ABSTRACT. This paper gives a plethysm formula on the characteristic map of the induced linear characters from $U_n(\mathbb{F}_q)$ to $GL_n(\mathbb{F}_q)$, the general linear group over finite field $\mathbb{F}_q$. The result turns out to be a multiple of a twisted version of the Hall-Littlewood symmetric functions $\tilde{P}_n(Y, q)$. A recurrence relation is also given which makes it easy to carry out the computation.

1. Introduction

Let $\mathbb{F}_q$ be a fixed finite field and $GL_n(\mathbb{F}_q)$ the finite general linear group over $\mathbb{F}_q$. The representation theory of $GL_n(\mathbb{F}_q)$ over $\mathbb{C}$ has been thoroughly studied by J.A.Green [4]. He also constructed the characteristic map which builds a connection between the character spaces of $GL_n(\mathbb{F}_q)$ for $n \geq 0$ and the Cartesian product over infinitely indexed sets of rings of symmetric functions. In character theory, the study of induced linear characters from subgroups is very useful to understand the character ring of the larger group. In this paper, we consider certain induced linear characters from the group of unipotent upper-triangular matrices $U_n(\mathbb{F}_q)$ to $GL_n(\mathbb{F}_q)$. The representations of these induced linear characters are known as Gelfand-Graev modules, which play an important role in the representation theory of finite groups of Lie type (3, 10). The formula for the characteristic map of the induced linear characters is given by Thiem [7]. We then apply plethysms on the image of the characteristic map. There are two advantages in doing so: to get a simpler formula and to express the result as a multiple of a twisted version of the Hall-Littlewood symmetric functions $\tilde{P}_n(y, q)$. We hope this method could contribute to the study on the irreducible decomposition of the induced characters from $U_n(\mathbb{F}_q)$ to $GL_n(\mathbb{F}_q)$.

In section 2 we give some background knowledge on symmetric functions and representation theory of $GL_n(\mathbb{F}_q)$ and $U_n(\mathbb{F}_q)$. Since the character theory of $U_n(\mathbb{F}_q)$ is known as a wild problem, supercharacter theory is built up as an approximation of the ordinary character theory. The linear characters of $U_n(\mathbb{F}_q)$ that we are considering are part of the category of supercharacters of $U_n(\mathbb{F}_q)$. We introduce further questions about the induction of all supercharacters in section 4. In Section 3 we give our main result about the plethysm formula. A recurrence relation is obtained naturally so that we can carry out the computation of plethysms on the characteristic map of the induced linear characters more easily. We also give a relation between the characteristic map of the induced characters from $U_n(\mathbb{F}_q)$ to $GL_n(\mathbb{F}_q)$ and the plethysms.
on those characteristics. This is depicted in the following diagram
\[
\otimes_{\varphi \in \Theta} \Lambda_{C}(Y^{\varphi}) \xrightarrow{T} \otimes_{f \in \Phi} \Lambda_{C}(X_{f}) \\
\rho \downarrow \quad \Pi|_{\Lambda_{C}(X_{f=x-1})}\\
\Lambda_{C}(Y) \xrightarrow{\circ \omega} \Lambda_{C}(X_{x-1})
\]
where the notation is explained in Theorem 3.16. From the above commutative
diagram we show that our simplified plethysm formula does not lose any information
on the characteristic map of the induced characters from \(U_{n}(F_{q})\) to \(GL_{n}(F_{q})\).

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2. **Background**

2.1. **Symmetric functions.** The notation in this paper follows closely the book of
Macdonald [6].

**Definition 2.1.** A partition \(\lambda\) of \(n \in \mathbb{N}\), is a sequence
\(\lambda = (\lambda_{1}, \lambda_{2}, \ldots, \lambda_{l})\) of positive
integers in weakly decreasing order: \(\lambda_{1} \geq \lambda_{2} \geq \cdots \geq \lambda_{l}\), such that
\(\lambda_{1} + \lambda_{2} + \cdots + \lambda_{l} = n\). We denote this by \(\lambda \vdash n\). Here, each \(\lambda_{i}\) \((1 \leq i \leq l)\) is called a part
of \(\lambda\). We say
the length of the partition \(\lambda\) is \(l\), which is the number of parts of \(\lambda\). We use \(|\lambda|\) to
denote the sum of all parts \(\lambda_{1} + \lambda_{2} + \cdots + \lambda_{l}\), and we call \(|\lambda|\) the size of the partition.
Sometimes we also use the notation:
\(\lambda = (1^{m_{1}}, 2^{m_{2}}, \ldots, n^{m_{n}}, \ldots)\)
where each \(m_{i}\) means there are \(m_{i}\) parts in \(\lambda\) equal to \(i\).

Let \(\Lambda_{C}(Y)\) denote the ring of symmetric functions with complex coefficients in
the variables \(Y = \{y_{1}, y_{2}, \ldots\}\). We denote the complete symmetric functions, ele-
mentary symmetric functions, monomial symmetric functions, power-sum symmetric
functions, and Schur symmetric functions by
\(h_{\lambda}(Y)\), \(e_{\lambda}(Y)\), \(m_{\lambda}(Y)\), \(p_{\lambda}(Y)\), and \(s_{\lambda}(Y)\)
respectively.

The generating function for \(h_{n}(Y)\) is
\[
H(Y; t) = \sum_{n \geq 0} h_{n}(Y)t^{n} = \prod_{j \geq 1} (1 - y_{j}t)^{-1}.
\]

Let \(X = \{x_{1}, x_{2}, \ldots\}\) be another set of finite or infinite variables. We have the
following identity:
\[
(2.1) \quad \Omega[XY] := \prod_{i,j} (1 - x_{i}y_{j})^{-1} = \sum_{\lambda} m_{\lambda}(X)h_{\lambda}(Y)
\]
summed over all partitions \(\lambda\).

There is a scalar product defined on \(\Lambda_{C}(Y)\), which makes \((m_{\lambda})\) and \((h_{\lambda})\) dual to
each other:
\[
\langle h_{\lambda}, m_{\mu} \rangle = \delta_{\lambda \mu}
\]
for all partitions \(\lambda, \mu\), where \(\delta_{\lambda \mu}\) is the Kronecker delta.
We use $P_{\lambda}(Y; t)$ to denote the Hall-Littlewood symmetric functions as defined in [6]. If we define

$$q_r = q_r(Y; t) = (1 - t)P_{(r)}(Y; t) \quad \text{for } r \geq 1,$$
$$q_0 = q_0(Y; t) = 1,$$

then the generating function for $q_r(Y; t)$ is

$$(2.2) \quad Q(u) = \sum_{r \geq 0} q_r(Y; t) u^r = \prod_i \frac{1 - y_i tu}{1 - y_i u}.$$  

For each partition $\lambda$ let $n(\lambda) = \sum_{i \geq 1} (i - 1)\lambda_i$. Define

$$\tilde{P}_{\lambda}(Y; q) = q^{-n(\lambda)}P_{\lambda}(Y; q^{-1})$$

and we call $\tilde{P}_{\lambda}(Y; q)$ the twisted Hall-Littlewood symmetric functions.

From [6] it is well known that the plethysm can be defined by

$$(2.3) \quad h_a[p_b] = h_a(y_1^b, y_2^b, \ldots),$$

which is the coefficient of $t^{ab}$ in $\prod_{j \geq 1} (1 - y_j^b t^b)^{-1}$.

2.2. Representation theory of $GL_n(F_q)$. The representation theory of the finite general linear group $G_n = GL_n(F_q)$ over $\mathbb{C}$ can be found in J.A.Greene [7], Macdonald [6] and Thiem [8]. Here we give a short description on the characteristic map constructed by J.A.Greene.

Let $\mathbb{F}_q$ denote the algebraic closure of the finite field $F_q$. The multiplicative group of $\mathbb{F}_q$ is denoted by $\mathbb{F}_q^\times$. Let $\mathbb{F}_q^* = \{\phi : \mathbb{F}_q^\times \to \mathbb{C}^\times\}$ be the group of complex-valued multiplicative characters of $\mathbb{F}_q^\times$. The Frobenius automorphism of $\mathbb{F}_q$ over $F_q$ is given by $F : x \rightarrow x^q$, where $x \in \mathbb{F}_q$.

We then define

$$\Phi = \{F\text{-orbits of } \mathbb{F}_q^\times\} \quad \text{and} \quad \Theta = \{F\text{-orbits of } \mathbb{F}_q^*\}.$$  

Since every $F$-orbits of $\mathbb{F}_q^\times$ is in one-to-one correspondence with irreducible polynomial $f$ over $F_q$, we can also use $f$ to denote each $F$-orbit in $\Phi$. A partition-valued function $\mu$ on $\Phi$ is a function which maps each $f \in \Phi$ to a partition $\mu(f)$. The size of $\mu$ is

$$\|\mu\| = \sum_{f \in \Phi} d(f) |\mu(f)|,$$

where $d(f)$ is equal to the degree of $f \in \Phi$.

Let $\mathbb{P}$ denote the set of all partitions and

$$\mathcal{P}^\Phi = \bigcup_{n \geq 0} \mathcal{P}^\Phi_n, \quad \text{where} \quad \mathcal{P}^\Phi_n = \{\mu : \Phi \to \mathbb{P} ; \|\mu\| = n\}.$$  

We use $K^\mu$ to denote the conjugacy classes in $G_n$ parameterized by $\mu \in \mathcal{P}^\Phi_n$ [6]. The characteristic function of the conjugacy class $K^\mu$ is denoted by $\pi_\mu$. 


Similarly, for each partition-valued function $\lambda : \Theta \to \mathbb{P}$, the size of $\lambda$ is

$$\|\lambda\| = \sum_{\varphi \in \Theta} d(\varphi) |\lambda(\varphi)|,$$

where $d(\varphi)$ is equal to the number of elements in $\varphi$. Let

$$\mathcal{P}^\Theta = \bigcup_{n \geq 0} \mathcal{P}^\Theta_n,$$

where $\mathcal{P}^\Theta_n = \{ \lambda : \Theta \to \mathbb{P}; \|\lambda\| = n \}$.

We use $G_n^\lambda$ to denote the irreducible $G_n$-modules indexed by $\lambda \in \mathcal{P}^\Theta_n$. The character of the irreducible $G_n$-modules $G_n^\lambda$ is denoted by $\chi^\lambda$.

For every $f \in \Phi$, let $X_f := \{ X_{1,f}, X_{2,f}, \ldots \}$ be a set of infinitely many variables. Each $X_{i,f}$ has degree $d(f)$.

Let

$$\tilde{P}_\eta(f) = \tilde{P}_\eta(X_f; q^{d(f)}) = q^{-d(f)n(\eta)} P_\eta(X_f; q^{-d(f)})$$

where $\tilde{P}_\eta(X_f; q^{d(f)})$ is the twisted Hall-Littlewood symmetric function. Define

$$\tilde{P}_\mu = \prod_{f \in \Phi} \tilde{P}_{\mu(f)}(f).$$

For every $\varphi \in \Theta$, let $Y^\varphi := \{ Y_1^\varphi, Y_2^\varphi, \ldots \}$ be a set of infinitely many variables. Each $Y_i^\varphi$ has degree $d(\varphi)$. Define

$$S_\lambda = \prod_{\varphi \in \Theta} s_{\lambda(\varphi)}(Y^\varphi),$$

where $s_{\lambda(\varphi)}(Y^\varphi)$ is the Schur symmetric function.

Let

$$\Lambda_C = \otimes_{f \in \Phi} \Lambda_C(X_f) = \otimes_{\varphi \in \Theta} \Lambda_C(Y^\varphi)$$

where $\Lambda_C(X_f)$ is the ring of symmetric functions in $X_f$, and $\Lambda_C(Y^\varphi)$ is the ring of symmetric functions in $Y^\varphi$. As a graded ring, we have

$$\Lambda_C = \mathbb{C}\text{-span}\{ \tilde{P}_\mu | \mu \in \mathcal{P}_n^\Phi \}$$

$$= \mathbb{C}\text{-span}\{ S_\lambda | \lambda \in \mathcal{P}_n^\Theta \}$$

The transformation between the symmetric functions in the variables $\{ X_f : f \in \Phi \}$ and those in the variables $\{ Y^\varphi : \varphi \in \Theta \}$ is given by the following identity:

$$p_k(Y^\varphi) = (-1)^{n-1} \sum_{x \in M_n} \xi(x) p_{n/d(\varphi)}(X_{fx}),$$

where $\xi \in \varphi$, $x \in f_x$ and $n = k \cdot d(\varphi)$. Here $p_k(Y^\varphi)$ and $p_{n/d(\varphi)}(X_{fx})$ are power-sum symmetric functions.

From [6] we know that the conjugacy classes $K^\mu$ of $G_n$ are parameterized by $\mu \in \mathcal{P}_n^\Phi$, and the irreducible characters $\chi^\lambda$ of $G_n$ are indexed by $\lambda \in \mathcal{P}_n^\Theta$. The following theorem gives the characteristic map of $G_n$. 

4.

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Theorem 2.2. (Green [7], Macdonald [6], Zelevinski [12]) Let $A_n$ denote the space of complex-valued class functions on $G_n$ and $A = \oplus_{n \geq 0} A_n$. The linear map
\[ ch : A \longrightarrow \Lambda_C \]
\[ \chi^\lambda \mapsto S_\lambda, \quad \text{for } \lambda \in \mathcal{P}^\Theta, \]
\[ \pi_\mu \mapsto \tilde{P}_\mu, \quad \text{for } \mu \in \mathcal{P}^\Phi, \]
is a Hopf algebra isomorphism.

2.3. Supercharacter theory. Let $U_n$ be the group of unipotent upper-triangular matrices with entries in the finite field $\mathbb{F}_q$ and ones on the diagonal. This group is the subgroup of the finite general linear group $G_n$. Although the character theory on $U_n$ is a wild problem, another slightly coarse version called superclass and supercharacter theory (André [1], Yan [11]) makes it easier to study and compute. Superclasses are certain unions of conjugacy classes and supercharacters are sums of irreducible characters. They are compatible in the sense that supercharacters are constant on superclasses. The supercharacter theory has a rich combinatorial structure (ref. [8]) and connects to some other algebra structures as well (ref. [2]).

The superclasses of $U_n$ can be indexed by the $\mathbb{F}_q^\times$-labeled set partitions, and a supercharacter becomes an irreducible character if the corresponding indexed $\mathbb{F}_q^\times$-labeled set partition has no crossing arcs. For the strict definitions and more details on supercharacters please see [8] or [2].

In this paper we consider the linear supercharacters of $U_n$ indexed by

\[
\begin{array}{cccccc}
 q_1 & q_2 & q_3 & \cdots & q_{n-1} \\
 1 & 2 & 3 & & n \\
\end{array}
\]

where $q_1, \ldots, q_{n-1} \in \mathbb{F}_q^\times$ (Thiem [8]). Let $\chi_{(q_1, \ldots, q_{n-1})}^{(n)}$ denote the above character. We induce $\chi_{(q_1, \ldots, q_{n-1})}^{(n)}$ from $U_n$ to $G_n$ by the formula
\begin{equation}
(2.5) \quad \chi_{(q_1, \ldots, q_{n-1})}^{(n)} \uparrow_{U_n}^{G_n} (g) = \frac{1}{|U_n|} \sum_{h \in G_n} \bar{\chi}_{(q_1, \ldots, q_{n-1})}^{(n)}(hgh^{-1}),
\end{equation}
where $\bar{\chi}(s) = \chi(s)$ if $s \in U_n$ and $\bar{\chi}(s) = 0$ if $s \not\in U_n$.

The induced character $\chi_{(q_1, \ldots, q_{n-1})}^{(n)} \uparrow_{U_n}^{G_n}$ is a character of $G_n$, which is known as the character of Gelfand-Greav module. Apply plethysms on the characteristic map of $\chi_{(q_1, \ldots, q_{n-1})}^{(n)} \uparrow_{U_n}^{G_n}$, and we get a multiple of the twisted Hall-Littlewood symmetric function $\tilde{P}_n$. A formula on this result together with a recurrence relation is given in Section 3.

3. Plethysm Formula for the induced character

We start from the formula of the characteristic map of $\chi_{(q_1, \ldots, q_{n-1})}^{(n)} \uparrow_{U_n}^{G_n}$, which is given by Thiem [7].
Theorem 3.1. (Thiem [7])

\[(3.1) \quad ch(\chi_{(q_1, \ldots, q_{n-1})}^{(n)} U_n) = \sum_{\lambda \in P_n^\theta, \lambda \vdash l_\theta(\lambda) = 1} S_\lambda, \]

where \(ht(\lambda) = \max \{l(\lambda(\varphi)) \mid \varphi \in \Theta \}\).

Notice that \(ht(\lambda) = 1\) implies for every \(\varphi \in \Theta\) we have \(l(\lambda(\varphi)) \leq 1\), which means \(\lambda(\varphi)\) contains at most one part. From the definition of \(S_\lambda\), we can write (3.1) as

\[(3.2) \quad ch(\chi_{(q_1, \ldots, q_{n-1})}^{(n)} U_n) = \sum_{\lambda \in P_n^\theta, \lambda \vdash l_\theta(\lambda) = 1} \prod_{\varphi \in \Theta} s_{\lambda(\varphi)}(Y^{\varphi}) \sum_{a_1 h_1 + \cdots + a_k b_k = n} \sum_{\lambda(\Theta) = (a_1, \ldots, a_k) \in P_n^\theta} \prod_{\varphi \in \Theta} h_{a_1} Y^{\varphi_1} h_{a_2} Y^{\varphi_2} \cdots h_{a_k} Y^{\varphi_k}, \]

where \(Y^{\varphi_1}, Y^{\varphi_2}, \ldots, Y^{\varphi_k}\) are different sets of variables. For \(i\) from 1 to \(k\), each variable in the set \(Y^{\varphi_i} = \{Y_1^{\varphi_i}, Y_2^{\varphi_i}, \ldots\}\) has degree \(b_i\).

We give an example to better understand formula (3.1) and (3.2).

**Example 3.2.** For \(n = 3\), we have

\[(3.3) \quad ch(\chi_{(q_1, q_2)}^{(3)} U_3) = \sum_{\varphi_1, \varphi_2, \varphi_3 \text{distinct}} h_1(Y^{\varphi_1}) h_1(Y^{\varphi_2}) h_1(Y^{\varphi_3}) + \sum_{\psi_1, \psi_2 \text{distinct}} h_2(Y^{\psi_1}) h_1(Y^{\psi_2}) + h_1(Y^{\varphi_1}) h_1(Y^{\varphi_2}) + \sum_{\deg(\varphi) = 3} h_3(Y^\varphi) + \sum_{\deg(\psi) = 3} h_1(Y^\psi). \]

From the above example we see that the expansion on the right-hand side of (3.2) becomes more complicated as \(n\) increases. This inspires us to use plethysm to simplify the computation.

For each term in equation (3.2), we have a two-rowed array \(\begin{pmatrix} b_1 & b_2 & \cdots & b_k \\ a_1 & a_2 & \cdots & a_k \end{pmatrix}\) where \(b_i = d(\varphi_i)\) and it satisfies the condition \(\sum_{i=1}^k a_i b_i = n\). We arrange the pairs \((b_i, a_i)\) such that:

1. \(b_1 \leq b_2 \leq \ldots \leq b_k\),
2. \(a_j \leq a_{j+1}\) if \(b_j = b_{j+1}\) for \(1 \leq j < k\).

Once the array is sorted, we can denote it as follows:

\[
\begin{pmatrix}
1_{m_1} & 2_{m_1} & \cdots & n_{m_1} & 1_{m_2} & 2_{m_2} & \cdots & n_{m_2} & \cdots & 1_{m_n} & 2_{m_n} & \cdots & n_{m_n}
\end{pmatrix}
\]

where \(\sum_{i,j=1}^n m_{i,j} \times j \times i = n\) and \(m_{i,1} + m_{i,2} + \ldots + m_{i,n} = m_i\) for \(1 \leq i \leq n\). Each \(m_{i,j}\) counts the number of different sets of variables appearing in the term with the same degree \(i\). Each \(m_{i,j}\) counts the number of complete symmetric functions \(h_j\) in variables with degree \(i\).
For a given $i$, let $l_q(i)$ denote the number of all different sets of variables with the same degree $i$. We know that $l_q(i)$ is equal to the number of irreducible polynomials $f$ over finite field $\mathbb{F}_q$ with degree $i$ and satisfying $f(0) \neq 0$. The number of irreducible polynomials of degree $i$ over $\mathbb{F}_q$ is given by the formula

$$L_q(i) = \frac{1}{i} \sum_{d|i} \mu(d)q^{\frac{i}{d}},$$

where $\mu$ is the Möbius function. Then we have

$$l_q(i) = \begin{cases} L_q(1), & \text{for } i = 1; \\ L_q(i), & \text{for } i \geq 2. \end{cases}$$

Thus for a given $i$ and a list of numbers $(m_{i,1}, m_{i,2}, \ldots, m_{i,n})$ where $m_{i,1} + m_{i,2} + \ldots + m_{i,n} = m_i$, the number of products in the form

$$h_1(Y^{\varphi_{i,1}})h_1(Y^{\varphi_{i,2}}) \cdots h_1(Y^{\varphi_{i,m_{i,1}}})$$

$$\times h_2(Y^{\varphi_{i,m_{i,1}+1}})h_2(Y^{\varphi_{i,m_{i,1}+2}}) \cdots h_2(Y^{\varphi_{i,m_{i,2}}})$$

$$\times \cdots$$

$$\times h_n(Y^{\varphi_{i,m_{i}+\cdots+m_{i,n-1}+1}})h_n(Y^{\varphi_{i,m_{i}+\cdots+m_{i,n-2}+2}}) \cdots h_n(Y^{\varphi_{i,m_{i}}})$$

is equal to

$$l_q(i)(l_q(i) - 1) \cdots (l_q(i) - m_i + 1)$$

$$\frac{m_{i,1}!m_{i,2}! \cdots m_{i,n}!}{m_{i,1}!m_{i,2}! \cdots m_{i,n}!},$$

where $Y^{\varphi_{i,1}}, Y^{\varphi_{i,2}}, \ldots, Y^{\varphi_{i,m_i}}$ are $m_i$ different sets of variables with the same degree $i$. Notice that when $n$ increases, we get more terms on the right-hand side of equation (3.2).

In order to simplify the computation, we apply plethysms on (3.2) which means replacing each set of variables $Y^{\varphi_{i}}$ by $\{y_1^{b_1}, y_2^{b_2}, \ldots\}$. In doing so we don’t differentiate the sets of variables. It seems that we lose information by applying the plethysms on the characteristic map. However this is not the case as we see later on in Theorem 3.16 and Corollary 3.17.

**Definition 3.3.** Define the plethysm map $\rho : \mathbb{C}\text{-span}\{S_\lambda | \lambda \in \mathcal{P}^\Theta\} \rightarrow \Lambda_\mathbb{C}(Y)$ as follows:

$$\rho(h_a(Y^\varphi)) = h_a[p_b(Y)], \quad \forall \varphi \in \Theta, b = \deg(\varphi).$$

Since $\text{ch}(\chi_{(q_1, \ldots, q_{n-1})}^{(n)} \uparrow_{U_n}^{G_n})$ is independent from $q_1, \ldots, q_{n-1}$, for $n \geq 1$ we simply denote $\rho(\text{ch}(\chi_{(q_1, \ldots, q_{n-1})}^{(n)} \uparrow_{U_n}^{G_n}))$ by $\rho_\Theta$ and set $\rho_0 = 1$. We also use $\rho_{(m_{i,1}, \ldots, m_{i,n})}$ to denote the results of taking plethysms on the sum of all different products in the form of (3.3) for the same index list $(m_{i,1}, \ldots, m_{i,n})$, i.e.

$$\rho_{(m_{i,1}, \ldots, m_{i,n})} := \frac{l_q(i)(l_q(i) - 1) \cdots (l_q(i) - m_i + 1)}{m_{i,1}!m_{i,2}! \cdots m_{i,n}!} (h_1[p_i])^{m_{i,1}} \cdots (h_n[p_i])^{m_{i,n}}.$$
Theorem 3.4. Let \( CH(t) \) denote the generating function for \( \rho_n \) as follows:

\[
CH(t) = 1 + \rho_1 t + \rho_2 t^2 + \cdots = \sum_{n \geq 0} \rho_n t^n.
\]

Then we have

\[
CH(t) = \prod_{i \geq 1} \left( \prod_{j \geq 1} (1 - y_j^i t^i)^{-1} \right)^{lq(i)} = \prod_{i \geq 1} \prod_{j \geq 1} (1 - y_j^i t^i)^{-lq(i)}.
\]

Proof. Since for every \( i \geq 1, \)

\[
\prod_{j \geq 1} (1 - y_j^i t^i)^{-1} = \sum_{a \geq 0} h_a(y_1^i, y_2^i, \ldots) t^{a i} = 1 + (h_1[p_i]) \cdot t^i + (h_2[p_i]) \cdot t^{2i} + \cdots.
\]

We have

\[
\left( \prod_{j \geq 1} (1 - y_j^i t^i)^{-1} \right)^{lq(i)} = (1 + (h_1[p_i]) t^i + (h_2[p_i]) t^{2i} + \cdots)^{lq(i)} = \sum_{m_{i,1} + m_{i,2} + \cdots + m_{i,n} = m_i} \binom{lq(i)}{m_i} \binom{m_i}{m_{i,1} m_{i,2} \cdots m_{i,n}} \\
\times (h_1[p_i])^{m_{i,1}} (h_2[p_i])^{m_{i,2}} \cdots (h_n[p_i])^{m_{i,n}} \cdot t^{(m_{i,1} + 2m_{i,2} + \cdots + n m_{i,n})i} = \sum_{m_{i,1} + m_{i,2} + \cdots + m_{i,n} = m_i, 0 \leq m_i \leq lq(i)} \rho(m_{i,1}, \ldots, m_{i,n}) \cdot t^{(m_{i,1} + 2m_{i,2} + \cdots + n m_{i,n})i}.
\]

From (3.4) we see that the coefficient of \( t^n \) in the product \( \prod_{i \geq 1} \left( \prod_{j \geq 1} (1 - y_j^i t^i)^{-1} \right)^{lq(i)} \) is exactly equal to \( \rho_n \) for \( n \geq 1 \). Thus we get the Theorem. \( \square \)

Theorem 3.5.

(3.5) \[
\prod_{i \geq 1} \prod_{j \geq 1} (1 - y_j^i t^i)^{-lq(i)} = \prod_{j \geq 1} \frac{(1 - y_j^i t^i)^{-1}}{(1 - y_j t)^{-1}}.
\]

Proof. The above identity is equivalent to the identity

(3.6) \[
\prod_{i \geq 1} \prod_{j \geq 1} (1 - y_j^i t^i)^{Lq(i)} = \prod_{j \geq 1} (1 - y_j^i t^i)
\]

where \( Lq(i) \) denotes the number of irreducible polynomials over \( \mathbb{F}_q \) for \( i \geq 1 \) as we stated before. To prove (3.6), we take the logarithm on both sides of (3.6) and show
they are equal.

\[
\ln \left( \prod_{i \geq 1} \prod_{j \geq 1} (1 - y_j t_i)^{Lq(i)} \right) = \sum_{j \geq 1} \left( \sum_{i \geq 1} Lq(i) \ln(1 - y_j t_i) \right) \\
= \sum_{j \geq 1} \left( \sum_{i \geq 1} \frac{Lq(i)}{\sum_{r \geq 1} \frac{y_j^{(i)} t_r}{i \cdot r}} \right) \\
= \sum_{j \geq 1} \left( \sum_{i \geq 1} \sum_{r \geq 1} \frac{Lq(i)}{\sum_{N \geq i \cdot r} \frac{N}{r}} y_j^{(N)} t_r^{(N)} \right) \\
= \sum_{j \geq 1} \left( \sum_{N \geq i \cdot r} \frac{y_j^{(N)} t_r^{(N)}}{N} \right) \cdot q^N \\
= \sum_{j \geq 1} (\ln(1 - y_j t)) = \ln \left( \prod_{j \geq 1} (1 - y_j t) \right)
\]

Thus we get (3.6). □

Theorem (3.4) and Theorem (3.5) together yield the formula for the generating function of \( \rho_n \) as follows:

\[
(3.7) \quad CH(t) = \prod_{j \geq 1} \frac{1 - y_j t}{1 - y_j t q}.
\]

Before we link it to Hall-Littlewood polynomials, we give a recurrence relation for \( \rho_n \) using formula (3.7).

**Corollary 3.6.** For every \( n \geq 1 \), we have

\[
(3.8) \quad \rho_n = (q^n - 1)h_n - \rho_{n-1}h_1 - \rho_{n-2}h_2 \cdots - \rho_1h_{n-1}.
\]

**Proof.** From (3.7) we have

\[
CH(t) \times H(t) = \prod_{j \geq 1} (1 - y_j t)^{-1}.
\]

Compare the coefficients of \( t^n \) on both sides we get

\[
\rho_0 h_n + \rho_1 h_{n-1} + \cdots + \rho_n h_0 = q^n h_n,
\]

which yields the theorem. □
Example 3.7.

\[
\begin{align*}
\rho_1 &= (q - 1)h_1; \\
\rho_2 &= (q^2 - 1)h_2 - \rho_1h_1 \\
&= (q^2 - 1)h_2 - (q - 1)h_{1,1} \\
&= (q - 1)[(q + 1)h_2 - h_{1,1}]; \\
\rho_3 &= (q^3 - 1)h_3 - \rho_1h_2 - \rho_2h_1 \\
&= (q^3 - 1)h_3 - (q - 1)h_{2,1} - (q^2 - 1)h_{2,1} + (q - 1)h_{1,1,1} \\
&= (q - 1)\left[(q^2 + q + 1)h_3 - (q + 2)h_{2,1} + h_{1,1,1}\right].
\end{align*}
\]

From the above examples we notice that the coefficients of \( h_\lambda \) are in \( \pm \mathbb{N}[q] \times (q - 1) \). Let \( [h_\lambda]\rho_n \) denote the coefficients of \( h_\lambda \) in the expansion of \( \rho_n \). In particular we have \( [h_n]\rho_n = q^n - 1 \) for all \( n \geq 1 \). The following corollary gives the recurrence relation on the coefficients.

**Corollary 3.8.** For any \( \lambda = (a_1^l, a_2^l, \ldots, a_k^l) \vdash n \) with \( l_i \geq 1 \) for all \( 1 \leq i \leq k \) and \( l(\lambda) \geq 2 \), we have

\[
[h_\lambda]\rho_n = -[h_{(a_1^{l_1}, a_2^{l_2}, \ldots, a_k^{l_k})}]\rho_{n-a_1} - \cdots - [h_{(a_1^{l_1}, a_2^{l_2}, \ldots, a_k^{l_k-1})}]\rho_{n-a_k}. \tag{3.9}
\]

Here if \( l_i = 1 \) for some \( 1 \leq i \leq k \), then we set

\[
(a_1^{l_1}, \ldots, a_i^{l_i-1}, \ldots, a_k^{l_k}) := (a_1^{l_1}, \ldots, \hat{a}_i, \ldots, a_k^{l_k}),
\]

where \( \hat{a}_i \) means simply remove \( a_i \) from the partition \( \lambda \). In particular, \( [h_\lambda]\rho_n \in \pm \mathbb{N}[q] \times (q - 1) \) while the sign is given by \(-1)^{l(\lambda)-1}\).

**Proof.** Equation (3.9) follows directly from Corollary 3.6 by comparing the coefficients of \( h_\lambda \) from two sides. The claim that \( [h_\lambda]\rho_n \) is in \( \pm \mathbb{N}[q] \times (q - 1) \) together with the sign property can be proved easily by using induction method on equation (3.9). \( \square \)

**Remark 3.9.** Corollary 3.6 and Corollary 3.8 give an easy way of computing \( \rho_n \) for every \( n \geq 1 \) simply by knowing \( [h_i]\rho_i = q^i - 1 \) for every \( i \geq 1 \).

**Example 3.10.**

\[
[h_{2,1}]\rho_3 = -[h_1]\rho_1 - [h_2]\rho_2
= -(q - 1) - (q^2 - 1)
= -(q - 1)(q + 2)
\]

\[
[h_{1,1,1}]\rho_3 = -[h_{1,1}]\rho_2 = [h_1]\rho_1
= q - 1
\]
Now back to our formula (3.7). We rewrite it into the following form so that we can easily use the generating function for \(q_r\) as in equation (2.2).

\[
CH(t) = \prod_{j \geq 1} \frac{1 - y_j t}{1 - y_j q t} = \prod_{j \geq 1} \frac{1 - y_j \cdot \frac{1}{q} \cdot (qt)}{1 - y_j \cdot (qt)}
\]

\[
= \sum_{r \geq 0} q_r(Y; q^{-1})q^r t^r,
\]

where \(Y = \{y_1, y_2, \ldots\}\). Comparing the coefficients from two sides we get the following corollary.

**Corollary 3.11.**

\[
\rho_n = q_n(Y; q^{-1})q^n = (1 - q^{-1})P_n(Y; q^{-1}) q^n
\]

\[
= q^{n-1}(q - 1)P_n(Y; q^{-1}) = q^{n-1}(q - 1)\tilde{P}_n(Y; q).
\]

Corollary 3.11 gives the connection between the plethysm of the characteristic map of \(\chi^{(n)}_{(q_1, \ldots, q_{n-1})} \uparrow_{U_n}^{G_n}\) and the Hall-littlewood symmetric functions.

For any linear supercharacter \([6, 7, 2]\) of \(U_n\), there is a unique way to decompose the indexed set partition into connected components. For a linear supercharacter with \(k\) connected components, we can denote it by \(\chi^{n_1[n_2] \ldots [n_k]}_{q_1, \ldots, q_k}\) where for \(i\) from 1 to \(k\), each \(n_i\) counts the size of the \(i^{th}\) connected component and \(\vec{q}_i = (q_{i,1}, \ldots, q_{i,n_i-1}) \in (\mathbb{F}_q^\times)^{n_i-1}\) denotes the labels of the arcs for the \(i^{th}\) connected component. The following corollary follows from the property of the linear supercharacters \([6, 7, 2]\).

**Corollary 3.12.**

\[
\rho \circ ch(\chi^{n_1[n_2] \ldots [n_k]}_{q_1, \ldots, q_k} \uparrow_{U_n}^{G_n}) = \prod_{i=1}^{k} \rho_{n_i}.
\]

**Example 3.13.** For the following linear supercharacter of \(U_6\)

\[
\chi^{1[2]3}_{q_1, q_2, q_3} = \chi
\]

where \(\vec{q}_1 = 0, \vec{q}_2 = (q_{2,1}), \vec{q}_3 = (q_{3,1}, q_{3,2})\) and \(q_{2,1}, q_{3,1}, q_{3,2} \in \mathbb{F}_q^\times\), we have

\[
\rho \circ ch(\chi^{1[2]3}_{q_1, q_2, q_3} \uparrow_{U_6}^{G_6}) = \rho_1 \rho_2 \rho_3.
\]

Let the transition matrix between \(\{m_\lambda(X)\}_{\lambda \vdash n}\) and \(\{p_\mu(X)\}_{\mu \vdash n}\) be \(C_{\lambda, \mu}\), i.e.

\[
m_\lambda(X) = \sum_\mu C_{\lambda, \mu} p_\mu(X).
\]

Define \(m_\lambda(q - 1)\) by the following equation

\[
m_\lambda(q - 1) = \sum_\mu C_{\lambda, \mu} p_\mu(q - 1),
\]

where \(p_n(q - 1) = q^n - 1\) for every \(n \geq 1\), and \(p_\mu(q - 1) = p_{\mu_1}(q - 1) \cdots p_{\mu_\ell}(q - 1)\) for \(\mu = \{\mu_1, \ldots, \mu_\ell\}\).
Remark 3.14. Using the orthogonal relation between the bases \( \{ m_\lambda \} \) and \( \{ h_\mu \} \), we give another expression for \( \rho_n \) as follows:

\[
\rho_n = \sum_{\lambda \vdash n} m_\lambda (q - 1) \cdot h_\lambda (Y).
\]

Proof. Using the notation in Section 2.1, we have

\[
\Omega(Yqt) = \prod_{j \geq 1} \frac{1}{1 - y_jqt}, \quad \Omega(-Yt) = \prod_{j \geq 1} (1 - y_jt).
\]

\[
CH(t) = \prod_{j \geq 1} \frac{1 - y_jt}{1 - y_jqt}
\]

\[= \Omega[(q - 1)Yt]
\]

\[= \sum_{n \geq 0} \left( \sum_{\lambda \vdash n} m_\lambda (q - 1) \cdot h_\lambda (Y) \right) t^n.
\]

It seems that we lose much information by taking plethysms on the characteristic map of \( \chi^{(n)}_{(q_1, \ldots, q_{n-1})} \uparrow^{G_n}_{U_n} \). However if we only consider the induced characters from \( U_n \) to \( G_n \), we can express the characteristic map of the induced characters in basis \( \{ \tilde{P}_\mu | \mu \in \mathcal{P}^\Phi \} \) from the results of doing plethysms. To show this fact, we first introduce the following homomorphism defined in [6]:

\[
\omega : \Lambda_C(Y) \to \Lambda_C(Y)
\]

by

\[
\omega(e_r(Y)) = h_r(Y), \quad \text{for all } r \geq 0.
\]

Lemma 3.15. ([6]) \( \omega \) is an involution and automorphism on \( \Lambda_C(Y) \). Also, we have

\[
\omega(p_r(Y)) = (-1)^{r-1}p_r(Y), \quad \text{for all } r \geq 0.
\]

The following theorem illustrates the relation between the application of plethysms on the characteristic map in basis \( \{ S_\lambda | \lambda \in \mathcal{P}^\Theta \} \) and the characteristic map in basis \( \{ \tilde{P}_\mu | \mu \in \mathcal{P}^\Phi \} \).

Theorem 3.16. The following diagram commutes:

\[
\begin{array}{ccc}
\otimes_{\varphi \in \Theta} \Lambda_C(Y^\varphi) & \xrightarrow{T} & \otimes_{f \in \Phi} \Lambda_C(X_f) \\
\rho \downarrow & & \downarrow \Pi|_{\Lambda_C(X_{f=x-1})} \\
\Lambda_C(Y) & \xrightarrow{t_{\omega}} & \Lambda_C(X_{x-1}) \\
\end{array}
\]

where \( T \) is the map of transformation from basis \( \{ S_\lambda | \lambda \in \mathcal{P}^\Theta \} \) to basis \( \{ \tilde{P}_\mu | \mu \in \mathcal{P}^\Phi \} \), \( t \) is the map of changing variables \( y_i \) into \( X_{i,x-1} \) for \( i = 1, 2, \ldots \), and \( \Pi|_{\Lambda_C(X_{f=x-1})} \) is the projection to the space \( \Lambda_C(X_{x-1}) \).
Proof. We rewrite equation (2.4) as follows
\[ p_k(Y^\varphi) = (-1)^{n-1} \sum_{x \in M_n} \xi(x)p_{n/d(x)}(X_f), \]
where \( \xi \in \varphi, x \in f_x \) and \( n = k \cdot d(\varphi). \) If we apply plethysm on \( p_k(Y^\varphi) \) we get \( p_n(Y). \) Applying the projection map \( \Pi|_{\Lambda_C(x_{f-x-1})} \) on the right-hand side of equation (2.4) yields \((-1)^{n-1}p_n(X_{x-1}). \) Since \( \{p_n : n = 1, 2, \ldots \} \) are algebraically independent over \( \mathbb{C} \) and \( \{\rho_\lambda : \lambda \text{ a partition}\} \) form a basis for \( \Lambda_C, \) we get the theorem from Lemma 3.15.

Corollary 3.17. If we use \( ch(\chi_{(q_1, \ldots, q_{n-1})}^{\mu} U_n^\mu)(X_f : f \in \Phi) \) to denote the expression of the characteristic map of \( ch(\chi_{(q_1, \ldots, q_{n-1})}^{\mu} U_n^\mu) \) in terms of basis \( \{\tilde{P}_\mu | \mu \in \mathcal{P}^\Theta\}, then we have the following identity:
\[ ch(\chi_{(q_1, \ldots, q_{n-1})}^{\mu} U_n^\mu)(X_f : f \in \Phi) = t \circ \omega(\rho_n) = q^{n-1}(q-1)\omega(\tilde{P}_n(X_{x-1})). \]

Proof. From the definition of the induced character by equation (2.5) we know that
\[ \chi_{(q_1, \ldots, q_{n-1})}^{\mu} U_n^\mu (g) = 0 \]
for all \( g \in G_n \) which are not similar to any unipotent upper-triangular matrices. Notice that the characteristic polynomial for all matrices in \( U_n \) is \((x-1)^n. \) Since similar matrices have the same characteristic polynomial, \( \chi_{(q_1, \ldots, q_{n-1})}^{\mu} U_n^\mu \) could possibly take nonzero values only on those matrices in \( G_n \) with characteristic polynomials equal to \((x-1)^n. \) Then we have
\[ ch(\chi_{(q_1, \ldots, q_{n-1})}^{\mu} U_n^\mu) \subseteq \Lambda_C(X_{x-1}) \]
and so
\[ \Pi|_{\Lambda_C(x_{f-x-1})}[ch(\chi_{(q_1, \ldots, q_{n-1})}^{\mu} U_n^\mu)] = ch(\chi_{(q_1, \ldots, q_{n-1})}^{\mu} U_n^\mu). \]
By theorem 3.16 we obtain the corollary.

Remark 3.18. From the proof of Corollary 3.17 we conclude that for any character \( \chi \) of \( U_n, \) if we induce \( \chi \) from \( U_n \) to \( G_n, \) then we have
\[ ch(\chi U_n^\mu)(X_f : f \in \Phi) = t \circ \omega \circ \rho(ch(\chi U_n^\mu)(Y^\varphi : \varphi \in \Theta)). \]

For \( \lambda = \lambda_1, \ldots, \lambda_t \) let \( \rho_\lambda = \rho_{\lambda_1} \rho_{\lambda_2} \cdots \rho_{\lambda_t}. \) By Corollary 3.11 since \( \rho_n = q^{n-1}(q-1)P_n(Y; q^{-1}) \) we know that \( \{\rho_\lambda\} \) forms a basis for the symmetric function ring \( \Lambda_C(Y). \) Thus \( \rho(ch(\chi U_n^\mu)) \) can be written into \( \rho(ch(\chi U_n^\mu)) = \sum_{\lambda \vdash n} C_\lambda \rho_\lambda \) where \( C_\lambda \in \mathbb{C}. \) We then define a map as follows.

Definition 3.19. Define \( \hat{\rho} : \Lambda_C(Y) \rightarrow \mathbb{C}-\text{span}\{S_\lambda | \lambda \in \mathcal{P}^\Theta\} \) by
\[ \hat{\rho}(\rho_n) := \sum_{\lambda \in \mathcal{P}^\Theta, \, h(\lambda) = 1} S_\lambda \]
\[ = ch(\chi_{(q_1, \ldots, q_{n-1})}^{\mu} U_n^\mu). \]
and
\[ \hat{\rho}(\rho_\lambda) = \hat{\rho}(\rho_{\lambda_1})\hat{\rho}(\rho_{\lambda_2}) \cdots \hat{\rho}(\rho_{\lambda_l}), \]
where \( \lambda = (\lambda_1, \ldots, \lambda_l) \).

**Proposition 3.20.** For a fixed finite field \( \mathbb{F}_q \) and a character \( \chi \) of \( U_n \), we have
\[ (\hat{\rho} \circ \rho)(ch(\chi \uparrow^{G_n}_{U_n})) = ch(\chi \uparrow^{G_n}_{U_n}). \]

**Proof.** Since \( \omega \) is an automorphism and \( \rho \) is multiplicative, the proposition follows from Theorem 3.16 and Remark 3.18. \( \square \)

Suppose \( \rho(ch(\chi \uparrow^{G_n}_{U_n})) = \sum_{\lambda \vdash n} C_\lambda \rho_\lambda \) where \( C_\lambda \in \mathbb{C} \), from the definition of \( \hat{\rho} \) we get
\[ \hat{\rho} \circ \rho(ch(\chi \uparrow^{G_n}_{U_n})) = \sum_{\lambda \vdash n} C_\lambda (\hat{\rho}(\rho_\lambda)) \]
\[ = \sum_{\lambda \vdash n} C_\lambda \hat{\rho}(\rho_{\lambda_1})\hat{\rho}(\rho_{\lambda_2}) \cdots \hat{\rho}(\rho_{\lambda_l}). \]
\[ (3.11) \]

Using Proposition 3.20 we get the following corollary.

**Corollary 3.21.** For a fixed finite field \( \mathbb{F}_q \) and a character \( \chi \) of \( U_n \), suppose \( ch(\chi \uparrow^{G_n}_{U_n}) = \sum_{\lambda \vdash n} C_\lambda \rho_\lambda \) where \( C_\lambda \in \mathbb{C} \). We have
\[ ch(\chi \uparrow^{G_n}_{U_n}) = \sum_{\lambda \vdash n} C_\lambda \left( \sum_{\lambda^{(1)} \in \mathcal{P}^{(\lambda_1)}} S_{\lambda(1)} \right) \left( \sum_{\lambda^{(2)} \in \mathcal{P}^{(\lambda_2)}} S_{\lambda(2)} \right) \cdots \left( \sum_{\lambda^{(l)} \in \mathcal{P}^{(\lambda_l)}} S_{\lambda(l)} \right). \]

**Remark 3.22.** It is difficult to get an expression for \( ch(\chi \uparrow^{G_n}_{U_n}) \) in terms of basis \( \{ S_\lambda | \lambda \in \mathcal{P}^{(\lambda)} \} \), which gives the irreducible decomposition of the induced character. However if we know the plethysm of the characteristic map of \( \chi \uparrow^{G_n}_{U_n} \), we may use \( \hat{\rho} \) to get the irreducible decomposition of \( ch(\chi \uparrow^{G_n}_{U_n}) \). We hope the results could contribute to research in this problem and we list some open problems in Section 4.

### 4. Further Questions

The induced characters that we are studying in this paper are very special, so a natural question to ask is if we can give a nice formula for the characteristics of all the induced supercharacters from \( U_n \) to \( G_n \). Zelevinsky [12] and Thiem and Vinroot [9] have worked on the case of degenerate Gelfand-Graev characters. The question of how the generalized Gelfand-Graev representations of the finite unitary group decompose is still open. The generalized Gelfand-Graev representations, which are defined by Kawanaka [5], are obtained by inducing certain irreducible representations from a unipotent subgroup [9]. Here the supercharacters that we are considering are more general than the case of the generalized Gelfand-Graev representations. We hope that the ideas and results developed in this paper could help to work on this direction.
Let us compute plethysms of the characteristic map of some induced supercharacters.

**Example 4.1.** For $q = 2$, we have

$$
\rho \circ ch \left( \chi \uparrow_{U_3}^{G_3} \right) = (\rho_3 + \rho_2 \rho_1)_{|q=2}
$$

$$
\rho \circ ch \left( \chi \uparrow_{U_4}^{G_4} \right) = (\rho_4 + 2\rho_3 \rho_1 + \rho_2 \rho_2^2)_{|q=2}
$$

$$
\rho \circ ch \left( \chi \uparrow_{U_4}^{G_4} \right) = (2\rho_4 + \rho_2 \rho_2 + \rho_3 \rho_1)_{|q=2}
$$

Inspired from these results, we give the following conjecture and open questions.

**Conjecture 4.2.** For a fixed finite field $\mathbb{F}_q$ and a supercharacter $\chi$ of $U_n$, we have

$$
\rho \circ ch(\chi \uparrow_{U_n}^{G_n}) \in \mathbb{N}[\rho_1, \ldots, \rho_n].
$$

If the above conjecture is true, then the following remark is meaningful.

**Remark 4.3.** For a fixed finite field $\mathbb{F}_q$ and a character $\chi$ of $U_n$, suppose $ch(\chi \uparrow_{U_n}^{G_n}) = \sum_{\lambda \vdash n} C_{\lambda} \rho_{\lambda}$ where $C_{\lambda} \in \mathbb{C}$. We have

$$
(4.1) \quad \dim(\chi) = \sum_{\lambda \vdash n} C_{\lambda}.
$$

**Proof.** From Corollary 3.12 we have

$$
\chi \uparrow_{U_n}^{G_n} = \sum_{\lambda \vdash n} C_{\lambda}(x_{\lambda_1|\lambda_2|\ldots|\lambda_l} \uparrow_{U_n}^{G_n}) = \left( \sum_{\lambda \vdash n} C_{\lambda} x_{\lambda_1|\lambda_2|\ldots|\lambda_l} \right) \uparrow_{U_n}^{G_n},
$$

where $\vec{q}_i = (q_{i,1}, \ldots, q_{i,\lambda_i-1}) \in (\mathbb{F}_q^\times)^{\lambda_i-1}$. So we have

$$
\dim(\chi) = \dim \left( \sum_{\lambda \vdash n} C_{\lambda} x_{\lambda_1|\lambda_2|\ldots|\lambda_l} \right) = \sum_{\lambda \vdash n} C_{\lambda} \dim(\chi_{\lambda_1|\lambda_2|\ldots|\lambda_l}).
$$

Since $\dim(\chi_{\lambda_1|\lambda_2|\ldots|\lambda_l}) = 1$, we prove the remark.
Question 4.4. For a fixed finite field $\mathbb{F}_q$ and a supercharacter $\chi$ of $U_n$, try to find a formula for the plethysms of the characteristic map of $\chi^G_{U_n}$.

$$\rho \circ ch(\chi^G_{U_n}) = \sum_{\lambda \vdash n} C_\lambda \rho_\lambda,$$

where $\rho_\lambda = \rho_{\lambda_1} \rho_{\lambda_2} \ldots \rho_{\lambda_l}$ for $\lambda = \lambda_1, \ldots, \lambda_l$. It is nice to give a combinatorial formula for the coefficient $C_\lambda$ since the example above suggest a few possible rules.

Remark 4.5. If we have the formula of $\rho \circ ch(\chi^G_{U_n})$, we can easily get the expression for the characteristic map of $\chi^G_{U_n}$ in terms of basis $\{ \tilde{P}_\mu | \mu \in P^\Phi \}$ by Remark 3.18. We may also use $\hat{\rho}$ to get an expression in the basis $\{ S_\lambda | \lambda \in P^\Theta \}$ by Corollary 3.21.

Question 4.6. Up to now the induced representations that we are considering are in characteristic zero. Another problem we can think about is what happens in characteristic $p$ case.

References

[1] C. ANDRÉ, Basic characters of the unitriangular group, J. Algebra 175 (1995), 287–319.
[2] Marcelo Aguiar, Carlos Andre, Carolina Benedetti, Nantel Bergeron, Zhi Chen, Persi Diaconis, Anders Hendrickson, Samuel Hsiao, I. Martin Isaacs, Andrea Jedwab, Kenneth Johnson, Gizem Karaali, Aaron Lauve, Tung Le, Stephen Lewis, Hulian Li, Kay Magaard, Eric Marberg, Jean-Christophe Novelli, Amy Pang, Franco Saliola, Lenny Tevlin, Jean-Yves Thibon, Nathaniel Thiem, Vidya Venkateswaran, C. Ryan Vinroot, Ning Yan, Mike Zabrocki, Basic characters of the unitriangular group, To appear in Adv. Math., DOI:10.1016/j.aim.2011.12.024.
[3] I. M. GELFAND and M. I. GRAEV, Construction of irreducible representations of simple algebraic groups over a finite field, Dokl. Akad. Nauk SSSR 147 (1962), 529–532.
[4] J. A. GREEN, The Characters of the finite general linear groups, Transactions of the American Mathematical Society, 80 (1955), 402–447.
[5] N. KAWANAKA, Generalized Gelfand-Graev representations and Ennola duality, In Algebraic groups and related topics (Kyoto/Nagoya, 1983), 175–206, Adv. Stud. Pure Math., 6, North-Holland, Amsterdam, 1985.
[6] I.G. MACDONALD, Symmetric Functions and Hall-Polynomials, Oxford Mathematical Monographs, Oxford Univ. Press, second edition (1995) 488p.
[7] Nathaniel Thiem, Unipotent Hecke algebras: the structure, representation theory, and combinatorics, Ph.D Thesis, University of Wisconsin - Madison, 2004.
[8] Nathaniel Thiem, Branching rules in the ring of superclass functions of unipotent upper-triangular matrices, J. Algebraic Combin. 31 (2010), no. 2, 267–298.
[9] Nathaniel Thiem and C. Ryan Vinroot, Gelfand-Graev characters of the finite unitary groups, Electron. J. Combin. 16 (2009), no. 1, Research Paper 146, 37 pp.
[10] R. STEINBERG, Lectures on Chevalley groups, mimeographed notes, Yale University, 1968.
[11] N. YAN, Representation theory of the finite unipotent linear groups, Unpublished Ph.D. Thesis, Department of Mathematics, University of Pennsylvania, 2001.
[12] A. V. ZELEVINSKY, Representations of finite classical groups. A Hopf algebra approach, Lecture Notes in Mathematics 869, Springer-Verlag, Berlin/New York, 1981.

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