Explicit decay rate for the Gini index in the repeated averaging model

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We investigate the repeated averaging model for money exchanges: two agents picked uniformly at random share half of their wealth to each other. It is intuitively convincing that a Dirac distribution of wealth (centered at the initial average wealth) will be the long time equilibrium for this dynamics. In other words, the Gini index should converge to zero. To better understand this dynamics, we investigate its limit as the number of agents goes to infinity by proving the so-called propagation of chaos, which links the stochastic agent-based dynamics to a (limiting) nonlinear partial differential equation (PDE). This deterministic description has a flavor of the classical Boltzmann equation arising from statistical mechanics of dilute gases. We prove its convergence towards its Dirac equilibrium distribution by showing that the associated Gini index of the wealth distribution converges to zero with an explicit rate.

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agent-based model, econophysics, Gini index, propagation of chaos, repeated averaging

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following rule:

\[(X_i, X_j) \rightarrow \left( \frac{X_i + X_j}{2}, \frac{X_i + X_j}{2} \right).\]  \hspace{1cm} (1)

We refer to Figure 1-left for a simple illustration of the repeated averaging dynamics.

Despite of the simplicity of the model, there are actually quite a few manuscripts in the literature that are solely dedicated to it. To the best of our knowledge, the first systematic treatment of this model is carried out by Aldous and Lanoue,\(^{15}\) which is followed up by a very recent study presented in Chatterjee et al.\(^{16}\) In both works, the focus is related to the estimation of the so-called mixing times, and hence, the targeted audiences are mathematicians from the Markov chain mixing time community. In this manuscript, we intend to give a kinetic theory perspective of the model. Indeed, under the large population \(N \to \infty\) limit, we can rigorously show that the law of the wealth of a typical agent (say \(X_1\)) satisfies the following limit PDE in a weak sense:

\[\partial_t \rho(t, x) = 2(\rho * \rho)(t, 2x) - \rho(t, x),\] \hspace{1cm} (2)

and we show in Figure 1-right a numerical simulation of the PDE (2) with the initial distribution \(\rho(0, \cdot)\) being a gamma probability density with shape parameter \(\mu = 5\) and rate parameter equal to unity, that is, \(\rho(0, x) = \frac{x^{\mu-1} e^{-x}}{\Gamma(\mu)}\).

Once the limit PDE is identified from the interacting particle system, the natural next step is to study the problem of convergence to equilibrium of the PDE at hand. In the present work, we demonstrate that the Gini index of \(\rho(t)\) converges to 0 (its minimum value), whence showing that a Dirac distribution centered at the initial average wealth is the equilibrium distribution. Moreover, this model can be served as the first example for which quantitative estimates on the convergence of Gini index can be obtained, which is our primary motivation for writing this paper. An illustration of the general strategy used in this work is shown in Figure 2.

**FIGURE 1  **Left: Illustration of the repeated averaging dynamics. Each time a Poisson clock rings, we pick \(i\) and \(j\) independently and uniformly at random from \(\{1, \ldots, N\}\), then we update \((X_i, X_j)\) according to (1). Right: Simulation of the PDE (2) for \(0 \leq t \leq 10\). The initial data \(\rho(0, \cdot)\) is a gamma probability density with shape parameter \(\mu = 5\) and rate parameter equal to unity, that is, \(\rho(0, x) = \frac{x^{\mu-1} e^{-x}}{\Gamma(\mu)}\). [Colour figure can be viewed at wileyonlinelibrary.com]

**FIGURE 2  **Schematic illustration of the general strategy of our treatment of the repeated averaging dynamics, where \(\mu\) represents the initial average wealth [Colour figure can be viewed at wileyonlinelibrary.com]
Although only a very specific binary exchange model is explored in the present paper, other exchange rules can also be imposed and studied, leading to different models. To name a few, the so-called immediate exchange model introduced in Heinsalu and Patriarca\(^{17}\) assumes that pairs of agents are randomly and uniformly picked at each random time, and each of the agents transfers a random fraction of its money to the other agents, where these fractions are independent and uniformly distributed in \([0, 1]\). The so-called uniform reshuffling model investigated in other works\(^{2,18,19}\) suggests that the total amount of money of two randomly and uniformly picked agents possesses before interaction is uniformly redistributed among the two agents after interaction. For models with saving propensity and with bank or debt, we refer the readers to previous studies.\(^{20–23}\) Also, one can also modulate the rule of picking agents, leading to biased models of money exchange; see for instance Cao and Motsch and Cao and Jabin.\(^{24,25}\)

This manuscript is organized as follows: In Section 2, we briefly discuss the heuristic derivation of the limit Equation (2) and give convergence results for the solution of (2) in terms of variance of the distribution as well as the Gini index. Then we present a rigorous treatment of the propagation of chaos phenomenon in Section 3, by applying the martingale-based technique employed in Cao et al.\(^{18}\) Finally, we draw a conclusion in Section 4.

### 2 CONVERGENCE TO DIRAC DISTRIBUTION

We present a heuristic argument behind the derivation of the limit PDE (2) arising from the repeated averaging dynamics in Section 2.1, and we also record a useful stochastic representation behind the PDE (2) on which we will heavily rely. Section 2.2 is devoted to the exponential decay of the variance of the solution \(\rho(t)\) of (2). In Section 2.3, we establish a quantitative convergence result on the Gini index of the probability distribution \(\rho(t)\), by enforcing the log-concavity property of the initial datum.

#### 2.1 Formal derivation of the limit PDE

Introducing \(N^{(i,j)}_t\) independent Poisson processes with intensity \(1/N\), the dynamics can be written as

\[
\frac{dX_i(t)}{dt} = \sum_{j=1, j \neq i} \left( \frac{X_i(t-) + X_j(t-)}{2} - X_i(t-) \right) dN^{(i,j)}_t. \tag{3}
\]

As the number of players \(N\) goes to infinity, one could expect that the processes \(X_i(t)\) become independent and of same law. Therefore, the limit dynamics would be of the form:

\[
\frac{d\bar{X}(t)}{dt} = \left( \frac{\bar{X}(t-) + \bar{Y}(t-)}{2} - \bar{X}(t-) \right) d\bar{N}_t, \tag{4}
\]

where \(\bar{Y}(t)\) is an independent copy of \(\bar{X}(t)\) and \(\bar{N}_t\) a Poisson process with intensity 1. Taking a test function \(\varphi\), the weak formulation of the dynamics is given by:

\[
\int dE(\bar{X}(t)) = \mathbb{E}\left[ \varphi \left( \frac{\bar{X}(t) + \bar{Y}(t)}{2} \right) - \varphi(\bar{X}(t)) \right] dt. \tag{5}
\]

In short, the limit dynamics correspond to the jump process:

\[
\bar{X} \sim \bar{X} + \bar{Y}. \tag{6}
\]

Let us denote \(\rho(t,x)\) the law of the process \(\bar{X}(t)\). To derive the evolution equation for \(\rho(t,x)\), we need to translate the effect of the jump of \(\bar{X}(t)\) via (6) onto \(\rho(t,x)\).

**Lemma 1.** Suppose \(X\) and \(Y\) two independent random variables with probability density \(\rho(x)\) supported on \([0, \infty)\). Let \(Z = (X + Y)/2\), then the law of \(Z\) is given by \(Q_+[\rho]\) with:

\[
Q_+[\rho](x) = 2(\rho * \rho)(2x), \quad \forall x \geq 0. \tag{7}
\]
The proof of this lemma is quite elementary and will be omitted. We can now write the evolution equation for the law of $X(t)$, and the density $\rho(t,x)$ satisfies weakly:

$$\partial_t \rho(t,x) = L[\rho](t,x) \quad \text{for } t \geq 0 \text{ and } x \geq 0$$

with

$$L[\rho](x) := Q_+[\rho](x) - \rho(x) = 2(\rho * \rho)(2x) - \rho(x).$$

**Remark 1.** Suppose that $\rho(0,x)$ is a probability density on $[0, \infty)$ with mean $\mu > 0$. It is readily checked that the dynamics (8) preserves the total mass and the mean value. That is,

$$\frac{d}{dt} \int_{\mathbb{R}_+} \rho(t,x) \, dx = 0 \quad \text{and} \quad \frac{d}{dt} \int_{\mathbb{R}_+} x \rho(t,x) \, dx = 0.$$

For each test function $\varphi$, one can show that

$$\int_{\mathbb{R}_+} \varphi(x) G[\delta_\mu](dx) = 0,$$

implying that the Dirac distribution centered at $\mu$ is an equilibrium solution of (8).

**Remark 2.** The PDE (8) is a particular case of so-called generalized Kac equation:

$$\partial_t \rho(t, \cdot) = Q^+(\rho(t, \cdot), \rho(t, \cdot)),$$

and in the general model $Q^+(\mu, \nu)$ is the law of the generalized convolution

$$pX_1 + qX_2$$

for $X_1 \sim \mu$ and $X_2 \sim \nu$, in which $X_1$ and $X_2$ are independent, and $p, q \in [0, 1]$ are fixed constants. Generalized Kac equations have been extensively studied both with Fourier-based techniques and with probabilistic methods. The repeated model considered in the present paper corresponds to $p = q = 1/2$. This special collision rule is one of the various models proposed in the literature for a kinetic description of the distribution of wealth. The interaction is point-wise conservative (in the language of Matthes and Toscani), and hence, provided the initial condition has a finite mean, the stationary state is the degenerate distribution centered on the mean of the initial distribution. This suggests that in the limit as $t \to \infty$ the Gini index must go to zero.

We now present a stochastic representation of the evolution Equation (8), which is interesting in its own right.

**Proposition 1.** Assume that $\rho_t(x) := \rho(t,x)$ is a solution of (8) with initial condition $\rho_0(x)$ being a probability density function supported on $\mathbb{R}_+$ with mean $\mu$. Defining $(X_t)_{t \geq 0}$ to be a $\mathbb{R}_+$-valued continuous-time pure jump process with jumps of the form

$$X_t \xrightarrow{\text{distributions}} \frac{X_t + Y_t}{2},$$

where $Y_t$ is a i.i.d. copy of $X_t$ and the jump occurs according to a Poisson clock running at the unit rate. If $\text{Law}(X_0) = \rho_0$, then $\text{Law}(X_t) = \rho_t$ for all $t \geq 0$.

**Proof.** Taking $\varphi$ to be an arbitrary but fixed test function, we have

$$\frac{d}{dt} \mathbb{E}[\varphi(X_t)] = \mathbb{E}[\varphi((X_t + Y_t)/2)] - \mathbb{E}[\varphi(X_t)].$$

Denoting $\rho(t,x)$ as the probability density function of $X_t$, (12) can be rewritten as

$$\frac{d}{dt} \int_{\mathbb{R}_+} \rho(t,x) \varphi(x) \, dx = \int_{\mathbb{R}_+^2} \varphi((k + \ell)/2) \rho(k,t) \rho(\ell,t) \, dk \, d\ell - \int_{\mathbb{R}_+} \rho(t,x) \varphi(x) \, dx.$$
After a simple change of variables, one arrives at

$$\frac{d}{dt} \int_{\mathbb{R}_+} \rho(t,x)\varphi(x)\,dx = \int_{\mathbb{R}_+} (Q_+[\rho](x,t) - \rho(t,x))\varphi(x)\,dx.$$  

Thus, \( \rho \) has to satisfy \( \partial_t \rho = \mathcal{L}[\rho] \), and the proof is completed.

**Remark 3.** The stochastic representation given in Proposition 1 is not the first stochastic representation for this kind of equations. A very abstract representation in terms of a stochastic differential equation driven by Poisson point measure is discussed in section 1.5 of Cortez and Fontbona.\(^{39}\) This abstract stochastic representation is derived from ideas contained in Tanaka.\(^{40}\) A different explicit stochastic representation based on Gabetta and Regazzini and Wild\(^{38,41}\) is given in Proposition 1 of Bassetti et al.\(^{35}\)

### 2.2 Exponential decay of the variance

Our main goal in this subsection is the proof of the following.

**Theorem 1.** Assume that \( \rho(t,x) \) is a classical solution of (8) for each \( t > 0 \), with the initial condition \( \rho(0,x) \) being a probability density on \([0, \infty)\) with mean \( \mu > 0 \) and finite variance. Then the variance of \( \rho \) at time \( t \), denoted by \( V(t) \), decays exponentially in time. More specifically, we have \( V(t) = V(0)e^{-\frac{1}{2}t} \).

**Proof.** Thanks to the conservation of the mean value, we have

$$V(t) = \int_{\mathbb{R}_+} x^2 \rho(t,x)\,dx - \mu^2.$$  

Thus, we deduce

$$\frac{d}{dt} V(t) = 2 \int_{\mathbb{R}_+} x^2 (\rho \star \rho)(2x)\,dx - \int_{\mathbb{R}_+} x^2 \rho(x)\,dx$$  

$$= \int_{\mathbb{R}_+} 2x^2 \left( \int_0^{2x} \rho(y)\rho(2x-y)\,dy \right)\,dx - \int_{\mathbb{R}_+} x^2 \rho(x)\,dx$$  

$$= \int_{y \geq 0} \rho(y) \int_{x \geq y/2} 2x^2 \rho(2x-y)\,dx\,dy - \int_{\mathbb{R}_+} x^2 \rho(x)\,dx$$  

$$= \int_{y \geq 0} \rho(y) \int_{z \geq 0} \left( (y+z)/2 \right)^2 \rho(z)\,dz\,dy - \int_{\mathbb{R}_+} x^2 \rho(x)\,dx$$  

$$= -\frac{1}{2} \left( \int_{\mathbb{R}_+} x^2 \rho(t,x)\,dx - \mu^2 \right) = -\frac{1}{2} V(t).$$

A simple integration yields the advertised conclusion. \( \square \)

**Remark 4.** The proof of Theorem 1 can also be carried out from a purely stochastic point of view, by leveraging the stochastic representation of the PDE (8). Indeed, suppose that \((X_t)_{t \geq 0}\) and \((Y_t)_{t \geq 0}\) are defined as in the statement of Proposition 1. Then we can calculate

$$\frac{d}{dt} V(t) = \frac{d}{dt} \text{Var}[X_t] = \text{Var}[(X_t + Y_t)/2] - \frac{1}{2} \text{Var}[X_t] = -\frac{1}{2} V(t).$$
which leads us to the same result. Variants of Theorem 1 (in form of inequality) can be found in Lemma 5 in Cortez and Fontbona or Theorem 3.2 in Matthes and Toscani. It is also possible to derive the result of Theorem 1 by combining Proposition 1 and Lemma 2 in Bassetti et al.

### 2.3 Exponential decay of the Gini index

The widely used inequality indicator Gini index $G$ measures the inequality in the wealth distribution and ranges from 0 (no inequality) to 1 (extreme inequality). We recall the definition of $G$ here for the reader’s convenience.

**Definition 1.** Given a probability density function $\rho$ supported on $\mathbb{R}_+$ with mean value $\mu > 0$. The Gini index of $\rho$ is given by

$$G[\rho] = \frac{1}{2\mu} \int \int_{\mathbb{R}^2_+} \rho(x)\rho(y) |x - y| \, dx \, dy.$$

Alternatively, we can also rewrite

$$G[\rho] = \frac{1}{2\mu} \mathbb{E}[|X - Y|],$$

in which $X$ and $Y$ are i.i.d. random variables with law $\rho$.

In econophysics literature, analytical results on Gini index are comparatively rare. In certain models, the Gini index can be shown to converge to 1, which implies the emergence of the “rich-get-richer” phenomenon and the accentuation of the wealth inequality; see for instance Boghosian et al and references therein. There is also a recently proposed model known as the rich-biased model, in which the authors observe a numerical evidence for the convergence of Gini index to its maximum possible value, but analytical justification is still absent. As have been indicated earlier, the limit PDE (8) associated with the repeated averaging model can be served as the first example for which quantitative estimates on the behavior of Gini index can be hoped. We start with the following preliminary observation.

**Proposition 2.** Assume that $\rho(t, x)$ is a classical solution of (8) for each $t > 0$, with the initial condition $\rho(0, x)$ being a probability density on $[0, \infty)$ with mean $\mu > 0$. Then the Gini index $G[\rho]$ is non-increasing in time. Moreover, we have

$$\frac{d}{dt} G[\rho] = -\frac{1}{\mu} \int \int_{\mathbb{R}^2_+} \rho(v)\rho(w) \rho(y) \left( \frac{|v - y| + |w - y|}{2} - \left| \frac{v + w}{2} - y \right| \right) \, dv \, dw \, dy \leq 0. \quad (13)$$

**Proof.** By symmetry, we have

$$\frac{d}{dt} G[\rho] = \frac{1}{\mu} \int \int_{\mathbb{R}^2_+} \partial_t \rho(x) \rho(y) |x - y| \, dx \, dy$$

$$= \frac{1}{\mu} \int \int_{\mathbb{R}^2_+} 2(\rho * \rho)(2x) \rho(y) |x - y| \, dx \, dy - 2G[\rho]$$

$$= \frac{1}{\mu} \int \int_{\mathbb{R}^2_+} 2 \left( \int_0^{2x} \rho(z) \rho(2x - z) \, dz \right) \rho(y) |x - y| \, dx \, dy - 2G[\rho]$$

$$= \frac{1}{\mu} \int \int_{\mathbb{R}^2_+} \rho(v)\rho(w) \rho(y) \left| \frac{v + w}{2} - y \right| \, dv \, dw \, dy - 2G[\rho]$$

$$= -\frac{1}{\mu} \int \int_{\mathbb{R}^2_+} \rho(v)\rho(w) \rho(y) \left( \frac{|v - y| + |w - y|}{2} - \left| \frac{v + w}{2} - y \right| \right) \, dv \, dw \, dy,$$

whence the proof is finished.

**Remark 5.** In light of the previous remark and stochastic representation of the PDE (8). We can also provide an alternative proof of Proposition 2. Indeed, suppose that $(X_i)_{i \geq 0}$ and $(Y_i)_{i \geq 0}$ are defined as in the statement of Proposition 1.
Then we can compute
\[
\frac{d}{dt} G[\rho] = \frac{1}{2\mu} \frac{d}{dt} \mathbb{E}[(X_t - Y_t)] = \frac{1}{\mu} \mathbb{E}[(X_t + Z_t)/2 - Y_t] - \frac{1}{\mu} \mathbb{E}[|X_t - Y_t|] \\
= -\frac{1}{\mu} (\mathbb{E}[|X_t - Y_t|] - \mathbb{E}[|(X_t + Z_t)/2 - Y_t|]) \leq 0,
\]
in which \(Z_t\) is a fresh i.i.d. copy of \(X_t\) (independent of \(Y_t\) as well). This coincides with (13).

At this point, we may expect to bound \(G[\rho]\) in terms of \(- \frac{d}{dt} G[\rho]\) in order to extract some information on the rate of decay of \(G\). But unfortunately, inequalities of the form \(- \frac{d}{dt} G[\rho] \geq c \cdot G[\rho]\) can not be always fulfilled. For example, if we take \(\rho = \frac{1}{2} \delta_0 + \frac{1}{2} \delta_2\), then one can check that \(\frac{d}{dt} G[\rho] = 0\), whereas \(G[\rho] = \frac{1}{2} > 0\). However, not all hope is lost. Indeed, if we restrict the initial data \(\rho(0,x)\) to be log-concave, we can prove the following

**Theorem 2.** Assume that \(\rho(t,x)\) is a classical solution of (8) for each \(t > 0\), with the initial condition \(\rho(0,x)\) being a log-concave probability density on \([0, \infty)\) with mean \(\mu > 0\). Then the Gini index \(G[\rho]\) converges to 0 exponentially fast in time. Moreover, we have
\[
G[\rho(t)] \leq G[\rho(0)]e^{-\frac{t}{144}}.
\]

To facilitate the proof of Theorem 2, we need the following

**Lemma 2.** Assume that \(\rho(t,x)\) is a classical solution of (8) for each \(t > 0\), with the initial condition \(\rho(0)\) being a log-concave probability density on \([0, \infty)\) with mean \(\mu > 0\). Then \(\rho(t)\) is again log-concave for each \(t > 0\).

**Proof.** The proof is an immediate consequence of the stochastic representation of (8), together with the elementary fact that log-concavity is preserved by convolution. \(\square\)

**Remark 6.** Preservation of log-concavity can also be established for other PDEs, although the proofs are usually quite involved. For instance, it is well-known that evolution under the one-dimensional heat equation preserves the log-concavity of the initial datum.\(^{44}\)

We can now present the proof of Theorem 2.

**Proof of Theorem.** For notational simplicity, we write
\[
G := G[\rho] \quad \text{and} \quad H := -\frac{d}{dt} G[\rho].
\]

In fact, we will not need the restriction that the support of the distribution \(\rho\) is \([0, \infty)\).

By approximation, without loss of generality, we may assume that \(\rho(x) > 0\) for all real \(x\). For example, one may approximate \(\rho\) by its convolution \(\rho * \phi\) with the density \(\phi\) of a centered normal distribution with an arbitrarily small variance. Then \(\rho * g > 0\) on \(\mathbb{R}\) and \(\rho * \phi\) is arbitrarily close to \(\rho\) and log-concave, thanks to the preservation of log-concavity by convolution.

As \(\rho\) is a log-concave density, \(\rho\) is continuous and attains its maximum value, say \(\rho_*(> 0)\), at some point \(c \in \mathbb{R}\), so that \(\rho_*(x) = \rho(c) \geq \rho(x)\) for all real \(x\). Moreover, again because \(\rho\) is log-concave, there exist (unique) real \(a\) and \(b\) such that
\[
a < c < b \quad \text{and} \quad \rho(a) = \rho(b) = \rho_*/e.
\]

We define
\[
q(x) := \begin{cases} 
q_1(x) := \rho_* \exp\left\{-\frac{x-c}{a-c}\right\} & \text{if } x < a, \\
\rho_* & \text{if } a \leq x < b, \\
q_2(x) := \rho_* \exp\left\{-\frac{x-c}{b-c}\right\} & \text{if } x \geq b.
\end{cases}
\]

Thanks to the log-concavity of \(\rho\) again, we have \(\rho(x) \leq q(x)\). We refer to Figure 3 for an illustration.
FIGURE 3 For \( \rho(x) = xe^{-x} \mathbb{1}_{(x>0)} \), here are the graphs
\( \{(x, \rho(x))| -2 \leq x \leq 6\} \) (blue), \( \{(x, q(x))| -2 \leq x \leq 6\} \) (black),
\( \{(x, q_1(x))| a \leq x \leq c\} \) (dashed red), and \( \{(x, q_2(x))| c \leq x \leq b\} \) (dashed green). For this particular \( \rho \), we have
\( c = 1 \), \( a = -W_0 \left( -1/e^2 \right) \approx 0.1586 \), and \( b = -W_{-1} \left( -1/e^2 \right) \approx 3.1461 \),
where \( W_j \) is the \( j \)th branch of the Lambert \( W \) function.45 [Colour figure can be viewed at wileyonlinelibrary.com]

By shifting, we may assume with of loss of generality that \( a = 0 \). Thus,
\[
G \leq \int_{\mathbb{R}^2} q(x)q(y)|x-y| \, dx \, dy
= \rho^2 \frac{e^2 b^3 + 9 e b^3 + 3 b^3 - 12 e b^2 c - 3 b c^2 + 12 e b c^2}{3 e^2}
\leq \rho^2 \frac{(1 + 3 e + e^2/3) b^3}{e^2},
\]
since \( 0 < c < b \). Moreover, again by the log-concavity of \( \rho \), we have \( \rho \geq \rho_*/e \) on the interval \( [a,b] = [0,b] \), so that
\[
1 = \int_{\mathbb{R}} \rho \geq \int_{[0,b]} \rho_*/e = b \rho_*/e, \text{ whence } \rho_* \leq e/b \text{ and}
G \leq (1 + 3 e + e^2/3) b. \tag{16}
\]

On the other hand, because \( \rho \geq \rho_*/e \) on the interval \( [a,b] = [0,b] \) and the integrand in the definition of \( H \) is nonnegative, we have
\[
H \geq \left( \frac{\rho_*}{e} \right)^3 \int_{[0,b]^3} \left( \frac{|x-z| + |y-z|}{2} - \frac{|x-z+y-z|}{2} \right) \, dx \, dy \, dz
= \left( \frac{\rho_*}{e} \right)^3 \frac{b^4}{24}.
\]
Also, \( 1 = \int_{\mathbb{R}} \rho \leq \int_{\mathbb{R}} q = \rho_* b \left( 1 + 1/e \right) \), so that \( \rho_* \geq 1/(b(1+1/e)) \), and hence,
\[
H \geq \left( \frac{1}{(e+1)b} \right)^3 \frac{b^4}{24} = \frac{b}{24(e+1)^3}. \tag{17}
\]

Comparing (16) and (17), we deduce
\[
H \geq \frac{G}{24(e+1)^3(1 + 3 e + e^2/3)} \geq \frac{G}{14334},
\]
as claimed. \( \square \)

Remark 7. The assumption of log-concave initial data may look quite strong. A similar result can be proved assuming existence of the \( (1 + \epsilon) \)-th moment of a smooth initial condition as follows. Let \( (X_t)_{t \geq 0} \) and \( (Y_t)_{t \geq 0} \) be defined as in the
statement of Proposition 1, then thanks to triangle inequality, one has

$$2\mu \cdot G = \mathbb{E}[|X_t - Y_t|] \leq 2\mathbb{E}[|X_t - \mu|] = 2W_1(\rho_t, \delta_\mu),$$

where $W_1(\mu, \nu)$ represents the Wasserstein distance (of order 1) between the probability measures $\mu$ and $\nu$. Applying Theorem 5 in Bassetti et al. gives in this case

$$W_1(\rho_t, \delta_\mu) \leq C e^{-\frac{1}{12}(1-2^{-1})t}$$

for some fixed constant $C > 0$ depending only on $\epsilon \in (0, 1]$. Thus, the exponential decay of the Gini index $G$ follows from bounding $W_1(\rho_t, \delta_\mu)$ by $W_1(\rho_t, \delta_\mu)$. We emphasize that the proof of Theorem 2 presented here is rather elementary while the proof of Theorem 5 in Bassetti et al. will require much more sophisticated techniques.

Finally, we provide a numerical experiment in order to corroborate the relaxation of the Gini index guaranteed by Theorem 2; see Figure 4. For the initial condition, we use a gamma probability density with shape parameter $\mu = 5$ and rate parameter equal to unity, that is, $\rho(0, x) = \mathbb{1}_{[0, \infty)}(x) \cdot x^{\mu-1} e^{-x/\Gamma(\mu)}$. The standard forward Euler scheme (with the time step-size $\Delta t = 0.05$ and the space step-size $\Delta x = 0.01$) is enforced for the numerical solution of (8). Note that the Gini index of our choice of $\rho(0, x)$ has a nice closed expression $G(\rho(0)) = \frac{2^{1-2\epsilon}}{\mu(\Gamma(2\mu))^{\frac{3}{2}}}\frac{2\mu - 1}{\mu(\Gamma(\mu))^2}$, which reduces (approximately) to 0.2461 for $\mu = 5$.

### 3 | PROPAGATION OF CHAOS

In the last part of the manuscript, we sketch the proof of the so-called propagation of chaos, relying on a martingale-based technique developed in Merle and Salez. We emphasize that the proof presented here is a modification of Theorem 6 in Cao et al.

We equip the space $P(\mathbb{R}_+)$ with the Wasserstein distance with exponent 1, which is defined via

$$W_1(\mu, \nu) = \sup_{\|\varphi\|_{\infty} \leq 1} \langle \mu - \nu, \varphi \rangle$$

for $\mu, \nu \in P(\mathbb{R}_+)$. The propagation of chaos result is summarized in the following.

**Theorem 3.** Denote the empirical distribution of the repeated averaging $N$ particle system (1) at time $t$ as

$$\rho_{emp}(t) = \frac{1}{N} \sum_{i=1}^{N} \delta_{X_i(t)},$$
and let $\rho(t)$ be the solution of (8) with initial data $\rho(0)$. If
\[
\mathbb{E}[W_1(\rho_{\text{emp}}(0), \rho(0))] \to 0 \text{ as } N \to \infty,
\]
then we have that
\[
\mathbb{E}[W_1(\rho_{\text{emp}}(t), q(t))] \to 0 \text{ as } N \to \infty,
\]
holding for all $0 \leq t \leq T$ with any prefixed $T > 0$.

Proof. We recall that the map $Q_+[-] : \mathcal{P}(\mathbb{R}_+) \to \mathcal{P}(\mathbb{R}_+)$ is defined via
\[
Q_+[\rho](x) = 2(\rho \ast \rho)(2x), \quad \forall x \geq 0.
\]
Assume that a classical solution $\rho(t,x)$ of
\[
\rho(t,x) = \rho(0,x) + \int_0^t \mathcal{L}[\rho](s,x)\,ds
\]
exists for $0 \leq t < \infty$, where $\mathcal{L} = Q_+ - \text{Id}$ and $\rho(0,x)$ is a probability density function whose support is contained in $\mathbb{R}_+$. The map $Q_+$ is Lipschitz continuous in the sense that
\[
W_1(Q_+[f], Q_+[g]) \leq W_1(f, g)
\]
for any $f, g \in \mathcal{P}(\mathbb{R}_+)$. Indeed, we have
\[
W_1(Q_+[f], Q_+[g]) = \sup_{\|\varphi\|_{\infty} \leq 1} \mathbb{E} \left[ \varphi(X_1 + Y_1)/2 - \varphi(X_2 + Y_2)/2 \right],
\]
where $X_1, Y_1$ are i.i.d with law $f$, $X_2, Y_2$ are i.i.d with law $g$. By Lipschitz continuity of the test function $\varphi$, we obtain
\[
W_1(Q_+[f], Q_+[g]) \leq \mathbb{E}[|X_1 - X_2|].
\]
We now recall an alternative formulation of $W_1(f, g)$, given by
\[
W_1(f, g) = \inf \left\{ \mathbb{E}[|X - Y|] ; \text{Law}(X) = f, \text{Law}(Y) = g \right\},
\]
so in particular, we may take a coupling of $X_1$ and $X_2$ so that $W_1(f, g) = \mathbb{E}[|X_1 - X_2|]$. Assembling these pieces together, we arrive at (21). More generally, suppose we have two random probability measures $f$ and $g$ with bounded second moment, taking expectation on both sides of (21) gives rise to
\[
\mathbb{E} \left[ \sup_{\|\varphi\|_{\infty} \leq 1} \int \varphi(x)(Q_+[f] - Q_+[g]) \right] \leq \mathbb{E} \left[ \sup_{\|\varphi\|_{\infty} \leq 1} \int \varphi(x)(f(dx) - g(dx)) \right].
\]

We now observe that the empirical measure is a compound jump process: Define $N_t$ a homogeneous Poisson process with constant intensity $\lambda = (N - 1)/2$. Given $\tau_1, \ldots, \tau_k$ the times when $N_t$ jumps, we take the $Y_{\tau_k}$ independent: At each $\tau_k$, with uniform probability $2/N(N - 1)$, we choose a pair $i < j$ and take
\[
Y_{\tau_k} = \frac{1}{N} \left( 2\delta(x - (X_i(\tau_k-)) + X_j(\tau_k-)/2) \right.
\]
\[
- \delta(x - X_i(\tau_k-)) - \delta(x - X_j(\tau_k-)) \right).
\]
We immediately note that

\[
\lambda E[Y_t] = \frac{1}{N^2} \sum_{i<j} \mathbb{E} \left[ 2\delta(x - (X_i(t) + X_j(t)/2)) - \delta(x - X_i(t)) - \delta(x - X_j(t)) \right].
\]  

(23)

We now show that the empirical measure of the stochastic system satisfies an approximate version of (20). Fix a deterministic test function \( \varphi \) with \( \|\nabla \varphi\|_\infty \leq 1 \) and consider the time evolution of \( \langle \rho_{\text{emp}}, \varphi \rangle \) where for some probability measure \( \nu \), we denote by the duality bracket \( \langle \nu, \varphi \rangle = \int \varphi \, d\nu \). Then

\[
d\mathbb{E}[\langle \rho_{\text{emp}}, \varphi \rangle] = d\mathbb{E} \left[ \langle Y_t, dN_t, \varphi \rangle \right] = \lambda \langle E[Y_t], \varphi \rangle \, dt.
\]

Therefore, thanks to (23),

\[
d\mathbb{E}[\langle \rho_{\text{emp}}, \varphi \rangle] = \frac{1}{N^2} \sum_{i<j} \mathbb{E} \left[ 2\varphi \left( (X_i + X_j)/2 \right) - \varphi(X_i) - \varphi(X_j) \right] \, dt
\]

\[
= \frac{1}{N^2} \sum_{i,j=1 \ldots N, i \neq j} \mathbb{E} \left[ \varphi \left( (X_i + X_j)/2 \right) - \varphi(X_i) \right] \, dt
\]

\[
= \frac{1}{N^2} \sum_{i,j=1}^N \mathbb{E} \left[ \varphi \left( (X_i + X_j)/2 \right) - \varphi(X_i) \right] \, dt,
\]

where all \( X_i, X_j \) are taken at time \( t- \). On the other hand, we may calculate

\[
\langle Q_+, \rho_{\text{emp}} \rangle = \int \varphi(x) 2 \frac{1}{N^2} \sum_{i,j=1}^N \delta_{X_i+X_j}(2x) \, dx = \frac{1}{N^2} \sum_{i,j=1}^N \varphi \left( (X_i + X_j)/2 \right).
\]

Therefore,

\[
d\mathbb{E}[\langle \rho_{\text{emp}}, \varphi \rangle] = \mathbb{E} \left[ \langle \mathcal{L}[\rho_{\text{emp}}], \varphi \rangle \right] \, dt.
\]  

(24)

By Dynkin's formula, the compensated process

\[
M_\varphi(t) := \langle \rho_{\text{emp}}(t), \varphi \rangle - \langle \rho_{\text{emp}}(0), \varphi \rangle - \int_0^t \mathbb{E}[\langle \mathcal{L}[\rho_{\text{emp}}(s)], \varphi \rangle] \, ds
\]  

(25)

is a martingale. Furthermore, comparing with (20), we easily obtain that

\[
\langle \rho_{\text{emp}}(t) - \rho(t), \varphi \rangle = M_\varphi(t) + \langle \rho_{\text{emp}}(0) - \rho(0), \varphi \rangle
\]

\[
+ \int_0^t \mathbb{E}[\langle \mathcal{L}[\rho_{\text{emp}}(s)] - \mathcal{L}[\rho(s)], \varphi \rangle] \, ds.
\]

Taking the supremum over \( \varphi \), we therefore have that

\[
\mathbb{E} \sup_{\|\nabla \varphi\|_\infty \leq 1} \langle \rho_{\text{emp}}(t) - \rho(t), \varphi \rangle \leq \mathbb{E} \sup_{\|\nabla \varphi\|_\infty \leq 1} (|M_\varphi(t)| + \langle \rho_{\text{emp}}(0) - \rho(0), \varphi \rangle)
\]

\[
+ \int_0^t \mathbb{E} \sup_{\|\nabla \varphi\|_\infty \leq 1} \langle \mathcal{L}[\rho_{\text{emp}}(s)] - \mathcal{L}[\rho(s)], \varphi \rangle \, ds.
\]
By the definition of the $W_1$ distance, we deduce from (22) that
\[
\mathbb{E} W_1(\rho_{\text{emp}}(t), q(t)) \leq \eta(t) + 2 \int_0^t \mathbb{E} W_1(\rho_{\text{emp}}(t), q(t)) \, ds,
\]
in which we have set
\[
\eta(t) := \mathbb{E} \sup_{\|\nabla \varphi\|_{\infty} \leq 1} |M_\varphi(t)| + \mathbb{E} W_1(\rho_{\text{emp}}(0), q(0)). \tag{26}
\]
Thus, Gronwall's inequality gives rise to
\[
\mathbb{E} W_1(\rho_{\text{emp}}(t), \rho(t)) \leq \left( \sup_{t \in [0,T]} \eta(t) \right) e^{2T}.	ag{27}
\]
In order to establish propagation of chaos for $t \leq T$, it therefore suffices to show that
\[
\sup_{t \in [0,T]} \eta(t) \xrightarrow{p,N \to \infty} 0. \tag{28}
\]
To prove (28), we treat each term appearing in the definition of $\eta(t)$ separately. The second term in (26) approaches to 0 as $N \to \infty$ by our assumption. The treatment of the first term is more delicate but can be carried out in a similar fashion as the proof of Theorem 6 in Cao et al.\cite{Cao2018} In the end, we obtain estimates of the form
\[
\mathbb{E} \left[ \sup_{\|\nabla \varphi\|_{\infty} \leq 1} \left| \frac{\partial}{\partial t} M_\varphi(t) \right| \right] \leq C \frac{t^\theta}{N^\theta}
\]
for some $\theta > 0$, which allows to finish the proof of (28).

Remark 8. Even though rigorous and quantitative results of propagation of chaos for more general models of wealth distribution are available,\cite{Cortez2016} we believe our approach is less technical. Indeed, the treatment of propagation of chaos in Cortez and Fontbona\cite{Cortez2016} is built on a stochastic differential equation driven by Poisson point measures and a delicate optimal transport argument is also employed, while our (qualitative) result relies only on elementary theory of martingales.

4 CONCLUSION

In this manuscript, we have investigated the repeated averaging dynamics for money exchange originated from econophysics. Because of the model simplicity in its appearance, there is a comparative lack of mathematical literature that is purely dedicated to this model, although this model is a special case of the general dynamics studied in Matthes and Toscani.\cite{Matthes2009} We presented a propagation of chaos result, which links the stochastic $N$ particle system to a deterministic nonlinear evolution equation. Although certain convergence results of the Gini index are obtained for other econophysics models, we emphasize that no quantitative estimates on the long time behavior of Gini index are available in the current literature (at least to our best knowledge). Thus, this toy model may serve as a starting point for more systematic, quantitative investigation of the large-time asymptotic of Gini index arising from other models.

It would also be interesting to investigate the behavior of the Gini index for the stochastic agent-based model where the number of agents $N$ is arbitrary but fixed. We believe that it would be relatively simple (in this setting) to demonstrate the convergence of the Gini index towards zero, but the difficulty arises when we want to obtain an explicit rate of the aforementioned convergence.

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CONFLICT OF INTEREST

The authors declare no potential conflict of interests.

AUTHOR CONTRIBUTIONS

This manuscript studies the so-called repeated averaging model in econophysics: Two agents picked uniformly at random share half of their wealth to each other. We managed to prove a propagation of chaos type result in the limit of large population using a martingale technique, then we analyzed the large time behavior of a limit equation and established convergence to equilibrium results. In particular, we proved a quantitative exponential decay for the Gini index associated with the limit equation. To the author’s best knowledge, this is the first quantitative result obtained for the Gini index in models arising from econophysics. Numerical experiments are also provided to illustrate the model and our results.

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