ABSTRACT. We consider a fixed contact 3-manifold that admits infinitely many compact Stein fillings which are all homeomorphic but pairwise non-diffeomorphic. Each of these fillings gives rise to a closed contact 5-manifold described as a contact open book whose page is the filling at hand and whose monodromy is the identity symplectomorphism. We show that the resulting infinitely many contact 5-manifolds are all diffeomorphic but pairwise non-contactomorphic. Moreover, we explicitly determine these contact 5-manifolds.

1. INTRODUCTION

Recent advances in symplectic geometry and topology showed that while some closed contact 3-manifolds have only finitely many Stein fillings, others have infinitely many, up to diffeomorphism (see [10] for a recent survey). Among the 4-manifold topologists, it is common to call a Stein filling of a contact 3-manifold exotic compared to another filling, if these two fillings are homeomorphic but non-diffeomorphic.

The first examples of a closed contact 3-manifold admitting infinitely many exotic simply-connected Stein fillings were discovered in [2]. The Stein fillings in that article, and many others which appeared in the literature since then, were given as Lefschetz fibrations over the disk whose boundary is a fixed open book supporting the contact manifold in question.

Recently, Akbulut and Yasui [1] constructed an infinite family of exotic simply-connected Stein fillings of a fixed contact 3-manifold, where the fillings (with $b_2 = 2$) are described by explicit handlebody diagrams, rather than Lefschetz fibrations. In this paper, we consider the infinite set of contact open books each of which has a fixed exotic Stein filling in their family as its page and identity as its monodromy, and show that the resulting closed contact 5-manifolds are all diffeomorphic but pairwise non-contactomorphic.

Moreover, we give two alternative arguments to distinguish the contact 5-manifolds: one is based on the Barden’s classification [3] of simply-connected closed 5-manifolds, and the other is based on the diagrammatic language developed for contact 5-manifolds in [6]. The advantage of the latter approach is that we can explicitly identify the contact 5-manifolds.

2. SIMPLY-CONNECTED EXOTIC STEIN FILLINGS

We briefly review the infinite family of exotic simply-connected Stein fillings of a fixed contact 3-manifold due to Akbulut and Yasui [1]. Let $X$ be the 4-manifold with boundary
described by the handlebody diagram on the left in Figure 1. Note that there is an embedded \( T^2 \times D^2 \) in the interior of \( X \). Throughout this section, let \( p \) denote a positive integer and let \( X_p \) be the result of \( p \)-log transform on the \( T^2 \times D^2 \subset X \). A handlebody diagram of \( X_p \) is presented on the right in Figure 1.

First of all, one observes that \( X \) and \( X_p \) are simply-connected, for all \( p \). To see this, cancel the upper 1-/2-handle pair and the lower 1-/2-handle pair to get a diagram consisting of only two 2-handles and no 1-handles for both \( X \) and \( X_p \). This already implies that \( b_2(X) = b_2(X_p) = 2 \), and allows one to easily compute the intersection forms of \( X \) and \( X_p \), which turns out to be unimodular and indefinite, for all \( p \). Such forms are classified, up to isomorphism, by their rank, signature and parity. The signature of all the forms are zero, the form of \( X \) is even, and the form of \( X_p \) is even if and only if \( p \) is odd. Thus, by Boyer’s generalization \[4\] of Freedman’s celebrated theorem \[7\], one concludes that \( X_p \) is homeomorphic to \( X_{p'} \) if and only if \( p \) and \( p' \) have the same parity and \( X_p \) is homeomorphic to \( X \) if and only if \( p \) is odd. Moreover, for all \( p \), \( \partial X = \partial X_p \) is a homology 3-sphere.

Next, one shows that \( X \) and \( X_p \) admit Stein structures, for all \( p \), by turning the smooth handlebody diagrams in Figure 1 into Legendrian handlebody diagrams (see Figure 2), after cancelling the upper 1-/2-handle pairs for convenience.

In particular, \( \partial X = \partial X_p \) admits a Stein fillable contact structure. Finally, one uses the adjunction inequality coupled with the genus function to distinguish the smooth structures on \( X_p \):

**Proposition 2.1** (Akbulut and Yasui). (i) There exists a contact structure \( \eta \) on \( \partial X \) such that for infinitely many values of \( q \geq 1 \), \( X_{2q-1} \) is a Stein filling of \((\partial X, \eta)\) with the property that all of these fillings are homeomorphic but pairwise non-diffeomorphic.

(ii) There exists a contact structure \( \eta' \) on \( \partial X \) such that for infinitely many values of \( q \geq 1 \), \( X_{2q} \) is a Stein filling of \((\partial X, \eta')\) with the property that all of these fillings are homeomorphic but pairwise non-diffeomorphic.
Let $M_p := OB(X_p, id)$ (resp. $M := OB(X, id)$) denote the closed contact 5-manifold viewed as the contact open book with page $X_p$ (resp. $X$) and monodromy the identity map. Let $\xi_p$ (resp. $\xi$) denote the contact structure supported by this open book in the sense of Giroux [8].

We start with some basic observations about the contact 5-manifolds $(M, \xi)$ and $(M_p, \xi_p)$. First of all, $(M, \xi)$ and $(M_p, \xi_p)$ are subcritically Stein fillable, and in fact $W := X \times D^2$ and $W_p := X_p \times D^2$ are their subcritical Stein fillings, respectively [6, Prop. 3.1].

Since $X$ and $X_p$ are simply-connected for all $p$, so are $W$ and $W_p$. Moreover, $M$ and $M_p$ are simply-connected for all $p$, as a result of the following simple but useful lemma.

Note that a compact Stein filling (a.k.a. a Stein domain) is a Weinstein domain (cf. [5]).

**Lemma 3.1.** If $V^{2n}$ is a Weinstein domain, then the inclusion map $i : \partial V \to V$ induces an isomorphism on $\pi_1$, provided that $n \geq 3$.

**Proof.** Fix a base point $b$ in the boundary $\partial V$. Suppose that $\gamma$ is a loop in $W$. We claim that $\gamma$ is homotopic to a loop in $\partial V$, or in other words, that $i$ induces a surjective map on $\pi_1$. First we perturb $\gamma$, a 1-dimensional CW-complex, to make it disjoint from the isotropic skeleton of $V$, which has dimension at most $n$. Then apply the Liouville flow, cut off near the boundary, to push $\gamma$ into a collar neighborhood of the boundary.

For injectivity on $\pi_1$, suppose that $\gamma_1$ and $\gamma_2$ are loops in $\partial V$ that are homotopic in $V$. Such a homotopy is a 2-dimensional CW-complex, so we can make it disjoint from the isotropic skeleton if $n \geq 3$. Then apply the Liouville flow to push the entire homotopy into a collar neighborhood of the boundary.

**Remark 3.2.** When $n = 2$, the map $i_* : \pi_1(\partial V) \to \pi_1(V)$ is surjective [11, Prop. 1.10], but not necessarily injective as, for instance, $T^2 \times D^2$ is a Stein (hence Weinstein) filling of the standard contact $T^3$.

**Lemma 3.3.** Let $S^2 \times S^3$ denote the non-trivial $S^3$-bundle over $S^2$. Then $M_p$ is diffeomorphic to either $S^2 \times S^3 \# S^2 \times S^3$ or $S^2 \times S^3 \# S^2 \times S^3$. 

![Figure 2. Stein handlebody diagrams for X and X_p](image-url)
Proof. For the subcritical Stein filling $W_p$, the homology sequence
\[
H_3(W_p, \partial W_p) \longrightarrow H_2(\partial W_p) \xrightarrow{i_*} H_2(W_p) \longrightarrow H_2(W_p, \partial W_p),
\]
of the pair $(W_p, \partial W_p)$ implies that $i_*$ is an isomorphism, since the homology of subcritical Weinstein manifolds vanishes in degree at least half the dimension. As a consequence we have
\[
H_2(M_p) \cong H_2(W_p) \cong H_2(X_p) \cong \mathbb{Z} \oplus \mathbb{Z}.
\]
The statement in the lemma follows from Barden’s classification of diffeomorphism classes of simply-connected closed 5-manifolds. Note that $S^2 \times S^3 \# S^2 \times S^3$ is diffeomorphic to $S^2 \times S^3 \# S^2 \times S^3$.

Next, in order to identify the contact 5-manifold $(M_p, \xi_p)$, for all $p$, we would like to determine the first Chern class $c_1(\xi_p)$ using the following general results. Let $c$ denote the total Chern class.

**Lemma 3.4.** Suppose that $(V, \omega)$ is a strong symplectic filling of some closed contact manifold $(Y, \xi)$. If $i : Y \to V$ denotes the inclusion map, then $i^* c(TV) = c(\xi)$.

**Proof.** There is a Liouville vector field $Z$ defined near $\partial V$ with the Liouville form $\lambda := i_Z \omega$. Its restriction $\alpha := \lambda|_{\partial V}$ defines a contact form. We get a map $\left( (\epsilon, 0] \times Y, d(e^\epsilon \alpha) \right) \to (V, \omega)$ by sending $(t, p)$ to $\text{Fl}^Z_t(i(p))$, where $\text{Fl}^Z$ denotes the flow induced by $Z$. This map preserves the symplectic structure. We conclude that $TV|_{\partial V = Y}$ symplectically splits as
\[
(\text{span}_\mathbb{R}(Z, R) \oplus \xi, \omega_0 \oplus \omega|_\xi).
\]
Here $R$ is Reeb vector field on the contact manifold $(Y, \alpha)$, and $\omega_0$ defines the standard symplectic form. Note that $Z, R$ forms a symplectic frame as $\omega(Z, R) = 1$. The rank 2 symplectic vector bundle $\epsilon = \text{span}_\mathbb{R}(Z, R)$ is clearly trivial, so we see that
\[
c(i^* TV) = c(\epsilon \oplus \xi) = c(\epsilon)c(\xi) = c(\xi).
\]

**Lemma 3.5.** Suppose $V^{2n}$ is a Weinstein domain with $n \geq 3$. Assume in addition that $W$ is subcritical for $n = 3$. Then the inclusion map $i : \partial V \to V$ induces an isomorphism on $H^2$. In particular, if $(Y, \xi)$ is the contact boundary of $V$, then $c_1(\xi)$ determines and is determined by $c_1(TV)$.

**Proof.** We just consider the long exact sequence of cohomology groups
\[
H^2(V, \partial V) \longrightarrow H^2(V) \xrightarrow{i^*} H^2(\partial V) \longrightarrow H^3(V, \partial V)
\]
of the pair $(V, \partial V)$ to conclude that $i^*$ is an isomorphism, assuming in addition that $V$ is subcritical for the case $n = 3$. Lemma 3.4 then shows the last claim.

Finally, we are ready to prove the main result of the paper.
Theorem 3.6. If \( p \) is odd, then \( M_p \) (and hence \( M \)) is diffeomorphic to \( S^2 \times S^3 \# S^2 \times S^3 \). If \( p \) is even, then \( M_p \) is diffeomorphic to \( S^2 \times S^3 \# S^2 \times S^3 \). Furthermore, \((M_p, \xi_p)\) is contactomorphic to \((M_{p'}, \xi_{p'})\) if and only if \( p = p' \).

Proof. We know that the first Chern class of any Stein surface can be calculated using an explicit Legendrian handlebody diagram representing the surface \([9, \text{Prop. 2.3}]\). Let \( \alpha_p, \beta_p, \) and \( \gamma_p \) be the Legendrian curves in the handlebody diagram for \( X_p \) depicted in Figure 2 and let \( T_p = [\gamma_p] \) and \( R_p = [\alpha_p] - p[\beta_p] \in H_2(X_p) \). As it was observed in \([11]\), \( H_2(X_p) \) has a basis consisting of \( T_p \) and \( S_p \), where

\[
S_p = \begin{cases} 
R_{q-1} + ((2q - 1)^2 - q + 1)T_{2q-1} & : p = 2q - 1 \\
R_{2q} + ((2q)^2 - q + 1)T_{2q} & : p = 2q 
\end{cases}
\]

It is easy to see that \( c_1(TX_p) \) evaluates on these homology classes as:

\[
\langle c_1(TX_p), S_p \rangle = -1 - p, \quad \langle c_1(TX_p), T_p \rangle = 0.
\]

The first Chern class \( c_1(\xi_p) \) can be viewed as a linear map from \( H^2(M_p) \cong \mathbb{Z} \oplus \mathbb{Z} \) to \( \mathbb{Z} \). The cycles \( S_p \times \{1\} \), and \( T_p \times \{1\} \) in \( M_p = \partial(X_p \times D^2) \), where we think of \( 1 \in \partial D^2 \), generate \( H_2(M_p) \). Using Lemma 3.5, we conclude that \( c_1(\xi_p) \) evaluates as \((-1 - p, 0)\) with respect to the chosen basis of \( H_2(M_p) \), which is sufficient to mutually distinguish \( \xi_p \)’s.

The calculation above shows that \( M_p \) is spin for odd \( p \) and non-spin otherwise. We conclude, by Lemma 3.3, that \( M_p \) is diffeomorphic to \( S^2 \times S^3 \# S^2 \times S^3 \) for odd \( p \), and to \( S^2 \times S^3 \# S^2 \times S^3 \) if \( p \) is even. \( \square \)

We can say more explicitly what contact manifold we get with the following lemma.

Lemma 3.7. \([6, \text{Prop. 4.5}]\) Suppose that \( V \) is a Stein surface obtained by attaching a single 2-handle to the standard Stein domain \( D^4 \) along a Legendrian knot \( \delta \) in the standard tight \( S^3 \). Then \( OB(V, \text{id}) \) is diffeomorphic to

- \( S^2 \times S^3 \) if \( \text{rot}(\delta) \) is even
- \( S^2 \times S^3 \) if \( \text{rot}(\delta) \) is odd.

Moreover, if \( V' \) is another Stein surface obtained as above using a Legendrian knot \( \delta' \), then \( OB(V, \text{id}) \) and \( OB(V', \text{id}) \) are contactomorphic if and only if \( |\text{rot}(\delta)| = |\text{rot}(\delta')| \).

For each integer \( k \), choose a Legendrian unknot \( \delta_k \) in the tight \( S^3 \) with rotation number equal to \( k \). We view this \( S^3 \) as the boundary of the Stein domain \( D^4 \). Define \( V_k \) as the handlebody obtained by attaching a Weinstein 2-handle to \( D^4 \) along \( \delta_k \).

Then \( \partial(V_k \times D^2) \) is an \( S^3 \)-bundle over \( S^2 \), and it is diffeomorphic to \( S^2 \times S^3 \) if \( k \) is even and to \( S^2 \times S^3 \) if \( k \) is odd. Up to contactomorphism, the contact structure only depends on \( |k| \) by Lemma 3.7. We denote the resulting contact structure by \( \zeta_k \).

Proposition 3.8. The contact 5-manifold \((M_p, \xi_p)\) is contactomorphic to

- \((S^2 \times S^3, \zeta_0) \# (S^2 \times S^3, \zeta_{p+1})\) if \( p \) is odd
- \((S^2 \times S^3, \zeta_0) \# (S^2 \times S^3, \zeta_{p+1})\) if \( p \) is even.

Proof. A complete argument is given in the proof of Proposition 4.3. \( \square \)
4. An alternative argument

As in [6] we encode a closed contact 5-manifold (described as a contact open book) by a handlebody diagram for its page—which can be assumed to be a compact Stein domain. We will only consider situations where the symplectic monodromy is a product of Dehn twists along Lagrangian spheres.

Definition 4.1. Two Stein surfaces $X$ and $X'$ are called contact stably equivalent if there are handlebody diagrams for $X$ and $X'$, and a third handlebody diagram for some Stein surface $X''$ with the property that the handlebody diagrams for $X$ and $X'$ can be transformed into the one for $X''$ by a finite sequence of the following moves:

- usual handlebody moves for Stein surfaces (cf. [9])
- stabilizing the attaching circles for the 2-handles (move I)
- changing a crossing (move II)

See Figure 3 for a description of moves I and II.

![Figure 3. Moves I and II give contactomorphic 5-manifolds](image)

With the arguments from [6, Section 4.1] we obtain the following proposition.

Proposition 4.2. Suppose that $X$ and $X'$ are contact stably equivalent. Then $X \times D^2$ and $X' \times D^2$ are symplectically deformation equivalent with contactomorphic boundaries.

We briefly summarize [6, Section 4.1] and explain why moves I and II give rise to contactomorphic manifolds. Given a contact open book $OB(W, id)$ we first positively stabilize the open book. This is done by first taking any properly embedded Lagrangian disk $D$. We attach a Weinstein 2-handle along $\partial D$ to obtain a new page $\tilde{W}$, which contains a Lagrangian sphere $L$ formed by gluing the core of the 2-handle to the Lagrangian disk $D$. We can then perform a right-handed Dehn twist along $L$. We denote this Dehn twist by $\tau_L$. According to Giroux, the contact open book $OB(\tilde{W}, \tau_L)$ is contactomorphic to $OB(W, id)$. We use the extra 2-handle in the page to perform a handle slide. We then destabilize the open book by simply not performing the Dehn twist and removing the extra 2-handle. Since the monodromy on $W$ was assumed to be the identity this last step can be done. See [6, Section 4.1] for a detailed description of moves I and II and a proof that these moves do not change the symplectic deformation type of the filling.

The upshot is that move I changes the Thurston-Bennequin invariant of an attaching circle of a 2-handle while preserving its rotation number. Move II changes an overcrossing into an undercrossing. This move can be used to change the knot type of the attaching circles.
Proposition 4.3. The contact 5-manifolds \((M_p, \xi_p)\) and \((M_{p'}, \xi_{p'})\) are diffeomorphic if and only if \(p \equiv p' \pmod{2}\), and contactomorphic if and only if \(p = p'\).

Proof. In the following, we refer to [9, Figures 3 and 9] for Legendrian Reidemeister moves. We consider the handlebody diagram of \(X_p\) in Figure 2 and apply a Legendrian isotopy to obtain step 1 in Figure 4. Then we apply move II simultaneously to the two overcrossings in the shaded region in step 1. Next we apply a Legendrian Reidemeister move 2 to the indicated cusp in step 2 and another Legendrian Reidemeister move 2 to the indicated cusp in step 3. We do this twice more, and then apply Legendrian Reidemeister move 4 to move the Legendrian knot \(\gamma_p\) off the 1-handle. Finally, by several applications of Reidemeister move 2 we obtain step 5 of Figure 4.

We now perform some moves so that we will be able to cancel the 1-handle. Note that the dotted curve near \(\beta_p\) depicted in the initial diagram in Figure 5 is not part of the handlebody and it will be used for a handle-slide. First we perform an isotopy of \(\alpha_p\), and then in step 2, we slide the \(\alpha_p\) handle over the \(\beta_p\) handle using handle subtraction. We keep calling the resulting curve the \(\alpha_p\)-curve. Note that its rotation number has decreased by 1 after the handle subtraction.

Now we use move II to unlink the curves \(\alpha_p\) and \(\beta_p\). We get additional cusps which we remove with the inverse of move I. This move keeps the rotation number unchanged. Now we perform Legendrian Reidemeister move 4 to move one strand of the curve \(\alpha_p\) off the 1-handle.

We repeat this procedure until there are no strands of \(\alpha_p\) left which is going over the 1-handle. To do this, we apply Reidemeister move 2 to the indicated cusp and apply an isotopy to move this cusp in the position of step 1. With move II we can make the curve \(\alpha_p\) into an unknot without changing the rotation number, so we end up with an unknotted curve whose rotation number is equal to \(-1 - p\). With Legendrian Reidemeister move 4 we make the curve \(\alpha_p\) disjoint from the 1-handle, and then cancel the 1-handle with the 2-handle attached along \(\beta_p\). We end up with a handlebody diagram containing only 2-handles attached along the curve \(\gamma_p\) with rotation number 0 and to the curve \(\alpha_p\) which has rotation number \(-1 - p\).

Now apply Lemma 3.7 to see that \(M_p\) is contactomorphic to

- \((S^2 \times S^3, \xi_0)\#(S^2 \times S^3, \xi_{p+1})\) if \(p\) is odd
- \((S^2 \times S^3, \tilde{\xi}_0)\#(S^2 \times S^3, \xi_{p+1})\) if \(p\) is even.

By forgetting the contact structure we get the diffeomorphism claim.

\[\square\]

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Figure 4. Constructing a contactomorphism between 5-manifolds

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Figure 5. Sliding strands off the 1-handle

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Department of Mathematics, Koç University, Rumelifeneri Yolu, 34450, Sariyer, Istanbul, Turkey
E-mail address: bozbagci@ku.edu.tr

Department of Mathematics and Research Institute of Mathematics, Seoul National University, Building 27, room 402, San 56-1, Sillim-dong, Gwanak-gu, Seoul, South Korea, Postal code 151-747
E-mail address: okoert@snu.ac.kr