On \( p \)-frame potentials of determinantal point processes on the sphere

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Abstract

This study investigates the expectations of the \( p \)-frame potentials of three typical types of determinantal point processes on the \( d \)-sphere: (i) spherical ensembles on the 2-sphere; (ii) harmonic ensembles on the \( d \)-sphere; and (iii) jittered sampling point processes on the \( d \)-sphere. By the results, it can be expected that such determinantal point processes have asymptotically smaller expectations of \( p \)-frame potentials than those of Poisson point processes on the sphere with the same number of points.

Keywords \( p \)-frame potential, determinantal point process, spherical ensemble, harmonic ensemble, jittered sampling point process

Research Activity Group Discrete Systems

1. Introduction

This study investigates random point configurations generated by determinantal point processes (DPPs) on the \( d \)-sphere \( S^d = \{x = (x_1, \ldots, x_{d+1}) \in \mathbb{R}^{d+1} \mid \|x\| = 1\} \), where \( \|x\| = \sqrt{\langle x, x \rangle} \) for the Euclidean norm and \( \langle x, y \rangle = \sum_{i=1}^{d+1} x_i y_i \) for the standard inner product.

DPPs are models in which a repulsive force acts among distinct points, distributing points more evenly [1]. Because of this property, numerous studies have been conducted on DPPs. For example, recently, applications to quasi-Monte Carlo methods have attracted much attention in numerical analysis [2–4]. Furthermore, there is a trend to study finite subsets on the \( d \)-sphere from the viewpoint of discrete potential energies [5]. Therefore, this study focuses on the most interesting discrete potential energy called the \( p \)-frame potential introduced by Ehler and Okoudjou [6].

Definition 1 Let \( N \) be a positive integer and \( p \) be a positive real number. Given a finite subset \( \{x_i\}_{i=1}^{N} \) of \( S^d \), the \( p \)-frame potential is the functional

\[
FP_{p,N}(\{x_i\}_{i=1}^{N}) = \sum_{i,j=1}^{N} \|\langle x_i, x_j \rangle\|^p.
\]

These potentials have been studied regarding finite unit norm tight frames for \( p = 2 \) [7] and in the context of spherical designs, which is also called a spherical cubature, for even integers \( p \) [8]. Moreover, recently, Bilyk et al. [5] classified its minimizers for several dimensions and some ranges of \( p \).

This study calculates (or estimates) the expectations of \( p \)-frame potentials of three types of DPPs on the sphere: (i) spherical ensembles of \( S^2 \); (ii) harmonic ensembles of \( S^d \); and (iii) jittered sampling point processes on \( S^d \). Therefore, Section 2 defines a DPP. Section 3 calculates (or estimates) the expectations of \( p \)-frame potentials for such DPPs and Section 4 summarizes our results. By the results, it can be expected that such DPPs have asymptotically smaller expectations of \( p \)-frame potentials than those of Poisson point processes on the sphere with the same number of points. Finally, we note that this study is a subsequent research of our previous work [9] for \( p \)-frame potentials.

2. Determinantal point processes on \( S^d \)

In this section, we introduce the definition of a DPP on \( S^d \). Let \( \sigma_d \) be the normalized surface measure on \( S^d \) and \( K : S^d \times S^d \to \mathbb{C} \) be a measurable function.

Definition 2 (e.g., [1]) A simple point process on \( S^d \) is a DPP with kernel \( K \) if its \( k \)-point joint intensity functions \( \rho_k : (S^d)^k \to \mathbb{R}_{\geq 0} \) (with respect to the background measure \( \sigma_d \)) are given by

\[
\rho_k(x_1, \ldots, x_k) = \det(K(x_i, x_j))_{1 \leq i, j \leq k}
\]

for every \( k \geq 1 \) and \( x_1, \ldots, x_k \in S^d \).

Specifically, if \( \{x_i\}_{i=1}^{N} \) represents an \( N \)-point DPP on \( S^d \), for any measurable function \( f : S^d \times S^d \to [0, \infty) \), we have

\[
\mathbb{E} \left[ \sum_{i \neq j} f(x_i, x_j) \right] = \int_{S^d} \int_{S^d} \rho_2(x, y) f(x, y) d\sigma_d(x) d\sigma_d(y) = \int_{S^d} \int_{S^d} (K(x, x) K(y, y) - |K(x, y)|^2) \\
\times f(x, y) d\sigma_d(x) d\sigma_d(y).
\]

We address the three types of DPPs on the sphere. (i) The spherical ensemble is a well-known ensemble on \( S^2 \) [1, 10]. The kernel of this ensemble is given by \( K(x, x) = (1 + g(x)g(y))^{N-1}/(|1 + g(x)g(y)|^{(N-1)/2}) \), where \( g \) is
the stereographic projection of the sphere $S^2$ from the north pole onto the complex plane $\mathbb{C}$, that is, $g(x_1, x_2, x_3) = (x_1 + \sqrt{-1}x_2)/(1 - x_3)$ for $(x_1, x_2, x_3) \in S^2$. Alishahi and Zamani [11] calculated the 2-point joint intensity function $\rho_2$ as

$$\rho_2(x, y) = N^2 \left[ 1 - \left( 1 - \frac{|x - y|^2}{4} \right)^{N^{-1}} \right].$$

(ii) The harmonic ensemble on $S^d$ is a DPP with the $L$-th reproducing kernel $K(x, y) = R_L(x, y)$ for the polynomial space $P_L(S^d)$ of degree at most $L$ in $d + 1$ variables restricted to the sphere $S^d$. Here $R_L$ is a scaled Jacobi polynomial of parameters $d/2$ and $d/2 - 1$, that is,

$$R_L(x) = \frac{(2L + d)(d + L - 1)}{d(d + L)} P_L(\frac{x}{\sqrt{2}});$$

see [12]. Note that the number of points in the harmonic ensemble is almost surely equal to $N = \dim(P_L(S^d)) = R_L(1) = \frac{(d + L)(d + L - 1)}{d}$ points. Moreover, the 2-point correlation function is given by

$$\rho_2(x, y) = R_L(1)^2 - R_L(x, y)^2.$$

(iii) The following is the definition of a jittered sampling point process on $S^d$.

**Definition 3** Let $\{D_{i,N}\}_{i=1}^N$ be an equal area partition of $S^d$ into $N$ pairwise distinct subsets, that is, $\sigma_d(D_{i,N} \cap D_{j,N}) = 0$ for all $j, k = 1, \ldots, N$ with $j \neq k$ and $\sigma_d(D_{i,N}) = 1/N$. Moreover, assume that $\text{diam} D_{i,N} := \text{sup} \{||x - y|| : x, y \in D_{i,N}\} \leq c/(N^{1/d}) < \sqrt{2}$ for some $c$ depending on $N$. Let $x_i = x_{i,N}$ be a point chosen randomly from $D_{i,N}$ with respect to uniform measure on $D_{i,N}$, then $\{x_i\}_{i=1}^N$ is called a jittered sampling point process on $S^d$.

The jittered sampling point process is also a DPP with kernel $K(x, y) = N \sum_{i=1}^N \chi_{D_{i,N}}(x)\chi_{D_{i,N}}(y)$ [13]. Here, we omit the expression of the 2-point correlation because we use the same methods as in Brauchart et al. [2] in the proof of Theorem 10.

**Remark 4** The condition on the upper bound $\sqrt{2}$ of the diameter in Definition 3 is technical to prove Theorem 10. Leopardi [14] showed the existence of such partitions.

### 3. Main theorems

This section investigates the $p$-frame potentials for three types of DPPs on the sphere: (i) spherical ensembles on $S^2$; (ii) harmonic ensembles on $S^d$; and (iii) jittered sampling point processes on $S^d$.

Before stating our theorems, we prepare some notations. Let $|S^d|$ be the surface area of $S^d$. For $d$ and $p$, define the probabilistic frame potential of $\sigma_d$ by $\text{FPF}(\sigma_d, p) = \int_{S^d} \int_{S^d} |x, y|^p d\sigma_d(x) d\sigma_d(y)$. It is well known [6] that when $p$ is an even integer that

$$\text{FPF}_{p, N}(\{x_i\}_{i=1}^N) \geq \frac{N^2 \text{FPF}(\sigma_d, p)}{2^N}$$

for any finite subset $\{x_i\}_{i=1}^N$ of $S^d$.

#### 3.1 Spherical ensemble on $S^2$

**Theorem 5** If $\{x_i\}_{i=1}^N$ denotes an $N$-point spherical ensemble on $S^2$, we have

$$\mathbb{E}[\text{FPF}_{p, N}(\{x_i\}_{i=1}^N)] = N + N^2 \text{FPF}(\sigma_d, p) - \frac{N^2(-1)^{p+1}B(-1; p + 1, N) + B(p + 1, N)}{2^N},$$

where $B(z; a, b) = \int_0^z t^{a-1}(1 - t)^{b-1} dt$ is the incomplete beta function.

**Proof** Let $a = (0, 0, 1) \in S^2$ be the north pole. Then it follows that

$$\mathbb{E}[\text{FPF}_{p, N}(\{x_i\}_{i=1}^N)] = N + \mathbb{E} \left[ \frac{\sum_{i \neq j} |x_i, x_j|^p}{\sum_{i, j} |x_i - x_j|^p} \right] \times d\sigma_d(x) d\sigma_d(y),$$

$$= N + N^2 \int_{S^2} \int_{S^2} |x, y|^p \left[ 1 - \frac{1}{2} \frac{1 - (x, y)^2}{|x - y|^2} \right] d\sigma_d(x) d\sigma_d(y),$$

$$= N + N^2 \int_{S^2} \int_0^\pi \cos \theta^p \left[ 1 - \frac{1}{2} \frac{1 + \cos \theta}{|x - y|^2} \right] \sin \theta d\theta,$$

$$= N + N^2 \int_{S^2} \int_0^\pi \cos \theta^p \sin \theta d\theta,$$

$$= N + N^2 \text{FPF}(\sigma_d, p) - \frac{N^2}{2} \int_0^\pi \cos \theta^p \left( \frac{1 + \cos \theta}{2} \right)^{N-1} \sin \theta d\theta,$$

$$= N + N^2 \text{FPF}(\sigma_d, p) - \frac{N^2}{2^N} \int_0^1 t^p (1 + t)^{N-1} dt + \int_0^1 t^p (1 - t)^{N-1} dt,$$

$$= N + N^2 \text{FPF}(\sigma_d, p) - \frac{N^2(-1)^{p+1}B(-1; p + 1, N) + B(p + 1, N)}{2^N}.$$ (QED)

**Remark 6** When $p$ is a small integer, we can directly confirm that the following limit holds (see also [9]):

$$\mathbb{E}[\text{FPF}_{p, N}(\{x_i\}_{i=1}^N)] = N - N^2 \text{FPF}(\sigma_d, p) \rightarrow 2p \quad (N \rightarrow \infty).$$

Furthermore, by numerical calculation, it can be expected that the above limit holds for any integer $p$. When $0 < p < \infty$, whether the above equation holds or not will be a future work, because it is challenging to investigate the asymptotic behavior of the incomplete beta function.

#### 3.2 Harmonic ensemble on $S^d$

For this case, we first prepare two classical results of the Jacobi polynomials. The following asymptotic esti-
mate is a specific case in Szegő [15, Theorem 8.21.12]:
\[ p_L(\frac{\pi}{2} - 1) \cos \theta \leq \frac{k(\theta)}{\sqrt{L}} \left[ \cos((L + \lambda + 1)\theta + \gamma) + \frac{O(1)}{L \sin \theta} \right], \tag{2} \]
if \( c/L \leq \theta \leq \pi - (c/L) \) with a fixed positive constant \( c \), \( k(\theta) = \pi^{-1/2}(\sin(\theta/2))^{-(d+1)/2}(\cos(\theta/2))^{-(d-1)/2} \) and \( \gamma = -((d+1)/\pi)/4 \).

Next, we prepare asymptotic behaviors of Jacobi polynomials, known as the Mehler-Heine formulas (Szegő [15, p.192]):
\[
\lim_{{L \to \infty}} L^{-1-\lambda} p_L(\frac{\pi}{2} - 1) \left( \frac{z}{L} \right)^{-\frac{d}{2} - \frac{1}{2}} J_{\frac{d}{2} - 1}(z), \tag{3}
\]
\[
\lim_{{L \to \infty}} L^{-\frac{d}{2} - \frac{1}{2}} p_L(\frac{\pi}{2} - 1) \left( \frac{\cos(\pi - \frac{\pi}{L})}{L} \right) = \left( \frac{z}{2} \right)^{-\frac{d}{2} + \frac{1}{2}} J_{\frac{d}{2} - 1}(z), \tag{4}
\]
where the limits are uniform on compact subsets of \( C \), and the \( J_d \) are Bessel functions of the first kind.

We need Lemma 7 to prove Theorem 8.

**Lemma 7** \[
\frac{[S_d^{d-1}]}{[S_d^d]} \int_0^1 |t|^p \left( R_L(t) \right)^2 (1 - t^2)^{\frac{d}{2} - 1} dt = \frac{2}{\Gamma(d + 1)} L^d + o(L^d) \quad (L \to \infty). \]

**Proof** We use similar arguments used in Proposition 6 of Beltrán et al. [12]. We first split the integral as follows:
\[
\int_0^1 |t|^p \left( R_L(t) \right)^2 (1 - t^2)^{\frac{d}{2} - 1} dt = \left( \int_{-1}^{-\cos \frac{\pi}{2}} + \int_{-\cos \frac{\pi}{2}}^{\cos \frac{\pi}{2}} + \int_{\cos \frac{\pi}{2}}^1 \right) |t|^p dt \times \left( R_L(t) \right)^2 (1 - t^2)^{\frac{d}{2} - 1} dt
\]
\[
= A(c, L) + B(c, L) + C(c, L),
\]
where \( c > 0 \) is fixed and \( c/L < \pi \). For integral \( C(c, L) \), we change the variable \( t = \cos(x/L) \) to get
\[
C(c, L) = \int_{\cos \frac{\pi}{2}}^1 |t|^p \left( R_L(t) \right)^2 (1 - t^2)^{\frac{d}{2} - 1} dt
\]
\[
= \int_{\cos \frac{\pi}{2}}^1 |t|^p \left( R_L(\cos(x/L)) \right)^2 (1 - \cos(x/L))^2 (1 + \cos(x/L)) \left( \frac{x}{L} \right)^{d-1} dx.
\]
Using the Mehler-Heine formula (3) and noting that \( \lim_{L \to \infty} \cos(x/L) = 1 \) and \( \lim_{L \to \infty} \sin(x/L)/x/L = 1 \), we obtain
\[
\lim_{L \to \infty} C(c, L) = 2^d \int_0^c J_\frac{d}{2} - 1(x)^2 dx.
\]
The integral \( A(c, L) \) can be verified in the same way as \( C(d, L) \). Using the Mehler-Heine formula (4) instead of (3), we obtain
\[
\lim_{L \to \infty} A(c, L)/L^{-\frac{d}{2} - 1} = 2^{-d - 2} \int_0^c x J_{\frac{d}{2} - 1}(x)^2 dx < \infty.
\]

For integral \( B(c, L) \), after changing the variable \( t = -\cos \theta \) and using the asymptotic estimate for the Jacobi polynomials (2), we have
\[
0 \leq B(c, L) \leq \frac{2^{d-1}}{\pi L} \int_0^\pi |t|^p \left( \frac{x}{\sqrt{2}} \right)^{-2} dx
\]
\[
\leq \frac{2^{d-1}}{\pi L} \int_0^\pi \sin \left( \frac{x}{2} \right)^2 dx \leq \frac{2^{d-1}}{\pi L} \int_0^\pi \sin \theta dx = \frac{2^{d-1}}{d}. \]
Combining the three parts, we can show that for \( c > 0 \),
\[
2^d \int_0^c \frac{J_{\frac{d}{2} - 1}(x)^2}{x} dx \leq \lim_{L \to \infty} \left[ \int_{-1}^1 |t|^p \left( R_L(\frac{\pi}{2} - \frac{d}{2} - 1) \right)^2 (1 - t^2)^{\frac{d}{2} - 1} dt \right]
\]
\[
\leq \frac{R(d)}{c} + 2^d \int_0^c \frac{J_{\frac{d}{2} - 1}(x)^2}{x} dx,
\]
where \( R(d) \) is a constant independent of \( c \). Taking the limit as \( c \to \infty \), we obtain
\[
\lim_{L \to \infty} \left[ \int_{-1}^1 |t|^p \left( R_L(\frac{\pi}{2} - \frac{d}{2} - 1) \right)^2 (1 - t^2)^{\frac{d}{2} - 1} dt \right]
\]
\[
= 2^d \int_0^\infty \frac{J_{\frac{d}{2} - 1}(x)^2}{x} dx = \frac{2^d}{d}.
\]

Finally, by recalling (1), we obtain the desired result.

(QED)

**Theorem 8** Let \( L \) be a positive integer, and \( N = \text{dim}(P_L(S_d^d)) = (d+1)/d + (d-1)/d \) if \( \{x_i\}_{i=1}^N \) denotes an \( N \)-point harmonic ensemble on \( S_d^d \), we have
\[
\mathbb{E}[\mathbb{F}_{p,N} \{x_i \}_{i=1}^N] = N^2 \mathbb{F}_{p} \mathbb{F}_{p} \sigma_d + o(N) \quad (N \to \infty).
\]

**Proof** Let \( a = (0, \ldots, 0, 1) \in S_d^d \) be the north pole. We change the variable \( t = \cos \theta \) to obtain
\[
\mathbb{E}[\mathbb{F}_{p,N} \{x_i \}_{i=1}^N] = N + \mathbb{E} \left[ \sum_{i \neq j} |\langle x_i, x_j \rangle| \right]
\]
\[
= N + N^2 \int_{S_d^d} \int_{S_d^d} |\langle x, y \rangle|^p d\sigma_d(x) d\sigma_d(y)
\]
\[
- \int_{S_d^d} \int_{S_d^d} |\langle x, y \rangle|^p R_L(\langle x, y \rangle)^2 d\sigma_d(x) d\sigma_d(y)
\]
\[
= N + N^2 \int_{S_d^d} |\langle x, a \rangle|^p d\sigma_d(x)
\]
\[
- \int_{S_d^d} |\langle x, a \rangle|^p R_L(\langle x, a \rangle)^2 d\sigma_d(x)
\]
\[
= N + N^2 \frac{|S_d^d|}{|S_d^d|} \int_0^\pi |\cos \theta|^p (\sin \theta)^{d-1} d\theta
\]
\[
- \frac{|S_d^{d-1}|}{|S_d^d|} \int_0^\pi |\cos \theta|^p R_L(\cos \theta)^2 (\sin \theta)^{d-1} d\theta
\]
\[
= N^2 \mathbb{F}_{p} \mathbb{F}_{p} \sigma_d + N
\]
\[
- \frac{|S_d^{d-1}|}{|S_d^d|} \int_{-1}^1 |t|^p R_L(t)^2 (1 - t^2)^{\frac{d}{2} - 1} dt.
\]
Thus, by using Lemma 7 and noting that \( N = (2/\Gamma(d+1))L^d + O(L^{d-1}) \) \((L \to \infty)\), we obtain the desired result.

(QED)
Remark 9 When $p = 2$, the above result can be refined as in [9]. If $\{x_i\}_{i=1}^N$ denotes the $N$-point harmonic ensemble on $S^d$, we have

$$\mathbb{E} [\text{FP}_p(N, \{x_i\}_{i=1}^N)] = \frac{N^2}{d+1} + O\left(\frac{1}{N^2}\right) \quad (N \to \infty).$$

3.3 Jittered sampling point process on $S^d$

Theorem 10 If $\{x_i\}_{i=1}^N$ denotes the $N$-point jittered sampling point process on $S^d$ defined in Definition 3, we have

$$\mathbb{E} [\text{FP}_{p,N}(\{x_i\}_{i=1}^N)] \leq N^2 \text{PFP}(\sigma_d, p) + N - N \left(1 - \frac{c^2}{2N^2}\right)^p.$$

Proof We use the same process used in Brauchet et al. [2]; see also our previous work [9] for $p = 2$. Let $\{D_j\}_{j=1}^N$ be an equal area partition of $S^d$ defined in Definition 3. As each $D_{j,N}$ is equipped with the probability measure $\mu_{j,N}(E) := \sigma_d(E \cap D_{j,N})/\sigma_d(D_{j,N})$ for any measurable set $E \subseteq S^d$, we have

$$\mathbb{E} [\text{FP}_p(N, \{x_i\}_{i=1}^N)] = N + N^2 \sum_{j=1}^N \int_{D_{j,N}} \int_{D_{j,N}} \|\langle x, y \rangle\|^p d\mu_{j,N}(x) d\mu_{j,N}(y)$$

$$\geq N \sum_{j=1}^N \left[1 - \frac{1}{2} (\text{diam } D_{j,N})^2\right]^p$$

Combining (5) and (6), we obtain the desired result.

(QED)

4. Conclusion

We note that letting $\{y_i\}_{i=1}^N$ represent an $N$-point Poisson point process on $S^d$, we can easily verify that

$$\mathbb{E} [\text{FP}_p(N, \{y_i\}_{i=1}^N)] = N^2 \text{PFP}(\sigma_d, p) + N(1 - \text{PFP}(\sigma_d, p))$$

$$= N^2 \text{PFP}(\sigma_d, p) + O(N) \quad (N \to \infty).$$

We provide calculations (or estimations) of the expectations of the $p$-frame potentials of three typical types of DPPs in Section 3. As a byproduct, we can see numerically or directly (see Remarks 6 and 9) that the expectation of $p$-frame potentials of such DPPs $\{x_i\}_{i=1}^N$ is asymptotically smaller than that of Poisson point processes on the sphere, that is, it holds that

$$\mathbb{E} [\text{FP}_p(N, \{x_i\}_{i=1}^N)] = N^2 \text{PFP}(\sigma_d, p) + o(N) \quad (N \to \infty).$$

Furthermore, although this study focused only on three typical DPPs, recently, Beltrán and Etayo [16, 17] proposed various point processes on the sphere, which we will explore in future work.

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