New Universality Classes
in One–Dimensional $O(N)$–Invariant Spin–Models
with an $n$–Parametric Action

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Abstract

An action with $n$ parameters, which generalizes the $O(N)$–$RP^{N–1}$–model, is considered in one dimension for general $N$. We use asymptotic expansion techniques to determine where the model becomes critical and show that for the actions considered there exists a family of hypersurfaces whose asymptotic behaviour determines a one–parameter family of new universality classes. They interpolate between the $O(N)$–vector–model–class and the $RP^{N–1}$–model–class. Furthermore continuum limits are discussed, including the exceptional case $N = 2$.  

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1 Introduction

The question of universality is central for lattice field theories. It is generally tacitly assumed that it does not matter which lattice discretization of a classical action is employed, if one is interested only in the continuum limit. Recently Caracciolo et al. [1] have sown doubt about this universality for the two–dimensional (2D) $O(N)$ nonlinear $\sigma$–models. Their claim that there are different universality classes once one introduces in addition to the standard ‘isovector’ coupling an ‘isotensor’ term in the action has generated some controversy. Both Niedermayer et al. [2] and Hasenbusch [3] argue that this violation of the universality dogma is only apparent and that it disappears as soon as one is looking at the right observables.

Since a definite mathematical answer to this question for the 2D models is out of reach, it seems worthwhile to study the question in the exactly solvable 1D model. While we were working on this question, Cucchieri et al. [4] produced a lengthy paper on the subject; their conclusions agree to a large extent with our findings. But we find that their paper does not answer all the questions that come to mind. For instance they study mostly one–parameter families of rather general coupling functions as well as a 2–parameter family, but in much less detail and generality.

In this paper we consider $n$–parameter families of actions that are natural generalization of the actions studied in [1]. We still obtain only a one–parameter family of universality classes, just as in the cases examined in [4]. On the other hand it turns out that these different universality classes reflect the true spectral properties of the transfer matrix, whereas the reinterpretations proposed by [2] and [3], which reduce everything to the ‘standard’ universality class are unrelated to the transfer matrix.

This paper is organized as follows: in section 2 we introduce an $n$–parameter family of actions for 1D $O(N)$–invariant spin models taking values on the sphere $S^{N-1}$, with nearest–neighbour interactions. It generalizes the generic mixed isovector/isotensor–model. The main result is then presented in section 3: using asymptotic expansion techniques we find where and in which way the models become critical. Especially, there are hypersurfaces on which an infinite number of new universality classes appear. In the next section it is shown that the restrictions on the Hamiltonian made in section 3 are not essential (in the case of non–negative parameters). In section 4 we also discuss the continuum limit and give a supplement to the paper [4]. Finally, our conclusions are stated in section 5.

2 Preliminaries

We want to study the critical behaviour of spin models which are generalizations of the well–known $O(N)$–$RP^{N-1}$–model [1]. Therefore, we consider nearest–neighbour interac-
tions given by polynomials \( \sum_{k=1}^{n} \beta_k x^k \) in the \( O(N) \)-invariant scalar product \( \sigma \cdot \sigma' \), i.e.

\[
S := \sum_{x} \sum_{k=1}^{n} \beta_k (\sigma_x \cdot \sigma_{x+1})^k.
\]

The spin \( \sigma \) takes values on the sphere \( S^{N-1} \subset \mathbb{R}^N \) (with the \( O(N) \)-invariant, normalized measure \( d\Omega(\sigma) \)) and all parameters \( \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{R}_{\geq 0}^n \) are nonnegative. The Hilbert space \( L^2(S^{N-1}) \) can be decomposed into the eigenspaces \( H_l \) of the Laplace–Beltrami–operator \( \Delta_{LB} \), corresponding to the eigenvalues \(-l(l + N - 2)\); since \( \Delta_{LB} \) is a Casimir element of the Lie algebra of \( O(N) \), these eigenspaces are invariant subspaces under the right action of \( O(N) \). The projections onto the corresponding eigenspaces are given by the integral kernels

\[
P_l(\sigma, \sigma') = \frac{2l + N - 2}{N - 2} C_n^{N-1} (\sigma \cdot \sigma') \quad (2)
\]

where the integrals are to be taken with the measure \( d\Omega(\sigma) \), and \( C_n^{N-1} \) are the Gegenbauer–polynomials \[5\].

The transfer matrix corresponding to (1) is given by the integral kernel

\[
T(\sigma \cdot \sigma') := \exp \left( \sum_{k=1}^{n} \beta_k (\sigma \cdot \sigma')^k \right). \quad (3)
\]

Because \( T \) commutes with the \( O(N) \) rotations, it will also leave these eigenspaces invariant, and in fact it will act as multiplication by the eigenvalue \( \lambda_l \) on these subspaces. Hence we can evaluate the eigenvalues \( \lambda_l \) for \( N \in \mathbb{N} \setminus \{1\} \) as

\[
\lambda_l(\beta) := \frac{\text{tr} P_l T}{\text{tr} P_l} = \int_{-1}^{1} \exp \left( \sum_{k=1}^{n} \beta_k x^k \right) \left( 1 - x^2 \right)^{\frac{N-3}{2}} \frac{C_n^{N-1}(x)}{C_l^{N-1}(1)} dx
\]

\[
= \int_{0}^{\pi} \exp \left( \sum_{k=1}^{n} \beta_k \cos^k (t) \right) \sin^{N-2} (t) \frac{C_n^{N-1}(\cos (t))}{C_l^{N-1}(1)} \ dt \quad (4)
\]

with the substitution \( \sigma \cdot \sigma' =: x = \cos(t) \) and \( l \in \mathbb{N}_0 \) (\( n := 2 \): mixed isovector/isotensor–model); for \( N = 2 \), we use the Chebychev–polynomials of the first kind: \( T_0(x) = 1 \) and \( T_l(x) = \frac{1}{2} \lim_{N \to 2} \frac{2}{N-2} C_l^{N-1}(x) \), for \( l \geq 1 \).

For the case of nonnegative parameters \( \beta_k \) the transfer matrix is a positive operator due to reflection positivity \[3, 4\]. Therefore it is possible to define ‘masses’ in terms of the normalized eigenvalues \( \tilde{\lambda}_l(\beta) = \lambda_l \lambda_0^{-1} \), \( l \geq 1 \), as

\[
m_l(\beta) := \log \left( \frac{\lambda_l(\beta)}{\tilde{\lambda}_l(\beta)} \right) = - \log \left( \tilde{\lambda}_l(\beta) \right). \quad (5)
\]
Incidentally, there is a certain ‘gauge’ symmetry in this action: changing the sign of all $\beta_k$, $k$ odd, can be compensated by the substitution $y := -x$ in the Gegenbauer–polynomials; this in turn can be achieved by multiplying $\sigma_x$ by $(-1)^x$, which can be considered as a gauge transformation not affecting the physics.

We are now going to examine where this model becomes critical and has a (well–defined) continuum limit; that means that all masses have to vanish. In one dimension, this requires that at least one of the parameters goes to infinity. Finally we define the ratio of the masses (5) as

$$R_l(\beta) := \frac{m_l(\beta)}{m_1(\beta)}$$

with $m_1(\beta)$ as the reference mass.

### 3 Main result: the hypersurfaces for the new universality classes

In this section we will show how to get (in principle) all normalized eigenvalues $\tilde{\lambda}_l := \lambda_0^{-1}$ from $\lambda_0$. Because of the impossibility of an exact analytic formula in the general $n$–parameter case, we use the generalized Laplace–method of asymptotic expansion techniques [8, 9] to evaluate the leading term(s) of $\lambda_0$, and thereby also for all $\tilde{\lambda}_l$.

Using (4) it can be seen that the eigenvalues $\tilde{\lambda}_l$ are obtained from $\lambda_0$ as follows:

$$\tilde{\lambda}_l(\beta) = \frac{2^l}{\Gamma\left(\frac{N}{2} - 1\right) C_{\frac{N}{2} - 1}^l(1)} \sum_{m=0}^{\left[\frac{l}{2}\right]} (-1)^m \Gamma\left(\frac{N}{2} - 1 + l - m\right) \frac{\partial}{\partial \beta_l - 2m} (\log (\lambda_0(\beta))) \quad (N \geq 3)$$

$$\tilde{\lambda}_l(\beta) = \frac{2^l}{2} \sum_{m=0}^{\left[\frac{l}{2}\right]} (-1)^m (l - m - 1)! \frac{\partial}{\partial \beta_l - 2m} (\log (\lambda_0(\beta))) \quad (N = 2)$$

with $\frac{\partial}{\partial \beta_0} (\log (\lambda_0(\beta))) \equiv 1$. A priori these two equations are valid for $1 \leq l \leq n$, but we can apply them for all $l \geq 1$ by the following trick: First notice that $n$ can be arbitrarily large. We use this fact to modify the action by introducing additional couplings $\beta_r$ for all $r \leq l$ (if not already present); then we take the required derivatives and finally set the parameters not appearing in the action equal to 0.

Now we turn to the problem of obtaining an asymptotic expansion for $\lambda_0$. Let us define first some abbreviations for the expressions in (6) for the case of $\lambda_0$ (note that $C_{\frac{N}{2} - 1}^l(x) \equiv 1$):

$$f(x) := \sum_{k=1}^{n} \beta_k x^k, \quad g(x) := (1 - x^2)^{\frac{N-3}{2}}$$

$$\tilde{\lambda}_l(\beta) = \frac{2^l}{\Gamma\left(\frac{N}{2} - 1\right) C_{\frac{N}{2} - 1}^l(1)} \sum_{m=0}^{\left[\frac{l}{2}\right]} (-1)^m \Gamma\left(\frac{N}{2} - 1 + l - m\right) \frac{\partial}{\partial \beta_l - 2m} (\log (\lambda_0(\beta))) \quad (N \geq 3)$$

$$\tilde{\lambda}_l(\beta) = \frac{2^l}{2} \sum_{m=0}^{\left[\frac{l}{2}\right]} (-1)^m (l - m - 1)! \frac{\partial}{\partial \beta_l - 2m} (\log (\lambda_0(\beta))) \quad (N = 2)$$

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Laplace’s method for asymptotic expansion requires the knowledge of the absolute maxima of \( f(x) \), respectively \( F(t) \), in the corresponding interval. A maximum of \( f \) at the point \( x_0 \) is said to be of order \( j \), if \( f^{(r)}(x_0) = 0 \) for \( r = 0, 1, \ldots, j-1 \) and \( f^{(j)}(x_0) \neq 0 \). An internal maximum is thus at least of order 2, whereas a boundary maximum can have any order \( j \geq 1 \). Any maximum that contributes to the leading term is called a principal maximum. We will now consider the simplest case and show in the following section that these restrictions are unimportant. — The simplest (but not trivial) case is given by

\[
F(t) := \sum_{k=1}^{n} \beta_k \cos^k(t), \quad G(t) := \sin^{N-2}(t). \tag{10}
\]

Due to (10) we have (for \( N \in \mathbb{N}\setminus\{1\} \))

\[
\lambda_0(\beta) = \frac{\Gamma \left( \frac{N-1}{2} \right) G^{(N-2)}(0)}{2^{\frac{N-1}{2}} (N-2)!} \exp \left( \sum_{k=1}^{n} \beta_k \right) \left( 1 + \frac{(1+\epsilon) \sum_{k=1}^{n} k \beta_k}{(1+\eta) \sum_{k=1}^{n} (-1)^k k \beta_k} \right)^{\frac{N-1}{2}} \exp \left( -2 \sum_{k=1}^{n} \beta_{2k-1} \right) \tag{12}
\]

where \( \epsilon \) stands for a correction \( O \left( \left( \sum_{k=1}^{n} k \beta_k \right)^{-1} \right) \) and \( \eta \) for \( O \left( \left( \sum_{k=1}^{n} (-1)^k k \beta_k \right)^{-1} \right) \). Due to (10) we have (for \( N \geq 3 \))

\[
G^{(N-2)}(t) = \partial_t^{N-2}(\sin^{N-2}(t)) = \partial_t^{N-2}(t^{N-2} + O(t^N)) = (N-2)! + O(t^2), \tag{13}
\]

so that \( \frac{G^{(N-2)}(t_0)}{(N-2)!} \) is 1 for \( t_0 = 0 \) and, by way of the transformation \( t \to \pi - t \), also for \( t_0 = \pi \).

From these formulae it follows, as we will show below, that for all \( N \geq 2 \)

\[
\tilde{\lambda}_l(\beta) \sim 1 - \frac{l(l+N-2)}{2} \frac{1}{\sum_{k=1}^{n} k \beta_k} - \left( 1 - (-1)^l \right) \exp \left( -2 \sum_{k=1}^{n} \beta_{2k-1} \right) \left( \frac{\sum_{k=1}^{n} \beta_k}{\sum_{k=1}^{n} (-1)^k \beta_k} \right)^{\frac{N-1}{2}} + O \left( \left( \sum_{k=1}^{n} \beta_k \right)^{-2} \right) \tag{14}
\]
if at least one of the $\beta_{2k-1} \to \infty$. In this case all $\tilde{\lambda}_l$ tend to 1, i.e. all masses vanish, so that the model becomes critical. After presenting the proof to (14), we will turn back to this point to examine the opposite case, in which all $\beta_k, k$ odd, remain bounded from above.

The proof to (14) requires two steps: first, we will show that the essential coefficient of the ‘power’ term is the eigenvalue of $\Delta_{LB}$, then we will examine the exponential term.

The eigenvalue of $\Delta_{LB}$ arises from the following identities, (15) below for $N \geq 3$ and (16) below for $N = 2$, valid in each case for $l \in \mathbb{N}_0$:

$$l(l+N-2) = \frac{2^l}{N-1} \Gamma\left(\frac{N}{2} - 1\right) C^{\frac{N}{2}-1}_l \sum_{m=0}^{\lfloor \frac{l}{2} \rfloor} (-1)^m \frac{(l-2m)(l-m-1)!}{m! (l-2m)! 2^{2m}}$$

and in the limit $N \downarrow 2$

$$l^2 = 2^l \sum_{m=0}^{\lfloor \frac{l}{2} \rfloor} \frac{(-1)^m (l-2m)(l-m-1)!}{m! (l-2m)! 2^{2m}}$$

where $\frac{\ell}{2}$ is the normalization factor coming from $T_l(x) = \frac{\ell}{N+2} \lim_{N \downarrow 2} C^{\frac{N}{2}-1}_l(x), l \geq 1$, with $T_l$ denoting the Chebychev–polynomials of the first kind [5].

The strategy for the proof of (15) and (16) is to apply the following formulae for $r = 1$ and $y = \frac{1}{2}$ [5, 10]

$$\sum_{m=0}^{\lfloor \frac{l}{2} \rfloor} (-1)^m y^{2m} = \Gamma\left(\frac{N}{2} - 1\right) y^l C^{\frac{N}{2}-1}_l \left(\frac{1}{2y}\right),$$

$$\frac{d^r}{dy^r} C^{\frac{N}{2}-1}_l(y) = 2^r \left(\frac{N}{2} - 1\right)_r C^{\frac{N}{2}-1+r}_l(y)$$

$$\sum_{m=0}^{\lfloor \frac{l}{2} \rfloor} (-1)^m (l-m-1)! y^{2m} = \frac{2^l}{l!} T_l\left(\frac{1}{2y}\right), \quad \frac{d^r}{dy^r} T_l(y) = 2^{r-1} \Gamma(r) l C^{r-1}_l(y).$$

In addition we need the identity

$$C^{\frac{N}{2}-1}_l(1) = \binom{N + l - 3}{l}$$

which arises from the definition of the Jacobi–polynomials [5]. Because of the triviality of the identities (15) and especially in (16) for $l = 0$, we restrict ourselves to $l \geq 1$:

$$\left.-g^{l+1} \Gamma\left(\frac{N}{2} - 1\right) \frac{d}{dy} C^{\frac{N}{2}-1}_l\left(\frac{1}{2y}\right)\right|_{y=\frac{1}{2}}$$

$$= -\left.g^{l+1} \Gamma\left(\frac{N}{2} - 1\right) 2 \left(\frac{N}{2} - 1\right) C^{\frac{N}{2}-1}_l\left(\frac{1}{2y}\right)\left(-\frac{1}{2y^2}\right)\right|_{y=\frac{1}{2}}$$
\[
\begin{align*}
&\quad = \frac{2}{2^l} \left( \frac{N}{2} - 1 \right) \Gamma \left( \frac{N}{2} - 1 \right) C_{l-1}^{N} \left( 1 \right) \\
&\quad = \frac{N - 2}{2^l} \Gamma \left( \frac{N}{2} - 1 \right) \left( \frac{N + l - 2}{l - 1} \right) \\
&\quad = \frac{\Gamma \left( \frac{N}{2} - 1 \right)}{2^l} (N - 2) \frac{(N + l - 2)!}{(l - 1)! (N - 1)!} \\
&\quad = \frac{\Gamma \left( \frac{N}{2} - 1 \right)}{2^l} (N + l - 3)! \frac{N - 2 N + l - 2 l}{l! (N - 3)! (N - 2) - N - 1} \\
&\quad = \frac{\Gamma \left( \frac{N}{2} - 1 \right)}{2^l} C_{l-1}^{N} \left( 1 \right) \frac{l (l + N - 2)}{N - 1} \\
\end{align*}
\]

(20)

and

\[
\begin{align*}
\left. \left( -y^{l+1} \frac{d}{dy} T_{l} \left( \frac{1}{2y} \right) \right) \right|_{y=\frac{1}{2}} &= \left. \left( -y^{l+1} \frac{2}{l} l C_{l-1}^{l} \left( \frac{1}{2y} \right) \left( \frac{-1}{2y^2} \right) \right) \right|_{y=\frac{1}{2}} \\
&= \frac{2}{2^l} C_{l-1}^{l} \left( 1 \right) = \frac{2}{2^l} \left( \frac{l}{l - 1} \right) = \frac{l^2}{2^l} \frac{2}{l - 1}. 
\end{align*}
\]

(21)

So far, the first two terms of \( \tilde{\lambda}_l \) (14) are determined. Because in the representation of (12) the argument of the exponential term involves only the \( \beta_k, k \) odd, we distinguish the derivatives of this term with respect to \( \beta_k, k \) even and odd, and denote them by \( \tilde{\partial}_{j,\text{even}} \) and \( \tilde{\partial}_{j,\text{odd}} \) for \( 1 \leq j \leq n \). Again, we consider only the leading term.

\[
\tilde{\partial}_{j,\text{even}} \sim \frac{N - 1}{2} \left( \sum_{k=1}^{n} k \beta_k \right) \frac{N - 3}{2} \exp \left( -2 \left[ \sum_{k=1}^{n} \beta_{2k-1} \right] \right) \left( \frac{\sum_{k=1}^{n} k \beta_k}{\sum_{k=1}^{n} k \beta_k} \right) \\
+ \left( \sum_{k=1}^{n} k \beta_k \right) \frac{N - 1}{2} \exp \left( -2 \left[ \sum_{k=1}^{n} \beta_{2k-1} \right] \right) \\
\sim - \exp \left( -2 \left[ \sum_{k=1}^{n} \beta_{2k-1} \right] \right) \frac{N - 1}{2} \left( \frac{\sum_{k=1}^{n} (2k - 1) \beta_{2k-1}}{\sum_{k=1}^{n} (-1)^k k \beta_k} \right) \left( \frac{\sum_{k=1}^{n} k \beta_k}{\sum_{k=1}^{n} k \beta_k} \right) \frac{N - 3}{2}
\]

(22)

Because one of the odd \( \beta_k \to \infty \), this contribution is of order exponential times power
and therefore subleading, compared to the first two terms in (14).

\[
\tilde{d}_{j,\text{odd}} \sim \tilde{d}_{j,\text{even}} + \frac{\exp \left( -2 \sum_{k=1}^{n} \beta_{2k-1} \right) \left( -2 \left( \frac{\sum_{k=1}^{n} k\beta_k}{\sum_{k=1}^{n} (-1)^k k\beta_k} \right)^{\frac{N-1}{2}} \right)}{1 + \left( \frac{\sum_{k=1}^{n} k\beta_k}{\sum_{k=1}^{n} (-1)^k k\beta_k} \right)^{\frac{N-1}{2}} \exp \left( -2 \sum_{k=1}^{n} \beta_{2k-1} \right)}
\]

\[
\sim -2 \exp \left( -2 \sum_{k=1}^{n} \beta_{2k-1} \right) \left( \frac{\sum_{k=1}^{n} k\beta_k}{\sum_{k=1}^{n} (-1)^k k\beta_k} \right)^{\frac{N-1}{2}}
\]

(23)

We remark that the first line of (22) as well as (23) is valid in general (independent of the condition \( \beta_{2k-1} \to \infty \)).

If we look at eq. (14), we see that three different ways of sending \( \|\beta\| \to \infty \) have to be distinguished, depending on the relative importance of the second and third terms:

\[
\frac{\sum_{k=1}^{n} k\beta_k}{\exp \left( 2 \sum_{k=1}^{n} \beta_{2k-1} \right)} \left( \frac{\sum_{k=1}^{n} (-1)^k k\beta_k}{\sum_{k=1}^{n} k\beta_k} \right)^{\frac{N-1}{2}}
\]

\[
= \frac{\left( \sum_{k=1}^{n} (2k-1) \beta_{2k-1} \right)}{\exp \left( 2 \sum_{k=1}^{n} \beta_{2k-1} \right)} + \frac{2 \sum_{k=1}^{n} k\beta_k}{\exp \left( 2 \sum_{k=1}^{n} \beta_{2k-1} \right)} \left( \frac{\sum_{k=1}^{n} (-1)^k k\beta_k}{\sum_{k=1}^{n} k\beta_k} \right)^{\frac{N-1}{2}}
\]

\[
\|\beta\| \to \infty \to \begin{cases} 0 & : \text{Case (I)} \\ c \in ]0, \infty[ & : \text{Case (II)} \\ \infty & : \text{Case (III)} \end{cases}
\]

(24)

Here the first summand vanishes in the limit \( \|\beta\| \to \infty \) because of the assumption that at least one of the \( \beta_{2k-1} \to \infty \). The quotient of the linear forms in \( \beta_k \) converges always to a value of the interval \([0, 1]\) (this follows from condition (11) in connection with (12) and (13)).

We are interested in the limit of the mass ratios \( R_t \) (see (4)) in the three cases just defined. By expanding the logarithms of the eigenvalues using (14) we obtain

\[
R_t = \frac{m_t}{m_1} = \lim_{\|\beta\| \to \infty} \frac{-\log(\hat{\lambda}_t(\beta))}{-\log(\hat{\lambda}_1(\beta))} = \begin{cases} \frac{N-1}{2} c \left( \frac{l(l+N-2)}{l(l+N-2)+1} \right)^{l-1} & : \text{Case (I)} \\ \frac{1}{2} \left( \frac{l(l+N-2)}{l(l+N-2)+1} \right)^{l-1} & : \text{Case (II)} \\ \frac{1}{2} \left( \frac{l(l+N-2)}{l(l+N-2)+1} \right)^{l-1} & : \text{Case (III)} \end{cases}
\]

(25)
The result of (25) can be interpreted as follows: In Case (I) the ratio is in the class of the $O(N)$–vector–model, in Case (III) in the class of the $RP^{N−1}$–model, whereas in Case (II) we have found new universality classes lying between them, parametrized by $c$. By the way, if we set all $\beta_k = 0$ except of $\beta_1$, we recognize the pure $O(N)$–vector–model which belongs to Case (I).

We can summarize the main result of this paper as follows: the new universality classes are obtained by sending $||\beta|| \to \infty$ in such a way that

$$\frac{\sum_{k=1}^{n} k\beta_k}{\exp \left( 2 \sum_{k=1}^{[\frac{n+1}{2}]} \beta_{2k-1} \right) \left( \sum_{k=1}^{n} (-1)^k k\beta_k \right)^{\frac{N−1}{2}}} \to c \in [0, \infty[ \ (26)$$

or equivalently for $||\beta|| \to \infty$ on the hypersurfaces

$$2 \sum_{k=1}^{[\frac{n}{2}]} k\beta_{2k} = c \exp \left( 2 \sum_{k=1}^{[\frac{n+1}{2}]} \beta_{2k-1} \right) + h(\beta), \quad c \in [0, \infty[ , \ (27)$$

with

$$\lim_{||\beta|| \to \infty} h(\beta) \exp \left( -2 \sum_{k=1}^{[\frac{n+1}{2}]} \beta_{2k-1} \right) = 0. \ (28)$$

Of course, what matters is only the asymptotic behaviour of those hypersurfaces which is independent of the function $h$.

Let us return to (14) where we had assumed that at least one of the $\beta, k$ odd, will go to infinity. The converse is that all of them are bounded (from above). Then the contribution from $\tilde{\partial}_j, \text{even}$ (22) has purely power character and moreover is equal to the term whose coefficient is the eigenvalue of $\Delta_{LB}$. For $\tilde{\partial}_j, \text{odd}$ (23) the additional term tends to a constant. That means in the limit $||\beta|| \to \infty$ (here at least one of the $\beta, k$ even, has to go to infinity), only the $\tilde{\lambda}_l, l$ even, tend to 1, whereas

$$\tilde{\lambda}_{2r+1} \to 1 + \frac{-2 \exp \left( -2 \sum_{k=1}^{[\frac{n+1}{2}]} \beta_{2k-1} \right)}{1 + \exp \left( -2 \sum_{k=1}^{[\frac{n+1}{2}]} \beta_{2k-1} \right)} = \text{th} \left( \sum_{k=1}^{[\frac{n+1}{2}]} \beta_{2k-1} \right) < 1. \ (29)$$

Therefore, the model will not become critical in this case. Nevertheless, we can consider the ratio (note that there is no $N$–dependence)

$$\mathcal{R}_l = \frac{m_l}{m_1} \to \frac{1 - (-1)^l}{2}. \ (30)$$
So we end up in the universality class of the pure $RP^{N-1}$-model, which is the special case in which all $\beta_k = 0$ except of $\beta_2$.

In the following section we will see that this picture remains valid in the general case of the $n$-parameter model, defined by (4).

4 Generalization and continuum limit

In this section we drop the two restrictions on the action made in section 3 and discuss the consequences for a continuum limit.

Firstly, we want to point out that the one restriction made in the beginning of the last section, namely that $f'(-1) \to -\infty$ and $f'(1) \to \infty$, is irrelevant. Since $f'(1) = \sum_{k=1}^{n} k \beta_k$ and we are interested in the limit $|\beta| \to \infty$, the second condition is obviously automatically fulfilled. Let us assume that the first condition is not satisfied, i.e. $f'(-1)$ is bounded from below. Since we only have to consider the case that asymptotically $f$ has a maximum at $x = -1$, we may also assume that it is bounded from above, i.e. we have

$$|f'(-1)| = \left| \sum_{k=1}^{n} (-1)^{k-1} k \beta_k \right| = \left| 2 \sum_{k=1}^{[n\frac{1}{2}]} k \beta_{2k} - \sum_{k=1}^{[n\frac{1}{2}]} (2k - 1) \beta_{2k-1} \right| \leq K < \infty. \quad (31)$$

If now all $\beta_{2k-1}$ as well as all $\beta_{2k}$ remain bounded, no statement about any asymptotic behaviour is possible. Otherwise, both sums above in $\beta_{2k-1}$ and $\beta_{2k}$ have to go to infinity. But in this case, using (31), we get from

$$e^{f(-1) - f(1)} = \exp \left( -2 \sum_{k=1}^{[n\frac{1}{2}]} \beta_{2k-1} \right) \leq e^{-\frac{K}{n}} \exp \left( -\frac{1}{n} \sum_{k=1}^{n} k \beta_k \right) \quad (32)$$

that $e^{f(-1)}$ is exponentially suppressed in all $\beta_{2k-1}$ as well as in all $\beta_{2k}$. Therefore, this corresponds to the Case (I) of (25).

Next, we allow (principal) maxima lying inside of the interval: $x_0 \in [-1, 1]$, respectively $t_0 \in [0, \pi]$. First of all, we want to remark that such an internal maximum can appear for $f$, defined in (6), only for $n \geq 3$. As mentioned there, at $x = 1$, we have always a principal maximum for the action in question. So, if there is (at least) one internal maximum at $x_0$, notice that in (12) the exponential term in the parenthesis would be replaced by $\exp \left( -\sum_{k=1}^{n} (1 - x_0^2) \beta_k \right)$. This means that $e^{f(x_0)}$ is exponentially suppressed in each $\beta_k, 1 \leq k \leq n$, and therefore in the asymptotic expansion the contribution from $x_0$ would be subleading. This completes the proof of the statement that the restrictions made in section 3 are unimportant.
Let us now discuss the continuum limit. This and the problem for principal maxima of different orders is discussed in some detail in [4] for the case of a family with at most two parameters and a general action. They find that if the only maximum is at an internal point of the interval (a situation excluded by our assumption that all the parameters $\beta_k$ are nonnegative), there exists no continuum limit. For $N \geq 3$, it is shown that the normalized eigenvalues $\tilde{\lambda}_l$ of (4) cannot tend to 1 (so that the masses (5) would vanish), because of the fact that

$$\left| \frac{C_l^{N-1}(x_0)}{C_l^{N-1}(1)} \right| < 1 \quad \text{for } |x_0| < 1 \text{ and } l \in \mathbb{N}.$$  \hspace{1cm} (33)

Unfortunately, this does not cover the special case $N = 2$. But we want to give an alternative (simpler) proof for this fact which is also valid for $N = 2$.

We use the following formula for the Gegenbauer–polynomials in terms of the Chebychev–polynomials of the first kind [11] (valid for $N \geq 3$)

$$C_l^{N-1}(T_1(x)) = \frac{2}{\left( \Gamma \left( \frac{N}{2} - 1 \right) \right)^2} \sum_{m=0}^{\left[ \frac{l}{2} \right]} \frac{\Gamma \left( \frac{N}{2} - 1 + m \right) \Gamma \left( \frac{N}{2} - 1 + l - m \right)}{m! (l-m)!} T_{l-2m}(x).$$ \hspace{1cm} (34)

Note that all coefficients are positive! To prove (33), it suffices to show that for any $t_0 \in [0, \pi]$ one of the $T_{l-2m}(x_0)$ in (34) (with $x_0 = \cos(t_0)$) is less than 1. (Of course, for $l = 1$, we have only one term, $T_1(x) = x \equiv \cos(t)$, whose absolute value is always less than 1 for such a $t_0$.) Assume that $T_l(x_0) = \pm 1$ (only one sign to choose). Without loss of generality, consider for $l \geq 2$

$$T_{l-2}(x_0) = \cos((l-2)t_0) = \cos(lt_0) \cos(2t_0) - \sin(lt_0) \sin(2t_0) = \pm \cos(2t_0) = \pm T_2(x_0) \neq \pm 1.$$ \hspace{1cm} (35)

For $N = 2$, the inequality (33) is not true for certain points $t_0 = \pi q$, $q \in \mathbb{Q} \cap [0,1]$. But we can repeat the arguments used above: assume $T_l(x_0) = \pm 1$ (again, $l \geq 2$ and one sign to choose); this time, consider

$$T_{l-1}(x_0) = \pm T_l(x_0) \neq \pm 1.$$ \hspace{1cm} (36)

This means that not all masses would go to 0 if at such a $t_0$ the function $F$ (see eq. (10)) has a maximum. Consequently, in these cases, there exists no continuum limit. So, as far as the continuum limit is concerned, there is nothing special about the case $N = 2$.

Our conclusions are in agreement with those of [4], where they overlap. Our analysis is, however, more general in one respect: we allow for arbitrarily many parameters, whereas the authors of [4] allow only one or two. On the other hand, the form of interactions considered by us is more restricted since we only consider polynomials in the scalar products of two neighbouring spins with nonnegative coefficients.

It is easy to see that the continuum limits obtained in the new universality classes correspond to quantum–mechanical Hamiltonians of the form

$$H = -a \Delta_{LB} + b P + \text{const}$$ \hspace{1cm} (37)
where $\Delta_{LB}$ is the Laplace–Beltrami–operator on $S^{N-1}$ and $P$ is the ‘parity operator’ mapping every point of the sphere into its antipode (note that $P$ is a unitary, self-adjoint involution). Since $P$ corresponds to multiplication by $(-1)^l$ on the eigenspace $\mathcal{H}_l$ of $\Delta_{LB}$, it is not hard to check that we obtain the mass ratios of the new universality classes given in (25) by choosing $a = \frac{1}{N-1}$ and $b = -\frac{2c}{N-1}$ (and const = $-b$).

We do not get the more general continuum Hamiltonians discussed in [4] because we restricted ourselves to polynomials with nonnegative coefficients in order to ensure reflection positivity. It should be noted, however, that the most general quantum–mechanical Hamiltonian compatible with $O(N)$–invariance is still more general than the form given in [4] : it is given by

$$H = \sum_{l \geq 1} c_l \mathcal{P}_l$$

where the $\mathcal{P}_l$ are the projectors onto the eigenspaces, defined in (2), and $c_l$ arbitrary coefficients.

5 Conclusions

In this paper we determined the critical behaviour of the generalized $O(N)$–$RP^{N-1}$–model with an $n$–parametric action for the general case in one dimension and for general $N$ using asymptotic expansion techniques. There exist hypersurfaces on which an infinite number of new universality classes appears, which can be parametrized by a variable interpolating between the $O(N)$–vector–model–class and the $RP^{N-1}$–model–class. For the ratio of the masses, there is a difference between even and odd masses in form of an additional constant. We also examined the continuum limit and gave some relevant additional information for the case $N = 2$.

We found a one–parameter family of universality classes in the continuum limit, described in eq. (27), that arises in particular for the standard mixed $O(N)$–$RP^{N-1}$ models. These models were also studied in the papers [2] and [3]: these authors concluded, however, that there is only the universality class corresponding to the standard $O(N)$–model. At least the arguments of [2] that rely on the negligibility of vortices are clearly applicable in our 1D situation. So how is this apparent conflict to be resolved ?

The authors of [2] reach their conclusion by considering the decay properties of the correlations of new spin variables, and their claims for those is certainly correct. But these new spins are nonlocal functions of the original spins, and therefore the behaviour of their correlations cannot be related in an obvious way to spectral properties of the transfer matrix. We analyzed directly the spectrum of the transfer matrix and found that there are indeed new universality classes.

Finally, we would like to make a remark about a possible generalization of our results to negative values of some parameters. If negative (or even complex) values are allowed, we generally lose reflection positivity and therefore the quantum–mechanical interpretation
of the models. The most general Hamiltonians given in eq. (38) can certainly be obtained as continuum limits of such lattice models; we only have to choose as the transfer matrix $T = \exp(-aH)$ and send $a \to 0$. Such transfer matrices correspond, however, to very involved actions that depend on $a$ (the parameter controlling the approach to criticality) in a very nonobvious way. It remains an open question whether the Hamiltonians (38) can also be obtained by using more ‘natural’ actions.

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