Minimization of $\ell_2$-Norm of the KSOR Operator

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ABSTRACT

We consider the problem of minimizing the $\ell_2$-norm of the KSOR operator when solving a linear systems of the form $AX = b$ where, $A = I + B$ ($T_j = -B$, is the Jacobi iteration matrix), $B$ is skew symmetric matrix. Based on the eigenvalue functional relations given for the KSOR method, we find optimal values of the relaxation parameter which minimize the $\ell_2$-norm of the KSOR operators. Use the Singular Value Decomposition (SVD) techniques to find an easy computable matrix unitary equivalent to the iteration matrix $T_{KSOR}$. The optimum value of the relaxation parameter in the KSOR method is accurately approximated through the minimization of the $\ell_2$-norm of an associated matrix $\Delta(\omega^*)$ which has the same spectrum as the iteration matrix. Numerical example illustrating and confirming the theoretical relations are considered. Using SVD is an easy and effective approach in proving the eigenvalue functional relations and in determining the appropriate value of the relaxation parameter. All calculations are performed with the help of the computer algebra system “Mathematica 8.0”.

Keywords: KSOR Iterative Method, $\ell_2$-Norm, Singular Value Decomposition (SVD)

1. INTRODUCTION

We consider linear systems of the form Equation 1:

$$\sum_{j=1}^{m} a_{ij}x_j = b_i, a_{ii} \neq 0, i = 1, 2, \cdots, m$$

(1)

With $a_{ij} = -a_{ji}$ for $i \neq j$ and the system admits a unique solution. This system of equations can be written as Equation 2:

$$AX = b, A \in \mathbb{R}^{m \times n}$$

(2)

Such linear systems arise in many different applications for example in the finite difference treatment of the Korteweg de Vries equation, Buckley (1977). Also, similar linear systems appears in the treatment of coupled harmonic equations, Ehrlich (1972).

In the iterative treatment of linear systems, we use the splitting, $A = D - L + U$, where $D = \text{diag}(A)$ is the diagonal part of $A$, for some non-zero constant $d$, -L is the strictly lower-triangular part of $A$ and -U is the strictly upper-triangular part of $A$, Woznicki (2001).

1.1. Jacobi Method Equation 3:

$$x_i^{[n+1]} = \frac{1}{a_{ii}} \left( b_i - \sum_{j=1}^{n-1} a_{ij} x_j^{[n]} - \sum_{j=i+1}^{m} a_{ij} x_j^{[n]} \right)$$

(3)

The Jacobi Method in matrix form is Equation 4:

$$X^{[n+1]} = T_j X^{[n]} + D^{-1} b T_j = D^{-1}(L + U)$$

(4)

$T_j$ is the Jacobi iteration matrix, it is clear that $T_j$ in this case is a skew symmetric matrix.

1.2. The SOR Method is Equation 5:

$$x_i^{[n+1]} = x_i^{[n]} + \frac{\omega}{a_{ii}} \left( b_i - \sum_{j=1}^{n-1} a_{ij} x_j^{[n+1]} - \sum_{j=i+1}^{m} a_{ij} x_j^{[n]} \right)$$

(5)

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where, \( \omega \in (0,2) \) is a relaxation parameter, \( \omega = 1 \) gives the well-known Gauss-Seidel method.

The SOR Method in matrix form is Equation 6:

\[
X^{[n+1]} = T_{\text{SOR}} X^{[n]} + (D - \omega L)^{-1}\omega b \\
T_{\text{SOR}} = (D - \omega L)^{-1}[(1 - \omega)D + \omega U]
\]

where, \( T_{\text{SOR}} \) is the SOR iteration matrix.

The choice of the relaxation parameter \( \omega \) is very important for the convergence rate of the SOR method. For certain classes of matrices (2-cyclic consistently ordered) with property A, in the sense of Young (2003), for such systems there is a functional eigenvalue relation of the form Equation 7:

\[
(\lambda + \omega - 1) = \omega \mu \lambda^{1/2}
\]

where, \( \lambda \) is an eigenvalue of the \( T_{\text{SOR}} \) and \( \mu \) is a corresponding eigenvalue of the \( T_{\text{J}} \). Most work on the choice of \( \omega \) is to minimize \( \rho(T_{\text{SOR}}) \) which is only an asymptotic criteria of the convergence rate of linear stationary iterative method, Hadjidimos and Neumann (1998). In real computations, we have to consider average convergence rate Milleo et al. (2006). The determination of the optimal value of the relaxation parameter \( \omega_{\text{opt}} \) can be obtained with the help of the eigenvalue functional relation (7). Young (2003), determined \( \omega_{\text{opt}} \) when \( T_{\text{J}} \) has only real eigenvalues and \( \rho(T_{\text{J}}) \) (1). In this case we have:

\[
\omega_{\text{opt}} = \frac{2}{1 + \sqrt{1 - (\rho(T_{\text{J}}))^2}}
\]

where the optimality is understood in the sense of the minimization of \( \rho(T_{\text{SOR}}) \).

Maleev (2006) determined \( \omega_{\text{opt}} \) when \( T_{\text{J}} \) has only pure imaginary eigenvalues and \( \rho(T_{\text{J}}) \) (1). In this case we have:

\[
\omega_{\text{opt}} = \frac{2}{1 + \sqrt{1 + (\rho(T_{\text{J}}))^2}}
\]

1.3. The KSOR Method is

In a recent work Youssef (2012), introduced the KSOR method Equation 8 and 9:

\[
X^{[n+1]} = x^{[n]} + \frac{\omega}{a_{kk}}
\]

\[
= b - \sum_{j=1}^{n} a_{kj}x^{[n]} - \sum_{j=1}^{m} a_{kj}x^{[n+1]}_{\text{old}} \text{ updated}
\]

\[
= \omega \mu \lambda^{1/2}
\]

\[
\text{where, } b_i = 1, 2, \ldots, m, \quad \omega \epsilon B \in [-2,0]
\]

\[
X^{[n+1]} = (1 + \omega)D - \omega L)^{-1}\omega b
\]

\[
T_{\text{KSOR}} = [(1 + \omega)D - \omega L)^{-1}[D + \omega U]
\]

where, \( T_{\text{KSOR}} \) is the KSOR iteration matrix (operator).

As it was in the SOR the rate of convergence of the KSOR method depends on the choice of the relaxation parameter \( \omega^* \). For certain classes of matrices (2-cyclic consistently ordered with property A), Youssef (2012) established the eigenvalue functional relation Equation 11:

\[
(\beta + \omega \beta, -1) = \omega \mu \beta_{i+2}
\]

where, \( \beta \)’s are the eigenvalues of the \( T_{\text{KSOR}} \) and \( \mu \)’s are the eigenvalues of the Jacobi iteration matrix \( T_{\text{J}} \). The eigenvalue functional relation (11) can be proved by the use of the SVD technique.

1.4. Singular Value Decomposition

Singular Value Decomposition (SVD) of a matrix \( B \in \mathbb{R}^{m \times m} \) is a factorization:

\[
B = U\Sigma V^T, \Sigma = \text{diag}(s_1, s_2, \ldots, s_q) \in \mathbb{R}^{q \times q}, q = \text{mim}\{q, m\}
\]

where, \( s_1 \geq s_2 \geq \ldots \geq s_q \geq 0 \), \( U \) and \( V \) are orthogonal matrices such that:
We consider in this study the case studied by Yin and Yuan (2002) also by Milleo et al. (2006) in which the coefficient matrix takes the form Equation 12:

\[ A = \begin{pmatrix} I_p & -F \\ F^T & I_q \end{pmatrix} \]  

(12)

where, \( F \in \mathbb{R}^{p \times q} \) with \( p + q = m \) and \( p \geq q \).

In this case the Jacobi iteration matrix becomes Equation 13:

\[ T_J = \begin{pmatrix} 0 & F \\ -F^T & 0 \end{pmatrix} \]  

(13)

It is clear that \( T_J \) is skew symmetric and accordingly admits pure imaginary eigenvalues and the KSOR iteration matrix \( T_{KSOR} \) becomes Equation 14:

\[ T_{KSOR} = \begin{pmatrix} \frac{1}{\omega + 1} I_p & \frac{\omega^*}{\omega + 1} F \\ -\frac{\omega^*}{(\omega + 1)^2} F^T & \frac{1}{\omega + 1} I_q - \frac{\omega^{*2}}{(\omega + 1)^2} F^T F \end{pmatrix} \]  

(14)

Usually, researchers work on obtaining the optimum value of the relaxation parameter \( \omega^* \) which minimizes the \( \ell_2 \)-norm of the iteration matrix or an equivalent quantity. We use the SVD approach in proving the eigenvalue functional relation for the KSOR method. Also, we use the same argument of Golub and Pillis (1990) to define a matrix \( \Delta(\omega^*) \) which has the same spectrum as the iteration matrix \( T_{KSOR} \).

Our objective is to find the optimal value of the relaxation parameter \( \omega^* \) which minimizes the \( \ell_2 \)-norm of the KSOR operator and illustrate the theoretical results through applications to a numerical example.

2. MATERIALS AND METHODS

We use the SVD in proving the relation between the eigenvalues of the skew symmetric Jacobi iteration matrix \( T_J \) and the singular values of a block sub-matrix \( F \), theorem (1). We will prove the relation between the eigenvalue functional relation between the eigenvalues of \( T_J \) and \( T_{KSOR} \) by using SVD, theorem (2). We will find the spectral radius of \( (T_{KSOR})^T T_{KSOR} \), theorem (3). We will find the optimal value of the relaxation parameter \( \omega^* \) to minimize the \( \ell_2 \)-norms of the \( T_{KSOR} \) theorem (4).

Theorem 1

Let \( A \) be the matrix given by (12), then Equation 15:

\[ \{ \mu_i^2 \} = \sigma(T_J^2) = \sigma(-FF^T) = \{ -S_i^2 \} i=1,2,...,q \]  

(15)

where, \( \mu_i^2 \) are the eigenvalues of \( T_J^2 \), \( S_i^2 \) are the squares of the singular values of \( F \) and \( \sigma(T_J) \) is the set of squares of the eigenvalues of \( T_J \).

Proof

Using the SVD to decompose the corner block matrix \( F \), we obtain Equation 16:

\[ F = U \Sigma V^T \]  

(16)

where, \( p \times p \) matrix \( U \) and \( q \times q \) matrix \( V \) are orthogonal, i.e.:

\[ U^T U = I_p \] \[ V^T V = I_q \]

and \( \Sigma \) is the \( p \times q \) diagonal matrix (of singular values) defined in (17). The eigenvalues of the matrix:

\[ FF^T = U \Sigma \Sigma^T U^T \] \[ = \{ s_i^2 \} \]

And:

\[ FF^T U = U \Sigma \Sigma^T U^T = U \Sigma \Sigma^T \]

Accordingly, the eigenvectors of the matrix \( FF^T \) equal to the columns of orthogonal matrix \( U \). Similarly, \( F^T F = \Sigma \Sigma^T \) has its eigenvectors equal to the columns of orthogonal matrix \( V \). The number of nonzero singular values \( S_i \) of \( F \) is equal to the rank of \( F \).

Substituting the singular value decomposition (16) into the corner elements \( F, F^T \) of (13), we obtain (18) Equation 17 and 18:

\[ \Sigma = \begin{pmatrix} s_1 & 0 & \cdots & 0 \\ 0 & s_2 & 0 & \cdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & s_q \end{pmatrix} \quad (p-q) \times q \]  

(17)

\[ T_J = \begin{pmatrix} 0 & F \\ -F^T & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -V^T U & 0 \end{pmatrix} \]  

(18)
Now, we will find a relation between the singular values $S_i$ (diagonal of $\Sigma$) and the eigenvalues $\mu_i$ of $T_J$ where $i = 1, 2, ..., q$. For $\mu_i \neq 0$ an eigenvalues of $T_J$, we have Equation 19:

$$T_J\begin{bmatrix} x_i \\ y_i \end{bmatrix} = \mu_i \begin{bmatrix} x_i \\ -y_i \end{bmatrix} \text{ iff } T_J\begin{bmatrix} x_i \\ -y_i \end{bmatrix} = -\mu_i \begin{bmatrix} x_i \\ y_i \end{bmatrix} \quad i = 1, 2, ..., t$$

(19)

So that, the number of non-zero eigenvalues of $T_J$ equals $2t$ that’s come in pairs $\pm \mu_i$. To account for zero eigenvalues, we write Equation 20:

$$T_J\begin{bmatrix} z_i \\ z_i^* \end{bmatrix} = 0 \quad i = 1, 2, ..., r$$

(20)

We construct the $n \times n$ non-singular matrix $W$ whose columns are the orthogonal eigenvectors of (19) and (20):

$$W = \begin{bmatrix} X & X & Z \\ Y & -Y & Z^* \end{bmatrix} n = p + q = 2t + r$$

Note that the $t$ columns of $p \times t$ matrix $X$ and $q \times t$ matrix $Y$ are the $t$ respective eigenvectors of (19), the $r$ columns of $p \times r$ matrix $Z$ and $q \times r$ matrix $Z^*$ come from the $r$ null vectors of (20).

Ordinarily, we would scale the columns of $W$ to produce an orthogonal matrix as a technical convenience, however, we assume that the columns of $W$ are scaled so that Equation 21:

$$W^H W = 2I$$

(21)

Let the matrix $I$ denote the $t \times t$ matrix whose diagonal elements are the $t$ positive eigenvalues $\mu_i$ of (19). Then (19) and (20) can be combined to produce the single matrix equation:

$$T_J = \begin{bmatrix} X & X & Z \\ Y & -Y & Z^* \end{bmatrix} = \begin{bmatrix} J & 0 & 0 \\ 0 & -J & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Which, when multiplied through on the right by $W^H$ we get Equation 22:

$$T_J = \begin{bmatrix} 0 & XJY^H \\ YJX^H & 0 \end{bmatrix}$$

(22)

Comparing the block entries of $T_J$ in (18) and (22), we obtain the equalities:

$$F = XJY^H = U\Sigma V^T$$

And:

$$-F^T = YJX^H = -V\Sigma U^T$$

And we see Equation 23:

$$T_J^2 = \begin{bmatrix} XJX^H & 0 \\ 0 & YJY^H \end{bmatrix} = \begin{bmatrix} -F^TF & 0 \\ 0 & -F^TF \end{bmatrix}$$

(23)

Accordingly:

$$\{\mu_{i}^2\} = \sigma(T_J^2) = (\sigma(T_J))^2 = \sigma(-F^TF) = \{-S_i\} \quad i = 1, 2, ..., q$$

**Theorem 2**

Let $T_{KSOR}$ and $T_J$ be given, respectively, by (14) and (13). Then the eigenvalues $\mu_i \in \sigma(T_J)$ and $\beta_i \in \sigma(T_{KSOR})$ are linked by the functional relation Equation 24:

$$\{\beta_i + \omega \beta_i, -1\}^2 = \omega^2 \mu_i \beta_i$$

(24)

Moreover, the eigenvalues and $2$-norm of matrices $T_{KSOR}$ and $\Delta(\omega^*)$ are related as follows Equation 25-29:

$$\sigma(T_{KSOR}) = \sigma(\Delta(\omega^*))$$

(25)

$$\rho(T_{KSOR}) = \rho(\Delta(\omega^*)) = \max_{\lambda \in \mathbb{C}} \rho(\Delta(\omega^*))$$

(26)

$$\| T_{KSOR} \|_2 = \| \Delta(\omega^*) \|_2 = \max_{\lambda \in \mathbb{C}} ||\Delta(\omega^*)||_2$$

(27)

Where:

$$\Delta(\omega^*) = \text{diag}(\Delta_1(\omega^*), \ldots, \Delta_q(\omega^*)) \frac{1}{\omega^* + \frac{1}{\lambda}}$$

(28)

$$\Delta_i(\omega^*) = \begin{cases} \frac{1}{1 + \omega^*} - \omega^* \frac{\omega^*}{(1 + \omega^*)^2} & \text{ for } i = 1, q \end{cases}$$

(29)
where, $s_i$ are the singular values of $F$.

Proof

By using the singular value decomposition of the matrix $F$ we have $F = U\Sigma V^T$ where $U$ and $V$ are orthogonal matrices, then the matrix $T_{KSOR}$ has the form Equation (30):

$$T_{KSOR} = \begin{bmatrix} \frac{1}{1+\omega}I_p - \frac{\omega}{1+\omega}UV^T \end{bmatrix} \left(\begin{array}{c} \frac{\omega}{1+\omega} - \frac{1}{1+\omega}I_p \end{array}\right)$$

Equation (30)

Let the orthogonal matrices $U$ and $V$ be factored out then $T_{KSOR}$ has the form Equation (31):

$$T_{KSOR} = \left[ \begin{array}{c} \frac{1}{1+\omega}I_p - \frac{\omega}{1+\omega}UV^T \end{array} \right] \left[ \begin{array}{c} U^T \ U^T \ 0 \end{array} \right]$$

Equation (31)

Note that (31) reveals the unitarily equivalent matrix $\Gamma_{\omega}$ with four block submatrices, each of which is a diagonal sub-matrix where Equation (32):

$$\Gamma_{\omega} = \left[ \begin{array}{c} \frac{1}{1+\omega}I_p - \frac{\omega}{1+\omega}UV^T \end{array} \right]$$

Equation (32)

This mean that there is a permutation matrix $P$ which “pulls” the two corner diagonal matrices to the main diagonal, i.e., $P\Gamma_{\omega}P^T$ has only $2\times2$ or $1\times1$ matrices along its main diagonal. When $\Gamma_{\omega}$ of (32) is permuted into the block diagonal form, we obtain Equation (33):

$$\Delta(\omega') = P\Gamma_{\omega}P^T = \text{diag} \left( \Delta_1(\omega'), \ldots, \Delta_q(\omega'), \frac{1}{\omega+1}I_{q-p} \right)$$

Equation (33)

where each $2\times2$ matrix $\Delta_i(\omega')$ is given by Equation (34):

$$\Delta_i(\omega') = \left[ \begin{array}{c} \frac{1}{1+\omega} - \frac{\omega}{1+\omega} \ 1 - \frac{\omega}{1+\omega} \end{array} \right] \left[ \begin{array}{c} 1 \ \frac{\omega}{1+\omega} \end{array} \right]$$

Equation (34)

where, $s_i$ are the singular values of $F$.

We have seen that each member of the $\omega'$ family of KSOR iteration matrix $T_{KSOR}$ is unitarily equivalent to a matrix $\Delta(\omega')$ having only $2\times2$ or $1\times1$ matrices on the main diagonal. That is, from (31) and (33) Equation (35):

$$T_{KSOR} = QP^T\Delta(\omega')PQ$$

Equation (35)

Unitary equivalent (35) implies that both the eigenvalues and the 2-norms agree for both ($\omega'$ families of matrices $T_{KSOR}$ and $\Delta(\omega')$) then we have (25), (26) and (27). From (25) we have Equation (36):

$$\det(\beta_{1,p} - T_{KSOR}) = 0 \text{ iff } \det(\beta_{1,p} - \Delta(\omega')) = 0$$

Equation (36)

From the right-hand determinant above, we see, from (33), (34), that all $\beta_i$ are constrained by Equation (37):

$$\Delta(\omega') = \left[ \begin{array}{c} \frac{1}{\omega+1} \ 1 - \frac{\omega}{1+\omega} \end{array} \right]$$

Equation (37)

Thus we find:

$$\beta_i = \frac{1}{\omega+1} \text{ or } (\beta_i + \omega \beta_i - 1) = \omega s_i^2 \beta_i = 0 \text{ i.e. } 1, 2, \ldots, q$$

Accordingly, with the help of (15) we can write Equation (38):

$$\beta_i = \frac{1}{\omega+1} \text{ or } (\beta_i + \omega \beta_i - 1) = \omega s_i^2 \beta_i = 0 \text{ i.e. } 1, 2, \ldots, q$$

Equation (38)
Now the left-hand equation, $\beta_i = \frac{1}{\omega + 1}$, appears in (33), (34) once for each occurrence of a zero eigenvalue for $T_J$, but $\beta_i = \frac{1}{\omega + 1}$ is a special case of the right-hand side of (38), namely, when $\mu_i$ is set to zero. Therefore, (38) is described by the single relation (24).

From the previous theorem we see the $\ell_2$-norm of the KSOR iteration matrix is equivalent to the $\ell_2$-norm of the $\Delta(\omega^*)$ then, equivalent to the square root of the spectral radius of $(\Delta(\omega^*))^T \Delta(\omega^*)$. Then, the problem of minimizing the $\ell_2$-norm of the KSOR iteration matrix is equivalent to the problem of minimizing the square root of the spectral radius of $(\Delta(\omega^*))^T \Delta(\omega^*)$.

**Theorem 3**

Under the assumptions of the theorem 2, for $K = 1$ the minimum of the of $\ell_2$-norm of the $T_{KSOR}$ is equivalent to Equation 39-41:

$$
\delta^2 = \min_{\omega^* \in \mathbb{R}\backslash[-2,0]} \delta^2 = \min_{\omega^* \in \mathbb{R}\backslash[-2,0]} \max\left\{ \frac{1}{2}(T(t) + [T^2(t) - 4C]^T) + \frac{1}{(1 + \omega^*)^T} \right\}
$$

Where:

$$
T(\omega^*, t) := \frac{2}{(1 + \omega^*)^T} + \frac{\omega^*}{(1 + \omega^*)} \frac{t}{(1 + t)}
$$

And:

$$
C(\omega^*) := \frac{1}{(1 + \omega^*)^T}
$$

With $t$ is the square of the spectral radius of the Jacobi iteration matrix $T_J$.

**Proof**

From the theorem 2 we have Equation 42:

$$
\| T_{KSOR} \|_2 = \| \Delta(\omega^*) \|_2 = \max_{\omega^* \in \mathbb{R}\backslash[-2,0]} \| \Delta(\omega^*) \|_2
$$

$$
\| \Delta(\omega^*) \|_2 = \max_{\omega^* \in \mathbb{R}\backslash[-2,0]} \{ \rho_{\omega^*}^2(\Delta(\omega^*) \Delta(\omega^*)) \}
$$

Now we go to calculate Equation 43 and 44:

$$
\Delta_i^T(\omega^*)\Delta_i(\omega^*) = \begin{pmatrix}
\frac{1}{1 + \omega^*} & \frac{-\omega^*}{1 + \omega^*} \\
\frac{\omega^*}{1 + \omega^*} & \frac{1}{1 + \omega^*}
\end{pmatrix}
$$

$$
\Delta_i^T(\omega^*)\Delta_i(\omega^*) = \begin{pmatrix}
\frac{1}{1 + \omega^*} & \frac{-\omega^*}{1 + \omega^*} \\
\frac{\omega^*}{1 + \omega^*} & \frac{1}{1 + \omega^*}
\end{pmatrix}
$$

It is easy to see Equation 45:

$$
det(\Delta(\omega^*)^T \Delta(\omega^*)) = \beta^2 - \left( \frac{2}{(1 + \omega^*)^T} + \frac{\omega^*}{(1 + \omega^*)} \frac{t}{(1 + t)} \right) \beta + \frac{1}{(1 + \omega^*)^T}.
$$

Set $s^2 = t$, and define Equation 46 and 47:

$$
c(\omega^*) := \frac{1}{(1 + \omega^*)}
$$

$$
T(\omega^*, t) := \frac{2}{(1 + \omega^*)^T} + \frac{\omega^*}{(1 + \omega^*)} \frac{t}{(1 + t)}
$$

Therefore Equation 48:

$$
det(\Delta(\omega^*)^T \Delta(\omega^*)) = \beta^2 - T(t) \beta + C
$$

Solving this quadratic equation, we find that Equation 49:

$$
\beta = \frac{1}{2} \left( T(t) \pm \sqrt{T(t)^2 - 4C} \right)
$$

Note that, for any $t \geq 0$, $\omega^* \in \mathbb{R}\backslash[-2,0]$, we have Equation 50:
\( T(t) = T(t) - 4C \geq 0 \) \hspace{1cm} (50)

Note that: The eigenvalues of the matrix \( \Delta_i^\top(\omega^*) \Delta(\omega^*) \) are nonnegative numbers and form the roots of the characteristic Equation 51 and 52:

\[ \beta^2 - T(t_0) \beta + C = 0 \quad i = 1, 2, \ldots, q \] \hspace{1cm} (51)

And:

\[ \beta - \frac{1}{(1 + \omega^*)^2} = 0 \] \hspace{1cm} (52)

The largest of the two roots of (51) is given by:

\[ L_i := L_i(\omega^*) = \frac{1}{2} \left[ T(\omega^*, t_i) + [T^\top(\omega^*, t_i) - 4C(\omega^*)]^{1/2} \right] \quad i = 1, 2, \ldots, q \]

The maximum value of each \( L_i \) is obtained for the maximum value of the corresponding \( T(\omega^*, t_i) \).

Proof

From (40) we see Equation 56 and 57:

\[ T(\omega^*, t) := \frac{2}{(1 + \omega^*)^2} + \frac{\omega^* t}{(1 + \omega^*)^2} \] \hspace{1cm} (56)

\( (1 + t) \geq 0 \) for any \( \omega^* \in \mathbb{R} / [-2, 0] \)

And:

\[ \frac{dT(\omega^*, t)}{d\omega^*} := \frac{-4 \omega^* t}{(1 + \omega^*)^3 (1 + t^2)} \] \hspace{1cm} (57)

Then we find Equation 58:

\[ \frac{dT(\omega^*, t)}{d\omega^*} = 0 \] \hspace{1cm} (58)

The function \( T(\omega^*, t) \) increases strictly in the interval \((-\infty, -2)\).

Differentiating \( L(\omega^*, t) \) defined in (54) with respect to \( \omega^* \), and using (40) and (41) we find Equation 59:

\[ \frac{dL}{d\omega^*} = \frac{1}{2} \left[ \frac{dT}{d\omega^*} + \frac{8}{T^2 - 4(1 + \omega^*)^2} \right] \] \hspace{1cm} (59)

It is clear that Equation 60:

\[ \frac{dL}{d\omega^*} = 0 \] \hspace{1cm} (60)

So that, the function \( L(\omega^*, t) \) increases strictly in the interval \((-\infty, -2)\). We will take limit of the function \( \frac{dL}{d\omega^*} \) as \( \omega^* \rightarrow \infty \), we obtain Equation 61:

\[ \lim_{\omega^* \rightarrow \infty} \frac{dL}{d\omega^*} = \frac{1}{2} \left[ \lim_{\omega^* \rightarrow \infty} \frac{dT}{d\omega^*} + \frac{8}{T^2 - 4(1 + \omega^*)^2} \right] \] \hspace{1cm} (61)

Then when, \( \omega^* \rightarrow \infty \) we have \( \frac{dL}{d\omega^*} \rightarrow 0^+ \), now we take limit as \( \omega^* \rightarrow \infty \) and obtain Equation 62:
\[
\lim_{\omega \to 0} \frac{dL}{d\omega} = -2 \langle 0 \rangle
\]

Therefore, from (61) and (65) \( L(\omega^*, t) \) has an odd number of local minimum points in \((0, \infty)\).

For any fixed \( t \in (0, 1) \), the global minimum point of \( L(\omega^*, t) \) is a point in \((0, \infty)\) at which \( \frac{dL}{d\omega} \) vanishes.

Setting \( \frac{dL}{d\omega} = 0 \) then:

\[
\frac{dT}{d\omega} \left[ T^2 - \frac{4}{(1 + \omega^*)^2} \right] = -T \frac{dT}{d\omega} - \frac{8}{(1 + \omega^*)^3}
\]

Then:

\[
\left( \frac{dT}{d\omega} \right)^2 \left[ T^2 - \frac{4}{(1 + \omega^*)^2} \right] = -T \left( \frac{dT}{d\omega} \right)^2 - \frac{8}{(1 + \omega^*)^3}
\]

That is:

\[
\left( \frac{dT}{d\omega} \right)^2 \left[ T^2 - \frac{4}{(1 + \omega^*)^2} \right] = \left( \frac{dT}{d\omega} \right)^2 + T \frac{dT}{d\omega} \frac{4}{(1 + \omega^*)^3} + \frac{16}{(1 + \omega^*)^6} = 0
\]

Eliminating \( (T \frac{dT}{d\omega})^2 \) and dividing through -4, we obtain:

\[
\frac{1}{(1 + \omega^*)^4} \left( \frac{dT}{d\omega} \right)^2 + T \frac{dT}{d\omega} \frac{4}{(1 + \omega^*)^3} + \frac{16}{(1 + \omega^*)^6} = 0
\]

It now follows that:

\[
\left( \frac{dT}{d\omega} \right)^2 + T \frac{dT}{d\omega} \frac{4}{(1 + \omega^*)^3} + \frac{16}{(1 + \omega^*)^6} = 0
\]

Substituting (56) for \( T \) and (57) for \( \frac{dT}{d\omega} \), we obtain Equation 66:

\[
f(\omega^*) = (t^2 + t)\omega - \omega^* - 1
\]

Then, we have Equation 67 and 68:

\[
\begin{align*}
\tau_1(\omega^*) &= \frac{1 + 4(t^2 + t)}{2} = \frac{1}{t} \\
\tau_2(\omega^*) &= \frac{1 - 4(t^2 + t)}{2} = -\frac{1}{t + 1}
\end{align*}
\]

Therefore, \( f(\omega^*) \) has a unique zero \( \tau_1(\omega^*) \) in that interval. So the \( \tau_1(\omega^*) \) is a unique real positive root in \((0, \infty)\) of the equation (66), from that and (61), we note that Equation 69:

\[
L(\tau_1(\omega^*), t) \left( \lim_{\omega \to \infty} L(\omega^*, t) \right)
\]

So that \( \tau_1(\omega^*) \) is a unique real positive root in \((0, \infty)\) of the equation which has the minimum of \( L(\omega^*) \).

**Example**

Consider a system with:
The eigenvalues of the Jacobi iteration matrix $T_J$ are the roots of the equation:

$$
\mu^4 + 0.25\mu^2 + 0.015625 = 0
$$

The roots are:

$$
\mu_1 = \mu_2 = 0.353553i, \mu_3 = \mu_4 = -0.353553i
$$

For the matrix, $F = \begin{bmatrix} -0.25 & 0.25 \\ -0.25 & -0.25 \end{bmatrix}$, the Singular values are $s_1 = s_2 = 0.353553$.

It is clear that the Jacobi iteration matrix $T_J$ is a skew symmetric, accordingly their eigenvalues are pure imaginary complex numbers, and satisfies $\mu_i^2 = s_i^2$.

### 3. RESULTS

- We used the SVD in proving the eigenvalue functional relation for the KSOR operator
- The minimization of the $\ell_2$-norm is used as a good estimation for determining the optimum value of the relaxation parameter in the KSOR method as well as in the SOR method

From Fig. 1 and 2 we see that the calculated results agree with the theoretical results of Milleo et al. (2006)
• From Fig. 3 and 4 we see that the calculated results agree with our theoretical results
• Numerical example illustrating and confirming the theoretical relations is considered

4. DISCUSSION

Young (2003), considered the problem why convergence of the SOR method with the optimum $\omega_{opt}$ in the sense of minimizing the spectral radius of the iteration matrix is some what slower than what might expected, the spectral radius is only an asymptotic measure of the rate of convergence of a linear iterative method. In his treatment Young (2003), established a relation between the eigenvalues of certain matrices related to $A$ (the SOR iteration matrix, $T_{SOR}$) and those of certain block $2 \times 2$ matrices.

Golub and Pillis (1990) raised the question of determining, for each $k \geq 1$, a relaxation parameter $\omega \in (0, 2)$ which minimizes the Euclidean norm of the $k$th power of the SOR iteration matrix, associated with a real symmetric positive definite matrix with “property A”.

Hadjidimos and Neumann (1998), used the reduction of the SOR operator introduced by Golub and Pillis (1990), with the help of the SVD of the associated block Jacobi iteration matrix to obtain the minimizing relaxation parameter for the case $k = 1$. Yin and Yuan (2002), used the SVD to re-derive the eigenvalue functional relations for block skew symmetric matrices for the AOR method. Milleo et al. (2006), considered systems with block skew symmetric Jacobi iteration matrix and used the SVD in studying the behavior of the SOR operator from the $\ell_2$-norm point of view they determined theoretically the minimizing relaxation parameter of the $\ell_2$-norm. Youssef (2012), defined the KSOR operator, we used the SVD in re-prove the functional eigenvalue relation for the KSOR operator and the corresponding unitary block $2 \times 2$ matrix $\Delta(\omega^*)$. we employed the same argument as in Yin and Yuan (2002), also in Milleo et al. (2006) for systems with block skew symmetric Jacobi iteration matrix and used the SVD in studying the behavior of the $\ell_2$-norm of the KSOR operator. We determined theoretically the minimizing relaxation parameter of the $\ell_2$-norm for the KSOR operator. We confirmed our theoretical results by a numerical example. We will continue this study in a subsequent work in which we will consider a generalizations of the KSOR operator.

5. CONCLUSION

We used the same argument defined by Golub and Pillis (1990), used by Yin and Yuan (2002) also by Milleo et al. (2006), we proved that the KSOR iteration matrix $T_{KSOR}$ is unitary equivalent to a matrix $\Delta(\omega^*)$ having only $2 \times 2$ or $1 \times 1$ matrices on the diagonal. We minimize the $\ell_2$-norm of the KSOR operator for matrices whose Jacobi iteration matrix is skew symmetric. By our results, the optimal value of the relaxation parameter is:

$$\omega_{opt} = \frac{1}{(\rho(T_{KSOR}))^2}.$$

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