A collocation method for the Williams equation with Chebyshev polynomials

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Abstract. The linearized problem of gas flow in plane channel with infinite walls has been solved in the kinetic approximation. The flow in the channel is caused by a constant pressure gradient parallel to the walls of the channel. The Williams equation has been used as a basic equation, and the boundary condition has been set in terms of the diffuse reflection model. The collocation method for Chebyshev polynomials has been applied to construct the solution of the equation of Williams with the given boundary conditions. The mass flux of the gas in the channel has been calculated.

1. Introduction

The gas flow through a long channel is one of the most important application of the MEMS and vacuum technology [1]. In the intermediate range of Knudsen numbers, where the concepts of continuous medium and free-molecular regime are inapplicable, the function determining the macroscopic (mass) velocity can be obtained by integrating the Boltzmann kinetic equation or model kinetic equations. The BGK (Bhatnagar, Gross, Krook) and the S (Shahov) models of the Boltzmann kinetic equation, and the Williams equation are often used as model equations [1-7]. The basis for the construction of BGK and S models is the constancy of the collision frequency of gas molecules. However, the assumption on the independence of the speed of the frequency of gas molecule collisions is quite a strong simplification [8]. It would be more realistic to assume the constancy of the length of the free path of gas molecules, at least for molecules, whose mutual interaction can be approximated by the model of solid spheres. This assumption is equivalent to the assumption that the frequency of molecule collisions is proportional to the absolute value (module) of their heat velocity. The latter assumption leads to the following the Williams equation [8]. In this paper, the Williams equation is used to describe the kinetics of the processes under consideration.

At the present time a large number of methods and their variations are proposed to construct solutions of model kinetic equations. Thus, the method of discrete ordinates and velocities was used in [1]. The finite-volume method based on a TVD scheme is applied in [7]. In [4, 5] the results are obtained by the use the Neumann series. Here we propose an efficient approximate method for solving the Williams model kinetic equation by expansion of the solution in a series of Chebyshev polynomials [9, 10]. Using the Chebyshev collocation points, we transform the Williams integro-differential equation to a matrix equation which corresponds to a system of linear algebraic equations with unknown Chebyshev coefficients. Therefore
this allows us much to decrease the number of grid nodes in the physical space. Methods related to Chebyshev polynomial interpolation also have received much interest recently in the areas of signal processing and applied mathematics [11]. In [12] by applying the shifted Chebyshev polynomials and rational Chebyshev functions, the authors solved the fractional diffusion equation with initial-boundary conditions over a long time domain. Their approach was based on collocation method [10].

In this paper the collocation method for Chebyshev polynomials is used to construct the solution of the Williams equation in the problem of the slow isothermal flow of a rarefied gas due to a small pressure gradient in a plane channel with infinite parallel walls (Poiseuille flow). It is assumed that the walls of the channel are in the planes \( y' = \pm b' \) of a rectangular Cartesian coordinate system, the axis \( Oz' \) of which is parallel to the channel walls. The diffuse reflection model of the boundary condition on the walls of channel is used.

By following [4, 8] we write the Williams equation and the boundary conditions in the following form

\[
C_y \frac{\partial Z}{\partial y} \gamma Kn + CZ(y, C_y) + 1 = \frac{3C}{4\pi} \int C' \exp(-C'^2)C'^2Z(y, C'_y)d^3C',
\]

(1)

\[
Z(\pm 1, C_y) = 0, \quad \pm C_y < 0.
\]

(2)

Here \( C = \beta^{1/2}v \) is the dimensionless velocity of the gas molecules, \( \beta = m/(2k_B T_0) \), \( m \) is the molecular mass of the gas, \( k_B \) is the Boltzmann constant, \( T_0 \) is gas temperature at the origin of coordinate system, \( y = y'/b' \), and \( z = z'/b' \) are the coordinates of the dimensionless radius-vector \( r \), \( Kn = \frac{l_y}{b'} \) is Knudsen number, \( l_y \) is the mean free path of gas molecules, \( \gamma = 15\sqrt{\pi}/16 \). The walls of the channel situate in the planes \( y' = \pm b'/2 \) of the Cartesian coordinate system with the origin located in the mid-section \( z' = 0 \). The axis \( z' \) of is parallel to the channel walls. The mass velocity profile we will calculate according to [13]

\[
U_z(y) = \frac{\gamma Kn G_p}{\pi^{3/2}} \int \exp \left( -C'^2 \right) C'^2Z(y, C_y)d^3C,
\]

(3)

where \( G_p = p_0^{-1} dp/dz \) is the dimensionless pressure gradient \( (G_p \ll 1) \), \( p_0 \) is the gas pressure at the origin of coordinate system.

The mass flux through a channel of finite width \( 2a' \) is calculated as

\[
J_{M,2a} = 2\beta^{1/2}p_0b'^2 \int_{-a}^{1} U_z(x, y) dx dy,
\]

where \( a = a'/b' \), \( x = x'/b' \). In theoretical calculations, for channel with infinite width \( (a' \to \infty) \) the mass flux is determined per unit height, i.e., as limits [13]

\[
J_M = \lim_{a' \to \infty} \frac{1}{2a'} J_{M,2a} = 2\beta^{1/2}p_0b'^2 \int_{-1}^{1} U_z(y) dy.
\]

(4)

Taking into account (4) the reduced mass flux we can calculate as [13]:

\[
J_M = \frac{J'_M}{2b'\beta^{1/2}p_0} = \int_{-1}^{1} U_z(y) dy.
\]

(5)

To find the solution of the integro-differential equation (1) the unknown function \( Z(y, C_y) \) is expanded in the orthogonal functions and solution of the resulting equation is represented by Chebyshev polynomials in the matrix form.
2. Materials and methods

Let us introduce in the velocity space the scalar product of functions depending on the molecular velocity modulus as follows

$$(e_1, e_2) = \int_0^{+\infty} g(C)e_1(C)e_2(C)dC.$$  

Let us choose two orthogonal functions $e_1 = 1$ and $e_2 = 1/C - 3\sqrt{\pi}/8$ (here, orthogonality is meant as the vanishing of the above scalar product) and expand function $Z(y, C_y)$ by the orthogonal functions:

$$Z(y, C) = Z_1(y, \cos \theta) + \left(\frac{1}{C} - \frac{3\sqrt{\pi}}{8}\right)Z_2(y, \cos \theta). \quad (6)$$

In (6), we passed to the velocity space in the spherical coordinate system ($C_z = C \cos \varphi \sin \theta$, $C_x = C \sin \varphi \sin \theta$, $C_y = C \cos \theta$). Substituting expansion (6) into (1) and taking into account the orthogonality of functions $e_1(C)$ and $e_2(C)$, we come to a set of equations

$$\tau \frac{\partial Z_1}{\partial y} \gamma Kn + Z_1(y, \tau) + \frac{3\sqrt{\pi}}{8} = \frac{3}{4} \int_{-1}^{1} \left(1 - \tau'\right)^2 Z_1(y, \tau')d\tau'; \quad (7)$$

$$\tau \frac{\partial Z_2}{\partial y} \gamma Kn + Z_2(y, \tau) + 1 = 0, \quad \tau = \cos \theta. \quad (8)$$

From (2) and (6) it follows that boundary conditions for $Z_1(y, \tau)$ and $Z_2(y, \tau)$ have the form

$$Z_i(\pm 1, \tau) = 0, \quad -1 \leq \pm \tau < 0, \quad i = 1, 2. \quad (9)$$

The solution of equation (8) is sought by applying the method of characteristics. Taking into account (9), we obtain

$$Z_2(y, \tau) = \exp\left(-\frac{y - 1}{\gamma Kn\tau}\right)H_+(-\tau) + \exp\left(-\frac{y + 1}{\gamma Kn\tau}\right)H_+(\tau) - 1. \quad (10)$$

Here $H_+$ is the stepwise Heaviside function.

The solution of equation (7) responsible for boundary condition (9) is written as

$$Z_1(y, \tau) = W_1(y, \tau)H_+(-\tau) + W_2(y, \tau)H_+(\tau). \quad (11)$$

Substituting expression (11) into equation (7) and boundary condition (9), for $-1 \leq \tau \leq 0$ we obtain

$$\tau \frac{\partial W_1}{\partial y} \gamma Kn + W_1(y, \tau) + \frac{3\sqrt{\pi}}{8} = \frac{3}{4} \int_{-1}^{0} \left(1 - \tau'^2\right) W_1(y, \tau')d\tau' + \frac{3}{4} \int_{0}^{1} \left(1 - \tau'^2\right) W_2(y, \tau')d\tau'; \quad (12)$$

$$W_1(1, \tau) = 0, \quad -1 \leq \tau < 0. \quad (13)$$

For $0 \leq \tau \leq 1$ we obtain

$$\tau \frac{\partial W_2}{\partial y} \gamma Kn + W_2(y, \tau) + \frac{3\sqrt{\pi}}{8} = \frac{3}{4} \int_{0}^{1} \left(1 - \tau'^2\right) W_1(y, \tau')d\tau' + \frac{3}{4} \int_{0}^{1} \left(1 - \tau'^2\right) W_2(y, \tau')d\tau';$$
\[ W_2(-1, \tau) = 0, \quad 0 < \tau \leq 1. \]  

The functions \( W_1(y, \tau) \) and \( W_2(y, \tau) \) may be expanded in Chebyshev polynomials as

\[ W_i(y, \tau) = \sum_{j_1, j_2=0}^{\infty} a_{j_1j_2}^{(i)} T_{j_1}(y) T_{j_2}(z_i(\tau)), \quad z_i = 2\tau - (-1)^i, \quad y, z_i \in [-1, 1], \]  

where \( a_{j_1j_2}^{(i)} \) are unknown parameters, to be determined. The following recurrence relation, generate these polynomials as \[ T_0(y) = 1, \quad T_1(y) = y, \quad T_j(y) = 2yT_{j-1}(y) - T_{j-2}(y), \quad j \geq 2. \]  

The Chebyshev polynomials satisfy at the following properties \[ dT_j(y) dt = 2j \frac{j^2}{2} \sum_{k=1}^{(j-1)/2} T_{2k}(y), \quad j \text{ even}, \]  

\[ dT_j(y) dt = j + 2j \frac{j^2}{2} \sum_{k=1}^{(j-1)/2} T_{2k}(y), \quad j \text{ odd}; \]

\[ 2T_j(y)T_k(y) = T_{j+k}(y) + T_{|j-k|}(y); \]

\[ T_j(-y) = (-1)^j T_j(y); \]

\[ \int_{-1}^{1} T_j(y) dy = \begin{cases} \frac{2}{1 - j^2}, & j \text{ even}, \\ 0, & j \text{ odd}. \end{cases} \]

For \( 0 \leq \tau \leq 1 \) from equality (21) we deduce that

\[ W_1(1, -\tau) = \sum_{j_1, j_2=0}^{\infty} (-1)^{j_1+j_2} a_{j_1j_2}^{(1)} T_{j_1}(-1) T_{j_2}(z_2(\tau)). \]  

In this case from boundary conditions (13) and (15) we have the equality \( W_2(-1, \tau) = W_1(1, -\tau) \). Then, taking into account the uniqueness of the expansion \( W_2(-1, \tau) \), we have

\[ a_{j_1j_2}^{(2)} = (-1)^{j_1+j_2} a_{j_1j_2}^{(1)}, \quad j_{1,2} = 0, n_{1,2}. \]  

Let the approximation of \( W_i(y, \tau) \) be obtained by truncating the series (16) as

\[ W_i(y, \tau) = \sum_{j_1=0}^{n_1} \sum_{j_2=0}^{n_2} a_{j_1j_2}^{(i)} T_{j_1}(y) T_{j_2}(z_i(\tau)), \quad i = 1, 2. \]  

The matrix form of expression (25) can be written as

\[ W_i(y, \tau) = T(y)Q(z_i(\tau))A_i, \quad i = 1, 2. \]
Here \( T(y) \) is the \( 1 \times (n_1 + 1) \) matrix as
\[
T(y) = (T_0(y) \ T_1(y) \ldots T_{n_1}(y)),
\]
and \( A_i \) is the \((n_1 + 1)(n_2 + 1) \times 1 \) matrix as
\[
A_i = \left( a_{00}^{(i)} a_{01}^{(i)} \ldots a_{0n_2}^{(i)} a_{10}^{(i)} a_{11}^{(i)} \ldots a_{1n_2}^{(i)} \ldots a_{n_10}^{(i)} \ldots a_{n_1n_2}^{(i)} \right)^T, \quad i = 1, 2;
\]
\( Q(z_i(\tau)) \) is \((n_1 + 1) \times (n_1 + 1)(n_2 + 1) \) block matrix
\[
Q(z_i(\tau)) = \begin{pmatrix}
\hat{T}(z_i(\tau)) & 0 & 0 & \ldots & 0 \\
0 & \hat{T}(z_i(\tau)) & 0 & \ldots & 0 \\
& & & \ddots & \\
0 & 0 & \ldots & 0 & \hat{T}(z_i(\tau))
\end{pmatrix},
\]
\( \hat{T}(z_i(\tau)) = (T_0(z_i(\tau)) T_1(z_i(\tau)) \ldots T_{n_2}(z_i(\tau))) \),
and \( 0 \) is the \( 1 \times (n_2 + 1) \) zero matrix.

From equalities (18) and (19), we deduce that
\[
\frac{dT(y)}{dy} = T(y)J^T,
\]
in which \( J \) is the \((n_1 + 1) \times (n_1 + 1) \) matrix as [14]
\[
J = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 2 \cdot 2 & 0 & 0 & 0 & \ldots & 0 & 0 \\
3 & 0 & 2 \cdot 3 & 0 & 0 & \ldots & 0 & 0 \\
0 & 2 \cdot 4 & 0 & 2 \cdot 4 & 0 & \ldots & 0 & 0 \\
5 & 0 & 2 \cdot 5 & 0 & 2 \cdot 5 & \ldots & 0 & 0 \\
& & & \ddots & & & & \\
0 & 2 \cdot n_1 & 0 & 2 \cdot n_1 & 0 & \ldots & 2 \cdot n_1 & 0 \\
& & & \ddots & & & & \\
n_1 & 0 & 2 \cdot n_1 & 0 & 2 \cdot n_1 & \ldots & 2 \cdot n_1 & 0
\end{pmatrix}.
\]

In view of the designations (26)-(32), equation (12) can be rewritten in the form
\[
\tau \gamma K n T(y)J^T Q(z_1(\tau)) A_1 + T(y)Q(z_1(\tau))A_1 + \frac{3\sqrt{\pi}}{8} = \frac{3}{4}T(y) \sum_{i=1}^2 G_i A_i,
\]
where \( G_i \) is \((n_1 + 1) \times (n_1 + 1)(n_2 + 1) \) block matrix
\[
G_i = \begin{pmatrix}
K_i & 0 & 0 & \ldots & 0 \\
0 & K_i & 0 & \ldots & 0 \\
& & \ddots & \ddots & \\
0 & 0 & \ldots & 0 & K_i
\end{pmatrix},
\]
\( K_i = (K_{i,0} \ K_{i,1} \ldots K_{i,n_2}) \),
\[ K_{1,k} = \int_{-1}^{0} \left(1 - \tau'^2\right) T_k(2\tau' + 1)d\tau', \quad K_{2,k} = \int_{0}^{1} \left(1 - \tau'^2\right) T_k(2\tau' - 1)d\tau'. \] (36)

In (36), we passe variable \( z_i \) as (16)

\[ 2K_{i,k} = \int_{-1}^{1} \left(1 - \frac{(z_i + (-1)^i)^2}{4}\right) T_k(z_i)dz_i. \] (37)

The functions \( 1 - \frac{(z_i + (-1)^i)^2}{4} \) may be expanded in Chebyshev polynomials as

\[ 1 - \frac{(z_i + (-1)^i)^2}{4} = \frac{5}{8} T_0(z_i) - \frac{1}{2} T_1(z_i)(-1)^i - \frac{1}{8} T_2(z_i), \quad i = 1, 2. \] (38)

Substituting (38) into (37), from equalities (20)-(22) we deduce that

\[ K_{2,k} = (-1)^k K_{1,k}, \quad 2K_{1,k} = \begin{cases} \frac{12 - k^2}{k^2 - 10k^2 + 9} & k \text{ even}, \\ \frac{1}{4 - k^2} & k \text{ odd}, \end{cases} \quad k = 0, n_2. \] (39)

In view of the first equality (39) and taking into account (24), we can rewrite the equation (33) in the following way

\[ \tau \gamma K \mu T(y)J Q(z_1(\tau))A_1 + T(y)Q(z_1(\tau))A_1 + \frac{3\sqrt{\pi}}{8} = \frac{3}{2} T(y)G A_1. \] (40)

Here \( G \) is \((n_1 + 1) \times (n_1 + 1)(n_2 + 1)\) block matrix of the form

\[
G = \begin{pmatrix}
K_1 & 0 & 0 & 0 & \ldots & 0 \\
0 & K_1 & 0 & 0 & \ldots & 0 \\
0 & 0 & K_1 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & \ldots & 0 & K_1 \\
0 & 0 & \ldots & 0 & 0 & 0
\end{pmatrix}.
\] (41)

We choose the collocation points as the zeros of the Chebyshev polynomials \( T_{n_1+1}(y) \), \( T_{n_2+1}(2\tau + 1) \) from (25) for \( i = 1 \) \[10\]

\[ y_{j_1}^{(1)} = \cos \left( \frac{\pi \left( n_1 - j_1 + \frac{1}{2} \right)}{n_1 + 1} \right), \quad \tau_{j_2}^{(1)} = \frac{1}{2} \left( \cos \left( \frac{\pi \left( j_2 + \frac{1}{2} \right)}{n_2 + 1} \right) - 1 \right), \] (42)

\[ j_{1,2} = 0, n_{1,2} \text{ and } T_{n_1+1}(y), T_{n_2+1}(2\tau - 1) \text{ for } i = 2 \]

\[ y_{j_1}^{(2)} = \cos \left( \frac{\pi \left( j_1 + \frac{1}{2} \right)}{n_1 + 1} \right), \quad \tau_{j_2}^{(2)} = \frac{1}{2} \left( \cos \left( \frac{\pi \left( n_2 - j_2 + \frac{1}{2} \right)}{n_2 + 1} \right) + 1 \right). \] (43)
By substituting the collocation points (42) and (43) into (40), we obtain the system of matrix equations ($j_{1,2} = 0, n_{1,2}$) as follows ($i = 1, 2$)

\[
\Lambda_i(y_{k_1}, \tau_{k_2})A_i = -\frac{3\sqrt{\pi}}{8}, \quad (44)
\]

\[
\Lambda_i(y_{k_1}, \tau_{k_2}) = \left(\begin{array}{cc}
\Lambda_i(y_0, \tau_0) & \\
\Lambda_i(y_1, \tau_1) & \\
\vdots & \\
\Lambda_i(y_{n_{1}-1}, \tau_{n_{2}-1}) & \\
\Omega_i(y_0, \tau_0) & \\
\Omega_i(y_1, \tau_1) & \\
\vdots & \\
\Omega_i(y_{n_{1}-1}, \tau_{n_{2}-1}) & \\
\end{array}\right), \quad F = \left(\begin{array}{c}
\frac{3\sqrt{\pi}}{8} \\
\frac{3\sqrt{\pi}}{8} - \frac{3\sqrt{\pi}}{8} \\
\vdots & \\
\frac{3\sqrt{\pi}}{8} - \frac{3\sqrt{\pi}}{8} \\
0 & \\
0 & \\
\end{array}\right), \quad (49)
\]

Substituting (6) into (3) and taking into account (10) and (11), we have

\[
U_z(y) = \frac{\gamma KnG_p}{2\sqrt{\pi}} \left(1 - \frac{9\pi}{32}\right) \int_{-1}^{1} Z_2(y, \tau) \left(1 - \tau^2\right) d\tau + \frac{3\gamma KnG_p}{4} T(y) G A_1. \quad (50)
\]

Hence, the mass flux (50) is obtained as

\[
J_M = -\frac{\gamma KnG_p}{\sqrt{\pi}} \left(1 - \frac{9\pi}{32}\right) \left(\frac{3}{4} - \frac{1}{\sqrt{\pi}} \int_{-1}^{0} \left(1 - \tau^2\right) \exp\left(-\frac{y - 1}{\gamma Kn\tau}\right) d\tau\right) + \\
+ \frac{3\gamma KnG_p}{4} L G A_1. \quad (51)
\]

Here $L$ is the $1 \times (n_1 + 1)$ matrix as $L = \left(\begin{array}{c}
1 & 0 & \cdots & \frac{1}{3} \frac{1}{1 - n_1^2} \frac{1}{1 - (n_2 + 1)^2} 0
\end{array}\right)$ if $n_1$ is even and

$L = \left(\begin{array}{c}
1 & 0 & \cdots & \frac{1}{3} \frac{1}{1 - (n_1 - 1)^2} 0
\end{array}\right)$ if $n_1$ is odd.

Following [15, 16], we write the error of calculating the second term in (51) as

\[
E_n = I_n - I_{n/2}, \quad (52)
\]

where $I_n$ and $I_{n/2}$ are the values of this term for $n = (n_1, n_2)^T$ and $n/2$. 

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3. Results and discussion

The solution of the boundary value problem (1), (2) can also be solved by applying the method used in [4], which the expansion of the solution in the eigenfunctions can be reduced to solving the Riemannian boundary value problem. In [4] mass flux values were obtained depending on the parameter introduced as $D = 2b'/l_g$. This definition of $D$ is determined by the choice of the size scale $2b'$. In this case, the reduce dimensionless pressure gradient is calculated according to the formula $\hat{G}_p = \frac{2}{G_p} \frac{dp}{dz} = 2G_p$. The mass flow values $J_M/\hat{G}_p$ given by (51) with $Kn = 2/D$ are presented in the table 1. The comparison between values $J_M/\hat{G}_p$ calculated by the proposed method and method [4] are listed in the table 2. The results show that the values $J_M/\hat{G}_p$ obtained according to (51) for $D = 1$ and 10 with $n_1 = 2n_2 = 36$ and for $D = 0.1$ with $2n_1 = n_2 = 36$ are in good agreement with the corresponding values of mass flux given in [4]. The using the Chebyshev polynomials and collocation points (42) and (43) in the matrix equation (49) allows to reduce the calculations thus accelerating the convergence. In [4] for $D = 1$ a uniform grid consisting of 13500 cells was used. Numerical investigation according to (49) makes it possible to grid refinement near the ends of the interval of the definition of variables $y$ and $\tau$ thus minimizing the problem of Runge’s phenomenon [11]. It should also be noted that the proposed method can be used to solve the Williams equation in the Poiseuille flow problem in a long channel with a much complex cross section configuration. Analogous results of mathematical simulation of the mass transfer process in the channel between two parallel walls obtained in [2, 17, 18] by the discrete coordinate method using the BGK model of the Boltzmann kinetic equation (BGK), the S-model (S), the model with combined kernel (CES), and the Boltzmann linearized equation for hardsphere molecules (LBE). As can be seen from the table 2, the results obtained in this work are in good agreement with the analogous results from [2, 17, 18]. The differences can be explained by a weak dependence of the results on the collision integral model.

Table 1. The mass flux $-J_M/\hat{G}_p$ for various values of $D$

| $D$  | $n_1 = 2n_2 = 18$ | $n_1 = 2n_2 = 36$ | $2n_1 = n_2 = 18$ | $2n_1 = n_2 = 36$ |
|------|------------------|------------------|------------------|------------------|
| 0.5  | 1.6373           | 1.6372           | 0.1              | 2.1289           | 2.1295           |
| 1.0  | 1.5446           | 1.5446           | 0.2              | 1.8801           | 1.8807           |
| 2.0  | 1.5738           | 1.5738           |                 |                  |                  |
| 5.0  | 1.9448           | 1.9445           |                 |                  |                  |
| 10.0 | 2.7159           | 2.7151           |                 |                  |                  |

Table 2. Comparison of the mass flux $-J_M/\hat{G}_p$ for $D = 0.1, 1, 10$

| $D$  | W     | W          | BGK        | S     | CES   | LBE   |
|------|-------|------------|------------|-------|-------|-------|
|      | (51)  | [4]        | [2]        | [17]  | [18]  | [18]  |
| 0.1  | 2.1295| 2.130      | 2.0323     | 2.0780| 1.9259| 1.9499|
| 1.0  | 1.5446| 1.545      | 1.5387     | 1.5536| 1.4863| 1.5067|
| 10.0 | 2.7151| 2.715      | –          | 2.7799| 2.7220| 2.7296|
4. Conclusion
In the presented article we solved the Williams equation with diffuse boundary conditions by applying the Chebyshev polynomials. Thus, and after using a set of collocation points, the required numerical solution is found to be equivalent to the solution of a linear system of algebraic equations. The value of the mass flux per unit height of the channel has been calculated. The resultant expressions are analyzed numerically. From the conducted comparison, it turns out that the proposed method converges quickly and can be used to solve the Williams equation in the Poiseuille flow problem in a long channel with a more complex cross section configuration.

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