On classification of dynamical r-matrices

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Abstract

Using the gauge transformations of the Classical Dynamical Yang-Baxter Equation introduced by P. Etingof and A. Varchenko in [EV], we reduce the classification of dynamical r-matrices r on a commutative subalgebra l of a Lie algebra g to a purely algebraic problem, under some assumption on the symmetric part of r. We then describe, for a simple complex Lie algebra g, all non skew-symmetric dynamical r-matrices on a commutative subalgebra l ⊂ g which contains a regular semisimple element. This interpolates results of P. Etingof and A. Varchenko ([EV], when l is a Cartan subalgebra) and results of A. Belavin and V. Drinfel’d for constant r-matrices ([BD]). This classification is similar, and in some sense simpler than the Belavin-Drinfeld classification.

1 The Classical Yang-Baxter Equation

Let g be a Lie algebra. The CYBE is the following algebraic equation for an element r ∈ g ⊗ g:

\[ [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0. \] (1)

Solutions of this equation are called r-matrices. In the theory of quantum groups, one is mainly interested in r-matrices satisfying

\[ r + r^{21} \in (S^2 g)^g. \] (2)

See [CP] for the links with the theory of quantum groups, and [Che] for links with Conformal Field Theory and the Wess-Zumino-Witten model on \( \mathbb{P}^1 \). The geometric interpretation of the CYBE was given by Drinfeld in terms of Poisson-Lie groups ([Dr1]).

2 The Belavin-Drinfeld Classification

Notations: Let g be a simple complex Lie algebra with a nondegenerate invariant form \( (\cdot, \cdot) \), \( \mathfrak{h} \subset g \) a Cartan subalgebra and \( \Delta \) the root system. For \( \alpha \in \Delta \), let \( g_\alpha \) denote the root subspace associated to \( \alpha \). Let W be the Weyl group and
s_{\alpha}, \alpha \in \Delta \text{ the reflection with respect to } \alpha^\perp. \text{ Finally, let } \Omega \in S^2 \mathfrak{g} \text{ and } \Omega_0 \in S^2 \mathfrak{h} \text{ be the inverse elements to the form } ( , ) \text{. Notice that } (S^2 \mathfrak{g})^0 = \mathfrak{h} \mathfrak{g}.

For any polarization \( \mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ \), we denote by \( \Pi \) or \( \Pi(n_+) \) the corresponding set of simple positive roots, by \( \Delta_+ \) the set of positive roots and by \( \mathfrak{b}_\pm = \mathfrak{n}_+ \oplus \mathfrak{h} \) the Borel subalgebras. For \( \Gamma \subset \Pi \), set \( (\Gamma) = \mathbb{Z} \Gamma \cap \Delta \), and let \( \mathfrak{g}_\Gamma \) be the subalgebra generated by \( \mathfrak{g}_{\alpha}, \alpha \in (\Gamma) \). We will write \( \mathfrak{g}_\Gamma = \mathfrak{n}_+(\Gamma) \oplus \mathfrak{h}(\Gamma) \oplus \mathfrak{n}_-(\Gamma) \) for the induced polarization and \( W(\Gamma) \) for the subgroup of \( W \) generated by \( s_{\alpha}, \alpha \in \Gamma \).

Let us fix a polarization of \( \mathfrak{g} \).

**Definition:** A Belavin-Drinfeld triple is a triple \( (\Gamma_1, \Gamma_2, \tau) \) where \( \Gamma_1, \Gamma_2 \subset \Pi \) and \( \tau : \Gamma_1 \rightarrow \Gamma_2 \) is a norm-preserving bijection satisfying the following ”nilpotency” condition:

"For any \( \gamma_1 \in \Gamma_1 \), there exists \( n > 0 \) such that \( \tau^n(\gamma_1) \in \Gamma_2 \setminus \Gamma_1 \)."

Let \( (\Gamma_1, \Gamma_2, \tau) \) be a Belavin-Drinfeld triple. For each choice of Chevalley generators \( (e_\alpha, f_\alpha, h_\alpha)_{\alpha \in \Gamma_1}, i = 1, 2 \), the isomorphism \( \tau \) induces a Lie algebra isomorphism \( \mathfrak{g}_{\Gamma_1} \rightarrow \mathfrak{g}_{\Gamma_2} \) (by \( e_\alpha \rightarrow e_{\tau(\alpha)} \), \( f_\alpha \rightarrow f_{\tau(\alpha)} \), \( h_\alpha \rightarrow h_{\tau(\alpha)} \)).

Define a partial order on \( \Delta_+ \) by setting \( \alpha < \beta \) if there exists \( n > 0 \) such that \( \tau^n(\alpha) = \beta \) (in particular, \( \alpha \in \Gamma_1 \) and \( \beta \in \Gamma_2 \)).

**Definition:** A basis \( (x_\alpha)_{\alpha \in \Delta} \) of \( \mathfrak{n}_+ \oplus \mathfrak{n}_- \) is called admissible if \( (x_\alpha, x_{-\alpha}) = 1 \) and \( \tau(x_\alpha) = x_{\tau(\alpha)} \) for \( \alpha \in (\Gamma_1) \).

**Theorem 1 (Belavin-Drinfeld)** Let \( \mathfrak{g} \) be a simple complex Lie algebra.

1. Let \( (\Gamma_1, \Gamma_2, \tau) \) be a Belavin-Drinfeld triple, \( (x_\alpha) \) an admissible basis, and let \( r_0 \in \mathfrak{h} \oplus \mathfrak{h} \) be such that

\[
\begin{align*}
r_0 + r_0^{21} &= \Omega_0, \\
(\tau(\alpha) \otimes 1)r + (1 \otimes \alpha)r &= 0 \quad \text{for } \alpha \in \Gamma_1.
\end{align*}
\]

Then

\[
r = r_0 + \sum_{\alpha \in \Delta_+} x_{-\alpha} \otimes x_\alpha + \sum_{\alpha, \beta \in \Delta_+, \alpha < \beta} x_{-\alpha} \wedge x_\beta
\]

is an \( r \)-matrix satisfying \( r + r^{21} = \Omega \).

2. Any \( r \)-matrix satisfying \( r + r^{21} = \Omega \) is of the above type for a suitable polarization of \( \mathfrak{g} \).

This theorem is proved in [BD]. For instance, the standard \( r \)-matrix for a fixed polarization \( r = \frac{\Omega_0}{2} + \sum_{\alpha \in \Delta_+} x_{-\alpha} \otimes x_\alpha \) corresponds to \( \Gamma_1 = \Gamma_2 = \emptyset \).

**Remark:** Skew-symmetric \( r \)-matrices admit a well known interpretation in terms of nondegenerate 2-cocycles on Lie subalgebras of \( \mathfrak{g} \) ([Dr1]), but their classification is unavailable since it requires a classification of Lie subalgebras in \( \mathfrak{g} \).
3 The Dynamical Yang-Baxter Equation

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$ and $\mathfrak{l} \subset \mathfrak{g}$ a subalgebra. An element $x \in \mathfrak{g} \otimes \mathfrak{g}$ will be called $\mathfrak{l}$-invariant if

$$[k \otimes 1 + 1 \otimes k, x] = 0 \quad (\forall k \in \mathfrak{l}). \quad (6)$$

For $x \in \mathfrak{g}^3$, we let $\text{Alt}(x) = x^{123} + x^{231} + x^{312}$. Let $D \subset \mathfrak{l}^*$ be any open region.

The CDYBE is the following differential equation for a holomorphic $\mathfrak{l}$-invariant function $r : D \to \mathfrak{g} \otimes \mathfrak{g}$:

$$\text{Alt}(dr) + [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0, \quad (7)$$

where the differential of $r$ is considered as a holomorphic function

$$dr : D \to \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g}, \quad \lambda \mapsto \sum_i x_i \otimes \frac{\partial r^{23}}{\partial x_i}(\lambda), \quad (\lambda \in \mathfrak{l}^*),$$

for any basis $(x_i)$ of $\mathfrak{l}$. In this case,

$$\text{Alt}(dr) = \sum_i x_i^{(1)} \frac{\partial r^{23}}{\partial x_i} + \sum_i x_i^{(2)} \frac{\partial r^{31}}{\partial x_i} + \sum_i x_i^{(3)} \frac{\partial r^{12}}{\partial x_i}.$$ 

The solutions to this equation are called dynamical $r$-matrices. Dynamical $r$-matrices which are relevant to the theory of quantum groups are those satisfying the following condition, analogous to (2):

Generalized unitarity : $r(\lambda) + r^{21}(\lambda) \in (S^2 \mathfrak{g})^\mathfrak{l}. \quad (8)$

Remark: the CDYBE was first written down by G. Felder and C. Wiezcerkowski in connection with the Wess-Zumino-Witten model on elliptic curves ([F W]). The relation with elliptic quantum groups is explained in [F]. A geometric interpretation of the CDYBE analogous to the theory of Poisson-Lie groups for the CYBE is given in [E V].

4 Gauge transformations:

We recall some results from [E V]. We suppose here that $\mathfrak{l}$ is commutative and we let $D$ be the formal polydisc centered at the origin. Let $G$ be a complex Lie group such that $\text{Lie}(G) = \mathfrak{g}$, and let $L$ be the connected subgroup of $G$ such that $\text{Lie}(L) = \mathfrak{l}$. Let $G^L$ be the centralizer of $L$ in $G$ and $\mathfrak{g}^l$ its Lie algebra. We will denote by $(\mathfrak{g} \otimes \mathfrak{g})^\mathfrak{l}$ the space of all $\mathfrak{l}$-invariant elements in $\mathfrak{g} \otimes \mathfrak{g}$.

Let $g : D \to G^L$ be any holomorphic function; the 1-form $\eta = g^{-1} dg$ gives rise to a function $\tilde{\eta} : D \to \mathfrak{l} \otimes \mathfrak{g}$. If $r : D \to (\mathfrak{g} \otimes \mathfrak{g})^\mathfrak{l}$ is an $\mathfrak{l}$-invariant function satisfying (8), we set

$$r^g = (g \otimes g)(r - \tilde{\eta} + \tilde{\eta}^{21})(g^{-1} \otimes g^{-1}).$$

Proposition 1 The function $r$ is a dynamical $r$-matrix if and only if the function $r^g$ is.
Thus the group $\text{Map}(D, G^{L})$ is a gauge transformation group for the CDYBE. Notice that this group is not commutative if $G^{L}$ isn’t.

**Theorem 2** Let $\rho, r : D \to g^{\otimes 2}$ be two dynamical $r$-matrices satisfying (8) such that $r(0) = \rho(0)$. Then there exists $g \in \text{Map}(D, G^{L})$ such that $\rho = r^{g}$.

This shows that the space of dynamical $r$-matrices is, up to gauge equivalence, finite dimensional. Proofs of the above results can be found in [EV].

We will now prove a converse of Theorem 2 which reduces the CDYBE to a purely algebraic equation under some assumption on the symmetric part $\Omega^{2}$ of $r$: let $\Omega \in (S^{2}g)^{g}$, let $g_{\Omega}$ be the ideal in $g$ generated by the components of $\Omega$ and denote by $g_{\Omega} = \bigoplus_{\lambda} g_{\Omega}(\lambda)$ the generalized weight space decomposition of $g_{\Omega}$ with respect to the adjoint action of $l$. The condition we will need is the following:

$$g^{l} \text{ acts semisimply on } g_{\Omega}(0) \quad (*)$$

Suppose that $(*)$ is fulfilled and let $z(g^{l})$ denote the center of $g^{l}$. Then we have a decomposition $g_{\Omega}(0) = z_{0}(g^{l}) \oplus V$ where $z_{0}(g^{l}) = z(g^{l}) \cap g_{\Omega}(0)$ and $V$ is the sum of all non-trivial irreducible $g^{l}$-modules in $g_{\Omega}(0)$. It is clear that $l \cap V = \{0\}$. We will say that a complement $l'$ of $l$ in $g$ is admissible if $V \subset l'$, and write $\pi : g \to l$ for the projection along $l'$. Notice that by $g^{l}$-invariance of $\Omega$,

$$\Omega \in S^{2}z_{0}(g^{l}) \oplus S^{2}V \oplus \bigoplus_{\lambda \neq 0} g_{\Omega}(\lambda) \otimes g_{\Omega}(-\lambda). \quad (9)$$

We will denote by $CYB : g^{\otimes 2} \to g^{\otimes 3}$ the map:

$$r \mapsto [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}].$$

It is more convenient to work with the skew-symmetric part of $r$. If $r(\lambda) + r^{21}(\lambda) = \Omega \in (S^{2}(g))^{g}$, we set $s(\lambda) = r(\lambda) - \frac{\Omega}{2}$. It is easy to see that the CDYBE for $r$ is equivalent to the following equation for $s$:

$$\text{Alt}(ds) + CYB(s) + \frac{1}{4} CYB(\Omega) = 0. \quad (10)$$

Recall that as $\Omega$ is symmetric and invariant, $CYB(\Omega) = [\Omega_{13}, \Omega_{23}]$.

**Theorem 3** Let $G$ be a complex Lie group and $L \subset G$ a connected commutative subgroup. Let $g, l, g^{l}$ denote the Lie algebras of $G, L$ and $G^{L}$. Let $\Omega \in (S^{2}g)^{g}$. Then

1. Let $l'$ be any complement of $l$ in $g$. Any dynamical $r$-matrix $r(\lambda)$ on $l$ such that $r(\lambda) + r^{21}(\lambda) = \Omega$ is gauge equivalent to a dynamical $r$-matrix $\tilde{r}(\lambda) : D \to \frac{\Omega}{2} + (\Lambda^{2}(l'))^{l}$.
2. Suppose that condition (*) is true and let \( l' \) be any admissible complement of \( l \) in \( g \). Let \( r_0 \in \frac{\Omega}{2} + (\Lambda^2(l'))^1 \) satisfy

\[
CYB(r_0) \in \text{Alt}(l \otimes g \otimes g) \tag{11}
\]

such that \( s_0 = r_0 - \frac{\Omega}{2} \) is a regular point of the algebraic manifold

\[
M_\Omega = \{ s \in (\Lambda^2(l'))^1 \mid CYB(s + \frac{\Omega}{2}) \in \text{Alt}(l \otimes g \otimes g) \}.
\]

Then there exists a dynamical r-matrix \( r(\lambda) : D \to \frac{\Omega}{2} + (\Lambda^2(l'))^1 \) such that \( r(0) = r_0 \).

The condition (*) is satisfied in the following two interesting special cases: when \( \Omega = 0 \) (triangular case) or when \( g_l \) acts semisimply on \( g \) (for instance, \( G \) is reductive and \( L \) is contained in a maximal torus of \( G \) or more generally, if \( GL \) is reductive).

The proof of this theorem will occupy the rest of this section.

Let us first prove part 1:

**Lemma 1** Any dynamical r-matrix such that \( r(\lambda) + r^{21}(\lambda) = \Omega \) is gauge-equivalent to a dynamical r-matrix \( \tilde{r}(\lambda) \) such that \( \tilde{r}(0) \in \frac{\Omega}{2} + (\Lambda^2(l'))^1 \).

**Proof:** Let \( \eta \in l \otimes g^1 \) be such that \( r(0) - \eta + \eta^{21} \in \frac{\Omega}{2} + \Lambda^2(l') \). There exists a function \( g : D \to GL \) such that \( g^{-1}dg(0) = \eta \) (see [EV], Lemma 1.3). It is easy to see that \( \tilde{r} = r^g \) satisfies the desired conditions.

By Theorem 2, part 1. is proved. Let us now prove part 2. We will interpret the CDYBE (10) as a consistent system of differential equations defined on \( M_\Omega \).

For \( s \in M_\Omega \), (10) is equivalent to

\[
(\pi \otimes 1 \otimes 1)\text{Alt}(ds) = -(\pi \otimes 1 \otimes 1)(CYB(s) + \frac{1}{4}CYB(\Omega)).
\]

This reduces to

\[
ds = -(\pi \otimes 1 \otimes 1)((s^{12}, s^{13}) + \frac{1}{4}CYB(\Omega)), \tag{12}
\]

or, in coordinates \((x_i)\), where \((x_i)\) is a basis of \( l \),

\[
\frac{\partial s}{\partial x_i} = -(x_i \otimes 1 \otimes 1)((s^{12}, s^{13}) + \frac{1}{4}CYB(\Omega)).
\]

**Lemma 2** The system (12) is consistent.
Proof: Set $X : M_{\Omega} \rightarrow \mathfrak{l} \otimes \mathfrak{g} \otimes \mathfrak{g}$, $s \mapsto (\pi \otimes 1 \otimes 1)([s^{12}, s^{13}] + \frac{1}{4} CYB(\Omega))$. By definition, the curvature of (12) is given by

$$
\sum_{i,j} x_i \otimes x_j \otimes (\frac{\partial^2 s}{\partial x_i \partial x_j} - \frac{\partial^2 s}{\partial x_j \partial x_i})
$$

$$
= (\pi \otimes \pi \otimes 1 \otimes 1)(\{[s^{23}, [s^{12}, s^{14}]] + [s^{23}, \frac{1}{4} CYB(\Omega)]^{124}\}
+ [[s^{12}, s^{13}], s^{24}] + \frac{1}{4} CYB(\Omega)^{123}, s^{24}]\}
- \{[s^{13}, [s^{21}, s^{24}]] + [s^{13}, \frac{1}{4} CYB(\Omega)^{214}]\}
+ [[s^{21}, s^{23}], s^{14}] + \frac{1}{4} CYB(\Omega)^{213}, s^{14}]\}
= (\pi \otimes \pi \otimes 1 \otimes 1)(\{[s^{23}, [s^{12}, s^{14}]] + [[s^{12}, s^{13}], s^{24}] - [s^{13}, [s^{21}, s^{24}]] - [[s^{21}, s^{23}], s^{14}]\}
+ \frac{1}{4} \{[s^{13} + s^{23}, CYB(\Omega)^{124}] - [s^{14} + s^{24}, CYB(\Omega)^{123}]\}).
$$

By the Jacobi identity,

$$
[s^{23}, [s^{12}, s^{14}]] = [[s^{21}, s^{23}], s^{14}], \quad [[s^{12}, s^{13}], s^{24}] = [s^{13}, [s^{21}, s^{24}]].
$$

By $\mathfrak{g}$-invariance of $CYB(\Omega)$, we have

$$
[s^{13} + s^{23}, CYB(\Omega)^{124}] = [s^{34}, CYB(\Omega)^{124}],
$$
$$
[s^{14} + s^{24}, CYB(\Omega)^{123}] = -[s^{34}, CYB(\Omega)^{123}].
$$

Overall, we have the following expression for the curvature of (12):

$$
\frac{1}{4} (\pi \otimes \pi \otimes 1 \otimes 1)([CYB(\Omega)^{123} + CYB(\Omega)^{124}, s^{34}] = \frac{1}{4} ([\pi \otimes \pi \otimes 1 \otimes 1)CYB(\Omega), s]
$$

But (1) and the fact that $l'$ is admissible imply that $(\pi \otimes \pi \otimes 1)CYB(\Omega) = 0$. Thus, (12) is consistent.

\[ \square \]

Lemma 3 The system (12) is defined on $M_{\Omega}$, i.e. the vector fields defined by (12) are tangent to $M_{\Omega}$.

Proof: Let $x^* \in \mathfrak{l}' \otimes \mathfrak{g}'$, and set $h = (x^* \otimes 1 \otimes 1)([s^{12}, s^{13}] + \frac{1}{4} CYB(\Omega))$. Since $s \in \Lambda^2(l')$ we have $(x^* \otimes 1 \otimes 1)[s^{12}, s^{13}] \in \Lambda^2(l')$. Moreover, the admissibility of $l'$ and (1) together imply that $(x^* \otimes 1 \otimes 1)(CYB(\Omega)) \in (\Lambda^2 l')^t$ since $[l \otimes 1, S^{20}(g')] = 0$. Thus $h \in \Lambda^2 l'$.

To conclude the proof of Lemma 3 and Theorem 3, we now show that

$$
[s^{12}, h^{13}] + [s^{12}, h^{23}] + [s^{13}, h^{23}]
+ [h^{12}, s^{13}] + [h^{12}, s^{23}] + [h^{13}, s^{23}] \in \text{Alt}(l \otimes \mathfrak{g} \otimes \mathfrak{g}).
$$

(13)

To make the presentation more clear, we will use the pictorial technique to represent expressions and make computations: we associate to each morphism
from a $n$-tensor to a $m$-tensor a diagram in the following way: the operation of taking the commutator is represented by

![Diagram](a [a,b] b)

Applying a linear form $x^*$ will be denoted by

![Diagram](a $x^*$ (a))

Finally, we will represent $s$ and $\frac{\Omega}{2}$, which can be thought of as maps from a 0-tensor to a 2-tensor, by

$s = \Omega = \frac{\Omega}{2}$

For instance,

$\text{CYB}(s) = \Omega$

**Lemma 4** *We have $x^{*(3)}[\text{CYB}(s + \frac{\Omega}{2}, s^{123}, s^{34}] \in \text{Alt}(l \otimes g \otimes g)$ or, in pictures (modulo $\text{Alt}(l \otimes g \otimes g)$)*

![Diagram](Proof)

**Proof:** Recall that $\text{CYB}(s + \frac{\Omega}{2}) \in \text{Alt}(l \otimes g \otimes g)$. Thus the only part of the above expression which can lie outside of $\text{Alt}(l \otimes g \otimes g)$ is obtained from the $g \otimes g \otimes l$-part of $\text{CYB}(s)$. But if $y \in l$,

$$(x^* \otimes 1)[y \otimes 1, s] = -(x^* \otimes 1)[1 \otimes y, s]$$

by $l$-invariance of $s$. This last expression is zero since $s \in (\Lambda^2(\ell'))^l$. Lemma [] is proved.

It is clear how to generalize Lemma [] to other expressions of the form $x^{*(k)}[\text{CYB}(s + \frac{\Omega}{2}, s^{123}, s^{34})]$.  

7
Now, (13) can be drawn as

![Diagram](attachment:image.png)

but by Lemma (4) we have, modulo $\text{Alt}(I \otimes g \otimes g)$,

![Diagram](attachment:image.png)
It is easy to check that the sum of the terms of type $[CYB(s), s]$ in this last expression is zero by the Jacobi identity. Moreover, by $g$-invariance of $\Omega$, we have

Thus, modulo $\text{Alt}(l \otimes g \otimes g)$, (13) reduces to

The sums of terms in each column is zero by Jacobi Identity. This concludes the proof of Theorem 3. 

□
5 Classification of dynamical r-matrices

Let $\mathfrak{g}$ be a simple algebra. In that case, (8) becomes

$$r(\lambda) + r^{21}(\lambda) = \epsilon \Omega. \quad (14)$$

We will classify all solutions of equations (6, 7, 14) when $\epsilon \neq 0$ and when $l$ contains a semisimple regular element. In particular, in this case, the centralizer $h$ of $l$ is the unique Cartan subalgebra containing $l$. Notice that we can assume that $\epsilon = 1$ (since the assignment $r(\lambda) \rightarrow cr(\epsilon \lambda)$ is a gauge transformation of (14)). We can also assume that the restriction of $(,)$ to $l$ is nondegenerate.

Indeed, for any dynamical r-matrix, we can replace $l$ by the largest subspace of $h$ for which $r$ is invariant, and such a subspace is real. Let $h_0$ be the orthogonal complement of $l$ in $h$ and let $i : l \hookrightarrow h$ be the inclusion map. We will also write $(,)$ for the induced bracket on $l^\ast$. Let $\Omega_{h_0}$ denote the Casimir element of the restriction of $(,)$ to $h_0$.

5.1 Statement of the theorem

Let $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ be a polarization of $\mathfrak{g}$.

**Definition:** A generalized Belavin-Drinfeld triple is a triple $(\Gamma_1, \Gamma_2, \tau)$ where $\Gamma_1, \Gamma_2 \subset \Pi$, and $\tau : \Gamma_1 \rightarrow \Gamma_2$ is a norm-preserving bijection.

In other terms, in a generalized Belavin-Drinfeld triple, we drop the n-nilpotency condition. We will say that a generalized Belavin-Drinfeld triple is $l$-graded if $\tau$ preserves the decomposition of $\mathfrak{g}$ in $l$-weight spaces. If $(\Gamma_1, \Gamma_2, \tau)$ is a generalized Belavin-Drinfeld triple, we will denote by $\Gamma_3$ the largest subset of $\Gamma_1 \cap \Gamma_2$ which is stable under $\tau$, and $\Gamma_1 = \Gamma_1 \setminus \Gamma_3$, $\Gamma_2 = \Gamma_2 \setminus \Gamma_3$. It is clear that $(\Gamma_1, \Gamma_2, \tau)$ is a Belavin-Drinfeld triple. As before, for each choice of Chevalley generators $(e_\alpha, f_\alpha, h_\alpha)_{\alpha \in \Gamma_i}$, the map $\tau$ induces isomorphisms $\mathfrak{g}_{\Gamma_1} \rightarrow \mathfrak{g}_{\Gamma_2}$ and $\tau : \mathfrak{g}_{\Gamma_3} \rightarrow \mathfrak{g}_{\Gamma_3}$.

For $\lambda \in \Gamma^\ast$, consider the map:

$$K(\lambda) : n_+ (\Gamma_1) \rightarrow n_+ (\Gamma_2)$$

$$e_\alpha \mapsto \frac{1}{2} e_\alpha + e^{- (\alpha, \lambda)} \frac{\tau}{1 - e^{- (\alpha, \lambda)} \tau} (e_\alpha).$$

Notice that we have

$$K(\lambda)(e_\alpha) = \frac{1}{2} e_\alpha + \sum_{n>0} e^{- n (\alpha, \lambda)} \tau^n (e_\alpha).$$

This sum is finite for $\alpha \notin (\Gamma_3)$.

**Theorem 4** Let $\mathfrak{g}$ be a simple Lie algebra with nondegenerate invariant bilinear form $(,)$, $l \subset \mathfrak{g}$ a commutative subalgebra containing a regular semisimple element on which $(,)$ is nondegenerate, $h$ the Cartan subalgebra containing $l$ and $h_0$ the orthogonal complement of $l$ in $h$. Then

1. Any dynamical r-matrix is gauge-equivalent to a dynamical r-matrix $\tilde{r}$ such that

$$\tilde{r}(\lambda) - \tilde{r}(\lambda)^{21} \in (l^\perp)^\otimes 2 = \left( \bigoplus_{\alpha \neq 0} (\mathfrak{g}_\alpha \oplus h_0) \right)^\otimes 2. \quad (15)$$


2. Let $(\Gamma_1, \Gamma_2, \tau)$ be an $l$-graded generalized Belavin-Drinfeld triple and let $(e_\alpha, f_\alpha, h_\alpha)_{\Gamma_i}$ be a choice of Chevalley generators. Let $r_{\mathfrak{b}_0, \mathfrak{b}_0} \in \mathfrak{b}_0 \otimes \mathfrak{b}_0$ satisfy the equation

$$(\tau(\alpha) \otimes 1)r_{\mathfrak{b}_0, \mathfrak{b}_0} + (1 \otimes \alpha)r_{\mathfrak{b}_0, \mathfrak{b}_0} = \frac{1}{2}((\alpha + \tau(\alpha)) \otimes 1)\Omega_{\mathfrak{b}_0}.$$  

(16)

Then

$$r(\lambda) = \frac{1}{2}\Omega + r_{\mathfrak{b}_0, \mathfrak{b}_0} + \sum_{\alpha \in (\Gamma_1) \cap \Delta_+} K(\lambda)(e_\alpha) \wedge e_{-\alpha} + \sum_{\alpha \in \Delta_+, \alpha \notin (\Gamma_1)} \frac{1}{2}e_\alpha \wedge e_{-\alpha}$$

is a solution the CDYBE satisfying $[\mathfrak{B}_1]$.

3. Any solution of the CDYBE satisfying $[\mathfrak{B}_1]$ is of the above type for a suitable polarization of $\mathfrak{g}$.

The proof of this theorem will occupy the rest of this section. Our methods are greatly inspired by the paper $[BD]$. Notice that 1. follows from Theorem $[\mathfrak{B}_3]$ but we will describe the gauge transformations explicitly in this case.

**Notations:** Let $\Delta \subset \mathfrak{h}^*$ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$ and set $\Delta_1 = \iota^*(\Delta) \subset \Gamma$. We will denote by $\mathfrak{g}_{\mathfrak{b}_0}^\mathfrak{h}$ the weight subspace associated to $\mathfrak{b}_0^\mathfrak{h} = \iota^*(\alpha) \in \Delta_1$, and we set $\mathfrak{g}_{\mathfrak{b}_0}^\mathfrak{h} = \mathfrak{b}_0$. It is clear that

$$\mathfrak{g}_{\mathfrak{b}_0}^\mathfrak{h} = \bigoplus_{\beta \in \Delta, \iota^*(\beta) = \mathfrak{b}_0^\mathfrak{h}} \mathfrak{g}_{\beta}.$$  

In particular, $(\ , \ )$ is a pairing $\mathfrak{g}_{\mathfrak{b}_0}^\mathfrak{h} \times \mathfrak{g}_{\mathfrak{b}_0}^\mathfrak{h} \rightarrow \mathbb{C}$.

A vector space $V \subset \mathfrak{g}$ will be called $\mathfrak{h}$-graded (resp. $l$-graded) if it is an $\mathfrak{h}$-submodule (resp. $l$-submodule) of $\mathfrak{g}$. Finally, let $\Omega' \in (\mathfrak{l}^*)^2$ denote the Casimir (inverse element) of the restriction of $(\ , \ )$ to $\mathfrak{l}^* = \mathfrak{h}_0 \bigoplus \mathfrak{g}_{\mathfrak{b}_0}^\mathfrak{h}$.

Now let $r : \mathfrak{l}^* \supset \mathfrak{d} \rightarrow (\mathfrak{g} \otimes \mathfrak{g})^1$ be a formal power series satisfying $[\mathfrak{B}_1]$ (with $\epsilon = 1$). By $[\mathfrak{B}_1]$, we can write

$$r(\lambda) = \frac{1}{2}\Omega + r_{1,1}(\lambda) + r_{1,0}(\lambda) + r_{0,1}(\lambda) + (\varphi(\lambda) \otimes 1)\Omega,'$$  

(17)

where $r_{1,1}(\lambda) \in \mathfrak{l} \otimes \mathfrak{l}$, $r_{1,0}(\lambda) \in \mathfrak{l} \otimes \mathfrak{h}_0$, $r_{0,1}(\lambda) \in \mathfrak{h}_0 \otimes \mathfrak{l}$ and where $\varphi(\lambda) \in \text{End} (\mathfrak{h}_0 \bigoplus \mathfrak{g}_{\mathfrak{b}_0}^\mathfrak{h})$ is a sum of maps $\varphi(\lambda) \in \text{End} (\mathfrak{g}_{\mathfrak{b}_0}^\mathfrak{h})$. By the unitarity condition, $r_{1,1}(\lambda) \in \mathfrak{A}^2$, $r_{1,0}(\lambda) = -r_{0,1}^T(\lambda)$ and $\varphi_{-1}(\lambda) = -\varphi_{-2}(\lambda)$.

With these notations, the CDYBE splits into 4 components: the $\mathfrak{l} \otimes \mathfrak{l}$-part, the $\mathfrak{l} \otimes \mathfrak{h}_0 \otimes \mathfrak{l}$-part, the $\mathfrak{l} \otimes \mathfrak{g}_{\mathfrak{b}_0}^\mathfrak{h} \otimes \mathfrak{g}_{-1}^\mathfrak{b}$-part and the $\mathfrak{g}_{\mathfrak{b}_0}^\mathfrak{h} \otimes \mathfrak{g}_{\mathfrak{b}_0}^\mathfrak{h} \otimes \mathfrak{g}_{\mathfrak{b}_0}^\mathfrak{h}$-part where $\mathfrak{b}_0^\mathfrak{h} + \mathfrak{b}_0^\mathfrak{h} + \mathfrak{b}_0^\mathfrak{h} = 0$.

- The $\mathfrak{l} \otimes \mathfrak{l}$-part: let us set $r_{1,1} = \sum_{i,j} C_{i,j}(\lambda) x_i \otimes x_j$. This part of the CDYBE can then be written:

$$\frac{\partial C_{i,j}}{\partial x_i} + \frac{\partial C_{k,i}}{\partial x_j} + \frac{\partial C_{i,j}}{\partial x_k} = 0 \quad \forall i,j,k$$  

(18)

and says that $\sum_{i,j} C_{i,j} dx_i \wedge dx_j$ is a closed 2-form.
• The $\mathfrak{l} \otimes \mathfrak{l} \otimes \mathfrak{h}_0$-part: let us set $r_{l, h_0} = \sum_{i,j} D_{i,j}(\lambda)x_i \otimes y_j$ for some basis $(y_j)$ of $\mathfrak{h}_0$. This part of the CDYBE is

$$\frac{\partial D_{i,j}}{\partial x_k} = \frac{\partial D_{k,j}}{\partial x_i} \quad \forall i, k, j \quad (19)$$

and says that for any $j$, $\sum_i D_{i,j}(\lambda)dx_i$ is a closed 1-form.

Since $r$ is defined on a polydisc, the above forms are exact. Let $f : D \to \mathfrak{h}_0$ be such that $df(\lambda) = \sum_i D_{i,j}(\lambda)dx_i \otimes y_j$ and let $\xi$ be a 1-form on $D$ such that $d\xi = \sum_{i,j} C_{i,j}dx_i \wedge dx_j$. Then $\xi$ defines a function $\xi : D \to \mathfrak{l}$. The gauge transformation which should be applied to $r$ to make it satisfy (15) is easily seen to be the following:

$$r(\lambda) \rightarrow r(\lambda)^g = \frac{1}{2} \Omega + (e^{-ad f(\lambda)}\varphi(\lambda)e^{ad f(\lambda)} \otimes 1)\Omega'$$

where $g(\lambda) = e^{f(\lambda)}e^{-\xi(\lambda)}$.

Thus, we can assume that $r_{l, l} = r_{l, h_0} = 0$, in which case the remaining components of the CDYBE can be written in the following way:

• The $\mathfrak{l} \otimes \mathfrak{g}_{\bar{\tau}} \otimes \mathfrak{g}_{-\bar{\tau}}$ part:

$$d\varphi_{\bar{\tau}} + (\varphi_{\bar{\tau}}^2 - \frac{1}{4})dh_{\bar{\tau}} = 0. \quad (20)$$

In particular, $r_{h_0, h_0} \in \Lambda^2\mathfrak{h}_0$ is constant.

• The $\mathfrak{g}_{\bar{\tau}} \otimes \mathfrak{g}_{\bar{\tau}} \otimes \mathfrak{g}_{-\bar{\tau}}$ part where $\bar{\tau} + \bar{\beta} + \bar{\gamma} = 0$:

$$\Lambda(\varphi_{\bar{\tau}} \otimes \varphi_{\bar{\beta}} \otimes 1 + \varphi_{\bar{\tau}} \otimes 1 \otimes \varphi_{\bar{\gamma}} + 1 \otimes \varphi_{\bar{\beta}} \otimes \varphi_{\bar{\gamma}} + \frac{1}{4}Id) = 0 \quad (21)$$

where $\Lambda : \mathfrak{g}_{\bar{\tau}} \otimes \mathfrak{g}_{\bar{\beta}} \otimes \mathfrak{g}_{\bar{\gamma}} \to \mathbb{C}$, $x \otimes y \otimes z \mapsto ([x, y], z)$.

This set of equations is sufficient by skew-symmetry of the CDYBE.

5.2 The Cayley transform

Let us set $A_{\pm} = \text{Im}(\varphi(\lambda) \pm \frac{1}{2})$, $I_{\pm} = \text{Ker}(\varphi(\lambda) \pm \frac{1}{2})$. Notice that, by (20), $A_{\pm}$ and $I_{\pm}$ are indeed independent of $\lambda$. Furthermore, $A_{\pm}$, $I_{\pm}$ are $\mathfrak{g}$-graded by the weight-zero condition, $I_{\pm} \subset A_{\pm}$ and $A_{\pm} = I_{\pm}$ by the unitarity condition. Notice also that $A_{+} + A_{-} \otimes \mathfrak{l} = \mathfrak{g}$. Now consider

$$\psi(\lambda) = \frac{\varphi - \frac{1}{2}}{\varphi + \frac{1}{2}} : A_{+}/I_{+} \to A_{-}/I_{-}.$$  

Extend $\psi(\lambda)$ to $\psi(\lambda) : \mathfrak{l} \oplus A_{+}/I_{+} \to \mathfrak{l} \oplus A_{-}/I_{-}$ by setting $\psi_{l} = \text{Id}$. It is clear that $\psi$ is a well-defined linear isomorphism. The following proposition is crucial:
Proposition 2 The maps $\varphi$ satisfy (20, 21) if and only if the following hold:

(i) $A_+ \oplus I$ is a subalgebra of $g$ and $I_+ \oplus I$ is an ideal of $A_+ \oplus I$.

(ii) there exists a (constant) map $\psi_0 : I \oplus A_+/I_+ \to I \oplus A_-/I_-$ such that

$$\psi(\lambda)_{g^*} = e^{-(\alpha, \lambda)}\psi_0(g^*).$$

(iii) The map $\psi_0$ is a Lie algebra map:

$$[\psi_0(x), \psi_0(y)] = \psi_0[x, y].$$

(22)

Proof: Assume that $\varphi$ satisfies (20, 21) and let $a \in g^\sigma, b \in g^\sigma, c \in g^\tau$ with $\overline{\alpha} + \overline{\beta} + \overline{\gamma} = 0$. From (21), we have

$$([\varphi_{\overline{\alpha}} + \frac{1}{2}]a, (\varphi_{\overline{\beta}} + \frac{1}{2})b, c) + ([a, (\varphi_{\overline{\beta}} + \frac{1}{2})b], (\varphi_{\overline{\gamma}} - \frac{1}{2})c)
$$

$$+ ([\varphi_{\overline{\gamma}} - \frac{1}{2}]a, b, (\varphi_{\overline{\gamma}} - \frac{1}{2})c) = 0.$$

Since $\varphi_{\overline{\alpha}} = -\varphi_{\overline{\alpha}}$, and $(, , )$ is a nondegenerate pairing $g^\sigma \otimes g^\sigma \to \mathbb{C}$, this implies that $A_+ \oplus I$ is a Lie subalgebra of $g$. Note that the term in I is necessary here since $[g^\sigma, g^\sigma, g^\sigma] \not\subseteq g^\sigma = h_0$, but is not consequential as $A_+$ is 1-graded. The second claim of (i) follows from the relation

$$([\varphi_{\overline{\gamma}} - \frac{1}{2}]a, (\varphi_{\overline{\beta}} + \frac{1}{2})b, c) + ([a, (\varphi_{\overline{\beta}} + \frac{1}{2})b], (\varphi_{\overline{\gamma}} + \frac{1}{2})c)
$$

$$+ ([\varphi_{\overline{\gamma}} - \frac{1}{2}]a, b, (\varphi_{\overline{\gamma}} + \frac{1}{2})c) = 0.$$

The proof is the same for $A_-$ and $I_-$. The equivalence of (ii) and (20) follows from the equality

$$d\psi|_{\overline{\sigma}} = \frac{d\varphi_{\alpha}(\varphi_{\overline{\alpha}} + \frac{1}{2}) - (\varphi_{\alpha} - \frac{1}{2})d\varphi_{\overline{\alpha}}}{(\varphi_{\overline{\alpha}} + \frac{1}{2})^2}
$$

$$= \frac{(\varphi_{\overline{\alpha}} - \frac{1}{2})}{(\varphi_{\alpha} + \frac{1}{2})^2}d\varphi_{\overline{\alpha}}
$$

$$= -(\overline{\alpha}, \lambda)\psi_0|_{\overline{\sigma}}.$$
Applying \((\varphi - \frac{1}{2})\) and dropping the indices, we have
\[
(\varphi - \frac{1}{2})\left([(\varphi + \frac{1}{2})x, y] + [x, (\varphi + \frac{1}{2})y] - [x, y]\right) = [(\varphi - \frac{1}{2})x, (\varphi - \frac{1}{2})y].
\]
Thus,
\[
[(\varphi + \frac{1}{2})x, (\varphi + \frac{1}{2})y] - (\varphi + \frac{1}{2})\left([(\varphi - \frac{1}{2})x, y] + [x, (\varphi + \frac{1}{2})y]\right) = 0.
\]
which is equivalent to \((21)\).

\[\square\]

We will call the triple \((A_+, A_-, \psi_0)\) the Cayley transform of \(\varphi\). We are now reduced to the classification of all triples satisfying (i-iii) and which arise as a Cayley transform (Cayley triples).

### 5.3 Classification of Cayley triples

Let \((A_+, A_-, \psi_0)\) be a Cayley triple. If \(g = n_+ \oplus h \oplus n_-\) is a polarization of \(g\) and \(\Gamma \subset \Pi(n_+)\) we will denote by \(q^+_\Gamma\) (resp. \(q^-\Gamma\)) the subalgebra generated by \(n_+\) and \(g_{-\alpha}\), \(\alpha \in \Gamma\) (resp. generated by \(n_-\) and \(g_{\alpha}\), \(\alpha \in \Gamma\)). We denote by \(p^\pm\Gamma = h + q^\pm\Gamma\) the parabolic subalgebras associated to \(\Gamma\).

**Proposition 3** There exists a polarization \(g = n^1_+ \oplus h \oplus n^1_-\), two subsets \(\Gamma_+, \Gamma_- \subset \Pi(n^1_+)\) and two vector spaces \(V_+, V_- \subset h\) with \(V^+_\Gamma \subset V_+\) such that
\[
I \oplus A_+ = q^+_\Gamma \oplus V_+, \quad I \oplus A_- = q^-\Gamma \oplus V_-
\]

**Proof:** Notice that \((I \oplus A_+)^\perp = I_+ \subset I \oplus A_+\). It is known, (c.f [Bou, chap.VIII,§10, Thm. 1] or [BD]), that this implies that \(I \oplus A_+ = q^+_\Gamma \oplus V_+\) for some polarization \(g = n^+_1 \oplus h^\prime \oplus n^1_-\). Similarly, \(I \oplus A_- = q^-\Gamma \oplus V_-\) for some polarization \(g = n'^+_1 \oplus h^\prime \oplus n'^-_1\). Moreover, \(I\) acts semisimply on \(A_\pm\) so \(I \subset h^\prime\), \(I \subset h^\prime\). But \(I\) contains a regular element, thus \(I = h^\prime = h''\). Proposition 3 is now an easy consequence of the following lemma:

**Lemma 5** Let \(g\) be a simple Lie algebra and \(h\) a Cartan subalgebra. Let \(a_1\) and \(a_2\) be two parabolic subalgebras containing \(h\) such that \(a_1 + a_2 = g\). Then there exists a polarization \(g = n_+ \oplus h \oplus n_-\) and \(\Gamma_+, \Gamma_- \subset \Pi\) such that \(a_1 = p^+_\Gamma\) and \(a_2 = p^-\Gamma\).

**Proof:** Let \(n_+ \oplus h \oplus n_-\) be a polarization of \(g\) such that \(b_+ \subset a_1\) and for which \(\dim(n_+ \cap a_2)\) is minimal. We claim that \(b_- \subset a_2\). Suppose on the contrary that there exists a simple root \(\alpha \in \Pi\) such that \(g_{-\alpha} \not\subset a_2\). Then \(g_{-\alpha} \subset a_1\) since \(a_1 + a_2 = g\) and \(g_{\alpha} \subset a_2\) since \(a_2\) is parabolic. But then \(s_{\alpha}n_+ \oplus h \oplus s_{\alpha}n_-\) is a polarization of \(g\) for which \(s_{\alpha}b_\pm \subset a_1\) and \(\dim(s_{\alpha}n_+ \cap a_2) < \dim(n_+ \cap a_2)\). Contradiction.

\[\square\]
In particular, \( A_\pm, I_\pm \) are all \( \mathfrak{h} \)-graded and
\[
I_+ = (q_{I+}^\perp \oplus V_+)^\perp = \bigoplus_{\alpha \in \Delta_+ \setminus \{I_+\}} \mathfrak{g}_\alpha \oplus (V_+^\perp \cap \mathfrak{h}_0),
\]
\[
I_- = (q_{I-}^\perp \oplus V_-)^\perp = \bigoplus_{\alpha \in \Delta_- \setminus \{I_-\}} \mathfrak{g}_\alpha \oplus (V_-^\perp \cap \mathfrak{h}_0).
\]

Thus \( A_+/I_+ = \mathfrak{g}_{\mathfrak{r}_+} \oplus V_1 \) and \( A_-/I_- = \mathfrak{g}_{\mathfrak{r}_-} \oplus V_2 \) for some suitable \( V_1, V_2 \subset \mathfrak{h}_0 \).

Let \( L_{\pm,\frac{1}{2}}(\lambda) \) be the generalized eigenspace of \( \varphi(\lambda) \) associated to \( \pm \frac{1}{2} \). Since \( \varphi \) is a solution of an ordinary differential equation with stationary points at \( \frac{1}{2}, -\frac{1}{2} \), \( L_{\pm,\frac{1}{2}}(\lambda) \) is independent of \( \lambda \) and we will simply denote it by \( L_{\pm,\frac{1}{2}} \). Similarly, let \( L' \) be the sum of all other generalized eigenspaces so that \( \mathfrak{g} = l \oplus L_{\frac{1}{2}} \oplus L' \oplus L_{-\frac{1}{2}} \).

**Proposition 4** There exists a polarization \( \mathfrak{g} = \mathfrak{p}_+ \oplus \mathfrak{h} \oplus \mathfrak{p}_- \) and a subset \( \Gamma_3 \subset \Pi(\mathfrak{p}_+) \) such that \( L_{\pm,\frac{1}{2}} \subset \mathfrak{p}_\pm \), \( L' \subset \mathfrak{g}_{\mathfrak{r}_+} + \mathfrak{h} \) and \( \varphi(\mathfrak{p}_+) \subset \mathfrak{p}_+ \).

**Proof:** We will construct a polarization satisfying the above conditions in several steps.

**Lemma 6** We have:

(i) \( l \oplus L_{\pm,\frac{1}{2}} \) is an \( \mathfrak{h} \)-graded solvable subalgebra,

(ii) \( l \oplus L' \) is an \( \mathfrak{h} \)-graded subalgebra,

(iii) we have \( [L_{\pm,\frac{1}{2}}, L'] \subset l \oplus L_{\pm,\frac{1}{2}} \).

**Proof:** this follows from the proofs of Lemma 12.3 and Theorem 12.6 in [BL].

Notice that \( L_{\pm,\frac{1}{2}} \not\subset \mathfrak{b}_\pm \) in general. We first construct a polarization \( \mathfrak{g} = \mathfrak{n}_2^2 \oplus \mathfrak{h} \oplus \mathfrak{n}_2^2 \) such that \( L_{\pm,\frac{1}{2}} \subset \mathfrak{b}_\pm^2 \). We have \( I_\pm \subset L_{\pm,\frac{1}{2}} \). Notice that \( L_+ \cap \mathfrak{n}_2^1 \subset \mathfrak{g}_{\mathfrak{r}_+} \cap \mathfrak{g}_{\mathfrak{r}_-} = \mathfrak{g}_{\mathfrak{r}_+} \cap \mathfrak{g}_{\mathfrak{r}_-} \) since \( \mathfrak{n}_2^1 \subset (\mathfrak{g}_{\mathfrak{r}_+} \cap L_-) \) and \( L_+ \) is solvable. Similarly, \( L_- \cap \mathfrak{n}_2^1 \subset \mathfrak{g}_{\mathfrak{r}_-} \cap \mathfrak{g}_{\mathfrak{r}_-} \). Moreover, by Lemma 3, \( l \oplus (L_+ \cap \mathfrak{g}_{\mathfrak{r}_+} \cap \mathfrak{g}_{\mathfrak{r}_-}) \) and \( l \oplus (L_- \cap \mathfrak{g}_{\mathfrak{r}_-} \cap \mathfrak{g}_{\mathfrak{r}_-}) \) are disjoint, solvable, \( \mathfrak{h} \)-graded subalgebras. By Lemma 3 it follows that there exists an element \( s \) of the group \( W_{\mathfrak{r}_+} \cap \mathfrak{g}_{\mathfrak{r}_-} \) such that
\[
l \oplus (L_{\pm,\frac{1}{2}} \cap \mathfrak{g}_{\mathfrak{r}_+} \cap \mathfrak{g}_{\mathfrak{r}_-}) \subset sb_{\pm,\frac{1}{2}}^1.
\]

Notice that \( s \) permutes elements of \( \Delta^+ \setminus \{\Gamma_+ \cap \Gamma_+\} \), leaving it globally unchanged. Thus, \( l \oplus L_{\pm,\frac{1}{2}} \subset \mathfrak{b}_{\pm,\frac{1}{2}}^1 \). Set \( n_{\pm,\frac{1}{2}}^2 = sn_{\pm,\frac{1}{2}}^1 \).

Now we construct a polarization of \( \mathfrak{g} \) satisfying the other conditions of proposition 3. Recall that \( l \oplus L \subset \mathfrak{g}_{\mathfrak{r}_+} \cap \mathfrak{g}_{\mathfrak{r}_-} \cap (V_1 \cap V_2) \). Thus
\[
(L' \cap n_{\pm,\frac{1}{2}}^2) \oplus (L_{\pm,\frac{1}{2}} \cap n_{\pm,\frac{1}{2}}^2 (\Gamma_+ \cap \Gamma_-)) = n_{\pm,\frac{1}{2}}^2 (\Gamma_+ \cap \Gamma_-).
\]

Since \( (L', L_{\pm,\frac{1}{2}}) \subset l \oplus L_{\pm,\frac{1}{2}} \) by Lemma 3(iii), \( L_{\pm,\frac{1}{2}} \cap n_{\pm,\frac{1}{2}}^2 (\Gamma_+ \cap \Gamma_-) \) is an ideal of \( n_{\pm,\frac{1}{2}}^2 (\Gamma_+ \cap \Gamma_-) \). But \( L' \cap n_{\pm,\frac{1}{2}}^2 \) is a subalgebra. It is easy to see that this implies that \( L' \cap n_{\pm,\frac{1}{2}}^2 \) is generated by a set of simple root subspaces of \( n_{\pm,\frac{1}{2}}^2 (\Gamma_+ \cap \Gamma_-) \), i.e
\[ L' \cap n_3^2 = n_3^2(\Gamma) \text{ for some } \Gamma \subset \Pi(n_3^2). \text{ Moreover, the restriction of } (\ , \ ) \text{ to } L' \text{ is nondegenerate, hence } L' \cap n_3^2 = n_3^2(-\Gamma). \text{ Thus} \]
\[ I \oplus \mathfrak{g}_I \subset I \oplus L' \subset I \oplus \mathfrak{g}_I + (V_1 \cap V_2). \]

Since \( \varphi(\lambda) + \frac{1}{2} \) is invertible in \( L' \), \( \psi(\lambda) \) is a well-defined operator \( L' \to L' \), satisfying (22), and \( \psi(\lambda)(\mathfrak{h}_0 \cap L') \subset \mathfrak{h}_0 \cap L' \). Now, \( I \) contains a regular element. Thus there exists a polarization of \( \mathfrak{g} \) compatible with the \( I \)-weight decomposition. This induces a polarization of \( \mathfrak{g}_I \), compatible with the \( I \)-weight decomposition of \( \mathfrak{g}_I \). Hence, there exists \( s' \in W_I \subset W \) such that \( \psi(0|\mathfrak{h}_I) \) is compatible with the polarization \( s'\mathfrak{n}_3^2 + \mathfrak{h} + s'\mathfrak{n}_3^2 \). Since \( s' \) leaves \( \Delta_+ \setminus \{ \Gamma \} \) globally unchanged, the polarization \( \mathfrak{g} = \mathfrak{p}_+ \oplus \mathfrak{h} \oplus \mathfrak{p}_- \) with \( \mathfrak{p}_\pm = s'\mathfrak{n}_3^2 \) and \( \Gamma_3 = s'\Gamma \) satisfies the requirements of proposition 4.

\[ \square \]

To sum up, we have shown that there exists a polarization \( \mathfrak{g} = \mathfrak{p}_+ \oplus \mathfrak{h} \oplus \mathfrak{p}_- \), compatible with \( \varphi \), subsets \( \Gamma_1 = s's\Gamma_1 \), \( \Gamma_2 = s's\Gamma_2 \), and \( \Gamma_3 \subset \Pi(\mathfrak{p}_+) \) such that \((A_+/I_+) \cap \mathfrak{n}_+ = \mathfrak{p}_+(\Gamma_1), \ A_- \cap \mathfrak{n}_+ = \mathfrak{p}_+(\Gamma_2) \text{ and } L' \cap \mathfrak{n}_+ = \mathfrak{p}_+(\Gamma_3)\).

The map \( \psi_0 \) now restricts to a Lie algebra isomorphism \( \mathfrak{p}_+(\Gamma_1) \to \mathfrak{p}_+(\Gamma_2) \). This isomorphism maps weight spaces to weight spaces as \( \psi_0 \) preserves \( \mathfrak{h}_0 \) and \( \varphi \) is \( I \)-invariant. Define \( \tau : \Gamma_1 \to \Gamma_2 \) by \( \psi_0(\mathfrak{g}_\alpha) = \mathfrak{g}_\tau(\alpha) \). It is a norm-preserving bijection. Thus \( (\Gamma_1, \Gamma_2, \Gamma_3) \) is a generalized Belavin-Drinfeld triple. It is clear that \( \Gamma_3 \) is the largest subset of \( \Gamma_1 \cap \Gamma_2 \) stable under \( \tau \), and that \( \psi_0 : \mathfrak{p}_+(\Gamma_3) \to \mathfrak{p}_+(\Gamma_3) \) is a Lie algebra isomorphism. Finally, it is easy to see that the map \( \varphi \) is obtained from this data by formulas

\[
\varphi(\lambda)(e_\alpha) = \begin{cases} 
\frac{1}{2}e_\alpha & (\alpha \not\in \langle \Gamma_1 \rangle) \\
\psi_0(e_\alpha) & (\alpha \in \langle \Gamma_1 \rangle)
\end{cases}
\]

Conversely, it is clear how to construct from a generalized Belavin-Drinfeld triple \( (\Gamma_1, \Gamma_2, \tau) \) the subalgebras \( \mathfrak{n}_+(\Gamma_1), \ \mathfrak{n}_+(\Gamma_2), \ \mathfrak{n}_+(\Gamma_3) \) and, for each choice of Chevalley generators, a Lie algebra isomorphism \( \psi_0 : \mathfrak{n}_+(\Gamma_1) \to \mathfrak{n}_+(\Gamma_2) \), and the map \( \varphi(\lambda) \). Condition (16) on the \( \mathfrak{h}_0 \otimes \mathfrak{h}_0 \)-part comes from (22)-see [BD].

6 Examples

6.1 Constant \( \mathfrak{r} \)-matrices

Our results imply the following:

Corollary 1 A dynamical \( \mathfrak{r} \)-matrix associated to a generalized Belavin-Drinfeld triple \( (\Gamma_1, \Gamma_2, \tau) \) is gauge equivalent to a constant \( \mathfrak{r} \)-matrix if and only if \( \Gamma_3 = \emptyset \).

6.2 \( \mathfrak{h} \)-invariant dynamical \( \mathfrak{r} \)-matrices

When \( I = \mathfrak{h} \), our classification coincides with that given in [SV]: the only \( \mathfrak{h} \)-graded generalized Belavin-Drinfeld triple is of the form \( (\Gamma, \Gamma, \tau = Id) \). The
dynamical r-matrices obtained are then (up to gauge transformations and choice of Chevalley generators):

\[
r(\lambda) = \frac{\Omega}{2} + \sum_{\alpha \in \Delta^+, \alpha \notin (\ell)} \frac{1}{2}e_\alpha \wedge e_{-\alpha} + \sum_{\alpha \in \ell \cap \Delta^+} \frac{1}{2} \coth\left(\frac{1}{2}(\alpha, \lambda)\right)e_\alpha \wedge e_{-\alpha}.
\]

### 6.3 Example for \( \mathfrak{sl}_3 \) and \( \mathfrak{sl}_n \)

The first nontrivial example is for \( \mathfrak{g} = \mathfrak{sl}_3 \): fix a polarization \( \mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\gamma \in \Delta} \mathfrak{g}_\gamma \) where \( \Delta^+ = \{\alpha, \beta, \alpha + \beta\} \) and set \( \ell = \mathbb{C}h_\beta \). Consider the generalized Belavin-Drinfeld triple with \( \Gamma_1 = \Gamma_2 = \{\alpha, \beta\} \) and \( \tau : \alpha \mapsto \beta, \beta \mapsto \alpha \). In this case, we can choose the map \( \psi_0 \) to be the following

\[
e_\alpha \mapsto e_\beta, \quad h_\alpha \mapsto h_\beta, \quad e_{-\alpha} \mapsto e_{-\beta}
\]

\[
e_\beta \mapsto e_\alpha, \quad h_\beta \mapsto h_\alpha, \quad e_{-\beta} \mapsto e_{-\alpha}
\]

\[
e_{\alpha + \beta} \mapsto -e_{\alpha + \beta}, \quad e_{-\alpha - \beta} \mapsto -e_{-\alpha - \beta}.
\]

The corresponding dynamical r-matrix is given by:

\[
r(\lambda) = \frac{\Omega}{2} + r_{b_0, b_0} + \frac{1}{2} \coth(\alpha, \lambda)e_\alpha \wedge e_{-\alpha} + \frac{1}{2} \coth(\beta, \lambda)e_\beta \wedge e_{-\beta}
\]

\[
+ \frac{1}{2} \thetah(\alpha + \beta, \lambda)e_{\alpha + \beta} \wedge e_{-\alpha - \beta} + \frac{1}{2 \sinh(\alpha, \lambda)} e_\beta \wedge e_{-\alpha} \quad (23)
\]

\[
+ \frac{1}{2 \sinh((\alpha, \lambda))} e_\alpha \wedge e_{-\beta}.
\]

This dynamical r-matrix is gauge-equivalent to the dynamical r-matrix

\[
\tilde{r}(\lambda) = \frac{\Omega}{2} + r_{b_0, b_0} + r_{1, b_0} - r_{1, b_0}^{21} + \frac{1}{2} \coth(\alpha, \lambda)e_\alpha \wedge e_{-\alpha} + \frac{1}{2} \coth(\beta, \lambda)e_\beta \wedge e_{-\beta}
\]

\[
+ \frac{1}{2} \thetah(\alpha + \beta, \lambda)e_{\alpha + \beta} \wedge e_{-\alpha - \beta} + \frac{e(\alpha, \lambda)}{2 \sinh(\alpha, \lambda)} e_\beta \wedge e_{-\alpha}
\]

\[
+ \frac{e(-\alpha, \lambda)}{2 \sinh(\alpha, \lambda)} e_\alpha \wedge e_{-\beta}. \quad (24)
\]

when

\[
(\alpha \otimes 1 + 1 \otimes \tau(\alpha))(r_{b_0, b_0} + r_{1, b_0} - r_{1, b_0}^{21}) = \frac{1}{2}(\alpha + \tau(\alpha))\theta h_\beta.
\]

In particular, \( \tilde{r}(\lambda) \) interpolates the constant r-matrix obtained from the Belavin-Drinfeld triple \( (\Gamma_1 = \alpha, \Gamma_2 = \beta, \tau : \alpha \mapsto \beta) \) at \((\alpha, \lambda) \to \infty\) and the r-matrix obtained from \( (\Gamma_1 = \beta, \Gamma_2 = \alpha, \tau : \beta \mapsto \alpha) \) at \((\alpha, \lambda) \to -\infty\).

**Remark:** The generalization of this example to \( \mathfrak{g} = \mathfrak{sl}_{2n+1} \) is the following. Fix a polarization and let \( \ell = \mathbb{C}h_\beta \). Denote by \( \Delta \) the root system and by \( \Pi = \{\alpha_1, \ldots, \alpha_{2n}\} \) the set of positive simple roots. Let \( i : \alpha_k \mapsto \alpha_{2n+1-k} \) be the involution of the Dynkin diagram. The dynamical r-matrix obtained from the generalized Belavin-Drinfeld triple \( (\Gamma_1 = \Gamma_2 = \Pi, \tau = i) \) interpolates the constant r-matrices obtained from the Belavin-Drinfeld triples \( (\Gamma_1 = (\alpha_1, \ldots, \alpha_n), \Gamma_2 = (\alpha_{n+1}, \ldots, \alpha_{2n}), \tau = i) \) and \( (\Gamma_1 = (\alpha_{n+1}, \ldots, \alpha_{2n}), \Gamma_2 = (\alpha_1, \ldots, \alpha_n), \tau = i^{-1}) \).
6.4 Permutation dynamical r-matrices

Consider \( g = \mathfrak{sl}_{2n} \), and let \( \Pi = (\alpha_1, \ldots, \alpha_{2n-1}) \) denote a system of simple roots. For any \( \sigma \in S_n \), we can construct a generalized Belavin-Drinfeld triple by setting \( \Gamma_1 = \Gamma_2 = (\alpha_1, \alpha_3, \ldots, \alpha_{2n-1}) \) and \( \tau : \alpha_{2k-1} \mapsto \alpha_{2\sigma(k)-1} \).

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