A simpler proof of a Katsurada’s theorem and rapidly converging series for $\zeta(2n+1)$ and $\beta(2n)$

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Abstract In a recent work on Euler-type formulae for even Dirichlet beta values, i.e. $\beta(2n)$, I have derived an exact closed-form expression for a class of zeta series. From this result, I have conjectured closed-form summations for two families of zeta series. Here in this work, I begin by using a known formula by Wilton to prove those conjectures. As example of applications, some special cases are explored, yielding rapidly converging series representations for the Apéry constant, $\zeta(3)$, and the Catalan constant, $G = \beta(2)$. Interestingly, our series for $\zeta(3)$ converges faster than that used by Apéry in his irrationality proof (1978). Also, our series for $G$ converges faster than a celebrated one discovered by Ramanujan (1915). At last, I present a simpler, more direct proof for a recent theorem by Katsurada which generalizes the above results.

Keywords Riemann zeta function · Dirichlet beta function · Zeta series · Clausen function

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1 Introduction

The Riemann zeta function is defined, for $\Re(s) > 1$, as $\zeta(s) := \sum_{k=1}^{\infty} 1/k^s$. This function has a singularity (in fact, a simple pole) at $s = 1$, which corresponds to the divergence of the harmonic series. For real values of $s$, $s > 1$, the series converges to a real number between 1 and 2, according to the integral test. For positive integer values of $s$, $s > 1$, one has a well-known formula by F.M.S. Lima
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Euler (see Ref. [4] and references therein):

\[ \zeta(2n) = (-1)^{n-1} \frac{2^{2n-1} B_{2n}}{(2n)!} \pi^{2n}, \]  

(1)

\( n \) being a positive integer. Here, \( B_n \) are Bernoulli numbers, i.e. the rational coefficients of \( z^n/n! \) in the Taylor series expansion of \( z/(e^z - 1) \), \( 0 < |z| < 2\pi \). For \( \zeta(2n + 1) \), on the other hand, no analogous expression is currently known. This scenario has a ‘reverse’ counterpart on the values of the Dirichlet beta function, defined as \( \beta(s) := \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^s} \), \( s > 0 \), in the sense that, for integer values of \( s \), the following analogue of Eq. (1) is known:

\[ \beta(2n + 1) = (-1)^n \frac{E_{2n}}{2^{2n+2}(2n)!} \pi^{2n+1}, \]  

(2)

where \( E_n \) are Euler numbers, i.e. the (integer) coefficients of \( z^n/n! \) in the Taylor expansion of sech(\( z \)), \( |z| < \pi/2 \). For \( \beta(2) \), no analogous expression is known, not even for \( \beta(1) \), known as the Catalan’s constant \( G \). Some progress in this direction was reached by Kölbig (1996), who proved that

\[ \beta(2) = \frac{\psi^{(1)}(\frac{1}{4})}{2(2-1)! 4^{2n-1}} - \frac{\zeta(2) B_2}{2} \pi^{2n}, \]  

(2)

\( \psi^{(n)}(x) \) is the polygamma function, i.e. the \( n \)-th derivative of \( \psi(x) \), the digamma function. Although this identity resembles Euler’s formula, the arithmetic nature of \( \psi^{(2n-1)}(\frac{1}{4}) \) is currently unknown.

In a very recent paper, I have succeeded in applying the Dancs and He series expansion method, as introduced in Ref. [3], to find similar formulae for \( \beta(2n) \) [7]. I could then prove that

\[ (-1)^n \frac{2^{2n-2}}{\pi^{2n-1}} \beta(2n) + \frac{n}{(2n)!} \ln 2 + \frac{1}{2} \sum_{m=1}^{n-1} (-1)^m \frac{2^{2m-1} - 1}{\pi^{2m}} \zeta(2m + 1) \]  

(3)

Since both

\[ \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2k+2n-1)!} \frac{\zeta(2k)}{2^{2k}} \]  

(4a)

and

\[ \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2k+2n-1)!} \frac{\zeta(2k)}{4^{2k}} \]  

(4b)

1 Note that this formula remains valid for \( n = 0 \), since \( E_0 = 1 \) and \( \beta(1) = \pi/4 \).

2 The function \( \psi(x) \), in turn, is defined as the logarithm derivative of \( \Gamma(x) \), the classical gamma function.
converge absolutely, the zeta series in Eq. (3) equals the difference of these individual series. However, the best I could do there in Ref. [7] was to investigate the pattern of the analytical results found for the first series, Eq. (4a), for small values of \( n \). This did lead me to conjecture a formula for its summation which should be valid for any positive integer \( n \).

Here in this work, I make use of a classical Wilton’s formula to prove the above mentioned conjecture. The proof of another conjecture, involving the zeta series in Eq. (4b), follows from Eq. (3). Finally, on aiming at a generalization of these formulae I substitute the fractions \( (1/2)^{2k} \) and \( (1/4)^{2k} \) by \( x^{2k} \), \( x \) being any non-null real number with \( |x| \leq 1 \). This has led me to develop a simpler, more direct proof of a theorem by Katsurada (1999) [5].

2 Zeta series for \( \zeta(2n+1) \) and \( \beta(2n) \)

The proof of the first conjecture in Ref. [7], involving the zeta series in Eq. (4a), above, follows from a classical result by Wilton for a rapidly convergent series representation for \( \zeta(2n+1) \) [13]. Hereafter, we define \( H_n := \sum_{k=1}^{n} 1/k \) as the \( n \)-th harmonic number.

**Lemma 1 (Wilton’s formula)** Let \( n \) be a positive integer. Then

\[
\frac{\zeta(2n+1)}{(-1)^{n-1} \pi^{2n}} = H_{2n+1} - \ln \pi + \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n-2k+1)!} \frac{\zeta(2k+1)}{\pi^{2k}} + \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2n+2k+1)!} \frac{\zeta(2k)}{2^{2k-1}}.
\]

For a skeleton of the proof, see the original work of Wilton [13]. For a more complete proof, see Sec. 4.2 (in particular, pp. 412–413) of Ref. [12], a systematic collection of zeta series recently published by Srivastava and Choi.

This lemma allows us to prove the first conjecture of Ref. [7], namely that in its Eq. (29).

**Theorem 1 (First zeta series)** Let \( n \) be a positive integer. Then

\[
\sum_{k=1}^{\infty} \frac{(2k-1)!}{(2k+2n-1)!} \frac{\zeta(2k)}{2^{2k}} = \frac{1}{2} \left( \frac{\ln \pi - H_{2n-1}}{(2n-1)!} + \sum_{m=1}^{n-1} (-1)^{m+1} \frac{\zeta(2m+1)}{\pi^{2m} (2n-2m-1)!} \right).
\]

**Proof** From Lemma 1 we know that

\[
\frac{(-1)^n \zeta(2n+1)}{\pi^{2n}} = \frac{\ln \pi - H_{2n+1}}{(2n+1)!} - \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n-2k+1)!} \frac{\zeta(2k+1)}{\pi^{2k}} - 2 \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2n+2k+1)!} \frac{\zeta(2k)}{2^{2k}}.
\]

By isolating the last term, one has

\[
2 \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2n+2k+1)!} \frac{\zeta(2k)}{2^{2k}} = \frac{\ln \pi - H_{2n+1}}{(2n+1)!} - \sum_{k=1}^{n} \frac{(-1)^k}{(2n-2k+1)!} \frac{\zeta(2k+1)}{\pi^{2k}}.
\]
By substituting $n = \ell - 1$ in the above equation, one finds
\[
2 \sum_{k=1}^{\infty} \frac{(2k - 1)! \zeta(2k)}{(2k + 2\ell - 1)!} \cdot \frac{2}{2^{2k}} = \frac{\ln \pi - H_{2\ell - 1}}{(2\ell - 1)!} - \sum_{k=1}^{\ell - 1} \frac{(-1)^k}{(2\ell - 2k)!} \frac{\zeta(2k + 1)}{\pi^{2k}}.
\]
A division by 2 completes the proof. \(\Box\)

For instance, on putting $n = 2$ in this theorem we get a rapidly converging series representation for $\zeta(3)$, namely
\[
\zeta(3) = \pi^2 \sum_{k=1}^{\infty} \frac{\zeta(2k)}{k(2k + 1)(2k + 2)(2k + 3) 2^{2k}} + \frac{11}{36} \pi^2 - \frac{1}{6} \pi^2 \ln \pi. \tag{5}
\]

This formula converges to $\zeta(3)$ much faster than $\sum_{k=1}^{\infty} \frac{1}{k^3}$ and even than \(\sum_{k\geq 1} \frac{(-1)^{k+1}}{k^3} \zeta\left(\frac{3}{2} k\right)\), a central binomial series used by Apéry [1]. Numerical computation using Mathematica shows that only ten terms of the above zeta series are enough for a ten decimal places accuracy.

Now we can use Eq. (3) to prove the other conjecture raised in Ref. [7], namely that in its Eq. (30).

**Theorem 2 (Second zeta series)** Let $n$ be a positive integer. Then
\[
\sum_{k=1}^{\infty} \frac{(2k - 1)! \zeta(2k)}{(2k + 2n - 1)!} \cdot \left(\frac{1}{4}\right)^{2k} = \frac{1}{2} \left[ \frac{\ln (\pi/2) - H_{2n - 1}}{(2n - 1)!} - (-1)^n \left(\frac{2}{\pi}\right)^{2n-1} \beta(2n) \right.
\]
\[
- \left. \sum_{m=1}^{n-1} (-1)^m \left(\frac{2}{\pi}\right)^{2m} \frac{\zeta(2m + 1)}{(2n - 2m - 1)!} \right].
\]

**Proof** From Eq. (3), we know that
\[
\sum_{k=1}^{\infty} \frac{(2k - 1)! \zeta(2k)}{(2k + 2n - 1)!} \cdot \left(\frac{1}{4}\right)^{2k} = \sum_{k=1}^{\infty} \frac{(2k - 1)! \zeta(2k)}{(2k + 2n - 1)!} \cdot \left(\frac{1}{4^{2k}}\right)
\]
\[
= (-1)^n \frac{\beta(2n)}{\pi^{2n-1}} + \frac{n}{(2n)!} \ln 2 + \frac{1}{2} \sum_{m=1}^{n-1} (-1)^m \frac{\pi^{2m} - 1}{\pi^{2m} (2n - 2m - 1)!} \frac{\zeta(2m + 1)}{(2n - 2m - 1)!}.
\]

On substituting the first zeta series at the left-hand side by the result proved in Theorem 1, one has
\[
\frac{\ln \pi - H_{2n - 1}}{(2n - 1)!} - \sum_{m=1}^{n-1} (-1)^m \frac{\zeta(2m + 1)}{\pi^{2m} (2n - 2m - 1)!} - 2 \sum_{k=1}^{\infty} \frac{(2k - 1)! \zeta(2k)}{(2k + 2n - 1)!} \cdot \left(\frac{1}{4^{2k}}\right)
\]
\[
= (-1)^n \left(\frac{2}{\pi}\right)^{2n-1} \beta(2n) + \frac{\ln 2}{(2n - 1)!} + \sum_{m=1}^{n-1} (-1)^m \frac{\pi^{2m} - 1}{\pi^{2m} (2n - 2m - 1)!} \frac{\zeta(2m + 1)}{(2n - 2m - 1)!}.
\]
By isolating the remaining zeta series, one finds

\[ 2 \sum_{k=1}^{\infty} \frac{(2k-1)! \zeta(2k)}{(2k + 2n - 1)!} \left( \frac{1}{4^{2k}} \right) = \frac{\ln(\pi/2) - H_{2n-1}}{(2n-1)!} - (-1)^n \left( \frac{2}{\pi} \right)^{2n-1} \beta(2n) \]

\[ - \sum_{m=1}^{n-1} (-1)^m \left( \frac{2}{\pi} \right)^{2m} \zeta(2m+1) \frac{\zeta(2m+1)}{(2n-2m-1)!}. \]

A division by 2 completes the proof. \( \square \)

On putting \( n = 1 \) in Theorem 2 we get a rapidly converging series representation for \( \beta(2) = G \), namely

\[ G = \pi \sum_{k=1}^{\infty} \zeta(2k) \frac{1}{2k(2k+1)} \frac{1}{4^{2k}} - \frac{\pi}{2} \ln\left( \frac{\pi}{2} \right) + \frac{\pi}{2}. \]  

(6)

This formula converges much faster than \( \sum_{k=0}^{\infty} (-1)^k/(2k + 1)^2 \) and even faster than \( \frac{\pi}{8} \ln(2 + \sqrt{3}) \) + \( \frac{3}{8} \sum_{n=0}^{\infty} 1/[(2n + 1)^2(\binom{2n}{n})] \), a rapidly converging central binomial series discovered by Ramanujan [9]. Numerically, only six terms of the above zeta series are enough for a result accurate to ten decimal places. After an extensive search for similar zeta series in literature, I have found a formula by Srivastava and Tsumura (2000) in Ref. [11] (see also Ref. [12], p. 421, Eq. (30)). In fact, this formula could be taken into account for an independent proof of our Theorem 2 after some simple manipulations, as the reader can easily check.

On investigating the substitution of \((1/2)^{2k}\) and \((1/4)^{2k}\) by \(x^{2k}\), \(x\) being any non-null real number with \(|x| \leq 1\), in the zeta series treated in the previous theorems, I have found the following general result.

**Theorem 3 (Generalization)** Let \( n \) be a positive integer and \( x \) be any real number with \( 0 < |x| \leq 1 \). Then

\[ \zeta(2n+1) - \frac{1}{2\pi x} \sum_{\ell=1}^{\infty} \frac{\sin(2\pi \ell x)}{\ell^{2n+2}} = (-1)^{n-1} (2\pi x)^{2n} \left[ \frac{H_{2n+1} - \ln(2\pi |x|)}{(2n+1)!} \right] \]

\[ + \sum_{k=1}^{n-1} (-1)^k \frac{\zeta(2k+1)}{(2n-2k-1)! (2\pi x)^{2k}} + 2 \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \zeta(2k)}{(2n+2k+1)! x^{2k}}. \]

**Proof** From the Euler’s product formula for the sine function, we know that, for all non-null real \( z \) with \(|z| \leq 1\), the following identity holds:

\[ \frac{\sin(\pi z)}{\pi z} = \prod_{n=1}^{\infty} \left( 1 - \frac{z^2}{n^2} \right). \]  

(7)

3 For this, it will be useful to know that \( \zeta(s, a) := \sum_{k=0}^{\infty} \frac{1}{(k+a)^s} \) is the Hurwitz zeta function, for which it is well-known that \( \zeta(n + 1, a) = (-1)^{n+1} \psi^{(n)}(a)/n! \) (see, e.g., Eq. (25.11.12) of Ref. [13]). Then \( \zeta(2m, \frac{1}{2}) = \psi^{(2m-1)}(\frac{1}{2})/(2m-1)! \) and, from our Eq. (8), it follows that \( \psi^{(2m-1)}(\frac{1}{2})/(2m-1)! = 2^{4m-2} \beta(2m) + (-1)^{m-1} 2^{4m-2} (22m - 1) B_{2m} \pi^{2m}/(2m)! \), where \( |B_{2m}| \) was substituted by \((-1)^{m-1} B_{2m} \).
On taking the logarithm on both sides, we have

$$
\ln \left[ \sin \left( \frac{\pi z}{2} \right) \right] = \sum_{n=1}^{\infty} \ln \left( 1 - \frac{z^2}{n^2} \right) = -\sum_{n=1}^{\infty} \left( \sum_{k=1}^{\infty} \frac{z^{2k}}{k n^{2k}} \right),
$$

where the logarithm was expanded in a Taylor series in the last step. This implies that

$$
\ln \left[ \frac{2 \sin \left( \frac{\pi z}{2} \right)}{2 \pi z} \right] = -\sum_{k=1}^{\infty} \left( \sum_{n=1}^{\infty} \frac{1}{n^{2k}} \right) \frac{z^{2k}}{k} = -\sum_{k=1}^{\infty} \frac{z^{2k}}{k} \zeta(2k)
$$

(9)

and then

$$
\ln |2 \sin (\pi z)| - \ln |2 \pi z| = -\sum_{k=1}^{\infty} \frac{z^{2k}}{k} \zeta(2k).
$$

(10)

On multiplying both sides by $(x - z)^{2n}$, $n$ being a positive integer, and integrating from 0 to $x$, $x$ being any non-null real in the interval $[-1, 1]$, we have

$$
\int_{0}^{x} (x - z)^{2n} \ln |2 \sin (\pi z)| \, dz - \int_{0}^{x} (x - z)^{2n} \ln |2 \pi z| \, dz
$$

$$
= -\int_{0}^{x} (x - z)^{2n} \sum_{k=1}^{\infty} \frac{z^{2k}}{k} \zeta(2k) \, dz.
$$

(11)

Let us solve each definite integral carefully.

The first integral in Eq. (11) can be expanded in a trigonometric series as follows. Since, for all $\theta \in (0, 2\pi)$$^3$

$$
\ln \left[ 2 \sin \left( \frac{\theta}{2} \right) \right] = -\sum_{k=1}^{\infty} \frac{\cos (k \theta)}{k},
$$

(12)

then, on substituting $z = \theta/(2\pi)$ in the integral, one finds

$$
I_n(x) := \int_{0}^{x} (x - z)^{2n} \ln |2 \sin (\pi z)| \, dz
$$

$$
= -\frac{1}{(2\pi)^{2n+1}} \int_{0}^{2\pi x} (2\pi x - \theta)^{2n} \ln \left| 2 \sin \left( \frac{\theta}{2} \right) \right| \, d\theta
$$

$$
= -\frac{1}{(2\pi)^{2n+1}} \int_{0}^{2\pi x} (2\pi x - \theta)^{2n} \sum_{k=1}^{\infty} \frac{\cos (k \theta)}{k} \, d\theta
$$

$$
= -\frac{1}{(2\pi)^{2n+1}} \int_{0}^{2\pi x} (2\pi x - \theta)^{2n} d\left( \sum_{k=1}^{\infty} \frac{\sin (k \theta)}{k^2} \right).
$$

(13)

$^3$ This is a well-known Fourier series expansion.
On integrating by parts, one has

\[ I_n(x) = \frac{2^n}{(2\pi)^{2n+1}} \int_0^{2\pi x} (2\pi x - \theta)^{2n-1} d \left( \sum_{k=1}^{\infty} \frac{\cos (k \theta)}{k^3} \right) \]

\[ = -\frac{2^n}{(2\pi)^2} x^{2n-1} \zeta(3) + \frac{(2n)!}{(2\pi)^{2n+1} (2n-2)!} \int_0^{2\pi x} (2\pi x - \theta)^{2n-2} d \left( \sum_{k=1}^{\infty} \frac{\sin (k \theta)}{k^4} \right). \quad (14) \]

On integrating by parts again and again, we find, after some algebra, that

\[ I_n(x) = \sum_{j=1}^{n} \frac{(-1)^j (2n)! \zeta(2j+1)}{(2\pi)^{2j} (2n+1-2j)!} x^{2n+1-2j} + \frac{(-1)^{n-1} (2n)!}{(2\pi)^{2n+1}} \sum_{k=1}^{\infty} \frac{\sin (k \pi x)}{k^{2n+2}}. \quad (15) \]

The second integral in Eq. (11) is readily solved by noting that

\[ \int_0^x (x-z)^{2n} \ln |2\pi z| \, dz = -\frac{1}{2n+1} \int_0^x \ln |2\pi z| \, d [(x-z)^{2n+1} - x^{2n+1}] \]

\[ = \frac{x^{2n+1}}{2n+1} \left[ \ln |2\pi x| + \sum_{\ell=1}^{2n+1} \frac{(-1)^\ell}{\ell} \left( \frac{2n+1}{\ell} \right) \right]. \quad (16) \]

The third integral, i.e. the one at the right-hand side of Eq. (11), can be written in the form of a zeta series on integrating it by parts directly. This yields

\[ \int_0^x (x-z)^{2n} \sum_{k=1}^{\infty} \frac{2k}{k} \zeta(2k) \, dz = 2(2n)! \sum_{k=1}^{\infty} \frac{(2k-1)! \zeta(2k) x^{2k+2n+1}}{(2k+2n+1)!}. \quad (17) \]

Finally, on substituting the results in Eqs. (15), (16), and (17) on the integrals in Eq. (11), we find

\[ \sum_{j=1}^{n} \frac{(-1)^j (2n)! \zeta(2j+1)}{(2\pi)^{2j} (2n+1-2j)!} x^{2j} + \frac{(-1)^{n-1} (2n)!}{(2\pi x)^{2n+1}} \sum_{k=1}^{\infty} \frac{\sin (k \pi x)}{k^{2n+2}} \]

\[ -\frac{1}{2n+1} \left[ \ln |2\pi x| + \sum_{\ell=1}^{2n+1} \frac{(-1)^\ell}{\ell} \left( \frac{2n+1}{\ell} \right) \right] = -2(2n)! \sum_{k=1}^{\infty} \frac{(2k-1)! \zeta(2k) x^{2k}}{(2k+2n+1)!}. \quad (18) \]

On dividing both sides by \[-(2n)!\], we have

\[ 2 \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2k+2n+1)!} \zeta(2k) x^{2k} = -\sum_{j=1}^{n} \frac{(-1)^j \zeta(2j+1)}{(2\pi x)^{2j} (2n+1-2j)!} + \frac{\ln (2 \pi |x|) - H_{2n+1}}{(2n+1)!} \]

\[ + \frac{(-1)^n}{(2\pi x)^{2n+1}} \sum_{k=1}^{\infty} \frac{\sin (2k \pi x)}{k^{2n+2}}, \quad (19) \]

where we have made use of the binomial sum \( H_{2n+1} = -\sum_{k=1}^{2n+1} (-1)^k \left( \begin{array}{c} 2n+1 \\ k \end{array} \right \right) \),

which is easily proved by induction on \( n \). On extracting the last term of the sum involving odd zeta values (i.e., that for \( j = n \)) we bring up the desired result. \( \square \)
Our proof of Theorem 3 above, has allowed us to detect some typos in Theorem 2 of Ref. [5], as well as some mistakes in its proof, which is based upon the Mellin transform technique. In fact, the formula as printed in Katsurada’s paper cannot be correct because as \(x \to 0\) the left-hand side approaches \(2\zeta(2n+1)\) whereas the right-hand side approaches zero. Unfortunately, that incorrect formula is reproduced on p. 442 of Ref. [12], which is currently the main reference on zeta series in literature. Our Theorem 3 corrects both the absence of a modulus in the argument of the logarithm and the ‘+’ sign preceding the sine series in Eq. (1.6) of Ref. [5].

Interestingly, the formula in our Theorem 3 can be reduced to a more appropriate form for both symbolic and numerical computations. This is easily obtained by noting that
\[
\sum_{\ell=1}^{\infty} \sin \left( \frac{2\pi x \ell}{\ell^{2n}} \right) / \ell^{2n} = C_{2n}(2\pi x)
\]
where \(C_{2n}(\theta) := \Im\{\Li_{2n}(e^{i\theta})\}\) is the Clausen function of order \(2n\). By substituting \(n = m - 1\) and \(x = \theta/(2\pi)\) in Theorem 3, we readily find
\[
\sum_{k=1}^{\infty} \frac{(2k-1)! \zeta(2k)}{(2m+2k-1)!} \left( \frac{\theta}{2\pi} \right)^{2k} = \frac{1}{2} \left[ \frac{\ln|\theta| - H_{2m-1}}{(2m-1)!} - (-1)^{m} \frac{Cl_{2m}(\theta)}{\theta^{2m-1}} \right] - \sum_{k=1}^{m-1} (-1)^{k} \frac{\zeta(2k+1)}{(2m-2k-1)!} \left( \frac{\theta}{2\pi} \right)^{2k},
\]
which holds for any non-null real \(\theta\) with \(|\theta| \leq 2\pi\). Advantageously, this form remains valid for all positive integer values of \(m\), as long as the sum at the right-hand side is taken as null when \(m = 1\), as usual. Since \(Cl_{2m}(\pi) = 0\) and \(Cl_{2m}(\pi/2) = \beta(2m)\), Eq. (20) allows for prompt proofs of Theorems 1 and 2, respectively, showing that they are special cases of Theorem 3, as expected. This is indeed the case for a number of rapidly convergent zeta series in literature, as e.g. some of the zeta series given in Refs. [2,3,7,10,13] and many zeta series presented in Chaps. 3 and 4 of Ref. [12]. This reflects the generality of our Theorem 3. Equation (20) can thus be viewed as a source of rapidly converging zeta series for odd zeta values and Clausen functions of even order.

For instance, on taking \(\theta = \pi/3\) in Eq. (20), one finds
\[
2 \sum_{k=1}^{\infty} \frac{(2k-1)! \zeta(2k)}{(2m+2k-1)!} \frac{\zeta(2k)}{6^{2k}} = \frac{\ln(\pi/3) - H_{2m-1}}{(2m-1)!} - (-1)^{m} \frac{Cl_{2m}(\pi/3)}{(\pi/3)^{2m-1}}
\]
which reduces to
\[
\sum_{k=1}^{\infty} \frac{\zeta(2k)}{k (2k+1) (2k+2) (2k+3)} \frac{1}{6^{2k}} = \frac{1}{6} \ln\left( \frac{\pi}{3} \right) - \frac{11}{36} - \frac{27}{\pi^2} Cl_{4}\left( \frac{\pi}{3} \right) + \frac{9}{\pi^2} \zeta(3).
\]
Similarly, for $\theta = \pi/4$ one has
\[
\sum_{k=1}^{\infty} \frac{\zeta(2k)}{k(2k+1)} \frac{1}{8^{2k}} = \ln\left(\frac{\pi}{4}\right) - 1 + 4 \frac{2^{2k}}{\pi \text{Cl}_2\left(\frac{\pi}{4}\right)}
\]
for $m = 1$ and
\[
\sum_{k=1}^{\infty} \frac{\zeta(2k)}{k(2k+1)(2k+2)(2k+3)} \frac{1}{8^{2k}} = \frac{1}{6} \ln\left(\frac{\pi}{4}\right) - \frac{11}{36} \frac{64}{\pi^3} \text{Cl}_4\left(\frac{\pi}{4}\right) + \frac{16}{\pi^2} \zeta(3)
\]
for $m = 2$. I could not find these formulae explicitly in literature.

### 3 Rates of convergence

All zeta series investigated here belong to the class embraced by Theorem 3. Their rate of convergence can be analyzed as follows. For convenience, denote by $S_k$ the summand of the zeta series at the right-hand side of the formula in our Theorem 3. By applying Stirling’s formula, namely $k! \sim \left(\frac{k}{e}\right)^k \sqrt{2\pi k}$, and noting that $\zeta(2k) \to 1$ as $k \to \infty$, one finds
\[
S_k \sim \frac{(2k-1)2^{2k-1}}{(2n+2k+1)2^{2n+2k+1}} e^{2n+2k} \frac{\sqrt{2k-1}}{\sqrt{2n+2k+1}} x^{2k}
\]
\[
\sim \frac{1}{e^{2n+2}} \frac{1}{\sqrt{1+\frac{2n+2}{2k}} \sqrt{\frac{2k+2n+1}{2}} x^{2k}}
\]
\[
\sim \frac{1}{(2k+2n+1)^{2n+2}} x^{2k} \sim \frac{1}{(2k)^{2n+2}} x^{2k},
\]
where we have made use of $\lim_{y \to \infty} (1 + \alpha/y)^y = e^\alpha$ and the binomial approximation $(1 + y)^\alpha \approx 1 + \alpha y$. Therefore,
\[
S_k = O\left(\frac{x^{2k}}{(2k)^{2n+2}}\right) \quad (k \to \infty, n \in \mathbb{N}),
\]
which is valid for any real $x$ with $|x| \leq 1$. This allows for a direct comparison of rates of convergence for the zeta series investigated here in this paper. For instance, the zeta series in Wilton’s formula has $x = \frac{1}{2}$, hence $S_k = O(2^{-2k}(2k)^{-2n-2}) = O(2^{-2k-2n-2} k^{-2n-2}) = O(2^{-2k} k^{-2n-2})$, whereas the zeta series in our Theorem 2 has $x = \frac{1}{4}$, hence $S_k = O(4^{-2k} (2k)^{-2n-2}) = O(4^{-2k-n-1} k^{-2n-2}) = O(4^{-2k} k^{-2n-2})$. This is why the latter converges somewhat faster than the former.
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