New inflationary exact solution from Lie symmetries

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For the inflaton field we determine a new exact solution by using the Lie symmetry analysis. Specifically, we construct a second-order differential master equation for arbitrary scalar field potential by assuming that the spectral index for the density perturbations $n_s$ and the scalar to tensor ratio $r$ are related as $n_s - 1 = h(r)$. Function $h(r)$ is classified according to the admitted Lie symmetries for the master equation. The possible admitted Lie symmetries form the $A_2$, $A_{3,2}$, $A_{3,3}$ and $\text{sl}(3,R)$ Lie algebras. The new inflationary solution is recovered by the Lie symmetries of the $A_{3,3}$ algebra. Scalar field potential is derived explicitly, while we compare the resulting spectral indices with the observations.

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1. INTRODUCTION

Inflation is the main mechanism to explain the homogeneity and isotropy of the observable universe at present time. During the inflationary period, the universe was dominated by the inflaton. The inflaton drove the dynamics such that to provide the expansion \cite{1}. Nevertheless, the inflationary models are mainly defined on homogeneous spacetimes, or on background spaces with small inhomogeneities \cite{2,3}.

In \cite{4} it was found that the presence of a positive cosmological constant in Bianchi cosmologies leads to expanding Bianchi spacetimes, evolving toward an rapidly expanding de Sitter universe. That was the first result to support the cosmic “no-hair” conjecture \cite{5,6}.

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The conjecture states that all expanding universes with a positive cosmological constant admit as asymptotic expanding solution. The necessity of the rapid expansion is that it provides a rapid expansion for the size of the universe such that the latter effectively loses its memory on the initial conditions, which means that the rapid expansion solves the “flatness”, “horizon” and monopole problem [7, 8].

Because inflation occurred in a finite time period during the cosmological evolution, instead of the cosmological constant, the inflaton is assumed to be described by a scalar field, which provides a dynamical behavior for the evolution of the field equations. In the single scalar field inflationary models the expansion appears when the scalar field potential dominates. In these models, the inflationary period is determined explicitly by the nature of the scalar field potential. In the literature, there is a plethora of scalar field potentials which have been proposed the last decades, see for instance [9–23].

Moreover, single scalar field gravitational models attribute the additional degrees of freedom provided by modified theories of gravity, such in the quadratic gravity [24]. The quadratic gravity is an geometric approach for the construction of the scalar field potential, through a conformal transformation with the use of a Lagrange multiplier [25]. The quadratic gravity, is also known as Starobinsky model for inflation and it is the model which is mainly supported by the recent cosmological observations [26]. There is a plethora of inflationary models which have been proposed the last years, multifield inflationary models or inflationary models which follow from modifications of the Einstein-Hilbert Action Integral, see for instance [27–37]. A catalogue with the viable inflationary models published in [38], while the updated version published later in [39] presents the viable inflationary models after the release of Planck 2013 data.

The field equations for the inflaton model are that of a minimally coupled scalar field in the context of General Relativity with a spatially flat Friedmann–Lemaître–Robertson–Walker (FLRW) as a background space. The field equations are nonlinear differential equations of second-order and form a singular Lagrangian system with dynamical variables the scalar factor of the metric tensor. The existence of exact solutions for the field equations depends on the functional form of the scalar field potential [40, 41] which drives the scalar field dynamics. In [42, 43], the scalar field potential has been constraint such that the field equations to admit conservation laws. The existence of the conservation laws indicates the Liouville integrability for the field equations such that exact and analytic solutions to be
determined. However, an innovative approach for the determination of analytic solutions in scalar field cosmology applied in [44]. Because the field equations form a singular dynamical system, there exist infinity number of nonlocal conservation laws, which can be used to reduce the order for the field equations. Indeed, with the use of the nonlocal conservation laws, the generic algebraic solution has been found, for arbitrary potential function. The results of [44] applied for the derivation of new inflationary solutions in [22], while this approach applied for the reconstruction of the inflationary potential in [23].

Specifically, in [23] it was assumed that spectral index for the density perturbations, n_s, and the tensor-to-scalar ratio, r, are related by a function such that n_s − 1 = h (r). where h (r) is either constant, linear or quadratic function in r. By applying the results of [44] in each case a master differential equation has been defined which gives the resulting inflationary potential. The resulting inflationary solutions fits the cosmological observations, while the closed-form expressions for the scalar field potential were found. In this piece of work, we focus on the classification of function h (r) according to the admitted Lie symmetries for the master equations. This geometric selection rule is inspired by the results of the applications of the symmetry analysis in [42]. The classification scheme that we follow, it was established by Ovsiannikov on the classification of the unknown function for a nonlinear Schrödinger equation [45]. A similar reconstruction approach for the slow-roll inflation can be found in the series of studies [46, 47] where the scalar-to-tensor ratio has been assumed to be a specific function with independent variable the number of e-folds. The corresponding scalar-tensor theory and the resulting $f (R)$-theory determined.

The plan of the paper is as follows.

In Section 2 we define the gravitational model for a single scalar field inflationary theory and we present Einstein’s field equations. Moreover, we present the generic algebraic solution for arbitrary potential found in [44], and we define the master equation of our analysis. The Lie symmetry analysis for the master equation of our consideration is performed in Section 3. In Section 4 we present the new inflationary exact solution as also we calculate the spectral indices n_s and r. Finally, in Section 5 we draw our conclusions.
2. THE INFLATON FIELD

Consider the Action Integral of Einstein-Hilbert Action with a scalar field minimally coupled to gravity, that is,

\[
S = \int dx^4 \sqrt{-g} \left[ R - \frac{1}{2} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} + V(\phi) \right]
\]

(1)

in which \( R \) is the Ricci-scalar for the background space with metric tensor \( g_{\mu\nu}(x^\kappa) \), \( \phi(x^\kappa) \) is the scalar field and \( V(\phi(x^\kappa)) \) is the scalar field potential. Variation with respect to the metric tensor provides the Einstein-field equations

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T^{(\phi)}_{\mu\nu}
\]

(2)

where \( R_{\mu\nu} \) is the Ricci tensor, and \( T^{(\phi)}_{\mu\nu} \) is the energy-momentum tensor which corresponds to the scalar field. \( T^{(\phi)}_{\mu\nu} \) is defined as

\[
T^{(\phi)}_{\mu\nu} = \phi_{,\mu} \phi_{,\nu} - g_{\mu\nu} \left( \frac{1}{2} \phi_{,\kappa} \phi_{,\kappa} - V(\phi) \right).
\]

(3)

Furthermore, variation with respect to the scalar field in the Action Integral, provides the equation of motion for the scalar field, that is, the Klein-Gordon equation \( T^{(\phi)\mu\nu}_{\ ;\nu} = 0 \), that is,

\[
g^{\mu\nu} \phi_{;\mu\nu} - V(\phi) = 0.
\]

(4)

For the background space we consider a spatially flat FLRW universe with line element

\[
ds^2 = -dt^2 + a^2(t) \left( dx^2 + dy^2 + dz^2 \right).
\]

(5)

Function \( a(t) \) is the scale factor while the Hubble function is defined as \( H = \frac{\dot{a}}{a} \), an overdot denote differentials with respect to comoving proper time, \( t \).

Hence, the Einstein’s field equations (2) are

\[
3H^2 = \frac{1}{2} \dot{\phi}^2 + V(\phi),
\]

(6)

\[
2\dot{H} + 3H^2 = -\frac{1}{2} \ddot{\phi}^2 + V(\phi),
\]

(7)
while the Klein-Gordon equation reads

$$\ddot{\phi} + 3H\dot{\phi} + V_{,\phi} = 0. \quad (8)$$

We have assumed that field $\phi$ inherits the symmetries of the background space, that is, $\phi = \phi(t)$. Inflation occurs, when the scalar field potential dominates, $3H^2 \simeq V(\phi)$ and the scalar field approaches a stationary point $\dot{\phi} \simeq -\frac{V_{,\phi}}{3H} \quad [48]$.

Thus in order to understand the existence of an inflationary era provided by a potential function, the potential slow-roll parameters (PSR)

$$\varepsilon_V = \left(\frac{V_{,\phi}}{2V}\right)^2, \quad \eta_V = \frac{V_{,\phi\phi}}{2V}, \quad (9)$$

have been introduced \[48\].

The condition an inflationary era to exists is expressed as, $\varepsilon_V \ll 1$, while in order for the inflationary phase to last long enough it is required require the second PSR parameter also to be small, i.e. $\eta_V \ll 1$.

Some inflationary potentials which have been proposed in the literature are presented. The quadratic $R + \left(\frac{R}{6M^2}\right)^2$ inflationary model with potential function in the Einstein-frame \[24\]

$$V_A(\phi) = V_0 \left(1 - e^{-\frac{\phi}{\sqrt{2}M}}\right)^2, \quad (10)$$

the intermediate inflationary potential \[49\]

$$V_B(\phi) = \frac{48A^2}{(\Delta + 4)^2} \left(\frac{\phi - \phi_0}{\sqrt{2A\Delta}}\right)^{-\Delta} - \frac{4A\Delta}{(\Delta + 4)^2} \left(\frac{\phi - \phi_0}{\sqrt{2A\Delta}}\right)^{-\Delta-2}, \quad (11)$$

the natural inflation \[50\]

$$V_C(\phi) = V_0 \left(1 + \cos \left(\frac{\phi}{f}\right)\right), \quad (12)$$

the hyperbolic inflationary potential \[19\]

$$V_D(\phi) = V_0 \sinh^q(p), \quad (13)$$

and many others. For an extended list of the proposed inflationary models we refer the reader in the reviews \[38, 39\].
Similarly, with the PSR parameters we can define the corresponding Hubble slow-roll parameters (HSR) as

\[ \varepsilon_H = -\frac{d \ln H}{d \ln a} = \left( \frac{H \phi}{H} \right)^2, \quad (14) \]

\[ \eta_H = -\frac{d \ln H \phi}{d \ln a} = \frac{H \phi \phi}{H}. \quad (15) \]

The two different set of parameters, the HSR and PSR parameters are related by the expressions

\[ \varepsilon_V \simeq \varepsilon_H \quad \text{and} \quad \eta_V \simeq \varepsilon_H + \eta_H. \quad (16) \]

Therefore, inflation occurs when \( \varepsilon_H \ll 1 \). The limit \( \varepsilon_H = 1 \), is called the end of inflation where we have the exit from the acceleration era. In the following we work with the Hubble slow-roll parameters.

As far as the observable values for the spectral indices \( n_s \) and \( r \), are concerned, from the Planck 2018 collaboration \[51\] follow that the spectral index for the density perturbations is constraint as \( n_s = 0.9649 \pm 0.0042 \), while the tensor to scalar ratio, \( r \) is constraint as \( r < 0.10 \).

These indices are related with the HSR parameters in the first-order approximation as

\[ r = 10 \varepsilon_H, \quad (17) \]

\[ n_s = 1 - 4 \varepsilon_H + 2 \eta_H. \]

In the second-order approximation we shall consider the additional slow-roll parameter \( \xi_H \equiv \frac{H \phi \phi}{H^2} \), such that \( n_s \) and \( n_s' \) are expressed as

\[ n_s \equiv 1 - 4 \varepsilon_H + 2 \varepsilon_H - 8 (\varepsilon_H)^2 (1 + 2C) + \varepsilon_H \eta_H (10C + 6) - 2C \xi_H, \quad (18) \]

\[ n_s' \equiv 2 \varepsilon_H \eta_H - 2 \xi_H. \quad (19) \]

where \( C = \gamma_E + \ln 2 - 2 \simeq -0.7296 \) and the range of the scalar spectral index is \( n_s' = -0.005 \pm 0.013 \quad [51] \).

In the following we consider the spectral indices in the first-order approximation.
### 2.1. Algebraic solution

In [44], the generic algebraic solution for the field equations (6)-(8) is presented for arbitrary potential function. Indeed, with the use of new variables the unknown potential function can be introduced inside the metric tensor, while the dynamical variables for the field equations are expressed in terms of an arbitrary function. The field equations are reduced in algebraic equation. Such a solution is called algebraic solution.

We define the new variable \( dt = \exp \left( \frac{F(\omega)}{2} \right) d\omega \) with \( \omega = 6 \ln a \). The FLRW line element becomes

\[
    ds^2 = -e^{F(\omega)/2}d\omega^2 + e^{\omega/3}(dx^2 + dy^2 + dz^2).
\]

with Hubble function \( H(\omega) = \frac{1}{6}e^{-\frac{F}{2}} \).

Hence, from the field equations (6)-(8) we find that then the scalar field is expressed as \[44\]

\[
    \phi(\omega) = \pm \sqrt{\frac{6}{6}} \int \sqrt{F'(\omega)}d\omega \tag{21}
\]

while the scalar field potential reads

\[
    V(\omega) = \frac{1}{12}e^{-F(\omega)}(1 - F'(\omega)). \tag{22}
\]

in which \( F'(\omega) = \frac{dF(\omega)}{d\omega} \).

Function \( F(\omega) \) can be constraint by various approaches. In \[22\] closed-form scalar field solutions were found by assuming specific functional forms for the equation of state parameter \( P_\phi = P_\phi(\rho_\phi) \) of the scalar field, \( P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) \), \( \rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) \).

In the new set of variables, the HSR parameters are expressed as follows \[22\]

\[
    \varepsilon_H = 3F', \quad \eta_H = 3\frac{(F')^2 - F''}{F'}, \tag{23}
\]

while the number of e-folds is defined as

\[
    N_e = \int_{t_i}^{t_f} H(t) dt = \ln \frac{a_f}{a_i} = \frac{1}{6}(\omega_f - \omega_i). \tag{24}
\]
In [22] it was considered that the spectral indices are constraint as

\[ n_s - 1 = h(r) , \]  

where \( h(r) \) has been assumed to be \( h(r) = h_1 + h_2 r + h_3 r^2 \). For this latter assumption, new exact and analytic inflationary solutions were found.

In the first-order approximation, from (25) and (23) the master equation (25) reads

\[ F'' + G(F') = 0 , \quad G(F') = -\frac{1}{6} h(F') F' + (F')^2 . \]  

(26)

In the following we constrain function \( G(F') \), i.e. \( h(F') \), by the requirement the master equation (26) to admit Lie point symmetries. This selection rule has geometric characteristics. Symmetries are geometric objects in the space where the differential equation is defined. Such a geometric selection rule is in agreement with the geometric nature of gravitational physics, for a discussion we refer to [42].

We proceed with the presentation of the symmetry analysis for the master equation (26).

3. LIE SYMMETRY ANALYSIS

The theory of symmetries of differential equations established at the end of the 19th century by Sophus Lie [52]. The novelty of Lie’s approach is that the transformations group which leave invariant a differential equation can be used to simplify the given equation. In the case of ordinary differential equations the Lie point symmetries are used to reduce the order of the differential equation.

Let us demonstrate this on our master equation (26). By assuming the new variables \( f = F' \), the master equation (26) can be written in the equivalent form \( f' = G(f) \), that is \( \int \frac{df}{G} = \omega - \omega_0 \). This is the most common reduction process which holds for autonomous dynamical systems where the autonomous point symmetry vector \( \partial_\omega \) exists.

The derivation of Lie (point) symmetries for a given dynamical system is based on a simple algorithm. We briefly discuss the algorithm.
Consider the infinitesimal point transformation

\[
\bar{\omega} = \omega + \varepsilon \xi (\omega, F), \\
\bar{F} = F + \varepsilon \eta (\omega, F),
\]

with generator the vector field \( X = \xi (\omega, F) \partial_\omega + \eta (\omega, F) \partial_F \). Then, the master equation (26) is invariant under the Action of the later point transformation if and only if

\[
X^{[2]} (F'' - G (F')) = 0
\]

in which \( X^{[2]} = X + \eta^{[1]} \partial_{F'} + \eta^{[2]} \partial_{F''} \) is the second extension of \( X \) in the tangent space, where \( \eta^{[1]} \) and \( \eta^{[2]} \) are defined as

\[
\eta^{[1]} = \eta' - F' \xi', \\
\eta^{[2]} = \eta^{[1]}' - F'' \xi'.
\]

When for a given point transformation the symmetry condition (29) is valid, the vector field \( X \) will be called a Lie symmetry.

The symmetry condition (29) for the master equation (26) is expressed as follow

\[
0 = \eta_{\omega\omega} - F'(\xi_{\omega\omega} + (F')^2 (\eta_{FF} - 2 \xi_{\omega F}) - (F')^3 \xi_{,FF} + 2 F' \eta_{\omega F} + G_{,F} \eta_{\omega} +
\]

\[
+ (G_{,F'} F' - G) \eta_{,F} + (3 F' G' - G_{,F'} (F')^2) \xi_{,F} + (2 G' - G_{,F'} F') \xi_{,\omega}.
\]

From the later equation we can define systems of partial differential equations which shall constraint the unknown functions \( \xi (\omega, F) \) and \( \eta (\omega, F) \). However the constraint system it depends on the functional form of \( G (F') \). Indeed, for arbitrary function \( G (F') \) we find the generic solution \( \xi (\omega, F) = \alpha_1 \) and \( \eta (\omega, F) = \alpha_2 \), which means that the generic symmetry vector is \( X = \alpha_1 \partial_\omega + \alpha_2 \partial_F \). Hence, the two independent symmetries, \( X_1 = \partial_\omega \) and \( X_2 = \partial_F \), follow, with commutator \([X_1, X_2] = 0\).

However, there are special forms of \( G (F') \) in which the master equation (26) admits additional Lie symmetries. This classification problem is that we attempt to solve in this study. It is important to mention that the Lie point symmetries for second-order differential
equations have been widely studied in the literature see for instance [53, 54].

It is well known that a second-order differential equation can admits 0, 1, 2, 3 or 8 Lie symmetries. When eight Lie point symmetries exist, the the differential equation is called maximal symmetric and there exist a similarity transformation such that the equation to be written in the form of the free particle.

For arbitrary function \( G (F') \) the master equation (26) admits two Lie symmetries, thus we investigate for specific forms of \( G (F') \) in which the master equation admits 3 or 8 Lie symmetries.

Thus, the classification scheme provides for equation (26) gives the three functional forms

\[
G_A (F') = G_0 (F')^{\nu+1}, \quad \nu \neq -1, 0, 1, 2, \quad G_B (F') = G_0 \exp (\mu F'), \quad \mu \neq 0 \quad \text{and} \quad G_C (F') = G_3 (F')^3 + G_2 (F')^2 + G_1 F' + G_0,
\]

where \( G_{0-3} \) and \( \mu \) are constants parameters.

For \( G_A (F') \), equation (26) admits the three Lie symmetries \( \{X_1, X_2, X_3 = \nu \omega \partial_\omega + (\nu - 1) F \partial_F\} \) with commutators \([X_1, X_2] = 0, [X_1, X_2] = \nu X_1\) and \([X_2, X_3] = (\nu - 1) X_2\).

For the exponential function \( G_B (F') \), the Lie symmetries of equation (26) are \( \{X_1, X_2, \bar{X}_3 = \mu \omega \partial_x + (F \mu - \omega) \partial_F\} \) with commutators \([X_1, \bar{X}_3] = \mu X_1 - X_2\) and \([X_2, \bar{X}_3] = \mu X_2\).

Finally, for \( G_C (F') \) equation (26) is maximal symmetric and admits eight Lie symmetries. The representation of the admitted Lie algebra depends on the values of the free parameters thus we omit it. The later case has been widely studied before in [23]. Moreover, the similarity transformations which connect the different inflationary models which belong to the family of \( G_C (F') \) are presented in [55].

For the inflationary models provided by \( G_A (F') \) and \( G_B (F') \) we derive the corresponding functions \( h (r) \) are \( h_A (r) = -6 \left(h_0 r^{\nu} - \frac{1}{10} r\right) \) and \( h_B (r) = -6 \left(h_0 r^{-1} e^{\mu r} - \frac{1}{10} r\right) \).

From the cosmological observations we know that \( n_s - 1 \simeq 0 \), while \( r < 0.11 \). Thus, from these two models, \( h_A (r) \) and \( h_B (r) \), only model \( h_A (r) \) provides a behaviour \( \lim_{r \to 0} (h_A) \simeq 0 \) for \( \nu > 0 \). On the other hand, \( h_A (r) \ll 1 \) when \( r \ll 1 \), and for \( \nu < -1 \) if and only if \( h_0 \simeq \frac{1}{r^{\nu+1}} \), that is, for very large values of the free parameter \( h_0 \).

In the following section we focus our analysis on model \( h_A (r) \), where we investigate the closed-form solution, we discuss the physical properties of the model and we investigate the evolution for the inflationary parameters.
4. NEW INFLATIONARY EXACT SOLUTION

For $G = G_A (F')$, the master equation reads

$$F'' + G_0 (F')^{\nu+1} = 0,$$

with closed-form solution

$$F (\omega) = \frac{(G_0 \nu \omega)^{1-\frac{1}{\nu}}}{G_0 (\nu - 1)}.$$  \hspace{1cm} (34)

Thus, the physical parameters for the inflaton field, such as the energy density $\rho_\phi$, the pressure $P_\phi$ and the equation of state parameter $w_\phi = \frac{P_\phi}{\rho_\phi}$ can be constructed analytical

$$\rho_\phi (\omega) = \frac{1}{12} \exp\left(\frac{(G_0 \nu \omega)^{1-\frac{1}{\nu}}}{G_0 (\nu - 1)}\right),$$

$$P_\phi (\omega) = \frac{1}{12} \exp\left(\frac{(G_0 \nu \omega)^{1-\frac{1}{\nu}}}{G_0 (\nu - 1)}\right) \left(-1 + 2 (G_0 \nu \omega)^{-\frac{1}{\nu}}\right),$$

and

$$w_\phi (\omega) = \left(-1 + 2 (G_0 \nu \omega)^{-\frac{1}{\nu}}\right).$$

We remark that for $\nu > 0$ and for large values of $\omega$, $w_\phi (\omega) \rightarrow -1$, otherwise when $\nu < 0$, $w_\phi (\omega) \rightarrow -1$ for small values of $\omega$.

As far as the scalar field and the scalar field potential are concerned, we derive the following expressions

$$\phi (\omega) = \sqrt{\frac{2}{3}} \left(\frac{G_0 \nu \omega)^{1-\frac{1}{\nu}}}{G (2\nu - 1)}\right), \nu \neq \frac{1}{2},$$

$$\phi (\omega) = \sqrt{\frac{2 \ln \omega}{3 G_0}}, \nu = \frac{1}{2},$$

and

$$V (\omega) = \frac{1}{12} \left(1 - (G_0 \nu \omega)^{-\frac{1}{2}}\right) \exp\left(\frac{(G_0 \nu \omega)^{1-\frac{1}{\nu}}}{G_0 (\nu - 1)}\right).$$

In terms of the scalar filed $\phi$, the potential function $V (\phi)$ is given by the following
FIG. 1: Qualitative evolution for the scalar field potential (41) for $G_0 = 1$. Solid line is for $\nu = \frac{3}{2}$, dotted line is for $\nu = \frac{5}{2}$ and dashed line is for $\nu = 4$.

functional form

$$ V(\phi) = \frac{1}{12} \left( 1 - \left( \frac{3}{2} \right)^{-\frac{1}{1-2\nu}} \left((2\nu - 1) G_0 \phi \right)^{\frac{2}{1-2\nu}} \right) \exp \left( \frac{\left( \frac{2}{3} \right)^{\frac{1-\nu}{2\nu-1}}}{G_0 (\nu - 1)} \left((2\nu - 1) G_0 \phi \right)^{1+\frac{1}{2\nu-1}} \right), \nu \neq \frac{1}{2} $$

or

$$ V(\phi) = \frac{1}{12} \left( 1 - \frac{4}{G_0} \exp \left(-\sqrt{6}G_0 \phi \right) \right) \exp \left( \frac{-4}{G_0} \exp \left(-\sqrt{\frac{3}{2}}G_0 \phi \right) \right), \nu = \frac{1}{2}. \quad (42) $$

In Fig. 1 we present the qualitative evolution for the scalar field potential (41) for $G_0 = 1$ and for different values of the free parameter $\nu$.

According to our knowledge, this inflationary potential has not been presented before in the literature.
4.1. Spectral indices

For the closed-form solution of $F(\omega)$, the definition of the slow-roll parameters from expressions \[23\] we can define the spectral indices $n_s$ and $r$ in terms of the number of e-fold $N_e$. Recall, that the end of inflation occurs when $\varepsilon(\omega_f) = 1$, that is, $\omega_f = \frac{3^\nu}{G_0\nu}$.

Consequently, the spectral indices by using the HSR parameters are written as follows

$$n_s = 1 - 6 \left( (G_0\nu) \left( 6N_e + \frac{3^\nu}{G_0\nu} \right) \right)^{-\frac{1}{\nu}} + \frac{6G_0}{3^\nu + 6G_0N_e\nu}, \quad (43)$$

$$r = 30 \left( (G_0\nu) \left( 6N_e + \frac{3^\nu}{G_0\nu} \right) \right)^{-\frac{1}{\nu}}. \quad (44)$$

We observe that the indices $n_s$ and $r$ depend on the number of e-fold $N_e$, and on the free parameters $(\nu, G_0)$. From the observations the number of e-fold it is considered to be in the range $N_e = (50, 60)$.

In Figs. 2, 3 and 4 we present the qualitative evolution of the spectral indices according to the different values of the free parameters. In Fig. 2 we present the contour plot for the spectral indices $n_s$ and $r$ in the first-order approximation, in the space of the free parameter $(\nu, G_0)$ and for $N_e = 50$, $N_e = 55$ and $N_e = 60$. we observe that as power $\nu$ increases then the model fits the observations as $G_0$ increases exponentially. In Figs. 3 and 4 we present the parametric plot in the space $(n_s, r)$ for varying parameter the number of e-fold $N_e$, and for specific values of the free parameters of the model $(\nu, G_0)$. In Fig. 4 we assume the that $G_0 = \tilde{G}_0\nu^{-1}$ and plots are for specific values of $\nu$.

From the results of Figs. 2, 3 and 4 it is clear that the new inflationary model provided by the symmetry analysis provide values for the spectral indices according to the cosmological observations, see Section 2 for the values of the Planck 2018 collaboration, Table 3 in \[26\].

5. CONCLUSIONS

In this work we investigated the construction of a new exact inflationary models in the inflaton theory by using Lie symmetry analysis for the master equation \[26\]. The master equation of our consideration is constructed by the assumption the spectral index $n_s$ and the
FIG. 2: Qualitative evolution for the spectral indices \((n_s, r)\) in the first approximation in the two-dimensional space for the free variables \((\nu, G_0)\), for \(N_e = 50\) (left figs.), \(N_e = 55\) (center figs.) and \(N_e = 60\) (right figs.).

FIG. 3: Parametric plots for the spectral indices \((n_s, r)\) in the first approximation varying number of e-folds \(N_e = (50, 55, 60)\) and different values of \(G_0\) and \(\nu = \nu_0 + \varepsilon\). Left fig. is for \((G_0, \nu_0) = (10, 1.5)\), center fig. is for \((G_0, \nu_0) = (10^2, 2)\) and right fig. is for \((G_0, \nu_0) = (10^3, 3)\). Solid lines are for \(\varepsilon = 0.01\), dashed lines are for \(\varepsilon = 0.02\) and dotted lines are for \(\varepsilon = 0.03\).
tensor to scalar ratio $r$, to be related by a function $G$. From this assumption the second-order differential equation follows.

We performed a classification for the function which relates $n_s$ and $r$ by assuming the master equation to admit Lie symmetries. By applying the symmetry condition we found that the master equation admits two Lie symmetries for arbitrary function $G$, while for $G = G_A$ and $G = G_B$ admits three Lie symmetries and for $G = G_C$ the admitted Lie symmetries form the $sl(3,\mathbb{R})$ Lie algebra and the master equation is maximally symmetric. The case $G_C$ was investigated in details in a previous study [23], hence in this work we focused on $G_A$ and $G_B$.

For $G_A$ and $G_B$, the master equation admits three Lie symmetries, which form the Lie algebras $A_{3,3}$ and $A_{3,2}$ in the Patera et al. classification [56], respectively. However function $G_B$ does not provide a relation for the indices $n_s, r$ as provided by the observations, thus, we focused our analysis on the function $G_A$ and the Lie algebra $A_{3,3}$.

We solved the master equation and we were able to derive the closed-form solution for the scalar field and for all the physical variables of the inflaton model. Furthermore, we wrote the closed-form expressions for the spectral indices $n_s, r$, in the first-order approximation by using the HSR parameters. We found that the spectral indices depend on the number of e-fold and on two free parameters. By presenting the qualitative evolution of the spectral indices we remark that they can fit in the cosmological observations.
FIG. 5: Qualitative evolution for the spectral indices \((n_s, n'_s)\) in the second approximation in the two-dimensional space for the free variables \((\nu, G_0)\), for \(N_e = 50\) (left figs.) \(N_e = 55\) (center figs.) and \(N_e = 60\) (right figs.)

In Fig. 5 we present the qualitative evolution for the spectral indices \(n_s\) and \(n'_s\) in the second approximation for different values of the free variables \((\nu, G_0)\). We observe that \(n'_s\) is always positive valued, and takes small values when \(n_s - 1 \simeq 0\). According the cosmological observations, the running index is constraint \(n'_s = -0.005 \pm 0.013\), therefore, small positive values are inside the \(1\sigma\) region.

We conclude that the Lie symmetry approach is a powerful method for the derivation of exact solutions and in this study the symmetry method applied for the derivation of a new inflationary exact solution. A natural question which raised is if this approach can be applied in \(f (R)\)-theory or in scalar-tensor gravity \([58]\). These three theories are related under conformal transformations. The analytical solution determined in \([44]\) for the scalar field can be easily used to write the corresponding analytic solutions for the conformal equivalent theories, and extend the present symmetry analysis in the conformal frame. However, such an analysis overpass the scopus of the present work and will be discussed elsewhere.
In a future study we plan to investigate further this scalar field solution model and we want to use this model as dark energy candidate and investigate if it solves the $H_0$-tension.

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