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Abstract

Two conjectures about homology groups, \( K \)-groups and topological full groups of minimal \( \acute{e} \)tale groupoids on Cantor sets are formulated. We verify these conjectures for many examples of \( \acute{e} \)tale groupoids including products of \( \acute{e} \)tale groupoids arising from one-sided shifts of finite type. Furthermore, we completely determine when these product groupoids are mutually isomorphic. Also, the abelianization of their topological full groups are computed. They are viewed as generalizations of the higher dimensional Thompson groups.

1 Introduction

One can construct \( \acute{e} \)tale groupoids from various topological dynamical systems on Cantor sets. Minimal \( \mathbb{Z} \)-actions (i.e. minimal homeomorphisms) provide the most important example of dynamical systems on Cantor sets. The study of these dynamics was initiated by T. Giordano, I. F. Putnam and C. F. Skau [14], in which they classified minimal \( \mathbb{Z} \)-actions up to orbit equivalence. The associated \( \acute{e} \)tale groupoids and their topological full groups have been also studied extensively ([15], [26], [20]). Another fundamental example of dynamical systems on Cantor sets is the class of shifts of finite type (also called topological Markov shifts). These dynamical systems give symbolic representations of hyperbolic dynamical systems through Markov partitions. The \( \acute{e} \)tale groupoids of one-sided shifts of finite type (SFT) and the associated topological full groups were studied in [28].

With an \( \acute{e} \)tale groupoid \( \mathcal{G} \), we can associate the reduced groupoid \( C^* \)-algebra \( C^*_\mathcal{r}(\mathcal{G}) \). This construction has played an important role in the theory of \( C^* \)-algebras. When \( \mathcal{G} \) is a groupoid of a minimal \( \mathbb{Z} \)-action on a Cantor set, the \( C^* \)-algebra \( C^*_\mathcal{r}(\mathcal{G}) \) is known to be an AT algebra with real rank zero ([32] [14]). When \( \mathcal{G} \) is a groupoid of a one-sided SFT, the \( C^* \)-algebra \( C^*_\mathcal{r}(\mathcal{G}) \) is the so-called Cuntz-Krieger \( C^* \)-algebra ([10]). In general, when \( \mathcal{G} \) is minimal and essentially principal, \( C^*_\mathcal{r}(\mathcal{G}) \) is a simple \( C^* \)-algebra. It is a central problem to classify simple \( C^* \)-algebras by the \( K \)-groups. The study of the \( K \)-groups \( K_i(C^*_\mathcal{r}(\mathcal{G})) \) of the groupoid \( C^* \)-algebra has importance from this standpoint.

In the present paper, we discuss relationship between the \( K \)-groups \( K_i(C^*_\mathcal{r}(\mathcal{G}))) \) and the homology groups \( H_n(\mathcal{G}) \). More precisely, we conjecture that \( \bigoplus_n H_{2n+i}(\mathcal{G}) \) is isomorphic to \( K_i(C^*_\mathcal{r}(\mathcal{G}))) \) for any minimal essentially principal \( \acute{e} \)tale groupoid \( \mathcal{G} \) whose unit space is
a Cantor set (Conjecture 2.6), and verify it for many examples. Especially, the conjecture is true for minimal \( \mathbb{Z} \)-actions, one-sided SFT and any products of them (Theorem 2.8).

The other conjecture states that the sequence

\[
H_0(\mathcal{G}) \otimes \mathbb{Z}_2 \longrightarrow [[\mathcal{G}]]_{ab} \longrightarrow H_1(\mathcal{G}) \longrightarrow 0
\]

is exact for any minimal essentially principal étale groupoid \( \mathcal{G} \) whose unit space is a Cantor set (Conjecture 2.9), where \([[[\mathcal{G}]]]_{ab}\) is the abelianization of the topological full group \([[[\mathcal{G}]]]_{ab}\). Topological full groups provide many interesting examples of discrete groups from the viewpoint of combinatorial and geometric group theory. For minimal \( \mathbb{Z} \)-actions, various properties of the topological full groups were studied in [15]. Among others, it was shown that the isomorphism class of topological full groups is a complete invariant for flip conjugacy. Later, in [26], it was shown that the commutator subgroup \( D([[[\mathcal{G}]]]_{ab}) \) is simple and that \( D([[[\mathcal{G}]]]) \) is finitely generated if and only if the \( \mathbb{Z} \)-action is expansive. K. Juschenko and N. Monod [20] proved that \([[[\mathcal{G}]]]_{ab}\) is amenable. By these results, we obtained infinitely many countable infinite groups which are simple, finitely generated and amenable. When \( \mathcal{G} \) is a groupoid of one-sided SFT, in [28], it was proved that \([[[\mathcal{G}]]]_{ab}\) is of type \( F_{\infty} \) (in particular, finitely presented) and that \( D([[[\mathcal{G}]]]) \) is simple. They are regarded as generalizations of the Higman-Thompson groups (see [20][28]). Also, K. Matsumoto and the author [25] completely determined when these \([[[\mathcal{G}]]]_{ab}\) are mutually isomorphic. In this paper, we will prove that Conjecture 2.9 holds for many étale groupoids, which include almost finite groupoids (Theorem 3.6) and purely infinite groupoids with property TR (Theorem 4.4).

The latter half of the paper is devoted to the study of product groupoids \( \mathcal{G} \) of finitely many SFT groupoids. Topological full groups \([[[\mathcal{G}]]]_{ab}\) of these groupoids are thought of as generalizations of the higher dimensional Thompson groups \( nV_{k,r} \) (see Remark 5.13). First, we find a necessary and sufficient condition so that two such groupoids become isomorphic to each other (Theorem 5.12). This is an extension of the classification result of W. Dicks and C. Martínez-Pérez [11]. Furthermore, we compute the abelianization \([[[\mathcal{G}]]]_{ab}\) of the topological full group (Theorem 5.20). As a special case, the abelianization of \( nV_{k,r} \) is given explicitly (Theorem 5.23).

## 2 Preliminaries

### 2.1 Étale groupoids

The cardinality of a set \( A \) is written \#\( A \) and the characteristic function of \( A \) is written \( 1_A \).

The finite cyclic group of order \( n \) is denoted by \( \mathbb{Z}_n = \{ \bar{r} \mid r = 1, 2, \ldots, n \} \). We say that a subset of a topological space is clopen if it is both closed and open. A topological space is said to be totally disconnected if its topology is generated by clopen subsets. By a Cantor set, we mean a compact, metrizable, totally disconnected space with no isolated points. It is known that any two such spaces are homeomorphic. The homeomorphism group of a topological space \( X \) is written \( \text{Homeo}(X) \). The commutator subgroup of a group \( \Gamma \) is denoted by \( D(\Gamma) \). We let \( \Gamma_{ab} \) denote the abelianization \( \Gamma/D(\Gamma) \).

In this article, by an étale groupoid we mean a second countable locally compact Hausdorff groupoid such that the range map is a local homeomorphism. We refer the reader to [33][34] for background material on étale groupoids. For an étale groupoid \( \mathcal{G} \),
we let $G^{(0)}$ denote the unit space and let $s$ and $r$ denote the source and range maps. For $x \in G^{(0)}$, $G(x) = r(Gx)$ is called the $G$-orbit of $x$. When every $G$-orbit is dense in $G^{(0)}$, $G$ is said to be minimal. For a subset $Y \subset G^{(0)}$, the reduction of $G$ to $Y$ is $r^{-1}(Y) \cap s^{-1}(Y)$ and denoted by $G|Y$. If $Y$ is clopen, then the reduction $G|Y$ is an ´etale subgroupoid of $G$ in an obvious way. For $x \in G^{(0)}$, we write $G_x = r^{-1}(x) \cap s^{-1}(x)$ and call it the isotropy group of $x$. The isotropy bundle of $G$ is $G' = \{g \in G \mid r(g) = s(g)\} = \bigcup_{x \in G^{(0)}} G_x$. We say that $G$ is principal if $G' = G^{(0)}$. When the interior of $G'$ is $G^{(0)}$, we say that $G$ is essentially principal. The set of interior points $(G')^\circ$ of $G'$ forms an ´etale subgroupoid. The quotient of $G$ by the equivalence relation

$$g \sim h \iff gh^{-1} \in (G')^\circ$$

becomes an essentially principal ´etale groupoid. We call it the essentially principal part of $G$.

A subset $U \subset G$ is called a $G$-set if $r|U, s|U$ are injective. Any open $G$-set $U$ induces the homeomorphism $(r|U) \circ (s|U)^{-1}$ from $s(U)$ to $r(U)$. We write $\theta(U) = (r|U) \circ (s|U)^{-1}$. When $U, V$ are $G$-sets,

$$U^{-1} = \{g \in G \mid g^{-1} \in U\}$$

and

$$UV = \{gg' \in G \mid g \in U, g' \in V, s(g) = r(g')\}$$

are also $G$-sets. A probability measure $\mu$ on $G^{(0)}$ is said to be $G$-invariant if $\mu(r(U)) = \mu(s(U))$ holds for every open $G$-set $U$. The set of all $G$-invariant probability measures is denoted by $M(G)$.

Let $\varphi : \Gamma \curvearrowright X$ be an action of a countable discrete group $\Gamma$ on a locally compact Hausdorff space $X$ by homeomorphisms. We let $G_\varphi = \Gamma \times X$ and define the following groupoid structure: $(\gamma, x)$ and $(\gamma', x')$ are composable if and only if $x = \varphi^{\gamma'}(x')$, $(\gamma, \varphi^{\gamma'}(x')) \cdot (\gamma', x') = (\gamma \gamma', x')$ and $(\gamma, x)^{-1} = (\gamma^{-1}, \varphi^{\gamma}(x))$. Then $G_\varphi$ is an ´etale groupoid and called the transformation groupoid arising from $\varphi : \Gamma \curvearrowright X$.

We would like to recall the notion of topological full groups for ´etale groupoids.

**Definition 2.1** ([27] Definition 2.3). Let $G$ be an essentially principal ´etale groupoid whose unit space $G^{(0)}$ is a Cantor set. The set of all $\alpha \in \text{Homeo}(G^{(0)})$ for which there exists a compact open $G$-set $U$ satisfying $\alpha = \theta(U)$ is called the topological full group of $G$ and denoted by $[[G]]$.

For $\alpha \in [[G]]$ the compact open $G$-set $U$ as above uniquely exists, because $G$ is essentially principal. Obviously $[[G]]$ is a subgroup of $\text{Homeo}(G^{(0)})$. Since $G$ is second countable, it has countably many compact open subsets, and so $[[G]]$ is at most countable.

In [27 Section 7], we introduced the index map $I : [[G]] \to H_1(G)$, where $H_1(G)$ is the homology group of $G$ (see Section 2.2 or [27 Section 3]). For $\alpha \in [[G]]$, let $U \subset G$ be the compact open $G$-set such that $\alpha = \theta(U) = (r|U) \circ (s|U)^{-1}$. Then the characteristic function $1_U \in C_c(G, \mathbb{Z})$ is a cycle, and $I(\alpha)$ is the equivalence class of $1_U$ in $H_1(G)$. The index map $I$ is a homomorphism. We let $[[G]]_0$ denote the kernel of the index map $I$. Evidently $D([[G]])$ is contained in $[[G]]_0$.

The following theorem says that the groups $[[G]]$, $[[G]]_0$ and $D([[G]])$ ‘remember’ the ´etale groupoid $G$.
Thus, \( C \) strings of \( n \) with local homeomorphisms to the category of abelian groups with homomorphisms. \( C \) from \( \pi \). It is not so hard to see that \( f \) spaces. For \( \pi \) is compact, we simply write \( X \). Let \( \pi \) denote by \( H \) groupoid whose unit space is a Cantor set and suppose that \( G \) conditions are equivalent.

\[
\begin{align*}
\text{(1)} & \quad \mathcal{G}_1 \text{ and } \mathcal{G}_2 \text{ are isomorphic as \'{e}tale groupoids.} \\
\text{(2)} & \quad [[[\mathcal{G}_1]]] \text{ and } [[[\mathcal{G}_2]]] \text{ are isomorphic as discrete groups.} \\
\text{(3)} & \quad [[[\mathcal{G}_1]]_0] \text{ and } [[[\mathcal{G}_2]]_0] \text{ are isomorphic as discrete groups.} \\
\text{(4)} & \quad D(\mathcal{G}_1]) \text{ and } D(\mathcal{G}_2]) \text{ are isomorphic as discrete groups.}
\end{align*}
\]

Recently, V. V. Nekrashevych \cite{31} introduced two normal subgroups \( A(\mathcal{G}) \subset S(\mathcal{G}) \subset [[[\mathcal{G}]]] \). Roughly speaking, \( S(\mathcal{G}) \) is the subgroup generated by all elements of order two, and \( A(\mathcal{G}) \) is the subgroup generated by all elements of order three (see \cite{31} for the precise definitions). They are analogs of the symmetric and alternating groups. He proved that the same statement as the theorem above is true for \( A(\mathcal{G}) \) and \( S(\mathcal{G}) \). Furthermore, he proved that \( A(\mathcal{G}) \) is simple if \( \mathcal{G} \) is minimal, and that \( A(\mathcal{G}) \) is finitely generated if \( \mathcal{G} \) is expansive. When \( \mathcal{G} \) is minimal and almost finite (or purely infinite), we can verify \( A(\mathcal{G}) = D([[\mathcal{G}]])) \).

When \( \mathcal{G} \) is principal and almost finite, we can verify \( S(\mathcal{G}) = [[[\mathcal{G}]]]_0 \).

For an \'{e}tale groupoid \( \mathcal{G} \), we denote the reduced groupoid \( C^* \)-algebra of \( \mathcal{G} \) by \( C^*_r(\mathcal{G}) \) and identify \( C_0(\mathcal{G}^{(0)}) \) with a subalgebra of \( C^*_r(\mathcal{G}) \). J. Renault’s theorem \cite{34} Theorem 5.9 tells us that two essentially principal \'{e}tale groupoids \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) are isomorphic if and only if there exists an isomorphism \( \varphi : C^*_r(\mathcal{G}_1) \rightarrow C^*_r(\mathcal{G}_2) \) such that \( \varphi(C_0(\mathcal{G}^{(0)}_1)) = C_0(\mathcal{G}^{(0)}_2) \) (see also \cite{27} Theorem 5.1).

### 2.2 Homology groups and the K"{u}nneth theorem

The homology groups \( H_n(\mathcal{G}) \) of an \'{e}tale groupoid \( \mathcal{G} \) were first introduced and studied by M. Crainic and I. Moerdijk in \cite{9}. In the case that the unit space \( \mathcal{G}^{(0)} \) is a Cantor set, we investigated connections between the homology groups and dynamical properties of \( \mathcal{G} \) in \cite{27,28}. Let us recall the definition of \( H_n(\mathcal{G}) \).

Let \( A \) be a topological abelian group. For a locally compact Hausdorff space \( X \), we denote by \( C_c(X,A) \) the set of \( A \)-valued continuous functions with compact support. When \( X \) is compact, we simply write \( C(X,A) \). With pointwise addition, \( C_c(X,A) \) is an abelian group. Let \( \pi : X \rightarrow Y \) be a local homeomorphism between locally compact Hausdorff spaces. For \( f \in C_c(X,A) \), we define a map \( \pi_* : Y \rightarrow A \) by

\[
\pi_*(f)(y) = \sum_{\pi(x) = y} f(x).
\]

It is not so hard to see that \( \pi_* \) belongs to \( C_c(Y,A) \) and that \( \pi_* \) is a homomorphism from \( C_c(X,A) \) to \( C_c(Y,A) \). Besides, if \( \pi' : Y \rightarrow Z \) is another local homeomorphism to a locally compact Hausdorff space \( Z \), then one can check \((\pi' \circ \pi)_* = \pi'_* \circ \pi_* \) in a direct way. Thus, \( C_c(\cdot, A) \) is a covariant functor from the category of locally compact Hausdorff spaces with local homeomorphisms to the category of abelian groups with homomorphisms.

Let \( \mathcal{G} \) be an \'{e}tale groupoid. For \( n \in \mathbb{N} \), we write \( \mathcal{G}^{(n)} \) for the space of composable strings of \( n \) elements in \( \mathcal{G} \), that is,

\[
\mathcal{G}^{(n)} = \{(g_1, g_2, \ldots, g_n) \in \mathcal{G}^n \mid s(g_i) = r(g_{i+1}) \text{ for all } i = 1, 2, \ldots, n-1\}.
\]
For $i = 0, 1, \ldots, n$, we let $d_i : G^{(n)} \to G^{(n-1)}$ be a map defined by

$$d_i(g_1, g_2, \ldots, g_n) = \begin{cases} (g_2, g_3, \ldots, g_n) & i = 0 \\ (g_1, \ldots, g_i g_{i+1}, \ldots, g_n) & 1 \leq i \leq n-1 \\ (g_1, g_2, \ldots, g_{n-1}) & i = n. \end{cases}$$

When $n = 1$, we let $d_0, d_1 : G^{(1)} \to G^{(0)}$ be the source map and the range map, respectively. For $j = 0, 1, \ldots, n$, we let $s_j : G^{(n)} \to G^{(n+1)}$ be a map defined by

$$s_j(g_1, g_2, \ldots, g_n) = \begin{cases} (r(g_1), g_1, g_2, \ldots, g_n) & j = 0 \\ (g_1, \ldots, g_j, r(g_{j+1}), g_{j+1}, \ldots, g_n) & 1 \leq j \leq n-1 \\ (g_1, g_2, \ldots, g_n, s(g_n)) & j = n. \end{cases}$$

Clearly the maps $d_i$ and $s_j$ are local homeomorphisms. The spaces $(G^{(n)})_{n \geq 0}$ together with the face maps $d_i$ and the degeneracy maps $s_j$ form a simplicial space, which is called the Moore complex above, i.e., the Moore complex of the simplicial abelian group $(G^{(n)})_{n \geq 0}$. In addition, we define

$$\partial_n : C_c(G^{(n)}, A) \to C_c(G^{(n-1)}, A)$$

by

$$\partial_n = \sum_{i=0}^{n} (-1)^i d_{i*}.$$

The abelian groups $C_c(G^{(n)}, A)$ together with the boundary operators $\partial_n$ form a chain complex, which is called the Moore complex of the simplicial abelian group $(C_c(G^{(n)}, A))_{n \geq 0}$ (see [17] Chapter III.2 for instance).

**Definition 2.3 ([11] Section 3.1,[27] Definition 3.1).** We let $H_n(\mathcal{G}, A)$ be the homology groups of the Moore complex above, i.e. $H_n(\mathcal{G}, A) = \text{Ker} \ \partial_n / \text{Im} \ \partial_{n+1}$, and call them the homology groups of $\mathcal{G}$ with constant coefficients $A$. When $A = \mathbb{Z}$, we simply write $H_n(\mathcal{G}) = H_n(\mathcal{G}, \mathbb{Z})$. In addition, we define

$$H_0(\mathcal{G})^+ = \{ [f] \in H_0(\mathcal{G}) \mid f(x) \geq 0 \text{ for all } x \in \mathcal{G}^{(0)} \},$$

where $[f]$ denotes the equivalence class of $f \in C_c(\mathcal{G}^{(0)}, \mathbb{Z})$.

When $\mathcal{G} = \mathcal{G}_\varphi$ is the transformation groupoid arising from a group action $\varphi : \Gamma \curvearrowright X$ on a Cantor set $X$, $H_n(\mathcal{G})$ is naturally isomorphic to the group homology $H_n(\Gamma, C(X, \mathbb{Z}))$.

For the homology groups $H_n(\mathcal{G}, A)$, the following Künneth theorem holds.

**Theorem 2.4.** Let $\mathcal{G}$ and $\mathcal{H}$ be étale groupoids. For any $n \geq 0$, there exists a natural short exact sequence

$$0 \to \bigoplus_{i+j=n} H_i(\mathcal{G}) \otimes H_j(\mathcal{H}) \to H_n(\mathcal{G} \times \mathcal{H}) \to \bigoplus_{i+j=n-1} \text{Tor}(H_i(\mathcal{G}), H_j(\mathcal{H})) \to 0.$$ 

Furthermore these sequences split (but not canonically).
Proof. The family of abelian groups $C_c(G^{(m)}, \mathbb{Z}) \otimes C_c(H^{(n)}, \mathbb{Z})$ form a bisimplicial abelian group $\mathcal{A}$ (see [17, Chapter IV.2.2] for instance). We will apply the generalized Eilenberg-Zilber theorem [17, Theorem IV.2.4] to this bisimplicial abelian group $\mathcal{A}$. Its diagonal simplicial abelian group $d(\mathcal{A})$ is canonically identified with the simplicial abelian group $(C_c((G \times H)^{(n)}, \mathbb{Z}))_n$, because

$$C_c(G^{(n)}, \mathbb{Z}) \otimes C_c(H^{(n)}, \mathbb{Z}) \cong C_c(G^{(n)} \times H^{(n)}, \mathbb{Z}) \cong C_c((G \times H)^{(n)}, \mathbb{Z}).$$

On the other hand, we consider the Moore bicomplex for the bisimplicial abelian group $\mathcal{A}$. Evidently its total complex $\text{Tot}(\mathcal{A})$ is equal to the tensor product of the Moore complex of $(C_c(G^{(m)}, \mathbb{Z}))_m$ and the Moore complex of $(C_c(H^{(n)}, \mathbb{Z}))_n$. It follows from [17, Theorem IV.2.4] that the Moore complex of $d(\mathcal{A})$ is chain homotopy equivalent to $\text{Tot}(\mathcal{A})$. Hence the chain complex $(C_c((G \times H)^{(n)}, \mathbb{Z}))_n$ is chain homotopy equivalent to the tensor product of $(C_c(G^{(m)}, \mathbb{Z}))_m$ and $(C_c(H^{(n)}, \mathbb{Z}))_n$. Then, the usual Künneth formula applies and yields the desired conclusion. □

Corollary 2.5. Let $G = G_1 \times G_2 \times \cdots \times G_m$ be a product groupoid of étale groupoids $G_1, G_2, \ldots, G_m$. We have

$$H_0(G) \cong H_0(G_1) \otimes H_0(G_2) \otimes \cdots \otimes H_0(G_m)$$

and

$$H_1(G) \cong \bigoplus_{i_1 + i_2 + \cdots + i_m = 1} H_{i_1}(G_1) \otimes H_{i_2}(G_2) \otimes \cdots \otimes H_{i_m}(G_m) \oplus \bigoplus_{p=1}^{m-1} H_0(G_1) \otimes \cdots \otimes H_0(G_{p-1}) \otimes \text{Tor}(H_0(G_p), H_0(G_{p+1} \times \cdots \times G_m)).$$

Proof. Repeated use of the Künneth theorem yields the conclusions. We remark that (1) can be shown directly from the definition without appealing to the Künneth theorem. □

2.3 Conjectures

In this subsection, we formulate two conjectures, namely the HK conjecture (Conjecture 2.6) and the AH conjecture (Conjecture 2.9). The first conjecture says that the homology groups $H_n(G)$ agree with the $K$-groups of the $C^*$-algebra $C_r^*(G)$.

Conjecture 2.6 (HK conjecture). Let $G$ be an essentially principal minimal étale groupoid whose unit space $G^{(0)}$ is a Cantor set. Then we have

$$\bigoplus_{i=0}^{\infty} H_{2i}(G) \cong K_0(C_r^*(G))$$

and

$$\bigoplus_{i=0}^{\infty} H_{2i+1}(G) \cong K_1(C_r^*(G)).$$
Recall that two minimal étale groupoids \( G_1 \) and \( G_2 \) are said to be Morita equivalent (or Kakutani equivalent) if there exists a clopen subset \( Y_i \subset G_i^{(0)} \) for \( i = 1, 2 \) such that \( G_1|Y_1 \cong G_2|Y_2 \) (see [27, Definition 4.1] for instance).

**Proposition 2.7.** Let \( G \) and \( H \) be essentially principal minimal étale groupoids whose unit spaces are Cantor sets. If \( G \) and \( H \) are Morita equivalent and \( G \) satisfies the HK conjecture, then so does \( H \).

**Proof.** Let \( G \) be an essentially principal minimal étale groupoid and let \( Y \subset G^{(0)} \) be a clopen subset. Then \( C^*_r(G|Y) \) is canonically isomorphic to the hereditary subalgebra \( 1_Y C^*_r(G)1_Y \) of \( C^*_r(G) \). Therefore, if \( G \) and \( H \) are Morita equivalent, then \( C^*_r(G) \) is strongly Morita equivalent to \( C^*_r(H) \). It is well-known that \( K \)-groups of \( C^* \)-algebras are invariant under strong Morita equivalence. On the other hand, by [9, Corollary 4.6] (see also [27, Theorem 4.8]), homology groups of étale groupoids are invariant under Morita equivalence. Hence we get the conclusion. \( \square \)

As an immediate consequence of Theorem 2.4, we get the following.

**Theorem 2.8.** Let \( G \) and \( H \) be étale groupoids whose unit spaces are Cantor sets. Suppose that \( C^*_r(G) \) is nuclear and satisfies the UCT. If the HK conjecture is true for \( G \) and \( H \), then it is also true for the product groupoid \( G \times H \).

**Proof.** This follows from Theorem 2.4 (the Künneth theorem for groupoids) and [3, Theorem 23.1.3] (the Künneth theorem for \( C^* \)-algebras). \( \square \)

The second conjecture says that the abelianization \( [[G]]_{ab} \) has a close connection to the homology groups \( H_0(G) \) and \( H_1(G) \).

**Conjecture 2.9 (AH conjecture).** Let \( G \) be an essentially principal minimal étale groupoid whose unit space \( G^{(0)} \) is a Cantor set. Then there exists an exact sequence

\[
H_0(G) \otimes \mathbb{Z}_2 \longrightarrow [[G]]_{ab} \xrightarrow{I} H_1(G) \longrightarrow 0.
\]

Especially, if \( H_0(G) \) is 2-divisible, then we have \( [[G]]_{ab} \cong H_1(G) \).

Indeed, in many examples we can verify that there exists a short exact sequence

\[
0 \longrightarrow H_0(G) \otimes \mathbb{Z}_2 \longrightarrow [[G]]_{ab} \xrightarrow{I} H_1(G) \longrightarrow 0.
\]

In such a case, we say that \( G \) satisfies the strong AH property.

**Remark 2.10.** At the beginning of this research, I had expected that this stronger version of AH might be true in general. But V. V. Nekrashevych kindly informed me that this is not the case. For every locally expanding self-covering map \( f : M \to M \) of a compact path connected metric space \( M \), he introduced a finitely presented group \( V_f \) ([30]). The group \( V_f \) can be regarded as a topological full group of an étale groupoid on a Cantor set. The abelianization of \( V_f \) is computed in [30, Section 5.3], and it shows that the map \( H_0(G) \otimes \mathbb{Z}_2 \to [[G]]_{ab} \) is not always injective (see [30, Proposition 5.8]). The reader may find its detailed description in [31, Example 7.1]. After this communication with him, I noticed that product groupoids of SFT groupoids do not always have the strong AH property. We discuss it in Section 5.5.
The homomorphism $H_0(\mathcal{G}) \otimes \mathbb{Z}_2 \to [\mathcal{G}]_{ab}$ appearing in the AH conjecture is described as follows. Let $f \in C(\mathcal{G}^{(0)}, \mathbb{Z})$ and let $A = \{ x \in \mathcal{G}^{(0)} \mid f(x) \notin 2\mathbb{Z} \}$. Then $f - 1_A$ is in $2C(\mathcal{G}^{(0)}, \mathbb{Z})$. Since $\mathcal{G}$ is minimal, for any $x \in A$, there exists a compact open $\mathcal{G}$-set $U$ such that $x \in s(U) \subset A$ and $r(U) \cap s(U) = \emptyset$. Using the compactness of $A$, we can find compact open $\mathcal{G}$-sets $U_1, U_2, \ldots, U_n$ such that $\{s(U_i) \mid i = 1, 2, \ldots, n \}$ is a clopen partition of $A$ and $r(U_i) \cap s(U_i) = \emptyset$. Define $\tau_i \in [\mathcal{G}]_0$ by

$$\tau_i(x) = \begin{cases} 
\theta(U_i)(x) & x \in s(U_i) \\
\theta(U_i^{-1})(x) & x \in r(U_i) \\
x & \text{otherwise}.
\end{cases}$$

The homomorphism $H_0(\mathcal{G}) \otimes \mathbb{Z}_2 \to [\mathcal{G}]_{ab}$ is defined by sending the equivalence class of $f$ to the equivalence class of $\tau_1 \tau_2 \ldots \tau_n$. Notice that it is not clear at all if this is well-defined. One can find a proof of the well-definedness in [31].

We introduce two properties of $\mathcal{G}$ which are related to the AH conjecture. We call $\tau \in [\mathcal{G}]$ a transposition if $\tau^2 = 1$ and $\{ x \in \mathcal{G}^{(0)} \mid \tau(x) = x \}$ is clopen. By [27, Lemma 7.7 (3)], any transposition belongs to $[[G]]_0$.

**Definition 2.11.** Let $\mathcal{G}$ be an essentially principal étale groupoid whose unit space $\mathcal{G}^{(0)}$ is a Cantor set.

1. We say that $\mathcal{G}$ has cancellation if for any clopen sets $A_1, A_2 \subset \mathcal{G}^{(0)}$ with $[1_{A_1}] = [1_{A_2}]$ in $H_0(\mathcal{G})$ there exists a compact open $\mathcal{G}$-set $U \subset \mathcal{G}$ such that $s(U) = A_1$ and $r(U) = A_2$.

2. We say that $\mathcal{G}$ has property TR if the group $[[\mathcal{G}]]_0$ is generated by transpositions.

When $\mathcal{G}$ has cancellation, it is easy to see that for any $A_1, A_2$ as above we can find a transposition $\tau$ such that $\tau(A_1) = A_2$ and $\tau(x) = x$ for $x \in \mathcal{G}^{(0)} \setminus (A_1 \cup A_2)$.

### 3 Almost finite groupoids

In this section we show that Conjecture 2.9 holds for principal, minimal, almost finite groupoids.

Let us recall the notion of AF (approximately finite) groupoids ([33, Definition III.1.1], [16, Definition 3.7]).

**Definition 3.1.** Let $\mathcal{G}$ be an étale groupoid whose unit space is compact and totally disconnected.

(1) We say that $\mathcal{G}$ is elementary if $\mathcal{G}$ is compact and principal.

(2) We say that $\mathcal{G}$ is an AF groupoid if there exists an increasing sequence $\langle K_n \rangle_n$ of elementary subgroupoids of $\mathcal{G}$ such that $K_n^{(0)} = \mathcal{G}^{(0)}$ and $\bigcup_n K_n = \mathcal{G}$.

Let $\mathcal{G}$ be a (not necessarily minimal) AF groupoid. The topological full group $[[\mathcal{G}]]$ is written as an increasing union of finite direct sums of symmetric groups ([26, Proposition 3.3]). In particular, $[[\mathcal{G}]]$ is locally finite (actually, its converse also holds, i.e. if $[[\mathcal{G}]]$ is...
locally finite, then $\mathcal{G}$ is AF, see [26, Proposition 3.2]). The $C^*$-algebra $C^*_r(\mathcal{G})$ is an AF algebra. By [27, Theorem 4.10, 4.11], $H_n(\mathcal{G})$ is trivial for $n \geq 1$ and $H_0(\mathcal{G})$ is isomorphic to $K_0(C^*_r(\mathcal{G}))$ (as a unital ordered group). Hence, the HK conjecture holds for $\mathcal{G}$. When every $\mathcal{G}$-orbit contains more than one point, by [26, Lemma 3.5], $[\mathcal{G}]_{ab}$ is isomorphic to $H_0(\mathcal{G}) \otimes \mathbb{Z}_2$. Thus, $\mathcal{G}$ has the strong AH property. Moreover, when $\mathcal{G}$ is minimal, $D([\mathcal{G}])$ is simple ([26, Lemma 3.5]).

Next, we would like to recall the definition of almost finite groupoids.

**Definition 3.2** ([27, Definition 6.2]). Let $\mathcal{G}$ be an étale groupoid whose unit space is a Cantor set. We say that $\mathcal{G}$ is almost finite if for any compact subset $C \subset \mathcal{G}$ and $\varepsilon > 0$ there exists an elementary subgroupoid $K \subset \mathcal{G}$ such that

$$\frac{\#(CKx \setminus Kx)}{\#(Kx)} < \varepsilon$$

for all $x \in \mathcal{G}(0)$. We also remark that $\#(K(x)) = \#(Kx)$, because $K$ is principal.

The following are known for almost finite groupoids.

**Theorem 3.3.** Let $\mathcal{G}$ be an almost finite groupoid.

1. If $\mathcal{G}$ is minimal, then $D([\mathcal{G}])$ is simple.
2. The index map $I : [\mathcal{G}] \to H_1(\mathcal{G})$ is surjective.
3. If $\mathcal{G}$ is minimal, then $\mathcal{G}$ has cancellation.
4. If $\mathcal{G}$ is principal, then $\mathcal{G}$ satisfies property TR.

**Proof.** (1) is [28, Theorem 4.7]. (2) is [27, Theorem 7.5]. (3) is [27, Theorem 6.12]. (4) By [27, Theorem 7.13], any element of $[\mathcal{G}]_0$ is a product of four elements of finite order. Since $\mathcal{G}$ is principal, any element of finite order is elementary in the sense of [27, Definition 7.6 (1)]. Therefore it is a product of transpositions. 

For a $\mathcal{G}$-invariant probability measure $\mu \in M(\mathcal{G})$, we can define a homomorphism $\hat{\mu} : H_0(\mathcal{G}) \to \mathbb{R}$ by

$$\hat{\mu}([f]) = \int f \, d\mu.$$ 

It is clear that $\hat{\mu}(1_{\mathcal{G}(0)}) = 1$ and $\hat{\mu}(H_0(\mathcal{G})^+) \subset [0, \infty)$. Thus $\hat{\mu}$ is a state on $(H_0(\mathcal{G}), H_0(\mathcal{G})^+, [1_{\mathcal{G}(0)}])$ (see [3, Definition 6.8.1] for instance). It is also easy to see that the map $\mu \mapsto \hat{\mu}$ gives an isomorphism from $M(\mathcal{G})$ to the state space.

**Theorem 3.4.** Let $\mathcal{G}$ be a minimal almost finite groupoid.

1. $(H_0(\mathcal{G}), H_0(\mathcal{G})^+)$ is a simple, weakly unperforated, ordered abelian group with the Riesz interpolation property.
2. The homomorphism $\rho : H_0(\mathcal{G}) \to \text{Aff}(M(\mathcal{G}))$ defined by $\rho([f])(\mu) = \hat{\mu}([f])$ has uniformly dense range, where $\text{Aff}(M(\mathcal{G}))$ denotes the space of $\mathbb{R}$-valued affine continuous functions on $M(\mathcal{G})$. 

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Proof. (1) is \cite[Proposition 6.10]{27}. By (1), $H_0(G)/\text{Tor}(H_0(G))$ is a simple dimension group (see \cite[Section 7.4]{3} for instance). Besides, $H_0(G)/\text{Tor}(H_0(G))$ is not equal to $\mathbb{Z}$. It follows that $\rho$ has uniformly dense range in $\text{Aff}(M(G))$. \hfill $\Box$

By using these properties of almost finite groupoids, we can show the AH conjecture (Theorem \ref{theo 3.6}).

Lemma 3.5. Let $G$ be a minimal almost finite groupoid. Let $f \in C(G^{(0)}, \mathbb{Z})$. For any non-empty clopen set $A \subset G^{(0)}$, there exists a clopen subset $B \subset A$ such that $f$ and $1_B$ have the same equivalence class in $H_0(G) \otimes \mathbb{Z}_2$.

Proof. There exists $c > 0$ such that $\mu(A) \geq c$ for any $\mu \in M(G)$. By Theorem \ref{theo 3.4} (2), there exists $g \in C(G^{(0)}, \mathbb{Z})$ such that

$$\hat{\mu}(f) - c < 2\hat{\mu}(g) < \hat{\mu}(f)$$

for all $\mu \in M(G)$. It follows that $0 < \hat{\mu}(f - 2g) < c$ holds for all $\mu \in M(G)$. Hence, by \cite[Lemma 6.7]{27}, we can find a clopen subset $B \subset A$ such that $[1_B] = [f - 2g]$ in $H_0(G)$, which means that $1_B$ and $f$ have the same equivalence class in $H_0(G) \otimes \mathbb{Z}_2$. \hfill $\Box$

Theorem 3.6. Let $G$ be a principal, minimal, almost finite groupoid. Then, Conjecture \ref{conj 2.9} is true for $G$.

Proof. By Theorem \ref{theo 3.3} (2), the map $[[G]]_{ab} \to H_1(G)$ is surjective.

For a compact open $G$-set $U$ satisfying $r(U) \cap s(U) = \emptyset$, we define $\tau_U \in [[G]]$ by

$$\tau_U(x) = \begin{cases} 
\theta(U)(x) & x \in s(U) \\
\theta(U^{-1})(x) & x \in r(U) \\
x & \text{otherwise}
\end{cases}$$

First, we claim that the equivalence class $[[\tau_U]]$ of $\tau_U$ in $[[G]]_{ab}$ depends only on the equivalence class $[1_{s(U)}]$ of $1_{s(U)}$ in $H_0(G) \otimes \mathbb{Z}_2$. Suppose that $[1_{s(U)}]$ is zero in $H_0(G) \otimes \mathbb{Z}_2$. By the results of \cite[Section 6.2]{27}, we can find a clopen subset $U_1 \subset U$ such that $[1_{s(U_1)}] = [1_{s(U)}] \subset H_0(G)$. Let $U_2 = U \setminus U_1$. By Theorem \ref{theo 3.3} (3), there exists a compact open $G$-set $V$ such that $s(V) = s(U_1)$ and $r(V) = s(U_2)$. Put $W = V \cup U_2VU_1^{-1}$. Then we have

$$\tau_U = \tau_{U_1}\tau_{U_2} = \tau_{U_1}\tau_W\tau_{U_1}\tau_W \in D([[G]])$$

and so $[[\tau_U]]$ is trivial in $[[G]]_{ab}$. Suppose that two $G$-sets $U_1$ and $U_2$ satisfy $r(U_i) \cap s(U_i) = \emptyset$ and $[1_{s(U_1)}] = [1_{s(U_2)}]$ in $H_0(G) \otimes \mathbb{Z}_2$. We would like to prove that $\tau_{U_1}$ and $\tau_{U_2}$ have the same equivalence class in $[[G]]_{ab}$. In view of Lemma \ref{lem 3.5} we can find clopen $G$-sets $V_1 \subset U_i$ such that $[1_{s(U_i)}] = [1_{s(V_1)}]$ in $H_0(G) \otimes \mathbb{Z}_2$ and $[1_{s(V_1)}] = [1_{s(V_2)}]$ in $H_0(G)$. Since the equivalence class of $1_{s(U_i \setminus V_i)} = 1_{s(U_i \setminus s(V_i))}$ is trivial in $H_0(G) \otimes \mathbb{Z}_2$, by what we have shown above, $\tau_{U_i \setminus V_1} = \tau_{U_1}\tau_{V_1}$ belongs to the commutator subgroup $D([[G]])$. By Theorem \ref{theo 3.3} (3), there exists a compact open $G$-set $V'$ such that $s(V') = s(V_1)$ and $r(V') = s(V_2)$. Hence, in the same way as above, one can show that $\tau_{V_1}$ is conjugate to $\tau_{V_2}$. Therefore, we get $[[\tau_{V_1}]] = [\tau_{V_1}] = [\tau_{V_2}] = [\tau_{V_2}]$ in $[[G]]_{ab}$.

We define the homomorphism $j : H_0(G) \otimes \mathbb{Z}_2 \to [[G]]_{ab}$ as follows. Let $f \in C(G^{(0)}, \mathbb{Z})$. Using Lemma \ref{lem 3.5}, one can find a non-empty compact open $G$-set $U$ such that $r(U) \cap s(U) = \emptyset$, and so $[[\tau_V]] = [\tau_V]$. Therefore, we get $[[\tau_V]] = [\tau_V] = [\tau_V] = [\tau_V]$ in $[[G]]_{ab}$.\hfill $\Box$
property or not. In other words, we cannot prove that the homomorphism

\[ j([f]) = [\tau_U]. \]

By the claim above, the map \( j \) is well-defined. Moreover, for any \( f_1, f_2 \in C(G^{(0)}, \mathbb{Z}) \), we may choose the \( G \)-sets \( U_1, U_2 \) so that \( (r(U_1) \cup s(U_1)) \cap (r(U_2) \cup s(U_2)) = \emptyset \). Then

\[
\begin{align*}
j([f_1] + [f_2]) &= j([f_1 + f_2]) = [\tau_{U_1 \cup U_2}] \\
&= [\tau_{U_1} \tau_{U_2}] = [\tau_{U_1}] + [\tau_{U_2}] \\
&= j([f_1]) + j([f_2]),
\end{align*}
\]

which means that \( j \) is a homomorphism.

Let us prove that \( \text{Ker} I \) is contained in the image of \( j \). Assume that \( \gamma \in [G] \) is in \( \text{Ker} I = [\mathcal{G}]_0 \). By Theorem \ref{thm:ah_property} (4), \( \gamma \) is a product of transpositions. Consequently, the equivalence class of \( \gamma \) is contained in the image of \( j \).

In \cite{Harpe-Skandalis} Lemma 6.3, it was shown that when \( \varphi : \mathbb{Z}^N \curvearrowright X \) is a free action of \( \mathbb{Z}^N \) on a Cantor set \( X \), the transformation groupoid \( \mathcal{G}_\varphi \) is almost finite. When \( N = 1 \) and \( \varphi : \mathbb{Z} \curvearrowright X \) is minimal, it is well known that \( H_0(\mathcal{G}_\varphi) \cong K_0(C_\varphi^*(\mathcal{G}_\varphi)) \), \( H_1(\mathcal{G}_\varphi) \cong K_1(C_\varphi^*(\mathcal{G}_\varphi)) \cong \mathbb{Z} \) and \( H_n(\mathcal{G}_\varphi) = 0 \) for \( n \geq 2 \). Hence, Conjecture \ref{conj:ah_property} holds for \( \mathcal{G}_\varphi \). Furthermore, the strong AH property also holds (\cite{Guentner} Theorem 4.8). When \( N \) is greater than one, it is not known whether Conjecture \ref{conj:ah_property} is true for \( \mathcal{G}_\varphi \). We remark that the Chern character implies the isomorphisms

\[
\bigoplus_{i=0}^{\infty} H_{2i}(\mathcal{G}_\varphi) \otimes \mathbb{Q} \cong K_0(C_\varphi^*(\mathcal{G}_\varphi)) \otimes \mathbb{Q}
\]

and

\[
\bigoplus_{i=0}^{\infty} H_{2i+1}(\mathcal{G}_\varphi) \otimes \mathbb{Q} \cong K_1(C_\varphi^*(\mathcal{G}_\varphi)) \otimes \mathbb{Q}
\]

(see \cite{Brown-Douglas-Schwarz} Section 4, \cite{Harpe-Skandalis} Section 3.1).

By Theorem \ref{thm:ah_property}, the AH conjecture holds for \( \mathcal{G}_\varphi \). But, we do not know whether the groupoid \( \mathcal{G}_\varphi \) has the strong AH property.

In general, it is not known if every almost finite groupoid \( \mathcal{G} \) satisfies the strong AH property or not. In other words, we cannot prove that the homomorphism \( j : H_0(\mathcal{G}) \otimes \mathbb{Z}_2 \to [\mathcal{G}]_{ab} \) is always injective, and cannot find an example such that the map \( j \) has nontrivial kernel. (For purely infinite groupoids, we know that \( j \) is not always injective. See \cite{Winter} Example 7.1 and Section 5.5 of the present paper.) However, for a minimal free action \( \varphi : \mathbb{Z}^N \curvearrowright X \), one can prove that the kernel of \( j \) is at least ‘contained’ in the infinitesimal subgroup of \( H_0(\mathcal{G}_\varphi) \) (Proposition \ref{prop:kernel}).

Let us recall the definition of the de la Harpe-Skandalis determinant from \cite{Harpe-Skandalis}. Let \( A \) be a unital \( C^* \)-algebra and let \( T(A) \) be the space of tracial states on \( A \). The de la Harpe-Skandalis determinant is a homomorphism

\[ \Delta : U(A)_0 \to \text{Aff}(T(A))/D_A(K_0(A)), \]

where \( U(A)_0 \) is the connected component of the identity in the unitary group \( U(A) \) of \( A \) and \( D_A : K_0(A) \to \text{Aff}(T(A)) \) is the homomorphism defined by \( D_A([p]) = \sigma(p) \). For \( u \in U(A)_0 \), \( \Delta(u) \) is defined as follows. Let \( \xi : [0, 1] \to U(A)_0 \) be a piecewise smooth path such that \( \xi(0) = 1 \) and \( \xi(1) = u \). Then the function

\[ T(A) \ni \sigma \mapsto \frac{1}{2\pi \sqrt{-1}} \int_0^1 \sigma(\xi'(t)\xi(t)^*) \, dt \in \mathbb{R} \]
belongs to \( \text{Aff}(T(A)) \), and \( \Delta(u) \) is defined as the equivalence class of this affine function in \( \text{Aff}(T(A))/D_A(K_0(A)) \). It is known that \( \Delta \) is a well-defined homomorphism.

Let \( G \) be an essentially principal étale groupoid whose unit space is a Cantor set. For any \( \mu \in M(G) \), there exists a tracial state \( \sigma_\mu \) on \( C^*_r(G) \) such that \( \sigma_\mu(1_C) = \mu(C) \) for every clopen set \( C \subset G^{(0)} \). The map \( \mu \mapsto \sigma_\mu \) gives an isomorphism between \( M(G) \) and \( T(C^*_r(G)) \). The infinitesimal subgroup of \( H_0(G) \) is defined by

\[
\text{Inf}(H_0(G)) = \left\{ [f] \in H_0(G) \mid \int f \, d\mu = 0 \quad \forall \mu \in M(G) \right\}.
\]

**Proposition 3.7.** Let \( \varphi : \mathbb{Z}^N \curvearrowright X \) be a minimal free action of \( \mathbb{Z}^N \) on a Cantor set \( X \). The kernel of the homomorphism \( \varphi : H_0(G_\varphi) \otimes \mathbb{Z}_2 \rightarrow [[G_\varphi]]_{\text{ab}} \) is contained in \( \text{Inf}(H_0(G_\varphi)) \otimes \mathbb{Z}_2 \). In particular, when \( \text{Inf}(H_0(G_\varphi)) \otimes \mathbb{Z}_2 \) is trivial, \( G_\varphi \) has the strong AH property.

**Proof.** Because \( G_\varphi \) is principal, minimal and almost finite, Conjecture 2.9 holds for \( G_\varphi \) by Theorem 3.6. Put \( A = C^*_r(G_\varphi) \).

Let \( U \subset G_\varphi \) be a compact open \( G_\varphi \)-set such that \( r(U) \cap s(U) = \emptyset \). Set

\[
V = U \cup U^{-1} \cup (G^{(0)} \setminus (r(U) \cup s(U))),
\]

so that \( \theta(V) \in [[G_\varphi]] \) is the transposition corresponding to \( [1_{s(U)}] \). Suppose that \( \theta(V) \) is in \( D([[G_\varphi]]) \). It suffices to show that the equivalence class of \( 1_{s(U)} \) is in \( \text{Inf}(H_0(G_\varphi)) \otimes \mathbb{Z}_2 \).

By [27, Proposition 5.6], there exists a short exact sequence

\[
1 \longrightarrow U(C(X)) \longrightarrow N(C(X), A) \longrightarrow [[G_\varphi]] \longrightarrow 1,
\]

where \( N(C(X), A) \) denotes the group of unitary normalizers of \( C(X) \) in \( U(A) \). Moreover, the homomorphism \( N(C(X), A) \rightarrow [[G_\varphi]] \) has the right inverse \( \rho : [[G_\varphi]] \rightarrow N(C(X), A) \) defined by \( \rho(\theta(W)) = 1_W \). By Theorem 4.3, \( G_\varphi \) has property TR. Hence any element of \( [[G_\varphi]] \) is a product of transpositions, and so we get \( \rho([[G_\varphi]]) \subset U(A)_0 \). Thus

\[
\Delta \circ \rho : [[G_\varphi]] \rightarrow \text{Aff}(T(A))/D_A(K_0(A))
\]

is a well-defined homomorphism. Thanks to [28, Theorem 4.7], we can conclude that \( D([[G_\varphi]]) \) is contained in the kernel of \( \Delta \circ \rho \). Therefore we obtain \( 0 = \Delta(\rho(\theta(V))) = \Delta(1_V) \).

The unitary \( 1_V \in U(A)_0 \) is clearly conjugate to the unitary \( u = -1_{s(U)} + (1 - 1_{s(U)}) \in U(C(X)) \). Define \( \xi : [0, 1] \rightarrow U(A)_0 \) by \( \xi(t) = e^{\pi \sqrt{-1}} 1_{s(U)} + (1 - 1_{s(U)}) \) so that \( \xi(0) = 1 \) and \( \xi(1) = u \). One has

\[
\frac{1}{2\pi \sqrt{-1}} \int_0^1 \sigma_\mu(\xi(t)\xi(t)^*) \, dt = \frac{1}{2} \mu(s(U))
\]

for all \( \mu \in M(G) \). It follows that the affine function \( \sigma_\mu \mapsto \mu(s(U))/2 \) belongs to the image of the dimension map \( D_A \). The gap labeling theorem ([11, 22, 21]) tells us that \( D_A(K_0(A)) = D_A(K_0(C(X))) \).

As a result, there exists \( f \in C(X, \mathbb{Z}) \) such that

\[
\frac{1}{2} \mu(s(U)) = \int_X f \, d\mu
\]

for all \( \mu \in M(G) \). Thus, \( [1_{s(U)}] - 2[f] \) is in the infinitesimal subgroup \( \text{Inf}(H_0(G_\varphi)) \), and whence \( [1_{s(U)}] \otimes 1 \) is in \( \text{Inf}(H_0(G_\varphi)) \otimes \mathbb{Z}_2 \). \( \square \)
Remark 3.8. Let $\mathcal{G}$ be a principal, minimal, almost finite groupoid. It is natural to ask if the gap labeling theorem holds for $\mathcal{G}$. Put $A = C^*_r(\mathcal{G})$ and consider $D_A : K_0(A) \to \text{Aff}(T(A))$. Let $\iota : C(\mathcal{G}^0) \to A$ be the inclusion map. Evidently we have $D_A(\iota_*([K_0(C(\mathcal{G}^0))])) \subset D_A([K_0(A)])$. The gap labeling asks whether or not the other inclusion holds. If this is the case, the proposition above is true for $\mathcal{G}$.

4 Purely infinite groupoids

In this section we discuss purely infinite groupoids.

Definition 4.1 ([28 Definition 4.9]). Let $\mathcal{G}$ be an essentially principal étale groupoid whose unit space is a Cantor set.

1. A clopen set $A \subset \mathcal{G}^0$ is said to be properly infinite if there exist compact open $\mathcal{G}$-sets $U, V \subset \mathcal{G}$ such that $s(U) = s(V) = A$, $r(U) \cup r(V) \subset A$ and $r(U) \cap r(V) = \emptyset$.

2. We say that $\mathcal{G}$ is purely infinite if every clopen set $A \subset \mathcal{G}^0$ is properly infinite.

It is easy to see that if $\mathcal{G}$ is purely infinite, then for any clopen set $A \subset \mathcal{G}^0$ the reduction $\mathcal{G}|A$ is also purely infinite. It is also clear that if $\mathcal{G}^0$ is properly infinite, then there does not exist a $\mathcal{G}$-invariant probability measure. When $\mathcal{G}^0$ is properly infinite, the topological full group $[[\mathcal{G}]]$ contains a subgroup isomorphic to the free product $\mathbb{Z}_2 \ast \mathbb{Z}_3$ ([28 Proposition 4.10]). In particular, $[[\mathcal{G}]]$ is not amenable.

Theorem 4.2. Let $\mathcal{G}$ be a purely infinite groupoid.

1. If $\mathcal{G}$ is minimal, then $D([[\mathcal{G}]])$ is simple.

2. The index map $I : [[\mathcal{G}]] \to H_1(\mathcal{G})$ is surjective.

3. If $\mathcal{G}$ is minimal, then $\mathcal{G}$ has cancellation.

Proof. (1) is [28, Theorem 4.16]. (2) is [28, Theorem 5.2].

Let us prove (3). Suppose that two clopen sets $A, B \subset \mathcal{G}^0$ satisfy $[1_A] = [1_B]$ in $H_0(\mathcal{G})$. Since $\mathcal{G}$ is minimal, there exists a compact open $\mathcal{G}$-set $W \subset \mathcal{G}$ such that $r(W) \subset B$ and $s(W) \subset A$. Let $A' = A \setminus s(W)$ and $B' = B \setminus r(W)$. Then, $[1_{A'}] = [1_A] - [1_{s(W)}] = [1_B] - [1_{r(W)}] = [1_{B'}]$. Hence there exist compact open $\mathcal{G}$-sets $U_1, U_2, \ldots, U_n$ such that

$$1_{A'} - 1_{B'} = \sum_{i=1}^n (1_{r(U_i)} - 1_{s(U_i)}).$$

Put

$$f = 1_{A'} + \sum_{i=1}^n 1_{s(U_i)} = 1_{B'} + \sum_{i=1}^n 1_{r(U_i)}.$$

Let $m = \max\{f(x) \mid x \in \mathcal{G}^0\}$. We can choose compact open $\mathcal{G}$-sets $V_1, V_2, \ldots, V_m$ so that

$$s(V_j) = \{x \in \mathcal{G}^0 \mid j \leq f(x) \leq m\}$$

and $r(V_j)$ are mutually disjoint, because $\mathcal{G}$ is purely infinite and minimal. Let $E = \bigcup_j r(V_j)$. 

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For $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$, we define clopen sets $C_{i,j}$ and $D_{i,j}$ by

$$C_{i,j} = \left\{ x \in s(U_i) \mid \left( 1_A' + \sum_{k=1}^{i} 1_{s(U_k)} \right)(x) = j \right\}$$

and

$$D_{i,j} = \left\{ x \in r(U_i) \mid \left( 1_B' + \sum_{k=1}^{i} 1_{r(U_k)} \right)(x) = j \right\}.$$

For any $i$, it is easy to see that $\{C_{i,j} \mid j = 1, 2, \ldots, m\}$ is a clopen partition of $s(U_i)$ and that $\{D_{i,j} \mid j = 1, 2, \ldots, m\}$ is a clopen partition of $r(U_i)$. For any $j$, we can verify

$$C_{i,j} \cap C_{k,j} = D_{i,j} \cap D_{k,j} = \emptyset$$

whenever $i \neq k$. Moreover,

$$\bigcup_{i=1}^{n} C_{i,j} = \begin{cases} s(V_1) \setminus A' & j = 1 \\ s(V_j) & j > 1 \end{cases}$$

and

$$\bigcup_{i=1}^{n} D_{i,j} = \begin{cases} r(V_1) \setminus B' & j = 1 \\ r(V_j) & j > 1. \end{cases}$$

Consider

$$\tilde{V} = \bigcup_{i,k,l} V_i D_{i,l} U_i C_{i,k} V_k^{-1}.$$ 

Evidently $\tilde{V}$ is a $\mathcal{G}$-set, and

$$\tilde{V}^{-1} \tilde{V}^{-1} = \left( \bigcup_{i,k,l} V_i D_{i,l} U_i C_{i,k} V_k^{-1} \right) \left( \bigcup_{j,p,q} V_p C_{j,p} U_j^{-1} D_{j,q} V_q^{-1} \right)$$

$$= \bigcup_{i,j,k,l,q} V_i D_{i,l} U_i C_{i,k} s(V_k) C_{j,k} U_j^{-1} D_{j,q} V_q^{-1}$$

$$= \bigcup_{i,k,l,q} V_i D_{i,l} U_i C_{i,k} U_i^{-1} D_{i,q} V_q^{-1}$$

$$= \bigcup_{i,l,q} V_i D_{i,l} U_i s(U_i) U_i^{-1} D_{i,q} V_q^{-1}$$

$$= \bigcup_{i,l,q} V_i D_{i,l} r(U_i) D_{i,q} V_q^{-1}$$

$$= \bigcup_{i,l} V_i D_{i,l} V_l^{-1}$$

$$= \left( \bigcup_{l} r(V_l) \right) \setminus V_1 B' V_1^{-1} = E \setminus V_1 B' V_1^{-1}.$$

Similarly, we obtain $\tilde{V}^{-1} \tilde{V} = E \setminus V_1 A' V_1^{-1}$. Since $\mathcal{G}$ is purely infinite and minimal, there exists a compact open $\mathcal{G}$-set $T$ such that $s(T) \subset s(W)$ and $r(T) = s(\tilde{V})$. Define a compact open $\mathcal{G}$-set $U$ by

$$U = (B' V_1^{-1} \cup W T^{-1} \tilde{V}^{-1})(V_1 A' \cup T).$$
Therefore, we obtain isomorphisms

\[ H_n \cong H_n(H|s(U)) \cong H_n(H|r(U)) \cong H_n(\mathcal{H}|s(U)) \cong H_n(\mathcal{H}). \]

By \cite{27} Proposition 3.7, Theorem 3.8, there exists a spectral sequence

\[ E^2_{p,q} = H_p(Z, H_q(\mathcal{H})) \Rightarrow H_{p+q}(\mathcal{G}). \]

Hence, \( H_0(\mathcal{G}) \cong \text{Coker}(\text{id} - \delta_0) \), and for any \( n \geq 1 \) there exists a short exact sequence

\[ 0 \longrightarrow \text{Coker}(\text{id} - \delta_n) \longrightarrow H_n(\mathcal{G}) \longrightarrow \text{Ker}(\text{id} - \delta_{n-1}) \longrightarrow 0. \tag{4.1} \]

For \( f \in C(\mathcal{G}(0), Z) \), we denote its equivalence class in \( H_0(\mathcal{H}) \) by \([f]_\mathcal{H}\). When \( V \subset c^{-1}(n) \) is a compact open \( \mathcal{G} \)-set, we have \( \delta^n_0([1_{r(V)}]_\mathcal{H}) = [1_{s(V)}]_\mathcal{H} \).
**Proposition 4.5.** Let $\mathcal{G}$ be a minimal étale groupoid. Let $c : \mathcal{G} \to \mathbb{Z}$ be a continuous surjective homomorphism and let $\mathcal{H} = \text{Ker} \, c$. Assume either of the following conditions.

1. $\mathcal{H}$ is a principal, minimal, almost finite groupoid with $M(\mathcal{H}) = \{\mu\}$, and there exists a real number $0 < \lambda < 1$ such that, for any compact open $\mathcal{G}$-set $U \subset c^{-1}(1)$, $\mu(r(U)) = \lambda \mu(s(U))$ holds.

2. $\mathcal{H}$ is a minimal, purely infinite groupoid satisfying property TR.

Then, $\mathcal{G}$ is purely infinite and satisfies property TR.

In order to prove this proposition, we need two lemmas. Write $X = \mathcal{G}^{(0)}$. For $\alpha \in \text{Homeo}(X)$, let $\text{supp}(\alpha)$ be the closure of $\{x \in X \mid \alpha(x) \neq x\}$. Let $\mathcal{T} \subset [[\mathcal{G}]]_0$ denote the set of all transpositions and let $(\mathcal{T}) \subset [[\mathcal{G}]]_0$ denote the subgroup generated by $\mathcal{T}$.

**Lemma 4.6.** Under the same hypothesis as Proposition 4.5, the following holds. For any $n \in \mathbb{Z}$ and any non-empty clopen sets $A, B \subset X$, if $\delta^n_0([1_A]\mathcal{H}) = [1_B]$, then there exists a compact open $\mathcal{G}$-set $V \subset c^{-1}(n)$ such that $A = r(V), B = s(V)$.

**Proof.** Let us consider the case (1). Choose a non-empty compact open $\mathcal{G}$-set $U \subset c^{-1}(n)$. Since $\mathcal{H}$ is minimal, there exist compact open $\mathcal{H}$-sets $W_1, W_2, \ldots, W_k \subset \mathcal{H}$ such that \{r($W_1$), r($W_2$), ..., r($W_k$)\} is a partition of $A$ and $s(W_i) \subset r(U)$. We have

$$\sum_{i=1}^{k} [1_{s(W_i)U}] = \sum_{i=1}^{k} \delta_0^n([1_{r(W_i)U}]\mathcal{H}) = \delta_0^n([1_A]\mathcal{H}) = [1_B]\mathcal{H}. $$

It follows from [27] Lemma 6.7 and Theorem 3.3 (3) that we can find compact open $\mathcal{H}$-sets $W'_1, W'_2, \ldots, W'_k \subset \mathcal{H}$ such that \{s($W'_1$), s($W'_2$), ..., s($W'_k$)\} is a partition of $B$ and $s(W'_i U) = r(W'_i)$. Then $V = \bigcup_i W_i U W'_i$ is a compact open $\mathcal{G}$-set such that $V \subset c^{-1}(n)$, $r(V) = A$ and $s(V) = B$.

In the case (2), almost the same proof works by replacing [27] Lemma 6.7 and Theorem 3.3 (3) with [28] Proposition 4.11 and Theorem 4.2 (3).

**Lemma 4.7.** Under the same hypothesis as Proposition 4.5, the following holds. If $\alpha \in [[\mathcal{H}]]$ satisfies $I(\alpha) = 0$ in $H_1(\mathcal{G})$, then $\alpha$ is in $(\mathcal{T})$.

**Proof.** Let us consider the case (1). Choose a non-empty compact open $\mathcal{G}$-set $U \subset c^{-1}(1)$. By replacing $U$ with its suitable clopen subset, we may assume that $r(U) \cap s(U) = \emptyset$. We let $I_{\mathcal{H}} : [[\mathcal{H}]] \to H_1(\mathcal{H})$ denote the index map for $\mathcal{H}$. Since $I(\alpha) = 0$ in $H_1(\mathcal{G})$, from the short exact sequence [11], we conclude that $I_{\mathcal{H}}(\alpha)$ is trivial in $\text{Coker}(\text{id} - \delta_1)$. Thus, there exists $z \in H_1(\mathcal{H})$ such that $I_{\mathcal{H}}(\alpha) = z - \delta_1(z)$. It follows from Theorem 3.3 (2) that there exists $\beta \in [[\mathcal{H}]]$ such that $I_{\mathcal{H}}(\beta) = z$. By [27] Lemma 7.10 and its proof, we can find an elementary homeomorphism $\beta_0 \in [[\mathcal{H}]]$ (see [27] Definition 7.6 (1)) such that $\mu(\text{supp}(\beta_0\beta)) < \mu(r(U))$. It follows from [27] Lemma 6.7 that, by replacing $U$ again, we may assume $\text{supp}(\beta_0\beta) \subset r(U)$. Define the transposition $\tau_U \in \mathcal{T}$ as in the proof of Theorem 3.6. Then, by the definition of $\delta_1$, one has $I_{\mathcal{H}}(\tau_U(\beta_0\beta)^{-1}\tau_U) = I_{\mathcal{H}}(\beta_0\beta) - \delta_1(I_{\mathcal{H}}(\beta_0\beta)) = I_{\mathcal{H}}(\beta) - \delta_1(I_{\mathcal{H}}(\beta)) = z - \delta_1(z)$. Then, the definition of $\delta_1$, one has $I_{\mathcal{H}}((\beta_0\beta)\tau_U(\beta_0\beta)^{-1}\tau_U) = I_{\mathcal{H}}(\beta_0\beta) - \delta_1(I_{\mathcal{H}}(\beta_0\beta)) = I_{\mathcal{H}}(\beta) - \delta_1(I_{\mathcal{H}}(\beta)) = z - \delta_1(z).$
Clearly $\gamma = (\beta_0\beta)\tau_U (\beta_0\beta)^{-1}\tau_U$ belongs to $\langle T \rangle$, and $I_H(\alpha\gamma^{-1}) = I_H(\alpha) - (z - \delta_1(z)) = 0$ in $H_1(\mathcal{H})$. Since $\mathcal{H}$ satisfies property TR, $\alpha\gamma^{-1}$ is in $\langle T \rangle$. Therefore $\alpha$ is in $\langle T \rangle$.

In the case (2), the same proof works by replacing Theorem 3.3 (2) and [27, Proposition 4.11].

**Proof of Proposition 4.5.** First, let us consider the case (1).

We will show that $\mathcal{G}$ is purely infinite. Let $A, B \subset X$ be non-empty clopen subsets. For sufficiently large $n \in \mathbb{N}$, we have $\lambda^n\mu(A) < \mu(B)$. By [27, Lemma 6.7], there exists $C \subset B$ such that $[1_C]_H = \delta_0^{-n}([1_A]_H)$. By Lemma 4.6 there exists a compact open $\mathcal{G}$-set $U \subset \mathcal{G}$ such that $s(U) = A, r(U) = C \subset B$. By [28, Proposition 4.11], $\mathcal{G}$ is purely infinite.

We will show that $\mathcal{G}$ has property TR. The proof goes exactly as that of [28, Lemma 6.10]. For $\alpha \in \langle [\mathcal{G}] \rangle$, we take the compact open $\mathcal{G}$-set $U \subset \mathcal{G}$ satisfying $\alpha = \theta(U) = (r|U) \circ (s|U)^{-1}$, and define clopen subsets $S(\alpha, n) \subset X$ by $S(\alpha, n) = s(U \cap c^{-1}(n))$. Note that $S(\alpha, n)$ is empty except for finitely many $n \in \mathbb{Z}$.

Suppose that $\alpha \in [[\mathcal{G}]]_0 \setminus \{1\}$ is given. In the same way as [28, Lemma 6.10], we may assume that $A = \text{supp}(\alpha)$ is not equal to $X$. We have

$$[1_X]_\mathcal{H} = \sum_{n \in \mathbb{Z}} [1_{S(\alpha, n)}]_\mathcal{H}$$

in $H_0(\mathcal{H})$, because $\{S(\alpha, n) \mid n \in \mathbb{Z}\}$ is a clopen partition of $X$. Since $\{\alpha(S(\alpha, n)) \mid n \in \mathbb{Z}\}$ is also a clopen partition of $X$, one obtains

$$[1_X]_\mathcal{H} = \sum_{n \in \mathbb{Z}} [1_{\alpha(S(\alpha, n))}]_\mathcal{H} = \sum_{n \in \mathbb{Z}} \delta_0^{-n} ([1_{S(\alpha, n)}]_\mathcal{H})$$

in $H_0(\mathcal{H})$. Therefore, we get

$$\sum_{n \in \mathbb{Z}} (\text{id} - \delta_0^{-n}) ([1_{S(\alpha, n)}]_\mathcal{H}) = 0. \quad (4.2)$$

For $n \in \mathbb{Z}$, we define homomorphisms $\delta_0^{(n)} : H_0(\mathcal{H}) \to H_0(\mathcal{H})$ by

$$\delta_0^{(n)} = \begin{cases} 
\text{id} + \delta_0 + \cdots + \delta_0^{n-1} & n > 0 \\
0 & n = 0 \\
-(\delta_0^{-1} + \delta_0^{-2} + \cdots + \delta_0^{n}) & n < 0,
\end{cases}$$

so that $(\text{id} - \delta_0)\delta_0^{(n)} = \text{id} - \delta_0^n$ hold. It follows from (4.2) that

$$\sum_{n \in \mathbb{Z}} (\text{id} - \delta_0) \left(\delta_0^{(n)} ([1_{S(\alpha, n)}]_\mathcal{H})\right) = 0,$$

which means that $\sum_{n \in \mathbb{Z}} \delta_0^{(n)} ([1_{S(\alpha, n)}]_\mathcal{H})$ is in $\text{Ker}(\text{id} - \delta_0)$. The proof of [27, Theorem 4.14] implies that, in the exact sequence (4.14), the element $I(\alpha) \in H_1(\mathcal{G})$ maps to $\sum_{n} \delta_0^{(n)} ([1_{S(\alpha, n)}]_\mathcal{H}) \in \text{Ker}(\text{id} - \delta_0)$ (see [28, Lemma 6.8]). Since we assumed $I(\alpha) = 0$, we get

$$\sum_{n \in \mathbb{Z}} \delta_0^{(n)} ([1_{S(\alpha, n)}]_\mathcal{H}) = 0$$
in $H_0(\mathcal{H})$. Set $P = \{n \in \mathbb{N} \mid S(\alpha, n) \neq \emptyset\}$ and $Q = \{n \in \mathbb{Z} \mid n < 0, \ S(\alpha, n) \neq \emptyset\}$. Then
\[
- \sum_{n \in P} \delta_0^{(-n)} ([S(\alpha, n)] \mathcal{H}) = \sum_{n \in Q} \delta_0^{(-n)} ([S(\alpha, n)] \mathcal{H}). \tag{4.3}
\]

Put
\[
z = [1_A] \mathcal{H} + \delta_0 \left( \sum_{n \in Q} \delta_0^{(-n)} ([S(\alpha, n)] \mathcal{H}) \right) \in H_0(\mathcal{H}).
\]

We have
\[
\hat{\mu}(z) = \mu(A) + \sum_{n \in Q} (\lambda^{-1} + \lambda^{-2} + \cdots + \lambda^n) \mu(S(\alpha, n)),
\]
where the homomorphism $\hat{\mu} : H_0(\mathcal{H}) \to \mathbb{R}$ is defined by
\[
\hat{\mu}([f] \mathcal{H}) = \int f \ d\mu
\]
for $f \in C(X, \mathbb{Z})$. Choose $m \in \mathbb{N}$ so that
\[
\lambda^m \mu(A) < \mu(X \setminus A) \quad \text{and} \quad \lambda^m \hat{\mu}(z) < 1
\]
hold. Then we can find a clopen set $B \subset X \setminus A$ such that $[1_B] \mathcal{H} = \delta_0^{-m}([1_A] \mathcal{H})$. It follows from Lemma 1.6 that there exists a compact open $\mathcal{G}$-set $V \subset c^{-1}(m)$ such that $r(V) = B$ and $s(V) = A$. Define the transposition $\tau_V \in \mathcal{T}$ as in the proof of Theorem 3.6. Set $\beta = \tau_V \alpha \tau_V$. It suffices to show that $\beta$ belongs to $\langle \mathcal{T} \rangle$. Notice that $\text{supp}(\beta) = \tau_V(A) = B$, $S(\beta, n) = \tau_V(S(\alpha, n))$ and
\[
[S(\beta, n)] \mathcal{H} = \delta_0^{-m} ([S(\alpha, n)] \mathcal{H})
\]
for every $n \in \mathbb{Z}$. Hence, by (4.3), we get
\[
- \sum_{n \in P} \delta_0^{(-n)} ([S(\beta, n)] \mathcal{H}) = \sum_{n \in Q} \delta_0^{(-n)} ([S(\beta, n)] \mathcal{H}). \tag{4.4}
\]

Then, in the same way as the proof of [28, Lemma 6.10], we can construct $\gamma \in \langle \mathcal{T} \rangle$ such that $S(\gamma, n) = S(\beta, n)$ for all $n \in \mathbb{Z}$. Then we obtain $S(\beta \gamma^{-1}, 0) = X$, which implies $\beta \gamma^{-1} \in [\mathcal{H}]$. Since $I(\beta \gamma^{-1}) = I(\beta) = I(\alpha) = 0$, by Lemma 4.7 we get $\beta \gamma^{-1} \in \langle \mathcal{T} \rangle$. It follows that $\beta$ belongs to $\langle \mathcal{T} \rangle$, as desired.

In the case (2), almost the same proof works by replacing [27, Lemma 6.7] and Theorem 3.3 (3) with [28, Proposition 4.11] and Theorem 4.2 (3). We omit it.

\section{Products of SFT groupoids}

\subsection{Preliminaries}

We first recall the definition of étale groupoids arising from one-sided shifts of finite type. Let $(\mathcal{V}, \mathcal{E})$ be a finite directed graph, where $\mathcal{V}$ is a finite set of vertices and $\mathcal{E}$ is a finite
set of edges. For \( e \in \mathcal{E} \), \( i(e) \) denotes the initial vertex of \( e \) and \( t(e) \) denotes the terminal vertex of \( e \). Let \( A = (A(\xi, \eta))_{\xi, \eta \in \mathcal{V}} \) be the adjacency matrix of \((\mathcal{V}, \mathcal{E})\), that is,

\[
A(\xi, \eta) = \#\{e \in \mathcal{E} \mid i(e) = \xi, \ t(e) = \eta\}.
\]

We assume that \( A \) is irreducible (i.e. for all \( \xi, \eta \in \mathcal{V} \) there exists \( n \in \mathbb{N} \) such that \( A^n(\xi, \eta) > 0 \)) and that \( A \) is not a permutation matrix. Define

\[
X_A = \{(x_k)_{k \in \mathbb{N}} \in \mathcal{E}^\mathbb{N} \mid t(x_k) = i(x_{k+1}) \quad \forall k \in \mathbb{N}\}.
\]

With the product topology, \( X_A \) is a Cantor set. Define a surjective continuous map \( \sigma_A : X_A \to X_A \) by

\[
\sigma_A(x)_k = x_{k+1} \quad k \in \mathbb{N}, \ x = (x_k)_{k \in \mathbb{N}} \in X_A.
\]

In other words, \( \sigma_A \) is the (one-sided) shift on \( X_A \). It is easy to see that \( \sigma_A \) is a local homeomorphism. The dynamical system \((X_A, \sigma_A)\) is called the one-sided irreducible shift of finite type (SFT) associated with the graph \((\mathcal{V}, \mathcal{E})\) (or the matrix \( A \)).

The étale groupoid \( \mathcal{G}_A \) for \((X_A, \sigma_A)\) is given by

\[
\mathcal{G}_A = \{(x, n, y) \in X_A \times \mathbb{Z} \times X_A \mid \exists k, l \in \mathbb{N}, \ n = k - l, \ \sigma_A^k(x) = \sigma_A^l(y)\}.
\]

The topology of \( \mathcal{G}_A \) is generated by the sets \( \{(x, k-l, y) \in \mathcal{G}_A \mid x \in P, \ y \in Q, \ \sigma_A^k(x) = \sigma_A^l(y)\} \), where \( P, Q \subset X_A \) are open and \( k, l \in \mathbb{N} \). Two elements \((x, n, y)\) and \((x', n', y')\) in \( \mathcal{G} \) are composable if and only if \( y = x' \), and the multiplication and the inverse are

\[
(x, n, y) \cdot (y, n', y') = (x, n+n', y'), \quad (x, n, y)^{-1} = (y, -n, x).
\]

We identify \( X_A \) with the unit space \( \mathcal{G}_A^{(0)} \) via \( x \mapsto (x, 0, x) \). We call \( \mathcal{G}_A \) the SFT groupoid associated with the matrix \( A \).

The following is a classification theorem of SFT groupoids. For an \( N \times N \) matrix \( A \) with entries in \( \mathbb{N} \cup \{0\} \), the Bowen-Franks group \( BF(A) \) is the abelian group \( \mathbb{Z}^N/(id - A)\mathbb{Z}^N \). We let \( u_A \) denote the equivalence class of \((1, 1, \ldots, 1) \in \mathbb{Z}^N \) in \( BF(A^t) \).

**Theorem 5.1** ([25] Theorem 3.6). Let \((X_A, \sigma_A)\) and \((X_B, \sigma_B)\) be two irreducible one-sided shifts of finite type. The following conditions are equivalent.

1. \((X_A, \sigma_A)\) and \((X_B, \sigma_B)\) are continuously orbit equivalent.

2. The étale groupoids \( \mathcal{G}_A \) and \( \mathcal{G}_B \) are isomorphic.

3. There exists an isomorphism \( \Phi : BF(A^t) \to BF(B^t) \) such that \( \Phi(u_A) = u_B \) and \( \text{sgn}(\det(id - A)) = \text{sgn}(\det(id - B)) \).

The following is an immediate consequence of the theorem above.

**Corollary 5.2.** Let \((X_A, \sigma_A)\) and \((X_B, \sigma_B)\) be two irreducible one-sided shifts of finite type. The following conditions are equivalent.

1. The étale groupoids \( \mathcal{G}_A \) and \( \mathcal{G}_B \) are Morita equivalent.

2. \( BF(A^t) \cong BF(B^t) \) and \( \text{sgn}(\det(id - A)) = \text{sgn}(\det(id - B)) \).
The following are known for the SFT groupoids.

**Theorem 5.3** (28). Let \((X_A, \sigma_A)\) be an irreducible one-sided shift of finite type and let \(G_A\) be the associated SFT groupoid.

1. \(G_A\) is purely infinite and minimal.
2. The homology groups of \(G_A\) are
   \[
   H_n(G_A) \cong \begin{cases} 
   BF(A^t) & n = 0 \\
   \ker(id - A^t) & n = 1 \\
   0 & n \geq 2
   \end{cases}
   \]
   and the equivalence class of \(1_{X_A}\) in \(H_0(G_A)\) is equal to \(u_A \in BF(A^t)\). In particular, the HK conjecture (Conjecture 2.6) holds for \(G_A\).
3. \(G_A\) has the strong AH property, and the group \([G_A]_{ab}\) is isomorphic to \((H_0(G_A) \otimes \mathbb{Z}_2) \oplus H_1(G_A)\).
4. \([G_A]\) has the Haagerup property.
5. \([G_A]\) is of type \(F_\infty\).

### 5.2 HK and AH

For \(i = 1, 2, \ldots, n\), let \((X_{A_i}, \sigma_{A_i})\) be an irreducible one-sided shift of finite type and let \(G_{A_i}\) be the associated SFT groupoid. We consider the product groupoid \(G = G_{A_1} \times G_{A_2} \times \cdots \times G_{A_n}\). It is clear that \(G\) is purely infinite and minimal.

The homology groups of \(G\) can be computed by using Theorem 5.3 (2) and the Künneth theorem (Theorem 2.4).

**Proposition 5.4.** Let \(G = G_{A_1} \times G_{A_2} \times \cdots \times G_{A_n}\) be a product groupoid of SFT groupoids. Then,
\[
H_k(G) \cong \left( \mathbb{Z}^{\binom{n-1}{k}} \otimes H_0(G_{A_1}) \otimes H_0(G_{A_2}) \otimes \cdots \otimes H_0(G_{A_n}) \right) \\
\quad \oplus \left( \mathbb{Z}^{\binom{n-1}{k-1}} \otimes H_1(G_{A_1}) \otimes H_1(G_{A_2}) \otimes \cdots \otimes H_1(G_{A_n}) \right),
\]
where \(\binom{n}{k}\) denote the binomial coefficients and they are understood as zero unless \(0 \leq k \leq n\). The equivalence class of the constant function \(1_{G(0)}\) in \(H_0(G) = H_0(G_{A_1}) \otimes \cdots \otimes H_0(G_{A_n})\) is \(u_{A_1} \otimes \cdots \otimes u_{A_n}\).

**Proof.** For an abelian group \(P\), we let \(P_{tor}\) denote the torsion subgroup of \(P\). When \(P\) and \(Q\) are finitely generated abelian groups, we have \(\text{Tor}(P, Q) \cong P_{tor} \otimes Q_{tor}\). By Theorem 5.3 (2), \(H_0(G_{A_i})\) is a finitely generated abelian group and \(H_1(G_{A_i})\) is isomorphic to the torsion free part of \(H_0(G_{A_i})\). Thus, \(H_0(G_{A_i}) \cong H_1(G_{A_i}) \oplus H_0(G_{A_i})_{tor}\).
The proof is by induction on \( n \). Assume that the proposition is true for \( n-1 \). Theorem 2.4 implies

\[
H_k(\mathcal{G}_A_1 \times \cdots \times \mathcal{G}_A_n) \cong (H_{k-1}(\mathcal{G}_A_1 \times \cdots \times \mathcal{G}_{A_{n-1}}) \otimes H_1(\mathcal{G}_{A_n})) \\
\oplus (H_k(\mathcal{G}_A_1 \times \cdots \times \mathcal{G}_{A_{n-1}}) \otimes H_0(\mathcal{G}_{A_n})) \\
\oplus \text{Tor}(H_{k-1}(\mathcal{G}_A_1 \times \cdots \times \mathcal{G}_{A_{n-1}}), H_0(\mathcal{G}_{A_n}))
\]

\[
\cong (H_{k-1}(\mathcal{G}_A_1 \times \cdots \times \mathcal{G}_{A_{n-1}}) \otimes H_1(\mathcal{G}_{A_n})) \\
\oplus (H_k(\mathcal{G}_A_1 \times \cdots \times \mathcal{G}_{A_{n-1}}) \otimes H_0(\mathcal{G}_{A_n})) \\
\oplus (H_{k-1}(\mathcal{G}_A_1 \times \cdots \times \mathcal{G}_{A_{n-1}})_{\text{tor}} \otimes H_0(\mathcal{G}_{A_n})_{\text{tor}})
\]

which is isomorphic to

\[
\left(\mathbb{Z}^{n-2}_{k-1} \otimes H_0(\mathcal{G}_{A_1}) \otimes \cdots \otimes H_0(\mathcal{G}_{A_{n-1}}) \otimes H_1(\mathcal{G}_{A_n})\right) \\
\oplus \left(\mathbb{Z}^{n-2}_{k-2} \otimes H_1(\mathcal{G}_{A_1}) \otimes \cdots \otimes H_1(\mathcal{G}_{A_{n-1}}) \otimes H_1(\mathcal{G}_{A_n})\right) \\
\oplus \left(\mathbb{Z}^{n-2}_{k} \otimes H_0(\mathcal{G}_{A_1}) \otimes \cdots \otimes H_0(\mathcal{G}_{A_{n-1}}) \otimes H_0(\mathcal{G}_{A_n})\right) \\
\oplus \left(\mathbb{Z}^{n-2}_{k-1} \otimes H_1(\mathcal{G}_{A_1}) \otimes \cdots \otimes H_1(\mathcal{G}_{A_{n-1}}) \otimes H_0(\mathcal{G}_{A_n})\right) \\
\oplus \left(\mathbb{Z}^{n-2}_{k-1} \otimes \left(H_0(\mathcal{G}_{A_1}) \otimes \cdots \otimes H_0(\mathcal{G}_{A_{n-1}})\right)_{\text{tor}} \otimes H_0(\mathcal{G}_{A_n})_{\text{tor}}\right)
\]

\[
\cong \left(\mathbb{Z}^{n-2}_{k-1} \otimes H_0(\mathcal{G}_{A_1}) \otimes \cdots \otimes H_0(\mathcal{G}_{A_{n-1}}) \otimes H_1(\mathcal{G}_{A_n})\right) \\
\oplus \left(\mathbb{Z}^{n-2}_{k-2} \otimes H_1(\mathcal{G}_{A_1}) \otimes \cdots \otimes H_1(\mathcal{G}_{A_{n-1}}) \otimes H_1(\mathcal{G}_{A_n})\right) \\
\oplus \left(\mathbb{Z}^{n-2}_{k} \otimes H_0(\mathcal{G}_{A_1}) \otimes \cdots \otimes H_0(\mathcal{G}_{A_{n-1}}) \otimes H_0(\mathcal{G}_{A_n})\right) \\
\oplus \left(\mathbb{Z}^{n-2}_{k-1} \otimes H_1(\mathcal{G}_{A_1}) \otimes \cdots \otimes H_1(\mathcal{G}_{A_{n-1}}) \otimes H_0(\mathcal{G}_{A_n})_{\text{tor}}\right) \\
\oplus \left(\mathbb{Z}^{n-2}_{k-1} \otimes \left(H_0(\mathcal{G}_{A_1}) \otimes \cdots \otimes H_0(\mathcal{G}_{A_{n-1}})\right)_{\text{tor}} \otimes H_0(\mathcal{G}_{A_n})_{\text{tor}}\right)
\]

\[
\cong \left(\mathbb{Z}^{n-2}_{k-1} \otimes H_0(\mathcal{G}_{A_1}) \otimes \cdots \otimes H_0(\mathcal{G}_{A_{n-1}}) \otimes H_1(\mathcal{G}_{A_n})\right) \\
\oplus \left(\mathbb{Z}^{n-2}_{k-2} \otimes H_1(\mathcal{G}_{A_1}) \otimes \cdots \otimes H_1(\mathcal{G}_{A_{n-1}}) \otimes H_1(\mathcal{G}_{A_n})\right) \\
\oplus \left(\mathbb{Z}^{n-2}_{k} \otimes H_0(\mathcal{G}_{A_1}) \otimes \cdots \otimes H_0(\mathcal{G}_{A_{n-1}}) \otimes H_0(\mathcal{G}_{A_n})\right) \\
\oplus \left(\mathbb{Z}^{n-2}_{k-1} \otimes H_1(\mathcal{G}_{A_1}) \otimes \cdots \otimes H_1(\mathcal{G}_{A_{n-1}}) \otimes H_0(\mathcal{G}_{A_n})_{\text{tor}}\right) \\
\oplus \left(\mathbb{Z}^{n-2}_{k-1} \otimes \left(H_0(\mathcal{G}_{A_1}) \otimes \cdots \otimes H_0(\mathcal{G}_{A_{n-1}})\right)_{\text{tor}} \otimes H_0(\mathcal{G}_{A_n})_{\text{tor}}\right)
\]
Since \( \binom{n-2}{k-1} + \binom{n-2}{k} = \binom{n-1}{k} \) and \( \binom{n-2}{k-2} + \binom{n-2}{k-1} = \binom{n-1}{k-1} \), we obtain the desired conclusion.

**Theorem 5.5.** Let \( \mathcal{G} = \mathcal{G}_{A_1} \times \mathcal{G}_{A_2} \times \cdots \times \mathcal{G}_{A_n} \) be a product groupoid of SFT groupoids. Then, \( \mathcal{G} \) satisfies Conjecture 2.6.

**Proof.** By Theorem 5.3 (2), the SFT groupoid satisfies Conjecture 2.6. The C*-algebra associated with the SFT groupoid is the so-called Cuntz-Krieger algebra, which is known to be nuclear and satisfy the UCT \([10]\). The theorem then follows from Theorem 2.8 by induction.

Next, let us consider the AH conjecture for \( \mathcal{G} = \mathcal{G}_{A_1} \times \mathcal{G}_{A_2} \times \cdots \times \mathcal{G}_{A_n} \).

A matrix \( A \) with entries in \( \mathbb{N} \cup \{0\} \) is said to be primitive if there exists \( k \in \mathbb{N} \) such that every entry of \( A^k \) is positive. The one-sided shift \((X_A, \sigma_A)\) is topologically mixing if and only if the matrix \( A \) is primitive \([23, \text{Proposition 4.5.10}]\).

**Lemma 5.6.** Any irreducible one-sided shift of finite type is continuously orbit equivalent to a one-sided shift of finite type which is topologically mixing.

**Proof.** The proof is by a slight modification of the construction given in \([25]\, \text{Lemma 3.7}]\). The \( N \times N \) matrix \( A \) considered there is clearly primitive. The equivalence class \((1,1,\ldots,1) \in \mathbb{Z}^N\) is zero in \( \text{BF}(A^t) \). For a given \( u \in \text{BF}(A^t) \), we choose \((c_1,c_2,\ldots,c_N) \in (\mathbb{N} \cup \{0\})^N\) whose equivalence class in \( \text{BF}(A^t) \) equals \( u \), and then construct a matrix \( B \). When choosing \((c_1,c_2,\ldots,c_N)\), we may assume that the greatest common divisor of \( c_i+1 \)'s is one, by adding a suitable multiple of \((1,1,\ldots,1)\). Then, the matrix \( B \) becomes primitive, which completes the proof.

**Lemma 5.7.** Let \( \mathcal{G} = \mathcal{G}_{A_1} \times \mathcal{G}_{A_2} \times \cdots \times \mathcal{G}_{A_n} \) be a product groupoid of SFT groupoids. Then, \( \mathcal{G} \) has property TR.

**Proof.** The proof is by induction on \( n \). We know that the SFT groupoid has property TR \([28, \text{Lemma 6.10}]\). Assume that the lemma is true for \( n-1 \). Suppose that we are given \( \mathcal{G} = \mathcal{G}_{A_1} \times \mathcal{G}_{A_2} \times \cdots \times \mathcal{G}_{A_n} \). By the lemma above, we may assume that the matrix \( A_n \) is primitive. The induction hypothesis implies that \( \mathcal{G}_{A_1} \times \mathcal{G}_{A_2} \times \cdots \times \mathcal{G}_{A_{n-1}} \) has property TR. Let \( c : \mathcal{G}_{A_n} \to \mathbb{Z} \) be the continuous homomorphism defined by \( c(x,m,y) = m \). Since \( A_n \) is primitive, \( \mathcal{H} = \text{Ker} \, c \) is a minimal AF groupoid. Therefore, \( \mathcal{G}_{A_1} \times \mathcal{G}_{A_2} \times \cdots \times \mathcal{G}_{A_{n-1}} \times \mathcal{H} \) is purely infinite, minimal, and has property TR. The kernel of the homomorphism

\[
\mathcal{G} = \mathcal{G}_{A_1} \times \cdots \times \mathcal{G}_{A_n} \longrightarrow \mathcal{G}_{A_n} \xrightarrow{c} \mathbb{Z}
\]

equals \( \mathcal{G}_{A_1} \times \mathcal{G}_{A_2} \times \cdots \times \mathcal{G}_{A_{n-1}} \times \mathcal{H} \). Hence, Proposition 4.5 applies and yields the conclusion.

This lemma, together with Theorem 4.4, implies the following.

**Theorem 5.8.** Let \( \mathcal{G} = \mathcal{G}_{A_1} \times \mathcal{G}_{A_2} \times \cdots \times \mathcal{G}_{A_n} \) be a product groupoid of SFT groupoids. Then, \( \mathcal{G} \) satisfies the AH conjecture (Conjecture 2.9).
### 5.3 Classification

In this subsection, we determine when two product groupoids of SFT groupoids are isomorphic to each other (Theorem 5.12).

Let $G$ be an étale groupoid. A closed subset $Y \subset G^{(0)}$ is called $G$-étale if the reduction $G|Y$ is an étale groupoid with the relative topology from $G$. The following lemmas are easy to prove and is left for the reader.

**Lemma 5.9.** Let $G$ be an étale groupoid whose unit space is a Cantor set and let $Y \subset G^{(0)}$ be a closed subset. The following are equivalent.

1. $Y$ is $G$-étale.
2. For any $g \in G|Y$, there exists a compact open $G$-set $U$ such that $g \in U$ and $(r|U)^{-1}(Y) = (s|U)^{-1}(Y)$.

**Lemma 5.10.** Let $G$ be an étale groupoid whose unit space is a Cantor set and let $U \subset G^\prime$ be a compact open $G$-set. Then $Y = r(U \cap G^\prime)$ is $G$-étale.

In [25, Section 3], we introduced the notion of attracting elements. Here we weakened its definition a little bit, namely we do not require that the limit set of the dynamics is a singleton.

**Definition 5.11.** Let $G$ be an étale groupoid whose unit space is a Cantor set and let $g \in G^\prime$, i.e. $r(g) = s(g)$. We say that $g$ is an attracting element if there exists a compact open $G$-set $U$ such that $g \in U$ and $r(U)$ is a proper subset of $s(U)$. For such $U$, we call the closed set

$$Y = \bigcap_{n=1}^{\infty} r(U^n)$$

a limit set of $g$. Notice that $r(g)$ is contained in every limit set of $g$.

For any SFT groupoid $G_A$ and $x \in X_A$, the isotropy group $(G_A)_x$ is either 0 or $\mathbb{Z}$, and $(G_A)_x \cong \mathbb{Z}$ if and only if $x \in X_A$ is eventually periodic (see [25, Lemma 3.3]). When $(G_A)_x \cong \mathbb{Z}$ and $g \in (G_A)_x$ is attracting, there exists a compact open $G$-set $U$ such that $\{x\}$ is the limit set and $U \cap G^\prime = \{g\}$.

**Theorem 5.12.** Let $G = G_{A_1} \times G_{A_2} \times \cdots \times G_{A_m}$ and $H = G_{B_1} \times G_{B_2} \times \cdots \times G_{B_n}$ be product groupoids of SFT groupoids. Then $G \cong H$ if and only if the following are satisfied.

1. $m = n$.
2. There exist a permutation $\sigma$ of $\{1, 2, \ldots, n\}$ and isomorphisms $\varphi_i : BF(A_i^1) \to BF(B_{\sigma(i)}^1)$ such that $\det(id - A_i) = \det(id - B_{\sigma(i)})$ and

$$\left(\varphi_1 \otimes \varphi_2 \otimes \cdots \otimes \varphi_n\right)(u_{A_1} \otimes u_{A_2} \otimes \cdots \otimes u_{A_n}) = u_{B_{\sigma(1)}} \otimes u_{B_{\sigma(2)}} \otimes \cdots \otimes u_{B_{\sigma(n)}}.$$

In particular, $G_{A_i}$ and $G_{B_{\sigma(i)}}$ are Morita equivalent.
Moreover, such a set of generators is unique. The same is also true for $\pi$ of $G$. Analogously we define $h_i$ for each $i \in [1, n]$. It follows from [28, Theorem 6.2] that $G_{A_i}$ is isomorphic to $G_{B_i}$. Hence $G$ is isomorphic to

$$
(G_{B_1}|Y_1) \times (G_{B_2}|Y_2) \times \cdots \times (G_{B_n}|Y_n)
$$

which is canonically identified with

$$(G_{B_1} \times G_{B_2} \times \cdots \times G_{B_n})(Y_1 \times Y_2 \times \cdots \times Y_n) = \mathcal{H}|(Y_1 \times Y_2 \times \cdots \times Y_n).$$

Since $[1_{Y_1 \times Y_2 \times \cdots \times Y_n}] = \varphi_1(u_{A_1}) \otimes \varphi_2(u_{A_2}) \otimes \cdots \otimes \varphi_n(u_{A_n}) = u_{B_1} \otimes u_{B_2} \otimes \cdots \otimes u_{B_n}$ in $H_0(\mathcal{H})$, Theorem 4.2(3) tells us that $\mathcal{H}|(Y_1 \times Y_2 \times \cdots \times Y_n)$ is isomorphic to $\mathcal{H}$. Therefore we get $G \cong \mathcal{H}$.

Let us prove the ‘only if’ part. Suppose that $\pi : G \to \mathcal{H}$ is an isomorphism. For

$$x = (x_1, x_2, \ldots, x_m) \in X_{A_1} \times X_{A_2} \times \cdots \times X_{A_m} = G^{(0)},$$

the isotropy group $G_x$ is isomorphic to $\mathbb{Z}^k$, where $k = \# \{i \mid x_i$ is eventually periodic$. For any $x \in G$, we have $\pi(G_x) = \mathcal{H}_{\pi(x)}$, and so $m$ must be equal to $n$. Take a point $x = (x_1, x_2, \ldots, x_n) \in G^{(0)}$ such that $G_x \cong \mathbb{Z}^n$. Let $y = (y_1, y_2, \ldots, y_n) = \pi(x)$. We have $\mathcal{H}_y \cong \mathbb{Z}^n$. Let $g_i \in (G_{A_i})_{y_i}$ be the unique attracting generator of $(G_{A_i})_{x_i} \cong \mathbb{Z}$. Define $\tilde{g_i} \in G_x$ by

$$\tilde{g_i} = (x_1, \ldots, x_{i-1}, g_i, x_{i+1}, \ldots, x_n).$$

Analogously we define $h_i \in (G_{B_i})_{y_i}$ and $\tilde{h_i} \in \mathcal{H}_y$. Consider the subset

$$P = \{ g \in G_x \mid g \text{ is attracting}\}$$

of $G_x \cong \mathbb{Z}$. Evidently one has

$$\pi(P) = \{ h \in \mathcal{H}_y \mid h \text{ is attracting} \}.$$ 

Clearly $P$ is generated by $\tilde{g_1}, \tilde{g_2}, \ldots, \tilde{g_n}$ as a semigroup, that is,

$$P = \left\{ \tilde{g_1}^{k_1} \tilde{g_2}^{k_2} \cdots \tilde{g_n}^{k_n} \mid k_1, k_2, \ldots, k_n \in \mathbb{N} \cup \{0\} \right\}.$$ 

Moreover, such a set of generators is unique. The same is also true for $\{\tilde{h_1}, \tilde{h_2}, \ldots, \tilde{h_n}\} \subset \pi(P)$. Hence we obtain $\{\pi(\tilde{g_1}), \pi(\tilde{g_2}), \ldots, \pi(\tilde{g_n})\} = \{\tilde{h_1}, \tilde{h_2}, \ldots, \tilde{h_n}\}$. To simplify notation, by permuting indices, we may assume $\pi(\tilde{g_i}) = \tilde{h_i}$ for every $i = 1, 2, \ldots, n$.

Since $h_i \in G_{B_i}$ is attracting, we can find a compact open $G_{B_i}$-set $V_i \subset G_{B_i}$ such that $h_i$ is in $V_i$, $r(V_i)$ is a proper subset of $s(V_i)$ and

$$\bigcap_{k=1}^{\infty} r(V_i^k) = \{y_i\}.$$ 

Let $D = s(V_1) \times s(V_2) \times \cdots \times s(V_n)$. For each $i = 1, 2, \ldots, n$, we define a compact open $\mathcal{H}$-set $\tilde{V}_i \subset \mathcal{H}$ by

$$\tilde{V}_i = V_1 \times \cdots \times V_{i-1} \times s(V_i) \times V_{i+1} \times \cdots \times V_n.$$
One has
\[ s(\tilde{V}_i) = D, \quad \tilde{h}_1 \ldots \tilde{h}_{i-1} \tilde{h}_{i+1} \ldots \tilde{h}_n \in \tilde{V}_i \]
and
\[ \bigcap_{k=1}^{\infty} r(\tilde{V}_i^k) = r(\tilde{V}_i \cap \mathcal{H}') = \{ y_1 \} \times \cdots \times \{ y_{i-1} \} \times s(V_i) \times \{ y_{i+1} \} \times \{ y_n \}. \]

By Lemma 5.10, \( r(\tilde{V}_i \cap \mathcal{H}') \) is \( \mathcal{H} \)-étale. It is easy to see that the reduction \( \mathcal{H}|r(\tilde{V}_i \cap \mathcal{H}') \) is isomorphic to \( \mathbb{Z}^{n-1} \times (\mathcal{G}_{B_1}|s(V_i)) \), and its essentially principal part (see Section 2.1) is \( \mathcal{G}_{B_1}|s(V_i) \).

In the same way, since \( g_i \in \mathcal{G}_{A_i} \) is attracting, we can find a compact open \( \mathcal{G}_{A_i} \)-set \( U_i \subset \mathcal{G}_{A_i} \) such that \( g_i \) is in \( U_i \), \( r(U_i) \) is a proper subset of \( s(U_i) \) and

\[ \bigcap_{k=1}^{\infty} r(U_i^k) = \{ x_i \}. \]

Let \( C = s(U_1) \times s(U_2) \times \cdots \times s(U_n) \). For each \( i = 1, 2, \ldots, n \), we define a compact open \( \mathcal{G} \)-set \( \tilde{U}_i \subset \mathcal{G} \) by

\[ \tilde{U}_i = U_1 \times \cdots \times U_{i-1} \times s(U_i) \times U_{i+1} \times \cdots \times U_n. \]

One has
\[ s(\tilde{U}_i) = C, \quad \tilde{g}_1 \ldots \tilde{g}_{i-1} \tilde{g}_{i+1} \ldots \tilde{g}_n \in \tilde{U}_i \quad (5.1) \]
and
\[ \bigcap_{k=1}^{\infty} r(\tilde{U}_i^k) = r(\tilde{U}_i \cap \mathcal{G}') = \{ x_1 \} \times \cdots \times \{ x_{i-1} \} \times s(U_i) \times \{ x_{i+1} \} \times \{ x_n \}. \quad (5.2) \]

By Lemma 5.10, \( r(\tilde{U}_i \cap \mathcal{G}') \) is \( \mathcal{G} \)-étale. It is easy to see that the reduction \( \mathcal{G}|r(\tilde{U}_i \cap \mathcal{G}') \) is isomorphic to \( \mathbb{Z}^{n-1} \times (\mathcal{G}_{A_i}|s(U_i)) \), and its essentially principal part is \( \mathcal{G}_{A_i}|s(U_i) \).

It follows from (5.1) and (5.2) that \( \pi(\tilde{U}_i) \) is a compact open \( \mathcal{H} \)-set satisfying

\[ s(\pi(\tilde{U}_i)) = \pi(C), \quad \tilde{h}_1 \ldots \tilde{h}_{i-1} \tilde{h}_{i+1} \ldots \tilde{h}_n \in \pi(\tilde{U}_i) \quad (5.1) \]
and
\[ \bigcap_{k=1}^{\infty} r(\pi(\tilde{U}_i)^k) = r(\pi(\tilde{U}_i) \cap \mathcal{H}'). \]

By replacing each \( U_i \) with a smaller subset if necessary, we may assume \( \pi(\tilde{U}_i) \subset \tilde{V}_i \).

Furthermore, we may also assume \([1_{s(U_i)}] = u_{A_i} \) in \( H_0(\mathcal{G}_{A_i}) \). We can find a clopen subset \( Y_i \subset s(V_i) \) such that

\[ r(\pi(\tilde{U}_i) \cap \mathcal{H}') = \{ y_1 \} \times \cdots \times \{ y_{i-1} \} \times Y_i \times \{ y_{i+1} \} \times \{ y_n \}. \]

The isomorphism \( \pi : \mathcal{G} \to \mathcal{H} \) induces an isomorphism between \( \mathcal{G}|r(\tilde{U}_i \cap \mathcal{G}') \) and \( \mathcal{H}|r(\pi(\tilde{U}_i) \cap \mathcal{H}') \), and hence induces an isomorphism \( \pi_i \) between their essentially principal parts \( \mathcal{G}_{A_i}|s(U_i) \) and \( \mathcal{G}_{B_1}|Y_i \). Therefore we obtain \( BF(A_i) \cong BF(B_i) \) and \( \det(id - A_i) = \det(id - B_i) \) by Corollary 5.2.
Let $\varphi_i : BF(A_i^t) \to BF(B_i^t)$ be the isomorphism induced by the isomorphism $\pi_i : G_{A_i}|s(U_i) \to G_{B_i}|Y_i$. One has $\varphi_i(u_{A_i}) = \varphi_i([1_{s(U_i)}]) = [1_{Y_i}]$. We would like to show $\pi(C) = Y_1 \times Y_2 \times \cdots \times Y_n$. For any $z = (z_1, z_2, \ldots, z_n) \in C$, it is easy to see that
\[
\lim_{k \to \infty} \theta(\tilde{U}_i)^k(z) = (x_1, \ldots, x_{i-1}, z_i, x_{i+1}, \ldots, x_n)
\]
holds. Similarly, for any $w = (w_1, w_2, \ldots, w_n) \in D$,
\[
\lim_{k \to \infty} \theta(\tilde{V}_i)^k(w) = (y_1, \ldots, y_{i-1}, w_i, y_{i+1}, \ldots, y_n)
\]
holds. We note that $\pi(\tilde{U}_i)$ is contained in $\tilde{V}_i$ and that $\pi(C)$ is contained in $D$. Hence, for $z = (z_1, z_2, \ldots, z_n) \in C$,
\[
\lim_{k \to \infty} \theta(\tilde{V}_i)^k(\pi(z)) = \lim_{k \to \infty} \theta(\pi(\tilde{U}_i))^k(\pi(z)) = \pi \left( \lim_{k \to \infty} \theta(\tilde{U}_i)^k(z) \right) = \pi(x_1, \ldots, x_{i-1}, z_i, x_{i+1}, \ldots, x_n) = (y_1, \ldots, y_{i-1}, \pi_1(z_i), y_{i+1}, \ldots, y_n),
\]
which implies $\pi(z) = (\pi_1(z_1), \pi_2(z_2), \ldots, \pi_n(z_n))$. Thus, $\pi(C) = Y_1 \times Y_2 \times \cdots \times Y_n$. Then we get
\[
\varphi_1(u_{A_1}) \otimes \varphi_2(u_{A_2}) \otimes \cdots \otimes \varphi_n(u_{A_n}) = [1_{Y_1}] \otimes [1_{Y_2}] \otimes \cdots \otimes [1_{Y_n}]
\]
\[
= [1_{Y_1} \times Y_2 \times \cdots \times Y_n]
\]
\[
= [1_{\pi(C)}]
\]
\[
= H_0(\pi)(1_{\mathcal{G}})
\]
\[
= H_0(\pi)([1_{s(U_1)}] \otimes [1_{s(U_2)}] \otimes \cdots \otimes [1_{s(U_n)}])
\]
\[
= H_0(\pi)(u_{A_1} \otimes u_{A_2} \otimes \cdots \otimes u_{A_n})
\]
\[
= H_0(\pi)(1_{\mathcal{G}(0)})
\]
\[
= [1_{\mathcal{G}(0)}]
\]
\[
= u_{B_1} \otimes u_{B_2} \otimes \cdots \otimes u_{B_n},
\]
which completes the proof. 

\[\square\]

**Remark 5.13.** Let $(\mathcal{V}, \mathcal{E})$ be a directed graph consisting of a unique vertex and $k$ edges (i.e. loops), where $k \geq 2$. The associated shift space is called the full shift over $k$ symbols. The adjacency matrix of $(\mathcal{V}, \mathcal{E})$ is the $1 \times 1$ matrix $[k]$. The groupoid $C^*$-algebra $C^*_r(G_{[k]})$ of the SFT groupoid $G_{[k]}$ is the Cuntz algebra $\mathcal{O}_k$. V.V. Nekrashevych proved that the Higman-Thompson group $V_k$ is identified with a certain subgroup of the unitary group $U(\mathcal{O}_k)$ (see [20 Proposition 9.6]). This identification yields an isomorphism between the Higman-Thompson group $V_{k,1}$ and the topological full group $[[G_{[k]}]]$. Therefore, when
(\(X_A, \sigma_A\)) is a shift of finite type which is not necessarily a full shift, \([G_A]\) can be thought of as a generalization of \(V_k,1\) (see [28, Remark 6.3]).

M. G. Brin introduced the notion of higher dimensional Thompson groups \(nV_{k,r}\) in [5, Section 4.2]. These groups can be considered as an \(n\)-dimensional analogue of the Higman-Thompson group \(V_{k,r} = 1V_{k,r}\). In [5, 6], he proved that \(V_{k,r}\) and \(2V_{2,1}\) are not isomorphic, \(2V_{2,1}\) is simple and \(2V_{2,1}\) is finitely presented. He also proved that \(nV_{2,1}\) is simple for all \(n \in \mathbb{N}\) in [7]. In [2], C. Bleak and D. Lanoue showed that \(nV_{2,1} \cong n'V_{2,1}\) if and only if \(n = n'\). W. Dicks and C. Martínez-Pérez [11] generalized this result and proved that \(nV_{k,r} \cong n'V_{k',r'}\) if and only if \(n = n',\ k = k'\) and \(\gcd(k-1, r) = \gcd(k'-1, r')\).

Define an \(r \times r\) matrix \(A_{k,r}\) by

\[
A_{k,r} = \begin{bmatrix}
0 & 0 & \ldots & 0 & k \\
1 & 0 & \ldots & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & 0 \\
\end{bmatrix}.
\]

The topological full group \([G_{A_{k,r}}]\) of the SFT groupoid \(A_{k,r}\) is naturally isomorphic to the Higman-Thompson group \(V_{k,r}\) (see [28, Section 6.7.1]). By Theorem 5.3 (2), \(H_0(G_{A_{k,r}}) \cong \mathbb{Z}_{k-1}, H_n(G_{A_{k,r}}) = 0\) for \(n \geq 1\) and the equivalence class of the characteristic function of \(X_{A_{k,r}}\) corresponds to \(\bar{r} \in \mathbb{Z}_{k-1}\). It is not so hard to see that the higher dimensional Thompson group \(nV_{k,r}\) is isomorphic to the topological full group \([G]\) of the product groupoid

\[
G = G_{A_{k,r}} \times G_{A_{k,1}} \times \cdots \times G_{A_{k,1}}.
\]

It follows from Theorem 1.12 (1) that the commutator subgroup \(D([G])\) is simple. By Proposition 1.14 we get \(H_1(G) \cong (\mathbb{Z}_{k-1})^{(n-1)}\). Therefore, Theorem 5.8 tells us that \([G]\) is simple if and only if \(k = 2\). This reproves the result of Brin [7]. Also, by applying Theorem 5.12 we get a new proof of the classification theorem by Dicks and Martínez-Pérez [11].

Finiteness conditions of (generalized) Higman-Thompson groups are studied by many authors. K. S. Brown [5] proved that \(V_{k,r}\) is of type \(F_\infty\). By using his method, we proved that the topological full group \([G_A]\) of any SFT groupoid \(G_A\) is of type \(F_\infty\) ([28, Theorem 6.21]). As for the higher dimensional Thompson groups, D. H. Kochloukova, C. Martínez-Pérez and B. E. A. Nucinkis [22] showed that \(2V_{2,1}\) and \(3V_{2,1}\) are of type \(F_\infty\). This was later generalized to all \(nV_{2,1}\) by M. Fluch, M. Marschler, S. Witzel and M. C. B. Zaremsky [12]. Recently, C. Martínez-Pérez, F. Matucci and B. E. A. Nucinkis [24] proved that \(nV_{k,1}\) (and many other relatives) are of type \(F_\infty\). We do not know if the same holds for topological full groups of product groupoids of SFT groupoids \(G = G_{A_1} \times G_{A_2} \times \cdots \times G_{A_n}\).

5.4 An embedding theorem

Let \((X_A, \sigma_A)\) be a one-sided irreducible SFT associated with a finite directed graph \((V, E)\) (\(A\) is the adjacency matrix of the graph). For every edge \(e \in E\), we define the clopen set \(C_e \subset X_A\) by

\[
C_e = \{(x_k)_{k \in \mathbb{N}} \in X_A \mid x_1 = e\},
\]
so that \( \{C_e \mid e \in E\} \) is a clopen partition of \( X_A \). Define a compact open \( G_A \)-set \( U_e \subset G_A \) by

\[
U_e = \{(x, 1, y) \in G_A \mid x \in C_e, \sigma_A(x) = y\}.
\]

One has

\[
r(U_e) = C_e, \quad s(U_e) = \bigcup_{t(e)=i(f)} C_f.
\]

Since

\[
1_{U_e}^r 1_{U_e}^r = 1_{r(U_e)} \quad \text{and} \quad 1_{U_e}^r 1_{U_e} = 1_{s(U_e)}
\]

in \( C_r^*(G_A) \), we get

\[
\sum_{e \in E} 1_{U_e}^r 1_{U_e} = 1_{X_A} \quad \text{and} \quad 1_{U_e}^r 1_{U_e} = \sum_{t(e)=i(f)} 1_{U_e}^r 1_{U_e}^r,
\]

which are called the Cuntz-Krieger relations. The Cuntz-Krieger algebra \( C_r^*(G_A) \) is characterized as the universal \( C^* \)-algebra generated by partial isometries \( \{1_{U_e} \mid e \in E\} \) subject to the Cuntz-Krieger relations \( (1) \). We remark that the Cuntz-Krieger algebra \( C_r^*(G_A) \) is simple and purely infinite.

**Proposition 5.14.** Let \( G_A \) be an SFT groupoid and let \( H \) be a minimal, purely infinite étale groupoid. Suppose that \( \varphi : H_0(G_A) \to H_0(H) \) is a homomorphism satisfying \( \varphi([1_{G_A}]) = [1_H] \). Then there exists a unital homomorphism \( \pi : C_r^*(G_A) \to C_r^*(H) \) such that the following hold.

1. \( \pi(C(G_A)) \subset C(H) \).
2. For any compact open \( G_A \)-set \( U \), there exists a compact open \( H \)-set \( V \) such that \( \pi(1_U) = 1_V \).
3. For any clopen set \( C \subset G_A \), \( [\pi(1_C)] = \varphi([1_C]) \) in \( H_0(H) \).

In particular, \( \pi \) induces an embedding of \( [G_A] \) into \( [H] \).

**Proof.** Let \( (V, E) \) be a finite directed graph whose adjacency matrix is \( A \). For \( e \in E \), let \( C_e \subset X_A \) and \( U_e \subset G_A \) as above. By Theorem 5.13 (2), the elements \( [1_{C_e}] \) generate \( H_0(G_A) \). We have

\[
\sum_{e \in E} \varphi([1_{C_e}]) = \varphi([1_{X_A}]) = [1_H] \).
\]

It follows from 28, Lemma 5.3] that there exists a clopen partition \( \{D_e \mid e \in E\} \) of \( H^{(0)} \) such that \( [1_{D_e}] = \varphi([1_{C_e}]) \) in \( H_0(H) \). Since

\[
[1_{D_e}] = \varphi([1_{C_e}]) = \sum_{t(e)=i(f)} \varphi([1_{C_f}]) = \sum_{t(e)=i(f)} [1_{D_f}],
\]

by means of Theorem 1.12 (3), we can find a compact open \( H \)-set \( V_e \subset H \) satisfying

\[
r(V_e) = D_e, \quad s(V_e) = \bigcup_{t(e)=i(f)} D_f.
\]

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Hence we obtain the Cuntz-Krieger relations
\[
\sum_{e \in E} 1_{V_e} 1^*_V = 1_{H^{(0)}} \quad \text{and} \quad 1^*_V 1_{V_e} = \sum_{t(e) = i(f)} 1_{V_f} 1^*_V.
\]
Therefore there exists a unital homomorphism \( \pi : C^*_r(G_A) \to C^*_r(H) \) such that \( \pi(1_{U_e}) = 1^*_V \).

The Cartan subalgebra \( C(G_A^{(0)}) = C(X_A) \) is generated by projections of the form
\[
(1_{U_e}, 1_{U_{e_2}} \cdots 1_{U_{e_k}})(1_{U_{e_1}} 1_{U_{e_2}} \cdots 1_{U_{e_k}})^*,
\]
which is mapped by \( \pi \) to
\[
(1_{V_1} 1_{V_{e_2}} \cdots 1_{V_{e_k}})(1_{V_{e_1}} 1_{V_{e_2}} \cdots 1_{V_{e_k}})^* \in \mathcal{H}^{(0)}.
\]
This proves (1).

Any compact open \( G_A \)-set can be written as a finite disjoint union of compact open \( G_A \)-sets of the form
\[
U = (U_{e_1} U_{e_2} \cdots U_{e_k})(U_{f_1} U_{f_2} \cdots U_{f_l})^{-1}.
\]
For such \( U \subset G_A \), put
\[
V = (V_{e_1} V_{e_2} \cdots V_{e_k})(V_{f_1} V_{f_2} \cdots V_{f_l})^{-1}.
\]
It is easy to see that the homomorphism \( \pi \) sends \( 1_U \) to \( 1_V \). This proves (2).

(3) is clear from the construction.

**Theorem 5.15.** Let \( \mathcal{G} = \mathcal{G}_{A_1} \times \mathcal{G}_{A_2} \times \cdots \times \mathcal{G}_{A_n} \) be a product groupoid of SFT groupoids. Let \( \mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_n \) be minimal, purely infinite étale groupoids and let \( \mathcal{H} = \mathcal{H}_1 \times \mathcal{H}_2 \times \cdots \times \mathcal{H}_n \). For each \( i = 1, 2, \ldots, n \), suppose that \( \varphi_i : H_0(G_{A_i}) \to H_0(H_i) \) is a homomorphism satisfying \( \varphi_i([1_{G_{A_i}}]) = [1_{H_i^{(0)}}] \). Then there exists a unital homomorphism \( \pi : C^*_r(\mathcal{G}) \to C^*_r(\mathcal{H}) \) such that the following hold.

1. \( \pi(C(G_A^{(0)})) \subset C(H^{(0)}) \).

2. For any compact open \( \mathcal{G} \)-set \( U \), there exists a compact open \( \mathcal{H} \)-set \( V \) such that \( \pi(1_U) = 1_V \).

3. For any clopen set \( C \subset \mathcal{G}^{(0)} \), \( \pi(1_C) = \varphi([1_C]) \) in \( H_0(\mathcal{H}) \), where \( \varphi = \varphi_1 \otimes \varphi_2 \otimes \cdots \otimes \varphi_n \).

In particular, \( \pi \) induces an embedding of \( \llbracket \mathcal{G} \rrbracket \) into \( \llbracket \mathcal{H} \rrbracket \).

**Proof.** By the proposition above, we can construct unital homomorphisms \( \pi_i : C^*_r(G_{A_i}) \to C^*_r(H_i) \) satisfying the properties described there. Then the unital homomorphism \( \pi = \pi_1 \otimes \pi_2 \otimes \cdots \otimes \pi_n \) from \( C^*_r(\mathcal{G}) \) to \( C^*_r(\mathcal{H}) \) meets the requirement.

**Corollary 5.16.** Let \( \mathcal{G} = \mathcal{G}_{A_1} \times \mathcal{G}_{A_2} \times \cdots \times \mathcal{G}_{A_n} \) be a product groupoid of SFT groupoids. Then the topological full group \( \llbracket \mathcal{G} \rrbracket \) is embeddable into the higher dimensional Thompson group \( nV_{2,1} \).

**Proof.** Let \( \mathcal{H} \) be the \( n \)-fold product of \( \mathcal{G}_{[n]} \). As discussed in Remark 5.13, \( \llbracket \mathcal{H} \rrbracket \) is canonically isomorphic to \( nV_{2,1} \). It follows from the theorem above that \( \llbracket \mathcal{G} \rrbracket \) is embeddable into \( nV_{2,1} \). 

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In this subsection, we would like to determine the abelianization $[[G]]_{ab}$ of the topological full group $[[G]]$ for product groupoids $G = G_{A_1} \times G_{A_2} \times \cdots \times G_{A_n}$ of SFT groupoids. By Theorem 5.8, we have already known that $G$ has the strong AH property when $n = 1$. We will show that $G$ may fail to have the strong AH property when $n \geq 3$.

Let $n \geq 2$ be a natural number and let $k : \{1, 2, \ldots, n\} \rightarrow \mathbb{N} \setminus \{1\}$ be a map. First, we would like to begin with the groupoid $G = G_{[k(1)]} \times G_{[k(2)]} \times \cdots \times G_{[k(n)]}$ ($[k(d)]$ is a $1 \times 1$ matrix). By Theorem 5.3 (2) and Proposition 5.4, we have

$$H_0(G) \cong \mathbb{Z}_g \text{ and } H_1(G) \cong (\mathbb{Z}_g)^{n-1},$$

where $g = \gcd(k(1)-1, k(2)-1, \ldots, k(n)-1)$. In later discussion, we actually need to work with the following three cases.

Case 0 $k$ is constant and $k(1) = k(2) = \cdots = k(n) = l$ for $l \in \mathbb{N} \setminus \{1\}$.

Case 1 $k(1) = 3$ and $k(2) = \cdots = k(n) = 5$.

Case 2 $k(1) = k(2) = 3$ and $k(3) = \cdots = k(n) = 5$.

So, the reader may restrict the attention to these special cases, but we proceed with the discussion for arbitrary $k$ for the moment. As mentioned in Remark 5.13, the topological full group of $G_{[2]} \times G_{[2]}$ is the two dimensional Thompson group $2V_{2,1}$. Presentations of the group $2V_{2,1}$ was given by M. G. Brin [6], and later it was extended to $nV_{2,1}$ by J. Hennig and F. Matucci [19]. We have to generalize some arguments of these works to $G = G_{[k(1)]} \times G_{[k(2)]} \times \cdots \times G_{[k(n)]}$.

Fix $n \geq 2$ and $k : \{1, 2, \ldots, n\} \rightarrow \mathbb{N} \setminus \{1\}$. Let $X_{[k(d)]} = \{0, 1, \ldots, k(d)-1\}^\mathbb{N}$ for $d = 1, 2, \ldots, n$. With the product topology, $X_{[k(d)]}$ is the Cantor set. The shift map $\sigma_{[k(d)]]} : X_{[k(d)]} \rightarrow X_{[k(d)]}$ is given by

$$\sigma_{[k(d)]}(x)_m = x_{m+1} \quad m \in \mathbb{N}, \quad x = (x_m)_m \in X_{[k(d)]}.$$

The dynamical system $(X_{[k(d)]}, \sigma_{[k(d)]})$ is the full shift over $k(d)$ symbols. Let $Z_{n,k}$ be the product space of $X_{[k(d)]}$, i.e.

$$Z_{n,k} = X_{[k(1)]} \times X_{[k(2)]} \times \cdots \times X_{[k(n)]}.$$

For $d \in \{1, 2, \ldots, n\}$, we define $\sigma_d : Z_{n,k} \rightarrow Z_{n,k}$ by

$$\sigma_d = \text{id} \times \cdots \times \text{id} \times \sigma_{[k(d)]} \times \text{id} \times \cdots \times \text{id},$$

where $\sigma_{[k(d)]}$ acts on the $d$-th coordinate. For all $i \in \mathbb{N}$ and $d \in \{1, 2, \ldots, n\}$, we define homeomorphisms $s_{i,d}, \tau_i \in \text{Homeo}(Z_{n,k} \times \mathbb{N})$ by

$$s_{i,d}(z,j) = \begin{cases} (z,j) & j < i \\ (\sigma_d(z), j + z_1^{(d)}) & j = i \\ (z, j + k(d)) & j > i, \end{cases}$$

and$$\tau_i(z,j) = (z, j + i).$$
Proof. We can prove this proposition exactly in the same way as \cite{6} and \cite{19}, using trees instead of binary trees.

\[ S(\text{\cite{6, Theorem 2}, \cite{19, Theorem 1.1}}) \]

Proposition 5.17 following set

\[ \tau_i(z, j) = \begin{cases} (z, j) & j \neq i, i+1 \\ (z, i+1) & j = i \\ (z, i) & j = i+1, \end{cases} \]

for \( z = (z(1), z(2), \ldots, z(n)) \in Z_{n,k} \) and \( j \in \mathbb{N} \), where \( z_1^{(d)} \) denotes the first coordinate of \( z^{(d)} \in X_{[k(d)]} \). Set \( S_{n,k} = \{ s_{i,d}, \tau_i \mid i \in \mathbb{N}, d = 1, 2, \ldots, n \} \). Let \( W_{n,k} \subset \text{Homeo}(Z_{n,k} \times \mathbb{N}) \) denote the subgroup generated by \( S_{n,k} \). It is easy to see that these generators satisfy the following set \( R_{n,k} \) of relations.

\[ R_{n,k} = \begin{cases} s_{i,d}s_{j,d'} = s_{j+k(d')-1,d'}s_{i,d} & i < j, 1 \leq d, d' \leq n \\ \tau_i^2 = 1 & i \in \mathbb{N}, \\ \tau_i\tau_j = \tau_j\tau_i & |i - j| \geq 2, \\ \tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1} & i \in \mathbb{N}, \\ s_{i,d}\tau_j = \tau_{j+k(d)-1}s_{i,d} & i < j, \\ s_{i,d}\tau_i = \tau_i\tau_{i+1},d' & i \in \mathbb{N}, \\ s_{i,d}\tau_j = \tau_{j+1}d' & i > j + 1, \\ s_{i,d'}s_{i+1,d'} \cdots s_{i+k(d')-1,d'}s_{i,d} \end{cases} \]

where \( \tilde{\tau}_{i,d} \) is \( \tau_{i+k(d)-1} \cdots \tau_{i+1} \tau_i \) and \( \alpha_{i,d,d'} \) is any word in \( \{ \tau_i \mid i \in \mathbb{N} \} \) satisfying

\[ \alpha_{i,d,d'}(z, j) = \begin{cases} (z, i+qk(d')+p) & j = i+pk(d')+q \text{ for some } 0 \leq p < k(d'), 0 \leq q < k(d) \\ (z, j) & \text{else} \end{cases} \]

Clearly one has

- the permutation on \( \mathbb{N} \) induced by \( \tilde{\tau}_{i,d} \) is odd if and only if \( k(d) \) is odd,
- when \( k(d) = k(d') = l \), the permutation on \( \mathbb{N} \) induced by \( \alpha_{i,d,d'} \) is odd if and only if \( l \in 4Z + 2 \) or \( l \in 4Z + 3 \),
- when \( \{ k(d), k(d') \} = \{ 3, 5 \} \), the permutation on \( \mathbb{N} \) induced by \( \alpha_{i,d,d'} \) is even.

**Proposition 5.17** (\cite{6} Theorem 2, \cite{19} Theorem 1.1). The group \( W_{n,k} \) is presented by using the generators \( S_{n,k} \) and the relations \( R_{n,k} \).

**Proof.** We can prove this proposition exactly in the same way as \cite{6} and \cite{19}, using \( k(d) \)-ary trees instead of binary trees. \( \square \)

In the same way as \( nV_{2,1} \), the topological full group \( [[G]] \) of

\[ G = G_{[A_{k(1),r}]} \times G_{[k(2)]} \times \cdots \times G_{[k(n)]} \]

is canonically isomorphic to the subgroup of \( W_{n,k} \) consisting of elements \( \gamma \) satisfying \( \gamma(z, j) = (z, j) \) unless \( j \in \{ 1, 2, \ldots, r \} \) (see Remark 5.13 for the definition of the matrix \( A_{l,r} \)). Especially, in Case 0, the subgroup of \( W_{n,k} \) consisting of elements \( \gamma \) satisfying
\( \gamma(z, j) = (z, j) \) unless \( j \in \{1, 2, \ldots, r\} \) is canonically isomorphic to the higher-dimensional Thompson group \( nV_{t, r} \). In what follows, we identify the topological full group \([G]\) with this subgroup of \( W_{n,k} \). It is easy to see that \( s_{1,d}^{-1}r_{1,1} \) and \( s_{1,d}^{-1}r_{1,1} \) are in \([G]\). The element \( s_{1,d}^{-1}r_{1,1} \) is the transposition corresponding to the generator of \( H_0(G_{k(d)}) \cong \mathbb{Z}_{k(d)}-1 \). For \( d \neq d' \), the element \( s_{1,d}^{-1}r_{1,1} \in \([G]\) is called the baker’s map. When \( n = 2 \) and \( k(1) = k(2) = l \), as observed in [5, Lemma 8.1], the homeomorphism \( s_{1,2}^{-1}r_{1,1} \) on \( 2V_{l,1} \) is conjugate to the two-sided full shift over \( l \) symbols.

**Remark 5.18.** The groupoid \( C^*\)-algebra \( C^*_r([G]) \) of \( G_{k(n)} \) is the Cuntz algebra, which is generated by \( l \) isometries \( t_1, t_2, \ldots, t_l \) satisfying

\[
\sum_{i=1}^{l} t_i^*t_i = 1.
\]

It is well-known that the unitary

\[ u = \sum_{i=1}^{l} t_i^* \otimes t_i \in C^*_r([G]) \otimes C^*_r([G]) \]

is a generator of \( K_1(C^*_r([G]) \otimes C^*_r([G])) \cong \mathbb{Z}_{l-1} \). The automorphism of \( C(X[l] \times X[g]) \) induced by \( u \) corresponds to the baker’s map \( s_{1,2}^{-1}r_{1,1} \in 2V_{l,1} \).

**Lemma 5.19.** Let \( G = G_{k(1),r} \times G_{k(2)} \times \cdots \times G_{k(n)} \).

1. In Case 0, suppose that \( k(1) = \cdots = k(n) = l \) is in \( 4\mathbb{Z} + 1 \). Then, there exists a homomorphism \( \rho : \[G]\to \mathbb{Z}_2 \) such that \( \rho(s_{1,d}^{-1}r_{1,1}) = 0 \) and \( \rho(s_{1,d}^{-1}r_{1,1}) = 1 \) for any \( d, d' \).

2. In Case 1, there exists a homomorphism \( \rho : \[G]\to \mathbb{Z}_2 \) such that \( \rho(s_{1,d}^{-1}r_{1,1}) = 0 \) and \( \rho(s_{1,d}^{-1}r_{1,1}) = 1 \) for any \( d, d' \).

3. In Case 2, there exists a homomorphism \( \rho : \[G]\to \mathbb{Z}_4 \) such that \( \rho(s_{1,d}^{-1}r_{1,1}) = 1 \) for any \( d \neq 1 \) and \( \rho(s_{1,d}^{-1}r_{1,1}) = 2 \) for any \( d \).

**Proof.** (1)(2) By the proposition above, we can find a homomorphism \( \rho : W_{n,k} \to \mathbb{Z}_2 \) such that \( \rho(s_{i,d}) = 0 \) and \( \rho(\tau_j) = 1 \).

(3) By the proposition above, we can find a homomorphism \( \rho : W_{n,k} \to \mathbb{Z}_4 \) such that \( \rho(s_{i,1}) = 1, \rho(s_{i,d}) = 0 \) for any \( d \neq 1 \) and \( \rho(\tau_j) = 2 \).

Let \( G \) be the product groupoid of \( n \) copies of \( G_{k(n)} \). Then, \([G]\) is isomorphic to \( nV_{l,1} \). When \( l \) is even, \( H_0(G) \otimes \mathbb{Z}_2 \cong \mathbb{Z}_{l-1} \otimes \mathbb{Z}_2 \) is trivial, and so \([G]\) is isomorphic to \( H_1(G) \cong (\mathbb{Z}_2)^{n-1} \) via the index map. When \( l \) is in \( 4\mathbb{Z} + 1 \), by Lemma 5.19 (1), the homomorphism \( j : H_0(G) \otimes \mathbb{Z}_2 \to \[G]\) is injective. Thus, \( G \) has the strong AH property. Moreover, Lemma 5.19 (1) tells us that the extension is trivial. Hence we obtain \([G]\) is isomorphic to \( (\mathbb{Z}_2)^{n-1} \otimes \mathbb{Z}_2 \). When \( n = 2 \) and \( l \in 4\mathbb{Z} + 3 \), in the same way as Lemma 5.19 (3), one can show that there exists a surjective homomorphism \( \[G]\to \mathbb{Z}_{2l-2} \). Therefore, \( G \) has the strong AH property, and the extension does not split. When \( n \geq 3 \) and \( l \in 4\mathbb{Z} + 3 \), by virtue of

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Lemma 5.21. In the setting above, one has the following. This may be thought of as a variant of the baker’s map associated with $g, a, b$. In order to state Lemma 5.21, we introduce some notations. Let $g \in \mathbb{N}$. Let $a \in H_0(\mathcal{G})$ and $b \in H_0(\mathcal{H})$ be such that $ga = 0$ and $gb = 0$. Choose nonempty clopen sets $E \subset \mathcal{G}(0)$ and $F \subset \mathcal{H}(0)$ so that $E \neq \mathcal{G}(0)$, $F \neq \mathcal{H}(0)$, $[1_E] = a$ and $[1_F] = b$. There exist compact open $G$-sets $U_0, U_1, \ldots, U_g$ such that $s(U_i) = E$ for every $i = 0, 1, \ldots, g$ and $\{r(U_i) \mid i = 0, 1, \ldots, g\}$ is a partition of $E$. In the same way, we can find compact open $\mathcal{H}$-sets $V_0, V_1, \ldots, V_g$ such that $s(V_i) = F$ for every $i = 0, 1, \ldots, g$ and $\{r(V_i) \mid i = 0, 1, \ldots, g\}$ is a partition of $F$. Then, we define a homeomorphism $\beta \in [[\mathcal{G} \times \mathcal{H}]]$ associated with $g, a, b$ by

$$\beta(x, y) = \begin{cases} \theta(U_i \times V_{j-1}^{-1})(x, y) & x \in E, \ y \in r(V_i) \\ (x, y) & \text{otherwise.} \end{cases}$$

This may be thought of as a variant of the baker’s map $s_{1,2}^{-1}s_{1,1} \in 2V_{1,1}$.

Lemma 5.21. In the setting above, one has the following.

1. If two proper nonempty clopen sets $F_1, F_2 \subset \mathcal{H}(0)$ satisfy $[1_{F_1}] = [1_{F_2}]$ in $H_0(\mathcal{H})$, then $\gamma \circ \text{id}_{F_1}$ and $\gamma \circ \text{id}_{F_2}$ are conjugate in $[[\mathcal{G} \times \mathcal{H}]]$. 

Theorem 5.20. For $n \geq 1$, $k \geq 2$ and $r \geq 1$, the following hold.

1. When $n = 1$, 

$$(1V_{k,r})_{ab} \cong \begin{cases} \mathbb{Z}_2 & k \in 2\mathbb{Z} \\ 0 & k \in 2\mathbb{Z} + 1. \end{cases}$$

2. When $n = 2$, 

$$(2V_{k,r})_{ab} \cong \begin{cases} \mathbb{Z}_{k-1} & k \in 2\mathbb{Z} \\ \mathbb{Z}_{k-1} \oplus \mathbb{Z}_2 & k \in 4\mathbb{Z} + 1 \\ \mathbb{Z}_{2k-2} & k \in 4\mathbb{Z} + 3. \end{cases}$$

3. When $n \geq 3$, 

$$(nV_{k,r})_{ab} \cong \begin{cases} (\mathbb{Z}_{k-1})^{n-1} & k \in 2\mathbb{Z} \text{ or } k \in 4\mathbb{Z} + 3 \\ (\mathbb{Z}_{k-1})^{n-1} \oplus \mathbb{Z}_2 & k \in 4\mathbb{Z} + 1. \end{cases}$$

We turn, now, to the consideration of the general case.

In order to state Lemma 5.21 we introduce some notations. Let $\mathcal{G}$ and $\mathcal{H}$ be minimal purely infinite groupoids. For $\gamma \in [[\mathcal{G}]]$ and a clopen set $F \subset \mathcal{H}(0)$, we define $\gamma \circ \text{id}_F \in [[\mathcal{G} \times \mathcal{H}]]$ by

$$(\gamma \circ \text{id}_F)(x, y) = \begin{cases} (\gamma(x), y) & y \in F \\ (x, y) & y \notin F. \end{cases}$$

It is easy to see that $I(\gamma \circ \text{id}_F) = I(\gamma) \otimes [1_F]$, where $H_1(\mathcal{G}) \otimes H_0(\mathcal{H})$ is regarded as a subgroup of $H_1(\mathcal{G} \times \mathcal{H})$.
(2) If $F \subset \mathcal{H}^{(0)}$ is a nonempty clopen set with $[1_F] = 0$ in $H_0(\mathcal{H})$, then $\gamma \circ \text{id}_F$ belongs to the commutator subgroup $D([\mathcal{G} \times \mathcal{H}])$.

(3) $\beta$ is uniquely determined by $g$, $a$ and $b$ up to conjugacy in $[[\mathcal{G} \times \mathcal{H}]]$.

(4) The equivalence class of $\beta^g$ in $[[\mathcal{G} \times \mathcal{H}]]_{ab}$ is equal to the image of

$$\frac{g(g+1)}{2}a \otimes b \otimes 1 \in H_0(\mathcal{G}) \otimes H_0(\mathcal{H}) \otimes \mathbb{Z}_2.$$

**Proof.** (1) By Theorem 4.2 (3), there exists $\alpha \in [[\mathcal{H}]]$ such that $\alpha(F_1) = F_2$. Then $(\text{id} \times \alpha)(\gamma \circ \text{id}_{F_1})(\text{id} \times \alpha^{-1})$ equals $\gamma \circ \text{id}_{F_2}$.

(2) Suppose $F \neq \mathcal{H}^{(0)}$. We can find disjoint nonempty clopen sets $F_1, F_2$ such that $F_1 \cup F_2 = F$ and $[1_{F_1}] = [1_{F_2}] = 0$. It follows from (1) that $\gamma \circ \text{id}_F$ and $\gamma \circ \text{id}_{F_i}$ are conjugate for each $i = 1, 2$. Since $(\gamma \circ \text{id}_{F_1})(\gamma \circ \text{id}_{F_2}) = \gamma \circ \text{id}_F$, $\gamma \circ \text{id}_F$ is in $D([\mathcal{G} \times \mathcal{H}])$, as desired. When $F = \mathcal{H}^{(0)}$, we can apply the same argument for $F_1, F_2$ as above.

(3) This can be proved in the same way as (1).

(4) Define a transposition $\tau \in [[\mathcal{G} \times \mathcal{H}]]$ by

$$\tau(x, y) = \begin{cases} (\theta(U_j \times V_i) \circ \theta(U_i \times V_j)^{-1})(x, y) & x \in r(U_i), \ y \in r(V_j), \ i \neq j \\ (x, y) & \text{otherwise}. \end{cases}$$

It is not so hard to see that $\tau \beta$ is a product of $g+1$ elements with mutually disjoint support and each of them is conjugate to $\beta$. This means $\tau \beta^{-g}$ is in $D([\mathcal{G} \times \mathcal{H}])$. We have

$$\sum_{i<j} [1_{r(U_i)} \times 1_{r(V_j)}] = \frac{g(g+1)}{2}a \otimes b$$

in $H_0(\mathcal{G} \times \mathcal{H})$, which implies the conclusion. \hfill $\square$

**Lemma 5.22.** Let $k$ be a natural number such that $k \in 4\mathbb{Z} + 3$ and let $\mathcal{G} = \mathcal{G}[k] \times \mathcal{G}[k] \times \mathcal{G}[k]$. The index map $I : [[\mathcal{G}]] \to H_1(\mathcal{G})$ induces an isomorphism $[[\mathcal{G}]]_{ab} \cong H_1(\mathcal{G})$. In particular, $\mathcal{G}$ does not have the strong AH property.

**Proof.** By Theorem 5.8

$$H_0(\mathcal{G}) \otimes \mathbb{Z}_2 \xrightarrow{j} [[\mathcal{G}]]_{ab} \xrightarrow{I} H_1(\mathcal{G}) \xrightarrow{\partial} 0$$

is exact. Since $k$ is odd, $H_0(\mathcal{G}) \otimes \mathbb{Z}_2 \cong \mathbb{Z}_{k-1} \otimes \mathbb{Z}_2 = \mathbb{Z}_2$. Let $X_{[k]} = \{0, 1, \ldots, k-1\}^\mathbb{N}$ and let $(X_{[k]}, \sigma_{[k]})$ be the full shift over $k$ symbols. As in Section 5.4, we define the clopen set $C_i \subset X_{[k]}$ by

$$C_i = \{(x_n)_{n \in \mathbb{N}} \in X_{[k]} \mid x_1 = i\}.$$

Define a compact open $\mathcal{G}[k]$-set $U_i \subset \mathcal{G}[k]$ by

$$U_i = \{(x, 1, y) \in \mathcal{G}[k] \mid x \in C_i, \ \sigma_{[k]}(x) = y\}.$$

Let $t \in [[\mathcal{G}]]_{ab}$ be the image of the generator of $H_0(\mathcal{G}) \otimes \mathbb{Z}_2 \cong \mathbb{Z}_2$. We would like to show $t = 0$. Define $\beta_{12} \in [[\mathcal{G}]]$ by

$$\beta_{12}(x, y, z) = \theta(U_i \times U_i^{-1} \times X_{[k]})(x, y, z) \quad \text{when} \ y \in C_i$$

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for \((x, y, z) \in G^{(0)} = X_{[k]} \times X_{[k]} \times X_{[k]}\). The homeomorphism \(\beta_{12}\) is the baker’s map acting on the first and second coordinates of \(X_{[k]} \times X_{[k]} \times X_{[k]}\). By Lemma 5.21 (4), one sees \((k-1)[\beta_{12}] = t\) in \([G]_{ab}\). We can define \(\beta_{23} \in [[G]]\) in the same way by

\[
\beta_{23}(x, y, z) = \theta(X_{[k]} \times U_i \times U_i^{-1})(x, y, z) \quad \text{when} \quad z \in C_i.
\]

Again one has 
\((k-1)[\beta_{23}] = t\). It is easy to see that \(\beta_{12}\beta_{23}\) is equal to the baker’s map acting on the first and third coordinates of \(X_{[k]} \times X_{[k]} \times X_{[k]}\). Therefore, we get \((k-1)[\beta_{12}\beta_{23}] = t\). Consequently, we obtain \(2t = t\), thus \(t = 0\).

Let \(G = G_{A_1} \times G_{A_2} \times \cdots \times G_{A_n}\) be a product groupoid of SFT groupoids. We compute the abelianization \([[G]]_{ab}\). By Corollary 2.5 \(H_1(G)\) is isomorphic to

\[
\bigoplus_{q_1+q_2+\cdots+q_n=1} H_{q_1}(G_{A_1}) \otimes H_{q_2}(G_{A_2}) \otimes \cdots \otimes H_{q_n}(G_{A_n})
\]

\[
\oplus \bigoplus_{p=1}^{n-1} H_0(G_{A_1}) \otimes \cdots \otimes H_0(G_{A_{p-1}}) \otimes \text{Tor}(H_0(G_{A_p}), H_0(G_{A_{p+1}} \times \cdots \times G_{A_n})�\).
\]

We write

\[
T_p = H_0(G_{A_1}) \otimes \cdots \otimes H_0(G_{A_{p-1}}) \otimes \text{Tor}(H_0(G_{A_p}), H_0(G_{A_{p+1}} \times \cdots \times G_{A_n})).
\]

For each \(d = 1, 2, \ldots, n\), we assume

\[
H_0(G_{A_d}) \cong \bigoplus_{i=1}^{h(d)} \mathbb{Z}_{m(d,i)}
\]

for \(h(d) \in \mathbb{N} \cup \{0\}\) and \(m(d, i) \in \{0, 2, 3, \ldots\}\), where \(\mathbb{Z}_0\) is understood as \(\mathbb{Z}\). Set

\[
J = \{(i_1, i_2, \ldots, i_n) \in \mathbb{N}^n \mid 1 \leq i_d \leq h(d)\}.
\]

The set \(J\) is empty if and only if there exists \(d\) such that \(H_0(G_{A_d}) = 0\). In such a case one has \(H_0(G) = H_1(G) = 0\) and \([[G]] = D([[G]])\). For \((i_1, i_2, \ldots, i_n) \in J\) and \(p \in \{1, 2, \ldots, n-1\}\), we write

\[
S(i_1, i_2, \ldots, i_n) = \mathbb{Z}_{m(1,i_1)} \otimes \mathbb{Z}_{m(2,i_2)} \otimes \cdots \otimes \mathbb{Z}_{m(n,i_n)} \subset H_0(G)
\]

and

\[
T_p(i_1, i_2, \ldots, i_n) = \mathbb{Z}_{m(1,i_1)} \otimes \cdots \otimes \mathbb{Z}_{m(p-1,i_{p-1})} \otimes \text{Tor}(\mathbb{Z}_{m(p,i_p)}, \mathbb{Z}_{m(p+1,i_{p+1})} \otimes \cdots \otimes \mathbb{Z}_{m(n,i_n)}) \subset T_p.
\]

Then one has

\[
H_0(G) = \bigoplus_{(i_1, i_2, \ldots, i_n) \in J} S(i_1, i_2, \ldots, i_n)
\]

and

\[
T_p = \bigoplus_{(i_1, i_2, \ldots, i_n) \in J} T_p(i_1, i_2, \ldots, i_n).
\]
Let \( J_0 \subset J \) be the subset consisting of \((i_1, i_2, \ldots, i_n)\) such that \( m(d, i_d) \in 2\mathbb{Z} \) for all \( d \) and \( \# \{d \mid m(d, i_d) \in 4\mathbb{Z}+2 \} \) is less than three. Let \( S_0 \subset H_0(\mathcal{G}) \) be the subgroup

\[
S_0 = \bigoplus_{(i_1, i_2, \ldots, i_n) \in J_0} S(i_1, i_2, \ldots, i_n).
\]

We have

\[
\Ext \left( \bigoplus_{p=1}^{n-1} T_p, S_0 \otimes \mathbb{Z}_2 \right)
\]

\[
= \bigoplus_{p=1}^{n-1} \bigoplus_{(i_1, i_2, \ldots, i_n) \in J} \bigoplus_{(i'_1, i'_2, \ldots, i'_n) \in J_0} \Ext \left( T_p(i_1, i_2, \ldots, i_n), S(i'_1, i'_2, \ldots, i'_n) \otimes \mathbb{Z}_2 \right)
\]

and each summand \( \Ext(T_p(i_1, \ldots, i_n), S(i'_1, \ldots, i'_n) \otimes \mathbb{Z}_2) \) is either 0 or \( \mathbb{Z}_2 \). We let

\[
e(A_1, A_2, \ldots, A_n) \in \Ext \left( \bigoplus_{p=1}^{n-1} T_p, S_0 \otimes \mathbb{Z}_2 \right)
\]

be the element whose \( \Ext(T_p(i_1, \ldots, i_n), S(i'_1, \ldots, i'_n) \otimes \mathbb{Z}_2) \) summand is nontrivial if and only if

- \( i_d = i'_d \) for all \( d \),
- \( m(d, i_d) \in 4\mathbb{Z} \) for all \( d \leq p-1 \),
- \( m(p, i_p) \in 4\mathbb{Z} + 2 \),
- \( \# \{d \mid d > p, m(d, i_d) \in 4\mathbb{Z}+2 \} = 1 \).

Making use of this notation, we can describe \([\mathcal{G}]_{ab}\) in the following way.

**Theorem 5.23.** Let \( \mathcal{G} = \mathcal{G}_{A_1} \times \mathcal{G}_{A_2} \times \cdots \times \mathcal{G}_{A_n} \) be a product groupoid of SFT groupoids. Let the notation be as above. Then

\[
0 \rightarrow S_0 \otimes \mathbb{Z}_2 \xrightarrow{j} [\mathcal{G}]_{ab} \xrightarrow{i} H_1(\mathcal{G}) \rightarrow 0
\]

is exact. The quotient map \([\mathcal{G}]_{ab} \rightarrow H_1(\mathcal{G})\) has a right inverse on the subgroup

\[
\bigoplus_{q_1+q_2+\cdots+q_n=1} H_{q_1}(\mathcal{G}_{A_1}) \otimes H_{q_2}(\mathcal{G}_{A_2}) \otimes \cdots \otimes H_{q_n}(\mathcal{G}_{A_n}) \subset H_1(\mathcal{G}).
\]

The extension of the remaining part

\[
\bigoplus_{p=1}^{n-1} H_0(\mathcal{G}_{A_1}) \otimes \cdots \otimes H_0(\mathcal{G}_{A_{p-1}}) \otimes \Tor\left(H_0(\mathcal{G}_{A_p}), H_0(\mathcal{G}_{A_{p+1}} \times \cdots \times \mathcal{G}_{A_n})\right)
\]

is given by the element \( e(A_1, A_2, \ldots, A_n) \) described above.
Proof. Assume

\[ \hat{H}_0(\mathcal{G}_{A_d}) \cong \bigoplus_{i=1}^{h(d)} \mathbb{Z}_{m(d,i)} \]

as above. Let \( a_{d,i} \in \mathbb{Z}_{m(d,i)} \subset H_0(\mathcal{G}_{A_d}) \) be a generator. Let \( Y_{d,i} \subset X_{A_d} \) be a proper nonempty clopen subset such that \( [1]_{Y_{d,i}} = a_{d,i} \).

First, we determine the kernel of the homomorphism \( j : H_0(\mathcal{G}) \otimes \mathbb{Z}_2 \to \mathbb{[G]}_{ab} \). Let \((i_1, i_2, \ldots, i_n) \in J \setminus J_0 \). We would like to show that \( S(i_1, \ldots, i_n) \otimes \mathbb{Z}_2 \) is contained in the kernel. If \( m(d, i_d) \) is odd for some \( d \), then \( S(i_1, \ldots, i_n) \otimes \mathbb{Z}_2 = 0 \). Suppose that \( m(d, i_d) \) is even for all \( d \) and that \#\{\(d\mid m(d, i_d) \in 4\mathbb{Z}+2\} \) is not less than three. To simplify notation, we assume \( m(d, i_d) \in 4\mathbb{Z}+2 \) for \( d = 1, 2, 3 \). Let \( \varphi_d \) be a homomorphism \( \varphi_d : \mathbb{Z}_2 \to H_0(\mathcal{G}_{A_d}) \) such that

\[ \varphi_d(\bar{1}) = \frac{m(d, i_d)}{2} a_{d,i} \in \mathbb{Z}_{m(d,i)} \]

for \( d = 1, 2, 3 \). Set

\[ \mathcal{H} = \mathcal{G}_{[3]} \times \mathcal{G}_{[3]} \times \mathcal{G}_{[3]} \times (\mathcal{G}_{A_d}|Y_{d,i_d}) \times \cdots \times (\mathcal{G}_{A_n}|Y_{n,i_n}) \]

and

\[ Y = Y_{1,i_1} \times Y_{2,i_2} \times \cdots \times Y_{n,i_n} \subset \mathcal{G}^{(0)} \].

Applying Theorem 5.15 to \( \mathcal{H} \) and \( \mathcal{G}|Y \), we get a unital homomorphism \( \pi : C^*_r(\mathcal{H}) \to C^*_r(\mathcal{G}|Y) \). Let \( \tau \in [[\mathcal{G}_{[3]} \times \mathcal{G}_{[3]} \times \mathcal{G}_{[3]}]] \) be a transposition corresponding to the nontrivial element of \( \mathbb{Z}_2 \). By Lemma 5.22 \( \tau \) is in the commutator subgroup. Consider the transposition \( \tau \times \text{id} \in \text{Homeo}(\mathcal{H}^{(0)}), \) which is in \( D([[\mathcal{H}])] \). The homomorphism \( \pi \) induces an embedding \([[\mathcal{H}]] \to [[\mathcal{G}|Y]] \), which maps \( \tau \times \text{id} \) to a transposition corresponding to \( a_{1,i_1} \otimes \cdots \otimes a_{n,i_n} \otimes \bar{1} \in H_0(\mathcal{G}) \otimes \mathbb{Z}_2 \). Therefore the image of \( a_{1,i_1} \otimes \cdots \otimes a_{n,i_n} \otimes \bar{1} \) by \( j \) is zero in \([[\mathcal{G}]]_{ab} \).

Assume \((i_1, i_2, \ldots, i_n) \in J_0 \). For each \( d \in \{1, 2, \ldots, n\} \), we define a homomorphism \( \varphi_d \) from \( H_0(\mathcal{G}_{A_d}) \) to a finite cyclic group. When \( m(d, i_d) \) is in \( 4\mathbb{Z} \), we define \( \varphi_d : H_0(\mathcal{G}_{A_d}) \to \mathbb{Z}_4 \) by

\[ \varphi_d(a_{d,i}) = \begin{cases} 1 & \text{if } i = i_d \\ 0 & \text{if } i \neq i_d \end{cases} \]

When \( m(d, i_d) \) is in \( 4\mathbb{Z}+2 \), we define \( \varphi_d : H_0(\mathcal{G}_{A_d}) \to \mathbb{Z}_2 \) by the same formula. Set

\[ \mathcal{H} = \mathcal{G}_{[k_1]} \times \mathcal{G}_{[k_2]} \times \cdots \times \mathcal{G}_{[k_n]} \],

where

\[ k_d = \begin{cases} 3 & m(d, i_d) \in 4\mathbb{Z}+2 \\ 5 & m(d, i_d) \in 4\mathbb{Z} \end{cases} \]

Applying Theorem 5.15 to \( \mathcal{G}|Y \) and \( \mathcal{H} \), we get a unital homomorphism \( \pi : C^*_r(\mathcal{G}|Y) \to C^*_r(\mathcal{H}) \). Let \( \tau \in [[\mathcal{H}]] \) be a transposition corresponding to the nontrivial element of \( H_0(\mathcal{H}) \otimes \mathbb{Z}_2 \cong \mathbb{Z}_2 \). Since \#\{\(d\mid m(d, i_d) \in 4\mathbb{Z}+2\} \) is less than three, by Lemma 5.19 there exists a homomorphism \( \rho \) from \([[\mathcal{H}]]_{ab} \) to a finite cyclic group such that \( \rho([\tau]) \) is nonzero. The homomorphism \( \pi \) induces an embedding \([[\mathcal{G}|Y]] \to [[\mathcal{H}]] \), and hence a homomorphism \( \pi_* : [[\mathcal{G}|Y]]_{ab} \to [[\mathcal{H}]]_{ab} \). The composition \( \rho \circ \pi_* \) satisfies

\[ \rho(\pi_*(j(a_{1,i_1} \otimes \cdots \otimes a_{n,i_n} \otimes \bar{1}))) \neq \bar{0} \iff (l_1, l_2, \ldots, l_n) = (i_1, i_2, \ldots, i_n). \]
Hence we can conclude

\[ \text{Ker } j = \bigoplus_{(i_1, i_2, \ldots, i_n) \notin J_0} S(i_1, i_2, \ldots, i_n) \otimes \mathbb{Z}_2, \]

which implies that

\[ 0 \longrightarrow S_0 \otimes \mathbb{Z}_2 \xrightarrow{j} [[G]]_{ab} \xrightarrow{\iota} H_0(G) \longrightarrow 0 \]

is exact.

Next, let us determine the equivalence class of the extension above. Let \( q_1 + q_2 + \cdots + q_n = 1 \) and consider the subgroup

\[ H_{q_1}(G_{A_1}) \otimes H_{q_2}(G_{A_2}) \otimes \cdots \otimes H_{q_n}(G_{A_n}) \subset H_1(G). \]

We would like to show that the quotient map \([G]_{ab} \to H_1(G)\) has a right inverse on this subgroup. To simplify the notation, we may assume \( q_1 = 1 \) and \( q_2 = q_3 = \cdots = q_n = 0 \). The group

\[ H_1(G_{A_1}) \otimes H_0(G_{A_2}) \otimes H_0(G_{A_3}) \otimes \cdots \otimes H_0(G_{A_n}) \]

is isomorphic to the direct sum of \( S(i_1, i_2, \ldots, i_n) \) for \((i_1, i_2, \ldots, i_n)\) such that \( m(1, i_1) = 0 \). If \( m(d, i_d) = 0 \) for all \( d \), then \( S(i_1, i_2, \ldots, i_n) \) is \( \mathbb{Z}_2 \), and so the exact sequence clearly splits on it. Suppose that \( m(d, i_d) > 0 \) for some \( d \geq 2 \). Let \( g \) be the greatest common divisor of \( \{m(d, i_d) \mid d \geq 2\} \). Then \( S(i_1, i_2, \ldots, i_n) \) is isomorphic to \( \mathbb{Z}_g \). Let \( \gamma \in [[G_{A_1}]] \) be an element whose index is equal to \( a_{1,i_1} \in \mathbb{Z}_{m(1,i_1)} \subset H_1(G_{A_1}) \). Set

\[ Y = Y_{2,i_2} \times Y_{3,i_3} \times \cdots \times Y_{n,i_n}. \]

It follows that \( \gamma \circ \text{id}_Y \in [[G]] \) (or its equivalence class in the abelianization) is a lift of the generator of \( S(i_1, i_2, \ldots, i_n) \). By Lemma 5.21 (1) and (2), \( \gamma \circ \text{id}_Y \in [[G]] \) is of order \( g \) in \([G]_{ab}\). Therefore, the exact sequence splits on \( S(i_1, i_2, \ldots, i_n) \).

Lastly, let us consider the subgroup \( T_p \subset H_1(G) \) for \( p = 1, 2, \ldots, n-1 \). Take \((i_1, i_2, \ldots, i_n) \in J\). When \( m(p, i_p) = 0 \) or \( m(d, i_d) = 0 \) for all \( d > p \), then \( T_p(i_1, i_2, \ldots, i_n) \) is zero. So, we may assume that \( m(p, i_p) > 0 \) and \( m(d, i_d) > 0 \) for some \( d > p \). Let \( g_0 \) be the greatest common divisor of \( \{m(d, i_d) \mid d > p\} \), so that \( \mathbb{Z}_{m(p,i_p+1)} \otimes \cdots \otimes \mathbb{Z}_{m(n,i_n)} \cong \mathbb{Z}_{g_0} \). Let

\[ g = \text{gcd}(m(p, i_p), g_0) \]

and let

\[ a = \frac{m(p, i_p)}{g} a_{p,i_p}, \quad b = \frac{g_0}{g} (a_{p+1,i_{p+1}} \otimes a_{p+2,i_{p+2}} \otimes \cdots \otimes a_{n,i_n}). \]

One has \( ga = 0 \) and \( gb = 0 \). Let \( \beta \in [[[G_{A_p} \times G_{A_{p+1}} \times \cdots \times G_{A_n}]]] \) be the element associated with \( g, a \) and \( b \) (see the construction stated before Lemma 5.21). Set

\[ Y = Y_{1,i_1} \times Y_{2,i_2} \times \cdots \times Y_{p-1,i_{p-1}}. \]

We can verify that \( \text{id}_Y \circ \beta \in [[G]] \) is a lift of the generator of \( T_p(i_1, i_2, \ldots, i_n) \). Let \( g_1 \) be the greatest common divisor of \( \{m(d, i_d) \mid d < p\} \) and let \( g_2 = \text{gcd}(g_1, g) \). Then \( T_p(i_1, i_2, \ldots, i_n) \) is isomorphic to \( \mathbb{Z}_{g_2} \). There exist \( u, v \in \mathbb{Z} \) such that \( g_2 = ug_1 + vg \). When \( g_1 > 0 \), \( g_1[1_Y] \) is zero in \( H_0(G_{A_1} \times \cdots \times G_{A_{p-1}}) \). It follows from Lemma 5.21 (1) and (2)

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that \( g_1[\text{id}_Y \circ \beta] \) is zero in \([\mathcal{G}]_{ab} \). When \( g_1 \) is zero, \( g_1[\text{id}_Y \circ \beta] \) is clearly zero. Hence, we have
\[
g_2[\text{id}_Y \circ \beta] = u g_1[\text{id}_Y \circ \beta] + v g[\text{id}_Y \circ \beta] = v g[\text{id}_Y \circ \beta]
\]
in \([\mathcal{G}]_{ab} \). This value is zero if and only if the extension splits on \( T_p(i_1, i_2, \ldots, i_n) \). By Lemma 5.21 (4), we get
\[
g[\text{id}_Y \circ \beta] = \frac{g(g + 1)}{2} j([1_Y] \otimes a \otimes b \otimes \bar{1})
\]
\[
= \frac{g(g + 1)}{2} \frac{m(p, i_p) g_0}{g} j(a_{1,i_1} \otimes a_{2,i_2} \otimes \cdots \otimes a_{n,i_n} \otimes \bar{1}).
\]

We have already shown that \( j(a_{1,i_1} \otimes a_{2,i_2} \otimes \cdots \otimes a_{n,i_n} \otimes \bar{1}) \neq 0 \) if and only if \( (i_1, i_2, \ldots, i_n) \in J_0 \), i.e.

- \( m(d, i_d) \in 2Z \) for all \( d \),
- \( \#\{d \mid m(d, i_d) \in 4Z+2\} \) is less than three.

Assume \( (i_1, i_2, \ldots, i_n) \in J_0 \). If \( m(p, i_p) \in 4Z \) and \( g_0 \in 4Z \), then \( g \) is also in 4Z. It follows that \( g(g+1)/2 \) is even, which implies \( g[\text{id}_Y \circ \beta] = 0 \). If \( m(p, i_p) \in 4Z \) and \( g_0 \in 4Z+2 \), then \( g \) is in 4Z+2. It follows that \( m(p, i_p)/g \) is even, which again implies \( g[\text{id}_Y \circ \beta] = 0 \). In the same way, if \( m(p, i_p) \in 4Z+2 \) and \( g_0 \in 4Z \), then \( g[\text{id}_Y \circ \beta] = 0 \). Assume \( m(p, i_p) \in 4Z+2 \) and \( g_0 \in 4Z+2 \). Then, \( \#\{d \mid d > p, m(d, i_d) \in 4Z+2\} = 1 \) and \( m(d, i_d) \in 4Z \) for every \( d < p \). Also, \( g \) is in 4Z+2, which implies
\[
g[\text{id}_Y \circ \beta] = \frac{g(g + 1)}{2} \frac{m(p, i_p) g_0}{g} j(a_{1,i_1} \otimes a_{2,i_2} \otimes \cdots \otimes a_{n,i_n} \otimes \bar{1}).
\]

Moreover, one has \( g_1 \in 4Z \) and \( g_2 \in 4Z+2 \). Therefore \( v \) is odd. Consequently, we obtain
\[
g_2[\text{id}_Y \circ \beta] = v g[\text{id}_Y \circ \beta] = v j(a_{1,i_1} \otimes \cdots \otimes a_{n,i_n} \otimes \bar{1})
\]
\[
= j(a_{1,i_1} \otimes \cdots \otimes a_{n,i_n} \otimes \bar{1}) \neq 0,
\]
which completes the proof.

\[\square\]

**Corollary 5.24.** Let \( \mathcal{G} = \mathcal{G}_{A_1} \times \mathcal{G}_{A_2} \times \cdots \times \mathcal{G}_{A_n} \) be a product groupoid of SFT groupoids.

1. If \( n = 1 \) or \( n = 2 \), then \( \mathcal{G} \) has the strong AH property.

2. When \( n \geq 3 \), \( \mathcal{G} \) has the strong AH property if and only if the number of \( d \in \{1, 2, \ldots, n\} \) such that \( H_0(\mathcal{G}_{A_d}) \) contains \( \mathbb{Z}_2 \) as a direct summand is less than three.

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