On the Inf-Sup Stability of Crouzeix-Raviart Stokes Elements in 3D

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Abstract

We consider non-conforming discretizations of the stationary Stokes equation in three spatial dimensions by Crouzeix-Raviart type elements. The original definition in the seminal paper by M. Crouzeix and P.-A. Raviart in 1973 is implicit and also contains substantial freedom for a concrete choice.

In this paper, we introduce basic Crouzeix-Raviart basis functions in 3D in analogy to the 2D case in a fully explicit way. We prove that this basic Crouzeix-Raviart element for the Stokes equation is inf-sup stable for polynomial degree \( k = 2 \) (quadratic velocity approximation). We identify spurious pressure modes for the conforming \((k, k - 1)\) 3D Stokes element and show that these are eliminated by using the basic Crouzeix-Raviart space.

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1 Introduction

In this paper, we investigate the discretization of the stationary Stokes problem by Crouzeix-Raviart elements in three dimensions. They were introduced in the seminal paper [11] in 1973 by Crouzeix and Raviart with the goal to obtain a stable discretization of the Stokes equation with relatively few unknowns. They can be considered as an non-conforming enrichment of conforming finite elements of polynomial degree \( k \) for the velocity and discontinuous pressure of degree \( k - 1 \). It is well known that the conforming pair of finite elements can be unstable; for two dimensions the proof of the inf-sup stability of Crouzeix-Raviart discretizations of

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general order $k$ has been developed over the last 50 years, the inf-sup stability for $k = 1$ has been proved in [11] and only recently the last open case $k = 3$, has been proved in [6]. We mention the papers [15], [19], [10], [2], [17], [5] which contain essential milestones for the proof of the inf-sup stability for general polynomial degree. There is a vast of literature on various further aspects of Crouzeix-Raviart elements; we omit to present a comprehensive review here but refer to the excellent overview article [4] instead.

In contrast to the analysis in 2D, the development of Crouzeix-Raviart elements for the Stokes equation in 3D is still in its infancy. The original definition in [11] is implicit: a finite element space is a Crouzeix-Raviart space if it satisfies certain jump conditions across the element interfaces. This leaves significant freedom for a concrete definition. In particular, the important question “What is a minimal Crouzeix-Raviart space?” i.e., “What is a Crouzeix-Raviart space with minimal dimension so that the inf-sup condition is satisfied?”, is completely open. For practical purpose, it is an even stronger obstruction that the basis functions for such a minimal Crouzeix-Raviart space are unknown. For quadratic Crouzeix-Raviart elements, an explicit basis has been introduced in [14]. In [8], a spanning set of functions is presented for a maximal Crouzeix-Raviart space of any order $k \in \mathbb{N}$ which allows for a local basis. However, the question of linear independence is subtle, in particular, the definition of a basis for a minimal Crouzeix-Raviart space.

In this paper, we make a step in these directions. After formulating the Stokes problem on the continuous level and introducing non-conforming finite element discretizations in Section 2, we define in Section 3 the basic Crouzeix-Raviart space in three dimensions for any polynomial degree $k \geq 1$. We call them “basic” because they are in full analogy as in the 2D case: for odd polynomial degree $k$ there is one and only one scalar non-conforming Crouzeix-Raviart function per inner facet (the analogue of a triangle edge in 2D) supported on the two adjacent simplices. As in the 2D case, this function can be expressed by a certain orthogonal polynomial composed with barycentric coordinates; for even polynomial degree $k$ there is one and only one scalar Crouzeix-Raviart function per tetrahedron (in analogy to one Crouzeix-Raviart function per triangle in 2D) supported on this tetrahedron. As in the 2D case, these functions can be expressed by an orthogonal polynomial composed with barycentric coordinates. The basic Crouzeix-Raviart spaces can be defined conceptually in the same manner also in higher dimensions. We have postponed the technical derivation of a fully explicit representation to Appendix A due to the lack of practical applications in spatial dimension larger than three. For the Stokes problem and the corresponding basic Crouzeix-Raviart elements, we prove the following results.

a) In Section 4, we show that the basic Crouzeix-Raviart space for $k = 2$ on simplicial finite element meshes in 3D is inf-sup stable.

b) In Section 5, we identify critical pressures for the conforming $(k, k-1)$ discretization of the Stokes problem. They are related to the presence of critical edges in the mesh, see Def. 5.1. As a consequence, this conforming discretization is not inf-sup stable if the mesh contains critical edges.

c) In Section 6, we show that these pressures are eliminated in the basic Crouzeix-Raviart space. Hence, an inf-sup stable Crouzeix-Raviart discretization should contain the basic
Crouzeix-Raviart space as a subspace while the question remains open, whether the basic Crouzeix-Raviart Stokes element (cf. Def. 3.6) is sufficient for an inf-sup stable discretization.

2 Setting

2.1 The continuous Stokes problem

Let $\Omega \subset \mathbb{R}^3$ denote a bounded polyhedral domain with boundary $\partial \Omega$. We consider the Stokes equation

$$\begin{align*}
-\Delta u - \nabla p &= f \quad \text{in } \Omega, \\
\text{div } u &= 0 \quad \text{in } \Omega,
\end{align*}$$

with Dirichlet boundary conditions for the velocity and a normalization condition for the pressure:

$$u = 0 \quad \text{on } \partial \Omega \quad \text{and} \quad \int_\Omega p = 0.$$ 

To state the classical existence and uniqueness result, we introduce the relevant function spaces and formulate this equation in a variational form. Throughout the paper we restrict to vector spaces over the field of real numbers.

For $s \geq 0$, $1 \leq p \leq \infty$, $W^{s,p}(\Omega)$ denotes the classical Sobolev space of functions with norm $\|\cdot\|_{W^{s,p}(\Omega)}$. As usual we write $L^p(\Omega)$ instead of $W^{0,p}(\Omega)$ and $H^s(\Omega)$ for $W^{s,2}(\Omega)$. For $s \geq 0$, we denote by $H^s_0(\Omega)$ the closure of the space of infinitely smooth functions with compact support in $\Omega$ with respect to the $H^s(\Omega)$ norm. Its dual space is denoted by $H^{-s}(\Omega)$.

The scalar product and norm in $L^2(\Omega)$ are denoted by

$$(u,v)_{L^2(\Omega)} := \int_\Omega uv \quad \text{and} \quad \|u\|_{L^2(\Omega)} := (u,u)^{1/2}_{L^2(\Omega)} \quad \text{in } L^2(\Omega).$$

Vector-valued and $3 \times 3$ tensor-valued analogues of the function spaces are denoted by bold and blackboard bold letters, e.g., $\mathbf{H}^s(\Omega) = (H^s(\Omega))^3$ and $\mathbb{H}^s = (H^s(\Omega))^{3\times3}$ and analogously for other quantities.

The $L^2(\Omega)$ scalar product and norm for vector valued functions are given by

$$(\mathbf{u},\mathbf{v})_{L^2(\Omega)} := \int_\Omega \langle \mathbf{u}, \mathbf{v} \rangle \quad \text{and} \quad \|\mathbf{u}\|_{L^2(\Omega)} := (\mathbf{u},\mathbf{u})^{1/2}_{L^2(\Omega)},$$

where $\langle \mathbf{u}, \mathbf{v} \rangle$ denotes the Euclidean scalar product in $\mathbb{R}^3$. In a similar fashion, we define for $\mathbf{G}, \mathbf{H} \in L^2(\Omega)$ the scalar product and norm by

$$(\mathbf{G},\mathbf{H})_{L^2(\Omega)} := \int_\Omega \langle \mathbf{G}, \mathbf{H} \rangle \quad \text{and} \quad \|\mathbf{G}\|_{L^2(\Omega)} := (\mathbf{G},\mathbf{G})^{1/2}_{L^2(\Omega)},$$

where $\langle \mathbf{G}, \mathbf{H} \rangle = \sum_{i,j=1}^3 G_{i,j} H_{i,j}$. Finally, let $L^2_0(\Omega) := \{ u \in L^2(\Omega) \mid \int_\Omega u = 0 \}$. 

3
We introduce the bilinear forms $a : H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{R}$ and $b : L^2_0(\Omega) \times H^1_0(\Omega) \to \mathbb{R}$ by
\[
a(u, v) := (\nabla u, \nabla v)_{L^2(\Omega)}; \quad b(p, v) := (p, \text{div} v)_{L^2(\Omega)},
\]
where $\nabla u$ and $\nabla v$ denote the derivatives of $u$ and $v$. The variational form of the Stokes problem is given by: Given $f \in H^{-1}(\Omega)$,
\[
\text{find } (u, p) \in H^1_0(\Omega) \times L^2_0(\Omega) \text{ s.t. } \begin{cases} a(u, v) + b(p, v) = f(v) & \forall v \in H^1_0(\Omega), \\ b(q, u) = 0 & \forall q \in L^2_0(\Omega). \end{cases}
\]
(2.2)

It is well-known, that (2.2) is well posed (see, e.g., [16]). Since we consider non-conforming discretizations, we restrict the space $H^{-1}(\Omega)$ for the right-hand side to a smaller space and assume for simplicity that $f \in L^2(\Omega)$, i.e. we write $f(v)$ as $(f, v)_{L^2(\Omega)}$.

### 2.2 Discretization

Given two finite-dimensional approximation spaces $V_h$ with an appropriate norm $\| \cdot \|_{V_h}$ for the velocity and $M_h$ for the pressure, a finite element approximation of (2.2) then reads: For given $f \in L^2(\Omega)$,
\[
\text{find } (u_h, p_h) \in V_h \times M_h \text{ s.t. } \begin{cases} a_h(u_h, v) + b_h(p_h, v) = (f, v)_{L^2(\Omega)} & \forall v \in V_h, \\ b_h(q, u_h) = 0 & \forall q \in M_h. \end{cases}
\]
(2.3)

Here, $a_h(\cdot, \cdot) : V_h \times V_h \to \mathbb{R}$ is a discrete version of the bilinear form $a(\cdot, \cdot)$ in (2.1) which is defined on the discrete space $V_h$ and $b_h(\cdot, \cdot) : M_h \times V_h \to \mathbb{R}$ is a discrete version of $b(\cdot, \cdot)$ in (2.1). For the choice of Crouzeix-Raviart elements, we will give the concrete definition of $\| \cdot \|_{V_h}$ and the bilinear forms $a_h, b_h$ in Definition 3.6. It is well known that if $a_h(\cdot, \cdot), b_h(\cdot, \cdot)$ are continuous, $a_h(\cdot, \cdot)$ is symmetric and $V_h$-coercive, and the spaces $V_h$ and $M_h$ satisfy the inequality
\[
\inf_{p \in M_h \setminus \{0\}} \sup_{v \in V_h \setminus \{0\}} \frac{b_h(p, v)}{\|v\|_{V_h} \|p\|_{L^2(\Omega)}} \geq \gamma > 0,
\]
then the discrete problem is well-posed. In this case, we call the pair $(V_h, M_h)$ inf-sup stable.

### 3 Basic Crouzeix-Raviart finite elements in 3D

In the following, we define Crouzeix-Raviart spaces in 3D for the velocity discretization in (2.2). Let $\mathcal{T}$ be a conforming finite element mesh for $\Omega$ consisting of closed tetrahedra $K \in \mathcal{T}$. We denote by $\hat{K}$ the reference tetrahedron with vertices
\[
\hat{z}_1 := 0, \quad \hat{z}_2 := (1, 0, 0)^T, \quad \hat{z}_3 := (0, 1, 0)^T, \quad \hat{z}_4 := (0, 0, 1).
\]
(3.1)

Moreover, let $\mathcal{F}$ ($\mathcal{E}$, $\mathcal{V}$, resp.) be the set of all two-dimensional facets (one-dimensional edges, vertices, resp.) in the mesh and let
\[
\mathcal{F}_{\partial \Omega} := \{ F \in \mathcal{F} \mid F \subset \partial \Omega \}, \quad \mathcal{F}_\Omega := \mathcal{F} \setminus \mathcal{F}_{\partial \Omega},
\]
\[
\mathcal{E}_{\partial \Omega} := \{ E \in \mathcal{E} \mid E \subset \partial \Omega \}, \quad \mathcal{E}_\Omega := \mathcal{E} \setminus \mathcal{E}_{\partial \Omega},
\]
\[
\mathcal{V}_{\partial \Omega} := \{ z \in \mathcal{V} \mid z \in \partial \Omega \}, \quad \mathcal{V}_\Omega := \mathcal{V} \setminus \mathcal{V}_{\partial \Omega}.
\]
For \( F \in \mathcal{F}, E \in \mathcal{E}, z \in \mathcal{V} \), we define facet, edge, nodal patches by

\[
\begin{align*}
T_F &:= \{ K \in T : F \subset K \}, \quad \omega_F := \bigcup_{K \in T_F} K, \\
T_E &:= \{ K \in T : E \subset K \}, \quad \omega_E := \bigcup_{K \in T_E} K, \\
T_z &:= \{ K \in T : z \in K \}, \quad \omega_z := \bigcup_{K \in T_z} K.
\end{align*}
\]

(3.2)

For a subset of tetrahedra \( M \subset T \), we define the patch

\[
\text{dom} M := \text{int} \left( \bigcup_{K \in M} K \right),
\]

where \( \text{int} (D) \) denotes the interior of a set \( D \subset \mathbb{R}^3 \). For a measurable subset \( D \subset \mathbb{R}^d \), we denote by \( |D|_d \) the \( d \)-dimensional volume of \( D \) and skip the index \( d \) if the dimension is clear from the context, e.g., \( |K| \) denotes the three-dimensional volume of a simplex \( K \in T \) and \( |F| \) the two-dimensional area of a facet \( F \in \mathcal{F} \).

For a conforming simplicial mesh \( T \) of the domain \( \Omega \), let \( H^1 (T) := \left\{ u \in L^2 (\Omega) \mid \forall K \in T : u|_K \in H^1 (K) \right\} \).

Let \( \mathbb{N} := \{ 1, 2, 3, \ldots \} \) and \( \mathbb{N}_0 := \mathbb{N} \cup \{ 0 \} \). For \( n \in \mathbb{N}_0 \) and a domain \( D \subset \mathbb{R}^3 \) we denote by \( \mathbb{P}_n (D) \) the space of polynomials of maximal degree \( n \) on \( D \) and set \( \mathbb{P}_{-1} (D) := \{ 0 \} \).

We introduce the following finite element spaces. For \( k \in \mathbb{N} \), let \( S_{k,0} (T) \) denote the space of globally continuous, piecewise polynomials of degree \( \leq k \) with vanishing trace on the boundary

\[
S_{k,0} (T) := \left\{ v \in C^0 (\text{dom} T) \mid v|_K \in \mathbb{P}_k (K) \ \forall K \in T \ \land \ \ v = 0 \text{ on } \partial (\text{dom} T) \right\}.
\]

Its vector valued version is \( \mathbf{S}_{k,0} (T) := (S_{k,0} (T))^3 \), which is a conforming subspace of \( H^1_0 (\Omega) \).

The space of discontinuous polynomials of maximal degree \( k - 1 \) is

\[
\mathbb{P}_{k-1} (T) := \left\{ p \in L^2 (\Omega) \mid p|_K \in \mathbb{P}_{k-1} (K) \ \forall K \in T \right\}
\]

and the subspace \( \mathbb{P}_{k-1,0} (T) \) is given by

\[
\mathbb{P}_{k-1,0} (T) := \left\{ p \in \mathbb{P}_{k-1} (T) \mid \int_{\text{dom} T} p = 0 \right\}.
\]

**Notation 3.1** For a simplex \( K \), its four vertices form the set \( \mathcal{V} (K) \) and its four facets form the set \( \mathcal{F}(K) \). For a triangular facet, its three vertices form the set \( \mathcal{V} (F) \), and for an edge \( E \), the two endpoints form the set \( \mathcal{V} (E) \). Let \( \delta_{x,y} \) denote Kronecker’s delta. The barycentric coordinate for a vertex \( z \in \mathcal{V} (K) \) is defined by the conditions

\[
\lambda_{K,z} \in \mathbb{P}_1 (K) \quad \text{and} \quad \lambda_{K,z} (y) = \delta_{z,y} \quad \text{for all vertices } y \in \mathcal{V} (K).
\]
For $\mu = (\mu_y)_{y \in V(K)} \in \mathbb{N}_0^4$, we set
\[
\lambda^K_\mu := \prod_{y \in V(K)} \lambda^K_{\mu_y}.
\]

For a facet $F \subset \partial K$ and $\mu = (\mu_y)_{y \in V(F)} \in \mathbb{N}_0^3$, we set
\[
\lambda^{\mu} _{K,F} := \prod_{y \in V(F)} \lambda^{\mu_y} _{K,y}.
\]

Finally, for an edge $E \subset \partial K$ and $\mu = (\mu_y)_{y \in V(E)} \in \mathbb{N}_0^2$, we set
\[
\lambda^{\mu_E} _{K,E} := \prod_{y \in V(E)} \lambda^{\mu_y} _{K,y}.
\]

We also set $1_K := (1)_{y \in V(K)}$, $1_F := (1)_{y \in V(F)}$, $1_E := (1)_{y \in V(E)}$. For $y \in V(K)$, we set $e^K_y := (\delta_y, x)_{x \in V(K)}$, for a facet $F$ and $y \in V(F)$, we set $e^F_y := (\delta_y, x)_{x \in V(F)}$, and for an edge $E$ and $y \in V(E)$ let $e^E_y := (\delta_y, x)_{x \in V(E)}$.

Next, we define the non-conforming Crouzeix-Raviart space. For a function $v \in H^1(T)$, we denote by $[v]_F$ the jump of $v \in \mathbb{P}_k(T)$ across the facet $F \in \mathcal{F}$ and $\mathbb{P}_{k-1}(F)$ is the space of polynomials of maximal degree $k - 1$ with respect to the local variables in $F$. For $k \geq 1$ and any $F \in \mathcal{F}$, let
\[
\mathbb{P}^\perp_{k,k-1}(F) := \{ q \in \mathbb{P}_k(F) \mid (q, r)_{L^2(F)} = 0 \ \forall r \in \mathbb{P}_{k-1}(F) \}.
\]

The scalar version of the Crouzeix-Raviart space of order $k$ is defined implicitly by
\[
\text{CR}^{\text{max}}_{k,0}(T) := \left\{ v \in \mathbb{P}_k(T) \mid \left( \begin{array}{c}
\forall F \in \mathcal{F}_\Omega \quad [v]_F \in \mathbb{P}^\perp_{k,k-1}(F) \\
\forall F \in \mathcal{F}_{\partial \Omega} \quad v \in \mathbb{P}^\perp_{k,k-1}(F)
\end{array} \right) \right\}. \quad (3.3)
\]

Its vector version is denoted by $\text{CR}^{\text{max}}_{k,0}(T) := (\text{CR}^{\text{max}}_{k,0}(T))^3$. We also define
\[
S'_{k,0}(T) := \{ v \in S_{k,0}(T) \mid v(z) = 0 \ \forall z \in V \} \quad (3.4)
\]
as the subspace of $S_{k,0}(T)$ consisting of functions which vanishes at the vertices of the mesh.

**Remark 3.2** In two spatial dimensions, local basis functions for Crouzeix-Raviart spaces have been defined in [22], [1], [7, for $p = 4, 6$], [9], [2], [5]. It turns out that the non-conforming Crouzeix-Raviart basis functions of odd polynomial degree $k$ are associated to the inner triangle edges while for even polynomial degree they are associated to the triangles in the mesh.

The situation is much more complicated in 3D. In [8] local shape functions are introduced which span the Crouzeix-Raviart space $\text{CR}^{\text{max}}_{k,0}(T)$ and it was shown that per inner facet $F$
there exist \( \lfloor \frac{k+2}{2} \rfloor \) linearly independent, non-conforming Crouzeix-Raviart functions and, in addition, per simplex, there exist \( \lfloor \frac{k}{2} \rfloor - \lfloor \frac{k-1}{2} \rfloor \) linearly independent, non-conforming Crouzeix-Raviart functions. We say that any space \( V \) with \( S_{k,0}(T) \subsetneq V \subset CR_{k,0}^\text{max}(T) \) is a Crouzeix-Raviart space.

A natural question is whether there is an analogous choice of \( V \) as in two dimensions: One Crouzeix-Raviart function per facet for odd polynomial degree and one Crouzeix-Raviart function per tetrahedron for even polynomial degree. In addition, these Crouzeix-Raviart functions should have a "similarly simple" representation as those in 2D, see [5, Def. 3.2]. We call the space \( S_{k,0}(T) \), enriched by those functions, the basic Crouzeix-Raviart space. Since this space is smaller than \( CR_{k,0}^\text{max}(T) \) we have used the superscript "max" in (3.3).

Next, we define the basic Crouzeix-Raviart functions on simplicial meshes in 3D. Let \( \alpha, \beta > -1 \) and \( n \in \mathbb{N}_0 \). The Jacobi polynomial \( P_n^{(\alpha,\beta)} \) is a polynomial of degree \( n \) such that

\[
\int_{-1}^{1} P_n^{(\alpha,\beta)}(x) q(x) (1-x)^\alpha (1+x)^\beta \, dx = 0
\]

for all polynomials \( q \) of degree less than \( n \), and (cf. [12, Table 18.6.1])

\[
P_n^{(\alpha,\beta)}(1) = \frac{(\alpha+1)n}{n!}, \quad P_n^{(\alpha,\beta)}(-1) = (-1)^n \frac{(\beta+1)n}{n!}.
\]

(3.5)

Here, the shifted factorial is defined by \( (a)_n := a(a+1)\ldots(a+n-1) \) for \( n > 0 \) and \( (a)_0 := 1 \). Note that \( P_k^{(0,0)} \) are the Legendre polynomials (see [12, 18.7.9]) and we set \( L_k := P_k^{(0,0)} \). For later use, we state an orthogonality relation on a tetrahedron for polynomials which are related to \( P_k^{(0,3)} \).

**Lemma 3.3** For a tetrahedron \( K \) with barycentric coordinates \( \lambda_{K,y}, y \in \mathcal{V}(K) \), the polynomial \( P_k^{(0,3)}(1-2\lambda_{K,y}) \) is orthogonal to \( P_{k-1}(K) \) with respect to the weight functions

\[
\lambda_{K,z}, \quad z \in \mathcal{V}(K) \setminus \{y\}.
\]

The assertion is a particular case of [13, Prop. 2.3.8]. Since the proof for our concrete case is very simple we give it here for completeness.

**Proof.** Let \( y, z \in \mathcal{V}(K) \) with \( y \neq z \) and let \( \chi_K : \hat{K} \to K \) be an affine pullback such that \( \lambda_{K,y} \circ \chi_K(x) = x_1 \) and \( \lambda_{K,z} \circ \chi_K(x) = x_2 \). For \( \alpha = (\alpha_j)_{j=1}^3 \in \mathbb{N}_0^3 \) with \( |\alpha| \leq k-1 \), let \( \hat{q}^\alpha(x) := x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3} \) and note that \( P_{k-1}(K) \) is spanned by the lifted versions \( \hat{q}^\alpha \circ \chi_K^{-1} \). Then,

\[
\int_K P_k^{(0,3)}(1-2\lambda_{K,y}) \lambda_{K,z} \hat{q}^\alpha \circ \chi_K^{-1} \]

\[
= \frac{|K|}{|\hat{K}|} \int_0^1 \int_0^{1-x_1} \int_0^{1-x_1-x_2} P_k^{(0,3)}(1-2x_1) x_1^{\alpha_1}x_2^{\alpha_2+1}x_3^{\alpha_3} dx_3 dx_2 dx_1
\]

\[
= \frac{1}{\alpha_3 + 1} \frac{|K|}{|\hat{K}|} \int_0^1 \int_0^{1-x_1} P_k^{(0,3)}(1-2x_1) x_1^{\alpha_1}x_2^{\alpha_2+1} (1-x_1-x_2)^{\alpha_3+1} dx_2 dx_1
\]
\[
\begin{align*}
&= \frac{(\alpha_2 + 1)!\alpha_3!}{(\alpha_2 + \alpha_3 + 3)!} \left| \frac{K}{K} \right| \int_0^1 P_k^{(0,3)} (1 - 2x_1) (1 - x_1)^3 x_1^{\alpha_1} (1 - x_1)^{\alpha_2 + \alpha_3} \, dx_1 \\
&= \frac{1 - 2x_1 = t}{16 (\alpha_2 + \alpha_3 + 3)!} \left| \frac{K}{K} \right| \int_{-1}^1 P_k^{(0,3)} (t) (t + 1)^3 \left( \frac{1 - t}{2} \right)^{\alpha_1} \left( \frac{1 + t}{2} \right)^{\alpha_2 + \alpha_3} \, dt \\
&= 0,
\end{align*}
\]

by the orthogonality properties of the Jacobi polynomials. 

We introduce univariate polynomials \( Q_k \in P_k \) by

\[
Q_k := \frac{1}{k + 1} (L_{k+1} - L_k)' .
\]  

**Definition 3.4** Let \( \mathcal{T} \) be a conforming simplicial finite element mesh in 3D.

1. For even \( k \geq 2 \), and any \( K \in \mathcal{T} \), the simplex-oriented Crouzeix-Raviart basis function \( B_{k, K}^{CR} \in P_k (\mathcal{T}) \) is given by

\[
B_{k, K}^{CR} := \begin{cases} 
(\sum_{z \in V(K)} Q_k (1 - 2\lambda_{K,z})) - 1 & \text{on } K, \\
0 & \text{otherwise}.
\end{cases}
\]  

2. For odd \( k \geq 1 \), and any \( F \in \mathcal{F}_\Omega \), the facet-oriented Crouzeix-Raviart basis function \( B_{k, F}^{CR} \in P_k (\mathcal{T}) \) is given by

\[
B_{k, F}^{CR} := \begin{cases} 
Q_k (1 - 2\lambda_{K,z}) & \text{for } K \in \mathcal{T}_F, \\
0 & \text{otherwise},
\end{cases}
\]  

where \( \lambda_{K,z} \) denotes the barycentric coordinate for the vertex \( z \in V(K) \) opposite to \( F \).

**Theorem 3.5**

(a) Let \( K \in \mathcal{T} \) and \( k \geq 2 \) be even. The function \( B_{k, K}^{CR} \) in (3.7) is \( L^2 (F) \)-orthogonal to \( P_{k-1} (F) \) on any facet \( F \) of \( K \) and belongs to \( \text{CR}_{k,0}^{\max} (\mathcal{T}) \).

(b) Let \( F \in \mathcal{F}_\Omega \) and \( k \geq 1 \) be odd. The function \( B_{k, F}^{CR} \) in (3.8) is \( L^2 (F) \)-orthogonal to \( P_{p-1} (F') \) on any outer facet \( F' \in \partial \omega_F \), continuous across \( F \), and belongs to \( \text{CR}_{k,0}^{\max} (\mathcal{T}) \).

**Proof.** (a): Let \( k \geq 2 \) be even. Let \( F \in \mathcal{F}_\Omega \) with \( F \subset \partial K \) and let \( z \in V(K) \) denote the vertex opposite to \( F \). Recall that the Legendre polynomial satisfies (cf. [12, Combine 18.9.19, 18.7.9 and Table 18.6.1])

\[
L'_k(\pm 1) = (\pm 1)^{k-1} \binom{k+1}{2}.
\]  

Since \( \lambda_{K,z} = 0 \) on \( F \), the (constant) values of \( B_{k, F}^{CR} \) on \( F \) from both sides coincide and are given by \( Q_k (1) = 1 \). This implies, that \( B_{k, F}^{CR} \) is continuous across \( F \).
Since \( \lambda_{K,z} = 0 \) on \( F \), the function \( B^{CR,K}_k \) can be expressed by

\[
B^{CR,K}_k|_F = \left( \sum_{y \in V(F)} Q_k \left(1 - 2 \lambda_{K,y}\right) \right) + \frac{L'_{k+1}(1) - L'_k(1)}{k+1} - 1
\]

\[
= \sum_{y \in V(F)} Q_k \left(1 - 2 \lambda_{K,y}|_F\right).
\]

For \( y \in V(F) \), let \( \chi_{K,y} : \hat{K} \to K \) denote an affine bijection with \( \chi_K : \hat{F} \to F \) for \( \hat{F} := \{ x \in \hat{K} | x_3 = 0 \} \) and \( \chi_{K,y}(1,0,0) = y \). Hence, it suffices to show that

\[
I^{\mu}_k = 0, \quad \forall \mu \in \mathbb{N}_0^3 \text{ with } |\mu| \leq k - 1,
\]

where

\[
I^{\mu}_k := \int_0^1 \int_0^{1-x_1} Q_k \left(1 - 2 x_1\right) (1 - x_1 - x_2)^{\mu_1} x_1^{\mu_2} x_2^{\mu_3} dx_2 dx_1
\]

\[
= \int_0^1 Q_k \left(1 - 2 x_1\right) g(x_1) dx_1,
\]

\[
g(x_1) := x_1^{\mu_2} \left(\int_0^{1-x_1} (1 - x_1 - x_2)^{\mu_1} x_2^{\mu_3} dx_2\right).
\]

It is easy to see that \( g \in \mathbb{P}_{\mu_2+\mu_1+\mu_3+1} \subset \mathbb{P}_k \) and \( g(1) = 0 \). Therefore, we get

\[
I^{\mu}_k = \frac{1}{k+1} \int_0^1 (L_{k+1} - L_k)' (1 - 2x) g(x) dx
\]

\[
= \frac{1}{2(k+1)} \int_0^1 (L_{k+1} - L_k) (1 - 2x) g'(x) dx - \frac{1}{2(k+1)} (L_{k+1} - L_k) (1 - 2x) g(x) \bigg|_{x=0}^{x=1}
\]

\[
= 0,
\]

by the orthogonality of the Legendre polynomials and by the properties \( L_n(1) = 1 \) for all \( n \in \mathbb{N}_0 \) and \( g(1) = 0 \). This proves (a).

(b): Let \( k \geq 1 \) be odd. Let \( F \in \mathcal{F}_\Omega \) and \( K \subset \mathcal{T}_F \). The vertex of \( K \) opposite to \( F \) is denoted by \( z \). Note that \( \lambda_{K,z} \neq 0 \) on any \( F' \subset \partial \omega_F \cap K \). The proof of orthogonality follows from a repetition of the arguments as in the proof of (a).

We now define the space

\[
B^{bc}_k(\mathcal{T}) := \begin{cases} 
\text{span} \left\{ B^{CR,K}_k | K \in \mathcal{T} \right\} & \text{if } k \text{ is even}, \\
\text{span} \left\{ B^{CR,F}_k | F \in \mathcal{F}_\Omega \right\} & \text{if } k \text{ is odd}.
\end{cases}
\]
Definition 3.6 The scalar basic Crouzeix-Raviart space of order $k$ for conforming simplicial finite element meshes $\mathcal{T}$ in 3D is given by (see (3.4))

$$CR_{k,0}(\mathcal{T}) := B_{k}^{nc}(\mathcal{T}) + \begin{cases} S_{k,0}(\mathcal{T}) & \text{if } k \text{ is even,} \\ S'_{k,0}(\mathcal{T}) & \text{if } k \text{ is odd.} \end{cases} \quad (3.10)$$

The basic Crouzeix-Raviart Stokes element is given by $V_h := CR_{k,0}(\mathcal{T}) := (CR_{k,0}(\mathcal{T}))^3$ and $M_h := P_{k-1,0}(\mathcal{T})$. The norm in $V_h$ is given by the broken $H^1$-seminorm,

$$\|v\|_{V_h} := \sqrt{\sum_{K \in \mathcal{T}} \int_{K} |\nabla v|^2},$$

which is a norm owing to the discrete Poincaré-Friedrichs inequality [3, Theorem (10.6.12)]. In this case, the bilinear forms $a_h (\cdot, \cdot)$, $b_h (\cdot, \cdot)$ in (2.3) are given by

$$a_h (u, v) := \sum_{K \in \mathcal{T}} \int_{K} \langle \nabla u, \nabla v \rangle \quad \forall u, v \in V_h,$$

$$b_h (q, v) := \sum_{K \in \mathcal{T}} \int_{K} q \, \text{div} \, v \quad \forall (q, v) \in M_h \times V_h.$$

Remark 3.7 For $k = 1$ and $F \in \mathcal{F}_{\partial \Omega}$, we obtain explicitly

$$B_{1}^{CR,F} := \begin{cases} 1 - 3 \lambda_{K,z} & \text{on } K \in \mathcal{T}_{F}, \\ 0 & \text{otherwise,} \end{cases}$$

where $z \in V(K)$ is the vertex opposite to $F$. This function is the basis function from the original paper [11, (5.1)].

Lemma 3.8 For any $k$, the sum in (3.10) is direct and we have

$$S_{k,0}(\mathcal{T}) \subset CR_{k,0}(\mathcal{T}) \subset CR_{k,0}^{\text{max}}(\mathcal{T}).$$

Proof. The proof that the sum in (3.10) for even $k$ is direct can be found in [8, Theorem 33]. For odd $k$, the proof in [8, Theorem 33] can be adapted: For $k = 1$, we have $S_{1,0}^{'}(\mathcal{T}) = \{0\}$ and the statement is trivial.

It remains to consider odd $k \geq 3$. We need to show that $S_{k,0}^{'}(\mathcal{T}) \cap B_{k}^{nc}(\mathcal{T}) = \{0\}$.

Assume that $u \in S_{k,0}^{'}(\mathcal{T}) \cap B_{k}^{nc}(\mathcal{T})$. By definition, $u \in C^{0}(\Omega)$ and $u$ vanishes on the boundary. Moreover, $u(z) = 0$ for all $z \in V$. We now consider a tetrahedron $K \in \mathcal{T}$ with facets $F_i$ (opposite to the vertices $z_i \in V(K)$, $1 \leq i \leq 4$) which has one facet that lies on the boundary, say $F_1 \in \mathcal{F}_{\partial \Omega}$. This implies that $u$ vanishes on $F_1$. Since $u \in B_{k}^{nc}(\mathcal{T})$ for odd $k$ we have

$$u|_{K} = \sum_{i=2}^{4} \alpha_i Q_k(1 - 2 \lambda_{K,z_i})$$

for some $\alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$. 

10
This leads to the system

\[
0 = u(z_2) = \alpha_1 Q_k(-1) + \alpha_2 Q_k(1) + \alpha_3 Q_k(1), \\
0 = u(z_3) = \alpha_1 Q_k(1) + \alpha_2 Q_k(-1) + \alpha_3 Q_k(1), \\
0 = u(z_4) = \alpha_1 Q_k(1) + \alpha_2 Q_k(1) + \alpha_3 Q_k(-1),
\]

since \( u(z_i) = 0 \) for \( i = 2, 3, 4 \). Moreover, we have (for odd \( k \))

\[
Q_k(\pm 1) = \frac{1}{k+1} (L_{k+1} - L_k)'(\pm 1) \equiv \frac{1}{k+1} \left( (\pm 1)^k \left( \frac{k+2}{2} \right) - (\pm 1)^{k-1} \left( \frac{k+1}{2} \right) \right)
\]

and, in turn, \( \alpha = (\alpha_2, \alpha_3, \alpha_4)^T \) is the solution of

\[
\begin{bmatrix}
-(k+1) & 1 & 1 \\
1 & -(k+1) & 1 \\
1 & 1 & -(k+1)
\end{bmatrix} \alpha = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.
\]

The determinant of this matrix is \( -(k-1)(k+2)^2 \); since \( k \geq 3 \) the matrix is regular and \( \alpha = (0, 0, 0) \) follows so that \( u|_K = 0 \). From an induction argument, we conclude that \( u = 0 \).

\section{Inf-sup stability for the quadratic velocity space}

We start this section with some remarks on the macroelement technique (see [20], [21]) which we employ for the analysis of the inf-sup stability.

\textbf{Definition 4.1} A macroelement \( \mathcal{M} \) is a connected set of elements \( K \in \mathcal{T} \). For a macroelement \( \mathcal{M} \), the spaces \( N_{k,\mathcal{M}}^{\text{CR}} \) and \( N_{k,\mathcal{M}} \) are the orthogonal complements in \( \mathbb{P}_{k-1}(\mathcal{M}) \) of the images \( \text{div}(\text{CR}_{k,0}(\mathcal{M})) \) and \( \text{div} S_{k,0}(\mathcal{M}) \):

\[
N_{k,\mathcal{M}}^{\text{CR}} = \left\{ p \in \mathbb{P}_{k-1}(\mathcal{M}) \mid \forall \mathbf{v} \in \text{CR}_{k,0}(\mathcal{M}) : (p, \text{div} \mathbf{v})_{L^2(\text{dom},\mathcal{M})} = 0 \right\},
\]

\[
N_{k,\mathcal{M}} = \left\{ p \in \mathbb{P}_{k-1}(\mathcal{M}) \mid \forall \mathbf{v} \in S_{k,0}(\mathcal{M}) : (p, \text{div} \mathbf{v})_{L^2(\text{dom},\mathcal{M})} = 0 \right\}.
\]

Non-zero elements in \( N_{k,\mathcal{M}} \cap L^2_0(\Omega) \) are critical pressures for \( \mathcal{M} \).

A direct consequence of [21, Thm. 2.1] is the following proposition for quadratic velocity spaces.

\textbf{Proposition 4.2} Let \( \mathcal{T} \) be a regular finite element simplicial mesh on a bounded polyhedral domain \( \Omega \subset \mathbb{R}^3 \). Let \( k = 2 \) and consider the macroelements consisting of one simplex, i.e. \( \mathcal{M} = \{ K \} \subset \mathcal{T} \). If

\[
\dim N_{k,\mathcal{M}}^{\text{CR}} = 1, \quad \forall K \in \mathcal{T},
\]

then the basic Crouzeix-Raviart element for the Stokes problem (cf. Def. 3.6) is inf-sup stable.
In the remaining part of this section, we prove the inf-sup stability of \((\text{CR}_{2,0}(T), \mathbb{P}_{1,0}(T))\) for the case \(k = 2\) by showing (4.1). Using \(P_1^{(0,3)}(x) = (5x - 3)/2\) and \(P_1^{(0,3)}(1 - 2\lambda_{K,z}) = 1 - 5\lambda_{K,z}\), it is easy to verify that the polynomials
\[
P_1^{(0,3)}(1 - 2\lambda_{K,z}), \quad z \in \mathcal{V}(K)
\]
form a basis for \(\mathbb{P}_{k-1}(K) = \mathbb{P}_1(K)\).

The following lemma will be used in Theorem 4.4 and can be easily proved using a transformation to the reference tetrahedron and the orthogonality properties of \(P_1^{(0,3)}(x)\) (see Lemma 3.3).

**Lemma 4.3** For \(y, z \in \mathcal{V}(K)\), it holds
\[
\int_K P_1^{(0,3)}(1 - 2\lambda_{K,y}) \lambda_{K,z} = \begin{cases} 
\frac{|K|}{4} & y = z, \\
0 & y \neq z.
\end{cases}
\] (4.2)

The goal of this section is to prove the stability of the pair \((\text{CR}_{2,0}(T), \mathbb{P}_{1,0}(T))\) using the macroelement technique (Proposition 4.2), where each macroelement consists of a single tetrahedron \(K \in T\).

**Theorem 4.4** The basic Crouzeix-Raviart Stokes element for \(k = 2\) is inf-sup stable with an inf-sup constant \(\gamma > 0\) independent of the mesh width \(h\).

**Proof.**
By a pullback to the reference element, it is straightforward to verify that for \(k = 2\) the Crouzeix Raviart basis functions for a tetrahedron satisfy
\[
B_{2,CR,K} = \frac{5}{3} \left( \sum_{y \in \mathcal{V}(K)} L_2(1 - 2\lambda_{K,y}) - 1 \right).
\]
By definition, \(B_{2,CR,K} t \in \text{CR}_{k,0}(T)\) for any constant vector \(t \in \mathbb{R}^3\). We will show that the space
\[
\mathcal{N}_{2,K}^{\text{CR}} = \left\{ p \in \mathbb{P}_1(K) \mid \forall t \in \mathbb{R}^3 : \left( p, \text{div} B_{2,CR,K} t \right)_{L^2(K)} = 0 \right\}
\]
is one-dimensional. Using the relation (see [12, 8.9.15])
\[
L'_n = \frac{n + 1}{2} P_n^{(1,1)}, \quad \forall n \in \mathbb{N}_0,
\]
we get for \(y \in \mathcal{V}(K)\)
\[
\int_K P_1^{(0,3)}(1 - 2\lambda_{K,y}) \text{div} \left( B_{2,CR,K} t \right) = -5 \sum_{z \in \mathcal{V}(K)} \partial_t \lambda_{K,z} \int_K P_1^{(0,3)}(1 - 2\lambda_{K,y}) P_1^{(1,1)}(1 - 2\lambda_{K,z}).
\] (4.3)
Let $\chi_K : \hat{K} \to K$ denote an affine pullback and $\hat{y} := \chi_{\hat{K}}^{-1}(y)$, $\hat{z} := \chi_{\hat{K}}^{-1}(z)$, and $\lambda_{\hat{z}} := \lambda_{K,z} \circ \chi_K$. Then,

\[
\int_K p_{1,3}^{(0,3)} (1 - 2\lambda_{K,y}) p_{1,1}^{(1,1)} (1 - 2\lambda_{K,x}) = \frac{|K|}{|\hat{K}|} \int_{\hat{K}} p_{1,3}^{(0,3)} (1 - 2\lambda_{\hat{y}}) p_{1,1}^{(1,1)} (1 - 2\lambda_{\hat{z}})
= 6 |K| \int_{\hat{K}} p_{1,3}^{(0,3)} (1 - 2\lambda_{\hat{y}}) p_{1,1}^{(1,1)} (1 - 2\lambda_{\hat{z}})
\tag{4.4}
\]

For a tetrahedron $K$, we fix a vertex $p \in \mathcal{V}(K)$ and set $t_v := v - p$, $v \in \mathcal{V}(K) \setminus \{p\}$. Then, it is easy to verify that

\[
\partial_{t_v} \lambda_{K,z} = \begin{cases} 
-1 & z = p, \\
\delta_{v,z} & z \in \mathcal{V}(K) \setminus \{p\}, \\
& \text{for all } v \in \mathcal{V}(K) \setminus \{p\}.
\end{cases}
\]

We combine this with (4.3), (4.4), and obtain for any $v \in \mathcal{V}(K) \setminus \{p\}$

\[
\int_K p_{1,3}^{(0,3)} (1 - 2\lambda_{K,y}) \text{div}(B_{2}^{CR,K} t_v) = -30 |K| \sum_{z \in \mathcal{V}(K)} \partial_{t_v} \lambda_{K,z} \int_{\hat{K}} p_{1,3}^{(0,3)} (1 - 2\lambda_{\hat{y}}) p_{1,1}^{(1,1)} (1 - 2\lambda_{\hat{z}})
= 30 |K| \left( \int_{\hat{K}} p_{1,3}^{(0,3)} (1 - 2\lambda_{\hat{y}}) \left( p_{1,1}^{(1,1)} (1 - 2\lambda_{\hat{p}}) - p_{1,1}^{(1,1)} (1 - 2\lambda_{\hat{y}}) \right) \right).
\]

The difference in the integrand can be simplified by using $p_{1,1}^{(1,1)} (x) = 2x$:

\[
p_{1,1}^{(1,1)} (1 - 2\lambda_{\hat{p}}) - p_{1,1}^{(1,1)} (1 - 2\lambda_{\hat{y}}) = 4 (\lambda_{\hat{v}} - \lambda_{\hat{p}}).
\]

Using Lemma 4.3, we have for any $v \in \mathcal{V}(K) \setminus \{p\}$

\[
\int_K p_{1,3}^{(0,3)} (1 - 2\lambda_{K,y}) \text{div}(B_{2}^{CR,K} t_v) = 120 |K| \int_{\hat{K}} p_{1,3}^{(0,3)} (1 - 2\lambda_{\hat{y}}) (\lambda_{\hat{v}} - \lambda_{\hat{p}})
= \begin{cases} 
-120 |K| \int_{\hat{K}} p_{1,3}^{(0,3)} (1 - 2\lambda_{\hat{y}}) \lambda_{\hat{y}} & p = y, \\
120 |K| \int_{\hat{K}} p_{1,3}^{(0,3)} (1 - 2\lambda_{\hat{y}}) \lambda_{\hat{y}} & p \neq y \text{ and } v = y, \\
0 & \text{otherwise}
\end{cases}
\tag{4.2}
\]

This implies that the matrix $\left( \text{div} B_{2}^{CR,K} t_v, p_{1,3}^{(0,3)} (1 - 2\lambda_{K,y}) \right)_{L^2(K)}$ is given by

\[
5 |K| \begin{pmatrix}
1 & -1 & 0 & 0 \\
1 & 0 & -1 & 0 \\
1 & 0 & 0 & -1
\end{pmatrix}.
\]

This matrix has full rank so that $N_{2,\{K\}}^{CR}$ is one-dimensional, containing only the constant pressures. Inf-sup stability follows by the macroelement technique (Proposition 4.2).
Remark 4.5 If $\mathcal{F}_\Omega \neq \emptyset$, then $\text{CR}_{2,0}(\mathcal{T}) \subseteq \text{CR}_{2,0}^{\text{max}}(\mathcal{T})$ since $\text{CR}_{2,0}^{\text{max}}(\mathcal{T})$ also contains one non-conforming Crouzeix-Raviart functions associated with each facet $F \in \mathcal{F}_\Omega$ (see Remark 3.2).

We have shown that it suffices to enrich $S_{2,0}(\mathcal{T})$ with certain local Crouzeix-Raviart functions (one scalar one per tetrahedron multiplied by three linearly independent constant vectors in $\mathbb{R}^3$) in order to stabilize the pair $(S_{2,0}(\mathcal{T}), P_{1,0}(\mathcal{T}))$.

5 Critical pressures for $(S_{k,0}(\mathcal{T}), P_{k-1,0}(\mathcal{T}))$ in 3D

In two dimensions, the existence of critical pressures (or also called spurious pressures in the literature) is related to the existence of critical vertices [19]. In this case, a vertex $\mathbf{z}$ is critical if all edges connected to $\mathbf{z}$ lie on at most two straight lines; in [19] the following dimension formula is proved for $k \geq 4$ and two-dimensional triangulations (recall Def. 4.1)

$$\dim N_{k,T} = 1 + \# \{ \mathbf{z}' \in V_\Omega \mid \mathbf{z}' \text{ is a critical vertex in } \mathcal{T} \}.$$ (5.1)

In [18, Def. 6.3], critical edges for conforming simplicial meshes in dimension 3 are introduced in analogy to critical vertices in two dimensions.

Definition 5.1 Let $\mathcal{T}$ be a conforming simplicial finite element mesh and let $\mathcal{E}$ denote the set of edges in $\mathcal{T}$. An edge $E \in \mathcal{E}$ is critical in $\mathcal{T}$ if all facets $F \in \mathcal{F}$ with $E \subset F$ lie in at most two flat planes.

An analogous dimension formula to (5.1) is not known for conforming simplicial meshes in dimension $d \geq 3$. In this section, we discuss the existence of critical pressures in the presence of critical edges (cf. Def. 5.1). In Section 6, we then prove that the non-conforming Crouzeix-Raviart functions remove these critical pressures.

Remark 5.2 At the current stage of research, a complete description of all critical pressures is still open and, hence, our result can be interpreted in the way that an inf-sup stable Crouzeix-Raviart space should contain the basic Crouzeix-Raviart space while the inf-sup stability of basic Crouzeix-Raviart elements for the Stokes equation is still not fully understood.

There are exactly two types of critical edges that can appear in a tetrahedral mesh.

a. The critical edge $E \in \mathcal{E}_\Omega$ is an inner edge of $\mathcal{T}$. In this case the edge patch $\omega_E$ consists of exactly four tetrahedra (see Fig. 1 for an illustration).

b. The critical edge $E \in \mathcal{E}_{\partial \Omega}$ is an outer edge of $\mathcal{T}$ and the edge patch $\omega_E$ consists of either one, two or three tetrahedra (see Fig. 2 for an illustration).

We now study the existence of critical pressures for those two types of critical edges.
Figure 1: Inner critical edge $E$ in a nodal patch. Red and green coloured facets lie in one plane, respectively.

Figure 2: The three cases of possible critical edges $E$ at the boundary of a mesh. The edge patch $\omega_E$ consist of one, two or three tetrahedra (from left to right). Red and green coloured facets lie in one plane, respectively.
### 5.1 Critical pressure for inner critical edges

Let $E \in \mathcal{E}_\Omega$ be a critical edge of the mesh $\mathcal{T}$. In the following, we construct pressures $p_{k-1}^{E, p} \in P_{k-1,0}(\mathcal{T})$, $p \in \mathcal{V}(E)$, supported on $\omega_E$ such that

$$
\left(p_{k-1}^{E, p}, \text{div } v\right)_{L^2(\Omega)} = 0 \quad \forall v \in S_{k,0}(\mathcal{T}).
$$

**Notation 5.3** For a set $M \subset \mathbb{R}^3$, we denote its convex hull by $[M]$. Let $E \in \mathcal{E}_\Omega$ be a critical edge. This implies that $E$ is shared by exactly four tetrahedra $K_j \in \mathcal{T}$, $1 \leq j \leq 4$ (see Figure 1). We employ a cyclic numbering convention and denote $K_{4+1} := K_1$ and $K_{1-1} := K_4$. We assume the numbering convention that $K_1, K_{i+1}$ share a facet (denoted by $F_i$) and $F_1, F_3$ lie in one plane and $F_2, F_4$ lie in one other plane.

**Definition 5.4 (Critical pressure for inner edges)** Assume the setting as in Notation 5.3 and let $k \geq 1$. Assume that there is some critical edge $E \in \mathcal{E}_\Omega$. For $p \in \mathcal{V}(E)$, the critical pressure $p_{k-1}^{E, p} \in P_{k-1,0}(\mathcal{T})$ is given by

$$
p_{k-1}^{E, p} := \sum_{i=1}^{4} \frac{(-1)^i}{|K_i|} \chi_{K_i} P_{k-1}^{(0,3)} (1 - 2\lambda_{K_i,p}),
$$

where $\chi_{K_i}$ denotes the characteristic function for $K_i$.

Next, we verify that (5.2) is satisfied. The restriction of $S_{k,0}(\mathcal{T})$ to $K \in \mathcal{T}$ belongs to the span of

$$
\chi^K(w_\ell) \left\{ \begin{array}{ll}
& \text{for } w_\ell \in \mathbb{R}^3, \ 1 \leq \ell \leq 3, \text{ linearly independent,} \\
& \mu = (\mu_v)_{v \in \mathcal{V}(K)} \in \mathbb{N}_0^d, \ |\mu| = k
\end{array} \right.
$$

For $p \in \mathcal{V}(E)$, we have for $K_i$ as in Notation 5.3

$$
\int_{K_i} p_{k-1}^{E, p} \text{div } (\chi^K(w_\ell)) = \int_{K_i} p_{k-1}^{E, p} \partial_{w_\ell} \chi^K = \frac{(-1)^i}{|K_i|} \int_{K_i} P_{k-1}^{(0,3)} (1 - 2\lambda_{K_i,p}) \partial_{w_\ell} \chi^K
$$

$$
= \frac{(-1)^i}{|K_i|} \sum_{v \in \mathcal{V}(K_i)} \mu_v \partial_{w_\ell} \chi_K, \ y \int_{K_i} P_{k-1}^{(0,3)} (1 - 2\lambda_{K_i,p}) \chi^K_{K_i}
$$

$$
= \frac{(-1)^i}{|K_i|} \left( \int_{K_i} P_{k-1}^{(0,3)} (1 - 2\lambda_{K_i,p}) \chi_{K_i}^{k-1} \right) \times \left\{ \begin{array}{ll}
& k \partial_{w_\ell} \chi_{K_i}, \\
& \mu = (k-1)e^K_{y}, \\
& y \in \mathcal{V}(K_i) \setminus \{p\}, \\
& 0 \text{ otherwise,}
\end{array} \right.
$$

where the last equality follows by the orthogonality of $P^{(0,3)}(1 - 2\lambda_{K_i,p})$ with respect to the weights $\lambda_{K_i,y}$, $y \in \mathcal{V}(K) \setminus \{p\}$ (cf. Lem. 3.3).

**Remark 5.5** We distinguish between four types of basis functions of $S_{k,0}(\mathcal{T})$:
(a) basis function supported on one tetrahedron $K \in \mathcal{T}$,

(b) basis functions supported on $\omega_F$ for some facet $F \in \mathcal{F}_\Omega$ (which do not belong to the span of basis functions of Type a),

(c) basis functions supported on $\omega_{E'}$ for some edge $E' \in \mathcal{E}_\Omega$ (which do not belong to the span of basis functions of Type a), b).

(d) basis functions associated with a node $z \in \mathcal{V}_\Omega$.

Note that only basis functions which are not identically zero on $\omega_E$ are relevant. Hence, it suffices to evaluate the integral $\int_{K} p_{k-1}^E \text{div} \ b$ for relevant basis functions $b$ of these different types.

Basis functions of Type (a): These basis functions only exist for $k \geq 4$. Let $K$ be a tetrahedron in the patch $\mathcal{T}_E$. In this case the basis functions are given by

$$B_{K}^{\mu,\ell} = \lambda_{K,F}^{\mu+1,F} w_{\ell} \quad \mu = (\mu_y)_{y \in \mathcal{V}(K)} \in \mathbb{N}_0^4, \ |\mu| = k - 4,$$

where $\{w_{\ell}, 1 \leq \ell \leq 3\}$ is a basis in $\mathbb{R}^3$. From (5.3) we conclude that

$$\int_{\Omega} p_{k-1}^E \text{div} B_{K}^{\mu,\ell} = \int_{K} p_{k-1}^E \text{div} B_{K}^{\mu,\ell} = 0.$$

Basis functions of Type (b): Let $K, K'$ be two adjacent tetrahedra with common facet $F$. Then the basis functions of this type are given by (cf. Notation 3.1)

$$B_{F}^{\mu,\ell} = w_{\ell} \begin{cases} 
\lambda_{K,F}^{\mu+1,F} & \text{on } K, \\
\lambda_{K',F}^{\mu+1,F} & \text{on } K', \\
0 & \text{otherwise},
\end{cases}$$

for $\mu = (\mu_y)_{y \in \mathcal{V}(F)} \in \mathbb{N}_0^3, \ |\mu| = k - 3$. Similar as before, we conclude that in this case

$$\int_{\Omega} p_{k-1}^E \text{div} B_{F}^{\mu,\ell} = \int_{\omega_F \cap \omega_E} p_{k-1}^E \text{div} B_{F}^{\mu,\ell} = 0.$$

Basis functions of Type (c): Recall that $p_{k-1}^E$ has support on the edge patch $\omega_E$ with respect to the critical edge $E$. The basis functions associated to an edge $E' \in \mathcal{E}_\Omega$, have support $\omega_{E'}$ and are defined, for $K' \in \mathcal{T}_{E'}$ by

$$B_{E'}^{\mu,\ell} \big|_{K'} = \lambda_{K',E'}^{\mu+1,E'} w_{\ell},$$

where $\mu = (\mu_y)_{y \in \mathcal{V}(E)} \in \mathbb{N}_0^2$ with $|\mu| = k - 2$. Hence, it holds

$$\int_{\Omega} p_{k-1}^E \text{div} B_{E'}^{\mu,\ell} = \int_{\omega_E \cap \omega_{E'}} p_{k-1}^E \text{div} B_{E'}^{\mu,\ell}. \quad (5.4)$$

Since $E$ is an inner edge, the edge patch $\omega_E$ consists of four tetrahedra and therefore $\omega_E \cap \omega_{E'}$
(i) is the union of two tetrahedra, or

(ii) is \( \omega_E \) (for \( E = E' \)).

We discuss these two cases separately.

**Case (i):** Without loss of generality we assume \( \omega_E \cap \omega_{E'} = K_1 \cup K_2 \) (cf. Notation 5.3). This implies \( E' = [z, q] \) for \( z \in \mathcal{V}(E) \) and \( q \notin \mathcal{V}(E) \). Then

\[
\int_{K_1 \cup K_2} p_{k-1}^E \operatorname{div} B^{\mu, \ell}_{E'} = \sum_{i=1}^2 \frac{(-1)^i}{|K_i|} \int_{K_i} P_{k-1}^{(0,3)} (1 - 2 \lambda_{K_i, p}) \partial_{w_\ell} \left( \lambda_{K_i, E'}^{\mu+1} \right).
\]

In view of (5.3), this integral can be different from zero only if \( p = z \) and \( \mu = (k-2) e_i^{E'} \). For \( \mu = (k-2) e_i^{E'} \) and \( \tilde{\mu} := (k-1) e_i^{E'} + e_i^{E'} \), we conclude from (5.3) that

\[
\int_{K_1 \cup K_2} p_{k-1}^E \operatorname{div} B^{\mu, \ell}_{E'} = \sum_{i=1}^2 \frac{(-1)^i}{|K_i|} \int_{K_i} P_{k-1}^{(0,3)} (1 - 2 \lambda_{K_i, z}) \partial_{w_\ell} \left( \lambda_{K_i}^{i} \right)
\]

A pullback to the reference element leads to

\[
\int_{K_1 \cup K_2} p_{k-1}^E \operatorname{div} B^{\mu, \ell}_{E'} = -\frac{1}{|K|} \int_{K} P_{k-1}^{(0,3)} (1 - 2 \lambda_{z}) \lambda_{k-1} (\partial_{w_\ell} \lambda_{K_1, q} - \partial_{w_\ell} \lambda_{K_2, q}). \tag{5.5}
\]

Since the edge \( E \) is critical the facet \( F_{i, q} \subset \partial K_i, i = 1, 2 \), opposite of \( q \) lies on one plane and the function

\[
\phi = \begin{cases} 
\lambda_{K_1, q} & \text{on } K_1, \\
\lambda_{K_2, q} & \text{on } K_2,
\end{cases}
\]

is globally affine on \( K_1 \cup K_2 \). As a direct consequence, the difference in the right in (5.5) vanishes and we conclude that \( \int_{\omega_E \cap \omega_{E'}} p_{k-1}^E \operatorname{div} B^{\mu, \ell}_{E'} = 0 \) also holds for \( z = p \) and \( \mu = (k-2) e_i^{E'} \).

**Case (ii):** Let \( \omega_E = \bigcup_{i=1}^4 K_i \) with the local enumeration as in Notation 5.3. We choose \( q \in \mathcal{V}(E) \) so that \( E = [q, p] \). Then we have

\[
\int_{\omega_q} p_{k-1}^E \operatorname{div} B^{\mu, \ell}_{E'} = \sum_{i=1}^4 \frac{(-1)^i}{|K_i|} \int_{K_i} P_{k-1}^{(0,3)} (1 - 2 \lambda_{K_i, p}) \partial_{w_\ell} \left( \lambda_{K_i, E'}^{\mu+1} \right).
\]

In view of (5.3) this integral can be different from zero only if \( \mu = (k-2) e_p^{E} \). For \( \mu =

18
Choose \( w \) so the sum in the right-hand side of (5.6) is zero. A similar argument can be used if we

and

\[ w \lambda_{K_i, q} \]

We choose \( w \) to be a unit vector perpendicular to \( K \) (and \( \hat{p} \)). Then again by continuity, we have that

\[ \partial_{w_2} \lambda_{K_1,q} = \partial_{w_2} \lambda_{K_4,q}, \]

and

\[ \partial_{w_2} \lambda_{K_2,q} = \partial_{w_2} \lambda_{K_3,q}, \]

so the sum in the right-hand side of (5.6) is zero. A similar argument can be used if we choose \( w_3 \) to be a unit vector perpendicular to \( w_1 \) which lies in the plane through \( F_4 \) (and \( F_2 \)). Then again by continuity, we have that

\[ \partial_{w_2} \lambda_{K_1,q} = \partial_{w_2} \lambda_{K_4,q}, \]

and

\[ \partial_{w_2} \lambda_{K_2,q} = \partial_{w_2} \lambda_{K_3,q}, \]

so the sum in the right-hand side of (5.6) is zero. A similar argument can be used if we choose \( w_3 \) to be a unit vector perpendicular to \( w_1 \) which lies in the plane through \( F_1 \) (and \( F_3 \)).

**Basis functions of Type (d):** In this case, the basis functions are given by

\[ B^{(d)}_{K|K} = w_i \lambda_{K,z}^k, \quad \forall K \in T_z, \]

for linearly independent vectors \( w_i \in \mathbb{R}^3, 1 \leq \ell \leq 3 \), to be fixed below. We distinguish the following two relevant cases; for all other cases the integral is zero due to \( |\omega_E \cap \omega_z| = 0 \).

(i) \( z \in \mathcal{V}_\Omega \) is an endpoint of \( E \): then the common support is the union of four tetrahedra

(ii) \( z \in \mathcal{V}_\Omega \) is not an endpoint of \( E \) but \( z \in \omega_E \): then the common support is the union of two tetrahedra

We will consider both cases by introducing the number \( t_z \) of tetrahedra in the common support. We use Notation 5.3 and assume w.l.o.g. that \( K_1 \cup K_2 \subset \omega_E \cap \omega_z \). Therefore

\[
\int_{\omega_z} p_{k-1}^{E,p} \operatorname{div} B^{(d)}_z = k \sum_{i=1}^{t_z} \left( \frac{(-1)^i}{|K_i|} \int_{K_i} P_{k-1}^{(0,3)} (1 - 2 \lambda_{K_i,p}) \lambda_{K_{i,z}}^{k-1} \right) \partial_{w_\ell} \lambda_{K_{i,z}}
\]

\[
= \left( \frac{k}{|K|} \int_K P_{k-1}^{(0,3)} (1 - 2 \lambda_{\hat{p}}) \lambda_{\hat{p}}^{k-1} \right) \sum_{i=1}^{t_z} (-1)^i \partial_{w_\ell} \lambda_{K_{i,z}}. \]  \hspace{1cm} (5.7)
From Lemma 3.3, we conclude that this is zero for \( z \neq p \). For \( p = z \), which implies \( \iota_z = 4 \), let \( w_1 \in \mathbb{R}^3 \) be the vector tangential to the critical edge \( E \) and let \( w_2, w_3 \in \mathbb{R}^3 \) be two unit vectors perpendicular to \( w_1 \) and such that they lie on the two planes, respectively. By continuity of \( B_z^2 \), the terms in the sum cancel in all cases due to changing signs.

The following proposition summarizes these findings.

**Proposition 5.6** Let \( k \geq 1 \). For any critical edge \( E \in \mathcal{E}_\Omega \) and any \( p \in \mathcal{V}(E) \), let \( p_{k-1}^{E,p} \in \mathcal{P}_{k-1,0}(\mathcal{T}) \) be as in Definition 5.4. Then

\[
\left( p_{k-1}^{E,p}, \text{div} \mathbf{v} \right)_{L^2(\Omega)} = 0, \quad \forall \mathbf{v} \in \mathcal{S}_{k,0}(\mathcal{T}).
\]

### 5.2 Critical pressure for outer critical edges

In this section, we consider critical edges that lie on the boundary of the domain \( \Omega \) and construct corresponding critical pressures.

**Definition 5.7** (Critical pressure for outer edges) Let \( k \geq 1 \). Let \( E \in \mathcal{E}_{\partial \Omega} \) be an outer critical edge in \( \mathcal{T} \) and let \( \mathcal{T}_E = \{ K_i : 1 \leq i \leq \iota_E \} \), for some \( \iota_E \in \{1, 2, 3\} \). For \( p \in \mathcal{V}(E) \), the critical pressure \( p_{k-1}^{E,p} \in \mathcal{P}_{k-1,0}(\mathcal{T}) \) is given by

\[
p_{k-1}^{E,p} := \sum_{i=1}^{\iota_E} \frac{(-1)^i}{|K_i|} \chi_{K_i} p_{k-1}^{(0,3)} \left( 1 - 2\lambda_{K_i,p} \right),
\]

where, again, \( \chi_{K_i} \) denotes the characteristic function on \( K_i \).

In the following, we prove that for any outer critical edge \( E \in \mathcal{E}_{\partial \Omega} \) and any \( p \in \mathcal{V}(E) \)

\[
\left( p_{k-1}^{E,p}, \text{div} \mathbf{v} \right)_{L^2(\Omega)} = 0, \quad \forall \mathbf{v} \in \mathcal{S}_{k,0}(\mathcal{T}).
\]

Similar as in the previous section, we evaluate the integral over \( \omega_E \) and consider the four types (a)-(d) of the basis functions of \( \mathcal{S}_{k,0}(\mathcal{T}) \) listed in Remark 5.5. As before, the cases (a) and (b) are straightforward and we omit to present this computation.

**Case (c):** Since edges in \( \mathcal{E}_{\partial \Omega} \) do not carry degrees of freedom for \( \mathcal{S}_{k,0}(\mathcal{T}) \) we can restrict to inner edges \( E' \in \mathcal{E}_\Omega \). In particular this implies \( E \neq E' \). We have to consider two non-trivial subcases

- (c.i) \( \mathcal{T}_E \) and \( \mathcal{T}_{E'} \) share two tetrahedra,
- (c.ii) \( \mathcal{T}_E \) and \( \mathcal{T}_{E'} \) share one tetrahedron.

**Case (c.i):** W.l.o.g. the edge \( E' \) is shared by \( K_1 \) and \( K_2 \) and we set \( q = E \cap E' \) for \( q \in \mathcal{V}(E) \), i.e. \( E' = [z, q] \) for some \( z \in \mathcal{V} \). The basis functions for the velocity for this edge are given by

\[
B_{E'}^\mu = w_\ell \begin{cases} 
\chi_{K_i}^{\mu+1} E' & \text{on } K \in \mathcal{T}_{E'}, \\
0 & \text{otherwise},
\end{cases}
\]

20
From the same analysis as in (5.3), we conclude that this integral vanishes unless \( p = q \) and \( \mu = (k - 2) e_{E'}^r \). In this case, we have

\[
\int_{\omega_E \cap \omega_{E'}} p_{k-1}^{E,p} \text{div} \, B_{E'}^{\mu,\ell} = \sum_{i=1}^{2} \frac{(-1)^i}{|K|} \left( \int_{K} p_{k-1}^{(0,3)} (1 - 2\lambda_{K,i,p}) \lambda_{K,i,p}^{k-1} \right) \partial_{w_i} \lambda_{K,i,z} = \frac{1}{|K|} \int_{K} p_{k-1}^{(0,3)} (1 - 2\lambda_{p}) \lambda_{p}^{k-1} \sum_{i=1}^{2} (-1)^i \partial_{w_i} \lambda_{K,i,z}. \tag{5.8}
\]

Since the facets \( F_i \subset \partial K \) opposite to \( z \), \( i = 1, 2 \), lie in one plane, the function \( \varphi : K_1 \cup K_2 \to \mathbb{R} \), \( \varphi|_{K_i} = \lambda_{K,i,z} \), \( i = 1, 2 \) is affine on \( K_1 \cup K_2 \) and the sum on the right-hand side in (5.8) vanishes due to the alternating sign.

**Case (c.ii):** Let \( K = \omega_E \cap \omega_{E'} \). In this case \( E, E' \) are edges of \( K \) with empty intersection. By repeating the arguments in (5.3), it follows that the integral \( \int_{K} p_{k-1}^{E,p} \text{div} \, B_{E'}^{\mu,\ell} \) vanishes.

**Case (d):** Let \( z \in \mathcal{V}_T \), in particular \( z \notin \mathcal{V}(E) \). In this case the basis function is given by

\[
B_{z}^{\ell} \big|_{K} = \mathbf{w}_i \lambda_{K,z}^{k}
\]

for \( K \in \mathcal{T}_z \). It follows that the integral \( \int_{\omega_{z}} p_{k-1}^{E,p} \text{div} \, B_{z}^{\ell} \) is zero by repeating the arguments in (5.3).

The following proposition summarizes the findings of Section 5.

**Proposition 5.8** Let \( k \geq 1 \) and let \( E \in \mathcal{E} \) be a critical edge in \( \mathcal{T} \). For \( p \in \mathcal{V}(E) \), the critical pressures \( p_{k-1}^{E,p} \) as in Def. 5.4, Def. 5.7 satisfy

\[
\left( p_{k-1}^{E,p}, \text{div} \, v \right)_{L^2(\Omega)} = 0, \quad \forall v \in S_{k,0}(\mathcal{T}).
\]

Consequently, if \( \mathcal{T} \) contains a critical edge \( E \in \mathcal{E}_\Omega \), then \( p_{k-1}^{E,p}, p \in \mathcal{V}(E) \), are critical pressures and the pair \((S_{k,0}(\mathcal{T}), \mathbb{P}_{k-1,0}(\mathcal{T}))\) is not inf-sup stable.

### 6 CR stabilization for critical edges

In this section, we consider critical edges \( E \in \mathcal{E} \) which are contained in the nodal patch \( \omega_z \) of some \( z \in \mathcal{V}_\Omega \). We show that the associated critical pressures \( p_{k-1}^{E,p}, p \in \mathcal{V}(E) \), are eliminated by testing \( b_h \left( p_{k-1}^{E,p}, \cdot \right) \) with some non-conforming Crouzeix-Raviart functions which are locally supported in \( \omega_z \). We distinguish between odd and even polynomial degree.
6.1 Stabilization for even polynomial degree

In the following, we prove for even \( k \geq 4 \) that those critical pressures for the conforming \((S_{k,0}(T), \mathbb{P}_{k-1,0}(T))\) Stokes element which have been defined in the previous section are “eliminated” by basic Crouzeix-Raviart elements.

**Theorem 6.1** Let \( k \geq 4 \) be even. Let \( E \in \mathcal{E} \) be a critical edge and assume that there is a tetrahedron \( K \in \mathcal{T}_E \), which has \( E \) as an edge. Let \( \mathbf{w}_\ell \in \mathbb{R}^3 \), \( \ell = 1, 2, 3 \), denote three linearly independent vectors. For \( p \in \mathcal{V}(E) \), consider the critical pressure function \( p_{E,p}^{k-1} \) defined in Definition 5.4 (if \( E \in \mathcal{E}_i \)) or Definition 5.7 (if \( E \in \mathcal{E}_{2i} \)). Then any function \( p \in \text{span} \{p_{E,p}^{k-1} : p \in \mathcal{V}(E)\} \) which satisfies

\[
(p, \text{div} \mathbf{v})_{L^2(\Omega)} = 0, \quad \forall \mathbf{v} \in \text{span} \{B^\text{CR,K}_k \mathbf{w}_\ell : 1 \leq \ell \leq 3\},
\]

is the zero function.

**Proof.** We choose \( K = K_1 \), where \( K_1 \) is as in Definition 5.4 and Definition 5.7, respectively. Then,

\[
\int_{\Omega} p_{E,p}^{k-1} \text{div} \left( B^\text{CR,K}_k \mathbf{w}_\ell \right) = -\frac{1}{|K|} \int_K p_{k-1}^{(0,3)} (1 - 2\lambda_{K,p}) \text{div} \left( \sum_{y \in \mathcal{V}(K)} Q_k(1 - 2\lambda_{K,y}) - 1 \right) \mathbf{w}_\ell \nonumber
\]

\[
= -\frac{1}{(k+1)|K|} \int_K p_{k-1}^{(0,3)} (1 - 2\lambda_{K,p}) \text{div} \left( \sum_{y \in \mathcal{V}(K)} (L_{k+1} - L_k)' (1 - 2\lambda_{K,y}) \mathbf{w}_\ell \right) 
\]

\[
= \frac{2}{(k+1)|K|} \int_K p_{k-1}^{(0,3)} (1 - 2\lambda_{K,p}) \sum_{y \in \mathcal{V}(K)} (L_{k+1} - L_k)'' (1 - 2\lambda_{K,y}) \partial_{w_\ell} \lambda_{K,y} 
\]

\[
= \frac{2}{(k+1)|K|} \sum_{y \in \mathcal{V}(K)} \partial_{w_\ell} \lambda_{K,y} \int_K p_{k-1}^{(0,3)} (1 - 2\lambda_{K,p}) ((L_{k+1} - L_k)'' (1 - 2\lambda_{K,y})).
\]

We employ an affine pullback \( \chi_{K,y} : \hat{K} \to K \) to the reference element which depends on the summation index \( y \in \mathcal{V}(K) \) such that \( \chi_{K,y}^{-1}(\mathbf{p}) = \hat{\mathbf{p}} = (1,0,0)^T \) and therefore \( \lambda_{K,p} \circ \chi_{K,y}(\mathbf{x}) = x_1 \). For \( y \in \mathcal{V}(K) \setminus \{\mathbf{p}\} \), we require in addition that \( \chi_{K,y} \) satisfies \( \chi_{K,y}^{-1}(\mathbf{y}) = \hat{\mathbf{y}} = (0,1,0)^T \) and \( \lambda_{K,y} \circ \chi_{K,y}(\mathbf{x}) = x_2 \). Then,

\[
\int_{\Omega} p_{E,p}^{k-1} \text{div} \left( B^\text{CR,K}_k \mathbf{w}_\ell \right) = \frac{2}{k+1} \frac{1}{|K|} \sum_{y \in \mathcal{V}(K)} \partial_{w_\ell} \lambda_{K,y} I_y,
\]

with

\[
I_y := \begin{cases} 
\int_{\hat{K}} p_{k-1}^{(0,3)} (1 - 2x_1) (L_{k+1} - L_k)'' (1 - 2x_1) & y = \mathbf{p}, \\
\int_{\hat{K}} p_{k-1}^{(0,3)} (1 - 2x_1) (L_{k+1} - L_k)'' (1 - 2x_2) & y \in \mathcal{V}(K) \setminus \{\mathbf{p}\}.
\end{cases}
\] (6.1)
These integrals are computed in Appendix B. From there, we get

\[ I_y = \begin{cases} 
\frac{k+1}{4} & y = p, \\
\frac{1}{2}(-1)^{k-1} & y \in \mathcal{V}(K) \setminus \{p\}.
\end{cases} \tag{6.2} \]

Hence, by taking into account that \( k \) is even, we have

\[
\int_{\Omega} p_{k-1}^{E,p} \text{div} \left( B_{k}^{CR,K} w_{\ell} \right) = \frac{1}{k+1} \frac{1}{|K|} \left( \frac{k+1}{2} \partial_{w_{\ell}} \lambda_{K,p} - \sum_{y \in \mathcal{V}(K) \setminus \{p\}} \partial_{w_{\ell}} \lambda_{K,y} \right).\
\]

We use \( \sum_{y \in \mathcal{V}(K)} \lambda_{K,y} = 1 \), and obtain

\[
\int_{\Omega} p_{k-1}^{E,p} \text{div} \left( B_{k}^{CR,K} w_{\ell} \right) = \frac{k+3}{k+1} \frac{1}{|K|} \partial_{w_{\ell}} \lambda_{K,p}.\
\]

The assertion is proved if there exist two vectors \( s, t \in \mathbb{R}^3 \) such that the Gram’s matrix

\[
m = (\partial_{r} \lambda_{K,p})_{r \in \mathcal{V}(E)} \in \mathbb{R}^{2 \times 2} \]

is regular. Let \( v \in \mathcal{V}(K) \setminus \mathcal{V}(E) \) and \( E = [p, q] \). We choose \( s = q - p \) and \( t = q - v \) to obtain \( \partial_{s} \lambda_{K,p} = -1, \partial_{s} \lambda_{K,q} = 1, \partial_{t} \lambda_{K,p} = 0, \partial_{t} \lambda_{K,q} = 1 \). This implies that \( m \) is regular and the theorem is proved. \( \blacksquare \)

### 6.2 Stabilization for odd polynomial degree

In this section, we prove an analogue to Theorem 6.1 for odd \( k \). However, we need more Crouzeix-Raviart functions in order to eliminate the critical pressures as can be seen from the following theorem.

**Theorem 6.2** Let \( k \geq 3 \) be odd. Let \( E \in \mathcal{E} \) be a critical edge and assume that there is a tetrahedron \( K \in \mathcal{T}_{E} \) such that there are two facets \( F, G \in \mathcal{F}(K) \) which satisfy

\[ F, G \in \mathcal{F}_{\Omega}, \quad E \subset F, \quad E \not\subset G. \]

For \( p \in \mathcal{V}(E) \), consider the critical pressure functions \( p_{k-1}^{E,p} \) defined in Definition 5.4 (if \( E \in \mathcal{E}_{\Omega} \)) or Definition 5.7 (if \( E \in \mathcal{E}_{\partial \Omega} \)). For any \( F' \in \mathcal{F}_{\Omega} \), let \( w_{\ell}^{F'} \in \mathbb{R}^3, 1 \leq \ell \leq 3 \), denote some basis in \( \mathbb{R}^3 \). Then, any function \( p \in \text{span} \left\{ p_{k-1}^{E,p} : p \in \mathcal{V}(E) \right\} \) which satisfies

\( (p, \text{div} v)_{L^2(\Omega)} = 0 \quad \forall v \in \text{span} \left\{ B_{k}^{CR,F'} w_{\ell}^{F'} : 1 \leq \ell \leq 3, F' \in \{F, G\} \right\} \)

is the zero function.

**Proof.** We will use the two facets \( F, G \) with corresponding Crouzeix-Raviart functions as test functions and derive the assertion. First, let \( F \in \mathcal{F}_{\Omega} \) be such that \( E \subset F \). We recall,
that $B^\text{CR,F}_k$ has support on two tetrahedra $K_1, K_2$ and $K_1 \cup K_2 = \omega_E \cap \omega_F$. For $s \in \mathbb{R}^3 \setminus \{0\}$, we compute

$$
\left( p_{k-1}^{E,p}, \text{div} \left( B^\text{CR,F}_k s \right) \right)_{L^2(\omega_s)} = \sum_{i=1}^{2} \frac{(-1)^i}{|K_i|} \int_{K_i} P_{k-1}^{(0,3)} (1 - 2\lambda_{K_i,p}) \text{div} (Q_k(1 - 2\lambda_{K_i,v_i})s),
$$

where $v_i$ is the vertex in $\mathcal{V}(K_i)$ opposite to $F$. For a single summand we get

$$
\int_{K_i} P_{k-1}^{(0,3)} (1 - 2\lambda_{K_i,p}) \text{div} (Q_k(1 - 2\lambda_{K_i,v_i})s) = -2 (\partial_\lambda \lambda_{K_i,v_i}) \int_{K_i} P_{k-1}^{(0,3)} (1 - 2\lambda_{K_i,p}) Q_k' (1 - 2\lambda_{K_i,v_i}) = -\frac{2}{k+1} (\partial_\lambda \lambda_{K_i,v_i}) \frac{|K_i|}{|K|} \int_{K} P_{k-1}^{(0,3)} (1 - 2\lambda_p) (L_{k+1} - L_k)^\prime\prime (1 - 2\phi),
$$

where we used an affine pullback $\chi_{K_i} : \hat{K} \to K_i$ which satisfies $\chi_{K_i}^{-1}(p) = \hat{p} = (1,0,0)^T$ and $\chi_{K_i}^{-1}(v_i) = \hat{v} = (0,1,0)^T$. Recall the computations (6.1), (6.2) from the previous section. Then,

$$
\int_{K} P_{k-1}^{(0,3)} (1 - 2\lambda_p) (L_{k+1} - L_k)^\prime\prime (1 - 2\phi) = \frac{1}{2} (-1)^{k-1}.
$$

The combination of these computations leads to

$$
\left( p_{k-1}^{E,p}, \text{div} B^\text{CR,F}_k s \right)_{L^2(\omega_s)} = \sum_{i=1}^{2} \frac{(-1)^i}{|K_i|} \int_{K_i} P_{k-1}^{(0,3)} (1 - 2\lambda_{K_i,p}) \text{div} (Q_k(1 - 2\lambda_{K_i,v_i})s) = \frac{(-1)^{k-1}}{(k+1)|K|} \partial_\lambda (\lambda_{K_1,v_1} - \lambda_{K_2,v_2}).
$$

Since the right-hand side does not depend on $p$, the test functions $B^\text{CR,F}_k s, s \in \mathbb{R}^3$, are not sufficient to eliminate both critical functions $p_{k-1}^{E,p}, p \in \mathcal{V}(E)$.

Next, we choose a facet for another Crouzeix-Raviart function. Let $K := K_1$ and $G \in \mathcal{F}_\Omega \setminus \{F\}$ be an inner facet which satisfies $E \not\subset G$. This implies

$$
\mathcal{T}_G \cap \mathcal{T}_E = \{K\}.
$$

Let $y$ denote the vertex in $K$ opposite to $G$ and hence $y \in \mathcal{V}(E)$. Next, we compute

$$
\left( p_{k-1}^{E,p}, \text{div} B^\text{CR,G}_k t \right)_{L^2(\Omega)} = \frac{1}{|K|} \int_{K} P_{k-1}^{(0,3)} (1 - 2\lambda_{K,p}) \text{div} (Q_k(1 - 2\lambda_{K,y})t).
$$

For $p = y$, we employ an affine transform $\chi_K : \hat{K} \to K$ with $\chi_K^{-1}(p) = \hat{p} = (1,0,0)^T$. Then,

$$
\left( p_{k-1}^{E,p}, \text{div} B^\text{CR,G}_k t \right)_{L^2(\Omega)} = \frac{2}{|K|} (\partial_t \lambda_{K,y}) \int_{K} P_{k-1}^{(0,3)} (1 - 2\lambda_p) Q_k'(1 - 2\lambda_p) \frac{1}{2} \frac{1}{|K|} (\partial_t \lambda_{K,y}).
$$

24
For \( p \neq y \), we employ an affine transform \( \chi_K : \hat{K} \to K \) with \( \chi_K^{-1}(p) = \hat{p} = (1, 0, 0)^T \) and \( \chi_K^{-1}(y) = \hat{y} = (0, 1, 0)^T \). Then,
\[
(p_{k-1}^{E,p}, \text{div} \ B_{k}^{CR,G,t})_{L^2(\Omega)} = \frac{2}{|\hat{K}|} \left( \partial_t \lambda_{K,y} \right) \int_{\hat{K}} P_{k-1}^{(0,3)} (1 - 2\lambda_{\hat{p}}) Q_k' (1 - 2\lambda_{\hat{y}})
\]
\[
\overset{(6.2)}{=} \frac{1}{(k+1)|\hat{K}|} \left( \partial_t \lambda_{K,y} \right).
\]

We define the Gram’s matrix
\[
m = \begin{bmatrix}
\partial_s (\lambda_{K_1,v_1} - \lambda_{K_2,v_2}) & \partial_t \lambda_{K,y} \\
\partial_s (\lambda_{K_1,v_1} - \lambda_{K_2,v_2}) & \frac{k+1}{2} \partial_t \lambda_{K,y}
\end{bmatrix},
\]
and choose \( s \) as the unit vector which is orthogonal to the facet \( F \) and points into \( K_2 \). In this way, \( \partial_s (\lambda_{K_1,v_1} - \lambda_{K_2,v_2}) =: \theta < 0 \). We choose \( t := y - u \) for some \( u \in V(K) \setminus \{y\} \).

Hence, \( \partial_t \lambda_{K,y} = 1 \) and
\[
m = \begin{bmatrix}
\theta & 1 \\
\theta & \frac{k+1}{2}
\end{bmatrix}.
\]

Since \( k \geq 3 \) this matrix is non-singular and this implies the claim. \( \blacksquare \)

\section*{A Non-conforming Crouzeix-Raviart functions in higher dimensions}

The construction of non-conforming Crouzeix-Raviart functions in three dimensions is based on the definition of the univariate polynomial \( Q_k \in \mathbb{P}_k([-1, 1]) \) in (3.6). In this section, we generalize this construction to arbitrary dimension \( d \geq 2 \) and \( Q_k \) in (3.6) will be a special case for \( d = 3 \).

Let \( K \subset \mathbb{R}^d \) be a closed simplex with vertices \( z_i, 1 \leq i \leq d+1 \), which form the set \( \mathcal{V}(K) \). The \((d-1)\)-dimensional facet in \( \partial K \) opposite to \( z \in \mathcal{V}(K) \) is denoted by \( F_z \) and the facets are collected in the set \( \mathcal{F}(K) \). The barycentric coordinates \( \lambda_{K,z}, z \in \mathcal{V}(K) \), are characterized by the conditions \( \lambda_{K,z} \in \mathbb{P}_1(K), \lambda_{K,z}(y) = \delta_{z,y} \) for all \( y, z \in \mathcal{V}(K) \).

Our goal is to define a polynomial \( Q_{d,k} \in \mathbb{P}_k([-1, 1]), d \geq 2, k \geq 1 \), such that the composition \( Q_{d,k}(1 - 2\lambda_{K,z}) \) satisfies
\[
Q_{d,k}(1 - 2\lambda_{K,z})|_{F_z} = 1 \text{ and } \forall F \in \mathcal{F}(K) \setminus \{F_z\} : \int_F Q_{d,k}(1 - 2\lambda_{K,z}) q = 0 \quad \forall q \in \mathbb{P}_{k-1}(F).
\]

(A.1)

Following the construction in Section 3, the non-conforming Crouzeix-Raviart functions for even polynomial degree \( k \geq 2 \) are supported on a single simplex \( K \) and given by
\[
B_{d,k}^{CR,K} := \begin{cases}
\left( \sum_{z \in \mathcal{V}(K)} Q_{d,k}(1 - 2\lambda_{K,z}) \right) - 1 & \text{on } K, \\
0 & \text{otherwise}.
\end{cases}
\]
For odd polynomial degree $k \geq 1$ they are supported on the two adjacent simplices $K_1, K_2$ of an inner facet $F$ and given by

$$B_{d,k}^{\text{CR},F} := \begin{cases} Q_{d,k}(1-2\lambda_{K,z}) & \text{for } K \in \{K_1, K_2\}, \\ 0 & \text{otherwise}, \end{cases} \quad (A.2)$$

where $\lambda_{K,z}$ denotes the barycentric coordinate for the vertex $z \in V(K)$ opposite to $F$.

Properties (A.1) allow us to repeat the arguments in the proof of Theorem 3.5 which then imply that $B_{d,k}^{\text{CR},K}$ and $B_{d,k}^{\text{CR},F}$ belong to the Crouzeix-Raviart space $CR_{\max,0}^k(T)$ for a conforming simplicial finite element mesh $T$ of a $d$-dimensional polytope $\Omega$.

Finally, we construct the polynomial $Q_{d,k}$. For ease of notation, we set $m = d - 2$ and define the polynomial $P_{k+m} \in \mathbb{P}_{k+m}([-1,1])$ as a linear combination of Legendre polynomials

$$P_{k+m} = \sum_{\ell=0}^{m} \beta_{k,\ell} L_{k+\ell}.$$ 

The coefficients $\beta_{k,\ell}$ are defined as the solutions of the linear system

$$\frac{1}{2^n n!} \sum_{\ell=0}^{m} \beta_{k,\ell}(\ell + k + n)! \ell! (\ell + k - n)! = \delta_{n,m}, \quad 0 \leq n \leq m, \quad (A.3)$$

where we use the convention that $\mu!/\nu! = 0$ for $\mu \geq 0$ and $\nu < 0$.

Lemma A.1 Let $d \geq 2$ and set $m = d - 2$. The polynomial

$$Q_{d,k} = P_{k+m}^{(m)}$$

belongs to $\mathbb{P}_k([-1,1])$ and satisfies the conditions in (A.1).

**Proof.** Let $C^{(n)}_n \in \mathbb{P}_n([-1,1])$ denote the Gegenbauer polynomials for $\alpha > -1/2$, $\alpha \neq 0$. We combine [12, 18.7.9] with [12, 18.9.19] to get for $\ell \leq n$

$$L_n^{(\ell)} = \frac{(2\ell)!}{2^\ell \ell!} C^{(1/2+\ell)}_{n-\ell}.$$ 

(A.4)

We use [12, Table 18.6.1] to obtain

$$L_n^{(\ell)}(1) = \frac{(2\ell)!}{2^\ell \ell!} \frac{(1 + 2\ell)_{n-\ell}}{(n-\ell)!} = \frac{(n + \ell)!}{2^\ell \ell! (n-\ell)!}.$$ 

Thus,

$$Q_{d,k}(1) = P_{k+m}^{(m)}(1) = \sum_{\ell=0}^{m} \beta_{k,\ell} L_{k+\ell}^{(m)}(1) = \sum_{\ell=0}^{m} \beta_{k,\ell} \frac{(k + \ell + m)!}{2^m m! (k + \ell - m)!}.$$ 

Condition (A.3) for $n = m$ shows $Q_{d,k}(1) = 1$. Since $Q_{d,k}((1-2\lambda_{K,z})|_{F_a}) = Q_{d,k}(1)$ the first condition in (A.1) follows.
Next, we prove the second condition in (A.1). For $z \in \mathcal{V}(K)$, let $F \in \mathcal{F}(K) \setminus \{F_z\}$. We employ an affine pullback $\chi_K : \tilde{K} \to K$ to the reference element

$$\tilde{K} := \left\{ \mathbf{x} = (x_i)_{i=1}^d \in \mathbb{R}_{\geq 0}^d \mid x_1 + \ldots + x_d \leq 1 \right\}$$

in such a way that $\tilde{F} := \left\{ \mathbf{x} = (x_i)_{i=1}^d \in \tilde{K} \mid x_d = 0 \right\}$ is mapped to $F$. Then, it is sufficient to prove

$$\int_{\tilde{F}} Q_{d,k} (1 - 2x_1) x_1^{\alpha_1} \ldots x_{d-1}^{\alpha_{d-1}} dx_{d-1} \ldots dx_1 = 0, \quad \forall \alpha = (\alpha_i)_{i=1}^{d-1} \in \mathbb{N}_0^{d-1}, \ |\alpha| \leq k - 1.

(A.5)

We set $\alpha' := (\alpha_i)_{i=2}^{d-1}$ and define

$$G(x_1) := \int_0^{1-x_1} \int_0^{1-x_1-x_2} \ldots \int_0^{1-x_1-x_2-\ldots-x_{d-2}} x_1^{\alpha_1} \ldots x_{d-1}^{\alpha_{d-1}} dx_{d-1} \ldots dx_2.

Hence, condition (A.5) is equivalent to

$$\int_0^1 Q_{d,k} (1 - 2x_1) G(x_1) \, dx_1 = 0.

By using integration by parts, one derives that this condition is equivalent to

$$0 = \int_0^1 Q_{d,k} (1 - 2x_1) G(x_1) \, dx_1 = -\sum_{\ell=1}^m 2^{-\ell} P_{k+m}(1 - 2x_1) G^{(\ell-1)}(x_1) \bigg|_0^1 + 2^{-m} \int_0^1 P_{k+m}(1 - 2x_1) G^{(m)}(x_1) \, dx_1.

(A.6)

The integral in the definition of $G$ can be evaluated explicitly and we get

$$G(x_1) = \frac{\alpha'!}{(m + |\alpha'|)!} x_1^{\alpha_1} (1 - x_1)^{m + |\alpha'|}.$$

Clearly, $G \in \mathbb{P}_{m+k-1}([0,1])$ and $G^{(\ell-1)}(1) = 0$ for all $1 \leq \ell \leq m$. We use this, $G^{(m)} \in \mathbb{P}_{k-1}([0,1])$, and the orthogonality relations of the Legendre polynomials in (A.6) to get the equivalent condition

$$0 = \sum_{\ell=1}^m 2^{-\ell} P_{k+m}(1) G^{(\ell-1)}(0).

However, the property $P_{k+m}(1) = 0$ for $1 \leq \ell \leq m$ follows from the first conditions $n = 0, 1, \ldots, m - 1$ in the definition of $\beta_{k,\ell}$ in (A.3). ■

Remark A.2 The $m$-th order derivative of $P_{k+m}$ in the definition of $Q_{d,k}$ can be avoided by employing the relation (A.4) for Gegenbauer and Legendre polynomials. We get with $m = d - 2$

$$Q_{d,k} = \frac{(2m)!}{2^m m!} \sum_{\ell=0}^m \beta_{k,\ell} C_{k+\ell-m}^{(1/2+m)}$$

27
where $C^{(\lambda)}_{\nu}$ is set to zero for $\nu < 0$ and we emphasize that the superscript $\left(\frac{1}{2} + m\right)$ does not denote a derivative but is the parameter in the Gegenbauer polynomial related to the corresponding weight function $\left(1 - x^2\right)^m$ in the orthogonality relation. The coefficients $\beta_{k,\ell}$ can be expressed explicitly

$$\beta_{k,\ell} = \frac{(-1)^{m-\ell} \frac{m!}{2^m (\frac{m}{2})!} (2k + 2\ell + 1)}{\prod_{r=\ell+1}^{m+\ell+1} (2k + r)}$$

(the verification that $\beta_{k,\ell}$ satisfy (A.3) is quite tedious and skipped) so that we obtain the fully explicit formula

$$Q_{d,k} = \frac{(2m)!}{m!} \sum_{\ell=0}^{m} \left( \prod_{r=\ell+1}^{m+\ell+1} (2k + r)^{-1} \right) (-1)^{m-\ell} \binom{m}{\ell} C^{(\frac{1}{2}+m)}_{k+\ell-m}.$$  

B Computing some integrals involving Jacobi polynomials

In this appendix, we evaluate the integrals $I_y$ defined in (6.1).

**Lemma B.1** For any $y \in V(K)$, the integral $I_y$ in (6.1) is explicitly given by

$$I_y = \begin{cases} \frac{k+1}{4} \left( \frac{k-1}{2} \right) & y = p, \\ \frac{1}{2} (-1)^{k-1} & y \in V(K) \setminus \{p\}. \end{cases}$$

**Proof.** We start with $y = p$. We evaluate the inner integral explicitly and apply integration by parts to get

$$I_p = \int_0^1 P_{k-1}^{(0,3)} (1 - 2x_1) (L_{k+1} - L_k)'' (1 - 2x_1) \left( \int_0^{1-x_1} \int_0^{1-x_1-x_2} 1 dx_3 dx_2 \right) dx_1$$

$$= \frac{1}{2} \int_0^1 (x_1 - 1)^2 P_{k-1}^{(0,3)} (1 - 2x_1) (L_{k+1} - L_k)'' (1 - 2x_1) dx_1$$

$$= -\frac{1}{4} \left( x_1 - 1 \right)^2 P_{k-1}^{(0,3)} (1 - 2x_1) (L_{k+1} - L_k)' (1 - 2x_1) \bigg|_0^1$$

$$- \frac{1}{4} \int_0^1 \left( (x_1 - 1)^2 P_{k-1}^{(0,3)} (1 - 2x_1) \right)' (L_{k+1} - L_k)' (1 - 2x_1) dx_1$$

$$= \frac{1}{4} P_{k-1}^{(0,3)} (1) (L_{k+1} - L_k)' (1)$$

$$+ \frac{1}{8} \left( (x_1 - 1)^2 P_{k-1}^{(0,3)} (1 - 2x_1) \right)' (L_{k+1} - L_k) (1 - 2x_1) \bigg|_0^1$$

$$- \frac{1}{8} \int_0^1 \left( (x_1 - 1)^2 P_{k-1}^{(0,3)} (1 - 2x_1) \right)'' (L_{k+1} - L_k) (1 - 2x_1) dx_1.$$
Since \( g \in \mathbb{P}_{k-1} \), the orthogonality properties of the Legendre polynomials imply that the last term vanishes. Hence,

\[
I_{p} = \frac{1}{4} P_{k-1}^{(0,3)} \left( 1 \right) \left( L_{k+1} - L_k \right)'(1)
- \frac{1}{8} \left. \left( (x_1 - 1)^2 P_{k-1}^{(0,3)} (1 - 2x_1) \right)' \left( L_{k+1} - L_k \right) (1 - 2x_1) \right|_{x_1=0}.
\]

The endpoint properties of the Legendre and Jacobi polynomials (cf. (3.9), (3.5)) imply that the second term vanishes and

\[
I_{p} = \frac{k + 1}{4}.
\]

Next, we consider the integral for \( y \neq p \). We get again by integration by parts

\[
I_{y} = \int_{0}^{1} P_{k-1}^{(0,3)} (1 - 2x_1) \left( \int_{0}^{1-x_1} \left( 1 - x_1 - x_2 \right) \left( L_{k+1} - L_k \right)'(1 - 2x_2) dx_2 \right) dx_1
- \frac{1}{2} \int_{0}^{1} P_{k-1}^{(0,3)} (1 - 2x_1) \left. \left( (1 - x_1 - x_2) \left( L_{k+1} - L_k \right)'(1 - 2x_2) \right) \right|_{0}^{1-x_1} dx_1
- \frac{1}{2} \int_{0}^{1} P_{k-1}^{(0,3)} (1 - 2x_1) \int_{0}^{1-x_1} \left( L_{k+1} - L_k \right)'(1 - 2x_2) dx_2 dx_1
= \frac{1}{2} \left( L_{k+1} - L_k \right)'(1) \int_{0}^{1} P_{k-1}^{(0,3)} (1 - 2x_1) (1 - x_1) dx_1
+ \frac{1}{4} \int_{0}^{1} P_{k-1}^{(0,3)} (1 - 2x_1) \left( L_{k+1} - L_k \right) (1 - 2x_2) dx_2 dx_1
= \frac{1}{2} \left( L_{k+1} - L_k \right)'(1) \int_{0}^{1} P_{k-1}^{(0,3)} (1 - 2x_1) (1 - x_1) dx_1
+ \frac{1}{4} \int_{0}^{1} P_{k-1}^{(0,3)} (1 - 2x_1) \left( L_{k+1} - L_k \right) (2x_1 - 1) dx_1,
\]

where we used \( (L_{k+1} - L_k) (1) = 0 \) for the last equality. Again by the orthogonality properties of the Legendre polynomials, the last summand is zero and we get

\[
I_{y} = \frac{k + 1}{8} t_{k-1} \quad \text{with} \quad t_{k} := \int_{-1}^{1} P_{k}^{(0,3)} (t) (t + 1) dt.
\]

We employ [12, 18.9.5] for \( \beta = 2, \alpha = 0, n = k \), i.e.,

\[
(2k + 3) P_{k}^{(0,2)} = (k + 3) P_{k}^{(0,3)} + k P_{k-1}^{(0,3)},
\]

to obtain

\[
t_{k} = -\frac{k}{k + 3} t_{k-1} + \frac{2k + 3}{k + 3} \int_{-1}^{1} P_{k}^{(0,2)} (t) (t + 1) dt.
\]
The last integral has been computed in [5, Lem. C.1], and we obtain
\[ \iota_k = -\frac{k}{k+3}\iota_{k-1} + 4(-1)^k\frac{(2k+3)}{(k+1)(k+2)(k+3)}. \]

For \( k = 0 \), it holds \( P_0^{(0,3)}(x) = 1 \) and \( \iota_0 = 2 \). It is easy to verify by induction that \( \iota_k := 4(-1)^k / (k+2) \) satisfies the initial value and the recurrence. The combination with (B.1) leads to the assertion. ■

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