A type of Feynman-like formulas for general Schrödinger equation

I. D. Remizov

Dedicated to the blessed memory of Alexander Abrosimov.

We take a family of self-adjoint operators which is Chernoff-equivalent to a semigroup with the generator $H$, and write a short formula for the another family of operators (they appear to be unitary), which is Chernoff-equivalent to a semigroup with the generator $iH$. Then we apply the Chernoff theorem to the latter of the semigroups and obtain its representation in the form of Chernoff approximations, which leads us to the new type of expressions similar to the Feynman formulas. This all gives us a way to solve Cauchy problem for the PDE $u'_t(t, x) = iHu(t, x)$. We call this PDE the general Schrödinger equation for two reasons. First, we allow $H$ to be more complicated than a second-order differential (with respect to spatial coordinate $x$) operator. Second, we admit that $x$ can range over some more complicated spaces than $\mathbb{R}^3$.

MSC2010 codes: 35C15, 47D06, 47D08.

1 Introduction

Feynman formula is a representation of the solution to the Cauchy problem for PDE in a form of the limit of the multiple integral where the multiplicity tends to infinity. First appeared in pioneering works of R.P. Feynman on the physical level of rigor, they were extremely useful for physicists for studying the Schrödinger equation. Later mathematicians managed to develop a consistent theory of such formulas and still continue finding more and more applications of Feynman’s idea to different equations. The history of Feynman formulas’ research and a sketch of results obtained up to 2009 can be found in [4].

2 Preliminaries

In this paper, we study so-called evolutionary equations (i.e. equations in the form $u'_t(t, x) = \ldots$), bright examples of such equations are heat equation and Schrödinger equation. Here $t \in [0, +\infty)$ is time, and the spatial variable $x$ ranges over the set $M$. In practice, $M$ is defined by the physical process that motivates mathematical setting of the problem. For example, if we study heat propagation in a ball $B \subset \mathbb{R}^3$, then $M = B$. Below we discuss a very general case, so the technique that follows can be potentially employed in a case when $M$ is $\mathbb{R}^n$ or some subset of $\mathbb{R}^n$, $\mathbb{C}^n$ or some subset of $\mathbb{C}^n$, linear (Hilbert, Banach, etc.) space or some subset of it, a lattice, a manifold of finite or infinite dimension, a group, an algebra, a graph etc.

Let $\mathcal{F}$ be a complex Hilbert space (separable or not separable, does not matter) of functions $f: M \to \mathbb{C}$. In this paper, we study equations for such functions $u: [0, +\infty) \times M \to \mathbb{C}$ that for every fixed moment of time $t \in [0, +\infty)$ the function $x \mapsto u(t, x)$ belongs to $\mathcal{F}$, and the function $t \mapsto u(t, \cdot)$ is continuous as a mapping $[0, +\infty) \to \mathcal{F}$. We work only with this class of functions $u$ and do not emphasize this each time when stating the uniqueness of the solution.

The main thing we need from $M$ and $\mathcal{F}$ is that

$$\mathcal{F} = \{ f | f: M \to \mathbb{C}, f \in \mathcal{F} \}$$

is a Hilbert space. Let us denote its scalar product by $\langle \cdot, \cdot \rangle$. The discussion that follows does not lean on the nature of $\langle \cdot, \cdot \rangle$. For example, it can originate from the fact that $\mathcal{F} = L^2(M, \mu)$

Email: ivremizov@yandex.ru. Everyone is welcome to contact the author to show possible mistakes in the article and/or to propose possible applications of the method presented.
for some measure $\mu$ on $M$, or it can be based on some more complicated structures. As a very particular yet important case let us mention $M = \mathbb{R}^3$ and $\mathcal{F} = L^2(\mathbb{R}^3)$ for usual Schrödinger equation.

We will use the symbol $L_b(\mathcal{F}, \mathcal{F})$ for a Banach space of all linear bounded operators in $\mathcal{F}$, endowed with its classical operator norm. Symbol $(H, Dom(H))$ stands for a linear (in most interesting cases unbounded), densely defined, closed operator $H : Dom(H) \to \mathcal{F}$, where $Dom(H) \subset \mathcal{F}$ is dense in $\mathcal{F}$. For example, $H = \Delta$ or $H = \Delta^2$ or $(H\psi)(x) = (\Delta\psi)(x) + V(x)\psi(x)$ or some other. The only thing we need is that coefficients of $H$ do not depend on $t$; nevertheless they may depend on $x \in M$.

$Hu$ stands for the result of applying the operator $H$ to a function $[x \mapsto u(t, x)] \in Dom(H)$ for fixed $t$. Symbol $u'_t$ stands for the result of differentiation (with respect to $t$) of the function $t \mapsto u(t, x)$ for fixed $x$.

In this notation, for example, the heat equation can be written in a form $u'_t = Hu$ for $(H\psi)(x) = (\Delta\psi)(x) + V(x)\psi(x)$ and the corresponding Schrödinger equation is $u'_t = iHu$.

If $H$ is a generator $[3,6,8]$ of a strongly continuous semigroup, we denote this semigroup by $e^{tH}$.

**Definition 2.1.** We call a mapping $G : [0, +\infty) \to L_b(\mathcal{F}, \mathcal{F})$ Chernoff-equivalent (the term was introduced in $[3]$) to the semigroup $e^{tH}$ if it satisfies the conditions of the Chernoff theorem:

**Theorem 2.1.** (Chernoff theorem, theorem 10.7.21 in $[2]$) Let $\mathcal{X}$ be a Banach space, and $L_b(\mathcal{X}, \mathcal{X})$ be the space of all linear bounded operators in $\mathcal{X}$ endowed with the operator norm. Suppose there is a function $G$ such that:

(C1). $G$ is defined on $[0, +\infty)$, takes values in $L_b(\mathcal{X}, \mathcal{X})$ and $t \mapsto G(t)x$ is continuous for every vector $x \in \mathcal{X}$.

(C2). $G(0) = I$.

(C3). There exists $\omega \in \mathbb{R}$ such that $\|G(t)\| \leq e^{\omega t}$ for all $t \geq 0$.

(C4). There exists a dense subspace $D \subset \mathcal{X}$ such that for every $x \in D$ there exists a limit $G'(0)x = \lim_{t \to 0} t^{-1}(G(t)x - x)$.

(C5). Operator $(G'(0), D)$ has a closure $(L, D_1)$.

(C6). Operator $(H, D_1)$ is the generator of a strongly continuous semigroup $(e^{tL})_{t \geq 0}$.

Then for every $x \in \mathcal{X}$ we have $G(t/n)x \to e^{tL}x$ as $n \to \infty$, and the limit is uniform with respect to $t$ from every segment $[0, t_0]$ for every fixed $t_0 > 0$.

### 3 Motivation

If there exists a strongly continuous semigroup $e^{tH}$, then $[3,6]$ Cauchy problem for the equation $u'_t = Hu$ with $u_0 \in Dom(H)$

$$
\begin{align*}
  \begin{cases}
    u'_t = Hu \\
    u(0, x) = u_0(x)
  \end{cases}
\end{align*}
$$

has the unique solution $u(t, x) = (e^{tH}u_0)(x)$. In a similar way, existence of a strongly continuous semigroup $e^{iuH}$ implies existence of the unique solution $u(t, x) = (e^{iuH}u_0)(x)$ of Cauchy problem

$$
\begin{align*}
  \begin{cases}
    u'_t = iHu \\
    u(0, x) = u_0(x)
  \end{cases}
\end{align*}
$$

for the equation $u'_t = iHu$ with $u_0 \in Dom(H)$. If we extend the area of the initial conditions from $Dom(H)$ to $\mathcal{F}$, then the Cauchy problems (1) and (2) are also solved by the same semigroups, but the solution in this case is called the mild solution.
This connection between semigroups of operators and solutions of the Cauchy problems is one of the reasons why the semigroups are so useful. Some other applications of semigroups see in [S] pp.237-239.

4 The method

The core idea (introduced in [9]) is the following: if a family \( S_t \) is Chernoff-equivalent to the semigroup \( e^{itH} \) and all the operators \( S_t \) are self-adjoint, then the family \( R_t = e^{i(S_t-I)} \) is Chernoff-equivalent to the semigroup \( e^{itH} \), where \( I \) stands for the identity operator. Then we apply the Chernoff theorem to the family \( R_t \) and obtain the representation of the semigroup \( e^{itH} \) by the limit of powers of operators \( e^{i(S_t-I)} \). These powers are the new type of a Feynman-like formulas, but not the Feynman formulas because in addition to the multiple integration (usually \( S_t \) is an integral operator) they include a summation. In a more formal way this is expressed in the following theorems.

**Theorem 4.1.** Suppose the mapping \( S \) satisfies all the conditions:

(i) \( S: [0, +\infty) \to L_b(\mathcal{F}, \mathcal{F}) \). (Denote \( \|S_t\| = m(t) \).)
(ii) For each \( f \in \mathcal{F} \) the mapping \( t \mapsto S_tf \in \mathcal{F} \) is continuous.
(iii). \( S_0 = I \), where \( I \) is the identity operator in \( \mathcal{F} \).
(iv). For each \( t \geq 0 \) operator \( S_t \) is self-adjoint.
(v). There exists a subspace \( \mathcal{D} \subset \mathcal{F} \) such that for every \( f \in \mathcal{D} \) the following limit exists with respect to norm in \( \mathcal{F} \)

\[
\lim_{t \to 0} \frac{S_tf - f}{t} \quad \text{denote} \quad S'_{0f} \quad \text{denote} \quad S'_0f.
\]

Then the mapping \( R \) satisfies all the conditions:

(I) \( R: [0, +\infty) \mapsto e^{i(S_t-I)} = R_t \in L_b(\mathcal{F}, \mathcal{F}) \).

(II). For each \( f \in \mathcal{F} \) the mapping \( t \mapsto R_tf \in \mathcal{F} \) is continuous.

(III). \( R_0 = I \).

(IV). \( \|R_t\| = 1 \) for all \( t \geq 0 \).

(V). For every \( f \in \mathcal{D} \) the following limit exists with respect to norm in \( \mathcal{F} \)

\[
\lim_{t \to 0} \frac{R_tf - f}{t} \quad \text{denote} \quad R'_{0f} \quad \text{denote} \quad R'_0f
\]

and \( R'_0 f|_\mathcal{D} = iS_0'|_\mathcal{D} \), i.e. \( R'_0f = iS_0f \) for each \( f \in \mathcal{D} \).

**Proof.**

(I). Item (i) implies that for each \( t \geq 0 \) operator \( i(S_t-I) \) is linear bounded (and \( \|i(S_t-I)\| \leq m(t) + 1 \)), so exponent \( e^{i(S_t-I)} \) is well-defined by a series \( e^{i(S_t-I)} = \sum_{n=0}^{\infty} \frac{(i(S_t-I))^n}{n!} \).

(II). Let \( f \in \mathcal{F} \) be fixed. For bounded operator \( A \) the mapping \( A \mapsto e^A \) is continuous. From (ii) we obtain that mapping \( t \mapsto S_tf \) is continuous, hence \( t \mapsto i(S_t-I)f \) is continuous too. Then \( t \mapsto e^{i(S_t-I)}f = R_tf \) is a composition of two continuous mappings and thus continuous.

(III). \( R_0 = e^{i(S_t-I)|_{t=0}} = e^{i(I-I)} = e^0 = I \).

(IV). Let \( t \geq 0 \) be fixed. By (iv) operator \( S_t \) is self-adjoint, so \( (S_t-I) \) is also self-adjoint. By Stone’s theorem if \( B \) is self-adjoint, then \( \|e^B\| = 1 \). So \( \|R_t\| = \|e^{i(S_t-I)}\| = 1 \).

(V). Suppose that linear operator \( A: \mathcal{F} \to \mathcal{F} \) is bounded. Then one can define exponent \( A \mapsto e^A \) as a mapping \( L_b(\mathcal{F}, \mathcal{F}) \to L_b(\mathcal{F}, \mathcal{F}) \) by equality

\[
e^A = I + A + A^2 \cdot \frac{1}{2!} + A^3 \cdot \frac{1}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{A^n}{n!} .
\]
This implies that \( e^0 = I \). Moreover, \( e^A - I = IA + A^2 \sum_{n=0}^{\infty} \frac{A^n}{(n+2)!} \) denote \( IA + A^2\Psi(A) \). One can see that
\[
\|\Psi(A)\| = \left\| \sum_{n=0}^{\infty} \frac{A^n}{(n+2)!} \right\| \leq \sum_{n=0}^{\infty} \frac{\|A\|^n}{(n+2)!} \leq \sum_{n=0}^{\infty} \frac{\|A\|^n}{n!} = e^\|A\|.
\]

This implies that \( A^2\Psi(A) = o(\|A\|) \) hence Fréchet derivative \([\mathbf{1}]\) of the function \( A \mapsto e^A \) at the point \( A = 0 \) is equal to \( I \).

Next, let us for fixed \( f \in D \) define a mapping \( G : [0, +\infty) \to \mathcal{F} \) by the formula \( G(t) = \frac{i(f-S(tf))}{t} \). The condition (iii) imply that \( G(0) = 0 \). As it follows from (v), for each \( f \in D \) the mapping \( G \) is Fréchet differentiable at the point \( t = 0 \) and \( \frac{d}{dt} G(t) \big|_{t=0} = iS_0' f \).

Now we have two Fréchet differentiable mappings \( t \mapsto G(t) \) and \( A \mapsto \exp(A) \) composed to make a mapping \( t \mapsto (\exp \circ G)(t) = R_t f \). So by the chain rule \([\mathbf{1}]\) we have
\[
(R_t f)' \big|_{t=0} = I \circ (iS_0' f) = iS_0' f.
\]

\[ \square \]

**Theorem 4.2.** (Corollary from theorem \([\mathbf{1}]\)) If
(a). All the conditions of the theorem \([\mathbf{1}]\) are fulfilled,
(b). The subspace \( D \) mentioned in the theorem \([\mathbf{1}]\) is dense in \( \mathcal{F} \).
(c). The operator \( iH \) is a generator of a strongly continuous semigroup \( (e^{itH})_{t \geq 0} \).
(d). \( S_0' f = H f \) for all \( f \in D \).

Then for each \( f \in \mathcal{F} \) with respect to norm in \( \mathcal{F} \)

\[
e^{itH} f = \left( \lim_{n \to \infty} \left( e^{i(S_{t/n} - I)} \right)^n \right) f \quad (4)
\]

\[
e^{itH} f = \left( \lim_{n \to \infty} e^{in(S_{t/n} - I)} \right) f \quad (5)
\]

\[
e^{itH} f = \left( \lim_{n \to \infty} \lim_{m \to \infty} \sum_{k=0}^{m} \frac{i^n k^n}{k!} (S_{t/n} - I)^k \right) f \quad (6)
\]

\[
e^{itH} f = \lim_{n \to \infty} \lim_{k \to \infty} \left( \left( 1 - \frac{in}{k} \right) I + \frac{in}{k} S_{t/n} \right)^k f \quad (7)
\]
uniformly with respect to \( t \) from every segment \([0, t_0]\) for every fixed \( t_0 > 0 \).

**Proof.** Items (a), (b), (c) and (d) with theorem \([\mathbf{1}]\) applied imply that \( R_t \) is Chernoff-equivalent to the semigroup \( e^{itH} \). Indeed, (C1) follows from (I) and (II), (C2) is (III), (C3) with \( \omega = 0 \) follows from (IV), (C4) follows from (V), (b) and (d), (C5) also follows from (V) and (b), (C6) is (c). Thus by the Chernoff theorem for each \( f \in \mathcal{F} \) one obtains \( \lim_{n \to \infty} R(t/n)^n x \to e^{itH} x \) uniformly with respect to \( t \) from every segment \([0, t_0]\) for every fixed \( t_0 > 0 \). Keeping in mind that \( R_t = e^{i(S_t - I)} \) one can see that

\[
e^{itH} = \lim_{n \to \infty} \left( e^{i(S_{t/n} - I)} \right)^n = \lim_{n \to \infty} e^{in(S_{t/n} - I)}.
\]

The second equality above holds because operators \( S_t \) and \( I \) commute. Recalling the following equality for bounded operator \( A \)

\[
e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = \lim_{k \to \infty} \left( I + \frac{A}{k} \right)^k
\]
and setting \( A = in(S_{t/n} - I) \) we obtain

4
\[ e^{itH} = \lim_{n \to \infty} \sum_{k=0}^{\infty} \left( \frac{i n (S_{t/n} - I)}{k!} \right)^k = \lim_{n \to \infty} \lim_{m \to \infty} \sum_{k=0}^{m} \frac{i^k n^k}{k!} (S_{t/n} - I)^k \]

and

\[ e^{itH} = \lim_{n \to \infty} \lim_{k \to \infty} \left( I + \frac{i n (S_{t/n} - I)}{k} \right)^k = \lim_{n \to \infty} \lim_{k \to \infty} \left( \left( 1 - \frac{i n}{k} \right) + \frac{i n}{k} S_{t/n} \right)^k. \]

Remark 4.1. Recall that Chernoff-equivalence includes the estimate \( \| S_t \| \leq e^{\omega t} \), which we never used in theorems 4.1 and 4.2. So the method introduced can be applied to the families of operators, which are not Chernoff-equivalent to any semigroup due to their growth of norm higher than \( e^{\omega t} \).

Theorem 4.3. (Corollary from theorem 4.2) If

(A). The family \( S_t \) is Chernoff-equivalent to the semigroup \( e^{itH} \).

(B). \( \left( S_t \right)^* = S_t \) for each \( t \geq 0 \).

(C). The operator \( iH \) is a generator of a strongly continuous semigroup \( (e^{itH})_{t \geq 0} \).

Then the family \( R_t = e^{i(S_t - I)} \) is Chernoff-equivalent to the semigroup \( e^{itH} \) and formulas (4) - (7) hold.

Proof. Let us check that all the conditions (a), (b), (c), (d) of theorem 4.2 hold.

(a) is (i)-(v); (i) and (ii) follows from (C1) for \( G(t) = S_t \), (iii) is (C2) for \( G(t) = S_t \), (iv) is (B), (v) follows from (C4) for \( G(t) = S_t \).

(b) follows from (C4) for \( G(t) = S_t \).

(c) is (C).

(d) follows from (A) applied to (C4) in the definition of the Chernoff-equivalence.

This means that theorem 4.2 can be employed here, so (4) - (7) hold. The last thing we need to check is (C1)-(C6) for \( G(t) = R_t \). Indeed, (C1) for \( G(t) = R_t \) follows from (I) and (II), (C2) for \( G(t) = R_t \) is (III), (C3) for \( \omega = 0 \) follows from (IV).

(C4) and (C5) for \( G(t) = R_t \) follows form the following: 1) (V) is true, 2) (A) imply C(4) and (C5) for \( G(t) = S_t \), 3) (C) is true. To see this recall that the generator of a semigroup is always a closed densely defined operator, and that operator \( iH \) is closed coincidently with \( H \).

Above, we based our constructions on the fact that strongly continuous semigroup \( e^{itH} \) already exists. The following variant of Chernoff theorem states its existence in the case when the image of the operator \( \lambda I - iH \) is dense in \( F \) for some \( \lambda > 0 \).

Theorem 4.4. (Chernoff-type theorem, [6], corollary 5.3 from theorem 5.2) Let \( X \) be a Banach space, and \( L_b(X, X) \) be the space of all linear bounded operators on \( X \) endowed with the operator norm. Suppose there is a function \( V : [0, +\infty) \to L_b(X, X) \), meeting the following conditions:

(a1). \( V(0) = I \), where \( I \) is the identity operator.

(a2). There exist numbers \( M \geq 1 \) and \( \omega \in \mathbb{R} \) such that \( \|(V_t)^k\| \leq Me^{\omega t} \) for every \( t \geq 0 \) and every \( k \in \mathbb{N} \).

(a3). The limit

\[ \lim_{t \downarrow 0} \frac{V_t \phi - \phi}{t} =: L \phi \]

meeting the following conditions:

(a1). \( V(0) = I \), where \( I \) is the identity operator.

(a2). There exist numbers \( M \geq 1 \) and \( \omega \in \mathbb{R} \) such that \( \|(V_t)^k\| \leq Me^{\omega t} \) for every \( t \geq 0 \) and every \( k \in \mathbb{N} \).

(a3). The limit

\[ \lim_{t \downarrow 0} \frac{V_t \phi - \phi}{t} =: L \phi \]

5
exists for every $\varphi \in \mathcal{D} \subset \mathcal{X}$, where $\mathcal{D}$ is a dense subspace of $\mathcal{X}$.

(a4). There exists a number $\lambda_0 > \omega$ such that $(\lambda_0 I - \mathcal{L})(\mathcal{D})$ is a dense subspace of $\mathcal{X}$.

Then the closure $\mathcal{L}$ of the operator $\mathcal{L}$ is a generator of a strongly continuous semigroup of operators $(T_t)_{t \geq 0}$ given by the formula

$$T_t \varphi = \lim_{n \to \infty} \left( V_t^n \right) \varphi$$

where the limit exists for every $\varphi \in \mathcal{X}$ and is uniform with respect to $t \in [0, t_0]$ for every $t_0 > 0$. Moreover $(T_t)_{t \geq 0}$ satisfies the estimate $\|T_t\| \leq M e^{\omega t}$ for every $t \geq 0$.

Applying this result, we can state the following theorem.

**Theorem 4.5.** Suppose the mapping $S$ satisfies all the conditions:

(i) $S : [0, +\infty) \to L_0(\mathcal{F}, \mathcal{F})$.

(iii). $S_0 = I$, where $I$ is the identity operator in $\mathcal{F}$.

(iv). For each $t \geq 0$ operator $S_t$ is self-adjoint.

(v). There exists a subspace $\mathcal{D} \subset \mathcal{F}$ such that for every $f \in \mathcal{D}$ the following limit exists with respect to norm in $\mathcal{F}$

$$\lim_{t \to 0} \frac{S_t f - f}{t} \text{ denote } = S'_t|_{t=0} f \text{ denote } = S'_0 f.$$ 

(d). $S'_0 f = H f$ for all $f \in \mathcal{D}$.

(e). There is a number $\lambda_0 > 0$ such that $(\lambda_0 I - iH)(\mathcal{D})$ is a dense subspace of $\mathcal{F}$.

Then there exists a strongly continuous semigroup $e^{itH}$ with the generator $iH$, and for each $f \in \mathcal{F}$ with respect to norm in $\mathcal{F}$ formulas (4), (5), (6), (7) hold uniformly with respect to $t$ from every segment $[0, t_0]$ for every fixed $t_0 > 0$. Moreover, $\|e^{itH}\| \leq 1$ for all $t \geq 0$.

**Proof.** We apply theorem 4.4 setting $\mathcal{X} = \mathcal{F}$ and $V(t) = R_t = e^{i(S_t - I)}$. The condition (a1) follows from (iii); in (a2) one can set $M = 1$ and $\omega = 0$ because $\|R_t\| = \|e^{i(S_t - I)}\| = 1$ due to (iv) and Stone’s theorem; (a3) follows from (v) and (d); (a4) follows from (e) united with the fact that one can set $\omega = 0$ in (a2) as discussed above.

□

**Remark 4.2.** All the constructions above can be repeated (mutatis mutandis) for the family $R_t^{-} = e^{-i(S_t - I)}$ to obtain the semigroup $e^{-itH}$ and it’s approximations in spirit of (4) - (7).

## 5 Applications

The first idea is to start from the usual heat equation $u'_t = Hu$ with self-adjoint second-order differential (with respect to spatial coordinate) operator $H$ and obtain (4) - (7) for the usual Schrödinger equation $u'_t = iH$. But the method described is applicable to more complicated cases, for instance, to $H = \Delta^2$. Let us show, how it works.

### 5.1 $M = \mathbb{R}^n$, heat and Schrödinger equation

Plyashechnik

### 5.2 $M = \mathbb{R}^n$, $H = \Delta^2$

Buzinov, Butko
5.3 $M$ is a $n$-dimensional compact manifold, heat and Schrödinger equation

Butko, Smolyanov.

5.4 $M$ is infinite-dimensional linear space

5.5 $M$ is a infinite-dimensional manifold

5.6 $M$ is a graph

6 Acknowledgements

It is a pleasure to thank Prof. D.V.Turaev for acquainting the author with the general Stone’s theorem and it’s applicability to the case studied.

References

[1] V.Beloshapka, A.Kalinin, M.Kuznetsov, K.Mayorov, G.Polotovskiy, I.Remizov, Ya.Sennikovskiy, G.Shabat, M.Tokman, A.Tumanov, G.Zhislin. Alexander Abrosimov.//Notices of the AMS, December 2012, vol. 59, No. 11, pp. 1569-1570.

[2] V.I. Bogachev, O.G. Smolyanov. Real and functional analysis: university course. (In Russian) — M. Izevsk: RCD, 2009.

[3] O.G. Smolyanov, H. v. Weizsäcker, O. Wittich. Chernoff’s Theorem and Discrete Time Approximations of Brownian Motion on Manifolds//Potential Analysis, February 2007, Volume 26, Issue 1, pp 1-29.

[4] O.G. Smolyanov. Feynman formulae for evolutionary equations. — Trends in Stochastic Analysis, London Mathematical Society Lecture Notes Series 353, 2009.

[5] A. Pazy. Semigroups of Linear Operators and Applications to Partial Differential Equations. — Springer-Verlag, 1983.

[6] K.-J. Engel, R. Nagel. One-Parameter Semigroups for Linear Evolution Equations. — Springer, 2000.

[7] H. Cartan. Differential Calculus. — Kershaw Publishing Company, 1971.

[8] K.-J. Engel, R. Nagel. A Short Course on Operator Semigroups. — Springer, 2006.

[9] I.D.Remizov. On the connection between resolving semigroups and families of operators Chernoff-equivalent to them for heat and Schrödinger equations in $L^2$ space (in Russian).// Proceedings of the Lomonosov-2014 conference, Moscow State University, April 2014. Online version http://lomonosov-msu.ru/archive/Lomonosov_2014/2588/2200_17603_5b4ae4.pdf