We show that the BMN operators in $D = 4$ $N = 4$ super Yang Mills theory proposed as duals of stringy oscillators in a plane wave background have a natural quantum group construction in terms of the quantum deformation of the $SO(6)$ $R$ symmetry. We describe in detail how a $q$-deformed $U(2)$ subalgebra generates BMN operators, with $q \sim e^{2\pi i \lambda}$. The standard quantum co-product as well as generalized traces which use $q$-cyclic operators acting on tensor products of Higgs fields are the ingredients in this construction. They generate the oscillators with the correct (undeformed) permutation symmetries of Fock space oscillators. The quantum group can be viewed as a spectrum generating algebra, and suggests that correlators of BMN operators should have a geometrical meaning in terms of spaces with quantum group symmetry.
1. Introduction

Type IIB String theory on a plane wave background with RR flux has recently been
discovered to be solvable \[1,2\]. This background is a limit of the \(ADS_5 \times S^5\) spacetime,
and a gauge theory dual has been proposed by Berenstein, Maldacena, Nastase (BMN) \[3\]. The proposal builds on the ADS/CFT duality \[4\] between type IIB string theory on
\(ADS_5 \times S^5\) and \(N = 4\) super-Yang Mills theory in four dimensions. BMN identified the
operator \(tr(\Phi_1^L)\) on the gauge theory side as corresponding to the vacuum of the string
theory on the plane wave background. Here \(\Phi_1\) is a complex Higgs field obtained from
combining two hermitian Higgs fields chosen from the six appearing in the super-Yang
Mills. Modifications of this operator are obtained by inserting, with some phase factors,
other fields in the \(SU(4|2, 2)\) multiplet of the gauge theory. The \(SU(4) \sim SO(6)\) subalgebra
allows the insertion of other Higgs fields. The phase factors are of the form \(e^{2\pi ip}\)
where \(p\) is a momentum carried by the corresponding stringy oscillator and \(J = L - n\), \(n\) being the
number of impurities. Correlation functions of these operators in the gauge theory and the
comparison with string theory have been discussed in many papers, see for example \[3-27\].
Discussions of the symmetries of this ppwave background have appeared in \[28,29,31,32\].

In this paper we start by observing that the construction of the BMN operator involv-
ing a set of impurities all of the same momentum \(p = -1\) can be viewed as the result
of the standard action of a generator of \(SO_q(6)\) on the \(J + n\) fold tensor product of Higgs
fields, followed by a trace. The \(q\)-deformed action on tensor products follows from the
quantum co-product which is necessary if we want the quantum group relations to be pre-
served on the tensor product. In other words, choosing the linear combinations of impurity
insertions weighted by \(q\) factors is equivalent to choosing a set of states which, along with
the vacuum, form a representation of the quantum group. The \(q\)-deformation parameter
is \(q = e^{2\pi iJ}\). For concreteness we describe this in detail for the case where the impurity
is another complex Higgs say \(\Phi_2\). For these insertions, we only need a \(q\)-deformed \(U(2)\)
or \(U_q(U(2))\). Relevant facts about \(U_q(U(2))\) are recalled in section 3 and this simplest
quantum group construction of a BMN operator is described in section 5.1. This suggests
that we should view the quantum group \(U_q(U(2))\), and more generally \(SU_q(4|2, 2)\) , for
\(q = e^{2\pi iJ}\) as a spectrum generating algebra for BMN operators.

A superficial look at phases \(e^{2\pi ip}\) which enter the construction of BMN operators
corresponding to stringy oscillators with generic momenta would suggest that a quantum
group \(SU_q(4|2, 2)\) depending on a single parameter \(q = e^{2\pi iJ}\) would not have enough struc-
ture to give the general BMN operator. One of main points of this paper is to show
that generic momenta are nevertheless obtained for a single $q$ deformation parameter in $SU_q(4|2, 2)$.

Physical applications of quantum groups in two-dimensional CFT and three-dimensional Chern Simons theory [33,34,35,36,37] show that, in addition to the quantum co-product an interesting role is played by quantum traces. In the Mathematics literature quantum traces with different choices of Cartan elements have been shown to have interesting properties [38]. Further, non-commutative geometry of spaces with quantum group symmetry, requires the use generalized traces [39]. These observations suggest that we should look for a construction of BMN operators involving the quantum group $SU_q(4|2, 2)$ with fixed $q$, and hence fixed quantum co-product, but using constructions which can be viewed as generalized traces.

Our generalized traces are constructed by composing the action of the co-product, with some $q$-cyclic operators, denoted by $\tau$ and then taking a trace. The $q$-cyclic operators act on a tensor product of Higgs fields and produce a sum of tensor products of Higgs fields, where each successive term in the sum involves a cycling of the Higgs fields accompanied by an additional phase factor which depends on the weight of the Higgs being cycled under a choice of element in the Cartan of $SU_q(4|2, 2)$. These $\tau$ operators are defined carefully in section 4. Our concrete calculations are done for $U_q(U(2))$ but the main ideas generalize to the full superalgebra.

An important property of the quantum group construction is that it automatically produces BMN operators with the correct permutation symmetries. Since they are dual to string theory oscillators which commute with each other, they should have the corresponding permutation symmetries. These symmetries have been discussed with some care in [8,22]. Note that we are here focusing on a $U(2)$ subalgebra of $SU(4|2, 2)$ which produces bosonic oscillators. More generally there will be fermionic oscillators which will involve anti-commutation properties. We review these symmetric BMN operators in section 2, and some other details about their properties are described in Appendix A. Our main technical result is that the $U_q(U(2))$ quantum group construction, with a single value of $q$, using the standard co-product, along with a sequence of $q$-cyclic ($\tau$) operators followed by a trace, automatically reproduces all stringy states with a fixed number of string oscillators, obeying the correct permutation symmetry. The different $q$ we need for different numbers of impurities differ by factors of $1/J$, where $J$ is large in the BMN limit, so in effect all symmetric BMN operators involving a single impurity type are produced by the $U_q(U(2))$
quantum group construction, with a fixed $q$. The construction of states involving general momenta using the co-product and generalized traces is described in section 5.

In section 6, we outline how our construction of BMN operators can lead to formulae for correlators as traces of quantum group operators in tensor spaces, generalizing the work of [40,41] where traces of projectors of classical groups in tensor spaces were related to correlators of SYM. We outline how the spectrum-generating quantum algebra acts on the super-Yang Mills action, showing that its action can be given a well-defined meaning but that, as expected, the SYM action is not invariant. We discuss the geometrical meaning of our algebraic quantum group construction of BMN operators in terms of quantum spaces, by using similarities between the $\tau$ we have used and some analogous operators that appear in the cyclic cohomology of quantum groups.

2. Review of BMN operators

We begin by reviewing the BMN correspondence between large R-charge operators of the $\mathcal{N} = 4$ super-Yang-Mills (SYM) theory and string states in a pp-wave background. The SYM theory contains six real scalar fields $X^i$ where $i = 1, \ldots, 6$. To express the theory in $\mathcal{N} = 1$ notation, these are combined into three complex combinations $\Phi_j = X^j + iX^{j+3}$ where $j = 1, 2, 3$. The theory contains an $SU(4)$ R-symmetry (subgroup of the $SU(4|2, 2)$ superalgebra symmetry) under which the scalars transform in the six-dimensional representation. To construct the BMN operators, one selects a $U(1)$ subgroup of the R-symmetry group, or equivalently chooses one of the complex scalars as the “background” scalar. For example, selecting the $\Phi_1$ scalar, then the ground state of the string theory in a pp-wave background $|0\rangle$ corresponds to the operator

$$|0\rangle \leftrightarrow \mathcal{N}_j \, TR(\Phi_1^J)$$

where the R-charge $J$ of the operator corresponds to the light cone momentum $p^+$ on the string theory side and the factor $\mathcal{N}_j$ is a normalization factor which will not be important for our purposes here.

Excitations above the ground state arise on the SYM side by inserting the other scalars $\Phi_2$ and $\Phi_3$ and their complex conjugates (there are other possibilities as well, generated by the superalgebra, that are in fact necessary to match with the string theory states, but as we shall not be discussing these other cases we refer the reader to the literature for more details, see eg. [3]). Moreover these “impurities” are accompanied by phase factors whose
powers correspond to oscillator number on the string theory side. Appendix A is devoted to a careful exposition of these operators. For those not interested in the details however we shall simply record the final result here. The BMN operator with $\Phi_{\beta_i}$ insertions, with momenta $p_l$, for $1 \leq l \leq n$ is given by

$$O_{\beta_n,p_n;\beta_1,p_1,\beta_2,p_2,\ldots,\beta_{n-1},p_{n-1}}$$

$$= N'_n \sum_{0 \leq k_1 \leq k_2 \leq \cdots \leq k_{n-1} \leq J} \sum_{\tau \in S_{n-1}} q^{\sum_{l=1}^{n-1} k_l p_{\tau(l)}}$$

$$TR[\beta_n, k_1, \beta_{\tau(1)}, k_2 - k_1, \beta_{\tau(2)}, \ldots, \beta_{\tau(n-1)}, J - k_{n-1}]$$

(2.2)

where we have introduced the convenient notation

$$TR\left(\Phi_{k_1}^{\beta_1}, \Phi_{k_2}^{\beta_2}, \ldots, \Phi_{k_n}^{\beta_n}, \Phi_{k_{n+1}}^{\beta_{n+1}}\right) \equiv TR[k_1, k_2, \ldots, k_n, \beta_n, k_{n+1}]$$

(2.3)

where $\beta_i = 2, 3$. In words, traces of products of operators are denoted as above with commas corresponding to impurities and the number above the comma indicating the type of impurity. The integers between the commas indicate the power of the background field. We have given the BMN operator for $\Phi_2$ and $\Phi_3$ impurities, but the extension to the complex conjugate fields is trivial. The sum on $\tau$ is over the permutation group $S_{n-1}$. We will often be interested in the case where all the impurities are a complex $\Phi_2$. In this case, we can drop the $\beta$ labels and just write

$$TR\left(\Phi_{k_1}^{k_1}, \Phi_{k_2}^{k_2}, \Phi_{k_3}^{k_3}, \ldots, \Phi_{k_n}^{k_n}, \Phi_{k_{n+1}}^{k_{n+1}}\right) \equiv TR[k_1, k_2, \ldots, k_n, k_{n+1}]$$

(2.4)

where now the commas are assumed to always correspond to $\Phi_2$ impurities.

Via the BMN correspondence this operator corresponds on the string side to the state

$$O_{\beta_n,p_n;\beta_1,p_1,\beta_2,p_2,\ldots,\beta_{n-1},p_{n-1}} \leftrightarrow \prod_{l=1}^{n} \alpha_{p_l}^{\beta_l}|0\rangle.$$  

(2.5)

The momenta $p_l$, for $l = 1 \cdots n - 1$ appear explicitly in (2.2) and $p_n$ is fixed by the constraint $p_1 + p_2 + \cdots + p_n = 0$ as follows from reparametrization invariance of the string worldsheet. The treatment of $p_n$ in (2.2) appears to break the $S_n$ symmetry of the state in (2.5) but cyclicity together with the condition $q^J = 1$ implies that it does not. We would like to point out that while $p_1 + p_2 + \cdots + p_n = 0$ is necessary for the correspondence (2.3) to make sense, it is possible to generalize the operator $O_{\beta_n,p_n;\beta_1,p_1,\ldots,\beta_{n-1},p_{n-1}}$ to the case where $p_1 + p_2 + \cdots + p_n \neq 0$. The correspondence (2.5) is then modified so that
$O_{\beta_n;p_n;\beta_1,p_1;\ldots;\beta_{n-1},p_{n-1}}$ corresponds to a linear superposition of single string states. This is discussed in detail in Appendix A.

In the case of just $\Phi_2$ insertions, \((2.2)\) simplifies to

$$O_{p_n;p_1\ldots p_{n-1}} = \mathcal{N}_n'^n \sum_{0 \leq k_1 \leq k_2 \leq \ldots \leq k_{n-1} \leq J} \left( \sum_{\tau \in S_{n-1}} q^{\sum_{l=1}^{n-1} k_l p_{\tau(l)}} \right) TR[1, k_1, k_2 - k_1, \ldots, J - k_{n-1}] \quad (2.6)$$

In particular in the sum over permutations, $\tau$ only enters the $q$-factor and not the operator. This is the form of the operator that we will be comparing to later. To get some feel for these operators we give some examples. If all $p_l$’s for $1 \leq l \leq n-1$ are equal, then the sum over permutations reduces to just one term, i.e., the $q$-factor becomes $q^{p(k_1+\ldots+k_{n-1})}$ with $p$ denoting the common values of the $p_l$’s. The correspondence \((2.5)\) then becomes

$$O_{-(n-1)p:p\ldots p} \leftrightarrow \alpha_{-(n-1)p}^\dagger (\alpha_{p}^\dagger)^{n-1} |0\rangle. \quad (2.7)$$

A slightly more non-trivial case is to let $p_1 = \ldots = p_{n-2} = p \neq p_{n-1}$. The sum over permutations of the phase factor in \((2.6)\) then reduces to

$$q^{p(k_1+\ldots+k_{n-1})} \sum_{l=1}^{n-1} q^{(p_{n-1}-p)k_l}. \quad (2.8)$$

Moreover the correspondence \((2.5)\) reduces to

$$O_{-(n-2)p-p_{n-1};p\ldots p,p_{n-1}} \leftrightarrow \alpha_{-(n-2)p-p_{n-1}}^\dagger (\alpha_{p}^\dagger)^{n-2} \alpha_{p_{n-1}}^\dagger |0\rangle. \quad (2.9)$$

A more detailed discussion of the BMN operators appears in Appendix A.

3. Review of quantum group facts

We begin by reviewing some facts about quantum algebras, focusing on the quantum deformation of $SU(2)$, which is the non-trivially deformed part of $U_q(U(2))$. For more details see for example [12] [13]. The quantum algebra $U_q(SU(2))$ is generated by $H, X_+, X_-$ with relations:

$$[H, X_\pm] = \pm 2X_\pm$$

$$[X_+, X_-] = \frac{q^H - q^{-H}}{q - q^{-1}} \quad (3.1)$$

$$1$$
In the limit $q \to 1$, this approaches the classical algebra:

$$\begin{align*}
[H, X_{\pm}] &= \pm 2X_{\pm} \\
[X_{+}, X_{-}] &= H
\end{align*} \quad (3.2)$$

An important property which the quantum algebra shares with the classical algebra, is that if $X_{\pm}, H$ are represented as operators acting on $V_1$ and $V_2$ obeying the quantum relations, then $V_1 \otimes V_2$ is also a representation. This is only true, however if the quantum group generators are taken to act on the tensor product using the quantum co-product. The quantum co-product can be viewed as a map $\Delta$ from the algebra $U_q$ to $U_q \otimes U_q$

$$\begin{align*}
\Delta(H) &= H \otimes 1 + 1 \otimes H \\
\Delta(q^H) &= q^H \otimes q^H \\
\Delta(X_{\pm}) &= X_{\pm} \otimes q^H + q^{-H} \otimes X_{\pm}
\end{align*} \quad (3.3)$$

One can check for example that

$$[\Delta(H), \Delta(X_{\pm})] = \pm 2\Delta(X_{\pm}) \quad (3.4)$$

Equivalently we may write

$$\begin{align*}
q^H X_{\pm} q^{-H} &= q^{\pm 2} X_{\pm} \\
\Delta(q^H) X_{\pm} \Delta(q^{-H}) &= q^{\pm 2} \Delta(X_{\pm})
\end{align*} \quad (3.5)$$

In the limit $q \to 1$ the quantum co-product leads to the ordinary action of the algebra on the tensor products.

An important point worth noting is that, given the normalizations used in (3.2) the eigenvalues of $H$ in the fundamental representation are 1 and $-1$. On the state with $H = 1$, we have $\frac{q^H - q^{-H}}{q - q^{-1}} = 1 = H$. On the state with $H = -1$, we have $\frac{q^H - q^{-H}}{q - q^{-1}} = -1 = H$. This means that the matrices representing $U_q SU(2)$ in the fundamental representation are the same as the ones representing the classical $SU(2)$. This fact is quite general, see for example the case of $SO_q(2n)$, which includes the $SO_q(6)$ of interest here, in [44]. Finite dimensional representations can be constructed from the tensor products of the fundamental one. The matrices in these tensor products differ because of the different co-products. In a sense, as far as finite dimensional representations are concerned, the essence of the quantum deformation is in the quantum co-product. The close relation between finite dimensional representations of the quantum group and the classical group is discussed in the physics literature in [45].
3.1. Quantum co-product

Using the co-product (3.3) we can consider the action on \( V \otimes V \) which we denote \( \Delta_2 \).

\[
\Delta_2(X_+) = X_+ \otimes q^{\frac{H}{2}} + q^{-\frac{H}{2}} \otimes X_+ \tag{3.6}
\]

Now consider the action of \( X_+ \) on a tensor product of three vector spaces. We can think of \( (V \otimes V \otimes V) \) as \( (V \otimes V) \otimes V \).

\[
\Delta_3(X_+) = \Delta_2(X_+) \otimes q^{\frac{H}{2}} + \Delta_2(q^{-\frac{H}{2}}) \otimes X_+ \\
= X_+ \otimes q^{\frac{H}{2}} \otimes q^{\frac{H}{2}} + q^{-\frac{H}{2}} \otimes X_+ \otimes q^{\frac{H}{2}} + q^{-\frac{H}{2}} \otimes q^{-\frac{H}{2}} \otimes X_+ \tag{3.7}
\]

By an induction argument, we can show that \( \Delta_n(X_+) \) is a sum where \( X_+ \) acts successively on each of the \( n \) factors, while \( q^{-\frac{H}{2}} \) acts on the factors to the left and \( q^{\frac{H}{2}} \) acts on the factors to the right. For the future use, we will express this by denoting the action of any generator \( X \) on the \( k \)'th factor as \( \rho_k(X) \). We have the formula

\[
\Delta_n(X_{\pm}) = \sum_{k=1}^{n} \rho_k(X_{\pm}) \left( \sum_{l=1}^{k-1} \rho_l(q^{-\frac{H}{2}}) + \sum_{l=k+1}^{n} \rho_l(q^{\frac{H}{2}}) \right) \tag{3.8}
\]

If we are considering the action on states where \( H = 1 \) this leads to the weighting of the action of \( X_+ \) by a power of \( q \) which depends on where the \( X_+ \) is acting. Acting on \( V^{\otimes 3} \) for example, one gets the sequence of \( q \)-factors \( q, 1, q^{-1} \). Acting on \( V^{\otimes 4} \), we get weights \( (q^{\frac{3}{2}}, q^{\frac{1}{2}}, q^{-\frac{1}{2}}, q^{-\frac{3}{2}}) \). More generally, when \( \Delta_n \) is acting on a product of states where \( H = 1 \) we get

\[
\Delta_n(X_{\pm}) = \sum_{k=1}^{n} q^{\frac{n-k}{2}} q^{-k} \rho_k(X_{\pm}) \tag{3.9}
\]

4. Embedding of \( U(2) \) in \( SO(6) \) and the \( q \)-cyclic operations

We describe with the \( SO(6) \) algebra and and the relevant \( U(2) \) subgroup which will be deformed according to the formulae in section 3..

Take the standard action of \( SO(6) \) on \( x_1, \ldots x_6 \). Let us form combinations

\[
\begin{align*}
    z_1 &= x_1 + ix_4 \\
    z_2 &= x_2 + ix_5 \\
    z_3 &= x_3 + ix_6 \tag{4.1}
\end{align*}
\]
The Cartan subalgebra is spanned by

\[ H_1 = z_1 \frac{\partial}{\partial z_1} - \bar{z}_1 \frac{\partial}{\partial \bar{z}_1} \]
\[ H_2 = z_2 \frac{\partial}{\partial z_2} - \bar{z}_2 \frac{\partial}{\partial \bar{z}_2} \]
\[ H_3 = z_3 \frac{\partial}{\partial z_3} - \bar{z}_3 \frac{\partial}{\partial \bar{z}_3} \]  

(4.2)

Additional generators of the \(SO(6)\) Lie algebra are, for \(i \neq j\) running from 1 to 3:

\[ E_{ij} = z_i \frac{\partial}{\partial z_j} - \bar{z}_j \frac{\partial}{\partial \bar{z}_i} \]  

(4.3)

We also take, for \(i < j\),

\[ P_{ij} = z_i \frac{\partial}{\partial \bar{z}_j} - z_j \frac{\partial}{\partial z_i} \]  

(4.4)

\[ Q_{ij} = -\bar{z}_i \frac{\partial}{\partial z_j} + \bar{z}_j \frac{\partial}{\partial \bar{z}_i} \]

It is easy to check that the above operators preserve \(z_1 \bar{z}_1 + z_2 \bar{z}_2 + z_3 \bar{z}_3\). and that

\[ E_{21}(z_1) = z_2 \]
\[ E_{21}(z_2) = 0 \]  

(4.5)

In the context of maximally supersymmetric SYM with \(U(N)\) gauge group, these generators act on complex fields \(\Phi_1, \Phi_2, \Phi_3\), which are matrices of size \(N\), instead of complex variables \(z_1, z_2, z_3\). For the dual of string theory on plane waves [3] we are interested in operators which are close to \(tr(\Phi_1^J)\) for some large \(J \sim \sqrt{N}\). The action of \(SO(6) \sim SU(4)\) can be used to generate insertions of other scalar operators. Note that if we use the full \(SO(6)\) algebra we would generate insertions of \(\Phi_2, \Phi_3, \Phi_2^+, \Phi_3^+\) as well as \(\Phi_1^+\). Operators which include insertions of \(\Phi_1^+\) are actually not of interest in the BMN limit, because strong coupling effects give them infinite dimensions. If we work with \(U(3)\) generated by \(H_1, H_2, H_3, E_{21}, E_{12}, E_{32}, E_{23}\) we can get insertions of the holomorphic \(\Phi_2, \Phi_3\) but not the \(\Phi_2^+, \Phi_3^+\) nor the \(\Phi_1^+\). The supersymmetric version of this will be a superalgebra \(SU(3|2,1)\). In this paper we will focus on a \(U(2)\) subgroup of this \(SU(3)\) and describe in detail the connection between the \(q\)-deformed \(U(2)\) and the BMN operators involving insertions of \(\Phi_2\).

The \(SU(2)\) subgroup of interest is generated by \(E_{12} = X_+, X_- = E_{21}\) and \(H = H_1 - H_2\) which obey the relations (3.2). The extra \(U(1)\) which gives \(U(2)\) is generated by \(H_1 + H_2\). The quantum group relations are (3.1). The co-products of the diagonal generators \(H_1, H_2\) are unchanged.
4.1. $q$-cyclic operations

To construct BMN operators in the next section we will require the use of a generalized trace which we define in this section. To construct this operator consider the algebra $\mathcal{A}$ of Higgs fields (for simplicity just $\Phi_1$ and $\Phi_2$ for the purposes of this paper) and its tensor products $\mathcal{A}^\otimes L$ acted on by the quantum group described in the previous section. We define $\tau(a,b)$ as a map from $\mathcal{A}^\otimes L$ to $\mathcal{A}^\otimes L$ by

$$\tau(a,b)(\Phi_{\beta_1} \otimes \Phi_{\beta_2} \otimes \cdots \otimes \Phi_{\beta_L}) = \sum_{i=1}^{L} \Phi_{\beta_{i+1}} \otimes \cdots \otimes \Phi_{\beta_L} \otimes \Delta_i(q^{aH_1+bH_2})(\Phi_{\beta_1} \otimes \cdots \otimes \Phi_{\beta_i}), \quad (4.6)$$

i.e., it is a sum of cyclic permutations of $\Phi_{\beta_1} \otimes \cdots \otimes \Phi_{\beta_L}$ weighted by $q$-dependent factors. The factor is easy to determine because the $\Phi_k$’s are eigenstates of the $H_i$’s with eigenvalues $\delta_{k,i}$. Therefore the weighting factor is simply $q^{an_1+bn_2}$ where $n_1$ and $n_2$ are the number of $\Phi_1$ and $\Phi_2$ fields cycled respectively. As an example consider the operator $\Phi_1 \otimes \Phi_2 \otimes \Phi_2$. Applying $\tau(a,b)$ we find

$$\tau(a,b)(\Phi_1 \otimes \Phi_2 \otimes \Phi_2) = q^a\Phi_2 \otimes \Phi_2 \otimes \Phi_1 + q^{a+b}\Phi_2 \otimes \Phi_1 \otimes \Phi_2 + q^{a+2b}\Phi_1 \otimes \Phi_2 \otimes \Phi_2$$

Note that we have the relations

$$\tau(a,b)(\Phi_1 \otimes \Phi_2 \otimes \Phi_2) = q^a\tau(a,b)(\Phi_2 \otimes \Phi_2 \otimes \Phi_1) = q^{a+b}\tau(a,b)(\Phi_2 \otimes \Phi_1 \otimes \Phi_2) = q^{a+2b}\tau(a,b)(\Phi_1 \otimes \Phi_2 \otimes \Phi_2) \quad (4.7)$$

as follows from the definition of $\tau(a,b)$. In other words, each time we cycle a Higgs field, we pick up a phase which is determined by the charge of the Higgs field under $q^{aH_1+bH_2}$. For the first and last lines of (4.7) to be consistent, we require that $q^{a+2b} = 1$. More generally, given an element of the tensor product algebra with $J \Phi_1$’s and $n \Phi_2$’s, we must demand that $q^{Ja+nb} = 1$. Moreover if we demand that $q^J = 1$, as will be done in this paper, then we must further require that $b = 0(\text{mod} J)$. If $a = b = 0$ we have the standard cyclicity of traces, except that it is here expressed as a property of a map from $\mathcal{A}^\otimes L$ to $\mathcal{A}^\otimes L$.

In the next section it will be more convenient to rewrite the $q$-cyclic operator defined above (4.6) as

$$\tau_{a,b} = \sum_{k=1}^{L} c^k q^{\sum_{i=1}^{k} \rho_i(aH_1+bH_2)} \quad (4.8)$$

The operator $c$ cycles one Higgs field through the left. The operator $c^k$ has the effect of performing a $k$-step cycling operation. The sum over $l$ at fixed $k$ is an instruction to pick up a factor of $q^{aH_1+bH_2}$ for each Higgs field cycled.
5. The construction of the BMN operators

5.1. With coproduct and trace

We will be interested in the action of $\Delta_q(E_{21}^n)$ on $\Phi_1^\otimes L$ which will lead to BMN operators with $J \Phi_1$ operators (where $J = L - n$) and $n$ copies of the $\Phi_2$ operator. The simplest way to get a class of BMN operators from this action of the quantum group is to multiply the $\Phi$’s in the tensor product and then take a trace of the resulting matrix. We will denote the result of this combined multiplication and tracing operation applied to $\Delta_q(E_{21}^n)\Phi_1^\otimes L$ as $\text{TR}(\Delta_q(E_{21}^n)\Phi_1^\otimes L)$. More general operators can be obtained by considering generalized traces such as the ones defined by combining the ordinary trace with the $q$-cyclic operators described in section 4. Before analysing these more general cases, we shall first consider just the ordinary trace.

To begin we derive a convenient expression for the operator $\Delta_q(E_{21}^n)$ using its definition given in (3.8). Substituting (3.8) into $\Delta_q(E_{21}^n)$ we obtain

$$
\Delta_q(E_{21}^n) = \sum_{i_1=1}^{L} \rho_{i_1}(E_{21})Q\left( \sum_{j_1=1}^{i_1-1} \rho_{j_1}(-\frac{H_1 - H_2}{2}) + \sum_{j_1=i_1+1}^{L} \rho_{j_1}(\frac{H_1 - H_2}{2}) \right) + \sum_{i_2=1}^{L} \rho_{i_2}(E_{21})Q\left( \sum_{j_2=1}^{i_2-1} \rho_{j_2}(-\frac{H_1 - H_2}{2}) + \sum_{j_2=i_2+1}^{L} \rho_{j_2}(\frac{H_1 - H_2}{2}) \right)
$$

(5.1)

where for clarity we have used a definition $Q(\rho_j(H)) \equiv q^{\rho_j(H)}$. It is convenient to bring all the $H$ factors to the right. In so doing we have to compute some commutators. The commutator terms coming from the $j_1$ sum are non-trivial when the $j_1$ is equal to $i_2$ or $i_3$ up to $i_n$. The commutator terms from $j_2$ are non-trivial when $j_2$ is equal to $i_3, i_4 \cdots i_n$. Let us focus on the terms we get when $j_1$ is equal to $i_2$. Using $q^{\frac{H_1 - H_2}{2}} E_{21} = q^{-1} E_{21} q^{\frac{H_1 - H_2}{2}}$ we find that these commutator terms are

$$
Q \left( \sum_{j_1=1}^{i_1-1} \delta(j_1, i_2) - \sum_{j_1=i_1+1}^{L} \delta(j_1, i_2) \right)
$$

(5.2)
If we define $\theta_+(x) = 1$ for integers $x \geq 1$ and zero otherwise, then we can write the above as

$$Q \left( \theta_+(i_1 - i_2) - \theta_+(i_2 - i_1) \right) = Q \left( \theta(i_1 - i_2) \right)$$  \hspace{1cm} (5.3)

We have also defined $\theta(x) \equiv \theta_+(x) - \theta_+(-x)$.

Now we can write

$$\Delta_q(E_{21}^n) = \sum_{i_1, i_2, \ldots, i_n} \rho_{i_1} (E_{21}) \cdots \rho_{i_n} (E_{21})$$

$$Q \left( \sum_{j_1 = 1}^{i_1 - 1} \rho_{j_1} \left( -\frac{H_1 - H_2}{2} \right) + \sum_{j_1 = i_1 + 1}^{L} \rho_{j_1} \left( \frac{H_1 - H_2}{2} \right) \right)$$

$$\vdots$$

$$Q \left( \sum_{j_n = 1}^{i_n - 1} \rho_{j_n} \left( -\frac{H_1 - H_2}{2} \right) + \sum_{j_n = i_n + 1}^{L} \rho_{j_n} \left( \frac{H_1 - H_2}{2} \right) \right)$$

$$Q \left( \sum_{1 \leq k < l \leq n} \theta(i_k - i_l) \right)$$  \hspace{1cm} (5.4)

When we act on $\Phi_1^L$ the $q^H$ factors which have been commuted through to the right are easily evaluated to give $q^{\frac{(L+1)n}{2}-(i_1+\cdots+i_n)}$. In the sums above we have a restriction $i_1 \neq i_2 \cdots \neq i_n$. This follows because $E_{21}^2 \Phi_1 = 0$. The sum includes all possible orderings of the $i_1 \cdots i_n$ which can be described using permutations $\sigma$ in $S_n$.

$$\sum_{i_1 \neq i_2 \cdots \neq i_n} = \sum_{\sigma} \sum_{i_{\sigma(1)} < i_{\sigma(2)} \cdots < i_{\sigma(n)}}$$  \hspace{1cm} (5.5)
From (5.4) we can write (5.4) as

\[
\Delta_q(E_{21}^n) \Phi_1^{\otimes L} = \sum_{\sigma} \sum_{i_{\sigma(1)} < i_{\sigma(2)} < \cdots < i_{\sigma(n)}} q^{(L+1)n \over 2} -(i_1 + i_2 + \cdots + i_n) \\
\rho_{i_1}(E_{21}) \cdots \rho_{i_n}(E_{21}) \Phi_1^{\otimes L} \\
Q \left( \sum_{1 \leq k < l \leq n} \theta(\sigma^{-1}(k) - \sigma^{-1}(l)) \right) \\
= \sum_{\sigma} Q \left( \sum_{1 \leq k < l \leq n} \theta(\sigma^{-1}(k) - \sigma^{-1}(l)) \right) \sum_{i_1 < i_2 < \cdots < i_n} q^{(L+1)n \over 2} -(i_1 + i_2 + \cdots + i_n) \\
\rho_{i_1}(E_{21}) \cdots \rho_{i_n}(E_{21}) \Phi_1^{\otimes L} \\
= [n]! \sum_{1 \leq i_1 < i_2 < \cdots < i_n \leq L} q^{(L+1)n \over 2} -(i_1 + i_2 + \cdots + i_n) \rho_{i_1}(E_{21}) \cdots \rho_{i_n}(E_{21}) \Phi_1^{\otimes L} \\
= [n]! \sum_{1 \leq i_1 < i_2 < \cdots < i_n \leq L} q^{(L+1)n \over 2} -(i_1 + i_2 + \cdots + i_n) \rho_{i_1}(E_{21}) \cdots \rho_{i_n}(E_{21}) \Phi_1^{\otimes L} \\
(5.6)
\]

In the second equality we have recognized that the phase factor coming from the commutations only depends on the ordering on the \(i\)'s and hence only on the permutations \(\sigma\), so they can be factored out of the \(i\) sum. In the next equality, we use the symmetry of the summand of the sum over \(i\) in order to replace \(i\) with \(i_{\sigma}\). In the next line we renamed the summation variable. Finally the sum over \(\sigma\) was evaluated to give a constant \(q\)-factorial \([n]_q!\) which is defined as \([n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q\) where \([k]_q = \frac{q^k - q^{-k}}{q - q^{-1}}\). Note that it is invariant under \(q \rightarrow q^{-1}\) which is as it should be since, in the sum, changing \(q\) to \(q^{-1}\) is equivalent to exchanging \(\sigma\) with \(\sigma^{-1}\).

With this form of the \(\Delta_q(E_{21}^n) \Phi_1^{\otimes L}\) we have a sequence of operators in tensor space. We multiply them and take a trace. Denoting the combined operation as \(TR\) we find

\[
TR \left( \Delta_q(E_{21}^n) \Phi_1^{\otimes L} \right) \\
= q^{(L+1)n \over 2} [n]_q! \sum_{1 \leq i_1 < i_2 < \cdots < i_n \leq L} q^{-(i_1 + i_2 + \cdots + i_n)} \rho_{i_1}(E_{21}) \cdots \rho_{i_n}(E_{21}) \Phi_1^{\otimes L} \\
= q^{(L+1)n \over 2} [n]_q! \sum_{1 \leq j_1 < j_2 < \cdots < j_n \leq J} q^{-(j_1 + j_2 + \cdots + j_n)} TR[j_1, j_2 - j_1, j_3 - j_2, \cdots, j_n - j_{n-1}, J - j_n] \\
(5.7)
\]

Here we have defined \(j_l = i_l - l\) for \(l = 1, \cdots, n\) and have used the notation in (2.4) in the last line. The upper limit of \(j_n\) is now \(J = L - n\). We now introduce variables \(k_1 = j_2 - j_1\)
and \( k_l - k_{l-1} = j_{l+1} - j_l \) for \( 2 \leq l \leq n - 1 \). We can rewrite the previous formula as

\[
q^{Jn}[n]q! \sum_{1 \leq k_1 \leq k_2 \cdots \leq k_{n-1} \leq J} \sum_{j_1=0}^{J-k_{n-1}} q^{-nj_1} q^{-(k_1 + k_2 + \cdots + k_{n-1})} \]  

(5.8)

where the upper limit on the \( j_1 \) sum is easily fixed by requiring that the sums in (5.8) and (5.7) have the same number of terms. After doing the sum over \( j_1 \) we get

\[
TR(\Delta_q(E_{21}^n) \Phi_1^{\otimes L}) = q^{Jn}[n]q! \sum_{1 \leq k_1 \leq k_2 \cdots \leq k_{n-1} \leq J} \frac{(1-q^{-n(1-k_{n-1})})}{(1-q^{-n})} q^{-(k_1 + k_2 + \cdots + k_{n-1})} \]  

(5.9)

The term involving \( q^{-n(1-k_{n-1})} \) does not look symmetric but actually is symmetric after we use cyclicity. We show this in more detail for the two \( \tau \) case later. The BMN operator corresponding to \( TR(\Delta_q(E_{21}^n) \Phi_1^{\otimes L}) \) is therefore \((-1)^n[n]q! \alpha_{n-1}^\dagger \alpha_{n-1}^\dagger (\Phi_{\dagger}^{\otimes L})^{n-1} |0\rangle\), where the denominator \((1-q^{-n})\) has cancelled after we combined contributions from the two terms in the numerator, and we used \( q^{Jn} = (-1)^n \).

5.2. The case of a single \( \tau \)

We now move on to consider the insertion of a single \( \tau \) operator of the type defined in section 4, i.e., we want to compute \( TR_{\tau_{a,0}} \Delta_q(E_{21}^n) \) The first step is to consider the operator \( \tau_{a,0} \Delta_q(E_{21}^n) \), which is an element of \( A^{\otimes L} \) where \( A \) is the algebra of Higgs fields. To evaluate the trace we apply the multiplication map to get an element of \( A \) from the element in \( A^{\otimes L} \). Then we take a trace of this element. As we saw in (4.8), the \( q \)-cyclic \( \tau \) operator can be written as a sum of cycling operations weighted by \( q \) factors which depend on the \( U_q(2) \) quantum numbers of the elements cycled. We are now composing \( \tau_{a,0} \) from the left with \( \Delta_q(E_{21}^n) \). It is useful to keep the cycling operators on the left but to commute the \( H \)-factors to the right. Since we are calculating a trace at the end, cycling operations on the left can be set to 1. On the other hand since we are acting on \( \Phi_{\dagger}^{\otimes L} \) on the right the
$H$ factors are easy to evaluate. With this in mind and using (5.6) and (4.8) we expand:

$$\tau_{a,0} \Delta_q (E_{21}^n) \Phi_1^{\otimes L}$$

$$= [n]_q! q^{(L+1)n} \sum_{k=1}^{L} c^k Q \left( \sum_{l=1}^{k} \rho_l \left( aH_1 \right) \right)$$

$$\sum_{1 \leq i_1 < i_2 \ldots < i_n \leq L} q^{-(i_1+i_2+\cdots+i_n)} \rho_{i_1}(E_{21}) \cdots \rho_{i_n}(E_{21}) \Phi_1^{\otimes L}$$

$$= [n]_q! q^{(L+1)n} \sum_{k=1}^{L} c^k \sum_{1 \leq i_1 < i_2 \ldots < i_n \leq L} q^{-(i_1+i_2+\cdots+i_n)} \rho_{i_1}(E_{21}) \cdots \rho_{i_n}(E_{21})$$

$$Q \left( \sum_{l=1}^{k} \rho_l \left( aH_1 \right) - a \sum_{l=1}^{k} \left( \delta(l, i_1) + \cdots + \delta(l, i_n) \right) \right) \Phi_1^{\otimes L}$$

$$= [n]_q! q^{(L+1)n} \sum_{k=1}^{L} c^k \sum_{1 \leq i_1 < i_2 \ldots < i_n \leq L} q^{-(i_1+i_2+\cdots+i_n)} \rho_{i_1}(E_{21}) \cdots \rho_{i_n}(E_{21})$$

$$Q \left( ak - a \sum_{l=1}^{n} \theta_+(k - i_l + 1) \right) \Phi_1^{\otimes L}$$

Acting with the trace we get

$$TR \left( \tau_{a,0} \Delta_q (E_{21}^n) \Phi_1^{\otimes L} \right)$$

$$= q^{(L+1)n} [n]_q! \sum_{1 \leq i_1 < i_2 \ldots < i_n \leq L} q^{-(i_1+i_2+\cdots+i_n)} \left( TR \left( \rho_{i_1}(E_{21}) \cdots \rho_{i_n}(E_{21}) \Phi_1^{\otimes L} \right) \right)$$

$$\sum_{k=1}^{L} q^{ak} Q \left( -a \left( \theta_+(k - i_1 + 1) + \cdots + \theta_+(k - i_n + 1) \right) \right)$$

(5.11)

The sum over $k$ can be written out as

$$\sum_{k=1}^{L} q^{ak} Q \left( -a \left( \theta_+(k - i_1 + 1) + \cdots + \theta_+(k - i_n + 1) \right) \right)$$

$$\sum_{k=i_1}^{i_1-1} q^{ak} + q^{-a} \sum_{k=i_1}^{i_2-1} q^{ak} + q^{-2a} \sum_{k=i_2}^{i_3-1} q^{ak} + \cdots$$

$$+ q^{-(n-1)a} \sum_{k=i_{n-1}}^{i_n-1} q^{ak} + q^{-na} \sum_{k=i_n}^{L} q^{ak}$$

$$= q^a \frac{(1 - q^{a(L-n)})}{(1 - q^a)} + (q^{a(i_1-1)}) + q^{a(i_2-2)} + \cdots + q^{a(i_n-n)})$$

$$= q^a \frac{(1 - q^{a(L-n)})}{(1 - q^a)} + \sum_{l=1}^{n} q^{aj_l}$$

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In the last line we have used $j_i \equiv i_t - l$. When we are constructing BMN operators we use $q^{L-n} = q^J = 1$, which means that the constant term is zero.

We can now write a simpler expression for the result of acting on $\Phi_1^\otimes L$ with the quantum group generators, the q-cyclic operator and the trace:

$$TR (\tau_{a,0} \Delta_q(E_2^m \Phi_1^\otimes L))$$

$$= [n]_q! \frac{1}{q^{J/2}} \sum_{0 \leq j_1 \leq j_2 \cdots \leq j_n \leq J} (\sum_{l=1}^{n} q^{aj_l}) q^{-(j_1 + j_2 + \cdots + j_n)}$$

$$TR (\rho_{j_1+1}E_{21} \cdots \rho_{j_n+n}E_{21} \Phi_1^\otimes L)$$

$$= [n]_q! (-1)^n \sum_{0 \leq j_1 \leq j_2 \cdots \leq j_n \leq J} (\sum_{l=1}^{n} q^{aj_l}) q^{-(j_1 + j_2 + \cdots + j_n)}$$

$$TR \left[ j_1, j_2 - j_1, \cdots, j_n - j_{n-1}, J - j_n \right]$$

In the final line we have used the notation of (2.4). It is useful to define a new set of summation variables $(j_1, k_1, k_2, \cdots, k_{n-1})$ to replace the set $(j_1, j_2, \cdots, j_n)$. They are defined as follows

$$k_1 = j_2 - j_1$$
$$k_2 - k_1 = j_3 - j_2$$
$$\vdots$$
$$k_{n-1} - k_{n-2} = j_n - j_{n-1}$$

which imply

$$k_1 = j_2 - j_1$$
$$k_2 = j_3 - j_1$$
$$\vdots$$

$$k_{n-1} = j_n - j_1$$

Now the sum can be manipulated to

$$\sum_{j_1=0}^{J} \sum_{j_2=j_1}^{J} \sum_{j_3=j_2}^{J} \cdots \sum_{j_{n}=j_{n-1}}^{J}$$

$$= \sum_{k_1=0}^{J} \sum_{k_2=k_1}^{J} \cdots \sum_{k_{n-1}=k_{n-2}}^{J} \sum_{j_1=0}^{J-k_{n-1}}$$

(5.14)

(5.15)

(5.16)
After doing the sum over $j_1$ we are left with

$$TR \left( \tau_{a,0} \Delta_q(E_{21}^n) \Phi_1^{\otimes L} \right)$$

$$= \frac{[n]_q! q^{\underline{n+2}}}{(1 - q^{a-n})} \sum_{0 \leq k_1 \leq k_2 \ldots \leq k_{n-1} \leq J} (1 - q^{(a-n)(1-k_{n-1})})(1 + q^{ak_1} + q^{ak_2} + \ldots + q^{ak_{n-1}})$$

$$q^{-k_1-k_2-\ldots-k_{n-1}} TR[1, k_1 - k_1, \ldots, k_{n-1} - k_{n-2}, J - k_{n-1}]$$

(5.17)

where we have written the trace of the operator piece of the expression in the notation introduced in section 2. When we look at the term $(1 + q^{ak_1} + q^{ak_2} + \ldots + q^{ak_{n-1}})$ it is clearly symmetric under permutations of $k_1$ to $k_{n-1}$. It corresponds to the string state

$$\alpha_{n-1} \alpha_{n-2} + \alpha_{n-1+a} \alpha_{n-2}$$

by the BMN map. The term

$$\sum_{0 \leq k_1 \leq k_2 \ldots \leq k_{n-1} \leq J} q^{(a-n)q^{-(a-n)k_{n-1}}(1 + q^{ak_1} + q^{ak_2} + \ldots + q^{ak_{n-1}})} q^{-k_1-\ldots-k_{n-1}}$$

(5.18)

does not appear manifestly symmetric, but we can, by using cyclicity, write it as

$$\sum_{0 \leq k_1 \leq k_2 \ldots \leq k_{n-1} \leq J} \frac{q^{(a-n)}}{(n-1)} q^{-k_1-\ldots-k_{n-1}} (q^{-q^{-(a-n)k_{n-1}}} + q^{-q^{-(a-n)k_{n-2}}} + \ldots + q^{-q^{-(a-n)k_1}})$$

$$(1 + q^{ak_1} + \ldots + q^{ak_{n-1}}) TR[1, k_2 - k_1, \ldots, k_{n-1} - k_{n-2}, J - k_{n-1}]$$

(5.19)

In the next section we elaborate on the cycling manipulations in the two $\tau$ case. Now the result is obviously symmetric and the entire operator corresponds to a sum of BMN states of the form

$$\alpha_{n-1} \alpha_{n-2} + \alpha_{n-1+a} \alpha_{n-2} |0\rangle.$$

5.3. Two $\tau$-operators

We now consider the action of two $\tau$-operators on (5.6), using the form (4.6) for $\tau$. 
operators. Explicitly we have

\[ \tau_{a_2,0} \tau_{a_1,0} \Delta_q (E_{21}^n) \Phi_1^{\otimes L} \]

\[ = [n]_q! \sum_{k=2}^{L} \sum_{l_1=1}^{k} c^{k_2} Q \left( \sum_{l_2=1}^{k_2} \rho_{l_2} \left( a_2 H_1 \right) \right) \sum_{k_1=1}^{L} c^{k_1} Q \left( \sum_{l_1=1}^{k_1} \rho_{l_1} \left( a_1 H_1 \right) \right) \]

\[ \sum_{1 \leq i_1 < i_2 \cdots < i_n \leq L} q^{-(i_1+i_2+\cdots+i_n)} \rho_{i_1} (E_{21}) \cdots \rho_{i_n} (E_{21}) \Phi_1^{\otimes L} \]

\[ = [n]_q! \sum_{k=2}^{L} \sum_{l_1=1}^{k} c^{k_2+k_1} \sum_{l_2=1}^{k_2} q^{-(i_1+i_2+\cdots+i_n)} \rho_{i_1} (E_{21}) \cdots \rho_{i_n} (E_{21}) \]

\[ Q \left( \sum_{l_1=1}^{k_1} \rho_{l_1} \left( a_1 H_1 \right) \right) - \sum_{l_1=1}^{k_2} \left( \delta(l_1, i_1) + \cdots + \delta(l_1, i_n) \right) \]

\[ Q \left( \sum_{l_2=1}^{k_2} \rho_{r_L(l_2+k_1)} \left( a_2 H_1 \right) \right) - \sum_{l_2=1}^{k_2} \left( \delta(r_L(k_1+l_2), i_1) + \cdots + \delta(r_L(k_1+l_2), i_n) \right) \Phi_1^{\otimes L} \]

\[ = [n]_q! \sum_{k=2}^{L} \sum_{l_1=1}^{k} c^{k_2+k_1} \sum_{l_2=1}^{k_2} q^{-(i_1+i_2+\cdots+i_n)} \rho_{i_1} (E_{21}) \cdots \rho_{i_n} (E_{21}) \]

\[ Q \left( \sum_{l_1=1}^{k_1} \rho_{l_1} \left( a_1 H_1 \right) \right) - a_1 \mathcal{F}(k_1, 0; \vec{i}) \]

\[ Q \left( \sum_{l_2=1}^{k_2} \rho_{r_L(l_2+k_1)} \left( aH_1 \right) \right) - a_2 \mathcal{F}(k_2, k_1; \vec{i}) \Phi_1^{\otimes L} \]

\[ (5.20) \]

The manipulations are similar to those in (5.10). One new thing is that when the \( \rho_{l_2} (a_2 H_1) \)

is commuted past the \( c^{k_2} \) it becomes \( \rho_{l_2+k_1} (a_2 H_1) \) if \( 1 \leq l_2 + k_1 \leq L \)

but \( \rho_{l_2+k_1-L} (a_2 H_1) \) if \( l_2 + k_1 > L \). This has been conveniently written as \( \rho_{r_L(l_2+k_1)} (a_2 H_1) \)

where \( r_L(m) \) for an integer \( m \) is defined as one plus the residue modulo \( L \) of \( m \). There is a consequent change in the sum over delta’s. We have introduced functions \( \mathcal{F}(p, q; \vec{i}) \) which depend on two positive integers \( p, q \) and a fixed set of integers \( i_1, \ldots, i_n \) between 1 and L. It is defined by

\[ \mathcal{F}(p, q ; \vec{i}) = \sum_{l=1}^{n} \sum_{m=q+1}^{p} \delta \left( i_l, r_L(m) \right). \]

\[ (5.21) \]

The function \( \mathcal{F}(p, q ; \vec{i}) \) counts the number of \( i_l \)'s satisfying \( r_L(q+1) \leq i_l \leq r_L(q+p) \).

The sums of delta’s are naturally written in terms of these functions. The functions \( \mathcal{F} \)

satisfy a useful property

\[ \mathcal{F}(p+q, 0 ; \vec{i}) = \mathcal{F}(p, 0 ; \vec{i}) + \mathcal{F}(q, p ; \vec{i}) \]

\[ (5.22) \]
In (5.24) the $H$ factors on the right are easily evaluated to give $q^a_2 k_2 + a_1 k_1$, moreover, taking the trace allows the $e^{k_1 + k_2}$ to be set to one due to cyclicity.

\[
TR \left( \tau_{a_2,0} \tau_{a_1,0} \Delta_q (E_{21}^n) \right) \Phi_1^{\otimes L} \\
= q^{- \frac{(L+1)n}{2}} [n] q! \sum_{1 \leq i_1 < i_2 \cdots < i_n \leq L} q^{- (i_1 + i_2 + \cdots + i_n)} TR \left( \rho_{i_1}(E_{21}) \cdots \rho_{i_n}(E_{21}) \right) \Phi_1^{\otimes L}
\]

Performing the sum over $k_2$ we find

\[
\sum_{k_2=1}^L q^{a_2 k_2 - a_2 F(k_2, k_1 ; \vec{i})} = q^{- a_2} (k_1 - F(k_1, 0 ; \vec{i}) \right) S(a_2 ; \vec{i})
\]

where $S(a ; \vec{i})$ is defined to be the sum

\[
S(a ; \vec{i}) = \sum_{l=1}^n q^{a(i_l - l)}.
\]

While the summation index $k_2$ in (5.24) is constrained by $1 \leq k_2 \leq L$, the integer $k_1$ can be outside this range. The same basic sums will be used over and over again as we increase the number of $\tau$'s in the next section. They are evaluated by similar methods to those used in the previous subsection, taking advantage of $q^J = 1$.

The $k_1$ sum now follows as a special case of (5.24) and is given by

\[
\sum_{k_1=1}^L q^{a_1 - a_2} (k_1 - F(k_1, 0 ; \vec{i}) \right) = S(a_1 - a_2 ; \vec{i}).
\]

The result is therefore

\[
TR \left( \tau_{a_2,0} \tau_{a_1,0} \Delta_q (E_{21}^n) \right) \Phi_1^{\otimes L} \\
= q^{- \frac{(L+1)n}{2}} [n] q! \sum_{1 \leq i_1 < i_2 \cdots < i_n \leq L} q^{- (i_1 + i_2 + \cdots + i_n)} S(a_1 - a_2 ; \vec{i}) S(a_2 ; \vec{i})
\]

\[
TR[i_1 - 1, i_2 - i_1 - 1, \ldots, i_n - i_{n-1} - 1, L - i_n]
\]

\[
= q^{- \frac{Jn}{2}} [n] q! \sum_{1 \leq j_1 \leq j_2 \cdots \leq j_n \leq J} q^{- (j_1 + j_2 + \cdots + j_n)} S(a_1 - a_2 ; j_1 + 1, \ldots, j_n + n)
\]

\[
S(a_2 ; j_1 + 1, \ldots, j_n + n) \right) TR[j_1, j_2 - j_1, \ldots, j_n - j_{n-1}, J - j_n]
\]
In the last line we changed variables $j_l = i_l - l$. This resulting expression is not quite of the BMN form given in (2.6), however we can make the same basic manipulations described in the formulae (5.15) and (5.16) to reach such a form. First we use cyclicity of the trace to move the $j_1$ powers of $\Phi_1$ to the right, and then redefine summation indices as $k_1 = j_2 - j_1$ and $k_l - k_{l-1} = j_{l+1} - j_l$ for $2 \leq l \leq n - 1$. Equivalently we find $j_{l+1} = k_l + j_1$ for $1 \leq l \leq n - 1$. The operator (5.27) becomes

$$TR(\tau_{a_2,0\tau_{a_1,0}}\Delta_q(E_{21}^n)\Phi_1^\otimes L) = q^{Jn/2[n]}q! \sum_{0 \leq k_1 \leq k_2 \leq \cdots \leq k_{n-1} \leq J} \sum_{j_1=0}^{J-k_{n-1}} q^{(a_1-n)(j_1)} q^{-(k_1+k_2+\cdots+k_{n-1})}$$

$$T(a_1-a_2;\vec{k})T(a_2;\vec{k}) TR[k_1, k_2 - k_1, \ldots, k_{n-1} - k_{n-2}, J - k_{n-1}]$$

where we have introduced the function $T(a;\vec{k})$ defined as

$$T(a;\vec{k}) = 1 + \sum_{l=1}^{n-1} q^{a_{k_l}}$$

and which is related to $S$ by

$$S(a; j_1 + 1, \cdots, j_n + n) = q^{a_{j_1}}T(a;\vec{k})$$

given the change of variables above. The $j_1$ sum can now be done trivially and is proportional to $1 - q^{(a_1-n)(1-k_{n-1})}$. Up to the factor of $T(a_1-a_2;\vec{k})$, this is exactly what we found in the single $\tau$ case in the previous section in (5.17) if one identifies $a$ with $a_2$. Exactly as in that case, we find that the coefficient of the operator $TR[k_1, \ldots, J - k_{n-1}]$ is symmetric under permutations of the $k_l$’s except for the factor of $q^{(a_1-n)(1-k_{n-1})}$ which arises from the $i_1$ sum. This term nevertheless can be made symmetric by using cyclicity of the trace and redefining summation indices. To see this, cyclicity allows us to rewrite the operator as

$$TR[k_1, k_2 - k_1, \ldots, J - k_{n-1}] = TR[k_2 - k_1, k_3 - k_2, \ldots, J - k_{n-1}, k_1].$$

To put this operator back into the standard form we define a new set of summation indices

$$\tilde{k}_1 = k_2 - k_1$$
$$\tilde{k}_l - \tilde{k}_{l-1} = k_{l+1} - k_l, \; 2 \leq l \leq n - 2$$
$$\tilde{k}_{n-1} - \tilde{k}_{n-2} = J - k_{n-1},$$

and we have introduced the function $T(a;\vec{k})$ defined as
or equivalently solving for the $\tilde{k}$ variables

$$\tilde{k}_l = k_{l+1} - k_1, \ 1 \leq l \leq n - 2$$

$$\tilde{k}_{n-1} = J - k_1.$$  \hspace{1cm} (5.33)

Under this change of variables the various $q$-factors transform as

$$q^{-(k_1+\cdots+k_{n-1})} = q^{nk_{n-1}}q^{-(k_1+\cdots+k_{n-1})}$$

$$\mathcal{T}(a;\vec{k}(\vec{k})) = q^{-a\tilde{k}_{n-1}}\mathcal{T}(a;\tilde{k})$$

$$q^{(a_1-n)(1-k_{n-1})} = q^{(a_1-n)(1-\tilde{k}_{n-2}+\tilde{k}_{n-1})}$$  \hspace{1cm} (5.34)

so that the $q$-factor transforms as

$$q^{(a_1-n)(1-k_{n-1})}q^{-(k_1+\cdots+k_{n-1})}\mathcal{T}(a_1 - a_2 ; \vec{k}) \mathcal{T}(a_2 ; \tilde{k})$$

$$= q^{(a_1-n)(1-k_{n-2})}q^{-(k_1+\cdots+k_{n-1})}\mathcal{T}(a_1 - a_2 ; \vec{k}) \mathcal{T}(a_2 ; \tilde{k}).$$  \hspace{1cm} (5.35)

In the end the only effect of these operations is to change the $k_{n-1}$ in the $q$-factor to $k_{n-2}$. Since these two different forms are equal, it means that this term is actually symmetric under exchange of $k_{n-1}$ and $k_{n-2}$. Repeating this procedure $n - 3$ more times, we can rewrite (5.28) in a manifestly symmetric form

$$TR\left( \tau_{a_2,0}\tau_{a_1,0} \Delta_q(E_{21}^n) \Phi_1^{\otimes L} \right)$$

$$= q^{j/2} \frac{1}{1-q^{a_1-n}[n]_q!} \sum_{0 \leq k_1 \leq k_2 \cdots \leq k_{n-1} \leq J} \left( 1 - \frac{1}{n-1} \sum_{l=1}^{n-1} q^{(a_1-n)(1-k_l)} \right)$$

$$q^{-(k_1+k_2+\cdots+k_{n-1})} \mathcal{T}(a_1 - a_2 ; \vec{k}) \mathcal{T}(a_2 ; \tilde{k}) \mathcal{T}(a_2 ; \tilde{k})$$

$$TR[1, k_1, k_2 - k_1, \ldots, k_{n-1} - k_{n-2}, J - k_{n-1}].$$  \hspace{1cm} (5.36)

This is our final form for the two $\tau$ operator. Comparing to the BMN operator (2.8) we see that (5.36) consists of a sum of BMN operators, specifically it consists of

$$TR\left( \tau_{a_2,0}\tau_{a_1,0} \Delta_q(E_{21}^n) \Phi_1^{\otimes L} \right) \leftrightarrow$$

$$\alpha_{n-a_1-1}^+\alpha_{a_1-a_2-1}^+\alpha_{a_2-1}^+(\alpha_{-1}^+)^{n-3}$$

$$+ \alpha_{n+a_2-a_1-1}^+\alpha_{a_1-a_2-1}^+\alpha_{a_2-1}^+(\alpha_{-1}^+)^{n-2}$$

$$+ \alpha_{n-a_2-1}^+\alpha_{a_2-1}^+(\alpha_{-1}^+)^{n-2}$$

$$+ \alpha_{n-a_1-1}^+\alpha_{a_1-1}^+(\alpha_{-1}^+)^{n-2}$$

$$+ \alpha_{n-1}^+(\alpha_{-1}^+)^{n-1}$$  \hspace{1cm} (5.37)

where the operator content on the right-hand-side is meant to be schematic in that we have not tried to get the constants right. Comparing to the single $\tau$ case the important
point to note is that there is now an operator on the RHS which has two generic oscillator numbers. In the single \( \tau \) case the best one could do was to get an operator with only one generic oscillator number. We now show that this pattern continues - letting \( P \tau \)'s act on the operator \((5.6)\), we find a dual string state containing in particular a state with \( P \) generic oscillator numbers. Consequently, acting with \( n - 1 \) \( \tau \)'s will produce a dual string state with completely generic oscillator numbers.

5.4. Three and more \( \tau \)'s

We now turn to the generic case of many \( \tau \)'s. For simplicity we shall sketch the three \( \tau \) case and simply state the end result for the \( n - 1 \) \( \tau \) case. The manipulations from the previous subsections are more or less identical in these higher \( \tau \) cases. Consider now the three \( \tau \) case:

\[
TR \left( \tau_{a_3,0} \tau_{a_2,0} \tau_{a_1,0} \Delta_q(E_{21}^n) \Phi_1^\otimes L \right) = q^{(L+1)n} [n]_q! \sum_{1 \leq i_1 < i_2 \cdots < i_n \leq L} q^{-(i_1+i_2+\cdots+i_n)} TR \left( \rho_{i_1}(E_{21}) \cdots \rho_{i_n}(E_{21}) \Phi_1^\otimes L \right)
\]

\[
= q^{a_1 k_1 + a_2 k_2 + a_3 k_3} q^{-a_1 \mathcal{F}(k_1,0 \, \vec{i})-a_2 \mathcal{F}(k_2,k_1 \, \vec{i})-a_3 \mathcal{F}(k_3,k_1+k_2 \, \vec{i})}
\]

The \( k_3 \) sum gives

\[
\sum_{k_3=1}^L q^{a_3} \left( k_3 - \mathcal{F}(k_3,k_1+k_2 \, \vec{i}) \right) = q^{-a_3} \left( k_1 + k_2 - \mathcal{F}(k_1,k_2,0 \, \vec{i}) \right) S(a_3 \, \vec{i})
\]

\[
= q^{-a_3} \left( k_1 + k_2 - \mathcal{F}(k_1,0 \, \vec{i}) - \mathcal{F}(k_2,k_1 \, \vec{i}) \right) S(a_3 \, \vec{i})
\]

Note that the sum we have to do is of the same form as in the two-\( \tau \) case \((5.24)\) with \( a_2 \) in that equation replaced by \( a_3 \) and the \( \mathcal{F}(k_2,k_1 \, \vec{i}) \) in \((5.24)\) replaced by \( \mathcal{F}(k_3,k_1+k_2 \, \vec{i}) \). In the last line we used the property of \( \mathcal{F} \) given in \((5.22)\). Now we do the sum over \( k_2 \) in \((5.38)\)

\[
\sum_{k_2=1}^L q^{(a_2-a_3)(k_2-\mathcal{F}(k_2,k_1 \, \vec{i}))} = q^{-(a_2-a_3)(k_1-\mathcal{F}(k_1,0 \, \vec{i}))} S(a_2-a_3 \, \vec{i})
\]
Again this is the same sum as \((5.24)\) with \(a_2\) replaced by \(a_2 - a_3\). Collecting the \(k_1\) dependent terms, the final sum over \(k_1\) is

\[
\sum_{k_1=1}^{L} q^{(a_1+(a_3-a_2)-a_3)(k_1-\mathcal{F}(k_1,0 ; \vec{i}))} = \sum_{k_1=1}^{L} q^{(a_1-a_2)(k_1-\mathcal{F}(k_1,0 ; \vec{i}))} = S(a_1 - a_2 ; \vec{i})
\]

This sum is again of the same form as \((5.24)\) except that \(a_2\) is now replaced by \(a_1 - a_2\) and \(\mathcal{F}(k_2, k_1 ; \vec{i})\) by \(\mathcal{F}(k_1,0 ; \vec{i})\). Combining the result of the sums we get

\[
TR(\tau_{a_3,0}\tau_{a_2,0}\tau_{a_1,0} \bigtriangleup q(E_{21}^n) \Phi_1^{\otimes L}) = q^{-(\frac{L+1)n}{2}}[n]_q! \sum_{1 \leq i_1 < i_2 \cdots < i_n \leq L} q^{-(i_1+i_2+\cdots+i_n)} TR(\rho_{i_1}(E_{21})\cdots\rho_{i_n}(E_{21}) \Phi_1^{\otimes L}) S(a_1 - a_2 ; \vec{i}) S(a_2 - a_3 ; \vec{i}) S(a_3 ; \vec{i})
\]

where \(S\) was defined in \((5.25)\).

In the case of \(P\) \(\tau\)'s the same kinds of manipulations lead to

\[
TR(\tau_{a_P,0}\cdots\tau_{a_2,0}\tau_{a_1,0} \bigtriangleup q(E_{21}^n) \Phi_1^{\otimes L}) = q^{-(\frac{L+1)n}{2}}[n]_q! \sum_{1 \leq i_1 < i_2 \cdots < i_n \leq L} q^{-(i_1+i_2+\cdots+i_n)} TR(\rho_{i_1}(E_{21})\cdots\rho_{i_n}(E_{21}) \Phi_1^{\otimes L}) S(a_1 - a_2 ; \vec{i}) \cdots S(a_{p-1} - a_p ; \vec{i}) S(a_p ; \vec{i})
\]

The final step is to rewrite this operator in a form that can be compared to the BMN operators given in \((2.6)\). The idea is exactly as described in the two \(\tau\) case in the discussion of formulae \((5.28)\) to \((5.36)\). Using the definition \((5.29)\) we find

\[
TR(\tau_{a_P,0}\cdots\tau_{a_2,0}\tau_{a_1,0} \bigtriangleup q(E_{21}^n) \Phi_1^{\otimes L}) = \frac{(-1)^n}{1 - q^{a_1-n}[n]_q!} \sum_{0 \leq k_1 \leq k_2 \cdots \leq k_{n-1} \leq L} \left( 1 - \frac{1}{n-1} \sum_{l=1}^{n-1} q^{(a_1-n)(1-k_l)} \right) q^{-(k_1+k_2+\cdots+k_{n-1})} \mathcal{T}(a_1 - a_2 ; \vec{k}) \cdots \mathcal{T}(a_{p-1} - a_p ; \vec{k}) \mathcal{T}(a_p ; \vec{k})
\]

This is our final expression for the trace which uses \(P\) \(\tau\) operators. Comparing to \((2.6)\) one sees that it is a linear combination of many BMN operators. However, it is important
to note that in this combination there is only one occurrence of an operator with the most
generic oscillator numbers, in this case $P$ generic oscillator numbers. It is of the form

$$TR \left( \tau_{a_{1}}\cdots\tau_{a_{P}} \Delta q(E_{21}^{n}) \Phi_{1}^{\otimes L} \right) \leftrightarrow$$

$$\alpha_{n-1-a_{1}}^{\dagger} \cdots \alpha_{a_{P}-1}^{\dagger} \prod_{l=1}^{P-1} \alpha_{a_{l}-a_{l+1}-1}^{\dagger} + \cdots$$

(5.45)

where the $\cdots$ denotes operators with $n - P$ or more $\alpha_{-1}^{\dagger}$’s. It is straightforward in practice
therefore to take linear combinations of the $P$ $\tau$ operator with lower $\tau$ operators to isolate
this generic oscillator number string state. Moreover by taking $P = n - 1$ one can obtain
the most general oscillator number state involving insertions of $\Phi_{2}$ operators.

6. Implications of the quantum group construction of BMN operators

We discuss some physical and mathematical implications of the quantum group con-
struction of BMN operators presented in the earlier sections.

6.1. Correlators as traces of quantum group generators

Since we have expressed the BMN operators in terms of the co-product of the quantum
group and $q$-cyclic operators built from quantum group generators, we may expect that we
should be able to express the correlators of BMN operators in terms of traces of quantum
group generators and $\tau$ operators acting on tensor space. For operators in half BPS
representations, this step of expressing correlators as traces of group theoretic quantities
in tensor space was described in [40,41] and was used to derive factorization equations and
to exhibit relations between correlators in the four dimensional theory and classical ( large
$k$ ) Chern Simons theory. We outline some steps in this direction for BMN operators.

Let us focus on the case which has been the main focus of the previous section, namely
where the impurities are all one complex $\Phi_{2}$. Both $\Phi_{1}$ and $\Phi_{2}$ are matrices which transform
an $N$-dimensional space $V$. It is useful to consider an operator which collects both of them
into one object. In a sense we are thinking of a $U(N)$ theory with two flavors as a $U(2N)$
theory broken to $U(N)$. We define a matrix $\Phi = \Phi_{1} \oplus \Phi_{2}$ or in matrix notation

$$\Phi = \begin{pmatrix} \Phi_{1} & 0 \\ 0 & \Phi_{2} \end{pmatrix}$$

(6.1)
which acts on two copies of $V$, i.e $W = V \oplus V$. Projection projectors $P_1, P_2$

\[
P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]

\[
P_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\]

(6.2)

project to the first and second copy of $V$ respectively. This allows us to write $\Phi_1 = \Phi P_1$ and $\Phi_2 = \Phi P_2$. The basic free field two point functions of $\Phi_1$ with $\Phi_1^\dagger$, and of $\Phi_2$ with $\Phi_2^\dagger$ and the vanishing of the two-point function of $\Phi_1$ with $\Phi_2^\dagger$ or $\Phi_2$ with $\Phi_1^\dagger$ are all encoded in the formula

\[
\langle \Phi \Phi^\dagger \rangle = (P_1 \otimes P_1 + P_2 \otimes P_2) \circ \gamma
\]

(6.3)

where the $\Phi$ and $\Phi^\dagger$ are viewed as operators in $W \otimes W$ and $\gamma$ is a twist which permutes one $W$ with the other. More generally we think of $\Phi$ and $\Phi^\dagger$ each as operators acting on the tensor product $W^{\otimes n}$, and the two-point function can be written as a sum of insertions of $P_1 \otimes P_1$ and $P_2 \otimes P_2$. The evaluation of correlation functions can then be mapped, using formulas we have developed, into the evaluation of traces of sequences of operators. The operators will include the $P_1, P_2$ projectors, as well as the quantum group generators $\Delta_q(E_{21}), \Delta_q(E_{12})$ and the $q$-cyclic operators.

In [40,41] the operators involved after evaluating the two-point functions of the Higgs fields were all permutations, essentially because a general multitrace highest weight half-BPS operator could be written as $tr(\sigma \Phi_1)$, or as $tr(P_R \Phi_1)$ where $\Phi_1$ was defined to act in $V^{\otimes n}$ and the trace was taken in the $n$-fold tensor product. $P_R$ is a projection operator onto Young Diagrams, and orthogonality of these projectors allowed one to diagonalize the two-point functions. The diagonalization of BMN operators is now a question related to finding projection operators in tensor space $W^{\otimes n}$. This new perspective on the BMN operators should be useful in further studies of their correlators. It is interesting that the expression of half BPS operators in the form $tr(\sigma \Phi_1)$ also plays a role in the string bit model [46,47].

One of the interesting features of physical applications of quantum groups at roots of unity is that they capture vanishing properties of correlation functions or fusion rules [33,44,36]. It will be interesting to look for signatures of such vanishings in this context. For example, in all the calculations of section 5, BMN operators emerge from the quantum group construction with the $q$-factorial $[n]_q !$ which vanishes at $q^n = e^{\frac{2\pi i n}{J}} = \pm 1$. So at $n = J$ (and $n = J/2$ for $J$ even), we have a qualitative change from the point of view of the quantum group construction. It will be interesting to see if this is reflected in the correlators computed either from the super-Yang Mills or the string field theory.
6.2. Remark on the quantum group transformation of the action

It is interesting to ask if the type of quantum deformation of the global symmetry group of SYM can be given meaning as a transformation of the action. We do not expect it to be a symmetry since it is a spectrum generating algebra which does not commute with the Hamiltonian. But we would like to see if a consistent definition can be given of the transformation rule. We will not explore this in detail here, except to indicate that a well-defined transformation is indeed possible. Consider for example the term

$$\int d^4x \text{TR}([\Phi_1, \Phi_1^\dagger] + [\Phi_2, \Phi_2^\dagger])^2$$

in the action. Expanding it out one finds many terms. There is a well defined action of the quantum group using the quantum co-product on a product of Φ’s (the algebra of Φ’s can be given the structure of a module algebra, as defined for example in [42]). But we have traces, which only determine a product up to cyclicity. We can use the cyclicity to write the trace in a manifestly cyclic symmetric form and then act on the sequence of products thus obtained. For example, consider the term in the expansion of the operator (6.4) above,

$$\text{TR}(\Phi_1^\dagger \Phi_2 \Phi_2^\dagger \Phi_2^\dagger) = \frac{1}{4} \text{TR}(\Phi_1^\dagger \Phi_2^\dagger \Phi_2^\dagger + \Phi_1^\dagger \Phi_2 \Phi_1^\dagger + \Phi_2 \Phi_2^\dagger \Phi_1^\dagger + \Phi_2^\dagger \Phi_2 \Phi_2^\dagger).$$  (6.5)

Applying this procedure to all the terms appearing in the expansion of (6.4) produces many more terms. Now we can act on each term appearing in this expansion using the quantum co-product. For example the action of $E_{21}$ on $\Phi_1^\dagger \Phi_2 \Phi_2^\dagger \Phi_2^\dagger$ gives

$$\frac{1}{2}(q^{-3/2} + 2q^{-1/2} + q^{1/2})\text{TR}(\Phi_1^\dagger \Phi_2^\dagger \Phi_2^\dagger \Phi_2^\dagger) - \frac{1}{2}(q^{-1/2} + 2q^{1/2} + q^{3/2})\text{TR}(\Phi_2 \Phi_2^\dagger \Phi_1^\dagger)$$  (6.6)

where we have applied the trace in arriving at this form. Computing the action of $E_{21}$ on all the terms in the expansion of (6.4), we find

$$\frac{1}{2}((−q^{-3/2} + q^{-1/2} + q^{1/2} - q^{3/2})\text{TR}(\Phi_2 \Phi_2^\dagger \Phi_1^\dagger)$$
$$+ (q^{-1/2} - q^{3/2})\text{TR}(\Phi_2 \Phi_1^\dagger \Phi_1^\dagger)^2 + (−q^{-3/2} + q^{1/2})\text{TR}(\Phi_2^\dagger \Phi_1^\dagger)^2 \Phi_1)$$
$$+ (q^{-3/2} - q^{-1/2} - q^{1/2} + q^{3/2})\text{TR}(\Phi_1^\dagger \Phi_2^\dagger \Phi_2^\dagger)$$
$$+ (−q^{-1/2} + q^{3/2})\text{TR}(\Phi_1^\dagger \Phi_2^\dagger \Phi_2^\dagger)^2 + (q^{-3/2} - q^{1/2})\text{TR}(\Phi_1^\dagger \Phi_2 \Phi_2^\dagger))$$  (6.7)

So the action is not invariant (although it becomes invariant as $q \to 1$ as is easily checked in the term above) but transforms in a specified way. It will be interesting to see if Ward identities can be developed using these transformations of the action, and if they have useful information for correlators of BMN operators.
6.3. Quantum group symmetry and quantum geometries

It is tempting to conjecture that the quantum group construction has a geometrical meaning in terms of quantum spaces. While we have explicitly shown the construction of BMN operators with correct symmetry using the class of single impurity insertions generated by an $U_q(U(2))$ subgroup, many of our considerations should apply to the construction of the most general operators using the full $q$-deformed superalgebra $SU_q(4|2,2)$. We have also commented that the $q$-deformed superalgebra $SU_q(3|2,1)$ can be expected to play a special role related to holomorphic insertions. Suggestions that $SU_q(4|2,2)$ might be relevant to $N = 4$ SYM were made in the context of a conjecture that quantum $ADS \times S$ spacetimes are relevant to finite $N$ effects [48][49]. The $q$ in those discussions was also a root of unity, but a different one $q = e^{\frac{2\pi i}{N}}$, chosen to capture certain truncations in the spectrum of chiral primaries associated with finite $N$. These truncations are related to the stringy exclusion principle [50] and giant gravitons [51]. In this context, $q$-deformed spectrum generating algebras have also been discussed [52]. The exploration of the connection between the algebraic constructions here and the stringy exclusion principle, giant gravitons and non-commutative spacetimes is an interesting problem we leave for the future.

The idea that there is some geometrical meaning to the quantum group construction of the BMN operators is also suggested by the technical similarities between the $q$-cyclic operators we have used and analogous operators that appear in cyclic cohomology of quantum groups. For example in [39] a map is found between cyclic cohomology of quantum groups and that for module algebras which are equipped with $q$-cyclic traces. Finding concrete connections between the work of [39] and the work of BMN is a fascinating direction. At least some known properties of cyclic cohomology may be taken to suggest the existence of a connection to the physical context of strings on plane waves. For example, Connes shows that the cyclic cohomology of the group ring of a finite group $\Gamma$ is related to the $S^1$ equivariant cohomology of the loop space related to the classifying space of $\Gamma$ (see section 2. of [53]). Assuming analogous results exist for the cyclic cohomology of quantum groups at roots of unity, the appearance of loop spaces would be mirrored on the physical side by presence of maps from a string. The $S^1$ equivariance is suggestive of residual diffeomorphism invariance. The appearance of the classifying space of the quantum group is suggestive of quantum homogeneous spaces which include quantum deformations of $ADS \times S$ or of the pp-wave background. Finding a more concrete formulation of ideas in this direction would be interesting, especially since they may give insight into the quantum geometrical meaning of correlators of stringy states in a plane wave background.
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Appendix A : Constructing BMN operators
Motivated by the discussions in [8,22], we construct the BMN operators in the following way. First we define an intermediate set of fields
\[ \tilde{\Phi}_{\beta,k} \equiv \Phi_1^{-k} \beta \Phi_1 q^{kp} \]  
(6.8)
The BMN operators can then be constructed as
\[ O_{\beta,n;p,n} = N_n \sum_{0 \leq i_1 \leq i_2 \leq \cdots \leq i_n \leq J} \sum_{\sigma \in S_n} \text{TR} \left( \tilde{\Phi}_{\beta(1)} p_{\sigma(1)} i_1 \cdots \tilde{\Phi}_{\beta(n)} p_{\sigma(n)} i_n \right) \]  
(6.9)
This expression treats all \( \tilde{\Phi} \) fields symmetrically, and as such it provides a simple generalization of the original BMN prescription for the two-impurity case [3,8,6]. We must however show that this operator reduces to that of the two-impurity case when \( n = 2 \).

Indeed, the two-impurity case as originally constructed in [3] comes with only one summation index, whereas the form proposed above comes with two. Our task in this appendix is to show that (6.9) does indeed reduce to the more familiar form given in section 2.

We begin by rewriting the operator (6.9) in terms of the notation (2.3) introduced in section 2. We find
\[ O_{\beta,n;p,n} = N_n \sum_{0 \leq i_1 \leq i_2 \leq \cdots \leq i_n \leq J} \sum_{\sigma \in S_n} q \sum_{l=1}^{n} i_l p_{\sigma(l)} \text{TR}[i_1^{\beta(1)} i_2 - i_1 \beta(2) i_3 - i_2 \beta(3) \cdots \beta(2) i_n J - i_n]. \]  
(6.10)
This form can be simplified somewhat by using cyclicity of the trace and redefining summation variables. In particular one notes that cycling the term \( i_1 \) in the trace to the end of that operator produces the term \( J + i_1 - i_n \). Therefore there are only \( n - 1 \) different variables in the operator that are being summed over. Consequently one of the sums can be done explicitly. One makes this manifest in the following way. For any given permutation \( \sigma \in S_n \), we use cyclicity of the trace to cycle the \( \beta_n \) operator insertion to the first position. For example, if \( \sigma(s) = n \), then we cycle the operator into the form
\[ \text{TR} \left[ \beta_n i_{s+1} - i_s \beta(s+1) i_{s+2} - i_{s+1} \beta(s+2) \cdots \beta(n) J + i_1 - i_n \beta(1) \cdots \beta(s-1) i_s - i_{s-1} - 1 \right]. \]  
(6.11)
This form of the operator suggests redefining the summation indices in the following way:

\[
k_1 = i_{s+1} - i_s \\
k_l - k_{l-1} = i_{s+l} - i_{s+l-1}, \quad 2 \leq l \leq (n - s) \\
k_{n-s+1} - k_{n-s} = J + i_1 - i_n \\
k_l - k_{l-1} = i_{l-(n-s)} - i_{l-(n-s)-1}, \quad n - s + 2 \leq l \leq n - 1.
\]

Equivalently one can solve for the \( k_l \)'s as

\[
k_l = i_{s+l} - i_s, \quad 1 \leq l \leq n - s \\
k_l = J + i_{l-(n-s)} - i_s, \quad n - s + 1 \leq l \leq n - 1.
\]

The new set of summation variables now consists of \( i_s \) and the \( k_l \)'s for \( 1 \leq l \leq n - 1 \). In terms of these variables the operator \((6.11)\) becomes

\[
TR[\beta_n^{\sigma(s+1)} k_1 \beta_{\sigma(s+2)} \beta_{\sigma(n)} \beta_{\sigma(1)} \beta_{\sigma(s-1)} J - k_{n-1}].
\]

In particular the \( i_s \) dependence drops out of the operator.

To do the \( i_s \) sum we make note of the following facts. First the sum over the permutation group \( S_n \) can be rewritten as

\[
\sum_{\sigma \in S_n} = \sum_{s=1}^{n} \sum_{\tau \in S_{n-1}}
\]

That is, given a permutation \( \sigma \in S_n \) satisfying \( \sigma(s) = n \), we can construct a permutation \( \tau \in S_{n-1} \) satisfying

\[
\tau(l) = \sigma(s + l), \quad 1 \leq l \leq n - s \\
= \sigma(l - n + s), \quad n - s + 1 \leq l \leq n - 1
\]

For a given \( s \), the \( S_{n-1} \) sum is simply over all \( \tau \) constructed in this way. The sum on \( s \) then fills out the remaining elements of the \( S_n \) permutation group. Secondly we note that the sums over the \( i_l \) indices becomes

\[
\sum_{0 \leq i_1 \leq i_2 \leq \cdots \leq i_n \leq J} = \sum_{0 \leq k_1 \leq k_2 \leq \cdots \leq k_{n-1} \leq J} \sum_{i_s = J - k_{n-s+1}}^{J-k_{n-s}}
\]
in the $k_l$ indices. This follows simply from the index redefinitions given in (6.13). The last fact that we need is to rewrite the phase factor as

$$q\sum_{i=1}^n i_{\sigma(i)}p_i = q\sum_{i=1}^{n-1} k_{r-1(i)}p_i + i_s(p_1 + \cdots + p_n) = q\sum_{i=1}^{n-1} k_i p_{r(i)} + i_s(p_1 + \cdots + p_n). \quad (6.18)$$

The first and third equality signs follow trivially. The second equality follows from the definition of $\tau$ given in (6.16) and the assumption that $q^J = 1$. If we take $p_1 + \cdots + p_n = 0$, as one usually does to obtain an operator that corresponds to a string state, then all dependence on the summation index $i_s$ drops out. However we shall not make this assumption here. As we shall see in a moment, keeping all $p_l$'s generic merely results on the string side to considering a string state which is a linear superposition of different ($n$ for generic $p_l$'s) single string states.

Now we combine these basic facts to reproduce the operator given in section 2. From the permutation sum, the $i_l$ sums, and the $q$-factor discussed in the preceding paragraph, all $s$ and $i_s$ dependent factors reduce to

$$\sum_{s=1}^n \sum_{i_s=J-k_{n-s}+1}^{J-k_{n-s}} q^i_s(p_1 + \cdots + p_n) \quad (6.19)$$

where we have introduced the new $k$ indices $k_0$ and $k_n$ which are fixed to 0 and $J$ respectively. These arise from the special cases $\sigma(n) = n$ and $\sigma(1) = n$ respectively and are easily checked to have the values just quoted. This sum is straightforward to evaluate. The $s$ and $i_s$ sums together fill out a sum (replacing $i_s$ by $i$) $\sum_{i=0}^J$, but neighboring $i_s$ sums overlap exactly at the endpoints, therefore (6.19) can be rewritten as

$$\sum_{i=0}^J q^i(p_1 + \cdots + p_n) + \sum_{l=1}^{n-1} q^{(J-k_l)(p_1 + \cdots + p_n)}. \quad (6.20)$$

Depending on whether $p_1 + \cdots + p_n$ vanishes or not, this sum evaluates to

$$= 1 + \sum_{l=1}^{n-1} q^{-k_l(p_1 + \cdots + p_n)}, \quad p_1 + \cdots + p_n \neq 0 \quad (6.21)$$

$$= J + n, \quad p_1 + \cdots + p_n = 0.$$

Putting everything together reproduces the operator (2.2) given in section 2 provided that $p_1 + \cdots + p_n = 0$ and where the factor of $J + n$ in (6.21) has been absorbed into the
normalization. If instead we take $p_1 + \cdots + p_n \neq 0$ we obtain the operator

$$O_{\beta_n,p_n;\beta_1,p_1,\ldots,\beta_{n-1},p_{n-1}} = N_n \sum_{0 \leq k_1 \leq k_2 \leq \cdots \leq k_{n-1} \leq J} \sum_{\tau \in S_{n-1}} q^{\sum_{l=1}^{n-1} k_l p_{\tau(l)}} (1 + \sum_{l=1}^{n-1} q^{-k_l(p_1 + \cdots + p_n)}) \text{TR}[\beta_n, k_1, \beta_{\tau(1)}(k_2 - k_1, \beta_{\tau(2)}(\ldots, \beta_{\tau(n-1)}(J - k_{n-1}))]

(6.22)$$

where now an extra sum of $q$-factors appears as compared to (2.2). The interpretation however is clear. This operator corresponds not to a single string state, but rather to a linear superposition of string states. Explicitly we find the correspondence

$$O_{\beta_n,p_n;\beta_1,p_1,\ldots,\beta_{n-1},p_{n-1}} \leftrightarrow \sum_{l=1}^{n} \alpha^\dagger_{-(p_1 + \cdots + p_n) + p_l} \prod_{k=1,k \neq l}^{n-1} \alpha^\dagger_{p_k}

(6.23)$$

Such a formula is expected as the starting point (6.11) treats the $n$ oscillator numbers $p_l$ symmetrically.

Finally we would like to emphasize the importance of the sum over the permutation group $S_n$ in (6.9) in comparing to the dual string state in (2.5) or (6.23). The creation operators in (2.5) and (6.23) commute; therefore, this property should be evident in the dual operator. The sum over the permutation group makes this property manifest, and moreover is necessary for the correspondence in (2.5) and (6.23) to make sense. In the two-impurity case, this property is satisfied trivially, and as such was not an issue in [3]. Already at the three-impurity level however this sum is necessary as was noted in [8] as part of a construction for $n$-impurities. The $n$-impurity case was discussed further in [22].

**Appendix B : The degenerate case $a_1 = n$**

In section 5.2 we described how to get BMN operators using $\text{TR}(\tau_{a_1}(\Delta(E_2^n)(\Phi_1^{J+n})}$. In doing the sums we assumed that $a_1 \neq n$. The case $a_1 = n$ has to be treated separately. Specializing (5.13) to the case $a_1 = n$ gives the result

$$\text{TR}n\Delta_q(E_2^n)[J + n] = (-1)^n[n]_q! \sum_{0 \leq j_1 \leq j_2 \cdots \leq j_n \leq J} (q^{nj_1} + q^{nj_2} + \cdots + q^{nj_n})

(6.24)$$

$$\text{TR}[j_1, j_2 - j_1, \cdots, j_n - j_{n-1}, J - j_n] q^{-j_1 - j_2 - \cdots - j_n}$$

We cycle the first $j_1$ operators to the right and change variables

$$k_1 = j_2 - j_1$$

$$k_2 = j_3 - j_2$$

$$\vdots$$

$$k_{n-1} = j_n - j_{n-1}

(6.25)$$
which implies

\[ k_1 = j_2 - j_1 \]
\[ k_2 = j_3 - j_1 \]
\[ \vdots \]
\[ k_{n-1} = j_n - j_1 \]

We now write the sum in terms of the variables \((j_1; k_1 \cdots k_{n-1})\). We write the sum in (6.24) as a sum of \(n\) copies of the same thing with an overall factor of \(1/n\). The first term is

\[
(-1)^n[n]_q! \sum_{0 \leq k_1 \cdots k_{n-1} \leq J}^{J-k_{n-1}} \sum_{j_1=0}^{k_1} TR([, k_1, k_2 - k_1, \cdots, k_{n-1} - k_{n-2}, J - k_{n-1}])
\]

\[ q^{-k_1 \cdots - k_{n-1}} (1 + q^{n k_1} + q^{n k_2} + \cdots q^{n k_{n-1}}) \]

\[ = (-1)^n[n]_q! \sum_{0 \leq k_1 \leq k_2 \cdots \leq k_{n-1} \leq J}^{J-k_{n-1}} TR([, k_1, k_2 - k_1, \cdots, k_{n-1} - k_{n-2}, J - k_{n-1}])))
\]

\[ (J - k_{n-1} + 1)(1 + q^{n k_1} + q^{n k_2} + \cdots q^{n k_{n-1}})q^{(-k_1-k_2-\cdots-k_{n-1})} \]

In the second term we will cycle one \(\phi_2\) impurity as well to get

\[
(-1)^n[n]_q! \sum_{0 \leq k_1 \leq k_2 \cdots \leq k_{n-1} \leq J}^{J-k_{n-1}} TR([, k_2 - k_1, k_3 - k_2, \cdots, k_{n-1} - k_{n-2}, J - k_{n-1}, k_1])
\]

\[ (1 + q^{n k_1} + q^{n k_2} + \cdots q^{n k_{n-1}})q^{(-k_1-k_2-\cdots-k_{n-1})} \]

Now we do a relabelling

\[ \tilde{k}_1 = k_2 - k_1 \]
\[ \tilde{k}_2 - \tilde{k}_1 = k_3 - k_2 \]
\[ \vdots \]
\[ \tilde{k}_{n-2} - \tilde{k}_{n-3} = k_{n-1} - k_{n-2} \]
\[ J - \tilde{k}_{n-1} = k_1 \]

(6.29)

to get operator in the summand to be of the same form as (6.27).

This leads the \(q\) factors to transform as

\[ q^{-k_1 \cdots - k_{n-1}} = q^{n \tilde{k}_{n-1}} q^{-k_1 \cdots - \tilde{k}_{n-1}} \]
\[ (1 + q^{n k_1} + \cdots + q^{n k_{n-1}}) = q^{-n \tilde{k}_{n-1}} (1 + q^{n \tilde{k}_1} + \cdots + q^{n \tilde{k}_{n-1}}) \]

(6.30)
which implies

\[ q^{-k_1 - \cdots - k_{n-1}} (1 + q^{nk_1} + \cdots + q^{nk_{n-1}}) = q^{-\tilde{k}_1 - \cdots - \tilde{k}_{n-1}} (1 + q^{n\tilde{k}_1} + \cdots + q^{n\tilde{k}_{n-1}}) \quad (6.31) \]

Thus the \( q \) factors as well as the operator are identical in the tilde variables as can be seen by comparing to \((6.27)\). The only difference is that the coefficient is \((\tilde{k}_{n-1} - \tilde{k}_{n-2} + 1)\). After renaming the tilded variables back to untilded variables and collecting the first two terms we get a coefficient \((J - k_{n-1} + 1) + (k_{n-1} - k_{n-2} + 1)\). Another cycling step produces a sum of the same form with the coefficient of \((k_{n-2} - k_{n-3} + 1)\). Continuing this procedure and collecting terms we get \((J - k_1 + 1) + (k_{n-1} - k_{n-2} + 1) + (k_{n-2} - k_{n-3} + 1) + (k_{n-3} - k_{n-4} + 1) + \cdots + (k_2 - k_1 + 1) + (k_1 + 1) = (J + n)\). This leads to the result

\[
TRn(\Delta q(E_{21}^n)\Phi_1^{(J+n)})
= \frac{J+n}{n} \sum_{0 \leq k_1 \leq k_2 \cdots \leq k_{n-1} \leq J} TR(\Phi_2 \Phi_1^{k_1} \Phi_2 \Phi_1^{k_2-k_1} \Phi_2 \cdots \Phi_2 \Phi_1^{k_{n-1}-k_{n-2}} \Phi_2 \Phi_1^{J-k_{n-1}})
q^{-k_1 - \cdots - k_{n-1}} (1 + q^{nk_1} + \cdots + q^{nk_{n-1}})
\rightarrow \alpha_2^{(2)} \alpha_1^{(2)} \cdots \alpha_1^{(2)} \alpha_1^{(2)} |0 >
\]

(6.32)
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