DIRECT PRODUCTS OF FINITE GROUPS
AS UNIONS OF PROPER SUBGROUPS

MARTINO GARONZI AND ANDREA LUCCHINI

Abstract. We determine all the ways in which a direct product of two finite groups can be expressed as the set-theoretical union of proper subgroups in a family of minimal cardinality.

1. Introduction

If $G$ is a non-cyclic finite group, then there exists a finite collection of proper subgroups whose set-theoretical union is all of $G$; such a collection is called a cover for $G$. A minimal cover is one of least cardinality and the size of a minimal cover of $G$ is denoted by $\sigma(G)$ (and for convenience we shall write $\sigma(G) = \infty$ if $G$ is cyclic). The study of minimal covers was introduced by J.H.E. Cohn [4]. If $N$ is a normal subgroup of $G$, then $\sigma(G) \leq \sigma(G/N)$; indeed a cover of $G/N$ can be lifted to a cover of $G$.

In particular, as it was noticed in [4], if $G = H_1 \times H_2$ is the direct product of two finite groups, then $\sigma(H_1 \times H_2) \leq \min\{\sigma(H_1), \sigma(H_2)\}$.

It is easy to prove that if $|H_1|$ and $|H_2|$ are coprime numbers, then $\sigma(H_1 \times H_2) = \min\{\sigma(H_1), \sigma(H_2)\}$ (see [4, Lemma 4]). The situation is different if $|H_1|$ and $|H_2|$ are not coprime; for example if $H_1 \cong H_2 \cong C_p$ are cyclic groups of order $p$ and $p$ is a prime, then $\sigma(H_1 \times H_2) = p + 1$ since a minimal cover of $H_1 \times H_2$ must contain all the $p + 1$ maximal subgroups. Some partial results are contained in [5]. In this paper we obtain a complete and general answer to the question how $\sigma(H_1 \times H_2)$ is related with $\sigma(H_1)$ and $\sigma(H_2)$.

Theorem 1. If $G = H_1 \times H_2$ is the direct product of two finite groups, then either $\sigma(G) = \min\{\sigma(H_1), \sigma(H_2)\}$ or $\sigma(G) = p + 1$ and the cyclic group of order $p$ is a homomorphic image of both $H_1$ and $H_2$.

This is a consequence of a more general result, describing all the possible minimal covers of a direct product.

Theorem 2. Let $\mathcal{M}$ be a minimal cover of a direct product $G = H_1 \times H_2$ of two finite groups. Then one of the following holds:

1. $\mathcal{M} = \{X \times H_2 \mid X \in \mathcal{X}\}$ where $\mathcal{X}$ is a minimal cover of $H_1$. In this case $\sigma(G) = \sigma(H_1)$.

2. $\mathcal{M} = \{H_1 \times X \mid X \in \mathcal{X}\}$ where $\mathcal{X}$ is a minimal cover of $H_2$. In this case $\sigma(G) = \sigma(H_2)$.

3. There exist $N_1 \trianglelefteq H_1$, $N_2 \trianglelefteq H_2$ with $H_1/N_1 \cong H_2/N_2 \cong C_p$ and $\mathcal{M}$ consists of the maximal subgroups of $H_1 \times H_2$ containing $N_1 \times N_2$. In this case $\sigma(G) = p + 1$.

1991 Mathematics Subject Classification. 20D60.
Key words and phrases. Covers; maximal subgroups; direct products.
Research partially supported by MIUR-Italy via PRIN Group theory and applications.
2. Proofs of the theorems

First we recall some elementary results on the minimal covers.

Lemma 3. If $X$ is a minimal cover of a finite group $Y$ and $F$ is a normal subgroup of $Y$ such that $FX \neq Y$ for each $X \in X$, then $F \leq X$ for each $X \in X$.

Proof. Let $X = \{X_1, \ldots, X_n\}$. By our assumption, $\{X_1F, \ldots, X_nF\}$ is also a minimal cover of $Y$. In particular, for each $i$, there exists $x_i \in X_iF$ such that $x_i \notin X_jF$ if $j \neq i$. Assume by contradiction $F \notin X_i$ and take $f \in F \setminus X_i$; we have $x_i \cdot g \in X_i$ for some $g \in F$ and consequently $x_i \cdot g \cdot f \notin X_i$: this implies $x_i \cdot g \cdot f \in X_j$ for some $j \neq i$, but then $x_i \in X_jF$, a contradiction. □

Lemma 4. [7 Lemma 3.2] Let $N$ be a proper normal subgroup of the finite group $G$. Let $U_1, \ldots, U_h$ be proper subgroups of $G$ containing $N$ and $V_1, \ldots, V_k$ be proper subgroups such that $V_iN = G$ with $|G : V_i| = \beta_i$ and $\beta_1 \leq \cdots \leq \beta_k$. If

\[
G = U_1 \cup \cdots \cup U_h \cup V_1 \cup \cdots \cup V_k \quad \text{but} \quad G \neq U_1 \cup \cdots \cup U_h,
\]

then $\beta_1 \leq k$. Furthermore, if $\beta_1 = k$, then $\beta_1 = \cdots = \beta_k$ and $V_i \cap V_j \leq U_1 \cup \cdots \cup U_h$ for all $i \neq j$.

The other tool that we need in the proof is a description of the maximal subgroups of a direct product $H_1 \times H_2$.

- We will say that a maximal subgroup $M$ of $H_1 \times H_2$ is of standard type if either $M = X_1 \times H_2$ with $X_1$ a maximal subgroup of $H_1$ or $M = H_1 \times X_2$ with $X_2$ a maximal subgroup of $H_2$.
- We will say that a maximal subgroup $M$ of $H_1 \times H_2$ is of diagonal type if there exist a maximal normal subgroup $N_1$ of $H_1$, a maximal normal subgroup $N_2$ of $H_2$ and an isomorphism $\phi : H_1/N_1 \rightarrow H_2/N_2$ such that

\[
M = \{(h_1, h_2) \in H_1 \times H_2 \mid \phi(h_1N_1) = h_2N_2\}.
\]

By [6 Chap. 2, (4.19)], the following holds.

Lemma 5. A maximal subgroup of $H_1 \times H_2$ is either of standard type or of diagonal type.

Lemma 6. Let $\mathcal{M} = \{M_1, \ldots, M_\sigma\}$ be a minimal cover of $G = H_1 \times H_2$. If all the subgroups in $\mathcal{M}$ are maximal and $\mathcal{M}$ contains a subgroup of diagonal type whose index is a prime number $p$, then $\sigma(G) = p + 1$ and all the subgroups in $\mathcal{M}$ are normal of index $p$.

Proof. First notice that if $\mathcal{M}$ contains a maximal subgroup of diagonal type and index $p$, then $C_p \times C_p$ is an epimorphic image of $G$ and consequently

\[
\sigma(G) \leq \sigma(C_p \times C_p) = p + 1.
\]

We argue by induction on the order of $G$. We may assume that there exists no nontrivial normal subgroup $N$ of $G$ such that $N \leq M$ for all $M \in \mathcal{M}$ and $N \leq H_1$. Otherwise $\{M_1/N, \ldots, M_\sigma/N\}$ would be a minimal cover of $(H_1/N) \times H_2$ containing a maximal diagonal subgroup of index $p$ and the conclusion follows by induction. For the same reason, there is no nontrivial normal subgroup $N$ of $G$ such that $N \leq M$ for all $M \in \mathcal{M}$ and $N \leq H_2$. In particular

\[
\text{Frat} \ G = \text{Frat} \ H_1 \times \text{Frat} \ H_2 = 1.
\]
First assume that \( Z(G) \) has order divisible by \( p \). This implies that there exists a central subgroup, say \( N \), of order \( p \), which is contained either in \( H_1 \) or in \( H_2 \). Let \( \mathcal{U} \) be the set of subgroups in \( \mathcal{M} \) not containing \( N \). By our assumption \( \mathcal{U} \neq \emptyset \), moreover if \( M \in \mathcal{U} \), then \( G = M \times N \) and in particular \( M \) is a normal subgroup of \( G \) and has index \( p \). By Lemma [1] \( p \leq |\mathcal{U}| \leq \sigma(G) \leq p + 1 \). Moreover \( N \) is not contained in the union of the subgroups in \( \mathcal{U} \), so we must have \( |\mathcal{U}| = p \) and \( \sigma(G) = p + 1 \). Let \( M \) be unique element of \( \mathcal{M} \setminus \mathcal{U} \). By Lemma [4] \( M \) contains the intersection \( M_i \cap M_j \) of any two different subgroups in \( \mathcal{U} \), but \( G/(M_i \cap M_j) \cong C_p \times C_p \), so \( M \) is a normal subgroup of index \( p \).

Now assume that \( p \) does not divide \( |Z(G)| \). Write \( \text{soc}(G) = N_1 \times \cdots \times N_t \) as a product of minimal normal subgroups. We may assume that each \( N_i \) is contained either in \( H_1 \) or in \( H_2 \) and that \( N_i \) is abelian if and only if \( i < u \). Let \( C = \bigcap_{1 \leq i \leq t} C_G(N_i) \). Since \( \text{Frat} G = 1 \), the socle of \( G \) coincides with the generalized Fitting subgroup of \( G \) and, by the Bender \( F^* \)-Theorem (see for example [1] (31.13)),

\[
C = C_G(\text{soc} G) = Z(\text{soc} G) = \prod_{i < u} N_i.
\]

Since \( p \) does not divide \( |Z(G)| \), if \( N_i \) is a \( p \)-group, then \( N_i = [N_i, G] \leq G' \cap C \).

In particular \( p \) does not divide \( |C : G' \cap C| = |CG' : G'| \). On the other hand \( p \) divides \( |G : G'| = |G : CG'| |CG' : G'| \), hence \( G/C \) has \( C_p \) as an epimorphic image. Since \( G/C \) is a subdirect product of \( \prod_{1 \leq i \leq t} G/C_G(N_i) \), there must exist a minimal normal subgroup \( N \) which is contained in either \( H_1 \) or \( H_2 \) and with the property that \( A = G/C_G(N) \) has a chief factor of order \( p \). By our assumption the set \( \mathcal{U} \) of the subgroups in \( \mathcal{M} \) not containing \( N \) is non empty. By Lemma [4] \( p + 1 \geq \sigma(G) \geq |\mathcal{U}| \geq \beta \), with \( \beta = \min_{M \in \mathcal{U}} |G : M| \). Fix a maximal subgroup \( M \) in \( \mathcal{U} \) with \( |G : M| = \beta \).

If \( N \) is abelian, then the subgroups in \( \mathcal{U} \) are complements of \( N \), hence \( \beta = |N| \). Moreover \( N \) is not contained in the union of the subgroups in \( \mathcal{U} \), hence \( p + 1 \geq \sigma(G) \geq |N| + 1 \). However \( p \) must be a prime divisor of \( |A| \), but \( A \leq \text{GL}(N) \) and this implies \( p < |N| \), a contradiction.

If \( N \) is a non-abelian simple group, then \( C_p \) is isomorphic to a chief factor of a subgroup of \( \text{Out}(N) \) hence \( p \leq |\text{Out}(N)| \). However \( \beta = |G : M| = |N : M \cap N| \) is the index of a proper subgroup of \( N \) so in particular \( \beta > 2p \) (see e.g. [2] Lemma 2.7). But then \( p + 1 \geq \beta > 2p \), a contradiction.

We are left with the case \( N = S_1 \times \cdots \times S_r \cong S^r \) where \( S \) is a nonabelian simple group. Let \( \pi_i : N \to S_i \) the projection to the \( i \)-th factor of \( N \). Since \( MN = G \) and \( N \) is a minimal normal subgroup of \( G \), the maximal subgroup \( M \) permutes transitively the minimal normal subgroups \( S_1, \ldots, S_r \) of \( N \) and normalizes \( M \cap N \). This implies that \( \pi_i(M \cap N) \cong \cdots \cong \pi_r(M \cap N) \) so either \( M \cap N \leq T_1 \times \cdots \times T_r \) with \( T_i < S_i \) for each \( i \in \{1, \ldots, r\} \) or (see for example [3] Proposition 1.1.44) \( M \cap N \cong S^u \) with \( u \) a proper divisor of \( r \). Therefore, by [2] Lemma 2.7,

\[
p + 1 \geq \beta = |N : M \cap N| \geq \min\{2^r q^r, |S|^{r/2}\}
\]

with \( q \) the largest prime divisor of \( |\text{Out}(S)| \). Moreover \( C_p \) is isomorphic to a chief factor of a subgroup of \( \text{Out}(N) \cong \text{Out}(S) \setminus \text{Sym}(r) \), so either \( p \) divides \( |\text{Sym}(r)| \), in which case \( p \leq r \), or \( p \) divides \( |\text{Out}(S)| \) and consequently \( p \leq q \). Both these cases lead to a contradiction. \( \square \)
Proposition 7. Let $X = \{X_1, \ldots, X_\sigma\}$ be a minimal cover of $G = H_1 \times H_2$. If a subgroup of $X$ is contained in a maximal subgroup of diagonal type whose index is a prime number, then $\sigma(G) = p + 1$, $X_i$ is a normal subgroup of index $p$ for each $i \in \{1, \ldots, \sigma\}$ and $\bigcap_i X_i$ has index $p^2$ in $G$.

Proof. For each $i \in \{1, \ldots, \sigma\}$, let $M_i$ be a maximal subgroup of $G$ containing $X_i$, chosen in such a way that $M_i$ is a maximal subgroup of diagonal type and index $p$ when $X_i$ is contained in such a maximal subgroup. The cover $\mathcal{M} = \{M_1, \ldots, M_\sigma\}$ satisfies the hypothesis of Lemma 8 so $\sigma = p + 1$ and $M_i$ is a maximal normal subgroup of index $p$ for each $i \in \{1, \ldots, \sigma\}$. Let $N = M_1 \cap M_2$. If, by contradiction, there exists $i \in \{3, \ldots, \sigma\}$ such that $M_i$ does not contain $N$, then, by Lemma 8, $\sigma \geq 2 + p$. So for each $i \in \{1, \ldots, \sigma\}$, we have $N \leq M_i$ but then $X_iN \neq M_i$, hence $N \neq X_i$ by Lemma 5. In particular $\{X_1/N, \ldots, X_\sigma/N\}$ is a cover of $G/N \cong C_p \times C_p$. Since $\sigma(C_p \times C_p) = p + 1 = \sigma$, $X_i \neq N$ for each $i \in \{1, \ldots, \sigma\}$. \hfill $\square$

Proposition 8. Let $X = \{X_1, \ldots, X_\sigma\}$ be a minimal cover of $G = H_1 \times H_2$. If $X$ contains no subgroup of diagonal type whose index is a prime number, then either $H_1 \times 1$ or $1 \times H_2$ is contained in $\bigcap_{1 \leq i \leq \sigma} X_i$.

Proof. For each $i \in \{1, \ldots, \sigma\}$, let $M_i$ be a maximal subgroup of $G$ containing $X_i$. We have that $\mathcal{M} = \{M_1, \ldots, M_\sigma\}$ is a minimal cover of $G$ given by $\sigma = \sigma(G)$ maximal subgroups of $G$. We set:

$\mathcal{M}_1 = \{M \in \mathcal{M} \mid M \geq H_2\} = \{L \times H_2 \mid L \text{ a maximal subgroup of } H_1\}$,

$\mathcal{M}_2 = \{M \in \mathcal{M} \mid M \geq H_1\} = \{H_1 \times L \mid L \text{ a maximal subgroup of } H_2\}$,

$\mathcal{M}_3 = \mathcal{M} \setminus (\mathcal{M}_1 \cup \mathcal{M}_2)$.

Then we define the two sets

$$\Omega_1 = H_1 \setminus \left( \bigcup_{L \times H_2 \in \mathcal{M}_1} L \right), \quad \Omega_2 = H_2 \setminus \left( \bigcup_{H_1 \times L \in \mathcal{M}_2} L \right)$$

If $\Omega_1 = \emptyset$, then $G = H_1 \times H_2 = \bigcup_{L \times H_2 \in \mathcal{M}_1} L \times H_2$, hence $\mathcal{M} = \mathcal{M}_1$. In the same way, if $\Omega_2 = \emptyset$, then $\mathcal{M} = \mathcal{M}_2$.

So we may assume $\Omega_1 \times \Omega_2 \neq \emptyset$. For $i \in \{1, 2\}$, let $K_i$ be the intersection of the maximal normal subgroups of $H_i$. Notice that $H_i/K_i$ is isomorphic to a direct product of simple groups and $K_i$ is the smallest subgroup of $H_i$ with this property. To fix our notation assume

$$H_1/K_1 = \prod_{1 \leq a \leq \alpha} S_a, \quad H_2/K_2 = \prod_{1 \leq b \leq \beta} T_b$$

with $S_a$, $T_b$ simple groups. To each $a \in A = \{1, \ldots, \alpha\}$ there corresponds the projection $\pi_{1,a}: H_1 \to S_a$ and to each $b \in B = \{1, \ldots, \beta\}$ there corresponds the projection $\pi_{2,b}: H_2 \to T_b$. For $i \in \{1, 2\}$, consider the projection $\rho_i: H_i \to H_i/K_i$ and the image

$$\Delta_i = \{\rho_i(\omega) \mid \omega \in \Omega_i\}$$

of $\Omega_i$ under this projection.

By Lemma 4 to any $M \in \mathcal{M}_3$ we may associate a triple $(a, b, \phi)$ with $a \in A$, $b \in B$ and $\phi: S_a \to T_b$ a group isomorphism such that

$$M = M(a, b, \phi) = \{(h_1, h_2) \in H_1 \times H_2 \mid \phi(\pi_{1,a}(h_1)) = \pi_{2,b}(h_2)\}.$$
Now let \( \Lambda \) be the set of the triples \((a, b, \phi)\) such that \( M(a, b, \phi) \in \mathcal{M}_3 \). By hypothesis, \( \mathcal{M}_3 \) contains no subgroup of index a prime number; this implies that if \((a, b, \phi) \in \Lambda\), then \( S_a \cong T_b \) is a nonabelian simple group.

Now fix an element \((s_1, \ldots, s_3) \in \Delta_3\) and an element \( x \in \Omega_3\) with \( \rho_1(x) = (s_1, \ldots, s_3) \) and for each \((a, b, \phi) \in \Lambda\) let
\[
U(a, b, \phi) = \{ h \in H_2 \mid \pi_{2,b}(h) \in \langle \phi(s_a) \rangle \}.
\]
Clearly, since \( T_b \) is a nonabelian simple group, \( \langle \phi(s_a) \rangle \neq T_b \) and \( U(a, b, \phi) \) is a proper subgroup of \( H_2 \). Consider the following family of subgroups of \( H_2 \):
\[
T = \{ M \mid H_1 \times M \in \mathcal{M}_3 \} \cup \{ U(a, b, \phi) \mid (a, b, \phi) \in \Lambda \}.
\]
We claim that \( T \) is a cover of \( H_2 \). We have to prove that if \( h_2 \in \Omega_2\), then \( h_2 \in U(a, b, \phi) \) for some \((a, b, \phi) \in \Lambda\). Observe that the elements of the set \( \Omega_1 \times \Omega_2 \) do not belong to any of the subgroups in \( \mathcal{M}_1 \) or \( \mathcal{M}_2 \), thus the set \( \Omega_1 \times \Omega_2 \) has to be covered by the subgroups in \( \mathcal{M}_3 \). In particular if \( h_2 \in \Omega_2\), then \((x, h_2) \in M(a, b, \phi)\) for some \((a, b, \phi) \in \Lambda\). This implies that \( \pi_{2,b}(h_2) = \phi(\pi_{1,a}(x)) = \phi(s_a) \in \langle \phi(s_a) \rangle\), hence \( h_2 \in U(a, b, \phi) \) and the claim is proved.

But this implies \( |\mathcal{M}_1| + |\mathcal{M}_2| + |\mathcal{M}_3| = \sigma(G) \leq \sigma(H_2) \leq |T| \leq |\mathcal{M}_2| + |\mathcal{M}_3| \) and, consequently, \( \mathcal{M}_1 = \emptyset \). With a similar argument we deduce \( \mathcal{M}_2 = \emptyset \). So if \( \mathcal{M}_3 \neq \emptyset\), then \( \mathcal{M} = \mathcal{M}_3 \). By [4, Lemma 1], there exists \( M \in \mathcal{M}_3 \) with \( \sigma(G) \geq |G : M| + 1\); however \( |G : M| = |S| \) for some nonabelian simple group \( S \) which is an epimorphic image of \( G \). This implies \( \sigma(G) \leq \sigma(S) \leq |S| = |G : M| \leq \sigma(G) - 1\), a contradiction.

Let \( \overline{H_1} = H_1 \times 1 \) and \( \overline{H_2} = 1 \times H_2 \). We have proved that there exists \( j \in \{1, 2\} \), such that \( \overline{H}_j \leq \bigcap_{i \leq \sigma} M_i \). In particular \( \overline{H}_j X_i \leq M_i \) for each \( i \in \{1, \ldots, \sigma\} \) hence, by Lemma [3] we can conclude \( \overline{H}_j \leq \bigcap_{i \leq \sigma} X_i \). \( \square \)

REFERENCES

1. M. Aschbacher, Finite group theory, Second edition, Cambridge Studies in Advanced Mathematics 10, Cambridge University Press, Cambridge, (2000).
2. M. Aschbacher and R. Guralnick, On abelian quotients of primitive groups, Proc. Amer. Math. Soc., 107 (1989) 89–95.
3. A. Ballester-Bolinches and L. M. Ezquerro, Classes of finite groups, Mathematics and Its Applications (Springer), vol. 584, Springer, Dordrecht, 2006.
4. J. H. E. Cohn, On n-sum groups, Math. Scand. 75 (1) (1994) 44–58.
5. E. Detomi and A. Lucchini, On the structure of primitive n-sum groups, Cubo 10 (2008), no. 3, 195–210.
6. M. Suzuki, Group theory, I, Grundlehren der Mathematischen Wissenschaften 247, Springer-Verlag, Berlin, (1982).
7. M. J. Tomkinson, Groups as the union of proper subgroups, Math. Scand. 81 (2) (1997) 191–198.

Dipartimento di Matematica Pura ed Applicata, Via Trieste 63, 35121 Padova, Italy.
E-mail address: mgaronzi@math.unipd.it

Dipartimento di Matematica Pura ed Applicata, Via Trieste 63, 35121 Padova, Italy.
E-mail address: lucchini@math.unipd.it