On spinless null propagation in five dimensional space-times with approximate
space-like Killing symmetry

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Abstract
In previous work we discussed the problem of how a 4D observer perceives particle propagation in a 5D
space-time, and postulated that 4D perception is greatly facilitated if the 5D space-time is symmetric.
In particular, if the 5D geometry is independent of the fifth coordinate then the 5D physics can be
interpreted as 4D quantum mechanics. In this work we address the case where symmetry is approximate,
 focusing on the case where the 5D geometry depends weakly on the fifth coordinate. We show that
 concepts developed for the case of exact symmetry approximately hold when other concepts such as
decaying quantum states, resonant quantum scattering and friction are adopted as well. We briefly
comment on the optical model of the nuclear interactions and Millikan’s oil drop experiment.

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1 Introduction
In previous work [1], we discussed the problem of how a 4D observer perceives a 5D space-time. We
postulated that 4D perception of the 5D space-time is greatly facilitated by symmetry. In particular, if the
5D geometry is independent of the fifth coordinate then the 5D physics can be interpreted as a 4D quantum
mechanics, while if the 5D geometry is independent of time then the 5D geometry can be interpreted as
a 4D statistical mechanics. In this work we briefly address the case where symmetry is approximate. We
postulate how concepts developed for the case where the symmetry is exact are used for the case where the
symmetry is approximate. Of course, the concepts will work in a small region where space-time is essentially
flat. However, they can be extended over larger regions if other concepts such as decaying quantum states,
resonant quantum scattering and Stokes drag are adopted, as well. In particular, we focus on the case where
the 5D geometry depends weakly on time leads to the statistical physics of time-dependent systems near thermodynamical equilibrium. We defer this to further work.

Other works on field equations (e.g., see Refs. [2–4]) and geodesic motion [2] are also based on the idea that
the existence of extra dimensions brings new physics to ordinary 4D space-time. Particularly, a discussion
of geodesic motion and classical tests of 4D general relativity adapted to the 5D space-time geometry are
summarized in Ref. [2]. We follow previous work where 5D space-time is assumed to have a space-like fifth
dimension which is neither compact nor Planckian [2,3].

2 Five-dimensional space-time
We consider a 5D space-time with the metric \( h_{AB} \) where \( A, B, ... = 0, 1, 2, 3, 5 \) and postulate that all quan-
tum propagation takes place on 5D null paths. A complete description of particle propagation in the 5D
space-time requires measurements of additional entities than 4D space-time events. We discuss a special
class of 5D space-times that can be foliated into conformally flat 4D space-times (i.e., the 4D conformal
factor is the inverse square lapse of the foliation) with superimposed electromagnetic fields, fit for describing
many experimental setups. We think of these space-times as local approximations of more realistic geometric
constructs including non-trivial gravitational fields, whose metrics satisfy suitable field equations [1,2].
Removing the conformal factor from the metric since it is irrelevant for null-path counting, we have

\[
\tilde{h}_{AB} = \left( \eta_{\mu\nu} + \frac{q^2}{c^2} A_\mu A_\nu \right), \quad \tilde{h}^{AB} = \left( \eta^{\mu\nu} - \frac{q^2}{c^2} A^\mu A^\nu \right),
\]

where the \( \mu, \nu, ... \) indices go over 0, 1, 2, 3 and are raised with \( \eta^{\mu\nu} \).

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3 Path integrals

Consider any two causally ordered events 1 and 2, with 1 in the past of 2, which we write as \( 1 \prec 2 \). Denote the coordinates of 1 and 2 by \( x^A_{(1)} \) and \( x^A_{(2)} \), respectively. Then, the sum over 5D null paths between 1 and 2, denoted here by \( R(x^A_{(1)}, x^A_{(2)}) \), is positively defined, conformally invariant, and has the status of a microcanonical sum, determining the particle propagation between 1 and 2 \([3]\). Null path integrals satisfy a selfconsistency relation virtue to their geometric interpretation

\[
R(x^A_{(1)}, x^A_{(2)}) = \int_{1 \prec 2} \sqrt{|h|} d^5x \mathcal{R}(x^A_{(1)}, x^A_{(3)}) R(x^A_{(3)}, x^A_{(2)}). \tag{2}
\]

The 5D infinitesimal null path element \( ds_5 = \sqrt{|h| AB} dx^A dx^B = 0 \) can be rewritten as

\[
d s_4 \equiv dx^5 = \pm \sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu} - \frac{q}{c^2} A_\rho dx^\rho. \tag{3}
\]

If \( A_\rho \) is independent of \( x^5 \), then we obtain a traditional quantum mechanical picture \([4]\). \( ds_4 \) can be regarded as a 4D metric in a 4D non-Lorentzian curved manifold, where the distance is \( ds_4 \) if \( 1 \prec 2 \), and \(-ds_4 \) if \( 2 \prec 1 \). Computing the path integral \( R(x^A_{(1)}, x^A_{(2)}) \) in the 5D Lorentzian manifold with the metric \( h_{AB} \) is equivalent to computing the following integral

\[
\Psi_\pm(\lambda^{-1}; x^\mu_{(1)}, x^\mu_{(2)}) = \int_{1 \prec 2} [d^4x] e^{i\lambda^{-1} s_4} \tag{4}
\]

of paths between \( x^\mu_{(1)} \) and \( x^\mu_{(2)} \), the 4D projections of \( x^A_{(1)} \) and \( x^A_{(2)} \), respectively. Hence, computing the path integral \( R(x^A_{(2)}, x^A_{(1)}) \) is equivalent to computing \( \Psi_\pm(\lambda^{-1}; x^\mu_{(1)}, x^\mu_{(2)}) \). \( \Psi_\pm \) does not result as a scalar field on the 4D manifold since a transformation of coordinates that reverses causality implies a complex conjugation of \( \Psi_\pm \).

If \( A_\rho \) is not independent of \( x^5 \), then the traditional quantum mechanics picture no longer holds. The mass of the particle is no longer a constant of motion. We discuss a possible interpretation of the case where we have weak electromagnetic fields of the form \( A_\rho + x^5 A'_\rho \) with \( \partial_5 A_\rho = 0 \), and \( \partial_5 A'_\rho \neq 0 \). We obtain

\[
dx^5 = \pm \sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu} - \frac{q}{c^2} A_\rho dx^\rho - \frac{q}{c^2} x^5 A'_\rho dx^\rho. \tag{5}
\]

Hence, it is not possible to naturally define a 4D path element. In fact, by separating the fifth coordinate and integrating, we obtain a non-local action

\[
s_4 \equiv x^5_{(2)} - x^5_{(1)} \equiv \int_{1 \prec 2} \left( \pm \sqrt{-\eta_{\mu\nu} dx^\mu dx^\nu} - \frac{q}{c^2} A_\rho dx^\rho \right) \exp \left( \int_{3 \prec 2} dx^\rho \frac{q}{c^2} A'_\rho \right), \tag{6}
\]

whose path integral

\[
\Psi'_\pm(\lambda^{-1}; x^\mu_{(1)}, x^\mu_{(2)}) = \int_{1 \prec 2} [d^4x] e^{i\lambda^{-1} s_4} \tag{7}
\]

satisfies the following differential equation obtained according to Feynman’s procedure \([5]\)

\[
\pm \frac{\hbar}{i} \frac{\partial \Psi'_\pm}{\partial t} = \frac{1}{2m} \left[ \frac{\hbar}{i} \nabla - \frac{mq}{c^2} \frac{\gamma A}{\gamma} \right]^2 \Psi'_\pm \mp mq A_0 \Psi'_\pm + mc^2 \Psi'_\pm
\]

\[
+ \frac{\hbar}{i} \left( -\frac{c}{2mc^2} \gamma A'_\rho \pm \frac{q}{2mc^2} \gamma \frac{\gamma A}{\gamma} \right) \nabla \Psi'_\pm. \tag{8}
\]

\( \Psi'_- \) describes the forward propagation of a particle of mass \( m \), while \( \Psi'_+ \) describes the forward propagation of an antiparticle of mass \( m \). Since the electromagnetic field depends on \( x^5 \), mass is, in fact, no longer a
constant of motion. However, the propagation is considered for a particle of mass $m$, where terms in $A'_\rho$ account for the change in the dynamics due to the change in mass.

The term in round brackets is non-hermitian and problematic for the probabilistic interpretation of quantum mechanics. Still, non-hermitian hamiltonians are used to describe key processes such as decaying quantum states [6] and resonant quantum scattering [7]. In particular, we emphasize the success of the phenomenological model of nuclear interactions in the range of 10-100 MeV known as the optical model [8].

Summation over paths in space-times with complex coordinates, involving actions with complex potential, proved successful in further developing some of these theories [9]. Later, it was shown that paths in manifolds with real coordinates suffice [10]. Our approach uses both real interaction potentials and paths in manifolds with real coordinates.

4 Geodesic propagation

We discuss the classical limit where $\lambda \to 0$ and the propagation takes place on a single path where the action achieves local extremum and which contributes most to the path integral. We denote by $\tau$ the parameter of a path and introduce the lagrangian at event 3 for the non-local action $s'_4 \pm (1 \prec 2)$, with $1 \prec 3 \prec 2$,

$$L \equiv L + L' \int_{1 \prec 3} d\tau L \exp \left( \int_{4 \prec 3} L'd\sigma \right),$$

where

$$L = \pm \sqrt{-\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu} - \frac{q}{c^2} A_\rho \dot{x}^\rho, \quad L' = - \frac{q}{c^2} A'_\rho \dot{x}^\rho,$$

and $\dot{x}^\mu \equiv dx^\mu/d\tau$. Assuming that on the classical path

$$\left| \int_{1 \prec 2} L'd\sigma \right| \ll 1,$$

we have

$$L \approx L + L' \int_{1 \prec 3} d\tau L,$$

and the equations Euler-Lagrange are

$$\left[ \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{x}^\mu} \right) - \frac{\partial L}{\partial x^\mu} \right] + \int_{\tau_1}^\tau L'd\sigma + \frac{dL'}{d\tau} \int_{\tau_1}^\tau \frac{\partial L}{\partial x^\mu} d\sigma - \frac{\partial L'}{\partial x^\mu} \int_{\tau_1}^\tau L'd\sigma - L' \int_{\tau_1}^\tau \frac{\partial L'}{\partial \dot{x}^\mu} d\sigma + \frac{\partial L'}{\partial \dot{x}^\mu} L + L' \frac{\partial L}{\partial \dot{x}^\mu} \right] = 0.$$ (13)

4.1 A simple non-relativistic case

To illustrate the 4D interpretation of 5D geodesic motion, we consider Eq. (13) under the following conditions. First, we assume a simple form of the electromagnetic potential

$$A_\rho + x^5 A'_\rho = (-zE, 0, 0, 0) + x^5 (A'_0, 0, 0, 0),$$

An alternate way to arrive at Eq. (12) is as follows. Equation (5), providing the infinitesimal action $dx^5$ as a function of $x^5$, may be read as an equation of type Dyson. Hence, for the first order in $L'$, we choose $x^5_{(1)} = 0$ and use $ds'_4 \pm (1 \prec 3)$ for a nested approximation of $ds'_4 \pm (1 \prec 2)$.  

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where \( E \) and \( A'_0 \) are constants. Second, the path parameter \( \tau \) represents the proper time of the time-like geodesics in the 4D manifold with the metric \( \eta_{\mu\nu} \); i.e., \( d\tau^2 = -\eta_{\mu\nu}dx^\mu dx^\nu \). Third, we take the non-relativistic limit where \( \tau \to ct \) and \( |dx^\beta dx_j/c^2| \ll 1 \). Fourth, we multiply Eq. (13) by \( mc \). \(^3\) Hence, we obtain

\[
m\ddot{z} - \frac{mq}{c} A'_0 \dot{z} \pm (mq)E \left[ 1 - \frac{q}{c} A'_0 (t_2 - t_1) \right] = 0,
\]

where we further neglect the term \( |(q/c)A'_0 (t_2 - t_1)| = | \int_{t_1}^{t_2} L' dt | \ll 1 \) \([c.f., Eq. (11)]\), meaning that the equation of motion holds for short time intervals \( (t_2 - t_1) \ll |c/(q A'_0)| \).

In the case where \( (mq)A'_0/c < 0 \), Eq. (15) may stand for the equation of motion of a particle in constant electric field, subject to drag of type Stokes. This corresponds to the 5D picture whereby a particle with momentum \( p^A = (E/c, p, mc) \) propagates with increasing mass, until eventually achieving the momentum \( \hat{p}^A = (E/c, 0, \hat{m}c) \), at rest.

The 5D setup with the electromagnetic potential given by Eq. (14) has yet another interpretation \([1\text{]}\). Since \( A_p + x^5 A'_p \) is time independent, the particle propagation in this setup may be described by the Langevin equation

\[
m\ddot{z} + \zeta \dot{z} \pm (mq)E + \eta(t) = 0,
\]

where \( \zeta \) is the drag coefficient and \( \eta(t) \) is a stochastic force with \( \langle \eta(t)\eta(t') \rangle = 2\zeta k_B T \delta(t - t') \). In the limit where thermal fluctuations go to zero (i.e., \( T \to 0 \)), the stochastic force vanishes. Hence, by direct comparison between Eqs. (15) and (16), we identify \(- (mq) A'_0/c \) with \( \zeta \). Furthermore, we rewrite the condition \( |(q/c)A'_0 (t_2 - t_1)| = | \int_{t_1}^{t_2} L' dt | \ll 1 \) as \( 2(t_2 - t_1)/u \ll 1 \), where \( u \equiv 2m/\zeta \) is a quantum of physical time \([1\text{]}\), defined for the case where \( E = 0 \).

Particle motion described by equations (15) and (16) was first studied experimentally by Brown \([11\text{]}\) and Millikan \([12,13\text{]}\). In particular, Millikan studied the motion on charged oil drops in constant electric field. By tuning \( E \), while keeping everything else unchanged, one could in principle measure \( mq \). However, to estimate \( mq \), Millikan observed oil drops whose charge suddenly changed during the experiment, due to ionization. Hence, according to our theory, this corresponds to several single particle trajectories. Explaining the experimental results obtained by Millikan requires additional physical principles than those described here.

Kaluza’s ansatz to set up \( \partial_5 \) to zero lead to a unified formulation of field equation for the gravitational and electromagnetic fields \([14\text{]}\). However, even in the case where the metric is \( x^5 \)-dependent, one may absorb the terms which contain \( \partial_5 \) into sources of the electromagnetic field. Hence, formally, a split of the 5D field equations between equations for the gravitational and the electromagnetic fields is possible even in the absence of the space-like symmetry. Therefore, one may conclude that experimental access to the electric field \( E \) is possible even though the 5D space-time is not symmetric.

### 5 Conclusion

In previous work, we discussed an interpretation of a symmetric 5D space-time and introduced physical concepts such as mass and temperature to 4D geometry. In the case where the 5D space-time is approximatively symmetric, the these concepts hold either approximatively or in a small region of space-time. The description of the 5D geometry may be improved by additional concepts such as decaying states, resonant quantum scattering and Stokes drag. These concepts correct the 4D picture and extend the description of particle propagation for longer durations of proper time. Furthermore, we link newly obtained equations of motion with previously studied phenomenology.

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\(^3\)Because the translational symmetry along \( x^5 \) holds only approximately, \( \lambda \) changes slowly with proper time. Still, we consider \( \lambda \) as an approximate constant of motion. However, as a precaution for the physical interpretation, we use \( \lambda^{-1} s'_4 \pm \) rather than \( s'_4 \pm \) as physical action.
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