Special points on products of modular curves.

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In memory of my mother, Lida Edixhoven-van Elzakker (1930–2003).

Abstract

We prove the André–Oort conjecture on special points of Shimura varieties for arbitrary products of modular curves, assuming the Generalized Riemann Hypothesis. More explicitly, this means the following. Let $n \geq 0$, and let $\Sigma$ be a subset of $\mathbb{C}^n$ consisting of points all of whose coordinates are $j$-invariants of elliptic curves with complex multiplications. Then we prove (under GRH) that the irreducible components of the Zariski closure of $\Sigma$ are special sub-varieties, i.e., determined by isogeny conditions on coordinates and pairs of coordinates. A weaker variant (Thm. 1.3) is proved unconditionally.

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1 Introduction.

The main goal of this article is to prove the André-Oort conjecture for arbitrary products of modular curves, assuming the generalized Riemann hypothesis (GRH) for imaginary quadratic fields. This conjecture is usually formulated for arbitrary Shimura varieties; see [8] for a precise statement in the general case, and the end of this introduction for a list of results that have been proved so far. The conjecture in question says that the irreducible components of the Zariski closure of any set of special points in a Shimura variety are sub-varieties of Hodge type. In order to be reasonably elementary in this article, we do not use the general formalism of Shimura varieties and their sub-varieties of Hodge type but rather state our results in more explicit terms. In fact, we will use the same terminology as in [7], which deals with the case of products of two modular curves. (In Section 2 we do use some Shimura variety formalism, but the result in that section is only included to show that our explicit result is in fact equivalent to the André-Oort conjecture.)

Let $\mathbb{H}$ denote the complex upper half plane, with its $\text{SL}_2(\mathbb{R})$-action given by fractional linear transformations. For $\Gamma$ a congruence subgroup of $\text{SL}_2(\mathbb{Z})$, we denote by $X_\Gamma$ the complex

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modular curve $\Gamma \backslash \mathbb{H}$, or, more precisely, the complex algebraic curve associated to this complex analytic variety, and we let $\pi_\Gamma$ be the quotient map from $\mathbb{H}$ to $X_\Gamma$. We view $X_\Gamma$ as the set of isomorphism classes of complex elliptic curves with a level structure of type $\Gamma$. The endomorphism ring $\text{End}(E)$ of a complex elliptic curve $E$ is either $\mathbb{Z}$ or an order in an imaginary quadratic extension of $\mathbb{Q}$; in the second case $E$ is said to be a CM elliptic curve (CM meaning complex multiplication). A point on some $X_\Gamma$ is called a CM point if the corresponding elliptic curve has CM. A point on a product of curves of the form $X_\Gamma$ is called a CM point if all its coordinates are CM points.

1.1 Definition. Let $S$ be a finite set. For every $s$ in $S$, let $\Gamma_s$ be a congruence subgroup of $\text{SL}_2(\mathbb{Z})$, and let $X$ be the product of the $X_{\Gamma_s}$. A closed irreducible algebraic sub-variety $Z$ of $X$ is called special if $S$ has a partition $(S_1, \ldots, S_r)$ such that $X$ is the product of sub-varieties $Z_i$ of the $X_i := \prod_{s \in S_i} X_{\Gamma_s}$, each of one of the forms:

1. $S_i$ is a one element set, and $Z_i$ a CM point;

2. the image of $\mathbb{H}$ in $X_i$ under the map sending $\tau$ in $\mathbb{H}$ to $\pi_{\Gamma_s}(g_s \tau)$ for every $s$ in $S_i$, with the $g_s$ elements of $\text{GL}_2(\mathbb{Q})$ with positive determinant.

In Section 2 we will show that our ad hoc notion of “special sub-variety of $X$” is the same as that of “sub-variety of Hodge type of $X$”. We note that a point in $X$ as above is special if and only if it is a CM point. We say that two points $x$ and $x'$ in $X$ are isogeneous if the corresponding products of elliptic curves are isogeneous. (We could have asked the isogenies to preserve the product structure, but for Theorems 1.2 and 1.3 below that would not change anything; just note that the category of elliptic curves up to isogeny is semi-simple, and that the symmetric group $S_n$ is finite.) The main results of this article are the following two (the second is motivated by possible applications in transcendence theory, see [5], and by work of Vatsal and Cornut, see [6] and also [9]).

1.2 Theorem. Let $\Sigma$ be a set of special points in a finite product of modular curves. Assume GRH for imaginary quadratic fields. Then all irreducible components of the Zariski closure of $\Sigma$ are special.

1.3 Theorem. Let $\Sigma$ be a set of special points in a finite product of modular curves, lying in one isogeny class. Then all irreducible components of the Zariski closure of $\Sigma$ are special.

1.4 Question. It seems very probable that the conclusion of Theorem 1.3 remains true if one replaces the hypothesis that the elements of $\Sigma$ be special by the existence of just one special point on each irreducible component of the Zariski closure of $\Sigma$. The idea is that the proof we give actually becomes easier if the Galois orbits in $\Sigma$ are bigger, and that is just what happens if instead of special points we take non-special points.

We end this introduction with some words on the history of our proof, on how it relates to the proof of more general cases, and on perspectives of future research. In [1] Andr´e has proved Theorem 1.2 unconditionally in the case of a product of two modular curves. It may be possible.
to use his method to give an unconditional proof of Theorem 1.2. We have not tried to do so, as our main goal was to test our approach to the André–Oort conjecture for higher dimensional sub-varieties in at least one situation. The proofs of Theorems 1.2 and 1.3 were obtained in March 1999, but writing it all up has been delayed for some time. One reason for that was that more important cases of the André–Oort conjecture have been dealt with first: Hilbert modular surfaces in [8] and a result on curves in arbitrary Shimura varieties in [10]. The importance of the last result is its application to transcendence theory. Yafaev has extended [10] to the case of arbitrary curves in Shimura varieties, assuming GRH (see [22]), and he has generalised a result of Moonen to arbitrary Shimura varieties in [23]. In the mean time, Breuer has succeeded in adapting the arguments of this article to the case of Drinfel’d modular curves in positive characteristic, see [2] and [3].

We hope that the more or less explicit methods of this article can be generalized and combined with the more abstract ones of [10] in order to treat the general case of the André–Oort conjecture, i.e., higher dimensional cases in general Shimura varieties. An interesting problem that suggests itself is to generalize Proposition 4.2, i.e., to get an effective criterion for irreducibility for images under Hecke correspondences. Is there an effective version of the theorem by Nori that is used in [10]? Another important problem is to find good enough lower bounds for Galois orbits. This is the main subject of [22]. Can one make use of reduction modulo \( p \), as in [16]? For relations between the André–Oort conjecture and equidistribution properties we refer to [20]. Particularly interesting is the main result of [4] concerning the equidistribution of “strongly special” sub-varieties: in our case the special curves in \( \mathbb{C}^n \) that project surjectively to all coordinates are “strongly special”.

## 2 Determination of the sub-varieties of Hodge type.

The definition of the notion “sub-variety of Hodge type of a Shimura variety” that we use is that of Moonen, see [8, Def. 1.1], or [16, 6.2] and [17, Prop. 2.8].

### 2.1 Proposition. The sub-varieties of Hodge type of a finite product of modular curves are precisely the special sub-varieties as defined in Definition 1.1.

**Proof.** Let \( X \) be a product of modular curves (indexed by some finite set \( S \)) as in Definition 1.1. Let \( G \) denote the algebraic group \( \text{PGL}_{2,\mathbb{Q}} \) over \( \mathbb{Q} \), and let \( \mathbb{H}^\pm \) denote the double half plane \( \mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R}) \). Then \( X \), with its modular interpretation, is a component of the Shimura datum associated to the Shimura datum \( (G^S, (\mathbb{H}^\pm)^S) \), together with a suitable compact open subgroup \( K \) of \( G^S(\mathbb{A}^f) \).

By definition, the sub-varieties of Hodge type of \( X \) are given by triplets \((H, Y, g)\), with \((H, Y)\) a sub Shimura datum of \((G^S, (\mathbb{H}^\pm)^S)\), and \( g \) an element of \( G^S(\mathbb{A}^f) \). Here \( H \) is a reductive subgroup of \( G^S \), and \( Y \) is an \( H(\mathbb{R}) \)-orbit in \((\mathbb{H}^\pm)^S \), consisting of \( h: S \to G^S_\mathbb{R} \) that factor through \( H_\mathbb{R} \) (here \( S \) is the real algebraic group \( \mathbb{C}^* \), and \( \mathbb{H}^\pm \) is to be viewed as the \( G(\mathbb{R}) \)-conjugacy class of the morphism \( a + bi \to (\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}) \) from \( S \) to \( G \)). To be precise, the sub-varieties of Hodge type associated to such a triplet \((H, Y, g)\) are the irreducible components of the image of \( Y \).
under the map $Y \to X$, $y \mapsto \pi(y, g)$, where $\pi$ is the quotient map from $(\mathbb{H}^\pm)^S \times G^S(\mathbb{A}_f)$ to $X \subset G^S(\mathbb{Q}) \setminus ((\mathbb{H}^\pm)^S \times G^S(\mathbb{A}_f)/K)$.

Let $Z$ be a special sub-variety of $X$. We want to show that it is of Hodge type. Since products of sub-varieties of Hodge type are again of Hodge type, we may assume that $Z$ is of one of the two forms as in Definition 1.1. If $Z$ is a CM point $x$, one can take $H$ to be the torus with $\mathbb{Q}$-points $K_x^0/\mathbb{Q}^*$, where $K_x$ is the endomorphism algebra of an elliptic curve corresponding to $x$ (the choice of an element $h : S \to G_\mathbb{R}$ lying over $x$ gives $H$ as the smallest $\mathbb{Q}$-subgroup of $G$ through which $h$ factors). In the second case, one can take $H$ to be $G$, embedded in $G^S$ by the morphism that sends $g$ to the $g_x g g_x^{-1}$.

Suppose now that $Z$ is a sub-variety of Hodge type of $X$. We want to show that $Z$ is special. Let $(H, Y, g)$ be a triplet as above that gives rise to $Z$. Since we are only interested in connected components, we may and do assume $H$ to be connected (replace $H$ by its connected component $H^0$, note that the elements of $Y$ factor through $H^0_\mathbb{R}$, and replace $Y$ by one of the finitely many $H^0(\mathbb{R})$-orbits of $Y$). The connected reductive algebraic subgroups of $G$ are $G$ itself, the trivial subgroup, and the one-dimensional tori. Hence the image of $H$ under any of the projections $p_s$ from $G^S$ to $G$ is of one of these three kinds. We note that the trivial subgroup cannot occur, because for any $h$ in $(\mathbb{H}^\pm)^S$, the morphism $p_s h$ from $S$ to $G_\mathbb{R}$ is non-trivial. If $p_s H$ is all of $G$, then $p_s Y = \mathbb{H}^\pm$, and if $p_s H$ is a one-dimensional torus, then $p_s Y$ is a point, because it is an orbit for the action of $H(\mathbb{R})$ under conjugation; such a point is necessarily a CM point (it is a sub-variety of Hodge type of dimension zero). Hence $S$ is the disjoint union of $S'$ and $S''$, with $S'$ the set of $s$ with $p_s H = G$. We have $Z = Z' \times Z''$, with $Z''$ a CM point, and with $Z'$ a sub-variety of Hodge type in the product of the $X_{\Gamma_s}$ for $s$ in $S'$, because images of a sub-variety of Hodge type under a morphism of Shimura varieties induced by a morphism of Shimura data are again of Hodge type. Hence we have reduced the problem of showing that $Z$ is of Hodge type to the case where $p_s H = G$ for all $s$.

Suppose that we have $p_{s,t} H \neq G^2$ for some pair $(s, t)$ with $s \neq t$. Then Goursat’s lemma (which says that the subgroups of a product $A \times B$ are the inverse images of graphs of isomorphisms from sub-quotients of $A$ to sub-quotients of $B$) implies that $p_{s,t} H$ is $G$, embedded by the map $x \mapsto (g_s x g_s^{-1}, g_t x g_t^{-1})$, for some $g_s$ and $g_t$ in $G(\mathbb{Q})$. It follows that $p_{s,t} Z$ is the one dimensional sub-variety of Hodge type of $X_{\Gamma_s} \times X_{\Gamma_t}$, associated to the Shimura datum $(G, \mathbb{H}^\pm)$ with embedding into $(\mathbb{H}^\pm)^2$ via $\tau \mapsto (g_s \tau, g_t \tau)$. Hence $p_{s,t} Z$ is itself a modular curve of the form $X_{\Gamma_u}$, embedded as a Hecke correspondence in $X_{\Gamma_s} \times X_{\Gamma_t}$. So we can view $Z$ as a sub-variety of Hodge type in the product of this $X_{\Gamma}$ and the $X_{\Gamma_u}$ with $u$ not in $\{s, t\}$. But then induction on the number of elements of $S$ finishes the proof.

Finally suppose that $p_{s,t} H = G^2$ for all $s \neq t$. Then we have $H = G^S$ (induction on the cardinality of $S$, Goursat’s lemma and the fact that the normal subgroups of $G^S$ are the $G^T$ with $T$ a subset of $S$), and $Z = X$, hence $Z$ is special.

3 Some general principles.

In this section we list and prove some results that we use in the proofs of Theorems 1.2 and 1.3.

We begin by giving a more intuitive description of the notion of special sub-variety. Defini-
tion implies that the special sub-varieties of a product of two modular curves $X_1$ and $X_2$ are the following: CM points $(x, y)$, fibers of a projection $X_1 \times \{y\}$ or $\{x\} \times X_2$ over a CM point, or the graph of a Hecke correspondence between $X_1$ and $X_2$, or $X_1 \times X_2$ itself. In particular, the special sub-varieties of $\mathbb{C}^2$, with $\mathbb{C}$ viewed as the $j$-line $SL_2(\mathbb{Z})/\mathbb{H}$, are the CM points, fibers of a projection over a CM point, $\mathbb{C}^2$ itself, or the image in $\mathbb{C}^2$ of the modular curve $Y_0(n)$ (parameterizing elliptic curves with a cyclic subgroup of order $n$) for some $n \geq 1$, under the map sending $(E, G)$ to $(j(E), j(E/G))$.

Let us now look at the special sub-varieties of a product $X$ of any number of modular curves $X_s = X_{\Gamma_s}$, in the notation of Lemma 3.1. Let $Z$ be a special sub-variety of $X$, arising from a partition $(S_1, \ldots, S_r)$ of $S$. Let $S''$ be the subset of $S$ consisting of those $s$ such that $p_s Z$ is a CM point, and let $S'$ be its complement. Then $Z$ decomposes as a product $Z' \times Z''$, with $Z''$ a CM point, and $Z'$ projecting dominantly (surjectively, in fact) to all $X_s$ (with $s$ in $S'$ of course). Now consider projections $p_T: Z' \to X_T := \prod_{s \in T} X_s$ for two element subsets $T$ of $S'$. Then $p_T Z'$ is either all of $X_T$, or it is the graph of a Hecke correspondence, depending on whether $T$ meets two or only one of the $S_i$. Obviously, $Z'$ is contained in the intersection $Z''$ of the $p_T^{-1} p_T Z'$, for $T$ ranging over the two element subsets of $S'$. If we take one element $s_i$ in each $S_i$ contained in $S'$, then the projections of both $Z'$ and $Z''$ to the product of the $X_{s_i}$ are finite and surjective. Hence $Z'$ and $Z''$ have the same dimension, and $Z'$ is actually an irreducible component of $Z''$. Let us state this conclusion in the following proposition.

3.1 Proposition. Let $n \geq 0$ be an integer. A closed irreducible sub-variety $Z$ of a product $X$ of $n$ modular curves $X_1, \ldots, X_n$ is special if and only if (1) all images of $Z$ under projection to one or two factors are special, and (2) $Z$ is an irreducible component of the intersection of the inverse images of its images under these projections. Equivalently, the special sub-varieties of $X$ are the irreducible components of loci defined by conditions that demand certain coordinates to be CM points, and by the existence of an isogeny of a given degree between certain pairs of coordinates.

The following two lemmas follow directly from this proposition.

3.2 Lemma. Let $n \geq 0$ be integer, and let $\Gamma_i$ and $\Gamma'_i$ be congruence subgroups of $SL_2(\mathbb{Z})$ for $i$ in $\{1, \ldots, n\}$, such that $\Gamma'_i$ is contained in $\Gamma_i$ for every $i$. Let $X$ be the product of the $X_{\Gamma_i}$, and $X'$ the product of the $X_{\Gamma'_i}$. Let $\pi$ be the morphism from $X'$ to $X$ induced by the inclusions of the $\Gamma'_i$ in the $\Gamma_i$. Let $Z$ be a closed irreducible sub-variety of $X$. Then the following statements are equivalent:

1. $Z$ is special;
2. every irreducible component of $\pi^{-1} Z$ is special;
3. at least one irreducible component of $\pi^{-1} Z$ is special.

3.3 Lemma. Let $n$, the $\Gamma_i$ and $X$ be as in the preceding proposition. Let $Z_1$ and $Z_2$ be two special sub-varieties of $X$. Then all irreducible components of $Z_1 \cap Z_2$ are special.
The notion introduced in the next definition will allow us to reduce the proof of our main results to the case where the $\Gamma_i$ are just $SL_2(\mathbb{Z})$, and where the Zariski closure of $\Sigma$ is a hyper-surface all of whose projections to products of all but one of the $X_i$ are dominant.

3.4 Definition. Let $k$ be a field, $n \geq 0$ an integer, $X_1, \ldots, X_n$ curves over $k$ (i.e., $k$-schemes of finite type, everywhere of dimension one). For $I$ a subset of $\{1, \ldots, n\}$, let $p_I$ be the projection from $X := X_1 \times \cdots \times X_n$ to $X_I := \prod_{i \in I} X_i$. Let $Z$ be a closed irreducible sub-variety of $X$. A subset $I$ of $\{1, \ldots, n\}$ is said to be minimal for $Z$ if $\dim(p_I Z) < |I|$, but $\dim(p_J Z) = |J|$ for all $J$ strictly contained in $I$; in this case, $p_I$ is called a minimal projection for $Z$.

3.5 Lemma. Notation as in Definition 3.4. Then $Z$ is an irreducible component of the intersection of the $p_I^{-1} p_I Z$, with $I$ minimal for $Z$.

Proof. First of all, note that the problem is only about closed subsets, hence we may and do replace all schemes here by their reduced sub-schemes. We replace each $X_i$ by an irreducible component of it that contains the image of $Z$ under $p_i$. Let $U_i$ be affine open in $X_i$, such that $U_i$ meets $p_i Z$. For each $i$, let $t_i$ be a regular function on $U_i$ that is transcendental over $k$. After renumbering the $X_i$, the elements $p_1^* t_1, \ldots, p_n^* t_n$ form a transcendence basis over $k$ of the function field of $Z$. Then, for every $j > d$, $p_j^* t_j$ is algebraic over the first $d$, and we find a minimal set $I_j$ consisting of $j$ and the $i \leq d$ that occur in the minimal dependence relation of $p_j^* t_j$. It follows that the intersection of the $p_I^{-1} p_I Z$, with $I$ ranging over the $I_j$, is the union of a $d$-dimensional closed part containing $Z$, and another closed part whose image in $X_1 \times \cdots \times X_d$ has dimension less than $d$. Hence $Z$ is an irreducible component of that intersection. The intersection of all $p_I^{-1} p_I Z$ contains $Z$ and is contained in the intersection that we just considered, hence has $Z$ as an irreducible component.

3.6 Proposition. Let $Z$ be an irreducible closed sub-variety of a product $X = X_1 \times \cdots \times X_n$ of complex modular curves. Then $Z$ is special if and only if for every subset $I$ of $\{1, \ldots, n\}$ that is minimal for $Z$ we have: $|I| \leq 2$ and $p_I Z$ is special.

Proof. This follows immediately from Proposition 3.1.

4 Special sub-varieties and Hecke correspondences.

For integers $m \geq 1$ and $n \geq 1$ we let $T_m$ be the Hecke correspondence on $\mathbb{C}^n$ that sends a point $(j(E_1), \ldots, j(E_n))$ to the formal sum of the $(j(E'_1), \ldots, j(E'_n))$ with each $E'_i$ a quotient of $E_i$ by a cyclic subgroup of order $m$. In other words, $T_m$ is the correspondence induced by the sub-variety of $\mathbb{C}^n \times \mathbb{C}^n$ consisting of the $(x, y)$ such that, for every $i$, $x$ and $y$ are $j$-invariants of elliptic curves related by an isogeny with kernel isomorphic to $\mathbb{Z}/m \mathbb{Z}$. The aim of this section is to prove the following theorem.

4.1 Theorem. Let $n \geq 0$ be integer. Let $Y$ be a closed algebraic sub-variety of $\mathbb{C}^n$ all of whose irreducible components contain a special point and are of the same dimension, $d$, say. Suppose
that $Y$ is contained in $T_m Y$ for some integer $m > 1$ composed of prime numbers $l$ greater than 3 and the degrees of the projections from the irreducible components of $Y$ to sub-products $\mathbb{C}^d$ of $\mathbb{C}^n$. Then every irreducible component of $Y$ is special.

Before proving this theorem, we will establish some ingredients for it. The idea is of course to use Proposition 3.6. The following proposition will be used to show that the subsets $I$ of $\{1, 2, \ldots, n\}$ that are minimal for an irreducible component $Z$ of $Y$ consist of at most two elements.

4.2 Proposition. Let $n \geq 3$ be integer. Let $Z$ be a closed irreducible hyper-surface in $\mathbb{C}^n$, and suppose that all projections $p_I$ from $Z$ to $\mathbb{C}^{n-1}$ are dominant. Then for every integer $m > 1$ composed of prime numbers $l > 3$ such that $l > \deg(p_I)$ for all $I$, the image $T_m Z$ of $Z$ is irreducible.

Proof. Let $m$ be as in the proposition; we write it as $m = l_1^{e_1} \cdots l_r^{e_r}$ with the $l_i$ distinct prime numbers, and with the $e_i > 0$. Let $G_i := \text{SL}_2(\mathbb{Z}/l_i^{e_i} \mathbb{Z})/\{1, -1\}$, and let $G := G_1 \times \cdots \times G_r$. Let $X$ be the modular curve corresponding to this quotient $G$ of $\text{SL}_2(\mathbb{Z})$. This curve $X$ parametrizes elliptic curves with, for each $i$, a symplectic level $l_i^{e_i}$ structure given up to sign. The group $G$ acts faithfully on $X$ with quotient $\mathbb{C}$. We let $G^n$ act on $X^n$ and denote the quotient map to $\mathbb{C}^n$ by $\pi_n$. Since $T_m Z$ is an image of $\pi_n^{-1} Z$, it suffices to show that $\pi_n^{-1} Z$ is irreducible.

Let $V$ be an irreducible component of $\pi_n^{-1} Z$, and let $H$ be its stabilizer in $G^n$ (i.e., $H$ is the subgroup of $g$ in $G^n$ such that $gV = V$). It suffices now to show that $H = G^n$, since then $V = \pi_n^{-1} Z$, as $G^n$ acts transitively on the set of irreducible components of $\pi_n^{-1} Z$. Lemma 4.3 below says that it is enough to prove that all projections from $H$ to $\mathbb{C}^{n-1}$ are surjective. By symmetry, it suffices to consider the projection on the first $n - 1$ factors. We consider the two diagrams:

\[
\begin{array}{cccccc}
G^{n-1} & \rightarrow & G^{n-1} & \rightarrow & G^n & \rightarrow & G^{n-1} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
P & \rightarrow & X^{n-1} & \rightarrow & X^n & \rightarrow & X^{n-1} \\
p_{n-1} & \rightarrow & \pi_n^{-1} Z & \rightarrow & \pi_n^{-1} Z & \rightarrow & \pi_n^{-1} Z \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
Z & \rightarrow & \mathbb{C}^{n-1} & \rightarrow & \mathbb{C}^n & \rightarrow & \mathbb{C}^{n-1} \\
p_{n-1} & \rightarrow & \mathbb{C}^{n-1} & \rightarrow & \mathbb{C}^n & \rightarrow & \mathbb{C}^{n-1}
\end{array}
\]

with $P$ the fibered product, $\pi_n^{-1}$ the quotient for the action of $G^{n-1}$, and $p_{n-1}$ the projection from $Z$ to the first $n - 1$ factors. All four morphisms in the Cartesian square defining $P$ are generically finite, and dominant. As the most right square is Cartesian, $P$ is the inverse image of $Z$ under the morphism from $X^{n-1} \times \mathbb{C}$ to $\mathbb{C}^n$. The canonical morphism of $\pi_n^{-1} Z$ to $P$ is the quotient for the action by $\{1\} \times G$; let $\overline{V}$ be the image of $V$ in $P$ under this morphism. The morphism from $P$ to $Z$ is the quotient by $G^{n-1}$. The fact the morphism $\pi_n^{-1} Z \rightarrow P$ is $G^{n-1} \times \{1\}$-equivariant implies that the stabilizer in $G^{n-1}$ of $\overline{V}$ is the image $\overline{H}$ of $H$ in $G^{n-1}$.

But now our hypothesis that $l_i$ is greater than the degree $d$ of the projection from $Z$ to $\mathbb{C}^{n-1}$ imply that $P$ is irreducible. Indeed, the $G^{n-1}$-set $\text{Irr}(P)$ of irreducible components of $P$ has at most $d$ elements, because each element has degree at least one over $X^{n-1}$. On the other hand, $G^{n-1}$ has no proper subgroups of index at most $d$ because each factor $G_i$ is generated by its $l_i$-subgroups. Hence $\overline{V} = P$ and $\overline{H} = G^{n-1}$, which is just what we had to prove. \qed
4.3 Lemma. Let $G = G_1 \times \cdots \times G_r$ be as above. Let $H$ be a subgroup of $G^n$ with $n \geq 2$ such that $p_i H = G^2$ for all projections $p_i : G^n \to G^2$. Then $H = G^n$.

Proof. Induction on $n$. We may and do assume that $n \geq 3$. We view $H$ as a subgroup of the product of $G$ by $G^{n-1}$. Then, by the induction hypothesis, $H$ projects surjectively to both factors. Let $H_1 := H \cap G^{n-1}$ and $H_2 := H \cap G$ (these are the kernels of the two projections); these are normal subgroups of $G^{n-1}$ and $G$, respectively. Goursat’s Lemma then says that $H$ is the inverse image of the graph of an isomorphism between $G^{n-1}/H_1$ and $G/H_2$.

The normal subgroups of $G$ are the kernels of the reduction morphisms from $G$ to the products $\prod_i \ SL_2(\mathbb{Z}/l_i^e \mathbb{Z})/\{1, -1\}$ with $t_i \leq e_i$. To prove this, one first notes that $SL_2(\mathbb{F}_l)/\{1, -1\}$ is simple for any prime $l \geq 5$ (see [15, VIII, Thm. 8.4]). Then one lets $V_{l,e}$ be the kernel of the reduction from $SL_2(\mathbb{Z}/l^e \mathbb{Z})/\{1, -1\}$ to $SL_2(\mathbb{Z}/l^{e-1} \mathbb{Z})/\{1, -1\}$. As a representation of $SL_2(\mathbb{F}_l)$, $V_{l,e}$ is isomorphic to $\text{Sym}^2(\mathbb{F}_l^2)$, hence is irreducible for $l > 2$. Finally, one uses that the $l$-torsion of $SL_2(\mathbb{Z}/l^e \mathbb{Z})$ is contained in $V_{l,e}$ for $l \geq 5$. Similarly, the normal subgroups of $G^{n-1}$ are products of normal subgroups of $G$.

Suppose that $H_2 \neq G$. We take $i$ such that $H_2 \cap G_i$ is not equal to $G_i$. Then for a unique $j$ with $2 \leq j \leq n$ the intersection of the factor $G_i$ in the $j$th factor $G$ in $G^{n-1}$ with $H_1$ is not equal to $G_i$. It follows that the projection $p_{(1,j)}$ from $H$ to $G^2$ is not surjective, contradicting the hypotheses of the Lemma. Hence $H_2 = G$, $H_1 = G^{n-1}$, and $H = G^n$.

4.4 Lemma. Let $m \geq 2$ be an integer, $n \geq 0$, and $x$ in $\mathbb{C}^n$. Then the $T_m$-orbit $\cup_{i \geq 0} T^i_{m} x$ is dense in $\mathbb{C}^n$ for the Archimedean topology.

Proof. Since $T_m$ is the product of the correspondence (also denoted) $T_m$ on each factor $\mathbb{C}$, the proof is reduced to the case $n = 1$. The inverse image of $\cup_{i \geq 0} T^i_{m} x$ under $j : \mathbb{H} \to \mathbb{C}$ is an orbit of the subgroup $H$ of $\text{GL}_2(\mathbb{Z}[1/m])$ generated by $SL_2(\mathbb{Z})$ and the element $(\begin{smallmatrix} m & 0 \\ 0 & 1 \end{smallmatrix})$. The little computation:

$$(m^{-k} 0)(1 0^{-1})(m^k 0 \ 1) = (m^{-k} 0 \ 1)$$

shows that $H \cap SL_2(\mathbb{R})$ is dense in $SL_2(\mathbb{R})$. (In fact, $H$ contains $SL_2(\mathbb{Z}[1/m])$.)

Proof. (Of Theorem 4.1) Let $n$, $Y$, $d$ and $m$ be as in the statement of the theorem, and let $Z$ be an irreducible component of $Y$.

We consider the correspondence $T_{m,Y}$ from $Y$ to itself induced by $T_m$: if we view $T_m$ as a closed sub-variety of $\mathbb{C}^n \times \mathbb{C}^n$, then $T_{m,Y}$ is given by the Cartesian diagram:

$$
\begin{array}{ccc}
T_{m,Y} & \rightarrow & Y \times Y \\
\downarrow & & \downarrow \\
T_m & \rightarrow & \mathbb{C}^n \times \mathbb{C}^n.
\end{array}
$$

The two projections from $T_{m,Y}$ to $Y$ are finite, because the projections from $T_m$ to $\mathbb{C}^n$ are finite. As $Y$ is contained in $T_mY$, the two projections from $T_{m,Y}$ to $Y$ are surjective. We let $T_{m,Y}$ be the union of the irreducible components of dimension $d$ of $T_{m,Y}$. Then both projections from $T_{m,Y}$ to $Y$ are finite and surjective.
Let $\text{Irr}(Y)$ be the set of irreducible components of $Y$. Then $T_{m,Y}$ induces a correspondence $T_m$ on $\text{Irr}(Y)$. We replace $Y$ by the union of the irreducible components of it that lie in the $T_m$-orbit of the element $Z$ of $\text{Irr}(Y)$. If $Z'$ is an irreducible component of $T_m Z$, then for any $I$, $\dim p_I Z = \dim p_I Z'$. Thus a subset $I$ of $\{1, 2, \ldots, n\}$ is minimal for $Z$ if and only if it is minimal for all irreducible components of $Y$.

Let $I$ be a subset of $\{1, 2, \ldots, n\}$ that is minimal for $Z$, hence for all irreducible components $Z'$ of $Y$. We want to apply Proposition 4.2 to the $p_I Z'$, the closure in $\mathbb{C}^l$ of $p_I Z'$, for all $Z'$. For every $i$ in $I$, the degree of the projection from $p_I Z'$ to $\mathbb{C}^{l-\{i\}}$ is at most that of the projection from $Z'$ to $\mathbb{C}^J$ where $J \supset I - \{i\}$ is such that $|J| = d$ and $\dim(p_J Z') = d$. Hence the hypotheses on $m$ in Proposition 4.2 are satisfied.

Suppose that $|I| \geq 3$. Proposition 4.2 shows that all $T_m p_I Z'$ are irreducible. But then $p_I Y$ and $T_m p_I Y$ have the same number of irreducible components, all these components are of the same dimension, and $p_I Y$ is contained in $T_m p_I Y$. It follows that $p_I Y = T_m p_I Y$. This contradicts the density of all $T_m$-orbits in $\mathbb{C}^l$ (Lemma 4.4). Hence we have $|I| \leq 2$.

In order to prove that $Z$ is special, it suffices to show that $p_I Z$ is special (Proposition 3.6). If $|I| = 1$ then $p_I Z$ is a special point because $Z$ contains a special point.

Suppose now that $|I| = 2$. We replace the $Y$ that we have by $p_I Y$, so our new $Y$ is a closed curve in $\mathbb{C}^2$, with quasi finite projections to both factors $\mathbb{C}$, of degrees $d_1$ and $d_2$ that are less than each prime number $l$ dividing $m$. We let $T_{m,Y}$ be the correspondence from $Y$ to itself induced by $T_m$ as above. We would like to apply [7, Theorem 6.1], but that result only applies to irreducible curves in $\mathbb{C}^2$, and to $m$ that are square free. We generalize the proof of that result to the present situation. We start with [7, Lemma 6.3]. Consider the commutative diagram:

$$
\begin{align*}
& T_{m,Y} \xrightarrow{p_1} Y \\
& \downarrow \\
& Y_0(m) \xrightarrow{p_1} \mathbb{C}
\end{align*}
$$

All four maps in this diagram are quasi finite and dominant, and the horizontal ones are finite and surjective. The hypotheses on the $l$ dividing $m$ imply that for each irreducible component $Z$ of $Y$, the fibered product of $Z$ and $Y_0(m)$ over $\mathbb{C}$ is irreducible ($G$ does not have a proper subgroup of index at most $d_1$ or $d_2$). It follows that $T_{m,Y}$ maps surjectively to the fibered product of $Y$ and $Y_0(m)$ over $\mathbb{C}$. This means that for every $(x, y)$ on $Y$, $T_{m,Y}(x, y)$ surjects to $T_m(x)$. For $n \geq 1$ let $T_{m,Y,n}$ be the correspondence on $Y$ obtained by taking in $T_{m,Y}$ the irreducible components that correspond to isogenies with cyclic kernel on the first coordinate, i.e., that send $(x, y)$ to the sum of those $(x', y')$ in $T_{m,Y}(x, y)$ that correspond to isogenies $x \to x'$ with cyclic kernel (of order $m^n$).

Let now $Z_0$ be in $\text{Irr}(Y)$. We can then choose elements $Z_1$, $Z_2$, etc. in $\text{Irr}(Y)$ such that $Z_i \subset T_{m,Y} Z_{i-1}$, such that moreover $Z_i \subset T_{m,Y,i} Z_0$ (here we use that $T_{m,Y}(x, y)$ surjects to $T_m(x)$; when choosing the $Z_i$ we just make sure that the isogeny on the first coordinate is cyclic). As there are only finitely many possibilities for the $Z_i$, it follows that for some $n \geq 1$ and some $Z$ in $\text{Irr}(Y)$ we have:

$$Z \subset T_{m,Y,n} Z.$$
Let \( T \) be the correspondence on \( Z \) induced by \( T_{m,y,n}Z \). Then, by the same irreducibility argument as above for \( T_{m,y} \), for each \((x,y)\) in \( Z \) the set \( T(x,y) \) surjects to \( T_{m^n}(x) \), with \( T_{m^n} \) the correspondence on \( \mathbb{C} \) given by isogenies with cyclic kernel of order \( m^n \). By Lemma 4.4 all \( T_{m^n} \)-orbits in \( \mathbb{C} \) are dense. It follows that all \( T \)-orbits in \( Z \) are not discrete, as their projection to \( \mathbb{C} \) is dense.

Now the rest of the proof of [7, Theorem 6.1] can be applied almost without change. Let \( X \) be an irreducible component of the complex analytic variety \( \pi^{-1}Z \), where \( \pi : \mathbb{H}^2 \rightarrow \mathbb{C}^2 \) is the quotient for the action of \( \text{SL}_2(\mathbb{Z})^2 \). Let \( G_X \) be the stabilizer of \( X \) in \( G := \text{SL}_2(\mathbb{R})^2 \). Then Lemmas 6.6, 6.7, 6.8 and 6.9 of [7] can be applied to \( X \). (In the proof of Lemma 6.9 we do not know the second coordinate of the elements \( g_{i,j} \), but that information was not used anyway.) Please note the erratum at the end of this article for a correction to the end of the proof of Theorem 6.1 of [7]. The proof of Theorem 4.1 is now finished (we have shown that the irreducible component \( Z \) of \( Y \) is special, but as \( T_{m,y} \) acts transitively on \( \text{Irr}(Y) \), all irreducible components are special). \( \square \)

5 Galois action.

We recall very briefly some facts about the action of \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \) on the set of \( j \)-invariants of elliptic curves with complex multiplications, i.e., on the special points of \( \mathbb{C} \). For some more details, see [7, §2]. Let \( E \) be an elliptic curve over \( \mathbb{C} \) with complex multiplications by a quadratic imaginary field \( K \). Then \( \text{End}(E) = \mathcal{O}_{K,f} = \mathbb{Z} + f\mathcal{O}_K \) for some unique integer \( f \geq 1 \) (the conductor of the order \( \mathcal{O}_{K,f} \) in the maximal order \( \mathcal{O}_K \)). For each automorphism \( \sigma \) of \( \mathbb{C} \) we have \( \text{End}(\sigma E) \cong \mathcal{O}_{K,f} \). The set \( S_{K,f} \) of isomorphism classes of complex elliptic curves with endomorphism ring isomorphic to \( \mathcal{O}_{K,f} \) is a \( \text{Pic}(\mathcal{O}_{K,f}) \)-torsor, hence finite. It follows that \( \text{Aut}(\mathbb{C}) \) acts on \( S_{K,f} \) via \( \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \). The action of \( \text{Gal}(\overline{\mathbb{Q}}/K) \) is given by the morphism from \( \text{Gal}(\overline{\mathbb{Q}}/K) \rightarrow \text{Pic}(\mathcal{O}_{K,f}) \) that is unramified outside \( f \) and that sends the Frobenius element at a maximal ideal \( m \) not containing \( f \) to the inverse of the class \([m]\) of \( m \) in \( \text{Pic}(\mathcal{O}_{K,f}) \). This morphism is surjective, hence we have, by the Brauer-Siegel theorem [14, Ch. XVI]:

\[
|\text{Gal}(\overline{\mathbb{Q}}/K) \cdot j(E)| = |\text{Pic}(\text{End}(E))| = |\text{discr}(\text{End}(E))|^{1/2+o(1)}, \quad |\text{discr}(\text{End}(E))| \rightarrow \infty.
\]

Let \( l \) be a prime number that is split in \( \text{End}(E) \), i.e., for which \( \mathbb{F}_l \otimes \text{End}(E) \) is isomorphic as a ring to \( \mathbb{F}_l \times \mathbb{F}_l \). Let \( m \) be one of the two ideals in \( \text{End}(E) \) of index \( l \). Then \( E \) is a quotient of its Galois conjugate \([m]E \) via an isogeny of degree \( l \) (if \( E \cong \mathbb{C}/\Lambda \), then \([m]E \cong \mathbb{C}/m\Lambda \)). It follows that we have the inclusion of subsets of \( \mathbb{C} \):

\[
\text{Gal}(\overline{\mathbb{Q}}/K) \cdot j(E) \subset T_l(\text{Gal}(\overline{\mathbb{Q}}/K) \cdot j(E)).
\]

6 Existence of small split primes.

The effective Chebotarev theorem of Lagarias, Montgomery and Odlyzko, assuming GRH, as stated in [18, Thm. 4] and the second remark following that theorem, plus a simple computation (see Section 5 of [7]) give the following result.
6.1 Proposition. For $M$ a finite Galois extension of $\mathbb{Q}$, let $n_M$ be its degree, $d_M = |\text{discr}(O_M)|$ its absolute discriminant, and for $x \in \mathbb{R}$, let $\pi_{M,1}(x)$ be the number of primes $p \leq x$ that are unramified in $M$ and such that the Frobenius conjugacy class $\text{Frob}_p$ contains just the identity element of $\text{Gal}(M/\mathbb{Q})$. Then for $M$ a finite Galois extension of $\mathbb{Q}$ for which GRH holds and for $x$ sufficiently big (i.e., bigger than some absolute constant) such that:

$$x > 2(\log d_M)^2 (\log(\log d_M))^2,$$

one has:

$$\pi_{M,1}(x) \geq \frac{x}{3n_M \log(x)}.$$

We will apply this result in the following situation. Let $n \geq 1$ be an integer, and let $K_1, \ldots, K_n$ be quadratic sub-fields of $\mathbb{Q}$ for which GRH holds. Let $M := K_1 \cdots K_n$ be the composite of the $K_i$. Then:

$$d_M \leq |\text{discr}(O_{K_1} \otimes \cdots \otimes O_{K_n})| = (d_{K_1} \cdots d_{K_n})^{2^{n-1}}.$$

On the other hand, for each $i$ we have embeddings $K_i \to M$, hence:

$$d_M = |\text{Norm}_{K_i/\mathbb{Q}}(\text{discr}(O_M/O_{K_i}))| \cdot d_{[M:K_i]} \geq d_{K_i}.$$

The preceding two inequalities mean that, for our purposes, $\log d_M$ is of the same order of magnitude as the maximum of the $\log d_{K_i}$.

For each $i$, let $R_i$ be an order of $K_i$. The number of primes dividing the discriminant of $R_i$ is of order $o(\log(\text{discr} R_i))$ (indeed, if $P(n)$ denotes the number of primes dividing a positive integer $n$, then one has $P(n) \log P(n) \leq n$). It follows that if $\max\{|\text{discr}(R_i)| \mid 1 \leq i \leq n\}$ is bigger than some absolute constant, then there are primes $l$ split in each $R_i$, such that:

$$l \leq (\log \max\{|\text{discr}(R_i)| \mid 1 \leq i \leq n\})^{2+o(1)},$$

where the $o(1)$ does not depend on the fields $K_i$.

7 The case of a curve.

In this section we prove Theorems 1.2 and 1.3 for the one-dimensional irreducible components of the Zariski closure of a set of special points on a product of modular curves. By Proposition 3.6 and Lemma 3.2 it suffices to consider closed irreducible curves $Z$ in $\mathbb{C}^2$ that contain infinitely many special points. Assume one of the following two conditions: the generalized Riemann hypothesis is true for imaginary quadratic number fields, or the special points can be taken in one isogeny class. Then we will prove that $Z$ is special.

Even though Theorem 1.2 has been proved in [7] for curves in a product of two modular curves, we reprove it here in a somewhat simplified way. Namely, it turns out that the first step of the proof given in [7] can be skipped, i.e., the arguments of [7] §3 are not needed. We also prove the variant Theorem 1.3 in this section (this variant was not treated in [7]). We should also mention that the next section reproves the results of this section, but we think that this section...
serves well as a kind of warming up exercise for the more complicated arguments of the next section.

If one of the two projections from $Z$ to $\mathbb{C}$ is not dominant, then $Z$ is the inverse image under that projection of a special point, hence special. So we assume that both projections are dominant. As $Z$ contains a dense set of points with coordinates in $\mathbb{Q}$ (the special points), $Z$ is defined over a finite extension of $\mathbb{Q}$, and therefore has only finitely many Galois conjugates. Let $Z_{\mathbb{Q}}$ be the closed irreducible algebraic curve in $\mathbb{A}^2_{\mathbb{Q}}$ that by base change to $\mathbb{C}$ gives the union of the finitely many Galois conjugates of $Z$. Let $d_1$ and $d_2$ be the degrees of the two projections to $\mathbb{A}^1_{\mathbb{Q}}$.

At this point, we proceed directly to the arguments of [7 §4]. Let $x = (x_1, x_2)$ be a special point in $Z_{\mathbb{Q}}(\mathbb{C})$. Let $l$ be a prime number that is split in both $\text{End}(x_i)$. Then we have (see Section 5):

$$\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x \subset Z_{\mathbb{Q}}(\mathbb{C}) \cap T_l Z_{\mathbb{Q}}(\mathbb{C}), \quad |\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x| \geq \max\{|\text{Pic}(\text{End}(x_i))| \mid 1 \leq i \leq 2\}.$$  

On the other hand, the intersection number in $\mathbb{P}^1_{\mathbb{Q}} \times \mathbb{P}^1_{\mathbb{Q}}$ of the closures of $Z_{\mathbb{Q}}$ and $T_l Z_{\mathbb{Q}}$ is $2d_1d_2(l + 1)^2$ (the bidegree of $T_l Z_{\mathbb{Q}}$ is $((l + 1)d_1, (l + 1)d_2)$). Hence:

$$|Z_{\mathbb{Q}}(\mathbb{C}) \cap T_l Z_{\mathbb{Q}}(\mathbb{C})| \leq 2d_1d_2(l + 1)^2, \quad \text{if the intersection is finite.}$$

7.1 Lemma. With the notation of this section, there exists a special point $x = (x_1, x_2)$ in $Z_{\mathbb{Q}}(\mathbb{C})$ and a prime number $l$ such that:

1. $l > \max\{3, d_1, d_2\}$;
2. $l$ splits in $\text{End}(x_i)$ for both $i$;
3. $2d_1d_2(l + 1)^2 < \max\{|\text{Pic}(\text{End}(x_i))| \mid 1 \leq i \leq 2\}$.

Proof. Let $\Sigma$ be the set of special points in $Z_{\mathbb{Q}}(\mathbb{C})$. The function $\Sigma \to \mathbb{Z}$ sending $x$ to $\max\{|\text{discr}(\text{End}(x_i))| \mid 1 \leq i \leq 2\}$ is not bounded, because for each possible value for the discriminant there are only finitely many elliptic curves. We recall that we have assumed that either GRH holds for imaginary quadratic fields, or that $Z$ contains infinitely many special points in one isogeny class. Let us first deal with the second case. Then we have two imaginary quadratic fields $K_1$ and $K_2$ (possibly the same) and an infinite set $\Sigma$ of $x$ in $Z_{\mathbb{Q}}(\mathbb{C})$ with $\text{End}(x_i)$ an order in $K_i$. Then $|\text{Pic}(\text{End}(x_i))| = |\text{discr}(\text{End}(x_i))|^{1/2 + o(1)}$ by a simple argument. The classical Chebotarev theorem (see for example [14 Ch. VIII, §4]) asserts that the set of primes $l$ that are split in $M = K_1K_2$ has natural density $1/n_M$ (actually, Dirichlet density is good enough here). We note that the number of primes $l$ that divide $\text{discr}(\text{End}(x_i))$ is at most $\log_2 |\text{discr}(\text{End}(x_i))|$. Hence there do exist $x$ and $l$ as claimed.

Let us now assume that GRH holds for imaginary quadratic fields. Then we use the Brauer-Siegel theorem (see Section 5), and the application of the effective Chebotarev theorem from Section 6. □
Let now $x$ and $l$ be as in Lemma 7.1. Then the intersection $Z_Q \cap T_l Z_Q$ cannot be finite. As $Z_Q$ is irreducible, we have:

$$Z_Q \subseteq T_l Z_Q.$$ 

Theorem 4.1 now implies that all components of $(Z_Q)_C$ are special, hence in particular that $Z$ is special. This finishes the proof of Theorems 1.2 and 1.3 in the case of a curve.

8 Producing special curves from special points.

We start the proof of Theorems 1.2 and 1.3 in the case of sub-varieties of arbitrary dimension. This proof will also reprove the case of curves that was treated in the previous section. By Lemma 3.2 it suffices to consider closed irreducible sub-varieties $Z$ of $\mathbb{C}^n$ of dimension $d \geq 1$, that contain a dense set $\Sigma$ of special points.

8.1 Theorem. Let $Z$ be a closed irreducible sub-variety of dimension $d \geq 1$ of $\mathbb{C}^n$. Assume that $Z$ contains a dense set $\Sigma$ of special points, and that at least one of the following conditions holds: GRH is true for imaginary quadratic fields, or $\Sigma$ can be taken to lie in one isogeny class. Then for all but finitely many $x$ in $\Sigma$, there is a special curve $C$ contained in $Z$ with $x$ in $C(\mathbb{C})$.

The curves $C$ will be obtained via repeated intersections of sub-varieties with their image under a suitable Hecke correspondence, until we get an inclusion as in Theorem 4.1. In order to control the degrees of the sub-varieties in question, we review some facts on intersection theory before starting the proof. Appendix A of [12] is a good reference for what we need. It may help to note that we only need intersections with divisors, as in [11] (see also [13]).

Let $k$ be a field, and $n \geq 0$ an integer. Let $\mathbb{P} := (\mathbb{P}_k^1)^n$. As the Chow ring of $\mathbb{P}_k^1$ is $\mathbb{Z}[x]/(x^2)$, with $x$ the class of a rational point, the Chow ring of $\mathbb{P}$ is $A := \mathbb{Z}[\varepsilon_1, \ldots, \varepsilon_n]$, with $\varepsilon_i^2 = 0$ for all $i$, and with $\mathbb{Z}$-basis the family of $\varepsilon_I = \prod_{i \in I} \varepsilon_i$ indexed by subsets $I$ of $\{1, \ldots, n\}$. Let $Z$ be a closed $k$-irreducible sub-variety of $\mathbb{P}$, and let $d$ be its dimension. We write its class $[Z]$ in $A$ as $\sum_I a_I(Z) \varepsilon_I$ (of course, for $|I| \neq n - d$ we have $a_I(Z) = 0$). The coefficient $a_I(Z)$ is the degree of the projection $\rho_T: Z \to (\mathbb{P}_k^1)^T$, with $T$ the complement of $I$ in $\{1, \ldots, n\}$. We let $a(Z) := \max_I a_I(Z)$. Let us suppose that $x$ is a closed point of $\mathbb{P}$ that does not lie on $Z$, and that $k$ is not finite. Then we want to produce a hyper-surface $H$ of $\mathbb{P}$ that contains $Z$, avoids $x$, and has all $a_I(H)$ suitably bounded in terms of the $a_I(Z)$. (The exact bound does not matter much; what is important is that the bound is polynomial in the $a_I(Z)$.) The line bundle $\mathcal{L} := \mathcal{O}(1, \ldots, 1)$ on $\mathbb{P}$ is very ample, and its space of sections gives an embedding of $\mathbb{P}$ in some projective space $\mathbb{P}_k^N$. The class $[\mathcal{L}]$ of $\mathcal{L}$ in $A$ is $\sum \varepsilon_i$. The image of $Z$ in $\mathbb{P}_k^N$ has degree:

$$[Z] \cdot [\mathcal{L}]^d = [Z] \cdot (\varepsilon_1 + \cdots + \varepsilon_n)^d = [Z] \cdot d! \sum_{|J| = d} \varepsilon_J = d! \sum_{|I| = n - d} a_I(Z).$$
In $\mathbb{P}^N_k$ we can project $Z$ birationally onto a hyper-surface in some $\mathbb{P}^{d+1}_k$ of the same degree, that avoids the image of $x$. It follows that we can take $H$ such that:

$$[H] = \left(d! \sum_{|I|=n-d} a_I(Z) \right) \sum_i \varepsilon_i.$$

For $l$ prime, let $[T_l]$ be the class in $A \otimes A = \mathbb{Z}[\varepsilon_1, \ldots, \varepsilon_n, \eta_1, \ldots, \eta_n]$ of the correspondence $T_l$ on $\mathbb{P} \times \mathbb{P}$. As $T_l$ is the product of the usual Hecke correspondences $T_l$ on each coordinate we get:

$$[T_l] = (l+1)^n \prod_i (\varepsilon_i + \eta_i).$$

It follows that for $Z$ and $l$ as above we have:

$$[T_lZ] = (l+1)^n[Z], \quad a_I(T_lZ) = (l+1)^n a_I(Z) \quad \text{for all } I.$$

**Proof.** (Of Theorem 8.1.) As in the previous section, $Z$ has only finitely many Galois conjugates, and we let $Z_\mathbb{Q}$ be the closed irreducible sub-variety of $\mathbb{A}^n_\mathbb{Q}$ such that $(Z_\mathbb{Q})_C$ is the union of these Galois conjugates. Let $x = (x_1, \ldots, x_n)$ be in $\Sigma$, and let $l$ be a prime number that is split in each of the $\text{End}(x_i)$. Then we have:

$$\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x \subset Z_\mathbb{Q}(\overline{\mathbb{Q}}) \cap T_l Z_\mathbb{Q}(\overline{\mathbb{Q}}).$$

Let $m_x := \max\{|\text{discr}(\text{End}(x_i))| : 1 \leq i \leq n\}$. As we only need to prove a statement for all but finitely many $x$, we may suppose that $m_x$ is sufficiently large in terms of the $a_I(Z_\mathbb{Q})$. By the results of Section 6 if we assume GRH, and by some simple argument if $\Sigma$ lies in one isogeny class, we can take $l$ such that:

$$l < (\log m_x)^3, \quad l > 3, \quad \text{and } l > a_I(Z_\mathbb{Q}) \text{ for all } I.$$

If $Z_\mathbb{Q}$ is contained in $T_l Z_\mathbb{Q}$ then $Z$ is special by Theorem 4.1 and Definition 1.1 implies the existence of a special curve $C$ as desired. Suppose now that $Z_\mathbb{Q}$ is not contained in $T_l Z_\mathbb{Q}$. The results in Section 5 tell us that, ignoring finitely many of the $x$:

$$|\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x| > m_x^{1/3}.$$

The discrepancy between $\log m_x$ and $m_x$ will be heavily exploited in the sense that, for $m_x$ large enough, any fixed power of $\log m_x$ is less than $m_x^{1/3}$. In particular, if $m_x$ is sufficiently large with respect to $n$ and $\alpha(Z)$, then the size of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x$ is larger than all intersection numbers that we will encounter in the rest of the proof.

Let $H$ be a hyper-surface in $(\mathbb{P}^1_\mathbb{Q})^n$ that contains $T_l Z_\mathbb{Q}$, that does not contain $Z_\mathbb{Q}$ and that satisfies:

$$[H] = \left( (l+1)^n d! \sum_{|I|=n-d} a_I(Z) \right) \sum_i \varepsilon_i.$$
Let $Z_1$ be a $\mathbb{Q}$-irreducible component of $Z_{\mathbb{Q}} \cap H$ that contains $x$. We note that $\dim(Z_1) = d - 1$. Equations (8.1.1)–(8.1.4) imply that $Z_{\mathbb{Q}} \cap H$ is not finite: it contains more points, namely the Galois orbit of $x$, than the intersection number. Hence $d > 1$ (so if $d = 1$ then we have proved that $Z$ is special).

The idea is now to apply to $Z_1$ the same constructions as we have just applied to $Z_{\mathbb{Q}}$. At this point we do not know whether $Z_1$ has a dense subset of special points, but we know that $Z_1$ contains $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot x$. We get a prime number $l_1$ of size about $a(Z_1)$ that is split in each of the $\text{End}(x_i)$. If $Z_1$ is contained in $T_i Z_1$ then we get a special curve $C$ in $Z$ as desired. If not, then we take a hyper-surface $H_1$ in $(\mathbb{P}_{\mathbb{Q}}^1)^n$ that contains $T_i Z_1$ but does not contain $Z_1$ and that has suitably bounded degree as above, and let $Z_2$ be an irreducible component of $Z_1 \cap H_1$, etc. As the dimension drops by one at each intersection, we need to repeat this process at most $d - 1$ steps. One easily computes that $a(Z_i)$ is of order of magnitude at most the $(3n)^{\text{th}}$ power of $\log m_x$. Hence the intersection $Z_i \cap H_i$ is never finite, which means that at some point we will have $Z_i \subset T_i Z_i$, with $Z_i$ of dimension at least one. The irreducible components of this $(Z_i)_C$ are then special by Thm.4.1 and we get a special curve $C$ as desired in $(Z_i)_C$. □

9 End of the proof.

We will now finish the proof of Theorems 1.2 and 1.3. So let $Z$ be an irreducible component of the Zariski closure of a set $\Sigma$ of special points in a finite product of modular curves, and assume either GRH for imaginary quadratic fields or that $\Sigma$ lies in one isogeny class. As explained in the introduction, in the last case we may and do assume that the isogenies preserve the product structure, i.e., the $n$-tuples of elliptic curves corresponding to the elements of $\Sigma$ are isogeneous coordinate-wise.

We have to prove that $Z$ is special. Let $I$ be a subset of $\{1, \ldots, n\}$ which is minimal for $Z$ (see Definition 3.4). By Proposition 3.6 it suffices to prove that $|I| \leq 2$ and that $p_I Z$ is special. If $|I| = 1$ then $p_I Z$ is a special point, hence special. If $|I| = 2$ then $p_I Z$ is a special curve as was proved in Section 7 and also in Section 8.

So let us assume that $|I| \geq 3$. We have to get a contradiction now. We replace $Z$ by $p_I Z$, $n$ by $|I|$, and renumber $I$ as $\{1, 2, \ldots, n\}$. By Lemma 3.2 it suffices to consider the case where the congruence subgroups are maximal, i.e., where $Z$ is contained in $\mathbb{C}^n$. So now $n \geq 3$, and $Z$ is an irreducible hyper-surface in $\mathbb{C}^n$ all of whose projections to coordinate hyperplanes are dominant.

Theorem 8.1 tells us that there is a Zariski dense subset $\mathcal{Y}$ of special curves in $Z$. For each $Y$ in $\mathcal{Y}$ we let $I_Y$ be the set of $i$ in $\{1, \ldots, n\}$ such that the projection $p_i : Y \to \mathbb{C}$ is surjective (note that as the $Y$ in $\mathcal{Y}$ are special, projecting surjectively or dominantly under $p_i$ is equivalent). The $I_Y$ are non-empty because each $Y$ is a curve. As there are only finitely many possibilities for $I_Y$, there is a subset $I$ of $\{1, \ldots, n\}$ such that the set of $Y$ with $I_Y = I$ is Zariski dense. We pick such a subset $I$ and replace $\mathcal{Y}$ by the set of $Y$ with $I_Y = I$. We renumber the set $\{1, \ldots, n\}$ such that $I = \{1, \ldots, j\}$, with $j \geq 1$.

We claim that $j \geq 3$. Indeed, if $j = 1$ then each $Y$ in $\mathcal{Y}$ is of the form $\mathbb{C} \times \{x\}$ with $x$ in $\mathbb{C}^{n-1}$ special; the set of these $x$ is then Zariski dense in $\mathbb{C}^{n-1}$, contradicting the fact that $Z \neq \mathbb{C}^n$. Let us suppose then that $j = 2$. Then for each $Y$ in $\mathcal{Y}$ we have $Y = p_{1,2} Y \times p_{>2} Y$, with $p_{1,2} Y$ a
special curve in $\mathbb{C}^2$, and $p_{>2}Y$ a special point in $\mathbb{C}^{n-2}$ (here $p_{>2}$ denotes the projection on the last $n - 2$ coordinates). The set of the special points $p_{>2}Y$ is Zariski dense in $\mathbb{C}^{n-2}$. But, over a non-empty Zariski open subset of $\mathbb{C}^{n-2}$ the fibers of $Z$ under the projection to the last $n - 2$ coordinates are curves in $\mathbb{C}^2$, of a fixed degree. Hence, after shrinking $\mathcal{Y}$, we may assume that the degrees of the $p_{1,2}Y$ are all equal. As the set of special curves in $\mathbb{C}^2$ that project surjectively under the two projections and that have a fixed degree is finite, this contradicts the Zariski density of the union of the $p_{1,2}Y$. Hence we have $j \geq 3$.

Let $x_1$ be any special point in $\mathbb{C}$. For simplicity we suppose that the CM-field of $x_1$ is different from $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{-3})$. Let $Z'$ be the Zariski closure of the union of the intersections $Y \cap \{x_1\} \times \mathbb{C}^{n-1}$, $Y$ ranging over $\mathcal{Y}$. We note that $Z'$ is contained in $Z \cap \{x_1\} \times \mathbb{C}^{n-1}$, and that $Z'$ is the Zariski closure of a set of special points, contained in one isogeny class if $\Sigma$ is contained in one isogeny class: the first $j$ coordinates of an element of $Y \cap \{x_1\} \times \mathbb{C}^{n-1}$ are isogeneous to $x_1$, and $p_{>2}Y$ is a special point whose coordinates lie in fixed isogeny classes. Hence, by induction on $n$, all irreducible components of $Z'$ are special.

Let $Z'_1, \ldots, Z'_r$ be the irreducible components of $Z'$. For $\tilde{Y}$ in $\mathcal{Y}$ let $\tilde{Y} \to Y$ denote the normalization morphism. After suitably renumbering the $Z'_i$ a Zariski dense subset of the $\tilde{Y}$ in $\mathcal{Y}$ have the property that the number of points on $\tilde{Y}$ mapping to $Y \cap Z'_i$ is at least $1/r$ times the number of points on $\tilde{Y}$ mapping to $Y \cap Z'$. We replace $\mathcal{Y}$ by such a subset. As $Z'_i$ is contained in $Z$, we have $Z'_i \neq \{x_1\} \times \mathbb{C}^{n-1}$ (recall that the dimension of $Z$ is $n - 1$ and that $Z$ projects surjectively to all coordinate hyperplanes). As $Z'_i$ is special and not equal to $\{x_1\} \times \mathbb{C}^{n-1}$, there are $i$ and $j$ with $1 < i < j$ such that $p_{i,j}Z'_i$ is a strict special sub-variety $S$ of $\mathbb{C}^2$. We renumber the indices so that this is so for $i = 2$ and $j = 3$.

The $p_{<3}Y$, for $Y$ ranging through $\mathcal{Y}$, form a set of special curves in $\mathbb{C}^3$ with the property that under projections to all coordinate hyperplanes they give a Zariski dense set of special sub-varieties (curves or points) in $\mathbb{C}^2$. For each $Y$ in $\mathcal{Y}$, let $N_Y: p_{<3}Y \to p_{<3}Y$ be the normalization map. As both $\tilde{Y}$ and $p_{<3}Y$ are quotients of $\mathbb{H}$ by a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, the morphism $N_Y$ is finite and locally free, and unramified at all points $z$ with $p_1(N_Yz) = x_1$ (note that $x_1$ is different from 0 and 1728). It follows that, for all $Y$ in $\mathcal{Y}$, the number of $x$ on $p_{<3}Y$ with $p_1(N_Yx) = x_1$ and $p_{2,3}N_Yx \in S$ is at least $1/r$ times the number of points $x$ on $p_{<3}Y$ with $p_1(N_Yx) = x_1$. The following lemma shows that this is not the case, and the proof of Theorems 1.2 and 1.3 is finished.

9.1 Lemma. Let $\mathcal{Y}$ be a set of special curves in $\mathbb{C}^3$ that map surjectively to $\mathbb{C}$ under projection to the first coordinate, and dense in some sub-variety of $\mathbb{C}^3$ (possibly equal to $\mathbb{C}^3$) that projects dominantly to all coordinate hyperplanes. For $Y$ in $\mathcal{Y}$, let $N_Y: \tilde{Y} \to Y$ be the normalization map. Let $x_1$ be a special point in $\mathbb{C}$ whose CM-field is not $\mathbb{Q}(i)$ or $\mathbb{Q}(\sqrt{-3})$, let $S \subset \mathbb{C}^2$ be a special point or curve, and let $r \geq 1$. Suppose that for each $Y$ in $\mathcal{Y}$ the number of $z$ in $\tilde{Y}$ with $p_1(N_Yz) = x_1$ and $p_{2,3}N_Yz \in S$ is at least $1/r$ times the number of $z$ in $\tilde{Y}$ with $p_1(N_Yz) = x_1$. Then one has a contradiction.

Proof. We assume all hypotheses of the lemma, and will arrive at a contradiction. For $Y$ in $\mathcal{Y}$, we let $I_Y$ be as before. Then we may suppose that there is a subset $I$ of $\{1, 2, 3\}$ such that $I_Y = I$ for all $Y$. By our assumptions, $I$ contains 1.
Let us suppose that \( I = \{1\} \). Then \( p_{2,3}Y \) is in \( S \) for each \( Y \) in \( \mathcal{Y} \), contradicting the assumption that the union of the \( p_{2,3}Y \) is dense in \( \mathbb{C}^2 \).

Let us then suppose that \( I \) has cardinality 2, say \( I = \{1, 2\} \) (if necessary, we renumber the last two coordinates). Then, for each \( Y \) in \( \mathcal{Y} \), \( p_3Y \) is a one element subset of \( \mathbb{C} \), and \( p_{2,3}Y = \mathbb{C} \times p_3Y \). It follows that \( S \) is a special curve, surjecting to \( \mathbb{C} \) under the projection \( p_2 \) (note that, with our notation, we have \( p_2 \circ p_{2,3} = p_3 \)). Then, for all \( Y \) in \( \mathcal{Y} \), the cardinality of \( S \cap p_{2,3}Y \) is bounded by the degree, \( m \) say, of \( p_2: S \to \mathbb{C} \). On the other hand, \( \widetilde{Y} \) is of the form \( Y_0(n) \), mapped to \( \mathbb{C}^3 \) by sending the isomorphism class of an isogeny \( f: E_1 \to E_2 \) with \( \ker(f) \) isomorphic to \( \mathbb{Z}/n\mathbb{Z} \) to \( (j(E_1), j(E_2), x_3) \) with \( \{x_3\} = p_3Y \). Let \( E_i \) be such that \( j(E_i) = x_i \). Then there are \( \psi(n) := |\mathbb{P}^1(\mathbb{Z}/n\mathbb{Z})| \) subgroups of \( E_i \) that are isomorphic to \( \mathbb{Z}/n\mathbb{Z} \). Two such subgroups of \( E_i \) give the same point on \( \widetilde{Y} \) if and only if they are equal (recall that \( \text{Aut}(E_i) = \{1, -1\} \)). Hence there are at least \( \psi(n) \) points on \( \widetilde{Y} \) with first coordinate \( x_1 \). Suppose now that two such subgroups both lead to isogenies \( f_1 \) and \( f_2 \) from \( E_1 \) to the same \( E_2 \). Let \( E' \) be the quotient of \( E_1 \) by \( \ker(f_1) \cap \ker(f_2) \), and \( f_1', f_2': E' \to E_2 \) the resulting cyclic isogenies, of degree \( d \), say. Then \( f_2' \circ (f_1')^Y \) is an endomorphism of \( E_2 \), with kernel isomorphic to \( \mathbb{Z}/d\mathbb{Z} \). This endomorphism, together with \( f_1 \), determines \( f_2 \). The number of endomorphisms of \( E_2 \) with kernel isomorphic to \( \mathbb{Z}/d\mathbb{Z} \) is at most \( 2^\pi(d) \), where, for an integer \( i \), \( \pi(i) \) is the number of (distinct) prime numbers dividing \( i \). As \( d \) divides \( n \), we see that the number of \( j(E_2) \) arising like this is at least \( \psi(n)/2^\pi(n) \). It follows that \( \psi(n)/2^\pi(n)r \leq m \). But then there are only finitely many possibilities for \( n \), contradicting that the union of the \( p_{1,2}Y \) is dense in \( \mathbb{C}^2 \).

Finally, let us suppose that \( I = \{1, 2, 3\} \). Let \( Y \) be in \( \mathcal{Y} \). Then, for some integers \( n_{1,2} \) and \( n_{1,3} \), \( p_{1,2}Y \) is the image of \( Y_0(n_{1,2}) \), and \( p_{1,3}Y \) is the image of \( Y_0(n_{1,3}) \). Considering the intersection of the kernels of the corresponding isogenies of degrees \( n_{1,2} \) and \( n_{1,3} \) shows that there are unique positive integers \( n_1, n_2 \) and \( n_3 \) such that \( \widetilde{Y} \) has the following moduli interpretation: \( \widetilde{Y} \) is the set of isomorphism classes of \( (E, H_1, H_2, H_3) \) with \( E \) a complex elliptic curve, \( H_i \) a subgroup of \( E \) isomorphic to \( \mathbb{Z}/n_i\mathbb{Z} \), such that for \( i \neq j \) one has \( H_i \cap H_j = \{0\} \). The map \( \widetilde{Y} \to \mathbb{C}^3 \) sends \( (E, H_1, H_2, H_3) \) to the point with coordinates \( j(E/H_i) \). In particular, we have \( n_{i,j} = n_in_j \). A good way to see what happens here is to consider, for each prime number \( p \), the tree of lattices in \( \mathbb{Q}_p^2 \) up to \( \mathbb{Q}_p^2 \), and to note that three points in a tree define a unique “center”: the point from which the paths to the three given points have disjoint edges. Also, one uses that the action of \( \text{PGL}_2(\mathbb{Z}_p) \) on the set of infinite non self-intersecting paths from the class of the standard lattice \( \mathbb{Z}_p^2 \) is 3-transitive, in order to see that the triplet \( (n_1, n_2, n_3) \) determines \( Y \).

For each \( (E, H_1, H_2) \) as above with \( j(E/H_1) = x_1 \) there is the same number of possibilities for \( H_3 \). The number of \( p_2(N_{Y,z}) \), with \( z \) on \( \widetilde{Y} \) such that \( p_1(N_{Y,z}) = x_1 \) is at least \( \psi(n_1n_2)/2^{\pi(n_1n_2)} \). As the set of \( n_1n_2 \), for \( Y \) varying in \( \mathcal{Y} \), is not finite, we may suppose that \( \psi(n_1n_2)/2^{\pi(n_1n_2)} > r \). It follows that \( p_1S \) is not a point. Similarly, \( p_2S \) is not a point. Hence \( S \) is a special curve that projects surjectively to both factors, say with degree \( m \). We will now show that this contradicts the fact that the set of \( n_2n_3 \), when \( Y \) varies, is not bounded. Indeed, there exists an \( (E, H_1, H_2) \) with \( j(E/H_1) = x_1 \), such that at least \( 1/r \) of the possibilities for \( H_3 \) give a \( (j(E/H_2), j(E/H_3)) \) in \( S \). This leads to at least \( \phi(n_3)/2^{\pi(n_3)} \) points \( (j(E/H_2), j(E/H_3)) \) in \( S \), with the same first coordinate. Hence we have \( \phi(n_3)/2^{\pi(n_3)} \leq r \), and \( n_3 \) is bounded as \( Y \) varies in \( \mathcal{Y} \). Similarly, \( n_2 \) is bounded. But then \( n_2n_3 \) is bounded, contradicting the fact that the union
9.2 Remark. The case \(|I| = 3\) in the proof of Theorems \([1,2]\) and \([13]\) above admits a simpler argument. In that case, \(Y\) is a special curve in \(\mathbb{C}^3\), hence of one of the two types treated in the proof of Lemma \([9,1]\). As \(Y\) is contained in \(Z\), the projection of \(Y\) to its image under a projection to \(\mathbb{C}^2\) has a degree that is bounded by the degree of the projection of \(Z\) to \(\mathbb{C}^2\). It follows that only finitely many \(Y\) of the first type (i.e., with all three projections surjective) are possible: if \(Y\) corresponds to \((n_1, n_2, n_3)\), then the degree of the projection from \(Y\) to its image under \(p_{i,j}\) is at least \(\phi(n_k)\), with \(\{1, 2, 3\} = \{i, j, k\}\). For \(Y\) with one constant projection, say the image of \(Y_0(a)\) in \(\mathbb{C}^2\), embedded in \(\mathbb{C}^3\) with third coordinate \(x_3\), it follows that \(a\) is bounded. This gives a contradiction with the fact that the set \(\mathcal{Y}\) is Zariski dense in the hypersurface \(Z\).

It is not hard to generalize the description given in the proof of Lemma \([9,1]\) of special curves in \(\mathbb{C}^3\) that project surjectively to \(\mathbb{C}\) under all (three) projections to the case of curves in \(\mathbb{C}^n\) with that property in \(\mathbb{C}^n\) with \(n\) arbitrary. One finds that the set of such curves is in bijection with the set \(\text{PGL}_2(\mathbb{Q}) \setminus (\text{PGL}_2(\mathbb{Q}) / \text{PGL}_2(\mathbb{Z}))^n\), which one can interpret as the set of relative positions of \(n\) lattices in \(\mathbb{Q}^2\); the bijection is induced by the elements \(g_i\) given in Definition \([\text{I.1}]\). For details see [3, §1.3]. It is also interesting to observe that there are such curves that are not contained in their image under Hecke correspondences \(T_m\) of small level (compared to their degree).

10 Erratum to [7].

There are two minor things to be dealt with, both of which do not invalidate the main result.

Serre has pointed out to me that it is used, in the proof of [7, Lemma 6.3], that for \(p\) prime and at least \(5\), \(\text{SL}_2(\mathbb{F}_p)\) has no proper subgroup of index at most \(p\). This is wrong, as for example \(\text{SL}_2(\mathbb{F}_{11})\) contains a subgroup isomorphic to \(A_5\), which has index 11. But, as Galois wrote in his “lettre testament”, it is true for all \(p > 11\). Hence the 5 in Theorem 6.1 should be replaced with 13. Another way to fix this is to consider only subgroups of index less than \(p\), as we do in this article.

When refereeing Yafaev’s thesis, Pink has observed that there is a gap at the end of the proof of [7, Theorem 6.1]. In the notation of that proof, there is an element \(g\) in \(\text{GL}_2(\mathbb{Q})\) such that the stabilizer in \(\text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R})\) of the irreducible complex analytic sub-variety of \(\mathbb{H} \times \mathbb{H}\) is the graph of the automorphism of \(\text{SL}_2(\mathbb{R})\) given by conjugation by \(g\). The problem is that in the last six lines of the proof it is assumed, without justification, that \(g\) has positive determinant. To repair this, we replace the last twelve lines of the proof, i.e., starting at “Let \(x\) be an element of \(X\)”, by what follows.

Let \(x = (x_1, x_2)\) be an element of \(X\) such that the two projections from \(X\) to \(\mathbb{H}\) induce isomorphisms on tangent spaces \(T_X(x) \rightarrow T_{\mathbb{H}}(x_1)\) and \(T_X(x) \rightarrow T_{\mathbb{H}}(x_2)\). These tangent spaces are naturally isomorphic to the quotients \(\text{Lie}(G_X) / \text{Lie}(G_{X,x})\), \(\text{Lie}(\text{SL}_2(\mathbb{R})) / \text{Lie}(\text{SL}_2(\mathbb{R}),x_1)\) and \(\text{Lie}(\text{SL}_2(\mathbb{R})) / \text{Lie}(\text{SL}_2(\mathbb{R}),x_2)\). Since \(X\) is a complex analytic sub-variety of \(\mathbb{H} \times \mathbb{H}\), the isomorphisms between the tangent spaces are compatible with the complex structures. Write \(x_2 = g'x_1\), with \(g'\) in \(\text{SL}_2(\mathbb{R})\). Then
$g'$ also induces an isomorphism $T_H(x_1) \to T_H(x_2)$ of one-dimensional complex vector spaces. It follows that conjugation by $g^{-1}g'$ on $SL_2(\mathbb{R})$ induces an automorphism of $\text{Lie}(SL_2(\mathbb{R}))/\text{Lie}(SL_2(\mathbb{R})_{x_1})$ that preserves orientation. A simple computation shows that this implies that $g^{-1}g'$ is in the connected component of identity of the normalizer in $GL_2(\mathbb{R})$ of $SL_2(\mathbb{R})_{x_1}$, hence that $g$ has positive determinant. (Note that $SL_2(\mathbb{R})_{x_1}$ is a conjugate of $SO_2(\mathbb{R})$, whose normalizer in $GL_2(\mathbb{R})$ is $\mathbb{R}^+O_2(\mathbb{R})$, a group with exactly two connected components.) Hence $g$ has positive determinant, and we have $x_2 = gx_1$. This means that $X = \{ (\tau, g\tau) \mid \tau \in \mathbb{H} \}$. We may replace $g$ by multiples $ag$ of it, with $a$ a non-zero rational number. So we can and do suppose that $g\mathbb{Z}^2$ is contained in $\mathbb{Z}^2$ and that $\mathbb{Z}^2/g\mathbb{Z}^2$ is cyclic, say of order $m$. It is now clear that $C$ is $Y_0(m)$.

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