GERBES, CLIFFORD MODULES AND THE INDEX THEOREM

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ABSTRACT. The use of bundle gerbes and bundle gerbe modules is considered as a replacement for the usual theory of Clifford modules on manifolds that fail to be spin. It is shown that both sides of the Atiyah-Singer index formula for coupled Dirac operators can be given natural interpretations using this language and that the resulting formula is still an identity.

1. Introduction

If $M$ is an even-dimensional, oriented, spin-manifold with spin-bundles $S^\pm$ and $W$ is a vector bundle with unitary connection $A$ then the coupled Dirac operator is a natural first-order differential operator

$$D^+_A : C^\infty(M, E^+) \to C^\infty(M, E^-).$$

where

$$E^+ := S^+ \otimes W, \quad E^- := S^- \otimes W.$$ 

If $M$ is compact, the Atiyah-Singer index theorem \cite{2,3} gives a topological formula for the index of $D^+_A$,

$$\text{ind}(D^+_A) := \dim \ker(D^+_A) - \dim \coker(D^+_A) = \langle \hat{A}(M) \operatorname{ch}(W), [M] \rangle.$$ 

If $M$ is not a spin-manifold, ‘generalized Dirac operators’ can be introduced, even though the Dirac operator itself is not well-defined. This can be done by observing that $E = E^+ \oplus E^-$ is a Clifford module (Clifford multiplication is extended to act as the identity on $W$) so that a compatible connection on $E$ leads to a Dirac operator as before. If $M$ is not spin, there is an index theorem for Dirac operators defined on (hermitian) Clifford modules $E$ which can be expressed in the form \cite{4}

$$\text{ind}(D^+_A) = \langle \hat{A}(M) \operatorname{ch}(E/S), [M] \rangle$$

where the notation will be explained below. The definition is such that if $M$ is spin and (1.2) holds, then the relative Chern character $\operatorname{ch}(E/S)$ reduces to $\operatorname{ch}(W)$, the Chern character of $W$.

In this note we explore an alternative approach to this index theorem, using bundle gerbes and bundle gerbe modules \cite{7,5}. The point is that there is a so-called spin bundle gerbe $\Gamma$, defined on any manifold, which is trivial if $M$ is spin, and modules for $\Gamma$ are defined for any representation of the spin group. In particular, the spin $\Gamma^1$-module is always defined, and this reduces, in a suitable sense, to the spin-bundle, if $M$ happens to be spin. We construct a ‘twisted Dirac operator’ $\mathcal{D}$ between the two spin $\Gamma^1$-modules, and show that when this is coupled to any suitable auxiliary $\Gamma^{-1}$-module $W$ then the resulting differential operator reduces to a Dirac-type operator between Clifford modules.

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Furthermore, we show how to define the Chern character \( \text{ch}(W) \) of the \( \Gamma^{-1} \)-module \( W \) and that the index formula (1.3) holds with this definition. Once all the definitions are in place, the proof of the index theorem is mainly a question of matching up the formalism of Clifford modules with that of \( \Gamma^d \)-modules.

Some of our constructions are relevant, and familiar, in relation to anomalies in quantum field theory. The failure of \( M \) to be spin can be regarded as an anomaly in the global definition of spinor fields. As is typically done in anomaly theory our solution is to introduce some additional fields which also have an anomaly in their global definition and choose these so that the two anomalies cancel in the coupled theory.

We note two recent preprints. The first [6] discusses the interesting situation of the families index theorem for projective families of pseudo-differential operators. In that case the index takes value in the twisted \( K \)-theory of the parameter space. Although this is related in spirit to this note, the situation considered there is rather different. The second [1] presents a general Atiyah-Singer index theorem for gerbes of which our result is an example.

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2. Background on lifting bundle gerbes

In this section we recall the definition of the lifting bundle gerbe and the corresponding bundle gerbe modules. The references for this material are [7, 5].

2.1. Generalities on central extensions of Lie groups. Let

(2.1) \[ 0 \to \mathbb{Z}_k \to \hat{G} \xrightarrow{\pi} G \to 1 \]

be a central extension of finite-dimensional Lie groups, where \( \mathbb{Z}_k \) is the cyclic subgroup \( \mathbb{Z}/k\mathbb{Z} \) of the circle \( U(1) \). The example of ultimate interest in this note is

(2.2) \[ 0 \to \mathbb{Z}_2 \to \text{Spin}(n) \xrightarrow{\pi} \text{SO}(n) \to 1, \]

where \( \text{Spin}(n) \) is the universal cover of \( \text{SO}(n) \), but it is clearer to work with the general case to begin with.

We will be interested in spaces on which \( \hat{G} \) and \( G \) act. Notice that if \( G \) acts on a space then so also does \( \hat{G} \) via the projection. We shall often assume this action in the discussion below without further comment. In such a case the center \( \mathbb{Z}_k \) of course acts trivially. To begin our discussion let \( X \) and \( \hat{X} \) be right principal sets for \( G \) and \( \hat{G} \) respectively. As sets, \( X \) and \( \hat{X} \) are just \( G \) and \( \hat{G} \): in particular we can choose a map \( \pi: \hat{X} \to X \) which is compatible with the respective actions of \( \hat{G} \) on \( \hat{X} \) and of \( G \) on \( X \). Notice that \( \pi: \hat{G} \to G \) induces an action of \( \hat{G} \) on \( X \).

Let \( \rho: \hat{G} \to GL(V) \) denote a (finite-dimensional) representation of \( \hat{G} \) and let \( V_X \) denote the bundle \( X \times V \). With the \( \hat{G} \) action \( \hat{g}(x, v) = (x\hat{g}^{-1}, \rho(\hat{g})v) \) this is an equivariant \( \hat{G} \) bundle. As the isotropy subgroup of the \( \hat{G} \) action at any \( x \in X \) is \( \mathbb{Z}_k \) we must have that \( \mathbb{Z}_k \) acts on the fibres of \( V_X \to X \) and it is clear that it does so by the restriction of the representation \( \rho \).

The product \( X^2 := X \times X \) has a canonical map

\[ \tau: X^2 \to G, \quad x_1 \tau(x_1, x_2) = x_2, \]

so that \( \tau(x_1, x_2) \) is the unique element of \( G \) that translates \( x_1 \) to \( x_2 \). Let \( \Gamma = \tau^*\hat{G} \to X^2 \) be the pull-back of the \( \mathbb{Z}_k \) bundle \( \hat{G} \to G \). By definition

\[ \Gamma_{(x_1, x_2)} = \{ \hat{g} \in \hat{G} : \pi(\hat{g}) = \tau(x_1, x_2) \} \]
is the set of all lifts to $\hat{G}$ of $\tau(x_1, x_2)$.

The $\mathbb{Z}_k$-principal bundle $\Gamma$ contains almost the same information as $\hat{X}$: if we denote by $\hat{X}_1$ and $\hat{X}_2$ the pull-backs of $\hat{X} \to X$ by the two projections $X^2 \to X$, then the principal bundle of $\mathbb{Z}_k$-equivariant maps $\hat{X}_1 \to \hat{X}_2$ is canonically isomorphic to $\Gamma$. Indeed, at $(x_1, x_2)$, the fibres of $\hat{X}_1$ and $\hat{X}_2$ are by definition

$$\{\xi : \pi(\xi) = x_1\} \text{ and } \{\xi : \pi(\xi) = x_2\}$$

respectively and it is clear that equivariant maps between these sets correspond precisely to elements of $\Gamma_{x_1,x_2}$.

The $G$-equivariant bundle $V_X \to X$ is a ‘module’ for $\Gamma$ in the following sense. Let $V_1$ and $V_2$ be the pull-backs of $V_X$ by the two projections $X^2 \to X$. Then $\Gamma_{(x_1, x_2)} \subset \hat{G}$ acts through $\rho$ to map the fibre at $(x_1, x_2)$ of $V_1$ to that of $V_2$.

Notice that if $\rho(\mathbb{Z}_k) = \{1\}$ then $\rho$ is really a representation of $G$ and one can divide $V_X$ by $G$ to get back the representation space $V$. In particular, $\text{End}(V_X)$ is always $G$-equivariant so that $\text{End}(V_X)/G = \text{End}(V)$.

Finally we notice that, because $\mathbb{Z}_k$ is finite, there is a natural $\hat{G}$-equivariant flat connection $\nabla_X$ on any of the bundles $V_X$.

2.2. The lifting bundle gerbe and associated modules. Let $M$ be a smooth manifold of dimension $n$. We now globalize the considerations of the previous section so that they correspond to the case when $M$ is a point.

We stay with the central extension $[2.1]$ and replace $X$ by a given principal $G$-bundle $\pi : P \to M$. Correspondingly $X^2$ is replaced by the fibre product $P^{[2]} \to M$. Extending the notation from the previous section, the map $\tau : P^{[2]} \to G$ is defined exactly as before and we put $\Gamma = \tau^*\hat{G}$. This is a $\mathbb{Z}_k$ bundle over $P^{[2]}$ and in the language of $[7]$ is called the lifting bundle gerbe associated to $P$ and the central extension $[2.1]$. Note that $\Gamma$ is a $\mathbb{Z}_k$ bundle gerbe whereas $[7]$ was mostly concerned with $U(1)$ bundle gerbes.

We will be interested in $\hat{G}$ equivariant bundles over $P$. Recall first that a finite-dimensional representation $U$ of $\mathbb{Z}_k$ decomposes into a direct sum of weight spaces $U_d$ defined to be the largest subspaces on which the action of every $z \in \mathbb{Z}_k$ is just multiplication by $z^d$. We say that $\mathbb{Z}_k$ acts on $U_d$ with weight $d$. Notice that if $m$ is any integer and $z \in \mathbb{Z}_k$ then $z^{d+mk} = z^d$ so that the weight $d$ is an element of $\mathbb{Z}_k$.

Let $W \to P$ be a $\hat{G}$ equivariant bundle for the action of $\hat{G}$ on $P$. The isotropy subgroup $\mathbb{Z}_k$ acts on the fibres of $W \to P$. If it acts with weight $d$ then we call $W$ a $\Gamma^d$-module. In the language of $[2]$ a $\Gamma^d$-module is precisely a bundle gerbe module for the bundle gerbe $\Gamma^d = \Gamma \otimes d$.

We note, as an aside, that if $W$ is any $\hat{G}$-equivariant bundle then at every point $p \in P$ we can decompose the fibre of $W$ into weight spaces for the $\mathbb{Z}_k$ action. The action of $\Gamma$ maps fibres to fibres and, because $\mathbb{Z}_k$ is central, preserves these weight spaces. Hence $W$ is a vector bundle direct sum of $\Gamma^d$-modules so we lose nothing by concentrating attention on $\Gamma^d$-modules.

Of particular importance for us are $\Gamma^0$-modules. Then every element of $\mathbb{Z}_k$ acts by the identity on $W$ so that $W$ is $G$-equivariant and hence it descends to a bundle on $M$ after quotienting by $G$.

If $\rho : \hat{G} \to GL(V)$ is a representation then $V_P = P \times V$ defines a $\hat{G}$-equivariant vector bundle, with $\hat{G}$ action $g(p,v) = (pg^{-1}, gv)$ and such that the restriction of $V_P$ to each fibre of $P$ is isomorphic to $V_X \to X$. If the action $\mathbb{Z}_k$ defined by $\rho$ is of weight $d$ then $V_P$ is a $\Gamma^d$-module. Note that $V_P$ is a trivial module if and only if the representation of $\hat{G}$ factors.
through a representation $\rho'$, say, of $G$. Then

$$V_P = \pi^{-1}(V_M), \text{ where } V_M := P \times_G V$$

where $G$ acts by $\rho'$ on $V$ (so that $V_M$ is the vector bundle over $M$ associated by $\rho'$ to $P$).

Define a group $\hat{G}_c$ by

$$\hat{G}_c = \left(U(1) \times \hat{G}\right)/\mathbb{Z}_k$$

where $\mathbb{Z}_k \subset U(1) \times \hat{G}$ is included as the anti-diagonal subgroup $\{(z, z^{-1}) \mid z \in \mathbb{Z}_k\}$. The group $\hat{G}_c$ is a central extension of $G$ by $U(1)$ and there is a short exact sequence

$$0 \to U(1) \to \hat{G}_c \to G \to 0$$

induced by $2.3$. A $\Gamma^d$-module $W$ has a natural $\hat{G}_c$ bundle action defined by $(z, g)w = z^dgw$. If it happens that there exists a principal $\hat{G}_c$-bundle $\hat{P}_c \to M$ covering $P$, then one can pull back any $\Gamma$-module $W$ to $\hat{P}$ and divide by $\hat{G}$ to get the associated vector bundle $W_M := \hat{P}_c \times_{\hat{G}} W$. Notice that we can repeat the construction of $\Gamma$ to define a $U(1)$-bundle $\Gamma_c = \tau^{-1}(\hat{G}_c) \to P[2]$. In [7] it is shown that the lift of $P$ to the $\hat{G}_c$ bundle $\hat{P}_c \to M$ exists exactly when the $U(1)$-bundle gerbe $\hat{G}_c$ is trivial.

Notice that there are thus two ways in which a $\Gamma^d$-module $W$ may be a legitimate vector bundle on $M$. Either $\Gamma_c$ is a trivial bundle gerbe or $W$ is a $\Gamma^0$-module, i.e. $d = 0$ modulo $k$.

In the next few sections we shall make definitions (for example of connections and curvature) for $\Gamma^d$-modules on $M$ that run parallel to the usual theory of vector bundles on $M$. The reader should have no difficulty in verifying that these definitions always reduce to the standard ones in the case that $d = 0$ is trivial or $\Gamma_c$ is trivial.

If $W$ is a $\Gamma$-module then $\text{End}(W)$ is a trivial module. This situation will occur frequently below, and we shall write

$$\mathcal{E}(W) = \text{End}(W)/G.$$

### 2.3. Concrete representation.

We can obtain a Čech description of the above structures by choosing a good cover $\{U_\alpha\}$ of $M$ and local cross-sections $s_\alpha: U_\alpha \to P$. Then the transition functions

$$g_{\alpha\beta}: U_\alpha \cap U_\beta \to G$$

are defined by the conditions $s_\alpha = s_\beta g_{\alpha\beta}$. The problem of constructing a covering principal bundle $\hat{P}$ is the same as that of making a consistent choice of lifts $\hat{g}_{\alpha\beta}$ of the $g_{\alpha\beta}$. Indeed, if we make an arbitrary choice of such lifts, then we obtain a $\mathbb{Z}_k$-valued Čech 2-cocycle $e_{\alpha\beta\gamma} \in H^2(M, \mathbb{Z}_k)$ defined as

$$e_{\alpha\beta\gamma} := \hat{g}_{\alpha\beta} \hat{g}_{\beta\gamma} \hat{g}_{\gamma\alpha}$$

(2.3)

We can also regard $e_{\alpha\beta\gamma}$ as a $U(1)$ 2-cocycle by inclusion of $\mathbb{Z}_k$ in $U(1)$. Using the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{R} \to U(1)$$

we obtain, in the usual way a class $[e] \in H^3(M, \mathbb{Z}) \simeq H^2(M, U(1))$. This class is the obstruction to the existence of $\hat{P}_c$.

Similarly, given a $\Gamma^d$-module $W$, set $W_\alpha = s_\alpha^{-1}(W)$. The lifted transition map $\hat{g}_{\alpha\beta}$ induces a map $\phi_{\alpha\beta}: W_\beta \to W_\alpha$. The $\phi$’s determine a cocycle

$$\phi_{\alpha\beta}\phi_{\beta\gamma}\phi_{\gamma\alpha} = e_{\alpha\beta\gamma}^d$$
If this cocycle is a coboundary then the locally defined bundles $W_\alpha$ patch together to form a genuine bundle on $M$ whose pull-back is $W$. This is the case if $(e_{\alpha\beta\gamma})$ is a coboundary (as a $U(1)$ 2-cocycle) or if $d = 0$.

An alternative description of the local structure of $W$ can be obtained by defining $U(1)$ bundles $L_{\alpha\beta} = g_{\alpha\beta}^{-1}(\hat{G}_c) \to U_\alpha \cap U_\beta$. Then there are isomorphisms $W_\alpha = L_{\alpha\beta}^d \otimes W_\beta$.

### 2.4. Operations on $\Gamma^d$-modules

The following observations are easy but important:

1. There is a bijection between $\Gamma^0$-modules and bundles on $M$. (Induced by pull back to $P$ in one direction, division by $G$ in the other.)
2. The tensor product of a $\Gamma^d_1$-module and a $\Gamma^d_2$-module is a $\Gamma^{d_1+d_2}$-module.
3. The endomorphism bundle of an $\Gamma^d$-module is a $\Gamma^0$-module.
4. The direct sum of two $\Gamma^d$-modules is a $\Gamma^d$-module.

These properties (especially the last one) lead to a definition of twisted $K$-theory, by introducing the $K$-group of the semi-group of $\Gamma$-modules. In this case the ‘twist’ is by $d.\delta[e] \in H^3(M, \mathbb{Z})$ where $[e]$ was defined in (2.3).

### 2.5. The spin-bundle gerbe

The main example in this paper arises by taking the central extension:

$$0 \to \mathbb{Z}_2 \to \text{Spin}(n) \to SO(n) \to 1,$$

and $P$ to be the bundle of oriented orthonormal frames in $TM$. The lifting bundle gerbe $\Gamma$ in this case is called the *spin-bundle gerbe*.

In the spin case the construction of $\hat{G}_c$ is the well-known construction of the group $\text{Spin}_c(n)$. The manifold $M$ is spin when $\Gamma$ is a trivial $\mathbb{Z}_2$ bundle gerbe and spin-c when $\Gamma_c$ is a trivial $U(1)$ bundle gerbe.

If (as we assume throughout) $n$ is even then we have the two half-spin representations $\Sigma^\pm$ of $\text{Spin}(n)$ as well as the full spin representation $\Sigma = \Sigma^+ \oplus \Sigma^-$. The $\Gamma^1$-modules associated to these representations are called the *spin* $\Gamma^1$-modules, and will be our replacement for honest spin-bundles on $M$.

### 3. Connections and curvature for $\Gamma^d$-modules

#### 3.1. Definition of $\Gamma^d$-module connection

In this note we shall generally be identifying connections with the corresponding covariant derivative operators. In particular a connection on $W$ is a linear differential operator

$$\nabla_A : \Omega^0(P, W) \to \Omega^1(P, W)$$

which satisfies the Leibnitz rule $\nabla_A(fs) = df \otimes s + f \otimes \nabla_As$ for all functions $f$ and sections $s$.

**Definition 3.1.** The connection $\nabla_A$ on a $\Gamma$-module is called a $\Gamma$-module connection if it is $\hat{G}_c$-equivariant and restricts to the flat connection $\nabla_X$ on the fibres of $P$.

If $U$ is a sufficiently small open set of $M$ then we can identify the restriction of $P$ to $U$ with $U \times X$ and the restriction of $W$ with $W_X$ (or rather with the pull-back of this bundle by the projection $U \times X \to X$). With these identifications, a $\Gamma$-module connection takes the form

$$\nabla_A = \nabla_X + \nabla_B$$
where $\nabla_B$ is the pull-back of a connection from the base $U$. It is straightforward to use a partition of unity argument to patch such local descriptions to show that $\Gamma$-module connections exist on any $\Gamma$-module. The following proposition gathers some basic information about $\Gamma$-module connections.

**Proposition 3.2.** Let the notation be as above and $W$ be a $\Gamma^d$-module.

1. The set of all $\Gamma^d$-module connections on $W$ is an affine space relative to the space $\Omega^1(M, \mathcal{E}(W))$.
2. The curvature of a $\Gamma^d$-module connection descends to define an element $\mathcal{F}_A \in \Omega^2(M, \mathcal{E}(W))$. If now $\nabla_1$ and $\nabla_2$ are $\Gamma^d$-module connections it follows that $a$ is $\hat{G}$-equivariant. Since the centre acts trivially here, $a$ is actually $G$-equivariant. But from the local description it follows that $i_\xi(a) = 0$ for any vertical vector field $\xi$. Hence we can divide by $G$ to realize $a$ as an element of $\Omega^1(M, \mathcal{E}(W))$.
3. If $W$ and $W'$ are two $\Gamma^d$-modules, and $\nabla_A$ and $\nabla_A'$ are $\Gamma^d$-module connections on $W$ and $W'$, then the curvature of the tensor product connection $\nabla_B$ on $W \otimes W'$ is

$$
\mathcal{F}_B = \mathcal{F}_A \otimes 1 + 1 \otimes \mathcal{F}_A'.
$$

an element of $\Omega^2(M, \mathcal{E}(W \otimes W')) = \Omega^2(M, \mathcal{E}(W) \otimes \mathcal{E}(W'))$.

**Proof.** Let $\nabla_1$ and $\nabla_2$ be two connections on $W$. It follows as usual from the Leibnitz rule that $a = \nabla_2 - \nabla_1$ lies in the vector space $\Omega^1(P, \text{End}(W))$. If now $\nabla_1$ and $\nabla_2$ are $\Gamma^d$-module connections it follows that $a$ is $\hat{G}$-equivariant. Since the centre acts trivially here, $a$ is actually $G$-equivariant. But from the local description it follows that $i_\xi(a) = 0$ for any vertical vector field $\xi$. Hence we can divide by $G$ to realize $a$ as an element of $\Omega^1(M, \mathcal{E}(W))$. It is even easier to see that if $\nabla_A$ is a $\Gamma$-module connection and $a \in \Omega^1(M, \mathcal{E}(W))$, then $\nabla_A + a$ is another $\Gamma$-module connection.

The proof of the statement about the curvature is very similar. The curvature $\mathcal{F}_A$ of $\nabla_A$ is an element of the space $\Omega^2(P, \text{End}(W))$. This element is $G$-equivariant and satisfies $i_\xi(\mathcal{F}_A) = 0$ if $\xi$ is any vertical tangent vector (from the local description and the fact that $\nabla_X$ is flat). Hence it descends to an element of $\Omega^2(M, \mathcal{E}(W))$ as required.

The proof of the last statement is trivial.

\[ \square \]

If $d = 0$ and $W = \pi^{-1}(W_M)$ and $W_M = W/G$ then it is easy to see that there is a bijective correspondence between connections on $W_M$ and $\Gamma^d$-module connections on $W$. Similarly the descended curvature in (ii) above is just the curvature of the connection on $W_M$.

We remark that if a $\Gamma^d$-module $W$ is given any $\hat{G}$ invariant connection $\nabla$ we can find a $\Gamma^d$-module connection by choosing a connection $A$ on $P \to M$. Indeed $A$ is a one-form on $P$ with values in the Lie algebra of $G$ which equals the Lie algebra of $\hat{G}$ which acts on the fibres of $W$. The connection $\nabla - A$ is a $\Gamma^d$-module connection. In particular if $V$ is a representation of $\hat{G}$ and we take the flat connection $d$ on $W = P \times V$ then $d - \rho(A)$ is a $\Gamma^d$-module connection. Here we abuse notation: $\rho$ is a representation of $\hat{G}$ on $V$ and we use this to obtain a representation of the Lie algebra of $\hat{G}$ and hence the Lie algebra of $G$ on $V$.

### 3.2. (Twisted) Chern character.

Let the data be as before, so that $\nabla_A$ is a $\Gamma$-module connection on $W$, with (descended) curvature-form $\mathcal{F}_A$.

**3.2.1. Definition.** The (twisted) Chern character of $\mathcal{F}_A$ is defined to be

$$
\text{ch}(\mathcal{F}_A) = \text{tr} \exp(\mathcal{F}_A/2\pi i).
$$
As in [5] one shows that $\text{ch}(\mathcal{F}_A/2\pi i)$ is closed and that its de Rham cohomology class is independent of the choice of $\Gamma^d$-module connection $A$. Define $\text{ch}(W)$ to be this cohomology class, called the (twisted) Chern character of the $\Gamma^d$-module $W$. It is readily checked that if $W = \pi^{-1}(W_M)$ is a trivial $\Gamma^d$-module then $\text{ch}(W) = \text{ch}(W_M)$.

3.3. (Twisted) Dirac operator. We apply the above theory in our main example, to construct a replacement for the Dirac operator. Recall the notation of §2.5 for the spin-bundle gerbe, and denote by $\mathbb{R}^n_P = \pi^*(T_M)$ is the pull-back to the frame-bundle of the tangent bundle. Then Clifford multiplication is defined as a map between $\Gamma$-modules

$$c : (\mathbb{R}^n_P)^* \otimes \Sigma_P^+ \to \Sigma_P^-.$$ 

The Riemannian metric on $M$ induces a standard connection on the frame-bundle $P$ and this in turn induces $\Gamma$-module connections on all associated bundles. Using these data, we define a differential operator (the twisted Dirac operator)

$$\overline{D}^+ : C^\infty(P, \Sigma^+_P) \to C^\infty(P, \Sigma^-_P)$$ 

by composing covariant differentiation with Clifford multiplication. This operator is Spin($n$)-equivariant but not elliptic because it doesn’t differentiate in the vertical directions.

Now let $W$ be a $\Gamma^{-1}$-module with $\Gamma^{-1}$-module connection $A^1$. Then there is a coupled version of (3.1)

$$\overline{D}_A : C^\infty(P, \Sigma^+_P \otimes W) \to C^\infty(P, \Sigma^-_P \otimes W).$$ 

(Here Clifford multiplication is extended to act trivially in $W$.) Because $\Sigma^\pm_P \otimes W$ is a $\Gamma^0$-module it descends to a bundle $E^\pm$ on $M$ and so also does the Dirac operator to give the twisted Dirac operator:

$$D^+_A : C^\infty(M, E^+) \to C^\infty(M, E^-).$$ 

If $M$ is spin, then both (3.1) and (3.2) descend to $M$ and $E^\pm = \Sigma^\pm_M \otimes W_M$. In general, however, this tensor product decomposition exists only locally on $M$.

We can now state:

**Proposition 3.3** (Twisted Index Theorem). If $M$ is compact, then the Dirac operator defined above satisfies the index formula

$$\text{ind}(D^+_A) = \langle \hat{A}(M) \text{ch}(W), [M] \rangle.$$ 

The proof will be given in the next section, by showing how the present gerbe framework fits with the more standard formalism of Clifford modules.

4. Clifford modules and generalized Dirac operators

In the present section we recall the definition of Clifford modules, and show how they correspond to $\Gamma$-modules (from now on, $\Gamma$ is the spin-bundle gerbe). The index theorem for generalized Dirac operators, in the form stated in [4] then yields Proposition 3.3 at once.

\footnote{Of course, as $d = 2$, we have $1 = -1$.}
4.1. **Clifford algebras and their representations.** If $V$ is a real vector space of even dimension $n$, equipped with a positive-definite inner product, then $C(V)$ will denote the corresponding (complex) Clifford algebra. This is a *superalgebra* in the sense that the underlying vector space is $\mathbb{Z}_2$-graded,

$$C(V) = C_+(V) \oplus C_-(V)$$

and the algebra operations are compatible with this grading (even times even and odd times odd are even, even times odd is odd). If $C(V)$ is identified with the complex exterior algebra of $V$ (but with a different product) then $C_+(V)$ corresponds to the forms of even degree and $C_-(V)$ corresponds to the forms of odd degree. The *chirality operator* will be denoted by $\gamma$, so $\gamma = \pm 1$ on $C_\pm(V)$. Recall that there is an embedding $c : V \subset C(V)$ called *Clifford multiplication*.

A $\mathbb{Z}_2$-graded complex inner product space $E$ is called a hermitian Clifford module if there is an even action of $C(V)$ on $E$ (i.e. the grading is preserved by $C_+(V)$ and altered by $C_-(V)$) and $c(v)$ is skew-adjoint for all $v \in V$. The spin representation $\Sigma = \Sigma^+ \oplus \Sigma^-$ is a hermitian Clifford module which has the property

$$C(V) = \text{End}(\Sigma).$$

Moreover, it is a basic fact that every finite-dimensional hermitian Clifford module $E$ has the form

(4.1) $$E = \Sigma \otimes W$$

where $W$ is some complex vector space carrying a trivial $C(V)$-action. In this situation one has the basic formulae:

$$W = \text{Hom}_{C(V)}(S, E), \quad \text{End}(W) = \text{End}_{C(V)}(E).$$

The subscript $C(V)$ here means ‘commutes with the action of $C(V)$’.

4.2. **Clifford bundles and $\Gamma$-modules.** Let $M$ be an even-dimensional oriented Riemannian manifold. Introduce $C(M)$, the bundle of complex Clifford algebras of $T^*M$ and continue to denote Clifford multiplication by $c : T^*M \to C(M)$.

A complex $\mathbb{Z}_2$-graded hermitian vector bundle $E = E_+ \oplus E_-$ over $M$ is called a Clifford module if

1. the sub-bundles $E_+$ and $E_-$ are orthogonal with respect to the inner product;
2. for each point $x \in M$, $E_x$ is a hermitian Clifford module for $C_x(M)$.

Let $\Gamma$ be the spin-bundle gerbe of §2.5 and recall that $\pi : P \to M$ is the frame-bundle. Let $E_P = \pi^{-1}(E)$ be the pull-back of $E$ to $P$. Each point $p$ of $P$ is an isomorphism of $\mathbb{R}^n$ with the $T_{\pi(p)}M$ which maps the standard inner product to the metric on the tangent space. It follows that $p$ induces an isomorphism $(E_P)_p$ can be made into a $C(\mathbb{R}^n)$ Clifford module. We define

$$W_p = \text{Hom}_{C(\mathbb{R}^n)}(\Sigma, (E_P)_p)$$

and we have

$$(E_P)_p = \Sigma \otimes W_p$$

or globally

$$E_P = \Sigma_p \otimes W.$$
Proposition 4.1. Let $M$ be a Riemannian manifold with $\Gamma$ its Spin bundle gerbe. If $\Sigma_P$ is the spin bundle gerbe module then every Clifford module $E$ on $M$ has the form $E = \Sigma_P \otimes W$ for some $\Gamma^{-1}$-module $W$.

Remark In [4] it was shown that if $M$ is spin then every Clifford module is the tensor product of the spin bundle with an arbitrary bundle.

4.3. Generalized Dirac operators. Given a Clifford module $E$ for $C(M)$, the next step in the definition of a generalized Dirac operator is the introduction of a connection $\nabla_A$ on $E$ compatible with the grading, the metric $h$ and with the Clifford action in the sense that

$$\nabla_A[c(\xi)f] = c(\nabla\xi)e + c(\xi)\nabla_Ae$$

for all $\xi \in \Omega^1(M)$ and $e \in C^\infty(M, E)$. In other words, if $c$ is viewed as a smooth section of $TM \otimes \text{End}(E)$, then $c$ is parallel with respect to the tensor product connection on this bundle.

Given a Clifford module with a compatible connection $\nabla_A$, the associated Dirac operator $D_A$ is defined as the composite

$$C^\infty(M, E) \xrightarrow{\nabla_A} C^\infty(M, T^*M \otimes E) \xrightarrow{c} C^\infty(M, E).$$

Since Clifford multiplication by $T^*M$ is in the odd part of $C(M)$, and the connection is even, the composite $D_A$ is an odd operator. It can therefore be broken into two pieces, traditionally written

$$D_A^\pm : C^\infty(M, E^\pm) \to C^\infty(M, E^{\mp})$$

with the property that $D_A^+$ and $D_A^-$ are formal adjoints of each other with respect to the standard $L^2$ inner product on $C^\infty(M, E)$.

In order to write a cohomological formula for the index of $D_A^+$ we need to know a little about the curvature $F_A$ of our connection $\nabla_A$. Now $F_A$ is a 2-form with values in $\text{End}(E)$ and we can try to decompose $F_A$ according to the decomposition

$$\text{End}(E) = C(M) \otimes \text{End}_{C(M)}(E).$$

Because of the compatibility between $\nabla_A$ and $c$, we have

$$[F_A, c(\xi)] = c(R(\xi))$$

for any tangent vector $\xi$. It is easy to show that there is an element $c(R) \in C(M)$ such that

$$[c(R), c(\xi)] = c(R(\xi))$$

and hence if we define

$$F_{E/S} = F_A - c(R)$$

then $F_{E/S}$ automatically commutes with the action of $C(M)$. The quantity $F_{E/S}$ is called the twisting curvature in [4].

If $M$ is spin and $E = \Sigma \otimes W$, the connection on $E$ would be given by the tensor product of the spin connection on $\Sigma$ and an auxiliary unitary connection $A$ on $W$ and the curvature decomposes according to the tensor product; in this case, $F_{E/S} = 1 \otimes F_A$ where $F_A$ just the curvature of the auxiliary connection $A$.

If $M$ is not spin, we pass to the corresponding $\Gamma$-modules, $\Sigma_P$ and $W$. It is easy to check that a Clifford-module connection on $E$ always lifts as the tensor product of the standard $\Gamma$-module connection on $\Sigma_P$ with a unitary $\Gamma$-module connection on $W$. 
Conversely the tensor product of such connections always descends to define a Clifford-compatible connection on $E$. Moreover, if $\mathcal{F}$ is the (descended) curvature of $\Sigma_P \otimes W$ then we have $\mathcal{F} = R \otimes 1 + 1 \otimes F_A$ where $R$ is the (descended) curvature of $\Sigma_P$ (essentially the Riemann curvature of the given Riemannian metric). Hence $F_{E/S} = 1 \otimes F_A$.

The index formula for generalized Dirac operators now takes the form [4]:

\begin{equation}
\text{ind}(D^+_A) = \langle \hat{A}(M) \text{ch}(E/S), [M] \rangle
\end{equation}

where the relative Chern character is defined as the relative supertrace of the exponential of the twisting curvature:

$$\text{ch}(E/S) = \text{Str}_{E/S}[\exp(-F_{E/S}/2\pi i)].$$

Here the relative supertrace is a way of defining the trace of an endomorphism of the auxiliary bundle $W$, without having to separate it off. More precisely, for an endomorphism $\phi$ of $E$ (super-)commuting with the action of $C(M)$ we define

$$\text{Str}_{E/S}(\phi) = 2^{-n/2} \text{Str}_E(\gamma \phi)$$

where $\gamma$ is the chirality operator of $E$.

Using the fact that $F_{E/S} = 1 \otimes F_A$ we see that $\text{ch}(E/S) = \text{ch}(W)$ and hence our twisted index theorem follows.

5. Concluding remarks

In the discussion of lifting bundle gerbes we specialised to the case of a central extension with center $\mathbb{Z}_k$ as for the application to the Dirac operator we only needed $\mathbb{Z}_2$. If we allow central extensions with center $U(1)$ the only additional complication is that the bundle gerbe does not have a canonical flat connection and we need to introduce the notion of a bundle gerbe connection and its associated curving and three-curvature [7, 5] which are fed into the definition of bundle gerbe module and its Chern character. This more general case might be useful for constructing a twisted Dirac-Raymond operator on a non-string manifold.

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