SHARP INTERFACE LIMIT OF AN ANISOTROPIC GINZBURG–LANDAU EQUATION

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ABSTRACT. In this work, we study the co-dimensional 1 interface limit of an anisotropic Ginzburg–Landau equation under parabolic scalings. This is a semi-linear parabolic system of a planar vector field with a small parameter $\varepsilon > 0$ corresponding to the transition layer width. For well-prepared initial datum, as $\varepsilon$ tends to 0, we show the solution gradient will concentrate on a closed simple curve evolving by curve-shortening flow. Moreover, the limiting solution satisfies an anchoring boundary condition when restricted on the curve. These results hold as long as the limiting curvature flow remains smooth. The main ingredient of the proof is an estimate of the level set of the quasi-distance function corresponding to the solution.

1. Introduction

We consider the following anisotropic Ginzburg–Landau type energy
\begin{equation}
A_\varepsilon(u) = \int_\Omega \left( \frac{\varepsilon}{2} \mu |\text{div} u|^2 + \frac{\varepsilon}{2} |\nabla u|^2 + \frac{1}{\varepsilon} F(u) \right) \, dx,
\end{equation}
on a bounded domain $\Omega \subset \mathbb{R}^2$ with smooth boundary with order parameter $u : \Omega \mapsto \mathbb{R}^2$. The bulk potential is chosen as the Higgs potential function of $\phi^6$ type \cite{4, 14, 18} which permits the isotropic-nematic phase transition:
\begin{equation}
F(u) = \sigma_0 |u|^2(1 - |u|^2)^2 =: f(|u|)
\end{equation}
Here $\sigma_0 > 0$ is a physical constant which will be made precise later on. In (1.1), $\varepsilon > 0$ denotes the relative intensity of elastic and bulk energy, which is usually quite small. The parameter $\mu \in \mathbb{R}^+$ is material dependent which measures the degree of anisotropy.

To model the isotropic-nematic phase transitions in the framework of Landau–De Gennes theory, we shall investigate the small $\varepsilon$ limit of the $L^2$-gradient flow of (1.1) with initial data undergoing a sharp transition near a smooth interface. To be more precise, we consider the system
\begin{equation}
\partial_t u^\varepsilon = \mu \nabla \text{div} u^\varepsilon + \Delta u^\varepsilon - \frac{1}{\varepsilon^2} \partial F(u^\varepsilon) \text{ in } \Omega \times (0, T),
\end{equation}
with initial and boundary conditions
\begin{align}
u^\varepsilon(x, 0) = u^\varepsilon_m(x), & \quad \text{in } \Omega, & \text{(1.4a)} \\
u^\varepsilon(x, t) = 0, & \quad \text{on } \partial \Omega \times (0, T). & \text{(1.4b)}
\end{align}
Note that in (1.3) $\partial F(u)$ is the differential of $F(u)$.

To state the main result of this work, we assume
\begin{equation}
I = \bigcup_{t \in [0, T]} I_t \times \{t\} \text{ is a smoothly evolving simple closed curve in } \Omega,
\end{equation}
starting from a closed smooth curve $I_0 \subset \Omega$. Let $\Omega_t^+$ be the domain enclosed by $I_t$, and $d_I(x, t)$ be the signed-distance from $x$ to $I_t$ which takes positive values in $\Omega_t^+$, and negative values in $\Omega_t^- = \Omega \setminus \Omega_t^+$, where
\begin{equation}
\Omega_t^\pm := \{x \in \Omega \mid d_I(x, t) \geq 0\}. \quad \text{(1.6)}
\end{equation}
For $\delta > 0$, the $\delta$-neighborhood of $I_t$ is denoted by
\begin{equation}
I_t(\delta) := \{x \in \Omega : |d_I(x, t)| < \delta\}. \quad \text{(1.7)}
\end{equation}
We shall choose $\delta_0 \in (0, 1)$ sufficiently small so that the nearest point projection $P_I(\cdot, t) : I_t(4\delta_0) \to I_t$ is smooth for any $t \in [0, T]$, and the interface (1.3) stays at least $4\delta_0$ distance away from the physical boundary $\partial \Omega$. A more detailed description of the geometry can be found in the beginning of Section 2 or [5].

Our strategy is to construct a Lyapunov functional which encodes the distance between the approximate solution (1.3) and the set (1.5) where the solution gradient will concentrate, and to derive a differential inequality of the functional. To this end, we shall extend the inner normal vector $n$ of $I_t$ to the whole computational domain $\Omega$ by

$$\xi(x) = \phi \left( \frac{d_I(x, t)}{\delta_0} \right) \nabla d_I(x, t)$$

where $\phi(x) \geq 0$ is an even, smooth function on $\mathbb{R}$, decreases for $x \in [0, 1]$, and satisfies

$$\phi(x) > 0 \text{ for } |x| < 1, \phi(x) = 0 \text{ for } |x| \geq 1,$$

$$1 - 4x^2 \leq \phi(x) \leq 1 - \frac{1}{2}x^2 \text{ for } |x| \leq 1/2.$$  

To fulfill these requirements, we can simply choose

$$\phi(x) = e^{x^2 - 1 + 1} \text{ for } |x| < 1 \text{ and } \phi(x) = 0 \text{ for } |x| \geq 1.$$  

Following [11, 22], we introduce the modulated energy

$$E_\varepsilon[u^\varepsilon|I|](t) := \int_{\Omega} \frac{\varepsilon}{2} |\text{div} u^\varepsilon(\cdot, t)|^2 \, dx$$

$$+ \int_{\Omega} \left( \frac{\varepsilon}{2} |\nabla u^\varepsilon(\cdot, t)|^2 + \frac{1}{\varepsilon} F(u^\varepsilon(\cdot, t)) + (\text{div} \xi) \psi^\varepsilon(\cdot, t) \right) \, dx.$$  

With the definition of $f$ at (1.2), we define

$$\psi^\varepsilon(x, t) := d_F \circ u^\varepsilon(x, t), \quad d_F(u) := \int_0^{|u|} \frac{1}{\sqrt{2f(s)}} \, ds.$$  

For the convenience of computations,

$$\sigma_0 \text{ in (1.2) is chosen so that } \int_0^1 \frac{1}{\sqrt{2f(s)}} \, ds = 1.$$  

With these preparations, we state below the main result of this work:

**Theorem 1.1.** Assume $I_t$ (1.5) is a curve-shortening flow\(^1\) and the initial data $u^\varepsilon(x, 0) = u_{i_0}^\varepsilon(x)$ satisfies

$$u_{i_0}^\varepsilon(x) = u_{i_0}(x) \text{ in } \Omega_0^+ \setminus I_0(\delta_0), \text{ and } u_{i_0}^\varepsilon(x) = 0 \text{ in } \Omega_0^+ \setminus I_0(\delta_0), \quad (1.14)$$

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\(^1\)It is also called one-dimensional the mean curvature flow
for some \( u_{in} \in H^1(\Omega, S^1) \). If there is a constant \( c_1 > 0 \) which is independent of \( \varepsilon \) so that

\[
A_\varepsilon(u_{in}^\varepsilon) \leq c_1, \quad \|u_{in}^\varepsilon\|_{L^\infty} \leq 1, \tag{1.15a}
\]

\[
E_\varepsilon[u_{in}^\varepsilon | I_0] + \int_{I_0(\delta_0)} |\psi_{in}^\varepsilon - 1_{\Omega_t^\varepsilon}| |dI_0| \, dx \leq c_1 \varepsilon, \tag{1.15b}
\]

then there exists \( C_1 > 0 \) which depends only on the geometry of \( I_t \) so that

\[
\sup_{t \in [0, T]} E_\varepsilon[u^\varepsilon | I_t](t) + \sup_{t \in [0, T]} \int_{I_t(\delta_0)} |\psi^\varepsilon - 1_{\Omega_t^\varepsilon}| |dI_t| \, dx \leq C_1 \varepsilon. \tag{1.16}
\]

Moreover, for a subsequence \( \varepsilon_k \downarrow 0 \) as \( k \uparrow +\infty \), there holds

\[
\psi_{in}^\varepsilon_k \xrightarrow{k \to \infty} 1_{\Omega_t^\varepsilon} \text{ strongly in } L^\infty(0, T; L^1(\Omega)) \quad \tag{1.17a}
\]

\[
u_{in}^\varepsilon \xrightarrow{k \to \infty} \mathbf{1}_{\Omega_t^\varepsilon} \text{ strongly in } L^\infty(0, T; L^1(\Omega)) \cap C([0, T]; L^2_{loc}(\Omega_t^\varepsilon)), \quad \tag{1.17b}
\]

\[
\mathbf{u} \in H^1(\Omega^+; S^1), \quad \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial \Omega_t^+, \quad \tag{1.17c}
\]

We first make a few comments about the conditions (1.15) for the initial datum. The first condition in (1.15a) is finite energy condition, which is quite standard. The second one \( \|u_{in}^\varepsilon\|_{L^\infty} \leq 1 \) is imposed to avoid some technical issues but can be relaxed. Actually in the proof of Theorem 1.1 below, it is a simple condition to guarantee a well-prepared initial data for a Grönnwall inequality. The essential assumption is (1.15b), which is used to obtain the calibration inequality. Such an inequality is the heart of the work and is the source of various modulated inequalities. Concerning the initial data satisfying (1.15b), we have the following result whose proof will be done in the end of Section 2.

**Proposition 1.2.** Let \( I_0 \subset \mathbb{R}^2 \) be a smooth simple close curve with inner normal vector \( \mathbf{n} \). Then for every \( u_{in} \in H^1(\Omega, S^1) \) with trace \( u_{in} | I_0 \cdot \mathbf{n} = 0 \), there exists \( u_{in}^\varepsilon \in H^1(\Omega) \cap L^\infty(\Omega) \) such that \( u_{in}^\varepsilon \) is well-prepared in the sense of (1.15b).

There is a huge number of literature on the co-dimensional 1 scaling limit of Allen–Cahn type equation to the mean curvature flow. To list a few, we mention the convergence to a Brakke’s flow by [15] [16] [26] [27] [28] and the convergence to the viscosity solution by [8] [29] and the references therein. All these works consider the scalar equation and make heavy use of the maximum principle and the comparison principle.

In contrast, there are much fewer works on the vectorial Allen–Cahn equation, i.e. the time dependent Ginzburg–Landau equation. To the best of the author’s knowledge, there are mainly two methods for the vectorial cases, both assuming the limiting interface has a classical solution. One is the asymptotical expansion method introduced in [6], and used in [9] [10]. Another one is the calibration method introduced in [11], motivated by [19]. The generalization to the vectorial case has been done in [22] to study the isotropic–nematic transition in Landau–De Gennes model of liquid crystals.

The current work is intended to generalize the method of [11] [22] to the anisotropic system (1.3). On the other hand, Theorem 1.1 can be considered as a dynamical version of a recent work by Golovaty et al [13]. Concerning the method we use, at least in obtaining the anchoring boundary condition (1.17c) which is one of the main novelty in this work, we are inspired by a recent work of Lin–Wang [23]. There the authors studied isotropic-nematic transitions based on an anisotropic Ericksen’s model. As the gradient flow of Ericksen’s functional is very singular, we work with (1.3) which corresponds to a simplified Landau–De Gennes model [17].

We would like to mention that the choice of the potential (1.2) is both for the interest from physics and for simplicity in the presentation. For instance such a choice will ease the calculations relating to the projection operator (2.11) below. See also the Remark 2.1 below. However, we believe our method can be adapted to more general cases where \( \partial d_F(u) \) is less regular. This will involve a careful mollification of \( d_F(u) \) like in [22], or the generalized chain rule in [1].
This work will be organized as follows: In Section 2 we shall adapt the calibration method introduced in [11] for scalar Allen–Cahn equation to the vectorial and anisotropic system (1.3), and then derive a differential inequality, i.e. Proposition 2.3. Such a differential inequality was first derived in [22] for a matrix-valued Allen–Cahn equation. However, due to the additional div term in (1.3), such an inequality includes a term which does not have an obvious sign. This problem will be solved in Section 3 during the proof of the calibration inequality in Theorem 3.1. This theorem, in which we prove the first part of (1.16), is a major novelty of this work and is the main reason that we consider the planar vector field \( u^\varepsilon : \Omega \to \mathbb{R}^2 \). In Section 4 Theorem 4.1, we use the calibration inequality to derive an \( L^1 \)-convergence estimate of \( \psi^\varepsilon \) and thus prove the second part of (1.16) and (1.17a). This \( L^1 \)-estimate will lead to an estimate of the level sets of \( \psi^\varepsilon \) in Lemma 4.3. With this key lemma, we derive in Section 5 the convergence (1.17b) and the anchoring boundary condition (1.17c).

We shall adopt the convention that \( C > 0 \) will be a generic constant depending only on the geometry of the interface (1.5) but not on \( \varepsilon \) or \( t \in [0, T] \). In order to simplify the presentation, we shall sometimes abbreviate the estimates like \( X \lesssim Y \) for some non-negative quantities \( X, Y \). We shall also use the Frobenius inner product \( A : B \coloneqq \text{tr} A^T B \) for two square matrices \( A, B \). Finally, for the sake of simplicity, we shall employ the Einstein summation convention by summing over repeated Latin index.

2. THE MODULATED ENERGY INEQUALITY

As the gradient flow of (1.1), the system (1.3) has the following energy dissipation law

\[
A_\varepsilon(u^\varepsilon(\cdot, T)) + \int_0^T \int_{\Omega} \varepsilon |\partial_t u^\varepsilon|^2 \, dx \, dt = A_\varepsilon(u^\varepsilon_m(\cdot)), \quad \text{for all } T > 0.
\]  

(2.1)

For initial data undergoing a transition near \( I_t \), due to the concentration of \( \nabla u^\varepsilon \) near the interface \( I_t \), the dissipation law (2.1) is not sufficient to derive the convergence of \( u^\varepsilon \), even away from \( I_t \). Following a recent work of Fisher et al. [11] we shall develop in this section a calibrated inequality, which modulates the concentration and obtain the compactness of solutions in Sobolev spaces. To this aim, we first give a more detailed geometric setting based on the description after (1.5).

Under a local parametrization \( \varphi_t(s) : T^1 \to I_t \), the curve-shortening flow writes

\[
\partial_t \varphi_t(s) \cdot n(s,t) = H(\varphi_t(s), t) \cdot n(s,t)
\]  

(2.2)

where \( H \) is the (mean) curvature vector pointing to the inner normal \( n \). We assume for \( t \in [0, T] \) the nearest-point projection \( P_t(\cdot, t) : I_t(4\delta_0) \to I_t \) is smooth for some sufficiently small \( \delta_0 \in (0, 1) \) which only depends on the geometry of \( I_t \). Analytically we have \( P_t(x, t) = x - \nabla d_I(x,t)d_I(x,t) \). So for each fixed \( t \in [0, T] \), any point \( x \in I_t(4\delta_0) \) corresponds to a unique pair \( (r, s) \) with \( r = d_I(x,t) \) and \( s \in T^1 \), and thus the identity

\[
d_I(\varphi_t(s) + rn(s,t), t) \equiv r
\]

holds with independent variables \( (r, s, t) \). Differentiating this identity with respect to \( r \) and \( t \) leads to the following identities:

\[
\nabla d_I(x,t) = n(s,t), \quad -\partial_t d_I(x,t) = \partial_t \varphi_t(s) \cdot n(s,t) =: V(s,t).
\]  

(2.3)

This extends the normal vector and the normal velocity of \( I_t \) to a neighborhood of it. Recall in (1.8) that we extend the normal vector field \( n \) of the interface \( I_t \) to a neighborhood of it. We also need to extend the curvature of (1.5). To proceed, choose a cut-off function

\[
\eta_0 \in C^\infty_c(I_t(2\delta_0)) \quad \text{with} \quad \eta_0 = 1 \text{ in } I_t(\delta_0).
\]  

(2.4)

We extend the curvature vector \( H \) by

\[
H(x,t) = \kappa \nabla d_I(x,t) \quad \text{with} \quad \kappa(x,t) = -\Delta d_I(x,t))\eta_0(x,t).
\]  

(2.5)
Since $\mathbf{H}$ (2.5) is extended constantly in the normal direction, we have
\[
(n \cdot \nabla)\mathbf{H} = 0 \quad \text{and} \quad (\boldsymbol{\xi} \cdot \nabla)\mathbf{H} = 0
\]
for all $(x,t)$ such that $|d_I(x,t)| \leq \delta_0$.

Moreover, by (1.8) we have
\[
\boldsymbol{\xi} = 0 \text{ on } \partial \Omega \text{ and } \mathbf{H} = 0 \text{ on } \partial \Omega.
\]

We claim also the following identities which will be employed to prove the calibrated inequality:
\[
\partial_t d_I(x,t) + (\mathbf{H}(x,t) \cdot \nabla)d_I(x,t) = 0 \quad \text{in } I_t(\delta_0).
\]
(2.8a)
\[
\nabla \cdot \boldsymbol{\xi} + \mathbf{H} \cdot \boldsymbol{\xi} = O(d_I),
\]
(2.8b)
\[
\partial_t \phi - (\nabla \nabla d_I + (\nabla \mathbf{H})^T \nabla \phi) = O(d_I),
\]
(2.8c)
\[
\partial_t |\boldsymbol{\xi}|^2 + (\nabla \cdot \nabla |\boldsymbol{\xi}|^2) = O \left( \frac{d_I^2}{\delta_0} \right),
\]
(2.8d)

where $\nabla := \{\partial_j \mathbf{H}_i\}_{1 \leq i,j \leq 2}$ is a matrix with $i$ being the row index.

**Proof of (2.8).** Using (2.3) and (2.5), we can write (2.2) as the transport equation (2.8a). Recall (1.8), we denote $\phi_\delta(\tau) = \phi(\frac{\tau}{\delta_0})$ which is an even function. We compute from $\phi_\delta'(0) = 0$ that
\[
\nabla \cdot \boldsymbol{\xi} = |\nabla d_I|^2 \phi_\delta'(d_I) + \phi_\delta(d_I) \Delta d_I(x,t)
\]
\[
= O(d_I) + \phi_\delta(d_I) \Delta d_I(P_I(x,t), t)
\]
\[
= O(d_I) - \mathbf{H} \cdot \boldsymbol{\xi}
\]
and this leads to (2.8b). By (2.8a) we have the following identities in $I_t(\delta_0)$:
\[
\partial_t \nabla d_I + (\mathbf{H} \cdot \nabla) \nabla d_I + (\nabla \mathbf{H})^T \nabla d_I = 0,
\]
\[
\partial_t \phi_\delta(d_I) + (\mathbf{H} \cdot \nabla) \phi_\delta(d_I) = 0.
\]
These imply (2.8c). Testing (2.8c) by $\boldsymbol{\xi}$ and using (2.6) lead us to (2.8d).

We denote the phase-field analogs of the mean curvature and the normal vector by
\[
\mathbf{H}^\varepsilon(x,t) := -\left( \varepsilon \Delta u^\varepsilon - \frac{1}{\varepsilon} \nabla F(u^\varepsilon) \right) \cdot \frac{\nabla u^\varepsilon}{|\nabla u^\varepsilon|},
\]
(2.9a)
\[
n^\varepsilon(x,t) := \frac{\nabla \psi^\varepsilon}{|\nabla \psi^\varepsilon|}(x,t),
\]
(2.9b)
respectively, where $\psi^\varepsilon$ is defined by (1.12). Note that in (2.9a), the inner product is made with the column vectors of $\nabla u^\varepsilon = (\partial_1 u^\varepsilon, \partial_2 u^\varepsilon)$. By chain rule and (1.12)
\[
\partial_t \psi^\varepsilon(x,t) = \partial_t |u^\varepsilon|(x,t) \sqrt{2f(|u^\varepsilon|)}(x,t) \quad \text{for a.e. } (x,t) \in \Omega \times (0,T).
\]
(2.10)

Here and in the sequel, we use abbreviations
\[
\partial_0 = \partial_t, \quad \partial_1 = \partial_{x_1}, \quad \partial_2 = \partial_{x_2}.
\]

Equation (2.10) motivates the projection of $\partial_1 u^\varepsilon$ into the span of $\partial d_F(u^\varepsilon)$:
\[
\Pi_{u^\varepsilon} \partial_1 u^\varepsilon = \left( \partial_1 u^\varepsilon \cdot \frac{\partial d_F(u^\varepsilon)}{\partial d_F(u^\varepsilon)} \right) \frac{\partial d_F(u^\varepsilon)}{|\partial d_F(u^\varepsilon)|}, \quad \text{if } \partial d_F(u^\varepsilon) \neq 0,
\]
(2.11)
\[
\Pi_{u^\varepsilon} \partial_1 u^\varepsilon = 0, \quad \text{otherwise}.
\]

Hence, (2.10) implies
\[
|\nabla \psi^\varepsilon| = |\Pi_{u^\varepsilon} \nabla u^\varepsilon| |\partial d_F(u^\varepsilon)| \quad \text{for a.e. } (x,t) \in \Omega \times (0,T),
\]
(2.12a)
\[
\Pi_{u^\varepsilon} \nabla u^\varepsilon = \frac{|\nabla \psi^\varepsilon|}{|\partial d_F(u^\varepsilon)|^2} \partial d_F(u^\varepsilon) \otimes n^\varepsilon \quad \text{for a.e. } (x,t) \in \Omega \times (0,T).
\]
(2.12b)
Remark 2.1. It follows from (1.12) and (1.12) that $\partial d_F(u) = 0$ if and only if $u = 0$ or $u \in S^1$. So we might encounter issues in the definition (2.11) on the set $U_t = \{x \in \Omega : |u^\varepsilon(x,t)| \in \{0,1\}\}. However, due to the space-time analyticity of parabolic systems (see for instance [20, 21, 25]), for each fixed $t > 0$, $U_t$ is a null set of $\mathbb{R}^2$ under 2-Lebesgue’s measure, and thus (2.11) is well-defined up to a null set of $\Omega$.

We deduce from Remark 2.1 that (2.11) can be simplified to

$$\Pi_u^\varepsilon \partial_t u^\varepsilon = \left( \partial_t u^\varepsilon \cdot \frac{u^\varepsilon}{|u^\varepsilon|} \right)$$
for a.e. $(x,t) \in \Omega \times (0,T)$ (2.13)

The following lemma establishes various lower bounds of $E_\varepsilon[u^\varepsilon|I]$ defined by (1.11). As we shall not integrate the time variable $t$ throughout this section, we shall abbreviate the spatial integration $\int_\Omega$ by $\int$ and sometimes we omit the $dx$ for simplicity.

Lemma 2.2. There exists a universal constant $C > 0$ which is independent of $t \in [0,T)$ and $\varepsilon$ such that the following estimates hold for every $t \in (0,T)$:

$$\int \left( \frac{\varepsilon}{2} |\nabla u^\varepsilon|^2 + \frac{1}{\varepsilon} F(u^\varepsilon) - |\nabla \psi^\varepsilon| \right) dx \leq E_\varepsilon[u^\varepsilon|I](t),$$

$$\varepsilon \int \left( \mu |\nabla u^\varepsilon|^2 + |\nabla \psi^\varepsilon - \Pi_u^\varepsilon \nabla u^\varepsilon|^2 \right) dx \leq 2E_\varepsilon[u^\varepsilon|I](t),$$

$$\int \left( \sqrt{\varepsilon} |\Pi_u^\varepsilon \nabla u^\varepsilon| - \frac{1}{\sqrt{\varepsilon}} |\partial d_F(u^\varepsilon)| \right)^2 dx \leq 2E_\varepsilon[u^\varepsilon|I](t),$$

$$\int \left( \frac{\varepsilon}{2} |\nabla u^\varepsilon|^2 + \frac{1}{\varepsilon} F(u^\varepsilon) + |\nabla \psi^\varepsilon| \right) (1 - \xi \cdot n^\varepsilon) dx \leq 4E_\varepsilon[u^\varepsilon|I](t),$$

$$\int \left( \frac{\varepsilon}{2} |\nabla u^\varepsilon|^2 + \frac{1}{\varepsilon} F(u^\varepsilon) + |\nabla \psi^\varepsilon| \right) \min(d_1^2, 1) dx \leq CE_\varepsilon[u^\varepsilon|I](t).$$

Proof. The case $\mu \equiv 0$ was proved in [22], and the proof carries out to the current case.

The last term in (1.11) can be written as $\xi \cdot \nabla \psi^\varepsilon$ after an integration by parts. Then using the orthogonality of (2.13), we can write

$$E_\varepsilon[u^\varepsilon|I](t) = \int \frac{\varepsilon}{2} |\nabla u^\varepsilon|^2 + \frac{\varepsilon}{2} |\nabla \psi^\varepsilon|^2 + \frac{1}{\varepsilon} F(u^\varepsilon) - |\nabla \psi^\varepsilon| + \int |\nabla \psi^\varepsilon|(1 - \xi \cdot n^\varepsilon)$$

$$= \int \left( \mu |\nabla u^\varepsilon|^2 + |\nabla \psi^\varepsilon - \Pi_u^\varepsilon \nabla u^\varepsilon|^2 \right)$$

$$+ \int \frac{\varepsilon}{2} |\Pi_u^\varepsilon \nabla u^\varepsilon|^2 + \frac{1}{\varepsilon} F(u^\varepsilon) - |\nabla \psi^\varepsilon| + \int |\nabla \psi^\varepsilon|(1 - \xi \cdot n^\varepsilon)$$

(2.15)

This implies (2.14a) and (2.14b). By (1.12), we have $|\partial d_F(u)| = \sqrt{2F(u)}$. Apply this and the chain rule in form of (2.12a) to the second integral of (2.15) and complete the square, we obtain (2.14c). Combining (2.14a) with the calibration control

$$E_\varepsilon[u^\varepsilon|I] \geq \int (1 - \xi \cdot n^\varepsilon) |\nabla \psi^\varepsilon|$$

(2.16)

and $1 - \xi \cdot n^\varepsilon \leq 1$ yields (2.14d). Finally, by (1.9) and $\delta_0 \in (0,1)$ we have

$$1 - \xi \cdot n^\varepsilon \geq 1 - \phi \left( \frac{d_I}{\delta_0} \right) \geq \min \left( \frac{d_I^2}{2\delta_0^2}, 1 - \phi \left( \frac{1}{2} \right) \right) \geq C_{\phi, \delta_0} \min(d_I^2, 1).$$

(2.17)

This together with (2.14d) implies (2.14e). \qed

The following result was first proved in [11] in the case of the scalar Allen-Cahn equation, and was generalized to the vectorial case in [22].
Proposition 2.3. There exists a generic constant $C > 0$ depending only on the geometry of the interface $I_t$ so that
\[
\frac{d}{dt} E_\varepsilon[u^\varepsilon|I] + \frac{1}{2\varepsilon} \int \left( \varepsilon^2 |\partial_t u^\varepsilon|^2 - |H^\varepsilon|^2 \right) dx + \frac{1}{2\varepsilon} \int |\varepsilon \partial_t u^\varepsilon - (\nabla \cdot \xi) \partial d_F(u^\varepsilon)|^2 dx \\
+ \frac{1}{2\varepsilon} \int \left| H^\varepsilon - \varepsilon |\nabla u^\varepsilon| H \right|^2 dx \leq CE_\varepsilon[u^\varepsilon|I].
\] (2.18)

We present the proof for the convenience of the readers. It is based on the following two lemmas, whose proofs are left to Appendix A.

Lemma 2.4. The following identity holds
\[
\int \nabla H : (\xi \otimes n^\varepsilon) |\nabla \psi^\varepsilon| dx = \int (\nabla \cdot H) \xi \cdot \nabla \psi^\varepsilon dx \\
= \int \nabla H : (\xi - n^\varepsilon) \otimes n^\varepsilon |\nabla \psi^\varepsilon| dx + \int H^\varepsilon \cdot H |\nabla u^\varepsilon| dx \\
+ \int \nabla \cdot H \left( \frac{\varepsilon}{2} |\nabla u^\varepsilon|^2 + \frac{1}{\varepsilon} F(u^\varepsilon) - |\nabla \psi^\varepsilon| \right) dx + \int \nabla \cdot H (|\nabla \psi^\varepsilon| - \xi \cdot \nabla \psi^\varepsilon) dx \\
- \int (\nabla H)_{ij} \varepsilon (\partial_i u^\varepsilon \cdot \partial_j u^\varepsilon) dx + \int \nabla H : (n^\varepsilon \otimes n^\varepsilon) |\nabla \psi^\varepsilon| dx
\] (2.19)

Here and in the sequel, we adopt Einstein summation convention by summing over repeated Latin index. We shall also use the Frobenius inner product $A : B \triangleq \text{tr} A^T B$ for two square matrices $A, B$. The second lemma gives the expansion of the time derivative of \( (1.11) \).

Lemma 2.5. Under the assumptions of Theorem 1.7, the following identity holds
\[
\frac{d}{dt} E[u^\varepsilon|I] + \frac{1}{2\varepsilon} \int \left( \varepsilon^2 |\partial_t u^\varepsilon|^2 - |H^\varepsilon|^2 \right) dx \\
+ \frac{1}{2\varepsilon} \int |\varepsilon \partial_t u^\varepsilon - (\nabla \cdot \xi) \partial d_F(u^\varepsilon)|^2 dx + \frac{1}{2\varepsilon} \int |H^\varepsilon - \varepsilon |\nabla u^\varepsilon| H |^2 dx
\] (2.20a)
\[
= \frac{1}{2\varepsilon} \int |(\nabla \cdot \xi) \partial d_F(u^\varepsilon)| n^\varepsilon + |\Pi_{ur} \nabla u^\varepsilon| H |^2 dx \\
+ \frac{\varepsilon}{2} \int |H|^2 (|\nabla u^\varepsilon|^2 - |\Pi_{ur} \nabla u^\varepsilon|^2) dx - \int \nabla H \cdot (\xi - n^\varepsilon)^{\otimes 2} |\nabla \psi^\varepsilon| dx
\] (2.20b)
\[
+ \int (\nabla \cdot H) \left( \frac{\varepsilon}{2} |\nabla u^\varepsilon|^2 + \frac{1}{\varepsilon} F(u^\varepsilon) - |\nabla \psi^\varepsilon| \right) dx
\] (2.20c)
\[
+ \int (\nabla \cdot H) (1 - \xi \cdot n^\varepsilon) |\nabla \psi^\varepsilon| dx + J_1^\varepsilon + J_2^\varepsilon,
\] (2.20d)

where $J_1^\varepsilon, J_2^\varepsilon$ are given by
\[
J_1^\varepsilon := \int \nabla H : n^\varepsilon \otimes n^\varepsilon (|\nabla \psi^\varepsilon| - \varepsilon |\nabla u^\varepsilon|^2) dx \\
+ \varepsilon \int \nabla H : (n^\varepsilon \otimes n^\varepsilon) (|\nabla u^\varepsilon|^2 - |\Pi_{ur} \nabla u^\varepsilon|^2) dx \\
- \varepsilon \int (\nabla H)_{ij} \left( (\partial_i u^\varepsilon - \Pi_{ur} \partial_i u^\varepsilon) \cdot (\partial_j u^\varepsilon - \Pi_{ur} \partial_j u^\varepsilon) \right) dx
\] (2.21)
\[
J_2^\varepsilon := - \int \left( \partial_i \xi + (H \cdot \nabla) \xi + (\nabla H)^T \xi \right) \cdot (n^\varepsilon - \xi) |\nabla \psi^\varepsilon| dx \\
- \int \left( \partial_i \xi + (H \cdot \nabla) \xi \right) \cdot \xi |\nabla \psi^\varepsilon| dx.
\] (2.22)

Proof of Proposition 2.3. The proof here is exactly the same as the case $\mu = 0$, done in [22, Lemma 4.4]. This is because the form of the energy dissipation law (2.1) remains unchanged in the presence
of the divergence term in (1.3). Note that in the statement of Lemma 4.4 of [22] the second term on the left hand side of (2.20) is missing, though the proof of the identity there is correct. See [22, equation (4.33)]

We first estimate the RHS of (2.20) by $E\epsilon[u^\epsilon|I]$ up to a constant that only depends on $I_t$. We start with (2.20a): it follows from the triangle inequality that

$$\frac{1}{2} \int \left| \frac{1}{\sqrt{\epsilon}}(\nabla \cdot \xi) \frac{\partial d_F(u^\epsilon)}{\partial \xi} \right| n^\epsilon + \sqrt{\epsilon} |\Pi_{u^\epsilon} \nabla u^\epsilon| |H|^2 \ dx$$

$$\leq \int \left| \frac{1}{\sqrt{\epsilon}}(\nabla \cdot \xi) \left( \sqrt{\epsilon} |\Pi_{u^\epsilon} \nabla u^\epsilon| - \frac{1}{\sqrt{\epsilon}} |\partial d_F(u^\epsilon)| \right) \right| n^\epsilon \ dx$$

$$+ \int \left| (\nabla \cdot \xi) \sqrt{\epsilon} |\Pi_{u^\epsilon} \nabla u^\epsilon| (n^\epsilon - \xi) \right| \ dx$$

$$+ \int \left| (H + (\nabla \cdot \xi) \xi) \sqrt{\epsilon} |\Pi_{u^\epsilon} \nabla u^\epsilon| \right|^2 \ dx.$$

The first integral on the RHS of the above inequality is controlled by (2.14c). Due to the elementary inequality $|\xi - n^\epsilon|^2 \leq 2(1 - n^\epsilon \cdot \xi)$, the second integral is controlled by (2.14d). The third integral can be treated using the relation $H = (H \cdot \xi) \xi + O(d_F(x,t))$ and (2.8d). So it can be controlled by (2.14c).

The integrals in (2.20c) can be controlled using (2.14c) and (2.14d). The integrals in (2.20b) can be controlled using (2.14d). The first term in (2.20d) can be controlled using (2.14d). It remains to estimate (2.21) and (2.22). The last two terms on the RHS of $J_\epsilon^1$ can be bounded using (2.14c), and the first integral can be rewritten using $n^\epsilon = n^\epsilon - \xi + \xi$:

$$J_\epsilon^1 \lesssim \int \nabla H : (n^\epsilon \otimes (\xi - \xi)) (||\nabla \psi^\epsilon|| - \epsilon |\nabla u^\epsilon|^2) \ dx$$

$$+ \int \nabla H : n^\epsilon \otimes \xi (||\nabla \psi^\epsilon|| - \epsilon |\nabla u^\epsilon|^2) \ dx + CE\epsilon[u^\epsilon|I]$$

$$\lesssim \int |n^\epsilon - \xi| \left( \frac{\epsilon |\nabla u^\epsilon|^2}{n^\epsilon - \xi} - |\Pi_{u^\epsilon} \nabla u^\epsilon|^2 \right) dx + \int |n^\epsilon - \xi| |\nabla \psi^\epsilon|| \ dx$$

$$+ \int \min (d_I, 1) (||\nabla \psi^\epsilon|| + \epsilon |\nabla u^\epsilon|^2) \ dx + E\epsilon[u^\epsilon|I].$$

(2.23)

Note that in the last step we employed $\nabla H : n^\epsilon \otimes \xi = (\xi \cdot \nabla) H \cdot n^\epsilon$ and (2.6). The first and the third integral in the last step can be treated by (2.14b) and (2.14c) respectively. Then we employ (2.12a) and yield

$$J_\epsilon^1 \lesssim \int |n^\epsilon - \xi| \left( |\Pi_{u^\epsilon} \nabla u^\epsilon|^2 - |\nabla \psi^\epsilon|| \ dx + E\epsilon[u^\epsilon|I]$$

$$= \int |n^\epsilon - \xi| \left| \sqrt{\epsilon} |\Pi_{u^\epsilon} \nabla u^\epsilon| - \frac{1}{\sqrt{\epsilon}} |\partial d_F(u^\epsilon)| \right| \ dx + E\epsilon[u^\epsilon|I].$$

(2.24)

Finally applying the Cauchy-Schwarz inequality and then (2.14c) and (2.14d), we obtain $J_\epsilon^1 \lesssim E\epsilon[u^\epsilon|I]$. As for $J_\epsilon^2$ (2.22), we employ (2.8c) and (2.8d) and deduce

$$J_\epsilon^2 \lesssim \int \left( |n^\epsilon - \xi|^2 + \min(d_I, 1) \right) |\nabla \psi^\epsilon| \ dx \lesssim E\epsilon[u^\epsilon|I],$$

(2.25)

after applying (2.14d) and (2.14c). So we proved that the RHS of (2.20) is bounded by $E\epsilon[u^\epsilon|I]$ up to a multiplicative constant which only depends on $I_t$. \qed

We end this section by the construction of initial data $u_{rin}^\epsilon$ satisfying (1.15b).

**Proof of Proposition 1.2.** Let $I_0 \subset \Omega$ be the initial interface and $\eta_0(z)$ be the cut-off function (2.4). Then we define

$$s_\epsilon(x) := \eta_0(x) \theta \left( \frac{d_{I_0}(x)}{\epsilon} \right) + \left( 1 - \eta_0(x) \right) 1_{\Omega^\dagger},$$

(2.26)
where $\theta(z)$ is the solution of the ODE
\begin{equation}
-\theta''(z) + f'(\theta) = 0, \quad \theta(-\infty) = 0, \quad \theta(+\infty) = 1.
\end{equation}
(2.27)

For every $u_m \in H^1(\Omega, S^1)$ with trace $u_m|_{I_0} \cdot n_{I_0} = 0$, the initial datum defined by
\begin{equation}
u^\varepsilon_m(x) := s_\varepsilon(x)u_m(x)
\end{equation}
(2.28)
satisfies $u^\varepsilon_m \in H^1(\Omega) \cap L^\infty(\Omega)$ and
\begin{equation}
u^\varepsilon_m(x) = \begin{cases} \theta \left( \frac{d_I(x)}{\varepsilon} \right) u_m & \text{if } x \in \Omega_0^+ \setminus I_0(2\delta_0), \\
\theta \left( \frac{d_I(x)}{\varepsilon} \right) u_m & \text{if } x \in I_0(\delta_0), \\
0 & \text{if } x \in \Omega_0^+ \setminus I_0(2\delta_0).
\end{cases}
\end{equation}
(2.29)

To verify (1.15b), we first compute the modulated energy (1.11) of the initial data $u^\varepsilon_m$. We write (2.26) by
\begin{equation}s_\varepsilon(x) = \theta \left( \frac{d_I(x)}{\varepsilon} \right) + \hat{s}_\varepsilon(x),
\end{equation}
(2.30)
where $\hat{s}_\varepsilon(x) := (1 - \eta_0(x)) \left( I_{\Omega_0^+} - \theta \left( \frac{d_I(x)}{\varepsilon} \right) \right)$. By (2.24) and the exponential decay of $\theta$, which solves (2.27), we deduce that
\begin{equation}
\| \hat{s}_\varepsilon \|_{L^\infty(\Omega)} + \| \nabla \hat{s}_\varepsilon \|_{L^\infty(\Omega)} \leq Ce^{-\frac{\varepsilon}{\delta}},
\end{equation}
(2.31)
for some constant $C > 0$ that only depends on $I_0$. Thus by (1.8)
\begin{equation}F(u^\varepsilon_m) = f(\theta + \hat{s}_\varepsilon) = f(\theta) + O(e^{-C/\varepsilon}).
\end{equation}
With (2.28) and (2.30), we write
\begin{equation}
\| \nabla u^\varepsilon_m \|^2 = \varepsilon^{-2}\theta^2 + \varepsilon^2 \left| \nabla u^\varepsilon_m \right|^2 + O(e^{-C/\varepsilon})(\left| \nabla u^\varepsilon_m \right|^2 + 1).
\end{equation}
Recall (1.2), we have $\sqrt{f(s)} = \sigma_0^{1/2} s(1 - s^2)$ for any $s \in [0, 1]$ and $\psi^\varepsilon = \int_0^{\theta + \hat{s}_\varepsilon} \sqrt{2f(s)} ds$. So the integrand of $E_\varepsilon[u^\varepsilon[I](0)$ can be written as
\begin{equation}
\frac{\varepsilon^2}{2} \left| \nabla u^\varepsilon_m \right|^2 + \varepsilon^{-1} \left| \nabla \psi^\varepsilon \right| + \frac{1}{\varepsilon} \int_0^{\theta + \hat{s}_\varepsilon} \left| \nabla u^\varepsilon_m \right|^2 + O(e^{-C/\varepsilon})(\left| \nabla u^\varepsilon_m \right|^2 + 1).
\end{equation}
(2.32)
By (1.8) we know $1 - \xi \cdot n_{I_0} = O(d_I^2)$. So we have
\begin{equation}\varepsilon^{-1} \xi \cdot n_{I_0} \theta' \sqrt{2f(\theta)} = \varepsilon^{-1} \theta' \sqrt{2f(\theta)} + O(e^{-C/\varepsilon}) + \varepsilon^{-1} O(d_I^2) \theta' \sqrt{2f(\theta)}.
\end{equation}
Note that the last term can be written as
\begin{equation}O(d_I^2) \theta' \sqrt{2f(\theta)} = O(\varepsilon) \zeta^2 \theta'(z) \sqrt{2f(\theta(z))} = d_I^2 \theta'(x).
\end{equation}
Substituting the above two equations into (2.32) yields
\begin{equation}
\frac{\varepsilon}{2} \left| \nabla u^\varepsilon_m \right|^2 + \varepsilon^{-1} \int_0^{\theta + \hat{s}_\varepsilon} \left| \nabla \psi^\varepsilon \right| + \frac{1}{\varepsilon} \int_0^{\theta + \hat{s}_\varepsilon} \left| \nabla u^\varepsilon_m \right|^2 + O(e^{-C/\varepsilon})(\left| \nabla u^\varepsilon_m \right|^2 + 1) + O(\varepsilon).
\end{equation}
(2.33)
Note that the integrand of the first integral vanishes because $\theta'(z) = 2f(\theta(z))$, a consequence of (2.27). Now we turn to the first term in (1.11). Using (2.31)
\begin{equation}
\| \nabla u^\varepsilon_m \|^2 \leq \| \nabla \theta \cdot u_m \|^2 + \theta^2 \| \nabla u^\varepsilon_m \|^2 + O(e^{-C/\varepsilon}).
\end{equation}
(2.34)
Using the exponential decay of $\theta'$ and $u_m \cdot n_{I_0} = 0$, the second term on the RHS can be written by
\begin{equation}
\| \nabla \theta \cdot u_m \| = \left| \frac{d_{I_0}(x)}{\varepsilon} \theta' \left( \frac{d_{I_0}(x)}{\varepsilon} \right) \frac{u_m \cdot n_{I_0}}{d_{I_0}(x)} \right| \leq C \left| \frac{u_m \cdot n_{I_0}}{d_{I_0}} \right|
\end{equation}
(2.35)
Applying Hardy’s inequality \([3]\) (in the normal direction) yields
\[
\int_{\Omega} |\nabla \theta \cdot \mathbf{u}_m|^2 \, dx \leq C \int_{\Omega} \left| \nabla \mathbf{u}_m \right|^2 \, dx.
\] (2.36)

Combining this with \((2.31)\) and \((2.33)\) leads to \(E_\varepsilon [\mathbf{u}^\varepsilon | I](0) \leq C \varepsilon\).

Now we verify the second integral in \((1.16)\). We shall only give the estimate in \(\Omega_1^+\) and the one for \(\Omega_1^-\) follows from exactly the same way. We use \((1.13)\) to deduce that at \(t = 0\), we have
\[
\int_{I_t(\delta_0) \cap \Omega_1^+} \left| \psi^\varepsilon - 1 \right| dt \, dx \bigg|_{t=0} = \int_{I_t(\delta_0) \cap \Omega_1^+} \left( \int_{s_t(\varepsilon)}^1 \sqrt{2f(s)} \, ds \right) \, dt \, dx \bigg|_{t=0}
\]
\[
= \varepsilon \int_{I_t(\delta_0) \cap \Omega_1^+} \left( \int_{s_t(\varepsilon)}^1 \sqrt{2f(s)} \, ds \right) \frac{dt}{\varepsilon} \, dx \bigg|_{t=0} + O(e^{-C/\varepsilon}) \quad (2.37)
\]

By the exponential decay of \(\beta(z) := z \int_{s_t(\varepsilon)}^1 \sqrt{2f(s)} \, ds\) to 0 as \(z \uparrow \infty\), we obtain \((1.15B)\).

---

3. Uniform estimates of solutions

The second term on the LHS of \((2.18)\) does not have an obvious sign. When \(\mu = 0\), it is non-negative by \((2.9a)\) and \((1.3)\). The main task of this section is to show that it is controllable for any fixed \(\mu > 0\) in space dimension two.

**Theorem 3.1.** Under the assumptions of Theorem 1.1, there exists a generic constant \(C > 0\) such that
\[
\sup_{t \in [0,T]} \varepsilon^{-1} E_\varepsilon[\mathbf{u}^\varepsilon | I] + \int_0^T \int_{\Omega} \left| \partial_t \mathbf{u}^\varepsilon + (\mathbf{H} \cdot \nabla) \mathbf{u}^\varepsilon \right|^2 + \left| \partial_t \mathbf{u}^\varepsilon - \Pi_{\mathbf{u}^\varepsilon} \partial_t \mathbf{u}^\varepsilon \right|^2 \, dx \, dt \leq e^{(1+T)C(I_0)}.
\] (3.1)

The crucial tool is the following lemma on the estimates of tangential derivatives.

**Lemma 3.2.** Let \(\eta_1 \in C_\infty(I_t(4\delta_0))\) be a cut-off function with \(\eta_1 = 1\) in \(I_t(3\delta_0)\). There exists a universal constant \(C > 0\) which is independent of \(t \in (0,T)\) and \(\varepsilon\) such that the following estimates hold for every \(t \in (0,T)\):
\[
\int \eta_1 \left| \nabla \mathbf{u}^\varepsilon (I_2 - \mathbf{n} \otimes \mathbf{n}) \right|^2 \leq C \varepsilon^{-1} E_\varepsilon[\mathbf{u}^\varepsilon | I].
\] (3.2)

**Proof.** Using \((2.12b)\), we can estimate the matrix product \((\Pi_{\mathbf{u}^\varepsilon} \nabla \mathbf{u}) (I_2 - \mathbf{\xi} \otimes \mathbf{n}^\varepsilon)\) by
\[
\left| \Pi_{\mathbf{u}^\varepsilon} \nabla \mathbf{u}^\varepsilon (I_2 - \mathbf{n}^\varepsilon \otimes \mathbf{\xi}) \right|^2
\]
\[
\leq \left( \frac{|\nabla \psi^\varepsilon|}{|\partial d_F(\mathbf{u}^\varepsilon)|^2} \right) \left| \partial d_F(\mathbf{u}^\varepsilon) \otimes (\mathbf{n}^\varepsilon - \mathbf{\xi}) \right|^2
\]
\[
\leq 2(1 - \mathbf{\xi} \cdot \mathbf{n}^\varepsilon) \left| \Pi_{\mathbf{u}^\varepsilon} \nabla \mathbf{u}^\varepsilon \right|^2
\]
\[
\leq 2(1 - \mathbf{\xi} \cdot \mathbf{n}^\varepsilon) \left| \nabla \mathbf{u}^\varepsilon \right|^2.
\]

This together with \((2.14d)\) implies
\[
\int \left| \Pi_{\mathbf{u}^\varepsilon} \nabla \mathbf{u}^\varepsilon (I_2 - \mathbf{n}^\varepsilon \otimes \mathbf{\xi}) \right|^2 \leq C \varepsilon^{-1} E_\varepsilon[\mathbf{u}^\varepsilon | I] .
\] (3.3)

In \(I_t(4\delta_0)\) where \(\mathbf{n}\) is well-defined, we have the decomposition
\[
I_2 - \mathbf{n}^\varepsilon \otimes \mathbf{n} = I_2 - \mathbf{n}^\varepsilon \otimes \mathbf{\xi} + \mathbf{n}^\varepsilon \otimes (\mathbf{\xi} - \mathbf{n})
\] (3.4)

It follows from \((1.8)\) and \((1.9)\) that
\[
|\mathbf{\xi} - \mathbf{n}|^2 \leq 2|\mathbf{\xi} - \mathbf{n}| = 2 \left( 1 - \phi(d_t^\varepsilon) \right) \lesssim \min \left( d_t^2, 1 \right).
\] (3.5)

This together with \((3.3)\) and \((2.14c)\) leads to
\[
\int \eta_1 \left| \Pi_{\mathbf{u}^\varepsilon} \nabla \mathbf{u}^\varepsilon (I_2 - \mathbf{n}^\varepsilon \otimes \mathbf{n}) \right|^2 \leq C \varepsilon^{-1} E_\varepsilon[\mathbf{u}^\varepsilon | I].
\] (3.6)
To prove (3.2), by the decomposition in \( I_t(4\delta_0) \)
\[
\mathbf{n}^e \otimes \mathbf{n} - I_2 = (\mathbf{n}^e - \mathbf{\xi}) \otimes \mathbf{n} + (\mathbf{\xi} - \mathbf{n}) \otimes \mathbf{n} + \mathbf{n} \otimes \mathbf{n} - I_2
\]
and (3.5) and (3.6), and it suffices to prove
\[
\int |\nabla \mathbf{u}^e|^2 \left( \min\left( d_t^2, 1 \right) + (1 - \mathbf{\xi} \cdot \mathbf{n}^e) \right) \leq C \varepsilon^{-1} E_\varepsilon[\mathbf{u}^e]|I|.
\]
Such an estimate is established by (2.14d) and (2.14e).

To proceed we need a higher integrability estimate of \( \mathbf{u}^e \). In contrast to the case when \( \mu = 0 \), the maximum modulus estimate in [22] Lemma 3.3 is no longer valid. However, we have the following lemma which implies \( L^6 \)-estimate of \( \mathbf{u}^e \) due to the choice (1.2).

**Lemma 3.3.** Under the assumption (1.15a), there exists a constant \( C_1 = C_1(c_1) > 0 \) such that
\[
\sup_{t \in [0,T]} A_\varepsilon(\mathbf{u}^e(\cdot, t)) + \sup_{t \in [0,T]} \|\nabla \psi^e\|_{L^1(\Omega)} \leq C_1,
\]
\[
\|\mathbf{u}^e(\cdot, t)\|_{L^2(\Omega)}^2 \leq C_1 \left( 1 + \varepsilon^{-1} E_\varepsilon[\mathbf{u}^e]|I|(t) \right).
\]

**Proof.** To prove (3.7a), we use (2.1), (2.13) and the Cauchy–Schwarz inequality to deduce that
\[
A_\varepsilon(\mathbf{u}^e) \geq \int_\Omega \left( \frac{\varepsilon}{2} |\Pi_{\mathbf{u}^e} \nabla \mathbf{u}^e|^2 + \frac{1}{\varepsilon} F(\mathbf{u}^e) \right) dx \geq \int_\Omega |\nabla \psi^e| dx.
\]
To prove (3.7b), we first prove an \( L^1 \)-estimate. We first note that if \( |\mathbf{u}^e| > 2 \), then by (1.13),
\[
\mathbf{\psi}^e = \int_0^2 \sqrt{2f(s)} ds + \int_2^{(|\mathbf{u}^e|)} \sqrt{2f(s)} ds \geq \int_0^2 \sqrt{2f(s)} ds + \sqrt{2f(2)}(|\mathbf{u}^e| - 2).
\]
This implies the \( L^1 \)-estimate by Sobolev’s imbedding:
\[
\int_\Omega |\mathbf{u}^e| dx = \int_{\{|x|:|\mathbf{u}^e| \leq 2\}} |\mathbf{u}^e| dx + \int_{\{|x|:|\mathbf{u}^e| > 2\}} |\mathbf{u}^e| dx
\]
\[
\leq 2|\Omega| + C \int_{\{|x|:|\mathbf{u}^e| > 2\}} (|\mathbf{\psi}^e| + 1) dx \leq 1 + \|\mathbf{\psi}^e\|_{L^1(\Omega)} \leq 1 + \|\nabla \mathbf{\psi}^e\|_{L^1(\Omega)}.
\]
Actually the above argument already leads to the \( L^2 \)-estimate thanks to the Sobolev’s imbedding in two dimensions, which controls \( \|\mathbf{\psi}^e\|_{L^2(\Omega)} \). However we would like to present a proof which also applies to higher dimensions and a more general potential (1.2).

To pass from \( L^1 \) to \( L^2 \), we shall prove a weighted Poincaré inequality on \( U = \Omega_t^\varepsilon \):
\[
\min_{\lambda \in \mathbb{R}} \int_U |u - \lambda|^2 dx \leq C \int_U \text{dist}^2(x, \partial U)|\nabla u|^2 dx, \quad \forall u \in W^{1,2}(U).
\]
Note that the minimum in (3.9) is achieved by \( \lambda = \frac{1}{|U|} \int_U u dx \). So (3.9), (3.8) and (2.14e) lead to the \( L^2 \)-estimate of \( \mathbf{u}^e \).

The estimate (3.9) is a consequence of the following inequality, proved in [24] Theorem 1.5:
\[
\int_U |g|^2 dx \leq C_U \int_U \left( |g|^2 + |\nabla g|^2 \right) \text{dist}^2(x, \partial U) dx \quad \forall g \in W^{1,2}_{\text{loc}}(U) \cap L^2(U).
\]
Actually, to pass from (3.10) to (3.9), we fix \( \delta > 0 \) such that \( C_U \delta^2 \leq \frac{1}{2} \), and let \( U_\delta = \{ x \in U : \text{dist}(x, \partial U) > \delta \} \). By the ordinary Poincaré inequality for \( U_\delta \), there exists \( a \in \mathbb{R} \) such that
\[
\int_{U_\delta} |u - a|^2 dx \leq C_\delta \int_{U_\delta} |\nabla u|^2 dx \leq C_\delta \frac{\delta^2}{2} \int_{U_\delta} |\nabla u|^2 \text{dist}^2(x, \partial U) dx
\]
Applying (3.10) with \( g = u - a \) and using \( \text{dist}^2(x, \partial U)C_U \leq \frac{1}{2} \) for \( x \in U \backslash U_\delta \) yield
\[
\int_U |u - a|^2 dx \leq \frac{1}{2} \int_{U \backslash U_\delta} |u - a|^2 dx + \left( \frac{C_\delta}{\delta^2} + 2 \right) C_U \int_U |\nabla u|^2 \text{dist}^2(x, \partial U) dx,
\]
and this implies (3.9).
Proof of Theorem 3.1: During the proof we shall use the following notation

\[ \nabla^\perp = (\partial_2, \partial_1), \]
\[ w^+ = (-w_2, w_1) \text{ for } w = (w_1, w_2) \in C^1(\Omega, \mathbb{R}^2), \]
\[ \text{rot } w = \nabla^\perp \cdot w = -\partial_2 w_1 + \partial_1 w_2. \]

We first establish a priori estimates of the solutions \( u^\varepsilon \) which are independent of \( \varepsilon \). It follows from (2.18) that

\[
\frac{2}{\varepsilon} \frac{d}{dt} E_\varepsilon[u^\varepsilon|I] + \frac{1}{\varepsilon^2} \int \left( \varepsilon^2 |\partial_t u^\varepsilon|^2 - |H^\varepsilon|^2 \right) + \left| H^\varepsilon - \varepsilon |\nabla u^\varepsilon| \right|^2 dx 
+ \frac{1}{\varepsilon^2} \int |\varepsilon \partial_t u^\varepsilon - \partial d_F(u^\varepsilon)(\nabla \cdot \varepsilon)|^2 dx \leq \frac{C}{\varepsilon} E_\varepsilon[u^\varepsilon|I] \tag{3.14}
\]

On the other hand, using the orthogonal projection (2.11), we can write

\[
|\varepsilon \partial_t u^\varepsilon - \varepsilon \Pi_{u^\varepsilon} \partial_t u^\varepsilon|^2 + |\varepsilon \Pi_{u^\varepsilon} \partial_t u^\varepsilon - \partial d_F(u^\varepsilon)(\nabla \cdot \varepsilon)|^2
\]

This together with (3.14) yields

\[
\frac{2}{\varepsilon} \frac{d}{dt} E_\varepsilon[u^\varepsilon|I] + \frac{1}{\varepsilon^2} \int \left( \varepsilon^2 |\partial_t u^\varepsilon|^2 - |H^\varepsilon|^2 \right) + \left| H^\varepsilon - \varepsilon |\nabla u^\varepsilon| \right|^2 dx 
+ \int |\partial_t u^\varepsilon - \Pi_{u^\varepsilon} \partial_t u^\varepsilon|^2 dx \leq \frac{C}{\varepsilon} E_\varepsilon[u^\varepsilon|I] \tag{3.15}
\]

To estimate the second term on the LHS, we use (1.3) and (2.9a)

\[
H^\varepsilon = -\varepsilon (\partial_t u^\varepsilon - \mu \nabla \text{div } u^\varepsilon) \cdot \frac{\nabla u^\varepsilon}{|\nabla u^\varepsilon|} \tag{3.16}
\]

Note that the inner product is made with the column vectors of \( \nabla u^\varepsilon = (\partial_1 u^\varepsilon, \partial_2 u^\varepsilon) \). Using the above formula, we expand the integrand of (3.15) and apply the Cauchy-Schwarz inequality:

\[
\varepsilon^2 |\partial_t u^\varepsilon|^2 - |H^\varepsilon|^2 + \left| H^\varepsilon - \varepsilon \nabla u^\varepsilon \right|^2
\]

\[
= \varepsilon^2 |\partial_t u^\varepsilon|^2 + \varepsilon^2 |H^\varepsilon|^2 |\nabla u^\varepsilon|^2 + 2 \varepsilon^2 \partial_t u^\varepsilon \cdot (H \cdot \nabla) u^\varepsilon - 2 \varepsilon^2 \mu \nabla \text{div } u^\varepsilon \cdot (H \cdot \nabla) u^\varepsilon
\]

\[
= \varepsilon^2 |\partial_t u^\varepsilon + (H \cdot \nabla) u^\varepsilon|^2 + \varepsilon^2 \left( |H^\varepsilon|^2 |\nabla u^\varepsilon|^2 - |(H \cdot \nabla) u^\varepsilon|^2 \right) - 2 \varepsilon^2 \mu \nabla \text{div } u^\varepsilon \cdot (H \cdot \nabla) u^\varepsilon
\]

This implies that

\[
\int |\partial_t u^\varepsilon + (H \cdot \nabla) u^\varepsilon|^2 dx
\]

\[
\leq \frac{1}{\varepsilon^2} \int \left( \varepsilon^2 |\partial_t u^\varepsilon|^2 - |H^\varepsilon|^2 \right) + \left| H^\varepsilon - \varepsilon |\nabla u^\varepsilon| \right|^2 dx + 2 \mu \int \nabla \text{div } u^\varepsilon \cdot (H \cdot \nabla) u^\varepsilon dx \tag{3.17}
\]

Adding the above inequality to (3.15) yields

\[
2 \varepsilon^{-1} \frac{d}{dt} E_\varepsilon[u^\varepsilon|I] + \int |\partial_t u^\varepsilon + (H \cdot \nabla) u^\varepsilon|^2 dx + \int |\partial_t u^\varepsilon - \Pi_{u^\varepsilon} \partial_t u^\varepsilon|^2 dx
\]

\[
\leq C \varepsilon^{-1} E_\varepsilon[u^\varepsilon|I] + 2 \mu \int \nabla \text{div } u^\varepsilon \cdot (H \cdot \nabla) u^\varepsilon \tag{3.18}
\]
In the sequel, we shall use the abbreviation $u_{i,j} = \partial_{x_j} u_i$. To estimate the last term we use integration by parts and (2.7)

$$- \int \nabla \div \mathbf{u}^\varepsilon \cdot (\mathbf{H} \cdot \nabla) \mathbf{u}^\varepsilon$$

$$= \int u_{j,j}^\varepsilon (\mathbf{H} \cdot \nabla) u_{k,k}^\varepsilon + \int \div \mathbf{u}^\varepsilon (\partial_j \mathbf{H} \cdot \nabla) u_j^\varepsilon$$

$$= -\frac{1}{2} \int (\div \mathbf{H})(\div \mathbf{u}^\varepsilon)^2 + \int (\div \mathbf{u}^\varepsilon) H_{j,k} u_{j,k}^\varepsilon + \int (\div \mathbf{u}^\varepsilon) (H_{k,j} - H_{j,k}) u_{j,k}^\varepsilon \quad (3.19)$$

Using (2.14b), the first integral in the line of (3.19) is bounded by $\varepsilon^{-1} E_\varepsilon[|\mathbf{u}^\varepsilon|]$. The second integral can be treated by decomposing $\nabla u_k$ along $\mathbf{n}$ and using (2.6):

$$\int (\div \mathbf{u}^\varepsilon) \nabla H_j \cdot \nabla u_j^\varepsilon$$

$$= \int (\div \mathbf{u}^\varepsilon) \nabla H_j \cdot ((I_2 - \mathbf{n} \otimes \mathbf{n}) \cdot \nabla u_j^\varepsilon) + \int (\div \mathbf{u}^\varepsilon) (\mathbf{n} \cdot \nabla) H_j (\mathbf{n} \cdot \nabla u_j^\varepsilon)$$

$$\lesssim \int |\div \mathbf{u}^\varepsilon|^2 + \int |\nabla H_j|^2 |(I_2 - \mathbf{n} \otimes \mathbf{n}) \cdot \nabla u_j^\varepsilon|^2 + \int |\nabla \mathbf{u}^\varepsilon|^2 \min (d_j^2, 1) \quad (3.20)$$

Finally by (3.2) and (2.14c), all these terms can be controlled by $\varepsilon^{-1} E_\varepsilon[|\mathbf{u}^\varepsilon|]$ up to a constant. To summarize we deduce from (3.18), (3.19) and (3.20) that

$$2\varepsilon^{-1} \frac{d}{dt} E_\varepsilon[|\mathbf{u}^\varepsilon|] + \int |\partial_t \mathbf{u}^\varepsilon + (\mathbf{H} \cdot \nabla) \mathbf{u}^\varepsilon|^2 dx + \int |\partial_t \mathbf{u}^\varepsilon - \Pi_{\mathbf{u}^\varepsilon} \partial_t \mathbf{u}^\varepsilon|^2 dx$$

$$\leq C \varepsilon^{-1} E_\varepsilon[|\mathbf{u}^\varepsilon|] - 2\mu \int (\div \mathbf{u}^\varepsilon)(H_{k,j} - H_{j,k}) u_{j,k}^\varepsilon \quad (3.21)$$

It remains to handle the last integral in (3.21). To this end, we use matrix decomposition \(^2\) and (3.13c) to write

$$(H_{k,j} - H_{j,k}) u_{j,k}^\varepsilon = -(\rot \mathbf{u}^\varepsilon) \rot \mathbf{H}$$

So integrating by parts and using (1.3) and (3.13), we obtain

$$\mu \int (\div \mathbf{u}^\varepsilon)(H_{k,j} - H_{j,k}) u_{j,k}^\varepsilon$$

$$= -\mu \int (\div \mathbf{u}^\varepsilon)(\rot \mathbf{u}^\varepsilon) \rot \mathbf{H}$$

$$\equiv \mu \int (\div \mathbf{u}^\varepsilon) \mathbf{u}^\varepsilon \cdot \nabla \perp \rot \mathbf{H} - \int \mu \nabla \div \mathbf{u}^\varepsilon \cdot (\mathbf{u}^\varepsilon) \perp \rot \mathbf{H}$$

$$\equiv \mu \int (\div \mathbf{u}^\varepsilon) \mathbf{u}^\varepsilon \cdot \nabla \perp \rot \mathbf{H} - \int (\partial_t \mathbf{u}^\varepsilon - \Delta \mathbf{u}^\varepsilon) \cdot (\mathbf{u}^\varepsilon) \perp \rot \mathbf{H}$$

$$= \mu \int (\div \mathbf{u}^\varepsilon) \mathbf{u}^\varepsilon \cdot \nabla \perp \rot \mathbf{H} - \int \left( \partial_t \mathbf{u}^\varepsilon + (\mathbf{H} \cdot \nabla) \mathbf{u}^\varepsilon \right) \cdot (\mathbf{u}^\varepsilon) \perp \rot \mathbf{H}$$

$$+ \int (\mathbf{H} \cdot \nabla) \mathbf{u}^\varepsilon \cdot (\mathbf{u}^\varepsilon) \perp \rot \mathbf{H} + \int \Delta \mathbf{u}^\varepsilon \cdot (\mathbf{u}^\varepsilon) \perp \rot \mathbf{H}$$

\(^2\)For a square matrix $A$, the decomposition $A = \frac{A + A^T}{2} + \frac{A - A^T}{2}$ is orthogonal under the Frobenius inner product $A : B \triangleq \text{tr} A^T B$. 

Submitting into (3.21), using Cauchy–Schwarz’s inequality, (3.7b) and integration by parts yields
\[
2\varepsilon^{-1} \frac{d}{dt} E_\varepsilon[u^\varepsilon |I|] + \frac{1}{2} \int |\partial_t u^\varepsilon + (H \cdot \nabla)u^\varepsilon|^2 \, dx + \int |\partial_t u^\varepsilon - \Pi_{u^\varepsilon} \partial_t u^\varepsilon|^2 \, dx
\]
\[
\leq C \left( 1 + \varepsilon^{-1} E_\varepsilon[u^\varepsilon |I|] \right) + \int (H \cdot \nabla)u^\varepsilon \cdot (u^\varepsilon) \cdot \text{rot} \, H + \int \Delta u^\varepsilon \cdot (u^\varepsilon) \cdot \text{rot} \, H
\]
\[
= C \left( 1 + \varepsilon^{-1} E_\varepsilon[u^\varepsilon |I|] \right) + \int H_k (\partial_k u^\varepsilon - \Pi_{u^\varepsilon} \partial_k u^\varepsilon) \cdot (u^\varepsilon) \cdot \text{rot} \, H
\]
\[
- \int (\partial_k \text{rot} \, H) (\partial_k u^\varepsilon - \Pi_{u^\varepsilon} \partial_k u^\varepsilon) \cdot (u^\varepsilon) \cdot \text{rot} \, H
\]
(3.22)

Note that in the last step we employed integration by parts and the equation \( \Pi_{u^\varepsilon} \partial_k u^\varepsilon \cdot (u^\varepsilon) \cdot \text{rot} \, H = 0 \), which follows from (2.13). Finally using the Cauchy–Schwarz inequality and then (2.14b) and (3.7b) to the last two integrals of (3.22) yields
\[
2\varepsilon^{-1} \frac{d}{dt} E_\varepsilon[u^\varepsilon |I|] + \frac{1}{2} \int |\partial_t u^\varepsilon + (H \cdot \nabla)u^\varepsilon|^2 \, dx + \int |\partial_t u^\varepsilon - \Pi_{u^\varepsilon} \partial_t u^\varepsilon|^2 \, dx
\]
\[
\lesssim 1 + \varepsilon^{-1} E_\varepsilon[u^\varepsilon |I|]
\]
(3.23)

This combined with (1.15b) and Gronwall’s inequality leads to the desired inequality (3.1). We thus prove the first part of (1.16). □

Using (2.14c) and (3.1), we immediately obtain the following

**Corollary 3.4.** There exists a generic constant \( C > 0 \) such that
\[
\sup_{t \in [0,T]} \int_{\Omega_t^+ \setminus \Gamma_t(\delta)} \left( |\nabla u^\varepsilon|^2 + \frac{1}{\varepsilon^2} F(u^\varepsilon) + \frac{1}{\varepsilon} |\nabla \psi^\varepsilon| \right) \, dx \leq C \delta^{-2},
\]
(3.24a)
\[
\int_0^T \int_{\Omega_t^+ \setminus \Gamma_t(\delta)} |\partial_t u^\varepsilon|^2 \, dx \, dt \leq C \delta^{-2},
\]
(3.24b)

holds for each fixed \( \delta \in (0, \delta_0) \).

Actually the estimate of \( \nabla u^\varepsilon \) in (3.24a) together with the bound on \( \partial_t u^\varepsilon + (H \cdot \nabla)u^\varepsilon \) in (3.1) lead to (3.24b). Another consequence of (3.1) is the following Lemma.

**Lemma 3.5.** We have the following estimates of \( \hat{u}^\varepsilon = u^\varepsilon / |u^\varepsilon| \):
\[
\sup_{t \in [0,T]} \int |u^\varepsilon|^2 |\nabla \hat{u}^\varepsilon|^2 \, dx + \sup_{t \in [0,T]} \int |\hat{u}^\varepsilon \cdot \nabla |u^\varepsilon||^2 \, dx \leq C,
\]
(3.25a)
\[
\sup_{t \in [0,T]} \int (\hat{u}^\varepsilon \cdot n)^2 |\nabla \psi^\varepsilon| \, dx \leq C \varepsilon
\]
(3.25b)

**Proof.** We first deduce from (3.1) and (2.14b) that
\[
\sup_{t \in [0,T]} \int (\mu \div u^\varepsilon)^2 + |\nabla u^\varepsilon - \Pi_{u^\varepsilon} \nabla u^\varepsilon|^2 \, dx \leq C
\]
(3.26)

As mentioned in Remark 2.1, for each fixed \( t \in [0, T] \), the subset of \( \Omega \) where \( u^\varepsilon = 0 \) is negligible. So we can write \( u^\varepsilon = |u^\varepsilon| \hat{u}^\varepsilon \) and using (2.13), we have the formula
\[
\nabla u^\varepsilon - \Pi_{u^\varepsilon} \nabla u^\varepsilon = |u^\varepsilon| \nabla \hat{u}^\varepsilon
\]
(3.27)

Substituting this formula into (3.26) yields the estimate of the first term in (3.25a). To obtain the second one, by (2.11) we obtain the following formula
\[
\text{tr} \nabla u^\varepsilon - \text{tr} (\Pi_{u^\varepsilon} \nabla u^\varepsilon) = \text{div} u^\varepsilon - \hat{u}^\varepsilon \cdot \nabla |u^\varepsilon|
\]
(3.28)

This together with the inequality \( |\text{tr} A|^2 \leq 2 |A|^2 \) for any \( 2 \times 2 \) matrix \( A \) and (3.26) yields
\[
\sup_{t \in [0,T]} \int |\hat{u}^\varepsilon \cdot \nabla |u^\varepsilon||^2 \, dx \leq C(\mu)
\]
(3.29)
Altogether, we have the desired estimate (3.25a).

Now we turn to the proof of (3.25b). Recall (2.9b), we have \( n^\varepsilon = \frac{\nabla |u^\varepsilon|}{|\nabla |u^\varepsilon|} \). Denoting \( \hat{u}^\varepsilon \cdot n^\varepsilon = \cos \theta^\varepsilon \), we obtain for any \( t \in [0, T] \) that

\[
\mu \int \cos^2 \theta^\varepsilon |\nabla |u^\varepsilon||^2 \, dx = \mu \int |\hat{u}^\varepsilon \cdot n^\varepsilon|^2 |\nabla |u^\varepsilon||^2 \, dx \leq C(\mu) \tag{3.30}
\]

This combined with (2.14a) and the obvious inequality \(|\nabla |u^\varepsilon|| \leq |\nabla u^\varepsilon|\) yields

\[
C(\mu) \geq \int \frac{\mu}{2} \cos^2 \theta^\varepsilon |\nabla |u^\varepsilon||^2 \, dx + \int \left( \frac{1}{2} |\nabla |u^\varepsilon||^2 + \frac{1}{\varepsilon^2} F(u^\varepsilon) - \frac{1}{\varepsilon} |\nabla \psi^\varepsilon| \right) \, dx
\]

\[
\geq \frac{1}{\varepsilon} \int \left( \sqrt{\mu \cos^2 \theta^\varepsilon + 1} \, |\nabla |u^\varepsilon|| \sqrt{2F(u^\varepsilon)} - |\nabla \psi^\varepsilon| \right) \, dx
\]

\[
= \frac{1}{\varepsilon} \int \left( \sqrt{\cos^2 \theta^\varepsilon + 1} - 1 \right) |\nabla \psi^\varepsilon| \, dx
\]

for any \( t \in [0, T] \). Note that in the last step we employed \(|\nabla |u^\varepsilon|| \sqrt{2F(u^\varepsilon)} = |\nabla \psi^\varepsilon|\). So we obtain (3.25b) by a conjugation. \( \square \)

We end this section by a technical lemmas which will be used in the next section for the estimates of the level sets of \( \psi^\varepsilon \). From the definition (1.12), if \( \psi^\varepsilon \) is away from \( \{0, 1\} \), it is obvious that \( u^\varepsilon \) is away from \( \{0\} \cup S^1 \). The following lemma is a quantitative version of it. Note that this result can be generalized to a wider class of double well potential (1.2) with some structural assumptions.

**Lemma 3.6.** There exists some generic constant \( C > 0 \) depending in \( \sigma_0 \) (1.3) so that the following statements hold\(^3\) for any \( \delta \in [0, 1/100] \)

\[
|u^\varepsilon(x, t)| \geq 1 + C\delta^{1/2} \quad \forall x \in \{ x : \psi^\varepsilon \geq 1 + \delta \}, \tag{3.31a}
\]

\[
|u^\varepsilon(x, t)| \leq 1 - C\delta^{1/2} \quad \forall x \in \{ x : \psi^\varepsilon \leq 1 - \delta \}, \tag{3.31b}
\]

\[
|u^\varepsilon(x, t)| \geq C\delta^{1/2} \quad \forall x \in \{ x : \psi^\varepsilon \geq \delta \}. \tag{3.31c}
\]

**Proof.** For \( \tau \geq 0 \) we deduce from (1.2) that

\[
(2\sigma_0)^{-1/2} \int_1^{1+\tau} \sqrt{2f(s)} \, ds \leq (2 + \tau)^2 \tau^2 \tag{3.32}
\]

By (1.12) and (1.13), we have \(|u^\varepsilon| \geq 1\) on the set \( \{ x : \psi^\varepsilon \geq 1 + \delta \} \). Moreover, by (3.32), we obtain

\[
\delta \leq \psi^\varepsilon - 1 = \int_1^{\max(1, |u^\varepsilon|)} \sqrt{2f(s)} \, ds \leq 2\sigma_0^{1/2}(|u^\varepsilon| + 1)^2(|u^\varepsilon| - 1)^2.
\]

So we have either \(|u^\varepsilon| \geq 2\), or \(|u^\varepsilon| \geq 1 + C\delta^{1/2}\). This proves (3.31a). For \( \tau \in [0, 1] \) we have

\[
(2\sigma_0)^{-1/2} \int_1^{1+\tau} \sqrt{2f(s)} \, ds \leq 2\int_1^{1+\tau} (1 - s) \, ds = \tau^2
\]

Applying it with \( \tau = 1 - |u^\varepsilon| \) yields the following inequalities on the set \( \{ x : \psi^\varepsilon \leq 1 - \delta \} \):

\[
\delta \leq 1 - \psi^\varepsilon = \int_{1 - (1 - |u^\varepsilon|)}^1 \sqrt{2f(s)} \, ds \leq (2\sigma_0)^{1/2}(|u^\varepsilon| - 1)^2 \tag{3.33}
\]

So \(|u^\varepsilon| \leq 1 - C\delta^{1/2}\) and this proves (3.31b). Finally on the set \( \{ x : \psi^\varepsilon \geq \delta \} \), we have either \(|u^\varepsilon| \geq 2\), or

\[
\delta \leq \psi^\varepsilon = \int_0^{\max(1, |u^\varepsilon|)} \sqrt{2f(s)} \, ds \leq C \int_0^{\max(1, |u^\varepsilon|)} 2s \, ds = C|u^\varepsilon|^2 \tag{3.34}
\]

Both of these two cases imply (3.31c). \( \square \)

\(^3\)Note that the number 1/100 here is far from being optimal. It is chosen merely to have a simple square root.
4. Estimates of level sets

The main result of this section is the following $L^1$-estimate of the characteristic functions. Our proof is inspired by the scalar case studied in [11].

**Theorem 4.1.** Under the assumptions of Theorem 1.1, there exists $C > 0$ independent of $\varepsilon$ so that

$$
\int_{\Omega} |\psi^\varepsilon - 1_{\Omega_t^\varepsilon}| \, dx \leq C\varepsilon^{1/3}
$$

(4.1)

**Proof of Theorem 4.1.** Our goal is to estimate $2\psi^\varepsilon - 1 - \chi$ where $\chi(x,t) = \pm 1$ in $\Omega_t^\varepsilon$.

**Step 1: Setup the proof**

Within this proof, we shall temporarily use $h^+$ (or $h^-$) to indicate the positive (or negative) part of a function $h$. By [11, pp. 153], for any $h \in W^{1,1}(\Omega)$, we have

$$
\partial_t(h(x))^+ = (\partial_t h(x))1_{\{x; h(x)>0\}}(x) \quad \text{a.e. } x
$$

(4.2)

Let $\eta(\cdot)$ be the truncation of the identity map:

$$
\eta(s) = \begin{cases} 
  s & \text{when } s \in [-\delta_0, \delta_0], \\
  \delta_0 & \text{when } s \geq \delta_0, \\
  -\delta_0 & \text{when } s \leq -\delta_0.
\end{cases}
$$

(4.3)

We denote its positive and negative parts by $\eta^+(s)$ respectively, and its absolute value by $|\eta|(s) =: \zeta(s)$. Using the formula $f = f^+ - f^-$, we can write

$$
2\psi^\varepsilon - 1 = 2(\psi^\varepsilon - 1^+) + (1 - 2(\psi^\varepsilon - 1^-))
$$

and we shall estimate its difference with $\chi$. This will be done by establishing differential inequalities of the following energies

$$
g_\varepsilon(t) := \int (\psi^\varepsilon - 1^+) \zeta(d_I) \, dx
$$

(4.5a)

$$
h_\varepsilon(t) := \int \left( \chi - [1 - 2(\psi^\varepsilon - 1^-)] \right) \eta(d_I) \, dx
$$

(4.5b)

It is obvious that (4.5a) has a non-negative integrand. Since $\psi^\varepsilon \geq 0$, we have $(\psi^\varepsilon - 1^-) \in [0, 1]$ and thus $[1 - 2(\psi^\varepsilon - 1^-)]$ ranges in $[-1, 1]$. As $\eta \chi = |\eta|$, we deduce that the integrand of (4.5b) is also non-negative. On the other hand, by (1.13) and (1.14), we have

$$
g_\varepsilon(0) = 0, \quad h_\varepsilon(0) \overset{(1.13)}{=} \int_{I_0(\delta_0)} \left( \chi(x,0) + 1 - 2\psi^\varepsilon(x,0) \right) d_{I_0}(x) \, dx \overset{(1.14)}{=} 2c_1\varepsilon.
$$

(4.6)

**Step 2: estimate of weighted difference.**

Using $\partial_t\psi^\varepsilon = \partial_t u^\varepsilon \cdot \frac{u^\varepsilon}{|u^\varepsilon|} \sqrt{2F(u^\varepsilon)}$, we have

$$
\partial_t\psi^\varepsilon = (\partial_t u^\varepsilon + (H \cdot \nabla)u^\varepsilon) \cdot \frac{u^\varepsilon}{|u^\varepsilon|} \sqrt{2F(u^\varepsilon)} - H \cdot \nabla \psi^\varepsilon
$$

(4.7)

Using (4.7), we can calculate

$$
g'_\varepsilon(t) \overset{(4.7)}{=} \int_{\{\psi^\varepsilon>1\}} (\partial_t u^\varepsilon + (H \cdot \nabla)u^\varepsilon) \cdot \frac{u^\varepsilon}{|u^\varepsilon|} \sqrt{2F(u^\varepsilon)} \zeta(d_I)
$$

$$
- \int_{\{\psi^\varepsilon<0\}} H \cdot \nabla \psi^\varepsilon \zeta(d_I) + \int (\psi^\varepsilon - 1^+) \partial_t \zeta(d_I)
$$

$$
= \int_{\{\psi^\varepsilon>1\}} (\partial_t u^\varepsilon + (H \cdot \nabla)u^\varepsilon) \cdot \frac{u^\varepsilon}{|u^\varepsilon|} \sqrt{2F(u^\varepsilon)} \zeta(d_I)
$$

$$
- \int H \cdot \nabla (\psi^\varepsilon - 1^+) \zeta(d_I) - \int (\psi^\varepsilon - 1^+) \partial_t \zeta(d_I)
$$

$$
+ \int (\partial_t \zeta(d_I) + H \cdot \nabla \zeta(d_I))(\psi^\varepsilon - 1^+). 
$$
Integrating by parts, we can combine the second and the third integral and use (2.8a) to obtain

\[ g_\varepsilon(t) = \int \left( \partial_t u^\varepsilon + (H \cdot \nabla) u^\varepsilon \right) \cdot \frac{u^\varepsilon}{|u^\varepsilon|} \sqrt{2F(u^\varepsilon)} \zeta(d_t) \]

\[ + \int (\operatorname{div} H)(\psi^\varepsilon - 1)^+ \zeta(d_t) + \int (\partial_t \zeta(d_t) + H \cdot \nabla \zeta(d_t)) (\psi^\varepsilon - 1)^+ \]

\[ \leq \int \varepsilon \left| \partial_t u^\varepsilon + (H \cdot \nabla) u^\varepsilon \right|^2 + \int \frac{1}{\varepsilon} F(u^\varepsilon) \zeta^2(d_t) + C g_\varepsilon(t) \]

Note that in the last step we employed the Cauchy–Schwarz inequality and then (2.14e). In view of (3.1) and (4.6), we can apply the Gronwall lemma and obtain \( g_\varepsilon(t) \leq C \varepsilon \) for some \( C \) which is independent of \( \varepsilon \) and \( t \in [0, T] \). Similar calculation shows \( h_\varepsilon(t) \leq C \varepsilon \). Now using (4.4) and the estimates of (4.5a) and (4.5b) yields,

\[ \int |2\psi^\varepsilon - 1 - \chi|^\varepsilon_\zeta(d_t) \]

\[ \leq \int 2(\psi^\varepsilon - 1)^+ \zeta(d_t) + \int |1 - 2(\psi^\varepsilon - 1)^- - \chi|^\varepsilon_\zeta(d_t) \]

\[ \leq g_\varepsilon + h_\varepsilon \leq C \varepsilon. \quad (4.8) \]

**Step 3: Remove the weight.** First note that (4.8) implies the estimate of integral (4.11) in \( \Omega \setminus I_t(\delta_0) \). So we shall focus on the estimate in \( I_t(\delta_0) \). In the sequel, we denote \( \chi^\varepsilon = 2\psi^\varepsilon - 1 \) and \( \delta_0 = \delta \). In order to pass to (4.11), we can apply (4.10) below with

\[ f_\varepsilon(r, p, t) = |\chi(p + r n, t) - \chi^\varepsilon(p + r n, t)| \quad (4.9) \]

for fixed \( t \in [0, T] \) and \( p \in I_t \) with normal vector \( n \):

\[ \left( \int_{I_t(\delta)} |\chi(x, t) - \chi^\varepsilon(x, t)| \, dx \right)^{3/2} = \left( \int_{I_t} \int_{-\delta}^{\delta} f_\varepsilon(r, p, t) \, dy \, dS(p) \right)^{3/2} \]

\[ \leq \varepsilon \int_{I_t} \left( \int_{-\delta}^{\delta} f_\varepsilon(r, p, t) \, dy \right)^{3/2} \, dS(p) \]

\[ \leq \varepsilon \int_{I_t} \|f_\varepsilon(\cdot, p, t)\|_{L^2(-\delta, \delta)} \sqrt{\int_{-\delta}^{\delta} f_\varepsilon(r, p, t) \, |r| \, dr \, dS(p)} \]

\[ \leq \varepsilon \int_{I_t} \|f_\varepsilon(\cdot, t)\|_{L^2(I_t(\delta))} \sqrt{\int_{I_t} \int_{-\delta}^{\delta} f_\varepsilon(r, p, t) \, |r| \, dr \, dS(p)}. \]

In view of (1.12) and (1.4b) we have \( \psi^\varepsilon|_{\partial \Omega} = 0 \). So by Sobolev’s inequality in dimension two,

\[ \left( \int_{I_t(\delta)} |\chi(x, t) - \chi^\varepsilon(x, t)| \, dx \right)^3 \lesssim \|\chi - \chi^\varepsilon\|_{L^2(\Omega)}^3 \int_{\Omega} |\chi^\varepsilon - \chi|^2 \zeta(d_t) \, dx \]

\[ \lesssim (1 + \|\nabla \psi^\varepsilon\|_{L^1(\Omega)}^2) \int_{\Omega} |\chi^\varepsilon - \chi|^2 \zeta(d_t) \, dx. \]

This together with (4.8) gives the desired estimate (4.11) in \( I_t(\delta_0) \). \( \square \)

**Lemma 4.2.** For any integrable function \( f : [-\delta, \delta] \to \mathbb{R}^+ \), we have

\[ \left( \int_{-\delta}^{\delta} f(r) \, dr \right)^3 \lesssim 8\|f\|_{L^2(-\delta, \delta)}^2 \int_{-\delta}^{\delta} |r| f(r) \, dr. \quad (4.10) \]
Proof. By the Cauchy–Schwarz inequality,
\[ \int_0^\delta \left( \int_{\{x, y \geq 0, x + y \leq r\}} f(x) f(y) \, dx \, dy \right) f(r) \, dr \leq \int_0^\delta \left( \int_{(x, y) \in [0, r]^2} f^2(x) f^2(y) \, dx \, dy \right) f(r) \, dr \leq \| f \|_{L^2(0, \delta)}^2 \int_0^\delta r f(r) \, dr. \]
Similarly,
\[ \int_0^\delta \left( \int_{\{0 \leq x, y \leq \delta, x + y \geq r\}} f(x) f(y) \, dx \, dy \right) f(r) \, dr \leq \| f \|_{L^2(0, \delta)}^2 \int_0^\delta r f(r) \, dr. \]
Adding up the above two inequalities yields
\[ \left( \int_0^\delta f(r) \, dr \right)^3 = \int_0^\delta \left( \int_{(x, y) \in [0, \delta]^2} f(x) f(y) \, dx \, dy \right) f(r) \, dr \leq 2 \| f \|_{L^2(0, \delta)}^2 \int_0^\delta r f(r) \, dr. \]
and this leads to (1.10). \( \square \)

In order to study the local convergence of level sets, we introduce the set
\[ S_{t}^{\epsilon, \delta} = \{ x \in \Omega : |2\psi^\epsilon(x, t) - 1| \leq 2\delta \}, \quad \forall \delta \in (0, 1/100). \] (4.12)

**Lemma 4.3.** For any \( \delta \in (0, 1/100) \), there exists \( b = b_\delta \in \left[ \frac{1}{2} - \delta, \frac{1}{2} + \delta \right] \) s.t. the set \( \{ x : \psi^\epsilon(x, t) > b \} \) has finite perimeter and
\[ |\mathcal{H}^1(\{ x : \psi^\epsilon(x, t) = b \}) - \mathcal{H}^1(I_t) | \leq C\epsilon^{1/6} \delta^{-1} \] (4.13)

**Proof.** First recall from (1.12) that \( \psi^\epsilon(x, t) = \int_0^{2\psi^\epsilon(x, t)} \sqrt{2f(s)} \, ds \). Recall also the co-area formula of BV function \([7, \text{section 5.5}]\) which states that \( \partial^* S_{t}^{\epsilon, \delta} \) has finite perimeter for almost every \( \delta \).
During the proof, we shall write \( \psi^\epsilon(x, t) = \psi^\epsilon \) for brevity.
Using (3.1) and (2.14d), we have for almost every \( \delta \in (0, 1/100) \) that
\[ C\epsilon \geq \int_{S_{t}^{\epsilon, \delta}} (|\nabla \psi^\epsilon| - \xi \cdot \nabla \psi^\epsilon) \, dx \geq 0 \]
\[ = \int_{\frac{1}{2} - \delta}^{\frac{1}{2} + \delta} \mathcal{H}^1(\{ x : \psi^\epsilon = s \}) \, ds - \int_{\partial^* S_{t}^{\epsilon, \delta}} \xi \cdot \nu \psi^\epsilon \, d\mathcal{H}^1 + \int_{S_{t}^{\epsilon, \delta}} (\div \xi) \psi^\epsilon \, dx \]
where \( \nu \) is the outward unit normal of the set \( S_{t}^{\epsilon, \delta} \), defined on its (measure-theoretic) boundaries.
By \([3.7a]\) and Sobolev’s imbedding, we can control
\[ \left| \int_{S_{t}^{\epsilon, \delta}} (\div \xi) \psi^\epsilon \, dx \right| \leq |S_{t}^{\epsilon, \delta}|^{\frac{1}{2}} \| \nabla \psi^\epsilon \|_{L^1(\Omega)} \lesssim |S_{t}^{\epsilon, \delta}|^{\frac{1}{2}} \]
So we have
\[ \int_{\frac{1}{2} - \delta}^{\frac{1}{2} + \delta} \mathcal{H}^1(\{ x : \psi^\epsilon = s \}) \, ds - \int_{\partial^* S_{t}^{\epsilon, \delta}} \xi \cdot \nu \psi^\epsilon \, d\mathcal{H}^1 \lesssim \epsilon + |S_{t}^{\epsilon, \delta}|^{\frac{1}{2}} \] (4.14)
Adding zero, the second integral on the LHS of (4.14) satisfies
\[ \int_{\partial^* S_{t}^{\epsilon, \delta}} \xi \cdot \nu \psi^\epsilon \, d\mathcal{H}^1 - \left( \frac{1}{2} - \delta \right) \int_{\{x: \psi^\epsilon < \frac{1}{2} - \delta\}} \div \xi \, dx - \left( \frac{1}{2} + \delta \right) \int_{\{x: \psi^\epsilon > \frac{1}{2} + \delta\}} \div \xi \, dx \]
\[ + \left( \frac{1}{2} - \delta \right) \int_{\Omega_t} \div \xi \, dx + \left( \frac{1}{2} + \delta \right) \int_{\Omega_t} \div \xi \, dx + 2\delta \mathcal{H}^1(I_t) \]
(4.15)
(4.16)
\[ = 0 \]
\[ \text{To get the constant 8 (which is not sharp), we need the elementary inequality } 4(a^3 + b^3) \geq (a + b)^3. \text{ This inequality can be obtained by multiplying } (a + b) \text{ on both sides of } 4(a^2 - ab + b^2) \geq (a + b)^2. \]
Note that by checking the orientation of $\nu$ and $\xi$ (1.8), we see the three terms in the line of (1.16) sum up to zero. So substituting the above equation into (4.14) yields

$$\left| \int_{\frac{1}{2} + \delta}^{2 + \delta} H^1 \left( \{ x : \psi^\varepsilon = s \} \right) \, ds - 2\delta H^1(I_t) \right| \lesssim \varepsilon + |S^\varepsilon_{t,\delta}|^{1/2} + \left| \Omega_t^+ \right| - \left| \{ x : \psi^\varepsilon > \frac{1}{2} + \delta \} \right| + \left| \Omega_t^- \right| - \left| \{ x : \psi^\varepsilon < \frac{1}{2} - \delta \} \right| \right| \tag{4.17}$$

Up to a bound constant, we rewrite the last two terms by

$$r_\varepsilon \equiv \left| \Omega_t^+ \right| - \left| \{ x : \psi^\varepsilon > \frac{1}{2} + \delta \} \right| + \left| \Omega_t^- \right| - \left| \{ x : \psi^\varepsilon < \frac{1}{2} - \delta \} \right|$$

$$= \left| \Omega_t^+ \right| - \left| \{ x \in \Omega_t^+ \ : \psi^\varepsilon > \frac{1}{2} + \delta \} \right| - \left| \{ x \in \Omega_t^- \ : \psi^\varepsilon > \frac{1}{2} + \delta \} \right|$$

$$+ \left| \Omega_t^- \right| - \left| \{ x \in \Omega_t^- \ : \psi^\varepsilon < \frac{1}{2} - \delta \} \right| - \left| \{ x \in \Omega_t^+ \ : \psi^\varepsilon < \frac{1}{2} - \delta \} \right|$$

$$\lesssim \left| \{ x \in \Omega_t^+ \ : \psi^\varepsilon \leq \frac{1}{2} + \delta \} \right| + \left| \{ x \in \Omega_t^- \ : \psi^\varepsilon > \frac{1}{2} + \delta \} \right|$$

$$+ \left| \{ x \in \Omega_t^- \ : \psi^\varepsilon < \frac{1}{2} - \delta \} \right| + \left| \{ x \in \Omega_t^+ \ : \psi^\varepsilon < \frac{1}{2} - \delta \} \right|$$

Now using Chebyshev’s inequality and (4.1) yields $r_\varepsilon \lesssim \varepsilon^{1/3}$. Similarly, $|S^\varepsilon_{t,\delta}| \lesssim \varepsilon^{1/3}$. Substitute these estimates into (4.17) leads

$$\left| \frac{1}{2\delta} \int_{\frac{1}{2} + \delta}^{2 + \delta} H^1 \left( \{ x : \psi^\varepsilon = s \} \right) \, ds - H^1(I_t) \right| \leq C\varepsilon^{1/6} \delta^{-1} \tag{4.18}$$

So the desired result follows by Fubini’s theorem. \hfill \Box

Similarly, we define

$$Q_t^{\varepsilon, \delta} = \{ x \in \Omega : |\psi^\varepsilon(x, t) - 2| \leq \delta \}, \quad \forall \delta \in (0, 1/100), \tag{4.19}$$

and we can prove the following:

**Lemma 4.4.** For any $\delta \in (0, 1/100)$, there exists $q = q_\delta \in [2-\delta, 2+\delta]$ s.t. the set $\{ x : \psi^\varepsilon(x, t) < q \}$ has finite perimeter and

$$|H^1(\{ x : \psi^\varepsilon(x, t) = q \})| \leq C\varepsilon^{1/6} \delta^{-1} \tag{4.20}$$

**Proof.** Using (3.1) and (2.14a), we have for almost every $\delta \in (0, 1/100)$ that

$$C\varepsilon \geq \int_{Q_t^{\varepsilon, \delta}} (|\nabla \psi^\varepsilon| - \xi \cdot \nabla \psi^\varepsilon) \, dx \geq 0$$

$$\geq \int_{2-\delta}^{2+\delta} H^1 \left( \{ x : \psi^\varepsilon = s \} \right) \, ds - \int_{\partial Q_t^{\varepsilon, \delta}} \xi \cdot \nu \psi^\varepsilon \, dH^1 + \int_{Q_t^{\varepsilon, \delta}} (\text{div } \xi) \psi^\varepsilon \, dx$$

where $\nu$ is the outward unit normal of the set $Q_t^{\varepsilon, \delta}$. By (3.1a) and Sobolev’s imbedding, we can control

$$\left| \int_{Q_t^{\varepsilon, \delta}} (\text{div } \xi) \psi^\varepsilon \, dx \right| \leq |Q_t^{\varepsilon, \delta}|^{1/2} \| \nabla \psi^\varepsilon \|_{L^1(\Omega)} \lesssim \|Q_t^{\varepsilon, \delta}\|^{1/2}$$

So we have

$$\int_{2-\delta}^{2+\delta} H^1 \left( \{ x : \psi^\varepsilon = s \} \right) \, ds \leq \int_{\partial Q_t^{\varepsilon, \delta}} \xi \cdot \nu \psi^\varepsilon \, dH^1 + C\varepsilon + C|Q_t^{\varepsilon, \delta}|^{1/2} \tag{4.21}$$

Use (2.7), we have $\int \text{div } \xi \, dx = 0$, and thus

$$\int_{\partial Q_t^{\varepsilon, \delta}} \xi \cdot \nu \psi^\varepsilon \, dH^1 \equiv - (2 - \delta) \int_{\{ x : \psi^\varepsilon < 2 - \delta \}} \text{div } \xi \, dx - (2 + \delta) \int_{\{ x : \psi^\varepsilon > 2 + \delta \}} \text{div } \xi \, dx$$

$$+ (2 - \delta) \int_{\{ x : \psi^\varepsilon > 2 - \delta \}} \text{div } \xi \, dx - (2 + \delta) \int_{\{ x : \psi^\varepsilon > 2 + \delta \}} \text{div } \xi \, dx$$

$$\equiv - (2 - \delta) \int_{\{ x : \psi^\varepsilon < 2 - \delta \}} \text{div } \xi \, dx - (2 + \delta) \int_{\{ x : \psi^\varepsilon > 2 + \delta \}} \text{div } \xi \, dx$$
Now using Chebyshev’s inequality and (4.1) we have

$$|Q_{t_1}^{x, \delta}|^{1/2} + \left| \int_{\partial Q_{t_1}^{x, \delta}} \xi \cdot \nu \psi d\mathcal{H}^1 \right| \lesssim \varepsilon^{1/6}. $$

This combined with (4.21) and Fubini’s theorem leads to the desired result. \qed

5. Convergence of solutions

With the uniform estimates obtained in previous sections, we can prove the following convergence result. We shall use $\wedge$ for the wedge product in $\mathbb{R}^2$:

$$a \wedge b = \det(a, b) = a^\perp \cdot b, \quad a^\perp = (-a_2, a_1) \quad (5.1)$$

**Proposition 5.1.** There exists a sequence of $\varepsilon_k \downarrow 0$ such that $u_k := u_{\varepsilon_k}$ satisfy

$$\partial_t u_k \wedge u_k \xrightarrow{k \to \infty} g_0(x, t) \text{ weakly in } L^2(0, T; L^{3/2}(\Omega)), \quad (5.2a)$$

$$\partial_t u_k \wedge u_k \xrightarrow{k \to \infty} g_i(x, t) \text{ weakly-star in } L^\infty(0, T; L^{3/2}(\Omega)) \quad (5.2b)$$

for $1 \leq i \leq 2$. Moreover, there exists a unit vector field $u = u(x, t)$ such that

$$u \in L^\infty(0, T; H^1(\Omega^+_i, S^1)) \cap H^1(0, T; L^2(\Omega^+_i; S^1)) \cap C([0, T]; L^2(\Omega^+_i; S^1)), \quad (5.3)$$

$$g_0 = \partial_t u \wedge u, \quad g_i = \partial_t u \wedge u, \quad 1 \leq i \leq 2, \text{ a.e. } t \in (0, T), \ x \in \Omega^+_i. \quad (5.4)$$

$$\partial_t u_k \xrightarrow{k \to \infty} \partial_t u \text{ weakly in } L^2(0, T; L^2_{loc}(\Omega^+_i)), \quad (5.5a)$$

$$\nabla u_k \xrightarrow{k \to \infty} \nabla u \text{ weakly in } L^\infty(0, T; L^2_{loc}(\Omega^+_i)), \quad (5.5b)$$

$$u_k \xrightarrow{k \to \infty} u \text{ strongly in } C([0, T]; L^2_{loc}(\Omega^+_i)), \quad (5.5c)$$

Such a result was first proved in [22] in a more sophisticated setting and can be simplified in the current case. We present the proof in the Appendix for the convenience of the readers.

Recall Lemma 4.3 and Lemma 4.4, we choose $\delta_k = \varepsilon_k^{1/2}$, then there exists

$$b_k = b_{\delta_k} \in \left[\frac{1}{2} - \delta_k, \frac{1}{2} + \delta_k\right], \quad q_k = q_{\delta_k} \in \left[2 - \delta_k, 2 + \delta_k\right]$$

with $(b_k, q_k) \xrightarrow{k \to \infty} (\frac{1}{2}, 2)$. Moreover, the set

$$\Omega^+_k := \{ x \in \Omega : b_k < \psi_{\varepsilon_k}(x, t) < q_k \} \quad (5.6)$$

has finite perimeter, and

$$|\mathcal{H}^1(\partial^* \Omega^+_k) - \mathcal{H}^1(\Omega_t)| \leq C \varepsilon_k^{1/2}. \quad (5.7)$$

**Proposition 5.2.** For any fixed $t \in [0, T]$, there exists a subsequence of $k$ (not relabeled) such that

$$v_k(x, t) := \hat{u}_k(x, t) 1_{\Omega^+_k} \xrightarrow{k \to \infty} v = 1_{\Omega^+_i} u \text{ weakly-star in } BV(\Omega). \quad (5.8)$$

**Proof.** It follows from Lemma 3.6 that in $\Omega^+_k$ we have $|u_k| \geq 0.1$ and thus $\nabla |u_k|$ make sense. Moreover, by (3.25a) we have

$$\sup_{t \in [0, T]} \int_{\Omega^+_k} |\nabla \hat{u}_k|^2 \, dx \leq C. \quad (5.9)$$

It is obvious that $v_k(\cdot, t)$ is uniformly bounded in $L^\infty(\Omega)$ and the distributional derivative has no Cantor parts. Moreover, its absolute continuous part and jump part enjoy the estimates (5.9) and (5.7) respectively. So according to [2] Section 4.1, $\{v_k(\cdot, t)\}$ is compact in $SBV(\Omega)$, the class of special function of bounded variation on $\Omega$. More precisely, there exists $v \in SBV(\Omega)$ s.t.

$$v_k \xrightarrow{k \to \infty} v \text{ weakly-star in } BV(\Omega), \quad (5.10)$$

$$\int_{\Omega^+_k} |\nabla v|^2 \, dx \leq \liminf_{k \to \infty} \int_{\Omega^+_k} |\nabla \hat{u}_k|^2 \, dx \leq C. \quad (5.11)$$
To identity \( \mathbf{v} \), we use (3.1) to deduce that \( \mathbf{1}_{\Omega_t^k} \overset{k \to \infty}{\rightarrow} \mathbf{1}_{\Omega_t^1} \). So combined with (5.5c), we have \( \mathbf{v} = \mathbf{1}_{\Omega_t^1} \mathbf{u} \).

\[ \square \]

**Theorem 5.3.** For any fixed \( t \in [0,T] \), we have \( \mathbf{v} \cdot \mathbf{\xi} = 0 \) for \( \mathcal{H}^1 \)-a.e. \( x \in I_t \).

The proof here is inspired by the density argument in [23]. It is worth mentioning that the use of density argument to prove weakly lower-semicontinuity like (5.20) was first employed in [12]. To proceed we define the following measures for any Borel set \( A \subset \Omega \):

\[
\theta(A) = \mathcal{H}^1(A \cap I_t), \\
\theta_k(A) = \int_{A \cap \Omega_t^k} |\nabla \psi_k| \, dx, \\
\mu_k(A) = \int_A (\mathbf{v}_k \cdot \mathbf{n}_k)^2 \, d\theta_k. 
\]

(5.12)

**Lemma 5.4.** There exists a Radon measure \( \mu \) so that

\[
\theta_k \overset{k \to \infty}{\rightarrow} \frac{1}{2} \theta, \quad \mu_k \overset{k \to \infty}{\rightarrow} \mu \quad \text{weakly-star as Radon measures.}
\]

(5.13)

**Proof.** By (3.7a) and (3.25b), the families of measures \( \{\theta_k\}, \{\mu_k\} \) have bounded total variations. So their weak convergences follow from weak compactness [2 Theorem 1.59]. It remains to show the limit of \( \theta_k \) is \( \frac{1}{2} \theta \).

To this end, we define truncation functions

\[
T_k(s) = \begin{cases} 0 & \text{when } s \leq b_k \\
 s - b_k & \text{when } b_k \leq s \leq q_k \\
 q_k - b_k & \text{when } s \geq q_k 
\end{cases} \quad T(s) = \begin{cases} 0 & \text{when } s \leq \frac{1}{2} \\
 s - 1/2 & \text{when } \frac{1}{2} \leq s \leq 2 \\
 3/2 & \text{when } s \geq 2 
\end{cases}
\]

(5.14)

It is obvious that \( T_k \overset{k \to \infty}{\rightarrow} T \) uniformly in \( \mathbb{R} \). Moreover,

\[
\nabla T_k(\psi_k) = \nabla \psi_k \mathbf{1}_{\Omega_t^k} \quad \text{a.e. } x \in \Omega, \\
T_k(\psi_k) \overset{k \to \infty}{\rightarrow} \frac{1}{2} \mathbf{1}_{\Omega_t^1} \quad \text{strongly in } L^p(\Omega) \quad \text{for any fixed } p \neq \infty.
\]

(5.15)

(5.16)

Actually (5.15) follows from [7, pp. 153] and (5.16) follows from (4.1) and the dominated convergence theorem.

Now using (3.1) and (2.14d), for any \( g(x) \in C^1_c(\Omega) \), we have

\[
\int g \, d\theta_k = \int_{\Omega_t^k} g |\nabla \psi_k| \, dx \overset{(2.14d)}{=} O(\varepsilon) + \int_{\Omega_t^k} g \, \xi \cdot \nabla \psi_k \, dx \\
\overset{(5.14)}{=} O(\varepsilon) + \int_{\Omega} g \, \xi \cdot \nabla T_k(\psi_k) \, dx \\
= O(\varepsilon) - \int_{\Omega} \text{div}(g\xi) T_k(\psi_k) \, dx.
\]

(5.17)

Using (5.16) and \( \xi \) being the inner normal, we get

\[
\lim_{k \to \infty} \int g \, d\theta_k = -\frac{1}{2} \int_{\Omega_t^1} \text{div}(g\xi) \, dx = \frac{1}{2} \int_{I_t} g \, d\mathcal{H}^1 = \frac{1}{2} \int_{I_t} g \, d\theta, \quad \forall g(x) \in C^1_c(\Omega).
\]

By approximation, one can pass from \( C^1_c(\Omega) \) to \( C_c(\Omega) \), and this completes the proof of (5.13). \( \square \)

**Proof of Theorem 5.3.** In view of (3.25b) and (5.8), it suffices to prove the following lower-semicontinuity inequality

\[ \frac{1}{2} \int_{I_t} (\mathbf{v} \cdot \mathbf{\xi})^2 \, d\mathcal{H}^1 \leq \liminf_{k \to \infty} \int (\mathbf{v}_k \cdot \mathbf{n}_k)^2 |\nabla \psi_k| \, dx. \]

(5.17)
Recall the Radon measures $\mu, \theta$ in (5.13). By the Besicovitch derivation theorem [2, Theorem 2.22], there exists a singular non-negative measure $\mu^s$ so that

$$
\mu = (D_\theta \mu) \theta + \mu^s, \quad \text{with } \mu^s \perp \theta,
$$

(5.18)

$$
D_\theta \mu = \lim_{r \downarrow 0} \frac{\mu(B_r(x_0))}{\theta(B_r(x_0))}, \quad \text{for } \theta - \text{a.e. } x_0 \in \text{supp}(\theta) = I_t.
$$

(5.19)

By the weakly lower-semicontinuity for the convergence of Radon measures and (5.18), we obtain

$$
\int_A D_\theta \mu \, d\theta \leq \mu(A) \leq \liminf_{k \to \infty} \mu_k(A)
$$

(5.20)

for any Borel set $A \subset \Omega$. So we can prove (5.17) by proving the following density estimate. $\square$

**Lemma 5.5.** For $\theta$-a.e. $x_0 \in \text{supp}(\theta) = I_t$, there holds

$$
\frac{1}{2} (v \cdot \xi)^2(x_0) \leq (D_\theta \mu)(x_0).
$$

(5.21)

**Proof.** During the proof, we shall omit the dependence on $t$. For instance we denote $v_k(x,t)$ by $v_k(x)$. To prove (5.21), we first use convexity to write, for some $a_m, c_m \in \mathbb{R}$, that

$$
f(s) = s^2 = \sup_m (a_m s + c_m).
$$

(5.22)

See [2, Proposition 2.31] for the proof. For any $x_0 \in I_t$ and any $R > 0$, it follows from Fubini’s theorem, (5.8) and (4.1) that

$$
\lim_{k \to \infty} \int_0^R \int_{\partial B_r(x_0)} \hat{u}_k(x) \cdot \frac{x - x_0}{|x - x_0|} T_k(\psi_k) \, dH^1(x) \, dr
$$

$$
= \lim_{k \to \infty} \int_{B_R(x_0)} 1_{\Omega_k^+} \hat{u}_k(x) \cdot \frac{x - x_0}{|x - x_0|} T_k(\psi_k) \, dx
$$

$$
= \frac{1}{2} \int_{B_R(x_0)} v(x) \cdot \frac{x - x_0}{|x - x_0|} 1_{\Omega_k^+} \, dx
$$

(5.8), (5.10)

So we can find $r_j \downarrow 0$ such that for each $j$, it holds that

$$
\lim_{k \to \infty} \int_{\partial B_{r_j}(x_0)} \hat{u}_k(x) \cdot \frac{x - x_0}{|x - x_0|} T_k(\psi_k) \, dH^1(x) = \frac{1}{2} \int_{\partial B_{r_j}(x_0) \cap \Omega_k^+} v(x) \cdot \frac{x - x_0}{|x - x_0|} \, dH^1(x).
$$

(5.23)

Moreover, we can arrange $r_j$ so that $\theta(\partial B_{r_j}(x_0)) = 0$. Due to the absolute continuity of $\mu$ with respect to $\theta$ [5.18], we also have $\mu(\partial B_{r_j}(x_0)) = 0$. This combined with (5.13) allows one to apply [2, Proposition 1.62] and yield

$$
\theta_k(B_{r_j}(x_0)) \xrightarrow{k \to \infty} \frac{1}{2} \theta(B_{r_j}(x_0)), \quad \mu_k(B_{r_j}(x_0)) \xrightarrow{k \to \infty} \mu(B_{r_j}(x_0)).
$$

(5.24)

Recall that for $\theta$-a.e. $x_0 \in I_t$, we have

$$
D_\theta \mu(x_0) \xrightarrow{j \to \infty} \lim_{j \to \infty} \frac{\mu(B_{r_j}(x_0))}{\theta(B_{r_j}(x_0))},
$$

(5.19)

$$
\lim_{j \to \infty} \frac{1}{\theta(B_{r_j}(x_0))} \int_{I_t \cap B_{r_j}(x_0)} v \cdot \xi \, dH^1 = (v \cdot \xi)(x_0).
$$

(5.26)
So it follows from (5.23), (5.24), (5.12c) and (5.22) that
\[
D_{\theta\mu}(x_0) = \lim_{j \to \infty} \frac{\mu(B_{r_j}(x_0))}{\theta(B_{r_j}(x_0))} = \lim_{j \to \infty} \lim_{k \to \infty} \frac{\mu_k(B_{r_j}(x_0))}{\theta(B_{r_j}(x_0))}
\]
\[
= \lim_{j \to \infty} \lim_{k \to \infty} \frac{1}{\theta(B_{r_j}(x_0))} \int_{B_{r_j}(x_0)} (v_k \cdot n_k)^2 \, d\theta_k
\]
\[
\geq \lim_{j \to \infty} \lim_{k \to \infty} \frac{1}{\theta(B_{r_j}(x_0))} \int_{B_{r_j}(x_0)} (a_{\alpha\beta}v_k \cdot n_k + c_m) \, d\theta_k.
\] (5.27)

In view of (5.12b) and (5.24), we obtain
\[
D_{\theta\mu}(x_0) \geq \lim_{j \to \infty} \lim_{k \to \infty} \left[ \frac{a_m}{\theta(B_{r_j}(x_0))} \int_{B_{r_j}(x_0)} v_k \cdot n_k |\nabla \psi_k| \, dx + c_m \frac{\theta_k(B_{r_j}(x_0))}{\theta(B_{r_j}(x_0))} \right]
\]
\[
= \lim_{j \to \infty} \lim_{k \to \infty} \frac{a_m}{\theta(B_{r_j}(x_0))} \int_{B_{r_j}(x_0)} v_k \cdot n_k |\nabla \psi_k| \, dx + \frac{c_m}{2}
\] (5.28)

It remains to compute the last integral under the limit $k \to \infty$ for fixed $j, m$. To this end, we first recall from (2.9b) that $n_k|\nabla \psi_k| = \nabla \psi_k$. So we integrate by parts and employ (4.12) and (5.8):
\[
\int_{B_{r_j}(x_0)} v_k \cdot n_k |\nabla \psi_k| \, dx
\]
\[
= \int_{B_{r_j}(x_0)} \hat{u}_k \cdot \nabla \psi_k 1_{\Omega^k_t} \, dx
\]
\[
= \int_{B_{r_j}(x_0)} \hat{u}_k \cdot \nabla T_k(\psi_k) \, dx
\]
\[
= \int_{\partial B_{r_j}(x_0)} \hat{u}_k \cdot \nu_{\partial B_{r_j}(x_0)} T_k(\psi_k) \, dH^1 - \int_{B_{r_j}(x_0)} 1_{\Omega^k_t}(\div \hat{u}_k) T_k(\psi_k) \, dx.
\] (5.29)

The convergence of the first integral in the last step is proved by (5.23). Concerning the second integral, by (5.9) and (5.8), we have
\[
1_{\Omega^k_t}(\div \hat{u}_k) \xrightarrow{k \to \infty} 1_{\Omega^k_t}(\div v) \text{ weakly in } L^2(\Omega).
\] (5.30)

This together with (5.16) yields the convergence of the second integral:
\[
\lim_{k \to \infty} \int_{B_{r_j}(x_0)} 1_{\Omega^k_t}(\div \hat{u}_k) T_k(\psi_k) \, dx = \frac{1}{2} \int_{B_{r_j}(x_0) \cap \Omega^k_t} (\div v) \, dH^1
\] (5.31)

Using (5.23) and (5.31), we can compute the last two integrals in (5.29) and obtain:
\[
\lim_{k \to \infty} \int_{B_{r_j}(x_0)} v_k \cdot n_k |\nabla \psi_k| \, dx
\]
\[
= \frac{1}{2} \int_{\partial B_{r_j}(x_0) \cap \Omega^k_t} v \cdot \nu_{\partial B_{r_j}(x_0)} \, dH^1 - \frac{1}{2} \int_{B_{r_j}(x_0) \cap \Omega^k_t} (\div v) \, dH^1
\]
\[
= -\frac{1}{2} \int_{B_{r_j}(x_0) \cap \Omega^k_t} v \cdot \xi \, dH^1.
\] (5.32)

Note that $\xi$ is the inner normal according to (1.8). Substituting (5.32) into (5.28) yields
\[
D_{\theta\mu}(x_0) \geq \lim_{j \to \infty} \frac{a_m}{\theta(B_{r_j}(x_0))} \frac{1}{2} \int_{B_{r_j}(x_0) \cap \Omega^k_t} v \cdot \xi \, dH^1 + \frac{c_m}{2} = \frac{a_m}{2} (v \cdot \xi)(x_0) + \frac{c_m}{2}
\] (5.33)

This together with (5.22) implies (5.21). \hfill \Box

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Appendix A. Proof of technical lemmas

Proof of Lemma 2.4. The LHS of (2.19) can be written as
\[
\int \nabla H : (\xi \otimes n^\varepsilon) |\nabla \psi^\varepsilon| \, dx - \int (\nabla \cdot H) \xi \cdot \nabla \psi^\varepsilon \, dx
\]
= \int \nabla H : (\xi - n^\varepsilon) \otimes n^\varepsilon |\nabla \psi^\varepsilon| \, dx + \int \nabla H : (n^\varepsilon \otimes n^\varepsilon) |\nabla \psi^\varepsilon| \, dx - \int (\nabla \cdot H) \xi \cdot \nabla \psi^\varepsilon \, dx. \tag{A.1}
\]
To treat the second term on the RHS of (2.19), we introduce the energy stress tensor \(T_\varepsilon\)
\[
(T_\varepsilon)_{ij} = \left(\frac{\varepsilon}{2} |\nabla u^\varepsilon|^2 + \frac{1}{\varepsilon} F(u^\varepsilon)\right) \delta_{ij} - \varepsilon \partial_i u^\varepsilon \cdot \partial_j u^\varepsilon. \tag{A.2}
\]
In view of (2.9a), we have the identity
\[
\nabla \cdot T_\varepsilon = -\varepsilon \nabla u^\varepsilon \cdot \Delta u^\varepsilon + \frac{1}{\varepsilon} \partial F(u^\varepsilon) \cdot \nabla u^\varepsilon = H^\varepsilon |\nabla u^\varepsilon|. \tag{A.3}
\]
Testing this identity by \(H\), integrating by parts and using (2.7), we obtain
\[
\int H^\varepsilon \cdot H |\nabla u^\varepsilon| \, dx = -\int \nabla H : T_\varepsilon \, dx,
\]
= \[- \int \nabla \cdot H \left(\frac{\varepsilon}{2} |\nabla u^\varepsilon|^2 + \frac{1}{\varepsilon} F(u^\varepsilon)\right) \, dx + \int (\nabla H)_{ij} \varepsilon (\partial_i u^\varepsilon \cdot \partial_j u^\varepsilon) \, dx. \tag{A.4}\]
So adding zero leads to
\[
\int \nabla H : n^\varepsilon \otimes n^\varepsilon |\nabla \psi^\varepsilon| \, dx
\]
= \[\int H^\varepsilon \cdot H |\nabla u^\varepsilon| \, dx + \int \nabla \cdot H \left(\frac{\varepsilon}{2} |\nabla u^\varepsilon|^2 + \frac{1}{\varepsilon} F(u^\varepsilon) - |\nabla \psi^\varepsilon|\right) \, dx + \int \nabla \cdot H |\nabla \psi^\varepsilon| \, dx \tag{A.5}\]
- \[\int (\nabla H)_{ij} \varepsilon (\partial_i u^\varepsilon \cdot \partial_j u^\varepsilon) \, dx + \int (\nabla H) : (n^\varepsilon \otimes n^\varepsilon) |\nabla \psi^\varepsilon| \, dx.
\]
Substituting this identity into (A.1) leads to (2.19). \(\Box\)

Proof of Lemma 2.5. Using the energy dissipation law (2.1) and adding zero, we compute the time derivative of the energy (1.11) by
\[
\frac{d}{dt} E_\varepsilon[u^\varepsilon[I] + \varepsilon \int |\partial_t u^\varepsilon|^2 \, dx - \int (\nabla : \xi) \partial d_F(u^\varepsilon) \cdot \partial_t u^\varepsilon \, dx
\]
= \[\int (H \cdot \nabla) \xi \cdot \nabla \psi^\varepsilon \, dx + \int (\nabla H)^T \xi \cdot \nabla \psi^\varepsilon \, dx
\]
- \[\int (\partial_t \xi + (H \cdot \nabla) \xi + (\nabla H)^T \xi) \cdot \nabla \psi^\varepsilon \, dx \tag{A.6}\]
Note that the last integral can be handled by (2.5). Due to the symmetry of the Hessian of \(\psi^\varepsilon\) and the boundary conditions (2.7), we have
\[
\int \nabla \cdot (\xi \otimes H) \cdot \nabla \psi^\varepsilon \, dx = \int \nabla \cdot (H \otimes \xi) \cdot \nabla \psi^\varepsilon \, dx.
\]
Hence, the first integral on the RHS of (A.6) can be rewritten as
\[
\int (H \cdot \nabla) \xi \cdot \nabla \psi^\varepsilon \, dx = \int \nabla \cdot (\xi \otimes H) \cdot \nabla \psi^\varepsilon \, dx - \int (\nabla \cdot H) \xi \cdot \nabla \psi^\varepsilon \, dx
\]
= \[\int (\nabla \cdot \xi) H \cdot \nabla \psi^\varepsilon \, dx + \int (\xi \cdot \nabla) H \cdot \nabla \psi^\varepsilon \, dx - \int (\nabla \cdot H) \xi \cdot \nabla \psi^\varepsilon \, dx.
\]
Using (2.19) in Lemma 2.4 to replace the third and fourth integrals on the RHS of the above identity and rewriting the last integral, we arrive at

\[
\frac{d}{dt} E_\varepsilon[u^\varepsilon|I] + \varepsilon \int |\partial_t u^\varepsilon|^2 \, dx - \int (\nabla \cdot \xi) \partial d_F(u^\varepsilon) \cdot \partial_t u^\varepsilon \, dx
\]

\[
= \int (\nabla \cdot \xi) H \cdot \nabla \psi^\varepsilon \, dx + \int (\xi \cdot \nabla) H \cdot \nabla \psi^\varepsilon \, dx - \int (\nabla \cdot H) \xi \cdot \nabla \psi^\varepsilon \, dx
\]

\[
+ \int \nabla H : (\xi \otimes n^\varepsilon) |\nabla \psi^\varepsilon| \, dx - \int \left( \partial_t \xi + (H \cdot \nabla) \xi + (H H)^T \xi \right) \cdot \nabla \psi^\varepsilon \, dx
\]

(A.7)

Note that the last integral vanishes due to (2.26). Moreover, the second to last and third to last integrals combine to \(J^1_\varepsilon\). Furthermore, by the property (2.12b) of the orthogonal projection (2.11), we can also find \(J^1_\varepsilon\) in the third to last line of (A.7). Indeed,

\[
- \int (\nabla H)_{ij} \varepsilon (\partial_i u^\varepsilon \cdot \partial_j u^\varepsilon) \, dx + \int \nabla H : n^\varepsilon \otimes n^\varepsilon |\nabla \psi^\varepsilon| \, dx
\]

\[
= \int \nabla H : n^\varepsilon \otimes n^\varepsilon |\nabla \psi^\varepsilon| \, dx - \int \varepsilon (\nabla H)_{ij} (\Pi u^\varepsilon \partial_i u^\varepsilon \cdot \Pi u^\varepsilon \partial_j u^\varepsilon) \, dx
\]

\[
- \int (\nabla H)_{ij} \varepsilon \left( (\partial_i u^\varepsilon - \Pi u^\varepsilon \partial_i u^\varepsilon) \cdot (\partial_j u^\varepsilon - \Pi u^\varepsilon \partial_j u^\varepsilon) \right) \, dx
\]

\[
= \int \nabla H : n^\varepsilon \otimes n^\varepsilon (|\nabla \psi^\varepsilon| - \varepsilon |\nabla u^\varepsilon|^2) \, dx + \int \varepsilon \nabla H : (n^\varepsilon \otimes n^\varepsilon) (|\nabla u^\varepsilon|^2 - |\Pi u^\varepsilon \nabla u^\varepsilon|^2) \, dx
\]

\[
- \int (\nabla H)_{ij} \varepsilon \left( (\partial_i u^\varepsilon - \Pi u^\varepsilon \partial_i u^\varepsilon) \cdot (\partial_j u^\varepsilon - \Pi u^\varepsilon \partial_j u^\varepsilon) \right) \, dx = J^1_\varepsilon.
\]

Using the definition (2.9b) of \(n^\varepsilon\), we may merge the second, third, and the last integral on the RHS of (A.7) to obtain

\[
\frac{d}{dt} E_\varepsilon[u^\varepsilon|I] = - \varepsilon \int |\partial_t u^\varepsilon|^2 \, dx + \int (\nabla \cdot \xi) \partial d_F(u^\varepsilon) \cdot \partial_t u^\varepsilon \, dx
\]

\[
+ \int (\nabla \cdot \xi) H \cdot \nabla \psi^\varepsilon \, dx + \int H \cdot \nabla |\nabla u^\varepsilon| \, dx - \int \nabla H : (\xi - n^\varepsilon) \otimes |\nabla \psi^\varepsilon| \, dx
\]

\[
+ \int (\nabla \cdot H) \left( \frac{\varepsilon}{2} |\nabla u^\varepsilon|^2 + \frac{1}{\varepsilon} F(u^\varepsilon) - |\nabla \psi^\varepsilon| \right) \, dx
\]

\[
+ \int (\nabla \cdot H) (1 - \xi \cdot n^\varepsilon) |\nabla \psi^\varepsilon| \, dx + J^1_\varepsilon + J^2_\varepsilon
\]

(A.8)
Now we complete squares for the first four terms on the RHS of (A.8). Reordering terms, we have

\[-\varepsilon|\partial_t u^\varepsilon|^2 + (\nabla \cdot \xi)\partial_d F(u^\varepsilon) \cdot \partial_t u^\varepsilon + (\nabla \cdot \xi)H \cdot \nabla u^\varepsilon + |H|\nabla u^\varepsilon|\]

\[= -\frac{1}{2\varepsilon}\left(\varepsilon|\partial_t u^\varepsilon|^2 - 2(\nabla \cdot \xi)\partial_d F(u^\varepsilon) \cdot \varepsilon\partial_t u^\varepsilon + (\nabla \cdot \xi)^2|\partial_d F(u^\varepsilon)|^2\right)
- \frac{1}{2\varepsilon}\varepsilon|\partial_t u^\varepsilon|^2 + \frac{1}{2\varepsilon}(\nabla \cdot \xi)^2|\partial_d F(u^\varepsilon)|^2 + (\nabla \cdot \xi)H \cdot \nabla \psi^\varepsilon
- \frac{1}{2\varepsilon}\left(|H|^2 - 2\varepsilon|\nabla u^\varepsilon|H \cdot \nabla u^\varepsilon + \varepsilon^2|\nabla u^\varepsilon|^2|H|^2\right) + \frac{1}{2\varepsilon}\left(|H|^2 + \varepsilon^2|\nabla u^\varepsilon|^2|H|^2\right)
- \frac{1}{2\varepsilon}\varepsilon|\partial_t u^\varepsilon - (\nabla \cdot \xi)\partial_d F(u^\varepsilon)|^2 - \frac{1}{2\varepsilon}|H| - \varepsilon|\nabla u^\varepsilon|H| - \frac{1}{2\varepsilon}|\partial_t u^\varepsilon|^2 + \frac{1}{2\varepsilon}|H|^2
+ \frac{\varepsilon}{2}\left(|\nabla u^\varepsilon|^2 - |\Pi_{u^\varepsilon}\nabla u^\varepsilon|^2\right)|H|^2.\]  

Using the definition (2.9b) of the normal \(n^\varepsilon\) and the chain rule in form of (2.12a), the terms in (A.9) form the last missing square. Integrating over the domain \(\Omega\) and substituting into (A.8) we arrive at (2.20).

**Proof of Proposition 5.1.** Now in view of (2.13), we have

\[\Pi_{u^\varepsilon}(\partial_t u^\varepsilon(x, t) \land u^\varepsilon(x, t) = 0 \text{ a.e.} (x, t) \in \Omega \times (0, T)\]  

for \(0 \leq i \leq 2\) where \(\partial_\delta := \partial_t\). Using (3.7b), (2.1) and (1.2), we deduce

\[\|u^\varepsilon\|_{L^\infty(0, T; L^q(\Omega))} \leq C, \quad \forall q \in [2, 6].\]  

This together with (3.7b) and (3.1) implies

\[\|\partial_t u^\varepsilon \land u^\varepsilon\|_{L^2(0, T; L^3/2(\Omega))} + \|\nabla u^\varepsilon \land u^\varepsilon\|_{L^\infty(0, T; L^{3/2}(\Omega))} \leq C\]

\[\leq \|(\partial_t u^\varepsilon - \Pi_{u^\varepsilon}\partial_t u^\varepsilon) \land u^\varepsilon\|_{L^2(0, T; L^{3/2}(\Omega))} + \|\nabla u^\varepsilon - \Pi_{u^\varepsilon}\nabla u^\varepsilon\} \land u^\varepsilon\|_{L^\infty(0, T; L^{3/2}(\Omega))} \leq C\]  

for some \(C\) independent of \(\varepsilon\). Combining this estimate with weak compactness implies (5.2).

It follows from (3.24a), (3.24b), (3.7b), and the Aubin-Lions lemma that, for any \(\delta > 0\), there exists a subsequence \(\varepsilon_k = \varepsilon_k(\delta) \to 0\) such that

\[\partial_t u^{\varepsilon_k} \to \partial_t \bar{u}_C, \text{ weakly in } L^2(0, T; L^2(\Omega_t^\pm \setminus I_t(\delta))),\]  

\[\nabla u^{\varepsilon_k} \to \nabla \bar{u}_C, \text{ weakly-star in } L^\infty(0, T; L^2(\Omega_t^\pm \setminus I_t(\delta))),\]  

\[u^{\varepsilon_k} \to \bar{u}_C, \text{ weakly-star in } L^\infty(\Omega \times (0, T)),\]  

\[u^{\varepsilon_k} \to \bar{u}_C, \text{ strongly in } C([0, T]; L^2(\Omega_t^\pm \setminus I_t(\delta))).\]  

By a diagonal argument, we infer there exists

\[u \in L^2(0, T; H^1_{loc}(\Omega^\pm)) \cap L^\infty(\Omega^\pm), \text{ with } \partial_t u \in L^2(0, T; L^2_{loc}(\Omega^\pm))\]  

such that

\[u(x, t) = \bar{u}_C(x, t) \text{ in } L^\infty(0, T; H^1(\Omega_t^\pm \setminus I_t(\delta))) \text{ hold for } \forall \delta > 0\]  

and thus the convergence (5.5) is proved.

Moreover, by (A.13), the interpolation theory and (A.12c), we have

\[u \in C([0, T]; L^2(\Omega_t^\pm)) \cap L^\infty(\Omega \times (0, T)).\]  

To prove \(u\) maps \(\Omega^+\) to \(S^1\), we first deduce that \(F(u)\) has the same regularity as \(u\) in (A.13), and thus by interpolation theory we obtain

\[F(u) \in C([0, T]; L^2(\Omega_t^\pm)).\]
So we use (5.5a), (3.24a), and Fatou’s lemma to deduce that
\[ F(u(x, t)) = 0, \; \forall t \in [0, T] \; \text{and a.e. in } x \in \Omega_t^\pm. \quad (A.17) \]
This together with (1.2) implies
\[ |u|(x, t) \in \{0, 1\}, \; \forall t \in [0, T] \; \text{and a.e. in } x \in \Omega_t^\pm. \quad (A.18) \]
By taking the $L^2$-norm, we obtain two continuous functions taking at most three values for each $t$:
\[ f^\pm(t) := \|u(\cdot, t)\|_{L^2(\Omega_t^\pm)} \in C([0, T] ; \{0, \sqrt{\Omega_t^\pm}\}). \quad (A.19) \]
On the other hand, by the choice of the initial condition (2.28) and the convergence (A.12d), we deduce that
\[ u(x, 0) = u_m(x), \; \text{a.e. in } \Omega_0^\pm \setminus I_0(\delta) \]
for any $\delta > 0$ and thus for $\delta = 0$. This implies $f^+(0) = \sqrt{|\Omega_0^+|}$, $f^-(0) = 0$ and thus
\[ f^+(t) = \sqrt{|\Omega_t^+|}, \; f^-(t) = 0, \; \forall t \in [0, T]. \]
This together with (A.18) implies
\[ u(x, t) = 0, \; \forall t \in [0, T] \; \text{and a.e. in } x \in \Omega_t^-, \quad (A.20) \]
\[ u(x, t) \in S^1, \; \forall t \in [0, T] \; \text{and a.e. in } x \in \Omega_t^+, \quad (A.21) \]
This combined with (A.13) yields
\[ u \in L^\infty(0, T; H^1_{loc}(\Omega_t^+; S^1)) \quad \text{with} \quad \partial_t u \in L^2(0, T; L^2_{loc}(\Omega_t^+; S^1)). \quad (A.22) \]
It remains to improve the integrability of $\nabla_x u$. To this end, we choose a sequence
\[ \eta^\ell(x, t) \in C^\infty_c(\Omega_t^+) \quad \text{such that} \quad \eta^\ell(x, t) \xrightarrow{\ell \to \infty} 1_{\Omega_t^+}(x). \quad (A.23) \]
Since $|u| = 1$ a.e., by (5.2a), (5.2b) and (5.3), we deduce that for a.e. $t \in (0, T)$ and $x \in \Omega_t^+$, there holds
\[ \eta^\ell g_i = \eta^\ell \partial_i u \wedge u, \; 0 \leq i \leq 2. \quad (A.24) \]
Since $g_i$ are $L^2$ integrable in $\Omega_T$, sending $\ell \to \infty$ and applying the dominated convergence theorem to (A.24) lead us to
\[ \partial_i u \wedge u \in L^\infty(0, T; L^{3/2}(\Omega_t^+)), \quad (A.25a) \]
\[ \partial_i u \wedge u \in L^2(0, T; L^{3/2}(\Omega_t^+)), \; 1 \leq i \leq 2. \quad (A.25b) \]
Retaining that $u$ maps into $S^1$, we deduce
\[ |\partial_i u|^2 = |\partial_i u \wedge u|^2, \quad |\partial_i u|^2 = |\partial_i u \wedge u|^2 \; \text{a.e. in } \Omega^+, \; 1 \leq i \leq 2. \quad (A.26) \]
So we improve (A.22) to (5.3). \hfill \square

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