Lower Complexity Bounds for Minimizing Regularized Functions

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Abstract

In this paper, we establish lower bounds for the oracle complexity of the first-order methods minimizing regularized convex functions. We consider the composite representation of the objective. The smooth part has Hölder continuous gradient of degree $\nu \in [0, 1]$ and is accessible by a black-box local oracle. The composite part is a power of a norm. We prove that the best possible rate for the first-order methods in the large-scale setting for Euclidean norms is of the order $O\left(k^{-p(1+3\nu)/(2(p-1-\nu))}\right)$ for the functional residual, where $k$ is the iteration counter and $p$ is the power of regularization. Our formulation covers several cases, including computation of the Cubically regularized Newton step by the first-order gradient methods, in which case the rate becomes $O(k^{-6})$. It can be achieved by the Fast Gradient Method. Thus, our result proves the latter rate to be optimal. We also discover lower complexity bounds for non-Euclidean norms.

Keywords: Lower Bounds, Convex Optimization, First-order Methods, Optimal Methods

1. Introduction

The modern complexity theory of Convex Optimization originated in the book of Nemirovski and Yudin (1983), where the first lower bounds were constructed for many different classes of optimization problems and algorithms. After this work, it became clear that the best possible rate of convergence for a given family of methods is fundamentally limited by a particular problem class.

For example, in the case of minimizing the convex functions with bounded subgradient, by black-box local methods, the best rate for decreasing the functional residual is $O(k^{-1/2})$, assuming the problem dimension is sufficiently large. This is the rate of the classical Subgradient Method (Shor (2012)), so the method is optimal. At the same time, for the differentiable convex functions with Lipschitz continuous gradient, the best rate is $O(k^{-2})$, which is much better. The optimal Fast Gradient Method with this rate of convergence was developed by Nesterov (1983).

During the last years, we started to see more and more examples of problem classes that significantly vary the standard picture of complexity theory. Thus, there were established lower complexity bounds for the functions with Hölder continuous gradient (w.r.t. different norms) by Guzmán and Nemirovski (2015), for the functions with second- and high-order derivatives being Lipschitz continuous by Agarwal and Hazan (2018) and Arjevani et al. (2019), for the relatively smooth functions by Dragomir et al. (2021). Different extensions of the complexity theory to the randomized algorithms were presented by Bubeck et al. (2019), Diakonikolas and Guzmán (2019), and Garg et al. (2021).

Meanwhile, the framework of composite optimization, that was proposed by Nesterov (2013), Beck and Teboulle (2009), provided us with much more flexibility in formulation of problem parameters. Indeed, in composite problems, we can add to the smooth part of the objective additional terms such as indicator of a convex set, or regularizers of a different kind. Consequently, smoothness characteristics of the objective (e.g. the Lipschitz constant) need not necessary be consistent with uniformly convex properties of the composite part.

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In this work, we investigate the complexity of the optimization problems with composite regularizer that is given by a power of a norm. While having a variety of examples for the problems of this type, it appears that the lower complexity bounds are not really covered in the literature.

Let us consider convex minimization problems with the following structure,

\[
\min_{x \in \mathbb{R}^n} \left\{ F(x) \overset{\text{def}}{=} f(x) + \frac{\sigma}{p} \|x\|^p \right\},
\]

with some parameters \( p \geq 2, \sigma > 0 \), and the standard Euclidean norm:

\[
\|x\| \overset{\text{def}}{=} \left( \sum_{i=1}^n (x^{(i)})^2 \right)^{1/2}.
\]

Function \( f \) is the main source of complexity of solving (1). We assume that its gradient is Hölder continuous, for some \( \nu \in [0, 1] \) and \( H_\nu > 0 \)

\[
\|\nabla f(x) - \nabla f(y)\| \leq H_\nu \|x - y\|^\nu, \quad \forall x, y \in \mathbb{R}^n.
\]

When \( p = 2 \), we obtain the classical \( \ell_2 \)-regularization by the squared Euclidean norm. Finding the minimum of (1) is equivalent to computing the value of the proximal operator of \( f \) (Moreau (1965)). We know that the problem is strongly convex. The rate of convergence becomes linear for the functions with Lipschitz continuous gradient (\( \nu = 1 \)), and \( O(k^{-1}) \) for bounded variation of gradient (\( \nu = 0 \)) (see, e.g. Nesterov (2018)). Thus, the cost of solving the regularized problem is much cheaper than minimizing \( f \) solely. It happens that these observations work for \( p > 2 \) as well.

For \( p = 3 \), minimization problem (1) appears to be a subproblem for computing one step of the Newton Method with Cubic Regularization (Nesterov and Polyak (2006)). The smooth part then is a convex second-order Taylor polynomial, which is a quadratic function,

\[
f(x) = \frac{1}{2} \langle Ax, x \rangle - \langle b, x \rangle, \quad A \succeq 0,
\]

and condition (2) is satisfied with \( \nu = 1 \) and constant \( H_1 = \lambda_{\max}(A) \). In the following years, the Cubic Newton Method received a substantial interest with the development of inexact, stochastic, and adaptive schemes (see the papers of Cartis et al. (2011); Grapiglia and Nesterov (2017); Doikov and Richtárik (2018); Cartis and Scheinberg (2018); Hanzely et al. (2020)). In all of these methods, it is required to solve the subproblem with our structure (1). The Fast Gradient Methods with restarts that have the rate of convergence \( O(k^{-3}) \) for this problem were constructed by Roulet and d’Aspremont (2017) and by Nesterov (2019) with applications to the Cubic Newton. In our paper we justify that the latter rate is optimal, i.e. the best that can be achieved for the considered class of functions. Note that in the methods of Grapiglia and Nesterov (2017, 2019), it is also required to minimize the quadratic function alongside the regularizer with arbitrary \( p \in [2, 3] \).

In a recent paper by Nesterov (2022), there were considered new second-order schemes based on regularization of degree \( p = 4 \). Then, the rate of the Fast Gradient Method for solving the corresponding subproblem is \( O(k^{-4}) \). We prove that this rate is optimal, by justifying the corresponding lower bound.

Finally, let us mention the framework of High-Order Proximal-Point Methods (Nesterov (2021a)) that became influential for the development of super-fast second-order schemes (see also the works of Kamzolov and Gasnikov (2020); Nesterov (2021b)). By generalizing the construction of the
classical Proximal-Point Algorithm (Rockafellar (1976)) onto arbitrary order \( p \geq 2 \), the methods assume that the subproblem of the form (1) can be solved somehow at each iteration. Thus, it becomes an important question to study the complexity of problems with arbitrary degree of regularization.

We are interested to analyse the worst-case behaviour of the first-order algorithms on such problem classes. Each optimization algorithm \( \mathcal{A} \) can be associated with a sequence of mappings

\[
\mathcal{A} = (A_0, A_1, A_2, \ldots).
\]

At each iteration \( k \geq 0 \), mapping \( A_k \) takes as input oracle information from the previous \( k \) points, and returns the next point \( x_{k+1} \). In other words, \( x_1 := A_0(\emptyset) \), and

\[
x_{k+1} := A_k(I(x_1), \ldots, I(x_k)), \quad k \geq 1.
\]

In this paper, we consider so called first-order local oracle of \( f \), that is

\[
I(x) = \{ f(x), \nabla f(x) \}.
\]

The parameters of the regularizer are directly available for the method.

When we provide a lower bound for the rate of convergence, for a fixed method we need to build an example of problem from our class such that the residual after the first \( k \) iterations is bounded from below:

\[
F(x_k) - F^* \geq R_k,
\]

by some quantity \( R_k \) that is called the risk. The inverse of it, i.e. the minimum number of iterations required to have an \( \varepsilon \)-precision for the solution is called the complexity:

\[
C_\varepsilon = \min \{ k : R_k < \varepsilon \}.
\]

In the following table, we list the new lower bounds that we prove in our work alongside the already known ones for the risk and complexity. The initial distance to the solution is denoted by \( D \). Numerical constants are hidden.

Euclidean norms

| Specification                         | Risk, \( R_k \)                                                                 | Complexity, \( C_\varepsilon \)                                               | Reference                      |
|--------------------------------------|--------------------------------------------------------------------------------|--------------------------------------------------------------------------------|--------------------------------|
| \( p > 1 + \nu \)                    | \( \left( \frac{H_\nu}{\sigma^{1+\nu/p}} k^{1+3\nu/p} \right)^{\frac{p}{p+1}} \) | \( \left( \frac{H_\nu}{\sigma^{1+\nu/p} \varepsilon^{p-1-\nu/p}} \right)^{\frac{2}{p+1}} \) | Our result: Theorem 4          |
| Lipschitz gradient (\( \nu = 1, p > 2 \)) | \( \left( \frac{H_1}{\sigma^{2/p} k^2} \right)^{\frac{p}{p+2}} \)          | \( \sqrt{\frac{H_1}{\sigma^{2/p} \varepsilon^{(p-2)/p}}} \)                    | \( \triangle \)                 |
| Bounded gradient (\( \nu = 0, p \geq 2 \)) | \( \left( \frac{H_0}{\sigma^{1/p} k^{1/2}} \right)^{\frac{p}{p+2}} \)     | \( \left( \frac{H_0}{\sigma^{1/p} \varepsilon^{(p-1)/p}} \right)^2 \)          | Juditsky and Nesterov (2014)   |
| Strongly convex functions with Lipschitz gradient (\( \nu = 1, p = 2 \)) | \( \sigma D^2 \exp \left( -k \sqrt{\frac{p}{\pi \tau}} \right) \)               | \( \sqrt{\frac{H_\nu}{\sigma^{2/\pi} \varepsilon^{\frac{2p^2}{\pi}}} \log \left( \frac{\sigma D^2}{\varepsilon} \right)} \) | Nesterov (2018)                |
| No composite part (\( \sigma = 0 \)) | \( \frac{H_\nu D^{1+\nu} k^{(3\nu+1)/2}}{\varepsilon} \)                  | \( \left( \frac{H_\nu D^{1+\nu}}{\varepsilon} \right)^{\frac{2}{p+1}} \)     | Guzmán and Nemirovski (2015)   |
Note that the complexity lower bounds of the same order for similar problem classes appeared for the first time in the work of Nemirovski and Nesterov (1985) (see their equation (1.21)). However, it seems that the corresponding proof was never published for a wide audience.

Lower bounds for the cubic subproblem with a (possibly nonconvex) quadratic smooth part were constructed by Carmon and Duchi (2018). Comparing with that work, we were interested in dependence of the complexity on the free parameter $\sigma$ in our sublinear rates. Thus, we admit arbitrary choice of the regularization constant.

Besides, one significant novelty of our analysis is the use of the composite formulation for (1) that provides a complete flexibility for the regularizer and smoothness parameters. Exploring this possibility, we also construct lower bounds for the regularization by non-Euclidean norms (see the table below). The employing of arbitrary norms as regularizers for the methods with Taylor’s polynomials of different order was considered in a recent paper by Gratton and Toint (2021). One step of their second-order algorithm requires to have an inexact solution to the problem of our form with $p = 3$.

Regularization by $\| \cdot \|_q$-norm, $q \geq 1$

| Specification | Risk, $R_k$ | Complexity, $C_\varepsilon$ | Reference |
|---------------|-------------|-----------------------------|-----------|
| $p > 1 + \nu$ | $\left( \frac{H_{\nu}}{\sigma^{1/p\kappa^{(2+q)/q}}} \right)^{\frac{p}{p-1}}$ | $\left( \frac{H_{\nu}}{\sigma^{1/p\kappa^{(2+q)/q}}} \right)^{\frac{q}{q-1}}$ | Our result: Theorem 5 |
| Lipschitz gradient ($\nu = 1, p > 2$) | $\left( \frac{H_1}{\sigma^{2/p\kappa(2+q)/q}} \right)^{\frac{p}{p-2}}$ | $\left( \frac{H_1}{\sigma^{2/p\kappa(p-2)/p}} \right)^{\frac{p}{p-2}}$ | △ |
| Bounded gradient ($\nu = 0, p \geq 2$) | $\left( \frac{H_0}{\sigma^{1/p\kappa^{1/q}}} \right)^{\frac{p}{p-1}}$ | $\left( \frac{H_0}{\sigma^{1/p\kappa(p-1)/p}} \right)^{\frac{q}{q-1}}$ | △ |

It appears that for $\ell_1$-norm we obtain the best lower bounds for the rate of convergence. However, it is not clear how far these estimates from the upper bounds that can be achieved by some optimization schemes. At the same time, the rates for the Euclidean case are tight. They can be reached by the Fast Gradient Methods with restarts (see, e.g. the monograph of d’Aspremont et al. (2021)).

The rest of the paper is organized as follows. In Section 2 we review the basic properties of the local smoothing. We use them in our analysis. Section 3 contains the proof of our lower bounds for the Euclidean norm, and in Section 4 we discuss generalizations to non-Euclidean norms. Section 5 contains some final remarks.

2. Local Smoothing

In this section, we summarise well known facts about the local smoothing, that is the core component of the worst-case example.

Let $g : \mathbb{R}^n \to \mathbb{R}$ be a convex 1-Lipschitz functions, not-necessary differentiable. Thus

$$g(x) - g(y) \leq \|x - y\|, \quad \forall x, y \in \mathbb{R}^n.$$
For a fixed parameter $\mu > 0$, we denote the local smoothing of $g$ by

$$S_\mu[g](x) \overset{\text{def}}{=} \min_{y \in \mathbb{R}^n} \left\{ g(y) + \frac{\mu}{2} \| y - x \|^2 \right\}, \quad x \in \mathbb{R}^n.$$  

Let us review the basic properties of the function $G(x) := S_\mu[g](x)$. The proofs can be found in the Appendix.

**Lemma 1** Function $G$ is convex and differentiable. It is $1$-Lipschitz, and its gradient is $\mu$-Lipschitz continuous. Thus, for all $x, y \in \mathbb{R}^n$:

$$G(x) - G(y) \leq \| x - y \|, \quad (3)$$

and

$$\| \nabla G(x) - \nabla G(y) \| \leq \mu \| x - y \|. \quad (4)$$

As a direct consequence of this lemma, we can conclude that $G$ has a Hölder continuous gradient for any $\nu \in [0, 1]$ with constant

$$H_\nu(G) = 2^{1-\nu} \mu^\nu. \quad (5)$$

Indeed, (3) implies that the gradients are bounded: $\| \nabla G(x) \| \leq 1$, $\forall x \in \mathbb{R}^n$. Hence,

$$\| \nabla G(x) - \nabla G(y) \| = \| \nabla G(x) - \nabla G(y) \|^\nu \cdot \| \nabla G(x) - \nabla G(y) \|^{1-\nu} \leq \mu^\nu \| x - y \|^{\nu} \cdot 2^{1-\nu} = H_\nu(G) \| x - y \|^{\nu}, \quad \forall x, y \in \mathbb{R}^n.$$

The following lemma shows that $G$ is also close to $g$.

**Lemma 2** For any $x \in \mathbb{R}^n$, it holds

$$g(x) \geq G(x) \geq g(x) - \frac{1}{2\mu}. \quad (6)$$

We see that parameter $\mu$ provides a trade-off between the quality of approximation and the level of smoothness for the new function. The effect of varying this parameter is shown in Figure 1.

![Figure 1: Local smoothing of a piecewise linear function.](image-url)
Finally, we state the following important property that $G$ depends on $g$ in a local way: the value and the gradient of $G$ at $x$ depend only on the restriction $g|_{U_{x,\mu}}$ of $g$ onto the ball

$$U_{x,\mu} \overset{\text{def}}{=} \left\{ y \in \mathbb{R}^n : \|y - x\| \leq \frac{2}{\mu} \right\}.$$ 

More formally, the following proposition holds.

**Lemma 3** Let $z := \arg\min_{y \in \mathbb{R}^n} \left\{ g(y) + \frac{\mu}{2} \|y - x\|^2 \right\}$ for some fixed $x \in \mathbb{R}^n$ and $\mu > 0$. Then

$$x - \frac{1}{\mu} \nabla G(x) \equiv z = \arg\min_{y \in \text{int } U_{x,\mu}} \left\{ g(y) + \frac{\mu}{2} \|y - x\|^2 \right\}.$$ 

Therefore, in order to compute the oracle information $\{G(x), \nabla G(x)\}$ at some given point $x$, it is enough to have an access only to the neighbourhood of function $g$ around this point. This fact is crucial for proving the lower bound.

### 3. Lower Complexity Bound

Now, employing the construction introduced in Guzmán and Nemirovski (2015), we are going to provide a resisting oracle strategy to establish the lower complexity bound for problem (1).

Let us fix arbitrary $\nu \in [0, 1]$ such that $\nu < p - 1$, and some constant $H_\nu > 0$. We prove the following result.

**Theorem 4** For any $T$-step algorithm with $T \leq n$, there exists a convex differentiable function $f(\cdot)$ whose gradient is Hölder continuous of degree $\nu$ with constant $H_\nu$, such that for problem (1), we have

$$F(x_T) - F^* \geq \left( \frac{p - 1}{p} \right) \left( \frac{1}{2} \right) \left( \frac{2p - 1}{p - 1 - \nu} \right) \left( \frac{H_\nu}{\sigma^{1+\nu}} \right) \frac{1}{\nu^{1-\nu}}.$$ 

The size of the solution to that problem is bounded as

$$\|x^*\| \leq \left( 3(p - 1)2^{p-3} \right)^{\frac{1}{p}} \left( \frac{1}{2} \right) \left( \frac{p - 1}{4p} \right)^{\frac{1}{p - 1 - \nu}} \left( \frac{H_\nu}{\sigma^{1+\nu}} \right)^{\frac{1}{\nu^{1-\nu}}}.$$ 

**Proof**

Let us fix some positive $\delta, \mu, \text{ and } \beta$.

Resisting oracle chooses a set of numbers $\xi_1, \ldots, \xi_T \in \{-1, 1\}$ and a permutation $k \mapsto \alpha(k) \in \{1, 2, \ldots, T\}$.

Having these parameters, we consider the family of convex functions, $x \in \mathbb{R}^n$:

$$g_t(x) = \max_{1 \leq k \leq T} \left[ \xi_k \langle e_{\alpha(k)}, x \rangle - (k - 1)\delta \right], \quad 1 \leq t \leq T,$$

where $e_i$ is the standard basis vector in $\mathbb{R}^n$ (recall that we assume that $n \geq T$). It is clear that all $g_t(\cdot)$ are Lipschitz continuous with constant 1.
Then, we are going to take as our final objective the function

\[
    f(x) := \beta S_\mu[g_T](x)
\]

Let us estimate the minimum \( F^* \) and the size of the solution \( \|x^*\| \). Note that we have, for any \( x \in \mathbb{R}^n \)

\[
    F(x) = \beta S_\mu[g_T](x) + \frac{\sigma}{p} \|x\|^p \leq \beta g_T(x) + \frac{\sigma}{p} \|x\|^p \tag{6}
\]

At the same time, we can also bound \( F \) by \( H \) from below, as follows

\[
    F(x) \geq \beta g_T(x) + \frac{\sigma}{p} \|x\|^p - \frac{\beta}{2\mu} \geq H(x) - \frac{\beta}{2\mu} - \beta(T - 1)\delta. \tag{10}
\]

We can compute the minimum of \( H(\cdot) \) by using the symmetry within the problem,

\[
    F^* \leq H^* = \min_{x \in \mathbb{R}^n} H(x) = \min_{\gamma > 0} \left\{ -\beta \gamma + \frac{\sigma T^2}{p} \gamma^p \right\} \tag{11}
\]

From the uniform convexity of the regularizer (see, e.g. Lemma 2.5 in Doikov and Nesterov (2021)) we can bound the size of the solution,

\[
    \left(\frac{1}{2}\right)^{p-2} \frac{\sigma}{p} \|x^*\|^p \leq F(0) - F^* \leq H(0) - H^* + \frac{\beta}{2\mu} + \beta(T - 1)\delta \tag{12}
\]

Now, let us present a strategy for choosing \( \xi_k \) and \( \alpha(k) \).

- At first step, the algorithm returns point \( x_1 \), which does not depend on the objective. Let us pick

\[
    \alpha(1) \in \text{Argmax}_{1 \leq k \leq T} \|\langle e_k, x_1 \rangle\|.
\]

In other words, \( \alpha(1) \) is an index of a maximal element (in absolute value) among first \( T \) coordinates of \( x_1 \). Then, we specify \( \xi_1 \in \{-1, 1\} \) in a way that

\[
    \xi_1 \langle e_{\alpha(1)}, x_1 \rangle = \|\langle e_{\alpha(1)}, x_1 \rangle\|,
\]

hence \( \xi_1 = \text{sign}(\langle e_{\alpha(1)}, x_1 \rangle) \). So,

\[
    g_1(x) = \xi_1 \langle e_{\alpha(1)}, x \rangle.
\]
At step $2 \leq t \leq T$, assume that we have built function $g_{t-1}(x)$. Let $x_t$ be the point of the trajectory of $\mathcal{A}$ at iteration $t-1$, applied to the current objective:

$$x_t := A_{t-1}(\mathcal{I}_{t-1}(x_1), \ldots, \mathcal{I}_{t-1}(x_{t-1})),$$

where $\mathcal{I}_{t-1}(x) \overset{\text{def}}{=} \{\beta \mathcal{S}_\mu[g_{t-1}](x), \beta \nabla \mathcal{S}_\mu[g_{t-1}](x)\}$. Let us choose as $\alpha(t)$ the index of a maximal element of $x_t$ (in absolute value) among first $T$ coordinates, except $\alpha(t-1)$. Thus,

$$\alpha(t) \in \arg\max_{1 \leq k \leq t, k \notin \{\alpha(1), \ldots, \alpha(t-1)\}} \langle e_k, x_t \rangle.$$

Then, we specify $\xi_t \in \{-1, 1\}$ such that

$$\xi_t \langle e_{\alpha(t)}, x_t \rangle = |\langle e_{\alpha(t)}, x_t \rangle|,$$

hence $\xi_t = \text{sign} (\langle e_{\alpha(t)}, x_t \rangle)$, and thus we obtain the next $g_t(x)$.

We need to prove that for any $2 \leq t \leq T$, function $g_t(\cdot)$ coincides with $g_s(\cdot)$, $1 \leq s < t$, in the ball of radius $\frac{2}{\mu}$ around $x_s$:

$$g_t(x) = g_s(x), \quad x \in U_{x_s, \mu} \overset{\text{def}}{=} \left\{x \in \mathbb{R}^n : \|x - x_s\| \leq \frac{2}{\mu}\right\},$$

which means that $g_t(\cdot)$ is indistinguishable from $g_s(\cdot)$ during the first $s$ steps, and hence by Lemma 3 their smoothings $\mathcal{S}_\mu[g_t]$ and $\mathcal{S}_\mu[g_s]$ are also indistinguishable for the algorithm.

Indeed,

$$g_t(x) = \max \left\{g_s(x), \max_{s < k \leq t} \left[\xi_k \langle e_{\alpha(k)}, x \rangle - (k - 1)\delta\right]\right\}.$$ 

By the definition of $g_s(x)$, we have

$$g_s(x_s) \geq \left[\xi_k \langle e_{\alpha(k)}, x_s \rangle - (k - 1)\delta\right] + \delta, \quad s < k \leq t.$$

Hence, due to the Lipschitz continuity, it holds for all $x$ such that $\|x - x_s\| \leq \frac{\delta}{2}$:

$$g_s(x) \geq \left[\xi_k \langle e_{\alpha(k)}, x \rangle - (k - 1)\delta\right], \quad s < k \leq t.$$

Choosing

$$\delta := \frac{4}{\mu}$$

we conclude that (13) is true.

Thus, we established correctness for the construction of the resisting oracle. Namely, it holds, for all $s \leq t \leq T$:

$$\mathcal{I}_t(x_s) = \mathcal{I}_T(x_s) \equiv \mathcal{I}(x_s),$$

so the oracles $\mathcal{I}_t(\cdot)$ and $\mathcal{I}(\cdot)$ are identical along the trajectory of the method.

It remains to bound from below the residual in the function value for the last iteration of the method. From simple observations, we notice that

$$g_T(x_T) \geq |x_T^{(\alpha(T))}| - (T - 1)\delta \geq -(T - 1)\delta \overset{(14)}{=} -(T - 1)\frac{4}{\mu}. \quad (15)$$
Hence, for the last function value, we have
\[ F(x_T) \geq f(x_T) = \beta S_{\mu}[g_T](x_T) \geq -(T-1) \frac{4\beta}{\mu} - \frac{\beta}{2\mu} \geq -\frac{4\beta T}{\mu}. \] (16)

Therefore, the residual can be bounded as follows,
\[ F(x_T) - F^* \geq \left( \frac{p-1}{p} \right) \left( \frac{\beta p}{\sigma^{2/p}} \right)^{\frac{1}{p-1}} - \frac{4\beta T}{\mu} = \frac{p-1}{2p} \left( \frac{\beta p}{\sigma^{2/p}} \right)^{\frac{1}{p-1}}, \] (17)
where the last equality holds by the following choice of the smoothing parameter:
\[ \mu := \frac{8p}{p-1} \left( \frac{\sigma^{2/p}}{\beta} \right)^{\frac{1}{p-1}}. \] (18)

The only parameter which remains to determine is \( \beta \). We know that \( f \) must have Hölder continuous gradient with constant
\[ H_\nu = \beta 2^{1-\nu} \mu^{\nu} \] (5) (18)

Therefore, we get the following value for the last free parameter:
\[ \beta := \left( \frac{1}{2} \left( \frac{p-1}{4p} \right)^{\nu} H_\nu \right)^{\frac{1}{p-1}} \left( \sigma T^{(3p-2)/2} / \beta \right)^{\frac{1}{p-1}} \beta^{\frac{p-1-\nu}{p-1}}. \] (19)

Substituting this value into (17), we obtain the final lower bound on the convergence rate (7).

The bound (8) for the size of the solution follows immediately from
\[ \|x^*\| \leq \left( \frac{\|x\|^2}{\sigma} \left( \frac{p-1}{p} \cdot \left( \frac{\beta p}{\sigma^{2/p}} \right)^{\frac{1}{p-1}} + \frac{\beta}{2\mu} + \beta(T-1)\delta \right) \right)^{1/p} \] (12)

\[ \leq \left[ \frac{\|x\|^2}{\sigma} \left( \frac{p-1}{p} \cdot \left( \frac{\beta p}{\sigma^{2/p}} \right)^{\frac{1}{p-1}} + \frac{4\beta T}{\mu} \right) \right]^{1/p} \] (14)

\[ \leq \left[ \frac{\|x\|^2}{\sigma} \left( \frac{p-1}{p} \cdot \left( \frac{\beta p}{\sigma^{2/p}} \right)^{\frac{1}{p-1}} + \frac{4\beta T}{\mu} \right) \right]^{1/p} \] (18)

by substituting the expression for \( \beta \) (19).

Let us consider the case \( \nu = 1 \). Then, Theorem 4 states that for any \( T \)-step algorithm there is an instance of problem with bounded solution,
\[ \|x^*\| \leq O\left( \left( \frac{H_1}{\sigma T^2} \right)^p \right), \] (20)

such that the residual is bounded from below,
\[ F(x_T) - F^* \geq O\left( \left( \frac{H_1}{\sigma T^2} \right)^{\frac{p}{p-2}} \right). \] (21)

At the same time, the standard rate of the composite Fast Gradient Method Nesterov (2013) starting from \( x_1 = 0 \) is
\[ F(x_T) - F^* \leq O\left( \frac{H_1 \|x^*\|^2}{T^2} \right) \leq O\left( \left( \frac{H_1}{\sigma T^2} \right)^{\frac{p}{p-2}} \right), \]

that matches the lower bound (21).
4. Regularization with Non-Euclidean Norms

In this section, we study a generalization of problem (1) to non-Euclidean norms. For arbitrary \( q \geq 1 \), we denote by \( \| \cdot \|_q \) the standard \( \ell_q \)-norm:

\[
\| x \|_q \overset{\text{def}}{=} \left( \sum_{i=1}^{n} \left| x^{(i)} \right|^q \right)^{1/q}.
\]

Now, let us consider the problem:

\[
\min_{x \in \mathbb{R}^n} \left\{ F(x) \overset{\text{def}}{=} f(x) + \frac{\sigma}{p} \| x \|^p_q \right\}, (22)
\]

for some \( p \geq 2, \sigma > 0 \), and convex differentiable \( f \) that has Hölder continuous gradient (w.r.t. the standard Euclidean norm\(^1\)). As before, we assume that \( \nu < p - 1 \). We can prove the following result.

**Theorem 5** For any \( T \)-step algorithm with \( T \leq n \), there exists a convex differentiable function \( f(\cdot) \) whose gradient is Hölder continuous of degree \( \nu \) with constant \( H_\nu \), such that for problem (22), we have

\[
F(x_T) - F^* \geq \left( \frac{p-1}{p} \right)^{(p-1)(1+\nu)} \left( \frac{1}{2} \right)^{(2p-1)(1+\nu)} \left( \frac{H_\nu}{\sigma^{1+\nu}} \right) \frac{p}{p-1-\nu}. (23)
\]

**Proof** The proof is very similar to that one of Theorem 4. We use the same strategy for the resisting oracle, and the candidate for the smooth part is \( f(x) := \beta S_\mu[\gamma](x) \), with \( \delta := \frac{4}{\mu} \) (see definition of \( g_T(\cdot) \) in the previous theorem).

For the minimum, we have the following estimate,

\[
F^* = \min_{x \in \mathbb{R}^n} F(x) \leq \min_{x \in \mathbb{R}^n} \left\{ \max_{1 \leq k \leq T} \beta \xi_k(e_\alpha(k), x) + \frac{\sigma}{p} \| x \|^p_q \right\}
\]

\[
= \min_{\gamma > 0} \left\{ -\beta \gamma + \frac{\sigma T^p}{p} \gamma^p \right\} = -\frac{p-1}{p} \left( \frac{\beta p}{\sigma T^p/q} \right)^{1/p-1}.
\]

Thus, we get the bound for the residual,

\[
F(x_T) - F^* \geq \frac{p-1}{p} \left( \frac{\beta p}{\sigma T^p/q} \right)^{1/p-1} - 4\beta T \mu = \frac{p-1}{2p} \left( \frac{\beta p}{\sigma T^p/q} \right)^{1/p-1}, (24)
\]

where we made the following choice of the smoothing parameter to balance the two terms,

\[
\mu := \frac{8p}{p-1} \left( \frac{\sigma^{p+q-p-q}}{\beta} \right)^{1/p-1}
\]

Substituting this expression into the equation for the constant of Hölder continuity, \( H_\nu = \beta 2^{1-\nu} \mu^{\nu} \), we get the value for the last unknown parameter,

\[
\beta := \left( \frac{1}{2} \left( \frac{p-1}{4p} \right)^{\nu} H_\nu \right)^{\frac{p-1}{p-1-\nu}} \left( \sigma T^p \frac{p+q-p-q}{q} \right)^{\nu} \left( \frac{p}{p-1-\nu} \right)
\]

Plugging it into (24) completes the proof. \( \square \)

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\(^1\) Hence, we use different norms for the regularizer and for defining the smoothness properties of \( f \). Generalizations to non-Euclidean norms for \( f \) are also possible, see (Guzmán and Nemirovski (2015)).
5. Discussion

In this paper, we proved new lower bounds on the oracle complexity for minimizing regularized convex functions by first-order methods. As a particular case, we established that the best possible rate of convergence for minimizing a function with Lipschitz continuous gradient regularized by cube of the Euclidean norm is $O(k^{-6})$, and for the fourth power of the norm we have $O(k^{-4})$. We know that the Fast Gradient Method achieves these rates. It remains to be an interesting open question: whether we can construct the lower bounds with a quadratic function for the smooth part.

Another interesting observation that we discovered is a change in the rate of convergence for the regularizers with different norms. Thus, by using the third power of $\ell_1$-norm, we obtain the lower bound of order $O(k^{-9})$. However, it is not clear how far this estimate from the upper bound that can be reached by some optimization method. We keep these questions for further investigation. At the same time, taking a power of $\ell_1$-norm might preserve its sparsity properties (see Figure 2), which is desirable for applications.

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Appendix A. Properties of Local Smoothing

In this section, we provide the proofs for basic properties of local smoothing (Section 2).

For any fixed \( x \in \mathbb{R}^n \), let us denote by \( z_x \) the solution to the smoothing problem, i.e.
\[
z_x \overset{\text{def}}{=} \arg\min_{y \in \mathbb{R}^n} \left\{ g(y) + \frac{\mu}{2} \|y - x\|^2 \right\}.
\]

Hence, \( z_x \) satisfies the following optimality condition (see, e.g. Theorem 3.1.23 in Nesterov (2018)),
\[
\langle \mu (z_x - x), y - z_x \rangle + g(y) \geq g(z_x), \quad \forall y \in \mathbb{R}^n.
\]
(25)

In other words, \( g'(z_x) \overset{\text{def}}{=} -\mu (z_x - x) \in \partial g(z_x) \).

A.1. Proof of Lemma 1

We know a simple formula for the gradient of \( G \), that is
\[
\nabla G(x) = -\mu (z_x - x)
\]
(see, e.g. Proposition 12.30 in Bauschke et al. (2011)). Therefore, since \( g \) is \( 1 \)-Lipschitz, we have
\[
\|\nabla G(x)\| = \|g'(z_x)\| \leq 1, \quad \forall x \in \mathbb{R}^n,
\]
and hence \( G \) is also \( 1 \)-Lipschitz.

Now, let us fix two arbitrary points \( x, y \in \mathbb{R}^n \). Then,
\[
0 \leq \langle g'(z_x) - g'(z_y), z_x - z_y \rangle = \langle \nabla G(x) - \nabla G(y), z_x - z_y \rangle
\]
\[
= \langle \nabla G(x) - \nabla G(y), x - \frac{1}{\mu} \nabla G(x) - y + \frac{1}{\mu} \nabla G(y) \rangle
\]
\[
= -\frac{1}{\mu} \|\nabla G(x) - \nabla G(y)\|^2 + \langle \nabla G(x) - \nabla G(y), x - y \rangle.
\]
Applying the Cauchy-Schwartz inequality for the second term completes the proof. \( \Box \)

A.2. Proof of Lemma 2

Indeed, by the definition of smoothing, we have for any \( y \in \mathbb{R}^n \),
\[
G(x) \leq g(y) + \frac{\mu}{2} \|y - x\|^2.
\]
Substituting \( y := x \), we obtain \( G(x) \leq g(x) \).

On the other hand, by using that \( g \) is \( 1 \)-Lipschitz, we conclude
\[
G(x) \overset{\text{def}}{=} \min_{y \in \mathbb{R}^n} \left\{ g(y) + \frac{\mu}{2} \|y - x\|^2 \right\}
\]
\[
\geq g(x) + \min_{y \in \mathbb{R}^n} \left\{ -\|y - x\| + \frac{\mu}{2} \|y - x\|^2 \right\}
\]
\[
= g(x) - \frac{1}{2\mu}.
\]
\( \Box \)
A.3. Proof of Lemma 3

Let us denote the objective of smoothing operator by

\[ h(y) \overset{\text{def}}{=} g(y) + \frac{\mu}{2} \|y - x\|^2. \]

For any point \( y \) outside interior of the ball, we have \( \|y - x\| \geq \frac{2}{\mu} \), and hence

\[
\begin{align*}
    h(y) &\geq g(x) - \|y - x\| + \frac{\mu}{2} \|y - x\|^2 \\
    &= g(x) + \|y - x\| \cdot \left( \frac{\mu}{2} \|y - x\| - 1 \right) \\
    &\geq g(x) = h(x).
\end{align*}
\]

So, the value of \( h(\cdot) \) outside the interior is always greater than or equal to the value at the center of the ball. Therefore, due to strong convexity, the minimum is in the interior. \( \square \)