On inhomogeneity parameters for Backus average

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Abstract

In this paper, we discuss five parameters that indicate the inhomogeneity of a stack of parallel isotropic layers. We show that, in certain situations, they provide further insight into the intrinsic inhomogeneity of a Backus medium, as compared to the Thomsen parameters. Additionally, we show that the Backus average of isotropic layers is isotropic if and only if $\gamma = 0$. This is in contrast to parameters $\delta$ and $\epsilon$, whose zero values do not imply isotropy.

1 Introduction

In this paper, we consider an inhomogeneous stack of thin, isotropic and parallel layers. We examine several parameters, which, by using the Backus (1962) and Voigt (1910) averages, indicate the strength of inhomogeneity. Using the former, we consider an inhomogeneous stack of thin isotropic layers, as a homogeneous transversely isotropic medium. In other words, the Backus average is a homogenization of inhomogeneity. The Voigt average represents an anisotropic medium, as the closest—in a Frobenius sense—isotropic counterpart. Among the parameters that we consider, we include the Thomsen (1986) parameter $\gamma$. In addition to indicating anisotropy of the resulting transversely isotropic medium, $\gamma$ shows the inhomogeneity of the stack of layers. Specifically, we emphasize two parameters that refer to different methods of homogenization of isotropic layers to their isotropic counterparts.

2 Background

2.1 Backus and Voigt averages

According to Backus (1962), a sequence of thin parallel isotropic layers can be considered as a transversely isotropic medium. One of the few restrictions imposed by Backus (1962) is that of long wavelengths and fine layering. The following elasticity parameters constitute the elasticity tensor that characterizes the medium resulting from the averaging process.

$$
\begin{align*}
\mathbb{T} c_{1111}^{\text{TI}} &= \left( \frac{c_{1111} - 2c_{2323}}{c_{1111}} \right)^2 \left( \frac{1}{c_{1111}} \right)^{-1} + \left( \frac{4(c_{1111} - c_{2323})c_{2323}}{c_{1111}} \right), \\
\mathbb{T} c_{1133}^{\text{TI}} &= \left( \frac{c_{1111} - 2c_{2323}}{c_{1111}} \right) \left( \frac{1}{c_{1111}} \right)^{-1}, \\
\mathbb{T} c_{1212}^{\text{TI}} &= c_{2323},
\end{align*}
$$

(1)

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\[
c_{2323} = \left( \frac{1}{c_{2323}} \right)^{-1},
\]
\[
c_{3333} = \left( \frac{1}{c_{1111}} \right)^{-1},
\]
as shown by Backus (1962) and discussed by Slawinski (2016, Section 4.2).

The Voigt (1910) average results in an isotropic medium; its parameters are
\[
c_{1111}^{\text{iso}} = \frac{1}{9} \left( 2(c_{1111} + c_{2222} + c_{3333} + c_{1212} + c_{1313} + c_{2323}) + c_{1122} + c_{1133} + c_{2233} \right)
\]
and
\[
c_{2323}^{\text{iso}} = \frac{1}{18} \left( c_{1111} + c_{2222} + c_{3333} + 4(c_{1212} + c_{1313} + c_{2323}) - (c_{1122} + c_{1133} + c_{2233}) \right),
\]
where \(c_{ijk}\) refers to a generally anisotropic tensor. We use Frobenius-21 norm, \(F_{21}\), because, according to Danek et al. (2015), it lends itself to statistical analysis more easily than Frobenius-36 norm, \(F_{36}\). Parameters \(c_{1111}^{\text{iso}}\) and \(c_{2323}^{\text{iso}}\) represent the closest isotropic tensor—in the \(F_{21}\) sense—to the anisotropic one.

For a transversely isotropic tensor that results from the Backus (1962) average, the Voigt average results in
\[
c_{1111}^{\text{iso}} = \frac{1}{9} \left( 5c_{1111} + 2c_{1133} + 4c_{2323} + 2c_{3333} \right),
\]
and
\[
c_{2323}^{\text{iso}} = \frac{1}{18} \left( c_{1111} - 2c_{1133} + 6c_{1212} + 8c_{2323} + c_{3333} \right).
\]
Herein, expressions (3) and (4) represent an isotropic counterpart to a stack of layers. To distinguish \(c_{1111}^{\text{iso}}\) and \(c_{2323}^{\text{iso}}\) from parameters obtained by arithmetic averaging, which are denoted below by \(c_{1111}\) and \(c_{2323}\), we let
\[
c_{1111}^{\text{iso}} =: c_{1111}^{\text{BV}}
\]
and
\[
c_{2323}^{\text{iso}} =: c_{2323}^{\text{BV}},
\]
where \(\text{BV}\) denotes the Backus-Voigt homogeneization process. In Section 3.1, we use \(c_{1111}^{\text{BV}}\) and \(c_{2323}^{\text{BV}}\) to define parameters measuring inhomogeneity of a stack of layers.

### 2.2 Thomsen parameters

To examine the strength of anisotropy of a transversely isotropic medium, we invoke Thomsen (1986) parameters
\[
\gamma = \frac{c_{1212} - c_{2323}}{2c_{2323}},
\]
\[
\delta = \frac{\left( c_{1133} + c_{2323} \right)^2 - \left( c_{3333} - c_{2323} \right)^2}{2c_{3333} \left( c_{3333} - c_{2323} \right)},
\]
\[
\epsilon = \frac{c_{1111} - c_{3333}}{2c_{3333}}.
\]
As shown by Adamus et al. (2018), by increasing their values, these parameters indicate an increase of inhomogeneity of a stack of isotropic layers.
2.3 Stability conditions

Stability conditions (e.g., Slawinski, 2015, Section 4.3) originate from the necessity of expending energy to deform a material. This necessity is mathematically expressed by the positive definiteness of the elasticity tensor. In general, a tensor is positive definite if and only if all its eigenvalues are positive. For an isotropic elasticity tensor, this entails that

\[ c_{1111} > \frac{4}{3} c_{2323}. \]  

(6)

According to Backus (1962), a medium obtained by Backus averaging is positive definite if the layers, prior to averaging, are also positive definite. Also, according to Gazis et al. (1963), a Frobenius-norm counterpart of a positive-definite tensor is positive definite. Thus, it suffices to ensure condition (6) for each layer.

3 Parameters indicating inhomogeneity

3.1 Inhomogeneity parameters for Backus average

In this paper, we consider five parameters that measure the inhomogeneity of a stack of isotropic layers. To obtain them, we use the averaging processes and expressions stated in Section 2.1. The Backus average allows us to relate wellbore information to seismic data.

As stated by Backus (1962), isotropic layers whose \( c_{2323} \) is constant result in an isotropic Backus medium. To examine the inhomogeneity of such layers, we introduce

\[ \mathcal{I} := \frac{c_{1111} - c_{TT}^{1111}}{2c_{TT}^{3333}} \]  

(7)

and

\[ \mathcal{I}_{BV} := \frac{c_{1111} - c_{BV}^{1111}}{2c_{BV}^{1111}}. \]

Equation (7) relates the elasticity parameters of the layers to those of a transversely isotropic medium resulting from the Backus average. For an isotropic medium, \( c_{TT}^{3333} = c_{TI}^{1111} \). Thus, \( \mathcal{I} \) indicates only the differences among \( c_{1111} \) within the stack of layers, as compared to \( \mathcal{I}_{BV} \), which provides more complex information about inhomogeneity, since \( c_{1111} \) depends on both \( c_{1111} \) and \( c_{2323} \). \( \mathcal{I}_{BV} \) shows the difference between two methods of homogeneization of an inhomogeneous stack of isotropic layers to its isotropic counterpart. In the inverse problem—where we only know Backus parameters provided by seismic information—\( \mathcal{I} \) and \( \mathcal{I}_{BV} \) cannot be used.

Another two parameters to measure inhomogeneity are

\[ \gamma = \frac{c_{2323} - c_{TT}^{2323}}{2c_{TT}^{2323}} = \frac{c_{1212} - c_{TT}^{2323}}{2c_{TT}^{2323}} \]

and

\[ \gamma_{BV} := \frac{c_{2323} - c_{BV}^{2323}}{2c_{BV}^{2323}} = \frac{c_{1212} - c_{BV}^{2323}}{2c_{BV}^{2323}}, \]

where \( \gamma \) is parameter (5). As shown in Theorem A.1 in Appendix A, the Backus average of isotropic layers is isotropic if and only if \( \gamma = 0 \), in contrast to parameters \( \delta \) and \( \epsilon \), whose zero values do not imply isotropy. Thus, in this paper, we do not use \( \delta \) and \( \epsilon \).
Parameter $\gamma_{BV}$—in comparison to $\gamma$—gives different information about inhomogeneity of a stack, since $c^{BV}_{2323}$ depends on both parameters $c_{1111}$ and $c_{2323}$, not only on $c_{2323}$. Also, $I_{BV}$ distinguishes between two methods of homogeneization, whereas $\gamma$ does not.

The last parameter we use is

$$\mathcal{N} := ||C_{TT}||_{F_{21}} - ||C_{BV}||_{F_{21}},$$

which indicates the difference between the transversely isotropic tensor resulting from Backus average and the effective isotropic tensor resulting from the Backus-Voigt average, where

$$||C_{TT}||_{F_{21}} = \left( 2 \left( c_{1111}^{TT} \right)^2 + 2 \left( c_{1133}^{TT} \right)^2 + \left( c_{1111}^{TT} - 2c_{1212}^{TT} \right)^2 + \left( c_{3333}^{TT} \right)^2 + 2 \left( 2c_{2323}^{TT} \right)^2 + \left( 2c_{1212}^{TT} \right)^2 \right)^{\frac{1}{2}}$$

and

$$||C_{BV}||_{F_{21}} = \left( 3 \left( c_{1111}^{BV} \right)^2 + 3 \left( c_{1111}^{BV} - 2c_{2323}^{BV} \right)^2 + 3 \left( 2c_{2323}^{BV} \right)^2 \right)^{\frac{1}{2}}.$$

Herein, $\mathcal{N}$ indicates inhomogeneity of a stack of layers, as well as the anisotropy of the medium. Similarly to the Voigt average, we use the $F_{21}$ norm.

### 3.2 Constant rigidity: Isotropic medium

To illustrate parameters $I$ and $I_{BV}$, let us consider a stack of isotropic layers with elasticity parameters shown in Table 1.

| $c_{1111}$  | $c_{2323}$ |
|-------------|-------------|
| 10          | 2           |
| 10          | 2           |
| 10          | 2           |
| 10x         | 2           |
| 10x         | 2           |
| 10x         | 2           |
| 10x         | 2           |
| 10x         | 2           |
| 10x         | 2           |
| 10          | 2           |
| 10          | 2           |

**Table 1:** Elasticity parameters for ten isotropic layers; factor $x$ controls the inhomogeneity of the stack.

Figure 1—for $x = 1$—represents homogeneous stack, where $c_{1111} = 10$ and $c_{2323} = 2$. As $x$ increases, the inhomogeneity of $c_{1111}$ increases.

Only $I$ and $I_{BV}$ indicate growing inhomogeneity of $c_{1111}$. The other parameters are zero; they indicate no inhomogeneity and no anisotropy. $I$ and $I_{BV}$ are equal to each other, because, for isotropy, $c_{3333}^{TT} = c_{1111}^{BV}$. For the case of constant rigidity, the medium is isotropic, as a consequence $\gamma = 0$; herein, Thomsen parameters $\delta$ and $\epsilon$ are also zero.
3.3 Near constant rigidity: Anisotropic medium

Let us consider an example of a non–significantly varying $c_{2323}$.

| $c_{1111}$ | $c_{2323}$ |
|-----------|-----------|
| 10x       | 3         |
| 10        | 2         |
| 10x       | 3         |
| 10        | 2         |
| 10x       | 3         |
| 10        | 2         |
| 10x       | 3         |
| 10        | 2         |
| 10x       | 3         |
| 10        | 2         |

Table 2: Elasticity parameters for ten isotropic layers; factor $x$ controls the inhomogeneity of the stack.

As shown on Figure 2, in general, $I$ exhibits larger values than $I_{BV}$. This stems from the exclusive dependance of inhomogeneity of $c_{1111}$ for $I$. However, for very low values of $x$—where inhomogeneity of $c_{1111}$ is weaker than that of $c_{2323}$—$I$ has lower values than $I_{BV}$. This results from the dependance of inhomogeneity of $c_{2323}$ for $I_{BV}$. $\gamma$ is approximately twice as large as $\gamma_{BV}$; the inhomogeneity of $c_{1111}$ does not influence parameter $\gamma$ and, for the case of low inhomogeneity of $c_{2323}$, has a negligible effect on $\gamma_{BV}$, due to the nature of equation (4). $N$ represents the inhomogeneity of $c_{1111}$ and $c_{2323}$, as expected.
3.4 Equally-scaled elasticity parameters: Anisotropic medium

Let us consider an example to illustrate that every parameter indicates inhomogeneity, and to exhibit the relationship between them. In Table 3, the inhomogeneity grows equally for both elasticity parameters; Figure 3 represents such a situation.

For weak inhomogeneity, all five parameters have similar values. Also, $\mathcal{I}$ and $\gamma$ have the same values for strong inhomogeneity. This comes from the fact that, in this example, the inhomogeneity of $c_{1111}$ and $c_{2323}$ grows proportionally, and $\mathcal{I}$ indicates only inhomogeneity of $c_{1111}$ while $\gamma$ of $c_{2323}$. Thus, we conclude that for similar inhomogeneity of $c_{1111}$ and $c_{2323}$, $\mathcal{I}$ and $\gamma$ have similar values. For strong inhomogeneity, $\mathcal{N}$ has much larger values than $\mathcal{I}$, $\mathcal{I}_{BV}$, $\gamma$ and $\gamma_{BV}$. Comparing Figures 2 and 3, we conclude that $\mathcal{N}$ is more sensitive to the inhomogeneity of $c_{2323}$ as opposed to that of $c_{1111}$. As the value of $x$ increases, the difference between $\gamma$ and $\gamma_{BV}$ also increases. For $x = 5$, $\gamma$ is approximately three times as large as $\gamma_{BV}$. Hence, a large difference between $\gamma$ and $\gamma_{BV}$ indicates strong inhomogeneity of $c_{2323}$ and—as shown in a similar example in Appendix B—strong inhomogeneity of $c_{1111}$. 

| $c_{1111}$ | $c_{2323}$ |
|-----------|-----------|
| 10$x$     | 2$x$      |
| 10        | 2         |
| 10$x$     | 2$x$      |
| 10        | 2         |
| 10$x$     | 2$x$      |
| 10        | 2         |
| 10$x$     | 2$x$      |
| 10        | 2         |

Table 3: Elasticity parameters for ten isotropic layers; factor $x$ controls the inhomogeneity of the stack.
Figure 3: Horizontal axis exhibits values of $x$. Values of $I$ are shown by a dashed black line, $I_{BV}$ by a dashed grey line, $\gamma$ by a dotted black line, $\gamma_{BV}$ by a dotted grey line and $N$ by a solid black line; $I$ and $\gamma$ are overlain.

4 Conclusions

The five parameters stated in Section 3.1 allow us to examine the inhomogeneity of a stack of layers resulting in a Backus medium. In the case of isotropic layers with constant $c_{2323}$, we require $I$ or $I_{BV}$ to measure inhomogeneity using the Backus average. In this special case, the resulting medium is isotropic; hence the Thomsen parameters are equal to zero and they do not indicate the intrinsic inhomogeneity of a Backus medium.

$N$ appears to be particularly useful in measuring inhomogeneity as it relies on both $c_{1111}$ and $c_{2323}$. By combining the properties of three Thomsen parameters, it shows complex inhomogeneity. It can be used in the inverse problem—where we only know the Backus parameters provided by seismic information—the same way as $\gamma$ and $\gamma_{BV}$.

Also, the relationship between $\gamma$ and $\gamma_{BV}$ indicates the inhomogeneity of $c_{2323}$, alongside the minor auxiliary influence of the inhomogeneity of $c_{1111}$. For the case of near-constant rigidity, the relationship is approximately 2:1; the influence of $c_{1111}$ on this relationship is very small. Stronger inhomogeneity of $c_{2323}$ affects this relationship. In such a case, the influence of the inhomogeneity of $c_{1111}$ also increases; the relationship can reach 3:1 or more.

The relationship between $I$ and $I_{BV}$ may be insightful. Larger values of $I$ are characteristic for strong inhomogeneity of $c_{1111}$. The case, where $I_{BV}$ is larger, indicates low inhomogeneity of $c_{1111}$ and stronger influence of $c_{2323}$.

Similar values of parameters $I$ and $\gamma$ indicate the case of similarly scaled $c_{1111}$ and $c_{2323}$, as shown in Section 3.4.

In summary, the five parameters may be used to show the inhomogeneity, beyond Thomsen parameters, especially in the case of near-constant rigidity.


Acknowledgements

We wish to acknowledge the supervision of Michael A. Slawinski, the discussions with Ayiaz Kaderali, the proofreading and editing of Theodore Stanoev, the proofreading of Elena Patarini, the computer support of Izabela Kudela and the graphic support of Elena Patarini. This research was performed in the context of The Geomechanics Project supported by Husky Energy. Also, this research was partially supported by the Natural Sciences and Engineering Research Council of Canada, grant 238416-2013.

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A Theorem A.1

Theorem A.1. The Backus average of isotropic layers is isotropic if and only if its $\gamma = 0$, in contrast to parameters $\delta$ and $\epsilon$, whose zero values do not imply isotropy.

Proof. 

Lemma A.1. If $\gamma = 0$, then the Backus average of isotropic layers is isotropic.

Proof. If $\gamma = 0$, then $c_{1212}^{TT} = c_{2323}^{TT}$, and in accordance with expressions (1) and (2),

$$c_{1212}^{TT} = c_{2323}^{TT}$$

and

$$c_{2323} = \left(\frac{1}{c_{2323}}\right)^{-1}$$

we obtain,

$$c_{2323} = \left(\frac{1}{c_{2323}}\right)^{-1}. \tag{8}$$
Equation (8) is true, if and only if, $c_{2323}$ is constant. As stated by Backus (1962) and discussed by Adamus et al. (2018), layers whose $c_{2323}$ is constant result in an isotropic Backus (1962) medium.

If Backus average is isotropic, then $\mathbf{c}^{1111} = \mathbf{c}^{3333}$, and, hence, $\gamma = 0$.

**Lemma A.2.** If $\delta = 0$, it does not follow that the Backus average of isotropic layers is isotropic.

**Proof.** If $\delta = 0$, then

$$
\left(\mathbf{c}^{1111} + \mathbf{c}^{2323}\right)^2 - \left(\mathbf{c}^{3333} - \mathbf{c}^{2323}\right)^2 = \left(\mathbf{c}^{1111}\right)^2 - \left(\mathbf{c}^{3333}\right)^2 + 2\mathbf{c}^{2323} \left(\mathbf{c}^{1111} + \mathbf{c}^{3333}\right) = 0.
$$

Let us consider anisotropic Backus medium (from Proposition A.1); $\mathbf{c}^{3333} = 2\mathbf{c}^{1111} = 4\mathbf{c}^{2323}$. In such a case, equation (9) becomes

$$
4\left(c_{2323}\right)^2 - 16\left(c_{2323}\right)^2 + 8\left(c_{2323}\right)^2 = 0,
$$

which remains true for an anisotropic Backus average.

**Lemma A.3.** If $\epsilon = 0$, it does not follow that the Backus average of isotropic layers is isotropic.

**Proof.** If $\epsilon = 0$, then

$$
c_{1111} = c_{3333}.
$$

Let us consider an anisotropic Backus, where $c_{2323} \neq c_{1212}$, $c_{1111} \neq c_{1111} - 2c_{2323}$ and $c_{1111} = c_{3333}$. Equation (10) becomes

$$
c_{3333} = c_{3333},
$$

which remains true for an anisotropic Backus average.

**Proposition A.1.** A transversely isotropic tensor—with $\mathbf{c}^{3333} = 2\mathbf{c}^{1111} = 4\mathbf{c}^{2323}$—remains transversely isotropic.

**Proof.** Consider

$$
\begin{array}{cccccccc}
\begin{bmatrix}
\mathbf{c}_{1111} & \mathbf{c}_{1111} - 2\mathbf{c}_{1212} & 2\mathbf{c}_{2323} & 0 & 0 & 0 \\
\mathbf{c}_{1111} - 2\mathbf{c}_{1212} & \mathbf{c}_{1111} & 2\mathbf{c}_{2323} & 0 & 0 & 0 \\
2\mathbf{c}_{2323} & 2\mathbf{c}_{2323} & 4\mathbf{c}_{2323} & 0 & 0 & 0 \\
0 & 0 & 0 & 2\mathbf{c}_{2323} & 0 & 0 \\
0 & 0 & 0 & 0 & 2\mathbf{c}_{2323} & 0 \\
0 & 0 & 0 & 0 & 0 & 2\mathbf{c}_{1212}
\end{bmatrix}
\end{array}
$$

Its eigenvalues are

$$
\lambda_1 = \mathbf{c}_{1111} - \mathbf{c}_{1212} + 2\mathbf{c}_{2323} - \sqrt{\mathbf{c}_{1111}^2 - 4\mathbf{c}_{1111}\mathbf{c}_{2323} - 2\mathbf{c}_{1111}\mathbf{c}_{1212} + 12\mathbf{c}_{2323}\mathbf{c}_{1212} + 4\mathbf{c}^2_{2323}},
$$

$$
\lambda_2 = \mathbf{c}_{1111} - \mathbf{c}_{1212} + 2\mathbf{c}_{2323} + \sqrt{\mathbf{c}_{1111}^2 - 4\mathbf{c}_{1111}\mathbf{c}_{2323} - 2\mathbf{c}_{1111}\mathbf{c}_{1212} + 12\mathbf{c}_{2323}\mathbf{c}_{1212} + 4\mathbf{c}^2_{2323}},
$$

$$
\lambda_3 = \lambda_4 = 2\mathbf{c}_{2323},
$$

$$
\lambda_5 = \lambda_6 = 2\mathbf{c}_{1212},
$$

which—due to the eigenvalue multiplicities—implies that $C$ is a transversely isotropic tensor (Bóna et al., 2007), as required.

This completes the proof.
To numerically exemplify Lemma A.2, let us consider a stack of two isotropic layers; herein, the parameters for the first layer are $c_{1111} = 4$ and $c_{2323} = 1$, while for the second $c_{1111} = 1$ and $c_{2323} = 0.250$. For such a case, the Backus average—wherein $c_{1111}^{\text{TT}} = 2.275$, $c_{1133}^{\text{TT}} = 0.800$, $c_{1212}^{\text{TT}} = 0.625$, $c_{2323}^{\text{TT}} = 0.400$ and $c_{3333}^{\text{TT}} = 1.600$—is not isotropic. Specifically, $\delta = 0$, $\epsilon = 0.211$ and $\gamma = 0.281$.

To numerically exemplify Lemma A.3, let us consider a stack of two isotropic layers; herein, the parameters for the first layer are $c_{1111} = 2$ and $c_{2323} = 1$, while for the second $c_{1111} = 1.200$ and $c_{2323} = 0.200$. For such a case, the Backus average—wherein $c_{1111}^{\text{TT}} = 1.500$, $c_{1133}^{\text{TT}} = 0.500$, $c_{1212}^{\text{TT}} = 0.600$, $c_{2323}^{\text{TT}} = 0.333$ and $c_{3333}^{\text{TT}} = 1.500$—is not isotropic. Specifically, $\epsilon = 0$, $\delta = -0.190$ and $\gamma = 0.400$.

**Proposition A.2.** The Backus average of isotropic layers is isotropic if and only if its $\delta = 0$ and $\epsilon = 0$.

**Proof.** Let us consider a stack of two isotropic layers. We denote elasticity parameters for the first layer as $c_{1111} = a$ and $c_{2323} = c$, and for the second as $c_{1111} = b$ and $c_{2323} = d$. For the Backus average, $\delta = 0$ if and only if

$$c_{1133}^{\text{TT}} = c_{3333}^{\text{TT}} = 2c_{2323}^{\text{TT}}.$$  \hspace{1cm} (11)

Considering equation (11) for two layers and assuming arithmetic average, we obtain

$$\left(\frac{1}{2} \left(\frac{a - 2c}{a} + \frac{b - 2d}{b}\right)\right)^{-1} + 2 \left(\frac{1}{2} \left(\frac{1}{a} + \frac{1}{b}\right)\right)^{-1} - \left(\frac{1}{2} \left(\frac{1}{a} + \frac{1}{b}\right)\right)^{-1} = 0.$$ \hspace{1cm} (12)

After laborious algebraic computation, equation (12) simplifies to

$$(c - d) (bc - ad) = 0.$$ \hspace{1cm} (13)

For the Backus average, $\epsilon = 0$ if and only if $c_{1111}^{\text{TT}} = c_{3333}^{\text{TT}}$, which for two layers is equal to

$$\left(\frac{1}{2} \left(\frac{1}{a} + \frac{1}{b}\right)\right)^{-1} \left(\frac{1}{2} \left(\frac{a - 2c}{a} + \frac{b - 2d}{b}\right)\right)^2 + \left(\frac{1}{2} \left(\frac{4(a - c)c}{a} + \frac{4(b - d)d}{b}\right)\right) = \left(\frac{1}{2} \left(\frac{1}{a} + \frac{1}{b}\right)\right)^{-1}.$$ \hspace{1cm} (14)

After laborious algebraic computation, equation (14) simplifies to

$$(c - d) (c - d + b - a) = 0.$$ \hspace{1cm} (15)

To receive $\delta = 0$ and $\epsilon = 0$, we need to solve equations (13) and (15). Both equations are satisfied by $c = d$, which means that $c_{2323}$ is constant and the medium is isotropic. If $c \neq d$, then both equations are satisfied by a system of equations,

$$\begin{align*}
bc &= ad \\
c - d &= b - a
\end{align*}.$$  

We obtain $b = d$ and $a = c$, which do not satisfy stability conditions, namely, $b > \frac{4}{3} d$ and $a > \frac{4}{3} c$. \hfill \Box
### B Relation between $\gamma$ and $\gamma_{BV}$: Anisotropic medium

Let us consider a case of stronger inhomogeneity than that of Section 3.3. As shown in Table 4, the differences among $c_{2323}$ within the stack of layers are greater.

| $c_{1111}$ | $c_{2323}$ |
|------------|------------|
| 10$x$      | 4          |
| 10         | 1          |
| 10$x$      | 4          |
| 10         | 1          |
| 10$x$      | 4          |
| 10         | 1          |
| 10$x$      | 4          |
| 10         | 1          |
| 10$x$      | 4          |

**Table 4:** Elasticity parameters for ten isotropic layers; factor $x$ controls the inhomogeneity of the stack.

As shown on Figure 4, the relationship between $\gamma$ and $\gamma_{BV}$ is more sensitive to increasing values of $c_{1111}$, as compared to Figure 2. In other words, increasing inhomogeneity of $c_{1111}$ has a larger impact on the relationship between $\gamma$ and $\gamma_{BV}$ for strong inhomogeneity of $c_{2323}$, than for the weak one. Also, the relationship is larger than 2:1, due to stronger inhomogeneity of $c_{2323}$.