STRUCTURE OF APPROXIMATE SOLUTIONS OF BOLZA VARIATIONAL PROBLEMS ON LARGE INTERVALS

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Abstract. In this paper we study the structure of approximate solutions of autonomous Bolza variational problems on large finite intervals. We show that approximate solutions are determined mainly by the integrand, and are essentially independent of the choice of time interval and data.

1. Introduction. The study of variational and optimal control problems defined on infinite (large) intervals has recently been a rapidly growing area of research [1, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 16, 17, 20, 22, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36]. These problems arise in engineering [8, 27], in models of economic growth [8, 19, 21, 27], in infinite discrete models of solid-state physics related to dislocations in one-dimensional crystals [3, 23] and in the theory of thermodynamical equilibrium for materials [15, 18].

In this paper we consider the following Bolza variational problems
\[
\int_0^T f(z(t), z'(t))dt + h(z(0), z(T)) \to \min, \quad (P_B)
\]
where \( T > 0 \) is sufficiently large, \( f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^1 \) is an integrand and \( h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^1 \) belongs to a space of functions to be described below.

In our previous research [24, 25, 26, 27, 28, 29, 30, 33] we analyze the structure of approximate solutions of the Lagrange variational problems
\[
\int_0^T f(z(t), z'(t))dt \to \min, \quad z(0) = x, \quad z(T) = y, \quad \text{ (P\text{\textsubscript{1}})}
\]
where \( z : [0, T] \to \mathbb{R}^n \) is an absolutely continuous (a. c.) function, \( f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^1 \) is an integrand and \( f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^1 \) is a space of functions to be described below.

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\[
\int_0^T f(z(t), z'(t))dt \to \min, \quad z(0) = x, \quad \text{ (P\text{\textsubscript{2}})}
\]
where \( z : [0, T] \to \mathbb{R}^n \) is an absolutely continuous (a. c.) function, \( f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^1 \) is an integrand and \( f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^1 \) is a space of functions to be described below.

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\[
\int_0^T f(z(t), z'(t))dt \to \min, \quad z(0) = x, \quad \text{ (P\text{\textsubscript{3}})}
\]
where \( z : [0, T] \to \mathbb{R}^n \) is an absolutely continuous (a. c.) function, \( f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^1 \) is an integrand and \( f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^1 \) is a space of functions to be described below.

More precisely, in our research which was summarized in [27] we were interested in turnpike properties of the approximate solutions of problem (P\text{\textsubscript{1}}) which are independent of the length of the interval, for all sufficiently large intervals. To have this

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property means, roughly speaking, that the approximate solutions of the variational problems are determined mainly by the integrand, and are essentially independent of the choice of an interval and endpoint conditions, except in regions close to the endpoints of the time interval.

It is clear that an optimal solution \( v : [0, T] \to \mathbb{R}^n \) of the variational problem \((P_1)\) always depends on the integrand \( f \) and on \( x, y, T \).

We say that the integrand \( f \) has the turnpike property if for any \( \epsilon > 0 \) there exist constants \( L_1 > L_2 > 0 \) which depend only on \( |x|, |y|, \epsilon \) such that for each \( \tau \in [L_1, T - L_1] \) the set \( \{v(t) : t \in [\tau, \tau + L_2]\} \) is equal to a set \( H(f) \) up to \( \epsilon \) in the Hausdorff metric where \( H(f) \subset \mathbb{R}^n \) is a compact set depending only on the integrand \( f \).

Thus if the integrand \( f \) has the turnpike property, then for large enough \( T \) the dependence on \( x, y, T \) is not essential. In [26] this turnpike property was established for a certain large class of integrands.

Turnpike properties are well known in mathematical economics. The term was first coined by Samuelson in 1948 (see [21]) where he showed that an efficient expanding economy would spend most of the time in the vicinity of a balanced equilibrium path (also called a von Neumann path). This property was further investigated for optimal trajectories of models of economic dynamics [19, 27]. Many turnpike results can be found in [27].

In [33] we studied the structure of approximate solutions of the problems \((P_2)\) and \((P_3)\) in regions close to the endpoints of the time intervals. We showed that in regions close to the right endpoint \( T \) of the time interval these approximate solutions are determined only by the integrand, and are essentially independent of the choice of interval and endpoint value \( x \). For the problems \((P_3)\), approximate solutions are determined only by the integrand also in regions close to the left endpoint 0 of the time interval.

More precisely, in [33] we define \( \bar{f}(x, y) = f(x, -y) \) for all \( x, y \in \mathbb{R}^n \) and consider the set \( \mathcal{P}(\bar{f}) \) of all solutions of a corresponding infinite horizon variational problem associated with the integrand \( \bar{f} \). For given positive constants \( \epsilon, \tau \), we show that if \( T \) is large enough and \( v : [0, T] \to \mathbb{R}^n \) is an approximate solution of the variational problem \((P_2)\), then \( |v(T - t) - w(t)| \leq \epsilon \) for all \( t \in [0, \tau] \), where \( w \in \mathcal{P}(\bar{f}) \). Moreover, using the Baire category approach, we showed that for most integrands \( f \) the set \( \mathcal{P}(\bar{f}) \) is a singleton.

In this paper we establish our results on the structure of approximate solutions of Bolza problems \((P_B)\).

Denote by \( |\cdot| \) the Euclidean norm in \( \mathbb{R}^n \). Let \( a \) be a positive constant and let \( \psi : [0, \infty) \to [0, \infty) \) be an increasing function such that \( \psi(t) \to \infty \) as \( t \to \infty \). Denote by \( \mathcal{A} \) the set of all continuous functions \( f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^1 \) which satisfy the following assumptions:

A(i) for each \( x \in \mathbb{R}^n \) the function \( f(x, \cdot) : \mathbb{R}^n \to \mathbb{R}^1 \) is convex;

A(ii) \( f(x, u) \geq \max\{\psi(|x|), \psi(|u|)|u|\} - a \) for each \( (x, u) \in \mathbb{R}^n \times \mathbb{R}^n \);

A(iii) for each \( M, \epsilon > 0 \) there exist \( \Gamma, \delta > 0 \) such that

\[
|f(x_1, u_1) - f(x_2, u_2)| \leq \epsilon \max\{f(x_1, u_1), f(x_2, u_2)\}
\]

for each \( u_1, u_2, x_1, x_2 \in \mathbb{R}^n \) which satisfy

\[
|x_i| \leq M, \ i = 1, 2, \ |u_i| \geq \Gamma, \ i = 1, 2, \ |x_1 - x_2|, |u_1 - u_2| \leq \delta.
\]

It is easy to show that an integrand \( f = f(x, u) \in C^1(\mathbb{R}^{2n}) \) belongs to \( \mathcal{A} \) if \( f \) satisfies assumptions A(i), A(ii) and if there exists an increasing function \( \psi_0 : \]
for each $x,u \in R^n$. Here
\[
\frac{\partial f}{\partial x} = (\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}), \quad \frac{\partial f}{\partial u} = (\frac{\partial f}{\partial u_1}, \ldots, \frac{\partial f}{\partial u_n}).
\]
For the set $A$ we consider the uniformity which is determined by the following base:
\[
E(N, \epsilon, \lambda) = \{(f,g) \in A \times A : |f(x,u) - g(x,u)| \leq \epsilon \text{ for all } x,u \in R^n \text{ satisfying } |x|, |u| \leq N \}
\]
\[
\cap \{(f,g) \in A \times A : (|f(x,u)| + 1)(|g(x,u)| + 1)^{-1} \in [\lambda^{-1}, \lambda] \text{ for all } x,u \in R^n \text{ satisfying } |x| \leq N \},
\]
where $N, \epsilon > 0$ and $\lambda > 1$. It is known \[27\] that the uniform space $A$ is metrizable and complete.

We consider functionals of the form
\[
I^f(T_1, T_2, x) = \int_{T_1}^{T_2} f(x(t), x'(t)) dt
\]
where $f \in A$, $-\infty < T_1 < T_2 < \infty$ and $x : [T_1, T_2] \to R^n$ is an absolutely continuous (a.c.) function.

For $f \in A$, $y,z \in R^n$ and real numbers $T_1, T_2$ satisfying $T_1 < T_2$ we set
\[
U^f(T_1, T_2, y, z) = \inf\{I^f(T_1, T_2, x) : x : [T_1, T_2] \to R^n \text{ is an a.c. function satisfying } x(T_1) = y, x(T_2) = z \}.
\]
It is easy to see that $-\infty < U^f(T_1, T_2, y, z) < \infty$ for each $f \in A$, each $y,z \in R^n$ and all numbers $T_1, T_2$ satisfying $T_1 < T_2$.

A function $x(\cdot)$ defined on an unbounded interval with the values in a finite-dimensional Euclidean space is called absolutely continuous (a.c.) if it is absolutely continuous on any bounded subinterval of its domain.

Let $f \in A$. For any a.c. function $v : [0, \infty) \to R^n$ we set
\[
J(v) = \liminf_{T \to \infty} T^{-1} I^f(0, T, v).
\]

The real number
\[
\mu(f) = \inf\{J(v) : v : [0, \infty) \to R^n \text{ is an a.c. function} \}
\]
is called the minimal long-run average cost growth rate of $f$.

Clearly, $-\infty < \mu(f) < \infty$. By Theorems 3.6.1 and 3.6.2 of \[27\],
\[
U^f(0, T, x, y) = T \mu(f) + \pi^f(x) - \pi^f(y) + \theta^f_T(x,y)
\]
for all $x,y \in R^n$ and all $T \in (0, \infty)$, where $\pi^f : R^n \to R^1$ is a continuous function and
\[
(T, x, y) \to \theta^f_T(x,y) \in R^1 \text{ is a continuous nonnegative function}
\]
defined for all $T > 0$ and all $x,y \in R^n$,
\[
\pi^f(x) = \inf_{T \to \infty} T \liminf_{T \to \infty} [I^f(0, T, v) - \mu(f)T] : v : [0, \infty) \to R^n
\]
is an a.c. function satisfying $v(0) = x$, $x \in R^n$.

for every $T > 0$, every $x \in R^n$ there is $y \in R^n$ satisfying $\theta^f_T(x,y) = 0$.
An a.c. function $x : [0, \infty) \to \mathbb{R}^n$ is called $(f)$-good if the function
\[ T \to I^J(0, T, x) - \mu(f)T, \; T \in (0, \infty) \]
is bounded.

By Theorem 3.6.3 of [27], for each $f \in \mathcal{A}$ and each $z \in \mathbb{R}^n$ there exists an
$(f)$-good function $v : [0, \infty) \to \mathbb{R}^n$ satisfying $v(0) = z$.

In the sequel we use the following result (Proposition 4.1.1 of [27]).

**Theorem 1.1.** For any a.c. function $x : [0, \infty) \to \mathbb{R}^n$ either $I^J(0, T, x) - T\mu(f) \to \infty$ as $T \to \infty$ or
\[ \sup\{|I^J(0, T, x) - T\mu(f)| : T \in (0, \infty)\} < \infty. \]
Moreover any $(f)$-good function $x : [0, \infty) \to \mathbb{R}^n$ is bounded.

We denote $d(x, B) = \inf\{|x - y| : y \in B\}$ for $x \in \mathbb{R}^n$ and $B \subset \mathbb{R}^n$ and denote by\dist(A, B) the distance in the Hausdorff metric for two sets $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^n$.

For every bounded a. c. function $x : [0, \infty) \to \mathbb{R}^n$ define
\[ \Omega(x) = \{y \in \mathbb{R}^n : \text{there exists a sequence } \{t_i\}_{i=1}^\infty \subset (0, \infty) \]
for which $t_i \to \infty, x(t_i) \to y$ as $i \to \infty$. \hfill (1.9)

We say that an integrand $f \in \mathcal{A}$ has an asymptotic turnpike property, or briefly
(ATP), if $\Omega(v_i) = \Omega(v_1)$ for all $(f)$-good functions $v_i : [0, \infty) \to \mathbb{R}^n$, $i = 1, 2$.

By Theorem 3.1.1 of [27], there exists a set $\mathcal{F} \subset \mathcal{A}$ which is a countable intersection
of open everywhere dense subsets of $\mathcal{A}$ such that each integrand $f \in \mathcal{F}$ possesses
(ATP). In other words, (ATP) holds for a typical (generic) integrand $f \in \mathcal{A}$.

By Theorem 1.1 for each integrand $f \in \mathcal{A}$ which possesses (ATP) there exists
a compact set $H(f) \subset \mathbb{R}^n$ such that $\Omega(v) = H(f)$ for each $(f)$-good function $v : [0, \infty) \to \mathbb{R}^n$. In this case we say that the set $H(f)$ is the turnpike of $f$.

The following turnpike result was obtained in [24]. For its proof see also Theorem
3.1.4 of [27].

**Theorem 1.2.** Assume that an integrand $f \in \mathcal{A}$ has the asymptotic turnpike
property and that $M_0, M_1, \epsilon > 0$. Then there exist a neighborhood $\mathcal{U}$ of $f$ in $\mathcal{A}$, numbers
$l, S > 0$ and integers $L, Q_*, \geq 1$ such that for each $g \in \mathcal{U}$, each pair of numbers
$T_1 \geq 0$, $T_2 \geq T_1 + L + LQ_*$ and each a.c. function $v : [T_1, T_2] \to \mathbb{R}^n$ which satisfies
\[ |v(T_i)| \leq M_1, \; i = 1, 2, \]
\[ I^J(T_1, T_2, v) \leq U^J(T_1, T_2, v(T_1), v(T_2)) + M_0 \]
the inequality $|v(t)| \leq S$ holds for all $t \in [T_1, T_2]$ and there exist sequences of numbers $\{b_i\}_{i=1}^l, \{c_i\}_{i=1}^Q \subset [T_1, T_2]$ such that\[ Q \leq Q_*, \; 0 \leq c_i - b_i \leq l, \; i = 1, \ldots, Q \]
and that\[ \text{dist}(H(f), \{v(t) : t \in [T, T + L]\}) \leq \epsilon \]
for each $T \in [T_1, T_2 - L] \setminus \cup_{i=1}^Q [b_i, c_i]$.

Denote by $\mathcal{M}$ the set of all functions $f \in C^1(\mathbb{R}^{2n})$ which satisfy the following assumptions:
\[ \partial f/\partial u_i \in C^1(\mathbb{R}^{2n}) \quad \text{for} \; i = 1, \ldots, n; \]
the matrix\[ (\partial^2 f/\partial u_i \partial u_j)(x, u), \; i, j = 1, \ldots, n \]
is positive definite for all \((x, u) \in \mathbb{R}^{2n}\);
\[
f(x, u) \geq \max \{\psi(|x|), \psi(|u|)|u|\} - a \quad \text{for all} \quad (x, u) \in \mathbb{R}^n \times \mathbb{R}^n;
\]
there exist a number \(c_0 > 1\) and monotone increasing functions \(\phi_i : [0, \infty) \to [0, \infty)\), \(i = 0, 1, 2\) such that
\[
\phi_0(t)/t \to \infty \quad \text{as} \quad t \to \infty,
\]
\[
f(x, u) \geq \phi_0(c_0|u|) - \phi_1(|x|), \quad x, u \in \mathbb{R}^n,
\]
\[
\max \{\|\partial f/\partial x_i(x, u)\|, \|\partial f/\partial u_i(x, u)\|\} \leq \phi_2(|x|)(1 + \phi_0(|u|)),
\]
\(x, u \in \mathbb{R}^n, \ i = 1, \ldots, n\).

It is easy to see that \(M \subset A\).

In [26] we established the following result which shows that for an integrand \(f \in M\), (ATP) implies the turnpike property described above with the turnpike \(H(f)\) (for its proof see also Theorem 5.11 of [27]).

**Theorem 1.3.** Assume that an integrand \(f \in M\) has the asymptotic turnpike property and that \(\epsilon, K > 0\). Then there exist a neighborhood \(U\) of \(f\) in \(A\) and numbers \(M > K, l_0 > l > 0, \delta > 0\) such that for each \(g \in U\), each \(T \geq 2l_0\) and each a.c. function \(v : [0, T] \to \mathbb{R}^n\) which satisfies
\[
|v(0)|, |v(T)| \leq K, \quad I^g(0, T, v) \leq U^g(0, T, v(0), v(T)) + \delta
\]
the inequality \(|v(t)| \leq M\) holds for all \(t \in [0, T]\) and
\[
dist(H(f), \{v(t) : t \in [\tau, \tau + l]\}) \leq \epsilon \quad (1.10)
\]
for each \(\tau \in [l_0, T - l_0]\). Moreover, if \(d(v(0), H(f)) \leq \delta\), then (1.10) holds for each \(\tau \in [0, T - l]\) and if \(d(v(T), H(f)) \leq \delta\), then (1.10) holds for each \(\tau \in [l_0, T - l]\).

Let \(k \geq 1\) be an integer. Denote by \(A_k\) the set of all integrands \(f \in A \cap C^k(\mathbb{R}^{2n})\). For any \(p = (p_1, \ldots, p_{2n}) \in \{0, \ldots, k\}^{2n}\) set \(|p| = \sum_{i=1}^{2n} p_i\). For each \(f \in C^k(\mathbb{R}^{2n})\) and each \(p = (p_1, \ldots, p_{2n}) \in \{0, \ldots, k\}^{2n}\) satisfying \(|p| \leq k\) define
\[
D^p f = \partial^{|p|} f / \partial y_1^{p_1} \cdots \partial y_2^{p_{2n}}.
\]
Here \(D^p f = f\).

For the set \(A_k\) we consider the uniformity which is determined by the following base:
\[
E_k(N, \epsilon, \lambda) = \{(f, g) \in A_k \times A_k : \|D^p f(x, u) - D^p g(x, u)\| \leq \epsilon
\]
for all \(x, u \in \mathbb{R}^n\) satisfying \(|x|, |u| \leq N\)

and each \(p \in \{0, \ldots, k\}^{2n}\) satisfying \(|p| \leq k\)
\[
\cap \{(f, g) \in A_k \times A_k : \|(f(x, u)) + 1\|(g(x, u)) + 1\|^{-1} \in [\lambda^{-1}, \lambda]
\]
for all \(x, u \in \mathbb{R}^n\) satisfying \(|x| \leq N\),
where \(N, \epsilon > 0\) and \(\lambda > 1\). It is known (see Chapter 5 of [27]) that the uniform space \(A_k\) is metrizable and complete.

Set \(A_0 = A, M_0 = M\). For each integer \(k \geq 1\) set \(M_k = M \cap A_k\).

Let \(k \geq 0\) be an integer. Denote by \(M_k\) the closure of \(M_k\) in \(A_k\) and consider the topological subspace \(M_k \subset A_k\) equipped with the relative topology.

Denote by \(L\) the set of all \(f \in M \cap C^2(\mathbb{R}^{2n})\) such that
\[
\delta f / \partial u_i \in C^2(\mathbb{R}^{2n}) \quad \text{for} \quad i = 1, \ldots, n.
\]
For any \(k \in \{0, 1, 2\}\) denote by \(L_k\) the closure of \(L\) in the space \(A_k\) and consider the topological subspace \(L_k \subset A_k\) equipped with the relative topology.
In [26] we established the following generic turnpike result which shows that most integrands possess the turnpike property described above (for its proof see also Theorem 5.1.2 of [27]).

**Theorem 1.4.** Let $\mathcal{M}$ be one of the following spaces:

\[ \mathcal{L}_k, \, k = 0, 1, 2, \, \tilde{\mathcal{M}}_q, \, q \geq 0. \]

Then there exists a set $\mathcal{F} \subset \mathcal{M}$ which is a countable intersection of open everywhere dense subsets of $\mathcal{M}$ such that each $f \in \mathcal{F}$ has (ATP) and the following property.

For each $\epsilon, K > 0$ there exist a neighborhood $\mathcal{U}$ of $f$ in $\mathcal{A}$ and numbers $M > K$, $l_0 > l > 0$, $\delta > 0$ such that for each $g \in \mathcal{U}$, each $T \geq 2l_0$ and each a.c. function $v : [0, T] \to \mathbb{R}^n$ which satisfies

\[ |v(0)|, |v(T)| \leq K, \, I^g(0, T, v) \leq U^g(0, T, v(0), v(T)) + \delta \]

the inequality $|v(t)| \leq M$ holds for all $t \in [0, T]$ and

\[ \text{dist}(H(f), \{v(t) : t \in [\tau, \tau + \delta]\}) \leq \epsilon \] \hspace{1cm} (1.11)

holds for each $\tau \in [l_0, T - l_0]$. Moreover, if $d(v(0), H(f)) \leq \delta$, then (1.11) holds for each $\tau \in [0, T - l_0]$ and if $d(v(T), H(f)) \leq \delta$, then (1.11) holds for each $\tau \in [l_0, T - l]$.

Note that in [26, 27] the result stated above was proved in the case when $\mathcal{M}$ in any of the spaces $\tilde{\mathcal{M}}_q$, $q \geq 0$. In the case when $\mathcal{M}$ is in any of the spaces $\mathcal{L}_q$, $q = 0, 1, 2$ Theorem 1.4 is proved with the same proof.

Our paper is organized as follows. Uniform boundedness of approximate solutions of problems $(P_1)$, $(P_2)$ and $(P_3)$ is considered in Section 2. Section 3 contains preliminaries. Section 4 contains a turnpike result for problem $(P_1)$. In Section 5 we begin to study Bolza problems $(P_B)$. Uniform boundedness of approximate solutions of problems $(P_B)$ is established in Section 6. Sections 7 and 8 contain two turnpike results for problems $(P_B)$. Section 9 contains results on the the structure of approximate solutions of Bolza problems $(P_B)$ in the regions close to the endpoints of time intervals (Theorems 9.2-9.4). Auxiliary results are collected in Section 10. In Section 11 we prove auxiliary results for Theorem 9.2 which is proved in Sections 12. Section 13 contains auxiliary results for Theorem 9.4 which is proved in Section 14. Auxiliary results for Theorem 9.3 are collected in Section 15. Theorem 9.3 is proved in Section 16.

2. Uniform boundedness of approximate solutions. For $f \in \mathcal{A}$, $x \in \mathbb{R}^n$ and a real number $T > 0$ set

\[ U^f(T, x) = \inf \{I^f(0, T, v) : v : [0, T] \to \mathbb{R}^n \text{ is an a.c. function satisfying } v(0) = x \}, \] \hspace{1cm} (2.1)

\[ U^f(T) = \inf \{I^f(0, T, v) : v : [0, T] \to \mathbb{R}^n \text{ is an a.c. function}\}. \] \hspace{1cm} (2.2)

The following result plays an important role in our study.

**Theorem 2.1.** Let $f \in \mathcal{A}$ and let $M_1, M_2, c > 0$. Then there exist a neighborhood $\mathcal{U}$ of $f$ in $\mathcal{A}$ and $S > 0$ such that for each $g \in \mathcal{U}$, each $T_1 \in [0, \infty)$ and each $T_2 \in [T_1 + c, \infty)$ the following properties hold:

(i) if an a. c. function $v : [T_1, T_2] \to \mathbb{R}^n$ satisfies

\[ |v(T_i)| \leq M_i, \, i = 1, 2, \, I^g(T_1, T_2, v) \leq U^g(T_1, T_2, v(T_1), v(T_2)) + M_2, \]

then

\[ |v(t)| \leq S, \, t \in [T_1, T_2]; \] \hspace{1cm} (2.3)
(ii) if an a. c. function \( v : [T_1, T_2] \to \mathbb{R}^n \) satisfies
\[
|v(T_1)| \leq M_1, \quad I^\beta(T_1, T_2, v) \leq U^\beta(T_2 - T_1, v(T_1)) + M_2,
\]
then (2.3) holds;

(iii) if an a. c. function \( v : [T_1, T_2] \to \mathbb{R}^n \) satisfies
\[
I^\beta(T_1, T_2, v) \leq U^\beta(T_2 - T_1) + M_2,
\]
then (2.3) holds.

The properties (i) and (ii) were established in [25]. See also Theorem 1.2.3 of
[27]. The property (iii) is proved analogously to the properties (i) and (ii).

3. Preliminaries. For each \( f \in \mathcal{A} \) define
\[
\bar{f}(x, y) = f(x, -y), \quad x, y \in \mathbb{R}^n. \quad (3.1)
\]
It is clear that for each \( f \in \mathcal{A}, \bar{f} \in \mathcal{A}, \) if \( f \in \mathcal{M}, \) then \( \bar{f} \in \mathcal{M}, \) the mapping \( f \to \bar{f}, f \in \mathcal{A} \) is continuous. This implies that \( \bar{f} \in \mathcal{M} \) for each \( f \in \mathcal{M}. \) It is easy to see that for each integer \( k \geq 1, \bar{f} \in \mathcal{A}_k \) for each \( f \in \mathcal{A}_k, \) for each integer \( k \geq 0, \bar{f} \in \mathcal{M}_k \) for all \( f \in \mathcal{M}_k \) and that the mapping \( f \to \bar{f}, f \in \mathcal{A}_k \) is continuous. This implies that for each integer \( k \geq 0, \bar{f} \in \mathcal{M}_k \) for each \( f \in \mathcal{M}_k. \) Evidently, \( \bar{f} \in \mathcal{L} \) for all \( f \in \mathcal{L} \) and for any \( k \in \{0, 1, 2\} \) and any \( f \in \mathcal{L}_k, \bar{f} \in \mathcal{L}_k. \)

Let \( f \in \mathcal{A}. \) For any \( T > 0 \) and any a. c. function \( v : [0, T] \to \mathbb{R}^n, \) put
\[
\bar{v}(t) = v(T - t), \quad t \in [0, T]. \quad (3.2)
\]
Clearly, for each \( T > 0 \) and each a. c. function \( v : [0, T] \to \mathbb{R}^n,
\[
\int_0^T \bar{f}(\bar{v}(t), \bar{v}'(t))dt = \int_0^T f(v(T - t), v'(T - t))dt = \int_0^T f(v(t), v'(t))dt. \quad (3.3)
\]
The next result easily follows from (3.3).

Proposition 1. Let \( f \in \mathcal{A}, T > 0, \quad M \geq 0 \) and \( v_i : [0, T] \to \mathbb{R}^n, i = 1, 2 \) be a. c. functions. Then
\[
I^\beta(f, 0, T, v_1) \leq I^\beta(f, 0, T, v_2) + M \quad \text{if and only if} \quad I^\beta(\bar{f}, 0, T, \bar{v}_1) \leq I^\beta(\bar{f}, 0, T, \bar{v}_2) + M.
\]

For each \( f \in \mathcal{A}, \) each \( x \in \mathbb{R}^n \) and each real number \( T > 0 \) set
\[
U_f(T, x) = \inf \{I^\beta(f, 0, T, v) : v : [0, T] \to \mathbb{R}^n \}
\]
is an a.c. function satisfying \( v(T) = x, \)
\[
\text{is an a.c. function satisfying } v(T) = x. \quad (3.4)
\]

Proposition 1 implies the following result.

Proposition 2. Let \( f \in \mathcal{A}, T > 0, \quad M \geq 0 \) and \( v : [0, T] \to \mathbb{R}^n \) be an a. c. function. Then
\[
\text{if } I^\beta(f, 0, T, v) \leq U^\beta(T) + M, \quad \text{then } I^\beta(\bar{f}, 0, T, \bar{v}) \leq U^\beta(T) + M;
\]
\[
\text{if } I^\beta(f, 0, T, v) \leq U^\beta(f, 0, T, v(0), v(T)) + M,
\]
\[
\text{then } I^\beta(\bar{f}, 0, T, \bar{v}) \leq U^\beta(\bar{f}, 0, T, \bar{v}(0), \bar{v}(T)) + M;
\]
\[
\text{if } I^\beta(f, 0, T, v) \leq U_f(T, v(T)) + M, \quad \text{then } I^\beta(\bar{f}, 0, T, \bar{v}) \leq U_f(T, \bar{v}(0)) + M;
\]
\[
\text{if } I^\beta(f, 0, T, v) \leq U^\beta(f, 0, T, v(0)) + M, \quad \text{then } I^\beta(\bar{f}, 0, T, \bar{v}) \leq U_f(T, \bar{v}(T)) + M.
\]

The next result follows from Proposition 2, Theorem 2.1 and the continuity of
the mapping \( f \to \bar{f}, f \in \mathcal{A}. \)
**Proposition 3.** Let \( f \in \mathcal{A} \) and let \( M_1, M_2, c > 0 \). Then there exist a neighborhood \( \mathcal{U} \) of \( f \) in \( \mathcal{A} \) and \( S > 0 \) such that for each \( g \in \mathcal{U} \), each \( T \geq c \) and each a. c. function \( v : [0, T] \rightarrow \mathbb{R}^n \) which satisfies
\[
I^g(0, T, v) \leq U_g(T, v(T)) + M_2, \quad |v(T)| \leq M_1
\]
the inequality \(|v(t)| \leq S\) holds for all \( t \in [0, T] \).

The following result is proved in [33].

**Proposition 4.** Assume that \( f \in \mathcal{A} \) has \( (ATP) \). Then \( \bar{f} \) has \( (ATP) \) and \( H(\bar{f}) = H(f) \).

The following result is proved in [27] (see Chapter 4, Proposition 4.2.1).

**Proposition 5.** Let \( f \in \mathcal{A} \). Then \( \pi^f(x) \rightarrow \infty \) as \( |x| \rightarrow \infty \).

Let \( f \in \mathcal{A} \). Define
\[
\mathcal{D}(f) = \{ x \in \mathbb{R}^n : \pi^f(x) \leq \pi^f(y) \text{ for all } y \in \mathbb{R}^n \}.
\]
Since the function \( \pi^f \) is continuous it follows from Proposition 5 that the set \( \mathcal{D}(f) \) is nonempty, bounded and closed.

For each \( \tau_1 \in \mathbb{R}^1, \tau_2 > \tau_1 \), each \( r_1, r_2 \in [\tau_1, \tau_2] \) satisfying \( r_1 < r_2 \) and each a.c. function \( u : [\tau_1, \tau_2] \rightarrow \mathbb{R}^n \) set
\[
\Gamma^f(r_1, r_2, u) = I^f(r_1, r_2, u) - \pi^f(u(r_1)) + \pi^f(u(r_2)) - (r_2 - r_1)\mu(f).
\]
In view of (1.2), (1.5), (1.6) and (3.6),
\[
\Gamma^f(r_1, r_2, u) \geq 0 \text{ for each } \tau_1 \in \mathbb{R}^1, \tau_2 > \tau_1, \text{ each } r_1, r_2 \in [\tau_1, \tau_2]
\]
satisfying \( r_1 < r_2 \) and each a.c. function \( u : [\tau_1, \tau_2] \rightarrow \mathbb{R}^n \).

**Proposition 6.** (Theorem 3.6.3 of [27]) Let \( f \in \mathcal{A} \). For every \( x \in \mathbb{R}^n \) there exists an \( (f)\)-good function \( v : [0, \infty) \rightarrow \mathbb{R}^n \) such that \( v(0) = x \) and \( \Gamma^f(T_1, T_2, v) = 0 \) for each \( T_1 \geq 0 \) and each \( T_2 > T_1 \).

Let \( f \in \mathcal{A} \). An a. c. function \( v : [0, \infty) \rightarrow \mathbb{R}^n \) is called \( (f)\)-perfect if \( \Gamma^f(T_1, T_2, v) = 0 \) for all \( T_1 \geq 0 \) and all \( T_2 > T_1 \).

Proposition 5 and Theorem 1.1 imply the following result.

**Proposition 7.** Let \( f \in \mathcal{A} \) and \( v : [0, \infty) \rightarrow \mathbb{R}^n \) be an \( (f)\)-perfect function. Then the function \( v \) is bounded and \( (f)\)-good.

For each \( f \in \mathcal{A} \) and each \( x \in \mathbb{R}^n \) denote by \( \mathcal{P}(f, x) \) the set of all \( (f)\)-perfect functions \( v : [0, \infty) \rightarrow \mathbb{R}^n \) such that \( v(0) = x \). In view of Proposition 6 this set is nonempty.

Let \( f \in \mathcal{A} \) and \( x \in \mathbb{R}^n \). By Proposition 7 any function belonging to \( \mathcal{P}(f, x) \) is bounded and \( (f)\)-good.

4. **A turnpike result.** In the sequel we use the following turnpike result which was proved in [36].

**Theorem 4.1.** Assume that an integrand \( f \in \mathcal{M} \) has the asymptotic turnpike property and that \( \epsilon, M_0 > 0 \). Then there exist a neighborhood \( \mathcal{U} \) of \( f \) in \( \mathcal{A} \) and numbers \( M_1 > M_0, l_1 > l > 0, \delta > 0 \) such that for each \( g \in \mathcal{U} \), each \( T \geq 2l_1 + l \) and each a.c. function \( v : [0, T] \rightarrow \mathbb{R}^n \) which satisfies
\[
|v(0)|, |v(T)| \leq M_0,
\]
Moreover, if \( d = 5 \), lower semicontinuous functions determine subsets of \( \mathbb{R}^n \times \mathbb{R}^n \) which is determined by the following base: For simplicity we set 

\[
\text{dist}(H(f), \{v(t) : t \in [\tau, \tau + l]\}) \leq \epsilon.
\]

Moreover, if \( d(v(0), H(f)) \leq \delta \), then \( \tau_1 = 0 \) and if \( d(v(T), H(f)) \leq \delta \), then \( \tau_2 = T \).

5. **Bolza variational problems.** Let \( a_1 > 0 \). Denote by \( \mathfrak{A}(R^n \times R^n) \) the set of all lower semicontinuous functions \( h : R^n \times R^n \to R^1 \) which are bounded on bounded subsets of \( R^n \times R^n \) and satisfy

\[
h(z_1, z_2) \geq -a_1 \text{ for all } z_1, z_2 \in R^n.
\]

For simplicity we set \( \mathfrak{A} = \mathfrak{A}(R^n \times R^n) \). We equip the set \( \mathfrak{A} \) with the uniformity which is determined by the following base:

\[
E(N, \epsilon) = \{(h_1, h_2) \in \mathfrak{A} \times \mathfrak{A} : |h_1(z) - h_2(z)| \leq \epsilon \}
\]

for each \( z \in R^n \times R^n \) satisfying \( |z| \leq N \), where \( N, \epsilon > 0 \). (Here \( | \cdot | \) is the Euclidean norm in the space \( R^{2n} \).) It is not difficult to see that the uniform space \( \mathfrak{A} \) is metrizable and complete. We consider the following Bolza variational problems

\[
I^g(T, T, v) \leq U^g(0, T, v(0), v(T)) + M_0
\]

and

\[
I^g(S, S + l_1, v) \leq U^g(S, S + l_1, v(S), v(S + l_1)) + \delta
\]

for each \( S \in [0, T - l_1] \), the inequality \( |v(t)| \leq M_1 \) holds for all \( t \in [0, T] \) and that there exist \( \tau_1 \in [0, l_1], \tau_2 \in [T - l_1, T] \) such that for all \( \tau \in [\tau_1, \tau_2 - l] \),

\[
\text{dist}(H(f), \{v(t) : t \in [\tau, \tau + l]\}) \leq \epsilon.
\]

Moreover, if \( d(v(0), H(f)) \leq \delta \), then \( \tau_1 = 0 \) and if \( d(v(T), H(f)) \leq \delta \), then \( \tau_2 = T \).

6. **A uniform boundedness result for Bolza variational problems.** We begin our study of Bolza variational problems with the following uniform boundedness result.

**Theorem 6.1.** Let \( f \in \mathfrak{A}, h \in \mathfrak{A} \) and let \( M_1, c > 0 \). Then there exist a neighborhood \( \mathcal{U} \) of \( f \) in \( \mathfrak{A} \), a neighborhood \( \mathcal{V} \) of \( h \) in \( \mathfrak{A} \) and \( S > 0 \) such that for each \( g \in \mathcal{U} \), each \( \xi \in \mathcal{V} \), each \( T_1 \in [0, \infty) \), each \( T_2 \in [T_1 + c, \infty) \) and each a. c. function \( v : [T_1, T_2] \to R^n \) which satisfies

\[
I^g(T_1, T_2, v) + \xi(v(T_1), v(T_2)) \leq \sigma(g, \xi, T_1, T_2) + M_1
\]

the inequality \( |v(t)| \leq S \) holds for all \( t \in [T_1, T_2] \).

**Proof.** By Theorem 2.1, there exist a neighborhood \( \mathcal{U}_1 \) of \( f \) in \( \mathfrak{A} \) and \( S_1 > 0 \) such that for each \( g \in \mathcal{U}_1 \), each \( T_1 \in [0, \infty) \) and each \( T_2 \in [T_1 + c, \infty) \) the following property holds:

(i) if an a. c. function \( v : [T_1, T_2] \to R^n \) satisfies

\[
I^g(T_1, T_2, v) \leq U^g(T_2 - T_1) + 1,
\]

then \( |v(t)| \leq S_1 \) for all \( t \in [T_1, T_2] \);
In view of (5.2), there exist a neighborhood \( V \) of \( h \) in \( \mathfrak{H} \) and \( S_2 > 0 \) such that for all \( \xi \in V \) and each \( z_1, z_2 \in \mathbb{R}^n \) satisfying \( |z_i| \leq S_1, i = 1, 2 \), we have

\[
|\xi(z_1, z_2)| \leq S_2. \tag{6.1}
\]

By Theorem 2.1, there exist a neighborhood \( U \subset U_1 \) of \( f \) in \( A \) and \( S > S_1 + S_2 \) such that for each \( g \in U \), each \( T_1 \in [0, \infty) \) and each \( T_2 \in [T_1 + c, \infty) \) the following property holds:

(ii) if an a. c. function \( v : [T_1, T_2] \to \mathbb{R}^n \) satisfies

\[
I^g(T_1, T_2, v) \leq U^g(T_2 - T_1) + 1 + M_1 + 2S_2 + 2a_1,
\]

then \( |v(t)| \leq S \) for all \( t \in [T_1, T_2] \).

Assume that

\[
g \in U, \ \xi \in V, \ T_1 \in [0, \infty), \ T_2 \geq T_1 + c \tag{6.2}
\]

and that an a. c. function \( v : [T_1, T_2] \to \mathbb{R}^n \) satisfies

\[
I^g(T_1, T_2, v) + \xi(v(T_1), v(T_2)) \leq \sigma(g, \xi, T_1, T_2) + M_1. \tag{6.3}
\]

There exists an a. c. function \( u : [T_1, T_2] \to \mathbb{R}^n \) such that

\[
I^g(T_1, T_2, u) \leq U^g(T_2 - T_1) + 1. \tag{6.4}
\]

It follows from (6.2), (6.4) and property (i) that

\[
|u(t)| \leq S_1, \ t \in [T_1, T_2]. \tag{6.5}
\]

By (6.2), (6.5) and the choice of \( V \) (see (6.1)),

\[
|\xi(u(T_1), u(T_2))| \leq S_2. \tag{6.6}
\]

In view of (5.1), (6.3), (6.4) and (6.6),

\[
I^g(T_1, T_2, v) - a_1 \leq I^g(T_1, T_2, v) + \xi(v(T_1), v(T_2))
\leq M_1 + \sigma(g, \xi, T_1, T_2) \leq M_1 + I^g(T_1, T_2, u) + \xi(u(T_1), u(T_2))
\leq M_1 + S_2 + I^g(T_1, T_2, u) \leq M_1 + S_2 + U^g(T_2 - T_1) + 1
\]

and

\[
I^g(T_1, T_2, v) \leq U^g(T_2 - T_1) + 1 + M_1 + S_2 + a_1. \tag{6.7}
\]

Property (ii), (6.2) and (6.7) imply that \( |v(t)| \leq S, \ t \in [T_1, T_2] \). Theorem 6.1 is proved.

7. **The first turnpike result for Bolza problems.** Let \( f \in A \) have (ATP). By Theorem 2.1, there exist a neighborhood \( U_f \) of \( f \) in \( A \) and \( S_f > 0 \) such that the following properties hold:

- (P1) for each \( g \in U_f \), each \( T \geq 1 \) and each a. c. function \( u : [0, T] \to \mathbb{R}^n \) satisfying \( I^g(0, T, u) \leq U^g(T) + 1 \) we have \( |u(t)| \leq S_f, \ t \in [0, T] \);

- (P2) for each \( g \in U_f \), each \( T \geq 1 \) and each a. c. function \( u : [0, T] \to \mathbb{R}^n \) satisfying

\[
d(u(0), H(f)) \leq 1, \ I^g(0, T, u) \leq U^g(T, u(0)) + 1
\]

we have \( |u(t)| \leq S_f, \ t \in [0, T] \).

The following turnpike result for Bolza variational problems shows that the turnpike phenomenon, for approximate solutions on large intervals, is stable under small perturbations of the objective functions.
Theorem 7.1. Assume that an integrand $f \in \mathcal{A}$ has the asymptotic turnpike property and that $M_1, M_2, \epsilon > 0$. Then there exist a neighborhood $\mathcal{U}$ of $f$ in $\mathcal{A}$, numbers $l, S > 0$ and integers $L, Q_* \geq 1$ such that for each $g \in \mathcal{U}$, each $T \geq L + lQ_*$, each $h \in \mathcal{A}$ satisfying
\[
|h(z_1, z_2)| \leq M_2 \text{ for all } z_1, z_2 \in \mathbb{R}^n \text{ such that } |z_i| \leq S, \ i = 1, 2 \quad \text{and each a.c. function } v : [0, T] \to \mathbb{R}^n \text{ which satisfies}
\]
the inequality $|v(t)| \leq S$ holds for all $t \in [0, T]$ and there exist sequences of numbers $\{a_i\}_{i=1}^n, \{b_i\}_{i=1}^n \subset [0, T]$ such that $q \leq Q_*$, $0 \leq b_i - a_i \leq l, \ i = 1, \ldots, q$ and that
\[
\text{dist}(H(f), \{v(t) : t \in [\tau, \tau + L]\}) \leq \epsilon
\]
for each $\tau \in [0, T - L] \setminus \cup_{i=1}^{L} [a_i, b_i]$.

Proof. By Theorems 1.2 and 2.1, there exist a neighborhood $\mathcal{U} \subset \mathcal{U}_f$ of $f$ in $\mathcal{A}$, numbers $l, S \geq 1$ and integers $L \geq 1, Q_* \geq 1$ such that the following property holds:

(i) for each $g \in \mathcal{U}$, each $T \geq L + lQ_*$ and each a.c. function $v : [0, T] \to \mathbb{R}^n$ which satisfies
\[
I^g(0, T, v) + h(v(0), v(T)) \leq \sigma(g, h, 0, T) + M_1
\]
the inequality $|v(t)| \leq S$ holds for all $t \in [0, T]$ and there exist sequences of numbers $\{b_i\}_{i=1}^Q, \{c_i\}_{i=1}^Q \subset [0, T]$ such that
\[
Q \leq Q_*, \ 0 \leq c_i - b_i \leq l, \ i = 1, \ldots, Q
\]
and that
\[
\text{dist}(H(f), \{v(t) : t \in [\tau, \tau + L]\}) \leq \epsilon
\]
for each $\tau \in [0, T - L] \setminus \cup_{i=1}^{Q} [b_i, c_i]$.

Assume that
\[
g \in \mathcal{U}, \ T \geq L + lQ_*, \ h \in \mathcal{A},
\]
\[
|h(z_1, z_2)| \leq M_2 \text{ for all } z_1, z_2 \in \mathbb{R}^n \text{ such that } |z_i| \leq S, \ i = 1, 2
\]
and that $v : [0, T] \to \mathbb{R}^n$ is an a. c. function which satisfies
\[
I^g(0, T, v) + h'(v(0), v(T)) \leq \sigma(g, h, 0, T) + M_1.
\]

There exists an a. c. function $u : [0, T] \to \mathbb{R}^n$ such that
\[
I^g(0, T, u) \leq U^g(T) + 1.
\]

Property (P1), (7.3) and (7.6) imply that
\[
|u(t)| \leq S, \ t \in [0, T].
\]

By (5.1) and (7.4)-(7.7),
\[
I^g(0, T, v) - a_1 \leq I^g(0, T, v) + h(v(0), v(T))
\]
\[
\leq \sigma(g, h, 0, T) + M_1 \leq I^g(0, T, u) + h(u(0), u(T)) + M_1
\]
\[
\leq I^g(0, T, u) + M_2 + M_1 \leq 1 + U^g(T) + M_2 + M_1
\]
and
\[
I^g(0, T, v) \leq U^g(T) + M_2 + M_1 + a_1 + 1.
\]

Property (i), (7.3) and (7.8) imply that the inequality $|v(t)| \leq S$ holds for all $t \in [0, T]$ and there exist sequences of numbers $\{b_i\}_{i=1}^Q, \{c_i\}_{i=1}^Q \subset [0, T]$ such that
(7.1) holds and (7.2) holds for each $\tau \in [0, T - L] \setminus \cup_{i=1}^{Q} [b_i, c_i]$. This completes the proof of Theorem 7.1. \qed
8. The second turnpike result for Bolza problems. The following turnpike result for Bolza variational problems shows that the turnpike phenomenon, for approximate solutions on large intervals, is stable under small perturbations of the objective functions.

**Theorem 8.1.** Assume that an integrand \( f \in \mathcal{M} \) has the asymptotic turnpike property and that \( c, M_1, M_2 > 0 \). Then there exist a neighborhood \( U \) of \( f \) in \( \mathcal{A} \) and numbers \( S > 0, L > l > 0, \delta > 0 \) such that for each \( g \in U \), each \( h \in \mathfrak{A} \) satisfying

\[
|h(z_1, z_2)| \leq M_2 \text{ for all } z_1, z_2 \in \mathbb{R}^n \text{ such that } |z_i| \leq S_f, i = 1, 2,
\]
each \( T \geq 2L + l \) and each a.c. function \( v : [0, T] \to \mathbb{R}^n \) which satisfies

\[
I^g(0, T, v) + h(v(0), v(T)) \leq \sigma(g, h, 0, T) + M_1
\]
and such that for each \( \tau \in [0, T - L] \),

\[
I^g(\tau, \tau + L, v) \leq U^g(\tau, \tau + L, v(\tau), v(\tau + L)) + \delta
\]
the inequality \( |v(t)| \leq S \) holds for all \( t \in [0, T] \) and that there exist \( \tau_1 \in [0, L], \tau_2 \in [T - L, T] \) such that for all \( \tau \in [\tau_1, \tau_2 - l] \),

\[
\text{dist}(H(f), \{v(t) : t \in [\tau, \tau + l]\}) \leq \epsilon.
\]
Moreover, if \( d(v(0), H(f)) \leq \delta \), then \( \tau_1 = 0 \) and if \( d(v(T), H(f)) \leq \delta \), then \( \tau_2 = T \).

**Proof.** Denote by \( \text{mes}(E) \) the Lebesgue measure of a Lebesgue measurable set \( E \subset \mathbb{R}^1 \).

By Theorem 4.1, there exist a neighborhood \( U_1 \) of \( f \) in \( \mathcal{A} \) and numbers \( l_1 > l > 0, \delta \in [0, 1] \) such that the following property holds:

(i) for each \( g \in U_1 \), each \( T \geq 2l_1 + l \) and each a.c. function \( v : [0, T] \to \mathbb{R}^n \) which satisfies

\[
d(v(0), H(f)) \leq \delta, \quad d(v(T), H(f)) \leq \delta, \quad I^g(0, T, v) \leq U^g(0, T, v(0), v(T)) + M_1
\]
and such that for each \( S \in [0, T - l_1] \),

\[
I^g(S, S + l_1, v) \leq U^g(S, S + l_1, v(S), v(S + l_1)) + \delta
\]
the inequality

\[
\text{dist}(H(f), \{v(t) : t \in [\tau, \tau + l]\}) \leq \epsilon
\]
holds for each \( \tau \in [0, T - l] \).

By Theorem 7.1, there exist a neighborhood \( U \subset U_1 \) of \( f \) in \( \mathcal{A} \) and numbers \( S > 0, L_0 > 2l_1 + 1 \) such that the following property holds:

(ii) for each \( g \in U \), each \( h \in \mathfrak{A} \) satisfying

\[
|h(z_1, z_2)| \leq M_2 \text{ for all } z_1, z_2 \in \mathbb{R}^n \text{ such that } |z_i| \leq S_f, i = 1, 2,
\]
each \( T \geq L_0 \) and each a.c. function \( v : [0, T] \to \mathbb{R}^n \) which satisfies

\[
I^g(0, T, v) + h(v(0), v(T)) \leq \sigma(g, h, 0, T) + M_1
\]
the inequality \( |v(t)| \leq S \) holds for all \( t \in [0, T] \) and there exists a set \( E \subset [0, T] \) which is a finite union of closed subintervals of \([0, T]\) such that \( \text{mes}(E) < L_0 \) and for each \( t \in [0, T] \setminus E \), we have \( d(v(t), H(f)) \leq \delta \).

Set \( L = 4L_0 \).
Assume that
\[ g \in \mathcal{U}, \ h \in \mathfrak{A}, \]  
(8.5) holds, \( T \geq 2L + l \) and that \( v : [0, T] \to \mathbb{R}^n \) is an a. c. function which satisfies (8.3) and for each \( \tau \in [0, T - L] \),
\[ I^\sigma(\tau, \tau + L, v) \leq U^\sigma(\tau, \tau + L, v(\tau), v(\tau + L)) + \delta. \]  
(8.6)
Property (ii) and (8.2)-(8.6) imply that \( |v(t)| \leq S, \ t \in [0, T] \) and that there exist \( \tau_1 \in [0, L_0], \ \tau_2 \in [T - L_0, T] \) such that
\[ d(v(\tau_1), H(f)) \leq \delta, \ i = 1, 2. \]  
(8.8)
We may assume that if \( d(v(0), H(f)) \leq \delta \), then \( \tau_1 = 0 \) and if \( d(v(T), H(f)) \leq \delta \), then \( \tau_2 = T \). Property (i), (8.3), (8.4) and (8.6)-(8.8) imply that (8.1) is true for all \( \tau \in [\tau_1, \tau_2 - l] \). Theorem 8.1 is proved.

9. Structure of solutions of Bolza problems near the endpoints. For each nonempty set \( X \) and each \( \eta : X \to \mathbb{R}^l \) define
\[ \inf(\eta) = \inf\{\eta(x) : x \in X\}. \]

Let \( f \in \mathcal{A} \) and \( h \in \mathfrak{A} \). For all \( z_1, z_2 \in \mathbb{R}^n \) define
\[ \psi_{f,h}(z_1, z_2) = \pi^f(z_1) + \pi^f(z_2) + h(z_1, z_2). \]
(9.1)
Proposition 5, (5.1) and (9.1) imply the following result.

**Lemma 9.1.** The function \( \psi_{f,h} \) is lower semicontinuous, for every \( M > 0 \) the set
\[ \{z \in \mathbb{R}^n \times \mathbb{R}^n : \psi_{f,h}(z) \leq M\} \]
is bounded, \( \inf(\psi_{f,h}) \) is finite and the function \( \psi_{f,h} \) has a point of minimum.

The next result is proved in Section 12.

**Theorem 9.2.** Suppose that an integrand \( f \in \mathcal{M} \) has the asymptotic turnpike property and that \( h \in \mathfrak{A} \). Let \( \epsilon, L_0 > 0 \). Then there exist a neighborhood \( \mathcal{U} \) of \( f \) in \( \mathcal{A} \), a neighborhood \( \mathcal{V} \) of \( h \) in \( \mathfrak{A} \) and numbers \( \delta \in (0, \epsilon) \) and \( T_0 > L_0 \) such that for each \( T \geq T_0 \), each \( g \in \mathcal{U} \), each \( \xi \in \mathcal{V} \) and each a.c. function \( v : [0, T] \to \mathbb{R}^n \) which satisfies
\[ I^\sigma(0, T, v) + \xi(v(0), v(T)) \leq \sigma(g, \xi, 0, T) + \delta \]
there exists an \((f)\)-perfect function \( \tilde{w}_1 : [0, \infty) \to \mathbb{R}^n \) and an \((\tilde{f})\)-perfect function \( \tilde{w}_2 : [0, \infty) \to \mathbb{R}^n \) such that
\[ \psi_{f,h}(\tilde{w}_1(0), \tilde{w}_2(0)) = \inf(\psi_{f,h}) \]
and for all \( t \in [0, L_0], |v(t) - \tilde{w}_1(t)| \leq \epsilon, |v(T - t) - \tilde{w}_2(t)| \leq \epsilon. \]

In the next theorem \( \mathfrak{M} \) is one of the spaces
\[ \mathcal{L}_k, \ k = 0, 1, 2, \ \mathcal{M}_q, \ q \geq 3 \] is an integer
and product \( \mathfrak{M} \times \mathfrak{A} \) is equipped with the product topology.
Theorem 9.3. Then there exists a set $F \subset \mathcal{M} \times \mathcal{A}$ which is a countable intersection of open everywhere dense subsets of $\mathcal{M} \times \mathcal{A}$ such that for each $(f, h) \in F$ there exist a unique pair of $(f)$-perfect function $w_{f,h} : [0, \infty) \rightarrow \mathbb{R}^n$ and an $(f)$-perfect function $w_{f,h} : [0, \infty) \rightarrow \mathbb{R}^n$ such that
\[
\psi_{f,h}(w_{f,h}(0), w_{f,h}(0)) = \inf(\psi_{f,h})
\]
and such that the following assertion holds.

Let $\epsilon, M, \tau_0 > 0$. Then there exist a neighborhood $U$ of $(f, h)$ in $\mathcal{M} \times \mathcal{A}$ and numbers $\delta \in (0, \epsilon)$ and $T_0 > \tau_0$ such that for each $(g, \xi) \in U$, each $T \geq T_0$ and each a.c. function $\nu : [0, T] \rightarrow \mathbb{R}^n$ satisfying
\[
I^\nu(0, T, v) + \xi(v(0), v(T)) \leq \sigma(g, \xi, 0, T) + \delta,
\]
for all $t \in [0, \tau_0]$, $|v(t) - w_{f,h}(t)| \leq \epsilon$, $|v(T) - w_{f,h}(t)| \leq \epsilon$.

Theorem 9.3 is proved in Section 16. The next result is proved in Section 14.

Theorem 9.4. Suppose that an integrand $f \in \mathcal{M}$ has the asymptotic turnpike property and that $h \in \mathcal{A}$. Let $\epsilon, L_0 > 0$. Then there exist a neighborhood $U$ of $f$ in $\mathcal{M}$, a neighborhood $V$ of $h$ in $\mathcal{A}$ and $\delta \in (0, \epsilon)$ such that for each $g \in U$, each $\xi \in V$ and each pair of a $(g)$-perfect function $w_1 : [0, \infty) \rightarrow \mathbb{R}^n$ and a $(\tilde{g})$-perfect function $w_2 : [0, \infty) \rightarrow \mathbb{R}^n$ satisfying
\[
\psi_{g,\xi}(w_1(0), w_2(0)) \leq \inf(\psi_{g,\xi}) + \delta
\]
there exist an $(f)$-perfect function $\tilde{w}_1 : [0, \infty) \rightarrow \mathbb{R}^n$ and an $(\tilde{f})$-perfect function $\tilde{w}_2 : [0, \infty) \rightarrow \mathbb{R}^n$ such that
\[
\psi_{f,h}(\tilde{w}_1(0), \tilde{w}_2(0)) = \inf(\psi_{f,h})
\]
and that for all $t \in [0, L_0]$, $|w_i(t) - \tilde{w}_i(t)| \leq \epsilon$, $i = 1, 2$.

10. Auxiliary results.

Lemma 10.1. (Lemma 4.2.8 of [27]) Let $f \in \mathcal{A}$ possess (ATP). Then
\[
\sup \{ \pi^f(z) : z \in H(f) \} = 0.
\]

Theorem 10.2. (Theorem 4.1.1 of [27]) Assume that $f \in \mathcal{A}$ has (ATP). Then $f$ is a continuity point of the mapping $g \rightarrow (\mu(g), \pi^g) \in \mathcal{R}^1 \times C(\mathbb{R}^n)$, $g \in \mathcal{A}$, where $C(\mathbb{R}^n)$ is the space of all continuous functions $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^1$ with the topology of the uniform convergence on bounded sets.

Theorem 10.3. (Theorem 1.2.2 of [27]) For each $f \in \mathcal{A}$ there exists a neighborhood $U$ of $f$ in $\mathcal{A}$ and a number $M > 0$ such that for each $g \in U$ and each $(g)$-good function $x : [0, \infty) \rightarrow \mathbb{R}^n$ the relation $\limsup_{t \rightarrow \infty} |x(t)| < M$ holds.

Lemma 10.4. (Proposition 1.3.5 of [27]) Assume that $f \in \mathcal{A}$, $M_1 > 0$, $0 \leq T_1 < T_2 < \infty$ and that $x : [T_1, T_2] \rightarrow \mathbb{R}^n$, $i = 1, 2, \ldots$ is a sequence of a.c. functions such that $I^f(T_1, T_2, x_i) \leq M_1$ for all integers $i \geq 1$. Then there exist a subsequence $(x_{i_k})_{k=1}^{\infty}$ and an a.c. function $x : [T_1, T_2] \rightarrow \mathbb{R}^n$ such that $I^f(T_1, T_2, x) \leq M_1$, $x_{i_k}(t) \rightarrow x(t)$ as $k \rightarrow \infty$ uniformly on $[T_1, T_2]$ and $x'_{i_k} \rightarrow x'$ as $k \rightarrow \infty$ weakly in $L^1(\mathbb{R}^n; (T_1, T_2))$.

Lemma 10.5. (Corollary 1.3.1 of [27]) For each $f \in \mathcal{A}$, each pair of numbers $T_1, T_2$ satisfying $0 \leq T_1 < T_2$ and each $z_1, z_2 \in \mathbb{R}^n$ there is an a.c. function $x : [T_1, T_2] \rightarrow \mathbb{R}^n$ such that $x(T_1) = z_1$, $i = 1, 2$ and $I^f(T_1, T_2, x) = U^f(T_1, T_2, z_1, z_2)$. 
Lemma 10.6. (Proposition 1.3.8 of [27]) Let \( f \in \mathcal{A} \), \( 0 < c_1 < c_2 < \infty \) and let \( D, \epsilon > 0 \). Then there exists a neighborhood \( V \) of \( f \) in \( \mathcal{A} \) such that for each \( g \in V \), each \( T_1, T_2 \geq 0 \) satisfying \( T_2 - T_1 \in [c_1, c_2] \) and each a. c. function \( x : [T_1, T_2] \rightarrow \mathbb{R}^n \) satisfying \( \min\{I^f(T_1, T_2, x), I^f(T_1, T_2, x)\} \leq D \) the inequality \( I^f(T_1, T_2, x) - I^f(T_1, T_2, x) \leq \epsilon \) holds.

Lemma 10.7. (Proposition 8 of [30]) Let \( g \in \mathcal{M} \) possess (ATP) and \( v : [0, \infty) \rightarrow \mathbb{R}^n \) be an a. c. function such that \( \sup\{v(t) : t \in [0, \infty)\} < \infty \) and \( I^g(0, T, v) = U^g(0, T, v(0), v(T)) \) for all \( T > 0 \). Then the function \( v \) is \((g)\)-perfect.

Lemma 10.8. (Theorem 1.2 of [28]) Let \( g \in \mathcal{L} \) and \( v_1, v_2 : [0, \infty) \rightarrow \mathbb{R}^n \) be \((g)\)-perfect functions such that \( v_1(0) = v_2(0) \). If there exist \( t_1, t_2 \in [0, \infty) \) such that \( (t_1, t_2) \neq (0, 0) \) and \( v_1(t_1) = v_2(t_2) \), then \( v_1(t) = v_2(t) \) for all \( t \in [0, \infty) \).

The following lemma is a particular case of Lemma 3.3 of [28].

Lemma 10.9. Let \( f \in \mathcal{A} \) have (ATP) and \( h \in H(f) \). Then there exists an a. c. function \( v : R^1 \rightarrow H(f) \) such that \( v(0) = h \) and \( \Gamma^f(-T, T, v) = 0 \) for all \( T > 0 \).

The following lemma is a particular case of Lemma 5.1 of [28].

Lemma 10.10. Let \( f \in \mathcal{L} \), \( v_1, v_2 : [0, \infty) \rightarrow \mathbb{R}^n \) be \((f)\)-perfect functions, \( 0 \leq T_1 < T_2 \) and let \( v_1(t) = v_2(t) \) for all \( t \in [T_1, T_2] \). Then \( v_1(t) = v_2(t) \) for all \( t \in [0, \infty) \).

Lemma 10.11. (Lemma 6.12 of [33]) Let \( f \in \mathcal{A} \) have (ATP) and \( S_0 > 0 \). Then there exist \( K_0 > 0 \) and a neighborhood \( \mathcal{U} \) of \( f \) in \( \mathcal{A} \) such that for each \( g \in \mathcal{U} \) and each \( x \in \mathbb{R}^n \) satisfying \( |x| > K_0 \) the inequality \( \pi^g(x) > S_0 \) holds.

Lemma 10.12. (Lemma 6.13 of [33]) Let \( f \in \mathcal{M} \) have (ATP) and \( \epsilon > 0 \). Then there exist numbers \( q \geq 8 \) and \( \delta > 0 \) such that for each \( h_1, h_2 \in \mathbb{R}^n \) satisfying \( d(h_1, H(f)) \leq \delta \), \( i = 1, 2 \) and each \( T \geq q \) there exists an a. c. function \( v : [0, T] \rightarrow \mathbb{R}^n \) which satisfies

\[ v(0) = h_1, \ v(T) = h_2, \ \Gamma^f(0, T, v) \leq \epsilon. \]

11. An auxiliary result for Theorem 9.2.

Lemma 11.1. Let \( f \in \mathcal{M} \) have the asymptotic turnpike property, \( h \in \mathfrak{X} \), \( \epsilon \in (0, 1) \) and \( T_0 > 0 \). Then there exists \( \delta \in (0, \epsilon) \) such that for each pair of a. c. functions \( u_i : [0, T_0] \rightarrow \mathbb{R}^n \), \( i = 1, 2 \) which satisfy

\[ \psi_{f,h}(u_1(0), u_2(0)) \leq \inf(\psi_{f,h}) + \delta, \]

\[ \Gamma^f(0, T_0, u_1) \leq \delta, \ \Gamma^f(0, T_0, u_2) \leq \delta \]

there exist an \((f)\)-perfect function \( w_1 : [0, \infty) \rightarrow \mathbb{R}^n \) and an \((\bar{f})\)-perfect function \( w_2 : [0, \infty) \rightarrow \mathbb{R}^n \) such that \( \psi_{f,h}(w_1(0), w_2(0)) = \inf(\psi_{f,h}) \) and for all \( t \in [0, T_0] \) and \( i = 1, 2 \), \( |u_i(t) - w_i(t)| \leq \epsilon \).

Proof. Assume that the lemma does not hold. Then there exist \( \{\delta_k\}_{k=1}^{\infty} \subset (0, 1) \) and sequences of a. c. functions \( u_{k,i} : [0, T_0] \rightarrow \mathbb{R}^n \), \( k = 1, 2, \ldots \), \( i = 1, 2 \) such that

\[ \lim_{k \to \infty} \delta_k = 0 \quad (11.1) \]

and that for each integer \( k \geq 1, \)

\[ \psi_{f,h}(u_{k,1}(0), u_{k,2}(0)) \leq \inf(\psi_{f,h}) + \delta_k, \quad (11.2) \]

\[ \Gamma^f(0, T_0, u_{k,1}) \leq \delta_k, \ \Gamma^{\bar{f}}(0, T_0, u_{k,2}) \leq \delta_k \quad (11.3) \]
and the following property holds:

(i) for each pair of $(f)$-perfect function $w_1 : [0, \infty) \to \mathbb{R}^n$ and an $(\bar{f})$-perfect function $w_2 : [0, \infty) \to \mathbb{R}^n$ satisfying

\[
\psi_{f,h}(w_1(0), w_2(0)) = \inf(\psi_{f,h})
\]  

we have

\[
\max\{\max\{|w_i(t) - u_{k,i}(t)| : t \in [0, T_0]\} : i = 1, 2\} > \epsilon. 
\]  

By (11.1), (11.2) and Lemma 9.1, extracting a subsequence and re-indexing if necessary we may assume without loss of generality that the sequences $\{u_{k,i}(0)\}_{k=1}^{\infty}$, $i = 1, 2$ converge and

\[
\psi_{f,h}(\lim_{k \to \infty} u_{k,1}(0), \lim_{k \to \infty} u_{k,2}(0)) = \inf(\psi_{f,h}).
\]  

Let $k \geq 1$ be an integer. By Proposition 6, there exist an $(f)$-good and $(\bar{f})$-perfect function $y_{k,1} : [0, \infty) \to \mathbb{R}^n$ and an $(\bar{f})$-good and $(\bar{f})$-perfect function $y_{k,2} : [0, \infty) \to \mathbb{R}^n$ such that

\[
y_{k,i}(0) = u_{k,i}(T_0), \quad i = 1, 2.
\]  

In view of (11.7) there exist a.c. functions $v_{k,i} : [0, \infty) \to \mathbb{R}^n$, $i = 1, 2$ such that for $i = 1, 2,$

\[
v_{k,i}(t) = u_{k,i}(t), \quad t \in [0, T_0], \quad v_{k,i}(t) = y_{k,i}(t - T_0), \quad t \in (T_0, \infty).
\]  

Since the function $y_{k,1}$ is $(\bar{f})$-perfect and the function $y_{k,2}$ is $(\bar{f})$-perfect it follows from (11.3) that for any $T > 0$,

\[
\Gamma^{\bar{f}}(0, T, v_{k,1}) \leq \Gamma^{\bar{f}}(0, T_0, u_{k,1}) \leq \delta_k, \quad \Gamma^{\bar{f}}(0, T, v_{k,2}) \leq \Gamma^{\bar{f}}(0, T_0, u_{k,2}) \leq \delta_k.
\]  

In view of Theorem 1.1, Proposition 5 and (3.6), the function $v_{k,1}$ is $(\bar{f})$-good and the function $v_{k,2}$ is $(\bar{f})$-good. By Theorem 10.3, there exists a number $S_1 > 0$ such that for all integers $k \geq 1$ and $i = 1, 2$,

\[
\limsup_{t \to \infty} |v_{k,i}(t)| \leq S_1.
\]  

It follows from (11.1), (11.2), (11.8)-(11.10), Theorem 2.1 and Lemma 9.1 that

\[
\sup\{\sup\{|u_{k,i}(t)| : t \in [0, T_0]\} : k = 1, 2, \ldots, i = 1, 2\} < \infty.
\]  

By (3.6) and (11.3), for each natural number $k$,

\[
I^{\bar{f}}(0, T_0, u_{k,1}) = \Gamma^{\bar{f}}(0, T_0, u_{k,1}) + T_0 \mu(f) + \pi^{\bar{f}}(u_{k,1}(0)) - \pi^{\bar{f}}(u_{k,1}(T_0)) \\
\leq \delta_k + T_0 \mu(f) + \pi^{\bar{f}}(u_{k,1}(0)) - \pi^{\bar{f}}(u_{k,1}(T_0)),
\]  

\[
I^{\bar{f}}(0, T_0, u_{k,2}) = \Gamma^{\bar{f}}(0, T_0, u_{k,2}) + T_0 \mu(f) + \pi^{\bar{f}}(u_{k,2}(0)) - \pi^{\bar{f}}(u_{k,2}(T_0)) \\
\leq \delta_k + T_0 \mu(f) + \pi^{\bar{f}}(u_{k,2}(0)) - \pi^{\bar{f}}(u_{k,2}(T_0)).
\]  

By (11.11)-(11.13) and the continuity of the functions $\pi^{\bar{f}}, \pi^f$ the sequences

\[
\{I^{\bar{f}}(0, T_0, u_{k,1})\}_{k=1}^{\infty}, \quad \{I^{\bar{f}}(0, T_0, u_{k,2})\}_{k=1}^{\infty}
\]  

are bounded. By Lemma 10.4, there exist subsequences $\{u_{ik,j}\}_{k=1}^{\infty}$, $j = 1, 2$ and a.c. functions $u_j : [0, T_0] \to \mathbb{R}^n$, $j = 1, 2$ such that

\[
I^{\bar{f}}(0, T_0, u_1) \leq \liminf_{k \to \infty} I^{\bar{f}}(0, T_0, u_{ik,1}),
\]  

\[
u_{ik,1}(t) \to u_1(t) \quad \text{as} \quad k \to \infty \quad \text{uniformly on} \quad [0, T_0].
\]  

\[
I^{\bar{f}}(0, T_0, u_2) \leq \liminf_{k \to \infty} I^{\bar{f}}(0, T_0, u_{ik,2}),
\]  

\[
u_{ik,2}(t) \to u_2(t) \quad \text{as} \quad k \to \infty \quad \text{uniformly on} \quad [0, T_0].
\]
In view of (11.6), (11.15) and (11.17),
\[
\psi_{f,h}(u_1(0), u_2(0)) = \inf(\psi_{f,h}) \tag{11.18}
\]

By (3.6), (11.1), (11.3), (11.14)-(11.18) and the continuity of \(\pi^{f}, \pi^\tilde{f}\)
\[
\Gamma^{f}(0, T_0, u_1) = I^{f}(0, T_0, u_1) - \pi^{f}(u_1(0)) + \pi^{f}(u_1(T_0)) - T_0 \mu(f)
\]
\[
\leq \liminf_{k \to \infty} I^{f}(0, T_0, u_{i_k,1}) - \pi^{f}(u_{i_k,1}(0)) + \pi^{f}(u_{i_k,1}(T_0)) - T_0 \mu(f)
\]
\[
= \liminf_{k \to \infty} \Gamma^{f}(0, T_0, u_{i_k,1}) \leq \lim_{k \to \infty} \delta_{i_k} = 0,
\]
\[
\Gamma^\tilde{f}(0, T_0, u_2) = I^\tilde{f}(0, T_0, u_2) - \pi^\tilde{f}(u_2(0)) + \pi^\tilde{f}(u_2(T_0)) - T_0 \mu(f)
\]
\[
\leq \liminf_{k \to \infty} I^\tilde{f}(0, T_0, u_{i_k,2}) - \pi^\tilde{f}(u_{i_k,2}(0)) + \pi^\tilde{f}(u_{i_k,2}(T_0)) - T_0 \mu(f)
\]
\[
= \liminf_{k \to \infty} \Gamma^\tilde{f}(0, T_0, u_{i_k,2}) \leq \lim_{k \to \infty} \delta_{i_k} = 0.
\]
Together with (3.7) this implies that \(\Gamma^{f}(0, T_0, u_1) = 0, \Gamma^\tilde{f}(0, T_0, u_2) = 0\). Combined with Proposition 6 this implies that there exist an \((f)\)-perfect function \(\tilde{u}_1 : [0, \infty) \to \mathbb{R}^n\) and an \((\tilde{f})\)-perfect function \(\tilde{u}_2 : [0, \infty) \to \mathbb{R}^n\) such that
\[
\tilde{u}_j(t) = u_j(t), \ t \in [0, T_0], \ j = 1, 2. \tag{11.20}
\]

It follows from (11.18) and (11.20) that
\[
\psi_{f,h}(\tilde{u}_1(0), \tilde{u}_2(0)) = \inf(\psi_{f,h}) \tag{11.21}
\]

By (11.15), (11.17) and (11.20), for all sufficiently large natural numbers \(k\),
\[
\sup\{ |u_{i_k,j}(t) - \tilde{u}_j(t)| : t \in [0, T_0], \ j = 1, 2 \} \leq \epsilon/2.
\]
Together with (11.21) this contradicts property (i). The contradiction we have reached proves Lemma 11.1. \(\square\)

12. **Proof of Theorem 9.2.** By Lemma 11.1, there exist \(\delta_0 \in (0, \epsilon)\) such that the following property holds:

(i) for each pair of a.c. functions \(u_i : [0, L_0] \to \mathbb{R}^n, \ i = 1, 2\) which satisfy
\[
\psi_{f,h}(u_1(0), u_2(0)) \leq \inf(\psi_{f,h}) + 8\delta_0,
\]
\[
\Gamma^{f}(0, L_0, u_1) \leq 8\delta_0, \ \Gamma^\tilde{f}(0, L_0, u_2) \leq 8\delta_0
\]

there exist an \((f)\)-perfect function \(w_1 : [0, \infty) \to \mathbb{R}^n\) and a \((\tilde{f})\)-perfect function \(w_2 : [0, \infty) \to \mathbb{R}^n\) such that
\[
\psi_{f,h}(w_1(0), w_2(0)) = \inf(\psi_{f,h})
\]

and
\[
|w_i(t) - v_i(t)| \leq \epsilon \text{ for all } t \in [0, L_0] \text{ and } i = 1, 2.
\]

By Lemma 10.12 and Proposition 4, there exist a number \(q \geq 8\) and \(\delta_1 \in (0, \delta_0)\) such that the following property holds:

(ii) for each \(z_1, z_2 \in \mathbb{R}^n\) satisfying \(d(z_1, H(f)) \leq \delta_1, \ i = 1, 2\) and each \(T \geq q\) there exist an a.c. function \(u : [0, T] \to \mathbb{R}^n\) which satisfies
\[
u(0) = z_1, \ u(T) = z_2, \ \Gamma^{f}(0, T, u) \leq 4^{-1}\delta_0.
and an a.c. function \( v : [0, T] \to R^n \) which satisfies
\[
v(0) = z_1, \quad v(T) = z_2, \quad \Gamma^f(0, T, v) \leq 4^{-1}\delta_0.
\]
By Proposition 6 and Lemma 9.1, there exist an \((f)\)-perfect function \( w_{*,1} : [0, \infty) \to R^n \) and an \((f)\)-perfect function \( w_{*,2} : [0, \infty) \to R^n \) such that
\[
\psi_{f,h}(w_{*,1}(0), w_{*,2}(0)) = \inf(\psi_{f,h}). \tag{12.1}
\]
Proposition 7 implies that
\[
\sup\{|w_{*,i}(t)| : t \in [0, \infty), \ i = 1, 2\} \leq \infty. \tag{12.2}
\]
It follows from (ATP) and Proposition 4 and 7 that there exists \( S_0 > 1 \) such that
\[
d(w_{*,i}(t), H(f)) \leq \delta_1 \text{ for all } t \geq S_0 \text{ and } i = 1, 2. \tag{12.3}
\]
By Theorem 8.1, there exist a neighborhood \( U_1 \) of \( f \) in \( A \), a neighborhood \( V_1 \) of \( h \) in \( \mathfrak{A} \) and numbers \( L_1 > l_1 > 0, S_1 > 0, \delta_2 \in (0, \delta_1) \) such that the following property holds:

(iii) for each \( g \in U_1 \), each \( \xi \in V_1 \), each \( T \geq 2L_1 + l_1 \) and each a.c. function \( v : [0, T] \to R^n \) which satisfies
\[
I^f(0, T, v) + \xi(v(0), v(T)) \leq \sigma(g, \xi, 0, T) + \delta_2
\]
the inequality \(|v(t)| \leq S_1, \ t \in [0, T] \) holds and for each \( \tau \in [L_1, T - L_1 - l_1] \),
\[
\text{dist}(H(f), \{v(t) : t \in [\tau, \tau + l_1]\}) \leq \delta_1.
\]
Set
\[
\Delta_0 = 8 + |\pi^f(w_{*,1}(0))| + |\pi^f(w_{*,2}(0))| + 2\sup\{|\pi^f(z)| + |\pi^f(z)| : z \in R^n \text{ and } |z| \leq S_1\} + 2|\mu(f)|(S_0 + L_0 + L_1 + q).
\tag{12.4}
\]
By Lemma 10.6 there exists a neighborhood \( U \subset U_1 \) of \( f \) in \( A \) such that the following property holds:

(iv) for each \( g \in U \), each \( \tau \geq 0 \) and each a.c. function \( u : [\tau, \tau + S_0 + L_0 + L_1 + q] \to R^n \) satisfying
\[
\min\{I^f(\tau, \tau + S_0 + L_0 + L_1 + q, u), I^f(\tau, \tau + S_0 + L_0 + L_1 + q, u)\} \leq \Delta_0
\]
we have \(|I^f(\tau, \tau + S_0 + L_0 + L_1 + q, u) - I^f(\tau, \tau + S_0 + L_0 + L_1 + q, u)| \leq \delta_0. \tag{12.5}\)

Clearly, there exists a neighborhood \( V \subset V_1 \) of \( h \) in \( \mathfrak{A} \) such that the following property holds:

(v) for each \( \xi \in V \) and each \( z_1, z_2 \in R^n \) satisfying
\[
|z_i| \leq 2 + 2S_1 + |w_{*,1}(0)| + |w_{*,2}(0)|, \ i = 1, 2
\]
we have \(|\xi(z_1, z_2) - h(z_1, z_2)| \leq \delta_0. \tag{12.6}\)

Choose \( \delta > 0, T_0 > 0 \) such that
\[
\delta \leq 4^{-1}\delta_2(S_0 + L_0 + L_1 + q + 8)^{-1}, \tag{12.5}
\]
\[
T_0 \geq 8(L_1 + L_0 + q + S_0 + 8). \tag{12.6}
\]
Assume that
\[
g \in U, \ \xi \in V, \ T \geq T_0
\tag{12.7}
\]
and that an a. c. function \( v : [0, T] \to R^n \) satisfies
\[
I^f(0, T, v) + \xi(v(0), v(T)) \leq \sigma(g, \xi, 0, T) + \delta. \tag{12.8}
\]
Property (iii), (12.5), (12.7) and (12.8) imply that
\[
|v(t)| \leq S_1, \ t \in [0, T], \tag{12.9}
\]
\[
d(v(t), H(f)) \leq \delta_1, \ t \in [L_1, T - L_1], \tag{12.10}
\]
In view of (12.3),
\[ d(w_{*,i}(S_0 + L_0), H(f)) \leq \delta_1, \ i = 1, 2. \]  
(12.11)

Property (ii), (12.6), (12.7), (12.10) and (12.11) imply that there exist a. c. functions \( w_i : [S_0 + L_0, S_0 + L_0 + q + L_1] \to \mathbb{R}^n, \ i = 1, 2 \) such that
\[ w_1(S_0 + L_0) = w_{*,1}(S_0 + L_0), \ w_1(S_0 + L_0 + q + L_1) = v(S_0 + L_0 + q + L_1), \]  
(12.12)
\[ \Gamma^f(S_0 + L_0, S_0 + L_0 + q + L_1, w_1) \leq \delta_0/4, \]  
(12.13)
\[ w_2(S_0 + L_0) = w_{*,2}(S_0 + L_0), \ w_2(S_0 + L_0 + q + L_1) = v(T - S_0 - L_0 - q - L_1), \]  
(12.14)
\[ \Gamma^f(S_0 + L_0, S_0 + L_0 + q + L_1, w_2) \leq \delta_0/4. \]  
(12.15)

For \( t \in [0, S_0 + L_0] \) and \( i = 1, 2 \) set
\[ w_i(t) = w_{*,i}(t). \]  
(12.16)

In view of (12.12), (12.14) and (12.16), \( w_i : [0, S_0 + L_0 + q + L_1] \to \mathbb{R}^n, \ i = 1, 2 \) is an a. c. function. By (3.6), (12.13), (12.15) and the choice of \( w_{*,i}, i = 1, 2, \)
\[ \Gamma^f(0, S_0 + L_0 + q + L_1, w_1) \]
\[ = \Gamma^f(0, S_0 + L_0, w_{*,1}) + \Gamma^f(S_0 + L_0, S_0 + L_0 + q + L_1, w_1) \leq \delta_0/4, \]  
(12.17)
\[ \Gamma^f(0, S_0 + L_0 + q + L_1, w_2) \]
\[ = \Gamma^f(0, S_0 + L_0, w_{*},2) + \Gamma^f(S_0 + L_0, S_0 + L_0 + q + L_1, w_2) \leq \delta_0/4. \]  
(12.18)

Set
\[ \hat{w}_2(t) = w_2(T - t), \ t \in [T - (S_0 + L_0 + q + L_1), T]. \]  
(12.19)

It follows from (12.14) and (12.19) that
\[ \hat{w}_2(T - (S_0 + L_0 + q + L_1)) = v(T - S_0 - L_0 - L_1 - q). \]  
(12.20)

By (12.8), (12.12) and (12.20),
\[ I^f(0, S_0 + L_0 + L_1 + q, v) + I^f(T - S_0 - L_0 - L_1 - q, T, v) + \xi(v(0), v(T)) \]
\[ \leq I^f(0, S_0 + L_0 + L_1 + q, w_1) + I^f(T - S_0 - L_0 - L_1 - q, T, \hat{w}_2) \]
\[ + \xi(w_1(0), \hat{w}_2(T)) + \delta. \]  
(12.21)

It follows from (3.6), (12.12) and (12.16)-(12.18) that
\[ I^f(0, S_0 + L_0 + L_1 + q, w_1) = \Gamma^f(0, S_0 + L_0 + L_1 + q, w_1) + \pi^f(w_1(0)) \]
\[ - \pi^f(w_1(S_0 + L_0 + L_1 + q)) + (S_0 + L_0 + L_1 + q)\mu(f) \]
\[ \leq \delta_0/4 + \pi^f(w_1(0)) - \pi^f(w_1(S_0 + L_0 + L_1 + q)) + (S_0 + L_0 + L_1 + q)\mu(f) \]
\[ = \delta_0/4 + \pi^f(w_{*,1}(0)) - \pi^f(v(S_0 + L_0 + L_1 + q)) + (S_0 + L_0 + L_1 + q)\mu(f). \]  
(12.22)

It follows from (3.3), (3.6), (12.14), (12.16), (12.18) and (12.20) that
\[ I^f(T - S_0 - L_0 - L_1 - q, T, \hat{w}_2) = \Gamma^f(0, S_0 + L_0 + L_1 + q, w_2) \]
\[ = \Gamma^f(0, S_0 + L_0 + L_1 + q, w_2) + \pi^f(w_2(0)) \]
\[ - \pi^f(w_2(S_0 + L_0 + L_1 + q)) + (S_0 + L_0 + L_1 + q)\mu(f) \]
\[ \leq \delta_0/4 + \pi^f(w_2(0)) - \pi^f(w_2(S_0 + L_0 + L_1 + q)) + (S_0 + L_0 + L_1 + q)\mu(f) \]
\[ \leq \delta_0/4 + \pi^f(w_{*,2}(0)) - \pi^f(v(T - S_0 - L_0 - L_1 - q)) + (S_0 + L_0 + L_1 + q)\mu(f). \]  
(12.23)

In view of (12.9) and (12.23),
\[ I^f(0, S_0 + L_0 + L_1 + q, w_1), \ I^f(T - S_0 - L_0 - L_1 - q, T, \hat{w}_2) \]
\[ \leq \delta_0/4 + |\pi^f(w_{*,1}(0))| + |\pi^f(w_{*,2}(0))| + 2\sup \{ |\pi^f(z)| \}
\[ + |\pi^f(z)| : z \in \mathbb{R}^n \text{ and } |z| \leq S_1 \} + \mu(f)((S_0 + L_0 + L_1 + q). \]  
(12.24)
Property (iv), (12.4), (12.7) and (12.24) imply that
\[ |I^f(0, S_0 + L_0 + L_1 + q, w_1) - I^g(0, S_0 + L_0 + L_1 + q, w_1)| \leq \delta_0, \]  
(12.25)\[ |I^f(T - S_0 - L_0 - L_1 - q, T, \hat{w}_2) - I^g(T - S_0 - L_0 - L_1 - q, T, \hat{w}_2)| \leq \delta_0. \]  
(12.26)By (12.16) and (12.19),
\[ \hat{w}_2(T) = w_2(0) = w_{*,2}(0). \]  
(12.27)Property (v), (12.7), (12.16) and (12.27) imply that
\[ |h(w_1(0), \hat{w}_2(T)) - \xi(w_1(0), \hat{w}_2(T))| \leq \delta_0. \]  
(12.28)It follows from (12.5), (12.16), (12.21)-(12.23), (12.25)-(12.28) that
\[ I^g(0, S_0 + L_0 + L_1 + q, v) + I^g(T - S_0 - L_0 - L_1 - q, T, v) + \xi(v(0), v(T)) \]
\[ \leq I^g(0, S_0 + L_0 + L_1 + q, w_1) + I^g(T - S_0 - L_0 - L_1 - q, T, \hat{w}_2) \]
\[ + \xi(w_1(0), \hat{w}_2(T)) + \delta \]
\[ \leq I^f(0, S_0 + L_0 + L_1 + q, w_1) + I^f(T - S_0 - L_0 - L_1 - q, T, \hat{w}_2) \]
\[ + h(w_1(0), \hat{w}_2(T)) + 4\delta_0 \]
\[ \leq \pi^f(w_{*,1}(0)) - \pi^f(v(S_0 + L_0 + L_1 + q)) + (S_0 + L_0 + L_1 + q)\mu(f) + \delta_0/4 \]
\[ + \pi^f(w_{*,2}(0)) - \pi^f(v(T - S_0 - L_0 - L_1 - q)) + (S_0 + L_0 + L_1 + q)\mu(f) + \delta_0/4 \]
\[ + h(w_{*,1}(0), w_{*,2}(0)) + 4\delta_0. \]  
(12.29)Property (v), (12.7) and (12.9) imply that
\[ |\xi(v(0), v(T)) - h(v(0), v(T))| \leq \delta_0. \]  
(12.30)By (5.1), (12.4), (12.9) and (12.29),
\[ \max\{I^g(0, S_0 + L_0 + L_1 + q, v), I^g(T - S_0 - L_0 - L_1 - q, T, v)\} \]
\[ \leq a_1 + a(S_0 + L_0 + L_1 + q) + |\pi^f(w_{*,1}(0))| + |\pi^f(w_{*,2}(0))| \]
\[ + 2\sup\{|\pi^f(z)| + |\pi^f(z)|: z \in \mathbb{R}^n \text{ and } |z| \leq S_1\} \]
\[ + 2|\mu(f)|(S_0 + L_0 + L_1 + q) + 8 + |h(w_{*,1}(0), w_{*,2}(0))| = \Delta_0. \]
Property (iv), the relation above and (12.7) imply that
\[ |I^g(0, S_0 + L_0 + L_1 + q, v) - I^f(0, S_0 + L_0 + L_1 + q, v)| \leq \delta_0, \]
\[ |I^g(T - S_0 - L_0 - L_1 - q, T, v) - I^f(T - S_0 - L_0 - L_1 - q, T, v)| \leq \delta_0. \]
It follows from the relations above, (12.29) and (12.30) that
\[ I^f(0, S_0 + L_0 + L_1 + q, v) + I^f(T - S_0 - L_0 - L_1 - q, T, v) + h(v(0), v(T)) \]
\[ \leq I^g(0, S_0 + L_0 + L_1 + q, v) + I^g(T - S_0 - L_0 - L_1 - q, T, v) + \xi(v(0), v(T)) + 3\delta_0 \]
\[ \leq \pi^f(w_{*,1}(0)) + \pi^f(w_{*,2}(0)) - \pi^f(v(S_0 + L_0 + L_1 + q)) \]
\[ - \pi^f(v(T - S_0 - L_0 - L_1 - q)) \]
\[ + 2(S_0 + L_0 + L_1 + q)\mu(f) + h(w_{*,1}(0), w_{*,2}(0)) + 8\delta_0. \]  
(12.31)Set
\[ \hat{v}(t) = v(T - t), \ t \in [0, T]. \]  
(12.32)By (3.3) and (12.32),
\[ I^f(0, S_0 + L_0 + L_1 + q, \hat{v}) = I^f(T - S_0 - L_0 - L_1 - q, T, v). \]  
(12.33)It follows from (12.31) and (12.33) that
\[ I^f(0, S_0 + L_0 + L_1 + q, v) - \mu(f)(S_0 + L_0 + L_1 + q) + \pi^f(v(S_0 + L_0 + L_1 + q)) \]

In view of (3.6), (3.7) and (12.34),

\[ +f^f(0, S_0 + L_0 + L_1 + q, \bar{v}) - \mu(f)(S_0 + L_0 + L_1 + q) + \pi^f(\bar{v}(S_0 + L_0 + L_1 + q)) \]

\[ +h(v(0), \bar{v}(0)) \leq \pi^f(w_{*,1}(0)) + \pi^f(w_{*,2}(0)) + h(w_{*,1}(0), w_{*,2}(0)) + 8\delta_0. \]  

(12.34)

In view of Proposition 5, there exist a neighborhood \( U \) such that for all \( x, \bar{x} \) in \( U \) and each \( \xi \) in \( V \),

\[ \psi_{f,h}(w_1(0), w_2(0)) = \inf(\psi_{f,h}) \]

and for all \( t \in [0, L_0] \),

\[ \epsilon \geq |w_1(t) - v(t)|, \epsilon \geq |w_2(t) - \bar{v}(t)| = |w_2(t) - v(T - t)|. \]

Theorem 9.2 is proved.

13. Auxiliary results for Theorem 9.4. Let \( f \in \mathcal{M} \) have (ATP) and \( h \in \mathfrak{A} \). Proposition 4, Theorem 10.2 and Lemma 10.11 imply the following result.

Lemma 13.1. Let \( \epsilon \in (0, 1) \). Then there exists a neighborhood \( \mathcal{U} \) of \( f \) in \( \mathcal{A} \) such that for each \( g \in \mathcal{U} \),

\[ |\inf(\pi^g) - \inf(\pi^f)| \leq \epsilon, |\inf(\pi^{\hat{g}}) - \inf(\pi^f)| \leq \epsilon. \]

Lemma 13.2. Let \( \epsilon \in (0, 1) \). Then there exist a neighborhood \( \mathcal{U} \) of \( f \) in \( \mathcal{A} \) and a neighborhood \( \mathcal{V} \) of \( h \) in \( \mathfrak{A} \) such that for each \( g \in \mathcal{U} \) and each \( \xi \in \mathcal{V} \),

\[ |\inf(\psi_{f,h}) - \inf(\psi_{g,\xi})| \leq \epsilon. \]

Proof. In view of Proposition 5, there exist \( c_0 > 0 \) such that for all \( x \in \mathbb{R}^n \),

\[ \pi^f(x) \geq -c_0, \pi^\hat{f}(x) \geq -c_0. \]  

(13.1)

Lemma 10.1 implies that there exist a neighborhood \( \mathcal{U}_1 \) of \( f \) in \( \mathcal{A} \) and \( M_0 > 0 \) such that for each \( x \in \mathbb{R}^n \) satisfying \( |x| \geq M_0 \) and each \( g \in \mathcal{U}_1 \),

\[ \min\{\pi^g(x), \pi^{\hat{g}}(x)\} \geq |\inf(\psi_{f,h})| + 4 + 2c_0 + 2\alpha_1. \]  

(13.2)

By Proposition 4, Theorem 10.2 and Lemma 13.1, there exist a neighborhood \( \mathcal{U} \) of \( f \) in \( \mathcal{A} \) and a neighborhood \( \mathcal{V} \) of \( h \) in \( \mathfrak{A} \) such that \( \mathcal{U} \subset \mathcal{U}_1 \) and that the following properties hold:

(i) for each \( g \in \mathcal{U} \) and each \( x \in \mathbb{R}^n \) satisfying \( |x| \leq M_0 + 2 \),

\[ |\pi^f(x) - \pi^g(x)| \leq \epsilon/8, |\pi^f(x) - \pi^{\hat{g}}(x)| \leq \epsilon/8; \]

(ii) for each \( z_1, z_2 \in \mathbb{R}^n \) satisfying \( |z_1|, |z_2| \leq M_0 + 2 \) and each \( \xi \in \mathcal{V} \),

\[ |h(z_1, z_2) - \xi(z_1, z_2)| \leq \epsilon/8; \]
Lemma 9.1 implies that there exist

\[ |\inf(\pi^g) - \inf(\pi^f)| \leq \varepsilon/8, \quad |\inf(\pi^g) - \inf(\pi^f)| \leq \varepsilon/8. \]

Assume that

\[ g \in \mathcal{U}, \; \xi \in \mathcal{V}. \]  

(13.3)

Properties (i) and (ii) and (13.3) imply that for all \( z_1, z_2 \in \mathbb{R}^n \) satisfying \( |z_1|, |z_2| \leq M_0 + 2, \)

\[ |\pi^g(z_1) + \pi^g(z_2) + \xi(z_1, z_2) - \pi^f(z_1) - \pi^f(z_2) - h(z_1, z_2)| \leq 3\varepsilon/8. \]  

(13.4)

Lemma 9.1 implies that there exist \( z_{*,1}, z_{*,2} \in \mathbb{R}^n \) such that

\[ \psi_{f,h}(z_{*,1}, z_{*,2}) = \inf(\psi_{f,h}). \]  

(13.5)

By (5.1), (13.1)-(13.3) and (13.5),

\[ |z_{*,1}|, |z_{*,2}| < M_0. \]  

(13.6)

In view of (13.4)-(13.6),

\[ \inf(\psi_{g,\xi}) \leq \psi_{g,\xi}(z_{*,1}, z_{*,2}) \leq \psi_{f,h}(z_{*,1}, z_{*,2}) + 3\varepsilon/8 \leq \inf(\psi_{f,h}) + 3\varepsilon/8. \]  

(13.7)

Lemma 9.1 implies that there exist \( x_1, x_2 \in \mathbb{R}^n \) such that

\[ \psi_{g,\xi}(x_1, x_2) = \inf(\psi_{g,\xi}). \]  

(13.8)

By (13.7) and (13.8),

\[ \psi_{g,\xi}(x_1, x_2) \leq \inf(\psi_{f,h}) + 3\varepsilon/8. \]  

(13.9)

Property (iii) and (13.1) imply that

\[ \pi^g(x_1) \geq \inf(\pi^g) \geq \inf(\pi^f) - 1 \geq -c_0 - 1, \]

\[ \pi^g(x_2) \geq \inf(\pi^g) \geq \inf(\pi^f) - 1 \geq -c_0 - 1. \]  

(13.10)

By (5.1), (9.1), (13.9) and (13.10),

\[ \pi^g(x_1) \leq \inf(\psi_{f,h}) + 3\varepsilon/8 - \pi^g(x_2) - \xi(x_1, x_2) \leq \inf(\psi_{f,h}) + 3\varepsilon/8 + c_0 + 1 + a_1, \]  

(13.11)

\[ \pi^g(x_2) \leq \inf(\psi_{f,h}) + 3\varepsilon/8 - \pi^g(x_1) - \xi(x_1, x_2) \leq \inf(\psi_{f,h}) + 3\varepsilon/8 + c_0 + 1 + a_1. \]  

(13.12)

It follows from (13.3), (13.11), (13.12) and the choice of \( M_0 \) (see (13.2)) that

\[ |x_1|, |x_2| \leq M_0. \]  

(13.13)

In view of (9.1), (13.4) and (13.13),

\[ |\psi_{g,\xi}(x_1, x_2) - \psi_{f,h}(x_1, x_2)| \leq 3\varepsilon/8. \]  

(13.14)

By (13.8) and (13.14), \( \inf(\psi_{g,\xi}) \geq \psi_{f,h}(x_1, x_2) - 3\varepsilon/8 \geq \inf(\psi_{f,h}) - 3\varepsilon/8. \) Lemma 13.2 is proved. \( \square \)

Define

\[ \mathcal{D}(f, h) = \{(x_1, x_2) \in \mathbb{R}^n \times \mathbb{R}^n : \psi_{f,h}(x_1, x_2) = \inf(\psi_{f,h})\}. \]

(13.15)

For each \( z_1, z_2 \in \mathbb{R}^n \) set

\[ d_1((z_1, z_2), \mathcal{D}(f, h)) = \inf\{|z_1 - x_1| + |z_2 - x_2| : (x_1, x_2) \in \mathcal{D}(f, h)\}. \]

Lemma 13.3. Let \( \epsilon \in (0, 1) \). Then there exist a neighborhood \( \mathcal{U} \) of \( f \) in \( \mathcal{A} \), a neighborhood \( \mathcal{V} \) of \( h \) in \( \mathcal{A} \) and \( \delta \in (0, \epsilon) \) such that for each \( g \in \mathcal{U} \) and each \( \xi \in \mathcal{V} \) and each \( z_1, z_2 \in \mathbb{R}^n \) satisfying \( \psi_{g,\xi}(z_1, z_2) \leq \inf(\psi_{g,\xi}) + \delta \) the inequality \( \inf\{|z_1 - x_1| + |z_2 - x_2| : (x_1, x_2) \in \mathcal{D}(f, h)\} \leq \epsilon \) holds.
Proof. In view of Proposition 5, there exist \( c_0 > 0 \) such that for all \( x \in \mathbb{R}^n \),

\[
\pi^f(x) \geq -c_0, \quad \pi^\bar{f}(x) \geq -c_0.
\]  

(13.16)

Lemma 10.1 implies that there exist a neighborhood \( \mathcal{U}_1 \) of \( f \) in \( \mathcal{A} \) and \( M_0 > 0 \) such that for each \( x \in \mathbb{R}^n \) satisfying \( |x| \geq M_0 \) and each \( g \in \mathcal{U}_1 \),

\[
\min\{\pi^g(x), \pi^\bar{g}(x)\} \geq |\inf(\psi_{f,h})| + 4 + 2c_0 + 2a_1.
\]  

(13.17)

By Lemma 9.1, there exists \( \delta \in (0, \epsilon/4) \) such that the following property holds:

(i) for each \( z_1, z_2 \in \mathbb{R}^n \) satisfying \( \psi_{f,h}(z_1, z_2) \leq \inf(\psi_{f,h}) + 8\delta \) we have

\[
\inf\{|z_1 - x_1| + |z_2 - x_2| : (x_1, x_2) \in \mathcal{D}(f, h)\} \leq \epsilon.
\]

By Proposition 4, Theorem 10.2, Lemmas 13.1 and 13.2 and (5.2), there exist a neighborhood \( \mathcal{U} \subset \mathcal{U}_1 \) of \( f \) in \( \mathcal{A} \) and a neighborhood \( \mathcal{V} \) of \( h \) in \( \mathcal{F} \) such that for each \( g \in \mathcal{U} \), each \( \xi \in \mathcal{V} \) and each \( z_1, z_2 \in \mathbb{R}^n \) satisfying \( |z_1| \leq M_0 + 2, i = 1, 2, \)

\[
|h(z_1, z_2) - \xi(z_1, z_2)| \leq \delta,
\]  

(13.18)

\[
|\pi^f(z_1) - \pi^g(z_1)| \leq \delta, \quad |\pi^f(z_2) - \pi^g(z_2)| \leq \delta,
\]  

(13.19)

\[
|\inf(\psi_{f,h}) - \inf(\psi_{g,\xi})| \leq \delta,
\]  

(13.20)

\[
|\inf(\pi^g) - \inf(\pi^f)| \leq \delta, \quad |\inf(\pi^g) - \inf(\pi^f)| \leq \delta.
\]  

(13.21)

Assume that

\[
g \in \mathcal{U}, \quad \xi \in \mathcal{V}, \quad z_1, z_2 \in \mathbb{R}^n, \quad \psi_{g,\xi}(z_1, z_2) \leq \inf(\psi_{f,\xi}) + \delta.
\]  

(13.22)

In view of (13.20) and (13.22),

\[
\psi_{g,\xi}(z_1, z_2) \leq \inf(\psi_{f,h}) + 2\delta.
\]  

(13.23)

By (5.1), (9.1), (13.16), (13.21) and (13.23),

\[
\pi^g(z_1) = \psi_{g,\xi}(z_1, z_2) - \pi^\bar{g}(z_2) - \xi(z_1, z_2) \leq \inf(\psi_{f,h}) + 2\delta - \inf(\pi^\bar{g}) + a_1
\]

\[
\leq \inf(\psi_{f,h}) + 2 + a_1 - \inf(\pi^f) + 1 \leq \inf(\psi_{f,h}) + 3 + a_1 + c_0,
\]  

(13.24)

\[
\pi^\bar{g}(z_2) = \psi_{g,\xi}(z_1, z_2) - \pi^g(z_1) - \xi(z_1, z_2) \leq \inf(\psi_{f,h}) + 2\delta - \inf(\pi^g) + a_1
\]

\[
\leq \inf(\psi_{f,h}) + 2 + a_1 - \inf(\pi^f) + 1 \leq \inf(\psi_{f,h}) + 3 + a_1 + c_0.
\]  

(13.25)

It follows from (13.22), (13.24), (13.25) and the choice of \( M_0 \) (see (13.17)) that

\[
|z_1|, |z_2| \leq M_0.
\]  

(13.26)

In view of (13.22), (13.23) and (13.26),

\[
\psi_{f,h}(z_1, z_2) \leq \psi_{g,\xi}(z_1, z_2) + 3\delta \leq \inf(\psi_{f,h}) + 5\delta.
\]

Together with property (i) this implies that

\[
\inf\{|z_1 - x_1| + |z_2 - x_2| : (x_1, x_2) \in \mathcal{D}(f, h)\} \leq \epsilon.
\]

Lemma 13.3 is proved. \( \Box \)
14. **Proof of Theorem 9.4.** By Lemma 11.1, there exists $\delta_0 \in (0, \varepsilon)$ such that the following property holds:

(i) for each pair of a. c. functions $w_i : [0, L_0] \to \mathbb{R}^n$, $i = 1, 2$ which satisfy

$$\psi_{f,h}(w_1(0), w_2(0)) \leq \inf(\psi_{f,h}) + 8\delta_0,$$

$$\Gamma^f(0, L_0, w_1) \leq 8\delta_0, \quad \Gamma^g(0, L_0, w_2) \leq 8\delta_0$$

there exist an $(f)$-perfect function $\tilde{w}_1 : [0, \infty) \to \mathbb{R}^n$ and an $(g)$-perfect function $\tilde{w}_2 : [0, \infty) \to \mathbb{R}^n$ such that $\psi_{f,h}(\tilde{w}_1(0), \tilde{w}_2(0)) = \inf(\psi_{f,h})$ and for all $t \in [0, L_0]$ and $i = 1, 2, |w_i(t) - \tilde{w}_i(t)| \leq \varepsilon$.

By Lemmas 13.1-13.3, there exist a neighborhood $U_0$ of $f$ in $\mathbb{A}$, a neighborhood $V_0$ of $h$ in $\mathbb{A}$ and $\delta \in (0, 4^{-1}\delta_0)$ such that the following properties hold:

(ii) for each $g \in U_0$, each $\xi \in V_0$ and each $z_1, z_2 \in \mathbb{R}^n$ satisfying

$$\psi_{g,\xi}(z_1, z_2) \leq \inf(\psi_{g,\xi}) + 8\delta$$

the inequality $d_1((z_1, z_2), D(f, h)) \leq 8^{-1}\delta_0$ holds;

(iii) for each $g \in U_0$,

$$|\inf(\pi^g) - \inf(\pi^f)| \leq 16^{-1}\delta, \quad |\inf(\pi^g) - \inf(\pi^\tilde{f})| \leq 16^{-1}\delta$$

and for each $g \in U_0$, each $\xi \in V_0$

$$|\inf(\psi_{f,h}) - \inf(\psi_{g,\xi})| \leq 16^{-1}\delta.$$

By Theorem 10.3, there exists a number

$$S_1 > \sup\{|z_1| + |z_2| : (z_1, z_2) \in D(f, h)\} + 8$$

(14.1)

and a neighborhood $U_1 \subset U_0$ of $f$ in $\mathbb{A}$ such that for each $g \in U_1$ and each $(g)$-good function $v : [0, \infty) \to \mathbb{R}^n$ we have

$$\limsup_{t \to \infty} |v(t)| < S_1$$

(14.2)

and that for each $g \in U_1$ and each $(\tilde{g})$-good function $v : [0, \infty) \to \mathbb{R}^n$ inequality (14.2) holds.

By Theorem 2.1, there exist a neighborhood $U_2 \subset U_1$ of $f$ in $\mathbb{A}$ and $S_2 > S_1 + 1$ such that the following property holds:

(iv) for each $g \in U_2$, each $\tau \geq 1$ and each a. c. function $v : [0, \tau] \to \mathbb{R}^n$ which satisfies

$$|v(0)|, \quad |v(0)| \leq S_1 + 1, \quad F^g(0, \tau, v) \leq U^g(0, \tau, v(0), v(\tau)) + 4$$

we have $|v(t)| \leq S_2, \quad t \in [0, \tau]$.

Choose

$$S_3 > \sup\{|\pi^f(z_1)| + |\pi^\tilde{f}(z_2)| + |h(z_1, z_2)| : z_1, z_2 \in \mathbb{R}^n \text{ and } |z_i| \leq S_2 + 1, \ i = 1, 2\}.$$

(14.3)

By Proposition 4, Theorem 10.2, (5.2) and (9.1), there exist a neighborhood $U_3 \subset U_2$ of $f$ in $\mathbb{A}$ and a neighborhood $V \subset V_0$ of $h$ in $\mathbb{A}$ such that the following properties hold:

(v) for each $\xi \in V$ and each $z_1, z_2 \in \mathbb{R}^n$ satisfying $|z_i| \leq 8(S_1 + S_2 + S_3 + 1), \ i = 1, 2$ we have $|h(z_1, z_2) - \xi(z_1, z_2)| \leq 64^{-1}\delta$;

(vi) for each $g \in U_3$, each $\xi \in V$ and each $z_1, z_2 \in \mathbb{R}^n$ satisfying $|z_i| \leq 8(S_1 + S_2 + S_3 + 1), \ i = 1, 2$

$$|\pi^f(z_1) - \pi^g(z_1)| \leq 64^{-1}\delta, \quad |\pi^\tilde{f}(z_2) - \pi^\tilde{g}(z_2)| \leq 64^{-1}\delta, \quad |\psi_{f,\xi}(z_1, z_2) - \psi_{g,\xi}(z_1, z_2)| \leq 64^{-1}\delta, \quad |\mu(f) - \mu(g)| \leq 64^{-1}\delta(L_0 + 1)^{-1}. \quad \square$$
By Lemma 10.6, there exists a neighborhood $U \subset U_3$ of $f$ in $\mathcal{A}$ such that the following property holds:

(vii) for each $g \in U$ and each a.c. function $u : [0, L_0] \to R^n$ satisfying
\[ \min\{I^g(0, L_0, u), I^f(0, L_0, u)\} \leq 2(S_2 + S_3) + 20 + (L_0 - 1)|\mu(f)| \]
we have $|I^f(0, L_0, u) - I^g(0, L_0, u)| \leq 64^{-1}\delta$.

Assume that
\[ g \in U, \xi \in \mathcal{V}, \] (14.4)
a function $w_1 : [0, \infty) \to R^n$ is $(g)$-perfect, a function $w_2 : [0, \infty) \to R^n$ is $(\bar{g})$-perfect and that
\[ \psi_{g,\xi}(w_1(0), w_2(0)) \leq \inf(\psi_{g,\xi}) + \delta. \] (14.5)

Property (ii), (14.4) and (14.5) imply that
\[ d_1((w_1(0), w_2(0)), D(f, h)) \leq 8^{-1}\delta_0. \] (14.6)

In view of (14.1) and (14.6),
\[ |w_1(0), |w_2(0)| < S_1. \] (14.7)

By (14.4), (14.5), (14.7) and properties (iii) and (vi),
\[ \psi_{f,h}(w_1(0), w_2(0)) \leq \psi_{g,\xi}(w_1(0), w_2(0)) + 64^{-1}\delta \leq \inf(\psi_{g,\xi}) + \delta + 64^{-1}\delta \leq \inf(\psi_{f,h}) + 3\delta/2. \] (14.8)

Let us show that $\Gamma^f(0, L_0, w_1) \leq \delta$ and $\Gamma^\bar{f}(0, L_0, w_2) \leq \delta$. It follows from Proposition 7 and the choice of $S_1$ (see (14.1) and (14.2)) that $w_1$ is a $(g)$-good function, $w_2$ is a $(\bar{g})$-function and that
\[ \limsup_{t \to \infty} |w_i(t)| < S_1, \quad i = 1, 2. \] (14.9)

Property (iv), (14.4), (14.7) and (14.9) imply that
\[ |w_1(t)| \leq S_2, \quad t \in [0, \infty), \quad i = 1, 2. \] (14.10)

By properties (v) and (vi), (14.4) and (14.10), for all $t \in [0, \infty)$,
\[ |\pi^f(w_1(t)) - \pi^g(w_1(t))| \leq 64^{-1}\delta, \quad |\pi^\bar{f}(w_2(t)) - \pi^\bar{g}(w_2(t))| \leq 64^{-1}\delta, \] (14.11)
\[ |\xi(w_1(t), w_2(t)) - h(w_1(t), w_2(t))| \leq 64^{-1}\delta, \] (14.12)
\[ |\psi_{g,\xi}(w_1(t), w_2(t)) - \psi_{f,h}(w_1(t), w_2(t))| \leq 64^{-1}\delta. \] (14.13)

Since the function $w_1$ is $(g)$-perfect it follows from (3.6), (14.3), (14.4), (14.10), (14.11) and property (vi) that
\[ I^{\bar{g}}(0, L_0, w_1) = \Gamma^{\bar{g}}(0, L_0, w_1) + L_0\mu(g) + \pi^g(w_1(0)) - \pi^\bar{g}(w_1(L_0)) \leq L_0\mu(f) + 64^{-1}\delta + 2\sup\{|\pi^f(z) - \pi^g(z)| : z \in R^n \text{ and } |z| \leq S_2\} + 32^{-1}\delta \leq L_0\mu(f) + 2S_3 + 1. \] (14.14)

Since the function $w_2$ is $(\bar{g})$-perfect it follows from (3.6), (14.3), (14.4), (14.10), (14.11) and property (vi) that
\[ I^\bar{g}(0, L_0, w_2) = \Gamma^\bar{g}(0, L_0, w_2) + L_0\mu(g) + \pi^\bar{g}(w_2(0)) - \pi^\bar{g}(w_2(L_0)) \leq L_0\mu(f) + 64^{-1}\delta + 2\sup\{|\pi^f(z) - \pi^\bar{g}(z)| : z \in R^n \text{ and } |z| \leq S_2\} + 32^{-1}\delta \leq L_0\mu(f) + 2S_3 + 1. \] (14.15)

Properties (vi) and (vii), (14.4), (14.14) and (14.15) imply that
\[ |I^f(0, L_0, w_1) - I^g(0, L_0, w_1)| \leq 64^{-1}\delta, \quad |I^\bar{f}(0, L_0, w_2) - I^\bar{g}(0, L_0, w_2)| \leq 64^{-1}\delta. \] (14.16)
Assume that an \((f)\)-perfect function \(w_1\) is \((g)\)-perfect it follows from (3.6), (14.4), (14.11) and (14.16) that
\[
\Gamma^f(0, L_0, w_1) = I^f(0, L_0, w_1) - \pi^f(w_1(0)) + \pi^f(w_1(L_0)) - L_0\mu(f)
\leq I^g(0, L_0, w_1) + 64^{-1}\delta - \pi^g(w_1(0)) + 64^{-1}\delta + \pi^g(w_1(L_0)) + 64^{-1}\delta - L_0\mu(g)
+ 64^{-1}\delta \leq \Gamma^g(0, L_0, w_1) + \delta = \delta.
\] (14.17)
Since the function \(w_2\) is \((g)\)-perfect it follows from (3.6), (14.4), (14.11) and (14.16) that
\[
\Gamma^f(0, L_0, w_2) = I^f(0, L_0, w_2) - \pi^f(w_2(0)) + \pi^f(w_2(L_0)) - L_0\mu(f)
\leq I^g(0, L_0, w_2) + 64^{-1}\delta - \pi^g(w_2(0)) + 64^{-1}\delta + \pi^g(w_2(L_0)) + 64^{-1}\delta - L_0\mu(g)
+ 64^{-1}\delta \leq \Gamma^g(0, L_0, w_2) + \delta = \delta.
\] (14.18)
By property (i), (14.8), (14.17) and (14.18), there exist an \((f)\)-perfect function \(\tilde{w}_1: [0, \infty) \to \mathbb{R}^n\) and an \((f)\)-perfect function \(\tilde{w}_2: [0, \infty) \to \mathbb{R}^n\) such that
\[
\psi_{f,h}(\tilde{w}_1(0), \tilde{w}_2(0)) = \inf(\psi_{f,h})
\]
and for all \(t \in [0, L_0]\) and \(i = 1, 2\), we have \(|w_i(t) - \tilde{w}_i(t)| \leq \epsilon\). Theorem 9.4 is proved. \(\Box\)

15. **Auxiliary results for Theorem 9.3.**

**Lemma 15.1.** Let \(f \in \mathcal{L}\) have \((ATP)\) and \(z \in H(f)\). Then there exists a unique \((f)\)-perfect function \(u: [0, \infty) \to \mathbb{R}^n\) satisfying \(u(0) = z\). Moreover, \(u(t) \in H(f)\) for all \(t \geq 0\).

**Proof.** By Lemma 10.9, there exists an a.c. function \(v: \mathbb{R}^1 \to H(f)\) such that \(v(0) = z\) and
\[
\Gamma^f(-T, T, v) = 0 \text{ for all } T > 0.
\] (15.1)

Assume that an \((f)\)-perfect function \(u: [0, \infty) \to \mathbb{R}^n\) satisfies
\[
u(0) = z.
\] (15.2)

Define
\[
\tilde{u}(t) = u(t), \ t \in [0, \infty), \ \tilde{u}(t) = v(t), \ t \in (-\infty, 0).
\] (15.3)

In view of (15.1) and (15.3), \(\tilde{u} : \mathbb{R}^1 \to \mathbb{R}^n\) is an a.c. function. Since the function \(u\) is \((f)\)-perfect it follows from (3.6), (15.2) and (15.3) that \(\Gamma^f(S, T, \tilde{u}) = 0\) for all \(S < T\). By Lemma 10.10, for all \(t \geq 0\), \(u(t) = \tilde{u}(t) = v(t)\). This completes the proof of Lemma 15.1. \(\Box\)

Let \(\mathcal{M}\) be one of the following spaces: \(\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2, \bar{\mathcal{M}}_q, q \geq 3\) is an integer. If \(\mathcal{M} = \bar{\mathcal{M}}_q\), where \(q \geq 3\) is an integer, then we set \(\mathcal{M} = \mathcal{M}_q\); if \(\mathcal{M} = \mathcal{L}_q\), where \(q \in \{0, 1, 2\}\), then we set \(\mathcal{M} = \mathcal{L}\). Denote by \(E_0\) the set of all \(f \in \bar{\mathcal{M}}\) which has \((ATP)\).

**Lemma 15.2.** [33] The set \(E_0\) is an everywhere dense subset of \(\mathcal{M}\).

This lemma was proved on page 170 of [27]. It is easy to see that the following lemma is true.

**Lemma 15.3.** \(E_0 \times \mathfrak{A}\) is an everywhere dense subset of \(\mathcal{M} \times \mathfrak{A}\).

The following result was proved in Section 3 of Chapter 2 of [2] (see also Proposition 3.7.1 of [27]).
Lemma 15.4. Let $\Omega$ be a closed subset of $\mathbb{R}^n$. Then there exists a bounded nonnegative function $\phi \in C^\infty(\mathbb{R}^n)$ such that $\Omega = \{x \in \mathbb{R}^n : \phi(x) = 0\}$ and for each sequence of nonnegative integers $p_1, \ldots, p_s$, the function $\partial^{p_1} \phi / \partial x_1^{p_1} \cdots \partial x_n^{p_s} : \mathbb{R}^n \to \mathbb{R}^1$ is bounded, where $|p| = \sum_{i=1}^n p_i$.

Denote by $E_1$ the set of all $(f, h) \in E_0 \times \mathfrak{A}$ such that the function $\psi_{f, h}$ has a unique point of minimum.

Denote by $E$ the set of all $(f, h) \in E_0 \times \mathfrak{A}$ for which there exists a a unique pair of an $(f)$-perfect function $w_1 : [0, \infty) \to \mathbb{R}^n$ and an $(f)$-perfect function $w_2 : [0, \infty) \to \mathbb{R}^n$ such that $\psi_{f, h}(w_1(0), w_2(0)) = \inf(\psi_{f, h})$.

Lemma 15.5. Let $f \in \mathcal{L}$ have (ATP), $v : [0, \infty) \to \mathbb{R}^n$ be an $(f)$-perfect function and let $t_0 \geq 0$ satisfy $v(t_0) \in H(f)$. Then $v(t) \in H(f)$ for all $t \geq 0$.

Proof. By Lemma 15.1, we may assume without loss of generality that $t_0 > 0$. Lemma 10.9 implies that there exists an a. c. function $u_0 : \mathbb{R}^3 \to H(f)$ such that $u_0(0) = v(t_0)$ and $\Gamma(f, T, u_0) = 0$ for all $T > 0$. Define

$$u(t) = v(t), \quad t \in [0, t_0], \quad u(t) = u_0(t - t_0), \quad t \in (t_0, \infty). \quad (15.4)$$

It is not difficult to see that the a. c. function $u : [0, \infty) \to \mathbb{R}^n$ is $(f)$-perfect. By Lemma 10.10 and (15.4), for all $t \geq 0$, $v(t) = u(t) = u_0(t - t_0)$. This completes the proof of Lemma 15.5. \qed

Lemma 15.6. Assume that $f \in E_0$, $u : [0, \infty) \to \mathbb{R}^n$ is an $(f)$-perfect function, $v : [0, \infty) \to \mathbb{R}^n$ is an $(f)$-perfect function,

$$v(0) \not\in H(f) \quad (15.5)$$

and that

$$v([0, \infty)) \subset u([0, \infty)) \cup H(f). \quad (15.6)$$

Then for every $t \geq 0$ there exists a unique number $S(t) \geq 0$ such that $v(t) = u(S(t))$, $
lim_{t \to \infty} S(t) = \infty$ and that the function $S : [0, \infty) \to [0, \infty)$ is continuous and strictly increasing.

Proof. Lemma 15.5, (15.5) and (15.6) imply that

$$v([0, \infty)) \subset u([0, \infty)) \backslash H(f). \quad (15.7)$$

In view of (15.7), for every $t \geq 0$ there exists $S(t) \geq 0$ such that

$$v(t) = u(S(t)). \quad (15.8)$$

Assume that $\tau \in [0, \infty)$, $S_1, S_2 \geq 0$, $S_2 > S_1$ and that

$$v(\tau) = u(S_1) = u(S_2). \quad (15.9)$$

Lemma 10.8 implies that for all $t \geq 0$ we have $u(S_1 + t) = u(S_2 + t)$ and that for all integers $k \geq 1$, $v(\tau) = u(S_1 + k(S_2 - S_1))$. Together with (ATP) this implies that $v(\tau) \in H(f)$. This contradicts (5.7). The contradiction we have reached proves that for every $t \geq 0$ there exists a unique number $S(t) \geq 0$ satisfying (15.8).

We show that $S : [0, \infty) \to [0, \infty)$ is injective. Assume the contrary. Then there exist $\tau_1 \geq 0$, $\tau_2 > \tau_1$ such that $S(\tau_1) = S(\tau_2)$. Then

$$v(\tau_1) = u(S(\tau_1)) = u(S(\tau_2)) = v(\tau_2).$$

The relations above and Lemma 10.8 imply that for all $t \geq 0$, we have $v(\tau_1 + t) = v(\tau_2 + t)$. Hence $v(\tau_1) \not\in H(f)$. This contradicts (15.7). The contradiction we have reached proves that $S : [0, \infty) \to [0, \infty)$ is injective.
We show that \( \lim_{t \to \infty} S(t) = \infty \). Assume the contrary. Then there exists a sequence \( \{t_k\}_{k=1}^{\infty} \subset (0, \infty) \) such that
\[
\lim_{k \to \infty} t_k = \infty, \quad \sup\{S(t_k) : k = 1, 2, \ldots\} < \infty. \tag{15.10}
\]
Extracting a subsequence and re-indexing if necessary we may assume without loss of generality that there exists \( \lim_{k \to \infty} S(t_k) \in [0, \infty) \). By (15.8), (15.10) and (ATP), \( u(\lim_{k \to \infty} S(t_k)) = \lim_{k \to \infty} u(S(t_k)) = \lim_{k \to \infty} v(t_k) \in H(f) \). Together with Lemma 15.5 this implies that \( u([0, \infty)) \subset H(f) \). This inclusion contradicts (15.7). The contradiction we have reached proves that
\[
\lim_{t \to \infty} S(t) = \infty. \tag{15.11}
\]
We show that the function \( S \) is continuous. Assume that \( \{\tau_i\}_{i=1}^{\infty} \subset [0, \infty) \) and that
\[
\lim_{i \to \infty} \tau_i = \tau. \tag{15.12}
\]
We show that the sequence \( \{S(\tau_i)\}_{i=1}^{\infty} \) is bounded.
Assume the contrary. Then extracting a subsequence and re-indexing, if necessary, we may assume that
\[
\lim_{i \to \infty} S(\tau_i) = \infty. \tag{15.13}
\]
In view of (15.8), (15.12) and (15.13), \( v(\tau) = \lim_{i \to \infty} v(\tau_i) = \lim_{i \to \infty} u(S(\tau_i)) \in H(f) \). This contradicts (15.7). Therefore the sequence \( \{S(\tau_i)\}_{i=1}^{\infty} \) is bounded.
We show that \( \lim_{i \to \infty} S(\tau_i) = S(\tau) \).
Assume the contrary. Then there exists a subsequence \( \{\tau_{i_k}\}_{k=1}^{\infty} \) such that there exists
\[
\lim_{k \to \infty} S(\tau_{i_k}) \neq S(\tau). \tag{15.14}
\]
By (15.8) and (15.12),
\[
u(S(\tau)) = v(\tau) = \lim_{k \to \infty} v(\tau_{i_k}) = \lim_{k \to \infty} u(S(\tau_{i_k})) = u(\lim_{k \to \infty} S(\tau_{i_k})). \tag{15.15}\]
Since in view of Lemma 10.8, Lemma 15.5, (15.7) and (ATP), the function \( u \) is injective, the relation above implies that \( S(\tau) = \lim_{k \to \infty} S(\tau_{i_k}) \). This contradicts (15.14). The contradiction we have reached proves that \( S(\tau) = \lim_{i \to \infty} S(\tau_i) \) and the function \( S \) is continuous.
We show that the function \( S \) is strictly monotone. Assume the contrary. Then there exist \( \tau_1 \geq 0 \) and \( \tau_2 > \tau_1 \) such that
\[
S(\tau_2) < S(\tau_1). \tag{15.16}\]
Since the function \( S \) is continuous and satisfies (15.11) there exists \( \tau_3 > \tau_2 \) such that \( S(\tau_3) = S(\tau_1) \). This contradicts the injectivity of \( S \). Therefore the function \( S \) is strictly monotone. Lemma 15.6 is proved.

Lemmas 15.6 and 10.8 imply the following result.

**Lemma 15.7.** Assume that \( f \in E_0 \), a function \( u : [0, \infty) \to R^n \) is \((f)\)-perfect, functions \( v_i : [0, \infty) \to R^n \), \( i = 1, 2 \) are \((f)\)-perfect and that
\[
v_1(0) = v_2(0) \notin H(f), v_i([0, \infty)) \subset u([0, \infty)) \cup H(f), \quad i = 1, 2.
\]
Then \( v_1(t) = v_2(t) \) for all \( t \geq 0 \).

**Lemma 15.8.** \( E_1 \) is an everywhere dense subset of \( \mathcal{M} \times A \).
Proof. Let \( \mathcal{V} \) be a nonempty open subset of \( \mathfrak{M} \times \mathfrak{A} \). Lemma 15.3 implies that there exists \((f, h) \in (E_0 \times \mathfrak{A}) \cap \mathcal{V} \). By Lemma 9.1, there exist \( z_{*,1}, z_{*,2} \in \mathbb{R}^n \) such that 
\[
\psi_{f,h}(z_{*,1}, z_{*,2}) = \inf(\psi_{f,h}).
\]
For any \( r > 0 \) set 
\[
h_r(z_1, z_2) = h(z_1, z_2) + r|z_1 - z_{*,1}|^2 + |z_2 - z_{*,2}|^2, \ z_1, z_2 \in \mathbb{R}^n.
\]
It is clear that for any \( r > 0 \), \( h_r \in \mathfrak{A} \), \( (z_{*,1}, z_{*,2}) \) is a unique point of minimum of the function \( \psi_{f,h} \) and that \((f, h_r) \in E_1 \). Clearly, \( h = \lim_{r \to 0^+} h_r \) in \( \mathfrak{A} \). Thus there exists \( r > 0 \) such that \((f, h_r) \in \mathcal{V} \). Lemma 15.8 is proved.

Lemma 15.1 implies the following auxiliary result.

**Lemma 15.9.** Assume that \((f, h) \in E_1, z_{*,1}, z_{*,2} \in H(f) \) and that 
\[
\psi_{f,h}(z_{*,1}, z_{*,2}) = \inf(\psi_{f,h}).
\]
Then \((f, h) \in E \).

**Lemma 15.10.** Let \((f, h) \in E_1, z_{*,1}, z_{*,2} \in \mathbb{R}^n \) and 
\[
\psi_{f,h}(z_{*,1}, z_{*,2}) = \inf(\psi_{f,h}),
\]
and \( \mathcal{V} \) be an open neighborhood of \((f, h) \) in \( \mathfrak{M} \times \mathfrak{A} \). Then \( \mathcal{V} \cap E \neq \emptyset \).

**Proof.** There are three cases:

(a) there exist an \((f)\)-perfect function \( w_{*,1} : [0, \infty) \to \mathbb{R}^n \) and an \((\bar{f})\)-perfect function \( w_{*,2} : [0, \infty) \to \mathbb{R}^n \) such that \( w_{*,i}(0) = z_{*,i}, \ i = 1, 2 \) and 
\[
w_{*,2}([0, \infty)) \subset w_{*,1}([0, \infty)) \cup H(f);
\]

(b) there exist an \((f)\)-perfect function \( w_{*,1} : [0, \infty) \to \mathbb{R}^n \) and an \((\bar{f})\)-perfect function \( w_{*,2} : [0, \infty) \to \mathbb{R}^n \) such that \( w_{*,i}(0) = z_{*,i}, \ i = 1, 2 \) and 
\[
w_{*,1}([0, \infty)) \subset w_{*,2}([0, \infty)) \cup H(f);
\]

(c) cases (a) and (b) do not hold.

By Proposition 6, there exist an \((f)\)-perfect function \( w_{*,1} : [0, \infty) \to \mathbb{R}^n \) and an \((\bar{f})\)-perfect function \( w_{*,2} : [0, \infty) \to \mathbb{R}^n \) such that 
\[
w_{*,i}(0) = z_{*,i}, \ i = 1, 2.
\]

In the case (a) we assume that (15.18) holds and in the case (b) we assume that (15.19) holds. In view of Propositions 4 and 7 and (ATP),
\[
\Omega(w_{*,i}) = H(f), \ i = 1, 2.
\]

Since the function \( w_{*,1} \) is \((f)\)-perfect and the function \( w_{*,2} \) is \((\bar{f})\)-perfect it follows from (3.6), (15.20), Proposition 4, Lemma 10.1 and (ATP) that 
\[
\lim_{T \to \infty} \inf[I^f(0, T, w_{*,1}) - T\mu(f)] = \lim_{T \to \infty} \inf[\pi^f(w_{*,1}(0)) - \pi^f(w_{*,1}(T))] = \pi^f(z_{*,1}),
\]
\[
\lim_{T \to \infty} \inf[I^\bar{f}(0, T, w_{*,2}) - T\mu(f)] = \lim_{T \to \infty} \inf[\pi^\bar{f}(w_{*,2}(0)) - \pi^\bar{f}(w_{*,2}(T))] = \pi^\bar{f}(z_{*,2}).
\]

By Lemma 15.4, Proposition 7 and (ATP), there exists a bounded nonnegative function \( \phi \in C^\infty(\mathbb{R}^n) \) such that the function \( \partial^{|p|}\phi/\partial x_1^{p_1} \ldots \partial x_n^{p_n} : \mathbb{R}^n \to \mathbb{R}^1 \) is bounded, for each sequence of nonnegative integers \( p_1, \ldots, p_n, \) where \( |p| = \sum_{i=1}^n p_i \) and 
\[
\{x \in \mathbb{R}^n : \phi(x) = 0\} = H(f) \cup \bigcup_{i=1}^2 \{w_{*,i}(t) : t \in [0, \infty)\}.
\]

For any \( r \in (0, 1) \) define a function \( f_r : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^1 \) by 
\[
f_r(x, y) = f(x, y) + r\phi(x), \ x, y \in \mathbb{R}^n.
\]
Let $r \in (0, 1)$. Clearly, $f_r \in \hat{M}$. In view of (15.24), $\mu(f_r) \geq \mu(f)$. Since the function $w_{*,1}$ is $(f)$-good it follows from (15.23) and (15.24) that

$$\mu(f_r) \leq \liminf_{T \to \infty} T^{-1} I^f_r(0, T, v) = \liminf_{T \to \infty} T^{-1} I^f(0, T, v) = \mu(f).$$

Thus $\mu(f_r) = \mu(f)$. Together with (15.23), (15.24), Theorem 1.1 and the inclusion $f \in E_0$ this implies that if $u : [0, \infty) \to R^n$ is an $(f_r)$-good function, then it is an $(f)$-good function and $\Omega(u) = H(f)$. Thus $f_r$ has (ATP). Therefore we have shown that

$$\mu(f_r) = \mu(f), \quad f_r \in E_0, \quad H(f_r) = H(f). \tag{15.25}$$

By (15.23) and (15.24), for each $x, y \in R^n$,

$$\pi^f_r(x) \leq \pi^f_r(x), \quad \pi^f_r(y) \leq \pi^f_r(y), \quad \psi_{f,h}(x, y) \leq \psi_{f,h}(x, y). \tag{15.26}$$

It follows from (15.20)-(15.22),

$$\pi^f_r(z_{*,1}) \leq \liminf_{T \to \infty} [I^f_r(0, T, w_{*,1}) - T T \mu(f)] = \liminf_{T \to \infty} [I^f(0, T, w_{*,1}) - T T \mu(f)] = \pi^f_r(z_{*,1}),$$

$$\pi^f_r(z_{*,2}) \leq \liminf_{T \to \infty} [I^f_r(0, T, w_{*,2}) - T T \mu(f)] = \liminf_{T \to \infty} [I^f(0, T, w_{*,2}) - T T \mu(f)] = \pi^f_r(z_{*,2}).$$

Now it is clear that

$$\pi^f_r(z_{*,1}) = \pi^f_r(z_{*,1}), \quad \pi^f_r(z_{*,2}) = \pi^f_r(z_{*,2}), \tag{15.27}$$

$$\psi_{f,h}(z_{*,1}, z_{*,2}) = \psi_{f,h}(z_{*,1}, z_{*,2}). \tag{15.28}$$

In view of (15.17), (15.26), (15.28) and the inclusion $(f, h) \in E_1$, $(z_{*,1}, z_{*,2})$ is a unique minimizer of $\psi_{f,h}$.

Assume that $v_1 : [0, \infty) \to R^n$ is an $(f_r)$-perfect function, $v_2 : [0, \infty) \to R^n$ is an $(f_r)$-perfect function and that

$$\psi_{f,h}(v_1(0), v_2(0)) = \inf(\psi_{f,h}). \tag{15.29}$$

By (15.29),

$$v_1(0) = z_{*,1}, \quad v_2(0) = z_{*,2}. \tag{15.30}$$

We show that $v_i(t) = w_{*,i}(t)$, $i = 1, 2$ for all $t \geq 0$. It is not difficult to see that it is enough to prove that $v_1(t) = w_{*,1}(t)$ for all $t \geq 0$. Proposition 7 implies that

$$\sup\{|w_{*,1}(t)| : t \in [0, \infty)\} < \infty, \quad \sup\{|v_1(t)| : t \in [0, \infty)\} < \infty. \tag{15.31}$$

By (15.23) and (15.24), for all $T > 0$,

$$I^f_r(0, T, w_{*,1}) = I^f(0, T, w_{*,1}) = U^f(0, T, w_{*,1}(0), w_{*,1}(T)) \leq U^f_r(0, T, w_{*,1}(0), w_{*,1}(T)).$$

In view of the relation above, (15.31) and Lemma 10.7, the function $w_{*,1}$ is $(f_r)$-perfect. Analogously, we can show that the function $w_{*,2}$ is $(f_r)$-perfect. Since the function $v_1$ is $(f_r)$-perfect it follows from Lemma 10.1 and (3.6) that

$$\liminf_{T \to \infty} [I^f_r(0, T, v_1) - T T \mu(f)] = \liminf_{T \to \infty} [\pi^f_r(z_{*,1}) - \pi^f_r(v_1(T))] = \pi^f_r(z_{*,1}).$$

By the relation above, (15.24) and (15.27),

$$\pi^f_r(z_{*,1}) = \pi^f_r(z_{*,1}) = \liminf_{T \to \infty} [I^f_r(0, T, v_1) - T T \mu(f)]$$
Lemma 15.11. \( E \) is an everywhere dense subset of \( \mathfrak{M} \times \mathfrak{A} \).
Proof. Let $\mathcal{V}$ be a nonempty open subset of $M \times A$. It is sufficient to show that $\mathcal{V} \cup E \neq \emptyset$. Lemma 15.8 implies that there exists $(f, h) \in E_1 \cap \mathcal{V}$. By Lemma 15.10, $\mathcal{V} \cap E \neq \emptyset$. Lemma 15.11 is proved. \hfill \Box

16. **Proof of Theorems 9.3.** Recall that $E_0$ is the set of all $f \in M$ which has (ATP) and that $E$ is the set of all $(f, h) \in E_0 \times A$ for which there exists a unique pair of an $(f)$-perfect function $w_{f,h} : [0, \infty) \rightarrow R^n$ and an $(f)$-perfect function $w_{f,h}^1 : [0, \infty) \rightarrow R^n$ such that $\psi_{f,h}(w_{f,h}(0), w_{f,h}(0)) = \inf(\psi_{f,h})$.

Let $(f, h) \in E$ and $k \geq 1$ be an integer. By Theorems 9.2 and 9.4, there exist an open neighborhood $U(f, h, k)$ of $(f, h)$ in $A \times A$ and numbers $L(f, h, k) > k$, $\delta(f, h, k) \in (0, k^{-1})$ such that the following properties hold:

(i) for each $(g, \xi) \in U(f, h, k)$, each $T \geq 1$ and each a.c. function $u : [0, T] \rightarrow R^n$ which satisfies

$$I^g(0, T, u) + \xi(u(0), u(T)) \leq \sigma(g, \xi, 0, T) + \delta(f, h, k)$$

we have for all $t \in [0, 1]$, $|u(t) - u_{f, h}(t)| \leq k^{-1}$, $|u(T - t) - u_{f, h}(t)| \leq k^{-1}$;

(ii) for each $(g, \xi) \in U(f, h, k)$ and each pair of a $(g)$-perfect function $w_1 : [0, \infty) \rightarrow R^n$ and a $(g)$-perfect function $w_2 : [0, \infty) \rightarrow R^n$ satisfying

$$\psi_{g, \xi}(w_1(0), w_2(0)) \leq \inf(\psi_{g, \xi}) + \delta(f, h, k)$$

we have $|w_1(t) - w_{f, h}(t)| \leq k^{-1}$, $|w_2(t) - w_{f, h}(t)| \leq k^{-1}$, $t \in [0, 1]$.

Define

$$F = \cap_{n=1}^{\infty} \cup \{U(f, h, k) : (f, h) \in E, \text{ an integer } k \geq p\}.$$  

(16.1)

In view of the construction and Lemma 15.11, $F$ is a countable intersection of open everywhere dense subsets of $M \times A$.

Let

$$(f, h) \in F, \epsilon, \tau_0, M > 0.$$  

(16.2)

Assume that $v_1, w_1 : [0, \infty) \rightarrow R^n$ are $(f)$-perfect functions and $v_2, w_2 : [0, \infty) \rightarrow R^n$ are $(f)$-perfect functions satisfying

$$\psi_{f, h}(v_1(0), v_2(0)) = \psi_{f, h}(w_1(0), w_2(0)) = \inf(\psi_{f, h}).$$  

(16.3)

Let a natural number $p$ satisfy

$$p > \tau_0, M, 2p^{-1} < \epsilon.$$  

(16.4)

By (16.1) and (16.3), for each integer $q \geq p$, there exist $(f_q, h_q) \in E$ and a natural number $k_q \geq q$ such that

$$(f, h) \in U(f_q, h_q, k_q).$$  

(16.5)

By (16.3), (16.5) and the property (ii), for all integers $q \geq p$ and all $t \in [0, q]$, the following inequalities

$$|w_1(t) - w_{f_q, h_q}(t)|, |v_1(t) - w_{f_q, h_q}(t)| \leq k_q^{-1} \leq q^{-1},$$

$$|w_2(t) - w_{f_q, h_q}(t)|, |v_2(t) - w_{f_q, h_q}(t)| \leq k_q^{-1} \leq q^{-1}.$$  

(16.6)

These inequalities imply that for all $t \in [0, q]$, $|w_1(t) - v_1(t)| \leq 2q^{-1}$, $|w_2(t) - v_2(t)| \leq 2q^{-1}$.

(16.7)

for any integer $q \geq p$. Therefore $v_1(t) = w_1(t)$ and $v_2(t) = w_2(t)$ for all $t \in [0, \infty)$ and there exists a unique pair of an $(f)$-perfect function $v_{*,1} : [0, \infty) \rightarrow R^n$ and an $(f)$-perfect function $v_{*,2} : [0, \infty) \rightarrow R^n$ such that $\psi_{f, h}(v_{*,1}(0), v_{*,2}(0)) = \inf(\psi_{f, h})$.

Clearly,

$$v_{i} = w_{i} = v_{*,i}, \ i = 1, 2.$$  

(16.8)
By (16.6)-(16.8),
\[ |v_{*,1}(t) - w_{f_p,h_p}(t)| \leq p^{-1}, \quad |v_{*,2}(t) - w_{f_p,h_p}(t)| \leq p^{-1}, \quad t \in [0,p]. \]  
(16.9)

Let
\[ T_0 = L(f_p,h_p,k_p), \quad \mathcal{U} = \mathcal{U}(f_p,h_p,k_p), \quad \delta = \delta(f_p,h_p,k_p). \]  
(16.10)

Assume that
\[ T \geq T_0, \quad (g, \xi) \in \mathcal{U} \]  
(16.11)
and that an a. c. function \( v : [0,T] \rightarrow \mathbb{R}^n \) satisfies
\[ I^g(0,T,v) + \xi(v(0),v(T)) \leq \sigma(g,\xi,0,T) + \delta(f_p,h_p,k_p). \]  
(16.12)

By (16.10)-((16.12) and the property (i),
\[ |v(t) - w_{f_p,h_p}(t)| \leq k_p^{-1} \leq p^{-1}, \quad |v(T-t) - w_{f_p,h_p}(t)| \leq k_p^{-1} \leq p^{-1} \]  
(16.13)
for all \( t \in [0,k_p] \). By (16.4), (16.9) and (16.13), for all \( t \in [0,p] \),
\[ |v_{*,1}(t) - v(t)| \leq 2p^{-1} < \epsilon, \quad |v(T-t) - v_{*,2}(t)| \leq 2p^{-1} < \epsilon. \]

Theorem 9.3 is proved.

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