Solving Tantrix via Integer Programming

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Abstract. Tantrix is a puzzle to make a loop by connecting lines drawn on hexagonal tiles, and the objective of this research is to solve it by a computer. For this purpose, we give a problem setting of solving Tantrix as arranging tiles in an appropriate shape and making a loop at the same time within a given hexagonal lattice board. We then formulate it as an integer program by expressing the rules of Tantrix as its constraints, and solve it by a mathematical programming solver to have a solution. As a result, we establish a formulation that solves Tantrix of moderate sizes, and even when the solutions are invalid only by elementary constraints, we achieved it by introducing additional constraints and an artificial objective function to avoid flaws in invalid solutions. By this approach we are successful in solving Tantrix of size up to 50.

1 Introduction

Games and puzzles are entertainments invented for human beings, and solving puzzles or playing games are lots of fun for everybody. Such puzzles and games are often logical enough, and they have long been attracted interests of mathematicians and computer scientists not only for the pleasure but for their research \cite{7}. Those may include Nim, Hex, Peg Solitaire, Tetris, Geography, Sudoku, Rubik’s Cube, Chess, Othello, Go, and so on \cite{6,7,12,14,15,16}.

There are a lot of directions and objectives when puzzles and games are treated as research topics \cite{14,7,8}. Some of those are investigating their mathematical structures \cite{6,11}, computational complexities \cite{5,12}, winning strategies \cite{17}, and so on. As computers evolve and expand their applicational utilization, they are rapidly incorporated into these research areas \cite{6,14,15,16}. Typical example is to develop a computer program that can solve puzzles faster than humans or can beat humans in playing games.

In this paper, we focus on a puzzle called Tantrix\textsuperscript{‡} that makes a loop by connecting lines drawn on hexagonal tiles \cite{17,18,19}, and the objective of this research is to solve it by a computer\textsuperscript{§}. More precisely, we first give a problem setting of solving Tantrix as arranging tiles in an appropriate shape and making a loop at the same time within a given hexagonal lattice board. We then formulate it as an integer program (IP) \cite{13,21} by expressing the rules or properties that solutions of Tantrix satisfy (necessary conditions) as its constraints, and we attempt to obtain a solution by solving that IP using a commercial mathematical programming solver. However, since we may not have valid solutions only by elementary constraints, we develop some new additional constraints and introduce an artificial objective function in order to derive valid solutions. We show the current best solution of Tantrix obtained by the proposed approach. Our approach of using IP formulation to solving puzzles seems novel and entertaining, and to the best of our knowledge, this is one of few cases that IP meets solving puzzles successfully.

In Section\textsuperscript{2} we introduce a puzzle called Tantrix and explain its rules. Next in Section\textsuperscript{3} we give a problem setting for solving Tantrix by a computer and terminology for IP formulations. Then

\textsuperscript{‡} Tantrix\textsuperscript{R} is a registered trademark of Colour of Strategy Ltd. in New Zealand, and of TANTRIX JAPAN in Japan, respectively, under the license of M. McManaway, the inventor.

\textsuperscript{§} Solutions obtained by computers are not authorized as official records.
in Section 4, we describe an elementary integer programming formulations of Tantrix and show some computational results. In Section 5, we develop novel ideas to obtain valid solutions and show our current best result. Finally Section 6 presents some future work and concludes the paper.

2 Tantrix

Tantrix is a puzzle originally invented in 1988 by Mike McManaway of New Zealand [17]. Several variants of commercial Tantrix products have been sold so far, and among those a solitaire version is named “Tantrix Discovery” [17,18,19,20]. Throughout this paper, we focus only on this solitaire version, Tantrix Discovery, and we simply call it “Tantrix”.

Tantrix is played with 10 sorts of hexagonal tiles of the same size. A tile has two surfaces, which we call a top (surface) and a back (surface). On a top surface three lines are drawn in red, blue and yellow (Fig. 1(a)), and on a back surface one of the numbers from 1 to 10 is drawn in either one of the three colors (Fig. 1(b)). The 10 patterns of lines drawn on tops are all different from each other.

Fig. 1. (a) Tops of 10 sorts of tiles (with their orientations 1), and (b) their corresponding backs.

Tantrix is played by putting tiles in the form of hexagonal lattice and the goal is to make a loop in one designated color according to the following rules:

1. determine the number of tiles for the challenge (challenge number), which is greater than 2,
2. prepare tiles as many as the challenge number by starting from a tile with number 1 and consecutively up to that number (in case it is more than 10, start from 1 again),
3. the designated color of the challenge number is the one in which its lowest digit is written (on the back of the tile),
4. connect all the lines of the designated color (drawn on prepared tiles) so that they form a single loop,
5. connect the lines of the other colors so that their touching colors match.

Here arrangements of tiles in which there is a hole (places without tiles surrounded by 6 tiles or more) (Fig. 2(a)), or any one of the lines of the designated color is not a part of a loop (Fig. 2(b)) are not allowed. When one completes making a loop according to the above rules, we say that she/he cleared that challenge number of Tantrix. In addition to the above rules, to avoid that a solution becomes a repetition of a specified pattern, its arrangement is required to be, intuitively speaking, ‘round’ (not flattened). (This leads our board setting explained later.)

To be authorized as an official record, an arrangement must satisfy the following condition: in the arrangement let one of the three directions (axes) that has the most number $x$ of tiles be $A$, and the other two directions that cross $A$ be $B$ and $C$. Then there must be more tiles than 30% of $x$ in more rows than 75% of those in directions $B$ and $C$, respectively. This condition is set to exclude the so-called “4-tiles’ equation” discovered by C. Fraser of England. This information was once posted in [17], however, we can no longer find it on Jan. 1, 2012. See Appendix A for details.
Suppose we try Tantrix of challenge number 5. The designated color of that number is red (Fig. 1(b)). In Fig. 3, a single loop of the designated color red is made by using all the red lines drawn on five tiles that have numbers from 1 to 5, and the colors of touching lines of the other colors also match (in this case blue only) at the same time. In addition there is no hole in this arrangement of tiles. Therefore, we say that we cleared Tantrix of challenge number 5.

A solution (of challenge number $n$) is an arrangement how $n$ tiles are placed, and a Tantrix solution is the one that satisfies all the above conditions (Fig. 3). A shape of a solution (or a solution shape) is its boundary formed by $n$ hexagonal tiles.

### 3 A Problem Setting and Terminology for Formulations

Our approach to solving Tantrix is to formulate it as an integer program and to solve it by mathematical programming solver. In this section, we give a problem setting for this purpose and some terminology for IP formulations.

#### 3.1 A Problem Setting for Solving Tantrix by a Computer

Humans may play Tantrix on the table, on the floor or at any other places they like. For computers, however, we have to prepare artificially and appropriately a space where solutions are made. Consider an infinite hexagonal lattice plane (Fig. 4), where the size of each hexagon is as the same as a single tile. When Tantrix is played by a computer, a tile is placed to fit on each hexagon, which we call a place (Fig. 4(left)), and we call a collection of multiple places where a solution is supposed to be made a board (Fig. 4(right)). The size of a board is the number of hexagons that constitute it.

**Fig. 4.** A place (left) and an example of a board (of size 5) (right) on a hexagonal lattice plane.
so that they do not create a hole and the shape become round at the same time. To realize these situations for a computer, especially to guarantee the freedom of solution shapes, we prepare a finite but sufficiently larger size of a board than the challenge number. That is, we describe the problem to be solved as follows:

TANTRIX (Free Board TANTRIX)
Input: a challenge number \( n \) and a board of size \( m \) (> \( n \)),
Output: a Tantrix solution on (within) the input board.

Here, since solutions are required to be arranged in round shapes, we give boards on which they are made in the following manner: number places from 1 in a spiral way starting from a single place (Fig. 5), and pick consecutively numbered places (from 1) so that they become some symmetric shapes as shown in Fig. 5(a) and (b) (we call them, for convenience, types A and B, respectively). Therefore, we can prepare boards of sizes 7, 19, 37, . . . , and 3, 12, 27, 48, . . . , of types A and B, respectively. When we try Tantrix of challenge number 20, for example, we prepare a board of size 27 of type B or size 37 of type A, and so on.

![Fig. 5. A numbering of places on infinite hexagonal plane, and boards of sizes (a) 19 of type A and (b) 27 of type B with their place numbers.](image)

3.2 Terminology and Definitions for IP Formulations
We call a tile with number \( i \) on its back tile \( i \). Remind that the designated color of a number is the one in which the number is written. We define an angle of a line as its central angle when we regard it as a circular arc of a circle, where we define the angle of a straight line to be 0°. Then an angle of a line is either one of 0°, 60° or 120° (Fig. 6). For example, tile 2 is said to have a red line of an angle 120°, 0° blue line and 120° yellow line (Fig. 1(a)).

To distinguish places and orientations of tiles to be placed, we assign numbers to them. For places of an input board of size \( m \), we give numbers from 1 to \( m \) as explained above (Fig. 5). We also give numbers from 1 to 6 to the edges of each place counter-clockwise as shown in Fig. 7. The orientations of a tile to be placed take values from 1 to 6. We define it in the following way (Fig. 8): place a tile showing its back and its number in the upright position; flip it horizontally, and it is orientation 1; every time we rotate it by 60° clockwise, its orientations will be 2, 3, 4, 5 and 6, respectively. Fig. 1(a) shows each tile in its orientation 1.

Two places are adjacent if they share an edge. Let \( a(j, \ell) \) be a function that returns the number of the place to which place \( j \) is adjacent with its edge \( \ell \), and returns 0 if such a place does not exist (out of a board). For a board shown in Fig. 5(a), for example, \( a(4, 6) = 12, a(8, 5) = 0, \) and so on.
For simplicity, once a tile is (or supposed to be) placed on a certain place with a certain orientation, we allow to identify the tile with that place. That is, we can say: adjacent tiles; we can alternatively say that the color of a line of an edge of a place instead of the color of a line (of a placed tile) appearing on the corresponding edge (of the place where the tile is placed in a certain orientation), adjacent tiles, and so on. We say that a place is empty if no tile is placed on it.

4 An Integer Programming Formulation

We solve this problem TANTRIX by utilizing a mathematical programming solver after formulating it as an integer program. We first introduce its variables, and explain constraints and an objective function. Then we show preliminary experimental results of solving this formulated IP.

4.1 Variables

Remind here that \( n \) is a challenge number, \( m (> n) \) is the size of a prepared board and that we use 10 sorts of tiles numbered from 1 to 10 on their backs. When the challenge number is \( n \), the number \( N \) of sorts of tiles to be used is \( N = \min\{n, 10\} \leq 10 \), which implies that \( N = 10 \) if \( n \geq 10 \). We assume that we try Tantrix of challenge number no less than 10 and that \( N \) is always 10. Then we set the following three integral variables for our IP formulation.

First, for each tile \( i \) \((1 \leq i \leq 10)\), for each place \( j \) \((1 \leq j \leq m)\) and for each orientation \( k \) \((1 \leq k \leq 6)\) of tiles, let a 0-1 variable \( x_{ijk} \) be

\[
x_{ijk} = \begin{cases} 
1, & \text{tile } i \text{ is placed on place } j \text{ with orientation } k, \\
0, & \text{otherwise (i.e., tile } i \text{ is not placed on place } j \text{ with orientation } k). 
\end{cases}
\]

Next for each place \( j \) \((1 \leq j \leq m)\) and its edge \( \ell \) \((1 \leq \ell \leq 6)\), let a variable \( y_{j\ell} \) that expresses the color of its corresponding line (of a tile) be

\[
y_{j\ell} = \begin{cases} 
0, & \text{for place } j \text{ and its edge } \ell, \text{ there is no color,} \\
1, & \text{for place } j, \text{ the color of the line corresponding to edge } \ell \\
 & \text{is neither the designated color nor color 2,} \\
2, & \text{for place } j, \text{ the color of the line corresponding to edge } \ell \\
 & \text{is neither the designated color nor color 1,} \\
3, & \text{for place } j, \text{ the color of the line corresponding to edge } \ell \\
 & \text{is the designated color.}
\end{cases}
\]
Here we use values 1, 2 and 3 to indicate three colors of lines, where 3 is intended to denote the designated color, and value 0 is prepared for empty places (thus no color is defined for their edges). Notice here that if we define a function \( c(i, k, \ell) \) that returns the color of the line corresponding to edge \( \ell \) when tile \( i \) is placed in orientation \( k \) (we can know it according to the top of tiles), the variable \( y_{j\ell} \) is represented by using \( x_{ijk} \) and \( c(i, k, \ell) \) as follows:

\[
y_{j\ell} = \sum_{i=1}^{n} \sum_{k=1}^{6} c(i, k, \ell)x_{ijk} \quad (j = 1, 2, \ldots, m; \; \ell = 1, \ldots, 6).
\]

To describe some constraints succinctly, we introduce for convenience a 0-1 variable \( u_{jj'} \) for each pair of adjacent places \( j \) and \( j' \):

\[
u_{jj'} = \begin{cases} 1, & \text{a tile is placed on either one of the places } j \text{ and } j', \\ 0, & \text{otherwise (i.e., tiles are placed both or neither on places } j \text{ and } j'). \end{cases}
\]

That is, \( u_{jj'} \) is simply the exclusive OR of two (0-1) variables \( \sum_{i=1}^{10} \sum_{k=1}^{6} x_{ijk} \) and \( \sum_{i=1}^{10} \sum_{k=1}^{6} x_{ijk} \) which imply if a tile is placed on place \( j \) and \( j' \), respectively. The possible combinations of values of these two variables and \( u_{jj'} \) are \((\sum_{i=1}^{10} \sum_{k=1}^{6} x_{ijk}, \sum_{i=1}^{10} \sum_{k=1}^{6} x_{ijk}, u_{jj'}) \in \{(0, 0, 0), (1, 1, 0), (1, 0, 1), (0, 1, 1)\}\), and we can confirm that these conditions are expressed only by using variable \( x_{ijk} \) in the following system of four inequalities: \( u_{jj'} \leq -\sum_{i=1}^{10} \sum_{k=1}^{6} x_{ijk} - \sum_{i=1}^{10} \sum_{k=1}^{6} x_{ijk} + 2, u_{jj'} \leq \sum_{i=1}^{10} \sum_{k=1}^{6} x_{ijk} + \sum_{i=1}^{10} \sum_{k=1}^{6} x_{ijk}, u_{jj'} \geq \sum_{i=1}^{10} \sum_{k=1}^{6} x_{ijk} - \sum_{i=1}^{10} \sum_{k=1}^{6} x_{ijk} + 2, \) and \( u_{jj'} \geq -\sum_{i=1}^{10} \sum_{k=1}^{6} x_{ijk} + \sum_{i=1}^{10} \sum_{k=1}^{6} x_{ijk} \).

4.2 Constraints and an Objective Function

We now describe the (necessary) conditions that have to be satisfied by Tantrix solutions as constraints of an integer program. Based on the rules of Tantrix, we introduce the following five constraints.

**Constraint 1** (C1). At most one tile is placed on each place.

This is required to hold for any place on a board that two or more tiles cannot be placed, and it is represented by the following formula:

\[
C1: \quad \sum_{i=1}^{10} \sum_{k=1}^{6} x_{ijk} \leq 1 \quad (j = 1, 2, \ldots, m).
\]

**Constraint 2** (C2). The number of places each on which a tile is placed equals the challenge number.

This implies that we use exactly the same number of places as the challenge number, and is formulated as follows:

\[
C2: \quad \sum_{i=1}^{10} \sum_{j=1}^{m} \sum_{k=1}^{6} x_{ijk} = n.
\]

**Constraint 3** (C3). Each tile is used the number of times defined by the challenge number.

Once we determine the challenge number to be \( n \), the number of tiles \( i \) used for it is described by the following formula:

\[
C3: \quad \sum_{j=1}^{m} \sum_{k=1}^{6} x_{ijk} = \left\lfloor \frac{n+i}{10} \right\rfloor \quad (i = 1, \ldots, 10).
\]

**Constraint 4** (C4). The color of a line of an edge that is adjacent to no other tile is not the designated color.

**Constraint 5** (C5). The colors of lines whose corresponding edges are touching each other have to match.
These constraints tell about two adjacent places $j$ and $j'$ via their edges $\ell$ and $\ell'$, respectively. For $C_4$, if exactly one tile is placed on either of two places $j$ and $j'$, then either one of $y_{j\ell}$ or $y_{j'\ell'}$ is 0 and the other is 1 or 2, i.e., if $u_{jj'} = 1$ then $1 \leq |y_{j\ell} - y_{j'\ell'}| \leq 2$. Similarly for $C_5$, $y_{j\ell} = y_{j'\ell'} = 1, 2$ or 3 if tiles are placed both on $j$ and $j'$ and $y_{j\ell} = y_{j'\ell'} = 0$ otherwise, i.e., if $u_{jj'} = 0$ then $0 \leq |y_{j\ell} - y_{j'\ell'}| \leq 0$ in any case. By paying attention to that $y_{j\ell} \neq y_{j'\ell'}$ if $u_{jj'} = 1$, we can formulate these two constraints together as follows:

$$C_4 + C_5: -2u_{jj'} \leq y_{j\ell} - y_{j'\ell'} \leq 2u_{jj'} \quad (a(j, \ell) = j'; a(j', \ell') = j).$$

Remark here that by $C_4$ and $C_5$, any line of the designated color must be a part of a loop, that is, they constitute necessary (but not sufficient) conditions for that all the lines of the designated color form loops, but that they do not imply sufficient conditions of Tantrix solutions in the sense that they may have holes or that a loop of the designated color may not be unique. We call such a loop of the designated color consists of less than $n$ tiles a subloop. We do not expect to have complete necessary and sufficient conditions in our elementary formulation, and we will deal with these issues (ad-hoc or try and error, in some sense) later in subsequent sections.

In our approach, since a solution is formulated as constraints of an integer program, it suffices to find one of its feasible solutions. In such a case, an objective function is enough to be virtual, and therefore, we set it as $x_{1,1,1} \rightarrow \min$, for a descriptive purpose.

### 4.3 Elementary Experiments and the Results

We solve the integer program formulated so far by a mathematical programming solver, to obtain a solution and to examine the solution time. We use IBM ILOG CPLEX 12.2 as a solver, which is installed on a single PC of Intel Pentium Dual E2200 processor (2.2GHz) with 1GB RAM. Since the computational time spent for solving a fixed formulation by a solver on a same computer is always the same in principle, we adopt the one by a single computation throughout our experiments.

We performed experiments for challenge numbers 10, 15 and 20 with appropriate board sizes. (We sometimes denote the input challenge number $n$ and its board size $m$ for experiments by $(n, m)$, for short.) Table 1 shows its computational time, the number of holes and the number of loops in each solution. We see that we can solve our formulation within 30 seconds up to challenge number 20. As we mentioned, constraints $C_1$–$C_5$ cannot eliminate holes and subloops in their solutions, and in fact, for challenge numbers 10, 15 and 20 all the solutions include holes and one solution includes a subloop as well (Fig. 9). These results imply that it is not so easy even to obtain (valid) Tantrix solutions only by these elementary constraints. Therefore, we concentrate on deriving such formulations that can produce valid solutions rather than obtain them in less computational time.

| $n$ | $m$ | time (sec.) | #holes | #loops |
|-----|-----|-------------|--------|--------|
| 10  | 12  | 0.27        | 1      | 1      |
| 19  | 1   | 1.64        | 1      | 1      |
| 15  | 19  | 0.84        | 1      | 2      |
| 27  | 9   | 9.64        | 1      | 1      |
| 20  | 27  | 27.41       | 1      | 1      |
| 37  | 12  | 12.36       | 1      | 1      |

Table 1. Computational results of solving IP formulations of TANTRIX using constraints $C_1$–$C_5$: computational time (sec.), the number of holes and loops for each challenge number $n$ with its board size $m$.
5 Improvement of Formulations for Valid Tantrix Solutions

The IP formulations which we proposed in the previous section cannot always lead Tantrix solutions even for relatively small challenge numbers mainly due to holes and subloops. To deal with these issues and to obtain valid solutions, in this section, we consider additional constraints for (i) arranging solution shapes to be ‘round’ (and thus holes may disappear) and (ii) eliminating subloops in advance.

5.1 Arranging Solution Shapes to be Round and Holeless

Rules of Tantrix implicitly require the solution shapes to be round and this automatically implies that there can hardly be holes in solutions. Hence, to avoid holes, we propose two ideas for keeping solution shapes to be round.

Restrict the number of adjacent tiles A hole is a set of adjacent empty places surrounded by 6 or more tiles. By utilizing this observation, we first consider the following three constraints.

Constraint 6a (C6a). At most 5 tiles are placed on adjacent places to an empty place.
Constraint 6b (C6b). At most 4 tiles are placed on adjacent places to an empty place.
Constraint 6c (C6c). At most 3 tiles are placed on adjacent places to an empty place.

These are expressed in the following formulas:

\[ \sum_{j=1}^{6} \sum_{x=1}^{10} \sum_{k=1}^{6} x_{i,j,x,k} \leq \sum_{j=1}^{n} \sum_{x=1}^{6} x_{ij} + 5 \quad (j = 1, \ldots, m), \]
\[ \sum_{j=1}^{6} \sum_{x=1}^{10} \sum_{k=1}^{6} x_{i,j,x,k} \leq 2 \sum_{j=1}^{n} \sum_{x=1}^{6} x_{ij} + 4 \quad (j = 1, \ldots, m), \]
\[ \sum_{j=1}^{6} \sum_{x=1}^{10} \sum_{k=1}^{6} x_{i,j,x,k} \leq 3 \sum_{j=1}^{n} \sum_{x=1}^{6} x_{ij} + 3 \quad (j = 1, \ldots, m). \]

The left-hand side of the formulas counts the number of tiles placed on adjacent places to a place \( j \), and each right-hand side becomes 5, 4, or 3, respectively if a tile is not placed on place \( j \), and 6 otherwise. Thus the inequalities directly imply the constraints.

Since C6c implies C6b, and C6b implies C6a (i.e., C6c \( \Rightarrow \) C6b \( \Rightarrow \) C6a), we use either one of them together with C1–C5, and adopt an effective one (in a way, try and error). We show in Table 2 computational results of solving formulations using C1–C5 with adding each one of C6a, C6b or C6c, and Fig. 10 shows some of their solutions. As a result, we see that we can obtain Tantrix solutions only for challenge numbers 10 and 15, and cannot yet for challenge number 20. In these solutions, not only there are subloops but the shapes seem irregular (not round (Fig. 10(a))) or a hole exists (Fig. 10(b)), and even worse, the shape becomes disconnected unexpectedly (Fig. 10(c)) although it does not have a hole. According to these observations, we see that C6’s are not effective enough to avoid holes or subloops.

Utilize the objective function The second idea for arranging solution shapes is to change the objective function so that it forces solution shapes to be round. For this purpose, we define a weight \( w(j) \) for each place \( j \) on a board; that is, if place \( j \) is on \( r \)-th round from its center (which is defined to be 0-th round), then \( w(j) = -r \). For a board of type A (Fig. 5(a)) and for its place 8, for example, since it is on 2nd round, \( w(8) = -2 \). Then we set a new objective function to be

\[ \sum_{j=1}^{m} \sum_{i=1}^{10} \sum_{k=1}^{6} w(j)x_{ij} \rightarrow \max. \]

(instead of \( x_{1,1,1} \rightarrow \min. \)). This is intended for solutions to use inner places more likely than outer places, and therefore we expect that solution shapes become round and are unlikely to have holes.
Table 2. Computational results of solving IP formulations using C1–C5 with C6a, C6b or C6c for challenge numbers 10, 15 and 20. Here * implies that the solution is disconnected.

|       | C6a |       |       | C6b |       |       | C6c |       |       |
|-------|-----|-------|-------|-----|-------|-------|-----|-------|-------|
|       | n   | m     | time  | #holes | #loops | time  | #holes | #loops | time  | #holes | #loops |
| 10    | 12  | 0.27  | 0     | 1      | ———   | ———   | ——— | ———   | ———   |
| 19    | 2.19| 1     | ———   | ———   | ———   | ———   | ——— | ———   | ———   |
| 15    | 17.72| 2    | 16.14 | 0      | 1      | 5.27  | 0      | 2      | ———   | ———   |
| 27    | 17.24| 1    | 4.51  | 1      | 1      | 63.89 | 0      | 2      | ———   | ———   |
| 20    | 5.69| 1     | 35.83 | 0      | 2      | 10.47 | 0      | 2      | ———   | ———   |
| 37    | 12.59| 1    | 98.67 | 0      | 2      | 509.11| 0      | 3      | ———   | ———   |

Fig. 10. The solution for (a) (15, 19) with C6a, (b) (15, 27) with C6b and (c) (15, 27) with C6c.

Table 3 shows computational results of solving formulations using this objective function with constraints C1–C5, and Fig. 11 shows some of these solutions. We can see that their shapes seem to become rather round compared to those obtained by the previous idea, however, we still have holes in some cases.

Incorporating two ideas simultaneously As we observed, the two proposed ideas for arranging solution shapes are not always effective when each of them is used alone. Therefore, we try to implement these simultaneously; that is, we adopt the weighted objective function and also add either one of constraints C6a, C6b or C6c. We show computational results for challenge numbers 15 and 20 in Table 4. Then the effect of combining two ideas seems remarkable as holes disappear completely from all solutions, and as a result, we can have at least one Tantrix solution for challenge numbers up to 20.

Then we try some larger challenge numbers, 25 and 30, under this formulation, and we show those computational results also in Table 4 and Fig. 12. The solution shapes seem to be well arranged and there are no holes in any solutions at all, which again confirms the effect of combining two ideas. However, we cannot have Tantrix solutions for challenge number 30 due to subloops. We approach this issue in the next subsection.

5.2 Eliminate Short Subloops in Advance

So far, we achieved to arrange solution shapes to be round and holeless to some extent. On the other hand, subloops can appear in solutions since we do not express the uniqueness of a loop as constraints of our formulations. It is, of course, not impossible to introduce constraints that eliminates
Table 3. Computational results of solving IP formulations using C1–C5 with a weighted objective function for challenge numbers 15 and 20.

| n  | m  | time | #holes | #loops |
|----|----|------|--------|--------|
| 15 | 19 | 27.44| 0      | 1      |
| 27 |    | 6.70 | 1      | 1      |
| 20 | 27 | 40.17| 1      | 1      |
| 37 |    | 13.20| 0      | 3      |

Table 4. Computational results of solving IP formulations using C1–C6 with a weighted objective function for challenge numbers 15, 20, 25 and 30.

| n  | m  | C6a  | C6b  | C6c  |
|----|----|------|------|------|
|    |    | time | #holes #loops | time | #holes #loops | time | #holes #loops |
| 15 | 19 | 3.28 | 2      29.58 | 0    1 | 5.92 | 0    2 |
| 27 |    | 36.75| 1      12.53 | 0    1 | 29.84| 0    2 |
| 20 | 27 | 5.73 | 2      7.34 | 0    2 | 10.13 | 0    1 |
| 37 |    | 405.61| 0      26.13 | 0    1 | 18.70 | 0    1 |
| 25 | 37 | 243.33| 0      79.16 | 1    525.28 | 0    1 |
| 30 | 37 | 74.39 | 3      415.72 | 0    4 | 26.19 | 0    3 |
| 48 |    | 72.55| 4      624.32 | 0    2 | 21.58 | 0    2 |

all the probable subloops as introduced and well studied for TSP formulations [23,13,21]. However, it may require exponential number of constraints, which may also cause a long computational time. Furthermore, since a solution shape is not determined to be unique due to the surplus of places prepared for a challenge number to guarantee the freedom of solution shapes, the cut which has to be crossed by a loop in a solution is no longer a cut for (a shape of) another solution. This makes it not so easy to take an approach that we add constraints for eliminating subloops every time they appear and re-solve it to obtain a unique loop of the designated color [10].

Now observing solutions that have subloops, we can see those consists of 3 or 4 tiles. Therefore, we attempt to eliminate these types of short subloops. More specifically, we eliminate subloops consists of 3, 4 or 5 tiles and describe it as the following constraints:

**Constraint 7** (C7). There are no loops consists of 3 tiles.
**Constraint 8** (C8). There are no loops consists of 4 tiles.
**Constraint 9** (C9). There are no loops consists of 5 tiles.

We achieve them by embedding these constraints for every possible places on a board in advance. It is easy to see that C7 is equivalent to that none of the arrangements of lines of the designated color shown in Fig. 13 (a) appears on any two adjacent places on a board. Therefore, this constraint is realized by forbidding these types of tiles (having 120° lines of the designated color) to become one of those arrangements.

Suppose now that the designated color is red. We can see that tiles 2 and 3 have 120° red lines (Fig. 1). To forbid these two tiles to be placed in one of those arrangements simultaneously, we
Fig. 12. The solutions for (a) (20, 27) with C6a and (b) (30, 37) with C6c.

Fig. 13. Forbidden arrangements of lines of the designated color (for challenge number greater than 5). These will force subloops consist of (a) 3, (b) 4 and (c) 5 tiles, respectively.

force the sum of $x_{ijk}$ corresponding to being these arrangements not to be greater than 1. Therefore, in this concrete case, the formulations become as follows. We can consider similarly in the case that the designated color is blue or yellow:

\[
\begin{align*}
\sum_{j=1}^{m} \left( x_{2,j,2} + x_{3,j,4} + x_{2,\alpha(j,1),6} + x_{3,\alpha(j,1),2} \right) & \leq 1, \\
\sum_{j=1}^{m} \left( x_{2,j,3} + x_{3,j,5} + x_{2,\alpha(j,1),5} + x_{3,\alpha(j,1),1} \right) & \leq 1, \\
\sum_{j=1}^{m} \left( x_{2,j,1} + x_{3,j,3} + x_{2,\alpha(j,2),5} + x_{3,\alpha(j,2),1} \right) & \leq 1, \\
\sum_{j=1}^{m} \left( x_{2,j,2} + x_{3,j,4} + x_{2,\alpha(j,2),4} + x_{3,\alpha(j,2),6} \right) & \leq 1, \\
\sum_{j=1}^{m} \left( x_{2,j,6} + x_{3,j,2} + x_{2,\alpha(j,3),4} + x_{3,\alpha(j,3),6} \right) & \leq 1, \\
\sum_{j=1}^{m} \left( x_{2,j,1} + x_{3,j,3} + x_{2,\alpha(j,3),3} + x_{3,\alpha(j,3),5} \right) & \leq 1.
\end{align*}
\]

For constraint C8, we can deal with it in a similar way to C7. In fact, C8 is equivalent to that none of the three arrangements of lines of the designated color in Fig. 13 (b) appears on any two adjacent places on a board. Therefore, it is realized by forbidding these types of tiles (having 60° lines of the designated color) to become one of those arrangements. It is also the case that we can represent this constraint by the corresponding variables $x_{ijk}$, and in the case of the designated color
is red, for example, it is formulated as follows (similarly for blue and yellow):

\[
\begin{align*}
&x_{1,j,2} + x_{4,j,1} + x_{6,j,4} + x_{7,j,2} + x_{8,j,3} + x_{10,j,3} + x_{1,at(j,1),5} + x_{4,at(j,1),4} \\
&+ x_{6,at(j,1),1} + x_{7,at(j,1),5} + x_{8,at(j,1),6} + x_{10,at(j,1),6} \leq 1 \quad (j = 1, \ldots, m), \\
&x_{1,j,1} + x_{4,j,6} + x_{6,j,3} + x_{7,j,1} + x_{8,j,2} + x_{10,j,2} + x_{1,at(j,2),4} + x_{4,at(j,2),3} \\
&+ x_{6,at(j,2),6} + x_{7,at(j,2),4} + x_{8,at(j,2),5} + x_{10,at(j,2),5} \leq 1 \quad (j = 1, \ldots, m), \\
&x_{1,j,6} + x_{4,j,5} + x_{6,j,2} + x_{7,j,6} + x_{8,j,1} + x_{10,j,1} + x_{1,at(j,3),3} + x_{4,at(j,3),2} \\
&+ x_{6,at(j,3),5} + x_{7,at(j,3),3} + x_{8,at(j,3),4} + x_{10,at(j,3),4} \leq 1 \quad (j = 1, \ldots, m).
\end{align*}
\]

For C9, it is equivalent to that for lines of the designated color none of the six arrangements in Fig. 13(c) appears on every mutually adjacent three places on a board. Suppose the designated color is red, then tiles 5 and 9 have 0° red lines and tiles 1, 3, 6, 7, 8 and 10 have 60° red lines. Therefore, we can express by using variables \(x_{i,j,k}\) corresponding to these tiles that they do not form none of the arrangements, and in the case of the designated color is red, for example, it is formulated as follows (similarly for blue and yellow):

\[
\begin{align*}
&x_{5,j,2} + x_{5,j,5} + x_{9,j,2} + x_{9,j,5} + x_{1,at(j,1),5} + x_{4,at(j,1),4} + x_{6,at(j,1),1} + x_{7,at(j,1),5} \\
&+ x_{9,at(j,1),6} + x_{10,at(j,1),6} + x_{1,at(j,2),4} + x_{4,at(j,2),3} + x_{6,at(j,2),6} + x_{7,at(j,2),4} \\
&+ x_{9,at(j,2),5} + x_{10,at(j,2),5} \leq 2 \quad (j = 1, \ldots, m), \\
&x_{1,j,1} + x_{4,j,6} + x_{6,j,3} + x_{7,j,1} + x_{8,j,2} + x_{10,j,2} + x_{1,at(j,1),6} + x_{4,at(j,1),5} \\
&+ x_{9,at(j,1),1} + x_{7,at(j,1),6} + x_{8,at(j,1),1} + x_{10,at(j,1),1} + x_{5,at(j,2),1} + x_{5,at(j,2),4} \\
&+ x_{9,at(j,2),1} + x_{9,at(j,2),4} \leq 2 \quad (j = 1, \ldots, m), \\
&x_{1,j,2} + x_{4,j,1} + x_{6,j,4} + x_{7,j,2} + x_{8,j,3} + x_{10,j,3} + x_{5,at(j,1),3} + x_{5,at(j,1),3} \\
&+ x_{9,at(j,1),3} + x_{9,at(j,1),6} + x_{1,at(j,2),4} + x_{4,at(j,2),3} + x_{6,at(j,2),6} + x_{7,at(j,2),4} \\
&+ x_{9,at(j,2),5} + x_{10,at(j,2),5} \leq 2 \quad (j = 1, \ldots, m), \\
&x_{5,j,1} + x_{5,j,4} + x_{9,j,1} + x_{9,j,4} + x_{1,at(j,2),4} + x_{4,at(j,2),3} + x_{6,at(j,2),6} + x_{7,at(j,2),4} \\
&+ x_{9,at(j,2),5} + x_{10,at(j,2),5} + x_{1,at(j,3),3} + x_{4,at(j,3),2} + x_{6,at(j,3),5} + x_{7,at(j,3),3} \\
&+ x_{8,at(j,3),4} + x_{10,at(j,3),4} \leq 2 \quad (j = 1, \ldots, m), \\
&x_{1,j,6} + x_{4,j,5} + x_{6,j,2} + x_{7,j,6} + x_{8,j,1} + x_{10,j,1} + x_{1,at(j,2),5} + x_{4,at(j,2),4} \\
&+ x_{9,at(j,2),1} + x_{7,at(j,2),5} + x_{8,at(j,2),6} + x_{10,at(j,2),6} + x_{5,at(j,3),3} + x_{5,at(j,3),6} \\
&+ x_{9,at(j,3),3} + x_{9,at(j,3),6} \leq 2 \quad (j = 1, \ldots, m), \\
&x_{1,j,1} + x_{4,j,6} + x_{6,j,3} + x_{7,j,1} + x_{8,j,2} + x_{10,j,2} + x_{1,at(j,3),2} + x_{5,at(j,2),5} \\
&+ x_{9,at(j,2),2} + x_{9,at(j,2),5} + x_{1,at(j,3),2} + x_{4,at(j,3),1} + x_{6,at(j,3),4} + x_{7,at(j,3),2} \\
&+ x_{8,at(j,3),3} + x_{10,at(j,3),3} \leq 2 \quad (j = 1, \ldots, m).
\end{align*}
\]

For challenge number 30, all of whose solutions have subloops, we show the results of solving formulations using C1–C9 in Table 5 where we omit the number of holes since all the solutions no longer have holes. We can see that we succeed to have Tantrix solutions for challenge number 30 by eliminating short subloops.

### 5.3 Further Challenge

To summarize all the ideas, observations and experiments presented so far, it gives good results that a formulation using constraints C1–C9 with weighted objective function is solved. Hence, under this formulation, we try to solve TANTRIX by increasing its challenge numbers as 35, 40, 45 and 50. Table 5 also shows these experimental results. As a consequence, the current best result (in the challenge number) is 50 which is solved in total time 5549.20 seconds. We show the solution in Fig. 14.
Table 5. Computational results of solving IP formulations using C1–C9 with a weighted objective function for challenge numbers 30, 35, 40, 45 and 50.

| n  | m  | time  | #loops | time  | #loops | time  | #loops |
|----|----|-------|--------|-------|--------|-------|--------|
| 30 | 37 | 89.63 | 1      | 37.47 | 1      | 48.28 | 1      |
| 48 | 444.09 | 1     | 1482.56| 1     | 4520.75| 1     |        |
| 35 | 48 | 25571.17 | 1    | 48.67 | 2      | 917.42| 1      |
| 40 | 48 | 1400.55 | 2     | 327.80 | 1      | 665.02| 2      |
| 45 | 48 | ——     | ——    | ——    | ——    | 4698.13| 3      |
| 61 | —— | ——     | ——    | 12747.89| 1      | ——    | ——    |
| 50 | 61 | 408.09 | 3      | 704.01 | 4      | 2459.06| 2      |
| 75 | —— | ——     | ——    | 5549.20| 1      | ——    | ——    |

Fig. 14. A Tantrix solution of challenge number 50 on a board of size 75, which is obtained by solving IP formulations.

6 Conclusion

In this paper, we tried to solve Tantrix by a computer. The approach we adopted here is to formulate it as an IP and to solve it by a mathematical programming solver. We believe that this approach to solving puzzles is quite unique and entertaining as well. As a result, we could successfully solve it for challenge numbers up to 50, which is more or less larger than we expected. The results in this paper show that an approach using IP to solving puzzles appears promising and we believe that it may be valid for solving other puzzles. One of the important future work, of course, is to develop more effective formulations to solve Tantrix of larger challenge numbers in less computational time.

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APPENDIX

A The Four-Tiles’ Equation

To be authorized as an official record, an arrangement must satisfy the following condition: in the arrangement let one of the three directions (axises) that has the most number \( x \) of tiles be \( A \), and the other two directions that cross \( A \) be \( B \) and \( C \). Then there must be more tiles than 30% of \( x \) in more rows than 75% of those in directions \( B \) and \( C \), respectively. This condition is set to exclude the so-called “4-tiles’ equation” discovered by C. Fraser of England.

The 4-tiles’ equation says that we can have a solution for an arbitrary large challenge number \( 10x + 4 \ (x = 0, 1, \ldots) \). In these cases the designated color is red. Fig. 15 (a)(i) shows a red loop consists of four tiles numbered from 1 to 4, and (a)(ii) shows two red lines, which will be a part of a longer loop, consists of ten tiles numbered from 1 to 10. By combining these two parts, we will arrange infinitely large number of tiles into a single red loop, a solution for challenge numbers 14 and 24 in Fig. 15(b) and (c), respectively.

![Fig. 15. The 4-tiles’ equation.](image)

This information was once posted in [17], however, we can no longer find it on Jan. 1, 2012.