Perturbative QCD at finite temperature and density

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Abstract

This is a comprehensive review on the perturbative hot QCD including the recent developments. The main body of the review is concentrated upon dealing with physical quantities like reaction rates.

Contents: §1. Introduction, §2. Perturbative thermal field theory: Feynman rules, §3. Reaction-rate formula, §4. Hard-thermal-loop resummation scheme in hot QCD, §5. Effective action, §6. Hard modes with $|P^2| \leq O(g^2T^2)$, §7. Application to the computation of physical quantities, §8. Beyond the hard-thermal-loop resummation scheme, §9. Conclusions.

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1 Introduction

Among various predictions of QCD, a distinguished one is the existence of the QCD phase transition \( T_c \approx 150 \text{MeV} \). The phase at the high-temperature side, \( T > T_c \), is called the quark-gluon plasma (QGP) phase. According to the standard big bang scenario, the QGP has lived in the early days of the Universe. Efforts of reviving the QGP in the present day are making and it is expected to be realized soon.

Theoretical approaches to the QGP physics fall into four main categories; lattice simulation, perturbative approach, effective-theory approach and phenomenological approach. These approaches are complementary to each other and each of them has its own advantage.

The purpose of this paper is to give a comprehensive review of the perturbative approach. Thanks to the asymptotic freedom of QCD the coupling constant \( g \) decreases with temperature \( T \) and/or density and then at high temperature/density the perturbative approach comes to be a powerful device.

Thermodynamic properties of a QGP system in thermal and chemical equilibrium is characterized by

\[
Z = \text{Tr} e^{-\beta (H - \mu Q)} \quad (\beta = 1/T),
\]

\[
\ln Z = PV/T,
\]

where \( Z, H, P \) and \( V \) are, in respective order, the grand-partition function, QCD Hamiltonian, pressure and volume of the system, and \( \mu \) is the chemical potential being conjugate to the quark number \( Q \). [For a concise review of the properties of a QGP, I refer to Ref. 8.)] The thermal average of a quantity \( \Omega \) is defined by

\[
\langle \Omega \rangle \equiv \text{Tr} \Omega e^{-\beta (H - \mu Q)/Z}.
\]

Rates of reactions taking place in a QGP are computed through thermal Green functions, which are defined as the thermal average of the relevant products of field operators.
2 Perturbative thermal field theory: \cite{1, 2, 3, 4, 5, 6} Feynman rules

Traditional approach starts with taking in-fields in vacuum ($T = 0$) theory as a basis of a Fock space.

2.1 Imaginary-time or Matsubara formalism

This formalism is convenient for calculating no-leg amplitudes (free energy, grand-partition function etc.) and two-point functions.

- Propagator:
  \[
  \frac{\mathcal{N}(P)}{p_0^2 + p^2 + m^2}.
  \]
  Here $P = (p_0, p)$ with $p_0 = 2\pi n T [\pi (2n + 1) T - i\mu]$ for gluon or FP ghost [quark] ($n = \cdots, -2, -1, 0, 1, 2, \cdots$). The form of the “numerator factor” $\mathcal{N}(P)$ is the same as in Euclidean vacuum theory.

- Vertex: To an $N$-particle vertex, is assigned
  \[
  \mathcal{V}_\ldots (2\pi)^3 \frac{1}{T} \delta_{n,0} \delta(p),
  \]
  where $n = \sum_{i=1}^{N} n_i$, $p = \sum_{i=1}^{N} p_i$, and $\mathcal{V}_\ldots$ stands for the factor that can be read off from the interaction Lagrangian.

- Internal momentum:
  \[
  T \sum_n \int \frac{d^3p}{(2\pi)^3}.
  \]

2.2 Real-time formalism

This formalism allows us to directly compute $N$-point functions. The theory is formulated by introducing a contour $C = C_1 + C_2 + C_3 + C_4$ in a complex time plane: $t_i \to t_f (C_1)$, $t_f \to t_f - i\sigma (C_3)$, $t_f - i\sigma \to t_i - i\sigma (C_2)$ and $t_i - i\sigma \to t_i - i\beta (C_4)$, where $0 \leq \sigma \leq \beta$. Then, the limit $t_{i/f} \to \mp\infty$ is taken. As far as thermal amplitudes are concerned, the contributions from the contour segments $C_3$ and $C_4$ may, in a sense, be ignored. \cite{6, 7, 8} Thus the formalism turns out to be a two-component theory; the field whose time argument is in the segment $C_1 (C_2)$, $\phi_1 (\phi_2)$, is called the type-1
(type-2) or physical (thermal-ghost) field. Then, in this formalism, propagators, vertices and self-energy parts enjoy $2 \times 2$ matrix structure. Theories with different $\sigma$’s constitute an equivalent class of theories. Physical-field amplitudes are independent of $\sigma$.

- Propagator: $i \mathcal{N}(P) \hat{\Delta}(P)$.

Here $P = (p_0, \mathbf{p})$ with $p_0$ real and [upper (lower) suffix refers to gluon and FP ghost (quark)]

$$
\hat{\Delta}(P) = \hat{M}_\pm(P) \hat{\Delta}_F(P) \hat{M}_\pm(P),
\hat{M}_\pm(P) = \begin{pmatrix} \sqrt{1 \pm n_\pm(p_0)} & e^{\sigma p_0} \frac{\theta(-p_0) \pm n_\pm(p_0)}{\sqrt{1 \pm n_\pm(p_0)}} \\ e^{-\sigma p_0} \frac{\theta(p_0) \pm n_\pm(p_0)}{\sqrt{1 \pm n_\pm(p_0)}} & \sqrt{1 \pm n_\pm(p_0)} \end{pmatrix},
\hat{\Delta}_F(P) = \text{diag} [\Delta_F(P), -\Delta_F^*(P)]
\left( \Delta_F(P) = 1/(P^2 - m^2 + i0^+) \right),
\right), \tag{2.4}
\right)

\begin{align*}
n_+(p_0) &= \frac{1}{e^{\beta|p_0|} - 1}, \\
n_-(p_0) &= \frac{1}{e^{\beta(|p_0| - \epsilon(p_0) \mu)} + 1}.
\end{align*}

$i \Delta_{11}, i \Delta_{12}, i \Delta_{21}$ and $i \Delta_{22}$ are Fourier transforms with respect to $(\text{Re}(x_0 - y_0), x - y)$ of, in respective order, $\langle T \phi(x) \bar{\phi}(y) \rangle$, $\tau \langle \bar{\phi}(y) \phi(x) \rangle$, $\langle \phi(x) \bar{\phi}(y) \rangle$ and $\langle T \phi(x) \bar{\phi}(y) \rangle$, where $\bar{\phi}$ is the adjoint of $\phi$, $\tau = +(-)$ for gluon and FP-ghost (quark) and $\overline{T}$ is the anti-time-ordering symbol.

- Vertex

$$
i \mathcal{V}_-(2\pi)^4 \delta^4(P) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
$$

where $P = \sum_{i=1}^{N} P_i$. Note that the vertex matrix is diagonal. The $(1,1)$ component is called the type-1 vertex, which is same as the vacuum-theory counterpart, and the $(2,2)$ component is called the type-2 vertex.

- Internal momentum:

$$
\int \frac{d^4P}{(2\pi)^4}.
$$

Notes:

1) Due to thermal radiative corrections, full gluon- and quark-propagators split into several pieces. However their structure remains essentially to be the same as
the corresponding bare propagator, Eqs. (2.2) - (2.5) (cf. below). Incidentally, (each piece of) the self-energy part takes the form

$$\hat{\Sigma}(P) = \hat{M}^{-1}_\pm(P) \begin{pmatrix} \Sigma_F(P) & 0 \\ 0 & -\Sigma^*_F(P) \end{pmatrix} \hat{M}^{-1}_\pm(P).$$  \hfill (2.6)

2) Thermo field dynamics [3, 4, 6] (A two-component theory formulated with canonical quantization): For each field $\phi(x)$ in the original Hamiltonian, introduce its “copy” $\tilde{\phi}(x)$. $\tilde{\phi}$ is essentially the field obtained from $\phi$ through time-reversal operation. $[\phi (\tilde{\phi})$ corresponds to $\phi_1 (\phi_2)$ in the time-path ordered formalism outlined above. Note that $\phi_2$ is in the contour segment $C_2$, in which Re $t$ flies backward, $+\infty \rightarrow -\infty$.] Then, a quasiparticle field $\varphi_i (i = 1, 2)$ is introduced, in momentum space, through

$$\hat{\varphi}(P) = \hat{M}^{-1}_\tau(P) \hat{\phi}(P) \hat{\phi}(P), \quad \hat{\phi}(P) = \begin{pmatrix} \phi(P) \\ \tilde{\phi}(P) \end{pmatrix},$$

$$\hat{\varphi}(P) = \hat{\phi}(P) \hat{M}^{-1}_\tau(P), \quad \hat{\tilde{\phi}}(P) = (\overline{\phi}(P), \phi(P)),$$  \hfill (2.7)

where $\tau = +[-]$ for gluon and FP ghost [quark]. Note that, in general, $\overline{\tau}_i$ is not the adjoint of $\varphi_i$. The so-called thermal vacuum is introduced by $\varphi^{(\xi_1)}_i(x)|0\rangle = \langle 0|\overline{\varphi}^{(-\xi_1)}_i(x) = 0 (\xi_1 = +, \xi_2 = -)$, where “(+)/(-)” indicates the positive/negative frequency part. With this preliminaries, we see that $\langle 0|T\hat{\phi}\hat{\tilde{\phi}}|0\rangle = i\mathcal{N}\Delta$, Eq. (2.1). Thus, as far as the perturbation scheme is concerned, both thermo field dynamics and the time-path formalism summarized above are equivalent.

3 Reaction-rate formula [10, 11]

To avoid inessential complications, I take a heat bath composed of massless scalar fields. Reactions taking place in the heat bath are of the following generic type;

$$\Phi(P_1) + \cdots + \Phi(P_n) + \text{heat bath} \rightarrow \Phi(Q_1) + \cdots + \Phi(Q_m) + \text{anything}. \hfill (3.1)$$

Here $\Phi$ is a nonthermalized heavy scalar particle. [Generalization to other cases is straightforward.]
The reaction rate $R$ reads

\[
\frac{1}{V} \left( \prod_{j=1}^{m} \frac{2 q_{j0}}{d q_{j}/(2 \pi)^3} \right) R = \left( \prod_{i=1}^{n} \frac{1}{2 p_{i0} V} \right) A \left( P_{1}^{(2)}, \ldots, P_{n}^{(2)}, Q_{1}^{(1)}, \ldots, Q_{m}^{(1)}; P_{1}^{(1)}, \ldots, P_{n}^{(1)}, Q_{1}^{(2)}, \ldots, Q_{m}^{(2)} \right),
\]

where $p_{i0} = E_{i} = \sqrt{p_{i}^2 + m^2}$ etc. In Eq. (3.2), $A$ is an amplitude evaluated in the Keldish variant ($\sigma = 0$ in § 2.2) of real-time formalism for the “process,”

\[
\Phi_{1}(P_{1}) + \cdots + \Phi_{1}(P_{n}) + \Phi_{2}(Q_{1}) + \cdots + \Phi_{2}(Q_{m})
\]

\[
\rightarrow \Phi_{2}(P_{1}) + \cdots + \Phi_{2}(P_{n}) + \Phi_{1}(Q_{1}) + \cdots + \Phi_{1}(Q_{m}),
\]

where the suffix ‘i’ ($i = 1, 2$) refers to type-i field.

Addenda:

1) $A$ in Eq. (3.2) is not an absolute square of some amplitude, in contrast to the case of vacuum theory.

2) When $\Phi(P_{i})$ [$\Phi(Q_{j})$] in the reaction (3.1) is a thermalized particle, the factor $n_{B}(E_{i}) [1 + n_{B}(E_{j})]$ is to be multiplied to the right-hand side (rhs) of Eq. (3.2). Here $n_{B}$ is the Bose distribution function.

3) The formula (3.2) is valid even for a finite cube system as far as the periodic boundary condition is employed for the single-particle wave function basis.

4) In the limit $T \rightarrow 0$, the formula (3.2) reduces to the formula obtained through Cutkosky or cutting rules. In particular, for $n = 2$ and $m = 0$, the formula goes to the optical theorem and, for $n = 2$ and $m = 1$, the formula goes to the Mueller formula for the corresponding inclusive reaction.

5) Applying the formula (3.2) to the reaction (3.1), where $\Phi$’s are constituent particles of the heat bath (cf. the second item above), one can derive the detailed-balance formula. Namely, the rate (3.2) is equal to the rate for the inverse process to (3.1).

6) Cutting rules: The reaction-rate formula (3.2) is derived from the “first-principle formula”

\[
R \propto \text{Tr} e^{-\beta (H - \mu Q)} S^{*} S / \text{Tr} e^{-\beta (H - \mu Q)},
\]

(3.3)
where $S$ is the $S$-matrix element in vacuum theory for the process

$$
\Phi(P_1) + \cdots + \Phi(P_n) + \{\phi\text{'s}\} \rightarrow \Phi(Q_1) + \cdots + \Phi(Q_m) + \{\phi\text{'s}\}.
$$

Here $\phi$'s are constituent particles of the heat bath.

In what follows, we depart from the scalar theory and keep in mind some general theory. The $(1, 1)$ component of the thermal propagator $\Delta_{11}(P)$ in $A$, Eq. (3.2), has the following three roots; (i) the $T = 0$ propagator connecting the vertices $v_1$ and $v_2$ in $S$, Eq. (3.3), which carry the momentum $P$ from $v_1$ to $v_2$, (ii) a particle of momentum $p$ is absorbed from the heat bath into the vertex $v_2$ and a particle of the same momentum $p$ is emitted from the vertex $v_1$ into the heat bath and (iii) an antiparticle of momentum $-p$ is absorbed from the heat bath into the vertex $v_1$ and an antiparticle of the same momentum $-p$ is emitted from the vertex $v_2$ into the heat bath. Both end-point vertices of $\Delta_{11}(P)$ are of type-1, which come from the vertices in $S$ in Eq. (3.3).

The roots of the $(2, 2)$ component of the thermal propagator $\Delta_{22}(P)$ in $A$ are obtained from above by $S \rightarrow S^\ast$. Both end-point vertices of $\Delta_{22}(P)$ are of type-2, which come from the vertices in $S^\ast$ in Eq. (3.3).

The $(2, 1)$ component of the thermal propagator $\Delta_{21}(P)$ in $A$ has the two roots; (i) a particle of momentum $p$ is emitted into the heat bath from a vertex $v_1$ in $S$, Eq. (3.3), and a particle of the same momentum $p$ is absorbed from the heat bath into a vertex $v_2$ in $S^\ast$, Eq. (3.3), and (ii) an antiparticle of momentum $-p$ is absorbed from the heat bath into $v_1$ and an antiparticle of the same momentum $-p$ is emitted from $v_2$ into the heat bath. The vertex $v_1$ ($v_2$) in $S$ ($S^\ast$) goes to the type-1 (type-2) end-point vertex of $\Delta_{21}(P)$.

The roots of the $(1, 2)$ component of the thermal propagator $\Delta_{12}(P)$ are obtained from those for $\Delta_{21}(P)$ by $S \leftrightarrow S^\ast$.

This inspection leads us to introduce the thermal cutting rules: [10] Cut all the lines $\Delta_{12}$’s, $\Delta_{21}$’s, $\Delta_{11}^{(T)}$’s, and $\Delta_{22}^{(T)}$’s. [The superscript “($T$)” refers to the $T$-dependent part.]

Through an application of the above cutting rules to some reaction-rate formula, $A$ in Eq. (3.2) is divided into several subparts, each of which corresponds either to $S$ or
to $S^*$, Eq. (3.3), and the interpretation of them in physical terms is straightforward.

Finally it is worth mentioning that the calculational rules of evaluating absorptive part of a generic thermal amplitudes are settled in Ref. 12). Finite-temperature generalizations of cutting rules are discussed in Refs. 12) and 13).

4 Hard-thermal-loop resummation scheme in hot QCD [1, 14]

When formally higher order correction to an (one-particle irreducible) amplitude is of the same order of magnitude as the lowest-order counterpart, a resummation of the “correction” is necessary. This is the case for classes of amplitudes whose all external momenta are soft, $P^\mu = O(gT)$. The relevant diagrams are the one-loop diagrams with hard loop momentum, $Q^\mu_{\text{loop}} = O(T)$, so is named the hard-thermal loop (HTL).

The computation of 2-point amplitudes or the self-energy parts has been carried out long ago. Let us summarize the result.

Gluon: In a covariant gauge, the full gluon propagator may be decomposed as

$$
\Delta_F^{\mu\nu}(P) = -\mathcal{P}_T^{\mu\nu} \Delta^T_F(P) - \mathcal{P}_L^{\mu\nu} \Delta^L_F(P) - \frac{1}{\lambda P^2 + i0^+} - c(P) \frac{C^{\mu\nu}}{P^2 + i0^+},
$$

where $\lambda$ is the gauge parameter and $\mathcal{P}_T [\mathcal{P}_L]$ is the projection operator onto the transverse [longitudinal] or chromomagnetic [chromoelectric] sector. The third term on the rhs of Eq. (4.1) is the gauge term. Explicit form of $\mathcal{P}_T^{\mu\nu}, D^{\mu\nu}$ and $C^{\mu\nu}$ is given, e.g., in Ref. 6).

The HTL contribution reads

$$
\Pi_T(P) = -\frac{3}{2} m_g^2 \left[ \frac{P_0 P^2}{2 p^3} \ln \left( \frac{P_0 + p}{P_0 - p} \right) - \frac{p_0^2}{p^2} \right],
$$

$$
\Pi_L(P) = \frac{3}{2} m_g^2 \frac{P^2}{p^3} \left[ \frac{P_0}{p} \ln \left( \frac{P_0 + p}{P_0 - p} \right) - 2 \right],
$$

$$
c(P) = 0,
$$

$$
m_g^2 = \frac{g^2}{3} T^2 \left[ 1 + \frac{N_f}{6} \left( 1 + \frac{3}{\pi^2} \frac{\mu^2}{T^2} \right) \right],
$$

where $P$ is soft and $N_f$ is the number of quark flavors. We have assumed the common chemical potential $\mu$ for all $N_f$ (anti)quarks. Note that $\Pi_T(P)$ and $\Pi_L(P)$ are even
functions of $p_0$. Observe that $\Pi(P)$’s are of $O(g^2 T^2)$, the same order of magnitude as the bare counterpart $P^2$.

Characteristic features:

G1) Landau damping. $\text{Im} \ln[(p_0 + p)/(p_0 - p)] \neq 0$ for space-like $P^\mu$, $P^2 < 0$.

G2) Static limit. Debye screening mass appears in the chromoelectric sector, $\Pi_L(p_0 = 0, p) = 3m_g^2$. On the other hand, no screening mass appears in the chromomagnetic sector, $\Pi_T(p_0 = 0, p) = 0$. The last fact indicates that, for some amplitudes that diverge (in naive perturbative calculation) due to the infrared singularity in the chromomagnetic sector, the screening is not sufficient for the amplitudes to converge.

G3) Dispersion curve [the (positive) solutions, $p_0 = \omega_{T/L}(p)$, to $P^2 - \Pi_{T/L}(P) = 0$]. The mode with $P^2 - \Pi_L(P) = 0$, being absent in vacuum theory, is called the plasmon.

$\omega_{T/L}(p) > 0$, $\omega_{T/L}(p) = m_g$ and, for $p >> m_g$, $\omega_T(p) \sim p + 3m_g^2/4p$ and $\omega_L(p) \sim p + 2pe^{-2p^2/3m_g^2}$. The group velocities $v_{T/L}(p) \equiv d\omega_{T/L}(p)/dp$ are positive.

$\omega_{T/L}(p) < m_g$: The solutions $p_0 = \omega_{T/L}(p)$ exist for pure imaginary $p$, showing the damping modes.

Quark: The HTL-resummed soft-quark propagator takes the form,

$$\hat{S}_F(P) = -\frac{1}{2} \left[ \frac{\gamma^0 - \vec{\gamma} \cdot \vec{p}/p}{D_+(P)} + \frac{\gamma^0 + \vec{\gamma} \cdot \vec{p}/p}{D_-(P)} \right],$$

$$D_\pm(P) = -(p_0 \mp p) + \frac{m_q^2}{2p} \left[ \left(1 \mp \frac{p_0}{p}\right) \ln \left(\frac{p_0 + p}{p_0 - p}\right) \pm 2 \right],$$

$$m_q^2 = \frac{g^2}{6} T^2 \left(1 + \frac{1}{\pi^2} \frac{\mu^2}{T^2}\right).$$

Note that $D_+(-p_0, p) = -D_-(p_0, p)$. Observe that the HTL contribution, the second term on the rhs of Eq. (4.5), is of $O(gT)$, the same order of magnitude as the bare counterpart $-p_0 \pm p$. The $2 \times 2$ matrix propagator is related to $\hat{S}_F(P)$ through $\hat{M}_-(P)\hat{S}_F(P)\hat{M}_-(P)$ (cf. § 2.2), where $\hat{S}_F(P) = \text{diag}[\hat{S}_F(P), -(\hat{S}_F(P))]$. Taking the complex conjugate, in obtaining the $(2, 2)$ component of $\hat{S}_F(P)$, does not apply to the Dirac matrices.
Characteristic features:

Q1) Landau damping as in the case of gluon.

Q2) Static limit. Debye-like screening mass, \( D_\pm(p_0 = 0, p) = \pm [p + m_q^2/p] \).

Q3) Dispersion curve [the (positive) solutions, \( p_0 = \omega_\pm(p) \), to \( D_\pm(P) = 0 \)]. The mode with \( D_-(P) = 0 \), which is absent in vacuum theory, is called the plasmino. Both modes are the propagating modes. \( \omega_\pm(p) = p + m_q^2/p \) and, for \( p > m_q \), \( \omega_+(p) \sim p + m_q^2/p \) and \( \omega_-(p) \sim p + 2p e^{-2p^2/m_q^2} \). The group velocity of the + mode, \( v_+(p) \equiv d\omega_+(p)/dp \), is positive. The group velocity \( v_-(p) \) of the plasmino shows an odd behavior. At \( p = 0 \), \( v_- \) is negative and, as \( p \) increases, \( v_-(p) \) increases across \( v_- = 0 \) and approaches \( v_-(p) = 1 \). Furthermore, at large \( p \), the residue \( Z_- \) of the plasmino pole damps exponentially, \( Z_- \sim 2(p^2/m_q^2)e^{-2p^2/m_q^2} \).

Let me summarize the prominent features of HTL amplitudes.

1) The HTL contributions, i.e., the contributions that are of the same order of magnitude as the lowest-order counterparts, arise in an \( N \)-gluon amplitude \((N \geq 2)\) and a quark–antiquark–\( N \)-gluon amplitude \((N \geq 0)\). For an amplitude including external FP-ghost lines, which appears in a covariant gauge, the HTL contribution does not appear.

2) The HTL amplitudes are gauge independent.

3) Kinetic-theory approach leads to the same result. \([15]\)

4) In vacuum massless QCD, no HTL has arisen. This is because the gauge (chiral) invariance of the theory protects a gluon (quark) from getting corrections to the mass.

5) The free part of the Lagrangian is modified so that the modified one (cf. §5) yields the HTL-resummed amplitudes. This means, among others, that, for soft modes, the in-field basis in vacuum theory, which is taken as the basis of perturbation theory, is not the good basis. It should be noted that the in-fields are irreducible representations of Poincaré group, which is a symmetry group of vacuum theory. However thermal field theory does not enjoy the Poincaré symmetry, so that the above result is not unnatural at all. In this relation I refer to Ref. 16).

6) Let \( G \) be the exact amplitude with soft external momenta and \( H \) be the HTL contribution to \( G \). In contrast to the case of vacuum theory, \( (G - H)/H = O(g) \).
7) HTL $N$-point amplitudes satisfy the Ward-Takahashi relation in a stronger sense than in vacuum theory.

From these properties, especially from the last one, one can construct the $N$-point HTL amplitude $H^{(N)}$ ($N \geq 3$) from $H^{(2)}$.

5 Effective action [17, 18, 19]

Having obtained HTL $N$-point functions, one can construct an effective action $^*S$, which is a generating functional of HTL $N$-point amplitudes. $^*S$ is the leading contribution to $S_{\text{eff}}$ defined by

$$e^{iS_{\text{eff}}} \equiv \int_{\text{hard modes}} \mathcal{D}\psi \mathcal{D}\overline{\psi} \mathcal{D}A e^{iS},$$

where $S$ is the QCD action. Various forms for $^*S$ are available, from which I reproduce here the one obtained in Ref. 19):

$$^*S = -\frac{3}{4} m_g^2 \int d^4x F_a^{\mu\alpha}(x) \langle \frac{Y^\alpha Y^\beta}{(Y \cdot D_g)_{ab}^2} F_b^{\beta}(x) \rangle - m_q^2 \int d^4x \overline{\psi}(x) \langle \frac{Y^\mu}{i Y \cdot D_q} \gamma_\mu \psi(x) \rangle,$$

$$\langle f \rangle \equiv \int \frac{d\Omega_k}{4\pi} f(K),$$

$$Y^\mu \equiv (1, k/k),$$

where $D_{g/q}$ is the covariant derivative acting on the gluon/quark field.

$^*S$ is also deduced [20] from the kinetic-theory approach.

Various properties of $^*S$ have been disclosed. For interested readers, I refer to the literature.

1) Equation of motion and its solution. [21]
2) Conserved quantities via Neother’s theorem or other means. [22]
3) Similarity to Chern-Simons theory. [18, 23, 24]
4) Classical nature of $^*S$. [24, 25]
6 Hard modes \[26\] with \(|P^2| \leq O(g^2 T^2)\)

For illustration of the point, let me cut out the portion from a HTL gluon N-point amplitude with a quark loop,

\[ S_{ij}(P + K)(-)^{j-1} \gamma^\mu S_{jk}(P). \]  

(6.1)

Here \(S_{ij}\) is the \((i,j)\) component of the bare quark propagator constituting the HTL, so that \(P\) is hard \(\sim T\). \(K\) is the momentum of an external gluon and is soft \(\sim gT\). Consider, e.g., the sector \(i = j = 1\). Equation (6.1) contains \(\delta(P^2)/(K + P)^2\). When the external momentum \(K\) is on the mass shell \(k_0 = \pm k\), this term develops well-known mass singularity at \(p \cdot k = \pm pk\), \(1/(K + P)^2 \propto 1/(1 + p \cdot k/pk)\), which, upon integration over \(p\), reflects on the logarithmic divergence of the HTL amplitude. This is the well-known mass singularity, which appears when the momentum \((K + P)^\mu\) can kinematically reach the light cone, \((P + K)^2 = 0\), on which the bare propagator \(1/(K + P)^2\) diverges. This observation leads us to analyze hard propagators near the light cone.

Let us analyze the one-loop self-energy parts, \(P \rightarrow Q + (P - Q) \rightarrow P\), with \(P\) the hard external momentum. Recall that, in the case of self-energy part with \(P\) soft, the hard \(Q\) region (HTL) had yielded the dominant contribution. The soft modes and the hard modes are “different” modes. By contrast, for hard \(P\), \(Q\) and/or \(P - Q\) are hard. When \(Q\) \([\((P - Q)\)]\) is hard, one should use the self-energy-part-resummed propagator for the \(Q\) \([\((P - Q)\)]\) line, the self-energy part which we are to evaluate. [As a matter of course, when \(Q\) \([\((P - Q)\)]\) is soft, the HTL-resummed effective propagator should be used for the \(Q\) \([\((P - Q)\)]\) line.] Thus, we are lead to compute the self-energy part in a self-consistent manner.

Here I display the result of the calculation, which is valid to leading order at logarithmic accuracy \(\ln 1/g \gg 1\).

**Gluon:** The self-energy-part resummed hard-gluon propagators \(\overset{\circ}{\Delta}_F (P)\)'s in the covariant gauge read, with obvious notations, (cf. Eqs. (4.1) and (4.2)):

\[ \overset{\circ}{\Delta}^T_F (P) \simeq \frac{\epsilon(\rho_0)}{2p} \frac{1}{p_0 - \epsilon(\rho_0)|p + 3m_2^2/4p| + i\epsilon(\rho_0)\gamma_T} \]  

(6.2)

\[ \overset{\circ}{\Delta}^L_F (P) \simeq \overset{\circ}{\Delta}^g_F (P) \simeq \frac{1}{P^2 + i0^+} \]  

(6.3)
\[ \Delta_F^T(P) \simeq \frac{1}{\lambda} \frac{1}{P^2 + i0^+} \]  
\[ \Delta_F^C(P) \simeq 0, \]  
\[ \gamma_T = \frac{g^2}{4\pi} N_c T \ln(g^{-1}) \left[ 1 + O\left(\frac{\ln \ln g^{-1}}{\ln g^{-1}}\right)\right] + O(g^2T). \]

Here \( \Delta_F^T(P) \) is the FP-ghost propagator. Above forms are valid in the following regions: \( \text{Im} \Delta_F^T(P); \ ||p_0| - p| \leq O(g^2T \ln g^{-1}), \text{Re} \Delta_F^T(P), \Delta_F^C(P), \Delta_F^o(P); \) \( O(g^3T) < ||p_0| - (p + 3m^2_g/4p)\leq O(g^2T \ln g^{-1}). \) \( 2 \times 2 \) matrix propagators \( \hat{\Delta} \)'s are related to \( \Delta_F(P) \)'s through \( \hat{\Delta} = \hat{\Delta}_+ \Delta_F \hat{\Delta}_+^\dagger. \)

Let us see how the bare propagators are changed through resummation of the self-energy part. I take \( \text{Im} \Delta_F^T(P). \) The bare form \( \text{Im} \Delta_F^T(P) = -\pi/2p \delta(\|p_0| - p) \) turns out to be the “smeared” function \( \text{Im} \Delta_F^T(P), \) which is peaked at \( |p_0| = p + 3m^2_g/4p \) with width \( \gamma_T. \) Note that \( m^2_g/p = O(g^2T) \) while \( \gamma_T = O(g^2T \ln g^{-1}), \) so that \( \gamma_T > m^2_g/p \) at logarithmic accuracy. For \( \text{Re} \Delta_F^T(P), \) similar observation may be made.

Equations (5.2) - (6.4) show that, as in the case of soft modes, the in-field basis in vacuum theory is not adequate for the transverse-gluon mode of hard \( P^\mu \) with \( P^2 \simeq 0. \)

**Quark** (\( \mu = 0 \)): The self-energy-part resummed \( 2 \times 2 \) quark propagators reads

\[ \bar{\phi}_{j\tau} \phi_{i\tau} \simeq \sum_{\tau = \pm} \hat{P}_{\tau} \bar{\phi}_{j\tau}(P) \phi_{i\tau}(P) \quad (\hat{P}_{\tau} = (1, \tau\hat{p})) \quad (j, i = 1, 2) \]

\[ \text{Re} \bar{\phi}_{11}(P) = -\text{Re} \bar{\phi}_{22}(P) \]

\[ \simeq \frac{1}{2} \frac{p_0 - \epsilon(p_0)(p + m^2_g/p)}{[p_0 - \epsilon(p_0)(p + m^2_g/p)]^2 + \gamma_q^2} \]

\[ \text{Im} \bar{\phi}_{11}(P) = \text{Im} \bar{\phi}_{22}(P) \]

\[ \simeq -\pi \epsilon(p_0) \left[ \frac{1}{2} - n_F(p) \right] \gamma_{\tau}(P), \]

\[ \bar{\phi}_{12/21}(P) \simeq -i \pi \epsilon(p_0) \left[ \theta(\mp p_0) - n_F(p) \right] \gamma_{\tau}(P), \]

\[ \gamma_{\tau}(P) = \frac{1}{\pi} \frac{\gamma_q}{[p_0 - \epsilon(p_0)(p + m^2_g/p) + \gamma_q^2]}. \]

\[ \gamma_q = \frac{g^2}{4\pi} C_F T \ln(g^{-1}) \left[ 1 + O\left(\frac{\ln \ln g^{-1}}{\ln g^{-1}}\right)\right] + O(g^2T), \]

where the Keldish variant of the real-time contour has been used.

Similar observation to that in the case of gluon may be made.
Substitution of the self-energy-part-resummed propagators screen the above-mentioned mass singularities and renders divergent integral finite.

7 Application to the computation of physical quantities

Various physical quantities like thermal reaction rates are classified as follows.

a) Computation in naive perturbation theory yields a finite result.

b) Naive perturbation theory leads to a diverging result due to infrared (IR) singularity, which turns out to be finite within the HTL resummation scheme.

c1) Same as above b) but it still diverges due to the IR singularity.

c2) Same as above c1) but the divergence is due to the mass singularity.

Examples of b) are the rate of hard photon and of hard photon-pair productions, energy loss of a particle, damping rate of a particle at rest etc. A typical example of c1) is the damping rate of moving particle and an example of c2) is the soft-photon production rare. A particle at rest feels only chromoelectric field and the IR singularity present in the computation within the naive perturbation theory is screened by the Debye mass. On the contrary, a moving particle feels chromomagnetic field also and, due to the absence of chromomagnetic screening mass, the screening at the IR region is not sufficient to render the diverging integral finite (cf. item G2) in § 4).

Hot QCD \((m_{\text{quark}} \ll T)\) has only two parameters \(g\) and \(T\). Then, there arises natural hierarchy of scales: \(T\) (hard), \(gT\) (soft), \(g^2T\) (super soft), ... .

Naive perturbation theory is valid at the hard region. HTL-resummation scheme deals with the soft region. Noting that the quantities classified into a) above receive a little contribution from the IR region, one can say that such quantities detect the “physics” in the hard region. The quantities classified into b) detect the “physics” at the soft as well as hard regions. The quantities belonging to c1) detect the super-soft, soft and hard regions. The scheme that deals with super-soft region in a consistent manner is not settled yet. In view of the fact that \((G - H)/H = O(g)\) (cf. item 6) at the end of § 4), settlement of this scheme is an urgent issue but there is still a long way to go toward the solution.
On the other hand, mass-singularity issue seems to be relatively easy to resolve. Let me mention, in turn, the damping rate of a moving particle and the soft-photon production rate (cf. c1) and c2) above).

**Damping rate:** Within the HTL-resummation scheme, the rate diverges at the IR end. It is expected that, at the next-to-leading order, the self-energy part acquires screening mass of $O(g^2T)$ or “something” which screens the IR singularity. If this is the case, the diverging factor $\ln(gT/0^+)$ turns out to be $\ln(gT/O(g^2T)) \simeq \ln g^{-1}$.

In hot QED, however, it is generally believed that, in any order of perturbation series, no magnetic mass is induced. In the IR region, the Bloch-Nordsieck approximation, $\gamma^\mu \rightarrow u^\mu$ (with $u^\mu$ the four velocity), is known to work. Employing this approximation scheme, it has been shown that a moving hard electron damps according to $\propto e^{-\alpha T \ln(mt)}$ ($t$: time, $m_e = eT/3$: the QED counterpart of Eq. (4.3)).

**Soft-photon production rate:** The dominant contribution to $g^{\mu\nu} \Pi_{\mu\nu}$ ($\Pi_{\mu\nu}$ the photon polarization tensor) comes from an one-loop diagram with soft loop momentum. Since all the relevant momenta are soft, one should use HTL-resummed effective quark propagators and HTL-resummed photon-quark vertices. As seen at the beginning of §6, the HTL photon-quark vertex diverges logarithmically $\sim \ln(gT/0^+)$, because the external photon momentum is on the mass shell.

According to the general argument in §6, substitution of self-energy-part-resummed hard-quark propagators $\S'$s for the bare ones make diverging result finite, $\ln(gT/0^+) \rightarrow \ln(gT/O(g^2T)) \simeq \ln g^{-1}$.

However, the above substitution violates the Ward-Takahashi relation, which indicates that there must be important vertex corrections. Lebedev and Smilga have shown that the corresponding diagrams are the (resummation of) ladder diagrams. This yields the additional contribution to the soft-photon production rate, which coincides with the above contribution to leading order at logarithmic accuracy.

Whether or not this analysis can be generalized to a generic reaction rate or thermal amplitude that belongs to the category c2) above is an open question.
8 Beyond the hard-thermal-loop resummation scheme

As mentioned in the last section (cf. also the item 6) at the end of § 4), toward establishing a next-to-leading-order resummation scheme is still a long way. Here I simply enumerate, without comment, some of the work made toward this end.

1) Next-to-leading order computation of (chromoelectric) Debye mass. [32]
2) Next-to-leading order computation of plasmon frequency. [33]
3) Next-to-leading order computation of the gluon vacuum polarization tensor $\Pi_{\mu\nu}(P)$ with soft $P$.
4) Next-to-leading order correction to the dispersion laws (cf. items G3) and Q3) in §4). [35]
5) Self-consistent determination of chromomagnetic mass. [36]
6) Improved effective action. [37]

Although not directly related to the subject of this review, I enumerate the following important achievements in the field of thermal field theory.

1) Hot QED and hot scalar QED. [27, 38]
2) Calculational scheme of grand partition function or pressure. [39]

Finally I mention the extensions to the nonequilibrium thermal field theory. For dealing with systems that are quasiuniform near equilibrium or quasistationary, traditional approach uses [3, 10] the Keldish variant of real-time formalism. As far as the computation of reaction rates are concerned, almost all the machineries of equilibrium thermal field theory hold as they are. An important one that does not hold is Eq. (2.6), which causes the appearance of pinch singularity in self-energy-part-inserted propagators. [11] Two approaches are devoted to this issue. 1) It has been shown [12] that such singular contributions can be resummed (see also Ref. 5)). Application of this result to the hard-photon production rate is made. [13] 2) A renormalization theory constructed through renormalizing number densities is proposed. [14] This theory is same in structure as the equilibrium thermal field theory, so that no pinch singularity appears.

Thermo field dynamics as mentioned in §2 (cf. item 2) after Eq. (2.3)) is generalized to the nonequilibrium case. [8] Recall that the Bogoliubov matrix $\hat{M}_\pm$ in Eq.
(2.7) defines the quasiparticle fields \( \varphi \)'s. Now \( \hat{M}_\pm \) and then the quasiparticle picture depend on space-time coordinates. From Eq. (2.7), we see that the time derivative of \( \varphi \)'s receives two contributions, one coming from the time derivative of the original fields \( \phi \)'s is governed by the Hamiltonian and the other comes from the time derivative of \( \hat{M}_\pm \), through which the thermal energy is introduced. Then, through renormalization procedure of propagators, time dependence of \( \hat{M}_\pm \) or the number density is determined. The determining equation turns out to be a generalized Boltzmann equation.

9 Conclusion

The structure of perturbative hot QCD is far more complicated than the perturbative vacuum QCD. Naive perturbation scheme, which is formulated using in-field basis (in vacuum theory) in a Fock space, is valid only at the low level of the “QCD mountain” (the short wave-length or the hard region \( \lambda \sim 1/T \)). At the high level of the mountain (the long wave-length or the soft region \( \lambda \sim 1/gT \)), naive perturbation scheme breaks down, which means that the in-fields are not the good basis at this level. The perturbation scheme that works here is the HTL-resummation scheme. Again this scheme does not apply at yet higher level of the mountain (the longer wave-length region). Continuous efforts aiming at establishing the new resummation scheme that works at this level are making. The goal is, however, still far a way.

There are “ravines” along the light cone (mass singularities) here and there in the QCD mountain. Techniques of pass over these ravines are not completely settled yet.

Comprehensive analysis of rates of various reaction taking place in nonequilibrium system, as well as the development of the theoretical framework of nonequilibrium quantum-field theory per se, have begun.

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