Statistically steady turbulence in thin films: Direct numerical simulations with Ekman friction

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Abstract. We present a detailed direct numerical simulation (DNS) of the two-dimensional Navier–Stokes equation with the incompressibility constraint and air-drag-induced Ekman friction; our DNS has been designed to investigate the combined effects of walls and such a friction on turbulence in forced thin films. We concentrate on the forward-cascade regime and show how to extract the isotropic parts of velocity and vorticity structure functions and hence the ratios of multiscaling exponents. We find that velocity structure functions display simple scaling, whereas their vorticity counterparts show multiscaling, and the probability distribution function of the Weiss parameter \( \Lambda_1 \), which distinguishes between regions with centers and saddles, is in quantitative agreement with experiments.

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1. Introduction

The pioneering work of Kraichnan [1]–[3] showed that fluid turbulence in two dimensions (2D) is qualitatively different from that in 3D: in the former, we have an infinite number of extra conserved quantities, in the inviscid, unforced case; the first of these is the enstrophy. It turns out, therefore, that 2D turbulence displays an inverse cascade of energy, from the length scale at which the force acts to larger length scales, and a forward cascade of enstrophy, from the forcing length scale to smaller ones; by contrast, 3D turbulence is characterized by a forward cascade of energy [4]. Kraichnan’s predictions were first confirmed in atmospheric experiments in quasi-2D, stratified flows [5]; subsequent experiments have studied systems ranging from large-scale geophysical flows to soap films [5]–[12]. The latter have proved to be especially useful in characterizing 2D turbulence.

We present the first direct numerical simulation (DNS) that has been designed specifically to explore the combined effects of the air-drag-induced Ekman friction $\alpha$ and walls on the forward cascade in 2D turbulence, and we employ Kolmogorov forcing used in many soap-film experiments [9]–[12]. Since we use the 2D Navier–Stokes equation with the incompressibility constraint, we cannot explore the effects of changes in the thickness of soap films, Marangoni stresses and compressibility [13, 14]. Nevertheless, as we show in detail below, our study is able to reproduce several results that have been obtained in the soap-film experiments [9]–[12]. In particular, if we use values of $\alpha$ that are comparable to those in experiments, we find that the energy dissipation rate because of the Ekman friction is comparable to the energy dissipation rate that arises from the conventional viscosity. We show how to extract the isotropic parts [15] of velocity and vorticity structure functions and, then, using the extended self-similarity (ESS) procedure [16], we obtain the ratios of multiscaling exponents from which we conclude that velocity structure functions show simple scaling whereas their vorticity counterparts display multiscaling. Most important, our probability distribution function (PDF) of the Weiss parameter $\Lambda$ [17, 18] is in quantitative agreement with that found in experiments [10, 11].

The remaining part of the paper is organized as follows. Section 2 contains a description of the model we use and the numerical methods we employ. Section 3 is devoted to our results: In section 3.1, we examine the temporal evolution of the energy and the dissipation rates because of the viscosity and Ekman friction; in section 3.2, we study the structure functions of the velocity and the vorticity in the forward-cascade regime; and in section 3.3, we study the topological properties of 2D turbulence and their dependence on the Ekman friction.
The concluding section 4 contains a discussion of our results and suggestions for some experiments that should be conducted to confirm our numerical findings.

2. Model and numerical technique

Soap-film dynamics is governed by the equations derived in [13, 14]. These equations account for mass, momentum and soap-film-concentration conservation and the boundary condition for the free, air-film interfaces. However, for low-Mach-number flows, which are relevant to the experiments of [6], [9]–[12], these equations reduce to the incompressible, 2D Navier–Stokes equations [13, 14] albeit with an extra Ekman friction term. The experiments of [9] showed the validity of these equations by testing the Karman–Howarth–Monin relation.

Thus we use the 2D, incompressible Navier–Stokes equations with an additional Ekman friction term to model soap-film dynamics [9, 14]:

\[(\partial_t + \mathbf{u} \cdot \nabla)\omega = \nu \nabla^2 \omega - \alpha \omega + F_\omega / \rho,\]

\[\nabla^2 \psi = \omega.\]  

Here \(\mathbf{u} \equiv (-\partial_t, \psi, \partial_y, \psi)\), \(\psi\) and \(\omega \equiv \nabla \times \mathbf{u}\) are, respectively, the velocity, stream function and vorticity at the position \(x, t\); we have chosen the uniform density \(\rho = 1\); \(\alpha\) is the Ekman friction coefficient, \(\nu\) is the kinematic viscosity and \(F_\omega \equiv k_{inj} F_0 \cos(k_{inj} y)\) is a Kolmogorov-type forcing term, with amplitude \(F_0\) and injection wave vector \(k_{inj}\) (the injection length scale \(\ell_{inj} \equiv 2\pi/k_{inj}\)). We impose no-slip (\(\psi = 0\)) and no-penetration (\(\nabla \psi \cdot \hat{n} = 0\)) boundary conditions on the walls, where \(\hat{n}\) is the outward normal to the wall. If we non-dimensionalize \(x\) by \(k_{inj}^{-1}\), \(t\) by \(k_{inj}^{-2}/\nu\) and \(F_\omega\) by \(2\pi / (k_{inj} ||F_\omega||_2\)\), where \(||F_\omega||_2 \equiv (\int_A |F_\omega|^2 \, dx)^{1/2}\) and \(A\) is the area of the film, then we have two control parameters, namely, the Grashof [19] number \(\mathcal{G} = 2\pi ||F_\omega||_2 / (k_{inj}^3 \rho \nu^2)\) and the non-dimensionalized Ekman friction \(\gamma = \alpha / (k_{inj}^2 \nu)\). For a given set of values of \(\mathcal{G}\) and \(\gamma\), the system attains a non-equilibrium statistical steady state after a time \(t/\tau \simeq 2.8\), where \(\tau = L/\mu_{rms}\) is the box-size time, \(L\) the side of our square simulation domain and \(\mu_{rms}\) the root-mean-square velocity. In this state, the Reynolds number \(Re \equiv \mu_{rms} / (k_{inj} \nu)\), the energy, etc fluctuate; their mean values, along with one-standard-deviation error bars, are given in tables 1 and 2 that list the values of the parameters in our runs R1–R7.
with constant $\epsilon$ rate. We then carry out a detailed study of the topological properties of 2D turbulence and vorticity structure functions with a view to elucidating their scaling and multiscaling differences, respectively, for points adjacent to the walls and for points inside the domain. The Poisson equation (equation (2)) is solved by using a fast-Poisson solver [20] and $\omega$ is calculated at the boundaries by using Thom’s formulae [21] that are given below:

$$\omega_{i,1} = 2\psi_{1,2}/\delta_x^2 \quad \text{(bottom wall)},$$

$$\omega_{i,N} = 2\psi_{1,N-1}/\delta_x^2 \quad \text{(top wall)},$$

$$\omega_{1,j} = 2\psi_{2,j}/\delta_x^2 \quad \text{(left wall)},$$

$$\omega_{N,j} = 2\psi_{N-1,j}/\delta_x^2 \quad \text{(right wall)},$$

where $1 \leq (i, j) \leq N$ are the Cartesian indices of points in our simulation domain with $N \times N$ grid points.

To evaluate spatiotemporal averages, we store $\psi(x, t_n)$ and $\omega(x, t_n)$, with $t_n = (4 + n\Delta)\tau$, $n = 0, 1, 2, \ldots, n_{\text{max}}$, and $96 \leq n_{\text{max}} \leq 200$; $\Delta = 0.28$ for runs R1–6 and $\Delta = 0.13$ for run R7.

### 3. Results

Our results are of three types and are given, respectively, in sections 3.1, 3.2 and 3.3. We begin with a short overview of these before we present details. In section 3.1, we study the time evolutions of the kinetic energy $E(t) \equiv \langle \int_A u^2 dx \rangle / A$, the viscous energy-dissipation rate $\epsilon_v(t) \equiv -\nu\langle \int_A |\omega|^2 dx \rangle / A$, and the energy-dissipation rate because of the Ekman friction $\epsilon_e(t) = -2\alpha E(t)$ and their time averages $E \equiv \langle E(t) \rangle$, $\epsilon_v \equiv \langle \epsilon_v(t) \rangle$ and $\epsilon_e \equiv \langle \epsilon_e(t) \rangle$. We show that there are important qualitative differences, not emphasized earlier, between runs in which $\mathcal{G}$ is held fixed and those in which $Re$ is held fixed (by varying $\mathcal{G}$ and $\gamma$). In particular, for runs with constant $\mathcal{G}$, $\epsilon_v$ turns out to be independent of the Ekman friction, whereas for runs in which $Re$ is held fixed, $E$ remains fixed. In section 3.2, we present a detailed analysis of velocity and vorticity structure functions with a view to elucidating their scaling and multiscaling properties. We then carry out a detailed study of the topological properties of 2D turbulence in section 3.3 and compare our simulations with experimental results. In particular, we examine

| $\epsilon_v$ | $\epsilon_e$ | $\Delta$ (x 10^3) | $b$ | $\delta_b$ (x 10^{-3}) |
|-------------|-------------|-----------------|------|-----------------|
| R1          | $-28 \pm 2$ | $-13.6 \pm 0.5$ | $5.3 \pm 0.3$ | $0.32 \pm 0.01$ | $3.1 \pm 0.1$ |
| R2          | $-28 \pm 1$ | $-26.8 \pm 0.9$ | $4.8 \pm 0.2$ | $0.33 \pm 0.01$ | $3.1 \pm 0.1$ |
| R3          | $-40 \pm 2$ | $-39.9 \pm 1.4$ | $7.2 \pm 0.4$ | $0.33 \pm 0.01$ | $2.8 \pm 0.1$ |
| R4          | $-28 \pm 2$ | $-13.6 \pm 0.4$ | $5.3 \pm 0.3$ | $0.31 \pm 0.01$ | $3.2 \pm 0.2$ |
| R5          | $-28 \pm 1$ | $-27.0 \pm 1.0$ | $4.8 \pm 0.2$ | $0.33 \pm 0.01$ | $3.1 \pm 0.1$ |
| R6          | $-40 \pm 2$ | $-40.0 \pm 1.5$ | $7.2 \pm 0.4$ | $0.33 \pm 0.01$ | $3.1 \pm 0.1$ |
| R7          | $-26 \pm 2$ | $-17.8 \pm 0.6$ | $5.0 \pm 0.4$ | $0.31 \pm 0.01$ | $3.7 \pm 0.3$ |

We use a fourth-order Runge–Kutta scheme with step size $\delta t = 10^{-4}$ for time marching in equation (1) and evaluate spatial derivatives via second-order and fourth-order, centered, finite differences, respectively, for points adjacent to the walls and for points inside the domain.
Figure 1. Representative plots from runs R1 (red circles), R2 (black lines) and R3 (black squares), showing the time evolution of $E(t)/N_E$ ((a) and (b)), $\epsilon_\nu(t)/N$ ((c) and (d)) and $\epsilon_e(t)/N$ ((e) and (f)). In (a), (c) and (e) we keep $G$ fixed and vary $\gamma$ ($\gamma = 0.25$ (red circles) and $\gamma = 0.71$ (black line)). In (b), (d) and (f) we maintain $Re \simeq 21.2$ by varying $\gamma$ ($\gamma = 0.25$ (red circles) and $\gamma = 0.71$ (black squares)) and $G$.

the dependence of the PDF $P(\Lambda)$ on $\gamma$ and we study the joint PDFs of velocity and vorticity differences with $\Lambda$. We obtain excellent agreement with experiments.

3.1. Energy and dissipation

Figures 1(a)–(f) show the time evolution of $E(t)$, normalized by $N_E \equiv (\nu k_{inj})^2$, and that of $\epsilon_\nu(t)$ and $\epsilon_e(t)$, normalized by $N \equiv -k_{inj}^4 \nu^3$. The mean values $E$, $\epsilon_\nu$ and $\epsilon_e$, given in table 1, are comparable to those in experiments; note that $\epsilon_\nu$ and $\epsilon_e$ are of similar magnitude. Comparing data from runs R1 (red circles) and R2 (black lines) in figures 1(a), (c) and (e), we see that, if we fix $G$ and increase $\gamma$, $E$ decreases, $\epsilon_\nu$ remains unchanged (within error bars) and $\epsilon_e$ increases. If we change both $G$ and $\gamma$, we can keep the mean $Re$ fixed, as in runs R1 and R3 in table 1, by compensating for an increase in $\gamma$ with an increase in $G$ (cf [10]); in figures 1(b), (d) and (f), comparing runs R1 (red circles) and R3 (black squares) we see that $E$ remains unchanged (within error bars), whereas both $\epsilon_\nu$ and $\epsilon_e$ increase as $\gamma$ and $G$ increase in such a way that $Re$ is held fixed.

3.2. Structure functions of the velocity and the vorticity

Since Kolmogorov forcing is inhomogeneous, we use the decomposition $\psi = \langle \psi \rangle + \psi'$ and $\omega = \langle \omega \rangle + \omega'$, where the angular brackets denote a time average and the prime the fluctuating part. The inhomogeneous forcing $F_\omega$ and the no-slip boundary conditions that we use generate
Figure 2. Representative pseudocolor plots of (a) the time-averaged stream function $\langle \psi \rangle$ and (b) the time-averaged vorticity $\langle \omega \rangle$ for our run $R7$.

Figure 3. Pseudocolor plots of (a) $S_2(r_c, R)$, for $r_c = (2, 2)$, (b) $S_2(R)$ (average of $S_2(r_c, R)$ over $r_c$), (c) $S_2^\omega(r_c, R)$, for $r_c = (2, 2)$, and (d) $S_2^\omega(R)$ (average of $S_2^\omega(r_c, R)$ over $r_c$).

the patterns shown via the pseudocolor plots of the time averages of $\langle \psi \rangle$ and $\langle \omega \rangle$ (figures 2(a) and (b)), respectively. We use $u'_x \equiv -\partial_y \psi'$, $u'_y \equiv \partial_x \psi'$ and $\omega'$ to calculate the order-$p$ velocity and vorticity structure functions $S_p(r_c, R) \equiv \langle |(u'(r_c + R) - u'(r_c)) \cdot R|^{p} \rangle$ and $S_p^\omega(r_c, R) \equiv \langle |\omega'(r_c + R) - \omega'(r_c)|^{p} \rangle$, respectively, where $R$ has magnitude $R$ and $r_c$ is an origin. Figures 3(a) and (c) show pseudocolor plots of $S_2(r_c, R)$ and $S_2^\omega(r_c, R)$, respectively, for $r_c = (2, 2)$; other values of $r_c$ yield similar results so long as they do not lie near the boundary layer (table 1) of thickness $\delta_p$, ($r_c$ is chosen at least $5\delta_p$ away from all boundaries). We now calculate $S_2(R) \equiv \langle S_2(r_c, R) \rangle_{r_c}$ and $S_2^\omega(R) \equiv \langle S_2^\omega(r_c, R) \rangle_{r_c}$, where the subscript $r_c$ denotes an average over the origin (we use $r_c = (i, j)$, $2 \leq i, j \leq 5$, where $r_c$ indicates the displacement vector relative to

Experiments [9]–[11] achieve homogeneity via a periodic, square-wave forcing with amplitude $F_0$; this introduces another timescale in the problem; to avoid this complication, we work with a time-independent force.
the origin of the simulation domain); these averaged structure functions (figures 3(b) and (d)) are nearly isotropic for $R < \ell_{\text{inj}}$ but not so for $R > \ell_{\text{inj}}$.

To obtain the isotropic parts in an $SO(2)$ decomposition of these structure functions [15], we integrate over the angle $\theta$ that $\mathbf{R}$ makes with the $x$-axis to obtain $S_p(R) \equiv \int_0^{2\pi} S_p(\mathbf{R}) \, d\theta$ and $S_{\omega p}(R) \equiv \int_0^{2\pi} S_{\omega p}(\mathbf{R}) \, d\theta$. Given $S_p(R)$ and $S_{\omega p}(R)$, we use the ESS procedure [16] to extract the multiscaling-exponent ratios $\zeta_p/\zeta_2$ and $\zeta_{\omega p}/\zeta_{\omega 2}$, respectively, from the slopes (in the forward-cascade inertial range) of log–log plots of $S_p(R)$ versus $S_\omega^2(R)$ (figure 4) and $S_{\omega p}(R)$ versus $S_\omega^{\omega 2}(R)$ (figure 5).

The insets figures 4(a) and 5(a) show, respectively, plots of the local slopes $\chi_p \equiv d \log_{10} S_p(R) / d \log_{10} S_\omega^2(R)$ versus $\log_{10} S_\omega^2(R)$ and $\chi_{\omega p} \equiv d \log_{10} S_{\omega p}(R) / d \log_{10} S_\omega^{\omega 2}(R)$ versus $\log_{10} S_\omega^{\omega 2}(R)$ in the forward-cascade regime; the mean values of $\chi_p$ and $\chi_{\omega p}$, over the ranges shown, yield the exponent ratios $\zeta_p/\zeta_2$ and $\zeta_{\omega p}/\zeta_{\omega 2}$ that are plotted versus $p$ in figures 4 and 5, respectively, in which the error bars indicate the maximum deviations of $\chi_p$ and $\chi_{\omega p}$ from their mean values. The Kraichnan–Leith–Batchelor (KLB) predictions [1]–[3] for these exponent ratios, namely, $\zeta_{p,KLB} / \zeta_2 \sim p^{p/2}$ and $\zeta_{\omega p,KLB} / \zeta_{\omega 2} \sim p^0$, agree with our values for $\zeta_p/\zeta_2$ but not $\zeta_{\omega p}/\zeta_{\omega 2}$: velocity structure functions do not display multiscaling (figure 4(b)), whereas their vorticity analogues do (note the curvature of the plot in figure 5(b)). This is in consonance with

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**Figure 4.** Log–log ESS plots of the isotropic parts of the order-$p$ velocity structure functions $S_p(R)$ versus $S_\omega^2(R)$; $p = 3$ (blue line with circles), $p = 4$ (green line with triangles), $p = 5$ (red line with squares), and $p = 6$ (cyan line with stars); plots of the local slope $\chi_p$ (see text), in the forward-cascade inertial range: (a) $\chi_p$ versus $\log_{10} S_\omega^2(R)$ and (b) the plots versus $p$ of the exponent ratios $\zeta_p/\zeta_2$ and error bars from the local slopes (see text), along with the KLB prediction (red line). All plots are for run R7.

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5 We employ ESS since forward-cascade inertial ranges have very modest extents even in the largest DNS studies [24, 25] that use periodic domains and hyperviscosity.
the results of DNS studies with periodic boundary conditions [24]–[27]. Indeed, if we use the same values of \( \gamma \) as in [24], we obtain the same exponent ratios (within error bars); thus our method for the extraction of the isotropic parts of the structure functions suppresses boundary and anisotropy effects efficiently.

3.3. Topological properties of soap-film turbulence

For an inviscid, incompressible 2D fluid, the local flow topology can be characterized via the Weiss criterion [17, 18] that uses the invariant

\[
\Lambda \equiv (\omega^2 - \sigma^2)/4, \tag{3}
\]

where \( \sigma^2 \equiv \sum_{i,j} \sigma_{ij} \sigma_{ji} \) and \( \sigma_{ij} \equiv (\partial_i u_j + \partial_j u_i) / \sqrt{2} \). This criterion provides a useful measure of flow properties even if \( \nu > 0 \) as noted in the experiments of [10]: regions with \( \Lambda > 0 \) and \( \Lambda < 0 \) correspond, respectively, to centers and saddles as we have shown in figure 6 by superimposing, at a representative time, a pseudocolor plot of \( \Lambda \) on contours of \( \psi \); an animated version of this plot is given as a multimedia file, mov_lam.mpg (MPEG file, 3.5 MB). This result is in qualitative accord with experiments (see e.g. figure 1 of [10] and also earlier DNS studies [17, 18], which do not use the Ekman friction). In figure 7, we compare the scaled PDFs \( P_2(\Lambda/\Lambda_{\text{rms}}) \) with data obtained from points near the walls (black curve) and from points in the bulk (red curve); the clear difference between these, not highlighted before, indicates that the regions of large \( \Lambda \) are suppressed in the boundary layers.
Figure 6. (a) Representative pseudocolor plot of \( \Lambda \) superimposed on a contour plot of the stream function \( \psi \) in the statistical steady state. Contours of \( \psi > 0 \) are shown as continuous lines whereas contours of \( \psi < 0 \) are indicated by dashed lines. Regions with \( \Lambda > 4000 \) are shown in dark red color and with \( \Lambda < -2400 \) in dark blue color. For intermediate values of \( \Lambda \), the colors used are as indicated in the color bar. For the temporal evolution of the \( \Lambda \) field, see the movie mov_lam.mpg (MPEG file, 3.5 MB, available from stacks.iop.org/NJP/11/073003/mmedia) from our DNS. (b) A representative plot of the velocity field in the left corner of the simulation domain; the border with the red boundary is of width \( 2\delta_b \) and the two square boxes show one center and one saddle; here \( x \) and \( y \) are given in grid spacings. Both plots are for run R7.

Figure 7. The PDF \( P_2(\Lambda/\Lambda_{\text{rms}}) \) obtained from points in the bulk \( \delta_b < x, y < L - \delta_b \) (red line) and from points within a distance \( \delta_b \) from the boundaries (black line) for our run R7. One-standard-deviation error bars are indicated by lightly shaded regions that straddle the curves of \( P_2(\Lambda/\Lambda_{\text{rms}}) \).
Figure 8. Representative pseudocolor plot of $\log_{10}(\sigma^2)$ (top frame), $\log_{10}(\omega^2)$ (middle frame) and $\Lambda$ (bottom frame) in a region that is one-boundary-layer thick, i.e. of width $\delta_b$, and which lies near the bottom wall. The plot shows that, although $\log_{10}(\sigma^2)$ and $\log_{10}(\omega^2)$ have very similar profiles, in most of the region $\log_{10}(\sigma^2) > \log_{10}(\omega^2)$. This explains the skewness of the near-wall $P_2(\Lambda/\Lambda_{rms})$. Regions with $\Lambda > 1000$ and $\log_{10}(\sigma^2), \log_{10}(\omega^2) > 4$ are shown in dark red, and regions with $\Lambda < -1000$ and $\log_{10}(\sigma^2), \log_{10}(\omega^2) < -4$ are shown in dark blue. For intermediate values of $\log_{10}(\sigma^2), \log_{10}(\omega^2)$ and $\Lambda$, the colors used are as in the color bar. For the temporal evolution of the above plots, see the movie mov_snw_sig2_omg2_lam.mpg (MPEG file, 1.3 MB, available from stacks.iop.org/NJP/11/073003/mmedia).

This is because near-wall regions (say, within one-boundary-layer thickness $\delta_b$ from the walls) are dominated by the strain, i.e. $\sigma^2 > \omega^2$ as shown in figure 8. For the temporal evolution of the $\sigma^2$, $\omega^2$ and $\Lambda$ fields in the region near the wall, see the movie mov_snw_sig2_omg2_lam.mpg, available from stacks.iop.org/NJP/11/073003/mmedia. This explains the skewness of the near-wall $P_2(\Lambda/\Lambda_{rms})$ in figure 7.

For the following discussion, to analyze the topological properties of the flow in the bulk, we evaluate $\Lambda$ from the fluctuating part of the stream function.

Figures 9(a) and (d) show the PDF $P_1(\Lambda)$ and the scaled PDF $P_2(\Lambda/\Lambda_{rms})$ for runs R4 (red line) and R5 (blue dashed line), with $\gamma = 0.25$ and 0.71, respectively, and $G = 3.5 \times 10^4$; comparing these figures, we see that both $P_1$ and $P_2$ overlap (within error bars) for runs R4 and R5. We believe that this is because, in fixed-$G$ runs like R4 and R5, $\epsilon_v$ does not change (table 1) even though $\gamma$ changes. By contrast, comparing $P_1$ and $P_2$ (figures 9(c) and (d)) for runs R4 (red line) and R6 (blue dashed line), in which the mean $Re$ is held fixed by tuning both $\gamma$ and $G$, we find, in agreement with experiments [10], that the PDFs $P_1$ do not agree for these runs.
but the PDFs $P_2$ overlap within error bars. Our results for $P_2$ in figure 9(d) are in quantitative agreement with experiments: we have obtained the points in this plot by digitizing the data points (see http://www.frantz.fi/software/g3data.php) in figure 2(d) of [10]; the errors in these points are comparable to the spread of data in [10]. The differences between figures 9(a) and (b) can be understood by the following heuristic argument: for homogeneous, isotropic turbulence $\bar{\Lambda} = (\bar{\omega^2} - \bar{\sigma^2})/4 = 0$, because the spatial averages of $\omega^2$ and $\sigma^2$ are both $2|\epsilon_v|/\nu$. Even for the flow we consider, the PDFs of figures 9(a)–(d) yield $\bar{\Lambda} \approx 0$ whence $\bar{\omega^2} \approx \bar{\sigma^2}$. On taking the square and then the spatial average of equation (3), we obtain $\bar{\Lambda^2} = [\omega^4 + \sigma^4 - 2\omega^2\sigma^2]/16$; and if we make the approximations $\omega^4 \approx 3\omega^2\sigma^2$ and $\sigma^4 \approx 3\sigma^2\omega^2$, then

$$\bar{\Lambda^2} \approx \frac{3|\epsilon_v|^2}{2\nu^2} \left( 1 - \frac{2\omega^2\sigma^2}{\omega^4 + \sigma^4} \right),$$

$$\bar{\Lambda^2} \approx 0.33 \frac{3|\epsilon_v|^2}{2\nu^2},$$

whence

$$\Lambda_{rms} = \sqrt{\bar{\Lambda^2} - (\bar{\Lambda})^2} \approx |\epsilon_v|/\nu,$$

(4)

where the second line in equation (4) follows from the last column of table 3; this table shows the degree to which the approximations made above agree with the results from our DNS.
Table 3. The ratios $\frac{\omega^2}{\sigma^2}$, $\frac{\omega^4}{3\omega^2}$, $\frac{\sigma^4}{3\sigma^2}$ and $\frac{\omega^2\sigma^2}{(\omega^4 + \sigma^4)}$ obtained from our DNS.

|.run| $\frac{\omega^2}{\sigma^2}$| $\frac{\omega^4}{3\omega^2}$| $\frac{\sigma^4}{3\sigma^2}$| $\frac{\omega^2\sigma^2}{(\omega^4 + \sigma^4)}$ |
|---|---|---|---|---|
|R4| $1 \pm 10^{-5}$| $1.4 \pm 0.2$| $1.0 \pm 0.2$| $0.32 \pm 0.03$ |
|R5| $1 \pm 10^{-5}$| $1.1 \pm 0.1$| $0.8 \pm 0.1$| $0.34 \pm 0.03$ |
|R6| $1 \pm 10^{-5}$| $1.3 \pm 0.1$| $0.9 \pm 0.1$| $0.32 \pm 0.02$ |
|R7| $0.99 \pm 10^{-4}$| $1.7 \pm 0.3$| $1.3 \pm 0.1$| $0.34 \pm 0.03$ |

Figure 10. Plots of conditional expectation values, with one-standard-deviation error bars, of $\sigma^2$ (black dots) and $\omega^2$ (blue circles) for a given $\Lambda$ for our run R5. A comparison of this figure with figure 3 of [10] shows that our results are in excellent qualitative agreement with the experiments.

Note that in all our runs R1–7, $\nu = 0.016$. So, if we hold the Grashof number $G$ fixed (runs R4 and R5), then $|\epsilon_v|$ is independent of $\gamma$ (figure 1(c) and table 2) and, therefore, $\Lambda_{\text{rms}}$ (equation (4)) is also independent of $\gamma$. In contrast, if we hold $Re$ fixed (runs R4 and R6), $|\epsilon_v|$ increases with increasing $\gamma$ (figure 1(d) and table 2), so $\Lambda_{\text{rms}}$ (equation (4)) also increases as $\gamma$ increases. This explains why the unscaled PDFs $P_1$ of figure 9(a) overlap ($G$ fixed) but those in figure 9(c) do not ($Re$ fixed). Only when we normalize $\Lambda$ by $\Lambda_{\text{rms}}$ do the scaled PDFs $P_2$ overlap (figures 9(b) and (d)).

Conditional expectation values of $\overline{\sigma^2}$ and $\overline{\omega^2}$, for a given value of $\Lambda$, also agree well with experiments as can be seen by comparing figure 10 with figure 3 of [10].

We also present in figures 11(a)–(c) pseudocolor plots of the joint PDFs of

$$
\delta \omega(r) \equiv \omega'(x + r \hat{e}_x) - \omega'(x),
\delta u_L(r) \equiv u'_x(x + r \hat{e}_x) - u'_x(x),
$$
or

$$
\delta u_T(r) \equiv u'_y(x + r \hat{e}_x) - u'_y(x)
$$

(5)
Figure 11. Pseudocolor plots of (a) the joint PDF $P(\delta\omega(r = 0.12), \Lambda'/\Lambda'_{rms})$, (b) the joint PDF $P(\delta u_L(r = 0.12), \Lambda'/\Lambda'_{rms})$ and (c) the joint PDF $P(\delta u_T(r = 0.12), \Lambda'/\Lambda'_{rms})$ for our run R7. The contours and the shading are for the logarithms of the joint PDFs. A comparison of (b) and (c) with figures 1(b) and 2 of [11] show that our results agree very well with experiments.

with $\Lambda' \equiv \det(M)$. Here

$$M^{\alpha\beta} \equiv \frac{1}{A_r} \int_{\Omega} m^{\alpha\beta} \, d\mathbf{r},$$

$$A_r \equiv \int_{\Omega} d\mathbf{r}, \quad m^{\alpha\beta} \equiv \partial_\alpha' u_\beta', \quad \text{(6)}$$

with $\Omega$ a circular disc with the center at $x + (r/2)\hat{e}_x$ and radius $r/2$, and $r$ in the forward-cascade regime. In figures 11(a)–(c), we present pseudocolor plots of the joint PDFs $P(\delta\omega(r = 0.12), \Lambda'/\Lambda'_{rms})$, $P(\delta u_L(r = 0.12), \Lambda'/\Lambda'_{rms})$ and $P(\delta u_T(r = 0.12), \Lambda'/\Lambda'_{rms})$. Figures 11(b) and (c) are in striking agreement with figures 1 and 2 of [11]. Figure 11(a) predicts that regions of large $\delta\omega$ and small $\Lambda'/\Lambda'_{rms}$ (and vice versa) are correlated; this result awaits experimental confirmation.

Finally, we calculate $\tilde{\Lambda} \equiv \sqrt{\langle (\partial_x u'_x)^2 (\partial_y u'_y)^2 \rangle}$ and $b \equiv -\langle \partial_x u'_x \partial_y u'_y \rangle / \tilde{\Lambda}$ (see table 2). Our simulations yield $b \simeq 0.3$, which is the same as that obtained in the experiments for the Kolmogorov forcing in table I of [10]. We find $530 \lesssim \tilde{\Lambda} \lesssim 720$, which is close to the experimental range $712 \lesssim \tilde{\Lambda} \lesssim 1282$; our values of $\tilde{\Lambda}$ are somewhat smaller than those in experiments since our Reynolds numbers are not as large as in these experiments. We find $\delta_b$, the boundary-layer thickness, to be small and it does not depend significantly on $\alpha$ ($\delta_b \sim 0.031 \pm 0.001$), which suggests that the bulk-flow properties are only weakly affected by the boundaries in such a soap film.

4. Conclusion

Some earlier numerical studies of 2D, wall-bounded, statistically steady turbulent flows [22, 23] use forcing functions that are not of the Kolmogorov type; furthermore, they do not include the air-drag-induced Ekman friction. Other numerical studies, which include the Ekman friction and Kolmogorov forcing, employ periodic boundary conditions [24, 25, 28]. To the best of our knowledge, our study of 2D turbulent flows is the first one that accounts for the Ekman friction, realistic boundary conditions and Kolmogorov forcing. We show that, for values of $\alpha$ that are comparable to those in experiments, the energy dissipation rate because of the Ekman
friction is comparable to the energy dissipation rate that arises from the conventional viscosity. We extract the isotropic part of the structure functions in the forward-cascade regime. We find that velocity structure function exponent ratios show simple scaling, whereas their vorticity counterparts show multiscaling. We also study the topological properties of 2D turbulence by using the Weiss criterion and we find excellent agreement with PDFs that have been obtained experimentally. We hope our results will stimulate experimental studies designed to extract (a) the isotropic parts of structure functions (and thereby to probe the multiscaling of vorticity structure functions (figure 5(b))) or (b) the PDF $P_2(\Lambda/\Lambda_{rms})$ (figure 7) near soap-film boundaries.

In [29] it was argued that if the Ekman friction is nonzero and in the limit of vanishing viscosity, the third-order velocity structure function shows an anomalous behavior. In our calculations of structure functions of odd orders, we have employed moduli of velocity increments; without these moduli the error bars are too large in our wall-bounded DNS to obtain good statistics for structure functions of odd order. Thus, we cannot compare our results directly with those of [29]. The main point of our study is to mimic, as closely as possible, parameters and boundary conditions in soap-film experiments such as those of [12]. Hence our viscosity is much higher (and the Reynolds number much lower) than in the DNS of [27], which was designed to investigate some of the issues raised in [29]. Therefore, a direct comparison of our structure-function results with those of [27] is not possible, especially for odd orders because, as mentioned above, we use moduli of velocity increments. We have, however, checked that our velocity structure functions show simple scaling as in the experiments of [12]; it would be interesting to explore whether these experiments can be extended to confirm the multiscaling of vorticity structure functions that we describe above; such experimental studies might well benefit from the procedures we have used to extract the isotropic parts of structure functions.

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