KAKEYA SETS AND DIRECTIONAL
MAXIMAL OPERATORS IN THE PLANE

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ABSTRACT. We completely characterize the boundedness of planar directional maximal operators on $L^p$. More precisely, if $\Omega$ is a set of directions, we show that $M_\Omega$, the maximal operator associated to line segments in the directions $\Omega$, is unbounded on $L^p$, for all $p < \infty$, precisely when $\Omega$ admits Kakeya-type sets. In fact, we show that if $\Omega$ does not admit Kakeya sets, then $\Omega$ is a generalized lacunary set, and hence $M_\Omega$ is bounded on $L^p$, for $p > 1$.

§0 Introduction

Given a closed set $\Omega \subset [0, 1]$ of slopes in the plane, we let $B_\Omega$ be the collection of all rectangles so that one of the sides has slope in $\Omega$, and we define

$$M_\Omega f(x) = \sup_{x \in R \in B_\Omega} \frac{1}{|R|} \int_R f.$$ 

The study of such operators dates at least to Cordoba’s paper [C], in which he considered the case $\Omega = [\frac{1}{2^N}, \frac{2}{2^N}, \ldots, 1]$, with the restriction that the rectangles in $B_\Omega$ have dimensions $1 \times N$. In the case where $\Omega$ is a lacunary sequence, i.e., when there is a $\lambda \in (0,1)$ such that $\Omega = \{\omega_0, \omega_1, \omega_2, \ldots\}$, and $\omega_{j+1} \leq \lambda \omega_j$ for $j = 0, 1, 2, \ldots$, Strömberg [S], and Cordoba and R. Fefferman [CF1] used covering arguments to show that $M_\Omega$ is bounded on $L^p$ when $p \geq 2$, and Nagel, Stein, and Wainger [NSW] followed with a Fourier analytic proof for boundedness on $L^p$ when $p > 1$. Let us say that a set $\Omega$ is lacunary of order $N$ if it is covered by the union of a lacunary sequence $L$ of order $N - 1$ with lacunary sequences converging to every point of $L$. Sjörgen and Sjölin [SS] iterated the proof in [NSW] to improve the result to include lacunary sequences of finite order.

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On the other hand, the existence of the Besicovitch set yields unboundedness of $M_\Omega$ on $L^p$, $p < \infty$, when $\Omega = [0, 1]$. A further negative result comes when $\Omega$ is the Cantor set: unboundedness in this case was shown in [K] for $p \leq 2$, and in [BK] for $p < \infty$.

Now let us say that $\Omega$ admits Kakeya sets if there is a collection $R_\Omega$ of rectangles, each pointed in a direction in $\Omega$ so that $|\bigcup_{R \in R_\Omega} R|$ is small relative to, say, $|\bigcup_{R \in R_\Omega} 3R|$, where $3R$ is the rectangle with the same center and width as $R$ and three times the length. In this paper we will prove

**Theorem 0.1.** Fix $1 < p < \infty$. The following are equivalent:

- $A$: $M_\Omega$ is bounded on $L^p$
- $B$: $\Omega$ does not admit Kakeya sets
- $C$: There exist $N_1, N_2 < \infty$ such that $\Omega$ is covered by $N_1$ lacunary sets of order $N_2$.

To prove this theorem we will view $\Omega$ as being the boundary of a subtree of the binary tree. Then we will introduce the splitting number of a tree, which measures, loosely speaking, to what degree the tree has a subtree that looks like the binary tree. This will allow us to categorize all such $\Omega$ as looking like either a lacunary-type set or a Cantor-type set.

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§ Outline

The goal of this paper is to provide a proof of Theorem 0.1. The proof that $A \Rightarrow B$ is simple. For suppose $\Omega$ admits Kakeya sets in the sense above: then for any $N$, we have sets

$$E_N := \bigcup_t R_t^{(N)} \quad \text{and} \quad E_N^* := \bigcup_t 3R_t^{(N)},$$

where the slopes of the $R_t^{(N)}$ are in $\Omega$, such that

$$\frac{|E_N|}{|E_N^*|} \to 0 \quad \text{as} \quad N \to \infty,$$

and such that $M_\Omega \chi_{E_N}(x) > \frac{1}{2}$ when $x \in E_N^*$. Hence

$$\frac{\int_{\mathbb{R}^n} (M_\Omega \chi_{E_N})^p}{\int_{\mathbb{R}^n} (\chi_{E_N})^p} \gtrsim \frac{|E_N|}{|E_N^*|} \to \infty \quad \text{as} \quad N \to \infty,$$

where, of course, by $\alpha \lesssim \beta$ we mean $\alpha \leq c\beta$ for some constant $c$.

Our contribution is the proof that $B \Rightarrow C$, and the majority of the paper is devoted to this. For completeness, we will review in the final section a proof that $C \Rightarrow A$. 
§1 Splitting Number and Notation for Trees

We begin by constructing the binary tree $B$: fix a vertex, $v_{\text{origin}}$, called the origin, and define

$$B_0 = \{v_{\text{origin}}\}.$$  

(Here we will use the word “origin”, since the more commonly used word “root” will be used frequently as a verb.) Then for $n = 0, 1, 2, \ldots$, connect each vertex $v \in B_n$ to two new vertices $c_0(v)$ and $c_1(v)$, called the 0th and 1st children of $v$, and define

$$B_{n+1} = \bigcup_{v \in B_n} \{c_0(v), c_1(v)\}.$$  

Then $\hat{B}$ is the tree with vertices

$$B := \bigcup_{n=0}^{\infty} B_n$$

and edges connecting a vertex $v$ with its children $c_0(v)$ and $c_1(v)$. We will refer to the tree $\hat{B}$ by its vertex set $B$, and we will do the same for other trees considered in this paper, which will all be subtrees of $B$. If $v \in B_n$, define the height of $v$, $h(v) = n$. Further, if $T \subseteq B$, then by $T_k$ we mean all vertices $v \in T$ such that $h(v) = k$.

Now given a vertex $v \in T \subseteq B$, we define a ray $R$ rooted at $v$ to be an ordered set of vertices $(v_1 = v, v_2, v_3, \ldots)$ such that $v_{j+1}$ is a child of $v_j$ for $j = 1, 2, \ldots$ Loosely speaking, a ray rooted at $v$ is a path from $v$ to infinity that always moves (strictly) away from the origin of the tree. The boundary of $T$ is the set of all rays in $T$ rooted at the origin, and will be denoted $\partial T$. Define the shadow, $U(v)$, of a vertex $v$, to be the set of all rays $R$ such that $v \in R$.

We identify the vertices of the binary tree with the dyadic intervals contained in $[0, 1]$ as follows:

1. Identify the origin with $[0, 1]$.
2. If $v$ is identified with the dyadic interval $I$, then identify $c_0(v)$ with the left half of $I$, and identify $c_1(v)$ with the right half of $I$.

We will write $v_I$ to indicated the vertex identified with the interval $I$, and $I_v$ to indicate the interval identified with the vertex $v$. Similarly, we can identify the boundary of the binary tree with the interval $[0, 1]$ in the following natural way: identify $a_1a_2\ldots$, where $a_j \in \{0, 1\}$, with the ray $(v_0 = v_{\text{origin}}, v_1, v_2, \ldots)$ if $v_{j+1}$ is the $a_{j+1}$th child of $v_j$, i.e., if $v_{j+1} = c_{a_{j+1}}(v_j)$ for every $j = 1, 2, \ldots$. 
If $\Omega$ is closed, then $[0, 1] - \Omega$ is the union of open intervals $Q_j$. Each $Q_j$ is the union of dyadic intervals, so we may write

$$[0, 1] - \Omega = \bigcup_j I_j,$$

where each $I_j$ is a dyadic interval. We define $T_{\Omega}$ to be the subtree of $B$ obtained by removing the subtrees of $B$ rooted at $v_{I_j}$ for $j = 1, 2, \ldots$. Alternatively, $T_{\Omega}$ is the subtree of $B$ with boundary

$$\partial T_{\Omega} = \partial B - \left( \bigcup_{j=1}^{\infty} U(v_{I_j}) \right).$$

Earlier, we defined what it means for a ray $R$ to be rooted at a vertex $v$. The collection of such rays depends on the tree $T$, and will be denoted by $R_T(v)$. If $u \in R$ for some $R_T(v)$, we will write $u \subseteq v$, and say that $u$ is a descendant of $v$, or that $v$ is an ancestor of $u$.

We will say that a vertex $v$ splits, or call $v$ a splitting vertex, if $v$ has two children, and define the splitting number $\text{split}(R)$ of a ray $R$ to be the number of splitting vertices along $R$. Then the splitting number of a vertex $v$ with respect to a tree $S$ rooted at $v$ is defined to be

$$\text{split}_S(v) = \min_{R \in R_S(v)} \text{split}(R),$$

and the splitting number of $v$ is defined to be

$$\text{split}(v) = \sup_S \text{split}_S(v),$$

where the sup is taken over all subtrees $S$ of $T$ rooted at $v$. Finally, for a tree $T$, we define

$$\text{split}(T) = \sup_{v \in T} \text{split}(v).$$

Before we state a theorem using this new language, we give a definition of lacunarity that is more suitable for trees: a subtree $L \subseteq T$ is said to be lacunary of order 0 if $L$ consists of a single ray in the boundary of $T$, and $L$ is said to be lacunary of order $N$ if all splitting vertices of $L$ lie along a lacunary tree of order $N - 1$.

**Remark 1.1.** If $L$ is a lacunary tree of order 1, then, loosely speaking, the directions associated with $L$ can be covered by four lacunary sequences in the traditional sense. More precisely, define $\alpha(R)$ to be the real number in $[0, 1]$ identified with the ray $R$. Note that all
splitting vertices of \( L \) lie on a single ray, call it \( \lim L \). We claim there exist four lacunary sequences \( \{a_j^{(i)}\}_{j=1}^\infty, i = 1, \ldots, 4 \), such that
\[
a_j^{(i)} \to \alpha(\lim L) \quad \text{as} \quad j \to \infty
\]
for each \( i \), and such that
\[
\alpha(R) \in \left( \bigcup_{i,j} a_j^{(i)} \right) \cup \{\alpha(\lim L)\}
\]
for every \( R \in \partial L \).

**Proof.** For each \( j = 0, 1, 2, \ldots \) there is at most one \( R \in \partial L \) such that \( d(\alpha(R), \alpha(\lim L)) = 2^{-j} \), where \( d \) denotes the dyadic distance on real numbers. (That is, \( d(\beta_1, \beta_2) \) is defined to be the size of the smallest dyadic interval containing both \( \beta_1 \) and \( \beta_2 \).) Now consider the set
\[
A := \{ R \in \partial L : \alpha(R) > \alpha(\lim L) \}.
\]
Finally, observe that if \( R_0, R_1, R_2, \ldots \in A \) are such that
\[
d(\alpha(R_j), \alpha(\lim L)) = 2^{-2j},
\]
then
\[
0 < \alpha(R_{j+1}) - \alpha(\lim L) \leq \frac{1}{2} (\alpha(R_j) - \alpha(\lim L)),
\]
i.e., \( \{\alpha(R_j)\}_{j=1}^\infty \) is lacunary in the traditional sense. An identical claim can be made if \( R_0, R_1, R_2, \ldots \in A \) are such that \( d(\alpha(R_j), \alpha(\lim L)) = 2^{-2j-1} \), hence \( A \) is covered by two lacunary sequences in the sense described above. Of course this implies \( L \) is covered by four lacunary sequences since we could similarly show that the set \( B := \{ R \in \partial L : \alpha(R) < \alpha(\lim L) \} \) is covered by two lacunary sequences. \( \Box \)

**Theorem 1.2.**

\( A: \) If \( \text{split}(T_\Omega) = N < \infty \), then \( T_\Omega \) is lacunary of order \( N \), and hence \( M_\Omega \) is bounded on \( L^p \) for \( 1 < p < \infty \).

\( B: \) Conversely, if \( \text{split}(T_\Omega) = \infty \), then \( \Omega \) admits Kakeya sets, and hence \( M_\Omega \) is unbounded on \( L^p, p < \infty \).

**Remark 1.3.** Let \( \Omega \) be such that \( \text{split}(T_\Omega) = N \). In \( \S 5 \) we will see that there exists a constant \( C \) such that
\[
\|M_\Omega f\|_p \leq CN\|f\|_p.
\]

Theorem 1.2 automatically yields the “\( B \Rightarrow C \)” part of Theorem 0.1, and we are already able to dispense with part \( A \) of Theorem 1.2. The following lemma records an easy observation that will help with the proof.
Lemma 1.4. If $T$ is a tree, and $u \neq v$ are vertices of $T$ with $\text{split}(u) \geq N$, and $\text{split}(v) \geq N$, and if $h(u) \geq h(v)$, then either $\text{split}(T) \geq N + 1$, or there exists $R \in \mathcal{R}_T(v)$ such that $u \in R$, i.e., $u \subseteq v$.

Proof. First note that $u$ and $v$ must have a common ancestor. If there is no $R \in \mathcal{R}_T(v)$ such that $u \in R$, then the common ancestor is some other vertex $w$, and $v \neq w \neq u$. (Of course $u$ cannot be the common ancestor since $h(u) \geq h(v)$.) But then $\text{split}(w) \geq N + 1$: since there are subtrees $T_v$ and $T_u$ for which $\text{split}_{T_v}(v) = N = \text{split}_{T_u}(u)$, we define $T_w$ to be the tree formed by joining $T_u$ with $T_v$ through $w$, and we have $\text{split}_{T_w}(w) = N + 1$. □

Proof of Theorem 1.2 part A. If $\text{split}(T_\Omega) = 0$, then $T_\Omega$ has only one ray rooted at the origin. Hence $T_\Omega$ is lacunary of order zero. Now we induct on the splitting number: suppose $\text{split}(T_\Omega) = N$. By Lemma 1.4, all vertices $v$ with $\text{split}(v) = N$ lie along a single ray. So if $v^*$ is a child of $v$ not lying on the ray $R$, and if $T_{v^*}$ is a subtree of $T_\Omega$ rooted at $v^*$, then $\text{split}_{T_{v^*}}(v^*) \leq N - 1$, and hence is lacunary of order $N - 1$ by the induction hypothesis. But we can repeat this process, which results in $T_\Omega$ being lacunary of order $N$. □

Since we suppose now that $\text{split}(T_\Omega) = \infty$, to prove Theorem 1.2 part B it suffices to exhibit, when $\text{split}(T_\Omega) \geq N$, a collection of parallelograms $\{P_t\}$, each of which is pointed in one of the directions in $\Omega$, such that

\((\clubsuit) \, 1\) \quad |\bigcup_t P_t| \lesssim \frac{1}{N}

and such that

\((\clubsuit) \, 2\) \quad |\bigcup_t 3P_t| \gtrsim \frac{\log N}{N},

where $3P_t$ is the parallelogram with the same center and width as $P_t$, but three times the length. To do this, we divide the interval $[0, 1]$ on the $y$-axis into small intervals, each of which will serve as a base for one of the parallelograms $P_t$. The difficult part of the construction is to specify a slope for each of the parallelograms so that they satisfy the properties $(\clubsuit) \, 1$ and $(\clubsuit) \, 2$. In fact, we will not give an explicit choice of slopes; instead, we will use the probabilistic method to show that such a choice exists.

§2 Pruned Trees and Sticky Maps

It will actually be to our advantage to limit the possible slopes to a subset of $\Omega$, represented by a pruned subtree $\mathcal{P}$ of $T_\Omega$, and to restrict our attention to a certain class of slope functions, called sticky maps. We now define these terms. Suppose $T \subseteq \mathcal{B}$ is a tree such that $\text{split}(T) = N$. Then there is a vertex $v_0 \in T$ such that $\text{split}(v_0) = N$. Without loss
of generality, suppose \( v_0 \) is the origin. We say that \( T \) is pruned if for every \( R \in \mathcal{R}_T(v_0) \), and every \( j = 1, 2, ..., N \), \( R \) contains exactly one vertex \( v_j \) such that \( \text{split}(v_j) = j \). If \( T \) is not necessarily pruned, then we can find a pruned subtree \( P \) of \( T \) that still has splitting number \( N \) by the following recursive procedure:

1. Let \( v_0 \) (the origin) be in \( P \).
2. Assign \( j := 0 \).
3. While \( 0 \leq j < N \), if \( v \in P \) has splitting number \( N - j \), choose a pair \( u, w \) such that 
   \[
   \text{split}(u) \geq N - j - 1, \quad \text{and} \quad \text{split}(w) \geq N - j - 1,
   \]
   and add \( u, w \) to \( P \).
   Also add all vertices and edges connecting \( v \) to \( u \) and \( w \).
4. Assign \( j := j + 1 \).

We call the vertices added to \( P \) at the \( j \)th iteration the \( j \)th generation, and denote the collection of vertices in the \( j \)th generation \( G_j(T) \). If \( T \) is already pruned, then \( G_j(T) \) still makes sense. We will denote by \( P(T) \) the subtree of \( T \) formed by \( \cup_j G_j(T) \) and the edges and vertices connecting \( G_j(T) \) to \( G_{j+1}(T) \). Note that it is not necessary for any \( v \in G_j \) to have \( h(v) = j \). (Except for \( j = 0 \), because we have assumed \( h(v_0) = 0 \), and \( G_0 = \{v_0\} \).) Also note that for a general tree \( T \) with splitting number \( N \), there may exist several different subtrees \( P_1, P_2, P_3, ... \) each with splitting number \( N \), and each pruned. The method above yields one of them.

We now consider maps \( \sigma : B \to S \subseteq B \). Such a map is said to be sticky if whenever \( u \subseteq v \in B \), then \( \sigma(u) \subseteq \sigma(v) \) in \( S \). In addition, all sticky maps considered here will be assumed to satisfy \( h(\sigma(v)) = h(v) \) for all \( v \in B \). Recall that \( I_v \) is the dyadic interval identified with \( v \), and note that \( |I_v| = 2^{-h(v)} \), where \( | \cdot | \) denotes the standard Euclidean measure. Note that if \( V \) is a collection of vertices, and if \( v_1, v_2, ... \) are the disjoint maximal elements in \( V \), then

\[
| \bigcup_{v \in V} I_v | = \sum |I_{v_j}|.
\]

The following lemma gives an elementary fact about sticky maps into pruned trees that will be useful in completing the proof of Theorem 1.2 part B.

**Lemma 2.1.** Suppose \( \sigma : B \to P \) is a sticky map into a pruned tree \( P \) with generations \( G_k(P) \). Then for each \( k = 0, 1, ..., N \),

\[
\sum_{v \in G_k(P)} \sum_{u \in \sigma^{-1}(v)} |I_u| = 1.
\]

**Proof.** Of course, for \( v_1 \neq v_2 \in G_k(P) \), the sets \( \sigma^{-1}(v_1) \) and \( \sigma^{-1}(v_2) \) are disjoint. Hence the sum in the statement of the lemma is over a collection of vertices \( u_1, u_2, ... \in B \) such
that \( u_i \not\subseteq u_l \) whenever \( i \neq l \). But then
\[
\sum_{v \in G_k(P)} \sum_{u \in \sigma^{-1}(v)} |I_u| = \left| \bigcup_{u \in \sigma^{-1}(G_k(P))} I_u \right| \leq 1,
\]
since \(|[0, 1]| = 1\). In fact, equality holds, since \( \bigcup_{u \in \sigma^{-1}(G_k(P))} I_u = [0, 1] \).

§3 Geometric Construction

We can now be more specific about how to construct the collection of parallelograms mentioned above. We define the height of a tree \( T \) by
\[
h(T) = 1 + \sup_{v \text{ splitting}} h(v)
\]
where the sup is only taken over vertices \( v \) that split. Since we suppose \( \text{split}(T_{\Omega}) \geq N \), there exists a pruned subtree \( P := \mathcal{P}(T_{\Omega}) \) such that \( \text{split}(P) = N \). We will ignore all vertices \( v \in \mathcal{P} \) such that \( h(v) > h(P) \). For each \( t = 0, \frac{1}{2h(P)}, \frac{2}{2h(P)}, ..., \frac{2^{h(P)} - 1}{2h(P)} \), we will have a parallelogram \( P_t = P_{t, \sigma} \) with corners \((0, t), (0, t + 2h(P)), (2, t + 2\sigma(t)), \) and \((2, t + 2^{h(P)} + 2\sigma(t))\), where \( \sigma : \mathcal{B} \to \mathcal{P} \) is a sticky map to be determined. We will write
\[
K_\sigma = \bigcup_t P_{t, \sigma}.
\]

To finish the proof of Theorem 1.2 part B, it remains to prove the following.

**Claim 3.1.**

A: If \( \sigma : \mathcal{B} \to \mathcal{P} \) is sticky, then
\[
|K_\sigma \cap ([0, 1] \times \mathbb{R})| \gtrsim \frac{\log N}{N}.
\]

B: There exists a sticky map \( \sigma : \mathcal{B} \to \mathcal{P} \) such that
\[
|K_\sigma \cap ([1, 2] \times \mathbb{R})| \lesssim \frac{1}{N}.
\]

We begin by proving A. For \( j = 0, 1, 2, ..., \) define \( X_j \) to be the vertical strip
\[
X_j = [2^{-j}, 2^{-j+1}] \times \mathbb{R}.
\]
We will show that for \( j = 0, 1, ..., \log N \), we have the estimate
\[
|K_\sigma \cap X_j| \gtrsim \frac{1}{N},
\]
and Claim 3.1 A will follow. To do this we will control the intersections of the rectangles \( P_{t, \sigma} \) in the strip \( X_j \). A more precise statement is given below in the setting of a measure space:
Lemma 3.2. Let $(X, \mathcal{M}, |\cdot|)$ be a measure space, and let $A_1, A_2, \ldots, A_K$ be sets with $|A_i| = \alpha$ for every $i$. If

$$\sum_{i=1}^K \sum_{l=1}^K |A_i \cap A_l| \leq M,$$

then

$$|\bigcup_{i=1}^K A_i| \geq \frac{\alpha^2 K^2}{16M}.$$

Proof of Lemma 3.2. By pigeonholing, we obtain a set $E \subseteq 1, \ldots, K$ such that $\#(E) \geq \frac{K}{2}$, and

$$\sum_{l=1}^K |A_i \cap A_l| \leq \frac{2M}{K}$$

for $i \in E$. But this implies that

$$\frac{1}{\alpha} \int_{A_i} \sum_{l=1}^K \chi_{A_l} \leq \frac{2M}{\alpha K}$$

for $i \in E$, and hence that

$$\sum_{l=1}^K \chi_{A_l}(x) \leq \frac{4M}{\alpha K}$$

for $x$ in a set $B_i \subseteq A_i$, with $B_i \geq \frac{\alpha}{2}$ for $i \in E$. But then

$$|\bigcup_{i=1}^K A_i| \geq |\bigcup_{i \in E} B_i| \geq \frac{\alpha K}{4M} \sum_{i \in E} |B_i| = \frac{\alpha^2 K^2}{16M}.$$

□

Writing $P_{t,\sigma,j} := P_{t,\sigma} \cap X_j$, we have

$$|P_{t,\sigma,j}| = 2^{-h(P)-j}.$$

Then by Lemma 3.2, we only need to show

$$(\exists) \quad \sum_{t_1=1}^{2^{h(P)}} \sum_{t_2=1}^{2^{h(P)}} |P_{t_1,\sigma,j} \cap P_{t_2,\sigma,j}| \lesssim \frac{N}{2^{2j}}.$$
Since the diagonal term is
\[ \sum_{t_1=1}^{2^h(P)} |P_{t_1,\sigma,j}| = \frac{1}{2^j}, \]
we will only be able to show \((\forall)\) when \(j = 0, 1, \ldots, \log N\). Let us introduce some notation that will be helpful in decomposing the sum in \((\forall)\). For any two vertices \(u\) and \(v\) in \(B_{h(P)}\), let \(D(u, v)\) be the minimal vertex containing both \(u\) and \(v\), i.e., let \(D(u, v)\) be the vertex \(w\) with largest height satisfying \(u \subseteq w\) and \(v \subseteq w\). (Of course this notion could be defined on all pairs of vertices in \(B\), but we only need to use the restriction to pairs of vertices in \(B_{h(P)}\).) Then of course for a vertex \(w\), we will write
\[ D^{-1}(w) := \{(u, v) \in B_{h(P)} \times B_{h(P)} : D(u, v) = w\}, \]
and if \(W\) is a collection of vertices, we will write \(D^{-1}(W) := \cup_{w \in W} D^{-1}(w)\). Now suppose \(t_1 \neq t_2\), and note that if \(P_{t_1,\sigma,j} \cap P_{t_2,\sigma,j} \neq \emptyset\), then
\[ 2^{-j} |I_{D(t_1,t_2)}| \gtrsim 2^{-j} |\sigma(t_1) - \sigma(t_2)| \gtrsim |t_1 - t_2|. \]
(Note that \(|I_{D(t_1,t_2)}|\) is just the usual dyadic distance, except that here it is defined on vertices in \(B\) identified with \(h(P)\)–digit binary expansions.) In light of this, we introduce for a vertex \(w \in B\),
\[ \Gamma_j^w := \{(t_1, t_2) \in D^{-1}(w) : 2^j |t_1 - t_2| \lesssim |I_w|\} \]
and
\[ \Gamma_{j,l}^w := \{(t_1, t_2) \in D^{-1}(w) : 2^{j+l} |t_1 - t_2| \sim |I_w|\} \]
so that
\[ \Gamma_j^w = \bigcup_{l \geq 0} \Gamma_{j,l}^w. \]
Observe that
\[ \# \left( \Gamma_{j,l}^w \right) \sim 2^{2h(P)-2l-2j-2h(w)} \]
and
\[ \# \left( \Gamma_j^w \right) \sim 2^{2h(P)-2j-2h(w)}. \]
Now we write the off-diagonal part of (\(\text{\textbullet}\)) as
\[
\sum_{t_1 \in B_{h(P)}} \sum_{t_2 \neq t_1} |P_{t_1, \sigma, j} \cap P_{t_2, \sigma, j}| = \sum_{v \in P} \sum_{(t_1, t_2) \in D^{-1}(\sigma^{-1}(v))} |P_{t_1, \sigma, j} \cap P_{t_2, \sigma, j}|
\]
\[
= \sum_{v \in P} \sum_{u \in \sigma^{-1}(v)} \sum_{(t_1, t_2) \in D^{-1}(u)} |P_{t_1, \sigma, j} \cap P_{t_2, \sigma, j}|
\]
\[
=: (\star).
\]

By (\(\diamond\)), we have that if \((t_1, t_2) \in D^{-1}(u)\) is to contribute to the sum, then

(\(\diamond\diamond\)) \[|I_u| \gtrsim 2^j |t_1 - t_2|,\]
i.e., \((t_1, t_2) \in \Gamma^j(u)\). Further, if \((t_1, t_2) \in \Gamma^j(u)\) contributes, then
\[|\sigma(t_1) - \sigma(t_2)| \gtrsim 2^j |t_1 - t_2|,\]
so that in this case,
\[|P_{t_1, \sigma, j} \cap P_{t_2, \sigma, j}| \lesssim \frac{1}{2^{2h(P)} 2^j |t_1 - t_2|} \sim \frac{1}{2^{2h(P)} 2^{-l}|I_u|}.\]

Then because of (\(\diamond\diamond\)), we may compute the innermost sum in (\(\star\)) as
\[
\sum_{(t_1, t_2) \in D^{-1}(u)} |P_{t_1, \sigma, j} \cap P_{t_2, \sigma, j}| = \sum_{(t_1, t_2) \in \Gamma^j(u)} |P_{t_1, \sigma, j} \cap P_{t_2, \sigma, j}|
\]
\[
= \sum_{l \geq 0} \sum_{(t_1, t_2) \in \Gamma^j(u)} |P_{t_1, \sigma, j} \cap P_{t_2, \sigma, j}|
\]
\[
\lesssim \sum_{l \geq 0} \#(\Gamma^j(u)) 2^{2h(P) - l |I_u|}
\]
\[
\lesssim \frac{|I_u|}{2^{2j}}.
\]

To finish estimating (\(\star\)), we state and prove a technical-looking proposition, whose proof requires little more than counting exponents.

**Proposition 3.4.** Fix \(w \in B\). Then for any \(l^* \geq 1\),
\[
\sum_{l=0}^{l^*-1} \sum_{\{u \subseteq w : h(u) = h(w) + l\}} \#(\Gamma^j_{l^*-l}(u)) \leq 2 \sum_{\{u \subseteq w : h(u) = h(w) + l^*\}} \#(\Gamma^j(u)).
\]
Proof. There are $2^l$ vertices $u \subseteq w$ such that $h(u) = h(w) + l$, so the estimates on $\Gamma_j^l(w)$ and $\Gamma_j^l(w)$ allow us to control the left hand side in the statement of the proposition by

$$
\sum_{l=0}^{l^* - 1} 2^l 2^{2h(\mathcal{P}) - 2(l^* - l) - 2(h(w) + l)} \lesssim 2^l 2^{2h(\mathcal{P}) - 2j - 2h(w)},
$$

which is controlled by the right hand side. □

This Proposition allows us to restrict attention in the outer sum in (*) to splitting vertices $v \in \mathcal{P}$, i.e., to vertices $v \in G_k(\mathcal{P})$ for some $k = 1, 2, \ldots, N$. For if $v_1, v_2, \ldots \in \mathcal{P}$ are such that $v_{j+1}$ is a child of $v_j$, with $v_1, v_2, \ldots, v_{n-1}$ not splitting and $v_n$ splitting, then by Proposition 3.4,

$$
\sum_{l=1}^{n-1} \sum_{u \in \sigma^{-1}(v_l)} \sum_{(t_1, t_2) \in D^{-1}(u)} |P_{t_1, \sigma, j} \cap P_{t_2, \sigma, j}| \lesssim \sum_{u \in \sigma^{-1}(v_n)} \frac{|I_u|}{2^{2j}}.
$$

Using the computation above with Proposition 3.4 and Lemma 2.1 yields

$$
(*) \lesssim \sum_{k=1}^{N} \sum_{v \in G_k(\mathcal{P})} \sum_{u \in \sigma^{-1}(v)} \frac{|I_u|}{2^{2j}} \lesssim \frac{N}{2^{2j}},
$$

which completes the proof of Claim 3.1 A.

§ 4 The Probabilistic Argument and Percolation on Trees

Now we prove probabilistically that there is some sticky map $\sigma : \mathcal{B} \to \mathcal{P}$ such that $|K_\sigma \cap ([1,2] \times \mathbb{R})| \lesssim \frac{1}{N}$. In fact, if we denote by $Pr(x, y)$ the probability (over sticky $\sigma$) that $(x, y) \in P_{t, \sigma}$ for some $t$, it is enough to show that given $(x, y) \in [1, 2] \times [0, 3]$, we have $Pr(x, y) \lesssim \frac{1}{N}$. (Of course, if $x \in [1, 2]$, and $y \notin [0, 3]$, then $(x, y)$ cannot possibly be covered by $K_\sigma$.) Then by the linearity of expectations, we would have

$$
\int \left( \int_1^2 \int_0^3 \chi_{K_\sigma}(x, y) dy dx \right) d\sigma = \int_1^2 \int_0^3 \left( \int \chi_{K_\sigma}(x, y) d\sigma \right) dy dx = \int_1^2 \int_0^3 Pr(x, y) dy dx \lesssim \frac{1}{N}.
$$
This, of course, would imply the existence of a sticky map $\sigma$ satisfying Claim 3.1 B.

Recall that there will be one parallelogram for each vertex $t \in B_{h}(P)$. Since $x > 1$, for each $t$, there is at most one possible value for $\sigma(t)$ in $P_{h}(P)$ such that $(x, y) \in P_{t, \sigma}$. If such a slope exists, call it $S_{(x, y)}(t)$; if it does not exist, we will say $S_{(x, y)}(t) = \infty$. The set of $t \in B_{h}(P)$ for which $S_{(x, y)}(t) < \infty$, call it the possible set of $(x, y)$, $\text{Poss}(x, y)$, will have at most $2N$ elements. Given a set of vertices $V \subseteq B$, we denote by $< V >$ the tree generated by $V$, i.e., the subtree of $B$ consisting of $V$ and all the ancestors of elements in $V$ (and all the edges connecting these vertices). Now consider the subtree $< \text{Poss}(x, y) > \subseteq B$. Given $t \in \text{Poss}(x, y)$, there are at least $N$ ancestors of $t$, say $t_{1}, ..., t_{N} \supseteq t$ such that $t_{j}$ is an ancestor of $t_{j+1}$ and such that $\sigma(t_{j})$ is a splitting vertex in $P$. Call such vertices choosing vertices.

Now let $C$ be the tree formed by all the choosing vertices of $< \text{Poss}(x, y) >$, and edges connecting any pair of choosing vertices $u, v \in < \text{Poss}(x, y) >$ such that $u \subseteq v$ with no choosing vertex $w$ such that $u \subsetneq w \subsetneq v$. Similarly, let $B_{N}^{*}$ be the tree formed by all the splitting vertices of $P$, with edges connecting all the splitting vertices $u, v \in P$ such that $u \subseteq v$ and there is no splitting vertex $w$ such that $u \subsetneq w \subsetneq v$. Now $B_{N}^{*}$ is the binary tree of height $N$, i.e.,

$$B_{N}^{*} = B \cap \left( \bigcup_{k=1}^{N} B_{k} \right),$$

and $C$ is a subtree of $B_{N}^{*}$.

So now we construct the sticky maps $\sigma : B \rightarrow P$ randomly as follows: to each edge $e$ in $C$, assign a random variable $r = r(e)$ that takes on the values 0 and 1 with probabilities $\frac{1}{2}$. We will write $e_{v, u}$ to denote the edge connecting $v$ to one of its children $u$. If $r(e_{v, u}) = l$, we set

$$\sigma(u) = c_{l}(\sigma(v)),$$

where, again, we use $c_{l}(w)$ to denote the $l$th child of a vertex $w$.

Let $k \in \{0, 1, ..., N\}$. Given $v \in B_{k}$, for $j = 0, 1, ..., k$, define $A_{j}(v)$ to be the ancestor of $v$ at height $j$. So if $(x, y) \in P_{t, \sigma}$, and if

$$A_{j+1}(S_{(x, y)}(t)) = c_{b_{j}(t)}(A_{j}(S_{(x, y)}(t))),$$

for some sequence $b_{j}$ of zeros and ones depending on $t$, then we must have

$$r(e_{A_{j}(t), A_{j}(t+1)}) = b_{j}(t).$$

Similarly, if we are to have $(x, y) \in K_{\sigma}$, then we must find a $t \in \text{Poss}(x, y)$ such that $r(e_{A_{j}(t), A_{j}(t+1)}) = b_{j}(t)$ for all $j = 1, ..., N$. Since $r$ takes on each value 0 or 1 with probability $\frac{1}{2}$, this requirement is equivalent to the following: if we remove each edge of $C$
with probability $\frac{1}{2}$, we require that a path remains from the root to $C_N$. In the probability literature, the probability of this outcome is called the survival probability of Bernoulli($\frac{1}{2}$) percolation on $C$, which we discuss below.

Given a tree $T \subseteq B$ of height $N$, remove each edge with probability $\frac{1}{2}$. Denote by $P(T)$ the probability that a path remains from the origin to $T_N$. A convenient way to compute this quantity is to view $T$ as an electrical circuit. Accordingly, we define the resistance of the tree $T$ as follows: place the positive node of a battery at the root of $T$, then identify all vertices in $T_N$, and place the negative node of the battery at this new vertex. For each edge at distance $k$ from the root, place a resistor of strength $2^k$. The resistance of the tree $T$, call it $R(T)$, is defined to be the resistance of this circuit. The following result of R. Lyons relates the resistance of $T$ to the survival probability of Bernoulli ($\frac{1}{2}$) percolation on $T$. We state and prove a special case to keep the paper self-contained. For a more general result, see [L]. For more about probability on trees, see [LP]. The proof given here is from [BK] and actually holds when $T$ is a subset of the ternary tree, but it works for our purposes since the binary tree is a subtree of the ternary tree.

**Theorem 4.1 (Lyons).** We have that

$$P(T) \lesssim \frac{1}{2 + R(T)}.$$ 

**Proof of Claim 3.1 B.** Assuming Theorem 4.1, it remains to show that the resistance of the tree $C$ is $\gtrsim \frac{1}{N}$. First recall that $C$ is a subtree of the truncated binary tree $B^*_N$, so $R(C) \geq R(B^*_N)$. Now to compute a lower bound for $R(B^*_N)$, connect all vertices at height $k$ by an ideal conductor to make one node $V_k$. (This only decreases the resistance of the circuit.) Now there are $2^k$ connections between $V_k^*$ and $V_{k+1}^*$, each with resistance $2^k$. If $R_k$ is the resistance between $V_k^*$ and $V_{k+1}^*$, then

$$\frac{1}{R_k} = \sum_{1}^{2^k} \frac{1}{2^k} = 1,$$

so $R_k = 1$ for all $k = 1, 2, \ldots, N$. Summing over $k$ results in $R(B^*_N) \geq N$. □

**Proof of Theorem 4.1.** We prove this by induction on $n$. Clearly it is true for constant 2, when $n = 0$. We assume up to $n - 1$, we have

$$P(T) \leq \frac{12}{2 + R(T)}.$$

We observe that we may view $T$ as the root together with up to 3 edges connected to 3 trees $T_1, T_2,$ and $T_3$. (If some of these trees are empty, we assign them probability zero and infinite resistance.) We denote

$$P(T) = P_j,$$
and

\[ R(\mathcal{T}) = R_j. \]

Then we have the recursive formulae

\[ P(\mathcal{T}) = \frac{1}{2}(P_1 + P_2 + P_3) - \frac{1}{4}(P_1P_2 + P_1P_3 + P_2P_3) + \frac{1}{8}P_1P_2P_3 \]

and

\[ \frac{1}{R(\mathcal{T})} = \frac{1}{2 + 2R_1} + \frac{1}{2 + 2R_2} + \frac{1}{2 + 2R_3}. \]

Now we break into two cases. In the first case, we have \( \frac{12}{2 + R_j} > 2 \) for some \( j \). Then we have \( R_j < 4 \). This implies \( R(\mathcal{T}) < 10 \) which implies \( \frac{12}{2 + R(\mathcal{T})} > 1 \), so that we certainly have

\[ P(\mathcal{T}) \leq \frac{12}{2 + R(\mathcal{T})}. \]

We define

\[ Q_j = \frac{12}{2 + R_j}. \]

We may assume each \( Q_j \leq 2 \). Observe that if we define

\[ F(x, y, z) = 1 - (1 - \frac{1}{2}x)(1 - \frac{1}{2}y)(1 - \frac{1}{2}z), \]

on the domain \([0, 2] \times [0, 2] \times [0, 2]\) then \( F \) is monotone increasing in each variable. Therefore we have that

\[ P(\mathcal{T}) = F(P_1, P_2, P_3) \]

\[ \leq F(Q_1, Q_2, Q_3) \]

\[ \leq \frac{1}{2}(Q_1 + Q_2 + Q_3) - \frac{1}{6}(Q_1Q_2 + Q_1Q_3 + Q_2Q_3) \]

Note that the equality is (4.1), while for the two inequalities we have used that the \( Q \)'s are \( \leq 2 \).

Now plugging into (4.3), the definition of the \( Q \)'s, we obtain

\[ P(\mathcal{T}) \leq \frac{12}{2} \left[ \frac{(R_1 + 2)(R_2 + 2) + (R_1 + 2)(R_3 + 2) + (R_2 + 2)(R_3 + 2) - \frac{12}{6}(R_1 + R_2 + R_3 + 6)}{(R_1 + 2)(R_2 + 2)(R_3 + 2)} \right] \]

\[ \leq \frac{12}{2} \left[ \frac{(R_1 + 2)(R_2 + 2) + (R_1 + 2)(R_3 + 2) + (R_2 + 2)(R_3 + 2) - \frac{12}{6}(R_1 + R_2 + R_3 + 6)}{(R_1 + 2)(R_2 + 2)(R_3 + 2) - 4R_1 - 4R_2 - 4R_3 - 13} \right] \]

\[ \leq \frac{12}{2} \left[ \frac{(R_1 + 1)(R_2 + 1)(R_3 + 1) + (R_1 + 1)(R_3 + 1) + (R_2 + 1)(R_3 + 1)}{(R_1 + 2)(R_2 + 2)(R_3 + 2) - 4R_1 - 4R_2 - 4R_3 - 13} \right] \]

\[ = \frac{12}{R(\mathcal{T}) + 2}. \]
Here the second inequality is by decreasing the denominator and the third inequality is by increasing the numerator.

\[ \square \]

§5 The Lacunary Case

To complete the proof of Theorem 0.1, it remains to show the following proposition:

**Proposition 5.1.** If \( T_\Omega \) is lacunary of order \( N \), then there exists a constant \( C \) such that

\[ ||M_\Omega f||_p \leq CN||f||_p. \]

**Remark 5.2.** As noted earlier, if \( T_\Omega \) has splitting number \( N \), then \( T_\Omega \) is lacunary of order \( N \), and hence

\[ ||M_\Omega f||_p \leq CN||f||_p. \]

As mentioned earlier, the result in Proposition 5.1 was published in [SS]. The proof given here extends ideas present in [NSW], and follows Alfonseca [A].

Recall that each ray \( R \in T_\Omega \) is identified with a real number \( \alpha(R) \in [0,1] \). If \( T \) is a tree, we will define

\[ \alpha(T) = \{ \alpha(R) : R \in \partial T \}. \]

Now write \( v_\theta \) to denote the unit vector with slope \( \theta \), and define

\[ M_\theta f(x) = \sup_{h>0} \frac{1}{2h} \int_{-h}^{h} f(x + v_\theta t) dt. \]

If a tree \( L \) is lacunary of order 1, then there is a ray, called \( \lim L \) as in Remark 1.1, such that every splitting vertex in \( L \) lies along \( \lim L \). Also define \( \beta_j(L) \) to be the (unique, if it exists) element of \( \alpha(L) \) such that \( d(\alpha(\lim L), \beta_j) = 2^{-j} \), where again \( d \) denotes the dyadic distance on real numbers, and let

\[ \Omega_j(L) := \{ \beta \in \alpha(L) : d(\alpha(\lim L), \beta) = 2^{-j} \}. \]

**Proposition 5.3.** Fix \( 1 < p \leq \infty \). Let \( \Omega^* \subseteq \Omega \). If there exists a lacunary tree \( L \) of order 1 such that \( L = T_{\Omega^*} \), then there exists a constant \( C \), depending only on \( p \), such that

\[ ||M_\Omega f||_p \leq C||f||_p \left( 1 + \sup_j ||M_{\Omega_j}||_{L^p \rightarrow L^p} \right). \]

A simple iteration of Proposition 5.3 will give us Proposition 5.1: For if \( T_\Omega \) is lacunary of order \( N \), then there exists \( \Omega^* \subseteq \Omega \) and a lacunary tree \( L \) of order 1 such that \( T_{\Omega^*} = L \).
Since $\mathcal{T}_\Omega$ is lacunary of order $N$, we may actually choose such an $L$ so that $\mathcal{T}_{\Omega_j}(L)$ is lacunary of order $N - 1$ for all $j$. But then we may apply the proposition again to the sets $\Omega_j$ and repeat to get $\|M_\Omega f\|_p \leq CN\|f\|_p$. It remains to prove Proposition 5.3.

**Proof of Proposition 5.3.** For convenience, we will rotate the plane so that $\alpha(\lim L) = 0$. Let $\delta_j = \frac{12}{20}2^{-j}$. Let $A_j$ be an interval centered around the dyadic interval containing the sets $\Omega_j$ such that $|A_j| = \delta_j$ and $\text{dist}(A_j^c, \Omega_j) \geq \frac{1}{30}2^{-j}$. Also let $\tilde{A}_j = \frac{13}{14}A_j$ and $\overline{A}_j = \frac{14}{13}A_j$.

Then define

$$
\Delta_j = \{(x, y) \in \mathbb{R}^2; \frac{y}{x} \in A_j\}
$$

$$
\tilde{\Delta}_j = \{(x, y) \in \mathbb{R}^2; \frac{y}{x} \in \tilde{A}_j\}
$$

$$
\overline{\Delta}_j = \{(x, y) \in \mathbb{R}^2; \frac{y}{x} \in \overline{A}_j\},
$$

and let $\omega_j$ be a $C^\infty$ function away from the origin, homogeneous of degree zero, such that $\omega_j \equiv 1$ on $\Delta_j$ and $\omega_j \equiv 0$ outside $\tilde{\Delta}_j$. Similarly, define $\tilde{\omega}_j$ with respect to the sectors $\tilde{\Delta}_j$ and $\overline{\Delta}_j$. Define

$$
\tilde{S}_j f = \omega_j \hat{f}, \quad \overline{S}_j f = \tilde{\omega}_j \hat{f}.
$$

Now let $\psi : \mathbb{R} \to \mathbb{R}$, $\psi \in C^\infty$, $\psi \geq 0$, be such that $\psi \equiv 1$ on $[-1, 1]$ and $\psi \equiv 0$ outside $[-2, 2]$. For $j \geq 1$ and $\theta \in \Omega_j$, define

$$
N_{h,j,\theta} f(x) = \frac{1}{2h} \int_{-\infty}^{\infty} \psi\left(\frac{t}{h}\right) f(x + v_\theta t) dt.
$$

We will only consider nonnegative $f$, and for such $f$, $\sup_{h>0} N_{h,j,\theta} f(x) \sim M_\theta f(x)$. Let $m = \hat{\psi}$ and let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be $C^\infty$ with $\phi(x) = 1$ when $|x| \leq 1$. Now we write

$$
\tilde{N}_{h,j,\theta} f(x) = m(h\xi_1 + h\xi_2\theta) \hat{f}(\xi)
$$

$$
= \phi(h\delta_j \xi) m(h\xi_1 + h\xi_2\theta) \hat{f}(\xi)
$$

$$
+ (1 - \phi(h\delta_j \xi))(1 - \omega_j(h\xi)) m(h\xi_1 + h\xi_2\theta) \hat{f}(\xi)
$$

$$
+ (1 - \phi(h\delta_j \xi))\omega_j(h\xi) m(h\xi_1 + h\xi_2\theta) \hat{f}(\xi)
$$

$$
=: I_{h,j,\theta} + II_{h,j,\theta} + III_{h,j,\theta}.
$$

To control $I_{h,j,\theta}$, we note that $\phi$ is Schwartz, and hence $I_{h,j,\theta}$ is controlled by the strong maximal function $M_{\beta_j}^S$ with respect to the axes with slopes $\beta_j$ and $\beta_j + \pi$, with constants independent of $j$. The term $II_{h,j,\theta}$ can be estimated in the same way, giving us

\[
(*) \quad \sup_h \sup_{\theta \in \Omega_j} I_{h,j,\theta} f(x) + II_{h,j,\theta} f(x) \leq CM_{\beta_j}^S f(x).
\]
Hence
\[(**\) \quad \| \sup_{j,h,\theta \in \Omega_j} (I_{h,j,\theta} + II_{h,j,\theta})f \|_p \leq C\|M_{\Omega^*}f \|_p \leq C\|f \|_p,\]
by the result in [NSW], since $T_{\Omega^*}$ is lacunary of order 1. It remains to control $III_{h,j,\theta}$.

We will assume, for now, that $\Omega$ is a finite set, and obtain a bound independent of the size of $\Omega$. Since $\Omega$ is finite, there is some minimal constant $C(\Omega)$ such that for $f \in L^p$,
\[
\| \sup_{j,h,\theta \in \Omega_j} |III_{h,j,\theta}f| \|_p \leq C(\Omega)\|f \|_p.
\]
Note that $N_{h,j,\theta}(g) \leq N_{h,j,\theta}(|g|)$ for any $g, h, j, \theta$, and let $\{g_j\}$ be a sequence of functions. Then by (\textit{*}), (**\) and the decomposition of $N_{h,j,\theta}$, we have
\[
\| \sup_{j,h,\theta \in \Omega_j} |III_{h,j,\theta}(g_j)| \|_p \leq \| \sup_{j,h,\theta \in \Omega_j} |III_{h,j,\theta}(\sup_j |g_j|)| \|_p + C(\Omega)\sup_{j} |g_j| \|_p.
\]

In fact, if $C(\Omega) \leq c_0$ independent of $\Omega$, we are already finished with Proposition 5.3, so we can assume otherwise and estimate the quantity above by $2C(\Omega)\|f \|_p$. In addition, it is clear that
\[
\left\| \left( \sum_{j} \sup_{h,\theta \in \Omega_j} |III_{h,j,\theta}(g_j)|^p \right)^{\frac{1}{p}} \right\|_p \leq \left( \sup_j \| \sup_{h,\theta \in \Omega_j} III_{h,j,\theta} \|_{L^p \to L^p} \right) \left\| \left( \sum_j |g_j|^p \right)^{\frac{1}{p}} \right\|_p.
\]
Interpolating yields
\[
\left\| \left( \sum_{j} \sup_{h,\theta \in \Omega_j} |III_{h,j,\theta}(g_j)|^2 \right)^{\frac{1}{2}} \right\|_p \leq C(\Omega)^{1-\frac{2}{p}} \left( \sup_j \| \sup_{h,\theta \in \Omega_j} III_{h,j,\theta} \|_{L^p \to L^p} \right) \left\| \left( \sum_j |g_j|^2 \right)^{\frac{1}{2}} \right\|_p.
\]
Recall that $III_{h,j,\theta}f$ has frequency support in $\tilde{\Delta}_j$ so
\[
\| \sup_{j,h,\theta \in \Omega_j} |III_{h,j,\theta}f| \|_p \leq \left\| \left( \sum_{j} \sup_{h,\theta \in \Omega_j} |III_{h,j,\theta}(\tilde{S}_j f)|^2 \right)^{\frac{1}{2}} \right\|_p \leq \left( \sum_j |\tilde{S}_j f|^2 \right)^{\frac{1}{2}}.
\]

\[
\leq C(\Omega)^{1-\frac{2}{p}} \left( \sup_j \| \sup_{h,\theta \in \Omega_j} III_{h,j,\theta} \|_{L^p \to L^p} \right) \left\| \left( \sum_j |\tilde{S}_j f|^2 \right)^{\frac{1}{2}} \right\|_p.
\]
But $C(\Omega)$ is minimal, and one can show $\left\| \left( \sum_j |\tilde{S}_j f|^2 \right)^{\frac{1}{2}} \right\|_p \leq C\|f\|_p$ by using Rademacher functions and the Marcinkiewicz multiplier theorem, as in [NSW], so

$$C(\Omega) \lesssim C(\Omega)^{1-\frac{1}{p}} \left( \sup_j \sup_{h,\theta \in \Omega_j} III_{h,j,\theta} \right)^{\frac{1}{p}},$$

and hence

$$C(\Omega) \lesssim \sup_j \sup_{h,\theta \in \Omega_j} III_{h,j,\theta} \left\| \right\|_{L^p \to L^p}.$$

However,

$$III_{h,j,\theta}(f) \lesssim N_{h,j,\theta}(|\tilde{\omega}_j * f|) \lesssim M_{\Omega_j}(|\tilde{\omega}_j * f|) \lesssim M_{\Omega_j} \left( \left( \sum_j |\tilde{S}_j f|^2 \right)^{\frac{1}{2}} \right),$$

so

$$\sup_j \sup_{h,\theta \in \Omega_j} III_{h,j,\theta} \left\| \right\|_{L^p \to L^p} \lesssim \sup_j \sup_{h,\theta \in \Omega_j} M_{\Omega_j} \left\| \right\|_{L^p \to L^p},$$

and this proves Proposition 5.2. □

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