A NEW CHARACTERIZATION OF DISCRETE DECOMPOSABLE GRAPHICAL MODELS

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Abstract. Decomposable graphical models, also known as perfect DAG models, play a fundamental role in standard approaches to probabilistic inference via graph representations in modern machine learning and statistics. However, such models are limited by the assumption that the data-generating distribution does not entail strictly context-specific conditional independence relations. The family of staged tree models generalizes DAG models so as to accommodate context-specific knowledge. We provide a new characterization of perfect discrete DAG models in terms of their staged tree representations. This characterization identifies the family of balanced staged trees as the natural generalization of discrete decomposable models to the context-specific setting.

1. Introduction

A graphical model is a collection of joint probability distributions of a vector of random variables \((X_1, \ldots, X_p)\) that satisfy conditional independence constraints specified by a graph with set of nodes \(p := \{1, \ldots, p\}\). Within the class of graphical models there are those whose underlying graph is undirected and those whose underlying graph is a directed acyclic graph (DAG). Both classes of models are usually defined in two different ways, either parametrically by means of a clique factorization for undirected models and the recursive factorization property for directed acyclic models, or implicitly via the undirected or directed Markov properties associated to each graph. The class of models for which these two perspectives coincide is the set of decomposable graphical models. These models have several different equivalent combinatorial, statistical and geometric characterizations. In combinatorial terms, a model is decomposable if and only if its underlying undirected graph is chordal or, equivalently, its underlying DAG is perfect [9]. In statistics, decomposable models are the subclass of DAG models that are linear exponential families [4]. In terms of algebra and geometry, discrete decomposable models are toric varieties defined by quadratic binomials which correspond to global separation statements in [5].

The main theorem of this paper, Theorem 3.1, provides a novel combinatorial characterization of discrete decomposable graphical models using staged tree models. Staged tree models are statistical models that encode context-specific conditional independence relations among events by the use of a combinatorial object called a staged tree [10]. We show that a discrete DAG model is decomposable if and only if its staged tree representation is balanced [3]. This result translates...
the combinatorial condition of a perfect DAG to staged trees and establishes balanced staged tree models as a natural generalization of discrete decomposable DAG models to the context-specific setting.

2. STAGED TREES AND DAG MODELS

The class of staged tree models was first introduced in [10]. We refer the reader to [2] for a detailed introduction to this model class. Given a directed graph \( G = (V, E) \) with node set \( V \) and collection of edges \( E \), for \( v \in V \) we call \( w \in V \) a parent of \( v \) (in \( G \)) if \( w \rightarrow v \in E \), and we let \( \text{pa}_G(v) \) denote the collection of all parents of \( v \) in \( G \). Conversely, \( v \) is a called a child of \( w \) (in \( G \)), and we let \( \text{ch}_G(w) \) denote the collection of all children of \( w \) in \( G \). A rooted tree \( T = (V, E) \) is a directed graph whose skeleton (i.e., underlying undirected graph) is a tree containing a unique node, \( r \), for which \( \text{pa}_T(r) = \emptyset \). It follows that for each \( v \in V \) there is a unique directed path from \( r \) to \( v \) in \( T \), which we denote by \( \lambda(v) \); that is, \( \lambda(v) \) denotes the collection of all edges constituting this unique directed path. We also denote the set of edges from a node \( v \) to each of its children by \( E(v) \).

**Definition 2.1.** For a rooted tree \( T = (V, E) \), a finite set of labels \( \mathcal{L} \), and a map \( \theta : E \rightarrow \mathcal{L} \) labeling the edges \( E \) with elements of \( \mathcal{L} \), the pair \((T, \theta)\) is a staged tree if

1. \( |\theta(E(v))| = |E(v)| \) for all \( v \in V \), and
2. for any two \( v, w \in V \), \( \theta(E(v)) \) and \( \theta(E(w)) \) are either equal or disjoint.

We typically refer to \( T \) as a staged tree whenever the labeling \( \theta \) is understood. The second condition of Definition 2.1 partitions the vertices of \( T \) into disjoint sets, called stages, defined by the property that, \( v, w \in V \) are in the same stage if and only if \( \theta(E(v)) = \theta(E(w)) \). The partition of \( V \) into its stages is the staging of \( T \). The space of canonical parameters of a staged tree \( T \) is the set

\[
\Theta_T := \left\{ \alpha \in \mathbb{R}^{|\mathcal{L}|} : \forall e \in E, \alpha_{\theta(e)} \in (0, 1) \text{ and } \forall v \in V, \sum_{e \in E(v)} \alpha_{\theta(e)} = 1 \right\}.
\]

**Definition 2.2.** Let \( \mathcal{I}_T \) be the collection of all leaves of \( T \). The staged tree model \( M_{(T, \theta)} \) for \((T, \theta)\) is the image of the map \( \psi_T : \Theta_T \rightarrow \Delta_{|\mathcal{I}_T| - 1}^0 \) where

\[
\psi_T : \alpha \mapsto f_v := \left( \prod_{e \in E(v)} \alpha_{\theta(e)} \right)_{v \in \mathcal{I}_T},
\]

and \( \Delta_{|\mathcal{I}_T| - 1}^0 \) denotes the \((|\mathcal{I}_T| - 1)\)-dimensional (open) probability simplex. We say that \( f \in \Delta_{|\mathcal{I}_T| - 1}^0 \) factorizes according to \( T \) if \( f \in M_{(T, \theta)} \).

Typically, we will identify the leaves of a staged tree \( T \) with the possible outcomes of some jointly distributed random variables – which is why the model \( M_{(T, \theta)} \) is defined to live in the probability simplex \( \Delta_{|\mathcal{I}_T| - 1}^0 \). To make this identification formal, let \( X_{[p]} = (X_1, \ldots, X_p) \) denote a vector of discrete random variables with joint state space \( \mathcal{R} \). Given a subset \( S \subset [p] \), we let \( \mathcal{R}_S \) denote the restricted state space of the random subvector \( X_S = (X_i : i \in S) \) of \( X_{[p]} \). For a permutation \( \pi_1 \cdots \pi_p \in \mathcal{S}_p \) of \([p]\), construct a rooted tree \( T = (V, E) \) where \( V := \{r\} \cup \bigcup_{j \in [p]} \mathcal{R}_{\{\pi_1, \ldots, \pi_j\}} \), and \( E \) is the set \( \{r \rightarrow x_{\pi_1} : x_{\pi_1} \in \mathcal{R}_{\{\pi_1\}} \} \cup \{x_{\pi_1} \cdots x_{\pi_{k-1}} \rightarrow x_{\pi_1} \cdots x_{\pi_k} : x_{\pi_1} \cdots x_{\pi_k} \in \mathcal{R}_{\{\pi_1, \ldots, \pi_k\}} \} \).
For $v \in V$, we call the number of edges in $\lambda(v)$ the level of $v$, and for $k \in \{0, \ldots, p\}$, we refer to the set of all nodes with level $k$, denoted $L_k$, as the $k^{th}$ level of $\mathcal{T}$. As the root node is the only element in $L_0$, we typically ignore this level. For trees defined from a vector of random variables $X[p]$ as above, the $k^{th}$ level of $\mathcal{T}$, for $k > 0$, is simply the set of outcomes $\mathcal{R}_{\{\pi_1, \ldots, \pi_k\}}$. Hence, we associate the variable $X_{\pi_k}$ with level $L_k$ and denote this association by $(L_1, \ldots, L_k) \sim (X_{\pi_1}, \ldots, X_{\pi_p})$. We call the permutation $\pi$ the causal ordering of $\mathcal{T}$.

Note that a tree $\mathcal{T}$ defined for the vector $X[p]$ is uniform; that is, $|E(v)| = |E(w)|$ for any $v, w \in L_k$, for all $k \geq 0$. They are also stratified; meaning that all of their leaves have the same level, and if any two nodes are in the same stage then they are also in the same level. For a uniform and stratified staged tree model $\mathcal{M}(\mathcal{T}, \theta)$, the parameter values on the edges $E(v)$ for $v \in \mathcal{R}$ abide by the chain rule:

**Lemma 2.1.** Let $\mathcal{T}$ be a uniform, stratified staged tree with levels $(L_1, \ldots, L_p) \sim (X_1, \ldots, X_p)$, let $f \in \mathcal{M}(\mathcal{T}, \theta)$, and fix $v \in i_\mathcal{T}$. If $e \in \lambda(v)$ is the edge $u \rightarrow w$ between levels $L_{k-1}$ and $L_k$, where $u = x_1 \cdots x_{k-1}$ and $w = x_1 \cdots x_{k-1} x_k$, then

$$\alpha_{\theta(e)} = f(x_k \mid x_1 \cdots x_{k-1}).$$

**Proof.** The proof is by induction on $p$, the number of levels of $\mathcal{T}$. For the base case, let $p = 1$. Given $v \in i_\mathcal{T}$, we have that $f_v = \alpha_{\theta(e)} = f(X_1 = x_1)$, where $v = x_1 \in \mathcal{R}_{\{1\}}$. Suppose now that the claim holds for $p - 1$ for some $p > 1$. Then, $f_v = f(x_1 \cdots x_p)$, where $v = x_1 \cdots x_p$, and by definition of $f_v$ (via Definition 2.2), we have

$$f_v = \prod_{e \in E(\lambda(v))} \alpha_{\theta(e)}. \quad (1)$$

Since $v = x_1 \cdots x_p$, then by the chain rule

$$f_v = f(x_1 \cdots x_p) = \prod_{k \in [p]} f(x_k \mid x_1 \cdots x_{k-1}). \quad (2)$$

Let $e$ denote the unique edge in $\lambda(v)$ between a node in level $L_{p-1}$ of $\mathcal{T}$ and a node in level $L_p$. By the inductive hypothesis, the equality of equations (1) and (2) reduces to $\alpha_{\theta(e)} = f(x_p \mid x_1 \cdots x_{p-1})$, which completes the proof. \hfill $\square$

A uniform, stratified staged tree $\mathcal{T}$ with levels $(L_1, \ldots, L_k) \sim (X_{\pi_1}, \ldots, X_{\pi_k})$ is called compatibly labeled if

$$\theta(x_{\pi_1} \cdots x_{\pi_{k-1}} \rightarrow x_{\pi_1} \cdots x_{\pi_{k-1}} x_{\pi_k}) = \theta(y_{\pi_1} \cdots y_{\pi_{k-1}} \rightarrow y_{\pi_1} \cdots y_{\pi_{k-1}} x_{\pi_k})$$

for all $x_{\pi_k} \in \mathcal{R}_{\{\pi_k\}}$ whenever $x_{\pi_1} \cdots x_{\pi_{k-1}}$ and $y_{\pi_1} \cdots y_{\pi_{k-1}}$ are in the same stage. This condition ensures that edges emanating from $x_{\pi_1} \cdots x_{\pi_{k-1}}$ and $y_{\pi_1} \cdots y_{\pi_{k-1}}$ with endpoints corresponding to the same outcome $x_{\pi_k}$ of the next variable encode the invariance in conditional probabilities:

$$f(x_{\pi_k} \mid x_{\pi_1} \cdots x_{\pi_{k-1}}) = f(x_{\pi_k} \mid y_{\pi_1} \cdots y_{\pi_{k-1}}). \quad (3)$$

These invariances, when considered in terms of the staging of the tree, collectively encode context-specific conditional independence relations satisfied by distributions in the model $\mathcal{M}(\mathcal{T}, \theta)$. The most relevant examples of this are DAG models.
2.1. Staged tree representations of DAG models. Given a DAG $\mathcal{G} = ([p], E)$ and a vector $X_p$ with state space $\mathcal{R}$, the DAG model associated to $\mathcal{G}$, denoted $\mathcal{M}(\mathcal{G})$, is the set of all distributions $f \in \Delta^p_{|\mathcal{R}| - 1}$ that satisfy the recursive factorization

$$f(X) = \prod_{k=1}^{p} f(X_k \mid X_{pa_G(k)}).$$

A DAG model $\mathcal{M}(\mathcal{G})$ is called a decomposable model if the DAG $\mathcal{G}$ is perfect; i.e., for all $k \in [p]$ the induced subDAG of $\mathcal{G}$ on $pa_G(k)$ is complete. To derive our characterization of decomposable graphical models in terms of their associated staged trees, we first characterize those staged trees that represent DAG models.

A permutation $\pi = \pi_1 \cdots \pi_p \in \mathfrak{S}_p$ of $[p]$ is called a linear extension (or topological ordering) of a DAG $\mathcal{G} = ([p], E)$ if $\pi_i - 1 < \pi_j - 1$ whenever $i \rightarrow j \in E$. Given a linear extension $\pi$ of a DAG $\mathcal{G}$, we construct a staged tree $T^\pi_\mathcal{G}$ for the vector $X_p$ with causal ordering $\pi$ by labeling the edges emanating from level $L_k = \mathcal{R}_{[\pi_1, \ldots, \pi_k]}$ as follows: Let

$$\mathcal{L} := \{f(x_{\pi_{k+1}} \mid x_{pa_G(\pi_{k+1})}) : k \in [p-1], x_{\pi_{k+1}} \in \mathcal{R}_{[\pi_{k+1}]}, x_{pa_G(\pi_{k+1})} \in \mathcal{R}_{pa_G(\pi_{k+1})}\},$$

and the labeling map $\theta : E \rightarrow \mathcal{L}$ where

$$\theta(x_{\pi_1} \cdots x_{\pi_k} \rightarrow x_{\pi_1} \cdots x_{\pi_k} x_{\pi_{k+1}}) = f(x_{\pi_{k+1}} \mid x_{pa_G(\pi_{k+1})}).$$

It follows that $(T^\pi_\mathcal{G}, \theta)$ is stratified and uniform, and hence, for any $f \in \mathcal{M}(T^\pi_\mathcal{G}, \theta)$,

$$f(x_{\pi_{k+1}} \mid x_{\pi_1} \cdots x_{\pi_k}) = \alpha \theta(e) = \alpha f(x_{\pi_{k+1}} \mid x_{pa_G(\pi_{k+1})}),$$

where $e = x_{\pi_1} \cdots x_{\pi_k} \rightarrow x_{\pi_1} \cdots x_{\pi_k} x_{\pi_{k+1}}$. Moreover, $T^\pi_\mathcal{G}$ is compatibly labeled with a stage

$$S_{\theta_{pa_G(k+1)}} := \{x = x_{\pi_1} \cdots x_{\pi_k} \in L_k : x_{[\pi_1, \ldots, \pi_k]} \cap pa_G(k+1) = y_{pa_G(k+1)}\}$$

in level $L_k$ for each $k \in [p-1]$ and each $y_{pa_G(k+1)} \in \mathcal{R}_{pa_G(k+1)}$. Hence, the invariances in equation (3) implied by $T^\pi_\mathcal{G}$ being compatibly labeled imply that

$$f(x_{\pi_{k+1}} \mid x_{\pi_1} \cdots x_{\pi_k}) = \alpha f(x_{\pi_{k+1}} \mid x_{[\pi_1, \ldots, \pi_k]} \cap pa_G(k+1)) = \alpha f(x_{\pi_{k+1}} \mid x_{pa_G(k+1)}),$$

where the last equality follows from the assumption that $\pi$ is a linear extension of $\mathcal{G}$. Thus, $\mathcal{M}(T^\pi_\mathcal{G}, \theta)$ consists of all distributions $f \in \Delta^p_{|\mathcal{R}| - 1}$ that satisfy equation (4); that is, $\mathcal{M}(\mathcal{G}) = \mathcal{M}(T^\pi_\mathcal{G}, \theta)$. Hence, $T^\pi_\mathcal{G}$ is a stage tree representation of a DAG model, which we call the staged tree associated to $\mathcal{G}$ and $\pi$.

Remark 2.1. From Lemma 2.1, the value $f(x_{\pi_{k+1}} \mid x_{\pi_1} \cdots x_{\pi_k})$ indicates a transition probability for any staged tree that is compatibly labeled with levels $(L_1, \ldots, L_p) \sim (X_{\pi_1}, \ldots, X_{\pi_p})$. The notation for this transition probability is different from the labels in a staged tree representation of a DAG model: $\mathcal{L} := \{f(x_{\pi_{k+1}} \mid x_{pa_G(\pi_{k+1})}) : k \in [p-1], x_{\pi_{k+1}} \in \mathcal{R}_{[\pi_{k+1}]}, x_{pa_G(\pi_{k+1})} \in \mathcal{R}_{pa_G(\pi_{k+1})}\}$. In the former case the notation indicates a parameter in the latter case the notation indicates a symbol. For the proof of the main theorem we will manipulate the elements in $\mathcal{L}$ as indeterminates.

Example 2.1. Consider the DAGs $\mathcal{G}_1 = ([4], \{1 \rightarrow 3, 2 \rightarrow 3, 3 \rightarrow 4, 2 \rightarrow 4\})$ and $\mathcal{G}_2 = ([4], \{1 \rightarrow 2, 2 \rightarrow 3, 2 \rightarrow 4, 3 \rightarrow 4\})$. The trees $T_1, T_2$ in Figure 1 are staged tree representations of the DAG models $\mathcal{M}(\mathcal{G}_1)$ and $\mathcal{M}(\mathcal{G}_2)$ with respect to the linear extension $\pi = 1234$ where $X_1, X_2, X_3, X_4$ are binary random
variables. An upwards arrow in Figure 1 represents the outcome 0 and a downwards arrow represents the outcome 1. Two vertices with the same color (except white) in any of the trees represent a context-specific conditional independence relation. For instance, the four colours in level three of $T_{G_1}$, $T_{G_2}$, respectively, represent $f(X_4 | (X_1, X_2, X_3) = (0, i, j)) = f(X_4 | (X_1, X_2, X_3) = (1, i, j))$ for each $i, j \in \{0, 1\}$, with a different stage/color for each possible pair $i, j$. Together these four context-specific conditional independence relations combine to make the CI relation $X_4 \perp \perp X_1 | (X_2, X_3)$.

Let $T_G$ denote the collection of all staged trees $T_G$ associated to any DAG and any of its linear extensions. The following proposition characterizes the elements of $T_G$ in terms of the stages of the trees.

**Proposition 2.2.** A staged tree $T$ is in $T_G$ if and only if it is compatibly labeled with levels $(L_1, \ldots, L_p)$ of $(X_{\pi_1}, \ldots, X_{\pi_p})$ for some $\pi \in \Pi_p$, and if for all $k \in [p-1]$ the level $L_k = \mathcal{R}_{\{\pi_1, \ldots, \pi_k\}}$ is partitioned into stages

$$\bigsqcup_{y \in \mathcal{R}_{\Pi_k}} S_y$$

for some subset $\Pi_{k+1} \subset \{\pi_1, \ldots, \pi_k\}$, where $S_y = \{x \in L_k : x_{\Pi_{k+1}} = y\}$.

**Proof.** The “only if” direction ($\Rightarrow$) follows from the construction of $T_G$. So it only remains to show that any compatibly labeled staged tree $T$ with levels $(L_1, \ldots, L_p) \sim (X_{\pi_1}, \ldots, X_{\pi_p})$ and the specified stages $S_y$ in each level $L_k$ is a staged tree associated to some DAG $G$ and one of its linear extensions. However, this follows in a straightforward way by taking $G$ to be the DAG $G = ([p], E)$ where $\text{pa}_G(k + 1) := \Pi_{k+1}$ for all $k \in [p-1]$ and $\text{pa}_G(\pi_1) := \emptyset$. It is immediate that $\pi$ is a linear extension of $G$ and that the staging of $T$ coincides with the staging of $T_G$. \hfill $\Box$

**Remark 2.2.** Note that, by definition, if $T \in T_G$ then $\mathcal{M}_{(T, \theta)} = \mathcal{M}(G)$ for some DAG $G$. It is likely that the converse holds, in particular if $T$ is compatibly labeled and $\mathcal{M}_{(T, \theta)} = \mathcal{M}(G)$ then $T \in T_G$. 
3. Balanced Models

The family of balanced staged tree models was introduced in [3] to characterize those staged tree models for which a certain pair of associated polynomial ideals coincide (see [3, Theorem 3.1]). This observation suggests that the balanced staged tree models are a generalization of decomposable models to more general staged trees. In this section, we prove that this is indeed the case (see Theorem 3.1).

For a staged tree $T = (V,E)$ and a node $v \in V$, we let $T_v$ denote the rooted subtree of $T$ whose root node is $v$. If we let $\Lambda_v$ denote the set of root-to-leaf paths in $T_v$, then the interpolating polynomial of $T_v$ is

$$t(v) := \sum_{\lambda \in \Lambda_v} \prod_{e \in \lambda} \theta(e)$$

The polynomial $t(v)$ is an element of the polynomial ring $\mathbb{R}[\Theta_T] := \mathbb{R}[\theta(e) : e \in E]$ with one indeterminate for each edge label in $L$. When $v$ is the root of $T$, $t(v)$ is called the interpolating polynomial of $T$.

Definition 3.1. Let $(T, \theta)$ be a staged tree and $v,w \in V$ be two vertices in the same stage with children $\text{ch}_T(v) = \{v_0, \ldots, v_k\}$ and $\text{ch}_T(w) = \{w_0, \ldots, w_k\}$, respectively. After a possible reindexing, we may assume that $\theta(v \to v_i) = \theta(w \to w_i)$ for all $i \in [k]_0$. The pair of vertices $v,w$ is balanced if

$$t(v_i)t(w_j) = t(w_i)t(v_j)$$

in $\mathbb{R}[\Theta_T]$ for all $i \neq j \in [k]_0$.

The staged tree $(T, \theta)$ is called balanced if every pair of vertices in the same stage is balanced.

Example 3.1. Two vertices $v,w$ in a staged tree $(T, \theta)$ are in the same position if $t(v) = t(w)$. The staged tree $(T, \theta)$ is simple if every pair of vertices in the same stage is also in the same position. A simple and compatibly labeled staged tree $(T, \theta)$ with levels $(L_1, \ldots, L_k) = (X_1, \ldots, X_p)$ is always balanced (see [1, Lemma 2.12]). The staged tree $T_{G_2}$ in Figure 1 is simple and therefore balanced. The Theorem 3.1, states that the staged trees that represent decomposable models are balanced. However, balanced and compatibly labeled staged trees are a larger class. The two staged trees in Figure 2 are balanced, compatibly labeled, and do not represent a DAG model.

In [3], the authors suggested balanced staged tree models as the natural generalization of perfect DAG models based on the fact that balanced staged tree models exhibit similar algebraic properties to perfect DAG models (see Section 4 for more details). The following theorem shows that balanced staged tree models do indeed generalize perfect DAG models.

Theorem 3.1. Let $G = ([p], E)$ be a DAG, and let $\pi$ be a linear extension of $G$. Then the following are equivalent:

1. $\mathcal{M}(G)$ is decomposable,
2. $G$ is a perfect DAG, and
3. the staged tree $T_{G^\pi}$ is balanced.

3.1. Proof of Theorem 3.1. Before we present the proof of the theorem we will prove a technical lemma:
Lemma 3.2. Let $G = ([p], E)$ be a DAG and assume that $\pi = 12 \cdots p$ is a linear extension of $G$. Then $T_G$ is balanced if and only if for every pair of vertices $v, w$ in the same stage with $v = x_i \cdots x_i, w = x_{i'} \cdots x_{i'} \in R[i]$ there exists a bijection

$$\Phi : R[p] \setminus [i+1] \times R[p] \setminus [i+1] \rightarrow R[p] \setminus [i+1] \times R[p] \setminus [i+1],$$

$$(y_{i+2} \cdots y_p, y'_{i+2} \cdots y'_{p}) \mapsto (z_{i+2} \cdots z_p, z'_{i+2} \cdots z'_{p})$$

such that for all $k \geq i + 2$ and all $s \neq r \in [d_i+1]$

$$f(y_k | (x_1 \cdots x_i, s, y_{i+2} \cdots y_p)_{pa_G(k)}) f(y'_{k} | (x'_{i} \cdots x'_{i}, r, y'_{i+2} \cdots y'_{p})_{pa_G(k)}) = f(z_k | (x_1 \cdots x_i, s, z_{i+2} \cdots z_p)_{pa_G(k)}) f(z'_{k} | (x'_{i} \cdots x'_{i}, r, z'_{i+2} \cdots z'_{p})_{pa_G(k)}).$$

(5)

Proof. Suppose $T_G$ is balanced and let $v, w$ be two vertices in the same stage as in the statement of this lemma. Since the pair $v, w$ is balanced,

$$t(x_1 \cdots x_i) t(x'_1 \cdots x'_i) = t(x_1 \cdots x_i) t(x'_1 \cdots x'_i)$$

(6)

for all $s \neq r \in [d_i+1]$. Next we rewrite the two factors on each side of the equality (6) using the definition of $t(\cdot)$ and multiply out the two expressions on each side. For the leftmost factor we have

$$t(x_1 \cdots x_i) = \sum_{x_{i+2} \cdots x_p \in R[p] \setminus [i+1]} \prod_{k = i+2}^p f(x_k | (x_1 \cdots x_i, s, x_{i+2} \cdots x_p)_{pa_G(k)}).$$

Using a similar expression for the other three factors, multiplying out and writing as a double summation, the equation (6) becomes

$$\sum_{x_{i+2} \cdots x_p \in R[p] \setminus [i+1]} \prod_{x_{i+2} \cdots x_p \in R[p] \setminus [i+1]} f(x_k | (x_1 \cdots x_i, s, x_{i+2} \cdots x_p)_{pa_G(k)}) f(x'_k | (x'_{i} \cdots x'_{i}, r, x'_{i+2} \cdots x'_p)_{pa_G(k)})$$

$$= \sum_{x_{i+2} \cdots x_p \in R[p] \setminus [i+1]} \prod_{x_{i+2} \cdots x_p \in R[p] \setminus [i+1]} f(x_k | (x_1 \cdots x_i, s, x_{i+2} \cdots x_p)_{pa_G(k)}) f(x'_k | (x'_{i} \cdots x'_{i}, r, x'_{i+2} \cdots x'_p)_{pa_G(k)}).$$

(7)

Where the product inside both summations in (7) is taken from $k = i + 2$ to $k = p$. The expression (7) is an equality in the polynomial ring where the labels in $L$ are treated as indeterminates. Since each of the terms in the sum is a monomial of
degree \(2(p - i - 1)\), (7) is an equality of homogeneous polynomials. Moreover, each side of (7) has the same number of terms and each term has coefficient equal to one. Hence there exists a bijection between terms in the left-hand-side of (7) and terms in its right-hand-side. We denote this bijection by \(\Phi : \mathcal{R}_{[p]\setminus[i+1]} \times \mathcal{R}_{[p]\setminus[i+1]} \rightarrow \mathcal{R}_{[p]\setminus[i+1]} \times \mathcal{R}_{[p]\setminus[i+1]}\) as in the statement of the lemma. Under this bijection it is true that

\[
\prod_{k=i+2}^{p} f(y_k)(x_1 \cdots x_i, s, y_{i+2} \cdots y_p)_{\pi_G}(k) f(y_k')(x_1' \cdots x_i', r, y_{i+2}' \cdots y_p')_{\pi_G}(k) = \\
\prod_{k=i+2}^{p} f(z_k)(x_1 \cdots x_i, r, z_{i+2} \cdots z_p)_{\pi_G}(k) f(z_k')(x_1' \cdots x_i', s, z_{i+2}' \cdots z_p')_{\pi_G}(k).
\]  

(8)

Since \(\mathcal{T}_G\) is stratified, any two vertices in the same stage must be in the same level. Thus from (8) we obtain the desired equality

\[
f(y_k)(x_1 \cdots x_i, s, y_{i+2} \cdots y_p)_{\pi_G}(k) = f(z_k)(x_1 \cdots x_i, r, z_{i+2} \cdots z_p)_{\pi_G}(k).
\]  

(9)

To check the other direction, it sufficient to note that we can trace backwards the steps in the proof to conclude that the pair \(v, w\) is balanced provided there exists a bijection \(\Phi\) that satisfies (9). \(\square\)

We can then prove Theorem 3.1.

3.1.1. Proof of Theorem 3.1. For all necessary graph theory terminology, we refer the reader to [9]. The equivalence (1) \(\iff\) (2) is [9, Proposition 3.28]. (3) \(\iff\) (2): Let \(G\) be a perfect DAG with linear extension \(\pi = 12 \cdots p\). Then (1, 2, \ldots, \(p\)) is a perfect elimination ordering for the skeleton of \(G\). Suppose that \(v, w \in V\) are in the same stage. Since \(\mathcal{T}_G\) is stratified, \(v\) and \(w\) are in the same level, say level \(i\). Therefore, \(v = x_1 \cdots x_i\) and \(w = x_i' \cdots x^i\) for some outcomes \(x_k, x'_k \in \mathcal{R}(k)\) for \(k \in [i]\). Using the characterization in Lemma 3.2 we must find a bijection

\[
\Phi : \mathcal{R}_{[p]\setminus[i+1]} \times \mathcal{R}_{[p]\setminus[i+1]} \rightarrow \mathcal{R}_{[p]\setminus[i+1]} \times \mathcal{R}_{[p]\setminus[i+1]}, \\
(y_{i+2} \cdots y_p, y_{i+2}' \cdots y_p') \mapsto (z_{i+2} \cdots z_p, z_{i+2}' \cdots z_p')
\]

such that for all \(k \geq i + 2\)

\[
f(y_k)(x_1 \cdots x_i, s, y_{i+2} \cdots y_p)_{\pi_G}(k) = f(z_k)(x_1' \cdots x_i', s, z_{i+2} \cdots z_p)_{\pi_G}(k)
\]  

whenever \(s \neq r \in [d_{i+1}]\). To this end, we define the bijection \(\Phi\) in the following way: Given a pair of outcomes \((y_{i+2} \cdots y_p, y_{i+2}' \cdots y_p') \in \mathcal{R}_{[p]\setminus[i+1]} \times \mathcal{R}_{[p]\setminus[i+1]}\) define the pair \((z_{i+2} \cdots z_p, z_{i+2}' \cdots z_p')\) by the rule: For \(k \geq i + 2\),

- if \(i + 1\) is an ancestor of \(k\) then set \(z_k := y_k\) and \(z_k' := y_k'\), and
- if \(i + 1\) is not an ancestor of \(k\) then set \(z_k := y_k'\) and \(z_k' := y_k\).

Notice that \(\Phi\) is a bijection since it is an involution. To prove that (10) is satisfied with respect to the chosen bijection \(\Phi\), we must check that it holds for all \(k \geq i + 2\) whenever \(s \neq r \in [d_{i+1}]\). In the following, suppose that \(s \neq r \in [d_{i+1}]\), and let \(k \geq i + 2\). It follows that \(i + 1\) is either an ancestor of \(k\) or it is not. We will show that (10) holds in both of these two cases, which will complete the proof.

In the first case, suppose that \(i+1 \in \text{ang}_G(k)\). To show that (10) holds in this case, it suffices to show that \((x_1 \cdots x_i, s, y_{i+2} \cdots y_p)_{\pi_G}(k) = (x_1' \cdots x_i', s, z_{i+2} \cdots z_{i+p})_{\pi_G}(k)\)
and \((x_1 \cdots x_i, s, y_{i+2} \cdots y_p)_{pa_G(k)} = (x'_1 \cdots x'_i, s, z'_{i+2} \cdots z'_p)_{pa_G(k)}\). Breaking these two equalities into equalities of subsequences, it suffices to prove \((x_1 \cdots x_i)_{pa_G(k)} = (x'_1 \cdots x'_i)_{pa_G(k)}\), \((y_{i+2} \cdots y_p)_{pa_G(k)} = (z_{i+2} \cdots z_p)_{pa_G(k)}\) and \((y'_{i+2} \cdots y'_p)_{pa_G(k)} = (z'_{i+2} \cdots z'_p)_{pa_G(k)}\).

To see that \((x_1 \cdots x_i)_{pa_G(k)} = (x'_1 \cdots x'_i)_{pa_G(k)}\) holds, we first note that for \(k = i + 1\), since \(x_1 \cdots x_i\) and \(x'_1 \cdots x'_i\) are in the same stage then
\[
f(x_{i+1} | (x_1 \cdots x_i)_{pa_G(i+1)}) = f(x_{i+1} | (x'_1 \cdots x'_i)_{pa_G(i+1)})
\]
for all \(x_{i+1} \in [d_{i+1}]\), and so \((x_1 \cdots x_i)_{pa_G(k)} = (x'_1 \cdots x'_i)_{pa_G(k)}\) for \(k \geq i + 2\), it suffices to show that \(pa_G(k) \cap [i] \subset pa_G(i+1)\). To this end, suppose that \(j \in pa_G(k)\) and that \(j < i + 1\). Since \(\pi = 12 \cdots p\) is a linear extension of \(\mathcal{G}\), we know that any descendant \(\ell \in [p]\) of \(i + 1\) (including \(k\)) satisfies \(\ell > i + 1\). Therefore, since \(i + 1 \in an_G(k)\), \(j \in pa_G(k)\) and \(j < i + 1\), we know that there exists \(k' \in pa_G(k)\) with \(k' \neq j\) satisfying \(i + 1 \geq k' > k\). If \(k' = i + 1\) we are done, otherwise iterating this argument shows that \(j \in pa_G(i+1)\). Thus, we conclude that \((x_1 \cdots x_i)_{pa_G(k)} = (x'_1 \cdots x'_i)_{pa_G(k)}\).

To see that \((y_{i+2} \cdots y_p)_{pa_G(k)} = (z_{i+2} \cdots z_p)_{pa_G(k)}\), it suffices to show the slightly stronger statement that
\[
(y_{i+2} \cdots y_p)_{an_G(k) \cup [p] \setminus [i+1]} = (z_{i+2} \cdots z_p)_{an_G(k) \cup [p] \setminus [i+1]}.
\]
By construction of the bijection \(\Phi\), we know that
\[
(y_{i+2} \cdots y_p)_{de_G(i+1)} = (z_{i+2} \cdots z_p)_{de_G(i+1)}.
\]
Hence, to prove the desired statement, it suffices to show that \(an_G(k) \cap [p] \setminus [i+1] \subset de_G(i+1)\). To see this, suppose that \(k'' \in an_G(k) \cap [p] \setminus [i+1]\). Let \([i+1, k]\) denote all nodes in \(\mathcal{G}\) that lie on a directed path from \(i + 1\) to \(k\). If \(k'' \in [i+1, k]\), then \(k'' \in de_G(i+1)\). To see this, suppose that \(k'' \not\in [i+1, k]\). Then there exists \(k''' \in de_G(k'')\) such that \(k''' \in [i+1, k]\). (For instance, \(k\) is one such node.) Pick such a \(k'''\) so that, over all such choices, the minimal length directed path from \(k''\) to an element of \([i+1, k]\) is of shortest length. Further pick \(k'''\) such that, over all such choices satisfying the previous condition, \(k'''\) has minimum value in the natural order on \([p]\).

Since \(k''' \in de_G(k'')\) then there exists a directed path in \(\mathcal{G}\)
\[
k'' \rightarrow a_0 \rightarrow a_1 \rightarrow \cdots \rightarrow a_m \rightarrow k'''.
\]
We assume that this path is the shortest possible path from \(k''\) to \([i+1, k]\), based on our previous assumptions. Since \(k''' \in [i+1, k]\), there also exists a directed path in \(\mathcal{G}\)
\[
i + 1 \rightarrow b_0 \rightarrow b_1 \rightarrow \cdots \rightarrow b_t \rightarrow k'''.
\]
Since \(b_t, a_m \in pa_G(k''')\) and \(\mathcal{G}\) is perfect, we know that \(b_t\) and \(a_m\) are adjacent in \(\mathcal{G}\). Since our path from \(k''\) to \([i+1, k]\) was chosen to have shortest possible length, and since \(k'''\) has minimum value over all such paths, we know that \(b_t < a_m\). Thus, \(b_t \rightarrow a_m\) is an edge of \(\mathcal{G}\). Since \(b_t, a_{m-1} \in pa_G(a_m)\), and since \(\mathcal{G}\) is perfect, we know that \(b_t \rightarrow a_{m-1}\) is also an edge of \(\mathcal{G}\). Otherwise, we did not pick the shortest path to \([i+1, k]\). Iterating this argument shows that \(b_t \rightarrow k''\) is an edge of \(\mathcal{G}\). This implies that \(k'' \in [i+1, k]\), which is a contradiction.

Hence, we conclude that \(an_G(k) \cap [p] \setminus [i+1] \subset de_G(i+1)\), and \((y_{i+2} \cdots y_p)_{pa_G(k)} = (z_{i+2} \cdots z_p)_{pa_G(k)}\). The same argument proves \((y'_{i+2} \cdots y'_p)_{pa_G(k)} = (z'_{i+2} \cdots z'_p)_{pa_G(k)}\).
It now only remains to check that (10) holds whenever \( i + 1 \notin \text{ang}(k) \). Since \( i + 1 \notin \text{ang}(k) \), it follows that \( i + 1 \notin \text{pa}_G(k) \). So to prove the desired equality, it suffices to show that \((y_{i+2} \cdots y_p)_{\text{pa}_G(k)} = (z_i' + z_{i+2}' \cdots z_p')_{\text{pa}_G(k)}\) and \((y_{i+2} \cdots y_p')_{\text{pa}_G(k)} = (z_i + z_{i+2} \cdots z_p)_{\text{pa}_G(k)}\). However, since \( i + 1 \notin \text{ang}(k) \), it also follows that no \( j \in \text{pa}_G(k) \) is in \( \text{de}_G(i + 1) \). Hence, by the definition of \( \Phi \), for all \( j \in \text{pa}_G(k) \cap [p] \setminus [i + 1] \) we have that \( z_j = y_{j}' \) and \( z_{j}' = y_j \), which completes the first direction of the proof.

(3) \( \Rightarrow \) (2): We prove the contrapositive, i.e. if \( G \) is not perfect then \( T_G \) is not balanced. Without loss of generality, we assume the nodes of \( G \) have a topological order. That is, if \( u \rightarrow v \) is an arrow in \( G \) then \( u \) is less than \( v \). If \( G \) is not a perfect DAG then \( G \) has a collider \( i \rightarrow l \leftarrow j \). By assumption \( i < l \) and \( j < l \), and we further assume \( i < j \). Since \( i \notin \text{pa}_G(j) \), there exist two outcomes \( x_1 \cdots x_p, x'_1 \cdots x'_p \in R \) such that \( x_i \neq x'_i \) and \( (x_1 \cdots x_p)_{\text{pa}_G(j)} = (x'_1 \cdots x'_p)_{\text{pa}_G(j)} \). The latter equality implies the vertices \( x_1 \cdots x_{j-1} \) and \( x'_1 \cdots x'_{j-1} \) in \( T_G \) are in the same stage. We show that the balanced condition cannot possibly hold for these two vertices.

As in the proof of (3) \( \Leftarrow \) (2), using Lemma 3.2, the balanced condition for the vertices \( x_1 \cdots x_{j-1} \) and \( x'_1 \cdots x'_{j-1} \) holds if and only if there exist a bijection \( \Phi: R_{[p] \setminus [j]} \times R_{[p] \setminus [j]} \rightarrow R_{[p] \setminus [j]} \times R_{[p] \setminus [j]} \) such that for all \( k \geq j + 1 \) and all \( s, r \in [d_j] \)

\[
\begin{align*}
&f(y_k \mid x_1 \cdots x_{j-1}, s, y_{j+1} \cdots y_p)_{\text{pa}_G(k)} f(y_k \mid x'_1 \cdots x'_{j-1}, r, y_{j+1} \cdots y_p)_{\text{pa}_G(k)} \\
&= f(z_k \mid x_1 \cdots x_{j-1}, s, z_{j+1} \cdots z_p)_{\text{pa}_G(k)} f(z_k' \mid x'_1 \cdots x'_{j-1}, r, z_{j+1} \cdots z'_p)_{\text{pa}_G(k)}.
\end{align*}
\]

(11)

Thus to show \( T_G \) is not balanced, we show that (11) cannot hold for \( k = l \). The only two ways to satisfy (11) are if

\[
\begin{align*}
&f(y_l \mid x_1 \cdots x_{j-1}, s, y_{j+1} \cdots y_p)_{\text{pa}_G(k)} = f(z_l \mid x'_1 \cdots x'_{j-1}, s, z_{j+1} \cdots z_p)_{\text{pa}_G(k)},
\end{align*}
\]

with \( y_l = z_l \) or

\[
\begin{align*}
&f(y_l \mid x_1 \cdots x_{j-1}, s, y_{j+1} \cdots y_p)_{\text{pa}_G(k)} = f(z'_l \mid x_1 \cdots x_{j-1}, r, z_{j+2} \cdots z_p)_{\text{pa}_G(k)},
\end{align*}
\]

with \( y_l = z'_l \). The first equation cannot hold for any choice of \( z_{j+1} \cdots z_p \in R_{[p] \setminus [j]} \) because \( i \in \text{pa}_G(l) \) and by construction \( x_i \neq x'_i \) hence \( x_1 \cdots x_{j-1} s y_{j+1} \cdots y_p)_{\text{pa}_G(l)} \neq (x'_1 \cdots x'_{j-1} s z_{j+1} \cdots z_p)_{\text{pa}_G(l)} \). The second equation cannot hold for any choice of \( z'_{j+1} \cdots z'_p \in R_{[p] \setminus [j]} \) because \( j \in \text{pa}_G(l) \) and \( s \neq r \) so \( x_1 \cdots x_{j-1} s y_{j+1} \cdots y_p)_{\text{pa}_G(l)} \neq (x_1 \cdots x_{j-1} r z'_{j+1} \cdots z'_p)_{\text{pa}_G(l)} \). Thus, \( T_G \) is not balanced.

\[
\square
\]

4. Algebraic consequences for toric models

A major endeavour in algebraic statistics is to identify which statistical models can be implicitly defined as the intersection of the space of model parameters with an algebraic variety [11] and, for such models, identify when the defining algebraic variety admits special properties. One commonly investigated question is: when is the defining algebraic variety toric? In this case the model is called a toric model.

A discrete DAG model \( M(G) \) is toric when \( G \) is a perfect DAG [9, Proposition 3.28]. Analogously, a staged tree model \( M(\mathcal{T}, \theta) \) is toric if \( (\mathcal{T}, \theta) \) is a balanced staged tree [3], Theorem 3.1, establishes that the condition for being toric for staged tree models is equivalent to the condition of being toric for DAG models when we restrict to staged trees in \( T_G \). This implies equality between certain ideals associated to a discrete decomposable model. In this section we explain the relation between these ideals and summarize them in Corollary 4.1.
Given a discrete random vector $X_{\{p\}}$ with state space $\mathcal{R}$ and a DAG $\mathcal{G} = ([p], E)$, we define two collections of indeterminates: $D = \{p_x : x \in \mathcal{R}\}$ and $U = \{q_{i,x} : x_i \in \mathcal{R}_{(i)}\} \cup \{x_{pa_q(i)}(i) \in \mathcal{R}_{pa_q(i)}\}$. It follows from equation (4) that the Zariski closure of $\mathcal{M}(\mathcal{G})$ is the algebraic variety defined by the vanishing of the kernel of the map of polynomial rings

$$\Phi_{\mathcal{G}} : \mathbb{R}[D] \to \mathbb{R}[U]/\bar{q}$$

$$\Phi_{\mathcal{G}} : p_x \mapsto \prod_{i \in \{p\}} q_{i,x} : x_{pa_q(i)}$$

where

$$\bar{q} = \{1 - \sum_{x_i \in \mathcal{R}_{(i)}} q_{i,x} : x_{pa_q(i)}(i) \in \mathcal{R}_{pa_q(i)}\}.$$ 

That is, $\mathcal{M}(\mathcal{G}) = \Delta^{|\mathcal{R}|-1} \cap V(\ker(\Phi_{\mathcal{G}}))$, we refer to [6] for more details about the defining ideal of $\mathcal{M}(\mathcal{G})$. Let $\tilde{\mathcal{G}}$ denote the skeleton (i.e. the underlying undirected graph of $\mathcal{G}$), and $C_{\tilde{\mathcal{G}}}$ its set of maximal cliques. We define another the set of indeterminates $W = \{\phi_{x_C} : C \in C_{\tilde{\mathcal{G}}}, x \in \mathcal{R}\}$, and a second map of polynomial rings

$$\Phi_{\tilde{\mathcal{G}}} : \mathbb{R}[D] \to \mathbb{R}[W]; \quad \Phi_{\tilde{\mathcal{G}}} : p_x \mapsto \prod_{C \in C_{\tilde{\mathcal{G}}}} \phi_{x_C}$$

whose kernel is a toric ideal. It follows from [5, Theorem 4.4] that $\ker(\Phi_{\tilde{\mathcal{G}}}) = \ker(\Phi_{\mathcal{G}})$ whenever $\mathcal{G}$ is perfect, and hence $\mathcal{M}(\mathcal{G})$ is toric.

Given a staged tree $(T, \theta)$ with $T = (V, E)$ and labeling $\theta : E \to \mathcal{L}$, we define the polynomial ring $\mathbb{R}[z, \mathcal{L}]$ and the ideal $q' = \{1 - \sum_{v \in E(x)} \theta(v) : v \in V\}$. In [3], the authors showed that the Zariski closure of $\mathcal{M}(T, \theta)$ is defined by the vanishing of the kernel of the map

$$\Psi_{T} : \mathbb{R}[D] \to \mathbb{R}[z, \mathcal{L}]/\bar{q}$$

$$\Psi_{T} : p_x \mapsto z \cdot \prod_{e \in \mathcal{L}(x)} \theta(e).$$

They also considered the kernel of the toric map

$$\Psi_{T}^{\text{toric}} : \mathbb{R}[D] \to \mathbb{R}[z, \mathcal{L}]; \quad \Psi_{T}^{\text{toric}} : p_x \mapsto z \cdot \prod_{e \in \mathcal{L}(x)} \theta(e),$$

and they showed that $T$ is balanced if and only if $\ker(\Phi_{T}) = \ker(\Psi_{T}^{\text{toric}})$. Hence, $\mathcal{M}(T, \theta)$ is toric whenever $T$ is balanced.

While both the result of [5] and [3] show that perfect DAG models and balanced staged tree models, respectively, are toric via a coincidence of ideals, it is not a priori clear that the coincidence of ideals $\ker(\Psi_{T}) = \ker(\Psi_{T}^{\text{toric}})$ implies the coincidence of ideals $\ker(\Phi_{\mathcal{G}}) = \ker(\Phi_{\tilde{\mathcal{G}}})$; i.e., that the identified toric staged tree models generalize the identified toric DAG models. The next corollary establishes this to be the case.

**Corollary 4.1.** Let $\mathcal{G}$ be a DAG and $T_{\mathcal{G}}$ its staged tree representation with respect to some linear extension. The following are equivalent:

1. $\mathcal{G}$ is perfect;
2. $\ker(\Phi_{T_{\mathcal{G}}}) = \ker(\Phi_{\tilde{\mathcal{G}}})$ and $\ker(\Psi_{T_{\mathcal{G}}}) = \ker(\Psi_{\tilde{\mathcal{G}}}^{\text{toric}})$.

If any of (1), (2) or (3) hold, then $\ker(\Phi_{T_{\mathcal{G}}}) = \ker(\Phi_{\mathcal{G}}) = \ker(\Psi_{T_{\mathcal{G}}}) = \ker(\Psi_{\mathcal{G}}^{\text{toric}})$.

**Proof.** (1) $\iff$ (2): This follows from [5, Theorem 4.4]. (1) $\iff$ (3): By an application of Theorem 3.1 and [3, Theorem 3.1], we see that $\mathcal{G}$ is perfect if and only if $\ker(\Phi_{T_{\mathcal{G}}}) = \ker(\Psi_{T_{\mathcal{G}}}^{\text{toric}})$. \qed
5. Future Directions in Statistics

The family of decomposable models plays an important role in probabilistic inference via DAGs. When one wishes to answer a probabilistic query such as, “What is $P(X_i)$?” given data drawn from a joint distribution $P(X_1, \ldots, X_p)$ Markov to a DAG $\mathcal{G}$, the standard approach is to identify a chordal covering for the DAG (i.e., a chordal graph constructed by adding additional edges to the skeleton of $\mathcal{G}$), and then form a clique tree which can be used to dynamically answer the query [8]. The nodes of the clique tree correspond to cliques in the chordal covering of $\mathcal{G}$, and the number of nodes in each of these cliques gives an upper bound on the complexity of answering the probabilistic query. Hence, it is best to identify a chordal covering of the DAG that has smallest possible clique sizes. The complexity bound for probabilistic inference via clique trees for the given DAG, called the treewidth of the DAG, is defined to be one less than the size of a maximal clique in such a minimal chordal covering.

It follows that decomposable models (i.e., chordal graphs) with small treewidth are desirable from the perspective of complexity of probabilistic inference. In this work, we have established the balanced staged trees as both a combinatorial and algebraic generalization of decomposable models. It would therefore be interesting to know if this generalization also generalizes the nice statistical properties of decomposable models in regards to probabilistic inference; that is, do balanced staged trees play the same role as decomposable models when conducting probabilistic inference in context-specific settings via staged trees? Does the notion of treewidth naturally generalize to balanced staged tree models in such a way as to offer analogous complexity bounds for context-specific probabilistic inference? Exploring such questions would be very natural statistical follow-up work to the result of Theorem 3.1.

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