PRICE OF ANARCHY IN BERNOULLI CONGESTION GAMES WITH AFFINE COSTS

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ABSTRACT. We consider an atomic congestion game in which each player participates in the game with an exogenous and known probability \( p_i \in [0, 1] \), independently of everybody else, or stays out and incurs no cost. We first prove that the resulting game is potential. Then, we compute the parameterized price of anarchy to characterize the impact of demand uncertainty on the efficiency of selfish behavior. It turns out that the price of anarchy as a function of the maximum participation probability \( p = \max_i p_i \) is a nondecreasing function. The worst case is attained when players have the same participation probabilities \( p_i = p \). For the case of affine costs, we provide an analytic expression for the parameterized price of anarchy as a function of \( p \). This function is continuous on \((0, 1]\), is equal to \( 4/3 \) for \( 0 < p \leq 1/4 \), and increases towards \( 5/2 \) when \( p \to 1 \). Our work can be interpreted as providing a continuous transition between the price of anarchy of nonatomic and atomic games, which are the extremes of the price of anarchy function we characterize. We show that these bounds are tight and are attained on routing games—as opposed to general congestion games—with purely linear costs (i.e., with no constant terms).

1. INTRODUCTION

Atomic congestion games, as first introduced by Rosenthal (1973), have been extensively studied as a prominent class of potential games. In fact, Monderer and Shapley (1996) showed that every finite potential game is isomorphic to a congestion game. Applications of congestion games can be found for instance in routing (Rosenthal, 1973), network formation (Anshelevich et al., 2008), and market entry games (Erev and Rapoport, 1998, Duffy and Hopkins, 2005).

In the past twenty years, the effects of selfishness over social welfare in congestion games have been widely addressed. Initially, the focus was on the simplest case of complete information congestion games where either the demand is infinitely divisible (Roughgarden and Tardos, 2004) or the cost functions are linear (Christodoulou and Koutsoupias, 2005). More recently, there has been ample interest in understanding the stochastic aspects of the game, both on the supply and demand sides. The interest comes from the need to understand how the uncertainty about costs and demand might affect the strategies and behavior of agents playing the game.

In this work, we start from a congestion game with atomic players and add demand uncertainty in the form of players who may or may not take an active part in the game with a given probability, as described below. One motivation comes from routing games: suppose a commuter is planning a trip from home to work. From experience, she probably knows the possible routes and might also

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have an idea of the potential number of commuters she may encounter on average. However, the identity and the actual number of commuters to be found on the road is unknown and is likely to vary around its average value. Such variability implies that for choosing the optimal route, players must anticipate the consequences of the uncertainty. Although our analysis applies to the whole class of congestion games, we will often use the language of routing games to provide a more vivid representation of our model. An instance of the game can be thought as the representation of the network conditions at a given time, when only a random subset of the population is present.

We model this situation as a game of incomplete information. As is common in the literature, we assume that planning happens before getting any signal of the realized congestion. The key assumption of our work is that each player $i \in \{1, \ldots, n\}$ participates in the game with probability $p_i$, independently of other players, and otherwise stays out of the game. The entire set of players (prior to the random entry) and their $p_i$’s are common knowledge, but the actual realization of the uncertainty is unknown to players at the time they make their strategic choices in the game. We allow for heterogenous probabilities across players, which, given our motivation, is a natural setting to consider. If all players happen to have the same probabilities $p_i = p$, the number of active players is a binomial random variable with parameters $n$ and $p$.

We assume that players (randomly) staying out of the game incur a cost equal to 0. Since staying out is determined exogenously and thus non-strategic, we could assign any cost to staying out of the game. The choice of cost for the outside option does not affect the set of Nash equilibria. Furthermore, a positive cost of non-participation would improve the efficiency of equilibria. For this reason, one does not need to study positive costs of non-participation to determine worst-case bounds for equilibria.

The participation probabilities may be different for instances capturing different times because they are affected by factors such as weather, road conditions, failures of telecommunication equipment, or events that influence the interest of internet users to be connected. While such external random factors provide coordination signals to the game, we consider the game conditional on the realizations of those factors. After conditioning on these factors, the individual participation events are assumed to be stochastically independent. In addition, our setup does not preclude having another step in which participation is a strategic decision. We do not explicitly mention the strategic participation because it can be easily represented by including the outside option in the strategy set, therefore not changing the structure of the game.

1.1. Our contribution. We study the general framework of Bernoulli congestion games and how players should strategize in this setting. In answering this, we explore the impact caused by both the individual optimization of routes and the uncertainty in who shows up. For this, we rely on the concept of price of anarchy (PoA) which is defined as the worst-case ratio between the social cost of an equilibrium and the minimum cost achievable by a social planner.

We start by proving that Bernoulli congestion games are exact potential games. Our goal is to shed light on how uncertainty impacts the efficiency of the system. Consequently, we parameterize the computations using the participation probabilities. In Theorem 3.4, we show that the PoA as a function of $p = \max_i p_i$ is nondecreasing, and the worst-case scenario arises when all the $p_i$’s are equal. We then focus on games in which all players have homogeneous probabilities, keeping in mind that the results also apply to games in which $p_i \leq p$ for all $i \in N$.

Even though we have a game of incomplete information, when players have homogeneous probabilities it can be transformed into an equivalent deterministic game by appropriately adjusting the cost functions. This allows us to analyze it as a (regular) complete information game.
For these transformed costs, one can derive tight bounds for the parametrized PoA. We prove that in the limit when $p \to 0$, the PoA converges to 1 if we keep the number of players bounded, whereas this is not necessarily the case when the number of players is allowed to grow.

To provide concrete expressions, we focus on the case of nondecreasing and nonnegative affine costs. Theorem 4.1 establishes tight upper bounds for the PoA as a function of $p \in (0, 1]$. The resulting bound is continuous and nondecreasing, and exhibits three distinct regions with kinks at $\bar{p}_0 = 1/4$ and $\bar{p}_1 \approx 0.3774$ (the real root of $8p^3 + 4p^2 = 1$), as shown in Fig. 1. In the lower region $0 < p \leq \bar{p}_0$, the PoA is at most $4/3$, which coincides with the PoA for nonatomic congestion games with affine costs. In the middle region $\bar{p}_0 < p \leq \bar{p}_1$, the PoA is at most

$$\frac{1 + p + \sqrt{p(2 + p)}}{1 - p + \sqrt{p(2 + p)}},$$

whereas in the upper region $\bar{p}_1 < p \leq 1$, the PoA is at most $1 + p + p^2/(1 + p)$.

![Figure 1. Tight bounds for the PoA (red) and PoS (blue) for Bernoulli congestion games with affine costs as a function of $p$.](image)

For $p = 1$, we recover the $5/2$-bound known for deterministic atomic congestion games with affine costs, whereas for all $p < 1$ we get a smaller bound. Note the (perhaps unexpected) fact that for $p \leq 1/4$ we have a significantly smaller and constant bound of $4/3$, independently of the structure of the congestion game and for any number of players. Thus, in contrast to the case of a bounded number of players where the PoA is close to 1, for low values of $p$ and an arbitrarily large number of players, the PoA is similar to that in nonatomic congestion games.

Our upper bound proofs are based on a non-trivial use of the $(\lambda, \mu)$-smoothness framework (Roughgarden, 2015a). Technically, this idea boils down to solving a linear program in which the number of constraints grows with the number of players in the game. In particular, this implies that in order to obtain a tight bound, a linear program with infinitely many constraints has to be solved. In the upper region ($p \geq \bar{p}_1$), the optimal solution is induced by two tight constraints for any finite number of players. In the middle and lower region ($p \leq \bar{p}_1$) the two tight constraints hold only asymptotically; therefore, the analysis is technically more involved.

The three bounds for the PoA turn out to be tight and are attained in the subclass of routing games (as opposed to general congestion games) and with purely linear costs (as opposed to
Example 4.1 shows that the bound is asymptotically tight for $p$ in the lower region by considering a sequence of bypass networks. Note that although costs are linear, due to the randomness in demand, the PoA is not equal to one as it would be expected in nonatomic congestion games with purely linear costs. In fact, the equivalent deterministic game has adjusted cost functions with a constant term that pushes the PoA up to $4/3$. Tightness in the lower region can also be obtained from a sequence of Pigou networks with affine costs (see Example 3.1). Example 4.2 shows that the bound is asymptotically tight for values of $p$ in the middle region by considering a sequence of roundabout networks. Finally, Example 4.3 shows that the bound is tight for $p$ in the upper region by considering a variant of the game in Awerbuch et al. (2013).

For completeness, Fig. 1 also depicts the price of stability (PoS) as a function of $p$, which is analogue to the PoA but with the best equilibrium instead of the worst. We see that the function is increasing and smooth, except for one kink at $p = 1/4$. For $0 < p \leq 1/4$ the PoS is at most $4/3$, while for $1/4 \leq p \leq 1$ we have $\text{PoS} \leq 1 + \sqrt{p/(2 + p)}$. The tight bound for the upper region ($p \geq 1/4$) was established in Kleer and Schäfer (2019), so we will not discuss it further here.

Some of the bounds in Fig. 1 can be derived from previous results that apply to different models, but which turn out to be mathematically equivalent to ours under some regimes. We discuss those results in the next section.

1.2. Organization of the paper. Section 2 frames our results with respect to the existing literature. We introduce Bernoulli congestion games and present results for general cost functions in Section 3. Next, Section 4 presents tight bounds for the price of anarchy and stability in the case of affine costs and heterogeneous participation probabilities. We present our final thoughts in Section 5. To improve the reading flow, we moved the most technical proofs to Appendix A.

2. Related work

The inefficiency induced by self-minded behavior in congestion games has been studied extensively in the past twenty years. The systematic study of the inefficiency of equilibria started with the work of Koutsoupias and Papadimitriou (1999, 2009). The impact of the self-mindedness of commuters is typically evaluated through worst-case bounds for the price of anarchy (PoA) and the price of stability (PoS) (first defined by Schulz and Stier-Moses 2003 and coined by Anshelevich et al. 2008), representing the worst and best equilibrium respectively. Different papers made different calls about what aspect to highlight, hence fixing some input parameters and taking the worst case among chosen families of instances. Some results have been given parametrically as a function of an appropriate scalar to shed light on the dependence between the scalar and the efficiency of equilibria.

Bounds for the PoA are substantially different for atomic and nonatomic congestion games. In nonatomic games, where the demand is perfectly divisible, the equilibrium concept is due to Wardrop (1952), and its properties have been thoroughly studied since Beckmann et al. (1956). Bounds for the PoA in these games were obtained in Roughgarden and Tardos (2002, 2004), Roughgarden (2003, 2005) and Correa et al. (2004, 2008). These papers provided tight bounds for the PoA when the cost functions are restricted to specific classes—such as polynomials—and showed that the bounds depend only on the cost functions but not on the topology of the network. We refer to Roughgarden (2007), Roughgarden and Tardos (2007), and Correa and Stier-Moses (2011) for surveys of these results.

On the other hand, atomic congestion games (a superset of atomic routing games) were introduced by Rosenthal (1973). The PoA for these games was examined in Christodoulou and
Koutsoupias (2005), Suri et al. (2007), and Awerbuch et al. (2013), for weighted and unweighted atomic players. Aland et al. (2011) provided exact bounds for the PoA when costs are polynomial functions. In a different direction, Chen et al. (2014) studied the PoA for congestion games where players can be altruistic.

Correa et al. (2008) computed the parameterized PoA as a function of the level of congestion in a game, to shed light on why lightly congested networks have low PoA. Other papers also provided parameterized PoA results. For instance, some recent papers have studied the behavior of the PoA in nonatomic routing games as a function of the total traffic demand. Colini-Baldeschi et al. (2017), Colini-Baldeschi et al. (2019), and Colini-Baldeschi et al. (2020) studied the asymptotic behavior of the PoA in light and heavy traffic regimes, and showed, that under mild conditions, efficiency is achieved in both limit cases. Similar results for congestion games in heavy traffic were obtained by Wu et al. (2021a), using a slightly different technique. A non-asymptotic analysis of the behavior of the PoA as a function of the demand has been provided by Cominetti et al. (2021), and Wu and Möhring (2019). These papers studied the behavior of the PoA for a single game as a parameter representing the demand changes. Instead, the present paper provides tight bounds for the PoA as a function of the maximal participation probability \( p \), for the whole class of congestion games with affine costs and for every possible \( p \).

While most previous results concern the PoA and its bounds for games with complete information, attention has recently turned to incomplete information games. Gairing et al. (2008) studied the PoA for congestion games on a network with capacitated parallel edges, where players are of different types—the type of each agent being the traffic that she moves—and types are private information. Ashlagi et al. (2006) and Ukkusuri and Waller (2010) considered network games in which agents have incomplete information about the demand. Ordóñez and Stier-Moses (2010), Nikolova and Stier-Moses (2014), Cominetti and Torrico (2016), and Lianeas et al. (2019) studied the consequences of risk aversion on models with stochastic cost functions. Penn et al. (2009) and Penn et al. (2011) dealt with congestion games with failures. Wang et al. (2014) considered nonatomic routing games with random demand and examined the behavior of the PoA as a function of the demand distribution. Macault et al. (2022) considered learning in repeated nonatomic routing games where the costs functions are unknown and the demands are stochastic. Roughgarden (2015b) showed that, whenever player types are independent, the PoA bounds for complete information games extend to Bayesian-Nash equilibria of the incomplete information game. In particular, for congestion games with affine costs, the bound of 5/2 holds also for games of incomplete information. The definition of smoothness in that framework is more restrictive than the definition we use, and has stronger implications since the quality of Bayesian-Nash equilibria is compared to an adaptive definition of optimum. However, because Roughgarden’s bounds are not derived for specific probabilities for players, they are not as sharp as ours. Recently, Correa et al. (2019) proved a similar result for smooth games with stochastic demand and arbitrary correlations in the participation probability, but again their bounds are not tight for a fixed \( p \in (0, 1) \).

Wrede (2019) considered the same model as ours, but restricted attention to games with a small number of players and gave a precise characterization of the price of anarchy for two players.

Our proofs are most closely related to the work of Christodoulou and Koutsoupias (2005) who computed the PoA for atomic unsplittable congestion games with affine costs by finding two coefficients for which an inequality for equilibria and optima holds. Related ideas also appeared around the same time in Harks and Végvári (2007) and Aland et al. (2011) for a variety of settings. Collectively this set of techniques became known as the \((\lambda, \mu)\)-smoothness framework, as coined
and systematized by Roughgarden (2015a) in a work that surveyed past uses and extended the framework to several other types of games. The bounds obtained by this technique not only hold for pure equilibria, but also for mixed, correlated, and coarse-correlated equilibria. The upper bound proofs in our paper are a non-trivial and technical application of this framework; moreover, they are shown to be tight by means of lower bound examples.

There are two key features that set our work apart from previous PoA analysis with similar motivations. First, Roughgarden (2015b) showed that PoA bounds for smooth games also apply to Bayesian extensions of the game as long as types are drawn independently. His goal was to find bounds that are independent of the distribution over types, which means that taking a worst-case approach reduces to the case of $p_i = 1$ for all players, and does not provide specific information for the case of stochastic players. Instead, we consider a weaker definition of smoothness and we parameterize our bounds explicitly with respect to $p$.

Second, we assume that the social-planner solution—the yardstick that all the PoA papers have used to evaluate the efficiency of equilibria—is exposed to the same uncertainty as the commuters themselves. We compute the social optimum, assigning strategies (or routes for network games) to players before knowing whether they will be active or not. For the players who show up, the corresponding strategies are used; for the others, the strategies are discarded. This sets the bar for a fairer comparison since there is similar uncertainty under both situations.

Congestion games with random players were already considered by Meir et al. (2012) in a model where player’s participation events can be correlated, and resource costs can be either increasing or decreasing (for a related model with both player and resource failures see Li et al. 2017). Most of their results concern the case of Bernoulli games with homogeneous probabilities $p_i \equiv p$. They show how uncertainty eliminates bad equilibria when $p \approx 1$, and establish the lower semi-continuity of PoA at $p = 1$, and (full) continuity for routing games on parallel networks. For fixed $p < 1$ they also show that PoA can grow with the number of players. Our results complement the above by showing that for heterogeneous probabilities with $p_i \leq p$ the worst PoA occurs in fact in the homogeneous case $p_i \equiv p$, and that it increases with $p$. Moreover, by restricting to affine costs, we are able to provide explicit tight bounds for the PoA as a function of $p$.

Angelidakis et al. (2013) also studied routing games with Bernoulli players. Specifically, they considered parallel link networks with risk-averse players that minimize the value-at-risk of the travel times (for some player-specific quantile). Their paper is mostly concerned with the existence and computation of equilibria. However, they show that PoA $\leq n$ for parallel networks with affine costs and $n$ players, with specific examples attaining PoA $\approx n/8$. In contrast, here we consider risk-neutral players that evaluate strategies according to their expected costs. This yields bounds on the PoA that do not grow with the number of players, and which hold for general Bernoulli congestion games beyond routing on parallel networks.

A few other papers have studied alternative models from which one can partially recover some of the bounds in Fig. 1. Although those works are concerned with different settings that do not encompass congestion games with Bernoulli players, they turn out to be analytically equivalent to our model for the special case of homogeneous players with $p_i \equiv p$ for all $i$. We stress that for heterogeneous players there is no clear connection between our model and the ones below.

Piliouras et al. (2016) considered an atomic congestion game where players using a given resource are randomly ordered and their costs depend on their position in this order. For risk-neutral players, their model exhibits the same mathematical structure as ours with $p_i \equiv 1/2$. Considering a different context with link failures in which players select robust strategies that
comprise a fixed number \( \rho \) of edge-disjoint routes, Bilò et al. (2018) established tight bounds that can be interpreted in our framework by taking \( p_1 \equiv \rho = 1/\rho \) with integer \( \rho \in \{1, 2, \ldots\} \). Unfortunately, their bounds give little insight into what happens to the PoA for \( 1/4 < p < 1/2\), except at the point \( p = 1/3 \). As mentioned earlier, we show that the PoA has two kinks in this region: one at \( \hat{p}_0 = 1/4 \) and a second one at \( \hat{p}_1 \approx 0.3774 \). On a different direction, Kleer and Schäfer (2019) studied routing games with affine costs and with two additional parameters \( \rho \) and \( \sigma \) that affect respectively the costs perceived by the players and by the central planner. Our bounds for affine costs and homogeneous players \( (p_i \equiv p \text{ for all } i) \) are formally equivalent to Kleer and Schäfer’s with \( \rho = \sigma = p \). Focusing on this case, their results on the PoA only cover the interval \( p \in [\frac{1}{2}, 1] \), naturally with the same bound as ours. They showed that this bound is tight by using a sequence of symmetric routing games adapted from Correa et al. (2015), whereas we use a simpler non-symmetric routing game with just 3 players. Moreover, we provide tight bounds in the full interval \( p \in [0, 1] \), not only for \( p \geq \frac{1}{2} \), and prove that these bounds apply more generally to Bernoulli congestion games with heterogenous players. Incidentally, we note that the analytic expression of the bound for \( p \in [\frac{1}{2}, 1] \) remains valid and tight on the larger interval \( p \in [\hat{p}_1, 1] \). As mentioned earlier, Kleer and Schäfer (2019) also studied the PoS and provided tight bounds for \( p \geq 1/4 \). We complete the bound in the region \( 0 \leq p < 1/4 \).

3. Bernoulli congestion games

3.1. Atomic congestion games. Consider a finite set of players \( N = \{1, \ldots, n\} \) and a finite set of resources \( E \). Each player \( i \in N \) has a set of feasible strategies \( S_i \subseteq 2^E \). Given a profile \( s \in S := \times_{i \in N} S_i \), the cost for player \( i \) is

\[
C_i(s) = \sum_{e \in S_i} c_e(N_e(s)).
\]

Here, \( N_e(s) \) is the load of resource \( e \in E \), defined as the number of players using the resource, that is,

\[
N_e(s) = |\{ j \in N | e \in s_j \}|.
\]

and \( c_e : \mathbb{N} \to \mathbb{R}_+ \) is the cost of the resource \( e \), with \( c_e(k) \) being the cost experienced by each player using that resource when the load is \( k \).

The tuple \( \Gamma = (N, E, S, (c_e)_{e \in E}) \) defines an atomic congestion game (ACG). Recall that a pure Nash equilibrium (PNE) is a strategy profile \( \hat{s} \) such that no player \( i \in N \) can benefit by unilaterally deviating to \( s_i \in S_i \), that is, for every player \( i \in N \) and every \( s_i \in S_i \), we have

\[
C_i(\hat{s}) \leq C_i(s_i, \hat{s}_{-i}),
\]

where \( \hat{s}_{-i} \) denotes the strategy profile of all players except \( i \). The set of PNE of this game is denoted by \( \text{NE}(\Gamma) \). Since \( \Gamma \) is an exact potential game, the set \( \text{NE}(\Gamma) \) is nonempty (see Rosenthal, 1973).

We define the social cost (SC) to be the sum of all players’ costs:

\[
C(s) = \sum_{i \in N} C_i(s) = \sum_{e \in E} N_e(s) \cdot c_e(N_e(s)) \quad \text{(3.4)}
\]

and we call a social optimum (SO) any strategy profile \( s^* \) that minimizes this aggregate cost. The price of anarchy and price of stability are then defined as

\[
\text{PoA}(\Gamma) = \max_{s \in \text{NE}(\Gamma)} \frac{C(s)}{C(s^*)} \quad \text{and} \quad \text{PoS}(\Gamma) = \min_{s \in \text{NE}(\Gamma)} \frac{C(s)}{C(s^*)} \quad \text{(3.5)}
\]
comparing the worst and best equilibria, respectively, to the best that a social planner can achieve. These concepts quantify the quality of equilibria, when one takes a pessimistic or optimistic perspective for equilibrium selection, respectively.

So far, we considered particular instances \( \Gamma \). Our main goal is to find the quality of equilibria across all possible games belonging to a class. More precisely, given a family \( \mathcal{C} \) of cost functions, we call \( \mathcal{G}(\mathcal{C}) \) the class of all atomic congestion games \( \Gamma \) with costs \( c_e \in \mathcal{C} \), and we set

\[
\text{PoA}(\mathcal{C}) := \sup_{\Gamma \in \mathcal{G}(\mathcal{C})} \text{PoA}(\Gamma) \quad \text{and} \quad \text{PoS}(\mathcal{C}) := \sup_{\Gamma \in \mathcal{G}(\mathcal{C})} \text{PoS}(\Gamma).
\] (3.6)

3.2. Bernoulli demand. Given an ACG \( \Gamma \), we consider situations where each player \( i \in \mathcal{N} \) participates in the game with probability \( p_i \), and otherwise remains inactive. An inactive player incurs no cost. The random variables \( W_i \sim \text{Bernoulli}(p_i) \) that indicate whether player \( i \) is active are assumed to be independent across players. As in most prior literature, we assume that players choose their strategies before observing the actual realization of these random variables, so that no player knows for sure who will be present in the game.\(^1\) Given \( \mathbf{p} = (p_i)_{i \in \mathcal{N}} \), the resulting Bernoulli congestion game (BCG) is denoted by \( \Gamma^\mathbf{p} \).

This can be framed as a game with incomplete information, for which the standard solution concept is Bayesian-Nash equilibrium. For a strategy profile \( \mathbf{s} = (s_i)_{i \in \mathcal{N}} \), we call \( \mathbb{1}_{\{e \in s_i\}} \) the indicator of the event \( e \in s_i \), and we define the random resource loads

\[
N_e(s) = \sum_{i \in \mathcal{N}} W_i \mathbb{1}_{\{e \in s_i\}}, \quad \text{and} \quad N_e^{-i}(s) = \sum_{j \neq i} W_j \mathbb{1}_{\{e \in s_j\}} \quad \text{for all} \quad i \in \mathcal{N},
\] (3.7)

considering all the players or excluding player \( i \), respectively. The total expected cost for player \( i \in \mathcal{N} \) is then given by

\[
C_i(s) = \mathbb{E} \left[ W_i \sum_{e \in s_i} c_e(N_e(s)) \right] = \sum_{e \in s_i} p_i \mathbb{E} \left[ c_e(1 + N_e^{-i}(s)) \right].
\] (3.8)

With a slight overload of notation, we use the same notation as in Eq. (3.1) for the total expected cost for a player. More explicitly, setting \( p(\mathcal{I}) = \prod_{j \in \mathcal{I}} p_j \prod_{j \notin \mathcal{I}} (1 - p_j) \) for each \( \mathcal{I} \subseteq \mathcal{N} \), we have

\[
C_i(s) = \sum_{e \in s_i} \sum_{\mathcal{I} \ni i} p(\mathcal{I}) c_e(\{|j \in \mathcal{I} : e \in s_j\}|).
\] (3.9)

Clearly, in the deterministic case with \( p_i = 1 \) for all \( i \in \mathcal{N} \), the definition of \( N_e(s) \) in Eq. (3.7) coincides with the one in Eq. (3.2), and the cost in Eq. (3.8) is the same as the one in Eq. (3.1). The notions of equilibrium, SO, PoA, and PoS are extended as follows.

**Definition 3.1.** A strategy profile \( \hat{s} \) is a Bayesian-Nash equilibrium (BNE) for \( \Gamma^\mathbf{p} \) if, for each \( i \in \mathcal{N} \) and every \( s_i \in \mathcal{S}_i \), we have \( C_i(\hat{s}) \leq C_i(s_i, \hat{s}_{-i}) \). The set of all such BNE is denoted by \( \text{NE}(\Gamma^\mathbf{p}) \). A social optimum (SO) is any strategy profile \( \mathbf{s}^* \) that minimizes the expected social cost (ESC)

\[
C(s) = \sum_{i \in \mathcal{N}} C_i(s) = \sum_{e \in \mathcal{E}} \mathbb{E} [N_e(s) c_e(N_e(s))],
\] (3.10)

while PoA(\( \Gamma^\mathbf{p} \)) and PoS(\( \Gamma^\mathbf{p} \)) are defined as in Eq. (3.5) considering this expected social cost.

\(^1\)In the context of routing games, Nguyen and Pallottino (1988), Miller-Hooks (2001), Marcotte et al. (2004) considered a richer set of strategies called hyperpaths in which players are allowed to update their priors along their journey and switch to alternative routes.
Example 3.1. Consider a routing game with an arbitrary number of players \( n = 2k \) on the Pigou network shown in Fig. 2 in which all players share a common origin \( O \) and destination \( D \). Assume that \( p_i = p > 0 \) for all \( i \in \mathcal{N} \). The cost function on the top link is linear while the cost function in the bottom link is constant. Note that the constant depends on the number of players and on the probability. It is suitably chosen to generate the lower bound on the price of anarchy.

\[
\Phi(s) := \mathbb{E} \left[ \sum_{e \in \mathcal{E}} \sum_{k=1}^{N_e(s)} c_e(k) \right],
\]

we have

\[
\Phi(s) - C_i(s) = \mathbb{E} \left[ \sum_{e \in \mathcal{E}} \sum_{k=1}^{N_e(s)} c_e(k) \right].
\]

We claim that for each player \( i = 1, \ldots, 2k \) the upper path is a strictly dominant strategy. Indeed, in every strategy profile there are at most \( 2k \) players on the upper link, and thus its expected cost is at most \( p \cdot (1 + (2k-1) \cdot p) \). This is strictly less than the constant cost of the lower link multiplied by \( p \). Hence in the unique Bayesian-Nash equilibrium \( \bar{s} \) all players use the upper path, whereas an optimal profile \( s^* \) is achieved by sending \( k \) players on each path. This implies

\[
\text{PoA}(\Gamma^P) = \text{PoS}(\Gamma^P) = \frac{2kp \cdot (1 + (2k-1) \cdot p)}{kp \cdot (1 + (k-1)p) + kp \cdot (1 + 2kp)} = \frac{4kp + 2 - 2p}{3kp + 2 - p},
\]

which increases towards \( 4/3 \) as \( k \) grows to \( \infty \), for any \( p \).

We will prove that the set of Bayesian-Nash equilibria is nonempty by showing that every BCG is a potential game. We recall that a cost game \( \Gamma \) is called an exact potential game if there exists a potential function \( \Phi: \mathcal{S} \to \mathbb{R} \) such that, for each strategy profile \( s \) and any unilateral deviation \( s_0 = (s'_i, s_{-i}) \) by a player \( i \), we have

\[
C_i(s') - C_i(s) = \Phi(s') - \Phi(s).
\]

As shown by Monderer and Shapley (1996), every finite exact potential game admits a pure equilibrium. The next result shows that BCGs are exact potential games.

Proposition 3.2. Every BCG \( \Gamma^P \) is an exact potential game. In particular \( \text{NE}(\Gamma^P) \) is nonempty.

The idea of the proof is basically the following: Every congestion game with a deterministic number of players is an exact potential game. So, if we condition on \( \mathcal{I} \subseteq \mathcal{N} \) being the set of active players, we still have an exact potential game as the inactive players do not affect the costs of the active players and incur a cost which is identically zero. Since each such subgame is played with a fixed probability, the average of exact potential games having the same set of players is also an exact potential game. Hence, the result.

Proof. Considering the expected Rosenthal’s potential

\[
\Phi(s) := \mathbb{E} \left[ \sum_{e \in \mathcal{E}} \sum_{k=1}^{N_e(s)} c_e(k) \right],
\]

Figure 2. A Pigou network. The arcs are annotated with their cost functions \( c_e(x_e) \).

\[
\text{PoA}(\Gamma^P) = \text{PoS}(\Gamma^P) = \frac{4kp + 2 - 2p}{3kp + 2 - p},
\]

we have

\[
\Phi(s) - C_i(s) = \mathbb{E} \left[ \sum_{e \in \mathcal{E}} \sum_{k=1}^{N_e(s)} c_e(k) \right].
\]
Since the latter does not depend on \( s_i \), it follows that for each unilateral deviation \( s' = (s'_i, s_{-i}) \) by a player \( i \) we have \( \Phi(s) - C_i(s) = \Phi(s') - C_i(s') \).

3.3. **Homogeneous probabilities.** A specially relevant case is that of homogeneous players with \( p_i = p \) for all \( i \in \mathcal{N} \). In this case, the expected cost \( c^p_e : \mathbb{N} \to \mathbb{R}_+ \) for a player using the resource \( e \) is given by

\[
c^p_e(x_e) = p \cdot E[c_e(1 + X_e)],
\]

where \( X_e \sim \text{Binomial}(x_e - 1, p) \) and \( x_e = \sum_{i \in \mathcal{N}} \mathbb{1}_{\{e \in s_i\}} \) is the number of players—active and inactive—who choose to use resource \( e \). It follows that a BCG \( \Gamma^p \) with homogeneous probabilities is equivalent to a standard deterministic ACG in which the costs \( c_e(\cdot) \) are replaced by \( c^p_e(\cdot) \).

To illustrate the construction of the expected cost function, consider the game in Example 3.1. If \( p_i = p \) for all \( i \in \mathcal{N} \), then the modified cost of the upper edge \( e \) is

\[
c^p_e(x_e) = p \cdot E[c_e(1 + X_e)] = p(1 + (x_e - 1)p).
\]

This is upper bounded by \( p(1 + 2kp) \) for every \( x_e \leq 2k \), which shows that, whenever the number of players is at most \( 2k \), the upper edge is a strictly dominant strategy.

The relevance of the homogeneous case is made clear in our next result. It shows that in order to obtain tight bounds for the PoA in BCGs, it suffices to focus on the case of homogeneous players with \( p = \max_{i \in \mathcal{N}} p_i \), and to check the \((\lambda, \mu)\)-smoothness condition (A.2) for the costs \( c^p_e(\cdot) \) in the equivalent deterministic game.

**Definition 3.3.** Given a class of cost functions \( \mathcal{C} \), we call \( \mathcal{C}^p \) the set of all the corresponding functions \( c^p(\cdot) \) defined as in (3.14) for each \( c \in \mathcal{C} \).

**Theorem 3.4.** Let \( \Gamma^p \) be a BCG with \( p = (p_i)_{i \in \mathcal{N}} \) and let \( q \geq \max_{i \in \mathcal{N}} p_i \). If the resource costs \( c_e(\cdot) \) belong to the class \( \mathcal{C} \), then \( \text{PoA}(\Gamma^p) \leq \text{PoA}(\mathcal{C}^q) \). In particular, this worst-case bound is attained when all the players are homogeneous with \( p_i = q \).

As a consequence of the previous result we obtain the following monotonicity of the PoA with respect to \( p \).

**Corollary 3.5.** For every family of cost functions \( \mathcal{C} \), the map \( p \mapsto \text{PoA}(\mathcal{C}^p) \) is nondecreasing.

**Proof.** Let \( 0 \leq p \leq q \leq 1 \). As noted before, every deterministic ACG \( \Gamma \in \mathcal{G}(\mathcal{C}^p) \) is equivalent to a Bernoulli congestion game \( \Gamma^p \) with homogeneous probabilities \( p_i = p \), so that the previous result yields \( \text{PoA}(\Gamma) = \text{PoA}(\Gamma^p) \leq \text{PoA}(\mathcal{C}^q) \). The conclusion follows by taking the supremum over \( \Gamma \in \mathcal{G}(\mathcal{C}^p) \).

A further consequence of Theorem 3.4 is that for a fixed number of players \( n \) and under a mild growth condition on the family of costs \( \mathcal{C} \), the price of anarchy converges to 1 as the probabilities \( p_i \) tend to 0. In the next section we will see that this is no longer the case if the number of players is not fixed.

**Proposition 3.6.** Let \( \text{PoA}(\mathcal{C}, n, q) \) denote the supremum of \( \text{PoA}(\Gamma^p) \) over all BCGs \( \Gamma^p \) with cost functions in \( \mathcal{C} \), a fixed number of players \( n \), and \( p_i \leq q \) for \( i = 1, \ldots, n \). Suppose that there exists a constant \( \sigma \) such that

\[
c(k) \leq \sigma c(1) \quad \forall k = 1, \ldots, n, \forall c \in \mathcal{C}.
\]

Then \( \text{PoA}(\mathcal{C}, n, q) \to 1 \) as \( q \to 0 \).
The condition in Eq. (3.16) holds trivially when the family $C$ is finite. This is the case when we consider a fixed graph $G$ with given costs $c_e(\cdot)$ and a fixed number of players, and we study the behavior of the PoA when $\max_{i \in \mathcal{N}} p_i \to 0$. Another interesting case is when $C$ is the class of all polynomials with nonnegative coefficients and maximum degree $d$. Indeed, if $c(k) = \sum_{i=0}^{d} a_i k^i$ with $a_i \geq 0$, then for $k = 1, \ldots, n$ we have $k^i \leq n^d$. Therefore

$$c(k) \leq n^d \sum_{i=0}^{d} a_i = n^d c(1),$$

so that Eq. (3.16) holds with $\sigma = n^d$.

4. Bernoulli congestion games with affine costs

For the rest of this paper we focus on atomic BCGs with nondecreasing and nonnegative affine costs, that is, we restrict the attention to the class $C_0$ of costs of the form $c(x) = ax + b$ with $a, b \geq 0$.

4.1. Tight upper bounds for PoA. The following theorems are the main results of our work.

**Theorem 4.1.** Let $C_0^p$ consist of all functions $c^p(\cdot)$ defined in Eq. (3.14) for each $c$ in the class $C_0$ of nondecreasing, nonnegative and affine functions. For the class of atomic BCGs, we define $\text{PoA}(p) = \text{PoA}(C_0^p)$ for each $p \in (0, 1]$. Moreover, let $\bar{p}_0 = 1/4$ and let $\bar{p}_1 \sim 0.3774$ be the real root of $8p^3 + 4p^2 = 1$. Then,

$$\text{PoA}(p) = \begin{cases} 
4/3 + \frac{p^2}{1 + p} & \text{if } 0 < p \leq \bar{p}_0, \\
1 + p + \sqrt{p(2 + p)} & \text{if } \bar{p}_0 \leq p \leq \bar{p}_1, \\
1 - p + \sqrt{p(2 + p)} & \text{if } \bar{p}_1 \leq p \leq 1.
\end{cases}$$

(4.1)

![Figure 3](image-url) The three curves correspond to the different bounds for PoA in the three regions. The upper envelope gives the global upper bound as a function of $p$. 

Our main result extends the previous theorem to the case of heterogeneous probabilities and shows that the price of anarchy is bounded by that corresponding to the maximum participation probability of all players. The result follows from combining the previous theorem with Theorem 3.4.

**Theorem 4.2.** Consider the class $\Gamma^p$ of atomic BCG with nondecreasing, nonnegative and affine costs and players with heterogeneous participation probabilities $p = (p_i)_{i \in \mathcal{N}}$. Then, $\text{PoA}(\Gamma^p) \leq \text{PoA}(q)$, for $q = \max_{i \in \mathcal{N}} p_i$.

4.2. **Routing games with purely linear costs are tight.** The following examples show that the upper bounds in Theorem 4.1 are tight and are in fact attained (at least asymptotically) by network routing games with purely linear costs (as opposed to affine costs as in Example 3.1). We proceed in order with three examples that address the three regimes

- $\bar{p}_0 = \frac{1}{4}$
- $\bar{p}_1 = \frac{1}{3774}$

**Example 4.1.** Let $k \in \mathbb{N}$ and consider a routing game with $n = 2k$ players on the bypass network $B_k$ shown in Fig. 4. Assume that $p_i = p > 0$ for all $i \in \mathcal{N}$. Players $i = 1, \ldots, k$ have two strategies, $s_i$ and $\bar{s}_i$, to travel from origin $O_i$ to destination $D_i$. Strategy $s_i$ consists of an exclusive direct link $e_i$ with cost $c_i(x) = x$, while the bypass strategy $\bar{s}_i$ uses a faster shared link $\bar{e}$ with cost

$$c(\bar{e}) = \frac{1}{1 + 2kp} \cdot x$$

connected to $O_i$ and $D_i$ by zero cost links (dashed). The remaining players $i = k + 1, \ldots, 2k$ have a common origin $\bar{O}$ and destination $\bar{D}$ with a unique strategy $\bar{s}_i$ using the shared link $\bar{e}$.

![Figure 4. The bypass network $B_5$. Dashed links have zero cost.](image)

We claim that for each player $i = 1, \ldots, k$ the bypass $\bar{s}_i$ is a strictly dominant strategy. Indeed, in every strategy profile there are at most $2k$ players on $\bar{e}$, and thus, for all $s'_{-i} \in S_{-i},$

$$c_i(\bar{s}_i, s'_{-i}) \leq p \cdot \frac{1}{1 + 2kp} \cdot (1 + (2k - 1)p) = c_i(s_i, s'_{-i}). \quad (4.2)$$

Hence, in the unique BNE all players use $\bar{s}_i$, whereas in the optimal profile $s^*$ players $i = 1, \ldots, k$ use their exclusive route $s^*_i = s_i$ and players $i = k + 1, \ldots, 2k$ use their only available strategy
where we recall that $\text{PoA}(p) = \text{PoA}(C_0^p)$ and define $\text{PoS}(p) = \text{PoS}(C_0^p)$ for all $p \in (0, 1]$. This quantity increases towards $4/3$ as $k$ grows to $\infty$. In particular, it follows that for $p \in (0, 1/2]$ we have in fact $\text{PoA}(p) = \text{PoS}(p) = 4/3$.

**Example 4.2.** Consider a pair of integers $m > k \geq 1$ and set $n = m + k$. We build a graph $G_{k,m}$ consisting of a roundabout with $n$ edges of the form $(A_i, B_i)$, with linear costs $h_i(x) = \gamma x$, where connected by zero-cost links $(B_i, A_{i+1})$ (modulo $n$). Notice that $\gamma > 0$. Additionally there are $n$ exit arcs $(B_i, F_i)$ with costs $g_i(x) = x$. Fig. 5 illustrates the roundabout network $G_{2,4}$.

**Figure 5.** The roundabout network $G_{2,4}$. For clarity only the origin and destination for player $i = 1$ are shown. The corresponding strategies are $s_i^* = \{h_1, h_2, g_2\}$ and $s_1 = \{h_3, h_4, h_5, g_6\}$. Dashed links have zero cost.

Consider players $i = 1, \ldots, n$ with $p_i = p > 0$. Players have origin nodes $O_i$, each of which has two outgoing links connecting to the roundabout at the nodes $A_i$ and $A_{i+k}$ (modulo $n$). Similarly, players have destination nodes $D_i$, each of which can be reached from the exit nodes $F_{i+k-1}$ and $F_{i+k+m-1}$ (modulo $n$). Each player $i$ has two undominated strategies that consist of entering the roundabout through one of the two available entrances and proceeding clockwise to the closest exit leading to $D_i$: (1) the short route $s_i^* = \{h_i, \ldots, h_{i+k-1}, g_{i+k-1}\}$, which uses $k$ resources of type $h_j$ and only one $g_j$, and (2) the long route $s_i = \{h_{i+k}, \ldots, h_{i+k+m-1}, g_{i+k+m-1}\}$, which uses $m$ resources of type $h_j$ and only one $g_j$. 

\[ s_i^* = s_i. \] This yields the lower bound

\[ \text{PoA}(p) \geq \frac{C(s) - C(s^*)}{C(s^*)} = \frac{2kp \cdot \frac{1}{1 + 2kp} \cdot (1 + (2k - 1)p)}{kp \cdot \frac{1}{1 + 2kp} \cdot (1 + (k - 1)p) + kp} = \frac{4kp + 2 - 2p}{3kp + 2 - p}, \]

\[ s_i^* = s_i. \] This yields the lower bound
If all players choose the long route $s_i$, then each $h_j$ has a load of $m$ players and each $g_j$ a load of 1, so that every player experiences the same cost

$$p[m\gamma(1 - p + pm) + 1].$$

Shifting individually to the short route $s_i^*$ implies the cost

$$p[k\gamma(1 + pm) + 1 + p],$$

so that, by the choice of $\gamma$, all players using $s_i$ constitutes an equilibrium. The social cost of this equilibrium is

$$C(s) = np[m\gamma(1 - p + pm) + 1].$$

Now, the feasible routing where all players use the short route $s_i^*$ gives an upper bound for the optimal social cost. In this case the loads are $k$ on each $h_j$ and again 1 on each $g_j$, so that

$$C(s^*) \leq np[k\gamma(1 - p + pk) + 1],$$

which yields the following lower bound for the PoA

$$\text{PoA}(p) \geq \frac{(1 + p)m(1 - p + pm) - k(1 + pm)}{pk(1 - p + pk) + m(1 - p + pm) - k(1 + pm)}.$$  \quad (4.4)

Take $z = 1 + p + \sqrt{p(2 + p)}$ and $m = \lceil zk \rceil$. Then $m > k$ for $k$ large enough. In fact,

$$\frac{m}{k} = \frac{\lceil zk \rceil}{k} \rightarrow z, \quad \text{as} \ k \rightarrow \infty.$$

With this choice of $m$ both the numerator and denominator in Eq. (4.4) grow quadratically with $k$, so that dividing by $k^2$ and letting $k \rightarrow \infty$ we get the asymptotic lower bound

$$\text{PoA}(p) \geq \frac{(1 + p)z^2 - z}{p + z^2 - z} = \frac{1 + p + \sqrt{p(2 + p)}}{1 - p + \sqrt{p(2 + p)}}.$$  \quad (4.5)

In particular, it follows that for $p \in [\tilde{p}_0, \tilde{p}_1]$, the previous bound (4.5) for $\text{PoA}(p)$ is tight.

**Example 4.3.** Consider the network congestion game of Fig. 6. The game contains 3 players, 6 costly resources $\{h_1, h_2, h_3, g_1, g_2, g_3\}$, and 15 connecting links (the dashed links). Assume that $p_i = p > 0$ for all $i \in \mathcal{N}$. The cost functions are $c_e(x) = p \cdot x$ for $e \in \{h_1, h_2, h_3\}$ and $c_e(x) = x$ for $e \in \{g_1, g_2, g_3\}$, whereas the dashed links have zero cost. Ignoring the dashed links, each player $i$ has two pure strategies $\{h_i, g_i\}$ and $\{h_{i-1}, h_{i+1}, g_{i+1}\}$ (all indices are modulo 3).

A strategy profile $s$ is a BNE if $s_i = \{h_{i-1}, h_{i+1}, g_{i+1}\}$ for all $i \in \mathcal{N}$, since

$$2p(p + 1) + 1 \leq p(2p + 1) + (p + 1).$$

The corresponding expected total costs are

$$C(s) = 3(p^2 \cdot 4p + 2p(1 - p) \cdot p) + 3p.$$ 

Second, the strategy profile $s^*$ in which $s_i = \{h_i, g_i\}$ yields an expected total cost of

$$C(s^*) = 3(p^2 + p).$$

Therefore,

$$\text{PoA}(p) \geq \frac{3p(1 + 2p + 2p^2)}{3p(1 + p)} = 1 + p \frac{p^2}{1 + p}.$$ 

In particular, it follows that the previous bound for $\text{PoA}(p)$ is tight for $p \in [\tilde{p}_1, 1]$. 


Figure 6. The triangle network. The pure strategies \( \{h_i, g_i\} \) and \( \{h_{i-1}, h_{i+1}, g_{i+1}\} \) for player \( i = 1 \) are highlighted in red and blue respectively. Dashed links have zero cost.

We now switch to bounding the price of stability to complement the prior upper and lower bounds on the PoA, and then conclude by putting these results in perspective in Section 5.

4.3. Price of stability. We conclude by establishing tight bounds for the PoS in the full range \( p \in [0, 1] \) (see Fig. 1). This result is a consequence of Kleer and Schäfer (2019, Theorem 5), except in the range \( p \leq 1/4 \), which follows from our results in the previous sections.

**Theorem 4.3.** Let \( C_0^p \) consist of all functions \( c^p(\cdot) \) defined in Eq. (3.14) for each \( c \) in the class \( C_0 \) of nondecreasing, nonnegative and affine functions. For the class of atomic BCGs, we define \( \text{PoS}(p) = \text{PoS}(C_0^p) \) for each \( p \in (0, 1] \). Then,

\[
\text{PoS}(p) = \begin{cases} 
4/3 & \text{if } 0 < p \leq 1/4, \\
1 + \sqrt{p/(2 + p)} & \text{if } p \geq 1/4.
\end{cases}
\]

**Proof.** Kleer and Schäfer (2019, Theorem 5) already established this tight bound for all \( p \geq 1/4 \). In the lower range \( p \leq 1/4 \), using Theorem 4.1, we have

\[
\text{PoS}(p) \leq \text{PoA}(p) \leq 4/3,
\]

and we conclude by noting that the family of linear cost routing games in Example 4.1 have a unique equilibrium with PoS arbitrarily close to 4/3.

5. Conclusions

Roughgarden (2015b) has shown that, when dealing with games of incomplete information, the bounds for the corresponding games of complete information are still valid. His framework for incomplete information games is very robust, but requires a smoothness definition that holds across different types (see Roughgarden, 2015b, Definition 3.1 and Remark 3.2). A result in the same spirit appears in Correa et al. (2019), where it is shown that the bound of 5/2 holds for
the PoA of BCGs with affine costs even if the events of players being active are not i.i.d. These authors consider a class of games and an information structure that makes these objects games of incomplete information; then they compute bounds for the PoA of games in this class over all possible probabilities that characterize the incomplete information. They show the remarkable result that the performance of the PoA does not decay in the presence of incomplete information.

Our results are in a different spirit. We fix not only the class of games and the information structure, but also the probability measure and examine the behavior of the PoA as this probability varies. In our case, when the probability is characterized by a single parameter \( p \), this is tantamount as studying the PoA as a univariate function of this parameter. This means that for a fixed value of \( p \), we consider the worst-case PoA among all possible instances where participation probabilities of players are bounded above by \( p \). The main results in this respect are:

1. The worst-case instances for the PoA occur when all the players have the same probabilities \( p_i = p \).
2. The map \( p \mapsto \text{PoA}(p) \) is monotone, which partially can be explained by the fact that a larger \( p \) implies a larger expected congestion.
3. The presence of two kinks in this function for affine costs.
4. The fact that for \( p \in (0, 1/4] \), the parameterized \( \text{PoA}(p) \) is constant and equal to \( 4/3 \), exactly as in nonatomic congestion games with affine costs, whereas the maximum of \( 5/2 \), which is the PoA in the atomic case, is only attained in the limit as \( p \to 1 \).

Several natural questions remain open. First, a precise characterization of the PoA as a function of \( p \) is only available for affine cost functions. What happens for more general cost functions, for example, polynomials with nonnegative coefficients and degree at most \( d \)? We know the PoA as a function of \( p \) is nondecreasing, but can we say more?

Second, our upper bound for heterogeneous probabilities depends on the maximum probability of each of the players. Are there any other natural parameters that this might depend on? The following two instances provide evidence that we cannot expect a monotonicity result if we consider the average number of players in the game as our parameter.

Considering Example 4.3 with \( p = 1 \), we compare the situation in which 3 players participate with probability 1, and 3 duplicate players participate with probability 0 to the situation in which all six player participate with probability 1/2. In expectation, the same number of players participate. However, in the former case, the PoA is 5/2, whereas in the latter case we know by Theorem 4.1 that the PoA is at most 5/3.

Now, consider a two-link parallel network with \( c_{e_1}(x) = x \) and \( c_{e_2}(x) = 5/2 \), and compare the situation in which 2 players participate with probability 1, and 2 players participate with probability 0 to the situation in which all four player participate with probability 1/2. In the former case, for both players it is a dominant strategy to choose \( e_1 \), yielding a PoA of \( 8/7 \), whereas in the latter case the equilibrium in which all players choose \( e_1 \) yields a PoA of at least 5/4.

Third, it may be interesting to consider other information regimes for the social optimum. We assume that the social optimum is the strategy profile that minimizes the social expected costs, but what if the central planner is able to relocate players after the realization of the random variables is known? This would imply a stronger version of the social optimum, the prophet optimum, and bounds against this social optimum would be interesting.

Fourth, an interesting direction would be to further look at convergence results as \( p \to 0 \) and see other kinds of limit games that might arise. We have already looked at the cases when the number of players is bounded or arbitrary. Cominetti et al. (2020) and more recently Wu et al.
(2021b) studied the convergence of sequences of atomic unsplittable congestion games with an increasing number of players and probabilities that tend to zero. In this case, the mixed equilibria converge to the set of Wardrop equilibria of another nonatomic game with suitably defined costs, which can be seen as a Poisson game. It shows that the price of anarchy of the sequence of games converges to the price of anarchy of the nonatomic limit.

**Appendix A. Proofs**

A handy tool for bounding the PoA is provided by the concept of smoothness. We recall that a game \( \Gamma \) is said to be \((\lambda, \mu)\)-smooth with \( \lambda \geq 0 \) and \( \mu \in [0,1) \) if

\[
\sum_{i \in N} C_i(s'_i, s_{-i}) \leq \lambda C(s') + \mu C(s) \quad \forall s, s' \in \mathcal{S}.
\]  

**Lemma A.1** (Roughgarden (2015a)). If a game \( \Gamma \) is \((\lambda, \mu)\)-smooth, then \( \text{PoA}(\Gamma) \leq \lambda/(1-\mu) \).

Our main goal is to use the above lemma to find the quality of equilibria across all possible games belonging to a class. For games in \( \mathcal{G}(C) \) the smoothness condition (A.1) can be translated in terms of the resource costs

\[
k c(m + 1) \leq \lambda k c(k) + \mu m c(m) \quad \forall k, m \in \mathbb{N}, \forall c \in C,
\]  

and, according to Roughgarden (2015a, Theorem 5.8), we have

\[
\text{PoA}(C) = \inf \left\{ \frac{\lambda}{1 - \mu} : \lambda \geq 0, \mu \in [0,1), \text{ subject to (A.2)} \right\}.
\]  

**Proofs of Section 3.3.**

**Proof of Theorem 3.4.** Let \( W_i \sim \text{Bernoulli}(p_i) \) be the indicator of the event that player \( i \) is active. Set \( r_i = p_i/q \in [0,1] \) and take independent random variables \( Y_i \sim \text{Bernoulli}(r_i) \) and \( Z_i \sim \text{Bernoulli}(q) \), so that \( Z_i Y_i \sim \text{Bernoulli}(p_i) \). Then, for each strategy profile \( s = (s_i)_{i \in N} \) the expected cost for player \( i \) is

\[
C_i(s) = E \left[ W_i \sum_{e \in s_i} c_e(N_e(s)) \right]
\]

\[
= p_i E \left[ \sum_{e \in s_i} c_e \left( 1 + \sum_{j \neq i} W_j 1_{\{e \in s_j\}} \right) \right]
\]

\[
= r_i \sum_{e \in s_i} E \left[ q c_e \left( 1 + \sum_{j \neq i} Z_j Y_j 1_{\{e \in s_j\}} \right) \right]
\]

\[
= r_i \sum_{e \in s_i} \left[ E \left[ q c_e \left( 1 + \sum_{j \neq i} Z_j Y_j 1_{\{e \in s_j\}} \right) \right] \left[ \sum_{j \neq i} Y_j 1_{\{e \in s_j\}} \right] \right],
\]  

where the last equality follows from the law of total expectation. Now, conditionally on the event \( \sum_{j \neq i} Y_j 1_{\{e \in s_j\}} = k \), we have that

\[
\sum_{j \neq i} Z_j Y_j 1_{\{e \in s_j\}} \sim \text{Binomial}(k, q)
\]  

(A.5)
and then the inner conditional expectation given \( \sum_{j \neq i} Y_j \mathbb{1}_{\{e \in s_j\}} = k \) becomes \( c^q_e(1 + k) \), so that

\[
C_i(s) = r_i \sum_{e \in s_i} E \left[ c^q_e \left( 1 + \sum_{j \neq i} Y_j \mathbb{1}_{\{e \in s_j\}} \right) \right] = E \left[ Y_i \sum_{e \in s_i} c^q_e \left( \sum_{j \neq i} Y_j \mathbb{1}_{\{e \in s_j\}} \right) \right].
\] (A.6)

This shows that the original game \( \Gamma^p \) is equivalent to a Bernoulli congestion game with probabilities \( (r_i)_{i \in N} \) and costs \( c^q_e(\cdot) \in C^q \). Now, from Correa et al. (2019, Theorem 5.3), it follows that any \((\lambda, \mu)\)-smoothness bound for the deterministic game with costs \( c^q_e(\cdot) \) remains valid for the BCG with probabilities \( (r_i)_{i \in N} \), and therefore also for the original game \( \Gamma^p \). The conclusion then follows from the identity (A.3).

\[\Box\]

**Proof of Proposition 3.6.** In view of Theorem 3.4, it suffices to show that \((\lambda, \mu)\)-smoothness holds with \( \mu = 0 \) and some \( \lambda = \lambda(q) \) such that \( \lambda(q) \to 1 \) when \( q \to 0 \), namely, we require

\[
k \ c^q(m + 1) \leq \lambda \ k \ c^q(k) \quad \forall k \geq 1, m \geq 0, \ \forall c \in C.
\]

Clearly, this inequality is equivalent to \( c^q(m + 1) \leq \lambda \ c^q(k) \). On the other hand, since the number of players \( n \) is fixed, in any strategy profile the load of any given resource is at most \( n \). Therefore it suffices to have this inequality for \( k \) and \( m + 1 \) smaller than \( n \). Now, taking \( X \sim \text{Binomial}(m, q) \) we have

\[
c^q(m + 1) = q \ E[c(1 + X)]
= q \ P(X = 0) \ c(1) + q \ P(X > 0) \ E[c(1 + X) \mid X > 0]
\leq q \ P(X = 0) \ c(1) + q \ P(X > 0) \ \sigma \ c(1)
\leq q \ c(1) + q \ (1 - (1 - q)^m) \ \sigma \ c(1)
\]

where the last inequality follows from \( P(X = 0) \leq 1 \) and \( P(X > 0) = 1 - (1 - q)^m \leq 1 - (1 - q)^n \). Similarly, since the costs \( c(\cdot) \) are non-negative, taking \( X \sim \text{Binomial}(k - 1, q) \) we get

\[
c^q(k) = q \ P(X = 0) \ c(1) + q \ P(X > 0) \ E[c(1 + X) \mid X > 0]
\geq q \ P(X = 0) \ c(1)
= q \ (1 - q)^{k-1} \ c(1)
\geq q \ (1 - q)^n \ c(1).
\]

The latter yields \( q \ c(1) \leq (1 - q)^n c^q(k) \), which, combined with the previous inequality for \( c^q(m + 1) \), yields the desired conclusion with \( \lambda(q) = (1 - q)^{-n} \left[ 1 + \sigma(1 - (1 - q)^n) \right] \). \[\Box\]

**Proofs of Section 4.1.** For simplicity, for each \( p \in [0, 1] \) we define \( G^p = G(C^p_0) \). The first fundamental proposition to prove Theorem 4.1 will show that the class of games \( G^p \) is \((\lambda, \mu)\)-smooth and will provide values for the parameters \( \lambda \) and \( \mu \).
Proposition A.2. Let \( \rho_0 = 1/4 \) and let \( \rho_1 \sim 0.3774 \) be the real root of \( 8p^3 + 4p^2 = 1 \). Any game in the class \( G^p \) is \((\lambda, \mu)\)-smooth with parameters

\[
(\lambda, \mu) = \begin{cases}
  \left( \frac{1}{4}, 1 \right) & \text{if } 0 < \rho_0 \leq \rho_0, \\
  \left( \frac{1 + p + \sqrt{p(2 + p)}}{2}, \frac{1 + p - \sqrt{p(2 + p)}}{2} \right) & \text{if } \rho_0 \leq \rho \leq \rho_0, \\
  \left( \frac{1 + 2p + 2p^2}{1 + 2p}, \frac{p}{1 + 2p} \right) & \text{if } \rho_0 \leq \rho \leq 1.
\end{cases}
\]

We split the proof of Proposition A.2 into three technical lemmas and three propositions, each one dealing with one of the three subregions of \([0, 1]\) determined by \( \rho_0 \) and \( \rho_1 \). We briefly sketch the main steps of the proof.

First we observe that, for each fixed \( p \), all games \( \Gamma \in G^p \) are \((\lambda, \mu)\)-smooth, so we proceed to optimize (A.3) over the parameters \( \lambda \) and \( \mu \) to get the best possible bounds for the class of all congestion games in \( G^p \). In fact, in Section 4.2 we showed that these bounds are tight, even when restricted to the subclass of routing games with purely linear costs.

We next show that the optimization over \((\lambda, \mu)\) can be reduced to the minimization of a one-dimensional convex function \( \psi_p(y) \) over the region \( y \geq 0 \). This auxiliary function \( \psi_p(\cdot) \) is an upper envelope of a countable family of affine functions, and for each \( p \) it has a minimizer \( y_p \), which takes different values, depending on where \( p \) is located with respect to \( \rho_0 \) and \( \rho_1 \). This optimal solution yields the three alternative expressions for PoA(\( p \)), with the corresponding optimal smoothness parameters \((\lambda, \mu)\).

Our starting point is the following simple observation.

Lemma A.3. A pair \((\lambda, \mu)\) with \( \lambda \geq 0 \) and \( \mu \in [0, 1) \) satisfies Eq. (A.2) for the class \( C = C_0^p \) iff

\[
k(1 + mp) \leq \lambda k(1 - p + pk) + \mu m(1 - p + pm) \quad \forall k, m \in \mathbb{N}.
\]

Proof. For an affine function \( c(x) = ax + b \), we have \( c^p(x) = p[a(1 - p + px) + b] \). It follows that Eq. (A.8) is just the special case of Eq. (A.2) with \( a = 1 \) and \( b = 0 \). Conversely, if we start from Eq. (A.8), from \( k = 1 \) and \( m = 0 \) we can conclude that \( \lambda \geq 1 \). This implies that multiplying by \( a \geq 0 \) and adding \( kb \geq 0 \) on both sides we readily get Eq. (A.2) for \( c^p(\cdot) \). \(\square\)

We next show that the minimization in (A.3) can be reduced to a one-dimensional problem. As already noted, the inequality in Eq. (A.2), or its equivalent Eq. (A.8), is trivially satisfied for \( k = 0 \) so hereafter we consider the set \( P \) of all pairs \((k, m) \in \mathbb{N}^2 \) with \( k \geq 1 \). Then, for any given \( \mu \in [0, 1) \), the smallest possible value of \( \lambda \) compatible with Eq. (A.8) is

\[
\lambda = \sup_{(k, m) \in P} \frac{k(1 + pm) - \mu m(1 - p + pm)}{k(1 - p + pk)}
\]

\[
= \sup_{(k, m) \in P} \mu \left[ \frac{k(1 + pm) - m(1 - p + pm)}{k(1 - p + pk)} \right] + (1 - \mu) \frac{1 + pm}{1 - p + pk},
\]

from which it follows that

\[
\text{PoA}(p) = \inf_{\lambda, \mu} \frac{\lambda}{1 - \mu} = \inf_{\mu \in [0, 1)} \sup_{(k, m) \in P} \mu \left[ \frac{k(1 + pm) - m(1 - p + pm)}{k(1 - p + pk)} \right] + \frac{1 + pm}{1 - p + pk}.
\]
Defining
\[ y := \frac{\mu}{1 - \mu} \in [0, \infty) \] (A.9)
and introducing the functions
\[ \psi_{p}^{k,m}(y) = y \left[ \frac{k(1 + pm) - m(1 - p + pm)}{k(1 - p + pk)} \right] + \frac{1 + pm}{1 - p + pk}, \] (A.10)
\[ \psi_{p}(y) = \sup_{(k,m) \in P} \psi_{p}^{k,m}(y), \] (A.11)
we obtain the following equivalent expression for the optimal bound in Eq. (A.3):
\[ \text{PoA}(p) = \inf_{y \geq 0} \psi_{p}(y). \] (A.12)
If this infimum is attained at a certain \( y_{p} \), then we get
\[ \text{PoA}(p) = \psi_{p}(y_{p}) \] together with the corresponding optimal parameters
\[ \mu = \frac{y_{p}}{1 + y_{p}} \quad \text{and} \quad \lambda = (1 - \mu) \text{PoA}(p) = \frac{\psi_{p}(y_{p})}{1 + y_{p}}. \]

To proceed, we need the auxiliary function \( \psi_{\infty}(y) \) defined in the next lemma.

**Lemma A.4.** For all \( p > 0 \) and \( y > 0 \) the following limit is well defined and does not depend on \( p \)
\[ \psi_{\infty}(y) = \lim_{k \to \infty} \sup_{m \geq 0} \psi_{p}^{k,m}(y) = \frac{(y + 1)^{2}}{4y}. \] (A.13)
This function is strictly decreasing for \( y \in (0, 1) \) and strictly increasing for \( y \in (1, \infty) \).

**Proof.** Fix \( y > 0 \) and \( p \in (0, 1] \). The maximum of \( \psi_{p}^{k,m}(y) \) for \( m \in \mathbb{N} \) is attained at the integer \( \hat{m} \) that is closest to the unconstrained (real) maximizer (because \( \psi_{p}^{k,m}(y) \) is quadratic in \( m \))
\[ \hat{m} = \frac{(y + 1)pk - y(1 - p)}{2yp}. \] (A.14)
For a large \( k \), we have \( \hat{m} \geq 0 \) and we may find \( f \in (-\frac{1}{2}, \frac{1}{2}] \) such that \( \hat{m} = \bar{m} + f \). Then,
\[ \sup_{m \in \mathbb{N}} \left( (y + 1)pk - y(1 - p) \right)m - ypm^{2} = yp(2\hat{m} - \bar{m})\bar{m} \]
\[ = yp(\hat{m} + f)(\hat{m} - f) \]
\[ = yp(\hat{m}^{2} - f^{2}) \]
\[ = \frac{(y + 1)pk - y(1 - p))^{2}}{4yp} - ypf^{2}, \]
from which it follows that
\[ \psi_{\infty}(y) = \lim_{k \to \infty} \frac{(y + 1)k + \frac{1}{4yp}((y + 1)pk - y(1 - p))^{2} - ypf^{2}}{k(1 - p + pk)} = \frac{(y + 1)^{2}}{4y}. \]
The monotonicity claims follow at once by computing the derivative \( \psi_{\infty}'(y) = (y^{2} - 1)/(4y^2) \). \quad \square

The following lemma gathers some basic facts about the function \( \psi_{p} : [0, \infty) \to \mathbb{R} \) and shows in particular that its infimum is attained.
Lemma A.5. For each \( p > 0 \) the function \( \psi_p(\cdot) \) is convex and finite over \((0, \infty)\), with \( \psi_p(y) \to \infty \) both when \( y \to 0 \) and \( y \to \infty \). In particular, the minimum of \( \psi_p(\cdot) \) is attained at a point \( y_p > 0 \).

Proof. Convexity is obvious since \( \psi_p(\cdot) \) is a supremum of affine functions. The infinite limits at 0 and \( \infty \) follow by noting that \( \psi_p(y) \geq \psi_\infty(y) \) for \( y > 0 \), together with the fact that \( \psi_p(0) = \infty \) which results from letting \( m \to \infty \) in the inequality \( \psi_p(0) \geq \psi_p^{1,m}(0) = 1 + pm \to \infty \).

To show that \( \psi_p(y) < \infty \) for \( y \in (0, \infty) \), we rewrite the expression of \( \psi_p(y) \) as

\[
\psi_p(y) = \sup_{k \geq 1} \frac{1}{k(1 - p + pk)} \left( (y + 1)k + \sup_{m \geq 0} \left[ ((y + 1)pk - y(1 - p))m - ypm^2 \right] \right). \tag{A.15}
\]

Relaxing the inner supremum and considering the maximum with \( m \in \mathbb{R} \) we get

\[
\psi_p(y) \leq \sup_{k \geq 1} \frac{1}{k(1 - p + pk)} \left( (y + 1)k + \frac{((y + 1)pk - y(1 - p))^2}{4yp} \right).
\]

The latter is a quotient of two quadratics in \( k \) so it remains bounded and the supremum is finite. Since \( \psi_p(\cdot) \) is convex and finite on \((0, \infty)\), it is continuous. Moreover, since it goes to \( \infty \) at 0 and \( \infty \), it is inf-compact and therefore its minimum is attained.

\[ \Box \]

Our next step is to find the exact expression for the optimal solution \( y_p \) for all \( p \in [0, 1] \). We will show that, for \( p \) large, the minimum of \( \psi_p(\cdot) \) is attained at a point \( y_p \) for which the supremum in Eq. (A.11) is reached with \( k = 1 \) and simultaneously for \( m = 1 \) and \( m = 2 \), that is,

\[
\psi_p(y_p) = \psi_p^{1,1}(y_p) = \psi_p^{1,2}(y_p).
\]

For smaller values of \( p \) the supremum is still reached at \( k = 1 \) with either \( m = 1 \) or \( m = 0 \), but also for \( k \) and \( m \) tending to \( \infty \). This suggests to consider the solutions of the equations

\[
\psi_\infty(y) = \psi_p^{1,0}(y) \quad \iff \quad y = y_{0,p} := 1/3 \tag{A.16}
\]

\[
\psi_\infty(y) = \psi_p^{1,1}(y) \quad \iff \quad y = y_{1,p} := \frac{1}{1 + 2p + 2\sqrt{p(2 + p)}} \tag{A.17}
\]

\[
\psi_p^{1,1}(y) = \psi_p^{1,2}(y) \quad \iff \quad y = y_{2,p} := \frac{p}{1 + p}. \tag{A.18}
\]

Note that these three solutions belong to \((0, 1)\). Let also \( \tilde{p}_0 = 1/4 \) be the point at which \( y_{0,p} = y_{1,p} \), and \( \tilde{p}_1 \approx 0.3774 \) the point where \( y_{1,p} = y_{2,p} \) which is the unique real root of \( 8p^3 + 4p^2 = 1 \).

Proposition A.6. The minimum of \( \psi_p(\cdot) \) is attained at \( y_{0,p} \) if and only if \( p \in (0, \tilde{p}_0] \).

Proof. We will prove that

\[
\psi_p(y_{0,p}) = \psi_\infty(y_{0,p}) = \psi_p^{1,0}(y_{0,p}) \quad \iff \quad p \leq \tilde{p}_0. \tag{A.19}
\]

Assuming this, since both \( \psi_p^{1,0}(\cdot) \) and \( \psi_\infty(\cdot) \) are minorants of \( \psi_p(\cdot) \), their slopes \( (\psi_p^{1,0})'(y_{0,p}) = 1 \) and \( \psi_\infty'(y_{0,p}) = -2 \) are subgradients of \( \psi_p(\cdot) \) at \( y_{0,p} \). Hence \( 0 \in [-2, 1] \subseteq \partial \psi_p(y_{0,p}) \) and \( y_{0,p} \) is indeed a minimizer, as claimed.

To prove Eq. (A.19), we observe that the second part of this equality stems from the definition of \( y_{0,p} \) in Eq. (A.16). To establish the first equality, we note that \( \psi_\infty(\cdot) \leq \psi_p(\cdot) \), so it suffices to show that \( \psi_p(y_{0,p}) \leq \psi_\infty(y_{0,p}) \), which is equivalent to

\[
y_{0,p} \left[ \frac{k(1 + pm) - m(1 - p + pm)}{k(1 - p + pk)} \right] + \frac{1 + pm}{1 - p + pk} \leq \frac{(1 + y_{0,p})^2}{4y_{0,p}} \quad \forall (k, m) \in \mathcal{P} \quad \text{iff} \quad p \leq \tilde{p}_0. \tag{A.20}
\]
Substituting $y_{0, p} = 1/3$, the left inequality can be written equivalently as

$$0 \leq p(2k - m - 1)^2 + m(1 - 3p) - p \quad \forall (k, m) \in \mathcal{P}.$$ 

This holds trivially for $m = 0$ so we just consider $m \geq 1$. Now, for $k = m = 1$ this requires $p \leq 1/4$. Conversely, if $p \leq 1/4$ we have $1 - 3p > 0$ and therefore $m(1 - 3p)$ increases with $m$ so that

$$p(2k - m - 1)^2 + m(1 - 3p) - p \geq m(1 - 3p) - p \geq 1 - 4p \geq 0. \quad \Box$$

**Proposition A.7.** The minimum of $\psi_p(\cdot)$ is attained at $y_{1, p}$ if and only if $p \in [\bar{p}_0, \bar{p}_1]$.

**Proof.** We will prove that

$$\psi_p(y_{1, p}) = \psi_\infty(y_{1, p}) = \psi_p^{1, 1}(y_{1, p}) \iff p \in [\bar{p}_0, \bar{p}_1]. \quad (A.21)$$

Assuming this, it follows that

$$(\psi_p^{1, 1})'(y_{1, p}) = p \quad \text{and} \quad \alpha := \psi_\infty'(y_{1, p}) \quad (A.22)$$

are subgradients of $\psi_p(\cdot)$ at $y_{1, p}$. Now, since $y_{1, p} < 1$, by Lemma A.4, we have $\alpha < 0$ so that $0 \in [\alpha, p] \subseteq \partial\psi_p(y_{1, p})$ and therefore $y_{1, p}$ is a minimizer.

To prove Eq. (A.21), we observe that the second equality stems from the definition of $y_{1, p}$ in Eq. (A.17). To establish the first equality, we note that $\psi_\infty(\cdot) \leq \psi_p(\cdot)$, so it suffices to show that $\psi_p(y_{1, p}) \leq \psi_\infty(y_{1, p})$, which is equivalent to

$$y_{1, p} \left[ \frac{k(1 + pm) - m(1 - p + pm)}{k(1 - p + pk)} \right] + \frac{1 + pm}{1 - p + pk} \leq \frac{(1 + y_{1, p})^2}{4y_{1, p}} \quad \forall (k, m) \in \mathcal{P}, \quad \text{iff} \ p \in [\bar{p}_0, \bar{p}_1]. \quad (A.23)$$

Dividing by $y_{1, p}$ and letting

$$z = \frac{1 + y_{1, p}}{2y_{1, p}} = 1 + p + \sqrt{p(2 + p)}, \quad (A.24)$$

the left inequality becomes

$$\left[ \frac{k(1 + pm) - m(1 - p + pm)}{k(1 - p + pk)} \right] + \frac{1 + pm}{1 - p + pk} (2z - 1) \leq z^2.$$ 

Multiplying by $k(1 - p + pk)$ and factorizing, this can be rewritten as

$$Q_p(k, m) := p \left( zk - m + \frac{(1 - p)z - 2}{2p} \right)^2 + ((1 - p)z - 1 - p)m - \frac{(1 - p)z - 2}{4p} \geq 0, \quad (A.25)$$

so that, the left inequality of Eq. (A.23) is equivalent to $Q_p(k, m) \geq 0$ for all $(k, m) \in \mathcal{P}$. We observe that

$$Q_p(1, 0) \geq 0 \iff z \geq 2 \iff p \geq \bar{p}_0 \quad (A.26)$$

$$Q_p(1, 2) \geq 0 \iff 8p^3 + 4p^2 - 1 \leq 0 \iff p \leq \bar{p}_1 \quad (A.27)$$

so that $p \in [\bar{p}_0, \bar{p}_1]$ is a necessary condition for Eq. (A.23). We now show that it is also sufficient.

**Case 1.** $m = 0$: The inequality $Q_p(k, 0) \geq 0$ is equivalent to $z(1 - p + pk) \geq 2$ so that the most stringent condition is for $k = 1$, which holds for all $p \geq \bar{p}_0$, as already noted in Eq. (A.26).
Case 2. \( m = 1 \): From the very definition of \( y_{1,p} \) we have that Eq. (A.23) holds with equality for \((k, m) = (1, 1)\), so that \( Q_p(1, 1) = 0 \). Since \( Q_p(k, 1) \) is quadratic in \( k \), in order to have \( Q_p(k, 1) \geq 0 \) for all \( k \geq 1 \), it suffices to check that \( Q_p(2, 1) \geq 0 \). The latter can be factorized as
\[
Q_p(2, 1) = 2(1 + p)z(z - 2) + 1,
\]
so that, substituting \( z \) and simplifying, the resulting inequality becomes
\[
4p(1 + p)\sqrt{p(2 + p)} + 4p^3 + 8p^2 + 2p - 1 \geq 0.
\]
The conclusion follows since this expression increases with \( p \) and the inequality holds for \( p = 1/4 \). 

Case 3. \( m = 2 \): As noted in Eq. (A.27) we have \( Q_p(1, 2) \geq 0 \) for all \( p \leq \bar{p}_1 \). On the other hand, since \( z > 1 \) we have that \( Q_p(k, 2) \) increases for \( k \geq 2 \), so that it suffices to show that \( Q_p(2, 2) \geq 0 \). Now, \( Q_p(2, 2) \) can be factorized as
\[
Q_p(2, 2) = 2(1 + p)(z - 1)^2 - 4pz
\]
and substituting \( z \) we get
\[
Q_p(2, 2) = 4p^2(1 + p + \sqrt{p(2 + p)}) \geq 0.
\]

Case 4. \( m \geq 3 \): Let \( \alpha = (1 - p)z - 1 - p \) be the slope of the linear term in \( Q_p(k, m) \). Neglecting the quadratic part we have
\[
Q_p(k, m) \geq \alpha m - \frac{((1 - p)z - 2)^2}{4p}
\]
and therefore it suffices to show that the latter linear expression is nonnegative. We claim that for all \( p \leq \bar{p}_1 \) we have \( \alpha \geq 0 \). Indeed, substituting \( z \) we get
\[
\alpha = (1 - p)\sqrt{p(2 + p)} - p(1 + p),
\]
so that \( \alpha \geq 0 \) if and only if \((1 - p)^2(2 + p) \geq p(1 + p)^2\) which simplifies as \( p^2 + 2p \leq 1 \) and holds for \( p \leq \sqrt{2} - 1 \), and in particular for \( p \leq \bar{p}_1 \). Thus, the right hand side in Eq. (A.28) increases with \( m \), so what remains to be shown is that it is nonnegative for \( m = 3 \). The latter amounts to
\[
3 \alpha \geq \frac{((1 - p)z - 2)^2}{4p},
\]
which is equivalent to
\[
2(6p + 1 + p^2)(1 - p)\sqrt{p(2 + p)} \geq 1 + 2p + 11p^2 + 12p^3 + 2p^4
\]
and can be seen to hold for all \( p \in [\bar{p}_0, \bar{p}_1] \). \( \square \)

**Proposition A.8.** The minimum of \( \psi_p(\cdot) \) is attained at \( y_{2,p} \) if and only if \( p \in [\bar{p}_1, 1] \).

**Proof.** For \( y = y_{2,p} \) and \( k = 1 \) the unconstrained maximizer in Eq. (A.14) is \( m = 3/2 \) so that \( \sup_{m \geq 0} \psi_p^{1,m}(y_{2,p}) \) is attained at \( m = 1 \) and \( m = 2 \). The slopes of the corresponding terms are
\[
(\psi_p^{1,m})'(y) = \begin{cases} p & \text{if } m = 1, \\ -1 & \text{if } m = 2. \end{cases}
\]
If the outer supremum \( \sup_{k \geq 1} \) in Eq. (A.15) is attained for \( k = 1 \) it follows that \( 0 \in [-1, p] \subseteq \partial \psi_p(y_{2,p}) \) and, as a consequence, \( y_{2,p} \) is a minimizer.
Considering the expression in Eq. (A.15), and substituting the value of \( y_{2,p} \) and using the fact that for \( k = 1 \) the \( \sup_{m \geq 0} (y_{2,p})^{1/m} \) is attained at \( m = 1 \), it follows that \( \sup_{k \geq 1} \) is attained at \( k = 1 \) if and only if

\[
(1+2p)k + \sup_{m \in \mathbb{N}} \left[ ((1+2p)k - (1-p))pm - p^2m^2 \right] \leq [(1+p)^2 + p^2]k(1-p+pk) \quad \forall \ k \geq 2. \tag{A.29}
\]

We claim that this holds if and only if \( p \in [\hat{p}_1, 1] \). To this end, we note that for all \( k \geq 1 \) the unconstrained maximum of the quadratic \( ((1+2p)k - (1-p))pm - p^2m^2 \) is attained at

\[
\hat{m} = \frac{(1+2p)k - (1-p)}{2p} > 1.
\]

Proceeding as in the proof of Lemma A.4 we may find an integer \( \bar{m} \geq 1 \) and \( f \in (-\frac{1}{2}, \frac{1}{2}] \) such that \( \hat{m} = \bar{m} + f \). Hence, the supremum for \( m \in \mathbb{N} \) is attained at \( \bar{m} \) and

\[
\sup_{m \in \mathbb{N}} \left[ ((1+2p)k - (1-p))pm - p^2m^2 \right] = p^2(\bar{m}^2 - f^2) = \frac{1}{4}((1+2p)k - (1-p))^2 - p^2f^2. \tag{A.30}
\]

Replacing this expression into Eq. (A.29) and after simplification, the condition becomes

\[
0 \leq [8p^3 + 4p^2 - 1]k^2 + [2 - 2p - 4p^2 - 8p^3]k + 4p^2f^2 - (1-p)^2 \quad \forall \ k \geq 2. \tag{A.31}
\]

It follows that a necessary condition is \( 8p^3 + 4p^2 - 1 \geq 0 \) which amounts to \( p \geq \hat{p}_1 \). It remains to be shown that, once \( p \geq \hat{p}_1 \), the inequality Eq. (A.31) holds automatically. Consider first the case \( k \geq 3 \). Ignore the nonnegative term \( 4p^2f^2 \) and define

\[
Q(x) = [8p^3 + 4p^2 - 1]x^2 + [2 - 2p - 4p^2 - 8p^3]x - (1-p)^2. \tag{A.32}
\]

For \( p \geq \hat{p}_1 \) this is quadratic and convex in \( x \) and we have

\[
Q'(3) = 40p^3 + 20p^2 - 2p - 4 \geq 0 \quad \forall \ p \geq \hat{p}_1.
\]

Hence \( Q(x) \) is increasing for \( x \in [3, \infty) \) and then Eq. (A.31) holds for all \( k \geq 3 \) since

\[
Q(k) \geq Q(3) = 48p^3 + 23p^2 - 4p - 4 \geq 0 \quad \forall \ p \geq \hat{p}_1.
\]

For \( k = 2 \) it is not always the case that \( Q(2) \geq 0 \) so we must consider also the role of the fractional residual \( 4p^2y^2 \). The inequality to be proved is

\[
2(1+2p) + \sup_{m \in \mathbb{N}} (1 + 5p)pm - p^2m^2 \leq [(1+p)^2 + p^2]2(1+p).
\]

The supremum for \( m \in \mathbb{N} \) is attained at the integer closest to

\[
\hat{m} = \frac{1 + 5p}{2p} = 2 + \frac{1}{2} + \frac{1}{2p},
\]

which can be either \( m = 3 \) or \( m = 4 \) depending on whether \( p \) is larger or smaller than \( 1/2 \). Now, for these values of \( m \), the inequalities to be checked are

\[
2(1+2p) + (1 + 5p)3p - 9p^2 \leq [(1+p)^2 + p^2]2(1+p),
\]

\[
2(1+2p) + (1 + 5p)4p - 16p^2 \leq [(1+p)^2 + p^2]2(1+p),
\]

which reduce, respectively, to

\[
0 \leq 4p^3 + 2p^2 - p,
\]

\[
0 \leq 4p^3 + 4p^2 - 2p,
\]

and are easily seen to hold for all \( p \in [\hat{p}_1, 1] \). \( \square \)
With all the previous ingredients the proof of our main result is straightforward.

*Proof of Proposition A.2.* Substituting the expressions for the optimal solution \( y_p \) obtained in Propositions A.6–A.8 we get the optimal bound \( \text{PoA}(p) = \psi_p(y_p) \) which gives Eq. (4.1), as well as the optimal parameters

\[
\mu = \frac{y_p}{1 + y_p} \quad \text{and} \quad \lambda = \frac{\psi_p(y_p)}{1 + y_p},
\]

which are shown in Eq. (A.7).

*Proof of Theorem 4.1.* This is an immediate consequence of Lemma A.1 and Proposition A.2.

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List of symbols

\( a_e \) slope of the affine cost function
\( b_e \) constant of the affine cost function
\( B \) optimal bound, defined in Eq. (A.12)
\( \mathcal{B}_k \) bypass network, shown in Fig. 4
\( c_e \) cost of using resource \( e \)
\( c_{i,e} \) cost of player \( i \) using resource \( e \)
\( C \) expected social cost, defined in Eq. (3.4) and Eq. (3.10)
\( C_i \) cost function of player \( i \), defined in Eq. (3.1)
\( \mathcal{C} \) class of cost functions
\( \mathcal{C}^p \) class of cost functions derived by the Bernoulli game, defined in Definition 3.3
\( D \) destination
\( e \) resource
\( f \) difference between maximizer and its closest integer
\( \mathcal{G} \) class of games
\( G \) graph
\( i \) player
\( \mathcal{I} \) subset of players
\( \tilde{m} \) unconstrained maximizer, defined in Eq. (A.14)
\( m \) closest integer
\( n \) number of players
\( \mathcal{N} \) set of players
\( N_e \) number of players who use resource \( e \), defined in Eq. (3.2) and Eq. (3.7)
\( N_{e-i} \) number of players different from \( i \) who use resource \( e \)
\( \text{NE} \) set of Bayesian-Nash equilibria
\( O \) origin
\( p_i \) probability that player \( i \) is active
\( \mathbf{p} \) vector of probabilities of being active
\( \tilde{p}_0 = 1/4 \)
\( \tilde{p}_1 \approx 0.3774 \) unique real root of \( 8p^3 + 4p^2 = 1 \)
\( \text{PoA} \) price of anarchy, defined in Eq. (3.5)
\( \text{PoS} \) price of stability, defined in Eq. (3.5)
\( q \) upper bound for \( p_i, i \in \mathcal{N} \)
\( Q \), defined in Eq. (A.32)
\( Q_p(k,m) \), defined in Eq. (A.25)
\( s \) strategy profile
\( \hat{s} \) equilibrium strategy profile, defined in Eq. (3.3)
\( s^* \) optimum strategy profile
\( \mathcal{S} \) set of strategy profiles
\( s_i \) strategy of player \( i \)
\( \mathcal{S}_i \) strategy set of player \( i \)
\( W_{e,s}(s) \) indicator that \( e \in s_i \)
\( x_e \) number of players who choose resource \( e \)
\( X_e \) random load on resource \( e \)
\( y = \frac{\mu}{1-\mu} \), defined in Eq. (A.9)
\[ Y_i \sim \text{Bernoulli}(r_i) \]
\[ z = \frac{1 + y_{i,p}}{2y_{i,p}}, \text{ defined in Eq. (A.24)} \]
\[ Z_i \sim \text{Bernoulli}(p) \]
\[ \alpha \text{ subgradient of } \psi_p, \text{ defined in Eq. (A.22)} \]
\[ \Gamma \text{ game} \]
\[ \Gamma^p \text{ Bernoulli congestion game} \]
\[ \lambda \text{ parameter of } (\lambda, \mu)\text{-smoothness, defined in Eq. (A.1)} \]
\[ \mu \text{ parameter of } (\lambda, \mu)\text{-smoothness, defined in Eq. (A.1)} \]
\[ W_i \text{ indicator of player } i \text{ being active} \]
\[ \Phi \text{ potential, defined in Eq. (3.12)} \]
\[ \psi_p \text{ defined in Eq. (A.11)} \]
\[ \psi_{j,m}^{k,m} \text{, defined in Eq. (A.10)} \]
\[ \psi_{\infty} \text{, defined in Eq. (A.13)} \]