On Line Graphs and 2-variegated graphs

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Abstract.
This note is on the structures of graphs of line graphs and 2-variegated graphs. It is natural to ask which 2-variegated graph is a line graph of some graph and furthermore when a line graph of a 2-variegated graph is a 2-variegated graph. In addition, a characterization of potentially 2-variegated line graphic degree sequences and a characterization of forcibly 2-variegated line graphic degree sequences are given.

Keywords: 2-Variegated graph, Line Graph, Potentially graphic degree sequences, Forcibly graphic degree sequences, Eigenvalues.

MSC codes: 05C07, 05C38, 05C50, 05C62, 05C75

1. Introduction

2-variegated (Bivariagated) graphs are due to Bednarek et al. in their paper [1] (1973) are 1-factorable graphs with some additional properties. K-variegated graphs, \( K \geq 2 \), and further results on 2-variegated graphs by other authors can be found in [2-5], [10-11], and [14]. The earliest work on line graph is from Whitney (1932) and Krausz (1943). In [7, page 80], Harary and Nash-Williams studied eulerian graphs, hamiltonian graphs and line graphs and answered which line graphs are eulerian and which line graphs are hamiltonian.

We will mostly follow the terminology of [7]. We consider only finite graphs without loops and without multiple edges. Let \( G = (p, q) \) where \( V \) is the vertex set and \( E \) is the edge set, \( |V| = p \) and \( |E| = q \). A graph is said to be a 2-variegated graph if its vertex set can be partitioned into two equal parts such that each vertex from one part is adjacent to exactly one vertex from the other part not containing it [1]. The following characterization of a 2-variegated graph is from [3]. A graph \( G \) of order \( 2n \) is a 2-variegated graph if and only if there exists a set \( S \) of \( n \) independent edges in \( G \) such that no cycle in \( G \) contains an odd number of edges from \( S \). We will call such independent edges in graph \( G \) “special edges”.

The line graph \( L(G) \) of \( G \) is one with \( V(L(G)) = E(G) \) and two vertices of \( L(G) \) are adjacent if and only if the corresponding two edges of \( G \) are adjacent in \( G \). The characterization of a line graph \( G \) due to Krausz [7] is that no vertex lies on more than two cliques (complete graphs) and every edge of \( G \) belongs to exactly one clique.
If G is a 2-variegated graph or not, then L(G) need not be a 2-variegated graph. The question is under what conditions, L(G) is a 2-variegated graph for a given graph G and when L(G) is a 2-variegated graph for a given 2-variegated graph G. In this direction, we have the following lemmas to address theorem 1.

**Lemma 1.** Let G be a graph. If G has an edge uv with degree d(u) ≥ 3 and degree d(v) ≥ 3 or G has an edge uv with degree d(u) ≥ 3 and degree d(v) = 1, then L(G) is not a 2-variegated graph.

**Proof.**
1. Since d(u) ≥ 3 and d(v) ≥ 3, L(G) has at least 2 cliques of size ≥ 3, the vertex w in L(G) corresponding to the edge uv in G does not have a special edge through w because the edges at w in L(G) are on a click of size ≥ 3. This implies, if L(G) is a 2-variegated graph, and uv is an edge in G, then d(u) < 3 or d(v) < 3.

2. The vertex w in L(G) corresponding to the edge uv in G does not have a special edge through w in L(G). It implies, if L(G) is a 2-variegated graph for some graph G and uv is an edge in G, then d(u) ≤ 2 or d(v) ≠ 1. This completes the proof.

**Lemma 2.** If G has a cycle of length n ≡ 1, 2, or 3 (mod 4), then L(G) is not a 2-variegated graph.

**Proof:** Suppose G has a cycle \( x_1x_2, x_2x_3, \ldots, x_nx_1 \) of length \( n \equiv 1, 2 \text{ or } 3 \pmod{4} \). These edges \( x_ix_j \) are vertices in L(G) and they are adjacent in L(G) in the same order as in G and form a cycle of length n. If \( n \equiv 2 \pmod{4} \), the alternate \( n/2 \) edges which are odd in number cannot be the special edges of the vertices on the cycle in L(G). If \( n \equiv 1 \text{ or } 3 \pmod{4} \), the alternate edges on the cycle can't be chosen as special edges of the vertices on the cycle in L(G). Let uv be the other edge at \( x_i \) other than the edges on the cycle in G. Let w be the corresponding vertex in L(G). The edges through w cannot be a special edge of any vertex on the cycle in L(G). It implies, if L(G) is a 2-variegated graph, then G has no cycle of length \( n \equiv 1, 2 \text{ or } 3 \pmod{4} \). This completes the proof.

Two paths are disjoint if they do not share edges. Let <xyz> denote the induced subgraph \( P_3 \), a path of length 2 on the vertices x, y and z with d(y) = 2.

**Theorem 1.** If G (p, q) \( p \geq 3 \), q even has q/2 mutually disjoint <xyz> paths and odd number of these paths do not lie on a cycle, then L(G) is a 2-variegated graph.

Conversely, if L(G) is a 2-variegated graph with p vertices, then G has p/2 mutually disjoint <xyz> paths and odd number of these paths do not lie on a cycle.

**Proof.** For the 1st part, the proof is by mathematical induction on p the number of vertices.

Base case: For \( p = 3 \), G is \( P_3 \), and L(G) is a 2-variegated graph.
Assume that the result is true for a graph $H$ with $p = k > 3$ satisfying the given conditions. $L(H)$ is a 2-variegated graph. Consider a graph $G(r, s)$ with $r \geq k + 1$ vertices and $s$ edges satisfying the given conditions. Since there are $s/2$ vertices of degree 2 from the $s/2 <xyz>$ paths in $G$, it is clear that the remaining vertices of $G$ are just adjacent to these $s/2$ vertices of degree 2.

Let $v$ be one of the $s/2$ vertices of degree 2. Let $vx$ and $vy$ be the adjacent edges at $v$. Let $S_x$ be the set of edges incident at $x$ and $S_y$ be the set of edges incident at $y$. Delete the vertex $v$ and the edges $vx$ and $vy$ incident at $v$ and also the isolated vertices if any. The resulting graph $H$ satisfies the conditions of the theorem and the graph $L(H)$ is a 2-variegated line graph. Let $V(H) = U_1 \cup V_1$ be a 2-variegation with the special edges $\{e_1, e_2, ... e_z\}$. Add a new edge $xy$ to $L(H)$ and join the vertex $x$ to the vertices in of the sets say $U_1$ corresponding to the edges from $S_x$ and join $y$ to the vertices in $U_2$ corresponding to the edges from $S_y$. The resulting graph $L(G)$ is a line graph of $G$ with a 2-variegation $V(G) = \{U_1 + x\} \cup \{V_1 + y\}$ with the special edges $\{e_1, e_2, ... e_z, xy\}$.

Conversely, let $L(G)$ be a 2-variegated graph with $p = 2n$ vertices. Since $L(G)$ has $2n$ vertices and $n$ special edges, these $2n$ vertices correspond to the $2n$ edges of $G$ and each special edge of $L(G)$ corresponds to a pair of adjacent edges say $v_1v_2$ and $v_2v_3$ of $G$ and such pairs have to be disjoint because the special edges in $L(G)$ are nonadjacent. By lemma 2, vertex $v_1$ cannot be adjacent to vertex $v_3$ in $G$. Let $v_4v_5$, and $v_5v_6$ be a pair of edges in $G$ corresponding to some other special edge in $L(G)$. By lemma 1, the edge $v_4v_5$ or $v_5v_6$ has to be different from the edges $v_1v_2$ or $v_2v_3$. All these pairs of edges in $G$ corresponding to the $n$ special edges in $L(G)$ are disjoint paths of length 2 induced on 3 vertices. So, $G$ has precisely $2n$ edges and $n$ disjoint paths $<xyz>$, $d(y) = 2$. Also, no odd number of these paths are on a cycle in $G$; otherwise $L(G)$ is not a 2-variegated graph. This completes the proof of theorem 1.

The above theorem identifies a graph $G$ for which $L(G)$ is a 2-variegated graph. The following corollary answers when graphs $G$ and $L(G)$ both are 2-variegated graphs.

**Corollary 1.** If $G$ is a graph with $2n$ vertices and $2n$ edges satisfying the conditions set in theorem 1, and $G$ has a set of $n$ nonadjacent edges such that each is exactly in one of the $n <xyz>$ paths of length 2, then both $G$ and $L(G)$ are 2-variegated.

**Proof.** It can be verified that the $n$ disjoint edges of $G$ are the $n$ special edges of $G$ where no odd number of them are on a cycle and thus $G$ is a 2-variegated graph and $L(G)$ is also a 2-variegated graph. This completes the proof of the corollary 1.

Note that since $G$ has $2n$ vertices and $2n$ edges, it has precisely one cycle of length, $k \equiv 0 \pmod{4}$.

Let $L^k(G)$ denote the $k^{th}$ iterated line graph of $G$.

**Theorem 2.** Only if $G$ is a cycle of length, $n \equiv 0 \pmod{4}$, then
\( L^k(G) \) is a 2-variegated graph.

**Proof.** Case 1: If the max degree of G is 2, then G is a cycle or a path.

If G is a cycle of length, \( n \equiv 1, 2, \text{ or } 3 \), then L(G) is not a 2-variegated graph. If G has a cycle of length \( n, n \equiv 0 \) (mod 4), then L(G) is a 2-variegated cycle of length n and hence \( L^k(G) \) is also a 2-variegated graph for \( k \geq 1 \).

If G is a path of length \( q, \) and \( q \) is odd, then L(G) is not a 2-variegated graph. If \( q \) is even and L(G) is a 2-variegated graph, then L(G) has odd number of edges and hence \( L^2(G) \) is not a 2-variegated graph.

Case 2: If G has a vertex \( u \) of degree \( > 2 \), then L(G) has a clique of size \( > 2 \). If L(G) is 2-variegated or not, \( L^2(G) \) is not a 2-variegated graph. This completes the proof.

**Corollary 2.** Let G be a 2-variegated graph. Then \( L(G) = G \) if and only if G is a cycle of length \( n, n \equiv 0 \) (mod 4).

**Proof:** If G is a graph, then \( L(G) = G \) if and only if G is a cycle [7]. But a cycle of length \( n \equiv 1, 2, \text{ or } 3 \) (mod 4) is not a 2-variegated graph. Hence, \( L(G) = G \) if and only if G is a cycle of length \( n, n \equiv 0 \) (mod 4). This completes the proof.

**2. Potentially 2-variegated line graphic degree sequence.**

A sequence of nonnegative integers, \( \pi = \{d_1, d_2, \ldots, d_p\}, d_1 \geq d_2 \geq \ldots d_p \geq 1 \) is graphic if there exists a graph with this sequence \( \pi \). If the degree sequence \( \pi \) is such that a degree \( d_i \) appears \( k_i \) times, \( 1 \leq i \leq l \), we may write \( \pi = \{d_1^{k_1}, d_2^{k_2}, \ldots, d_l^{k_l}\} \). Let \( P \) be a property of a graph \( G \). Nash-Williams in [12] called a graphic degree sequence \( \pi \) a potentially \( P \) if it has at least one realization with the property \( P \) and forcibly \( P \) if every realization of it has the property \( P \). S. B. Rao [13] has characterized forcibly line graphic degree sequences. The problems of characterizing potentially line graphic degree sequences and characterizing potentially 2-variegated degree sequences are unsolved problems in [13]. We characterize here potentially 2-variegated line graphic degree sequences.

In a 2-variegated line graph with \( 2n \) vertices, every vertex \( u \) belongs to exactly one special edge of the 2-variegation and if \( d(u) = n > 1 \), \( u \) is connected to \((n-1)\) vertices on one side of the 2-varigation making a clique of size \( n \).

Let \( 2n = n_1d_1 + n_2d_2 + \ldots + n_md_m, d_1 \geq d_2 \geq \ldots \geq d_m, 1 \leq d_i \leq n, n_i \geq 1 \) for \( i \geq 1 \). Further, let A and B be two disjoint sets of nd terms of \( 2n \) such that \( \sum_A nd = \sum_B nd = n \).

We will call such a partition of \( 2n \) as an admissible partition of \( 2n \).

**Theorem 3.** If \( n_1d_1 + n_2d_2 + \ldots + n_md_m \) is an admissible partition of \( 2n \), then the sequence,
\[
\pi = \{d_1^{n_1d_1}, d_2^{n_2d_2}, ..., d_m^{n_md_m}\}
\]
is a 2-variegated line graphic degree sequence.

Conversely, if \{d_1^{k_1}, d_2^{k_2}, ..., d_l^{k_l}\} is a 2-variegated line graphic degree sequence with \(2n\) vertices, then \(2n = k_1 + k_2 + ..., + k_l\), \(k_i = n_id_i\), \(1 \leq d_i \leq n\), \(n_i \geq 1\) for \(1 \leq i \leq l\) is an admissible partition of \(2n\).

**Proof.** We prove this theorem by mathematical induction on \(n\).

Base case: For \(n = 1\), the admissible partition of 2 is \(2 = 1(1) + 1(1)\), and the corresponding degree sequence \(\pi = \{1^2(1)\}\) is a 2-variegated line graphic. Note that in short \(1(1) + 1(1)\) is same as \(2(1)\). The theorem is true for \(n = 1\).

Let us assume that the theorem is true for \(n = k > 1\). Let \(n = k + 1\) and consider an admissible partition of \(2(k+1) = n_1d_1 + n_2d_2 + ... + n_md_m\). \hspace{1cm} (3.1)

Case 1: \(m = 1\), the degree sequence \(\pi = \{d_1^n1d_1\}\) is clearly a 2-variegated line graphic.

Case 2: \(m > 1\).

From (3.1), we get \(2k = n_1d_1 + n_2d_2 + ... + n_rd_r + ... + n_sd_s + ... n_md_m - 2\) \hspace{1cm} (3.2)

Let \(d_r\) and \(d_s\) be from the nd terms of the sets A and B respectively of the partition \(2(k+1)\) given above. From (3.2), we have,

\[2k = n_1d_1 + n_2d_2 + ... + (n_r - 1)d_r + 1(d_r - 1) + ... + (n_s - 1)d_s + 1(d_s - 1) + ... + n_md_m.\] \hspace{1cm} (3.3)

If \((n_r - 1) = 0\) or \((n_s - 1) = 0\), discard the terms involving them in (3.3). Also discard the terms involving \((d_s - 1)\), and \((d_r - 1)\) if they are zeros from (3.3). Clearly, there exists two disjoint sets \(A_1\) and \(B_1\) of the nd terms of (3.3) such that \(\sum_{A_1} nd = \sum_{B_1} nd = k\). Since the theorem is true for \(n = k\), there exists a 2-variegated line graph with the above partition of \(2k\). Let \(H\) be a 2-variegated line graph from the partition. Add an edge \(uv\) to \(H\) and join \(u\) to the \((d_r - 1)\) vertices of degree \((d_r - 1)\) on a clique and \(v\) to the \((d_s - 1)\) vertices of degree \((d_s - 1)\) on a clique. The resulting graph is a 2-variegated line graph with the degree sequence \(\pi = \{d_1^{n_1d_1}, d_2^{n_2d_2}, ..., d_m^{n_md_m}\}\).

Conversely, suppose \{\(d_1^{k_1}, d_2^{k_2}, ..., d_l^{k_l}\)\} is a 2-variegated line graphic degree sequence with \(2n\) vertices and \(G\) be its 2-variegated line graph. Obviously \(2n = k_1 + k_2 + ... + k_l\) and the largest clique of \(G\) could be on \(n\) vertices. Let \(C\) be the set of all clicks of sizes \(\geq 1\) except the \(n\) special edges; these clicks together span the \(2n\) vertices of \(G\). The special edges of \(G\) contribute \(1\) to the degree of each of the \(2n\) vertices. Let \(n_i\) be the number of cliques of the same size \(i \geq 1\); \(k_i = n_id_i\), \(d_i \geq 1\), for \(1 \leq i \leq l\). Since \(G\) is a 2-variegated graph, the cliques from \(C\) can
be put into two sets A and B such that the sum of the vertices covered on each side equals n. Therefore, \(2n = n_1d_1 + n_2d_2 + \cdots + n_md_m\) is an admissible partition. This completes the proof.

2.1 Forcibly 2-variegated line graphic degree sequences.

**Corollary 3.** A graphical degree sequence \(\pi\) is forcibly 2-variegated line graphic if it is one of the followings.

\[
\{ n^n, 1^n \} \\
\{ 1^{2n} \} \\
\{ 2^4 \}
\]

**Proof.** It can be verified using theorem 3 that the above sequences are potentially 2-variegated line graphic degree sequences. In [4], it is proved that these are the only forcibly 2-variegated degree sequences with unique realizations. This completes the proof.

4. The Spectra of complete 2-variegated line graphs.

The goal is to study the eigenvalues and the polynomials of the 2-variegated line graphs. Our findings are limited to a complete 2-variegated line graph \(G\) on \(2n\) vertices, \(n \geq 3\). The adjacency matrix \(A(G)\) of \(G\), is a square \((0-1)\) matrix of order \(2n\) whose rows and columns correspond to the vertices of \(G\), and whose \((i,j)\)\(^{th}\) entry is 1 if and only if vertex \(i\) and vertex \(j\) are adjacent, otherwise zero. The eigenvalues of \(G\) are the eigenvalues of its adjacency matrix \(A(G)\). Since \(G\) is a line graph of a complete bipartite graph \(K_{n,2}\), the eigenvalues of \(G\) are \(n, n-2, 0 \text{ and } -2\) [6]. Since \(G\) is a regular graph of degree \(n\), the eigenvector of \(G\) corresponding to the eigenvalue \(n\) is \(u = (1, 1, 1, \ldots, 1)\) of length \(2n\) and \(A(G)\) is a real symmetric matrix, the eigenvectors corresponding to different eigenvalues are orthogonal. The multiplicities of \(n\) and \((n-2)\) are 1 whereas the multiplicity of -2 is \((n-1)\) and the multiplicity of 0 is \((n-1)\).

If \(A = A(G)\), the polynomial \(P(G)\) in [9], it can be verified

\[
P(A) = \frac{A^3 + (4-n)A^2 - 2(n-2)A}{(n+2)} = J\] where \(J\) is a matrix with entries 1.

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