Abstract

Classical $\mathcal{W}_3$ transformations are discussed as restricted diffeomorphism transformations ($W$-Diff) in two-dimensional space. We formulate them by using Riemannian geometry as a basic ingredient. The extended $\mathcal{W}_3$ generators are given as particular combinations of Christoffel symbols. The defining equations of $W$-Diff are shown to depend on these generators explicitly. We also consider the issues of finite transformations, global $SL(3)$ transformations and $W$-Schwarzsians.
1 Introduction

Non-linear extensions of the Virasoro algebra are known as $W$-algebras \cite{1}. The classical counterparts of these algebras are obtained through Drinfel’d-Sokolov (DS) Hamiltonian reduction for Kac-Moody current algebras \cite{2} and the zero-curvature approaches \cite{3}. Classical $W$-algebras are also related to the theory of integrable systems through the Gel’fand-Dickey brackets \cite{4}. For example, classical $W_N$ transformations are the symmetries of the equation

$$LX \equiv \partial X + \sum_{j=0}^{N-2} U_{N-j} \partial^j X = 0,$$

(1.1)

where $X$ and $U_j$ fields are regarded as functions of a single variable $t$ \cite{5}. The reparametrizations of $t$ induce diffeomorphism transformations (Diff) on $X$ and $U_j$, whereas $W$-transformations cannot be understood as reparametrizations in one-dimensional base space. In order to understand all $W$-transformations as diffeomorphisms it is natural to extend the base space to a multi-dimensional one ($W$-space) \cite{6}.

In $W$-space we will consider multi-time extension equations associated with (1.1) given by:

$$LX = 0, \quad \partial_t k X = L_{k/N}^{k/N} X, \quad k = 2, \ldots, N - 1,$$

(1.2)

where $X$ and $U_j$ depend now on $N - 1$ time parameters: $t^1, \ldots, t^{N-1}$ and $L_{k/N}^{k/N}$ is the differential part of the pseudo-differential operator $L^{k/N}$ \cite{4}. Therefore $X$ is a truncation of the Baker-Akhiezer function associated to the generalized hierarchies of partial differential equations generated by $L$. In this framework the $W_N$ transformations are those leaving the whole set of multi-time equations (1.2) invariant. They are induced by some restricted class of Diff in $W$-space which we refer to as $W$-Diff.

We will also show how Riemannian geometry is useful for the formulation of the multi-time equations. We derive them from a set of covariant equations by imposing suitable gauge-fixing conditions on the Christoffel symbols. In particular for $N = 3$ we will explicitly see how $W_3$ generators appear as some combinations of the Christoffel symbols and how they transform under finite $W$-Diff. The $W$-Diff are defined by a set of differential equations involving the extended $W$-generators. $W$-space is in some sense a bosonic version of the superspace used in the covariant description of superconformal theories \cite{7}. However the dynamical fields are not present in the defining relations of superconformal transformations in superspace.

The organization of the paper is as follows: In sect. 2 we introduce a set of covariant equations used to construct the multi-time $W_N$ equations in $W$-space. In sect. 3 we consider the $N = 3$ case explicitly. Finite $W$-Diff and the finite $W$-transformations of
the extended generators are obtained in sect. 4. Conclusions and discussions are given in the last section.

2 Covariant Formalism for Multi-time Equations

Let us consider a system described by a set of fundamental covariant equations [8],

$$\nabla_\alpha \nabla_\beta X^A(t^1, t^2, \ldots, t^{N-1}) = 0, \quad \alpha, \beta = t^1, \ldots, t^{N-1}. \quad (2.1)$$

We assume (2.1) to have \(N\) independent solutions labeled by the index \(A = 1, \ldots, N\). Each \(X^A\) is a scalar density of weight \(h = 1/N\) under general coordinate transformations in \(N - 1\) dimensional space. The covariant derivative is defined by

$$\nabla_\rho X = \partial_\rho X + h \Gamma^\sigma_{\rho \sigma} X = (\partial_\rho + h \Gamma_\rho) X, \quad (2.2)$$

and the second derivative is

$$\nabla_\nu \nabla_\rho X = \partial_\nu \nabla_\rho X - \Gamma^\sigma_{\rho \nu} \nabla_\sigma X + h \Gamma^\sigma_{\rho \nu} \nabla_\rho X = \{\partial_\nu - (A_\nu)_\rho \} \nabla_{\sigma} X, \quad (2.3)$$

where

$$(A_\nu)_\rho = (\Gamma_\nu - h \Gamma_\nu)_\rho^\sigma, \quad (\Gamma_\alpha)_\beta^\gamma = \Gamma_\beta^\gamma_\alpha, \quad \Gamma_\alpha = \Gamma^\beta_\beta_\alpha = \text{tr}(\Gamma_\alpha). \quad (2.4)$$

The Christoffel symbol \(\Gamma_\alpha\) is regarded as \(gl(N - 1)\) valued gauge field and \(\Gamma_\alpha\) as its trace.

We can show that equation (2.1) is only consistent when the Riemann-Christoffel curvature tensor vanishes. Let us write the integrability condition of (2.1),

$$\nabla_{[\mu} \nabla_{\nu]} \nabla_\rho X = - (F^\lambda_{\mu \nu})^\rho_\lambda \nabla_\lambda X - \Gamma^\sigma_{\lambda \nu} \nabla_\sigma \nabla_\rho X = 0. \quad (2.5)$$

Here \(F\) is the field strength for \(A\),

$$F_{\alpha \beta} \equiv \partial_\alpha A_\beta - \partial_\beta A_\alpha - [A_\alpha, A_\beta] = \partial_\alpha \Gamma_\beta - \partial_\beta \Gamma_\alpha - [\Gamma_\alpha, \Gamma_\beta] - h \text{tr}(\partial_\alpha \Gamma_\beta - \partial_\beta \Gamma_\alpha - [\Gamma_\alpha, \Gamma_\beta]). \quad (2.6)$$

The assumption of \(N\) independent solutions for \(X^A\) requires zero field strength, \(F = 0\). By taking the trace of (2.6), the vanishing of \(F\) implies zero Riemann-Christoffel curvature,

$$R^\delta_{\alpha \beta} = \{\partial_\alpha \Gamma_\beta - \partial_\beta \Gamma_\alpha - [\Gamma_\alpha, \Gamma_\beta]\}_\gamma^\delta = 0. \quad (2.7)$$

Equation (2.1) also requires symmetric Christoffel connections. In fact from (2.3),

$$0 = [\nabla_\alpha, \nabla_\beta] X = h(\partial_\alpha \Gamma_\beta - \partial_\beta \Gamma_\alpha) X - \Gamma^\gamma_{[\beta \alpha]} \nabla_\gamma X. \quad (2.8)$$
The first term in the r.h.s. is just the trace of (2.7) and vanishes for the solutions. Thus the Christoffel symbols are symmetric and there is no torsion,
\[ \Gamma_{\beta\alpha}^{\gamma} = \Gamma_{\alpha\beta}^{\gamma}. \] (2.9)

Since the Riemann-Christoffel tensor is zero there exists a set of flat coordinates \( k^a \)'s \((a = 1, \ldots, N - 1)\) such that the Christoffel symbols vanish. Equation (2.1) has trivial solutions there. For a generic system of coordinates we can write the Christoffel symbols as
\[ \Gamma_{\nu\rho}^{\mu} = (\Gamma_{\mu})^{\rho}_{\nu} = (\partial_{\mu} J_{\nu}^{a})(J^{-1})^{\rho}_{a}, \quad J \in GL(N - 1). \] (2.10)
where \((J)_{\nu}^{a} = \partial_{\nu} k^{a}(t^{a})\). Due to the fact that \( X \) is a scalar density of weight \( h \) the solution in generic coordinates is
\[ X(t^{a}) = |J|^{-h} (C_{a} k^{a}(t^{a}) + C_{N}). \] (2.11)
where \( C_{a} \) and \( C_{N} \) are integration constants.

The relation between the covariant system of equations (2.1) and the multi-time equations for \( \mathcal{W}_{N} \) (1.2) is established through a suitable set of gauge conditions on the Christoffel symbols. The form of gauge-fixing conditions for general \( \mathcal{W}_{N} \) have been discussed in [8] and worked out for \( N = 3 \) and 4 explicitly. In the following sections we restrict ourselves to the \( \mathcal{W}_{3} \) case and make the analysis of \( \mathcal{W}\text{-Diff} \) in detail.

3 Multi-time \( \mathcal{W}_{3} \) equations

We consider two-dimensional space with the local coordinates \( t \equiv t^{1} \) and \( z \equiv t^{2} \). Let us impose the following gauge-fixing conditions [3]:
\[ \Gamma_{tt}^{t} = 1, \quad \Gamma_{tz}^{t} = 2\Gamma_{tz}^{t}. \] (3.1)

Zero-curvature condition (2.7) imposed on Riemann-Christoffel curvature gives 4 equations. It is used to express \( \Gamma_{tz}^{t} \) and \( \Gamma_{zz}^{t} \) in terms of independent Christoffel symbols, \( \Gamma_{tz}^{t} \) and \( \Gamma_{zz}^{t} \),
\[ \Gamma_{tz}^{t} = \Gamma_{tz}^{t} + \Gamma_{zz}^{t} - \partial_{t} \Gamma_{tz}^{t}, \] (3.2)
and
\[ \Gamma_{zz}^{t} = \Gamma_{zz}^{t} + \Gamma_{tz}^{t} \partial_{t} \Gamma_{tz}^{t} - \frac{1}{3} \Gamma_{tz}^{t} \partial_{t} \Gamma_{tz}^{t} - \frac{1}{3} \partial_{t} \Gamma_{zz}^{t} - \frac{1}{3} \partial_{t}^{2} \Gamma_{tz}^{t}. \] (3.3)
It also gives \( z \)-derivative of \( \Gamma_{tz}^{t} \) and \( \Gamma_{zz}^{t} \) as functions of \( \Gamma_{tz}^{t} \) and \( \Gamma_{zz}^{t} \) and their \( t \) derivatives:
\[ \partial_{z} \Gamma_{tz}^{t} = \frac{1}{3} \partial_{t} (\Gamma_{tz}^{t} + \Gamma_{zz}^{t}) = \frac{1}{3} (2\Gamma_{tz}^{t} \partial_{t} \Gamma_{tz}^{t} + 2 \partial_{t} \Gamma_{zz}^{t} - \partial_{t}^{2} \Gamma_{tz}^{t}), \] (3.4)
\[ \partial_{z} \Gamma_{zz}^{t} = \frac{1}{3} \left( 10 \Gamma_{tz}^{t} \partial_{t} \Gamma_{tz}^{t} + 6 \Gamma_{zz}^{t} \partial_{t} \Gamma_{tz}^{t} - 2 (\partial_{t} \Gamma_{tz}^{t})^{2} - 2 \partial_{t} \Gamma_{zz}^{t} + 2 \partial_{t} \Gamma_{zz}^{t} \partial_{t} \Gamma_{tz}^{t} + \partial_{t}^{2} \Gamma_{zz}^{t} - 2 \partial_{t}^{2} \Gamma_{tz}^{t} \right). \] (3.5)
In this gauge the covariant equations (2.1) are expressed as
\[ \partial_t^3 X(t, z) + T(t, z) \partial_t X(t, z) + \left( W_3(t, z) + \frac{1}{2} \partial_t T(t, z) \right) X(t, z) = 0 \] (3.6)
and
\[ \partial_z X(t, z) = \partial_t^2 X(t, z) + \frac{2}{3} T(t, z) X(t, z). \] (3.7)
They are the multi-time \( W_3 \) equations in which \( T \) and \( W_3 \) are defined as functions of Christoffel symbols:
\[ T(t, z) = -2 \Gamma_{tz}^2 - \Gamma_{zz} - 2 \partial_t \Gamma_{tz} = \Gamma_{zz} - 2 \Gamma_{tz}, \] (3.8)
\[ W_3(t, z) = -\Gamma_{tz}^3 - \Gamma_{zz} \Gamma_{tz} + \frac{1}{2} \partial_t \Gamma_{zz} = -\Gamma_{zt} - \frac{1}{6} \partial_t T(t, z). \] (3.9)
The \( z \)-extension equations (3.4) and (3.5) for \( \Gamma_{zt} \) and \( \Gamma_{zz} \) are translated to those for \( T \) and \( W_3 \),
\[ \partial_z T(t, z) = 2 \partial_t W_3(t, z), \] (3.10)
\[ \partial_z W_3(t, z) = -\frac{2}{3} T(t, z) \partial_t T(t, z) - \frac{1}{6} \partial_t^3 T(t, z). \] (3.11)
They are nothing but the integrability conditions of the multi-time \( W_3 \) equations (3.6) and (3.7).

In sect. 2 we have determined solutions of (2.1) for \( X^A \) and the Christoffel symbols in terms of \( N - 1 \) arbitrary functions \( k^a \). After imposing the gauge-fixing conditions (3.1) the functions \( k^a \) are no longer arbitrary but must satisfy a set of partial differential equations. The first gauge-fixing condition of (3.1), \( \Gamma_{tt} = 1 \), requires
\[ |J| \equiv \partial_t k^1 \partial_z k^2 - \partial_t k^2 \partial_z k^1 = \partial_t k^1 \partial_t^2 k^2 - \partial_t k^2 \partial_t^2 k^1 \equiv K_3. \] (3.12)
Using the second gauge-fixing condition of (3.1), \( \Gamma_{tt} = 2 \Gamma_{tz} \), we have
\[ \partial_z k^a = \partial_t^2 k^a + Q \partial_t k^a, \quad Q = -\frac{2}{3} \frac{\partial_t K_3}{K_3}, \quad (a = 1, 2). \] (3.13)
It is worth noticing that the extension equation for \( k^a \) (3.13) shows a global \( SL(3) \) invariance. It is non-linearly realized as
\[ k^a \rightarrow k^b B_{b}^{\ a} + \tilde{B}_{b}^{\ a}, \quad a = 1, 2, \quad \tilde{B} \in SL(3)_{\text{global}}. \] (3.14)
We also point out that the extension equation has a trivial set of solution \( k^1 = t, \ k^2 = z + \frac{t^2}{2} \) giving zero values for \( T \) and \( W_3 \).

The solutions of the multi-time \( W_3 \) equations (3.6), (3.7), (3.10) and (3.11) are now expressed in terms of \( k^a \) satisfying (3.13). Using (2.11), (3.8) and (3.9) we obtain:
\[ X^A(t, z) = \{ k^1 K_3^{-1/3}, \ k^2 K_3^{-1/3}, \ K_3^{-1/3} \}, \] (3.15)
and

\[ T(t, z) = \frac{\partial^2 K_3}{K_3} - \frac{4}{3} \left( \frac{\partial t K_3}{K_3} \right)^2 + \frac{1}{K_3} \left( \partial^2_t k^1 \partial^3_t k^2 - \partial^2_t k^2 \partial^3_t k^1 \right) \] (3.16)

and

\[ W_3(t, z) = \frac{1}{6} \frac{\partial^2 K_3}{K_3} + \frac{5}{6} \frac{\partial t K_3}{K_3} \left( \frac{\partial^2_t k^1}{K_3} \frac{\partial^3_t k^2}{K_3} - \frac{\partial^2_t k^2}{K_3} \frac{\partial^3_t k^1}{K_3} \right) \]
\[ - \frac{20}{27} \left( \frac{\partial t K_3}{K_3} \right)^3 + \frac{1}{2} \frac{\partial^2_t K_3}{K_3} - \frac{1}{2} \frac{\partial^2 t K_3}{K_3} - \frac{5}{12} \frac{\partial t \partial^2 K_3}{K_3} + \frac{1}{4} \frac{\partial \partial^2 t K_3}{K_3}. \] (3.17)

These formulas coincide for \( z = 0 \) with the well-known expressions of the \( W_3 \) generators as obtained using, for example, the Wronskian method \[9\]. If we use the extension equations for \( k^a \) (3.13) we can obtain an alternative expression for \( T(t, z) \) and \( W(t, z) \) in terms of \( K_3 \) in (3.12) only:

\[ T(t, z) = -\frac{2}{3} \left( \frac{\partial^2 K_3}{K_3} \right) + \frac{1}{2} \frac{\partial^2 K_3}{K_3} - \frac{1}{2} \frac{\partial^2 t K_3}{K_3}, \]
\[ W_3(t, z) = \frac{4}{27} \left( \frac{\partial t K_3}{K_3} \right)^3 - \frac{1}{3} \frac{\partial^2 t K_3}{K_3} + \frac{1}{2} \frac{\partial^2 K_3}{K_3} - \frac{5}{12} \frac{\partial t \partial^2 K_3}{K_3} + \frac{1}{4} \frac{\partial \partial^2 t K_3}{K_3}. \] (3.18)

The extension equation (3.13) is different from that discussed by Gervais and Matsuo \[12\]. They consider an equation corresponding to (3.13) with \( Q = 0 \). The condition that the two-dimensional infinitesimal Diff preserve (3.6) and \( \partial_a k^a = \partial^2_t k^a \) cannot be written in a local way in terms of \( T \) and \( W_3 \). It can be shown that (3.13) is the only possible form of the extension equation satisfying this local property.

4 \( \mathcal{W} \)-diffeomorphisms and Finite \( \mathcal{W} \)-Symmetry

We have shown that the covariant equations (2.1) in the gauge (3.1) are the multi-time \( \mathcal{W}_3 \) equations (3.6), (3.7), (3.10) and (3.11) for \( N = 3 \). In this section we will show how the general coordinate transformations that preserve the gauge conditions do generate the classical \( \mathcal{W}_3 \) transformations in the extended space.

Under general coordinate transformations \( t = f(\tilde{t}, \tilde{z}), z = g(\tilde{t}, \tilde{z}) \) the scalar density \( X \) and the Christoffel symbols transform according to:

\[ \tilde{X}(\tilde{t}, \tilde{z}) = J^{-\frac{1}{2}} X(t, z), \quad J = \frac{\partial t}{\partial \tilde{t}} \frac{\partial z}{\partial \tilde{z}} - \frac{\partial t}{\partial \tilde{z}} \frac{\partial z}{\partial \tilde{t}}. \] (4.1)
\[ \tilde{\Gamma}^{\alpha}_{\beta \gamma}(\tilde{t}, \tilde{z}) = \left( \frac{\partial t^\mu}{\partial \tilde{t}} \frac{\partial t^\nu}{\partial \tilde{z}} \Gamma^\rho_{\mu \nu}(t, z) + \frac{\partial^2 t^\rho}{\partial \tilde{t} \partial t^\gamma} \right) \frac{\partial \tilde{t}^\alpha}{\partial \tilde{t}^\rho}. \] (4.2)

From the requirement that general coordinate transformations keep the gauge conditions (3.1) invariant we find a set of equations to be satisfied by the transformation functions \( f(\tilde{t}, \tilde{z}) \) and \( g(\tilde{t}, \tilde{z}) \):

\[ f' = \tilde{f} - \frac{2}{3} \tilde{t}^\prime \tilde{t} - \frac{2}{3} \tilde{f} \tilde{g} T(t, z) - \tilde{g}^2 V_3(t, z), \]
\[ g' = \tilde{g} - \frac{2}{3} \tilde{g} \tilde{f} + \tilde{f}^2 + \frac{1}{3} \tilde{g}^2 T(t, z). \] (4.3)
Here $\dot{f} = \frac{\partial f(\tilde{t}, \tilde{z})}{\partial \tilde{t}}$, $f' = \frac{\partial f(\tilde{t}, \tilde{z})}{\partial \tilde{z}}$ and so on. Equations (3.8) and (3.9) have been used to give the expressions of $T$ and $V_3 \equiv W_3 + \frac{1}{6} \partial_\tilde{t} T$ in terms of Christoffel symbols. Then we define a $\mathcal{W}$-Diff as a general coordinate transformation $t = f(\tilde{t}, \tilde{z})$, $z = g(\tilde{t}, \tilde{z})$ satisfying equations (4.3) for given $T$ and $V_3$. In contrast to the conformal and superconformal transformations they depend on the connections $T$ and $V_3$. In other words, the two-dimensional coordinate $\mathcal{W}$-transformations cannot be performed independently of the $\mathcal{W}$-generators of the system.

The finite transformations of $X$ are given by (4.1) with the Jacobian determined from (4.3):

$$J(\tilde{t}, \tilde{z}) = \dot{f} \tilde{g} - \tilde{f} \dot{g} + \dot{f} \tilde{g}^2 T(t, z) + \tilde{g}^3 V_3(t, z).$$

(4.4)

The finite transformations of the extended $\mathcal{W}$-generators are obtained from those of Christoffel connections,

$$\tilde{T}(\tilde{t}, \tilde{z}) = \frac{1}{J} \left( \dot{f} g'' - f'' \tilde{g} - 2(f' \dot{g}' - f' \dot{g}') + 3 \dot{f} f' + \left( \dot{f} g'' + 2f' \dot{g}g' \right) T(t, z) + 3 \dot{g} \tilde{g}^2 V_3(t, z) \right),$$

$$\tilde{V}_3(\tilde{t}, \tilde{z}) = \frac{1}{J} \left( \dot{f} g'' - f'' \dot{g}' + f^3 + f' \dot{g} g'^2 T(t, z) + g^3 \tilde{V}_3(t, z) \right).$$

(4.5)

Notice that $f$ and $g$ are not arbitrary functions but satisfy (4.3) and depend implicitly on $T$ and $V_3$. Thus the $\tilde{T}$ and $\tilde{V}_3$ in (4.3) have non-linear dependence on $T$ and $V_3$.

Let us consider a coordinate system on which $T = 0$ and $W_3 = 0$ and consider a subset of $\mathcal{W}$-Diff which can be performed on this coordinate system. It can be constructed using solutions $k^a$ of the extension equations (3.13) by

$$f = k^1 \quad \text{and} \quad g = k^2 - \frac{1}{2}(k^1)^2.$$

(4.6)

The Jacobian $J$ of this particular transformation is $K_3$ given in (3.14), and the transformed generators $T$ and $W_3$ take the same form as (3.18). They can be rewritten in a more compact form as

$$T = \frac{3}{2} J^{1/3} (\partial_\tilde{z} - \partial_\tilde{t}^2) J^{-1/3}, \quad W_3 = \frac{1}{2} J^{1/3} (\partial_\tilde{t}^3 - 3 \partial_\tilde{t} \partial_\tilde{z}) J^{-1/3} + \frac{1}{2} \partial_\tilde{t} T.$$

(4.7)

Finite global $\mathcal{W}$-Diff are defined as those leaving the values $T = 0$ and $W_3 = 0$ invariant. They have the general form

$$t = \frac{a \tilde{t} + b(\tilde{z} + \frac{\tilde{r}}{2}) + c}{q \tilde{t} + r(\tilde{z} + \frac{\tilde{r}}{2}) + s}, \quad z + \frac{\tilde{t}^2}{2} = \frac{m \tilde{t} + n(\tilde{z} + \frac{\tilde{r}}{2}) + p}{q \tilde{t} + r(\tilde{z} + \frac{\tilde{r}}{2}) + s},$$

$$\begin{pmatrix} a & b & c \\ m & n & p \\ q & r & s \end{pmatrix} = \text{constant} \in SL(3).$$

(4.8)
The Jacobian of the transformations (4.7) gives vanishing $T$ and $W_3$ in (1.7). This property suggests to consider the expressions (1.7) as the associated $\mathcal{W}$-Schwarzians.

Note that (3.14) is a projective realization of an $SL(3)$ transformation law which is generalizing the projective $SL(2)$ (Möbius) transformation in the standard conformal theories. Therefore it is rather natural to consider the two-dimensional space where we describe $\mathcal{W}_3$ as being $RP^2$.

For the infinitesimal general coordinate transformations $\delta t = \epsilon^t(t,z)$ and $\delta z = \epsilon^z(t,z)$ the conditions of $\mathcal{W}$-Diff (1.3) are expressed as:

$$\partial_z \epsilon^z(t,z) = 2 \partial_t \epsilon^t(t,z) + \partial_t^2 \epsilon^z(t,z), \quad (4.9)$$

$$\partial_z \epsilon^t(t,z) = - \frac{2}{3} T(t,z) \partial_t \epsilon^z(t,z) - \frac{2}{3} \partial_t \partial_z \epsilon^z(t,z) + \frac{1}{3} \partial_t^2 \epsilon^t(t,z). \quad (4.10)$$

The infinitesimal $\mathcal{W}$-Diff transformations of the extended $\mathcal{W}_3$-generators are

$$\delta T = \epsilon^z \partial_z T + \epsilon^t \partial_t T + T \partial_z \epsilon^z + \partial_t^2 \epsilon^z + 3 V_3 \partial_t \epsilon^z - 2 \partial_t \partial_z \epsilon^t$$

$$\delta V_3 = \epsilon^z \partial_z V_3 + \epsilon^t \partial_t V_3 + T \partial_z \epsilon^z + 2 V_3 \partial_t \epsilon^z - \partial_t^2 \epsilon^t - V_3 \partial_t \epsilon^t. \quad (4.11)$$

To see the relations of these transformations with the standard $\mathcal{W}_3$ transformations in one dimension we express all $z$-derivatives in terms of $t$-derivatives using (3.10), (3.11), (1.9) and (4.10). The resulting transformations are:

$$\delta T(t,z) = \alpha \partial_t T + 2 \partial_t \alpha T + 2 \partial_t^2 \alpha + 2 \rho \partial_t W_3 + 3 \partial_t \rho W_3,$$

$$\delta W_3(t,z) = \alpha \partial_t W_3 + 3 \partial_t \alpha W_3 - \rho \left( \frac{2 T \partial_t T}{3} + \frac{\partial_t^2 T}{6} \right) - \partial_t \rho \left( \frac{2 T^2}{3} + \frac{\partial_t^2 T}{4} \right)$$

$$- \frac{1}{6} \partial_t^2 \rho \partial_t T - \frac{1}{3} \partial_t \partial_t \rho T - \frac{1}{6} \partial_t^3 \rho,$$  \quad (4.13)

$$\delta X(t,z) = \alpha \partial_t X - \partial_t \alpha X + \rho \left( \partial_t^2 X + \frac{2}{3} T X \right) - \frac{1}{3} \partial_t \rho \partial_t X + \frac{1}{3} \left( \partial_t^2 \rho \right) X.$$  \quad (4.14)

where $\alpha(t,z) \equiv \epsilon^t(t,z) + \frac{1}{2} \partial_t \epsilon^z(t,z)$ and $\rho(t,z) \equiv \epsilon^z(t,z)$.

The transformations (4.13)-(4.15) are reduced to the classical infinitesimal $\mathcal{W}_3$ transformations by putting $z = 0$ and considering $\alpha(t,0)$ and $\rho(t,0)$ as arbitrary transformation functions of $t$. The parameter $\alpha$ generates $t$ diffeomorphism transformations under which $T$, $W_3$ and $X$ transform respectively as weight 2 quasi-primary, weight 3 and weight $-1$ primary fields. The transformations generated by $\rho$ are the well-known $\mathcal{W}_3$ transformations.

The algebra of two infinitesimal $\mathcal{W}$-Diff is given by

$$[\delta_1, \delta_2] = \delta_3, \quad \epsilon_3^\mu = \epsilon_2^\nu \partial_\nu \epsilon_1^\mu - \epsilon_1^\nu \partial_\nu \epsilon_2^\mu + \delta_1 \epsilon_2^\mu - \delta_2 \epsilon_1^\mu.$$ \quad (4.16)
Here the last two terms of $\epsilon_3$ are contributions coming from the $T$ dependence of $\epsilon$’s through (4.9) and (4.10). They satisfy, for example:

$$
\partial_z (\delta \epsilon^t) = -\frac{2}{3} (T \partial_t (\delta \epsilon^z) + (\delta T) \partial_t \epsilon^z) - \frac{2}{3} \partial_t \partial_z (\delta \epsilon^z) + \frac{1}{3} \partial_t^2 (\delta \epsilon^t).
$$

(4.17)

Taking this into account it can be shown that the transformation with parameter $\epsilon_3$ is also a $\mathcal{W}$-Diff because it satisfies equations (4.9) and (4.10). Therefore we can say that the $\mathcal{W}$-Diff have a composition law, at least locally, forming a quasi-group.

The existence of a composition law enables us to define the $\mathcal{W}$-surface in general. Let us consider the manifold $M = \mathbb{RP}^2$ with a flat affine connection $\Gamma$. We define a $\mathcal{W}$-neighborhood as an ordinary one supplemented with the conditions $\Gamma^t_{tt} = 1$ and $\Gamma^t_{tt} - 2 \Gamma^z_{tz} = 0$. These conditions single out some special parametrizations of each patch of $M$. General two-dimensional Diff will not preserve these conditions but $\mathcal{W}$-Diff, as defined above, do preserve them. Hence we can define the $\mathcal{W}$-surface as a collection of $\mathcal{W}$-neighborhoods patched together by $\mathcal{W}$-Diff.

## 5 Conclusion

We have studied $\mathcal{W}_3$ transformations in the language of Riemannian geometry. The $\mathcal{W}$-generators are given as particular combinations of the Christoffel symbols in a suitable gauge. We have seen that $\mathcal{W}$-Diff preserving the gauge-fixing conditions depend in general on $T$ and $W_3$. In general the finite transformations of $T$ and $W_3$ are expressed as non-linear functions of the fields. Explicit expressions for the finite global transformations and the $\mathcal{W}$-Schwarzians have been given. We have also indicated how to recover the well-known one-dimensional $\mathcal{W}_3$-transformations from our results.

It is desirable that the gauge-fixing condition (3.1) is interpreted geometrically in order to have a better understanding of $\mathcal{W}$-surfaces. We would like to emphasize that the definition of $\mathcal{W}$-Diff (4.3) given above may be valid for more general cases and we believe that it will be useful, for example, in the study of the classical limit of the $\mathcal{W}_3$ string theory.

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