PROPAGATION OF THE GABOR WAVE FRONT SET FOR SCHRÖDINGER EQUATIONS WITH NON-SMOOTH POTENTIALS

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ABSTRACT. We consider Schrödinger equations with real-valued smooth Hamiltonians, and non-smooth bounded pseudo-differential potentials, whose symbols may be not even differentiable. The well-posedness of the Cauchy problem is proved in the frame of the modulation spaces, and results of micro-local propagation of singularities are given in terms of Gabor wave front sets.

1. INTRODUCTION

The authors in [12] and in collaboration with Gröchenig in [7] proposed a new approach to the calculus of the Fourier integral operators (FIOs) in terms of time-frequency localization, cf. [17] and [24], also named Gabor analysis. The FIOs under consideration were of the type of those appearing in the study of the Schrödinger equations, typically the phase function being a homogeneous function of degree 2 in the whole of the phase space variables. With respect to the standard representations of FIOs, the time-frequency representation looks more involved, since old and new phase-space variables appear simultaneously, and everything depends on the choice of the so-called window function. On the other hand, the problem of the caustics is automatically solved in this new setting, see [7], and the expression provides an excellent tool for the numerical analysis, see [12].

In the present paper we apply the aforesaid results to the analysis of the Schrödinger equation. With respect to the enormous existing literature, our results will be new in the following aspects. Fixed a real-valued Hamiltonian, homogeneous of degree 2, we allow a pseudo-differential perturbation (called also potential in the following) with a bounded, complex-valued, non-smooth symbol, for which even differentiability may be lost. A global-in-time propagator is constructed in the class of the FIOs in [7], and well-posedness of the Cauchy problem is deduced in suitable modulation spaces. About propagation of singularities, which is our main concern in this paper, the known results do not apply to such situation. We are

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then led to a new definition of Gabor wave front set, which allows the expression of optimal results of propagation in our context.

Let us be more precise. The aim of the paper is to study the representation in terms of time-frequency analysis of the propagator \( e^{itH} \),

\[
H = a(x, D) + \sigma(x, D),
\]

providing the solution to the Cauchy problem:

\[
\begin{aligned}
& i \frac{\partial u}{\partial t} + a(x, D)u + \sigma(x, D)u = 0 \\
& u(0, x) = u_0(x).
\end{aligned}
\]

The Hamiltonian \( a(x, D) \) is a pseudodifferential operator in the Kohn-Nirenberg form

\[
Af(x) = a(x, D)f(x) = \int_{\mathbb{R}^d} e^{2\pi i\langle x, \xi \rangle} a(x, \xi) \hat{f}(\xi) \, d\xi,
\]

where the symbol \( a(z), z = (x, \xi) \), is real-valued positively homogeneous of degree 2, i.e. \( a(\lambda z) = \lambda^2 a(z) \) for \( \lambda > 0 \), with \( a \in C^\infty(\mathbb{R}^{2d} \setminus 0) \). This implies \( a(x, D) \) is formally self-adjoint modulo 0-order perturbations. Basic examples are real-valued quadratic forms \( a(z) \), including the cases when \( i\partial_t + a(x, D) \) is the free particle or the harmonic oscillator operator. When \( a(z) \) is not a polynomial, we shall assume \( a(z) \) modified in a bounded neighborhood of the origin, in such a way that we have \( a \in C^\infty(\mathbb{R}^{2d}) \) keeping real values. As we shall see, cf. Example 4 below, the singularity at the origin of \( a(z) \) can be admitted as well, by absorbing it in a non-smooth potential. The pseudodifferential operator \( a(x, D) \) enters the classes of \([49]\), see also \([27]\), to which we address for the symbolic calculus and other properties, see also the next Section 2.

Concerning the potential \( \sigma(x, D) \), the regularity assumptions will be expressed in terms of the modulation spaces, introduced by Feichtinger in \([19]\), see also \([20]\), and in the last decades applied in many fields of mathematics, in particular in PDEs. We need first to recall some basic notations. The time-frequency shifts (phase-space shifts) are denoted by

\[
\pi(z)f(t) = M_\eta T_x f(t) = e^{2\pi i(t, \eta)} f(t - x), \quad z = (x, \eta).
\]

The short-time Fourier transform (STFT) of a function or distribution \( f \) on \( \mathbb{R}^d \) with respect to a Schwartz window function \( g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\} \) is defined by

\[
V_g f(x, \eta) = \langle f, \pi(z)g \rangle = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}^d} f(t) \overline{g(t - x)} e^{-2\pi i (t, \eta)} \, dt, \quad z = (x, \eta) \in \mathbb{R}^{2d}.
\]

Assuming for simplicity \( \|g\|_2 = 1 \), from \( V_g f \) we may reconstruct \( f \) by the formula

\[
f(t) = \int_{\mathbb{R}^{2d}} V_g f(x, \eta) M_\eta T_x g \, dx d\eta.
\]
Fix a non-null window function $\psi \in \mathcal{S}(\mathbb{R}^{2d})$ and perform the STFT $V_\psi \sigma(z, \zeta)$ of $\sigma(x, \xi)$ with respect to $z = (x, \xi) \in \mathbb{R}^{2d}$ with dual variables $\zeta \in \mathbb{R}^{2d}$.

**Definition 1.1.** We say that $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ belongs to the class $S^s_w$, $s \geq 0$, if

$$|V_\psi \sigma(z, \zeta)| \leq C \langle \zeta \rangle^{-s}, \quad z, \zeta \in \mathbb{R}^{2d},$$

for a suitable $C > 0$ independent of $z$ and $\zeta$, with $\langle \zeta \rangle = (1 + |\zeta|^2)^{1/2}$.

Our assumption on the potential will be $\sigma \in S^s_w$ with $s > 2d$. Observe that

$$\bigcap_{s \geq 0} S^s_w = S^0_{0,0},$$

where $S^0_{0,0}$ is the class of all $\sigma \in C^\infty(\mathbb{R}^{2d})$ satisfying

$$|\partial^\alpha \sigma(z)| \leq C_\alpha, \quad \alpha \in \mathbb{Z}^{2d}_+, \quad z = (x, \xi) \in \mathbb{R}^{2d}.$$

Whereas, for $s \rightarrow 2d+$, the symbols in $S^s_w$ have a smaller regularity. More precisely, if $s > 2d + m$, then $S^s_w \subset C^m(\mathbb{R}^{2d})$. In particular, for $s > 2d$, $S^s_w \subset C^0(\mathbb{R}^{2d})$, but the differentiability is lost in general as soon as $s \leq 2d + 1$.

It is worth to mention now the definition of the Sjöstrand class $S^s_w$, see [50], [51] and [23], given by all the symbols $\sigma$ for which

$$\int_{\mathbb{R}^d} \sup_{z \in \mathbb{R}^{2d}} |V_\psi \sigma(z, \zeta)| \, d\zeta < \infty.$$

Note that

$$\bigcup_{s > 2d} S^s_w \subset S_w \subset C^0(\mathbb{R}^{2d}).$$

In the present paper we shall not treat the case $\sigma \in S_w$, let us address to [8] where quadratic Hamiltonians with a Sjöstrand potential are studied.

Given any linear continuous operator $P : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d)$, its time-frequency representation is provided by the (continuous) Gabor matrix

$$k(w, z) := \langle P \pi(w) g, \pi(z) g \rangle, \quad w, z \in \mathbb{R}^{2d}$$

so that

$$V_g Pf(z) = \int_{\mathbb{R}^{2d}} k(w, z) V_g f(w) \, dw.$$

Time-frequency representations give a deep insight into the properties of relevant classes of operators, see for example [2, 6, 13, 23, 40, 53]. We want to study the
Gabor matrix \( k(t, w, z) \) of the propagator \( e^{itH} \). Its structure will be linked, as expected, to the Hamiltonian field of \( a(x, \xi) \). Namely, consider

\[
\begin{cases}
2\pi \dot{x} = -\nabla_\xi a(x, \xi) \\
2\pi \dot{\xi} = \nabla_x a_2(x, \xi) \\
x(0) = y, \quad \xi(0) = \eta,
\end{cases}
\]

(the factor \( 2\pi \) depends on our normalization of the STFT). Under our assumptions, the solution \( \chi(t, y, \eta) = (x(t, y, \eta), \xi(t, y, \eta)) \) exists for all \( t \in \mathbb{R} \) and defines a symplectic diffeomorphism \( \chi_t : \mathbb{R}^{2d} \to \mathbb{R}^{2d} \) homogeneous of degree 1 with respect to \( w = (y, \eta) \) for large \(|w|\), for every fixed \( t \in \mathbb{R} \).

**Theorem 1.2.** Let the preceding assumptions be satisfied, in particular let \( \sigma \in S^s_w, \ s > 2d \), and let \( k(t, w, z) \) be the Gabor matrix of the Schrödinger propagator \( e^{itH} \). Then there exists \( C = C(t, s) > 0 \) such that

\[
|k(t, w, z)| \leq C(z - \chi(t, w))^{-s}, \quad z = (x, \xi), \ w = (y, \eta) \in \mathbb{R}^{2d}.
\]

According to the notations of [7], this can be rephrased as \( e^{itH} \in \text{FI}O(\chi_t, s) \). For \( t \) sufficiently small our assumptions yield \( \det \frac{\partial}{\partial y}(t, y, \eta) \neq 0 \) in the expression of \( \chi_t \), and (14) is then equivalent to

\[
(e^{itH}u_0)(t, x) = \int_{\mathbb{R}^d} e^{2\pi i \Phi(t, x, \eta)} b(t, x, \eta) \hat{u}_0(\eta) \, d\eta,
\]

with the phase \( \Phi \) linked to \( \chi_t \) as standard and \( b(t, \cdot) \in S^s_w \), see [7, Theorem 4.3]. In the classical approach, cf. [1], the occurrence of caustics makes the validity of (15) local in time. So for \( t \in \mathbb{R} \) one is led to multiple compositions of local representations, with unbounded number of variables possibly appearing in the expression. Whereas \( k(t, w, z) \) obviously keeps life for every \( t \in \mathbb{R} \), and the estimates (14) hold for \( \chi_t \) with \( t \in \mathbb{R} \).

Under the assumption \( \sigma \in S^s_w, \ s > 2d \), natural functional frame to express boundedness and propagation results for \( e^{itH} \) is given by the modulation spaces (the classes \( S^s_w \) are special cases), see [19] and the short survey in Section 2.

We begin to recall here that for \( 1 \leq p \leq \infty, \ r \in \mathbb{R} \), the modulation space \( M_p^r(\mathbb{R}^d) \) is defined as the space of all \( f \in S'(\mathbb{R}^d) \) for which

\[
\int_{\mathbb{R}^{2d}} |V_g f(z)|^p |z|^{pr} < \infty
\]

(with obvious modifications for \( p = \infty \)). Let us now define the Gabor wave front set \( WF_{G}^{p,r}(f) \) under our consideration.

**Definition 1.3.** Let \( g \in S(\mathbb{R}^d), \ g \neq 0, \ r > 0 \). For \( f \in M^r_p(\mathbb{R}^d), \ z_0 \in \mathbb{R}^{2d}, \ z_0 \neq 0 \), we say that \( z_0 \notin WF_{G}^{p,r}(f) \) if there exists an open conic neighborhood \( \Gamma_{z_0} \subset \mathbb{R}^{2d} \).
containing \( z_0 \) such that for a suitable constant \( C > 0 \)
\[
\int_{\Gamma_{z_0}} |V_g f(z)|^p \langle z \rangle^{pr} < \infty
\]
(with obvious changes for \( p = \infty \)).

Then \( WF_G^{pr}(f) \) is well-defined as conic closed subset of \( \mathbb{R}^{2d} \setminus \{0\} \). Our main results are summarized as follows.

**Theorem 1.4.** Consider \( \sigma \in S_w^s, s > 2d, 1 \leq p \leq \infty \). Then
\[
e^{itH} : M^p_r(\mathbb{R}^d) \rightarrow M^p_r(\mathbb{R}^d)
\]
continuously, for \( |r| < s - 2d \). Moreover, for \( u_0 \in M^p_r(\mathbb{R}^d) \),
\[
WF_G^{pr}(e^{itH} u_0) = \chi_t(WF_G^{pr}(u_0)),
\]
provided \( 0 < 2r < s - 2d \).

Observe the more restrictive assumption on \( r \) for (19), with respect to that for (18).

As an elementary example consider the perturbed harmonic oscillator (studied in Example 4 in the sequel)
\[
i\partial_t u - \frac{1}{4\pi} \partial^2_x u + \pi x^2 u + |\sin x|^\mu u = 0
\]
\[
u(0, x) = u_0(x)
\]
with \( \mu > 1 \). We shall prove that \( |\sin x|^\mu \in S_w^{\mu+1} \) and from Theorem 1.4 we have that the Cauchy problem is well-posed for \( u_0 \in M^p_r(\mathbb{R}), |r| < \mu - 2 \) and the propagation of \( WF_G^{pr}(u(t, \cdot)) \) for \( t \in \mathbb{R} \) takes place as in Theorem 1.4 for \( 0 < r < \mu/2 - 1 \), where
\[
\chi(y, \eta) = \begin{pmatrix}
\cos t & -\sin t \\
\sin t & \cos t
\end{pmatrix}
\begin{pmatrix}
y \\
\eta
\end{pmatrix}
\]
with \( I \) being the identity matrix.

Using (8) and (19), we may recapture the known results for the propagation in the case of a smooth potential, i.e. \( \sigma \in S^0_w \). We define the wave front set \( WF_G(f) \) by stating \( z_0 \notin WF_G(f) \) if there exists an open conic set \( \Gamma_{z_0} \subset \mathbb{R}^{2d} \) containing \( z_0 \) such that for every \( r > 0 \)
\[
|V_g f(z)| \leq C_r(z)^{-r}, \quad z \in \Gamma_{z_0}
\]
for a suitable \( C_r > 0 \). Then the estimate (14) is satisfied for every \( s \) and from Theorem 1.4 we recapture for \( u_0 \in S^0(\mathbb{R}^d) \)
\[
WF_G(e^{itH} u_0) = \chi_t(WF_G(u_0)).
\]
This identity is contained in preceding results. Although it is impossible to do justice to the vast literature in this connection, let us mention some of the related
contributions. The pioneering work is that of Hörmander [29] 1991, who defined the wave front set in [22] as well as its analytic version, and proved (23) in the case of the metaplectic operators (cf. [21]). For subsequent results providing (23) and its analytic-Gevrey version for general smooth symbols, let us refer to [26, 30, 31, 37, 38, 41, 42, 43, 55]. The wave front sets introduced there under different names actually coincide with those of Hörmander 1991, cf. [47], [48] and [5]. Still concerning propagation of singularities in the case of smooth or analytic symbols we address to [16, 39, 45, 46, 54]. Besides, concerning global-in-time representations of $e^{itH}$, solving the problem of the caustics for smooth symbols, see [1, 3, 4, 22, 52].

Despite the abundance of contributions in the case when Hamiltonians and potentials are smooth, our study of propagation of singularities in the case of non-smooth potentials is new in literature, as far as we know. We hope, in future papers, to extend the analysis to non-smooth Hamiltonians as well, with applications to propagations for non linear Schrödinger equations. In such order of ideas, time-frequency methods represent an important tool. Beside [7, 12] see [2, 6, 8, 9, 10, 11, 13, 34, 35, 36, 40, 53].

The contents of the next sections are the following. In Section 2 after a survey on modulation spaces, Shubin classes and construction of propagators in their setting, we provide some improvements of the calculus in [7] for the classes $FIO(\chi, s)$, as preparation for the sequel. In Section 3 we treat the unperturbed equation, giving a global construction of the propagator in terms of time-frequency analysis. In Section 4 we add the non-smooth bounded perturbation, and we prove the main results of representation and continuity, stated before. The propagation result is proved in Section 5, where we also give some examples.

**Notation.** The Schwartz class is denoted by $\mathcal{S}(\mathbb{R}^d)$, the space of tempered distributions by $\mathcal{S}'(\mathbb{R}^d)$. The brackets $\langle \cdot, \cdot \rangle$ denote either the inner product on $\mathbb{R}^d$ or the extension to $\mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ of the inner product $\langle f, g \rangle = \int f(t) \overline{g(t)} dt$ on $L^2(\mathbb{R}^d)$. The Fourier transform is normalized to be $\hat{f}(\eta) = \mathcal{F}f(\eta) = \int f(t) e^{-2\pi i \langle t, \eta \rangle} dt$.

We shall use the notation $A \lesssim B$ to express the inequality $A \leq cB$ for a suitable constant $c > 0$, and $A \asymp B$ for the equivalence $c^{-1}B \leq A \leq cB$.

2. Preliminaries

We recall the basic concepts of time-frequency analysis and refer the reader to [24] for the full details.

2.1. The Short-time Fourier Transform. Consider a distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ and a Schwartz function $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ (the so-called window). The short-time Fourier transform (STFT) of $f$ with respect to $g$ is defined in [5]. The short-time Fourier transform is well-defined whenever the bracket $\langle \cdot, \cdot \rangle$ makes sense for dual
Lemma 2.1. If \( g_0, g_1, \gamma \in S(\mathbb{R}^d) \) such that \( \langle \gamma, g_1 \rangle \neq 0 \) and \( f \in S'(\mathbb{R}^d) \), then the inequality
\[
|V_{g_0}f(x, \xi)| \leq \frac{1}{|\langle \gamma, g_1 \rangle|} \langle |V_{g_1}f| * |V_{g_0}\gamma| \rangle(x, \xi)
\]
holds pointwise for all \((x, \xi) \in \mathbb{R}^{2d}\).

2.2. Modulation spaces and Shubin classes. Weighted modulation spaces measure the decay of the STFT on the time-frequency (phase space) plane and were introduced by Feichtinger in the 80’s [19].

**Weight Functions.** A weight function \( v \) is submultiplicative if \( v(z_1 + z_2) \leq v(z_1)v(z_2) \), for all \( z_1, z_2 \in \mathbb{R}^d \). We consider the weight functions
\[
v_s(z) = \langle z \rangle^s = (1 + |z|^2)^{\frac{s}{2}}, \quad s \in \mathbb{R},
\]
which are submultiplicative for \( s \geq 0 \).

For \( s \geq 0 \), we denote by \( M_{v_s}(\mathbb{R}^d) \) the space of \( v_s \)-moderate weights on \( \mathbb{R}^d \); these are measurable positive functions \( m \) satisfying \( m(z + \zeta) \leq Cv_s(z)m(\zeta) \) for every \( z, \zeta \in \mathbb{R}^d \).

**Definition 2.2.** Given \( g \in S(\mathbb{R}^d), s \geq 0 \), a weight function \( m \in M_{v_s}(\mathbb{R}^d) \), and \( 1 \leq p, q \leq \infty \), the modulation space \( M_{m}^{p,q}(\mathbb{R}^d) \) consists of all tempered distributions \( f \in S'(\mathbb{R}^d) \) such that \( V_g f \in L_m^{p,q}(\mathbb{R}^{2d}) \) (weighted mixed-norm spaces). The norm on \( M_m^{p,q}(\mathbb{R}^d) \) is
\[
\|f\|_{M_m^{p,q}} = \|V_g f\|_{L_m^{p,q}} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |V_g f(x, \xi)|^p m(x, \xi)^q \, dx \right)^{q/p} \, d\xi \right)^{1/q}
\]
(obvious changes if \( p = \infty \) or \( q = \infty \)).

When \( p = q \), we simply write \( M_m^{p}(\mathbb{R}^d) \) instead of \( M_m^{p,p}(\mathbb{R}^d) \). The spaces \( M_m^{p,q}(\mathbb{R}^d) \) are Banach spaces and every nonzero \( g \in M_{v_s}^1(\mathbb{R}^d) \) yields an equivalent norm in \( M_m^{p,q}(\mathbb{R}^d) \) and so \( M_{m}^{p,q}(\mathbb{R}^d) \) is independent on the choice of \( g \in M_{v_s}^1(\mathbb{R}^d) \).

In particular, we recover the Hörmander class
\[
S_{0,0}^0 = \bigcap_{s \geq 0} M_{s}^{1,1}(\mathbb{R}^d).
\]

Note that, for any \( 1 \leq p, q \leq \infty \),
\[
\bigcap_{s \geq 0} M_{v_s}^{p,q}(\mathbb{R}^d) = S(\mathbb{R}^d), \quad \bigcup_{s \geq 0} M_{v_s}^{p,q}(\mathbb{R}^d) = S'(\mathbb{R}^d).
\]
In the introduction we used the short notations $M^\infty_v(\mathbb{R}^d)$ for $M^\infty(\mathbb{R}^d)$ and $S^*_v$ for $M^\infty_{1\otimes v}(\mathbb{R}^{2d})$. Fix $g \in S(\mathbb{R}^d) \setminus \{0\}$. The adjoint operator of $V_g$, defined by $\langle V_g^* F, h \rangle = \langle F, V_g h \rangle$, can be written as

$$V_g^* F = \int_{\mathbb{R}^{2d}} F(x, \xi) \pi(x, \xi) g dx d\xi,$$

(28) $V_g^*$ maps the Banach space $L^p_v(\mathbb{R}^{2d})$ into $M^p_m(\mathbb{R}^d)$, in particular it maps $S(\mathbb{R}^{2d})$ into $S(\mathbb{R}^d)$ and the same for their dual spaces. In particular, if $F = V_g f$ we obtain the inversion formula for the STFT

$$\text{Id}_{M^p_m} = \frac{1}{\|g\|^2} V_g^* V_g$$

and the same holds when replacing $M^p_m(\mathbb{R}^d)$ by $S(\mathbb{R}^d)$ or $S'(\mathbb{R}^d)$.

In the subsequent Section 5 we shall use the following properties.

**Lemma 2.3.** Consider $\mu > 0$. Then the function $f(x) = |\sin x|^\mu \in M^\infty_{1\otimes v_{\mu+1}}(\mathbb{R})$.

**Proof.** Consider a window function $g \in C^\infty_0(\mathbb{R})$, with $\text{supp } g \subset [-\pi/4, \pi/4]$ to compute the STFT $V_g f$ with $f(x) = |\sin x|^\mu$. Then $|V_g f(x, \xi)|$ is a periodic function of period $\pi$ in the $x$ variable. So

$$\|f\|_{M^\infty_{1\otimes v_{\mu+1}}} = \sup_{|x| \leq \pi/2} \sup_{\xi \in \mathbb{R}} |\langle \xi \rangle^{\mu+1} |V_g f|(x, \xi)|.$$

Now observe that $\text{supp } T_x g \subset [-3\pi/4, 3\pi/4]$, for $x \in [-\pi/2, \pi/2]$, and on that interval $f(x) = |x|^\mu \varphi(x)$, with $\varphi \in C^\infty_0(\mathbb{R})$. We can write,

$$V_g f(x, \xi) = \int_0^{+\infty} e^{-2\pi it\xi} t^\mu \varphi(t) g(t-x) dt + \int_{-\infty}^0 e^{-2\pi it\xi} (-t)^\mu \varphi(t) g(t-x) dt := A + B.$$

So it suffices to estimate the integral $A$, the estimate of $B$ is analogous. Setting $F_x(t) = e^t \varphi(t) g(t-x) \in S(\mathbb{R})$ we observe that the family $\{F_x\}_{x \in [-\pi/2, \pi/2]}$ belongs to a bounded subset of $S(\mathbb{R})$. Now

$$A = \int_0^{+\infty} e^{-2\pi it\xi} t^\mu e^{-t} e^t \varphi(t) g(t-x) dt$$

$$= \frac{\Gamma(\mu+1)}{(1 + 2\pi i \xi)^{\mu+1}} \hat{F}_x(\xi),$$

and this yields

$$|\langle \xi \rangle^{\mu+1} A| \lesssim \frac{|\langle \xi \rangle^{\mu+1}|}{(1 + 2\pi i \xi)^{\mu+1}} \ast (|\langle \xi \rangle^{\mu+1} \hat{F}_x(\xi)|) \in L^\infty(\mathbb{R})$$

by Young’s inequality, since the first factor of the convolution product is bounded and the second one lies in a bounded subset of $S(\mathbb{R}) \subset L^1(\mathbb{R})$.  \qed
Corollary 2.4. Consider the symbol \( \sigma(x, \xi) = |\sin x|^{\mu} \) on \( \mathbb{R}^2 \). Then we have \( \sigma \in M^s_{1\otimes v\mu+1}(\mathbb{R}^2) \).

Proof. It is an immediate consequence of Lemma 2.3. Indeed, taking \( \psi(x, \xi) = g(x)\varphi(\xi) \), with \( g \) being the 1-dimensional window of the previous proof and \( \varphi \in \mathcal{S}(\mathbb{R}) \), we have \( V_{g}^{\mu} \sigma((x_1, x_2), (\xi_1, \xi_2)) = V_{\varphi}([\sin(\cdot)]^{\mu})(x_1, \xi_1)V_{\varphi}1(x_2, \xi_2) \) and the thesis follows since \( \langle (\xi_1, \xi_2) \rangle \leq \langle \xi_1 \rangle \langle \xi_2 \rangle \) and \( 1 \in S^0_{0, 0} \subset M^\infty_{1\otimes v\mu}(\mathbb{R}) \), for every \( s \geq 0 \). \( \Box \)

Proposition 2.5. Let \( h \in C^\infty(\mathbb{R}^d \setminus \{0\}) \) be positively homogeneous of degree \( r > 0 \), i.e. \( h(\lambda x) = \lambda^r h(x) \) for \( x \neq 0 \), \( \lambda > 0 \), and \( \chi \in C^\infty_0(\mathbb{R}^d) \). Set \( f = h\chi \). Then, for \( \psi \in \mathcal{S}(\mathbb{R}^d) \) there exists a constant \( C > 0 \) such that
\[
|V_{\psi}f(x, \xi)| \leq C(1 + |\xi|)^{-r-d}, \quad x, \xi \in \mathbb{R}^d.
\]

Proof. We know that the Fourier transform of \( h \) is a homogeneous distribution of degree \( -r - d \), smooth in \( \mathbb{R}^d \setminus \{0\} \) [28, Vol.1, Theorems 7.1.16, 7.1.18]. Hence, if \( \chi' \in C^\infty_0(\mathbb{R}^d) \), \( \chi = 1 \) in a neighborhood of the origin we have
\[
(30) \quad |(1 - \chi'(\xi))\hat{h}(\xi)| \leq C(1 + |\xi|)^{-r-d}, \quad \xi \in \mathbb{R}^d.
\]

On the other hand, by the very definition of the STFT we have
\[
|V_{\psi}f(x, \xi)| \leq |((\chi'\hat{h})*_{\xi} \hat{\chi})|*_{\xi} |\hat{\psi}| + |((1 - \chi'\hat{h})*_{\xi} \hat{\chi})|*_{\xi} |\hat{\psi}|.
\]

Since \( \hat{\psi}, \hat{\chi} \in \mathcal{S}(\mathbb{R}^d) \), the first term in the right-hand side has a rapid decay, because \( \mathcal{E}'*\mathcal{S} \subset \mathcal{S} \), whereas the second term is estimated using (30), as at the end of the proof of Lemma 2.3.

\( \Box \)

Here we are interested in operators with symbols in the Shubin classes (cf. [49], Helffer [27]); indeed, we shall use them as symbol and phase spaces for the unperturbed initial value problem for Schrödinger equations.

Definition 2.6. For \( m \in \mathbb{R} \), the class \( \Gamma^m(\mathbb{R}^{2d}) \) is the set of functions \( a \in C^\infty(\mathbb{R}^{2d}) \) such that for every \( \alpha \in \mathbb{Z}_+^{2d} \) there exists a constant \( C_\alpha > 0 \) such that
\[
|\partial^m_{\xi} a(z)| \leq C_\alpha v_{m-|\alpha|}(z), \quad z \in \mathbb{R}^{2d},
\]
where we recall \( v(z) = \langle z \rangle \) is defined in (24).

Consider \( a_j \in \Gamma^{m_j}(\mathbb{R}^{2d}) \) with \( m_j \) being a decreasing sequence tending to \(-\infty\). Then a function \( a \in C^\infty(\mathbb{R}^{2d}) \) satisfies
\[
(31) \quad a \sim \sum_{j=1}^{\infty} a_j
\]
if
\[ \forall r \geq 2 \quad a - \sum_{j=1}^{r-1} a_j \in \Gamma^m(\mathbb{R}^{2d}). \]

Namely, our symbol class will be a subclass of \( \Gamma^m(\mathbb{R}^{2d}) \), defined as follows \[27\], Sec. 1.5.

**Definition 2.7.** A function \( a \) is in the class \( \Gamma^{m,cl}(\mathbb{R}^{2d}) \) if \( a \in \Gamma^m(\mathbb{R}^{2d}) \) and admits an asymptotic expansion
\[
a \sim \sum_{j=0}^{\infty} a_{m-j},
\]
where \( a_{m-j} \in C^\infty(\mathbb{R}^{2d}) \) and satisfies \( a_{m-j}(\lambda z) = \lambda^{m-j} a_{m-j}(z) \), for \( |z| \geq 1 \) and \( \lambda \geq 1 \). The function \( a_m \) corresponding to \( j = 0 \) in the expansion (32) is called principal symbol of the symbol \( a \).

For \( a \in \Gamma^m(\mathbb{R}^{2d}) \), the corresponding pseudodifferential operator \( a(x, D) \) is defined by (3).

**Definition 2.8.** We say that \( A \in G^m \) (resp. \( A \in G^{m,cl} \)) if its symbol satisfies \( a \in \Gamma^m(\mathbb{R}^{2d}) \) (resp. \( a \in \Gamma^{m,cl}(\mathbb{R}^{2d}) \)).

A pseudodifferential operator \( A \in G^{m,cl} \) is called globally elliptic if there exist \( R > 0, C > 0 \) such that
\[
|a_m(z)| \geq C \langle z \rangle^m, \quad \text{for} \quad z \in \mathbb{R}^{2d}, \quad |z| \geq R,
\]
where \( a_m \) is the principal symbol.

### 2.3. Phase functions and canonical transformations.

Let \( a \in \Gamma^{2,cl}(\mathbb{R}^{2d}) \) with real principal symbol \( a_2 \). The related classical evolution, given by the linear Hamilton-Jacobi system, following our normalization can be written as
\[
\begin{align*}
2\pi \partial_t x(t, y, \eta) &= -\nabla_\xi a_2(x(t, y, \eta), \xi(t, y, \eta)) \\
2\pi \partial_t \xi(t, y, \eta) &= \nabla_x a_2(x(t, y, \eta), \xi(t, y, \eta)) \\
x(0, y, \eta) &= y, \\
\xi(0, y, \eta) &= \eta.
\end{align*}
\]

The solution \( (x(t, y, \eta), \xi(t, y, \eta)) \) exists for every \( t \in \mathbb{R} \). Indeed, setting \( u := (x, \xi), F(u) := (-\nabla_\xi a(u), \nabla_x a(u)) \), the initial value problem (31) can be rephrased as
\[
u'(t) = F(u(t)), \quad u(t_0) = u_0,
\]
in the particular case \( t_0 = 0 \). Observe that \( a \in \Gamma^{2,cl}(\mathbb{R}^{2d}) \) implies \( F_j \in \Gamma^{1,cl}(\mathbb{R}^{2d}) \), for \( j = 1, \ldots, 2d \) and \( \partial^\alpha F_j \in \Gamma^{0,cl}(\mathbb{R}^{2d}) \), for every \( |\alpha| > 0, j = 1, \ldots, 2d \), hence in
particular $F : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ is a Lipschitz continuous mapping. Thus the previous ODE is an autonomous ODE with a mapping $F \in C^\infty(\mathbb{R}^{2d} \to \mathbb{R}^{2d})$ having at most linear growth, hence $\|F(u)\| \lesssim 1 + \|u\|$. This implies that for each $u_0 \in \mathbb{R}^{2d}$ and $t_0 \in \mathbb{R}$ there exists a unique classical global solution $u : \mathbb{R} \to \mathbb{R}^{2d}$ (in this case $u \in C^\infty(\mathbb{R} \to \mathbb{R}^{2d})$ since $F \in C^\infty(\mathbb{R}^{2d} \to \mathbb{R}^{2d})$) to (35). Moreover the solution maps $S_{t_0}(t) : \mathbb{R}^{2d} \to C^\infty(\mathbb{R} \to \mathbb{R}^{2d})$, defined by $S_{t_0}(t_0)u_0 = u(t)$, and $S_{t_0}(t_0) = \text{Id}$, the identity operator on $\mathbb{R}^{2d}$, are Lipschitz continuous mappings, obey the time translation invariance $S_{t_0}(t) = S_{0}(t-t_0)$ and the group laws
\begin{equation}
S_{0}(t)S_{0}(t') = S_{0}(t+t'), \quad S_{0}(0) = \text{Id}.
\end{equation}
Observe that $S_{0}(t)$ is a bi-Lipschitz diffeomorphism with $S_{0}^{-1}(t) = S_{0}(-t)$. To be consistent with the notations of the earlier paper [7], we call the bi-Lipschitz diffeomorphism
\begin{equation}
\chi(t, y, \eta) := S_{0}(t)(y, \eta), \quad (y, \eta) \in \mathbb{R}^{2d}.
\end{equation}
The theory of Hamilton-Jacobi allows to find a $T > 0$ such that for $t \in [-T, T]$ there exists a phase function $\Phi(t, x, \eta)$, solution of the eiconal equation (cf. [27, (3.2.12),(3.2.13)])
\begin{equation}
\begin{cases}
2\pi \partial_t \Phi + a_2(x, \nabla_x \Phi) = 0 \\
\Phi(0, x, \eta) = x\eta
\end{cases}
\end{equation}
The phase $\Phi(t, x, \eta) \in C^\infty([-T, T] \times C^2(\mathbb{R}^{2d}))$ is real-valued since the principal symbol $a_2(x, \xi)$ is real-valued, moreover $\Phi$ fulfills the condition of non-degeneracy:
\begin{equation}
|\det \partial_{x,\eta}^2 \Phi(t, x, \eta)| \geq c > 0, \quad (t, x, \eta) \in [-T, T] \times (\mathbb{R}^{2d} \setminus \{0\}),
\end{equation}
after possibly shrinking $T > 0$ (cf. [27, Pages 142-143] and [13]).
The relation between the phase $\Phi$ and the canonical transformation $\chi$ is given by
\begin{equation}
(x, \nabla_x \Phi(t, x, \eta)) = \chi(t, \nabla_\eta \Phi(t, x, \eta), \eta), \quad t \in [-T, T].
\end{equation}
In particular,
\begin{equation}
\begin{cases}
y(t, x, \eta) = \nabla_\eta \Phi(t, x, \eta) \\
\xi(t, x, \eta) = \nabla_x \Phi(t, x, \eta),
\end{cases}
\end{equation}
and there exists $\delta > 0$ such that
\begin{equation}
|\det \frac{\partial x}{\partial y}(t, y, \eta)| \geq \delta \quad t \in [-T, T].
\end{equation}
Observe that each component of $\chi(t, \cdot)$ is a function in $C^\infty([-T, T] \times C^1(\mathbb{R}^{2d}))$, positively homogeneous of degree 1 for $(y, \eta)$ large. Moreover, using (36) we observe that the same holds in fact for every $t \in \mathbb{R}$. 
For \( t \in ] - T, T [ \), the phase function \( \Phi(t, \cdot) \) above is a \textit{tame} phase, and similarly for the canonical transformation \( \chi(t, \cdot) \), according to the following definition \([7, \text{Definition 2.1}]\):

\textbf{Definition 2.9.} A real and smooth phase function \( \Phi(x, \eta) \) on \( \mathbb{R}^{2d} \) is called \textit{tame} if:

(i) For \( z = (x, \eta) \),

\[ \left| \partial_z^\alpha \Phi(z) \right| \leq C_\alpha, \quad |\alpha| \geq 2; \]

(ii) There exists \( c > 0 \) such that the following condition of non-degeneracy holds:

\[ \left| \det \partial^2_{x, \eta} \Phi(x, \eta) \right| \geq c. \]

The mapping defined by \( (x, \xi) = \chi(y, \eta) \), which solves the system

\[ \begin{cases} 
  y(x, \eta) = \nabla_\eta \Phi(x, \eta) \\
  \xi(x, \eta) = \nabla_x \Phi(x, \eta),
\end{cases} \]

is called tame canonical transformation.

Note that in this general context we have no assumption of homogeneity for large \( (x, \eta) \), nevertheless the mapping \( \chi \) is well-defined by the global inverse function theorem, moreover \( \chi \) is a smooth bi-Lipschitz canonical transformation (i.e. it preserves the symplectic form) and satisfies, for \( (x, \xi) = \chi(y, \eta) \),

\[ \left| \partial^\alpha z_i(z) \right| + \left| \partial^\alpha \xi_i(z) \right| \leq C_\alpha, \quad |\alpha| \geq 1, \quad z = (y, \eta), \quad i = 1, \ldots, d. \]

Finally, the mapping \( \chi \) enjoys

\[ \left| \det \frac{\partial x}{\partial y}(y, \eta) \right| \geq \delta \]

(that is \((42)\) for the canonical transformations of the Hamilton-Jacobi theory), which allows to uniquely determine (up to a constant) the related tame phase function \( \Phi_\chi \) (see \([7, \text{Section 2}]\)).

We shall refine and apply results for tame canonical transformations in \([7]\) to the special case of the canonical transformations coming from \((34)\). First, we need to introduce the class of global FIOs which are the main ingredient of this study.

\textbf{2.4. The classes } FIO(\chi, s) \text{ of Fourier Integral Operators.} The definition of the class \( FIO(\chi, s) \) was introduced in \([7]\) and can be rephrased as follows.

\textbf{Definition 2.10.} Let \( g \in \mathcal{S}(\mathbb{R}^d) \) be a non-zero window function and \( s \in \mathbb{R} \). Consider a canonical transformation \( \chi \) which is a smooth bi-Lipschitz diffeomorphism and satisfies \((46)\). We say that a continuous linear operator \( T : \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d) \) is in the class \( FIO(\chi, s) \) if its (continuous) Gabor matrix satisfies the decay condition

\[ |\langle \pi(w)g, \pi(z)g \rangle| \leq C |z - \chi(w)|^{-s}, \quad \forall z, w \in \mathbb{R}^{2d}. \]
Note that we do not require (47) to be valid.

The class $FIO(\Xi, s) = \bigcup \chi FIO(\chi, s)$ is the union of these classes where $\chi$ runs over the set of all smooth bi-Lipschitz canonical transformations satisfying (46).

Gabor frames decompositions of FIOs in [7] produce the following issues.

(i) \textit{Boundedness of $T$ on $M^p(\mathbb{R}^d)$} ([7, Theorem 3.4]): If $s > 2d$ and $T \in FIO(\chi, s)$, then $T$ can be extended to a bounded operator on $M^p(\mathbb{R}^d)$ (in particular on $L^2(\mathbb{R}^d)$).

(ii) \textit{The algebra property} ([7, Theorem 3.6]): For $i = 1, 2$, $s > 2d$,

\[ T^{(i)} \in FIO(\chi_i, s) \Rightarrow T^{(1)} T^{(2)} = T^{(1)}(\chi_1 \circ \chi_2, s), \]

(iii) \textit{The Wiener property} ([7, Theorem 3.7]): If $s > 2d$, $T \in FIO(\chi, s)$ and $T$ is invertible on $L^2(\mathbb{R}^d)$, then $T^{-1} \in FIO(\chi^{-1}, s)$.

These three properties imply that the union $FIO(\Xi, s)$ is a Wiener subalgebra of $\mathcal{L}(L^2(\mathbb{R}^d))$, the class of linear bounded operators on $L^2(\mathbb{R}^d)$. Property (ii) can be refined as follows.

\begin{lemma}
For $s > 2d$, $T^{(i)} \in FIO(\chi_i, s)$, $i = 1, 2$, the continuous Gabor matrix of the composition $T^{(1)} T^{(2)}$ is controlled by
\[ |\langle T^{(1)} T^{(2)}(\pi(w)g, \pi(z)g) \rangle| \leq C_0 C_1 C_2 (z - \chi_1 \circ \chi_2(w))^{-s}, w, z \in \mathbb{R}^{2d}, \]

where $C_i > 0$ is the constant of $T^{(i)}$ in (18), $i = 1, 2$, whereas $C_0 > 0$ depends only on $s$ and on the Lipschitz constants of $\chi_1$ and $\chi_1^{-1}$.
\end{lemma}

\textit{Proof.} Consider $g \in \mathcal{S}(\mathbb{R}^d)$ with $\|g\|_2 = 1$. We write the product $T^{(1)} T^{(2)}$ as

\[ T^{(1)} T^{(2)} = V^*_g V^*_g T^{(1)}(\chi_i, s) V_g V_g = V^*_g (V^*_g V_g T^{(1)}(\chi_i, s) V_g V_g V_g). \]

Thus the composition of operators corresponds to the multiplication of their (continuous) Gabor matrices. Using the decay estimates for the continuous Gabor matrices of $T^{(i)}$, $i = 1, 2$,

\[ |\langle T^{(1)} T^{(2)}(\pi(w)g, \pi(z)g) \rangle| \leq \int_{\mathbb{R}^{2d}} |\langle T^{(1)}(\pi(w)g, \pi(y)g) \rangle| |\langle T^{(2)}(\pi(y)g, \pi(z)g) \rangle| dy \]

\[ \leq C_1 C_2 \int_{\mathbb{R}^{2d}} \langle z - \chi_1(y) \rangle^{-s} \langle y - \chi_2(w) \rangle^{-s} dy \]

\[ \leq C_1 C_2 C(\chi_1) \int_{\mathbb{R}^{2d}} \langle \chi_1^{-1}(z) - y \rangle^{-s} \langle y - \chi_2(w) \rangle^{-s} dy \]

\[ \leq C_1 C_2 C(\chi_1) C_s \langle \chi_1^{-1}(z) - \chi_2(w) \rangle^{-s} \]

\[ \leq C_1 C_2 C(\chi_1) C_s C(\chi_1^{-1}) \langle z - \chi_1 \circ \chi_2(w) \rangle^{-s} \]

for every $z, w \in \mathbb{R}^{2d}$, $s > 2d$, where $C_1$ and $C_2$ are the controlling constants in (18) of the operators $T^{(1)}$ and $T^{(2)}$, and the bi-Lipschitz property of $\chi_1$ gives

\[ \langle z - \chi_1(y) \rangle^{-s} \leq C(\chi_1) \langle \chi_1^{-1}(z) - y \rangle^{-s}, \quad \forall y, z \in \mathbb{R}^{2d} \]
and
\[ \langle \chi_1^{-1}(y) - z \rangle^{-s} \leq C(\chi_1^{-1}) \langle y - \chi_1(z) \rangle^{-s}, \quad \forall y, z \in \mathbb{R}^{2d}. \]
Furthermore, we used that \( v_s \) is subconvolutive for \( s > 2d \): \( v_s * v_{-s} \leq C_s v_{-s} \) \[21\] Lemma 11.1.1(d)]. If we call \( C_0 = C(\chi_1)C_sC(\chi_1^{-1}) \), the claim is proved. \( \square \)

By induction we immediately obtain

**Corollary 2.12.** For \( n \in \mathbb{N}, n \geq 2, s > 2d, T^{(i)} \in FIO(\chi_i, s), i = 1, \ldots, n, \) we have
\[ (T^{(1)}T^{(2)} \cdots T^{(n)}g, \pi(z)g) \leq C_0 C_1 \cdots C_n (z - \chi_1 \circ \chi_2 \circ \cdots \circ \chi_n(w))^{-s}. \]
where \( C_0 \) depends on \( s \) and on the Lipschitz constants of the mappings:
\[ \chi_1, \chi_1^{-1}, \chi_1 \circ \chi_2, (\chi_1 \circ \chi_2)^{-1}, \cdots, \chi_1 \circ \chi_2 \circ \cdots \circ \chi_{n-1}, (\chi_1 \circ \chi_2 \circ \cdots \circ \chi_{n-1})^{-1}. \]

Observe that, using Schur’s test and the same techniques as in the proof \[7\] Theorem 3.4, it is straightforward to obtain the following weighted version of \[7\] Theorem 3.4. Hence we omit the proof.

**Theorem 2.13.** Let \( 0 \leq r < s - 2d, \) and \( \mu \in M_{v_r} \). For every \( 1 \leq p \leq \infty, \) \( T \in FIO(\chi, s) \) extends to a continuous operator from \( M^p_{\mu \circ \chi} \) into \( M^p_{\mu} \).

Let us underline that \( \mu \circ \chi \in M_{v_r}, \) since \( v_r \circ \chi \asymp v_r, \) due to the bi-Lipschitz property of \( \chi. \)

If \( \chi = \text{Id}, \) the identity operator, then the corresponding Fourier integral operators are simply pseudodifferential operators, as already shown in \[25\].

The characterization below is written for pseudodifferential operators in the Kohn-Nirenberg form \( \sigma(x, D), \) but it works the same for any \( \tau \)-form (in particular Weyl form \( \sigma^w(x, D) \)) in which is written a pseudodifferential operator.

**Proposition 2.14.** Fix \( g \in S(\mathbb{R}^d) \) and let \( \sigma \in S'(\mathbb{R}^{2d}). \) For \( s \in \mathbb{R}, \) the symbol \( \sigma \in M^\infty_{1 \otimes v_s}(\mathbb{R}^{2d}) \) if and only if
\[ (\sigma(x, D)\pi(w)g, \pi(z)g) \leq C \langle z - w \rangle^{-s} \quad \forall w, z \in \mathbb{R}^{2d}. \]

Similarly, under additional assumptions on the classes \( FIO(\chi, s) \), their operators can be written in the following integral form, called FIOs of type I:
\[ I(\sigma, \Phi)f(x) = \int_{\mathbb{R}^d} e^{2\pi i \Phi(x, \eta)} \sigma(x, \eta) \hat{f}(\eta) d\eta, \quad f \in S(\mathbb{R}^d), \]
where \( \sigma \in M^\infty_{1 \otimes v_s}(\mathbb{R}^{2d}) \) and \( \Phi \) a tame phase function. This particular form is allowed starting from the class \( FIO(\chi, s) \) whenever the mapping \( \chi \) enjoys the additional property \[47\] as explained in the following characterization \[7\] Theorem 4.3.\]
Theorem 2.15. Consider \( g \in \mathcal{S}(\mathbb{R}^d) \) and \( s \geq 0 \). Let \( I \) be a continuous linear operator \( \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d) \) and \( \chi \) be a tame canonical transformation satisfying \((\gamma)\). Then the following properties are equivalent.

(i) \( I = I(\sigma, \Phi_\chi) \) is a FIO of type I for some \( \sigma \in M^\infty_{I_\otimes v_s}(\mathbb{R}^{2d}) \).

(ii) \( I \in \text{FIO}(\chi, s) \).

Moreover, gluing together the results [6, Theorem 3.3] and [7, Theorem 4.3] we observe that the constant \( C \) in \((\delta)\) satisfies

\[
C \asymp \| \sigma \|_{M^\infty_{I_\otimes v_s}}.
\]

For \( \chi = \text{Id} \) we recapture the characterization for pseudodifferential operators of Proposition 2.14.

Since we shall apply our results to the mappings \( \chi(t, x, \eta) \) coming from the Hamilton-Jacobi system \((\zeta)\), we need to be more precise on the estimate \((\delta)\): it is important to see how the constants involved in the equivalence depend on the time variable \( t \). It amounts rewriting the proofs of the results cited above for the special case of a phase function \( \Phi \in \mathcal{C}^\infty([-T, T], \Gamma^2(\mathbb{R}^{2d})) \) and following the time variable \( t \). We state the result here and we refer to the Appendix for a sketch the main points of the proofs, leaving the details to the interested reader.

Theorem 2.16. Consider \( g \in \mathcal{S}(\mathbb{R}^d) \), \( s \geq 0 \), and \( T > 0 \) such that in \([-T, T] \) the equation \((\zeta)\) is solved by the tame phase \( \Phi \in \mathcal{C}^\infty([-T, T], \Gamma^2(\mathbb{R}^{2d})) \). Let \( \chi \) be the related tame canonical transformation in \((\zeta)\). Let \( I \) be a continuous linear operator \( \mathcal{S}(\mathbb{R}^d) \rightarrow \mathcal{S}'(\mathbb{R}^d) \). Then the following are equivalent:

(i) \( I = I(\sigma_t, \Phi_\chi) \) is a FIO of type I for some \( \sigma_t \in M^\infty_{I_\otimes v_s}(\mathbb{R}^{2d}) \) such that

\[
\| \sigma_t \|_{M^\infty_{I_\otimes v_s}} \leq H(t) \in \mathcal{C}([-T, T]).
\]

(ii) \( I \in \text{FIO}(\chi, s) \) and the constant \( C = C(t) \) in \((\delta)\) is in \( \mathcal{C}([-T, T]) \).

### 3. Unperturbed Schrödinger Equations

The previous theory applies in the study of the Cauchy problem for linear Schrödinger equations. First, consider the unperturbed case:

\[
\begin{cases}
  i\partial_t u + Au = 0 \\
  u(0, x) = u_0(x),
\end{cases}
\]

with \( x \in \mathbb{R}^d, u_0 \in \mathcal{S}(\mathbb{R}^d) \). The operator \( A = a(x, D) \in \Gamma^{2,cl} \) is a formally self-adjoint pseudodifferential operator in the Kohn-Nirenberg form. This means that the symbol \( a \in \Gamma^{2,cl}(\mathbb{R}^{2d}) \) has the expansion

\[
a(x, \xi) \sim \sum_{j=0}^{\infty} a_{2-j}(x, \xi),
\]
where the principal symbol \( a_2(x, \xi) \) is real-valued, since \( A \) is self-adjoint. The problem (57) is forward and backward well-posed in \( \mathcal{S}(\mathbb{R}^d) \) and the corresponding evolution operator \( e^{itA} \), acting from \( \mathcal{S}(\mathbb{R}^d) \) into \( \mathcal{S}(\mathbb{R}^d) \), extends to \( L^2 \)-isometries [27].

The classical evolution (34) has the solution \( (x(t), \xi(t)) = \chi(t, y, \eta) \) in (37) and for a suitable \( T > 0 \) and \( t \in ]-T, T[ \) the evolution operator \( e^{itA} \) can be well approximated by a FIO of type I, as expressed in [27, Proposition 3.1] for the special case of elliptic operators (that is operators whose corresponding principal symbols satisfy (33)), but still valid without the assumption (33), as observed in [13, Section 5.3]). In our framework the result [27, Proposition 3.1] can be rephrased as follows.

**Proposition 3.1.** Given the Cauchy problem (57) with \( a(x, D) \) as above, then there exists a \( T > 0 \), a symbol \( \sigma(t, x, \eta) \in C^\infty(] - T, T[, \Gamma^0(\mathbb{R}^{2d})) \) a real-valued phase function \( \Phi \in C^\infty(] - T, T[, \Gamma^2(\mathbb{R}^{2d})) \) satisfying (38) and (39) such that the evolution operator can be written as

\[
(e^{itA}u_0)(t, x) = (F_tu_0)(t, x) + (R_tu_0)(t, x),
\]

where \( F_t \) is the FIO of type I

\[
(F_tu_0)(t, x) = \int_{\mathbb{R}^d} e^{2\pi i \Phi(t, x, \eta)} \sigma(t, x, \eta) \hat{u}_0(\eta) d\eta
\]

and the operator \( R_t \) has kernel in \( C^\infty(] - T, T[, \mathcal{S}(\mathbb{R}^{2d})) \) (thus \( R_t \) is regularizing, i.e., \( R_t : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d) \).)

This result says that in an interval \( ] - T, T[ \) the propagator \( e^{itA} \) can be represented by a type I FIO \( F_t \) up to an error, which however is a regularizing operator.

**Remark 3.2.** We observe that the function \( \Phi(t, \cdot) \) of Proposition 3.1 and the related canonical transformation \( \chi(t, \cdot) \) in (37) are tame, with Lipschitz constants of \( \chi(t, \cdot) \) and its inverse that can be controlled by a continuous function of \( t \) on the interval \( ] - T, T[ \) and so can be chosen uniform with respect to \( t \) on \( ] - T, T[ \).

We will show that if we replace the type I FIO \( F_t \) by a more general operator in the classes \( \text{FIO}(\chi(t, \cdot), s) \), we are able to remove the error \( R_t \) in (59). Precisely, we can state the following issue.

**Proposition 3.3.** Under the assumptions of Proposition 3.1 we have

\[
e^{itA} \in \cap_{s \geq 0} \text{FIO}(\chi(t, \cdot), s), \quad t \in ] - T, T[,
\]

where \( \chi \) is defined in (37). Moreover for every \( s \geq 0 \) there exists \( C(t) = C_s(t) \in C(] - T, T[) \) such that, for every \( g \in \mathcal{S}(\mathbb{R}^d) \) the Gabor matrix satisfies

\[
|\langle e^{itA} \pi(w)g, \pi(z)g \rangle| \leq C(t)|z - \chi(t, w)|^{-s}, \quad w, z \in \mathbb{R}^{2d}.
\]
Proof. By Proposition 3.1 there exists a \( T > 0 \) such that the evolution \( e^{itA} \) can be written as (59), where \( F_t \) is a type I FIO with symbol in \( \sigma(t, x, \eta) \in C^\infty(-T, T, \Gamma^0) \) and phase \( \Phi(t, \cdot) \) in (58). Since

\[
C^\infty(-T, T, \Gamma^0) \subset C^\infty(-T, T, S^0_{0,0}) = C^\infty(-T, T, \cap_{s \geq 0} M^\infty_{1,\otimes w_s}),
\]

we can find \( C > 0 \) and \( k_1 = k_1(s) \in \mathbb{N} \) such that

\[
\|\sigma(t, \cdot)\|_{M^\infty_{1,\otimes w_s}} \leq C \sum_{|\alpha| \leq k_1} \|\partial^\alpha \sigma(t, \cdot)v_{|\alpha|}\|_{\infty}
\]

where \( \sum_{|\alpha| \leq k_1} \|\partial^\alpha \sigma(t, \cdot)v_{-|\alpha|}\|_{\infty} \in C([-T, T]) \) by assumption. Hence the characterization of Theorem 2.16 gives \( F_t \in \cap_{s \geq 0} FIO(\chi(t, \cdot), s, t \in [-T, T], \) where the canonical transformation \( \chi \) is defined in (37) and related with \( \Phi(t, \cdot) \) by (40) and

\[
|\langle F_t \pi(w)g, \pi(z)g \rangle| \leq C(t) \langle z - \chi(t, w) \rangle^{-s}
\]

with \( C(t) \in C([-T, T]) \). Fix now \( g \in S(\mathbb{R}^d) \) with \( \|g\|_2 = 1 \) so that the inversion formula (29) becomes \( \text{Id} = V^*_g V_g \) and we can write \( R_t = V^*_g V_g R_t V^*_g V_g \). Since \( R_t \) is a regularizing operator, for \( T_t := V_g R_t V^*_g \), the following diagram is commutative:

\[
\begin{array}{ccc}
S'(\mathbb{R}^{2d}) & \xrightarrow{T_t} & S(\mathbb{R}^{2d}) \\
V^*_g \downarrow & & \downarrow V_g \\
S'(\mathbb{R}^d) & \xrightarrow{R_t} & S(\mathbb{R}^d)
\end{array}
\]

(see the definition and properties of \( V_g \) and its adjoint \( V^*_g \) in Subsection 2.2). This means that the linear operator \( T_t : S'(\mathbb{R}^{2d}) \to S(\mathbb{R}^{2d}) \) is regularizing as well and so its kernel \( k_t(w, z) = \langle R_t \pi(w)g, \pi(z)g \rangle \in S(\mathbb{R}^{4d}) \) satisfies

\[
|k_t(w, z)| = |\langle R_t \pi(w)g, \pi(z)g \rangle| \leq \langle z \rangle^{-N} \langle w \rangle^{-N}, \quad \forall z, w \in \mathbb{R}^{2d}, \forall N \in \mathbb{N}.
\]

The previous estimates yields \( R_t \in FIO(\chi, s) \), for every bi-Lipschitz mapping \( \chi \) and every \( s \geq 0 \). Indeed,

\[
\langle z - \chi(w) \rangle \leq \langle z \rangle \langle \chi(w) \rangle \asymp \langle z \rangle \langle w \rangle
\]

and choosing \( N \geq s \) in (63) we obtain

\[
|k_t(w, z)| \lesssim \langle z \rangle^{-N} \langle w \rangle^{-N} \lesssim \langle z - \chi(w) \rangle^{-s}.
\]

Finally, if \( \sigma(R_t)(z) \) is the Kohn-Nirenberg symbol of \( R_t \), using the fact that \( S(\mathbb{R}^{2d}) \subset S^0_{0,0} \) with continuous embedding for every \( s \geq 0 \) we find \( C > 0 \) and \( k_2 \in \mathbb{N} \) such
that
\[ \|\sigma(R_t)\|_{M^\infty_{100\pi}} \leq C \sum_{|\alpha+\beta| \leq k_2} \|z^\alpha \partial_z^\beta \sigma(R_t)(z)\|_\infty \in C(]-T,T[). \]

Using Theorem 2.16 with \( \chi(t, \cdot) \) in (37) which is tame for \( t \in ]-T,T[ \), we find \( C(t) \in C(]-T,T[) \) such that
\[ |\langle R_t \pi(w)g, \pi(z)g \rangle| \leq C(t)\langle z - \chi(t,w) \rangle^{-s}. \]

Finally the thesis follows since \( FIO(\chi, s) \) are linear spaces:
\[ |\langle e^{itA} \pi(w)g, \pi(z)g \rangle| \leq |\langle F_t \pi(w)g, \pi(z)g \rangle| + |\langle R_t \pi(w)g, \pi(z)g \rangle| \leq C(t)\langle z - \chi(t,w) \rangle^{-s}. \]

which gives (62).

The previous proposition gives an approximation of \( e^{itA} \) for \( |t| < T \). Using the group property of the propagator \( e^{itA} \) Helffer in [27], page 139] describes how to obtain an approximation of \( e^{itA} \) for every \( t \in \mathbb{R} \). Indeed, a classical trick, jointly with the group property of \( e^{itA} \), applies. We consider \( T_0 < T/2 \) and define
\[ I_h = \lfloor hT_0, (h+2)T_0 \rfloor, \quad h \in \mathbb{Z}. \]

For \( t \in I_h \), by the group property of \( e^{itA} \):
\[ e^{itA} = e^{i(t-hT_0)A} (e^{ihT_0} A)^h |^{|h|} \]
and using Proposition 3.3 one can write
\[ e^{itA} = F_t-hT_0 (F_{hT_0} |^{h|}) \in C^\infty(I_h, \mathcal{L}(S', S)). \]

In general, \( e^{itA} \) or even the composition \( F_{t-hT_0} (F_{hT_0} |^{h|}) \) cannot be represented as a type I FIO in the form (60). We shall prove below that the evolution \( e^{itA} \) is in the class \( \cap_{s \geq 0} FIO(\chi(t, \cdot), s) \) for every \( t \in \mathbb{R} \), with \( \chi \) defined in (37), so that this class is proven to be the right framework for describing the evolution \( e^{itA} \).

**Theorem 3.4.** Given the Cauchy problem (57) with \( A = a(x, D) \) as above. Consider the mapping \( \chi \) defined in (37). Then
\[ \|e^{itA}\| \in \cap_{s \geq 0} FIO(\chi(t, \cdot), s), \quad t \in \mathbb{R} \]
and for every \( s > 2d \) there exists \( C(t) \in C(\mathbb{R}) \) such that
\[ |\langle e^{itA} \pi(w)g, \pi(z)g \rangle| \leq C(t)\langle z - \chi(t,w) \rangle^{-s}, \quad w, z \in \mathbb{R}^{2d}, \quad t \in \mathbb{R}. \]

**Proof.** We fix \( T_0 < T/2 \) as above. For \( t \in \mathbb{R} \), there exists a \( h \in \mathbb{Z} \) such that \( t \in I_h \). Using Proposition 3.3 for \( t_1 = t-hT_0 \in ]-T,T[ \) we have that \( e^{it_1 A} \in FIO(\chi(t_1, \cdot), s) \) and for \( t_2 = \frac{1}{|h|} T_0 \in ]-T,T[ \), \( e^{i t_2 A} \in FIO(\chi(t_2, \cdot), s) \), for every \( s \geq 0 \), and there
exists a continuous function $C(t)$ on $]-T,T[$ such that (62) is satisfied for $t = t_1$ and $t = t_2$. Using the algebra property (49), for every $s > 2d$, 

$$e^{it_1A}(e^{it_2A})^{|h|} \in FIO(\chi(t_1,\cdot) \circ (\chi(t_2,\cdot))^{|h|}, s)$$

and the group law (36) for $\chi(t,y,\eta) = S_0(t)(y,\eta)$ gives 

$$\chi(t_1,\cdot) \circ (\chi(t_2,\cdot))^{|h|} = \chi(t_1 + |h|t_2,\cdot) = \chi(t,\cdot),$$

as expected and using (52) we obtain that the Gabor matrix of the product $e^{it_1A}(e^{it_2A})^{|h|}$ is controlled by a continuous function $C_h(t)$ on $I_h$. Finally, from the estimate 

$$|\langle e^{itA}\pi(w)g, \pi(z)g \rangle| \leq C_h(t) \langle z - \chi(t,w) \rangle^{-s}, \quad t \in I_h,$$

with $C_h \in C(I_h)$, it is easy to construct a new continuous controlling function $C(t)$ on $\mathbb{R}$ such that (67) is satisfied. 

4. Schrödinger Equations with bounded perturbations

We now study the Cauchy problem for linear Schrödinger equations of the type 

\begin{equation}
\begin{cases}
  \frac{i}{\partial t} u + Hu = 0 \\
  u(0,x) = u_0(x),
\end{cases}
\end{equation}

with $t \in \mathbb{R}$ and the initial condition $u_0 \in S(\mathbb{R}^d)$. We consider a Hamiltonian of the form 

\begin{equation}
H = a(x,D) + \sigma(x,D),
\end{equation}

where $A = a(x,D)$ is the pseudodifferential operator satisfying (57), whose corresponding propagator $e^{itA} \in \cap_{s \geq 0} FIO(\chi(t,\cdot), s)$, for $t \in \mathbb{R}$, as shown in the preceding section.

The perturbation $B = \sigma(x,D)$ is a pseudodifferential operator with a symbol $\sigma \in M^\infty_{1,\infty} (\mathbb{R}^{2d}), s > 2d$. This last requirement implies the boundedness of $B$ on $M_\mu(\mathbb{R}^d)$ for a weight $\mu$ as in the assumptions of Theorem 2.13 (with $\chi = 1d$), (see also [24] using $M^\infty_{1,\infty} (\mathbb{R}^{2d}) \subset M^{\infty,1} (\mathbb{R}^{2d}), s > 2d$) and in particular on $L^2(\mathbb{R}^d)$. Hence, $H = A + B$ is a bounded perturbation of the generator $A$ of a unitary group by [44], and $H$ is the generator of a well-defined (semi-)group. We shall heavily use the theory of operator semigroups, addressing to the textbooks [44] and [18] for an introduction on the topic. Our result, containing Theorem 1.2, is as follows.

**Theorem 4.1.** Let $s > 2d$. Consider the Cauchy problem (68) with $A = a(x,D)$ and $B = \sigma(x,D)$ as above. Let $\chi$ be the mapping defined in (37). Then the solution can be written as 

$$e^{itH} = e^{itA}Q(t) = \tilde{Q}(t)e^{itA} \in FIO(\chi(t,\cdot), s), \quad t \in \mathbb{R},$$
where \( Q(t) \) and \( \tilde{Q}(t) \) are pseudodifferential operators with symbols in \( M^\infty_{1\otimes v_s}(\mathbb{R}^{2d}) \) and the continuous Gabor matrix satisfies

\[
|\langle e^{itH} \pi(w)g, \pi(z)g \rangle| \leq C(t) |z - \chi(t, w)|^{-s}, \quad w, z \in \mathbb{R}^{2d},
\]

for a suitable positive continuous function \( C(t) \) on \( \mathbb{R} \).

**Proof.** The pattern is similar to [8, Theorem 4.1]. We show the result on the interval \([0, +\infty[\), for the interval \([-\infty, 0]\) the result is obtained by the previous case by replacing \( t \) with \(-t\).

The operator \( A \) is the generator of a strongly continuous one-parameter group on \( L^2(\mathbb{R}^d) \) and \( T(t) = e^{itA} \) is the corresponding (semi)group that solves the evolution equation \( \frac{dT(t)}{dt} = AT(t) \). Then \( e^{itA} \) is a strongly continuous one-parameter group on \( L^2(\mathbb{R}^d) \). As already observed, by the assumptions on the symbol of \( B \), it follows that \( B \) is a bounded operator on \( L^2(\mathbb{R}^d) \), hence \( H = A + B \) is the generator of a strongly continuous one-parameter group \( S(t) \) [15]. The perturbed semigroup \( S(t) = e^{itH} \) satisfies an abstract Volterra equation

\[
S(t)f = T(t)f + \int_0^t T(t-s)BS(s)f \, ds = T(t)\left(\text{Id} + \int_0^t T(-s)BT(s)T(-s)S(s) \, ds\right)f
\]

for every \( f \in L^2(\mathbb{R}^d) \) and \( t \geq 0 \). If we define by \( Q(t) = T(-t)S(t) \), then by (70) \( Q(t) \) satisfies the Volterra equation

\[
Q(t) = \text{Id} + \int_0^t T(-s)BT(s)Q(s) \, ds,
\]

where the integral is to be understood in the strong sense. Now write \( B(s) = T(-s)BT(s) \), then the solution of (71) can be written as a so-called *Dyson-Phillips expansion* ([14, X.69] or [13, Ch. 3, Thm. 1.10])

\[
Q(t) = \text{Id} + \sum_{n=1}^\infty (-i)^n \int_0^t \int_0^{t_1} \ldots \int_0^{t_{n-1}} B(t_1)B(t_2) \ldots B(t_n) \, dt_1 \ldots dt_n := \sum_{n=0}^\infty Q_n(t).
\]

We shall show that \( Q(t) \) is a pseudodifferential operator with symbol in \( M^\infty_{1\otimes v_s}(\mathbb{R}^{2d}) \). For \( \tau \in [0, t] \), the algebra property (19) gives

\[
B(\tau) = e^{i(-\tau)A}Be^{i\tau A} \in FIO(\chi(-\tau, \cdot) \circ \text{Id} \circ \chi(\tau, \cdot), s) = FIO(\text{Id}, s)
\]

since \( \chi(-\tau, \cdot) \circ \text{Id} \circ \chi(\tau, \cdot) = \chi(-\tau, \cdot) \circ \chi(\tau, \cdot) = S_0(0) = \text{Id} \) by (35). Moreover, \( e^{i(\pm A)} \) satisfies (67), so that using (52) with \( n = 3 \), \( T(1) = e^{i(-\tau)A} \), \( T(2) = B \), \( T(3) = e^{i\tau A} \) and \( \chi = \text{Id} \) we can write

\[
|\langle B(\tau)\pi(w)g, \pi(z)g \rangle| \leq C(\tau) |z - w|^{-s},
\]
for a new continuous function $C(\tau)$ on $\mathbb{R}$. Using (52) again for the composition of pseudodifferential operators $\prod_{j=1}^{n} B(t_j)$ we obtain

$$|\langle \prod_{j=1}^{n} B(t_j) \pi(w) g, \pi(z) g \rangle| \leq C_0 C(t_1) \cdots C(t_n) (z - w)^{-s},$$

with $C(t) \in C(\mathbb{R})$ in (73).

We now show that $Q_n(t)$ is a pseudodifferential operator with symbol in $M_{1\otimes v_{s}}^{\infty}(\mathbb{R}^{2d})$. We control the Gabor matrix of $Q_n(t)$ as follows:

$$|\langle Q_n(t) \pi(w) g, \pi(z) g \rangle| \leq C_0 \int_{0}^{t} \int_{0}^{t_1} \cdots \int_{0}^{t_{n-1}} C(t_1) \cdots C(t_n) dt_1 \cdots dt_n (z - w)^{-s}.$$ 

If we define

$$H(t) = \max_{\tau \in [0,t]} C(\tau) \in C(\mathbb{R}),$$

we obtain

$$|\langle Q_n(t) \pi(w) g, \pi(z) g \rangle| \leq C_0 H(t)^n \frac{t^n}{n!} (z - w)^{-s}.$$ 

Finally, setting $\tilde{H}(t) = tH(t) \in C(\mathbb{R})$,

$$|\langle Q(t) \pi(w) g, \pi(z) g \rangle| \leq \sum_{n=0}^{\infty} |\langle Q_n(t) \pi(w) g, \pi(z) g \rangle| (z - w)^{-s}$$

$$\leq C_0 \sum_{n=0}^{\infty} \frac{\tilde{H}(t)^n}{n!} (z - w)^{-s}$$

$$= C(t) (z - w)^{-s},$$

for a new function $C(t) \in C(\mathbb{R})$. This gives by Theorem 2.16 that $Q(t) \in FIO(\text{Id}, s)$ that is $Q$ is a pseudodifferential operator with symbol in $M_{1\otimes v_{s}}^{\infty}(\mathbb{R}^{2d})$. Finally, the algebra property again gives

$$e^{itH} = T(t) C(t) \in FIO(\chi(t, \cdot), s),$$

and the estimate (51) gives that the Gabor matrix of $e^{itH}$ is controlled by a continuous function $C(t)$ on $\mathbb{R}$. \hfill \square

Consequently, the Schrödinger equation preserves the phase-space concentration, as expressed by the following issue.

\textbf{Corollary 4.2.} Let $0 \leq r < s - 2d$, and $\mu \in M_{v_{r}}$. If the initial condition $u_0 \in M_{p_{\chi}}^{p}$, $1 \leq p \leq \infty$, then $u(t, \cdot) = e^{itH} u_0 \in M_{\mu}^{p}$, for all $t \in \mathbb{R}$. 

Proof. It follows immediately from Theorems 4.1 and 2.13.

Using \( v_r \circ \chi \approx v_r \), we observe that the Schrödinger evolution preserves the phase space concentration \( M_{v_r}^p \) of the initial condition \( u_0 \). In other words, the time evolution leaves \( M_{v_r}^p \) invariant.

Corollary 4.3. Let \( |r| < s - 2d \). If the initial condition \( u_0 \in M_{v_r}^p \), \( 1 \leq p \leq \infty \), then \( u(t, \cdot) = e^{itH}u_0 \in M_{v_r}^p \), for all \( t \in \mathbb{R} \).

Proof. The result is a special case of Corollary 4.2 once we prove, for \( r > 0 \), \( \psi_q \) is \( v_r \)-moderate if and only if \( |q| \leq r \). But this is an easy consequence of Peetre’s inequality

\[
\langle z + \zeta \rangle^q \leq \langle z \rangle^{|q|} \langle \zeta \rangle^q.
\]

From Corollaries 4.2 and 4.3 we recapture (18) in Theorem 1.4.

5. Propagation of singularities

In what follows we shall use \( \chi_t \) for \( \chi(t, \cdot) \) when it is more convenient.

Proposition 5.1. Let \( f \in \mathcal{S}'(\mathbb{R}^d) \), \( r > 0 \). Then:

(i) The definitions of \( WF_{p,r}^{G} (f) \) and \( WF_{G}(f) \) do not depend on the choice of the window \( g \).

(ii) \( f \in M_{v_0}^p(\mathbb{R}^d) \) if and only if \( WF_{G}^{p,r}(f) = \emptyset \). Similarly, \( f \in \mathcal{S}(\mathbb{R}^d) \) if and only if \( WF_{G}(f) = \emptyset \).

The proof of (i) will be given later, as a consequence of more general arguments. The proof of (ii) follows easily from the compactness of the sphere \( S^{2d-1} \) and (27).

The following statement gives the second part of Theorem 1.4.

Theorem 5.2. Under the assumptions of Theorem 4.1, for \( u_0 \in M_{v_r}^p(\mathbb{R}^d) \), \( 1 \leq p \leq \infty \), \( 0 < 2r < s - 2d \), we have

\[
WF_{G}^{p,r}(e^{itH}u_0) = \chi_t(WF_{G}^{p,r}(u_0)).
\]

Proof. We shall prove that \( WF_{G}^{p,r}(e^{itH}u_0) \subset \chi_t(WF_{G}^{p,r}(u_0)) \) for any \( t \in \mathbb{R} \). Then, by applying the inclusion to \( v_0 = e^{-itH}u_0 \), the opposite inclusion will follow, and (74) will be proved.

Fixed \( t \in \mathbb{R} \), we assume \( z_0 \notin \chi_t(WF_{G}^{p,r}(u_0)) \). Since \( \chi_t \) is a homogeneous diffeomorphism for large \( |z| \), this is equivalent to say that \( w_0 = \chi_t^{-1}(z_0) \) does not belong to \( WF_{G}^{p,r}(u_0) \). Therefore for a sufficiently small open conic neighborhood \( \Gamma_{w_0} \subset \mathbb{R}^{2d} \setminus 0 \) of \( w_0 \) we have

\[
\int_{\Gamma_{w_0}} |V_z u_0(w)|^p \langle w \rangle^{pr} < \infty.
\]
Note also that, in view of the assumption \( u_0 \in M^p_{\mathcal{L},r}(\mathbb{R}^d) \), we have
\[
\int_{\mathbb{R}^d} |V_g u_0(w)|^p \langle w \rangle^{-pr} < \infty.
\]
Now from Theorem 4.1, we have
\[
\int_{\mathbb{R}^d} \langle \cdot \rangle^r V_g(e^{itH}u_0)(z) = \int_{\mathbb{R}^d} k(t, w, z) V_g u_0(w) dw
\]
with
\[
|k(t, w, z)| \leq C(t) \langle z - \chi(t, w) \rangle^{-s}, \quad w, z \in \mathbb{R}^d.
\]
We have to show that \( z_0 \notin WF^r_G(e^{itH}u_0) \). To this end, take an open conic neighborhood \( \Gamma'_{z_0} \) of \( z_0 \), such that \( \Gamma'_{z_0} \subset \chi_t(\Gamma_{w_0}) \). This implies that for \( z \in \Gamma'_{z_0} \) and \( w \notin \Gamma_{w_0} \) we have
\[
\langle z - \chi(t, w) \rangle \gtrsim \max\{\langle z \rangle, \langle w \rangle\},
\]
since \( \chi_t \) is a Lipschitz diffeomorphism. Using (77) and (78) we estimate
\[
|\langle z \rangle^r V_g(e^{itH}u_0)(z)| \lesssim \int_{\mathbb{R}^d} I(z, w) dw,
\]
with
\[
I(z, w) = \langle z \rangle^r \langle z - \chi(t, w) \rangle^{-s} |V_g u_0(w)|.
\]
To show \( z_0 \notin WF^r_G(e^{itH}u_0) \) it will be sufficient to show that
\[
\left\| \int_{\mathbb{R}^d} I(\cdot, w) dw \right\|_{L^p(\mathbb{R}^d)} < \infty.
\]
First, we estimate \( \int_{\mathbb{R}^d} I(z, w) dw \) for \( z \in \Gamma'_{z_0} \). We split the domain of integration into two domains \( \Gamma_{w_0} \) and \( \mathbb{R}^d \setminus \Gamma_{w_0} \). In \( \mathbb{R}^d \setminus \Gamma_{w_0} \) we use (77) to obtain
\[
\int_{\mathbb{R}^d \setminus \Gamma_{w_0}} I(z, w) dw \leq \int_{\mathbb{R}^d \setminus \Gamma_{w_0}} \langle z \rangle^r \langle w \rangle^r \langle z - \chi(t, w) \rangle^{-s} |V_g u_0(w)| \langle w \rangle^{-r} dw
\]
\[\lesssim \int_{\mathbb{R}^d \setminus \Gamma_{w_0}} \langle z - \chi(t, w) \rangle^{2r-s} |V_g u_0(w)| \langle w \rangle^{-r} dw\]
\[\lesssim \left( \langle \cdot \rangle^{2r-s} \ast |V_g u_0(\cdot)| \right)(z)\].
So by (76) and using \( 2r - s < -2d \),
\[
\left\| \int_{\mathbb{R}^d \setminus \Gamma_{w_0}} I(\cdot, w) dw \right\|_{L^p(\Gamma'_{z_0})} \lesssim \left\| \langle \cdot \rangle^{2r-s} \right\|_{L^1(\mathbb{R}^d)} \left\| V_g u_0 \right\|_{L^p(\mathbb{R}^d)} < \infty.
\]
In the domain $\Gamma_{w_0}$, we have

$$\int_{\Gamma_{w_0}} I(z, w) \, dw \leq \int_{\Gamma_{w_0}} \langle z \rangle^r \langle w \rangle^{-r} \langle z - \chi(t, w) \rangle^{-r} \langle z - \chi(t, w) \rangle^{-s} |V_g u_0(w)| \langle w \rangle^r \, dw$$



$$\lesssim \int_{\Gamma_{w_0}} \langle z - \chi(t, w) \rangle^{-r} |V_g u_0(w)| \langle w \rangle^r \, dw$$



$$\lesssim \langle \chi^{-1}(\cdot) \rangle^{-s} * (\text{Char}_{\Gamma_{w_0}} \cdot |V_g u_0(\cdot)|^r) (z)$$

where $\text{Char}_{\Gamma_{w_0}}$ is the characteristic function of the set $\Gamma_{w_0}$. The assumption (75) yields to the estimate

$$\left\| \int_{\Gamma_{w_0}} I(\cdot, w) \, dw \right\|_{L^p(\Gamma_{w_0})} \lesssim \| \langle \chi^{-1}(\cdot) \rangle^{-s} \|_{L^1(\mathbb{R}^{2d})} \| |V_g u_0(\cdot)|^r \|_{L^p(\Gamma_{w_0})}$$

for $\chi_t$ is a bi-Lipschitz diffeomorphism and $r - s < 2r - s < -2d$. This concludes the proof.

The preceding arguments apply with small changes in the proof of (23). Let us detail the proof for sake of clarity.

**Proof of (23).** As in the previous proof, it is enough to show $WF_G(e^{itH}u_0) \subset \chi_t(WF_G(u_0))$ for any $t \in \mathbb{R}$. We have to prove that for every $u_0 \in S'(\mathbb{R}^d)$ and $z_0 \in \mathbb{R}^d$, $z_0 \neq 0$, the assumption $z_0 \notin \chi_t(WF_G(u_0))$ implies $z_0 \notin WF_G(e^{itH}u_0)$. Arguing as before, we have that the estimates (75) are satisfied for every $r > 0$ in a cone $\Gamma_{w_0}$ independent of $r$. Now recall from (27) that $S'(\mathbb{R}^d) = \bigcup_{s \geq 0} M^\infty_{1,s}(\mathbb{R}^d)$. Therefore $u_0 \in M^\infty_{1,s_0}(\mathbb{R}^d)$ for some $s_0 \geq 0$. Since $\sigma \in S^0_{0,0} = \bigcap_{s \geq 0} M^\infty_{1,s}(\mathbb{R}^{2d})$ by (20), we have $\sigma \in M^\infty_{1,s}(\mathbb{R}^{2d})$ for every $s \geq 0$. We may then apply the arguments in the preceding proof with $s > 2r + 2d > 2r_0 + 2d$ and obtain the expected estimates (22) for any $r > 0$. By observing that the choice of the cone $\Gamma_{z_0}$ does not depend on $r$, the proof is concluded.

**Proof of Proposition 5.1, (i).** We prove the independence of the definition of $WF^p_G(f)$ on the choice of the window $g$. The independence of $WF_G(f)$ is attained similarly.

We assume the estimate for $V_g f$ (17) satisfied, for some fixed $g \in S(\mathbb{R}^d) \setminus \{0\}$ and some conic neighborhood $\Gamma_{z_0}$ and we want to prove that the estimate holds for $V_h f$, where $h \in S(\mathbb{R}^d) \setminus \{0\}$ is fixed arbitrary, after possibly shrinking $\Gamma_{z_0}$. To this end, we use Lemma 2.3 which gives

$$|V_h f(z)| \lesssim (|V_g f| * |V_h g|)(z).$$
Since $V_hg \in \mathcal{S}(\mathbb{R}^d)$ for $g, h \in \mathcal{S}(\mathbb{R}^d)$, we have that for every $s \geq 0$

$$|V_h f(z)| \lesssim \int_{\mathbb{R}^d} \langle z - w \rangle^{-s} |V_g f|(w) \, dw.$$ 

We know that $f \in M^p_{\nu-\sigma_0}(\mathbb{R}^d)$ for some $\sigma_0 > 0$. Taking then $s > \max\{r, \sigma_0 + 2\}$, the arguments in the proof of Theorem 5.2 apply with $\chi(t, \cdot) = \text{Id}$, $w_0 = z_0$.

**Proposition 5.3.** Let $\sigma \in M_{1_{\mathbb{R}^d}}(\mathbb{R}^d)$, $s > 2d$ and $0 < 2r < s - 2d$. Then for every $f \in M^p_{\nu-\sigma}(\mathbb{R}^d)$ we have

$$(82) \quad WF_{p,r}^G(\sigma(x, D)f) \subset WF_{p,r}^G(f).$$

If $\sigma \in S^{0}_{0,0}$, then for every $f \in \mathcal{S}'(\mathbb{R}^d)$,

$$(83) \quad WF_G(\sigma(x, D)f) \subset WF_G(f).$$

**Proof.** If $\sigma \in M_{1_{\mathbb{R}^d}}(\mathbb{R}^d)$, then from Proposition 2.14 we have that the Gabor matrix $k(w, z)$ of $\sigma(x, D)$ satisfies

$$|k(w, z)| \lesssim \langle z - w \rangle^{-s}, \quad w, z \in \mathbb{R}^d,$$

so that $\sigma(x, D) \in FIO(\chi, s)$ with $\chi = \text{Id}$. The arguments of the proof of the Theorem 5.2 then apply with $w_0 = z_0$. The proof of (83) is similar.

We end the paper with some examples of Schrödinger equations.

Addressing first to non-expert readers, we present some properties of $WF_G(f)$ and treat in this frame the free particle and the harmonic oscillator with smooth potentials, cf. Examples 1, 2, 3. The conclusive Example 4 concerns non-smooth potentials.

**Proposition 5.4.** Let $f \in \mathcal{S}'(\mathbb{R}^d)$. Then

(i) $WF_G(\pi(z_0)f) = WF_G(f)$ for every $z_0 = (x_0, \xi_0) \in \mathbb{R}^d$.

(ii) Let $\delta_{x_0}$ be the Dirac distribution at the point $x_0 \in \mathbb{R}^d$. Then

$$WF_G(\delta_{x_0}) = \{z = (x, \xi), x \neq 0, \xi \neq 0\}$$

independently of $x_0$.

(iii) Let $\xi_0$ be fixed in $\mathbb{R}^d$. Then

$$WF_G(e^{2\pi i (x, \xi_0)}) = \{z = (x, \xi), x \neq 0, \xi = 0\}$$

independently of $\xi_0$.

(iv) Let $c \in \mathbb{R}, c \neq 0$, be fixed. Then

$$WF_G(e^{\pi i c|x|^2}) = \{z = (x, \xi), x \neq 0, \xi = cx\}.$$
Proof. The proof of (i) is a consequence of Proposition \[5.3\] since \(\pi(z_0) = M_{\xi_0} T_{x_0} = \sigma(x, D)\) with \(\sigma(x, D)\) being a pseudodifferential operator with symbol
\[
\sigma(x, \xi) = e^{2\pi i (\langle x, \xi \rangle - \langle x_0, \xi \rangle)} \in S^0_{0,0}.
\]
Concerning (ii), we are reduced to compute \(WF_G(\delta)\) since
\[
WF_G(\delta_{x_0}) = WF_G(T_{x_0} \delta) = WF_G(\delta)
\]
by item (i). On the other hand, \(V_g(\delta)(x, \xi) = \sqrt{g(-x)}\). Hence in a small conic neighborhood \(\Gamma \subset \mathbb{R}^{2d}\) of the ray \(x = t\xi, \ t \in \mathbb{R}, \ \xi \neq 0\), we have rapid decay of \(g(-t\xi)\) but for \(t = 0\), giving the claim.

To prove (iii) we proceed similarly as before. From item (i) we obtain that
\[
WF_G(e^{2\pi i (x, \xi_0)}) = WF_G(M_{\xi_0} 1) = WF_G(1).
\]
On the other hand \(|V_g 1(x, \xi)| = |M_{-x} \hat{g}(-\xi)|\) so that \(|V_g 1(x, \xi)| = |\hat{g}(-\xi)|\) and the arguments of item (ii) give the desired result.

We now prove (iv). We use the Gaussian \(g(x) = e^{-\pi|x|^2}\) as a window for the STFT \(V_g f\) with \(f(x) := e^{\pi i |x|^2}\). Then standard computations (see also [2, Theorem 14]) give
\[
|V_g f(x, \xi)| = (1 + e^{2d/4} e^{-\pi |k - cx|^2/(1+c^2)}).
\]
The right-hand side is rapidly decaying in any open cone of \(\mathbb{R}^{2d}\) excluding the line \(\xi - cx = 0\). This concludes the proof of the proposition.

Example 1. The free particle.

Consider the Cauchy problem for the Schrödinger equation
\[
\begin{cases}
i \partial_t u + \Delta u = 0 \\
u(0, x) = u_0(x),
\end{cases}
\]
with \(x \in \mathbb{R}^d, \ d \geq 1\). The explicit formula for the solution in terms of the kernel is
\[
u(t, x) = (K_t * u_0)(x),
\]
where
\[
K_t(x) = \frac{1}{(4\pi i t)^{d/2}} e^{i|x|^2/(4t)}.
\]
whereas in terms of classical FIO:
\[
u(t, x) = \int_{\mathbb{R}^d} e^{2\pi i (\langle x, \eta \rangle - 2t|\eta|^2)} \hat{u}_0(\eta) d\eta.
\]
The Gabor matrix with window function \(g(x) = e^{-\pi |x|^2}\) can be controlled (see [15, Theorem 5.3] even for more general operators):
\[
|k(w, z)| \leq C e^{-c|z - \chi_t(w)|^2},
\]
for suitable constants $C > 0$ and $\epsilon > 0$ and where, for $w = (y, \eta)$,

\[(x, \xi) = \chi_t(y, \eta) = (y + 4\pi t \eta, \eta).\]

Beside the effectiveness in numerical analysis, cf. [12, Section 6.1], this expression emphasizes the microlocal properties of the propagator. Let us test the propagator of the Gabor wave front set on some particular initial data. If $u_0 = \delta$ then $u(t, x) = K_t(x)$ by (85). This is coherent with (23) and (89), since from Proposition 5.4, (iv) and (ii), we have

\[WF_G(u(t, x)) = WF_G(K_t) = \left\{(x, \xi), x = 4\pi t \xi, \xi \neq 0\right\}\]

We remark a similar propagation for the initial datum $u_0 = K_{-1}(t) = (-4\pi i)^{-d/2} e^{-|x|^2/4}$ for which we have $u_{t=1} = \delta$. Instead, for $u_0 = e^{2\pi i(x, \xi_0)}$, with $\xi_0 \in \mathbb{R}^d$, we have

\[u(t, x) = e^{-4\pi^2 t |\xi_0|^2} e^{2\pi i(x, \xi_0)}\]

and in this case the Gabor wave front set is stuck:

\[WF_G(u(t, x)) = WF_G(u_0) = \{(x, 0), x \neq 0\},\]

by Proposition 5.4 (iii) and (89).

**Example 2. The harmonic oscillator.**

Consider the Cauchy problem

\[
\begin{align*}
\begin{cases}
i\partial_t u - \frac{1}{4\pi} \Delta u + \pi |x|^2 u &= 0 \\
u(0, x) &= u_0(x).
\end{cases}
\end{align*}
\]

The solution in terms of a FIO type (15) is

\[
u(t, x) = (\cos t)^{-d/2} \int_{\mathbb{R}^d} e^{2\pi i [\frac{1}{2} (x, \eta) + \frac{1}{2} (x^2 + \eta^2)]} \hat{u}_0(\eta) \, d\eta, \quad t \neq \frac{\pi}{2} + k\pi, \ k \in \mathbb{Z}.
\]

The Gabor matrix with Gaussian window $g(x) = e^{-\pi |x|^2}$ can be explicitly computed as

\[
|k(w, z)| = 2^{-\frac{d}{2}} e^{-\frac{\pi}{2} |z - \chi_t(w)|^2},
\]

where the canonical transformation is defined in (21). Observe that the expression (92) is meaningful for every $t \in \mathbb{R}$. Let us address to [12, Section 6.2] for applications to numerical experiments.

We may test (23) on the initial datum $u_0(x) = 1$, giving for $t < \pi/2$,

\[u(t, x) = (\cos t)^{-d/2} e^{\pi i \tan t |x|^2}.
\]
From Proposition 5.4 (iii) and (iv), we have coherently with (21)
\[ WF_G(u(t, x)) = \{(x, \xi), x = (\cos t)y, \xi = (\sin t)y, y \neq 0\} \]
\[ = \chi_t(WF_G(1)) = \chi_t(\{(y, \eta), y \neq 0, \eta = 0\}). \]

**Example 3. Smooth potentials.**

We now consider the presence in Example 1 of a potential with symbol in the class \(S^0_{0,0}\). Consider the case
\[ e^{-2\pi i(x_0, \xi)}, \quad x_0 \in \mathbb{R}^d \text{ fixed.} \]
The related pseudodifferential operator \(\sigma(D)\) is the translation operator
\[ \sigma(D)f(x) = T_{x_0}f(x) = f(x - x_0), \]
which does not preserve the singular support. Consider first the equation
\[ \begin{cases} 
  i\partial_t u + \sigma(D)u = 0 \\
  u(0, x) = u_0(x). 
\end{cases} \]
The solution is given by
\[ u(t, x) = e^{itT_{x_0}}u_0(x) = \int_{\mathbb{R}^d} e^{2\pi i(x, \xi)} \exp(i\xi e^{-2\pi i(x_0, \xi)}) \hat{u}_0(\xi) \, d\xi. \]
Despite the nasty oscillations, the symbol of the solution operator belongs to \(S^0_{0,0}\) and from Proposition 25.4 we have for every fixed \(t \in \mathbb{R}\),
\[ WF_G(e^{itT_{x_0}}u_0) = WF_G(u_0), \]
the identity being granted by the fact that \(T_{x_0}^{-1} = T_{-x_0}\). Note that the singular support can be expanded. In fact, taking \(u_0 = \delta\) we have
\[ e^{itT_{x_0}}\delta = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \delta_{nx_0} \in \mathcal{S}'(\mathbb{R}^d) \]
so that \(\text{sing supp } e^{itT_{x_0}}\delta = \{nx_0\}_{n \in \mathbb{Z}_+}\) as soon as \(t \neq 0\), whereas
\[ WF_G(e^{itT_{x_0}}\delta) = WF_G(\delta) = \{(0, \xi), \xi \neq 0\}. \]
Adding now the potential \(\sigma(D)\) to the free particle in Example 1, we have the Schrödinger equation with space-delay
\[ \begin{cases} 
  i\partial_t u + \Delta u + T_{x_0}u = 0 \\
  u(0, x) = u_0(x). 
\end{cases} \]
Since the operators $e^{it\Delta}$ and $T_{x_0}$ commute, the arguments of of Section 4 provide as propagator $e^{itT_{x_0}}e^{it\Delta}$, that is the convolution with

$$
\sum_{n=0}^{\infty} \frac{(it)^n}{n!} K_t(x - nx_0) \in S'(\mathbb{R}^d),
$$

where $K_t$ is defined in (86). The Gabor propagation is the same as in Example 1. From a physical point of view, it is perhaps most natural to consider the case when the potential depends on $x$ alone, for example

$$
\left\{
\begin{array}{l}
i\partial_t u + \Delta u + M_{\xi_0}u = 0 \\
u(0, x) = u_0(x),
\end{array}
\right.
$$

with $M_{\xi_0}u_0 = e^{2\pi i (x, \xi_0)}u_0$, $\xi_0$ fixed in $\mathbb{R}^d$. Notice that now the operators $e^{it\Delta}$ and $M_{\xi_0}$ do not commute and, proceeding as in Section 4 with the perturbation $Bu = M_{\xi_0}u$, we have first to consider

$$
B(t) = e^{-it\Delta}e^{2\pi i (x, \xi_0)}e^{it\Delta}.
$$

Omitting further explicit computations, we obtain

$$
B(t) = e^{4\pi^2 i t^\xi_0^d}M_{\xi_0}T_{-4\pi t\xi_0}.
$$

In principle, one could then continue the computation of the pseudodifferential operator $Q(t)$ in (72) explicitly, and the solution operator will be $e^{it\Delta}Q(t)$.

Observe in (99) the presence of the translation factor $T_{4\pi t\xi_0}$, providing same phenomena as before.

**Example 4. Non-smooth potentials.**

As examples of admissible non-smooth potentials, consider first a non-polynomial homogeneous function $h(z)$, $z = (x, \xi)$, $h(\lambda z) = \lambda^r h(z)$ for $z \neq 0$, $\lambda > 0$, $r > 0$, with $h \in C^\infty(\mathbb{R}^{2d} \setminus \{0\})$, and take then as potential any function $\sigma(z) = h(z)$, for $|z| \leq 1$, and $h(z) \in S_{0,0}^0$ for $|z| \geq 1$. This potential satisfies $\sigma \in M^\infty_{1\otimes n_{r+2d}}(\mathbb{R}^{2d})$. In fact, we may limit the analysis to the singularity at the origin. From Proposition 2.5 we have, for $\psi \in S(\mathbb{R}^{2d})$,

$$
|V_\psi \sigma(z, \zeta)| \leq C\langle \zeta \rangle^{-r-2d}, \quad z, \zeta \in \mathbb{R}^{2d}.
$$

We may now return to the discussion about the smoothness at the origin of the Hamiltonian $a(z)$ in the Introduction. Consider $h(z)$ real-valued non-polynomial homogeneous of degree 2, $h \in C^\infty(\mathbb{R}^{2d} \setminus \{0\})$, just to give an example

$$
h(x, \xi) = (|x|^4 + |\xi|^4)^{1/2}.
$$
We can include in our analysis the equation

\begin{equation}
\begin{aligned}
    i\partial_t u + h(x, D)u &= 0 \\
    u(0, x) &= u_0(x),
\end{aligned}
\end{equation}

by absorbing the singularity at the origin into the potential. Namely, take \( \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^d) \), \( 0 \leq \varphi(z) \leq 1 \), \( \varphi(z) = 1 \) for \( |z| \leq 1 \), \( \varphi(z) = 0 \) for \( |z| \leq 2 \), and split

\[ h(z) = a(z) + \sigma(z), \quad a(z) = (1 - \varphi(z))h(z), \quad \sigma(z) = \varphi(z)h(z). \]

At this moment \( a(z) \) satisfies the assumptions in the Introduction and the potential \( \sigma \) belongs to \( M_{i\mathbb{S}^{n+1}_\omega}(\mathbb{R}^2d) \), in view of \( \text{(100)} \). We may then apply Theorem \( \text{(1.4)} \) to the Cauchy problem \( \text{(100)} \). Note that the result of propagation should be limited to \( u_0 \in M^p_{\nu_\omega}(\mathbb{R}^d) \) and \( WF_G^{p,r}(e^{itH}u_0) \) with \( 0 < r < 1 \).

Finally, we present an example of non-smooth potential depending on \( x \) alone, namely in dimension \( d = 1 \)

\begin{equation}
    \sigma(x, \xi) = |\sin x|^\mu, \quad \mu > 1, \quad x, \xi \in \mathbb{R}.
\end{equation}

By Corollary \( \text{2.3} \), \( \sigma \in M_{i\mathbb{S}^{n+1}_\omega}(\mathbb{R}^2) \). So, consider for instance the perturbed harmonic oscillator in \( \text{(20)} \). From Theorem \( \text{1.4} \) we have that the Cauchy problem is well-posed for \( u_0 \in M^p_{\nu_\omega}(\mathbb{R}) \), \( |r| < \mu - 2 \) and the propagation of \( WF_G^{p,r}(u(t, \cdot)) \) for \( t \in \mathbb{R} \) takes place as in Example \( 2 \) for \( 0 < r < \mu/2 - 1 \).

**Appendix**

**Proof of Theorem \( \text{2.15} \)** First we prove \( (ii) \Rightarrow (i) \). Assume \( I \in FIO(\chi(t, \cdot), s) \) and

\begin{equation}
    |(I\pi(w)g, \pi(z))g| \leq C(t)\langle z - \chi(t, w) \rangle,
\end{equation}

with \( C(t) \) positive continuous function on \( [-T, -T[ \). Setting \( w = (x, \eta) \) and \( z = (x', \eta') \), using the fact that each component of the mapping \( \chi(t, y, \eta) \) and its inverse is in \( C^\infty([-T, T[; \Gamma^1(\mathbb{R}^{2d})) \) we can control the Lipschitz constants of \( \chi(t, \cdot) \) and \( \chi^{-1}(t, \cdot) \) by continuous constants of \( t \) so that the equivalence of \( \text{[7, Lemma 4.2]} \) becomes

\begin{equation}
    |\nabla_x \Phi(t, x', \eta') - \eta'| + |\nabla_\eta \Phi(x', \eta) - x'| \asymp t |\chi_1(t, x, \eta) - x'| + |\chi_2(t, x, \eta) - \eta'| \quad \text{for every } x, x', \eta, \eta' \in \mathbb{R}^d
\end{equation}

and the implicit constants in the equivalence \( \asymp t \) are continuous with respect to \( t \in [-T, T[ \). This reduces the study to showing that if the operator \( I(\sigma_t, \Phi(t, \cdot)) \), with \( \Phi(t, \cdot) \) being the phase related to \( \chi(t, \cdot) \) in \( \text{(11)} \) and satisfying \( \text{(38)} \), fulfils the estimate

\begin{equation}
    |I(\sigma_t, \Phi(t, \cdot))\pi(x, \eta)g, \pi(x', \eta')g| \leq C(t)|\nabla_x \Phi(t, x', \eta) - \eta'|\nabla_\eta \Phi(t, x', \eta) - x'|^{-s}
\end{equation}

for every \( x, \eta, \eta' \in \mathbb{R}^d \) and the implicit constants in the equivalence \( \asymp t \) are continuous with respect to \( t \in [-T, T[ \), then

\begin{equation}
    ||\sigma_t||_{M_{i\mathbb{S}^{n+1}_\omega}^\infty} \leq C(t), \quad t \in [-T, T[.
\end{equation}
For $z, w \in \mathbb{R}^d$, let $\Phi_{z,w}(t, \cdot)$ be the remainder in the second order Taylor expansion of the phase $\Phi(t, \cdot)$, i.e.,

$$\Phi_{z,w}(t, w) = 2 \sum_{|\alpha| = 2} \int_0^1 (1 - \tau) \partial^\alpha \Phi(t, z + \tau w) d\tau \frac{w^\alpha}{\alpha!}.$$  

For a given window $g \in \mathcal{S}(\mathbb{R}^d)$, we set

$$\Psi_g(t, w) = e^{2\pi i \Phi_{z,w}(t, w)} (\mathcal{F} \otimes \hat{g})(w).$$

Then, the fundamental relation between the Gabor matrix of a FIO and the STFT of its symbol from [10, Prop. 3.2] and [11, Section 6] can be rephrased in this framework as

$$|\langle I \pi(x, \eta)g, \pi(x', \eta')g \rangle| = |V_{\Psi_g}(x', \eta), (\eta' - \nabla_x \Phi(t, x', \eta), x - \nabla_\eta \Phi(t, x', \eta))|.$$  

Writing $u = (x', \eta), v = (\eta', x)$, (105) translates into

$$|V_{\Psi_g}(x, \eta)\sigma_t(u, v - \nabla \Phi(t, u))| \leq C(t) \langle v - \nabla \Phi(t, u) \rangle^{-s},$$

and then into the estimate

$$\sup_{(u, v) \in \mathbb{R}^d \times \mathbb{R}^d} \langle w \rangle^s \langle V_{\Psi_g}(x, \eta)\sigma_t(u, v) \rangle \leq C(t).$$

The main technical work done in [6] for the time independent case $\Psi_u(t, \cdot) = \Psi_u(\cdot)$ is to show that the set of windows $\Psi_u$ possesses a joint time-frequency envelope. This property allows to write $\sigma \in M^\infty \otimes_{\sigma_u}(\mathbb{R}^d)$ if and only if $\sup_{u \in \mathbb{R}^d} |V_{\Psi_u} \sigma| \in L^\infty \otimes_{\sigma_u}(\mathbb{R}^d)$ with

$$\|\sigma\|_{M^\infty \otimes_{\sigma_u}} \simeq \sup_{u \in \mathbb{R}^d} |V_{\Psi_u} \sigma|_{L^\infty \otimes_{\sigma_u}}.$$  

The proof of the previous equivalence passes through several lemmas. We point out that the crucial element of the equivalence is a control of $|V_{\Psi} e^{2\pi i \Phi_{z,w}(u, w)}|$, with $\Psi \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$ fixed, by a polynomial $p_\alpha(\partial \Phi_{z,w}(\zeta))$ of derivatives of $\Phi_{z,w}$ of degree at most $|\alpha|$ times a factor that does not depend on $t$. Since $\Phi(t, \cdot) \in \mathcal{C}(\mathbb{R}^d)$, we can control the polynomial by a continuous function of $t$ and in the end obtaining that the equivalence (108) depends continuously on $t$, which together with (108) gives (109).

(i) $\Rightarrow$ (ii). If $I = I(\sigma_t, \Phi_\chi)$ is a FIO of type I for $\Phi(t, \cdot)$ and $\chi(t, \cdot)$ in (10) and $\sigma_t \in M^\infty \otimes_{\sigma_u}(\mathbb{R}^d)$ which satisfies (109), then essentially reading backwards the arguments above give $I(\sigma_t, \Phi_\chi) \in FIO(\chi(t, \cdot), s)$ with $C(t)$ being a continuous function of $t$.  

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