Calibration of tri-axial magnetometer using vector observations and inner products

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Abstract. This paper introduces a novel calibration method for tri-axial magnetometers in navigation systems, which is based on the observations of geomagnetic and gravity vectors as well as their inner products. The proposed method can eliminate the inter-triad misalignment between magnetometer and other sensors in the navigation system, and it shows robustness when dealing with raw measurements with high noise level. The effectiveness and reliability of the proposed method are demonstrated by numerical simulations and experiments.

1. Introduction
There has been widespread use of tri-axial magnetometer in the field of navigation and attitude estimation in recent years, since it can provide the information of horizontal azimuth (i.e. the heading or yaw angle) with respect to geomagnetic field. However, various error sources can easily interfere with the measurements of tri-axial magnetometer, such as the soft-iron and hard-iron interferences, bias, scale factor, non-orthogonality and misalignment, etc. Thus the calibration techniques of tri-axial magnetometer are indispensable.

Most often, the error model of tri-axial magnetometer is expressed in matrix form [1], as shown in (1). In (1), \( h = (h_1, h_2, h_3)^T \) is the geomagnetic vector, and \( v = (v_1, v_2, v_3)^T \) is the measurement provided by magnetometer. Note that both \( h \) and \( v \) are in the sensor frame. The 3x3 matrix \( K \) describes the linear errors, such as the first-order term of soft-iron interference, scale factors, and misalignment. On the other hand, the 3x1 vector \( b \) stands for the fixed errors, including the bias and hard-iron interference. Moreover, \( \epsilon \) represents the measurement noise and higher order terms of errors.

\[
v = Kh + b + \epsilon \tag{1}
\]

Let \( L = K^{-1} \), we can write the inverse form of (1) that can be used to compensate the error of tri-axial magnetometer, as shown in (2). The noise and higher order term \( \epsilon \) is usually omitted if it can be treated as a small quantity compared to the geomagnetic field. Therefore, the calibration of tri-axial magnetometer is essentially to determine (or estimate) \( L \) and \( b \).

\[
h = L(v - b) \tag{2}
\]

In most cases, the actual value of \( h \) is unknown. But fortunately, we can assume that \( h \) is a constant vector in the earth frame during the calibration process. Therefore, if the magnetometer performs in situ rotations, calibration can be carried out using the change of \( h \) in the sensor frame. This approach is similar to the multi-position calibration scheme that is commonly used for the calibration of inertial measurement unit (IMU). However, this type of calibration method is attitude-dependent, i.e. we need attitude information to perform the calibration. In practice, the attitude-dependent calibration scheme...
relies on auxiliary devices or tools to determine the rotation angle, e.g. a precision non-magnetic cube [2].

Without auxiliary instrument or reference, the only information we can use in calibration process is the constant norm of \( h \). It leads to the scalar checking method [3], i.e. \( L \) and \( b \) can be determined according to the principle that the norm of \( h \) is a constant scalar during any in situ rotations, as described in (3).

\[
|h|^2 = h^T h = v^T L_v L_v - 2v^T L_v L_b + b^T L_b L_b = \text{constant}
\]  

The calibration method based on (3) is also called ellipsoid fitting (EF) method, because (3) stands for an ellipsoid if \( L^T L \) is a positive definite matrix [1, 4], and this holds true if the off-diagonal elements of \( L \) are small. However, it is noteworthy that the ellipsoid fitting method cannot determine all the elements of \( L \). From mathematical point of view, ellipsoid fitting can determine only 9 independent parameters at most, but \( L \) and \( b \) have 12 elements altogether. Therefore, any attempt to calibrate tri-axial magnetometer using ellipsoid fitting technique will result in a mathematically underdetermined problem. This defect of EF method will be further discussed in the following section.

To overcome the shortcoming of ellipsoid fitting method, a practical way is to use the observation of gravity vector \( g \), since it can be directly measured by tri-axial accelerometer that available in most navigation applications. The accelerometer does not suffer from the interference by ferromagnetic materials, and thus a well-calibrated accelerometer can provide reliable gravity observation [5]. Note that both \( h \) and \( g \) are constant vectors in the earth frame, and thus their dot product (also referred to as inner product in mathematics) is a constant scalar, as described in (4).

\[
g^T h = g^T L_v - g^T L_b = \text{constant}
\]  

According to (4), we can directly determine \( L \) and \( b \), and it is the dot-product-invariance (DPI) method introduced in [6]. Alternatively, we can use (4) to modify the result of ellipsoid fitting method and form a two-step calibration scheme [7-10]. The DPI method is more straightforward in mathematics compared with ellipsoid fitting. However, according to follow-up study, DPI method shows less robustness when processing raw measurements with high level of noise [7, 10].

In this paper, we present a novel calibration method for tri-axial magnetometer. This method makes use of the two different inner products in (3) and (4) simultaneously, in order to acquire more reliable and robust calibration results. Thus it can be named as dual-inner-product (DIP) method. Simulation and experiment results are also presented to show the effectiveness and superiority of the proposed method.

2. Discussion of EF and DPI method

The principle of ellipsoid fitting (EF) method has already been described above. It is clear that EF method cannot determine the matrix \( L \) directly. Instead, it gives the solution of \( L^T L \). Hence, we need certain decomposition algorithm to solve \( L \) from \( L^T L \). For instance, the singular value decomposition (SVD) algorithm can give a solution of \( L \) in symmetric form.

However, it is certain that the artificially chosen solution of \( L \) will not be the exact solution in most cases. Actually, the solution of \( L \) given by EF method usually contains an undetermined orthogonal matrix that stands for a 3D rotation of the sensor frame. As a result, the axes of magnetometer will not coincide with the axes of other sensors (e.g. accelerometer and/or gyroscope) in the navigation system. This is called the inter-triad misalignment. Besides, misalignment can also appear between the sensor frame and the body frame (i.e. the coordinates of the host platform that the sensor is mounted to) [11]. Actually, these two types of misalignment cannot be eliminated by any calibration method (including EF) that based on the invariance of vector norm only [5].

The dot-product-invariance (DPI) method can directly solve the matrix \( L \), and thus it can eradicate the mathematically underdetermined problem caused by EF method. Nevertheless, the performance of DPI method will become worse than EF method when the noise level is high, as it has been revealed in [10]. This disadvantage will also be demonstrated by simulation and experiment results in the following sections.
3. Dual-inner-product method

Note that both EF and DPI methods involve the inner product of vector observations (i.e. $h^T h$ and $g^T h$). Hence, they can be incorporated with each other. That leads to the dual-inner-product (DIP) method. It should be emphasized that DPI method is by no means a two-step algorithm. Instead, this method utilized the two inner products in (3) and (4) simultaneously.

DIP method is a nonlinear optimization problem in essence, since (3) is not a linear expression of the elements in $L$ and $b$. We can use Levenberg-Marquardt (L-M) algorithm, a damped variation of Gauss-Newton method, to solve this problem. Another critical issue is the constant scalars appear on the right sides of (4). Theoretically, this constant scalar equals $|g||h|\cos(90^\circ - \gamma)$, with $\gamma$ denoting the magnetic dip angle (geomagnetic inclination). However, if the accurate value of $\gamma$ is unknown, we should introduce an additional parameter $\lambda=\cos(90^\circ - \gamma)$ to be estimated. Then the parameter vector $x$ can be written as (5). In (5), $l_g$ is the element in $i$-th row and $j$-th column of $L$, and $b_j$ is the $i$-th element of $b$. There are 13 parameters to be estimated in total.

$$x = (l_{i1} \ldots l_{i3} \ b_1 \ b_2 \ b_3 \ \lambda)^T$$

(5)

Then we introduce the cost function $e$ based on (3) and (4), as shown in (6). Note that $e$ has $2r$ rows if there are $r$ different samples (each sample consists of $v$ and $g$). Besides, while $v$ is the raw measurement of magnetometer that needs to be calibrated, $g$ should be provided by pre-calibrated inertial sensor, so that it can be utilized as a reliable reference.

$$e = \begin{pmatrix}
    v_1^T L^T L v_1 - 2v_1^T L^T Lb + b^T L^T L b - |h|^2 \\
    \vdots \\
    v_r^T L^T L v_r - 2v_r^T L^T Lb + b^T L^T L b - |h|^2 \\
    g_1^T L v_1 - g_1^T L b - |g||h|^2 \\
    \vdots \\
    g_r^T L v_r - g_r^T L b - |g||h|^2
\end{pmatrix}$$

(6)

Let $a_r=\lambda(v_r-b)$ and $b_r=v_r-b$, in which $1 \leq r \leq r$, then the Jacobian matrix $J$ can be written as (7).

$$J = \frac{\partial(e_1, e_2, \ldots, e_r)}{\partial(x_1, x_2, \ldots, x_13)} = \begin{pmatrix}
    2a_{r,1} \beta_{r,1} & \cdots & 2a_{r,3} \beta_{r,3} & -2\sum_{m=1}^{3} a_{r,m} l_{m1} & -2\sum_{m=1}^{3} a_{r,m} l_{m2} & -2\sum_{m=1}^{3} a_{r,m} l_{m3} & 0 \\
    \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\
    2a_{r,1} \beta_{r,1} & \cdots & 2a_{r,3} \beta_{r,3} & -2\sum_{m=1}^{3} a_{r,m} l_{m1} & -2\sum_{m=1}^{3} a_{r,m} l_{m2} & -2\sum_{m=1}^{3} a_{r,m} l_{m3} & 0 \\
    g_{r,1} \beta_{r,1} & \cdots & g_{r,3} \beta_{r,3} & -\sum_{m=1}^{3} g_{r,m} l_{m1} & -\sum_{m=1}^{3} g_{r,m} l_{m2} & -\sum_{m=1}^{3} g_{r,m} l_{m3} & -|g||h| \\
    \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\
    g_{r,1} \beta_{r,1} & \cdots & g_{r,3} \beta_{r,3} & -\sum_{m=1}^{3} g_{r,m} l_{m1} & -\sum_{m=1}^{3} g_{r,m} l_{m2} & -\sum_{m=1}^{3} g_{r,m} l_{m3} & -|g||h|
\end{pmatrix}$$

(7)

At last, the implementation of L-M algorithm is as follows:

1) Initialize: $k=0$, $\lambda_k=1$, $L_k=I_{3\times 3}$, $b_k=0_{3\times 1}$, as well as the damping factor $\mu_k=10^{-3}$;
2) Calculate $x_k$, $e_k$, and $J_k$ according to (5)–(7);
3) Calculate $x_{k+1} = x_k - (J_k^T J_k + \mu_k I_{3\times 3})^{-1} J_k^T e_k$, then update $L_{k+1}$, $b_{k+1}$, and $\lambda_{k+1}$ according to $x_{k+1}$;
4) Calculate $e_{k+1}$;
5) If $|e_{k+1}|^2 < |e_k|^2$, accept $L_{k+1}$, $b_{k+1}$, and $\lambda_{k+1}$, then $\mu_{k+1} = 0.1 \mu_k$; otherwise rollback $L_{k+1} = L_k$, $b_{k+1} = b_k$, and $\lambda_{k+1} = \lambda_k$, then $\mu_{k+1} = 10 \mu_k$;
6) If $|e_{k+1}|^2 < |e_k|^2 < 10^{-4} |e_k|^2$ or $k \geq 100$, output $L_{k+1}$ and $b_{k+1}$, then finish; otherwise $k = k+1$ and go to 2).

4. Numerical simulation

In this section, we assume that the tri-axial magnetometer has the error model in (8). The standard deviation of the 3x1 Gaussian white noise vector $e$ is denoted as $\sigma_e$. 

$$h = \begin{pmatrix}
    a_{1,1} & a_{1,2} & a_{1,3} \\
    a_{2,1} & a_{2,2} & a_{2,3} \\
    a_{3,1} & a_{3,2} & a_{3,3}
\end{pmatrix}$$

(8)
\[
\nu = \begin{pmatrix} 1.100 & 0.015 & -0.010 \\ -0.020 & 1.000 & 0.025 \\ 0.030 & -0.035 & 0.950 \end{pmatrix} \ h + \begin{pmatrix} 1.100 \\ 0.700 \\ -0.900 \end{pmatrix} + \varepsilon
\]

(8)

Meanwhile, the gravity vector also has Gaussian white noise, with the standard deviation \( \sigma_g \). But it has no sensor error. The geomagnetic vector and gravity vector in the earth frame (north-east-down coordinates) are \((40 \ 0 \ 30) \, \mu T\) and \((0 \ 0 \ 9.8) \, \text{m/s}^2\), respectively.

Assuming that the host platform of sensors is set to five different attitudes in sequence: completely level, as well as 30° inclined to the front, back, left and right. In another word, we have 5 different sets of the pitch and roll angles (denoted as \( \theta \) and \( \varphi \), respectively): (0° 0°), (−30° 0°), (30° 0°), (0° −30°), and (0° 30°). Moreover, the heading angle \( \psi \) is varied from 0° to 360° in each attitude with the increment of 5°. The geomagnetic and gravity data are generated with the above settings. Figure 1 illustrates the geomagnetic data in the sensor frame.

![Geomagnetic data in sensor frame.](image)

Table 1. Heading error (RMS) before and after calibration with different noise level.

| Calibration method | Noise level  | Noise level  | Noise level  | Noise level  | Noise level  |
|--------------------|--------------|--------------|--------------|--------------|--------------|
|                    | \( \sigma_h = 0.05 \, \mu T \) | \( \sigma_h = 0.10 \, \mu T \) | \( \sigma_h = 0.20 \, \mu T \) | \( \sigma_h = 0.50 \, \mu T \) | \( \sigma_h = 1.00 \, \mu T \) |
| No calibration     | 2.621°       | 2.634°       | 2.640°       | 2.745°       | 3.010°       |
| EF                 | 1.257°       | 1.270°       | 1.279°       | 1.485°       | 2.109°       |
| DPI                | 0.268°       | 0.276°       | 0.411°       | 1.303°       | 4.451°       |
| EF+DPI             | 0.119°       | 0.202°       | 0.348°       | 0.838°       | 2.624°       |
| DIP                | 0.082°       | 0.150°       | 0.313°       | 0.810°       | 1.557°       |
According to table 1, DPI method shows much better performance than EF method when the noise level is low ($\sigma_h=0.05\mu T$). As stated above, this is mainly because DPI method can eliminate the inter-triad misalignment but EF method cannot. However, when the noise level is high ($\sigma_h=1.00\mu T$), the performance of DPI method become poorer than EF method. The combination of EF and DPI (two-step calibration scheme) shows better results than that of DPI method, but its performance is still inferior to EF method at high noise level. By contrast, the DIP method proposed in this paper shows the best performance in all cases.

5. Experiment

We apply all the above mentioned methods to an attitude and heading reference system (AHRS) that contains a tri-axial magnetometer (Honeywell® HMC5883) and a tri-axial accelerometer (Analog Devices Inc. ADXL345). There are 5 AHRS modules being tested in the experiment altogether, in order to guarantee the repeatability. Besides, we use Xsens® MTi-300-2A5G4 AHRS module to provide precision attitude and heading information in the experiment. The configuration of experimental system is shown in figure 2, and the calibration results are demonstrated in figure 3.

![Figure 2.](image)

From figure 3, we can see that the performance of DPI method is no better than that of EF method. This is mainly because that the resolution of HMC5843 is relatively low (~0.7\mu T according to its datasheet), which can be equivalent to a high level of quantizing noise. As a result, the performance of DPI and EF methods are quite close to each other. Moreover, the two-step calibration scheme (i.e. EF+DPI) only brings in slight improvement of the calibration results. Once again, the proposed DIP method has the best performance.

The experiment results on AHRS module #1 are shown in figure 4, which further demonstrates the performance of each method as well as the superiority of the proposed DIP method.
6. Conclusions
We introduce a novel calibration method for tri-axial magnetometer in navigation applications. This method makes use of two constant inner products: the square norm of geomagnetic vector itself, as well as the dot product of geomagnetic and gravity vectors. The proposed dual-inner-product (DIP) method combines the advantages of both ellipsoid fitting (EF) and dot-product-invariance (DPI) methods, and thus it can eliminate the inter-triad misalignment between magnetometer and other sensors while being robust against measurement noise. Simulation and experiment results demonstrate the effectiveness and superiority of DIP method.

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References

[1] Vasconcelos J F, Elkaim G, Silvestre C, Oliveira P and Cardeira B 2011 Geometric approach to strap-down magnetometer calibration in sensor frame IEEE Trans. Aerosp. Electron. Syst. 47 1293–306

[2] Yang D, You Z, Li B, Duan W and Yuan B 2017 Complete tri-axis magnetometer calibration with a gyro auxiliary Sensors 17 1223

[3] Merayo J M G, Brauer P, Primdahl F, Petersen J R and Nielsen O V 2000 Scalar calibration of vector magnetometers Meas. Sci. Technol. 11 120–32

[4] Fang J, Sun H, Cao J, Zhang X and Tao Y 2011 A novel calibration method of magnetic compass based on ellipsoid fitting IEEE Trans. Instrum. Meas. 60 2053–61

[5] Gheorghe M V 2016 Advanced calibration method, with thermal compensation, for 3-axis MEMS accelerometers Rom. J. Inf. Sci. Tech. (ROMJIST) 19 255–68

[6] Li X and Li Z 2012 A new calibration method for tri-axial field sensors in strap-down navigation systems Meas. Sci. Technol. 23 105015

[7] Liu Y, Li X, Zhang X and Feng Y 2014 Novel calibration algorithm for a three-axis strap-down magnetometer Sensors 14 8485–504

[8] Zhu X, Zhao T, Cheng D and Zhou Z 2017 A three-step calibration method for tri-axial field sensors in a 3D magnetic digital compass Meas. Sci. Technol. 28 055106

[9] Liu Z and Song J 2018 A Low-Cost Calibration Strategy for Measurement-While-Drilling System IEEE Trans. Ind. Electron. 65 3559–67

[10] Li X, Wang Y and Li Z 2015 Calibration of tri-axial magnetometer in magnetic compass using vector observations Proc. IEEE 28th Canadian Conf. Electrical and Computer Engineering (Halifax) pp 123–7

[11] Gheorghe M V 2017 Calibration for tilt-compensated electronic compasses with alignment between the magnetometer and accelerometer sensor reference frames Proc. 2017 IEEE Int. Instrumentation and Measurement Technology Conf. (Turin, Italy) 7969813