Torsional vibrations of circular poroelastic plates

S Ahmed Shah¹ and C Nageswara Nath²
¹ Department of Mathematics, Deccan College of Engineering and Technology, Hyderabad 500 001, India
² Department of Mathematics, CMR Technical Campus, Hyderabad - 501 401, India
E-mail: ahmed_shah67@yahoo.com, nagesh.nath@rediffmail.com

Abstract. Torsional vibrations of annular poroelastic circular plates are studied in the framework of Biot’s theory of wave propagation in porous solids. The frequency equation of torsional vibrations is same for pervious and impervious surfaces. The frequency equation of torsional vibrations is obtained by using the traction free boundaries and edges of the annular circular plate. Non-dimensional frequency of annular plate is computed as a function of aspect ratio. The frequency equation is discussed for first two modes. Frequency equation of torsional vibrations of uniform circular poroelastic plate is obtained as a particular case of the annular plate. Resonant frequency of infinite poroelastic plate is obtained as a limiting case of annular plate. The expressions for phase velocity and attenuation are obtained and these are computed for two different poroelastic materials as a function of frequency in presence of dissipation. Phase velocity is almost same for the considered poroelastic materials and dissipations. Results of previous study are obtained as a particular case of the present investigation.

1. Introduction
Propagation of waves and vibrations in elastic plates has been extensively studied in monograph prepared by Mindlin [8]. The study of torsional vibrations of an elastic solid is important in several fields of engineering, for example, soil mechanics, transmission of power through shafts with flange at the end as integral part of the shaft. It is now recognized that virtually no high-speed equipment can be properly designed without obtaining solution to torsional vibrations problems. Examples of torsional vibrations are vibrations in gear train and motor-pump shafts. Thus, from engineering point of view the study of torsional vibrations has greater interest. Such vibrations, for example, are used in delay lines. The least mode of torsional vibration is dispersionless and is used for storing information in form of pulses. It receives a pulse from a transmitter and returns the pulse to the receiving end, essentially undistorted but delayed by a time interval which is determined by the path length and velocity. The memory properties of delay lines are used extensively in computers. Further, based on reflections and refractions during the propagation of a pulse imperfection can be identified. Other use of torsional vibrations is the measurement of the shear modulus of a crystal. These modes find applications in the design of resonators, transducers and sensors. Tajuddin and Ahmed Shah [9] studied torsional vibrations of hollow poroelastic cylinders in presence of dissipation. They obtained phase velocity, group velocity and attenuation of torsional vibrations for different dissipations. Ahmed Shah [1,2] studied the axially symmetric vibrations of fluid-filled poroelastic circular cylindrical shells and spherical shells of various wall-thicknesses. Gupta et al.[6] investigated the effect of presence of...
rigid boundary on the propagation of torsional surface wave in a homogeneous layer over a semi infinite heterogeneous half space with an irregularity at the interface. Ahmed Shah and Tajuddin [3] studied torsional vibrations of poroelastic spheroidal shells. Torsional vibrations of circular elastic plates with thickness steps are investigated by Kang et al.[7]. Zhou et al.[11] studied three dimensional vibrations analysis of circular and annular plates by using Chebyshev-Ritz method.

In the present analysis, the frequency equation of torsional vibrations of a homogeneous and isotropic poroelastic annular circular plate of uniform thickness is derived and then discussed. Let the boundaries of the annular plate are free from stress. The frequency equation is discussed for first two modes of vibrations. Frequency is computed as a function of aspect ratio for the considered poroelastic materials. The expressions for non-dimensional phase velocity and attenuation are obtained in presence of dissipation for an infinite poroelastic plate. Phase velocity and attenuation are computed as a function of frequency for two types of poroelastic materials and then discussed. The phase velocity is almost same for the considered poroelastic materials and different dissipations. Attenuation varies widely for the considered dissipation for the considered poroelastic materials and different dissipations. Attenuation varies widely for the considered dissipations.

2. Basic equations

The equations of motion of a homogeneous, isotropic poroelastic solid [4] in presence of dissipation \( b \) are

\[
N \nabla^2 \vec{u} + (A + N) \nabla e + Q \nabla \epsilon = \frac{\partial^2}{\partial t^2} \left( \rho_{11} \vec{u} + \rho_{12} \vec{U} \right) + b \frac{\partial}{\partial t} (\vec{u} - \vec{U}),
\]

\[
Q \nabla e + R \nabla \epsilon = \frac{\partial^2}{\partial t^2} \left( \rho_{12} \vec{u} + \rho_{22} \vec{U} \right) - b \frac{\partial}{\partial t} (\vec{u} - \vec{U}).
\]

(1)

where \( \nabla^2 \) is the Laplace operator, \( \vec{u}(u, v, w) \) and \( \vec{U}(U, V, W) \) are displacements of solid and liquid respectively, \( e \) and \( \epsilon \) are the dilatations of solid and liquid; \( A, N, Q, R \) are all poroelastic constants and \( \rho_{ij} \) \((i,j=1,2)\) are the mass coefficients [4]. The poroelastic constants \( A, N \) correspond to familiar Lame constants in purely elastic solid. The coefficient \( N \) represents the shear modulus of the solid. The coefficient \( R \) is a measure of the pressure required on the liquid to force a certain amount of the liquid into the aggregate while total volume remains constant. The coefficient \( Q \) represents the coupling between the volume change of the solid to that of liquid. The stresses \( \sigma_{ij} \) and the liquid pressure \( s \) of the poroelastic solid are

\[
\sigma_{ij} = 2N e_{ij} + (Ae + Q \epsilon) \delta_{ij}, (i,j = r, \theta, z),
\]

\[s = (Qe + R \epsilon) \] (2)

where \( \delta_{ij} \) is the well-known Kronecker delta function and \( e_{ij} \) are the strain components of poroelastic solid.

3. Secular equation

Let \((r, \theta, z)\) be the cylindrical polar coordinates. Consider a homogeneous, isotropic annular circular poroelastic plate with inner and outer radii \( r_1 \) and \( r_2 \), respectively, whose axis is in the direction of z-axis and uniform thickness \( h_2 \). The flat surfaces of the plate lie at \( z=0 \) and \( z=h_2 \). The boundaries of the isotropic poroelastic plate are free from stress. The only non-zero displacement component of the solid and liquid media are \( v \) and \( V \) respectively. These displacements are functions of \( r, z \) and time, \( t \). Then the equations of motion of poroelastic solid [4], that is equation (1) reduces to

\[
N(\nabla^2 - \frac{1}{r^2})v = \frac{\partial^2}{\partial t^2} (\rho_{11} v + \rho_{12} V) + b \frac{\partial}{\partial t} (v - V),
\]

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\begin{equation}
0 = \frac{\partial^2}{\partial t^2} (\rho_{12} v + \rho_{22} V) - b \frac{\partial}{\partial t} (v - V).
\end{equation}

Let the propagation mode shapes of solid and liquid \( v \) and \( V \) are
\begin{align}
v &= f_1(r)f_2(z)e^{i\omega t}, \\
V &= F_1(r)F_2(z)e^{i\omega t},
\end{align}

where \( t \) is time, \( \omega \) is circular frequency, and \( i \) is the complex unity. Substitution of equation (4) into equation (3) yield
\begin{equation}
N[f''_1 f_2 + \frac{1}{r} f'_1 f_2 + f'_1 f''_2 - \frac{1}{r^2} f_1 f_2] = -\omega^2[K_{11} f_1 f_2 + K_{12} F_1 F_2],
\end{equation}

\begin{equation}
0 = -\omega^2[K_{12} f_1 f_2 + K_{22} F_1 F_2].
\end{equation}

In equation (5), a dash over \( f_1 \) represents differentiation with respect to \( r \), and a dash over \( f_2 \) represents differentiation with respect to \( z \), and
\begin{align}
K_{11} &= \rho_{11} - \frac{ib}{\omega}, \\
K_{12} &= \rho_{12} + \frac{ib}{\omega}, \\
K_{22} &= \rho_{22} - \frac{ib}{\omega}.
\end{align}

From second equation of (5), we get
\begin{equation}
F_1 F_2 = -\frac{K_{12}}{K_{22}} f_1 f_2.
\end{equation}

Substitution of equation (7) into the first equation of (5) we obtain
\begin{equation}
f''_1 f_2 + \frac{1}{r} f'_1 f_2 + f'_1 f''_2 - \frac{1}{r^2} f_1 f_2 + \xi_3^2 f_1 f_2 = 0,
\end{equation}

where
\begin{equation}
\xi_3^2 = \frac{\omega^2}{V_3^2} = \frac{\omega^2(K_{11} K_{22} - K_{12}^2)}{N K_{22}}.
\end{equation}

In equation (9), \( V_3 \) is shear wave velocity [4]. By using the method of separation of variables, equation (8) is simplified to
\begin{equation}
\frac{f''_1}{f_1} + \frac{1}{r} \frac{f'_1}{f_1} + \frac{\xi_3^2 f_1}{f_1} = -\frac{f''_2}{f_2} = k^2 (\text{let}).
\end{equation}

Then equation (10) is reduced to two separate equations given below
\begin{align}
\left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} + (\alpha_3^2 - \frac{1}{r^2}) \right] f_1 &= 0, \\
\left[ \frac{d^2}{dz^2} + k^2 \right] f_2 &= 0.
\end{align}

In equation (11), \( \alpha_3^2 \) is defined as
\begin{equation}
\alpha_3^2 = \xi_3^2 - k^2.
\end{equation}
Solution of equation (11) is [bounded solution in case of \( \alpha_3 = 0 \)]

\[
f_1(r) = \begin{cases} 
  [C_1 J_1(\alpha_3 r) + C_2 Y_1(\alpha_3 r)], & \text{when } \alpha_3 \neq 0, \\
  [C_1 r], & \text{when } \alpha_3 = 0.
\end{cases}
\] (14)

where \( C_1 \) and \( C_2 \) are the constants, \( J_1 \) and \( Y_1 \) are Bessel functions of first and second kind respectively each of order one. The solution of equation (12) is

\[
f_2(z) = \begin{cases} 
  [D_1 \cos(kz) + D_2 \sin(kz)], & \text{when } k \neq 0, \\
  [D_1 z + D_2], & \text{when } k = 0.
\end{cases}
\] (15)

where \( k \), \( D_1 \) and \( D_2 \) are the constants. Substituting equations (14) and (15) into the first equation of (4), the propagation modes shapes are given by

\[
v(r, z, t) = [C_1 J_1(\alpha_3 r) + C_2 Y_1(\alpha_3 r)][D_1 \cos(kz) + D_2 \sin(kz)]e^{i\omega t},
\] (16)

for \( \alpha_3 \neq 0 \) and \( k \neq 0 \). From equation (4), it can be seen that the normal strains \( e_{rr}, e_{\theta\theta}, e_{zz} \) all are zero. Therefore the dilatations of solid and liquid media each is zero. Hence the liquid pressure \( p \) is identically zero. Thus for torsional vibrations no distinction between a pervious and an impervious surface is made. Considering the edges and surfaces of the poroelastic plate to be stress free, the frequency equation obtained for torsional vibrations is same for both pervious and impervious surfaces. Following equation (2), the only non-zero stresses are

\[
\sigma_{r\theta} = N\left( \frac{\partial v}{\partial r} - \frac{v}{r} \right), \\
\sigma_{z\theta} = N\frac{\partial v}{\partial z}.
\] (17)

Substituting equation (16) into equation (17), these stresses are simplified to

\[
\sigma_{r\theta} = -N[C_1 J_2(\alpha_3 r) + C_2 Y_2(\alpha_3 r)][D_1 \cos(kz) + D_2 \sin(kz)]e^{i\omega t},
\] (18)

\[
\sigma_{z\theta} = Nk[C_1 J_1(\alpha_3 r) + C_2 Y_1(\alpha_3 r)][-D_1 \sin(kz) + D_2 \cos(kz)]e^{i\omega t}.
\] (19)

4. Boundary conditions and frequency equation

The stress free boundary conditions for torsional vibrations at the inner and outer edges and flat surfaces of the poroelastic annular circular plate are

\[
\sigma_{r\theta} = 0, \text{ at } r = r_1, r_2,
\] (20)

\[
\sigma_{z\theta} = 0, \text{ at } z = 0, h_2,
\] (21)

\[
s = 0, \frac{\partial s}{\partial r} = 0, \text{ at } r = r_1, r_2 \text{ and } z = 0, h_2.
\] (22)

Equations (20) and (21) together with first equation of (22) are to be satisfied for a pervious surface while equations (20) and (21) together with second equation of (22) are to be satisfied for an impervious surface. Since the considered vibrations are shear vibrations, the dilatations of solid and liquid media each is zero, thereby liquid pressure \( s \) developed in solid-liquid aggregate will be identically zero and no distinction between pervious and impervious surface is made. Thus equation (22) is satisfied identically. Equations (18) and (20) together yield the frequency equation of torsional vibrations

\[
J_2(\alpha_3 r_1)Y_2(\alpha_3 r_2) - J_2(\alpha_3 r_2)Y_2(\alpha_3 r_1) = 0,
\] (23)
by eliminating the constants $C_1$ and $C_2$. Equation (23) is the frequency equation of torsional vibrations of hollow poroelastic cylinder studied by Tajuddin and Ahmed Shah [9]. Similarly, employing equation (19) into equation (21), after elimination of the constants $D_1$ and $D_2$, we get the equation $\sin(kh_2) = 0$.

Now we consider $\sin(kh_2) = 0$, or $kh_2 = q\pi$, $q = 1, 2, 3, \ldots$ then we have

$$k = \frac{q\pi}{h_2}, q = 1, 2, 3, \ldots \tag{24}$$

Let $r_2 - r_1 = h_1$ and $g = r_2/r_1$ so that $h_1/r_1 = g - 1$ and $h_1/r_2 = (g - 1)/g$. Also let the aspect ratio be $h_2/h_1 = h$. Let

$$R_n^2 = \frac{\alpha h_1^2}{\beta}, \tag{25}$$

then equation (23) is reduced to

$$J_2\left(\frac{R_n g}{g - 1}\right)Y_2\left(\frac{R_n g}{g - 1}\right) - J_2\left(\frac{R_n g}{g - 1}\right)Y_2\left(\frac{R_n g}{g - 1}\right) = 0. \tag{26}$$

Substituting equation (24) into equation (13) and after necessary simplification, the resonant frequencies of the torsional vibrations of poroelastic annular plate are given by the equation

$$\omega = V_3\sqrt{\frac{R_n^2}{h_1^2} + \frac{q^2\pi^2}{h_2^2}}, \tag{27}$$

In equation (27), $R_n$ is the $n$th root of equation (26) for different values of $g$. By eliminating liquid effects from equation (27), the results of Kang et al. [7] are recovered as a particular case. Under the limiting conditions when $r_1 \to 0$ then $r_2 \to h_1$ and the poroelastic annular circular plate reduces to circular uniform poroelastic plate of thickness $h_2$. Under these conditions, equation (23) reduce to

$$J_2(\alpha h_1) = 0. \tag{28}$$

And the resonant frequencies are given by the equation

$$\omega = V_3\sqrt{\frac{R_n^2}{r_2^2} + \frac{q^2\pi^2}{h_2^2}}, \tag{29}$$

where $R_n$ is the $n$th root of equation (28). When $r_2 \to \infty$ the uniform circular poroelastic plate reduces to infinite poroelastic plate of thickness $h_2$. Thus frequency equation (29) is reduced to

$$\omega = \frac{V_3q\pi}{h_2}, \tag{30}$$

which is independent of $R_n$. Equation (30) gives the resonant frequencies of torsional vibrations of an infinite poroelastic plate of thickness $h_2$. From equation (24), it is clear that $k$ is wavenumber. Thus substituting equation (24) into the equation (30), it reduces to

$$\omega = kV_3. \tag{31}$$

To analyze the frequency equations (27) and (31), it is convenient to introduce the following non-dimensional parameters:

$$a_4 = \frac{N}{H} = \left(\frac{V_0}{V_3}\right)^2, h = \frac{h_2}{h_1},$$
\[ m_{11} = \frac{\rho_{11}}{\rho}, \quad m_{12} = \frac{\rho_{12}}{\rho}, \quad m_{22} = \frac{\rho_{22}}{\rho}, \quad b_1 = \frac{bh_2}{\rho C_0}, \quad \Omega = \frac{\omega h_2}{C_0}, \quad (32) \]

where \( b_1, \omega \) and \( h \) are non-dimensional dissipation, frequency and aspect ratio respectively, \( V_0^2 = \frac{H}{\rho}, C_0^2 = \frac{N}{\rho} \) are the reference velocities, \( \rho = \rho_{11} + 2\rho_{12} + \rho_{22} \) and \( H=P+2Q+R \) with \( P=A+2N \).

By using equation (32) into equation (27), it is simplified to

\[ \Omega = \frac{(R_3^2 h_2^2 + q^2 \pi^2)^{\frac{1}{2}}}{(a_4^2)^{\frac{1}{2}}}. \quad (33) \]

Now we employ equation (32) into equation (31) and with the help of equations (6) and (9), simplify it in the form

\[ \frac{N k^2}{\rho \omega^2} = E_r - iE_i, \quad (34) \]

where \( E_r \) and \( E_i \) are

\[ E_r = \frac{\Omega^2 m_{22} (m_{11} m_{22} - m_{12}^2) + b_1^2}{\Omega^2 m_{22}^2 + b_1^2}, \]

\[ E_i = \frac{\Omega b_1 (m_{12} + m_{22})^2}{\Omega^2 m_{22}^2 + b_1^2}. \quad (35) \]

5. Phase velocity and attenuation

Poroelastic medium is dissipative in nature and thus the wavenumber \( k \) is complex. The waves generated obey diffusion type process and therefore get attenuated. Let \( k = k_r + ik_i \), where \( k_r \) is real and \( k_i \) is the imaginary part of the wavenumber \( k \). The real and imaginary part of the wavenumber corresponds to propagation and attenuation of waves. Hence the phase velocity \( C_p \) and attenuation \( X_h \) [4] are

\[ C_p = \frac{\omega}{|k_r|}, \quad X_h = \frac{1}{|k_i|}. \quad (36) \]

By substituting \( k = k_r + ik_i \) in equation (34) and using equation (32), we can separate the real and imaginary parts of the wavenumber as

\[ k_r = \frac{1}{\sqrt{2h_2}} (B_3 + B_4)^{\frac{1}{2}}, \quad k_i = \frac{1}{\sqrt{2h_2}} (B_3 - B_4)^{\frac{1}{2}}, \quad (37) \]

where the expressions for \( B_3 \) and \( B_4 \) are

\[ B_3 = \Omega^2 (E_r^2 + E_i^2)^{\frac{1}{2}}, \quad B_4 = \Omega^2 E_r. \quad (38) \]

In equation (38), \( E_r \) and \( E_i \) are defined in equation (35). Thus the non-dimensional phase velocity \( \frac{C_p}{C_0} \) and attenuation \( \frac{x h_2}{h_2} \) are

\[ \frac{C_p}{C_0} = \sqrt{2} \Omega (B_3 + B_4)^{-\frac{1}{2}}, \quad \frac{x h_2}{h_2} = \sqrt{2} (B_3 - B_4)^{-\frac{1}{2}}, \quad (39) \]

where \( B_3 \) and \( B_4 \) are defined in equation (38).
6. Results and discussion

Two types of poroelastic materials are considered to carry out the computational work, one is sandstone saturated with kerosene, say Material-I [5], the other one is sandstone saturated with water, say Material-II [10], whose non-dimensional physical parameters are given in Table 1. For a given poroelastic material, the non-dimensional frequency is computed as a function of aspect ratio, phase velocity and attenuation are determined as a function of non-dimensional frequency. The different dissipation parameters ($b_1$) chosen are 0.01, 0.1 and 1. Three values of $g$ have been considered for the purpose of computation. These values are $g=1.034, 3$ and $g \to \infty$. These three cases physically represent the results related to poroelastic thin annular plate, poroelastic thick annular plate and uniform circular poroelastic plate, respectively. The first two roots of equation (26) for thin annular plate, thick annular plate and uniform circular plate are 3.1423, 6.2835; 3.736, 6.6477; and 5.1356, 8.4172, respectively. The frequency is obtained for first two roots (modes) of the equation (26) for the considered poroelastic materials with $q=1, 2, 3, 4$ and 5 and presented in figure 1 -figure 6. Phase velocity and attenuation of an infinite poroelastic plate are presented as a function of frequency in figure 7 and figure 8 respectively, for different dissipations.

| Material/Parameter | $a_4$  | $\tilde{z}$ | $m_{11}$ | $m_{12}$ | $m_{22}$ |
|--------------------|--------|-------------|----------|----------|----------|
| Material-I         | 0.234  | 3.851       | 0.901    | -0.001   | 0.101    |
| Material-II        | 0.412  | 2.129       | 0.877    | 0        | 0.123    |

Figure 1. Frequency as a function of aspect ratio (Mat-I, Thin annular plate).

Figure 2. Frequency as a function of aspect ratio (Mat-I, Thick annular plate).

Figure 1 shows the frequency as a function of aspect ratio for thin annular poroelastic plate made of Material-I. The frequency is presented for first two modes of vibrations. From figure 1 it is clear that as the aspect ratio increases, the frequency increases gradually otherwise it is almost linear. The frequency for second mode is higher than that of the frequency of first mode. But it is interesting to note that frequency of first and second mode is same in for all the case. Variation of frequency in case of thick annular plate (Fig.2) and uniform circular plate...
(Fig.3) made of Material-I is similar to that of thin annular plate. The variation of frequency for thin, thick annular plates and uniform circular plate made of Material-II (Fig.4-Fig.6) is same as that of respective plates made of Material-I. The frequency is same for considered plates made of Material-I and Material-II. Hence, it is noted that the presence of mass-coupling parameter does not affect the frequency for torsional vibrations.

The phase velocity of torsional vibrations of an infinite poroelastic plate for the Materials-I and II for different dissipations is presented in figure 7. It is clear that, the phase velocity in case of Material-I is slightly higher for the dissipation $b_1=0.01$. And as the dissipation increase, the phase velocity decrease to a small extent. The same is true in case of Material-II also. There is increase in phase velocity in $(0, 2)$ and beyond it is almost constant. The phase velocity is same for Materials-I and II for different dissipations. Again here we see that the presence of mass-coupling parameter is not affecting the phase velocity. The attenuation of torsional vibrations of an infinite poroelastic plate for Materials-I and II is presented in figure 8 for different dissipations. The attenuation increase in $(0, 2)$ that then it is almost linear in case of Material-I and $b_1=0.01$. The same trend is observed for $b_1=0.1$ and 1. But as the dissipation
increases, the attenuation decreases. A similar trend is seen in the variation of attenuation in case of Material-II for different dissipations. The attenuation is same for Materials-I and II for $b_1=0.1$. When $b_1=0.01$, the attenuation in poroelastic plate made of Material-I is higher than that of poroelastic plate made of Material-II for $\Omega \geq 2$. A similar behavior is observed in $(0, 5)$ for $b_1=1$. Also we see that the presence of mass-coupling parameter is increasing the attenuation in particular range of frequency.

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