ON THE NON-VANISHING OF MODULAR $L$-VALUES AND FOURIER COEFFICIENTS OF CUSP FORMS

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Abstract. We prove a non-vanishing result of modular $L$-values with quadratic twists, where the quadratic discriminants are in a short interval. Using this fact and Waldspurger’s theorem, we improve the results of Balog-Ono[The chebotarev density theorem in short intervals and some questions of Serre, Journal of number theory. 91(2):356-371(2001)] on the non-vanishing of Fourier coefficients of half-integral weight eigenform.

1. Introduction and statement of results

Let $f$ be a Hecke eigenform of level $N$ and weight $2k$ where $k$ is a positive integer, and let $\chi_d$ be a primitive quadratic character associated to the quadratic field $\mathbb{Q}(\sqrt{d})$. The study of the non-vanishing of $L(k, f, \chi_d)$ arose from the study of the rank of modular elliptic curves. The celebrated theorem of Kolyvagin asserted that if $L(s, E)$ has a simple zero at $s = 1$ and there exists a quadratic discriminant $d < 0$ coprime to $4N$ such that $L(1, E, \chi_d) \neq 0$, then the rank $E(\mathbb{Q}) = 1$ (See the introduction of [13]). Murty-Murty [9] showed that one can get rid of the non-vanishing condition, by showing that there exists an infinite family of quadratic discriminants with $L(1, E, \chi_d) \neq 0$. Even further, the quantitative estimates on the number of quadratic discriminants with $L(1, E, \chi_d) \neq 0$ has gained huge interests.

Let $N_f(X) := \# \{ d : d \text{ is a square-free quadratic discriminant, } |d| \leq X, L(k, f, \chi_d) \neq 0 \}.$

The well-known conjecture on the non-vanishing of $L(k, f, \chi_d)$ asserts that $N_f(X) \gg X$. The best known lower bound is that of Ono-Skinner [10], when $f$ is an even weight newform of trivial nebentype,

$$N_f(X) \gg X/\log X.$$

We are interested in a short interval analogue of $N_f(X)$, where $f(z) \in S_{2k}(N, \psi)$ is a normalized Hecke eigenform. Here, the set of quadratic discriminants are defined modulo $4N$, Namely

$$\mathcal{D} := \{ 0 < (-1)^k d : d \equiv v^2 \mod 4N \text{ for some } (v, 4N) = 1 \}.$$

We denote $\gamma(4N) := \# \mathcal{D}$. The counting function of the number of nonzero $L$-values in short interval is defined by

$$N_f(X, h) := \# \{ d \in \mathcal{D} : d \text{ square-free, } X \leq |d| \leq X + h, L(k, f, \chi_d) \neq 0 \}.$$

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To bound $N_f(X,h)$, it is necessary to estimate the first moment of $L$-values in a short interval. That is,

$$S_f(X,h) := \sum_{d \in D, X \leq |d| \leq X+h} L(k, f, \chi_d).$$

Here, the superscript ’ denotes that the sum is taken over the square-free numbers $d$.

**Theorem 1.1.** For $X^{3/4+\epsilon} \leq h \leq X$, we have

$$S_f(X,h) = C_N L_f(k) h + O_{f,\epsilon}(hX^{-\epsilon}),$$

where the constants $C_N$ and $L_f(k)$ are given by

$$C_N = \frac{3\gamma(4N)}{\pi^2N} \prod_{p \nmid 4N} (1 - p^{-2})^{-1}$$

and

$$L_f(k) = \sum_{n=rj^2 \atop \gcd(n,4N)=1} \frac{a_n}{n^k} \prod_{p \mid n} (1 + p^{-1})^{-1} \neq 0,$$

respectively. In the R.H.S of (1.1), the indices $r$ and $j$ are positive integers with $r \mid (4N)^\infty$ and $(j,4N) = 1$, respectively. Note $L_f(s)$ is the $L$-function $L_{f,1}(s)$ given in [8, p.385].

To obtain Theorem 1.1, we need to make use of the second moment of $L$-values, given in [12] and [5].

**Theorem 1.2.** [12, 5] If $f$ is an eigenform, then

$$\sum_{|d| \leq X} |L(k, f, \chi_d)|^2 \ll_{\epsilon} X^{1+\epsilon}.$$

The authors in [12] and [5] proved Theorem 1.2 for newforms, but the discussion goes through for eigenforms as well.

By Cauchy’s inequality, we have an immediate corollary.

**Corollary 1.3.** For $X^{3/4+\epsilon} \leq h \leq X$, we have

$$N_f(X,h) \gg \frac{h^2}{X^{1+\epsilon}}.$$

The implied constant only depends on $f,\epsilon$.

As far as we know, the only present result on non-vanishing of $L(k, f, \chi_d)$ in short intervals other than Corollary 1.3 is that of Balog-Ono [1]. Their result is as follows.

**Theorem 1.4.** [1] Let $f \in S_{2k}^{\text{new}}(\Gamma_0(N), \psi_{\text{triv}})$. If $f(z)$ is not a linear combination of weight-$3/2$ theta functions, then there exists a positive integer $k_f$ such that for $X^{1-1/k_f+\epsilon} \leq h \leq X$,

$$\# \{d : d \text{ square-free}, X \leq |d| \leq X + h, L(k, f, \chi_d) \neq 0\} \gg \frac{h}{\log X}.$$
Note that in (1.2), \(d\) runs through not only \(D\) but also all square-free quadratic discriminants. Unlike Theorem 1.4, Corollary 1.3 gives a non-vanishing result for the eigenforms of non-trivial nebentypus. In case of trivial nebentype, although Corollary 1.3 gives a weaker result when \(h \geq X^{3/4+\epsilon}\), it is worth noting that it gives further information on \(N_f(X,h)\) when \(X^{3/4+\epsilon} \leq h < X^{1-1/k_f+\epsilon}\). The advantage of having a non-vanishing result in a shorter interval is more visible in view of the non-vanishing of Fourier coefficients.

Theorem 1.4 is a consequence of the following corollary and Waldspurger’s theorem.

**Theorem 1.5.** [1, Corollary 4] Let \(g \in S_{k+1/2}(M,\psi)\), i.e., a cuspform of level \(M\) and weight \(k+1/2\) with the nebentype \(\psi\). If \(g(z)\) is not a linear combination of weight-3/2 theta functions, there exists a positive integer \(k_f\) such that for \(X^{1-1/k_f+\epsilon} \leq h \leq X\),

\[
\#\{X \leq n \leq X + h : a_g(n) \neq 0\} \gg \frac{h}{\log X}.
\]

**Theorem 1.6.** [11] Let \(N\) be even, \(f \in S_{2k}(N,\psi^2)\) and \(g \in S_{k+1/2}(2N,\psi)\) where \(f\) is the Shimura correspondent of \(g\). Then

\[
a_g(d)^2 = \kappa_f L(k,f,\psi_0^{-1}\chi_d)\psi(d)d^k,
\]

where \(\psi_0(n) = \psi(n)(\frac{1}{n})^k\).

Even further, Theorem 1.5 answers to a stronger form of Serre’s questions. Serre initiated the study on the sizes of gaps between non-zero Fourier coefficients. He defined the following gap function.

\[
i_f(n) = \begin{cases} 
\max\{i : a_f(n + j) = 0 \text{ for all } 0 \leq j \leq i\} & \text{if } a_f(n) = 0, \\
0 & \text{otherwise}.
\end{cases}
\]

Serre’s question asks us to estimate this gap function with regards to the cusp forms of integral or half-integral weights. Combining Theorem 1.6 and Corollary 1.3, our second corollary immediately follows.

**Corollary 1.7.** Assume the conditions in Theorem 1.6 with \(\psi = \psi_{\text{triv}}\). Then for \(X^{3/4+\epsilon} \leq h \leq X\),

\[
\#\{X \leq n \leq X + h : a_g(n) \neq 0\} \gg_{f,\epsilon} \frac{h^2}{X^{1+\epsilon}}.
\]

In particular, \(i_g(n) \lesssim n^{3/4+\epsilon}\).

Corollary 1.7 along with Theorem 1.5 are the only results on \(i_g(n)\) where \(g\) is of half-integral weight. On the other hand, many mathematicians have studied \(i_f(n)\) where \(f\) is of integral weight. Unfortunately, most of the approaches in the study of integral weight case are not available in half-integral weight case. Let us briefly review the works on \(i_f(n)\).

Rankin-Selberg estimates, the multiplicativity of Hecke operators, or the arithmetic of Galois representations have been useful tools to study \(i_f(n)\) for the eigenforms of integral weight. For example, we have a classical result on \(L(f \otimes \overline{f})\) such that there exists an integer \(c_f\) for which

\[
\sum_{n \leq X} |a_f(n)|^2 n^{1-\frac{k}{2}} = c_f X + O(X^{3/2}),
\]
It immediately follows that \( i_f(n) \ll n^{3/5} \). Another approach appeals to the theory of \( \mathfrak{B} \)-free numbers, especially when \( f \) is not of CM type. Let \( \mathfrak{B} = \{ b_i \} \) be a set of integers such that

\[
\sum_{b \in \mathfrak{B}} \frac{1}{b} < \infty \quad \text{and} \quad (b_i, b_j) = 1 \quad \text{whenever} \quad i \neq j.
\]

We say that a natural number \( n \) is \( \mathfrak{B} \)-free if it is not divisible by any of the elements of \( \mathfrak{B} \). Specifically, one can define \( \mathfrak{B} \) as follows.

\[
\mathfrak{B} = \{ p \text{ prime : } a_f(p) = 0 \} \cup \{ p | N \text{ prime} \}.
\]

\( \mathfrak{B} \) has a zero density due to the following result of Serre\[14, \text{p.174, Cor.2}\]. Let \( f \) be a newform with integral weight \( 2k \geq 2 \) which is not of CM type. Then

\[
\# \{ p \leq X \text{ prime : } a_f(p) = 0 \} \ll f,\epsilon \frac{X}{(\log X)^{3/2 - \epsilon}}. \tag{1.3}
\]

In view of (1.3) and the multiplicativity of Hecke eigenvalues, \( a_f(n) \neq 0 \) if \( n \) is square-free and \( \mathfrak{B} \)-free. Thus estimating \( i_f(n) \) becomes a problem of counting \( \mathfrak{B} \)-free numbers in the short intervals. Balog-Ono were the first to take this approach and they deduced that

\[
i_f(n) \ll n^{17/41 + \epsilon}.
\]

Later, the estimates of \( i_f(n) \) has been refined several times. The best bound for \( i_f(n) \) is due to Kowalski-Robert-Wu\[6\]. They proved that for any holomorphic non-CM cuspidal eigenform \( f \) on general congruence groups,

\[
i_f(n) \ll n^{7/17 + \epsilon}.
\]

If \( f \) is of CM type, there are no similar general results on \( i_f(n) \). The major difficulty in this case is that the density estimate in (1.3) is valid only for non-CM type forms. Thus the previous works on CM type forms took different approaches other than distribution of \( \mathfrak{B} \)-free numbers. For example, Das-Ganguly\[2\] showed that for all nonzero cuspforms \( f \) of level one,

\[
i_f(n) \ll n^{1/4}. \tag{1.4}
\]

The main ingredient of their work were the congruence relation of Hecke eigenvalues due to Hatada\[3\] and the distribution of sum of two squares in short intervals. (1.4) can be extended to the eigenforms of higher levels, under some conditions on \( f \). Let \( E/\mathbb{Q} \) be an elliptic curve which has a cyclic rational 4-isogeny and \( f_E \) be a newform corresponding to \( E \) by the modularity theorem. Kumar\[7\] first proved that \( f_E \) satisfies (1.3), by showing that there exists a positive integer \( m \) such that \( a_{f_E}(m) \neq 0 \) and \( m \) is a sum of two squares in intervals \( (X, X + cX^{1/4}) \). From this, he deduced that if \( f \) is 2-adically close enough to \( f_E \), \( f \) also satisfies (1.4).

Let us recall some necessary facts for the future discussion. Let \( e(z) = e^{2\pi i z} \) and

\[
f(z) = \sum_{n \geq 1} a_n e(nz)
\]

be the Fourier expansion of \( f \) at the cusp \( \infty \). The associated modular \( L \)-function

\[
L(s, f) = \sum_{n \geq 1} a_n n^{-s}
\]

absolutely converges in \( \Re(s) > k+1/2 \). It can be analytically continued to an entire function and satisfies the following functional equation

\[
\left( \frac{\sqrt{N}}{2\pi} \right)^s \Gamma(s)L(s, f) = w \left( \frac{\sqrt{N}}{2\pi} \right)^{2k-s} \Gamma(2k-s)L(2k-s, f),
\]
where \( w = \pm 1 \). In addition, the twisted \( L \)-function \( L(s, f, \chi_d) \) satisfies the functional equation
\[
\left( \frac{d\sqrt{N}}{2\pi} \right)^s \Gamma(s)L(s, f, \chi_d) = w_d \left( \frac{d\sqrt{N}}{2\pi} \right)^{2k-s} \Gamma(2k-s)L(2k-s, f, \chi_d),
\]
where \( w_d = w\chi_d(-N) = 1 \) for all \( d \in \mathcal{D} \).

If \( f \) is an eigenform, we have the Euler product
\[
L(s, f) = \prod_p \left( 1 - \frac{\alpha_p}{p^s} \right)^{-1} \left( 1 - \frac{\beta_p}{p^s} \right)^{-1}
\]
and it satisfies the Ramanujan-Petersson conjecture (i.e. \( |\alpha_p| = |\beta_p| = p^{k-1/2} \) for all \( p \nmid N \) and \( |\alpha_p|, |\beta_p| \leq p^{k-1/2} \) otherwise). Given \( \alpha_p, \beta_p \) for all \( p \), we define \( n \mapsto \alpha_n, n \mapsto \beta_n \) as totally multiplicative functions on \( \mathbb{N} \).

2. Proof of Theorem

We modify the methods of [4] and [8]. Necessary changes will be described in detail.

Set
\[
V(x) := \frac{1}{2\pi i} \int_{(4/5)} \frac{\Gamma(k + s)}{\Gamma(k)} x^{-s} ds.
\]
Here, note that the integral \( \int_{(4/5)} \) denotes \( \int_{4/5 - i\infty}^{4/5 + i\infty} \). By the Mellin transform and the integration by parts, we have
\[
V(x) = \frac{1}{\Gamma(k)} \int_x^\infty e^{-y} y^{k-1} dy = (1 + x + \cdots + \frac{x^{k-1}}{(k-1)!}) e^{-x}.
\]

Next, we will define \( L(k, f, \chi_d) \) in terms of the rapidly convergent sums. Let
\[
A(Q, \chi_d) = \frac{1}{2\pi i} \int_{(4/5)} L(f, \chi_d, k + s) \frac{\Gamma(k + s)}{\Gamma(k)} \left( \frac{2\pi Q}{Q} \right)^{-s} ds.
\]
We have
\[
A(Q, \chi_d) = \sum_{n \geq 1} a_n n^{-k} \chi_d(n) V \left( \frac{2\pi n}{Q} \right).
\]
By shifting the contour of integration to \( \Re(s) = -4/5 \), we obtain
\[
L(k, f, \chi_d) = A(Q, \chi_d) + A(d^2 N Q^{-1}, \chi_d),
\]
for any \( Q > 0 \) and square-free \( d \in \mathcal{D} \). In particular,
\[
L(k, f, \chi_d) = 2A(|d|\sqrt{N}, \chi_d).
\]

By Abel’s summation formula,
\[
A(Q, \chi_d) \ll_f Q^{2}.\]
Combining (2.1) and (2.3) gives
\[
L(k, f, \chi_d) = A(Q, \chi_d) + O_f (|d|Q^{-\frac{1}{2}}) \text{ for all } Q > 0.
\]
As in [8], we have an upper bound of the fourth moment of \( L \)-values.
\[
\sum'_{d \in D, |d| \leq X + h} |L(k, f, \chi_d)|^4 \ll (X + h)^{2+\varepsilon}.
\]

The first moment we are considering is
\[
S_f(X, h) := \sum'_{d \in D, \text{Re}[\chi_d] \leq X + h} L(k, f, \chi_d) = 2 \sum'_{d \in D, \text{Re}[\chi_d] \leq X + h} A(|d| \sqrt{N}, \chi_d).
\]

We introduce the Möbius function to relax the square-free condition, so that
\[
S_f(X, h) = 2 \sum_{d \in D, \text{Re}[\chi_d] \leq X + h} \sum_{n \mid d} \mu(n) A(n \sqrt{N}, \chi_d).
\]

We split the sum into two parts, say, \( S_f(X, h) = S + R \), where
\[
S = 2 \sum_{a \leq A, (a, 4N) = 1} \mu(a) \sum_{d \in D, \text{Re}[\chi_d] \leq X + h/a^2} A(a^2 |d| \sqrt{N}, \chi_{a^2 d})
\]
and
\[
R = 2 \sum_{a > A, (a, 4N) = 1} \mu(a) \sum_{d \in D, \text{Re}[\chi_d] \leq X + h/a^2} A(a^2 |d| \sqrt{N}, \chi_{a^2 d})
\]

Now we first estimate the partial sum \( R \).

**Proposition 2.1.** \( R \ll \varepsilon, (X + h)^{1+\frac{1}{2}} h^\frac{1}{2} + (X + h)^{\frac{1}{2}} h A^{-3+\varepsilon} \)

**Proof.** From \([13], [24]\) and by introducing the factors \( \sum_{d_1 \mid b} \mu(d_1), \sum_{d_2 \mid b} \mu(d_2) \), as in \([3]\) we have
\[
A(Q, \chi_{b^2 d}) = \sum_{d_1 \mid b} \sum_{d_2 \mid b} \mu(d_1) \mu(d_2) \frac{\alpha_{d_1} \beta_{d_2}}{(d_1 d_2)^k} \chi_d(d_1 d_2) A\left(\frac{Q}{d_1 d_2}, \chi_d\right)
\]
\[
= \sum_{d_1 \mid b} \sum_{d_2 \mid b} \mu(d_1) \mu(d_2) \frac{\alpha_{d_1} \beta_{d_2}}{(d_1 d_2)^k} \chi_d(d_1 d_2) \left( L(k, f, \chi_d) + O((d_1 d_2)^\frac{1}{2} |d| Q^{\frac{k}{2}}) \right).
\]

We split the sum into \( A, B \), where
\[
A = \sum_{d_1 \mid b} \sum_{d_2 \mid b} \mu(d_1) \mu(d_2) \frac{\alpha_{d_1} \beta_{d_2}}{(d_1 d_2)^k} \chi_d(d_1 d_2) L(k, f, \chi_d)
\]
and
\[
B = \sum_{d_1 \mid b} \sum_{d_2 \mid b} \mu(d_1) \mu(d_2) \frac{\alpha_{d_1} \beta_{d_2}}{(d_1 d_2)^k} \chi_d(d_1 d_2) O\left((d_1 d_2)^\frac{1}{2} |d| Q^{-\frac{k}{2}}\right).
\]
From Deligne’s bound \( \alpha_n, \beta_n \leq d(n)n^{12k-1/2} \), we have
\[
A \leq |L(k, f, \chi_d)| \sum_{d \mid b} \sum_{d_1 \mid d_2} (d_1 d_2)^{-\frac{1}{2}+\epsilon} \ll |L(k, f, \chi_d)|
\]
and
\[
B = O \left( |d| Q^{-\frac{1}{2}} \sum_{d_1 \mid d_2} (d_1 d_2)^{\frac{1}{2}} \right) = O \left( b^\epsilon |d| Q^{-\frac{1}{2}} \right).
\]
Thus
\[
A(Q, \chi_{\beta^2 d}) \ll_f |L(k, f, \chi_d)| + b^\epsilon |d| Q^{-\frac{1}{2}}.
\]
Collecting \( A \) and \( B \) together, and with \( Q = b^2 |d| \sqrt{N} \), \( R \) has a bound
\[
R \ll_f \sum_{(b, 4N) = 1} \left( \sum_{a \mid b, a > A} 1 \right) \sum_{d \in D} \sum_{b^2 \leq |d| \leq (X+h)/b^2} \left( |L(k, f, \chi_d)| + b^{-1+\epsilon} |d|^\frac{1}{2} \right).
\]
By the Hölder inequality and Theorem 1.2
\[
\sum_{d \in D} \sum_{b^2 \leq |d| \leq (X+h)/b^2} |L(k, f, \chi_d)| \ll \left( \sum_{d \in D} \sum_{b^2 \leq |d| \leq (X+h)/b^2} |L(k, f, \chi_d)|^2 \right)^{\frac{1}{2}} \left( \sum_{b^2 \leq |d| \leq (X+h)/b^2} 1 \right)^{\frac{1}{2}}
\]
and
\[
b^{-1+\epsilon} \sum_{d \in D} \sum_{b^2 \leq |d| \leq (X+h)/b^2} |d|^\frac{1}{2} \ll b^{-1+\epsilon} \frac{(X+h)^\frac{1}{2}}{b} + (X+h)^\frac{1}{2} hb^{-4+\epsilon}.
\]
In sum,
\[
R \ll_f \sum_{(b, 4N) = 1} \left( \sum_{a \mid b, a > A} 1 \right) \left( b^{-2-2\epsilon}(X+h)^{\frac{1}{2}} h^\frac{1}{2} + (X+h)^{\frac{3}{2}} h A^{-3+\epsilon} \right)
\]
\[
\ll A^{-1-\epsilon}(X+h)^{\frac{1}{2}} h^\frac{1}{2} + (X+h)^{\frac{3}{2}} h A^{-3+\epsilon}.
\]
We now evaluate \( S \). For \((a, 4N) = 1\) and \(d \in D\), we have
\[
A(a^2 |d| \sqrt{N}, \chi_{a^2 d}) = \sum_{(n,a) = 1} a_n n^{-k} \chi_d(n) V(a^2 |d| \sqrt{N}).
\]
Write \( n = rj^2m \), where \( r \mid (4N)^\infty \), \((jm, 4N) = 1\), and \(m\) is square-free. From \( \chi_d(n) = \chi_d(m) \) for \((d,j) = 1\),
we obtain
\[
\begin{align*}
S &= 2 \sum_{\substack{a \leq A \\ (a,4N)=1}} \mu(a) \sum_{\substack{n=rj^2m \\ (n,a)=1}} a_n n^{-k} \sum_{\substack{(d,j)=1 \\ d \in D}} \chi_d(m) V \left( \frac{2\pi n}{a^2|d|\sqrt{N}} \right) \\
&= 2 \sum_{\substack{a \leq A \\ (a,4N)=1}} \mu(a) \sum_{\substack{n=rj^2m \\ (n,a)=1}} a_n n^{-k} \sum_{\substack{q|j}} \mu(q) \sum_{\substack{dq \in D \\ X/a^2 \leq |d|q \leq (X+h)/a^2}} \chi_{dq}(m) V \left( \frac{2\pi n}{a^2|d|\sqrt{N}} \right) 
\end{align*}
\]

In the second inequality, we introduced a M"obius factor to relax the coprimality condition on \(d\).

By the Gauss inversion formula,
\[
S = 2 \sum_{\substack{a \leq A \\ (a,4N)=1}} \mu(a) \sum_{\substack{n=rj^2m \\ (n,a)=1}} a_n n^{-k} \sum_{\substack{q|j}} \mu(q) \sum_{\substack{dq \in D \\ X/a^2 \leq |d|q \leq (X+h)/a^2}} \chi_{dq}(m) e \left( \frac{4Nd}{m} \right) V \left( \frac{2\pi n}{a^2|d|\sqrt{N}} \right),
\]

where
\[
\overline{\epsilon}_m = \begin{cases} 
1 & \text{if } m \equiv -1 \pmod{4} \\
0 & \text{if } m \equiv 1 \pmod{4}.
\end{cases}
\]

Set \(\Delta = \min \left( \frac{1}{2}, a^2q(X+h)^{\epsilon-1} \right)\). We split the sum \(S\) into
\[
S = S_0 + S_1 + S_2,
\]
where the three partial sums are restricted by the conditions \(b = 0, 0 < |b| < \Delta m, \Delta m \leq |b| < m/2\), respectively. Evaluating \(|S_1|\) and \(|S_2|\) only requires minor modifications of Iwaniec’s method.

First of all, note that as in [4], \(S_2 \ll_f 1\).

**Proposition 2.2.** \(S_1 \ll_f A^2(X+h)^{\epsilon-\frac{1}{2}}h\)

**Proof.** We have
\[
S_1 = 2 \sum_{\substack{a \leq A \\ (a,4N)=1}} \mu(a) \sum_{\substack{n=rj^2m \\ (n,a)=1}} \mu(q) \sum_{\substack{(d,j)=1 \\ d \in D \\ X/a^2 \leq |d|q \leq (X+h)/a^2}} \chi_{dq}(m) \epsilon_m m^{\frac{1}{2}} e \left( \frac{4Nd}{m} \right) V \left( \frac{2\pi n}{a^2|d|\sqrt{N}} \right).
\]

Just as in section 8 of [4],
\[
\sum_b \sum_{m} \ll a^3 q^2 r^{-\frac{3}{2}} j^{-3}(X+h)^{\epsilon-\frac{1}{2}}.
\]
With this estimate, it is straightforward to show

\[ S_1 \ll \sum_{a \leq A \atop (a, 4N) = 1} \sum_{r, j} \sum_{q | j} \sum_{X/a^2 q \leq |d| \leq (X + h)/a^2 q} a^3 q^2 r^{-2} j^{-3} (X + h)^{-2} \]

\[ \ll (X + h)^{-2} h \sum_{a \leq A} a \sum_{r, j} r^{-2} j^{-3} \sum_{q | j} q \]

\[ \ll A^2 (X + h)^{-2} h. \]

\( \boxed{} \)

It remains to evaluate \( S_0 \). Because of the condition \( b = 0 \), \( S_0 \) is written in the following simplified form

\[ S_0 = 2 \sum_{a \leq A \atop (a, 4N) = 1} \mu(a) \sum_{n = r, j^2 \atop (n, a) = 1} a_n n^{-k} \sum_{q | j} \mu(q) \]

\[ \times \sum_{d q \in D \atop X/a^2 q \leq |d| \leq (X + h)/a^2 q} V \left( \frac{2\pi n}{a^2 |d| q \sqrt{N}} \right). \]

We then evaluate the innermost sum. We split the inner sum into residue classes mod \( 4N \). Each class contributes

\[ \frac{1}{4N} \int_{X/a^2 q}^{(X + h)/a^2 q} V \left( \frac{2\pi n}{a^2 t q \sqrt{N}} \right) dt + O \left( \left( 1 + \frac{n}{X} \right)^{-\epsilon} \right) \]

by Euler's summation formula. Hence

\[ S_0 = 2 \sum_{a \leq A \atop (a, 4N) = 1} \mu(a) \sum_{n = r, j^2 \atop (n, a) = 1} a_n n^{-k} \sum_{q | j} \mu(q) \]

\[ \times \left[ \frac{\gamma(4N)}{4N} \frac{h}{a^2 q} \int_0^1 V \left( \frac{2\pi n}{(ht + X) \sqrt{N}} \right) dt + O \left( \left( 1 + \frac{n}{X} \right)^{-\epsilon} \right) \right]. \]

Here, \( \gamma(4N) \) is the order of \( D \). The second term in the inner sum contributes \( O(AX^{1/2+\epsilon}) \), by trivial summation over \( r, j \).

As in section 9 of [4], the first term in the innermost sum contributes

\[ (2.5) \quad \gamma(4N) h \sum_{n = r, j^2} a_n \Phi(j) \sum_{a \leq A \atop (a, 4Nj) = 1} \frac{\mu(a)}{a^2} \int_0^1 V \left( \frac{2\pi n}{(ht + X) \sqrt{N}} \right) dt. \]

Now, using the identity

\[ \sum_{a \leq A \atop (a, 4Nj) = 1} \frac{\mu(a)}{a^2} = \frac{6}{\pi^2} \prod_{p | 4Nj} (1 - p^{-2})^{-1} + O(A^{-1}), \]
we split (2.5) into two. The first term is

\[
\begin{align*}
3\gamma(4N) & \frac{\prod_{p\mid 4N} (1-p^{-2})^{-1}}{\pi^2 N} \sum_{n=rj^2} \frac{a_n}{n^{k+\frac{s}{2}}} \prod_{p\mid j} (1-p^{-2})^{-1} \int_0^1 V\left(\frac{2\pi n}{(ht+X)\sqrt{N}}\right) dt \\
& = 3\gamma(4N) \frac{\prod_{p\mid 4N} (1-p^{-2})^{-1}}{\pi^2 N} \int_0^1 \left[ \sum_{n=rj^2} \frac{a_n}{n^{k+\frac{s}{2}}} \prod_{p\mid j} (1+p^{-1})^{-1} V\left(\frac{2\pi n}{(ht+X)\sqrt{N}}\right) \right] dt.
\end{align*}
\]

Also, the second term contributes \(O(A^{-1}hX^\epsilon)\), by trivial summation over \(r, j\). Thus, finally,

\[
S_0 = 3\gamma(4N) \frac{\prod_{p\mid 4N} (1-p^{-2})^{-1}}{\pi^2 N} \int_0^1 B(ht+X) dt + O(A^{-1}hX^\epsilon + AX^{1/2+\epsilon}),
\]

where

\[
B(x) = \sum_{n=rj^2} \frac{a_n}{n^{k+\frac{s}{2}}} \prod_{p\mid j} (1+p^{-1})^{-1} V\left(\frac{2\pi n}{x}\right).
\]

By shifting the line of integration to \((-\frac{1}{5})\),

\[
B(x) = \frac{1}{2\pi i} \int_{(-1/5)} \frac{\Gamma(k+s)}{\Gamma(k)} \frac{a_n}{n^{k+\frac{s}{2}}} \prod_{p\mid j} (1+p^{-1})^{-1} \left(\frac{2\pi}{x}\right)^{-s} ds.
\]

Note that \(L_f(s)\) appeared in \([3]\) as \(L(s)\) for \(k = 1\), and it was later generalized in \([8]\) as \(L_{f,1}(s)\) for all positive integer \(k\). Recall that \(L_f(k) \neq 0\) and \(L_f(s)\) is polynomially bounded if \(\Re(s) > k-1/4\). Thus we have

(2.6) \quad B(x) = L_f(k) + O_f(x^{-\frac{1}{2}}).

From (2.6), we obtain

\[
S_0 = 3\gamma(4N) \frac{\prod_{p\mid 4N} (1-p^{-2})^{-1}}{\pi^2 N} L_f(k) h + O(h^{\frac{1}{2}} + A^{-1}hX^\epsilon + AX^{\frac{1}{2}+\epsilon}).
\]

Now we collect all the error terms of \(S_f(X, h)\). They are

\[
O_{f,\epsilon}(h^{\frac{1}{2}} + A^{-1}hX^\epsilon + AX^{\frac{1}{2}+\epsilon} + A^2(X+h)^{\epsilon-\frac{1}{2}} h + A^{-1-\epsilon}(X+h)^{\epsilon+\frac{1}{2}} h X^3 h A^{-3+\epsilon}).
\]

Let \(A = X^{\frac{1}{2}-\epsilon}\). Then for \(X^{3/4+\epsilon} \leq h \leq X\), the main term dominates the error terms. We have

\[
S_f(X, h) = 3\gamma(4N) \frac{\prod_{p\mid 4N} (1-p^{-2})^{-1}}{\pi^2 N} L_f(k) h + O_{f,\epsilon}(hX^{-\epsilon}).
\]
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