Graphs which their certain polynomials have few distinct roots- a survey

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ABSTRACT

Let $G = (V, E)$ be a simple graph. We consider domination polynomial, matching polynomial and edge cover polynomial of $G$. Graphs which their polynomials have few roots can give sometimes a very surprising information about the structure of the graph. In this paper we study graphs which their domination polynomial, independence polynomial and edge cover polynomial have few roots.

Keywords: Domination polynomial, Edge cover polynomial; Matching polynomial; Independence polynomial; Root.

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1 Introduction

Let $G = (V, E)$ be a simple graph. Graph polynomials are a well-developed area useful for analyzing properties of graphs. The study of graphs which their polynomials have few roots can give sometimes a very surprising information about the structure of the graph.

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We consider domination polynomial, matching polynomial (and independence polynomial) and edge cover polynomial of a graph $G$. For convenience, the definition of these polynomials will be given in the next sections.

The corona of two graphs $G_1$ and $G_2$, as defined by Frucht and Harary in [24], is the graph $G = G_1 \circ G_2$ formed from one copy of $G_1$ and $|V(G_1)|$ copies of $G_2$, where the ith vertex of $G_1$ is adjacent to every vertex in the ith copy of $G_2$. The corona $G \circ K_1$, in particular, is the graph constructed from a copy of $G$, where for each vertex $v \in V(G)$, a new vertex $v'$ and a pendant edge $vv'$ are added. The join of two graphs $G_1$ and $G_2$, denoted by $G_1 \vee G_2$ is a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv|u \in V(G_1) \text{ and } v \in V(G_2)\}$.

The characterization of graphs with few distinct roots of characteristic polynomials (i.e. graphs with few distinct eigenvalues) have been the subject of many researches. Graphs with three adjacency eigenvalues have been studied by Bridges and Mena [12] and Klin and Muzychuk [31]. Also van Dam studied graphs with three and four distinct eigenvalues [17, 18, 19, 20, 21]. Graphs with three distinct eigenvalues and index less than 8 studied by Chuang and Omidi in [15].

In this paper we study graphs which their domination polynomial, matching polynomial (and so independence polynomial) and edge cover polynomial have few roots. In Section 2 we investigate graphs with few domination roots. In Section 3 we characterize graphs which their matching polynomials (and so their independence polynomials) have few roots. Finally in Section 4 we characterize graphs with few edge cover roots.

As usual we use $\lfloor x \rfloor$, $\lceil x \rceil$ for the largest integer less than or equal to $x$ and for the smallest integer greater than or equal to $x$, respectively. In this paper we denote the set $\{1, 2, \ldots, n\}$ simply by $[n]$. 

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2 Graphs with few domination roots

Let $G = (V, E)$ be a graph of order $|V| = n$. For any vertex $v \in V$, the open neighborhood of $v$ is the set $N(v) = \{u \in V | uv \in E\}$ and the closed neighborhood of $v$ is the set $N[v] = N(v) \cup \{v\}$. For a set $S \subseteq V$, the open neighborhood of $S$ is $N(S) = \bigcup_{v \in S} N(v)$ and the closed neighborhood of $S$ is $N[S] = N(S) \cup S$. A set $S \subseteq V$ is a dominating set if $N[S] = V$, or equivalently, every vertex in $V \setminus S$ is adjacent to at least one vertex in $S$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in $G$. For a detailed treatment of this parameter, the reader is referred to [29]. Let $\mathcal{D}(G, i)$ be the family of dominating sets of a graph $G$ with cardinality $i$ and let $d(G, i) = |\mathcal{D}(G, i)|$. The domination polynomial $D(G, x)$ of $G$ is defined as $D(G, x) = \sum_{i=\gamma(G)}^{|V(G)|} d(G, i) x^i$, where $\gamma(G)$ is the domination number of $G$ ([8, 9]). The path $P_4$ on 4 vertices, for example, has one dominating set of cardinality 4, four dominating sets of cardinality 3, and four dominating sets of cardinality 2; its domination polynomial is then $D(P_4, x) = x^4 + 4x^3 + 4x^2$.

A root of $D(G, x)$ is called a domination root of $G$. In this section, the set of distinct roots of $D(G, x)$ is denoted by $Z(D(G, x))$.

We recall a formula for the computation of the domination polynomial of join of two graphs.

**Theorem 1.** ([2]) Let $G_1$ and $G_2$ be graphs of orders $n_1$ and $n_2$, respectively. Then

$$D(G_1 \vee G_2, x) = \left((1 + x)^{n_1} - 1\right)\left((1 + x)^{n_2} - 1\right) + D(G_1, x) + D(G_2, x).$$

The following theorem is an easy result about roots of $K_n$ and $K_{1,n}$.

**Theorem 2.** ([1])
(i) For every $n \in \mathbb{N}$,

$$Z\left(D(K_n, x)\right) = \left\{ \exp\left(\frac{2k\pi i}{n}\right) - 1 \mid k = 0, 1, \ldots, n - 1 \right\}.$$

(ii) For every $n \in \mathbb{N}$, $D(K_{1,n}, x)$ has exactly two real roots for odd $n$ and exactly three real roots for even $n$.

The following result is about the domination roots of $K_{n,n}$:

**Theorem 3.** For every even $n$, no nonzero real numbers is domination root of $K_{n,n}$.

**Proof.** By Theorem 1 for $G_1 = G_2 = K_n$, we have

$$D(K_{n,n}, x) = \left((1 + x)^n - 1\right)^2 + 2x^n.$$

If $D(K_{n,n}, x) = 0$, then $\left((1 + x)^n - 1\right)^2 = -2x^n$. Obviously this equation does not have real nonzero solution for even $n$. □

In [1, 8] we characterized graphs with one, two and three distinct domination roots. Since 0 is a root of any domination polynomial of graph $G$, we have the following theorem.

**Theorem 4.** ([2]) A graph $G$ has one domination root if and only if $G$ is a union of isolated vertices.

The following theorem characterize graphs with two distinct domination roots.

**Theorem 5.** ([8]) Let $G$ be a connected graph with exactly two distinct domination roots. Then there exists natural number $n$ such that $D(G, x) = x^n(x + 2)^n$. Indeed $G = H \circ K_1$ for some graph $H$ of order $n$. Moreover, for every graph $H$ of order $n$, $D(H \circ K_1, x) = x^n(x + 2)^n$. 
Theorem 6. ([8]) Let $G$ be a connected graph of order $n$. Then, $Z(D(G, x)) = \{0, \frac{-3 \pm \sqrt{5}}{2}\}$, if and only if $G = H \circ K_2$, for some graph $H$. Indeed $D(G, x) = x^\frac{4}{3}(x^2 + 3x + 1)^\frac{2}{3}$.

Theorem 6 characterize graphs with $Z(D(G, x)) = \{0, \frac{-3 \pm \sqrt{5}}{2}\}$ (see Figure ??). The following theorem shows that roots of graphs with exactly three distinct domination roots can not be any numbers.

Theorem 7. ([8]) For every graph $G$ with exactly three distinct domination roots

$$Z(D(G, x)) \subseteq \left\{0, \frac{-3 \pm \sqrt{5}}{2}, -2 \pm \sqrt{2i}, \frac{-3 \pm \sqrt{3i}}{2}\right\}.$$ 

Remark. Since $Z(D(K_3, x)) = \{0, \frac{-3 \pm \sqrt{5}}{2}\}$, $Z(D(P_3, x)) = \{0, \frac{-3 \pm \sqrt{5}}{2}\}$ and $Z(D(C_4, x)) = \{0, -2 \pm \sqrt{2i}\}$, for every number of set $\left\{0, \frac{-3 \pm \sqrt{5}}{2}, -2 \pm \sqrt{2i}, \frac{-3 \pm \sqrt{3i}}{2}\right\}$, there exist a graph which its domination polynomial have exactly three distinct roots. Since cycles are determined by their domination polynomials ([4]), so graphs with exactly three domination roots from $\{0, \frac{-3 \pm \sqrt{5}}{2}, -2 \pm \sqrt{2i}\}$ are $C_4$ and graphs of the form $H \circ K_2$, for some graph $H$.

By Theorems 4, 5 and 7 we have the following corollary:

Corollary 1. For every graph $G$ with at most three distinct domination roots

$$Z(D(G, x)) \subseteq \left\{0, -2, \frac{-3 \pm \sqrt{5}}{2}, -2 \pm \sqrt{2i}, \frac{-3 \pm \sqrt{3i}}{2}\right\}.$$ 

Now we shall study graphs with exactly four distinct domination roots.

Let $G_n$ be an arbitrary graph of order $n$. Let to denote the graph $G_n \circ K_1$ simply by $G_n^*$. Here we consider the labeled $G_n^*$ as show in Figure ?? (the graph in this figure is $P_n^*$ which called centipede). We denote the graph obtained from $G_n^*$ by deleting the vertex labeled $2n$ as $G_n^* - \{2n\}$. 
The following theorem state a recursive formula for the domination polynomial of $G_n^* - \{2n\}$.

**Theorem 8.** ([6]) For every $n \geq 5$,

$$D(G_n^* - \{2n\}, x) = x \left[ D(G_{n-1}^*, x) + D(G_{n-2}^*, x) \right] + x^2 D(G_{n-2}^*, x).$$

The following theorem give the formula for $D(G_n^* - \{2n\}, x)$.

**Theorem 9.** ([6]) For every $n \geq 2$, $D(G_n^* - \{2n\}, x) = (x^2 + 3x + 1)x^{n-1}(x + 2)^{n-2}.$

The following theorem characterize graphs with four domination roots $-2, 0, \frac{-3 + \sqrt{5}}{2}$.

**Theorem 10.** ([6]) Let $G$ be a connected graph of order $n$. Then, $Z(D(G, x)) = \{-2, 0, \frac{-3 + \sqrt{5}}{2}\}$, if and only if $G = G_n^* - \{n\}$, for some graph $G_n^*$ of order $\frac{n}{2}$. Indeed

$$D(G, x) = (x^2 + 3x + 1)x^{\frac{n}{2} - 1}(x + 2)^{\frac{n}{2} - 2}.$$

Using tables of domination polynomials (see [7]), we think that numbers which are roots of graphs with exactly four distinct domination roos are finite and are about nine numbers, but we are not able to prove it. So complete characterization of graphs with exactly four distinct domination roots remains as open problem.

It is natural to ask about the domination roots of paths and cycles. First we recall the following theorem:

**Theorem 11.** For every $n \geq 3$,

\begin{enumerate}[(i)]
    \item ([10]) $D(P_n, x) = x \left[ D(P_{n-1}, x) + D(P_{n-2}, x) + D(P_{n-3}, x) \right]$,
\end{enumerate}
\[(ii) \quad D(C_n, x) = x \left[ D(C_{n-1}, x) + D(C_{n-2}, x) + D(C_{n-3}, x) \right],\]

We think that real roots of the families \(D(P_n, x)\) and \(D(C_n, x)\) are dense in the interval \([-2, 0]\), for \(n \geq 4\), but we couldn’t to prove it yet.

3 Graphs with few matching roots

In this section we study graphs which their matching polynomials have few roots. First we state the definition of matching polynomial. Let \(G = (V, E)\) be a graph of order \(n\) and size \(m\). An \(r\)-matching of \(G\) is a set of \(r\) edges of \(G\) which no two of them have common vertex. The maximum number of edges in a matching of a graph \(G\) is called the matching number of \(G\) and denoted by \(\alpha'(G)\). The matching polynomial is defined by

\[
\mu(G, x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k m(G, k) x^{n-2k},
\]

where \(m(G, k)\) is the number of \(k\)-matching of \(G\) and \(m(G, 0) = 1\). The roots of \(\mu(G, x)\) are called the matching roots of \(G\). As an example the matching polynomial of path \(P_5\) is \(\mu(P_5, x) = (x-1) x(x+1)(x^2-3)\). For more details of this polynomial refer to [22, 23, 26].

Two following theorems may are the first results on matching roots.

**Theorem 12.** ([30]) *The roots of matching polynomial of any graph are all real numbers.*

**Theorem 13.** ([27]) *If \(G\) has a Hamiltonian path, then all roots of its matching polynomial are simple (have multiplicity 1).*

We need the following definition to study graphs with few matching roots.
Add a single vertex \( u \) to the graph \( rK_{1,k} \cup tK_1 \) and join \( u \) to the other vertices by \( p + q \) edges so that the resulting graph is connected and \( u \) is adjacent with \( q \) centers of the stars (for \( K_{1,1} \) either of the vertices may be considered as center). We denote the resulting graph by \( S(r,k,t;p,q) \) (see Figure ??). Clearly \( r + t \leq p + q \leq r(k + 1) + t \) and \( 0 \leq q \leq r \) (see [25]).

For any \( G \in S(r,3,t;p,q) \), we add \( s \) copies of \( K_3 \) to \( G \) and join them by \( l \) edges to the vertex \( u \) of \( G \). Clearly \( s \leq l \leq 3s \). We denote the set of these graphs by \( H(r,s,t;p,q,l) \).

The following theorem gives the matching polynomial of graph \( G \) in the family \( S(r,k,t;p,q) \).

**Theorem 14.** ([25]) For every \( G \in S(r,k,t;p,q) \),
\[
\mu(G, x) = x^{r(k-1)+t-1}(x^2-k)^{r-1}(x^4-(p+k)x^2+(p-q)(k-1)+t).
\]

**Theorem 15.** ([25]) For every \( G \in H(r,s,t;p,q,l) \),
\[
\mu(G, x) = x^{2r+s+t-1}(x^2-3)^{r+s-1}(x^4-(p+l+3)x^2+3t+2(p-t-q)+l).
\]

Similar to [25] we distinguish some special graphs in the families \( S \) and \( H \) which are important for our study.

We denote the family \( S(r,1,0;s,q) \) which consists a single graph by \( S(r,s) \). Note that in this case \( q \) is determined by \( r \) and \( s \), namely \( q = s - r \). Its matching polynomial is
\[
\mu(S(r,s), x) = x(x^2-s-1)(x^2-1)^{r-1}.
\]

The family \( S(r,k,0;r,r) \) consists of a single graph which is denoted by \( T(r,k) \). Its matching polynomial is
\[
\mu(T(r,k), x) = x^{r(k-1)+1}(x^2-r-k)(x^2-k)^{r-1}.
\]
We also denote the unique graphs in \( S(1, k; t + t, 0) \) and \( S(1, k; t + t + 1, 1) \) by \( K(k; t) \) and \( K'(k; t) \), respectively. Their matching polynomials are

\[
\mu(K(k; t), x) = x^{k + t - 2}(x^4 - (k + t + l)x^2 + (l + t)(k - 1) + t);
\mu(K'(k; t), x) = x^{k + t - 2}(x^4 - (k + t + l + 1)x^2 + (l + t)(k - 1) + t).
\]

Moreover, we denote the unique graph in \( H(0, 1; t; 0, l) \) by \( L(t; l) \) for \( l = 1, 2, 3 \). We have \( \mu(L(t; l), x) = x^t(x^4 - (t + l + 3)x^2 + 3t + l) \). Typical graphs from the above families are shown in Figure ??.

**Theorem 16.** ([25]) Let \( G \) be a connected graph and \( z(G) \) be the number of its distinct matching roots.

(i) If \( z(G) = 2 \), then \( G \simeq K_2 \).

(ii) If \( z(G) = 3 \), then \( G \) is either a star or \( K_3 \).

(iii) If \( z(G) = 4 \), then \( G \) is a non-star graph with 4 vertices.

(iv) If \( z(G) = 5 \), then \( G \) is one of the graphs \( K(k; t), K'(k; t), L(t; l), T(r, k), S(r, s) \), for some integers \( k, r, s, t, l \) or a connected non-star graph with 5 vertices.

Using above theorem we would like to study graphs with few independence roots. First we recall the definition of independence polynomial.

An independent set of a graph \( G \) is a set of vertices where no two vertices are adjacent. The independence number is the size of a maximum independent set in the graph and denoted by \( \alpha(G) \). For a graph \( G \), let \( i_k \) denote the number of independent sets of cardinality \( k \) in \( G \) \((k = 0, 1, \ldots, \alpha)\). The independence polynomial of \( G \),

\[
I(G, x) = \sum_{k=0}^{\alpha} i_k x^k,
\]
is the generating polynomial for the independent sequence \((i_0, i_1, i_2, \ldots, i_\alpha)\). For more study on independence polynomial and independence root refer to [11] [13] [14].

The path \(P_4\) on 4 vertices, for example, has one independent set of cardinality 0 (the empty set), four independent sets of cardinality 1, and three independent sets of cardinality 2; its independence polynomial is then \(I(P_4, x) = 1 + 4x + 3x^2\).

Here we recall the definition of line graph. Given a graph \(H = (V, E)\), the line graph of \(H\), denoted by \(L(H)\), is a graph with vertex set \(E\), two vertices of \(L(H)\) are adjacent if and only if the corresponding edges in \(H\) share at least one endpoint. We say that \(G\) is a line graph if there is a graph \(H\) for which \(G = L(H)\).

**Theorem 17.** ([28]) For every graph \(G\), \(\mu(G, x) = x^n I(L(G), -\frac{1}{x^2})\).

The following corollary is an immediate consequence of Theorem 17.

**Corollary 2.** If \(\alpha \neq 0\) is a matching root of \(G\), then \(-\frac{1}{\alpha^2}\) is an independence root of \(L(G)\).

Now we are ready to state a theorem for graphs which its independence polynomial have few roots. The following theorem follows from Theorem 17 and Corollary 2.

**Theorem 18.** Let \(G\) be a connected graph and \(z(L(G))\) be the number of distinct independence non-zero roots of \(L(G)\).

(i) If \(z(L(G)) = 2\), then \(L(G) \cong K_2\).

(ii) If \(z(L(G)) = 3\), then \(L(G)\) is either a star or \(K_3\).

(iii) If \(z(L(G)) = 4\), then \(L(G)\) is a non-star graph with 4 vertices.
(iv) If $z(L(G)) = 5$, then $L(G)$ is one of the graphs then $G$ is one of the graphs $K(k, t; l), K'(k, t; l), L(t; l), T(r, k), S(r, s)$, for some integers $k, r, s, t, l$ or a connected non-star graph with 5 vertices.

**Theorem 19.** ([11])

(i) For any integer $n$, $I(P_n, x)$ has the following zeros,

$$p_s^{(n)} = -\frac{1}{2 \left(1 + \cos \frac{2s\pi}{n+2}\right)}, \quad s = 1, 2, \ldots, \left\lfloor \frac{n+1}{2} \right\rfloor.$$

(ii) For any integer $n \geq 3$, $I(C_n, x)$ has the following zeros,

$$c_s^{(n)} = -\frac{1}{2 \left(1 + \cos \frac{(2s-1)\pi}{n}\right)}, \quad s = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor.$$

**Corollary 3.** ([11]) The independence roots of the family $\{P_n\}$ and $\{C_n\}$ are real and dense in $(-\infty, -\frac{1}{4}]$.

In [5] authors studied graphs whose independence roots are rational.

**Theorem 20.** ([5]) Let $G$ be a graph with rational polynomial $I(G, x)$. If $i(G, r - 1) > i(G, r)$, for some $r$, $1 \leq r \leq \alpha(G)$, then $G$ has an independence root in the interval $(-1, -\frac{1}{\alpha(G)})$.

The following theorem characterize graphs with exactly one independence roots:

**Theorem 21.** ([5]) Let $G$ be a graph of order $n$. Then $I(G, x)$ has exactly one root if and only if $G = rK_s$, where $n = rs$ for some natural $r, s$. 

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4 Graphs with few edge-cover roots

In this section we characterize graphs which their edge cover polynomials have one and two distinct roots. First we state the definition of edge-cover polynomial of a graph.

For every graph $G$ with no isolated vertex, an edge covering of $G$ is a set of edges of $G$ such that every vertex is incident to at least one edge of the set. A minimum edge covering is an edge cover of the smallest possible size. The edge covering number of $G$ is the size of a minimum edge cover of $G$ and denoted by $\rho(G)$. The edge cover polynomial of $G$ is the polynomial $E(G, x) = \sum_{i=1}^{m} e(G, i)x^i$, where $e(G, i)$ is the number of edge covering sets of $G$ of size $i$. Note that if graph $G$ has isolated vertex then we put $E(G, x) = 0$ and if $V(G) = E(G) = \emptyset$, then $E(G, x) = 1$. For more detail on this polynomial refer to [3, 16].

As an example the edge cover polynomial of path $P_5$ is $E(P_5, x) = 2x^3 + x^4$. Also $E(K_{1, n}, x) = x^n$.

The following results are about edge cover polynomial of a graph:

**Lemma 1.** ([3]) Let $G$ be a graph of order $n$ and size $m$ with no isolated vertex. If the edge cover polynomial of $G$ is $E(G, x) = \sum_{i=\rho(G)}^{m} e(G, i)x^i$, then the following hold:

(i) $n \leq 2\rho(G)$.

(ii) If $i_0 = \min\{i|e(G, i) = \binom{m}{i}\}$, then $\delta = m - i_0 + 1$.

(iii) If $G$ has no connected component isomorphic to $K_2$, then $a_{\delta(G)} = \binom{m}{m-\delta} - e(G, m-\delta)$.

**Theorem 22.** ([16]) Let $G$ be a graph. Then $E(G, x)$ has at least $\delta(G) - 2$ non-real roots (not necessary distinct). In particular, when $\delta(G) = 3$, $E(G, x)$ has at least two distinct non-real roots.
Now we shall characterize graphs with few edge cover roots. Note that zero is one of the roots of $E(G, x)$ with multiplicity $\rho(G)$. The next theorem characterize all graph $G$ whose edge cover polynomials have exactly one distinct root. Note that $E(K_{1,n}, x) = x^n$.

**Theorem 23.** ([16]) *Let $G$ be a graph. Then $E(G, x)$ has exactly one distinct root if and only if every connected component of $G$ is star.*

We need the following definition to study graphs with two distinct edge cover roots.

Let $H$ be a graph of order $n$ and size $m$. Suppose $\{v_1, \ldots, v_n\}$ is the vertex set of $H$. By $H(r)$ we mean the graph obtained by joining $r_i \geq 1$ pendant vertices to vertex $v_i$, for $i = 1, \ldots n$ such that $\sum r_i = r$. If $m$ is the size of $H$, then $H(r)$ is a graph of order $n + r$ and size $m + r$. The graph $C_4(9)$ shown in Figure ??.

**Theorem 24.** ([16]) *Let $G$ be a graph. Then $E(G, x) = x^r(x+1)^m$, for some natural numbers $r$ and $m$, if and only if there exists a graph $H$ with size $m$ such that $G = H(r)$.*

The next theorem characterizes all graphs $G$ for which $E(G, x)$ has exactly two distinct roots.

**Theorem 25.** ([16]) *Let $G$ be a connected graph whose edge cover polynomial has exactly two distinct roots. Then one of the following holds:

(i) $G = H(r)$, for some connected graph $H$ and natural number $r$.

(ii) $G = K_3$.

(iii) $\delta(G) = 1$, $E(G, x) = x^{\frac{m-s}{2}}(x+2)^{\frac{m+s}{2}}$, where $s$ is the number of pendant vertices of $G$.***
\( (iv) \, \delta(G) = 2, \, E(G, x) = x^m(x + 2)^{\frac{m}{2}} , \alpha_2(G) = \frac{m}{2}, \) and \( G \) has cycle of length 3 or 5.

By Theorems 24 and 25 we have the following corollaries:

**Corollary 4.** \([3]\) Let \( G \) be a connected graph. If \( E(G, x) \) has exactly two distinct roots, then \( Z(E(G, x)) = \{-1, 0\}, \, Z(E(G, x)) = \{-2, 0\} \) or \( Z(E(G, x)) = \{-3, 0\}. \) Also \( \delta(G) = 1 \) or \( \delta(G) = 2. \)

**Corollary 5.** \([3]\) Let \( G \) be a connected graph. If \( Z(E(G, x)) = Z(E(K_3, x)) = \{-3, 0\} \) if and only if \( G = K_3. \)

We have the following corollary:

**Corollary 6.** Let \( G \) be a graph with at most two distinct edge cover roots. Then \( Z(E(G, x)) \subseteq \{-3, -2, -1, 0\}. \)

There are infinite graphs \( G \) with \( \delta(G) = 1 \) and \( Z(E(G, x)) \subseteq \{-2, -1, 0\}. \) To see some algorithms for constructing of these kind of graphs refer to \([3]\). For the case \( \delta(G) = 2, \) we can see that if \( G \) is a graph with \( \delta(G) = 2 \) and \( Z(E(G, x)) = \{-2, 0\}, \) then \( G \) has cycle with length 3 or 5 (see \([3]\)). For this case there is the following conjecture:

**Conjecture 1.** \([3]\) There isn’t any graph \( G \) with \( \delta(G) = 2 \) and \( Z(E(G, x)) = \{-2, 0\}. \)

It is proven that if \( G \) is a graph without cycle of length 3 or 5 and \( \delta(G) = 2, \) then \( E(G, x) \) has at least three distinct roots (see \([3]\)). Theorem 25 and above conjecture implies that if \( \delta(G) = 2 \) and \( G \neq K_3, \) then \( E(G, x) \) has at least three distinct roots.

The following theorem gives the roots of edge cover polynomial of paths and cycles:
Theorem 26.\textsuperscript{(16)}

(i) For any integer \(n\), \(E(P_n, x)\) has the following non-zero roots,

\[
-2 \left( 1 + \cos \frac{2s\pi}{n-1} \right), \quad s = 1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor - 1.
\]

(ii) For any integer \(n \geq 3\), \(E(C_n, x)\) has the following zeros,

\[
c_s^{(n)} = -2 \left( 1 + \cos \frac{(2s + 1)\pi}{n} \right), \quad s = 0, 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor - 1.
\]

Also we have the following theorem for roots of \(E(P_n, x)\) and \(E(C_n, x)\).

Theorem 27.\textsuperscript{(16)} The roots of the family \(\{E(P_n, x)\}\) and \(\{E(C_n, x)\}\) are real and dense in \((-4, 0]\).

Remark. It is interesting that the non zeros roots of \(I(C_n, x)\) are exactly the inverse of (non zeros) roots of \(E(C_n, x)\).

Note that in \textsuperscript{[16]} proved that if every block of the graph \(G\) is \(K_2\) or a cycle, then all real roots of \(E(G, x)\) are in the interval \((-4, 0]\).

For most of graphs polynomials such as chromatic polynomial, matching polynomial, independence polynomial and characteristic polynomial there is no constant bound for the roots (complex) of these polynomials. Surprisingly, in the following theorem we observe that there is a constant bound for the roots of the edge cover polynomials.

Theorem 28.\textsuperscript{(16)} All roots of the edge cover polynomial lie in the ball

\[
\left\{ z \in \mathbb{C} : |z| < \frac{(2 + \sqrt{3})^2}{1 + \sqrt{3}} \simeq 5.099 \right\}.
\]
5 Open problems and conjectures

In this section we state and review some open problems and conjectures related to the subject of paper.

**Problem 1.** Characterize all graphs with exactly three distinct domination roots \( \{0, -\frac{3\pm\sqrt{3}i}{2}\} \).

**Problem 2.** Characterize all graphs with exactly four distinct domination roots.

**Problem 3.** Characterize all graphs with no real domination roots except zero.

**Conjecture 2.** The set of integer domination roots of any graphs is a subset of \( \{-2, 0\} \).

**Conjecture 3.** Real roots of the families \( D(P_n, x) \) and \( D(C_n, x) \) are dense in the interval \([-2, 0]\), for \( n \geq 4 \).

In this paper we obtained Theorem [18] for graphs which its independence polynomials have few roots which is for line graphs. But complete characterization remain as open problem:

**Problem 4.** Characterize all graphs with few independence roots.

**Conjecture 4.** If \( \delta(G) = 2 \) and \( G \neq K_3 \), then \( E(G, x) \) has at least three distinct roots.

**Conjecture 5.** Let \( G \) be a graph. Then \( E(G, x) \) has at least \( \delta(G) \) distinct roots.

**Conjecture 6.** Let \( G \) be a graph with \( \delta(G) = 2 \). If \( E(G, x) \) has only real roots, then all connected components of \( G \) are cycles.
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