CAUCHY PROBLEM FOR THE ELLIPSOIDAL BGK MODEL FOR POLYATOMIC PARTICLES

SA JUN PARK, SEOK-BAE YUN

Abstract. We establish the existence and uniqueness of mild solutions for the polyatomic ellipsoidal BGK model, which is a relaxation type kinetic model describing the evolution of polyatomic gaseous system at the mesoscopic level.

1. Introduction

The derivation of the celebrated Boltzmann equations relies heavily on the assumption that the gas consists of monatomic particles, which is not the case for most of the realistic gases. Efforts to derive Boltzmann type kinetic models soon confront with the difficulty that it is virtually impossible to write the pre- and post-collision velocities in an explicit form, since polyatomic molecules can possess arbitrarily complicated structures. In search of tractable model equation for polyatomic gases that avoids such difficulties, a BGK type model was suggested as a generalization of the ellipsoidal BGK model [2, 6, 7, 9, 32, 36]:

\[ \partial_t f + v \cdot \nabla_x f = A_{\nu, \theta}(M_{\nu, \theta}(f) - f) \]

\[ f(0, x, v, I) = f_0(x, v, I). \]

Unlike the monatomic case, a new variable \( I \) related to the internal energy due to the rotational and vibrational motions of the molecules is introduced so that the velocity distribution function \( f(t, x, v, I) \) represents the number density on \((x, v) \in T^2_\nu \times \mathbb{R}^3_v\) at time \( t \) with internal energy \( I^{2/\delta} \in \mathbb{R}^+ \), where \( \delta \) is the number of degrees of freedom except for the translational motion. We consider the fixed collision frequency \( A_{\nu, \theta} = 1/(1 - \nu + \nu\theta) \) throughout this paper. Two relaxation parameters \(-1/2 < \nu < 1\) and \(0 \leq \theta \leq 1\) are chosen in such a way that Prandtl number and the second viscosity coefficient computed through the Chapman-Enskog expansion, agrees with the physical data. (See [11, 12, 13, 36]).

The polyatomic Gaussian \( M_{\nu, \theta}(f) \) reads

\[ M_{\nu, \theta}(f) = \frac{\rho \Lambda_\delta}{\sqrt{\det(2\pi T_{\nu, \theta})(I_{\theta})}} \exp \left( -\frac{1}{2} (v - U)^\top T_{\nu, \theta}^{-1} (v - U) - \frac{I_{\theta}^2}{T_{\theta}} \right) \]

with normalizing factor

\[ \Lambda_\delta^{-1} = \int_{\mathbb{R}^+} \exp(-I_{\theta}^2) dI. \]

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The macroscopic local density $\rho(t, x)$, bulk velocity $U(t, x)$, stress tensor $\Theta(t, x)$ and internal energy $E_\delta(t, x)$ are defined respectively by

\[
\rho(t, x) = \int_{\mathbb{R}^3} f(t, x, v, I) \, dv \, dI,
\]
\[
U(t, x) = \frac{1}{\rho} \int_{\mathbb{R}^3} v f(t, x, v, I) \, dv \, dI,
\]
\[
\Theta(t, x) = \frac{1}{\rho} \int_{\mathbb{R}^3} f(t, x, v, I) \left( v - U(t, x) \right) \otimes \left( v - U(t, x) \right) \, dv \, dI,
\]
\[
E_\delta(t, x) = \int_{\mathbb{R}^3} \left( \frac{1}{2} |v - U(t, x)|^2 + I^2 \delta \right) f(t, x, v, I) \, dv \, dI.
\]

(1.3)

We split the internal energy $E_\delta$ into the internal energy from the translational motion $E_{tr}$ and the one from the non-translational motion $E_{I,\delta}$:

\[
E_{tr} = \int_{\mathbb{R}^3} \frac{1}{2} |v - U|^2 f \, dv \, dI,
\]
\[
E_{I,\delta} = \int_{\mathbb{R}^3} I^2 f \, dv \, dI,
\]

and define the corresponding temperatures $T_{\delta}, T_{tr}$ and $T_{I,\delta}$ by the equi-partition principle:

\[
E_\delta = \frac{3 + \delta}{2} \rho T_{\delta}, \quad E_{tr} = \frac{3}{2} \rho T_{tr}, \quad E_{I,\delta} = \frac{\delta}{2} \rho T_{I,\delta}.
\]

Note that $T_{\delta}$ is a convex combination of $T_{tr}$ and $T_{I,\delta}$:

\[
T_{\delta} = \frac{3}{3 + \delta} T_{tr} + \frac{\delta}{3 + \delta} T_{I,\delta}.
\]

(1.4)

Then, the relaxation temperature $T_\theta$ and the corrected temperature tensor $\mathcal{T}_{\nu, \theta}$ are defined as follows:

\[
T_\theta = \theta T_{\delta} + (1 - \theta) T_{I,\delta},
\]
\[
\mathcal{T}_{\nu, \theta} = \theta T_{\delta} \text{Id} + (1 - \theta) \left\{ (1 - \nu) T_{tr} \text{Id} + \nu \Theta \right\}.
\]

(1.5)

The relaxation operator satisfies the following cancellation properties:

\[
\int_{\mathbb{T}^3 \times \mathbb{R}^3 \times \mathbb{R}^+} (\mathcal{M}_{\nu, \theta}(f) - f) \, dx \, dv \, dI = 0
\]
\[
\int_{\mathbb{T}^3 \times \mathbb{R}^3 \times \mathbb{R}^+} v (\mathcal{M}_{\nu, \theta}(f) - f) \, dx \, dv \, dI = 0
\]
\[
\int_{\mathbb{T}^3 \times \mathbb{R}^3 \times \mathbb{R}^+} \left( \frac{1}{2} |v|^2 + I^2 \right) (\mathcal{M}_{\nu, \theta}(f) - f) \, dx \, dv \, dI = 0,
\]

(1.6)

yielding the conservation of mass, momentum and energy respectively. The entropy dissipation for the polyatomic gas was proved by Andries and Perthame et al. [2]. (See also [9, 28])

\[
\frac{d}{dt} \int_{\mathbb{T}^3 \times \mathbb{R}^3 \times \mathbb{R}^+} f(t) \ln f(t) \, dx \, dv \, dI \leq 0.
\]
2. MAIN RESULT

Definition 2.1. Let $T > 0$. $f \in C_+([0, T]; \| \cdot \|_{L^\infty_q})$ is said to be a mild solution for $\text{[1.1]}$ if it satisfies

$$f(t, x, v, I) = e^{-A_{\nu, \theta}t}f_0(x - vt, v, I) + A_{\nu, \theta} \int_0^t e^{-A_{\nu, \theta}(t-s)}M_{\nu, \theta}(f)(x - (t-s)v, v, s, I)ds,$$

where the weighted norm $\| \cdot \|_{L^\infty_q}$ is defined by

$$\|f(t)\|_{L^\infty_q} = \text{ess sup}_{x, v, I} |f(t, x, v, I)(1 + |v|^2 + I^2)^{\frac{q}{2}}|.$$

Our main result is as follows:

Theorem 2.2. Let $0 < \theta \leq 1$, $-1/2 < \nu < 1$, $\delta > 0$ and $q > 5 + \delta$. Suppose there exist positive constants $C_u, C_l$ and $C_1$ such that

$$\|f_0\|_{L^\infty_q} < C_u, \quad \int_{\mathbb{R}^3 \times \mathbb{R}^+} f_0(x - vt, v, I)dvdI \geq C_l > 0.$$

Then, for any final time $T > 0$, there exists a unique mild solution $f \in C_+([0, T]; \| \cdot \|_{L^\infty_q})$ for $\text{[1.1]}$ such that

1. $f$ is bounded on $t \in [0, T]$ as

$$\|f(t)\|_{L^\infty_q} = \text{ess sup}_{x, v, I} \{f(t, x, v, I)(1 + |v|^2 + I^2)^{\frac{q}{2}}\}.$$

2. There exist positive constants $C_{T, f_0}, C_{T, f_0, \delta}$ and $C_{T, f_0, \delta, q}$ such that

$$\rho(x, t) \geq C_{T, f_0}, \quad T_\delta(x, t) \geq C_{T, f_0, \delta},$$

$$\rho(x, t) + |U(x, t)| + T_\delta(x, t) \leq C_{T, f_0, \delta, q}.$$

3. Conservation laws of mass, momentum and energy hold:

$$\frac{d}{dt} \int_{\mathbb{R}^3 \times \mathbb{R}^+} f\left(1, v, \frac{1}{2}|v|^2 + I^2\right)dxvdI = 0.$$

4. $H$-theorem holds:

$$\frac{d}{dt} \int_{\mathbb{R}^3 \times \mathbb{R}^+} f \ln f dxvdI = \int_{\mathbb{R}^3 \times \mathbb{R}^+} (M_{\nu, \theta}(f) - f) \ln f dxvdI \leq 0.$$

Remark 2.3. When $\theta = 0$, all the above estimates break down. Therefore, this case should be considered separately. See Section 7 for the discussion of this case.

Ever since it was introduced in $\text{[4, 42]}$, the BGK model has seen huge applications in engineering and physics. The first mathematical study was carried out by Perthame in $\text{[29]}$, where the existence of weak solutions was proven under the assumption of finite mass, momentum, energy and entropy. Perthame and Pulvirenti $\text{[30]}$ then considered the class of solution space in which the uniqueness is guaranteed. It was later extended to the whole space $\text{[26]}$, and to $L^p$ solutions $\text{[50]}$. The Cauchy problem in the presence of external force or mean field was considered in $\text{[5, 43, 49]}$. Ukai studied a stationary problem on a bounded interval in $\text{[39]}$. The existence and asymptotic behavior near a global maxwellian were studied in $\text{[3, 44, 49]}$. For various macroscopic limits of BGK type models, see $\text{[15, 23, 24, 25, 34, 35]}$. Recently, Holway’s ellipsoidal generalization of the original BGK model (ES-BGK model) was re-suggested in $\text{[2]}$ with the first proof of $H$-theorem, and studied analytically in a series of paper $\text{[8, 10, 16, 27, 45, 46, 47, 48]}$. Mathematical study on the polyatomic BGK model
Lemma 3.1. Let \(\nu, \theta\) satisfy the following equivalence type estimates:

\[
\begin{align*}
(1) \quad & \theta T_\delta Id \leq T_{\nu, \theta} \leq \frac{1}{3} C_\nu \{3 + \delta(1 - \theta)\} T_\delta Id, \\
(2) \quad & \theta T_\delta \leq T_\theta \leq \frac{1}{\delta} \{\delta + 3(1 - \theta)\} T_\delta,
\end{align*}
\]

where \(C_\nu = \max_\nu \{1 - \nu, 1 + 2\nu\}\).

Proof. (1) (a) Upper bound: Recalling the definition of \(T_{\nu, \theta}\), we write

\[
\rho T_{\nu, \theta} = \theta \rho T_\delta Id + (1 - \theta) \{(1 - \nu)\rho T_{\nu, \theta} Id + \nu \rho \Theta\}
\]

\[
= \theta \rho T_\delta Id + (1 - \theta) \left\{(1 - \nu)\rho T_{\nu, \theta} Id + \nu \int_{\mathbb{R}^3 \times \mathbb{R}^+} f(v - U) \otimes (v - U) \, dv \right\}.
\]

From the identity

\[
k^T \{ (v - U) \otimes (v - U) \} k = \{(v - U) \cdot k\}^2, \quad \text{for} \ k \in \mathbb{R}^3,
\]

we derive

\[
(3.1) \quad k^T \{ \rho T_{\nu, \theta} \} k = \theta \rho T_\delta |k|^2 + (1 - \theta) \left\{(1 - \nu)\rho T_{\nu, \theta} |k|^2 + \nu \int_{\mathbb{R}^3 \times \mathbb{R}^+} f \{(v - U) \cdot k\}^2 \, dv \right\}.
\]

If \(0 \leq \nu < 1\), using the Cauchy-Schwartz inequality, we get

\[
\int_{\mathbb{R}^3 \times \mathbb{R}^+} f \{(v - U) \cdot k\}^2 \, dv \leq \int_{\mathbb{R}^3 \times \mathbb{R}^+} f |v - U|^2 |k|^2 \, dv \leq 3 \rho T_{\nu, \theta} |k|^2,
\]

For the numerical results of BGK model - monatomic, or polyatomic - we refer to [11, 17, 18, 20, 22, 31, 33, 51] and references therein. A nice survey on various mathematical and physical issues on kinetic equations can be found in [12, 13, 14, 19, 32, 37, 38, 40, 41].

The paper is organized as follows: In the following Section 3, we establish several estimates for macroscopic variables. In Section 4, we define our solution space and show that the approximate solutions lie in that space for all steps of iterations. Section 5 is devoted to showing that the relaxation operator is Lipschitz continuous in the solution space. In Section 6, we combine all the previous results to complete the existence proof. The reason why the case \(\theta = 0\) should be treated independently is briefly discussed in Section 7. In the appendix, we prove the cancellation property of the relaxation operator.

3. Estimates on Macroscopic Fields

Let \(\delta > 0\), \(-1/2 < \nu < 1\) and \(0 < \theta \leq 1\). Suppose \(\rho > 0\), \(T_{\nu, \theta} > 0\) and \(T_{\nu, \theta} > 0\). Then temperature tensor \(T_{\nu, \theta}\) and the relaxation temperature \(T_\theta\) satisfy the following equivalence type estimates:

\[
\begin{align*}
(1) \quad & \theta T_\delta Id \leq T_{\nu, \theta} \leq \frac{1}{3} C_\nu \{3 + \delta(1 - \theta)\} T_\delta Id, \\
(2) \quad & \theta T_\delta \leq T_\theta \leq \frac{1}{\delta} \{\delta + 3(1 - \theta)\} T_\delta,
\end{align*}
\]

where \(C_\nu = \max_\nu \{1 - \nu, 1 + 2\nu\}\).
so that
\[
k^\top \{ \rho T_{\nu, \theta} \} k \leq \theta \rho T_{\delta} |k|^2 + (1 - \theta) \left\{ (1 - \nu) \rho T_{\nu} |k|^2 + 3 \nu \rho T_{\nu} |k|^2 \right\} \\
= \theta \rho T_{\delta} |k|^2 + (1 - \theta)(1 + 2\nu) \rho T_{\nu} |k|^2 \\
\leq (1 + 2\nu) \rho \{ \theta T_{\delta} + (1 - \theta) T_{\nu} \} |k|^2.
\]

In the case of \(-1/2 < \nu < 0\), the last term in (3.1) is non-positive. Thus
\[
k^\top \{ \rho T_{\nu, \theta} \} k \leq \theta \rho T_{\delta} |k|^2 + (1 - \theta)(1 - \nu) \rho T_{\nu} |k|^2 \\
\leq (1 - \nu) \rho \{ \theta T_{\delta} |k|^2 + (1 - \theta) T_{\nu} \} |k|^2
\]
Combining these two cases, we arrive at
\[
k^\top \{ \rho T_{\nu, \theta} \} k \leq \max \{ 1 - \nu, 1 + 2\nu \} \rho \{ (1 - \theta) T_{\nu} + \theta T_{\delta} \} |k|^2.
\]

Now, we recall (1.4) to see
\[
T_{\delta} = \frac{3}{3 + \delta} T_{\nu} \quad \text{or} \quad T_{\nu} \leq \frac{3 + \delta}{3} T_{\delta}
\]
to derive from (3.2) that
\[
k^\top \{ \rho T_{\nu, \theta} \} k \leq \frac{1}{3} \max \{ 1 - \nu, 1 + 2\nu \} \rho \{ 3 + \delta(1 - \theta) \} T_{\delta} |k|^2.
\]
This implies the desired estimate, since we assumed \( \rho > 0 \).
\[(b) \textbf{Lower bound:} \] Denote the last term in (3.1) by \( A \):
\[
A = (1 - \nu) \rho T_{\nu} |k|^2 + \nu \int_{\mathbb{R}^3 \times \mathbb{R}^+} f \left\{ (v - U) \cdot k \right\}^2 dv dI.
\]
Then, when \( 0 < \nu < 1 \), \( A \) satisfies
\[
A \geq (1 - \nu) \rho T_{\nu} |k|^2,
\]
whereas we have
\[
A \geq (1 - \nu) \rho T_{\nu} |k|^2 + \nu \left( \int_{\mathbb{R}^3} f |v - U|^2 dv \right) |k|^2 = (1 + 2\nu) \rho T_{\nu} |k|^2,
\]
for \(-1/2 < \nu \leq 0\). Therefore, we conclude from our assumption on \( \rho \) and \( T_{\nu} \) that \( A \geq 0 \).

Thus, we deduce from (3.1)
\[
k^\top \{ \rho T_{\nu, \theta} \} k \geq \theta \rho T_{\delta} |k|^2 + (1 - \theta) A \geq \theta \rho T_{\delta} |k|^2,
\]
which gives the desired result.

(2) From the definition of \( T_{\delta} \) (1.4), we have
\[
T_{\delta} = \frac{3}{3 + \delta} T_{\nu} + \frac{\delta}{3 + \delta} T_{I, \delta} \geq \frac{\delta}{3 + \delta} T_{I, \delta},
\]
so that
\[
T_{I, \delta} \leq \frac{3 + \delta}{\delta} T_{\delta}.
\]
Therefore,
\[ T_\theta = (1 - \theta)T_{1,\delta} + \theta T_\delta \]
\[ \leq (1 - \theta) \left( \frac{3 + \delta}{\delta} T_\delta \right) + \theta T_\delta \]
\[ = \frac{1}{\delta} \{ \delta + 3(1 - \theta) \} T_\delta. \]
The lower bound comes directly from the definition:
\[ T_\theta = (1 - \theta)T_{1,\delta} + \theta T_\delta \geq \theta T_\delta. \]
\[ \square \]

\textbf{Lemma 3.2.} Assume \( \rho > 0 \) and \( \| f \|_{L^\infty} < \infty \). Then we have
\[ \rho \leq C_\delta \| f \|_{L^\infty}^{\frac{3+\delta}{\delta}} \]
for
\[ C_\delta = 2^{\frac{7}{2}} \pi^2 (3 + \delta) \frac{1}{2+\delta} \delta. \]
\textbf{Proof.} We divide the integral domain as
\[ \rho = \int_{\mathbb{R}^3 \times \mathbb{R}^+} f \, dv \, dI \]
(3.6)
\[ \leq \int_{\frac{1}{\delta + \delta} |v - U|^2 + \frac{2}{\delta |v - U|^2} I^\frac{3}{2} > R^2} f \, dv \, dI + \int_{\frac{1}{\delta + \delta} |v - U|^2 + \frac{2}{\delta |v - U|^2} I^\frac{3}{2} \leq R^2} f \, dv \, dI \]
\[ \equiv I_1 + I_2. \]
From the definition of \( T_\delta \), we see that
\[ I_1 \leq \frac{1}{R^2} \int_{\frac{1}{\delta + \delta} |v - U|^2 + \frac{2}{\delta |v - U|^2} I^\frac{3}{2} > R^2} \left( \frac{1}{3 + \delta} |v - U|^2 + \frac{2}{3 + \delta} I \right) f \, dv \, dI \]
\[ \leq \frac{1}{R^2} \rho T_\delta. \]
For \( I_2 \), we estimate
\[ I_2 \leq \left( \int_{\frac{1}{\delta + \delta} |v - U|^2 + \frac{2}{\delta |v - U|^2} I^\frac{3}{2} \leq R^2} d\nu \right) \| f \|_{L^\infty}, \]
and make a change of variable:
\[ \sqrt{\frac{1}{3 + \delta}} (v_1 - U_1) = r \sin \varphi \cos \theta \sin k, \]
\[ \sqrt{\frac{1}{3 + \delta}} (v_2 - U_2) = r \sin \varphi \sin \theta \sin k, \]
\[ \sqrt{\frac{1}{3 + \delta}} (v_3 - U_3) = r \cos \varphi \sin k, \]
\[ \sqrt{\frac{2}{3 + \delta}} I = r \cos k, \]
for \( 0 \leq r \leq R, 0 \leq \varphi \leq \pi, 0 \leq \theta \leq 2\pi, 0 \leq k \leq \frac{\pi}{2} \). The the Jacobian
\[ J = \frac{\partial(v_1, v_2, v_3, I)}{\partial(r, \varphi, \theta, k)}. \]
is computed as
\[ |J| = (3 + \delta)^\frac{3}{2} \left( \frac{3 + \delta}{2} \right)^{\frac{3}{2}} \]
\[
\times \det \left( \begin{array}{cccc}
\cos \theta \sin \varphi \sin k & r \cos \theta \sin k \cos \varphi & -r \sin \varphi \sin \theta \sin k & r \sin \varphi \cos \theta \cos k \\
\sin \varphi \sin \theta \sin k & r \cos \varphi \sin \theta \sin k & r \sin \varphi \sin \theta \sin k & r \sin \varphi \sin \theta \cos k \\
\cos \varphi \sin k & -r \sin \varphi \sin k & r \sin \varphi \cos \theta \sin k & r \cos \varphi \sin \theta \sin k \\
\delta r^{\delta-1} \cos^k k & 0 & 0 & \delta r^{\delta-1} \cos^{k} k
\end{array} \right)
\]
so that
\[
= \delta (3 + \delta)^\frac{3}{2} \left( \frac{3 + \delta}{2} \right)^{\frac{3}{2}} r^{\delta+2} |\sin \varphi \cos^{\delta-1} k \sin^2 k|.
\]
Thus, (3.6) can be estimated as follows:
\[
I_2 \leq \|f\|_{L^\infty_q} \left\{ \frac{2}{2+\delta} \pi^2 (3 + \delta)^{\frac{1+\delta}{\delta}} \right\} R^{3+\delta} \|f\|_{L^\infty_q}.
\]
We optimize this by setting
\[
R^{5+\delta} = \rho T_\delta \left\{ \frac{2}{2+\delta} \pi^2 (3 + \delta)^{\frac{1+\delta}{\delta}} \right\} \|f\|_{L^\infty_q}
\]
to get
\[
\rho \leq 2 \left\{ \frac{2}{2+\delta} \pi^2 (3 + \delta)^{\frac{1+\delta}{\delta}} \right\} \frac{R^{5+\delta}}{\|f\|_{L^\infty_q} T_\delta},
\]
which implies
\[
\rho \leq \left\{ \frac{2}{2+\delta} \pi^2 (3 + \delta)^{\frac{1+\delta}{\delta}} \right\} \|f\|_{L^\infty_q} T_\delta^{\frac{3+\delta}{\delta}}.
\]
This completes the proof. \( \Box \)

**Lemma 3.3.** Assume \( \rho > 0 \) and \( \|f\|_{L^\infty_q} > 0 \). Then, for \( q > 5 + \delta \), we have
\[
\rho(T_\delta + |U|^2)^{\frac{q-4}{2}} \leq C_{\delta,q} \|f\|_{L^\infty_q},
\]
where constant \( C_{\delta,q} \) is given by
\[
C_{\delta,q} = \left\{ 1 + \frac{2^{\frac{q-4}{2}} \pi^2 (3 + \delta)^{\frac{3+\delta}{\delta}}}{q-\delta-5} \right\} .
\]

**Proof.** From the definition of \( T_\delta \), we write
\[
\rho \left( T_\delta + \frac{1}{3 + \delta}|U|^2 \right) = \int_{\mathbb{R}^3} \left( \frac{1}{3 + \delta} |v|^2 + \frac{2}{3 + \delta} f^\frac{q}{2} \right) f dv dI.
\]
We then split the integral into the following two parts as
\[
\rho \left( T_\delta + \frac{1}{3 + \delta} |U|^2 \right) = \int_{\frac{1}{3 + \delta} |v|^2 + \frac{2}{3 + \delta} f^2 > R^2} \left( \frac{1}{3 + \delta} |v|^2 + \frac{2}{3 + \delta} f^2 \right) f dv dI \\
+ \int_{\frac{1}{3 + \delta} |v|^2 + \frac{2}{3 + \delta} f^2 \leq R^2} \left( \frac{1}{3 + \delta} |v|^2 + \frac{2}{3 + \delta} f^2 \right) f dv dI
\]
(3.7)
= I_1 + I_2.

The estimate for \( I_2 \) is simple:
\[
I_2 \leq R^2 \int_{\frac{1}{3 + \delta} |v|^2 + \frac{2}{3 + \delta} f^2 \leq R^2} f dv dI \leq R^2 \rho.
\]

For \( I_1 \), we extract \( \|f\|_{L^\infty_q} \) out of the integral:
\[
I_1 \leq \int_{\frac{1}{3 + \delta} |v|^2 + \frac{2}{3 + \delta} f^2 \geq R^2} \frac{\left( \frac{1}{3 + \delta} |v|^2 + \frac{2}{3 + \delta} f^2 \right)^{\frac{3}{2}}}{\left( \frac{3}{3 + \delta} |v|^2 + \frac{2}{3 + \delta} f^2 \right)^{\frac{3}{2}} + 1} f dv dI \\
\leq \|f\|_{L^\infty_q} \int_{\frac{1}{3 + \delta} |v|^2 + \frac{2}{3 + \delta} f^2 > R^2} \frac{1}{\left( \frac{3}{3 + \delta} |v|^2 + \frac{2}{3 + \delta} f^2 \right)^{\frac{3}{2}} + 1} dv dI,
\]
and use the same change of variable as in the proof of the previous lemma to estimate
\[
I_1 \leq \|f\|_{L^\infty_q} \int_0^{\frac{3}{2}} \int_0^\pi \int_0^{2\pi} \int_0^R \delta (3 + \delta)^{\frac{3}{2}} \left( \frac{3 + \delta}{q} \right)^{\frac{2}{3}} \sin \varphi \cos \delta^{\frac{1}{2}} k \sin^2 k \frac{1}{r^{q-2}} dr d\theta d\varphi dk \\
\leq \|f\|_{L^\infty_q} \left\{ \frac{2\pi^2 \delta (3 + \delta)^{\frac{3}{2}}}{q - \delta - 5} \right\} R^{\delta+5-q} \\
= \|f\|_{L^\infty_q} \left\{ \frac{2\pi^2 \delta (3 + \delta)^{\frac{3}{2}}}{q - \delta - 5} \right\} R^{\delta+5-q}.
\]

Inserting these computations into (3.7), we get
\[
\rho \left( T_\delta + \frac{1}{3 + \delta} |U|^2 \right) \leq \rho R^2 + \left\{ \frac{2\pi^2 \delta (3 + \delta)^{\frac{3}{2}}}{q - \delta - 5} \right\} R^{\delta+5-q} \|f\|_{L^\infty_q}.
\]

Now, take
\[
R^{\delta+3-q} = \left\{ \frac{q - \delta - 5}{2\pi^2 \delta (3 + \delta)^{\frac{3}{2}}} \right\} \frac{\rho}{\|f\|_{L^\infty_q}},
\]
to get
\[
\rho \left( T_\delta + \frac{1}{3 + \delta} |U|^2 \right) \leq 2 \left\{ \frac{2\pi^2 \delta (3 + \delta)^{\frac{3}{2}}}{q - \delta - 5} \right\} \frac{r^{\frac{q}{2} - 3}}{\rho^{\frac{q}{2} - 3}} \|f\|_{L^\infty_q}^{-\frac{q}{2} - 3}.
\]
This implies
\[
\rho(T_\delta + |U|^2)^\frac{q-\delta}{2} \leq \left\{ 2(3+\delta) \right\}^\frac{q-\delta}{2} \left\{ \frac{2^{2+\delta}}{q-\delta-5} \pi^2 (3+\delta)^{\frac{q+\delta}{2}} \delta \right\} \|f\|_{L^\infty_q}
\]
\[
= \left\{ \frac{2^{2+\delta}}{q-\delta-5} \pi^2 (3+\delta)^{\frac{q+\delta}{2}} \delta \right\} \|f\|_{L^\infty_q},
\]
which completes the proof. \(\square\)

**Lemma 3.4.** Assume \(\|f\|_{L^\infty_q}, \rho, T_\delta > 0\). Then we have
\[
\frac{\rho|U|^{3+\delta+q}}{(T_\delta + |U|^2)^\frac{q-\delta}{2}} \leq C_{\delta,q} \|f\|_{L^\infty_q}
\]
where \(C_{\delta,q} = 2^{\frac{11+2q+2\delta}{2}} \pi^2 (3+\delta)^{2+\delta}\).

**Proof.** For simplicity, we set
\[
A(v, I) = \sqrt{\frac{1}{3+\delta}|v - U|} + \sqrt{\frac{2}{3+\delta} I^I}.
\]
We split the macroscopic momentum as
\[
\rho|U| \leq \int_{A(v, I) \leq R} f|v|dvdI + \int_{A(v, I) > R} f|v|dvdI
\]
\[
\equiv I_1 + I_2.
\]
By Hölder’s inequality,
\[
I_1 \leq \left\{ \int_{A(v, I) \leq R} f|v|dvdI \right\}^{1-\frac{q}{2}} \left\{ \int_{A(v, I) \leq R} \left( f^{\frac{q}{2}} |v| \right)^q dvdi \right\}^{\frac{1}{q}}
\]
\[
\leq \left\{ \int_{R^3 \times R^+} f|v|dvdI \right\}^{1-\frac{q}{2}} \left\{ \int_{A(v, I) \leq R} f|v|^qdvdi \right\}^{\frac{1}{q}}
\]
\[
\leq \rho^{1-\frac{q}{2}} \|f\|^\frac{1}{q}_{L^\infty_q} \left\{ \int_{A(v, I) \leq R} dvdi \right\}^{\frac{1}{q}}.
\]
Then, computing similarly as in the previous lemma, we have
\[
\int_{A(v, I) \leq R} dvdi \leq \int_{\sqrt{\frac{1}{3+\delta}|v - U|} + \sqrt{\frac{2}{3+\delta} I^I} \leq R^2} dvdi \leq \left\{ \frac{2^{2+\delta}}{2} \pi^2 (3+\delta)^{\frac{q+\delta}{2}} \right\} R^{3+\delta}.
\]
Therefore, we bound \(I_1\) by
\[
\rho^{1-\frac{q}{2}} \|f\|^\frac{1}{q}_{L^\infty_q} \left\{ \frac{2^{2+\delta}}{2} \pi^2 (3+\delta)^{\frac{q+\delta}{2}} \right\} R^{3+\delta}.
\]
On the other hand, we observe
\[
I_2 \leq \frac{1}{R} \int_{A(v, I) > R} f|v| \left\{ \sqrt{\frac{1}{3+\delta}|v - U|} + \sqrt{\frac{2}{3+\delta} I^I} \right\} dvdi.
\]
Applying Hölder’s inequality again,
\[
I_2 \leq \frac{\sqrt{2(3 + \delta)}}{R} \left\{ \int_{\mathbb{R}^3 \times \mathbb{R}^+} f \left( \frac{1}{3 + \delta} |v|^2 + \frac{2}{3 + \delta} \right) dv \right\}^{\frac{1}{2}} \times \left\{ \int_{\mathbb{R}^3 \times \mathbb{R}^+} f \left( \frac{1}{3 + \delta} |v - U|^2 + \frac{2}{3 + \delta} |\partial_t v| \right) dv \right\}^{\frac{1}{2}} = \frac{\sqrt{2(3 + \delta)}}{R} \left\{ \frac{1}{3 + \delta} \rho |U|^2 + \rho T_\delta \right\}^{\frac{1}{2}} \{\rho T_\delta\}^{\frac{1}{2}}.
\]
In conclusion,
\[
\rho |U| \leq \rho^{1 - \frac{1}{q}} \|f\|_{L_q}^{\frac{1}{q}} \left\{ 2^{\frac{2+\delta}{q}} \pi^2 (3 + \delta) \frac{1+\delta}{2} \right\}^{\frac{1}{2}} R^{\frac{3+\delta}{q}} + \frac{\sqrt{2(3 + \delta)}}{R} \rho \| |U|^2 + T_\delta T_\delta | |_{L_q}^{\frac{1}{2}}.
\]
The optimizing choice for \( R \) then is
\[
R^{1+\delta+q} = \left\{ \frac{2(3 + \delta)}{\rho} \rho \| |U|^2 + T_\delta T_\delta | |_{L_q}^{\frac{1}{2}} \right\} \left\{ 2^{\frac{2+\delta}{q}} \pi^2 (3 + \delta) \frac{1+\delta}{2} \right\}^{\frac{1}{2}} \|f\|_{L_q}^{\frac{1}{q}}
\]
for which the right hand side of (3.8) becomes
\[
2 \left\{ 2^{\frac{2+\delta}{q}} \pi^2 (3 + \delta) \frac{1+\delta}{2} \right\}^{\frac{1}{2}} \left\{ 2(3 + \delta) \right\}^{\frac{1+\delta}{2}} \rho \| |U|^2 + T_\delta T_\delta | |_{L_q}^{\frac{1}{2}} \|f\|_{L_q}^{\frac{1}{q}}.
\]
This gives
\[
\frac{\rho |U|^{3+\delta+q}}{\| |U|^2 + T_\delta T_\delta | |_{L_q}^{\frac{1}{2}}} \leq 2^{3+\delta+q} \left\{ 2(3 + \delta) \right\}^{\frac{1+\delta}{2}} \left\{ 2^{\frac{2+\delta}{q}} \pi^2 (3 + \delta) \frac{1+\delta}{2} \right\}^{\frac{1}{2}} \|f\|_{L_q}^{\frac{1}{q}}
\]
\[
= 2^{\frac{2+\delta}{2}} e^{2\pi \delta} \pi^2 (3 + \delta)^{2+\delta} \|f\|_{L_q}^{\frac{1}{q}}.
\]
\[
\square
\]

4. Solution Space and Approximate Scheme

We set up our solution space \( \Omega \):
\[
\Omega = \{ f \in C_{+}([0, T]; \| \cdot \|_{L_q}) \mid f \text{ satisfies (A1) and (A2)} \},
\]
where properties (A1) and (A2) are

- (A1): There exists a constant \( C_1 > 0 \) such that
  \[
  \| f(t) \|_{L_q} \leq e^{C_1 t} \| f_0 \|_{L_q}, \quad \text{for } t \in [0, T].
  \]

- (A2): There exist positive constants \( C_{T,f_0}, C_{T,f_0,\delta} \) and \( C_{T,f_0,\delta,q} \) such that
  \[
  \begin{align*}
  (1) \quad & \rho(x, t) \geq C_{T,f_0}, \\
  (2) \quad & T_{\delta}(x, t) \geq C_{T,f_0,\delta}, \\
  (3) \quad & \rho + |U| + T_\delta \leq C_{T,f_0,\delta,q}.
  \end{align*}
  \]

We consider the following iteration scheme: \((n \geq 1)\)
\[
\begin{aligned}
\partial_t f^{n+1} + v \cdot \nabla_x f^{n+1} &= A_{\nu, \theta}(M_{\nu, \theta}(f^n) - f^{n+1}), \\
\partial_t f^{n+1}(0) &= f_0.
\end{aligned}
\]
We set \( f^0 = 0 \) and \( \mathcal{M}(f^0) = 0 \), so that
\[
\partial_t f^1 + v \cdot \nabla_x f^1 + A_{\nu, \theta} f^1 = 0,
\]
\[
f^1(0) = f_0.
\]

Our first goal is to show that \( \{f^n\} \) lies in \( \Omega \) for all \( n \geq 0 \). We start with the following estimates on the polyatomic Gaussian.

**Proposition 4.1.** Suppose \( f \in \Omega \), there exists a constant \( C_M \) depending on \( \nu, \delta, \theta \) and \( q \) such that
\[
\|\mathcal{M}_{\nu, \theta}(f)\|_{L^\infty_q} \leq C_M \|f\|_{L^\infty_q}.
\]

**Remark 4.1.** \( C_M \) blows up as \( \theta \) tends to 0. See the end of the proof.

**Proof.** We will show that \( \mathcal{M}_{\nu, \theta}(f) \), \( |v|^q \mathcal{M}_{\nu, \theta}(f) \) and \( I_{2q}^\theta \mathcal{M}_{\nu, \theta}(f) \) are controlled by \( \|f\|_{L^\infty_q} \).

(a) **The estimate for \( \mathcal{M}_{\nu, \theta}(f) \):** We first recall Lemma 3.1 to observe
\[
\frac{1}{2} (v - U)^T T_{\nu, \theta}^{-1} (v - U) + I_{2q}^\theta \geq \frac{3}{2C_\nu (1 - \theta)} |v - U|^2 + I_{2q}^\theta \geq 0
\]
for \( f \in \Omega \). Hence we have
\[
\exp \left( -\frac{1}{2} (v - U)^T T_{\nu, \theta}^{-1} (v - U) - I_{2q}^\theta \right) \leq 1.
\]
Using this and Lemma 3.1 and Lemma 3.2, we have
\[
\mathcal{M}_{\nu, \theta}(f) \leq \frac{\rho \Lambda_\delta}{\sqrt{\det(2\pi T_{\nu, \theta})(T_{\theta})}} \leq \frac{1}{(2\pi)^{3/2} \theta^{1/2}} \frac{\rho}{T_{\delta}^{3/2}} \leq \frac{1}{(2\pi)^{3/2} \theta^{1/2}} \left( 2^q (3 + \delta) \frac{\pi^2 \delta}{q - \delta} \right) \|f\|_{L^\infty_q} \equiv C_0 \frac{\rho}{\theta^{1/2}} \|f\|_{L^\infty_q}.
\]

(b) **The estimate for \( |v|^q \mathcal{M}_{\nu, \theta}(f) \):** We divide it into the estimates of \( |U|^q \mathcal{M}_{\nu, \theta}(f) \) and \( |v - U|^q \mathcal{M}_{\nu, \theta}(f) \).

(b1) **\( |U|^q \mathcal{M}_{\nu, \theta}(f) \):** We use (4.3) and Lemma 3.1 to compute
\[
|U|^q \mathcal{M}_{\nu, \theta}(f) \leq \frac{\rho \Lambda_\delta}{\sqrt{\det(2\pi T_{\nu, \theta})(T_{\theta})}} \leq \frac{1}{(2\pi)^{3/2} \theta^{1/2}} |U|^q \frac{\rho}{T_{\delta}^{3/2}}.
\]
We divide this estimate into two cases. In the case of \( |U| < T_{\delta}^2 \), we have from Lemma 3.3 that
\[
|U|^q \frac{\rho}{T_{\delta}^{3/2}} \leq \rho(T_{\delta} + |U|^2)^{\frac{3}{4}} \leq \left\{ \frac{2^{\frac{4q+4}{q - \delta}} \pi^2 (3 + \delta) \frac{\delta}{q - \delta}}{\rho(T_{\delta} + |U|^2)^{\frac{3}{4}}} \right\} \|f\|_{L^\infty_q}.
\]
On the other hand, in the case of $|U| \geq T_\delta^{1/2}$, we have from Lemma 3.3 that

$$|U|^q \frac{\rho}{T_\delta^{3/2}} \leq \frac{\rho |U|^{q+3+\delta}}{|U|^{3+\delta} T_\delta^{1/2}} \leq 2^{\frac{q+3+\delta}{2}} \|f\|_{L_\infty^q}$$

These two estimates give

$$|U|^q \mathcal{M}_{\nu, \theta}(f) \leq \frac{C_1}{\theta^{3+\delta}} \|f\|_{L_\infty^q}.$$

for

$$C_1 = \left\{ \frac{2^{\frac{q-3-4}{2}} \sqrt{\pi} (3+\delta)^{\frac{3}{2}} \delta}{q-\delta-5} \right\} + 2^{\frac{11+3q+2q}{2}} \sqrt{\pi} (3+\delta)^{2+\delta} \delta.$$

(b) $|v - U|^q \mathcal{M}_{\nu, \theta}(f)$: From (4.2) and Lemma 3.1, we have

$$|v - U|^q \mathcal{M}_{\nu, \theta}(f) \leq \frac{1}{(2\pi)^{3/2}} \frac{1}{\theta^{3+\delta}} |v - U|^q \frac{\rho}{T_\delta^{3/2}} \exp \left( -\frac{3}{2C_{v'} (3+\delta (1-\theta))} \frac{|v - U|^2}{T_\delta} \right)$$

$$= \frac{1}{(2\pi)^{3/2}} \frac{1}{\theta^{3+\delta}} T_\delta^q \frac{\rho}{T_\delta^{3/2}} \left( \frac{|v - U|^2}{T_\delta} \right)^{\frac{q}{2}} \exp \left( -\frac{3}{2C_{v'} (3+\delta (1-\theta))} \frac{|v - U|^2}{T_\delta} \right)$$

$$= \frac{C_2}{\theta^{3+\delta}} \rho T_\delta^{\frac{q-3-\delta}{2}},$$

where

$$C_2 = \frac{1}{(2\pi)^{3/2}} \sup_{x \geq 0} \left( x^{q/2} e^{-x} \right) \left\{ \frac{2C_{v'} (3+\delta (1-\theta))}{3} \right\}^{q/2}.$$

This, combined with Lemma 3.3 implies

$$|v - U|^q \mathcal{M}_{\nu, \theta}(f) \leq \frac{C_2}{\theta^{3+\delta}} \rho (T_\delta + |U|^2)^{\frac{q-3-\delta}{2}} \|f\|_{L_\infty^q}$$

$$\leq \frac{C_2}{\theta^{3+\delta}} \left\{ \frac{2^{\frac{q-3-4}{2}} \sqrt{\pi} (3+\delta)^{\frac{3}{2}} \delta}{q-\delta-5} \right\} \|f\|_{L_\infty^q}$$

$$= \frac{C_3}{\theta^{3+\delta}} \|f\|_{L_\infty^q}.$$

(c) The estimate for $I^\sharp \mathcal{M}_{\nu, \theta}(f)$: Again from (4.2), we have

$$\frac{1}{2} (v - U)^\top T_\nu^{-1} (v - U) + \frac{I^\sharp}{T_\delta} \geq \frac{\delta}{\delta + 3(1-\theta)} \frac{I^\sharp}{T_\delta},$$

so that $I^\sharp \mathcal{M}_{\nu, \theta}(f)$ is estimated as follows:

$$I^\sharp \mathcal{M}_{\nu, \theta}(f) \leq \frac{1}{\sqrt{(2\pi)^3}} I^\sharp \frac{1}{\theta^{3+\delta}} T_\delta^{\frac{3+\delta}{2}} \rho \exp \left( -\frac{\delta}{\delta + 3(1-\theta)} \frac{I^\sharp}{T_\delta} \right)$$

$$= \frac{1}{\sqrt{(2\pi)^3}} I^\sharp \frac{1}{\theta^{3+\delta}} T_\delta^{\frac{3+\delta}{2}} \rho \frac{I^\sharp}{T_\delta^{3/2}} \left( \frac{I^\sharp}{T_\delta} \right)^{\frac{3+\delta}{2}} \exp \left( -\frac{\delta}{\delta + 3(1-\theta)} \frac{I^\sharp}{T_\delta} \right)$$

$$= \frac{C_4}{\theta^{3+\delta}} \rho T_\delta^{\frac{q-3-\delta}{2}},$$
where
\[ C_4 = \frac{1}{(2\pi)^{3/2}} \sup_{x \geq 0} (x^{3/2} e^{-x}) \left( \frac{\delta + 3(1 - \theta)}{\delta} \right)^{3/2}. \]

Then, in view of Lemma 3.3, we derive
\[
I^2 \mathcal{M}_{\nu,0}(f) \leq \frac{C_4}{\theta^{1/2}} \rho(T_0 + |U|^2) \frac{2^{q-2}}{q - \delta - 5} \|f\|_{L^\infty}
\]
\[
= \frac{C_5}{\theta^{1/2}} \|f\|_{L^\infty}.
\]

Finally, we combine (a), (b) and (c) to conclude that
\[
\|\mathcal{M}_{\nu,0}(f)\|_{L^\infty} \leq \frac{C_{\nu,\delta, \theta, q}}{\theta^{1/2}} \|f\|_{L^\infty},
\]

where
\[
C_{\nu,\delta, \theta, q} = C_0 + C_1 + C_3 + C_5.
\]

Note that \( \max_{0 \leq \theta \leq 1} C_{\nu,\delta, \theta, q} < \infty. \)

**Proposition 4.2.** \( f^n \) lies in \( \Omega \) for all \( n > 0 \). That is, \( f^n \) satisfies

- (A1): \( f^n \) is uniformly bounded in \( \| \cdot \|_{L^\infty} \)
\[
\|f^n\|_{L^\infty} \leq e^{C_1 t}\|f_0\|_{L^\infty},
\]

where \( C_1 = A_{\nu, \theta}(C_M - 1) \).

- (A2): There exist positive constants \( C_{T,f_0}, C_{T,f_0,\delta} \) and \( C_{T,f_0,\delta,q} \) such that
  1. \( \rho^n(x,t) \geq C_{T,f_0} \)
  2. \( T^n_s(x,t) \geq C_{T,f_0,\delta} \)
  3. \( \rho^n + |U^n| + T^n_s \leq C_{T,f_0,\delta,q} \)

**Proof.** We proceed by induction. The properties are trivially satisfied for \( n = 0 \). Assume that \( f^n \in \Omega \). We prove that \( f^{n+1} \) also satisfies (A1) and (A2).

(A1) We write (4.1) in the mild form:
\[
f^{n+1}(t, x, v, I) = e^{-A_{\nu,\theta} t} f_0(x - vt, v, I) + A_{\nu, \theta} \int_0^t e^{-A_{\nu, \theta}(t-s)} \mathcal{M}_{\nu,0}(f^n)(x - (t-s)v, v, s, I) ds
\]
and take \( \| \cdot \|_{L^\infty} \) on both sides,

(4.4) \[
\|f^{n+1}(t)\|_{L^\infty} \leq e^{-A_{\nu, \theta} t}\|f_0\|_{L^\infty} + A_{\nu, \theta} \int_0^t e^{-A_{\nu, \theta}(t-s)} \|\mathcal{M}_{\nu,0}(f^n)(s)\|_{L^\infty} ds.
\]

Since \( f^n \in \Omega \), we can apply Proposition 4.1 to estimate

\[
A_{\nu, \theta} \int_0^t e^{-A_{\nu, \theta}(t-s)} \|\mathcal{M}_{\nu,0}(f^n)(s)\|_{L^\infty} ds \leq A_{\nu, \theta} \int_0^t e^{-A_{\nu, \theta}(t-s)} C_M \|f^n(s)\|_{L^\infty} ds
\]
\[
\leq A_{\nu, \theta} \int_0^t e^{-A_{\nu, \theta}(t-s)} C_M e^{C_1 s}\|f_0\|_{L^\infty} ds
\]
\[
= \frac{A_{\nu, \theta} C_M}{C_1 + A_{\nu, \theta}} e^{C_1 t} \|f_0\|_{L^\infty},
\]

where
\[
C_M = \sup_{s \geq 0} e^{-A_{\nu, \theta} s} \|\mathcal{M}_{\nu,0}(f^n)(s)\|_{L^\infty}.
\]
where we used $\|f^n\|_{L_q^\infty} \leq e^{Ct} \|f_0\|_{L_q^\infty}$. Plugging this estimate into (A.4), we get
\[ \|f^{n+1}(t)\|_{L_q^\infty} \leq e^{Ct} \|f_0\|_{L_q^\infty}, \]
since $(A_{\nu, \theta} C_M)/(C_1 + A_{\nu, \theta}) = 1$.

(A.2) By the nonnegativity of polyatomic Gaussian $M_{\nu, \theta}(f^n)$, we have from the above mild form
\[ f^{n+1} \geq e^{-A_{\nu, \theta} t} f_0(x - vt, v, I). \]
Integrating in $v$ and $I$ on both sides, and recalling the lower bound assumption imposed on $f_0$,
\[ \rho^{n+1} = \int_{\mathbb{R}^3} f^{n+1} dv dI \geq e^{-A_{\nu, \theta} t} \int_{\mathbb{R}^3} f_0(x - vt, v, I) dv dI \geq C_{f_0} e^{-A_{\nu, \theta} t}. \]
Hence, combining the above results and Lemma 3.2 gives
\[ C_{f_0} e^{-A_{\nu, \theta} t} \leq \rho^{n+1} \leq C_0 \|f^{n+1}\|_{L_q^\infty} \left\{ T_\delta^{n+1} \right\} + \frac{C_0 e^{Ct} \|f_0\|_{L_q^\infty}}{\|M_{\nu, \theta}\|_{L_q^\infty}} \left\{ T_\delta^{n+1} \right\} \frac{3+\delta}{2}. \]
Therefore,
\[ T_\delta^{n+1} \geq \left( \frac{C_{f_0} e^{-A_{\nu, \theta} t}}{C_0 e^{Ct} \|f_0\|_{L_q^\infty}} \right) \frac{2}{3+\delta} \geq C_{T, f_0, \delta}. \]
The estimate (A.2) (3) follows immediately from the above lower bound for $\rho^{n+1}$ and Lemma 3.3. This completes the proof.

5. Lipschitz Continuity of $M_{\nu, \theta}$

**Proposition 5.1.** Let $f$ and $g$ lie in $\Omega$. Then $M_{\nu, \theta}$ satisfies the following continuity property:
\[ \|M_{\nu, \theta}(f) - M_{\nu, \theta}(g)\|_{L_q^\infty} \leq C_{Lip} \|f - g\|_{L_q^\infty} \]
for some constant $C_{Lip}$ depending on $T, \delta, \theta, q$ and $f_0$.

**Proof.** For the proof of this proposition, we set the transitional macroscopic fields $\rho_\eta, U_\eta, T_{\nu, \theta \eta}, T_{I, \delta \eta}$:
\[ (\rho_\eta, U_\eta, T_{\nu, \theta \eta}, T_{I, \delta \eta}) = (\rho_f, U_f, T_{\nu, \theta f}, T_{I, \delta f}) + (1 - \eta)(\rho_g, U_g, T_{\nu, \theta g}, T_{I, \delta g}) \]
and define the transitional polyatomic Gaussian:
\[ M_{\nu, \theta}(\eta) = \frac{\rho_\eta A_\delta}{\sqrt{\det(2\pi T_{\nu, \theta \eta}(T_{\theta \eta})^\frac{1}{2}}} \exp \left( -\frac{1}{2}(v - U_\eta)^T T_{\nu, \theta \eta}^{-1}(v - U_\eta) - \frac{I_3}{T_{\theta \eta}} \right). \]
Applying Taylor’s theorem, we expand
\[
M_{\nu, \theta}(f) - M_{\nu, \theta}(g) = (\rho_f - \rho_g) \int_0^1 \frac{\partial M_{\nu, \theta}(\eta)}{\partial \rho_\eta} d\eta \\
+ (U_f - U_g) \int_0^1 \frac{\partial M_{\nu, \theta}(\eta)}{\partial U_\eta} d\eta \\
+ (T_{\nu, \theta f} - T_{\nu, \theta g}) \int_0^1 \frac{\partial M_{\nu, \theta}(\eta)}{\partial T_{\nu, \theta \eta}} d\eta \\
+ (T_{I, \delta f} - T_{I, \delta g}) \int_0^1 \frac{\partial M_{\nu, \theta}(\eta)}{\partial T_{I, \delta \eta}} d\eta \\
= I_1 + I_2 + I_3 + I_4.
\]
We only consider $I_3$. Other terms can be treated in a similar and simpler manner. Recalling the definition of $T_{v,\theta}$, we see that

$$\rho_f T_{v,\theta f} - \rho_g T_{v,\theta g} = (1 - \theta)\rho_f \{ (1 - \nu)T_{v_f}Id + \nu\Theta_f \} + \theta \rho_f T_{\delta_f}Id - (1 - \theta)\rho_g \{ (1 - \nu)T_{v_g}Id + \nu\Theta_g \} - \theta \rho_g T_{\delta_g}Id$$

$$= (1 - \theta) \int_{\mathbb{R}^3 \times \mathbb{R}^+} f \left\{ \frac{(1 - \nu)}{3} |v - U_f|^2 Id + \nu(v - U_f) \otimes (v - U_f) \right\} dvdI$$

$$+ \frac{\theta}{3 + \delta} \int_{\mathbb{R}^3 \times \mathbb{R}^+} f \left\{ |v - U_f|^2 + 2I^2 \right\} Id dvdI$$

$$- (1 - \theta) \int_{\mathbb{R}^3 \times \mathbb{R}^+} g \left\{ \frac{(1 - \nu)}{3} |v - U_g|^2 Id + \nu(v - U_g) \otimes (v - U_g) \right\} dvdI$$

$$- \frac{\theta}{3 + \delta} \int_{\mathbb{R}^3 \times \mathbb{R}^+} g \left\{ |v - U_g|^2 + 2I^2 \right\} Id dvdI,$$

which can be rearranged as

$$\left\{ (1 - \theta) \frac{1 - \nu}{3} + \frac{\theta}{3 + \delta} \right\} \left( \int_{\mathbb{R}^3 \times \mathbb{R}^+} (f |v - U_f|^2 - g |v - U_g|^2) dvdI \right) Id$$

$$+ (1 - \theta) \nu \int_{\mathbb{R}^3 \times \mathbb{R}^+} \{ f(v - U_f) \otimes (v - U_f) - g(v - U_g) \otimes (v - U_g) \} dvdI$$

$$+ \frac{2\theta}{3 + \delta} \left( \int_{\mathbb{R}^3 \times \mathbb{R}^+} (f - g) I^2 dvdI \right) Id$$

$$\equiv T_1 + T_2 + T_3.$$

For $T_1$, we note that

$$\int_{\mathbb{R}^3 \times \mathbb{R}^+} f |v - U_f|^2 - g |v - U_g|^2 dvdI = \int_{\mathbb{R}^3 \times \mathbb{R}^+} (f - g)|v|^2 dvdI - (\rho_f |U_f|^2 - \rho_g |U_g|^2).$$

The first term is clearly bounded by $C \|f - g\|_{L^q_{\infty}}$ ($q > 3$). For the second term, we compute

$$\left| \rho_f |U_f|^2 - \rho_g |U_g|^2 \right| = \left| \frac{\rho_f^2 |U_f|^2 - \rho_g^2 |U_g|^2}{\rho_f} + \frac{\rho_g^2 |U_g|^2 - \rho_g |U_g|^2}{\rho_f} \right|$$

$$\leq \frac{1}{\rho_f} \left( |\rho_f U_f| + |\rho_g U_g| \right) |\rho_f U_f - \rho_g U_g| + \frac{1}{\rho_f \rho_g} |\rho_g U_g|^2 |\rho_f - \rho_g|.$$

Now, since $f, g \in \Omega$, we have from (A1)

$$|\rho_f U_f| + |\rho_g U_g| \leq C_{\delta, q} (\|f\|_{L^q_{\infty}} + \|g\|_{L^q_{\infty}}) \leq C_{\delta, q} e^{C_{1,t} \|f_0\|_{L^q_{\infty}}}$$

and $\rho_f, \rho_g \geq C_{T, f_0}$. Therefore,

$$\left| \rho_f |U_f|^2 - \rho_g |U_g|^2 \right| \leq C_{T, f_0, \delta, q} \left( |\rho_f U_f - \rho_g U_g| + |\rho_f - \rho_g| \right)$$

$$\leq C_{T, f_0, \delta, q} \int_{\mathbb{R}^3 \times \mathbb{R}^+} |f - g|(1 + |v|^2) dvdI$$

$$\leq C_{T, f_0, \delta, q} \|f - g\|_{L^q_{\infty}}.$$
We now move on to the estimate of the integral in (5.2)

Before proceeding further, we establish the following claims: For \( f, g \)

\[
|\rho_f \nu_f - \rho_g \nu_g| \leq C_{T,f,u,\delta,\theta,q} ||f - g||_{L^\infty}.
\]

Therefore,

\[
|\nu_f \nu_f - \nu_g \nu_g| \leq C_{T,f,u,\delta,\theta,q} ||f - g||_{L^\infty},
\]

where we used Lemma 3.1 and Property (A2)(1) of \( \Omega \) as

\[
\rho_f \geq C_{T,f,u},
\]

and

\[
|T_{\nu,\nu}^{ij}| \leq 1 - \frac{\nu}{\rho} \int_{\mathbb{R}^3 \times \mathbb{R}^+} g \left\{ \frac{(1 - \nu)}{3} |v - U_g|^2 \delta_{ij} + \nu(v - U_g)(v - U_g) \right\} dv \]

\[
+ \frac{\nu}{\rho} \int_{\mathbb{R}^3 \times \mathbb{R}^+} g \left\{ \frac{1}{3 + \delta} (|v - U_g|^2 + 2 I_2) \delta_{ij} \right\} dv \]

\[
\leq (3 + \delta - 2\theta - \delta \theta) T_{\delta g}.
\]

We now move on to the estimate of the integral in \( I_3 \). A straightforward computation gives

\[
\int_{0}^{1} \frac{\partial M_{\nu,\theta}(\eta)}{\partial T_{\nu,\theta}} d\eta
\]

\[
= \int_{0}^{1} \left[ \frac{1}{\det T_{\nu,\theta}} \right] \frac{\partial \det T_{\nu,\theta}}{\partial T_{\nu,\theta}} + (v - U_{\eta})^T T_{\nu,\theta}^{-1} \frac{\partial T_{\nu,\theta}}{\partial T_{\nu,\theta}} T_{\nu,\theta}^{-1} (v - U_{\eta}) \right] M_{\nu,\theta}(\eta) d\eta.
\]

Before proceeding further, we establish the following claims: For \( f, g \in \Omega \), we have

\( F_1 \) : \( (v - U_{\eta})^T T_{\nu,\theta}^{-1} \left( \frac{\partial T_{\nu,\theta}}{\partial T_{\nu,\theta}} \right) T_{\nu,\theta}^{-1} (v - U_{\eta}) \leq \frac{|v - U_{\eta}|^2}{[\theta (\eta T_{\delta f} + (1 - \eta) T_{\delta g})]^2}
\]

\( F_2 \) : \( |\det T_{\nu,\theta}| \geq \theta^3 \{ \eta T_{\delta f} + (1 - \eta) T_{\delta g} \}^3
\]

\( F_3 \) : \( \frac{\partial \det T_{\nu,\theta}}{\partial T_{\nu,\theta}} \leq 2(3 + \delta - 2\theta - \delta \theta)^2 \{ \eta T_{\delta f} + (1 - \eta) T_{\delta g} \}
\]

* (\( F_1 \)): Let \( D_{ij} \) denote a \( n \times n \) matrix whose \( ij \)th and \( j \)th entries are 1 and the remaining entries are 0. Then,

\[
|X^T \left( \frac{\partial T_{\nu,\theta}}{\partial T_{\nu,\theta}} \right) Y| = |X^T D_{ij} Y| \leq |X_i Y_j + X_j Y_i| \leq |X||Y|.
\]
Thus we have
\[\left|(v - U_\eta)^\top T_{\nu, \theta \eta}^{-1} \frac{\partial T_{\nu, \theta \eta}}{\partial T_{\nu, \theta \eta}} T_{\nu, \theta \eta}^{-1} (v - U_\eta)\right| \leq |(v - U_\eta)^\top T_{\nu, \theta \eta}^{-1} |T_{\nu, \theta \eta}^{-1} (v - U_\eta)|.\]

Now we use Lemma 3.1:
\[T_{\nu, \theta \eta} = \eta T_{\nu, \theta f} + (1 - \eta) T_{\nu, \theta g} \geq \left[\theta \{\eta T_{\delta f} + (1 - \eta) T_{\delta g}\}\right] I_d\]

(5.3)

to compute
\[|(v - U_\eta)^\top T_{\nu, \theta \eta}^{-1} (v - U_\eta)| \leq \sup_{|Y| \leq 1} |(v - U_\eta)^\top T_{\nu, \theta \eta}^{-1} Y| \leq \sup_{|Y| \leq 1} \frac{|v - U_\eta||Y|}{\theta \{\eta T_{\delta f} + (1 - \eta) T_{\delta g}\}} \leq \frac{|v - U_\eta|}{\theta \{\eta T_{\delta f} + (1 - \eta) T_{\delta g}\}}.
\]

Likewise,
\[|T_{\nu, \theta \eta}^{-1} (v - U_\eta)| \leq \frac{|v - U_\eta|}{\theta \{\eta T_{\delta f} + (1 - \eta) T_{\delta g}\}},\]

which gives the desired estimate.

- (F2): By (5.3), we have
  \[\det T_{\nu, \theta \eta} \geq \theta^3 \{\eta T_{\delta f} + (1 - \eta) T_{\delta g}\}^3.\]

- (F3): We only prove the case: \((i, j) = (1, 2)\). An explicit calculation gives
  \[\frac{\partial \det T_{\nu, \theta \eta}}{\partial T_{12}^{\nu, \theta \eta}} = T_{23}^{\nu, \theta \eta} T_{31}^{\nu, \theta \eta} - T_{33}^{\nu, \theta \eta} T_{21}^{\nu, \theta \eta}.
\]

Then we recall (5.2) to derive
\[|T_{ij}^{\nu, \theta \eta}| = |\eta T_{ij}^{\nu, \theta f} + (1 - \eta) T_{ij}^{\nu, \theta g}| \leq (3 + \delta - 2\theta - \delta\theta)|\eta T_{\delta f} + (1 - \eta) T_{\delta g}|.
\]

Therefore, (5.4) leads to
\[\left|\frac{\partial \det T_{\nu, \theta \eta}}{\partial T_{ij}^{\nu, \theta \eta}}\right| \leq 2(3 + \delta - 2\theta - \delta\theta)^2 \{\eta T_{\delta f} + (1 - \eta) T_{\delta g}\}^2.
\]

This ends the proof of the claims.

Using these claims, we estimate the integral in \(I_3\) as \((T_{\delta \eta} = \eta T_{\delta f} + (1 - \eta) T_{\delta g})\)
\[\int_0^1 \frac{\partial M_{\nu, \theta \eta}(\eta)}{\partial T_{\nu, \theta \eta}} d\eta \leq C_{\delta, \theta} \int_0^1 \left[\frac{1}{T_{\delta \eta}} + \frac{|v - U_\eta|^2}{T_{\delta \eta}^2}\right] M_{\nu, \theta}(\eta) d\eta.
\]

Now, since \(f, g \in \Omega\), we have
\[C_{T_f, \delta, 1} \leq T_{\delta \eta} = \eta T_{\delta f} + (1 - \eta) T_{\delta g} \leq C_{T_f, \delta, 2},\]
and
\[
\mathcal{M}_{\nu,\theta}(\eta) \leq C_{T,f_0,\delta,q} \frac{\rho_{\eta}}{\delta^{2+\frac{1}{2}}} T^{\frac{3}{2}} \delta_{\eta} \exp \left( -\frac{3}{2C_{\nu}(3 + \delta(1 - \theta)) T_{\delta\eta}} |v - U_{\eta}|^2 \right) \exp \left( -\frac{\delta}{\delta + 3(1 - \theta) T_{\delta\eta}} I_{\frac{7}{2}} \right)
\]
\[
\leq C_{T,f_0,\delta,q,g} e^{-CT_{f_0,\delta,q,g} \left( |v - U_{\eta}|^2 + I_{\frac{7}{2}} \right)}.
\]

Therefore, we can proceed further from (5.5) as
\[
\int_0^1 \frac{\partial \mathcal{M}_{\nu,\theta}(\eta)}{\partial t_{\nu,\theta}} d\eta \leq C_{T,f_0,\delta,q,g} \int_0^1 (1 + |v - U_{\eta}|^2) e^{-CT_{f_0,\delta,q,g} \left( |v - U_{\eta}|^2 + I_{\frac{7}{2}} \right)} d\eta
\]
\[
\leq C_{T,f_0,\delta,q,g} \int_0^1 e^{-CT_{f_0,\delta,q,g} \left( |v - U_{\eta}|^2 + I_{\frac{7}{2}} \right)} d\eta
\]
\[
\leq C_{T,f_0,\delta,q,g} \int_0^1 e^{-CT_{f_0,\delta,q,g} \left( |v|^2 + I_{\frac{7}{2}} \right)} d\eta
\]
\[
\leq C_{T,f_0,\delta,q,g} e^{-CT_{f_0,\delta,q,g} \left( |v|^2 + I_{\frac{7}{2}} \right)}.
\]

In the last line, we have used
\[
|U_{\eta}| \leq \eta |U_f| + (1 - \eta) |U_g| \leq C_{T,f_0,\delta,q,\eta}
\]
which holds when \( f, g \in \Omega \).

Finally, we turn to the proof of the proposition, which is almost done. Plugging the above inequalities into (5.5), we have
\[
|\mathcal{M}_{\nu,\theta}(f) - \mathcal{M}_{\nu,\theta}(g)|
\]
\[
\leq C \left\{ |\rho_f - \rho_g| + |U_f - U_g| + |T_{\nu,\theta} f - T_{\nu,\theta} g| + |T_{I,\delta} f - T_{I,\delta} g| \right\} e^{-C(|v|^2 + I_{\frac{7}{2}})}
\]
\[
\leq C \|f-g\|_{L^{\infty}_q} e^{-C(|v|^2 + I_{\frac{7}{2}})}.\]

Multiplying \((1 + |v|^2 + I_{\frac{7}{2}})^{\frac{5}{2}}\) and taking supremum on both sides, we get the desired estimate. \( \square \)

6. PROOF OF THE MAIN THEOREM

In the mild form, (4.3) reads
\[
f^{n+1}(x,v,t,I) = e^{-A_{\nu,\theta} t} f_0(x - vt, v, I) + A_{\nu,\theta} \int_0^t e^{-A_{\nu,\theta}(t-s)} M_{\nu,\theta}(f^n)(x - (t-s)v, v, s, I) ds,
\]
\[
f^n(x,v,t,I) = e^{-A_{\nu,\theta} t} f_0(x - vt, v, I) + A_{\nu,\theta} \int_0^t e^{-A_{\nu,\theta}(t-s)} M_{\nu,\theta}(f^{n-1})(x - (t-s)v, v, s, I) ds.
\]
Taking difference and applying Proposition 5.1, we obtain
\[
\|f^{n+1}(t) - f^n(t)\|_{L^\infty_q} \leq A_{\nu,\theta} \int_0^t e^{-A_{\nu,\theta}(t-s)} \|M_{\nu,\theta}(f^n(t)) - M_{\nu,\theta}(f^{n-1}(t))\|_{L^\infty_q} ds
\]
\[
\leq A_{\nu,\theta} C_{\text{Lip}} \int_0^t \|f^n(t) - f^{n-1}(t)\|_{L^\infty_q} ds.
\]
Iterating this inequality,
\[
\|f^{n+1}(t) - f^n(t)\|_{L^\infty} \leq A_{\nu,\theta} C_{Lip} \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} \|f^1(s_n) - f^0(s_n)\|_{L^\infty} ds_n \cdots ds_1 \\
\leq A_{\nu,\theta} C_{Lip} \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} e^{-A_{\nu,\theta}s_n} \|f_0\|_{L^\infty} ds_n \cdots ds_1 \\
\leq A_{\nu,\theta} C_{Lip} \frac{n^m}{n!} \|f_0\|_{L^\infty}.
\]

This immediately gives for \( n > m \)
\[
\sup_{0 \leq t \leq T} \|f^n(t) - f^m(t)\|_{L^\infty} \leq \left( e^{A_{\nu,\theta}T} - \sum_{k=0}^{m-1} \frac{(A_{\nu,\theta}C_{Lip}T)^k}{k!} \right) \|f_0\|_{L^\infty}.
\]

Therefore, we conclude that \( \{f^n\} \) is a Cauchy sequence and converges to an element, say \( f \), in \( \Omega \). It is standard to check that \( f \) is the mild solution:
\[
f(t, x, v, I) = e^{-A_{\nu,\theta}t} f_0(x - vt, v, I) + A_{\nu,\theta} \int_0^t e^{-A_{\nu,\theta}(t-s)} \mathcal{M}_{\nu,\theta}(f)(s, x - (t-s)v, v, I) ds.
\]

This proves the existence and estimates (1) and (2).

For the proof of conservation laws, we find that Proposition (4.1) and the Lebesgue differentiation theorem give from the above mild form,
\[
\frac{d}{dt} f(t, x + tv, v, I) = A_{\nu,\theta} \{ \mathcal{M}_{\nu,\theta}(f) - f \}(t, x + tv, v, I)
\]

for almost all \( t \). Multiplying \( 1, v, \frac{1}{2}|v|^2 + I^2 \) on both sides and integrating with respect to \( x, v, I \), we obtain (3).

The entropy dissipation estimate in the form of (4) was established in [2, 8, 28] at the formal level. But the lower and upper bounds for \( f \) and the macroscopic fields justify all those formal computations given in [2, 8, 28]. This completes the proof.

### 7. When \( \theta = 0 \)

Recall that the l.h.s of equivalence estimates in Lemma 3.1 vanish, and the r.h.s of the inequality in Proposition 4.1 blows up, as \( \theta \) tends to zero. Therefore, the argument we’ve developed so far does not work for \( \theta = 0 \). We, however, observe that the polyatomic Gaussian \( \mathcal{M}_{\nu,\theta}(f) \) in this case is completely split into the translational internal energy part and the non-translational internal energy part as follows:

\[
\mathcal{M}_{\nu,\theta}(f) = \frac{\rho \Lambda_\delta}{\sqrt{\det(2\pi T_{\nu,0})}} e^{-\frac{1}{2}(v-U)^\top \mathcal{T}_{\nu,0}(v-U) - \frac{1}{2} \mathcal{L}}
\]

\[
= \left( \frac{\rho}{\sqrt{\det(2\pi T_{\nu,0})}} e^{-\frac{1}{2}(v-U)^\top \mathcal{T}_{\nu,0}(v-U)} \right) \left( \frac{\Lambda_\delta}{T_{\nu,0}^{3/2}} e^{-\frac{1}{2} \mathcal{L}} \right)
\]

in the sense that \( \mathcal{T}_{\nu,0} \) and \( T_{\nu,0} \) does not share the common factor \( T_\delta \). Now, if we define
\[
g(t, x, v) = \int_{\mathbb{R}^+} f(t, x, v, I) dI
\]
and integrate (1.1) with respect to $I$, we get

$$\partial_t g + v \cdot \nabla_x g = A_0 \{ \mathcal{M}_\nu(g) - g \}$$

(7.1)

$$g_0(x, v) = \int_{\mathbb{R}_+} f_0(x, v, I) dI,$$

where

$$\mathcal{M}_\nu(g) = \int_{\mathbb{R}_+} \mathcal{M}_{\nu, 0}(f) dI = \frac{\rho}{\sqrt{\text{det}(2\pi T_\nu)}} e^{-\frac{1}{2} (v-U)^\top T_{\nu, \theta}^{-1} (v-U)}$$

and

$$T_\nu \equiv T_{\nu, 0} = (1 - \nu) T_{\nu, \theta} + \nu \Theta.$$

Note that $\rho$, $U$, $T_\nu$ are naturally interpreted as macroscopic fields of $g$, and hence, so is $\mathcal{M}_\nu(g)$. This is exactly the ellipsoidal BGK model for monatomic particles [2, 8, 21]. Relevant existence result for (7.1) can be found in [27, 45, 46]. Thus, in the case that $\theta = 0$, our problem (1.1) should be understood in the form (7.1). This dichotomy between the two case, $\theta = 0$ and $0 < \theta \leq 1$, was also observed in [28, 47].

8. Appendix: Conservation laws

In this appendix, we prove the cancellation property (1.6) for reader’s convenience.

- Mass conservation: Note that

$$\int_{\mathbb{R}^3 \times \mathbb{R}^+} \mathcal{M}_{\nu, \theta}(f) dv dI = \rho \int_{\mathbb{R}^3 \times \mathbb{R}^+} f dv dI.$$

Make a change of variable $X = \frac{1}{\sqrt{2}} T_{\nu, \theta}^{-\frac{1}{2}} (v - U)$, so that

$$dX = (\sqrt{2})^{-\frac{3}{2}} \det T_{\nu, \theta}^{-\frac{1}{2}} dv = \frac{\sqrt{\pi}}{\sqrt{\text{det}(2\pi T_{\nu, \theta})}} dv,$$

$$\int_{\mathbb{R}^3} e^{-\frac{X^\top X}{\sigma^2}} dX = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}^3} e^{-|X|^2} dX = 1.$$
• Momentum conservation: We write

\[
\int_{\mathbb{R}^3 \times \mathbb{R}^+} \nu \mathcal{M}_{\nu, \theta}(f) dv dI = \int_{\mathbb{R}^3 \times \mathbb{R}^+} (v - U) \mathcal{M}_{\nu, \theta}(f) dv dI
\]

\[
= \int_{\mathbb{R}^3 \times \mathbb{R}^+} (v - U) \mathcal{M}_{\nu, \theta}(f) dv dI + \rho U,
\]

and make the same change of variable: \( X = \frac{1}{\sqrt{2}} T_{\nu, \theta}^{-\frac{1}{2}} (v - U) \), \( I/(T_{\theta})^{\frac{3}{2}} = J \) to get

\[
\int_{\mathbb{R}^3 \times \mathbb{R}^+} (v - U) \mathcal{M}_{\nu, \theta}(f) dv dI
\]

\[
= \rho \int_{\mathbb{R}^3} (v - U) \sqrt{\text{det}(2\pi T_{\nu, \theta})} e^{-\frac{1}{2} (v-U)^\top T_{\nu, \theta}^{-1} (v-U)} dv \int_{\mathbb{R}^+} \frac{\Lambda_\delta}{T_{\theta}^{\frac{3}{2}}} e^{-\frac{\Lambda_\delta^2}{2 T_{\theta}}} dI
\]

\[
= \frac{\sqrt{2 \rho}}{\sqrt{\pi^3}} \int_{\mathbb{R}^3} X e^{-|X|^2} dX
\]

\[
= 0.
\]

Therefore, (8.2) yields

\[
\int_{\mathbb{R}^3 \times \mathbb{R}^+} \nu \mathcal{M}_{\nu, \theta}(f) dv dI = \rho U = \int_{\mathbb{R}^3 \times \mathbb{R}^+} v f dv dI.
\]

• Energy conservation: To compute the translational part, we again set \( X = \frac{1}{\sqrt{2}} T_{\nu, \theta}^{-\frac{1}{2}} (v - U) \). Then, since

\[
|v - U|^2 = (v - U)\top (v - U) = (\sqrt{2} T_{\nu, \theta}^{-\frac{1}{2}} X)\top (\sqrt{2} T_{\nu, \theta}^{-\frac{1}{2}} X) = 2 X\top T_{\nu, \theta} X,
\]

we have

\[
\int_{\mathbb{R}^3 \times \mathbb{R}^+} \frac{1}{2} |v - U|^2 \mathcal{M}_{\nu, \theta}(f) dv dI = \frac{\rho}{\sqrt{\pi^3}} \int_{\mathbb{R}^3} \{ X\top T_{\nu, \theta} X \} e^{-|X|^2} dX
\]

\[
= \frac{\rho}{\sqrt{\pi^3}} \int_{\mathbb{R}^3} \{ \sum_{ij} X_i X_j T_{\nu, \theta}^{ij} \} e^{-|X|^2} dX
\]

\[
= \frac{\rho}{\sqrt{\pi^3}} \int_{\mathbb{R}^3} \{ \sum_{i=1}^3 X_i^2 T_{\nu, \theta}^{ii} \} e^{-|X|^2} dX
\]

\[
= \frac{\rho}{\sqrt{\pi^3}} \sum_{i=1}^3 T_{\nu, \theta}^{ii} \left( \int_{\mathbb{R}^3} X_i^2 e^{-|X|^2} dX \right)
\]

\[
= \frac{\rho}{2} \text{tr} T_{\nu, \theta}
\]

\[
= \frac{3}{2} \rho \{ (1 - \theta) T_{tr} + \theta T_{\delta} \},
\]

where we used

\[
\left( \int_{\mathbb{R}^3} X_i^2 e^{-|X|^2} dX \right) = \frac{\sqrt{\pi^3}}{2}.
\]
For the non-translational part, one finds
\[
\int_{\mathbb{R}^3 \times \mathbb{R}^+} I^{2/\delta} M_{\nu,\theta}(f) dv dI = \rho \Lambda_\delta \int_{\mathbb{R}^+} \frac{I^{2/\delta}}{T_\theta^{3/2}} e^{-\frac{I^{2/\delta}}{2T_\theta}} dI \int_{\mathbb{R}^3} \frac{1}{\sqrt{\det(2\pi T_{\nu,\theta})}} e^{-\frac{1}{2} (v - U)^T T_{\nu,\theta}^{-1} (v - U)} dv
\]
\[
= \rho \Lambda_\delta \int_{\mathbb{R}^+} \frac{I^{2/\delta}}{T_\theta^{3/2}} e^{-\frac{I^{2/\delta}}{2T_\theta}} dI.
\]
Let \( X = I^{2/\delta}/T_\theta \), then \( dI = \frac{\delta}{2} T_\theta^{3/2} X^{\delta/2 - 1} dX \), and thus,
\[
\int_{\mathbb{R}^3 \times \mathbb{R}^+} I^{2/\delta} M_{\nu,\theta}(f) dv dI = \rho \Lambda_\delta \int_{\mathbb{R}^+} \frac{I^{2/\delta}}{T_\theta^{3/2}} e^{-\frac{I^{2/\delta}}{2T_\theta}} dI
\]
\[
= \frac{\delta}{2} \rho \Lambda_\delta T_\theta \int_{\mathbb{R}^+} X^{\delta/2} e^{-X} dX
\]
\[
= \frac{\delta}{2} \rho T_\theta,
\] where we have used
\[
\frac{\delta/2 \rho T_\theta \int_{\mathbb{R}^+} X^{\delta/2} e^{-X} dX}{\int_{\mathbb{R}^+} e^{-I^{2/\delta}} dI} = \frac{\delta/2 \rho T_\theta \int_{\mathbb{R}^+} X^{\delta/2} e^{-X} dX}{\delta/2 \int_{\mathbb{R}^+} X^{2/2 - 1} e^{-X} dX}
\]
\[
= \rho T_\theta \left[ (-X^{\delta/2} e^{-X}) \bigg|_{X=0}^{X=\infty} + \frac{\delta}{2} \int_{\mathbb{R}^+} X^{2/2 - 1} e^{-X} dX \right]
\]
\[
= \frac{\delta}{2} \rho T_\theta.
\]
Combining (8.4) with (8.3) and recalling the definition of \( T_\delta \) in (1.4), we get
\[
\int_{\mathbb{R}^3 \times \mathbb{R}^+} \left( \frac{|v - U|^2}{2} + I^{2/\delta} \right) M_{\nu,\theta}(f) dv dI
\]
\[
= 3 \rho \left\{ (1 - \theta) T_{tr} + \theta T_\delta \right\} + \frac{\delta}{2} \rho \left\{ (1 - \theta) T_{l,\delta} + \theta T_\delta \right\}
\]
\[
= 3 + \frac{\delta}{2} (1 - \theta) \rho \left\{ \frac{3}{3 + \delta} T_{tr} + \frac{\delta}{3 + \delta} T_{l,\delta} \right\} + \frac{3 + \delta}{2} \rho T_\delta
\]
\[
= \frac{3}{2} \rho T_\delta
\]
\[
= \int_{\mathbb{R}^3 \times \mathbb{R}^+} \left( \frac{|v - U|^2}{2} + I^{2/\delta} \right) f dv dI.
\]
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DEPARTMENT OF MATHEMATICS, SUNGKYUNKWAN UNIVERSITY, SUWON 440-746, REPUBLIC OF KOREA
E-mail address: parksajune@skku.edu

DEPARTMENT OF MATHEMATICS, SUNGKYUNKWAN UNIVERSITY, SUWON 440-746, REPUBLIC OF KOREA
E-mail address: sbyun012@skku.edu