PRESYMPLECTIC GEOMETRY AND LIOUVILLE SECTORS
WITH CORNERS AND THEIR INTRINSIC CHARACTERIZATION

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Abstract. We provide a presymplectic characterization of Liouville sectors introduced by Ganatra-Pardon-Shende in [GPS20] in terms of the characteristic foliation of the boundary, which we call Liouville \( \sigma \)-sectors. We extend this definition to the case with corners using the presymplectic geometry of null foliations of the coisotropic intersections of transversal coisotropic collection of hypersurfaces which appear in the definition of Liouville sectors with corners. We identify its automorphism group which enables one to give a natural definition of bundles of Liouville sectors. As a byproduct, we affirmatively answer to a question raised in [GPS20, Question 2.6], which asks about the optimality of their definition of Liouville sectors in [GPS20].

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1. Introduction

Ganatra-Pardon-Shende introduced a flexible framework of Liouville sectors (with corners) and established the local-to-global principle of wrapped Fukaya categories in a series of papers [GPS20, GPS18b, GPS18a].

In [OT20], Tanaka and the present author constructed an unwrapped Floer theory for bundles of Liouville manifolds and Liouville sectors. The output was a collection of unwrapped Fukaya categories associated to fibers of a Liouville bundle of Liouville sectors, along with a compatibility between two natural constructions of continuation maps. This set-up enabled them to make the construction of Floer-theoretic invariants of smooth group actions on Liouville manifolds, and they exploited these constructions in [OT19] to construct homotopically coherent actions of Lie groups on wrapped Fukaya categories, thereby proving a version of a conjecture from Teleman’s 2014 ICM address.

1.1. Presymplectic characterization of Liouville sectors. The original definition of Liouville sectors given in [GPS20] makes it somewhat clumsy to identify the structure group of a bundle of Liouville sectors, and to define the bundle of Liouville sectors with corners as in [OT20], which was the starting point of current investigation. In this paper, we introduce a more intrinsic definition but equivalent definition of Liouville sector which skirts this issue: We say it is more intrinsic in that our definition is closer to one in the sense of $G$-structures. (See [Che66] or [Ste83, Chapter VII] for a general introduction to $G$-structures.)

We start with our discussion of $M$ for the case without corners.

Let $(M, \omega)$ be a symplectic manifold with boundary. The boundary $\partial M$ (or more generally any coisotropic submanifold $H$) then carries a natural structure of a presymplectic manifold in the sense that the restriction two form

$$\omega_\partial := \iota^* \omega$$

has constant nullity. (See [Got82], [OP05] for some detailed explanation on presymplectic manifolds.) Here $\iota : \partial M \to M$ is the inclusion map.

**Notation 1.1** $(D_{\partial M}, N_{\partial M} \text{ and } \pi : \partial M \to N_{\partial M})$. We denote the characteristic distribution of $(\partial M, \omega_\partial)$ by

$$D_{\partial M} = \ker \omega_\partial.$$  

With a slight abuse of notation, we also denote by $D_{\partial M}$ the associated integrable foliation, and let $\pi_{\partial M} : \partial M \to N_{\partial M}$ be its leaf map.

Now consider a Liouville manifold $(M, \lambda)$ with boundary and denote by

$$(\partial_\infty M, \xi_\infty)$$
its ideal boundary as a contact manifold equipped with the contact distribution $\xi_\infty$ canonically induced by the Liouville form $\lambda$. (See [Gir17]. We recall that there is no contact form on $\partial_\infty M$ canonically induced from $\lambda$.)

We will always assume that $\partial_\infty M$ is compact. (See [GPS20, Section 1.3] for the strategy of studying the noncompact case of $\partial_\infty M$ through the process of taking a colimit of the compact cases over a homology hypercover.)

**Definition 1.2 (Liouville $\sigma$-sectors).** We say a Liouville manifold with boundary $(M, \lambda)$ is a Liouville $\sigma$-sector if the following holds:

(a) The Liouville vector field $Z$ of $M$ is tangent to $\partial M$ near infinity.
(b) $\partial_\infty M \cap \partial M$ is the boundary of $\partial_\infty M$, and is convex (as a hypersurface of the contact manifold $\partial_\infty M$).
(c) The canonical projection map $\pi : \partial M \to N_{\partial M}$ (to the leaf space of the characteristic foliation) admits a continuous section, and has fibers abstractly homeomorphic to $\mathbb{R}$.

The condition (c) in this definition is the difference from that of the Liouville sector of [GPS20] and is responsible for our naming of Liouville $\sigma$-sectors where $\sigma$ stands either for ‘section’ or for ‘sectional’. It can be replaced by the contractibility of fibers. (See Corollary 4.12.) We will also show in Corollary 3.5 that the line bundle $\mathcal{D}_{\partial M}$ appearing in this definition is trivial.

**Remark 1.3.**
1. Our definition is closer to the one given in the spirit of G-structure (with integrability condition) [Che66], [Ste83]. (See also Corollary 6.2 for a similar characterization of convexity near infinity imposed in Definition (b).) In this sense, the choice of a section corresponds to a reduction of the structure group from $\text{Diff}(\mathbb{R})$ to $\text{Diff}(\mathbb{R}, \{0\})$ of the $\mathbb{R}$-bundle associated to the null foliation.
2. It is worthwhile to mention that the presymplectic structure on $(\partial M, \omega_\partial)$ uniquely determines a symplectic structure on the germ of a neighborhood up to symplectic diffeomorphism. (See [Got82] on Gotay’s coisotropic embedding theorem of presymplectic manifolds [Got82], applied to a germ of neighborhoods of the boundary $\partial M$ or more generally of coisotropic submanifolds of $(M, d\lambda)$.)
3. The condition (c) depends only on the presymplectic geometry of $(\partial M, d\lambda)$ with $\lambda_\partial = i_{\partial M}^* \lambda$ while the conditions (a) and (b) depend on the Liouville geometry of the ideal contact boundary $\partial_\infty M$ the details of $\lambda$ on which matter only “near infinity”. The two geometries are connected by the global topological triviality of the characteristic foliation implied by (c). (See Theorem 1.7.)

Note that a Liouville $\sigma$-sector $M$ is a smooth manifold (possibly with noncompact corners) and the Liouville flow determines a well-defined contact manifold $\partial_\infty M$ “near infinity” (possibly with boundary). We will informally write

$$\partial_\infty M \cap \partial M = \partial(\partial_\infty M)$$

(1.1)

to mean the boundary of $\partial_\infty M$ and call it the ceiling corner of the Liouville sector. (When $\partial_\infty M$ has corners, “boundary” means the union of all boundary strata.)

Throughout this paper, by “near infinity,” we mean “on the complement on some compact subset of $M$.”
Theorem 1.4 (Theorem 3.20 for $H = \partial M$). Under the above definition of Liouville $\sigma$-sector, the following holds:

1. $N_{\partial M}$ carries the structure of Hausdorff smooth manifold such that $\pi : \partial M \to N_{\partial M}$ is a smooth submersion.
2. There exists a smooth section $\sigma^{sm}$ of $\pi : \partial M \to N_{\partial M}$ which can be $C^0$-approximated to the given continuous section $\sigma$ as close as we want.
3. $N_{\partial M}$ carries a canonical symplectic structure denoted by $\omega_{N_{\partial M}}$ as a coisotropic reduction of $\partial M \subset M$: We set $F := \text{Image } \sigma^{sm}$. Then there is a diffeomorphism $\Psi : \partial M \to F \times \mathbb{R}$ and a commutative diagram

\[
\begin{array}{ccc}
\partial M & \xrightarrow{\Psi} & F \times \mathbb{R} \\
\downarrow \pi & & \downarrow \pi_F \\
N_{\partial M} & \xrightarrow{\psi} & F
\end{array}
\]

such that $\pi$ is a smooth map which admits a smooth section $\sigma : N_{\partial M} \to \partial M$ for which $\sigma$ satisfies $\sigma^* \omega_\partial = \omega_{N_{\partial M}}$, and $\pi_F$ is the canonical projection.

4. $(N_{\partial M}, \omega_{N_{\partial M}})$ carries a canonical Liouville one-form $\lambda_{N_{\partial M}}$. The map $\psi$ is a Liouville diffeomorphism between $(N_{\partial M}, \lambda_{N_{\partial M}})$ and the $(F, \lambda|_F)$ with the Liouville form $\lambda|_F$ on $F$, which is given by $\psi(\ell) = \sigma(\ell)$ for $\ell \in N_{\partial M}$.

The existence result of a smooth section $\sigma^{sm}$ is a kind of a smoothing result of the given continuous section $\sigma : N_H \to H$. In the literature, we could not locate such a smoothing result of a section of the leaf space projection of the foliation, and so provide its full proof in Subsection 3.3 for our current circumstance. We refer to Section 3 for the precise description on the dependence of various structures and maps on the choice of section $\sigma$.

Remark 1.5. Other than the existence of the contact vector field transversal to the contact distribution, which is the defining property of the convexity of hypersurfaces, the contact geometry of ideal boundary $\partial_\infty M$ does not enter in the proof of this theorem: It is mainly about the presymplectic geometry of coisotropic submanifold $\partial M$, which makes our affirmative answer to the question [GPS20, Question 2.6] plausible. See Remark 2.9, 3.3 below for a further elaboration.

The following can be also derived in the course of proving the above theorem. (In fact the argument deriving this proposition is nearly identical to that of the proof of [GPS20, Lemma 2.5].)

Proposition 1.6. Let $(M, \lambda)$ be a Liouville $\sigma$-sector. Then

1. Each choice of smooth section $\sigma$ of $\pi$ and a constant $0 < \alpha \leq 1$ canonically provides a smooth function $I : \partial M \to \mathbb{R}$ such that $Z(I) = \alpha I$.
2. There is a germ of neighborhood $\text{Nbhd}(\partial M)$ (unique up to a symplectomorphism fixing $\partial M$) on which the natural extension of $I$, still denoted by $I$, admits a unique function $R : \text{Nbhd}(\partial M) \to \mathbb{R}$ satisfying $\{R, I\} = 1$.

1.2. Solution to [GPS20, Question 2.6]: presymplectic geometry versus Liouville geometry. Another interesting consequence, when combined with Gotay’s normal form theorem of neighborhoods of coisotropic submanifolds, is the following affirmative answer to a question raised by Ganatra-Pardon-Shende in [GPS20].

Theorem 1.7 (Theorem 6.1; Question 2.6 [GPS20]). Suppose $M$ is a Liouville manifold-with-boundary such that
(1) the Liouville vector field is tangent to $\partial M$ near infinity, and
(2) there is a diffeomorphism $\partial M = F \times \mathbb{R}$ sending the characteristic foliation
to the foliation by leaves $\mathbb{R} \times \{p\}$.

Then $\partial_\infty M \cap \partial M$ is convex in $\partial_\infty M$. In particular $M$ is a Liouville sector in the
sense of [GPS20].

The main task is to construct a contact vector field transversal t o the ceiling corner

$$\partial_\infty M \cap \partial M =: F_\infty$$

in the contact manifold $\partial_\infty M$. Subtlety of the proof again lies in the question
how to compromise the difference between the two flows, the Liouville flow and the
characteristic flow, given on $H$ near infinity, the former arising from the presym-
plectic geometry of $\partial M$ while the latter from the Liouville geometry near $\partial_\infty M$.

The theorem claims that the presymplectic geometry of the characteristic foliation
constrains the Liouville geometry near infinity, i.e., triviality of the characteristic
foliation of $\partial M$ implies convexity of the intersection

$$\partial_\infty M \cap \partial M \subset \partial_\infty M$$

near infinity. We make our construction of the aforementioned contact vector field
by utilizing Gotay’s normal form theorem of the neighborhoods of $\partial M \subset M$, whose
details we refer readers to the proof of Theorem 6.1 in Section 6.

The following equivalence theorem is an immediate corollary of Theorem 1.4 and
Theorem 1.7.

**Theorem 1.8.** Let $(M, \lambda)$ be a Liouville manifold with boundary. Suppose the
Liouville vector field $Z$ of $\lambda$ is tangent to $\partial M$ near infinity. Then the followings are
equivalent:

(1) $(M, \lambda)$ is a Liouville sector in the sense of [GPS20].
(2) $(M, \lambda)$ is a Liouville $\sigma$-sector.
(3) There is a diffeomorphism $\partial M = F \times \mathbb{R}$ sending the characteristic foliation
to the foliation by leaves $\mathbb{R} \times \{p\}$.

1.3. Transversal coisotropic collections and Liouville $\sigma$-sectors with cor-
ners. The definition of Liouville $\sigma$-sector can be extended to the case with corners.

Here we start with the giving another equivalent definition of that of the sectorial hy-
persurface from [GPS18b]. Our definition is intrinsic in that it utilizes only the
canonical presymplectic geometry of null foliation of the hypersurface in the sym-
plectic manifold $(M, \omega)$, which is coisotropic. Now the existence of the defining data
of function $I$ or of the diffeomorphism $\partial M \to F \times \mathbb{R}$ appearing in the definition of
Liouville sectors in [GPS20] is a ‘property’ of Liouville $\sigma$-sector in our definition.

We start with giving the aforementioned equivalent definitions.

**Definition 1.9 (\(\sigma\)-sectorial hypersurface).** Let $(M, \lambda)$ be a Liouville manifold with
boundary (without corners). Let $H \subset M$ be a cooriented smooth hypersurface such that
its completion $\overline{H}$ has the union

$$(\partial_\infty M \cap \overline{H}) \cup (\overline{H} \cap \partial M) =: \partial_\infty H \cup \partial \overline{H}$$
as its (topological) boundary. $H$ is a $\sigma$-sectorial hypersurface if it satisfies the following:

(1) $Z$ is tangent to $H$ near infinity,
(2) $H_\infty = \partial_\infty H = \partial_\infty M \cap H \subset \partial_\infty M$ is a convex hypersurface of the contact manifold $\partial_\infty M$.

(3) The canonical projection map $\pi : H \to N_H$ has a continuous section and each of its fiber is homeomorphic to $\mathbb{R}$.

The definition of Liouville $\sigma$-sectors with corners strongly relies on the general intrinsic geometry of the transversal coisotropic collection. Study of this geometry in turn strongly relies on the coisotropic calculus and Gotay’s coisotropic embedding theorem of general presymplectic manifolds [Got82].

**Definition 1.10** (Transversal coisotropic collection). Let $(M, \lambda)$ be a Liouville manifold with corners. Let $H_1, \ldots, H_m \subset M$ be a collection of cooriented hypersurfaces $Z$-invariant near infinity, that satisfies

1. The $H_i$ transversally intersect,
2. All pairwise intersections $H_i \cap H_j$ are coisotropic.

Denote the associated codimension $m$ corner by

$$C = H_1 \cap \cdots \cap H_m$$

and by $N_C$ the leaf space of the null-foliation of the coisotropic submanifold $C$.

Then we prove in Subsection 5.2 that for each choice of sections $\sigma = \{\sigma_1, \cdots, \sigma_m\}$,

- there is a natural fiberwise $\mathbb{R}^m$-action on $C$ which is a simultaneous linearization of the characteristic flows of the sectorial hypersurfaces $H_i$’s.
- each fiber is diffeomorphic to $\mathbb{R}^m$ utilizing the standard construction of action-angle variables in the integrable system.

(See [Arn88] and Corollary 4.12 for the relevant discussion.) This leads us to the final definition of Liouville $\sigma$-sectors with corners.

**Definition 1.11** (Liouville $\sigma$-sectors with corners). Let $M$ be a manifold with corners equipped with a Liouville one-form $\lambda$. We call $(M, \lambda)$ a Liouville $\sigma$-sector with corners if at each corner $\delta$ of $\partial M$, the corner can be expressed as

$$C_\delta := H_{\delta, 1} \cap \cdots \cap H_{\delta, m}$$

for a collection $\{H_{\delta, 1}, \cdots, H_{\delta, m}\}$ such that

1. it is a transversally coisotropic,
2. each fiber of the canonical projection $\pi_{C_\delta} : C_\delta \to N_{C_\delta}$ is contractible.

We call such a corner a $\sigma$-sectorial corner of codimension $m$.

We will show that each choice of $\sigma$ will canonically provide an equivariant splitting data

$$(F, \{(R_i, I_i)\}_{i=1}^m), \quad d\lambda = \omega_F \oplus \sum_{i=1}^m dR_i \wedge dI_i$$
on $\text{Nbd}(C_\delta) \cong F \times \mathbb{C}^m_{\geq 0}$ for $\sigma$-sectorial corners that is equipped with the Hamiltonian $\mathbb{R}^m$-action whose moment map is precisely the coordinate projection $\text{Nbd}(C) \to \mathbb{R}^m_{\geq 0}; \quad x \mapsto (R_1(x), \ldots, R_m(x))$.

(See Theorem 5.4 for the precise statement.)

We also prove the following equivalence result.
Theorem 1.12. Definition 1.11 is equivalent to that of Liouville sectors with corners from [GPS18b].

We refer to Definition 4.1 for the comparison between Definition 1.11 and the definition of Liouville sectors with corners from [GPS18b].

1.4. Automorphism group of Liouville $\sigma$-sectors with corners. Thanks to Theorem 1.4 or Theorem 1.8, our definition of Liouville $\sigma$-sectors with corners enables us to give a natural notion of Liouville automorphisms of Liouville sectors from [GPS20] which is similar to the case without boundary and which does not depend on choices of auxiliary defining function $I$ that appear in the original definition of [GPS20, Definition 2.4].

We start with the following observation

Lemma 1.13 (Lemma 7.3). Fix a diffeomorphism $\phi : (M, \partial M) \to (M, \partial M)$ and suppose $\phi^*\lambda = \lambda + df$ for a function $f : M \to \mathbb{R}$, not necessarily compactly supported. Then the restriction $\phi|_{\partial M} = \phi_\partial : \partial M \to \partial M$ is a presymplectic diffeomorphism, i.e., satisfies $\phi_\partial^*\omega_\partial = \omega_\partial$. In particular, it preserves the characteristic foliation of $\partial M$.

Remark 1.14. Recall that a manifold with corners $X$ is (pre)symplectic if there is a stratawise (pre)symplectic form $\omega$, i.e., a collection of (pre)symplectic forms $\{\omega_\alpha\}_{\alpha \in I}$ that is compatible under the canonical inclusion map of strata

$$i_{\alpha\beta} : X_\alpha \hookrightarrow X_\beta, \quad \alpha < \beta$$

i.e., $\omega_\alpha = i_{\alpha\beta}^*\omega_\beta$. Here $I$ is the POSET that indexes the strata of the stratified manifold $X$. By definition, a diffeomorphism between two manifolds with corners preserves dimensions of the strata.

Lemma 1.13 enables us to define the “structure” of Liouville $\sigma$-sectors (Definition 7.1), and to identify its automorphism group $\text{Aut}(M, \lambda)$ in the same way as for the Liouville manifold case.

Definition 1.15 (Automorphisms group $\text{Aut}(M, \lambda)$). Let $(M, \lambda)$ be a Liouville $\sigma$-sector, possibly with corners. We call a diffeomorphism $\phi : (M, \partial M) \to (M, \partial M)$ a Liouville automorphism if $\phi$ satisfies the following:

$$\phi^*\lambda = \lambda + df$$

for a compactly supported function $f : M \to \mathbb{R}$. We denote by $\text{Aut}(M, \lambda)$ the set of automorphisms of $(M, \lambda)$.

Obviously $\text{Aut}(M, \lambda)$ forms a topological group which is a subgroup of $\text{Symp}(M, d\lambda)$, the group of symplectic diffeomorphisms of $(M, d\lambda)$.

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Conventions:
• Hamiltonian vector field \( X_H \): \( X_H \omega = dH \),
• Canonical one-form \( \theta_0 \) on \( T^* Q \): \( \theta_0 = \sum_{i=1}^n p_i dq_i \),
• Canonical symplectic form \( \omega_0 \) on \( T^* Q \): \( \omega_0 = d(-\theta) = \sum_{i=1}^n dq_i \wedge dp_i \),
• Liouville one-form on \( (T^* Q, \omega_0) \): \( \lambda = -\theta = -\sum_{i=1}^n p_i dq_i \),
• Symplectization \( SC \) of contact manifold \( (C, \theta) \): \( SC = C \times \mathbb{R} \) with \( \omega = d(e^s \pi^* \theta) \). Here note that we write the \( \mathbb{R} \)-factor after the \( C \)-factor.
• Contact Hamiltonian: The contact Hamiltonian of contact vector field \( X \) on a contact manifold \( (M, \theta) \) is given by \( -\theta(X) \). (See [Oh21a] for the same convention adopted in the general framework of contact dynamics.)

Notations:
• \( \partial M \): the asymptotic boundary of \( M \).
• \( \overline{M} \): the completion of \( M \) which is \( \partial M \sqcup \partial M \).
• \( DM \): the union \( \partial M \cup \partial M \) in \( \overline{M} \).
• \( F_\infty := \partial M \cap \partial M \): the ideal boundary of \( \partial M \).
• \( \partial^\text{Liou} \infty M \): the ideal boundary of a Liouville manifold \( (M, \lambda) \).
• \( \text{Aut}(M, \lambda) \): The group of Liouville diffeomorphisms of Liouville \( \sigma \)-sector \( (M, \lambda) \).
• \( \omega_\partial = d\lambda_\partial \): The induced presymplectic form on \( \partial M \) with \( \lambda_\partial := i^* \lambda \).
• \( \text{Aut}(M, \lambda_\partial) \): The group of pre-Liouville diffeomorphisms of exact presymplectic manifolds \( (M, d\lambda_\partial) \).
• \( H \): a \( \sigma \)-sectorial hypersurface \( H \subset M \).
• \( H_\infty = \partial M \cap H \): the ideal boundary \( H \).
• Constants \( N \) and \( C \): We consistently use the letter \( N \) to write the level of symplectization radial function \( s \) and the letter \( C \) for the level of the characteristic flow of the sectorial hypersurface or for the \( \mathbb{R} \)-coordinate in the product \( F \times \mathbb{R} \).

2. Preliminaries

We start with the case without corners but with nonempty boundary \( \partial M \), postponing the study of the case with corners till Section 4.

For the comparison, we recall the definition of Liouville sectors in [GPS20]. In fact we will consider the definition of sectorial hypersurfaces in [GPS18b, Definition 9.2] and restrict that to the sectorial boundary of a Liouville domain.

To facilitate our exposition, we utilize Giroux’s notion of the \textit{ideal completion} of the Liouville domain \( (M, \lambda) \).

**Definition 2.1** (Ideal completion \( \overline{M} \)). [Gir17]

1. An \textit{ideal Liouville domain} \( (W, \omega) \) is a domain endowed with an ideal Liouville structure \( \omega \).
2. The \textit{ideal Liouville structure} is an exact symplectic form on \( \text{Int} W \) admitting a primitive \( \theta \) such that: For some (and then any) function \( u : W \to \mathbb{R}_{\geq 0} \) with regular level set \( \partial_\infty W = \{ u = 0 \} \), the product \( u \theta \) extends to a smooth one-form on \( W \) which induces a contact form on \( \partial W \).
3. When a Liouville manifold \( (M, \lambda) \) is Liouville isomorphic to \( (\text{Int} W, \theta) \), we call \( W \) the ideal completion of \( M \) and denote it by \( \overline{M} \).
Remark 2.2. First, this definition provides a natural topology and smooth structure on the completion $\overline{M}$ and a Liouville structure on $M (= \text{Int } W)$ as an open Liouville manifold. Secondly it also provides a natural class of Liouville diffeomorphisms on $M$ as the restriction of diffeomorphisms of $\overline{M} = W$. (See [Gir17].)

For a (noncompact) Liouville manifold $(M, \lambda)$ (without boundary) its ideal boundary, denoted by $\partial_{\infty} M$, is defined to be the set of asymptotic rays of Liouville vector field $Z$. Then the ideal completion is the coproduct

$$\overline{M} = M \coprod \partial_{\infty} M$$

equipped with the obvious topology.

2.1. Liouville manifolds with boundary and orientations. When $(M, \lambda)$ is a Liouville sector with boundary $\partial M$, its ideal boundary is still well-defined by the $Z$-invariance requirement near infinity put on $\partial M$ in the definition of Liouville sectors [GPS20] and so is its completion $\overline{M}$. Then we have the formula for the topological boundary

$$\partial \overline{M} = \partial_{\infty} M \cup \partial M.$$ 

To ease our exposition, we often abuse our notation

$$DM := \partial_{\infty} M \cup \partial M$$

for the coproduct $\partial_{\infty} M \coprod \partial M$ after the present section, as long as there is no danger of confusion. Likewise we also abuse the notation like

$$\partial_{\infty} M \cap H := \partial_{\infty} M \cap \overline{H}$$

for ideal boundary of $\sigma$-sectorial hypersurface $H$ where the intersection is actually taken as a subset of $\overline{M}$. For the simplicity of notation, we will also use

$$H_{\infty} := \partial_{\infty} M \cap \overline{H}$$

similarly as we denoted $F_{\infty} = \partial_{\infty} M \cap \partial M$ when $H = \partial M$.

2.1.1. Null foliation. We recall the well-known fact that each hypersurface $H \subset M$ in a symplectic manifold $(M, \omega)$ carries the canonical characteristic foliation $\mathcal{D}$. The definition of this foliation is based on the fact that any hypersurface $S$ of $(M, \omega)$ is a coisotropic submanifold in that

1. We have

$$(T_x H)^{\omega_x} \subset T_x H,$$

for any $x \in H$, where $(T_x H)^{\omega_x}$ is the $\omega_x$-orthogonal complement

$$(T_x H)^{\omega_x} := \{ v \in T_x M \mid \omega_x(v, w) = 0 \forall w \in T_x H \}.$$ 

2. Let $\iota_H : H \to M$ be the inclusion map and

$$\ker \iota_H^* \omega_x := \{ v \in T_x H \mid \omega_x(v, w) = 0 \forall w \in T_x H \}$$

has constant rank 1 for all $x \in H$.

Then we denote $\mathcal{D} = \ker \iota_H^* \omega$ which defines a 1-dimensional (integrable) distribution of $H$, and call it the characteristic distribution or the null distribution of $H$. We denote by $\mathcal{N}_H$ the leaf space of the associated foliation. It is also well-known that $\mathcal{D}$ carries a transverse symplectic structure which induces one on the leaf space

$$\mathcal{N}_H := H / \sim$$

(2.2)
chart-wise. With slight abuse of notation, we will also denote by $\mathcal{D}$ the associated foliation. Of course, the quotient topology of a leaf space may not be Hausdorff in general. We will show that under the conditions laid out in Definition 1.2, the aforementioned transverse symplectic form, as well as its smooth structure, descends to the leaf space.

We denote the ideal boundary of $H$ (relative to $Z$) by $\partial_\infty H =: H_\infty$. Then

$$H_\infty = \partial_\infty M \cap \overline{H}.$$ 

At each point $x \in \overline{H} \cap \text{Nbd}(\partial_\infty M) \supset H_\infty$, we have a natural exact sequence

$$0 \to \mathcal{D}_x \to T_x H \to T_x H/\mathcal{D}_x \to 0. \quad (2.3)$$

The quotient carries a canonical symplectic bilinear form and so carries a natural symplectic orientation.

**Choice 2.3** (Orientation of $\mathcal{D}$). Let $H \subset M$ be a proper $\sigma$-sectorial hypersurface. Make a choice of orientation on the trivial line bundle $\mathcal{D} \rightarrow H$.

**Definition 2.4** (Presymplectic orientation on $H$). Let $\mathcal{D} \rightarrow H$ be given an orientation $\omega_{\mathcal{D}}$ on a neighborhood of $H_\infty$ in $\partial_\infty M$. We call the orientation on $T_H|_{H_\infty} = (T_x H/\mathcal{D}_x) \oplus \mathcal{D}_x$, $x \in H \cap \text{Nbd}(\partial_\infty M)$ the *presymplectic orientation* of $H$ relative to $\omega_{\mathcal{D}}$.

**Example 2.5** ($F_{\pm,\infty}$ on $T^*[0,1]$). Now consider the case of the cotangent bundle $M = T^*[0,1]$ of the closed interval $[0,1]$ equipped with the Liouville form

$$\lambda = -p \, dq. \quad (2.4)$$

(This is the negative of the standard Liouville one-form $pdq$ in the cotangent bundle.) The standard orientation of the interval induces a diffeomorphism $M \cong [0,1]_q \times \mathbb{R}_p$ which carries the symplectic orientation induced by the symplectic form

$$dq \wedge dp.$$ 

(We alert the readers that this is the negative of the convention $dp \wedge dq$ used by [GPS20].) The boundary $\partial M \cong \{0,1\} \times \mathbb{R}_p$ has 2 connected components. The characteristic foliation’s orientation is compatible with the vector field $\frac{\partial}{\partial p}$. Note that the Liouville vector field of the Liouville form (2.4) on $T^*[0,1] \cong [0,1]_q \times \mathbb{R}_p$ is given by the Euler vector field

$$\vec{E} : = p \frac{\partial}{\partial p} \quad (2.5)$$

on $T^*M$ which vanishes at $p = 0$. So each leaf $\{q\} \times \mathbb{R}_p$ of the foliation consists of 3 different orbit sets of the Liouville vector field

$$\mathbb{R}_+ = (0, \infty), \quad \{0\}, \mathbb{R}_- = (-\infty, 0).$$

We may identify $\partial_\infty M$ with two disjoint copies of $[0,1]$ at “$p = \pm \infty$.” $F_\infty$ consists of four points, which we will denote by $(0,\pm\infty)$ and $(1,\pm\infty)$ again using the informal notation allowing $p$ to attain $\pm\infty$. Under this notation, we have that

$$F^+_{\infty} = \{(0, -\infty), (1, \infty)\}, \quad \text{and} \quad F^-_{\infty} = \{(0, \infty), (1, -\infty)\}. \quad (2.6)$$
Example 2.6 (dim $Q \geq 2$). More generally, let $Q = Q^n$ be a connected $n$-manifold with boundary and let $M = T^*Q$. The inclusion $T(\partial Q) \hookrightarrow TQ$ induces a quotient map $T^*Q|_{\partial Q} \to T^*(\partial Q)$ of bundles on $\partial Q$; the kernel induces the characteristic foliation on $T^*Q|_{\partial Q} = \partial M$.

Informally: At a point $(q, p) \in \partial M$, the oriented vector defining the characteristic foliation is the symplectic dual to an inward vector normal to $\partial Q$. For example, identifying $Q$ near $\partial Q$ with the right half plane with final coordinate $p_n$, in standard Darboux coordinate $(q, p)$, the characteristic foliation is generated by $\partial \overline{\partial p}$. 

2.2. Convexity of $H_\infty = \partial_\infty M \cap H$ and contact vector field. By applying the notion of $\sigma$-sectorial hypersurface from Definition 1.9 to the boundary $\partial M \subset M$, we introduce the following definition. This is the counterpart of the definition of sectorial hypersurface given in [GPS18b, Definition 9.2].

Definition 2.7 (Liouville $\sigma$-sector). Let $M$ be a noncompact manifold with boundary such that its completion $\overline{M}$ has (topological) boundary given by the union $\partial_\infty M \cup \partial M = DM$ and $\partial_\infty M \cap \partial M$ is the codimension two corner of $\overline{M}$. $M$ is called a Liouville $\sigma$-sector if its boundary $\partial M \subset M$ is a $\sigma$-sectorial hypersurface in the sense of Definition 1.9.

To avoid some confusion with the corners in $\partial M$, we call the intersection $\partial_\infty M \cap \partial \overline{M}$ the ceiling corner. This is the corner of the ideal completion $\overline{M}$ of $M$ of codimension 2. (We will call the genuine corners of $M$ the sectorial corners in Section 4 when we consider the Liouville sectors with corners.)

Recall that $\partial_\infty M$ is naturally oriented as the ideal boundary of symplectic manifold $M$ with $Z$ pointing outward along $\partial_\infty M$.

We take a contact-type hypersurface $S_0 \subset M$ and identify a neighborhood $\text{Nbhd}(\partial_\infty M)$ with the (half) of the symplectization $S_0 \times [0, \infty)$ of the contact manifold $(S_0, \iota^*\lambda)$. We denote $H_0 = S_0 \cap H$. (2.7)

Then considering the Liouville embedding $S_0 \times [0, \infty) \hookrightarrow M$, we can decompose $M$ into $M = (M \setminus \text{Nbhd}(\partial_\infty M)) \cup \text{Nbhd}(\partial_\infty M)$ so that

- $Z = \frac{\partial}{\partial s}$ for the symplectization form $d(e^s \pi^*\iota_{S_0}^*\lambda)$ of the contact manifold $(S_0, \iota_{S_0}^*\lambda)$ on $S_0 \times [0, \infty)$,

- we may identify the one-form $\theta := \pi^*\iota_{S_0}^*\lambda$ as a contact form of $\partial_\infty M$.

By the convexity hypothesis of $H_\infty := H \cap \partial_\infty M$ in $\partial_\infty M$, there exists a contact vector field $\eta$ of the contact structure $(\partial_\infty M, \xi_\infty)$ on a neighborhood of $H_\infty$ in $\partial_\infty M$ that is transverse to $H_\infty$.

Since there are different sign conventions in the literature in defining the contact Hamiltonian associated to a contact vector field, we set our sign convention as follows by adopting the one used by the present author in [Oh21a] and its sequels, which also coincides with that of [dLLV19].
Definition 2.8 (Contact Hamiltonian). We call the function

\[ h := -\theta(\eta) \]

the contact Hamiltonian associated to the contact vector field \( \eta \).

Remark 2.9. It is well-known that a choice of contact vector field \( \eta \) transverse to \( H_\infty \) in \( \partial_\infty M \), gives rise to a decomposition of \( H_\infty \) into

\[ H_\infty = H_\infty^+ \cup \Gamma_\eta \cup H_\infty^- \] (2.8)

where \( H_\infty^\pm \) and \( \Gamma_\eta \) are defined by

\[ H_\infty^\pm = \{ x \in H_\infty \mid \pm \theta(\eta(x)) > 0 \}, \quad \Gamma_\eta = \{ x \in H_\infty \mid \theta(\eta(x)) = 0 \}. \]

(Recall that \( \Gamma_\eta \) is called the dividing set of \( \eta \) on \( H_\infty \). See [Gir91] for a general study of convex hypersurface.) Other than the existence of the contact vector field transversal to the contact distribution, which is the defining property of the convexity of hypersurfaces, this contact geometry of ideal boundary \( \partial_\infty M \) does not enter in our study of presymplectic geometry of coisotropic submanifold, \( \partial M \), which makes our affirmative answer to the question [GPS20, Question 2.6] plausible. See Remark 3.3 below for a further elaboration.

3. Sectional characterization of sectorial hypersurfaces

Let \( H \subset M \) be a \( \sigma \)-sectorial hypersurface of a Liouville \( \sigma \)-sector \( (M, \lambda) \). Equip the leaf space \( N_H \) with the quotient topology induced by the projection \( \pi = \pi_H : H \to N_H \). The main goal of this section is to equip this quotient space with a canonical Liouville structure induced from that of \( M \).

3.1. The leaf space is a topological manifold. Before providing a smooth atlas on \( N_H \), our first order of business is to prove the existence of topological manifold structure thereon. This is the most technical step towards the goal of the section as common in the study of general topology argument. The proof of this proposition occupies the rest of this subsection.

Theorem 3.1. Let \( H \) be a \( \sigma \)-sectorial hypersurface. The leaf space \( N_H \) is a topological manifold. (In particular, \( N_H \) is second countable and Hausdorff.)

We start with the following lemma.

Lemma 3.2. There exists a neighborhood \( \text{Nbhd}(\partial_\infty M \cap H) \) of the ceiling corner \( \partial_\infty M \cap H \) in \( M \) and a smooth function

\[ G : \text{Nbhd}(\partial_\infty M \cap H) \to [0, \infty) \] (3.1)

on \( \text{Nbhd}(\partial_\infty M \cap H) \) of \( M \) that has the following properties:

1. \( Z[G] = G \),
2. its Hamiltonian vector field \( X_G \) is transversal to \( H \) and represents the given coorientation of \( H \) at each point \( x \in H \cap \text{Nbhd}(\partial_\infty M \cap H) \).

Proof. By the defining data of Liouville \( \sigma \)-sectors, we have

- \( H_\infty \) is convex in \( \partial_\infty M \),
- \( Z \) is tangent to \( H \) near infinity.
the second requirement enables us to choose a contact-type hypersurface $S_0$ far out close to $\partial_\infty M$ so that $S_0 \cap H$. Write the smooth hypersurface $H_0 := S_0 \cap H$ of $H$.

We take a symplectization neighborhood of $\partial_\infty M$ obtained by the Liouville embedding

$$\phi_{Z,S_0} : S_0 \times [0, \infty) \hookrightarrow M$$

defined by $\phi_{Z,S_0}(y,t) := \phi^Z_t(y)$. We denote by $s$ the associated radial function defined by $s(y,t) := t$. Then we have the splitting

$$TM|_{S_0 \times [0, \infty)} \cong TS_0 \oplus \mathbb{R} \left\{ \frac{\partial}{\partial s} \right\}$$

and satisfies

$$s^{-1}(0) = S_0, \quad \pi = \frac{\partial}{\partial s}, \quad S_0 \cong \partial_\infty M. \tag{3.3}$$

We also have the contact form $\theta \cong \iota^*_S \lambda$ on $S_0$ so that we can express the Liouville form as

$$\lambda = e^s \pi^* \theta$$
on a neighborhood $\text{Nbhd}(\partial_\infty M)$.

Using the convexity hypothesis of $H_\infty \subset \partial_\infty M$, we can take a contact vector field $\eta$ on a neighborhood of $H_\infty$ in $\partial_\infty M$ such that $\eta \cap H_\infty$. Take its contact Hamiltonian $h = -\theta(\eta)$ on a neighborhood of $H_\infty$ in $\partial_\infty M$. (Recall the sign convention from Definition 2.8 adopted in the present paper.) By considering the function $\pi^* h$ on a neighborhood of $H_\infty$ in $M$, we take the associated homogeneous Hamiltonian function on the symplectization in a neighborhood of $H_\infty$ in $M$, which we denote it by

$$G := e^s \pi^* h$$

which is defined on a neighborhood $H_\infty = H \cap \partial_\infty M$ in $M$, say, on

$$V \times [0, \infty) \subset s^{-1}([0, \infty)) \subset M,$$

where $V \subset \partial_\infty M$ is an open neighborhood of $H_\infty$ in $\partial_\infty M$. Through the symplectization end Liouville embedding $S_0 \times [0, \infty) \hookrightarrow M$, we may identify the function $h : H_\infty \rightarrow \mathbb{R}$ with $\pi^* h|_{\{s=0\}} : H \cap S_0 \times \{0\} \rightarrow \mathbb{R}$. Then the Hamiltonian vector field $X_G|_{H \cap \{s\geq 0\}} : H \cap S_0 \times \{0\} \rightarrow \mathbb{R}$ represents the coorientation of $H$ compatible with the one on $H_\infty \subset \partial_\infty M$ given by $\eta$.

Clearly it satisfies $Z[G] = G$ since $Z = \frac{\partial}{\partial s}$ thereon. This finishes the proof. \qed

**Remark 3.3.** One of the consequences of the convexity of $H_\infty$ in $\partial_\infty M$ is the presence of the function $G$ on $\text{Nbhd}(\partial_\infty M \cap H)$ which gives rise to this taming of the behavior of the characteristic foliation of $H$ in a neighborhood $\text{Nbhd}(\partial_\infty M) \cap H$. Indeed, such a taming is also a sufficient condition for $H_\infty$ to be convex, which is precisely what [GPS20, Question 2.6] is asking about which we provide its affirmative answer in Theorem 6.1 of the present paper.

We fix a Riemannian metric $g$ on $M$ that is $Z$-invariant near infinity $\partial_\infty M$. More explicitly we require the metric to satisfy

- Near $H$, we require it to have the form

$$g = g_H \oplus dt^2$$
on the neighborhood $H \times (-\epsilon, \epsilon) \hookrightarrow M$ where $t$ is the coordinate of $(-\epsilon, \epsilon)$.  

• Near $\partial_\infty M$ on the symplectization end, we require the metric to satisfy
\[ g = g_{S_0} \oplus ds^2 \]
on $S_0 \times [0, \infty)$ where $g_{S_0}$ is any Riemannian metric on $S_0$, recalling $Z = \frac{\partial}{\partial s}$ on this region.

• Near $H_\infty = H \cap \partial_\infty M$, we require that the above two choices are compatible in that $g_{H|H_\infty} = g_{S_0|H_\infty}$ and has the form
\[ g = g_{H_\infty} \oplus dt^2 \oplus ds^2. \]

In addition, using the coorientation hypothesis on $H \subset M$, we fix a coorientation. (For the case of $H = \partial M$, we use the canonical outward coorientation.) Then we choose the aforementioned contact vector field $\eta$ so that it defines the same coorientation as that of the coorientation on $H \subset M$ induced by $X_G|H$. With the above Riemannian metric equipped with the neighborhood of $H$, we require
\[ d\lambda \left( D_H, \frac{\partial}{\partial t} \right) > 0 \quad (3.4) \]
with $D_H$ equipped with the one given in Definition 2.3: Note that we have the exact sequence of symplectic vector bundle
\[ 0 \to \text{span} \left\{ D_H, \frac{\partial}{\partial t} \right\} \to TM|_H \to TM/\text{span} \left\{ D_H, \frac{\partial}{\partial t} \right\} \to 0 \quad (3.5) \]
where we have
\[ TM/\text{span} \left\{ D_H, \frac{\partial}{\partial t} \right\} \cong TH/D_H. \]

**Proposition 3.4.** There exists a unique vector field $Z'$ on $H$ that satisfies the following:

1. $g(Z', Z') \equiv 1$ and in particular $Z'$ is nowhere vanishing,
2. $Z'$ is tangent to the foliation $\mathcal{D}$, and
3. The choice of $Z'$ is compatible with the orientation (2.3) of the leaves and satisfies $d\lambda \left( Z', \frac{\partial}{\partial t} \right) > 0$.

**Proof.** We first recall that the Liouville vector field $Z$ is tangent to $H$ near infinity. Take a contact-type hypersurface $S_0$ and the symplectization Liouville embedding $S_0 \times [0, \infty) \hookrightarrow M$ as before.

We start with defining the vector field $Z' \in \mathcal{D} \subset TH$ on $H$ along the hypersurface $s^{-1}(N) = S_0 \times \{N\} \cong S_0$ by expressing it as the sum
\[ Z' = Y' + a \frac{\partial}{\partial s}, \]
for some function $a = a(y)$ on $S_0$, and $Y'$ tangent to $s^{-1}(N) \cap H$ for all $N \geq 0$. Then we have
\[ 0 < d\lambda \left( Y' + a \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right) \]
and
\[ d\lambda \left( Y' + a \frac{\partial}{\partial s}, X \right) = 0 \quad \text{for all } X \in TH, \]
\[ d\lambda \left( Y' + a \frac{\partial}{\partial s}, \frac{\partial}{\partial s} \right) = 0 \quad \text{on } s^{-1}(N) \cap H \]
The second equation also implies $d\lambda(Y', \frac{\partial}{\partial s}) = 0$ since $Z = \frac{\partial}{\partial s}$ is tangent to $H$ for all sufficiently large $N > 0$.

Using the property that $Z$ is tangent to $H$ near infinity, we will choose $Z'$ near infinity, say for $s \geq N$ for a sufficiently large $N > 0$ so that
\[ Z'(y, s) := d\phi_Z(Z'(y, N)) = Y'(y) \oplus a(y) \frac{\partial}{\partial s} \] (3.6)
i.e., it is just the $s$-translation of the initial vector $Z'(y, 0)$. By normalizing $Z'$ to $Z'/|Z'|$, we may assume that $Z'$ has unit norm which makes its choice unique among the vector fields tangent to $D$ in the orientation given in Choice 2.3.

Next we would like to extend the vector field $Z'$ to everywhere on $H$ that still satisfies the standing requirements (1) - (3). For this purpose, we consider equation for $Z' \in TH$ in the orientation from Choice 2.3 to satisfy
\[ Z' \in D \subset TH, \quad d\lambda(Z', TH) = 0, \quad d\lambda\left(Z', \frac{\partial}{\partial t}\right) > 0. \] (3.7)

By further requiring $|Z'| = 1$, the equation is uniquely solvable at each point of $y \in H$. This finishes the proof of Proposition 3.4. \hfill \Box

In the course of proving the above proposition, we have also proved the following.

**Corollary 3.5.**

1. The line bundle $D \to H$ is trivial.
2. Denote by $R : \text{Nbhd}(H) \to \mathbb{R}$ the defining function $R := t$ on a neighborhood of $H$ in $M$. Then $Z' = \frac{\partial}{\partial R}$.

**Proof.** Statement (1) is obvious since $Z'$ is nowhere vanishing section of the line bundle $D \to H$. For Statement (2), we note that $X_R$ satisfies $d\lambda(X_R, TH) = dR(TH)dt(TH) \equiv 0$ and hence is tangent to $D$. Furthermore we have
\[ d\lambda\left(X_R, \frac{\partial}{\partial t}\right) \equiv 1 > 0 \]
Then by the aforementioned uniqueness, we derive $Z' = \frac{\partial}{\partial R}$. \hfill \Box

This corollary will be useful for the later study of intrinsic characterization of Liouville sectors with corners. (Of course this is a tautological property with the original definition of Liouville sectors from [GPS20].)

**Definition 3.6 (Leaf-generating vector field $Z'$ of $D_H$).** We call the above constructed vector field $Z'$ on $H$ a leaf-generating vector field of $D_H$.

The next lemma shows that the presence of continuous section implies that line bundle $D$ is trivial and that the leaf space $N_H$ is Hausdorff with respect to the quotient topology of $\pi : H \to N_H$.

**Lemma 3.7.** Take a continuous section $\sigma_{\text{ref}} : N_H \to H$ guaranteed by Definition 1.2. We write
\[ F_{\text{ref}} := \text{Image} \sigma_{\text{ref}} \subset H. \] (3.8)
Then the flow map
\[ \Phi_{\text{ref}} : F_{\text{ref}} \times \mathbb{R} \to H; \quad \Phi_{\text{ref}}(y, t) = \phi_{Z'}(\sigma_{\text{ref}}(\pi(x))) \] (3.9)
is a homeomorphism.

**Proof.** We will first show
(1) Any trajectory of $Z'$ eventually exits from any given compact subset $K \subset M$ both forward and backward.

(2) Moreover every leaf is a flow orbit of $Z'$ and vice versa.

It is a standard fact that each leaf is second countable because the manifold $M$ is assumed to be second countable. (This rules out the possibility for a leaf becomes a ‘Long line’ [SS78, pp. 71-72].) Note that since $Z'$ is regular, each leaf of $H$ of the characteristic foliation is a flow line of the regular vector field $Z'$. (See [CC00, Section 2.1].) Furthermore no leaf can be a point. By the condition stated in Definition 1.2 (d), $Z'$ cannot have a nontrivial periodic orbit either. Therefore each flow trajectory $t \mapsto \Phi_t^\ref(y)$ in $H$ defined on $\mathbb{R}$ is one-one, and hence $\Phi^\ref$ is a one-one map.

Furthermore there is a uniquely defined $T \in \mathbb{R}$ such that $\phi^{\ref}_t(\sigma^\ref(\pi(x))) = x$ for each $x \in H$. We define a continuous function $T : M \to \mathbb{R}$ by

$$T(x) := \text{“the reaching time of the flow of } Z' \text{ issued at } \sigma(\pi(x)) \text{”} \quad (3.10)$$

By definition, $H$ is an increasing union

$$H = \bigcup_{C} T^{-1}[-C,C]$$

of open subset $T^{-1}[-C,C]$. Since $|Z'| = 1$ and $Z'$ is tangent to the leaf $\ell_y$ through $y$ of the characteristic foliation, any point $y \in F$ has its forward (resp., backward) flow of $Z'$ goes out the given compact subset. This proves the aforementioned claims.

Then, combining this with the aforementioned completeness, we can define another map

$$\Psi^\ref : H \to F \times \mathbb{R}; \quad \Psi^\ref(x) = (\sigma^\ref(\pi(x)),T(x)). \quad (3.11)$$

By construction, $\Psi^\ref$ is continuous and satisfies

$$\Psi^\ref \circ \Phi^\ref = id_{H}, \quad \Phi^\ref \circ \Psi^\ref = id_{F \times \mathbb{R}}$$

This finishes the proof of Lemma 3.7. \hfill $\Box$

Consider the leaf map $\pi^\ref : F \to N_H$ where $F$ equipped with the subspace topology of $H$ and $N_H$ is the quotient topology of the projection $\pi : H \to N_H$.

**Corollary 3.8.** The leaf map $\pi^\ref : F \to N_H$ is a homeomorphism.

**Proof.** Since $\sigma^\ref : N_H \to H$ is a continuous section, we have $\pi^\ref \circ \sigma^\ref = id_{N_H}$ which shows $\pi^\ref$ is surjective.

On the other hand, if $y_1 \neq y_2$ in $F$, then $\ell_{y_1} \neq \ell_{y_2}$ since otherwise we would have

$$y_1 = \sigma^\ref(\ell_{y_1}) = \sigma^\ref(\ell_{y_2}) = y_2$$

which is a contradiction. This shows that $\pi^\ref$ is a bijective continuous map.

By construction, the map

$$\ell \mapsto \sigma^\ref(\ell) ; N_H \to F$$

defines a continuous map which also satisfies $\pi^\ref \circ \sigma^\ref = id_{N_H}$, and $\sigma^\ref \circ \pi^\ref = id_{F}$. This proves that $N_H$ is homeomorphic to $F$. \hfill $\Box$

Now we go back to the proof of Theorem 3.1.

**Wrap-up of the proof of Theorem 3.1.** First we show the following.
Lemma 3.9. \( F_{\text{ref}} \) with the subspace topology of \( H \) is Hausdorff and locally Euclidean (and in particular, locally compact).

Proof. Consider the map \( T : H \to \mathbb{R} \) given in (3.10). Since the function \( T : H \to \mathbb{R} \) is continuous and \( F_{\text{ref}} = T^{-1}(0) \), \( F_{\text{ref}} \) is a closed subset of a smooth manifold \( H \). In particular \( F_{\text{ref}} \) with the subspace topology of \( H \) is Hausdorff.

Furthermore since \( T \) is monotonically increasing along the trajectory of \( Z' \). To see the locally Euclidean property of \( F_{\text{ref}} \), let \( x_0 \in F_{\text{ref}} \) be any given point. We have only to note that (3.11) induces a homeomorphism

\[
U/\sim \to F_{\text{ref}} \cap U
\]

for a sufficiently small foliation chart \( U \) containing \( x_0 \) where \( \sim \) is the orbit equivalence with respect to \( Z' \). Since \( U/\sim \) is homeomorphic to \( \mathbb{R}^{2n-1} \), so is \( F_{\text{ref}} \cap U \). This proves that \( F_{\text{ref}} \cap U \) is locally Euclidean. \( \square \)

Now combination of Corollary 3.8 and Lemma 3.9 finish the proof of Theorem 3.1. \( \square \)

3.2. Smooth structure on the leaf space. When the leaf space is Hausdorff and locally Euclidean, the well-known construction of coisotropic reduction (or symplectic reduction) applies to prove existence of the symplectic structure on the leaf space once the smooth structure on the leave space is equipped. (See [AM78] for example.) Since we also need to construct the map \( \Psi \) appearing in the statement of Theorem 1.4 and will also use the details of the proof later, we provide the full details of the existence proofs of both structures below along the way partly for readers’ convenience.

The goal of this section is to prove the first item of Theorem 1.4. We start with the following proposition whose proof will occupy entirety of this subsection.

Proposition 3.10. The leaf space \( N_H \) carries a canonical smooth manifold structure such that

1. \( \pi : H \to N_H \) is a smooth submersion, and
2. there is a smooth diffeomorphism \( \Psi : H \to N_H \times \mathbb{R} \) which makes the following diagram commute

\[
\begin{array}{ccc}
H & \xrightarrow{\Psi} & N_H \times \mathbb{R} \\
\downarrow{\pi_H} & & \downarrow{\pi_1} \\
N_H & & 
\end{array}
\]  

(3.12)

We follow the standard notation of [CC00] in our discussion of foliations. It follows from a well-known result in foliation theory that the foliation \( \mathcal{F} \) is determined by its holonomy cocycle \( \gamma = \{\gamma_{\alpha\beta}\}_{\alpha,\beta \in \mathcal{U}} \) with

\[
\gamma_{\alpha\beta} : y_\beta(U_\alpha \cap U_\beta) \to y_\alpha(U_\alpha \cap U_\beta),
\]

arising from the transverse coordinate map \( y_\alpha : U_\alpha \to \mathbb{R}^{2n-2} = \mathbb{R}^{2n-2} \text{ or } \mathbb{H}^{2n} \).

Each \( y_\alpha \) is a submersion and \( \gamma_{\alpha\beta} \) is given by \( y_\alpha = y_\alpha(y_\beta) \) in coordinates. (See e.g., [CC00, Definition 1.2.12].) Furthermore for the null foliation \( \mathcal{F} \) of the coisotropic submanifold \( H \), we can choose a foliated chart \( \mathcal{U} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{U}} \) so that the associated cocycle elements \( \gamma_{\alpha\beta} \) become symplectic, i.e., the foliation \( \mathcal{F} \) carries a transverse symplectic structure. We refer readers to the proof of Proposition 3.10 below for the details.
Remark 3.11. When $H$ has corners, the foliated chart $B = B_x \times B_y$ means that the tangential factor $B_x$ of the foliated chart has no boundary but the transverse factor $B_y$ has a boundary. (See e.g., [CC00, Definition 1.1.18] for the definition.)

We will first show that the above holonomy cocycle naturally descends to a smooth atlas on $N_H$ under the defining condition of $\sigma$-sectorial hypersurface above, especially in the presence of a continuous section of the projection $\pi_H : H \to N_H$.

For this purpose, we consider a coherent regular foliated atlas $\{\varphi_{\alpha} : U_{\alpha} \to \mathbb{R}^{2n-1}\}$, and its associated foliation cocycle $\gamma = \{\gamma_{\alpha\beta}\}$ (see e.g., [CC00, Section 1.2.A]).

By considering a refinement $\{U_{\alpha'}\}$ of the given covering, we can choose a collection of foliated charts $\varphi_{\alpha'} : U_{\alpha'} \to \mathbb{R}^{2n-2} \times \mathbb{R}$ of the form

$$(y_1, \ldots, y_{2n-2}, t)$$

whose transversal coordinate $(y_1, \ldots, y_{2n-2})$ satisfies

$$dt(Z') \equiv 1.$$  

We take a maximal such collection which we denote by

$$\mathcal{O}' = \{\varphi_{\alpha'}, U_{\alpha'}\}.$$  

By the definition of transverse coordinates $(y_1, \ldots, y_{2n-2})$ of the foliated chart, it follows that the collection thereof defines a smooth atlas of $N_H$. We write the resulting atlas of $N_H$ by

$$[\mathcal{O}'] := \{[\varphi_{\alpha'}] : [U_{\alpha'}] \to \mathbb{R}^{2n-2}\}.$$  

Lemma 3.12. The projection map $\pi : H \to N_H$ is a smooth submersion.

Proof. To show smoothness of $\pi$, we will show that for any smooth function $f : N_H \to \mathbb{R}$ the composition $f \circ \pi$ is smooth. For this purpose, at any point $x$, we consider the foliated chart $\varphi_{\alpha} : U_{\alpha} \to \mathbb{R}^{2n-1}$ given above in (3.13).

Let $f : N_H \to \mathbb{R}$ be any smooth function on $N_H$. With respect to the aforementioned foliated atlas of $H$, we will show that $f \circ \pi$ is smooth at every point $x \in H$.

If $x$ is contained in $U_{\alpha'}$, we have

$$(f \circ \pi) \circ (\varphi_{\alpha'})^{-1}(y_1, \ldots, y_{2n-2}, t) = f \circ (\varphi_{\alpha'})^{-1}(y_1, \ldots, y_{2n-2})$$

The right hand side is smooth in the variables $y_1, \ldots, y_{2n-2}$ by the hypothesis on $f$, and does not depend on $t$-variable. This in particular implies that the left hand map $(f \circ \pi) \circ (\varphi_{\alpha'})^{-1}$ is smooth at $x$.

Otherwise, let $(\varphi_{\beta}, U_{\beta})$ be a foliation chart at $x$. We take a flow map $\phi_t^{U_{\beta}}$, satisfying $y := \phi_t^{U_{\beta}}(x) \in U_{\beta}$ for some chart $(\varphi_{\beta'}, U_{\beta'}) \in \mathcal{O}'$ at $y$ given by

$$(U_{\beta'} = \phi_t^{U_{\beta}}(U_{\beta}), \quad \varphi_{\beta'} = \varphi_{\beta} \circ (\phi_t^{U_{\beta}})^{-1})$$

which is contained in $\mathcal{O}'$ by the maximality of the collection $\mathcal{O}'$.

Therefore the map $(f \circ \pi) \circ \varphi_{\beta'}^{-1}$ is smooth at $y = \phi_t^{U_{\beta}}(x) \in U_{\beta'}$. We can factorize $f \circ \pi$ into

$$f \circ \pi = \left((f \circ \pi) \circ \varphi_{\beta'}^{-1}\right) \circ \left((\varphi_{\beta'} \circ \phi_t^{U_{\beta}})|_{U_{\beta}}\right)$$

which is a composition of two smooth maps and so smooth at $x$. This implies $f \circ \pi$ is smooth at $x$. This finishes the proof of smoothness $\pi \circ f$ for all smooth function $f : N_H \to \mathbb{R}$ and hence proves that $\pi$ is smooth.

Submersivity of $\pi$ is obvious by the above construction. \qed
3.3. Construction of a smooth section: smoothing. Finally, we would like to improve the existence of continuous section of \( \pi : H \to N_H \) to a smooth one \( \sigma^{sm} : N_H \to H \). For this purpose, we apply the ‘standard mollifier smoothing and a partition of unity’. However a priori the set of sections of the fibration \( H \to N_H \) is not a linear space which prevents us from directly implementing the smoothing of the sections.

The first order of business for our purpose is to reduce the problem of smoothing to that of smoothing a section of certain smooth line bundle. For this purpose, we need to choose a collection of the atlas of foliated atlases of \( N_H \) that is compatible with the flow of the leaf-generating vector field \( Z' \) on \( H \).

For the simplicity of notation and exposition, we write the maximal atlas \( O' \) chosen in the previous subsection back as \( O \) and the atlas of \( N_H \) given in (3.16) without prime.

Thanks to the property (3.14), the transition map
\[
\varphi_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\alpha \cap U_\beta) \to \varphi_\alpha(U_\alpha \cap U_\beta)
\]
associated to the covering \( O \) has the form
\[
\varphi_{\alpha\beta}(y, t) = (\psi_{\alpha\beta}(y), t + c_{\alpha\beta}(y)) \quad (3.17)
\]
where \( y = (y_1, \ldots, y_{2n-2}) \) on \( \varphi_\alpha(U_\alpha \cap U_\beta) \) for some smooth functions \( c_{\beta\alpha} \) and \( \psi_{\beta\alpha} \) on \( \varphi_\alpha(U_\alpha \cap U_\beta) \).

A direct translation of the cocycle condition of \( \{\varphi_{\alpha\beta}\} \) gives rise to the following identities for \( c_{\alpha\beta} \).

**Lemma 3.13.** Let \( \psi_{\alpha\beta} \) be the transition map for the transverse coordinate charts of \( \varphi_\alpha \) which is given by
\[
\psi_\alpha \circ \psi_\beta^{-1} : \psi_\beta([U_\alpha] \cap [U_\beta]) \to \psi_\alpha([U_\alpha] \cap [U_\beta]).
\]
Then the collection \( c_{\alpha\beta} \) satisfies
\[
c_{\alpha\gamma} = c_{\beta\gamma} + c_{\alpha\beta} \circ \psi_{\beta\gamma} \quad (3.18)
\]
In particular, \( c_{\alpha\alpha} \equiv 0 \) for all \( \alpha \).

The rest of this subsection will be occupied by the proof of the following.

**Proposition 3.14.** There exists a smooth section \( \sigma^{sm} : N_H \to H \) and a diffeomorphism \( \Psi : H \to N_H \times \mathbb{R} \) such that
\[
\sigma^{sm}(\ell) = \Psi^{-1}(\ell, 0). \quad (3.19)
\]
which makes the diagram (3.12) commute.

We first provide some general discussion on the coordinate representation of sections of \( \pi : H \to N_H \). Let \( \sigma : N_H \to H \) be a continuous section of \( \pi \) and \( T_\sigma : H \to \mathbb{R} \) be the continuous function associated to \( \sigma \) given in (3.10). Then we have
\[
F_{ref} := \text{Image} \sigma = T_{\sigma}^{-1}(0) \quad (3.20)
\]
and a homeomorphism \( \Psi_\sigma : H \to N_H \times \mathbb{R} \) of the type
\[
\Psi_\sigma(x) = (\pi_H(x), T_\sigma(x))
\]
whose inverse \( \Phi_\sigma : N_H \times \mathbb{R} \to H \) is given by the flow map
\[
\Phi_\sigma(\ell, t) = \phi_{Z'}^t(\sigma(\ell))
\]
such that $\mathcal{T}_\alpha(\phi^\alpha_{2\beta}(x)) = t$ for all $x \in F_{ref}$.

We take a collection $\{(U_\alpha, \varphi^\alpha)\}$ with $\varphi^\alpha : U_\alpha \to \mathbb{R}^{2n-1}$ of foliated charts of $H$ that covers $F_{ref} = T^{-1}(0)$ each element of which is centered at a point in $F_{ref}$. We write

$$\varphi^\alpha = (y^\alpha, t^\alpha) = (y^\alpha_1, \ldots, y^\alpha_{2n-2}, t^\alpha).$$

Thanks to the requirement (3.14), we must have

$$t^\alpha = t^\beta + c_{\alpha\beta}(y^\beta)$$

(3.21)
on $U_\alpha \cap U_\beta$. (See (3.17).)

Let $\sigma$ be the given continuous section. On each such a chart $(U_\alpha, \varphi^\alpha)$ with $\varphi^\alpha = (y^\alpha, t^\alpha)$, the level set $T^{-1}_\sigma(0)$ of the continuous function $T_\sigma$ can be locally represented as

$$F_{ref} \cap U_\alpha = \{ x \in U_\alpha \mid t^\alpha = f_\alpha(y^\alpha) \}, \quad y^\alpha = (y_1, \ldots, y_{2n-2}) \in V_\alpha \subset \mathbb{R}^{2n-2}$$

for some continuous function $f_\alpha = f_\alpha(y_1, \ldots, y_{2n-2})$ that satisfies

$$\begin{cases}
T \circ \varphi^{-1}_{\alpha\beta}(y, t) = t - f_\alpha(y), \\
\tilde{f}_\alpha(0, \ldots, 0) = 0
\end{cases}$$

(3.22)

The transverse coordinates $(V_\alpha, \psi_\alpha)$ induce a smooth chart on $[U_\alpha] \subset \mathcal{N}_H$, and the function $f_\alpha$ induces a continuous function $f^\alpha_\beta$ thereon. Note that the section $\sigma$ can be expressed in terms of its local representatives $\{\sigma_\alpha := \sigma|_{U_\alpha}\}$: we require them to satisfy

$$\varphi_\alpha(\sigma_\alpha(\ell)) = (\psi_\alpha(\ell), f_\alpha(\ell))$$

in terms of the coordinate charts $([U_\alpha], \psi^\alpha)$ of $\mathcal{N}_H$ and $(U_\alpha, (y^\alpha, t^\alpha))$ of $H$. It follows from the above discussion that to define a global section out of the collection $\{\sigma^\alpha\}$, the collection should satisfy

$$g_\alpha \circ \psi_{\alpha\beta} = g_\beta + c_{\alpha\beta}$$

(3.23)

by (3.21).

We summarize the above discussion into the following.

**Lemma 3.15.** A section of $\pi_H : H \to \mathcal{N}_H$ is characterized by the collection of maps $\{g_\alpha\}$ and $\{c_{\alpha\beta}\}$ with $g_\alpha : \psi_\alpha([U_\alpha]) \to \mathbb{R}$, $c_{\alpha\beta} : \psi_\alpha([U_\alpha] \cap [U_\beta]) \to \mathbb{R}$ that satisfy (3.23), or equivalently

$$g_\beta = g_\alpha \circ \psi_{\alpha\beta} - c_{\alpha\beta}$$

(3.24)
on $\psi_\beta([U_\alpha] \cap [U_\beta])$ and vice versa.

**Proof.** For the proof of (3.24), we apply Lemma 3.13 to (3.23) and get

$$c_{\beta\alpha} \circ \psi^{-1}_{\beta\alpha} = c_{\beta\alpha} \circ \psi_{\alpha\beta} = c_{\beta\beta} - c_{\alpha\beta} = -c_{\alpha\beta}.$$

Then we rewrite (3.23) into

$$g_\beta = (g_\alpha + c_{\beta\alpha}) \circ \psi^{-1}_{\beta\alpha} = (g_\alpha + c_{\beta\alpha}) \circ \psi_{\alpha\beta}$$

$$= g_\alpha \circ \psi_{\alpha\beta} + c_{\beta\alpha} \circ \psi_{\alpha\beta}$$

$$= g_\alpha \circ \psi_{\alpha\beta} - c_{\alpha\beta}$$

which finishes the proof. \qed
By exponentiating (3.24), we get $e^{g_{\alpha}} \circ \psi_{\alpha} = e^{c_{\alpha \beta}} e^{g_{\beta}}$ which is equivalent to
\begin{equation}
e^{g_{\alpha}} \circ \psi_{\alpha} = e^{c_{\alpha \beta} \circ \psi_{\beta} e^{g_{\beta}} \circ \psi_{\beta}}. \quad (3.25)
\end{equation}
If we set $s_{\alpha} = e^{g_{\alpha}} \circ \psi_{\alpha}$ and $g_{\alpha \beta} = e^{c_{\alpha \beta} \circ \psi_{\beta}}$, the equation becomes $s_{\alpha} = g_{\alpha \beta} s_{\beta}$ on $[U_{\alpha}] \cap [U_{\beta}]$.

**Lemma 3.16.** The collection $\{g_{\alpha \beta}\}$ is a $\mathbb{R}_+^+$-valued smooth cocycle.

**Proof.** By definition of $c_{\alpha \beta}$, it is a smooth function. The equation (3.18) is equivalent to
\begin{equation}
c_{\alpha \gamma} \circ \psi_{\gamma} = c_{\beta \gamma} \circ \psi_{\gamma} + c_{\alpha \beta} \circ \psi_{\beta}.
\end{equation}
By exponentiating this equation, we obtain
\begin{equation}
g_{\alpha \gamma} = g_{\beta \gamma} g_{\alpha \beta} = g_{\alpha \beta} g_{\beta \gamma}.
\end{equation}
Furthermore since $c_{\alpha \alpha} = 0$, we have $g_{\alpha \alpha} = 1$. This finishes the proof. □

This shows that the collection $\{g_{\alpha \beta}\}$ defines a real oriented smooth line bundle on $N_H$, and $\{s_{\alpha}\}$ associated to the local representatives $\{f_{\alpha}\}$ of the given section $\sigma$ defines a nowhere vanishing continuous section thereof.

**Remark 3.17.** This line bundle can be also described as follows. The presence of leaf-generating vector field $Z'$ equips each leaf with the structure of an oriented 1-dimensional real affine space. A choice of section of $\pi : H \rightarrow N_H$ then identifies each leaf with the real line $\mathbb{R}$. Then the bundle is nothing but the tautological line bundle of $N_H$.

We denote this smooth oriented line bundle by $L$. Lemma 3.15 shows that this collection also provides $L$ with a trivializing cover and hence defines a smooth trivialization
\begin{equation}
L \rightarrow N_H \times \mathbb{R}.
\end{equation}
We summarize the above discussion into the following.

**Lemma 3.18.** Consider the collections $\{g_{\alpha \beta}\}$ and $\{s_{\alpha}\}$ defined by
\begin{equation}
g_{\alpha \beta} = e^{c_{\alpha \beta} \circ \psi_{\beta}}, \quad s_{\alpha} = e^{f_{\alpha} \circ \psi_{\alpha}}
\end{equation}
of continuous $\mathbb{R}_+^+$-valued functions respectively. Then the collection $\{s_{\alpha}\}$ defines a nowhere vanishing continuous section of the smooth oriented line bundle $L$. We denote the associated global section of $L$ by $s_{\sigma}$.

We are now ready to complete the proof of Proposition 3.14.

**Wrap-up of the proof of Proposition 3.14.** We would like to construct a smooth section or the collection $\{g_{\alpha}\}$ satisfying (3.24), knowing the existence of this continuous section $\sigma$. For this purpose, we have only to find a smooth approximation of the section $s_{\sigma}$ of the line bundle $L$, which is a standard process by taking the mollifier smoothing whose details is now in order.

We denote by $s_{\sigma \alpha}$ the local representative of $s_{\sigma}$ determined by $e^{f_{\alpha}} \circ \psi_{\alpha}$, i.e., we will characterize the section $s_{\sigma}$ by the collection $\{s_{\alpha} : [U_{\alpha}] \rightarrow \mathbb{R}\}$ that satisfy
\begin{equation}
s_{\alpha} = g_{\alpha \beta} s_{\beta}.
\end{equation}
For this purpose, without loss of any generality, we assume
\( \psi_\alpha([U_\alpha]) = I^{2n-2} \) with
\( I = (-1,1) \) for all \( \alpha \), and take a family of mollifier \( \{\rho_\delta\}_{\delta > 0} \) supported in \( I^{2n-2} \). We then take the collection \( \{s_\alpha\} \) by setting
\[
s_\alpha = h_\alpha^\delta \circ \psi_\alpha
\]
for the mollifier smoothing of the functions \( \{e^{f_\alpha}\} \) which are defined by
\[
h_\alpha^\delta = e^{f_\alpha} \ast \rho_\delta
\]
for all \( \alpha \). Here \( \ast \) is the standard convolution product defined by
\[
a \ast b(x) := \int_{\mathbb{R}^{2n-2}} a(x-y)b(y) \, dy
\]
for two real-valued functions \( a, b : \mathbb{R}^{2n-2} \to \mathbb{R} \). Then we take the sum
\[
s_{\text{sm}} := \sum_\alpha \chi_\alpha s_{\sigma,\alpha}
\]
for a partitions of unity \( \{\chi_\alpha\} \) subordinate to \( \{[U_\alpha]\} \) which defines a global smooth section of \( L \).

It follows from the general property of the mollifier smoothing that \( h_\alpha^\delta \to e^{f_\alpha} \) as \( \delta \to 0 \) in compact open topology or in \( C^0 \) topology. This is easy to check (or see [GS68] for example). Therefore \( h_\alpha^\delta \) is nowhere vanishing for a sufficiently small \( \delta = \delta_\alpha > 0 \). Therefore we can take the logarithm \( g_\alpha = \log h_\alpha^\delta \) so that \( h_\alpha^\delta = e^{g_\alpha} \) unambiguously.

Reading back the above explicit correspondence between a section of \( H \to N_H \) and a nowhere-vanishing section of the line bundle \( \mathcal{L} \), we conclude that the collection \( \{g_\alpha \circ \psi_\alpha\} \) associated to \( \{[U_\alpha]\} \) represents a smooth section of the projection \( \pi : H \to N_H \). We denote by \( \sigma_{\text{sm}} \) the corresponding smooth section.

Now we consider the flow map of the vector field \( Z' \Phi \sigma_{\text{sm}} \): \( N_H \times \mathbb{R} \to H \) given by \( \Phi_{H_{\sigma_{\text{sm}}}}(\ell,t) = \phi_{Z'}(\sigma_{\text{sm}}(\ell)) \), and define the map \( \Psi : H \to N_H \times \mathbb{R} \) to be its inverse
\[
\Psi(x) = (\pi_H(x), T_{\sigma_{\text{sm}}}(x)).
\]
By construction, \( \Psi \) now satisfies all the properties required in Proposition 3.10. This finally completes the proof of Proposition 3.10.

This will finish the proof of the diagram (3.28) required in the proof of Theorem 1.4.

### 3.4. Symplectic structure on the leaf space.

Now we turn to the construction of symplectic structure. Using Proposition 3.10, we fix a smooth section \( \sigma_{\text{sm}} : N_H \to H \) and write \( F := \text{Image} \sigma_{\text{sm}} \).

When we choose the above used coherent atlas, we can choose them so that the associated cocycle \( \gamma_{\alpha,\beta} \) becomes symplectic by requiring the chart \( (U_\alpha, \varphi_\alpha) \) also to satisfy the defining equation
\[
(y^\alpha)^* \omega_0 = \iota_H^* \omega, \quad \omega = d\lambda
\]
(3.27)
of the general coisotropic reduction (see [AM78, Theorem 5.3.23] for example) where \( \iota_H : H \to M \) is the inclusion map and \( \omega_0 \) is the standard symplectic form on \( \mathbb{R}^{2n-2} \).

(See also [Got82], [OP05].) By using such a foliated chart satisfying (3.27), the associated holonomy cycles define symplectic atlas and so a symplectic structure on
\(N_H\), when the holonomy is trivial as in our case where we assume the presence of smooth section. This will then finish construction of reduced symplectic structures on \(N_H\). (We refer to [OP05, Section 5] for a detailed discussion on the construction of transverse symplectic structure for the null foliation of general coisotropic submanifolds.)

An immediate corollary of the above construction of diffeomorphism \(\Psi : H \to N_H \times \mathbb{R}\) is that any Liouville \(\sigma\)-sector is a Liouville sector in the sense of [GPS20].

**Remark 3.19.** On the other hand, the converse is almost a tautological statement in that [GPS20, Lemma 2.5] shows that any of their three defining conditions given in [GPS20, Definition 2.4] is equivalent to the condition

- There exists a diffeomorphism \(\Psi : H \to F \times \mathbb{R}\) making (3.12) commute

Once this is in our disposal, \(\Psi\) induces a diffeomorphism \([\Psi] : N_H \to F\). Therefore we can choose a continuous section \(\sigma_{\text{ref}} : N_H \to H\) required for the definition of \(\sigma\)-sectorial hypersurface to be

\[\sigma_{\text{ref}}(\ell) := [\Psi]^{-1}(\ell), \quad \ell \in N_H.\]

Now we wrap up the proof of Theorem 1.4 as the special case \(H = \partial M\) of the following theorem. We will postpone the proof of Statement (3) till the next subsection.

**Theorem 3.20.** Under the above definition of \(\sigma\)-sectorial hypersurface \(H \subset M\), the following holds:

1. \(N_H\) carries the structure of Hausdorff smooth manifold such that \(\pi : H \to N_H\) is a smooth submersion.
2. There exists a smooth section \(\sigma^{\text{sm}}\) of \(\pi : H \to N_H\) which can be \(C^0\)-approximated to the given continuous section \(\sigma\) as close as we want.
3. \(N_H\) carries a canonical symplectic structure denoted by \(\omega_{N_H}\) as a coisotropic reduction of \(H \subset M\): We set \(F := \text{Image } \sigma^{\text{sm}}\). Then there is a diffeomorphism \(\Psi : H \to F \times \mathbb{R}\) and a commutative diagram

\[
\begin{array}{ccc}
H & \xrightarrow{\Psi} & F \times \mathbb{R} \\
\downarrow{\pi} & & \downarrow{\pi_F} \\
N_H & \xrightarrow{\psi} & F
\end{array}
\]

such that \(\pi\) is a smooth map which admits a smooth section \(\sigma : N_H \to H\) for which \(\sigma\) satisfies \(\sigma^* \omega_{\partial M} = \omega_{N_H}\), and \(\pi_F\) is the canonical projection.

4. \((N_H, \omega_{N_H})\) carries a canonical Liouville one-form \(\lambda_{N_H}\): The map \(\psi\) is a Liouville diffeomorphism between \((N_H, \lambda_{N_H})\) and the \((F, \lambda_F)\) with the Liouville form \(\lambda_F\) on \(F\), which is given by \(\psi(\ell) = \sigma(\ell)\) for \(\ell \in N_H\).

3.5. **Induced Liouville structure on the leaf space.** Finally we prove Statement (3) of Theorem 3.20 by extracting some consequences on the above constructed symplectic structure on \(N_H\) derived from the given property of the characteristic foliation \(D\) near infinity. Recall the definitions \(F = \text{Image } \sigma^{\text{sm}}\) and the smooth flow map \(\Phi_H := (\Psi_H^{\text{sm}})^{-1}\)

\[
\Phi_H : F \times [0, \infty) \to H
\]

where \(\Psi_H^{\text{sm}}\) is given in (3.26). By the convexity hypothesis on \(H_\infty\), we have a contact vector field \(\eta\) on \(\partial_\infty M\) that is transverse to \(H_\infty\).
Lemma 3.21. The symplectic manifold \((N_H, \omega_{N_H})\) is exact.

Proof. Note that \(F = \text{Image} \sigma^{sm}\) is a symplectic submanifold of \(M\) and the symplectic form \(d\lambda\) induces an exact symplectic form \(d(\iota_F^* \lambda) = \iota_F^* d\lambda\) for the inclusion map

\[
\iota_F : F \hookrightarrow H \hookrightarrow (M, \lambda).
\]

Therefore it follows from (3.27) and \(\pi^* F = \pi^* F \circ \Phi_H\), \(\iota_F = \iota_F \circ \Phi_H\)

\[
\omega_{N_H} = (\pi_F \circ \sigma^{sm})^* \omega_{N_H} = (\sigma^{sm})^* (\pi_F^* \omega_{N_H})
\]

\[
= (\sigma^{sm})^* (\iota_F^* d\lambda) = (\sigma^{sm})^* \iota_F^* d\lambda = d((\sigma^{sm})^* \iota_F^* \lambda)
\]

which proves exactness of \(\omega_{N_H}\): Here the third equality follows from the defining equation (3.27) and the equalities

\[
\pi_F = \pi_H \circ \Phi, \quad \iota_F = \iota_H \circ \Phi
\]

with the map \(\Phi_H\) given in (3.29). \(\square\)

This leads us to the following reduced Liouville structure on \(N_H\).

Definition 3.22 (Reduced Liouville structure). We call the primitive \(\lambda_{N_H}\) of \(\omega_{N_H}\) defined as above the canonical Liouville structure on \((N_H, \omega_{N_H})\).

4. Geometry of transversal coisotropic collections

Recall that [GPS18b] requires the following properties on the boundary strata when studying Liouville sectors:

Definition 4.1 (Definition 9.2 & Lemma 9.4 & Definition 9.14 [GPS18b]). A sectorial collection is a collection of \(m\) hypersurfaces \(H_1, \ldots, H_m \subset M\), cylindrical near infinity, such that:

(S1) The \(H_i\) transversally intersect,
(S2) All pairwise intersections \(H_i \cap H_j\) are coisotropic, and
(S3) There exist functions \(I_i : \text{Nbd}(\partial M) \to \mathbb{R}\), linear near infinity, satisfying the following on the characteristic foliations \(D_i\) of \(H_i\):

\[
dI_i|_{D_i} \neq 0, \quad dI_i|_{D_j} = 0 \quad \text{for } i \neq j, \quad \{I_i, I_j\} = 0.
\]

(4.1)

A Liouville sector \((M, \lambda)\) with corners is a Liouville manifold-with-corners whose codimension one boundary strata form a sectorial collection.

We will introduce another definition of sectorial collection by replacing Condition (S3) in the spirit of Definition 1.9.

For this purpose, we need some preparations. We start with introducing the following definition

Definition 4.2 (Transversal coisotropic collection). Let \((M, \lambda)\) be a Liouville manifold with boundary and corners. Let \(H_1, \ldots, H_m \subset M\) be a collection hypersurfaces cylindrical near infinity, that satisfies Conditions (S1), (S2) of Definition 4.1.

In the remaining section, we first study the underlying geometry and prove a general structure theorem of such a collection. In the next section, based on the theorem, we will provide an intrinsic characterization of the sectorial collection and Liouville sectors with corners above purely in terms of geometry of coisotropic submanifolds. We call the resulting structure the structure of Liouville \(\sigma\)-sectors with corners.
4.1. **Gotay’s coisotropic embedding theorem of presymplectic manifolds.**

For a finer study of the neighborhood structure of the sectorial corner $C$, we first recall below some basic properties of the coisotropic submanifolds and the coisotropic embedding theorem of Gotay [Got82]. See also [Wei79], [OP05] for relevant material on the geometry of coisotropic submanifolds. We will mostly adopt the notations used in [Got82], [OP05, Section 3].

Let $(Y, \omega_Y)$ be any presymplectic manifold. The null distribution on $Y$ is the vector bundle

$$ E := (TY)^{\omega_Y} \subset TY, \quad E_y = \ker \omega_Y|_y. $$

This distribution is integrable since $\omega_Y$ is closed. We call the corresponding foliation the *null foliation* on $Y$ and denote it by $\mathcal{F} = \mathcal{F}_Y$.

(Then $E$ is nothing but the total space of the foliation tangent bundle $T\mathcal{F}$.) We now consider the dual bundle $\pi : E^* \to Y$ which is the foliation cotangent bundle $E^* = T^*\mathcal{F}$.

The tangent bundle $TE^*$ of the total space $E^*$ has its restriction to the zero section $Y \hookrightarrow E^*$; this restriction carries a canonical decomposition

$$ TE^*|_Y \cong TY \oplus E^*. $$

**Example 4.3.** A typical example of a presymplectic manifold is given by

$$(Y, \omega_Y) = (H, \omega_H), \quad \omega_H := \iota_H^* \omega$$

arising from any coisotropic submanifold $H \subset^{\omega_H} (X, \omega)$. Then $E = D_H$, the null distribution of $(H, \omega_H)$. It is easy to check that the isomorphism $TX \to T^*X$ maps $TY^{\omega}$ to the conormal $N^*Y \subset T^*X$, and induces an isomorphism between $NY = (TX)|_Y/TY$ and $E^*$.

Gotay [Got82] takes a transverse symplectic subbundle $G$ of $TY$ and associates to each splitting $\Gamma : TY = G \oplus E$, $E = T\mathcal{F}$ (4.2) the zero section map

$$ \Phi_\Gamma : Y \hookrightarrow T^*\mathcal{F} = E^* $$

as a coisotropic embedding with respect to a ‘canonical’ two-form $\omega_{E^*}$ on $E^*$ which restricts to a symplectic form on a neighborhood of the zero section of $E^*$ such that

$$ \omega_Y = \Phi_\Gamma^* \omega_{E^*}. $$

**Remark 4.4.** When $\omega_Y = 0$, Gotay’s embedding theorem reduces to the well-known Weinstein’s neighborhood theorem of Lagrangian submanifolds $L$ in which case $E^* = T^* L$ with $Y = L$.

We now describe the last symplectic form closely following [Got82].

We denote the aforementioned neighborhood by $V \subset T^*\mathcal{F} = E^*$.

Using the splitting $\Gamma$, which may be regarded as an ‘Ehresmann connection’ of the ‘fibration’ $T\mathcal{F} \to Y \to N_Y$,
we can explicitly write down a symplectic form $\omega_{E^*}$ as follows.

First note that as a vector bundle, we have a natural splitting

$$TE^*|_Y \cong TY \oplus E^* \cong G \oplus E \oplus T^*F$$

on $Y$, which can be extended to a neighborhood $V$ of the zero section $Y \subset E^*$ via the ‘connection of the fibration’ $T^*F \to Y$. (We refer readers to [OP05] for a complete discussion on this.)

We denote

$$p_\Gamma : TY \to T^*F$$

the (fiberwise) projection to $E = T^*F$ over $Y$ with respect to the splitting (4.2). We have the bundle map

$$TE^* \xrightarrow{T\pi} TY \xrightarrow{p_\Gamma} E$$

over $Y$.

**Definition 4.5** (Canonical one-form $\theta_\Gamma$ on $E^*$). Let $\zeta \in E^*$ and $\xi \in T_\zeta E^*$. We define the one form $\theta_\Gamma$ on $E^*$ whose value is to be the linear functional

$$\theta_\Gamma|_\zeta : T^*_\zeta E^* \to \mathbb{R}$$

at $\zeta$ that is determined by its value

$$\theta_\Gamma|_\zeta (\xi) := \zeta (p_\Gamma \circ T\pi(\xi)) \quad (4.3)$$

against $\xi \in T_\zeta (T^*F)$.

(We remark that this is reduced to the canonical Liouville one-form $\theta$ on the cotangent bundle $T^*L$ for the case of Lagrangian submanifold $L$ in which case $\omega_L = 0$ and the splitting is trivial and not needed.)

Then we define the closed (indeed exact) two form on $E^* = T^*F$ by

$$-d\theta_\Gamma.$$ Together with the pull-back form $\pi^*\omega_Y$, we consider the closed two-form $\omega_{E^*,\Gamma}$ defined by

$$\omega_{E^*,\Gamma} := \pi^*\omega_Y - d\theta_\Gamma \quad (4.4)$$

on $E^* = T^*F$. It is easy to see that $\omega_{E^*,\Gamma}$ is non-degenerate in a neighborhood $V \subset E^*$ of the zero section (See the coordinate expression [OP05, Equation (6.6)] of $d\theta_\Gamma$ and $\omega_V$.)

**Definition 4.6** (Gotay’s symplectic form [Got82]). We denote the restriction of $\omega_{E^*,\Gamma}$ to $V$ by $\omega_V$, i.e.,

$$\omega_V := (\pi^*\omega_Y - d\theta_\Gamma)|_V.$$ We call this two-form *Gotay’s symplectic form* on $V \subset E^*$.

The following theorem ends the description of Gotay’s normal form for the neighborhood of a coisotropic submanifold $C \subset (M, \omega)$ of any symplectic manifold $(M, \omega)$ as a neighborhood $V$ of the zero section of $T^*F_C$ of its null foliation $F_C$ on $C$ equipped with the symplectic form.

**Theorem 4.7** (See [Got82, OP05]). Let $Y \subset (X, \omega_X)$ be any coisotropic submanifold. Fix a splitting $\Gamma$ in (4.2). Then there is a neighborhood $\text{Nbd}(Y) := U \subset X$ and a diffeomorphism

$$\Phi_\Gamma : U \to V \subset E^*$$

such that the following hold:
(1) $\omega_X = \Phi^* \omega_{E^*} \Gamma$ on $U \subset X$.

(2) For two different choices, $\Gamma$ and $\Gamma'$, of splitting of $TY$, the associated two forms $\omega_{E^*} \Gamma$ and $\omega_{E^*} \Gamma'$ are diffeomorphic relative to the zero section $Y \subset E^*$, on a possibly smaller neighborhood $V' \subset E^*$ of $Y$.

Proof. The first statement is proved in [Got82]. Statement (2) is then proved in [OP05, Theorem 10.1].

We have the natural projection map

$$\pi_Y : \text{Nbd}(Y) \to Y$$

(4.5)
defined by

$$\pi_Y := \pi_{E^*} \circ \Phi \circ \iota_Y,$$

(4.6)
for the inclusion map $\iota_Y : Y \hookrightarrow \text{Nbd}(Y) =: U \subset X$, which is induced by restricting the canonical projection $E^* \to Y$ to the neighborhood $V \subset E^*$ of the zero section $Y$. In particular, we have

$$\ker d_x \pi_Y = E_x = \mathcal{D}_Y|_x.$$

4.2. Structure of the null foliation of $\sigma$-sectorial corners. We apply the discussion in the previous subsection to general transversal coisotropic collection

$$\{H_1, \cdots, H_m\}.$$

For any given subset $I \subset \{1, \cdots, m\}$, we denote

$$H_I = \bigcap_{i \in I} H_i$$

and $\pi_{H_i} : H_I \to \mathcal{N}_{H_i}$ be the canonical projection. We also denote the full intersection by

$$C = \bigcap_{i=1}^m H_i.$$

Furthermore, by the transversal intersection property of the coisotropic collection, we can choose the collection $\{\sigma_{C,1}, \cdots, \sigma_{C,m}\}$ to have the complete intersection property in that their images form a collection of transversal intersection. More precisely, we fix the following choice of smooth sections for a finer study of the neighborhood structure of further constructions we will perform.

Choice 4.8 (Choice of sections $\sigma_i : \mathcal{N}_{H_i} \to H_i$). For each $i = 1, \cdots, m$, we choose a smooth section

$$\sigma_i : \mathcal{N}_{H_i} \to H_i$$

for each $i = 1, \cdots, m$. Denote the set of sections $\sigma_i : \mathcal{N}_{H_i} \to H_i$ by

$$\sigma = \{\sigma_1, \cdots, \sigma_m\}.$$  

(4.7)

Recall from Section 3 that for each $i$ a choice of smooth section

$$\sigma_i : \mathcal{N}_{H_i} \to H_i$$

provides the trivialization map

$$\Psi_i^\sigma : H_i \to \mathcal{N}_{H_i} \times \mathbb{R}, \quad \Psi_i^\sigma(x) = (\pi_{H_i}(x), t_i^\sigma(x))$$

given in (3.12). We choose each $\sigma_i$ to be $\sigma_i = \sigma_{H_i}$ as defined in (3.19) For the given choice of $\sigma = \{\sigma_1, \cdots, \sigma_m\}$, we collectively write

$$\Psi_i = \Psi_i^\sigma, \quad i = 1, \cdots, m.$$  

(4.8)
The following theorem is the generalization of Theorem 1.4 whose proof also extends the one used in Section 3 to the case with corners. The main task for this extension is to establish compatibility of the null foliations of various coisotropic intersections arising from taking a sub-collection \( I \subset \{1, \ldots, m\} \): This compatibility condition and construction of relevant strata is in the same spirit as the combinatorial construction of a toric variety out of its associated fan. (See [Ful93] for example.)

**Theorem 4.9.** Let \((M, \lambda)\) be a Liouville \(\sigma\)-sector with corners, and let \(Z\) be the Liouville vector field of \((M, \lambda)\). Let

\[
\sigma = \{\sigma_1, \ldots, \sigma_m\}
\]

be a collection of smooth sections \(\sigma_i : NH_i \to H_i\) for \(i = 1, \ldots, m\). Then the leaf space \(N_C\) carries a canonical structure \(\lambda_{NC}\) of a Liouville manifold with boundary and corners.

We also define the function \(t^{C,\sigma}_i : C \to \mathbb{R}\) to be the restriction

\[
t^{C,\sigma}_i = t_i^{\sigma_i}|_C \tag{4.9}
\]

where \(t_i^{\sigma_i}\) is the function appearing in (3.13). The collection \(\sigma = \{\sigma_i\}\) also induces a surjective map \(\Psi_C : C \to N_C \times \mathbb{R}^m\),

\[
\Psi_C(x) := \left(\pi_C(x), \left(\lambda^{C,\sigma}_1(x), \ldots, \lambda^{C,\sigma}_m(x)\right)\right) \tag{4.10}
\]

which is also smooth with respect to the induced smooth structure on \(N_C\). (The functions \(\lambda^{C,\sigma}_i\) correspond to \(t_i\) appearing in [Arn88, Section 49] in the discussion following below.)

**Proposition 4.10.** There is an \(\mathbb{R}^m\)-action on \(C\) that is free, proper and discontinuous and such that \(C\) is foliated by the \(\mathbb{R}^m\)-orbits. In particular the map

\[
\Psi_C^\sigma : C \to N_C \times \mathbb{R}^m
\]

is an \(\mathbb{R}^m\)-equivariant diffeomorphism with respect to the \(\mathbb{R}^m\)-action on \(C\) and that of linear translations on \(\mathbb{R}^m\).

**Proof.** Let \((s_1, \ldots, s_m)\) be the standard coordinates of \(\mathbb{R}^m\). We set

\[
Z_i := (\Psi_C^\sigma)^* \left(\delta_{NC} \oplus \frac{\partial}{\partial s_i}\right). \tag{4.11}
\]

Then \(Z_i \in T_C\), and \([Z_i, Z_j] = 0\) since \(\left[\frac{\partial}{\partial s_i}, \frac{\partial}{\partial s_j}\right] = 0\). On \(C\), we also have

\[
t_j^{C,\sigma}(Z_i) = d(s_j \circ \Psi_C) \left(\left(\Psi_C^\sigma\right)^* \left(\delta_{NC} \oplus \frac{\partial}{\partial s_i}\right)\right) = ds_j\left(\frac{\partial}{\partial s_i}\right) = \delta_{ij}.
\]

In particular \(Z_i\) is tangent to all level sets of \(\lambda^{C,\sigma}_i\) with \(j \neq i\), and is transversal to the level sets of \(\lambda^{C,\sigma}_i\) for each \(i\).

The so-constructed global frame \(\{Z_1, \ldots, Z_m\}\) of \(TC\) on \(C\) are commuting vector fields. Therefore we have an \(\mathbb{R}^m\)-action on \(C\) induced by the flows of commuting vector fields \(\{Z_1, \ldots, Z_m\}\).

**Lemma 4.11.** This \(\mathbb{R}^m\)-action is also proper and discontinuous. In particular, its isotropy subgroup is a discrete subgroup of \(\mathbb{R}^m\).
Proof. The Liouville vector field $Z$ is tangent to every $H_i$ near infinity. Since $Z$ is tangent to $H_i$ for all $i$ near infinity, the flag

$$H_1 \cap \cdots \cap H_m \subset H_1 \cap \cdots \cap H_{m-1} \subset \cdots \subset H_1$$

is $Z$-invariant near infinity, and in particular we have

$$Z \in TC$$

near infinity of $C$. Since $Z[s] = 1$, $Z$ is also transversal to $s^{-1}(r)$ for all sufficiently large $r > 0$. Therefore the $\mathbb{R}^m$-action induces a free $\mathbb{R}^m/\mathbb{R}$-action on the set $\partial_{\infty} C = \partial_{\infty} M \cap C$ of asymptotic Liouville rays tangent to $C$. Since the latter set is compact, it follows that the $\mathbb{R}^m/\mathbb{R}$-action is proper and discontinuous. Since the flow of $Z$ or the $\mathbb{R}$-action induced by $Z$ moves the level of $s$ by 1 as time varies by 1, we conclude that the $\mathbb{R}^m$-action on $C$ is proper and discontinuous.

Once the action is proved to be proper and discontinuous, the second statement of the lemma follows e.g. from the proof in [Arn88, Section 49, Lemma 3], to which we refer. This finishes the proof. □

With Lemma 4.11 in our disposal, the standard argument in the construction of action-angle coordinates proves that each orbit of the $\mathbb{R}^m$-action is homeomorphic to $\mathbb{R}^n_1 \times T^n_2$ for some $n_1, n_2$ with $n_1 + n_2 = n$. (See [Arn88, Section 49, Lemma 3] and its proof.)

Now we immediately conclude the following

**Corollary 4.12.** Suppose $\pi_C : C \to N_C$ has contractible fibers. Then

1. The $\mathbb{R}^m$-action is free and its fiber is naturally diffeomorphic to $\mathbb{R}^m$, i.e., it is a principle $\mathbb{R}^m$ bundle over $N_C$.
2. The map $\Psi$ is an $\mathbb{R}^m$-equivariant diffeomorphism with respect to the translations of $\mathbb{R}^m$.

The inverse of $\Psi_C$, denoted by

$$\Phi^C_C : N_C \times \mathbb{R}^m \to C$$

is also easy to explicitly write down as follows. First we note

$$t^{C,\sigma}_i (\sigma_{C,i}(\pi_C(x))) = 0$$

for all $i = 1, \ldots, m$ by the definitions of $\sigma_{C,i}$ and $t^{C,\sigma}_i$. Now let a point

$$(\ell, (t_1, \ldots, t_m)) \in N_C \times \mathbb{R}^m$$

be given. Then there is a unique point $x \in C$ satisfying

$$\begin{cases} 
\pi_C(x) = \ell \\
x = \bigcap_{i=1}^m (t_i^{C,\sigma})^{-1}(t_i)
\end{cases}$$

(See (4.9) for the definition of $t_i^{C,\sigma}$ and Proposition 3.4 for the definition of the vector field $Z_i$ respectively.) Then we define $\Phi^C_C(\ell, (t_1, \ldots, t_m))$ to be this unique point. It is easy to check from definition that $\Phi^C_C$ is indeed the inverse of $\Psi_C$. This finishes the proof of Proposition 4.10. □

By applying the above proof and Proposition 4.10 to any sub-collection $I \subset \{1, \cdots, m\}$ including the full collection itself, we also obtain the following stronger form of Theorem 4.9
**Theorem 4.13.** Let $I \subset \{1, \cdots, m\}$ be any sub-collection, and define

$$H_I = \bigcap_{i \in I} H_i.$$ 

Assume $\pi_{H_I} : H_I \to \mathcal{N}_{H_I}$ has contractible fibers. Let $\lambda_{\mathcal{N}_{H_I}}$ be the canonical induced Liouville form as before. Then the following hold:

(1) There is an $\mathbb{R}^{\#I}$-action on $H_I$ that is free, proper and discontinuous and such that $H_I$ is foliated by the $\mathbb{R}^{\#I}$-orbits. In particular the map

$$\Psi^\sigma_{H_I} : H_I \to \mathcal{N}_{H_I} \times \mathbb{R}^{\#I}$$

is an $\mathbb{R}^{\#I}$-equivariant diffeomorphism with respect to the $\mathbb{R}^{\#I}$-action on $H_I$ and that of linear translations on $\mathbb{R}^{\#I}$.

(2) The leaf space $\mathcal{N}_{H_I}$ carries a canonical structure of Liouville manifold with boundary and corners.

By applying the above to the full collection $C = H_{\{1, \cdots, m\}}$, we have finished the proof of Theorem 4.9.

**4.3. Compatibility of null foliations of transversal coisotropic intersections.** Let $C_\delta = C$ as in the previous section and let $\{\sigma_1, \cdots, \sigma_m\}$ a collection of sections $\sigma_i : \mathcal{N}_{H_i} \to H_i$ made in Choice 4.8. For each subset $I \subset \{1, \cdots, m\}$, we have the following section

$$\sigma_I : \mathcal{N}_{H_I} \to H_I$$

defined by

$$\sigma_I([\ell]) := \Phi^\sigma_{H_I}([\ell], (0, \cdots, 0)) = (\Psi^\sigma_{H_I})^{-1}([\ell], (0, \cdots, 0))$$

(4.14)

for the diffeomorphism $\Phi_{H_I}$ given in (4.12) applied to $C = H_I$.

Then for each pair of subsets $I \subset J$ of $\{1, \cdots, n\}$, we have $H_J \subset H_I$ and the map

$$\psi^\sigma_{IJ} : \mathcal{N}_{H_J} \to \mathcal{N}_{H_I}$$

given by

$$\psi^\sigma_{IJ}([\ell]) := \pi_{\mathcal{N}_{H_J}}(\Phi^\sigma_{H_J}([\ell], (0, \cdots, 0))).$$

(4.15)

In particular consider the cases with $I = \{i\}$, $J = \{i, j\}$ and $K = \{i, j, k\}$. Then we prove the following compatibility of the collection of maps $\psi_{IJ}$: For each $i \neq j$, we consider the maps

$$\psi^\sigma_{ij,i} : \mathcal{N}_{H_i \cap H_j} \to \mathcal{N}_{H_i}$$

defined by $\psi^\sigma_{ij,i} := \psi_{(ij)\{i\}}$, and the inclusion maps

$$\iota_{ij,i} : H_i \cap H_j \to H_i.$$ 

**Proposition 4.14.** Let $\{H_1, \cdots, H_m\}$ be a collection of hypersurfaces satisfying only (S1) and (S2). Then the maps $\psi^\sigma_{ij,i}$ satisfy the following:

(1) They are embeddings.

(2) The diagram

$$\begin{array}{ccc}
H_i \cap H_j & \xrightarrow{} & H_i \\
\pi_{ij} \downarrow & & \pi_i \\
\mathcal{N}_{H_i \cap H_j} & \xrightarrow{\psi^\sigma_{ij,i}} & \mathcal{N}_{H_i}
\end{array}$$

(4.16)

commutes for all pairs $1 \leq i, j \leq n$. 
(3) The diagrams are compatible in the sense that we have

\[ \psi^\sigma_{ij,i} \circ \psi^\sigma_{ijk,ij} = \psi^\sigma_{ijk,i} \]

for all triples \( 1 \leq i, j, k \leq n \).

**Proof.** We first show that the map \( \psi^\sigma_{ij,i} \) is an embedding. Let \( \ell_1, \ell_2 \) be two leaves of the null-foliation of \( H_i \cap H_j \) such that

\[ \ell_1 \cap H_i = \ell_2 \cap H_i. \]

By definition of leaves, we have only to show that \( \ell_i \cap \ell_j \neq \emptyset \).

Let \( x \in H_i \) be in the above two common intersection which obviously implies \( x \in \ell_1 \cap \ell_2 \subset H_i \cap H_j \).

This proves \( \psi^\sigma_{ij,i} \) is a one-one map. Then smoothness and the embedding property of \( \psi^\sigma_{ij,i} \) follow from the definition of smooth structures given on the leaf spaces.

For the commutativity, we first note

\[ \psi^\sigma_{ij,i}(\pi_{ij}(x)) = \pi_i(\Phi^\sigma_{ij}((\pi_{ij}(x), 0, 0))) \quad (4.17) \]

by the definition of the maps \( \psi^\sigma_{ij,i} \). But by the definition (4.12) of \( \Phi^\sigma_{ij} \), the point

\[ y := \Phi^\sigma_{ij}((\pi_{ij}(x), 0, 0)) \]

is the intersection point \( y \in \text{Image } \sigma_i \cap \text{Image } \sigma_j \).

Since \( x \in H_i \cap H_j \), we can express it as

\[ x = \Phi^\sigma_{ij}(\pi_{ij}(x), t_1, t_2) \]

for some \( t_1, t_2 \in \mathbb{R} \). In other words, it is obtained from \( y \) by the characteristic flows of \( H_i \) and \( H_j \) by definition of \( \Phi^\sigma_{ij} \) in (4.12). In particular, we have

\[ \pi_i(\iota_{ij,i}(x)) = \pi_i(y). \]

On the other hand, the definition of the null foliation of \( N_{H_i} \) implies

\[ \pi_i(y) = \psi^\sigma_{ij,i}(\pi_{ij}(x)) \quad (4.18) \]

for all \( x \in H_i \cap H_j \). Combining the last two equalities and commutativity of the diagram \( \pi_i \circ \iota_{ij,i} = \psi^\sigma_{ij,i} \circ \pi_{ij} \), we have proved the commutativity of (4.16).

Finally we show that \( \psi^\sigma_{ij,i} \) is a symplectic map. Consider the pull-back

\[ \omega^\sigma_{ij} := (\psi^\sigma_{ij,i})^*(\omega_{N_{H_i}}). \]

We will show that \( \omega^\sigma_{ij} \) satisfies the defining property

\[ \pi^*_{H_i \cap H_j} \omega^\sigma_{ij} = \iota^*_{H_i \cap H_j} \omega, \quad \omega = d\lambda \]

of the reduced form on \( N_{H_i \cap H_j} \) under the coisotropic reduction on the coisotropic submanifolds \( H_i \cap H_j \subset M \). We compute

\[
\begin{align*}
\pi^*_{H_i \cap H_j} \omega^\sigma_{ij} & = \pi^*_{H_i \cap H_j} ((\psi^\sigma_{ij,i})^*(\omega_{N_{H_i}})) \\
& = ((\psi^\sigma_{ij,i} \circ \pi_{H_i \cap H_j})^*(\omega_{N_{H_i}})) \\
& = ((\pi_{H_i} \circ \iota_{H_i \cap H_j,H_i})^*(\pi^*_H \omega_{N_{H_i}})) \\
& = ((\iota_{H_i \cap H_j,H_i})^*(\pi^*_H \omega_{N_{H_i}})) \\
& = ((\iota_{H_i \cap H_j,H_i})^*(\iota^*_H \omega)) = \iota^*_H \omega
\end{align*}
\]
where we use the defining condition of the reduced form $\omega_{N_{H_i}}$ of $\omega_{\partial H_i}$

$$\pi^*_H \omega_{N_{H_i}} = \iota^*_H \omega$$

for the penultimate equality. Therefore we have proved

$$\pi^*_H \omega_{\partial H_i} \omega^{ij}_{\eta_i} = \iota^*_H \omega_{\partial H_i} \omega.$$ 

This shows that the form $\omega^{ij}_{\eta_i}$ satisfies the defining equation (3.27) of the reduced form $\omega_{H_i \cap H_j}$. Then by the uniqueness of the reduced form, we have derived

$$\omega^{ij}_{\eta_i} = \omega_{H_i \cap H_j}.$$ 

This proves $$(\psi^{ij}_{\sigma_{ij}},i)^* \omega_{H_i} = \omega_{H_i \cap H_j},$$ which finishes the proof of Statement (1).

Statement (2) also follows by a similar argument this time from the naturality of the coisotropic reduction by stages: Consider $H_i$, $H_j$, $H_k$ in the given coisotropic collection and consider the two flags

$$H_i \cap H_j \cap H_k \subset H_i \cap H_j \subset H_i \quad (4.19)$$

and

$$H_i \cap H_j \cap H_k \subset H_i.$$ 

(4.20)

The composition $\psi^{ij}_{\sigma_{ij}} \circ \psi^{ijk}_{\sigma_{ij}}$ is the map obtained by the coisotropic reductions in two stages and $\psi^{ijk}_{\sigma_{ij}}$ is the one obtained by the one stage reduction performed in the proof of Statement 1 with the replacement of the pair $(H_i \cap H_j, H_i)$ by $(H_i \cap H_j \cap H_k, H_i)$. Then by the naturality of the coisotropic reduction, we have proved Statement 2. This finishes the proof of the proposition.

The following is an immediate corollary of the above proposition and its proof. (See Remark 1.14 for the relevant remark on the stratified presymplectic manifolds.)

**Corollary 4.15.** The collection of maps

$$\{\psi_I\}_{I \subset \{1, \ldots, m\}}$$

are compatible in that the leaf space $N_{H_i}$ carries the structure of symplectic manifold with boundary and corners.

5. **Liouville $\sigma$-sectors and canonical splitting data**

Let $\{H_1, \ldots, H_m\}$ be a transversal coisotropic collection as in Definition 4.2. We denote their intersection by

$$C = H_1 \cap \cdots \cap H_m$$

as before, which is a coisotropic submanifold of codimension $m$ associated thereto.

5.1. **Definition of Liouville $\sigma$-sectors with corners.** Denote by $\iota_{C_{H_i}} : C \to H_i$ the inclusion map, and $\sigma = \{\sigma_1, \ldots, \sigma_m\}$ be the collection as before. This induces the diagram

$$\begin{array}{ccc}
C & \xrightarrow{\iota_{C_{H_i}}} & H_i \\
\pi_C & & \pi_i \\
N_C & \xrightarrow{\psi^{\partial H_i}} & N_{H_i}
\end{array}$$

(5.1)

for all $i$ which are compatible in the sense of Statement (2) of Proposition 4.14. In fact, we have

$$\mathcal{D}_C = \mathcal{D}_{H_1}|_C + \mathcal{D}_{H_2}|_C + \cdots + \mathcal{D}_{H_m}|_C$$ 

(5.2)
which canonically induces the leaf map \( \psi^\sigma_{C,H} \) in the bottom arrow that makes the diagram commute.

With these preparations, we are finally ready to provide the sectional characterization of Liouville sectors with corners.

**Definition 5.1** (Liouville \( \sigma \)-sectors with corners). Let \( M \) be a manifold with corners equipped with a Liouville one-form \( \lambda \). We call \((M,\lambda)\) a **Liouville \( \sigma \)-sector with corners** if at each sectoral corner \( \delta \) of \( \partial M \), the corner can be expressed as \( C_{\delta} := H_{\delta,1} \cap \cdots \cap H_{\delta,m} \) for a transversal coisotropic collection \( \{H_{\delta,1}, \cdots, H_{\delta,m}\} \) of \( \sigma \)-sectorial hypersurfaces such that fibers of the map \( \pi_{C_{\delta}} : C_{\delta} \to N_{C_{\delta}} \) are contractible. We call such a corner \( C_{\delta} \) a \( \sigma \)-sectorial corner of codimension \( m \).

In the remaining section, we will derive the consequences of this definition.

### 5.2. Integrable systems and canonical splitting data.

By applying Theorem 4.7 to the coisotropic submanifold \( C \), we will obtain a neighborhood \( \text{Nbd}(C) \subset M \) and the projection \( \tilde{\pi}_C : \text{Nbd}(C) \to C \).

**Choice 5.2** (Splitting \( \Gamma^\sigma_C \)). Let \( \sigma = \{\sigma_1, \cdots, \sigma_m\} \) be a choice of sections of transversal coisotropic collection \( \{H_1, \cdots, H_m\} \). Then we associate the splitting \( \Gamma = \Gamma^\sigma_C : T_C = G^\sigma_C \oplus D_C \) (5.3) thereto given by the transversal symplectic subspace \( G^\sigma_C|_x := (d\Psi^\sigma_C|_x)^{-1}(T_{\pi_C(x)}N_C \oplus \{0\}_R^m) \). (5.4)

Applying Theorem 4.7, we obtain a diffeomorphism \( \Psi^\sigma_{t} : \text{Nbd}(C) \to V \subset E^* = T^*\mathcal{F}_C \) where \( \mathcal{F}_C \) is the null foliation of \( C \). Furthermore the pushforward of symplectic form \( d\lambda \) on \( U \) is given by the canonical Gotay’s symplectic form on \( V \subset E^* \)

\[ (\Psi^\sigma_{t})_*(d\lambda) = \pi^*\omega_C - dt \Gamma \]

for the presymplectic form \( \omega_C = \iota^*_C(d\lambda) \) on \( C \). (See Theorem 4.7.)

Note that we have

\[ D_C|_x = \text{span}_R\{Z_1(x), \cdots, Z_m(x)\} \]

by definition of \( Z_i \) above. Therefore the aforementioned \( R^m \)-action induces an \( R^m \)-equivariant bundle isomorphism

\[ D_C \cong C \times R^m \]

over \( C \). (This isomorphism does not depend on the choice of \( \sigma \) but depends only on the Liouville geometry of \( \text{Nbd}(C \cap \partial_{\infty} M) \). )

Then we have made the aforementioned splitting \( TC = G^\sigma_C \oplus D_C \) given in (5.4) \( R^m \)-equivariant. In other words, for each group element \( t = (t_1, \ldots, t_m) \in R^m \), we have the equality

\[ dt(G^\sigma_{x}) = G^\sigma_{t \cdot x}. \]
For a fixed $\alpha > 0$, we put
\[ I_\sigma^i = \pm e^{\alpha t_i C_i} \quad (5.5) \]
which then satisfies $dI_\sigma^i(Z_i) = \alpha I_\sigma^i$ on $C$.

Noting that the induced $\mathbb{R}^m$-action on $TC$ preserves the subbundle
\[ T\mathcal{F}_C = \mathcal{D}_C \subset TC, \]
the canonically induced action on $T^*C$ also preserves the subbundle
\[ \mathcal{D}^\perp_C \subset T^*C \]
for which we have the isomorphism
\[ T^*_F \cong \mathcal{D}^\perp_C. \]
Therefore the $\mathbb{R}^m$-action on $C$ can be lifted to $T^*_F$ which is the restriction of the canonical induced action on $T^*C$ of the one on $C$.

**Lemma 5.3.** We can lift the vector fields $Z_j$'s to $Z_j'$ on $T^*_F$ which are the generators of the induced $\mathbb{R}^m$-action such that

1. $Z_j'|_C = Z_j$,
2. The collection $\{Z_j'\}$ are commuting.

**Proof.** Let $\phi^t_{Z_j}$ be the flow of $Z_j$ on $C$. Since the $\mathbb{R}^m$-action is abelian, the vector fields $Z_j$'s are pairwise commuting. Then lifting $Z_j'$ is nothing but the vector field generating the isotopy of canonical derivatives maps
\[ ((d\phi^t_{Z_j})^{-1} : T^*C \to T^*C \]
on $T^*C$. Since the flows $\phi^t_{Z_j}$ are commuting, their derivatives are also commuting. Then obviously their dual flows $((d\phi^t_{Z_j})^{-1}$ on $T^*C$ are also commuting and hence $Z_j'$'s too. The first condition also follows since any derivative maps zero vector to a zero vector. This finishes the proof. \qed

We now define
\[ \tilde{I}_\sigma^i = I_\sigma^i \circ \pi_{T^*F}. \]
Then $\{d\tilde{I}_\sigma^1, \ldots, d\tilde{I}_\sigma^m\}$ are linearly independent on a neighborhood of the zero section of $T^*F$ if we choose the neighborhood small enough. This is because $\{dI_\sigma^1, \ldots, dI_\sigma^m\}$ are linearly independent on $C$. By suitably adjusting the parametrization $t_i^{C,\sigma}$ of the $\mathbb{R}^m$-action, we can make the equation
\[ d\tilde{I}_\sigma^i(Z_j') = \alpha \delta_{ij} \tilde{I}_\sigma^i \quad (5.6) \]
hold.

This is precisely the situation of completely integrable system to which we can apply the standard construction of action-angle coordinates. (See [Arn88, Section 49] for example.) Therefore, regarding $\{\tilde{I}_\sigma^1, \ldots, \tilde{I}_\sigma^m\}$ as the (fiberwise) angle coordinates, we can find a unique choice of (fiberwise) action coordinates $\{\tilde{R}_\sigma^1, \ldots, \tilde{R}_\sigma^m\}$ over $\mathcal{N}_C$ satisfying
\[ \{\tilde{R}_\sigma^i, \tilde{I}_\sigma^j\} = \delta_{ij}, \quad \tilde{R}_\sigma^i \circ \Phi_C^i|_H = 0 \]
on a neighborhood $V \subset T^*F_C$ of the zero section $0_{T^*F_C} \cong C$. Now we define the pull-back functions
\[ R_\sigma^i := \tilde{R}_\sigma^i \circ \Phi_C^i, \quad I_\sigma^i := \tilde{I}_\sigma^i \circ \Phi_C^i \]
on $U = \text{Nbhd}(C)$. We also pull-back the vector fields $Z^i_j$ to $\text{Nbhd}(C)$ by $\Phi^\sigma_C$, and denote them by $Z_j$. (Note that the notations $I^\sigma_j$ and $Z_j$ are consistent in that their restrictions to $C$ are nothing but the above already given $I^\sigma_j$ or $Z_j$ respectively on $C$.) Furthermore, we have the relationship

$$Z_j = X_{R^\sigma_j}.$$

(See the definition (4.11) of $Z_i$ on $C$.)

Then we have

$$\{R^\sigma_i, R^\sigma_j\} = \omega(X_{R^\sigma_i}, X_{R^\sigma_j}) = \omega(Z_i, Z_j) = 0$$

on $U = \text{Nbhd}^Z(C)$. This is precisely the defining equation of the moment map $\Phi^\sigma_{G,C}$.

Then we have

$$\{R^\sigma_i, R^\sigma_j\} = \omega(X_{R^\sigma_i}, X_{R^\sigma_j}) = \omega(Z_i, Z_j) = 0$$

on $U = \text{Nbhd}^Z(C)$. Since $Z_i := X_{R^\sigma_i}$, we have

$$Z_i|_\omega = dR^\sigma_i$$

on $U = \text{Nbhd}(C)$. This is precisely the defining equation of the moment map $\phi^\sigma_{G,C} : \text{Nbhd}(C) \to g^* \cong \mathbb{R}^m$ with $G = \mathbb{R}^m$ given by

$$\phi^\sigma_{G,C}(x) = (R^\sigma_1(x), \cdots, R^\sigma_m(x))$$

for the above $G = \mathbb{R}^m$-action. Recall that the hypersurfaces $H_i$ are $Z$-invariant near infinity. Therefore we can choose the neighborhood $\text{Nbhd}(C)$ so that it is $Z$-invariant near infinity. Then by the requirement put on the Liouville vector field $Z$ which is pointing outward along $\partial M$, we can choose the whole neighborhood $\text{Nbhd}(C)$ $Z$-invariant. Together with the normalization condition of $R_i$'s

$$R^\sigma_i|_{H_i} = \tilde{R}_i \circ \Phi^\sigma_C|_{H_i} = 0,$$

it also implies $R^\sigma_i \geq 0$ on $\text{Nbhd}(C)$ for all $i$. We now take the neighborhood $U \subset M$ to be this $Z$-invariant neighborhood

$$U = \text{Nbhd}^Z(C).$$

The content of the above discussion can be summarized into the following intrinsic derivation of the splitting data.

**Theorem 5.4 ($\sigma$-Splitting data).** Let $C \subset \partial M$ be a sectorial corner of codimension $n$ associated to the sectorial coisotropic collection \{\(H_1, \ldots, H_m\)\} on $\partial M$. Then for each choice

$$\sigma = \{\sigma_1, \cdots, \sigma_m\}$$

of sections $\sigma_i : \mathcal{N}_{H_i} \to H_i$ of $\pi_{H_i}$ for $i = 1, \cdots, n$, there is a diffeomorphism

$$\Psi^\sigma_C : \text{Nbhd}^Z(C) \cap \partial M \to F \times \mathbb{R}^m$$

and

$$\psi^\sigma_C : \mathcal{N}_C \to F^\sigma_C$$

such that

1. $F^\sigma_C = \text{Image} \sigma_1 \cap \cdots \cap \text{Image} \sigma_m$,
2. $(\Psi^\sigma_C)_* \omega_{\partial} = \pi_F^* \omega_F$,
3. The following diagram

$$\begin{array}{ccc}
\partial M|_C & \xrightarrow{\Psi^\sigma_C} & F^\sigma_C \times \mathbb{R}^m \\
\downarrow \pi_{\partial M} & & \downarrow \pi_F \\
\mathcal{N}_{\partial M|_C} & \xrightarrow{\psi^\sigma_C} & F^\sigma_C.
\end{array}$$

(5.8)
commutes for the map
\[ \Psi_C^\sigma = (\sigma_C \circ \pi_{FC\delta}(I_1^\sigma, \ldots, I_m^\sigma)). \]

(4) The \( G \)-action with \( G = \mathbb{R}^m \) has the moment map \( \phi_{G,C}^\sigma : \text{Nbhd}_C^Z(C) \to \mathbb{R}^m \) given by
\[ \phi_{G,C}^\sigma = (R_1^\sigma, \ldots, R_m^\sigma) \]
for a collection of Poisson-commuting \( R_i \)'s satisfying the simultaneous normalization condition
\[ R_i^\sigma|_{H_i} = 0, \quad R_i^\sigma \geq 0 \]
for all \( i \) on \( \text{Nbhd}_C^Z(C) \).

(5) The map \( \text{Nbhd}(\partial M) \to F_C \times \mathbb{C}^m_{\mathbb{R}^m} \) is given by the formula
\[ \tilde{\Psi}_C^\sigma(x) = (\sigma_C(\pi_{F_C}(x)), I_1^\sigma(x) + \sqrt{-1}I_1^\sigma(x), \ldots, I_m^\sigma(x) + \sqrt{-1}I_m^\sigma(x)) \] for a collection of Poisson-commuting \( R_i \)'s satisfying the simultaneous normalization condition
\[ R_i^\sigma|_{H_i} = 0, \quad R_i^\sigma \geq 0 \]
for all \( i \) on \( \text{Nbhd}_C^Z(C) \).

We call these data a \( \sigma \)-splitting data of \( \text{Nbhd}(C) \) associated to the choice \( \sigma = \{\sigma_1, \ldots, \sigma_m\} \) of sections \( \sigma_i : \mathcal{N}_{H_i} \to H_i \).

We also gather the following consequences of the above discussion separately. The first one, in particular, states that Proposition 1.6 still holds for the Liouville \( \sigma \)-sectors with corners.

**Theorem 5.5.**

(1) Each Liouville \( \sigma \)-sector with corners is a Liouville sector in the sense of Definition 4.1.

(2) The leaf space \( \mathcal{N}_C^\delta \) carries a natural structure of manifold with corners at each sectorial corner \( \delta \) such that the map \( \pi_{FC} : \partial M \to \mathcal{N}_C^\delta \) is a morphism of manifolds with corners.

**Proof.** We have already constructed a diffeomorphism
\[ \Psi_\delta^\sigma : \partial M|_{C_{\delta}} \to F_\delta^\sigma \times \mathbb{R}^m \]
given by
\[ \Psi_\delta^\sigma(x) = (\pi_{F_\delta^\sigma}(x), I_1^\sigma(x), \ldots, I_m^\sigma(x)). \]
Each \( I_i^\sigma \) defined on \( \partial M \) is extended to the function \( \bar{I}_i^\sigma \circ \Phi_{C_{\delta}}^\sigma \) on a symplectic neighborhood \( U_{\delta} := \text{Nbhd}_C^Z(C_{\delta}) \subset M \) via Gotay's coisotropic neighborhood map
\[ \Phi_{C_{\delta}}^\sigma : \text{Nbhd}(C_{\delta}) \hookrightarrow \mathcal{T}^*\mathcal{F}_{C_{\delta}} \]
where the function \( \bar{I}_i^\sigma \) is canonically defined on a neighborhood
\[ V \subset E^* = \mathcal{T}^*\mathcal{F}_{C_{\delta}}. \]
This diffeomorphism \( \Phi_{C_{\delta}} \) onto \( V_{\delta} \subset \mathcal{T}^*\mathcal{F}_{C_{\delta}} \) also induces a splitting of the tangent bundle \( \mathcal{T}C_{\delta} \)
\[ \Gamma_{\delta}^\sigma : \mathcal{T}C_{\delta} = G_\delta^\sigma \oplus \mathcal{T}\mathcal{F}_{C_{\delta}} = G_\delta^\sigma \oplus \mathcal{D}_{C_{\delta}} \]
such that \( G_\delta^\sigma \) is a transverse symplectic subbundle of \( \mathcal{T}C_{\delta} \)
\[ G_\delta^\sigma|_x := d\Psi^{-1}(T_{\pi_{F_\delta^\sigma}(x)}F_\delta^\sigma \oplus \{0\}) \]
at each \( x \in C_{\delta} \). Theorem 5.4 then finishes the construction of the data laid out in Definition 4.1.
For the proof of Statement (2), we start with the observation that for each $H = H_i$ the canonical smooth structure on $N_H$ carries the natural structure of a manifold with boundary and corners through a choice of smooth section made in Choice 4.8, whose existence relies on the defining hypothesis of $\sigma$-sectorial hypersurfaces that the projection map $\pi_H : H \to N_H$ admits a continuous section. For each choice of smooth section, by the same construction as in Subsection 3.5, we have a symplectic structure $(N_H, \omega_{N_H})$, and a smooth map $\sigma_\infty : N_H \to \partial_\infty M$ which is a symplectic diffeomorphism onto the convex hypersurface $F_\infty$ of the contact manifold $(\partial_\infty M, \xi)$. For two different choices of splittings, the resulting structures are diffeomorphic.

Finally it remains to verify the property of $N_C$ carrying the structure of the Liouville manifolds with corners. But this immediately follows from the compatibility result, Proposition 4.14: The moment map $\phi_{G,\delta} : Nbhd(\partial_\infty M) \to \mathbb{R}^m$ provides local description of the codimension $k$-corner of $NC_k$. This finishes the proof. □

6. Solution to [GPS20, Question 2.6] and convexity near infinity

As an application of our arguments used to derive the canonical splitting data, we can now provide the affirmative answer to a question raised by Ganatra-Pardon-Shende in [GPS20].

**Theorem 6.1** (Question 2.6 [GPS20]). Suppose $(M, \lambda)$ is a Liouville manifold-with-boundary that satisfies the following:

1. Its Liouville vector field $Z$ is tangent to $\partial M$ near infinity.
2. There is a diffeomorphism $\partial M = F \times \mathbb{R}$ sending the characteristic foliation to the foliation by leaves $\mathbb{R} \times \{p\}$.

Then $M$ is a Liouville $\sigma$-sector.

The proof will be divided into two parts: we first examine the presymplectic geometry component of the proof, and then combine the discussion with that of the Liouville geometry.

In the mean time, the following is an immediate corollary of Theorem 6.1.

**Corollary 6.2.** In the presence of other conditions, the convexity condition (b) in Definition 1.2 is equivalent to the existence of a diffeomorphism

$$\partial M \cap Nbhd(\partial_\infty M) \cong F_0 \times [C, \infty)$$

sending the characteristic foliation to the foliation by leaves $\{p\} \times [C, \infty)$ for a sufficiently large $C > 0$.

**6.1. Presymplectic geometry.** Denote by $\iota_{\partial M} : \partial M \to M$ the inclusion map. Then the one-form $\lambda_\partial := \iota_{\partial M}^* \lambda$ induces the structure of presymplectic manifold

$$(\partial M, d\lambda_\partial).$$

By definition, $D_{\partial M} = \ker d\lambda_\partial$. Denote by $\Psi : \partial M \to F \times \mathbb{R}$ the diffeomorphism entering in Condition (2) of the hypothesis. Then the hypothesis implies that we have a commutative diagram

$$
\begin{array}{ccc}
\partial M & \xrightarrow{\Psi} & F \times \mathbb{R} \\
\downarrow^{\pi_{\partial M}} & & \downarrow^{\pi_F} \\
N_{\partial M} & \xrightarrow{\psi} & F
\end{array}
$$

"
where $\psi := [\Psi] : N_{BM} \to F$ the obvious quotient map, which becomes a diffeomorphism. In particular, Condition (2) implies that the foliation is a fibration and the induced smooth structure $N_{BM}$ from the presymplectic structure is nothing but the pulled-back of that of $F$.

Obviously the map $\sigma : N_{BM} \to \partial M$ defined by

$$\sigma(l) := \Psi^{-1}(\psi(l),0) \quad (6.2)$$

defines a continuous section of $\pi_{BM} : \partial M \to N_{BM}$, one of the defining data of Liouville $\sigma$-sectors. This section is in fact already smooth with respect to the afore-mention smooth structure equipped with $N_{BM}$.

Next, by Condition (1), we have

$$\partial_{\infty}M \cap \partial M = \partial_{\infty}(\partial M).$$

Therefore it remains to show convexity of $\partial_{\infty}M \cap \partial M$ in $\partial_{\infty}M$, i.e., that there exists a contact vector field defined on $\text{Nbhd}(\partial_{\infty}M \cap \partial M) \subset \partial_{\infty}M$ that is transversal to the hypersurface

$$\partial(\partial_{\infty}M) = \partial_{\infty}M \cap \partial M.$$ 

We denote the reduced symplectic form on $N_{BM}$ of the presymplectic form $d\lambda_{BM}$ by

$$\omega_{BM}.$$ 

Next we prove

**Lemma 6.3.** Suppose that $Z$ is tangent to $\partial M$ outside a compact subset $K \subset M$. Consider the pull-back $\lambda_{\theta} := \iota_{\partial M}^* \lambda$ which is a presymplectic form on $\partial M$. Let $X$ be a vector field tangent to $\ker \iota_{\partial M}^* \lambda = \mathcal{D}_{BM}$. Then we have

$$\mathcal{L}_X \lambda_{\theta} = 0$$

on $\partial M \cap (M \setminus K)$.

**Proof.** Since $X$ spans the characteristic distribution of $(\partial M, \lambda_{\theta})$, we have

$$X[d\lambda_{\theta}] = 0$$

on $\partial M$. On the other hand, since $Z$ is tangent to $\partial M \cap (M \setminus K)$ and $X \in \ker \omega_{\theta} = d\lambda_{\theta}$, we also have

$$0 = d\lambda_{\theta}(Z, X) = \lambda_{\theta}(X)$$

where the second equality follows by definition of Liouville vector field $Z$. Therefore on $\partial M \cap (M \setminus K)$, we compute

$$\mathcal{L}_X \lambda_{\theta} = (d(X] \lambda) + X[d\lambda])|_{\partial M} = 0$$

which finishes the proof. \hfill \Box

We push-forward the presymplectic structure and the one-form $\lambda_{\theta}$ on $\partial M$ to $F \times \mathbb{R}$ by $\Psi$, and write

$$\lambda_{\theta}^\text{pre} := \Psi_*(\lambda_{\theta})$$

on $F \times \mathbb{R}$. By applying Gotay’s embedding theorem, Theorem 4.7, we extend the presymplectic map $\Psi : \partial M \to F \times \mathbb{R}$ to a symplectic thickening

$$\tilde{\Psi} : \text{Nbhd}(\partial M) \to F \times \mathbb{C}.$$ 

We have the natural Liouville form $\lambda_{NBM}$ from $\lambda_{\theta}$ so that

$$\lambda_{\theta} = \pi_{BM}^* \lambda_{NBM}. \quad (6.3)$$
Lemma 6.4. Let $t$ be the standard coordinate of $\mathbb{R}$. Suppose that $Z$ is tangent to $\partial M$ on $(M \setminus K) \cap \partial M$ for a compact subset $K \subset M$. Then there exists a sufficiently large constant $C = C(K) > 0$ such that $\mathcal{L}_Z \lambda^\text{pre} = 0$ on $F \times [-C, C]$ and so

$$
\lambda^\text{pre} = \pi_F^* \lambda_F
$$

for some one-form $\lambda_F$ on $F$, where $\pi_F : F \times \mathbb{R} \to F$ is the projection.

Proof. Since $K$ is compact and $\Psi$ is continuous, $\Psi(K)$ is compact and so there exists a sufficiently large $C > 0$ such that $\Psi(K) \subset F \times [-C, C]$. In particular, we have

$$
F \times ([C, \infty) \cup (-\infty, -C)] \subset \Psi(\partial M \setminus K)
$$

for a sufficiently large $C > 0$. We now newly set

$$
H := F \times ([C, \infty) \cup (-\infty, -C]).
$$

Then Lemma 6.3 applied to $H$ implies

$$
\mathcal{L}_Z \lambda^\text{pre} = 0
$$
on $H$, i.e., $\lambda^\text{pre}$ is $\partial_Z$-invariant and hence there exists a one-form $\lambda_F$ on $F$ such that $\pi_F^* \lambda_F = \lambda^\text{pre}$ thereon. This finishes the proof. \qed

We denote by

$$(Y, d\lambda^\text{pre}), \quad Y := F \times \mathbb{R}$$

the resulting presymplectic manifold $(F \times \mathbb{R}, d\lambda^\text{pre})$. In this case, we have natural identification $N_Y = F$. Then we have the reduced symplectic form of $d\lambda^\text{pre}$ on $N_Y = F$ is given by

$$
\psi^*(\omega_{N_{\partial M}}) = d\lambda_F := \omega_F.
$$

We equip a tubular neighborhood $V = \text{Nbhd}(F \times \mathbb{R}) \subset F \times \mathbb{C}$ of $F \times \mathbb{R} = Y$ with the pushforward symplectic form

$$
\omega_V = \tilde{\Psi}^*(d\lambda).
$$

(6.4)

Theorem 4.7 reads that we have the canonical isomorphism

$$
T^*\mathcal{F} \cong Y \times \mathbb{R} = F \times \mathbb{R} \times \mathbb{R} \cong F \times \mathbb{C}
$$

and that $Y$ is contained in $V \subset T^*\mathcal{F}$ as the zero section of $T^*\mathcal{F}$. We denote by

$$
\text{pr} : V \subset T^*\mathcal{F} \to Y
$$

the tubular projection.

The following lemma shows that we can convert $\omega_V$ to Gotay’s symplectic form which will have the form

$$
\pi_F^* \omega + dR \wedge dI.
$$

Lemma 6.5. Let $u + \sqrt{-1}v$ be the standard coordinates of $\mathbb{C}$, and let $C > 0$ be the constant given in Lemma 6.4. We define

$$
R = u \circ \pi_C \circ \tilde{\Psi}, \quad I = v \circ \pi_C \circ \tilde{\Psi}
$$
on $F \times \mathbb{C}$. Then there is a neighborhood $V' \subset V$ and a diffeomorphism

$$
\Upsilon : V' \cap \{I > C\} \to V
$$
onto its image that preserves \((Y, d\lambda_{pre})\) and so that
\[
\omega_{V'} := \Upsilon^* \omega_V = \pi_F^* \omega_F + dR \wedge dI
\]
on \(\{ I > C \} \cap V'\).

**Proof.** Recall from (6.3) that
\[
pr^* \lambda_{pre} = \pi^* \lambda_F
\]
on \(\{ I \geq C \}\) for a sufficiently large \(C > 0\). Then the inclusion map
\[
(F \times \{ I \geq C \}, d\lambda_{pre}) \hookrightarrow (F \times \mathbb{C}, \pi_F^* \omega_F + dR \wedge dI)
\]
is a coisotropic embedding and the lemma follows by the uniqueness of Gotay’s coisotropic embedding theorem. □

From now on, based on Lemma 6.5, we may and will assume
\[
\tilde{\Psi}_* d\lambda = \omega_V = \pi_F^* \omega_F + dR \wedge dI, \quad \omega_F = \psi_* \omega_{\pi M}
\]
on a neighborhood \(\text{Nbd}(F \times \mathbb{R}) = V' \subset F \times \mathbb{C}_{R \geq 0}\).

6.2. Liouville geometry. In this subsection, we will work with the trivial fibration \(F \times \mathbb{R} =: Y \to F\) and regard \(Y\) as a hypersurface embedded in the Liouville manifold
\[
V \subset F \times \mathbb{C} \cong T^* \mathcal{F},
\]
an open neighborhood of the zero section of the foliation cotangent bundle \(T^* \mathcal{F}\).

We equip \(V\) with Gotay’s normal form \(\omega_V = \pi_F^* \omega_F + dR \wedge dI\) as its symplectic form.

Now we take the Liouville form of \((V, \omega_V)\) given by
\[
\lambda_V = \tilde{\Psi}_* \lambda
\]
and let \(Z_V = \tilde{\Psi}_* Z\) be its associated Liouville vector field. By definition, \(Z_V\) satisfies
\[
Z_V \mid d\lambda_V = \lambda_V
\]
where we have
\[
d\lambda_V = \pi_F^* \omega_F + dR \wedge dI
\]
from (6.6). By decomposing the Liouville vector field \(Z_V\) into
\[
Z_V = X_F + a \frac{\partial}{\partial R} + b \frac{\partial}{\partial I}
\]
in terms of the splitting \(TV = TF \oplus TC\), we compute
\[
Z_V \mid d\lambda_V = X_F \mid \pi_F^* \omega_F + a dI - b dR
\]
for some coefficient functions \(a = a(y, R, I), \ b = b(y, R, I)\) for \((y, R, I) \in F \times \mathbb{C}\). Then (6.7) becomes
\[
X_F \mid \pi_F^* \omega_F + a dI - b dR = \lambda_V.
\]

**Proposition 6.6.** Regard \(Y \cong \{ R = 0 \} =: H\) as a hypersurface of \(V\). Let \(b\) be the coefficient function appearing in (6.9). Then we have \(b \neq 0\) on \(V' \cap \{ I > C' \}\) on a possibly smaller neighborhood \(V' \subset V\) of \(H\) for a sufficiently large constant \(C' > 0\).
Proof. We denote by \( \iota_H : H \to V \) the inclusion map. We first recall from Lemma 6.3 that \( X = \Psi^* \frac{\partial}{\partial R} \in \ker d\lambda_\beta \). Therefore we have
\[
\frac{\partial}{\partial I} | \Psi_* d\lambda_\beta = 0
\]
on \( Y = F \times \mathbb{R} \), since \( \Psi_* \lambda_\beta = \lambda_{\preceq} = \iota_H^* \lambda_V \). Since \( \omega_F = d\lambda_F \), (6.8) implies
\[
d\lambda_V = \pi_F^* d\lambda_F + dR \wedge dI
\]
which in turn implies
\[
d(\lambda_V - \pi_F^* \lambda_F - IdR) = 0.
\]
Since the choice of \( \sigma \) made above implies
\[
\pi_F^* \lambda_F = \Psi_* \lambda_\beta = \iota_H^* \lambda_V
\]
we have \( \iota_H^*(\lambda_V - \pi_F^* \lambda_F - IdR) = 0 \) on \( \{ R = 0 \} \) recalling \( V \to F \times \mathbb{C} \) is a codimension zero embedding. In particular the form \( \lambda_V - \pi_F^* \lambda_F - IdR \) is exact on any neighborhood \( V \) of \( \{ R = 0 \} \) which deformation retracts to \( \{ R = 0 \} \). Therefore we have
\[
\lambda_V - \pi_F^* \lambda_F - IdR = dh_V
\]
on such a neighborhood \( V \) for some function \( h_V : V \to \mathbb{R} \), i.e.,
\[
\lambda_V = \pi_F^* \lambda_F + IdR + dh_V. \tag{6.11}
\]
Since \( \ker \iota_H^*(d\lambda_V) = \text{span}\{ \frac{\partial}{\partial I} \} \), we have
\[
\frac{\partial}{\partial I} | d\lambda_V = 0
\]
on \( H \). Then since \( Z \) is tangent to \( H \) near infinity, we have
\[
\lambda_V \left( \frac{\partial}{\partial I} \right) = d\lambda \left( Z, \frac{\partial}{\partial I} \right) = 0.
\]
Obviously we have \( (\pi_F^* \lambda_F + IdR)(\frac{\partial}{\partial I}) = 0 \). Therefore we have derived
\[
\frac{\partial h_V}{\partial I} \bigg|_{R=0} = 0
\]
by evaluating (6.11) against \( \frac{\partial}{\partial I} \). Therefore \( h_V \big|_{\{ R=0 \}} \) does not depend on \( I \). In particular, we have
\[
\| h_V \big|_{\{ R=0 \}} \|_{C^1} \leq C
\]
for some constant \( C > 0 \). In turn, since \( h_V \) is a smooth function, we have
\[
\| h_V \|_{C^1} \leq C'
\]
by precompactness of \( V/\mathbb{R} \) on a sufficient small neighborhood \( V' \subset V \) of \( H = \{ R = 0 \} \) for some constant \( C' \) choosing \( C' \) larger, if necessary. In particular we have
\[
\left\| \frac{\partial h_V}{\partial R} \right\|_{C^{0,V'}} \leq C'. \tag{6.12}
\]
Substituting (6.11) into (6.10), we obtain the equation
\[
X_F \big| d\lambda_V + a dI - b dR = \pi_F^* \lambda_F + IdR + dh_V.
\]
By evaluating this equation against \( \frac{\partial}{\partial R} \) after substituting (6.8) thereinto, we obtain
\[
b = - \left( I + \frac{\partial h_V}{\partial R} \right)
\]
on \( V \). Therefore we have \( b(y, R, I) \neq 0 \) for all \((y, R, I) \in V'\), if \( |I| > C'\). This finishes the proof. \( \square \)

From (6.9), we have also derived
\[
Z_V[I] = b(y, R, I).
\]
In particular \( Z_V[I] \neq 0 \) on
\[
I^{-1}(I) \cap V
\]
for any \( I \geq C' \) and hence any such level set is a contact-type hypersurface in \( V \).

### 6.3. Combining the two

The following lemma then will finish the proof of Theorem 6.1.

**Lemma 6.7.** The Hamiltonian vector field \( X_I \) induces a contact vector field on the contact-type hypersurface \( I^{-1}(I_0) \) that is transversal to \( F_{I_0} := I^{-1}(I_0) \cap \partial M \) for all \( I_0 \geq C' \).

**Proof.** By the expression of symplectic form in (6.5) we have
\[
\frac{\partial}{\partial R} = X_I, \quad \frac{\partial}{\partial I} = -X_R.
\]
This implies by (6.9)
\[
1 = d\lambda(X_R, X_I) = -b d\lambda(Z, X_I) = -b \lambda(X_I).
\]
Therefore \( \lambda(X_I) \neq 0 \). Since \( X_I \) is tangent to the level set \( I^{-1}(I_0) \), \( X_I \) defines a contact vector field transversal to the contact distribution of \( I^{-1}(I_0) \) induced by the contact form \( \theta_{I_0} := \iota^*_I \lambda \).

The same discussion applies to \( F^{-\infty} \), which finishes the proof. \( \square \)

This finishes the proof of Theorem 6.1.

### 7. Structure of Liouville \( \sigma \)-sectors and their automorphism groups

Our definition of Liouville \( \sigma \)-sectors with corners enables us to give a natural notion of Liouville automorphisms which is the same as the case without boundary and which does not depend on the choice of splitting data.

We first recall the following well-known definition of automorphisms of Liouville manifold (without boundary)

**Definition 7.1.** Let \((M, \lambda)\) be an Liouville manifold without boundary. We call a diffeomorphism \( \phi : M \to M \) a Liouville automorphism if \( \phi \) satisfies
\[
\phi^* \lambda = \lambda + df
\]
for a compactly supported function \( f : M \to \mathbb{R} \). We denote by \( \text{Aut}(M) \) the set of automorphisms of \((M, \lambda)\).

It is easy to check that \( \text{Aut}(M) \) forms a topological group. Now we would like extend this definition of automorphisms to the case of Liouville \( \sigma \)-sectors. For this purpose, we need some preparations by examining the universal geometric structures inherent on the boundary \( \partial M \) of a Liouville manifold with boundary and corners.
7.1. **Some presymplectic geometry of** $\partial M$. We start with the observation that $(\partial M, \omega_{\partial M})$ carries the structure of *presymplectic manifolds* as usual for any coisotropic submanifold mentioned as before. We first introduce automorphisms of presymplectic manifolds $(Y, \omega)$ in general context.

**Definition 7.2.** Let $(Y, \omega)$ and $(Y', \omega')$ be two presymplectic manifolds. A diffeomorphism $\phi: Y \to Y'$ is called *presymplectic* if $\phi^*\omega' = \omega$. We denote by $\mathcal{PSymp}(Y, \omega)$ the set of presymplectic diffeomorphisms.

(We refer to [OP05] for some detailed discussion on the geometry of presymplectic manifolds and their automorphisms and their application to the deformation problem of coisotropic submanifolds.)

Then we note that any diffeomorphism $\phi: (M, \partial M) \to (M, \partial M)$ satisfying

$$\phi^*\lambda = \lambda + df$$  \hspace{1cm} (7.1)

for some function $f$, *not necessarily compactly supported*, induces a presymplectic diffeomorphism

$$\phi_{\partial} := \phi|_{\partial M}$$

on $\partial M$ equipped with the presymplectic form

$$\omega_{\partial} := d\lambda_{\partial}, \quad \lambda_{\partial} := \iota^*\lambda$$

for the inclusion map $\iota: \partial M \to M$.

**Lemma 7.3.** The presymplectic diffeomorphism $\phi_{\partial}: \partial M \to \partial M$ preserves the characteristic foliation of $\partial M$.

**Proof.** We have

$$\mathcal{D}_{\partial M} = \ker \omega_{\partial}.$$  

Since any Liouville automorphism $\phi$ of $(M, \partial M)$ satisfies (7.1), we have

$$\phi^*_\partial \omega_{\partial} = \omega_{\partial}.$$  

Therefore we have

$$\phi^*(\mathcal{D}_{\partial M}) = \mathcal{D}_{\partial M}$$

which finishes the proof. \hfill \Box

In fact, for the current case of our interest $Y = \partial M$, the presymplectic form $\omega_{\partial}$ is exact in that

$$\omega_{\partial} = d\lambda_{\partial}, \quad \lambda_{\partial} := \iota^*\lambda.$$  

Furthermore (7.1) implies that $\phi$ actually restricts to an exact presymplectic diffeomorphism

$$\phi_{\partial} : (\partial M, \omega_{\partial}) \to (\partial M, \omega_{\partial})$$

on $\partial M$ in that

$$\phi^*_{\partial} \lambda_{\partial} - \lambda_{\partial} = dh, \quad h = f \circ \iota$$

where the function $h: \partial X \to \mathbb{R}$ is not necessarily compactly supported.

We have a natural restriction map

$$\text{Aut}(M, \lambda) \to \mathcal{PSymp}(\partial M, \omega_{\partial}): \quad \phi \mapsto \phi_{\partial}.$$  \hspace{1cm} (7.2)
**Definition 7.4** (Pre-Liouville automorphism group $\text{Aut}(\partial M, \lambda_\partial)$). We call a diffeomorphism $\phi : (\partial M, \lambda_\partial) \to (\partial M, \lambda_\partial)$ a pre-Liouville diffeomorphism if the form $\phi^* \lambda_\partial - \lambda_\partial$ is exact. We say $\phi$ is a pre-Liouville automorphism if it satisfies

$$\phi^* \lambda_\partial = \lambda_\partial + dh$$

for a compactly supported function $h : M \to \mathbb{R}$. We denote by $\text{Aut}(\partial M, \lambda_\partial)$ the set of pre-Liouville automorphisms of $(\partial M, \lambda_\partial)$.

The following is an immediate consequence of the definition.

**Corollary 7.5.** The restriction map (7.2) induces a canonical group homomorphism

$$\text{Aut}(M, \lambda) \to \text{Aut}(\partial M, \lambda_\partial).$$

We recall that $\partial M$ carries a canonical transverse symplectic structure arising from the presymplectic form $d\lambda_\partial$. (See [OP05, Section 4].)

**Proposition 7.6.** The induced pre-Liouville automorphism $\phi_\partial : \phi|_{\partial M} : \partial M \to \partial M$ descends to a (stratawise) symplectic diffeomorphism $\phi_{N\partial M} : N\partial M \to N\partial M$

and satisfies

$$\pi_{\partial M} \circ \phi_\partial = \phi_{N\partial M} \circ \pi_{\partial M}$$

when we regard both $\partial M$ and $N\partial M$ as manifolds with corners.

### 7.2. Automorphism group of Liouville $\sigma$-sectors.

Now we are ready give the geometric structure of Liouville $\sigma$-sectors.

**Definition 7.7** (Structure of Liouville $\sigma$-sectors). We say two Liouville $\sigma$-sectors $(M, \lambda)$ and $(M', \lambda')$ are isomorphic, if there exists a diffeomorphism $\psi : M \to M'$ (as a manifold with corners) such that $\psi^* \lambda' = \lambda + df$ for some compactly supported function $f : M \to \mathbb{R}$. A structure of Liouville $\sigma$-sectors is defined to be an isomorphism class of Liouville $\sigma$-sectors.

With this definition of the structure of Liouville $\sigma$-sectors in our disposal, the following is an easy consequence of the definition and Proposition 7.6, which shows that the definition of an automorphism of a Liouville sector $(M, \lambda)$ is in the same form as the case of Liouville manifold given by the defining equation

$$\psi^* \lambda = \lambda + df$$

for some compactly supported function $f : M \to \mathbb{R}$, except that $\psi$ is a self diffeomorphism of $M$ as a stratified manifold and the equality of the above equation as in the sense of Remark 1.14.

**Theorem 7.8** (Automorphism group). Let $(M, \lambda)$ be a Liouville $\sigma$-sector. Suppose a diffeomorphism $\psi : M \to M$ satisfies

$$\psi^* \lambda = \lambda + df$$

for some compactly supported function $f : M \to \mathbb{R}$. Then $\psi$ is an automorphism of the structure of Liouville $\sigma$-sectors.

**Proof.** We first discuss how the action of diffeomorphisms $\psi$ satisfying $\psi^* \lambda = \lambda + df$ affects the structure of Liouville $\sigma$-sectors, when the function $f$ is compactly supported. In particular it implies
• $\psi^*d\lambda = d\lambda$
• $\psi^*\lambda = \lambda$ near infinity.

Then $\psi$ restricts to a presymplectic diffeomorphism $\psi_\partial : \partial M \to \partial M$ which is also pre-Liouville, i.e., satisfies

$$(\psi|_{\partial M})^*\lambda_\partial = \lambda_\partial + dh$$

for a compactly supported function $h$ on $\partial M$.

We need to show that the structure of Liouville $\sigma$-sectors with respect to

$$(M, \psi^*\lambda) = (M, \lambda + df)$$

is isomorphic to that of $(M, \lambda)$. For this, we make a choice of $\sigma = \{\sigma_1, \ldots, \sigma_m\}$ associated to a transversal coisotropic collection $\{H_1, \ldots, H_m\}$ for each sectorial corner $\delta$ of $M$ with

$C_\delta = H_1 \cap \cdots \cap H_m$.

Such a collection exists by definition for $(M, \lambda)$ being a Liouville $\sigma$-sector.

Now we consider the pushforward collection of hypersurfaces

$$\{H'_1, \ldots, H'_m\} := \{\psi(H_1), \ldots, \psi(H_m)\}.$$

Since smooth diffeomorphisms between two manifolds with corners preserve strata dimensions by definition, we work with the defining data of $(M, \psi^*\lambda)$ stratawise of the fixed dimensional strata.

We first need to show that each $H'_i$ is $\sigma$-sectorial hypersurface by finding a collection

$$\sigma' = \{\sigma'_1, \ldots, \sigma_m\}$$

where each $\sigma'_i$ is a smooth section of $H'_i$ respectively. For this purpose, we prove the following

**Lemma 7.9.** Choose the sections $\sigma_i$s so that

$$\text{Image } \sigma_i \subset M \setminus \text{supp } df.$$

Then there exists a neighborhood $\text{Nbhd}(\partial_\infty M)$ such that the following hold:

1. The map $\psi : \text{Nbhd}(\partial_\infty M) \cap H_i \to H_i$ descends to a diffeomorphism $[\psi] : N_{H_i} \to N_{H_i}$.
2. The map $\sigma_i^\psi : N_{H_i} \to \psi(H_i)$ defined by

$$\sigma_i^\psi := \psi \circ \sigma_i \circ [\psi]^{-1}$$

is a section of the projection $\psi(H_i) \to N_{\psi(H_i)} = N_{H_i}$.

**Proof.** Since $\text{Image } \sigma_i \subset M \setminus \text{supp } df$, we have

$$\psi^*\lambda = \lambda$$
on $\text{Image } \sigma_i := F_i$. In particular, the projection $\pi_{H_i} : H_i \to N_{H_i}$ restricts to a bijective map on $F_i$. Furthermore since $\psi^*\lambda = \lambda$ on $\text{Nbhd}(\partial_\infty M)$, the associated Liouville vector field $Z_\lambda$ of $\lambda$ satisfies

$$\psi_* Z_\lambda = Z_\lambda$$
thereon. Recall that $\psi$ restricts to a diffeomorphism on $\partial M$ (as a map on manifold with corners). Then the equality $\psi^*\lambda = \lambda$ implies $\psi_\partial^* d\lambda_\partial = d\lambda_\partial$ and hence

$$d\psi_\partial (\ker d\lambda_\partial) = \ker d\lambda_\partial.$$
on \( \text{Nbhd}(\partial M) \cap H_i \). Therefore \( \psi \) descends to a diffeomorphism \([\psi] : N_{H_i} \to N_{H_i}\) so that we have the commutative diagram

\[
\begin{array}{c}
H_i \xrightarrow{\psi} \psi(H_i) \\
\downarrow \pi_{H_i} \quad \quad \quad \downarrow \pi_{\psi(H_i)} \\
N_{H_i} \xrightarrow{[\psi]} N_{H_i}.
\end{array}
\]

By composing \( \sigma'_i = \psi \circ \sigma_i \) with \( \pi_{\psi(H_i)} \) to the left, we obtain

\[
\pi_{\psi(H_i)} \sigma'_i = \pi_{\psi(H_i)} \circ \psi \circ \sigma_i = [\psi] \circ \pi_{H_i} \circ \sigma_i = [\psi]
\]

which is a diffeomorphism. Therefore the map

\[
\sigma_i^\psi := \psi \circ \sigma'_i = \psi \circ \sigma_i \circ [\psi]^{-1}
\]

is a section of the projection \( H'_i \to N_{H'_i} \). This finishes the proof. \( \square \)

Clearly any diffeomorphism preserves the transversal intersection property. This proves that any diffeomorphism \( \psi \) satisfying \( \psi^* \lambda = \lambda + df \) with compactly supported \( f \) is an automorphism of the structure of Liouville \( \sigma \)-sectors. (See Definition 4.1 and 7.7.) This finishes the proof of the theorem. \( \square \)

Based on this discussion, we will unambiguously denote by \( \text{Aut}(M) \) the automorphism group of Liouville \( \sigma \)-sector \((M,\lambda)\) as in the case of Liouville manifolds.

Remark 7.10.  
(1) The above proof shows that the group \( \text{Aut}(M,\lambda) \) is manifestly the automorphism group of the structure of Liouville \( \sigma \)-sectors. We alert the readers that this is not manifest in the original definition of Liouville sectors from \([GPS20], [GPS18b] \).

(2) This simple characterization of the automorphism groups of Liouville \( \sigma \)-sectors with corners enables one to define the bundle of Liouville sectors with corners in the same way for the case of Liouville manifolds (with boundary) \textit{without corners}. See \([OT20] \) for the usage of such bundles in the construction of continuous actions of Lie groups on the wrapped Fukaya category of Liouville sectors (with corners).

(3) Recall that the Liouville structure \( \lambda \) on \( M \) induces a natural contact structure on its ideal boundary \( \partial_\infty M \). We denote the associated contact structure by \( \xi_\infty \). Then we have another natural map

\[
\text{Aut}(M,\lambda) \to \text{Cont}(\partial_\infty M, \xi_\infty)
\]

where \((\partial_\infty M, \xi_\infty)\) is the group of contactomorphisms of the contact manifold \((\partial_\infty M, \xi_\infty)\). (See \([Gir17], [OT22] \) for the details.)

(4) The different geometric nature of \((\partial_\infty M, \xi_\infty)\) and \((\partial M, \lambda_\partial)\) is partially responsible for the difficulty of the constructions of a pseudoconvex pair \((\psi, J)\) in a neighborhood

\[
\text{Nbhd}(\partial_\infty M \cup \partial M)
\]

such that the almost complex structures \( J \) is amenable to the (strong) maximum principle for the (perturbed) pseudoholomorphic maps into the Liouville sectors as manifested in \([Oh21b] \).
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