A FAMILY OF NON-COCYCLE CONJUGATE $E_0$-SEMIGROUPS OBTAINED FROM BOUNDARY WEIGHT DOUBLES

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Abstract. We have seen that if $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a unital $q$-positive map and $\nu$ is a type II Powers weight, then the boundary weight double $(\phi, \nu)$ induces a unique (up to conjugacy) type $\Pi_0$ $E_0$-semigroup. Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ and $\psi : M_{n'}(\mathbb{C}) \to M_{n'}(\mathbb{C})$ be unital rank one $q$-positive maps, so for some states $\rho \in M_n(\mathbb{C})^*$ and $\rho' \in M_{n'}(\mathbb{C})^*$, we have $\phi(A) = \rho(A)I_n$ and $\psi(D) = \rho'(D)I_{n'}$ for all $A \in M_n(\mathbb{C})$ and $D \in M_{n'}(\mathbb{C})$. We find that if $\nu$ and $\eta$ are arbitrary type II Powers weights, then $(\phi, \nu)$ and $(\psi, \eta)$ induce non-cocycle conjugate $E_0$-semigroups if $\rho$ and $\rho'$ have different eigenvalue lists. We then completely classify the $q$-corners and hyper maximal $q$-corners from $\phi$ to $\psi$, obtaining the following result: If $\nu$ is a type II Powers weight of the form $\nu(\sqrt{1 - \Lambda(1)} B \sqrt{1 - \Lambda(1)}) = (f, B f)$, then the $E_0$-semigroups induced by $(\phi, \nu)$ and $(\psi, \nu)$ are cocycle conjugate if and only if $n = n'$ and $\phi$ and $\psi$ are conjugate.

1. Introduction

An $E_0$-semigroup $\alpha = \{\alpha_t\}_{t \geq 0}$ is a semigroup of unital $*$-endomorphisms of $B(H)$ which is weakly continuous in $t$. $E_0$-semigroups are divided into three types, depending on the existence and structure of their units. More specifically, if $\alpha$ is an $E_0$-semigroup acting of $B(H)$ and there is a strongly continuous semigroup $U = \{U_t\}_{t \geq 0}$ of bounded operators acting on $H$ such that $\alpha_t(A)U_t = U_tA$ for all $A \in B(H)$ and $t \geq 0$, then we say that $U$ is a unit for $\alpha$. An $E_0$-semigroup is said to be spatial if it has at least one unit, and a spatial $E_0$-semigroup is called completely spatial if, in essence, its units can reconstruct $H$. We say an $E_0$-semigroup $\alpha$ is type I if it is completely spatial and type II if it is spatial but not completely spatial. If $\alpha$ has no units, we say it is of type III. Every spatial $E_0$-semigroup $\alpha$ is assigned an index $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ which corresponds to the dimension of a particular Hilbert space associated to its units. The type I $E_0$-semigroups are classified up to cocycle conjugacy by their index: If $\alpha$ is of type $I_n$ (type I, index $n$) for $n \in \mathbb{N} \cup \{\infty\}$, then $\alpha$ is cocycle conjugate to the CAR flow of rank $n$ ([3]), while if $\alpha$ is of type $I_0$, then it is a semigroup of $*$-automorphisms.

However, uncountably many examples of non-cocycle conjugate $E_0$-semigroups of types II and III are known (see, for example, [6], [7], [12], [13], [14], and [15]). Bhat’s dilation theorem ([1]) and developments in the theory of $CP$-flows ([11] and [12]) have led to the introduction of boundary weight doubles and related cocycle conjugacy results for $E_0$-semigroups in [9]. A boundary weight double is a pair $(\phi, \nu)$, where $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is $q$-positive (that is, $\phi(I + t\phi)^{-1}$ is completely positive for all $t \geq 0$) and $\nu$ is a positive boundary weight over $L^2(0, \infty)$. If $\phi$ is unital and $\nu$
is normalized and unbounded (in which case we say $\nu$ is a type II Powers weight), then $(\phi, \nu)$ induces a unital $CP$-flow whose Bhat minimal dilation is a type II$_0$ $E_0$-semigroup $\alpha^d$. If $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is unital and $q$-positive and $U \in M_n(\mathbb{C})$ is unitary, then the map $\phi_U(A) = U^*\phi(UAU^*)U$ is also unital and $q$-positive. The relationship between $\phi$ and $\phi_U$ is analogous to the definition of conjugacy for $E_0$-semigroups. With this in mind, we say that $q$-positive maps $\phi, \psi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ are conjugate if $\psi = \phi_U$ for some unitary $U \in M_n(\mathbb{C})$. If $\nu$ is a type II Powers weight of the form $\nu(\sqrt{I - \Lambda(1)}B\sqrt{I - \Lambda(1)}) = (f, Bf)$, then $(\phi, \nu)$ and $(\phi_U, \nu)$ induce cocycle conjugate $E_0$-semigroups (for details, see Proposition 2.11 of [8] and the discussion preceding it).

Suppose $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ and $\psi : M_{n'}(\mathbb{C}) \to M_{n'}(\mathbb{C})$ are unital rank one $q$-positive maps, so for some states $\rho \in M_n(\mathbb{C})^*$ and $\rho' \in M_{n'}(\mathbb{C})^*$, we have $\phi(A) = \rho(A)I_n$ and $\psi(D) = \rho'(D)I_{n'}$ for all $A \in M_n(\mathbb{C}), D \in M_{n'}(\mathbb{C})$. Let $\nu$ and $\eta$ be type II Powers weights. We prove three main results. First, we find that if $(\phi, \nu)$ and $(\psi, \eta)$ induce cocycle conjugate $E_0$-semigroups, then $\rho$ and $\rho'$ have identical eigenvalue lists (Definition 2.13 and Proposition 3.3). We then find all $q$-corners and hyper maximal $q$-corners from $\phi$ to $\psi$ (see Remark 1 and Theorems 3.6 and 3.7). With this result in hand, we complete the cocycle conjugacy comparison theory for $E_0$-semigroups $\alpha^d$ and $\beta^d$ induced by $(\phi, \nu)$ and $(\psi, \nu)$ in the case that $\nu$ is of the form $\nu(\sqrt{I - \Lambda(1)}B\sqrt{I - \Lambda(1)}) = (f, Bf)$, finding that $\alpha^d$ and $\beta^d$ are cocycle conjugate if and only if $n = n'$ and $\phi$ is conjugate to $\psi$ (Theorem 3.8).

2. Background

2.1. $q$-positive and $q$-pure maps. Let $\phi : \mathfrak{A} \to \mathfrak{B}$ be a linear map between unital $C^*$-algebras. For each $n \in \mathbb{N}$, define $\phi_n : M_n(\mathfrak{A}) \to M_n(\mathfrak{B})$ by

$$
\phi_n \left( \begin{array}{ccc}
A_{11} & \cdots & A_{1n} \\
\vdots & \ddots & \vdots \\
A_{n1} & \cdots & A_{nn}
\end{array} \right) = \left( \begin{array}{ccc}
\phi(A_{11}) & \cdots & \phi(A_{1n}) \\
\vdots & \ddots & \vdots \\
\phi(A_{n1}) & \cdots & \phi(A_{nn})
\end{array} \right).
$$

We say that $\phi$ is completely positive if $\phi_n$ is positive for all $n \in \mathbb{N}$. From the work of Choi ([5]) and Arveson ([2]), we know that every normal completely positive map $\phi : B(H) \to B(K)$ ($H, K$ separable Hilbert spaces) can be written in the form

$$
\phi(A) = \sum_{i=1}^{n} S_iAS_i^* 
$$

for some $n \in \mathbb{N} \cup \{\infty\}$ and bounded operators $S_i : H \to K$ which are linearly independent over $\ell_2(\mathbb{N})$.

We will be interested in a particular kind of completely positive map:

**Definition 2.1.** Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a linear map with no negative eigenvalues. We say $\phi$ is $q$-positive (and write $\phi \geq_q 0$) if $\phi(I + t\phi)^{-1}$ is completely positive for all $t \geq 0$.

We make two observations in light of this definition. First, it is not uncommon for a completely positive map to have negative eigenvalues. Second, there is no “slowest rate of failure” for $q$-positivity: For every $s \geq 0$, there exists a linear map $\phi$ with no
negative eigenvalues such that \(\phi(I + t\phi)^{-1} (t \geq 0)\) is completely positive if and only if \(t \leq s\). These observations are discussed in detail in section 2.1 of [8].

There is a natural order structure for \(q\)-positive maps. If \(\phi, \psi : M_n(\mathbb{C}) \to M_m(\mathbb{C})\) are \(q\)-positive, we say \(\phi\) \(q\)-dominates \(\psi\) (i.e. \(\phi \geq_q \psi\)) if \(\phi(I + t\phi)^{-1} - \psi(I + t\psi)^{-1}\) is completely positive for all \(t \geq 0\). It is not always true that \(\phi \geq_q \lambda\phi\) if \(\lambda \in (0, 1)\) (for a large family of counterexamples, see Theorem 6.11 of [9]). However, if \(\phi\) is \(q\)-positive, then for every \(s \geq 0\), we have \(\phi \geq_q \phi(s\phi)^{-1} \geq_q 0\) (Proposition 4.1 of [9]). If these are the only nonzero \(q\)-subordinates of \(\phi\), we say \(\phi\) is \(q\)-pure. The unital \(q\)-pure maps which are either rank one or invertible have been classified (Proposition 5.2 and Theorem 6.11 of [9]).

If \(\phi\) is a unital \(q\)-positive map, then as \(t \to \infty\), the maps \(t\phi(I + t\phi)^{-1}\) converge to an idempotent completely positive map \(L_\phi\) which has interesting properties (see Lemma 3.1 of [8]):

**Lemma 2.2.** Suppose \(\phi : M_n(\mathbb{C}) \to M_m(\mathbb{C})\) is \(q\)-positive and \(||t\phi(I + t\phi)^{-1}|| < 1\) for all \(t \geq 0\). Then the maps \(t\phi(I + t\phi)^{-1}\) have a unique norm limit \(L_\phi\) as \(t \to \infty\), and \(L_\phi\) is completely positive. Furthermore,

(i) \(\phi = \phi \circ L_\phi = L_\phi \circ \phi\),
(ii) \(I^2_\phi = L_\phi\),
(iii) range\((L_\phi) = \text{range}(\phi)\), and
(iv) nullspace\((L_\phi) = \text{nullspace}(\phi)\).

2.2. \(E_0\)-semigroups and \(CP\)-flows. From a celebrated result of Wigner ([16]), we know that every one-parameter group \(\alpha = \{\alpha_t\}_{t \in \mathbb{R}}\) of \(*\)-automorphisms of \(B(H)\) arises from a strongly continuous unitary group \(\{V_t\}_{t \in \mathbb{R}}\) in the sense that \(\alpha_t(A) = V_tAV_t^*\) for all \(t \in \mathbb{R}\) and \(A \in B(H)\).

**Definition 2.3.** Let \(H\) be a separable Hilbert space. We say a family \(\alpha = \{\alpha_t\}_{t \geq 0}\) of \(*\)-endomorphisms of \(B(H)\) is an \(E_0\)-semigroup if:

(i) \(\alpha_s \circ \alpha_t = \alpha_{s+t}\) for all \(s, t \geq 0\) and \(\alpha_0(A) = A\) for all \(A \in B(H)\);
(ii) For each \(f, g \in H\) and \(A \in B(H)\), the inner product \((f, \alpha_t(A)g)\) is continuous in \(t\);
(iii) \(\alpha_t(I) = I\) for all \(t \geq 0\).

We have two notions of equivalence for \(E_0\)-semigroups:

**Definition 2.4.** Let \(\alpha\) and \(\beta\) be \(E_0\)-semigroups acting on \(B(H_1)\) and \(B(H_2)\), respectively, are said to be conjugate if there is a \(*\)-isomorphism \(\theta\) from \(B(H_1)\) onto \(B(H_2)\) such that \(\theta \circ \alpha \circ \theta^{-1} = \beta\).

We say \(\alpha\) and \(\beta\) are cocycle conjugate if \(\alpha\) is conjugate to \(\beta'\), where \(\beta'\) is an \(E_0\)-semigroup of \(B(H_2)\) satisfying the following condition: For some strongly continuous family of unitaries \(W = \{W_t\}_{t \geq 0}\) acting on \(H_2\) and satisfying \(W_t\beta_t(W_s) = W_{t+s}\), we have \(\beta'_t(A) = W_t\beta_t(A)W_t^*\) for all \(A \in B(H_2)\) and \(t \geq 0\).

Let \(K\) be a separable Hilbert space, and form \(H = K \otimes L^2(0, \infty)\), which we identify with the space of \(K\)-valued measurable functions on \((0, \infty)\) which are square integrable. Let \(U = \{U_t\}_{t \geq 0}\) be the right shift semigroup on \(H\), so for all \(t \geq 0\), \(f \in H\), and \(x > 0\), we have

\[(U_tf)(x) = f(x-t)\text{ if }x > t,\quad (U_tf)(x) = 0\text{ if }x \leq t.\]
A strongly continuous semigroup \( \alpha = \{ \alpha_t \}_{t \geq 0} \) of completely positive contractions from \( B(H) \) into itself is called a CP-flow over \( K \) if \( \alpha_t(A)U_t = U_tA \) for all \( A \in B(H) \) and \( t \geq 0 \). A result of Bhat in [4] shows that if \( \alpha \) is unital, then it minimally dilates to a unique (up to conjugacy) \( E_0 \)-semigroup \( \alpha^d \). We may naturally construct a CP-flow \( \beta = \{ \beta_t \}_{t \geq 0} \) over \( K \) using the right shift semigroup by defining

\[
\beta_t(A) = U_tAU_t^*
\]

for all \( A \in B(H) \), \( t \geq 0 \). In fact, if \( \alpha \) is any CP-flow over \( K \), then \( \alpha \) dominates \( \beta \) in the sense that \( \alpha_t - \beta_t \) is completely positive for all \( t \geq 0 \).

Define \( \Lambda : B(K) \to B(H) \) by

\[
(\Lambda(A)f)(x) = e^{-x}Af(x)
\]

for all \( A \in B(K) \), \( f \in H \), and \( x \in (0, \infty) \), and let \( \mathfrak{A}(H) \) be the algebra

\[
\mathfrak{A}(H) = \sqrt{I - \Lambda(I_K)B(H)}\sqrt{I - \Lambda(I_K)}.
\]

We say a linear functional \( \tau \) acting on \( \mathfrak{A}(H) \) is a boundary weight (denoted \( \tau \in \mathfrak{A}(H)_* \)) if the functional \( \ell \) defined on \( B(H) \) by

\[
\ell(A) = \tau\left(\sqrt{I - \Lambda(I_K)}A\sqrt{I - \Lambda(I_K)}\right)
\]

satisfies \( \ell \in B(H)_* \). For a discussion of boundary weights and their properties, we refer the reader to Definition 1.10 of [10] and the remarks that follow it.

Every CP-flow over \( K \) corresponds to a boundary weight map \( \rho \to \omega(\rho) \) from \( B(K)_* \) to \( \mathfrak{A}(H)_* \) ([11]). On the other hand, it is an extremely important and non-trivial fact that, under certain conditions, a map from \( B(K)_* \) to \( \mathfrak{A}(H)_* \) can induce a CP-flow (see Theorem 3.3 of [12]):

**Theorem 2.5.** If \( \rho \to \omega(\rho) \) is a completely positive mapping from \( B(K)_* \) into \( \mathfrak{A}(H)_* \) satisfying \( \omega(\rho)(I - \Lambda(I_K)) \leq \rho(I_K) \) for all positive \( \rho \), and if the maps

\[
\tilde{\pi}_t := \omega_t(I + \hat{\Lambda}\omega_t)^{-1}
\]

are completely positive contractions from \( B(K)_* \) into \( B(H)_* \) for all \( t > 0 \), then \( \rho \to \omega(\rho) \) is the boundary weight map of a CP-flow over \( K \). The CP-flow is unital if and only if \( \omega(\rho)(I - \Lambda(I_K)) = \rho(I_K) \) for all \( \rho \in B(K)_* \).

If \( \alpha \) is a CP-flow over \( \mathbb{C} \), then we identify its boundary weight map \( c \to \omega(c) \) with the single positive boundary weight \( \omega := \omega(1) \), so \( \omega \) has the form

\[
\omega(\sqrt{I - \Lambda(1)}A\sqrt{I - \Lambda(1)}) = \sum_{i=1}^{k}(f_i, A_{f_i})
\]

for some mutually orthogonal nonzero \( L^2 \)-functions \( \{ f_i \}_{i=1}^k \) \( (k \in \mathbb{N} \cup \{ \infty \}) \) with \( \sum_{i=1}^{k}||f_i||^2 < \infty \). We call \( \omega \) a positive boundary weight over \( L^2(0, \infty) \), and, following the notation of [10], we write \( \omega \in \mathfrak{A}(L^2(0, \infty))^+ \). We say \( \omega \) is bounded if there exists some \( r > 0 \) such that \( ||\omega(B)|| \leq r||B|| \) for all \( B \in \mathfrak{A}(H) \). Otherwise, we say \( \omega \) is unbounded. Suppose \( \omega(I - \Lambda(1)) = 1 \) (i.e. \( \omega \) is normalized), so \( \alpha \) is unital and therefore dilates to an \( E_0 \)-semigroup \( \alpha^d \). Results from [11] show that \( \alpha^d \) is of type \( I_k \) if \( \omega \) is bounded but of type \( II_0 \) if \( \omega \) is unbounded, leading us to make the following definition:
Definition 2.6. A boundary weight $\nu \in \mathfrak{A}(L^2(0, \infty))_*$ is called a Powers weight if $\nu$ is positive and normalized. We say a Powers weight $\nu$ is type I if it is bounded and type II if it is unbounded.

We note that if $\nu$ is a type II Powers weight, then both $\nu(I)$ and $\nu(A(1))$ approach infinity as $t \to 0^+$. We can combine unital $q$-positive maps with type II Powers weights to obtain $E_0$-semigroups (see Proposition 3.2 and Corollary 3.3 of [9]):

Proposition 2.7. Let $H = C^n \otimes L^2(0, \infty)$. Let $\phi : M_n(C) \to M_n(C)$ be a unital $q$-positive map, and let $\nu$ be a type II Powers weight. Let $\Omega_\nu : \mathfrak{A}(H) \to M_n(C)$ be the map that sends $A = (A_{ij}) \in M_n(\mathfrak{A}(L^2(0, \infty))) \cong \mathfrak{A}(H)$ to the matrix $(\nu(A_{ij})) \in M_n(C)$. The map $\rho \mapsto \omega(\rho)$ from $M_n(C)^*$ into $\mathfrak{A}(H)_*$ defined by

$$\omega(\rho)(A) = \rho(\phi(\Omega_\nu(A)))$$

is the boundary weight map of a unital CP-flow $\alpha$ over $C^n$ whose Bhat minimal dilation $\alpha^d$ is a type $II_0$ $E_0$-semigroup.

In the notation of the previous proposition, we say that $\alpha^d$ is the $E_0$-semigroup induced by the boundary weight double $(\phi, \nu)$.

Definition 2.8. Suppose $\phi : B(H_1) \to B(K_1)$ and $\psi : B(H_2) \to B(K_2)$ are normal completely positive maps. Write each $A \in B(H_1 \oplus H_2)$ as $A = (A_{ij})$, where $A_{ij} \in B(H_j, H_i)$ for each $i, j = 1, 2$. We say a linear map $\gamma : B(H_2, H_1) \to B(K_2, K_1)$ is a corner from $\alpha$ to $\beta$ if $\Theta : B(H_1 \oplus H_2) \to B(K_1 \oplus K_2)$ defined by

$$\Theta \left( \begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right) = \left( \begin{array}{cc} \phi(A_{11}) & \gamma(A_{12}) \\ \gamma^*(A_{21}) & \psi(A_{22}) \end{array} \right)$$

is normal and completely positive.

Suppose $H_1 = K_1 = C^n$ and $H_2 = K_2 = C^m$. We say $\gamma : M_{n,m}(C) \to M_{n,m}(C)$ is a $q$-corner from $\phi$ to $\psi$ if $\Theta \geq_q 0$. A $q$-corner $\gamma$ is hyper maximal if, whenever

$$\Theta \geq_q \Theta' = \left( \begin{array}{cc} \phi' & \gamma' \\ \gamma^* & \psi' \end{array} \right) \geq_q 0,$$

we have $\Theta = \Theta'$.

Hyper maximal $q$-corners between unital $q$-positive maps $\phi$ and $\psi$ allow us to compare $E_0$-semigroups induced by $(\phi, \nu)$ and $(\psi, \nu)$ if $\nu$ is a particular kind of type II Powers weight:

Proposition 2.9. Let $\phi : M_n(C) \to M_n(C)$ and $\psi : M_k(C) \to M_k(C)$ be unital $q$-positive maps, and let $\nu$ be a type II Powers weight of the form

$$\nu(\sqrt{I - \Lambda(1)}B\sqrt{I - \Lambda(1)}) = (f, Bf).$$

The boundary weight doubles $(\phi, \nu)$ and $(\psi, \nu)$ induce cocycle conjugate $E_0$-semigroups if and only if there is a hyper maximal $q$-corner from $\phi$ to $\psi$.

From [9], we know that a unital rank one map $\phi : M_n(C) \to M_n(C)$ is $q$-positive if and only if it has the form $\phi(A) = \rho(A)I$ for a state $\rho \in M_n(C)^*$, and that $\phi$ is $q$-pure if and only if $\rho$ is faithful. We also have the following comparison result (Theorem 5.4 of [9]), which we will extend in this paper to all unital rank one $q$-positive maps (Theorem 3.5):
Theorem 2.10. Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ and $\psi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be rank one unital q-pure maps, so for some faithful states $\rho \in M_n(\mathbb{C})^*$ and $\rho' \in M_n(\mathbb{C})^*$, we have

$$\phi(A) = \rho(A)I_n \quad \text{and} \quad \psi(D) = \rho'(D)I_{n'},$$

for all $A \in M_n(\mathbb{C})$ and $D \in M_{n'}(\mathbb{C})$. Let $\nu$ be a type II Powers weight of the form $\nu(\sqrt{I - \Lambda(1)}B\sqrt{I - \Lambda(1)}) = (f, Bf)$. The boundary weight doubles $(\phi, \nu)$ and $(\psi, \nu)$ induce cocycle conjugate $E_0$-semigroups if and only if $n = n'$ and for some unitary $U \in M_n(\mathbb{C})$ we have $\rho'(A) = \rho(UAU^*)$ for all $A \in M_n(\mathbb{C})$.

2.3. Conjugacy for q-positive maps. We will only be concerned with the identity of a q-positive map up to an equivalence relation we will call conjugacy. More specifically, if $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a unital q-positive map and $U \in M_n(\mathbb{C})$ is any unitary matrix, the map $\phi_U(A) := U^*\phi(UAU^*)U$ is also unital and q-positive. We have the following definition from [3]:

Definition 2.11. Let $\phi, \psi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be q-positive maps. We say $\phi$ is conjugate to $\psi$ if $\psi = \phi_U$ for some unitary $U \in M_n(\mathbb{C})$.

Conjugacy is clearly an equivalence relation, and its definition is analogous to that of conjugacy for $E_0$-semigroups. Indeed, since every $*$-isomorphism of $M_n(\mathbb{C})$ is implemented by unitary conjugation, two q-positive maps $\phi, \psi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ are conjugate if and only if $\psi = \theta \circ \phi \circ \theta^{-1}$ for some $*$-isomorphism $\theta$ of $M_n(\mathbb{C})$. If $\nu$ is a type II Powers weight of the form $\nu(\sqrt{I - \Lambda(1)}B\sqrt{I - \Lambda(1)}) = (f, Bf)$, then conjugacy between unital q-positive maps $\phi$ and $\psi$ is always a sufficient condition for $(\phi, \nu)$ and $(\psi, \nu)$ to induce cocycle conjugate $E_0$-semigroups. Indeed, it is straightforward to verify that if $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is unital and q-positive, then the map $\gamma : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ defined by $\gamma(A) = \phi(AU^*)U$ is a hyper maximal $q$-corner from $\phi$ to $\phi_U$ (for details, see the discussion preceding Proposition 2.11 of [3]), whereby Proposition 2.11 gives us:

Proposition 2.12. Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be unital and q-positive, and suppose $\psi$ is conjugate to $\phi$. If $\nu$ is a type II Powers weight of the form $\nu(\sqrt{I - \Lambda(1)}B\sqrt{I - \Lambda(1)}) = (f, Bf)$, then $(\phi, \nu)$ and $(\psi, \nu)$ induce cocycle conjugate $E_0$-semigroups.

In the case that $\phi$ and $\psi$ are unital rank one q-pure maps and $\nu$ is a type II Powers weight of the form $\nu(\sqrt{I - \Lambda(1)}B\sqrt{I - \Lambda(1)}) = (f, Bf)$, Theorem 2.10 states that conjugacy between $\phi$ and $\psi$ is both necessary and sufficient for $(\phi, \nu)$ and $(\psi, \nu)$ induce cocycle conjugate $E_0$-semigroups.

Let $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a unital linear map of rank one. It is not difficult to see that $\phi$ is q-positive if and only if it has the form $\phi(A) = \rho(A)I$ for some state $\rho \in M_n(\mathbb{C})^*$. It is well-known that we can write $\rho$ in the form

$$\rho(A) = \sum_{i=1}^{k \leq n} \lambda_i (g_i, Ag_i),$$

for some mutually orthogonal unit vectors $\{g_i\}_{i=1}^k \subset \mathbb{C}^n$ and some positive numbers $\lambda_1 \geq \cdots \geq \lambda_k > 0$ such that $\sum_{i=1}^k \lambda_i = 1$. With the conditions of the previous
sentence satisfied, the number \( k \) and the monotonically decreasing set \( \{\lambda_i\}_{i=1}^k \) are unique.

**Definition 2.13.** Assume the notation of the previous paragraph. We call \( \{\lambda_i\}_{i=1}^k \) the eigenvalue list for \( \rho \).

We should note that our definition differs from a previous definition of eigenvalue list in the literature (see, for example, [1]) in that our eigenvalue lists do not include zeros. By our definition, it is possible for states \( \rho \) and \( \rho' \) acting on \( M_n(\mathbb{C}) \) and \( M_{n'}(\mathbb{C}) \) to have identical eigenvalue lists if \( n \neq n' \).

Let \( \{e_i\}_{i=1}^n \) be the standard basis for \( \mathbb{C}^n \). If \( \rho \) has the form (1) and \( U \in M_n(\mathbb{C}) \) is any unitary matrix such that \( U e_i = g_i \) for all \( i = 1, \ldots, k \), then

\[
\rho(UAU^*) = \sum_{i=1}^k \lambda_i(g_i, UAU^* g_i) = \sum_{i=1}^k \lambda_i(U^* g_i, AU^* g_i) = \sum_{i=1}^k \lambda_i(e_i, Ae_i)
\]

and

\[
\phi_U(A) = U^* \phi(UAU^*) U = U^* \left( \left( \sum_{i=1}^k \lambda_i(e_i, Ae_i) \right) I \right) U = \left( \sum_{i=1}^k \lambda_i a_{ii} \right) I
\]

for all \( A \in M_n(\mathbb{C}) \). We will use this fact repeatedly.

3. Our Results

We begin with the following observation:

**Lemma 3.1.** Let \( \phi : M_n(\mathbb{C}) \to M_n(\mathbb{C}) \) and \( \psi : M_{n'}(\mathbb{C}) \to M_{n'}(\mathbb{C}) \) be unital \( q \)-positive maps, and let \( \nu \) and \( \eta \) be type II Powers weights. If the boundary weight doubles \( (\phi, \nu) \) and \( (\psi, \eta) \) induce cocycle conjugate \( E_0 \)-semigroups, there is a corner \( \gamma \) from \( L_\phi \) to \( L_\psi \) such that \(|\gamma|| = 1\) and 1 is an eigenvalue of \( \gamma \).

**Proof.** This is a slight generalization of Lemma 5.3 of [9] (where \( \phi \) and \( \psi \) were assumed to have rank one and be \( q \)-pure), but its proof is identical. Indeed, the exact same argument as in the proof of Lemma 5.3 shows that there is a corner \( \gamma \) from \( \lim_{t \to 0^+} \nu(\Lambda(1)) \phi(I + \nu(\Lambda(1)) \phi)^{-1} \) to \( \lim_{t \to 0^+} \eta(\Lambda(1)) \psi(I + \eta(\Lambda(1)) \psi)^{-1} \) (provided the limits exist) such that \(|\gamma|| = 1\) and 1 is an eigenvalue of \( \gamma \). We observe that the former limit is \( L_\phi \) and the latter limit is \( L_\psi \). Indeed, the values \( \{\nu\Lambda(1)\}_{t>0} \) and \( \{\eta(\Lambda(1))\}_{t>0} \) are monotonically decreasing in \( t \), and since \( \nu \) and \( \eta \) are unbounded, we have

\[
\lim_{t \to 0^+} \nu(\Lambda(1)) = \lim_{t \to 0^+} \eta(\Lambda(1)) = \infty.
\]

We have the following lemma (see Lemma 3.5 of [9]):

**Lemma 3.2.** Let \( \phi : M_n(\mathbb{C}) \to M_n(\mathbb{C}) \), \( \psi : M_{n'}(\mathbb{C}) \to M_{n'}(\mathbb{C}) \) be completely positive maps, so for some \( k, k' \in \mathbb{N} \) and sets of linearly independent matrices \( \{S_i\}_{i=1}^k \subset M_{r,n}(\mathbb{C}) \) and \( \{T_i\}_{i=1}^{k'} \subset M_{r',n'}(\mathbb{C}) \), we have

\[
\phi(A) = \sum_{i=1}^k S_i A S_i^*, \quad \psi(D) = \sum_{i=1}^{k'} T_i A T_i^*
\]

for all \( A \in M_n(\mathbb{C}) \), \( D \in M_{n'}(\mathbb{C}) \).
A linear map \( \gamma : M_{n,n'}(\mathbb{C}) \to M_{r,r'}(\mathbb{C}) \) is a corner from \( \phi \) to \( \psi \) if and only if, for some \( C = (c_{ij}) \in M_{k,k'}(\mathbb{C}) \) with \( \|C\| \leq 1 \), we have

\[
\gamma(B) = \sum_{i=1}^{k} \sum_{j=1}^{k'} c_{ij} S_i B T_j^*
\]

for all \( B \in M_{n,n'}(\mathbb{C}) \).

**Remark 1:** Suppose \( \gamma \) is a \( q \)-corner from \( \phi \) to \( \psi \). Let \( U \in M_n(\mathbb{C}) \) and \( V \in M_{n'}(\mathbb{C}) \) be arbitrary unitary matrices, and let

\[
\vartheta = \begin{pmatrix} \phi & \gamma \\ \gamma^* & \psi \end{pmatrix} \geq_q 0.
\]

For the unitary matrix

\[
Z = \begin{pmatrix} U & 0_{n,n'} \\ 0_{n',n} & V \end{pmatrix} \in M_{n+n'}(\mathbb{C}),
\]

we have \( \vartheta Z \geq_q 0 \) (since \( \vartheta \geq_q 0 \)), where

\[
\vartheta Z \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \varphi_U(A) & U^* \gamma(UBV^*) \nu_V(D) \\ V^* \gamma^*(VCU^*) & \psi_V(D) \end{pmatrix}.
\]

Therefore, \( B \to U^* \gamma(UBV^*) \nu_V \) is a \( q \)-corner from \( \varphi_U \) to \( \psi_V \). By Proposition 4.5 of \([9]\), there is an isomorphism between the \( q \)-subordinates of \( \vartheta \) and the \( q \)-subordinates of \( \vartheta Z \). In particular, if \( \Phi : M_{n+n'}(\mathbb{C}) \to M_{n+n'}(\mathbb{C}) \) is a linear map, then \( \vartheta \geq_q \Phi \geq_q 0 \) if and only if \( \vartheta Z \geq_q \Phi Z \geq_q 0 \). It follows that \( \gamma \) is a hyper maximal \( q \)-corner from \( \phi \) to \( \psi \) if and only if \( B \to U^* \gamma(UBV^*) V \) is a hyper maximal \( q \)-corner from \( \varphi_U \) to \( \psi_V \). The same argument just used also gives us a bijection between norm one corners from \( \phi \) to \( \psi \) and norm one corners from \( \varphi_U \) to \( \psi_V \).

**Proposition 3.3.** Let \( \phi : M_n(\mathbb{C}) \to M_n(\mathbb{C}) \) and \( \psi : M_{n'}(\mathbb{C}) \to M_{n'}(\mathbb{C}) \) be unital rank one \( q \)-positive maps, so for some states \( \ell \in M_n(\mathbb{C})^* \) and \( \ell' \in M_{n'}(\mathbb{C})^* \) with eigenvalue lists \( \{\lambda_i\}_{i=1}^k \) and \( \{\mu_i\}_{i=1}^{k'} \), respectively, we have

\[
\phi(A) = \ell(A) I_n, \quad \psi(D) = \ell'(D) I_{n'}
\]

for all \( A \in M_n(\mathbb{C}) \) and \( D \in M_{n'}(\mathbb{C}) \). Let \( \nu \) and \( \eta \) be type II Powers weights.

If the boundary weight doubles \( \langle \phi, \nu \rangle \) and \( \langle \psi, \eta \rangle \) induce cocycle conjugate \( E_0 \)-semigroups \( \alpha^d \) and \( \beta^d \), then \( k = k' \) and \( \lambda_i = \mu_i \) for all \( i = 1, \ldots, k \).

**Proof.** Our proof is similar to the proof of Theorem 5.4 of \([9]\). Suppose \( \alpha^d \) and \( \beta^d \) are cocycle conjugate. For some unitaries \( U \in M_n(\mathbb{C}) \) and \( V \in M_{n'}(\mathbb{C}) \), we have

\[
\phi_U(A) = \left( \sum_{i=1}^{k} \lambda_i a_{ii} \right) I_n, \quad \psi_V(D) = \left( \sum_{i=1}^{k'} \mu_i b_{ii} \right) I_{n'}
\]

for all \( A \in M_n(\mathbb{C}) \) and \( D \in M_{n'}(\mathbb{C}) \). Let \( \{e_i\}_{i=1}^n \) and \( \{e'_i\}_{i=1}^{n'} \) be the standard bases for \( \mathbb{C}^n \) and \( \mathbb{C}^{n'} \), respectively, and let \( \rho \in M_n(\mathbb{C})^* \) and \( \rho' \in M_{n'}(\mathbb{C})^* \) be the functionals

\[
(4) \quad \rho(A) = \sum_{i=1}^{k} \lambda_i e^*_i A e_i = \sum_{i=1}^{k'} \lambda_i a_{ii}, \quad \rho'(D) = \sum_{i=1}^{k'} \mu_i e^*_i D e'_i = \sum_{i=1}^{k'} \mu_i d_{ii},
\]
hence from (6) we have

\[
(5)
\]

where \( |\psi| \) is a functional defined by

\[L(\phi) = \phi A, \quad L(\psi) = \psi, \quad \text{so by Lemma 3.1 there is a norm one corner from } \phi \text{ to } \psi.\]

Therefore, by Remark 1, there is a norm one corner \( \gamma \) from \( \phi \) to \( \psi \), so the map \( \Theta : M_{n+n'}(\mathbb{C}) \to M_{n+n'}(\mathbb{C}) \) defined by

\[
\Theta \left( \begin{array}{cc} A_{n,n} & B_{n,n'} \\ C_{n',n} & D_{n',n'} \end{array} \right) = \left( \begin{array}{cc} \rho(A) & \gamma(B) \\ \gamma^*(C) & \rho'(D) \end{array} \right)
\]

is completely positive.

Since \( ||\gamma|| = 1 \), there is some \( X \in M_{n,n'}(\mathbb{C}) \) with \( ||X|| = 1 \) and some unit vector \( g \in \mathbb{C}^n \) such that \( ||\gamma(X)g||^2 = (\gamma(X)g, \gamma(X)g) = 1 \). Let \( \tau \in M_{n,n'}(\mathbb{C})^* \) be the functional defined by

\[\tau(B) = (\gamma(X)g, \gamma(B)g).\]

Letting

\[S = \left( \begin{array}{cc} \gamma(X)g & 0_{n,1} \\ 0_{n',1} & g \end{array} \right) \in M_{n+n',2}(\mathbb{C}),\]

we observe that

\[
\begin{pmatrix} \rho(A) & \tau(B) \\ \tau^*(C) & \rho'(D) \end{pmatrix} = S^* \Theta \begin{pmatrix} A & B \\ C & D \end{pmatrix} S \quad \text{for all } \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_{n+n'}(\mathbb{C}),
\]

hence \( \tau \) is a corner from \( \rho \) to \( \rho' \). Note that \( ||\tau|| = ||\tau(X)|| = 1 \).

Let \( D_\lambda \in M_k(\mathbb{C}) \) and \( D_\mu \in M_k(\mathbb{C}) \) be the diagonal matrices whose \( ii \) entries are \( \sqrt{\lambda_i} \) and \( \sqrt{\mu_i} \), respectively. Since \( \tau \) is a corner from \( \rho \) to \( \rho' \), equation (4) and Lemma 3.2 imply that \( \tau \) has the form \( \tau(B) = \sum_{i,j} c_{ij} \sqrt{\lambda_i \mu_j} (e_i, B e_j) \) for some \( C = (c_{ij}) \in M_{k,k}(\mathbb{C}) \) such that \( ||C|| \leq 1 \). For each \( B \in M_{n,n'}(\mathbb{C}) \), let \( \tilde{B} \in M_{k',k}(\mathbb{C}) \) be the top left \( k' \times k \) minor of \( B^T \), observing that

\[
\tau(B) = \sum_{i=1}^k \sum_{j=1}^{k'} c_{ij} \sqrt{\lambda_i \mu_j} b_{ij} = \text{tr}(CD_\mu \tilde{B} D_\lambda) = \text{tr} \left( CD_\mu (D_\lambda \tilde{B})^* \right).
\]

Let \( M = \tilde{X} \in M_{k',k}(\mathbb{C}) \). Applying the Cauchy-Schwarz inequality to the inner product \( \langle A, B \rangle = \text{tr}(BA^*) \) on \( M_{k,k'}(\mathbb{C}) \), we see

\[
1 = ||\tau(X)||^2 = ||\text{tr}(CD_\mu (D_\lambda M^*)^*)||^2 = ||D_\lambda M^* CD_\mu||^2
\]

\[
\leq ||CD_\mu||_{tr}^2 ||D_\lambda M^*||_{tr}^2 = \text{tr}(D_\mu C^* CD_\mu) \text{tr}(D_\lambda M^* M D_\lambda)
\]

\[
(5) \quad \leq \text{tr}(D_\mu I_{k'} D_\mu) \text{tr}(D_\mu I_k D_\lambda) = \left( \sum_{i=1}^{k'} \mu_i \right) \left( \sum_{i=1}^k \lambda_i \right) = 1 * 1 = 1.
\]

Since equality holds in Cauchy-Schwarz, it follows that for some \( m \in \mathbb{C} \),

\[
(6) \quad m CD_\mu = D_\lambda M^*,
\]

where \( |m| = 1 \) since \( ||CD_\mu||_{tr} = ||D_\lambda M^*||_{tr} = 1 \). In fact, \( m = 1 \) since \( \tau(X) = 1 \).

Since equality holds in (5) and the trace map is faithful, we have \( C^* C = I_{k'} \) and \( M^* M = I_k \). Note that

\[\min\{k, k'\} \geq \text{rank}(C) = k', \quad \min\{k, k'\} \geq \text{rank}(M) = k,\]

hence \( k = k' \) and the previous sentence shows that \( C \) and \( M \) are unitary. Therefore, from (6) we have

\[D_\mu = C^* D_\lambda M^* = C M^* (MD_\lambda M^*),\]
whereby uniqueness of the right polar decomposition for the invertible positive matrix \( D_\mu \) implies \( D_\mu = MD_\lambda M^* \). Since the eigenvalues of \( D_\mu \) and \( D_\lambda \) are listed in decreasing order, we have \( D_\mu = D_\lambda \), hence \( \lambda_i = \mu_i \) for all \( i = 1, \ldots, k \). \( \square \)

**Remark 2:** If \( \phi : M_k(\mathbb{C}) \rightarrow M_k(\mathbb{C}) \) is a unital rank one \( q \)-pure map, and if \( \gamma \) is a nonzero \( q \)-corner from \( \phi \) to \( \phi \), then by Lemma 2.2 \( \sigma := \lim_{t\to\infty} t\gamma(I + t\gamma)^{-1} \) is a corner from \( \phi \) to \( \phi \) satisfying \( \sigma^2 = \sigma \). We note that \( ||\sigma|| = 1 \). Indeed, since \( \sigma^2 = \sigma \) and \( \text{range}(\sigma) = \text{range}(\gamma) \supseteq \{0\} \), we have \( ||\sigma|| \geq 1 \), while the fact that \( \sigma \) is a corner between norm one completely positive maps implies \( ||\sigma|| \leq 1 \), hence \( ||\sigma|| = 1 \).

The following lemma gives us the form of \( \sigma \):

**Lemma 3.4.** Let \( \phi : M_k(\mathbb{C}) \rightarrow M_k(\mathbb{C}) \) be a unital \( q \)-positive map of the form \( \phi(A) = \rho(A)I \). Assume \( \rho \) is a faithful state of the form

\[
\rho(A) = \sum_{i=1}^{k} \mu_{ii} a_{ii},
\]

where \( \mu_1, \ldots, \mu_k \) are positive numbers and \( \sum_{i=1}^{k} \mu_i = 1 \). Let \( D_\mu \) be the diagonal matrix with \( ii \) entries \( \sqrt{\mu_i} \) for \( i = 1, \ldots, k \), so \( \Omega := (D_\mu)^2 \) is the trace density matrix for \( \rho \).

Let \( \sigma : M_k(\mathbb{C}) \rightarrow M_k(\mathbb{C}) \) be a nonzero linear map such that \( \sigma^2 = \sigma \). Then \( \sigma \) is a corner from \( \phi \) to \( \phi \) if, and only if, some unitary \( X \in M_k(\mathbb{C}) \) that commutes with \( \Omega \), we have

\[
\sigma(B) = \text{tr}(X^*B\Omega)X
\]

for all \( B \in M_k(\mathbb{C}) \).

**Proof.** For the forward direction, suppose that \( \sigma \) is a nonzero corner from \( \phi \) to \( \phi \) and \( \sigma^2 = \sigma \). Note that \( ||\sigma|| = 1 \) by Remark 2. We first show that \( \sigma \) has rank one. If \( \text{rank}(\sigma) \geq 2 \), then there is a non-invertible element \( A \in \text{range}(\sigma) \). Scaling \( A \) if necessary, we may assume \( ||A|| = 1 \). Let \( P \) be the orthogonal projection onto the range of \( A \), so \( PA = A \) and \( A^* = A^*P \). Since \( P \neq I \) and \( \rho \) is faithful, we have \( \phi(P) = \rho(A)I = aI \) for some \( a < 1 \). We note that

\[
\begin{pmatrix}
  P & 0 \\
  0 & I
\end{pmatrix}
\begin{pmatrix}
  I & A \\
  A^* & I
\end{pmatrix}
\begin{pmatrix}
  P & 0 \\
  0 & I
\end{pmatrix} = \begin{pmatrix}
  P & PA \\
  A^*P & I
\end{pmatrix} = \begin{pmatrix}
  P & A \\
  A^* & I
\end{pmatrix} \geq 0,
\]

so by complete positivity of \( \Theta \) and the fact that \( \sigma^2 = \sigma \), we have

\[
\begin{pmatrix}
  \phi(P) & \sigma(A) \\
  \sigma^*(A^*) & \phi(I)
\end{pmatrix} = \begin{pmatrix}
  aI & A \\
  A^* & I
\end{pmatrix} \geq 0,
\]

which is impossible since \( a < 1 \) and \( ||A|| = 1 \). This shows that not only does \( \sigma \) have rank one, but that every non-zero element of its range is invertible. In other words, for some linear functional \( \tau \in M_k(\mathbb{C})^* \) and some invertible matrix \( X \in M_k(\mathbb{C}) \) with \( ||X|| = 1 \), we have \( \sigma(B) = \tau(B)X \) for all \( B \in M_k(\mathbb{C}) \). Since \( \sigma \) fixes its range and \( ||\sigma|| = 1 \), we have \( ||\tau|| = ||\sigma|| = 1 \).

Let \( g \in \mathbb{C}^k \) be a unit vector such that \( ||Xg|| = 1 \). We observe that \( \tau \) is merely the functional \( \tau(B) = (\sigma(X)g, \sigma(B)g) \) for all \( B \in M_k(\mathbb{C}) \), and an argument analogous to the one given in the proof of Proposition 3.3 shows that \( \tau \) is a corner \( \rho \) to \( \rho \). By Lemma 3.2 there is some \( C \in M_k(\mathbb{C}) \) with \( ||C|| \leq 1 \) such that

\[
\tau(B) = \sum_{i,j=1}^{k} c_{ij} \sqrt{\mu_i \mu_j} (e_i, Be_j) = \text{tr}(CD_\mu B^T D_\mu)
\]
Thus, \( \eta \to M \) is positive for every \( \eta, \eta, \tau \). Uniqueness of the polar decomposition for the invertible positive matrix \( D_\mu \) gives us \( C^*(X^T)^* = I \) and \( X^TD_\mu X = D_\mu \), where the transpose of the last equality is \( X^*D_\mu X = D_\mu \). Therefore, \( C = (X^*)^T \) and \( X \) commutes with \( \Omega \), so for all \( B \in M_k(\mathbb{C}) \) we have
\[
\tau(B) = \text{tr} \left( (X^*)^T D_\mu B^T D_\mu \right) = \text{tr}(D_\mu B D_\mu X^*) = \text{tr}(X^* B \Omega)
\]
and \( \sigma(B) = \tau(B)X = \text{tr}(X^* B \Omega)X \).

Now assume the hypotheses of the backward direction and define \( \tau \in M_k(\mathbb{C})^* \) by \( \tau(B) = \text{tr}(X^* B \Omega) \), noting that \( \sigma^2 = \sigma \) and \( \sigma(B) = \tau(B)X \) for all \( B \in M_k(\mathbb{C}) \). Let \( \eta, \eta' : M_{2k}(\mathbb{C}) \to M_{2k}(\mathbb{C}) \) be the maps
\[
\eta \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) = \left( \begin{array}{cc} \rho(A)I & \tau(B)X \\ \tau^*(C)X^* & \rho(D)I \end{array} \right), \quad \eta' \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) = \left( \begin{array}{cc} \rho(A)I & \tau(B)I \\ \tau^*(C)I & \rho(D)I \end{array} \right).
\]

Define \( \Upsilon : M_{2k}(\mathbb{C}) \to M_{2k}(\mathbb{C}) \) by
\[
\Upsilon \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) = \left( \begin{array}{cc} X^* & 0 \\ 0 & I \end{array} \right) \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \left( \begin{array}{cc} X & 0 \\ 0 & I \end{array} \right).
\]

Note that \( \Upsilon \) and \( \Upsilon^{-1} \) are completely positive, \( \Upsilon \circ \eta = \eta' \), and \( \Upsilon^{-1} \circ \eta' = \eta \). Therefore, \( \eta \) is completely positive if and only if \( \eta' \) is completely positive. Since a complex matrix \( (m_{ij}) \in M_r(\mathbb{C}) \) (\( r \in \mathbb{N} \)) is positive if and only if \( (m_{ij}I_n) \in M_r(M_n(\mathbb{C})) = M_{rn}(\mathbb{C}) \) is positive for every \( n \in \mathbb{N} \), it follows that \( \eta' \) is completely positive if and only if \( \eta'' \) below is completely positive:
\[
\eta'' \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) = \left( \begin{array}{cc} \rho(A) & \tau(B) \\ \tau^*(C) & \rho(D) \end{array} \right).
\]

Thus, \( \eta \) is completely positive if and only if \( \eta'' \) is. In other words, \( \sigma \) is a corner from \( \phi \) to \( \phi \) if and only if \( \tau \) is a corner from \( \rho \) to \( \rho \). But for all \( B \in M_k(\mathbb{C}) \), we have
\[
\tau(B) = \sum_{i,j=1}^k c_{ij}\sqrt{\mu_i \mu_j}(e_i, Be_j)
\]
for the unitary matrix \( C = (X^*)^T \), so \( \tau \) is a corner from \( \rho \) to \( \rho \) by Lemma 3.2.

We will make use of the following standard result regarding completely positive maps, providing a proof here for the sake of completeness:

**Lemma 3.5.** Let \( \phi : M_n(\mathbb{C}) \to M_n(\mathbb{C}) \) be a completely positive map. If \( \phi(E) = 0 \) for a projection \( E \), then \( \phi(A) = \phi(F A F) \) for all \( A \in M_n(\mathbb{C}) \), where \( F = I - E \).

**Proof.** We know from [5] and [2] that \( \phi \) can be written \( \phi(A) = \sum_{i=1}^p S_i A S_i^* \) for some \( p \leq n^2 \) and \( \{S_i\}_{i=1}^p \subset M_n(\mathbb{C}) \). If \( \phi(E) = 0 \) for a projection \( E \), then
\[
0 = S_i E S_i^* = S_i E E S_i^* = (S_i E)(S_i E)^*
\]
for all \( i \), so \( S_i E = E S_i^* = 0 \) for all \( i \). Therefore, \( \phi(E A E) = \phi(E A F) = \phi(F A E) = 0 \) for every \( A \in M_n(\mathbb{C}) \). Letting \( F = I - E \), we observe that for every \( A \in M_n(\mathbb{C}) \),
\[
\phi(A) = \phi(E A E + E A F + F A E + F A F) = \phi(F A F).
\]
Proposition 3.3 implies $k$ φ $σ$ where by Lemma 2.2, the map below is We observe that Proof. for all 12CHRISTOPHER JANKOWSKI

If so, can we find all such q-corners, and, even further, determine which q-corners are hyper maximal? The following two theorems give us a complete answer to both questions when φ and ψ are implemented by diagonal states. This suffices, since for any unital rank one $q$-positive maps φ and ψ, there are always unitaries $U ∈ M_n(ℂ)$ and $V ∈ M_{n′}(ℂ)$ such that $φ_U$ and $ψ_V$ are implemented by diagonal states, where Remark 1 tells us exactly how to transform the q-corners and hyper maximal q-corners from $φ_U$ to $ψ_V$ into those from φ to ψ.

**Theorem 3.6.** Let $\{μ_i\}_{i=1}^k$ and $\{r_i\}_{i=1}^{k′}$ be monotonically decreasing sequences of strictly positive numbers such that $∑_{i=1}^k μ_k = ∑_{i=1}^{k′} r_i = 1$. Define unital q-positive maps $φ : M_n(ℂ) → M_n(ℂ)$ and $ψ : M_{n′}(ℂ) → M_{n′}(ℂ)$ (where $n ≥ k$ and $n′ ≥ k′$) by

\[
φ(A) = \left( ∑_{i=1}^k μ_i a_{ii} \right) I_n \quad \text{and} \quad ψ(D) = \left( ∑_{i=1}^{k′} r_i d_{ii} \right) I_{n′}
\]

for all $A = (a_{ij}) ∈ M_n(ℂ)$ and $D = (d_{ij}) ∈ M_{n′}(ℂ)$. Let $Ω ∈ M_k(ℂ)$ be the trace density matrix for the faithful state $ψ ∈ M_k(ℂ)^*$ defined by $ρ(Ω) = ∑_{i=1}^k μ_i a_{ii}$. If there is a nonzero q-corner from φ to ψ, then $k = k′$ and $μ_i = r_i$ for all $i = 1, \ldots, k$. In that case, a linear map $γ : M_{n,n′}(ℂ) → M_{n,n′}(ℂ)$ is a q-corner from φ to ψ if and only if: for some unitary $X ∈ M_k(ℂ)$ that commutes with $Ω$, some contraction $E ∈ M_{n−k,n′−k}(ℂ)$, and some $λ ∈ ℂ$ with $|λ|^2 ≤ Re(λ)$, we have

\[
γ \left( \begin{array}{cc}
B_{k,k} & W_{n,n′−k} \\
Q_{n−k,k} & Y_{n−k,n′−k}
\end{array} \right) = λ \, tr(X^* B_{k,k} Ω) \left( \begin{array}{cc}
X & 0_{n,n′−k} \\
0_{n−k,k} & E
\end{array} \right)
\]

for all

\[
\left( \begin{array}{cc}
B_{k,k} & W_{n,n′−k} \\
Q_{n−k,k} & Y_{n−k,n′−k}
\end{array} \right) ∈ M_{n,n′}(ℂ).
\]

**Proof.** Suppose that $γ$ is a nonzero q-corner from φ to ψ, so $φ : M_{n+n′}(ℂ) → M_{n+n′}(ℂ)$ below is q-positive:

\[
φ \left( \begin{array}{cc}
A_{n,n} & B_{n,n′} \\
C_{n,n′} & D_{n,n′}
\end{array} \right) = \left( \begin{array}{cc}
φ(A_{n,n}) & γ(B_{n,n′}) \\
γ^*(C_{n,n′}) & ψ(D_{n,n′})
\end{array} \right).
\]

We observe that

\[
L_φ \left( \begin{array}{cc}
A_{n,n} & B_{n,n′} \\
C_{n,n′} & D_{n,n′}
\end{array} \right) = \left( \begin{array}{cc}
φ(A_{n,n}) & σ(B_{n,n′}) \\
σ^*(C_{n,n′}) & ψ(D_{n,n′})
\end{array} \right),
\]

where by Lemma 2.2 the map $σ := \lim_{t→∞} tγ(I + tγ)^{-1}$ is a corner of norm one from φ to ψ satisfying $σ^2 = σ$, range($σ) = range(γ)$, and $γ ∘ σ = σ ∘ γ = γ$. Since $||σ|| = 1$, Proposition 3.3 implies $k = k′$ and $r_i = μ_i$ for all $i = 1, \ldots, k$.

We observe that $L_φ(E) = 0$ for the projection

\[
E = \left( ∑_{i=k+1}^n e_{ii} + ∑_{i=n+k′+1}^{n+n′} e_{ii} \right) ∈ M_{n+n′}(ℂ).
\]
Therefore, \( L_\theta(A) = L_\theta(I - E)A(I - E) \) for all \( A \in M_{n+n'}(\mathbb{C}) \) by Lemma 3.5. In particular, \( \sigma \) satisfies
\[
\sigma \left( \begin{array}{cc} 0_{k,k} & W_{k,n'-k} \\ Q_{n-k,k} & Y_{n-k,n'-k} \end{array} \right) = 0.
\]
In other words, \( \sigma \) depends only on its top left \( k \times k \) minor, so for some \( \tilde{\sigma} : M_k(\mathbb{C}) \to M_k(\mathbb{C}) \) and some maps \( \ell_i \) from \( M_k(\mathbb{C}) \) into the appropriate matrix spaces, we have
\[
\sigma \left( \begin{array}{cc} B_{k,k} & W_{k,n'-k} \\ Q_{n-k,k} & Y_{n-k,n'-k} \end{array} \right) = \left( \begin{array}{cc} \tilde{\sigma}(B_{k,k}) & \ell_1(B_{k,k}) \\ \ell_2(B_{k,k}) & \ell_3(B_{k,k}) \end{array} \right).
\]
From the facts \( \sigma^2 = \sigma \) and \( ||\sigma|| = 1 \), it follows that \( \tilde{\sigma}^2 = \tilde{\sigma} \) and \( ||\tilde{\sigma}|| = 1 \).

Let \( \hat{\phi} : M_k(\mathbb{C}) \to M_k(\mathbb{C}) \) be the map
\[
\hat{\phi}(A) = \rho(A)I_k = \left( \sum_{i=1}^{k} \mu_ia_{ii} \right)I_k
\]
for all \( A = (a_{ij}) \in M_k(\mathbb{C}) \). Define \( \Theta : M_{2k}(\mathbb{C}) \to M_{2k}(\mathbb{C}) \) by
\[
\Theta \left( \begin{array}{cc} A_{k,k} & B_{k,k} \\ C_{k,k} & D_{k,k} \end{array} \right) = \left( \begin{array}{cc} \hat{\phi}(A_{k,k}) & \tilde{\sigma}(B_{k,k}) \\ \tilde{\sigma}^*(C_{k,k}) & \hat{\psi}(D_{k,k}) \end{array} \right),
\]
and let
\[
S = \left( \begin{array}{ccc} I_{k,k} & 0_{k,n-k} & 0_{k,n'-k} \\ 0_{k,k} & I_{k,k} & 0_{k,n'-k} \end{array} \right) \in M_{2k,n+n'}(\mathbb{C}).
\]
Note that
\[
\Theta(N) = SL_\theta(S^*NS)S^*
\]
for all \( N \in M_{2k}(\mathbb{C}) \), so \( \Theta \) is completely positive. Therefore, \( \tilde{\sigma} \) is a norm one corner from \( \hat{\phi} \) to \( \tilde{\phi} \). Since \( ||\tilde{\sigma}|| = 1 \) and \( \tilde{\sigma}^2 = \tilde{\sigma} \), Lemma 3.4 implies that for some unitary \( X \in M_k(\mathbb{C}) \) that commutes with \( \Omega \), we have
\[
\tilde{\sigma}(B) = \text{tr}(X^*B\Omega)X
\]
for all \( B \in M_k(\mathbb{C}) \). For simplicity of notation in what follows, let \( \tau \in M_k(\mathbb{C})^* \) be the functional \( \tau(B) = \text{tr}(X^*B\Omega) \).

We claim that \( \ell_1 = \ell_3 \equiv 0 \). For this, let
\[
M = \left( \begin{array}{cc} B_{k,k} & W_{k,n'-k} \\ Q_{n-k,k} & Y_{n-k,n'-k} \end{array} \right) \in M_{n,n'}(\mathbb{C})
\]
be arbitrary. We will suppress the subscripts for \( B, Q, W, \) and \( Y \) for the remainder of the proof. From (2) and the fact that \( \sigma^2(M) = \sigma(M) \), we have
\[
\ell_i(B) = \ell_i(\tilde{\sigma}(B)) = \ell_i(\tau(B)X) = \tau(B)\ell_i(X)
\]
for \( i = 1, 2, 3 \). Since \( \sigma \) is a contraction, it follows that
\[
1 \geq \left\| \sigma \left( \begin{array}{cc} X & 0 \\ 0 & 0 \end{array} \right) \right\| = \left\| \left( \begin{array}{cc} X & \ell_1(X) \\ \ell_2(X) & \ell_3(X) \end{array} \right) \right\|.
\]
But \( X \) is unitary, so the line above implies that \( \ell_1(X) = \ell_2(X) = 0 \), hence \( \ell_1 = \ell_2 \equiv 0 \) by (10). Let \( E = \ell_3(X) \in M_{n-k,n'-k}(\mathbb{C}) \), noting that \( ||E|| \leq 1 \) since \( \sigma \) is a contraction. Therefore, \( \sigma \) has the form
\[
\sigma \left( \begin{array}{cc} B & W \\ Q & Y \end{array} \right) = \tau(B) \left( \begin{array}{cc} X & 0_{k,n'-k} \\ 0_{n-k,k} & E \end{array} \right).
\]
Since \( \gamma = \gamma \circ \sigma \) and
\[
\text{range}(\gamma) = \text{range}(\sigma) = \{c \begin{pmatrix} X & 0 \\ 0 & E \end{pmatrix} : c \in \mathbb{C}\},
\]
we have
\[
\gamma \left( \begin{array}{cc} B & W \\ Q & Y \end{array} \right) = \gamma(\sigma \left( \begin{array}{cc} B & W \\ Q & Y \end{array} \right)) = \gamma \left( \tau(B) \left( \begin{array}{cc} X & 0 \\ 0 & E \end{array} \right) \right) = \tau(B) \gamma \left( \begin{array}{cc} X & 0 \\ 0 & E \end{array} \right)
\]
\[
= \tau(B) \left[ \lambda \left( \begin{array}{cc} X & 0 \\ 0 & E \end{array} \right) \right] = \lambda \tau(B) \left( \begin{array}{cc} X & 0 \\ 0 & E \end{array} \right)
\]
for some \( \lambda \in \mathbb{C} \). Since \( \gamma \) is a nonzero \( q \)-corner between unital completely positive maps and is thus necessarily a contraction with no negative eigenvalues, we have \( \lambda \leq 0 \) and \( |\lambda| \leq 1 \).

In summary: we have proved that if \( \gamma \) is a nonzero \( q \)-corner, then it is of the form
\[
\gamma \left( \begin{array}{cc} B & W \\ Q & Y \end{array} \right) = \lambda \text{tr}(X^*B\Omega) \left( \begin{array}{cc} X & 0 \\ 0 & E \end{array} \right)
\]
for some \( \lambda \leq 0 \) with \( |\lambda| \leq 1 \), where \( X \) and \( E \) satisfy the conditions stated in the theorem. To complete the proof, we show that such a map \( \gamma \) is a \( q \)-corner if and only if \( |\lambda|^2 \leq \text{Re}(\lambda) \).

Straightforward computations show that for all \( t \geq 0 \),
\[
(I + t\gamma)^{-1} \left( \begin{array}{cc} B & W \\ Q & Y \end{array} \right) = \left( \begin{array}{cc} B - \frac{t\lambda \tau(B)}{1 + t\lambda}X \\ Q \end{array} \right) \left( \begin{array}{cc} Y - \frac{t\lambda \tau(B)}{1 + t\lambda}E \end{array} \right)
\]
and
\[
\gamma(I + t\gamma)^{-1} \left( \begin{array}{cc} B & W \\ Q & Y \end{array} \right) = \left( \begin{array}{cc} \frac{\lambda \tau(B)}{1 + t\lambda}X \\ 0 \end{array} \right) \left( \begin{array}{cc} \frac{\lambda \tau(B)}{1 + t\lambda}E \end{array} \right) = \frac{\lambda}{1 + t\lambda} \gamma \left( \begin{array}{cc} B & W \\ Q & Y \end{array} \right).
\]

For each \( t \geq 0 \), define maps \( \Theta_t : M_{2k}(\mathbb{C}) \to M_{2k}(\mathbb{C}) \), \( L_t : M_{2k}(\mathbb{C}) \to M_{n+n'-2k}(\mathbb{C}) \), and \( \Upsilon_t : M_{2k}(\mathbb{C}) \to M_{n+n'-2k}(\mathbb{C}) \) by
\[
\Theta_t \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) = \left( \begin{array}{cc} \frac{1}{1 + t\lambda} \rho(A)I_{k,k} \\ \frac{\lambda}{1 + t\lambda} \tau^*(C)X^* \end{array} \right) \left( \begin{array}{cc} \frac{1}{1 + t\lambda} \rho(D)I_{k,k} \end{array} \right),
\]
\[
L_t \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) = \left( \begin{array}{cc} \frac{1}{1 + t\lambda} \rho(A)EE^* \\ \frac{\lambda}{1 + t\lambda} \tau^*(C)E^* \end{array} \right) \left( \begin{array}{cc} \frac{1}{1 + t\lambda} \rho(D)I_{n' - k,n' - k} \end{array} \right),
\]
and
\[
\Upsilon_t \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) = L_t \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) + \left( \begin{array}{cc} \frac{1}{1 + t\lambda} \rho(A)(I_{n - k,n - k} - EE^*) \\ 0_{n' - k,n' - k} \end{array} \right) \left( \begin{array}{cc} 0_{n - k,n - k} \\ 0_{n' - k,n' - k} \end{array} \right).
\]
Let
\[
T = \left( \begin{array}{cccc} 0_{n - k,k} & I_{n - k,n - k} & 0_{n - k,k} & 0_{n - k,n' - k} \\ 0_{n' - k,k} & 0_{n' - k,n - k} & 0_{n' - k,k} & I_{n' - k,n' - k} \end{array} \right) \in M_{n+n'-2k,n+n'}(\mathbb{C}),
\]
and let \( M \in M_{n+n'}(\mathbb{C}) \) be arbitrary, writing
\[
M = \begin{pmatrix}
A_{k,k} & q_{k,n-k} & B_{k,k} & r_{k,n'-k} \\
ŝ_{n-k,k} & t_{n-k,n-k} & u_{n-k,k} & v_{n-k,n'-k} \\
C_{k,k} & w_{k,n-k} & D_{k,k} & c_{k,n'-k} \\
d_{n'-k,k} & e_{n'-k,n-k} & f_{n'-k,k} & g_{n'-k,n'-k},
\end{pmatrix},
\]
so \( SMS^* = \begin{pmatrix} A_{k,k} & B_{k,k} \\ C_{k,k} & D_{k,k} \end{pmatrix} \).

For every \( t \geq 0 \), we have
\[
\vartheta(I + t\vartheta)^{-1}(M) = \begin{pmatrix}
\frac{1}{1+i\lambda}\rho(A)I_{k,k} & 0_{k,n-k} & \frac{1}{1+i\lambda}\rho(A)I_{n-k,n-k} & 0_{k,n-k} \\
0_{n-k,k} & \frac{1}{1+i\lambda}\tau(B)X & 0_{n-k,k} & \frac{1}{1+i\lambda}\tau(B)E \\
\frac{\lambda}{1+i\lambda}\tau^*(C)X^* & 0_{k,n-k} & \frac{1}{1+i\lambda}\rho(D)I_{k,k} & 0_{k,n-k} \\
0_{n'-k,k} & \frac{\lambda}{1+i\lambda}\tau^*(C)E^* & 0_{n'-k,k} & \frac{1}{1+i\lambda}\rho(D)I_{n'-k,n'-k},
\end{pmatrix}
\]
(11) \( = S^*\Theta_t(SMS^*)S + T^*\Upsilon_t(SMS^*)T. \)

Note also that for all \( N \in M_{2k}(\mathbb{C}), \)
\[
\Theta_t(N) = S \left( \vartheta(I + t\vartheta)^{-1}(S^*NS) \right)S, \quad \Upsilon_t(N) = T \left( \vartheta(I + t\vartheta)^{-1}(S^*NS) \right)T^*.
\]

It follows from (11) and (12) that \( \vartheta \) is \(|\vartheta|\)-positive if and only if \( \Theta_t \) and \( \Upsilon_t \) are completely positive for all \( t \geq 0 \).

We may easily argue as in the proof of Lemma 3.4 to conclude that \( \Theta_t \) is completely positive for all \( t \geq 0 \) if and only if the maps \( \eta''_t : M_{2k}(\mathbb{C}) \to M_{2}(\mathbb{C}) \) below are completely positive for all \( t \geq 0 \):
\[
\eta''_t \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) = \begin{pmatrix}
\frac{1}{1+i\lambda}\rho(A) & \frac{\lambda}{1+i\lambda}\tau(B) \\
\frac{\lambda}{1+i\lambda}\tau^*(C) & \frac{1}{1+i\lambda}\rho(D) \end{pmatrix}.
\]

Recall that in the proof of Lemma 3.4 we showed that \( \tau \) is a corner from \( \rho \) to \( \rho \). Since \(||\rho|| = ||\tau|| = 1\), it follows from Lemma 3.2 that \( c\tau \) is a corner from \( \rho \) to \( \rho \) if and only if \(|c| \leq 1\). Since
\[
(1 + t)\eta''_t \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) = \begin{pmatrix}
\rho(A) & \frac{\lambda}{1+i\lambda}\tau^*(C) \\
\frac{\lambda(1+t)}{1+i\lambda}\tau^*(C) & \rho(D) \end{pmatrix},
\]
we see that \( \eta''_t \) is completely positive for all \( t \geq 0 \) if and only if
\[
\left| \frac{\lambda(1 + t)}{1 + t\lambda} \right| \leq 1 \quad \text{(where we already know \( \lambda \leq 0 \) and \(|\lambda| \leq 1\)}
\]
for all \( t \geq 0 \). Squaring both sides of the above equation and then cross multiplying gives us
\[
|\lambda|^2(1 + 2t + t^2) \leq 1 + 2t \text{Re}(\lambda) + |\lambda|^2 t^2, \quad (\lambda \leq 0, \ |\lambda| \leq 1)
\]
which is equivalent to
\[
|\lambda|^2 \leq \frac{1 + 2t \text{Re}(\lambda)}{1 + 2t} \quad (\lambda \leq 0, \ |\lambda| \leq 1)
\]
for all nonnegative \( t \). Note that if \(|\lambda|^2 \leq \text{Re}(\lambda)\), then \( \text{Re}(\lambda) \leq 1 \) and equation (13) holds for \( t \geq 0 \). On the other hand, suppose that \( \lambda \) is any complex number that satisfies (13) for all \( t \geq 0 \). We conclude immediately that \( \text{Re}(\lambda) > 0 \), whereby the fact that \(|\lambda| \leq 1 \) implies \( \text{Re}(\lambda) \in (0, 1] \). A computation shows that the net \( \{ \frac{1 + 2t\text{Re}(\lambda)}{1 + 2t} \}_{t \geq 0} \) is monotonically decreasing and converges to \( \text{Re}(\lambda) \), hence \(|\lambda|^2 \leq \text{Re}(\lambda)\) by (13). We
have now shown that $\eta''_t$ (and thus $\Theta_t$) is completely positive for all $t \geq 0$ if and only if $|\lambda|^2 \leq \text{Re}(\lambda)$. Therefore, if $|\lambda|^2 > \text{Re}(\lambda)$ then (12) implies that $\vartheta$ is not $q$-positive, which is to say that $\gamma$ is not a $q$-corner from $\phi$ to $\psi$.

Suppose that $|\lambda|^2 \leq \text{Re}(\lambda)$. Then from above, the maps $\{\Theta_t\}_{t \geq 0}$ are all completely positive. Let

$$G = \begin{pmatrix} E & 0_{n-k,n'-k} \\ 0_{n'-k,n'-k} & I_{n'-k} \end{pmatrix} \in M_{n+n'-2k,2n'-2k}(\mathbb{C}).$$

We observe that

$$(1 + t) L_t \begin{pmatrix} A & B \\ C & D \end{pmatrix} = G \begin{pmatrix} \rho(A)I_{n'-k} & \frac{\lambda(1+t)}{1+\tau^2} \tau(C)I_{n'-k} \\ \frac{1+\tau^2}{1+t^2} \tau(B)I_{n'-k} & \rho(D)I_{n'-k} \end{pmatrix} G^*, $$

where we have already shown that the map in the middle is completely positive since $|\lambda|^2 \leq \text{Re}(\lambda)$. Thus, $L_t$ is completely positive for every $t \geq 0$. Also, $\Upsilon_t - L_t$ has the form

$$ (\Upsilon_t - L_t) \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \rho(A)(I_{n-k} - EE^*) & 0_{n-k,n'-k} \\ 0_{n'-k,n-k} & 0_{n'-k,n'-k} \end{pmatrix}, $$

where the right hand side is completely positive since $||E|| \leq 1$. Therefore, the maps $\{\Upsilon_t\}_{t \geq 0}$ are all completely positive, so (11) implies that $\vartheta(I + t\vartheta')^{-1}$ is completely positive for all $t \geq 0$, hence $\gamma$ is a $q$-corner from $\phi$ to $\psi$. \hfill $\Box$

**Theorem 3.7.** Assume the notation of the previous theorem, and suppose that $k = k'$ and $\mu_i = \tau_i$ for all $i = 1, \ldots, k$. A $q$-corner $\gamma : M_{n,n'}(\mathbb{C}) \to M_{n,n'}(\mathbb{C})$ from $\phi$ to $\psi$ is hyper maximal if and only if $n = n'$, $0 < |\lambda|^2 = \text{Re}(\lambda)$, and $E$ is unitary.

**Proof.** We first show that $\gamma$ is not hyper maximal if $n \neq n'$, regardless of the assumptions for $\lambda$ or $E$. If $n > n'$, then $EE^* \in M_{n-k}(\mathbb{C})$ is a positive contraction of rank at most $n' - k$, so $EE^* \neq I_{n-k}$.

Define $\phi' : M_n(\mathbb{C}) \to M_0(\mathbb{C})$ by

$$\phi'(R) = \phi(R) \begin{pmatrix} I_{k,k} & 0_{k,n-k} \\ 0_{n-k,k} & EE^* \end{pmatrix},$$

observing that $\phi'(I + t\phi')^{-1} = (1/(1 + t))\phi'$ for all $\geq 0$. Define $\vartheta' : M_{n+n'}(\mathbb{C}) \to M_{n+n'}(\mathbb{C})$ by

$$\vartheta'(\gamma) = \begin{pmatrix} \phi' \\ \gamma^* \\ \psi \end{pmatrix},$$

noting that $\vartheta'$ has no negative eigenvalues. Writing each $M \in M_{n+n'}(\mathbb{C})$ in the form (19), we see

$$\vartheta'(I + t\vartheta')^{-1}(M) = \begin{pmatrix} \frac{1}{1+t^2}\rho(A)I_{k,k} & 0_{k,n-k} & 0_{k,n-k} \\ 0_{n-k,k} & \frac{1}{1+t^2}\rho(A)EE^* & \frac{\lambda}{1+\tau^2} \tau(B)X \\ 0_{n-k,k} & \frac{1}{1+t^2} \tau(C)X^* & \frac{1}{1-t^2} \tau(D)I_{k,k} \end{pmatrix} \begin{pmatrix} \frac{1}{1+t^2} \tau(B) \rho(D) \frac{1}{1+t^2} \rho(D)I_{n'-k,n'-k} \end{pmatrix} $$

$$= S^* \Theta_t(SMS^*)S + T^* L_t(SMS^*)T,$$

for every $t \geq 0$, hence $\vartheta'$ is $q$-positive. By (11) and (14), we have

$$\left(\vartheta(I + t\vartheta')^{-1} - \vartheta'(I + t\vartheta')^{-1}\right)(M) = T\left(\left(\Upsilon_t - L_t\right)(S^*MS)\right)T^*.$$
Since $\Upsilon_t - L_t$ is completely positive for all $t \geq 0$ (as shown in the previous proof), the above equation implies that $\vartheta \geq \vartheta'$. However, $\vartheta \neq \vartheta'$ since $EE^* \leq I_{n-k}$, so $\gamma$ is not hyper maximal.

If $n < n'$, then since $E^*E \leq I_{n' - k}$, we may replace $\{L_t\}_{t=0}^{\infty}$ with the maps $\{R_t\}_{t=0}^{\infty}$ below and argue analogously (this time cutting down $\psi$ using $E^*E$) to show that $\gamma$ is not hyper maximal:

$$R_t \begin{pmatrix} A_{k,k} & B_{k,k} \\ C_{k,k} & D_{k,k} \end{pmatrix} = \left( \begin{array}{cc} \frac{1}{1+t\alpha} \rho(A)I_{n-k} & \frac{\lambda}{1+t\alpha} \tau(C)E^* \\ \frac{1}{1+t\alpha} \tau(D)E^* & \frac{\lambda}{1+t\alpha} \end{array} \right).$$

Of course, if $n = n'$ but $E$ is not unitary, then $EE^* \leq I_{n-k}$, and the same argument given in the case that $n > n'$ shows that $\gamma$ is not hyper maximal.

Therefore, we may suppose for the remainder of the proof that $n = n'$ and $E$ is unitary. Note that $\gamma = \psi$ since $n = n'$. For some $a \in (0, 1]$, we have $|\lambda|^2 = a \Re(\lambda)$. We first show that $\gamma$ is not hyper maximal if $a \neq 1$. We claim that the map $\vartheta'' : M_{2n}(\mathbb{C}) \to M_{2n}(\mathbb{C})$ defined by

$$\vartheta'' \begin{pmatrix} A_{n,n} & B_{n,n} \\ C_{n,n} & D_{n,n} \end{pmatrix} = \left( \begin{array}{cc} a\phi(A_{n,n}) & \gamma(B_{n,n}) \\ \gamma^*(C_{n,n}) & a\phi(D_{n,n}) \end{array} \right)$$

satisfies $\vartheta'' \geq 0$. For each $t \geq 0$, let $\eta_{t}^{(a)} : M_{2k}(\mathbb{C}) \to M_{2}(\mathbb{C})$ be the map

$$\eta_{t}^{(a)} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \left( \begin{array}{cc} \frac{a}{1+t\alpha} \rho(A) & \lambda \frac{a}{1+t\alpha} \tau(D) \\ \frac{\lambda}{1+t\alpha} \tau^*(C) & \frac{a}{1+t\alpha} \end{array} \right).$$

It is routine to check that since $\tau$ is a corner from $\rho$ to $\rho$, the condition $|\lambda|^2 = a \Re(\lambda)$ implies that $\frac{\lambda}{1+t\alpha} \tau$ is a corner from $\frac{a}{1+t\alpha} \rho$ to $\frac{a}{1+t\alpha} \rho$ for every $t \geq 0$, so $\eta_{t}^{(a)}$ is completely positive for all $t \geq 0$. Defining $\Theta_{t}^{(a)}$ and $\Upsilon_{t}^{(a)}$ for each $t \geq 0$ by

$$\Theta_{t}^{(a)} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \left( \begin{array}{cc} \frac{a}{1+t\alpha} \rho(A)I_k & \frac{\lambda}{1+t\alpha} \tau^*(C)X^* \\ \frac{\lambda}{1+t\alpha} \tau(C)X^* & \frac{a}{1+t\alpha} \rho(D)I_k \end{array} \right)$$

and

$$\Upsilon_{t}^{(a)} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = G \left( \begin{array}{cc} \frac{a}{1+t\alpha} \rho(A)I_{n-k} & \frac{\lambda}{1+t\alpha} \tau(B)I_{n-k} \\ \frac{\lambda}{1+t\alpha} \tau^*(C)I_{n-k} & \frac{a}{1+t\alpha} \rho(D)I_{n-k} \end{array} \right) G^*$$

we observe that the maps $\{\Theta_{t}^{(a)}\}_{t \geq 0}$ and $\{\Upsilon_{t}^{(a)}\}_{t \geq 0}$ are all completely positive since $\eta_{t}^{(a)}$ is completely positive for all $t \geq 0$. Note that

$$(a\phi)(I + ta\phi)^{-1} = \frac{a}{1+at} \phi$$

for all $t \geq 0$, so for every $M \in M_{2n}(\mathbb{C})$, we have

$$\vartheta''(I + t\vartheta'')^{-1}(M) = S \left( \Theta_{t}^{(a)}(S^*MS) \right) S^* + T^* \left( \Upsilon_{t}^{(a)}(S^*MS) \right) T.$$

Therefore, $\vartheta'' \geq 0$, and trivially $\vartheta \geq \vartheta''$. If $a \neq 1$, then $\vartheta'' \neq \vartheta$, hence $\gamma$ is not hyper maximal. To finish the proof, it suffices to show that $\gamma$ is hyper maximal if $a = 1$ (of course, maintaining our assumption that $E$ is unitary).
Suppose $a = 1$, and let $\phi'$ be any $q$-subordinate of $\phi$ such that
\[
\chi := \begin{pmatrix} \phi' & \gamma \\ \gamma^* & \phi \end{pmatrix} \geq_q 0.
\]

If $L_{\phi'}(I) \neq I$, then $L_{\phi'}(I) = R \leq I$ for some positive $R \in M_n(\mathbb{C})$. Letting $Z$ be the unitary matrix
\[
Z = \begin{pmatrix} X & 0_k \cdot n-k \\ 0_{n-k,k} & E \end{pmatrix} \in M_{n-k}(\mathbb{C}),
\]
we observe that
\[
0 \leq L_X \begin{pmatrix} I & Z \\ Z^* & I \end{pmatrix} = \begin{pmatrix} R & Z \\ Z^* & I \end{pmatrix}.
\]

Since $R \leq I$, we have $(f, Rf) < 1$ for some unit vector $f \in \mathbb{C}^n$. A quick calculation shows that
\[
\langle \begin{pmatrix} f \\ -Z^*f \end{pmatrix}, \begin{pmatrix} R \\ Z^* \\ I \end{pmatrix} \begin{pmatrix} f \\ -Z^*f \end{pmatrix} \rangle = (f, Rf) - 1 < 0,
\]
contradicting (15).

Therefore, $L_{\phi'}(I) = I$. Since $\phi \geq_q \phi'$, it follows that $L_{\phi} - L_{\phi'}$ is completely positive, so
\[
||L_{\phi} - L_{\phi'}|| = ||L_{\phi}(I) - L_{\phi'}(I)|| = 0,
\]
hence $L_{\phi'}(A) = L_{\phi}(A) = \phi(A) = \ell(A)I$ for the state $\ell \in M_n(\mathbb{C})^*$ defined by $\ell(A) = \sum_{i=1}^k \mu_i a_{kk}$. But range$(\phi') = $ range$(L_{\phi'}) = \{cI : c \in \mathbb{C}\}$ and $\phi' = \phi' \circ L_{\phi'}$, so $\phi'(I) = rI$ for some $r \leq 1$ and
\[
\phi'(A) = \phi'((L_{\phi'}(A)) = \phi(\ell(A)I) = \ell(A)\phi'(I) = r\ell(A)I = r\phi(A)
\]
for all $A \in M_n(\mathbb{C})$.

We claim that $r = 1$. To prove this, we define $V_t : M_{2k}(\mathbb{C}) \to M_{2k}(\mathbb{C})$ for each $t \geq 0$ by
\[
V_t \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) = S \left( \chi(I + t\chi)^{-1} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right) S^* = \begin{pmatrix} r \frac{\overline{\tau(A)k}}{1 + \lambda t} & \frac{\overline{\tau(B)k}}{1 + \lambda t} \\ \frac{\overline{\tau(C)k}}{1 + \lambda t} & \frac{\overline{\tau(D)k}}{1 + \lambda t} \end{pmatrix}.
\]

Since $\chi \geq 0$, each $V_t$ is completely positive. Therefore,
\[
0 \leq \begin{pmatrix} X^* & 0 \\ 0 & I \end{pmatrix} \left[ V_t \left( \begin{array}{cc} I & X \\ X^* & I \end{array} \right) \right] \left( \begin{array}{cc} X & 0 \\ 0 & I \end{array} \right) = \begin{pmatrix} r \frac{\overline{\lambda}}{1 + \lambda t} & \frac{\overline{\lambda}}{1 + \lambda t} \\ \frac{\overline{\lambda}}{1 + \lambda t} & \frac{\overline{\lambda}}{1 + \lambda t} \end{pmatrix},
\]
hence
\[
\frac{r}{(1 + rt)(1 + t)} \geq \frac{|\lambda|^2}{1 + (t^2 + 2t)\mathrm{Re}(\lambda)} = \frac{\mathrm{Re}(\lambda)}{1 + t\mathrm{Re}(\lambda)}
\]
for all $t \geq 0$. This is equivalent to
\[
(16) \quad r \geq \frac{(1 + t)\mathrm{Re}(\lambda)}{1 + t\mathrm{Re}(\lambda)}
\]
for all $t \geq 0$. We take the limit as $t \to \infty$ in (16) and observe $r \geq 1$. Since $r \leq 1$ we have $r = 1$, so $\phi' = \phi$.  


We have shown that if 
\[
\left( \begin{array}{cc}
\phi & \gamma \\
\gamma^* & \phi
\end{array} \right) \geq_q \left( \begin{array}{cc}
\phi' & \gamma \\
\gamma^* & \phi'
\end{array} \right) \geq_q 0,
\]
then \( \phi = \phi' \). An analogous argument shows that if 
\[
\left( \begin{array}{cc}
\phi & \gamma \\
\gamma^* & \phi
\end{array} \right) \geq_q \left( \begin{array}{cc}
\phi' & \gamma \\
\gamma^* & \phi'
\end{array} \right) \geq_q 0,
\]
then \( \phi = \phi' \). Therefore, \( \gamma \) is hyper maximal.

We are now ready to prove the following:

**Theorem 3.8.** Let \( \phi : M_n(\mathbb{C}) \to M_n(\mathbb{C}) \) and \( \psi : M_{n'}(\mathbb{C}) \to M_{n'}(\mathbb{C}) \) be rank one unital \( q \)-positive maps, and let \( \nu \) be a type II Powers weight of the form 
\[ \nu(\sqrt{I - \Lambda(1)}B\sqrt{I - \Lambda(1)}) = (f, Bf). \]

The \( E_0 \)-semigroups induced by \( (\phi, \nu) \) and \( (\psi, \nu) \) are cocycle conjugate if and only if \( n = n' \) and \( \phi \) is conjugate to \( \psi \).

**Proof.** The backward direction follows trivially from Proposition 2.12. For the forward direction, suppose \( (\phi, \nu) \) and \( (\psi, \nu) \) induce cocycle conjugate \( E_0 \)-semigroups \( \alpha^d \) and \( \beta^d \). For some sets \( \{\mu_i\}_{i=1}^k \) and \( \{\nu_i\}_{i=1}^{k'} \) satisfying the conditions of Theorem 3.6 and some unitaries \( U \in M_n(\mathbb{C}) \) and \( V \in M_{n'}(\mathbb{C}) \), \( \phi_U \) and \( \psi_V \) have the form of (7). Let \( \alpha_U^d \) and \( \beta_V^d \) be the \( E_0 \)-semigroups induced by \( (\phi_U, \nu) \) and \( (\psi_V, \nu) \), respectively. Since \( \alpha_U^d \simeq \alpha^d \) and \( \beta_V^d \simeq \beta^d \simeq \alpha^d \), we have \( \alpha_U^d \simeq \beta_V^d \), so by Proposition 2.9 there is a hyper maximal \( q \)-corner from \( \phi_U \) to \( \psi_V \). Theorems 3.6 and 3.7 imply that \( n = n' \), \( k = k' \), and \( \mu_i = \nu_i \) for all \( i = 1, \ldots, k \). In other words, \( \phi_U = \psi_V \). Therefore, \( \phi = \psi(\phi_U^* \gamma) \), so \( \phi \) and \( \psi \) are conjugate. \( \square \)

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