LOCAL-GLOBAL PRINCIPLE FOR NORM ONE TORI

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Abstract. Let \( K \) be a complete discretely valued field with residue field \( \kappa \). Let \( F \) be a function field in one variable over \( K \) and \( \mathcal{X} \) a regular proper model of \( F \) with reduced special fibre \( X \) a union of regular curves with normal crossings. Suppose that the graph associated to \( \mathcal{X} \) is a tree (e.g. \( F = K(t) \)). Let \( L/F \) be a Galois extension of degree \( n \) with Galois group \( G \) and \( n \) coprime to \( \text{char}(\kappa) \). Suppose that \( \kappa \) is algebraically closed or a finite field containing a primitive \( n \)-th root of unity. Then we show that an element in \( F^* \) is a norm for the extension \( L/F \) if it is a norm from the extensions \( L \otimes_F F_\nu/F_\nu \) for all discrete valuations \( \nu \) of \( F \).

1. Introduction

Let \( F \) be a field and \( \Omega_F \) be the set of all discrete valuations on \( F \). For \( \nu \in \Omega_F \), let \( F_\nu \) denote the completion of \( F \) at \( \nu \). Let \( G \) be a linear algebraic group over \( F \). One says that the local-global principle holds for \( G \)-torsors if for any \( G \)-torsor \( X \), \( X \) has a rational point over \( F \) if and only if it has a rational point over \( F_\nu \) for all \( \nu \in \Omega_F \). If \( F \) is a number field, then it is known that the local-global principle holds for torsors of various classes of linear algebraic groups ([18, Chapter 6]), including semisimple simply connected groups. In particular, it is well-known that if \( T_{L/F} \) is the norm one torus defined by a cyclic extension \( L/F \), then the local-global principle holds for \( T_{L/F}-\)torsors i.e. an element \( \lambda \in F \) is a norm from the extension \( L/F \) if and only if \( \lambda \) is a norm from \( F \otimes_F F_\nu/F_\nu \) for all \( \nu \in \Omega_F \) ([2, Chapter 11]). However, very little is known for general fields.

Let \( K \) be a complete discretely valued field with residue field \( \kappa \). Let \( F \) be a function field of a smooth, projective, geometrically integral curve over \( K \). Harbater, Hartman and Krashen ([12]) developed patching techniques to study \( G \)-torsors over \( F \) and proved that if \( G \) is connected and \( F \)-rational, then a \( G \)-torsor over \( F \) has a rational point over \( F \) if and only if it has a rational point over certain overfields of \( F \) which are defined using patching. As a consequence of this result, Colliot-Thélène, Parimala and Suresh ([4, Theorem 4.3.]) showed that if \( G \) is reductive, \( F \)-rational and defined over the ring of integers of \( K \), then the local-global principle holds for \( G \)-torsors. Similar local-global principles are proved for \( G \)-torsors for various linear algebraic groups \( G \) over \( F \) if the residue field of \( K \) is either finite or algebraically closed field ([4], [5], [10], [15], [17]).

The first example of a linear algebraic group \( G \) over \( F \) where such a local-global principle fails was given by Colliot-Thélène, Parimala and Suresh ([5, Section 3.1. & Proposition 5.9.]). In their example, the residue field of \( K \) is the field of complex numbers, \( G \) is the norm one torus of a Galois extension \( L/F \) with Galois group \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) and the field \( F \) has a regular proper model with the associated graph not a tree. Suppose that \( F \) has a regular proper model with the associated graph a tree. If \( L/F \) is a Galois extension with Galois group \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) and \( \kappa \) is algebraically closed, then they also proved that the local-global principle holds for torsors of the norm one torus of \( L/F \) ([5, Section 3.1. & Corollary 6.2.]).
The main aim of this paper is to prove the following theorem (6.4, 6.5):

**Theorem 1.1.** Let $K$ be a complete discretely valued field with residue field $\kappa$ and $F$ be the function field of a curve over $K$. Let $\mathcal{X}$ be a regular proper model of $F$ with reduced special fibre $X$ a union of regular curves with normal crossings. Let $L/F$ be a Galois extension over $F$ of degree $n$ with Galois group $G$. Suppose that $\kappa$ is either a finite field or an algebraically closed field, $n$ is coprime to $\text{char}(\kappa)$ and $K$ contains a primitive $n^{th}$ root of unity. If the graph associated to $X$ is a tree, then an element $\lambda \in F$ is a norm from the extension $L/F$ if and only if $\lambda$ is a norm from the extensions $L \otimes_F F_\nu/F_\nu$ for all $\nu \in \Omega_F$.

In fact one can restrict to divisorial discrete valuations in the above theorem.

For a finite extension $L/F$, let $T_{L/F}$ denote the norm one torus associated to $L/F$. For any extension $N/F$, let $RT_{L/F}(N)$ be subgroup of $T_{L/F}(N)$ consisting of $R$-trivial elements (c.f. section 2). The above theorem follows from the following more general theorem (6.3):

**Theorem 1.2.** Let $K$ be a complete discretely valued field with residue field $\kappa$ and $F$ be the function field of a curve over $K$. Let $\mathcal{X}_0$ be a regular proper model of $F$ with reduced special fibre $X_0$ a union of regular curves with normal crossings. Let $L/F$ be a Galois extension over $F$ of degree $n$. Suppose

- $n$ is coprime to $\text{char}(\kappa)$,
- $K$ contains a primitive $n^{th}$ root of unity $\rho$,
- for all finite extensions $\kappa'/\kappa$ and for all finite Galois extensions $l/\kappa'$ of degree $d$ dividing $n$, $T_{l/\kappa'}(\kappa') = RT_{l/\kappa'}(\kappa') < \rho^\frac{n}{d}$,
- the graph associated to $\mathcal{X}_0$ is a tree.

Then the local-global principle holds for $T_{L/F}$-torsors.

In the last section of the paper, we also give counterexamples to the local-global principle for torsors of certain norm one tori and multi-norm tori over a semiglobal field.

We now briefly describe the structure of the paper. Let $K$ be a complete discretely valued field with residue field $\kappa$ and $F$ be the function field of a curve over $K$. Let $\mathcal{X}$ be a regular proper model of $F$ with reduced special fibre $X$ a union of regular curves with normal crossings. For any point $P \in X$, let $F_P$ be the fraction field of the completion of the local ring at $P$ on $\mathcal{X}$. Let $G$ be a linear algebraic group over $F$ and Let $\Xi_X(F,G)$ be the kernel of the natural map $H^1(F,G) \to \prod_{P \in X} H^1(F_P,G)$ and $\Xi(F,G)$ be the kernel of the natural map $H^1(F,G) \to \prod_{\nu \in \Omega_F} H^1(F_\nu,G)$. It is known that $\Xi_X(F,G) \subseteq \Xi(F,G)$ ([12, Proposition 8.2.]).

First we prove the following (6.2):

**Theorem 1.3.** Let $K$ be a complete discretely valued field with residue field $\kappa$ and $F$ be the function field of a curve over $K$. Let $\mathcal{X}$ be a regular proper model of $F$ with reduced special fibre $X$ a union of regular curves with normal crossings. Let $L/F$ be a Galois extension over $F$ of degree $n$. Suppose

- $n$ is coprime to $\text{char}(\kappa)$,
• $K$ contains a primitive $n^{th}$ root of unity $\rho$,
• for all finite extensions $\kappa'/\kappa$ and for all finite Galois extensions $l/\kappa'$ of degree $d$ dividing $n$, 

$$T_{l/\kappa'}(\kappa') = RT_{l/\kappa'}(\kappa') < \rho^d >,$$

• the graph associated to $\mathcal{X}$ is a tree.

Then $\mathfrak{X}_X(F,T_{L/F}) = 0$.

We conclude our main theorem, by proving that for $K$, $F$ and $L$ as in (1.3), $\mathfrak{X}(F,T_{L/F}) = \bigcup_{X} \mathfrak{X}_X(F,G)$ (5.4), where $X$ is running over the reduced special fibre of regular proper models of $\mathcal{X}$ of $F$ which are obtained as a sequence of blow-ups of $\mathcal{X}_0$ centered at closed points of $\mathcal{X}_0$.

2. Preliminaries

In this section we recall a few basic definitions and facts about patching and $R$-equivalence on algebraic groups ([3], [8], [11], [12]) which will be used in this paper.

2.1. Patching and various Shas.

Let $F$ be a field and $\Omega_F$ the set of all equivalence classes of discrete valuations $\nu$ on $F$. For $\nu \in \Omega_F$, let $F_\nu$ denote the completion of $F$ at $\nu$ and $\kappa(\nu)$ the residue field at $\nu$. For an algebraic group $G$ over $F$, let

$$\mathfrak{X}(F,G) = \ker(H^1(F,G)) \to \prod_{\nu \in \Omega_F} H^1(F_\nu,G)).$$

Let $T$ be a complete discrete valued ring with fraction field $K$, residue field $\kappa$ and $t \in T$ a parameter. Let $F$ be a function field in one variable over $K$. Then there exists a regular 2-dimensional integral scheme $\mathcal{X}$ which is proper over $T$ with the function field $F$. We call such a scheme $\mathcal{X}$ a regular proper model of $F$. Further there exists a regular proper model of $F$ with the reduced special fibre a union of regular curves with only normal crossings. Let $\mathcal{Y}$ be a regular proper model of $F$ with the reduced special fibre $X$ a union of regular curves with only normal crossings.

For any point $x$ of $\mathcal{Y}$, let $R_x$ be the local ring at $x$ on $\mathcal{Y}$, $\hat{R}_x$ the completion of the local ring $R_x$, $F_x$ the fraction field of $\hat{R}_x$ and $\kappa(x)$ the residue field at $x$.

For any subset $U$ of $X$ that is contained in an irreducible component of $X$, let $R_U$ be the subring of $F$ consisting of the rational functions which are regular at every point of $U$. Let $\hat{R}_U$ be the $t$-adic completion of $R_U$ and $F_U$ the fraction field of $\hat{R}_U$.

Let $\eta \in X$ be a codimension zero point and $P \in X$ a closed point such that $P$ is in the closure $X_\eta$ of $\eta$. Such a pair $(P, \eta)$ is called a branch. For a branch $(P, \eta)$, we define $F_{P,\eta}$ to be the completion of $F_P$ at the discrete valuation of $F_P$ associated to $\eta$. If $\eta$ is a codimension zero point of $X$, $U \subset X_\eta$ an open subset and $P \in X_\eta$ a closed point, then we will use $(P,U)$ to denote the branch $(P, \eta)$ and $F_{P,U}$ to denote the field $F_{P,\eta}$. With $P, U, \eta$ as above, there are natural inclusions of $F_P$, $F_U$ and $F_\eta$ into $F_{P,\eta} = F_{P,U}$. Also, there is a natural inclusion of $F_U$ into $F_\eta$. Let $\mathcal{P}$ be a non-empty finite set of closed points of $X$ that contains all the closed points of $X$ where distinct irreducible components of $X$ meet. Let $\mathcal{U}$ be the set of connected components of the complement of $\mathcal{P}$ in $X$ and let $\mathcal{B}$ be the set of branches $(P,U)$ with $P \in \mathcal{P}$ and $U \in \mathcal{U}$ with $P$ in the closure of $U$. 
Let $G$ be a linear algebraic group over $F$. Let
\[
\bigoplus_{\mathcal{P}} H^1(F, G) = \ker(H^1(F, G) \rightarrow \prod_{\xi \in \mathcal{P} \cup \mathcal{U}} H^1(F_{\xi}, G)).
\]
If $\mathcal{X}$ is understood, we will just use the notation $\bigoplus_{\mathcal{P}} H^1(F, G)$. Let
\[
\bigoplus_{\mathcal{P}} H^1(F_{\mathcal{P}}, G) = \ker(H^1(F, G) \rightarrow \prod_{P \in \mathcal{P}} H^1(F_{P}, G)).
\]
Again, if $\mathcal{X}$ is understood, we will just use the notation $\bigoplus_{\mathcal{P}} H^1(F, G)$.

We have a bijection([12, Corollary 3.6.]):
\[
\prod_{U \in \mathcal{U}} G(F_U) \setminus \prod_{(P, U) \in \mathcal{B}} G(F_{P, U})/ \prod_{P \in \mathcal{P}} G(F_P) \rightarrow \bigoplus_{\mathcal{P}} H^1(F, G)
\]

By ([12, Corollary 5.9.]), we have $\bigoplus_{\mathcal{P}} H^1(F, G) = \bigcup \bigoplus_{\mathcal{P}} H^1(F, G)$ where union ranges over all finite subsets $\mathcal{P}$ of closed points of $\mathcal{X}$ which contain all the singular points of $X$. We also have $\bigoplus_{\mathcal{P}} H^1(F, G) \subseteq \bigoplus_{\mathcal{F}, \mathcal{G}} H^1(F, G)$ ([12, Proposition 8.2.]).

2.2. The associated graph.

**Lemma 2.1.** Let $\Gamma$ be a bipartite graph and $G$ be an abstract group. Let $V$ be the set of vertices with parts $V_1$ and $V_2$. For each edge $\theta$ of $\Gamma$, let $g_\theta \in G$. If $\Gamma$ is a tree, then for every vertex $v \in V$, there exists $g_v \in G$ such that if $e$ is an edge joining two vertices $v_1 \in V_1$, then $g_{v_1} g_{v_2} = g_e = g_{v_1} g_{v_2}$.

**Proof.** Suppose $\Gamma$ is a tree. Without loss of generality, we may assume that $\Gamma$ is a connected graph. We prove the lemma by the induction on number of vertices. Suppose $\Gamma$ has one one vertex. Then there is nothing to prove.

Suppose that $\Gamma$ has more than one vertex. Since $\Gamma$ is a connected tree, there exists a vertex $v_0 \in V$ with exactly one edge $\theta$ at $v_0$. Without loss of generality, we may assume $v_0 \in V_1$. Let $\Gamma'$ be the graph obtained from $\Gamma$ by removing the vertex $v_0$ and the edge $\theta$. Then $\Gamma'$ is again a bipartite graph which is a tree. Then, by induction hypothesis, for every vertex $v$ of $\Gamma'$, there exists $g_v \in G$ such that if $e$ is an edge in $\Gamma'$ joining $v_1 \in V_1 \setminus \{v_0\}$ and $v_2 \in V_2$, then $g_{v_1} g_{v_2} = g_e = g_{v_1} g_{v_2}$. Let $v'_1 \in V_1 \setminus \{v_0\}$ and $v'_2 \in V_2$. Then it follows that $g_{v_0} = g_{v'_1} g_{v'_2}^{-1}$.

Let $T$ be a complete discrete valued ring with fraction field $K$, residue field $\kappa$ and $t \in T$ a parameter. Let $F$ be a function field in one variable over $K$ and $\mathcal{X}$ a regular proper model of $F$ with the reduced special fibre $X$ a union of regular curves with only normal crossings. Let $\mathcal{P}$ be a non-empty finite set of closed points of $X$ that contains all the closed points of $X$ where distinct irreducible components of $X$ meet. Let $\mathcal{U}$ be the set of connected components of the complement of $\mathcal{P}$ in $X$ and let $\mathcal{B}$ be the set of branches $(P, U)$ with $P \in \mathcal{P}$ and $U \in \mathcal{U}$ with $P$ in the closure of $U$.

We have a graph $\Gamma(\mathcal{X}, \mathcal{P})$ associated to $\mathcal{X}$ and $\mathcal{P}$ whose vertices are elements of $\mathcal{P} \cup \mathcal{U}$ and edges are elements of $\mathcal{B}$. Since there are no edges between any vertices which are in $\mathcal{P}$ (resp. $\mathcal{U}$), $\Gamma(\mathcal{X}, \mathcal{P})$ is a bipartite graph with parts $\mathcal{P}$ and $\mathcal{U}$. If $\mathcal{P}'$ is another finite set of closed points of $X$ containing all the closed points of $X$ where distinct irreducible components of $X$ meet, then $\Gamma(\mathcal{X}, \mathcal{P})$ is a tree is and only if $\Gamma(\mathcal{X}, \mathcal{P}')$ is a tree ([12, Remark 6.1(b)]). Hence if $\Gamma(\mathcal{X}, \mathcal{P})$ is a tree for some $\mathcal{P}$ as above, then we say that the graph $\Gamma(\mathcal{X})$ associated to $\mathcal{X}$ is a tree.
Corollary 2.2. Let $F$, $\mathcal{X}$, $X$, $\mathcal{P}$, $\mathcal{U}$ and $\mathcal{B}$ be as above. Let $G$ be an abstract group and for each branch $b \in \mathcal{B}$, let $g_b \in G$. Suppose that the graph $\Gamma(\mathcal{X})$ associated to $\mathcal{X}$ is a tree. Then for every $\zeta \in \mathcal{U} \cup \mathcal{P}$, there exists $g \in G$ such that if $b = (P, U) \in \mathcal{B}$, then $g_b = g_{\zeta} g_U$.

2.3. $R$-equivalence and $R$-trivial elements.

Notation 2.3. Let $F$ be a field and $L$ be an étale algebra over $F$. Throughout this paper, we will denote the norm 1 torus $R^1_{L/F} \mathbb{G}_m$ by $T_{L/F}$.

Let $X$ be a variety over a field $F$. For a field extension $L$ of $F$, let $X(L)$ be the set of $L$-points of $X$. We say that two points $x_0, x_1 \in X(L)$ are elementary $R$-equivalent, denoted by $x_0 \sim x_1$, if there is a rational map $f : \mathbb{P}^1 \cdots \to X_L$ such that $f(0) = x_0$ and $f(1) = 1$. The equivalence relation generated by $\sim$ is called $R$-equivalence. When $X = G$ an algebraic group defined over $F$ with the identity element $e$, we define $RG(L) = \{ x \in G : x$ is $R$-equivalent to $e \}$. The elements of $RG(L)$ are called $R$-trivial elements. Let $L/F$ be a Galois extension with the Galois group $G$, and $T_{L/F}$ the norm 1 torus associated to the extension $L/F$. Then for any extension $N/F$, $RT_{L/F}(N)$ is the subgroup generated by the set $\{ a^{-1} \sigma(a) : a \in (L \otimes_F N)^*, \sigma \in G \}$ ([3, Proposition 15]).

The following statements will be used in the later sections:

Proposition 2.4. Let $F$ be a field and $L_0/F$ a finite extension. Let $L$ be the product of $r$ copies of $L_0$. Then the homomorphism $T_{L/F} \to T_{L_0/F}$ given by $(a_1, \cdots, a_r) \mapsto a_1 \cdots a_r$ induces an isomorphism $T_{L/F}(F)/RT_{L/F}(F) \to T_{L_0/F}(F)/RT_{L_0/F}(F)$.

Proof. In fact the isomorphism $(R_{L_0/F}(\mathbb{G}_m))^r \to (R_{L_0/F}(\mathbb{G}_m))^r$ given by sending $(b_1, \cdots, b_r)$ to $(b_1, \cdots, b_{r-1}, b_1b_2 \cdots b_r)$ induces an isomorphism of algebraic groups $T_{L/F} \to (R_{L_0/F}(\mathbb{G}_m))^{-1} \times T_{L_0/F}$ ([7, Lemma 1.1]). Since $R_{L_0/F}(\mathbb{G}_m)$ is rational, it is $R$-trivial by ([8, Corollary 1.6]). Hence the homomorphism $T_{L/F} \to T_{L_0/F}$ given by $(a_1, \cdots, a_r) \mapsto a_1 \cdots a_r$ induces an isomorphism $T_{L/F}(F)/RT_{L/F}(F) \to T_{L_0/F}(F)/RT_{L_0/F}(F)$.  □

Corollary 2.5. Let $F$ be a field and $L_0/F$ a finite extension of degree $d$ and $L$ the product of $r$ copies of $L_0$. Suppose that $F$ contains $\rho$, a primitive $(dr)^{th}$ root of unity. If $T_{L_0/F}(F) = RT_{L_0/F}(F) < \rho^r$, then $T_{L/F}(F) = RT_{L/F}(F) < \rho^r$.

Proof. Since $(\rho, \rho, \cdots, \rho)$ maps to $\rho^r$ under the isomorphism given in (2.4), the corollary follows from (2.4). □

Lemma 2.6. Let $L/F$ be a finite Galois extension of degree $m$ and $N/F$ any field extension. If $\alpha \in (L \otimes_F N)^*$, then $N_{L \otimes_F N/N}(\alpha)^{-1} \alpha^m \in RT_{L/F}(N)$.

Proof. Let $G$ be the Galois group of $L/F$. Since $N_{L \otimes_F N/N}(\alpha) = \prod_{\sigma \in G} \sigma(\alpha)$, we have

$$N_{L \otimes_F N/N}(\alpha)^{-1} \alpha^m = \prod_{\sigma \in G} \sigma(\alpha)^{-1} \alpha^m = \prod_{\sigma \in G} \frac{\alpha^m}{\sigma(\alpha)} = \prod_{\sigma \in G} \frac{\alpha}{\sigma(\alpha)}.$$

Hence $N_{L \otimes_F N/N}(\alpha)^{-1} \alpha^m \in RT_{L/F}(N)$. □

3. Norm one elements - complete discretely valued fields

Lemma 3.1. Let $F$ be a complete discretely valued field with residue field $\kappa$ and $L/F$ a finite Galois extension of degree $n$ with residue field $l$. Suppose that $n$ is coprime to $\text{char}(\kappa)$. Let $z \in T_{L/F}(F)$. If the image of $z$ in $\mathfrak{l}$ is 1, then $z \in RT_{L/F}(F)$.?
Proof. Let $S$ be the integral closure of $R$ in $L$. Then $S$ is a complete discrete valuation ring with residue field $l$. Let $z \in T_{L/F}(F)$ with the image of $z$ in $l$ is 1. Since $n$ is coprime to char($\kappa$), by Hensel’s lemma, there is a $w \in S$ with $\overline{w} = 1$ and $z = w^n$. Since $N_{L/F}(z) = 1$, $N_{L/F}(w)^n = 1$ and hence $\rho = N_{L/F}(w)$ is an $n^{th}$ root of unity. Since $\overline{w} = 1$, $N_{L/F}(w) = N_{\sigma}(\overline{w})^e = 1$, where $e$ is the ramification index of the extension $L/F$. Hence $\overline{\rho} = 1$. Since $n$ is coprime to char($\kappa$), by Hensel’s lemma, the quotient map $S \rightarrow l$ induces a bijection from the set of $n^{th}$ roots of unity in $S$ to the set of $n^{th}$ roots of unity in $l$. Hence $\rho = 1$ and $w \in T_{L/F}(F)$. Since $z = w^n$, $z \in RT_{L/F}(F)$ by (2.6).

\section*{Lemma 3.2.} Let $F$ be a complete discretely valued field with residue field $\kappa$. Let $L/F$ be a Galois extension of degree $n$. Suppose that $(n,\text{char}(\kappa)) = 1$. Suppose that $F$ contains a primitive $n^{th}$ root of unity $\rho_n$. Let $l$ be the residue field of $L$ and $f = [l : \kappa]$. If $T_{l/\kappa}(\kappa) = RT_{l/\kappa}(\kappa) < \rho_n^{\rho_l/f}$, then $T_{L/F}(F) = RT_{L/F}(F) < \rho_n$.

Proof. Let $R$ be the discrete valuation ring of $F$ and $S$ be the integral closure of $R$ in $L$. Let $e$ be the ramification index of the extension $L/F$. Then $n = ef$. For any element $y \in S$ (resp. $R$), we will use $\overline{y}$ to denote its image in the residue field $l$ (resp. $\kappa$).

Let $x \in L$ with $N_{L/F}(x) = 1$. Then, $N_{l/\kappa}(\overline{x})^e = N_{L/F}(x) = 1$. Hence $N_{l/\kappa}(\overline{x}) = \rho_n^{f_i}$ for some $i$ with $0 \leq i < e$. Let $y = \rho_n^{-i}x$. Then $N_{L/F}(y) = 1$ and $N_{l/\kappa}(\overline{y}) = N_{l/\kappa}(\rho_n^{-i})N_{l/\kappa}(\overline{x}) = \rho_n^{-i}\rho_n^{f_i} = 1$. Thus $\overline{y} \in T_{l/\kappa}(\kappa)$ and hence, by the assumption, $\overline{y} = \theta \rho_n^{j}$ for some $\theta \in RT_{l/\kappa}(\kappa)$ and $j$ an integer. Write

$$\theta = \prod_{\sigma \in \text{Gal}(l/\kappa)} (a_\sigma)^{-1}\sigma(a_\sigma)$$

for some $a_\sigma \in l^*$. Since $\text{Gal}(l/\kappa)$ is a quotient of $\text{Gal}(L/F)$, for every $\sigma \in \text{Gal}(l/\kappa)$ we choose a lift $\tilde{\sigma} \in \text{Gal}(L/F)$ of $\sigma$. Let $b_\sigma \in S$ with $\overline{b_\sigma} = a_\sigma$ and

$$z = y^{-1}\rho_n^{-ej} \prod_{\sigma \in \text{Gal}(l/\kappa)} (b_\sigma)^{-1}\tilde{\sigma}(b_\sigma).$$

Then $z \in T_{L/F}(F)$ and $\overline{z} = 1$. Thus, by (3.1), $z \in RT_{L/F}(F)$. Therefore $y \in RT_{L/F}(F) < \rho_n$ and hence $x \in RT_{L/F}(F) < \rho_n$.

\section*{Definition 3.3.} A local field $K$ is called a 1-local field. For $m \geq 1$, a complete discretely valued field $K$ with $m$-local residue field $k$ is called a $m + 1$-local field. If $K$ is a 1-local field, the residue field of $K$ is called the first residue field of $K$. If $K$ is a $m + 1$-local field with residue field $k$, then the first residue field of $k$ is called the first residue field of $K$.

\section*{Corollary 3.4.} Let $K$ be a $m$-local field with first residue field $\kappa$ and $L/K$ a finite Galois extension of degree $n$. If $n$ is coprime to char($\kappa$) and $K$ contains a primitive $n^{th}$ root of unity $\rho_n$. Then $T_{l/K}(K) = RT_{l/K}(K) < \rho_n$.

Proof. Suppose $K$ is 1-local. Then $\kappa$ is a finite field. Hence every finite extension $l/\kappa$ is cyclic and by Hilbert 90, $T_{l/\kappa}(\kappa) = RT_{l/\kappa}(\kappa)$. Thus, by (3.2), $T_{L/K}(F) = RT_{L/K}(K) < \rho_n$. The corollary follows by induction on $m$ and by (3.2).

4. Extensions of two dimensional complete fields

Let $K$ be a field with a discrete valuation $v$. Let $\kappa(v)$ be the residue field of $v$. Let $L/K$ be a finite extension and $w$ a discrete valuation on $L$ extending $v$. Let $e(w/v)$
be the ramification index of $w$ over $v$. For any field $E$, $a \in E^*$ and $n \geq 1$, let $E(\sqrt[n]{a})$ denote the field generated by $E$ and $\sqrt[n]{a}$ in a fixed algebraic closure of $E$.

We begin with the following well-known fact:

**Lemma 4.1.** Let $F$ be a with a discrete valuation $v$, $\pi \in F^*$ with $v(\pi) = 1$. Let $L/F$ be a finite extension of degree coprime to $\text{char}(\kappa(v))$ and $w$ be a discrete valuation of $L$ extending $v$. Let $\ell$ be a prime not equal to the characteristic $\kappa(v)$. Then there is a unique discrete valuation $\tilde{v}$ on $F(\sqrt{\pi})$ extending $v$. Let $\tilde{w}$ on $L(\sqrt{\pi})$ be a discrete valuation extending $w$. Then the following holds:

- if $\ell$ does not divide $e(w/v)$, $e(\tilde{w}/\tilde{v}) = e(w/v)$.
- if $\ell$ divides $e(w/v)$, then $e(\tilde{w}/\tilde{v}) = e(w/v)/\ell$.

**Proof.** Since $v(\pi) = 1$, $v$ is totally ramified in $F(\sqrt{\pi})$ and hence there is a unique extension $\tilde{v}$ of $v$ to $F(\sqrt{\pi})$.

For the ramification index calculations, we can replace $F$ by $F_v$, the completion of $F$ with respect to the valuation $v$ and assume that $F$ is complete. Let $L^{nr}$ be the maximal unramified subextension of $L/F$. Since the ramification index of $L/L^{nr}$ is same as the ramification index of $L/F$, replacing $F$ by $L^{nr}$, we assume that $L/F$ is totally ramified. Since $n = e = [L:F]$ is coprime to $\text{char}(\kappa(v))$, we have $L = F(\sqrt{u\pi})$ for some $u \in F$ with $v(u) = 0$ ([2, Proposition 1, p-32]).

Suppose $\ell$ does not divide $n$. Then $L(\sqrt{\pi})/F$ is totally ramified extension of degree $n\ell$ and hence $L(\sqrt{\pi})/F(\sqrt{\pi})$ is a totally ramified extension of degree $n$.

Suppose $\ell$ divides $n$. Suppose $u \in F^{*\ell}$. Then $F(\sqrt{\pi}) \subseteq L$ and hence $L/F(\sqrt{\pi})$ is a totally ramified extension of degree $n/\ell$. Suppose $u \notin F^{*\ell}$. Then $L(\sqrt{\pi}) = F(\sqrt{\pi})(\sqrt{u})^i$. Hence the ramification index of the extension $L(\sqrt{\pi})/F(\sqrt{\pi})$ is $n/\ell$.

**Notation 4.2.** Let $A$ be a complete regular local ring of dimension 2 with residue field $\kappa$ and field of fractions $F$. Let $\mathfrak{m} = (\pi_1, \pi_2) \subset A$ be the maximal ideal of $A$. Then we denote by $\hat{A}_{(\pi_i)}$ the completion of the local ring $A_{(\pi_i)}$ with respect to the ideal $(\pi_i)$ and by $F_{\pi_i}$ the fraction field of $\hat{A}_{(\pi_i)}$.

**Lemma 4.3.** Let $A$ be a complete regular local ring of dimension 2 with residue field $\kappa$ and field of fractions $F$. Let $L/F$ be a field extension of degree $n$ where $n$ is coprime to the $\text{char}(\kappa)$. Let $\mathfrak{m} = (\pi_1, \pi_2) \subset A$ be the maximal ideal of $A$. Suppose that $L/F$ unramified on $A$ except possibly at $\pi_1$ and $\pi_2 \in A$. Then $L \otimes_F F_{\pi_i}$ is a field.

**Proof.** Let $v_i$ be the discrete valuations of $F$ given by $\pi_i$. To show that $L \otimes_F F_{\pi_i}$ is a field, it is enough to show that there is a unique extension of $v_i$ to a discrete valuation on $L$.

Let $w_i$ be the extensions of the valuations $v_i$ to $L$. Let $m$ be the maximum of $e(w_i^j/v_i)$. Since each $e(w_i^j/v_i) \geq 1$, $m \geq 1$. We prove the result by induction on $m$.

Suppose $m = 1$. Then $e(w_i^j/v_i) = 1$ for all $i$ and $j$ and hence $L/F$ is unramified at $\pi_i$ for $i = 1, 2$. Since $L/F$ is unramified on $A$ except possibly at $\pi_1, \pi_2$, $L/F$ is unramified on $A$. Let $\hat{A}$ be the integral closure of $A$ in $L$. Then $\hat{A}$ is again a complete regular local ring of dimension 2 with $(\pi_1, \pi_2)$ as maximal ideal and fraction field $L$. 
Thus $\pi_i$ remains a prime over $\tilde{A}$. Hence there is a unique extension of $v_i$ to a discrete valuation of $L$. Hence $L \otimes_F F_{\pi_i} \cong L_{\pi_i}$ is a field.

Suppose $m > 1$. Let $\ell$ be a prime which divides $m$. Let $E = F(\sqrt[p]{\pi_1})$ and $M = L(\sqrt[p]{\pi})$. Let $B$ be the integral closure of $A$ in $E$. Then $B$ is a regular local ring with maximal ideal $(\sqrt[p]{\pi_1}, \pi_2)$ by ([16, Lemma 3.2.]). Let $\pi'_1 = \sqrt[p]{\pi_1}$ and $\pi'_2 = \pi_2$. Then $M/E$ is unramified on $B$ except possibly at $\pi'_1$ and $\pi'_2$.

Let $\tilde{v}_i$ be the unique extension of $v_i$ to $E$ (c.f. 4.1). Let $\omega$ be a discrete valuation of $M$ extending $\tilde{v}_i$ for some $i$. Then the restriction of $\omega$ to $L$ is equal to $w_i^j$ for some $j$. Then $e(\omega/\tilde{v}_i) \leq e(w_i^j/v_i)$ (c.f. 4.1). Suppose $e(w_i^j/v_i) = m$. Since $\ell$ divides $e(w_i^j/v_i)$, by (4.1), $e(\omega/\tilde{v}_i) = e(w_i^j/v_i)/\ell = m/\ell < m$. Hence, by induction hypothesis, for each $i = 1, 2$, there is a unique extension of $\tilde{v}_i$ to $M$. Since $L$ is a subfield of $M$, there is a unique extension of $v_i$ to $L$. Hence $L \otimes_F F_{\pi_i}$ is a field. □

**Lemma 4.4.** Let $A$ be a complete regular local ring of dimension 2 with residue field $\kappa$ and field of fractions $F$. Let $L/F$ be a Galois extension of degree $n$ where $n$ is coprime to the $\text{char}(\kappa)$. Let $m = (\pi_1, \pi_2) \subset A$ be the maximal ideal of $A$. Suppose that $L/F$ is unramified on $A$ except possibly at $\pi_1$ and $\pi_2 \in A$ and totally ramified at $\pi_2$. Then $L = F(\sqrt[p]{\nu\pi_1^m\pi_2})$ for some $u \in A$ a unit.

**Proof.** Let $G$ be the Galois group of $L/F$. Since the degree of $L/F$ is coprime to $\text{char}(\kappa)$ and $L/F$ is unramified on $A$ except possibly at $\pi_1$ and $\pi_2$, by (4.3), $L \otimes_F F_{\pi_2}$ is a field. Since $L \otimes_F F_{\pi_2}$ is a totally tamely ramified extension, $F_{\pi_2}$ contains a primitive $n^{th}$ root of unity and we have $L \otimes_F F_{\pi_2} = F_{\pi_2}(\sqrt[p]{\theta \pi_2})$ for some $\theta \in F_{\pi_2}$ which is a unit in the discrete valuation ring of $F_{\pi_2}$ by ([2, Proposition 1, p-32]). In particular $G$ is a cyclic group. Since $F_{\pi_2}$ contains a primitive $n^{th}$ root of unity, the residue field $\kappa(\pi_2)$ of $F_{\pi_2}$ contains a primitive $n^{th}$ root of unity. Since $\kappa$ is the residue field of $\kappa(\pi_2)$, $\kappa$ also contains a primitive $n^{th}$ root of unity. Since $A$ is complete, by Hensel’s lemma, $F$ contains a primitive $n^{th}$ root of unity.

Hence $L = F(\sqrt[p]{a})$ for some $a \in F$. Since $L/F$ is unramified on $A$ except possibly at $\pi_1, \pi_2$, we can choose $a = u\pi_1^m\pi_2^d$ for some $u \in A$ a unit and integers $m, d$. Since $L/F$ is totally ramified at $\pi_2$, $d$ is coprime to $n$ and hence we can assume that $d = 1$. □

**Lemma 4.5.** Let $A$ be a complete regular local ring of dimension 2 with residue field $\kappa$ and field of fractions $F$. Let $L/F$ be a Galois extension of degree coprime to the $\text{char}(\kappa)$. Let $m = (\pi_1, \pi_2) \subset A$ be the maximal ideal of $A$. Suppose that $L/F$ is unramified on $A$ except possibly at $\pi_1$. Then there exists a subextension $L_1/F$ of $L/F$ such that

- $L_1/F$ is unramified on $A$
- $L = L_1(\sqrt[p]{u\pi_1})$ for some unit $u$ in the integral closure of $A$ in $L_1$.

**Proof.** Let $G$ be the Galois group of $L/F$. Since $L \otimes_F F_{\pi_1}/F_{\pi_1}$ is a field extension (4.3), the Galois group $\text{Gal}(L \otimes_F F_{\pi_1}/F_{\pi_1}) \cong G$. We will identify these two groups. We consider the inertia group $H$ of the extension $L \otimes_F F_{\pi_1}/F_{\pi_1}$ which is a subgroup of $G$. Let $L_1 = L^H$. Then $(L \otimes_F F_{\pi_1})^H = L_1 \otimes_F F_{\pi_1}$ is unramified over $F_{\pi_1}$ by ([2, Theorem 2, p-27]). Hence $L_{1,\pi_1} \cong L_1 \otimes_F F_{\pi_1}$ is unramified over $F_{\pi_1}$ and $L_1/F$ is unramified at $\pi_1$. Since $L/F$ is unramified on $A$ except possibly at $\pi_1$, $L_1/F$ is
unramified on \( A \). Then the integral closure \( B \) of \( A \) in \( L_1 \) is a regular local ring with maximal ideal \((\pi_1, \pi_2)\). Let \( e = [L:L_1] \). Since \( L/F \) is unramified on \( A \) except possibly at \( \pi_1 \), \( L/L_1 \) is unramified on \( B \) except possibly at \( \pi_1 \). Hence by (4.4), we have \( L = L_1(\sqrt[2]{u_1} \pi_1) \) for some \( u \in B \) a unit. Since \( L/L_1 \) is unramified on \( B \) except possibly at \( \pi_1, m \) is divisibly by \( f \) and hence \( L = L_1(\sqrt[2]{u_1}) \).

**Theorem 4.6.** Let \( A \) be a complete regular local ring of dimension 2 with residue field \( \kappa \) and field of fractions \( F \). Let \( L/F \) be a Galois extension of degree coprime to the \( \text{char}(\kappa) \). Let \( \mathfrak{m} = (\pi_1, \pi_2) \subset A \) be the maximal ideal of \( A \). Suppose that \( L/F \) unramified on \( A \) except possibly at \( \pi_1, \pi_2 \in A \). Then there exists subfields \( L_1 \) and \( L_2 \) of \( L \) such that

- \( F \subseteq L_1 \subseteq L_2 \subseteq L \)
- \( L_1/F \) is unramified on \( A \)
- \( L_2 = L_1(\sqrt[2]{u_1}) \) for some unit \( u \) in the integral closure of \( A \) in \( L_1 \)
- \( L = L_2(\sqrt[2]{v(\sqrt[2]{u_1})^s \pi_2}) \) for some unit \( v \) in the integral closure of \( A \) in \( L_2 \).

**Proof.** Let \( G \) be the Galois group of \( L/F \). Since \( L \otimes F \pi_2/F \pi_2 \) is a field extension (4.3), the Galois group \( \text{Gal}(L \otimes F \pi_2/F \pi_2) \cong G \). We identify these two groups. We consider the inertia group \( H \) of the extension \( L \otimes F \pi_2/F \pi_2 \) which is a subgroup of \( G \). Let \( L_2 = L^H \). Then, as in (4.5), \( L_2/F \) is unramified on \( A \) except possibly at \( \pi_1 \). Hence, by (4.5), there exists a sub extension \( L_1/F \) of \( L_2/F \) such that \( L_1/F \) is unramified on \( A \) and \( L_2 = L_1(\sqrt[2]{u_1}) \) for some unit \( u \) in the integral closure of \( A \) in \( L_1 \). Let \( B \) be the integral closure of \( A \) in \( L_2 \). Then \( B \) is a regular local ring with maximal ideal \((\sqrt[2]{u_1}, \pi_2)\) by ([16, Lemma 3.2.]). Since \( L_1/L_2 \) is unramified on \( B \) except possibly at \( \sqrt[2]{u_1}, \pi_2 \) and totally ramified at \( \pi_2 \), by (4.4), \( L = L_2(\sqrt[2]{v(\sqrt[2]{u_1})^s \pi_2}) \) for some unit \( v \in B \). \( \boxdot \)

5. III vs \( \text{III}_X \)

**Lemma 5.1.** Let \( A \) be a complete regular local ring of dimension 2 with residue field \( \kappa \) and fraction field \( F \). Let \( \mathfrak{m} = (\pi_1, \pi_2) \) be the maximal ideal of \( A \). Let \( m \geq 1 \) be an integer coprime to \( \text{char}(\kappa) \). Let \( F_{\pi_1} \) be the completion of \( F \) at the discrete valuation of \( F \) given by \( \pi_1 \). Then every element in \( F_{\pi_1} \) can be written as \( u_1^s \pi_2^m b^m \) for some \( u \in A \) unit, \( b \in F_{\pi_1} \) and integers \( s, t \).

**Proof.** Let \( \widehat{A}_{(\pi_1)} \) be the completion of the local ring \( A_{(\pi_1)} \). Then \( F_{\pi_1} \) is the fraction field of \( \widehat{A}_{(\pi_1)} \). Let \( x \in F_{\pi_1} \). Then \( x = v_1^s \) for some unit \( v \in \widehat{A}_{(\pi_1)} \) and integer \( s \). Let \( \pi \) be the image of \( v \) in the residue field \( \kappa(\pi_1) \) of \( F_{\pi_1} \). Since \( \kappa(\pi_1) \) is the fraction field of \( A/(\pi_1) \) and \( A/(\pi_1) \) is a discrete valuation ring with the image \( \pi_2 \) as a parameter, we can write \( \pi = z_1^t \) for some unit \( z \in A/(\pi_1) \) and integer \( t \). Let \( \omega \) be the image of \( z \) in \( \kappa(\pi_1) \). Since \( z \) is a unit in \( A/(\pi_1) \), \( \omega \) is a unit in \( A \). Hence \( x^{-1} \omega_1^{s_1} \pi_2^m \) maps to 1 in \( \kappa(\pi_1) \). Since \( m \) is coprime to \( \text{char}(\kappa) \), by Hensel’s lemma, \( x = \omega_1^{s_1} \pi_2^m \) maps to 1 in \( \kappa(\pi_1) \). \( \boxdot \)

**Lemma 5.2.** Let \( A \) be a complete regular local ring of dimension 2 with residue field \( \kappa \) and fraction field \( F \). Let \( L/F \) be a Galois field extension of degree \( n \) where \( n \) is coprime to \( \text{char}(\kappa) \). Let \( \mathfrak{m} = (\pi_1, \pi_2) \) be the maximal ideal of \( A \). Suppose that \( L/F \)
is unramified on $A$ except possibly at $\pi_1$. Let $\lambda = u\pi_1^r\pi_2^s$ for some unit $u \in A$ and integers $r, s$. If $\lambda$ is a norm from the extension $L \otimes_F F_{\pi_1}/F_{\pi_1}$, then $\lambda$ is a norm from the extension $L/F$.

Proof. Let $\lambda = u\pi_1^r\pi_2^s$ for some unit $u \in A$ and integers $r, s$. Suppose that $\lambda$ is a norm from $L \otimes_F F_{\pi_1}/F_{\pi_1}$. Let $\mu \in L \otimes_F F_{\pi_1}$ be such that $N_{L \otimes_F F_{\pi_1}/F_{\pi_1}}(\mu) = \lambda$.

Since $L/F$ is a Galois extension which is unramified on $A$ except possibly at $\pi_1$, by (4.5), we have a subfield $F \subseteq L_1 \subseteq L$ such that $L_1/F$ is unramified on $A$ and $L = L_1(\sqrt[\nu]{\Delta})$ where $\nu = [L : L_1]$ and $\nu$ is a unit in the integral closure of $A$ in $L_1$. Let $B$ be the integral closure of $A$ in $L$. Then $B$ is a regular local ring with maximal ideal $(\sqrt[\nu]{\Delta}, \pi_1, \pi_2)$ by ([16, Lemma 3.2.]). Hence, by (5.1), $\mu = w\sqrt[\nu]{\Delta}\pi_1^{i}\pi_2^{j}$ for some integers $i, j, b \in L \otimes F F_{\pi_1}$ and $w$ a unit in $B$. Let $\theta = w\sqrt[\nu]{\Delta}\pi_1^{i}\pi_2^{j} \in L$. Since $N_{L \otimes F F_{\pi_1}/F_{\pi_1}}(\mu) = \lambda$, we have $N_{L/F}(\theta^{-1})\lambda = N_{L \otimes F F_{\pi_1}/F_{\pi_1}}(b^{n}) \in F_{\pi_1}^{n}$. Since $N_{L/F}(\theta^{-1})\lambda = [N_{L/F}(w)\sqrt[\nu]{\Delta}\pi_1^{i}\pi_2^{j}]^{-1}u\pi_1^r\pi_2^s = [u(N_{L/F}(w))^{-1}]\pi_1^{-r}\pi_2^{-s-nj}$ and $u(N_{L/F}(w))^{-1}$ is a unit in $A$, by ([17, Corollary 5.5.]), $N_{L/F}(\theta^{-1})\lambda \in F_{\pi_1}^{n}$. In particular $N_{L/F}(\theta^{-1})\lambda$ is a norm from the extension $L/F$ and hence $\lambda$ is a norm from $L/F$. □

Theorem 5.3. Let $A$ be a complete regular local ring of dimension 2 with residue field $\kappa$ and fraction field $F$. Let $L/F$ be a Galois field extension of degree $n$ where $n$ is coprime to $\text{char}(\kappa)$. Let $m = (\pi_1, \pi_2)$ be the maximal ideal of $A$. Suppose that $L/F$ is unramified on $A$ except possibly at $\pi_1, \pi_2$. Let $\lambda = u\pi_1^r\pi_2^s$ for some unit $u \in A$ and integers $r, s$. If $\lambda$ is a norm from the extension $L \otimes_F F_{\pi_1}/F_{\pi_1}$ then $\lambda$ is a norm from the extension $L/F$.

Proof. Let $\lambda = u\pi_1^r\pi_2^s$ for some unit $u \in A$ and integers $r, s$. Suppose that $\lambda$ is a norm from $L \otimes_F F_{\pi_1}/F_{\pi_1}$. Let $\mu \in L \otimes_F F_{\pi_1}$ be such that $N_{L \otimes_F F_{\pi_1}/F_{\pi_1}}(\mu) = \lambda$. We show by induction on the degree of the field extension $L/F$ that $\lambda$ is a norm from the extension $L/F$.

Since $L/F$ is a Galois extension which is unramified on $A$ except possibly at $\pi_1$ and $\pi_2$, we have subfields $L_1$ and $L_2$ as in (4.6).

Let $B$ be the integral closure of $A$ in $L_2$. Then $B$ is a complete regular local ring with maximal ideal $(\sqrt[\nu]{\Delta}, \pi_1, \pi_2)$ by ([16, Lemma 3.2.]). By (5.1), we have $N_{L \otimes_F F_{\pi_1}/L_2 \otimes F F_{\pi_1}}(\mu) = w\sqrt[\nu]{\Delta}\pi_1^{i}\pi_2^{j}b^{n}$ for some integers $i, j, b \in L_2 \otimes F F_{\pi_1}$ and $w$ a unit in $B$. Then $\theta = w\sqrt[\nu]{\Delta}\pi_1^{i}\pi_2^{j}$ is a norm from $L \otimes_F F_{\pi_1}/L_2 \otimes F F_{\pi_1}$.

Suppose $F \neq L_2$. Then $[L : L_2] < [L : F]$ and by induction, $\theta$ is a norm from $L/L_2$. Write $\theta = N_{L/L_2}(\theta')$. Then

$$
\lambda = N_{L \otimes_F F_{\pi_1}/F_{\pi_1}}(\mu) = N_{L_2 \otimes F F_{\pi_1}/F_{\pi_1}}(N_{L \otimes_F F_{\pi_1}/L_2 \otimes F F_{\pi_1}}(\mu)) = N_{L_2 \otimes F F_{\pi_1}/F_{\pi_1}}(\theta'^{n}) = N_{L_2 \otimes F F_{\pi_1}/F_{\pi_1}}(\theta)N_{L_2 \otimes F F_{\pi_1}/F_{\pi_1}}(b^{n}) = N_{L_2/F}(N_{L/L_2}(\theta')N_{L_2 \otimes F F_{\pi_1}/F_{\pi_1}}(b^{n}) = N_{L/F}(\theta')N_{L_2 \otimes F F_{\pi_1}/F_{\pi_1}}(b^{n})
$$

Since $N_{L/F}(\theta') = N_{L_2/F}(\theta) = N_{L_2/F}(w\sqrt[\nu]{\Delta}\pi_1^{i}\pi_2^{j})$, $N_{L/F}(\theta')^{-1}\lambda$ is a product of a unit in $A$ with a power of $\pi_1$ and a power of $\pi_2$. Since $N_{L/F}(\theta')^{-1}\lambda = N_{L_2 \otimes F F_{\pi_1}/F_{\pi_1}}(b^{n}) \in F_{\pi_1}^{n}$ by ([17, Corollary 5.5.]), we conclude that $N_{L/F}(\theta')^{-1}\lambda$ is a $n^{th}$ power in $F$ and hence a norm from $L$ to $F$. Hence $\lambda$ is also a norm from $L$ to $F$.  

10 LOCAL-GLOBAL PRINCIPLE
Now suppose $F = L_2$. Then $L = F(\sqrt[n]{\pi/2})$ where $\nu$ is a unit in $A$ and hence $L/F$ is a cyclic extension of degree $n$. Let $\sigma$ be a generator of the Galois group of $L/F$ and $C$ be the cyclic algebra $(L, \sigma, \lambda)$. Since $L/F$ is unramified on $A$ except at $\pi_2$, $C$ is a unramified on $A$ except possibly at $\pi_1$ and $\pi_2$. Since $\lambda$ is a norm from $L \otimes_F F_{\pi_1}$, $C \otimes_F F_{\pi_1}$ is a split algebra. Thus, by ([17, Corollary 5.5.]), $C$ is a split algebra and hence $\lambda$ is a norm from the extension $L/F$ by ([1, Theorem 6, p-95]).

\begin{proof}
Let $K$ be a complete discretely valued field with residue field $\kappa$. Let $F/K$ be the function field of a curve and $\mathcal{X}_0$ a regular proper model of $F$ with reduced special fibre $X_0$. Let $L/F$ be a Galois field extension of degree coprime to $\text{char}(\kappa)$. Then $\bigcup_X \mathfrak{X}(F, T_{L/F}) = \mathfrak{X}(F, T_{L/F})$, where $X$ is running over the reduced special fibres of regular proper models of $\mathcal{X}$ of $F$ which are obtained as a sequence of blow-ups of $\mathcal{X}_0$ centered at closed points of $\mathcal{X}_0$.

Let $x \in \mathfrak{X}(F, T_{L/F}) \subseteq H^1(F, T_{L/F})$. Since $H^1(F, T_{L/F}) \simeq F^*/N_{L/F}(L^*)$, let $\lambda \in F^*$ be a lift of $x$. By ([13, p-193]), there exists a sequence of blow-ups $\mathcal{X} \to \mathcal{X}_0$ centered at closed points of $\mathcal{X}_0$ such that the union of $\text{supp}_x(\lambda)$, $\text{ram}_x(L/F)$ and the reduced special fibre $X$ of $\mathcal{X}$ is a union of regular curves with normal crossings. We show that $x \in \mathfrak{X}(F, T_{L/F})$.

Let $P \in X$. Suppose $P$ is a generic point of $X$. Then $P$ gives a discrete valuation $\nu$ of $F$ with $F_{\nu} = F_P$. Since $x \in \mathfrak{X}(F, T_{L/F})$, $x$ maps to 0 in $H^1(F_P, T_{L/F})$. Suppose that $P$ is a closed point. Let $\eta_1$ be the generic point of an irreducible component of $X$ containing $P$. Let $\mathcal{O}_{\mathcal{X}, P}$ be the local ring at $P$ and $m_{\mathcal{X}, P}$ be its maximal ideal. Then, by our choice of $\mathcal{X}$, $m_{\mathcal{X}, P} = (\pi_1, \pi_2)$ where $\pi_1$ is a prime defining $\eta_1$ at $P$, $\lambda = u\pi_1^r \pi_2^s$ for some unit $u \in A$ and integers $r, s$, and $L \otimes_F F_P/F_P$ is unramified on $\mathcal{O}_{\mathcal{X}, P}$ except possibly at $\pi_1, \pi_2$. Since $L/F$ is a Galois extension, $L \otimes_F F_P = \prod L_P$ for some Galois extension $L_P/F_P$. Since $L \otimes_F F_P/F_P$ is unramified on $\mathcal{O}_{\mathcal{X}, P}$ except possibly at $\pi_1, \pi_2$, $L_P/F_P$ is unramified on $\mathcal{O}_{\mathcal{X}, P}$ except possibly at $\pi_1, \pi_2$. Since $\lambda$ is a lift of $x \in \mathfrak{X}(F, T_{L/F})$, $\lambda$ is a norm from $L \otimes_F F_{\eta_1}/F_{\eta_1}$. Since $F_{\eta_1} \subseteq F_{P_{\eta_1}}$, $\lambda$ is a norm from $L \otimes_F F_{P_{\eta_1}}/F_{P_{\eta_1}}$. Hence $\lambda$ is a norm from $L \otimes_F F_{P_{\eta_1}}/F_{P_{\eta_1}}$. Thus, by (5.3), $\lambda$ is a norm from $L_P/F_P$ and $x$ maps to 0 in $H^1(F_P, T_{L/F})$. Therefore $x \in \mathfrak{X}(F, T_{L/F})$.

By ([12, Proposition 8.2.]), we have $\bigcup_X \mathfrak{X}(F, T_{L/F}) = \mathfrak{X}(F, T_{L/F})$, where $X$ is running over the reduced special fibres of regular proper models of $\mathcal{X}$ of $F$ which are obtained as a sequence of blow-ups of $\mathcal{X}_0$ centered at closed points of $\mathcal{X}_0$.

\end{proof}

\begin{remark}
The proof of (5.4) also works if we just consider divisorial discrete valuations instead of considering all discrete valuations on $F$.
\end{remark}

6. Local Global Principle

\begin{lemma}
Let $K$ be a complete discretely valued field with residue field $\kappa$ and $F$ be the function field of a curve over $K$. Let $\mathcal{X}$ be a regular proper model of $F$ with the reduced special fibre $X$ a union of regular curves with normal crossings. Let $L/F$ be a Galois extension over $F$ of degree $n$. Let $P \in X$ be a closed point and $U$ an irreducible open subset of $X$ with $P$ in the closure of $U$. Suppose

- $n$ is coprime to $\text{char}(\kappa)$,
- $K$ contains a primitive $n^{th}$ root of unity $\rho$,
- for all finite Galois extensions $l/\kappa(P)$ of degree $d$ dividing $n$,

\[ T_{l/\kappa(P)}(\kappa(P)) = RT_{l/\kappa(P)}(\kappa(P)) < \rho^d >. \]

Then $T_{L \otimes_F F_{P,U}/F_{P,U}}(F_{P,U}) = RT_{L \otimes_F F_{P,U}/F_{P,U}}(F_{P,U}) < \rho >$. 

\end{lemma}
Proof. Let $\kappa(U)$ be the function field of $U$. Since $X$ is a union of regular curves, $P$ gives a discrete valuation on $\kappa(U)$. Let $\kappa(U)_P$ be the completion of $\kappa(U)$ at $P$. Then, by definition, $F_{P,U}$ is a complete discretely valued field with residue field $\kappa(U)_P$. Since $L/F$ is Galois extension of degree $n$, $L \otimes_F F_{P,U} \simeq \prod L_0$ for some finite Galois extension $L_0/F_{P,U}$ of degree $d$ dividing $n$. Since $F_{P,U}$ is a complete discretely valued field with residue field $\kappa(U)_P$, $L_0$ is a complete discretely valued field with residue field $M_0$ a finite extension of $\kappa(U)_P$ of degree $d_1$ dividing $d$. Since $\kappa(U)_P$ is a complete discretely valued field with residue field $\kappa(P)$, $M_0$ is a complete discretely valued field with residue field $l_0$ a finite Galois extension of $\kappa(P)$ of degree $d_2$ dividing $d_1$. Hence, by the assumption on $\kappa(P)$ and (3.2), we have

$$T_{M_0/\kappa(U)_P}(\kappa(U)_P) = RT_{M_0/\kappa(U)_P}(\kappa(U)_P) < \rho^{\frac{d_2}{d_1}}.$$  

Hence, once again by (3.2), we have

$$T_{L_0/F_{P,U}}(F_{P,U}) = RT_{L_0/F_{P,U}}(F_{P,U}) < \rho^{n/d}.$$  

Since $L \otimes_F F_{P,U}$ is the product of $\frac{n}{d}$ copies of $L_0$, by (2.5), we have

$$T_{L \otimes_F F_{P,U}/F_{P,U}}(F_{P,U}) = RT_{L \otimes_F F_{P,U}/F_{P,U}}(F_{P,U}) < \rho^d.$$  

\[\square\]

Theorem 6.2. Let $K$ be a complete discretely valued field with residue field $\kappa$ and $F$ be the function field of a curve over $K$. Let $\mathcal{X}$ be a regular proper model of $F$ with reduced special fibre $X$ a union of regular curves with normal crossings. Let $L/F$ be a Galois extension over $F$ of degree $n$. Suppose

- $n$ is coprime to $\text{char}(\kappa)$,
- $K$ contains a primitive $n^{th}$ root of unity $\rho$,
- for all finite extensions $\kappa'/\kappa$ and for all finite Galois extensions $l/\kappa'$ of degree $d$ dividing $n$,

$$T_{l/\kappa'}(\kappa') = RT_{l/\kappa'}(\kappa') < \rho^{\frac{d}{n}},$$

- the graph associated to $\mathcal{X}$ is a tree.

Then $\mathbb{III}_X(F, T_{L/F}) = 0$.

Proof. Let $P$ be a finite set of closed points of $X$ containing all the nodal points of $X$. By ([12, Corollary 5.9.]), it is enough to show that $\mathbb{III}_P(F, T_{L/F}) = 0$. Let $X \setminus P = \bigcup U_i$. Then each $U_i$ is an irreducible open subset of $X$. By ([12, Corollary 3.6.1]), it is enough to show that the product map

$$\psi : \prod_i T_{L/F}(F_{U_i}) \times \prod_{P \in P} T_{L/F}(F_P) \to \prod_{(P,U_i)} T_{L/F}(F_{P,U_i})$$

is onto, where the product on the right hand side is taken over all pairs $(P,U_i)$ with $P \in P$ and $U_i$ such that $P$ is the closure of $U_i$.

Let $(\lambda_{P,U_i}) \in \prod_{(P,U_i)} T_{L/F}(F_{P,U_i})$. We show that $(\lambda_{P,U_i})$ is in the image of $\psi$. By (6.1), for each pair $(P,U_i)$ with $P$ in the closure of $U_i$, we have $\lambda_{P,U_i} = \rho^{j_{P,U_i}} \mu_{P,U_i}$ for some integer $j_{P,U_i}$ and $\mu_{P,U_i} \in RT_{L/F}(F_{P,U_i})$. Let $G$ be the Galois group of $L/F$. For each $\sigma \in G$, there exists $a_{\sigma,P,U_i} \in (L \otimes_F F_{P,U_i})^*$ such that

$$\mu_{P,U_i} = \prod_{\sigma \in G(L/F)} \sigma(a_{\sigma,P,U_i})(a_{\sigma,P,U_i})^{-1}.$$
Since the group $R_{L/F}(\mathbb{G}_m)$ is $F$-rational, by ([11, Theorem 3.6]),

$$\prod_i (L \otimes_F F_{U_i})^* \times \prod_{P \in \mathcal{P}} (L \otimes_F F_P)^* \rightarrow \prod_{(P,U_i)} (L \otimes_F F_{P,U_i})^*$$

is onto. Hence for each $\sigma \in G$, there exist $b_{\sigma,U_i} \in (L \otimes_F F_{U_i})^*$ and $b_{\sigma,P} \in (L \otimes_F F_P)^*$ such that $a_{\sigma,P,U_i} = b_{\sigma,U_i}b_{\sigma,P}$. We have

$$\mu_{P,U_i} = \prod_{\sigma \in G(L/F)} \sigma(a_{\sigma,P,U_i})(a_{\sigma,P,U_i})^{-1}$$

$$= \prod_{\sigma \in G(L/F)} \sigma(b_{\sigma,U_i}b_{\sigma,P})(b_{\sigma,U_i}b_{\sigma,P})^{-1}$$

$$= \prod_{\sigma \in G(L/F)} \sigma(b_{\sigma,U_i})(b_{\sigma,U_i})^{-1}\sigma(b_{\sigma,P})(b_{\sigma,P})^{-1}.$$

Since $\sigma(b_{\sigma,U_i})(b_{\sigma,U_i})^{-1} \in T_{L/F}(F_{U_i})$ and $\sigma(b_{\sigma,P})(b_{\sigma,P})^{-1} \in T_{L/F}(F_P)$, $(\mu_{P,U_i})$ is in the image of $\psi$.

Since $T_{L/F}(F)$ is a group and $\rho \in T_{L/F}(F)$, by (2.2), $(\rho^{\mu_{P,U_i}})$ is in the image of $\psi$. Since $\psi$ is a homomorphism, $(\lambda_{P,U_i})$ is in the image of $\psi$, hence proving that $\psi$ is onto.

We have the following:

**Theorem 6.3.** Let $K$ be a complete discretely valued field with residue field $\kappa$ and $F$ be the function field of a curve over $K$. Let $\mathcal{X}_0$ be a regular proper model of $F$ with reduced special fibre $X_0$ a union of regular curves with normal crossings. Let $L/F$ be a Galois extension over $F$ of degree $n$. Suppose

- $n$ is coprime to $\text{char}(\kappa)$,
- $K$ contains a primitive $n^{th}$ root of unity $\rho$,
- for all finite extensions $\kappa'/\kappa$ and for all finite Galois extensions $l/\kappa'$ of degree $d$ dividing $n$,

$$T_{l/\kappa'}(\kappa') = RT_{l/\kappa'}(\kappa') < \rho^{\frac{n}{d}} >,$$

- the graph associated to $\mathcal{X}_0$ is a tree.

Then $\mathbb{III}(F,T_{L/F}) = 0$.

**Proof.** Let $\mathcal{X}$ be a regular proper model of $F$ which is obtained as a sequence of blow-ups of $\mathcal{X}_0$ at closed points. Since the graph $\Gamma(\mathcal{X}_0)$ is a tree, $\Gamma(\mathcal{X})$ is also a tree ([12, Remark 6.1(b)]). Let $X$ be the reduced special fibre of $\mathcal{X}$. Then $\mathbb{III}_X(F,T_{L/F}) = 0$ (6.2). Thus, by (5.4), we have $\mathbb{III}(F,T_{L/F}) = 0$. \qed

**Corollary 6.4.** Let $K$ be a complete discretely valued field with residue field $\kappa$ algebraically closed. Let $F$ be the function field of a curve over $K$ and $L/F$ be finite Galois extension of degree $n$ with $(n,\text{char}(\kappa)) = 1$. Let $\mathcal{X}$ be a regular proper model of $F$ with reduced special fibre $X$ a union of regular curves with normal crossings. Suppose that the graph associated to $\mathcal{X}$ is a tree. Then $\mathbb{III}(F,T_{L/F}) = 0$.

**Corollary 6.5.** Let $K$ be a $m$-local field with residue field $\kappa$. Let $F$ be the function field of a curve over $K$ and $L/F$ be finite Galois extension of degree $n$ with $(n,\text{char}(\kappa)) = 1$. Let $\mathcal{X}$ be a regular proper model of $F$ with reduced special fibre $X$ a union of regular curves with normal crossings. Suppose that the graph associated to $\mathcal{X}$ is a tree. If $K$ contains a primitive $n^{th}$ root of unity, then $\mathbb{III}(F,T_{L/F}) = 0$. 
Proof. Follows from (3.4) and (6.3).

Remark 6.6. By (5.5), the result also holds true if we just consider divisorial discrete valuations instead of all discrete valuations on \( F \) in (6.3), (6.4) and (6.5).

7. Counterexamples

Let \( K \) be a complete discretely valued field with residue field algebraically closed. Colliot-Thélène, Parimala and Suresh ([5, Section 3.1. & Proposition 5.9.]) constructed a function field of a curve \( F \) over \( K \) and Galois extension \( L/F \) with Galois group \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) such that the local-global principle fails for the norm one torus \( T_{L/F} \) associated to \( L/F \). They use higher reciprocity laws to detect non-trivial elements in \( \mathbb{III}(F,T_{L/F}) \). In this section, we produce examples of Galois extensions \( L/F \) with Galois group \( \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \) and using patching techniques, we show that \( \mathbb{III}(F,T_{L/F}) \) is non-trivial.

Let \( k \) be a number field and \( L_1, L_2 \) be two Galois extension of \( k \). Let \( T \) be the \( k \)-torus given by \( N_{L_1/k}(z_1)N_{L_2/k}(z_2) = 1 \). If \( L_1 \) and \( L_2 \) are linearly disjoint, then Demarche and Wei ([6, Theorem 1]) proved that the local-global principle holds for \( T \). In this section, we also give an example to show that a similar result does not hold in general for function fields of curves over a complete discretely valued field.

**Proposition 7.1.** Let \( A \) be a unique factorization domain and \( F \) the fraction field of \( A \). Let \( L/F \) be a finite Galois extension and \( B \) the integral closure of \( A \) in \( L \). Suppose that \( B \) is a unique factorization domain. Then every element in \( T_{L/F}(F) \) can be written as \( \alpha s \theta \) for some \( s \in RT_{L/F}(L) \) and \( \theta \in B \) a unit.

**Proof.** Let \( \lambda \in T_{L/F}(F) \). Then \( \lambda \in L^* \) and \( N_{L/F}(\lambda) = 1 \). Since \( L \) is the fraction field of \( B \), \( \lambda = \frac{\alpha}{\beta} \) for some \( \alpha, \beta \in B \). Since \( N_{L/F}(\lambda) = 1 \), \( N_{L/F}(\alpha) = N_{L/F}(\beta) \). Let \( p \in B \) be a prime. Since \( A \) is a unique factorization domain, \( pB \cap A = qA \) for some prime \( q \in A \) and \( N_{L/F}(p) = vq^r \) for some unit \( v \in A \). Suppose that \( p \) divides \( \alpha \) in \( B \). Then \( N_{L/F}(p) \) divides \( N_{L/F}(\alpha) \) in \( A \) and hence \( q \) divides \( N_{L/F}(\alpha) \). Since \( N_{L/F}(\alpha) = N_{L/F}(\beta) \), there exists a prime \( p' \in B \) such that \( p' \) divides \( \beta \) and \( p'B \cap A = pA \). Since \( L/F \) is a Galois extension, there exists \( \sigma \in Gal(L/F) \) such that \( p = w \sigma(p') \) for some unit \( w \in B \).

Write \( \alpha = po' \) and \( \beta = p'o' \). Then \( \lambda = \frac{\alpha}{\beta} = \frac{p}{p'} \frac{\alpha'}{\beta'} = \frac{\sigma p' w o'}{p' \beta'} \). Since \( B \) is a unique factorization domain, the proposition follows by induction on the number of prime factors of \( \alpha \) in \( B \).

**Proposition 7.2.** Let \( A \) be a complete regular local ring of dimension 2 with the maximal ideal \((\pi,\delta)\), fraction field \( F \) and residue field \( \kappa \). Let \( n \) be a positive integer which is coprime to \( \text{char}(\kappa) \). Let \( L = F(\sqrt[3]{\pi}, \sqrt[3]{\delta}) \). Suppose that \( F \) contains a primitive \( n^2 \)-th root of unity \( \rho \). Then \( T_{L/F}(F) = RT_{L/F}(F) < \rho > \).

**Proof.** Let \( B \) be the integral closure of \( A \) in \( L \). Then \( B \) is a regular local ring of dimension 2 with the fraction field \( L \) and the residue field \( \kappa \) ([16, Corollary 3.3.]). Let \( \lambda \in T_{L/F}(F) \). Then \( \lambda \in L^* \) with \( N_{L/F}(\lambda) = 1 \). Then, by (7.1), there exists \( s \in RT_{L/F}(F) \) and a unit \( \theta \in B \) such that \( \lambda = s \theta \). Since the residue field of \( A \) and \( B \) are equal, there exists \( \theta_1 \in A \) such that \( \theta = \theta_1 \) modulo the maximal ideal of \( B \). Since \( n \) is coprime to \( \text{char}(\kappa) \), by Hensel’s lemma, we have \( \theta = \theta_1 \alpha^n \) for some unit \( \alpha \in B \). Let \( s_1 = N_{L/F}(\alpha)^{-1} \alpha^n \in L \). Then, by (2.6), \( s_1 \in RT_{L/F}(F) \). Let \( a = \theta_1 N_{L/F}(\alpha) \in F \). Then \( \theta = as_1 \). Thus \( \lambda = s \theta = sas_1 = ss_1 a \). Since \( N_{L/F}(\lambda) = \)
Let $F$ be a complete discretely valued field with residue field $\kappa$ and ring of integers $R$. Let $n$ be a positive integer coprime to $\text{char}(\kappa)$. Let $\pi \in R$ be a parameter and $u \in R$ a unit with $[F(\sqrt[n]{u}) : F] = n$. Let $L = F(\sqrt[n]{u}, \sqrt[n]{\pi})$. Suppose that $F$ contains a primitive $n^2$-th root of unity $\rho$. Then $\rho^t \in RT_{L/F}(F)$ if and only if $n$ divides $t$.

Proof. Let $\sigma$ be the automorphism of $L/F$ given by $\sigma(\sqrt[n]{\pi}) = \rho^n \sqrt[n]{\pi}$ and $\sigma(\sqrt[n]{u}) = \sqrt[n]{u}$ and $\tau$ be the automorphism $L/F$ given by $\tau(\sqrt[n]{u}) = \rho^n \sqrt[n]{u}$ and $\tau(\sqrt[n]{\pi}) = \sqrt[n]{\pi}$. Then the Galois group of $L/F$ is an abelian group of order $n^2$ generated by $\sigma$ and $\tau$ and hence $\text{RT}_{L/F}(F) = \{\frac{\sigma(a)\tau(b)}{a} \in L^* / a, b \in L^*\}$. Since $\rho^n = \tau(\sqrt[n]{u})/\sqrt[n]{u} \in \text{RT}_{L/F}(F)$, $\rho^n \in \text{RT}_{L/F}(F)$ for any integer $j$.

Conversely, suppose $\rho^t \in \text{RT}_{L/F}(F)$ for some integer $t$. Without loss of generality, we may assume that $1 \leq t \leq n^2$. Then $\rho^t = a^{-1}\sigma(a)b^{-1}\tau(b)$ for some $a, b \in L$. Let $L' = F(\sqrt[n]{\pi})$. Since $\rho \in F$ and $N_{L/L'}(b^{-1}\tau(b)) = 1$, we have $\rho^nt = N_{L/L'}(a^{-1}N_{L/L'}(\sigma(a)))$. Let $c = N_{L/L'}(a) \in L'$. Since $\sigma(c) = \sigma(N_{L/L'}(a)) = N_{L/L'}(\sigma(a))$, we have $\sigma(c) = \rho^n c$. Hence $\sigma(c)^n = (\sigma(c))^n = (\rho^nt)^n c^n = c^n$. Since $L'/F$ is Galois with Galois group generated by $\sigma, c^n \in F$. Thus $c = \sqrt[n]{\pi}^{m}$ for some integer $m$ and $\theta \in F$. Since $L'/L'$ is an unramified extension of degree $n$ and $c$ is a norm from $L'/L'$, the valuation of $c$ is divisible by $n$. Since $\theta \in F$ and $\sqrt[n]{\pi}$ is a parameter in $L'$, $m = nr$ for some $r$. Hence $c \in F$ and $\rho^nt = c^{-n}\sigma(c) = 1$. Since $\rho$ is a primitive $n^2$-th root of unity, $n$ divides $t$.

Notation 7.4. Let $A$ be a semi-local regular ring of dimension 2 with three maximal ideals $m_1, m_2, m_3$. Suppose that there exist three prime elements $\pi_1, \pi_2, \pi_3 \in A$ such that $m_1 = (\pi_1, \pi_2, \pi_3), m_2 = (\pi_1, \pi_3) \text{ and } m_3 = (\pi_1, \pi_2)$. Suppose that $\pi_i \notin m_i$ for all $i$. Let $n \geq 2$ be an integer coprime to $\text{char}(A/m_i)$ for all $i$. Let $F$ be the fraction field of $A$. For $1 \leq i \leq 3$, let $\widehat{A}_{m_i}$ be the completion of $A$ at $m_i$, $F_{m_i}$ be the fraction field $\widehat{A}_{m_i}$ and $F_{\pi_i}$ be the completion of $F$ at the discrete valuation given by $\pi_i$. Let $1 \leq i \neq j, k \leq 3$. Since $m_i = (\pi_j, \pi_k), \widehat{A}_{m_i}$ is a regular local ring with maximal ideal $(\pi_j, \pi_k)$. In particular, $\pi_j$ gives a discrete valuation on $F_{m_i}$ which extends the discrete valuation on $F$ given by $\pi_j$. Let $F_{m_i, \pi_j}$ be the completion of $F_{m_i}$ at the discrete valuation given by $\pi_j$. Then $F_{\pi_j} \subset F_{m_i, \pi_j}$. Let $L = F(\sqrt[n]{\pi_1\pi_2}, \sqrt[n]{\pi_2\pi_3})$. Suppose that $F$ contains $\rho$, a primitive $n^2$-th root of unity.

Corollary 7.5. With notation as in (7.4), we have $T_{L/F}(F_{m_i}) = \text{RT}_{L/F}(F_{m_i}) < \rho$.

Proof. Since $\pi_2$ is a unit at $m_2$, we have $m_2A_{m_2} = (\pi_1, \pi_2, \pi_3)$. Hence, by (7.2), we have $T_{L/F}(F_{m_2}) = \text{RT}_{L/F}(F_{m_2}) < \rho$. Since $\pi_1$ is a unit at $m_1, m_1A_{m_1} = (\pi_1, \pi_2, \pi_1^{-1}\pi_3)$. Since $L = F(\sqrt[n]{\pi_1\pi_2}, \sqrt[n]{\pi_1\pi_3}) = F(\sqrt[n]{\pi_1\pi_2}, \sqrt[n]{\pi_1^{-1}\pi_3})$, by (7.2), we have $T_{L/F}(F_{m_1}) = \text{RT}_{L/F}(F_{m_1}) < \rho$. Similarly, $T_{L/F}(F_{m_3}) = \text{RT}_{L/F}(F_{m_3}) < \rho$.

Corollary 7.6. With notation as in (7.4), we have $T_{L/F}(F_{\pi_i}) = \text{RT}_{L/F}(F_{\pi_i}) < \rho$.

Proof. Let $\kappa(\pi)$ be the residue field of $F_{\pi_i}$. The discrete valuation $\nu_{\pi_i}$ of $F$ given by $\pi_i$ has unique extension $\nu_{\pi_i}$. Since $F$ contains a primitive $n^2$-th root of unity, the residue field $\kappa(\tilde{\nu}_{m_i})$ of $L$ at $\tilde{\nu}_{m_i}$ is a cyclic extension of $\kappa(\pi)$ of degree $n$. In particular,
$T_{\kappa(\nu_{\sigma_i})/\kappa(\nu_{\pi_i})}(\kappa(\nu_{\pi_i})) = RT_{\kappa(\nu_{\sigma_i})/\kappa(\nu_{\pi_i})}(\kappa(\nu_{\pi_i})).$ Hence, by (3.2),

$$T_{L/F}(F_{\pi_i}) = RT_{L/F}(F_{\pi_i}) < \rho > .$$

\[ \square \]

**Corollary 7.7.** Let $F_{m_i, \pi_j}$ be as in (7.4). Then $\rho^i \in RT_{L/F}(F_{m_i, \pi_j})$ if and only if $n$ divides $t$.

**Proof.** Since the residue field of $F_{m_i, \pi_j}$ is a complete discretely valued field with the image of $\pi_k$ ($k \neq i, j$) as a parameter and the image of $\pi_i$ as a unit, it is easy to see that $L \otimes_F F_{m_i, \pi_j} \simeq F_{m_i, \pi_j}(\sqrt[\nu]{\pi_j}, \sqrt[\nu]{u})$ for some units $u$ and $v$ such that $[F_{m_i, \pi_j}(\sqrt[\nu]{u}) : F_{m_i, \pi_j}] = n$. Thus, the corollary follows from (7.3).

For each $1 \leq i \neq j \leq 3$, we have inclusions fields $F_{m_i} \supseteq F_{m_i, \pi_j}$ and $F_{\pi_j} \supseteq F_{m_i, \pi_j}$. Thus we have the induced homomorphisms $\alpha_{ij} : T_{L/F}(F_{m_i})/R \rightarrow T_{L/F}(F_{m_i, \pi_j})/R$ and $\beta_{ij} : T_{L/F}(F_{\pi_j})/R \rightarrow T_{L/F}(F_{m_i, \pi_j})/R$.

**Lemma 7.8.** The product map

$$\phi : \left( \prod_{i=1}^{3} T_{L/F}(F_{m_i})/R \times \prod_{j=1}^{3} T_{L/F}(F_{\pi_j})/R \right) \rightarrow \prod_{1 \leq i \neq j \leq 3} (T_{L/F}(F_{m_i, \pi_j})/R)$$

is not onto.

**Proof.** Let $y_{12} = \rho \in T_{L/F}(F_{m_i, \pi_j})$ and $y_{ij} = 1 \in T_{L/F}(F_{m_i, \pi_j})$ for all $i \neq j$ and $(i, j) \neq (1, 2)$. Then we show that $y = (y_{ij}) \in \prod_{1 \leq i \neq j \leq 3} (T_{L/F}(F_{m_i, \pi_j})/R)$ is not in the image of $\phi$.

Suppose $y$ is in the image of $\phi$. Then there exist $a_i \in T_{L/F}(F_{m_i})$ and $b_j \in T_{L/F}(F_{\pi_j})$ such that $\phi(a_1, a_2, a_3, b_1, b_2, b_3) = y$ modulo $R$-trivial elements. Then we have $\alpha_{12}(a_1)\beta_{21}(b_2) = x_{12} = \rho$ modulo $R$-trivial elements and $\alpha_{ij}(a_i)\beta_{ji}(b_j) \in RT_{L/F}(F_{m_i, \pi_j})$ for all $i \neq j$ and $(i, j) \neq (1, 2)$. By (7.5) and (7.6), we have $a_i = c_i\rho^{s_i}$ for some $c_i \in RT_{L/F}(F_{m_i})$ and $b_j = d_j\rho^{t_j}$ for some $d_j \in RT_{L/F}(F_{\pi_j})$. Hence $a_i = \rho^{s_i}$ and $b_j = \rho^{t_j}$ modulo $R$-trivial elements. Since $\rho \in F$, $\alpha_{ij}(\rho) = \rho$ and $\beta_{ji}(\rho) = \rho$ for all $i \neq j$. We have $\rho = y_{12} = \alpha_{12}(a_1)\beta_{21}(b_2) = \rho^{s_1+t_2}$ modulo $R$-trivial elements. Hence, by (7.7), $n$ divides $1 - s_1 - t_2$.

Let $1 \leq i \neq j \leq 3$ with $(i, j) \neq (1, 2)$. Then $1 = \alpha_{ij}(a_i)\beta_{ji}(b_j) = \rho^{s_i+t_j}$ modulo $R$-trivial elements. Hence $\rho^{s_i+t_j} \in RT_{L/F}(F_{m_i, \pi_j})$ and by (7.7), $n$ divides $s_i + t_j$. Since $n$ divides $s_2 + t_1$ and $s_3 + t_1$, $n$ divides $s_3 - s_2$. Since $n$ divides $s_1 + t_3$ and $s_2 + t_3$, $n$ divides $s_1 - s_2$. Hence $n$ divides $s_1 - s_3$. Since $n$ divides $s_3 + t_2$, $n$ divides $s_1 + t_2$, which contradicts the fact that $n$ divides $1 - s_1 - t_2$.

\[ \square \]

**Theorem 7.9.** Let $K$ be a complete discretely valued field with residue field $\kappa$ and ring of integers $R$. Let $\mathcal{X}$ be a regular integral surface proper over $R$ and $F$ the fraction field and $X$ its reduced special fibre. Suppose $X$ is a union of regular curves with normal crossings and there exist three three irreducible curves $X_1$, $X_2$ and $X_3$ regular on $\mathcal{X}$ such that $X_i \cap X_j$, $i \neq j$ has exactly one closed point. Let $n \geq 2$ be an integer coprime to char($\kappa$). Suppose that $K$ has a primitive $n^2$-th root of unity. Then there exists a Galois extension $L/F$ of degree $n^2$ with the Galois group isomorphic to $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ such that the local-global principle for $T_{L/F}$-torsors fails.

**Proof.** Let $P_1$, $P_2$ and $P_3$ be the points of $X_i \cap X_j$, $i \neq j$. Let $A$ be the semi local ring at $P_1$, $P_2$ and $P_3$ on $\mathcal{X}$. Then $A$ has three maximal ideals $m_1$, $m_2$ and $m_3$. 
Since \( X \) is regular and each \( X_i \) is regular on \( \mathcal{X} \), there exist primes \( \pi_1, \pi_2, \pi_3 \in A \) such that \( m_i = (\pi_j, \pi_k) \) for all distinct \( i, j, k \). Let \( L = F(\sqrt[n]{\pi_1}, \sqrt[n]{\pi_2}, \sqrt[n]{\pi_3}) \). Since \( K \) contains primitive \( n \)th root of unity, \( L/F \) is Galois with Galois group isomorphic to \( \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \). We claim that the local-global principle for \( T_{L/F} \)-torsors fails.

Let \( \mathcal{P} \) be finite set of closed points of \( X \) containing all the singular points of \( X \). Let \( X \setminus \mathcal{P} = \bigcup U_i \), with \( U_i \subset X_i \) for \( i = 1, 2, 3 \). By ([12, Corollary 3.6.1]), it is enough to show that the product map

\[
\prod_{P \in \mathcal{P}} T_{L/F}(F_P) \times \prod_i T_{L/F}(F_{U_i}) \rightarrow \prod_{P, U_i} T_{L/F}(F_{P, U_i})
\]

is not onto. Since \( X_1, X_2, X_3 \) are the only curves in \( X \) passing through \( P_1, P_2 \) or \( P_3 \), it is enough to show that

\[
\phi : \prod_{i=1}^3 T_{L/F}(F_{P_i}) \times \prod_{j=1}^3 T_{L/F}(F_{U_j}) \rightarrow \prod_{P_i, U_j} T_{L/F}(F_{P_i, U_j})
\]

is not onto. Since \( F_{U_j} \subset F_{\pi_j} \) and \( F_{P_i, U_j} = F_{P_i, \pi_j} \), \( \phi \) factors as

\[
\prod_{i=1}^3 T_{L/F}(F_{P_i}) \times \prod_{j=1}^3 T_{L/F}(F_{U_j}) \rightarrow \prod_{i=1}^3 T_{L/F}(F_{P_i}) \times \prod_{j=1}^3 T_{L/F}(F_{\pi_j}) \rightarrow \prod_{P_i, \pi_j} T_{L/F}(F_{P_i, \pi_j}).
\]

Since, by (7.8),

\[
\prod_{i=1}^3 (T_{L/F}(F_{\pi_i})/R) \times \prod_{j=1}^3 (T_{L/F}(F_{P_j})/R) \rightarrow \prod_{U_i, P_j} (T_{L/F}(F_{P_i, U_j})/R)
\]

is not onto, \( \phi \) is not onto.

\[\square\]

Remark 7.10. The above theorem for \( \kappa \) algebraically closed and \( n = 2 \) is proved by Colliot-Thélène, Parimala and Suresh ([5, Section 3.1. & Corollary 6.2.]).

Corollary 7.11. Let \( K \) be a complete discretely valued field with residue field \( \kappa \) and ring of integers \( R \). Let \( t \in R \) a parameter. Let \( \mathcal{X} = \text{Proj} R[x, y, z]/(xy(x + y - z) - tz^3) \). Let \( X \) be the special fibre of \( \mathcal{X} \). Then \( X = \text{Proj}(\kappa[x, y, z]/(xy(x + y - z))) \) which is reduced. Then \( X \) has three irreducible components \( X_1, X_2, X_3 \) and \( X_i \) intersects \( X_j, i \neq j \) at exactly one point. Let \( F \) be the function field of \( \mathcal{X} \). Then \( F \cong K(x)[y]/(xy(x + y - 1)) \). Let \( n \geq 2 \) be coprime to \( \text{char}(\kappa) \). Suppose that \( K \) contains a primitive \( n^2 \)-th root of unity. Let \( L = F(\sqrt[2n]{xy}, \sqrt[2n]{y(x-1)}) \). Then \( L/F \) is a Galois extension with Galois group isomorphic to \( \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z} \). By (7.9), the local-global principle for \( T_{L/F} \)-torsors fails.

Proof. Let \( U = \text{Spec} R[x, y]/(xy(x + y - 1) - t) \). Then \( U \) is an affine open subset of \( \mathcal{X} \). Let \( P_1 = (1, 0), P_2 = (0, 1) \) and \( P_3 = (0, 0) \) be the three closed points of \( U \). Let \( A \) be the semi local ring at \( P_1, P_2 \) and \( P_3 \) and let \( m_i \) be the maximal ideal of \( A \) corresponding to \( P_i \). Then \( m_1 = (x + y - 1, y), m_2 = (x, x + y - 1) \) and \( m_3 = (x, y) \). Hence, by (7.9), the local-global principle fails for \( T_{L/F} \)-torsors.

\[\square\]

Corollary 7.12. Let \( K \) be a complete discretely valued field with residue field \( \kappa \) and ring of integers \( R \). Let \( t \in R \) a parameter. Let \( \mathcal{X} = \text{Proj} R[x, y, z]/(xy(x + y - z)(x - 2z) - tz^4) \) and \( F \) be the function field of \( \mathcal{X} \). Then \( F = K(x)[y]/(xy(x + y - 1)(x - 2)) \). Let \( \theta_1 = (x - 2)/((x - 2 + xy(x + y - 1)) \) and \( \theta_2 = (y - 2)/(y - 2 + xy(x + y - 1)) \). Let \( n \geq 2 \) with \( 6n \) coprime to \( \text{char}(\kappa) \). Let \( L_1 = F(\sqrt[6n]{xy}, \sqrt[6n]{y(x + y - 1)}) \) and \( L_2 = \)
Then $L_1$ and $L_2$ are Galois extensions of $F$ that are linearly independent and the local-global principle fails for $T_{L_1 \times L_2}/F$.

Proof. To show that the local-global principle fails for $T_{L_1 \times L_2/f}$-torsors, by ([11, Theorem 3.6.]) and as in the proof of (7.9), it is enough to show that
\[
\phi : \prod_{i=1}^{3}(T_{L_1 \times L_2/F}(F_{\pi_i})/R) \times \prod_{j=1}^{3}(T_{L_1 \times L_2/F}(F_{P_j})/R) \to \prod_{U_i, P_j}(T_{L_1 \times L_2/F}(F_{P_i, U_j})/R)
\]
is not onto.

Let $U = \text{Spec}R[x, y]/(xy(x + y - 1)(x - 2) - t)$. Then $U$ is an affine open subset of $\mathcal{X}$. Let $P_1 = (1, 0)$, $P_2 = (0, 1)$, $P_3 = (0, 0)$ and $Q = (2, 2)$. Let $A$ be the semi local ring at $P_1$, $P_2$, $P_3$ and $Q$. Let $m_i$ the maximal ideals of $A$ corresponding to $P_i$ and $m$ the maximal ideal corresponding to $Q$. Let $\pi_1 = x$, $\pi_2 = y$ and $\pi_3 = x + y - 1$. Then $m_i = (\pi_2, \pi_3)$, $m_2 = (\pi_1, \pi_3)$, $m_3 = (\pi_1, \pi_2)$. We also have $m = (x - 2, y - 2)$. Since $2 \neq \text{char}(\kappa)$, $x - 2$ and $y - 2$ are units at $m_i$ and $\theta_i = 1$ modulo $m_i$ and $\pi_j$. Since $n$ is coprime to $\text{char}(\kappa)$, $\theta_i \in F_{\pi_i}^n$ and $\pi_i \in F_{\pi_i}^n$ for all $i$ and $j$. Hence $L_1 \otimes_F F_{\pi_i} \simeq L_2 \otimes_F F_{\pi_i}$ and $L_1 \otimes_F F_{P_j} \simeq L_2 \otimes_F F_{P_j}$. By (2.4), we have $T_{L_1 \times L_2/F}(F_{\pi_i})/R \simeq T_{L_1/F}(F_{P_j})/R$ and $T_{L_1 \times L_2/F}(F_{P_j})/R \simeq T_{L_1/F}(F_{U_i, P_j})/R$. Since, by (7.8),
\[
\prod_{i=1}^{3}(T_{L/F}(F_{\pi_i})/R) \times \prod_{j=1}^{3}(T_{L/F}(F_{P_j})/R) \to \prod_{U_i, P_j}(T_{L/F}(F_{P_i, U_j})/R)
\]
is not onto, $\phi$ is not onto. Hence the local-global principle fails for $T_{L/F}$-torsors.

Since $\pi_1 \pi_2 = xy = 4$ modulo $m$ and $\pi_2 \pi_3 = 6$ modulo $m$, we have $L_1 \otimes_F F_Q = F_Q(\sqrt[3]{4}, \sqrt[6]{6})$. Since $6n$ is coprime to $\text{char}(\kappa)$, $L_1 \otimes_F F_Q/F_Q$ is unramified. Since $xy(x + y - 1) = 12$ modulo $m$, $x - 2 + xy(x + y - 1) = 12a^n$ for some $a \in F_Q$. Similarly, $y - 2 + xy(x + y + 1) = 12b^n$ for some $b \in F_Q$. Hence $L_2 \otimes_F F_Q = F_Q(\sqrt[3]{x - 2}/3, \sqrt[6]{y - 2}/2)$. Since the maximal ideal $m = (x - 2, y - 2)$ and 3, 2 are units at $m$, $L_1 \otimes_F F_Q$ and $L_2 \otimes_F F_Q$ are linearly independent over $F_Q$. In particular, $L_1$ and $L_2$ are linearly independent over $F$. \[\]

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