The QCD string and generalized wave equation

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Abstract. The equation for QCD string proposed earlier is reviewed. This equation appears when we examine the gonihedric string model and the corresponding transfer matrix. Arguing that string equation should have a generalized Dirac form we found the corresponding infinite-dimensional gamma matrices as a symmetric solution of the Majorana commutation relations. The generalized gamma matrices are anticommuting and guarantee unitarity of the theory at all orders of $v/c$. In the second quantized form the equation does not have unwanted ghost states in Fock space. In the absence of Casimir mass terms the spectrum reminds hydrogen excitations. On every mass level $r = 2, 4, ..$ there are different charged particles with spin running from $j = 1/2$ up to $j_{\text{max}} = r - 1/2$, and the degeneracy is equal to $d_r = 2r - 1 = 2j_{\text{max}}$. This is in contrast with the exponential degeneracy in superstring theory.

Introduction. There is some experimental and theoretical evidence for the existence of a string theory in four dimensions which may describe strong interactions and represent the solution of QCD [1].

One of the possible candidates for that purpose is the gonihedric string which has been defined as a model of random surfaces with an action which is proportional to the linear size of the surface [4]

$$A(M) = m \sum_{<ij>} \lambda_{ij} \cdot \Theta(\alpha_{ij}), \quad \Theta(\alpha) = |\pi - \alpha|^\varsigma,$$

where $\lambda_{ij}$ is the length of the edge $<ij>$ of the triangulated surface $M$ and $\alpha_{ij}$ is the dihedral angle between two neighboring triangles of $M$ sharing a common edge $<ij>$. The model has a number of properties which make it very close to the Feynman path integral for a point-like relativistic particle. In the limit in which the surface degenerates into a single world line the action becomes proportional to the length of the path and the classical equation of motion for the gonihedric string is reduced to the classical equation of motion for a free relativistic particle. At the classical level the string tension is equal to zero and, as it was demonstrated in [4], quantum fluctuations generate the nonzero string tension $\sigma_{\text{quantum}} = \frac{d}{\alpha} (1 - \ln \frac{\beta}{\alpha})$, where $d$ is the dimension of the spacetime, $\beta$ is the coupling constant, $\alpha$ is the scaling parameter and $\varsigma = (d - 2)/d$ in (1).

It is natural therefore to ask what type of equation may describe this string theory in the continuum limit. The aim of the articles [5, 6] was to suggest a possible answer to this question. The analysis of the transfer matrix shows [5] that the desired equation should describe propagation of fermionic degrees of freedom distributed over the space contour. When this contour shrinks to a point,
the equation should describe propagation of a free Dirac fermion. Thus each particle in this theory should be viewed as a state of a complex fermionic system and the system should have a point-particle limit when there is no excitation of the internal motion. In the given case this restriction should be understood as a principle according to which the infinite sequence of particles should contain the spin one-half fermion and the equation should have the Dirac form [5, 6]

$$\{ i \Gamma_\mu \partial^\mu - M \} \Psi = 0.$$

(2)

The invariance of this equation under Lorentz transformations $x'_\mu = A^\nu_\mu x_\nu$, $\Psi'(x') = \Theta(A) \Psi(x)$ leads to the equation for the gamma matrices [8, 9] $\Gamma_\mu = A^\rho_\mu \Theta \Gamma_\rho \Theta^{-1}$. If we use the infinitesimal form of Lorentz transformations $A_{\mu\nu} = \eta_{\mu\nu} + \epsilon_{\mu\nu}$ it follows that gamma matrices should satisfy the Majorana commutation relation [9]

$$[\Gamma_\mu, I_{\lambda\rho}] = \eta_{\mu\lambda} \Gamma_\rho - \eta_{\mu\rho} \Gamma_\lambda,$$

(3)

where $I_{\mu\nu}$ are the generators of the Lorentz algebra. These equations allow us to find the $\Gamma_\mu$ matrices when the representation of the $I_{\mu\nu}$ is given.

Ettore Majorana in 1932 [9] found a solution of the equations (3) which differs from the Dirac gamma matrices [8]. The original Majorana solution for $\Gamma_\mu$ matrices is infinite-dimensional (see equation (14) in [9]) and is given by the formulas:

$$< j, m | \Gamma_0 | j, m > = j + 1/2$$

$$< j, m | \Gamma_z | j + 1, m > = \frac{i}{2} \sqrt{(j + m + 1)(j - m + 1)}$$

$$< j, m | \Gamma_z | j - 1, m > = -\frac{i}{2} \sqrt{(j + m)(j - m)}$$

$$< j, m | \Gamma_+ | j + 1, m - 1 > = \frac{i}{2} \sqrt{(j - m + 1)(j - m + 2)}$$

$$< j, m | \Gamma_+ | j - 1, m - 1 > = \frac{i}{2} \sqrt{(j + m)(j + m - 1)}$$

$$< j, m | \Gamma_- | j + 1, m + 1 > = -\frac{i}{2} \sqrt{(j + m + 1)(j + m + 2)}$$

$$< j, m | \Gamma_- | j - 1, m - 1 > = -\frac{i}{2} \sqrt{(j - m)(j - m - 1)}.$$

One can see from this solution that the mass spectrum of the theory is equal to

$$M_j = \frac{M}{j + 1/2}.$$

(4)

where $j = 1/2, 3/2, 5/2, ....$ in the fermion case and $j = 0, 1, 2, ....$ in the boson case. The main problems of Majorana theory are the decreasing mass spectrum (4), absence of antiparticles and troublesome tachyonic solutions - the problems common to high spin theories [10, 16].
An alternative way to incorporate the internal motion into the Dirac equation was suggested by Pierre Ramond in 1971 [11]. In his extension of the Dirac equation the internal motion is incorporated through the construction of operator-valued gamma matrices. The equations which define the $\Gamma_\mu$ matrices are

$$ < \Gamma_\mu(\tau) > = \gamma_\mu, \quad \{ \Gamma_\mu(\tau), \Gamma_\nu(\tau') \} = 2\eta_{\mu\nu} \delta \left( \frac{1}{2\pi\alpha'} (\tau - \tau') \right),$$

$$ \Gamma^+_\mu(\tau) \gamma_0 = \gamma_0 \Gamma_\mu(\tau),$$

where it is required that the proper-time average $< ... >$ over the periodic internal motion with period $2\pi\alpha' = 1/\sigma$ should coincide with the Dirac matrices. The solution has the form

$$ \Gamma_\mu(\tau) = \gamma_\mu + i\gamma_5 \sum_{n=1}^{\infty} [b^\mu_n \exp(-in\omega\tau) + b^{\mu+}_n \exp(in\sigma\tau)],$$

where $b$'s are operators obeying the anticommutation relations $\{b^\mu_n, b^{\nu+}_m\} = 2\delta_{\mu\nu} \delta_{nm}$ and all others are set equal to zero ($\omega = 1/\alpha'$). The mass spectrum lies on linear trajectories and the free Ramond string is a consistent theory in ten dimensions and the spectrum contains a massless ground state [2, 3].

In both cases one can see an effective extension of Dirac gamma matrices into the infinite-dimensional case, but these extensions are quiet different [5, 6]. For our purposes we shall follow Majorana’s approach to incorporate the internal motion in the form of an infinite-dimensional wave equation. Indeed in [5, 6] the Majorana theory was interpreted as a natural way to incorporate additional degrees of freedom into the relativistic Dirac equation. Unlike Majorana the authors consider the infinite sequence of high-dimensional representations of the Lorentz group with nonzero Casimir operators $(a \cdot b)$ and $(a^2 - b^2)$. These representations $(j_0, \lambda)$ and their adjoint $(j_0, -\lambda)$ are enumerated by the index $r = j_0 + 1/2$, where $r = 1, ..., N$ and $j_0 = 1/2, 3/2, ...$ is the lower spin in the representation $(j_0, \lambda)$, thus $j = j_0, j_0 + 1, ...$. We took the free complex parameter $\lambda$ in the real interval $-3/2 \leq \lambda \leq 3/2$ in order to have real matrix elements for the Lorentz boost operator $b$. These representations are infinite-dimensional except of the case $j_0 = 1/2, \lambda = \pm 3/2$.

The duality transformation $(j_0, \lambda) \rightarrow (\lambda; j_0)$, defined in [5], leads to a subsequent restriction on the free parameter $\lambda$ and requires $\lambda = 1/2$, so that the dual representations $\Theta_\tau, \Theta_\tau$ become finite-dimensional ($1/2, \pm(1/2 + r)$). The corresponding equation found in [5] is not in contradiction with no-go theorem of [16], because dual representations are finite-dimensional. However having a physically acceptable spectrum this equation admits unitarity only at zero order of $v/c$ [5]. In the article [6] we found a new equation which corresponds to a symmetric solution of the Majorana commutation relations and admits untiarity in all orders of $v/c$. It is based on the same dual representations ($1/2, \pm(1/2 + r)$) of the Lorentz algebra and is a natural extension of the previous equation [5] because the new gamma matrices have the same form as the old ones but with additional antidiagonal elements. The new gamma matrices are anticommuting.
and satisfy generalized Clifford algebra [6]

\[ \{ \Gamma_{\mu}, \Gamma_{\nu} \} = 2 \eta_{\mu\nu} \Gamma_{0}^2. \] (5)

They guarantee unitarity of the theory at all orders of \( v/c \). The anticommutation relations (5) are covariant, because the matrices \( \Gamma_{\mu} = A_{\mu} \Gamma_{\nu} \) satisfy the same relations as one can check by direct computation (notice that \( d \Gamma_{0}^2 = \sum_{\nu} \Gamma_{\nu}^2 \)).

The irreducible representations \( R_\lambda \) of the Lorentz algebra can be parameterized as the sum [15, 9, 8]:

\[ \sum_{j=0}^{\infty} \sum_{\lambda=-j}^{j} R(\lambda), \]

where \( j_{\lambda} \) defines the lower spin in the representation and \( \lambda \) is a free complex parameter. The \( \lambda \) appears as an essential dynamical parameter which cannot be determined solely from the kinematics of the Lorentz group.

The Majorana anticommutation relation (3) together with the last equations allow to find \( \Gamma_{\mu} \) matrices when a representation \( \Theta \) of the Lorentz algebra \( I_{\mu\nu} \) is given [9]. Because \( \Gamma_{0} \) commutes with spatial rotations \( a \) and Lorentz boosts \( b \) (\( a_x = iI^{23} \), \( a_y = iI^{31} \), \( a_z = iI^{12} \), \( b_x = iI^{10} \), \( b_y = iI^{20} \), \( b_z = iI^{30} \)) the algebra of the \( SO(3,1) \) generators can be rewritten as [9, 15, 8] (we use Majorana’s notations)

\[ [a_x,a_y] = ia_z \quad [a_x,b_y] = ib_z \quad [b_x,b_y] = -ia_z. \] (6)

The irreducible representations \( R^{(j)} \) of the \( SO(3) \) subalgebra (6) are well known. The dimension of \( R^{(j)} \) is 2\( j + 1 \) and \( j = 0, 1, 2, 3, ... \). The representation \( \Theta = (j_{0}, \lambda) \) of the Lorentz algebra can be parameterized as the sum [15, 9, 8]:

\[ \Theta \left( j_{0}, \lambda \right) = \sum_{j=0}^{\infty} \sum_{\lambda=-j}^{j} R(\lambda), \]

where \( j_{0} \) defines the lower spin in the representation and \( \lambda \) is a free complex parameter.

The superposition principle and physical constraints. The Majorana commutation relation (3) together with the last equations allow to find \( \Gamma_{\mu} \) matrices when a representation \( \Theta \) of the Lorentz algebra \( I_{\mu\nu} \) is given [9]. Because \( \Gamma_{0} \) commutes with spatial rotations \( a \) (see (3)) it should have the form

\[ \langle j, m | \Gamma_{0}^{rr'} | j', m' \rangle = \gamma^{rr'}_{j} \delta_{jj'} \delta_{mm'}, \quad r, r' = \hat{N}, ..., \hat{1}, 1, ..., N \] (7)

where we consider \( N \) pairs of adjoint representations \( \Theta = (\Theta_{N}, ..., \Theta_{1}, \Theta_{1}, ..., \Theta_{N}) \) with \( j_{0} = 1/2, ..., N - 1/2 \). Thus \( \gamma^{rr'} \) is a \( 2N \times 2N \) matrix which should satisfy the equation for \( \Gamma_{0} \) [9]

\[ \Gamma_{0}^{2}b_{z}^{2} - 2b_{z}\Gamma_{0}b_{z} + b_{z}^{2}\Gamma_{0} = -\Gamma_{0}. \] (8)

If \( \Gamma_{\mu}^{(1)} \) and \( \Gamma_{\mu}^{(2)} \) are two solutions of the equation (8), then their sum is also a solution of (8) [5] and we shall use this superposition property in order to find

\[ \Theta = j_{0} + r, r = 1, 2, 3, ... \]. The representations used in the Dirac equation are \( (1/2,-3/2) \) and \( (1/2,3/2) \) and in the Majorana equation they are \( (0,1/2) \) in the boson case and \( (1/2,0) \) in the fermion case. The infinite-dimensional Majorana representation \( (1/2,0) \) contains \( j = 1/2, 3/2, ... \) multiplets of the \( SO(3) \) while \( (0,1/2) \) contains \( j = 0, 1, 2, ... \) multiplets.
solution with required physical properties. In [5] the authors were searching the solution of the above equation in the form of Jacoby matrices

$$\gamma_j = \begin{pmatrix}
0, & \gamma_j^{NN-1}, & \ldots & \ldots & \gamma_j^{N-1N-2}, \\
\gamma_j^{N-1N}, & 0, & \gamma_j^{N-1N-2}, & \ldots & \ldots \\
\ldots & \ldots & \ldots & \gamma_j^{N-1N-2}, & 0, \\
\gamma_j^{NN-1}, & 0, & \gamma_j^{N-1N} & \ldots & \ldots \\
\end{pmatrix}, \quad \Psi_j = \begin{pmatrix}
\psi_j^N, \\
\psi_j^{N-1}, \\
\ldots \ldots \\
\psi_j^{N}, \\
\psi_j^{N-1} \\
\end{pmatrix}. \tag{9}
$$

In the subsequent work [6] we found the finale solution which has additional nonvanishing antidiagonal elements $\gamma_j^{rr}$ and represents a symmetric solution of the equation (8) of non-Jacobian form.

These solutions of the equation (8) are defined up to a set of constant factors which are independent from $j$. Indeed, because Jacoby matrices (9) have a specific form, the original equation (8) factorizes into separate equations for every element $\gamma_j^{rr+1}$ of the Jacoby matrix and the solution has the form [5]

$$\gamma_j^{rr+1} = \text{Const} \sqrt{\left(1 - \frac{r^2}{N^2}\right) \cdot \sqrt{\left(\frac{j^2 + \frac{1}{4r^2} - 1}{4r^2}\right)}}. \tag{10}
$$

It has $(4N - 2)$ $j$-independent free constant (this fact reflects the superposition property of the equation (8)). This freedom allows us to impose necessary physical constraints on a solution requiring [5, 6]:

- i) physical behaviour of the spectrum,
- ii) Hermitean property of the system,
- iii) reality and positivity of the probability density matrix

This means that we require sensible physical behaviour of the theory following one-particle interpretation of the solutions [8, 12].

In order to impose these constraints one should study the spectral properties of the matrices of infinite size with matrix elements $\gamma_j^{rr+1}$ and $\gamma_j^{rr}$, which have complicated "root" dependence. The first inspection of the solution (10) simply shows that every element $\gamma_j^{rr+1}$ grows like $\approx j$ and in general all eigenvalues $\epsilon_j$ will also grow with $j$. Therefore the mass spectrum $M_j = M/\epsilon_j$ will have Majorana-like behaviour (6) $M_j \approx M/j$. To avoid this general behaviour of the spectrum one should carefully inspect eigenvalues of the matrix $\Gamma_0$ for small values of $N$ and then for arbitrary $N$ [5]. The parameter $N$ plays the role of a natural regularization. The $B - H - \Sigma - \Sigma_1 - \Sigma_2$-solutions which appear (see [5] and below) have exceptional behaviour: half of the eigenvalues of the spectrum are increasing and the other half are decreasing. One can achieve this exceptional behaviour of the solution by tuning the free constants in the general solution (10). However solutions $B - H - \Sigma - \Sigma_1 - \Sigma_2$ cannot be accepted [5] because still half of the eigenvalues produce a mass spectrum which has an accumulation point at zero mass.
**Dual representations.** Last phenomenon can be understood on the example of the Dirac equation. For that let us define the dual representation as \([5] \Theta = (j_0; \lambda) \rightarrow (\lambda; j_0) = \Theta^{\text{dual}}\). This symmetry transformation imposes constraints on the free parameter \(\lambda\), so that it should be integer or half-integer. This is because in the representation \((j_0; \lambda)\) \(j_0\) must be integer or half-integer \([14]\). For the dual representations \(\Theta\) and \(\Theta^{\text{dual}}\) the matrix elements of Lorentz generators \(I_{\mu\nu}\) are precisely the same, the only difference between them is that the lower spin is equal to \(j_0\) for the representation \(\Theta\) and is equal to \(\lambda\) for its dual \(\Theta^{\text{dual}}\). Therefore any solution \(\Gamma_\mu\) of the Majorana commutation relations (3) for \(\Theta\) can be translated into the corresponding solution \(\Gamma_\mu^{\text{dual}}\) for \(\Theta^{\text{dual}}\) by exchanging \(j_0\) for \(\lambda\) \([5]\).

The dual transformation of the Dirac representations \((1/2, -3/2)\) and \((1/2, 3/2)\) would be infinite-dimensional \((3/2, -1/2)\) and \((3/2, 1/2)\) with \(j = 3/2, 5/2, \ldots\) and the corresponding solution \(\Gamma_0^{\text{dual}}\) has the form \(\gamma_j^1 1 = \gamma_j^1 1 = j + 1/2\) with the following mass spectrum

\[ M_j^{\text{Dirac dual}} = \frac{M}{j + 1/2}, \quad j = 3/2, 5/2, \ldots \]  

(11)

This Majorana-like mass spectrum is dual to the physical spectrum of the Dirac equation

\[ M_j^{\text{Dirac}} = M, \quad j = 1/2. \]  

(12)

The dual equation is simply unphysical, but we have to admit that the whole decreasing mass spectrum of the dual equation corresponds or is dual to a physical Dirac fermion. From this point of view we have to ask about physical properties of the equations which are dual to "unphysical" ones \(B - H - \Sigma - \Sigma_1 - \Sigma_2\).

The dual transformation completely improves the decreasing mass spectrum of these equations \([5]\), as it takes place in (11) and (12), and indeed the last \(\Sigma_2^{\text{dual}}\) equation has the spectrum of particles and antiparticles of increasing half-integer spin lying on quasilinear trajectories of different slope. However having a physically acceptable spectrum (see constraint i) \(\) this equation admits unitarity (see constraint ii) \) only at zero order of \(v/c\). The equation which admits unitarity at all orders have been found in \([6]\) and we shall review it below.

**Invariant product and probability density.** Let us introduce the invariant scalar product \(< \Theta \Psi_1 \mid \Theta \Psi_2 > = < \Psi_1 \mid \Psi_2 >\), where \(\Theta = 1 + \frac{1}{2} \epsilon_{\mu\nu} I^{\mu\nu}\) and the matrix \(\Omega\) is defined as \(< \Psi_1 \mid \Psi_2 > = \Psi_1^\dagger \Omega \Psi_2 = \Psi_1^\dagger \Omega_{j_1 m_1 j_2 m_2} \Psi_2^\dagger\) with the properties

\[ \Omega a_k = a_k \Omega, \quad \Omega b_k = b_k^+ \Omega, \quad \Omega = \Omega^+. \]  

(13)

From the first relation it follows that \(\Omega = \omega_j^r r \cdot \delta_{j j'} \cdot \delta_{m m'}\) and from the last two equations, for our choice of the representation \(\Theta\) and for a real \(\lambda\) in the interval \(-3/2 \leq \lambda \leq 3/2\), that \(\omega_j^r = \omega_j^r = 1, \quad \omega_j^2 = 1\), thus \(\Omega\) is an antidiagonal matrix. The conserved current density is equal to \(J_\mu = \bar{\Psi} \Gamma_\mu \Psi, \quad \partial^\mu J_\mu = 0.\)
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The current density \( J_0 \) should be \textit{real and positive definite} (see constraint iii), which is equivalent to the requirement that

\[
\Gamma_\mu^+ \Omega = \Omega \Gamma_\mu,
\]

and to the positivity of the eigenvalues of the matrix \( \rho = \Omega \Gamma_0 \).

\textbf{The \( B - H - \Sigma - \Sigma_1 - \Sigma_2 \)-solutions.} The basic solution (\textit{B-solution}) of the equation (8) for the \( \Gamma_0 \) has the form (9), (10) with all set of constant factors equal to \( Const = \pm i \) [5] and \( \gamma_j^1 = \gamma_j^1 = j + \frac{1}{2} \). The positive eigenvalues \( \epsilon_j \) can be found in [5]. They show that the coefficient of proportionality behind \( \epsilon \) drops \( N \) times compared with the one in the Majorana solution \( \epsilon_j = j + \frac{1}{2} \) in (6) and now many eigenvalues are less than unity and the corresponding masses \( M_j = M/\epsilon_j \) are bigger than the ground state mass \( M \). This actually means that increasing the number of representations in \( \Theta = (\Theta_0, \ldots, \Theta_1, \Theta_1, \ldots, \Theta_N) \) one can slow down the growth of the eigenvalues. To have the mass spectrum bounded from below one should have spectrum with all eigenvalues \( \epsilon_j \) less than unity.

In the limit \( N \to \infty \) the \( B \)-solution is being reduced to the form

\[
\gamma_j^{r+1} r = \gamma_j^r r + 1 = \gamma_j^r r + 1 i = \gamma_j^r r + 1 = i \sqrt{\frac{j^2 + j}{4r^2 - 1} - \frac{1}{4}} \quad j \geq r + \frac{1}{2} \quad (15)
\]

and \( \gamma_j^1 i = \gamma_j^1 = j + \frac{1}{2} \), where \( r = 1, 2, \ldots \). All eigenvalues \( \epsilon_j \) tend to unity when the number of representations \( N \to \infty \). The characteristic equation which is satisfied by the gamma matrix in this limit is

\[
(\gamma_j^2 - 1)j^{+1/2} = 0 \quad j = 1/2, 3/2, 5/2, \ldots \quad (16)
\]

with all eigenvalues \( \epsilon_j = \pm 1 \). Therefore all states have equal masses \( M_j = 1 \), but the Hamiltonian is not Hermitian (\( \Gamma_0^+ \neq \Gamma_0 \)). The matrix \( \Omega \Gamma_0 \) has the characteristic equation \( (\omega_j \gamma_j - 1)^{2j+1} = 0 \) with all eigenvalues equal to \( \rho_j = +1 \). Thus the matrix \( \Omega \Gamma_0 \) is positive definite and all its eigenvalues are equal to one, but the relations \( \Omega \Gamma_0 \neq \Gamma_0^{+} \Omega \), \( \Gamma_0^{+} \neq \Gamma_0 \) do not hold. What is crucial here is that we can improve the \( B \)-solution without disturbing its determinant which is equal to one (16) (\( Det\Gamma_0 = 1 \)). The last property of the determinant is necessary to keep, in order that the spectrum will be symmetrically distributed around unity.

The Hermitian solution (\textit{H-solution}) of (8) for \( \Gamma_0 \) can be found as a phase modification of the basic \( B \)-solution [5]

\[
\gamma_j^{r+1} r = -\gamma_j^r r + 1 = -\gamma_j^r r + 1 i = \gamma_j^r r + 1 = i \sqrt{\frac{j^2 + j}{4r^2 - 1} - \frac{1}{4}} \quad j \geq r + \frac{1}{2}. \quad (17)
\]

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\(^2\) The determinant and the trace are equal to \( Det \gamma_j = \pm 1 \), \( Tr \gamma_j^2 = 2j + 1 \), thus \( \epsilon_1^2 \cdot \epsilon_2^2 \cdot \ldots \cdot \epsilon_{j+1/2}^2 = 1 \), \( \epsilon_1^2 + \ldots + \epsilon_{j+1/2}^2 = j + 1/2 \).
These matrices are Hermitian $\Gamma^+_0 = \Gamma_0$, but the characteristic equations are more complicated now. These polynomials $p(\epsilon)$ have reflective symmetry and are even $p_{2j}(\epsilon) = \epsilon^{2j+1} p_j(1/\epsilon)$, $p_{2j}(-\epsilon) = p_j(\epsilon)$ therefore if $\epsilon_j$ is a solution then $1/\epsilon_j$, $-\epsilon_j$ and $-1/\epsilon_j$ are also solutions. The eigenvalues $\epsilon_j$ can be found in [5]. The changes in the phases of the matrix elements (17) result in different behaviour of eigenvalues. The half of the eigenvalues (decreasing eigenvalues) produce quasilinear trajectories with nonzero string tension and the other half (increasing eigenvalues) affect the low spin states on trajectories, so that the smallest mass on a given trajectory tends to zero (see [5]). The matrix $\Omega \Gamma_0$ has again the characteristic equation $(\omega_j \gamma_j - 1)^{2j+1} = 0$ and all eigenvalues are equal to one. Thus again the matrix $\Omega \Gamma_0$ is positive definite because all eigenvalues are equal to one, but the important relation $\Omega \Gamma_0 \neq \Gamma^+_0 \Omega$ does not hold.

The solution of (8) for $\Gamma_0$ with both properties $\Gamma^+_0 = \Gamma_0$ and $\Omega \Gamma_0 = \Gamma^+_0 \Omega$ can be found by using the basic solution rewritten with arbitrary phases of the matrix elements and then by requiring that $\Gamma_0$ should be Hermitian $\Gamma^+_0 = \Gamma_0$ and should satisfy the relation $\Omega \Gamma_0 = \Gamma^+_0 \Omega$. This solution, $\Sigma$-solution, is symmetric and has the form [5]

$$\gamma^{r+1}_j = \gamma^{r+1}_j = \gamma^{r+1}_j = \gamma^{r+1}_j = \sqrt{\frac{j^2 + j + 1}{4r^2 - 1} - \frac{1}{4}} \gamma \geq r+1/2. \quad (18)$$

In this case the Hermitian matrix $\Gamma^+_0 = \Gamma_0$ has the desired property $\Gamma^+_0 \Omega = \Omega \Gamma_0$. This means that the current density is equal to $\rho = \Omega \Gamma_0$. In addition, all of the gamma matrices now have this property (14) $\Gamma^+_k \Omega = \Omega \Gamma_k \quad k = x, y, z$ which follows from the equation $\Gamma_k = i[b_k \Gamma_0]$ (3) and equation (13) $\Omega b_k = b_k \Omega$.

The characteristic equations and the spectrum are the same for the Hermitian H-solution and symmetric $\Sigma$-solution, but the corresponding characteristic equations for the matrices $\rho_j$ are different and the eigenvalues are not positive definite any more [5], that is there are many ghost states. The positive norm physical states are lying on the quasilinear trajectories of different slope and the negative norm ghost states are also lying on the quasilinear trajectories. There are also tachyonic solutions [5], which appear in Majorana equation as well [9, 14, 16, 5]. In [5] it was suggested to ”protect” equation from tachyonic solutions by setting some of the transition amplitudes $\gamma^{r'}_j$ to zero.

In the case when some of the transition amplitudes $\gamma^{r'}_j$ in (18) are set to zero (we again use ”superposition” property of the equation (8) [5])

$$\gamma_{1}^{1} = \gamma_{2}^{2} = \gamma_{3}^{3} = \gamma_{4}^{4} = ... = 0 \quad \gamma_{1}^{1} = \gamma_{2}^{2} = \gamma_{3}^{3} = \gamma_{4}^{4} = ... = 0 \quad (19)$$

and all other elements of the $\Gamma_0$ matrix remain the same as in (18) we have a new $\Sigma_1$-solution with the important property that $\Gamma^+_0$ is a diagonal matrix and that

\[3\] Computing the traces and determinants of these matrices one can get the following general relation for the eigenvalues $\epsilon_j^2 \cdots \epsilon_{j+1/2} = 1$, $\epsilon_j^2 + ... + \epsilon_{j+1/2}^2 = j(2j + 1)$. 

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corporated in \( \xi \) to be half-integer and to Dual equations. (\( \Theta \) the dual transformation \( \Sigma \) to zero and the spectrum still has an accumulation point. We have to remark \( \Sigma \) spin is \( n \)

The difference between these last two solutions is that in the first case the lower solution is (18) with \( \gamma_j^1 \) and \( \gamma_j^4 \) where \( \gamma_j^1, \gamma_j^4 \rightarrow \frac{n+1}{2}, \frac{n+5}{2}, \ldots \). The string tension \( \sigma_n = 1/2\pi\alpha_n \) varies from one trajectory to another and is equal to

\[
2\pi\sigma_n = \frac{1}{\alpha_n} = \frac{2M^2}{n} \quad n = 1, 2, \ldots
\]

Thus we have the string equation which has trajectories with different string tension and that trajectories with large \( n \) are almost "free" because the string tension tends to zero. The smallest mass on a given trajectory \( n \) has spin \( j = n+1/2 \) and decreases as \( M_n^2(j = n+1/2) = 4M^2/(n(n+3)) \).

The other solution, \( \Sigma_2 \)-solution, which shares the above properties of \( \Sigma_1 \)-solution is (18) with

\[
\gamma_j^1 = \gamma_j^3 = \gamma_j^4 = \ldots = 0 \quad \gamma_j^1 = \gamma_j^3 = \ldots = 0.
\]

The difference between these last two solutions is that in the first case the lower spin is \( j = 3/2 \) and in the second case it is \( j = 1/2 \). The unwanted property of all these solutions \( \Sigma, \Sigma_1, \Sigma_2 \) is that the smallest mass \( M_n^2(min) \) tends to zero and the spectrum still has an accumulation point. We have to remark also that both equations, \( \Sigma_1 \) and \( \Sigma_2 \), which correspond to (19) and to (22) have natural constraints [5].

**Dual equations.** The unwanted property of the \( \Sigma \)-solutions, that is the decreasing of the smallest mass on a given trajectory, can be avoided by dual transformation of the system [5] (see section "Dual representations"). Under the dual transformation \( \Theta = (j_0; \lambda) \rightarrow (\lambda; j_0) = \Theta^{\text{dual}} \) the representation \( \Theta = (\Theta_N, \ldots, \Theta_1, \Theta_1, \ldots, \Theta_N) \) is transformed into its dual \( \Theta^{\text{dual}} = (\lambda; -5/2) \lambda; -3/2) \lambda; 1/2) \lambda; 3/2) \lambda; 5/2) \ldots \) and we are lead to take \( \lambda \) to be half-integer and to \( \lambda = 1/2 \) in order to have the Dirac representation incorporated in \( \Theta \). The solution which is dual to \( \Sigma_2 \) (18) and (22) is equal to [5]

\[
\gamma_j^{r+1} = \gamma_j^{r+1} = \gamma_j^{r+1} = \sqrt{(1/4 - j^2 + j/4n^2 + 1}) \quad r \geq j + 3/2
\]

where \( j = 1/2, 3/2, 5/2, \ldots \), \( r = 2, 4, 6, \ldots \) and the rest of the elements are equal to zero

\[
\gamma_j^{11} = \gamma_j^{12} = \gamma_j^{34} = \ldots = 0 \quad \gamma_j^{11} = \gamma_j^{12} = \gamma_j^{34} = \ldots = 0.
\]

These representations do not coincide with the ones in Ramond equation [11].
The Lorentz boost operators $b$ are antihermitian in this case $b_k^* = -b_k$ [5], and therefore the $\Gamma_k$ matrices are also antihermitian $\Gamma_k^+ = -\Gamma_k$. The matrix $\Omega$ changes and is now equal to the parity operator $P$, the relation $\Omega \Gamma_\mu^+ = \Gamma_\mu \Omega$ remains valid. The diagonal part of $\Gamma_k$ anticommutes with $\Gamma_0$ as it was before $\{\Gamma_0, \Gamma_k\} = 0$ if $k = x, y, z$. The mass spectrum is highly degenerated and is given by the formula

$$M_n^2 = \frac{2M^2}{n} \frac{(j + n)(j + n + 1)}{j + (n + 1)/2}$$

where $n = 1, 2, 3, \ldots$ and enumerates the trajectories. The lowest spin on a given trajectory is either 1/2 or 3/2 depending on $n$: if $n$ is odd then $j_{\text{min}} = 1/2$, if $n$ is even then $j_{\text{min}} = 3/2$. This is an essential new property of the dual equation because now we have an infinite number of states with a given spin $j$ instead of $j + 1/2$. The string tension is the same as in the dual system (21).

The lower mass on a given trajectory $n$ is given by the formula $(j = 1/2)$ is

$$M_0^2(j = 1/2) = \frac{4M^2(2n+1)(2n+3)}{n+2} \to (4M)^2$$

and the spectrum is bounded from below by positive mass.

**Generalized wave equation.** The last $\Sigma_{2}^{\text{dual}}$-equation has the property that only the diagonal matrix elements of the anticommutator $\{\Gamma_0, \Gamma_z\}$ are equal to zero $< j, m, r|\{\Gamma_0, \Gamma_z\}|r, j, m > = 0$, and that nondiagonal elements are not equal to zero $< j - 1, m, r|\{\Gamma_0, \Gamma_z\}|r, j, m > = \frac{i}{\sqrt{2}} - m^2 \gamma_j^r \gamma_j^{r+1}$, (28). Let us search the solution of the Majorana commutation relation (8) in the same $\Sigma_{2}^{\text{dual}}$-Jacoby form (9) but with additional nonvanishing antidiagonal matrix elements $\gamma_j^r \gamma_j^{r+1}$. The solution has the form [6]

$$\gamma_j^r \gamma_j^{r+1} = \gamma_j^r \gamma_j^{r+1} = \frac{j + 1/2}{\sqrt{4r^2 - 1}}$$

where $j = 1/2, 3/2, 5/2, \ldots$ and $r = 2, 4, 6, \ldots$ and $r \geq j + 3/2$ and one can check directly that $\Gamma_0$ is the solution of (8). These additional matrix elements in $\Gamma_0$ will not change the diagonal matrix elements of the anticommutator but will cancel nondiagonal matrix elements [6] $< j - 1, m, r|\{\Gamma_0, \Gamma_z\}|r, j, m > = 0$. One can check this fact also using the relation $\{\Gamma_0, \Gamma_z\} = i [\Gamma_z, \Gamma_0^2]$ which follows from (3). Using the relations (3) $\Gamma_y = -i [\Gamma_z, \Gamma_x]$ and $\{\Gamma_0, \Gamma_y\} = 0$ one can see that $\{\Gamma_0, \Gamma_0\} = 0$ and in the same way that $\{\Gamma_0, \Gamma_z\} = 0$. Finally using the relation (3) $\Gamma_k = -i [\Gamma_0, b_k]$ one can prove by direct calculation that $\{\Gamma_k, \Gamma_l\} = 0$ for $k \neq l$ and then using (8) and the fact that $[b_k, \Gamma_0^2] = 0$ one can prove that $\Gamma_k^2 = -\Gamma_0^2$, (27). The mass spectrum is highly degenerated and is given by the formula

$$M_j^2 = M^2(1 - \frac{1}{4r^2}) \quad r \geq j + 1/2.$$
New mass terms \((\mathbf{a} \cdot \mathbf{b}) \Gamma_5\) and \((\mathbf{a}^2 - \mathbf{b}^2)\) can be added into the string equation (2) in order to increase the string tension

\[
\{ i \Gamma_\mu \partial^\mu - M (\mathbf{a} \cdot \mathbf{b}) \Gamma_5 - gM (\mathbf{a}^2 - \mathbf{b}^2) \} \Psi = 0, \tag{29}
\]

where \((\mathbf{a} \cdot \mathbf{b})\) and \((\mathbf{a}^2 - \mathbf{b}^2)\) are the Casimir operators of the Lorentz algebra.

Now all trajectories acquire a nonzero slope

\[
M_j^2 = M^2 \frac{4r^2 - 1}{4r^2} (r - 1/2)^2 (1 + 2g(r - 1/2))^2 \quad r \geq j + 1/2. \tag{30}
\]

where \(r = 2, 4, 6, \ldots\), \(j = 1/2, 3/2, 5/2, \ldots\), thus \(M_j^2 \geq M^2(j + 1)^2\). Thus the spectrum of the theory consists of particles and antiparticles of increasing half-integer spin and masses. On every mass level there are particles with spin running from \(j = 1/2\) up to \(j_{\text{max}} = r - 1/2\), \(r = 2, 4, \ldots\) and the degeneracy is equal to \(d_r = 2r - 1 = 2j_{\text{max}}\). This is in contrast with the exponential degeneracy in case of superstrings. The tachyonic solutions which appear in Majorana equation (see (20) in [9]) do not show up here.

We can introduce now the interaction with gauge field using covariant derivative \(\Pi_\mu = i\partial_\mu + gA_\mu\) and see that there are no obvious inconsistencies which are characteristic to high spin equations in the background field (see W.Pauli and M.Fierz in [14]), because the equation (2) can be transformed to the form

\[
\{ \Pi^2 \Gamma_0^2 + \frac{ig}{2} \Gamma_\mu \Gamma_\nu F^{\mu\nu} - M^2 \} \Psi = 0 \tag{31}
\]

and does not produce additional constraints.

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