STATIONARY POLYHEDRAL VARIFOLDS MINIMIZE AREA

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Abstract. We prove that every stationary polyhedral varifold minimizes area in the following senses: (1) its area cannot be decreased by a one-to-one Lipschitz ambient deformation that coincides with the identity outside of a compact set, and (2) it is the varifold associated to a mass-minimizing flat chain with coefficients in a certain metric abelian group.

1. Introduction

The tangent cone to any 2-dimensional minimal variety (i.e., stationary varifold with a positive lower bound on density) is polyhedral (because the link is a stationary 1-varifold in the unit sphere, and thus is a finite union of geodesic arcs [AA76].) If the variety is area-minimizing in some sense, then the cone is also area-minimizing. Thus it is natural to ask: which 2-dimensional stationary cones, or, more generally, which m-dimensional stationary polyhedral cones, are area-minimizing in some sense?

In her celebrated 1976 paper [Tay76] about soap films, Jean Taylor classified all two-dimensional, multiplicity-one polyhedral cones in $\mathbb{R}^3$ that minimize area in the following sense: the area (without counting multiplicity) cannot be decreased by a compactly supported Lipschitz deformation. The Lipschitz deformation need not be one-to-one. She showed that there are only three such cones: the plane, the union of three halfplanes meeting at equal angles along their common edge, and the cone over a regular tetrahedron. However, there are many other stationary polyhedral cones. For example, in $\mathbb{R}^3$, there are seven other 2-dimensional, multiplicity-one stationary cones that have, away from the vertex, only triple-junction-type singularities [Tay76] pp. 501, 502. Taylor’s theorem leaves open the possibility that those other cones could be area-minimizing in some other sense. For example, one could ask

(1) Which stationary polyhedral cones minimize area with respect to deformations by compactly supported Lipschitz homeomorphisms of the ambient space?

(2) Which stationary polyhedral cones can be assigned orientations and multiplicities in some coefficient group in such a way that the cone is mass-minimizing as a flat chain for that coefficient group?

In this paper, we show that all stationary polyhedral cones have both of those properties. Indeed, the properties hold for every stationary polyhedral varifold, whether or not conical.

In particular, we prove

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Theorem 1. Suppose that $V$ is an $m$-dimensional, rectifiable varifold in an open subset $U$ of $\mathbb{R}^N$. Suppose that $V$ is stationary, that $V$ is supported in a finite union of affine $m$-planes, and that $V$ has finite mass. Then
\[
M(V) \leq M(f_# V)
\]
for every Lipschitz homeomorphism $f : U \to U$ such that $\{x : \phi(x) \neq x\}$ has compact closure in $U$.

Here $M(V)$ is the total mass of the varifold $V$. (Thus $M(V)$ is $\mu_V(\mathbb{R}^N)$ in the notation of [Sim83] or $\|V\|_m(\mathbb{R}^N)$ in the notation of [All72].)

Theorem 1 is a consequence of the following:

Theorem 2. Suppose that $\Gamma$ is a closed subset of $\mathbb{R}^N$ consisting of a finite union of $(m-1)$-dimensional polyhedra. Suppose that $V$ is an $m$-dimensional, compactly supported, rectifiable varifold in $\mathbb{R}^N$ such that

1. $V$ is supported in a finite union of affine $m$-planes, and
2. $V$ is stationary in $\mathbb{R}^N \setminus \Gamma$.

Then there is a metric abelian group $G$ and a polyhedral $m$-chain $A$ with coefficients in $G$ such that

1. $V$ is the varifold associated to $A$.
2. $\partial A$ is supported in $\Gamma$.
3. $A$ is mass-minimizing: if $A'$ is any other flat $m$-chain (with coefficients in $G$) such that $\partial A' = \partial A$, then $M(A) \leq M(A')$.

We can choose the coefficient group $G$ to be a certain finite-dimensional Euclidean space (namely $\Lambda_n \mathbb{R}^N$) with its associated norm. Alternatively, we can choose $G$ to be a discrete group with the property:

(*) \[ \{g \in G : |g| \leq \lambda\} \text{ is finite for every } \lambda < \infty. \]

If the varifold is an integral varifold, then we can also require that $|g|$ is an integer for every $g \in G$.

(Theorem 1 and Theorem 2 with $G = \Lambda_n \mathbb{R}^N$ are proved in §6. Theorem 10 in §7 shows how one can then construct from $\Lambda_n \mathbb{R}^N$ a suitable coefficient group that has Property (*).)

For either choice of coefficient group, the resulting space of flat chains satisfies the compactness theorem: given any sequence of flat chains supported in a compact set with mass and boundary mass bounded above, there is a subsequence that converges in the flat topology. However, if $G$ is a normed vector space over $\mathbb{R}$, then there will be finite-mass flat chains with coefficients in $G$ that fail to be rectifiable. On the other hand, if the coefficient group has Property (*), then every finite mass flat chain is rectifiable. (These assertions about rectifiability and lack of rectifiability follow from [Whi99, Theorem 7.1].) Thus coefficient groups with Property (*) provide a nice setting for the Plateau problem: mass minimizing surfaces (for any boundary) exist, and they must be rectifiable. This paper shows that every stationary polyhedral cone arises as the varifold associated to a mass-minimizing cone for such a coefficient group.

This paper is meant to be largely self-contained. See [Whi96] for a brief overview of flat chain with coefficients in a metric abelian group, or Fleming’s original paper [Fle66] for a thorough treatment.
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2. Abstract Calibrations

Let \( C \) be a collection (e.g., of surfaces), let \( M : C \to \mathbb{R} \) be a real-valued function (e.g., “mass” or “weighted area”), and let \( \sim \) be an equivalence relation on \( C \) (e.g., the relation “is homologous to”). An **abstract calibration** on \((C, M, \sim)\) is a function

\[
\Phi : C \to \mathbb{R}
\]

such that

1. \( \Phi \) is constant on each equivalence class, and
2. \( \Phi(S) \leq M(S) \) for every \( S \in C \).

If \( \Phi(S) = M(S) \), we say that \( S \) is calibrated by \( \Phi \).

Abstract calibrations are of interest because of the following trivial theorem:

**Theorem 3.** Suppose that \( \Phi : C \to \mathbb{R} \) is a calibration on \((C, M, \sim)\) and that \( S \in C \) is calibrated by \( \Phi \). Then \( S \) minimizes \( M \) in its equivalence class:

\[
M(S) \leq M(S') \quad \text{for all } S' \sim S.
\]

Furthermore, if \( S' \sim S \), then \( S' \) minimizes \( M \) in its equivalence class if and only if \( S' \) is also calibrated by \( \Phi \).

Now suppose that \( C \) is an abelian group and that \( B \) is a subgroup. (Think of \( C \) as a collection of \( m \)-chains and \( B \) as the chains in \( C \) that are boundaries of \((m+1)\)-chains.) Then we have the equivalence relation on \( C \) given by

\[
S \sim S' \quad \text{if and only if } \quad S - S' \in B.
\]

Now suppose that

\[
\Phi : C \to \mathbb{R}
\]

is a homomorphism that vanishes on \( B \). Then \( \Phi \) is constant on equivalence classes for the equivalence relation \( \sim \). Thus if \( \Phi(\cdot) \leq M(\cdot) \), then \( \Phi \) is an abstract calibration for \((C, M, \sim)\). In that case, we will call \( \Phi \) a (generalized) calibration.

3. Polyhedral Chains

Fix an ambient Euclidean space \( \mathbb{R}^N \) and an abelian group \((G, +)\) with a norm \(| \cdot |\), i.e., with a function

\[
| \cdot | : G \to \mathbb{R}
\]

such that the distance function \( d(g, g') := |g - g'| \) makes \( G \) into a metric space.

A **formal polyhedral \( m \)-chain** (in \( \mathbb{R}^N \), with coefficients in \( G \)) is a formal sum

\[
\sum_{i=1}^{\ell} g_i \sigma_i
\]

where \( g_i \in G \) and \( \sigma_i \) is an oriented \( m \)-dimensional polyhedron in \( E \). Let \( P_m^{\text{formal}}(G) \) be the abelian group of formal polyhedral \( m \)-chains in \( \mathbb{R}^N \) with coefficients in \( G \).

Consider the equivalence relation on formal polyhedral chains generated by

\[
g \sigma \equiv g \sigma' + g \sigma''
\]
if \( \sigma' \) and \( \sigma'' \) are obtained from \( \sigma \) by subdivision, and by
\[
g\sigma \equiv -g\tilde{\sigma}
\]
is \( \tilde{\sigma} \) is obtained from \( \sigma \) by reversing the orientation.

A polyhedral \( m \)-chain is an equivalence class in \( P_{\text{formal}} \) under this equivalence relation. We let \( P_m = P_m(G) \) be the abelian group of polyhedral \( m \)-chains with coefficients in \( G \).

If \( \sum_{i=1}^k g_i \sigma_i \) is a formal polyhedral \( m \)-chain, we let
\[
\sum_{i=1}^k g_i [\sigma_i]
\]
denote the corresponding equivalence class, i.e., the corresponding polyhedral chain. Every polyhedral chain has a representation \( \sum_i g_i [\sigma_i] \) in which the \( \sigma_i \) are non-overlapping. For such a representation, the mass (or weighted area) of the chain is given by
\[
M \left( \sum_i g_i [\sigma_i] \right) := \sum |g_i| \mathcal{H}^m(\sigma_i).
\]

4. A Canonical Calibration

In this section, we fix nonnegative integers \( m \) and \( N \) with \( m < N \). Let \( G \) be the additive group \( \Lambda_m \mathbb{R}^N \) of \( m \)-vectors in \( \mathbb{R}^N \). We give \( \Lambda_m \mathbb{R}^N \) the Euclidean norm. That is, we give \( \Lambda_m \mathbb{R}^N \) the norm associated to the Euclidean structure for which the \( m \)-vectors \( e_{i_1} \wedge \cdots \wedge e_{i_m} (i_1 < i_2 < \cdots < i_m) \) form an orthonormal basis of \( \Lambda_m \mathbb{R}^N \).

(For the study of non-rectifiable chains, it would probably be better to use the mass norm [FF60, page 461] on \( \Lambda_n \mathbb{R}^N \). However, in this paper we are primarily interested in rectifiable chains, and whether one uses the Euclidean norm or the mass norm does not affect the masses of rectifiable chains.)

If \( \sigma \) is an oriented \( m \)-dimensional polyhedron in \( \mathbb{R}^N \), we let \( \eta(\sigma) \) be the corresponding simple unit \( m \)-vector.

Now define a map
\[
\Phi : P_{\text{formal}}(G) \to \mathbb{R}
\]
by
\[
\Phi \left( \sum_i g_i \sigma_i \right) = \sum_i (g_i \cdot \eta(\sigma_i)) \mathcal{H}^m(\sigma_i).
\]
Note that \( \Phi \) is an additive homomorphism.

If \( \tilde{\sigma} \) is obtained from \( \sigma \) by reversing the orientation, then \( \eta(\tilde{\sigma}) = -\eta(\sigma) \), so
\[
(-g) \cdot \eta(\tilde{\sigma}) = g \cdot \sigma,
\]
and therefore
\[
\Phi(-g\tilde{\sigma}) = \Phi(g\sigma).
\]
Similarly,
\[
\Phi(g\sigma) = \Phi(g\sigma') + \Phi(g\sigma'')
\]
if \( \sigma' \) and \( \sigma'' \) are obtained by subdividing \( \sigma \).
Hence \( \Phi \) induces a well-defined homomorphism

\[
\Phi : \mathcal{P}_m(G) \to \mathbb{R},
\]

\[
\Phi \left( \sum_i g_i[\sigma_i] \right) = \sum (g_i \cdot \eta(\sigma_i)) \mathcal{H}^m(\sigma_i).
\]

**Theorem 4.** Suppose that \( G = \Lambda_m \mathbb{R}^N \) and that

\[
A = \sum g_i[\sigma_i]
\]

is a polyhedral chain in \( \mathcal{P}_m(G) \), where the \( \sigma_i \) are non-overlapping. Then

\[
\Phi(A) \leq M(A),
\]

with equality if and only if each \( g_i \) is a nonnegative scalar multiple of \( \eta(\sigma_i) \).

**Proof.** By Cauchy-Schwartz, \( g_i \cdot \eta(\sigma_i) \leq |g_i| \), with equality if and only if \( g_i \) is a nonnegative scalar multiple of \( \eta(\sigma_i) \). Hence

\[
\Phi(A) = \sum (g_i \cdot \eta(\sigma_i)) \mathcal{H}^m(\sigma_i)
\]

\[
\leq \sum |g_i| \mathcal{H}^m(\sigma_i)
\]

\[
= M(A).
\]

with equality if and only if each \( g_i \) is a nonnegative scalar multiple of \( \eta(\sigma_i) \). \( \square \)

**Theorem 5.** Suppose that \( G = \Lambda_m \mathbb{R}^N \) and that \( A \in \mathcal{P}_m(G) \) is the boundary of an \((m+1)\)-chain. Then \( \Phi(A) = 0 \).

**Proof.** Let \( g \in G \), let \( \tau \) be an oriented \((m+1)\)-dimensional polyhedron in \( E \), and let \( \sigma_1, \ldots, \sigma_k \) be the \( m \)-dimensional faces of \( \tau \) with the induced orientations. It suffices to show that

(1) \[
\Phi \left( \sum_i g[\sigma_i] \right) = 0.
\]

Since any \( m \)-vector is a sum of simple \( m \)-vectors, it suffices to prove (1) when \( g \) is a simple \( m \)-vector. Let \( V \) be the oriented \( m \)-plane associated to \( g \). Let \( \omega \) be the volume form on \( V \), \( \Pi : \mathbb{R}^N \to V \) be orthogonal projection, and let \( \Omega = \Pi^* \omega \). Note that

\[
\Phi(g[\sigma_i]) = |g| \int_{\sigma_i} \Omega.
\]

Thus

\[
\Phi \left( \sum_i g[\sigma_i] \right) = |g| \sum \int_{\sigma_i} \Omega
\]

\[
= |g| \int_{\partial \tau} \Omega
\]

\[
= 0
\]

by Stokes Theorem (since \( \Omega \) is constant and therefore \( d\Omega = 0 \)). \( \square \)

Recall that the **flat norm** of \( A \in \mathcal{P}_m(G) \) is given by

\[
\mathcal{F}(A) = \inf_{Q \in \mathcal{P}_{m+1}(G)} (M(A + \partial Q) + M(Q)),
\]

which is trivially \( \leq M(A) \).
Theorem 6. If $G = \Lambda_m R^N$ and $A \in \mathcal{P}_m(G)$, then

$$\Phi(A) \leq \mathcal{F}(A) \leq M(A),$$

where $\mathcal{F}(A)$ is the flat norm of $A$.

Proof. Let $Q$ be a polyhedral $(m+1)$-chain. By Theorem 5,

$$\Phi(A) = \Phi(A + \partial Q) \leq M(A + \partial Q) \leq M(A + \partial Q) + M(Q).$$

Hence

$$\Phi(A) \leq \inf_Q (M(A + \partial Q) + M(Q)) = \mathcal{F}(A).$$

□

Corollary 7. The map $\Phi$ extends continuously to an additive homomorphism $\Phi : \mathcal{F}_m(G) \to \mathbb{R}$ such that

$$\Phi(A) \leq \mathcal{F}(A) \leq M(A)$$

for every flat $m$-chain $A$.

The corollary follows immediately from the theorem since $\mathcal{F}_m(G)$ is the metric space completion of $\mathcal{P}_m(G)$ with respect the flat norm.

5. Polyhedral Varifolds

If $\sigma$ is an $m$-dimensional polyhedron in $R^N$, we let $\text{var}(\sigma)$ be the associated $m$-dimensional, multiplicity-1 rectifiable varifold, i.e., the rectifiable varifold whose associated Radon measure is $\mathcal{H}^m_{\cdot \sigma}$. An $m$-dimensional polyhedral varifold in $R^N$ is a varifold of the form

$$\sum_{i=1}^k c_i \text{var}(\sigma_i),$$

where each $\sigma_i$ is a polyhedron and each $c_i \geq 0$. By subdividing, we can assume that if $i \neq j$, then $\sigma_i \cap \sigma_j$ is either empty or is a common face of $\sigma_i$ and $\sigma_j$ of dimension $< m$.

As in §4 we let $G = \Lambda_m R^N$. If $\sigma$ is an oriented $m$-dimensional polyhedron in $R^N$, we let $\eta(\sigma)$ be the simple unit $m$-vector associated with the orientation. Let $\langle \sigma \rangle$ be the polyhedral $m$-chain in $\mathcal{P}_m(G)$ given by

$$\langle \sigma \rangle = \eta(\sigma)[\sigma].$$

Note that if $\tilde{\sigma}$ is obtained from $\sigma$ by reversing the orientation, then $\langle \tilde{\sigma} \rangle = \langle \sigma \rangle$. Thus given an unoriented $m$-dimensional polyhedron $\sigma$, we have a well-defined polyhedral $m$-chain $\langle \sigma \rangle$ in $\mathcal{P}_m(G)$.

If

$$V = \sum_{i=1}^k c_i \text{var}(\sigma_i)$$

is an $m$-dimensional polyhedral varifold, we let

$$\langle V \rangle = \sum_{i=1}^k c_i \langle \sigma_i \rangle.$$
If we give the $\sigma_i$ orientations, then

$$\langle V \rangle = \sum_{i=1}^{k} c_i \eta(\sigma_i)[\sigma_i],$$

from which it follows (by Theorem 4) that $V$ is calibrated by $\Phi$. Note that $M(\langle V \rangle) = M(V)$. Thus $\langle \cdot \rangle$ is a mass-preserving homomorphism from the additive semigroup of $m$-dimensional polyhedral varifolds to the additive group $P_m(G)$ of $m$-dimensional polyhedral chains.

**Theorem 8.** Let $V$ be an $m$-dimensional polyhedral varifold in $\mathbb{R}^N$, and let $\Gamma$ be the union of some of the $(m-1)$-dimensional faces of polyhedra in $V$. Then $V$ is stationary in $\mathbb{R}^N \setminus \Gamma$ if and only if $\partial \langle V \rangle$ is supported in $\Gamma$.

**Proof.** Since the result is essentially local, it suffices to prove it for

$$V = \sum_{i} c_i \text{var}(\sigma_i)$$

where the $\sigma_i$ are polyhedra with a common $(m-1)$-dimensional face $\tau$ and where $\Gamma$ is the union of the faces $\neq \tau$ of the various $\sigma_i$.

Give $\tau$ an orientation, and then give each $\sigma_i$ the orientation that induces the chosen orientation on $\tau$. Let $\eta(\tau)$ be the simple unit $(m-1)$-vector associated to the orientation of $\tau$ and let $\eta(\sigma_i)$ be the simple unit $m$-vector associated with the orientation of $\sigma_i$. Then

$$\eta(\sigma_i) = \eta(\tau) \wedge \nu_i,$$

where $\nu_i$ is the unit normal to $\tau$ that lies in the $m$-plane containing $\sigma_i$ and that points out from $\sigma_i$.

Note that

$$V \text{ is stationary in } \Gamma^c \iff \sum_{i} c_i \nu_i = 0$$

$$\iff \sum_{i} c_i \eta(\sigma_i) = 0.$$  

On the other hand,

$$(\partial \langle \text{var}(\sigma_i) \rangle)_{\Gamma^c} = (\partial \eta(\sigma_i)[\sigma_i])_{\Gamma^c} = \eta(\sigma_i)[\tau],$$

so

$$\partial \langle V \rangle)_{\Gamma^c} = \left( \sum_{i} c_i \eta(\sigma_i) \right)[\tau].$$

The theorem follows immediately from (3) and (4). □

6. The Main Theorem

**Theorem 9.** Let $G = \Lambda_m \mathbb{R}^N$. If $V$ is an $m$-dimensional polyhedral varifold, then $\langle V \rangle$ is a mass-minimizing polyhedral chain in $P_m(G)$. If $\Gamma$ is closed set (such as the union of some of the faces of polyhedra in $V$) and if $V$ is stationary in $\Gamma^c$, then $\partial \langle V \rangle$ is supported in $\Gamma$ and

$$M(V) \leq M(\phi \# V)$$

for any lipschitz homeomorphism $\phi : \mathbb{R}^N \rightarrow \mathbb{R}^N$ such that $\phi(x) = x$ for all $x \in \Gamma$. 

Proof. By construction, $\langle V \rangle$ is calibrated by $\Phi$. Hence $\langle V \rangle$ is homologically mass-minimizing. Since the ambient space $\mathbb{R}^N$ is homologically trivial, that means that $\langle V \rangle$ minimizes mass, i.e., that $M(\langle V \rangle) \leq M(A)$ for any flat chain $A$ in $\mathcal{F}_m(G)$ with $\partial A = \partial(\langle V \rangle)$.

Now suppose that $V$ is stationary in $\Gamma^c$. Then $\partial(\langle V \rangle)$ is supported in $\Gamma$ (by Theorem 3), so

$$M(\langle V \rangle) \leq M(f_\# \langle V \rangle)$$

for any Lipschitz map $f : \mathbb{R}^N \to \mathbb{R}^N$ such that $f(x) = x$ for all $x \in \Gamma$. Now $M(\langle V \rangle) = M(V)$, and, if $f$ is one-to-one, then $M(f_\# \langle V \rangle) = M(f_\# V)$. Thus

$$M(V) \leq M(f_\# V).$$

$\square$

7. Changing the Coefficient Group

Theorem 10. Let $G$ be an abelian group with norm $|\cdot|_G$. Let $S = \{g_1, \ldots, g_k\}$ be a finite subset of $G$, and let $H$ be the subgroup of $G$ generated by $S$. Define a norm $|\cdot|_H$ on $H$ by

$$|g|_H = \min \left\{ \sum_{i=1}^k |n_i| |g_i| : g = \sum_{i=1}^k n_i g_i \right\},$$

where the $n_i$ are integers. Suppose that $A$ is a rectifiable flat chain in $\mathcal{F}_m(G)$ and that all of its multiplicities are in $S$. Then $A$ may also be regarded as a rectifiable flat chain in $\mathcal{F}_m(H)$ (with the same mass). Furthermore, if $A$ is mass-minimizing in $\mathcal{F}_m(G)$, then it is also mass-minimizing in $\mathcal{F}_m(H)$.

Note that $\{g \in H : |g|_H \leq \lambda\}$ is finite if $\lambda < \infty$. Note also that $|\cdot|_H$ is the largest norm on $H$ such that $|g_i|_H = |g_i|_G$ for each of the generators $g_1, \ldots, g_k$.

Proof. Let $M_G$ and $\mathcal{F}_G$ denote mass and flat norm on chains (polyhedral or flat) with coefficients in the group $G$ with the norm $|\cdot|_G$. Let $M_H$ and $\mathcal{F}_H$ denote mass and flat norm on chains with coefficients in the group $H$ with the norm $|\cdot|_H$. Since $M_H \geq M_G$ on $\mathcal{P}_m(H)$, it follows that $\mathcal{F}_H \geq \mathcal{F}_G$ on $\mathcal{P}_m(H)$. Consequently, the inclusion

$$\mathcal{P}_m(H) \subset \mathcal{P}_m(G)$$

extends to an inclusion

$$\mathcal{F}_m(H) \subset \mathcal{F}_m(G)$$

such that $M_G(T) \leq M_H(T)$ and $\mathcal{F}_G(T) \leq \mathcal{F}_H(T)$ for $T \in \mathcal{F}_m(H)$.

Now suppose that $A$ is an $M_G$-minimizing rectifiable chain in $\mathcal{F}_m(G)$ and that the multiplicities of $A$ lie in $S$. Then $A$ is also a rectifiable chain in $\mathcal{F}_m(H)$, and $M_H(A) = M_G(A)$. If $A'$ is a chain in $\mathcal{F}_m(H)$ with $\partial A' = \partial A$, then

$$M_G(A) \leq M_G(A')$$

since $A$ is $M_G$-minimizing. Since $M_G(A) = M_H(A)$ and $M_G(A') \leq M_H(A')$, it follows that

$$M_H(A) \leq M_H(A').$$

Hence $A$ is $M_H$-minimizing in $\mathcal{F}_m(H)$. $\square$
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