Adaptive Path Interpolation for Sparse Systems: Application to a Simple Censored Block Model

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Abstract—A new adaptive path interpolation method has been recently developed as a simple and versatile scheme to calculate exactly the asymptotic mutual information of Bayesian inference problems defined on dense factor graphs. These include random linear and generalized estimation, superposition codes, or low rank matrix and tensor estimation. For all these systems the method directly proves in a unified manner that the replica symmetric prediction is exact. When the underlying factor graph of the inference problem is sparse the replica prediction is considerably more complicated and rigorous results are often lacking or obtained by rather complicated methods. In this contribution we extend the adaptive path interpolation method to sparse systems. We concentrate on a Censored Block Model, where hidden variables are measured through a binary erasure channel, for which we fully prove the replica prediction.

I. INTRODUCTION

Much progress has been achieved recently in Bayesian inference of high dimensional problems. It has been possible to develop rigorous methods in order to derive exact “single letter” variational formulas for the mutual information (or free energy) in the asymptotic limit of the number of variables tending to infinity, when the prior and all hyperparameters of the problem are assumed to be known. Such formulas have often been first conjectured on the basis of the replica and cavity methods of statistical mechanics and are also known as “replica symmetric” formulas [1]. Examples where full proofs have been achieved are random linear estimation and compressed sensing [2–4], generalized estimation and learning (for single layer networks) [5], low-rank matrix and tensor estimation [6–10]. Invariably, the Guerra-Toninelli interpolation method [11] has been used to derive one-sided bounds. For the converse bounds typically other ideas have usually been necessary, such as spatial coupling [2, 3, 6] or the Aizenman-Sims-Starr principle [7–9]. Recently two of us introduced a new interpolation scheme, called adaptive path interpolation method, that allows to derive the replica symmetric formulas in a more straightforward and unified manner [12]. The new method is quite generic once the mean field solution has been identified and is directly applicable when the concentration of the “overlap” can be proved. Overlap concentration itself follows from variants of Ghirlanda-Guerra identities [13] adapted to Bayesian inference combined with the so-called Nishimori identities (see [12]). The successes of the adaptive interpolation method have so far been limited to inference models with a dense underlying factor graph. It is therefore desirable to see to what extent the method can be developed when the factor graph is instead sparse. Typical examples of such systems are Low-Density Parity-Check codes, Low-Density Generator-Matrix (LDGM) codes, or the Stochastic and Censored Block Models [14] (the latter can be viewed as a particular LDGM code). It is long to say that the replica symmetric formulas for the mutual information is much more complicated in such models. On one hand, besides the measurements (or channel outputs), the graph is also random, and on the other hand the single letter variational problem involves a functional over a set of measures (instead of scalars). Existing rigorous derivations of the replica formulas have so far been achieved using a combination of the interpolation method (first developed by [15] for sparse models) and spatial coupling [16] or the Aizenman-Sims-Starr principle [17, 18]. In this work we consider a simple version of the Censored Block Model for which we fully develop the adaptive interpolation method. We believe that this constitutes a first step towards an analysis of more complicated models via this relatively simple method.

Our aim here is to present the general structure of the adaptive interpolation method. Some of the technical lemmas (mainly the overlap concentrations) are only stated and their proofs can be found in the long version of this article [19].

II. SETTING AND MAIN RESULT

A. The Censored Block Model: Setting and properties

We denote binary variables by $\sigma_i \in \{-1, +1\}, i = 1, \ldots, n$ and vectors $\sigma = (\sigma_1, \ldots, \sigma_n) \in \{-1, +1\}^n$. Subsets $S \subset \{1, \ldots, n\}$ with at least two elements are denoted by capital letters. For the product of binary variables in a subset $S$ we use $\sigma_S \equiv \prod_{i \in S} \sigma_i$. Below, the integer $K \geq 2$ and the rate $R \in \mathbb{R}_+$ are fixed numbers independent of $n$.

In the (symmetric) Censored Block Model $n$ hidden binary variables $\sigma_0^0, \ldots, \sigma_0^K$ are i.i.d. uniform. A noiseless measurement is a product $\sigma_0^0, \sigma_1^0, \ldots, \sigma_K^0$ of a $K$-tuple of variables drawn uniformly at random. The $K$-tuple is identified with a subset $J_K \equiv \{i_1, \ldots, i_K\} \subset \{1, \ldots, n\}$ and we set $\sigma_A^0 \equiv \sigma_0^{i_1} \sigma_0^{i_2} \ldots \sigma_0^{i_K} (\sigma_0^0 = \pm 1)$. The true observations $J_A \in \mathbb{R}$ are noisy versions of these products obtained through a binary input memoryless channel described by some transition probability $Q(J_A | \sigma_A^0)$. In this work we consider the binary erasure channel (BEC) $Q(J_A | \sigma_A^0) = (1 - q)\delta_{J_A, \sigma_A^0} + q \delta_{J_A, 0}$. 


0 \leq q \leq 1. For large \( n \) the total number of observations \( m \) asymptotically follows a Poisson distribution with mean \( n/R \), i.e., \( m \sim \text{Poi}(n/R) \). We shall also index the observations as \( A = 1, \ldots, m \).

Let us now describe the Bayesian setting used here to determine the information theoretic limits for reconstructing the hidden variables. From the Bayes rule we have that the posterior can be rewritten as

\[
P(\sigma | \tilde{J}) = \frac{\exp \sum_{A=1}^{m} \tilde{J}_A(\sigma_A - 1)}{\sum_{\sigma' \in \{-1,+1\}^n} \exp \sum_{A=1}^{m} \tilde{J}_A(\sigma'_A - 1)},
\]

which is equivalently expressed in terms of the log-likelihood ratios \( \tilde{J}_A \equiv \frac{1}{2} \ln \frac{Q(\sigma_A | \tilde{J}_A)}{Q(-\sigma_A | \tilde{J}_A)} \) (that are more convenient to work with instead of the noisy observations \( \tilde{J} \)). It is easy to show from the distribution of the observations that the log-likelihood ratios are distributed according to \( c(\tilde{J}_A | \sigma_A^0) = (1-q)\delta_{\sigma_A} \tilde{J}_A,-\infty + q\delta_{\tilde{J}_A,0} \). The denominator in expression (1) which serves as normalization is denoted \( Z \) and is also called the partition function.

The bipartite factor graph \( G \) underlying (1) contains variable nodes \( i = 1, \ldots, n \), which carry the variables \( \sigma_i \), and constraint nodes \( A = 1, \ldots, m \) carrying the log-likelihoods \( \tilde{J}_A \). Each constraint node uniformly connects to \( K \) variable nodes \( i_1, \ldots, i_K \). Distribution (1) can be interpreted as the Gibbs distribution of a random spin system (or spin glass). The expectation with respect to (1) will be denoted by a random average form (2).

\[
\mathbb{E}_{\sigma | \tilde{J}} f(\sigma) = \frac{1}{n} \mathbb{E}_{\tilde{J}} \mathbb{E}_{\sigma | \tilde{J}} f(\sigma),
\]

where \( f = \prod_{x \in B} f(x) \). We assume \( \sigma_i^0 = 1, i = 1, \ldots, n \) and that \( \tilde{J} \) have distribution \( c(\tilde{J}_A | 1) = 1, \ldots, m \). Since \( c(\tilde{J}_A | 1) = (1-q)\Delta_\infty + q\Delta_0 \) (we adopt the coding theory notation where \( \Delta_0, \Delta_\infty \) denote point masses at 0 and \( +\infty \)) the Gibbs distribution (1) has non-negative coupling constants \( \tilde{J}_A, A = 1, \ldots, m \). Therefore the Gibbs distribution satisfies the Griffiths-Kelly-Sherman (GKS) inequalities [22,24]: For any subsets \( S,T \subset \{1, \ldots, n\} \) of variable indices we have

\[
\langle \sigma_S \rangle \geq 0, \quad \langle \sigma_S \sigma_T \rangle - \langle \sigma_S \rangle \langle \sigma_T \rangle \geq 0.
\]

- (Nishimori identity) Let \( C \) be any collection of subsets of \( \{1, \ldots, n\} \). A general identity, from which a number of other ones follow, is

\[
\mathbb{E}_{\tilde{J}} \prod_{S \in C} \langle \sigma_S \rangle = \mathbb{E}_{\tilde{J}} \left[ \prod_{S \in C} \langle \sigma_S \rangle \right] \prod_{S \in C} \langle \sigma_S \rangle.
\]

This identity follows from the knowledge of all parameters in the Bayesian inference setting. This is key in proving the overlap concentration Lemmas III.3 and III.6 (see [19]).

B. The replica symmetric formula for the conditional entropy

The cavity method [1] predicts that the asymptotic average conditional entropy density is accessible from the “replica symmetric” (RS) functional, which is an “average form” of the Bethe free entropy (see, e.g., [25, Appendix VII] for details).

The set of distributions with point masses at \( 0, +\infty \) plays a special role and will be called \( B \). Thus any distribution \( x \in B \) is of the form \( x = x\Delta_0 + (1-x)\Delta_\infty, x \in [0,1] \).

**Definition II.1** (RS free functional). Let \( V \sim x \in B \), and \( V_i, i = 1, \ldots, K \) be i.i.d. copies of \( V \). Let

\[
U = \tanh^{-1} \left( \tanh \tilde{J} \sum_{i=1}^{K-1} \tanh V_i \right),
\]

and \( U_B, B = 1, \ldots, l \) be i.i.d. copies of \( U \) where \( l \) is a \( \text{Poi}(K/2) \) random integer. The RS free functional is

\[
h_{RS}(x) = \mathbb{E}_{\tilde{J},\sigma^0,\tilde{J}_0,x|\sigma^0,\tilde{J}_0} \left[ -\frac{1}{R} \ln(1 + \tanh \tilde{J}) + \ln \left( \prod_{B=1}^{l} (1 + \tanh U_B) + \prod_{B=1}^{l} (1 - \tanh U_B) \right) \right] - \frac{K-1}{R} \ln \left( 1 + \tanh \tilde{J} \prod_{i=1}^{K} \tanh V_i \right).
\]

Previous literature considered a Censored Block Model for a uniform Bernoulli prior, a binary symmetric channel, and \( K = 2 \) corresponding to pairwise measurements. In particular the replica formula has been proved in that case by combining the interpolation bound of [15] with a rigorous version of the cavity method for sparse graphs [18].

Our main result is the proof, through the adaptive interpolation method for sparse graphs, of the following theorem:

**Theorem II.2** (The RS formula is exact). For a Censored Block Model with observations obtained through a binary erasure channel as described above we have

\[
\lim_{n \to \infty} \frac{1}{n} \mathbb{E}_{\tilde{J}} \mathbb{E}_{\sigma | \tilde{J}} H(\sigma | \tilde{J}) = \sup_{x \in B} h_{RS}(x).
\]
III. THE ADAPTIVE PATH INTERPOLATION METHOD

For \( t = 1, \ldots, T \) let \( Y_t^{(t)} \) be i.i.d. r.v. distributed according to \( x^{(t)} \in \mathcal{B} \). We set \( x = (x^{(1)}, \ldots, x^{(T)}) \). Consider the r.v.

\[
U^{(t)} = \tanh^{-1} \left( \tan \frac{1}{R} \sum_{i=1}^{K} h_i V_i^{(t)} \right)
\]

and independent copies denoted \( U^{(t)}_B \) where \( B \) is a subgraph which runs over \( l^{(t)} \sim \text{Poi} \left( \frac{K}{R^2} \right) \) of these copies. Later on, we call \( \tilde{x}^{(t)} \) the distribution of \( U^{(t)}_B \) induced by \( x^{(t)} \) and \( c \). Let also \( H \) be a r.v. such that \( H = 0 \) with probability \( 1 - \epsilon \) and \( H = \infty \) with probability \( \epsilon \). We define

\[
\tilde{h}_e(x) \equiv \mathbb{E}\left[ -\frac{1}{R} \ln(1 + \tanh \tilde{J}) + \ln \left( \prod_{t=1}^{T} \prod_{i=1}^{K} (1 + \tanh U^{(t)}_B) \right) + e^{-2H} \prod_{t=1}^{T} \prod_{i=1}^{K} (1 - \tanh U^{(t)}_B) \right] - \frac{K-1}{RT} \sum_{t=1}^{T} \ln \left( 1 + \tanh J \sum_{i=1}^{K} \tanh V_i^{(t)} \right).
\]

One can easily check that if \( x^{(t)} = x \) for all \( t \) then \( \tilde{h}_e = 0 \). More is true as the following lemma shows (see [19]):

**Lemma III.1.** Let \( B^T = B \times B \times \ldots \times B \). For \( \epsilon = 0 \) we have

\[
\sup_{x \in B^T} \tilde{h}_e(x) = \sup_{x \in B} h_{R,S}(x).
\]

A. The \((t, s)\)-interpolating model

Consider the construction of an interpolating factor graph ensemble \( G_{t,s} \) for \( t \in \{1, 2, \ldots, T\} \) and \( s \in [0, 1] \) with Algorithm 1 (the sparse graph counterpart of the interpolation developed for dense graphs in [12]). The distribution of \( \tilde{x}^{(t)} \) is a function of \( x^{(t)} \) and \( c \) according to (5). Therefore \( \tilde{x}^{(t)} \in B \) when \( x^{(t)}, c \in B \). The interpolating graph is designed such that \( G_{t,1} \) is statistically equivalent to \( G_{t+1,0} \); in addition, \( G_{t,s} \) maintains the degree distribution of variable nodes invariant for different \( t \) and \( s \). The Hamiltonian associated with \( G_{t,s} \) is

\[
\mathcal{H}_{t,s}(\mathcal{J}, \tilde{U}, \tilde{m}, \epsilon) = -\sum_{i=1}^{n} \sum_{B=1}^{T} U^{(t)}_{B,i}(\sigma_i - 1) - \sum_{t'=1}^{T-1} \sum_{A=1}^{m^{(t')}} \tilde{J}_{t',A}(\sigma_{t',A} - 1) - \sum_{i=1}^{n} \sum_{B=1}^{T} U^{(t)}_{B,i}(\sigma_i - 1) - \sum_{A=1}^{m} \tilde{J}_{t,A}^{(t)}(\sigma_{t,A} - 1).
\]

We further consider a generalized version of it by adding a perturbation \( H_t \) to each variable node, where \( H_i = 0 \) with probability \( 1 - \epsilon \) and \( H_i = \infty \) with probability \( \epsilon \). Thus our final interpolating Hamiltonian is

\[
\mathcal{H}_{t,s,\epsilon}(\mathcal{J}, \tilde{U}, \tilde{m}, \epsilon; H) \equiv \mathcal{H}_{t,s}(\mathcal{J}, \tilde{U}, \tilde{m}, \epsilon) + H,
\]

**Algorithm 1:** Construction of \( G_{t,s} \)

\[
\begin{align*}
&\text{for } \ell = 1, \ldots, t - 1 \text{ do } \\
&\quad \text{for } i = 1, \ldots, n \text{ do } \\
&\quad \quad \text{draw a random number } e^{(t)}_i \sim \text{Poi} \left( \frac{K}{R^2} \right) \\
&\quad \quad \text{for } B = 1, \ldots, e^{(t)}_i \text{ do } \\
&\quad \quad \quad \text{connect variable node } i \text{ with a half edge with weight } \tilde{J}_{t,B} \sim \tilde{x}^{(t)} \\
&\quad \text{for } t' = t + 1, \ldots, T \text{ do } \\
&\quad \quad \text{draw a random number } m^{(t')} \sim \text{Poi} \left( \frac{m}{RT} \right) \\
&\quad \quad \text{for } A = 1, \ldots, m^{(t')} \text{ do } \\
&\quad \quad \quad \text{associate factor node } (t', A) \text{ with weight } \tilde{J}_{t',A} \sim \tilde{x}^{(t')} \\
&\quad \text{for } i = 1, \ldots, n \text{ do } \\
&\quad \quad \text{draw a random number } e^{(t)}_{i,s} \sim \text{Poi} \left( \frac{Ks}{RT} \right) \\
&\quad \quad \text{for } B = 1, \ldots, e^{(t)}_{i,s} \text{ do } \\
&\quad \quad \quad \text{connect variable node } i \text{ with a half edge with weight } U^{(t)}_{B,i} \sim \tilde{x}^{(t)} \\
&\quad \quad \quad \text{draw a random number } m^{(t)}_{s} \sim \text{Poi} \left( \frac{m(1-s)}{RT} \right) \\
&\quad \quad \text{for } A = 1, \ldots, m^{(t)}_{s} \text{ do } \\
&\quad \quad \quad \text{associate factor node } (t, A) \text{ with weight } \tilde{J}_{t,A} \sim \tilde{x}^{(t)} \text{ uniformly and randomly connect factor node } (t, A) \text{ to } K \text{ variable nodes}
\end{align*}
\]

The unperturbed and perturbed free energies are linked:

**Lemma III.2.** Let \( c \in B \) and \( x \in B^T \). For any sequence \( \epsilon \to 0_+ \) we have

\[
\lim_{n \to \infty} |h_{t,s,n} - h_{t,s,0}| = 0.
\]

**Proof.** Simple algebra allows to show that \( |\frac{d}{d\epsilon} h_{t,s,\epsilon}| \leq \ln 2 \). Thus by the mean value theorem \( |h_{t,s,n} - h_{t,s,0}| \leq \epsilon_n \ln 2 \) which proves the result.

B. Free entropy change along the \((t, s)\)-interpolation

By interpolating \( h_{t,s} \) from \((t = 1, s = 0)\) to \((t = T, s = 1)\), we have

\[
h_{1,0,\epsilon} = h_{T,1,\epsilon} - \sum_{t=0}^{T-1} \int_{0}^{1} ds \frac{d}{ds} h_{t,\epsilon,s} = \frac{d}{ds} h_{T,\epsilon,s}.
\]

One can see from Algorithm 1 that the initial interpolating Hamiltonian equals in distribution

\[
\mathcal{H}_{1,0,\epsilon} \equiv \mathcal{H}_{T,1,\epsilon} - \sum_{A=1}^{m} \tilde{J}_{A}(\sigma_A - 1) - \sum_{i=1}^{n} H_i(\sigma_i - 1).
\]
due to the fact that a sum of independent Poisson-distributed r.v. is also Poisson-distributed (with mean equal to the sum of their means). Therefore
\[ h_{1,0,x=0} = \frac{1}{n} \mathbb{E} H(\sigma_1 | \tilde{J}). \quad (8) \]

On the other hand \( h_{T,1,\varepsilon} \) corresponds to a part of expression (6). Computations starting from (7) lead to the fundamental sum rule (see [15, 21, 22, 26] for similar computations)
\[ h_{1,0,\varepsilon} = \tilde{h}_{\varepsilon}(x) + \frac{1}{RT} \sum_{t=1}^{n} \int_0^1 ds \mathcal{R}_{t,s,\varepsilon}, \quad (9) \]
\[ \mathcal{R}_{t,s,\varepsilon} = \sum_{p=1}^{\infty} \frac{\mathbb{E}[(\tanh J)^{2p}]}{2p(2p-1)} \mathbb{E} \left[ (\mathbb{Q}_p)^{2p} - (\mathbb{Q}_p^{(t)})^{2p} \right] - Kq_2^{(t)K-1} (\mathbb{Q}_p - (\mathbb{Q}_p^{(t)})_{t,s,\varepsilon}), \quad (10) \]
with \( Q_p = \frac{1}{n} \sum_{i=1}^{n} \sigma_i^{(1)} \cdots \sigma_i^{(p)} \) the overlap for \( p \) independent replicas \( \sigma_i^{(1)}, \ldots, \sigma_i^{(p)} \) and \( q_p^{(t)} = \mathbb{E}[(\tanh V^{(t)})^p] \).

The Gibbs average \( \langle \cdot \rangle_{t,s,\varepsilon} \) over a function of \( p \) replicas must be understood as average over the product measure \( \prod_{t=1}^{\infty} z_{\tilde{t},s} e^{-\mathbb{R}_{\mathcal{T},t,*} \mathbb{C}^{(t)} \pm \tilde{J} \varepsilon} \) where the quenched variables are the same for all replicas. We still denote this Gibbs average by \( \langle \cdot \rangle_{t,s,\varepsilon} \) for simplicity.

C. Lower bound

In order to show the lower bound we need the following concentration lemma:

**Lemma III.3** (Concentration of \( Q_p^K \) on \( \langle Q_p \rangle_{t,s,\varepsilon}^{K} \)). For any \( c \in B, \varepsilon \in B^T \), we have
\[ \lim_{n \to \infty} \int_0^1 ds \mathbb{E} \left[ (\mathbb{Q}_p)^K - (\mathbb{Q}_p_{t,s,\varepsilon})^K \right]_{t,s,\varepsilon} = 0. \]

**Proposition III.4** (Lower bound). For \( c \in B \) we have
\[ \lim_{n \to \infty} -\mathbb{E} H(\sigma \mid \tilde{J}) \geq \sup_{x \in B} h_{RS}(x). \]

**Proof.** Lemma III.3 implies that there exists a sequence \( \epsilon_n \to 0_+, n \to +\infty \) such that
\[ \lim_{n \to \infty} \mathbb{E} \left[ (\mathbb{Q}_p)^K - (\mathbb{Q}_p_{t,s,\varepsilon})^K \right]_{t,s,\varepsilon} = 0. \] From (9) and (10) this implies
\[ \mathcal{R}_{t,s,\varepsilon} = \sum_{p=1}^{\infty} \frac{\mathbb{E}[(\tanh J)^{2p}]}{2p(2p-1)} \mathbb{E} \left[ (\mathbb{Q}_p)^K - (\mathbb{Q}_p_{t,s,\varepsilon})^K \right] - Kq_2^{(t)K-1} (\mathbb{Q}_p - (\mathbb{Q}_p_{t,s,\varepsilon})^K) + o_n(1) \quad (11) \]
where \( o_n(n) = 0. \) It is easy to check that \( (\mathbb{Q}_p_{t,s,\varepsilon})^K \geq 0 \), so that due to convexity of the function \( x^K \) for \( x \in \mathbb{R}^+ \) we get that the quantity inside the second expectation \( \mathbb{E}[-\cdot] \) in (11) is non-negative and thus \( \mathcal{R}_{t,s,\varepsilon} \geq o_n(1). \)

Combining this fact with (9) we conclude
\[ \lim_{n \to \infty} h_{1,0,\varepsilon} \geq \lim_{n \to \infty} \tilde{h}_{\varepsilon}(x) = \tilde{h}_{\varepsilon}(x). \quad (12) \]

Recall Lemma III.2. Since \( \lim \inf \) is the infimum of the set of accumulation points of a sequence, and the only accumulation point of the sequence \( h_{1,0,\varepsilon} - h_{1,0,\varepsilon=0} \) is 0, we have
\[ \lim_{n \to \infty} h_{1,0,\varepsilon} = \lim \inf_{n \to \infty} h_{1,0,\varepsilon} = \lim \inf_{n \to \infty} h_{1,0,\varepsilon=0}. \]
Thus (12) becomes
\[ \lim \inf_{n \to \infty} h_{1,0,\varepsilon} \geq \tilde{h}_{\varepsilon}(x). \]

Finally one can take the supremum over \( x \in B^T \) of the right hand side and use Lemma III.1 as well as (8) to obtain the result.

D. Upper bound

In this paragraph we crucially use the specificities of the BEC. In particular we use the following lemma:

**Lemma III.5.** For any \( c \in B, \varepsilon \in B^T, \) and any \( A \subseteq \{1, \ldots, n\} \) we have \( \langle \sigma_{\varepsilon} \rangle_{t,s,\varepsilon} \in [0,1]. \)

**Proof.** The first GKS inequality (3) implies \( \langle \sigma_{\varepsilon} \rangle_{t,s,\varepsilon} \geq 0. \) Nishimori’s identity (4) implies \( \mathbb{E}[\langle \sigma_{\varepsilon} \rangle_{t,s,\varepsilon} - \langle \sigma_{\varepsilon} \rangle_{t,s,\varepsilon}] = 0. \) Thus \( \langle \sigma_{\varepsilon} \rangle_{t,s,\varepsilon} \) equals 0 or 1.

Notice that \( \mathbb{E}[(\mathbb{Q}_p)_{t,s,\varepsilon} = \frac{1}{n} \sum_{i=1}^{n} \sigma_i^{(p)} \varepsilon \mathbb{E}[(\tanh V^{(t)})^p] \).

**Lemma III.5 then implies** \( \langle \mathbb{Q}_p \rangle_{t,s,\varepsilon} = \langle \mathbb{Q}_p \rangle_{t,s,\varepsilon} \) for all \( p \in \mathbb{N}. \) We also have \( V^{(t)} \) equals either 0 or 1 because \( x \in B \) thus \( q^{(t)} = q^{(t)} \) for \( p \in \mathbb{N}. \) Finally recall that \( c(\tilde{J}) = (1-q)\Delta + q\Delta_0. \) These facts reduce (11) to
\[ \mathcal{R}_{t,s,\varepsilon} = (1-q)(\ln 2) \mathbb{E} \left[ (\mathbb{Q}_1)_{t,s,\varepsilon}^K - q^{(t)K} - Kq^{(t)K-1} (\mathbb{Q}_1^K - q^{(t)K}) + o_n(1). \right. \quad (13) \]

To further simplify this expression we need the next lemmas:

**Lemma III.6** (Concentration of \( \langle \mathbb{Q}_1 \rangle_{t,s,\varepsilon} \) on \( \mathbb{E}[\langle \mathbb{Q}_1 \rangle_{t,s,\varepsilon}] \)). For any \( c \in B \) and \( \varnothing \in B^T \), we have
\[ \lim_{n \to \infty} \mathbb{E} \left[ (\mathbb{Q}_1)_{t,s,\varepsilon}^K - \mathbb{E}[\langle \mathbb{Q}_1 \rangle_{t,s,\varepsilon}]^K \right] = 0 \]
uniformly in \( 0 < \varepsilon < 1. \) In particular this is also true with \( \varepsilon \) replaced by the sequence \( \epsilon_n \to 0. \)

**Proof.** Using the fundamental theorem of calculus,
\[ \mathbb{E} \left[ (\mathbb{Q}_1)_{t,s,\varepsilon} - \mathbb{E}[\langle \mathbb{Q}_1 \rangle_{t,s,\varepsilon}] \right] = \frac{1}{n} \int_0^\varnothing ds \sum_{i=1}^{n} d \mathbb{E}(\sigma_i)_{t,s,\varepsilon} \]
\[ = \frac{1}{n} \int_0^\varnothing ds \sum_{i=1}^{n} \frac{K}{RT} \sum_{j=1}^{n} \mathbb{E}(\sigma_i)_{(j),\varepsilon}^j - \mathbb{E}(\sigma_i)_{e,j}^j \]
\[ - \frac{n}{RT} (\mathbb{E}(\sigma_i)_{m,j}^j - \mathbb{E}(\sigma_i)_{m,j}^j) \quad (14) \]
where (14) follows from \( \frac{d \mathbb{E}(X)}{d \varepsilon} = \mathbb{E}f(X + 1) - \mathbb{E}f(X) \) if \( X \) is a Poisson r.v. with mean \( \nu. \) The result then follows from \( \mathbb{E}(\sigma_i)_{e,j}^j \leq 1 \).

\[ \square \]
Using Lemma III.6 the expression (13) becomes
\[
\mathcal{R}_{t,s,n} = (1 - q)(\log 2) \left\{ \mathbb{E}[\mathcal{Q}(t,s,n)] - q \mathbb{E}[\mathcal{Q}(t,s,n)]^K - q \mathbb{E}[\mathcal{Q}(t,s,n)]_{K-1} - q \mathbb{E}[\mathcal{Q}(t,s,n)]_{K-1} - q \right\} + o_n(1). 
\]
Furthermore note that since \( T \) is a free parameter (controlling the mean of \( \epsilon^{(t)}_s \) and \( m^{(t)}_s \)) we can set it significantly larger than \( n \) when using Lemma III.7. Therefore, we can remove the \( s \)-dependence in the average overlaps in \( \mathcal{R}_{t,s,n} \) and write it as
\[
\mathcal{R}_{t,s,n} = (1 - q)(\log 2) \left\{ \mathbb{E}[\mathcal{Q}(t,s,n)] - q \mathbb{E}[\mathcal{Q}(t,s,n)]^K - q \mathbb{E}[\mathcal{Q}(t,s,n)]_{K-1} - q \mathbb{E}[\mathcal{Q}(t,s,n)]_{K-1} - q \right\} + o_n(1). 
\]
Recall that \( q^{(t)}_1 \equiv \mathbb{E}\tanh V^{(t)} \). The use of Lemma III.7 is critical because \( \mathbb{E}[\mathcal{Q}(t,s,n)] \) is independent of \( \{x^{(t)}_s\}_{t \geq 1} \). Thus, this allows us to sequentially choose a distribution \( \tilde{x}^{(t)}_n \) for \( V^{(t)} \) such that
\[
q^{(t)}_1 = \mathbb{E}[\mathcal{Q}(t,s,n)], \quad t = 1, \ldots, T. 
\]
In other words the interpolation path is adapted so that (16) holds in order to cancel the remainder (15), thus allowing us to choose the “optimal interpolation path.” The proof of the existence of a solution \( \tilde{x}^{(t)}_n \), \( t = 1, \ldots, T \) for equations (16) is not difficult and can be found in [19].

### Lemma III.8
(Existence of the “optimal interpolation path”). Equation (16) has a unique solution \( \tilde{x}^{(t)}_n = \{x^{(t)}_n\}_{t = 1, \ldots, T} \in \mathcal{B}^T \).

### Proposition III.9
(Upper bound). For any \( c \in \mathcal{B} \) we have
\[
\limsup_{n \to \infty} \frac{1}{n} \mathbb{E} H(\sigma|x^{(t)}_n|) \leq \sup_{x\in\mathcal{B}} h_{RS}(x).
\]
Proof. The specific choice \( \tilde{x}^{(t)}_n \) (essentially) eliminates \( \mathcal{R}_{t,s,n} \) and reduces (9) to
\[
h_{t,0,n} = h_{t,0}(\tilde{x}^{(t)}_n) + o_n(1) \leq \sup_{x\in\mathcal{B}^T} h_{t,0}(x) + o_n(1).
\]
Directly from (6) we see that \( h_{t,0}(x) \) is of the form \( \epsilon_n g(x) + \tilde{h}_{t,0}(x) \). Thus
\[
\sup_{x\in\mathcal{B}^T} h_{t,0}(x) \leq \epsilon_n \sup_{x\in\mathcal{B}^T} g(x) + \sup_{x\in\mathcal{B}^T} \tilde{h}_{t,0}(x)
\]
and passing to the limit (and recalling that \( \epsilon_n \to 0 \))
\[
\limsup_{n \to \infty} h_{t,0,n} \leq \limsup_{n \to \infty} h_{t,0}(x) \leq \sup_{x\in\mathcal{B}^T} \tilde{h}_{t,0}(x).
\]
Because of Lemma III.2 we have
\[
\limsup_{n \to \infty} h_{t,0,0} = \limsup_{n \to \infty} h_{t,0,0} = \limsup_{n \to \infty} \frac{1}{\mathbb{E} H(\sigma|x^{(t)}_n|)} (8).
\]
This combined with Lemma III.1 concludes the proof. \( \square \)

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