The tensor Dirac equation in Riemannian space

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Abstract

We suggest a tensor equation on Riemannian manifolds which can be considered as a generalization of the Dirac equation for the electron. The tetrad formalism is not used. Also we suggest a new form of the tensor Dirac equation with a Spin$(1,3)$ gauge symmetry in Minkowski space.

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In this paper, following [2], [3], we consider the tensor Dirac equation (16) on Riemannian manifolds. Our approach leads to a new form of the tensor Dirac equation with a Spin(1,3) gauge symmetry in Minkowski space. The research was carried out while the author was visiting at Bath University. The author is grateful to Professor D.Vassiliev, Dr. A.King, and Dr. F.Burstall for useful discussions and for hospitality.

1 Differential forms on Riemannian manifolds.

Let \( \mathcal{M} \) be a four dimensional differentiable manifolds covered by a system of coordinates \( x^\mu \). Greek indices run over \((0,1,2,3)\). Summation convention over repeating indices is assumed. We consider atlases on \( \mathcal{M} \) consisting of one chart. Suppose that there is a smooth twice covariant tensor field (a metric tensor) with components \( g_{\mu\nu} = g_{\mu\nu}(x), \ x \in \mathcal{M} \) such that

- \( g_{\mu\nu} = g_{\nu\mu} \);
- \( g = \det \|g_{\mu\nu}\| < 0 \) for all \( x \in \mathcal{M} \);
- The signature of the matrix \( \|g_{\mu\nu}\| \) is equal to \(-2\).

The matrix \( \|g^{\mu\nu}\| \) composed from contravariant components of the metric tensor is the inverse matrix to \( \|g_{\mu\nu}\| \). The full set of \( \{\mathcal{M}, g_{\mu\nu}\} \) is called an
**elementary Riemannian manifolds** (with one chart atlases) and is denoted by \( \mathcal{V} \).

Let \( \Lambda^k \) be the sets of exterior differential forms of rank \( k = 0, 1, 2, 3, 4 \) on \( \mathcal{V} \) (covariant antisymmetric tensor fields) and

\[
\Lambda = \Lambda^0 \oplus \ldots \oplus \Lambda^4 = \Lambda^{\text{even}} \oplus \Lambda^{\text{odd}},
\]

\[
\Lambda^{\text{even}} = \Lambda^0 \oplus \Lambda^2 \oplus \Lambda^4, \quad \Lambda^{\text{odd}} = \Lambda^1 \oplus \Lambda^3.
\]

Elements of \( \Lambda \) are called (nonhomogeneous) **differential forms** and elements of \( \Lambda^k \) are called **\( k \)-forms** or differential forms of rank \( k \). The set of smooth scalar functions on \( \mathcal{V} \) (invariants) is identified with the set of 0-forms \( \Lambda^0 \). A \( k \)-form \( U \in \Lambda^k \) can be written as

\[
U = \frac{1}{k!} u_{\nu_1 \ldots \nu_k} dx^{\nu_1} \wedge \ldots \wedge dx^{\nu_k} = \sum_{\mu_1 < \ldots < \mu_k} u_{\mu_1 \ldots \mu_k} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_k},
\]

where \( u_{\nu_1 \ldots \nu_k} = u_{\nu_1 \ldots \nu_k}(x) \) are real valued components of a covariant antisymmetric \((u_{\nu_1 \ldots \nu_k} = u_{\nu_1 \ldots \nu_k})\) tensor field. Differential forms from \( \Lambda \) can be written as linear combinations of the 16 basis differential forms

\[
1, dx^\mu, dx^{\mu_1} \wedge dx^{\mu_2}, \ldots, dx^0 \wedge \ldots \wedge dx^3, \quad \mu_1 < \mu_2 < \ldots.
\]

The exterior multiplication of differential forms is defined in the usual way. If \( U \in \Lambda^r, V \in \Lambda^s \), then

\[
U \wedge V = (-1)^{rs} V \wedge U \in \Lambda^{r+s}.
\]

In this paper we consider changes of coordinates with positive Jacobian and do not distinguish tensors and pseudotensors.

Consider the Hodge star operator \( \star : \Lambda^k \to \Lambda^{4-k} \). If \( U \in \Lambda^k \) has the form (\( \Box \)), then

\[
\star U = \frac{1}{k!(4-k)!} \sqrt{-g} \varepsilon_{\mu_1 \ldots \mu_k} u^{\mu_1 \ldots \mu_k} dx^{\mu_{k+1}} \wedge \ldots \wedge dx^{\mu_4},
\]

where \( u^{\mu_1 \ldots \mu_k} = g^{\mu_1 \nu_1} \ldots g^{\mu_k \nu_k} u_{\nu_1 \ldots \nu_k}, \varepsilon_{\mu_1 \ldots \mu_4} \) is the sign of the permutation \((\mu_1 \ldots \mu_4)\), and \( \varepsilon_{0123} = 1 \). It is easy to prove that for \( U \in \Lambda^k \)

\[
\star(\star U) = (-1)^{k+1} U.
\]
Further on we consider the bilinear operator \( \text{Com} : \Lambda^2 \times \Lambda^2 \to \Lambda^2 \) such that

\[
\text{Com}(\frac{1}{2}a_{\mu_1\mu_2}dx^{\mu_1} \wedge dx^{\mu_2}, \frac{1}{2}b_{\nu_1\nu_2}dx^{\nu_1} \wedge dx^{\nu_2}) = \frac{1}{2}a_{\mu_1\mu_2}b_{\nu_1\nu_2}(-g^{\mu_1\nu_1}dx^{\mu_2} \wedge dx^{\nu_2} - g^{\mu_2\nu_2}dx^{\mu_1} \wedge dx^{\nu_1} + g^{\mu_1\nu_2}dx^{\mu_2} \wedge dx^{\nu_1} + g^{\mu_2\nu_1}dx^{\mu_1} \wedge dx^{\nu_2})
\]

Evidently, \( \text{Com}(U, V) = -\text{Com}(V, U) \).

Now we define the Clifford multiplication of differential forms with the aid of the following formulas (see formulas for the space dimensions 2 and 3 in [2]):

\[
\begin{align*}
0^k U V &= VU = U \wedge V = V \wedge U, \\
1^k U V &= U \wedge V - \star (U \wedge V), \\
0^k U^1 V &= U \wedge V + \star (U \wedge V), \\
2^2 U V &= U \wedge V + \star (U \wedge V) + \frac{1}{2}\text{Com}(U, V), \\
2^3 U V &= \star U \wedge V - \star (U \wedge V), \\
2^4 U V &= \star U \wedge V, \\
3^2 U V &= -\star U \wedge V - \star (U \wedge V), \\
3^3 U V &= \star U \wedge V + \star (U \wedge V), \\
3^4 U V &= \star U \wedge V, \\
4^2 U V &= \star U \wedge V, \\
4^3 U V &= -\star U \wedge V, \\
4^4 U V &= -\star U \wedge V,
\end{align*}
\]

where ranks of differential forms are denoted as \( k \in \Lambda^k \) and \( k = 0, 1, 2, 3, 4 \).

From this definition we may obtain some properties of the Clifford multiplication of differential forms.

1. If \( U, V \in \Lambda \), then \( UV \in \Lambda \).

2. The axioms of associativity and distributivity are satisfied for the Clifford multiplication.
3. \(dx^\mu dx^\nu = dx^\mu \wedge dx^\nu + g^{\mu\nu}, \quad dx^\mu dx^\nu + dx^\nu dx^\mu = 2g^{\mu\nu}\).

4. If \(U, V \in \Lambda^2\), then \(\text{Com}(U, V) = UV - VU\).

Let us define the trace of a differential form as a linear operation \(\text{Tr} : \Lambda \to \Lambda^0\) such that
\[
\text{Tr}(1) = 1, \quad \text{Tr}(dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_k}) = 0 \quad \text{for} \quad k = 1, 2, 3, 4.
\]
The reader can easily prove that
\[
\text{Tr}(UV - VU) = 0, \quad \text{Tr}(V^{-1}UV) = \text{Tr}U, \quad U, V \in \Lambda.
\]

Let us define an involution \(* : \Lambda^k \to \Lambda^k\). By definition, put
\[
U^* = (-1)^{k(k-1)/2}U, \quad U \in \Lambda^k.
\]
It is readily seen that
\[
U^{**} = U, \quad (UV)^* = V^*U^*, \quad U, V \in \Lambda.
\]

Now we can define the spinor group
\[
\text{Spin}_V = \{S \in \Lambda^{\text{even}} : S^*S = 1\}.
\]

2. **Tensors with values in \(\Lambda^k\).**

Let
\[
u_{\lambda_1\ldots\lambda_r}^{\mu_1\ldots\mu_k\nu_1\ldots\nu_s}(x) = u_{\lambda_1\ldots\lambda_r, \mu_1\ldots\mu_k}^{\nu_1\ldots\nu_s}(x), \quad x \in V
\]
be components of a tensor field of rank \((r, k+s)\) antisymmetric with respect to the first \(k\) covariant indices. One may consider the following objects:
\[
U_{\nu_1\ldots\nu_s}^{\lambda_1\ldots\lambda_r} = \frac{1}{k!}u_{\lambda_1\ldots\lambda_r, \mu_1\ldots\mu_k}^{\nu_1\ldots\nu_s} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_k}
\]
which are formally written as \(k\)-forms. Under a change of coordinates \((x) \to (\tilde{x})\) the values (3) transform as components of a tensor field of rank \((r, s)\)
\[
\tilde{U}_{\alpha_1\ldots\alpha_r}^{\beta_1\ldots\beta_s} = q_{\beta_1}^{\alpha_1} \ldots q_{\beta_s}^{\alpha_s} p_{\lambda_1}^{\alpha_1} \ldots p_{\lambda_r}^{\alpha_r} U_{\nu_1\ldots\nu_s}^{\lambda_1\ldots\lambda_r},
\]
\[
q_{\beta}^{\alpha} = \frac{\partial x^{\nu}}{\partial \tilde{x}^{\beta}}, \quad p_{\lambda}^{\alpha} = \frac{\partial \tilde{x}^{\alpha}}{\partial x^{\lambda}}.
\]
The objects \( \mathbf{3} \) are called tensors of rank \((r, s)\) with values in \(\Lambda^k\). We write this as

\[ U_{\mu_1...\nu_s}^{\lambda_1...\lambda_r} \in \Lambda^k \mathcal{T}^r_s. \]

Elements of \(\Lambda^0 \mathcal{T}^r_s\) are ordinary tensors of rank \((r, s)\) on \(\mathcal{V}\). For \(U_\mu \in \Lambda^k \mathcal{T}^1_1\) we have

\[ dx^\mu U_\mu \in \Lambda^{k+1} \oplus \Lambda^{k-1}. \]

3 The covariant derivatives \(\nabla_\mu\).

On Riemannian manifolds \(\mathcal{V}\) the Christoffel symbols \(\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}\) (Levi-Civita connectedness components) are defined with the aid of the metric tensor

\[ \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa} \left( \partial_\mu g_{\kappa\nu} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu} \right). \]  

(5)

Let us remind the definition of covariant derivatives \(\nabla_\mu\) acting on tensor fields on \(\mathcal{V}\) by the following rules \((\partial_\mu = \partial/\partial x^\mu)\):

1. If \(t = t(x), \ x \in \mathcal{V}\) is a scalar function (invariant), then
   \[ \nabla_\mu t = \partial_\mu t. \]

2. If \(t^\nu\) is a vector field on \(\mathcal{V}\), then
   \[ \nabla_\mu t^\nu \equiv t^\nu_{\ \ \mu} = \partial_\mu t^\nu + \Gamma^\nu_{\mu\lambda} t^\lambda. \]

3. If \(t_\nu\) is a covector field on \(\mathcal{V}\), then
   \[ \nabla_\mu t_\nu \equiv t_\nu_{\ \ \mu} = \partial_\mu t_\nu - \Gamma^\lambda_{\mu\nu} t^\lambda. \]

4. If \(u = u^{\nu_1...\nu_k}, \ v = v^{\nu_1...\nu_s}\) are tensor fields on \(\mathcal{V}\), then
   \[ \nabla_\mu (u \otimes v) = (\nabla_\mu u) \otimes v + u \otimes \nabla_\mu v. \]

With the aid of these rules it is easy to calculate covariant derivatives of arbitrary tensor fields. Also, it is easy to check the correctness of the following formulas:

\[ \nabla_\mu g_{\nu\lambda} = 0, \ \ \ nabla_\mu g^{\nu\lambda} = 0, \ \ \ nabla_\mu \delta^\nu_{\lambda} = 0. \]
4 The Clifford derivatives $\Upsilon_\mu$.

Let us define the Clifford derivatives $\Upsilon_\mu$ (Upsilon), which act on tensors from $\Lambda^r \Lambda^s$ by the following rules:

1. If $t^{\nu_1 \ldots \nu_s}$ is a covariant tensor field on $\mathcal{V}$ of rank $(r, s)$, then
   \[ \Upsilon_\mu t^{\nu_1 \ldots \nu_s} = \partial_\mu t^{\nu_1 \ldots \nu_s}. \]

2. $\Upsilon_\mu dx^\nu = -\Gamma^\nu_{\mu \lambda} dx^\lambda$.

3. If $U, V \in \Lambda$ and $UV$ is the Clifford product of differential forms, then
   \[ \Upsilon_\mu (UV) = (\Upsilon_\mu U)V + U \Upsilon_\mu V. \]

With the aid of these rules it is easy to calculate how operators $\Upsilon_\mu$ act on arbitrary tensor from $\Lambda^r \Lambda^s$.

If $U \in \Lambda^k$, written as (6), then
\[ \Upsilon_\mu U = \frac{1}{k!} u_{\nu_1 \ldots \nu_k :\mu} dx^{\nu_1} \wedge \ldots \wedge dx^{\nu_k}. \] (6)

That means $\Upsilon_\mu : \Lambda^k \rightarrow \Lambda^{k \top}$. The formula (6) indicate the connection between operators $\Upsilon_\mu$ and $\nabla_\mu$.

If $U^{\nu_1 \ldots \nu_s} \in \Lambda^r \Lambda^s$ and $r + s > 0$, then the values $\Upsilon_\mu U^{\nu_1 \ldots \nu_s}$ are not the components of a tensor (when the curvature is nonzero). In what follows we do not use the Clifford derivatives $\Upsilon_\mu$ as isolated operators acting on tensors from $\Lambda^r \Lambda^s$, $r + s > 0$. But we use them as building blocks of operators, that map tensors to tensors. For example, if $B_\mu \in \Lambda^2 \Lambda^1$, then the expression
\[ \Upsilon_\mu B_\nu - \Upsilon_\nu B_\mu - [B_\mu, B_\nu] \]
is a tensor from $\Lambda^2 \top \Lambda^2$.

Consider the change of coordinates $(x) \rightarrow (\tilde{x})$
\[ p_\nu^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^\nu}, \quad q_\nu^\mu = \frac{\partial x^\mu}{\partial \tilde{x}^\nu}, \quad dx^\mu = q_\nu^\mu d\tilde{x}^\nu, \]
where $p_\nu^\mu, q_\nu^\mu$ are functions of $x \in \mathcal{V}$. Then the Clifford derivatives $\Upsilon_\nu$ in coordinates $x^\mu$ related to the Clifford derivatives $\tilde{\Upsilon}_\mu$ in coordinates $\tilde{x}^\mu$ by the formula
\[ \Upsilon_\nu = p_\nu^\mu \tilde{\Upsilon}_\mu \] (7)
exact the same as formula for partial derivatives $\partial_\nu = p_\mu^\nu \tilde{\partial}_\mu$, where $\partial_\nu = \frac{\partial}{\partial x^\nu}$, $\tilde{\partial}_\nu = \frac{\partial}{\partial \tilde{x}^\nu}$. The proof of this formula is followed from the transformation rule of Christoffel symbols.

The main properties of the operators $\Upsilon_\mu$ are listed below.

1) $\Upsilon_\mu(U^*) = (\Upsilon_\mu U)^*$ for $U \in \Lambda$.
2) $\Upsilon_\mu(*U) = *(\Upsilon_\mu U)$ for $U \in \Lambda$.
3) $\Upsilon_\mu(\text{Tr } U) = \text{Tr}(\Upsilon_\mu U)$ for $U \in \Lambda$.

From the formula $\Upsilon_\mu dx^\lambda = -\Gamma_\mu^\nu_\lambda dx^\nu$ we get

$$\begin{align*}
(\Upsilon_\mu \Upsilon_\nu - \Upsilon_\nu \Upsilon_\mu) dx^\lambda &= -R^\lambda_{\mu\nu\rho} dx^\rho, \\
\text{where} \\
R^\kappa_{\lambda\mu\nu} &= \partial_\lambda \Gamma^\kappa_{\nu\mu} - \partial_\nu \Gamma^\kappa_{\lambda\mu} + \Gamma^\kappa_{\mu\eta} \Gamma^\eta_{\nu\lambda} - \Gamma^\kappa_{\nu\lambda} \Gamma^\eta_{\mu\eta} \\
\text{(9)}
\end{align*}$$

is the rank (1,3) tensor, known as the curvature tensor (or Riemannian tensor). Consider the antisymmetric tensor from $\Lambda^2 \Upsilon_2$ such that

$$C_{\mu\nu} = \frac{1}{2} R^\kappa_{\alpha\beta\mu\nu} dx^\alpha \wedge dx^\beta.$$

Theorem 1. For all $U \in \Lambda$

$$\begin{align*}
(\Upsilon_\mu \Upsilon_\nu - \Upsilon_\nu \Upsilon_\mu) U &= \frac{1}{2} [C_{\mu\nu}, U]. \\
\text{The proof is by direct calculation.}
\end{align*}$$

If $U$ is invertible (w.r.t. Clifford multiplication) differential form from $\Lambda$, then the relation (10) can be written as

$$\begin{align*}
\frac{1}{2} C_{\mu\nu} &= U^{-1}(\frac{1}{2} C_{\mu\nu}) U - U^{-1}(\Upsilon_\mu \Upsilon_\nu - \Upsilon_\nu \Upsilon_\mu) U. \\
\text{(11)}
\end{align*}$$

Let $B_\mu \in \Lambda^2 \Upsilon_1$ be a tensor such that

$$\begin{align*}
\Upsilon_\mu B_\nu - \Upsilon_\nu B_\mu - [B_\mu, B_\nu] &= \frac{1}{2} C_{\mu\nu}. \\
\text{(12)}
\end{align*}$$

The existence of solutions of this equation must be investigated.
Theorem 2. If $B_\mu \in \Lambda^2 \mathcal{T}_1$ satisfy (12) and $S \in \text{Spin}_\mathcal{V}$, then $B'_\mu = S^{-1} B_\mu S - S^{-1} \Upsilon_\mu S$ also satisfy (12).

Proof. It is easily shown that the formula (12) is invariant under the following gauge transformation with Spin$_\mathcal{V}$ symmetry group:

$$
\begin{align*}
B_\mu &\to B'_\mu = S^{-1} B_\mu S - S^{-1} \Upsilon_\mu S, \\
\frac{1}{2} C_{\mu\nu} &\to \frac{1}{2} C'_{\mu\nu} = S^{-1} \left( \frac{1}{2} C_{\mu\nu} \right) S - S^{-1} (\Upsilon_\mu \Upsilon_\nu - \Upsilon_\nu \Upsilon_\mu) S,
\end{align*}
$$

where $S \in \text{Spin}_\mathcal{V}$. Substituting $U = S$ to the formula (11), we obtain $C'_{\mu\nu} \equiv C_{\mu\nu}$. This completes the proof.

Now we may connect our construction to the following linear system of differential equations:

$$
\Upsilon_\mu U - [B_\mu, U] = 0, \quad (14)
$$

where $U \in \Lambda$ is an unknown differential form and $B_\mu$ satisfies (12). There are three things to be said about this system of equations.

The first. The equations (14) are invariant under the gauge transformation

$$
U \to S^{-1} US, \quad B_\mu \to S^{-1} B_\mu S - S^{-1} \Upsilon_\mu S, \quad S \in \text{Spin}_\mathcal{V}.
$$

The second. From (14) we may get the following equalities:

$$
\Upsilon_\mu (\Upsilon_\nu U - [B_\nu, U]) - \Upsilon_\nu (\Upsilon_\mu U - [B_\mu, U]) = 0. 
$$

If $U \in \Lambda$ satisfies (14), then $U$ also satisfies (13). Hence equalities (15) can be considered as necessary conditions for the existence of a solution of the equations (14). We see that equalities (13) are equivalent to the equalities (10), which are valid for any $U \in \Lambda$.

The third. If $U_1$ and $U_2$ satisfy (14), then $U_1 U_2$ and $U_1 + U_2$ also satisfy (14).

5 Equations for the electron on Riemannian manifolds.

In [1] we prove that in Minkowski space the Dirac equation for the electron can be written as a tensor equation (the tensor Dirac equation). Now we
present the following system of tensor equations on Riemannian manifolds, which can be considered as a generalization of the tensor Dirac equation:

$$dx^\mu (\Upsilon_\mu \Psi + \Psi I a_\mu + \Psi B_\mu) + m \Psi HI = 0,$$

$$\Upsilon_\mu H = [B_\mu, H], \quad \Upsilon_\mu I = [B_\mu, I], \quad H^2 = 1, \quad I^2 = -1, \quad [H, I] = 0,$$  \hspace{1cm} (16)

where

$$\Psi \in \Lambda^{\text{even}}, \quad I \in \Lambda^2, \quad H \in \Lambda^1, \quad a_\mu \in \Lambda^{0+1}, \quad B_\mu \in \Lambda^{2+1},$$  \hspace{1cm} (17)

$m \geq 0$ is a real constant, and $B_\mu$ satisfies (12). We suppose that in (16) the differential forms $\Psi, H, I$ are unknown and the tensors $a_\mu, B_\mu$ are known.

**Theorem 3.** The system of equations (16) is invariant under the gauge transformation

$$\Psi \rightarrow \Psi \exp(\lambda I), \quad a_\mu \rightarrow a_\mu - \partial_\mu \lambda, \quad (H, I, B_\mu) \rightarrow (H, I, B_\mu),$$  \hspace{1cm} (18)

where $\lambda = \lambda(x) \in \Lambda^0$ and $\exp(\lambda I) = \cos \lambda + I \sin \lambda$.

**Proof.** Denote $S = \exp(\lambda I)$. We have

$$\Upsilon_\mu S = (\Upsilon_\mu I) \sin \lambda + I \partial_\mu \lambda (\cos \lambda + I \sin \lambda)$$  \hspace{1cm} (19)

Multiplying the first equation in (16) from the right by $S$ and denoting $\Psi' = \Psi S$, we obtain

$$dx^\mu (\Upsilon_\mu \Psi' + \Psi' (-S^{-1} \Upsilon_\mu S + a_\mu I + S^{-1} B_\mu S)) + m \Psi' IH = 0.$$ 

If we substitute $\Upsilon_\mu S$ from (19) to this equation, then we get

$$-S^{-1} \Upsilon_\mu S + a_\mu I + S^{-1} B_\mu S = (a_\mu - \partial_\mu \lambda) I + B_\mu.$$ 

This completes the proof.

**Theorem 4.** The system of equations (16) is invariant under the gauge transformation

$$\Psi \rightarrow \Psi S, \quad H \rightarrow S^{-1} HS, \quad I \rightarrow S^{-1} IS, \quad B_\mu \rightarrow S^{-1} B_\mu S - S^{-1} \Upsilon_\mu S, \quad a_\mu \rightarrow a_\mu,$$  \hspace{1cm} (20)

where $S \in \text{Spin}_\nu$.

The proof is evident.
The conservative law and the Lagrangian.

With the aid of the 1-form $H$ we define the operation of conjugation

$$\bar{U} = HU^*, \quad U \in \Lambda.$$  

**Lemma.** Suppose $\Psi, H, I, a_\mu, B_\mu$ are chosen as in (17) and

$$L = \Psi^*(dx^\mu(\Upsilon_\mu\Psi + \Psi Ia_\mu + \Psi B_\mu) + m\Psi H).$$

Then the conjugated differential form $\bar{L}$ can be written as

$$\bar{L} = (\Upsilon_\mu\bar{\Psi} - a_\mu I\bar{\Psi} - B_\mu \bar{\Psi})dx^\mu - mIH\bar{\Psi}. $$

Proof is by direct calculation.

**Theorem 5.** Let $\Psi, H, I, a_\mu, B_\mu$ satisfy (14), (17) and $j^\mu = \text{Tr}(\bar{\Psi}dx^\mu\Psi)$. Then

$$\partial_\mu(\sqrt{-g} j^\mu) = 0. \quad (21)$$

The identity (21) is called a conservative law for the equation (16). The vector $j^\mu$ is called a current.

Proof. It can be checked that

$$\text{Tr}(H(L + L^*)) = \frac{\partial_\mu(\sqrt{-g} j^\mu)}{\sqrt{-g}}.$$  

For a solution of the equation (16) we have $L = 0$ and so we obtain the conservative law (21). This completes the proof.

Let us define the Lagrangian (the Lagrangian density) from which the main equation (16) can be derived

$$\mathcal{L} = \text{Tr}(\sqrt{-g} HLI) = \text{Tr}(\sqrt{-g}\Psi(dx^\mu(\Upsilon_\mu\Psi + \Psi Ia_\mu + \Psi B_\mu)I - m\Psi H).$$

Note that this Lagrangian is invariant under the gauge transformations (18) and (20).

Using the variational principle [5] we suppose that in the Lagrangian $\mathcal{L}$ the differential forms $\Psi$ and $\bar{\Psi}$ are independent and as variational variables we take 8 functions which are the coefficients of the differential form $\bar{\Psi}$. The Lagrange-Euler equations with respect to these variables give us the system of equations, which can be written in the form (16).
7 The operators $d, \delta, \Upsilon$.

With the aid of Clifford derivatives one can define three differential operators of first order, which map $\Lambda$ into $\Lambda$. By definition, put

$$dV = dx^\mu \Upsilon_\mu V,$$

$$\Upsilon V = dx^\mu \Upsilon_\mu V,$$

$$\delta V = dV - \Upsilon V,$$

for $V \in \Lambda$. Some of the properties of these operators are listed below.

- $d : \Lambda^k \rightarrow \Lambda^{k+1}$;
- $d^2 = 0$;
- $d(U \wedge V) = dU \wedge V + (-1)^k U \wedge dV, U \in \Lambda^k, V \in \Lambda$;
- $\delta : \Lambda^k \rightarrow \Lambda^{k-1}$;
- $\delta^2 = 0$;
- $\delta U = \star d \star U, U \in \Lambda$;
- $\Upsilon : \Lambda^k \rightarrow \Lambda^{k-1} \oplus \Lambda^{k+1}$;

The operator $d$ is called the *exterior differential* (or generalized gradient), the operator $\Upsilon$ is called the *Clifford differential*, and the operator $\delta$ is called the *generalized divergence*.

8 The Maxwell equations and QED equations.

It is well known that the Maxwell equations on Riemannian manifolds have the form

$$dA = F, \quad \delta F = \alpha J,$$

(22)

where $A = a_\mu dx^\mu \in \Lambda^1, F = \frac{1}{2} f_{\mu\nu} dx^\mu \wedge dx^\nu \in \Lambda^2, J \in \Lambda^1$, and $\alpha$ is a constant. Using the properties $d^2 = 0, \delta^2 = 0$ we get from (22) that

$$dF = 0, \quad \delta J = 0.$$
If we consider the Lagrangian
\[ \mathcal{L}_{\text{Maxwell}} = \text{Tr}(\sqrt{-g} \, F^2) = -\frac{1}{2} \sqrt{-g} \, f_{\mu\nu} f^{\mu\nu} \]
and take as the variational variables \( a_\mu \), then we obtain the equations \( (22) \).

Now we can join the systems of equations \( (16) \) and \( (22) \) and obtain the system of equations

\[ dx^\mu (\Upsilon_\mu \Psi + \Psi B_\mu) + A \Psi I + m \Psi H I = 0, \]
\[ \Upsilon_\mu H = [B_\mu, H], \quad \Upsilon_\mu I = [B_\mu, I], \quad \Upsilon H = \sum_{\mu} \Upsilon_\mu H, \quad \Upsilon I = \sum_{\mu} \Upsilon_\mu I, \quad \Upsilon^2 = 1, \quad \Upsilon = \sum_{\mu} \Upsilon_\mu, \quad \Upsilon^2 = -1, \quad [H, I] = 0, \]
\[ H^2 = 1, \quad I^2 = -1, \quad \delta J = 0. \]

where \( J = \Psi \bar{\Psi} = \Psi H \Psi = j_\mu dx^\mu, \quad j_\mu = \text{Tr}(\bar{\Psi} dx^\mu \Psi) \), and conservative law \( (21) \) can be written with the aid of 1-form \( J \) as \( \delta J = 0 \).

In the system of equations \( (23) \) we consider the differential forms \( \Psi, H, I, A, F \) as unknown and the tensor \( B_\mu \) as known.

**Physical interpretation.** We suppose that the system of equations \( (23) \) describes the local interaction of two physical fields. Namely the field of matter \( \{\Psi, H, I\} \) (which is identified with the wave function of the electron) and the electromagnetic field \( \{A, F\} \). So the equations \( (23) \) is QED equations with presence of the gravity field \( \{B_\mu, C_{\mu\nu}\} \).

Let us define the differential operators of the first order \( \mathcal{D}_\mu \), which act on tensors from \( \Lambda^r T \)
\[ \mathcal{D}_\mu U = \Upsilon_\mu U - [B_\mu, U] \]
and put
\[ \mathcal{D} = dx^\mu \mathcal{D}_\mu. \]

Using the operators \( \mathcal{D}_\mu \) the gauge transformation \( B_\mu \rightarrow S^{-1} B_\mu - S^{-1} \Upsilon_\mu S, \quad S \in \text{Spin}_V \) can be written as
\[ B_\mu \rightarrow B_\mu - S^{-1} \mathcal{D}_\mu S, \quad S \in \text{Spin}_V. \quad (24) \]
and the equations (23), together with the (12), can be written as

\[ D \Psi + A\Psi I + B\Psi + m\Psi HI = 0, \]

\[ D_\mu B_\nu - D_\nu B_\mu + [B_\mu, B_\nu] = \frac{1}{2} C_{\mu\nu}, \]

\[ D_\mu H = 0, \quad D_\mu I = 0, \]

\[ H^2 = 1, \quad I^2 = -1, \quad [H, I] = 0, \]

\[ dA = F, \quad \delta F = \alpha J, \]

where \( \alpha \) is a constant, \( B = dx^\mu B_\mu \), and \( J = \Psi \bar{\Psi} = \Psi H \Psi^* \).

**Theorem 7.** The system of equations (23) is invariant under the gauge transformations

\[ \Psi \rightarrow \Psi S, \quad H \rightarrow S^{-1} HS, \quad I \rightarrow S^{-1} IS, \quad B_\mu \rightarrow S^{-1} B_\mu S - S^{-1} \Upsilon_\mu S, \]

\[ (A, F, C_{\mu\nu}) \rightarrow (A, F, C_{\mu\nu}) \]

and

\[ \Psi \rightarrow \Psi \exp(\lambda I), \quad A \rightarrow A - d\lambda, \quad (H, I, F, B_\mu, C_{\mu\nu}) \rightarrow (H, I, F, B_\mu, C_{\mu\nu}), \]

where \( S \in \text{Spin}_V \) and \( \lambda \in \Lambda^0 \).

A proof follows from the theorems 3,4,6.

We suppose that in the system of equations (23) the forms \( \Psi, H, I, A, F, B_\mu \) are unknown and the forms \( C_{\mu\nu} \) (the curvature tensor) are known. As a gravity Lagrangian we may take Einstein-Hilbert Lagrangian \( \sqrt{-g} R \), where \( R \) is the scalar curvature. Also, there is an interesting possibility to try to describe a gravity field using Lagrangian \( \text{Tr}(\sqrt{-g} C_{\mu\nu} C^{\mu\nu}) \). Further development for this part of the model is needed.

### 9 Equations in Minkowski space.

Let us suppose that Riemannian manifolds \( V \) under consideration is such that in coordinates \( x^\mu \) the metric tensor has the form

\[ \|g_{\mu\nu}\| = \|g^{\mu\nu}\| = \text{diag}(1, -1, -1, -1). \]
So we may consider this Riemannian manifolds as Minkowski space admitting only Lorentzian changes of coordinates. Let $e_{\mu}$ be basis coordinate vectors and $e^{\mu} = g^{\mu\nu} e_{\nu}$ be basis covectors. Differential forms $\frac{1}{k!} u_{\mu_1...\mu_k} dx^{\mu_1} \wedge ... \wedge dx^{\mu_k}$ can be written as the exterior forms

$$\frac{1}{k!} u_{\mu_1...\mu_k} e^{\mu_1} \wedge ... \wedge e^{\mu_k}.$$ 

In Minkowski space $\Gamma_{\alpha \mu \nu} \equiv 0$ and 

$$\Upsilon_\mu V = \partial_\mu V, \quad V \in \Lambda.$$

The curvature tensor $R_{\mu \nu \alpha \beta} \equiv 0$. The tensor $B_\mu \in \Lambda^2 \top_1$ is such that

$$\partial_\mu B_\nu - \partial_\nu B_\mu - [B_\mu, B_\nu] = 0. \quad (26)$$

The solutions of this equations are

$$B_\mu = U^{-1} \partial_\mu U, \quad U \in \text{Spin}(1, 3). \quad (27)$$

By the Theorem 2, the covector

$$B'_\mu = S^{-1} B_\mu S - S^{-1} \partial_\mu S = (U S)^{-1} \partial_\mu (U S)$$

also satisfy (26).

Now we may consider the system of equations (16) in Minkowski space

$$e^{\mu}(\partial_\mu \Psi + \Psi I a_\mu + \Psi B_\mu) + m \Psi H I = 0,$$

$$\partial_\mu H = [B_\mu, H], \quad \partial_\mu I = [B_\mu, I],$$

$$H^2 = 1, \quad I^2 = -1, \quad [H, I] = 0, \quad (28)$$

where $\Psi \in \Lambda^{\text{even}}$, $I \in \Lambda^2$, $H \in \Lambda^1$, $a_\mu \in \Lambda^0 \top_1$, and $B_\mu \in \Lambda^2 \top_1$ satisfy (27). According to the Theorem 4, the system of equations (28) is invariant under the gauge transformation (20) which depends on an exterior form $S = S(x) \in \text{Spin}(1, 3)$.

The equations (28) are the new form of the tensor Dirac equation with a Spin(1, 3) gauge symmetry in Minkowski space. A wave function of the electron is identified with the full set $\{\Psi, H, I\}$.

If we take $S = U^{-1}$, then the gauge transformation (20) has the form

$$\Psi \rightarrow \Psi' = \Psi S, \quad H \rightarrow H' = S^{-1} H S, \quad I \rightarrow I' = S^{-1} I S, \quad B_\mu \rightarrow 0, \quad a_\mu \rightarrow a_\mu.$$
and the system of equations (28) is identical to the tensor Dirac equation [1]

\begin{align*}
\epsilon^\mu (\partial_\mu \Psi' + \Psi' I' a_\mu) + m \Psi' H' I' &= 0, \\
\partial_\mu H' &= 0, \\
\partial_\mu I' &= 0, \\
(H')^2 &= 1, \\
(I')^2 &= -1, \\
[H', I'] &= 0,
\end{align*}

(29)

This system of equations is invariant under the global transformation

\begin{align*}
\Psi' &\rightarrow \Psi' V, \\
H' &\rightarrow V^{-1} H' V, \\
I' &\rightarrow V^{-1} I' V, \\
a_\mu &\rightarrow a_\mu,
\end{align*}

where \( V \in \text{Spin}(1, 3) \) and \( \partial_\mu V = 0. \)

10 A geometrical interpretation of the model.

Let \( \mathcal{M} \) be a four dimensional differentiable manifolds and let \( \mathcal{V} = \{\mathcal{M}, g_{\mu\nu}\} \) be the Riemannian manifolds with Levi-Civita connection \( \Gamma^\lambda_{\mu\nu} \), with the covariant derivatives \( \nabla_\mu \), with the Clifford derivatives \( \Upsilon_\mu \), and with the curvature tensor \( R_{\alpha\beta\mu\nu} \) defined in previous sections. Suppose that a new structure on \( \mathcal{V} \) is given. Namely the affine connection \( \tilde{\Gamma}^\lambda_{\mu\nu} \). We get definitions of the covariant derivatives \( \tilde{\nabla}_\mu \), the Clifford derivatives \( \tilde{\Upsilon}_\mu \), and the curvature tensor \( \tilde{R}_{\alpha\beta\mu\nu} \), replacing \( \Gamma^\lambda_{\mu\nu} \) by \( \tilde{\Gamma}^\lambda_{\mu\nu} \) in the corresponding definitions in previous sections. We suppose that the affine connection \( \tilde{\Gamma}^\lambda_{\mu\nu} \) is metric compatible

\[ \tilde{\nabla}_\kappa g_{\mu\nu} = 0, \quad \tilde{\nabla}_\kappa g^{\mu\nu} = 0. \]

It is convenient to introduce the tensor

\[ K^\lambda_{\mu\nu} = \tilde{\Gamma}^\lambda_{\mu\nu} - \Gamma^\lambda_{\mu\nu}, \]

which we, following [8], will call contorsion. It is easy to see that the affine connection is metric compatible iff \( K^\lambda_{\nu\mu\lambda} = -K^\lambda_{\lambda\mu\nu} \). Torsion is expressed via contorsion as

\[ T^\lambda_{\mu\nu} = K^\lambda_{\mu\nu} - K^\lambda_{\nu\mu}. \]

Conversely, the contorsion of a metric compatible connection is expressed via torsion as

\[ K^\lambda_{\mu\nu} = \frac{1}{2}(T^\lambda_{\mu\nu} + T^\lambda_{\nu\mu} + T^\lambda_{\nu\mu}). \]
So we arrive at the affine space \( \{ M, g_{\mu
u}, K^\lambda_{\mu\nu} \} \). Let us define the tensors

\[
\begin{align*}
b_{\alpha\beta\mu} &= -\frac{1}{2} K_{\alpha\mu\beta}, \\
B_\mu &= \frac{1}{2} b_{\alpha\beta\mu} dx^\alpha \wedge dx^\beta \in \Lambda^2 T_1, \\
C_{\mu\nu} &= \frac{1}{2} R_{\alpha\beta\mu\nu} dx^\alpha \wedge dx^\beta \in \Lambda^2 T_2.
\end{align*}
\]

**Theorem 8.** If \( U \in \Lambda \), then

\[
\tilde{\Upsilon}_\mu U = \Upsilon_\mu U - [B_\mu, U].
\]

**Proof** follows from the formula

\[
K^\nu_{\mu\lambda} dx^\lambda = [B_\mu, dx^\nu],
\]

which can be easily checked.

**Theorem 9.** (F.E.Burstall, A.D.King, N.G.Marchuk, D.G.Vassiliev) The following equality holds

\[
\Upsilon_\mu B_\nu - \Upsilon_\nu B_\mu - [B_\mu, B_\nu] = \frac{1}{2} C_{\mu\nu}
\]

iff

\[
\tilde{\mathcal{R}}_{\alpha\beta\mu\nu} = 0.
\]

**Proof.** Suppose that the tensors \( q_{\alpha\beta\mu\nu} \) and \( \tilde{\mathcal{R}}_{\alpha\beta\mu\nu} \) are such that

\[
\frac{1}{2} q_{\alpha\beta\mu\nu} dx^\alpha \wedge dx^\beta = \Upsilon_\mu B_\nu - \Upsilon_\nu B_\mu - [B_\mu, B_\nu] - \frac{1}{2} C_{\mu\nu}
\]

and

\[
\tilde{\mathcal{R}}_{\alpha\beta\mu\nu} = g_{\kappa\alpha}(\partial_\mu \tilde{\Gamma}^\kappa_{\nu\lambda} - \partial_\nu \tilde{\Gamma}^\kappa_{\mu\lambda} + \tilde{\Gamma}^\kappa_{\mu\eta} \tilde{\Gamma}^\eta_{\nu\lambda} - \tilde{\Gamma}^\kappa_{\nu\eta} \tilde{\Gamma}^\eta_{\mu\lambda}).
\]

Then it can be easily checked that

\[
\tilde{\mathcal{R}}_{\alpha\beta\mu\nu} = -2 q_{\alpha\beta\mu\nu}.
\]

This completes the proof.
This theorem leads us to the conclusion that the equations (25) can be considered as equations in the flat affine space ($\hat{R}_{\alpha\beta\mu\nu} = 0$, $D_\mu = Y_\mu$).

**Postulate** (The flat affine field model of gravitation). We suppose that the physical space-time is a flat affine space such that

(i) The metric tensor $g_{\mu\nu}$ satisfies conditions of the first section.

(ii) The affine connection $\hat{\Gamma}_{\lambda\mu\nu}^{\lambda}$ is metric compatible and defined via connection $K_{\lambda\mu\nu}^{\lambda}$ or, equivalently, by $B_\mu$.

(iii) The affine connection curvature $\hat{R}_{\alpha\beta\mu\nu} = 0$.

Then the gravitation field is the pair $\{B_\mu, C_{\mu\nu}\}$, where $B_\mu$ is identified with a potential of gravity and $C_{\mu\nu}$ is identified with a strength of gravity.

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