STABILITY EQUATION AND TWO-COMPONENT EIGENMODE FOR DOMAIN WALLS IN A SCALAR POTENTIAL MODEL

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Abstract

The connection between the supersymmetric quantum mechanics involving two-component eigenfunctions and the stability equation associated with two classical configurations is investigated and a matrix superpotential is deduced. The question of stability is ensured for the Bogomol’nyi-Prasad-Sommerfield (BPS) states on two domain walls in a scalar potential model containing up to fourth-order powers in the fields, which is explicit demonstrated using the intertwining operators in terms of two-by-two matrix superpotential in the algebraic framework of supersymmetry in quantum mechanics. Also, a non-BPS state is found to be non-stable via the fluctuation Hessian matrix.

Keywords: Matrix superpotential, BPS states on two domain walls, stability equation.

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I. INTRODUCTION

The algebraic framework of Supersymmetry in Quantum Mechanics (SUSY QM), as formulated by Witten [1, 2], may be elaborated from a 2-dimensional model. The SUSY QM generalization of the harmonic oscillator raising and lowering operators has been several applications [3, 4, 5, 6, 7]. The generalization of SUSY QM for the case of matrix superpotential, is well known in the literature for a long time. See, for example, for one-dimensional systems the works in Ref. [8, 9, 10, 11, 12, 13].

Recently, those has been investigated the superpotential associated with the linear classical stability from the static solutions for systems of two real scalar fields in (1+1) dimensions that present powers up to sixth-order [14] and one field [15]. In the case of two coupled scalar fields, the static field configurations were determined via Rajaraman’s trial orbit method [16, 17]. However, while Rajaraman has applied your method for the equation of motion, here one uses the trial orbit method for the first order differential equations [14, 18].

Also, recently, the reconstruction of a single real scalar field theory [15] from excitation spectra of inflation potential in framework of inflationary cosmology with the production of a topological defect has been implemented the via superpotentials [19], and a superfield formulation of the central charge anomaly in quantum corrections to soliton solutions with N=1 SUSY has been investigated [20].

In this work, for the time being, we will only our attention focus on SUSY QM for the two-component eigenmodes of the fluctuation operator; of course, for a zero mode we have a corresponding $\Psi_0(z) = \begin{pmatrix} \eta_0 \\ \xi_0 \end{pmatrix}$. Nevertheless, this is not necessarily true for an arbitrary non-trivial real eigenvalue $\omega^2_n$. Here, we shall determine the number of bound states. We also construct the two-by-two matrix superpotential for SUSY QM in the case of the stability equation associated with 2-dimensional potential model considered in literature [21]. We consider the classical configurations with domain wall solutions, which are bidimensional structures in 3+1 dimensions. They are static, non-singular, classically stable Bogomol’nyi [22] and Prasad-Sommerfield [23] (BPS) soliton (defect). Also, the non-BPS defect solution to field equations with finite localized energy associated with a real scalar field potential model is found.

The BPS states are classical configurations that satisfy the first order differential equa-
tions and the second order differential equations (equations of motion). On the other hand, non-BPS defect satisfies the equation of motion, but does not obey the first order differential equations.

Relativistic systems with topological defect appear to be extended objects such as strings or membranes, for which have been obtained a necessary condition for some stable stringlike intersections, providing the marginal stability curves [24]. Recently, marginal stability and the metamorphosis of BPS states have been investigated [25], the via SUSY QM, with a detailed analysis for a 2-dimensional $N = 2$–Wess-Zumino model in terms of two chiral superfields, and composite dyons in $N = 2$-supersymmetric gauge theories.

Domain walls have been recently exploited in a context that stresses their connection with BPS-bound states [26]. Let us point out that some investigations are interesting in connection with Condensed Matter, Cosmology, coupled field theories with soliton solutions [27, 28, 29, 30, 31, 32] and one-loop quantum corrections to soliton energies and central charges in the supersymmetric $\phi^4$ and sine-Gordon models in (1+1)-dimensions [33, 34]. The work of Ref. [34] reproduces the results for the quantum mass of the SUSY solitons previously obtained in Ref. [33]. Recently, the reconstruction of 2-dimensional scalar field potential models has been considered, and quantum corrections to the solitonic sectors of both potentials are pointed out [35]. The quantization of two-dimensional supersymmetric solitons is in fact a surprisingly intricate issue in many aspects [36, 37, 38, 39, 40, 41].

In the paper work, a connection between SUSY QM developments and the description of such a physical system with stability equation is expressed in terms of two-component wave functions. This leads to four-by-four supercharges and supersymmetric Hamiltonian matrices whose bosonic sectors possesses a fluctuation operator $(O_F)$ associated with two-component eigenstates in terms of BPS states.

This paper is organized as follows: In Section II, we investigate domain walls configurations for two coupled scalar fields; supersymmetric non-relativistic quantum mechanics with two-component wave functions is implemented in Section III. In Section IV, from a scalar potential model of two coupled scalar fields we analyze the stability of non-BPS and BPS states, and a new matrix superpotential to supersymmetric non-relativistic quantum mechanics with two-component wave functions is also implemented. Our Conclusions are presented in Section V.
II. DOMAIN WALLS FROM TWO COUPLED SCALAR FIELDS

In this section, we investigate a potential model in terms of two coupled real scalar fields in (1+1) dimensions that present classical soliton solutions known as domain walls.

The Lagrangian density for such a non-linear system, in natural units, is written as

\[ \mathcal{L} (\phi, \chi, \partial_\mu \phi, \partial_\mu \chi) = \frac{1}{2} (\partial_\mu \phi)^2 + \frac{1}{2} (\partial_\mu \chi)^2 - V(\phi, \chi), \]

where \( \eta^{\mu \nu} = \text{diag}(+, -) \) is the metric tensor. Here, the potential \( V = V(\phi, \chi) \) is any positive semidefinite function of \( \phi \) and \( \chi \), which can be written as a sum of perfect squares and must have at least two different zeros in order have present domain walls as possible solutions. The general classical configurations obey the equations:

\[ \Box \phi + \frac{\partial}{\partial \phi} V = 0, \quad \Box \chi + \frac{\partial}{\partial \chi} V = 0. \]

For static soliton solutions, the equations of motion become the following system of non-linear differential equations:

\[ \phi'' = \frac{\partial}{\partial \phi} V, \quad \chi'' = \frac{\partial}{\partial \chi} V, \]

where primes stand for differentiations with respect to the space variable. There appears in the literature a trial orbit method for the attainment of static solutions for certain positive potentials. This method yields, at best, some solutions to Eq. (3) and by no means to all classes of potentials \[16\]. In this work, the trial orbit method has been applied to systems of two coupled scalar fields containing up to fourth-order powers in the fields.

If the potential is a sum of perfect squares given by

\[ 2V(\phi, \chi) = \left( \frac{\partial}{\partial \phi} W \right)^2 + \left( \frac{\partial}{\partial \chi} W \right)^2 \]

one can deform the total energy

\[ E = \int_{-\infty}^{+\infty} dz \frac{1}{2} \left[ (\phi')^2 + (\chi')^2 + 2V(\phi, \chi) \right], \]

under the BPS form of the energy, consisting of a sum of squares and surface terms,
$$E = \int_{-\infty}^{+\infty} dz \left[ \frac{1}{2} \left( \phi' - \frac{\partial}{\partial \phi} W \right)^2 + \frac{1}{2} \left( \chi' - \frac{\partial}{\partial \chi} W \right)^2 + \frac{\partial}{\partial z} W \right]$$  \hspace{1cm} (6)$$

so that the first and second terms are always positive. In this case the lower bound of the energy (or classical mass) is given by the third term, viz.,

$$E \geq \left| \int_{-\infty}^{+\infty} dz \frac{\partial}{\partial z} W[\phi(z), \chi(z)] \right|,$$

where the superpotential \( W = W[\phi(z), \chi(z)] \) shall be discussed below.

The BPS mass bound of the energy result in a topological charge and is given by \( W_{ij} = W[M_j] - W[M_i] \), where \( M_i \) and \( M_j \) represent the vacuum states. It is required that \( \phi \) and \( \chi \) satisfy the BPS state conditions \([22]\):

$$\phi' = \frac{\partial W}{\partial \phi}$$

$$\chi' = \frac{\partial W}{\partial \chi}.$$  \hspace{1cm} (8)

Note that the BPS states saturate the lower bound so that \( E_{BPS} = T_{ij} = |W_{ij}| \), where \( W_{ij} \) is the central charge of the realization of \( N = 1 \) SUSY in 1+1 dimensions.

### III. SUSY QM AND LINEAR STABILITY

In this section, we present the connection between the SUSY QM and the stability of classical static domain walls against small quantum fluctuations, in which for BPS domain walls, the fluctuation Hessian is the bosonic part of a SUSY matrix fluctuation operator, and thus, it is positive definite.

Now, let us analyze the classical stability of the domain walls in the non-linear system considered in this work, by considering small perturbations around \( \phi(z) \) and \( \chi(z) \):

$$\phi(z, t) = \phi(z) + \eta(z, t)$$  \hspace{1cm} (9)$$

and

$$\chi(z, t) = \chi(z) + \xi(z, t).$$  \hspace{1cm} (10)
Next, let us expand the fluctuations $\eta(z, t)$ and $\xi(z, t)$ in terms of normal modes:

$$\eta(z, t) = \sum_n \epsilon_n \eta_n(z) e^{i\omega_n t}$$

and

$$\xi(z, t) = \sum_n c_n \xi_n(z) e^{i\omega_n t},$$

where $\epsilon_n$ and $c_n$ are chosen so that $\eta$ and $\chi$ are real (there are scattering states). Thus, considering a Taylor expansion of the potential $V(\phi, \chi)$ in powers of $\eta$ and $\xi$, by retaining the first order terms in the equations of motion (2), one gets a Schrödinger-like equation for two-component wave functions

$$O_F \Psi_n = \omega_n^2 \Psi_n, \quad \Psi_n = \begin{pmatrix} \eta_n(z) \\ \xi_n(z) \end{pmatrix}, \quad n = 0, 1, 2, \cdots$$

The frequency terms, $\omega_n^2$, arise from the 2-nd times derivatives and the matrix fluctuation operator becomes

$$O_F = \begin{pmatrix} -\frac{d^2}{dz^2} + \frac{\partial^2}{\partial \phi^2} V & \frac{\partial^2}{\partial \phi \partial \chi} V \\ \frac{\partial^2}{\partial \phi \partial \chi} V & -\frac{d^2}{dz^2} + \frac{\partial^2}{\partial \chi^2} V \end{pmatrix} \equiv \begin{pmatrix} -I & \frac{d^2}{dz^2} + V_F(z) \end{pmatrix},$$

where $I$ is the two-by-two identity matrix and

$$V_{F12}(z) = V_{F21}(z) \equiv \frac{\partial^2}{\partial \phi \partial \chi} V = \frac{\partial^2}{\partial \phi \partial \chi} V,$$

$$V_{F11}(z) \equiv \frac{\partial^2}{\partial \phi^2} V,$$

$$V_{F22}(z) \equiv \frac{\partial^2}{\partial \chi^2} V.$$

are the elements of fluctuation Hessian matrix, $V_F(z)$. Thus, the eigenvalues of the Hessian matrix at each stationary point $M_i$ provide a classification of gradient curves of the superpotential.

We can get the masses of the bosonic particles, using the results above, from the second derivatives of the potential:
\( m_\phi^2 \equiv \frac{\partial^2 V}{\partial \phi^2} \bigg|_{z \to \pm \infty} \)
\( m_\chi^2 \equiv \frac{\partial^2 V}{\partial \chi^2} \bigg|_{z \to \pm \infty} \) \hspace{1cm} (16)

Coming to the case of a single real scalar field \([15]\), we can realize, \textit{a priori}, the two-by-two matrix superpotential that satisfies the following Ricatti equation associated to the non-diagonal fluctuation Hessian \( V_F(z) \) given by Eq. (14):

\[
W^2(z) + W'(z) = V_F(z) = \begin{pmatrix} V_{F11}(z) & V_{F12}(z) \\ V_{F12}(z) & V_{F22}(z) \end{pmatrix}_{\phi = \phi(z), \chi = \chi(z)},
\]
whose solution is a Hermitian operator

\[
W(z) = \begin{pmatrix} f(\phi, \chi) & g(\phi, \chi) \\ g(\phi, \chi) & h(\phi, \chi) \end{pmatrix}_{\phi = \phi(z), \chi = \chi(z)} = W^\dagger,
\]

where we must use the BPS state conditions (8) and \( V_{Fij}(z) \) is given by the fluctuation Hessian. The Eqs. (17) and (18) are correct only for BPS states. Thus the bosonic zero-mode

\[
\Psi_-(0) = \Psi(z) = N \begin{pmatrix} \eta_0(z) \\ \xi_0(z) \end{pmatrix}, \quad \eta_0 = \frac{d\phi}{dz}, \quad \xi_0 = \frac{d\chi}{dz},
\]

satisfies the annihilation condition \( A^- \Psi_-(0) = 0 \), i.e., \( \frac{d}{dz} \Psi_-(0) = \mathbf{W} \Psi_-(0) \), where \( N \) is the normalization constant. According to the Witten’ SUSY model \([1, 3, 13]\), we have

\[
\mathcal{A}^\pm = \pm \mathbf{1} \frac{d}{dz} + \mathbf{W}(z), \quad \Psi^{(n)}_{\text{SUSY}}(z) = \begin{pmatrix} \psi_-^{(n)}(z) \\ \psi_+^{(n)}(z) \end{pmatrix},
\]

where \( \psi_\pm^{(n)}(z) \) are two-component eigenfunctions. In this case, the graded Lie algebra of the supersymmetry in quantum mechanics for the BPS states may be readily realized as

\[
H_{\text{SUSY}} = [Q_-, Q_+]_+ = \begin{pmatrix} \mathcal{A}^+ & 0 \\ 0 & \mathcal{A}^- \end{pmatrix}_{4 \times 4} = \begin{pmatrix} \mathcal{H}_- & 0 \\ 0 & \mathcal{H}_+ \end{pmatrix},
\]

\[
[H_{\text{SUSY}}, Q_\pm]_0 = 0 = (Q_-)^2 = (Q_+)^2,
\]

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where $Q_\pm$ are the 4 by 4 supercharges of Witten’s $N = 2$ SUSY model, viz.,

$$Q_- = \sigma_- \otimes A^-, \quad Q_+ = Q_+^\dagger = \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix} = \sigma_+ \otimes A^+,$$

(23)

with the intertwining operators, $A^\pm$ are given in terms of two-by-two matrix superpotential by Eq. (20) and $\sigma_\pm = \frac{1}{2}(\sigma_1 \pm i\sigma_2)$, where $\sigma_1$ and $\sigma_2$ are Pauli matrices. Of course, the bosonic sector of $H_{SUSY}$ is exactly the fluctuation operator given by $H_- = O_F = -\frac{d^2}{dz^2} + V_F(z)$, where $V_- = V_F(z)$ is the non-diagonal fluctuation Hessian. The SUSY partner of $H_-$ is $H_+ = -\frac{d^2}{dz^2} + V_+(z)$, where $V_+ = V_F(z) - W'$. 

Thus the $N=2$ SUSY algebra in (1+1)-dimensions in terms of real supercharges becomes

$$[Q_i^\mu, Q_j^\nu] = 2\delta^{ij}(\gamma^\mu \gamma^0)_{\nu,\beta} P_\mu + 2i(\gamma^5 \gamma^0)_{\nu,\beta} W_{ij}^\mu, \quad \nu, \mu, \alpha, \beta = 0, 1,$$

(24)

where $P_\mu = (P_0, P_1)$ is the energy-momentum operator, and the Majorana basis for two-by-two $\gamma$-matrices is realized in terms of the Pauli matrices, $\gamma^0 = \sigma_2, \gamma^1 = i\sigma_3$ and $\gamma^5 = \gamma^0 \gamma^1 = -\sigma_1$. In this case, the $SO(3,1)$ Lorentz symmetry in 3+1 dimensions reduces in 1+1 dimensions to the product of the Lorentz boost in 1+1 and a global symmetry associated with the fermion charge, viz., $SO(1,1) \otimes U(1)_R$.

In the rest frame $P_1 \to 0$ and $P_0 \to M = \sqrt{P^\mu P_\mu}$, the algebra (24) may be rewritten as

$$\begin{align*}
(Q_1^1)^2 &= (Q_2^1)^2 = M + |W_{ij}| \\
(Q_2^1)^2 &= (Q_1^2)^2 = M - |W_{ij}|
\end{align*}$$

(25)

with all other anticommutators vanishing, where the supercharge $W_{ij}$ is the lower bound for the classical mass, $M \geq |W_{ij}|$, so the BPS states saturate the bound states $E_{BPS} = |W_{ij}|$. Also, it is easy to show that the linear stability is satisfied, i.e.,

$$\omega_n^2 = \langle O_F \rangle = \langle A^+ A^- \rangle = \langle A^- \Psi_n \rangle^\dagger \langle A^- \Psi_n \rangle = |\bar{\Psi}_n|^2 \geq 0, \quad \bar{\Psi}_n = A^- \Psi_n.$$

(26)

as it has been anticipated. Note that we have set $O_F \equiv A^+ A^-$, where the intertwining operators $A^\pm$ of SUSY QM must be given in terms of the matrix superpotential, $W(z)$.

Therefore, the two-component normal modes in (13) satisfy $\omega_n^2 \geq 0$, so that the stability of the BPS domain wall is ensured, for $\phi \neq 0$ and $\chi \neq 0$. 

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A realization of SUSY QM model for this system must necessarily be modified \([13, 19]\). The essential reason for the necessity of modification is that the Ricatti equation given by \((17)\) is reduced to a set of first-order coupled differential equations. In this case, the superpotential is not necessarily according to the system described by one-component wave functions with SUSY in the context of non-relativistic Quantum Mechanics \([1, 3, 4, 5, 6, 7, 13]\), which is defined as
\[
W(x) = \frac{1}{\psi_0} \frac{d}{dx} \psi_0(x).
\]
Therefore, as the bosonic zero-mode is associated with a two-component eigenfunction, \(\Psi_0(z)\), one may write the matrix superpotential only in the form \(\frac{d}{dz} \Psi_0(z) = W \Psi_0(z)\) \([13]\). Also, we can find the eigenmodes of the supersymmetric partner \(H_-\) from those of \(H_- \equiv O_F\), and the spectral resolution of the hierarchy of matrix fluctuation operator may be achieved in an elegant way. In this case, the intertwining operators \(A^+(A^-)\) convert an eigenfunction of \(H_-(H_+)\) to an eigenfunction of \(H_+(H_-)\) with the same energy and simultaneously destroys (creates) a node of \(\Psi_n(z)\) \(\Psi_n^{(n+1)}(z)\).

IV. THE SCALAR POTENTIAL MODEL

Let us now consider a positive potential, \(V(\phi, \chi)\), with the explicit form:
\[
V(\phi, \chi) = \frac{1}{2} \lambda^2 \left( \phi^2 - \frac{m^2}{\lambda^2} \right)^2 + \frac{1}{2} \alpha^2 \chi^2 (\chi^2 + 4 \phi^2) + \alpha \lambda \chi^2 (\phi^2 - \frac{m^2}{\lambda^2}). \tag{27}
\]
This potential can be written as a sum of perfect squares. For static soliton solutions, the equations of motion become the following system of non-linear differential equations:
\[
\phi'' = \frac{\partial}{\partial \phi} V = 2 \lambda^2 \phi \left( \phi^2 - \frac{m^2}{\lambda^2} \right) + 2 \alpha \lambda^2 \phi (2 \alpha + \lambda)
\]
\[
\chi'' = \frac{\partial}{\partial \chi} V = -2 \alpha \chi (\frac{m^2}{\lambda} - \lambda \phi^2 - \alpha \lambda^2) + 4 \alpha^2 \chi \phi^2. \tag{28}
\]
The above potential contains only two free parameters, viz., \(\alpha\) and \(\lambda\). It is non-negative for all real \(\alpha\) and \(\lambda\). This potential is of interest because it has solutions like BPS and non-BPS, however, only for \(\alpha \lambda > 0\), does it have 4 minima in which \(V = 0\). Note that this potential has the discrete symmetry: \(\phi \rightarrow -\phi\) and \(\chi \rightarrow -\chi\), so that we have a necessary (but non-sufficient) condition that it must have at least two zeroes in order that domain walls can exist.
The superpotential $W(\Phi, \chi)$, as proposed in the literature, yields the component-field potential $V(\phi, \chi)$ of Eq. (27), which can be written as

$$W(\Phi, \chi) = \frac{m^2}{\lambda} \Phi - \frac{\lambda}{3} \Phi^3 - \alpha \Phi \chi^2,$$

where $\Phi$ and $\chi$ are chiral superfields which, in terms of bosonic $(\phi, \chi)$, fermionic $(\psi, \xi)$ and auxiliary fields $(F, G)$, are $\theta$—expanded as shown below:

$$\Phi = \phi + \bar{\theta} \psi + \frac{\theta \bar{\theta}}{2} F,$$
$$\chi = \chi + \bar{\theta} \xi + \frac{\theta \bar{\theta}}{2} G,$$

where $\theta$ and $\bar{\theta} = \theta^*$ are Grassmannian variables. The superpotential above, with two interacting chiral superfields, allows for solutions describing string like "domain Ribbon" defects embedded within the domain wall. It is energetically favorable for the fermions within the wall to populate the domain Ribbons [27].

It is required that $\phi$ and $\chi$ satisfy the BPS state conditions:

$$\phi' = -\lambda \phi^2 - \alpha \chi^2 + \frac{m^2}{\lambda},$$
$$\chi' = -2 \alpha \phi \chi.$$

Clearly, only the bosonic part of those superfields are relevant for our discussion and one should retain only them and the corresponding part of the $W$ function. Thus, the vacua are determined by the extrema of the superpotential, so that

$$\frac{\partial W}{\partial \phi} = 0$$

and

$$\frac{\partial W}{\partial \chi} = 0$$

providing four vacuum states $(\phi, \chi)$ whose values are listed below in what follows

$$M_1 = \left(-\frac{m}{\lambda}, 0\right)$$
\[ M_2 = \left( \frac{m}{\lambda}, 0 \right) \]
\[ M_3 = \left( 0, -\frac{m}{\sqrt{\lambda \alpha}} \right) \]
\[ M_4 = \left( 0, \frac{m}{\sqrt{\lambda \alpha}} \right). \] (34)

When the wall \( M_{13} \) is stable, the two vacuum states, \( M_1 \) and \( M_3 \), may be adjacent. They are energy-degenerated with the energies of the three walls \( M_{23}, M_{14}, M_{24} \). Of course, from (34), we can see that we have two more possible domain walls, viz., \( M_{12} \) and \( M_{34} \), with different energies. Indeed, in this work, the potential presents a \( Z_2 \times Z_2 \)-symmetry, so that one can build some intersections between the walls [29, 30, 31, 32].

The generalized system, given by Eq. (31), can be solved by the trial orbit method developed by Rajaraman [16]. Indeed, using the trial orbit below for the first order differential equations [14, 18]

\[ (\lambda - \beta)\varphi^2 + \alpha \chi^2 = \frac{m^2}{\lambda} - \gamma, \] (35)

we obtain a pair of defect solutions given by

\[ \varphi(z) = \sqrt{\frac{\gamma}{\beta}} \tanh(\sqrt{\beta \gamma} z) \]
\[ \chi(z) = \pm \frac{1}{\sqrt{\alpha}} \left( \frac{\gamma}{\beta} (\beta - \lambda) \tanh^2(\sqrt{\beta \gamma} z) + \frac{m^2}{\lambda} - \gamma \right)^\frac{1}{2}, \] (36)

where \( \beta \) and \( \gamma \) are two real parameters. Note that we have an elliptical orbit of the BPS domain wall only for the case \( \lambda > \beta \) and \( \gamma < \frac{m^2}{\lambda} \). When \( \lambda - \beta = \alpha \) and \( \gamma < \frac{m^2}{\lambda} \), the orbit becomes a semi-circle.

### A. Non-BPS Solutions

The non-BPS wall, for \( \varphi = 0 \), is described by the following equation of motion:

\[ \frac{d^2 \chi}{dz^2} = -2\alpha \chi \left( \frac{m^2}{\lambda} - \alpha \chi^2 \right), \] (37)

whose solution can not be associated with a first-order differential equation. However, the solution of the equation of motion connecting the vacua \( M_3 \) and \( M_4 \) is given by
\[ \chi(z) = \frac{m}{\sqrt{\lambda \alpha}} \tanh(\tilde{m} z), \quad \tilde{m} = \sqrt{\frac{\alpha}{\lambda}} m, \]

so, in this case, the fluctuation Hessian potential on the \( M_{34} \) wall becomes

\[ V_{NBPS}(z) = 2m^2 \begin{pmatrix} \frac{2\alpha}{\lambda} - (1 + \frac{2\alpha}{\lambda}) \text{sech}^2(\tilde{m} z) & 0 \\ 0 & \frac{\alpha}{2\lambda} - \frac{3\alpha}{\lambda} \text{sech}^2(\tilde{m} z) \end{pmatrix}. \]

Note that as \( \alpha \neq \lambda \) the tension of \( M_{34} \) is different from the \( M_{12} \) wall tension. But, if \( \alpha = \lambda = m \), we obtain

\[ T_{34} = \int_{-\infty}^{\infty} \left( \frac{1}{2}(\chi')^2 + V(0, \chi) \right) dz = m^2 \int_{-\infty}^{\infty} \text{sech}^4(mz)dz = \frac{4}{3} m, \]

which is equivalent to the tension \( T_{12} \). In the limit \( \alpha \ll \lambda \), we have \( \tilde{m} \ll m \), for very large range of \( z \), \( V_{NBPS} \sim \text{diag}(-2m^2, 0) \), which is constant and non-positive. Therefore, the corresponding Hessian in Eq. (39) must have negative energy states, so that in this range for the parameters \( \alpha \) and \( \lambda \), the non-BPS domain wall is not stable.

**B. BPS Solutions**

If we choose \( \gamma = \frac{m^2}{2\lambda} \) in the trial orbit above, we obtain the elliptical orbit

\[ \frac{\lambda^2}{m^2} \phi^2 + \frac{\lambda^2}{2m^2} \chi^2 = 1 \]

providing the following pair of BPS solutions

\[ \phi(z) = \frac{m}{\lambda} \tanh(\frac{m}{2} z) \]
\[ \chi(z) = \pm \sqrt{2} \frac{m}{\lambda} \text{sech}(\frac{m}{2} z), \]

for \( \beta = \frac{1}{2} \) and \( \alpha = \frac{1}{4} \).

In this case the four vacua become: \( (\pm \frac{m}{\lambda}, 0) \) and \( (0, \pm \frac{2m}{\lambda}) \). It is easy to show that these static two-field solutions of the BPS conditions are satisfied by the equations of motion given by Eq. (28).

This pair of BPS solutions represents a straight line segment between the vacuum states \( M_1 \) and \( M_2 \). The superpotential, in terms of the bosonic fields, leads to the correct value
for a Bogomol’nyi minimum energy, corresponding to the BPS-saturated state. Then, we see that, at classical level, according to Eq. (7), one may substitute

$$E_B^{\text{min}} = |W[\phi(z), \chi(z)]_{z=+\infty} - W[\phi(z), \chi(z)]_{z=-\infty}| = \frac{4m^3}{3\lambda^2},$$

(43)

which corresponds to the tension $T_{12}$ of the $M_{12}$ domain walls. Indeed, some tensions of the $M_{ij}$ domain walls are identical. Thus, the non-zero tensions of the domain walls are $T_{13} = T_{14} = T_{23} = T_{24} = \frac{1}{2}T_{12} = \frac{2m^3}{3\lambda^2}$. Note that these three walls satisfy a Ritz-like combination rule

$$T_{12} = T_{23} + T_{31},$$

(44)

which is a stability relation generally seen in N=2 SUSY. One has used the tension given by $T_{ij} = |W_{ij}|$ which is called the Bogomol’nyi mass bound.

The projection of the potential over the $\phi$ axis provides the double-well potential $V(\phi, 0) = \frac{\lambda^2}{2}(\phi^2 - m^2)$ for the kink-like domain walls given by Eq. (42).

In what follows we will discuss the stability of the two-field BPS solutions and how the Lie graded algebra in SUSY QM is readily realized via the factorization method. Note that, for the potential model considered in this work, according to Eq. (27), we can readily arrive at the following elements of fluctuation Hessian matrix, $V_F(z)$:

$$V_{F12}(z) = V_{F21}(z) \equiv \frac{\partial^2}{\partial \chi \partial \phi} V = \frac{\partial^2}{\partial \phi \partial \chi} V = 4\alpha(2\alpha + \lambda)\phi\chi,$$

$$V_{F11}(z) \equiv \frac{\partial^2}{\partial \phi^2} V = 6\lambda^2\phi^2 - 2m^2 + 2\alpha(2\alpha + \lambda)\chi^2,$$

$$V_{F22}(z) \equiv \frac{\partial^2}{\partial \chi^2} V = 6\alpha^2\chi^2 + 2\alpha(2\alpha + \lambda)\phi^2 - \frac{2\alpha m^2}{\lambda},$$

(45)

Thus, from the Ricatti equation (17) associated to the non-diagonal fluctuation Hessian $V_F(z)$ given by Eq. (45), we obtain

$$W(z) = -2\lambda \begin{pmatrix} \phi \\ \frac{1}{4}\chi \\ \frac{1}{4}\phi \end{pmatrix}_{|\phi=\phi(z), \chi=\chi(z)},$$

(46)

where we have used the BPS state conditions (31). Therefore the bosonic zero-mode becomes
\[ \Psi^{(0)}(z) = \Psi_0(z) = N \begin{pmatrix} \text{sech}^2(mz) \\ \text{sech} \frac{2\alpha}{m} (mz) \end{pmatrix}, \] (47)

which satisfies the annihilation condition \( A^- \Psi^{(0)}(z) = 0 \). The Eqs. (47) and (46) are correct only for BPS states, with \( \alpha = \frac{\lambda}{4} \).

For the sector \( \chi = 0 \), and \( z \to \pm \infty, m^2_\phi = 4m^2 \). Indeed, a possible soliton solution occurs even when we choose \( \beta = \lambda \) and \( \gamma = \frac{m^2}{\lambda} \), so that it implies \( \chi = 0 \) with a BPS domain wall, \( M_{12} \), which belongs to the topological soliton of the \( \phi^4 \) model, whose solution is

\[ \phi(z) = \frac{m}{\lambda} \tanh(mz). \] (48)

In this sector of the BPS states, the fluctuation potential Hessian becomes

\[ V_{BPS}(z) = m^2 \begin{pmatrix} 6 \tanh^2(mz) - 2 & 0 \\ 0 & 2\alpha \frac{2 \alpha + \lambda}{\lambda} \tanh^2(mz) - \frac{2\alpha}{\lambda} \end{pmatrix}. \] (49)

In this particular case, the \( \chi = 0 \) and \( \phi = \text{kink (or anti-kink) configurations are the only BPS domain wall, so that } W(z) \text{ becomes diagonal:} \]

\[ W(z) = -2 \begin{pmatrix} m \tanh(mz) & 0 \\ 0 & m^{\frac{\alpha}{\lambda}} \tanh(mz) \end{pmatrix} \bigg|_{\phi=\phi(z), \chi=0}. \] (50)

Indeed, the bosonic sector of \( H_{SUSY} \) is exactly given by \( O_F = A^+ A^- \), which, as obtained from the stability Eq. (13), has the following zero-eigenmode:

\[ \Psi^{(0)}_{-1} = \begin{pmatrix} c_{(1)} \text{sech}^2(mz) \\ 0 \end{pmatrix}, \quad \Psi^{(0)}_{-2} = \begin{pmatrix} 0 \\ c_{(2)} \text{sech} \frac{2\alpha}{m} (mz) \end{pmatrix}, \] (51)

c\(_{(i)}\) being, \( i = 1, 2 \) the normalization constants of the corresponding ground state. Note that, if \( \alpha = \lambda \), we see that the two SUSY representations of the superpotential diagonal component become equivalent. One should also remark that \( \alpha \) and \( \lambda \) have been chosen both positive, thus \( \frac{2\alpha}{\lambda} > 0 \), so that as \( \xi^{(0)}(z) \) is a normalizable configuration.

We can see that \( V_{BPS} \) is a diagonal Hermitian matrix, then \( O_F \) is also Hermitian. Hence, the eigenvalues \( \omega_n^2 \) of \( O_{F11} \) and \( \tilde{\omega}_n^2 \) of \( O_{F22} \) are all real. In this case, the fluctuation operator is diagonal, so we have two representations of SUSY in quantum mechanics. We shall explicitly
show that \( \omega_n^2 \) are non-negative, the proof of which takes us to a solution of the Pöschl-Teller potential \([42]\). Indeed, the mode equations are decoupled and may be of two kinds, given by

\[
O_{F11}\eta_n \equiv -\frac{d^2}{dz^2}\eta_n - m^2(6\text{sech}^2(mz) - 4)\eta_n = \omega_n^2\eta_n \tag{52}
\]

and

\[
O_{F22}\xi_n \equiv -\frac{d^2}{dz^2}\xi_n - \frac{2m^2\alpha}{\lambda^2}(2\alpha + \lambda)\text{sech}^2(mz)\xi_n + m^2\left(\frac{2\alpha}{\lambda^2}(2\alpha + \lambda) - \frac{2\alpha}{\lambda}\right)\xi_n = \tilde{\omega}_n^2\xi_n. \tag{53}
\]

Note that, according to Eqs. (27) and (15), if \( \alpha = 0 \), the potential becomes \( V(\phi) = \frac{\lambda^2}{2}(\phi^2 - m^2)^2 \), so the stability equation is given by Eq. (52) and, therefore, there exists only the wall \( M_{12} \).

Now, we can see that both types of solutions exist only for certain discrete eigenvalues. Let us perform the transformation, \( mz = y \), so that, upon comparison with the equation (12.3.22) in \([42]\), we obtain the following eigenvalues:

\[
\omega_n^2 = m^2 \left\{ 4 - \left[ \frac{5}{2} - (n + \frac{1}{2}) \right]^2 \right\}. \tag{54}
\]

In this case, we find only two bound states associated with the eigenvalues \( \omega_0^2 = 0 \) and \( \omega_1^2 = 3m^2 \) and, therefore, the BPS states are stable.

Similarly, for the second type of solutions, we obtain, from Eqs. (53) and (12.3.22) of Ref. [42], the following eigenvalues:

\[
\tilde{\omega}_n^2 = m^2 \left\{ \frac{2\alpha}{\lambda^2}(2\alpha + \lambda) - \frac{2\alpha}{\lambda} - \left[ \sqrt{\frac{2\alpha}{\lambda^2}(2\alpha + \lambda) + \frac{1}{4} - (n + \frac{1}{2})} \right]^2 \right\}. \tag{55}
\]

In this case, we find the number of bound states given as below:

\[
n = 0, 1, \ldots < \sqrt{\frac{2\alpha}{\lambda^2}(2\alpha + \lambda) + \frac{1}{4} - \frac{1}{2}}, \tag{56}
\]

from which, we get

\[
\tilde{\omega}_n^2 = m^2 \left\{ 16 - (4 - n)^2 \right\}, n = 0, 1, 2, 3, \tag{57}
\]

if we take \( \alpha = 2\lambda \).
We can see that, in this particular case, we have four bound states associated with the eigenvalues \( \tilde{\omega}_0^2 = 0 \), \( \tilde{\omega}_1^2 = 7m^2 \), \( \tilde{\omega}_2^2 = 12m^2 \) and \( \tilde{\omega}_3^2 = 15m^2 \). Also, the eigenvalue for the zero mode associated with the second type of solutions is given by

\[
\tilde{\omega}_0^2 = m^2 \left\{ \frac{2\alpha}{\lambda^2}(2\alpha + \lambda) - \frac{2\alpha}{\lambda} - \left[ \sqrt{\frac{2\alpha}{\lambda^2}(2\alpha + \lambda)} + \frac{1}{4} - \frac{1}{2} \right] \right\}, \tag{58}
\]

which is non-negative if we assume

\[
\frac{\alpha}{\lambda} \geq \frac{1}{\sqrt{2}} \left( \sqrt{\frac{2\alpha}{\lambda^2}(2\alpha + \lambda)} + \frac{1}{4} - \frac{1}{2} \right).
\]

Therefore we can conclude that the BPS state is stable.

V. CONCLUSIONS

In the present work, we investigate, in terms of fluctuation operators, the BPS domain wall states from a 2-dimensional model. The corresponding stability equations have been analyzed for both cases with and without supersymmetry. We have seen that domain walls associated with the two-field potentials have features that are not in the one-field models. However, if \( \phi \) is a kink or anti-kink configurations and \( \chi = 0 \), the second equation in (36) implies \( \beta = \lambda = \gamma = m \), so only this topological sector provides the interpolation between the (1,0) and (-1,0) vacua, which corresponds \( M_{12} \) BPS domain walls.

Also, we see that the elliptic orbit providing domain walls have internal structure, so that the \( \chi \) mesons are outside the host defect and the \( \phi \) mesons are inside. We present a particular pair of BPS defect that is mapped into an elliptic orbit, whose solutions map the Ising and Bloch walls in magnetic systems [43].

The connection between the (0,1) and (0,-1) vacua is realized via non-BPS solutions, for \( \alpha = \lambda = m \). In this case, both tensions \( T_{34} \) and \( T_{12} \) are equivalent. Our analysis of the Hessian (37) shows us that the non-BPS domain wall is not stable.

The stability equations associated with the soliton solutions of a simple model of two coupled real scalar fields have been investigated, by calculating the tensions of the domain walls. In all cases, the domain walls belonging to the BPS states, the zero-mode ground states become two-component eigenfunctions, for the particular case in which we have found the explicit form for the situations with different eigenvalues.
The result derived in this work for the matrix superpotential given by Eq. (46) in supersymmetry in Quantum Mechanics (SUSY QM) is valid only for BPS solutions with $\alpha = \lambda/4$.

Also, results on sn-type elliptic functions given in Ref. [44], for which boundary conditions on bound energy levels of a classical system defined by one single scalar field, and an extension to a relativistic system of two coupled real scalar fields in a finite domain in (1+1) dimensions have been investigated [45, 46]. According to our development, we can readily realize the SUSY QM algebra, in coordinate representation, for a 2-dimensional potential model in a finite domain.

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FIG. 1: The two coupled field potential model considered in this work.

FIG. 2: The polynomial superpotential that generates the BPS domain walls.
FIG. 3: Static classical configuration which represents the domain wall.

FIG. 4: The eigenfunction of the zero mode, for the topological sector $\chi = 0$ and $\phi=$kink.