FINITE-TIME STABILITY OF IMPULSIVE DIFFERENTIAL INCLUSION: APPLICATIONS TO DISCONTINUOUS IMPULSIVE NEURAL NETWORKS

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Abstract. In this article, we present several results on Finite-Time Stability (FTS) of impulsive differential inclusion. In order to investigate the FTS problem, a new concept of Finite-Time Stable Function Pair (FTSFP) is proposed. By virtue of average impulsive interval and FTSFP, two unified criteria on FTS of impulsive differential inclusion are obtained, which are effective for both the destabilizing impulses and the stabilizing impulses. In addition, the settling-time depends not only on the initial value, but also on the information of impulsive sequence. As an extension, a delay-independent FTS result of impulsive delayed differential inclusion is presented. Finally, the obtained results are applied to study the FTS of discontinuous impulsive neural networks.

1. Introduction. Differential equations (DE) possessing discontinuous effects arise in various areas of mechanical engineering, biology and economics [24, 31, 32, 37]. Taking interconnected perturbed double-integrators system with bang-bang control strategy as an example, the designed controllers varying between the minimum and maximum input values will lead to discontinuous change of the state’s derivative [37]. Due to the discontinuity, the traditional solution may not exist for discontinuous DE. Fortunately, Filippov put forward a framework named Differential Inclusion (DI), which is effective to deal with discontinuous differential equations [12]. In fact, by using Filippov regularization method, the investigation on the dynamics of discontinuous DE is transformed into the analysis on the dynamical behavior of...
its corresponding DI [3, 13, 14, 20, 26]. During the past several decades, there have been some results on analysis of DI. In 1984, Aubin and Cellina developed DI to deal with differential equation possessing discontinuity in right-side [6]. Later, Haddad presented some results on viable trajectories and topological properties of delayed differential inclusions (DDI) in [16, 17]. In 1999, Hu and Papageorgiou investigated the periodicity of non-convex DI [19]. Subsequently, Qin et al. studied the periodic solution of nonlinear DI with multi-valued perturbations [36]. Recently, some results on dynamics of DI or DDI have been obtained [2, 10, 25]. However, there is still not much work concerning on dynamical behaviors of impulsive DI or DDI.

The dynamics of many evolving processes are subject to sudden changes such as shocks, harvesting and natural disasters. Impulses are often encountered in real applications such as biological systems, power supplying systems, and financial market models [8, 21, 38, 41]. Impulsive system, which belongs to a class of hybrid system, exhibits discrete state jumps during the continuous dynamic evolution process [4, 7]. In the past decade, some unified criteria on the dynamics of nonlinear impulsive systems have been obtained by using average impulsive interval, which are effective for both destabilizing impulses and stabilizing impulses [33, 40]. In [9], the authors studied the FTS of hybrid systems involving both switching and impulsive effect. Furthermore, the FTS of closed set of hybrid systems possessing impulse was extensively studied in [23]. Recently, many models of actual processes and phenomena have been described by DI, in which the disturbance take the form of impulses [1, 11, 39]. In [1], the authors studied the existence and controllability of impulsive functional differential inclusion. Later, the periodicity of impulsive DI was obtained in [11]. However, as far as we know, almost no results on FTS of impulsive DI has been considered yet.

As a special case of stability, FTS has wide applications in the field of physics and aeronautics. For example, in a coupled-oscillator system, these oscillators are sometimes required to be stabilized to a fixed position in a finite time [15]. As it was pointed out in [28], FTS means the system state trajectory reaching a desirable target in a finite time, which depends on the initial value. In [28], the author discussed the FTS problem of differential inclusion systems. Subsequently, Moulay et al. established FTS criteria for a non-Lipschitz continuous system with or without time delay [29, 30]. Polyakov et al. studied finite/fixed-time stabilization of nonlinear system [34, 35]. The authors presented sufficient and necessary conditions for ensuring FTS of the linear impulsive system [4, 5]. Recently, Li et al. studied the FTS of nonlinear impulsive systems and investigated the influence of pulse on settling time [22]. However, these results were obtained by the traditional Lyapunov function method, in which the constructed Lyapunov function is required to possess negative-definite derivative and exist time-derivative everywhere. This means that the application range of the traditional Lyapunov function method is greatly limited. Thus, it is interesting to improve the Lyapunov function method, in which the derivative of the Lyapunov function can be estimated by time-varying coefficients and fails to exist at some instants. Therefore, how to study the FTS of impulsive DI by an improved Lyapunov function method is an crucial question to be addressed.

Section 2 presents some necessary definitions and lemmas and Section 3 shows the FTS analysis via Lyapunov function method with indefinite derivative. A special extension to impulsive DDI is given in Section 4 and the obtained result is applied to study FTS of discontinuous impulsive neural networks in Section 5. Finally, some concluding words are made in Section 6.
Notations: Denote \( \mathbb{N} = \{1, 2, \cdots\} \), \( \mathbb{R} = (-\infty, +\infty) \), \( \mathbb{R}_+ = [0, +\infty) \) and \( \|x\| = \sqrt{x^T x} \) for any \( x \in \mathbb{R}^n \). Let \( \mathcal{PC}([a, b] : \mathbb{R}^n) \) be the class of piecewise continuous functions defined on \([a, b]\), which are right-continuous and have finite numbers of jumps at most. \( \text{mod}(l, N_0) \) denotes the remainder of \( l \) over \( N_0 \) and \( <a, b> \) shows the inner product of vectors \( a \in \mathbb{R}^n \) and \( b \in \mathbb{R}^n \). For \( \varphi \in \mathcal{PC}([a, b] : \mathbb{R}^n) \), the norm is defined to be \( \|\varphi\|_{[a, b]} = \sup_{a \leq s \leq b} |\varphi(s)| \). For any \( X \subseteq \mathbb{R}^n \), if there exists a nonempty set \( \mathcal{F}(x) \subseteq \mathbb{R}^n \) corresponding to each point \( x \in X \), then \( x \mapsto \mathcal{F}(x) \) is said to be a set-valued map from \( X \) to \( \mathbb{R}^n \). Moreover, the set-valued map \( \mathcal{F} \) is upper semi-continuous at \( x_0 \in X \) if for any open set \( \mathcal{N} \) containing \( \mathcal{F}(x_0) \), there exists a neighborhood \( \mathcal{M} \) of \( x_0 \) satisfying \( \mathcal{F}(\mathcal{M}) \subseteq \mathcal{N} \). Let \( \mathcal{K} \) be the set of continuous strictly increasing function \( \varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) with \( \varphi(0) = 0 \). \( \mathcal{K}_\infty \) means the subset of \( \mathcal{K} \) functions that are unbounded, i.e., \( \lim_{s \rightarrow +\infty} \varphi(s) = +\infty \).

2. Preliminaries. Consider the following nonlinear discontinuous impulsive differential equation (DIDE):

\[
\begin{cases}
\dot{x}(t) = f(t, x(t)), & t \neq t_i, \\
x(t_i) = g(t_i, x(t_i^-)), & t = t_i, \quad l \in \mathbb{N}, \\
x(t_0) = x_0,
\end{cases}
\]

where the state \( x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T \in \mathbb{R}^n \) is assumed to be right-continuous and has left limits at every \( t \): \( f : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is Lebesgue measurable and essentially locally bounded. In addition, the nonlinear function \( f(t, x) \) may be discontinuous on variable \( x \). The impulsive function \( g : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is continuous and \( g(t, x) = 0 \Leftrightarrow x = 0 \). The impulsive sequence \( \{t_l : l \in \mathbb{N}\} \) satisfies \( 0 < t_1 < t_2 < \cdots \) and \( \lim_{l \rightarrow +\infty} t_l = +\infty \).

Remark 1. The dynamics of nonlinear impulsive systems has been extensively studied in [4, 7, 33, 40]. However, the requirement of \( f_i(t, x) \) is continuous or Lipschitz continuous in these articles. In this paper, we relax this requirement, which means that the nonlinear function can be discontinuous. So, the solution of system (1) is adopted in the Filippov sense.

Construct the following Filippov set-valued map \( \mathcal{F} : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow 2^{\mathbb{R}^n} \):

\[
\mathcal{F}(t, x) = \bigcap_{\delta > 0} \bigcap_{\mu(Z) = 0} \text{co}[f(t, x, \delta) \setminus Z],
\]

where \( \bigcap_{\mu(Z) = 0} \) denotes the intersection over all sets of \( Z \) of Lebesgue measure zero and \( B(x, \delta) \) is a ball with center of \( x \) and a radius of \( \delta > 0 \).

Definition 2.1. \( x(t) \) is called a Filippov solution of DIDE (1), if it is absolutely continuous on any subset of \( [t_l, t_{l+1}] \subseteq \mathbb{I} \), \( x(t) \) satisfies the following Impulsive Differential Inclusion (IDI):

\[
\begin{cases}
\dot{x}(t) \in \mathcal{F}(t, x(t)), & \text{a.e. } t \in \mathbb{I} - \{t_1, t_2, \cdots\}, \\
x(t_l) = g(t_l, x(t_l^-)), & t = t_l, \\
x(t_0) = x_0,
\end{cases}
\]

Because of the essential local boundedness of \( f \), the multi-valued map \( \mathcal{F} \) is nonempty and takes compact, convex values and \( \mathcal{F}(t, x(t)) \) is USC [12]. Therefore, for every \( (t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n \), the local solution existence of \( x(t_0, x_0)(t) \) can be guaranteed. According to the solution existence theorem in [7], system (1) has a solution
\(x(t_0, x_0)(t)\) defined on \([t_0, t_1]\) under some appropriate conditions. When \(t = t_1\), the impulse appears, which makes the solution \(x(t_0, x_0)(t)\) jump to \(g(t_1, x(t_1^-))\). By the same procedure, there exists a solution \(x(t_0, x_0)(t)\) defined on \([t_1, t_2]\). By repeating the above analysis, one obtains that for any \(x_0 \in \mathbb{R}^n\), system (1) has a global solution.

**Definition 2.2.** If for any \(t \in \mathbb{R}_+\), \(x^* \in F(t, x^*)\) and \(g(t, x^*) = x^*\), then \(x^*\) is an equilibrium point of DIDE (1) or IDI (3). Moreover, if \(0 \in F(t, 0)\), then \(x = 0\) is a trivial solution of DIDE (1) or IDI (3).

**Definition 2.3.** The trivial solution of DIDE (1) or IDI (3) is said to be finite-time stable, if the following two statements hold:

- Lyapunov stability: For any \(\epsilon > 0\) and \(t_0 \geq 0\), there exists \(\delta = \delta(\epsilon, t_0) > 0\) such that for any \(x_0 \in B(0, \delta)\), \(\|x(t_0, x_0)(t)\| < \epsilon\).

- Finite-time attractivity: There exists a \(T(t_0, x_0, \{t_i\}) > 0\) such that \(x(t_0, x_0)(t) = 0, \forall t \geq T(t_0, x_0, \{t_i\})\). Here \(T(t_0, x_0, \{t_i\})\) is called settling-time.

**Definition 2.4.** The trivial solution of DIDE (1) or IDI (3) is uniformly finite-time stable, if the second statement of Definition 2.3 and the following statement hold:

- Lyapunov uniform stability: For any \(\epsilon > 0\) and every \(t_0 \geq 0\), there exists a \(t_0\)-independent \(\delta = \delta(\epsilon) > 0\) such that for any \(x_0 \in B(0, \delta)\), \(\|x(t_0, x_0)(t)\| < \epsilon\) holds.

In order to further study the FTS of DIDE (1) or IDI (3), we introduce the following definitions of average impulsive interval and finite-time stable function pair.

**Definition 2.5.** [18] \(\Gamma_a\) and \(N_0\) are called the average impulsive interval and the elasticity number of a given impulsive sequence \(\{t_l : l \in \mathbb{N}\}\), if

\[
\frac{t-s}{\Gamma_a} - N_0 \leq N(t, s) \leq \frac{t-s}{\Gamma_a} + N_0 \tag{4}
\]

holds for \(\Gamma_a > 0\) and \(N_0 > 0\), where \(N(t, s)\) means the number of instant \(t_l\) in the semi-open interval \((s, t]\).

**Definition 2.6.** Two integrable functions \(\mu(t)\) and \(\rho(t)\) are said to be a Finite-Time Stable Function Pair (FTSFP) with parameters \(\lambda_1, \theta_1, \lambda_2, \theta_2, \) and \(\alpha\), if there exist \(0 < \alpha < 1, \lambda_1 \geq 0, \theta_1 \geq 0, \lambda_2 > 0, \theta_2 \geq 0\) and \(0 < \alpha < 1\) such that

\[
\int_{t_0}^{t} \mu(s) ds \leq -\lambda_1(t - t_0) + \theta_1 \quad \text{and} \quad \int_{t_0}^{t} \rho(s) G_\alpha(t, s) ds \leq -\lambda_2(t - t_0) + \theta_2 \tag{5}
\]

hold, where \(G_\alpha(t, s) = e^{(1-\alpha) \int_{s}^{t} \mu(\sigma)d\sigma}\) is called the Green’s function. We mark this class of function pair as \((\mu(t), \rho(t)) \in \mathcal{FP}\).

**Remark 2.** Inspired by the definition of uniformly exponentially stable function (UES) proposed in [42], we proposed the definition of FTSFP, which can be used to solve the FTS issue. If we assume \(\rho(t) = 0\), this definition will degenerate into UES with parameter \(\lambda_1\), which is effective to deal with exponential stability problem.

**Remark 3.** As pointed out in [33,42], it is not difficult to verify the former inequality of (5), which means that \(\mu(t)\) is UES with parameter \(\lambda_1\). Moreover, if \(\mu(t) > 0\) (taking \(\mu(t) = \frac{1}{t+1}\) as an example, the first inequality of (5) holds with \(\lambda_1 = 0\) and \(\theta_1 = \frac{\pi}{2}\) and \(\rho(t)\) is also UES with parameter \(\lambda_2 > 0\), then function pair \((\mu(t), \rho(t))\) belongs to \(\mathcal{FP}\).
It is well known that the non-smooth analysis theory is effective to deal with
dynamics of discontinuous systems [32]. As pointed out in [12], we just need to
construct a non-smooth Lyapunov function for a given discontinuous system. If
the generalized derivative of the constructed Lyapunov is negative, then the given
discontinuous system is stable. In the following, we will introduce some basic knowl-
edge on non-smooth analysis.

**Definition 2.7.** Let $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ be locally Lipschitz continuous (LLC) on
$(t, y) \to (t)$. Let $\Omega$ be a set of points, where
the derivative of $V(t, x)$ does not exist, then the Clark generalized gradient of $V$ at
$(t, x)$ is defined as
\[
\partial V(t, x) = \overline{\lim}_{k \to \infty} \nabla V(t_k, x_k) : (t_k, x_k) \to (t, x), (t_k, x_k) \notin \mathbb{Z} \bigcup \Omega,
\]
where $\mathbb{Z}$ denotes a set with $\mu(\mathbb{Z}) = 0$.

**Lemma 2.8.** [32] Suppose $V(t, x) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ is $C$-regular and $x(t)$ is
absolutely continuous, then $V(t, x)$ is differentiable for a.a. $t \in \mathbb{R}_+$ and
\[
\dot{V}(t, x) = \langle \dot{\zeta}(t, x), (1, (x)^T) \rangle
\]
where $\forall \zeta(t, x) \in \partial V(t, x)$.

3. **Finite-time stability analysis.** Here, we present the FTS criteria of DIDE
(1) via Filippov framework. Note that we can study the stability of $x^*$ of DIDE
(1) through studying the stability of $x = 0$ of the corresponding IDI (3) by a transformation. This section always assumes the following condition holds

- For any $t \in \mathbb{R}_+, 0 \in F(t, 0)$ and $g(t, x) = 0 \iff x = 0$.

**Theorem 3.1.** Suppose the impulsive sequence $\{t_l : l \in \mathbb{N}\}$ possesses an average
impulsive interval $\Gamma_a$ and an elasticity number $N_0$. If there exist $\kappa_1 \in K_b$ and a locally
Lipschitz continuous $C$-regular function $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ with $V(t, 0) = 0$ for
every $t \in \mathbb{R}_+$, one indefinite integral function $\mu(\cdot)$ and one negative integral function
$\rho(\cdot)$ such that

$$(H_1)$$
\[
\kappa_1(||x||) \leq V(t, x(t)), \ \forall (t, x(t)) \in \mathbb{R}_+ \times \mathbb{R}^n.
\]

$(H_2)$ the derivative of $V(t, x(t))$ along the trajectories of system (3) satisfies
\[
\dot{V}(t, x(t)) \leq \mu(t)V(t, x(t)) + \rho(t)V^\alpha(t, x(t))
\]
for a.e. $t \in [t_l, t_{l+1})$, where $0 < \alpha < 1$.

$(H_3)$ when $t = t_l,$
\[
V(t_l, g(t_l, x(t_l)))) \leq \eta \frac{1}{\alpha} V(t_l^-, x(t_l^-))
\]
holds for $\eta > 0$.

$(H_4)$ $\mu(\cdot)$ and $\rho(\cdot)$ belong to FTSFP with parameters $\lambda_1, \theta_1, \lambda_2, \theta_2$ and $\alpha,$ where
$\mu_\alpha(t) = \mu(t) + \frac{\ln(t)}{(1-\alpha)\lambda_\alpha},$ then the trivial solution of IDI (3) is finite-time stable.
Moreover, the settling time $T(t_0, x_0)$ is estimated by
\[
T(t_0, x_0) = t_0 + \frac{\frac{V(t_0^1)}{V(t_0^1)} e^{(1-\alpha)\theta_1 - N_0 \ln \eta} + (1-\alpha)\theta_2 e^{N_0 \ln \eta}}{(1-\alpha)\lambda_2 e^{N_0 \ln \eta}}
\]
when $0 < \eta < 1$ and
\[
T(t_0, x_0) = t_0 + \frac{V_0^{1-\alpha} e^{(1-\alpha)\theta_1 + N_0 \ln \eta} + (1 - \alpha) \theta_2 e^{-N_0 \ln \eta}}{(1-\alpha) \lambda_2 e^{-N_0 \ln \eta}}
\]
when $\eta \geq 1$, where $V_0 = V(0)$.

**Proof.** For any given initial value $(t_0, x_0)$, we denote the solution of IDI (3) by $x(t_0, x_0)(\cdot)$. Multiplying both sides of (7) by $V^{-\alpha}(t, x(t))$, one has
\[
V^{-\alpha}(t, x(t)) \dot{V}(t, x(t)) \leq \mu(t) V^{1-\alpha}(t, x(t)) + \rho(t)
\]
for a.e. $t \in [t_0, +\infty)$ and $t \neq t_i$. From the statement of condition ($H_3$), we have
\[
V^{1-\alpha}(t_1, g(t_1, x(t_1^-))) \leq \eta V^{1-\alpha}(t_1^-, x(t_1^-))
\]
for $t = t_1$.

Define $W(t) = V^{1-\alpha}(t, x(t))$ and $W(0) = V^{1-\alpha}(0) \triangleq W_0$. According to (9), (10) and $0 < \alpha < 1$, one obtains
\[
W(t) \leq (1 - \alpha) \mu(t) W(t) + (1 - \alpha) \rho(t), \quad t \neq t_i
\]
and
\[
W(t_i) \leq \eta W(t_i^-), \quad t = t_i.
\]
For all $t \in [t_0, t_1)$, based on (11), we obtain
\[
W(t) e^{-\int_{t_0}^t \mu(s) ds} \leq W_0 + (1 - \alpha) \int_{t_0}^t \mu(s) e^{-\alpha \int_{t}^s \mu(\sigma) d\sigma} ds,
\]
which leads to
\[
W(t) \leq W_0 e^{\int_{t_0}^t \mu(s) ds} + (1 - \alpha) \int_{t_0}^t \mu(s) e^{\alpha \int_{t}^s \mu(\sigma) d\sigma} ds.
\]
Because of the continuity of $W(t)$, one has
\[
W(t^-) \leq W_0 e^{\int_{t_0}^t \mu(s) ds} + (1 - \alpha) \int_{t_0}^t \mu(s) e^{\alpha \int_{t}^s \mu(\sigma) d\sigma} ds.
\]
When $t = t_1$, according to (12), one has
\[
W(t_1) \leq \eta W(t^-_1) \leq \eta W_0 e^{\int_{t_0}^t \mu(s) ds} + (1 - \alpha) \int_{t_0}^{t_1} \eta \mu(s) e^{\alpha \int_{t}^s \mu(\sigma) d\sigma} ds.
\]
For all $t \in [t_1, t_2)$, by using the same method as that used in (13) and (14), one gets
\[
W(t) \leq W(t_1) e^{\int_{t_1}^t \mu(s) ds} + (1 - \alpha) \int_{t_1}^t \rho(s) e^{\alpha \int_{t}^s \mu(\sigma) d\sigma} ds.
\]
Due to the continuity of $W(t)$ and formula (16), (17), it follows that
\[
W(t) \leq \eta W_0 e^{\int_{t_0}^t \mu(s) ds} + (1 - \alpha) \int_{t_0}^t \eta \mu(s) e^{\alpha \int_{t}^s \mu(\sigma) d\sigma} ds + (1 - \alpha) \int_{t}^t \rho(s) e^{\alpha \int_{t}^s \mu(\sigma) d\sigma} ds.
\]
When $t = t_2$, according to (12) and (18), one has
\[
W(t_2) \leq \eta^2 W_0 e^{\int_{t_0}^t \mu(s) ds} + (1 - \alpha) \int_{t_0}^{t_1} \eta^2 \mu(s) e^{\alpha \int_{t}^s \mu(\sigma) d\sigma} ds + (1 - \alpha) \int_{t_1}^{t_2} \eta \mu(s) e^{\alpha \int_{t}^s \mu(\sigma) d\sigma} ds.
\]
By the method of mathematical induction, one derives
\[ W(t) \leq \eta^k W_0 e^{(1-\alpha) \int_0^t \mu(s)ds} + (1 - \alpha) \int_{t_0}^t \eta^k \rho(s) e^{(1-\alpha) \int_0^t \mu(s)ds} ds \]
\[ + (1 - \alpha) \sum_{j=1}^{k-1} \int_{t_j}^{t_{j+1}} \eta^{j-2} \rho(s) e^{(1-\alpha) \int_0^t \mu(s)ds} ds + (1 - \alpha) \int_{t_k}^t \rho(s) e^{(1-\alpha) \int_0^t \mu(s)ds} ds \]
holds for \( t \in [t_k, t_{k+1}) \) and \( k = 2, 3, \ldots \). Therefore, one obtains that
\[ W(t) \leq W_0 \Phi(t, t_0) + (1 - \alpha) \int_{t_0}^t \rho(s) \Phi(t, s) ds, \quad (20) \]
where \( \Phi(t, s) = \eta^N(t,s) e^{(1-\alpha) \int_0^t \mu(s)ds} \).

For further studying finite-time stability of IDI (3), we will consider the following two cases.

**Case I:** When \( 0 < \eta < 1 \), we will prove that the trivial solution is Lyapunov stable first.

Because of the negativity of \( \rho(t) \), one obtains that
\[ W(t) \leq W_0 \Phi(t, t_0) \leq W_0 \eta^N(t,t_0) e^{(1-\alpha) \int_0^t \mu(s)ds}. \quad (21) \]

According to the definition of average impulsive interval and formula (21), one derives that
\[ W(t) \leq W_0 \eta^{\frac{t_1}{t_0} - N_0 e^{(1-\alpha) \int_0^t \mu(s)ds}} = W_0 \eta^{N_0 e^{(1-\alpha) \int_0^t \mu(s)ds + \frac{\ln \eta}{\alpha \ln N_\alpha}} \sigma} \]
\[ \leq W_0 \eta^{N_0 e^{(1-\alpha) \theta_1}} \]
holds for \( t \in [t_0, +\infty) \), where \( L = \theta_1 - \frac{N_0 \ln \eta}{1-\alpha} \). Since \( W(t) = V^{1-\alpha}(t, x(t)) \), one gets
\[ V(t, x(t)) \leq e^L V(t_0, x_0). \quad (23) \]
Because of the continuity of \( V(t_0, x_0) \) at \( (t_0, 0) \) and \( V(t_0, 0) = 0 \), for all \( \varepsilon > 0 \) and \( t_0 \geq 0 \), there exists a \( \delta = \delta(\varepsilon, t_0) > 0 \) such that for all \( x_0 \in B(0, \delta) \)
\[ V(t_0, x_0) \leq \frac{\kappa_1(\varepsilon)}{e^L} \]
holds. Under the assumption \((H_1)\), it follows from (23) and (24) that
\[ \|x(t)\| \leq \kappa_1^{-1}(V(t, x)) \leq \kappa_1^{-1}(e^L V(t_0, x_0)) \leq \kappa_1^{-1}(e^L \frac{\kappa_1(\varepsilon)}{e^L}) = \varepsilon. \quad (25) \]
Based on the first statement of Definition 2.3, we have proved the Lyapunov stability of the trivial solution of IDI (3).

Next, we will show that the trivial solution of IDI (3) is finite-time attractive.

According to \((H_4)\) and Definition 2.5, from definition \( \Phi(t, t_0) \), one obtains
\[ \Phi(t, t_0) = \eta^N(t,t_0) e^{(1-\alpha) \int_0^t \mu(s)ds} \leq \eta^{N_0 e^{(1-\alpha) \int_0^t \mu(s)ds}} \leq e^{(1-\alpha) \theta_1 - N_0 \ln \eta}. \quad (26) \]
Since \( \rho(t) < 0 \), according to Condition \((H_4)\) and Definition 2.6, one gets
\[ \int_{t_0}^t \rho(s) \Phi(t, s) ds \leq \int_{t_0}^t \rho(s) \eta^N(t,s) e^{(1-\alpha) \int_0^t \mu(s)ds} ds \]
\[ \leq \int_{t_0}^t \rho(s) \eta^{\frac{t_1}{t_0} - N_0 e^{(1-\alpha) \int_0^t \mu(s)ds}} ds \]
\[ \leq \eta N_0 \int_{t_0}^t \rho(s) e^{(1-\alpha) \int_0^t \mu(s)ds + \frac{\ln \eta}{\alpha \ln N_\alpha}} ds \leq e^{N_0 \ln \eta} (-\lambda_2 (t - t_0) + \theta_2). \]
Therefore, one obtains
\[ W(t) \leq W_0 e^{(1-\alpha) \theta_1 - N_0 \ln \eta} + (1 - \alpha) e^{N_0 \ln \eta} (-\lambda_2 (t - t_0) + \theta_2), \quad (27) \]
which implies that $W(t) \equiv 0$ for all $t \geq t_0 + \frac{W_0 e^\left(1 - \alpha\right) \theta_1 - N_0 l_{\text{in} \eta} + (1 - \alpha) \theta_2 e^{-N_0 l_{\text{in} \eta}}}{\lambda_2 (1 - \alpha) e^{-N_0 l_{\text{in} \eta}}}$. Since $W(t) = V(t,x)$ and the positivity of $V(t,x)$, it implies that $V(t,x) \equiv 0$ for all $t \geq t_0 + \frac{W_0 e^\left(1 - \alpha\right) \theta_1 - N_0 l_{\text{in} \eta} + (1 - \alpha) \theta_2 e^{-N_0 l_{\text{in} \eta}}}{\lambda_2 (1 - \alpha) e^{-N_0 l_{\text{in} \eta}}}.$

**Case II:** When $\eta \geq 1$, one obtains

$$W(t) \leq W_0 \eta \int_{t_0}^{t} (1 - \alpha) \mu(\sigma) d\sigma \leq W_0 e^\left(1 - \alpha\right) L (28)$$

holds for $t \in [t_0, +\infty)$, where $L = \theta_1 + \frac{N_0 l_{\text{in} \eta}}{1 - \alpha}$. Using the same method as in the Case I, we can obtain the Lyapunov stability of the trivial solution of IDI (3).

Next, we will prove the finite-time attractiveness of the trivial solution of IDI (3) briefly.

According to ($H_4$) and Definition 2.5, we get

$$\Phi(t, t_0) \leq \eta \int_{t_0}^{t} (1 - \alpha) \mu(\sigma) d\sigma \leq e^\left(1 - \alpha\right) \eta_0 (29)$$

Since $\rho(t) < 0$, according to Condition ($H_4$) and Definition 2.6, one gets

$$\int_{t_0}^{t} \rho(s) \Phi(t, s) ds \leq \eta \int_{t_0}^{t} \rho(s) (1 - \alpha) \mu(\sigma) d\sigma ds \leq e^{-\lambda_2 (t - t_0) + \theta_2} (30)$$

In conclusion, one has

$$W(t) \leq W_0 e^\left(1 - \alpha\right) \theta_1 + N_0 l_{\text{in} \eta} + (1 - \alpha) e^{-N_0 l_{\text{in} \eta}} (-\lambda_2 (t - t_0) + \theta_2) \leq e^{-N_0 l_{\text{in} \eta}} (-\lambda_2 (t - t_0) + \theta_2)$$

which implies that $W(t) \equiv 0$ for all $t \geq t_0 + \frac{W_0 e^\left(1 - \alpha\right) \theta_1 + N_0 l_{\text{in} \eta} + (1 - \alpha) \theta_2 e^{-N_0 l_{\text{in} \eta}}}{\lambda_2 (1 - \alpha) e^{-N_0 l_{\text{in} \eta}}}.$ Since $W(t) = V(t,x)$ and the positivity of $V(t,x)$, it implies that $V(t,x) \equiv 0$ for all $t \geq t_0 + \frac{W_0 e^\left(1 - \alpha\right) \theta_1 + N_0 l_{\text{in} \eta} + (1 - \alpha) \theta_2 e^{-N_0 l_{\text{in} \eta}}}{\lambda_2 (1 - \alpha) e^{-N_0 l_{\text{in} \eta}}}.$

**Remark 4.** As far as we know, there are few results on FTS involving differential inclusion and Lyapunov-like function. Moreover, the Lyapunov function in the reported articles such as [18, 22, 40] is required to be differentiable, even its derivative is required to be non-positive definite. In contrast, Theorem 3.1 relaxes the condition that the derivative (if it exists) of the Lyapunov-like function is allowed to be indefinite for a.e. $t$ because of the indefiniteness of integral function $\mu(t)$ in ($H_2$). Thus, this result is more general.

**Remark 5.** In [22], the authors studied the FTS of nonlinear impulsive systems. However, the conditions require that $\dot{V}(x(t)) \leq -\alpha V^\eta(x(t))$, which means that the continuous dynamics of nonlinear impulsive systems are stable. In this theorem, we relaxed this assumption. Obviously, the system (1) without impulses may be unstable due to the indefiniteness of the derivative of Lyapunov function (7).

**Remark 6.** In this theorem, we can get the following two statements from condition ($H_4$). First, if the impulsive gain parameters $\eta$ is bigger than 1, condition ($H_4$) implies that $\Gamma_{\text{a}}$ should not be too small. Second, if the impulsive gain parameters $\eta$ is smaller than 1, condition ($H_4$) shows that the average impulsive interval of stabilizing impulse should not be too large. Therefore, we provide a unified FTS criterion on impulsive systems with either destabilizing impulse or stabilizing impulse with the help of average impulsive interval and FTSFP.
Remark 7. In [22], the authors presented an excellent result on FTS of nonlinear impulsive system. The authors also pointed out that it is not easy to discuss the FTS for nonlinear impulsive systems with destabilizing flow and stabilizing jump. In this theorem, a FTS result of nonlinear impulsive systems with destabilizing flow is obtained. In other words, this means that the FTS could be achieved by an impulse control strategy. Factually, FTS is realized under the joint action of stabilizing impulsive jump and function parameters $\rho(t)$.

**Theorem 3.2.** Assume that the impulsive sequence $\{t_i: i \in \mathbb{N}\}$ possesses an average impulsive interval $\Gamma_n$ and elasticity number $N_0$. If there exist $\kappa_1, \kappa_2 \in \mathcal{K}_\infty$ and a locally Lipschitz continuous $C$-regular function $V: \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}_+$ with $V(t,0) = 0$ for every $t \in \mathbb{R}_+$, one indefinite integral function $\mu(t)$ and one negative integral function $\rho(t)$ such that

\[
(\mathcal{H}_1) \quad \kappa_1(||x||) \leq V(t,x(t)) \leq \kappa_2(||x||), \quad \forall (t,x(t)) \in \mathbb{R}_+ \times \mathbb{R}^n. \tag{32}
\]

$\mathcal{H}_2$ the derivative of $V(t,x(t))$ along the trajectories of system (3) satisfies

\[
\dot{V}(t,x(t)) \leq \mu(t)V(t,x(t)) + \rho(t)V^\alpha(t,x(t)) \tag{33}
\]

for a.e. $t \in [t_i, t_{i+1})$, where $0 < \alpha < 1$.

$\mathcal{H}_3$ when $t = t_i$,

\[
V(t_i, g(t_i,x(t_i))) \leq \eta \frac{1}{1-\alpha} V(t_i^- - x(t_i^-)) \tag{34}
\]

holds for $\eta > 0$.

$\mathcal{H}_4$ $\mu_\alpha(t)$ and $\rho(t)$ belong to FTSFP with parameters $\lambda_1, \theta_1, \lambda_2, \theta_2$ and $\alpha$, where $\mu_\alpha(t) = \mu(t) + \frac{\ln}{(1-\alpha)1/\alpha}$, then the trivial solution of IDI (3) is finite-time uniformly stable.

**Proof.** Based on Theorem 3.1, we obtain that the trivial solution to IDI (3) is finite-time attractive. According to Definition 2.4, we just need to prove the Lyapunov uniform stability of the trivial solution. From $\mathcal{H}_1$, we have

\[
V(t_0,x_0) \leq \kappa_2(||x_0||). \tag{35}
\]

Based on (23), one has

\[
V(t,x(t)) \leq e^t \kappa_2(||x_0||), \tag{36}
\]

which implies that

\[
||x(t)|| \leq \kappa_1^{-1}(V(t,x(t))) \leq \kappa_1^{-1}(e^t \kappa_2(||x_0||)). \tag{37}
\]

For $\forall \varepsilon > 0$, there exists a $\delta = \kappa_2^{-1}(\frac{\kappa_1(\varepsilon)}{e^\varepsilon}) > 0$ such that for all $x_0 \in B(0,\delta)$, it follows that

\[
||x(t)|| \leq \kappa_1^{-1}(e^t \kappa_2(||x_0||)) \leq \varepsilon. \tag{38}
\]

Based on Definition 2.4, we have shown that the trivial solution of IDI (3) is uniformly Lyapunov stable.

If we do not consider the effects of impulses, one obtains the following corollary.

**Corollary 1.** Suppose $\kappa_1, \kappa_2 \in \mathcal{K}_\infty$. If there exists a $C$-regular and LLC function $V: \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}_+$ with $V(t,0) = 0$ for every $t \in \mathbb{R}_+$, one indefinite integral function $\mu(t)$ and one negative integral function $\rho(t)$ such that

\[
(\mathcal{H}_1) \quad \kappa_1(||x||) \leq V(t,x(t)) \leq \kappa_2(||x||), \quad \forall (t,x(t)) \in \mathbb{R}_+ \times \mathbb{R}^n. \tag{39}
\]
Impulsive Functional Differential Inclusion (IFDI):

\[ \dot{V}(t, x(t)) \leq \mu(t)V(t, x(t)) + \rho(t)V^\alpha(t, x(t)) \]  

(40)

for a.e. \( t \), where \( 0 < \alpha < 1 \).

(H1) \( \mu(t) \) and \( \rho(t) \) belong to FTSFP with parameters \( \lambda_1, \theta_1, \lambda_2, \theta_2 \) and \( \alpha \).

then the trivial solution of IDI (3) without impulsive effects is uniformly finite-time stable. Moreover, the setting time \( T(t_0, x_0) \) is estimated by

\[ T(t_0, x_0) = t_0 + \frac{V_0^{1-\alpha}e^{(1-\alpha)t_0} + (1-\alpha)\theta_2}{(1-\alpha)\lambda_2} \]

Remark 8. If \( \mu(t) \equiv 0 \) and \( \rho(t) = -\lambda \) in corollary 1, which means that \( \lambda_1 = \theta_1 = \theta_2 = 0 \) and \( \lambda_2 = \lambda \), then this corollary becomes the traditional finite-time stability theorem in [35]. Moreover, the finite-time stability theorem in [35] is only suitable for continuous differential systems. In this paper, our corollary is effective to deal with discontinuous differential system.

4. A special extension to impulsive delayed differential inclusion. Consider the discontinuous impulsive delayed differential equation (DIDDE):

\[
\begin{cases}
\dot{x}(t) = f(t, x_t), & t \neq t_i, \\
x(t_i) = g(t_i, x(t_i^-)), & t = t_i, \quad l \in \mathbb{N}, \\
x_{t_0}(s) = \phi(s), & s \in [-\tau, 0],
\end{cases}
\]

where \( x_t \in PC([-\tau, 0] : \mathbb{R}^n) \) is defined by \( x_t(s) = x(t + s), x \in [-\tau, 0] \) and \( \phi \in PC([-\tau, 0] : \mathbb{R}^n) \), \( \tau \) shows the time delay. The nonlinear function \( f : \mathbb{R}_+ \times PC([-\tau, 0] : \mathbb{R}^n) \rightarrow \mathbb{R}^n \) is assumed to be Lebesgure measurable and essentially locally bounded, which may be discontinuous on variable \( x_t \).

Construct a multi-valued map \( \mathcal{F} : \mathbb{R}_+ \times PC([-\tau, 0] : \mathbb{R}^n) \rightarrow 2^{\mathbb{R}^n} \) as

\[ \mathcal{F}(t, x_t) = \bigcap_{\delta > 0} \bigcap_{\mu(Z) = 0} \mathcal{C}(f(t, B(x_t, \delta) \setminus Z)), \]

where \( \bigcap_{\mu(Z) = 0} \) means the intersection over all sets and \( B(x_t, \delta) = \{ x_t^\ast \in C([\tau, 0], \mathbb{R}^n) | \| x_t^\ast - x_t \|_C \leq \delta \} \) with \( \| \varphi \|_C = \sup_{s \in [-\tau, 0]} \| \varphi(s) \| \). Then one obtains the following Impulsive Functional Differential Inclusion (IFDI):

\[
\begin{cases}
\dot{x}(t) \in \mathcal{F}(t, x_t), & t \neq t_i, \\
x(t_i) = g(t_i, x(t_i^-)), & t = t_i, \\
x_{t_0}(s) = \phi(s), & s \in [-\tau, 0],
\end{cases}
\]

where \( \phi \in PC([-\tau, 0] : \mathbb{R}^n) \). Obviously, the multi-valued map \( \mathcal{F}(t, x_t) \) is nonempty, compact, convex and USC. Then according to [1], the local and global existence of \( x(t_0, x_{t_0})(t) \) of (43) can be guaranteed under appropriate conditions.

Based on Theorem 3.2, one derives the following delay-independent FTS result. The proof is straight forward and leave it out here.

Theorem 4.1. Suppose the impulsive sequence \( \{ t_l : l \in \mathbb{N} \} \) possesses an average impulsive interval \( \Gamma_\alpha \) and elasticity number \( N_0, \kappa_1, \kappa_2 \in K_\infty \). If there exists a \( C \)-regular and LLC Lyapunov function \( V : \mathbb{R}_+ \times \mathbb{R}^n \rightarrow [0, \infty) \) with \( V(t, 0) = 0 \) for every \( t \in \mathbb{R}_+ \), one indefinite integral function \( \mu(t) \) and one negative integral function \( \rho(t) \)
such that \(( H_1), (H_3), (H_4)\) of Theorem 3.2 hold and 
\(( H_2)\) the derivative of \(V(t, x(t))\) along the trajectories of system \((43)\) satisfies
\[
\dot{V}(t, x(t)) \leq \mu(t)V(t, x(t)) + \rho(t)V^{\alpha}(t, x(t))
\]
for a.e. \(t \in [t_l, t_{l+1})\), where \(0 < \alpha < 1\), then the trivial solution of IFDI \((43)\) is
uniformly finite-time stable.

5. Application to impulsive discontinuous neural networks. In this section, we present a FTS result for impulsive
discontinuous neural networks. Before that, we introduce a special impulsive sequence \(\{t_l : l \in \mathbb{N}\}\).

Lemma 5.1. [27] If the given impulsive sequence \(\{t_l : l \in \mathbb{N}\}\) is described in the
following form:
\[
t_l - t_{l-1} = \begin{cases} 
\theta & \text{if } \text{mod}(l, N_0) \neq 0, \\
N_0(\Gamma_a - \theta) & \text{if } \text{mod}(l, N_0) = 0,
\end{cases}
\]
where \(\theta\) and \(\Gamma_a\) are positive numbers satisfying \(\theta \leq \Gamma_a \in \mathbb{N}\), then \(\Gamma_a\) and
\(N_0\) are the average impulsive interval and elasticity number of \(\{t_l : l \in \mathbb{N}\}\).

We consider the following discontinuous neural networks with impulses:
\[
\begin{align*}
\dot{x}_i(t) &= d_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}f_j(x_j(t)) + u_i(t), \quad t \neq t_l, \\
x_i(t_l) &= \eta_i(t_l^-)x_i(t_l^-), \quad t = t_l, \quad l \in \mathbb{N},
\end{align*}
\]
where \(i \in \{1, 2, \ldots, n\}\); \(x_i(t)\) means the \(i\)-th neuron state; \(d_i(t)\) denotes the self-
inhibition and \(a_{ij}(t)\) is the connection strength between neurons \(i\) and \(j\); \(f_j(\cdot)\) shows
the \(j\)-th neuron activation function; \(u_i(t)\) means external input and \(\eta_i(t_l^-)\) shows
the bounded impulsive gain at instant \(t_l\). The discontinuous neuron activations in
\((46)\) should satisfy the following properties:
\((A_1)\) For every discontinuous function \(f_i : \mathbb{R} \rightarrow \mathbb{R}\) is discontinuous on a set of
isolate points \(\rho_k\), where the right and left limits \(f_i^+(\rho_k), f_i^-(\rho_k)\) exist. Moreover, \(f_i\)
has at most a finite number of jump discontinuities in every compact interval of \(\mathbb{R}\).
\((A_2)\) For every \(j, 0 \in \text{co}[f_j(\emptyset)]\) and there exist nonnegative constants \(k_j\) and \(l_j\)
such that
\[
\sup_{\gamma_j \in \text{co}[f_j(x_j)]} |\gamma_j| \leq k_j \|x_j\| + l_j
\]
where for \(\theta \in \mathbb{R}\) and \(\text{co}[f_j(\emptyset)] = [\min\{f_j^-(\emptyset), f_j^+(\emptyset)\}, \max\{f_j^-(\emptyset), f_j^+(\emptyset)\}]\).

Through Filippov regularization, if \(x(t)\) is a Filippov solution of \((46)\), then it satisfies the following differential inclusion
\[
\begin{align*}
\dot{x}_i(t) &\in d_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)\text{co}[f_j(x_j)] + \text{co}[u_i(t)] \triangleq \mathcal{F}_i(t, x) \quad \text{for a.e. } t \neq t_l, \\
x_i(t_l) &= \eta_i(t_l^-)x_i(t_l^-), \quad t = t_l, \quad l \in \mathbb{N}.
\end{align*}
\]
(47)
Here, we assume that the external input is given in the following form
\[
u_i(t) = \alpha_i(t)x_i(t) + \beta_i(t)\text{sign}(x_i(t)),
\]
where \(\alpha_i(t), \beta_i(t)\) are time-varying parameters. Obviously, \(x = 0\) is a zero solution in
sense of Filippov. Our goal is proposing the conditions to stabilize the discontinuous
impulsive differential inclusion \((47)\) to equilibrium point \(x = 0\) in finite time.
**Theorem 5.2.** Suppose that the impulsive sequence \( \{t_l : l \in \mathbb{N}\} \) is expressed in the form of Lemma 5.1 and conditions \((A_1)\) and \((A_2)\) are satisfied. If the system parameters and control gains are satisfying
\[
\mu_i(t) = \alpha_i(t) + d_i(t) + \frac{1}{2} \sum_{j=1}^{n} (k_{ij}|a_{ij}(t)| + k_i|a_{ji}(t)|) \leq \frac{1}{2(1+ \rho_a)} - \frac{1}{\rho_a},
\]
\[
\rho_i(t) = \beta_i(t) + \sum_{j=1}^{n} l_j|a_{ij}(t)| \leq -\frac{1}{2},
\]
where \( \eta = \max_{i \in \{1,2,\ldots,n\}} \sup\{|\eta_i(t)|\} \), then the trivial solution of system (47) is uniformly finite-time stable.

**Proof.** Construct a Lyapunov function to be \( V(t, x) = \sum_{i=1}^{n} x_i^2(t) \). When \( t \neq t_l \), estimating the derivative of \( V(t, x) \) along the solution of (47), one has
\[
\dot{V}(t, x)(47) = 2 \sum_{i=1}^{n} x_i[d_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}\gamma_j(t) + u_i(t)]
\]
\[
\leq 2 \sum_{i=1}^{n} (d_i(t) + \alpha_i(t))x_i^2(t) + 2 \sum_{i=1}^{n} \sum_{j=1}^{n} k_{ij}|a_{ij}(t)||x_i(t)||x_j(t)|
\]
\[
+ 2 \sum_{i=1}^{n} (\beta_i(t) + \sum_{j=1}^{n} l_j|a_{ij}(t)||x_i(t)|)
\]
\[
\leq 2 \sum_{i=1}^{n} \mu_i(t)x_i^2(t) + 2 \sum_{i=1}^{n} \rho_i(t)|x_i(t)|
\]
\[
\leq 2(\mu(t)V(t, x) + 2\rho(t)V^{\frac{1}{2}}(t, x)), \text{ for a.e. } t \geq t_0,
\]
where \( \mu(t) = \max_{1 \leq i \leq n} \{\mu_i(t)\} \) and \( \rho(t) = \max_{1 \leq i \leq n} \{\rho_i(t)\} \).

When \( t = t_l \), one has
\[
V(t_l, x(t_l)) = \sum_{i=1}^{n} \eta_i^2(t_l)x_i^2(t_l) \leq \eta^2V(t_{l^-}, x(t_{l^-})).
\]

Recalling the conditions \((A_1)-(A_3)\), we derive that all the conditions in Theorem 5.2 are satisfied with \( \alpha = \frac{1}{2} \), and \( \mu_\alpha(t) = 2\mu(t) + 2\frac{\ln 2}{\rho_a} \leq \frac{1}{1+ \rho_a} \) and \( \rho(t) = 2\rho_1(t) \leq -1 \).

According to Remark 3, \((\mu_\alpha(t), \rho(t)) \in \mathbb{F}\mathbb{P}\) with \( \lambda_1 = 0, \theta_1 = \frac{\pi}{2}, \lambda_2 = -1, \theta_2 = 0 \) and \( \alpha = \frac{1}{2} \). So based on Theorem 5.2, the trivial solution of (46) is uniformly finite-time stable.

**Example 1.** Consider a 2-dimensional discontinuous neural networks (46) with impulsive effects:
\[
\left\{
\begin{array}{l}
\dot{x}_1(t) = (1 - \frac{3}{4\sqrt{2}})x_1(t) + \frac{1}{1+ \sqrt{2}}f_1(x_1(t)) + \frac{1}{4+ \sqrt{2}}f_2(x_2(t)) + u_1(t), \quad t \neq t_l, \\
\dot{x}_2(t) = (1 - \frac{1}{4\sqrt{2}})x_2(t) + \frac{1}{1+ \sqrt{2}}f_1(x_1(t)) + \frac{1}{4+ \sqrt{2}}f_2(x_2(t)) + u_2(t), \quad t \neq t_l, \\
x_1(t_l) = e^{-\frac{1}{2}}x_1(t_{l^-}), \quad t = t_l, \\
x_2(t_l) = e^{-\frac{1}{2}}x_2(t_{l^-}), \quad t = t_l.
\end{array}
\right.
\]
The discontinuous neuron activation functions are given as \( f_i(\theta) = (\theta - 0.5)\text{sign}(\theta) \).

It is easy to check that the discontinuous activation function satisfy \((A_1)\) and \((A_2)\) with \( k_i = l_i = 1 \) and \( \eta = e^{-\frac{1}{2}} \). The coefficients of (48) are selected to be \( \alpha_i(t) = -\frac{1}{1+ \sqrt{2}} \) and \( \beta_i(t) = -\frac{1}{2} - \frac{1}{1+ \sqrt{2}} \). Let the impulsive sequence \( \{t_l\} \) be given in the form of (45) with \( \Gamma_a = \frac{1}{2} \). It is not difficult to verify that \((A_3)\) in Theorem 5.2 holds. Then we can stabilize discontinuous impulsive neural networks (46) with external input (48) in a finite time. If we selected the impulsive sequence \( \{t_l : l \in \mathbb{N}\} \)
with $\Gamma_a = \frac{1}{2}, N_0 = 2, \theta = 0.1$ and initial values $x_1(0) = 35, x_2(0) = 30$, the FTS of system (50) is realized and the settling time is $T(0, 35, 30) = 45.48$. Moreover, if the impulsive sequences are chosen to be $\{t_l : l \in \mathbb{N}\}$ with $\Gamma_a = \frac{1}{2}, N_0 = 3, \theta = 0.2$ and $\Gamma_a = \frac{1}{2}, N_0 = 1, \theta = 0.5$, and the initial values are selected to be $x_1(0) = 50, x_2(0) = -20$ and $x_1(0) = -50, x_2(0) = 80$, the settling times are $T(0, 50, -20) = 28.57$ and $T(0, -50, 80) = 53.51$, respectively. The trajectories of states with different initial values are depicted in Fig. 1(b), which shows that the nonlinear discontinuous impulsive system is FTS. In addition, the trajectories of the discontinuous neural networks (50) without impulsive effects with different initial values are shown in Fig.1(a).

**Remark 9.** From Theorem 5.2, one obtains that Condition $(A_3)$ gives a balance of system parameters, impulsive gains and average impulsive interval $\Gamma_a$. Moreover, Fig. 1(b) shows that the settling time of FTS also depends on elasticity number $N_0$ and impulsive gains $\eta$.

**Remark 10.** From Fig. 1(a), we see that the trajectories of discontinuous neural networks without impulsive effects are unstable. Fig.1 shows that the trajectories of discontinuous neural networks can be forced to be zero in finite time by impulsive control protocol. In fact, whether an unstable nonlinear system can achieve FTS via impulsive control method depends not only on the impulsive control protocol but also on the system itself.

![Figure 1. The state trajectories of $x_i(t)$ ($i = 1, 2$) without impulsive effects in Example 1.](image)

6. **Conclusions.** In this paper, we obtained a theoretical result on FTS of impulsive differential inclusion systems with the help of FTSFP, average impulsive interval and an improved Lyapunov function method. Different from traditional LF method with negative definite or semi-negative definite derivative, the developed LF method of this paper is allowed to possess indefinite derivative (when it exists). Finally, we presented a FTS result of discontinuous impulsive neural networks. In future, we will study the finite-time input-to-state stability or fixed time stability...
The trajectories of states $x_i(t)$ ($i = 1, 2$) with different impulsive sequences in Example 1.

for impulsive discontinuous system or stochastic discontinuous differential system based on differential inclusion theory.

REFERENCES

[1] N. Abada, M. Benchohra and H. Hammouche, Existence and controllability results for non-densely defined impulsive semilinear functional differential inclusions, J. Differential Equations, 246 (2009), 3834–3863.

[2] J. Abderrahim and E. Vilches, A differential equation approach to implicit sweeping processes, J. Differential Equations, 266 (2019), 5168–5184.

[3] W. Allegretto, D. Papini and M. Forti, Common asymptotic behavior of solutions and almost periodicity for discontinuous, delayed, and impulsive neural networks, IEEE Trans. Neural Netw., 21 (2010), 1110–1125.

[4] F. Amato, G. De Tommasi and A. Pironti, Necessary and sufficient conditions for finite-time stability of impulsive dynamical linear systems, Automatica J. IFAC, 49 (2013), 2546–2550.

[5] R. Ambrosino, F. Calabrese, C. Cosentino and G. Tommasi, Sufficient conditions for finite-time stability of impulsive dynamical systems, IEEE Trans. Automat. Control, 54 (2009), 861–865.

[6] J.-P. Aubin and A. Cellina. Differential Inclusions. Set-Valued Functions and Viability Theory, Grundlehren der Mathematischen Wissenschaften, 264. Springer-Verlag, Berlin, 1984.

[7] G. Ballinger and X. Z. Liu, Existence and uniqueness results for impulsive delay differential equation, Dyn. Contin. Discrete Impuls. Syst., 5 (1999), 579–591.

[8] J. Cao, G. Stamov, I. Stamova and S. Simeonov, Almost periodicity in impulsive fractional-order reaction-diffusion neural networks with time-varying delays, IEEE Trans. Cybern., (2020), http://dx.doi.org/10.1109/TCYB.2020.2987625.

[9] G. Chen, Y. Yang and J. Li, Finite time stability of a class of hybrid dynamical systems, IET Control Theory Appl., 6 (2012), 8–13.

[10] G. Craciun, Polynomial dynamical systems, reaction networks, and toric differential inclusions, SIAM J. Appl. Algebra Geometry, 3 (2019), 87–106.

[11] S. Djebali, L. Gorniewicz and A. Ouahab, First-order periodic impulsive semilinear differential inclusions: Existence and structure of solution sets, Math. Comput. Modelling., 52 (2010), 683–714.

[12] A. F. Filippov, Differential Equations with Discontinuous Righthand Sides Mathematics and its Applications (Soviet Series), 18. Kluwer Academic Publishers Group, Dordrecht, 1988.

[13] M. Forti and P. Nistri, Global convergence of neural networks with discontinuous neuron activations, IEEE Trans. Circuits Systems I Fund. Theory Appl., 50 (2003), 1421–1435.
[14] M. Forti and D. Papini, Global exponential stability and global convergence in finite time of delayed neural network with infinite gain, IEEE Trans. Neural Netw., 16 (2005), 1449–1463.
[15] H. Fujisaka and T. Yamada, Stability theory of synchronized motion in coupled-oscillator systems, Progr. Theoret. Phys., 69 (1983), 32–47.
[16] G. Haddad, Monotone viable trajectories for functional differential inclusions, J. Differential Equations, 41 (1981), 1–24.
[17] G. Haddad, Topological properties of the sets of solutions for functional differential inclusion, Nonlinear Anal., 39 (1981), 1349–1366.
[18] J. P. Hespanha, D. Liberzon and A. R. Teel, Lyapunov conditions for input-to-state stability of impulsive systems, Automatica J. IFAC, 44 (2008), 2735–2744.
[19] S. C. Hu, D. A. Kandilakis and N. S. Papageorgiou, Periodic solutions for nonconvex differential inclusions, Proc. Amer. Math. Soc., 127 (1999), 89–94.
[20] L. Huang, Z. Guo and J. Wang, Theory and Applications of Differential Equations with Discontinuous Right-hand Sides, Science Press, Beijing, 2011.
[21] P. Hur, B. Duiser, S. Salapaka and E. Weckster, Measuring robustness of the postural control system to a mild impulsive perturbation, IEEE Trans. Neur. Syst. Rehab. Engin., 18 (2010), 461–467.
[22] X. D. Li, D. W. C. Ho and J. D. Cao, Finite-time stability and settling-time estimation of nonlinear impulsive systems, Automatica J. IFAC, 99 (2019), 361–368.
[23] Y. C. Li and R. G. Sanfelice, Finite time stability of sets for hybrid dynamical systems, Automatica J. IFAC, 100 (2019), 200–211.
[24] J. X. Liu, L. G. Wu, C. W. Wu, W. S. Luo and L. Franquelo, Event-triggering dissipative control of switched stochastic systems via sliding mode, Automatica J. IFAC, 103 (2019), 261–273.
[25] K.-Z. Liu, X.-M. Sun, J. Liu and R. Andrew, Stability theorems for delayed differential inclusions, IEEE Trans. Autom. Control., 61 (2016), 3215–3220.
[26] W. L. Lu and T. P. Chen, Almost periodic dynamics of a class of delayed neural networks with discontinuous activations, Neural Comput., 20 (2008), 1065–1090.
[27] J. Q Lu, D. W. C. Ho and J. D. Cao, A unified synchronization criterion for impulsive dynamical networks, Automatica J. IFAC, 46 (2010), 1215–1221.
[28] E. Moulay and W. Perruquetti, Finite time stability of differential inclusions, IMA J. Math. Control Inform., 22 (2005), 465–475.
[29] E. Moulay and W. Perruquetti, Finite time stability and stabilization of a class of conituous systems, J. Math. Anal. Appl., 323 (2006), 1430–1443.
[30] E. Moulay, M. Dambrine, N. Yeganefar and W. Perruquetti, Finite time stability and stabilization of time-delayed systems, Systems Control Lett., 57 (2008), 561–566.
[31] J. Nygren and K. Pelckmans, A stability criterion for switching Lu’re systems with switching-path restrictions, Automatica J. IFAC, 96 (2018), 337–341.
[32] B. E. Paden and S. S. Sastry, A calculus for computing Filippov’s differential inclusion with application to the variable structure control of robot manipulator, IEEE Trans. Circuits Syst., 34 (1987), 73–82.
[33] S. G. Peng, F. Q. Deng and Y. Zhang, A unified Razumikhin-type criteria on input-to-state stability of time-varying impulsive delayed system, Systems Control Lett., 216 (2018), 20–26.
[34] A. Polyakov, Nonlinear feedback design for fixed-time stabilization of linear control systems, IEEE Trans. Auto. Contr., 57 (2012), 2106–2109.
[35] A. Polyakov, D. Efimov and W. Perruquetti, Finite-time and fixed-time stabilization: Implicit Lyapunov function approach, Automatica J. IFAC, 51 (2015), 332–340.
[36] S. T. Qin and X. P. Xue, Periodic solutions for nonlinear differential inclusions with multivalued perturbations, J. Math. Anal. Appl., 424 (2015), 988–1005.
[37] E. Serpelloni, M. Maggiore and C. Damaren, Bang-bang hybrid stabilization of perturbed double-integrators, Automatica J. IFAC, 69 (2016), 315–323.
[38] S. Vaddi, K. Alfriend, S. Vadali and P. Sengupta. Formation establishment and reconfiguration using impulsive control, J. Guid Control. Dynam., 28 (2005), 262–268.
[39] A. Vinodkumar and A. Anguraj, Existence of random impulsive abstract neutral nonautonomous differential inclusions with delays, Nonlinear Anal. Hybrid Syst., 5 (2011), 413–426.
[40] X. T. Wu, Y. Tang and W. B. Zhang, Input-to-state stability of impulsive stochastic delayed systems under linear assumptions, Automatica J. IFAC, 66 (2016), 195–2014.
[41] T. Yang, *Impulsive Control Theory*, Lecture Notes in Control and Information Sciences, 272. Springer-Verlag, Berlin, 2001.

[42] B. Zhou, On asymptotic stability of linear time-varying systems, *Automatica J. IFAC*, 68 (2016), 266–276.

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