Dynamic properties of piecewise linear systems with fractional time-delay feedback

Jianchao Zhang¹, Jun Wang¹,², Jiangchuan Niu¹,² and Yufei Hu²

Abstract
The forced vibration of a single-degree-of-freedom piecewise linear system containing fractional time-delay feedback was investigated. The approximate analytical solution of the system was obtained by employing an averaging method. A frequency response equation containing time delay was obtained by studying a steady-state solution. The stability conditions of the steady-state solution, the amplitude–frequency results, and the numerical solutions of the system under different time-delay parameters were compared. Comparison results indicated a favorable goodness of fit between the two parameters and revealed the correctness of the analytical solution. The effects of the time-delay and fractional parameters, piecewise stiffness, and piecewise gap on the principal resonance and bifurcation of the system were emphasized. Results showed that fractional time delay occurring in the form of equivalent linear dampness and stiffness under periodic variations in the system and influenced the vibration characteristic of the system. Moreover, piecewise stiffness and gap induced the nonlinear characteristic of the system under certain parameters.

Keywords
Fractional time-delay feedback, principal resonance, bifurcation, piecewise linear system

Introduction
As a branch of nonlinear dynamics, nonsmooth systems extensively exist in engineering applications, including circuits,¹,² vehicle suspensions,³,⁴ and vibration isolators.⁵ Each piecewise system in nonsmooth systems is linear and can be easily processed, while the global solution of the system cannot be obtained under most circumstances.⁶–⁸ Moreover, many complicated dynamic phenomena are generated due to the existence of gaps, which may exert adverse effects on the system. Hence, piecewise linear systems have attracted increasing research attention. Shaw and Holmes⁹ used a central manifold theorem to analyze the local bifurcation of a piecewise linear impact oscillator under simple harmonic excitation force and studied chaotic motion under homoclinic transverse conditions. Hu¹⁰,¹¹ found that the nonsmoothness of the vector field in a continuous piecewise linear system destroys the second-order differentiability of Poincare mapping at a fixed point and thus results in system singularity. Luo¹² analyzed all kinds of unstable and stable periodic motion morphologies of a piecewise linear system under periodic excitation. Jin and Guan¹³ evaluated the stability of the periodic motion of an asymmetric piecewise linear model with multiple degrees of freedom. The selection of proper piecewise stiffness or dampness can improve the performance of the original dynamic system by improving the stability and buffer performance of the system.¹⁴–¹⁶

¹State Key Laboratory of Structural Mechanical Behavior and System Safety for Traffic Engineering Structures, Shijiazhuang Tiedao University, Shijiazhuang, China
²Mechanical Engineering School, Tiedao University, Shijiazhuang, China

Corresponding author:
Jianchao Zhang, State Key Laboratory of Structural Mechanical Behavior and System Safety for Traffic Engineering Structures, Shijiazhuang Tiedao University, Beierhuandong Road 17#, Shijiazhuang, 050043 Hebei, China.
Email: zhangjianchao@yeah.net

Creative Commons CC BY: This article is distributed under the terms of the Creative Commons Attribution 4.0 License (https://creativecommons.org/licenses/by/4.0/) which permits any use, reproduction and distribution of the work without further permission provided the original work is attributed as specified on the SAGE and Open Access pages (https://us.sagepub.com/en-us/nam/open-access-at-sage).
Based on the extensive application of fractional calculus and interdisciplinary development, fractional calculus has appeared in models in numerous disciplinary fields and practical problems, such as the fractal Micro-Electro-Mechanical System (MEMS) and Toda oscillators. Zuo designed a gecko-inspired fractal-like receptor system for the three-dimensional printing process. He et al. investigated the possible working mechanism of ink slab-like materials by using two-scale fractal derivatives. Given that fractional models can effectively describe essential system characteristics, an increasing number of studies have been conducted on fractional order. Moreover, studies on fractional differential systems have achieved considerable progress, especially in the analysis of the equilibrium points of systems, quantities of periodic solutions, limit cycles, and stability.

However, in actual systems, particularly control systems, the time-delay phenomenon is ubiquitous in various fields, and time delays usually lead to changes in vibration form and system instability or oscillation. Even simple systems become unstable when time-delay factors are neglected in the modeling process. Time delays also cause difficulties in the theoretical analysis of systems. The analysis of systems with time delays is imperative in studying actual systems, particularly control systems. Therefore, systems with time delays, especially fractional systems with time delays, should be investigated comprehensively.

Integer-order systems with time delays have been extensively investigated, but fractional systems with time delays have been largely ignored because of their complexity. With regard to the control of fractional systems with time delays, Kaslik and Sivasundaram and Li and Zhang conducted a stability study on a fractional linear system with time delay. Deng et al. investigated the stability of a linear fractional differential system with multiple time delays. Babakhani et al. determined a solution for the fractional differential equation with time delay and hope bifurcation. Ye et al. also determined a positive solution for the fractional differential equation with time delay. Zhang proved that for a specific type of a fractional system with time delay, sufficient conditions for stability within finite time can be obtained. Čermák et al. studied the stability and progressivity of the fractional differential equation with time delay.

A few studies have controlled piecewise smooth systems by using fractional time-delay feedback. The dynamic behaviors of fractional time-delay feedback systems became complicated due to the existence of nonlinear factors caused by piecewise stiffness. Steady-state solutions, system stability, and the effects of all kinds of system parameters on dynamic system characteristics are yet to be studied comprehensively.

The fractional time-delay feedback of a single-degree-of-freedom (SDOF) piecewise linear system was studied in the current work. An averaging method was used to analyze the approximate analytical solution of the system and obtain the principal resonance amplitude–frequency response equation with time-delay parameters and stability conditions of the system. The effects of time-delay and fractional differential parameters and gaps on the principal resonance and bifurcation of the system were investigated in detail through a numerical analysis.

**Model of system dynamics and approximate solution**

The SDOF piecewise linear oscillator with fractional time-delay feedback mentioned in this study consists of a mass block, external excitation, linear spring, linear dampness, and fractional time-delay feedback term. Its physical model is shown in Figures 1 and 2.

\[
mx + cx + k_1x + g(x) = K_1 D^p [x(t - \tau)] + F\cos \omega t
\]

\[
g(x) = \begin{cases} 
  k_2(x - a_0) & x > a_0 \\
  0 & -a_0 \leq x \leq a_0 \\
  k_2(x + a_0) & x < -a_0 
\end{cases}
\]
where \( m \) is the oscillator mass; \( k_1x \) is the linear spring force; \( g(x) \) is the piecewise spring force; \( c \) is the damping coefficient; \( K_1D^p[x(t - \tau)] \) is the fractional time-delay feedback term; \( K_1 \) is the coefficient of fractional time-delay feedback; \( p \) is the fractional order whose physical meaning and geometric interpretation have been explained in References 35–37; \( \tau \) is the time delay of the system; \( F \) is the excitation force; and \( \omega \) is the excitation frequency.

Many analytical methods, such as fractional complex transform,\(^{38}\) variational iteration,\(^{39}\) homotopy perturbation method,\(^{40}\) and He’s frequency method,\(^{21}\) could be used for the fractional differential equation as shown in equation (1).

The following parameters are introduced.

\[
\omega_0 = \sqrt{\frac{k_1}{m}}, \quad 2\varepsilon\mu = \frac{c}{m}, \quad \varepsilon k_3 = \frac{k_2}{m}, \quad \varepsilon K = \frac{K_1}{m}, \quad \varepsilon f = \frac{F}{m}
\]

Equations (1) and (2) can be transformed as follows

\[
\ddot{x} + 2\varepsilon\mu \dot{x} + \omega_0^2 x + \varepsilon g_1(x) = \varepsilon K D^p [x(t - \tau)] + \varepsilon f \cos \omega t
\]

\[
g_1(x) = \begin{cases} 
  k_3(x - a_0) & x > a_0 \\
  0 & -a_0 \leq x \leq a_0 \\
  k_3(x + a_0) & x < -a_0 
\end{cases}
\]

The principal resonance of the system was studied. \( \omega_0^2 = \omega^2 - \varepsilon \sigma \) was set. Equation (3) was transformed into the expression

\[
\ddot{x} + \omega^2 x = \varepsilon(F_1 \cos \omega t + \sigma x - g_1(x) + K D^p [x(t - \tau)] - 2\mu \dot{x})
\]

Equation (5) can be organized as follows

\[
g_1(x) = g_1(acos \phi) = \begin{cases} 
  k_3(acos \phi - acos \phi_0) & 0 < \phi < \phi_0, 2\pi - \phi_0 < \phi < 2\pi \\
  0 & \text{else} \\
  k_3(acos \phi + acos \phi_0) & \pi - \phi_0 < \phi < \pi + \phi_0 
\end{cases}
\]

\[
\ddot{x} + \omega^2 x = \varepsilon P(x, \dot{x}) = \varepsilon(P_1 + P_2 + P_3)
\]

Figure 2. SDOF piecewise linear oscillator.
\[ P_3 = KD^p[x(t - \tau)] = KD^p[\cos(\phi - \omega t)] \]

According to the averaging method

\[
\dot{a} = -\frac{1}{T}\int_0^T e(P_1 + P_2 + P_3)\sin\phi \, d\phi
\]  

(7a)

\[
\dot{\theta} = -\frac{1}{T}\int_0^T e(P_1 + P_2 + P_3)\cos\phi \, d\phi
\]

(7b)

where \( P_1 \) and \( P_2 \) are periodic functions, \( T = 2\pi \), and fractional time-delay term \( P_3 \) is a periodic function, \( T = \infty \).

\[
\dot{a}_1 = -\frac{1}{2\pi\omega} \int_0^{2\pi} eP_1 \sin\phi \, d\phi = -\frac{a}{2\pi\omega} \int_0^{2\pi} e[\cos\phi + 2\mu \sin\phi + f \cos(\phi - \theta)] \sin\phi \, d\phi = -\varepsilon \mu a - \frac{e\sigma}{2\omega} \sin\theta
\]  

(8a)

\[
\dot{\theta}_1 = -\frac{1}{2\pi\omega} \int_0^{2\pi} eP_1 \cos\phi \, d\phi = -\frac{a}{2\pi\omega} \int_0^{2\pi} e[\cos\phi + 2\mu \sin\phi + f \cos(\phi - \theta)] \cos\phi \, d\phi = -\frac{e\omega}{2\pi} a - \frac{e\sigma}{2\omega} \cos\theta
\]

(8b)

\[
\dot{a}_2 = -\frac{1}{2\pi\omega} \int_0^{2\pi} -eP_2 \sin\phi \, d\phi = \frac{ak_3}{2\pi\omega}
\]

\[
\dot{\theta}_2 = -\frac{1}{2\pi\omega} \int_0^{2\pi} -eP_2 \cos\phi \, d\phi = \frac{ak_3}{2\pi\omega}
\]

\[
\left\{ \int_0^{\text{arccos} \frac{a}{\sqrt{1 - a^2}}} e \left[ \cos\phi - \cos \left( \text{arccos} \frac{a}{\sqrt{1 - a^2}} \right) \right] \sin\phi \, d\phi + \int_{\text{arccos} \frac{a}{\sqrt{1 - a^2}}}^{\pi - \text{arccos} \frac{a}{\sqrt{1 - a^2}}} e \left[ \cos\phi + \left( \text{arccos} \frac{a}{\sqrt{1 - a^2}} \right) \right] \sin\phi \, d\phi + \int_{\pi - \text{arccos} \frac{a}{\sqrt{1 - a^2}}}^{2\pi} e \left[ \cos\phi - \left( \text{arccos} \frac{a}{\sqrt{1 - a^2}} \right) \right] \sin\phi \, d\phi \right\} = \frac{e\left(-a\sqrt{1 - \frac{a^2}{2}} + a \times \text{arccos} \frac{a}{\sqrt{2}} \right) k_3}{\pi\omega}
\]

(9a)

\[
\left\{ \int_0^{\text{arccos} \frac{a}{\sqrt{1 - a^2}}} e \left[ \cos\phi - \cos \left( \text{arccos} \frac{a}{\sqrt{1 - a^2}} \right) \right] \cos\phi \, d\phi + \int_{\text{arccos} \frac{a}{\sqrt{1 - a^2}}}^{\pi - \text{arccos} \frac{a}{\sqrt{1 - a^2}}} e \left[ \cos\phi + \left( \text{arccos} \frac{a}{\sqrt{1 - a^2}} \right) \right] \cos\phi \, d\phi + \int_{\pi - \text{arccos} \frac{a}{\sqrt{1 - a^2}}}^{2\pi} e \left[ \cos\phi - \left( \text{arccos} \frac{a}{\sqrt{1 - a^2}} \right) \right] \cos\phi \, d\phi \right\} = \frac{e\left(-a\sqrt{1 - \frac{a^2}{2}} + a \times \text{arccos} \frac{a}{\sqrt{2}} \right) k_3}{\pi}\]

(9b)

The following equation is obtained in accordance with the Caputo fractional differential definition.\(^{41}\)

\[
D^\delta [x(t)] = \frac{1}{\Gamma(1 - \delta)} \int_0^t \frac{x'(u)}{(t - u)^\delta} \, du
\]

Then

\[
\dot{a}_3 = -\frac{1}{T\omega} \lim_{T \to \infty} \int_0^T eP_3 \sin\phi \, d\phi = -\frac{aK}{T\omega} \lim_{T \to \infty} \int_0^T eD^p[\cos(\phi - \omega t)] \sin\phi \, d\phi
\]

\[
= -\frac{eaK}{T\omega} \times \frac{1}{\Gamma(1 - P)} \lim_{T \to \infty} \int_0^T \left[ \int_0^t \frac{\sin(\omega u + \theta - \omega t)}{(t - u)^P} \, du \right] \sin(\omega t + \theta) \, dt
\]

(10)

\( s = t - u \) and \( ds = -du \) are set, and

\[
\dot{a}_3 = \frac{eaK}{T \times \Gamma(1 - P)} \times \lim_{T \to \infty} \int_0^T \left[ \int_0^t \frac{\sin(\omega u + \theta - \omega t)}{s^P} \, ds \right] \sin(\omega t + \theta) \, dt
\]

\[
= \frac{eaK}{T \times \Gamma(1 - P)} \times \lim_{T \to \infty} \int_0^T \left[ \int_0^t \frac{\cos \omega s}{s^P} \, ds \right] \sin(\omega t + \theta) \, dt
\]
in equation (11) is expressed as follows

\[
\lim_{T \to \infty} \int_0^T \frac{\cos \omega t}{T^p} \, dt = \omega^{p-1} \Gamma(1-P) \sin \frac{P\pi}{2}
\]  

(12a)

\[
\lim_{T \to \infty} \int_0^T \frac{\sin \omega t}{T^p} \, dt = \omega^{p-1} \Gamma(1-P) \cos \frac{P\pi}{2}
\]  

(12b)

After the integration of the parts and fractional order calculation in equation (12), the first part of the integral in equation (11) is expressed as follows

\[
\hat{a}_{31} = - \frac{\varepsilon a K}{T \times \Gamma(1-P)} \times \lim_{T \to \infty} \int_0^T \left[ \frac{\sin \omega t}{T^p} \int_0^t \frac{\cos \omega s}{s^p} \, ds \right] \cos(\omega t + \theta - \omega \tau) \sin(\omega t + \theta) \, dt
\]

\[
\hat{a}_{31} = - \frac{\varepsilon a K}{4\omega T \times \Gamma(1-P)} \times \lim_{T \to \infty} \int_0^T \left[ \frac{\sin (2\omega t + 2\theta - \omega \tau) - 2\omega \cos \omega \tau}{T^p} \right] \cos(\omega t + \theta - \omega \tau) \sin(\omega t + \theta) \, dt
\]

\[
\hat{a}_{31} = - \frac{\varepsilon a K}{4\omega T \times \Gamma(1-P)} \times \lim_{T \to \infty} \int_0^T \left[ \frac{\sin (2\omega t + 2\theta - \omega \tau) - 2\omega \cos \omega \tau}{T^p} \right] \cos(\omega t + \theta - \omega \tau) \sin(\omega t + \theta) \, dt = 0
\]  

(13)

Similarly, the second part of the integral in equation (11) is expressed as follows

\[
\hat{a}_{32} = - \frac{\varepsilon a K}{T \times \Gamma(1-P)} \times \lim_{T \to \infty} \int_0^T \left[ \frac{\sin \omega t}{T^p} \int_0^t \frac{\cos \omega s}{s^p} \, ds \right] \cos(\omega t + \theta - \omega \tau) \sin(\omega t + \theta) \, dt = 0
\]  

(14)

Hence

\[
\hat{a}_3 = \hat{a}_{31} + \hat{a}_{32} = - \frac{\varepsilon a K \omega^{p-1}}{2} \sin \left( \frac{P\pi}{2} - \omega \tau \right)
\]  

(15a)

The process is repeated, and the following expression is obtained.

\[
\hat{a}_3 = - \frac{\varepsilon a K \omega^{p-1}}{2} \cos \left( \frac{P\pi}{2} - \omega \tau \right)
\]  

(15b)

In sum

\[
\hat{a} = \hat{a}_1 + \hat{a}_2 + \hat{a}_3 = -\varepsilon \mu a - \frac{ef}{2\omega} \sin \theta + \frac{\varepsilon a K \omega^{p-1}}{2} \sin \left( \frac{P\pi}{2} - \omega \tau \right)
\]  

(16a)

\[
\hat{a} = \hat{a}_1 + \hat{a}_2 + \hat{a}_3 = -\frac{\varepsilon a}{2\omega} \sin \theta - \frac{ef}{2\omega} \cos \theta + \frac{e}{2\omega} \left( \frac{-a_0 \sqrt{1 - \frac{\omega^2}{\omega^2}} \sin \theta + a \times \arccos \frac{\omega}{\omega} \right) k_3 = - \frac{\varepsilon a K \omega^{p-1}}{2} \cos \left( \frac{P\pi}{2} - \omega \tau \right)
\]  

(16b)

The following expression can be obtained by substituting the original parameters

\[
\hat{a} = - \frac{F}{2m\omega} \sin \theta - \frac{a_0 - \omega_0}{2m} \sin (\frac{P\pi}{2} - \omega \tau) K_1
\]  

(17a)
\[
\begin{align*}
\ddot{a} &= -F \frac{2m\omega}{2m} \cos \theta - \frac{a_0}{2} + \frac{ak_H}{2m} - \frac{a_{0t}^2 + p \cos \left( \frac{\pi}{2} - \omega \tau \right)}{2m} K_1 
\end{align*}
\] (17b)

where

\[
K_H = k_1 + k_2 L(a)
\]

\[
L(a) = \frac{2}{\pi} \arccos \left( \frac{a_0}{a} \right) - \frac{2a_0}{a_0} \sqrt{1 - \frac{a_0^2}{a^2}}
\]

**Steady-state solution and its stability conditions**

The steady-state solution of the system was analyzed, \( \dot{a} = 0 \) and \( \ddot{a} = 0 \) were set, and

\[
\begin{align*}
- F \frac{2m\omega}{2m} \sin \theta &= \frac{ac}{2m} + \frac{a_{0t}^2 + p \sin \left( \frac{\pi}{2} - \omega \tau \right)}{2m} K_1 = 0 \quad (18a) \\
- F \frac{2m\omega}{2m} \sin \theta &= \frac{ac}{2m} + \frac{a_{0t}^2 + p \sin \left( \frac{\pi}{2} - \omega \tau \right)}{2m} K_1 = 0 \quad (18b)
\end{align*}
\]

The trigonometric function term was eliminated, and the amplitude–frequency response equation containing the time delay in the system was obtained through the simplification

\[
a^2 \left[ K_1 \omega^p \sin \left( \frac{\pi}{2} - \omega \tau \right) - c_0 \right]^2 + a^2 \left[ K_H - m\omega^2 - K_1 \omega^p \cos \left( \frac{\pi}{2} - \omega \tau \right) \right]^2 - F^2 = 0 \quad (19)
\]

The stability of the steady-state solution was analyzed.

Linear processing of equation (18) was performed to obtain the differential equations of two disturbing quantities, \( \Delta a \) and \( \Delta \theta \), which are, respectively, related to steady-state solutions for \( \Delta a \) and \( \Delta \theta \) as follows

\[
\begin{align*}
\frac{d\Delta a}{dt} &= \left[ - \frac{c}{2m} \frac{K_1 \omega^p \sin \left( \frac{\pi}{2} - \omega \tau \right)}{2m} \Delta \dot{a} - \frac{F \cos \theta}{2m\omega} \Delta \theta 
\end{align*}
\] (20a)

\[
\frac{d\Delta \theta}{dt} = \left[ \frac{F \cos \theta}{2\Delta m\omega} \frac{k_2 L'(\ddot{a})}{2m\omega} \Delta \ddot{a} \right] \frac{F \sin \theta}{2m\omega} \Delta \theta
\] (20b)

where

\[
L'(\ddot{a}) = \frac{4a_0^2 \sqrt{\ddot{a}^2 - a_0^2}}{a^4 \pi}
\]

\( \ddot{\theta} \) in equation (20) was eliminated, and the following characteristic equation was obtained in accordance with equation (18).

\[
\begin{align*}
\det \left[ \begin{array}{cc}
- \frac{c}{2m} & \frac{K_1 \omega^p \sin \left( \frac{\pi}{2} - \omega \tau \right)}{2m} \\
- \frac{\omega}{2a} & \frac{k_H(a)}{2m\omega} + \frac{k_2 L'(\ddot{a})}{2m\omega} - \frac{K_1 \omega^p \cos \left( \frac{\pi}{2} - \omega \tau \right)}{2m\ddot{a}} - \frac{c}{2m} - \frac{\omega \ddot{a}}{2m} - \frac{ak_H(a)}{2m\omega} - \frac{\omega K_1 \omega^p \cos \left( \frac{\pi}{2} - \omega \tau \right)}{2m} - \lambda
\end{array} \right] = 0 \quad (21)
\]
Equation (20) was manipulated and simplified, and the following expression was derived.

\[ k^2 - 2B_1 \lambda + B_1^2 - B_2B_3 = 0 \]  

(22)

where

\[ B_1 = -\frac{c}{2m} + \frac{K_1 \omega^{p-1} \sin \left(\frac{\omega^p}{2} - \omega t\right)}{2m} \]

\[ B_2 = \frac{\omega a}{2} - \frac{\bar{a}k_1(a)}{2m_0a} + \frac{\bar{a}K_1 \omega^{p-1} \cos \left(\frac{\omega^p}{2} - \omega t\right)}{2m} \]

\[ B_3 = -\frac{\omega}{2a} + \frac{k_2L'(\tilde{a})}{2m_0a} - \frac{k_1 \omega^{p-1} \cos \left(\frac{\omega^p}{2} - \omega t\right)}{2ma} \]

The stability condition of the steady-state solution of the system was obtained through equation (22).

\[ B_1 > 0 \]  

(23a)

\[ B_1^2 - B_2B_3 > 0 \]  

(23b)

**Numerical verification**

To verify the correctness of the steady-state solution obtained previously, we compared the solution of the amplitude–frequency equation (19) and the result obtained through the numerical simulation of equation (1). The selected parameters were as follows: \( m = 5, k_1 = 5, k_2 = 6, c = 1, \rho = 0.5, K_1 = 5, F = 2, a_0 = 0.8 \).

To analyze the effects of time delay on the amplitude–frequency curve, we selected two time-delay values for comparison: \( \tau = 0.5 \) and \( \tau = 5.2 \). Numerical simulation was performed using the power series method, and the numerical calculation formula (1) of the system can be expressed as follows\(^{18,42} \)

\[ x(t_n) = y(t_{n-1})h - \sum_{j=1}^{n} C_j^p x(t_{n-j}) \]  

(24a)

\[ y(t_n) = \frac{h}{m} \left[ F \cos(\omega h^p i) - cy(t_{n-1}) - k_1 x(t_n) - g(x(t_n)) - K_1 z(t_{n-1-i}) - \sum_{j=1}^{n} C_j^{1-p} y(t_{n-j}) \right] \]  

(24b)

\[ z(t_n) = [y(t_n)]h^{1-p} - \sum_{j=1}^{n} C_j^{1-p} z(t_{n-j}) \]  

(24c)

where \( x(t_n) \) is the oscillator displacement, \( y(t_n) \) is the oscillator speed, \( z(t_n) \) is the fractional differential term of displacement, \( t_n = nh \) is the time sampling point, \( h \) is the sampling time step size, and \( C_j^p \) is the fractional binomial coefficient that satisfies the following formula

\[ C_0^p = 1 \]  

(25a)

\[ C_j^p = \left(1 - \frac{1 + p}{j}\right) C_{j-1}^p \]  

(25b)

In the calculation process, the time step was set to \( h = 0.004 \), and calculation time was set to 500 s. The average value of the last 10% of the maximum response values was interpreted as the steady-state response amplitude. The numerical calculation results at \( \tau = 0.5 \) and \( \tau = 5.2 \) are shown as circles in Figures 3 and 4, and the analytical
solution obtained through the amplitude–frequency equation (19) of the steady-state solution is shown as black solid lines in Figures 3 and 4. The numerical and approximate analytical solutions had a high goodness of fit.

Figure 3 shows a comparison of the numerical and analytical solutions at $\tau = 5.2$. The amplitude–frequency response of the system under this circumstance had a linear relationship. Figure 4 shows a comparison of the numerical and analytical solutions at $\tau = 0.5$. In this situation, only the time delay was changed, and the other parameters stayed the same. The amplitude–frequency response curve presented a major change with a multi-solution phenomenon and an obvious nonlinear characteristic. This change was attributed to the change in time delay. Therefore, to thoroughly understand the effects of time delay on the vibration response of the system, we used a fixed excitation frequency in the follow-up analysis and emphasized the effects of time-delay changes on the vibration response amplitude.

Effects of system parameters on the vibration response

**Effects of time delay on steady-state solution amplitude and time-delay response**

Figures 5 and 7 show the response time delay–amplitude relation curves of the steady-state solution of the system under different excitation frequencies.

Figure 5 shows the response time delay–amplitude relation chart of the steady-state solution of the system under excitation frequency $\omega = 0.8$. As the time delay increased, the system response amplitude presented a periodic change, which was obviously caused by the trigonometric function related to the time delay in equation (18). Figure 5 also shows that for the gaps of the piecewise system, the time-delay change caused a continuous change in the vibration response between the vibration amplitude, which is smaller than the gap value (solid blue
and the system, which is higher than the gap value (solid black line) in each period of time delay-dependent amplitude change. Hence, when the other conditions are fixed, the system vibration mode can be changed by regulating the time delay.

Figure 6 shows the response time delay–amplitude relation chart of the steady-state solution under excitation frequency $\omega = 1.27$. When the external excitation frequency increased, the system dynamic response became increasingly
complicated under the dual effects of piecewise nonlinearity and time delay, and the situation shown in Figure 6(a) occurred. Multiple solutions appeared in the time delay-dependent change of the system response amplitude and triggered piecewise stiffness. Moreover, no response curve with piecewise stiffness can be observed in Figure 6(b). The system response with time delay is a linear curve presenting a periodic change without piecewise stiffness. Hence, this multisolution phenomenon is a nonlinear phenomenon induced by the piecewise nature of the system.

With the continuous increase in external excitation, the topological structure of the time delay-dependent change curve of the system response amplitude underwent a major change. Figure 7 shows the change curve of \( a-t \) under \( \omega = 1.32 \). The response curve is divided into two parts: one part is below the gap, which is a linear curve presenting a periodic change with time delay, and the other part is above the gap, which is a nonlinear curve presenting a periodic change with time delay.

In summary, the system vibration response presented three states under different excitation frequencies. The vibration responses under the three states exhibited time delay-dependent periodic changes, and the period duration was decided by excitation frequency. For a simplified analysis, the most typical state shown in Figure 6 was considered for evaluation in the subsequent investigation of the effects of other system parameters on the system vibration response.

**Effects of piecewise parameters on the amplitude–time delay response of the steady-state solution**

The effect of piecewise stiffness on the system vibration characteristic was first analyzed. Figure 8 shows the vibration amplitude–time delay relation curve of the steady-state solution under \( k_2=0, 6, 8, \) and 10. When \( k_2=0 \) and piecewise stiffness was absent, the system vibration amplitude presented a complete linear characteristic and...
gradually increased with $k_2$. When $k_2=6$, 8, and 10 and the system presented a nonlinear vibration characteristic, the system jump amplitude increased with $k_2$, indicating that the system nonlinearity was gradually enhanced. As the contribution of piecewise stiffness to system stiffness continuously increased, the peak response amplitude gradually decreased, but the system response period did not change.

Figure 9 shows the amplitude–time delay relation chart of the steady-state solution of the system under fractional orders 0.1, 0.5, and 0.9. When fractional order $p$ increased, the amplitude response curve of the steady-state solution became skewed toward the direction of the increase in time delay, and the peak response amplitude was gradually enlarged, which can also be explained by equation (17). We considered the weakening effect of the equivalent stiffness generated by the fractional time-delay feedback term on system stiffness and the resulting periodic change. The peak response amplitude of the system within a period gradually increased with $p$.

The response amplitude–time delay relation curve obtained through equation (19) under $K_1=-0.4$, $-0.7$, and $-1$ is shown in Figure 10. When $K_1$ gradually increased, the resonance amplitude of the system was gradually enhanced because the damping effect of the fractional time-delay term on the system weakened step by step. Given that the equivalent linear stiffness generated by the fractional time-delay term to the system gradually weakened, the nonlinear characteristic of the system also gradually weakened, and the nonlinearity was stronger under $K_1=-0.4$ than under the two other circumstances with the islanding phenomenon.

**Effects of piecewise parameters on the amplitude–time delay response of the steady-state solution**

Figure 11 shows a graph of the effect of the piecewise gap on the amplitude–time delay relation of the steady-state solution. When $a_0$ was small and the system presented a linear vibration, vibration occurred at the position
increase. In general, the change in the peak response amplitude enhanced the system nonlinearity and increased the peak response amplitude. Moreover, the time delay corresponding to the peak amplitude changed, but the period of the periodic response amplitude remained unchanged.

**Singularity analysis**

To further study the frequency response characteristics of the fractional-order primary resonance of the piecewise system, we applied singularity theory to analyze the bifurcation of the system and determine the influence of the system parameters on the response characteristics of the fractional-order piecewise system. The amplitude-frequency response equation was expanded by Taylor expansion at point \( a_0 \), and the higher order terms were omitted.

\[
p_4a^4 + p_3a^3 + p_2a^2 + p_1a + p_0 = 0
\]

where

\[
p_4 = \pi^2 \left[ k_1 + k_2 - m\omega^2 + K_1\omega^\rho \cos \left( \frac{p\pi}{2} - \omega t \right) \right]^2
\]
\[
p_3 = -2a_0k_2\pi(2 + \pi) \left[ k_1 + k_2 - m\omega^2 + K_1\omega^\rho \cos \left( \frac{p\pi}{2} - \omega t \right) \right]
\]
\[
p_2 = \pi^2 (-F^2 + c_1^2\omega^2) + a_0^2k_2 \left( k_2 \left( 4 + \pi(8 + \pi) \right) + 4\pi(K_1 - \omega^2) \right)
\]
\[
+ K_1\pi\omega^\rho \left( 4a_0^2k_2^2 \cos \left( \frac{p\pi}{2} - \omega t \right) + \pi\sin \left( \frac{p\pi}{2} - \omega t \right) \left[ 2c_1\omega + K_1\omega^\rho \sin \left( \frac{p\pi}{2} - \omega t \right) \right] \right)
\]
\[
p_1 = -4a_0^2k_2^2(2 + \pi)
\]
\[
p_0 = 4a_0^4k_2^2
\]

where \( a = z - \frac{p_2}{4p_4} \). Equation (26) is rewritten as follows

\[
z^4 + d_2z^2 + d_1z + \eta = 0
\]

where \( d_2 = \frac{p_0}{p_4} - \frac{3p^2}{8p_4} \), \( d_1 = \frac{p_2^2 - 4p_4p_4^2 + 8p_4^2p_1^2}{8p_4^2} \), \( \mu = \frac{p_2}{p_4} - \frac{p_1(3p_1^2 - 16p_4p_4^2 + 64p_4^2p_1^2)}{256p_4^2} \)
\( z^4 + d_2 z^2 + d_1 z + \eta = 0 \) is the universal unfolding of normal form \( z^4 + \eta = 0 \), which has two unfolding parameters \( d_1 \) and \( d_2 \). In accordance with singularity theory, the persistence of this universal unfolding is analyzed below.

The bifurcation equations of the system are as follows

\[
\begin{align*}
T(z, \eta, d_1, d_2) &= z^4 + d_2 z^2 + d_1 z + \eta \\
T(z, \eta, d_1, d_2) &= 0
\end{align*}
\]

The point sets corresponding with the system were obtained through the corresponding calculation. The calculation processes are as follows

1. The set of bifurcation points is

\[
\begin{align*}
T = 0 &\Rightarrow z^4 + d_2 z^2 + d_1 z + \eta = 0 \\
T_y = 0 &\Rightarrow 4z^3 + 2d_2 z + d_1 = 0 \\
T_\mu = 0 &\Rightarrow \Phi
\end{align*}
\]

That is, the set of bifurcation points \( B = \Phi \) (\( \Phi \) is an empty set).

2. The set of hysteresis points is

\[
\begin{align*}
T = 0 &\Rightarrow z^4 + d_2 z^2 + d_1 z + \eta = 0 \\
T_y = 0 &\Rightarrow 4z^3 + 2d_2 z + d_1 = 0 \\
T_{yy} = 0 &\Rightarrow 6z^2 + d_2 = 0
\end{align*}
\]

That is, the set of hysteresis points \( H = \{d_2^3 + \frac{27}{8} d_1^2 = 0, d_2 \leq 0\} \).

3. The set of double-limit points is

\[
\begin{align*}
T = 0 &\Rightarrow z^4 + d_2 z^2 + d_1 z + \eta = 0 \\
T_y = 0 &\Rightarrow 4z^3 + 2d_2 z + d_1 = 0
\end{align*}
\]

That is, the set of double-limit points \( D = \Phi \).

4. The set of transition points is

\[
\sum = B \cup H \cup D
\]

That is, the set of transition points \( \sum = \begin{cases} B = \Phi \\ H = \{d_2^3 + \frac{27}{8} d_1^2 = 0, d_2 \leq 0\} \\ D = \Phi \end{cases} \).

The spatial distribution of the transition sets is shown in Figure 12.

The topological structure diagrams of the bifurcation curves corresponding to different regions of the transition sets are shown in Figure 13.

Figure 13 indicates that in spatial distribution regions \( B_1 \), (1), \( B_2 \), and (2), the structure of the solution is one \( d_1 \) corresponding to two \( d_2 \). In regions \( H_1 \), (3), \( H_2 \), and (4), the structural form of the solution changes from one \( d_1 \) value corresponding to two \( d_2 \) solutions and gradually develops into the case of one \( d_1 \) value corresponding to four \( d_2 \) solutions.
Effects of system parameters on system bifurcation

The changes in unfolding parameters $d_1$ and $d_2$ affect the transition set of the system, and the change in bifurcation parameter $\mu$ affects the principal resonance response of the system. By analyzing the effects of time delay, piecewise gap, and fractional order on unfolding parameters $d_1$ and $d_2$ and bifurcation parameter $\mu$, the influences of these system parameters on the bifurcation of the system were studied.

Figure 12. Spatial distribution of the transition sets.

Figure 13. Topological structure diagrams of the bifurcation curves.

Effects of system parameters on system bifurcation
Effect of time delay on system bifurcation

The selected parameters were as follows: \( m = 5, \ c = 1, \ F = 2, \ k_1 = 5, \ k_2 = 6, \ K_1 = -1, \ p = 0.5, \) and \( \omega = 1.27. \) Piecewise gap \( a_0 \) was set to 0.2, 0.4, 0.6, 0.8, and 1.0. Then, the relationship curves of unfolding parameters \( d_1 \) and \( d_2 \) and bifurcation parameter \( \mu \) with time delay \( \tau \) were drawn based on equation 28, as shown in Figure 14(a) to (c).

Figure 14(a) to (c) indicates that for each given piecewise gap value, when time delay \( \tau \leq 0.7, \) the unfolding and bifurcation parameters gradually increased with the increase in time delay \( \tau \). The positions of the transition set also changed with the increase of time delay \( \tau \). The range of transition spaces (1) and (2) kept increasing, and the range of transition spaces (3) and (4) gradually decreased, indicating that the complex response areas of the principal resonance of the system gradually became small. When the time delay value was \( 0.7 < \tau < 3, \) as time delay \( \tau \) increased, unfolding parameter \( d_1 \) and bifurcation parameter \( \mu \) decreased, and unfolding parameter \( d_2 \) increased. Time delay \( \tau \) had a much greater influence on unfolding parameter \( d_1 \) than on unfolding parameter \( d_2 \).

We conclude that with the increase in time delay \( \tau \), the range of transition spaces (1) and (2) continued to decrease, and the range of transition spaces (3) and (4) continued to increase, showing that the complex areas of the principal resonance responses of the system gradually widened. When time delay \( \tau > 3, \) the periodic processes were repeated as time delay \( \tau \) increased. Time delay \( \tau \) had a much greater influence on unfolding parameter \( d_1 \) and bifurcation parameter \( \mu \) than on unfolding parameter \( d_2 \); hence, the influence of time delay on unfolding parameter \( d_2 \) could be ignored.
Effect of piecewise gap on system bifurcation

Piecewise gap $a_0$ was set to 0.2, 0.4, 0.6, 0.8, and 1.0. Then, the change curves of unfolding parameters $d_1$ and $d_2$ and bifurcation parameter $\mu$ with time delay $\tau$ are shown in different colors in Figure 14(a) to (c). As the piecewise gap gradually increased, unfolding parameter $d_1$ gradually increased, and unfolding parameter $d_2$ gradually decreased. As piecewise gap $a_0$ increased, the range of transition spaces (1) and (2) continued to decrease, and the range of transition spaces (3) and (4) continued to increase, that is, the complex area of the principal resonance responses of the system widened.

Figure 12 indicates that the change in piecewise gap $a_0$ exerted a great influence on unfolding parameter $d_1$ and bifurcation parameter $\mu$. Therefore, when analyzing the transition set change, the influence of piecewise gap $a_0$ on unfolding parameter $d_2$ can be ignored.

Effect of fractional order on system bifurcation

The variation relations of unfolding parameters $d_1$ and $d_2$ and bifurcation parameter $\mu$ with time delay $\tau$ are shown in curves of different colors in Figure 15(a) to (c) when piecewise gap $a_0$ was 0.2, the fractional order was 0.1, 0.3, 0.5, 0.7, and 0.9, and the other parameters were unchanged.

![Figure 15](image-url)

**Figure 15.** (a) Influence of fractional order $p$ on unfolding parameter $d_1$. (b) Influence of fractional order $p$ on unfolding parameter $d_2$. (c) Influence of fractional order $p$ on unfolding parameter $\mu$. 
Figure 15(a) to (c) indicates that the change trends of unfolding parameters $d_1$ and $d_2$ and bifurcation parameter $\mu$ in different time-delay parameter regions varied with fractional order $p$. When time delay $\tau$ was in a small range, unfolding parameters $d_1$ and $d_2$ and bifurcation parameter $\mu$ increased with the increase in fractional order $p$. Moreover, the ranges of transition spaces (1) and (2) continued to increase, and the ranges of transition spaces (3) and (4) continued to decrease, that is, the complex response areas of the principal resonance of the system narrowed. When time delay $\tau$ was in a large range, unfolding parameters $d_1$ and $d_2$ and bifurcation parameter $\mu$ decreased with the increase in fractional order $p$. Moreover, as time delay $\tau$ increased, the ranges of transition spaces (1) and (2) continued to decrease, and the ranges of transition spaces (3) and (4) continued to increase, that is, the complex response areas of the principal resonance of the system widened.

As fractional order $p$ increased, the curves of unfolding parameters $d_1$ and $d_2$ and bifurcation parameter $\mu$ with time delay $\tau$ gradually shifted to the right. When fractional order $p$ changed, the time delay values corresponding to the peak values of unfolding parameters $d_1$ and $d_2$ and bifurcation parameter $\mu$ also changed. The time delay values corresponding to the peak values should be avoided as much as possible to make the system vibration stable.

**Conclusion**

The forced vibration of an SDOF piecewise linear system with fractional time-delay feedback was investigated in this paper. An averaging method was used to obtain the approximate analytical solution and stability condition of the system. Then, the analytical solution was compared with a numerical solution iterated. The two solutions showed high goodness of fit, indicating the correctness of the analytical solution. The effects of system parameters on the system vibration response and system bifurcation were further analyzed, and the results implied that the fractional time delay influenced the system dynamic properties in the forms of equivalent stiffness and dampness with periodic changes. The system was converted from linear to nonlinear under the effects of certain parameters due to the existence of piecewise stiffness and gaps. The proposed method and the results of this work can provide insights into other piecewise linear systems containing fractional time-delay feedback.

**Declaration of conflicting interests**

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

**Funding**

The author(s) disclosed receipt of the following financial support for the research, authorship, and/or publication of this article: This work was supported by the National Natural Science Foundation of China (No. 11872256, 11802183, and 11872254).

**ORCID iD**

Jianchao Zhang https://orcid.org/0000-0003-4267-1999

**References**

1. Balcerzak M, Chudzik A and Stefanski A. Properties of generalized synchronization in smooth and non-smooth identical oscillators. *Eur Phys J Spec Top* 2020; 229: 2151–2165.
2. Proskurnikov AV and Smirnova VB. Constructive estimates of the pull-in range for synchronization circuit described by integro-differential equations. In: 2020 IEEE international symposium on circuits and systems (ISCAS) (ed M Ley), Seville, Spain, 17–20 May 2020, pp. 1–5. Piscataway: IEEE.
3. Zhang T and Dai H. On the nonlinear dynamics of a high-speed railway vehicle with nonsmooth elements. *Appl Math Model* 2019; 76: 526–544.
4. Sheng W, Lin H, Yang C, et al. Nonlinear vibrations of a piecewise-linear quarter-car truck model by incremental harmonic balance method. *Nonlinear Dyn* 2018; 92: 1719–1732.
5. Gao X, Chen Q and Liu XB. Nonlinear dynamics design for piecewise smooth vibration isolation system. *Nonlinear Dyn* 2016; 48: 192–200.
6. Giannakopoulos F and Pliete K. Planar systems of piecewise linear differential equations with a line of discontinuity. *Nonlinearity* 2001; 14: 1611–1632.
7. Carmona V, Freire E, Ponce E, et al. The continuous matching of two stable linear systems can be unstable. *Discrete Cont Dyn-A* 2006; 16: 689–703.
8. Sobamowo G. Finite element thermal analysis of a moving porous fin with temperature-variant thermal conductivity and internal heat generation. *Rep Mech Eng* 2020; 1: 110–127.
9. Shaw SW and Holmes PJ. A periodically forced piecewise linear oscillator. *J Sound Vib* 1983; 90: 129–155.
10. Hu HY. Detection of grazing orbits and incident bifurcations of a forced continuous piecewise-linear oscillator. *J Sound Vib* 1995; 187: 485–493.
11. Hu HY. Nonsmooth analysis of dynamics of a piecewise linear system. *Chinese J Theor Appl Mech* 1996; 28: 483–488.
12. Luo ACJ. The mapping dynamics of periodic motions for a three-piecewise linear system under a periodic excitation. *J Sound Vib* 2005; 283: 723–748.
13. Jin JD and Guan LZ. Fourier series solution of forced vibration of a multi-degree-of-freedom system with piecewise-linear elastic elements. *J Vib Eng* 2003; 16: 273–278.
14. Gao X and Chen Q. Static and dynamic analysis of a high static and low dynamic stiffness vibration isolator utilising the solid and liquid mixture. *Eng Struct* 2015; 99: 205–213.
15. Jazar GN, Mahinfalah M and Deshpande S. Design of a piecewise linear vibration isolator for jump avoidance. *P I Mech Eng K-J Mul* 2007; 221: 441–449.
16. Hu HY. Designing elastic constraints in a vibration isolation system from the viewpoint of nonlinear dynamics. *Acta Aeronaut Astronaut Sin* 1996; 17: 18–22.
17. Oldham K and Spanier J. The fractional calculus theory and applications of differentiation and integration to arbitrary order. New York: Academic Press, 1974.
18. Podlubny I. *Fractional differential equations*. London: Academic Press, 1999.
19. Jamil M and Ahmed A. Traveling wave solutions of 3D fractionalized MHD Newtonian fluid in porous medium with heat transfer. *J Appl Comput Mech* 2020; 6: 968–984.
20. Zuo YT. A gecko-like fractal receptor of a three-dimensional printing technology: a fractal oscillator. *J Math Chem* 2021; 59: 735–744.
21. He CH, Liu C, He JH, et al. Passive atmospheric water harvesting utilizing an ancient Chinese ink slab. *FU Mech Eng*. Epub December 2020. DOI: 10.22190/FUME201203001H.
22. Sun HL, Jin C, Zhang WM, et al. Modeling and tests for a hydro-pneumatic suspension based on fractional calculus. *J Vib Shock* 2014; 33: 167–172.
23. Kumar D, Agarwal RP and Singh J. A modified numerical scheme and convergence analysis for fractional model of Liénard’s equation. *J Comput Appl Math* 2018; 339: 405–413.
24. He JH, Kou SJ, He CH, et al. Fractal oscillation and its frequency-amplitude property. *Fractals*. Epub ahead of print 2021. DOI: 10.1142/S0218348X2150105X.
25. Jena RM, Chakraverty S, Jena SK, et al. Analysis of time-fractional fuzzy vibration equation of large membranes using double parametric based Residual power series method. *ZAMM-J Appl Math Mech*. Epub ahead of print 2020. DOI: 10.1002/zamm.202000165.
26. Mohebbi A and Saffarian M. Implicit RBF meshless method for the solution of two-dimensional variable order fractional cable equation. *J Comput Appl Math*. 2020; 6: 235–247.
27. Jena RM, Chakraverty S, Jena SK, et al. On the wave solutions of time-fractional Sawada-Kotera-Ito equation arising in shallow water. *Math Meth Appl Sci* 2021; 44: 583–592.
28. Kaslik E and Sivasundaram S. Analytical and numerical methods for the stability analysis of linear fractional delay differential equations. *J Comput Appl Math* 2012; 236: 4027–4041.
29. Li CP and Zhang FR. A survey on the stability of fractional differential equations. *J Comput Appl Math* 2017; 309: 405–413.
30. Babakhani A, Baleanu D and Khanbabaie R. Hopf bifurcation for a class of fractional differential equations with delay. *Nonlinear Dyn* 2012; 69: 721–729.
31. Ye H, Ding Y and Gao J. The existence of a positive solution of a fractional differential equation with delay. *Positivity* 2007; 11: 341–350.
32. Zhang XY. Some results of linear fractional order time-delay system. *Appl Math Comput* 2008; 197: 407–411.
33. Čermák J, Došlá Z and Kisela T. Fractional differential equations with a constant delay: stability and asymptotics of solutions. *Appl Math Comput* 2017; 298: 336–350.
34. He JH. A tutorial review on fractal spacetime and fractional calculus. *Int J Theor Phys* 2014; 53: 3698–3718.
35. He JH. Fractal calculus and its geometrical explanation. *Results Phys* 2018; 10: 272–276.
36. He JH and Ain QT. New promises and future challenges of fractal calculus: from two-scale thermodynamics to fractal variational principle. *Therm Sci* 2020; 24: 65–65.
37. He JH, Elagan SK and Li ZB. Geometrical explanation of the fractional complex transform and derivative chain rule for fractional calculus. *Phys Lett A* 2012; 376: 257–259.
38. Wang KL, Wang KJ and He CH. Physical insight of local fractional calculus and its application to fractional Kdv–Burgers–Kuramoto equation. *Fractals* 2019; 27: 1950122.
39. He JH and El-Dib YO. Periodic property of the time-fractional Kundu–Mukherjee–Naskar equation. *Results Phys* 2020; 19: 103345.
40. Caponetto R, Dongola G, Fortuna L, et al. *Fractional order systems: modeling and control applications*. Singapore: World Scientific, 2010.
41. Petrás I. *Fractional-order nonlinear systems*. Beijing: Higher Education Press, 2011.