On Negacyclic MDS-Convolutional Codes

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Abstract

New families of classical and quantum optimal negacyclic convolutional codes are constructed in this paper. This optimality is in the sense that they attain the classical (quantum) generalized Singleton bound. The constructions presented in this paper are performed algebraically and not by computational search.

1 Introduction

Much effort have been paid in order to construct good quantum error-correcting codes (QECC) [4, 9, 22, 24, 25, 36, 44] as well as quantum convolutional codes with good parameters [1–3, 13–16, 27, 37, 38]. On the other hand, the investigation of the class of (classical) convolutional codes and their corresponding properties as well as constructions of maximum-distance-separable (MDS) convolutional codes (i.e., codes attaining the generalized Singleton bound [41]) have also been presented in the literature [11, 12, 17, 26–34, 39, 41–43].

In this paper, we utilize the class of negacyclic codes [5–8, 10, 23] in order to construct classical and quantum MDS convolutional codes. More precisely, we apply the famous method proposed by Piret [39] (generalized recently by Aly et al. [2]), which consists in the construction of (classical) convolutional codes derived from block codes. An advantage of our techniques of construction lie in the fact that all new (classical and quantum) convolutional codes are generated algebraically and not by computational search, in contrast with many works where only exhaustively computational search or even specific codes are constructed.

Our classical convolutional MDS codes constructed here have parameters

• \((n, n - 2i + 1, 2; 1, 2i + 2)_{q^2}\), where \(q \equiv 1 \pmod{4}\) is a power of an odd prime, \(n = q^2 + 1\) and \(2 \leq i \leq n/2 - 1\);

• \((n, n - 2i + 2, 2; 1, 2i + 1)_{q^2}\), where \(q\) is a power of an odd prime, \(n = (q^2 + 1)/2\) and \(2 \leq i \leq (n - 1)/2\);

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• \((n, n - 2i + 1, 2; 1, 2i + 2)_{q^2}\), where \(q \geq 5\) is a power of an odd prime, 
\(n = (q^2 + 1)/2\) and \(2 \leq i \leq (n - 1)/2 - 1\).

The new convolutional stabilizer MDS codes have parameters

• \([(n, n - 4i + 2, 1; 2, 2i + 2)]_{q^2}\), where \(q \equiv 1 \pmod{4}\) is a power of an odd prime, 
\(n = q^2 + 1\) and \(2 \leq i \leq (q - 1)/2\);

• \([(n, n - 4i + 4, 1; 2, 2i + 1)]_{q^2}\), where \(q \geq 7\) is a power of an odd prime, 
\(n = (q^2 + 1)/2\) and \(2 \leq i \leq (q - 1)/2\).

We observe that the order between the degree and the memory are changed when comparing the parameters of classical and quantum convolutional codes. We adopt this notation to keep the same notation utilized in [2].

The paper is organized as follows. In Sections 2 we review basic concepts on negacyclic codes. In Sections 3 and 4, we review of concepts concerning classical and quantum convolutional codes, respectively. In Section 5, we propose constructions of new families of classical MDS convolutional derived from negacyclic codes. In Section 6 we construct new optimal (MDS) quantum convolutional codes and, in Section 7, a brief summary of this work is described.

### 2 Negacyclic codes

The class of negacyclic codes [5–8, 10, 21, 23] have been studied in the literature. This class of codes are a particular class of a more general class of constacyclic codes [8]. In this section we review the basic concepts of these codes.

Throughout this paper, we always assume that \(q\) is a power of an odd prime, \(F_q\) is a finite field with \(q\) elements and \(n\) is a positive integer with \(\gcd(n, q) = 1\). Analogously to cyclic codes, if we consider the quotient ring \(R_n = F_q/(x^n + 1)\), then a negacyclic code is a principal ideal of \(R_n\) under the usual correspondence \(c = (c_0, c_1, \ldots, c_{n-1}) \rightarrow c_0 + c_1x + \ldots + c_{n-1}x^{n-1} \pmod{x^n + 1})\). The generator polynomial \(g(x)\) of a negacyclic code \(C\) satisfies \(g(x)|(x^n + 1)\). The roots of \((x^n + 1)\) are the roots of \((x^{2n} - 1)\) which are not roots of \((x^n - 1)\) in some extension field of \(F_{q^2}\) (since we will work with codes endowed with the Hermitian inner product).

Consider that \(m = \text{ord}_{2n}(q^2)\) and let \(\beta\) be a primitive \(2^m\)th root of unity in \(F_{q^2}\) (so \(\alpha = \beta^2 \in F_{q^2}\) is a primitive \(n\)th root of unity). Then the roots of \(x^n + 1\) are given by \(\beta^{2i+1}\ 0 \leq i \leq n − 1\). Put \(\Omega_{2n} = \{1, 3, \ldots, 2n − 1\}\), the defining set of a negacyclic code \(C\) of length \(n\) generated by \(g(x)\) is given by \(Z = \{i \in \Omega_{2n}|\beta^i \text{ is root of } g(x)\}\). The defining set is a union of \(q^2\)-ary cyclotomic cosets given by \(\mathcal{C}_i = \{i, iq^2, \ldots, iq^2(m_i-1)\}\), where \(m_i\) is the smallest positive integer such that \(iq^{2m_i} \equiv i \pmod{2n}\). The minimal polynomial (over \(F_{q^2}\)) of \(\beta^j \in F_{q^{2m}}\) is denoted by \(M^{(j)}(x)\) and it is given by \(M^{(j)}(x) = \prod_{j \in \mathcal{C}_i} (x-\beta^j)\).

The dimension of \(C\) is given by \(n - |Z|\). The BCH bound for Constacyclic codes (see [5, 23]) asserts that is \(C\) is a \(q^2\)-ary negacyclic code of length \(n\) with
generator polynomial $g(x)$ and if $g(x)$ has the elements \( \{ \beta^{2i+1} | 0 \leq i \leq d - 2 \} \) as roots, where $\beta$ is a primitive $2n$th root of unity, then the minimum distance $d_C$ of $C$ satisfies $d_C \geq d$.

### 3 Classical Convolutional Codes

The class of (classical) convolutional codes is a well-studied class of codes [2, 3, 12, 18, 19, 39]. We assume the reader is familiar with the theory of convolutional codes. For more details, see [19]. Recall that a polynomial encoder matrix \( G(D) = (g_{ij}) \in \mathbb{F}_q[D]^{k \times n} \) is called basic if $G(D)$ has a polynomial right inverse. A basic generator matrix is called reduced (or minimal [18, 32, 43]) if the overall constraint length $\gamma = \sum g_{ij}$, where $\gamma_i = \max_{1 \leq j \leq n} \{ \deg g_{ij} \}$, has the smallest value among all basic generator matrices (in this case the overall constraint length $\gamma$ will be called the degree of the resulting code).

**Definition 3.1** [3] A rate $k/n$ convolutional code $C$ with parameters $(n, k, \gamma; \mu, d_f)_q$ is a submodule of $\mathbb{F}_q[D]^n$ generated by a reduced basic matrix $G(D) = (g_{ij}) \in \mathbb{F}_q[D]^{k \times n}$, that is, $C = \{ u(D)G(D) \mid u(D) \in \mathbb{F}_q[D]^k \}$, where $n$ is the length, $k$ is the dimension, $\gamma = \sum g_{ij}$ is the degree, $\mu = \max_{1 \leq i \leq k} \{ \gamma_i \}$ is the memory and $d_f = \min \{ \wt(v(D)) \mid v(D) \in C, v(D) \neq 0 \}$ is the free distance of the code.

The Hermitian inner product is defined as $\langle u(D) \mid v(D) \rangle_h = \sum_i u_i \cdot v_i^q$, where $u_i, v_i \in \mathbb{F}_q^n$ and $v_i^q = (v_{i1}^q, \ldots, v_{in}^q)$. The Hermitian dual of the code $C$ is defined by $C^\perp = \{ u(D) \in \mathbb{F}_q[D]^n : \langle u(D) \mid v(D) \rangle_h = 0 \mbox{ for all } v(D) \in C \}$.

Let $C$ an $[n, k, d_f]_q$ block code with parity check matrix $H$. We split $H$ into $\mu + 1$ disjoint submatrices $H_i$ such that $H = \begin{bmatrix} H_0 \\ H_1 \\ \vdots \\ H_\mu \end{bmatrix}$, where each $H_i$ has $n$ columns, obtaining the polynomial matrix $G(D) = \tilde{H}_0 + \tilde{H}_1 D + \tilde{H}_2 D^2 + \ldots + \tilde{H}_\mu D^\mu$, where the matrices $\tilde{H}_i$, for all $1 \leq i \leq \mu$, are derived from the respective matrices $H_i$ by adding zero-rows at the bottom in such a way that the matrix $\tilde{H}_i$ has $\kappa$ rows in total, where $\kappa$ is the maximal number of rows among the matrices $H_i$. As it is well known, the matrix $G(D)$ generates a convolutional code. Note that $\mu$ is the memory of the resulting convolutional code generated by the matrix $G(D)$.

**Theorem 3.1** [2, Theorem 3] Let $C \subseteq \mathbb{F}_q^n$ be a linear code with parameters $[n, k, d_f]_q$ and assume also that $H \in \mathbb{F}_q^{(n-k) \times n}$ is a parity check matrix for $C$ partitioned into submatrices $H_0, H_1, \ldots, H_\mu$ as above such that $\kappa = r_0 H_0$ and
\(\text{rk} H_i \leq \kappa\) for \(1 \leq i \leq \mu\) and consider the polynomial matrix \(G(D)\) as given above. Then we have:

(a) The matrix \(G(D)\) is a reduced basic generator matrix;

(b) If \(C^\perp \subset C\) (resp. \(C^{\perp h} \subset C\)), then the convolutional code \(V = \{v(D) = u(D)G(D) \mid u(D) \in \mathbb{F}_q^{-k}[D]\}\) satisfies \(V \subset V^\perp\) (resp. \(V \subset V^{\perp h}\));

(c) If \(d_f\) and \(d_f^\perp\) denote the free distances of \(V\) and \(V^\perp\), respectively, \(d_i\) denote the minimum distance of the code \(C_i = \{v \in \mathbb{F}_q^n \mid vH_i^T = 0\}\) and \(d^\perp\) is the minimum distance of \(C^\perp\), then one has \(\min\{d_0, d_\mu, d\} \leq d_f^\perp \leq d\) and \(d_f \geq d^\perp\).

Recall that the (classical) generalized Singleton bound [41, Theorem 2.2] of an \((n, k, \gamma; \mu, d_f)_q\) convolutional code is given by

\[
d_f \leq (n - k)[\lceil \gamma/k \rceil + 1] + \gamma + 1
\]

If the parameters of a convolutional code \(C\) satisfies (1) with equality then \(C\) is said maximum-distance-separable (MDS).

### 4 Quantum Convolutional Codes

A quantum convolutional code is defined by means of its stabilizer, which is a subgroup of the infinite version of the Pauli group, consisting of tensor products of generalized Pauli matrices acting on a semi-infinite stream of qudits. The stabilizer can be defined by a stabilizer matrix of the form

\[
S(D) = (X(D) \mid Z(D)) \in \mathbb{F}_q[D]^{(n-k) \times 2n}
\]

satisfying \(X(D)Z(1/D)^t - Z(D)X(1/D)^t = 0\) (symplectic orthogonality). More precisely, consider a quantum convolutional code \(C\) defined by a full-rank stabilizer matrix \(S(D)\) given above. Then \(C\) is a rate \(k/n\) code with parameters \([(n, k, \mu; \gamma, d_f)]_q\), where \(n\) is the frame size, \(k\) is the number of logical qudits per frame, \(\mu = \max_{1 \leq i \leq n-k, 1 \leq j \leq n} \{\max\{\deg X_{ij}(D), \deg Z_{ij}(D)\}\}\) is the memory, \(d_f\) is the free distance and \(\gamma\) is the degree of the code. Similarly as in the classical case, the constraint lengths are defined as \(\gamma_i = \max_{1 \leq i \leq n} \{\max\{\deg X_{ij}(D), \deg Z_{ij}(D)\}\}\), and the overall constraint length is defined as

\[
\gamma = \sum_{i=1}^{n-k} \gamma_i.
\]

Next, let \(\mathbb{H} = \mathbb{C}^n = \mathbb{C}^q \otimes \ldots \otimes \mathbb{C}^q\) be the Hilbert space and \(|x\rangle\) be the vectors of an orthonormal basis of \(\mathbb{C}^q\), where the labels \(x\) are elements of \(\mathbb{F}_q\). Consider \(a, b \in \mathbb{F}_q\) and take the unitary operators \(X(a)\) and \(Z(b)\) in \(\mathbb{C}^q\) defined by \(X(a)|x\rangle = |x + a\rangle\) and \(Z(b)|x\rangle = w^{tr(bx)}|x\rangle\), respectively, where \(w = \exp(2\pi i/p)\) is a primitive \(p\)-th root of unity, \(p\) is the characteristic of \(\mathbb{F}_q\) and \(tr\) is the trace map from \(\mathbb{F}_q\) to \(\mathbb{F}_p\). Considering the error basis \(E = \{X(a), Z(b)\mid a, b \in \mathbb{F}_q\}\), one defines the set \(P_{\infty}\) (according to [3]) as the set of all infinite tensor products of matrices \(N \in \{M \mid M \in \mathbb{E}\}\), in which all but finitely many tensor components are equal to \(I\), where \(I\) is the \(q \times q\) identity matrix. Then one defines the weight
wt of $A \in P_\infty$ as its (finite) number of nonidentity tensor components. In this context, one says that a quantum convolutional code has free distance $d_f$ if and only if it can detect all errors of weight less than $d_f$, but cannot detect some error of weight $d_f$. The code $C$ is pure if does not exist errors of weight less than $d_f$ in the stabilizer of $C$.

5 The New Convolutional MDS Codes

In this section we propose the construction of new classical convolutional codes. In order to proceed further, let us recall some results shown in the literature:

**Lemma 5.1** [20, Lemma 4.1] Let $n = q^2 + 1$, where $q \equiv 1 \pmod{4}$ is a power of an odd prime and suppose that $\beta$ is an odd prime and suppose that $s = n/2$. Then the $q^2$-ary cosets modulo $2n$ are given by: $C_s = \{s\}$, $C_{3s} = \{3s\}$ and $C_{s-2i} = \{s-2i, s+2i\}$, where $1 \leq i \leq s-1$.

**Lemma 5.2** [20, Lemma 4.4] Let $n = (q^2 + 1)/2$, where $q$ is a power of an odd prime. Then the $q^2$-ary cosets modulo $2n$ containing all odd integers from 1 to $2n - 1$ are given by: $C_n = \{n\}$, and $C_{2i-1} = \{2i - 1, 1 - 2i\}$, where $1 \leq i \leq (n-1)/2$.

Recall the concept of negacyclic BCH codes:

**Definition 5.1** (Negacyclic BCH codes) Let $q$ be a power of an odd prime with $\gcd(n, q) = 1$. Let $\beta$ be a primitive $2n$th root of unity in $\mathbb{F}_q$. A negacyclic code $C$ of length $n$ over $\mathbb{F}_q$ is a BCH code with designed distance $\delta$ if, for some odd integer $b \geq 1$, we have

$$g(x) = \text{lcm}\{M^{(b)}(x), M^{(b+2)}(x), \ldots, M^{(b+2(\delta-2))}(x)\},$$

i.e., $g(x)$ is the monic polynomial of smallest degree over $\mathbb{F}_q$ having $\alpha^b, \alpha^{b+2}, \ldots, \alpha^{b+2(\delta-2)}$ as zeros. Therefore, $c \in C$ if and only if $c(\alpha^b) = c(\alpha^{(b+2)}) = \ldots = c(\alpha^{b+2(\delta-2)}) = 0$. Thus the code has a string of $\delta - 1$ consecutive odd powers of $\beta$ as zeros.

**Remark 5.3** Let $\mathcal{B} = \{b_1, \ldots, b_l\}$ be a basis of $\mathbb{F}_q^n$ over $\mathbb{F}_q$. If $u = (u_1, \ldots, u_n) \in \mathbb{F}_q^n$, then one can write the vectors $u_i$, $1 \leq i \leq n$, as linear combinations of the elements of $\mathcal{B}$, that is, $u_i = u_{i1}b_1 + \ldots + u_{il}b_l$. Consider that $u^{(j)} = (u_{1j}, \ldots, u_{nj})$ are vectors in $\mathbb{F}_q^n$ with $1 \leq j \leq l$. Then, if $v \in \mathbb{F}_q^n$, one has $v \cdot u = 0$ if and only if $v \cdot u^{(j)} = 0$ for all $1 \leq j \leq l$.

In the following theorem we construct a parity-check matrix for negacyclic codes:

**Theorem 5.4** Assume that $q$ is a power of an odd prime, $\gcd(n, q) = 1$, and $m = \ord_{2n}(q)$. Let $\beta$ be a primitive $2n$th root of unity in $\mathbb{F}_q$. Let $b$ be an odd positive integer with $1 \leq b \leq 2n - 1$. Then a parity-check matrix for the
BCH negacyclic code $C$ of length $n$ and designed distance $\delta$, generated by the polynomial $g(x) = \text{lcm}\{M^{(b)}(x), M^{(b+2)}(x), \ldots, M^{(b+2(\delta-2))}(x)\}$, is the matrix

$$H_{b,b} = \begin{bmatrix}
1 & \beta^b & \beta^{2b} & \cdots & \beta^{(n-1)b} \\
1 & \beta^{(b+2)} & \beta^{2(b+2)} & \cdots & \beta^{(n-1)(b+2)} \\
1 & \beta^{(b+4)} & \beta^{2(b+4)} & \cdots & \beta^{(n-1)(b+4)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \beta^{[b+2(\delta-2)]} & \beta^{2[b+2(\delta-2)]} & \cdots & \beta^{(n-1)[b+2(\delta-2)]}
\end{bmatrix},$$

where each entry is replaced by the corresponding column of $m$ elements from $\mathbb{F}_q$ and then removing any linearly dependent rows.

**Proof:** Assume that $c = (c_0, c_1, \ldots, c_{n-1}) \in C$. Thus we have $c(\beta^b) = c(\beta^{b+2}) = c(\beta^{b+4}) = \cdots = c(\beta^{[b+2(\delta-2)]}) = 0$, hence

$$\begin{bmatrix}
1 & \beta^b & \beta^{2b} & \cdots & \beta^{(n-1)b} \\
1 & \beta^{(b+2)} & \beta^{2(b+2)} & \cdots & \beta^{(n-1)(b+2)} \\
1 & \beta^{(b+4)} & \beta^{2(b+4)} & \cdots & \beta^{(n-1)(b+4)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \beta^{[b+2(\delta-2)]} & \beta^{2[b+2(\delta-2)]} & \cdots & \beta^{(n-1)[b+2(\delta-2)]}
\end{bmatrix}\begin{bmatrix}
c_0 \\
c_1 \\
c_2 \\
\vdots \\
c_{n-1}
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
\vdots \\
0
\end{bmatrix}.$$

From Remark 5.3 and from the definition of BCH negacyclic codes, the result follows. \qed

Now we are ready to show one of the main results of this section:

**Theorem 5.5** Let $n = q^2 + 1$, where $q \equiv 1 \mod 4$ is a power of an odd prime and suppose that $s = n/2$. Then there exist MDS convolutional codes with parameters $(n, n - 2i + 1, 1, 2i + 2)_{q^2}$, where $2 \leq i \leq n/2 - 1$.

**Proof:** First, note that $\gcd(n, q) = 1$ and $\text{ord}_{2s}(q^2) = 2$. Let $\beta$ be a primitive $2n$th root of unity in $\mathbb{F}_{q^{2s}}$. Consider that $C_2$ is the negacyclic BCH code of length $n$ over $\mathbb{F}_{q^{2s}}$ generated by the product of the minimal polynomials

$$C_2 = \langle g_2(x) \rangle = \langle M^{(s)}(x)M^{(s+2)}(x) \ldots M^{(s+2i)}(x) \rangle,$$

where $2 \leq i \leq s - 1$.

By Theorem 5.4, a parity check matrix of $C_2$ is obtained from the matrix

$$H_2 = \begin{bmatrix}
1 & \beta^s & \beta^{2s} & \cdots & \beta^{(n-1)s} \\
1 & \beta^{(s+2)} & \beta^{2(s+2)} & \cdots & \beta^{(n-1)(s+2)} \\
1 & \beta^{(s+4)} & \beta^{2(s+4)} & \cdots & \beta^{(n-1)(s+4)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \beta^{(s+2i)} & \beta^{2(s+2i)} & \cdots & \beta^{(n-1)(s+2i)}
\end{bmatrix}.$$
by expanding each entry as a column vector (containing 2 rows) with respect to some $\mathbb{F}_{q^2}$-basis $\beta$ of $\mathbb{F}_{q^2}$, and then removing one linearly dependent row. From Lemma 5.1, this new matrix $H_{C_2}$ has rank $2i + 1$, so $C_2$ has dimension $n - 2i - 1$. From the BCH bound for negacyclic codes it follows that the minimum distance $d_2$ of $C_2$ satisfies $d_2 \geq 2i + 2$. Thus, from the (classical) Singleton bound, one concludes that $C_2$ is a MDS code with parameters $[n, n - 2i + 1, 2i + 2]_{q^2}$ and, consequently, its Hermitian dual code has dimension $2i + 1$.

Next we assume that $C_1$ is the negacyclic BCH code of length $n$ over $\mathbb{F}_{q^2}$, generated by the product of the minimal polynomials

$$C_1 = \langle g_1(x) \rangle = \langle M^{(s)}(x)M^{(s+2)}(x) \cdots M^{[s+2(i-1)]}(x) \rangle,$$

Similarly, by Theorem 5.4, $C_1$ has a parity check matrix derived from the matrix

$$H_1 = \begin{bmatrix} 1 & \beta^s & \beta^{2s} & \ldots & \beta^{(n-1)s} \\ 1 & \beta^{(s+2)} & \beta^{2(s+2)} & \ldots & \beta^{(n-1)(s+2)} \\ 1 & \beta^{(s+4)} & \beta^{2(s+4)} & \ldots & \beta^{(n-1)(s+4)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \beta^{(s+2(i-1))} & \beta^{2(s+2(i-1))} & \ldots & \beta^{(n-1)(s+2(i-1))} \end{bmatrix}$$

by expanding each entry as a column vector with respect to some $\mathbb{F}_{q^2}$-basis $\beta$ of $\mathbb{F}_{q^2}$ (already done, since $H_1$ is a submatrix of $H_2$) and then removing one linearly dependent row. From Lemma 5.1, this new matrix $H_{C_1}$ has rank $2i + 1$, so $C_1$ has dimension $n - 2i + 1$. From the BCH bound for negacyclic codes, the minimum distance $d_1$ of $C_1$ satisfies $d_1 \geq 2i$, so $C_1$ is an $[n, n - 2i + 1, 2i]_{q^2}$ MDS code. Thus, its Hermitian dual code has dimension $2i + 1$.

Now, let $C_0$ be the negacyclic BCH code of length $n$ over $\mathbb{F}_{q^2}$ generated by the minimal polynomial $M^{(s+2i)}(x)$. Then $C_0$ has parameters $[n, n - 2, d_0 \geq 2]_{q^2}$. A parity check matrix $H_{C_0}$ of $C_0$ is given by expanding the entries of the matrix

$$H_0 = \begin{bmatrix} 1 & \alpha^{(s+2i)} & \alpha^{2(s+2i)} & \ldots & \alpha^{(n-1)(s+2i)} \end{bmatrix}$$

with respect to $\beta$ (already done, since $H_0$ is a submatrix of $H_2$).

Further, let us construct the convolutional code $V$ generated by the reduced basic (according to Theorem 3.1 Item (a)) generator matrix

$$G(D) = \tilde{H}_{C_1} + \tilde{H}_{C_0}D,$$

where $\tilde{H}_{C_1} = H_{C_1}$ and $\tilde{H}_{C_0}$ is obtained from $H_{C_0}$ by adding zero-rows at the bottom such that $\tilde{H}_{C_0}$ has the number of rows of $H_{C_0}$ in total. By construction, $V$ is a unit-memory convolutional code of dimension $2i - 1$ and degree $\delta_V = 2$. We know that the Hermitian dual $V^\perp$ of $V$ has dimension $n - 2i + 1$ and degree $2$. By Theorem 3.1 Item (c), the free distance of $V^\perp$ is bounded by $\min\{d_0 + d_1, d_2\} \leq d_1 \leq d_2$, where $d_1$ is the minimum distance of the code $C_1 = \{v \in \mathbb{F}_q^n \mid vH_{C_1}^\perp = 0\}$. From construction one has $d_2 = 2i + 2$, $d_1 = 2i$ and $d_0 \geq 2$, so $V^\perp$ has parameters $(n, n - 2i + 1, 2i + 2)_{q^2}$. It is easy to see
that the parameters of $V \perp h$ satisfies (1) with equality, so $V \perp h$ is MDS.

Theorem 5.6 given in the following is the second main result of this section:

**Theorem 5.6** Let $n = (q^2 + 1)/2$, where $q$ is a power of an odd prime. Then there exist MDS convolutional codes with parameters $(n, n - 2i + 2, 2; 1, 2i + 1)_{q^2}$, where $2 \leq i \leq (n - 1)/2$.

**Proof:** It suffices to consider $C_2$ to be the code generated by $\langle M^{(1)}(x)M^{(3)}(x)\ldots M^{(2i-1)}(x) \rangle$, where $2 \leq i \leq (n - 1)/2$, $C_1$ be the negacyclic BCH code generated by $\langle g_1(x) \rangle = \langle M^{(1)}(x)M^{(3)}(x)\ldots M^{(2i-3)}(x) \rangle$ and $C_0$ be the negacyclic BCH code generated by $M^{(2i-1)}(x)$. Proceeding similarly as in the proof of Theorem 5.5, the result follows. $\square$

**Theorem 5.7** Let $n = (q^2 + 1)/2$, where $q \geq 5$ is a power of an odd prime. Then there exist MDS convolutional codes with parameters $(n, n - 2i + 1, 2; 1, 2i + 2)_{q^2}$, where $2 \leq i \leq (n - 1)/2 - 1$.

**Proof:** Consider that $C_2$, $C_1$ and $C_0$ are negacyclic BCH codes of length $n$ over $\mathbb{F}_{q^2}$ generated, respectively by $\langle g_2(x) \rangle = \langle M^{(n)}(x)M^{(n+2)}(x)\ldots M^{(n+2i)}(x) \rangle$, $\langle g_1(x) \rangle = \langle M^{(n)}(x)M^{(n+2)}(x)\ldots M^{(n+2i-2)}(x) \rangle$, and $\langle g_0(x) \rangle = \langle M^{(n+2i)}(x) \rangle$. Applying the same procedure given in the proofs of Theorems 5.5 and 5.6, the result follows. $\square$

### 6 New Quantum MDS-Convolutional codes

As in the classical case, the construction of MDS quantum convolutional codes is a difficult task. This task is performed in [3, 13, 14, 16] but only in [3, 14] the constructions are made algebraically. Here, we propose the construction of MDS convolutional stabilizer codes derived from the convolutional codes constructed in Section 5. To proceed further, let us recall some results available in the literature:

**Lemma 6.1** [2, Proposition 2] Let $C$ be an $(n, (n - k)/2, \gamma; \mu)_{q^2}$ convolutional code such that $C \subseteq C \perp h$. Then there exists an $[(n, k, \mu; \gamma, d_f)]_{q^2}$ convolutional stabilizer code, where $d_f = \text{wt}(C \perp h \setminus C)$.

**Theorem 6.2** [3] (Quantum Singleton bound) The free distance of an $[(n, k, \mu; \gamma, d_f)]_{q^2}$ $\mathbb{F}_{q^2}$-linear pure convolutional stabilizer code is bounded by

$$d_f \leq \frac{n - k}{2} \left( \left\lfloor \frac{2\gamma}{n + k} \right\rfloor + 1 \right) + \gamma + 1.$$
Lemma 6.3 [20] Let \( n = q^2 + 1 \), where \( q \equiv 1 \mod 4 \) is a power of an odd prime
and suppose that \( s = n/2 \). If \( C \) is a \( q^2 \)-ary negacyclic code of length \( n \) with
defining set \( Z = \bigcup_{i=0}^{\delta} \mathbb{C}_{s-2i} \), where \( 0 \leq \delta \leq (q-1)/2 \), then \( C_{\perp h} \subseteq C \).

Lemma 6.4 [20] Let \( n = (q^2 + 1)/2 \), where \( q \) is a power of an odd prime. If \( C \)
is a \( q^2 \)-ary negacyclic code of length \( n \) with defining set \( Z = \bigcup_{i=1}^{\delta} \mathbb{C}_{2i} \), where
\( 1 \leq \delta \leq (q-1)/2 \), then \( C_{\perp h} \subseteq C \).

Now, we are able to show the following two results, in which new families of
quantum convolutional MDS codes are constructed:

Theorem 6.5 Let \( n = q^2 + 1 \), where \( q \equiv 1 \mod 4 \) is a power of an odd prime
and suppose that \( s = n/2 \). Then there exist quantum MDS convolutional codes
with parameters \( [(n, n-4i+2, 1; 2i+2)]_q \), where \( 2 \leq i \leq (q-1)/2 \).

Proof: We consider the same notation utilized in Theorem 5.5. From Theorem
5.5, there exists a classical convolutional MDS code \( V_{\perp h} \) with parameters
\( (n, n-2i+1, 2; 2i-1, d_f)_q \), for each \( 2 \leq i \leq n/2 - 1 \). This code is the Hermitian
dual of the code \( V \) with parameters \( (n, 2i-1, 2; 1, d_f)_q \). From Lemma 6.3 and
from Theorem 3.1 Item (b), one has \( V \subseteq V_{\perp h} \). Applying Lemma 6.1, there
exists an \( [(n, n-4i+2, 1; 2, d_f \geq 2i+2)]_q \) convolutional stabilizer code \( Q \), for
each \( 2 \leq i \leq (q-1)/2 \). Replacing the parameters of \( Q \) in Theorem 6.2, the
result follows. \( \square \)

Theorem 6.6 Let \( n = (q^2 + 1)/2 \), where \( q \geq 7 \) is a power of an odd prime. Then
there exist quantum MDS convolutional codes with parameters \( [(n, n-4i+4, 1; 2,
2i+1)]_q \), where \( 2 \leq i \leq (q-1)/2 \).

Proof: From Theorem 5.6, there exists a classical convolutional MDS code \( V_{\perp h} \)
with parameters \( (n, n-2i+2, 2; 1, 2i+1)_q \), for each \( 2 \leq i \leq (n-1)/2 \). This
code is the Hermitian dual of the code \( V \) with parameters \( (n, 2i-2, 2; 1, d_f)_q \).
From Lemma 6.4 and from Theorem 3.1 Item (b), one has \( V \subseteq V_{\perp h} \). Applying
Lemma 6.1, there exists a convolutional stabilizer code \( Q \) with parameters
\( [(n, n-4i+4, 1; 2, d_f \geq 2i+1)]_q \), for each \( 2 \leq i \leq (q-1)/2 \). Replacing the
parameters of \( Q \) in Theorem 6.2, the result follows. \( \square \)

In the following we present Tables 1 and 2, containing the parameters of
some new convolutional codes and some new quantum convolutional codes,
respectively, constructed in this paper. Recall the these codes are optimal in the
sense they attain the classical (quantum) generalized Singleton bound.

7 Summary

In this paper we have constructed new families of classical and quantum MDS-
convolutional codes derived from negacyclic codes. All the constructions pre-
presented here are performed algebraically and not by exhaustively computa-

| Table 1: Classical MDS New convolutional codes |
|---------------------------------------------|
| \((n, n - 2i + 1, 2; 1, 2i + 2)_{q^2}, q \equiv 1 \mod 4, n = q^2 + 1, 2 \leq i \leq n/2 - 1\) |
| \((26, 23, 2; 1, 6)_{25}\) |
| \((26, 21, 2; 1, 8)_{25}\) |
| \((26, 19, 2; 1, 10)_{25}\) |
| \((26, 9, 2; 1, 20)_{25}\) |
| \((26, 7, 2; 1, 22)_{25}\) |
| \((26, 5, 2; 1, 24)_{25}\) |
| \((82, 63, 2; 1, 22)_{81}\) |
| \((82, 53, 2; 1, 32)_{81}\) |
| \((82, 43, 2; 1, 42)_{81}\) |
| \((82, 23, 2; 1, 62)_{81}\) |
| \((82, 13, 2; 1, 72)_{81}\) |
| \((n, n - 2i + 2, 2; 1, 2i + 1)_{q^2}, n = (q^2 + 1)/2, 2 \leq i \leq (n - 1)/2\) |
| \((5, 3; 2, 1, 5)_{9}\) |
| \((25, 23, 2; 1, 5)_{49}\) |
| \((25, 21, 2; 1, 7)_{49}\) |
| \((25, 19, 2; 1, 9)_{49}\) |
| \((25, 17, 2; 1, 11)_{49}\) |
| \((25, 15, 2; 1, 13)_{49}\) |
| \((25, 13, 2; 1, 15)_{49}\) |
| \((25, 11, 2; 1, 17)_{49}\) |
| \((25, 7, 2; 1, 21)_{49}\) |
| \((n, n - 2i + 1, 2; 1, 2i + 2)_{q^2}, q \geq 5, n = (q^2 + 1)/2, 2 \leq i \leq (n - 1)/2 - 1\) |
| \((13, 10, 2; 1, 6)_{25}\) |
| \((13, 8, 2; 1, 8)_{25}\) |
| \((13, 6, 2; 1, 10)_{25}\) |
| \((13, 4, 2; 1, 12)_{25}\) |
| \((25, 16, 2; 1, 12)_{49}\) |
| \((25, 10, 2; 1, 18)_{49}\) |
| \((25, 4, 2; 1, 24)_{49}\) |
| \((41, 38, 2; 1, 6)_{81}\) |
| \((41, 32, 2; 1, 12)_{81}\) |
| \((41, 24, 2; 1, 20)_{81}\) |
| \((41, 4, 2; 1, 40)_{81}\) |
| \((61, 32, 2; 1, 32)_{121}\) |
| \((61, 22, 2; 1, 42)_{121}\) |
| \((61, 4, 2; 1, 60)_{121}\) |
Table 2: Quantum MDS

New convolutional stabilizer codes

\[(n, n - 4i + 2, 1; 2, 2i + 2)_{q^6}, \ n = q^2 + 1, \ 2 \leq i \leq (q - 1)/2\]

\begin{tabular}{l}
\hline
\((26, 20, 2; 1, 6)_{9}\) \\
\((82, 80, 2; 1, 4)_{9}\) \\
\((82, 76, 2; 1, 6)_{9}\) \\
\((82, 72, 2; 1, 8)_{9}\) \\
\((82, 68, 2; 1, 10)_{9}\) \\
\((170, 168, 2; 1, 4)_{13}\) \\
\((170, 164, 2; 1, 6)_{13}\) \\
\((170, 160, 2; 1, 8)_{13}\) \\
\((170, 156, 2; 1, 10)_{13}\) \\
\((170, 152, 2; 1, 12)_{13}\) \\
\((170, 148, 2; 1, 14)_{13}\) \\
\hline
\((n, n - 4i + 4, 2; 1, 2i + 1)_{q^6}, \ n = (q^2 + 1)/2, \ 2 \leq i \leq (q - 1)/2)\]
\((25, 21, 2; 1, 5)_{7}\) \\
\((25, 17, 2; 1, 7)_{7}\) \\
\((61, 57, 2; 1, 5)_{11}\) \\
\((61, 53, 2; 1, 7)_{11}\) \\
\((61, 49, 2; 1, 9)_{11}\) \\
\((61, 45, 2; 1, 11)_{11}\) \\
\((145, 141, 2; 1, 5)_{17}\) \\
\((145, 137, 2; 1, 7)_{17}\) \\
\((145, 133, 2; 1, 9)_{17}\) \\
\((145, 129, 2; 1, 11)_{17}\) \\
\((145, 125, 2; 1, 13)_{17}\) \\
\((145, 121, 2; 1, 15)_{17}\) \\
\((145, 117, 2; 1, 17)_{17}\) \\
\hline
\end{tabular}
search. The results obtained in this paper show that the class of negacyclic codes is also a good source in the search for optimal codes.

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