Multi-agent formation based on self-loop Laplacian

D W Djamari¹,* and M R Fikri²

¹ Department of Mechanical Engineering, Faculty of Engineering Technology, Sampoerna University, Jakarta Indonesia
² Department of Information Systems, Faculty of Engineering Technology, Sampoerna University, Jakarta Indonesia

*djati.wibowo@sampoernauniversity.ac.id

Abstract. This work studies formation control of Multi-Agent Systems (MASs) where its formation size is scalable via a scaling factor (scalable formation). Literature on scalable formation where the design of communication weight is based on the formation pattern is limited to the case of k-rooted graph and generic configuration of formation pattern. Different from the literature, scalable formation in this work is achieved under a weak assumption on graph and the formulation allows for a more general formation pattern. In this work, scalable formation is achieved by designing a self-loop communication weight for each agent which results in a new matrix called self-loop Laplacian. The self-loop weight is a function of the desired formation pattern. The convergence of the proposed algorithm is shown by standard linear system theory by first deriving the spectral properties of the self-loop Laplacian. The effectiveness of the proposed method is illustrated by numerical example. The proposed method is shown to achieve scalable formation for any initial position of agents.

1. Introduction

Scalable multi-agent formation is a challenging problem in cooperative control of MASs. Scalable formation allows for the change in formation size during operation triggered by auxiliary states [1-3] or by initial states of agents [4,5].

The works Coogan et al. [1], Coogan et al. [2], and Tran et al. [3] achieve scalable formation by introducing an auxiliary state that acts as a scaling factor, while Lin et al. [4] and Han et al. [5] achieve scalable formation by designing the communication weight according to the desired formation. In Lin et al. [4] and Han et al. [5], the formation pattern must be of generic configuration, and the formation is realizable under k-rooted graph condition. This work is similar Lin et al. [4] and Han et al. [5] in that the communication weight in the formulation is designed based on the desired formation. However, different from Lin et al. [4] and Han et al. [5], the necessary and sufficient condition to achieve scalable formation in this work is the spanning tree condition of the graph. In addition, there is no restriction in the configuration of the formation pattern.

Non-negative and positive integer sets are indicated by \( \mathbb{Z}_0^+ \) and \( \mathbb{Z}^+ \) respectively. Let \( M, L \in \mathbb{Z}^+ \) with \( M > L \). Then \( \mathbb{Z}^M = \{1,2,\cdots,M\} \) and \( \mathbb{Z}_L^M = \{L,L+1,\cdots,M\} \). While \( \mathbb{R}, \mathbb{R}^n, \mathbb{R}^{n \times m} \) refer to the sets of real numbers, \( n \)-dimensional real vectors and \( m \) by \( m \) real matrices respectively. \( I_n \) is the \( n \times n \) identity matrix with \( 1_n \) being the \( n \)-column vector of all ones (subscript omitted when the dimension is clear). The transpose of matrix \( M \) and vector \( v \) are indicated by \( M^T \) and \( v^T \) respectively. Eigenvalues of \( M \) is denoted by \( \text{spec}(M) \).
2. Preliminaries

2.1. Graph theory
A directed graph $\mathcal{G}$ is defined by a pair of finite set of nodes $\mathcal{V} = \{1, \ldots, N\}$ and a set of edges $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$, thus $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. An edge is represented by an ordered pair of nodes, i.e. $e_q = (j, i) \in \mathcal{E}$. The pair $(j, i) \in \mathcal{E}$ if node $j$ points towards node $i$, or that node $i$ receives information from node $j$. The set of neighbors of node $i$ is $\mathcal{N}_i = \{ j \in \mathcal{V} : (j, i) \in \mathcal{E}, i \neq j \}$.

A directed spanning tree is a directed graph where there is one node called the root which has no neighbor node, and there is a directed path from the root node to every other node, either a direct path or an indirect path. A directed graph $\mathcal{G}$ is said to contain a directed spanning tree if graph $\mathcal{G}$ has a directed spanning tree as its subgraph.

The element of adjacency matrix $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N}$ associated with the graph $\mathcal{G}$ is $a_{ij} = 1$ if $j \in \mathcal{N}_i$ and $a_{ij} = 0$ otherwise. The Laplacian matrix, $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{N \times N}$ is defined as $l_{ii} = \sum_{j \in \mathcal{N}_i} a_{ij}$ and $l_{ij} = -a_{ij}$ for $i \neq j$.

Following are standard results on spectral properties of the Laplacian matrix and the convergence of consensus algorithm of single integrator that we will use in this paper:

- **Lemma 1.** ([6], Lemma 3.3) Let $\mathcal{L}$ be the Laplacian matrix associated with directed graph $\mathcal{G}$. $\mathcal{L}$ has the following properties:
  - $0$ is an eigenvalue of $\mathcal{L}$ with eigenvector $1_N$ and all the nonzero eigenvalues of $\mathcal{L}$ have positive real parts,
  - $0$ is a simple eigenvalue of $\mathcal{L}$ if and only if $\mathcal{G}$ contains a directed spanning tree.

- **Lemma 2.** ([7], Theorem 3.12) Let $\mathcal{L}$ be the Laplacian matrix associated with directed graph $\mathcal{G}$ and $\mathbf{p} \in \mathbb{R}^N$. Let $\mathbf{y}^T \mathbf{L} = 0, l_1 \mathbf{N} = 0$ and $1_N^T \mathbf{y} = 1$. The system $\dot{\mathbf{p}}(t) = -\mathcal{L} \mathbf{p}(t)$ has trajectories that satisfy: $\lim_{t \to \infty} \mathbf{p}(t) = (\mathbf{y}^T \mathbf{p}(t_0))1_N$ for any $\mathbf{p}(t_0)$ if and only if the directed graph $\mathcal{G}$ contains a directed spanning tree.

The assumption on graph needed in this work is

**A1** The graph $\mathcal{G}$ contains a directed spanning tree.

2.2. Problem formulation
We consider $N$ single integrator agent with the following state space model:

$$\dot{x}_i(t) = u_i(t) \quad (1)$$

where $\dot{x}_i(\cdot) \in \mathbb{R}$ and $u_i(\cdot) \in \mathbb{R}$ are the velocity and the control input of agent $i$ respectively. The dynamics of (1) can be expanded for higher dimension by using Kronecker product operation.

Let $x_i^d \in \mathbb{R}$ and $x_j^d \in \mathbb{R}$ be the desired position of agents $i$ and $j$ on a local coordinate system, the problem defined in this paper is to design a distributed controller $u_i$ in (1) such that **P1** For any $x_i(t_0)$, $i \in \mathbb{Z}^N$,

$$\lim_{t \to \infty} (x_i(t) - x_j(t)) = c(t_0)(x_i^d - x_j^d) \quad (2)$$

for all $i, j \in \mathbb{Z}^N$, where $c(t_0) \in \mathbb{R}$ is a function of the initial starting time.

3. Main results
This section starts with the introduction of the self-loop Laplacian matrix. We first explain its desired properties followed by the method to construct such matrix and we prove that the constructed matrix indeed has the desired properties. This section is ended with the construction of $u_i$ based on the self-loop Laplacian matrix.
3.1. Self-loop Laplacian desired properties

Let the system of interest be

$$\dot{x}(t) = -\Phi x(t)$$  \hfill (3)

where $x = [x_1, \ldots, x_N]^T \in \mathbb{R}^N$ is the stacked vector of agents’ position, $\Phi \in \mathbb{R}^{N \times N}$ is the self-loop Laplacian associated to graph $\mathcal{G}$, and let $x^d = [x_1^d, \ldots, x_N^d]^T \in \mathbb{R}^N$ be the desired formation pattern defined on a local coordinate system. Following are the desired properties of $\Phi$:

- 0 is an eigenvalue of $\Phi$ with eigenvector of $x^d$ and all the nonzero eigenvalues of $\Phi$ has positive real parts,

- 0 is a simple eigenvalue of $\Phi$ if and only if $\mathcal{G}$ contains a directed spanning tree.

Properties (i) and (ii) above are exactly the properties (i) and (ii) from Lemma 1, only that $1_N$ is now replaced by $x^d$. For easy reference in further discussion, the following Lemma is stated:

**Lemma 3.** Consider system (3) where $\Phi$ is the self-loop Laplacian associated to graph $\mathcal{G}$. Suppose $\Phi$ has the desired properties (i) and (ii). Then the solution of (3) has the properties: $lim_{t \to \infty} x(t) = (y^T x(t_0))x^d$ for any $x(t_0)$, where $y$ satisfies $y^T \Phi = 0$ and $y^T x^d = 1$, if and only if graph $\mathcal{G}$ contains a directed spanning tree.

Lemma 3 can be shown using the same argument in the proof of Lemma 2 by replacing $1_N$ with $x^d$. Thus, the proof is omitted. From Lemma 3, we can see that the asymptotic value of the solution of (3) is a scaled version of the desired formation, where $y^T x(t_0)$ is the scaling factor of the formation. Here, referring to (P1), $y^T x(t_0) = c(t_0)$. Next, we will discuss the construction of $\Phi$.

3.2. The construction of $\Phi$

Let $\Phi$ be decomposed into:

$$\Phi = \mathcal{L} + R,$$

where $\mathcal{L}$ is the Laplacian matrix associated with graph $\mathcal{G}$ and $R = \text{diag}[r_1, \ldots, r_N] \in \mathbb{R}^{N \times N}$ is a diagonal matrix. To make $\Phi x^d = 0$, we compute $R$ in the following way:

$$\Phi x^d = (\mathcal{L} + R)x^d = 0$$

$$Rx^d = -\mathcal{L}x^d$$

$$\begin{bmatrix} r_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & r_N \end{bmatrix} \begin{bmatrix} x_1^d \\ \vdots \\ x_N^d \end{bmatrix} = \begin{bmatrix} l_{11} & \cdots & l_{1N} \\ \vdots & \ddots & \vdots \\ l_{N1} & \cdots & l_{NN} \end{bmatrix} \begin{bmatrix} x_1^d \\ \vdots \\ x_N^d \end{bmatrix}.$$  \hfill (4)

From every line in (3), we can express

$$r_i = -\left( l_{ii} + \frac{1}{x_i^d} \sum_{j=1, i \neq j}^N l_{ij} x_j^d \right).$$  \hfill (5)

for all $i \in \mathbb{Z}^N$. By noting that $l_{ij} = -a_{ij}$ and $a_{ij} > 0$ only when $j \in \mathcal{N}_i$ and $a_{ij} = 0$ otherwise, we can write $r_i$ as $r_i = -l_{ii} + s$ for all $i \in \mathbb{Z}^N$, where

$$s_i = \frac{1}{x_i^d} \sum_{j \in \mathcal{N}_i} a_{ij} x_j^d, \forall i \in \mathbb{Z}^N.$$  \hfill (6)

Thus, $\Phi = [\phi_{ij}]$ can be written as

$$\phi_{ij} = \begin{cases} s_i, & i = j; \\ -a_{ij}, & i \neq j. \end{cases}$$  \hfill (7)
with $s_i$ given by (6). By the construction, it is clear that $\Phi$ has properties of $\Phi x^d = 0$. Next, we discuss the spectral properties of $\Phi$.

3.3. Spectral properties of $\Phi$

We first make the following assumption

(A2) $x^d_i > 0$ for all $i \in \mathbb{Z}^N$.

(A2) does not make the formulation lose its generality since $x^d_i$ for all $i$ is defined on the local coordinate system. We now state the following theorem on the spectral properties of $\Phi$:

**Theorem 1.** Let $\Phi$ be given by (7) where the associated graph to $\Phi$ is $\mathcal{G}$, then:
- $0$ is an eigenvalue of $\Phi$ with eigenvector of $x^d$ and all of the nonzero eigenvalues of $\Phi$ has positive real parts,
- $0$ is a simple eigenvalue of $\Phi$ if and only if $\mathcal{G}$ contains a directed spanning tree.

**Proof.** (i) $\Phi x^d$ is obvious from the construction of $\Phi$ in the previous section. To show that the nonzero eigenvalues of $\Phi$ has positive real parts, consider the following coordinate transformation

$$\bar{\Phi} = T \Phi T^{-1}$$

where $T = \text{diag} \left[ \frac{1}{x^d_1}, \cdots, \frac{1}{x^d_N} \right]$ and $T^{-1} = \text{diag} [x^d_1, \cdots, x^d_N]$, then we have $\bar{\Phi} = [\bar{\phi}_{ij}]$:

$$\bar{\phi}_{ij} = \begin{cases} s_i, & i = j; \\ -a_{ij} \left( \frac{x^d_i}{x^d_j} \right), & i \neq j. \end{cases} (8)$$

We can see from (8) that $\bar{\Phi}$ is the Laplacian matrix associated to weighted graph $\mathcal{G}^d = (\mathcal{V}, \mathcal{E}, \mathcal{A}^d)$, where $\mathcal{A}^d = [a^d_{ij}]$ is the weight matrix with its entries $a^d_{ij} = a_{ij}(x^d_i/x^d_j)$. Under (A2), the sign of $(x^d_i/x^d_j)$ is always positive and thus $a^d_{ij} > 0$ ($a^d_{ij} = 0$) if and only if $a_{ij} = 1$ ($a_{ij} = 0$). Therefore, graph $\mathcal{G}^d$ contains a directed spanning tree if and only if graph $\mathcal{G}$ contains a directed spanning tree. In conclusion, the spectral properties of $\Phi$ follow properties (i) and (ii) from Lemma 1. Since $\Phi$ is obtained from similarity transformation $\Phi = T \Phi T^{-1}$, eigenvalues of $\Phi$ are also the eigenvalues of $\bar{\Phi}$, thus the nonzero eigenvalue of $\Phi$ has positive real parts.

- This point is clear since graph $\mathcal{G}^d$ contains a directed spanning tree if and only if graph $\mathcal{G}$ contains a directed spanning tree, and eigenvalues of $\Phi$ are also eigenvalues of $\bar{\Phi}$.

Now that we have established the spectral properties of the self-loop Laplacian based on the condition of graph $\mathcal{G}$, we state the control input $u_i$ in the next section.

3.4. The control input

Following the desired system (3), the control input $u_i$ in (1) is the following

$$u_i(t) = \left( \sum_{j \in N_i} a_{ij} (x_i(t) - x_j(t)) \right) + \left( \frac{1}{x^d_i} \sum_{j \in N_i} a_{ij} (x^d_i - x^d_j) \right) x_i(t), i \in \mathbb{Z}^N \quad (9)$$

The result is stated below

**Corollary 1.** Consider $N$ systems of (1) with control input $u_i$ given by (9). Suppose assumption (A2) is satisfied, then the problem (P1) is solved if and only if (A1) holds.

Corollary 1 is a consequence of Theorem 1 and Lemma 3.
4. Extension to self-loop Laplacian with distributed observer

From control input (9), each agent must know its absolute position. A distributed observer will be designed in this section, such that only relative position between agents and its neighbour is needed in the control input. The control input $u_i$ is now expressed as

$$u_i(t) = \left( -\sum_{j \in N_i} a_{ij} \left( x_i(t) - x_j(t) \right) \right) + \left( \frac{1}{x_i^d} \sum_{j \in N_i} a_{ij} (x_i^d - x_j^d) \right) \dot{x}_i(t), \quad i \in \mathbb{Z}^N, \quad (10)$$

where $\dot{x}_i$ is the estimate of $x_i$, and its dynamics is

$$\dot{\hat{x}}_i(t) = \left( -\sum_{j \in N_i} a_{ij} \left( \hat{x}_i(t) - \hat{x}_j(t) \right) \right) + \left( \frac{1}{x_i^d} \sum_{j \in N_i} a_{ij} (x_i^d - x_j^d) \right) \hat{x}_i(t) \quad (11)$$

We now state the following result

**Theorem 2.** Consider $N$ systems of (1) with $u_i$ given by (10) and observer dynamics given by (11). Let (A2) be satisfied. The problem (P1) is solved if and only if (A1) holds.

Proof. Let $\bar{x} = [\hat{x}_1, \cdots, \hat{x}_N]^T$. The $N$ systems of (10)-(11) can be expressed as

$$\begin{bmatrix} \dot{x}(t) \\ \dot{\hat{x}}(t) \end{bmatrix} = \begin{bmatrix} -L & -R \\ 0 & -\Phi \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}. \quad (12)$$

Let $e = x - \hat{x}$, then the dynamics (12) can be written as

$$\begin{bmatrix} \dot{x}(t) \\ \dot{e}(t) \end{bmatrix} = \begin{bmatrix} -\Phi & R \\ 0 & -L \end{bmatrix} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} = \Omega \begin{bmatrix} x(t) \\ e(t) \end{bmatrix}. \quad (13)$$

From (13) we know that the eigenvalues of $\Omega$ are $\text{spec} \{-\Phi\} \cup \text{spec}\{-L\}$.

($\Rightarrow$) Suppose (A1) holds, then from Theorem 1 and Lemma 2, we know that $\Omega$ has two semisimple eigenvalues of 0 and the nonzero eigenvalues of $\Omega$ has negative real parts. Following are the equations showing eigenvectors of $\Omega$ corresponding to its zero eigenvalues

$$\begin{bmatrix} -\Phi & R \\ 0 & -L \end{bmatrix} \begin{bmatrix} x^d \\ 0 \end{bmatrix} = 0, \quad \begin{bmatrix} -\Phi & R \\ 0 & -L \end{bmatrix} \begin{bmatrix} 1_N \\ 1_N \end{bmatrix} = 0.$$

Meanwhile, for its left eigenvectors we have

$$\begin{bmatrix} y^T \\ -y^T \end{bmatrix} \begin{bmatrix} -\Phi & R \\ 0 & -L \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 \\ y^T \end{bmatrix} \begin{bmatrix} -\Phi & R \\ 0 & -L \end{bmatrix} = 0,$$

where $y^T$ and $y^2T$ are the left eigenvectors of $\Phi$ and $L$ respectively corresponding to their zero eigenvalues, and they satisfy $y^T \Phi = 0$, $y^T x^d = 1$ and $y^2T L = 0$, $y^2T 1_N = 1$. Therefore, the asymptotic solution to (13) is

$$\lim_{t \to \infty} \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} = \left( \begin{bmatrix} y^T \\ -y^T \end{bmatrix} \begin{bmatrix} x(t_0) \\ e(t_0) \end{bmatrix} \right) \begin{bmatrix} x^d \\ 0 \end{bmatrix}$$

$$+ \left( \begin{bmatrix} y^T \\ 0 \end{bmatrix} \begin{bmatrix} x(t_0) \\ e(t_0) \end{bmatrix} \right) \begin{bmatrix} 1_N \\ 0 \end{bmatrix}$$

$$= c_1(t_0) \begin{bmatrix} x^d \\ 0 \end{bmatrix} + c_2(t_0) \begin{bmatrix} 1_N \\ 1_N \end{bmatrix},$$

$$\text{where} \quad c_1(t) = \frac{e(t)}{e(t_0)}, \quad c_2(t) = \frac{x(t)}{x(t_0)}.$$
and thus we have \( \lim_{t \to \infty} x(t) = c_1(t_0)x^d + c_2(t_0)1_N \), and for each \( i \) we can write \( \lim_{t \to \infty} x_i(t) = c_1(t_0)x_i^d + c_2(t_0) \), where \( c_2(t_0) \) is a common term among all agent \( i \). (P1) is shown by evaluating \( (x_i - x_j) \) for all \( i, j \in \mathbb{Z}^N \).

(⇐) Suppose (A1) is not satisfied, then each \( \Phi \) and \( L \) has more than one zero eigenvalues. Therefore, the asymptotic solution of (13) will be (14) plus additional term that also depend on the initial states of \( x \) and \( e \). This way, we cannot guarantee that for any \( x_i(t_0) \) \( i \in \mathbb{Z}^N \), \( \lim_{t \to \infty} x_i(t) = c_1(t_0)x^d_i + \bar{c}(t_0) \) holds for all \( i \in \mathbb{Z}^N \), where \( \bar{c}(t_0) \) is a common term among all agent \( i \). In other words, there exist an index \( i \) such that the asymptotic solution of (13) for \( i \) can be written as \( \lim_{t \to \infty} x_i(t) = c_1(t_0)x^d_i + c_i(t_0) \), where \( c_i(t_0) \) is a term only for index \( i \).

5. Numerical example

The simulation is done on two-dimensional plane with

\[
\begin{bmatrix}
  x^d_1 \\
  x^d_2 \\
  x^d_3 \\
  x^d_4 \\
\end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 3 \end{bmatrix} \quad \text{and} \quad
\begin{bmatrix}
  x^d_1 \\
  x^d_2 \\
  x^d_3 \\
  x^d_4 \\
\end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 3 \end{bmatrix}
\]

Being the desired formation vector on \( x \)-axis and \( y \)-axis respectively, satisfying assumption (A2). We can see that when the formation scaling factor on both \( x \)-axis and \( y \)-axis are equal, the achieved formation forms a square.

In this example, the adjacency matrix entries are \( a_{12} = a_{23} = a_{34} = a_{41} = a_{12} = 1 \), and the rest of the entries are zeros. Two simulations are done, each with different initial position, and the initial position is set to be arbitrary. The simulation results are given in Figure 1, that shows the trajectories of agents' position. Figure 1 shows that the scalable formation is achieved and different initial position results in different formation scaling factor. We see that the formation scaling factor on \( x \)-axis and \( y \)-axis is different. Thus, the achieved formation forms a rectangle rather than a square.

![Figure 1](image.png)

Figure 1. Plot of agents’ trajectories with different initial position of example 1.

6. Conclusion

This paper presented a method to achieve scalable formation on single integrator agents. Scalable formation is achieved by designing the communication weight according to the desired formation. A distributed observer is designed such that only relative position of agents is needed in the control input. The extension of the approach into linear system agents will be discussed in future publication.
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