Low energy 2+1 string gravity; black hole solutions

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(Dated: December 19, 2014)

In this report a detailed derivation of the dynamical equations for an $n$–dimensional heterotic string theory of the Horowitz type is carried out in the string frame and in the Einstein frame too. In particular, the dynamical equations of the three dimensional string theory are explicitly given. The relation of the Horowitz–Welch and Horne–Horowitz string black hole solution is exhibited. The Chan–Mann charged dilaton solution is derived and the subclass of string solutions field is explicitly identified. The stationary generalization, via $SL(2, R)$ transformations, of the static (2+1) Horne–Horowitz string black hole solution is given.

PACS numbers: 04.20.Jb, 04.50.+h

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I. \textit{n–DIMENSIONAL HETEROTIC STRING DYNAMICAL EQUATIONS}

Following Horowitz [1], in this contribution we reproduce the field equations “for a part of the low energy action” to a \textit{n–dimensional heterotic string theory described by a metric g_{\mu\nu}, a scalar field }\Phi, a Maxwell field F_{\mu\nu}, and a three–form H_{\mu\nu\lambda}. The three–form H is related to the two–form potential B and a gauge field A_{\mu} through \( H = dB - a A \wedge dF \), where \( a \) is a constant to be adjusted at the end for final results. In this text to denote the number of dimensions is used \( n \) instead of \( D \). Moreover \( \Lambda \) is reserved for the standard cosmological constant, whereas \( \Lambda_{H} \), and \( \Lambda_{CM} = -\Lambda \), denote the \( \Lambda \)'s used by Horowitz [1] and Chan and Mann [5] respectively.

A. String frame

The corresponding heterotic string action for dimension \( n \), \( S = \int d^{n}x \sqrt{g} \), is given by

\[
S = \int d^{n}x \sqrt{g} e^{-2\Phi} \left[ R - 2\Lambda + U(\Phi) + 4(\nabla \Phi)^{2} - F^{2} - \frac{1}{12} H^{2} \right].
\]

Variations with respect to the metric give:

\[
\frac{\delta \sqrt{-g}}{\delta g_{\mu\nu}} = -\frac{1}{2} \sqrt{-g} g_{\mu\nu},
\]

\[
\frac{\delta \sqrt{-g} R}{\delta g_{\mu\nu}} = \frac{\delta \sqrt{-g} g^{\alpha\beta} R_{\alpha\beta}}{\delta g_{\mu\nu}} = \sqrt{-g}(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) + \sqrt{-g} g^{\alpha\beta} \frac{\delta R_{\alpha\beta}}{\delta g_{\mu\nu}} = \sqrt{-g} G_{\mu\nu} + \sqrt{-g} g^{\alpha\beta} \frac{\delta R_{\alpha\beta}}{\delta g_{\mu\nu}},
\]

\[
\frac{\delta}{\delta g_{\mu\nu}} \sqrt{-g} (\nabla \Phi)^{2} = \frac{\delta}{\delta g_{\mu\nu}} \sqrt{-g} g^{\alpha\beta} \nabla_{\alpha} \Phi \nabla_{\beta} \Phi = \sqrt{-g} (\nabla_{\mu} \Phi \nabla_{\nu} \Phi - \frac{1}{2} g_{\mu\nu} (\nabla \Phi)^{2}),
\]

\[
\frac{\delta}{\delta g_{\mu\nu}} \sqrt{-g} F^{2} = \frac{\delta}{\delta g_{\mu\nu}} \sqrt{-g} g^{\alpha\lambda} g^{\beta\rho} F_{\alpha\beta} F_{\lambda\rho} = 2 \sqrt{-g} (g^{\beta\rho} F_{\mu\beta} F_{\nu\rho} - \frac{1}{4} g_{\mu\nu} F^{2}),
\]

\[
\frac{\delta}{\delta g_{\mu\nu}} \sqrt{-g} H^{2} = \frac{\delta}{\delta g_{\mu\nu}} (\sqrt{-g} g^{\alpha\sigma} g^{\beta\rho} g^{\gamma\lambda} H_{\alpha\beta\gamma} H_{\sigma\rho\lambda}) = 3 \sqrt{-g} (H_{\alpha\beta\gamma} H_{\nu}^{\alpha\beta} - \frac{1}{6} g_{\mu\nu} H^{2}).
\]

As far as to the term \( \sqrt{-g} g^{\alpha\beta} \delta R_{\alpha\beta} \) in Eq. (1.2b) is concerned, it is easy to establish its proportionality to a divergence, \( \nu_{\mu\nu} \): in standard textbooks one find the variation of the Riemann tensor in terms of the variation of the Christoffel symbols

\[
\delta R^{\lambda}_{\mu\nu\kappa} = (\delta \Gamma^{\lambda}_{\mu\nu\kappa})_{;\kappa} - (\delta \Gamma^{\lambda}_{\mu\kappa})_{;\nu}
\]
then, since \( R_{\mu \nu} = R^\kappa_{\mu \kappa \nu} \), one has
\[
g^{\mu \nu} \delta R_{\mu \nu} = (g^{\mu \lambda} \delta \Gamma^\nu_{\mu \lambda})_{, \nu} - (g^{\mu \nu} \delta \Gamma^\lambda_{\mu \lambda})_{, \mu} = (g^{\mu \lambda} \delta \Gamma^\nu_{\mu \lambda} - g^{\mu \nu} \delta \Gamma^\lambda_{\mu \lambda})_{, \nu} =: \upsilon^\nu_{, \nu} =
\]
\[
= \frac{1}{\sqrt{-g}} \left[ \sqrt{-g} \left( g^{\mu \lambda} \delta \Gamma^\nu_{\mu \lambda} - g^{\mu \nu} \delta \Gamma^\lambda_{\mu \lambda} \right) \right]_{, \nu}.
\] (1.4)

This term, taking into account the multiplicative factor \( \exp(-2\Phi) \), will give rise in the Einstein’s equation to second order derivatives of \( \Phi \) due first to the covariant derivative acting on \( \delta \Gamma \) when forming a new divergence, and second to the covariant derivative acting on variations of the metric \( \delta g \) for \( \delta \Gamma \) expressed in terms of \( \delta g \) according to 1,
\[
\delta \Gamma^\mu_{\alpha \beta} = \frac{1}{2} g^{\mu \nu} (\nabla_\alpha \delta g_{\nu \beta} + \nabla_\beta \delta g_{\alpha \nu} - \nabla_\nu \delta g_{\alpha \beta}),
\] (1.5)

which implies
\[
\upsilon^\nu = g^{\nu \alpha} g^{\beta \gamma} (\nabla_\alpha \delta g_{\beta \gamma} - \nabla_\gamma \delta g_{\alpha \beta}) = \delta g_{\alpha \beta} (g^{\nu \alpha} g^{\beta \gamma} - g^{\nu \beta} g^{\alpha \gamma})_{, \mu},
\] (1.6)

therefore,
\[
e^{-2\Phi} \sqrt{-g} g^{\mu \nu} \delta R_{\mu \nu} = e^{-2\Phi} \sqrt{-g} \upsilon^\nu_{, \nu}
\]
\[
e^{-2\Phi} \left( \sqrt{-g} \upsilon^\nu \right)_{, \nu} = \left( e^{-2\Phi} \sqrt{-g} \upsilon^\nu \right)_{, \nu} + 2 \Phi \upsilon_{, \nu} e^{-2\Phi} \sqrt{-g} \upsilon^\nu,
\] (1.7)

consequently
\[
2 \Phi_{, \nu} e^{-2\Phi} \sqrt{-g} \upsilon^\nu = 2 \sqrt{-g} \left[ e^{-2\Phi} \Phi_{, \nu} \delta g_{\alpha \beta} (g^{\nu \alpha} g^{\beta \gamma} - g^{\nu \beta} g^{\alpha \gamma})_{, \mu} - 2 \left( e^{-2\Phi} \Phi_{, \nu} \right)_{, \mu} \sqrt{-g} g^{\mu \nu} g^{\alpha \beta} \delta g_{\alpha \beta} + 2 \left( e^{-2\Phi} \Phi_{, \nu} \right)_{, \mu} \sqrt{-g} g^{\mu \nu} g^{\alpha \beta} \delta g_{\alpha \beta} \right]_{, \mu}
\]
\[
= 2 \sqrt{-g} \text{Div} + 2 \sqrt{-g} \left[ \left( e^{-2\Phi} \Phi_{, \nu} \right)_{, \mu} g^{\mu \nu} g^{\alpha \beta} \delta g_{\alpha \beta} \right]_{, \mu} = 2 \sqrt{-g} \text{Div} + 2 \sqrt{-g} \left( e^{-2\Phi} \Phi_{, \nu} \right)_{, \mu} g^{\mu \nu} g^{\alpha \beta} \delta g_{\alpha \beta},
\] (1.8)

where it has been used \( \delta g_{\mu \nu} = -g_{\mu \alpha} g_{\nu \beta} \delta g^{\alpha \beta} \). Gathering all the divergence terms, by applying the Stokes theorem, the integral of the total derivative terms becomes an integral over the boundary; imposing conditions on the variations \( \delta g_{\alpha \nu} \) and on the covariant derivatives of these variations, \( \nabla_\mu \delta g_{\alpha \nu} \), one can avoid the contributions of the boundary terms. On this respect see Wald [9], Appendix E, and also Ortín [10], Part I, Chapter 4. The second term in the last line of Eq. (1.8) contributes to Einstein equations. Therefore \( \frac{\partial \delta \Phi}{\partial g_{\mu \nu}} = 0 \) gives rise to Einstein equations in the form
\[
G_{\mu \nu} + \frac{1}{2} (2\Lambda - U(\Phi)) g_{\mu \nu} + 2 \Phi_{, \nu ; \mu} - 2 g_{\mu \nu} g^{\alpha \beta} \Phi_{, \alpha ; \beta} + 2 g_{\mu \nu} g^{\alpha \beta} \Phi_{, \alpha} \Phi_{, \beta}
\]
\[
- 2 \left( g^{\beta \rho} F_{\beta \rho} F_{\nu \rho} - \frac{1}{4} g_{\mu \nu} F^2 \right) - \frac{1}{4} \left( H_{\mu \alpha \beta} H_{\nu \alpha \beta} g_{\mu \nu} H \right) = 0.
\] (1.9)

---
1 Acting with the variation operation \( \delta \) on the metric tensor and its derivatives in the expression \( \Gamma^\mu_{\alpha \beta} \), taking into account the commutativity of \( \delta \) and partial derivatives \( \partial_i \), and \( g^{\mu \nu} \delta g_{\sigma \nu} = -\delta g^{\mu \nu} g_{\sigma \nu} \), one has
\[
\Gamma^\mu_{\alpha \beta} = \frac{1}{2} g^{\mu \nu} [g_{\nu \beta, \alpha} + g_{\alpha \nu, \beta} - g_{\alpha \beta, \nu}] = \frac{1}{2} g^{\mu \nu} [\alpha \beta, \nu], \quad \delta \Gamma^\mu_{\alpha \beta} = \frac{1}{2} \delta g^{\mu \nu} [\alpha \beta, \nu] + \frac{1}{2} g^{\mu \nu} \delta [\alpha \beta, \nu] = g^{\mu \nu} \delta g_{\sigma \nu} \Gamma^\sigma_{\alpha \beta} + \frac{1}{2} g^{\mu \nu} \delta [\alpha \beta, \nu] = g^{\mu \nu} \delta g_{\sigma \nu} \Gamma^\sigma_{\alpha \beta} [1.5].
Finally, the electromagnetic Maxwell equations are derived for $A_\alpha$ together with $B_{\mu\nu}$, consequently

\[
\frac{\partial}{\partial x^\alpha} \left( e^{-2\Phi} \sqrt{-g} \frac{\partial}{\partial \Phi_\alpha} (g^{\mu\nu} \Phi_\mu \Phi_\nu) \right) = 2 \frac{\partial}{\partial x^\alpha} \left( e^{-2\Phi} \sqrt{-g} g^{\alpha\nu} \Phi_\nu \right) \\
= 2e^{-2\Phi} \left( -2\sqrt{-g} \Phi_\alpha g^{\alpha\nu} \Phi_\nu + \frac{\partial}{\partial x^\alpha} (\sqrt{-g} g^{\alpha\nu} \Phi_\nu) \right) \\
= -4e^{-2\Phi} \sqrt{-g} \Phi_\alpha g^{\alpha\nu} \Phi_\nu + 2e^{-2\Phi} \sqrt{-g} (g^{\alpha\nu} \Phi_\nu)_{\alpha},
\]

yields the dynamical equation

\[
4g^{\alpha\nu} \Phi_{\nu,\alpha} - 4g^{\mu\nu} \Phi_\mu \Phi_\nu + R - 2\Lambda - F^2 - \frac{1}{12} H^2 + U(\Phi) - \frac{1}{2} \frac{dU}{d\Phi} = 0. \quad (1.10)
\]

The field equation for $B_{\mu\nu}$ arises from the variation of $H^2$, where

\[
H_{\mu\nu\lambda} = 3 \left[ B_{[\mu\nu,\lambda]} - a A_{[\mu} F_{\nu]\lambda} \right],
\]

consequently

\[
\frac{\partial}{\partial B_{\mu\nu,\sigma}} H^2 = \frac{\partial H_{\alpha\beta\gamma}}{\partial B_{\mu\nu,\sigma}} \partial \frac{\partial g^{\tau\sigma} g^{\kappa\rho} \xi H_{\tau\kappa\xi} H_{\sigma\rho\lambda}}{\partial \partial B_{\mu\nu,\sigma}} = 2 \frac{\partial H_{\alpha\beta\gamma}}{B_{\mu\nu,\sigma}} H_{\alpha\beta\gamma} = 3! H^{\mu\nu\sigma} \quad (1.11)
\]

therefore, for $\frac{\delta}{\delta B_{\mu\nu}} \mathcal{L} = (\frac{\partial}{\partial B_{\mu\nu}} - \frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial B_{\mu\nu,\sigma}}) \mathcal{L}$, one gets

\[
\frac{\delta}{\delta B_{\mu\nu}} \sqrt{-g} e^{-2\Phi} H^2 = -3! \frac{\partial}{\partial x^\sigma} \left( \sqrt{-g} e^{-2\Phi} H^{\mu\nu\sigma} \right) \rightarrow \nabla_\sigma (e^{-2\Phi} H^{\mu\nu\sigma}) = 0. \quad (1.12)
\]

Finally, the electromagnetic Maxwell equations are derived for $A_\alpha$ related with the electromagnetic field through $F_{\mu\nu} = F_{\mu\nu}(A_\alpha) = A_{\nu,\mu} - A_{\mu,\nu} = 2\partial_{[\mu} A_{\nu]}$,

\[
\frac{\delta}{\delta A_\xi} \mathcal{L} = -\left( \frac{\partial}{\partial A_\xi} - \frac{\partial}{\partial x^\sigma} \frac{\partial}{\partial A_{\xi,\sigma}} \right) [\sqrt{-g} e^{-2\Phi} (F^2 + \frac{1}{12} H^2)] = -\frac{1}{12} \sqrt{-g} e^{-2\Phi} \frac{\partial H_{\alpha\beta\gamma}}{\partial A_\xi} \frac{\partial}{\partial H_{\alpha\beta\gamma}} H^2 \\
+ \frac{\partial}{\partial x^\sigma} (\sqrt{-g} e^{-2\Phi} \delta F_{\alpha\beta} \frac{\partial}{\partial F_{\alpha\beta}} F^2) + \frac{1}{12} \frac{\partial}{\partial x^\sigma} (\sqrt{-g} e^{-2\Phi} \delta H_{\alpha\beta\gamma} \frac{\partial}{\partial H_{\alpha\beta\gamma}} H^2) \quad (1.13)
\]

taking into account that

\[
\frac{\partial}{\partial F_{\alpha\beta}} F^2 = 2F_{\alpha\beta}, \quad \frac{\partial F_{\alpha\beta}}{\partial A_{\xi,\lambda}} = -2\delta_{[\alpha}^{\xi} \delta_{\beta]}^{\lambda}, \quad \frac{\partial}{\partial H_{\alpha\beta\gamma}} H^2 = 2H_{\alpha\beta\gamma} \quad (1.14)
\]

together with

\[
\frac{\partial}{\partial A_\xi} H_{\mu\nu\lambda} = -a \delta_{[\mu}^{\xi} F_{\nu]\lambda} \\
\frac{\partial}{\partial A_{\xi,\sigma}} H_{\alpha\beta\gamma} = -a \frac{\partial}{\partial A_{\xi,\sigma}} A_{[\alpha} F_{\beta]\gamma} = 2a (A_{[\alpha} \delta_{\beta]}^{\xi} \delta_{\gamma]}^{\xi})
\]
one has for $\frac{\delta \Omega}{\delta A_\lambda}$ the expression
\[
\frac{\delta \Omega}{\delta A_\lambda} = -4 \frac{\partial}{\partial x^\lambda} (\sqrt{-g} e^{-2\Phi} F^{\nu\lambda}) + a \frac{1}{6} \sqrt{-g} e^{-2\Phi} F_{\nu\lambda} H^{\nu\lambda}
\]
\[
+ a \frac{1}{3} \frac{\partial}{\partial x^\sigma} \left( A_{[\alpha} \delta^\sigma_{\beta]} \sqrt{-g} e^{-2\Phi} H^{\alpha\beta \gamma} \right)
\]
\[
= 4 \sqrt{-g} (e^{-2\Phi} F^{\nu\lambda})_{,\lambda} + a \frac{1}{3} \sqrt{-g} e^{-2\Phi} F_{\nu\lambda} H^{\nu\lambda} = 0,
\]
where the equation (1.12) has been used.

Summarizing, the field dynamical equations are
\[
G_{\mu\nu} = -\frac{1}{2} g_{\mu\nu} (2\Lambda + U) + 2 \nabla_\mu \nabla_\nu \Phi - 2 g_{\mu\nu} \nabla^2 \Phi + 2 g_{\mu\nu} (\nabla \Phi)^2
\]
\[
- 2 \left( F_{\mu\alpha} F_{\nu\beta} g^{\alpha\beta} - \frac{1}{4} g_{\mu\nu} F^2 \right) - \frac{1}{4} \left( H_{\mu\alpha\beta} H^{\nu \alpha\beta} - \frac{1}{6} g_{\mu\nu} H^2 \right) = 0,
\]
(1.16a)
\[
4 \nabla^2 \Phi - 4 (\nabla \Phi)^2 + R - 2 \Lambda - F^2 - \frac{1}{6} H^2 + U(\Phi) - \frac{1}{2} \frac{dU}{d\Phi} = 0,
\]
(1.16b)
\[
\nabla_\sigma (e^{-2\Phi} H^{\mu\nu \sigma}) = 0,
\]
(1.16c)
\[
\nabla_\lambda (e^{-2\Phi} F^{\nu\lambda}) + a \frac{1}{12} e^{-2\Phi} F_{\nu\lambda} H^{\nu\lambda} = 0.
\]
(1.16d)

The last equation differs in sign from the corresponding Horowitz’s equation (2.10b).

By constructing (1.16a) one evaluates $R$,
\[
R = \frac{2}{n - 2} \left( n\Lambda - \frac{n^2}{2} U - 2(n - 1) \nabla^2 \Phi + 2n(\nabla \Phi)^2 + \frac{n - 4}{2} F^2 + \frac{n - 6}{24} H^2 \right),
\]
(1.17)

which replaced again into (1.16a) and (1.16b) allows one to rewrite the set of dynamical equations as
\[
R_{\mu\nu} = -2 \nabla_\mu \nabla_\nu \Phi + 2 F_{\mu\alpha} F_{\nu\beta} g^{\alpha\beta} + \frac{1}{4} H_{\mu\alpha\beta} H^{\nu \alpha\beta}
\]
\[
+ \frac{2}{n - 2} g_{\mu\nu} \left( \Lambda - \frac{1}{2} U - \nabla^2 \Phi + 2(\nabla \Phi)^2 - \frac{1}{2} F^2 - \frac{1}{12} H^2 \right),
\]
(1.18a)
\[
2 \nabla^2 \Phi - 4 (\nabla \Phi)^2 - 2\Lambda - F^2 + \frac{1}{6} H^2 + U + \frac{n - 2}{4} \frac{dU}{d\Phi} = 0,
\]
(1.18b)
\[
\nabla_\sigma (e^{-2\Phi} H^{\mu\nu \sigma}) = 0,
\]
(1.18c)
\[
\nabla_\lambda (e^{-2\Phi} F^{\nu\lambda}) + a \frac{1}{12} e^{-2\Phi} F_{\nu\lambda} H^{\nu\lambda} = 0.
\]
(1.18d)
B. Einstein frame

On the other hand, to pass to the Einstein frame description of this low energy string theory, one accomplishes a conformal transformation of the form

\[ \tilde{g}_{\mu\nu} = e^{2\sigma} g_{\mu\nu}, \tilde{g}^{\mu\nu} = e^{-2\sigma} g^{\mu\nu} \to \tilde{g} = e^{2n\sigma} g, \text{dim} = n, \]  

(1.19)

which transforms the action \( S \) considered as the barred one, taking into account that

\[ \tilde{F}^2 = \tilde{g}^{\mu\alpha} \tilde{g}^{\nu\beta} F_{\mu\nu} F_{\alpha\beta} = e^{-4\sigma} g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu} F_{\alpha\beta} = e^{-4\sigma} F^2, \]  

and \( \tilde{H}^2 = e^{-6\sigma} H^2 \), to the form

\[ S = \int d^n x \sqrt{-\tilde{g}} e^{(n\sigma - 2\sigma - 2\phi)} \left[ e^{2\sigma} \tilde{R} + e^{2\sigma} (-2\Lambda + U) + 4(\nabla \Phi)^2 - e^{-2\sigma} F^2 - \frac{1}{12} e^{-4\sigma} H^2 \right]. \]  

(1.20)

Thus, one may use the conformal transformed curvature scalar

\[ e^{2\sigma} \tilde{R} = R - 2(n-1) g^{\mu\nu} \nabla_\mu \nabla_\nu \sigma - (n-1)(n-2) g^{\mu\nu} \nabla_\mu \sigma \nabla_\nu \sigma, \]  

(1.21)

and chose

\[ n\sigma - 2\sigma - 2\Phi = 0 \to \sigma = \frac{2}{n-2} \Phi, \Phi = \frac{n-2}{2} \sigma. \]  

(1.22)

Substituting these relations in the action above \( S \), one arrives at

\[ S = \int d^n x \sqrt{-\tilde{g}} \left[ R - \frac{4}{n-2} (\nabla \Phi)^2 - 2e^{4\Phi/(n-2)} \Lambda + V(\Phi) - e^{-4\Phi/(n-2)} F^2 - \frac{1}{12} e^{-8\Phi/(n-2)} H^2 \right], \]  

(1.23)

where it has been dropped from this action the divergence

\[ -2(n-1) \sqrt{-\tilde{g}} g^{\mu\nu} \nabla_\nu \nabla_\mu \sigma = -2(n-1)(\sqrt{-g} \sigma^{\nu})_{;\nu}, \]  

and denoted

\[ e^{2\sigma} U(\Phi) = V(\Phi). \]

The extremum of \( S \) is achieved along the dynamical equations:

\[ G_{\mu\nu} = -g_{\mu\nu} e^{4\Phi/(n-2)} \Lambda + \frac{1}{2} g_{\mu\nu} V(\Phi) + \frac{4}{n-2} \left( \nabla_\mu \Phi \nabla_\nu \Phi - \frac{1}{2} g_{\mu\nu} \nabla_\alpha \Phi \nabla^\alpha \Phi \right) + 2 e^{-4\Phi/(n-2)} (F_{\mu\sigma} F_{\nu}^{\sigma} - \frac{1}{4} g_{\mu\nu} F^2) - \frac{1}{4} e^{-8\Phi/(n-2)} (H_{\mu\alpha\beta} H_{\nu}^{\alpha\beta} - \frac{1}{6} g_{\mu\nu} H^2), \]  

(1.24a)

\[ 8 \nabla_\nu \nabla_\mu \Phi - 8 \Lambda e^{4\Phi/(n-2)} + 4 e^{-4\Phi/(n-2)} F^2 + \frac{2}{3} e^{-8\Phi/(n-2)} H^2 + \frac{n-2}{2} \frac{dV}{d\Phi} = 0, \]  

(1.24b)

\[ \nabla_\sigma \left( e^{-8\Phi/(n-2)} H^{\alpha\beta\sigma} \right) = 0, \]  

(1.24c)
\[
\n\nabla_{\lambda} \left( e^{-4\Phi/(n-2)} F^{\lambda \epsilon} \right) + \frac{6}{12} e^{-8\Phi/(n-2)} F_{\alpha \beta} H^{\alpha \beta \epsilon} = 0,
\]

Replacing in (1.24a) the scalar curvature \( R \),

\[
R = \frac{4}{n-2} (\nabla \Phi)^2 + \frac{n-4}{n-2} e^{-4\Phi/(n-2)} F^2 + \frac{1}{12} \frac{n-6}{n-2} e^{-8\Phi/(n-2)} H^2 - \frac{n}{n-2} \left( -2\Lambda e^{4\Phi/(n-2)} + V \right),
\]

one rewrites (1.24a) as

\[
R_{\mu \nu} = 2\Lambda g_{\mu \nu} e^{4\Phi} + \frac{4}{n-2} \nabla_{\mu} \Phi \nabla_{\nu} \Phi + e^{-4\Phi} \left( 2 F_{\mu \sigma} F_{\nu}^{\sigma} - \frac{1}{n-2} g_{\mu \nu} F^2 \right)
+ \frac{1}{4} e^{-8\Phi} \left( 2 H_{\mu \alpha \beta} H_{\nu}^{\alpha \beta} - \frac{2}{3} \frac{n}{n-2} g_{\mu \nu} H^2 \right) - \frac{1}{n-2} g_{\mu \nu} V(\Phi).
\]

(1.25)

II. DYNAMICAL EQUATIONS IN 2 + 1 STRING GRAVITY

In the three dimensional case, the above Einstein action (1.23) reduces to

\[
S = \int d^3 x \sqrt{-g} \left[ R - 2 e^{4\Phi} \Lambda - 4 (\nabla \Phi)^2 - e^{-4\Phi} F^2 - \frac{1}{12} e^{-8\Phi} H^2 + V(\Phi) \right],
\]

(2.1)

and the Einstein frame dynamical equations (1.24) become

\[
R_{\mu \nu} = 2\Lambda g_{\mu \nu} e^{4\Phi} + 4 \nabla_{\mu} \Phi \nabla_{\nu} \Phi + e^{-4\Phi} \left( 2 F_{\mu \sigma} F_{\nu}^{\sigma} - g_{\mu \nu} F^2 \right)
+ \frac{1}{2} e^{-8\Phi} \left( H_{\mu \alpha \beta} H_{\nu}^{\alpha \beta} - \frac{1}{3} g_{\mu \nu} H^2 \right) - g_{\mu \nu} V(\Phi).
\]

(2.2a)

\[
8 \nabla^2 \Phi - 8 \Lambda e^{4\Phi} + 4 e^{-4\Phi} F^2 + \frac{2}{3} e^{-8\Phi} H^2 + \frac{1}{2} \frac{d V}{d \Phi} = 0,
\]

(2.2b)

\[
\nabla_{\sigma} \left( e^{-8\Phi} H^{\alpha \beta \sigma} \right) = 0,
\]

(2.2c)

\[
\nabla_{\lambda} \left( e^{-4\Phi} F^{\lambda \epsilon} \right) + \frac{6}{12} e^{-8\Phi} F_{\alpha \beta} H^{\alpha \beta \epsilon} = 0.
\]

(2.2d)

One can recover the string dynamical equations by using the conformal inverse relations, see [8],

\[
\sigma,_{ij} = \tilde{\sigma},_{ij} + 2 \sigma,_{i} \sigma,_{j} - \tilde{g}_{ij} (\tilde{\nabla} \sigma)^2, \quad e^{-2\sigma} \sigma,_{i}^{k} = \sigma,_{i}^{k} - (n-2)(\tilde{\nabla} \sigma)^2,
\]

(2.3)
where tilde is used to denote that covariant differentials are constructed with \( \tilde{\Gamma} \)'s or contravariant tensor components are build with \( \tilde{g}^{\mu\nu} \). For \( \sigma = 2\Phi \) and \( n = 3 \) one gets

\[
\begin{align*}
\Phi_{,\mu;\nu} &= \tilde{\Phi}_{,\mu;\nu} + 4\Phi_{,\mu}\Phi_{,\nu} - 2\tilde{g}_{\mu\nu}(\tilde{\nabla}\Phi)^2, \\
\Phi^{\alpha\beta} &= e^{4\Phi}\left(\tilde{\Phi}^{\alpha\beta} - 2(\tilde{\nabla}\Phi)^2\right), \\
R_{\mu\nu} &= \tilde{R}_{\mu\nu} + 2\tilde{\Phi}_{,\mu;\nu} + 2\tilde{g}_{\mu\nu}\tilde{\Phi}_{,\alpha}^{\alpha} + 4\Phi_{,\mu}\Phi_{,\nu} - 4\tilde{g}_{\mu\nu}(\tilde{\nabla}\Phi)^2, \\
R &= e^{4\Phi}\left(\tilde{R} + 8\tilde{\Phi}^{\alpha}_{,\alpha} - 8(\tilde{\nabla}\Phi)^2\right),
\end{align*}
\]

and conformally transforming the above dynamical equations (2.2), using relations (2.4), one gets the barred dynamical equations of the 2 + 1 string theory under consideration

\[
S = \int d^3x \sqrt{-\tilde{g}} e^{-2\Phi} \left[ \tilde{R} - 2\Lambda + 4(\tilde{\nabla}\Phi)^2 - \tilde{F}^2 - \frac{1}{12}\tilde{H}^2 + U(\phi) \right],
\]

\[ (2.5a)
\]

\[
\begin{align*}
\tilde{R}_{\mu\nu} + 2\nabla_\mu \nabla_\nu \Phi - 2F_{\mu\alpha}F_{\nu\beta}\tilde{g}^{\alpha\beta} - \frac{1}{4}H_{\mu\alpha\beta}H_{\nu\gamma\lambda}\tilde{g}^{\gamma\alpha}\tilde{g}^{\lambda\beta} \\
+ \tilde{g}_{\mu\nu}\left(-2\Lambda + U + \tilde{F}^2 + \tilde{H}^2 + 2\tilde{\nabla}^2\Phi - 4(\tilde{\nabla}\Phi)^2\right) &= 0, \quad (2.5b)
\end{align*}
\]

\[
2\tilde{\nabla}^2\Phi - 4(\tilde{\nabla}\Phi)^2 + \tilde{F}^2 - 2\Lambda + \frac{1}{6}\tilde{H}^2 + U(\Phi) + \frac{1}{4}\frac{dU}{d\Phi} = 0, \quad (2.5c)
\]

\[
\tilde{\nabla}_\alpha (e^{-2\Phi}\tilde{H}^{\mu\nu\alpha}) = 0, \quad (2.5d)
\]

\[
\tilde{\nabla}_\lambda \left(e^{-2\Phi}\tilde{F}^{\mu\nu}\right) + a\frac{1}{12}e^{-2\Phi}\tilde{F}_{\nu\lambda}\tilde{H}^{\mu\nu\lambda} = 0. \quad (2.5e)
\]

### III. HORNE-HOROWITZ BLACK STRING

In 1991, Horne and Horowitz published an exact string black hole solution in three dimensions for the full string theory of Sec. I, endowed with mass, axion charge per unit length, the asymptotic value of the dilaton, and a cosmological constant. This string solution is given by

\[
\begin{align*}
\frac{s}{g} &= -(1 - \frac{M}{r})dt^2 + (1 - \frac{Q^2}{Mr})dx^2 + (1 - \frac{M}{r})^{-1}(1 - \frac{Q^2}{Mr})^{-1}\frac{k}{8}\frac{dr^2}{r^2}, \\
H_{rtx} &= \frac{Q}{r^2}, \quad \phi = \ln r + \frac{1}{2}\ln\frac{k}{2}.
\end{align*}
\]

The identification of the functions appearing in the action (1.1) and equations (1.16) with the ones of [2] corresponds to \( \phi = -2\Phi \), and \( \frac{s}{g} = -2\Lambda = \frac{4}{l^2}, k = 2l^2 \), where \( l \) has dimension of length.
Accomplishing in the above-mentioned metric a conformal transformation \( E \tilde{g}_{\mu\nu} = e^{-2\sigma} \tilde{g}_{\mu\nu} = \frac{k r^2}{2} \tilde{g}_{\mu\nu} \), where superscripts \( E \) and \( S \) stand for Einstein and String respectively, one arrives at the corresponding solution in the Einstein frame, namely

\[
E g = -\frac{k r^2}{2} (1 - \frac{M}{r}) dt^2 + \frac{k r^2}{2} (1 - \frac{Q^2}{M r}) dx^2 + (1 - \frac{M}{r})^{-1} (1 - \frac{Q^2}{M r})^{-1} \frac{k^2 dr^2}{16},
\]

\[
H_{rtx} = \frac{Q}{r^2}, \quad \Phi = -\frac{1}{2} \ln r - \frac{1}{4} \ln k,
\]

fulfilling the dynamical equations (1.24) for dimension \( n = 3 \).

By accomplishing a \( SL(2, R) \) transformation of the Killing coordinates

\[
t = \alpha T + \beta \phi, \quad \Delta := \alpha \delta - \beta \gamma.
\]

\[
x = \gamma T + \delta \phi,
\]

one arrives at a stationary HH-string solution

\[
E g = -\frac{k r^2}{2} \left[ (1 - \frac{M}{r}) \alpha^2 - (1 - \frac{Q^2}{M r}) \gamma^2 \right] dt^2 + \frac{k r^2}{2} \left[ (1 - \frac{Q^2}{M r}) \gamma \delta - (1 - \frac{M}{r}) \alpha \beta \right] dT d\phi
\]

\[
+ \frac{k r^2}{2} \left[ (1 - \frac{Q^2}{M r}) \delta^2 - (1 - \frac{M}{r}) \beta^2 \right] d\phi^2 + (1 - \frac{M}{r})^{-1} (1 - \frac{Q^2}{M r})^{-1} \frac{k^2 dr^2}{16},
\]

\[
H_{rtx} = \frac{Q}{r^2}, \quad \Phi = -\frac{1}{2} \ln r - \frac{1}{4} \ln k
\]

(3.2)

In the literature one frequently encounters the \( SL(2, R) \) transformation

\[
t = \frac{T}{\sqrt{1 - \frac{\omega^2}{\tau^2}}} - \omega \frac{\phi}{\sqrt{1 - \frac{\omega^2}{\tau^2}}}, \quad x = -\frac{\omega}{l^2} \frac{T}{\sqrt{1 - \frac{\omega^2}{\tau^2}}} + \frac{\phi}{\sqrt{1 - \frac{\omega^2}{\tau^2}}},
\]

(3.3)

which yields

\[
g = -\frac{(r - M) (r M - Q^2)}{2 l^2 (\omega^2 - r M + \omega^2 M^2 + r M - Q^2)} \frac{dT^2}{l^2} + (1 - \frac{M}{r})^{-1} (1 - \frac{Q^2}{M r})^{-1} \frac{k^2 dr^2}{16}
\]

\[
+ \frac{l^2 k r}{2 M (l^2 - \omega^2)} \left( \frac{d\phi}{l^2} - \frac{\omega}{l^2} \frac{dT}{r} \left( \frac{l^2 M r + l^2 M^2 + r M - Q^2}{l^2 (\omega^2 - r M + \omega^2 M^2 + r M - Q^2)} \right) \right)^2,
\]

\[
H_{rtx} = \frac{Q}{r^2}, \quad \Phi = -\frac{1}{2} \ln r - \frac{1}{4} \ln k.
\]

(3.4)

The evaluation of the quasilocal mass, energy and momentum is done using the Brown–York approach [3], this yields

\[
J(r \to \infty) = 2 \omega l^2 \frac{(M^2 - Q^2)}{(l^2 - \omega^2) M},
\]

(3.7a)

\[
\epsilon(r \to \infty) = -1/2 \frac{1}{r \pi l^2} - 1/4 \frac{(M - \omega Q) (M + \omega Q)}{Ml^2 \pi (\omega - 1) (\omega + 1) r^2} - \epsilon_0,
\]

(3.7b)
\[ E(r \to \infty) = \frac{2\sqrt{1 - \omega^2}}{\sqrt{1 - \omega^2}} - 2\pi K(r)\epsilon_0, \quad (3.7c) \]

\[ M_{BY}(r \to \infty) = 4r + 4 l (-2r^2 + M_0 l^2), \quad (3.7d) \]

\[ \epsilon_0 = -\frac{1}{\pi l} + \frac{1}{2\pi r^2}, \quad (3.7e) \]

\( \omega \) is interpreted as a rotating parameter, the mass function increases as the radial coordinate approach spatial infinity. Moreover various pathologies take place at this location.

### IV. HOROWITZ–WELCH BLACK STRING

In 1993, Horowitz and Welch published an exact string black hole solution in three dimensions \[ [6] \] for the low energy string theory

\[
S = \int d^3 x \sqrt{-g} e^{-2\Phi} \left[ R - 2\Lambda + 4(\nabla \Phi)^2 - \frac{1}{12} H^2 \right], \quad \Lambda = -2/k_{HW}, \quad (4.1)
\]

of Sec. \[ \| \] endowed with mass, angular momentum, axion charge per unit length, and a negative cosmological constant. This string solution is given by a modified BTZ black hole to a 2 + 1 string theory with vanishing scalar \( \Phi \) and electromagnetic \( F_{\alpha\beta} \) fields; in this last case the field equations (2.5) become

\[
R_{\mu\nu} - \frac{1}{4} H_{\mu\alpha\beta} H_{\nu}^{\alpha\beta} = 0, \quad (4.2a)
\]

\[- 2\Lambda + \frac{1}{6} H^2 = 0, \quad (4.2b)\]

\[
\nabla_\sigma (H_{\mu\nu\sigma}) = 0. \quad (4.2c)
\]

The totally anti–symmetric tensor \( H_{\mu\nu\alpha} \) has to be proportional to the volume three form \( \epsilon_{\mu\nu\alpha} \), because of the equation (1.2c), the proportionality factor ought to be a constant, hence one may chose

\[
H_{\mu\nu\sigma} = \alpha \frac{1}{l} \epsilon_{\mu\nu\sigma}. \quad (4.3)
\]

Taking into account the properties of \( \epsilon \)

\[
\epsilon_{\alpha\nu\sigma} \epsilon^{\beta\nu\sigma} = -2\delta^{\beta}_{\alpha}, \quad (4.4)
\]

therefore

\[
H_{\alpha\nu\sigma} H^{\beta\nu\sigma} = -2\frac{\alpha^2}{l^2} \delta^{\beta}_{\alpha}, \quad H^2 = -6\frac{\alpha^2}{l^2}. \quad (4.5)
\]
Consequently, from (4.6) one has

\[ R_{\mu\nu} - \frac{1}{4} H_{\mu\alpha\beta} H_{\nu}{}^{\alpha\beta} = 0 \rightarrow R_{\mu\nu} = -\frac{2}{l^2} g_{\mu\nu}, \alpha = 2 \quad (4.6) \]

\[ -2\Lambda + \frac{1}{6} H^2 = 0 \rightarrow \Lambda = -\frac{\alpha^2}{2l^2} = -\frac{2}{l^2}, k_{HW} = l^2 \quad (4.7) \]

On the other hand, \( H = dB \), thus

\[ H_{\alpha\beta\gamma} = 3B_{[\alpha\beta,\gamma]} = 2l\epsilon_{\alpha\beta\gamma}. \quad (4.8) \]

As concluded in [6]: “Thus every solution to three dimensional general relativity with negative cosmological constant is a solution to low energy string theory with: \( \Phi = 0, H_{\alpha\beta\gamma} = \frac{2}{l}\epsilon_{\alpha\beta\gamma}, \) and \( \Lambda = -\frac{2}{l^2} \).

In particular in [6] is established that the BTZ black hole metric

\[ g = -(\frac{r^2}{l^2} - M) dt^2 - J dt d\phi + r^2 d\phi^2 + \left(\frac{r^2}{l^2} - M + \frac{J^2}{4l^2}\right)^{-1} dr^2, \quad (4.9) \]

in the presence of an anti–symmetric \( B \) field

\[ B_{\phi t} = \frac{r^2}{l}, H = dB, \quad (4.10) \]

is a solution of the string theory with a zero scalar field \( \Phi \).

By a target space duality transformation (13) of [6], which referred to [7], which means that from a given solution \( (g_{\mu\nu}, B_{\mu\nu}, \Phi) \) independent on one variable, say \( x \), one generates a new solution \( (\tilde{g}_{\mu\nu}, \tilde{B}_{\mu\nu}, \tilde{\Phi}) \) with

\[ \tilde{g}_{xx} = \frac{1}{g_{xx}}, \quad \tilde{g}_{x\alpha} = \frac{B_{x\alpha}}{g_{xx}}, \]
\[ \tilde{g}_{\alpha\beta} = g_{\alpha\beta} - \frac{1}{g_{xx}} (g_{x\alpha} g_{x\beta} - B_{x\alpha} B_{x\beta}), \]
\[ \tilde{B}_{x\alpha} = \frac{g_{x\alpha}}{g_{xx}}, \quad \tilde{B}_{\alpha\beta} = B_{\alpha\beta} - 2 \frac{g_{[\alpha\beta] x}}{g_{xx}}, \]
\[ \tilde{\Phi} = \Phi - \frac{1}{2} \ln g_{xx}, \quad (4.11) \]

where \( \alpha \) and \( \beta \) run over all directions except \( x \). Applying this transformation to expressions (4.9) and (4.10), along the coordinate symmetry \( \phi \), one gets (14) of [6], namely

\[ s g = (M - \frac{J^2}{4r^2}) dt^2 + \frac{2}{l} dt d\phi + \frac{1}{r^2} d\phi^2 + \left(\frac{r^2}{l^2} - M + \frac{J^2}{4l^2}\right)^{-1} dr^2, \quad (4.12) \]

which, once diagonalized by means of a \( SL(2, R) \) coordinate transformation

\[ t = -\frac{l}{\sqrt{r_+^2 - r_-^2}} \tilde{t} + \frac{l}{\sqrt{r_+^2 - r_-^2}} \tilde{x}, \]
\[ \phi = \frac{r_+}{\sqrt{r_+^2 - r_-^2}} \tilde{t} - \frac{r_-}{\sqrt{r_+^2 - r_-^2}} \tilde{x}, \quad (4.13) \]
and the $r$–coordinate transformation

$$r^2 = l\tilde{r}$$

(4.14)

yields the string solution derived in [2], see also Sec. [IV] Dropping primes, it becomes

$$S_g = -(1 - \frac{M}{r})dt^2 + (1 - \frac{Q^2}{M r})dx^2 + (1 - \frac{M}{r})^{-1}(1 - \frac{Q^2}{M r})^{-1}\frac{l^2}{4r^2} dr^2,$$

$$B_{xt} = \frac{Q}{r}, \; \phi = -\frac{1}{2} \ln (r l),$$

(4.15)

where $M = r^2/l$ and $Q = J/2$.

The identification of the functions appearing in the action (1.1) and equations (1.16) with the ones of [2],

$$S = \int d^3x \sqrt{-g} c^0 \left[ R - 2\Lambda + (\nabla \Phi)^2 - \frac{1}{12} H^2 \right], \; \Lambda = -4/k_{HH},$$

(4.16)

requires that $\phi = -2\Phi_{HW}$, and $k_{HH} = 2 k_{HW}$, $k_{HW} = l^2$, $\Lambda = -2/l^2$, where the subscripts are in correspondence with the initial of the author’s family name.

V. CHAN–MANN STRING SOLUTION

Chan and Mann [4], see also [5], derived a class of solutions to dilaton minimally coupled to 2 + 1 Einstein–Maxwell gravity. There is a subclass of solutions allowing an interpretation from the viewpoint of the low energy 2 + 1 string theory for specific values of the charged dilaton solution. First, we derive the dilaton solution. Next, assigning specific values to constant characterizing the charged dilaton, the correspondence with the string theory developed in the previous section is established.

A. Einstein–Maxwell–scalar field equations

The action considered in [4], CM (1), for a (2+1)-dimensional gravity is given by

$$S = \int d^3x \sqrt{-g} \left[ R - \frac{B}{2} \nabla_\mu \Psi \nabla^\mu \Psi + 2 e^{b\Psi} \Lambda_{CM} - e^{-4a\Psi} F^2 \right],$$

(5.1)

where $\Lambda_{CM}, b$ are arbitrary at this stage parameters, $\Psi$ is the massless minimally coupled scalar field, $R$ is the scalar curvature, and $F^2 = F_{\mu \nu} F^{\mu \nu}$ the electromagnetic invariant. The variations of this action yield the dynamical equations

$$R_{\mu \nu} = \frac{B}{2} \nabla_\mu \Psi \nabla_\nu \Psi - 2g_{\mu \nu} e^{b\Psi} \Lambda_{CM} + 2 e^{-4a\Psi} \left( F_{\mu}^\alpha F_{\nu \alpha} - g_{\mu \nu} F^2 \right),$$

$$\frac{B}{2} \nabla^\mu \nabla_\mu \Psi + b e^{b\Psi} \Lambda_{CM} + 2 a e^{-4a\Psi} F^2 = 0,$$

$$\nabla^\mu \left( e^{-4a\Psi} F_{\mu \nu} \right) = 0.$$  

(5.2)
B. General static cyclic symmetric black hole solution coupled to a scalar field

\( \Psi(r) = k \ln(r) \)

The static cyclic symmetric metric in the 2 + 1 Schwarzschild coordinate frame is given by

\[
g = -N(r)^2 dt^2 + \frac{dr^2}{L(r)^2} + r^2 d\phi^2. \tag{5.3}
\]

The electromagnetic field equations for the tensor field \( F_{\mu\nu} = 2 F_{tr} \delta^t_{[\mu} \delta^r_{\nu]} \), and the dilaton \( \Phi(r) = k \ln(r) \) becomes

\[
EQ_F = \frac{d}{dr} F_{tr} L r^{-4ak+1} N \rightarrow F_{tr} = Q N L^{4ak-1}. \tag{5.4}
\]

The simplest Einstein equations occurs to be \( R_{tt} + R_{rr} L^2 N^2 \), which yields

\[
\frac{1}{N} \frac{d}{dr} N - \frac{1}{L} \frac{d}{dr} L - \frac{1}{2} \frac{B k^2}{r} = 0, \tag{5.5}
\]

thus one gets

\[
N(r) = C_N L(r) r^{Bk/2}. \tag{5.6}
\]

On the other hand, the equation \( R_{\phi\phi} \) gives a first order equation for \( Y(r) = L^2 \), namely

\[
\frac{d}{dr} Y(r) + \frac{1}{2} \frac{B k^2 Y(r)}{r} + 2 \frac{r^{4ak} Q^2}{r} - 2 \Lambda_{CM} r^{bk+1} = 0 \tag{5.7}
\]

integrating one obtains

\[
L(r)^2 = Y(r) = -4 \frac{r^{4ak} Q^2}{Bk^2 + 8ak} + 4 \frac{\Lambda_{CM} r^{2+bk}}{4 + Bk^2 + 2bk} + r^{-1/2 Bk^2} C_I \tag{5.8}
\]

Substituting this expression of \( Y(r) \) into the remaining scalar field equation

\[
\frac{d}{dr} Y(r) + \frac{1}{2} \frac{B k^2 Y(r)}{r} - 8 \frac{r^{4ak} a Q^2}{Bk^2} + 2 \frac{b \Lambda_{CM} r^{bk+1}}{Bk} = 0, \tag{5.9}
\]

one arrives at relationships between constants, namely

\[
a = -\frac{1}{4Bk}, b = -Bk. \tag{5.10}
\]

Therefore, the general charged dilaton static solution can be given as

\[
g = -C_N^2 r^{Bk^2} L(r)^2 dt^2 + \frac{dr^2}{L(r)^2} + r^2 d\phi^2,
\]

\[
L(r)^2 = \left( r^{Bk^2/2} C_I + 4 \frac{\Lambda_{CM} r^{2+bk}}{4 - Bk^2} + 4 \frac{Q^2}{Bk^2} \right) r^{-Bk^2}, \quad k^2 \neq \frac{4}{B},
\]

\[
F_{\mu\nu} = 2 F_{tr} \delta^t_{[\mu} \delta^r_{\nu]}, \quad F_{tr} = Q C_N r^{-1/2 Bk^2 - 1} = -A_{t,r} \rightarrow A_t = 2 \frac{Q C_N}{Bk^2} r^{-Bk^2/2},
\]

\[
\Psi(r) = k \ln(r), \tag{5.11}
\]
endowed with four relevant parameters: in particular, one may identify the mass $M = -C_1$, cosmological constant $\Lambda_{CM} \to \pm \frac{1}{l^2}$, dilaton parameter $k$, and the charge $Q$. The constant $C_N$ can be absorbed by scaling the coordinate $t$, thus it can be equated to unit, $C_N \to 1$. Moreover, one has to set the charge $Q$ to zero, $Q = 0$, when looking for the limiting solutions for vanishing dilaton $k = 0$, which are just the dS and AdS solutions with parameters $C_1 = \pm M$ respectively, and $C_N = 1$. There is no static electrically charged limit of this solution for vanishing dilaton field.

The constant $\Lambda_{CM}$ can be equated to minus the standard cosmological constant $\Lambda_s = \pm \frac{1}{l^2}$; indeed, by setting in (5.11)

$$\Lambda_{CM} = \pm \frac{1}{l^2} \alpha^2, \quad r \to r \alpha^{2/(B k^2)}, \quad \phi \to \phi \alpha^{-2/(B k^2)}, \quad Q \to Q \alpha^{(1+2/(B k^2))},$$

$$C_1 \to C_1 \alpha^{1+4/(B k^2)}, \quad C_N \to C_N \alpha^{-(1+2/(B k^2))},$$

(5.12)
one arrives at the metric (5.11) with $\Lambda_{CM} = \pm \frac{1}{l^2}$. Notice that the $\Lambda_{CM}$ used by in Chan–Mann works, when considered as a cosmological constant, differs from the standard cosmological constant $\Lambda_s = \pm \frac{1}{l^2} = -\Lambda_{CM}$, where $+$ and $-$ stand correspondingly for de Sitter and Anti de Sitter (AdS) in $\Lambda_s$.

The string solution derived in [4], for $B = 8, k = -1/2, a = 1, b = 4$, which fulfills the Einstein string equations (2.2) for $\Lambda = \Lambda_s = \pm \frac{1}{l^2} = -\Lambda_{CM}$, in the case of vanishing $H$, can be given as

$$g_E = -r^2 L(r)^2 dt^2 + \frac{dr^2}{L(r)^2} + r^2 d\phi^2,$$

$$L(r)^2 = (r C_1 - 2 r^2 \Lambda_s + 2 Q^2) r^{-2},$$

$$F_{t\mu} = 2 F_{tr} \delta_{\mu}^t \delta_{\nu}^r, \quad F_{tr} = Q r^{2} = -A_{t,r} \to A_t = \frac{Q}{r}, \quad \Psi(r) = -1/2 \ln(r).$$

(5.13)

Under the conformal transformation

$$\tilde{g}_{\mu\nu} = e^{4 \Psi(r)} g_{\mu\nu} = \frac{1}{r^2} g_{\mu\nu},$$

(5.14)

it becomes

$$g_s = -(r C_1 - 2 r^2 \Lambda_s + 2 Q^2) r^{-2} dt^2 + \frac{dr^2}{(r C_1 - 2 r^2 \Lambda_s + 2 Q^2)} + d\phi^2,$$

$$F_{t\mu} = 2 F_{tr} \delta_{\mu}^t \delta_{\nu}^r, \quad F_{tr} = \frac{Q}{r^2} = -A_{t,r} \to A_t = \frac{Q}{r}, \quad \Psi(r) = -1/2 \ln(r),$$

(5.15)

dis is a solution of the equations (2.5) of the 2 + 1 string theory.

Moreover, subjecting the metric (5.15) to $SL(2, R)$ transformations of the Killing coordinates

$$t = \alpha \tau + \beta \theta, \quad \phi = \gamma \tau + \delta \theta$$

(5.16)
one arrives at a rotating charged string solution, namely

\[ g_S = -\left(\alpha^2 \mathcal{L}^2 \, r^2 - \gamma^2\right) d\tau^2 - 2 \left(\alpha \beta \mathcal{L}^2 \, r^2 - \gamma \delta\right) d\theta \, d\tau + \left(\delta^2 - \beta^2 \mathcal{L}^2 \, r^2\right) d\theta^2 + \frac{dr^2}{\mathcal{L}^2}, \]

\[ F_{\mu \nu} = 2\alpha F_{tr} \delta_{[\mu} \delta_{\nu]} - 2\beta F_{tr} \delta_{[\mu} \delta_{\nu]} \quad \text{with} \quad F_{tr} := \frac{Q}{r^2} = -A_{t,r} \rightarrow A_t = \frac{Q}{r}, \]

\[ \Psi(r) = -1/2 \ln(r), \quad \mathcal{L}^2 := r C_1 - 2r^2 \Lambda_s + 2Q^2. \]  

(5.17)

In particular when \( \Lambda_{CM} = 1/l^2 \), AdS branch, a usual choice of the \( SL(2, R) \) transformations is given by

\[ t = \frac{\tau}{\sqrt{1 - \omega^2/l^2}} - \omega \frac{\theta}{\sqrt{1 - \omega^2/l^2}}, \quad \phi = -\omega \frac{\tau}{\sqrt{1 - \omega^2/l^2}} + \frac{\theta}{\sqrt{1 - \omega^2/l^2}}, \]  

(5.18)

where \( \omega \) stands for the rotation parameter. This metric can be written as

\[ g_S = -\mathcal{L}^2 \frac{1 - \omega^2/l^2}{1 - \omega^2/\mathcal{L}^2} d\tau^2 + \frac{1 - \omega^2\mathcal{L}^2/r^2}{1 - \omega^2/l^2} \left( d\theta - \frac{\omega}{l^2} \frac{1 - \omega^2/l^2}{1 - \omega^2/\mathcal{L}^2} d\tau \right)^2 + \frac{dr^2}{\mathcal{L}^2}, \]  

(5.19)

and is endowed with four parameters: the mass \( M = -C_1 \), charge \( Q \), rotation \( \omega \), and cosmological constant \( \Lambda_s = \pm 1/l^2 \).

It is worthy to point out that string solutions one can find in various dimensions, for instance, the works [11] [12], and [13], among others.

VI. ACKNOWLEDGMENTS

This work has been partially carried out during a sabbatical year at the Physics Department–UAMI, and has been supported by Grant CONACyT 178346.

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