NONCONVEX QUADRATIC REFORMULATIONS AND
SOLVABLE CONDITIONS FOR MIXED INTEGER QUADRATIC
PROGRAMMING PROBLEMS

YE TIAN
Department of Industrial and Systems Engineering
North Carolina State University
Raleigh, NC 27606, USA

CHENG LU
Department of Mathematical Sciences
Tsinghua University
Beijing 100084, China

Abstract. In this paper, we study a mixed integer constrained quadratic
programming problem. This problem is NP-Hard. By reformulating the
problem to a box constrained quadratic programming and solving the reformulated
problem, we can obtain a global optimal solution of a sub-class of the original
problem. The reformulated problem may not be convex and may not be
solvable in polynomial time. Then we propose a solvability condition for the
reformulated problem, and discuss methods to construct a solvable reformula-
tion for the original problem. The reformulation methods identify a solvable
subclass of the mixed integer constrained quadratic programming problem.

1. Introduction. In this paper, we consider a mixed integer constrained quadratic
programming problem, which is defined as follow.

$$\min \quad F(x) = \frac{1}{2} x^T Q x + c^T x$$
$$\text{s.t.} \quad x_i \in \{0, 1\}, i \in I,$$
$$\quad x_i \in [0, 1], i \in J, \quad \tag{1}$$

where $Q$ is an $n \times n$ real symmetric matrix, $c$ is a vector in $\mathbb{R}^n$, $x \in \mathbb{R}^n$, sets $I$ and
$J$ satisfy $I \cup J \subseteq \{1, 2, \ldots, n\}$ and $I \cap J = \emptyset$. The feasible domain of MQP is denoted
by $\mathcal{F} = \{x \in \mathbb{R}^n | x_i \in \{0, 1\} \text{ for } i \in I, x_i \in [0, 1] \text{ for } i \in J\}$.

This mixed integer constrained quadratic programming problem (MQP in short)
is important in both theoretical researches and real applications. We can find many
industrial and management applications for this general problem. For example,
in [5] Bienstock showed that MQP could play very important role in the portfolio optimization. And two subclasses of MQP such as box constrained quadratic programming and 0-1 constrained quadratic programming also have many applications. For example, in [3] [4] Billionnet et al. studied the quadratic 0-1 knapsack problem which is a quadratic 0-1 problem; in [14] [21] researchers used quadratic box constrained problem in numerical simulation of friction problems in rigid body mechanics and image reconstruction from projections.

Since MQP is an NP-Hard problem (see [13]), there is no polynomial algorithm for solving it if \( NP \neq P \). Many studies try to identify polynomial time solvable subclass of MQP. For example, for 0-1 constrained quadratic programming problem, which can be denoted as QIP. Many researchers have done great works and provide different global optimality conditions. For example, Allemand et al. discussed a polynomial time solvable case in [1]. They adopted a well-designed enumeration algorithm to achieve a global optimal solution in polynomial time. Jeyakumar et al. provided global optimality conditions for Non-convex quadratic minimization problems with quadratic constraints in [15]. They obtained necessary global optimality conditions for weighted least squares with ellipsoidal constraints, and quadratic minimization with binary constraints. Fang et al. used a canonical duality approach to provide a polynomial-time solvable sub-class of QIP in [10]. Further, Wang et al. extended the result and proposed an approximation algorithm for solving the max-cut problem in [25]. Burer et al. modeled nonconvex quadratic program having a mix of binary and continuous variables as a linear program over the dual of the cone of copositive matrices in [8]. They provided a relationship among some convex sets of symmetric matrices. Based on this relationship, they showed that these two problems are equivalent. In this way, they packaged all the difficulty completely inside the convex cones of completely positive matrices. Besides this, they also showed the possibility to reduce the dimension of the completely positive representation and established extension to complementarity constraints over bounded variables. Moreover, Bonze et al. interpreted Burer’s work from a topological point of view in [7]. They defined a weak key condition and showed that under this condition the Minkowski sum of the lifted feasible set and the lifted recession cone will give exactly the closure of the former. Sun et al. gave new results on the duality gap between the binary quadratic optimization problem and its Lagrangian dual or semidefinite programming relaxation in [24]. They derived a necessary and sufficient condition for the zero duality gap and discussed its relationship with the polynomial solvability of the primal problem.

Our work, motivated by these works, identifies a new solvable sub-class of MQP and provides a practical algorithm to get a global optimal solution of a subclass of MQP.

We study MQP from the viewpoint of quadratic reformulations, that is, we reformulate MQP to another quadratic programming problem, and by solving the reformulated problem to obtain an optimal solution of MQP. Since the MQP problem is NP-Hard in general, there is no polynomial-time algorithm for solving it, unless \( P=NP \). However, for some subclasses of MQP, it may be transformed to a polynomial-time solvable convex programming problem. Then we can design algorithms for solve these subclasses with hidden convexity. Besides, for another subclass, the problem may not be transformed to a convex problem. Instead, it can only be transformed to an equivalent nonconvex quadratic programming problem.
Since finding a local optimal solution of a nonconvex quadratic programming problem is also not hard, then if the local optimal solution of the problem is unique, and by conventional local-search algorithms, the problem is still solvable. In this paper, we will give conditions that cases of MQP can be transformed to a solvable problem.

This paper is arranged as follows. Firstly we introduce the definition of quadratic reformulation from the viewpoint of Lagrangian dual theory. Then based on the quadratic reformulation, we discuss conditions for solvability of the reformulations. Next, we discuss the relationship between the Lagrangian multiplier and a conic programming problem to design algorithms for obtaining a solvable reformulation.

Lastly, we provide an algorithm to compute solvable reformulations.

In this paper, we adopt the following notations: For an $n \times n$ matrix $Q$ and a set $K \subseteq \{1, 2, \ldots, n\}$, $Q_{KK}$ denotes the sub-matrix of $Q$, with sub-rows and sub-columns of $Q$ whose index belonging to $K$. For two matrices $A$ and $B$, $A \cdot B$ denotes $\text{trace}(A^T \cdot B)$. Let $e_i$ be a vector in $\mathbb{R}^n$, with its $i$-th element being equal to 1 and other elements being equal to 0, and $e$ be a vector in $\mathbb{R}^n$ with all elements being equal to 1. For a given optimization problem $(\ast)$, its optimal objective value is denoted by $V(\ast)$.

2. Lagrangian dual and quadratic reformulation. The Lagrangian function for MQP is defined as

$$L(x, \lambda) = \frac{1}{2} x^T Q x + c^T x + \sum_{i \in I} \lambda_i (x_i^2 - x_i) + \sum_{i \in J} \lambda_i (x_i^2 - x_i)$$

$$= \frac{1}{2} x^T (Q + 2\Lambda) x + (c - \lambda)^T x,$$

where $\Lambda$ denotes the $n \times n$ diagonal matrix with $\lambda_i$ being the $i$-th diagonal element, and $\lambda_i \geq 0$ for $i \in J$.

Since MQP is non-convex, the duality gap between MQP and its conventional Lagrangian dual problem is not always zero. So we consider the extended Lagrangian duality. The extended Lagrangian dual function is defined as

$$P^e(\lambda) = \min_{x \in [0, 1]^n} L(x, \lambda).$$

By this definition, the value of the extended Lagrangian dual function is no larger than the primal optimal value for any $\lambda \in \mathcal{G}$, as stated in the following theorem.

**Theorem 2.1.** Let $V((1))$ be the optimal value of problem (1), then $P^e(\lambda) \leq V((1))$ for any $\lambda \in \mathcal{G}$.

**Proof.** By definition, we have $P^e(\lambda) = \min_{x \in [0, 1]^n} L(x, \lambda) \leq \min_{x \in \mathcal{F}} L(x, \lambda)$ (noting $\mathcal{F} \subset [0, 1]^n$). For any $x \in \mathcal{F}$, it is easy to verify that $L(x, \lambda) = \frac{1}{2} x^T Q x + c^T x + \sum_{i \in I} \lambda_i (x_i^2 - x_i) + \sum_{i \in J} \lambda_i (x_i^2 - x_i) = \frac{1}{2} x^T Q x + c^T x + \sum_{i \in J} \lambda_i (x_i^2 - x_i)$. Since $\lambda_i \geq 0$ and $x_i^2 - x_i \leq 0$ for any $i \in J$, we have $L(x, \lambda) \leq F(x)$ for any $x \in \mathcal{F}$. Hence $P^e(\lambda) \leq \min_{x \in \mathcal{F}} L(x, \lambda) \leq \min_{x \in \mathcal{F}} F(x) = V((1))$. \qed

Then the extended Lagrangian dual problem is defined as

$$\max_{\{\lambda \in \mathbb{R}^n, \lambda_i \geq 0 \text{ for } i \in J\}} P^e(\lambda).$$

(2)

Different from the conventional Lagrangian dual problem, the extended Lagrangian dual problem always satisfies the strong dual principle, which can be stated as the following theorem.
Theorem 2.2. The gap between problem (2) and problem (1) is always zero.

Proof. As Theorem 2.1 shown, \( P^e(\lambda) \leq V((1)) \) is satisfied. We only need to show the equality is attainable. Now, we choose a \( \bar{\lambda} \in \mathbb{R}^n \) satisfying \((Q + 2\Lambda)_{II} \prec 0\) and \( \bar{\lambda}_i = 0 \) for any \( i \in J \), and let \( \bar{x} \in [0, 1]^n \) be an optimal solution of the problem \( \min_{x \in [0, 1]^n} L(x, \bar{\lambda}) \). For any \( i \in I \), let \( g_i(t) = L(\bar{x} + te_i, \bar{\lambda}) \) be a univariate quadratic function defined on \( t \in [0 - \bar{x}_i, 1 - \bar{x}_i] \), we have \( \frac{dg_i(t)}{dt} = c_i^T(Q + 2\Lambda)e_i < 0 \). So \( g_i(t) \) is strict concave on \( t \in [0 - \bar{x}_i, 1 - \bar{x}_i] \) and its minimizer can be attainable only at \( t = 0 - \bar{x}_i \) or \( t = 1 - \bar{x}_i \). As we assumed, \( \bar{x} \) is an optimal solution for \( \min_{x \in [0, 1]^n} L(x, \bar{\lambda}) \), which implies \( t = 0 \) is a minimizer of \( g_i(t) \), so \( \bar{x}_i = 0 \) or \( \bar{x}_i = 1 \). Hence \( P^e(\lambda) = \min_{x \in [0, 1]^n} L(x, \bar{\lambda}) = \min_{x \in F} L(x, \bar{\lambda}) = \min_{x \in F} F(x) = V((1)) \), and the gap between (2) and (1) is zero. \( \square \)

Then we have the next corollary.

Corollary 1. For any optimal solution \( \lambda^* \) of problem (2), the problem
\[
\begin{align*}
\min & \quad L(x, \lambda^*) \\
\text{s.t.} & \quad x_i \in [0, 1], i = 1, 2, ..., n,
\end{align*}
\]

is a reformulation of problem (1) in the sense that any optimal solution of problem (1) is optimal to problem (3).

Proof. Let \( x^* \) be an optimal solution of problem (1). By the definition of the Lagrangian function \( L(x, \lambda) \) and \( P^e(\lambda) \), \( P^e(\lambda^*) = \min_{x \in [0, 1]^n} L(x, \lambda^*) \leq \min_{x \in [0, 1]^n} F(x) = V((1)) \). Besides, from Theorem 2.2, we also have \( P^e(\lambda^*) = V((1)) \). Hence \( \min_{x \in [0, 1]^n} L(x, \lambda^*) = L(x^*, \lambda^*) \), which proves that \( x^* \) is optimal to problem (3). \( \square \)

This corollary implies that the Lagrangian function \( L(x, \lambda^*) \), with \( \lambda^* \) being an optimal solution of problem (2), is a quadratic reformulation of problem (1). This reformulation transforms the original problem to a continuous optimization problem. Since the optimal solution of problem (2) is not unique, there are many reformulations in such a way. Our object is to find a solvable reformulation to solve the original problem.

3. Solvable conditions. We have discussed reformulations of problem (1). In this section, we will discuss the solvability of these reformulations.

The simplest solvable case is the convex case, i.e., if \( L(x, \lambda^*) \) is a convex function, then the reformulation is solvable. Hence we have the next theorem:

Theorem 3.1. (Convex Solvable Condition.) Let \( \lambda^* \in \mathbb{R}^n \) be an optimal solution of problem (2), if \( Q + 2\Lambda^* \succeq 0 \), then the reformulated problem (3) is solvable. Besides, if \( Q + 2\Lambda^* > 0 \), then problem (3) has a unique optimal solution \( x^* \), which is also an optimal solution of problem (2).

Proof. If \( Q + 2\Lambda^* \succeq 0 \), then problem (3) is a convex quadratic programming problem, which is polynomial-time solvable. Besides, if \( Q + 2\Lambda^* > 0 \), then the objective function of problem (3) is strictly convex, and there exists a unique optimal solution for problem (3). \( \square \)

The Convex Solvable Condition is the simplest solvable condition for reformulations.
For problem (1), any optimal solution \(x^*\) must satisfy the following KKT condition, that is, there exists a Lagrangian multiplier \(\lambda^*\)

\[
\nabla_x L(x, \lambda^*)|_{x=x^*} = (Q + 2\Lambda^*)x^* + c - \lambda^* = 0,
\]

\[
x^*_i - x_i \leq 0, \quad \lambda^*_i \geq 0, \quad \text{for } i \in J
\]

\[
x^*_i - x_i = 0, \quad \text{for } i \in I
\]

\[
\lambda^*_i (x^*_i - x_i) = 0, \quad i = 1, 2, ..., n.
\]

The KKT condition is not a sufficient optimality condition for problem (1). In the literature, we can find the following Positive Semidefinite Condition:

**Condition 1. (Positive Semidefinite Condition)** The problem (1) has a KKT solution \(x^*\) with its Lagrangian multiplier \(\lambda^*\) satisfy \(Q + 2\Lambda^* \succeq 0\).

Then we have the following theorem.

**Theorem 3.2.** If problem (1) has a KKT solution \(x^*\) with its Lagrangian multiplier \(\lambda^*\) such that \(Q + 2\Lambda^* \succeq 0\), then \(\lambda^*\) satisfies the Convex Solvable Condition.

**Proof.** Since \(Q + 2\Lambda^* \succeq 0\) and \((Q + 2\Lambda^*)x^* + c - \lambda^* = 0\), we have \(\min_{x \in \mathbb{R}^n} L(x, \lambda^*) = L(x^*, \lambda^*)\). Besides, it is easy to verify \(L(x^*, \lambda^*) = F(x^*)\) and \(\min_{x \in \mathbb{R}^n} L(x, \lambda^*) \leq P^e(\lambda^*) \leq F(x^*)\). Hence \(P^e(\lambda^*) = F(x^*)\), \(\lambda^*\) is optimal for problem (2) and \(L(x, \lambda^*)\) is convex for \(x\). Thus the Convex Solvable Condition is satisfied.

Hence, the Lagrangian multiplier in the Positive Semidefinite Condition can be interpreted as a convex reformulation of problem (1).

Now we consider nonconvex cases. Let \(x^*\) be a KKT solution of problem (1) with its corresponding Lagrangian multiplier \(\lambda^*\). Define the tangent cone \(T(x^*)\) at \(x^*\) as \(T(x^*) = \{d \in \mathbb{R}^n | 0 \leq d_i \text{ if } x^*_i = 0; \ d_i \leq 0 \text{ if } x^*_i = 1\}\). We give the next condition to verify the global optimality of \(x^*\) for problem (1) and optimality of \(\lambda^*\) for problem (2).

**Theorem 3.3.** Let \(x^*\) be a KKT solution of problem (1) with its corresponding Lagrangian multiplier \(\lambda^*\), if \(\lambda^*\) satisfies \(d^T(Q + 2\Lambda^*)d \geq 0 \text{ for all } d \in T(x^*)\), then \(x^*\) is a global optimal solution of both problem (1) and the problem \(\min_{x \in [0,1]^n} L(x, \lambda^*)\), and \(\lambda^*\) is an optimal solution of problem (2).

**Proof.** For any \(x \in [0,1]^n\), let \(d = x - x^*\) and define \(f_d(t) = L(x^* + td, \lambda^*)\). We can verify \(d \in T(x^*)\). Then for any \(t \in [0,1]\), we have \(x^* + td \in [0,1]^n\). The function \(f_d(t)\) satisfies \(\frac{d^2 f_d(t)}{dt^2} = d^T(Q + 2\Lambda^*)d \geq 0\) and \(\frac{df_d(t)}{dt}|_{t=0} = d^T((Q + 2\Lambda^*)x^* + c - \lambda^*) = 0\). Hence \(f_d(t)\) is convex and \(t = 0\) is a minimizer of \(f_d(t)\) on \([0,1]\). So \(f_d(1) \geq f_d(0)\), which is equivalent to \(L(x, \lambda^*) \geq L(x^*, \lambda^*)\). Thus \(x^*\) is an optimal solution of \(L(x, \lambda^*)\) on \([0,1]^n\). Meanwhile, we have \(V((2)) \geq P^e(\lambda^*) = L(x^*, \lambda^*) = F(x^*) \geq V((1))\), and by weak duality principle, we have \(V((2)) \leq V((1))\). Hence \(P^e(\lambda^*) = F(x^*) = V((1))\), \(x^*\) is optimal to problem (1) and \(\lambda^*\) is optimal to problem (1).

Theorem 3.3 presents optimality conditions for both problem (1) and problem (2). Besides, if there exists a pair \((x^*, \lambda^*)\) satisfying conditions in Theorem 3.3, then the problem \(\min_{x \in [0,1]^n} L(x, \lambda^*)\) is a reformulation for problem (1).

Under Theorem 3.3, it is easy to verify that the reformulation problem is solvable, since any local optimal solution is global optimal for the reformulated problem. However, Under conditions of Theorem 3.3, there is no guarantee to obtain the optimal solution \(x^*\) by local search, since there may be multiple local optimal solutions.
The following stronger condition guarantees the uniqueness of local optimal solution for reformulation.

**Theorem 3.4.** Let \( x^* \) be a KKT solution of problem (1) with its corresponding Lagrangian multiplier \( \lambda^* \). If \( d^T (Q + 2\lambda^*)d \geq 0 \) for any \( d \in T(x^*) - \{0\} \), then \( x^* \) is the unique local optimal solution of the reformulated problem \( \min_{x \in [0,1]^n} L(x, \lambda^*) \).

**Proof.** The proof is similar to that of Theorem 3.2. For any \( x \neq x^* \), let \( d = x - x^* \) and \( f_d(t) \) being defined the same as that in Theorem 3.2, and using a similar method, we can prove \( f_d(1) > f_d(0) \) and \( f_d(t) \) is a strictly increasing function on \([0,1]\), which implies \( L(x, \lambda^*) > L(x^*, \lambda^*) \), and \( x \) is not a local optimal solution of \( L(x, \lambda^*) \) (since \(-d\) is a decreasing feasible direction at \( x \) for the reformulated problem). Hence \( x^* \) is the unique local optimal solution for \( \min_{x \in [0,1]^n} L(x, \lambda^*) \). \( \square \)

Under condition of Theorem 3.4, the reformulation may not be convex, but it has only one local optimal solution, so it can be solved by any local optimization algorithms to find its unique local optimal solution. This condition is also a sufficient condition for global optimality. Here, we give a name for it.

**Condition 2. (Second order solvability condition)** There exists a KKT solution \( x^* \) of problem (1), with its corresponding Lagrangian multiplier \( \lambda^* \), such that \( d^T (Q + 2\lambda^*)d \geq 0 \) for all \( d \in T(x^*) \), then we say problem (1) satisfies the second order solvability condition. If \( d^T (Q + 2\lambda^*)d > 0 \) for all \( d \in T(x^*) \), then we say problem (1) satisfies the second order strong solvability condition.

This solvable condition is more general than Positive Semidefinite Condition.

In [6], I.M.Bomze provided a global optimality condition for quadratic programming problem with a polyhedron feasible set. The above solvable condition on problem (1) situation is similar to their conditions.

In our work, we will not only give the solvability condition in theory, but also provide methods to find reformulations that satisfies the above conditions to design practical algorithms.

4. KKT system and Conic reformulation problem. To obtain a solvable reformulation, we will discuss the relationship between the Lagrangian multiplier and a conic programming problem, and compute the reformulation by solving the conic programming problem.

Define a cone
\[
D_{n+1} = \{ U \in M_{n+1} \mid \begin{bmatrix} 1 \\ x \end{bmatrix}^T U \begin{bmatrix} 1 \\ x \end{bmatrix} \geq 0, \forall x \in [0,1]^n \}.
\]

For a KKT pair \((x^*, \lambda^*)\), define an corresponding matrix
\[
D(x^*, \lambda^*) = \begin{bmatrix} -F(x^*) & \frac{1}{2}(c - \lambda^*)^T \\ \frac{1}{2}(c - \lambda^*) & \frac{1}{2}Q + \Lambda \end{bmatrix}
\]

**Theorem 4.1.** For a KKT pair \((x^*, \lambda^*)\) of problem (1), \((x^*, \lambda^*)\) satisfies the Second order solvable condition if and only if \( D(x^*, \lambda^*) \in D_{n+1} \).

**Proof.** Since \( x^* \) is a KKT solution of problem (1), thus \( x_i^* = 0 \) or 1 for all \( i \in I \), \( 0 \leq x_i^* \leq 1 \) for all \( i \in J \). For any \( x \in [0,1]^n \), let \( d = x - x^* \). We can verify that
$d \in T(x^*)$. And for any $d \in T(x^*)$, we always can multiply a positive scale to make the new $d$ satisfy that $0 \leq x^* + d \leq 1$. Then we have that
\[
\begin{bmatrix}
1 \\
x
\end{bmatrix}^T \begin{bmatrix}
-F(x^*) \\
\frac{1}{2}(c - \lambda^*)^T \\
\frac{1}{2}Q + \Lambda^*
\end{bmatrix} \begin{bmatrix}
1 \\
x
\end{bmatrix} = -F(x^*) + L(x^*, \lambda^*) + d^T((Q + 2\Lambda^*)x^* + c - \lambda^*) + d^T(Q + 2\Lambda^*)d = d^T(Q + 2\Lambda^*)d.
\]
So $d^T(Q + 2\Lambda^*)d \geq 0$ for all $d \in T(x^*)$ if and only if $D(x^*, \lambda^*) \in D_{n+1}$.

The above theorem transformed the second order solvable condition to the condition $D(x^*, \lambda^*) \in D_{n+1}$. Now we give more discussions for this condition.

Denote $D'_{n+1}$ as the dual cone of $D_{n+1}$. We define the conic programming
\[
\begin{align*}
&\min \begin{bmatrix} 0 \\ \frac{1}{2}(c - \lambda) \\ \frac{1}{2}Q + \Lambda \end{bmatrix} \cdot Y \\
s.t. \begin{bmatrix} 1 \\ x \end{bmatrix}^T X = Y \\
&\quad X_{ii} - x_i = 0, \ i \in I \\
&\quad X_{ii} - x_i \leq 0, \ i \in J \\
&\quad Y \in D'_{n+1} \tag{5}
\end{align*}
\]
with its dual problem
\[
\begin{align*}
&\max -\sigma \\
&s.t. \begin{bmatrix} \sigma \\ \frac{1}{2}(c - \lambda) \\ \frac{1}{2}Q + \Lambda \end{bmatrix} \in D_{n+1} \\
&\quad \lambda_i \geq 0, \forall i \in J. \tag{6}
\end{align*}
\]
Then we have the next theorem.

**Theorem 4.2.** Let $x^*$ be a KKT solution of problem (1) with its corresponding Lagrangian multiplier $\lambda^*$ such that the corresponding matrix $D(x^*, \lambda^*) \in D_{n+1}$, then $x^*$ is a global optimal solution of problem (1), and $(\sigma^*, \lambda^*)$ is an optimal solution of problem (6), where $\sigma^* = -F(x^*)$.

**Proof.** Since $x^*$ is a KKT solution, Let $Y^* = \begin{bmatrix} 1 \\ x^* \end{bmatrix}^T$, then we can easily verify $Y^* \in D'_{n+1}$. And we have that $D(x^*, \lambda^*) \cdot Y^* = -F(x^*) + L(x^*, \lambda^*) = 0$. Thus the complementary condition satisfies. From the optimality theory of conic programming, we know that $x^*$ is a global optimal solution of problem (5), and $(\sigma^*, \lambda^*)$ is an optimal solution of problem (6), where $\sigma^* = -F(x^*)$. And since problem (5), problem (6) and problem (1) are equivalent, we also know that $x^*$ is a global optimal solution of problem (1).

Hence, if there exists a KKT solution $x^*$ with its corresponding Lagrangian multiplier $\lambda^*$ satisfy $D(x^*, \lambda^*) \in D_{n+1}$, then $(\sigma^*, \lambda^*)$ with $\sigma^* = -F(x^*)$ is an optimal solution of problem (6). However, by solving problem (6), there is no guarantee to obtain $\lambda^*$.\]
The above theorem also provides a sufficient condition of global optimality for problem (1). We give a name here.

**Condition 3. (Extended global optimality condition)** The problem (1) has a KKT solution \( x^* \) with its Lagrangian multiplier \( \lambda^* \) such that \( D(x^*, \lambda^*) \in D_{n+1} \).

**Lemma 4.3.** Under the Extended Global Optimality Condition with \( \lambda^* \) and \( x^* \) being defined, if \( (\sigma_D, \lambda_D) \) is an optimal solution of the problem (6), then \( \lambda_D \leq \lambda^* \).

**Proof.** Since \( x^* \) is a KKT solution, we let \( Y^* = \begin{bmatrix} 1 \\ x^* \end{bmatrix} \), \( Y^* \in D_{n+1} \). Because \( x^* \) is an optimal solution of problem (1), \( F(x^*) = V((1)) \). We have that \( V((1)) = -\sigma_D \leq \left[ \frac{1}{2}(c - \lambda_D) \frac{1}{2} Q + \Lambda_D \right] \cdot Y^* - \sigma_D = L(x^*, \lambda_D) \leq F(x^*) = V((1)) \). And we know that \( \left[ \begin{array}{c} \sigma_D \\ \frac{1}{2}(c - \lambda_D) \frac{1}{2} Q + \Lambda_D \end{array} \right] \cdot \begin{bmatrix} 1 \\ x \end{bmatrix} = 0 \) for any \( x \in [0, 1]^n \). Hence \( L(x, \lambda_D) = -\sigma_D = V((1)) \) for any \( x \in [0, 1]^n \). So \( x^* \) is optimal for the problem \( \min_{x \in [0, 1]^n} L(x, \lambda_D) \). Notice that the gradient of the function \( L(x, \lambda_D) \) at \( x^* \) is \( (Q + 2\Delta_D)x^* + (c - \lambda_D) \cdot \). If \( [x^*]_i = 1 \), then \( -\epsilon_i \) is a feasible direction for \( L(x, \lambda_D) \) at \( x^* \). If \( [x^*]_i = 0 \), then \( \epsilon_i \) is a feasible direction for \( L(x, \lambda_D) \) at \( x^* \). Because \( x^* \) is a global optimal solution for the problem \( \min_{x \in [0, 1]^n} L(x, \lambda_D) \), its optimality condition results in \( d^T[(Q + 2\Delta_D)x^* + (c - \lambda_D)], \) where \( d \) is any feasible direction at \( x^* \). Therefore, we have that if \( [x^*]_i = 1 \), then \( (Q + 2\Delta_D)x^* + (c - \lambda_D) \leq 0 \); if \( [x^*]_i = 0 \), then \( (Q + 2\Delta_D)x^* + (c - \lambda_D) \geq 0 \). Let \( r = [(Q + 2\Delta_D)x^* + (c - \lambda_D)] - [(Q + 2\Delta_D)x^* + (c - \lambda_D)] \). From KKT condition, we know that \( (Q + 2\Delta_D)x^* + (c - \lambda_D) = 0 \). Moreover, from above result, if \( x^* = 0, r_i \geq 0 \); if \( x^* = 1, r_i \leq 0 \). We notice that \( r_i = (4x^* - 2)(\lambda_D - \lambda^*)_i \). Thus whenever \( x^*_i \) is 0 or 0, we have \( [\lambda_D]_i \leq [\lambda^*]_i \). For any \( 0 < x^*_i < 1 \), from KKT condition, we have that \( \lambda_i = 0 \). Because \( L(x, \lambda_D) = F(x^*) \), we have that \( [\lambda_D]_i([x^*_i]^2 - x^*_i] = 0 \), hence for any \( 0 < x^*_i < 1 \) we have that \( [\lambda_D]_i = 0 \). Above all, we have that \( \lambda_D \leq \lambda^* \).

Due to the special property of optimal solution \( \sigma_D, \lambda_D \) and the relationship between \( \lambda_D, \lambda^* \), we have the following key result:

**Theorem 4.4.** Under the Extended Global Optimality Condition with \( \lambda^* \) and \( x^* \) being defined, if \( (\sigma_D, \lambda_D) \) is an optimal solution of the problem (6), we have that \( \sigma_D = -V((1)) \), and \( \lambda^* \) is the unique optimal solution of the following linear conic programming problem:

\[
\begin{align*}
\max & \quad e^T \lambda \\
\text{s.t.} & \quad \left[ \begin{array}{c} \sigma_D \\ \frac{1}{2}(c - \lambda) \frac{1}{2} Q + \Lambda \end{array} \right] \in D_{n+1} \\
& \quad \lambda_i \geq 0, \text{ for any } i \in J.
\end{align*}
\]

**Proof.** We notice that any feasible solution \( (\sigma_D, \lambda) \) of problem (7) is optimal to problem (6) with \( \lambda \leq \lambda^* \) implied by lemma 1. Because \( (\sigma_D, \lambda^*) \) is feasible to problem (7), and the linear independence for every constraint for \( x_i, \lambda^* \) is the unique optimal solution of problem (7).
The above theorem indicates that under the Extended Global Optimality Condition with \( \lambda^* \) being defined, \( \lambda^* \) is the unique optimal solution of problem (7). Hence, if we can obtain the optimal solution \( \lambda^* \) of problem (7), then we get a solvable reformulation
\[
\min_{x \in [0,1]^n} L(x, \lambda^*)
\]
for problem (1).

5. **Proposed algorithm and numerical example.** Because problem (6) is equivalent to problem (1), we pack the whole difficulty of the original problem into the cone \( D_{n+1} \) and \( D^*_{n+1} \), thus there is no polynomial time algorithm for solving problem (1) unless P=NP. Then in order to solve the problem (6), we need to choose a computable cone \( C_{n+1} \) which satisfy \( C_{n+1} \subseteq D_{n+1} \) to substitute the cone \( D_{n+1} \) which is difficult to compute. Given such a cone \( C_{n+1} \), we define the following conic programming problem:

\[
\begin{align*}
\max & \quad \sigma \\
\text{s.t.} & \quad \begin{bmatrix} -\sigma \\ \frac{1}{2}(c - \lambda) \end{bmatrix} \begin{bmatrix} \frac{1}{2}Q + \Lambda \\ \frac{1}{2}(c - \lambda)^T \\ \frac{1}{2}(c - \lambda) \end{bmatrix} \in C_{n+1} \\
& \quad \lambda_i \geq 0, \text{ for any } i \in J.
\end{align*}
\]

(8)

Assuming that \( \sigma_d \) is the optimal solution of C1, we define an additional conic programming problem:

\[
\begin{align*}
\max & \quad e^T \lambda \\
\text{s.t.} & \quad \begin{bmatrix} \sigma_d \\ \frac{1}{2}(c - \lambda) \end{bmatrix} \begin{bmatrix} \frac{1}{2}Q + \Lambda \\ \frac{1}{2}(c - \lambda)^T \\ \frac{1}{2}(c - \lambda) \end{bmatrix} \in C_{n+1} \\
& \quad \lambda_i \geq 0, \text{ for any } i \in J.
\end{align*}
\]

(9)

Correspondingly, we have the following result.

**Theorem 5.1.** If there exists a KKT pair \( (x^*, \lambda^*) \) such that \( D(x^*, \lambda^*) \in C_{n+1} \), then \( \lambda^* \) is the unique optimal solution of problem (9).

**Proof.** Since \( D(x^*, \lambda^*) \in C_{n+1} \subseteq D_{n+1} \), we know that \( \sigma_d \) in problem (9) is equal to \( V((1)) \). Besides, by theorem 4.4, \( \lambda^* \) is optimal for problem (7) and feasible for problem (9). Moreover, any feasible solution of problem (9) is also feasible for problem (7), \( \lambda^* \) must be the unique optimal solution of problem (9). \( \square \)

**Remark 1.** The primary purpose of this paper is theoretical—to provide a more general global optimality condition for problem (1) and identify a bigger solvable subclass of problem (1). We do not aim to discuss how to find a computable cone \( C_{n+1} \) to approximate the cone \( D_{n+1} \). This is a big issue in recent research ([9] [20] [22] [26]). The performance of the lower bound is determined by the choice of \( C_{n+1} \). A tighter inner approximation of the cone \( D_{n+1} \) not only provides a tighter lower bound, but also a higher chance of getting the global optimal solution of problem (1).

With problem (8) and problem (9) being defined, we design the following algorithm to compute a reformulation of problem (1).

**Algorithm 1.** MQP Algorithm:

Step 1: For a given problem (1), construct the conic relaxation problem (8).

Step 2: Solve the problem (8) to get its optimal value \( \sigma_d \).
Step 3: Construct the conic relaxation problem (9) using $\sigma_d$.

Step 4: Solve the problem (9) to get its optimal value $\lambda^*$ and the reformulation $L(x, \lambda^*)$.

Step 5: Solve $\min_{x \in [0,1]^n} L(x, \lambda^*)$ to obtain a local optimal solution $x^*$. If $F(x^*) = \sigma_d$, then return $x^*$ as an optimal solution of problem (1); otherwise, the algorithm fails to obtain a solvable reformulation.

The next theorem validates Algorithm 1.

**Theorem 5.2.** If Algorithm 1 returns a solution $x^*$ successfully, then $x^*$ is a global optimal solution of the problem (1).

**Proof.** At step 5 of Algorithm, if $x^*$ is returned successfully, then $F(x^*) = \sigma_d$ guarantees that $x^*$ is a global optimal solution of problem (1). \qed

**Remark 2.** For all known methods in literature, the largest solvable subclasses of mixed integer constrained quadratic programming problem are those satisfying the Positive Semidefinite condition with $Q + 2\Lambda$ being invertible. But for Algorithm 1, a solvable subclass of nonconvex quadratically constrained quadratic programs are those satisfying the condition of Theorem 5.1 with $Q + 2\Lambda$ being invertible. Under this condition, we can obtain a reformulation $L(x, \lambda^*)$ satisfies the second order solvability condition. If $C_{n+1} = S_{n+1}$, then these two conditions become equivalent. Whereas, the cone $C_{n+1}$ can be better chosen than $S_{n+1}$, based on the structure of the feasible domain $F$. For example we can choose $C_{n+1} = S_{n+1} + N_{n+1}$ which is bigger. In this way, the performance of Algorithm 1 could be improved. And in this sense, the newly proposed Algorithm 1 extends the known solvable subclass to more general cases.

The next example shows that the Algorithm 1 indeed works. Consider the following problems:

**Example 1.**

$$\min \frac{1}{2} x^T Q x + c^T x$$

s.t. $x \in \{0, 1\}^n$.

with

$$Q = \begin{bmatrix}
417 & 716 & -4 & 732 & -69 & 478 & 40 \\
716 & 919 & 73 & 1131 & -71 & 104 & -37 \\
-4 & 73 & 161 & 54 & -182 & 11 & 49 \\
732 & 1131 & 54 & 123 & -25 & 620 & -49 \\
-69 & -71 & -182 & -25 & 412 & 16 & -133 \\
478 & 104 & 11 & 620 & 16 & 353 & 34 \\
40 & -37 & 49 & -49 & -133 & 34 & 159
\end{bmatrix}$$

and $c = [-3 58 -56 57 -90 -123 -34]^T$.

Here we choose $C_{n+1} = S_{n+1} + N_{n+1}$. Using Algorithm 1, we find the optimal solution $\lambda^* = [-36 23 28 37 -7 -62 -41]^T$ with $Q + 2\Lambda^*$ is not positive semidefinite. The reformulation problem $\min_{x \in [0,1]^n} L(x, \lambda^*)$ is not convex but has unique local optimal solution $x^* = [0 0 1 0 1 0 1]^T$, which is the global optimal solution of our problem.
Example 2.

\[
\begin{align*}
\text{min} & \quad \frac{1}{2} x^T Q x + c^T x \\
\text{s.t.} & \quad x_i \in [0, 1], \ i = 1, \ldots, n
\end{align*}
\]

with

\[
Q = \begin{bmatrix}
263 & -97 & 62 & 217 & 52 & 621 & 935 & 258 & -61 & -10 \\
-97 & 299 & -17 & 9 & -4 & -123 & -17 & -40 & -3 & 37 \\
62 & -17 & 178 & 71 & -118 & -83 & -110 & 9 & -56 & 42 \\
217 & 9 & 71 & 143 & -5 & 842 & 228 & 42 & 58 & -41 \\
52 & -4 & -118 & -5 & 177 & 102 & -15 & 120 & 13 & -52 \\
621 & -123 & -83 & 842 & 102 & 219 & 574 & 22 & 73 & -53 \\
935 & -17 & -110 & 228 & -15 & 574 & 457 & 154 & -25 & 84 \\
258 & -40 & 9 & 42 & 120 & 22 & 154 & 473 & 18 & -29 \\
-61 & -3 & -56 & 58 & 13 & 73 & -25 & 18 & -4 & -79 \\
-10 & 37 & 42 & -41 & -52 & -53 & 84 & -29 & -79 & 224
\end{bmatrix}
\]

\[
c = [-20 \ -314 \ 46 \ -83.45 \ -128.7 \ 41.3 \ 43.85 \ -341.8 \ -34.05 \ -34.6]^{T}.
\]

Here we choose \(C_{n+1} = S_{n+1} + N_{n+1}\). Using Algorithm 1, we find the optimal solution \(\lambda^* = [41.1732 \ 35.1504 \ 0 \ 20.0594 \ 0 \ 13.1491 \ 70.1486 \ 0 \ 88.0332 \ 0]^{T}\) with \(Q + 2\Lambda^*\) is not positive semidefinite. The reformulation problem \(\min_{x \in [0, 1]} L(x, \lambda^*)\) is not convex, too. Solving the reformulation by local search method, we can obtain a local optimal solution \(x^* = [0.0001 \ 0.9996 \ 0 \ 0.7501 \ 0 \ 0.0001 \ 0.5999 \ 0.9998 \ 0.5]^{T}\). This is indeed the global optimal solution of our problem.

The above examples demonstrate that Algorithm 1 is applicable for a larger subclasses of solvable MQP problems. Based on the different structures of some given problems, we may find some tighter computable cones for relaxation. Then we may improve the performance of our algorithm.

6. Conclusion. In this paper, we studied quadratic reformulation methods for mixed integer constrained quadratic programming problem. We gave a more general global optimality condition based on KKT system. It is the generalization of the known Positive Semidefinite Condition in Literature. Under this condition, a bigger sub-class of the problems can be polynomial-time solved. Then based on this condition, we reformulated the original problem into a conic problem. And for the solvable sub-class problems, we proposed a practical algorithm to get the corresponding optimal KKT pair. The proposed Algorithm 1 sheds some light on designing effective algorithms for a new solvable sub-classes of MQP.

Some researchers have studied the relationship between quadratic programming and conic programming in literature. For instance, Burer reformulated the mixed-binary QPs in copositive representation in [8]; Strum and Zhang reformulated the general nonconvex quadratic programming problem as a conic programming problem in [23]. Their works mainly discussed on the relationship between the solutions of the original quadratic optimization problem and the reformulated conic programming problem. Compared to their works, the main difference of this paper is that we consider the relationship between the Lagrangian multipliers of the quadratic optimization problem and several corresponding conic programming problems. The Lagrangian multipliers gave us some new insights, which is not easy to see from the original problem and its conic reformulation directly, and led us to discover a larger solvable sub-class of MQP.
The relationship between the Lagrangian multipliers and the SDP relaxation method can also be found in [19]. In this paper, we have refined and extended this relationship to more general cones which include the positive semidefinite cones. In this sense, our work is an extension of the previously known results. Our simple numerical example clearly indicates that the proposed Algorithm 1 indeed can solve some new solvable subclass which cannot be solved by using previously reported algorithms in literature.

The proposed algorithm is based on the choice of the computable cone $C_{n+1}$. We are currently investigating new techniques to approximate the cone $D_{n+1}$ from inside.

REFERENCES

[1] K. Allemand, K. Fukuda, T. M. Liebling and E. Steiner, A polynomial case of unconstrained zero-one quadratic optimization, Math. Program, 91 (2001), 49–52.

[2] A. Ben-Israel and T. N. E. Greville, “Generalized Inverses: Theory and Applications,” 2nd edition, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 15, Springer-Verlag, New York, 2003.

[3] A. Billionnet and F. Calmels, Linear programming for the 0-1 quadratic knapsack problem, European Journal of Operational Research, 92 (1996), 310–325.

[4] A. Billionnet, A. Faye and E. Soutif, A new upper bound for the 0-1 quadratic knapsack problem, European Journal of Operational Research, 113 (1999), 664–672.

[5] D. Bienstock, Computational study of a family of mixed-integer quadratic programming problems, Math. Program, 74 (1996), 121–140.

[6] I. M. Bomze, Global optimization: A quadratic programming perspective, in “Nonlinear Optimization,” Lecture Notes in Mathematics, 1989, Springer, Berlin, (2010), 1–53.

[7] I. M. Bomze and F. Jarre, A note on Burer’s copositive representation of mixed-binary QPs, Optimization Letter, 4 (2010), 465–472.

[8] S. Burer, On the copositive representation of binary and continuous nonconvex quadratic programs, Math. Program., 120 (2009), 479–495.

[9] S. Bundfuss and M. Diir, “An Adaptive Linear Approximation Algorithm for Copositive Programs,” Manuscript, Department of Mathematics, Technische Universitat Darmstadt, Darmstadt, Germany, 2008.

[10] S.-C. Fang, D. Y. Gao, R.-L. Sheu and S.-Y. Wu, Canonical dual approach to solving 0-1 quadratic programming problems, Journal of Industrial and Management Optimization, 4 (2008), 125–142.

[11] D. Y. Gao, Canonical dual transformation method and generalized triality theory in nonsmooth global optimization, J. Global Optimization, 17 (2000), 127–160.

[12] D. Y. Gao, Advances in canonical duality theory with applications to global optimization, Available from: http://www.math.vt.edu/people/gao/papers/focapo08.pdf.

[13] M. R. Garey and D. S. Johnson, “Computers and Intractability: A Guide to the Theory of NP-Completeness,” A Series of Books in the Mathematical Sciences, W. H. Freeman and Co., San Francisco, CA, 1979.

[14] G. T. Herman, “Image Reconstruction from Projections: The Fundamentals of Computerized Tomography,” Computer Science and Applied Mathematics. Academic Press, Inc., New York-London, 1980.

[15] V. Jeyakumar, A. M. Rubinov and Z. Y. Wu, Non-convex quadratic minimization problems with quadratic constraints: Global optimality conditions, Math. Program., 110 (2007), 521–541.

[16] E. de Klerk and D. V. Pasechnik, Approximation of the stability number of a graph via copositive programming, SIAM J. Optim., 12 (2002), 875–892.

[17] C. Lu, S.-C. Fang, Q. Jin, Z. Wang and W. Xing, KKT solution and conic relaxation for solving quadratically constrained quadratic programming problems, working paper, 2010.

[18] C. Lu, Z. Wang, W. Xing and S.-C. Fang, Extended canonical duality and conic programming for solving 0-1 quadratic programming problems, Journal of Industrial and Management Optimization, 6 (2010), 779–793.
[19] C. Lemaréchal and F. Oustry, *SDP relaxations in combinatorial optimization from a Lagrangian viewpoint*, in “Advances in Convex Analysis and Global Optimization” (eds. N. Hadijsavvas and P. M. Paradalos), Nonconvex Optim. Appl., 54, Kluwer Acad. Publ., Dordrecht, (2001), 119–134.

[20] J. B. Lasserre, *Global optimization with polynomials and the problem of moments*, SIAM J. Optimization, 11 (2001/01), 796–817.

[21] P. Lotstedt, *Solving the minimal least squares problem subject to bounds on the variables*, BIT, 24 (1984), 206–224.

[22] P. Parrilo, “Structured Semidefinite Programs and Semi-Algebraic Geometry Methods in Robustness and Optimization,” Ph.D. Thesis, California Institute of Technology, 2000.

[23] J. F. Strum and S. Zhang, *On cones of nonnegative quadratic functions*, Mathematics of Operations Research, 28 (2003), 246–267.

[24] X. Sun, C. Liu, D. Li and J. Gao, *On duality gap in binary quadratic programming*, Available from : http://www.optimization-online.org/DB_FILE/2010/01/2512.pdf.

[25] Z. Wang, S.-C. Fang, D. Y. Gao and W. Xing, *Global extremal conditions for multi-integer quadratic programming*, J. Industrial and Management Optimization, 4 (2008), 213–225.

[26] L. F. Zuluage, J. Vera and J. Peña, *LMI approximations for cones of positive semidefinite forms*, SIAM J. Optimization, 16 (2006), 1076–1091.

Received October 2010; 1st revision May 2011; 2nd revision July 2011.

E-mail address: ytian4@ncsu.edu
E-mail address: c-lu06@mails.tsinghua.edu.cn