Para-complex geometry and gravitational instantons

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Received 31 May 2013, in final form 2 August 2013
Published 2 September 2013
Online at stacks.iop.org/CQG/30/195001

Abstract
We give a complete classification of supersymmetric gravitational instantons in Euclidean \(N = 2\) supergravity coupled to vector multiplets. An interesting class of solutions is found which corresponds to the Euclidean analogue of stationary black hole solutions of \(N = 2\) supergravity theories.

PACS numbers: 02.40.Dr, 04.65.+e

1. Introduction

Instantons are of particular importance in theoretical physics and mathematics. For example, instantons are an essential ingredient in the non-perturbative analysis of non-Abelian gauge theories and quantum mechanical systems [1]. Moreover, the existence of spin-1/2 zero-mode of the instanton is linked to the Atiyah–Singer index theorem [2], a fact reflecting the intimate relation of non-Abelian gauge theory to the field of fibre bundles and differential geometry. An important example of Yang–Mills instanton solutions are those given in [3]. In finding the instanton solutions of [3], the self-duality (or anti-self-duality) is imposed on the Yang–Mills field strength, this leads to the fact that the Bianchi identity implies the Yang–Mills field equations. This considerably simplifies finding solutions, as instead of solving second order differential equations, one solves the Bianchi identities containing only first derivatives of the vector potential. Gravitational instantons are in general defined as non-singular complete solutions to the Euclidean Einstein equations of motion. Notable early examples of gravitational instantons are the Eguchi–Hanson instantons [4], which are the first examples of the family of the Gibbons–Hawking instanton solutions [5].

In finding gravitational instantons [4, 5] and in analogy with the Yang–Mills case, the spin connection 1-form is assumed to be self-dual (or anti-self-dual), which leads to a self-dual curvature 2-form. This property together with the cyclic identity ensures that Einstein’s equations of motion are satisfied. The equations coming from the self-duality of the spin connection are simpler as they contain only first derivatives of the spacetime metric.

In recent years, a good deal of work has been done on the classification of solutions preserving fractions of supersymmetry in supergravity theories in various dimensions. It is clear that the quest of finding solutions admitting some supersymmetry is easier as one in
these cases is simply dealing with first order Killing spinors differential equations rather than Einstein’s equations of motion. Following the results of [6], a systematic classification for all metrics admitting Killing spinors in $D = 4$ Einstein–Maxwell theory, was performed in [7]. The solutions with time-like Killing spinors turn out to be the Israel–Wilson–Perjés (IWP) solutions [8] whose static limit is given by the Majumdar–Papapetrou solutions [9]. It was shown by Hartle and Hawking that all the non-static solutions suffered from naked singularities [10, 11]. Using the two-component spinor calculus [12], the instanton analogue of the IWP metric was constructed in [13]. These solutions were also recovered in the complete classification of instanton solutions admitting Killing spinors using spinorial geometry techniques [14]. Spinorial geometry, partly based on [15–17], was first used in [18] and has also been a very powerful tool in the classification of solutions in lower dimensions (see for example [19]) and in the classification of supersymmetric solutions of Euclidean $N = 4$ super Yang–Mills theory [20].

Some time ago general stationary solutions of $N = 2$ supergravity action coupled to $N = 2$ matter multiplets were found in [21]. These can be thought of as generalizations of the IWP solutions of Einstein–Maxwell theory to include more gauge and scalar fields. The symplectic formulation of the underlying special geometry played an important role in the construction of these solutions. The stationary solutions found are generalizations of the double-extreme and static black hole solutions found in [22]. It was also shown in [23] that the solutions of [21] are the unique half-supersymmetric solutions with time-like Killing vector. The $N = 2$ solutions are covariantly formulated in terms of the underlying special geometry. The solution is defined in terms of the symplectic sections satisfying the so-called stabilization equations.

In the present work we extend the construction of [14] to $N = 2$ Euclidean supergravities with gauge and scalar fields. A class of these theories were recently derived in [24] as a reduction of the five-dimensional $N = 2$ supergravity theories coupled to vector multiplets [25] on a time-like circle. The paper is organized as follows. In the next section, we will collect some formulae and expressions of $N = 2$ supergravity which will be important for the following discussion. Section three contains a derivation of the gravitational instantons using spinorial geometry methods. The solutions found are the Euclidean analogues of the stationary black hole solutions of [21]. Section 4 contains a summary and some future directions. We include an appendix containing a linear system of equations obtained from the Killing spinor equations.

2. Special geometry

In this section we review some of the structure and equations of the original theory of special geometry when formulated in $(1, 3)$ signature. We then briefly discuss the modifications one introduces for the Euclidean $(0, 4)$ signature. For further details on the subject the reader is referred to [26]. The bosonic Lagrangian of the four-dimensional $N = 2$ supergravity theory coupled to vector multiplets can be written as

$$e^{-1}L = \frac{1}{2}R - g_{A\bar{B}}\partial_\mu z^A \partial^\mu \bar{z}^B + \frac{1}{4} \text{Im} N_{IJ} F^I \cdot F^J + \frac{1}{4} \text{Re} N_{IJ} \tilde{F}^I \cdot \tilde{F}^J. \quad (1)$$

The $n$ complex scalar fields $z^A$ of $N = 2$ vector multiplets are coordinates of a special Kähler manifold. $F^I$ are $n + 1$ 2-forms representing the gauge field strength 2-forms and we have used the notation $F \cdot F = F_{\mu\nu} F^{\mu\nu}$.

A special Kähler manifold is a Kähler–Hodge manifold with conditions on the curvature

$$R_{ABCD} = g_{AB} g_{CD} + g_{AD} g_{CB} - C_{ACE} C_{BDE} g^{EL}.$$ \quad (2)

Here $g_{A\bar{B}} = \partial_A \partial_{\bar{B}} K$ is the Kähler metric, $K$ is the Kähler potential and $C_{ABC}$ is a completely symmetric covariantly holomorphic tensor. A Kähler–Hodge manifold has a $U(1)$ bundle...
whose first Chern class coincides with the Kähler class, thus locally the $U(1)$ connection $A$ can be written as

$$A = -\frac{i}{2} (\bar{\partial}_A K \, dz^A - \partial_A K \, d\bar{z}^A).$$

A useful definition of a special Kähler manifold can be given by introducing a $(2n+2)$-dimensional symplectic bundle over the Kähler–Hodge manifold with the covariantly holomorphic sections

$$V = \begin{pmatrix} L^I \\ M_I \end{pmatrix}, \quad I = 0, \ldots, n$$

$$D_A V = (\partial_A - \frac{i}{2} \partial_A K) V = 0.$$  \hspace{1cm} (4)

These sections obey the symplectic constraint

$$i(V, \bar{V}) = i(\bar{L}^M M_I - L^I \bar{M}_I) = 1.$$  \hspace{1cm} (5)

One also defines

$$U_A = D_A V = \left( \partial_A + \frac{1}{2} \partial_A K \right) V = \begin{pmatrix} f_A^I \\ h_{AI} \end{pmatrix}.$$  \hspace{1cm} (6)

In general one can write

$$M_I = N_{IJ} L^J, \quad h_{AI} = \tilde{N}_{IJ} f_J^A$$

where $N_{IJ}$ is a symmetric complex matrix. It can be demonstrated that the constraint (2) can be obtained from the integrability conditions on the following differential constraints

$$U_A = D_AV,$$

$$D_A U_B = iC_{ABC} \bar{g}^{BD} \bar{U}_D,$$

$$D_A \bar{U}_B = g_{AB} \bar{V},$$

$$D_A \bar{V} = 0,$$

$$\langle V, U_A \rangle = 0.$$  \hspace{1cm} (8)

The Kähler potential is introduced via the definition of the holomorphic sections

$$\Omega = e^{-K/2} V = \begin{pmatrix} \chi^I \\ F_I \end{pmatrix}, \quad \bar{\partial}_A \Omega = 0,$$

$$\bar{D}_A \Omega = (\partial_A - \partial_A K) \Omega,$$

$$F_I(z) = N_{IJ} X^J(z), \quad \bar{D}_A F_I(z) = \tilde{N}_{IJ} \bar{D}_A X^J(z).$$  \hspace{1cm} (9)

Using (4) we obtain

$$e^{-K} = i(\bar{X}^I F_I - X^I \bar{F}_I).$$  \hspace{1cm} (10)

Here we list some equations coming from special geometry

$$g_{AB} = \partial_A \partial_B K = -i(U_A, \bar{U}_B) = -2 \operatorname{Im} N_{IJ} f^I_A \bar{f}^J_B,$$  \hspace{1cm} (11)

$$g^{AB} f_A^I \bar{f}_B^J = -\frac{1}{2} (\operatorname{Im} N_{IJ})^I_J - \bar{L}^I L^J,$$  \hspace{1cm} (12)

$$F_I \partial_\alpha X^I - X^I \partial_\alpha F_I = 0.$$  \hspace{1cm} (13)

Recently, Euclidean versions of special geometry have been investigated in the context of Euclidean supergravity theories [24]. The Euclidean theories are found by replacing $i$ with $e$ in the corresponding Lorentzian versions of the theories, where $e$ has the properties $e^2 = 1$ and $\bar{e} = -e$. Thus one has, in the Euclidean theory, the para-complex fields

$$L^I = \operatorname{Re} L^I + e \operatorname{Im} L^I, \quad \bar{L}^I = \operatorname{Re} L^I - e \operatorname{Im} L^I.$$  \hspace{1cm} (14)
and as a result all quantities expressed in terms of $L^I$ become para-complex. One can also introduce the so-called adapted coordinates which are defined as

$$L^I_\pm = \text{Re } L^I \pm i \text{Im } L^I$$

and also set

$$M^I_\pm = \text{Re } M^I \pm i \text{Im } M^I.$$  

It should be noted that the replacement of $i$ by $e$ was first done in the context of finding D-instanton solutions in type IIB supergravity [27]. This replacement is effectively the replacement of the complex structure by a para-complex structure. Details on para-complex geometry, para-holomorphic bundles, para-Kähler manifolds and affine special para-Kähler manifolds can be found in [28]. The Killing spinor equations in the Euclidean $N = 2$ supergravity theory were recently obtained in [29] by reducing those of the five-dimensional theory given in [25].

The equations of special geometry for either signatures can be considered in a unified manner by introducing the symbol $i_\epsilon, \bar{i}_\epsilon = -i_\epsilon$, where $i^2_\epsilon = \epsilon$, with $\epsilon = -1$ for theories with $(1, 3)$ signature and $\epsilon = +1$ for theories with $(0, 4)$ signature. Note for the adapted coordinates one uses the definition given in (15).

From the above equations one can derive some useful relations which will be needed in our analysis. Using (4) and (6), we write

$$\partial_\mu L^I = \bar{\partial}_\mu \bar{L}^A \bar{\partial}_\mu \bar{z}^A + \partial_\mu \bar{A}_\mu \bar{z}^A$$

which implies the relation

$$\partial_\mu L^I = \bar{\partial}_\mu \bar{L}^A - \epsilon i_\epsilon L^I \partial_\mu$$

after using

$$A = -\frac{i_\epsilon}{2} (\partial_\mu K \partial^A - \partial_\mu K \partial^A).$$

Moreover from (10) we have

$$\partial_\mu K \partial^A = -i_\epsilon \epsilon^K (\dot{X}^I dF_I - \bar{X}^I d\bar{F}_I).$$

Using (20), we obtain from (3)

$$A = \epsilon (L^I dM_I - M^I d\bar{L}^I).$$

Also using (4), (6) and (7), one obtains

$$\partial_\mu M_I = \bar{\partial}_\mu \bar{M}_I \partial^A + \partial_\mu \bar{M}_I \partial^A - \frac{i}{2} M_I \partial_\mu K \partial^A + D_A \partial_\mu \bar{L}^A - \frac{i}{2} \bar{M}_I \partial_\mu K \partial^A$$

$$= \epsilon i_\epsilon M_I A_\mu + \bar{N}_{IJ} D_A \partial_\mu \bar{L}^A$$

$$= \epsilon i_\epsilon M_I A_\mu + \bar{N}_{IJ} (\partial_\mu L^J - \epsilon i_\epsilon L^J A_\mu).$$

This implies that

$$\partial_\mu M_I = 2 \text{Im } \bar{N}_{IJ} L^J A_\mu = \bar{N}_{IJ} \partial_\mu L^J.$$  

It will be convenient to rewrite a number of these conditions in terms of adapted coordinates, which will be used for the Euclidean calculation in the following section. In particular, the relationship between $M_I$ and $L^I$ given in (7) is equivalent to

$$M^I_\pm = (\text{Re } N^I_{IJ} \pm \text{Im } N^I_{IJ}) L^J_\pm.$$  

$$4$$
The condition (12) is equivalent to
\[(\text{Re}(g^{AB}D_{\bar{A}}\bar{L}^I) \pm \text{Im}(g^{AB}D_{\bar{A}}\bar{L}^I)) (\text{Re}(D_{\mu}L_j^I) \pm \text{Im}(D_{\mu}L_j^I)) = -\frac{1}{2}(\text{Im}(N)^{IJ} - L^I_{\pm}L^J_{\pm}),\]  
(25)

and (13) is equivalent to
\[M_{\pm I} dL^I_{\pm} - L^I_{\pm} dM_{\pm I} = 0.\]  
(26)

Also, (18) is equivalent to
\[\partial_{\mu}L^I_{\pm} = (\text{Re}(D_{\mu}L^I) \pm \text{Im}(D_{\mu}L^I)) A_{\mu}L^I_{\pm},\]  
(27)

and (23) is equivalent to
\[\partial_{\mu}M_{\pm I} = (\text{Re}(N)^{IJ} \pm \text{Im}(N)^{IJ}) A_{\mu} M^{IJ}_{\pm}.\]  
(28)

Also, note that on contracting (11) with \(g^{AB}\), and using (12), one finds that
\[n^2 = g^{AB}g_{AB} = \text{Im}(N)^{IJ} \text{Im}(N)^{IJ} + 2 \text{Im}(N)^{IJ} \bar{L}^I L^J.\]  
(29)

This implies that
\[\text{Im}(N)^{IJ} \bar{L}^I L^J = -\frac{1}{2} - \frac{n}{4}.\]  
(30)

3. Gravitational instantons

In this section we classify the gravitational instanton solutions by solving the Killing spinor equations for the Euclidean theory [29]:
\[\left(\nabla_{\mu} + \frac{1}{2} A_{\mu} \Gamma_5 + \frac{1}{4} \Gamma^I F^I (\text{Im} L^I + \text{Re} \text{Im} L^I) (\text{Im} N)^{IJ} \Gamma_{IJ} \Gamma_{\mu}\right) \epsilon = 0\]  
(32)

\[\frac{i}{2} (\text{Im}N)^{IJ} \Gamma^I F^J [\text{Im}(D_{\bar{A}} \bar{L}^I g^{AB}) + \text{Re}(D_{\bar{A}} \bar{L}^I g^{AB})] \epsilon + \Gamma^A \delta_{\mu}[\text{Re} \epsilon^A - \Gamma_3 \text{Im} \epsilon^A] \epsilon = 0.\]  
(33)

We remark that if \(\epsilon\) is a Killing spinor satisfying (32) and (33), then so is \(C^* \epsilon\), which is moreover linearly independent of \(\epsilon\) (over \(\mathbb{C}\)); here \(C^*\) is a charge conjugation operator whose construction is defined in terms of spinorial geometry techniques in [14]. It follows that the complex space of Killing spinors must be of even dimension, i.e. the supersymmetric solutions must preserve either four or eight real supersymmetries. If a solution is maximally supersymmetric, then the gaugino Killing spinor equation (33) implies that the scalars \(\epsilon^A\) are constant. It then follows that the gravitino Killing spinor equation reduces to the gravitino Killing spinor equation of the minimal theory. As the maximally supersymmetric solutions of the minimal theory have already been fully classified in [14], for the remainder of this paper we shall consider solutions preserving half of the supersymmetry.

In order to proceed with the analysis, we define the spacetime basis \(e^1, e^2, \bar{e}^1, \bar{e}^2\), with respect to which the spacetime metric is
\[ds^2 = 2(e^1 \bar{e}^1 + e^2 \bar{e}^2).\]  
(34)

The space of Dirac spinors is taken to be the complexified space of forms on \(\mathbb{R}^2\), with basis \([1, e_1, e_2, e_1 \wedge e_2]\); a generic Dirac spinor \(\epsilon\) is a complex linear combination of these basis elements. In this basis, the action of the Dirac matrices \(\Gamma_m\) on the Dirac spinors is given by
\[\Gamma_m = \sqrt{2} i e_m, \quad \bar{\Gamma}_{\bar{m}} = \sqrt{2} e_{\bar{m}} \wedge\]  
(35)
for \( m = 1, 2 \). We also define
\[
\Gamma_3 = \Gamma_{122}
\]
which acts on spinors via
\[
\Gamma_3 1 = 1, \quad \Gamma_3 e_{12} = e_{12}, \quad \Gamma_3 e_m = -e_m \quad m = 1, 2.
\]
With this representation of the Dirac matrices acting on spinors, the resulting linear system obtained from (32) and (33) is listed in appendix A.

There are three non-trivial orbits of Spin(4) = Sp(1) \times Sp(1) acting on the space of Dirac spinors. In our notation, one can use SU(2) transformations to rotate a generic spinor \( \epsilon \) into the canonical form [14, 30]
\[
\epsilon = \lambda 1 + \sigma \epsilon_1,
\]
where \( \lambda, \sigma \in \mathbb{R} \). The three orbits mentioned above correspond to the cases \( \lambda = 0, \sigma \neq 0; \lambda \neq 0, \sigma = 0 \) and \( \lambda \neq 0, \sigma \neq 0 \). The orbits corresponding to \( \lambda = 0, \sigma \neq 0 \) and \( \lambda \neq 0, \sigma = 0 \) are equivalent under the action of Pin(4). We shall treat these orbits separately.

### 3.1. Solutions with \( \lambda \neq 0 \) and \( \sigma \neq 0 \)

For solutions with \( \lambda \neq 0 \) and \( \sigma \neq 0 \), the analysis of the linear system (A.3) obtained from the gravitino equation produces the following geometric conditions
\[
\omega_{1,22} = \partial_1 \log \frac{\sigma}{\lambda} + A_1, \quad \omega_{2,11} = \partial_2 \log \frac{\lambda}{\sigma} - A_2, \quad 
\omega_{1,1} = -\partial_1 \log \lambda \sigma, \quad \omega_{2,2} = \partial_2 \log \lambda \sigma, \\
\omega_{1,21} = 2 \partial_2 \log \lambda - A_2, \quad \omega_{2,21} = 2 \partial_1 \log \sigma + A_2, \\
\omega_{2,21} = -2 \partial_1 \log \sigma - A_1, \quad \omega_{2,21} = -2 \partial_1 \log \lambda + A_1, \\
\omega_{1,21} = \omega_{2,21} = \omega_{1,21} = \omega_{2,21} = 0,
\]

as well as the following conditions involving the gauge field strengths
\[
(\text{Im} N)_{ij} F_{22}^{I_j} + F_{11}^{I_j} L_+^I = -\frac{\sqrt{2\lambda}}{\lambda \sigma} (\partial_1 - A_1) \lambda^2, \\
(\text{Im} N)_{ij} F_{22}^{I_j} - F_{11}^{I_j} L_-^I = \frac{\sqrt{2\lambda}}{\lambda \sigma} (\partial_1 + A_1) \sigma^2, \\
(\text{Im} N)_{ij} F_{21}^{I_1} L_+^I = \frac{i}{\sqrt{2\lambda \sigma}} (\partial_2 - A_2) \lambda^2, \\
(\text{Im} N)_{ij} F_{21}^{I_1} L_-^I = -\frac{i}{\sqrt{2\lambda \sigma}} (\partial_2 + A_2) \sigma^2.
\]

The geometric constraints (39) imply the following:
\[
\text{d} e^1 = -\partial_1 \log \lambda \sigma e^1 \land e^1 + (2 \partial_2 \log \sigma + A_2) e^1 \land e^2 + (2 \partial_2 \log \lambda - A_2) e^1 \land e^3 + \\
\bigg( \partial_2 \log \frac{\lambda}{\sigma} + A_2 \bigg) e^1 \land e^2 + \\
\bigg( \partial_2 \log \frac{\lambda}{\sigma} - A_2 \bigg) e^1 \land e^3 + 2 \bigg( \partial_1 \log \frac{\lambda}{\sigma} + A_1 \bigg) e^2 \land e^3
\]
\[
\text{d} e^2 = -\text{d}(\log \lambda \sigma) \land e^2
\]

implying that
\[
\text{d}(\lambda \sigma (e^1 + e^1)) = 0.
\]
Thus we introduce three real local coordinates $x, y$ and $z$, such that
\[
(e^1 + e^3) = \sqrt{2} \frac{\xi}{\lambda} dx, \quad e^2 = \frac{1}{\sqrt{2} \lambda} (dy + i dz).
\]
Furthermore, the vector defined by
\[
V = i \xi \sigma (e^1 - e^3)
\]
is a Killing vector, and so we introduce a local-coordinate $\tau$ such that
\[
V = \sqrt{2} \frac{\partial}{\partial \tau}
\]
and
\[
\sqrt{2} (e^1 - e^3) = -2i \xi \sigma (d\tau + \phi)
\]
where $\phi = \phi_i dx + \phi_y dy + \phi_z dz$ is a 1-form. We remark that the conditions imposed on the geometry by the gravitino Killing spinor equations imply that
\[
\left( A + d \left( \log \frac{\sigma}{\lambda} \right) \right) \tau = 0.
\]
So, in the coordinates $\tau, x, y, z$, the metric is
\[
dx^2 = (\lambda \sigma)^2 (d\tau + \phi)^2 + \frac{1}{(\lambda \sigma)^2} (dx^2 + dy^2 + dz^2)
\]
where $\lambda \sigma$ and $\phi$ are independent of $\tau$, and $\phi$ satisfies
\[
d\phi = \frac{2}{(\lambda \sigma)^2} \hat{\ast} \left( A + d \left( \log \frac{\sigma}{\lambda} \right) \right).
\]
Next consider we consider the linear system (A.4) derived from (33). First, one finds that
\[
\mathcal{L}_V z^A = 0.
\]
This condition implies that
\[
A_\tau = 0
\]
and hence the $\tau$-independence of $\lambda \sigma$, together with (47), imply that both $\lambda$ and $\sigma$ are independent of $\tau$.

Next, using (25), together with (40), one obtains
\[
\begin{align*}
\text{if} & F^I_{21} - L^I_{21} \sqrt{\frac{\xi}{\lambda \sigma}} (\partial_2 - A_2) \lambda^2 - \sqrt{2} \frac{\sigma}{\lambda} (\partial_2 - A_2) L^I_{1+} \lambda^2 = 0, \\
\text{if} & F^I_{21} + L^I_{21} \sqrt{\frac{\xi}{\lambda \sigma}} (\partial_2 + A_2) \sigma^2 - \sqrt{2} \frac{\lambda}{\sigma} (\partial_2 + A_2) L^I_{1+} \lambda^2 = 0, \\
\text{if} & -\frac{i}{2} (F^I_{22} - F^I_{11}) + L^I_{1+} \sqrt{\frac{\xi}{\lambda \sigma}} (\partial_1 + A_1) \sigma^2 + \sqrt{2} \frac{\lambda}{\sigma} (\partial_1 + A_1) L^I_{1+} \lambda^2 = 0, \\
\text{if} & \frac{i}{2} (F^I_{22} + F^I_{11}) + L^I_{1+} \sqrt{\frac{\xi}{\lambda \sigma}} (\partial_1 - A_1) \lambda^2 + \sqrt{2} \frac{\sigma}{\lambda} (\partial_1 - A_1) L^I_{1+} \lambda^2 = 0,
\end{align*}
\]
from which we obtain
\[
\begin{align*}
F^I_{11} &= i \partial_1 (\sigma^2 L^I_{1+} + \lambda^2 L^I_{1-}), \\
F^I_{22} &= i \partial_2 (\lambda^2 L^I_{1+} - \sigma^2 L^I_{1-}) + 2i \sigma^2 (\partial_2 - A_2) L^I_{1+} - 2i \lambda^2 (\partial_2 + A_2) L^I_{1-}, \\
F^I_{21} &= i L^I_{1+} ((\partial_2 - i \partial_2) \sigma^2 + \sigma^2 (A_2 - iA_2)) + i (\lambda^2 (\partial_2 - i \partial_2) + \lambda^2 (A_2 - iA_2)) L^I_{1+}, \\
F^I_{21} &= i L^I_{1-} ((\partial_2 - i \partial_2) \lambda^2 + \lambda^2 (A_2 - iA_2)) - i (\sigma^2 (\partial_2 - i \partial_2) - \sigma^2 (A_2 - iA_2)) L^I_{1-}.
\end{align*}
\]
In terms of the local coordinates \(\tau, x, y, z\), the gauge field strengths are

\[
F^I = -d[(\sigma^2 L^I + \lambda^2 L^I_\sigma)(d\tau + \phi)] + \hat{\phi} \left[ \frac{L^I_+}{\lambda^2} - \frac{L^I_-}{\sigma^2} \right].
\]  

(54)

Thus the Bianchi identity implies that

\[
\hat{\nabla}^2 \left[ \frac{L^I_+}{\lambda^2} - \frac{L^I_-}{\sigma^2} \right] = 0.
\]  

(55)

where \(\hat{\nabla}^2\) is the Laplacian on \(\mathbb{R}^3\). The dual gauge field strength \(\tilde{F}^I\) is given by

\[
\tilde{F}^I_{\mu_1 \mu_2} = \frac{1}{2} \epsilon_{\mu_1 \mu_2 \nu_1 \nu_2} F^{I \nu_1 \nu_2}.
\]  

(56)

where the volume form satisfies \(\epsilon_{123} = 1\). Hence

\[
\tilde{F}^I_{23} = -F^I_{11}, \quad \tilde{F}^I_{11} = -F^I_{23}, \quad \tilde{F}^I_{12} = F^I_{12}, \quad \tilde{F}^I_{12} = -F^I_{12}.
\]  

(57)

In terms of the local coordinates \(\tau, x, y, z\) one finds

\[
\tilde{F}^I = (L^I_- d\lambda^2 - \lambda^2 d\sigma^2 - \sigma^2 dL^I_+ + 2(\lambda^2 L^I_+ + \sigma^2 L^I_-)A) \wedge (d\tau + \phi)
\]

\[- \frac{1}{\lambda^2 \sigma^2} \hat{\phi} d(\lambda^2 L^I_+ + \sigma^2 L^I_-).\]  

(58)

Evaluating \(\text{Re} \mathcal{N}_{IJ} F^J + \text{Im} \mathcal{N}_{IJ} \tilde{F}^J\) and making use of (28) we obtain

\[
\text{Re} \mathcal{N}_{IJ} F^J + \text{Im} \mathcal{N}_{IJ} \tilde{F}^J = -d[(\sigma^2 M_{+I} + \lambda^2 M_{-I})(d\tau + \phi)] + \hat{\phi} \left[ \frac{M_{+I}}{\lambda^2} - \frac{M_{-I}}{\sigma^2} \right].
\]  

(59)

The gauge field equations are

\[
d[\text{Re} \mathcal{N}_{IJ} F^J + \text{Im} \mathcal{N}_{IJ} \tilde{F}^J] = 0,
\]  

(60)

which implies that

\[
\hat{\nabla}^2 \left[ \frac{M_{+I}}{\lambda^2} - \frac{M_{-I}}{\sigma^2} \right] = 0.
\]  

(61)

Therefore Bianchi identities and Maxwell’s equations imply

\[
\frac{M_{+I}}{\lambda^2} - \frac{M_{-I}}{\sigma^2} = H_I, \quad \frac{L^I_+}{\lambda^2} - \frac{L^I_-}{\sigma^2} = H^I.
\]  

(62)

These are the Euclidean version of the stabilization conditions. Also (62) implies

\[
\frac{1}{\alpha^2} = L^I_- H_I - M_{+I} H^I, \quad \frac{1}{\lambda^2} = L^I_+ H_I - M_{-I} H^I.
\]  

(63)

From the stabilization conditions (62) and (21) we obtain

\[
A = \frac{\lambda^2 \alpha^2}{2} (H_I dH^I - H^I dH_I) + d \log \frac{\lambda}{\alpha}.
\]  

(64)

Returning to (49), (64) implies that

\[
d\phi = \hat{\phi}[(H_I dH^I - H^I dH_I)].
\]  

(65)

This system of equations for Euclidean instantons can be analysed in a similar way to the analysis performed for the corresponding black hole solutions in the Lorentzian theory [31, 32].
3.2. Solutions with $\sigma = 0$, $\lambda \neq 0$ and $\lambda = 0$, $\sigma \neq 0$

The analysis of the linear system in (A.3) for the case of $\sigma = 0$, $\lambda \neq 0$, implies that the spin connections satisfy

$$\omega_{\mu, 12} = 0, \quad \omega_{\mu, 1\bar{1}} + \omega_{\mu, 2\bar{2}} = 0.$$  (66)

The conditions on the spin connection (66) imply that the (anti-self-dual) almost complex structures $I^1$, $I^2$, $I^3$ defined by

$$I^1 = (e^{12} + e^{1\bar{2}}), \quad I^2 = i(e^{12} - e^{1\bar{2}}), \quad I^3 = i(e^{1\bar{1}} + e^{2\bar{2}})$$  (67)

and which satisfy the algebra of the imaginary unit quaternions, are covariantly constant with respect to the Levi-Civita connection, i.e. the manifold is hyper-Kähler. The remaining conditions from (A.3) are

$$A = 2 \, \text{d} \, \log \lambda,$$  (68)

and

$$\text{Im} \, N_{JI} (F^I_{1\bar{1}} - F^I_{2\bar{2}}) L^J_- = 0,$$

$$\text{Im} \, N_{IJ} F^J_{1\bar{1}} L^I_- = 0.$$  (69)

It is also straightforward to show that (A.4) implies that

$$F^N_{1\bar{1}} + F^N_{2\bar{2}} = -2 \, \text{Im} \, N_{JI} L^J_- (F^I_{1\bar{1}} + F^I_{2\bar{2}}) L^N_-,$$

$$F^N_{1\bar{1}} = -2 \, \text{Im} \, N_{IJ} L^J_+ F^I_{2\bar{2}} L^N_-.$$  (70)

In the non-minimal theory, (70) implies that $F^I$ are self-dual. To see this, consider first $F^I_{1\bar{1}} + F^I_{2\bar{2}}$; the first condition in (70) implies that

$$F^I_{1\bar{1}} + F^I_{2\bar{2}} = h L^I_-$$  (71)

where

$$h = -2 \, \text{Im} \, N_{JI} L^J_- (F^I_{1\bar{1}} + F^I_{2\bar{2}})$$  (72)

and hence

$$h = -2h \, \text{Im} \, N_{JI} L^J_- = -2h \left( \frac{1}{2} - \frac{n}{4} \right).$$  (73)

where we have made use of (31). So, for the case of the non-minimal theory, $n \geq 1$, and this condition implies that $h = 0$, and hence

$$F^I_{1\bar{1}} + F^I_{2\bar{2}} = 0.$$  (74)

Similar reasoning applied to the second condition in (70) also implies that $F^I_{1\bar{1}} = 0$. It follows that $F^I \cdot I^1 = F^I \cdot I^2 = F^I \cdot I^3 = 0$, so as the hypercomplex structures are anti-self-dual, the $F^I$ must be self-dual.

Next, consider the Einstein field equations; as the manifold is hyper-Kähler there is no contribution from the curvature terms. Also, as the $F^I$ are self-dual, the contribution from the gauge field strengths also vanishes. So, on taking the trace of the Einstein equations, one obtains

$$g_{AB} \partial_x^A \partial^B = 0.$$  (75)

Assuming that the scalar manifold metric is positive definite, this implies that the scalars are constant, and (68) then implies that $\lambda$ is constant as well.

The analysis of the case $\lambda = 0$, $\sigma \neq 0$ proceeds in exactly the same fashion. In particular, the conditions on the geometry, and on the gauge field strengths, are identical to those of the $\lambda \neq 0$, $\sigma = 0$ case, modulo the interchange $L^I_+ \leftrightarrow L^I_-$ and (spacetime frame index) $1 \leftrightarrow \bar{1}$ throughout. Hence, it follows that the manifold is again hyper-Kähler, though now with self-dual hyper-complex structures, and the gauge field strengths must (for the non-minimal theory) be anti-self-dual. The scalars are again constant, as is $\sigma$. 


4. Summary

In this paper we have classified the instanton solutions admitting Killing spinors for Euclidean $N = 2$ supergravity theory coupled to vector multiplets. The half-supersymmetric solutions are either hyper-Kähler with constant scalars and self-dual (or anti-self-dual) gauge field strengths, or they are Euclidean analogues of the black hole solutions found in [21]. The stationary black holes of [21] were shown to be the unique solutions with time-like Killing vector, and admitting half of supersymmetry, in the systematic analysis of [23]. They can also be obtained using spinorial geometry techniques. Employing the results of [33], the time-like Killing spinors can be written in the canonical form

$$\epsilon = 1 + \beta e^2$$

which is obtained by gauge fixing the generic spinor of the form

$$\epsilon = \lambda 1 + \mu^i e^i + \sigma e^{12}$$

where $e^1, e^2$ are 1-forms on $\mathbb{R}^2$, and $i = 1, 2; e^{12} = e^1 \wedge e^2$. $\lambda, \mu^i$ and $\sigma$ are complex functions.

The analysis of the Killing spinor equations then gives the solutions

$$ds^2 = -|\beta|^2 ((dt + \sigma)^2 + \frac{1}{|\beta|^2} ((dx)^2 + (dy)^2 + (dz)^2))$$

$$d\sigma = -\ast_3 [(H_I dH^I - H^I dH_I)]$$

where $\beta$ is a complex $t$-independent function. The gauge field strengths and scalars are given by

$$F^I = -\hat{d} \left[ \left( \frac{L^I}{\beta} + \frac{\bar{L}^I}{\bar{\beta}} \right) (dt + \sigma) \right] - \ast_3 \hat{d} H^I$$

and

$$i \left( \frac{L^I}{\beta} - \frac{\bar{L}^I}{\bar{\beta}} \right) = H^I, \quad i \left( \frac{M_I}{\beta} - \frac{\bar{M}_I}{\bar{\beta}} \right) = H_I,$$

where $H^I$ and $H_I$ are harmonic functions. The equations (80) are the so-called generalized stabilization equations for the scalar fields.

Recently, instanton solutions of Einstein–Maxwell theory with non-zero cosmological constant were considered in [34]. In the analysis of the particular case with anti-self-dual Maxwell field, the field equations of supersymmetric solutions were shown to reduce to the Einstein–Weyl system in three dimensions [35] which is integrable by a twistor construction. Also, it was demonstrated that the Maxwell field anti-self-duality implies Weyl tensor anti-self-duality. Moreover, interesting relations were discovered between gravitational instantons and the $SU(\infty)$ Toda equation. Following on from this, the anti-self-duality condition on the Maxwell field was relaxed, and supersymmetric gravitational instanton solutions were classified using spinorial geometry techniques [36]. An important generalization of our work is the construction of Euclidean gauged supergravity theories and the analysis of their gravitational instanton solutions and their relations to Toda theories and integrable models. Lifting the solutions of this paper to higher dimensions as well as the analysis of the instanton moduli spaces are also left for future investigation.

Acknowledgment

JG is supported by the STFC grant, ST/1004874/1.
Appendix A. Linear systems

The (non-vanishing) actions of the Dirac matrices on the spinors are given by

\[ \Gamma_1 e_1 = \Gamma_2 e_2 = \sqrt{2} e_1, \]
\[ \Gamma_1 l = -\Gamma_2 e_{12} = \sqrt{2} e_1, \]
\[ \Gamma_1 e_2 = -\Gamma_2 e_{12} = \sqrt{2} e_2, \]
\[ \Gamma_1 l = \Gamma_1 e_{12} = \sqrt{2} e_2, \]

and one also obtains, for a 2-form \( T \),

\[ T^{ab} \Gamma_{ab} \epsilon_1 = 2(T^{22} + T^{11}) \epsilon_1 - 4T^{21} \epsilon_{12}, \]
\[ T^{ab} \Gamma_{ab} \epsilon_1 = 2(T^{22} - T^{11}) \epsilon_1 + 4T^{21} \epsilon_{12}, \]
\[ T^{ab} \Gamma_{ab} \epsilon_{12} = -2(T^{22} - T^{11}) \epsilon_2 - 4T^{21} \epsilon_1, \]
\[ T^{ab} \Gamma_{ab} \epsilon_{12} = -2(T^{22} + T^{11}) \epsilon_2 + 4T^{21} \epsilon_{12}. \]

The linear system obtained from (32) is

\[ \partial_1 \lambda = -\frac{\lambda}{2} (\omega_{1,22} + \omega_{1,11}) - \frac{i}{\sqrt{2}} (\text{Im} N)_{ij} L_i^j (F_{22}^j + F_{11}^j) - \frac{\lambda}{2} A_1 = 0, \]
\[ \partial_1 \sigma = -\frac{\lambda}{2} (\omega_{1,22} - \omega_{1,11}) + \frac{\sigma}{2} A_1 = 0, \]
\[ \partial_2 \lambda = -\frac{\lambda}{2} (\omega_{2,22} + \omega_{2,11} + \lambda A_2) = 0, \]
\[ \partial_2 \sigma = -\frac{\lambda}{2} (\omega_{2,22} - \omega_{2,11} - \lambda A_2) = 0, \]
\[ \partial_3 \lambda = -\frac{\lambda}{2} (\omega_{3,1,22} + \omega_{3,2,11} + \sqrt{2i} \lambda (\text{Im} N)_{ij} L_i^j F_{22}^j), \]
\[ \partial_3 \sigma = -\frac{\lambda}{2} (\omega_{3,1,22} - \omega_{3,2,11}) + \sqrt{2i} \lambda (\text{Im} N)_{ij} L_i^j F_{22}^j, \]
\[ \lambda A_{1,21} = 0, \]
\[ \lambda A_{2,21} = 0, \]
\[ \sigma A_{1,2i} = 0, \]
\[ \sigma A_{2,2i} = 0, \]
\[ \sigma A_{1,2i} - \sqrt{2i} \lambda (\text{Im} N)_{ij} F_{23}^j L_i^j = 0, \]
\[ \sigma A_{2,2i} - \frac{i}{\sqrt{2}} (\text{Im} N)_{ij} (F_{22}^j - F_{11}^j) L_i^j = 0, \]
\[ \lambda A_{2,1i} + \sqrt{2} i \sigma (\text{Im} N)_{ij} F_{21}^j L_i^j = 0, \]
\[ \lambda A_{3,2i} + \frac{\sigma}{\sqrt{2}} i (\text{Im} N)_{ij} (F_{22}^j + F_{11}^j) L_i^j = 0. \]

The linear system obtained from (33) is

\[ -i \lambda \text{ Im} N_{ij} (F_{11}^j + F_{22}^j) (\text{Im} (D_g L^j) g^{A\bar{B}} + \text{Re} (D_g L^j) g^{A\bar{B}}) + \sqrt{2} \sigma \partial_1 (\text{Re} z^4 + \text{Im} z^4) = 0, \]
\[ i \sigma \text{ Im} N_{ij} (F_{11}^j - F_{22}^j) (\text{Im} (D_g L^j) g^{A\bar{B}} - \text{Re} (D_g L^j) g^{A\bar{B}}) + \sqrt{2} \lambda \partial_1 (\text{Re} z^4 - \text{Im} z^4) = 0, \]
\[2\bar{\sigma}\Im\mathcal{N}_{ij}F_{2}^{I}(\Im(D_g\vec{L}^I_{g^A}) - \Re(D_g\vec{L}^I_{g^A})) + \sqrt{2}\lambda\bar{\sigma}_{2}(\Re\varepsilon^{A} - \Im\varepsilon^{A}) = 0,
\]
\[2\bar{\lambda}\Im\mathcal{N}_{ij}F_{2}^{I}(\Im(D_g\vec{L}^I_{g^A}) + \Re(D_g\vec{L}^I_{g^A})) - \sqrt{2}\bar{\sigma}_{2}(\Re\varepsilon^{A} + \Im\varepsilon^{A}) = 0. \]  

(A.4)

References

[1] Vandoren S and van Nieuwenhuizen P 2008 Lectures on instantons arXiv:0802.1862 [hep-th]
[2] Atiyah M F and Singer I M 1963 Bull. Am. Math. Soc. 69 322
[3] Belavin A A, Polyakov A M, Schwarz A S and Tyupkin Y S 1975 Phys. Lett. B 59 85
[4] Eguchi T and Hanson A J 1978 J. Phys. A 24 7249
[5] Atiyah M F and Singer I M 1963 Bull. Am. Math. Soc. 69 322
[6] Belavin A A, Polyakov A M, Schwarz A S and Tyupkin Y S 1975 Phys. Lett. B 59 85
[30] Bryant R 2000 Sémin. Congr. 4 53
[31] Denef F 2000 J. High Energy Phys. JHEP08(2000)050
[32] Bates B and Denef F 2011 J. High Energy Phys. JHEP11(2011)127
[33] Grover J, Gutowski J B and Sabra W A 2007 Class. Quantum Grav. 24 3259
[34] Dunajski M, Gutowski J B, Sabra W A and Tod P 2011 Class. Quantum Grav. 28 025007
[35] Hitchin N 1982 Complex manifolds and Einstein’s equations Twistor Geometry and Non-Linear Systems (Springer Lecture Notes in Mathematics vol 970) ed D H Doebner and T D Palev (Berlin: Springer) pp 73–99
Jones P and Tod K P 1985 Class. Quantum Grav. 2 565
Dunajski M 2009 Solitons, Instantons & Twisters (Oxford Graduate Texts in Mathematics vol 19) (Oxford: Oxford University Press)
[36] Dunajski M, Gutowski J B, Sabra W A and Tod P 2011 J. High Energy Phys. JHEP03(2011)131