Spin-liquid behavior is one of the most striking manifestations of strong correlations in condensed matter. Spin liquids exhibit strong short-range spin correlations but do not show any long-range magnetic order. Such a behavior has been observed in the magnetic insulator Cs$_2$CuCl$_4$ in a substantial range of temperatures and magnetic fields, for example in inelastic neutron scattering experiments [1]. Experimentally, the boundary between the spin-liquid phase and the conventional paramagnetic phase has been characterized by broad peaks in the specific heat [2] and in the magnetic susceptibility [3]. Given the fact that Cs$_2$CuCl$_4$ can be modeled by a spin-1/2 Heisenberg antiferromagnet on a spatially anisotropic triangular lattice with nearest neighbor exchange couplings $J = 4.34$ K along the crystallographic $b$-axis and $J' = 1.49$ K $\approx J/3$ along the diagonal links within the $bc$-plane [4] (see Fig. 1), one would naively expect an anisotropic two-dimensional spin liquid state. However, several independent calculations [5-15] found that the spin excitations in the anisotropic triangular lattice antiferromagnet are quasi-one-dimensional and propagate dominantly along the direction corresponding to the largest exchange coupling, which is the crystallographic $b$-axis in Cs$_2$CuCl$_4$. In this work we shall give further evidence for this dimensional reduction scenario [16] by showing that ultrasound experiments probing the sound propagation along the $b$-axis can be quantitatively explained using a one-dimensional Heisenberg chain which is coupled to lattice vibrations via the usual exchange-striction mechanism [17].

The spin-phonon interaction and the ultrasonic attenuation in two-dimensional spin liquids have recently been discussed by Zhou and Lee [18], and by Serbyn and Lee [19]. In one-dimensional Heisenberg [20] and XY-chains [21] the interaction between the spin degrees of freedom and the phonons have been studied long time ago, but these older works mainly focused on the spin-Peierls transition and treated the spin-phonon interaction in an adiabatic approximation. Moreover, only the first derivative of the exchange coupling with respect to the phonon coordinates were considered, which turns out to be insufficient to explain our ultrasound experiments for the $c_{22}$-mode in Cs$_2$CuCl$_4$.

From the exact Bethe ansatz solution of the spin-1/2 antiferromagnetic Heisenberg chain we know that the groundstate is a spin liquid, exhibiting algebraic correlations but no long-range magnetic order. The elementary excitations above this ground state are spinons carrying spin 1/2. A combination of numerical and analytical methods has lead to an excellent understanding of this model [22]; for example, exact numerical results for the the magnetization [23], the magnetic susceptibility [24], the specific heat [25], and the dynamic structure factor
are available. However, a proper microscopic calculation of ultrasound propagation and attenuation in the Heisenberg chain cannot be found in the literature. Below we shall present a simple solution of this problem using the Jordan-Wigner representation of the spin-algebra in terms of spinless fermions \[28\]. We shall also present some new data of the elastic constant \(c_{22}\) and the corresponding ultrasound damping rate in the spin-liquid phase of \(\text{Cs}_2\text{CuCl}_4\) which agree very well with our theory. For details concerning the experiment and the sample preparation we refer to Ref. \[29\] and \[30\], respectively. The ultrasound physics of \(\text{Cs}_2\text{CuCl}_4\) has previously been studied for magnetic fields along the \(a\)-axis in the ordered phase using spin-wave theory \[29\], and for magnetic fields along the \(b\)-axis in the spin-liquid phase by combining phenomenological expressions for the ultrasound propagation and attenuation with calculations for two-dimensional spin models \[31\].

Assuming that the relevant spin excitations can propagate only along the crystallographic \(b\)-axis in \(\text{Cs}_2\text{CuCl}_4\), we expect that ultrasound experiments probing the \(c_{22}\) mode can be explained by the following one-dimensional spin-phonon Hamiltonian,

\[
\mathcal{H} = \sum_n J_n (S_n \cdot S_{n+1} - 1/4) - \hbar \sum_n S_n^z + \mathcal{H}_2^p, \tag{1}
\]

where \(S_n\) are spin-1/2 operators localized at the positions \(x_n\) on a chain with \(N\) spins and periodic boundary conditions, and \(\hbar = g \mu_B \mathbf{B}\) (with \(g \approx 2.2\ [4]\)) is the Zeeman energy associated with an external magnetic field \(\mathbf{B}\) along the crystallographic \(a\)-axis, see Fig. 1. The spin-phonon coupling arises from the fact that for a vibrating lattice the spins are located at \(x_n = nb + X_n\), where \(nb\) (with \(n = 1, \ldots, N\)) are the points of a one-dimensional lattice with lattice spacing \(b\), and \(X_n\) denote the deviations from the lattice points. Since the exchange coupling \(J_n\) between a pair of spins \(S_n\) and \(S_{n+1}\) located at \(x_n\) and \(x_{n+1}\) depends on the actual distance between the spins, \(J_n\) is a function of \(X_{n+1} - X_n\). Assuming that this difference is small, we may expand to second order,

\[
J_n \approx J + J^{(1)}(X_{n+1} - X_n) + \frac{J^{(2)}}{2}(X_{n+1} - X_n)^2, \tag{2}
\]

where \(J^{(1)}\) and \(J^{(2)}\) are the first and second derivative of the exchange coupling along the \(b\)-axis with respect to the phonon coordinates. As usual, we quantize the lattice vibrations by introducing the conjugate momenta \(P_n\) and demanding that \([X_n, P_m] = i\hbar \delta_{nm}\), where we have set \(\hbar = 1\). The last term in Eq. (1) describes non-interacting phonons with dispersion \(\omega_q = c|q|\),

\[
\mathcal{H}_2^p = \sum_q \left[ \frac{P_q P_q}{2M} + \frac{M}{2} \omega_q^2 X_q X_q \right], \tag{3}
\]

where \(M\) is the mass attached to the vibrating sites, and the operators \(X_q\) and \(P_q\) are defined via the Fourier expansions

\[
S_n^+ = (S_n^-)^\dagger = c_n^\dagger (-1)^n e^{i\pi \sum_{j<n} c_j c_{j+n}} - c_n^\dagger c_n - 1/2, \tag{4}
\]

where \(c_n\) annihilates a fermion at site \(x_n\) and the phase factor \((-1)^n\) is introduced for convenience. Our spin-phonon Hamiltonian (1) then assumes the form

\[
\mathcal{H} = -\frac{1}{2} \sum_n J_n (c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n) + \sum_n J_n c_n^\dagger c_n c_{n+1}^\dagger c_{n+1} - \hbar \sum_n c_n^\dagger c_n + N \hbar/2 + \mathcal{H}_2^p. \tag{5}
\]

In this work, we shall treat the two-body interaction in the second line of Eq. (5) within the self-consistent Hartree-Fock approximation, which amounts to approximating the two-body term by

\[
e_n c_n^\dagger c_{n+1}^\dagger c_{n+1} \approx \rho (c_n^\dagger c_{n+1}^\dagger c_{n+1} + c_n^\dagger c_n) - \rho^2 - \tau (c_n^\dagger c_{n+1}^\dagger c_{n+1} + c_n^\dagger c_n) + \tau^2, \tag{6}
\]

where the dimensionless variational parameters \(\rho\) and \(\tau\) satisfy the self-consistency conditions

\[
\rho = \langle c_n^\dagger c_n \rangle, \quad \tau = \langle c_n^\dagger c_{n+1} \rangle. \tag{7}
\]

In the absence of phonons, the solution of these equations has been worked out long time ago by Bulaevskii \[32\]. Within the Hartree-Fock approximation the fermion dispersion is \(\xi_k = -ZJ \cos k + 2sJ - h\), where \(Z = 1 + 2\tau\) is the dimensionless renormalization factor of the nearest neighbor hopping, \(s = \rho - 1/2\) is the dimensionless

![FIG. 2. (Color online) Comparison of the Hartree-Fock result for the magnetization curve \(s(h)\) of the Heisenberg chain (without phonons) at \(T = 0\) with the exact Bethe ansatz \[23\]. Inset: Renormalization \(Z(h)\) of the nearest neighbor hopping.](image-url)
magnetization, and $k$ is the fermion lattice momentum in units of the inverse lattice spacing $1/b$. In Fig. [3] we show the numerical result for $Z(h)$ at $T = 0$ and we compare our mean-field result for $s(h)$ with the exact magnetization curve of the Heisenberg chain obtained via the Bethe ansatz [29].

To obtain the change of the elastic constant and the sound attenuation, we should calculate the self-energy of the phonons due to the coupling to the spins. Substituting the gradient expansion (2) for the exchange coupling and the Hartree-Fock decoupling (6) into Eq. (5), we arrive at the approximate spin-phonon Hamiltonian

$$\mathcal{H} = \mathcal{H}_0 + \sum_k \xi_k c_k^\dagger c_k + \mathcal{H}_2^p + \mathcal{H}_3^p + \mathcal{H}_4^p, \quad (8)$$

where $\mathcal{H}_0/N = h/2 + J(\tau^2 - \rho^2)$ and

$$\delta \mathcal{H}_2^p = 2J(2)(\tau^2 - \rho^2) \sum_q \sin^2(q/2) X_{-q} X_q, \quad (9a)$$

$$\mathcal{H}_3^p = \frac{1}{\sqrt{N}} \sum_{k'q} \delta_{k',k+q}^* \Gamma_3(k,q)c_k^\dagger c_k X_q, \quad (9b)$$

$$\mathcal{H}_4^p = \frac{1}{2N} \sum_{k'k,q_1,q_2} \delta_{k',k+q_1+q_2}^* \Gamma_4(k,q_1,q_2)c_k^\dagger c_k X_{q_1} X_{q_2}. \quad (9c)$$

Here $\delta_{k',k} = \sum_n \delta_{k'k+2\pi n/b}$ enforces momentum conservation modulo a vector of the reciprocal lattice, $c_k = N^{-1/2} \sum_n e^{-i k \cdot n/c}$, and the cubic and quartic interaction vertices are for small phonon momenta given by

$$\Gamma_3(k,q) \approx -iqJ(1)[Z \cos k - 2s], \quad (10a)$$

$$\Gamma_4(k,q_1,q_2) \approx q_1 q_2 J(2)[Z \cos k - 2s]. \quad (10b)$$

The coupling to the fermions gives rise to a momentum- and frequency dependent self-energy correction $\Pi(q,i\omega)$ to the propagator of the phonon field $X_q$, which is proportional to $[\omega^2 + 2J^2 + \Pi(q,i\omega)]^{-1}$. To second order in the derivatives of the exchange coupling the phonon self-energy has three contributions, $\Pi(q,i\omega) = \Pi_2(q) + \Pi_3(q,i\omega) + \Pi_4(q)$, where

$$\Pi_2(q) = \frac{[J(2)/M][\tau^2 - \rho^2]4\sin^2(q/2)}{15}, \quad (11a)$$

$$\Pi_3(q,i\omega) = \frac{1}{MN} \sum_k \frac{f_k - f_{k+q}}{\xi_k - \xi_{k+q} + i\omega} |\Gamma_3(k,q)|^2, \quad (11b)$$

$$\Pi_4(q) = \frac{1}{MN} \sum_k f_k \Gamma_4(k,q,-q), \quad (11c)$$

and $f_k = [e^{\beta \xi_k} + 1]^{-1}$ is the occupation of the fermion state with momentum $k$ in self-consistent Hartree-Fock approximation. From the analytic continuation of the self-energy $\Pi(q,i\omega)$ to real frequencies we obtain the renormalized phonon energy and the phonon damping [29],

$$\tilde{\omega}_q = \omega_q + \frac{\text{Re}\Pi(q,\omega_q + i0)}{2\omega_q}, \quad \gamma_q = -\frac{\text{Im}\Pi(q,\omega_q + i0)}{2\omega_q}, \quad (12)$$

The renormalized phonon velocity can be obtained from $\tilde{c}/c = \lim_{q \to 0} \tilde{\omega}_q/\omega_q$, which yields for the shift $\Delta c = \tilde{c} - c$,

$$\Delta c/c = g_1 \epsilon^{(1)} + g_2 \epsilon^{(2)}, \quad (13a)$$

$$\epsilon^{(1)} = \int_{-\pi}^{\pi} \frac{dk}{2\pi} J'(\xi_k)\frac{v_k}{v_k - c} [2s - Z \cos k]^2, \quad (13b)$$

$$\epsilon^{(2)} = s^2 - Z^2/4, \quad (13c)$$

where $\int_1$ denotes the Cauchy principal value, $J'(\xi_k) = -\beta f_k[1 - f_k]$ is the derivative of the Fermi function, $v_k = Z J b \sin k$ is the group velocity of the fermionic excitations, and we have introduced the dimensionless coupling constants $g_1 = [J(3) h^2]/(2 N c^2 J)$ and $g_2 = J(2) h^2/(2 N c^2 J)$. In principle it should be possible to calculate these coupling using $ab$ initio methods, but here we simply determine $g_1$ and $g_2$ by fitting our theoretical prediction (13) to our experimental data. In Fig. [3] we show a comparison between theory and experiment as a function of the magnetic field. We find that the relative change of the sound velocity $\Delta c/c$ is best fit by $g_1 \approx 0$ and $g_2 \approx -1.1 \times 10^{-3}$. Only for our experimental data at the highest temperature $T = 1.15$ K, we get a

FIG. 3. (Color online) Comparison of theory and experiment for the relative change of the sound velocity of the $c_{22}$-mode: a) in the ordered phase ($T = 50$ mK) with coupling constants $g_1 = 0$ and $g_2 = -1.2 \times 10^{-3}$, b) in the spin-liquid phase ($T = 1$ K) with coupling constants $g_1 = 0$ and $g_2 = -1.1 \times 10^{-3}$. In the fitting procedure we have allowed a constant offset for $\Delta c/c$ which is necessary due to a small anomaly in the experimental data close to zero magnetic field.
finite value $g_1 = 0.85 \times 10^{-3}$. Because of the large value of $c/(Jb) \approx 6.8$, the term $c^{(1)}$ is more than one order of magnitude smaller than $c^{(2)}$ for $\text{Cs}_2\text{CuCl}_4$. While in the ordered phase ($T = 50\text{ mK}$, upper panel) our theoretical prediction (13) agrees only for magnetic fields up to $5\text{T}$ with our experimental data, in the spin-liquid phase ($T = 1\text{ K}$, lower panel) we obtain excellent agreement between theory and experiment for magnetic fields up to $7\text{T}$. A natural explanation for the deviations at larger fields is that in this regime the fluctuations are controlled by the dilute Bose gas quantum critical point at $B_c = 8.5\text{T}$ [2] which of course cannot be described by our one-dimensional model.

Finally, let us discuss the ultrasound attenuation of the $c_{22}$-mode in $\text{Cs}_2\text{CuCl}_4$. Our experimental data for three different temperatures of the magnetic field are shown in Fig. 4. In the regime $B \lesssim 7\text{T}$ where the fluctuations controlled by the quantum critical point are negligible and our theoretical prediction for the renormalization of the phonon velocity agrees with experiment, the sound attenuation is very small and practically constant. This can easily be explained within our one-dimensional model. Using Eqs. (11b) and (12) we obtain for the damping

$$
\gamma_q = \frac{\pi}{2M\omega_q} \int_{-\pi}^{\pi} \frac{dk}{2\pi} (f_k-f_{k+q})|\Gamma_3(k,q)|^2 \delta(\xi_k-\xi_{k+q}+\omega_q).
$$

(14)

This expression is only finite if the absolute value of the maximal group velocity $v_* = ZJb$ of the fermions exceeds the phonon velocity $c$. Because in $\text{Cs}_2\text{CuCl}_4$ this condition is never satisfied ($c/(Jb) \approx 6.8$), the attenuation of the $c_{22}$-mode vanishes in our approximation. Higher orders in perturbation theory will give a finite result, but it will involve more than two derivatives of the exchange coupling which are expected to be small.

On the other hand, the condition $v_* > c$ can possibly be realized in some other quasi-one-dimensional quantum antiferromagnet. Let us therefore evaluate Eq. (14) in the regime $v_* > c$. In the long-wavelength limit $q \rightarrow 0$ we obtain

$$
\frac{\gamma_q}{\omega_q} \sim \frac{g_1 c J \Theta(v_*^2 - c^2)}{2v_* \sqrt{1 - c^2/v_*^2}} \left[ -f^\prime(\xi_+) V_+^2 - f^\prime(\xi_-) V_-^2 \right],
$$

(15)

where $\xi_\pm = JV_{\pm} - h$, and $V_{\pm} = 2s \pm Z \sqrt{1 - c^2/v_*^2}$. A numerical evaluation of this expression is shown in the upper panel of Fig. 5. The damping exhibits strong peaks as function of the magnetic field, corresponding to the resonance conditions $\xi_\pm = 0$ imposed by the broadened delta-functions $-f^\prime(\xi_{\pm})$ in Eq. (14). In the lower panel of Fig. 5 we show that close to the resonance the corresponding shift $c^{(1)}$ in the phonon velocities can exhibit a sign change depending on the value of $r = c/(Jb)$.

In summary, we have developed a simple microscopic theory which explains ultrasound experiments probing the propagation and the attenuation of the $c_{22}$-mode in the spin-liquid phase of $\text{Cs}_2\text{CuCl}_4$. Our basic assumption is that in the spin-liquid phase the elementary excitations are one-dimensional fermions. The excellent agreement between theory and experiments shown in Fig. 3 gives further support to the dimensional reduction scenario advanced by Balents [16]. It would be interesting to test our theoretical predictions for the ultrasound attenuation shown in Fig. 5 using suitable antiferromagnetic spin chains with sufficiently small phonon velocities.

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