On scalar radiation

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We discuss radiation in theories with scalar fields. Even in flat spacetime, the radiative fields depend qualitatively on the coupling of the scalar field to the Ricci scalar: for non-minimally coupled scalars, the radiative energy density is not positive definite, and the radiated power is not Lorentz invariant. We explore implications of this observation for radiation in conformal field theories. We find evidence that for a probe coupled to $\mathcal{N} = 4$ super Yang-Mills, and following an arbitrary trajectory, the angular distribution of radiated power is independent of the Yang-Mills coupling.

I. INTRODUCTION

The study of the creation and propagation of field disturbances by sources is one of the basic questions in any field theory. In classical electrodynamics, emission of electromagnetic waves by charged particles is of paramount importance, both at the conceptual and practical level.\textsuperscript{1} Similarly, the recent detection of gravitational waves\textsuperscript{2} provides a striking confirmation of General Relativity, and opens a new way to explore the Universe.

Undoubtedly, radiation of massless scalar fields due to accelerated probes coupled to them, has received much less attention.\textsuperscript{3} An exception is the study of radiation in scalar-tensor theories of gravity, since the radiation pattern can differ from General Relativity\textsuperscript{4}.

The comments above refer to classical field theories. Recent formal developments, like holography and supersymmetric localization, have allowed to explore radiation in the strong coupling regime of conformal field theories (CFTs), which if they admit a Lagrangian formulation, very often include scalar fields.

In field theory, radiation is determined from the one point function of the energy-momentum tensor of the field theory in the presence of an accelerated probe, which is described by a Wilson line $W$. In any CFT, a special conformal transformation maps a worldline with constant proper acceleration to a static one, for which $\langle T_{\mu\nu} \rangle$ is fixed up to a coefficient\textsuperscript{5}

$$\frac{\langle WT^{00} \rangle}{\langle W \rangle} = \frac{h}{r^4}$$  \hspace{1cm} (1)

so $h$ should capture the radiated power, at least for a probe with constant proper acceleration.\textsuperscript{6} Much of the recent literature has assumed that for generic CFTs, the radiated power satisfies a Larmor type formula

$$\mathcal{P} = -2\pi B a^3 a_\lambda$$  \hspace{1cm} (2)

where $B$ is called the Bremsstrahlung function.\textsuperscript{7} For Lagrangian CFTs with $\mathcal{N} = 2$ supersymmetry this function can be computed using supersymmetric localization\textsuperscript{8-9}. For $\mathcal{N} = 2$ SCFTs it was argued\textsuperscript{10} and then proved\textsuperscript{11} that $B = 3h$. This relation is not satisfied in Maxwell’s theory\textsuperscript{10}, proving that no universal relation between $B$ and $h$ exists that is valid for all CFTs.

In holography, radiation by accelerated charges in a CFT is studied by first introducing a holographic probe, a string or a D-brane. Computations can be done at the world-sheet/world-volume level, or taking into account the linear response of the gravity solution due to the presence of the holographic probe. Intriguingly, these two methods do not fully agree. At the holographic probe level, the computation of\textsuperscript{12} indicated that for a 1/2 BPS probe coupled to $\mathcal{N} = 4$ super Yang-Mills, in the large $N$, large $\lambda$ limit, the total radiated power is indeed of the form given by (2). The beautiful work\textsuperscript{15,16} dealt with the backreacted holographic computations, see also\textsuperscript{17-19}. The work\textsuperscript{15} considered only a probe in circular motion, and found agreement with (2). However, the work\textsuperscript{16} dealt with arbitrary trajectories, and found

$$\mathcal{P} = -2\pi B \left( a^3 a_\lambda + \frac{1}{9} \frac{a^0}{\gamma} \right)$$  \hspace{1cm} (3)

The additional term in (3) would imply that the radiated power in $\mathcal{N} = 4$ SYM is not Lorentz invariant. The work\textsuperscript{15} was restricted to circular motion in a particular frame where $a^0 = 0$, so by construction, it was not sensitive to the presence of the additional term in (3).

The angular distribution of radiated power is a more refined quantity than the total radiated power. At strong coupling it has been studied in\textsuperscript{15,16}, where the angular distribution of radiation emitted by a 1/2 BPS probe coupled to $\mathcal{N} = 4$ super Yang-Mills was determined holographically. Some of the features of the angular distribution of radiation found in\textsuperscript{15,16} were unexpected, like regions with negative energy density, or its dependence on the derivative of the acceleration, eq. (3). This prompted\textsuperscript{16} to consider them artifacts of the supergravity approximation.

In this work we revisit the issue of radiation in scalar field theory, bringing new insights to many of the issues reviewed above. Our key observation is rather elementary: scalar fields couple to the scalar curvature of spacetime via the term\textsuperscript{20} $\xi R a^2$ so, even in flat spacetime, the energy-momentum tensor\textsuperscript{21} and therefore the pattern of radiation, depend on $\xi$. In particular, radiation in conformal field theories requires considering conformally coupled scalars ($\xi = 1/6$) instead of minimally coupled ones, $\xi = 0$.

In section II, we revisit by probes coupled to free field theories. We show that for non-minimally coupled scalars, the radiative energy density is not positive definite, which is just a manifestation of the more
general fact that non-minimally coupled scalars can violate energy conditions even classically\textsuperscript{22}. Furthermore, for non-minimally coupled scalars, the radiated power $\mathcal{P}$ is not Lorentz invariant, so equation (2) can’t be true for all CFTs, without qualifications. Our results explain the additional term in (3) as coming from the improvement term of the energy-momentum tensor of the conformal scalars. The new term that we find in the rate of 4-momentum loss is formally similar to the Schott term that appears in the Lorentz-Dirac equation in electrodynamics\textsuperscript{1}. We will argue however that in theories with non-minimally coupled scalars its origin and meaning are different than the Schott term in classical electrodynamics.

In section III we discuss radiation by 1/2 BPS probes coupled to $\mathcal{N} = 2$ SCFTs. Quite remarkably, for a 1/2 BPS probe coupled to $\mathcal{N} = 4$ super Yang Mills following an arbitrary trajectory, the classical computation with conformally coupled scalars matches exactly the angular distribution found holographically\textsuperscript{15,16}. We also discuss how to reinterpret the $B = 3h$ relation in the light of our findings, and present a variant of this relation that holds for all the probes considered in this work.

In section IV we mention some open questions. Our conventions are as follows: we work with a mostly minus metric, so the 4-velocity $u$ and the 4-acceleration $a$ satisfy $u^2 = 1$, $a^2 < 0$. Dots have different meaning for vectors and 4-vectors: $\dot{a} = da/d\tau$, but $\ddot{a} = d\dot{a}/dt$. Our overall normalization of the energy-momentum tensor for scalars is not the usual one; it has been chosen for convenience when we add scalar and vector contributions in supersymmetric theories.

II. RADIATION IN FREE FIELD THEORIES

Consider a probe coupled to a field theory, following an arbitrary, prescribed, timelike trajectory $z^\mu(\tau)$. One first solves the equations of motion for the field theory, in the presence of this source, choosing the retarded solution. Let $x^\mu$ be the point where the field is being measured; define $\tau_{ret}$ by the intersection of the past light-cone of $x^\mu$ and the world-line of the probe, and the null vector $\ell = x - z(\tau_{ret})$.

One then evaluates the energy-momentum tensor with the retarded solution. Usually one defines the radiative part of the energy-momentum tensor $T^{\mu\nu}_{r}$ as the piece that decays as $1/r^2$ so it yields a nonzero flux arbitrarily far away from the source. A more restrictive definition of $T^{\mu\nu}_{r}$ was introduced in\textsuperscript{23,24}, who required that
\begin{itemize}
  \item $\partial_\mu T^{\mu\nu}_{r} = 0$ away from the source.
  \item $\ell_\mu T^{\mu\nu}_{r} = 0$ so flux through the light-cone emanating from the source is zero.
  \item $T^{\mu\nu}_{r} = \frac{A}{(\ell \cdot u)^2} \ell^\mu \ell^\nu$ with $A$ a Lorentz scalar.
  \item $A \geq 0$ so the radiative energy density is nonnegative.
\end{itemize}
In this work we will consider theories that don’t satisfy the weak energy condition classically; for these theories, the requirement that the radiative energy density is nonnegative is less well motivated. In this work we use the first definition of $T^{\mu\nu}_{r}$, but we will discuss the implications of considering the second one. From $T^{\mu\nu}_{r}$ we define
\begin{equation}
\frac{dP^\mu}{dt} = r^2 T^{\mu\nu}_{r} u_\nu
\end{equation}
and integrating over the solid angle we obtain $dP^\mu/d\tau$. It is a 4-vector that gives the rate of energy and momentum emitted by the probe. From it one can define two quantities. The first one is the radiated power $\mathcal{P}$,
\begin{equation}
\mathcal{P} = \frac{dP^0}{dt}
\end{equation}
which is not manifestly Lorentz invariant. Following Rohrlich\textsuperscript{1}, we define a second quantity, the invariant radiation rate $\mathcal{R}$ as
\begin{equation}
\mathcal{R} = u_\mu \frac{dP^\mu}{d\tau}
\end{equation}
which is manifestly Lorentz invariant.

A. Maxwell field

The energy-momentum tensor is
\begin{equation}
T^{\mu\nu} = \frac{1}{4\pi} \left( F^{\mu\lambda} F_{\lambda\nu} + \frac{1}{4} \eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right)
\end{equation}
It is traceless, without using the equations of motion. Consider a probe coupled to the Maxwell field, with charge $q$, following an arbitrary trajectory. The full energy-momentum tensor evaluated on the retarded solution is\textsuperscript{25}
\begin{equation}
T^{\mu\nu} = \frac{q^2}{4\pi} \left( \frac{\ell^\mu u^\nu + \ell^\nu u^\mu}{(\ell \cdot u)^2} (1 - \ell \cdot a) + \frac{\ell^\mu a^\nu + \ell^\nu a^\mu}{(\ell \cdot u)^4} \right) - \frac{a^2}{(\ell \cdot u)^4} \eta^{\mu\nu} \ell^\mu \ell^\nu - \frac{(1 - \ell \cdot a)^2}{(\ell \cdot u)^8} \ell^\mu \ell^\nu - \frac{1}{2} \frac{\eta^{\mu\nu}}{(\ell \cdot u)^4}
\end{equation}
The part of (8) decaying as $1/r^2$ is
\begin{equation}
T^{\mu\nu}_{r} = -\frac{q^2}{4\pi} \left( \frac{a^2}{(\ell \cdot u)^4} + \frac{(\ell \cdot u)^2}{(\ell \cdot u)^8} \right) \ell^\mu \ell^\nu
\end{equation}
It satisfies all criteria of \(^{23,24}\), so it is the radiative part according to both definitions. Integration over angular variables yields

\[
\frac{dP^\mu}{d\tau} = -\frac{2}{3}q^2 a^\lambda a_\lambda u^\mu
\]  

(11)

It is a future-oriented timelike 4-vector, guaranteeing that all inertial observers agree that the particle is radiating away energy. The relativistic Larmor’s formula follows

\[
P = R = -\frac{2}{3}q^2 a^\lambda a_\lambda
\]  

(12)

recall that \(a^2 < 0\) in our conventions. From (12) we derive the Bremsstrahlung coefficient for Maxwell’s theory,

\[
B = \frac{q^2}{3\pi}
\]  

(13)

It follows from (9) and (13) that

\[
B = \frac{8}{3} \hbar.
\]  

(14)

### B. Scalar fields

Consider a free massless scalar field, with arbitrary coupling \(\xi\) to the Ricci scalar. The energy-momentum tensor is\(^{21}\)

\[
4\pi T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - \frac{1}{2} \eta^{\mu\nu} \partial_\alpha \phi \partial^\alpha \phi - \xi (\partial^\mu \partial^\nu - \eta^{\mu\nu} \Box) \phi^2
\]  

(15)

In general, the trace of (15) does not vanish, even when applying the equations of motion. For the conformal value \(\xi = \frac{1}{6}\) it vanishes away from the sources, if we apply the equations of motion. For \(\xi \neq 0\), this energy-momentum tensor can violate the weak energy condition at the classical level\(^{22}\), even in Minkowski space.

Now consider a probe coupled to the scalar field, following an arbitrary trajectory. The energy-momentum tensor (15) evaluated on the retarded solution of the equation of motion is evaluated at retarded time. It depends on \(\dot{a} = da/d\tau\), because the improved energy-momentum tensor (15) involves second derivatives of the field, and the solution depends on the velocity of the probe.

In the conformal case \(\xi = 1/6\) the terms independent or linear in the acceleration are the same as in (8), up to an overall factor. The reason is that they are fixed by conformal invariance. The full energy-momentum tensor of a CFT in the presence of a static probe is fixed by conformal invariance\(^{5}\), up to an overall coefficient. By applying a boost, it is then also fixed for a probe with constant velocity. This determines all the acceleration independent terms.

Furthermore, by applying a special conformal transformation to a static world-line, one obtains a world-line with constant proper acceleration. Therefore, for any CFT, the full energy-momentum tensor for this trajectory is completely determined up to an overall constant. Since a world-line with constant proper acceleration satisfies \(\dot{a} = -a^2 u\), terms that are not universal in \(T^{\mu\nu}\) and change from one CFT to another, must be such that they collapse to the same universal expression when \(\dot{a} = -a^2 u\), as one can check for (8) and (16). But terms linear in \(a\) don’t depend on \(\dot{a}\) or \(a^2\), so they must be universal for all CFTs.

Evaluating (16) on a static probe for the conformal value \(\xi = 1/6\), we derive

\[
T^{00}_{\mid v=\delta} = \frac{1}{8\pi} \frac{1 - \frac{1}{4}\frac{a^2}{r^2}}{q^2} \Rightarrow h = \frac{1}{24\pi} q^2
\]  

(17)

The part of (16) decaying as \(1/r^2\) is

\[
T^\mu_{\nu} = \frac{q^2}{4\pi} \left(1 - 8\xi\right) \left(\frac{\ell \cdot a}{\ell \cdot u}\right)^2 + 2\xi \left(\frac{\ell \cdot a}{\ell \cdot u}\right) \ell^\mu \ell^\nu
\]  

(18)

It satisfies the first three criteria of\(^{24}\) to be the radiative part. It also satisfies \(|T^{00}| = |T_{0i}|\). As a check, for \(\xi = 0\), it reduces to the energy density found in\(^{15}\), which is manifestly positive definite. However, for \(\xi \neq 0\), \(T^{00}\) is not guaranteed to be positive. After integration over the angular variables, we find

\[
\frac{dP^\mu}{d\tau} = -\frac{1}{3} q^2 a^\lambda a_\lambda u^\mu + \frac{2\xi}{3} q^2 \dot{a}^\mu
\]  

(19)

The improvement term in the energy-momentum tensor of the scalar field (15) induces a qualitatively new term in \(dP^\mu/d\tau\), compared with the electrodynamics case. The additional term in (19) is a total derivative, and it is formally identical to the Schott term in classical
electrodynamics\textsuperscript{1}. However, the origin is different. In classical electrodynamics, the Schott term appears in the Lorentz-Dirac equation of motion of the probe, and it can be deduced from the fields created by the probe, in the zone near its worldline. It does not appear from evaluating the radiative part of the energy-momentum tensor (10). On the other hand, in (19) the new term appears directly from evaluating the energy-momentum tensor of the fields that decay like $1/r^2$, away from the probe.

This additional term that we have encountered in (19) in a free theory computation is precisely the additional term found holographically by\textsuperscript{16}. In that context, the works\textsuperscript{18,19} have advocated using the more restrictive definition of $T^{\mu\nu}$, thus setting $\xi = 0$ in (18, 19). An argument in favor of doing so is that the new term in (19) is a total derivative so, for instance, its contribution vanishes for any periodic motion when integrated over a full period. This clashes with the intuition of radiated energy as something irretrievably lost by the particle. However, we think this intuition is built on the idea that the energy density is positive definite, which is not the case for non-minimally coupled fields.

For a minimally coupled scalar field, $\xi = 0$, $dP^\mu / d\tau$ is again a future-oriented, timelike 4-vector, and $\mathcal{P} = \mathcal{R}$, as in Maxwell's theory\textsuperscript{3,15}. On the other hand, for $\xi \neq 0$, this 4-vector is no longer guaranteed to be time-like. This is related with $T^{00}$ no longer being positive definite. In the instantaneous rest frame,

$$
d P^\mu \bigg|_{\mathcal{E} = 0} = \left(1 - \frac{2\xi}{3} q^2 \bar{a}^2, \frac{2\xi}{3} q^2 \bar{a}\right) \quad (20)
$$

So for $\xi < 1/2$, in the instantaneous rest frame, there is energy loss. However, if $dP^\mu / d\tau$ is space-like, the sign of its zeroth component is no longer the same in all inertial frames.

For a non-minimally coupled scalar, $\mathcal{P}$ and $\mathcal{R}$ no longer coincide, and $\mathcal{P}$ is not Lorentz invariant. Indeed,

$$
\mathcal{P} = \frac{1}{3} q^2 a^\lambda a_\lambda - \frac{2\xi}{3} q^2 \bar{a}^0 / \gamma 
$$

and

$$
\mathcal{R} = -\frac{1}{3} q^2 a^\lambda a_\lambda 
$$

For non-minimally coupled scalars, we will still define $2\pi B$ as the coefficient in front of the $-a^\lambda a_\lambda$ term in (21). We furthermore introduce a new coefficient $B_\xi$, as the coefficient in $\mathcal{R} = -2\pi B_\xi a^\lambda a_\lambda$. We obtain

$$
B_\xi = \frac{1 - 2\xi}{6\pi} q^2 
$$

Notice that $B_{\xi=0} = B$; we also define $B_{\xi=1/6}$. In particular, for the conformally coupled scalar it follows that $B_{cc} = \frac{2}{3} \hbar$. This ratio is the same as in Maxwell's theory, eq. (14). The work\textsuperscript{10} failed to reproduce this ratio for scalars, because it compared $B_{\xi=0}$ to $\hbar_{\xi=1/6}$.

III. RADIATION IN $\mathcal{N} = 2$ SUPERCONFORMAL THEORIES

The discussion in the previous section was completely classical. In this section we consider $\mathcal{N} = 2$ SCFTs, for which powerful techniques to study the strong coupling regime are available.

Consider the radiative energy-momentum tensor created by a 1/2 BPS probe coupled to a Lagrangian $\mathcal{N} = 2$ SCFT in the classical limit. The probe is coupled to a vector and a scalar in the adjoint representation of the gauge group. As argued in\textsuperscript{15,16}, at very weak coupling this amounts to adding the contribution of the Maxwell (10) and free scalar (18) terms, with an effective charge. However\textsuperscript{15,16} considered a free minimally coupled scalar. In CFTs, the correct computation amounts to adding (10) and (18) with the conformal value, $\xi = 1/6$. We obtain

$$
T^{\mu\nu}_{\mathcal{N}=2} = \frac{B}{6} \left( -\frac{3a^2}{(\ell \cdot u)^4} + \frac{\ell \cdot \bar{a}}{(\ell \cdot u)^5} - 4(\ell \cdot a)^2 \right) \ell^\mu \ell^{\nu} 
$$

In three-dimensional language, with $\bar{n} = \frac{\bar{r} - \bar{z}}{\bar{r} \cdot \bar{z}}$, the radiative energy density is

$$
T^{00}_{\mathcal{N}=2} = \frac{B}{6r^2} \left( \frac{4|\bar{a}|^2 + 3\gamma^2 (\beta \cdot \bar{a})^2 + \beta \cdot \bar{a}}{1 - \beta \cdot \bar{n} \gamma^2 (\beta \cdot \bar{a})^2 + \beta \cdot \bar{a} \gamma^2 (\beta \cdot \bar{n})^2} \right) + \frac{5(\beta \cdot \bar{a})(\bar{n} \cdot \bar{a}) - \gamma^2 \bar{n} \cdot \bar{a}}{(1 - \beta \cdot \bar{n})^5} - 4 \frac{\gamma^2 (\bar{n} \cdot \bar{a})^2}{(1 - \beta \cdot \bar{n})^6} 
$$

where we have written our result in terms of the Bremstrahlung function (recall that $B = B_{\xi=0}$, the coefficient evaluated for minimally coupled scalars). Our free classical computation only guarantees (24, 25) at leading order in $\lambda$, for small $\lambda$. Strikingly, (25) is exactly the same result found by a rather elaborate holographic computation for a 1/2 BPS probe in the fundamental represenation of $\mathcal{N} = 4$ SU($N$) super Yang-Mills in\textsuperscript{15,16}, in the planar limit and at strong 't Hooft coupling where\textsuperscript{12} $B = \sqrt{8}/4\pi^2$ ! We stress that this exact agreement is at the level of radiative energy density, before performing any time average. This agreement prompts us to conjecture that (24) is true for all values of $\lambda$, in the planar limit. It is tempting to conjecture that (24) is true even
at finite $N$ and finite $\lambda$, but we currently don’t have evidence for this stronger claim. Conformal symmetry alone is not enough to explain this agreement: comparing (10), (18) and (24) it is clear that the radiative energy density of a probe in arbitrary motion is not the same for different conformal field theories. Notice also that while the probe is 1/2 BPS, it is following an arbitrary trajectory, so the Wilson line does not preserve any supersymmetry globally.

Many of the unexpected features of (25) have simple classical explanations that arise from properties of conformally coupled scalars: the fact that (25) is not positive definite everywhere, was interpreted in $15$ as an inherently quantum effect. In fact, it’s a feature already present at the classical level, reflecting that conformally coupled scalar fields can violate energy conditions even classically. As first noticed in $16$, (25) depends on the derivative of the acceleration; now we understand that this follows from the fact that the improved tensor (15) involves second derivatives of the field. Another puzzle raised in $16$ is that in $\mathcal{N} = 4$ SYM, radiation was isotropic at weak coupling; as our classical derivation of (25) shows, this isotropy is just an artifact of considering minimally coupled scalars, instead of conformally coupled ones.

In $15$ it was noticed that for circular motion, while the angular distribution of radiated power computed holographically did not match the classical computation of Maxwell plus minimally coupled scalar, the respective time averages over a period did match. The reason is now easy to understand: the details of the angular distribution depend on $\xi$, but after averaging over a period, the averaged angular distribution is independent of $\xi$.

Let’s discuss now the total radiated power in $\mathcal{N} = 2$ SCFTs. Integration of (25) over angular variables yields

$$\frac{dP^\mu}{d\tau} = -2\pi B\left(a^\lambda a_\lambda u^\mu + \frac{1}{9} i^{\mu}\right)$$

(26)

Our computation ensures that this formula is valid at the classical level. At strong coupling, the only evidence is the $\mathcal{N} = 4$ SYM holographic computation of $16$.

To conclude, let’s comment on the relation $B = 3h$ conjectured in $8, 10$ and proved in $11$. It should be clear that the $B$ that appears in this relation is $B_{\xi=0}$. For starters, the prescription to compute $B$ in $10$ involves adding an extra term to a traceless energy-momentum tensor (thus ‘deteriorating’ it, in the terminology of $26$). We can illustrate this point rather explicitly for a free $U(1) \mathcal{N} = 2$ SCFT, using the values computed in section II,

$$B^{N=2}_{cc} = B_{EM} + B_{\xi=1/6} = \frac{8}{3}(h_{EM} + h_{\xi=1/6}) = \frac{8}{3} h_{N=2}$$

(27)

On the other hand,

$$B^{N=2} = B_{EM} + B_{\xi=0} = 3(h_{EM} + h_{\xi=1/6}) = 3h_{N=2}$$

(28)

The $B = 3h$ relation has been proved in $11$ for generic $\mathcal{N} = 2$ SCFTs, not necessarily Lagrangian. $B$ and $h$ are defined in terms of different energy-momentum tensors (not traceless for $B$, traceless for $h$). It must be the case that the energy-momentum tensor obtained by acting with the supersymmetry algebra on the lowest weight operator of its multiplet does not come out to be automatically traceless.

Finally, if we contract (26) with $u_\mu$ and use $B = 3h$, we obtain the value of $B_{cc}$ for $\mathcal{N} = 2$ SCFTs

$$B_{cc} = \frac{8}{9} B = \frac{8}{3} h$$

(29)

which is again the relation found for Maxwell’s theory and for a free conformal scalar. So if (26) holds, (29) would be true for all the probes coupled to CFTs considered in this paper. Furthermore, this relation involves quantities evaluated with the same energy-momentum tensor, the traceless one.

**IV. DISCUSSION AND OUTLOOK**

In this work we have discussed radiation for theories with scalar fields. We have found that for non-minimally coupled scalars, the energy density is no longer positive definite, and the radiated power is not Lorentz invariant.

In the introduction we mentioned that holographic computations of radiation do not completely agree with each other. Our work suggests the following scenario: probe string/brane computations$12$-$14$ reproduce the results of a ‘minimally coupled’ version of the dual CFT, where the improvement term in the energy-momentum tensor is ‘turned-off’, while backreacted supergravity computations$15, 16$ yield results that fully agree with the field theory ones when the scalars are conformally coupled.

In this work we have not discussed radiation reaction on the probe coupled to the scalar field. It would be interesting to discuss it in the non-minimal case.

We have presented evidence that the relation (29) holds for 1/2 BPS probes in $\mathcal{N} = 2$ SCFTs, and it will be interesting to study if it holds for less supersymmetric probes.

The fact that (25) holds both at weak and strong $\lambda$ in the planar limit of $\mathcal{N} = 4$ super Yang-Mills is rather mysterious, as it is not a BPS quantity. It will be important to prove if (25) holds for any $\lambda$, in the planar limit, or event at finite $N$. An even stronger conjecture is that it holds for generic $\mathcal{N} = 2$ superconformal theories.

Finally, this note has only considered radiation of scalar fields in Minkowski spacetime. It will be interesting to revisit the issue in generic spacetimes.

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