ASYMPTOTICS OF SCHWARTZ FUNCTIONS: NONARCHIMEDEAN

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Abstract. Let $G$ be a split, simply connected, simple group, and let $P \leq G$ be a maximal parabolic. Braverman and Kazhdan in [BK02] defined a Schwartz space on the affine closure $X_P$ of $X_P^o := P^\text{der} \setminus G$. An alternate, more analytically tractable definition was given in [GHL21], following several earlier works. In the nonarchimedean setting when $G$ is a classical group or $G_2$, we show the two definitions coincide and prove several previously conjectured properties of the Schwartz space that will be useful in applications. In addition, we prove that the quotient of the Schwartz space by the space of compactly supported smooth functions on the open orbit is of finite length and we describe its subquotients. Finally, we use our work to study the set of possible poles of degenerate Eisenstein series under certain assumption at archimedean places.

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1. Introduction

Let $G$ be a split, simple, simply connected algebraic group over a local field $F$. Let $\psi : F \to \mathbb{C}^\times$ be a nontrivial additive character. For a parabolic subgroup $P$ of $G$, let $X_P^o := P^\text{der} \setminus G$ and $X_P := \text{Spec}(F[X_P^o])$ be its affine closure. Generalizing earlier work in [BK99], Braverman and Kazhdan in [BK02] constructed isometries

$$\mathcal{F}_{P|Q} := \mathcal{F}_{P|Q,\psi} : L^2(X_P(F)) \xrightarrow{\sim} L^2(X_Q(F))$$

such that

$$\mathcal{F}_{Q|R} \circ \mathcal{F}_{P|Q} = \mathcal{F}_{P|R} \quad \text{and} \quad \mathcal{F}_{P|P} = \text{Id}$$

for any parabolic subgroups $P, Q, R$ of $G$ with the same Levi component $M$. These operators generalize the standard Fourier transform on vector spaces. A nice exposition of their construction is given in [Sha18, §4].

Let $M$ be a Levi subgroup of $G$ and $P$ be a parabolic subgroup of $G$ with Levi component $M$. Braverman and Kazhdan defined the Schwartz space to be

$$\mathcal{S}_{BK}(X_P(F)) := \mathcal{S}(G, M) := \sum_Q \mathcal{F}_{Q|P} (\mathcal{S}(X_Q^o(F))) < L^2(X_P(F)),$$

where the sum is taken over all parabolic subgroups $Q$ with Levi component $M$. Here $\mathcal{S}(X_Q^o(F))$ is simply $C_0^\infty(X_Q^o(F))$ when $F$ is nonarchimedean (see e.g. [ES18, Remark 3.2] for the definition

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when $F$ is archimedean). That is, $\mathcal{S}_{\text{BK}}(X_P(F)) < C^\infty(X_P^0(F))$ is the smallest space of functions that contains $\mathcal{S}(X_P^0(F))$ and is stable under Fourier transforms. While this definition of Schwartz space is succinct and natural, it is not known in general how to characterize functions in $\mathcal{S}_{\text{BK}}(X_P(F))$. One main difficulty is that the original construction of the operator $\mathcal{F}_{Q|P}$ was only defined on an inexplicit subspace of $C_c^\infty(X_Q^0(F))$ and extended to an $L^2$-isometry by unitarity. In particular, little analytic information about functions in $\mathcal{S}_{\text{BK}}(X_P(F))$ is understood aside from the case where $X_P$ is a vector space.

Let $B \leq P$ be a Borel subgroup of $G$ and $P^{\text{op}}$ be the opposite of $P$. In the case $G$ is symplectic and $P$ is the Siegel parabolic, using the work of Ikeda [Ike92], Getz and Liu defined in [GL21] another Schwartz space $\mathcal{S}(X_P(F)) < C^\infty(X_P^0(F))$. Roughly, it is defined to be a space of functions whose Mellin transforms have poles no worse than those of certain local $L$-functions. This definition was later modified in [GH20] and extended by Getz, Leslie, and the author in [GHL21] to all split, simple, simply connected groups $G$ and maximal parabolics $P$. In loc. cit. we showed that functions in $\mathcal{S}(X_P(F))$ lie in $L^1 \cap L^2(X_P(F))$ and have good analytic behavior towards infinity. For instance, they are compactly supported in $X_P(F)$ if $F$ is nonarchimedean. Moreover, the isometry $\mathcal{F}_{P|P^{\text{op}}}$ descends to an isomorphism

$$\mathcal{F}_{P|P^{\text{op}}}: \mathcal{S}(X_P(F)) \xrightarrow{\simeq} \mathcal{S}(X_{P^{\text{op}}}(F)) \tag{1.1}$$

and comes with an explicit formula. What is not clear is the following locality property:

**Conjecture 1.1.** Suppose $F$ is a nonarchimedean local field. The Schwartz space $\mathcal{S}(X_P(F))$ is a $C^\infty(X_P(F))$-module (under multiplication of functions).

For $F$ archimedean, $C^\infty(X_P(F))$ is replaced with the subspace of tempered functions (see e.g. [AG08, Definition 4.2.1]). When $X_P$ is a vector space, the locality property follows from the fact that the usual Schwartz space is the subalgebra of $C^\infty(X_P(F))$ consisting of rapidly decreasing functions, which plays an important role in Fourier analysis.

**Remark 1.2.** The notion of locality of a function space was introduced in [BK00, §5]. Their definition in the nonarchimedean setting used continuous sections instead of smooth sections; however, under their definition, the Schwartz space $\mathcal{S}(F)$ will not be local, for it is not stable under the multiplication by the norm function (which is continuous).

1.1. **Properties of Schwartz spaces $\mathcal{S}(X_P(F))$.** Suppose $F$ is a nonarchimedean local field and $G$ is not of type $E$ or type $F$. Let $P$ be a maximal parabolic subgroup of $G$. Our main result is the following.

**Theorem 1.3.** We have

$$\mathcal{S}(X_P^0(F)) < \mathcal{S}(X_P(F)) \tag{1.2}$$

and the space $\mathcal{S}(X_P(F))$ is local. Moreover, $\mathcal{S}(X_P(F))/\mathcal{S}(X_P^0(F))$ is a smooth $G(F)$-module of finite length.

The inclusion (1.2) will follow from an analysis of poles of the intertwining operator $M_{\text{int}}$ associated to the long Weyl element (see §1.2). We explain how the locality property of $\mathcal{S}(X_P(F))$ follows from the inclusion (1.2).

**Proof of Conjecture 1.1 assuming (1.2).** Choose a $G$-equivariant closed embedding $X_P \hookrightarrow V_P$ into a $F$-vector space such that the point $X_P(F) - X_P^0(F)$ is mapped to the origin (see §3). Let $\mathcal{S} := \text{Im}(\mathcal{S}(V_P(F))) \hookrightarrow C^\infty(X_P^0(F)))$, where the implicit map is the restriction. Note that $\mathcal{S}(X_P^0(F)) < \mathcal{S}$ and $\mathcal{S}/\mathcal{S}(X_P^0(F)) = \mathbb{C}$. Since each function in $\mathcal{S}(X_P(F))$ is compactly supported in $X_P(F)$, we have $C^\infty(X_P(F)) \cdot \mathcal{S}(X_P(F)) = \mathcal{S} \cdot \mathcal{S}(X_P(F))$. As $\mathcal{S}(X_P^0(F)) \cdot \mathcal{S}(X_P(F)) = \mathcal{S}(X_P(F)) < \mathcal{S}(X_P(F))$ by (1.2), it suffices to show $\mathcal{S}(X_P(F))/\mathcal{S}(X_P^0(F))$ is a $\mathcal{S}/\mathcal{S}(X_P^0(F))$-module, i.e., a $\mathbb{C}$-vector space, which is clear. \(\square\)
The quotient $S(X_P(F))/S(X_P^\circ(F))$ is roughly the kernel of $M_{w_0}$ at points of reducibility on the left half-plane. A precise description is given in Theorem 4.10 together with an example in §4.3. As a result, the Schwartz space $S(X_P(F))$ admits a natural inductive limit topology under which it is nuclear, barreled, and separable (see [Li18, §4.1]). These topological properties of $S(X_P(F))$ are stated as axioms in [Li18] to validate Weil’s interpretation of zeta integrals as a unique family of tempered distributions. We expect an analogous result in the archimedean setting can be derived similarly.

1.2. Poles of intertwining operators. Let $T \leq B$ be a maximal torus of $G$. Let $\Phi := \Phi_G, \Delta, \text{and } W := W_G$ denote the set of roots, simple roots, and the Weyl group of $(G, B, T)$ respectively. Let $W/W_M$ collect the unique element of minimal length in each left coset of $W_M$ in $W$. Let $\{\omega_\alpha\}_{\alpha \in \Delta}$ be fundamental weights of $T$, and let $\omega_P$ be the fundamental weight associated to the simple root corresponding to $P$.

We briefly recall the parameters $\Lambda = \{(s_i, \lambda_i)\}$ considered in [BK02] to define $\mathcal{F}_{P|P_{op}}$. Let $P = MN_P$ be a Levi decomposition. On the dual side, we have a natural action of

$$\hat{M}^{ab} \times \hat{m}$$
on
on $\hat{n}_P$ induced by the adjoint action. Fix a principal $\mathfrak{sl}_2$-triple $\{e, h, f\}$ in $\hat{m}$, and let $\hat{n}_P^e$ be the vector space spanned by highest weight vectors. Decompose $\hat{n}_P^e = \oplus_{i=1}^k L_i$ into one dimensional modules. Let $\lambda_i > 0$ be the $\hat{M}^{ab}$-character and $s_i \geq 0$ be $\frac{1}{2}\log h$ times the $h$-eigenvalue on $L_i$. It is proved in [GHL21, Appendix A] that one can rearrange the indices so that $s_k > \frac{\lambda_i}{h}$ for all $k > i$ and $\lambda_k = 1$, which we assume henceforth.

Let $\mathcal{O}$ be the ring of integers of $F$ and $q$ be the cardinality of the residue field. Fixing a uniformizer, we identify groups of quasi-characters $\hat{F}^\times = \mathbb{C}/\mathbb{Z} \times \overline{\mathcal{O}} \times \overline{\mathcal{O}}$ (see §2.1 for conventions). For $\chi \in \overline{\mathcal{O}}^\times$, put $\chi_s := |\chi| \cdot |s|$ for $s \in \mathbb{C}$. Identifying $M^{ab}(F) \rightarrow F^\times$ via $\omega_P$, we view $\chi_s$ as a quasi-character of $M(F)$. To prove the inclusion (1.2), by the definition of $S(X_P(F))$ it suffices to show the intertwining operator

$$M_{w_0}(\chi_s) \prod_{i=1}^k L(-s_i, \chi_s^i)^{\lambda_i}$$
on
is holomorphic, where $w_0 \in W/W_M$ is the long Weyl element.

Let $\Phi^+$ be the set of coroots and $(\Phi^+)^+$ (resp. $(\Phi^-)^-$) be the set of positive (resp. negative) coroots. For each $w \in W/W_M$, let

$$\Phi_w^+ := \{\beta^+ : \beta^+ \in (\Phi^+)^+, w\beta^+ \in (\Phi^-)^-\} \subset (\Phi^-)^+ - (\Phi^+)^+$$

Let $\langle \cdot, \cdot \rangle : X^*(T) \times X_*(T) \rightarrow \mathbb{Z}$ denote the canonical pairing on characters and cocharacters. For $\chi \in \overline{\mathcal{O}}^\times$ and $s \in \mathbb{C}$, define

$$c_w(\chi_s) := \prod_{\beta^+ \in \Phi_w^+} L\left(-\sum_{\alpha \in \Delta} \langle \omega_\alpha, \beta^+ \rangle, \chi_s^{\langle \omega_\alpha, \beta^+ \rangle+1}\right),$$

$$d(\chi_s) := \prod_{i=1}^k L(s_i + 1, \chi_s^i)^{\lambda_i}. \tag{1.3}$$

Our notation is consistent with [Ike92]. Following the idea of [Ike92], we show

**Theorem 1.4.** Suppose $F$ is nonarchimedean. For $w \in W/W_M$ and $\chi \in \overline{\mathcal{O}}^\times$, $d(\chi_s)c_w(\chi_s)$ is a product of $L$-functions. The intertwining operator

$$(d(\chi_s)c_w(\chi_s))^{-1} M_w(\chi_s)$$

is a product of $L$-functions.
is holomorphic.

Therefore, (1.2) is a consequence of the identity
\[
d(\chi) c_w(\chi) = \prod_{i=1}^{k} L(-s_i, \chi_s^\lambda),
\]
which can be directly verified (see Lemma 5.4). Moreover,

**Corollary 1.5.** We have \( S(X_P(F)) = S_{\text{BR}}(X_P(F)) \). For any \( w \in W/W_M \) and \( f \in S(X_P(F)) \),
\[
(d(\chi) c_w(\chi))^{-1} M_w(\chi) f_{\chi}\text{ is holomorphic, where } f_{\chi}\text{ is the Mellin transform of } f \text{ along } \chi \text{ defined in (3.4)}.
\]

With Theorem 1.4 and Corollary 1.5 in hand, one can also determine the set of possible poles of an Eisenstein series (see e.g. [PSR87] for a detailed argument). Let \( E \) be a global field and \( \mathbb{A}_E \) be its ring of adele. Define the global Schwartz space
\[
S(X_P(\mathbb{A}_E)) := \bigotimes_v S(X_P(E_v))
\]
where the restricted tensor product is taken with respect to the basic function \( b_v \ (v < \infty) \). Here \( b_v \) is the right \( G(O_v) \)-invariant function in \( S(X_P(E_v)) \) whose Mellin transform (along the trivial character) is \( d(1_s) \phi_s \), where \( \phi_s \in I_P(1_s) := \text{Ind}_{G(E_v)}^{G(E_v)}(1_s) \) is the unique right \( G(O_v) \)-invariant section such that \( \phi_s|_{G(O_v)} = 1 \). For \( f \in S(X_P(\mathbb{A}_E)) \) and an idelic unitary character \( \omega \), the degenerate Eisenstein series
\[
\text{Eis}(g; f_{\omega}) := \sum_{\gamma \in X_P(E)} f_{\omega}(\gamma g)
\]
converges absolutely for \( g \in G(\mathbb{A}_E) \) and \( \text{Re}(s) \gg 0 \) and admits an analytic continuation.

**Corollary 1.6.** Suppose \( E \) is a function field or \( E \) is a number field and Corollary 1.5 holds in the archimedean setting. Let
\[
d_0 := \max \{ \lambda : (\omega \rho, \beta^\lambda) = \lambda, \beta^\lambda \in \Phi_{\omega_0}^\gamma = (\Phi^\gamma)^+ - (\Phi^\gamma)^+ \} = \max_{1 \leq i \leq k} \lambda_i \leq 6.
\]
Let \( f \in S(X_P(\mathbb{A}_E)) \) and \( \omega \) be an idelic unitary character. For \( g \in G(\mathbb{A}_E) \), the Eisenstein series \( \text{Eis}(g; f_{\omega}) \) is entire unless \( \omega^d \) is a consequence of the identity
\[
\sum_{w \in W/W_M} M_w(\omega) f_{\omega} \text{ is holomorphic.}
\]

By Gindikin-Karpelevič formula [Lai80, Proposition 4.6] for unramified places \( v \) with \( \psi_v \) unramified
\[
(d(1_s) c_w(1_s))^{-1} M_w(1_s) (b_v)_{1_s}|_{G(O_v)} = 1.
\]
Therefore by Corollary 1.5 and the assumption at the archimedean places

\[(d(\omega_s)c_w(\omega_s))^{-1} M_w(\omega_s)f_{\omega_s}\]

is entire for all \(w \in W/W_M\). Consequently the order of poles of \(\text{Eis}(g; f_{\omega_s})\) is dominated by the maximum of the order of poles of \(d(\omega_s)c_w(\omega_s)\) ranging over \(w \in W/W_M\). By Theorem 1.4 \(d(\omega_s)c_w(\omega_s)\) are products of L-functions. Since \(d(\omega_s)c_w(\omega_s)\) is entire if \(\omega^d \neq 1\) for any \(d \leq d_0\), the first assertion follows. Take \(\Gamma\) to be the set of poles of \(d(\omega_s)c_w(\omega_s)\) and \(N\) be the maximal order of poles of \(d(\omega_s)c_w(\omega_s)\), where \(w\) ranges over all \(W/W_M\) and \(\omega\) ranges over all \(\omega^d = 1\) for some \(d \leq d_0\). By (1.3) and (1.4) and the fact that \(s_k > \frac{s}{\beta}\) for all \(i < k\), we have

\[0 < \sum_{\alpha \in \Delta(\omega_{\alpha}, \beta')} \|\omega P, \beta'\| \leq s_k + (s + 1)\]

for all \(\beta' \in \Phi_w = (\Phi')^* - (\Phi_M^*)^*\). Therefore, poles of \(d(\omega_s)c_w(\omega_s)\) are contained in the vertical strip \(\{s : \text{Re}(s) \leq s_k + 1\}\). The rest is clear. 

In [CG21], Choie and Getz establish the Poisson summation conjecture for certain Schubert varieties under mild assumptions on poles of Eisenstein series. Corollary 1.6 partially justifies [CG21, Conjecture 1.4] and reduces their assumption to the analysis of poles of \(M_w(\chi_s)\) in the archimedean case. More work is required to locate the exact set of possible poles and orders, which will give rise to an explicit form of Poisson summation formula on \(X_P\) and the Schubert varieties. Several cases have been treated in [Ike92, Han18, HM15]. See [CG21] for a detailed discussion.

Finally, we remark that our proofs of Theorem 1.3 and Corollary 1.5 rely only on Theorem 1.4 but not on the type of \(G\). However, proving Theorem 1.4 eventually boils down to certain combinatorics statements on the set of coroots in \((\Phi')^* - (\Phi_M^*)^*\), which requires a case-by-case analysis. Since the proof of type \(E\) and type \(F\) is more complicated, we do not carry out the proof.

1.3. Outline of the paper. Conventions are stated in §2. We recall in §3 some facts about Braverman-Kazhdan spaces. In §4, we will assume the inclusion (1.2) and study the quotient \(S(X_P(F))/S(X^*_P(F))\). Motivated by the approach in [JLZ20, §4], we begin with rephrasing the definition of \(S(X_P(F))\) in terms of its asymptotic behavior towards the origin via the work of Igusa [Igu78, §1.5]. Then we restate the asymptotics in terms of representations in Theorem 4.10, which will justify the latter part of Theorem 1.3.

In §5, we prove Theorem 1.4 and Corollary 1.5. We start with a general setup, regardless of the type of \(G\), to prove Theorem 5.3, which is a more precise (and stronger) version of Theorem 1.4. Theorem 5.3 allows us to rephrase in Corollary 5.5 our definition of Schwartz space in the sense of [Yam14], from which we deduce Corollary 1.5, restated as Corollary 5.6 and Corollary 5.7 below. The rest of the paper is a case-by-case proof of Theorem 5.3. The case \(G = G_2\) is proved in §5.1, and classical groups are discussed in §5.2–§5.5. We also offer a list of \(L(d) \cap \mathbb{R}\) and \(s_k\) (see (4.2)) in the Appendix.

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2. Preliminaries

Throughout the paper, \(F\) will be a nonarchimedean local field. We denote by \(\mathcal{O}\) the ring of integers of \(F\) and fix, once and for all, a choice of uniformizer \(\varpi\). Let \(q\) be the cardinality of
the residue field. Let \( \text{ord} : F \to \mathbb{Z} \cup \{\infty\} \) be the discrete valuation with \( \text{ord}(\varpi) = 1 \) and let \( |\cdot| \) be the induced norm on \( F \) with \( |\varpi| = q^{-1} \).

Let \( G \) be a split, simply connected, simple group over \( F \). Fix throughout the paper a maximal torus \( T < G \) and a pinning \( (\Delta, (e_\alpha)_{\alpha \in \Delta}) \) of \( (G, T) \). Let \( B \) be the Borel subgroup of \( G \) standard with respect to the pinning. For each \( \alpha \in \Delta \), let \( \omega_\alpha \in X^*(T) \) be the fundamental weight such that

\[
\langle \omega_\alpha, \beta^\vee \rangle = \delta_{\alpha, \beta}
\]

for all \( \beta \in \Delta \), where \( \delta_{\alpha, \beta} \) is the Kronecker \( \delta \).

Let \( P \geq B \) be a maximal parabolic subgroup with Levi decomposition \( P = MN \) and \( \omega_P \) be the fundamental weight corresponding to the simple root associated to \( P \). The attached Braverman-Kazhdan space (a.k.a. pre-flag variety)

\[
X_P^\circ := P_{\text{der}} \setminus G
\]

admits a natural right \( M_{\text{ab}} \times G \) action, given on points in an \( F \)-algebra \( R \) by

\[
X_P^\circ(R) \times M_{\text{ab}}(R) \times G(R) \to X_P^\circ(R)
\]

\[
(x, m, g) \mapsto m^{-1}xg.
\]

Let \( X_P := \text{Spec}(F[X_P^\circ]) \) be the affine closure of \( X_P^\circ \).

2.1. Quasi-characters. We have a group isomorphism

\[
F^\times \longrightarrow \mathbb{Z} \times \mathcal{O}^\times
\]

\[
a \mapsto (\text{ord}(a), \tilde{a}),
\]

where

\[
\tilde{a} := a \cdot \varpi^{-\text{ord}(a)}.
\]

Let \( \widehat{\mathbb{F}}^\times, \widehat{\mathcal{O}}^\times \) denote groups of quasi-characters of \( F^\times \) and \( \mathcal{O}^\times \) respectively. The isomorphism (2.2) induces a group isomorphism

\[
\mathbb{C}/2\pi \sqrt{-1} \mathbb{Z} \times \widehat{\mathcal{O}}^\times \longrightarrow \widehat{\mathbb{F}}^\times
\]

\[
(s, \chi) \mapsto (a \mapsto |a|^s \chi(\tilde{a})).
\]

We henceforth identify \( \widehat{\mathcal{O}}^\times \) as a subgroup of \( \widehat{\mathbb{F}}^\times \) under this isomorphism, and thus \( \chi(a) = \chi(\tilde{a}) \) for \( \chi \in \widehat{\mathcal{O}}^\times \). Let \( \text{ord}(\chi) \) be the order of \( \chi \), i.e., the smallest positive integer \( d \) such that \( \chi^d = 1 \). For \( \chi \in \widehat{\mathbb{F}}^\times \) and \( s \in \mathbb{C}/2\pi \sqrt{-1} \mathbb{Z} \), define \( \chi_s := \chi |\cdot|^s \). Let \( \text{Re}(\chi) \) be the unique real number such that \( \chi - \text{Re}(\chi) \) is unitary.

2.2. Measures. We fix once and for all a nontrivial additive character \( \psi : F \to \mathbb{C}^\times \). The Haar measure \( dt \) on \( F \) will always be normalized so that the Fourier transform on \( S(F) \) defined by \( \psi \) is self-dual. We set \( d^\times x := \frac{\zeta(1)dx}{|x|} \), where \( \zeta \) is the usual local zeta function. It is a Haar measure on \( F^\times \).

Choose a Chevalley basis \((e_\alpha)_{\alpha \in \Phi}\) from the pinning \((e_\alpha)_{\alpha \in \Delta}\) (see [Mil17, §23.h]). The Chevalley basis determines group isomorphisms \( \mathbb{G}_\alpha \to N_\alpha \), where \( N_\alpha \) is the root subgroup of \( \alpha \in \Phi \). We endow \( N_\alpha(F) \) with the measure transferred from that on \( F = \mathbb{G}_\alpha(F) \) under these isomorphisms. This in turn induces measures on unipotent subgroups of \( B(F) \) and its opposite \( B^{op}(F) \).
2.3. Multisets. A multiset $L$ consists of a set $U$ and a function $m(L) : U \to \mathbb{Z}_{>0}$. Equivalently, we can understand $L$ as the set

$$\{(r, m_r) : r \in U\}$$

where $m_r := m_r(L) \in \mathbb{Z}_{>0}$ is the multiplicity of $r$ in $L$. We will write $\text{Supp}(L) := U$ as the underlying set of $L$. Given two multisets $L_1, L_2$, their sum $L_1 + L_2$ is the multiset such that

$$m_r(L_1 + L_2) = m_r(L_1) + m_r(L_2),$$

where the domain of $m(L_i)$ is extended to $\text{Supp}(L_i) \cup \text{Supp}(L_2)$ by setting $m_r(L_i) = 0$ if $r \notin \text{Supp}(L_i)$.

3. Braverman-Kazhdan spaces

We briefly recall the Schwartz space and the Fourier transform on $X_P$. For a detailed discussion, we refer one to [GHL21, §3-§6], and we follow the notation therein below.

Let $V_P$ be the right representation of $G$ of highest weight $-\omega_P$, and fix a highest weight vector $v_P \in V_P(F)$. By [GHL21, Lemma 3.4], we have a well defined embedding

$$P_l := P_l : X_P \hookrightarrow V_P$$

induced by the map (given on points in an $F$-algebra $R$ by)

$$G(R) \longrightarrow V_P(R)$$
$$g \longmapsto v_P g.$$

The map $\omega_P$, originally a character of $T$, extends to a character of $M$ and induces an isomorphism

$$\omega_P : M^{\text{ab}} \longrightarrow \mathbb{G}_m.$$

For $m \in M^{\text{ab}}(R)$, one has

$$P_l(m^{-1}g) = \omega_P(m)P_l(g). \quad (3.1)$$

Moreover, by [VP73, Theorem 1 and 2] the embedding $P_l$ extends to a closed immersion $X_P \hookrightarrow V_P$, and $X_P - X_P^\circ$ is a closed point mapped to 0 under $P_l$.

Let $V_P^\vee$ be the dual representation of $V_P$ and $P^{\text{pop}}$ be the parabolic subgroup opposite to $P$. Let $v_{p^{\text{pop}}} \in V_P^\vee(F)$ be the lowest weight vector such that $\langle v_p, v_{p^{\text{pop}}} \rangle = 1$. Similarly, we have a $G$-equivariant closed embedding $P_l v_{p^{\text{pop}}} : X_{p^{\text{pop}}} \hookrightarrow V_P^\vee$. The restriction of the canonical pairing between $V_P$ and $V_P^\vee$ yields a pairing

$$\langle \cdot, \cdot \rangle_{P|P^{\text{pop}}} : X_P \times X_{p^{\text{pop}}} \longrightarrow \mathbb{G}_a \quad (3.2)$$

via the embeddings $P_l v_P, P_l v_{p^{\text{pop}}}$. This pairing is independent of the choice of $v_P$. We drop the subscript and simply write $\langle \cdot, \cdot \rangle$ whenever the context is clear.

3.1. The Schwartz space. For a quasi-character $\chi \in \hat{F}^\times$, let

$$I(\chi) := I_P(\chi) := \text{Ind}_{P(F)}^G(\chi \circ \omega_P), \quad \overline{I}(\chi) := \overline{I}_{P^{\text{pop}}(F)}(\chi \circ \omega_P) \quad (3.3)$$
be the normalized inductions in the category of smooth representations. Let \( \delta_P \) be the modular character of \( P \). The Mellin transforms along \( \chi \) are defined as
\[
S(X_P^\sigma(F)) \longrightarrow I(\chi)
\]
\[
f \mapsto f_\chi(\cdot) := f_{\chi,P}(\cdot) := \int_{M^\ab(F)} \delta_P^{1/2}(m) \chi(\omega_P(m)) f(m^{-1}) dm,
\]
(3.4)
\[
S(X_{\text{pop}}^\sigma(F)) \longrightarrow \overline{T}(\chi)
\]
\[
f \mapsto f_\chi^\op(\cdot) := f_{\chi,\text{pop}}^\op(\cdot) := \int_{M^\ab(F)} \delta_P^{1/2}(m) \chi(\omega_P(m)) f(m^{-1}) dm.
\]
Here \( dm \) is the Haar measure on \( \mathcal{M}^\ab(F) \) transferred from that on \( F^\times \) via \( \omega_P \). In the notation \( \overline{T}_{\text{pop}}(\chi) \) and \( f_{\chi,\text{pop}}^\op \), the bar and the superscript \( \text{op} \) indicate that we are inducing from \( \chi \circ \omega_P \) instead of \( \chi \). The same notation is used for functions in \( C^\infty(X_P^\sigma(F)) \) and \( C^\infty(X_{\text{pop}}^\sigma(F)) \) whenever the integrals defining \( f_\chi, f_{\chi,\text{pop}}^\op \) exist for \( \Re(\chi) \) in a subset of \( \mathbb{R} \), and in some cases are extended to larger complex domains by analytic continuation.

For \( \chi \in \tilde{O}^\times \), a section \( f(s) \in I(\chi_s) \) is said to be holomorphic if for all \( g \in G(F) \) the complex function
\[
\mathbb{C} \longrightarrow \mathbb{C}
\]
\[
s \mapsto f(s)(g)
\]
lies in \( \mathbb{C}[q^{-s}, q^s] \). It is meromorphic if there exists a nonzero \( a(s) \in \mathbb{C}[q^{-s}, q^s] \) such that \( a(s)f(s) \) is holomorphic.

For functions \( f \in C^\infty(X_P^\sigma(F)) \) and \( x^s = P_{\text{pop,der}}(F)g \in X_{\text{pop}}^\sigma(F) \), we define the unnormalized intertwining operator
\[
\mathcal{R}_{P|\text{pop}}(f)(x^s) := \int_{N_{\text{pop}}(F)} f(ug) du = \int_{N_{\text{pop}}(F)} f(ux^s) du,
\]
whenever this integral is absolutely convergent (or obtained via some regularization procedure). Here the measure \( du \) on \( N_{\text{pop}}(F) \) is defined as in §2.2. Similarly, one also has
\[
\mathcal{R}_{P|\text{pop}} : I(\chi_s) \longrightarrow \overline{T}(\chi_s)
\]
for \( \chi \in \tilde{O}^\times \) and \( \Re(s) \) sufficiently large, and the map extends meromorphically to \( \mathbb{C}/\mathbb{Z} \). Moreover, the diagram [GHL21, Lemma 5.9]
\[
\begin{array}{ccc}
S(X_P(F)) & \xrightarrow{\mathcal{R}_{P|\text{pop}}} & \mathcal{R}_{P|\text{pop}}(S(X_P(F))) \subset C^\infty(X_{\text{pop}}^\sigma(F)) \\
I(\chi_s) & \xrightarrow{\mathcal{R}_{P|\text{pop}}} & \overline{T}(\chi_s)
\end{array}
\]
is commutative for \( \Re(s) \) sufficiently large.

Recall the multiset \( \Lambda := \{ (s_i, \lambda_i) \}_{1 \leq i \leq k} \) in \( \mathbb{R}_{\geq 0} \times \mathbb{Z}_{> 0} \) defined in §1.2. For \( \chi \in \tilde{F}^\times \), we set
\[
a_{P|\text{pop}}(\chi) := \prod_{i=1}^k L(s_i + 1, \chi^{\lambda_i}) \quad \text{and} \quad a_{P|\text{pop}}(\chi) := \prod_{i=1}^k L(-s_i, \chi^{\lambda_i}).
\]
(3.6)

A section \( f(s) \in I(\chi_s) \) is said to be good if it is meromorphic, and if both sections
\[
\frac{f(s)}{a_{P|\text{pop}}(\chi_s)}, \quad \frac{\mathcal{R}_{P|\text{pop}}f(s)}{a_{P|\text{pop}}(\chi_s)}
\]
of \( I(\chi_s) \) and \( \overline{T}(\chi_s) \) are holomorphic.

Fix a maximal compact open subgroup \( K \) of \( G(F) \) such that the Iwasawa decomposition \( G(F) = P(F)K \) holds.
Definition 3.1. The Schwartz space $S(X_P(F))$ is defined to be the space of right $K$-finite functions $f \in C^\infty(X_P(F))$ such that for each $g \in G(F)$ and $\chi \in \widehat{O}^\times$, the integral (3.4) defining $f_{\chi,s}(g)$ is absolutely convergent for $\text{Re}(s)$ large enough and defines a good section.

3.2. The Fourier transform $\mathcal{F}_{P|P^{\text{op}}}$. Since $(s,\lambda) \in \mathbb{R}_{\geq 0} \times \mathbb{Z}_{>0}$, $\Lambda$ admits a good ordering, i.e., it can be reindexed so that

$$\frac{s_{i+1}}{\lambda_{i+1}} \geq \frac{s_i}{\lambda_i}$$

for all $i$. We henceforth assume $\Lambda$ is in such ordering. It is shown in [GHL21, Appendix A] that the highest data $(s_k,\lambda_k)$ is unique in the following sense.

Proposition 3.2. We have $\lambda_k = 1$ and $s_k > \frac{\lambda_k}{s_k}$ for all $i < k$. □

Moreover, by [GHL21, Proposition 6.2] we have

$$\delta_P = |\omega_P|^{2(s_k+1)}. \quad (3.7)$$

For each $(s,\lambda) \in \mathbb{C} \times \mathbb{Z}$, define a linear map

$$\lambda!(\mu_s) : S(X^{\text{op}}_P(F)) \to C^\infty(X^{\text{op}}_P(F))$$

given by

$$\lambda!(\mu_s)(f)(x) := \int_{M^{\text{ab}}(F)} \psi(\omega_P(m))|\omega_P(m)|^{s+1} \delta_P(m) f(m^{-\lambda}x) \, dm \frac{1}{\zeta(1)}.$$

The same notation is used if the integral exists by regularization. We define

$$\mu^\text{aug}_P := \lambda!(\mu_{s_1}) \circ \cdots \circ \lambda!(\mu_{s_{k-1}}) \circ \lambda!(\mu_{s_k}) \quad \text{and} \quad \mu^\text{geo}_P := 1!(\mu_{s_k}).$$

The Fourier transform $\mathcal{F}_{P|P^{\text{op}}}$ is defined by

$$\mathcal{F}_{P|P^{\text{op}}} := \mu^\text{aug}_P \circ \mathcal{F}^\text{geo}_{P|P^{\text{op}}},$$

where

$$\mathcal{F}^\text{geo}_{P|P^{\text{op}}} := \mu^\text{geo}_P \circ R_{P|P^{\text{op}}}.$$

It descends to an isomorphism $\mathcal{F}_{P|P^{\text{op}}} : S(X_P(F)) \to S(X^{\text{op}}_P(F))$, and by [GHL21, Theorem 6.5] $\mathcal{F}^\text{geo}_{P|P^{\text{op}}}$ has an explicit formula:

Theorem 3.3. For $f \in S(X_P(F))$ and $x^* \in X^{\text{op}}_P(F)$, we have

$$\mathcal{F}^\text{geo}_{P|P^{\text{op}}}(f)(x^*) = \int_{X^{\text{op}}_P(F)} f(x) \psi(\langle x, x^* \rangle_{P|P^{\text{op}}}) \, dx.$$ 

Here the pairing $\langle \cdot, \cdot \rangle_{P|P^{\text{op}}}$ is defined as in (3.2), and $dx$ is the unique right $G(F)$-invariant Radon measure such that

$$d(mu) = \delta_{P^{\text{op}}}(m) \, dm \, du = \frac{1}{\zeta(1)} \, \delta_{P^{\text{op}}}(m) \, dm \, du \quad \text{for} \ (m,u) \in M^{\text{ab}}(F) \times N_{P^{\text{op}}}(F).$$

□
4. Characterization of $\mathcal{S}(X_P(F))$

Throughout this section, we will assume the inclusion
\begin{equation}
C_c^\infty(X_P^0(F)) = \mathcal{S}(X_P^0(F)) < \mathcal{S}(X_P(F))
\end{equation}
holds. If $G$ is not of type $E$ or $F$, this follows from Theorem 1.4 to be proved in §5.

By the definition of $\mathcal{S}(X_P(F))$ and (3.5), a smooth function $f \in C^\infty(X_P^0(F))$ lies in $\mathcal{S}(X_P(F))$ if and only if it satisfies the following two conditions.

**Condition 4.1.**
1. The function $f$ is right $K$-finite.
2. For each $g \in G(F)$ and $\chi \in \mathcal{O}^\times$, the integral defining $f_{\chi,s}(g)$ is absolutely convergent for $\text{Re}(s)$ sufficiently large, and
3. the section
\[
\frac{f_{\chi,s}}{\prod_{i=1}^k L(s_i + 1, \chi_s^{\lambda_i})}
\]
is holomorphic.

**Condition 4.2.** The section
\[
\frac{\left(R_{P|P^0}(f)\right)_{\chi,s}^{\text{op}}}{\prod_{i=1}^k L(-s_i, \chi_s^{\lambda_i})}
\]
is holomorphic for each $\chi \in \mathcal{O}^\times$.

Choose a norm $|\cdot|$ on $V_P(F)$ that is right invariant under $K$, which induces a norm on $X_P(F)$
\[
|\cdot| : X_P(F) \to \mathbb{R}_{\geq 0}
\]
x $\mapsto |\pi(x)|.

Normalize the norm so that
\[
|mk| = |\omega_P(m)|^{-1} \quad \text{for } m \in P(F), k \in K.
\]

Note that by the proof of [GHL21, Lemma 5.7 and Corollary 5.8], if $f$ satisfies Condition 4.1, then $f \in (L^1 \cap L^2)(X_P(F))$ and has compact support in $X_P(F)$, i.e., $f(x) = 0$ for $|x|$ sufficiently large (depending on $f$).

We will reformulate Conditions 4.1 and 4.2 in terms of asymptotics of $f(x)$ as $|x| \to 0$ in §4.1. Then restate these asymptotic conditions in terms of representations of $G(F)$ in §4.2. Our main result is Theorem 4.10 that gives a detailed description of the $G(F)$-module $\mathcal{S}(X_P(F))/\mathcal{S}(X_P^0(F))$. An illustration of Theorem 4.10 is given in §4.3.

4.1. Asymptotics toward the origin. We adopt the following notation: For $a, b \in \mathbb{R}_{\geq 0}$ and a (finite) set of elements $?$, we write $a \ll_? b$ if there exists a constant $C_? > 0$ depending on $?$ such that $a \leq C_? b$.

Recall that we have fixed a good ordering of $\Lambda = \{(\lambda_i, s_i)\}_{1 \leq i \leq k}$ and thus $\lambda_k = 1$. For each $d \in \mathbb{Z}_{\geq 0}$, define multisets, whose underlying sets are subsets of $\mathbb{C}/2\pi \sqrt{-1} \mathbb{Z},$
\[
L(d) := \sum_{\lambda, \lambda_i = d} \left\{ \frac{s_i + 1}{\lambda_i} - (s_k + 1) + \frac{2\pi \sqrt{-1}}{\log q} \mathbb{Z} : 0 \leq \mu < \lambda_i \right\},
\]
\[
L_d := \sum_{\lambda \vdash d} L(\lambda),
\]
where sums are taken in the sense of multisets (see §2.3 for conventions). From Proposition 3.2 we deduce
Lemma 4.3. For $r \in \text{Supp}(L)$, $0 \geq \text{Re}(r) > -(s_k + 1)$. Moreover, $\text{Re}(r) = 0$ if and only if $r = 0$, and $m_0(L_1) = 1$.

Let $m : \mathbb{G}_m \to M$ be a section of $\omega_P$ and $X_P^1 := \{x \in X_P(F) : |x| = 1\}$.

**Proposition 4.4.** Let $f \in C^\infty(X_P^1(F))$. Then $f$ satisfies Condition 4.1 if and only if

1. $f$ has compact support in $X_P(F)$, and
2. for $n \gg f$ and $x \in X_P^1$, we have

$$f(m(\varpi^n)^{-1}x) = \sum_{d=1}^\infty \sum_{\chi \in \hat{\mathbb{O}}^\times} \sum_{r \in \text{Supp}(L_d)} \sum_{j=1}^{m_r(L_d)} c_{r,j,\chi}(x) q^{-nr}n^j,$$

for some $c_{r,j,\chi} \in C^\infty(X_P^1)$ satisfying $c_{r,j,\chi}(m(a)^{-1}x) = \chi(a)c_{r,j,\chi}(x)$ for $a \in \mathcal{O}^\times$.

**Remark 4.5.** Since $\Lambda$ is a finite multiset, the sum in (4.3) is actually finite. Indeed, $L(d)$ and $L_d$ are empty for $d > 6$ (see charts in §A).

The key ingredient to establish this equivalence is the work of Igusa in [Igu78, §1.5], which we now review. Let $L$ be a finite multiset whose underlying set is a subset of $\mathbb{C}/\mathbb{R}_{\text{log} q} \mathbb{Z}$. We define $C_L^\infty(F^\times)$ to be the subspace of $C^\infty(F^\times)$ that consists of functions $f$ satisfying

1. $f$ has compact support in $F$, and
2. for $|a| \ll f$, we have

$$f(a) = \sum_{r \in \text{Supp}(L)} \sum_{j=1}^{m_r(L)} c_{r,j}(a)|a|^r \text{ord}(a)^j,$$

for some $c_{r,j} \in C^\infty(\mathcal{O}^\times)$.

Note that we can further write

$$c_{r,j} = \sum_{\chi \in \hat{\mathcal{O}}^\times} c_{r,j,\chi} \chi$$

for some constants $c_{r,j,\chi} \in \mathbb{C}$ that are zero for all but finitely many $\chi$. On the other hand, let $Z_L(F^\times)$ be the set of complex-valued functions $Z$ on $\hat{F}^\times$ for which

1. for every $\chi \in \hat{\mathcal{O}}^\times$, there exist constants $b_{r,j,\chi} \in \mathbb{C}$ such that

$$Z(\chi^s) - \sum_{r \in \text{Supp}(L)} \sum_{j=1}^{m_r(L)} b_{r,j,\chi}(s + r)^j$$

is a function in $\mathbb{C}[q^{-s}, q^s]$, and
2. for all but finitely many $\chi \in \hat{\mathcal{O}}^\times$, $Z(\chi^s) = 0$ for all $s$.

Igusa showed in [Igu78, Theorem 1.5.3] the following:

**Theorem 4.6.** For $f \in C_L^\infty(F^\times)$, the Mellin transform

$$M : f \mapsto \left\{ \chi \mapsto \int_{F^\times} f(a) \chi(a) d^\times a \right\},$$

originally defined on $\chi \in \hat{F}^\times$ with $\text{Re}(\chi) \gg_L 0$, extends analytically to whole $\hat{F}^\times$ and gives rise to an isomorphism between $C_L^\infty(F^\times)$ and $Z_L(F^\times)$. Moreover, the Mellin inversion $M^{-1}$ is given as follows. Given $Z \in Z_L(F^\times)$, for each $\chi \in \hat{\mathcal{O}}^\times$, let $Z_\chi(z)$ be the complex function obtained from $Z(\chi^s)$ by substituting $z$ for $q^{-s}$. Then for $a \in \mathcal{F}^\times$, $M^{-1}(Z)(a) = \frac{1}{d^\times x(\mathcal{O}^\times)} \sum_{\chi \in \hat{\mathcal{O}}^\times} \text{Res}_{z=0}(Z_\chi(z)z^{-\text{ord}(a)-1}) \chi^{-1}(a).$ (4.4)
Proof of Proposition 4.4. Clearly, both conditions imply \( f \) is right \( K \)-finite and has compact support in \( X_P(F) \), which we assume henceforth. Let \( K' \leq K \) be a compact open subgroup such that \( f \) is right \( K' \)-invariant. Let \( \{ g_i \} \) be a set of representatives of left cosets of \( K' \) in \( K \). By Iwasawa decomposition, (2)(3) in Condition 4.1 is satisfied if and only if they are satisfied for all \( g_i \). On the other hand, since the set \( \{ g_i \} \) is finite, asymptotics (4.3) can be checked at each \( g_i \) separately.

Therefore, it suffices to fix \( g \in G(F) \) and study the smooth function

\[
f_g : F^\infty \rightarrow \mathbb{C} \\
a \mapsto f(m(a)^{-1}g),
\]

Note that \( f_g \) has compact support in \( F \) and is \( \mathcal{O}^\infty \)-finite by our assumption. Moreover, for \( \chi \in \widehat{\mathcal{O}^\infty} \) by (3.7) we can write

\[
f_{\chi_s}(g) = M(f_g)(\chi_{s+s_k+1}).
\]

Then Condition 4.1 is reduced to

(2') The integral defining \( M(f_g)(\chi_s) \) is absolutely convergent for \( \text{Re}(s) \) sufficiently large, and

(3') the complex function

\[
\prod_{i=1}^{k} L(s_i + 1 - \lambda_i(s_k + 1), \chi_s^{\lambda_i})
\]

is a function in \( \mathbb{C}[q^{-s}, q^s] \).

To prove the proposition, we need to show \( f_g \) satisfies conditions (2') and (3') if and only if there exist constants \( c_{r,j,\chi} \in \mathbb{C} \) such that for \( |a| \ll f_g 1 \) we have

\[
f_g(a) = \sum_{d=1}^{\infty} \sum_{\chi \in \mathcal{O}^\infty / \text{ord}(\chi) = d} \sum_{r \in \text{Supp}(L_d)} \sum_{j=1}^{m_r(L_d)} c_{r,j,\chi} \chi(a) |a|^j \text{ord}(a)^j^{-1}.
\]

Observe that for \( \chi \in \widehat{\mathcal{O}^\infty} \) of order \( d \), we have

\[
\prod_{i=1}^{k} L(s_i + 1 - \lambda_i(s_k + 1), \chi_s^{\lambda_i}) = \prod_{r \in \text{Supp}(L_d)} \zeta(s + r)^{m_r(L_d)}.
\]

Therefore, the set

\[
\{ M(f_g) : f_g \text{ satisfying conditions (2') and (3')} \}
\]

is the subset of functions in \( Z_{L_1}(F^\infty) \) such that the associated constants \( b_{r,j,\chi} = 0 \) unless there is some \( d \in \mathbb{Z}_{>0} \) such that \( r \in \text{Supp}(L_d) \), \( j \leq m_r(L_d) \), and \( \text{ord}(\chi) = d \). By Theorem 4.6 and (4.4), this is equivalent to saying the associated constants \( c_{r,j,\chi}^{-1} \) of \( f_g \in C_{L_1}^\infty(F^\infty) \) are zero unless there is some \( d \in \mathbb{Z}_{>0} \) such that \( r \in \text{Supp}(L_d) \), \( j \leq m_r(L_d) \), and \( \text{ord}(\chi) = d \). Thus the assertion follows as \( \text{ord}(\chi) = \text{ord}(\chi^{-1}) \).

\[\square\]

For \( j \in \mathbb{Z}_{>0} \), let \( h_j \in \mathbb{Z}[z] \) be the monic polynomial of degree \( j - 1 \) such that

\[
\frac{h_j(z)}{(1-z)^j} = \sum_{i=0}^{\infty} i^{j-1} z^i.
\]
Proposition 4.7. Let $f \in C^\infty(X_1^0(F))$ satisfy equivalent conditions in Proposition 4.4. Then $f \in S(X_P(F))$ if and only if for every $d \in \mathbb{Z}_{>0}$, $r \in \text{Supp}(L_d), \chi \in \hat{O}^\times$ of order $d$, and $x^* \in X_{pop}^0(F)$, the meromorphic function

$$
m_r(L_d) \sum_{j=1}^{m_r(L_d)} h_j(q^{-s}) \zeta(s) j^{m_r(L_d)} \int_{F^x} |t|^{s-r-2k-2} \int_{X_1^P} c_{r,j,\chi}(x) \psi(|x|^{-ord(t)} \langle x, x^* \rangle) dx dt$$

has zeros at $s = 0$ of order at least $m_r(L_d)$.

To prove the proposition, we first give an alternative definition of $S(X_P(F))$ in terms of $\mathcal{F}^{geo}_{P, pop}$ instead of $\mathcal{R}_{P, pop}$.

Lemma 4.8. Suppose $f \in C^\infty(X_1^0(F))$ satisfies Condition 4.1. Then Condition 4.2 is equivalent to

1. For each $\chi \in \hat{O}^\times$ and $g \in G(F)$, the integral defining $(\mathcal{F}^{geo}_{P, pop}(f))_{\chi, s}^\text{op}(g)$ is absolutely convergent for $\frac{s_k - 1}{\chi_s} < \text{Re}(s) < s_k + 1$, and
2. the section $(\mathcal{F}^{geo}_{P, pop}(f))_{\chi, s}^\text{op} / L(s_k + 1, \chi_s^{-1}) \prod_{i=1}^{k-1} L(-s_i, \chi_s^{\lambda_s})$ is holomorphic.

Proof. Recall by definition $\mathcal{F}^{geo}_{P, pop} = 1_{(\mu_{s_k})} \circ \mathcal{R}_{P, pop}$. The sufficiency follows from the work in [GHL21, §4-§6]. For the converse, in view of [GHL21, Proposition 4.8] and its proof, it suffices to check the integral defining $(\mathcal{R}_{P, pop}(f))_{\chi, s}^\text{op}$ is holomorphic for $\text{Re}(s) > s_k + 1 - \epsilon$ for some $\epsilon > 0$ for all $\chi \in \hat{O}^\times$. Since $f \in L^1(X_P(F))$, by Fubini-Tonelli theorem one may reverse the proof of [GHL21, Theorem 6.5] to see the integral defining $(\mathcal{R}_{P, pop}(f))_{\chi, s}^\text{op}$ is absolutely convergent for $\text{Re}(s) = s_k + 1$. Since it is also absolutely convergent for $\text{Re}(s) > 0$, it is absolutely convergent for $\text{Re}(s) > s_k + 1$. As $(\mathcal{R}_{P, pop}(f))_{\chi, s}^\text{op}$ is meromorphic and vanishes for all but finitely many $\chi$, there exists $\epsilon > 0$ such that it is holomorphic for $\text{Re}(s) > s_k + 1 - \epsilon$ for all $\chi$. \hfill \Box

Proof of Proposition 4.7. Since $S(X_1^0(F)) \subset S(X_P(F))$, we may assume

$$f(m(\varpi^n)^{-1}x) = \sum_{d=1}^{\infty} \sum_{\chi \in \hat{O}^\times} \sum_{r \in \text{Supp}(L_d)} \sum_{j=1}^{m_r(L_d)} c_{r,j,\chi}(x) q^{-nr} n^{j-1}$$

for $n \geq 0$ and vanishes for $n < 0$ for all $x \in X_1^1$. We fix $(d, \chi)$ such that $\text{ord}(\chi) = d$ and further assume for $n \geq 0$ and $x \in X_1^1$

$$f(m(\varpi^n)^{-1}x) = \sum_{r \in \text{Supp}(L_d)} \sum_{j=1}^{m_r(L_d)} c_{r,j,\chi}(x) q^{-nr} n^{j-1}.$$ 

Let $x^* = P_{pop, \text{der}}(F) g \in X_{pop}^0(F)$. Viewing $c_{r,j,\chi}$ as a function in $S(X_1^0(F))$ supported on $X_1^1$, we have by Theorem 3.3

$$\mathcal{F}^{geo}_{P, pop}(c_{r,j,\chi})(x^*) = \int_{X_1^1} c_{r,j,\chi}(x) \psi(\langle x, x^* \rangle) dx.$$
By Lemma 4.3, Re\((r + s_k + 1)\) > 0 for all \(r \in \text{Supp}(L_d)\). Therefore, taking a change of variables \(t \mapsto at\), the integral
\[
\int_{F^x} \left| t^{s-k-1}a^{r+2s_k+\text{ord}(a)} \mathcal{F}_{P|P_{\text{op}}}(c_{r,j,\chi})(m(a^{-1}t)^{-1}x^*) \right| d^x a^x t
\]
\[
= \int_{|a| \leq 1} |a|^{s-k+1} \text{ord}(a)^{-1} \int_{F^x} |t|^{s-k-1} \mathcal{F}_{P|P_{\text{op}}}(c_{r,j,\chi})(m(t)^{-1}x^*) d^x t d^x a
\]
converges for \(\frac{s}{s_k} < \text{Re}(s) < s_k + 1\) by Lemma 4.8.

Thus by Fubini-Tonelli theorem and Iwasawa decomposition, for \(\eta \in \mathcal{O}^\times\)
\[
(\mathcal{F}_{P|P_{\text{op}}}(f))_{\eta_s}(g) = \sum_{r \in \text{Supp}(L_d)} \sum_{j=1}^{m_r(L_d)} h_j(q^{s-r-s_k-1}) \zeta(s + r + s_k + 1)^j (\mathcal{F}_{P|P_{\text{op}}}(c_{r,j,\chi}))_{\eta_s}(g).
\]

By Lemmas 4.3 and 4.8,
\[
(\mathcal{F}_{P|P_{\text{op}}}(f))_{\eta_s}(g) \quad \zeta(s + r + s_k + 1)^j
\]
lies in \(\mathbb{C}[q^{-s}, q^s]\) if and only if
\[
\sum_{r \in \text{Supp}(L_d)} \sum_{j=1}^{m_r(L_d)} h_j(q^{s-r-s_k-1}) \zeta(s + r + s_k + 1)^j \int_{X_p^1} ^x |t|^{s-k-1} \int_{X_p^1} ^x c_{r,j,\chi}(x)\psi(\omega^{-\text{ord}(t)}(x, x^*)) d^x t d^x x
\]
has no poles at \(s = -r - s_k - 1\) for all \(r \in \text{Supp}(L_d)\), which is equivalent to
\[
\sum_{j=1}^{m_r(L_d)} h_j(q^{s-r-s_k-1}) \zeta(s + r + s_k + 1)^{m_r(L_d)} \int_{X_p^1} ^x |t|^{s-k-1} \int_{X_p^1} ^x c_{r,j,\chi}(x)\psi(\omega^{-\text{ord}(t)}(x, x^*)) d^x t d^x x
\]
has zeros at \(s = -r - s_k - 1\) of order at least \(m_r(L_d)\) for every \(r \in \text{Supp}(L_d)\). Observe that conditions among distinct \((d, \chi)\) are independent. The assertion then follows from linearity and Lemma 4.8. \(\square\)

4.2. **An explicit characterization.** Fix \(d \in \mathbb{Z}_{\geq 0}\) with \(L_d \neq \emptyset\), \(r \in \text{Supp}(L_d)\) and \(\chi \in \mathcal{O}^\times\) of order \(d\). For \(n \in \mathbb{Z}_{\geq 0}\), consider smooth \(G(F)\)-modules \(I_{r,n,\chi}\) defined by
\[
\left\{ \begin{array}{l}
 f \in C^\infty(X_P^1(F)) : \\
 f(x) = 0 \text{ for } |x| \gg f 1,
 \end{array} \right\} / \mathcal{S}(X_P^0(F)).
\]

For ease of notation, we will write interchangeably \(f\) as the vector \((c_{r,j,\chi})_{1 \leq j \leq n}\).

**Lemma 4.9.** For \(n \in \mathbb{Z}_{\geq 0}\), we have canonical \(G(F)\)-equivariant isomorphisms
\[
\varphi_{r,n,\chi} : I_{r,n,\chi} \xrightarrow{\sim} I_P(\chi_{-r-s_k-1}).
\]
by sending \(f\) to the unique function in \(I_P(\chi_{-r-s_k-1})\) that equals \(c_{r,n,\chi}\) on \(X_P^1\).
Let $f = (c_{r,j,\chi})_{1 \leq j \leq n} \in I_{r,n,\chi}$ and $g \in G(F)$, we have
\[
R(g)f(x) := f(xg) = \sum_{j=1}^{n} |x|^r \left( \log_{1/q}(|x|) + \log_{1/q} \left( \left| \frac{|xg|}{|x|} \right| \right) \right)^{j-1} \left( \frac{|xg|}{|x|} \right)^r c_{r,j,\chi}(m(\varpi \log_q |xg|)^{-1}xg)
\]
\[
= \sum_{j=1}^{n} |x|^r \log_{1/q}^{j-1}(|x|) \sum_{i=j}^{n} \binom{i-1}{j-1} \log_{1/q}^{i-j} \left( \frac{|xg|}{|x|} \right) \left( \frac{|xg|}{|x|} \right)^r c_{r,i,\chi}(m(\varpi \log_q |xg|)^{-1}xg)
\]
for $|x|$ sufficiently small. In other words, $R(g)f = (\tilde{c}_{r,j,\chi})_{1 \leq j \leq n}$, where
\[
\tilde{c}_{r,j,\chi}(x) := \sum_{i=j}^{n} \binom{i-1}{j-1} \log_{1/q}^{i-j} \left( \frac{|xg|}{|x|} \right) \left( \frac{|xg|}{|x|} \right)^r c_{r,i,\chi}(m(\varpi \log_q |xg|)^{-1}xg).
\]
Therefore for $x \in X_P^0(F)$,
\[
\varphi_{r,n,\chi}(R(g)f)(x) = |xg|^r \varphi_{r,n,\chi}(m(\varpi \log_q |xg|)^{-1}xg) = R(g)\varphi_{r,n,\chi}(f)(x).
\]

For $n \leq m_r(L_d)$, define $G(F)$-submodules
\[
\begin{align*}
A_{r,n,\chi} &:= \{ f \in I_{r,n,\chi} : (c_{r,j,\chi})_{1 \leq j \leq n} \text{ satisfies } \text{ord}_{s=0}(4.5) \geq m_r(L_d) \}, \\
A_{r,\chi} &:= A_{r,m_r(L_d),\chi}.
\end{align*}
\]

Now we are ready to state our main result.

**Theorem 4.10.** We have an exact sequence of smooth $G(F)$-modules
\[
0 \rightarrow \mathcal{S}(X_P^0(F)) \rightarrow \mathcal{S}(X_p(F)) \rightarrow \bigoplus_{d \geq 1, L_d \neq 0} \bigoplus_{\chi \in \mathcal{O}^\times} \bigoplus_{r \in \text{Supp}(L_d)} A_{r,\chi} \rightarrow 0.
\]

Moreover, $A_{r,\chi}$ admits a natural filtration of $G(F)$-submodules
\[
0 = A_{r,0,\chi} < A_{r,1,\chi} < \cdots < A_{r,m_r(L_d),\chi} = A_{r,\chi}
\]
together with canonical $G(F)$-equivariant injections
\[
0 \neq A_{r,n,\chi}/A_{r,n-1,\chi} \hookrightarrow \cdots \hookrightarrow A_{r,2,\chi}/A_{r,1,\chi} \hookrightarrow A_{r,1,\chi} \cong \text{Ker}(R_P|_{\text{pop}}(\overline{\chi}_{r-s_k-1})) \subseteq I_P(\overline{\chi}_{r-s_k-1}).
\]
The exact sequence follows directly from Proposition 4.7 and the definition of $A_{r,\chi}$. Therefore, we are left to inspect the filtration $0 = A_{r,0,\chi} \leq A_{r,1,\chi} \leq \cdots \leq A_{r,m_r(L_d),\chi} = A_{r,\chi}$.

Define the map
\[
I_{r,1,\chi} \xrightarrow{\Phi} I_{P|_{\text{pop}}}(\overline{\chi}_{r+s_k+1})
\]
\[
(c_{r,1,\chi}) \mapsto \int_{F^s} \left| t \right|^{s-r-2s_k-2} \int_{X_P} c_{r,1,\chi}(x) \psi(\varpi^{-\text{ord}}(t)(x, \cdot)) dx \, dt \bigg|_{s=0}.
\]

Here the integral is viewed as a meromorphic section (see the proof of Proposition 4.7), and $\Phi$ is well defined by Lemma 4.8.

**Lemma 4.11.** We have a commutative diagram
\[
\begin{array}{ccc}
I_{r,1,\chi} & \xrightarrow{\varphi_{r,1,\chi}} & I_P(\overline{\chi}_{r-s_k-1}) \\
\downarrow c \Phi & & \downarrow R_P|_{\text{pop}} \\
I_{P|_{\text{pop}}}(\overline{\chi}_{r+s_k+1}) & \xrightarrow{c \Phi} & I_{P|_{\text{pop}}}(\overline{\chi}_{r+s_k+1})
\end{array}
\]
where $c$ is a nonzero constant. In particular, $\Phi$ is $G(F)$-equivariant and $\varphi_{r,1,\chi}$ restricts to an isomorphism

$$A_{r,1,\chi} = A_{r,\chi} \cap I_{r,1,\chi} \xrightarrow{\sim} \text{Ker} \left( R_{P|\text{pop}}(\overline{\chi}_{r-s_k-1}) \right).$$

**Proof.** By [GHL21, Proposition 4.8] and (3.5), we have an identity of meromorphic sections

$$\int_{P \times X_p} (t)|t|^{s-r-2s_k-2} \int_{X_p} c_{r,1,\chi}(x) \psi(t^{-1} \langle x, \gamma \rangle) dx dt$$

$$= \left( [\psi]_{\mu_{s_k}} \circ R_{P|\text{pop}} \left( c_{r,1,\chi}^1 X_p^1 \right) \right)_{\overline{\chi}_{r-s_k-1}}$$

$$= \gamma(-s_k, \overline{\chi}_{r-s_k-1}, \psi)^{-1} R_{P|\text{pop}} \left( c_{r,1,\chi}^1 X_p^1 \right)_{\overline{\chi}_{r-s_k-1}}$$

where $\gamma(-s_k, \overline{\chi}_{r-s_k-1}, \psi)$ is the Tate $\gamma$-factor. By Lemma 4.3, $\gamma(-s_k, \overline{\chi}_{r-s_k-1}, \psi)$ is a nonzero constant, and evaluating at $s = 0$ we have

$$\Phi((c_{r,1,\chi})) = \gamma(-s_k, \overline{\chi}_{r-s_k-1}, \psi)^{-1} R_{P|\text{pop}} \circ \varphi_{r,1,\chi}((c_{r,1,\chi})).$$

□

**Lemma 4.12.** The map $\Phi$ is nonzero.

**Proof.** We may assume the conductor of $\psi$ is $\mathcal{O}$. It suffices to show there exists $(c_{r,1,\chi}) \in I_{r,1,\chi}$ such that $\Phi((c_{r,1,\chi})(\text{Id}) \neq 0$. Choose a compact open subgroup $K' \leq K$ sufficiently small such that $\omega_P(K' \cap P(F)) \leq (\mathcal{O}^\times)^d$ and $(v_P g, v_P^{\text{pop}}) \in \ker(\chi) \cap \mathcal{O}^\times$ for all $g \in K'$. Let $c_{r,1,\chi}$ be the unique function supported on $P_{\text{der}}(F)m(\mathcal{O}^\times)K'$ such that for $a \in \mathcal{O}^\times$ and $g \in K'$,

$$c_{r,1,\chi}(m(a)^{-1} g) = \chi(a).$$

Then we have

$$\int_{X_p} c_{r,1,\chi}(x) \psi((\varpi^{-\text{ord}(t)} \langle x, \text{Id} \rangle) dx$$

is up to a nonzero constant

$$\int_{\mathcal{O}^\times} \chi(a) \psi((\varpi^{-\text{ord}(t)} a) d \chi$$

which is the well-known Gauss sum

$$\mathcal{G}(\varpi^{-\text{ord}(t)} \chi) = \begin{cases} 1 & \text{if } \chi = 1, |t| \geq 1, \\ \zeta(-1) & \text{if } \chi = 1, |t| = q^{-1}, \\ 0 & \text{if } c(\chi) = \text{ord}(t) > 0, \\ 0 & \text{otherwise}, \end{cases}$$

where $c(\chi)$ is the conductor of $\chi$. Therefore, $\Phi((c_{r,1,\chi})(\text{Id})$ is up to a nonzero constant

$$\begin{cases} \zeta(r + 2s_k + 2) + \zeta(-1) q^{-2s_k + 2} & \text{if } \chi = 1, \\ q^{-c(\chi)(r-s_k-2)} \mathcal{G}(\varpi^{-c(\chi)} \chi) & \text{if } \chi \neq 1, \end{cases}$$

which is nonzero since $r + 2s_k + 2 > 1$. □

**Proof of Theorem 4.10.** The nontriviality statement

$$A_{r,m_r(L_d),1,\chi}/A_{r,m_r(L_d)-1,\chi} \neq 0$$

will be proved in Corollary 5.6. By Lemmas 4.11 and 4.12, it remains to show for $m_r(L_d) \geq n \geq 2$ there is a (canonical) $G(F)$-equivariant injection

$$A_{r,n,\chi}/A_{r,n-1,\chi} \hookrightarrow A_{r,n-1,\chi}/A_{r,n-2,\chi}.$$
Suppose \( (c_{r,j,\chi})_{1 \leq j \leq n} \in A_{r,n,\chi} \), i.e.,
\[
\sum_{j=1}^{n} h_j(q^{-s}) \zeta(s)^j \int_{F^x} |t|^{s-r-2s_k-2} \int_{X_{S_k}^1} c_{r,j,\chi}(x) \psi(\omega^{\ord(t)}(x,x^*)) dx dt
\]
has at least order \( n \) at \( s = 0 \) for all \( x^* \in X_{P_{\text{op}}}(F) \). Therefore,
\[
\sum_{j=2}^{n} h_j(q^{-s}) \zeta(s)^j \int_{F^x} |t|^{s-r-2s_k-2} \int_{X_{S_k}^1} c_{r,j,\chi}(x) \psi(\omega^{\ord(t)}(x,x^*)) dx dt
\]
has at least order \( n-1 \) at \( s = 0 \) for all \( x^* \in X_{P_{\text{op}}}(F) \). Note that for \( j \geq 2 \), we have the identity
\[
h_j(z) = z(1-z) \frac{dh_{j-1}}{dz}(z) + (j-1)zh_{j-1}(z).
\]
Thus, (4.8) can be written as
\[
\sum_{j=1}^{n-1} \left( 1 + \frac{h_j'(q^{-s})\zeta(s)^{-1}}{j h_j(q^{-s})} \right) q^{-s} h_j(q^{-s}) \zeta(s)^j \int_{F^x} |t|^{s-r-2s_k-2} \int_{X_{S_k}^1} j c_{r,j+1,\chi}(x) \psi(\omega^{\ord(t)}(x,x^*)) dx dt
\]
As \( h_j \) has positive coefficients (which follows from (4.9) and induction),
\[
\frac{h_j'(q^{-s})\zeta(s)^{-1}}{j h_j(q^{-s})} \in \mathbb{C}(q^{-s})
\]
has a simple zero at \( s = 0 \). Therefore, by considering the expansion of the function (4.10) in \( \zeta(s)^{-1} \) at \( s = 0 \), we can find inductively (unique) constants \( b_{m,j} \in \mathbb{C} \) such that
\[
c'_{r,n-1,\chi} := (n-1)c_{r,n,\chi} \quad \text{and} \quad c'_{r,m,\chi} := mc_{r,m+1,\chi} + \sum_{j=m+1}^{n-1} j b_{m,j} c_{r,j+1,\chi} \quad \text{for } 1 \leq m \leq n-2
\]
satisfies \( (c'_{r,m,\chi})_{1 \leq m \leq n-1} \in A_{r,n-1,\chi} \).

Therefore, we have a linear map
\[
\Psi : A_{r,n,\chi} \to A_{r,n-1,\chi}
\]
\( (c_{r,j,\chi})_{1 \leq j \leq n} \mapsto (c'_{r,j,\chi})_{1 \leq j \leq n-1} \).

that induces an injection \( A_{r,n,\chi}/A_{r,n-1,\chi} \hookrightarrow A_{r,n-1,\chi}/A_{r,n-2,\chi} \). We claim the map is \( G(F) \)-equivariant. Indeed, by (4.6) and (4.11) we have for \( g \in G(F) \),
\[
\Psi(R(g)(c_{r,j,\chi})_{1 \leq j \leq n}) = R(g)\Psi((c_{r,j,\chi})_{1 \leq j \leq n}) \in A_{r,n-2,\chi}.
\]
Thus the theorem follows. \( \square \)

4.3. The Lagrangian Grassmannian. We close this section with an example of Theorem 4.10 applying to the case \( G = \text{Sp}_{2n} \) and \( P \) is the Siegel parabolic. In this case, by Appendix A we have
\[
L(1) = \{0\},
\]
\[
L(2) = \{-i + j \frac{n}{2 \log q} \mathbb{Z} : 1 \leq i \leq \lfloor \frac{n}{2} \rfloor, 0 \leq j \leq 1\},
\]
\[
s_k = \frac{n+1}{2}.
\]
Therefore, \( L_1 \) and \( L_2 \) are indeed sets. By Theorem 4.10, we have
\[
S(X_P(F))/S(X_{P_{\text{op}}}(F)) \cong \bigoplus_{d=1}^{2} \bigoplus_{\chi \in \mathbb{O}_{\chi}^*} \bigoplus_{r \in L_d} \text{Ker}(R_{P_{\text{op}}}(\mathbb{X}_{r-n+1})).
\]
When \( F \) is of characteristic zero, by the work of Kudla and Rallis [KR92], the locus

\[
\{ \chi_{-r-n+1/2} : \text{ord}(\chi) = d, r \in L_d \}
\]

is precisely the points of reducibility of \( \mathcal{R}_{P|P_{op}} \) in the left half-plane. Moreover, we have

\[
\bigoplus_{d=1}^2 \bigoplus_{\chi \in \mathcal{O}_K \atop \text{ord}(\chi) = d} \ker(\mathcal{R}_{P|P_{op}}(\chi_{-r-n+1/2})) \cong \bigoplus_{V}(\mathcal{S}(V^n(F))_{O(V)(F)}
\]

where \( V \) ranges over all equivalence classes of nondegenerate even dimensional quadratic spaces over \( F \) of dimension not larger than \( n \), and \( \mathcal{S}(V^n(F))_{O(V)(F)} \) is the space of \( O(V)(F) \)-coinvariants of the Weil representation of \( G = \text{Sp}_{2n} \) realized on \( \mathcal{S}(V^n(F)) \).

5. Poles of Intertwining Operators

In this section we prove the inclusion \( \mathcal{S}(X^*_P(F)) < \mathcal{S}(X_P(F)) \) by studying more generally poles of intertwining operators following the approach of [Ike92, §1]. We first recall some general results on intertwining operators discussed in §1.2 in loc. cit. Our main references are [Sha81, Sha10].

Let \( B = TN \) be the standard Borel subgroup with respect to our fixed pinning in §2, \( N^{op} \) be the opposite of \( N \), and \( W := W_G \) be the Weyl group of \( (G, T) \). For each \( w \in W \), we choose a representative in \( K \), which we continue to denote by \( w \).

For a quasi-character \( \tilde{\chi} \) of \( T(F) \), let \( I_B(\tilde{\chi}) := \text{Ind}_{B(F)}^{G(F)}(\tilde{\chi}) \) be the normalized induced representation in the category of smooth representations. Here we have identified \( \tilde{\chi} \) as a quasi-character of \( B(F) \). For \( w \in W \), let \( \tilde{\chi}^w(t) := \tilde{\chi}(wt^w) \) and \( N_w := N \cap wN^{op}w^{-1} \). The (unnormalized) intertwining operator \( M_w \) is defined as

\[
M_w(\tilde{\chi}) : I_B(\tilde{\chi}) \rightarrow I_B(\tilde{\chi}^w)
\]

\[
f \mapsto \left( g \mapsto \int_{N_w(F)} f(w^{-1}ug)du \right).
\]

Here the measure \( du \) on \( N_w \) is defined as in §2.2. The integral defining \( M_w(\tilde{\chi})f(g) \) converges absolutely for all \( g \in G(F) \) for \( \tilde{\chi} \) in some open cone, and extends meromorphically to all \( \tilde{\chi} \).

For a simple root \( \alpha \in \Delta \), let \( \iota_\alpha : SL_2 \rightarrow G \) be the homomorphism determined by the fixed pinning. Let \( \tilde{w}_0 \) be the nonidentity element in \( W_{SL_2} \). We can and do assume \( \iota_\alpha(\tilde{w}_0) = s_\alpha \) is the simple reflection defined by \( \alpha \). Then for \( f \in I_B(\tilde{\chi}) \), we have

\[
\iota_\alpha^*(M_{s_\alpha}(\tilde{\chi})f) = M_{\tilde{w}_0}(\iota_\alpha^*(\tilde{\chi})(\iota_\alpha^*f)).
\]

For \( w \in W \), let \( \ell(w) \) be the length of \( w \). If \( w_1, w_2 \in W \) satisfies \( \ell(w_1w_2) = \ell(w_1) + \ell(w_2) \), then \( M_{w_1w_2}(\tilde{\chi}) = M_{w_1}(\tilde{\chi}^{w_2}) \circ M_{w_2}(\tilde{\chi}) \). Note that by definition \( M_w \) commutes with right translations, and thus to study poles of \( M_w \), it suffices to have a good understanding for the case \( G = SL_2 \), which we now recall.

**Theorem 5.1.** Let \( G = SL_2 \) and \( \tilde{w}_0 \) be the long Weyl element. Then the intertwining operator

\( M_{\tilde{w}_0}(\tilde{\chi}_s) : I_B(\tilde{\chi}_s) \rightarrow I_B(\tilde{\chi}_s^{-1}) \) has following properties.

(i) \( L(0, \tilde{\chi}_s)^{-1}M_{\tilde{w}_0}(\tilde{\chi}_s) \) is holomorphic and has nonzero images for all \( s \).

(ii) The kernel of \( M_{\tilde{w}_0}(1_{-1}) \) is the trivial representation.

(iii) \( \text{Res}_{s=0} M_{\tilde{w}_0}(1_s) \) is a nonzero scalar multiplication.

(iv) The image of \( M_{\tilde{w}_0}(1_{1}) \) is the trivial representation.

(v) \( \gamma(s, \tilde{\chi}, \psi)^\gamma(-s, \tilde{\chi}^{-1}, \psi)M_{\tilde{w}_0}(\tilde{\chi}_s^{-1})M_{\tilde{w}_0}(\tilde{\chi}_s) = 1_d. \)
(vi) Suppose the conductor of $\psi$ is $\mathcal{O}$. Let $\phi_s$ be the unique $\text{SL}_2(\mathcal{O})$-invariant section in $I_B(1_s)$ such that $\phi_s|_{\text{SL}_2(\mathcal{O})} = 1$. Then
\[ M_{\widetilde{w}_0}(1_s)\phi_s = \frac{L(0, 1_s)}{L(1, 1_s)}\phi_s. \]

**Proof.** See [Ike92, §1.2]. Note that there is a typo in (1.2.3) in loc. cit.. We refer one to [GH20, Lemma 3.11] for a corrected statement. 

**Remark 5.2.** Since $\gamma(s, \tilde{\chi}, \psi)$ equals $L(1, \tilde{\chi}_s^{-1})/L(0, \tilde{\chi}_s)$ up to a nowhere vanishing function, (v) implies
\[ \frac{L(1, \tilde{\chi}_s^{-1})L(1, \tilde{\chi}_s)}{L(0, \tilde{\chi}_s)L(0, \tilde{\chi}_s')} M_{\widetilde{w}_0}(\tilde{\chi}_s) \]
is a $\text{SL}_2(F)$-equivariant isomorphism at all $s$.

When considering the quotient $W/W_M$, we always choose representatives of minimal length in each left coset of $W_M$. Let $w_0 \in W/W_M$ be (the representative of) the long Weyl element. For $w \in W/W_M$, $\chi \in \mathcal{O}^\times$, and $s \in \mathbb{C}/2\pi \mathbb{Z}$, define
\[ \Phi_w \gamma := \{ \beta \gamma \in (\Phi \gamma)^+ : w\beta \gamma \in (\Phi \gamma)^- \} \subset (\Phi \gamma)^+ - (\Phi \gamma)_M^+, \]
\[ m_w(h, \lambda) := \left| \left\{ \beta \gamma \in \Phi_w : \sum_{\alpha \in \Delta} \langle \omega_\alpha, \beta \gamma \rangle = h, \langle \omega_P, \beta \gamma \rangle = \lambda \right\} \right| \text{ for } h, \lambda \in \mathbb{Z}, \]
\[ c_w(\chi_s) := \prod_{\beta \gamma \in \Phi_w} \frac{L(-\sum_{\alpha \in \Delta} \langle \omega_\alpha, \beta \gamma \rangle, \chi_{s+\ell+1})}{L(1 - \sum_{\alpha \in \Delta} \langle \omega_\alpha, \beta \gamma \rangle, \chi_{s+\ell+1})} \]
\[ = \prod_{\lambda=1}^{\infty} \prod_{h=1}^{\infty} \frac{L(-h, \chi_{s+\ell+1})^{\max(0, m_w(h, \lambda) - m_w(h+1, \lambda))}}{L(1 - h, \chi_{s+\ell+1})^{\max(0, m_w(h, \lambda) - m_w(h+1, \lambda))}}. \]
\[ a_w(\chi_s) := \prod_{\lambda=1}^{\infty} \prod_{h=1}^{\infty} L(-h, \chi_{s+\ell+1})^{\max(0, m_w(h, \lambda) - m_w(h+1, \lambda))}. \]

Note that $I_P(\chi_s) < I_B(\tilde{\chi}_s)$, where
\[ \tilde{\chi}_s := \delta_P^{1/2} \delta_B^{-1/2} \cdot \chi \circ \omega_P \cdot |\omega_P|^s = \left( \prod_{\alpha \in \Delta} |\omega_\alpha|^{-1} \right) \cdot \chi \circ \omega_P \cdot |\omega_P|^{s+\ell+1}, \quad (5.2) \]
and the restriction of $M_w(\tilde{\chi}_s)$ to $I_P(\chi_s)$, denoted by $M_w(\chi_s)$, is well defined outside a finite set of $s$. Also observe that, up to a conjugation by an element of $K$, $M_{w_0}$ is essentially equal to $\mathcal{R}_{P|P_P}$. In the case $P$ is self-associate, one may choose the element to be $w_0$ (see [GHL21, §5.3]).

The goal of this section is to prove

**Theorem 5.3.** The intertwining operator
\[ M_w(\chi_s) := a_w(\chi_s)^{-1} M_w(\chi_s). \]
is holomorphic. Moreover, for all $h, \lambda \in \mathbb{Z}_{>0}$,
\[ \max(0, m_w(h, \lambda) - m_w(h-1, \lambda)) \leq \max(0, m_{s+w}(h, \lambda) - m_{s+w}(h-1, \lambda)) \quad (5.4) \]
for every $w \in W/W_M$ and $\alpha \in \Delta$ such that $\ell(s_\alpha w) = \ell(w) + 1$. Consequently,
\[ d(\chi_s) := \prod_{\lambda=1}^{\infty} \prod_{h=1}^{\infty} L\left(1 - h, \chi_{s+\ell+1}\right)^{\max(0, m_{w_0}(h, \lambda) - m_{w_0}(h-1, \lambda))}. \]
is the least common denominator of $c_w(\chi_s)$ for all $w \in W/W_M$.

Our proof of Theorem 5.3 requires case-by-case discussion; we will prove the theorem for classical groups and $G = G_2$ in §5.1-§5.5. Theorem 5.3 together with the following lemma has several consequences.

**Lemma 5.4.** We have

$$a_{w_0}(\chi_s) = a_{P|P_{op}}(\chi_s) = \prod_{i=1}^k L(-s_i, \chi_s^{\lambda_i})$$

$$d(\chi_s) = a_{P|P}(\chi_s) = \prod_{i=1}^k L(1+s_i, \chi_s^{\lambda_i}).$$

**Proof.** Both identities can be easily verified which we leave to reader. For a list of positive (co)roots see e.g., [Bou02] for exceptional groups, and see §5.2-§5.5 for classical groups; for a list of multiset $\Lambda = \{(s_i, \lambda_i)\}$, see [GHL21, Appendix A]. We alert the reader that the result in loc. cit is stated in the dual side. \qed

First, we rephrase our definition of good sections in the sense of [Yam14, Definition 3.1].

**Corollary 5.5.** Let $\chi \in \hat{O}$. A meromorphic section $f^{(s)} \in I_P(\chi_s)$ is good if and only if $f^{(s)}$ has no poles for $\Re(s) > -\frac{1}{6}$ and $d(\chi_s^{-1})a_{w_0}(\chi_s)^{-1}M_{w_0}(\chi_s)f^{(s)}$ has no poles for $\Re(s) < 0$.

**Proof.** Sufficiency follows from Remark 4.5. For the converse, by the assumption of $f^{(s)}$ and Theorem 5.3, the meromorphic section

$$\frac{M_{w_0}(\chi_s)f^{(s)}}{a_{w_0}(\chi_s)}$$

has no poles for $\Re(s) > -\frac{1}{6}$ or $\Re(s) < 0$. Thus it is holomorphic. On the other hand, by (5.1) and (v) we can write

$$f^{(s)} = h(s) \frac{d(\chi_s)d(\chi_s^{-1})}{a_{w_0}(\chi_s)^{-1}a_{w_0}(\chi_s)}M_{w_0}(\chi_s)^{-1}M_{w_0}(\chi_s)f^{(s)}$$

for some nowhere vanishing function $h(s) \in \mathbb{C}[q^{-s}, q^s]$. Then by assumption of $f^{(s)}$ and Theorem 5.3

$$\frac{f^{(s)}}{d(\chi_s)} = h(s) \frac{M_{w_0}(\chi_s^{-1})}{a_{w_0}(\chi_s)^{-1}} \frac{d(\chi_s^{-1})}{a_{w_0}(\chi_s)}M_{w_0}(\chi_s)f^{(s)}$$

has no poles for $\Re(s) < 0$. Thus $d(\chi_s)^{-1}f^{(s)}$ is holomorphic, and $f^{(s)}$ is a good section. \qed

By the equivalent definitions of good sections, we obtain from [Yam14, Proposition 3.1] the following.

**Corollary 5.6.**

(a) $S(X_P(F)) = S_{\text{BK}}(X_P(F)) := S(X_P^{\circ}(F)) + F_{P_{op}|P}(S(X_P^{\circ}(F)))$.

(b) Let $d \in \mathbb{Z}_{>0}$ with $L_d \neq \emptyset$. The module $A_{r,m_r(L_d),\chi}/A_{r,m_r(L_d)-1,\chi}$ is nonzero for all ord$(\chi) = d$ and $r \in \text{Supp}(L_d)$.

(c) $\mathbb{C} \subseteq A_{0,1,1} = A_{0,1}$ and thus $S := S(V_P(F))|_{X_P^{\circ}(F)} \leq S(X_P(F))$.

**Proof.** By the Mellin inversion, sections that are holomorphic for all $\chi \in \hat{O}$ correspond bijectively to functions in $S(X_P^{\circ}(F))$. Therefore by [GHL21, Theorem 5.12] the first statement follows from the proof of [Yam14, Proposition 3.1 (4)]. The second statement is [Yam14, Proposition 3.1 (3)]. For the last statement, since the Schwartz space $S(X_P(F))$ is independent of $\psi$, we may assume the conductor of $\psi$ is $\mathcal{O}$. Let $f$ be the unique right $G(\mathcal{O})$-invariant function in $C^\infty(X_P^{\circ}(F))$ such that $(f)|_{G(\mathcal{O})} = d(1_s)$. Then by Gindikin-Karpelevič formula, we have
$f \in \mathcal{S}(X_P(F))$. Since $d(1_0) \neq 0$, the image of $f$ in $A_{0,1}$ under the exact sequence in Theorem 4.10 is nonzero. As $f$ is right $G(O)$-invariant, the image is a nonzero constant. \hfill \Box

The following corollary generalizes [Ike92, Lemma 1.2].

**Corollary 5.7.** For $f \in \mathcal{S}(X_P(F))$, the section

$$(d(\chi_s)c_w(\chi_s))^{-1} M_w(\chi_s)f_{\chi_s}$$

(5.5)

is holomorphic for any $w \in W/W_M$ and $\chi \in \hat{O}^\times$.

**Proof.** Replace $\chi_s$ with $\chi_s^{-1}$. By Theorem 5.3, Corollary 5.6(a), and [GHL21, Theorem 5.12], we may replace $f_{\chi_s}$ with

$$\frac{d(\chi_s^{-1})}{d(\chi_s^{-1})} M_w(\chi_s) f_{\chi_s} \text{ where } f \in C_c^\infty(X_{w_0}^{\varepsilon}(F)),$$

Write $w_0 = ww'$ such that $\ell(w_0) = \ell(w) + \ell(w')$. Then we can rewrite (5.5) as

$$\frac{1}{c_w(\chi_s^{-1})} M_w(\chi_s^{-1}) M_w(\chi_s) f_{\chi_s},$$

By (5.1) and (v), this section has the same poles (counting multiplicities) as

$$\frac{1}{d(\chi_s)c_w(\chi_s)} M_w(\chi_s)f_{\chi_s},$$

which is holomorphic by Theorem 5.3. \hfill \Box

**Remark 5.8.** In [GL21] their notion of a Schwartz function $f$ is defined so that

$$(d(\chi_s)c_w(\chi_s))^{-1} M_w(\chi_s)f_{\chi_s},$$

is holomorphic for all $w \in W/W_M$ and $\chi$. By Corollary 5.7 it is enough to check for $w = \text{Id}, w_0$, and thus the Schwartz spaces in [GL21] are equal to $\mathcal{S}(X_P(F))$ in the nonarchimedean case.

### 5.1. Proof outline of Theorem 5.3

In this subsection, we discuss in detail the proof of Theorem 5.3 for the case $G = G_2$ and explain how to adapt it for classical groups. We record the following lemma and notations for later use (see e.g., [Bou02]).

**Lemma 5.9.** Let $w \in W/W_M$ and $w = w_m \cdots w_1$ be a reduced expression in $W$. For each $i$, let $\alpha(i) \in \Delta$ be the simple root such that $w_i$ is the corresponding reflection. Then

$$\Phi^\vee_w = \{ \hat{\alpha}^\vee_i : = w_1 \cdots w_{i-2} w_{i-1} \alpha^\vee_i : 1 \leq i \leq m \}.$$

We say the coroot $\hat{\alpha}^\vee_i$ corresponds to the simple reflection $w_i$ (under the reduced expression).

Recall $M_w(\chi_s)$ defined in (5.3).

**Lemma 5.10.** Let $w \in W/W_M$. Let $\alpha$ be a simple root such that $\ell(s_{\alpha} w) = \ell(w) + 1$, and $(h, \lambda)$ be the unique pair such that $m_{s_{\alpha}w}(h, \lambda) - m_{w}(h, \lambda) = 1$. Suppose $M'_w(\chi_s)$ is holomorphic.

Assume

$$m_w(h, \lambda) \geq m_w(h + 1, \lambda).$$

Then $M'_{s_{\alpha} w}(\chi_s)$ is holomorphic if either $\chi^\lambda \neq 1$ or $\chi^\lambda = 1$ and one of the following holds.

1. $m_w(h, \lambda) \geq m_w(h - 1, \lambda)$.
2. $M'_w(\chi_s)f^{(s)}|_{\lambda^{(s_{\alpha} h_{s_k} + 1) = h - 1}}$ is left $\iota_{\alpha}(SL_2)$-invariant for all holomorphic sections $f^{(s)}$ of $I_P(\chi_s)$.
Proof. By (5.1), (5.2), and (i),

\[ L \left( - \sum_{\beta \in \Delta} \langle w\omega_\beta, \alpha^\vee \rangle, \chi_{s+k+1} \right)^{-1} M_{s\alpha} (\chi_s) \left( M'_{w}(\chi_s) f^{(s)} \right) \]

is holomorphic. By Lemma 5.9

\[ L \left( - \sum_{\beta \in \Delta} \langle w\omega_\beta, \alpha^\vee \rangle, \chi_{s+k+1} \right) = L \left( - \sum_{\beta \in \Delta} \langle \omega_\beta, w^{-1}\alpha^\vee \rangle, \chi_{s+k+1} \right) = L(-h, \chi_{s+k+1}). \]

Thus by definition of \( a_{s\alpha}(\chi_s) \) we may assume \( \chi = 1 \) and (I) fails, so that by assumption (5.6)

\[ a_{s\alpha}(\chi_s) = \frac{\zeta (\lambda(s+k+1) - (h-1))}{\zeta (\lambda(s+k+1) - h)} a_w(\chi_s). \]

By (II) and (ii),

\[ M_{s\alpha}(\chi_s) M'_{w}(\chi_s) f^{(s)} |_{\lambda(s+k+1)=h-1} = 0, \]

and thus \( M'_{s\alpha}(\chi_s) f^{(s)} \) is holomorphic. \( \square \)

Note that for every \( w \in W/W_M \) there is a reduced expression \( w_m \cdots w_2w_1 \) of \( w \) such that \( w = w_r \cdots w_2w_1 \) for some \( r \leq m \), and a reduced expression can be transformed to another expression by performing a sequence of defining relations of the Weyl group \( W \). Explicitly, for two distinct simple (co)roots \( \alpha \) and \( \alpha' \), let \( s_\alpha, s_{\alpha'} \) be the corresponding simple reflections and let \( n_{\alpha\alpha'} \) be the number of edges between the corresponding nodes in the Dynkin diagram.

(a) If \( n_{\alpha\alpha'} = 0 \), then replace \( s_\alpha s_{\alpha'} \) with \( s_{\alpha'} s_\alpha \);
(b) if \( n_{\alpha\alpha'} = 1 \), then replace \( s_\alpha s_{\alpha'} s_\alpha \) with \( s_{\alpha'} s_\alpha s_{\alpha'} \);
(c) if \( n_{\alpha\alpha'} = 2 \), then replace \( (s_\alpha s_{\alpha'})^2 \) with \( (s_{\alpha'} s_\alpha)^2 \);
(d) if \( n_{\alpha\alpha'} = 3 \), then replace \( (s_\alpha s_{\alpha'})^3 \) with \( (s_{\alpha'} s_\alpha)^3 \).

The strategy to prove Theorem 5.3 is as follows. We will choose a reduced expression \( w_m \cdots w_1 \) of \( w_0 \) and list corresponding coroots \( \tilde{\alpha}'_{(i)} \) (see Lemma 5.9). For each case, we will first verify the combinatorics inequalities (5.4) and (5.6) for any \( w \in W/W_M \) by studying the effects of operations above on the order of \( \tilde{\alpha}'_{(i)} \). Then we apply Lemma 5.10 repeatedly to show, inductively on the length of \( w \), that the invariance property (II) holds whenever (I) fails.

In the rest of the section, we will use the Bourbaki numbering of the Dynkin diagram and (co)roots. Let \( P := P_\ell \) be the maximal parabolic subgroup associated to the \( \ell \)th node of the Dynkin diagram of \( (G, B, T) \). Let \( \alpha_i \) denote the simple root attached to the \( i \)th node and let \( s_i \) be the corresponding simple reflection. However, note that we also write \( s_k \) for the highest data. This should lead to little or no confusion as the subscript \( k \) is only used as the size of the multiset \( \Lambda \) throughout the paper.

Proof of Theorem 5.3 for \( G = G_2 \). Observe that \( M_{\text{der}} = \text{SL}_2 \) for both cases and \( |W/W_M| = 6 = \ell(w_0) + 1 \), so that \( w_0 \in W/W_M \) has a unique reduced expression. Inequalities (5.4) and (5.6) will be clear from the order of coroots below.

For \( \ell = 1 \), the reduced expression of \( w_0 \) is

\[ s_1s_2s_1s_2s_1; \]

corresponding coroots \( \tilde{\alpha}'_{(i)} \) are

\[ \alpha_1^\vee, \alpha_1^\vee + \alpha_2^\vee, 2\alpha_1^\vee + 3\alpha_2^\vee, \alpha_1^\vee + 2\alpha_2^\vee, \alpha_1^\vee + 3\alpha_2^\vee. \]

Since (I) holds for \( w = \text{Id}, s_2s_1 \), we may assume \( \chi = 1 \), and we are left to verify (II) holds for \( w = s_1, s_1s_2s_1, s_2s_1s_2s_1 \).
A holomorphic section $f^{(s)}$ of $I_{P_1}(\chi_s)$ is left $\iota_{\alpha_2}(\text{SL}_2)$-invariant, and so is

$$M'_{s_1}(1_s)f^{(s)}|_{s+s_k+1=1}$$

by (iii), which justifies the case $w = s_1$. For $w = s_1s_2s_1$, by (iv)

$$M'_{s_1}(\chi_s)f^{(s)}|_{s+s_k+1=2}$$

is left $\iota_{\alpha_1}(\text{SL}_2)$-invariant, and so is

$$M'_{s_2s_1}(\chi_s)f^{(s)}|_{s+s_k+1=2}$$

by (iii). As (I) holds for $s_2s_1$, by (ii)

$$M'_{s_1s_2s_1}(1_s)f^{(s)}|_{s+s_k+1=2} = 0.$$ 

This proves the case $w = s_1s_2s_1$. Finally for $w = s_2s_1s_2s_1$, by (iv)

$$M'_{s_1s_2s_1}(1_s)f^{(s)}|_{s+s_k+1=3}$$

is left $\iota_{\alpha_1}(\text{SL}_2)$-invariant, and thus so is

$$M'_{s_2s_1s_2s_1}(1_s)f^{(s)}|_{s+s_k+1=3}$$

by (iii).

For $\ell = 2$, the reduced expression of $w_0$ is

$$s_2s_1s_2s_1s_2;$$

corresponding coroots $\tilde{\alpha}^{\vee}_{(i)}$ are

$$\alpha^\vee_2, \alpha^\vee_1 + 3\alpha^\vee_2, \alpha^\vee_1 + 2\alpha^\vee_2, 2\alpha^\vee_1 + 3\alpha^\vee_2, \alpha^\vee_1 + \alpha^\vee_2.$$ 

Since (I) holds for $w = \text{Id}, s_2, s_1s_2$, we may assume $\chi^3 = 1$, and we are left to verify (II) holds for $w = s_2s_1s_2, s_1s_2s_1s_2$.

A holomorphic section $f^{(s)}$ of $I_{P_3}(\chi_s)$ is left $\iota_{\alpha_1}(\text{SL}_2)$-invariant. Since

$$L(-4, \chi^3_{s+s_k+1})^{-1}M_{s_1}(\chi^s_s)|_{s+(s+s_k+1)=4}$$

is a nonzero scalar multiplication by (iii) and $s_2s_2 = \text{Id}$, we have

$$M'_{s_2s_1s_2}(\chi_s)f^{(s)}|_{s+(s+s_k+1)=4}$$

is left $\iota_{\alpha_1}(\text{SL}_2)$-invariant, which justifies the case $w = s_2s_1s_2$. For $w = s_1s_2s_1s_2$, it suffices to consider $\chi = 1$. By (iii)

$$M'_{s_2}(\chi_s)f^{(s)}|_{s+s_k+1=1}$$

is left $\iota_{\alpha_1}(\text{SL}_2)$-invariant. As (I) holds for $s_2$ and $s_1s_2$, by (ii)

$$M'_{s_1s_2}(\chi_s)f^{(s)}|_{s+s_k+1=1} = 0,$$

and hence

$$M'_{s_1s_2s_1s_2}(\chi_s)f^{(s)}|_{s+s_k+1=1} = 0.$$ 

Arguments are more complicated for classical groups (also for groups of type $E, F$) as in general there are more than one reduced expression of $w_0$. We closed this subsection by studying the effects of (a), (b) and (c) on the order of coroots $\tilde{\alpha}^{\vee}_{(i)}$.

Let $w_0 = w_m \cdots w_1$ be a reduced expression.

(a') If $n_{\alpha_{(i+1)}}\alpha_{(i)} = 0$ and $w_{i+1}w_i$ is replaced with $w_iw_{i+1}$, then the position of the coroots $\tilde{\alpha}^{\vee}_{(i)}$ and $\tilde{\alpha}^{\vee}_{(i+1)}$ are swapped.
If \( n_{\alpha(i+1)\alpha(i)} = 1 \), \( \alpha(i) = \alpha(i+2) \), and \( w_{i+2}w_{i+1}w_i \) is replaced with \( w_{i+1}w_iw_{i+1} \), then the position of the coroots \( \tilde{\alpha}_i \) and \( \tilde{\alpha}_{i+2} \) are swapped.

Moreover, \( \alpha^\vee_{i+1} = \alpha^\vee_{i+2} + \alpha^\vee_{(i)} \), so
\[
\langle \omega_p, \alpha^\vee_{(i+1)} \rangle \geq 2
\]
and thus such operations will not occur if \( G \) is of type \( A \).

If \( n_{\alpha(i+1)\alpha(i)} = 2 \), \( \alpha(i) = \alpha(i+2) \) and \( \alpha(i+1) = \alpha(i+3) \), then
\[
\Phi^\vee_{w_{i+3} \cdots w_1} - \Phi^\vee_{w_{i+1} \cdots w_1} = \{ \beta^\vee, \beta^\vee + \gamma^\vee, \beta^\vee + 2\gamma^\vee, \gamma^\vee \}
\]
for some coroots \( \beta^\vee, \gamma^\vee \in \Phi^+ - \Phi^+_M \). In particular, we have \( \langle \omega_p, \beta^\vee + 2\gamma^\vee \rangle \geq 3 \), which is impossible for classical groups.

### 5.2. Type \( A_n \)

A reduced expression of \( w_0 \) is
\[
(s_{n-\ell+1} \cdots s_{n-1}s_n) \cdots (s_{r-\ell+1} \cdots s_{r-1}s_r) \cdots (s_1 \cdots s_{\ell-1}s_\ell)
\]
Here the parentheses are only present in this expression so that the reader can follow the pattern. The corresponding order of coroots \( \tilde{\alpha}^\vee_{(i)} \) is
\[
\alpha^\vee_\ell, \quad \cdots \quad \sum_{j=1}^\ell \alpha^\vee_j, \quad \cdots \quad \sum_{j=1}^\ell \alpha^\vee_j, \\
\vdots \quad \cdots \quad \vdots \\
\sum_{j=1}^r \alpha^\vee_j, \quad \cdots \quad \sum_{j=1}^r \alpha^\vee_j, \quad \cdots \quad \sum_{j=1}^r \alpha^\vee_j, \\
\vdots \quad \cdots \quad \vdots \\
\sum_{j=1}^n \alpha^\vee_j, \cdots \quad \sum_{j=1}^n \alpha^\vee_j, \quad \cdots \quad \sum_{j=1}^n \alpha^\vee_j.
\]
The \( i \)th row of coroots (read from the top left) corresponds to the \( i \)th parenthesis of (5.7) (read from the right). Each positive coroot in \( \Phi^\vee_{w_0} \) appears exactly once and there are no other coroots in \( \Phi^\vee - \Phi^\vee_{w_0} \) in the above list, so (5.7) is indeed a reduced expression of \( w_0 \). The same justification of reduced expressions will be used in the rest of the section without mention.

We denote
\[
\beta^\vee \leftrightarrow \beta'^\vee
\]
if the order of two coroots \( \beta^\vee \) and \( \beta'^\vee \) can be reversed under a series of operations (a') and (b').

Denote coroots
\[
\sum_{j=t}^r \alpha^\vee_j \quad \text{for} \quad 1 \leq t \leq \ell \leq r \leq n
\]
between \( (r, t) \). Since operations (b') cannot be applied if \( G \) is of type \( A \), via operations (a') we have
\[
\text{for} \quad r \leq r', (r, t) \leftrightarrow (r', t') \quad \text{if and only if} \quad r < r' \quad \text{and} \quad t < t'.
\]
(R1)

Note that coroots share the same \( h \)-value if they lie on the same 45° line in the table above. Consequently for \( w \in W/W_M, m_w(h, 1) \geq c \) only if \( m_w(h - 1, 1) \geq c \) unless
\[
m_w(h, 1) = m_{w_0}(h, 1) = m_{w_0}(h - 1, 1) + 1 = m_w(h - 1, 1) + 1.
\]
This verifies inequalities (5.4) and (5.6).

We prove the holomorphy of \( M'_w(\chi_s) \) by induction on both \( n \) and length of \( w \). It suffices to prove the holomorphy when \( \chi = 1 \). For \( \ell = 1 \), the reduced expression (5.7) is unique. As (I) only holds for \( \text{Id} \), we need to check
\[
M'_{s, \cdots s_1(1_s)f^{(s)}} \text{is left } \iota_{s_{\ell+1}}(\text{SL}_2) \text{-invariant.}
\]
for \( 1 \leq r < n \). Since the coroot \( (r, 1) \) has \( h \)-value \( r \) and \( f^{(s)} \) is left \( \iota_{s_{\ell+1}}(\text{SL}_2) \)-invariant, the assertion follows from (iii).
For the general case, by symmetry we may assume \( n \geq 3 \) and \([n/2] \geq \ell \geq 2\). By induction hypothesis, we may assume \((r,1) \in \Phi_w^\vee\) for some \( r \). Choose \( r \) to be maximal. By \((R1)\) we can write \( w = s_{r-\ell+1}w_1 \) such that \( s_{r-\ell+1} \) corresponds to the coroot \((r,1)\). Assume first \( r > \ell \). We need to show

\[
M'_{w_1}(1_s) f^{(s)} \mid_{s+s_k+1=r-1} \text{ is left } \iota_{\alpha_{r-\ell+1}}(\text{SL}_2)\text{-invariant.}
\] (5.8)

Write \( w_1 = s_{r-\ell} w' \) such that \( s_{r-\ell} \) corresponds to the coroot \((r-1,1)\). We have

\[
w'^{-1} \alpha_{r-\ell+1} = w_1^{-1} \alpha_{r-\ell+1} - w'^{-1} \alpha_{r-\ell} = \sum_{j=1}^r \alpha_j - \sum_{j=1}^{r-1} \alpha_j = \alpha_r.
\]

Since \( f^{(s)} \) is left \( \iota_{\alpha_r}(\text{SL}_2)\)-invariant, \( M'_w(1_s) f^{(s)} \) is left \( \iota_{\alpha_{r-\ell+1}}(\text{SL}_2)\)-invariant. As the coroot \((r-1,1)\) has \( h \)-value \( r-1 \), (5.8) follows by (iii).

Suppose \( r = \ell \). By induction hypothesis, we may assume \((n,t) \in \Phi_w^\vee\) for some \( t \). Choose \( t \) to be minimal. A similar argument as above justifies the holomorphy of \( M'_w(1_s) \) if \( t < \ell \), and thus we can assume \( t = \ell \). By \((R1)\) we can also write \( w = s_n w_1 \) such that \( s_n \) corresponds to the coroot \((n,\ell)\), which has \( h \)-value \( n-\ell+1 \). Therefore by induction hypothesis and (i), if \( n-\ell \neq \ell-1 \) then \( M'_w(1_s) f^{(s)} = M'_w(1_s) f^{(s)} \) is holomorphic at \( s + s_k + 1 = \ell - 1 \) and hence is holomorphic at all \( s \). For the same reason, to justify \( M'_w(1_s) f^{(s)} \) is holomorphic at \( s + s_k + 1 = n - \ell = \ell - 1 \), it suffices to consider the case where no coroot in \( \Phi_w^\vee \) has \( h \)-value greater than \( n - \ell + 1 = \ell \).

Note that when \( n = 2\ell - 1 \), (1) fails except when \( m_w(\ell,1) = \ell \). Assume \( m_w(\ell,1) < \ell \) and thus there exists an integer \( 0 \leq i < \ell - 2 \) such that \((n-i,\ell-i) \in \Phi_w^\vee \) but \((n-i-1,\ell-i-1) \notin \Phi_w^\vee \). By \((R1)\) we can rewrite \( w = s_{n-2i}s_{n-2i-1}w' \) such that \( s_{n-2i} \) and \( s_{n-2i} \) correspond to coroots \((n-i-1,\ell-i)\) and \((n-i,\ell-i)\) respectively. We need to show

\[
M'_w(1_{s-2i} w_{s-2i-1}) f^{(s)} \mid_{s+s_k+1=n-\ell} \text{ is left } \iota_{\alpha_{n-2i}}(\text{SL}_2)\text{-invariant.}
\]

We have

\[
w'^{-1} \alpha_{n-2i} = (s_{n-2i-1}w')^{-1} \alpha_{n-2i} - w'^{-1} \alpha_{n-2i-1} = \sum_{j=\ell-i}^{n-i} \alpha_j - \sum_{j=\ell-i}^{n-i-1} \alpha_j = \alpha_{n-i}.
\]

Since \( f^{(s)} \) is left \( \iota_{\alpha_{n-i}}(\text{SL}_2)\)-invariant and the coroot \((n-i-1,\ell-i)\) has \( h \)-value \( n-\ell \), the assertion follows by (iii).

\[
\square
\]

5.3. Type \( B_n \) \((n \geq 2)\). We break down the discussion into two cases: \( n = \ell \) and \( n > \ell \). Consider first \( 1 \leq \ell < n \). A reduced expression of \( w_0 \) is

\[
(s_1 \cdots s_{n-1})(s_n) \cdots (s_{\ell-1+1} \cdots s_{\ell-r})(s_{\ell-1+1} \cdots s_n)(s_{\ell-r+1} \cdots s_{n-1})(s_{\ell-r+1} \cdots s_1) \cdots (s_1 \cdots s_{\ell-1}s_n); \tag{5.9}
\]

corresponding coroots \( \tilde{\alpha}^\vee_i \) are

\[
\begin{align*}
\tilde{\alpha}^\vee_1, & \quad \cdots \quad \sum_{j=1}^\ell \alpha^\vee_j, & \quad \cdots \quad \sum_{j=1}^\ell \alpha^\vee_j, \\
\vdots & \quad \vdots & \quad \vdots \\
\sum_{j=\ell}^r \alpha^\vee_j, & \quad \cdots \quad \sum_{j=\ell}^r \alpha^\vee_j, & \quad \cdots \quad \sum_{j=\ell}^r \alpha^\vee_j, \\
\vdots & \quad \vdots & \quad \vdots \\
\sum_{j=1}^{n-1} \alpha^\vee_j, & \quad \cdots \quad \sum_{j=1}^{n-1} \alpha^\vee_j, & \quad \cdots \quad \sum_{j=1}^{n-1} \alpha^\vee_j,
\end{align*}
\]
\[
\begin{align*}
\alpha_n^\vee + 2 \sum_{j=t}^{n-1} \alpha_j^\vee, & \quad \alpha_n^\vee + 2 \sum_{j=t}^{n-1} \alpha_j^\vee + \sum_{j=t}^{r-1} \alpha_j^\vee, & \quad \alpha_n^\vee + 2 \sum_{j=t}^{n-1} \alpha_j^\vee + \sum_{j=t}^{r-1} \alpha_j^\vee, \\
\alpha_n^\vee + \sum_{j=t}^{r-1} \alpha_j^\vee, & \quad \alpha_n^\vee + 2 \sum_{j=t}^{n-1} \alpha_j^\vee + \sum_{j=t}^{r-1} \alpha_j^\vee, & \quad \alpha_n^\vee + 2 \sum_{j=t}^{n-1} \alpha_j^\vee + \sum_{j=t}^{r-1} \alpha_j^\vee, \\
& \quad \vdots & \quad \vdots \\
& \quad \vdots & \quad \vdots \\
\alpha_n^\vee + \sum_{j=t}^{n-1} \alpha_j^\vee, & \quad \alpha_n^\vee + 2 \sum_{j=t}^{n-1} \alpha_j^\vee + \sum_{j=t}^{m-1} \alpha_j^\vee, & \quad \alpha_n^\vee + 2 \sum_{j=t}^{n-1} \alpha_j^\vee + \sum_{j=t}^{m-1} \alpha_j^\vee, \\
& \quad \alpha_n^\vee + \sum_{j=t}^{n-1} \alpha_j^\vee, & \quad \alpha_n^\vee + 2 \sum_{j=t}^{n-1} \alpha_j^\vee + \sum_{j=t}^{m-1} \alpha_j^\vee, & \quad \alpha_n^\vee + 2 \sum_{j=t}^{n-1} \alpha_j^\vee + \sum_{j=t}^{m-1} \alpha_j^\vee.
\end{align*}
\]

Here \((*)\) corresponds to the second row of \((5.9)\) (see §5.2). For rows of coroots below \((*)\), the \(i\)th odd (resp. even) row corresponds to the former (resp. latter) parenthesis in the \(i\)th big parenthesis (reading from the right).

We retain the terminology in §5.2. Denote coroots
\[
\begin{align*}
\alpha_n^\vee + 2 \sum_{j=r}^{n-1} \alpha_j^\vee &= \sum_{j=t}^{r-1} \alpha_j^\vee \quad \text{for } 1 \leq t \leq r \leq \ell, \quad \text{and} \\
\alpha_n^\vee + 2 \sum_{j=m}^{n-1} \alpha_j^\vee &= \sum_{j=r}^{m-1} \alpha_j^\vee \quad \text{for } 1 \leq r \leq \ell < m \leq n
\end{align*}
\]
by \((2, r, t)\) and \((1, m, r)\) respectively. Note that neither the sum nor the difference of two coroots of the same type is a coroot in \(\Phi_{w_0}^\vee\). Therefore, \((b')\) has no effects on the order of two coroots of the same type, so by \((a')\) additional to \((R1)\) we have
\[
\begin{align*}
\text{for } r \geq r', (2, r, t) &\iff (2, r', t') \quad \text{if and only if } r > r' \text{ and } t' > t; & (R2) \\
\text{for } r \geq r', (1, m, r) &\iff (1, m', r') \quad \text{if and only if } r > r' \text{ and } m' > m. & (R3)
\end{align*}
\]
Arguing similarly as in §5.2, we have \((R2)\) implies \((5.4)\) and \((5.6)\) for \(\lambda = 2\). We also remark that since \((c')\) is not applicable, the coroot \((2, r, r)\) always corresponds to the \((\ell - r + 1)\)th \(s_n\) in a reduced expression of \(w_0\).

Observe that \((2, r, t)\) can be written as the sum
\[
(1, m', r') + (r'', t''')
\]
if and only if either \(r'' = r\) and \(t''' = t\) or \(r'' = t\) and \(t''' = r\). We claim the operation \((b')\) can only be applied to triples \((2, r, t), (1, m, r), (m - 1, t)\) for \(t < r\) due to the occurrence of \(s_n\). Indeed, if \(r = t = r'\), then \((2, r, r)\) corresponds to \(s_n\); if \(r > t = r'\), then by \((R2)\) the coroot \((2, t, t)\), which corresponds to \(s_n\), must come after \((2, r, t)\), while the original position of \((1, m, t)\) is behind \((2, t, t)\).

Consequently, together with operations \((a')\),
\[
\begin{align*}
(2, r, t) &\iff (r', t') \quad \text{if and only if } t' < r; & (B1) \\
(1, m, r) &\iff (r', t') \quad \text{if and only if } t' < r; & (B2) \\
(2, r, t) &\iff (1, m', r') \quad \text{if and only if } r < r' \text{ or } t < r = r'. & (B3)
\end{align*}
\]
Inequalities \((5.4)\) and \((5.6)\) for \(\lambda = 1\) follow from \((R1), (R3),\) and \((B2)\) by an argument similar to that in §5.2. To see this, flip the above list of coroots of type \((1, m, r)\) along the diagonal and attach it below \((*)\), so one obtains a table of coroots (with \(\lambda\)-value equal to 1) of size \(2(n - \ell)\) by \(\ell\). Then rules \((R1), (R3),\) and \((B2)\) combined are essentially the same as the rule \((R1)\) for \(G\) of type \(A_{2n-\ell-1}\) with node \(\ell\).

We prove the holomorphy of \(M'_w(\chi_s)\) by induction on both \(n\) and length of \(w\). Only holomorphy of \(M'_w(\chi_s)\) for \(\chi^2 = 1\) requires a proof. Suppose \(\ell = 1\) and \(n \geq 2\). Then the reduced expression \((5.9)\) of \(w_0\) is unique. Note that \((1)\) only holds for \(\text{Id}\) and \(s_{n-1}s_{n-2}\cdots s_1\). As type \(A\)
is justified, we may assume \( w = s_{m-1} \cdots s_{n-1} s_{n} s_{n-1} \cdots s_{1} \) for some \( 2 \leq m \leq n \). If \( m < n \), then we need to show
\[ M'_{s_{m-1} \cdots s_{n-1} s_{n} \cdots s_{1}}(1_s)f(s)|_{s+s_{k}+1=2m-1} \text{ is left } \tau_{\alpha_{m-1}}(SL_2)-\text{invariant.} \]
Since \( M'_{s_{m-1} \cdots s_{n-1} s_{n} \cdots s_{1}}f(s) \) is left \( \tau_{\alpha_{m-1}}(SL_2)-\text{invariant} \) and the coroot \((1, m + 1, 1)\) has \( h \)-value \( 2n - m - 1 \), this follows from (iii). For \( m = n \), note that \( f(s) \) is left \( \tau_{\alpha_{n}}(SL_2)-\text{invariant} \). Since the coroot \((n - 1, 1)\) has \( h \)-value \( n - 1 \), by (iii) \( M'_{s_{n-1} \cdots s_{2} s_{1}}(1_s)f(s)|_{s+s_{k}+1=n-1} \) is both left \( s_{n-1}\tau_{\alpha_{n}}(SL_2)-\text{invariant} \) and left \( \tau_{\alpha_{n}}(SL_2)-\text{invariant} \). As
\[ s_{n}\alpha_{n-1} = (\alpha_{n} + \alpha_{n-1}) + \alpha_{n} = s_{n-1}\alpha_{n} + \alpha_{n}, \]
we have \( s_{n}\tau_{\alpha_{n-1}}(SL_2) \) is contained in the group generated by \( s_{n-1}\tau_{\alpha_{n}}(SL_2) \) and \( \tau_{\alpha_{n}}(SL_2) \), and thus
\[ M'_{s_{n-1} \cdots s_{1}}(1_s)f(s)|_{s+s_{k}+1=n-1} \text{ is left } \tau_{\alpha_{n-1}}(SL_2)-\text{invariant.} \]

Now consider general \( n > \ell \geq 2 \). We will use rules (R1)-(R3) and (B1)-(B3) without further mention below. As type A is justified, by induction hypothesis it suffices to justify the assertion for \( w \) such that
\[ \Phi'_{w} \cap \{(r, 1), (1, m, 1), (2, r', 1) : \ell \leq r \leq n - 1, \ell < m \leq n, 1 \leq r' \leq \ell \} \]
is nonempty and \( \Phi'_{w} \) does not only consist of coroots of type \((r, t)\). Write \( w = s_{\alpha}w_{1} \) and let \( \beta'_{w} \) be the coroot in \( \Phi'_{w} - \Phi'_{w_{1}} \).

**Lemma 5.11.** There exists a reduced expression \( w = s_{\alpha}w_{1} \) such that \( \beta'_{w} \) is either \((r, 1), (1, m, 1), \) or \((2, r', 1)\).

**Proof.** Observe that the coroot \((1, m, 1)\) is not a part of any triple of coroots on which an operation \((b')\) can be applied. Therefore, if \((1, m, 1) \in \Phi'_{w} \) for some \( m \), then by performing a series of \((a')\), one can take \( \beta'_{w} \) to be \((1, m, 1) \in \Phi'_{w} \) with \( m' \) minimal.

Thus we assume no coroots \((1, m, 1)\) are contained in \( \Phi'_{w} \). Consider first the case that \((r, 1) \in \Phi'_{w} \) for some \( r \). Choose \( r \) to be maximal. Let \( c_{1} \) (resp. \( c_{2} \)) be the number of coroots of the form \((2, r', 1)\) (resp. \((1, r + 1, r')\)) contained in \( \Phi'_{w} \). Note that coroots \((2, r', 1), (1, r + 1, r'), (r, 1)\) form a triple on which the operation \((b')\) can be applied. Therefore, if \( c_{1} \leq c_{2} \) (resp. \( c_{1} > c_{2} \)), then \( \beta'_{w} \) can be taken as \((r, 1)\) (resp. \((2, \ell - c_{2} + 1, 1)\)). The case \((2, r', 1) \in \Phi'_{w} \) for some \( r' \) can be argued similarly.

\[ \square \]

Suppose \( \beta'_{w} = (2, r, 1) \). Note that (1) fails for \( w_{1} \) unless
\[ \ell = r \text{ is odd and (2, r - c, 1 + c) } \in \Phi'_{w} \text{ for all } 0 \leq c \leq (\ell - 1)/2. \tag{5.10} \]

We need to show
\[ M'_{s_{n-1} s_{n} w_{1}}(1_s)f(s)|_{2(s+s_{k}+1)=2(n-1)+1-r} \text{ is left } \tau_{\alpha}(SL_2)-\text{invariant} \tag{5.11} \]
except when (5.10) occurs. By (i) and (R2) we can assume coroots in \( \Phi'_{w} \) with \( \lambda \)-value 2 have \( h \)-value at most \( 2(n - 1) + 2 - r \).

**Case r = 1:** We have \( \alpha = \alpha_{n} \) and we can write \( w_{1} = w'' s_{n-1} s_{n} w' \) where \( s_{n} \) and \( s_{n-1} \) correspond to coroots \((2, 2, 2)\) and \((2, 2, 1)\) respectively, and \( \Phi'_{w_{1}} - \Phi'_{s_{n-1} s_{n} w'} \) consists of coroots of the form \((1, m, 2)\). Since coroots \((2, 2, 2)\) and \((2, 2, 1)\) have \((h, \lambda)\)-value \((2(n - 2) + 1, 2)\) and \((2(n - 1), 2)\) respectively, we have
\[ M'_{s_{n-1} s_{n} w_{1}}(1_s)f(s)|_{2(s+s_{k}+1)=2(n-1)} \text{ is left } \tau_{\alpha}(SL_2)-\text{invariant} \]
by (iii) and (iv). As coroots in $\Phi_{w_1}^\vee - \Phi_{s_{n-1}s_n w'}^\vee$ correspond to reflections $s_{\alpha_i}$ where $i < n - 1$, to prove (5.11) by (i) it suffices to show

$$\frac{a_{s_{n-1}s_n w'}(\chi_s)}{a_{w_1}(\chi_s)} M_{w'}(\chi_{s_{n-1}s_n w'})$$

is holomorphic at $s + s_k + 1 = n - 1$. Observe that the coroot $(1, n - 1, 2)$ has $h$-value $n$, and the coroot $(1, n, 2)$ must come before $(1, n - 1, 2)$. Thus the assertion follows from the proof of type $A$ with node 1.

**Case $r$ is even, $\ell > r$:** Since we are to check holomorphy of $M_{w'}(\chi_s)$ at $2(s + s_k + 1) = 2(n - 1) + 1 - r$, which is an odd integer, by (i) and (v) we may add or remove from $\Phi_w^\vee$ coroots with $\lambda$-value 1 as long as rules $(R1)$-$(R3)$ and $(B1)$-$(B3)$ are obeyed. In particular, we can take $w$ such that $\alpha = \alpha_{n-r+1}$ and $w_1 = s_{n-r}w'$ where $s_{n-r}$ corresponds to the coroot $(2, r + 1, 1)$. Then we have

$$w'^{-1}\alpha_{n-r+1} = w_1^{-1}\alpha_{n-r+1} - w'^{-1}\alpha_{n-r}$$

$$= (2 \sum_{j=r}^{n} \alpha_j + \sum_{j=1}^{r-1} \alpha_j) - (2 \sum_{j=r+1}^{n} \alpha_j + \sum_{j=1}^{r} \alpha_j) = \alpha_r.$$

As the coroot $(2, r + 1, 1)$ has $h$-value $2(n - 1) + 1 - r$ and $f(s)$ is left $\iota_{\alpha_r}(SL_2)$-invariant, we deduce (5.11) from (iii).

**Case $r = \ell$ is even:** Let $0 \leq c \leq \ell/2 - 1$ be the largest integer such that $(2, \ell - c, 1 + c) \in \Phi_w^\vee$. As in the previous case, we can assume $w$ can be rewritten as $w = s_{n-\ell+1+2c}s_{n-\ell+2+2c}w'$, where $s_{n-\ell+2+2c}$ and $s_{n-\ell+1+2c}$ correspond to coroots $(2, \ell - c, c + 2)$ and $(2, \ell - c, c + 1)$ respectively. We claim

$$M'_{w_1}(\chi_s)f(s)|_{2(s+s_k+1)=2(n-1)+1-\ell}$$

is left $\iota_{\alpha_{n-\ell+1+2c}}(SL_2)$-invariant. We have

$$w'^{-1}\alpha_{n-\ell+1+2c} = (s_{n-\ell+2+2c}w')^{-1}\alpha_{n-\ell+1+2c} - (1 + \delta c/2-1)w'^{-1}\alpha_{n-\ell+2+2c}$$

$$= (2 \sum_{j=\ell-c}^{n} \alpha_j + \sum_{j=c+1}^{\ell-c} \alpha_j) - (2 \sum_{j=\ell-c}^{n} \alpha_j + \sum_{j=c+1}^{\ell-c} \alpha_j) = \alpha_{c+1}.$$ Since $(2, \ell - c, c + 2)$ has $h$-value $2(n - 1) + 1 - \ell$ and $f(s)$ is left $\iota_{\alpha_{c+1}}(SL_2)$-invariant, our claim follows from (iii).

**Case $r$ is odd, $\ell > r > 1$:** Say $\alpha = \alpha_{n-r+1-i}$ where $n - \ell - i \geq 0$ is the number of coroots $(m, 1)$ not in $\Phi_w^\vee$. By our assumption on coroots, we have $(2, t, t) \not\in \Phi_w^\vee$ for $2t < r + 1$ and thus $(1, m, t) \not\in \Phi_w^\vee$ for $t < (r + 1)/2$. Write $w_1 = w''s_{n-r-i}w'$ where $s_{n-r-i}$ corresponds to the coroot $(2, r + 1, 1)$, and $\Phi_{w_1}^\vee - \Phi_{s_{n-r-i}w'}^\vee$ consists of coroots of the form $(1, m, t)$ where $t \geq (r + 1)/2$. Choose a reduced expression such that reflections corresponding to coroots in $\Phi_{w_1}^\vee - \Phi_{s_{n-r-i}w'}^\vee$ fixes $\alpha_{n-r+1-i}$. Then we have

$$w'^{-1}\alpha_{n-r+1-i} = (s_{n-r-i}w')^{-1}\alpha_{n-r+1-i} - w'^{-1}\alpha_{n-r-i}$$

$$= w_1^{-1}\alpha_{n-r+1-i} - w'^{-1}\alpha_{n-r-i}$$

$$= (2 \sum_{j=r}^{n} \alpha_j + \sum_{j=1}^{r-1} \alpha_j) - (2 \sum_{j=r+1}^{n} \alpha_j + \sum_{j=1}^{r} \alpha_j) = \alpha_r.$$

As the coroot $(2, r + 1, 1)$ has $h$-value $2(n - 1) + 1 - r$ and $f(s)$ is left $\iota_{\alpha_r}(SL_2)$-invariant, we conclude by (iii)

$$M'_{s_{n-r-i}w'}(\chi_s)f(s)|_{2(s+s_k+1)=2(n-1)+1-r}$$

is left $\iota_{\alpha_{n-r+1-i}}(SL_2)$-invariant.
Therefore (5.11) follows if
\[
\frac{a_{s_{n-r-i}w'}(\chi_s)}{a_{w_1}(\chi_s)} \Phi^{s_{n-r-i}w'}
\]
is holomorphic at \(2(s + s_k + 1) = 2(n - 1) + 1 - r\). By the proof of type \(A\), this is true if \(\chi \not= 1\) or \(\chi = 1\) and the number of coroots of the form \((1, m, t)\) in \(\Phi^w\) with \(h\)-value \(\frac{2(n-1)+1-r}{2}\) is at most that with \(h\)-value \(\frac{2(n-1)+1-r}{2}\).

We assume this is not the case, so \(\chi = 1, 2(n-\ell) \geq \frac{2(n-1)+1-r}{2} + 1\), and \((1, n, c, \frac{r+1}{2} + c) \in \Phi^w\) for all \(c\). Since \((2, \frac{r+1}{2}, \frac{r-1}{2}) \not\in \Phi^w\) by assumption and \((1, n, \frac{r+1}{2}) \not\in \Phi^w\), we have \((n-1, \frac{r-1}{2}) \not\in \Phi^w\) but \((n-1, \frac{r+1}{2}) \in \Phi^w\). We have \(s_{n-1} = s_{n-\ell+(r-1)}\) and \((n-1, \frac{r+1}{2})\) correspond to coroots \((1, n, \frac{r+1}{2})\) and \((n-1, \frac{r-1}{2})\) respectively, and coroots of type \((m, t)\) in \(\Phi^w\) all lie in \(\Phi^w\). We need to show
\[
M'_{s_{n-\ell+(r-1)}}(1_s f(s))|_{s+s_k+1=\frac{2(n-1)+1-r}{2}} \text{ is left } \iota_{\alpha_n}(\text{SL}_2)-\text{invariant}. \quad (5.12)
\]

**Lemma 5.12.** We have
\[
M'_{s_{n-\ell+(r-1)}}(1_s f(s))|_{s+s_k+1=\frac{2(n-1)+1-r}{2}} \text{ is left } v'w\alpha_n(\text{SL}_2)-\text{invariant}.
\]

**Proof.** Since the coroot \((n-1, \frac{r+1}{2})\) has \(h\)-value \(\frac{2(n-1)+1-r}{2}\) and \(f(s)\) is left \(\iota_{\alpha_n}(\text{SL}_2)\)-invariant, by (iii) to justify the lemma it suffices to show
\[
\frac{a_{s_{n-\ell+(r-1)}}(1_s f(s))}{a_{s_{n-\ell+(r-1)}}(1_s f(s))} \Phi^{s_{n-\ell+(r-1)}}
\]
is holomorphic at \(s + s_k + 1 = \frac{2(n-1)+1-r}{2}\). By (R2) and (R3), the number of coroots in \(\Phi^w\) \(s_{n-\ell+(r-1)}\) \(v - \Phi^w\) \(s_{n-\ell+(r-1)}\) \(v\) with \((h, \lambda)\)-value \((2(n-1) + 2 - r, 2)\) (resp. \(\frac{2(n-1)+1-r}{2}, 1))\) is at most that with \((h, \lambda)\)-value \((2(n-1) + 1 - r, 2)\) (resp. \(\frac{2(n-1)+1-r}{2}, 1))\). Therefore, the holomorphy follows by the induction on length (and proofs).

Clearly, \(M'_{s_{n-\ell+(r-1)}}(1_s f(s))\) is left \(v' s_{n-\ell+(r-1)} v \alpha_n(\text{SL}_2)\)-invariant. We have
\[
v\alpha_n = \alpha_n + \alpha_{n-1} + \ldots + \alpha_{n-\ell+\frac{r+1}{2}},
\]
and
\[
v' v\alpha_n = s_n \alpha_n, \quad v' s_{n-\ell+(r-1)} v \alpha_n = s_n (\alpha_{n-1} + \alpha_n).
\]

Since \(s_n \alpha_n + s_n (\alpha_{n-1} + \alpha_n) = s_n s_n \alpha_{n-1} = \alpha_{n-1}\), we deduce (5.12).

**Case \(\ell = r\) is odd:** Since (5.10) does not hold, there exists smallest \(0 \leq c \leq \frac{(\ell - 1)/2 - 1}\) such that \(2, \ell - c - 1, 2 + c) \not\in \Phi^w\). We can rewrite \(w = s_{n-\ell+2c-i} s_{n-\ell+2c-i} w'\) for some \(i \geq 0\), where \(s_{n-\ell+2c-i} \) and \(s_{n-\ell+2c-i}\) correspond to coroots \((2, \ell - c, c + 2)\) and \((2, \ell - c, c + 1)\) respectively. A similar argument as in the case \(r = \ell\) even justifies the holomorphy.

For the rest of the proof we may assume \(\chi = 1\). Consider \(\beta^w = (1, m, 1)\) and thus \(\alpha = \alpha_{m-1}\). By (B2) \(\Phi^w\) contains all coroots of type \((r, t)\). Therefore, we can choose a reduced expression of \(w\) with no operations \((w')\) carried out. Note that (I) fails for \(w_1\) in this case. Suppose \(\ell < m < n\). We can write \(w_1 = s_m w'\) where \(s_m\) corresponds to the coroot \((1, m + 1, 1)\). We have
\[
w'^{-1} \alpha_{m-1} = w'^{-1} \alpha_{m-1} - w'^{-1} \alpha_m
\]
\[
= (2 \sum_{j=m} a_j + \sum_{j=1}^{m-1} \alpha_j) - (2 \sum_{j=m+1}^{n} \alpha_j + \sum_{j=1}^{m} \alpha_j) = \alpha_m.
\]
Since $f(s)$ is left $\iota_{\alpha_m}(\SL_2)$-invariant and the coroot $(1, m + 1, 1)$ has $h$-value $2n - m - 1$, by (iii)

$$M'_{w_1}(1_s)f(s)|_{s+s_k+1=2n-m-1}$$

is left $\iota_{\alpha_{m-1}}(\SL_2)$-invariant.

Suppose $m = n$. We break down the discussion into two cases.

**Case** $n < 2\ell$: Let $0 \leq c < n - \ell$ be the largest integer such that $(1, n - c, 1 + c) \in \Phi_w^{\vee}$. Rewrite $w = s_{n-1-2c}s_{n-2-2c}w'$ where $s_{n-2-2c}$ and $s_{n-1-2c}$ correspond to coroots $(1, n - c, 2 + c)$ and $(1, n - c, 1 + c)$ respectively. We claim

$$M'_{s_{n-2-2c}w'}(1_s)f(s)|_{s+s_k+1=n-1}$$

is left $\iota_{\alpha_{n-1-2c}}(\SL_2)$-invariant.

We have

$$w'^{-1}\alpha_{n-1-2c} = (s_{n-2-2c}w')^{-1}\alpha_{n-1-2c} - w'^{-1}\alpha_{n-2-2c} = (2\sum_{j=n-c}^{n-c-1}\alpha_j + \sum_{j=n-c}^{n-c+1}\alpha_j) = \alpha_{c+1}$$

Since the coroot $(1, n - c, 2 + c)$ has $h$-value $n - 1$ and $f(s)$ is left $\iota_{\alpha_{c+1}}(\SL_2)$-invariant, our claim follows from (iii).

**Case** $n \geq 2\ell$: Suppose there exists $0 \leq c < \ell$ such that $(1, n - c - 1, 2 + c) \not\in \Phi_w^{\vee}$. Choose $c$ to be minimal, and rewrite $w = s_{n-1-2c}s_{n-2-2c}w'$ where $s_{n-2-2c}$ and $s_{n-1-2c}$ correspond to coroots $(1, n - c, 2 + c)$ and $(1, n - c, 1 + c)$ respectively. The holomorphy of $M'_{w}(1_s)$ can be justified similarly as the previous case. Therefore, we assume $(1, n - c, 1 + c) \in \Phi_w^{\vee}$ for all $c$. We claim

$$M'_{w_1}(1_s)f(s)|_{s+s_k+1=n-1}$$

is left $\iota_{\alpha_{n-1}}(\SL_2)$-invariant.

**Lemma 5.13.** $M'_{w_1}(1_s)f(s)|_{s+s_k+1=n-1}$ is left $w''w'\iota_{\alpha_n}(\SL_2)$-invariant.

**Proof.** Write $w_1 = w''s_{n-\ell}w'$ where $s_{n-\ell}$ corresponds to the coroot $(n-1,1)$ and $\Phi_{s_{n-\ell}w'}^{\vee}$ consists of all coroots of type $(r,t)$. Since the coroot $(n-1,1)$ has $h$-value $n-1$, by (iii)

$$M'_{s_{n-\ell}w'}(1_s)f(s)|_{s+s_k+1=n-1}$$

is left $w'\iota_{\alpha_n}(\SL_2)$-invariant.

Write $w'' = v'v$ where $\Phi_{vs_{n-\ell}w'}^{\vee} - \Phi_{s_{n-\ell}w'}^{\vee}$ consists of all coroots of type $(2,r,t)$. Since $(2,1,1)$ is the only coroot of type $(2,r,t)$ with $h$-value $2(n-1) + 1$ and coroots $(2,2,2)$ and $(2,2,1)$ are in $\Phi_{vs_{n-\ell}w'}^{\vee} - \Phi_{s_{n-\ell}w'}^{\vee}$, it follows from the proof of the case $\beta^{\vee} = (2,1,1)$ that

$$\frac{a_{s_{n-\ell}w'}(1_s)}{a_{vs_{n-\ell}w'}(1_s)}M_r(1_s)\iota_{\alpha_n}(w_{n-\ell}w')$$

is holomorphic at $s + s_k + n - 1$. Therefore, the lemma follows once we show

$$\frac{a_{vs_{n-\ell}w'}(1_s)}{a_{w_1(1_s)}M_r(1_s)w_{n-\ell}w'}$$

is holomorphic at $s + s_k + n - 1$. Note that $\Phi_{vs_{n-\ell}w'}^{\vee} - \Phi_{s_{n-\ell}w'}^{\vee}$ consists of coroots of type $(1, r, t)$. By (R3) the number of coroots in $\Phi_{vs_{n-\ell}w'}^{\vee} - \Phi_{s_{n-\ell}w'}^{\vee}$ with $h$-value $n$ is at most that with $h$-value $n - 1$. Therefore, the holomorphy follows from the proof of type A.

Clearly, $M'_{w_1}(1_s)f(s)$ is left $w_1\iota_{\alpha_n}(\SL_2)$-invariant. Since $(1, n - c, 1 + c) \in \Phi_w^{\vee}$ for all $c$, we have $(1, n, r) \in \Phi_w^{\vee}$ for all $r$, and thus a direct computation gives

$$w_1\alpha_n = \alpha_n + \alpha_{n-1}, \quad w''w'\alpha_n = -\alpha_n.$$

As $(\alpha_n + \alpha_{n-1}) + (-\alpha_n) = \alpha_{n-1}$, our claim follows from the above lemma.

Consider now $\beta^{\vee} = (r,1)$. By (i), (R1), (R3) and (B2), we may assume every coroot with $\lambda$-value 1 in $\Phi_w^{\vee}$ has $h$-value at most $r$. 

Case $n - 1 \geq r > \ell$: Note that (I) fails for $w_1$ in this case. If $(1, r + 1, \ell) \not\in \Phi_w^\vee$, then $\alpha = \alpha_{r-\ell+1}$ and we can write $w_1 = s_{r-\ell}w'$ where $s_{r-\ell}$ corresponds to the coroot $(r-1, 1)$. We have

$$w'^{-1}\alpha_{r-\ell+1} = w_1^{-1}\alpha_{r-\ell+1} - w'^{-1}\alpha_{r-\ell} = \sum_{j=1}^r \alpha_j - \sum_{j=1}^{r-1} \alpha_j = \alpha_r.$$ 

Since the coroot $(r-1, 1)$ has $h$-value $r-1$ and $f^{(s)}$ is left $\iota_{\alpha_r}(\text{SL}_2)$-invariant, by (iii) $M'_{w_1}(1_s)f^{(s)}|_{s+s_k+1=r-1}$ is left $\iota_{\alpha_r}(\text{SL}_2)$-invariant.

Therefore, we assume $(1, r + 1, \ell) \in \Phi_w^\vee$. Let $\ell > c \geq 1$ be the number of coroots $(1, r + 1, \ell)$ in $\Phi_w^\vee$. Suppose the coroot $(1, r + 1, \ell+1-c)$ has $h$-value less than $r$. If $(1, r, \ell+1-c) \not\in \Phi_w^\vee$, then we can rewrite $w = s'w'$ such that $s'$ corresponds to the coroot $(1, r + 1, \ell+1-c)$. The holomorphy of $M'_{w_1}(1_s) = M'_{s'w'}(1_s)$ follows by induction hypothesis and (i). If $(1, r, \ell+1-c) \in \Phi_w^\vee$, let $m \leq r$ be the smallest integer such that $(1, m, \ell + 1 - c) \in \Phi_w^\vee$. Rewrite $w = s_i s_{r+1} w' (i < n - 1)$ such that $s_{i+1}$ and $s_i$ correspond to coroots $(1, m + 1, \ell + 1 - c)$ and $(1, m, \ell + 1 - c)$ respectively. The holomorphy of $M'_{w}(1_s)$ in this case follows by a similar argument as in the case $\beta^\vee = (1, m, 1)$.

Consequently, we assume the coroot $(1, r + 1, \ell+1-c)$ has $h$-value $r$, i.e., $2n - r - \ell - 1 + c = r$. If $c \geq 2$ and $(1, r, \ell + 2 - c) \not\in \Phi_w^\vee$, then we rewrite $w = s_i s_{i+1} w'$ ($i < n$) where $s_i$ and $s_{i-1}$ corresponds to coroots $(1, r + 1, \ell + 2 - c)$ and $(1, r + 1, \ell + 1 - c)$ respectively. We have

$$w'^{-1}\alpha_{i-1} = (s_i w_1)^{-1}\alpha_{i-1} - w'^{-1}\alpha_i = (2 \sum_{j=r+1}^n \alpha_j + \sum_{j=\ell+1}^{r-1} \alpha_j) - (2 \sum_{j=r+1}^n \alpha_j + \sum_{j=\ell+2}^{r-1} \alpha_j) = \alpha_{\ell+1-c}.$$ 

Since the coroot $(1, r + 1, \ell + 1 - c)$ has $h$-value $r-1$ and $f^{(s)}$ is left $\iota_{\alpha_{\ell+1-c}}(\text{SL}_2)$-invariant, we have $M'_{s_i w'}(1_s)f^{(s)}|_{s+s_k+1=r-1}$ is left $\iota_{\alpha_{\ell+1-c}}(\text{SL}_2)$-invariant.

Suppose either $c = 1$ or $c \geq 2$ and $(1, r, \ell + 2 - c) \in \Phi_w^\vee$. Let $s_i$ be a reflection such that $\ell(s_i w) = 1 + \ell(w)$ and the coroot corresponding to $s_i$ is $(1, r, \ell + 1 - c)$. Since the coroot $(1, r, \ell + 1 - c)$ has $h$-value $r + 1$, by (i) and (v) $M'_{s_i w'}(1_s)$ is holomorphic at $s + s_k + 1 = r - 1$ iff $M'_{s_i w'}(1_s)$ is. To see $M'_{s_i w'}(1_s)$ is holomorphic at $s + s_k + 1 = r - 1$, rewrite $s_i w = s_{r-\ell+c+1} s_{r-\ell+c} w'$ where $s_{r-\ell+c}$ and $s_{r-\ell+c+1}$ correspond to coroots $(r, 1)$ and $(r-1, 1)$ respectively. The holomorphy follows from a similar argument as the case $(1, r + 1, \ell) \not\in \Phi_w^\vee$.

Case $r = \ell, 2(n - \ell) < \ell$: Note that in this case (I) fails for $w_1$. For the case that $(1, \ell + 1 + t, 2n - 2\ell - t) \not\in \Phi_w^\vee$, for some $t$, the holomorphy in this case follows from an argument analogous to that of the previous case when $(1, r + 1, \ell + 1 - c) \in \Phi_w^\vee$ with $c \geq 2$ but $(1, r, \ell + 2 - c) \not\in \Phi_w^\vee$. Therefore, suppose $(1, \ell + 1 + t, 2n - 2\ell - t) \not\in \Phi_w^\vee$ for any $t$. If there is no coroot of type $(1, m, t)$ in $\Phi_w^\vee$, then we can write $w = w'' w'$, where $\Phi''_w^\vee$ consists of all coroots of type $(m, t)$ in $\Phi_w^\vee$. The holomorphy then follows from the proof of the case $\beta^\vee = (2, m, 1)$. If $\Phi_w^\vee$ contains some coroot of type $(1, m, t)$, we can rewrite $w = s_i w'$ where $s_i$ corresponds to a coroot with $(h, \lambda)$-value $(c, 1)$ where $c < \ell$. Then by induction hypothesis and (i) $M'_{w'}(1_s) = M'_{s_i w'}(1_s)$ is holomorphic at $s + s_k = 1 + \ell - 1$.

Case $r = \ell, 2(n - \ell) \geq \ell$: By the same argument as the previous case, it suffices to consider when $n - \ell < \ell$ and $(1, 2(n - \ell) + 1, \ell - t) \not\in \Phi_w^\vee$ for all $t$. Since $2(n - \ell) \geq \ell$, for (I) to fail for $w_1$, there exists smallest $n - \ell > c \geq 0$ such that $(\ell + c + 1, 2 + c) \not\in \Phi_w^\vee$ and $(\ell + c, 1 + c) \in \Phi_w^\vee$. Since $n < 2\ell$, we can rewrite $w = s_i s_{i+1} w' (i < n - 1)$ where $s_{i+1}$ and $s_i$ correspond to coroots.
\((\ell + c, 2 + c)\) and \((\ell + c, 1 + c)\). We have

\[
w'^{-1} \alpha_i = (s_{i+1}w')^{-1} \alpha_i - w'^{-1} \alpha_{i+1} = \sum_{j=1+c}^{\ell+c} \alpha_j - \sum_{j=2+c}^{\ell+c} \alpha_j = \alpha_{c+1}.
\]

Since the coroot \((\ell + c, 2 + c)\) has \(h\)-value \(\ell - 1\) and \(f(s)\) is left \(\iota_{\alpha_{c+1}}(\text{SL}_2)\)-invariant, by \(\text{(iii)}\)

\[
M'_{\ell+c} w(1_s) f(s)_{\ell+c+1 = \ell - 1}\]

is left \(\iota_{\alpha}(\text{SL}_2)\)-invariant.

This completes the proof for the case \(\ell < n\).

For \(\ell = n \geq 2\), a reduced expression of \(w_0\) is

\[
(s_n) \cdots (s_r \cdots s_1) (s_1 \cdots s_n);
\]

\(\text{(5.13)}\)

corresponding coroots \(\hat{\alpha}_i^{(\ell)}\) are

\[
\alpha_1^{\text{\text{v}}} + 2 \sum_{j=1}^{\ell-1} \alpha_j, \quad \cdots \quad \alpha_\ell^{\text{\text{v}}} + 2 \sum_{j=1}^{\ell-1} \alpha_j + \sum_{j=1}^{n-1} \alpha_j, \quad \alpha_{n+1}^{\text{\text{v}}} + 2 \sum_{j=1}^{n-1} \alpha_j + \sum_{j=1}^{n-1} \alpha_j, \quad \alpha_{n+2}^{\text{\text{v}}} + 2 \sum_{j=1}^{n-1} \alpha_j + \sum_{j=1}^{n-1} \alpha_j, \quad \cdots
\]

The inequalities \(\text{(5.4)}\) and \(\text{(5.6)}\) and the holomorphy of \(M'_{\ell}(\chi_s)\) (especially \(\chi = 1\)) follow from the same (and actually simpler) argument as the previous case for type \((2, r, t)\) coroots.

\(\square\)

5.4. Type \(C_n\) \((n \geq 2)\). For \(1 \leq \ell < n\), coroots \(\hat{\alpha}_i^{\text{\text{v}}}\) corresponding to the reduced expression \(\text{(5.9)}\) are

\[
\alpha_1^{\text{\text{v}}}, \quad \cdots \quad \sum_{j=1}^{\ell} \alpha_j, \quad \cdots \quad \sum_{j=1}^{\ell} \alpha_j, \quad \cdots
\]

\[
\sum_{j=1}^{\ell} \alpha_j, \quad \cdots \quad \sum_{j=1}^{\ell} \alpha_j, \quad \cdots \quad \sum_{j=1}^{\ell} \alpha_j, \quad \cdots
\]

\[
\sum_{j=1}^{\ell} \alpha_j, \quad \cdots \quad \sum_{j=1}^{\ell} \alpha_j, \quad \cdots \quad \sum_{j=1}^{\ell} \alpha_j, \quad \cdots
\]

By duality, the proof for type \(B_n\) carries over with minor modification. The major difference is that \(i\)th \(s_n\) in any reduced expression of \(w_0\) corresponds to the coroot \(\sum_{j=\ell+1}^{n} \alpha_j\), which has \(\lambda\)-value 1 instead of 2. If the last coroot is \(\sum_{j=1}^{n} \alpha_j\), one modifies the argument of the case \(\beta^{\text{\text{v}}} = (1, n, 1)\) in \(\text{§5.3}\). Some explanation is given in the case \(\ell = n\) below. If the last coroot is \(2\alpha_n + \sum_{j=1}^{n-1} \alpha_j\), one applies the argument of the case \(\beta^{\text{\text{v}}} = (1, r, 1)\) with \(r < n\) in \(\text{§5.3}\). We leave the details of the other cases to the reader.

For \(\ell = n\), coroots \(\hat{\alpha}_i^{\text{\text{v}}}\) corresponding to the reduced expression \(\text{(5.13)}\) are
As in the case $\ell < n$, except for the case where the last coroot is $\sum_{j=1}^{\ell} \alpha_j^\vee$, a similar inductive proof as in §5.3 for type $(2,r,\ell)$ coroots justifies the holomorphy in this case. Therefore, we only explain how to prove the holomorphy of $M'_{\alpha_0}(1_s)$ assuming $M'_{\alpha_0}(\chi_\alpha)$ is holomorphic for any $w \neq w_0$ and $\chi$.

Rewrite $w_0 = s_n s_{n-1} s_{n-2} s_n w'$ by switching the order of coroots $\sum_{j=2}^{n} \alpha_j^\vee$ and $2 \sum_{j=3}^{n} \alpha_j^\vee + \sum_{j=1}^{2} \alpha_j^\vee$. We need to show

$$M'_{s_n s_{n-1} s_{n-2} s_n w'}(1_s) f^{(s)} |_{s+\ell k + 1 = n-1}$$

is left $\iota_{\alpha_0}(\text{SL}_2)$-invariant.

We have $w' \alpha_{n-1} = s_n s_{n-1} \alpha_{n-2}$ and

$$s_n s_{n-1} \alpha_{n-2} + s_{n-1} \alpha_{n-2} = \alpha_n + 2 \alpha_n + 2 \alpha_{n-2} = s_n s_{n-1} \alpha_n.$$

Since $\sum_{j=2}^{n} \alpha_j^\vee$ has $h$-value $n - 1$ and $f^{(s)}$ is left $\iota_{\alpha_{n-1}}(\text{SL}_2)$-invariant, by (iii)

$$M'_{s_n w'}(1_s) f^{(s)} |_{s+\ell k + 1 = n-1}$$

is left $s_n s_{n-1} \iota_{\alpha_0}(\text{SL}_2)$-invariant.

Therefore, to justify the holomorphy it suffices to show

$$\frac{a_{s_n w'}(1_s)}{a_{s_n s_{n-1} s_{n-2} s_n w'}(1_s)} M_{s_n s_{n-1} s_{n-2}}(1_{s_n w'})$$

is holomorphic at $s + \ell k + 1 = n - 1$. Since coroots corresponding to $s_n - 2$ and $s_{n-1}$ have $(h,\lambda)$-value $(2(n-1), 2)$ and $(2(n-1) + 1, 2)$ respectively, the holomorphy follows again from (iii).

5.5. **Type $D_n$** ($n \geq 4$). For $1 \leq \ell < n-1$, a reduced expression of $w_0$ is

$$\begin{align*}
(s_\ell \cdots s_n s_{\ell+1} - 1 \mod 2), \\
(s_\ell \cdots s_n s_{\ell}) (s_{n-\ell} + 1 \mod 2), \\
\vdots \\
(s_4 \cdots s_{n-\ell}, s_{4} \cdots s_{n-4} + 1 \mod 2).
\end{align*}$$

(5.14)

corresponding coroots $\alpha_i^\vee$ are

$$\begin{align*}
\alpha_i^\vee, \quad \cdots \sum_{j=1}^{\ell} \alpha_j^\vee, \quad \sum_{j=1}^{\ell} \alpha_j^\vee, \\
\vdots \\
\sum_{j=\ell+1}^{\ell} \alpha_j^\vee, \quad \cdots \sum_{j=\ell+1}^{\ell} \alpha_j^\vee, \\
\vdots \\
\sum_{j=1}^{n-1} \alpha_j^\vee, \quad \cdots \sum_{j=1}^{n-1} \alpha_j^\vee, \\
\sum_{j=1}^{n-1} \alpha_j^\vee, \quad \cdots \sum_{j=1}^{n-1} \alpha_j^\vee, \\
\vdots \\
\sum_{j=1}^{n-1} \alpha_j^\vee, \quad \cdots \sum_{j=1}^{n-1} \alpha_j^\vee.
\end{align*}$$

(*')
We retain the terminology in §5.2 and follow the idea in §5.3. Denote coroots

\[
\sum_{j=r}^{n-2} \alpha_j^\vee + \sum_{j=t}^{n} \alpha_j^\vee = \alpha_n^\vee + \alpha_{n-1}^\vee + 2 \sum_{j=r}^{n-2} \alpha_j^\vee + \sum_{j=t}^{r-1} \alpha_j^\vee \quad \text{for } 1 \leq t < r \leq \ell, \text{ and }
\]

\[
\sum_{j=m}^{n} \alpha_j^\vee + \sum_{j=r}^{n-2} \alpha_j^\vee \quad \text{for } 1 \leq r \leq \ell < m \leq n
\]

by \((2, r, t)\) and \((1, m, r)\) respectively. As the sum of two coroots of the same type is not a coroot, so we still have rules \((R1)\), \((R2)\), \((R3)\). The rule \((R2)\) implies \((5.4)\) and \((5.6)\) for \(\lambda = 2\) similarly.

**Remark 5.14.** Rules of coroots are stable under the symmetry of the Dynkin diagram \(D_n\). For instance, \((R1)\) implies the coroot \((1, n, t)\) must come after the coroot \((n-2, t)\) for any \(1 \leq t \leq \ell\).

Observe that the coroot \((2, r, t)\) can be written as a sum of coroots

\((1, m', r') + (r'', t'')\)

if and only if either \(r' = r\) and \(t'' = t\) or \(r' = t\) and \(t'' = r\). In both cases, \(m' = r'' + 1\).

**Lemma 5.15.** The operation \((b')\) cannot be applied to the triple of coroots \((2, r, t), (1, m, t), (m - 1, r)\) if (and only if) \(m < n\).

**Proof.** By \((R2)\) and \((R3)\), it suffices to show for \(r = t + 1, m = n - 1\). By \((R1)\), an operation \((b')\) that reverses the order of coroots \((n - 1, t + 1)\) and \((2, t + 1, t)\) needs to be carried out first. However, in this case the coroot \((1, n, t + 1)\) comes before \((2, t + 1, t)\), but comes after \((n - 2, t + 1)\) by Remark 5.14. \(\square\)

Consequently, together with operations \((a')\), we have

\[(2, r, t) \leftrightarrow (r', t')\] if and only if \(t' < r\) or \(r' = n - 1\); \hspace{1cm} (D1)

\[(1, m, r) \leftrightarrow (r', t')\] if and only if \(t' < r\) or \(m = r' + 1 = n\); \hspace{1cm} (D2)

\[(2, r, t) \leftrightarrow (1, m', r')\] if and only if \(r < r'\) or \(t < r' \) or \(m' = n\). \hspace{1cm} (D3)

Inequalities \((5.4)\) and \((5.6)\) for \(\lambda = 1\) follow similarly from \((R1)\), \((R3)\), and \((D2)\).

As mentioned in Remark 5.14, the relaxation of rules, compared to rules of type \(B\), arises from the symmetry of \(D_n\). More precisely, the Dynkin diagram of \(D_n\) folds into that of \(B_{n-1}\). This is the reason why we have named our coroots the same way as in §5.3. We explain how one can modify the inductive proof of holomorphy of \(M'_w(\chi_\delta)\) of type \(B\) to that of type \(D\).

By induction hypothesis we have an analogue of Lemma 5.11 that asserts that \(w\) can be written as \(s_\alpha w_1\) where the corresponding coroot \(\beta^\vee\) of \(s_\alpha\) is either \((r, 1)\), \((1, m, 1)\), or \((2, r', 1)\).
For the case $\beta^\vee = (r, 1)$ for $r \neq n - 1$, $\beta^\vee = (1, m, 1)$ for $m \neq n$, or $\beta^\vee = (2, r', 1)$ for $1 < r'$, a similar proof for the same type of roots considered in §5.3 proves the holomorphy. Therefore, by the symmetry of $D_n$, it suffices to consider the case where $\beta^\vee = (n - 1, 1)$ and $\chi = 1$. We can assume any coroot in $\Phi_w$ with $\lambda$-value 1 has $h$-value at most $n - 1$. Note that (I) fails if $\ell \geq 2$ or $\ell = 1$ and $(1, n, 1) \not\in \Phi_w$. If $(1, n, 1) \not\in \Phi_w$, then the holomorphy follows from the argument in §5.3 of the case $\beta^\vee = (n - 1, 1)$. Therefore, we assume $(1, n, 1) \in \Phi_w$. Since (I) holds for $w_1$ if $\ell = 1$, it suffices to consider $\ell \geq 2$. In this case, a similar argument as in §5.3 of the case $\beta^\vee = (1, n, 1)$ justifies the holomorphy.

For $\ell = n, n - 1$, by symmetry it suffices to deal with either case. Let $\ell = n$. A reduced expression of $w_0$ is

$$
(s_{n-(\ell+1 \mod 2)}), \\
(s_{n-2s_{n-(\ell \mod 2)}}), \\
\vdots \\
(s_4 \cdots s_{n-3s_{n-2s_{n-1}}}), \\
(s_3 \cdots s_{n-3s_{n-2s_{n}}}), \\
(s_2 \cdots s_{n-3s_{n-2s_{n-1}}}), \\
(s_1 \cdots s_{n-3s_{n-2s_{n}}});
$$

the corresponding coroots $\alpha^\vee_{(i)}$ are

$$
\begin{align*}
\sum_{j=n-2}^n \alpha^\vee_j, \\
\sum_{j=\ell}^n \alpha^\vee_j, \\
\cdots \\
\sum_{j=r}^n \alpha^\vee_j + \sum_{j=t}^n \alpha^\vee_j, \\
\sum_{j=r}^n \alpha^\vee_j + \sum_{j=\ell}^n \alpha^\vee_j, \\
\cdots \\
\sum_{j=n} \alpha^\vee_j + \sum_{j=1}^n \alpha^\vee_j.
\end{align*}
$$

As mentioned above, the proof of holomorphy for coroots of type $(2, r, t)$ with $r > t$ in §5.3 can be modified to justify the holomorphy. We leave the details to the reader. \qed

### Appendix A.

For readers’ convenience, we list $s_k$ and the multisets $L(d) \cap \mathbb{R}$, which we denote by $L(d)$ for simplicity, in each case.

**$A_n$**

| Node | $L(1)$ | $s_k$ |
|------|--------|------|
| $\ell$ | $\{-i : 0 \leq i \leq \min(\ell - 1, n - \ell)\}$ | $\frac{n-1}{2}$ |

**$B_n$ ($n \geq 2$)**

| Node | $L(1)$ | $L(2)$ | $s_k$ |
|------|--------|--------|------|
| 1 | $\{0\}$ | $\{1 - n\}$ | $\frac{2n-3}{2}$ |
| $1 < \ell < n$ | $\{-i : 0 \leq i \leq \min(\ell - 1, 2n - 2\ell - 1)\}$ | $\{\ell - n - i : 0 \leq i \leq |(\ell - 1)/2|\}$ | $\frac{2n-\ell-2}{2}$ |
| $n$ | $\{-2i : 0 \leq i \leq [(n - 1)/2]\}$ | | $n - 1$ |
\[ C_n \ (n \geq 2) \]

| Node | \( L(1) \) | \( L(2) \) | \( s_k \) |
|------|-------------|-------------|---------|
| 1    | \{0\}      |             | \( n - 1 \) |

\[ 1 < \ell \ \{-i : 0 \leq i \leq \min(\ell - 1, 2n - 2\ell)\} \quad \{\ell - n - i : 1 \leq i \leq \lfloor \ell/2 \rfloor\} \quad \frac{2n - \ell - 1}{2} \]

\[ D_n \ (n \geq 3) \]

| Node | \( L(1) \) | \( L(2) \) | \( s_k \) |
|------|-------------|-------------|---------|
| 1    | \{0, 2 - n\} |             | \( n - 2 \) |

\[ 1 < \ell < n - 1 \ \{-i, \ell - n + 1 : 0 \leq i \leq \min(\ell - 1, 2n - 2\ell - 2)\} \quad \{\ell - n - i : 0 \leq i \leq \lfloor (\ell - 2)/2 \rfloor\} \quad \frac{2n - \ell - 3}{2} \]

\[ n - 1, n \ \{-2i : 0 \leq i \leq \lfloor (n - 2)/2 \rfloor\} \quad \frac{n - 2}{2} \]

\[ E_6 \]

| Node | \( L(1) \) | \( L(2) \) | \( L(3) \) | \( s_k \) |
|------|-------------|-------------|-------------|---------|
| 1, 6 | \{0, -3\}  |             |             | 10      |
| 2    | \{0, -2, -3\} | \{-5\}  |             | 9       |
| 3, 5 | \{0, -1, -2, -3\} | \{-3\}  |             | 7       |
| 4    | \{0, -1, -1, -2, -2\} | \{-2, -\frac{5}{2}, -3\} | \{-3\}  | 5       |

\[ E_7 \]

| Node | \( L(1) \) | \( L(2) \) | \( L(3) \) | \( L(4) \) | \( L(5) \) |
|------|-------------|-------------|-------------|-------------|-------------|
| 1    | \{0, -3, -5\} | \{-8\}      |             |             | 15          |
| 2    | \{0, -2, -3, -4, -6\} | \{-5\}  |             |             | 12          |
| 3    | \{0, -1, -2, -3, -4\} | \{-3, -4, -5\} | \{-5\}  |             | 9           |
| 4    | \{0, -1, -1, -2, -2, -3\} | \{-2, -\frac{5}{2}, -3, -3\} | \{-3, -\frac{10}{7}\} | \{-\frac{7}{7}\} | 6           |
| 5    | \{0, -1, -2, -2, -3, -4\} | \{-3, -\frac{7}{2}, -4\} | \{-4\}  |             | 8           |
| 6    | \{0, -1, -3, -4\} | \{-4, -6\} |             |             | 11          |
| 7    | \{0, -4, -8\} |             |             |             | 16          |
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|}
\hline
Node & $L(1)$ & $L(2)$ & $L(3)$ & $L(4)$ & $L(5)$ & $L(6)$ & $s_k$ \\
\hline
1 & $\{0, -3, -5\}$ & $\{-8, -11\}$ & & & & & 21 \\
\hline
2 & $\{0, -2, -3, -4, -5, -6\}$ & $\{-5, -6, -7, -8\}$ & $\{-7\}$ & & & & 15 \\
\hline
3 & $\{0, -1, -2, -3, -4, -5\}$ & $\{-3, -4, -9/2\}$ & $\{-5, -6\}$ & $\{-11/2\}$ & & & 11 \\
\hline
4 & $\{0, -1, -1, -2, -3, -4, -5\}$ & $\{-2, -3, -9/2, -4\}$ & $\{-3, -10/3, -11/3, -4\}$ & $\{-7/2, -15/4, -4\}$ & $\{-4, -21/5\}$ & $\{-4\}$ & 7 \\
\hline
5 & $\{0, -1, -2, -3, -4, -5\}$ & $\{-4, -9/2, -14/3, -5\}$ & $\{-9/2, -5\}$ & $\{-5\}$ & & & 9 \\
\hline
6 & $\{0, -1, -2, -3, -4, -5\}$ & $\{-4, -9/2, -5, -6\}$ & $\{-5, -6\}$ & $\{-13/2\}$ & & & 12 \\
\hline
7 & $\{0, -1, -4, -5, -8\}$ & $\{-5, -7, -9\}$ & $\{-9\}$ & & & & 17 \\
\hline
8 & $\{0, -5, -9\}$ & $\{-13\}$ & & & & & 27 \\
\hline
\end{tabular}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
Node & $L(1)$ & $L(2)$ & $L(3)$ & $L(4)$ & $s_k$ \\
\hline
1 & $\{0, -3\}$ & $\{-2\}$ & & & 6 \\
\hline
2 & $\{0, -1\}$ & $\{-1, -1/2\}$ & $\{-2\}$ & $\{-2\}$ & 3 \\
\hline
3 & $\{0, -1, -2\}$ & $\{-2, -3\}$ & $\{-3\}$ & & 5 \\
\hline
4 & $\{0, -3\}$ & $\{-5\}$ & & & 9 \\
\hline
\end{tabular}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
Node & $L(1)$ & $L(2)$ & $L(3)$ & $s_k$ \\
\hline
1 & $\{0\}$ & $\{-2\}$ & & 3 \\
\hline
2 & $\{0\}$ & $\{-1\}$ & $\{-1\}$ & 1 \\
\hline
\end{tabular}
\end{table}

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