
\gamma^c_q(p) = \frac{\pi}{\sin(\pi p)} \frac{(1 - q)^p}{\gamma_q(1 - p)}
$$

Write $1 - p = z = \rho e^{i\theta}$. If $\text{Re}(p) < 0$, then $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and we can use Moak’s asymptotic for the function $\Gamma_q$. When $\rho \to \infty$, we have

$$
|\sin(\pi z)| = |\sin(\pi \rho e^{i\theta})| \sim \frac{1}{2} e^{\rho |\sin(\theta)|}
$$

$$
|(1 - q)^p| = e^{\log(1-q)(1-\rho \cos(\theta))}
$$

$$
\text{Li}_2(1 - q^z) \to \frac{\pi^2}{6}
$$

$$
\left|\frac{(1 - q^{z-\frac{1}{2}})}{1 - q}\right| \sim \frac{1}{(1 - q)^{\rho \cos(\theta) - \frac{1}{2}}}
$$

Therefore,

$$
|\gamma^c_q(p)| \sim (\text{Constant}) e^{-\pi \rho |\sin(\theta)|}
$$

In the case of $X = \mathbb{P}^N$, we have

$$
\mathcal{MT}^{K-\text{th}, X = \mathbb{P}^N}_{c}\left(q, p_1\right) = \gamma^c_q(p_1)^{N+1}
$$

In that case, we can compute the inverse Mellin transform of this expression through a contour deformation identical to the one used in the proof of Lemma B.1.4:

We want to compute

$$
\int_{p_1 \in \mathbb{C} + i \mathbb{R}} \gamma^c_q(p_1)^{N+1} Q_1^{-p_1} dQ_1
$$

We deform this integral using the following contour: we pick $R_M > 0$ some large number and define the closed contour $\Gamma_{RM}$ as the union of the following pieces: a vertical line segment $L_1(R_M)$ from $(\varepsilon_1, -R_M)$ to $(\varepsilon_1, +R_M)$, a horizontal line $L_2(R_M)$ from $(\varepsilon_1, R_M)$ to $(0, R_M)$, a half circle $C(R_M)$ from $(0, R_M)$ to $(0, -R_M)$ of radius $R_M$ and origin 0 in the half place $\{\text{Re}(p_1) < 0\}$, and a horizontal line $L_3(R_M)$ from $(0, -R_M)$ to $(\varepsilon_1, -R_M)$.

We focus on the integral

$$
\int_{p_1 \in C(R_M)} \gamma^c_q(p_1)^{N+1} Q_1^{-p_1} dQ_1
$$

When $|Q_1| < 0$, this integrand has exponential decay for $\text{Re}(p_1) < 0$ as $|p_1| \to \infty$, therefore the integral along the arc of circle $C(R_M)$ will vanish at when $R_M \to \infty$. Applying the residue theorem to the deformed contour and using continuity for the vanishing of the other integrals along $L_{1,2}(R_M)$ as $\varepsilon_1 \to 0$, we obtain the formula of Conjecture C.3.2.

To obtain a proof of the comparison theorem C.2.6, it remains once again to identify the residue computed in Conjecture C.3.2 with a Jeffrey–Kirwan residue using Theorem 2.6 of [SV04] and apply Proposition 2.3 of [SV04].

Unfortunately, when the target space is not a projective space $\mathbb{P}^N$, we have not been able to express the inverse Mellin transform of the oscillatory integral as a sum of residues yet. We have
attempted to find a formal continuation (for the coordinate $q$) of the Mellin transform $\mathcal{M}_c^{K^{-\text{th}}}$ to \{\(q > 1\}\) that satisfies the same difference equation, however our contour deformation strategy does not work for that function either.

References

[AGV08] Dan Abramovich, Tom Graber, and Angelo Vistoli, *Gromov-Witten theory of Deligne-Mumford stacks*, Amer. J. Math. **130** (2008), no. 5, 1337–1398.

[Ask80] Richard Askey, *Ramanujan’s extensions of the gamma and beta functions*, Amer. Math. Monthly **87** (1980), no. 5, 346–359.

[Aud04] Michèle Audin, *Torus actions on symplectic manifolds*, revised ed., Progress in Mathematics, vol. 93, Birkhäuser Verlag, Basel, 2004.

[BF97] K. Behrend and B. Fantechi, *The intrinsic normal cone*, Invent. Math. **128** (1997), no. 1, 45–88. MR 1437495

[CR02] Weimin Chen and Yongbin Ruan, *Orbifold Gromov-Witten theory*, Orbifolds in mathematics and physics (Madison, WI, 2001), Contemp. Math., vol. 310, Amer. Math. Soc., Providence, RI, 2002, pp. 25–85.

[DSK05] Alberto De Sole and Victor G. Kac, *On integral representations of $q$-gamma and $q$-beta functions*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. **16** (2005), no. 1, 11–29.

[DVZ09] Lucia Di Vizio and Changgui Zhang, *On $q$-summation and confluence*, Ann. Inst. Fourier (Grenoble) **59** (2009), no. 1, 347–392.

[FBB06] A. Fitouhi, N. Bettaibi, and K. Brahim, *The Mellin transform in quantum calculus*, Constr. Approx. **23** (2006), no. 3, 305–323.

[Boy] Alexander Givental, *On the WDVV equation in quantum $K$-theory.*

[Giv96] Alexander B. Givental, *Equivariant Gromov-Witten invariants*, Internat. Math. Res. Notices (1996), no. 13, 613–663.

[Giv98] Alexander Givental, *A mirror theorem for toric complete intersections*, Topological field theory, primitive forms and related topics, Progress in Mathematics, vol. 160, Birkhäuser, Boston, MA, 1998, pp. 141–175.

[Giv15a] Alexander Givental, *Permutation-equivariant quantum $k$-theory II. fixed point localization*, 2015.

[Giv15b] Alexander Givental, *Permutation-equivariant quantum $K$-theory V. Toric $q$-hypergeometric functions*, Available online at https://math.berkeley.edu/~giventh/perm/perm.html.

[Giv15c] Alexander Givental, *Permutation-equivariant quantum $K$-theory VI. Mirrors*, Available online at https://math.berkeley.edu/~giventh/perm/perm.html.

[GR] George Gasper and Mizan Rahman, *Basic hypergeometric series*, second ed., Encyclopedia of Mathematics and its Applications, vol. 96.
[GT14] Alexander Givental and Valentin Tonita, *The Hirzebruch-Riemann-Roch theorem in true genus-0 quantum K-theory*, Symplectic, Poisson, and noncommutative geometry, Math. Sci. Res. Inst. Publ., vol. 62, Cambridge Univ. Press, New York, 2014, pp. 43–91.

[GY21] Alexander Givental and Xiaohan Yan, *Quantum K-theory of grassmannians and non-abelian localization*, SIGMA Symmetry Integrability Geom. Methods Appl. 17 (2021), Paper No. 018, 24.

[HSS] Charlotte Hardouin, Jacques Sauloy, and Michael F. Singer, *Galois theories of linear difference equations: an introduction*, Mathematical Surveys and Monographs, vol. 211.

[IMT15] Hiroshi Iritani, Todor Milanov, and Valentin Tonita, *Reconstruction and convergence in quantum K-theory via difference equations*, Int. Math. Res. Not. IMRN (2015), no. 11, 2887–2937.

[Iri09] Hiroshi Iritani, *An integral structure in quantum cohomology and mirror symmetry for toric orbifolds*, Adv. Math. 222 (2009), no. 3, 1016–1079.

[Iri20] ______, *Quantum D-modules of toric varieties and oscillatory integrals*, Handbook for mirror symmetry of Calabi-Yau & Fano manifolds, Adv. Lect. Math. (ALM), vol. 47, Int. Press, Somerville, MA, [2020] ©2020, pp. 131–147.

[JMNT21] Hans Jockers, Peter Mayr, Urmia Ninad, and Alexander Tabler, *Bps indices, modularity and perturbations in quantum k-theory*, 2021.

[Kaw79] Tetsuro Kawasaki, *The Riemann-Roch theorem for complex V-manifolds*, Osaka Math. J. 16 (1979), no. 1, 151–159.

[KK92] H.T. Koelink and T.H. Koornwinder, *q-special functions, a tutorial*, Deformation theory and quantum groups with applications to mathematical physics (Amherst, MA, 1990), Contemp. Math., vol. 134, Amer. Math. Soc., Providence, RI, 1992, pp. 141–142.

[Lee04] Y.-P. Lee, *Quantum K-theory. I. Foundations*, Duke Math. J. 121 (2004), no. 3, 389–424.

[ML86] O. P. Misra and J. L. Lavoine, *Transform analysis of generalized functions*, North-Holland Mathematics Studies, vol. 119, North-Holland Publishing Co., Amsterdam, 1986, Notas de Matemática [Mathematical Notes], 106.

[Moa84] Daniel S. Moak, *The q-analogue of Stirling’s formula*, Rocky Mountain J. Math. 14 (1984), no. 2, 403–413.

[Pal76] Victor Palamodov, *Deformations of complex spaces*, Uspekhi Mat. Nauk (1976), no. 31, 129–194.

[Roq19] Alexis Roquefeuil, *Confluence of quantum k-theory to quantum cohomology for projective spaces*, PhD thesis, Université d’Angers, 2019.

[SP78] H. M. Srivastava and Rekha Panda, *Certain multidimensional integral transformations. I*, Nederl. Akad. Wetensch. Proc. Ser. A 81=Indag. Math. 40 (1978), no. 1, 118–131.

[SV04] András Szénés and Michèle Vergne, *Toric reduction and a conjecture of Batyrev and Materov*, Invent. Math. 158 (2004), no. 3, 453–495.

[Ton14] Valentin Tonita, *Twisted orbifold Gromov-Witten invariants*, Nagoya Math. J. 213 (2014), 141–187.
[Tse10] Hsian-Hua Tseng, *Orbifold quantum Riemann-Roch, Lefschetz and Serre*, Geom. Topol. **14** (2010), no. 1, 1–81.

[Wen20] Yaoxiong Wen, *Difference equation for quintic 3-fold*, 2020.

[Xia21] Xiaokun Xia, *Gamma integral structure for the blowup of $\mathbb{P}^n$ at a point*, Master thesis.

KAVLI IPMU (WPI), UTIAS, THE UNIVERSITY OF TOKYO, KASHIWA, CHIBA 277-8583, JAPAN

Email address: toodor.milanov@ipmu.jp

KAVLI IPMU (WPI), UTIAS, THE UNIVERSITY OF TOKYO, KASHIWA, CHIBA 277-8583, JAPAN

Email address: alexis.roquefeuil@ipmu.jp