NONCOMMUTATIVE GEOMETRY AND A DISCRETIZED VERSION 
OF KALUZA-KLEIN THEORY WITH A FINITE FIELD CONTENT

Nguyen Ai Viet and Kameshwar C.Wali

Department of Physics, Syracuse University, 
Syracuse, NY 13244-1130, U.S.A.

Abstract

We consider a four-dimensional space-time supplemented by two discrete points 
assigned to a $\mathbb{Z}_2$ algebraic structure and develop the formalism of noncommutative 
geometry. By setting up a generalised vielbein, we study the metric structure. Metric 
compatible torsion free connection defines a unique finite field content in the model 
and leads to a discretized version of Kaluza-Klein theory. We study some special cases 
of this model that illustrate the rich and complex structure with massive modes and 
the possible presence of a cosmological constant.

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1 Introduction

Mathematical framework based on classical differential manifolds and the associated algebras of smooth functions and their differentiable structures has provided so far the necessary algebraic and geometric tools to construct quantum field theories to describe elementary particle interactions. However, in spite of great progress in our understanding of these interactions, problems remain. The inadequacies of the Standard Model as a fundamental theory are too well emphasized in the literature to merit repetition here. String theory’s claim as a fundamental theory that unifies all interactions including gravity is still far from being established. It certainly has not made a convincing contact with the experimental world and has not provided so far new insights into the successes of the Standard Model. These and other considerations beg for new mathematical ideas and new ways to explore physics at small scales.

Connes’ recent development of noncommutative geometry (NCG) has provided such new ideas and new tools to construct particle physics models based on a geometric picture in which Higgs fields can be introduced as geometric objects on an equal footing with the gauge fields. The Higgs fields trigger spontaneous symmetry breaking in a natural way in the Standard Model and Grand Unified Theories (GUT’s). Among several approaches to NCG, Connes’ approach that we will follow in this paper is based on enlarging the usual four-dimensional space-time by including additional discrete dimensions. This leads to more than one copies of space-time and enables one to introduce different symmetries on different copies.

Connes’ NCG can also be thought of as describing a discretized version of Kaluza-Klein theories that ordinarily aim at incorporating internal symmetries of elementary particles and unify their interactions. Instead of compactified continuous space degrees of freedom, in NCG we have a countable number of discrete points. With this in mind, one may ask how to introduce gravity in this space-time. The first step in this direction was taken by Chamseddine, Felder and Fröhlich, who gave a generalization of the basic notions - vierbein, spin connection and curvature - of Riemannian geometry in the context of the new framework. As a result, they obtained Einstein’s gravity along with Brans-Dicke scalar field. More recently, several others have introduced some refinements, but have obtained essentially the same results.

In this and a previous short paper, we have followed a different track. Guided by the Kaluza-Klein theory, we consider the two-point internal space as a discretization of the fifth dimension. We keep all the allowed fields by assuming the most general form for the vielbein and therefore, vector and scalar fields appear naturally along with tensor fields. With two discrete point internal space, these fields come in pairs and in each pair, one field is massless and the other is massive. In the conventional

* Klimčík et al. independently, have considered gravity together with a vector field (without the Brans-Dicke field). Madore has also discussed the possibility of having vectors fields together with gravity in a somewhat different NCG with matrix algebras $M_\mathbb{n}$.
Kaluza-Klein theories, the particle spectrum consists of infinite number of massive modes. Usually, one resorts to truncation of the spectrum, but then one runs into inconsistencies in constructing realistic theories [15].

In the previous paper [12], we had a restricted version of the theory due to the choice of a hermitian connection. We showed that the zero mode sector of the Kaluza-Klein theory emerged without truncation. In this extended version, we consider a general formalism dictated by NCG. We shall see that the torsion free and metric compatibility conditions impose a very strong constraint on the field content of the theory. In the more general case, these conditions allow, beside the tensor, vector and scalar fields, two new dynamical fields $\alpha(x)$ and $\beta(x)$. The $\beta(x)$ field rescales the metric on one sheet and therefore it acts like a dilaton field in conventional theories. However, we find it has a mass term unlike in the conventional theories. Furthermore, if we assumed $\beta$ to be a constant everywhere, it gives rise to a cosmological constant. Similarly, the $\alpha(x)$ field rescales the vector and the scalar fields on one sheet and when we assume it to be constant, it makes the vector field massive. From physical considerations, if we adhere to metric compatibility, then torsion is not arbitrary, but it is determined in terms of the metric and allows to have both massless and massive fields. In other words, a metric compatible torsion provides the raison d'etre for the massive modes in a theory that has originally only zero modes. This is a new and beautiful feature of NCG.

The paper is organized as follows: In the next section, we extend the formalism in [12] by including more details about the two-point internal space and noncommutative differential calculus. In Section 3, we set up the vielbein in an orthonormal basis and discuss the structure of the metric that follows. We also give the definitions of various inner products that are necessary in our computations. In Section 4, after defining the generalized connection, covariant derivative and torsion, we use the metric compatibility condition to obtain the torsion free connection which can be used to compute the generalised Ricci scalar curvature and the Lagrangian. Section 5. is devoted to some special cases and the final section to a summary and discussion of the results.

2 Two-point internal space, noncommutative differential calculus

2.1 Basic elements of noncommutative geometry

Let us consider a physical space-time manifold $\mathcal{M}$ extended by a discrete internal space of two points to which we assign a $\mathbb{Z}_2$-algebraic structure. Hence, besides the space-time variable, we will have a new discrete variable denoted by an element $h \in \mathbb{Z}_2 = \{e, r \mid e^2 = e, r^2 = e, er = re = r\}$. With this extended space-time, the customary
algebra of smooth functions $C^\infty(\mathcal{M})$ is generalized to $\mathcal{A} = C^\infty(\mathcal{M}) \oplus C^\infty(\mathcal{M})$ and any generalized function $F \in \mathcal{A}$ can be written as
\[ F(x) = f_+(x)e + f_-(x)r, \tag{2.1} \]
which can be viewed as a formal expansion by the $Z_2$ variable. We can represent the elements of the $Z_2$ algebra by $2 \times 2$ matrices:
\[ e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad r = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.2} \]
Then the function $F(x)$ is represented by a $2 \times 2$ matrix
\[ F = f_+(x)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + f_-(x)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} f_1(x) & 0 \\ 0 & f_2(x) \end{pmatrix}, \tag{2.3} \]
where $f_1, f_2$ are obvious combinations of $f_+, f_-$. To simplify notation, we will use the same mathematical symbols for abstract elements and their representations. In this paper we will use the small letters to denote the quantities of ordinary geometry and capital letters for generalized quantities of NCG.

The algebra $\mathcal{A}$ of smooth functions can be considered as the algebra of the generalized 0-forms $\Omega^0(\mathcal{M}) = C^\infty(\mathcal{M}) \oplus C^\infty(\mathcal{M})$. To go beyond the ordinary geometry, we must introduce a second important geometric ingredient, the Dirac operator $D \ [1, 2]$, that serves as an exterior derivative giving us the starting point of the noncommutative differential calculus. As a direct generalization of the usual exterior derivative $d = dx^\mu \partial_\mu$, $d^2 = 0$, we will assume that it has the form $D = d + Q$, where $Q$ is the part of the exterior derivative, that comes from derivations over the $Z_2$ internal variable. Connes has given it \([2]\) formally as
\[ D : (f_1, f_2) \longrightarrow (df_1, df_2, m(f_2 - f_1), m(f_1 - f_2)), \tag{2.4} \]
where $m$ is a parameter with dimension of mass or the inverse of length. Therefore, it is apparent that we can look upon the last two terms in Eq (2.4) as representing derivatives over the discrete dimensions.

We are seeking a realization of Eq (2.4) in which the operator $D$ appears more transparently as a derivative operator satisfying the Newton-Leibnitz rule. For this purpose, let us define derivatives as follows;
\[ D_\mu = \begin{pmatrix} \partial_\mu & 0 \\ 0 & \partial_\mu \end{pmatrix}, \quad \mu = 0, 1, 2, 3, \]
\[ D_5 = \begin{pmatrix} 0 & m \\ -m & 0 \end{pmatrix}, \tag{2.5} \]
and specify their action on the 0-form elements as
\[ D_N(F) = [D_N, F], \quad N = \mu, 5. \tag{2.6} \]
It is easy to verify that in the above representation, $D_N$ satisfies the Newton-Leibnitz rule,

$$D_N(FG) = D_N(F)G + FD_N(G).$$  \hspace{1cm} (2.7)

Hence, we can consider $D_N$ as derivations in the $\mathbb{Z}_2$-noncommutative geometry.

The generalized differential elements $DX^M$ have the following realizations

$$DX^\mu \doteq \begin{pmatrix} dx^\mu \\ 0 \end{pmatrix}, \quad \mu = 0, 1, 2, 3 ,$$

$$DX^5 \sigma^\dagger \doteq \begin{pmatrix} \theta \\ 0 \\ 0 \end{pmatrix},$$ \hspace{1cm} (2.8)

where $\theta$ is a Clifford element satisfying

$$\theta^2 = 1 , \quad \theta dx^\mu = -dx^\mu \theta.$$ \hspace{1cm} (2.9)

and

$$\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$ \hspace{1cm} (2.10)

The exterior derivative operator $D$ is given by

$$D \doteq (DX^\mu D_\mu + DX^5 \sigma^\dagger D_5 ) \equiv \begin{pmatrix} d \\ \theta_m & \theta_m \end{pmatrix},$$ \hspace{1cm} (2.11)

where $d$ denotes the exterior derivative on $\mathcal{M}$. The exterior derivative acts on $F = (f_1, f_2) \in \Omega^0(\mathcal{M})$ as follows:

$$DF \doteq (DX^\mu D_\mu + DX^5 \sigma^\dagger D_5 )F = \begin{pmatrix} df_1 \\ \theta_m(f_1 - f_2) \\ df_2 \end{pmatrix}. $$ \hspace{1cm} (2.12)

By placing the "discrete derivative" off-diagonal, we really mean that $Q$ is an outer automorphism. The importance of an outer automorphism was discussed by Balakrishna et al in Ref. [5]. Without this off-diagonal part, even with an extended algebra of 0-forms the geometry has only the commutative character as in the ordinary geometry [16].

Working in the Hilbert space of spinors, Connes [2, 3] choose the 'Γ-representation' of differential elements

$$\Gamma^\mu = \begin{pmatrix} \gamma^\mu \\ 0 \end{pmatrix}, \quad \Gamma^5 = \begin{pmatrix} \gamma^5 \\ 0 \end{pmatrix}.$$ \hspace{1cm} (2.13)

\hspace{1cm} \footnote{Here we have used a slightly different definition for $DX^5$ compared with the one in Ref [12]. This new choice for $DX^5$ is more suitable to be a basis of 1-forms discussed in the next subsection.}
To compare with papers that use the representation (2.13), let us note that our \( DX^\mu \) and \( DX^5 \sigma^i \) correspond to \( \Gamma^\mu \) and \( \Gamma^5 \) respectively. Hence, their Dirac operator in the \( Z_2 \)-noncommutative geometry has the self-adjoint realization

\[
\mathcal{D} \doteq \Gamma^N D_N \equiv \begin{pmatrix} \partial & \gamma^5 m \\ \gamma^5 m & \partial \end{pmatrix}.
\]

(2.14)

In our formalism \( \theta \) is not necessarily \( \gamma_5 \) but can be any Clifford element depending on the content of the matter field. As we are working only with the pure geometric sector, we will not refer to its concrete meaning. The triplet \((\mathcal{A}, D, \mathcal{H})\), where \( \mathcal{H} \) is a Hilbert space of matter fields is the basic ingredient of NCG. We have already assumed that \( \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \) in representing the algebra \( \mathcal{A} \) and the Dirac operator \( D \) by \( 2 \times 2 \) matrices. The algebra \( \mathcal{A} \) completely replaces the concept of an underlying manifold. In NCG, calculations can be done formally without explicit derivatives as in Eqs (2.5) and (2.6). We introduce such an object just to show that NCG is a direct generalisation of the ordinary geometry, that was traditionally founded on the notion of a tangent space at a point. It is worth noting that Dubois-Violette, Kerner and Madore [14] have also introduced derivatives when discussing a different NCG with matrix algebras \( M_n \).

2.2 Wedge product and generalized differential forms

We can extend the space of derivatives of 0-forms in Eq (2.12) to the space of 1-forms \( \Omega^1(\mathcal{M}) \), where any 1-form \( U \in \Omega^1(\mathcal{M}) \) is defined as

\[
U \doteq DX^N U_N = \begin{pmatrix} dx^\mu u_{1\mu}(x) & \theta u_2(x) \\ \theta u_1(x) & dx^\nu u_{2\nu} \end{pmatrix},
\]

(2.15)

where \( U^\mu, U_5 \) are elements of \( \Omega^0(\mathcal{M}) \).

Let us note that the hermitian conjugate of an 1-form is also an 1-form

\[
U^\dagger = U_N^\dagger DX^N = DX^\mu U^\mu + DX^5 \tilde{U}_5,
\]

(2.16)

where we have introduced the notation \( \tilde{F} \):

\[
\text{if } F \in \mathcal{A}, \quad F = \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix}, \quad \text{then } \tilde{F} = \begin{pmatrix} f_2 & 0 \\ 0 & f_1 \end{pmatrix}.
\]

(2.17)

It is straightforward to generalize the definition of the wedge product to construct higher differential forms and differential algebra by defining

\[
DX^\mu \wedge DX^\nu \doteq \begin{pmatrix} dx^\mu \wedge dx^\nu & 0 \\ 0 & dx^\mu \wedge dx^\nu \end{pmatrix} \equiv -DX^\nu \wedge DX^\mu,
\]

\[
DX^5 \wedge DX^\mu \doteq \begin{pmatrix} 0 & \theta dx^\mu \\ \theta dx^\mu & 0 \end{pmatrix} \equiv -DX^\mu \wedge DX^5,
\]

\[
DX^5 \wedge DX^5 \doteq 0
\]

(2.18)
Alternately, we could have postulated \( DX^5 \land DX^5 \neq 0 \) and recover the construction of Coquereaux et al. \[17\]. To do this, of course, we have to assume that the additional dimension represented by the discrete variables is an odd dimension to have a commuting differential element instead of the anti-commuting one. Considering our space-time as a discretized version of Kaluza-Klein theory, we continue to treat the internal space as an even dimension on an equal footing with the space-time coordinates and hence the wedge product \((2.18)\).

A general \( p \)-form \( W_p \in \Omega^p \) is defined as

\[
W_p = DX^{N_1} \land \ldots \land DX^{N_p} W_{N_1 \ldots N_p},
\]

(2.19)

where \( W_{N_1 \ldots N_p} \) are generalised 0-forms.

The exterior derivative \( DW_p \in \Omega^{p+1} \) of a \( p \)-form \( W_p \in \Omega^p \) is defined to be

\[
DW_p = (DX^\mu \land DX^{N_1} \land \ldots \land DX^{N_p} D_\mu + DX^5 \land DX^{N_1} \land \ldots \land DX^{N_p} D_5) W_{N_1 \ldots N_p},
\]

(2.20)

The wedge product \( W_{1p} \land W_{2q} \in \Omega^{p+q} \) of a \( p \)-form \( W_{1p} \in \Omega^p \) and a \( q \)-form \( W_{2q} \in \Omega^q \) is given as follows

\[
W_{1p} \land W_{2q} = DX^{N_1} \land \ldots \land DX^{N_p} \land DX^{N_{p+1}} \land \ldots \land DX^{M_{p+q}} (W_{1} \cdot W_{2})_{N_1 \ldots N_p N_{p+1} \ldots N_{p+q}},
\]

(2.21)

where

\[
(W_{1} \cdot W_{2})_{N_1 \ldots N_p N_{p+1} \ldots N_{p+q}} = W_{1N_1 \ldots N_p} W_{2N_{p+1} \ldots N_{p+q}},
\]

(2.22)

if there is one 5 index among \( N_1, \ldots, N_p \) indices, and

\[
(W_{1} \cdot W_{2})_{N_1 \ldots N_p N_{p+1} \ldots N_{p+q}} = \tilde{W}_{1N_1 \ldots N_p} W_{2N_{p+1} \ldots N_{p+q}},
\]

(2.23)

if there is no 5 index among \( N_1, \ldots, N_p \) indices.

We have the following essential properties for the exterior derivative,

\[
D^2 W_p = 0, \quad \forall \ p,
\]

\[
D(W_p \land W_q) = DW_p \land W_q + (-1)^p W_p \land DW_q.
\]

(2.24)

The noncommutative character of our geometry is reflected in the fact that \( W_p \land W_q \) and \( W_q \land W_p \) are not related in general to each other by a simple factor as in the case of ordinary commutative geometry.

Although, in what follows, the geometrical objects we construct resemble those of ordinary geometry, their noncommutative character dictates a specific order in their definitions.
2.3 Inner product and signature in "flat space-time"

In 'flat' NCG, we can define the signature as

\[ G^{MN} = G(DX^M, DX^N) = \langle DX^M, DX^N \rangle = \eta^{MN}, \]  

(2.25)

where \( \eta^{MN} = (-, +, +, +) \). In the 'Γ-representation' the inner product is simply a Clifford trace (not to be confused with the trace over the \( 2 \times 2 \) matrix indices),

\[ \eta^{MN} = \frac{1}{4} \text{Tr}(\Gamma^M \Gamma^N) \]  

(2.26)

3 The generalized metric and an orthonormal basis

In this section we discuss the setting up of a generalized vielbein and the ensuing metric structure of the assumed space-time.

3.1 The generalized vielbein

In Riemann-Cartan geometry, the existence of a metric structure on a manifold is equivalent to the assumption that there exists an orthonormal basis of vierbein, that are 1-forms. We extend this idea to NCG and assume, as in [12] that there exists a generalized vielbein \( \{ E^A \} (A = a, 5) \). \( E^A = DX^M E^A_M \) are 1-forms in the \( Z_2 \)-noncommutative geometry with the general form

\[ E^a = \left( \begin{array}{c} e^a_1 \\ \theta f^a_1 \\ e^a_2 \\ \end{array} \right) = DX^\mu E^a_\mu + DX^5 F^a, \quad a = 0, 1, 2, 3, \]

\[ E^5 = \left( \begin{array}{c} a_1 \\ \theta \phi_2 \\ a_2 \end{array} \right) = DX^\mu A_\mu + DX^5 \Phi, \]  

(3.1)

where \( e^a_1, e^a_2 \) are vielbein on \( \mathcal{M} \), \( a_1, a_2 \) are 1-forms on \( \mathcal{M} \) and \( f^a_1, f^a_2, \phi_1, \phi_2 \) are real functions on \( \mathcal{M} \). We use a \( 5 \) index in the orthonormal basis to distinguish it from the index \( 5 \) in the general one.

As in the usual Riemannian geometry, we still have a degree of freedom to choose the following forms for vielbein without any loss of generality:

\[ E^a = \left( \begin{array}{c} e^a_1 \\ 0 \\ e^a_2 \end{array} \right) = DX^\mu E^a_\mu \quad a = 0, 1, 2, 3, \]

\[ E^5 = \left( \begin{array}{c} a_1 \\ \theta \phi_2 \\ a_2 \end{array} \right) = DX^\mu A_\mu + DX^5 \Phi. \]  

(3.2)
In Ref [12] we have considered the self-adjoint vielbein
\[ E^a \doteq \begin{pmatrix} e^a & 0 \\ 0 & e^a \end{pmatrix} = DX^\mu e^a_\mu \]
\[ E^5 \doteq \begin{pmatrix} a & \theta \phi \\ \theta \phi & a \end{pmatrix} = DX^\mu a_\mu + DX^5 \phi(x). \] (3.3)

Here we will consider the general case (3.2).

### 3.2 \( DX^M \) basis and vielbein in two different representations

In the last subsection \( DX^\mu \) and \( DX^5 \) have been chosen as pure diagonal and pure off-diagonal respectively. Technically, it is convenient to choose such a representation, in which the rules of differential calculus in the previous section can still be used. The basis \( E^A \) can be used to formulate the structure equations and to read the field content of the theory conveniently. However, in NCG it is troublesome to use this basis to compute higher forms. The main reason is that \( E^A \) does not satisfy the anti-commutativity of the wedge product for all its components, hence higher forms do not have a unique expansion in this basis. Thus

\[ E^a \wedge E^b = - E^b \wedge E^a, \]
\[ E^5 \wedge E^5 = E^b \wedge E^c A_b E^a_\mu a_{-\mu} r - E^5 \wedge E^c E^a_\mu a_{-\mu} r, \]
\[ E^5 \wedge E^a = - E^b \wedge E^5 \tilde{E}^b_{\mu} E^a_\mu + E^b \wedge E^c A_c (\tilde{E}^b_{\mu} E^a_\mu - \delta^a_b). \] (3.4)

Spinors are defined locally in a locally flat basis of vielbein. Because of the local flatness of the orthonormal basis, \( E^a \) and \( E^5 \) can be represented by 'flat' \( \Gamma \) matrices as in Eq (2.13). The trace over the spinor indices can be taken only in this frame.

On the other hand, in a curved space the differential elements become curvilinear. In NCG, we will assume similarly that the basis of generalised 1-forms \( DX^\mu \), \( DX^5 \) will no longer be orthonormal in "curved" space time. In general, we can allow \( DX^\mu \) and \( DX^5 \) to mix with each other. So the basis vectors of the Hilbert space, in which \( DX^M \) are pure diagonal or pure off-diagonal will be combinations of two Hilbert spaces \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \). Hence, we cannot use the trace in the \( DX^M \) basis but always have to go back to the orthonormal basis to do so. In the basis \( DX^M \) the inner products defined in the paragraph 3.4 should be used to compute the metric as we will see later.

Finally, a comment about the existence of the vector \( A_\mu \) is in order. Intuitively, the existence of the vector field is related to the freedom to assign to a pair of point on different sheets the same coordinate system. The vector fields will appear if we mix \( DX^5 \) and \( DX^\mu \) to redefine \( DX^5 \) basis of the "curved" 1-forms as

\[ DX^5 \rightarrow DX^5 - DX^\mu D_\mu \Lambda(x), \] (3.5)

where \( \Lambda(x) \) is an arbitrary generalized function. This degree of freedom will guarantee the gauge invariance of the generalised interval defined in the next subsection.
3.3 The Metric and its structure

Having a vielbein we can always construct a metric tensor $G$. We will think of $G$ as a sesquilinear functional $G : \Omega^1 \times \Omega^1 \rightarrow A$, having the hermitian structure

$$G(UF, WH) = F^\dagger G(U, W) H, \quad \forall \ U, W \in \Omega^1, \ F, H \in \Omega^0. \quad (3.6)$$

In the $E^A$-basis, the metric is taken to be

$$G(E^A, E^B) = \eta^{AB}, \quad \eta^{AB} = diag(-1, 1, 1, 1). \quad (3.7)$$

In the $DX^M$-basis we will have

$$G^{MN} = G(DX^M, DX^N) = E^M_A \eta^{AB} E^N_B, \quad (3.8)$$

where $E^M_A$ are the inverses of $E^A_M$.

Explicitly the components of the metric are:

$$G^{\mu\nu} = \begin{pmatrix} g^{\mu\nu}_1 & 0 \\ 0 & g^{\mu\nu}_2 \end{pmatrix},$$

$$G^{\mu 5} = -A^{\mu} \Phi^{-1},$$

$$G^{5\mu} = -\Phi^{-1} A^{\mu},$$

$$G^{55} = \Phi^{-2}(1 + A^2). \quad (3.9)$$

where $g^{\mu\nu}_i = e^\mu_i \eta^{ab} e^\nu_b, \ i = 1, 2$ are the metrics on two sheets. The inverse metric satisfying $G^{MN}.G^{NK} = G^{KN}.G^{NM} = \delta^K_N$ is

$$G^{MN} = E^A_M \eta_{AB} E^B_N. \quad (3.10)$$

Explicitly,

$$G^{\mu\nu} = A_{\mu} A_{\nu} + \begin{pmatrix} g^{\mu\nu}_1 & 0 \\ 0 & g^{\mu\nu}_2 \end{pmatrix},$$

$$G^{\mu 5} = A^{\mu} \Phi,$$

$$G^{5\mu} = \Phi A^{\mu},$$

$$G^{55} = \Phi^2. \quad (3.11)$$

It is worth noting that the metric continues to be symmetric, although the vielbein is not Hermitian.

$$G_{MN} = G_{NM},$$

$$G^{MN} = G^{NM}. \quad (3.12)$$

Hence, we can define the generalised interval as

$$DS^2 = DX^M DX^N G_{MN}(x) \quad (3.13)$$
It is easy to check that the generalised interval is invariant under the gauge transformations $\text{(3.3)}$ and

$$A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \Lambda(x). \quad (3.14)$$

This feature is exactly the same as in Kaluza-Klein theory. In the locally flat frame, $E^A$ can be represented as flat $\Gamma$ matrices in Eq (2.13) and the metric can be computed by taking a trace over the local spinor indices to give $\text{(3.7)}$. This definition of metric in the locally flat basis $E^A$ is consistent with the metric defined in the diagonal representation of $DX^M$ via Eq (3.8).

### 3.4 Inner products of forms and the volume element

In our computations in the next section, we need to introduce inner product of one- and two-forms and their extensions. For later convenience, we define them here. First, we will give the definitions of the inner products in the diagonal representation of the $DX^M$ basis and then show that they are consistent with the definitions in the locally flat representation of the vielbein $E^A$.

To start with, we note that metric structure on a curved manifold defines an inner product in the algebra $\Omega^1(M)$ of generalized 1-forms. We denote the metric as the sesquilinear and hermitian inner product of two 1-forms,

$$G^{MN} = \langle DX^M, DX^N \rangle. \quad (3.15)$$

Then the inner product of two arbitrary 1-forms $U = DX^M U_M$ and $V = DX^N V_N$ can be computed from Eq (3.15) as

$$\langle U, V \rangle = U_M^\dagger \langle DX^M, DX^N \rangle V_N = U_M^\dagger G^{MN} V_N. \quad (3.16)$$

The first extension of the inner product $\text{(3.15)}$ we need is the inner product of one 1-form and a tensorial product of two 1-forms. As in the case of the metric, we will require that this inner product be sesquilinear and possesses a hermitian structure. That is to say,

$$\langle U \otimes V, W \rangle = V^\dagger G(U, W), \quad \langle U, W \otimes V \rangle = G(U, W) V, \quad (3.17)$$

where $U, V$ and $W$ are 1-forms. Hence, this inner product in the $DX^M$ basis is

$$\langle DX^M \otimes DX^N, DX^P \rangle = DX^N \langle DX^M, DX^P \rangle = DX^N G^{MP}, \quad (3.18)$$

The second extension of the inner product is for two 2-forms. It also has a hermitian structure and is sesquilinear, and again it is sufficient for our purposes to give it in the same basis as the first inner extension:

$$\langle UF, WG \rangle = F^\dagger \langle U, W \rangle G, \quad (3.19)$$
where $U, W$ are 2-forms and $F, G$ are two arbitrary $2 \times 2$ matrices. We shall define
\[
< DX^M \wedge DX^N, DX^R \wedge DX^S > = \frac{1}{2} (G^{MS} G^{NR} - G^{MR} G^{NS}). \tag{3.20}
\]
and calculate (3.19) by expanding the two-forms in the $DX^M$ basis. All three inner products defined above are direct generalizations of the corresponding products in the Riemannian geometry.

The above formulae are valid for calculations in the $DX^M$ basis. In the orthonormal basis $E^A$, we can use another representation, where $E^A$ are not in the representation (3.2), but in the flat $\Gamma$ representation (2.13). In that representation, the inner product can be taken as trace over the local spinor indices as discussed previously. This is consistent with our definition of inner product, since
\[
G(E^A, E^B) = < E^A, E^B >, \tag{3.21}
\]
and in the $\Gamma$-representation
\[
G(E^A, E^B) = Tr(\Gamma^A \Gamma^B) = \eta^{AB}, \tag{3.22}
\]
In the representation (3.2)
\[
G(E^A, E^B) = < DX^M E_M^A, DX^N E_N^B >= E_M^A G^{MN} E_N^B \tag{3.23}
\]
Due to Eq (3.8) the consistency is obvious.

The volume element is given by
\[
D^5X = D^4X \sqrt{-det|G|} \tag{3.24}
\]
Here $det|G|$ denotes the determinant of our generalized metric defined in Eq(3.24) and is given by
\[
det|G| = \frac{1}{5!} \epsilon_{N_1 N_2 N_3 N_4 N_5} \epsilon_{M_1 M_2 M_3 M_4 M_5} G^{N_1 M_1} G^{N_2 M_2} G^{N_3 M_3} G^{N_4 M_4} G^{N_5 M_5} = \frac{1}{4!} \epsilon_{\nu_1 \nu_2 \nu_3 \nu_4} \epsilon_{\mu_1 \mu_2 \mu_3 \mu_4} G^{\nu_1 \mu_1} G^{\nu_2 \mu_2} G^{\nu_3 \mu_3} G^{\nu_4 \mu_4} G^{55} \equiv det|g| \phi 1, \tag{3.25}
\]
where $det|g|$ is the determinant of the 4-dimensional metric and $\epsilon$'s are the fully antisymmetric Levi-Civita tensors. The expression of generalised determinant is rather simple as the metric is diagonal.

4 Generalized connection, torsion and curvature
4.1 Covariant derivative and generalized structure equations

Following Connes [2], we define the generalized connection through a covariant derivative $\nabla$. The covariant derivative as a direct generalization of the ordinary one is an operation which acts on a 1-form satisfying the properties

$$\nabla : \Omega^1 \rightarrow \Omega^1 \otimes \mathcal{A} \Omega^1,$$

$$\nabla(UF) = (\nabla U)F + U \otimes DF. \quad (4.1)$$

Here the tensor product $\Omega^1 \otimes \mathcal{A} \Omega^1$ is generated by the elements \{\(U_1 \otimes U_2; U_1, U_2 \in \Omega^1\}\) with the relation $U_1 F \otimes U_2 = U_1 \otimes FU_2$ for any $F \in \Omega^0$.

Due to the property (4.1), the covariant derivative of an arbitrary 1-form is known if its action on a basis is given. Hence, the generalized covariant derivative is equivalently given by a set of generalized connection one-forms $\Omega^A_B \in \Omega^1$, the relation being

$$\nabla E^A = E^B \otimes \Omega^A_B. \quad (4.2)$$

The connection is said to be a Levi-Civita or metric compatible connection, if it satisfies $\nabla G = 0$. By definition, the covariant derivative of the metric functional is given by the following rule

$$\nabla G(U, W) = D(< U, W >) + < \nabla U, W > + < U, \nabla W >, \quad \forall U, W \in \Omega^1(\mathcal{M}). \quad (4.3)$$

By imposing the metric compatibility condition in Eq (4.3) in the orthonormal basis \((3.2)\) gives

$$\Omega^A_B = - \eta^{AD} \Omega^C_D \eta_{BC}. \quad (4.4)$$

Explicitly,

$$\Omega^a_{b\mu} = - \eta^{ad} \Omega^c_{d\mu} \eta_{cb},$$

$$\Omega^a_{b5} = - \eta^{ad} \tilde{\Omega}^c_{d5} \eta_{cb},$$

$$\Omega^a_{5\mu} = - \eta^{ab} \Omega^5_{b\mu},$$

$$\Omega^a_{55} = - \eta^{ab} \tilde{\Omega}^5_{b5},$$

$$\Omega^5_{5\mu} = 0,$$

$$\Omega^5_{55} = f(r)r, \quad (4.5)$$

where $f(r)$ is an ordinary function.

As the connection has been introduced independently of the metric structure, it cannot be in general determined in terms of the vielbein defined from the metric. However, the metric compatibility condition \((4.4)\) can be used effectively as a supplementary condition. We shall employ it along with torsion free structure equation to determine the connection.
The generalized Cartan structure equations define torsion and curvature of a given connection as follows:

\[ T^A = DE^A - E^B \wedge \Omega^A_B, \]  

\[ R^A_B = D\Omega^A_B + \Omega^A_C \wedge \Omega^C_B, \]  

(4.6)  

(4.7)

where \( T^A \) and \( R^A_B \) are 2-forms.

### 4.2 Torsion free connection

As in the case of the ordinary Riemannian geometry, we can impose the torsion free condition \( T^A = 0 \). Then the structure equation (4.6) reduces to

\[ DE^A = E^B \wedge \Omega^A_B, \]  

(4.8)

Eq (4.8) and the condition (4.4) that follow from metric compatibility "overconstrain" the connection in our geometry. However, we can determine it uniquely provided the vielbein (i.e. the metric structure) satisfies some restrictive conditions. This is in contrast with the Riemannian geometry, where metric compatibility determines the connection uniquely for any given metric, if torsion vanishes.

The structure equation (4.8) gives us the connection 1-forms \( \Omega^a \), which has the following components

\[ \Omega^\hat{a}_{\nu} = E^\mu_\mu (\frac{1}{2} F_{[\mu \nu]} + X_{(\mu \nu)}) \]

\[ \Omega^\hat{a}_{5} = \tilde{E}^a_\mu (\partial_\mu \psi + m a_{-\mu} - \tilde{A}_\mu f r) \]

\[ \Omega^a_{\nu} = E^\mu_\mu (\partial_\nu E^a_\mu - \partial_\mu E^a_\nu) - \frac{1}{2} (A_\mu \Omega^a_{\mu 5} - A_\nu \Omega^a_{5 \nu}) + Y^a_{(\mu \nu)} \]

\[ \Omega^a_{5} = \tilde{E}^a_\mu (\Phi \Omega^a_{\mu 5} + me_{-\mu} r - \tilde{A}_\mu \Omega^a_{5 \mu}) \]  

(4.9)

where \( F_{[\mu \nu]} = \partial_\mu A_\nu - \partial_\nu A_\mu \), \( u_\pm = u_2 \pm u_1 \) for any function \( U \). The connection 1-forms \( \Omega^a_{\hat{\delta}} \) and the symmetric tensor functions \( X_{(\mu \nu)}, Y^a_{(\mu \nu)} \) are to be determined by the metric compatible conditions (4.5).

From Eq (4.5), we derive the equations, that determine \( X_{(\mu \nu)} \)

\[ \Phi X_{(\mu \nu)} + \Phi X_{(\mu \nu)} = - \frac{1}{2} (\tilde{F}_{[\mu \nu]} \Phi - F_{[\mu \nu]} \Phi) - mr (\tilde{E}^b_\rho e_{-b \nu} - E_{\nu \mu} e_{-\rho} - A_\nu a_{-\rho} + \tilde{A}_\mu a_{-\nu} + \tilde{A}_\rho \partial_\nu \Phi + A_\nu \partial_\rho \Phi) \]

This equation implies two independent equations. One of them is a constraint on vielbein

\[ e_1 = \beta e_2 = \beta e, \]

\[ a_1 = \alpha a_2 = \alpha a, \]
with the result

\[
\phi_1 = \frac{\phi_2}{\alpha} = \frac{\phi}{\alpha}, \quad (4.10)
\]

where \(\alpha\) and \(\beta\) are two arbitrary functions. In this case, \(e^{a}_\mu\), \(a_\mu\), \(\phi\), \(\beta\) and \(\alpha\) are independent dynamical variables. (As we shall see later, the function \(f\) in \(\Omega^{\hat{5}}\) without having a kinetic term, can be eliminated from the Lagrangian). The second equation determines \(X_{(\rho\nu)}\)

\[
X_{(\rho\nu)} = x_{(\rho\nu)} \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}, \quad (4.11)
\]

where

\[
x_{(\rho\nu)} = -\frac{1}{2\phi}m(\beta-1)^2g_{(\rho\nu)} + (a_\nu\frac{\partial_\rho\phi}{\phi} + a_\rho\frac{\partial_\nu\phi}{\phi}) - \frac{m}{2\phi}a_\rho a_\nu(\alpha-1)^2
\]

\[-\frac{1}{4}(a_\nu\frac{\partial_\rho\alpha}{\alpha} + a_\rho\frac{\partial_\nu\alpha}{\alpha})\]

With the vielbein satisfying (4.10) we can determine the connection 1-form uniquely with the result

\[
\Omega^{\hat{5}}_{\hat{5} \mu} = 0, \\
\Omega^{\hat{5}}_{\hat{5} 5} = f.r, \\
\Omega^{\hat{5}}_{\hat{5} \nu} = \frac{1}{2}e^\rho_\mu \left[ \left( f_{[\rho\nu]} - \frac{m}{\phi}(\beta-1)^2g_{(\rho\nu)} + (a_\nu\frac{\partial_\rho\phi}{\phi} + a_\rho\frac{\partial_\nu\phi}{\phi}) - \frac{m}{\phi}(\alpha-1)^2a_\rho a_\nu \\
- \frac{1}{4}(a_\nu\frac{\partial_\rho\alpha}{\alpha} + a_\rho\frac{\partial_\nu\alpha}{\alpha}) \right) \right], \\
\Omega^{\hat{5}}_{\nu 5} = e^\rho_\mu \left[ \left( \alpha^{-1} 0 \right) \right. \\
- \frac{1}{2}(a_\nu\frac{\partial_\rho\alpha}{\alpha} + a_\rho\frac{\partial_\nu\alpha}{\alpha}) \left( 0 0 \right) \\
\Omega^{\hat{5}}_{\hat{5} \nu} = \omega^{\hat{5}}_{\hat{5}\nu} + e^{a}\epsilon^b_\rho \left[ \left( g_{\tau\mu} - \frac{\partial_\mu\beta}{\beta} - g_{\rho\tau} \frac{\partial_\nu\beta}{\beta} \right) \left( 1 0 \right) \right. \\
+ \frac{1}{2}l_{[\rho\tau]} + \frac{m}{\phi}(\beta-1)^2(a_\tau g_{\mu\rho} - a_\rho g_{\mu\tau}) \left( 0 0 \right) \\
\Omega^{\hat{5}}_{\hat{5} 5} = \frac{1}{2\beta}e^{a_\tau}e^b_\rho \left[ m(1-\beta^2)g_{\rho\tau} r + a_\tau a_\rho r (m(\alpha^2-1) + f\phi) \right. \\
+ \frac{3}{2}\phi a_\tau\frac{\partial_\rho\alpha}{\alpha} r - \frac{3}{4}\phi a_\rho\frac{\partial_\alpha\phi}{\alpha} r + (a_\rho\partial_\rho\phi - a_\tau\partial_\alpha\phi) - \phi f_{[\tau\rho]} \right], \quad (4.12)
\]

where

\[
\omega^{a}_{b\nu} = \frac{1}{2}e^{a_\tau}e^b_\rho \left[ e^d_\tau(\partial_\rho e_{d\nu} - \partial_\nu e_{d\rho}) + e^d_\rho(\partial_\nu e_{d\tau} - \partial_\tau e_{d\rho}) \right. \\
+ e^d_\nu(\partial_\rho e_{d\tau} - \partial_\tau e_{d\rho}) \left. \right], \\
f_{[\tau\rho]} = \partial_\tau a_\rho - \partial_\rho a_\tau, 
\]

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\[
\begin{align*}
h_{\tau\rho} &= a_\tau \partial_\rho \phi - a_\rho \partial_\tau \phi, \\
l_{\tau\rho} &= a_\tau \partial_\rho \alpha - a_\rho \partial_\tau \alpha,
\end{align*}
\]
and \(\omega_a^b\) is the ordinary metric compatible and torsion free connection.

### 4.3 The generalized Ricci scalar curvature and the action

Eq (4.7) determines completely the curvature 2-forms \(R^A_B\) and hence the Ricci scalar once the set of the connection 1-forms \(\Omega^A_B\) are given. The Ricci scalar curvature in our case can be defined as follows

\[
R = \langle E^A \wedge E^B, R_{AB} \rangle
\]

where \(R_{AB} = \eta_{AC} R^C_B\). It is convenient to compute the Ricci scalar curvature in the \(DX^M\) basis. The curvature 2-form \(R_{AB}\) can be expanded in the form

\[
R_{AB} = DX^M \wedge DX^N R_{ABMN}
\]

The \(Z_2\) functions \(R_{ABPQ}\) are determined uniquely, due to the anti-commutativity of the diagonal basis \(DX^M\). The vielbein can be also expanded in terms of \(DX^M\).

Using the sesquilinearity and the hermicity of the inner product we can bring all the coefficients out and we are left with the inner products in \(DX^M\) basis as defined in Eq (3.20). It is straightforward to substitute this inner product into Eq (4.14) and compute the Ricci scalar curvature.

The action is defined as

\[
S = \frac{1}{m\kappa} Tr \left( \int dx^4 \sqrt{-\det G} R \right),
\]

where \(\kappa\) is a constant to be fixed later.

The integration over the discrete space follows naturally to be \(\frac{1}{m} Tr\).

### 5 Some particular cases

It is cumbersome but straightforward to compute the action based on Eq (4.16). The resulting action involves lengthy expressions including kinetic terms and cross-interaction terms for gravity, the vector field \(a_\mu(x)\) and the scalar fields \(\beta(x), \alpha(x)\) and \(\phi(x)\). The function \(f(x)\) in Eq (4.5) appears as an auxiliary field without a kinetic term. It can be eliminated from the action. Rather than present the full action, in what follows, we shall discuss some special cases that demonstrate the role \(\alpha\) and \(\beta\) play and their physical significance.
5.1 The zero modes sector of the Kaluza-Klein theory

In this case we choose $\beta = 1$, $\alpha = 1$. We recover the action of the previous paper \cite{12}. With the field redefinition $a \rightarrow a\phi$, the result is

$$L = \frac{1}{m\kappa} \int d^4x \sqrt{-\det g}(\phi r_4 - 2\Box \phi + \frac{\phi^3}{4}f_{\mu \nu}f^{\mu \nu}), \quad (5.1)$$

where $r_4$ is the four-dimensional Ricci scalar curvature. We see that the discretised version contains only the zero mode sector of the Kaluza-Klein theory.

As $\phi(x)$ and $a_\mu(x)$ are dimensionless we can introduce the dimensional parameters $v$ and $b$ with dimension of mass into the theory via

$$\phi(x) \rightarrow e^{\chi(x)}$$
$$a_\mu(x) \rightarrow \frac{a_\mu(x)}{b} \quad (5.2)$$

The Lagrangian (5.1) then assumes the form

$$L = \frac{1}{m\kappa} \int d^4x \sqrt{-\det g}e^{\chi(x)/v}(r_4 - \frac{2}{v^2}\partial_\mu \chi(x)\partial^\mu \chi(x) + \frac{e^{2\chi(x)}}{4b^2}f_{\mu \nu}f^{\mu \nu}), \quad (5.3)$$

and leads to the identification

$$\frac{m\kappa}{4\pi G^2 v^2} = \frac{16\pi G^2}{1}, \quad \frac{16\pi G^2 b^2}{1} = \frac{1}{1}, \quad (5.4)$$

in order to have standard expressions for the kinetic terms. The scalar field $\chi(x)$ and the vector field $a_\mu(x)$ are massless.

5.2 Massive dilaton and cosmological constant

In this subsection, we consider the case, where $a = 0, \alpha = \phi = 1$ to see the new features the dynamical variable $\beta(x)$ brings into the theory. The Lagrangian in this case reduces to

$$L = \frac{1}{16\pi G^2} \int d^4x \sqrt{-\det g}(\frac{1}{2}(\beta^2 + 1)r_4 + \frac{3}{2}\partial_\mu \beta \partial^\mu \beta$$
$$+ \frac{m^2}{4\beta^2}(\beta - 1)^3(2\beta^6 + 2\beta^4 + 7\beta^3 - 21\beta^2 + 9\beta - 5)) \quad (5.5)$$

By redefining $\beta \rightarrow \frac{\tilde{\beta}}{u}$, where $u$ is a parameter with dimension of mass, we obtain a dynamical scalar field $\tilde{\beta}(x)$. The Lagrangian now contains gravity and a dilaton with a highly nonlinear potential

$$L = \frac{1}{16\pi G^2} \int d^4x \sqrt{-\det g}(\frac{\tilde{\beta}^2 + u^2}{u^2}r_4 + \frac{3}{2u^2}\partial_\mu \tilde{\beta} \partial^\mu \tilde{\beta} + V(\tilde{\beta})) \quad (5.6)$$
To have the right factor for the kinetic term of $\tilde{\beta}$ the parameter $u$ should be given by

$$u = \sqrt{\frac{3}{m\kappa}}. \quad (5.7)$$

The potential is non-renormalisable as to be expected from a theory of gravity. The $V(\tilde{\beta})$ potential has a minimum at $\tilde{\beta} = u$. Expanding $\tilde{\beta}$ around $u$, we obtain a mass term for the $\tilde{\beta}$ field.

$$\mathcal{L}_{\text{mass}} = \frac{95}{4} m^2 \tilde{\beta}^2. \quad (5.8)$$

Hence, although on one sheet $\tilde{\beta}(x)$ rescales gravity and acts like a dilaton field of conventional theory, it has a mass in the present framework.

The mass of the $\beta$ dilaton is $\sim 4.87m$. If the theory is applied in the two left- and right-sheeted model of Connes and Lotts [3] with $m$ as the electroweak scale $\sim 246GeV$, then the $\beta$ dilaton has mass in the TeV range.

We can also imagine $\beta(x)$ to be just a constant. In that case $V(\beta)$ plays the role of the cosmological constant that vanishes at $\beta = 1$ (or $\tilde{\beta} = u$). The cosmological constant is positive (negative) for $\beta > 1$ ($\beta < 1$). It can be made arbitrarily small by taking its value arbitrarily close to unity. This feature has clearly important application in cosmology.

### 5.3 Mass term, quartic potential and higher derivative interactions for the vector field

In this subsection we consider the special case, where we have flat space-time, $\phi(x) = 1$. $\beta$ and $\alpha$ assumed to be constant other than unity. For simplicity we shall assume $\beta = \alpha$.

The Lagrangian in this case is given by

$$\mathcal{L} = \frac{1}{16\pi G^2} \int d^4x \sqrt{-\det|g|}(\mathcal{L}_{\text{kin}} + \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_{\text{high}} + \text{const}), \quad (5.9)$$

where

$$\mathcal{L}_{\text{kin}} = -\frac{1}{8\alpha^2}(2\alpha^3 - \alpha^2 - \alpha - 2)f_{\mu\nu} f^{\mu\nu},$$

$$\mathcal{L}_2 = m^2 a_{\mu} a^{\mu} (\alpha - 1)^4 (\alpha + 1)^3,$$

$$\mathcal{L}_4 = m^2 (a_{\mu} a^{\mu})^2 \frac{8\alpha}{(\alpha + 1)^3 (\alpha + 1)(\alpha^3 + 1)(\alpha - 3/2)},$$

$$\mathcal{L}_{\text{high}} = \frac{1}{8} \frac{(\alpha - 1)^2 (\alpha + 1)}{\alpha} (a_{\mu} f^{\mu\nu})^2. \quad (5.10)$$

The vector field $a_{\mu}(x)$ indeed has a nonvanishing mass term.
6 Summary and Conclusions

The noncommutative geometric approach à la Connes has provided new insights into the Standard Model of elementary particle interactions by providing a unified geometric description of gauge and Higgs particles and at the same time providing a Higgs potential with spontaneously broken symmetry. Since the approach is based on a fundamentally new structure of space-time, it is natural to ask how gravity fits into the picture and what are the consequences on the other interactions. Indeed as shown by one of the authors [18] of this paper, if one assumes the same underlying space-time structure for both gravity and electroweak interactions, one can predict the top quark and Higgs particle masses $m_t \sim 172 GeV$ and $m_H \sim 241 GeV$.

The present work is an extension of that in [12]. The noncommutative geometry is based on the algebra $\mathcal{A} = \mathcal{C}^\infty(M) \times Z_2$. The discrete $Z_2$ structure supplementing the smooth functions on the four-dimensional manifold allows one to introduce a Dirac operator that has a component corresponding to an outer automorphism of the algebra and develop a differential geometry that has close analogy with the usual Riemannian geometry. The discrete elements belonging to $Z_2$ may be considered as two discrete points of a fifth spatial dimension in Kaluza-Klein-type theories or alternately simply as giving rise to two independent copies of space-time. We have adhered to the first point of view in this paper.

We have considered in this paper the case of a torsion free, metric compatible connection. The ensuing lagrangian has a rich and complex structure with a finite field content including two additional dynamical scalar fields along with the fields introduced in the vielbein. To understand the structure in more physical terms, we have considered some special cases. First of all, if the fields $\alpha$ and $\beta$ are constants equal to unity, we obtain the previously studied case [12] in which there are only zero mass tensor, vector and scalar fields. Secondly, if we consider only gravity and the field $\beta(x)$, we find that while rescaling the metric, $\beta(x)$ acts as a dilaton field of conventional theories, it is different in that it has a mass term in the present framework. On the other hand if we treat $\beta(x)$ to be a constant everywhere, it gives rise to a cosmological constant term that vanishes at $\beta = 1$. It can be positive or negative depending upon whether $\beta > 1$ or $\beta < 1$ and can be made arbitrarily small by taking $\beta$ arbitrarily close to unity. Thus, the model can have application in Cosmology. As a third possibility, if we consider $\alpha$ and $\beta$ as constants we obtain a model in which the vector field is massive.

These special cases give rise to a rich variety of physical models. In the more general case when torsion is present the metric compatibility condition (4.4) is sufficient to determine uniquely a non-vanishing torsion in terms of the metric without imposing any constraints on the vielbein. (The Eq (4.10) now instead of being a constraint on the vielbein becomes a torsion determining equation). Hence, a pair of independent metric, vector and scalar fields can coexist and we expect that one field in each pair is massless and the other is massive. We defer the consideration of this case in a paper.
Finally, we note that such a general metric with two independent tensor, vector and scalar fields opens up new and intriguing possibilities for the Standard Model. In Connes-Lott model of right- and left-handed sheets, for instance, two different metrics on the two sheets implies that gravity couples differently to different chiralities. Distance measured by a left-handed beam is different from that measured by a right-handed one. Are these considerations relevant to the Standard Model? We do not know at present. But even the simplest version of NCG based on two sheets has all these possibilities suggests that it merits further study.

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