THE BASIN OF INFINITY OF TAME POLYNOMIALS

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Abstract. Let \( \mathbb{C}_v \) be a characteristic zero algebraically closed field which is complete with respect to a non-Archimedean absolute value. We provide a necessary and sufficient condition for two tame polynomials in \( \mathbb{C}_v[z] \) of degree \( d \geq 2 \) to be analytically conjugate on their basin of infinity. In the space of monic centered polynomials, tame polynomials with all their critical points in the basin of infinity form the tame shift locus. We show that a tame map \( f \in \mathbb{C}_v[z] \) is in the closure of the tame shift locus if and only if the Fatou set of \( f \) coincides with the basin of infinity.

Contents

1. Introduction 1
2. Preliminaries 4
3. Extendable and Analytic Conjugacies 10
4. Existence of extendable conjugacies 16
5. Polynomials with Julia critical points 21
Appendix A. A local lemma 24
References 25

1. Introduction

Let \( \mathbb{C}_v \) be a characteristic 0 algebraically closed field with residue characteristic \( p \geq 0 \) which is complete with respect to a non-Archimedean absolute value \( |\cdot| \), that is assumed to be non-trivial. We regard nonconstant polynomials \( f \in \mathbb{C}_v[z] \) as dynamical systems acting on the Berkovich analytic line \( A^1_{an} \) over \( \mathbb{C}_v \). The dynamics of a degree \( d \geq 2 \) polynomial \( f \) partitions \( A^1_{an} \) into two sets: the basin of infinity \( B(f) \) consisting of all \( x \in A^1_{an} \) with unbounded orbit and, its complement, the filled Julia set \( K(f) \), which is compact. The theme of this paper is to investigate when the actions of two polynomials on their basins of infinity are analytically conjugate. However, in this introduction we start discussing some consequences of our investigation and finish describing directly related results.

It is well known that the dynamics of a polynomial \( f \) is strongly influenced by the dynamical behavior of points \( x \) where \( f \) fails to be locally injective. The set of such points \( x \in A^1_{an} \) is called the ramification locus \( R(f) \subset A^1_{an} \) (c.f. [Fab13b]). We will focus on the more tractable class of tame polynomials consisting on maps \( f \) for which \( R(f) \) is a locally finite tree (c.f. [Tru14]). We say that a tame polynomial lies in the tame shift locus if \( R(f) \) is contained in the basin of infinity, equivalently, if the critical points of \( f \) are contained in the basin of infinity. In this case, the unique Fatou component of \( f \) is the basin of infinity and the Julia set \( J(f) := \partial K(f) \) is formed by type I points. That is, \( J(f) \) is contained in the (classical) affine line \( A^1 \). Moreover, \( f : J(f) \rightarrow J(f) \) is topologically conjugate to the one-sided shift on \( d \) symbols, (c.f. [Kiw06 Theorem 3.1]).

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In parameter space, for simplicity, we work in the space $\text{Poly}_d$ of *monic and centered polynomials of degree $d$* which, via coefficients, is naturally identified with $\mathbb{C}_v^d$. Unless otherwise stated, we work with the metric topology in $\text{Poly}_d$ induced by the (sup) norm on $\mathbb{C}_v^d$. Our investigation regarding the basin of infinity will allow us to characterize the location in parameter space of tame polynomials whose basin of infinity agree with the Fatou set or, equivalently, the Julia set is contained in the classical affine line $\mathbb{A}^1$.

**Theorem A.** Consider a tame polynomial $f \in \text{Poly}_d$. Then $\mathcal{J}(f) \subset \mathbb{A}^1$ if and only if $f$ is in the closure of the tame shift locus.

Note that each polynomial in the tame shift locus is expanding on its Julia set (c.f. [Ben01, Lee19]). Hence we show that tame polynomials with $\mathcal{J}(f) \subset \mathbb{A}^1$ can be perturbed to maps which, in a certain sense, have hyperbolic (expanding) dynamics over their Julia sets.

When a tame polynomial $f$ is in the closure of the tame shift locus, it is not difficult to deduce from Trucco’s work [Tru14] that $\mathcal{J}(f) \subset \mathbb{A}^1$. Thus, our contribution is to show that Julia critical points (if any) of a tame polynomial $f$ with $\mathcal{J}(f) \subset \mathbb{A}^1$ simultaneously become escaping under an appropriate arbitrarily small perturbation of $f$. We say that a Julia critical point $c \in \mathcal{J}(f)$ is *active at $f$* if there exists a polynomial $g$ arbitrarily close to $f$ with a critical point $c'$ close to $c$ such that $c'$ lies in the basin of infinity under iterations of $g$. (We must warn the reader that our definition of active/passive critical points differs from the one by Irokawa [Iro19].)

**Corollary B.** Assume that $f \in \text{Poly}_d$ is a tame polynomial such that $\mathcal{J}(f) \subset \mathbb{A}^1$. If a critical point $c$ lies in the Julia set $\mathcal{J}(f)$, then $c$ is active at $f$.

Without any assumption on $f \in \text{Poly}_d$, we conjecture that if a critical point $c$ lies in $\mathcal{J}(f)$, then $c$ is active at $f$. That is, we conjecture that bifurcations must occur at maps with Julia critical points. On the other hand, $J$-stability and “hyperbolicity” results for maps whose Julia set is critical point free have been obtained by Benedetto and Lee [Ben01, Lee19, BL22], and T. Silverman [Sil17, Sil19]. Although useful, analogies with open problems and results in complex dynamics should be taken with extreme caution. In fact, the hypothesis that $c$ is a Julia critical point is *stronger* in the non-Archimedean context than in the complex setting since it immediately implies a control on the geometry around $c$ (i.e. there exists a sequence of nested Fatou annuli around $c$ with divergent sum of moduli). So although the “analogue” conjecture in complex dynamics has proven to be hard and elusive, even for quadratic polynomials, it might be the case that establishing activity for Julia critical points is more accessible in the non-Archimedean setting.

To study perturbations of a given polynomial it is convenient to consider analytic families of critically marked monic and centered polynomials $\{(f_\lambda, c_1(\lambda), \ldots, c_{d-1}(\lambda))\}$ parametrized by a disk $\Lambda \subset \mathbb{C}_v$ (see §2.3). In contrast with T. Silverman’s work, for our purpose, we only consider parameters $\lambda \in \mathbb{C}_v$ and polynomials with coefficients in $\mathbb{C}_v$, that is we do not work with (non-classical) parameters in the Berkovich disk associated to $\Lambda$ (c.f. [Sil17, Sil19]). A critical point $c_i(\lambda)$ is called *passive in $\Lambda$* if either $c_i(\lambda) \in \mathcal{B}(\lambda) := \mathcal{B}(f_\lambda)$ for all $\lambda \in \Lambda$ or $c_i(\lambda) \notin \mathcal{B}(\lambda)$ for all $\lambda \in \Lambda$. Analytic conjugacies between actions on basins of infinity must respect critical orbits and agree, up to normalization, with a conjugacy furnished near infinity by the Böttcher coordinates. Thus, it is convenient to prescribe the locations of the critical points in $\mathcal{B}(\lambda)$ with the aid of the Böttcher coordinate $\phi_\lambda$ which conjugates $f_\lambda$ with $z \mapsto z^d$ in a neighborhood of $\infty$ (see §2.2). We say that the *Böttcher coordinate of a critical point $c_i(\lambda) \in \mathcal{B}(\lambda)$ is constant in $\Lambda* if $\phi_\lambda(f_\lambda^n(c_i(\lambda)))$ is a constant function of $\lambda \in \Lambda$ for some $n$ sufficiently large. Along analytic families with passive critical points having constant Böttcher coordinates, the analytic dynamics in the basin of infinity is constant:
**Theorem C.** Consider an analytic family \( \{ (f_\lambda, c_1(\lambda), \ldots, c_{d-1}(\lambda)) \} \) of critically marked tame monic and centered polynomials parametrized by an open disk \( \Lambda \subset \mathbb{C}^v \). Assume that all critical points are passive and that the Böttcher coordinates of escaping critical points are constant in \( \Lambda \). Then for all \( \lambda_1, \lambda_2 \in \Lambda \), the maps \( f_{\lambda_1} : \mathcal{B}(\lambda_1) \to \mathcal{B}(\lambda_1) \) and \( f_{\lambda_2} : \mathcal{B}(\lambda_2) \to \mathcal{B}(\lambda_2) \) are analytically conjugate. Moreover, if in addition, \( \mathcal{J}(f_{\lambda_0}) \subset \mathbb{A}^1 \) for some \( \lambda_0 \in \Lambda \), then \( f_{\lambda} = f_{\lambda_0} \) for all \( \lambda \in \Lambda \).

The moreover part of the statement above is a consequence of the fact from analytic geometry that compact subsets of \( \mathbb{C}^v \) are analytically removable (see Theorem [2.1] or [FvdP04, Proposition 2.7.13]). In complex polynomial dynamics DeMarco and Pilgrim [DP11] studied the map which assigns to each element of moduli space its dynamics on the basin of infinity (modulo analytic conjugacy). In this language, removability implies that, for maps in the closure of the tame shift locus, the analytic dynamics on their basins of infinity determines the map uniquely (modulo affine conjugacy). For a general \( f_0 \in \text{Poly}_d \) it would be interesting to describe the locus of maps \( f \in \text{Poly}_d \) whose action on \( \mathcal{B}(f) \) is analytically conjugate to \( \mathcal{B}(f_0) \) (c.f. [DP11]).

Given a tame polynomial \( f \), we define the dynamical core of \( f \) as the smallest forward invariant set \( \mathcal{A}_f \) containing the non-classical ramification points in \( \mathcal{B}(f) \). It is not difficult to show that \( \mathcal{A}_f \) is a locally finite tree. Then, given any pair of tame polynomials \( f \) and \( g \) in \( \text{Poly}_d \) we introduce the notion of an extendable conjugacy \( h : \mathcal{A}_f \to \mathcal{A}_g \). Loosely speaking, an extendable conjugacy is a diameter preserving isometric conjugacy that is locally a translation and agrees with a Böttcher coordinate change near infinity. For a precise definition see §3.1. Analytic geometry will imply that, modulo a root of unity, an analytic conjugacy \( \varphi : \mathcal{B}(f) \to \mathcal{B}(g) \) restricts to an extendable conjugacy \( h : \mathcal{A}_f \to \mathcal{A}_g \) and, conversely, via an analytic continuation argument we prove that extendable conjugacies upgrade to analytic conjugacies:

**Theorem D.** Let \( f, g \in \text{Poly}_d \) be tame polynomials. There exists an extendable conjugacy between \( f : \mathcal{A}_f \to \mathcal{A}_f \) and \( g : \mathcal{A}_g \to \mathcal{A}_g \) if and only if there exists an analytic conjugacy between \( f : \mathcal{B}(f) \to \mathcal{B}(f) \) and \( g : \mathcal{B}(g) \to \mathcal{B}(g) \) which extends the Böttcher coordinate change.

Let us now give an overview of the paper.

Section 2 is devoted to preliminaries. We start summarizing basic facts and notation related to the Berkovich affine line \( \mathbb{A}^1_\mathbb{C} \) in §2.1. The space of monic and centered polynomials \( \text{Poly}_d \) is introduced in §2.2. Here we also discuss basic (dynamical) objects associated to a given polynomial such as the base point and Böttcher coordinates, and introduce the dynamical core. Analytic families of polynomials is the topic discussed in §2.3 with emphasis on definitions such as passive/active critical points and constant Böttcher coordinates.

Section 3 is devoted to the proof of Theorem D. Extendable conjugacies are defined in §3.1. In §3.2, one direction of Theorem D is established, namely that analytic conjugacies restrict to extendable conjugacies between dynamical cores (Corollary 3.7). The other direction is obtained via an “analytic continuation” argument in §3.3 which employs a lemma proven in Appendix A.

In Section 4 we prove a slightly stronger version of Theorem C which relies on the notion of \( \rho \)-close Böttcher coordinates. The stronger version of Theorem C will be needed to establish Theorem A. Intuitively, given two polynomials \( f, g \) we introduce \( \rho \in [0, \infty] \) to quantify how close are the Böttcher coordinates of escaping critical points. With this...
2. Preliminaries

For future reference we introduce basic definitions, notation and results regarding Berkovich space and analytic maps in §2.1 regarding the dynamics of monic and centered polynomials in §2.2 and, regarding analytic families of such polynomials in §2.3.

2.1. Berkovich space and analytic maps. The elements of $\mathbb{A}^1_{an}$ are the multiplicative seminorms in $\mathbb{C}_v[z]$ that extend the absolute value $| \cdot |$ on $\mathbb{C}_v$. As customary we write elements of $\mathbb{A}^1_{an}$ simply by $x$ and, when necessary, the corresponding seminorm of $f(z) \in \mathbb{C}_v[z]$ is denoted by $\| f(z) \|_x$. Also, to ease notations, let $| x | := \| z \|_x$. Via the identification of $x \in \mathbb{A}^1$ with the seminorm $\| f(z) \|_x = | f(x) |$, we may regard $\mathbb{A}^1$ as a subset of $\mathbb{A}^1_{an}$. Most of our work occurs in $\mathbb{A}^1_{an}$; however sometimes it is convenient to regard $\mathbb{A}^1_{an}$ as naturally embedded in the Berkovich projective line $\mathbb{P}^1_{an}$ which (set theoretically) is obtained by adding one point, denoted by $\infty$, to $\mathbb{A}^1_{an}$. The default topology for $\mathbb{A}^1_{an}$ and $\mathbb{P}^1_{an}$ will be the Gel’fand topology. With this topology $\mathbb{A}^1_{an}$ is locally compact and $\mathbb{P}^1_{an}$ is the one-point compactification of $\mathbb{A}^1_{an}$. Moreover, $\mathbb{A}^1$ and $\mathbb{P}^1$ are dense in $\mathbb{A}^1_{an}$ and $\mathbb{P}^1_{an}$, respectively. We refer the reader to [BR10, Ben19] for general background on the Berkovich affine and projective line specially adapted to one dimensional non-Archimedean dynamics.

Given $x \in \mathbb{A}^1_{an}$, let

$$\overline{D}_x := \{ y \in \mathbb{A}^1_{an} : \| f(z) \|_y \leq \| f(z) \|_x, \forall f(z) \in \mathbb{C}_v[z] \}.$$  

According to Berkovich’s classification of seminorms, $\overline{D}_x \cap \mathbb{A}^1$ is a (possibly degenerate) $\mathbb{C}_v$-closed disk of radius $r \geq 0$ or the empty set. If $r = 0$ (resp. $r \in | \mathbb{C}_v^\times |$, $r \notin | \mathbb{C}_v^\times |$), then $x$ is called a type I (resp. II, III) point. When $\overline{D}_x \cap \mathbb{A}^1$ is empty, $x$ is called a type IV point. The Gauss point, denoted by $x_G$, is the unique point whose associated disk $\overline{D}_{x_G}$ is $\{ z \in \mathbb{C}_v : | z | \leq 1 \}$. It follows that $x_G$ is of type II.

Any point in $\mathbb{A}^1_{an}$ can be represented by the cofinal equivalence class of a decreasing sequence of $\mathbb{C}_v$-closed disks, so $\mathbb{A}^1_{an}$ is equipped with a partial order $\prec$ that extends the inclusion relation on the $\mathbb{C}_v$-closed disks, in particular, for $x_1, x_2 \in \mathbb{A}^1_{an}$ of type I, II or III, $x_1 \prec x_2$ if and only if $\overline{D}_{x_1} \cap \mathbb{A}^1 \subset \overline{D}_{x_2} \cap \mathbb{A}^1$. The partial order $\prec$ extends to $\mathbb{P}^1_{an}$ by declaring $\infty$ to be the unique maximal element in $\mathbb{P}^1_{an}$. This partial order $\prec$ endows $\mathbb{P}^1_{an}$ and $\mathbb{A}^1_{an}$ with a tree structure. Given two distinct points $y_1, y_2 \in \mathbb{P}^1_{an}$, denote by $[y_1, y_2]$ (resp. $| y_1, y_2 |$) the closed (resp. open) segment in $\mathbb{P}^1_{an}$ connecting $y_1$ and $y_2$.

For $a \in \mathbb{C}_v$ and $r > 0$, the Berkovich open and closed disks with radii $r$ containing $a$, respectively are

$$D(a, r) := \{ x \in \mathbb{A}^1_{an} : \| z - a \|_x < r \},$$

$$\overline{D}(a, r) := \{ x \in \mathbb{A}^1_{an} : \| z - a \|_x \leq r \}.$$
For any \( x \in \mathbb{A}^1_{an} \), after defining the diameter of \( x \) by
\[
\text{diam}(x) := \inf \{|z - a| : a \in \mathbb{C}_v\},
\]
we have \( D_x = \overline{D}(a, \text{diam}(x)) \) for all \( a \in \overline{D}_x \cap A^1 \) if \( x \) is of type I, II, or III, and therefore \( \text{diam}(x) \) coincides with the diameter \( r \) of \( \overline{D}_a \cap A^1 \); in this case we write \( x = x_{a,r} \). If \( x \) is of type IV, then \( \text{diam}(x) > 0 \), while \( \overline{D}_x \cap A^1 \) is empty.

The space \( \mathbb{H} := \mathbb{A}^1_{an} \backslash A^1 \) is equipped with a metric \( \text{dist}_\mathbb{H} \) called the hyperbolic metric which is invariant under affine transformations. With respect to this metric \( \mathbb{H} \) is an \( \mathbb{R} \)-tree. The subspace topology in \( \mathbb{H} \) is strictly coarser than the metric topology. We normalize \( \text{dist}_\mathbb{H} \) so that
\[
\text{dist}_\mathbb{H}(x_0,r,x_0,s) = \log(s/r)
\]
for all \( 0 < r < s \).

We say that a closed and connected subset \( \mathbb{A}^1_{an} \) or \( \mathbb{P}^1_{an} \) is a subtree. Unless explicitly stated, a subtree is equipped with neither vertices nor edges. However, for a subtree \( \Gamma \subset \mathbb{A}^1_{an} \), the valence of \( \Gamma \) at \( x \in \Gamma \) is the number (maybe \( \infty \)) of connected components of \( \Gamma \setminus \{x\} \). We denote by \( \partial \Gamma \) the set of points in \( \Gamma \) having valence 1. A nonempty connected subset \( S \subset \mathbb{A}^1_{an} \) is an open subtree if \( S = \overline{S}\setminus \partial \mathbb{S} \); so in this case \( S \) is contained in \( \mathbb{H} \) and any point in \( S \) has valence at least 2, moreover, for simplicity, set \( \partial S := \partial \mathbb{S} \). An (open) subtree in \( \mathbb{A}^1_{an} \) is locally finite if each point has finite valence and a neighborhood containing finitely many branch points of this (open) subtree. Given an open and connected set \( U \subseteq \mathbb{A}^1_{an} \), the skeleton of \( U \), denoted by \( \text{sk}U \), is the set formed by all \( x \in U \) that separate the complement of \( U \) in \( \mathbb{P}^1_{an} \). It follows that \( \text{sk}U \) is a locally finite open subtree in \( \mathbb{H} \).

Given a point \( x \in \mathbb{A}^1_{an} \), a connected component of \( \mathbb{A}^1_{an} \setminus \{x\} \) corresponds to a direction \( \vec{v} \) at \( x \). Denote by \( D_x(\vec{v}) \) the component of \( \mathbb{A}^1_{an} \setminus \{x\} \) corresponding to the direction \( \vec{v} \) at \( x \). When the point \( x \) is clear from context or irrelevant, we will sometimes write \( D(\vec{v}) \) for \( D_x(\vec{v}) \). The set of all directions at \( x \) forms the space of directions or (projectivized) tangent space at \( x \), denoted by \( T_x\mathbb{A}^1_{an} \). It will also be convenient to denote the unique direction at \( x \) containing \( z \in \mathbb{A}^1_{an} \) by \( \vec{v}_x(z) \) and denote by \( \vec{v}_x(\infty) \) the unique direction at \( x \) such that \( D(\vec{v}_x(\infty)) \) is unbounded. At type I and IV points, the tangent space is a singleton and at type III, consists of two directions. However, at type II points the tangent space is naturally endowed with the structure of a projective line over the residue field of \( \mathbb{C}_v \) with a distinguished point \( \infty \).

Given an open set \( U \subset \mathbb{A}^1_{an} \), an analytic map \( \psi : U \to \mathbb{A}^1_{an} \) is a morphism between the corresponding analytic structures. Analytic maps can be described in a more concrete manner as follows. An affinoid \( X \) in \( \mathbb{A}^1 \) is a subset of the form \( \overline{B}(a,r) \cup B(a_i, r_i) \) where \( r, r_i \in |\mathbb{C}_v^\times| \) and \( a, a_i \in A^1 \). The corresponding Banach \( \mathbb{C}_v \)-algebra \( \mathcal{A}(X) \) is formed by the uniform limits of rational maps with poles outside \( X \) endowed with the sup-norm \( \| \cdot \|_X \) on \( X \). Then \( X_{an} \) is the space of all multiplicative seminorms on \( \mathcal{A}(X) \) bounded by \( \| \cdot \|_X \) that agree with \( | \cdot | \) on the constants, endowed with the Gel’fand topology. Restriction to \( \mathbb{C}_v[z] \) identifies \( X_{an} \) with \( \overline{D}(a,r) \cup D(a_i, r_i) \subset \mathbb{A}^1_{an} \). Analytic maps \( \psi : X_{an} \to \mathbb{A}^1_{an} \) are naturally identified with \( \mathcal{A}(X) \). That is, given \( \varphi \in \mathcal{A}(X) \), \( x \in X_{an} \) and \( q(z) \in \mathbb{C}_v[z] \), then \( \| q(z) \|_{\psi(x)} = \| q(\varphi(z)) \|_x \) defines a morphism \( \varphi : X_{an} \to \mathbb{A}^1_{an} \). It follows that for any given arbitrarily large hyperbolic ball in \( \mathbb{H} \), the action of \( \varphi : X_{an} \to \mathbb{A}^1_{an} \) agrees with that of a rational map with poles outside \( X \). Given an open set \( U \subset \mathbb{A}^1_{an} \), we have that \( \psi : U \to \mathbb{A}^1_{an} \) is analytic if its restriction to every affinoid \( X \) contained in \( U \) is analytic. Assuming that \( U \) is connected, nonconstant analytic maps \( \psi : U \to \mathbb{A}^1_{an} \) are open maps with isolated zeros. Therefore, analytic maps that coincide on an open subset of \( U \) are equal. Furthermore, we have the following analytic removability result (c.f [PvdP04 Proposition 2.7.13]):
Theorem 2.1. Assume that $K \subset \mathbb{A}^1$ is a compact set contained in a disk $D \subset \mathbb{A}^1_{an}$. If $f : D \setminus K \to \mathbb{A}^1_{an}$ is a bounded analytic map, then $f$ extends to a analytic map $\bar{f} : D \to \mathbb{A}^1_{an}$.

Assume that $U$ is an open set containing $x \in \mathbb{A}^1_{an}$ and $\psi : U \to \mathbb{A}^1_{an}$ is a non-constant analytic map. Then $\psi$ has a well defined local degree at every $x \in U$ which we denote by $\deg_x \psi$. Write $y := \psi(x)$, and define the tangent map $T_x \psi : T_x \mathbb{A}^1_{an} \to T_y \mathbb{A}^1_{an}$ of $\psi$ at $x$ by the property that there exists a neighborhood $V$ of $x$ such that $\psi(D_x(\bar{v}) \cap V) \subset D_y(T_x \psi(\bar{v}))$ for any $\bar{v} \in T_x \mathbb{A}^1_{an}$. The direction $T_x \psi(\bar{v}) \in T_y \mathbb{A}^1_{an}$ exists and it is independent of the choice of $V$. When $x$ is a type II point, the map $T_x \psi$ is a non-constant rational map over the residue field of $\mathbb{C}_v$. Moreover, the local degree $\deg_x \psi$ of $\psi$ at $x$ coincides with the degree of $T_x \psi$.

An analytic map is piecewise linear with respect to the hyperbolic metric. The scaling factors or “slopes” are determined by the local degrees (c.f. [BR10] Theorem 9.35):

**Lemma 2.2.** Let $\psi$ be a non-constant analytic map on an open set $U \subset \mathbb{A}^1_{an}$. If $\psi$ has constant local degree on a segment $[x_1, x_2] \subset U$, then for any $x \in [x_1, x_2]$,

$$\text{dist}_H(\psi(x_1), \psi(x_2)) = \deg_x \psi \cdot \text{dist}_H(x_1, x_2).$$

2.2. Polynomial dynamics. We denote by $\text{Poly}_d$ the space of monic and centered polynomials of degree $d \geq 2$ which is naturally identified via coefficients with $\mathbb{C}_v^{d-1}$, since the elements of $\text{Poly}_d$ are of the form

$$f(z) = z^d + a_{d-2}z^{d-2} + \cdots + a_0 \in \mathbb{C}_v[z].$$

Observe that every degree $d$ polynomial is affinely conjugate to a unique element of $\text{Poly}_d$, modulo conjugacy given by multiplication by a $(d-1)$-th root of unity.

For $f$ as above,

$$f'(z) = d \cdot \prod_{i=1}^{d-1} (z - c_i),$$

where $\sum_{i=1}^{d-1} c_i = 0$ and we say that $(f, c_1, \ldots, c_{d-1})$ is a critically marked polynomial. In fact, $\text{Crit}(f) := \{c_1, \ldots, c_{d-1}\}$ is the set of critical points of $f$. Two critically marked polynomials $(f, c_1, \ldots, c_{d-1})$ and $(g, c_1', \ldots, c_{d-1}')$ are affinely conjugate if there exists an affine map $A$ such that $A \circ f = g \circ A$ and $A(c_i) = c_i'$ for all $i$. When the marking is clear from context, sometimes we simply say that $f$ is a critically marked polynomial.

Recall that the ramification locus of $f \in \text{Poly}_d$ is

$$\mathcal{R}(f) := \{x \in \mathbb{A}^1_{an} : \deg_x f \geq 2\}.$$

According to Faber [Fab13a] this set is unbounded, connected and $\text{Crit}(f) = \mathcal{R}(f) \cap \mathbb{A}^1$. If $\mathcal{R}(f) \cap \mathbb{H}$ is a locally finite open subtree, then we say that $f$ is a tame polynomial. In this case, $\mathcal{R}(f) \cup \{\infty\}$ is the convex hull of $\text{Crit}(f) \cup \{\infty\}$ in $\mathbb{P}^1_{an}$. Let $p$ be the residue characteristic of $\mathbb{C}_v$. If $p = 0$, then all polynomials are tame. If $p \neq 0$, then $f$ is tame if and only if for all $x \in \mathbb{A}^1_{an}$, we have $p \nmid \deg_x f$ or, equivalently, the degree of $f : D \mapsto f(D)$ is not divisible by $p$ for all disks $D \subset \mathbb{A}^1_{an}$. A thorough study of $\mathcal{R}(f)$ is contained in [Fab13a, Fab13b].

For tame polynomials, we have the following Riemann-Hurwitz formula:

**Lemma 2.3** ([Tru14 Propositions 2.6 and 2.8]). Let $f \in \text{Poly}_d$ be a tame polynomial. For any $x \in \mathbb{A}^1_{an}$,

$$\deg_x f - 1 = \sum_{z \in D_x(\cap \text{Crit}(f))} (\deg_z f - 1).$$
From \cite{Fab13} we deduce below that perturbations preserving multiplicities of critical points preserve tameness. Thus, sometimes it will be convenient to work, in the corresponding parameter space. Namely, consider \( k \geq 1 \) and \( \mathbf{d} := (d_1, \ldots, d_k) \) where \( d_i \geq 2 \) for all \( i \) and
\[
d_1 + \cdots + d_k - k = d - 1.
\]
We denote by \( \text{Poly}(\mathbf{d}) \) the space formed by all the pairs \((f, c_1, \ldots, c_k)\) where \( f \in \text{Poly}_d \), \( c_i \in \mathbb{A}^1 \) and \( \text{deg}_{c_i} f = d_i \) for all \( i = 1, \ldots, k \). The space \( \text{Poly}(\mathbf{d}) \) is naturally identified via \((f, c_1, \ldots, c_k) \mapsto (c_1, \ldots, c_k, f(0))\) with the space formed by the elements \((c_1, \ldots, c_k, b)\) of \( \mathbb{C}^k \times \mathbb{C}_v \) such that \( \sum d_i c_i = 0 \) and \( c_i \neq c_j \) if \( i \neq j \). Sometimes we abuse of notation and simply say that \( f \in \text{Poly}(\mathbf{d}) \) and denote by \( c_i(f) \) the corresponding marked critical points.

**Lemma 2.4.** Tame polynomials are an open (possibly empty) subset of \( \text{Poly}(\mathbf{d}) \) for any \( \mathbf{d} \).

**Proof.** Let us assume that there exists a tame polynomial in \( \text{Poly}(\mathbf{d}) \). First observe that if \( k = 1 \), then the elements of \( \text{Poly}(\mathbf{d}) \) are unicritical polynomials which are tame if and only if \( p \nmid d \). So we may assume that \( k \geq 2 \) and let \( f \in \text{Poly}(\mathbf{d}) \) be a tame polynomial. Set \( d_0 = d \) and \( r_i = \min\{|d_i^\ell| : 0 \leq \ell \leq d_i\} \) for \( i = 0, \ldots, k \). Note that \( d_i \geq 2 \) and \( 0 < r_i \leq 1 \). Let \( z_0 \) be a type II point of sufficiently large diameter so that all the critical points of \( f \) are contained in \( D_{z_0} \). From the Riemann-Hurwitz formula, it follows that \( \text{deg}_{z_0} f = d \). Consider an open disk \( B \subset \mathbb{H} \) of finite radius \( R \) centered at \( z_0 \) such that \( B \) contains all the branch points of \( \mathcal{R}(f) \) and for all \( 1 \leq i \leq k \), there exist \( x_i, z_i \in \mathcal{c}_i(f), z_0] \cap B \) such that \( c_i(f) \) is the unique critical point of \( f \) in \( \overline{D}_{x_i} \) and \( x_i < z_i \) with \( \text{diam}(x_i) < r_i \text{diam}(z_i) \). We may also assume that \( R > 1/r_0 \) and pick \( x_0 \in B \) such that \( x_0 > z_0 \) and \( \text{diam}(x_0) > \text{diam}(z_0)/r_0 \). Consider a neighborhood \( \mathcal{U} \subset \text{Poly}(\mathbf{d}) \) of \( f \) such that for all \( g \in \mathcal{U} \) we have \( g(x) = f(x) \) for all \( x \in B \) and \( |c_i(g) - c_i(f)| < \text{diam}(x_i) \) for all \( i \geq 1 \). It follows that \( \mathcal{R}(g) \cap B = \mathcal{R}(f) \cap B \). Moreover, \( c_i(g) < x_i \) for \( i \geq 1 \). We now compute the local degrees of \( g \in \mathcal{U} \) at points in \( \overline{\mathcal{D}}_{x_i} \) applying \cite{Fab13} Section 4. Observing that \( d_i = \text{deg}_{z_i} f = \text{deg}_{x_i} g \) and \( \text{deg}_{c_i(g)} g = d_i \), we conclude that \( \text{deg}_{y} g = d_i \) for all \( y \in [c_i(g), z_i] \). Changing coordinates on the source and target by affine maps, we may assume that \( c_i(g) = 0 = g(c_i(g)) \) and \( z_i = x_G \), which implies that \( x_i = x_{0,s_i} \), with \( s_i < r_i \). Then \( 0 \) is the unique critical point of \( g \) in \( \overline{\mathcal{D}}_{x_G} \) and \( \text{deg}_{y} g = d_i \) for all \( y \in [0, x_G] \). Since \( p \nmid d_i \), the conclusion in \cite{Fab13} Section 4.1 implies that \( \mathcal{R}(g) \cap \overline{\mathcal{D}}_{x_0,s_i} = [0, x_{0,s_i}] \). Let \( D = \mathbb{A}^1 \setminus D_{x_0} \). Via the change of coordinates of the form \( z \mapsto 1/(az) \), we may also conclude that \( \mathcal{R}(g) \cap D = [x_0, \infty[. \) Now let \( V = B \cup \overline{\mathcal{D}} \bigcup \bigcup_{1 \leq i \leq d-1} \overline{\mathcal{D}}_{x_i} \). Then \( \mathcal{R}(g) \subset V \), since \( \mathcal{R}(g) \) is connected. Hence, \( \mathcal{R}(g) \) is the union of the arcs \([c_i(g), \infty[ \) for \( 1 \leq i \leq k \). Therefore, \( g \) is tame for all \( g \in \mathcal{U} \).

**Remark 2.5.** The space \( \text{Poly}(\mathbf{d}) \) may contain no tame polynomials if \( \mathbb{C}_v \) has positive residue characteristic \( p > 0 \). For example, if an entry \( d_i \) of \( \mathbf{d} \) is divisible by \( p \), then all polynomials in \( \text{Poly}(\mathbf{d}) \) are not tame.

Given \( f \in \text{Poly}_d \), the **basin of infinity of \( f \)**
\[
\mathcal{B}(f) := \{ x \in \mathbb{A}^1_{an} : f^{on}(x) \to \infty \}
\]
is an open, connected and unbounded subset of \( \mathbb{A}^1_{an} \). Its complement
\[
\mathcal{K}(f) := \mathbb{A}^1_{an} \setminus \mathcal{B}(f)
\]
is the **filled Julia set of \( f \)** whose boundary
\[
\mathcal{J}(f) := \partial \mathcal{K}(f),
\]
is the Julia set of \( f \). The Fatou set is \( \mathcal{F}(f) := \mathbb{A}^1 \setminus \mathcal{J}(f) \). Every connected component of \( \mathcal{K}(f) \) is either a point or a closed disk. In the latter case, every maximal open disk contained in the component of \( \mathcal{K}(f) \) is a bounded Fatou component.

Let

\[
\mathcal{R}_\infty(f) := (\mathcal{R}(f) \cap \mathcal{B}(f)) \setminus \text{Crit}(f)
\]

be the set formed by the non-classical ramification points in the basin of infinity. We say that the forward invariant set

\[
\mathcal{A}_f := \bigcup_{j \geq 0} f^{\circ j}(\mathcal{R}_\infty(f))
\]

is the dynamical core of \( f \) in \( \mathcal{B}(f) \). If in addition we assume that \( f \) is tame, then after denoting the postcritical set by

\[
\text{Post}(f) := \bigcup_{j \geq 0} f^{\circ j}(\text{Crit}(f))
\]

and considering the open and connected subset of \( \mathbb{A}^1 \)

\[
\mathcal{B}_0(f) := \mathcal{B}(f) \setminus \text{Post}(f),
\]

we have that

\[
\mathcal{A}_f \subset \text{sk} \mathcal{B}_0(f).
\]

Thus, the dynamical core \( \mathcal{A}_f \) of a tame polynomial \( f \) is a locally finite open subtree of \( \mathbb{H} \) and any point in \( \partial \mathcal{A}_f \) is either a postcritical point in \( \mathcal{B}(f) \) or a point in \( \mathcal{J}(f) \cap \mathbb{H} \).

Given a monic and centered polynomial \( f(z) = a_0 + a_1 z + \cdots + z^d \) let

\[
R_f := \max\{1, |a_i|^{1/(d-i)} : i = 0, \ldots, d - 2\}.
\]

Then

\[
x_f := x_{0,R_f}
\]

is called the base point of \( f \). The base point is characterized by the fact that \( \overline{D}_{x_f} \) is the minimal closed Berkovich disk containing \( \mathcal{K}(f) \), see [RL00, Section 6.1]. It follows that \( \text{deg} f = d \) and \( x_f \in \mathcal{R}(f) \). If moreover \( f \) is tame, then every \( c \in \text{Crit}(f) \) is contained in \( \overline{D}_{x_f} \). Keep in mind that \( R_f = |x_f| \).

A polynomial \( f \in \text{Poly}_d \) is simple if its Julia set is a singleton; otherwise, we say \( f \) is nonsimple. For simple \( f \in \text{Poly}_d \), the unique point in \( \mathcal{J}(f) \) is the Gauss point which is also the base point of \( f \) and its dynamical core is the open segment connecting the base point to infinity. Moreover, a tame polynomial is simple if and only if its basin of infinity is critical point free. If \( f \in \text{Poly}_d \) is nonsimple, the base point \( x_f \) lies in \( \mathcal{B}(f) \) and hence in \( \mathcal{A}_f \), see [Tru14, Proposition 4.3].

In an appropriate coordinate near \( \infty \), called the Böttcher coordinate, a degree \( d \) monic and tame polynomial \( f \) acts as the monomial \( z \mapsto z^d \). More precisely, given a tame \( f \in \text{Poly}_d \) there exists an analytic isomorphism,

\[
\phi_f : \mathbb{A}^1 \setminus \overline{D}(0, R_f) \to \mathbb{A}^1 \setminus \overline{D}(0, R_f)
\]

such that \( \phi_f \circ f = \phi_f^d \) and \( \phi_f(z)/z \to 1 \) as \( \mathbb{A}^1 \ni z \to \infty \) (e.g. see [Ing13, DGK+19, SS20]). The germ of \( \phi_f \) at infinity is uniquely determined by the above properties. It follows that \( |\phi_f(z)| = |z| \) for all \( z \in \mathbb{A}^1 \setminus \overline{D}(0, R_f) \). Although \( \phi_f \) does not extend to \( \mathcal{B}(f) \), its absolute value \( |\phi_f| \) extends via the functional equation \( |\phi_f(z)| = |\phi(f(z))|^{1/d} \) to a well defined function on the whole basin of infinity \( \mathcal{B}(f) \). Provided that \( f \in \text{Poly}_d \) is tame and nonsimple, it is not difficult to show that

\[
R_f = \max\{|\phi_f(c)| : c \in \text{Crit}(f) \cap \mathcal{B}(f)\} > 1,
\]
In this case, we say that $T$ where $a = 2.3$. Analytic family.

$C_z$ to a polydisk (e.g. see [FvdP04]). For $n \geq 1$, considering $r = (r_1, \ldots, r_n)$ where $r_i \in |C_v^\times|$ for all $i$, denote by $P(r)$ the rational polydisk in $C_v^n$ formed by all $(\lambda_1, \ldots, \lambda_n) \in C_v^n$ such that $|\lambda_i| \leq r_i$ for all $i$; then the Tate algebra $T_n[r]$ is the one formed by all (formal) power series in $C_v[\lambda_1, \ldots, \lambda_n]$ convergent in $P(r)$. Endowed with the sup norm, the Tate algebra $T_n[r]$ is a Banach algebra over $C_v$. [FvdP04 Theorem 3.2.1]. Recall that $p \geq 0$ denotes the residue characteristic of $C_v$.

**Lemma 2.6.** Assume that $p \not\mid d$ and pick $r = (r_1, \ldots, r_{d-1})$ with $r_i \in |C_v^\times|$ for all $1 \leq i \leq d - 1$. Suppose that there exists $R \in |C_v^\times|$ with $R > 1$ such that $|x_f| < R$ for any $f \in P(r) \subset \text{Poly}_d \cong \mathbb{C}_d^{d-1}$. Then the map

$$\Phi : P(r) \times \{|z| \geq R\} \to \{|z| \geq R\},$$

sending $(f, z)$ to $\phi_f(z)$ is analytic. More precisely, setting $\Phi_\infty(f, z) = \Phi(f, z)/z$ we have $\Phi_\infty(f, 1/z) \in T_d[\overline{r}]$ where $\overline{r} = (r_1, \ldots, r_{d-1}, R^{-1})$.

**Proof.** First it follows from [Ing13, Lemma 8] the map $\Phi$ is well-defined. Now to prove the analyticity, we apply a standard convergence argument as in [Ing13 Lemma 7]. For given $f \in P(r)$ and $|z| \geq R$, define $\Phi_n(f, z)$ to be the $d^n$-th root of $f^\circ_n(z)/z^{d^n}$ such that $\Phi_n(f, z) - 1 \in C_v[z^{-1}]$. According to [Ing13] $\Phi_n(f, z) \to \phi_f(z)/z$ as $n \to \infty$. The lemma will follow after showing that the convergence is uniform.

Since $p \not\mid d$, $|\Phi_{n+1}(f, z) - \Phi_n(f, z)| = \left|\left(\Phi_{n+1}(f, z)\right)^{d^{n+1}} - (\Phi_n(f, z))^{d^{n+1}}\right|$. It follows that

$$|\Phi_{n+1}(f, z) - \Phi_n(f, z)| = \left|\frac{f^{\circ_{n+1}}(z)}{z^{d^{n+1}}} - \left(\frac{f^{\circ_n}(z)}{z^{d^n}}\right)^d\right| = |z^{-d^n}| |f(f^{\circ_n}(z)) - f^{\circ_n}(z)|$$

$$= |z^{-d^{n+1}}| \left| \sum_{j=0}^{d-2} a_j(f) (f^{\circ_n}(z))^j \right|,$$

where $a_j(f)$ is the coefficients of $z^j$ in $f$. Let $\|f\| := \max_{0 \leq j \leq d-2} \{|a_j(f)|\}$ and $M := \sup\{|\|f\| : f \in P(r)\} < \infty$, we conclude that for sufficiently large $n$,

$$|\Phi_{n+1}(f, z) - \Phi_n(f, z)| \leq |z|^{-d^{n+1}} \cdot \|f\| \cdot |z|^{d^n(d-2)} = |z|^{-2d^n} \cdot \|f\| \leq MR^{-2d^n}.$$

Note that $z\Phi_n(f, z) \to \phi_f(z)$ for all $f \in P(r)$ and $|z| \geq R$. Moreover, $\{\Phi_n(f, 1/z)\}$ is a Cauchy sequence in the Banach algebra $T_d[\overline{r}]$ and therefore its limit $\Phi_\infty(f, 1/z)$ is analytic. \qed

2.3. Analytic family. We consider analytic families parametrized by an open disk $\Lambda$ in $\mathbb{C}_v$ of radius in $|C_v^\times|$. A family $\{f_\lambda\}_{\lambda \in \Lambda} \subset \text{Poly}_d$ is analytic if

$$f_\lambda(z) = z^d + \sum_{i=0}^{d-2} a_i(\lambda) z^i$$

where $a_i(\lambda)$ is analytic for all $i$ (i.e. a power series in $\lambda$ which converges for all $\lambda \in \Lambda$). In this case, we say that $\{(f_\lambda, c_1(\lambda), \ldots, c_{d-1}(\lambda))\}$ is a critically marked analytic family if
constant:

\( f \)

dynamical space of polynomials parametrized by a disk

Also we will often omit the markings \( c_i(\lambda) \) from the notation and simply say that \( \{f_\lambda\} \) is a critically marked analytic family implicitly assuming that the markings are \( c_1(\lambda), \ldots, c_{d-1}(\lambda) \).

When \( \{f_\lambda\}_{\lambda \in \Lambda} \subset \text{Poly}_d \) is an analytic family and \( c : \Lambda \to \mathbb{C}_v \) is an analytic function such that \( c(\lambda) \) is a critical point of \( f_\lambda \) for all \( \lambda \) we say that \( c(\lambda) \) is a marked critical point of the family \( \{f_\lambda\} \).

**Definition 2.7.** Let \( \{f_\lambda\}_{\lambda \in \Lambda} \) be an analytic family with a marked critical point \( c(\lambda) \). We say that \( c(\lambda) \) is a passive critical point in \( \Lambda \) if \( \{\lambda \in \Lambda : c(\lambda) \in B_0(\lambda)\} = \Lambda \) or \( \emptyset \). We say that a critically marked analytic family parametrized by \( \Lambda \) is passive in \( \Lambda \) if all its critical points are passive in \( \Lambda \).

We emphasize that our definition of passive is not a local property, it depends on the domain \( \Lambda \) of the analytic family \( \{f_\lambda\} \).

For an analytic family \( \{f_\lambda\}_{\lambda \in \Lambda} \), denote by \( \phi_\lambda : \mathbb{A}_d^1 \setminus \overline{D}(0,R_\lambda) \to \mathbb{A}_d^1 \setminus \overline{D}(0,R_\lambda) \) the Böttcher coordinate of \( f_\lambda \). Given \( \lambda_0, \lambda_1 \in \Lambda \) and \( R = \max\{R_{\lambda_0}, R_{\lambda_1}\} \) we say that \( \phi_{\lambda_1}^{-1} \circ \phi_{\lambda_0} : \mathbb{A}_d^1 \setminus \overline{D}(0,R) \to \mathbb{A}_d^1 \setminus \overline{D}(0,R) \) is the Böttcher coordinate change between the dynamical space of \( f_{\lambda_0} \) and \( f_{\lambda_1} \).

**Definition 2.8.** Consider a passive marked critical point \( c(\lambda) \) in \( B(\lambda) \) of an analytic family \( \{f_\lambda\} \) of tame polynomials parametrized by \( \Lambda \). We say that the Böttcher coordinate of \( c(\lambda) \) is constant if for all \( \lambda_0, \lambda_1 \in \Lambda \), there exists \( n \) such that the corresponding Böttcher coordinate change sends \( f_{\lambda_0}^m(c(\lambda_0)) \) to \( f_{\lambda_1}^m(c(\lambda_1)) \).

As an immediate consequence of (2.1) we have that base points of passive families are constant:

**Lemma 2.9.** Let \( \{f_\lambda\}_{\lambda \in \Lambda} \) be a passive critically marked analytic family of tame polynomials parametrized by a disk \( \Lambda \). Then the base point of \( f_\lambda \) is independent of \( \lambda \in \Lambda \).

**Proof.** We may assume that the maps involved are nonsimple. Pick an element \( \lambda_0 \in \Lambda \) and apply (2.1) to find a critical point \( c_i(\lambda_0) \in B(\lambda_0) \) such that \( R_{\lambda_0} = |\phi_{\lambda_0}|(c_i(\lambda_0)) \). Keep in mind that \( |c_i(\lambda)| \leq R_\lambda \) for all \( \lambda \). Let \( v(\lambda) = f_\lambda(c_i(\lambda)) \). Then \( |v(\lambda)| = R^d_{\lambda_0} \). Observe that \( v(\lambda) - c_i(\lambda) \neq 0 \) for all \( \lambda \), since the critical points are all passive. Therefore, \( |v(\lambda) - c_i(\lambda)| \) has constant value \( R^d_{\lambda_0} \). Thus, for all \( \lambda \),

\[
R_{\lambda_0}^d = |v(\lambda) - c_i(\lambda)| \leq R^d_{\lambda_}\lambda
\]

where the last inequality follows from \( |c_i(\lambda)| \leq R_\lambda \) and \( |v(\lambda)| \leq R^d_\lambda \). Since this occurs for all \( \lambda_0 \in \Lambda \), we have that \( R_\lambda \) is constant. \( \square \)

Observe that if the base point \( x_\Lambda \) of an analytic family \( \{f_\lambda\} \) of tame polynomials is independent of \( \lambda \), then the Böttcher coordinate change \( \phi_{\lambda_0}^{-1} \circ \phi_{\lambda_0}(z) \) is also analytic in \( \Lambda \) for all \( z \) such that \( |z| > |x_\Lambda| \) (see Lemma 2.6). Hence, if a passive marked critical point \( c(\lambda) \) in \( B(\lambda) \) has constant Böttcher coordinate, then the Böttcher coordinate change sends \( f_{\lambda_0}^m(c(\lambda_0)) \) to \( f_{\lambda_1}^m(c(\lambda_1)) \) for all \( m \geq 1 \) such that \( |f_{\lambda_0}^m(c(\lambda_0))| > |x_\Lambda| \).

### 3. Extendable and Analytic Conjugacies

The aim of this section is to discuss the notion of extendable conjugacies and prove Theorem [D].
3.1. **Extendable conjugacies.** We shall investigate conditions under which the conjugacy $\phi_g^{-1} \circ \phi_f$ furnished near $\infty$ by the Böttcher coordinates of two polynomials $f, g \in \text{Poly}_d$ extends to an analytic conjugacy $\varphi : \mathcal{B}(f) \to \mathcal{B}(g)$.

**Definition 3.1.** Consider tame polynomials $f, g \in \text{Poly}_d$ and locally finite unbounded open sub-trees $\mathcal{T}_f, \mathcal{T}_g \subset \mathbb{H}$ such that $f(\mathcal{T}_f) \subset \mathcal{T}_f$ and $g(\mathcal{T}_g) \subset \mathcal{T}_g$. A distance-isometry $h : \mathcal{T}_f \to \mathcal{T}_g$ is an **extendable conjugacy** if

1. (conjugacy) $h \circ g = f \circ h$,
2. (Böttcher coordinate change) $\phi_g^{-1} \circ \phi_f(x) = h(x)$ for all $x \in \mathcal{T}_f$ in a neighborhood of $\infty$, and
3. (locally a translation) for all $x \in \mathcal{T}_f$, there exists a translation $\tau_x(z) = z + b_x$ such that $\tau_x(y) = h(y)$ for all $y \in \mathcal{T}_f$ in a neighborhood of $x$.

The following result implies that an extendable conjugacy has a well-defined tangent map.

**Lemma 3.2.** Let $\mathcal{T}_0, \mathcal{T}_1$ be two locally finite open sub-trees in $\mathbb{H}$ and let $i : \mathcal{T}_0 \to \mathcal{T}_1$ be locally a translation. Consider $x \in \mathcal{T}_0$ and two translations $\tau$ and $\tau'$ which agree with $i$ in a neighborhood of $x$. Then $T_x \tau = T_x \tau'$.

**Proof.** Noting that the valence of $\mathcal{T}_0$ at $x$ is at least 2, since $\tau = i = \tau'$ in a neighborhood of $x$, we obtain that $T_x \tau$ and $T_x \tau'$ agree in at least two directions in $T_x \mathcal{A}^1_{an}$ and therefore $T_x \tau = T_x \tau'$.

By Lemma 3.2 if $h$ is an extendable conjugacy as in Definition 3.1 given $x \in \mathcal{T}_f$ and setting $y = h(x)$, we denote by $T_x h : T_x \mathcal{A}^1_{an} \to T_y \mathcal{A}^1_{an}$ the unique map that agrees with the tangent map of a translation $\tau_x$ locally coincident with $h$, that is $T_x h = T_x \tau_x$.

3.2. **From analytic to extendable conjugacies.** In this subsection we discuss general results about analytic maps between open subsets of $\mathcal{A}^1_{an}$ and prove one direction of Theorem D (see Corollary 3.7).

An open annulus in $\mathcal{A}^1_{an}$ is a set $A$ of the form $D(a, r) \setminus \overline{D}(a, s)$ for some $0 < s < r$. The image of a non-constant analytic map $\psi : A \to \mathcal{A}^1_{an}$ is either a disk or an annulus (c.f. [BR10] Proposition 9.44 and Lemma 9.45). In the latter case, $\psi$ has a well defined degree, and in a certain sense, $T_x \psi$ is “constant” along type II points $x \in \text{sk} A$:

**Lemma 3.3.** Let $A_0$ and $A_1$ be two open annuli contained in $\mathcal{A}^1_{an}$, and suppose that $\psi : A_0 \to A_1$ is a surjective analytic map of degree $\delta$. Then there exist $a \in \{+\delta, -\delta\}$ and $0 \neq a \in \mathbb{C}_v$ such that for arbitrary $b \in \mathcal{A}^1$ in the bounded component of $\mathcal{A}^1_{an} \setminus A_1$ and arbitrary $c \in \mathcal{A}^1$ in the bounded component of $\mathcal{A}^1_{an} \setminus A_0$, setting $\gamma(z) = a(z - c)^{\sigma} + b$, we have that $\psi(x) = \gamma(x)$ and $T_x \psi = T_x \gamma$ for all $x \in \text{sk}(A_0)$.

**Proof.** Pick any $c \in \mathcal{A}^1$ in the bounded component of $\mathcal{A}^1_{an} \setminus A_0$ and any $b \in \mathcal{A}^1$ in the bounded component of $\mathcal{A}^1_{an} \setminus A_1$. Modulo the translation $z \mapsto z - c$ in the domain and the translation $z \mapsto z - b$ in the range, we may assume that $\text{sk}(A_0)$ and $\text{sk}(A_1)$ are contained in $|0, \infty|$. Then, from the Mittag-Leffler decomposition of $\psi$ (see [FvdP04] Proposition 2.2.6)), we have $\psi(z) = \sum_{n=0}^{+\infty} a_n z^n$. Moreover, for some $\sigma \in \{+\delta, -\delta\}$ and all $r$ such that $|z| = r$ is contained in $A_0$, we have that $|\psi(z)| = |a_\sigma| r^\sigma$ and $|a_j| r^j < |a_\sigma| r^\sigma$ for all $j \neq \sigma$ since $\psi$ maps $\text{sk}(A_0)$ onto $\text{sk}(A_1)$ and has degree $\delta$. Hence,

$$|\psi(z) - a_\sigma z^\sigma| < |a_\sigma| r^\sigma.$$ 

Let $\gamma(z) = a_\sigma z^\sigma$. It follows that $\psi(x) = \gamma(x)$ and $T_x \psi = T_x \gamma$ for all $x \in \text{sk}(A_0)$. □
Corollary 3.4. Let $A_0$ and $A_1$ be two open annuli in $A_{an}^1$, and suppose that $\psi_1, \psi_2 : A_0 \to A_1$ are surjective analytic maps. If there exists a type II point $x_0 \in sk A_0$ such that $\psi_1(x_0) = \psi_2(x_0)$ and $T_{x_0} \psi_1 = T_{x_0} \psi_2$, then $\psi_1(x) = \psi_2(x)$ and $T_x \psi_1 = T_x \psi_2$ for all $x \in sk A_0$.

Proof. Modulo affine maps in the domain and the target, we can assume that $sk(A_0)$ and $sk(A_1)$ are contained in $]0, \infty[$ and $x_0 = x_G = \psi_1(x_0) = \psi_2(x_0)$. Since the point 0 is contained in both the bounded component of $A_{an}^1 \setminus A_0$ and the bounded component of $A_{an}^1 \setminus A_1$, by Lemma 3.3 there exists $\gamma_1(z) = az^b$ and $\gamma_2(z) = a'z^{b'}$ such that $\gamma_1(x_0) = \psi_1(x_0)$, $\gamma_2(x) = \psi_2(x)$ and $T_x \gamma_1 = T_x \psi_1$, $T_x \gamma_2 = T_x \psi_2$ for all $x \in sk(A_0)$. Since $x_0 = x_G \in sk(A_0)$ and $\gamma_1(x_0) = x_0 = \gamma_2(x_0)$, we obtain $|a - a'| < |a| = |a'| = 1$ and $\delta = \delta'$. Then for any $r$ with $|z| = r$ contained in $A_0$, we have

$$|\gamma_1(z) - \gamma_2(z)| = |(a - a')z^{\delta}| < |\gamma_1(z)|.$$

It follows that $|(\gamma_1 - \gamma_2)(x)| < |\gamma_1(x)|$ and $T_x \gamma_1 = T_x \gamma_2$ for all $x \in sk(A_0)$. Thus the conclusion holds.

We will also need the following observation:

Lemma 3.5. Suppose that $\psi_1, \psi_2 : U \to A_{an}^1$ are analytic maps defined in a neighborhood $U$ of a type II point $x_0$. If $\psi_1(x_0) = \psi_2(x_0)$ and $T_{x_0} \psi_1 = T_{x_0} \psi_2$, then there exists $\varepsilon > 0$ such that for all $x \in U$ with $dist_{\mathbb{H}}(x, x_0) < \varepsilon$ we have that $\psi_1(x) = \psi_2(x)$ and $T_x \psi_1 = T_x \psi_2$.

Proof. Let $y_0 = \psi_1(x_0) = \psi_2(x_0)$. Taking into account that $x_0$ is a type II point, given such maps $\psi_1, \psi_2$ we have that $|\psi_1(z) - \psi_2(z)|_{x_0} < \operatorname{diam}(y_0)$. Since $\operatorname{diam}(y)$ is continuous with respect to the metric topology in $\mathbb{H}$ and $|\psi_1(z) - \psi_2(z)|_x$ is continuous in the Berkovich topology which is weaker than the metric topology, it follows that $|\psi_1(z) - \psi_2(z)|_x < \operatorname{diam}(\psi((x)))$ for all $x$ in a $\mathbb{H}$-metric disk around $x_0$. Hence, $\psi_2(x) = \psi_1(x)$ and $T_x \psi_2 = T_x \psi_1$ for all $x$ in this $dist_{\mathbb{H}}$-disk.

Analytic isomorphisms between open and connected sets which agree with a translation in a neighborhood of an skeleton point of the domain, are locally a translation at all skeleton points:

Proposition 3.6. For $b \in \mathbb{C}_v$, let $\tau_b(z) = z + b$. Consider two open and connected sets $U_0, U_1 \subseteq A_{an}^1$. Assume that $\psi : U_0 \to U_1$ is an analytic isomorphism such that $\psi(x_0) = \tau_b(x_0)$ and $T_{x_0} \psi = T_{x_0} \tau_b$ for some $x_0 \in sk U_0$ and some $b \in \mathbb{C}_v$. Then for any $x \in sk U_0$, there exist $b_x \in \mathbb{C}_v$ and a neighborhood $V \subset sk U_0$ of $x$ such that for all $y \in V$, $\psi(y) = \tau_{b_x}(y)$ and $T_y \psi = T_y \tau_{b_x}$.

Proof. Let $\Gamma \subset sk U_0$ be the set formed by all $x \in sk U_0$ with the following property: there exist $b_x \in \mathbb{C}_v$ and a neighborhood $V \subset sk U_0$ of $x$ such that for all $y \in V$ we have $\psi(y) = \tau_{b_x}(y)$ and $T_y \psi = T_y \tau_{b_x}$. By Lemma 3.3 if a branch point $x' \in sk U_0$ is contained in $\Gamma$ (resp. $sk U_0 \setminus \Gamma$), then there exists an open hyperbolic ball $V' \subset U_0$ around $x'$ such that $V' \cap sk U_0$ is contained in $\Gamma$ (resp. $(sk U_0) \setminus \Gamma$). Given any two branch points $x_1$ and $x_2$ of $sk U_0$ such that $]x_1, x_2[$ contains no branch point of $sk U_0$, by Lemma 3.3 and 3.5 the segment $]x_1, x_2[$ is either contained in $\Gamma$ or in $(sk U_0) \setminus \Gamma$. Therefore, $\Gamma$ is a non-empty clopen subset of $sk U_0$ and the proposition follows.

Given a tame polynomial $f \in \operatorname{Poly}_d$, recall from Section 2.2 that the dynamical core $A_f$ is contained in the skeleton of the open and connected set $B_0(f)$.

Corollary 3.7. Let $f, g \in \operatorname{Poly}_d$ be two tame polynomials. Assume that $\psi : B(f) \to B(g)$ is an analytic conjugacy between $f : B(f) \to B(f)$ and $g : B(g) \to B(g)$ that extends the corresponding Böttcher coordinate change. Then $\psi : A_f \to A_g$ is an extendable conjugacy between $f : A_f \to A_f$ and $g : A_g \to A_g$. 
Proof. By Definition 3.1, it suffices to show \( \psi \) is locally a translation on \( A_f \). Observe that \( c \) is a critical point of \( f \) in \( B(f) \) if and only if \( \psi(c) \) is a critical point of \( g \) in \( B(g) \). Hence, \( \psi : B_0(f) \to B_0(g) \) is an analytic isomorphism. Since \( A_f \subset \text{sk} B_0(f) \), by Proposition 3.6 \( \psi \) is locally a translation on \( A_f \).

Corollary 3.7 establishes one direction of Theorem D.

3.3 From extendable to analytic conjugacies. In this subsection, we discuss results about maps between open sets and prove the remaining direction of Theorem D.

Recall that \( p \geq 0 \) denotes the characteristic of the residue field of \( C_v \). We begin with the following well-known facts about prime-to-\( p \) étale maps between annuli. Proofs are supplied for the sake of completeness.

Lemma 3.8. Let \( \delta \geq 1 \) be an integer not divisible by \( p \), and let \( A_0 = \{ r^{1/\delta} < |z| < s^{1/\delta} \} \) and \( A_1 = \{ r < |z| < s \} \) be two open annuli in \( \mathbb{A}^{1}_{an} \) for some \( 0 < r < s \). Consider an analytic map \( \psi : A \to A_1 \) of degree \( \delta \), where \( A \subset \mathbb{A}^{1}_{an} \) is an open annulus. Then there exists an analytic isomorphism \( \varphi : A \to A_0 \) such that \( \varphi(z)^\delta = \psi(z) \). Moreover, \( \varphi \) is unique up to multiplication by \( \mu \in C_v \) where \( \mu^\delta = 1 \).

Proof. Write \( A = \{ r' < |z - a| < s' \} \) for some \( a \in C_v \). Then there exists \( b_n \in C_v \) such that

\[
\psi(z) = \sum_{n = -\infty}^{+\infty} b_n(z - a)^n
\]

where, for all \( r' < t < s' \), either \( |b_n| t^\delta > |a_n| t^n \) for all \( n \neq \delta \) or \( |b_{-\delta}| t^{-\delta} > |b_n| t^n \) for all \( n \neq -\delta \). Without loss of generality, assume the former. Then

\[
\psi(z) = b_0(z - a)^\delta (1 + \varepsilon(z))
\]

with \( |\varepsilon(z)| < 1 \) for all \( z \in A \). Consider \( \beta \in C_v \) such that \( \beta^\delta = b_0 \). Since \( p \) does not divide \( d \), there exists a unique function \( \gamma(z) \) such that \( \gamma(z)^\delta = 1 + \varepsilon(z) \) and \( |\gamma(z) - 1| < 1 \). Then \( \varphi(z) = \beta(z - a) \gamma(z) \) is such that \( \varphi(z)^\delta = g(z) \). The lemma follows.

Corollary 3.9. Suppose that \( \psi : A \to A_1 \) is an analytic map of degree \( \delta \geq 2 \) between two open annuli \( A, A_1 \subset \mathbb{A}^{1}_{an} \) such that \( \delta \) is not divisible by \( p \). Assume that \( \psi_1 : A \to A \) is an analytic isomorphism such that the following hold:

1. \[ \psi \circ \psi_1 = \psi. \]
2. \( \psi_1(x) = x \) and \( T_x \psi_1 = \text{id} \) for some type II point \( x \) in \( \text{sk}(A) \).

Then \( \psi_1 = \text{id} \).

Proof. After translation we may assume \( A_1 = \{ r < |z| < s \} \). Let \( A_0 \) and \( \varphi : A \to A_0 \) be as in Lemma 3.8. Then \( \varphi(z)^\delta = \psi(z) \). It follows that \( (\varphi(\psi_1(z)))^\delta = \psi(\psi_1(z)) \). By assumption (1), we have \( (\varphi(\psi_1(z)))^\delta = \psi(\psi_1(z)) = \psi(z) = \varphi(z)^\delta \). Thus \( \varphi \circ \psi_1(z) = (\mu \varphi)(z) \) for some \( \mu \in C_v \) with \( \mu^\delta = 1 \). By assumption (2), we conclude \( \varphi(x) = (\mu \varphi)(x) \) and \( T_x \varphi = T_x (\mu \varphi) \). Thus \( \mu = 1 \). It follows that \( \varphi \circ \psi_1 = \varphi \) which implies \( \psi_1 = \text{id} \).

Corollary 3.10. Let \( A, A' \) and \( A_1 \) be open annuli in \( \mathbb{A}^{1}_{an} \). Suppose that \( \psi_1 : A \to A_1 \) and \( \psi_2 : A' \to A_1 \) are degree \( \delta \geq 1 \) analytic maps such that \( \delta \) is not divisible by \( p \). Then there exist exactly \( \delta \) isomorphisms \( \varphi : A \to A' \) such that \( \psi_2 \circ \varphi = \psi_1 \). Moreover, if \( B_1 \) is a subannulus of \( A_1 \) with \( \text{sk}(B_1) \subset \text{sk}(A_1) \), then every isomorphism \( \psi : \psi_1^{-1}(B_1) \to \psi_2^{-1}(B_1) \) such that \( \psi_2 \circ \psi = \psi_1 \) extends to a unique isomorphism \( \varphi : A \to A' \) such that \( \psi_2 \circ \varphi = \psi_1 \).
Proof. After change of coordinates, unique modulo multiplication by a \( \delta \)-th root of unity, \( \psi_1 \) and \( \psi_2 \) become \( z \mapsto z^\delta \). It follows that there are exactly \( \delta \) isomorphisms \( \varphi \) as in the statement. Similarly, there are exactly \( \delta \) isomorphisms \( \psi \) as in the statement. Thus, every such \( \psi \) is the restriction of an isomorphism \( \varphi \). \( \square \)

Now we provide a criterium for the tangent maps of two analytic functions to agree at a type II point.

**Lemma 3.11.** Let \( D \subset \mathbb{A}^1_{an} \) be an open Berkovich disk and \( x_0 \in D \) be a type II point. Suppose that \( \psi_1, \psi_2 : D \to \mathbb{A}^1_{an} \) are analytic maps satisfying the following:

1. \( p \nmid \deg_x \psi_1 \) and \( p \nmid \deg_x \psi_2 \).
2. \( \deg_v \psi_1 = \deg_v \psi_2 \) for all \( v \in T_{x_0} \mathbb{A}^1_{an} \).
3. There exists \( x \in \mathbb{A}^1_{an} \) with \( x \leq x \) such that
   a. \( \psi_1(x_0) = \psi_2(x_0) \) for all \( x_0 < y_0 < x \), and
   b. \( T_{y_0} \psi_1 = T_{y_0} \psi_2 \) for all \( x_0 \leq y_0 < x \).
4. \( T_{x_0} \psi_1(v_0) = T_{x_0} \psi_2(v_0) \) for at least one direction \( v_0 \in T_{x_0} \mathbb{A}^1_{an} \setminus \{ v_{x_0}(\infty) \} \).

Then \( T_{x_0} \psi_1 = T_{x_0} \psi_2 \).

**Proof.** From (3a), continuity yields that \( \psi_1(x_0) = \psi_2(x_0) \). Using the same coordinate changes for \( \psi_1 \) and \( \psi_2 \), we may assume that \( x_0 \) and \( \psi_1(x_0) = \psi_2(x_0) \) are the Gauss point. Thus, it is sufficient to show that the reductions \( \tilde{\psi}_1 \) and \( \tilde{\psi}_2 \) of \( \psi_1 \) and \( \psi_2 \) coincide. While statement (1) implies that the polynomials \( \tilde{\psi}_1 \) and \( \tilde{\psi}_2 \) have finitely many critical points, statement (2) implies that \( \tilde{\psi}_1 \) and \( \tilde{\psi}_2 \) have the same critical points counting multiplicities. In particular, \( \deg_{x_0} \tilde{\psi}_1 = \deg_{x_0} \tilde{\psi}_2 \). From (3b), it is not difficult to conclude that the leading coefficients of \( \tilde{\psi}_1 \) and \( \tilde{\psi}_2 \) coincide. From (4), the constant terms of \( \tilde{\psi}_1 \) and \( \tilde{\psi}_2 \) agree. Therefore, \( \tilde{\psi}_1 = \tilde{\psi}_2 \). \( \square \)

Through an analytic continuation argument along \( A_f \), we will prove the following:

**Proposition 3.12.** Let \( f, g \in \text{Poly}_d \) be tame polynomials. If \( h : A_f \to A_g \) is an extendable conjugacy between \( f : A_f \to A_f \) and \( g : A_g \to A_g \), then there exists an analytic conjugacy \( \phi : B(f) \to B(g) \) between \( f : B(f) \to B(f) \) and \( g : B(g) \to B(g) \) which agrees with the Böttcher coordinate change near infinity and with \( h \) on \( A_f \).

Given a tame polynomial \( f \in \text{Poly}_d \), recall from Section[22] that \( \phi_f \) denotes the Böttcher coordinate and that \( |\phi_f| \) extends to \( B(f) \). For \( s > 1 \), let

\[
V_f(s) := \{ z \in \mathbb{A}^1_{an} : |\phi_f(z)| > s \}
\]

and

\[
A_f(s) := A_f \cap V_f(s).
\]

Observe that \( A_f(s) \) is forward invariant. Moreover, \( A_f(s) \) is contained in the skeleton of \( V'_f(s) := V_f(s) \setminus \text{Post}(f) \).

**Proof of Proposition 3.12.** For all \( z \in \mathbb{A}^1_{an} \) such that \( |z| \) is sufficiently large, we have \( |\phi_f(z)| = |z| = |\phi_g(z)| \). Hence, for \( s_0 \gg 1 \) sufficiently large, \( \phi(z) = \phi^{-1}_g \circ \phi(z) : V_f(s_0) \to V_g(s_0) \) is an analytic isomorphism and \( \phi = h \) on \( A_f(s_0) \). Moreover, \( \phi \) is an analytic conjugacy between \( f : V_f(s_0) \to V_f(s_0) \) and \( g : V_g(s_0) \to V_g(s_0) \).

We assume that there exists \( s_0 > 1 \) such that \( \phi \) extends to \( V_f(s_0) \), that is, we have an analytic isomorphism \( \phi : V_f(s_0) \to V_g(s_0) \) extending the Böttcher coordinate change and conjugating \( f : A_f(s) \to A_f(s) \) with \( g : A_g(s) \to A_g(s) \) such that \( \phi(x) = h(x) \) for all \( x \in A_f(s) \). To prove the proposition it is sufficient to show that \( \phi \) in fact extends to an analytic isomorphism \( \overline{\phi} : V_f(s) \to V_g(s) \) for some \( s < s_0 \) such that \( \overline{\phi}(x) = h(x) \) for all
Consider \( x_0 \in \partial V_f(s_*) \). Keep in mind that there are only finitely many such \( x_0 \). Moreover, \( |x_0, \infty| \subset V_f(s_*) \) and \( \phi(x_0, \infty) = |x_0', \infty| \) for some \( x_0' \in \partial V_f(s_*) \). The main step is to extend \( \phi \) to a neighborhood of \( x_0 \) in \( \mathbb{A}^n \). We consider three cases.

**Case 1.** \( \deg x_0 f = 1 \). Consider \( x \to x_0 \) close to \( x_0 \) such that \( x \) is not in the ramification tree of \( f \). Set \( x' := \phi(x) \). Then \( \deg x f = 1 \) and, therefore, \( \deg x' g = 1 \), since \( \phi \) is a conjugacy in \( V_f(s_*) \). Let \( U_0 \) (resp. \( V_0 \)) be the Berkovich open disk formed by points in the direction of \( x_0 \) (resp. \( x' \)) at \( x \) (resp. \( x' \)). We may denote by \( G \) the inverse of \( g : V_0 \to g(V_0) \). It follows that \( \phi := G \circ \phi \circ f : U_0 \cap V_f(s_*^{1/d}) \to V_0 \cap V_f(s_*^{1/d}) \) is a well defined analytic isomorphism that coincides with \( \phi \) in \( U_0 \cap V_f(s_*) \) and agrees with \( h \) on the points of \( A_f \) contained in \( U_0 \cap V_f(s_*^{1/d}) \).

**Case 2.** \( \deg x_0 f \geq 2 \) and \( x_0 \) is a type II point. In particular, \( x_0 \) lies in the ramification tree and therefore in \( A_f \). Our assumption that \( \phi = h \) on \( A_f(s_*) \) guarantees that \( \phi \) maps \( V'_f(s_*) \) onto \( V'_f(s_*) \). The Böttcher coordinate \( \phi \) is asymptotic to the identity at infinity, therefore, \( \phi(x) = x \) and \( T_x \phi = \text{id} \) for all \( x \in A_f(s_*) \) sufficiently close to \( \infty \). By Proposition 3.10 and Lemma 3.11 we conclude that \( T_x \phi = T_x h \) for all \( x \in A_f(s_*) \) and agrees with \( h \) on the points of \( A_f \) respectively. After checking that (1)–(4) of Lemma 3.11 hold for \( \psi_1 = g \circ \tau_0 \) and \( \psi_2 = \tau_1 \circ f \), we will conclude that \( T_{x_0}(g \circ \tau_0) = T_{x_0}(\tau_1 \circ f) \). Tameness of \( f \) and \( g \) guarantees that \( p \not\mid \deg x_0 f(g \circ \tau_0) \) and \( p \not\mid \deg x_0 f(\tau_1 \circ f) \), and hence (1) of Lemma 3.11 follows. Now for (2), we may denote by \( \hat{G} \) the Berkovich open disk formed by points in \( g \circ \tau_0 \) and \( \tau_1 \circ f \). We must show that \( \deg g \circ \tau_0 f = \deg g \circ \tau_1 f \). Since \( g \) agrees with \( \tau_0 \) on a neighborhood of \( x_0 \), the direction \( D(\hat{G}) \) is disjoint from \( A_f \) if and only if the direction \( D(\hat{G}) \) is disjoint from \( A_g \). When \( D(\hat{G}) \) and \( D(\hat{G}) \) are disjoint from \( A_f \) and \( A_g \), respectively, \( \deg g \circ \tau_0 f = 1 = \deg g \circ \tau_1 f \). When \( D(\hat{G}) \) intersects \( A_f \), consider a small arc \( [x, x_0] \subset A_f \) in the direction \( D(\hat{G}) \). Then the hyperbolic length of \( [x, x_0] \) under \( g \circ \tau_0 f \) (resp. \( g \circ \tau_1 f \)) is multiplied by a factor of \( \deg g \circ \tau_0 f \) (resp. \( \deg g \circ \tau_1 f \)), since \( h \) is an isometry. But \( h \) is also a conjugacy, so these factors must agree. That is, \( \deg g \circ \tau_0 f = \deg g \circ \tau_1 f \). Thus (2) of Lemma 3.11 holds. For \( x \in [x_0, \infty] \subset V_f(s_*) \) sufficiently close to \( x_0 \), we have \( T_x \phi = T_x h \). Therefore, \( T_x(\tau_1 \circ f) = T_x(g \circ \tau_0) \) since \( T_x \phi = T_x h \) and \( T_x h = T_x \tau_1 \tau_0 x \subset [x, x_0] \), which is a contradiction. That is, (3) of Lemma 3.11 holds. Finally (4) also holds since \( A_f \) has points in at least one bounded direction at \( x_0 \). From Lemma 3.11 now we have \( T_{x_0}(g \circ \tau_0) = T_{x_0}(\tau_1 \circ f) \).

By Lemma A.1 in the appendix, there exists an analytic map \( \psi \) defined in a neighborhood \( V_0 \) of \( y_0 \) such that \( g \circ \tau_0 \circ \psi = \tau_1 \circ f \) and \( \psi(y_0) = y_0 \). Thus, \( \hat{G} \) is disjoint from \( A_f \) if and only if \( \hat{G} \) is disjoint from \( A_g \). The proof is complete.
Thus for $x_0 \in \partial V_f(s_*)$, we obtain an extension $\tilde{\phi}$ of $\phi$ in a neighborhood of $x_0$ such that $\tilde{\phi} = h$ on the points of $\mathcal{A}_f$ in this neighborhood. \hfill \Box

**Proof of Theorem 4.2** Theorem 4.2 is an immediate consequence of Corollary 3.7 and Proposition 3.2. \hfill \Box

## 4. Existence of extendable conjugacies

The goal of this section is to prove Propositions 4.7 and 4.8 which together become a slightly stronger version of Theorem 4.3.

### 4.1. Trimmed dynamical core

Given a tame polynomial $f \in \text{Poly}_d$, recall that the dynamical core is

$$\mathcal{A}_f = \mathcal{B}(f) \cap \bigcup_{c \in \text{Crit}(f), n \geq 0} \{ f^n(c), \infty \}.$$

Sometimes it will be convenient to consider the subtree of $\mathcal{A}_f$ formed by the elements that escape to infinity through a hyperbolic $\rho$-neighborhood of the axis $\{ x_f, \infty \}$ for some $\rho > 0$ (possibly $\infty$). Note that since $f$ is tame, in $|z| > |x_f|$ the distance to the axis is preserved by dynamics. More precisely, for all $x \in \mathbb{H}$ such that $|x| > |x_f|$ we have that

$$\text{dist}_\mathbb{H}(x, |x_f, \infty|) = \text{dist}_\mathbb{H}(f(x), |x_f, \infty|).$$

**Definition 4.1.** Consider a tame polynomial $f \in \text{Poly}_d$. Given $\rho \in [0, \infty]$, we say that the $\rho$-trimmed dynamical core of $f$, denoted by $\mathcal{T}_f$, is the set formed by all $x \in \mathcal{A}_f$ such that $\text{dist}_\mathbb{H}(f^n(x), |x_f, \infty|) < \rho$ for all $n$ sufficiently large.

Keep in mind that $\mathcal{A}_f$ is the $\rho$-trimmed dynamical core of $f$ with $\rho = \infty$. Also, from the previous discussion, if $x \in \mathcal{A}_f$ with $|x| > |x_f|$ lies in the $\rho$-trimmed dynamical core $\mathcal{T}_f$ if and only if $\text{dist}_\mathbb{H}(x, |x_f, \infty|) < \rho$, equivalently, $\log(|x|/\text{diam}(x)) < \rho$.

**Lemma 4.2.** Let $f \in \text{Poly}_d$ be a tame polynomial. For any $\rho \in [0, \infty]$, the corresponding $\rho$-trimmed dynamical core $\mathcal{T}_f$ is a locally finite open subtree of $\mathbb{H}$ which is forward invariant (i.e. $f(\mathcal{T}_f) \subset \mathcal{T}_f$). Moreover, if $x \in \mathcal{T}_f$, then $|x, \infty| \subset \mathcal{T}_f$. Furthermore, if a critical point $c$ lies in a direction $D_x(\tilde{v}^i)$ at some $x \in \mathcal{T}_f$, then $D_x(\tilde{v}) \cap \mathcal{T}_f \neq \emptyset$.

**Proof.** Directly from its definition we conclude that $\mathcal{T}_f$ is forward invariant. Moreover, $\mathcal{T}_f$ is obtained from $\mathcal{A}_f$ by cutting off some of its “branches”. Indeed, if $x \in \mathcal{A}_f \setminus \mathcal{T}_f$ and $y < x$, then $f^n(y) < f^n(x)$ for all $n \geq 0$. Hence, for $n$ sufficiently large $f^n(y)$ is at distance at least $\rho$ from the axis and therefore $y \notin \mathcal{T}_f$. It follows that $\mathcal{T}_f$ is connected and relatively open in $\mathcal{A}_f$. Therefore $\mathcal{T}_f$ is also a locally finite open subtree of $\mathbb{H}$. If a direction $D_x(\tilde{v})$ contains a critical point $c$ for some $x \in \mathcal{T}_f$, then $D_x(\tilde{v}) \cap \mathcal{T}_f \neq \emptyset$, since $\mathcal{T}_f$ is open in $\mathcal{A}_f$ and $|c, \infty| \cap \mathcal{B}(f) \subset \mathcal{A}_f$. \hfill \Box

Given a $\rho$-trimmed dynamical core $\mathcal{T}_f$, we say that $x \in \mathcal{T}_f$ is a vertex of $\mathcal{T}_f$ if $f^n(x)$ is a branch point of $\mathcal{T}_f$ for some $n \geq 0$. The set formed by the vertices of $\mathcal{T}_f$ is denoted by $\mathcal{V}_f$. An edge of $\mathcal{T}_f$ is a connected component of $\mathcal{T}_f \setminus \mathcal{V}_f$.

**Proposition 4.3** (Vertices of $\mathcal{T}_f$). Consider a non-simple tame polynomial $f \in \text{Poly}_d$ with base point $x_f$ and $\rho \in [0, \infty]$. Let $\mathcal{V}_f$ be the set of vertices of the $\rho$-trimmed dynamical core $\mathcal{T}_f$. Then $x_n \in \mathcal{V}_f$ and $f(\mathcal{V}_f) \subset \mathcal{V}_f$. Moreover, the set of accumulation points of $\mathcal{V}_f$ in $\mathbb{H}^*$ is contained in the Julia set $\mathcal{J}(f)$.

**Proof.** We claim that if $x \in \mathcal{T}_f$ is a branch point of $\mathcal{T}_f$ then $f(x)$ is also a branch point. Indeed, let us proceed by contradiction assuming that $x$ is branch point and $f(x)$ is not. Then all bounded directions at $x$ containing elements of $\mathcal{T}_f$ are contained in the preimage of a single direction $\tilde{w}$ at $f(x)$. By the previous lemma, $\tilde{w}$ is the unique critical value
direction of $T_x f$ and therefore $T_x f$ is a monomial. Hence the preimage of $\bar{v}$ is a singleton and $x$ is not a branch point.

In view of the previous paragraph, if $f^{\circ n}(x)$ is a branch point of $T_f$, then $f^{\circ n}(f(x))$ is a branch point. Thus, $f(\mathcal{V}_f) \subset \mathcal{V}_f$.

Forward invariance of the axis $[x_f, \infty]$ implies that $x_f \in T_f$. Since $f$ is nonsimple and tame we may apply (2.1) to conclude that there exists a critical value $v$ of $f$ such that $|v| = |f(x_f)|$. It follows that $f(x_f)$ is a branch point of $A_f$. Therefore, $x_f \in \mathcal{V}_f$.

Let us say that two escaping critical points $c, c'$ are eventually in the same direction if there exists $m, n \geq 0$ such that $|x_f| < |f^{\circ n}(c)| = |f^{\circ m}(c')| = R$ and $|f^{\circ n}(c) - f^{\circ m}(c')| < R$ (i.e. they lie in the same direction at $x_0 R$). There exists $N \geq 0$ such that for all pairs of critical points $c, c'$ which are eventually in the same direction, then $n, m$ above can be chosen to be at most $N$. Now consider $X := \{x : |f^{\circ n}(x_f)| < |x| < |f^{\circ n+1}(x_f)|\}$. It follows that $A_f \cap X$ maps homoeomorphically onto its image under $f$. Moreover, $A_f \cap X$ is the union of finitely many arcs of the form $[z, f^{\circ n+1}(x_f)]$ with the arc $[f^{\circ n}(x_f), f^{\circ n+1}(x_f)]$. In particular, $A_f \cap X$ contains finitely many branch points. Also, $X$ is a "fundamental domain" for the action of $f$ on $B(f)$. Namely, $B(f)$ is the disjoint union of the sets $f^{\circ n}(X)$ for $n \in \mathbb{Z}$. Since every branch point of $T_f$ is the iterated image or preimage of one in $A_f \cap X$, the conclusion follows.

Now we introduce an increasing collection $\{T_f^{(j)}\}_{j \geq 0}$ of subtrees of $T_f$ that saturate $T_f$. Set $T_f^{(0)} := T_f \cap \{x : |x| > |x_f|\}$. Recursively, the level $j + 1$ subtree $T_f^{(j+1)}$ is obtained by adjoining to $T_f^{(j)}$ all the elements $x \in T_f$ such that there exists $y \in T_f^{(j)}$ with the property that $|x, y| \cap A_f$ contains at most one vertex of $T_f$.

**Corollary 4.4.** Consider a nonsimple tame polynomial $f \in \text{Poly}_d$ and $\rho \in [0, \infty]$. Let $T_f$ be the $\rho$-trimmed dynamical core of $f$. Then

$$T_f = \bigcup_{j \geq 0} T_f^{(j)}.$$

Moreover,

$$f(T_f^{(j+1)}) \subset T_f^{(j)},$$

for all $j \geq 0$.

**Proof.** For all $x \in T_f$, observe that $|x, x_f| \cap \mathcal{V}_f$ is finite, since $|x, x_f|$ is free of accumulation points of $\mathcal{V}_f$, by Proposition 4.3.

Given $x \notin T_f^{(0)}$ and $j \geq 1$, note that $x \in T_f^{(j)}$ if and only if $|x, x_f|$ contains at most $j$ elements of $\mathcal{V}_f$. By the previous paragraph, we have that every element of $T_f$ lies in some $T_f^{(j)}$.

By definition, $f(T_f^{(0)}) \subset T_f^{(0)}$. For all $j \geq 1$ and $x \in T_f^{(j)} \setminus T_f^{(0)}$, our map $f$ acts bijectively on $|x, x_f|$ and therefore $|f(x), f(x_f)|$ contains at most $j$ elements of $\mathcal{V}_f$. If $f(x) \notin T_f^{(0)}$, then $|f(x), f(x_f)|$ is the disjoint union of $|f(x), x_f|$ and $|x_f, f(x_f)|$. Hence, $|f(x), x_f|$ contains at most $j - 1$ elements of $\mathcal{V}_f$ and $f(x) \in T_f^{(j-1)}$. It follows that $f(T_f^{(j)}) \subset T_f^{(j-1)}$.

**Proposition 4.5** (Edges of $T_f$). Consider a nonsimple tame polynomial $f$ and $\rho \in [0, \infty]$. Denote by $T_f$ the $\rho$-trimmed dynamical core of $f$. Let $e$ be an edge of $T_f$. Then there exist $a \in \mathcal{V}_f \cup \partial T_f$ and $b \in \mathcal{V}_f$ such that $a \prec b$ and $e = [a, b]$. Moreover, $f(e) = [f(a), f(b)]$ is also an edge of $T_f$ and $\deg_x f = \deg_a f$, for all $x \in e$. Furthermore, for all $y, y' \in \mathcal{V}$,

$$\text{dist}_H(f(y), f(y')) = \deg_a f \cdot \text{dist}_E(y, y').$$
Proof. An edge $e$ is, by definition, a connected component of $\mathcal{T}_f \setminus \mathcal{V}_f$. If the extreme points are $a$ and $b$, then $|a, \infty|$ and $|b, \infty|$ are contained in $\mathcal{T}_f$ and, $[a, b]$ is branch point free. It follows that $a \prec b$ or $b \prec a$. Without loss of generality we assume that $a \prec b$. By definition if a preimage $w$ of a vertex lies in $\mathcal{T}_f$, then it is a vertex. Therefore, $f(e)$ is an edge. For all $x \in \mathbb{H}$, since $f$ is tame, $\text{deg}_x f - 1$ agrees with the number of critical points in the disk $\overline{D}_x$, counted with multiplicities (see Lemma 2.3). This number is constant along edges and therefore $\text{deg}_x f = \text{deg}_a f$ for all $x \in e$. Thus, the hyperbolic metric is expanded by a factor $\text{deg}_a f$ along $e$ (see Lemma 2.2).

4.2. Close Böttcher coordinates of escaping critical points. We start by introducing a notion that quantifies how close are the Böttcher coordinates of escaping critical points with the agreement that $\exp(-\infty) := 0$.

Definition 4.6. Consider two tame critically marked polynomials $f$ and $g$ in $\text{Poly}_d$ with the same base point $x_0$. Let $\phi_f, \phi_g : \mathbb{A}^1_{\text{an}} \setminus \overline{D}(a, |x_0|) \to \mathbb{A}^1_{\text{an}} \setminus \overline{D}(a, |x_0|)$ be the corresponding Böttcher coordinates. Given $\rho \in [0, \infty]$ we say that the escaping critical points of $f$ and $g$ have $\rho$-close Böttcher coordinates if $|f^{om}(c_i(f))| > |x_0|$ if and only if $|g^{om}(c_i(g))| > |x_0|$ and in this case:

$$ (4.1) \quad |\phi_f(f^{om}(c_i(f))) - \phi_g(g^{om}(c_i(g)))| \leq \exp(-\rho)|f^{om}(c_i(f))|. $$

For any passive critically marked analytic family of tame polynomials, observe that escaping critical points have $\infty$-close Böttcher coordinates if and only if they have constant Böttcher coordinates, by Definition 2.8.

Note that if $f$ and $g$ have $\rho$-close Böttcher coordinates and (4.1) holds for $i$ and $m$, then

$$ |f^{om}(c_i(f))| = |\phi_f(f^{om}(c_i(f)))| = |\phi_g(g^{om}(c_i(g)))| = |g^{om}(c_i(g))|. $$

For every escaping critical point $c_i(f)$ let $m_i \geq 1$ be such that $|x_0| < |f^{om_i}(c_i(f))| \leq |x_0|^d$, equivalently $m_i$ is the smallest integer such that $|x_0| < |f^{om_i}(c_i(f))|$. If for all such escaping $c_i(f)$ we have that $|g^{om_i}(c_i(g))| > |x_0|$ and (4.1) holds for $m = m_i$, then $f$ and $g$ have $\rho$-close Böttcher coordinates, since $z \mapsto z^d$ is a $\text{dist}_\mathbb{H}$-isometry on $D_{x_0}(\infty) \setminus \{|x_0|\}$. |x_0|.

Proposition 4.7. Let $\{f_\lambda\}$ be an analytic family of critically marked tame polynomials in $\text{Poly}_d$ parametrized by an open disk $\Lambda \subset \mathbb{C}$, with radius in $|\mathbb{C}^\times|$. Assume that all the critical points are passive in $\Lambda$. Then for every closed disk $\overline{\Omega}$ contained in $\Lambda$, there exists $\rho > 0$ such that for all $\lambda_1, \lambda_2 \in \overline{\Omega}$, the escaping critical points of $f_{\lambda_1}$ and $f_{\lambda_2}$ have $\rho$-close Böttcher coordinates.

Proof. From Lemma 2.9 we know that the base point $x_\Lambda$ is independent of $\lambda \in \Lambda$. Denote by $r_\Lambda$ the diameter of $x_\Lambda$. Consider a closed disk $\overline{\Omega} \subset \Lambda$ and pick a reference parameter $\lambda_1 \in \overline{\Omega}$. Assume that $|f^{om}_{\lambda_1}(c_i(\lambda_1))| > r_\Lambda$ where $n$ is the smallest number with this property. To simplify notation let $\alpha_i(\lambda) = f^{om}_{\lambda_1}(c_i(\lambda))$. Then $|\alpha_i(\lambda)|$ takes a constant value, say $r_i$, because otherwise $|\alpha_i(\lambda) - c_i(\lambda)|$ would vanish at some parameter. Moreover, for the same reason, the direction of $\alpha_i(\lambda)$ at $x_0r_i$ is also constant in $\Lambda$. Since the tangent map of $\phi_\lambda$ at $x_0r_i$ is the identity, in standard coordinates,

$$ |\phi_\lambda(\alpha(\lambda)) - \phi_{\lambda_1}(\alpha(\lambda_1))| < r_i $$

for all $\lambda \in \Lambda$. Since $\phi_\lambda(\alpha_i(\lambda))$ is analytic, there exists $\rho > 0$ such that for all $\lambda_2 \in \overline{\Omega}$ and all $i$:

$$ |\phi_{\lambda_2}(\alpha_i(\lambda_2)) - \phi_{\lambda_1}(\alpha_i(\lambda_1))| \leq \exp(-\rho) \cdot r_i. $$

$\Box$
4.3. From passive critical points to extendable conjugacies. To prove Theorem C, we first show the following result.

Proposition 4.8. Let \( \{f_\lambda\} \) be a passive analytic family of critically marked tame polynomials in \( \text{Poly}_d \) parametrized by a disk \( \Lambda \subset \mathbb{C}_v \). Consider \( \rho > 0 \) and denote by \( T_\lambda \) the \( \rho \)-trimmed dynamical core of \( f_\lambda \). Also let

\[
C := \{ \alpha : \Lambda \to \mathbb{R}^1 : \alpha(\lambda) = f_\lambda^\ell(c_i(\lambda)) \text{ for some } \ell \geq 0, 1 \leq i < d \}.
\]

Suppose that the Böttcher coordinates of escaping critical points of \( f_{\lambda_0} \) and \( f_\lambda \) are \( \rho \)-close for all \( \lambda \in \Lambda \). Then there exists a unique map \( h_\lambda : T_{\lambda_0} \to T_\lambda \) such that if \( x = x_{\alpha(\lambda_0),r} \in T_{\lambda_0} \) for some \( \alpha \in C \) and \( r > 0 \), then

\[
h_\lambda(x) = x_{\alpha(\lambda),r}.
\]

Moreover, \( h_\lambda \) is an extendable conjugacy between \( f_{\lambda_0} : T_{\lambda_0} \to T_{\lambda_0} \) and \( f_\lambda : T_\lambda \to T_\lambda \).

Proof. In view of Lemma 2.3 denote by \( x_\lambda \) the base point of \( f_\lambda \), which is independent of \( \lambda \).

Observe that for any \( \lambda \in \Lambda \), every point in \( T_\lambda \) has form \( x_{\alpha(\lambda),r} \) for some \( \alpha \in C \) and \( r > 0 \). So if \( h_\lambda \) exists, it is unique.

The existence of \( h_\lambda \) will follow once we prove by induction on \( j \geq 0 \) the following assertions:

1. if \( \alpha, \beta \in C \) and \( r > 0 \) are such that \( x_{\alpha(\lambda_0),r} = x_{\beta(\lambda_0),r} \in T_{\lambda_0}^{(j)} \), then \( x_{\alpha(\lambda),r} = x_{\beta(\lambda),r} \in T_{\lambda}^{(j)} \) for all \( \lambda \in \Lambda \). In this case, we let \( h_\lambda(x_{\alpha(\lambda_0),r}) := x_{\alpha(\lambda),r} \).

2. \( h_\lambda : T_{\lambda_0}^{(j)} \to T_{\lambda}^{(j)} \) is an extendable conjugacy.

To simplify notation we set \( \lambda_0 := 0 \) and employ subscripts accordingly.

We start with the case in which \( j = 0 \). Consider \( \alpha \in C \) such that \( |\alpha(0)| > |x_\lambda| \). Set \( r_\alpha := |\phi_0|(|\alpha(0)|) \) and \( R_\alpha := \exp(-\rho)r_\alpha \). Since the Böttcher coordinates of escaping critical points are \( \rho \)-close, it follows that \( r_\alpha = |\phi_\lambda|(|\alpha(\lambda)| = |\alpha(\lambda)| \) for all \( \lambda \). The inequality

\[
|\phi_\lambda(\alpha(\lambda)) - \phi_0(\alpha(0))| \leq R_\alpha
\]

which by definition holds for all \( \alpha \) such that \( |x_\lambda| < |\alpha(0)| \leq |x_\lambda|^d \), extends to all \( \alpha \in C \) such that \( |\alpha(0)| > |x_\lambda| \). This follows since \( \phi_\lambda \) and \( z \mapsto z^d \) are isometries in the hyperbolic metric on connected components of the complement of \( |x_\lambda, \infty| \) not containing \( z = 0 \). Note that \( x_{\alpha(\lambda),r} \in T_{\lambda}^{(0)} \) if and only if \( r > R_\alpha \). In this case, \( \phi_0(x_{\alpha(0),r}) = \phi_\lambda(x_{\alpha(\lambda),r}) \) since Böttcher coordinates are diameter preserving. Therefore, \( h_\lambda := \phi_\lambda^{-1} \circ \phi_0 : T_{\lambda_0}^{(0)} \to T_{\lambda}^{(0)} \) is well defined and (1) holds for all \( \alpha, \beta \) such that \( |\alpha(0)| > |x_\lambda| \) and \( |\beta(0)| > |x_\lambda| \). Statement (1) extends to arbitrary \( \alpha, \beta \in C \) since in the rest of the cases \( x_{\alpha(\lambda),r} = x_{0,r} \) for some \( r > |x_\lambda| \) whenever \( x_{\alpha(\lambda),r} \in T_{\lambda}^{(0)} \). Moreover, the open set with skeleton \( T_{\lambda}^{(0)} \) is mapped by \( \phi_\lambda^{-1} \circ \phi_0 \) onto the open set with skeleton \( T_{\lambda}^{(0)} \). Therefore, by Proposition 3.6 \( h_\lambda \) is an extendable conjugacy and (2) also holds for \( j = 0 \).

Now we assume that assertions (1) and (2) hold for some \( j \geq 0 \) and proceed to establish their validity for \( j + 1 \).

Consider an endpoint \( x_0 \in \partial T_{\lambda_0}^{(j)} \) such that \( x_0 \in T_{\lambda}^{(j+1)} \) and denote by \( r_0 \) its diameter. We will establish assertions (1) and (2) for \( x_0 \) and the points on the edges of \( T_{\lambda}^{(j+1)} \) growing from \( x_0 \).

A key observation will be the following. Assume that \( \alpha, \beta \in C \) are such that \( \alpha(0) \) and \( \beta(0) \) lie in the same bounded direction at \( x_0 \) (i.e., \( \alpha(0) \prec x_0, \beta(0) \prec x_0 \) and \( |\alpha(0) - \beta(0)| < \rho \)).
\[ r_0 \). Then
\begin{equation}
(4.2) \quad |\alpha(\lambda) - \beta(\lambda)| < r_0 \text{ for all } \lambda \in \Lambda.
\end{equation}
Indeed, \( x_{\alpha(\lambda),r} = x_{\beta(\lambda),r} \in T^{(j)}_{\lambda} \) for all \( r > r_0 \). Thus, \( |\alpha(\lambda) - \beta(\lambda)| \leq r_0 \) for all \( \lambda \in \Lambda \). By the Maximum Principle, the last inequality is strict, as claimed.

Now, let us suppose that there exists a point \( x' \neq x_0 \) such that \( e_0 = [x'_{x_0}, x_0] \) is an edge of \( T^{(j+1)}_0 \). Denote the diameter of \( x'_{x_0} \) by \( r'_0 \). By construction, there exists \( \alpha \in C \) such that \( \alpha(0) < x'_0 \). For all \( \lambda \in \Lambda \), let \( x_{\lambda} \) (resp. \( x'_{\lambda} \)) be the point of diameter \( r_0 \) (resp. \( r'_0 \)) such that \( \alpha(\lambda) < x'_{\lambda} < x_{\lambda} \).

Let us first prove that (1) holds if \( \alpha(\lambda) = e_{\lambda} \) is an edge of \( T^{(j+1)}_{\lambda} \). Indeed, \( x_{\alpha(\lambda),r} = x_{\beta(\lambda),r} \in e_{\lambda} \) for all \( \beta \in C \). Thus any analytic conjugacy between \( f \) and \( f \) is an immediate consequence of the combination of Proposition 4.8, Theorem D, and a classical removability result [FvdP04, Proposition 2.7.13].

**Proof of Theorem C.** The first assertion follows immediately from Proposition 4.8 and Theorem D. If \( J(\lambda_0) \subseteq A^1 \), then \( J(\lambda_0) \) is analytically removable, by Theorem 2.4 (see [FvdP04, Proposition 2.7.13]). Thus any analytic conjugacy between \( f_{\lambda_1} : B(\lambda_1) \to B(\lambda_1) \) and \( f_{\lambda_2} : B(\lambda_2) \to B(\lambda_2) \) extends to an analytic automorphism of \( A^1_{\lambda_0} \). Therefore \( f_{\lambda_0} \) and \( f_{\lambda} \) are affinely conjugate for all \( \lambda \in \Lambda \). Since there are finitely many elements in Poly_d.
that are affinely conjugate to \( f_{\lambda_0} \), we conclude that \( f_\lambda = f_{\lambda_0} \) for all \( \lambda \in \Lambda \). Thus the second assertion holds.

\[ \square \]

5. Polynomials with Julia critical points

The aim of this section is to prove Theorem [A] and Corollary [B]

5.1. Bounded Fatou components of tame polynomials. For a point \( x \in A_k^1 \) and a polynomial \( f \), denote by \( \omega(x) \) the \( \omega \)-limit set of \( x \) under \( f \). The following result, due to Trucco, shows that the orbit of \( x \in J(f) \cap \mathbb{H} \) accumulates inside a hyperbolic ball around the base point \( x_f \) of radius proportional to \( \text{dist}_\mathbb{H}(x,x_f) \).

**Proposition 5.1** ([Tru14, Proposition 7.1]). Suppose that \( f \) is a tame polynomial of degree at least 2. If \( x \in J(f) \cap \mathbb{H} \), then for all \( y \in \omega(x) \) we have that \( y \in \mathbb{H} \) and

\[
\text{dist}_\mathbb{H}(y,x_f) \leq d^{d-1} \text{dist}_\mathbb{H}(x,x_f).
\]

One direction of Theorem [A] is a consequence of the following:

**Corollary 5.2.** If \( f \in \text{Pol}_d \) is a tame polynomial in the closure of the tame shift locus, then \( J(f) \subset A^1 \).

**Proof.** Suppose on the contrary that \( J(f) \cap \mathbb{H} \neq \emptyset \). Pick \( x_0 \in J(f) \cap \mathbb{H} \). By Proposition 5.1 for all \( y \in \omega(x_0) \) we have that \( \text{dist}_\mathbb{H}(f^{\circ n}(y),x_f) \leq R \) for some \( R > 0 \). Let \( D \) be a neighborhood of \( f \) in \( \text{Pol}_d \) such that for all \( g \in D \) we have that \( f(x) = g(x) \) for all \( x \) such that \( \text{dist}_\mathbb{H}(x,x_f) \leq R \). It follows that every \( y \in \omega(x_0) \subset \mathbb{H} \) lies in the filled Julia set \( K(g) \) for all \( g \in D \). Therefore, \( D \) is free of polynomials in the tame shift locus.

A necessary and sufficient condition for the absence of bounded Fatou components is provided by our following result:

**Proposition 5.3.** Let \( f \in \text{Pol}_d \) be a tame polynomial. Then \( \text{Crit}(f) \subset B(f) \cup J(f) \) if and only if \( J(f) \subset A^1 \).

**Proof.** If \( J(f) \subset A^1 \), then \( A^1_{\text{pol}} = B(f) \cup J(f) \) clearly contains all the critical points of \( f \). We assume that \( \text{Crit}(f) \subset B(f) \cup J(f) \) and proceed by contradiction to show that \( J(f) \subset A^1 \). Hence, suppose that there exists \( x \in J(f) \setminus A^1 \). For each \( j \geq 0 \), set \( x_j := f^{\circ j}(x) \) and regard \( |x_j,x_f| \) as an open subtree equipped with the vertices \( \{x_j^{(n)}\}_{n \geq 1} \) formed by all iterated preimages of \( x_f \) in \( |x_j,x_f| \) in decreasing order (i.e. \( x_j^{(n)} \supset x_j^{(n+1)} \) for all \( n \)). Then \( f^{\circ n}(x_j^{(n)}) = x_f \) and \( x_j^{(n)} \to x_j \) as \( n \to \infty \), see [Tru14, Proposition 3.6].

We claim that there exists a uniform \( N \geq 1 \) such that the degree of \( f \) on \( |x_j,x_j^{(n)}| \) is 1 for all \( j \geq 0 \) and \( n \geq N \). For otherwise, there would exist a critical point \( c, n_k \to \infty \) and \( j_k \geq 0 \) such that \( \{x_j^{(n_k)}\}_{k \in [c,x_f]} \) is a decreasing sequence. Then both \( x_j^{(n_k)} \) and \( x_j \) would converge, as \( k \to \infty \), to some \( y \geq c \). By Proposition 5.1, we would have that \( y \in J(f) \cap \mathbb{H} \) and therefore \( D_y \subset K(f) \), which contradicts that \( \text{Crit}(f) \subset B(f) \cup J(f) \) if \( D_y \setminus \{y\} \) is contained in \( J(f) \setminus B(f) \).

Now from Lemma 2.2 it follows that \( f^{\circ (\ell+1)} \) maps \( |x_0^{(N+\ell+1)},x_0^{(N+\ell)}| \) isometrically onto \( |x_{\ell+1}^{(N)},x_{\ell+1}^{(N-1)}| \) for any \( \ell \geq 0 \). Moreover, there are only finitely many arcs of the form \( |x_{\ell+1}^{(N)},x_{\ell+1}^{(N-1)}| \). Therefore, the length of \( |x_0^{(N+\ell+1)},x_0^{(N+\ell)}| \) is uniformly bounded below for all \( \ell \). We conclude that

\[
\text{dist}_\mathbb{H}(x,x_f) \geq \sum_{\ell \geq 0} \text{dist}_\mathbb{H}(x_0^{(N+\ell+1)},x_0^{(N+\ell)}) = \infty,
\]

and hence \( x \in A^1 \), which contradicts the assumption that \( x \in J(f) \setminus A^1 \).

\[ \square \]
5.2. Moving critical points to the basin of infinity. The following result establishes a strong form of density of polynomials with all their critical points passive. To state the result let us agree that if $\Lambda \subset \mathbb{C}_v$ is a disk and $\lambda_0 \in \Lambda$, a maximal disk $\Lambda'$ in $\Lambda \setminus \{\lambda_0\}$ is an open disk of radius $s$ such that $s = |\lambda' - \lambda_0|$ for all $\lambda' \in \Lambda'$.

**Lemma 5.4.** Let $\Lambda$ be a disk of radius $r > 0$ parametrizing a critically marked analytic family $\{f_\lambda\}$ of tame polynomials in $\text{Poly}_d$ with the same base point $x_\Lambda$. Given $\lambda_0 \in \Lambda$, for any $s < r$ sufficiently close to $r$, there exist a maximal disk $\Lambda' \subset \Lambda \setminus \{\lambda_0\}$ of radius $s$ such that the following hold:

1. all the critical points of $\{f_\lambda\}$ are passive in $\Lambda'$; and
2. if $c_\lambda(\lambda)$ is active in $\Lambda$, then $c_\lambda(\lambda') \in B(\lambda')$ for all $\lambda' \in \Lambda'$.

**Proof.** Let $c_1(\lambda), \ldots, c_k(\lambda)$ be the active critical points in $\Lambda$. For all $j$ we have let $g_{j,n}(\lambda) := f_\lambda^n(c_j(\lambda))$. Note that $\sup\{|g_{j,n}(\lambda)| : \lambda \in \Lambda\}$ converges to $\infty$ as $n \to \infty$. Thus, there exists $N_j$ and $s_j < r$ such that $|x_\Lambda| < r_{j,n}(t) := \sup\{|g_{j,n}(\lambda)| : |\lambda - \lambda_0| \leq t\}$ for all $s_j < t < r$ and $n \geq N_j$. Pick $s < r$ arbitrarily close to $r$ in the value group so that $s > s_j$ for all $j = 1, \ldots, k$. Then $g_j, N_j$ maps all but finitely maximal open disks contained in $|\lambda - \lambda_0| = s$ onto a maximal open disk contained in the “sphere” $S_j := \{z \in A^1 : |z| = r_{j,N_j}(s)\}$. Since $S_j \subset B(\lambda)$ for all $\lambda \in \Lambda$, we may choose a maximal open disks $\Lambda'$ contained in $|\lambda - \lambda_0| = s$ such that for all $j = 1, \ldots, k$, we have $g_{j,N_j}(\Lambda') \subset S_j$ and the conclusions of the lemma hold.

**Corollary 5.5.** Suppose $\{f_\lambda\}$ is a critically marked analytic family of tame polynomials in $\text{Poly}_d$ parametrized by a disk $\Lambda$ with the same base point $x_\Lambda$. Assume that $c_1(\lambda), \ldots, c_k(\lambda)$ are passive critical points in $\Lambda$ and that $c_{k+1}(\lambda), \ldots, c_{d-1}(\lambda)$ are active critical points in $\Lambda$. Given $\lambda_0 \in \Lambda$ there exists a disk $\Lambda' \subset \Lambda$, such that the following hold:

1. all the critical points are passive in $\Lambda'$;
2. for all $k + 1 \leq j \leq d - 1$ and all $\lambda' \in \Lambda'$, we have $c_j(\lambda') \in B(\lambda')$; and
3. for all $1 \leq j \leq k$, if $c_j(\lambda_0) \in J(\lambda_0)$, then $c_j(\lambda') \in J(\lambda')$ for all $\lambda' \in \Lambda'$.

**Proof.** Given $\lambda_0$, consider a disk $\Lambda'$ furnished by Lemma 5.4. Hence (1) and (2) hold for $\Lambda'$. Assume that $c_j(\lambda_0) \in J(\lambda_0)$ and $c_j(\lambda)$ is passive in $\Lambda$. Then $c_j(\lambda) \in K(\lambda)$ for all $\lambda \in \Lambda$. Denote by $\delta(\lambda) \geq 0$ the diameter of the component of $K(\lambda)$ containing $c_j(\lambda)$. To establish (3) for $c_j$ we must show that $\delta(\lambda) = 0$ for all $\lambda \in \Lambda'$. We proceed by contradiction. Suppose that there exists $\lambda' \in \Lambda'$ so that $\delta(\lambda') > 0$. Then by Propositions 17 and 18 for every closed disk $\overline{\Lambda'} \subset \Lambda'$ with $\lambda' \in \Lambda''$, there exists $\rho > 0$ such that the dynamics of $f_\lambda$ over the corresponding $\rho$-trimmed dynamical core $T_\lambda$ is isometrically conjugate to the action of $f_\lambda$ on $T_\lambda$ for all $\lambda \in \Lambda''$. In particular, the iterated preimages of $x_\Lambda$ in $|c_j(\lambda), x_\Lambda|$ isometrically correspond to those of $x_\Lambda$ in $|c_j(\lambda'), x_\Lambda|$. It follows that $\delta(\lambda) = \delta(\lambda')$ for all $\lambda \in \overline{\Lambda'}$ for every closed disk $\overline{\Lambda'}$ contained in $\Lambda'$. Therefore, $\delta(\lambda) = \delta(\lambda')$ for all $\lambda \in \Lambda'$. Modulo change of coordinates, we may assume that $c_j(\lambda) = 0$ for all $\lambda \in \Lambda$. Then, given $y$ with $|y| \leq \delta(\lambda')$ and $n \geq 1$, we have $|f_\lambda^n(y)| \leq |x_\Lambda|$ for all $\lambda \in \Lambda'$. Since $\Lambda'$ is a maximal open disk in $B = \{\lambda : |\lambda - \lambda_0| \leq s\}$ and $\lambda \mapsto f_\lambda^n(y)$ is analytic on $B$, we have $|f_\lambda^n(y)| \leq |x_\Lambda|$ for all $\lambda \in B$. In particular, since $\lambda_0 \in B$, we have that $\delta(\lambda_0) \geq \delta(\lambda')$ which contradicts our assumption that $c_j(\lambda_0) \in J(\lambda_0)$.

Recall that the space of polynomials $\text{Poly}(d)$ with constant critical multiplicity was introduced in Section 2.2. Given $f \in \text{Poly}(d)$ with at least one non-escaping critical point, the following result guarantees the existence of a one parameter analytic family passing through $f$ with certain properties that will be need in the proofs of Theorem A and Corollary B.
Lemma 5.6. Assume that \((f, c_1, \ldots, c_k)\) is a nonsimple tame polynomial in \(\text{Poly}(d)\) such that \(c_i \in \mathcal{B}(f)\) for \(1 \leq i \leq j\). If \(j < k\), then there exists a nonconstant analytic family of tame polynomials \(\{(f_\lambda, c_1(\lambda), \ldots, c_k(\lambda))\}\) parametrized by a disk \(\Lambda \subset \mathbb{C}_v\) with \(0 \in \Lambda\) such that \((f, c_1, \ldots, c_k) = (f_0, c_1(0), \ldots, c_k(0))\), the base point of \(f_\lambda\) is independent of \(\lambda \in \Lambda\) and the Böttcher coordinates of \(c_1(\lambda), \ldots, c_j(\lambda)\) are constant.

Proof. For each \(i \leq j\), let \(n_i\) be such that \(|f^{n_i}(c_i)| > |x_f|\). Recall that \(\text{Poly}(d)\) is naturally identified with elements of a hyperplane in \(\mathbb{C}_v^k \times \mathbb{C}_v\) which are in the complement of a finite collection of linear subspaces of \(\mathbb{C}_v^k \times \mathbb{C}_v\) and that tame polynomials are open in \(\text{Poly}(d)\) (endowed with the sup-norm). Thus we may assume that there exists a polydisk \(P \subset \text{Poly}(d)\) of tame polynomials containing \(f\). Shrinking \(P\) if necessary, we may also assume that for all \(g \in P\) the basepoint of \(g\) is \(x_f\) and that for every \(i \leq j\) we have \(|g^{n_i}(c_i)| > |x_f|\). Without loss of generality we identify \(P\) with \(\overline{D}^k\) where \(\overline{D} = \{z \in \mathbb{C}_v : |z| \leq 1\}\).

For \(n \geq 1\), consider \(r = (r_1, \ldots, r_n)\) where \(r_i \in |\mathbb{C}_v|\) for all \(i\). Recall that \(P(r)\) denotes the corresponding polydisk and \(T_n[r]\) is the associated Tate algebra, as in the end of Section 2.2. If \(r_i = 1\) for all \(i\), we simply write \(T_n\) for the Tate algebra and \(\overline{D}^n\) for the corresponding polydisk. Note that the maximal spectrum \(\text{Sp}(T_n)\) is in natural bijection with \(\overline{D}^n\).

By Lemma 2.6 we have that \(F_i(g) := 1/\phi_i(g^{n_i}(c_i))\) lies in the Tate algebra \(T_k\) for all \(i \leq j\). Let \(I\) be the ideal generated by \(F_1(g) - F_1(f), \ldots, F_j(g) - F_j(f)\) in \(T_k\), which we may assume to be a radical ideal. Our aim is to find the analytic image of one-dimensional disk contained in the vanishing locus \(V \subset \overline{D}^k\) of \(I\). Since \(j < k\), it follows that \(T_k/I\) is a reduced affinoid algebra of dimension at least 1. Enlarge \(I\), if necessary, to obtain an ideal \(J\) such that \(A = T_k/I\) is reduced and has dimension 1. Let \(B\) be a normalisation of \(A\). That is, \(B\) is an integrally closed affinoid algebra of dimension 1 and \(A \hookrightarrow B\) is a finite morphism.

Take a maximal ideal \(\mathfrak{m}_f\) in \(B\) which maps onto \(f\) via \(\text{Sp}(B) \rightarrow \text{Sp}(A) \hookrightarrow \text{Sp}(T_k) \rightarrow \overline{D}^n\). Since the dimension of \(B\) is 1, the localization \(B_{\mathfrak{m}_f}\) is regular (e.g. see [AM69, Proposition 9.2]). By [PvdP04, Theorem 3.6.3], \(B\) can be represented by \(T_n/(G_1, \ldots, G_s)\) such that a \((n - 1) \times (n - 1)\)-minor of the Jacobian matrix \((\partial G_i/\partial \lambda_m)\), modulo \((G_1, \ldots, G_s)\), is not in \(\mathfrak{m}_f\). To fix ideas, let us assume that \(\mathfrak{m}_f = (\lambda_1, \ldots, \lambda_n)\) (i.e. it corresponds to the origin in \(\overline{D}^n\)) and let \(J' := (G_1, \ldots, G_s)\). By the Implicit Function Theorem (e.g. see [Ser06, II.III.10]), after relabeling if necessary, there exists \(\epsilon_i > 0\) in \(|\mathbb{C}_v|\) such that \(B_\epsilon = T_n[\epsilon_i]/J'\) is isomorphic to \(T_1[\epsilon_1]\). Thus, \(T_k \rightarrow A \rightarrow B \rightarrow B_\epsilon \rightarrow T_1[\epsilon_1]\) gives a nonconstant analytic map \(\lambda \mapsto f_\lambda\) from \(\{\lambda : |\lambda| \leq \epsilon_1\}\) into \(\overline{D}^n\) such that \(f_0 = f\) and \(f_\lambda \in V\) for all \(\lambda\) (i.e. the escaping critical points \(c_1(\lambda), \ldots, c_j(\lambda)\) of \(f_\lambda\) have constant Böttcher coordinates). \(\square\)

Now we can prove Theorem [A] and Corollary [B].

Proof of Theorem [A]. Since we have already established Corollary 5.2, now assume that \(f \in \text{Poly}_{d}\) is a tame polynomial such that \(\mathcal{J}(f) \subset \mathbb{A}^1\) and \(\text{Crit}(f) \cap \mathcal{J}(f) \neq \emptyset\). To prove that \(f\) is in the closure of the tame shift locus, it will be sufficient to show that there exists an arbitrarily close tame polynomial \(g \in \text{Poly}_{d}\) such that \(\mathcal{J}(g) \subset \mathbb{A}^1\) and \(|\text{Crit}(g) \cap \mathcal{J}(g)| < |\text{Crit}(f) \cap \mathcal{J}(f)|\).

By Lemma 5.6 we can consider a nonconstant one-dimensional critically marked analytic family \(\{f_\lambda\}_{\lambda \in \Lambda}\) in \(\text{Poly}_{d}\) parametrized by an open disk with \(0 \in \Lambda \subset \mathbb{C}_v\) such that \(f_0 = f\), the base point \(x_f\) is of \(f_\lambda\) is independent of \(\lambda \in \Lambda\) and the Böttcher coordinates of escaping critical points of \(f_0\) are constant. Since \(\mathcal{J}(f_0) \subset \mathbb{A}^1\), we conclude that at least one critical point of \(f_\lambda\) is active in \(\Lambda\); for otherwise, by Theorem [C], the map \(f_\lambda\) would be affine.
conjugate to $f_0$ for all $\lambda \in \Lambda$, which is a contradiction since affine conjugacy classes in $\text{Poly}_d$ are finite. By Corollary 5.5 there exists $\lambda' \in \Lambda$ such that all the critical points of $f_{\lambda'}$ are in $B(f_{\lambda'})$ or in $\mathcal{J}(f_{\lambda'})$ and the number of Julia critical points of $f_{\lambda'}$ is strictly smaller than the number of Julia critical points of $f_0$. It follows that $g = f_{\lambda'}$ is such that $\mathcal{J}(g) \subset \mathbb{A}^1$ and $\#(\text{Crit}(g) \cap \mathcal{J}(g)) < \#(\text{Crit}(f) \cap \mathcal{J}(f))$. □

Proof of Corollary B. Suppose on the contrary that $c_i$ is passive. Corollary 5.5 implies that the critical point $c_i(g)$ is contained in $\mathcal{J}(g)$ for all $g$ in a sufficiently small neighborhood of $f$, which contradicts Theorem A. □

APPENDIX A. A LOCAL LEMMA

For the sake of completeness we present here the proof of a result which easily follows along the lines of Hensel’s Lemma.

Lemma A.1. Consider two polynomials $f, g \in \mathbb{C}[z]$ and suppose that

1. $f(x_G) = x_G = g(x_G)$,
2. $T_{x_G} f = T_{x_G} g$, and
3. $f'(x_G) = x_G$.

Then there exist an affinoid $V$ containing $x_G$ and an analytic function $h : V \to h(V)$ such that

(a) $f \circ h = g$ in $V$, and
(b) $h(x_G) = x_G$ and $T_G h = \text{id}$.

Proof. Denote by $d \geq 2$ the degree of $f$. For $\rho > 1$ sufficiently close to 1, there exists $0 < s < 1$ such that $|f(z) - g(z)| \leq s$ for all $z \in \mathbb{A}^1$ with $|z| \leq \rho$. Choose $s < \mu < 1$ and consider $1 < r < \rho$ such that:

$$\mu r^d < 1 \quad \text{and} \quad sr^d < \mu.$$

Let $V$ be a closed (rational) affinoid contained in $D(0, r)$ and containing $x_G$ in its interior such that $\text{diam}(x) > \mu$ for all $x \in \partial V$ and the following hold for all $z \in V \cap \mathbb{A}^1$:

$$\mu r^d < |f'(z)| \quad \text{and} \quad sr^d < \mu \cdot |f'(z)|^2.$$

Observe that if $z \in V \cap \mathbb{A}^1$ and $w \in \mathbb{A}^1$ is such that $|w| \leq \mu$ then $z + w \in V$ and, moreover,

$$|f'(z + w)| = |f'(z)|;$$

indeed, for some polynomials $a_j(z)$ of degree at most $d$ with coefficients in the closed unit ball $\mathcal{O} \subset \mathbb{C}$,

$$|f'(z + w) - f'(z)| = \left| \sum_{j \geq 1} a_j(z) w^j \right| \leq s^d |w| < |f'(z)|.$$

Given $x \in V \cap \mathbb{A}^1$, for $n \geq 0$, let

$$z_0(x) = x,$$

$$w_n(x) = \frac{f(z_n(x)) - g(x)}{f'(z_n(x))},$$

$$z_{n+1}(x) = z_n(x) + w_n(x).$$

Note that $|w_0(x)| \leq s/|f'(x)| < s \sqrt{\mu/sr^d} < \mu$. We claim that for all $n \geq 0$,

$$|f(z_{n+1}(x)) - g(x)| < \mu |f(z_n(x)) - g(x)|,$$

$$|w_{n+1}(x)| < \mu |w_n(x)|.$$
Let us proceed by induction. We will omit the case \( n = 0 \) for the first inequality since the inductive step is very similar, so consider \( n \geq 1 \) and assume that the inequalities hold for all \( k < n \). Observe that

\[
 f(z_n(x) + w_n(x)) - g(x) = \sum_{j \geq 2} f_j(z_n(x))w_n(x)^j
\]

for some \( f_j(z) \in \mathcal{O}[z] \) of degrees at most \( d \). Hence

\[
 |f(z_{n+1}(x)) - g(x)| \leq r^d|w_n(x)|^2 = r^d |f(z_n(x)) - g(x)|^2 \frac{1}{|f'(z_n(x))|^2} \leq r^d |f(z_n(x)) - g(x)| \frac{s}{|f'(z_n(x))|^2} < \mu |f(z_n(x)) - g(x)|,
\]

where the last inequality follows from \( |f'(x)| = |f'(z_n(x))| \) since \( |w_k(x)| < \mu \) for all \( k < n \). Similarly, noting that \( |w_n(x)| < \mu \) and hence \( |f'(z_{n+1}(x))| = |f'(z_n(x))| \), we have

\[
 |w_{n+1}(x)| \leq r^d|w_n(x)|^2 \frac{1}{|f'(z_n(x))|} = |w_n(x)| \frac{r^d |f(z_n(x)) - g(x)|}{|f'(z_n(x))|^2} \frac{r^d s}{|f'(z_n(x))|^2} < \mu |w_n(x)|.
\]

To finish the proof observe that \( w_n(x) \) converges to 0 uniformly in \( V \) as \( n \to \infty \), and hence \( z_n(x) \) converges to an analytic function \( z(x) \) in \( V \). Moreover,

\[
 f(z(x)) - g(x) = \lim_{n \to \infty} f(z_n(x)) - g(x) = 0.
\]

Also \( |z(x) - x| < \mu \) since \( |z_n(x) - x| < \mu \) for all \( n \). The conclusion of the lemma follows immediately by letting \( h(x) = z(x) \).

\[
 \square
\]

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