Efficient proper embedding of a daisy cube

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Abstract

For a set $X$ of binary words of length $h$ the daisy cube $Q_h(X)$ is defined as the subgraph of the hypercube $Q_h$ induced by the set of all vertices on shortest paths that connect vertices of $X$ with the vertex $0^h$. A vertex in the intersection of all of these paths is a minimal vertex of a daisy cube. A graph $G$ isomorphic to a daisy cube admits several isometric embeddings into a hypercube. We show that an isometric embedding is proper if and only if the label $0^h$ is assigned to a minimal vertex of $G$. This result allows us to devise an algorithm which finds a proper embedding of a graph isomorphic to a daisy cube into a hypercube in linear time.

Keywords: daisy cube, partial cube, isometric embedding, proper embedding, algorithm

1. Introduction

Hypercube is one of the most important interconnection scheme for multicomputers. An obstacle for a direct application of a hypercube is the fact that the number of different hypercubes is very small with respect to the wanted (maximum) number of nodes, that is to say, the number of vertices of a hypercube is always equal to a power of two. For that reason, several other interconnection topologies for multicomputers based on hypercubes have been proposed. These classes of graphs have been devised with the intention of preserving most important properties of a hypercube while allowing more variety of resulting specific graphs. The corresponding families of graphs are mostly various subgraphs of a hypercube, of which its isometric subgraphs, i.e. its induced subgraphs that preserve distances, are of particular importance. A crucial problem in this scope is to find an embedding of a graph of this type to a hypercube (see for example [1, 2, 7]).
Quite recently, a new concept which led to the class of graphs called daisy cubes has been proposed in [3]. It has been shown that daisy cubes are isometric subgraphs of a hypercube, moreover, they include several other important classes of graphs, e.g. Fibonacci and, Lucas cubes. It is known that the cube-complement of a daisy cube is also a daisy cube [6]. It is also proven that a class of graphs, which is of significant importance in chemical graph theory, also belongs to daisy cubes [8].

In [5], daisy cubes are characterized in terms of an expansion procedure. For a given graph $G$ isomorphic to a daisy cube, but without the corresponding embedding into a hypercube, an algorithm which finds a proper embedding of $G$ into a hypercube in $O(mn)$ time is also presented.

Several challenging open problems with respect to daisy cubes have been proposed [3, 5]. In this paper, we focus our study to the following one.

**Problem 1.** *Is there a faster way of finding the vertex $0^h$ of a daisy cube $Q_h(X)$ than the one provided in [5]??*

In [5], it is also noted that a positive answer to Problem 1 would give a linear time algorithm for finding a proper embedding of a graph isomorphic to a daisy cube.

The paper is organized as follows. In the next section some basic definitions, concepts and results needed in the sequel are given. In Section 3, a notion of a minimal vertex of a daisy cube is introduced. Some necessary and sufficient conditions that a minimal vertex has to fulfill are also given. In Section 4, it is shown that an isometric embedding of a graph isomorphic to a daisy cube, but without the corresponding embedding into a hypercube, can be constructed in linear time even if a minimal vertex of a daisy cube is unknown. The last section shows that an isometric embedding devised in the Section 4 can be applied in order to find a proper embedding within the same time bound.

2. Preliminaries

Let $B = \{0, 1\}$. If $b$ is a word of length $h$ over $B$, that is, $b = (b_1, \ldots, b_h) \in B^h$, then we will briefly write $b$ as $b_1 \ldots b_h$. If $x, y \in B^h$, then the Hamming distance $H(x, y)$ between $x$ and $y$ is the number of positions in which $x$ and $y$ differ.

We will use $[n]$ for the set $\{1, 2, \ldots, n\}$. 

The hypercube of order \( h \) or simply \( h \)-cube, denoted by \( Q_h \), is the graph \( G = (V, E) \) where the vertex set \( V(G) \) is the set of all binary strings \( b = b_1 b_2 \ldots b_h, b_i \in \{0, 1\} \) for all \( i \in [h] \), and two vertices \( x, y \in V(G) \) are adjacent in \( Q_h \) if and only if the Hamming distance between \( x \) and \( y \) is equal to one.

For a binary string \( b = b_1 b_2 \ldots b_h \), let \( \overline{b}_i = 1 - b_i \) for \( i \in [h] \). The weight of \( u \in B^h \) is \( w(u) = \sum_{i=1}^{h} u_i \), in other words, \( w(u) \) is the number of 1s in the word \( u \). For the concatenation of bits the power notation will be used, for instance \( 0^h = 0 \ldots 0 \in B^h \).

If \( G \) is a connected graph, then the distance \( d_G(u, v) \) (or simply \( d(u, v) \)) between vertices \( u \) and \( v \) is the length of a shortest \( uv \)-path (that is, a shortest path between \( u \) and \( v \)) in \( G \). The set of vertices lying on all shortest \( u, v \)-paths is called the interval between \( u \) and \( v \) and denoted by \( I_G(u, v) \).

We will also write \( I(u, v) \) when \( G \) will be clear from the context.

If \( G \) is a graph and \( X \subseteq V(G) \), then \( G[X] \) denotes the subgraph of \( G \) induced by \( X \).

If \( u \) is a vertex of a graph \( G \), let \( N(u) \) denote the set of neighbors of \( u \). Moreover, let \( N[u] = N(u) \cup \{u\} \).

Let \( G = (V, E) \) be a graph. A mapping \( \alpha : V(G) \rightarrow V(Q_h) \) is an isometric embedding of \( G \) into \( Q_h \) if \( d_{Q_h}(\alpha(u), \alpha(v)) = d_G(u, v) \) for every \( u, v \in V(G) \). We will denote the \( i \)-th coordinate of \( \alpha \) with \( \alpha_{(i)} \), i.e. \( \alpha = (\alpha_{(1)}, \alpha_{(2)}, \ldots, \alpha_{(h)}) \).

Let \( G \) be a connected graph. The isometric dimension of \( G \) is the smallest integer \( h \) such that \( G \) admits an isometric embedding into \( Q_h \). Isometric subgraphs of hypercubes are called partial cubes.

Let \( \leq \) be a partial order on \( V(Q_h) \) defined with \( u_1 \ldots u_h \leq v_1 \ldots v_h \) if \( u_i \leq v_i \) holds for all \( i \in [h] \). For \( X \subseteq V(Q_h) \) the graph induced by the set \( \{v \in V(Q_h) \mid v \leq x \text{ for some } x \in X\} \) is a daisy cube generated by \( X \) or order \( h \) and denoted by \( Q_h(X) \).

Let also \( \lor \), \( \land \) and \( \oplus \) denote the bitwise OR, bitwise AND and bitwise exclusive OR operator, respectively.

By a slight abuse of definition, we will say that a graph \( G \) is a daisy cube if it is isomorphic to a daisy cube generated by some \( X \subseteq V(Q_h) \). If \( G \) is a daisy cube \( Q_h(X) \), then \( G \) may admit more than one isometric embedding of \( G \) into the \( h \)-cube. Let \( X_G \subseteq B^h \) be the set of labels of the vertices of \( G \) assigned by an isometric embedding \( \alpha \), i.e. \( X_G = \alpha(V(G)) \). We say that \( \alpha \) is a proper embedding of \( G \) if \( G \) is isomorphic to \( Q_h(X_G) \).

Let \( G \) be a graph isomorphic to a daisy cube of order \( h \) and let \( \alpha \) denote its proper embedding. Note that every permutation of indices of \( \alpha \) yields basically the “same” embedding. We say that proper embeddings \( \alpha \) and \( \beta \)
are equivalent if $\beta$ can be obtained from $\alpha$ by a permutation of its indices.

For a daisy cube $Q_h(X)$, let $\hat{X}$ denote the antichain consisting of the maximal elements of the poset $(X, \leq)$. It was shown in [3] that $Q_h(X) = Q_h(\hat{X})$. Hence, for a given set $X \subseteq B_h$ it is enough to consider the antichain $\hat{X}$. The vertices of $Q_h(X)$ from $\hat{X}$ the are called the maximal vertices of $Q_h(X)$. More generally, if $G$ is a daisy cube of dimension $h$ with a proper embedding $\alpha$ such that for exactly one $v \in V(G)$ we have $\alpha(v) = 0^h$, then $X \subseteq V(G)$ is the set of maximal vertices of $G$ with respect to $v$ if $G \cong Q_h(\alpha(X))$ and $\hat{\alpha}(X) = \alpha(X)$. We also say that $v$ is the minimal vertex of $G$ with respect to $\alpha$.

The following result shows that a daisy cube is a subgraph of $Q_h$ induced by the union of intervals between $0^h$ and the vertices from $\hat{X}$.

**Lemma 1.** Let $X \subseteq B_h$. Then $Q_h(X) = Q_h[\bigcup_{x \in \hat{X}} I(0^h, x)]$.

**3. Minimal vertices of a daisy cube**

If $u \in V(Q_h(X))$, then $I(0^n, u)$ induces a $w(u)$-cube in $Q_h(X)$. Note that if $x \in \hat{X}$, then the cube induced by $I(0^n, x)$ is maximal in $Q_h(X)$, i.e., it is not contained in any other cube that belongs to $Q_h(X)$.

If $x \in B_h$, let $S^x$ denote the set of indices of $v$ with $x_i = 1$, i.e., $S^x = \{i \mid x_i = 1, i \in [h]\}$.

Let $v \in B^h$ and let $v_\beta : B^h \to B^h$ be the function defined as

$$v_\beta(i)(u) = \begin{cases} u_i, & v_i = 0 \\ \bar{u}_i, & v_i = 1 \end{cases}$$

**Lemma 2.** Let $G$ be a graph isomorphic to a daisy cube of order $h$ with a proper embedding $\alpha$ such that $\alpha(v^0) = 0^h$ and $\hat{X} \subseteq V(G)$ is its corresponding maximal set. If $v \in \bigcap_{x \in \hat{X}} I(v^0, x)$, then

(i) $\beta$ restricted to $\alpha(V(G))$ is a bijection that maps to $\alpha(V(G))$,

(ii) $\beta \circ \alpha$ is a proper embedding of $G$ with the minimal vertex $v$ and the maximal vertex set $Y = \{y \mid v_\beta(\alpha(y)) = \alpha(x) ; x \in \hat{X}\}$.

**Proof.** (i) We have to show that if $v \in \bigcap_{x \in \hat{X}} I(v^0, x)$, then for every $u \in \alpha(V(G))$ there is exactly one $v_\beta(u) \in \alpha(V(G))$. Note that $\alpha^{-1}(u) \in I(v^0, x)$ and $v \in I(v^0, x)$ for some $x \in \hat{X}$. Thus, $S^u \subseteq S^{\alpha(x)}$ and $S^{\alpha(v)} \subseteq S^{\alpha(x)}$. It follows that $S^{\beta(u)} \subseteq S^{\alpha(x)}$. Since $\alpha$ is proper, $\alpha(V(G)) = \bigcup_{x \in \hat{X}} I(0^h, \alpha(x))$ by Lemma 1 and we obtain $v_\beta(u) \in V(\alpha(G))$. 
In order to see that \( v^b \) is injective, note that \( v^b(v^b(u)) = u \) for every \( u \in \alpha(V(G)) \). Suppose to the contrary that there exist \( u, z \in \alpha(V(G)) \), \( u \neq z \), such that \( v^b(u) = v^b(z) \). It follows that \( v^b(v^b(u)) = v^b(v^b(z)) \) and thus \( u = z \), which yields a contradiction.

(ii) By (i), \( v^b \) maps from \( \alpha(V(G)) \) to \( \alpha(V(G)) \). Let \( x \in \hat{X} \) and recall that \( v^b(\alpha(x)) = \alpha(x) \). Thus, if \( y \in V(G) \) such that \( \alpha(y) = v^b(\alpha(x)) \), we have \( v^b(\alpha(y)) = \alpha(x) \). Moreover, \( v^\beta(v) = 0^h \). It follows that \( Y = \{ y \mid v^b(\alpha(y)) = \alpha(x); x \in \hat{X} \} \) is the maximal vertex set of \( G \) with respect to \( v^b \circ \alpha \), while \( v \) is the corresponding minimal vertex.

![Figure 1: Two proper embeddings of a daisy cube.](image)

Fig. 1 shows two proper embeddings of a daisy cube \( G \). The embedding on the left hand side, say \( \alpha \), admits the set of maximal vertices \( \hat{X} = \{ x, y, z \} \) with labels \( \alpha(x) = 10011 \), \( \alpha(y) = 01011 \) and \( \alpha(z) = 00111 \). Let \( v^0 \in V(G) \) such that \( v^0 = \alpha^{-1}(00000) \). Then \( I(v^0, x) \cap I(v^0, y) \cap I(v^0, z) = \{ v^0, v^1, v^2, v^3 \} \), where \( \alpha(v^3) = 00011 \). The embedding on the right hand side of Fig. 1 is \( v^b \circ \alpha \) with the set of maximal vertices \( Y = \{ x', y', z' \} \), where the corresponding labels are \( \alpha(x') = 10000 \), \( \alpha(y') = 01000 \) and \( \alpha(z') = 00100 \). Note also that \( v^3(\alpha(x')) = 10011 \), \( v^3(\alpha(y')) = 01011 \) and \( v^3(\alpha(z')) = 00111 \).

If \( v \) is a vertex of a partial cube \( G \), then \( N_G^v(u) \) (or simply \( N^v(u) \)) is the set of neighbors of \( u \) which are closer to \( v \) than \( u \), more formally \( N_G^v(u) := \{ z \mid z \in N(u), d(v, z) = d(v, u) - 1 \} \).

Let \( u \in V(G) \) where \( G = Q_h(X) \) and let \( X' \) be the maximal subset of \( \hat{X} \) with the property \( u \in \cap_{x \in X'} I(0^h, x) \). Let \( G^u \) be the graph induced by the set \( \cup_{x \in X'} I(0^h, x) \), i.e. \( G^u = G[\cup_{x \in X'} I(0^h, x)] \). Note that by Lemma 2 and Lemma 1, \( G^u \) is a daisy cube of order \( h \) and \( u \) is its minimal vertex. Observe
for example the graph $Q_4(0111, 1011, 1101, 1110)$ on the right hand side of Fig. 2 if $u = 1100$, then $X' = \{1110, 1101\}$.

As noted in [5], an efficient way of finding a minimal vertex of a daisy cube $G$ would give a linear time algorithm for finding a proper embedding of $G$. It was also shown that if $G$ is a daisy cube of order $h$, then a minimal vertex of $G$ is of degree $h$. It is not difficult to see that a vertex of degree $h$ need not to be a minimal vertex of $G$. Note for example that $Q^{-}_h$ (that is a vertex deleted $Q_h$) admits $2^h - h - 1$ vertices of degree $h$ and exactly one minimal vertex (see also Fig. 2, where $Q^{-}_4$ is depicted).

**Proposition 1.** Let $u \in V(G)$, where $G = Q_h(X)$ and $d(u) = h$. If $G^u = G[\bigcup_{x \in X'} I(0^h, x)]$, then every minimal vertex of $G$ belongs to $\cap_{x \in X} I(0^h, x)$.

**Proof.** Let $v$ be a minimal vertex of $G$. Note that for every $x \in \hat{X}$ and every $u \in I(0^h, x)$ we have $d(v, u) \leq |S^x|$. Suppose to the contrary that $v \notin \cap_{x \in X'} I(0^h, x)$. It follows that there exists $x \in X'$ such that $v \notin I(0^h, x)$. If $u \in I(0^h, x)$, then $S^u \subseteq S^x$. Moreover, since $v \notin I(0^h, x)$, there exists an index $j \notin S^x$ such that $v_j = 1$. It follows that the string $u$ defined by

$$u_i = \begin{cases} 
\bar{v}_i, & i \in S^x \\
0, & \text{otherwise}
\end{cases}$$

is a vertex of $I(0^h, x)$ with $d(v, u) > |S^x|$ and we obtain a contradiction. $\square$

**Theorem 1.** If $G = Q_h(X)$ and $\hat{x} = \land_{x \in \hat{X}} x$, then $v$ is a minimal vertex of $G$ if and only if $v \in \cap_{x \in \hat{X}} I(0^h, x) = I(0^h, \hat{x})$.

**Proof.** By Lemma 2 and Proposition 1, $v$ is a minimal vertex of $G$, if and only if $v \in \cap_{x \in \hat{X}} I(0^h, x)$. Note that $v \in \cap_{x \in \hat{X}} I(0^h, x)$ if and only if $S^v \subseteq \cap_{x \in \hat{X}} S^x$. Since $S^{\hat{x}} = \cap_{x \in X} S^x$, for every $v \in V(G)$ we have $v \in \cap_{x \in \hat{X}} I(0^h, x)$ if and only if $v \leq \hat{x}$. It follows that $\cap_{x \in \hat{X}} I(0^h, x) = I(0^h, \hat{x})$ and the assertion follows. $\square$

### 4. Isometric embedding

If $G$ is a graph isomorphic to a hypercube (but without an embedding), then its isometric embedding is easy to obtain as shown in the next result.
Proposition 2. Let $G$ be a graph isomorphic to a $h$-cube, $v$ an arbitrary vertex of $G$ and $\alpha : V(G) \to V(Q_h)$ a function such that $\alpha(v) = 0^h$, the vertices of $N(v)$ obtain pairwise different labels of the form $0^{i-1}10^{h-i}$, $i \in [h]$, while for any other vertex $u \in V(G)$ we set $\alpha(u) = \bigvee_{z \in N^v(u)} \alpha(z)$. Then $\alpha$ is an isometric embedding of $G$ into $Q_h$. Moreover, when a labeling of vertices in $N[v]$ is chosen, $\alpha$ is unique.

Proof. Since a hypercube is vertex-transitive, we may choose an arbitrary vertex $v$ of $G$ and set $\alpha(v) = 0^h$. Moreover, for every $u \in V(G)$ with $d(v,u) = i$, $i \geq 1$, we must have $N^v(u) = \{z \mid \alpha(i)(z) = \alpha(i)(u) = 1$ for exactly one $i \in [h]$ and $\alpha(j)(z) = \alpha(j)(u)$ for every $j \in [h] \setminus \{i\}$. Thus, $\alpha(u) = \bigvee_{z \in N^v(u)} \alpha(z)$. Thus, for chosen labeling of vertices in $N[v]$, $\alpha$ is unique. 

Lemma 3. Let $G$ be partial cube of isometric dimension $h$, $u$ a vertex of degree $h$ in $G$ and let for every $v \in V(G) \setminus N[u]$ it holds that $|N^u(v)| \geq 2$. Then we may define a function $\alpha : V(G) \to V(Q_h)$ such that $\alpha(u) = 0^h$, the vertices of $N(u)$ obtain pairwise different labels of the form $0^{i-1}10^{h-i}$, $i \in [h]$, while for any other vertex $v \in V(G)$ we set $\alpha(v) = \bigvee_{z \in N^u(v)} \alpha(z)$. Moreover,

(i) $\alpha$ is an isometric embedding of $G$ into $Q_h$,

(ii) when a fixed embedding of vertices in $N[v]$ is chosen, $\alpha$ is unique.

Proof. Since $G$ is a partial cube of dimension $h$, we may assume that $G$ is an isometric subgraph of an (unlabeled) $h$-cube $H$. Let $\beta$ be an embedding of $H$ with respect to $v$ as defined in Proposition 2 and let $\alpha$ be an embedding of $G$ such that for every $z \in N[u]$ we set $\alpha(z) = \beta(z)$. Since $|N^u_G(v)| \geq 2$ and $N^v_G(v) \subseteq N^v_H(v)$ for every $v \in V(G) \setminus N[u]$, it follows that $\alpha(v) = \beta(v)$ for every vertex $v \in V(G)$. By Proposition 2 $\beta$ is an isometric embedding of $H$ into $Q_h$. Thus, $\alpha$ is an isometric embedding of $H$ into $Q_h$. Moreover, by Proposition 2 $\alpha$ is unique for a fixed embedding of vertices in $N[v]$.

Corollary 1. Let $G$ be a graph isomorphic to a daisy cube of order $h$. If $v$ is a minimal vertex of $G$ and $\alpha$ an isometric embedding with $\alpha(v) = 0^h$, then $\alpha$ is proper.

Proof. Since $v$ is a minimal vertex of $G$, there exist a proper embedding, say $\beta$, such that $\beta(v) = 0^h$. We may also assume w.l.o.g. that for every $u \in N(v)$ we have $\beta(u) = \alpha(u)$. From Lemma 3 then it follows that $\beta(u) = \alpha(u)$ for every $v \in V(G)$.
Remark
If $G$ is isomorphic to a daisy cube and $\alpha$ a proper embedding of $G$, then different selections of labels for vertices of $N(u)$ yield different but equivalent proper embeddings.

If $G$ is a partial cube and $\alpha$ its isometric embedding to $Q_h$, let $W_i(G)$ denote the set of vertices of $G$ with weight $i$, i.e. $W_i(G) = \{v \mid w(\alpha(v)) = i\}$.

We will also need the following result.

**Proposition 3.** If $G$ is a partial cube, $\alpha$ its isometric embedding to $Q_h$ and $v \in V(G)$ such that $w(\alpha(v)) = i$, then $|N(v) \cap W_{i-1}(G)| \leq i$.

**Proof.** Since $\alpha$ is isometric embedding of $G$ to $Q_h$, for every $v \in V(G)$ with $w(\alpha(v)) = i$, we have $N_G(v) \subseteq N_{Q_i}(v)$. Moreover, $|N(v) \cap W_{i-1}(Q_h)| = i$ and therefore $|N(v) \cap W_{i-1}(G)| \leq i$. □

**Proposition 4.** Let $G = Q_h(X)$, $x, y \in \hat{X}$ and $x \neq y$. If $u \in I(0^h, x)$ and $v \in I(0^n, y)$ such that $u, v \notin I(0^n, x) \cap I(0^h, y)$ then $uv \notin E(G)$.

**Proof.** Suppose to the contrary that there exist $u \in I(0^h, x)$ and $v \in I(0^h, y)$ such that $u, v \notin I(0^h, x) \cap I(0^h, y)$ and $d(u, v) = 1$. Since $\hat{X}$ is maximal, there exist at least two indices $i, j \in [h]$, such that $x_i \neq y_i$ and $x_j \neq y_j$ (otherwise we have either $x \leq y$ or $y \leq x$). Suppose w.l.o.g. $x_i = 1, y_j = 1$ and $u_k = v_k$ for every $k \neq i, j$. If $u_i = 0$ (resp. $v_j = 0$), then $u \in I(0^h, y)$ (resp. $v \in I(0^h, x)$). It follows that $u_i = v_j = 1$. But then $u = v$ and we obtain a contradiction. □

**Proposition 5.** Let $G = Q_h(X)$. If $u \in V(G)$ and $d(u) = h$, then $N(u) \subseteq G^u$.

**Proof.** Let $G^u = G[\cup_{x \in X'}I(0^h, x)]$. Suppose to the contrary that there exists $v \in N(u)$ such that $v \notin G[\cup_{x \in X'}I(0^h, x)]$. It follows that there exists $y \in \hat{X} - X'$ such that $v \in I(0^h, y)$. Since $u \in I(0^h, x)$ for some $x \in \hat{X}$ and $x \neq y$, Proposition 4 yields a contradiction. □

**Proposition 6.** Let $G = Q_h(X)$ and $u \in V(G)$. If $d(u) = h$ and $G^u = G[\cup_{x \in X'}I(0^h, x)]$, then $|\cup_{x \in X'}S^x| = h$.

**Proof.** Suppose $|\cup_{x \in X'}S^x| < h$. It follows that there exist $j \in [h]$ such that for all $v \in \cup_{x \in X'}I(0^h, x)$ we have $v_j = 0$. Since $d(u) = h$, there exists $z \in N(u)$ such that $z_j = 1$. It follows that $z \notin \cup_{x \in X'}I(0^h, x)$. Thus, there exists $y \in \hat{X} - X'$ such that $v \in I(0^h, y)$. Proposition 5 yields a contradiction. □
Lemma 4. Let $G = Q_h(X)$ and $u \in V(G)$ such that $d(u) = h$. Then $|N^u(v)| \geq 2$ for every $v \in V(G) \setminus N[u]$.

Proof. Let $G^u = G[\cup_{x \in X'} I(0^h, x)]$. By Lemma 2 and Lemma 1, $G^u$ is a daisy cube and $u$ its minimal vertex. It follows that the lemma holds for every $v \in V(G^u)$. Suppose then that $v \not\in \cup_{x \in X'} I(0^h, x)$. Thus, there exists $y \in \hat{X} - X'$, such that $v \in I(0^h, y)$. Note that $S^u \subseteq \cap_{x \in X'} S^x$.

Let $S^u_+ = \{i \mid u_i = 1, v_i = 0\}$ and $S^u_- = \{i \mid v_i = 1, u_i = 0\}$.

We first show that $|S^u_-| \geq 2$. Suppose to the contrary that $|S^u_-| = 1$, i.e., there exist exactly one index $i \in [h] \setminus S^u_+$, such that $v_i = 1$ and $u_i = 0$. Since $d(u) = h$, by Proposition 6, there exists $x \in X'$ such that $x_i = 1$. Note also that $S^u \subseteq S^x$ and since $x_i = 1$, we have $S^v \subseteq S^x$. It follows that $v \leq x$ and we obtain a contradiction.

If $|S^u_+| = 0$, then vertices of $I(u, v)$ induces a $|S^u_-|$-cube in $G$. Thus, $v$ admits $|S^u_-|$ neighbors at distance $d(u, v) - 1$ from $u$. Since $|S^u_-| \geq 2$, the case is settled.

If $|S^u_+| > 0$, we may find $i, j \in S^u_-$ such that $i \neq j$. Let $z$ and $z'$ be vertices obtained from $v$ by setting the $i$-th and $j$-th coordinate to zero, respectively. Obviously, $z, z' \in N^u(v)$. This assertion completes the proof. \hfill \qed

Figure 2: An isometric (left) and proper embeddings of a daisy cube isomorphic to $Q_4$. 

Lemma 4 is the basis for the next algorithm which finds an isometric embedding for an unlabeled graph isomorphic to a daisy cube of dimension $h$. 

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**Procedure** Embedding($G$, $h$, $\beta$, $u$);

1. $u$ is a vertex of degree $h$ in $G$;
2. $\beta(u) := 0^h$;
3. $i := 0$;
4. $Q := \emptyset$; \{$Q$ is an empty queue}\)
5. **for all** $v \in V(G)$ **do** $p(v) := 0$;
6. **for all** $v \in N(u)$ **do begin**
   $\beta(v) := 0^{i-1}10^{h-i}$;
   $i := i + 1$;
   $p(v) := u$;
   Insert $v$ in the end of $Q$;
**end**;
7. **while** $Q \neq \emptyset$ **do begin**
   7.1 Remove the first vertex $v$ from $Q$;
   7.2. **for all** $z \in N(v)$ **do**
      **if** $p(z) = 0$ **then begin**
      $p(z) := v$;
      Append $z$ to the end of $Q$;
      **end**
      **else** $\beta(z) := \beta(v) \lor \beta(p(z))$;
**end**.

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**Theorem 2.** If $G$ is a daisy cube, that the isometric embedding of $G$ can be found in linear time.

**Proof.** Note first that Lemma 3 defines the procedure to construct an isometric embedding of $G$ into $Q_h$. Let $\alpha$ and $\beta$ be isometric embeddings as defined in Lemma 3 and algorithm Embedding, respectively. Suppose that $u$ is the vertex being labeled $0^h$ both by the algorithm and by the construction of Lemma 3. Clearly, for every $v$ in $N[u]$ we could have $\alpha(v) = \beta(v)$. Note also that in the essence the algorithm performs a BFS search in $G$ (see for example [2, Section 17.3]). Thus, for every $z \in N(v)$ of Step 7.2 we have $d(u, z) = d(u, p(z)) + 1 = d(u, v) + 1$. It follows that $v, p(z) \in N^u(z)$. By Lemma 4 since $d(u) = h$, for every $v \in V(G) \setminus N[u]$ we have $|N^u_v(v)| \geq 2$. Therefore, $\alpha(z) = \beta(z)$ for every $z \in V(G) \setminus N[u]$.

For the time complexity of the algorithm, note that the number of the executions of the body of the loop in Step 7.2 is bounded by the number of edges of a graph. Since the time complexity of the body of the loop is
constant, the overall number of steps of the algorithm is linear in the number of the edges of the graph.

\[ \square \]

5. Proper embedding

**Lemma 5.** Let \( G \) be a daisy cube of order \( h \), \( v \) its minimal vertex and \( u \) a vertex of degree \( h \) of \( G \). If \( \beta \) is an isometric embedding of \( G \) such that \( \beta(u) = 0^h \), then \( \beta \circ \beta \) is a proper embedding of \( G \).

**Proof.** Note that \( v = \beta(u) \). Since \( \beta \) is isometric, it is easy to see that \( \beta \circ \beta \) is also isometric. Corollary 1 now yields the assertion. \[ \square \]

Let \( u \) be a vertex of degree \( h \) of \( G = Q_h(X) \). Recall that \( G^u \) is a daisy cube of order \( h \) and \( u \) its minimal vertex. If \( \beta \) is an isometric embedding of \( G \) such that \( \beta(u) = 0^h \), let \( Y^u \) be the set of maximal vertices of \( G^u \) with respect to \( u \). If \( z \in Z^u \) is the set of vertices \( z \) of \( V(G) \setminus V(G^u) \) with the property \( N^u(z) = N(z) \).

**Proposition 7.** Let \( u \) be a vertex of degree \( h \) of \( G = Q_h(X) \) and \( \beta \) is an isometric embedding of \( G \) such that \( \beta(u) = 0^h \). Then \( Y^u = \{ y \mid \beta(y) = x, x \in X' \} \).

**Proof.** As noted above, \( G^u \) is a daisy cube of order \( h \) and \( u \) its minimal vertex. Since \( u \) is of degree \( h \) and \( \beta(u) = 0^h \), the restriction of \( \beta \) to \( V(G^u) \) is a proper embedding of \( G^u \). Moreover, since every permutation of indices of a proper embedding yields an equivalent embedding, we may assume w.l.o.g. that for every \( z \in N(u) \) we have \( \beta(z) = 0^{i-1}10^{h-i} \) if and only if \( u_i \neq z_i \). It follows that for every \( w \in N(0^h) \) we have \( \beta(\beta(w)) = w \). By Lemma 2, \( \beta \circ \beta \) is proper. Moreover, by Lemma 3, \( \beta(\beta(v)) = v \) for every \( v \in V(G^u) \). From Lemma 2 then follows that \( Y^u = \{ y \mid \beta(y) = x, x \in X' \} \). \[ \square \]

**Proposition 8.** Let \( u \) be a vertex of degree \( h \) of \( G = Q_h(X) \), \( G^u = G[\cup_{x \in X'} I(0^h, x)] \) and \( z \in Z^u \). If \( \beta \) is an isometric embedding of \( G \) and \( \beta(u) = 0^h \), then there exists \( y \in X' \) such that \( z \in I(0^h, y) \). Moreover, \[ \beta(i)(z) = \begin{cases} 0, & i \in S^u \\ y_h, & i \notin S^u \end{cases} \]

**Proof.** Let \( G^u = G[\cup_{x \in X'} I(0^h, x)] \). By Lemma 1, there must be \( y \in X' \) such that \( z \in I(0^h, y) \). By \( N^u(z) = N(z) \), we
have \( d(u, z) \geq d(u, v) \) for every \( v \in I(0^h, y) \). If \( v_i = 1 \) for some \( i \in S^u \), then let \( v' \) be the vertex of \( G \) such that \( v'_j = v_j \) for every \( j \neq i \) and \( v'_i = 0 \).

Obviously, \( v' \leq y \), thus \( v' \in I(0^h, y) \). Moreover, since \( \beta_{(i)}(v') = 1 \), we have \( d(u, v') > d(u, v) \) and we obtain a contradiction. It follows that the assertion holds for every \( i \in S^u \). If \( i \notin S^u \), then \( \beta_{(i)}(v) = v_i \) for every \( v \in I(0^h, y) \).

Since \( y \) is maximal in \( I(0^h, y) \), the assertion follows. \( \square \)

**Theorem 3.** Let \( u \) be a vertex of degree \( h \) of \( G = Q_h(X) \). If \( \beta \) is an isometric embedding of \( G \) such that \( \beta(u) = 0^h \), \( \hat{y} = \wedge_{y \in Y^u} \beta(y) \) and \( \hat{z} = \wedge_{z \in Z^u} \beta(z) \wedge 1^h \), then \( \beta^{-1}(\hat{y} \wedge \hat{z}) \) is a minimal vertex of \( G \).

**Proof.** Note first that \( \beta = \beta^{-1} \), thus, for every \( b \in B^h \) and every \( i \in [h] \) it holds

\[
\beta_{(i)}(b) = \beta_{(i)}^{-1}(b) = \begin{cases} b_i, & i \in S^u \\ b_i, & i \notin S^u \end{cases} \tag{1}
\]

Let \( \hat{x} = \wedge_{x \in X'} x \). By Proposition [7], we have \( Y^u = \{ y \mid \beta(y) = x, x \in X' \} \).

Thus, \( \hat{x} = \hat{y} \). Note that by Proposition [11] every minimal vertex of \( G \) belongs to \( I(0^h, \hat{x}) \).

If \( X' = \bar{X} \), then \( Z^u = \emptyset \) and we get \( \beta^{-1}(\hat{y} \wedge \hat{z}) = \beta^{-1}(\hat{y}) = \beta^{-1}(\hat{x}) \). By [11], we have \( \beta^{-1}(\hat{x}) \leq x \). It follow that \( \beta^{-1}(\hat{x}) \in I(0^h, \hat{x}) \) and we are done.

Otherwise, let \( z \in Z^u \) such that \( z \in I(0^h, y) \) for some \( y \in \bar{X} - X' \). We have to show that \( \beta^{-1}(\hat{x} \wedge \beta(z)) \) is a minimal vertex of \( \cup_{x \in X} I(0^h, x) \cup I(0^h, y) \), i.e. \( S^{\beta^{-1}(\hat{x} \wedge \beta(z))} \subseteq S^{\hat{x} \wedge y} \).

By Proposition [8] we have

\[
\beta_{(i)}(z) = \begin{cases} 0, & i \in S^u \\ y_i, & i \notin S^u \end{cases}
\]

Since \( S^u \subseteq S^{\hat{x}} \), we have

\[
(\hat{x} \wedge \beta(z))_i = \begin{cases} y_i, & i \notin S^{\hat{x}} \setminus S^u \\ 0, & \text{otherwise} \end{cases}
\]

By [11], we have \( \beta^{-1}(\hat{x} \wedge \beta(z)) = 0 \) for every \( i \in [h] \setminus S^{\hat{x} \wedge y} \). Since we can repeat the above discussion for every \( z \in Z^u \), the proof is complete. \( \square \)

Fig. [2] shows two embeddings of a daisy cube \( G \) isomorphic to \( Q_4^- \). The embedding \( \beta \) on the left hand side is determined such that \( \beta(u) = 0000 \) (note that \( d(u) = 4 \)). Since \( u \) is not minimal in \( G \), the embedding \( \beta \) is
isometric but not proper. From $X' = Y^u = \{x, y\}$ and $Z^u = \{z\}$ we get $\hat{y} = 1110 \land 1101 = 1100$, $\hat{z} = 1111$ and $\hat{y} \land \hat{z} = 1100 \land 1111 = 1100$. Moreover, the minimal vertex of $G$ is $v = \beta^{-1}(1100)$ and $\beta \circ \beta$ is the proper embedding of $G$ as described in Lemma 5. That is to say, we obtain the proper embedding of $G$ by assigning $\beta(w) \oplus 1100$ to every $w \in V(G)$.

Theorem 3 is the basis for the next algorithm, which finds a proper embedding of a graph isomorphic to a daisy cube of dimension $h$.

**Procedure** Proper($G$, $h$, $\alpha$);

1. Embedding($G$, $h$, $\beta$, $u$);
2. for $i := 1$ to $h + 1$ do $W_i := \emptyset$;
3. for all $v \in V(G)$ do $W_{w(\beta(v))} := W_{w(\beta(v))} \cup \{v\}$;
4. for all $v \in V(G)$ do $q(v) := 0$;
5. for $i := 1$ to $h$ do begin
   5.1. for all $x \in W_i$ do
      5.1.1 if $\sum_{y \in N(x) \cap W_{i-1}} q(y) = i(i - 1)$ then begin
         $q(x) := i$;
         for all $y \in N(x) \cap W_{i-1}$ do $q(y) := 0$;
      end
      5.1.2 else if $N(x) \cap W_{i+1} = \emptyset$ then $q(x) := i$;
6. $s := 1^h$;
7. for all $v \in V(G)$ do
   7.1. if $q(v) \neq 0$ then $s := s \land \beta(v)$;
8. for all $v \in V(G)$ do $\alpha(v) := s \oplus \beta(v)$;
end.

**Theorem 4.** A proper embedding of an unlabeled graph isomorphic to a daisy cube can be found in linear time.

**Proof.** We first show that the algorithm Proper finds a proper embedding of $G$. As shown in Theorem 2, embedding $\beta$ provided by the algorithm Embedding is isometric. With respect to Theorem 3 and Step 7, we have to show that if $q(v) \neq 0$, then either $v \in Y^u$ or $v \in Z^u$. Clearly, in Step 3, all vertices at distance $i$ from $u$ are inserted in $W_i$, while in Step 4, $q(v)$ is set to 0 for every $v \in V(G)$. The value of $q(v)$ is altered either in Step 5.1.1 or in Step 5.1.2.

Let $w(x) = i$. We show that $q(x) = i$ in the $i$-th iteration of for loop if and only if either $I(u, x)$ induces an $i$-cube or $x \in Z^u$. Note that $I(u, x)$ induces an $i$-cube, if and only if $|N(x) \cap W_{i-1}| = i$ and for every $y \in N(x) \cap W_{i-1}$ the
set $I(u, y)$ induces a $(i - 1)$-cube. Moreover, if $x \in Y_u$, then $I(u, x)$ induces a maximal $i$-cube in $G^u$.

In the first iteration of Step 5, for every vertex of $W_1$ the value of $q$ is set to 1. In the next iteration, when a vertex $x$ of $W_2$ is considered, these values for two vertices of $W_1$, say $y$ and $y'$, are set to zero if \{u, y, y', x\} induce a 2-cube. Thus, for every $x, y \in W_1 \cup W_2$ we have

- $q(y) = 1$ if and only if $x \in N(u)$ and there is no vertex $y \in W_2$ such that $I(u, y) \subseteq I(u, x)$ and $I(u, x)$ induces $Q_2$.

- $q(x) = 2$ if and only if $I(u, x)$ induces $Q_2$.

Suppose now that for $i \geq 3$ and $y \in W_{i-1}$ it holds that $q(y) = i - 1$ if and only if $I(u, y)$ induces a maximal cube in $G[W_1 \cup W_2 \ldots \cup W_{i-1}]$ or $N^u(y) = N(y)$; otherwise, $q(y) = 0$. Let $w(x) = i$. Note that $|N(x) \cap W_{i-1}| \leq i$ by Proposition $\exists$. Thus, the condition of the if statement in Step 5.1.1 is fulfilled if and only if for every $y \in N(x) \cap W_{i-1}$ we have $q(y) = i - 1$, i.e. for every $y \in N(x) \cap W_{i-1}$ the set $I(u, y)$ induces an $(i - 1)$-cube. If the condition of the if statement returns true, then $q(x)$ obtains the value $i$ while for every $y \in N(x) \cap W_{i-1}$ the value of $q(y)$ is set to 0. If the condition of the if statement returns false, then $q(x)$ is set to $i$ if and only if $N(x) \cap W_{i+1} = \emptyset$, i.e. $x \in Z^u$. Thus, we showed that in the $i$-th iteration of the for loop $q(x) = i$ if and only if either $I(u, x)$ induces an $i$-cube or $x \in Z^u$. Since the claim holds for every $i$, we showed that if $q(v) \neq 0$, $v \in V(G)$, then either $v \in Y^u$ or $v \in Z^u$. From Theorem $\exists$ then it follows that the string $s$ computed in Step 7 is equal to $\hat{y} \land \hat{z}$, where $\hat{y} = \land_{x \in Y^u} \beta(y)$ and $\hat{z} = \land_{x \in Z^u} \beta(z)$. By Theorem $\exists$ $\beta^{-1}(s) = v$ is a minimal vertex of $G$ while the embedding $\alpha$ obtained in Step 8 is equal to $\nu \beta \circ \beta$. Moreover, $\alpha$ is proper by Lemma $\exists$.

In order to consider the time complexity of the algorithm, note first that all steps of the algorithm except Step 5 can be executed in $O(m)$ time, where $m$ is the number of edges of $G$. For the time complexity of Step 5 it is convenient to store the weights of vertices in a vector, which allows that the weight of a vertex and therefore its inclusion in a set $W_i$ can be determined in constant time. Thus, the time complexity of Steps 5.1.1 and 5.1.2 is linear in the number of edges incident with the vertex $x$. Since Step 5 is performed for every vertex of the graph, the total number of steps is bounded by the number of edges of $G$. This assertion concludes the proof. $\square$
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