\({\mathcal{PT}}\)-symmetric Quantum systems for position-dependent effective mass violate the Heisenberg uncertainty principle

Pinaki Patra

Department of Physics, Brahmananda Keshab Chandra College, Kolkata, India-700108

(Dated: August 23, 2022)

Abstract

We have studied a \({\mathcal{PT}}\)-symmetric quantum system for a class of position-dependent effective mass. Formalisms of supersymmetric quantum mechanics are utilized to construct the partner potentials. Since the system under consideration is not self-adjoint, the intertwining operators do not factorize the Hamiltonian. We have factorized the Hamiltonian with the aid of generalized annihilation and creation operators, which acts on a deformed coordinate and momentum space. The coherent state structure for the system is constructed from the eigenstates of the generalized annihilation operator.

It turns out that the self-adjoint deformed position and momentum operators violate the Heisenberg uncertainty principle for the \({\mathcal{PT}}\)-symmetric system. This violation depends solely on the \({\mathcal{PT}}\)-symmetric term, not on the choice of the inner product. For explicit construction, we have demonstrated, for simplicity, a constant mass \({\mathcal{PT}}\)-symmetric system Harmonic oscillator, which shows the violation of the uncertainty principle for a choice of acceptable parameter values. The result indicates that either \({\mathcal{PT}}\)-symmetric systems are a trivial extension of usual quantum mechanics or only suitable for open quantum systems.

Keywords: Position-Dependent Effective Mass; PT-symmetric quantum system, Coherent states, Uncertainty Principle

*Corresponding author; Electronic address: monk.jn@gmail.com
I. INTRODUCTION

Unitarity of the time-evolution operator and real eigenvalues of quantum observables are two fundamental cornerstones of quantum mechanics. For a self-adjoint Hamiltonian ($\hat{H}$), the condition of the real spectrum is trivially followed, whereas the evolution operator is given by $\hat{U}(t) = e^{-i\hat{H}t/\hbar}$ [1]. Nonetheless, non-Hermitian Hamiltonians are still useful for studying open quantum systems in nuclear physics or quantum optics, among others [2, 3]. These non-Hermitian Hamiltonians are considered an effective subsystem within a projective subspace of the total system, which obeys conventional quantum mechanics with a Hermitian Hamiltonian [2–4]. However, in 1988, Bender et al. had shown that the issues of the unitarity and real energy eigenvalues can be taken into account with the help of a weaker condition on the Hamiltonian, namely parity-time ($\mathcal{PT}$)-symmetric Hamiltonians, along with a deformed inner product [5, 6]. The space-reflection linear operator, namely the parity operator ($\hat{P}$), changes the sign of the position ($\hat{x}$) and momentum ($\hat{p}$) operators ($\hat{P}\hat{x}\hat{P}^{-1} = -\hat{x}$, $\hat{P}\hat{p}\hat{P}^{-1} = -\hat{p}$), keeping the fundamental commutation relation (the Heisenberg algebra $[\hat{x}, \hat{p}] = i\hbar$) invariant. The anti-unitary ($\hat{T}i\hat{T}^{-1} = -i$) time-reversal operator ($\hat{T}$) changes the sign of momentum ($\hat{T}\hat{p}\hat{T}^{-1} = -\hat{p}$), but leaves $\hat{x}$ invariant ($\hat{T}\hat{x}\hat{T}^{-1} = \hat{x}$), keeping the Heisenberg-algebra intact. Since, $\hat{P}$ and $\hat{T}$ are similar to the reflection operators, they are involutory operators ($\hat{P}^2 = \hat{T}^2 = \hat{1}$, where $\hat{1}$ is the identity operator). Moreover, $\hat{P}$ and $\hat{T}$ commutes with each other. We say that $\hat{H}$ is $\mathcal{PT}$-symmetric if $[\hat{H}, \hat{P}\hat{T}] = 0$. In order to define an inner product with a positive norm for a complex non-hermitian Hamiltonian having an unbroken $\mathcal{PT}$ symmetry, we will construct a new linear operator $\hat{C}$, which commutes with both $\mathcal{PT}$ and $\hat{H}$. $\hat{C}$ is similar to the charge-conjugation operator in particle physics [7]. However, since quantum mechanics is a single particle theory, $\hat{C}$ is not exactly the charge conjugation operator. The only purpose of $\hat{C}$ is to construct a meaningful inner product.

If the real spectrum and the unitarity are the concerns, then $\mathcal{PT}$-symmetric Hamiltonians had opened up the possibilities of incorporating a larger class of operators to be considered as a viable Hamiltonian for a quantum mechanical system. A wide class of systems had been studied for the last two decades [8–13]. For an operational foundations of $\mathcal{PT}$-symmetric and quasi-Hermitian quantum theory, one can see [12].

Complementary, another important issue in quantum mechanics is the concept of the
position-dependent effective mass (PDEM), which was incepted to describe the electronic properties and band structure of semiconductor Physics [14, 15]. Later, it was proved to be exist for various domain of Physics. Notably, frequent occurrence of effective mass in the study of nonlinear optical properties in quantum well, the asymmetric shape of crackling noise pulses emitted by a diverse range of noisy systems, in cosmological models, and even in quantum information theory, had made PDEM a topical issue [16–24]. Since PDEM and momentum do not commute with each other, there is ambiguity in the form of a viable kinetic part of PDEM Hamiltonian [25–28]. We shall consider a fairly accepted form of PDEM Hamiltonian in the present paper [29–32].

On the other hand, the construction of coherent states (CS) for a quantum system opens up the possibility of experimental verification with the help of interferometry [33–38]. Being a minimum uncertainty state, CS closely resembles the classical limit of a quantum system [39]. If the Hamiltonian can be factorized in terms of annihilation and creation operators, then the eigenstates of the annihilation operator will provide CS for the system [40–42]. For a Hermitian Hamiltonian, the factorization of the Hamiltonian can be done with the help of intertwining operator which connects the Hamiltonian with its supersymmetric (SUSY) partner [43, 44]. CS structures for PDEM systems are well studied in the literature [45–47]. In the present paper, we have shown that for $\mathcal{PT}$-symmetric case, the intertwining operators are not sufficient to factorize the Hamiltonian. We have factorized our Hamiltonian under consideration with the help of an ansatz, which is a deformed version of the intertwining operator connecting the SUSY partner of the original Hamiltonian. It turns out that the factorizing operators satisfy the algebra of the annihilation and creation operators in a deformed space (deformed position and momentum). The eigenstates of the deformed annihilation operator provide the CS structure for the system.

At the end of this paper, with the aid of the expectation values and variances, we have shown that the system violates the Heisenberg uncertainty principle. We have shown that this violation also holds for the constant mass system. Moreover, we have shown that the violation occurs solely due to the non-Hermitian $\mathcal{PT}$-symmetric term. When $\mathcal{PT}$-symmetric contribution is omitted, the uncertainty principle perfectly holds for the general PDEM system. These are the main findings of the present paper.

Not only the Heisenberg uncertainty principle is a building block of quantum philosophy, but also it is so profound that any violation of this will leads to the violation of the empiri-
cally established second law of thermodynamics [48]. Since I believe that the second law of thermodynamics is very unlike to be violated in nature, I would like to conclude that the \( \mathcal{PT} \)-symmetric system is either a trivial extension of usual quantum mechanics or likely to be false.

II. SYSTEM UNDER CONSIDERATION

Let us consider a \( \mathcal{PT} \)-symmetric Hamiltonian (\( \hat{H} \)) for the position-dependent effective mass \( m(x) \) as

\[
\hat{H} = \hat{p} \frac{1}{m} \hat{p} - \frac{1}{2m} \hat{p} \hat{m} \hat{p} \frac{1}{m} + V(x) + i(\beta_1 \hat{p} \hat{x} + \beta_2 \hat{x} \hat{p}).
\]  

(1)

For constant \( m \) and the absence of the \( \mathcal{PT} \)-symmetric term (dependent on the parameters \( \beta_1 \) and \( \beta_2 \)) will lead to the usual Hermitian Hamiltonian. For convenience, we shall consider the following sufficient conditions to (1) be \( \mathcal{PT} \)-symmetric.

i. \( \beta_1, \beta_2 \in \mathbb{R} \).

ii. If \( V(x) : \mathbb{R} \to \mathbb{R} \), then \( V(x) \) is an even function (\( V(-x) = V(x) \)).

Otherwise, \( V(x) = iV_0(x) \), with \( V_0(x) : \mathbb{R} \to \mathbb{R} \), and \( V_0(x) \) is an odd function (\( V_0(-x) = -V_0(x) \)).

iii. If \( m(x) : \mathbb{R} \to \mathbb{R} \), then \( m(-x) = m(x) \).

Otherwise, \( m(x) = im_0(x) \), with \( m_0(x) : \mathbb{R} \to \mathbb{R} \), and \( m_0(x) \) is an odd function.

The adjoint (\( \hat{H}^\dagger \)) of \( \hat{H} \) is given by

\[
\hat{H}^\dagger = \hat{p} \frac{1}{m} \hat{p} - \frac{1}{2m} \hat{p} \hat{m} \hat{p} \frac{1}{m} + V(x) - i(\beta_1 \hat{x} \hat{p} + \beta_2 \hat{p} \hat{x}),
\]  

(2)

which is not equal to \( \hat{H} \) for \( \beta_1 \neq -\beta_2 \). In other words, \( \hat{H} \) is Hermitian iff \( \beta_1 = -\beta_2 \).

If an invertible Hermitian operator \( \hat{\eta} : \mathcal{H} \to \mathcal{H} \) exists, such that

\[
\hat{H}^\dagger = \hat{\eta} \hat{H} \hat{\eta}^{-1},
\]  

(3)

then the linear operator \( \hat{H} : \mathcal{H} \to \mathcal{H} \) is said to be pseudo-Hermitian. To see \( \hat{H} \) in (1) is a pseudo-Hermitian operator, let us show the existence of a unitarity operator \( \hat{\eta} \) such that (3) holds. Let us use the ansatz

\[
\hat{\eta} = e^{\hat{\Lambda}(x)}
\]  

(4)
in (3) and use the Baker-Campbell-Hausdorff formula. We get the following set of equations.

\[ e^{\hat{\Lambda}} \frac{1}{m} \hat{p} e^{-\hat{\Lambda}} = \frac{1}{m} \hat{p} + i \hbar \{ \frac{1}{m} \hat{p}, \frac{1}{m} \Lambda' \} - \frac{\hbar^2}{m} \Lambda'^2. \]  

(5)

\[ e^{\hat{\Lambda}} \frac{1}{m} \hat{p} m \frac{1}{m} e^{-\hat{\Lambda}} = \frac{1}{m} \hat{p} m \frac{1}{m} + i \hbar (\Lambda' \hat{p} \frac{1}{m} m + \frac{1}{m} \hat{p} \Lambda') + \frac{i \hbar}{m} (\Lambda')^2. \]  

(6)

\[ e^{\hat{\Lambda}} V(x) e^{-\hat{\Lambda}} = V(x). \]  

(7)

\[ e^{\hat{\Lambda}} i (\beta_1 \hat{p} \hat{x} + \beta_2 \hat{x} \hat{p}) e^{-\hat{\Lambda}} = i (\beta_1 \hat{p} \hat{x} + \beta_2 \hat{x} \hat{p}) - \hbar (\beta_1 + \beta_2) x \Lambda'. \]  

(8)

Here prime denotes the derivative with respect to \( x \), and \( \{ u, v \} := uv + vu \) denotes the anti-commutator of \( u \) and \( v \). Now we can write the following transformation law of the Hamiltonian.

\[ e^{\hat{\Lambda}} \hat{H} e^{-\hat{\Lambda}} = \hat{H} + \frac{i \hbar}{2} \{ \frac{1}{m} \hat{p}, \frac{1}{m} \Lambda' \} - \frac{\hbar^2}{2m} (\Lambda')^2 - \hbar (\beta_1 + \beta_2) x \Lambda'. \]  

(9)

Comparing the equations (2), (3) and (9), we arrive at the following sufficient condition for \( \Lambda(x) \).

\[ \Lambda'(x) = -\frac{2}{\hbar} (\beta_1 + \beta_2) x m(x). \]  

(10)

The first order equation (10) always admits a solution for all locally integrable \( m(x) \). For \( \beta_1 = -\beta_2 \), \( \hat{H} \) is identical with its adjoint for the choice of integration constant of the solution of (10) to be zero, which will be assumed throughout this paper. In particular, the Hamiltonian (1) is pseudo-Hermitian, and \( \hat{H} \) is connected with its adjoint by the similarity transformation

\[ \hat{\eta} = \exp \left[ -\frac{2}{\hbar} (\beta_1 + \beta_2) \int x y m(y) dy \right]. \]  

(11)

Existence of \( \hat{\eta} \) suggests the following inner product

\[ \langle \zeta_1 | \zeta_2 \rangle = (\zeta_1, \hat{\eta} \zeta_2), \]  

(12)

where \( (u, v) \) is the usual inner-product in \( L^2 \) space. We shall use the inner product (12) in the subsequent sections.

III. SUSY FORMALISM

In supersymmetric quantum mechanics (SUSY) formalism, we seek for an intertwining operator \( \hat{A} \), which connects \( \hat{H} \) to its supersymmetric partner \( \hat{\tilde{H}} \) by the relation

\[ \hat{A} \hat{H} = \hat{\tilde{H}} \hat{A}. \]  

(13)
Kinetic part of $\hat{H}$ and $\hat{\tilde{H}}$ are the same, whereas the supersymmetric partner potential ($\tilde{V}(x)$) are different than that of the original potential ($V(x)$) of $\hat{H}$.

For illustration, let us work in the position representation ($\{|x\rangle\}$), in which the Hamiltonian (1) reads

$$\hat{H} = -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial x^2} - u \frac{\partial}{\partial x} \right) + V_e(x), \quad (14)$$

with

$$u(x) = \frac{m'}{m} + \frac{2}{\hbar} (\beta_1 + \beta_2) x m. \quad (15)$$

$$V_e(x) = \hbar \beta_1 + V(x) - \frac{\hbar^2}{2m} \left( \frac{m'}{m} \right)'. \quad (16)$$

If we consider the ansatz

$$\hat{A} = \frac{1}{\sqrt{2}} \left( a(x) \frac{d}{dx} a(x) + \phi(x) \right), \quad (17)$$

which connects the supersymmetric partner Hamiltonian

$$\hat{\tilde{H}} = -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial x^2} - u \frac{\partial}{\partial x} \right) + \tilde{V}(x), \quad (18)$$

through the equation (13), then from consistency conditions, we obtain the following set of equations.

$$2(1/m)(a^2)' = (1/m)'a^2. \quad (19)$$

$$(2ma^2/\hbar^2)(\tilde{V} - V_e) = (ua^2)' + 2\phi_a' + (a^2)''. \quad (20)$$

$$\frac{\hbar^2}{2m}(u\phi_a' - \phi_a'') + (\tilde{V} - V_e)\phi_a = a^2V_e''. \quad (21)$$

Where

$$\phi_a(x) = \phi + \frac{1}{2}(a^2)'. \quad (22)$$

Solving the equation (19), we get

$$a(x) = a_0 m^{-1/4}(x), \quad (23)$$

where the integration constant $a_0 \in \mathbb{R} \setminus \{0\}$. Using (23) and (15) in (20), the partner potential is reduced to

$$\tilde{V} = V_e + \frac{\hbar^2 \phi'}{a_0^2 \sqrt{m}} + \hbar(\beta_1 + \beta_2)(1 + \frac{m'}{2m} x). \quad (24)$$
Defining an auxiliary function \( K(x) \) by

\[
K(x)a^2 = \phi_a(x),
\]

the equation (21) is reduced to

\[
\frac{d}{dx} \left( \frac{1}{m} (K^2 - K' + uK - v_e) \right) = 0,
\]

where

\[
V_e = \frac{\hbar^2}{2m} v_e.
\]

Solving equation (26), we see that the auxiliary function \( K(x) \) obeys the Riccati equation

\[
K' - uK - K^2 + v_e + \mu m = 0.
\]

Where, the integration constant \( \mu \) is arbitrary. Solving the Riccati equation (28), we can determine the superpotential \( \phi(x) \) from

\[
\phi(x) = Ka^2 - \frac{1}{2}(a^2)'.
\]

For a Hermitian Hamiltonian, the intertwining operator (\( \hat{A} \)) and its adjoint (\( \hat{A}^\dagger \)) will factorize the Hamiltonian. However, for our \( \mathcal{PT} \)-symmetric Hamiltonian, one can see that \( \hat{H} \neq \hat{A}^\dagger \hat{A} + \lambda \) for some \( \lambda \in \mathbb{R} \). On the following section, we have proposed a factorization for our system.

**IV. COHERENT STATE STRUCTURE**

Since we are dealing with a non-hermitian Hamiltonian (\( \hat{H} \)), the intertwining operator \( \hat{A} \) will not be sufficient to factorize \( \hat{H} \). Let us consider the ansatz

\[
\hat{A}_\pm = \frac{1}{\sqrt{2}} \left( a_\pm(x) \frac{d}{dx} a_\pm(x) + \phi_\pm(x) \right),
\]

such that \( \hat{H} \) is factorized as

\[
\hat{H} = \hat{A}_+ \hat{A}_- + \lambda,
\]

for some constant \( \lambda \in \mathbb{C} \).

Using the ansatz (30) in (31) and comparing it with (14), we get the following set of equa-
\[ a^2 - a^2 = -\frac{\hbar^2}{m}. \] (32)
\[ \frac{\hbar^2}{m} u = a^2 (a^2 a^2)' + a^2 \tilde{a} + a^2 \phi_+. \] (33)
\[ 2V_e = a^2 \phi_+ + (\phi_+ + a^2 a^2) \phi_. \] (34)

Where \( \tilde{\phi}_- = \phi_+ + \frac{1}{2} (a^2)' \). From (32), we make the following choice for \( a_\pm \).
\[ a_+ = i a_-, \quad a_- = \sqrt{\hbar m}^{-\frac{1}{4}}. \] (35)

Using (35) in (33), we get
\[ u_0(x) = \phi_+ - \phi_- = 2(\beta_1 + \beta_2)x\sqrt{m}. \] (36)

Using (36) in (34) we get the following Riccati equation for \( \phi_- \).
\[ -\frac{\hbar}{\sqrt{m}} \phi_- + u_0 \phi_- + \phi_-^2 = \tilde{V}_e, \] (37)

where
\[ \tilde{V}_e = 2V_e + \frac{\hbar}{2} (\beta_1 + \beta_2)x \left( \frac{m'}{m} \right) + \frac{\hbar^2}{4m^2} \left( \frac{7m'^2}{4m} - m'' \right). \] (38)

Using the explicit form of \( \hat{A}_\pm \) in (31), one can see that
\[ \lambda = 0. \] (39)

That means, we set the ground state energy to zero. Let us define a deformed self-adjoint co-ordinate \( \hat{\Phi} \) and momentum \( \hat{\Pi} \) operators by
\[ \hat{\Phi} = \frac{1}{2} (\phi_- + \phi_+) = \frac{1}{\sqrt{2}} (\hat{A}_+ + \hat{A}_-), \] (40)
\[ \hat{\Pi} = \frac{i}{\sqrt{2}} (\hat{A}_+ - \hat{A}_-) - u_0) = \frac{i}{\sqrt{2}} (\hat{A}_+ - \hat{A}_-). \] (41)

The commutation relations
\[ [\hat{\Phi}, \hat{\Pi}] = \frac{i \hbar}{\sqrt{m}} \phi', \] (42)
\[ [\hat{A}_-, \hat{A}_+] = \frac{\hbar}{\sqrt{m}} \phi', \] (43)

suggest that \( \hat{A}_- \) and \( \hat{A}_+ \) are the generalized annihilation and creation operators for our \( \mathcal{PT} \) symmetric system, respectively. If \( |\alpha\rangle \) are the eigen-vectors of \( \hat{A}_- \), i.e., if
\[ \hat{A}_- |\alpha\rangle = \alpha |\alpha\rangle, \quad \alpha \in \mathbb{C}, \] (44)
then \( |\alpha\rangle \) provide a coherent state (CS) structure for the system. To verify whether \( |\alpha\rangle \) are indeed CS, we have to verify the uncertainty measure, which should be minimum for a CS.
V. EXPECTATION VALUES AND VARIANCES

To compute the expectation values, we first observe the followings.

\[ \hat{\mathcal{A}}_+ = \frac{1}{\sqrt{2}} u_0 + \hat{\mathcal{A}}_\dagger. \] (45)

\[ [u_0, \hat{\mathcal{A}}_\dagger] = \frac{\hbar}{\sqrt{2m}} u_0. \] (46)

\[ [\hat{\mathcal{A}}_-, \hat{\mathcal{A}}_\dagger] = \frac{\hbar}{\sqrt{m}} \phi'. \] (47)

\[ \hat{\mathcal{A}}_+^2 = (\hat{\mathcal{A}}_\dagger)^2 + \sqrt{2} \hat{\mathcal{A}}_\dagger u_0 + \frac{1}{2}(u_0^2 + \frac{\hbar}{\sqrt{m}} u_0'). \] (48)

\[ \hat{\mathcal{A}}_- \hat{\mathcal{A}}_+ + \hat{\mathcal{A}}_+ \hat{\mathcal{A}}_- = 2 \hat{\mathcal{A}}_\dagger \hat{\mathcal{A}}_- + \sqrt{2} u_0 \hat{\mathcal{A}}_- + \frac{\hbar}{\sqrt{m}} \Phi'. \] (49)

If we define the expectation value \( \langle \hat{O} \rangle_\alpha \) of an operator \( \hat{O} \) on a normalized state \( |\alpha\rangle \) by \( \langle \alpha|\hat{O}|\alpha\rangle \), then we have following expressions for the expectation values.

\[ \langle \hat{\mathcal{A}}_- \rangle = \alpha, \quad \langle \hat{\mathcal{A}}_\dagger \rangle = \alpha^*, \quad \langle \hat{\mathcal{A}}_-^2 \rangle = \alpha^2. \] (50)

\[ \langle \hat{\mathcal{A}}_+ \rangle = \alpha^* + \frac{1}{\sqrt{2}} \langle u_0 \rangle. \] (51)

Using these we can write

\[ \langle \Phi \rangle = \frac{1}{2} \langle u_0 \rangle + \sqrt{2} \alpha_r. \] (52)

\[ \langle \Pi \rangle = \sqrt{2} \alpha_i \] (53)

\[ \langle \Phi^2 \rangle = 2 \alpha_r^2 + \sqrt{2} \alpha_r \langle u_0 \rangle + \frac{1}{4} \langle u_0^2 \rangle + \frac{1}{2} \langle \frac{\hbar}{\sqrt{m}} \phi' \rangle. \] (54)

\[ \langle \Pi^2 \rangle = 2 \alpha_i^2 + \frac{1}{2} \langle \frac{\hbar}{\sqrt{m}} \phi' \rangle. \] (55)

Where

\[ \alpha_r = \text{Re}(\alpha), \quad \alpha_i = \text{Im}(\alpha). \] (56)

Therefore the variances \( \Delta \hat{O} = \sqrt{\langle \hat{O}^2 \rangle_\alpha - \langle \hat{O} \rangle^2_\alpha} \) reads

\[ (\Delta \hat{\Phi})^2 = \frac{\hbar}{2} \langle \frac{\Phi'}{\sqrt{m}} \rangle + \frac{\hbar}{4} \langle \frac{u_0'}{\sqrt{m}} \rangle + \frac{1}{4} \langle \Delta u_0 \rangle^2. \] (57)

\[ (\Delta \hat{\Pi})^2 = \frac{\hbar}{2} \langle \frac{\Phi'}{\sqrt{m}} \rangle - \frac{\hbar}{4} \langle \frac{u_0'}{\sqrt{m}} \rangle. \] (58)
It is evident that for a self-adjoint position-dependent mass Hamiltonian (i.e., for \( u_0 = 0 \)), the minimum uncertainty conditions

\[
\begin{align*}
(\Delta \hat{\Phi})^2 &= \frac{\hbar}{2} \langle \frac{\Phi'}{\sqrt{m}} \rangle, \\
(\Delta \hat{\Phi})(\Delta \hat{\Pi}) &= \frac{\hbar}{2} |\langle [\hat{\Phi}, \hat{\Pi}] \rangle| \\
(\Delta \hat{\Pi})^2 &= \frac{\hbar^2}{4} |\langle [\hat{\Phi}, \hat{\Pi}] \rangle|^2 + \frac{\hbar^2}{8} (\Delta u_0)^2 \langle \frac{\phi_-}{\sqrt{m}} \rangle - \frac{\hbar^2}{16} \langle \frac{u'_0}{\sqrt{m}} \rangle^2.
\end{align*}
\]

(59) (60) (61)

are satisfied in a straightforward manner. However, for in general \( \mathcal{PT} \)-symmetric system the following holds.

\[
(\Delta \hat{\Phi})^2 (\Delta \hat{\Pi})^2 = \hbar^2 \langle [\hat{\Phi}, \hat{\Pi}] \rangle^2 + \frac{\hbar^2}{8} (\Delta u_0)^2 \langle \frac{\phi_-}{\sqrt{m}} \rangle - \frac{\hbar^2}{16} \langle \frac{u'_0}{\sqrt{m}} \rangle^2.
\]

Contribution from the last two terms of (61) can be negative. For a demonstrative purpose, let us consider the constant mass system (i.e., \( m \) is constant) under the potential

\[
V(x) = \frac{1}{2} m \omega^2 x^2, \quad \omega \in \mathbb{R}.
\]

(62)

Let us fix the parameter values

\[
\omega^2 = 4 \beta_2 \geq 0, \quad \beta_1 = -1.
\]

(63)

Then from (37) we get

\[
\phi_- = 2 \sqrt{m} x.
\]

(64)

This leads to

\[
(\Delta \Pi)^2 = \hbar, \quad (\Delta \Phi)^2 = \hbar \beta_2 + (\beta_2 - 1)m(\Delta x)^2,
\]

(65) (66) (67)

Let us calculate \( \Delta x \) on the state \( |0 \rangle \). Since

\[
\hat{A}_- |0 \rangle = 0,
\]

(68)

we have the coherent state

\[
|0 \rangle = c_0 e^{-\frac{m}{\hbar} x^2}.
\]

(69)

Using the inner product (12), we have the normalization constant

\[
|c_0| = \sqrt{\frac{\beta}{\pi}}, \quad \beta = \frac{m}{\hbar} (1 + \beta_2).
\]

(70)
Using (69), we get
\[(\Delta x)^2 = \frac{1}{2\beta}.\]  
(71)

Therefore
\[(\Delta \Phi)^2(\Delta \Pi)^2 = \hbar^2[\beta_2 + \frac{(\beta_2 - 1)}{2(\beta_2 + 1)}].\]  
(72)

Now if we consider
\[(\Delta \Phi)^2(\Delta \Pi)^2 \geq |\langle[\Phi, \Pi]\rangle|^2,\]  
(73)

we get
\[(\beta_0 - \frac{1}{2})^2 + \frac{1}{4} \leq -\frac{1}{\beta_0}, \quad \beta_0 = 1 + \beta_2 \geq 1.\]  
(74)

Clearly (74) is false for all \(\beta_2 \geq 0\). That means (73) is not true. Therefore, the model \(\mathcal{P}\mathcal{T}\) symmetric system violates uncertainty principle. Since any violation of uncertainty principle operationally implies the violation of the second law of thermodynamics, it is very unlike that this type of system exists in nature.

VI. CONCLUSIONS

We have considered a general class of position dependent effective mass (PDEM) under a \(\mathcal{P}\mathcal{T}\) symmetric interaction. The formalism of supersymmetric quantum mechanics (SUSY) is utilized. Thus the result can be extended for the SUSY partner potentials and to construct the eigen-states in a straightforward manner. Since the Hamiltonian under consideration is not Hermitian, the supersymmetric intertwining operators in general will not factorize the system. In order to factorize the system, we have considered an ansatz for a generalized annihilation (\(\hat{A}_-\)) and a creation (\(\hat{A}_+\)) operators. From the consistency condition, the exact form of \(\hat{A}_-\) and \(\hat{A}_+\) are determined. From the algebra of the operators, it turns out that \(\hat{A}_-\) and \(\hat{A}_+\) acts as an annihilation and a creation operators on a deformed space. With the construction of a deformed inner product, under which the Hamiltonian becomes pseudo-Hermitian, we have shown that the uncertainty realtion for the deformed co-ordinate and momentum violates the Heisenberg uncertainty principle. Since, any violation of the uncertainty principle will leads to a violation of the second law of thermodynamics [48], it is very unlike to occur in the nature. Thus, we conclude that either \(\mathcal{P}\mathcal{T}\) -symmetric systems are a trivial extension of usual quantum mechanics, or it is only suitable for the open quantum
systems. We would like to mention that in [4] it was shown that local $\mathcal{PT}$ symmetry violates the no-signaling principle of relativity. It will be an interesting problem to establish a connection between [4], [48] and the findings of the present paper.

[1] D. J. Griffiths, “Introduction to quantum mechanics”(1st ed.). New Jersey: Prentice Hall. ISBN 978-0-13-124405-4 (1995).
[2] H. Feshbach, Ann. Phys. (N.Y.) 5, 357 (1958).
[3] M. B. Plenio and P. L. Knight, Rev. Mod. Phys. 70, 101 (1998).
[4] Yi-Chan Lee, Min-Hsiu Hsieh, S. T. Flammia, and Ray-Kuang Lee, Phys. Rev. Lett. 112, 130404 (2014).
[5] C. M. Bender and S. Boettcher, Phys. Rev. Lett. 80, 5243 (1998).
[6] C. M. Bender, D. C. Brody, and H. F. Jones, Phys. Rev. Lett. 89, 270401 (2002); Erratum Phys. Rev. Lett. 92, 119902 (2004).
[7] C. M. Bender, Rep. Prog. Phys. 70 947 (2007).
[8] C. E. Rüter, K. G. Makris, R. El-Ganainy, D. N. Christodoulides, M. Segev, and D. Kip, Nature Phys 6, 192-195 (2010).
[9] J. Schindler, A. Li, M. C. Zheng, F. M. Ellis, and T. Kottos, Phys. Rev. A 84, 040101(R) (2011).
[10] J. Zhang, L. Li, G. Wang, X. Feng, B-Ou Guan, and J. Yao, Nat Commun 11, 3217 (2020).
[11] S. Bittner, B. Dietz, U. Günther, H. L. Harney, M. Miski-Oglu, A. Richter, and F. Schäfer, Phys. Rev. Lett. 108, 024101 (2012).
[12] A. Alase, S. Karuvade, and C. M. Scandolo, J. Phys. A: Math. Theor. 55, 244003 (2022).
[13] A. Benbernou, N. Boussekkine, N. Mecherout, T. Ramond, and J. Sjöstrand, Lett Math Phys 106, 1817-1835 (2016).
[14] R. A. Morrow, Physical Review B 35, 8074 (1987).
[15] M. G. Silveirinha, and N. Engheta, Phys. Rev. B 86 161104(R) (2012).
[16] M. L. Cassou, S. H. Dong, and J. Yu, Phys. Lett. A 331, 45 (2004).
[17] F. A. de Saavedra, J. Boronat, A. Polls, and A. Fabrocini, Phys. Rev. B 50, 4248 (1994).
[18] L. C. N. Santos, and C. C. Barros Jr., Eur. Phys. J. C 78, 13 (2018).
[19] H. Hassanabadi, W. S. Chung, S. Zare, and M. Alimohammadi, The European Physical
[20] K. Li, K. Guo, X. Jiang, and M. Hu, Optik 132, 375 (2017).

[21] Q. Zhao, S. Aqiqi, J. F. You, M. Kria, K. X. Guo, E. Feddi, Z. H. Zhang, and J.H. Yuan, Physica E 115, 113707 (2020).

[22] S. Zapperi, C. Castellano, F. Colaiori, and G. Durin, Nat. Phys. 1, 46 (2005).

[23] F. A. Serrano, B. J. Falaye, and S. H. Dong, Physica A 446, 152 (2016).

[24] C. A. Onate, O. Adenimpe, A. F. Lukman, I. J. Adama, E. O. Davids, and K. O. Dopamu, Results Phys. 11, 1094 (2018).

[25] A. de Souza Dutra, J. Phys. A 39, 203 (2006).

[26] A. G. M Schmidt, Phys. Lett. A 353, 459 (2006).

[27] M. S. Abdalla, and H. Eleuch, AIP Adv. 6 055011 (2016).

[28] P. K. Jha, H. Eleuch, and Y. V. Rostovtsev, J. Mod. Optic. 58, 652 (2011).

[29] K. Biswas, J. P. Saha, and P. Patra, Eur. Phys. J. Plus 135, 457 (2020).

[30] V. C. Ruby, and M. Senthilvelan, J. Math. Phys. 51, 052106 (2010).

[31] G. Bastard, Phys. Rev. B 24, 4714-4722 (1981).

[32] H. Sari, E. Kasapoglu, S. Sakiroglu, I. Sökmen, and C.A. Duque, Phys. Status Solidi B 256, 1800758 (2019).

[33] Integrability, Supersymmetry and Coherent States, Editors: Ş. Kuru, J. Negro, and L. M. Nieto, Publisher: Springer Cham, eBook ISBN 978-3-030-20087-9 (2019).

[34] C. L. Mehta, P. Chand, E. C. G. Sudarshan, and R. Vedam, Phys. Rev. 157, 1198 (1967).

[35] R. J. Glauber, Phys. Rev. 131, 2766 (1963).

[36] D. Cocco, G. Cutler, M. S. del Rio, L. Rebuffi, X. Shi, and K. Yamauchi, Physics Reports 974, 1-40 (2022).

[37] M. Pitkin, S. Reid, S. Rowan and J. Hough, Living Rev. Relativ. 14, 5 (2011).

[38] U. D. Jentschura, Phys. Rev. A 98, 032508 (2018).

[39] W. Zhang, D. H. Feng, and R. Gilmore, Rev. Mod. Phys. 62, 867 (1990).

[40] E. Díaz-Bautista, and D. J. Fernández C., Eur. Phys. J. Plus 134, 61 (2019).

[41] M. Tchoffo, F.B. Migueu, M. Vubangsi, and L.C. Fai, Heliyon 5, e02395 (2019).

[42] R. Roknizadeh, and H. Heydari, International Journal of Geometric Methods in Modern Physics 10, 1350056 (2013).

[43] B. G. da Costa, G. A. C. da Silva, and I. S. Gomez, J. Math. Phys. 62, 092101 (2021).
[44] D. J. Fernández, V. Hussin, and O. Rosas-Ortiz, J. Phys. A: Math. Theor. 40 6491 (2007).
[45] R. N. Costa Filho, M. P. Almeida, G. A. Farias, and J. S. Andrade, Jr., Phys. Rev. A 84, 050102(R) (2011).
[46] S. H. Mazharimousavi, Phys. Rev. A 85, 034102 (2012).
[47] A. de Souza Dutra, and C. A. S. Almeida, Phys. Lett. A 275, 25 (2000).
[48] E. Härggi, and S. Wehner, Nat Commun 4, 1670 (2013).