The link between two-body model space and many-body model space

C. J. Yang\textsuperscript{1,2}

\textsuperscript{1}Institut de Physique Nucléaire, IN2P3-CNRS, Université Paris-Sud, Université Paris-Saclay, F-91406 Orsay Cedex, France

\textsuperscript{2}Institute for Nuclear Studies, Department of Physics, George Washington University, Washington DC 20052, USA

(Dated: October 16, 2018)

Abstract

An exact relation which links the ideal model space to be used in A-body calculations when the two-body interaction is given in a truncated model space is derived. Its implications on the effective field theory (EFT) approach to no-core-shell-model (NCSM)\textsuperscript{1} is analyzed. Some insights regarding whether details of two-body interaction becomes less important in the calculations of many-body system are given. The result suggests that there might be a way to establish an EFT expansion for heavy nuclei and nuclear matter with an effective interaction which has a much simpler form than the nucleon-nucleon (NN) interaction in the vacuum.

\textsuperscript{*}Electronic address: yangjerry@gwu.edu
I. INTRODUCTION

Due to the limitation of computational power and limited knowledge about the nature, calculations of physical systems always needed to be performed in certain model space. In ab-initio calculations, when the inter-particle interaction is defined through a transformation from the infinite space$^1$, one usually needs to increase the model space in the many-body calculations until a pattern of convergence is observed or can be extrapolated. Various extrapolation methods, especially the extrapolation regarding infrared cutoffs, have been studied recently$^2$. On the other hand, one could define the interaction directly in a given finite model space, effects generated by physics out of the model space can be included systematically through effective field theory (EFT), and are represented by the low energy constants (LECs) associated with counter terms after renormalization. In nuclear structure calculations, this direction has been advocated by the Arizona group$^1, 4–8$ and carried out recently in Refs.$^9–12$. In these approaches, since the interactions are built in a given finite model space, one expects that the results of many-body calculations become exact also in a finite model space. One then needs to answer the following question: Suppose the interaction between two particles$^2$ is built and renormalized within their center of mass (c.m.) momentum $|\mathbf{p}| \in [0, \Lambda]$ and disappears elsewhere, then, for a system consists of $A$ particles through the same interaction, to what maximum momentum (with respect to the c.m. of the system) will the system still respond to an outer probe? This maximum momentum is linked to the ideal model space to be used in any $A$-body calculation. In the following we derive an exact formula which links the two- and $A$-body model space through an analysis in Jacobi coordinate.

$^1$ This includes interactions which are regularized through regulators but only vanishing exactly to zero at infinite momentum and those have been tamed by unitary transformations such as similarity renormalization group (SRG)$^2$.

$^2$ Here the particles are just the basic degree of freedom considered in an EFT, and they need not to be elementary particles.
II. DERIVATION

We start with the usual definition of Jacobi momentums:

\[
\vec{\pi}_1 = \frac{1}{2}(\vec{p}_1 - \vec{p}_2), \quad (1)
\]

\[
\vec{\pi}_2 = \frac{2}{3}(\vec{p}_3 - \frac{1}{2}(\vec{p}_1 + \vec{p}_2)), \quad (2)
\]

\[
\vec{\pi}_3 = \frac{3}{4}(\vec{p}_4 - \frac{1}{3}(\vec{p}_1 + \vec{p}_2 + \vec{p}_3)), \quad (3)
\]

\[\vdots\]

\[
\vec{\pi}_{N-1} = \frac{N-1}{N} \left( \vec{p}_N - \frac{1}{N-1}(\vec{p}_1 + \vec{p}_2 + \cdots + \vec{p}_{N-1}) \right). \quad (4)
\]

Here \(\vec{p}_i\) denotes the momentum of particle \(i\) in the reference frame, and the Jacobi momentum \(\vec{\pi}_{N-1}\) denotes the relative momentum of particle \(N\) with respect to the sub-system with one less particle. Denote \(\vec{\Lambda}_{ij} = |\vec{\Lambda}_{ij}|\hat{n}_{ij} = \frac{1}{2}(\vec{p}_i - \vec{p}_j) = \frac{1}{2}\vec{p}_{ij}\), where \(|\vec{\Lambda}_{ij}| \in [0, \Lambda]\), with \(\Lambda\) the two-body cutoff, which defines the maximum model space the two body system lives, and \(\hat{n}_{ij}\) the unit vector pointing from particle \(i\) to \(j\), then from Eq. (1) and (2) one obtains:

\[
2\vec{\pi}_1 + 3\vec{\pi}_2 = \vec{p}_1 - \vec{p}_2 + 2\vec{p}_3 - (\vec{p}_1 + \vec{p}_2),
\]

\[
= 4 \cdot \frac{1}{2}(\vec{p}_3 - \vec{p}_2) = 4\vec{\Lambda}_{32}. \quad (5)
\]

Using \(\vec{\pi}_1 = \frac{1}{2}(\vec{p}_1 - \vec{p}_2) = \vec{\Lambda}_{12}\), one obtains

\[
2\vec{\Lambda}_{12} + 3\vec{\pi}_2 = 4\vec{\Lambda}_{32},
\]

\[
3\vec{\pi}_2 = 4\vec{\Lambda}_{32} - 2\vec{\Lambda}_{12}. \quad (6)
\]

Since there is no restriction on the alignment between \(\vec{\Lambda}_{32}\) and \(\vec{\Lambda}_{12}\), the maximum value \(|\vec{\pi}_2|\) can have is \(2\Lambda\).

The above maximum corresponds to the scenario that all 3 particles are aligned and moving in the same direction, where particle 2 has the maximum allowed relative momentum \(|\vec{p}_{12}|_{\text{max}} = |\vec{p}_1 - \vec{p}_2|_{\text{max}} = 2\Lambda\) with respect to particle 1, while particle 3 is moving toward particle 1 with momentum \(|\vec{p}_{13}| = 4\Lambda\). In this case, particle 3 will not interact with particle 1, but will still interact with particle 2 since \(|\vec{p}_{23}| = 2\Lambda\) is within the maximum allowed relative momentum between a pair. Actually, this extreme case can be generalized to the \(N\) particles case as listed in Fig. \(\square\) where all particles are aligned in the same direction with ascending increment of relative momentum \(\vec{q}\) \((|\vec{q}| = 2\Lambda)\) respect to particle 1. In this case, the \(N^{th}\) particle only interacts with \((N-1)^{th}\) particle
and nothing else, which guarantees that the largest possible model space has been taken. To obtain the maximum value of $\vec{\pi}_N$, one only needs to transfer the momentum \((N-1)|\Lambda|\) of $N^{th}$ particle from with respect to particle 1 to with respect to the (N-1)-body’s c.m. frame, which can be simply obtained through Eq. (4):

$$\vec{\pi}_{N-1} = \frac{N-1}{N} \left( p_N - \frac{1}{N-1} (p_1 + p_2 + ... + p_{N-1}) \right),$$

$$\vec{\pi}_{N-1}^{\text{max}} = \frac{N-1}{N} \left( N - 1 - \frac{1}{N-1} (0 + 1 + 2 + ... + N - 2) \right) \vec{q},$$

$$= \frac{N-1}{2} \vec{q}. \quad (7)$$

Thus, the maximum value of the $(N-1)^{th}$ Jacobi momentum $|\vec{\pi}_{N-1}^{\text{max}}| = (N-1)|\Lambda|$.}

![Graphical representation of the extreme case where $N$ particles interact within their maximum model space.](image)

**FIG. 1:** A visual illustration of the extreme case where $N$ particles interact within their maximum model space.

In ab-initio calculations, the total model space where a $N$-body system occupied corresponds exactly to the space spanned by the wavefunctions in the Jacobi coordinate representation. For example, when the wavefunctions are expanded in Harmonic oscillator (HO) basis, the maximum allowed number of HO excitations for an $A$-particle system $N_A$ can be defined as:

$$N_A = \sum_{i=1}^{A-1} (2n_{\pi_i} + l_{\pi_i}), \quad (8)$$
where $n_{\pi_i}, l_{\pi_i}$ are the principle and angular momentum quantum number associated with Jacobi coordinate $\pi_i$. Note that the link between $n_{\pi_i}, l_{\pi_i}$ and the corresponding momentum cutoff of each Jacobi coordinate, $|\pi_i|$ is provided by $|\pi_i| = \sqrt{M(2n_{\pi_i} + l_{\pi_i} + 7/2)\hbar\omega}$, with $M$ the nucleon mass, $\hbar$ the Planck constant and $\omega$ the oscillator strength. The total maximum model space for an $A$-particle ($A > 1$) system is then

$$N_A^{\text{max}} = \sum_{i=1}^{A-1} (2n_{\pi_i}^{\text{max}} + l_{\pi_i} + 7/2)$$

$$= \frac{\Lambda^2}{M\hbar\omega} \sum_{i=1}^{A-1} i^2$$

$$= \frac{A(A - 1)(2A - 1)}{6} \frac{\Lambda^2}{M\hbar\omega}$$

$$= \frac{A(A - 1)(2A - 1)}{6 N_A^{\text{max}}} \Lambda^2.$$  \hspace{1cm} (9)

### III. DISCUSSION

First, we note that Eq. (9) denotes the exact model space to be used in principle in the ab-initio calculations. In other words, if the inter-particle interaction is established or renormalized within a finite model space (with $\Lambda$ and $\omega$ specified), then, instead of extrapolating to infinity, one should stop at the finite size of model space specified in Eq. (9) in an $A$-body calculation. Additionally, even in cases where one suppresses the interaction (normally by exponential regulators) after the momentum $k > \Lambda$, so that the interaction still occupies the entire space, adopting a model space higher than the one defined in Eq. (9) will just capture the artifact of unimportant physics in the high momentum tail of the interaction. We note that, in usual ab-initio calculations, the contribution given by including model spaces higher than $N_A^{\text{max}}$ given in Eq. (9) does not vanish automatically. The kernel of the interaction $V_{ij}$ restricts the interaction between particle $i$ and $j$ up to the shells correspond to the maximum momentum $\Lambda$, but the rest of the configurations could go up to arbitrary high $n_{\pi_i}, l_{\pi_i}$ and still give contribution to the matrix element. Thus, in calculations where $N_A > N_A^{\text{max}}$ is practical, one should consider the results obtained up to $N_A = N_A^{\text{max}}$ only.

From the results of Eq. (9), one immediately sees that the HO-basis model space grows as cubic power in number of particles, i.e., $A^3$. Translating the space into the total accumulated momentum $P_A^{\text{max}}$,

$$P_A^{\text{max}} \equiv \sum_{i=1}^{A} (i - 1)\Lambda = \frac{A(A - 1)}{2} \Lambda,$$  \hspace{1cm} (10)
one still has an \( A^2 \) grows in the maximum momentum an \( A \)-body system could have. Thus, even starting with a small two-body cutoff \( \Lambda \), the model space where the \( A \)-body system lives is enriched quickly by the number of particles.

If one is only interested in probing physical phenomena up to a fixed energy scale for a wide range of particle number \( A \), then the required two-body model space would decrease drastically with the number of particle of the system. For example, suppose one starts with a two-body interaction \( V_2(p,p') \) with flexible form and parameters, where \( p^{(t)} \) is the incoming (outgoing) momentum in the c.m. frame. Then to encode physics up to the same energy scale into the parameters of \( V_2(p,p') \), the minimum required momentum space would range from \( p^{(t)} \in [0, \Lambda] \) for a 2-particle system to \( p^{(t)} \in [0, \frac{\Lambda}{A}] \) for an \( A \)-particle system. Whether there exists an EFT expansion on the parametrization of \( V_2(p,p') \) is another question.

In other words, if there exists a way to renormalize the interaction under an EFT to arbitrary low cutoffs \( \Lambda \), then the required (minimum) two-body cutoff to describe a physical phenomenon (up to a given energy scale) in many-body systems would be much smaller than the required cutoff to describe the few-body systems. In nuclear physics, this required cutoff quickly goes below the pion-exchange threshold. Take the case of pure neutron matter as an example. In this extreme case (where \( A \to \infty \)), it was demonstrated that the equation of state (EOS) up to twice of the saturation density is not far from a simple parametrization governed by the Bertch parameter\([14, 15]\). It also suggests that the effect of pion-exchange between nucleons can be integrated out and re-parametrized into an EFT-like expansion around the unitarity limit\([16–18]\), if one just wants to obtain a reasonable EOS up to twice of the saturation density. Moreover, by considering beyond mean field effects, it could be possible to establish an EFT-based, Skyrme-like interaction, to describe general properties for systems consist of many protons and neutrons\([19, 20]\).

Another case of interested is the recent attempts to apply an unitarity-based pionless EFT into the calculations of light nuclei properties\([21]\), where, at leading order (LO), one first adjusts the interaction at three-body level instead of fitting to two-body observables\(^3\). Based on the power counting analysis, at least in the fermionic case, the details of the two-body nucleon-nucleon interaction (i.e., corrections on top of unitarity) only starts to contribute at NLO. As one can see in Eq. (10), only 1/3 of \( \Lambda \) in the interaction is needed to describe a three-body observable than to describe a two-body observable at the same energy scale\(^4\). Therefore, on top of the advantages

---

\(^3\) The two-body interaction is kept at unitairity limit at LO.

\(^4\) This also suggests that the minimum cutoff for a three-body force is 1/3 of the two-body cutoff \( \Lambda \), provided that the renormalization is performed consistently and one does not probe physics of the three-body system higher than
analyzed in the power counting, an earlier conversion of the three-body observable into the low energy constants in the theory could capture important physics into the interaction within a smaller model space and potentially fasten the convergence for high-body calculations.

IV. CONCLUSION

In this work, a direct link between two- and A-body model space is derived. It is found that instead of infinity, one needs to extrapolate to a finite many-body model space in ab-initio calculations when interactions renormalized in a given model space are adopted. In addition, the results suggest that to describe a system equally well up to a fixed energy scale, it is possible to “integrated out” more ultra-violate physics in the many-body cases, provided that the errors is quantified by the EFT principle.

Acknowledgments

The authors thanks G. Hupin and H. W. Griesshammer for useful discussion at different stage of the work. This project has received funding from the European Unions Horizon 2020 research and innovation program under grant agreement No. 654002 and US Department of Energy under contract de-sc0015393.

---

[1] I. Stetcu, B.R. Barrett, P, Navratil, U. van Kolck, Phys. Lett. B 653, (2007) 358.
[2] R. J. Furnstahl, Nucl. Phys. Proc. Suppl. 228, 139 (2012); R. J. Furnstahl and K. Hebeler, Rep. Prog. Phys. 76, 126301 (2013).
[3] S.N. More, A. Ekstrom, R.J. Furnstahl, G. Hagen, T. Papenbrock, Phys. Rev. C 87, 044326 (2013); R.J. Furnstahl, T. Papenbrock and S.N. More, Phys. Rev. C 89 044301 (2014); S. König, S.K. Bogner, R.J. Furnstahl, S.N. More, T. Papenbrock, Phys. Rev. C 90 064007 (2014); S. A. Coon, M. I. Avetian, M.K.G. Kruse, U. van Kolck, P. Maris, J. P. Vary, Phys. Rev. C 86 054002 (2012).
[4] I. Stetcu, B.R. Barrett, P, Navratil, U. van Kolck, J.P. Vary, Phys. Rev. A 76, 063613 (2007).
[5] I. Stetcu, J. Rotureau, B.R. Barrett, U. van Kolck, J. Phys. G 37, 064033 (2010).
[6] J. Rotureau, I. Stetcu, B.R. Barrett, M.C. Birse, U. van Kolck, Phys. Rev. A 82, 032711 (2010).
[7] I. Stetcu, J. Rotureau, B.R. Barrett, U. van Kolck, Annals. Phys. 325, 1644-1666 (2010).

\[ k \geq \Lambda/3. \]
[8] J. Rotureau, I. Stetcu, B.R. Barrett, U. van Kolck, Phys. Rev. C 85, 034003 (2012).
[9] S. Binder, A. Ekstrom, G. Hagen, T. Papenbrock and K.A. Wendt, Phys. Rev. C 93 044332 (2016).
[10] W. C. Haxton, Phys. Rev. C. 77 034005 (2008).
[11] Kenneth S. McElvain and W.C. Haxton, arXiv:1607.06863 [nucl-th].
[12] C.-J. Yang, Phys. Rev. C 94 064004 (2016).
[13] P. Navratil, G.P. Kamuntavicius, B.R. Barrett, Phys. Rev. C 61, 044001 (2000).
[14] G.F. Bertsch, Proceedings of the tenth international conference on recent progress in many-body theories. in Bishop, R., Gernoth, K.A., Walet, N.R., Xian, Y. (eds.) Recent Progress in Many-Body Theories. World Scientific, Seattle (2000).
[15] U. van Kolck, Few Body Syst. 58 (2017) no.3, 112.
[16] D. Lacroix, Phys. Rev. A 94, 043614 (2016).
[17] D. Lacroix, A. Boulet, M. Grasso, C.-J. Yang, Phys. Rev. C 95 054306 (2017).
[18] A. Boulet and D. Lacroix, arXiv:1709.05160.
[19] C.-J. Yang, M. Grasso and D. Lacroix, Phys. Rev. C 96 034318 (2017); C.-J. Yang, M. Grasso, X. Roca-Maza, G. Colo and K. Moghrabi, Phys. Rev. C 94 034311 (2016); C.-J. Yang, M. Grasso, K. Moghrabi and U. van Kolck, Phys. Rev. C 95 054325 (2017).
[20] H. Gil, P. Papakonstantinou, C. H. Hyun, Tae-Sun Park, Yongseok Oh, Acta Phys. Polon. B48 (2017) 305; P. Papakonstantinou, Tae-Sun Park, Yeunhwan Lim, C. H. Hyun, arXiv:1606.04219 [nucl-th].
[21] S. König, H. W. Griesshammer, H.-W. Hammer, U. van Kolck, J. Phys. G: Nucl. Part. Phys. 43 055106 (2016); S. König, H. W. Griesshammer, H.-W. Hammer, U. van Kolck, Phys. Rev. Lett. 118, 202501 (2017).