EXCHANGE AND EXCLUSION IN THE NON-ABELIAN ANYON GAS

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Abstract. We review and develop the many-body spectral theory of ideal anyons, i.e. identical quantum particles in the plane whose exchange rules are governed by unitary representations of the braid group on \( N \) strands. Allowing for arbitrary rank (dependent on \( N \)) and non-abelian representations, and letting \( N \to \infty \), this defines the ideal non-abelian many-anyon gas. We compute exchange operators and phases for a common and wide class of representations defined by fusion algebras, including the Fibonacci and Ising anyon models. Furthermore, we extend methods of statistical repulsion (Poincaré and Hardy inequalities) and a local exclusion principle (also implying a Lieb–Thirring inequality) developed for abelian anyons to arbitrary geometric anyon models, i.e. arbitrary sequences of unitary representations of the braid group, for which two-anyon exchange is nontrivial.

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1. Introduction

Quantum statistics is fundamental to our understanding of the physical world. In three spatial dimensions this refers to the division of particles into either bosons, such as Higgs, photons, gluons and other force carriers as well as certain atoms such as $^4$He, or fermions, such as electrons, quarks and other ordinary matter particles including $^3$He atoms. Their collective behavior underlies the explanation of everyday phenomena such as conduction vs. insulation, lasing, as well as stability of large systems such as planets and stars. When quantum systems of many particles are confined to two dimensions, however, other possibilities emerge. Such intermediate quantum statistics of exchange phases [LM77, GMS81] (as opposed to the exclusion principle [Gen40, Gen42, Hal91]), with associated particles known as anyons [Wil82, Wu84], first appeared in theory in the 70’s and 80’s and filled a gap in the logical argument used since the 1920’s to derive the boson/fermion dichotomy [Gir65, Kla68, Sou70, SW70, LD71, Frö88]. They were later found to have an application within the fractional quantum Hall effect (FQHE), both in their abelian form [Lau99, ASW84] as well as in their non-abelian form [MR91, GN97]. The latter eventually spawned possible applications to quantum computing and complexity theory [Kit03, Llo02, FKLW03]. Rooted in configuration spaces and braid groups, these concepts naturally interact with pure mathematics and are e.g. entwined to knot theory, quantum groups, Chern-Simons theory, and conformal field theory (CFT) [Wit89, MS89]. In a quantum gravity context both abelian [tH88, DJ88, Car90, KS91] and non-abelian [PRE15] anyons appear, the former as a toy model for point masses in 2+1 dimensions or cosmic strings in 3+1 dimensions, and the latter proposed for the degrees of freedom on the spherical event horizon of ‘normal’ black holes. Reviews focusing on the physics of anyons are numerous, see e.g. [DMV03, For92, Frö90, IL92, Jac90, Kha05, Ler92, LR16, Myr99, NSS+08, Ouv07, Ste08, Wil90], while for introductions to the mathematical techniques involved we refer to [DRW16, DST01, FG90, LSSY05, Lun19, MS19, MS95, Rou16, RW18].

The concrete emergence of anyons from underlying systems of bosons and fermions has been studied recently [LR16, YGL+19, BLLY20]. On the experimental side one of the difficulties involved is in measuring phase interference for individual particles (see [NLGM20]}
for a recent proposal) while another, more robust, route is to observe indirect effects of statistics in density distributions [CDLR19]. A fundamental problem [CJ94, MS94, NW94] in this regard concerns the relationship between exchange phases and exclusion (which in the fermionic case was attributed to Pauli [Pau47]), and this is only beginning to be addressed in a mathematically precise manner for anyons.

1.1. Anyons — abelian vs. non-abelian. The most direct route to quantum statistics is to start from an $N$-body Schrödinger wave function $\Psi: (\mathbb{R}^2)^N \to \mathbb{C}$, where $x = (x_1, \ldots, x_N) \in (\mathbb{R}^2)^N$ are the positions of $N$ point particles in the plane. The square of its amplitude $|\Psi(x)|$ is normalized $\int_{\mathbb{R}^2^N} |\Psi|^2 = 1$ and carries the interpretation of a probability density to find the particle labeled $j$ at the position $x_j \in \mathbb{R}^2$, at the same instant in time for $j = 1, \ldots, N$. In the case that the particles are all identical (indistinguishable), this probability must be symmetric,

$$|\Psi(x_1, \ldots, x_j, \ldots, x_k, \ldots, x_N)|^2 = |\Psi(x_1, \ldots, x_k, \ldots, x_j, \ldots, x_N)|^2, \quad j \neq k.$$

(1.1)

This leaves the possibility for an exchange phase:

$$\Psi(x_1, \ldots, x_j, \ldots, x_k, \ldots, x_N) = e^{i\alpha \pi} \Psi(x_1, \ldots, x_k, \ldots, x_j, \ldots, x_N), \quad j \neq k.$$

By arguments of logical consistency, it turns out that this phase must be independent of which pair of particles are exchanged. Thus, if $\alpha = 0$ (or $\alpha \in 2\mathbb{Z}$) then $\Psi$ is symmetric and this defines bosons, subject to Bose-Einstein statistics [Bos24, Ein24], while if $\alpha = 1$ (or $\alpha \in 2\mathbb{Z} + 1$) then $\Psi$ is antisymmetric and this defines fermions, subject to Fermi-Dirac statistics [Fer26, Dir26]. In particular, bosons admit product states (Bose-Einstein condensate)

$$\Psi(x_1, \ldots, x_N) = \prod_{j=1}^N u(x_j),$$

(1.2)

having independent identical distribution in a one-body state $u \in L^2(\mathbb{R}^2)$, while fermions are necessarily correlated subject to Pauli’s exclusion principle [Pau25, Pau47]

$$\Psi(x) = 0 \quad \text{if } x_j = x_k \text{ for } j \neq k,$$

(1.3)

and are spanned by Slater determinants (exterior products) of one-body states

$$\Psi(x) = u_1 \wedge \ldots \wedge u_N(x) = (N!)^{-1/2} \det [u_k(x_j)].$$

(1.4)

If $\alpha \notin \mathbb{Z}$ then (1.1) defines anyons (as in ‘any phase’ [Wil82]) with statistics parameter $\alpha$, and requires that $\Psi$ is a multivalued function, since a double exchange is not the identity. The ideal $N$-partice Hamiltonian operator, which we take to be the non-relativistic kinetic energy,

$$\hat{T}_0 = -\Delta_x = -\sum_{j=1}^N \Delta x_j$$

is defined to act on these different functions $\Psi$, with sufficient regularity and suitable boundary conditions on the unit square $Q_0 = [0, 1]^2$, for example, to make the system finite. The ground states of the form (1.2) respectively (1.4) immediately produce calculable ground state energies, $E_N = \inf \text{spec } \hat{T}_0$; for bosons (i.e. the ideal Bose gas) simply

$$E_N^{(\alpha=0)} = N\lambda_0(-\Delta^N_{Q_0}) = 0, \quad \tilde{E}_N^{(\alpha=0)} = N\lambda_0(-\Delta^N_{Q_0}) = 2\pi^2 N,$$

(1.5)

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respectively for fermions (i.e. the ideal Fermi gas) the semiclassical Weyl’s law [Wey12]

\[ E_N^{(\alpha=1)} = \sum_{k=0}^{N-1} \lambda_k(-\Delta_{Q_0}^N) = 2\pi N^2 + o(N^2), \]

\[ E_N^{(\alpha=1)} = \sum_{k=0}^{N-1} \lambda_k(-\Delta_{DQ_0}^N) = 2\pi N^2 + o(N^2), \]

as \( N \to \infty \), where \( \lambda_0 \leq \lambda_1 \leq \ldots \) denote the eigenvalues of \(-\Delta_{Q_0}^{N/2}\), the Neumann/Dirichlet Laplacian on \( Q_0 \), ordered according to their multiplicity. On the other hand, the anyonic case presents a real difficulty as it is not directly reducible to simple product states but actually turns out to be equivalent to a system of interacting bosons or fermions with complicated many-body correlations. In fact, the operator \( \hat{T}_0 \) acting on multivalued \( \Psi \) subject to (1.1) is equivalent to the magnetic operator

\[ \hat{T}_0 = \sum_{j=1}^{N} \left( -i \nabla_{x_j} + \alpha \sum_{k \neq j} (x_j - x_k)^{-1} \right)^2, \quad \mathbf{x}^{-1} := \frac{(-y,x)}{x^2 + y^2} \text{ for } \mathbf{x} = (x,y) \in \mathbb{R}^2 \setminus \{0\}, \]

acting on bosonic \( \Psi \). Thus mathematical techniques for interacting Bose gases are essential.

If \( \Psi \) takes values in some Hilbert space \( \mathcal{F} \), then the condition (1.1) may be replaced by

\[ \Psi(x_1, \ldots, x_j, \ldots, x_k, \ldots, x_N) = U \Psi(x_1, \ldots, x_k, \ldots, x_j, \ldots, x_N), \quad j \neq k, \]

where \( U \in U(\mathcal{F}) \), the group of unitary operators on \( \mathcal{F} \). In general there will be topological consistency conditions on the possibilities for exchange (considered properly as continuous loops in the configuration space of particle positions) and one must consider a representation of the corresponding braid group, i.e. a homomorphism

\[ \rho : B_N \to U(\mathcal{F}). \]

We will later define precisely what we mean by this. In our context, where we consider ideal anyons, the most general model we may consider (referred to as a geometric anyon model) will be in one-to-one correspondence with such a representation \( \rho \). The case (1.1) is a special case where each generator \( \sigma_j \) of \( B_N \) is represented as the phase \( \rho(\sigma_j) = e^{i\pi \alpha} \).

Whereas phases are abelian, a representation (1.8) such that \( \rho(\sigma_j) \) do not all commute is non-abelian (non-abelian anyons are also known as nonabelions or plektons in the literature). We will focus attention on a particular family of non-abelian representations which arise naturally from the perspective of quantum field theory [FRS89, FG90] and which include a number of anyon models which have been proposed to be relevant in condensed matter contexts such as the FQHE. Some of these, such as one known as the Fibonacci anyon model, have even been proposed as good candidates for topologically protected quantum computing [FKLW03, Kit06, NSS+08, RW18]. We refer to this general class of representations as algebraic anyon models.

Our main aim in this work is to connect the relatively well-developed general algebraic theory of non-abelian anyons to the hitherto relatively undeveloped many-body spectral theory and analysis of corresponding geometrically defined Laplace operators \( \hat{T}_\rho \), in order to eventually be able to compute or estimate a physically essential property such as the ground state energy in the many-body (thermodynamic) limit \( N \to \infty \).
1.2. Some mathematically rigorous results for abelian anyons. Apart from a few special systems such as for $2 \leq N \leq 4$ [LM77, Wil82, ASWZ85, Sen91, MLBB91, SVZ91, SVZ92, Min93], strong external field [Ouv07], as well as a subspace of states in harmonic oscillator confinement for general $N$ [Wu84, Cho91b, Cho91a], the spectrum of the many-anyon Hamiltonian $\hat{T}_\alpha$ remains unknown for $\alpha \notin \mathbb{Z}$. It has been noted however that in a rotationally symmetric situation such as the harmonic oscillator potential the ground state (g.s.) in the bosonic representation must have a very specific total angular momentum [CS92]

$$L \sim -\alpha \left( \frac{N}{2} \right)$$

corresponding to a relative angular momentum $-\alpha$ for each pair of particles. This implies that the g.s. energy $E_N$ as a function of $\alpha$ will have to have level crossings between different suitable $L$ and is only likely to be smooth on intervals of lengths tending to zero as $N \to \infty$.

Statistical repulsion, generalizing the Pauli principle (1.3) for fermions, manifests concretely in three ways: primarily as an effective scalar repulsion between pairs of particles, making $|\Psi|^2$ smaller along the diagonals of the configuration space, secondarily as a non-trivial growth of the local Neumann energy with the number of particles, and tertiarily as a degeneracy pressure in the density (e.g. a Thomas–Fermi profile compared to a condensed one-body profile). The first effect has been observed already for $N = 2$ anyons in early works, manifested as a centrifugal-barrier repulsion due to the fractional relative angular momentum. In the abelian many-anyon gas it was brought forward by D. L. and Solovej in [LS13a, LS13b] and quantified by the ‘fractionality’ of $\alpha$

$$\alpha_N = \min_{p \in \{0, \ldots, N-2\}} \min_{q \in \mathbb{Z}} |(2p + 1)\alpha - 2q|,$$

which entered as a coupling constant in a many-anyon Hardy inequality (as quadratic forms)

$$\hat{T}_\alpha \geq \frac{4\alpha^2}{N} \sum_{1 \leq j \neq k \leq N} |x_j - x_k|^{-2}. \quad (1.9)$$
This type of inequality had been proven for fermions ($\alpha_N = 1 \forall N$) by Hoffmann-Ostenhoff et al. [HOHOLT08, Theorem 2.8] (improvements in higher dimensions were discussed in [FHOLS06]). They had in fact proven a similar bound also for anyons [HOHOLT08, Theorem 2.7] although in this case with a much weaker constant, replacing $4\alpha^2N/N$ by

$$C_{\alpha,N} = \min_{p \in \{1,\ldots,N-1\}} \left( \frac{1}{p} \min_{q \in \mathbb{Z}} |p\alpha - q| \right)^2,$$

which vanishes e.g. for fermions. For this result they used a many-body version of a magnetic Hardy inequality for an Aharonov-Bohm singular field due to Laptev and Weidl [LW99] and generalized by Balinsky [Bal03] to the case of many singularities, in a sense considering a single particle in the field of the others fixed (monodromy; cf. Figure 1.1, left). On the other hand, the validity of (1.9) for fermions boils down to the Poincaré inequality in pairwise relative coordinates, namely that the energy for an antipodal-antisymmetric and nonzero function on the unit circle $S^1$ (or unit sphere $S^{d-1}$) cannot be zero,

$$\int_0^{2\pi} |u'(\varphi)|^2 d\varphi \geq \int_0^{2\pi} |u(\varphi)|^2 d\varphi,$$

if $u(\varphi + \pi) = -u(\varphi)$.

Such a relative Poincaré inequality was generalized to the anyonic setting in [LS13a] in the form of a symmetry adaptation of the magnetic inequality of Laptev and Weidl. For multivalued functions this amounts to

$$\int_0^{2\pi} |u'(\varphi)|^2 d\varphi \geq \min_{q \in \mathbb{Z}} |\alpha - 2q|^2 \int_0^{2\pi} |u(\varphi)|^2 d\varphi,$$

if $u(\varphi + \pi) = e^{i\alpha \pi}u(\varphi)$.

This provides then the first manifestation of statistical repulsion (rooted in simple exchange or half-monodromy; cf. Figure 1.1, right).

Also importantly for the subsequent development, a more powerful local version of the Hardy inequality was introduced (it will be given below in its improved form). In the case that

$$\alpha_* := \inf_{N \geq 2} \alpha_N = \lim_{N \to \infty} \alpha_N$$

is positive, which is true iff $\alpha$ is an odd-numerator rational number, this local version of the inequality (1.9) was used to derive local energy estimates for $\hat{T}_\alpha$ (with Neumann b.c. on $Q^N$)

$$E_N \geq C\alpha_*^2(N-1)_+ \geq C\alpha_*^2(N-1)_+, \quad (1.10)$$

with an explicit numerical constant $C > 0$. This provides a local exclusion principle — a secondary manifestation of statistical repulsion — and it may be applied iteratively in a way which in the case of fermions goes back already to Dyson and Lenard [DL67] in their ingenious proof of the thermodynamic stability of fermionic matter with Coulomb interaction (see e.g. [LS10, Lun19]). Namely, the corresponding bound for fermions, which follows immediately from (1.6) with $\lambda_k \geq \pi^2$ for $k \geq 1$, is

$$E_N \geq \pi^2(N-1)_+.$$ 

It was shown in [LS13a, LS14] that such a linear bound in $N$ is sufficient to estimate the abelian anyon gas energy

$$E_N \geq \frac{1}{4} C\alpha_*^2N^2 + o(N^2)$$

as $N \to \infty$. The function $\alpha \mapsto \alpha_*$ appearing in these bounds and supported on odd-numerator rationals is a variant of the Thomae or ‘popcorn’ function; cf. Figure 1.2.
These bounds were subsequently improved in works of D.L. with Larson respectively Seiringer; first in [LL18] considering a refinement of the many-anyon Hardy inequality to yield
\[ E_N \geq f(j_{\alpha N}^2)(N - 1) + \]
where the function \( f(j_{\alpha}^2) \sim 4\pi\alpha \) as \( \alpha \to 0 \) has an expected linear behavior for small \( \alpha \) (its definition and uniform bounds will be given below), and then with a scale-covariant method in [LS18] to
\[ E_N \geq c(\alpha^2)N, \quad N \geq 2, \]
where \( c(\alpha^2) = \frac{1}{4} \min \{ f(j_{\alpha}^2), 0.147 \} \). The latter bound thus removes the dependence on \( \alpha_* \), although with a weaker constant; cf. Figure 1.2. In fact it was shown in [LS18] that the homogeneous ideal abelian anyon gas is subject to uniform linear bounds in \( \alpha \in [0, 1] \) (by periodicity and a conjugation symmetry \( \Psi \mapsto \overline{\Psi}, \alpha \mapsto -\alpha \), it suffices to study this range):

**Theorem 1.1** *(Uniform bounds for the ideal abelian anyon gas [LS18]).* For any sequence of abelian \( N \)-anyon models \( \rho_N : B_N \to U(1), \rho_N(\sigma_j) = e^{i\alpha\sigma_j} \), with statistics parameters \( \alpha = \alpha(N) \in [0, 1] \) and \( n \)-anyon exchange parameters \( \alpha_n = \alpha_n(N) \in [0, 1] \), we have the following uniform bounds for the ground-state energy, \( \inf \text{spec } T_\alpha \):
\[
\frac{1}{4} C(\rho_N) N^2 (1 - O(N^{-1})) \leq E_N \leq \bar{E}_N \leq 2\pi^2 N^2 (1 + O(N^{-1/2})),
\]
where
\[
C(\rho_N) := \max \{ C_4(\rho_N), f(j_{\alpha N}^2) \}, \quad \nu/3 \leq f(j_{\alpha N}^2) \leq 4\pi\nu(1 + \nu) \quad \forall \nu \in [0, 1],
\]
and
\[
C_4(\rho_N) := \frac{1}{4} \min\{ E_2, E_3, E_4 \} \geq c(\alpha) := \frac{1}{4} \min \{ f(j_{\alpha}^2), 0.147 \}.
\]
Further, there exist universal constants \( C_2 \geq C_1 > 0 \) such that, if \( \alpha \) is independent of \( N \),
\[
C_1 \alpha \leq \liminf_{N \to \infty} E_N/N^2 \leq \limsup_{N \to \infty} \bar{E}_N/N^2 \leq C_2 \alpha.
\]
The latter limits are the energy per particle and density, thus, the abelian anyon gas with \( \alpha \in (0,1] \) exhibits extensivity in the energy like the Fermi gas (1.6) (and unlike the Bose gas (1.5)), as a consequence of the statistical repulsion between pairs of particles.

Another development in [LS13a, LL18, LS18] concerns the Lieb-Thirring inequality
\[
\langle \Psi, \hat{T}_\alpha \Psi \rangle \geq C_\alpha \int_{\mathbb{R}^2} \varrho_\Psi(x)^2 \, dx, \quad \alpha \in [0,1],
\]
(1.11)
where the one-body density \( \varrho_\Psi \) is the marginal of the probability distribution
\[
\varrho_\Psi(x) := \sum_{j=1}^{N} \int_{\mathbb{R}^2(N-1)} |\Psi(x_1, \ldots, x_{j-1}, x, x_{j+1}, \ldots, x_N)|^2 \prod_{k \neq j} dx_k.
\]
Such an inequality was first proved for fermions by Lieb and Thirring [LT75, LT76] with the goal of simplifying Dyson and Lenard’s proof of stability of matter. It is a powerful combination of both the uncertainty principle and the exclusion principle, and manifests the tertiary form of statistical repulsion as a degeneracy pressure in the density. Namely the r.h.s. of (1.11) is on the form of the Thomas–Fermi approximation [Tho27, Fer27]
\[
\inf \text{spec } \hat{T}_\alpha = \inf_{\varrho} \varrho^{\frac{1}{2}} \mathcal{E}(\varrho)^{\frac{1}{2}} \varrho \approx \inf_{\varrho} \int_{\mathbb{R}^2} 2\pi \alpha \varrho(x)^2 \, dx,
\]
(1.12)
which employs the extensive ideal Fermi gas energy locally at \( x \in \mathbb{R}^2 \) (with a mean-field motivated guess for intermediate \( \alpha \); cf. [Sen91, CS92, LBM92, CDLR19]). In [LS13a] it was observed that the local exclusion principle (1.10) may be used to prove the Lieb-Thirring inequality, with a constant involving \( \alpha^2 \) and the successive improvements in [LL18, LS18] for the local energy led to corresponding improvements of the constant in (1.11). Furthermore, we should mention that a possibility of minimizing the energy due to statistical repulsion using certain clustering states (in a sense generalizing (1.2) and (1.4) for particular \( \alpha \)) was discussed in [LS13b, Lun17], although this still remains an open problem.

Another line of approach to understanding the many-anyon spectrum is to consider the limit \( \alpha \to 0 \), known as almost-bosonic anyons. In the corresponding mean-field (or ‘average-field’) theory (see e.g. [CWWH89, IL92]) of weakly magnetically interacting bosons, the Thomas–Fermi approximation (1.12) has been rigorously justified in a particular limit via regularized (extended, i.e. non-ideal) anyons [LR15, CLR17, CLR18, Gir20], although with a few surprises, such as a slightly bigger constant than \( 2\pi \) due to self-interactions and the emergence of a vortex lattice [CDLR19]. Note that this approach is typically not possible for non-abelian models, at least not the algebraic anyon models studied here whose associated exchange phases are distributed at discrete positions on the unit circle, and therefore it will not be our focus in this work.

1.3. Main new results. In this work we give, to our knowledge, a first mathematically rigorous study of the ground-state properties of the ideal non-abelian many-anyon gas, taking \( N \to \infty \). Two-particle energies, second virial coefficients and other pairwise statistics-dependent properties have been investigated in the past for certain non-abelian models, notably [Ver91, LO94] as well as [MTM13b, MTM13a] for non-abelian Chern-Simons (NACS) particles, and these could be argued to be dominant for the dilute gas.

After dealing with the basic tasks of defining an \( N \)-anyon model in sufficient generality, its kinetic energy operator and ground-state energy, the main aim is to find a non-abelian
equivalent of the statistical repulsion to generalize the above results for abelian anyons. Again we choose to take the route via a pairwise Poincaré inequality and a many-anyon Hardy inequality, however this requires first to define and compute for an arbitrary sequence of anyon models \( \rho_N : B_N \to U(F_N) \) two-anyon exchange operators \( U_p \), which are the unitary matrices in (1.7) given that \( p \) other anyons are encircled in the exchange of two anyons. It is a consequence of the uncertainty principle that \( U_0 \sim \rho_N(\sigma_j) \) does not suffice.

Our first main observation, Theorem 3.9, concerns the reduction of \( U_p \) in any algebraic anyon model to an exchange involving only three objects, thus giving a recipe on how to compute it in general. We illustrate this by computing explicit exchange operators \( U \) for an arbitrary sequence \( n \)-anyon models as well as upper bounds for the homogeneous non-abelian anyon gas ground-state energy given in Theorem 5.5 in a stronger but more technical local version. We also discuss a few counterexamples to such an inequality with any positive constant for models with \( \rho \) given in Theorem 5.5.

In Section 4.2 we define the kinetic energy operator \( \hat{T}_\rho \) for an arbitrary representation \( \rho \) (1.8). We then extend methods of local statistical repulsion — in all three of its manifestations — to general anyon models. In particular, we prove a many-anyon Hardy inequality given in Theorem 5.5 in a stronger but more technical local version. We also discuss a few counterexamples to such an inequality with any positive constant for models with \( \alpha_N = 0 \). A local exclusion principle is obtained in Lemma 5.17, and as a consequence uniform lower as well as upper bounds for the homogeneous non-abelian anyon gas ground-state energy are given in Theorem 6.3 and its Corollaries 6.6-6.8. We summarize:

**Theorem 1.2 (Uniform bounds for the ideal non-abelian anyon gas).**

For any sequence of \( N \)-anyon models \( \rho_N : B_N \to U(F_N) \) with \( n \)-anyon exchange parameters \( \alpha_n = \alpha_n(N) \in [0,1], 2 \leq n \leq N \), we have the uniform bounds

\[
\frac{1}{4} C(\rho_N) N^2(1 - O(N^{-1})) \leq E_N \leq \bar{E}_N \leq 2\pi^2 N^2(1 + O(N^{-1/2})),
\]

where

\[
C(\rho_N) := \max \left\{ C_4(\rho_N), f^{j_{\alpha_N}} \right\}
\]

and

\[
C_4(\rho_N) := \frac{1}{4} \min\{E_2, E_3, E_4\} \geq c(\alpha_2) \geq \frac{1}{4} \min \{\alpha_2/3, 0.147\}.
\]

In particular:

- For Fibonacci anyons the exchange parameters are \( \alpha_2 = \beta_0 = 3/5 \) and \( \alpha_N = \beta_1 = 1/5 \) for \( N \geq 3 \), and hence (1.14) holds with \( C(\rho_N) \geq 1/15 \) (numerically \( C(\rho_N) \geq 0.35 \)) for \( N \geq 3 \).
- For Ising anyons the exchange parameters are \( \alpha_N = \beta_0 = 1/8 \) for all \( N \geq 2 \), and hence (1.14) holds with \( C(\rho_N) \geq 1/24 \) (numerically \( C(\rho_N) \geq 0.25 \)) for \( N \geq 2 \).
The inhomogeneous gas is covered by the Lieb–Thirring inequality, Theorem 6.10, which generalizes (1.11) to
\[
\langle \Psi, \hat{T}_\rho \Psi \rangle \geq C \alpha_2 \int_{\mathbb{R}^2} \rho(x)^2 \, dx,
\]
for a universal constant \( C > 0 \). An immediate application is the thermodynamic stability of any Coulomb-interacting (2D in 3D, and otherwise ideal) anyon gas for which \( \alpha_2(N) > 0 \) uniformly in \( N \); cf. [LS14, Theorem 21] and [Lun19, Sec. 7.3.3].

Due to the necessity in this field of employing tools from all corners of mathematics, we take the style of a review and will introduce the necessary concepts as they arise. We will go from algebraic to geometric and magnetic models and, via functional inequalities of Poincaré, Hardy and Lieb–Thirring, finally to the ‘physical’/thermodynamic application. In Section 2 we recall the essentials of algebraic anyon models, which include models considered in FQHE contexts and of relevance to topological quantum computing. Section 3 is devoted to defining and computing general exchange operators and phases for these models of interest. Previous work for abelian anyons has relied on the existence of a magnetic representation, however as we will discuss in Section 4.3, it is not obvious that such a representation should exist for all non-abelian models. Therefore we take a different approach via covering spaces, as outlined in [FM89, MS95]. The technical machinery for this is recalled in Sections 4.1-4.2 and 4.8. In Section 5 statistical repulsion is discussed, based on our generalization (1.13) of the many-anyon Hardy inequality for an arbitrary geometric anyon model. Finally, in Section 6 these tools are applied to prove bounds for the homogeneous anyon gas energy as well as the potentially nonhomogeneous gas by means of the Lieb–Thirring inequality.

We use boldface to help the reader see whenever a new key concept arises. Although we have tried to uniformize and streamline the various notations found in the literature, beware e.g. that the popular symbol \( \sigma \) has multiple meanings (spectrum, anyon, braid, ...).

The results in this work are based in part on the MSc thesis of V. Q. [Qva17].

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2. Algebraic anyon models

A potential confusion for a newcomer to the vast literature on anyons is that, while a choice of an ideal $N$-anyon model is mathematically equivalent to a choice of a braid group representation (1.8), as we shall make precise in Section 4, for any physical realization it makes all the difference in the world whether the corresponding particles are treated as classical or quantum mechanical. That is, one could think of localizing the positions of the particles to a specific set of points and then perform braiding operations on them as if they were classical objects. This is convenient from the practical perspective of quantum computation and is sometimes the way that anyons are portrayed. We stress however that an anyonic object is a consequence of identity at a fundamental quantum/mathematical level and can never be classical, just like bosons and fermions cannot. The uncertainty principle forbids us to talk about precise positions unless we have access to infinite energy, and demands that we consider all possibilities that are not logically excluded. One way to clarify the differences in perspectives is on one hand to consider a realization of a braid group representation (such as an algebraic anyon model or a FQHE quasihole ansatz) as a kinematical framework, and on the other to require an associated geometric or magnetic anyon model in which also the dynamics (actual quantum Hamiltonian, Hilbert space etc.) of the particles — actual anyons — is specified (see [For91, LR16] for a related discussion).

For an algebraic anyon model we specify the following information:

1. Particle types / labels / topological charges.
2. Rules for fusion / splitting.
3. Rules for braiding.

Intuitively, fusion takes into account how any subsystem of particles looks from afar, and is convenient when there is a flexible/large number of particles. The fully consistent machinery of fusion and braiding, which was initially developed in the context of CFT and Chern-Simons [Wit89, MS89] as well as algebraic QFT [FRS89, FG90], led eventually to the notion of a unitary braided fusion category / modular tensor category [Kit06, Appendix E] [RSW09]. This is a highly technical subject and, luckily, over the years it has been simplified and distilled through a number of excellent lecture notes, reviews and theses into a form which is accessible to an average humanoid mathematical physicist. We have attempted to continue this effort and give here only a very brief summary of the parts that are necessary for our purposes. Thus we follow the notations of [Pre04, Kit06, Bon07, NSS+08, TTWL08, FdG10, Wan10, Pac12, DRW16, Sim16, Ton16, MG17, FS18] (although note convention differences!).

2.1. Fusion.

2.1.1. Fusion rules. We start with a finite set of labels $\mathcal{L} = \{a, b, c \ldots\}$, whose elements are interpreted as the different types of anyons (or the topological charges) present in the model. It must always contain a special element $1 \in \mathcal{L}$ called the vacuum. The elements of $\mathcal{L}$ are assumed to generate a commutative, associative fusion algebra with unit 1. This means that we may take formal linear combinations of labels and have a binary composition $\times$ such that for $a, b \in \mathcal{L}$,

$$a \times b = \sum_{c \in \mathcal{L}} N_{ab}^c c,$$

(2.1)
where the nonnegative integers $N^c_{ab} \in \mathbb{N}_0$ are called fusion multiplicities. If $N^c_{ab} \neq 0$ we write $c \in a \times b$ and say that $c$ is a valid result of fusion, and $N^c_{ab}$ then counts the number of distinguishable ways that this type of fusion may occur, with each one of them called a fusion channel. The model is formally called non-abelian if there exists some $a$ and $b$ in $L$ such that

$$\sum_{c \in L} N^c_{ab} \geq 2,$$

i.e. if there are several possible ways to fuse $a$ and $b$, and otherwise, i.e. if both the product and process of fusion are always unique, it is abelian. The typical models considered in the literature are actually multiplicity free, i.e. $N^c_{ab} \in \{0, 1\}$ for all $a, b, c \in L$, but instead there may be several possible fusion products $c \in a \times b$, each defining a unique fusion channel.

To each anyon type $a \in L$ is associated a quantum dimension $d_a \geq 1$ obtained as the largest eigenvalue of the matrix $N_a = [N^c_{ab}]_{b,c \in L}$. By the Perron-Frobenius theorem, these can also be found by replacing (2.1) by the corresponding system of polynomial equations:

$$d_ad_b = \sum_{c \in L} N^c_{ab} d_c.$$

Furthermore, the fusion algebra (2.1) has the property that to each $a \in L$ there is a unique inverse element, or charge conjugate, $\bar{a} \in L$ such that $1 \in a \times \bar{a}$. In particular, $\bar{1} = 1$. Also note that $1 \times a = a = a \times 1$ implies $N^c_{1a} = N^c_{a1} = \delta^c_a$.

2.1.2. Fusion diagrams and spaces. Given a possible fusion $c \in a \times b$, i.e. $N^c_{ab} \neq 0$, we construct a Hilbert space $V^c_{ab}$ by assigning an element $\langle a, b; c, \mu \mid$ of an abstract orthonormal basis to each fusion channel, enumerated by an index $\mu \in \{1, 2, \ldots, N^c_{ab}\}$:

$$V^c_{ab} := \text{Span}_C\{\langle a, b; c, \mu \mid\}_{\mu}.$$

The dual space to this fusion space is called a splitting space:

$$V^{ab}_c := \text{Span}_C\{|a, b; c, \mu\\}_{\mu}.$$

We picture each fusion resp. splitting channel basis state diagrammatically, with time running upwards, as:

$$\begin{array}{ccc}
\begin{tikzpicture}
\node (a) at (0,0) {$a$};
\node (b) at (1,0) {$b$};
\node (c) at (1.5,0) {$c$};
\draw (a) -- (b);
\draw (b) -- (c);
\end{tikzpicture}
\end{array}$$

respectively

$$\begin{array}{ccc}
\begin{tikzpicture}
\node (a) at (0,0) {$a$};
\node (b) at (1,0) {$b$};
\node (c) at (1.5,0) {$c$};
\draw (a) -- (b);
\draw (b) -- (c);
\end{tikzpicture}
\end{array}$$

or simply

$$\begin{array}{ccc}
\begin{tikzpicture}
\node (a) at (0,0) {$a$};
\node (b) at (1,0) {$b$};
\node (c) at (1.5,0) {$c$};
\end{tikzpicture}
\end{array}.$$

In the last diagram we have suppressed the channel index $\mu$ (anticipating multiplicity-free models) and the arrows and allow time to flow from down/right to up/left (this allows for tremendous typographical simplifications upon composing diagrams). If one prefers, one may turn the diagrams (or time) around and replace fusion spaces by splitting spaces, and vice versa. Beware that the conventions of the literature are mixed. Also note that by the properties of the inverse/conjugate/unit we have the simple diagrams

$$\begin{array}{ccc}
\begin{tikzpicture}
\node (a) at (0,0) {$a$};
\node (b) at (0,1) {$a$};
\node (c) at (1,1) {$1$};
\node (d) at (1,0) {$a$};
\draw (a) -- (b);
\draw (b) -- (c);
\draw (c) -- (d);
\end{tikzpicture}
\end{array}, \quad
\begin{tikzpicture}
\node (a) at (0,0) {$a$};
\node (b) at (0,1) {$1$};
\node (c) at (1,1) {$a$};
\node (d) at (1,0) {$a$};
\draw (a) -- (b);
\draw (b) -- (c);
\draw (c) -- (d);
\end{tikzpicture}, \quad
\begin{tikzpicture}
\node (a) at (0,0) {$a$};
\node (b) at (0,1) {$\bar{a}$};
\node (c) at (1,1) {$1$};
\node (d) at (1,0) {$a$};
\draw (a) -- (b);
\draw (b) -- (c);
\draw (c) -- (d);
\end{tikzpicture}, \quad
\begin{tikzpicture}
\node (a) at (0,0) {$a$};
\node (b) at (0,1) {$a$};
\node (c) at (1,1) {$1$};
\node (d) at (1,0) {$a$};
\draw (a) -- (b);
\draw (b) -- (c);
\draw (c) -- (d);
\end{tikzpicture}.$$

\(^{(1)}\)This terminology is justified in [RW16].
Fusions/splittings may be diagrammatically composed into more complicated processes; a typical one is shown in Figure 2.1, left. In case a single charge $d \in \mathcal{L}$ splits consecutively into charges $a, b, c$, there are two distinct ways to represent this by diagrams:

$$
\begin{array}{c}
\begin{aligned}
\frac{b}{a} & \frac{c}{e} & \frac{d}{a} & \frac{c}{e} \quad \text{respectively} \quad \frac{b}{a} & \frac{c}{e} \\
\end{aligned}
\end{array}
$$

with corresponding composed splitting spaces defined by

$$
V_d^{abc} := \bigoplus_e V_e^{ab} \otimes V_e^{ec} \quad \text{respectively} \quad \tilde{V}_d^{abc} := \bigoplus_e V_e^{ae} \otimes V_e^{bc}.
$$

Note that we must sum over all possible intermediate charges $e \in a \times b$ resp. $e \in b \times c$, and that there could also be a multiplicity $N_{ab}^c = \dim V_c^{ab}$ etc. in each splitting process. The diagrams (2.2) (possibly with additional suppressed labels $\mu \in \{1, \ldots, N_{ab}^c\}$, $\nu \in \{1, \ldots, N_{ec}^b\}$, etc.) represent the basis elements of the corresponding space. All bases are naturally ordered according to a choice of order of the elements in $\mathcal{L}$.

In the case of further successive splitting we define the standard splitting space

$$
V_c^{a_1a_2\ldots a_n} := \bigoplus_{b_1,b_2,\ldots,b_{n-2}} V_{b_1}^{a_1a_2} \otimes V_{b_2}^{b_3a_3} \otimes V_{b_3}^{b_4a_4} \otimes \cdots \otimes V_{b_{n-2}}^{b_{n-1}a_n}
$$

$$
= \left\{ \begin{array}{c}
\begin{aligned}
\frac{a_2}{a_1} & \frac{a_3}{b_1} & \frac{a_4}{b_2} & \cdots & \frac{a_{n-1}}{b_{n-3}} & \frac{a_n}{b_{n-2}} & \frac{e}{c} \\
\end{aligned}
\end{array} \right\}
$$

for all possible intermediate charges $b_1, b_2, \ldots, b_{n-2}$.

Again, in the case of nontrivial multiplicities one must also consider all intermediate labels $\mu_1, \ldots, \mu_{n-2}$ of the respective spaces. Also note that by fusion with the vacuum we may always shift the diagrammatic representation:

$$
\frac{a_2}{a_1} \frac{a_3}{b_1} \frac{b_2}{b_2} \cdots = \frac{a_2}{a_1} \frac{a_3}{b_1} \frac{a_3}{b_2} \frac{b_2}{b_2} \cdots
$$

2.1.3. $F$-symbols: Associativity of fusion. The requirement of associativity of the fusion algebra,

$$(a \times b) \times c = a \times (b \times c),$$

enforces natural isomorphisms on the fusion spaces $V_d^{abc}$ and $\tilde{V}_d^{abc}$.

**Definition 2.1** (F operator). The $F$ operator (or $F$ matrix) $F_d^{abc} : V_d^{abc} = \bigoplus_e V_e^{ab} \otimes V_c^{ec} \rightarrow V_d^{abc} = \bigoplus_e V_e^{ae} \otimes V_e^{bc}$ is an isomorphism, diagrammatically represented by

$$
F : \begin{array}{c}
\begin{aligned}
\frac{b}{a} & \frac{c}{e} & \frac{d}{a} \\
\end{aligned}
\end{array} \rightarrow \frac{b}{a} \frac{c}{d} = \sum_f F^{abc}_{d;fe} \frac{b}{a} \frac{c}{f},
$$

In the case of nontrivial fusion multiplicities one may write

$$
|a,e;d,\alpha) \otimes |b,c,e,\beta) = \sum_{f,\mu,\nu} [F^{abc}_{d}]_{(f,\mu,\nu),(e,\alpha,\beta)} |a,b;f,\mu) \otimes |f,c;d,\nu),
$$

where the components $F^{abc}_{d;fe} = [F^{abc}_{d}]_{f,e}$, expressed in the standard bases, are known as the F-symbols.
The following lemma will be useful when computing the $F$-symbols:

**Lemma 2.2.** Consider the splitting space $V_{d}^{abc}$ in a multiplicity-free model. When one of the particle types is trivial, i.e. $a, b, c$ or $d$ equals 1, then $\dim V_{d}^{abc} = 1$, and furthermore if $a, b$ or $c$ equals 1, then the corresponding $F$-matrix $F_{d}^{abc}$ is trivial. Explicitly that is

\[
F_{d}^{1bc} = F_{d;bd}^{1bc} = 1, \\
F_{d}^{a1c} = F_{d;ac}^{a1c} = 1, \\
F_{d}^{ab1} = F_{d;db}^{ab1} = 1.
\]

**Proof.** By the defining properties of the fusion algebra,

\[
\frac{b}{1} \frac{c}{d} = \frac{b}{1} \frac{c}{d} = \frac{b}{1} \frac{c}{d},
\]

and so on. \qed

**2.2. Braiding.** Now that we have defined the fusion spaces, we may consider braiding operators on these spaces. The simplest cases define the R and B symbols.

**2.2.1. R-symbols: Commutativity of fusion.** One requires that the result of fusing $a$ with $b$ must be the same as fusing $b$ with $a$. That is, fusion is commutative,

\[
a \times b = b \times a.
\]

This gives rise to a natural isomorphism between the corresponding fusion/splitting spaces for each possible fusion channel.

**Definition 2.3** (R operator). The **R operator** $R^{ab}$ is an isomorphism on each fusion channel $c \in a \times b$

\[
R^{ab}: V_{c}^{ba} \rightarrow V_{c}^{ab},
\]

or in terms of the basis states, unitary matrices $R_{c}^{ab} \in U(N_{a}^{c})$,

\[
R^{ab}: |b,a;c,\mu \rangle \mapsto \sum_{\nu} [R_{c}^{ab}]_{\nu\mu} |a,b;c,\nu \rangle,
\]

diagrammatically represented by

\[
R^{ab}: \begin{array}{c} a \\ \downarrow \end{array} \begin{array}{c} b \\ \downarrow \end{array} \begin{array}{c} c \\ \downarrow \end{array} = \sum_{\nu} \begin{array}{c} a \\ \downarrow \end{array} \begin{array}{c} b \\ \downarrow \end{array} \begin{array}{c} c \\ \downarrow \end{array}.
\]

In the case of trivial fusion multiplicities we thus see that the $R$ operator is diagonal,

\[
R_{c}^{ab} = R_{c}^{ab}: \begin{array}{c} a \\ \downarrow \end{array} \begin{array}{c} b \\ \downarrow \end{array} \begin{array}{c} c \\ \downarrow \end{array} = R_{c}^{ab} \begin{array}{c} a \\ \downarrow \end{array} \begin{array}{c} b \\ \downarrow \end{array} \begin{array}{c} c \\ \downarrow \end{array},
\]

with $R_{c}^{ab} \in U(1)$ the **R-symbols**. That is, there is no mixing of the $c$ label (and mixing of $a$ and $b$ cannot occur since these are fixed in the definition).

We require trivial braiding with the vacuum: $R_{c}^{1b} = 1$ and $R_{c}^{a1} = 1$, for all $a, b, c \in \mathcal{L}$. 

2.2.2. B-symbols: Braiding of standard fusion states. We note that
\[
\sum_{f} (F^{-1})^{abc}_{d;fe} \frac{b}{a} \frac{c}{d} = \sum_{f} \frac{R_{f}^{bc}}{b} (F^{-1})^{ac}_{d;fe} \frac{b}{a} \frac{c}{d},
\]
where the ordering of the symbols is natural for matrix multiplication (also with regard to any suppressed multiplicities). The above operation on splitting states in \(V^{abc}_{d;fe}\) defines the B operator.

**Definition 2.4** (B operator). The B operator \(B^{abc}_{d} \) is an isomorphism on the splitting space \(V^{abc}_{d}\), given by
\[
B^{abc}_{d} = \sum_{f} \sum_{g} F^{abc}_{d;gf} R_{f}^{bc} (F^{-1})^{ac}_{d;fe} \frac{b}{a} \frac{c}{d}.
\]
Symbolically we write this relationship as \(B = FRF^{-1}\).

**Lemma 2.5.** Consider the fusion space \(V^{abc}_{d}\) in a multiplicity-free model. When one of the particle types is trivial, i.e. \(a, b, c\) or \(d\) equals 1, then \(\dim V^{abc}_{d} = 1\) and the corresponding B-matrix \(B^{abc}_{d}\) is one-dimensional,
\[
\begin{align*}
B^{1bc}_{d} &= B^{1bc}_{d;bc} = R_{d}^{bc}, \\
B^{1ac}_{d} &= B^{1ac}_{d;ac} = R_{c}^{ac} = 1, \\
B^{abc}_{d} &= B^{abc}_{d;da} = R_{b}^{bc} = 1, \\
B^{abc}_{1;cb} &= B^{abc}_{1;cb} = R_{a}^{bc}.
\end{align*}
\]
**Proof.** This follows directly from the diagrams

\[
\begin{align*}
\frac{b}{a} \frac{c}{d} &= \frac{b}{a} \frac{c}{d} = R_{d}^{bc} \frac{b}{a} \frac{c}{d} = R_{d}^{bc} \frac{b}{a} \frac{c}{d}, \\
\frac{b}{a} \frac{c}{d} &= \frac{b}{a} \frac{c}{d} = R_{d}^{bc} \frac{b}{a} \frac{c}{d} = R_{d}^{bc} \frac{b}{a} \frac{c}{d},
\end{align*}
\]
and so on. \(\square\)

2.3. Pentagon and hexagon equations. There are certain consistency conditions to be met upon combining fusions and braidings. Apart from the geometrically more obvious relations that have been coded into the diagrammatic framework, such as e.g. Figures 2.1 and 3.1 (for their formal origins, see e.g. [MS89, Eq. (4.3)], [FG90, Eq. (4.89)-(4.91)] or [Kit06, Eq. (206)]), there are also a few less obvious ones.
One such family of constraints are the **Pentagon equations** (Figure 2.2)

\[
F_{wpzw}^{yzd} F_{u}^{xyz} = \sum_{r \in \mathcal{L}} F_{ywr}^{xyz} F_{q}^{xyz} F_{s}^{xyz},
\]

and another the **Hexagon equations** (Figure 2.3)

\[
R_{p}^{yz} F_{u}^{xyz} F_{wpzw}^{yzd} = \sum_{r \in \mathcal{L}} F_{ywr}^{xyz} R_{u}^{xyz},
\]

\[
(R_{p}^{yz})^{-1} F_{u}^{xyz} (R_{q}^{yz})^{-1} = \sum_{r \in \mathcal{L}} F_{ywr}^{xyz} (R_{u}^{xyz})^{-1} F_{u}^{xyz}.
\]

Furthermore, there is a remaining **gauge freedom** in the space of solutions to these equations; see e.g. [Bon07, Section 2.5] and [Kit06, Appendix E.6]. Namely, for states \( |\mu\rangle \in V_{c}^{ab} \) one may consider a change of basis

\[
|\tilde{\mu}\rangle = \sum_{\mu} \left[ u_{c}^{ab} \right]_{\mu\mu'} |\mu\rangle
\]
with $u_{ab}^{c} \in \mathbb{U}(N_{ab}^{c})$. In the multiplicity-free case these are simply phases, and amount to the transformations

$$
\tilde{F}_{d;ef}^{abc} = \frac{u_{d}^{a}u_{f}^{bc}}{u_{e}^{a}u_{d}^{c}} F_{d;ef}^{abc},
\tilde{R}_{c}^{ab} = \frac{u_{c}^{ba}}{u_{a}^{bc}} R_{c}^{ab}.
$$

With the totality of constraints it has been shown that there are only finitely many gauge equivalence classes of solutions with a given fusion algebra (2.1) (Ocneanu rigidity; see [Kit06, ENO05]).

We now consider a few central examples of algebraic anyon models which solve the above constraints.

2.4. Vacuum, bosons and fermions. A simplest example of an algebraic anyon model is to take $\mathcal{L} = \{1\}$, so that all symbols are trivial and there is only the vacuum state. This is the trivial/vacuum model.

The next-to-simplest case is two symbols $\mathcal{L} = \{1, \psi\}$, where $\bar{\psi} = \psi$ and $\psi \times \psi = 1$. Note that together with the default fusion relations $1 \times 1 = 1$ and $1 \times \psi = \psi$, these make up the relations of the group $\mathbb{Z}_2$. Here $F = 1$ up to a gauge phase, the pentagon equations are trivial while the hexagon equations reduce to

$$
R_{a+b+c}^{ac}R_{a+b}^{ab} = R_{a+b+c}^{a(b+c)},
$$

where addition is that in $\mathbb{Z}_2$, with the only non-trivial equation

$$
(R_{1}^{\psi\psi})^2 = 1.
$$

There are thus two such models: $R_{1}^{\psi\psi} = \pm 1 = e^{in\pi}$, $n = 0, 1$, denoted $\mathbb{Z}_2^{(n)}$ and corresponding to bosons and fermions respectively. Note that there is also an accompanying grading in the fusion whether we have even or odd numbers of particles,

$$
\psi^{N} = \begin{cases} 
1, & N \text{ even}, \\
\psi, & N \text{ odd}.
\end{cases}
$$
2.5. Abelian anyons. Recall that an abelian model is characterized by the fact that there is a unique result of fusion, \( a \times b = c = b \times a \).

As a generalization of the above \( \mathbb{Z}_2 \) abelian model, and in order to still only have a finite collection of charges \( \mathcal{L} \), we consider an elementar

\[
\alpha \in [0, 2) \text{ such that } q = e^{i\alpha} \text{ is a finite root of unity.}
\]

Either \( \alpha = \mu/\nu \) is a reduced fraction with \( \mu \in \{0, 2, \ldots, 2(\nu - 1)\} \)
and \( \nu \geq 1 \) odd, i.e. \( q^\nu = 1 \) an odd root of unity, or \( \alpha = \mu/\nu \) is a reduced fraction with \( \mu \in \{1, 3, \ldots, 2\nu - 1\} \) and \( \nu \geq 1 \), i.e. \( q^\nu = -1 \) an even root of unity. We fix the denominator \( \nu \) and define the fusion rules

\[
[\alpha] \times [\beta] = [\alpha + \beta \mod 2]
\]

or equivalently, with \( \alpha = \mu/\nu, \beta = \lambda/\nu \), we may use the notation

\[
[\mu\nu] \times [\lambda\nu] = [\mu + \lambda \mod 2\nu] \nu,
\]

the relations of \( \mathbb{Z}_{2\nu} \). However in the case that \( \nu \) is odd and \( \mu, \lambda \) even we obtain the subgroup \( \mathbb{Z}_\nu \) of even integers. The conjugate charge is \( \bar{\nu} \)
and define the fusion rules

\[
\nu_{\mu}^2 = 2\nu\mu = [0\nu] = 1, \text{ iteration of the hexagon equation (2.7) yields}
\]

\[
(R_{[2\mu]})^{2\nu} = R_{[(\nu+1)\mu]} R_{[(\nu+1)\mu]}. \quad R_{[\nu]\nu} \nu_{2\nu\nu} = 1,
\]

and similarly to the previous case we have a family of models satisfying these equations — all the \( 2\nu \) roots of unity. Let \( \mathcal{L} = \{1, \alpha, \alpha^2, \ldots, \alpha^{2\nu - 1}\} \) denote the one with \( R_{\alpha}^\alpha = e^{i\alpha} \), i.e. the elementary anyon \( \alpha \) has the statistics parameter \( \alpha \). In the case of even-numerator \( \alpha \) we have \( \alpha^{2\nu - 1} = 1 \) and a reduction to a group of \( \nu \) roots of unity \( \mathcal{L} = \{1, \alpha, \alpha^2, \ldots, \alpha^{2\nu - 1}\} \).

Note that the quantum dimension is \( d_\alpha = 1 \) for all \( \alpha \in \mathcal{L} \), since \( d_\alpha d_\alpha = d_1 = 1 \) and

\[
\prod_{\alpha \in \mathcal{L}} d_\alpha = d_{[\sum_{\alpha \in \mathcal{L}} \alpha]},
\]

and which would otherwise produce a contradiction.

2.6. Fibonacci anyons. The Fibonacci anyon model is determined by a single nontrivial anyon type \( \mathcal{L} = \{1, \tau\} \) (with \( \tau = \tau \)) and the fusion rules

\[
1 \times 1 = 1, \quad 1 \times \tau = \tau \times 1 = \tau, \quad \tau \times \tau = 1 + \tau.
\]

The quantum dimension \( d_\tau \) is the largest solution to \( d^2 = 1 + d_\tau \). Hence, \( d_\tau = \phi > 1 \), where \( \phi := (1 + \sqrt{5})/2 \) is the golden ratio.

Only fusion or splitting processes involving chains of nontrivial anyons \( \tau \) are of interest, and the corresponding standard splitting spaces \( V^{\tau^n}_{\tau^n} = V^{\tau^n}_{\tau^n} \oplus V^{\tau^n}_{\tau^n} \) are spanned by elements of the form

\[
\frac{\tau}{1 \tau}; \quad \frac{\tau \tau}{1 \tau \tau}; \quad \frac{\tau \tau \tau}{1 \tau \tau \tau}; \quad \frac{\tau \tau \tau \tau}{1 \tau \tau \tau \tau}; \quad \frac{\tau \tau \tau \tau \tau}{1 \tau \tau \tau \tau \tau}; \quad \frac{\tau \tau \tau \tau \tau \tau}{1 \tau \tau \tau \tau \tau \tau}; \quad \cdots;
\]

with dimensions growing according to the Fibonacci sequence. Namely, the total charge can be either 1 or \( \tau \) and if it is \( \tau \) then the next-to-last intermediate charge can be either 1 or \( \tau \), enumerating the states of \( V^{\tau^{n-1}}_{\tau^{n-1}} \) recursively, while if the total charge is 1 then the next-to-last must be \( \tau \), while the second next-to-last can then be either 1 or \( \tau \), enumerating the states of \( V^{\tau^{n-2}}_{\tau^{n-2}} \), and so on.
**Definition 2.6.** The \( n \):th **Fibonacci number** \( \text{Fib}(n) \) is defined by the recurrence relation

\[
\text{Fib}(0) = 0, \quad \text{Fib}(1) = 1, \quad \text{Fib}(n) = \text{Fib}(n - 1) + \text{Fib}(n - 2).
\]

For example:

\[
\begin{array}{ccccccccccc}
 n & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\text{Fib}(n) & 2 & -1 & 1 & 0 & 1 & 1 & 2 & 3 & 5 & 8 & 13 \\
\end{array}
\]

Furthermore, there is the closed-form formula

\[
\text{Fib}(n) = \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}}.
\]

From the above observations we have

\[
\dim V_1^{1^\tau n} = \text{Fib}(n - 1), \quad \dim V_1^{1^\tau n} = \text{Fib}(n), \quad \dim V_s^{1^\tau n} = \text{Fib}(n + 1).
\]

See e.g. [TTWL08] and [MG17, Examples 3.9 and 3.12] for explicit solutions of the constraints. The nontrivial pentagon equations read

\[
F_\tau^{cda} F_\tau^{cb} = F_\tau^{dce} F_\tau^{db} F_\tau^{kea},
\]

for all allowed \( a, b, c, d, e \) internal edges of fusion trees, and the hexagon equations

\[
R_\tau^{c} F_\tau^{ctt} R_\tau^{a} = \sum_b F_\tau^{ttt} R_\tau^{cb} F_\tau^{taa},
\]

for \( a, b, c \) internal edges of fusion trees.

One can show that the only potentially nontrivial \( F \)-symbols are \( F_\tau^{ttt} \) and \( F_\tau^{ttt} \) (all others allowed by fusion equal 1), and that the pentagon equations and unitarity imply

\[
F_\tau^{ttt} = 1 \quad \text{and} \quad F_\tau^{ttt} = \begin{bmatrix}
\phi^{-1} & \eta \phi^{-1/2} \\
\bar{\eta} \phi^{-1/2} & -\phi^{-1}
\end{bmatrix} =: F,
\]

(with a convenient abuse of notation), where the phase \( |\eta| = 1 \) is a remaining gauge parameter which we will put to \( \eta = 1 \) for simplicity (cf. [MG17]). Note that with this convention \( \det F = -1 \) and \( F^2 = 1 \), i.e. \( F^{-1} = F = \bar{F} \), as matrices. Hence,

\[
\begin{align*}
\tau \tau \tau \tau \tau & = \tau \tau \tau \tau \tau, \\
\tau \tau \tau \tau \tau & = \phi^{-1} \tau \tau \tau \tau \tau + \phi^{-1/2} \tau \tau \tau \tau \tau, \\
\tau \tau \tau \tau \tau & = \phi^{-1/2} \tau \tau \tau \tau \tau - \phi^{-1} \tau \tau \tau \tau \tau,
\end{align*}
\]

and the inverse transformations are found by a direct exchange of diagrams.

There are two sets of solutions for the nontrivial \( R \)-symbols (with continued abuse of notation):

\[
R := R_\tau^{tt} \oplus R_\tau^{tt} = \text{diag}(R_\tau^{tt}, R_\tau^{tt}),
\]

with

\[
R_\tau^{tt} = e^{i\pi i/5}, \quad R_\tau^{tt} = e^{-3i\pi i/5}, \quad (2.8)
\]
respectively their complex conjugates
\[ R_{1}^{\tau\tau} = e^{-4\pi i/5}, \quad R_{\tau}^{\tau\tau} = e^{3\pi i/5}. \quad (2.9) \]
Hence for the nontrivial B-symbols, \( B = FRF \):
\[ B_{a}^{1\tau\tau} = R \ (\text{i.e.} \ B_{a}^R = R^a \mathbb{1}, \ a \in \{1, \tau\}), \quad B_{1}^{\tau\tau\tau\tau} = R_{\tau}^{\tau\tau}, \quad (2.10) \]
and (yet another abuse of notation)
\[ B := B_{\tau}^{\tau\tau\tau} = \begin{bmatrix} B_{\tau,11}^{\tau\tau\tau\tau} & B_{\tau,1\tau}^{\tau\tau\tau\tau} \\ B_{1,\tau\tau}^{\tau\tau\tau\tau} & B_{1,\tau\tau\tau\tau} \end{bmatrix} = \begin{bmatrix} \phi^{-1}e^{-4\pi i/5} & \phi^{-1/2}e^{3\pi i/5} \\ \phi^{-1/2}e^{3\pi i/5} & -\phi^{-1} \end{bmatrix}, \quad (2.11) \]
i.e. diagrammatically
\[
\begin{align*}
\begin{tikzpicture}
\fill[black!20] (0.5,0) -- (0,1) -- (1,1) -- (1,0) -- cycle;
\draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle;
\end{tikzpicture} & = e^{4\pi i/5} \begin{tikzpicture}
\fill[black!20] (0.5,0) -- (0,1) -- (1,1) -- (1,0) -- cycle;
\draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle;
\end{tikzpicture}, \\
\begin{tikzpicture}
\fill[black!20] (0.5,0) -- (0,1) -- (1,1) -- (1,0) -- cycle;
\draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle;
\end{tikzpicture} & = e^{-3\pi i/5} \begin{tikzpicture}
\fill[black!20] (0.5,0) -- (0,1) -- (1,1) -- (1,0) -- cycle;
\draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle;
\end{tikzpicture}, \\
\begin{tikzpicture}
\fill[black!20] (0.5,0) -- (0,1) -- (1,1) -- (1,0) -- cycle;
\draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle;
\end{tikzpicture} & = e^{-3\pi i/5} \begin{tikzpicture}
\fill[black!20] (0.5,0) -- (0,1) -- (1,1) -- (1,0) -- cycle;
\draw (0,0) -- (1,0) -- (1,1) -- (0,1) -- cycle;
\end{tikzpicture}.
\end{align*}
\]
One could also take all symbols complex conjugated, corresponding to the choice (2.9). We will stick to the first choice (2.8) throughout, while the second (2.9) amounts to time reversal of all diagrams. Note further that \( B^{-1} = B^1 = \bar{B} \) and \( B^{10} = FR^{10}F = \mathbb{1} \).

2.7. Ising anyons. The Ising anyon model is described by the set of charges \( \mathcal{L} = \{1, \sigma, \psi\} \) (with \( \sigma = \sigma \) and \( \bar{\psi} = \psi \)), i.e. two nontrivial anyon types, with the nontrivial fusion rules
\[
\begin{align*}
\sigma \times \sigma & = 1 + \psi, \\
\sigma \times \psi & = \psi \times \sigma = \sigma, \\
\psi \times \psi & = 1.
\end{align*}
\]
The quantum dimensions are thus \( d_{\sigma} = 1 \) and \( d_{\psi} = \sqrt{2} \). The \( \sigma \) and \( \psi \) are called Ising anyon and Majorana fermion, respectively. Even though there is the additional particle type in this model, the possible fusion/splitting basis states involving the Ising anyon \( \sigma \) are actually less numerous than those of the Fibonacci anyon \( (d_{\sigma} < d_{\tau}) \), the first few being
\[
\begin{array}{cccccccc}
\sigma & \sigma & \sigma & \sigma & \sigma & \sigma & \sigma & \sigma \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1.
\end{array}
\]
There are two (see e.g. [MG17, Example 3.10]) solutions for the F-symbols:
\[
F_{\sigma\sigma\sigma}^{\sigma\sigma\sigma} = \pm \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} =: F, \\
F_{\psi;\sigma\sigma}^{\psi;\sigma\sigma} = F_{\sigma;\sigma\sigma}^{\psi;\sigma\sigma} = -1, \\
F_{\psi;\psi\sigma}^{\psi;\psi\sigma} = F_{\psi;\psi\sigma}^{\psi;\psi\sigma} = -1.
\]
all other symbols being trivial. Note that also in this case $F^{-1} = \bar{F}$. There are actually eight different anyon models with the above fusion algebra, characterized by the value of the “topological spin” (cf. [Kit06, Sec. 10.5-10.6], [BGN11])

$$\theta_\sigma := \frac{1}{\sqrt{2}} \left( R_1^{\sigma \sigma} + R_\psi^{\sigma \psi} \right) = e^{i2\pi(2j+1)/16}, \quad j = 0, 1, \ldots, 7.$$ 

We consider in this work one set of solutions for the $R$-symbols, with $\theta_\sigma = e^{\pi i/8}$:

\begin{align*}
R_1^{\psi \psi} &= e^{-\pi i/8}, \\
R_\sigma^{\psi \psi} &= e^{3\pi i/8}, \\
R_\sigma^{\sigma \psi} &= e^{-\pi i/2} = -i, \\
R_1^{\psi \psi} &= -1,
\end{align*}

and another is found by taking complex conjugates. Corresponding diagrams are given in [Kit06, Table 1]. Hence for the nontrivial B-symbols, we have directly from Lemma 2.5

\begin{align*}
B_1^{1 \psi \psi} &= B_1^{1 \psi \psi} = R_1^{\psi \psi} = -1, \\
B_\sigma^{1 \psi \psi} &= B_\sigma^{1 \psi \psi} = R_\sigma^{\psi \psi} = -i, \\
B_\sigma^{1 \sigma \psi} &= B_\sigma^{1 \sigma \psi} = R_\sigma^{\sigma \psi} = -i, \\
B_\sigma^{1 \psi \sigma} &= B_\sigma^{1 \psi \sigma} = R_\sigma^{\psi \sigma} = -i, \\
B_\sigma^{1 \sigma \sigma} &= B_\sigma^{1 \sigma \sigma} = R_\sigma^{\sigma \sigma} = e^{\pi i/8},
\end{align*}

(2.12)

and by simple computation one finds also

\begin{align*}
B_\psi^{\psi \psi} &= B_\psi^{\psi \psi} = R_1^{\psi \psi} = -1, \\
B_\sigma^{\psi \psi} &= B_\sigma^{\psi \psi} = R_1^{\psi \psi} = -1, \\
B_\sigma^{\psi \sigma} &= B_\sigma^{\psi \sigma} = -R_\sigma^{\psi \sigma} = i, \\
B_\sigma^{\psi \psi} &= B_\sigma^{\psi \psi} = -R_\sigma^{\psi \psi} = i, \\
B_\psi^{\psi \sigma} &= B_\psi^{\psi \sigma} = R_1^{\psi \sigma} = e^{-\pi i/8}, \\
B_\psi^{\sigma \psi} &= B_\psi^{\sigma \psi} = -R_\sigma^{\psi \psi} = i, \\
B_\psi^{\sigma \sigma} &= B_\psi^{\sigma \sigma} = -R_\sigma^{\psi \sigma} = i, \\
B_\psi^{\psi \sigma} &= B_\psi^{\psi \sigma} = -R_\sigma^{\psi \psi} = i,
\end{align*}

as well as

\begin{align*}
B &= FRF = B_\sigma^{\sigma \sigma} = \frac{1}{2} \left[ R_1^{\sigma \sigma} + R_\psi^{\sigma \psi} - R_1^{\sigma \sigma} - R_\psi^{\sigma \psi} \right] = \frac{1}{\sqrt{2}} \begin{bmatrix} e^{\pi i/8} & e^{-3\pi i/8} \\ e^{-3\pi i/8} & e^{\pi i/8} \end{bmatrix}.
\end{align*}

(2.13)

Note again that $B^{-1} = B^\dagger = \bar{B}$, while $B^4 = FR^4F = -i1$.

3. Exchange operators and phases

Here we show how to compute the operators and corresponding phases associated with an exchange of two anyons in an arbitrary algebraic anyon model, and illustrate the procedure for abelian, Fibonacci and Ising anyons. We also consider a couple of other common
representations that arise in a different way. For an introduction to the braid group and its representations we refer to [Bir74, KT08].

3.1. Braid group representations.

**Definition 3.1** (Braid/permutation group). Recall that the braid group $B_N$ on $N$ strands, $B_N$, may be defined as the group with generators $\sigma_1, \ldots, \sigma_{N-1}$ subject to the relations

\[
\sigma_j \sigma_k = \sigma_k \sigma_j, \quad \text{if } |j - k| \geq 2,
\]

\[
\sigma_j \sigma_{j+1} \sigma_j = \sigma_{j+1} \sigma_j \sigma_{j+1}, \quad j = 1, 2, \ldots, N-2,
\]

and similarly for their inverses $\sigma_j^{-1}$. The symmetric/permutation group $S_N$ may be defined by the same generators and relations as the braid group, with the additional relations $\sigma_j^2 = 1$ for $j = 1, 2, \ldots, N-1$, hence $\sigma_j^{-1} = \sigma_j$ in this case.

See Figure 3.1 for corresponding diagrams. An important example that we will return to frequently in this work is the exchange of two anyons, while enclosing exactly $p$ other anyons. This is represented by the group element and diagram (Figure 1.1, right):

\[
\Sigma_p := \sigma_1 \sigma_2 \ldots \sigma_p \sigma_{p+1} \sigma_p \ldots \sigma_2 \sigma_1 \quad \leftrightarrow \quad \sigma_1 \sigma_2 \ldots \sigma_p \sigma_{p+1} \sigma_p \ldots \sigma_2 \sigma_1,
\]

In fact, the number of generators of the braid group may be reduced to only two:

**Lemma 3.2.** For any $N \geq 3$, $B_N$ is generated by the two elements $\sigma_1$ and

\[
\Theta_N := \sigma_1 \sigma_2 \ldots \sigma_{N-1},
\]

and furthermore

\[
\sigma_k = \Theta_N^{k-1} \sigma_1 \Theta_N^{1-k},
\]

for any $k \in \{1, 2, \ldots, N-1\}$.

The proof is straightforward by induction; see e.g. [Wei15, Theorem 3.5]. One should note however that the group relations expressed in terms of $\{\sigma_1, \Theta_N\}$ are more complicated.

As a consequence of this alternative presentation we have also the following similarity property. We write as usual $\tilde{U} \sim U$ if $\tilde{U} = S^{-1}US$ for some isometry $S$. 

\[
\begin{align*}
\sigma_j \sigma_k &= \sigma_k \sigma_j, & \text{if } |j - k| \geq 2, \\
\sigma_j \sigma_{j+1} \sigma_j &= \sigma_{j+1} \sigma_j \sigma_{j+1}, & j = 1, 2, \ldots, N-2,
\end{align*}
\]

**Figure 3.1.** Braid diagrams corresponding to the generator $\sigma_j$ of $B_N$, i.e. a counterclockwise exchange of particles/strands $j$ and $j+1$ with time running upwards, and the relations $\sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2$ respectively $\sigma_3 \sigma_1 = \sigma_1 \sigma_3$ of $B_4$.
Lemma 3.3. For any $N \geq 2$, any representation $\rho: B_N \to \text{GL}(V)$ and any pair $j, k \in \{1, 2, \ldots, N-1\}$ of generators, we have similarity

$$\rho(\sigma_j) \sim \rho(\sigma_k).$$

Furthermore, any unitary representation $\rho: B_N \to U(V)$ is completely reducible into orthogonal blocks of irreducible unitary representations $\rho = \oplus_n \rho_n$, and if $\rho_n(\sigma_j)$ is scalar (a scalar multiple of the identity) for some $j$ then so is $\rho_n$.

Proof. This is a trivial consequence of the presentation (3.3):

$$\rho(\sigma_k) = \rho(\Theta_{N-1}) \rho(\sigma_1) \rho(\Theta^{-1}_{N-1}),$$

hence $\rho(\sigma_1) \sim \rho(\sigma_2) \sim \ldots \sim \rho(\sigma_{N-1})$. For the second statement see e.g. [Wei15, Propositions 4.2 and 4.7].

□

Definition 3.4 (Abelian vs. non-abelian representations). Naturally, we call a representation $\rho: B_N \to \text{GL}(V)$ abelian if $[\rho(\sigma_j), \rho(\sigma_k)] = 0$ for all $j, k \in \{1, \ldots, N-1\}$, and non-abelian otherwise.

Now look for concrete representations $\rho(\sigma_j)$ on the spaces of our algebraic anyon models. We will use the following notation for the standard fusion/splitting spaces:

$$V^a_{c,t} = \text{Span} \{ \frac{t}{a} \frac{t}{b_1} \frac{t}{b_2} \ldots \frac{t}{b_{N-2}} \frac{t}{b_{N-1}} \frac{t}{c} : \text{for all possible } b_j \}$$

$$V^a_{s,t} = \text{Span} \{ \frac{t}{a} \frac{t}{b_1} \frac{t}{b_2} \ldots \frac{t}{b_{N-2}} \frac{t}{b_{N-1}} : \text{for all possible } b_j \}$$

$$V^1_{s,t} = \text{Span} \{ \frac{t}{1} \frac{t}{b_1} \frac{t}{b_2} \ldots \frac{t}{b_{N-2}} \frac{t}{b_{N-1}} : \text{for all possible } b_j \} = V^t_{s,t}$$

$$V^*_{s,t} = \text{Span} \{ \frac{t}{b_1} \frac{t}{b_2} \frac{t}{b_3} \ldots \frac{t}{b_{N+1}} : \text{for all possible } b_j \}$$

Note how the only thing that differs are the fixing of left and right charge sectors. Also note the chain of inclusions $V^1_{c,t} \subseteq V^s_{c,t} \subseteq V^*_{s,t}$, and

$$V^*_{s,t} = \bigoplus_{a \in L} V^a_{c,t}.$$ (3.4)

Definition 3.5 (Splitting basis representation). Denote $\rho_n(\sigma_j)$ the representation of the braid generator $\sigma_j$ on the standard splitting space $V^s_{s,t}$, mapping a basis state to

$$\frac{t}{b_1} \frac{t}{b_2} \ldots \frac{t}{b_{N+1}} \quad , \quad \text{for all possible intermediate } b_k \in L, k \in \{1, \ldots, n+1\}.$$

Hence the $j$th anyon goes in front of the $j+1$th anyon, while in $\rho_n(\sigma_j^{-1})$ the $j+1$th anyon goes in front of the $j$th anyon. The action for an arbitrary braid $b = \sigma_{k_1}^{n_{k_1}} \ldots \sigma_{k_m}^{n_{k_m}}$ is defined
similarly by composition upwards (multiplication on the left corresponding to braiding later in time), thus obtaining a $D_n$-dimensional unitary representation $\rho_n : B_n \to U(V_{s,t}^\ast)$, with

$$D_n = \dim V_{s,t}^\ast = \sum_{b_2, \ldots, b_{n-1}} N_{b_1}^{b_2} N_{b_2}^{b_3} \cdots N_{b_{n-1}}^{b_n}$$

(thus typically $D_n \sim d_n^t$, as $n \to \infty$).

Subrepresentations may then also be obtained by restricting to $V_{c,a,t}^\ast$ for allowed values of $a = b_1$ and $c = b_{n+1}$, $\rho_n = \bigoplus \rho_{c,a,t}^\ast$. (Note that braiding does not change these values, cf. Lemma 3.7 below.)

**Theorem 3.6.** We have for the splitting basis representation $\rho_n$

$$\rho_n(\sigma_j) \sim \bigoplus R_{tft}^t,$$

where the sum is over the labels $a, c \in \mathcal{L}$ of subrepresentations $V_{c,a,t}^\ast$ in the decomposition (3.4) and the intermediate possibilities $d$ of splitting spaces $V_{d,t}^\ast$.

**Proof.** On each $V_{c,a,t}^\ast$ in (3.4) use Lemma 3.3 and the definition of the B-operator, $B \sim R$, $\rho_{c,a,t}^\ast(\sigma_1) = \bigoplus_{d \in a \times t} B_{d,ct}^t \otimes 1_c d t^{n-2} \sim \bigoplus_{d \in a \times t} R_{d,t}^t \otimes 1_c d t^{n-2}$.

□

In the case that consecutive braids are considered, the following diagram, allowing for an arbitrary central element $c$, will be particularly important.

**Lemma 3.7.** Useful diagram:

$$\begin{array}{c}
\begin{array}{cc}
t_1 & c \\
a & b \\
d & e
\end{array}
\end{array}
= \sum_{f,g,h} B_{d,ft}^t c B_{g,tf}^t c B_{e,td}^t c \begin{array}{c}
t_1 & c \\
a & h \\
g & e
\end{array}.$$

(3.5)

**Proof.** We can replace $\begin{array}{c}
\begin{array}{cc}
t_1 & c \\
a & b \\
d & e
\end{array}
\end{array}$ with $\sum_f B_{d,ft}^t c B_{a,tf}^t c$ and continue:

$$\begin{array}{c}
\begin{array}{cc}
t_1 & c \\
a & b \\
d & e
\end{array}
\end{array}
= \sum_f B_{d,ft}^t c \begin{array}{c}
t_1 & c \\
a & f \\
g & d
\end{array} = \sum_f B_{d,ft}^t \sum_g B_{e,td}^t c \begin{array}{c}
t_1 & c \\
a & g \\
e & d
\end{array}$$

$$= \sum_f B_{d,ft}^t \sum_g B_{e,td}^t \sum_h B_{g,tf}^t c \begin{array}{c}
t_1 & c \\
a & h \\
g & e
\end{array}.$$

□
3.2. Exchange operators. We may now generalize the simple exchange of two anyons, allowing for other anyons to be topologically enclosed in the exchange loop. Given a representation \( \rho: B_n \to U(\mathcal{F}) \), with \( n \geq p+2 \), one may thus consider the two-anyon exchange operator\(^{(2)}\) corresponding to the braid (3.2),

\[
U_p := \rho(\Sigma_p) = \rho(\sigma_1)\rho(\sigma_2)\ldots\rho(\sigma_p)\rho(\sigma_{p+1})\rho(\sigma_p)\ldots\rho(\sigma_2)\rho(\sigma_1).
\] (3.6)

However we will also define a more general class of exchange operators acting on the standard splitting states.

**Definition 3.8 (Exchange operator).** Let \( U_{t,c,t} \) denote counterclockwise exchange of a pair of anyons of type \( t \) around one anyon of type \( c \) acting on any corresponding standard splitting state, that is, according to (3.5),

\[
U_{t,c,t} = \sum_{d,e} B_{d,j}^{atc} B_{c,jd}^{bt} B_{g,h}^{tc} = \begin{array}{ccc}
  t & c & t \\
  a & b & d \\
  e & \end{array} \rightarrow \begin{array}{ccc}
  t & c & t \\
  a & b & d \\
  e & \end{array}.
\] (3.7)

More generally, if the exchange is done in such a way as to enclose \( p \) anyons of type \( t_1, \ldots, t_p \), we write \( U_{t,\{t_1,\ldots,t_p\},t} \),

\[
U_{t,\{t_1,\ldots,t_p\},t} = \begin{array}{cccccccccccc}
  t & t_1 & t_2 & \ldots & t_p & t \\
  a_1 & a_2 & a_3 & a_4 & \ldots & a_{p+1} & a_{p+2} & a_{p+3} \\
  \end{array} \rightarrow \begin{array}{ccc}
  t & t_1 & \ldots & t_p \\
  a_1 & a_2 & \ldots & a_{p+3} \\
  \end{array}.
\]

As usual, if an anyon \( t \) is repeated \( p \) times we write \( t^p \), and if the anyon type \( t \) is understood we denote for brevity \( U_p := U_{t,t^p,t} \). This is the representation of (3.6) obtained from the splitting basis representation \( \rho_{n=p+2} \) in Definition 3.5.

We shall only need to compute \( U_p \) up to isometries — in fact we only need to know its eigenvalues. Therefore, as it turns out, the order of the inner anyons is not important in \( U_{t,\{t_1,\ldots,t_p\},t} \), only their fusion products.

**Theorem 3.9 (Reduction of exchange operator \([Qva17]\)).** The exchange operator is given up to similarity by

\[
U_{t,\{t_1,\ldots,t_p\},t} \sim \bigoplus_{c \in t_1 \times \ldots \times t_p} U_{t,c,t}
\]

where the direct sum is taken over all fusion channels of the fusion \( t_1 \times t_2 \times \ldots \times t_p \). That is, \( c \) is a possible result of the fusion \( t_1 \times \ldots \times t_p \), counted with multiplicity.

\(^{(2)}\)\([FdG10]\) call the matrices \( \rho(\sigma_j) \) ‘elementary braiding operations’ (EBO), and we may refer to these as elementary exchange operators, however we have not seen a name convention given to \( U_p \) in the literature.
Proof. We make a change of basis at the base of the braid and use the diagrammatic rules. Namely, using a sequence of $F$-moves, where $t_2$ moves to the same tree as $t_1, t_3$ to the same as $t_1$ and $t_2$, etc:

\[
\begin{array}{cccccccc}
 t & t_2 & \ldots & t_p & t & F^{-1} \\
 a_1 & a_2 & \ldots & a_p+3 & a_p+3 \rightarrow \\
 t_1 & t_2 & \ldots & t_p & t & F^{-1-p} \\
\end{array}
\]

The states are then enumerated by the intermediate total charge $c \in t_1 \times t_2 \times \ldots \times t_p$. In other words we have decomposed the space

\[ V_{a_{p+3}}^{t_1 t_2 \ldots t_p} \cong \bigoplus_{c \in t_1 t_2 \ldots t_p} \tilde{V}_{c}^{t_1 \ldots t_p} \otimes V_{a_{p+3}}, \]

where $\tilde{V}$ is the space of states formed of successive splitting of the intermediate charge $c$ (cf. (2.3)) into the anyons $t_p, t_{p-1}, \ldots, t_1$, forming a standard splitting tree (in this case also known as a ‘staircase configuration’).

We may then use the diagrammatic identities of Figure 2.1 (which actually translate to the hexagon equations [MS89, p.196-197], [Kit06, p.81-82]) to place the braid at the base state, thereby reducing the operation to $U_{t,c,t}$ acting only on the factor $V_{a_{p+3}}^{t_1 t_2 \ldots t_p}$. Finally we may use the inverse isomorphism $F$-moves to transform the resulting unbraided state into the original standard form. Hence, in our shorthand notation,

\[ U_{t,1 \ldots t_p,t} = (F)^{p-1}(\oplus_c U_{t,c,t})(F^{-1})^{p-1}. \]

The procedure is similar to the one depicted in [MS89, Figure 33] corresponding to the first $p$ simple braids or first $B$-move of (3.7). \hfill \square

We illustrate this theorem with some important examples of anyon models.

3.3. Abelian anyons. In an arbitrary abelian anyon model with elementary exchange phase $\rho(\sigma_j) = e^{i\alpha_j}$, we have the one-dimensional exchange operator (complex phase factor)

\[ U_p = U_{\alpha,\alpha^p,\alpha} = \rho(\sigma_1) \ldots \rho(\sigma_p)\rho(\sigma_{p+1})\rho(\sigma_p) \ldots \rho(\sigma_1) = e^{i(1+2p)\alpha}. \]

An alternate derivation is using $B_{a+b+c}^{a b c} = R_{b+c}^{bc}$:

\[ U_p = U_{\alpha,\alpha^p,\alpha} = B_{\alpha}^{\alpha \alpha^p \alpha} B_{\alpha^p}^{\alpha^p \alpha} B_{\alpha}^{\alpha^p \alpha \alpha^p} B_{\alpha}^{\alpha^p \alpha \alpha^p} = R_{\alpha}^{\alpha \alpha^p} R_{\alpha^p}^{\alpha^p \alpha} R_{\alpha}^{\alpha^p \alpha \alpha^p} = e^{i(1+2p)\alpha}. \]

Note that in Theorem 3.9 we have always a unique result of fusion $\alpha^p = \alpha^{x \rho \rho}$.

More generally we could also have considered a reducible abelian model:

\[
\rho(\sigma_j) = S^{-1} \text{diag}(e^{i\alpha_1}, \ldots, e^{i\alpha_D}) S, \quad j = 1, \ldots, N - 1, \quad S \in U(D),
\]
i.e. $\rho \sim \rho^{a_1} \oplus \cdots \oplus \rho^{a_D}$, for which

\[ U_p = S^{-1} \text{diag}(e^{i(1+2p)\alpha_1}, \ldots, e^{i(1+2p)\alpha_D}) S, \quad p \geq 0. \]

Note that this formula also provides a way to test if a given model is non-abelian:


\[ e^{i\pi/5} (p \equiv 0) \]
\[ e^{i\pi} (p \equiv 2) \]
\[ e^{i7\pi/5} (p \equiv 4) \]

\[ e^{i3\pi/5} (p \equiv 0) \]
\[ e^{i\pi/5} (p \equiv 3) \]
\[ e^{i9\pi/5} (p \equiv 1) \]

**Figure 3.2.** Exchange phases \( U_p, p \text{ mod } 5, \) on the complex unit circle for abelian anyons with \( \alpha = 3/5. \)

**Proposition 3.10.** Let \( \rho: B_N \to U(D) \) be a representation with \( \rho(\sigma_j) \sim \text{diag}(e^{i\alpha_1\pi}, \ldots, e^{i\alpha_D\pi}) \) (the similarity here is not necessarily uniform in \( j \)). If, for some \( p \geq 1, \)
\[ U_p \sim \text{diag}(e^{i(1+2p)\alpha_1\pi}, \ldots, e^{i(1+2p)\alpha_D\pi}) \]
then \( \rho \) is non-abelian, i.e. there exist \( j \neq k \) so that \( [\rho(\sigma_j), \rho(\sigma_k)] \neq 0. \)

The proof is immediate from the above observation for arbitrary abelian \( \rho. \)

**3.4. Fibonacci anyons.** In the case of Fibonacci anyons the exchange operator is
\[ U_p = U_{\tau,\tau^p,\tau}. \]

Since \( p \) anyons may fuse either to 1 or \( \tau \) with multiplicities
\[ \tau^p = \text{Fib}(p-1) \cdot 1 + \text{Fib}(p) \cdot \tau, \]
by Theorem 3.9 we have
\[ U_{\tau,\tau^p,\tau} \sim U_{\tau,1,\tau}^{\oplus \text{Fib}(p-1)} \oplus U_{\tau,\tau,\tau}^{\oplus \text{Fib}(p)}. \]

We compute the respective blocks in this decomposition:

**3.4.1.** \( U_{\tau,1,\tau} \). Begin by observing \( \tau^1 \tau^1 = \tau^1 \tau^1 \). This exchange operator acts on the space with (ordered) basis
\[ \left\{ \tau^1 \tau^1, \tau^1 \tau^1, \tau^1 \tau^1, \tau^1 \tau^1, \tau^1 \tau^1 \right\}. \]

By (2.10)–(2.11) we find (let us again drop some indices for simplicity)
\[ U_0 = \rho_2(\sigma_1) = \begin{bmatrix} B_{1,\tau^p}^{1\tau} & B_{1,\tau^p}^{1\tau} \\ B_{1,\tau^p}^{1\tau} & B_{1,\tau^p}^{1\tau} \end{bmatrix} = \begin{bmatrix} R_1^{\tau} & R_1^{\tau} \\ R_1^{\tau} & R_1^{\tau} \end{bmatrix} \]

\[ = R_1^{\tau \tau} + R_1^{\tau \tau} + R_1^{\tau \tau} + B. \]
Taking the first conjugation convention (2.8), we observe by $B \sim R$ that
\[ \text{spec}(B) =: \sigma(B) = \sigma(R) = \{ R^{+}_{\tau} R^{+}_{\tau} \} = \{ e^{4\pi i/5}, e^{-3\pi i/5} \}, \]
and thus
\[ \sigma(U_0) = \{ R^{+}_{\tau}(\text{mult. 2}), R^{+}_{\tau}(\text{mult. 3}) \} = \{ e^{4\pi i/5}(\text{mult. 2}), e^{-3\pi i/5}(\text{mult. 3}) \}. \]

3.4.2. $U_{\tau,\tau,\tau}$. This exchange operator acts on the space with (ordered) basis
\[ \left\{ \begin{array}{cccc}
\tau \tau \tau \\
1 \tau 1 \\
1 \tau \tau \\
\tau 1 \tau \\
\tau 1 1 \\
1 \tau \tau \\
\tau \tau 1 \\
1 \tau \tau \\
\tau \tau \tau \\
\end{array} \right\}. \]
Grouping the basis into different charge sectors, we obtain
\[ \rho_3(\sigma_1) = R^{+}_{\tau} \oplus R \oplus B \oplus \begin{bmatrix} B_{11} & B_{1\tau} \\ B_{\tau 1} & B_{\tau\tau} \end{bmatrix}, \]
\[ \rho_3(\sigma_2) = R^{+}_{\tau} \oplus B \oplus R \oplus \begin{bmatrix} R^{+}_{\tau} \\ B_{11} & B_{1\tau} \\ B_{\tau 1} & B_{\tau\tau} \end{bmatrix}, \]
\[ U_1 = \rho_3(\sigma_1) \rho_3(\sigma_2) \rho_3(\sigma_1) = (R^{+}_{\tau})^3 \oplus (RBR) \oplus (BRB) \oplus M, \]
where $BRB \sim RFRFR$ and
\[ M = \begin{bmatrix} R^{+}_{\tau}(B_{11})^2 & B_{1\tau} & B_{1\tau} R^{+}_{\tau} & B_{1\tau} (B_{\tau\tau})^2 + B_{1\tau} B_{\tau\tau} R^{+}_{\tau} \\ B_{\tau 1} & R^{+}_{\tau} & B_{\tau 1} R^{+}_{\tau} & B_{\tau 1} (B_{\tau\tau})^2 + B_{\tau 1} B_{\tau\tau} R^{+}_{\tau} \\ B_{\tau 1} (B_{\tau\tau})^2 + B_{1\tau} B_{\tau\tau} R^{+}_{\tau} & B_{1\tau} B_{\tau\tau} R^{+}_{\tau} \end{bmatrix}. \]

In this case we obtain (as can be verified simply by computer algebra)
\[ (R^{+}_{\tau})^3 = e^{\pi i/5}, \]
\[ \sigma(RBR) = \sigma(BRB) = \{ e^{4\pi i/5}, e^{-\pi i/5} \}, \]
\[ \sigma(M) = \{ e^{4\pi i/5}, e^{\pi i/5}, e^{-\pi i/5} \}, \]
and hence
\[ \sigma(U_1) = \{ e^{4\pi i/5}(\text{mult. 3}), e^{\pi i/5}(\text{mult. 2}), e^{-\pi i/5}(\text{mult. 3}) \}. \]

Corollary 3.11 ([Qva17]). The eigenvalues of $U_p$ for Fibonacci anyons are (cf. Figure 3.3)
\[ \sigma(U_p) = \begin{cases} 
    e^{i4\pi/5} & \text{with multiplicity } \text{Fib}(p + 3), \\
    e^{i\pi/5} & \text{with multiplicity } 2\text{Fib}(p), \\
    e^{-i\pi/5} & \text{with multiplicity } 3\text{Fib}(p), \\
    e^{-i3\pi/5} & \text{with multiplicity } 3\text{Fib}(p - 1) 
\end{cases}. \]

Reducing to $U_p |_{V_{\tau,\tau+2+p}}$ yields the same eigenvalues with multiplicity $\text{Fib}(p + 1)$, $\text{Fib}(p)$, $\text{Fib}(p)$, respectively $\text{Fib}(p - 1)$.

Note that the above expressions are valid for all $p \geq 0$ since $\text{Fib}(p)$ is defined also for negative $p$. 
3.5. **Ising anyons.** Let us compute the exchange matrices $U_{\sigma, \{\sigma, \psi\}, \sigma}$ resp. $U_{\psi, \{\sigma, \psi\}, \psi}$. The fusion multiplicities are

$$
\sigma^{2n+1} = 2^n \cdot \sigma, \quad \sigma^{2n} = 2^{n-1} \cdot (1 + \psi),
$$

and hence by Theorem 3.9 we have (note that these act on differently sized spaces)

$$
U_{\sigma, \sigma^{2n+1}, \sigma} \sim U_{\sigma, \sigma, \sigma}^{\otimes 2n},
$$
$$
U_{\sigma, \sigma^{2n}, \sigma} \sim U_{\sigma, \sigma}^{\otimes 2n-1} \oplus U_{\sigma, \psi, \sigma}^{\otimes 2n-1},
$$
$$
U_{\psi, \sigma^{2n+1}, \psi} \sim U_{\psi, \sigma, \psi}^{\otimes 2n},
$$
$$
U_{\psi, \sigma^{2n}, \psi} \sim U_{\psi, \psi, \psi}^{\otimes 2n-1} \oplus U_{\psi, \psi, \psi}^{\otimes 2n-1}.
$$

The exchanges involving $\psi$ are easiest to compute as they are all diagonal:

$$
U_{\psi, 1, \psi} = R_{1}^{\psi} 1 = -1,
$$
$$
U_{\psi, \psi, \psi} = R_{1}^{\psi} 1 = -1,
$$
$$
U_{\psi, \sigma, \psi} = R_{\sigma}^{\psi} R_{1}^{\psi} R_{\sigma}^{\psi} 1 = 1,
$$

i.e. for all valid labels $a, b, c, d \in \{1, \sigma, \psi\}$,

$$
\frac{\psi_{a} \psi_{b} \psi_{c}}{a b c} = - \frac{\psi_{a} \psi_{b} \psi_{c}}{a b c}, \quad \frac{\psi_{a} \psi_{b} \psi_{c} \psi_{d}}{a b c d} = - \frac{\psi_{a} \psi_{b} \psi_{c} \psi_{d}}{a b c d}, \quad \frac{\psi_{a} \sigma_{b} \psi_{c}}{a b c} = + \frac{\psi_{a} \sigma_{b} \psi_{c}}{a b c}.
$$

(3.8)

For the ones involving braiding of $\sigma$’s we obtain off-diagonal matrices however:

3.5.1. **$U_{\sigma, 1, \sigma}$**. This exchange operator acts on the space $\sigma_{1}^{\sigma} = \sigma_{1}^{\sigma}$ with (ordered) basis

$$
\left\{ \sigma_{1} \sigma_{1}, \sigma_{1} \sigma_{1}, \sigma_{1} \sigma_{1}, \sigma_{1} \sigma_{1}, \sigma_{1} \sigma_{1}, \sigma_{1} \sigma_{1}, \sigma_{1} \sigma_{1}, \sigma_{1} \sigma_{1}, \sigma_{1} \sigma_{1}, \sigma_{1} \sigma_{1} \right\}.
$$
By (2.12)–(2.13) we have immediately

\[
U_{\sigma,1,\sigma} = \rho_2(\sigma_1) = \begin{bmatrix}
R_1^{\sigma}
& R_\psi^{\sigma}
& B_{\sigma;11}^{\sigma} & B_{\sigma;1\psi}^{\sigma}
& B_{\sigma;\psi 1}^{\sigma} & B_{\sigma;\psi \psi}^{\sigma}
\end{bmatrix} = R \oplus B \oplus \tilde{R},
\]

where \( \tilde{R} := R_\psi \oplus R_1 \). Since \( \sigma(B) = \sigma(R) = \sigma(\tilde{R}) = \{R_1, R_\psi\} \), we find

\[
\sigma(U_{\sigma,1,\sigma}) = \{R_1(\text{mult. 3}), R_\psi(\text{mult. 3})\} = \{e^{-\pi i/8} (\text{mult. 3}), e^{3\pi i/8} (\text{mult. 3})\}.
\]

3.5.2. \( U_{\sigma,\sigma,\sigma} \). This exchange operator acts on the space with (ordered) basis

\[
\left\{ \frac{\sigma \sigma \sigma}{1 \sigma 1 \sigma}, \frac{\sigma \sigma \sigma}{1 \sigma \psi \sigma}, \frac{\sigma \sigma \sigma}{\sigma 1 \sigma 1}, \frac{\sigma \sigma \sigma}{\sigma 1 \sigma \psi}, \frac{\sigma \sigma \sigma}{\sigma \psi \sigma 1}, \frac{\sigma \sigma \sigma}{\sigma \psi \sigma \psi} \right\}.
\]

Again, by grouping into charge sectors,

\[
\rho_3(\sigma_1) = \begin{bmatrix}
R_1^{\sigma}
& R_\psi^{\sigma}
& B_{\sigma;11}^{\sigma} & B_{\sigma;1\psi}^{\sigma}
& B_{\sigma;\psi 1}^{\sigma} & B_{\sigma;\psi \psi}^{\sigma}
\end{bmatrix} = R \oplus B \oplus B \oplus \tilde{R},
\]

\[
\rho_3(\sigma_2) = \begin{bmatrix}
B_{\sigma;11}^{\sigma} & B_{\sigma;1\psi}^{\sigma} & R_1^{\sigma} & R_\psi^{\sigma}
B_{\sigma;\psi 1}^{\sigma} & B_{\sigma;\psi \psi}^{\sigma} & R_1^{\sigma} & R_\psi^{\sigma}
\end{bmatrix} = B \oplus R \oplus \tilde{R} \oplus B.
\]

Hence

\[
U_{\sigma,\sigma,\sigma} = \rho_3(\sigma_1)\rho_3(\sigma_2)\rho_3(\sigma_1) = BBR \oplus BRB \oplus B\tilde{R}B \oplus \tilde{B}\tilde{R}\tilde{B},
\]

and using that \( BRB \sim BBR = RFRFR \) and that (by computation)

\[
\sigma(BRB) = \sigma(BRB) = \sigma(\tilde{B}\tilde{R}\tilde{B}) = \{e^{-\pi i/8}, e^{3\pi i/8}\},
\]

we obtain

\[
\sigma(U_{\sigma,\sigma,\sigma}) = \{e^{-\pi i/8} (\text{mult. 4}), e^{3\pi i/8} (\text{mult. 4})\}.
\]
Reducing to $U_p$ respectively, for $n \geq 1$, 

$$
\sigma(U_{\sigma,\sigma^n,\sigma}) = \{e^{-\pi i/8}, e^{-\pi i/8}, e^{3\pi i/8}, e^{7\pi i/8}\}, \quad \text{each with multiplicity } 3 \cdot 2^{n-1},
$$

respectively, for $n \geq 0$

$$
\sigma(U_{\sigma,\sigma^{2n+1},\sigma}) = \{e^{-\pi i/8}, e^{7\pi i/8}\}, \quad \text{each with multiplicity } 2^{n+2}.
$$

Reducing to $U_p|_{V_p^{2+p}}$ yields the same eigenvalues with multiplicity 1, $2^{n-1}$, respectively $2^n$. 

**Corollary 3.12.** The eigenvalues of $U_p = U_{\sigma,\sigma^n,\sigma}$ for Ising anyons are (cf. Figure 3.4) 

$$
\sigma(U_{\sigma,\sigma,\sigma}) = \{e^{-\pi i/8}, e^{3\pi i/8}\}, \quad \text{each with multiplicity } 3,
$$

In summary,
3.6. Clifford anyons. Another anyonic model that has been considered in the literature is obtained by taking a composition of Ising anyons with an abelian factor of $\alpha = -1/8$:

$$\rho_{\text{Clifford}}(\sigma_j) := \rho = -1/8(\sigma_j) \rho_{\text{Ising}}(\sigma_j).$$

Because the resulting representation may alternatively be defined using Clifford algebras and spinors [NW96, Iva01], these anyons may be referred to as Clifford anyons (or “half-quantum vortices from Majorana fermions”, see e.g. [Ton16, Ch. 4.2.2], and [DRW16, Theorem 4.8]).

Using

$$U_p = U_p^{\text{Clifford}} = e^{-(2p+1)\pi i/8} U_p^{\text{Ising}},$$

we find

$$\sigma(U_0) = \{e^{-\pi i/4}, e^{\pi i/4}\},$$
$$\sigma(U_1) = \{e^{-\pi i/2}, e^{\pi i/2}\} = \{\pm i\},$$
$$\sigma(U_2) = \{e^{-3\pi i/4}, e^{-\pi i/4}, e^{\pi i/4}, e^{3\pi i/4}\},$$
$$\sigma(U_3) = \{\pm 1\},$$
$$\sigma(U_4) = \{e^{-3\pi i/4}, e^{-\pi i/4}, e^{\pi i/4}, e^{3\pi i/4}\},$$
$$\sigma(U_5) = \{\pm i\},$$

and so on, with $\sigma(U_{4k+1}) = \sigma(U_l)$, $k \geq 1$, $l = 1, 2, 3, 4$. The first few of these $U_p$ have also been computed in [BS18, Wik18] using representations of Clifford algebras.

3.7. Burau anyons. The standard (unreduced) Burau representation [Bur35] is simply a deformation of the (defining) permutation representation on $\mathbb{C}^N$ with deformation parameter $z \in \mathbb{C} \setminus \{0\}$:

$$\rho : B_N \to \text{GL}(\mathbb{C}^N)$$
$$\rho(\sigma_j) = 1_j \oplus \begin{bmatrix} 1 - z & z \\ -1 & 0 \end{bmatrix} \oplus 1_{N-j-1}. \quad (3.9)$$

Thus it has rank $N$, but may, depending on $z$, be reduced either once or twice to an irreducible representation of rank $N - 1$ or $N - 2$. At the outset these are not unitary but for $z \in U(1)$ close enough to 1 they may be unitarized [Squ84]. For more details we refer to e.g. [Wei15], [DRW16, Sec. 2.3]. Note that these are different than the algebraic anyon models considered above since their dimension grows linearly with $N$.

As a simplest example we may consider $N = 3$, reduced and unitarized:

$$\rho : B_3 \to U(2)$$

$$\rho(\sigma_1) = \frac{1}{2} \begin{bmatrix} -w^2 + w + 1 & -w\sqrt{w + w^{-1} - 1}\sqrt{w + w^{-1} + 1} \\ -w\sqrt{w + w^{-1} - 1}\sqrt{w + w^{-1} + 1} & -w^2 - w + 1 \end{bmatrix},$$
$$\rho(\sigma_2) = \frac{1}{2} \begin{bmatrix} -w^2 + w + 1 & +w\sqrt{w + w^{-1} - 1}\sqrt{w + w^{-1} + 1} \\ +w\sqrt{w + w^{-1} - 1}\sqrt{w + w^{-1} + 1} & -w^2 - w + 1 \end{bmatrix},$$

where $w = \sqrt{z} = e^{i\pi\alpha}$ and $|\alpha| < 1/3$; see [Wei15]. We find that

$$\sigma(U_0) = \sigma(\rho(\sigma_1)) = \sigma(\rho(\sigma_2)) = \{1, -w^2\}.$$
while
\[ U_1 = \rho(\sigma_1 \sigma_2 \sigma_1) = \rho(\sigma_2 \sigma_1 \sigma_2) = \text{diag}(w^3, -w^3). \]

Proposition 3.10 then verifies that \( \rho \) is necessarily non-abelian for these \( w \) (except possibly at \( w = 1 \)), but it may also be verified directly (also at \( w = 1 \)) by the identity
\[
\rho(\sigma_1 \sigma_2) - \rho(\sigma_2 \sigma_1) = \begin{bmatrix}
0 & w^2 \sqrt{w + w^{-1}} - 1 \sqrt{w + w^{-1}} + 1 \\
-w^2 \sqrt{w + w^{-1}} - 1 \sqrt{w + w^{-1}} + 1 & 0
\end{bmatrix}.
\]

In fact, a central result in the theory of braid group representations is that the rank of any non-abelian representation must grow at least linearly with \( N \), and that the Burau representations are of the smallest rank:

**Theorem 3.13** (Formanek [For96]). Let \( \rho : B_N \to U(D) \) be a unitary braid group representation, and \( N > 6 \). If \( D < N - 2 \), then \( \rho \) is abelian. Furthermore, if \( D = N - 2 \) or \( D = N - 1 \) then \( \rho \) is either abelian or of Burau type (i.e. equivalent to the product of an abelian and a reduced Burau representation).

The proof is given in [For96, Theorem 23] for irreducible representations \( \rho : B_N \to \text{GL}(\mathbb{C}^D) \), and the above formulation follows from the complete reducibility of any unitary representation (see Lemma 3.3). Also the case \( D \leq N \) for arbitrary \( N \geq 3 \) is discussed in [FLSV03, Theorem 6.1]. See [Wei15] for some further remarks.

### 4. Geometric and magnetic anyon models

The general definition of the Hilbert space of anyon models was discussed in some detail in [MS95], and the choice of a self-adjoint Hamiltonian in [BCMS93, DFT97, LS14]. See also [Sou70, FM89, MD93], [KLS99, Appendix A], [DGH99, DŠT01, GM04], [Lun19, Sec. 3.7], [MS19] for further details concerning the motivations for these definitions. For background on fiber bundles and connections we refer to [Nak03, Tau11].

#### 4.1. Classical configuration space

Let us temporarily consider particles as points in \( \mathbb{R}^d \) for general \( d \in \mathbb{N} \), before fixing \( d = 2 \). The configuration space for \( N \) distinguishable particles in \( \mathbb{R}^d \) is simply the set \((\mathbb{R}^d)^N\) of \( N \)-tuples of coordinates. Let \( \Delta^N := \{(x_1, \ldots, x_N) \in (\mathbb{R}^d)^N : \exists j \neq k \text{ s.t. } x_j = x_k\} \)

denote the fat diagonal of this space, i.e. the coincidence set of at least two particles. The configuration space for \( N \) distinct particles in \( \mathbb{R}^d \) is then the set \((\mathbb{R}^d)^N \setminus \Delta^N\) of \( N \)-tuples of distinct coordinates. Consider the natural\(^{(3)} \) actions of permutations \( \sigma \in S_N \) on \((\mathbb{R}^d)^N\),
\[
(\sigma, x) \mapsto \sigma.x = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(N)}) \\
(x, \sigma) \mapsto x.\sigma = (x_{\sigma(1)}, \ldots, x_{\sigma(N)}).
\]

Identifying the orbits of \( S_N \) under this action on distinct particles, one obtains the configuration space for \( N \) identical particles in \( \mathbb{R}^d \),
\[
C^N := \left((\mathbb{R}^d)^N \setminus \Delta^N\right)/S_N = \mathcal{P}_N(\mathbb{R}^d) = \left\{ X = \{x_1, \ldots, x_N\} \subset \mathbb{R}^d : |X| = N \right\},
\]

\( ^{(3)} \)We may regard a tuple \( x = (x_j)_{j=1}^N \) as a map \( j \mapsto x_j \), so that \( x.\sigma = x \circ \sigma \) indeed acts on the right.
the space of $N$-point subsets of $\mathbb{R}^d$. For reference we fix a particular point in $C_N$,

$$x_0 = (1e_1, 2e_1, \ldots, Ne_1) \mapsto X_0 = \{1e_1, 2e_1, \ldots, Ne_1\},$$

where $e_1$ is the first unit basis vector in $\mathbb{R}^d$ (for $d = 2$ this fixes a real axis in $\mathbb{C} \cong \mathbb{R}^2$). This choice of base point will provide us with a reference ordering of the particles, and corresponds naturally to the arrangement of strands in Figure 3.1.

Consider now the continuous motion of identical particles in $\mathbb{R}^d$. Given a continuous path $\gamma : [0,1] \to C_N$ from a point $X = \gamma(0)$ to a point $Y = \gamma(1)$, there is an induced action on $\tilde{C}_N$:

$$(\gamma, X) \mapsto \gamma.X = Y$$

In the same way we obtain a (trivial) action of continuous loops $\gamma$ s.t. $\gamma.X = X$. Compositions $(\gamma, \eta) \mapsto \gamma\eta$ of paths and loops are associative under these actions. A continuous exchange of identical particles with initial configuration $X \in C_N$ is a loop $\gamma$ in $C_N$ based at $X$. In the case that we are interested in exchanges modulo homotopy equivalence, which is the case for free particles, the relevant group of loops is the fundamental group $\pi_1(C_N) = \pi_1(C_N, X_0)$.

**Theorem 4.1** (Configuration space topology). The space $C_N$ is path-connected and its fundamental group is

$$\pi_1(C_N) = \begin{cases} 1, & d = 1, \\ B_N, & d = 2, \\ S_N, & d \geq 3. \end{cases}$$

The proof amounts to Artin’s correspondence between geometric and algebraic braid groups [Art25, Art47], and the simple-connectedness of the configuration space of distinct particles for $d \geq 3$. For details we refer to e.g. [Bir74, Theorem 1.8], [KT08], or [Wei15].

The covering space of $C_N$ is a fiber bundle

$$\tilde{C}_N \to C_N \quad \text{with fiber } \pi_1(C_N)$$

and may be defined as the space of paths in $C_N$ based at $X_0$ up to homotopy equivalence,

$$\gamma \sim \gamma' \quad \text{iff} \quad \exists F : [0,1] \times [0,1] \to C_N \text{ cont. s.t. } F(0,\cdot) = \gamma, F(1,\cdot) = \gamma',$$

$$F(s,0) = \gamma(0), F(s,1) = \gamma(1) \quad \forall s \in [0,1].$$

Thus, with the usual composition of paths, the action of paths and loops $\gamma$ in $C_N$ lifts to a faithful action on $\tilde{C}_N$, $(\gamma, \tilde{X}) \mapsto \gamma.\tilde{X}$, with $\gamma.\tilde{X} = \gamma'.\tilde{X}$ for homotopic paths $\gamma \sim \gamma'$ in $C_N$ (with the same endpoints), where the image of the action of loops $\gamma$ at $X \in C_N$ is exactly the induced group $\pi_1(C_N, X)$ of equivalence classes $[\gamma]$ of homotopic loops based at $X$. The canonical projection to the endpoint of a path is denoted $\tilde{pr} : \tilde{C}_N \to C_N$. By definition $\tilde{C}_N$ is simply connected, $\pi_1(\tilde{C}_N, \tilde{X}_0) = 1$.

Hence, taking the reference points $X_0 \leftrightarrow \tilde{X}_0 \leftrightarrow (X_0,1)$, an arbitrary point $\tilde{X} \in \tilde{C}_N$ may be expressed

$$\tilde{X} = \tilde{\gamma}_0.\tilde{X}_0 = \gamma_0.\tilde{X}_0 = [\gamma_0].\tilde{X}_0$$

for some path $\tilde{\gamma}_0$ in $\tilde{C}_N$ based at $\tilde{X}_0$, or equivalently, using its projection $\gamma_0 = \tilde{pr} \circ \tilde{\gamma}_0$ in $C_N$ based at $X_0 = \tilde{pr}(\tilde{X}_0)$ and ending at $X = \tilde{pr}(\tilde{X})$, or using simply the equivalence class $[\gamma_0]$ of homotopic such paths.
Any two points $\tilde{X}, \tilde{X}' \in \tilde{C}^N$ with the same projection $X = \tilde{\pi}(\tilde{X}) = \tilde{\pi}(\tilde{X}')$ differ only by an element of $\pi_1(C^N) = \pi_1(\tilde{C}^N, X_0)$, since 

$$\tilde{X} = [\gamma_0].\tilde{X}_0 = [\gamma_0.\gamma_0^{-1}.\gamma_0].\tilde{X}_0 = ([\gamma_0]\gamma).\tilde{X}_0 = \tilde{X}'.[\gamma],$$

$[\gamma] = [\gamma_0.\gamma_0^{-1} \in \pi_1(C^N, X_0)$, where we have defined an action of equivalence classes of homotopic loops based at $X_0$ on the right,

$$(\tilde{X}, [\gamma]) \mapsto \tilde{X}.[\gamma] := [\gamma \gamma_0].\tilde{X}_0.$$ 

In other words we first do $\gamma$ at $X_0$ and then the path $\gamma_0$ taking us to $X = \tilde{\pi}(\tilde{X})$. The corresponding representation $L: \pi_1(C^N) \to Diff(\tilde{C}^N)$ on the left, s.t. $L_{[\gamma]}[0] = L_{[\gamma]}[1]$, is

$$L_{[\gamma]}(\tilde{X}) := \tilde{X}.[\gamma^{-1}] = [\gamma \gamma_0^{-1}].\tilde{X}_0.$$ 

The group $\pi_1(C^N)$ then acts in this way as a group of diffeomorphisms of $\tilde{C}^N$, and $C^N = \tilde{C}^N / \pi_1(\tilde{C}^N)$. By writing each point $\tilde{X} \in \tilde{C}^N$ using a simple path $\gamma_0$ from $X_0$ to $X$ and a loop $\gamma$ at $X_0$, we have a way of representing $\tilde{X} = ([\gamma_0].\tilde{X}_0).[\gamma]$ as a pair $(X, [\gamma]) \in C^N \times \pi_1(C^N)$ (this is not a canonical identification but rather a choice of local trivialization of the bundle). The corresponding set of points $(X, 1) \leftrightarrow [\gamma_0].\tilde{X}_0$, with $\gamma_0$ a simple path from $X_0$ to $X$, constitutes a fundamental domain in $\tilde{C}^N$ in bijection to $C^N$.

Further, one may interchange the above actions via the conjugation of paths

$$Ad_\eta[\gamma] := [\eta \gamma \eta^{-1}],$$

for compatible paths $\eta, \gamma$ in $C^N$. Namely, given $\tilde{X} = [\gamma_0].\tilde{X}_0$ and $[\gamma] \in \pi_1(C^N, X)$,

$$[\gamma].\tilde{X} = [\gamma \gamma_0].\tilde{X}_0 = [\gamma_0(\gamma^{-1} \gamma \gamma_0)].\tilde{X}_0 = \tilde{X}.Ad_{\gamma_0^{-1}}[\gamma],$$

with $Ad_{\gamma_0^{-1}}[\gamma] \in \pi_1(C^N, X_0)$. Conversely, if $[\gamma] \in \pi_1(C^N, X_0)$ then

$$\tilde{X}.[\gamma] = Ad_{\gamma_0}[\gamma].\tilde{X}$$

with $Ad_{\gamma_0}[\gamma] \in \pi_1(C^N, X)$.

For $d = 1$, trivially $\tilde{C}^N \cong C^N$ since no continuous exchange is possible and the particles may be ordered canonically along the real line according to the choice $X_0 \leftrightarrow x_0 \leftrightarrow \tilde{X}_0$. For $d \geq 3$ we have $C^N \cong C^N \times S_N$, i.e. each point $X$ in the configuration space has $N$! different representatives corresponding to each permutation $\sigma \in S_N$, and we may therefore reconstruct the space of distinct particles as precisely the covering space

$$(\mathbb{R}^d)^N \backslash \Delta^N \cong \tilde{C}^N \cong \{(X, \sigma) : X \in C^N, \sigma \in S_N\},$$

with the reference point $\tilde{X}_0 \leftrightarrow (X_0, 1) \leftrightarrow x_0$, the ordered $N$-tuple of particles.

For concreteness, we will from now on fix $d = 2$ so that the relevant group is the braid group $B_N$. In this case we deal with an infinite set of representatives for each point of $C^N$:

$$\tilde{C}^N \ni \tilde{X} \leftrightarrow (X, b),\ X \in C^N, \ b \in B_N.$$ 

It is now convenient to consider two projections:

$$\tilde{C}^N \xrightarrow{pr} C^N \xrightarrow{pr} (\mathbb{R}^2)^N \backslash \Delta^N$$

$$(X, b) \mapsto X = \{x_1, \ldots, x_N\} \mapsto x = (x_1, \ldots, x_N)$$
In fact there is also a canonical projection \( \hat{\text{pr}} : B_N \to S_N \) defined by quoting with the additional relations \( \{ \sigma_j^2 = 1 \} \) of \( S_N \) and thus we may in similarity to the higher-dimensional case consider

\[
\tilde{C}^N \xrightarrow{\hat{\text{pr}}} C^N \times S_N \cong (\mathbb{R}^2)^N \setminus \Delta^N \xrightarrow{\text{pr}} C^N
\]

so that the composition is \( \tilde{\text{pr}} = \text{pr} \circ \text{pr} \). Inverses may then be defined,

\[
\text{pr}^{-1}_\tilde{N}(X) := (X, 1), \quad \tilde{\text{pr}}^{-1}_0(X) := (X, 1),
\]

mapping \( C^N \) to fundamental domains in \((\mathbb{R}^2)^N\) respectively \( \tilde{C}^N \).

**Example 4.2.** In the case \( N = 2 \) we may identify

\[
C^2 \cong C_0 \times \mathbb{R}^+ \times \mathbb{R}^2,
\]

where the configuration space of the relative angle \( C_0 = S^1/\mathbb{Z}_2 = \{0, \pi\}_{\text{per.}} \) is a circle, \( \text{pr}_0^{-1}(C_0) = \{0, \pi\} \) is a single-cover of \( C_0 \) and a half-circle, \( \text{pr}_1^{-1}(C_0) = \{0, 2\pi\}_{\text{per.}} \) is a double-cover and a circle, while the full covering space \( \tilde{C}_0 = \mathbb{R} \) is the real line containing the fundamental domain \( \tilde{\text{pr}}_0^{-1}(C_0) = \{0, \pi\} \). Points on \( \tilde{C}_0 \) are indexed by \( C_0 \) and \( \pi_1(C_0) = \mathbb{Z} \), the winding number of the loop relative to a base point.

### 4.2. Hamiltonian and Hilbert space.

One means of quantizing a classical system is to find a formulation in terms of a Poisson algebra of observables and then look for representations as operators on a Hilbert space (see e.g. [Thi07, Lun19] for introductions). We consider quantizations of the classical non-relativistic kinetic energy, i.e. the Hamiltonian function

\[
T = \sum_{j=1}^N p_j^2,
\]

and thus seek representations of the momenta \( p = (p_j)_{j=1}^N = (p_{jk})_{j=1,...,N;k=1,2} \) as differentiation operators. The important point here is that the momenta are conjugate to the configuration variables \( X = \{x_1, \ldots, x_N\} \) of identical particles, which take values on the manifold \( C^N \). Locally, i.e. on any topologically trivial open subset \( \Omega \subseteq C^N \) the particles remain distinguishable, and may thus be given representatives \( x = (x_1, \ldots, x_N) \) on coordinate charts \( \text{pr}_0^{-1}(\Omega) \subseteq \mathbb{R}^{2N} \), by the correspondence above. We may then use that, locally w.r.t \( x \) in these charts, the classical canonical Poisson algebra

\[
\{x_{jk}, p_{j'k'}\} = \delta_{j,j'} \delta_{kk'}, \quad j, j' \in \{1, \ldots, N\}, \ k, k' \in \{1, 2\},
\]

can be given a Schrödinger representation as the quantum CCR algebra

\[
\frac{1}{i}[\hat{x}_{jk}, \hat{p}_{j'k'}] = \delta_{jj'} \delta_{kk'} \mathbb{1}, \quad j, j' \in \{1, \ldots, N\}, \ k, k' \in \{1, 2\},
\]

represented on a Hilbert space \( \mathcal{H}_\Omega = L^2(\Omega; F) \), where \( F \) is a representation Hilbert space for any internal degrees of freedom (observables other than these \( x \)'s and \( p \)'s), and then the different charts \( \Omega \) are to be patched together to a representation on all of the configuration space manifold \( C^N \). The natural geometric framework [MD93, DGH99, DST01], [Lun19, Remark 3.29] for this problem is to consider a hermitian fiber bundle \( E \to C^N \) over the

\[\text{(4)}\]For simplicity we put the mass equal to 1/2 throughout, as well as Planck’s constant \( \hbar = 1 \).
configuration space, with fiber $F$, endowed with a connection $\mathcal{A}$ and thus a covariant derivative $\nabla^A$. We are in this work, as a first step, interested in ideal, free anyon models so that the pure effects of statistics may be isolated. Physically, this means that upon restricting the particles to topologically trivial subsets $\Omega \subseteq \mathbb{C}^N$ they should behave as ideal, free and distinguishable particles with the usual momenta $\hat{p}_j = -i\nabla_{x_j}$ and the usual free kinetic energy $\hat{T} = \sum_{j=1}^N (-\Delta_{x_j})$ and no interactions. Geometrically, this means that the connection $\mathcal{A}$ should be (locally) flat. Furthermore, for simplicity we only consider finite-dimensional spaces $\mathcal{F}$. We have the following convenient classification of such bundles:

**Theorem 4.3** (Classification theorem of flat bundles). There is a 1-to-1 correspondence between flat hermitian vector bundles $E_\rho \to \mathbb{C}^N$ with fiber $\mathcal{F} = \mathbb{C}^D$ and unitary representations of $B_N$ on $\mathcal{F}$, $\rho : B_N \to U(\mathcal{F})$, up to conjugation (similarity) in $U(\mathcal{F})$.

In other words, given $N$ and fiber $\mathcal{F}$, the moduli space of flat connections is exactly

$$\text{Hom}(B_N, U(\mathcal{F}))/U(\mathcal{F})$$

where the action of $U(\mathcal{F})$ on representations $\rho \in \text{Hom}$ is the adjoint one,

$$(S, \rho) \mapsto \text{Ad}_S \rho = S\rho S^{-1} : b \mapsto S\rho(b)S^{-1}.$$ 

A detailed proof of this well-known classification theorem is given in [Tau11, Theorem 13.2], and a further discussion of the correspondence in [Mic13, Chapter 5] as well as [MS19]. This correspondence motivates the following definition of an arbitrary model of ideal, free anyons.

**Definition 4.4** (Geometric anyon model). By a geometric $N$-anyon model we mean here a flat hermitian vector bundle $E_\rho \to \mathbb{C}^N$, or equivalently, a unitary representation $\rho : B_N \to U(\mathcal{F})$. Similarly, a geometric many-anyon model is a family of geometric anyon models $\rho_N : B_N \to U(\mathcal{F}_N)$ for some sequence of $N \geq 1$ (possibly with restrictions on which $N$ are allowed; multiples, odd/even, etc.), where $B_1$ is the trivial group.

Further, the Hilbert space of a geometric $N$-anyon model is defined as the space of $L^2$ global sections on the bundle $E_\rho \to \mathbb{C}^N$, i.e. the $L^2$-closure of smooth global sections,

$$\mathcal{H}_\rho^N := \left\{ \Psi \in \Gamma(\mathcal{C}^N, E_\rho) : \int_{\mathbb{C}^N} |\Psi|^2 < \infty \right\},$$

where $\Gamma(M, E)$ denotes the set of smooth sections on the bundle $E \to M$.

**Example 4.5.** Using the map $\hat{\rho} : B_N \to S_N$ and the canonical representation $\hat{\rho} : S_N \to U(N)$ of $\sigma_j$ acting as permutation matrices ($z = 1$ in (3.9)), we may construct a bundle (the canonical permutation bundle) with fiber $\mathcal{F} = \mathbb{C}^N$ and the canonical open covering $\{\Omega_\sigma\}_{\sigma \in S_N}$ of $\mathbb{R}^{2N} \setminus \Delta^N \to \mathbb{C}^N$ with locally constant transition functions

$$t_{\sigma\sigma'} : \Omega_\sigma \cap \Omega_{\sigma'} \to U(N), \quad X \mapsto \hat{\rho}(\sigma\sigma'^{-1}), \quad \sigma, \sigma' \in S_N,$$

The connection is locally trivial but its holonomy nontrivial due to the transition functions.

The space $\Gamma(\mathcal{C}^N, E_\rho)$ of smooth global sections can equivalently be characterized as the space $C_\rho^*(\mathcal{C}^N; \mathcal{F})$ of smooth $\rho$-equivariant functions on the covering space $\mathcal{C}^N$, i.e. functions $\Psi_\rho : \mathcal{C}^N \to \mathcal{F}$ such that, given any two points $\tilde{X} = [\gamma_0] \cdot \tilde{X}_0$ and $\tilde{Y} = \tilde{\gamma} \cdot \tilde{X}$ in $\mathcal{C}^N$ connected by a path $\tilde{\gamma}$ in $\mathcal{C}^N$ and such that its projection $\gamma = \text{pr} \circ \tilde{\gamma}$ in $\mathbb{C}^N$, based at the point $X = \gamma(0) \in \mathbb{C}^N$, is a closed loop, $\gamma \cdot X = \gamma(1) = X$, we have

$$\Psi_\rho(\tilde{Y}) = \rho(\gamma^{-1}) \Psi_\rho(\tilde{X}).$$

(4.3)
where \( b = \text{Ad}_{-1}[\gamma] \in B_N \) is the equivalence class at \( X_0 \) to which the loop \( \gamma \) belongs, i.e. \( \tilde{Y} = \tilde{X}.b = L_{b^{-1}}(\tilde{X}) \). Thus we have an isomorphism \( H^N_\rho \cong L^2_b(\tilde{C}^N; \mathcal{F}) \) (see [MS95, Lemma 2.4]), where the inner product of the latter space is defined

\[
\langle \Phi_\rho, \Psi_\rho \rangle_{L^2_\rho} = \int_{\tilde{C}^N} \langle \Phi_\rho(\tilde{X}), \Psi_\rho(\tilde{X}) \rangle_\mathcal{F} \, dX.
\]

In the integrand \( \tilde{X} \) denotes any pre-image of \( X \in C^N \) under the projection \( \tilde{p}r: \tilde{C}^N \to C^N \). This integrand is indeed well defined thanks to the equivariance condition (4.3), because if \( \tilde{X}' = \tilde{X}.b, b \in B_N \), then

\[
\langle \Phi_\rho(\tilde{X}'), \Psi_\rho(\tilde{X}') \rangle_\mathcal{F} = \langle \rho(b^{-1})\Phi_\rho(\tilde{X}), \rho(b^{-1})\Psi_\rho(\tilde{X}) \rangle_\mathcal{F} = \langle \Phi_\rho(\tilde{X}), \Psi_\rho(\tilde{X}) \rangle_\mathcal{F},
\]

by the unitarity of the representation \( \rho \). The measure \( dX \) on \( C^N \) is derived from the usual Lebesgue measure \( dx \) on \( \mathbb{R}^{2N} \) via the projection \( \tilde{p}r \), such that \( \int_{\Omega} dX = \frac{1}{N} \int_{\tilde{p}r^{-1}(\Omega)} dx \).

The condition (4.3) is in fact used in the proof of Theorem 4.3 to explicitly construct the flat bundle \( E_\rho \) associated to \( \rho \); cf. [Tan11, Chapter 13.9.3]. A \( \rho \)-equivariant function may be considered a relation \( \Psi_\rho \subseteq \tilde{C}^N \times \mathcal{F} \) with a unique \( v \in \mathcal{F} \) to each \( \tilde{X} \in \tilde{C}^N \) and such that if \( (\tilde{X}, v) \in \Psi_\rho \), then \( (\tilde{X}.b, \rho(b^{-1})v) \in \Psi_\rho \) for all \( b \in B_N \). Then the bundle \( E_\rho \to C^N \) is precisely the set of orbits in \( \tilde{C}^N \times \mathcal{F} \) of the right action

\[
(\tilde{X}, v).b := (\tilde{X}.b, \rho(b^{-1})v), \quad \tilde{X} \in \tilde{C}^N, v \in \mathcal{F}, b \in B_N,
\]

with projection \([\tilde{X}, v] \to \tilde{p}r(\tilde{X})\). Conversely, given a smooth global section \( \Psi \in \Gamma(C^N, E_\rho) \), there is to each \( \tilde{X} \) a unique \( v \in \mathcal{F} \) such that \( \Psi(\tilde{p}r(\tilde{X})) = [(\tilde{X}, v)] \), which defines a \( \rho \)-equivariant function \( \Psi_\rho(\tilde{X}) := v \) (again, see [MS95, Lemma 2.4]).

Given an \( N \)-anyon model \( \rho \) and locally flat bundle \( E_\rho \to C^N \) we now have a means of defining a covariant derivative \( \nabla^\rho \Psi \) on smooth sections \( \Psi \in \Gamma(C^N, E_\rho) \) which on any topologically trivial subset \( \Omega \subseteq C^N \) reduces to the ordinary derivative on \( \Psi: \Omega \to \mathcal{F} \)

\[
\nabla^\rho \Psi(X) = (\nabla_j \Psi)_{j=1,\ldots,N} = (\partial \Psi/\partial x^j_{\rho})_{j=1,\ldots,N}^{p=1,2}
\]

and thus a kinetic energy operator \( \hat{T}_\rho = (-i\nabla^\rho)^*(-i\nabla^\rho) \) which on \( \Omega \) reduces to the ordinary one for distinguishable particles

\[
\hat{T}_\rho \Psi(X) = \sum_{j=1}^N (-i\nabla_j)^2 \Psi = -\sum_{j=1}^N \sum_{p=1,2} \partial^2 \Psi/\partial x^j_{\rho p}.
\]

The corresponding quadratic form

\[
T_\rho[\Psi] := \int_{C^N} |\nabla^\rho \Psi|^2_{\mathcal{F}^2N} \, dX \quad (4.4)
\]

may be extended to the \( L^2 \)-sections for which the integral is finite, by taking the closure of smooth sections with compact support on \( C^N \) (see below).

In the \( \rho \)-equivariant setting, a function \( \Psi_\rho \in C^\infty_\rho(\tilde{C}^N; \mathcal{F}) \) has a derivative at any point \( \tilde{X} \in \tilde{C}^N \), defined by

\[
\nabla^\rho \Psi = \left( \frac{\partial \Psi_\rho}{\partial x^j_{\rho 1}}, \frac{\partial \Psi_\rho}{\partial x^j_{\rho 2}} \right)_{j=1,\ldots,N}, \quad \frac{\partial \Psi_\rho}{\partial x^j_{\rho p}} = \lim_{t \to 0} \frac{1}{t}(\Psi_\rho(\gamma^t_{j \rho p} \tilde{X}) - \Psi_\rho(\tilde{X})) \quad (4.5)
\]
where $\gamma_{jp}^t$ is the path $[0,1] \ni s \mapsto \{x_1, \ldots, x_j + st e_p, \ldots, x_N\}$ in $C^N$ defining the corresponding tangent vector. The index $j$ as it stands here is artificial and depends on which ordering of particles we choose, and we may use the projection $\pi_r : X \mapsto x$ before applying the path on the $j$th entry of the tuple $x$ and then project again to $C^N$ with $\pi_r$. Hence the vector in (4.5) depends in fact on the pre-image of $X$ in $B_N$. In any case, this ambiguity is removed in the permutation-invariant sum

$$|\nabla \Psi_\rho|_{F2N}^2 = \sum_{j=1}^N \sum_{p=1,2} \left| \frac{\partial \Psi_\rho}{\partial x_{jp}} \right|^2.$$  

(4.6)

The kinetic energy

$$T_\rho[\Psi_\rho] := \int_{C^N} |\nabla \Psi_\rho(\tilde{X})|_{F2N}^2 dX$$  

(4.7)

is then again well defined because of equivariance,

$$\frac{\partial \Psi_\rho}{\partial x_{jp}}(\tilde{X}, b) = \lim_{t \to 0} \frac{1}{t} (\Psi(\gamma_{jp}^t, (\tilde{X}, b)) - \Psi(\tilde{X}, b)) = \rho(b^{-1}) \frac{\partial \Psi_\rho}{\partial x_{jp}}(\tilde{X}), \quad j' = \pi_r(b)[j],$$

(4.8)

$$\langle \nabla \Phi(\tilde{X}), \nabla \Psi(\tilde{X}) \rangle_{F2N} = \sum_{j=1}^N \sum_{p=1,2} \left\langle \rho(b^{-1}) \frac{\partial \Phi_\rho}{\partial x_{jp}}(\tilde{X}), \rho(b^{-1}) \frac{\partial \Psi_\rho}{\partial x_{jp}}(\tilde{X}) \right\rangle_{F} = \langle \rho(b^{-1}) \nabla \Phi(\tilde{X}), \rho(b^{-1}) \nabla \Psi(\tilde{X}) \rangle_{F2N} = \langle \nabla \Phi(\tilde{X}), \nabla \Psi(\tilde{X}) \rangle_{F2N},$$

and is identical to (4.4) by the above correspondence. We consider the closure of this form (4.7) on $\Psi_\rho \in \tilde{C}^\infty_{\rho,c}$, the $\rho$-equivariant smooth functions on $\tilde{C}^N$ such that $\tilde{\pi}_r(\supp \Psi_\rho) \subseteq C^N$ is compact, which then defines a Sobolev subspace $H^1_\rho \subseteq L^2_\rho$ of anyon wave functions with finite expected kinetic energy. This is also a Hilbert space with the inner product

$$\langle \Phi, \Psi \rangle_{H^1_\rho} := \int_{C^N} \left( \langle \Phi(\tilde{X}), \Psi(\tilde{X}) \rangle_F + \langle \nabla \Phi(\tilde{X}), \nabla \Psi(\tilde{X}) \rangle_{F2N} \right) dX.$$  

Definition 4.6 (Anyon Hamiltonian). We define the kinetic energy operator $\hat{T}_\rho$ on the full $N$-anyon Hilbert space $H^1_\rho$ as the unique self-adjoint operator corresponding to the closed non-negative quadratic form $T_\rho$ in (4.7), with operator domain $\mathcal{D}(\hat{T}_\rho) \subseteq H^1_\rho \subseteq H^N_\rho$ (see e.g. [Tes14, Theorem 2.14]). Considering $\hat{T}_\rho$ as initially defined on $C^\infty_{\rho,c}$, with supports away from diagonals $\Delta^N$, this is the **Friedrichs extension**.

Remark 4.7. One may consider other extensions, such as the **Krein extension**(5), however comparing to the two-anyon case where the extension theory has been carried through in detail [GHKL91, MT91, BS92, ACB95, CO18, CF20] these other extensions would all correspond to introducing additional attractive interactions between the anyons. Thus taking the Friedrichs extension is in alignment with our wish in this work to isolate the

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(5)This is the smallest (in terms of energy) non-negative extension while Friedrichs is the largest. The extension theory for the 2-anyon problem is actually somewhat analogous to that of two bosons in 3D, admitting a circle of extensions, with the functions of the Friedrichs extension vanishing as $r^{-\alpha}$ and those of Krein behaving like $r^{-\alpha}$ as the relative distance $r \to 0$. Indeed, just as in 3D, we anticipate that there could be good physical reasons to consider these other extensions whenever anyons arise as emergent particles along with other interactions. Non-Friedrichs extensions are sometimes referred to as soft-core anyons.
effects of statistics and study purely ideal anyons. See [DFT94] for a possible approach to the other extensions in the many-anyon case, and [LS14] for some further mathematical details in the abelian Friedrichs case. Note that the issue here concerns point interactions supported at \( \Delta^N \) while we could also have introduced ordinary scalar interactions by adding pair (or higher order) potentials to the Hamiltonian (4.1). Both these types of modifications would alter the behavior for distinguishable particles as well.

In the many-anyon context one may consider the group \( B_\infty \) defined as the direct limit of the \( N \)-strand groups with respect to the inclusion maps \( B_N \hookrightarrow B_{N+1} \) sending \( \sigma_j \mapsto \sigma_j \). It is also natural to consider sequences of representations \( \rho_N \colon B_N \rightarrow \text{U}(\mathcal{F}_N) \) such that the respective inclusions \( B_N \hookrightarrow B_{N+1} \) and \( \mathcal{C}\rho_N(B_N) \hookrightarrow \mathcal{C}\rho_{N+1}(B_{N+1}) \) (the group algebras) commute [RW12, Definition 2.1], although not all representations are of this type, such as the Burau representations [DRW16, Sec. 6.1].

For geometric many-anyon models we may also consider the Fock space

\[
\mathcal{H}^\infty := \bigoplus_{N=0}^{\infty} \mathcal{H}^N_{\rho_N},
\]

where \( \mathcal{H}^0 := \mathcal{F}_0 = \mathbb{C} \) is the vacuum sector. Recall that in order to have a non-abelian model it is necessary that \( \dim \mathcal{F}_N \geq N - 2 \) (cf. Theorem 3.13), and we cannot therefore simply fix a finite-dimensional \( \mathcal{F} \) as we take \( N \to \infty \), as one may do for bosons and fermions.

4.3. Statistics transmutation. Given \( N \) and fiber \( \mathcal{F} \), let the trivial or bosonic bundle be the bundle with trivial geometry, taking the trivial representation \( \rho_\sigma(\sigma_j) := +1 \) for all \( j \), thus \( E_{\rho_+} = C^N \times \mathcal{F} \). We note that any function \( \Psi_+ \in \mathcal{H}^N_+ \) of the standard bosonic \( N \)-body Hilbert space

\[
\mathcal{H}^N_+ := L^2_{\text{sym}} ((\mathbb{R}^2)^N; \mathcal{F}) := \{ \Psi \in L^2(\mathbb{R}^{2N}; \mathcal{F}) : \Psi(x, \sigma) = \Psi(x) \forall \sigma \in S_N \},
\]

may be considered an equivariant function \( \Psi_{\rho_+} \in \mathcal{H}^N_{\rho_+} \) of the bosonic bundle. Namely, by the canonical map \( \hat{\rho} \colon B_N \rightarrow S_N \), we have in \( \hat{\mathcal{C}}^N \) pre-images of \( C^N \) labeled by permutations \( \hat{\rho}(b) \). A symmetric function \( \Psi_+ \in \mathcal{H}^N_+ \) thus extends to a function \( \Psi_{\rho_+}(X, x, b) := \Psi_+(x, \hat{\rho}(b)) = \Psi_+(x), x = \hat{\rho}(X) \). Conversely, any function \( \Psi_{\rho_+} : \hat{\mathcal{C}}^N \rightarrow \mathcal{F} \) with symmetry

\[
\Psi_{\rho_+}(X, x, b) = \Psi_{\rho_+}(X, b) = \Psi_{\rho_+}(X), \quad b \in B_N,
\]

where \( \hat{X} \) is any pre-image in \( \hat{\mathcal{C}}^N \) of a point \( X \in \mathcal{C}^N \), may be identified with a function in \( \mathcal{H}^N_+ \) by defining \( \Psi_+(x) := \Psi_{\rho_+}(\hat{\rho}(x)) \) on \( \mathbb{R}^{2N} \setminus \Delta^N \) and using that \( L^2_{\text{sym}}(\mathbb{R}^{2N} \setminus \Delta^N) = L^2_{\text{sym}}(\mathbb{R}^{2N}) \) by the vanishing measure of \( \Delta^N \) in \( \mathbb{R}^{2N} \). Note however that to make it a probability distribution in \( L^2(\mathbb{R}^{2N}) \) (i.e. on the distinguishable configuration space as is conventional) it is necessary to normalize \( \Psi_+ := \Psi_{\rho_+}/\sqrt{N!} \). We note for example that, given any \( \rho \)-equivariant \( \Psi_\rho \), the scalar function \( \|\Psi_\rho\|^2_\mathcal{F} \) (amplitude) is \( \rho_+ \)-equivariant,

\[
|\Psi_\rho(X, b)|^2_\mathcal{F} = |\rho(b^{-1})\Psi_\rho(X)|^2_\mathcal{F} = |\Psi_\rho(X)|^2_\mathcal{F},
\]

which is to say that \( |\Psi_\rho|^2_\mathcal{F} \) defines a probability distribution on \( \mathcal{C}^N \) for \( L^2_\rho \)-normalized \( \Psi_\rho \). Furthermore, we have that \( H^1(\mathbb{R}^{2N} \setminus \Delta^N) = H^1(\mathbb{R}^{2N}) \) (see e.g. [LS14, Lemma 3] and, more...
generally, [LLN19, Appendix B]) so that we may as well identify the **bosonic Sobolev spaces**

\[ H^1_{\rho_+} \cong H^1_{\text{sym}} := C_1^\infty(\mathbb{R}^{2N} \setminus \Delta^N) \cap L^2_{\text{sym}} \mathcal{H}^1(\mathbb{R}^{2N}). \]

Thus, after taking the closure, the functions do not have to vanish at \( \Delta^N \), and indeed for example the Gaussian \( \Psi(x) = e^{-|x|^2} \in H^1_{\text{sym}} \).

There is also an analogous version for fermions, where the **fermionic bundle** is defined by \( \rho_-(\sigma_j) := -1 \) for all \( j \), i.e. \( \rho_- = \text{sign} \circ \text{pr} \otimes 1 \). Instead of \( \mathcal{H}^N_\rho \) we then consider the **fermionic \( N \)-body Hilbert space**

\[ \mathcal{H}^N := L^2_{\text{asym}}((\mathbb{R}^2)^N; \mathcal{F}) := \{ \Psi \in L^2(\mathbb{R}^{2N}; \mathcal{F}) : \Psi(x,\sigma) = (\text{sign} \sigma) \Psi(x) \forall \sigma \in S_N \}, \]

Taking \( \Psi_\rho_-(\tilde{X}.b) := \Psi_-(x, \text{pr}(b)) = \text{sign} \, \text{pr}(b^{-1}) \Psi_-(x) \), we obtain a section of the fermionic bundle, and vice versa \( \Psi_-(x) := \Psi_\rho_-(\tilde{X}) \) (modulo normalizations) with any lift \( \tilde{X} \to x \to X \) defines an antisymmetric function. We define analogously the **fermionic Sobolev space**

\[ H^1_{\rho_-} \cong H^1_{\text{asym}} := C_1^\infty(\mathbb{R}^{2N} \setminus \Delta^N) \cap L^2_{\text{asym}} \mathcal{H}^1(\mathbb{R}^{2N}), \]

and in this case we will see (by means of the Hardy inequality in Section 5) that the functions with finite kinetic energy necessarily vanish on \( \Delta^N \), as had been expected from (1.3).

Note that to any flat hermitian vector bundle \( E \to C^N \) with fiber \( \mathcal{F} \), or representation \( \rho : B_N \to \text{U}(\mathcal{F}) \), there is also the associated flat principal bundle \( P \to C^N \) with fiber \( \text{U}(\mathcal{F}) \) (and vice versa). Again the correspondence to \( \rho \)-equivariant functions carries over to \( P \), which may be defined as the set of orbits in \( \tilde{C}^N \times \text{U}(\mathcal{F}) \) of the right action

\[ (\tilde{X},g).b := (\tilde{X}.b, \rho(b^{-1})g), \quad \tilde{X} \in \tilde{C}^N, \quad g \in \text{U}(\mathcal{F}), \quad b \in b_N, \]

with the projection \([\tilde{X},g] \mapsto \text{pr}\tilde{X} = X \). Using a similar identification as for \( \Gamma(C^N, E) \) then, the set of smooth global sections \( \Gamma(C^N, P) \) is the set of smooth functions \( u_\rho : \tilde{C}^N \to \text{U}(\mathcal{F}) \) that are \( \rho \)-equivariant,

\[ u_\rho(\tilde{X}.b) = \rho(b^{-1})u_\rho(\tilde{X}). \quad (4.9) \]

Recall that a hermitian vector bundle of rank \( n \) is **trivial** iff there exist \( n \) global sections which are pointwise orthonormal thus defining a frame \( \{ u_1(X), \ldots, u_n(X) \} \) in \( \mathcal{F} \) and, equivalently, iff there exists a global section of the principal bundle, so that \( P \cong C^N \times \text{U}(\mathcal{F}) \). Thus, triviality of \( E \) and \( P \) is equivalent to the existence of \( u_\rho \) satisfying (4.9).

Now, given an anyon model \( \rho : B_N \to \text{U}(\mathcal{F}) \), assume that there exists a smooth function \( u_\rho : \tilde{C}^N \to \text{U}(\mathcal{F}) \) satisfying (4.9), i.e. a global section of \( P \). Then, given any symmetric function \( \Psi_+ \in \mathcal{H}^N_+ \) one obtains a \( \rho \)-equivariant function \( \Psi_\rho \in \mathcal{H}^N_\rho \) by taking

\[ \Psi_\rho(\tilde{X}) := u_\rho(\tilde{X})\Psi_+(\tilde{X}). \quad (4.10) \]

Conversely, given a \( \rho \)-equivariant function \( \Psi_\rho \in \mathcal{H}^N_\rho \), the function

\[ \Psi_+(\tilde{X}) := u_\rho(\tilde{X})^{-1}\Psi_\rho(\tilde{X}) \quad (4.11) \]

satisfies \( \Psi_+(\tilde{X}.b) = \Psi_+(\tilde{X}) \), where \( \tilde{X} \) is any pre-image in \( \tilde{C}^N \) of \( X \in C^N \), and may therefore be identified with a function in \( \mathcal{H}^N_+ \).
Considering the kinetic energy, we may write using (4.10)

$$T_\rho[\Psi] = \int_{\mathcal{C}^N} |\nabla(u_\rho \Psi_+)(\hat{X})|^2_{\mathcal{F}^2N} dX = \int_{\mathcal{C}^N} |\nabla^A \Psi_+(\hat{X})|^2_{\mathcal{F}^2N} dX = \int_{\mathcal{C}^N} |\nabla^A \Psi_+(X)|^2_{\mathcal{F}^2N} dX,$$

(4.12)

where the covariant derivative

$$\nabla^A = \nabla + \mathcal{A},$$

is given in terms of the connection \(\mathcal{A}: \tilde{\mathcal{C}}^N \to \mathfrak{u}(\mathcal{F})^2N\) \((\mathfrak{u}(\mathcal{F})\) is the Lie algebra of \(\mathbb{U}(\mathcal{F})\)),

$$\mathcal{A}(\hat{X}) := u_\rho(\hat{X})^{-1} \nabla u_\rho(\hat{X}).$$

This expression descends to \(\hat{\rho}(\tilde{\mathcal{C}}^N) = \mathbb{R}^{2N} \setminus \Delta^N\) thanks to the symmetry

$$\mathcal{A}_j(\hat{X}.b) = u_\rho(\hat{X})^{-1} \rho(b) \nabla_j [\rho(b^{-1}) u_\rho](\hat{X}) = \mathcal{A}_j'(\hat{X})$$

and thus the integrand to \(\mathcal{C}^N\), thanks to the permutation invariance of the sum, as in (4.6).

We call the above operation statistics transmutation as it takes us from one model of anyons to another, to the cost of introducing the non-trivial connection or gauge potential \(\mathcal{A}\). In fact in the abelian case \(\mathcal{F} = \mathbb{C}\), \(u(\mathcal{F}) = i\mathbb{R}\), one has \(\mathcal{A}_j = i \mathcal{A}_j\) for magnetic vector potentials \(\mathcal{A}_j: \mathbb{R}^{2N} \setminus \Delta^N \to \mathbb{R}\) with magnetic field

$$\text{curl} \mathcal{A}_j = \nabla u_\rho \wedge \nabla u_\rho + u_\rho \nabla \wedge \nabla u_\rho = 0,$$

as follows due to differentiating \(u_\rho^\dagger u_\rho = 1\). We have thus represented the geometric anyon model \(\rho\) in this way as a magnetic anyon model using bosons and magnetic potentials. Also in the non-abelian case one could talk about non-abelian “magnetic” gauge fields with zero curvature,

$$D\omega_\mathcal{A} = d\omega_\mathcal{A} + \omega_\mathcal{A} \wedge \omega_\mathcal{A} = 0, \quad \omega_\mathcal{A} = \sum_{j,k} \mathcal{A}_{jk} dx_{jk} = u_\rho^\dagger du_\rho.$$

One may do the corresponding transformations (4.10)-(4.11)-(4.12) also with a fermionic function as reference (and indeed there are good physical reasons for doing so), although for simplicity we will stick to the bosonic case in all that follows as it will be seen not to be a loss in generality.

**Definition 4.8** (Transmutable anyon model). A geometric \(N\)-anyon model, determined by a braid group representation \(\rho: B_N \to \mathbb{U}(\mathcal{F})\), will be called transmutable if its associated flat principal bundle \(P \to \mathcal{C}^N\) is trivial, i.e. if it admits a smooth global section in \(\Gamma(\mathcal{C}^N, P)\) or a \(\rho\)-equivariant function \(u_\rho: \tilde{\mathcal{C}}^N \to \mathbb{U}(\mathcal{F})\). The corresponding transmuted model with bosons and the gauge potential \(\mathcal{A} = u_\rho^\dagger \nabla u_\rho\) in the kinetic energy (4.12) will be called a magnetic anyon model corresponding to \(\rho\). The geometric model formulation is also referred to as the anyon gauge and the magnetic model as the magnetic gauge.

Bosons are then obviously transmutable with the global section \(u_\rho = 1\). Naturally this leads to the following question which we have not seen duly discussed in the literature (among the exceptions we find [Blo80, Dow85, SISI88, HMS89, ISIS90, DGH99, MS95], as well as very recently [MS19] which take an alternative approach via graph configuration spaces):

**Question 4.9. Which (geometric) anyon models are transmutable?**

Only in the abelian case do we find a complete answer:
Theorem 4.10. All abelian anyon models are transmutable.

A proof was given by [Dow85] and [MS95, Theorem 2.22] (see also e.g. [MS19, Example 5]) and involves an observation that torsion in homology $H_1(B_N, \mathbb{Z})$ is the same as torsion in cohomology $H^2(B_N, \mathbb{Z})$, with an explicit global section given below, which was also implicitly used in the earlier works [Wu84, FM91].

Since a fermionic model is abelian it is thus transmutable into a bosonic model. Furthermore, two-anyon models are trivially transmutable since they are also abelian (though possibly of higher rank and reducible). Also recall that for rank $D = \dim \mathcal{F} < N - 2$ and $N > 6$, any representation of $B_N$ is necessarily abelian (Theorem 3.13) and therefore the bundle trivializes to a sum of line bundles. Geometrically, the moduli space (4.2) of possible anyon models or 2D quantum statistics splits into components of non-transmutable models (i.e. isomorphism classes of topologically non-trivial bundles) and a component of mutually transmutable models (which in turn may a priori consist of several connected components of different flat connections on a topologically trivial bundle; see however [MS19, Section 3.3]) including the circle $\rho^a \otimes 1_{\mathcal{F}}$, $a \in [0, 2)$, of abelian models and the identity point $\rho^a=0 = \rho^+_+$ of the bosonic/trivial bundle.

The classification of isomorphism classes of higher-rank flat hermitian vector bundles on $\mathbb{C}^N$ for $N > 2$ is to our knowledge an open problem. However, as pointed out in [MS95, p.10], one possible way to obtain transmutable models given non-transmutable ones is to simply take the Whitney sum of the bundles:

Definition 4.11 (Whitney sum). Given two geometric $N$-anyon models, determined by representations $\rho_1: B_N \to U(\mathcal{F}_1)$ resp. $\rho_2: B_N \to U(\mathcal{F}_2)$, their Whitney sum is the geometric $N$-anyon model determined by $\rho = \rho_1 \oplus \rho_2: B_N \to U(\mathcal{F}_1 \oplus \mathcal{F}_2)$. This is the bundle $E_\rho = E_{\rho_1} \oplus E_{\rho_2}$.

Namely, as noted in [MS95, p.10], the bundles trivialize under the taking of their $k$-fold Whitney sum $\bigoplus^k E$ for large enough $k$, because of the following finiteness theorem for the cohomology ring $H^*(\mathbb{C}^N, \mathbb{Z}) = H^*(B_N, \mathbb{Z})$ [Arn70, Arn14, Vai78]:

Theorem 4.12 (Arnold’s Cohomology Theorems [Arn14, p.201]).
Finiteness: With the exception of $H^0 \cong H^1 \cong \mathbb{Z}$, the cohomology groups $H^j(B_N, \mathbb{Z})$ are all finite. Furthermore, $H^j(B_N, \mathbb{Z}) = 0$ for $j = 2$ and for $j \geq N$.
Repetition: $H^j(B_{2N+1}, \mathbb{Z}) = H^j(B_{2N}, \mathbb{Z})$ for all $j$ and $N$.
Stability: $H^j(B_N, \mathbb{Z}) = H^j(B_{2j-2}, \mathbb{Z})$ for all $N \geq 2j - 2$.

Theorem 4.13. Given any flat vector bundle $E \to \mathbb{C}^N$, there exists a $k \in \mathbb{N}$ such that the $k$-fold Whitney sum $\bigoplus^k E$ is trivial.

Proof. Hermitian vector bundles are classified up to stable equivalence by their Chern classes, $c_j \in H^{2j}$, which trivialize in $\bigoplus^k E$ for large enough $k$ if they are only torsion. Furthermore, a hermitian vector bundle of sufficiently high rank is trivial iff it is stably trivial [AE12].

Remark 4.14. As a relevant comparison, note that with the analogous definitions, a pair of spinless fermions in dimension $d = 3$ is not transmutable to bosons. Namely, in relative coordinates

$$C^2 \cong C_0 \times \mathbb{R}^+ \times \mathbb{R}^3,$$
where the relative angles are on $C_0 \cong S^2/\sim$, the sphere with antipodal points identified. Here $\tilde{C}_0 \cong S^2$ and hence a global section $u_\rho : \tilde{C}_0 \to U(1)$ is a smooth function on the sphere with the equivariance relation

$$u_\rho(-x) = \rho(\tau)u_\rho(x), \quad x \in S^2,$$

where $\tau$ is the generator of $\pi_1(\tilde{C}_0) = \mathbb{Z}_2$ and $\rho$ the representation. For bosons ($\rho_+ = 1$) we can take $u_{\rho_+} = 1$ while for fermions ($\rho_-(\tau) = -1$) there is no such function by the Borsuk-Ulam theorem (any continuous and odd complex-valued function on the sphere has a zero somewhere). However, taking the Whitney sum, $\rho_2 = \rho_- \oplus \rho_-$, there is e.g. the global section

$$u_{\rho_2} : S^2 \to U(2), \quad u_{\rho_2}(x, y, z) = x \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + y \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} + z \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad x^2 + y^2 + z^2 = 1,$$

that trivializes the corresponding bundle.\(^{(6)}\)

4.4. Abelian anyons. The fiber for an irreducible abelian model $\rho^\alpha(\sigma_j^{-1}) := e^{i\alpha}, \alpha \in (-1, 1]$, is $\mathcal{F} = V^N = \mathbb{C}$, where the total charge $c = \alpha^N$ corresponds to the total normalized magnetic flux of all particles (mod 2). The convention we use here on the sign of $\alpha$ (conjugation symmetry) will help to recover the familiar conventions with the equivariance condition (4.3) actually being an action of $B_N$ on the right.

A global section is given by the analytic continuation to $\tilde{C}^N$ of $u_{\rho^\alpha}(x) := U(x)^\alpha$, where we use $z_j := x_{j1} + ix_{j2} \in \mathbb{C}$ and the boson-fermion transmutation

$$U : L^2_{\text{sym/asy}} \to L^2_{\text{asy/sym}}, \quad (U\Psi)(x) := \prod_{1 \leq j < k \leq N} \frac{z_j - z_k}{|z_j - z_k|} \Psi(x) = \exp(\sum_{j < k} i\phi_{jk})\Psi(x),$$

where $\phi_{jk}$ is the phase of $z_j - z_k$ which is initially defined for $(-\pi, \pi]$ and extends to a multivalued function. In fact, the combination

$$\frac{1}{\pi} \sum_{j < k} \phi_{jk} = \frac{1}{2\pi} \sum_{j \neq k} \phi_{jk} := \frac{1}{2\pi} \sum_{j \neq k} d\phi_{jk}[\bar{X}] = \frac{1}{2\pi} \int \sum_{j \neq k} d\phi_{jk}, \quad \bar{X} = [\gamma].\bar{X}_0,$$

is well defined on $\tilde{C}^N$ and maps braids to winding numbers $B_N \to \mathbb{Z}$, hence abelianizes the braid group (see [Dow86, MS95]). We then obtain $U(\bar{X}, \sigma_j) = -U(\bar{X})$ and $u_{\rho^\alpha}(\bar{X}, \sigma_j) = \rho^\alpha(\sigma_j^{-1})u_{\rho^\alpha}(\bar{X})$. Thus by the above recipe, the abelian model may be converted to a magnetic model on $\mathcal{H}_+$ with gauge potential $A = u_{\rho}^{-1}\nabla u_{\rho} = iA_{\alpha}$:

$$A_{\alpha,j}(x) = \alpha \sum_{k \neq j} \nabla j \phi_{jk} = \alpha \sum_{k \neq j} \frac{(x_j - x_k)^+}{|x_j - x_k|^2}, \quad x^+ = (x, y)^+ := (-y, x).$$

Using that $\nabla x^+ / |x|^2 = \Delta \ln |x| = 2\pi \delta(x)$, its magnetic field w.r.t. $x_j \in \mathbb{R}^2$ is

$$\text{curl} A_{\alpha,j}(x_j) = 2\pi \alpha \sum_{k \neq j} \delta(x_j - x_k).$$

\(^{(6)}\)Thus from this perspective, one could say that spin emerges as a trivialization. Also, in $d = 3$ there is the notion of dyons [Sch69] which uses magnetic monopoles to transmute statistics, however this is beyond the scope of the present article.
The corresponding kinetic energy for bosonic $\Psi_+ = u_{\rho\alpha}^{-1} \Psi_{\rho\alpha}$ is

$$\hat{T}_\alpha := \sum_{j=1}^{N} (-i \nabla_j + A_{\alpha,j})^2, \quad T_\alpha[\Psi_+] = \frac{1}{N!} \int_{\mathbb{R}^{2N}} |(-i \nabla_j + A_{\alpha,j})\Psi_+|^2.$$

In this formulation there is the possibility to extend the statistics parameter $\alpha$ to all of $\mathbb{R}$ and consider the gauge-equivalent magnetic models

$$-i \nabla + A_{\alpha+2n} = U^{-2n}(-i \nabla + A_\alpha)U^{2n}, \quad \mathcal{D}(\hat{T}_{\alpha+2n}) = U^{-2n} \mathcal{D}(\hat{T}_\alpha), \quad n \in \mathbb{Z}.$$

For $\alpha = 0$ the domain of the Friedrichs extension is $\mathcal{D}(\hat{T}_{\rho_+}) = H^2_{\text{sym}} \cap H^2(\mathbb{R}^{2N})$ which defines free bosons, while for $\alpha = 1$ it is $\mathcal{D}(\hat{T}_{\rho_-}) = U^{-1} H^2_{\text{asym}} = U^{-1} L^2_{\text{asym}} \cap H^2(\mathbb{R}^{2N})$ which defines free fermions in a bosonic representation (see [LS14, Section 2.2]).

In the reducible case $\mathcal{F} = \mathbb{C}^D$, we take a basis of joint eigenvectors s.t.

$$\rho(\sigma^{-1}_j) = S \text{diag}(e^{i\gamma_1}, \ldots, e^{i\gamma_D}) S^{-1},$$

$S \in U(D)$, and the corresponding global section is the analytic continuation to $\mathcal{C}^N$ of

$$u_\rho(x) = S \text{diag}(U(x)\gamma_1, \ldots, U(x)\gamma_D) S^{-1}.$$

Thus, if $\{v_n\}$ is a basis in $\mathcal{F}$ of joint unit eigenvectors such that $\rho(\sigma^{-1}_j)v_n = e^{i\gamma_n}v_n$, then we may consider the subspace in $H^1_\rho$ consisting of functions

$$\Psi(X) = \sum_{n=1}^{D} U(X)\gamma_n \Phi_n(X)v_n,$$

where $\Phi_n \in C^\infty_{\text{sym}}(\mathbb{R}^{2N} \setminus \Delta^N)$, for which

$$\int_{\mathcal{C}^N} |\Psi|^2 = \frac{D}{N!} \int_{\mathbb{R}^{2N}} |\Phi_n|^2, \quad T_\rho[\Psi] = \sum_{n=1}^{D} T_\gamma[\Phi_n],$$

i.e. $(\Phi_n) \in (\mathcal{H}^N_{\rho})^D$ models a collection of $N$-anyon wave functions in the magnetic representation with statistics parameters $\alpha = \gamma_n, n \in \{1, \ldots, D\}$.

For a glimpse of how such abelian models may arise in a FQHE context, see e.g. the brief introduction [Tou20, Chapter 2], and [LR16] for a more precise application (accounting for certain ambiguities [For91, For92] in the original derivation [ASW84]). Another recent realization is via quantum impurity problems, polarons and angulons [YGL+19, BLLY20].

### 4.5. Fibonacci anyons.

We consider the representation $\rho_N : B_N \to U(\mathcal{F}^N)$ on the splitting space given in Definition 3.5. Imposing that we should have exactly $N$ anyons, as well as irreducibility, we may thus take $\mathcal{F}^{N}_\tau := V^{N}_\tau$, where there is a remaining choice to be made for the total charge $c \in \{1, \tau\}$: $\mathcal{F}^{N}_1 \cong \mathbb{C}^{\text{Fib}(N-1)}$ and $\mathcal{F}^{N}_\tau \cong \mathbb{C}^{\text{Fib}(N)}$.

Presently we do not know if this model is transmutable or not. However, there is a correspondence to FQHE and CFT (Read-Rezayi states [RR99, CGT01, AKS05, Lun17]) which is similar to that for the Ising model discussed below, and which seems to suggest transmutability [AS07, TA20].
4.6. Ising anyons. Here we could work with two types of particles (the Ising anyon \( \sigma \) and the Majorana fermion \( \psi \)) explicitly present in the system and having dynamics, but to simplify matters slightly we consider \( N \) anyons of only the more interesting type \( \sigma \), with fiber \( \mathcal{F}^N = V^N = \mathbb{C}^{2(N-1)/2} \) for \( N \) odd and \( \mathcal{F}_c^N = V_c^N = \mathbb{C}^{2N/2-1} \) for \( N \) even, where either \( c = \sigma \) or \( c = \psi \).

The corresponding model, also including the fermions \( \psi \), arises in the FQHE context, supposedly as an effective model for quasiparticles modeled by a family of electron (fermion) wave functions known as Moore-Read or Pfaffian states [MR91, NW96]. Another representation for these states are as correlators of a CFT related to the Ising model. This story is beyond the scope of the present article (see e.g. [HHSV17] and [Tou20, Chapter 3.4] for recent reviews), but it suffices to note that, given positions \( x_j \in \mathbb{R}^2 \), \( j = 1, \ldots, N \) lifting to \( \tilde{X} \in \tilde{\mathcal{C}}^N \) there exists a family of states \( \{ \Psi_n(\tilde{X}) \}_{n=1}^D \) with values in an \( M \)-fermion Hilbert space \( \mathbb{C}^D = \mathcal{F}_N \leq L^2_{\text{asym}}(\mathbb{R}^{2M}) \) and some constant \( C > 0 \) such that in the limit \( M \to \infty \)

\[
\langle \Psi_n(\tilde{X}), \Psi_m(\tilde{X}) \rangle_{\mathcal{F}_N} = C \delta_{nm} + O(e^{-|x_j-x_k|})
\]

if \( |x_j - x_k| \to \infty \) [GN97, BGN11]. Thus the model appears to be transmutable by the existence of this trivialization, at least in some limit and for \( N = 3 \) or \( 4 \), and \( D = 2 \). However the connection in this family of models is the Berry connection induced from the embedding \( \mathcal{F}_N \hookrightarrow L^2_{\text{asym}}(\mathbb{R}^{2M}) \) and which is not necessarily locally flat.

Another way to obtain its transmutability would be to use that is expected (as part of the CFT correspondence) to arise as subrepresentations of NACS with gauge group \( G = SU(2) \).

4.7. NACS. There is a certain family of representations which arise in the context of Non-Abelian Chern-Simons (NACS) theory [Koh87, GMM90, Ver91, Lo93, LO94, BJP94, MTM13b, MTM13a], These may be explicitly defined as magnetic models on the trivial bundle [Koh87], and are hence transmutable. Typically one formulates the models in a holomorphic gauge for convenience.

Take a compact Lie group \( G \) (for example \( G = SU(2) \)) and a unitary representation \( \rho_1: G \to U(\mathcal{F}_1) \) (for example \( \mathcal{F}_1 = \mathbb{C}^2 \)), such that the generators \( \hat{L}_a \) of the Lie algebra \( \mathfrak{g} \) of \( G \) are mapped to operators \( \hat{L}_a \in \mathcal{L}(\mathcal{F}_1) \). Using path-ordered exponentials, we may write

\[
g = \exp_p \int_\gamma \sum_a L_a \, dg^a, \quad \rho_1(g) = \exp_p \int_\gamma \sum_a \hat{L}_a \, dg^a.
\]

Let \( \mathcal{F}_N = \bigotimes^N \mathcal{F}_1 \), with \( \rho_1 \) acting canonically on each of the factors

\[
\rho_1^{(j)}(g) = \exp_p \int_\gamma \sum_a \hat{L}_a^{(j)} \, dg^a = 1 \otimes \rho_1(g) \otimes 1, \quad \hat{L}_a^{(j)} = 1 \otimes \hat{L}_a \otimes 1.
\]

Defining

\[
U_\rho(\tilde{X}) := \exp_p \int_\Gamma \frac{1}{4\pi K} \sum_{j \neq k} \sum_a \hat{L}_a^{(j)} \hat{L}_a^{(k)} d \log(z_j - z_k), \quad \tilde{X} = [\Gamma].\tilde{X}_0,
\]

then \( \mathcal{F}_N \) carries a representation \( \rho: B_N \to U(\mathcal{F}_N) \) with a globally defined section

\[
u_\rho(\tilde{X}) := U_\rho(\tilde{X})/\|U_\rho(\tilde{X})\|_{\mathcal{F}_N}.
\]

The parameter \( 4\pi K - 2 \in \mathbb{Z} \) is known as the level of the representation, and \( U_\rho \) formally solves the Knizhnik-Zamolodchikov equations [KZ84], [Koh87], [LO94, Eq. (39)].
The representation $\rho$ will typically be reducible into irreducible blocks ('conformal blocks') $\rho_n$ and one could say that, while $\rho_n$ may not be transmutable individually, their sum $\rho$ is.

4.8. Local kinetic energy. Given a subset $\Omega \subseteq \mathbb{R}^2$ we define the local configuration space of identical particles on $\Omega$

$$C^N(\Omega) := \left(\Omega^N \setminus \Delta^N\right) / S_N = \mathcal{P}_N(\Omega),$$

the set of $N$-point subsets of $\Omega$. Let now $\Omega$ be open and simply connected (homotopic to $\mathbb{R}^2$). Similarly to the full space we may again define the covering space

$$\tilde{C}^N(\Omega) \to C^N(\Omega)$$

as the space of continuous paths in $C^N(\Omega)$ modulo homotopy equivalence. We choose as reference base point for these paths a point $X_0(\Omega) \in C^N(\Omega)$, which is exactly the ordered set of points $X_0 \in C^N(\mathbb{R}^2)$ as before, only translated and scaled to fit the domain $\Omega$. Denote a path in $C^N(\mathbb{R}^2)$ which performs this scaling and translation by $\gamma$, so that $X_0(\Omega) = [\gamma \cdot X_0]$ and $\tilde{X}_0(\Omega) := [\gamma_0 \cdot \tilde{X}_0]$. With the corresponding scaling and translation $[\gamma \gamma^{-1}]$ to match the reference points $X_0(\Omega)$ and $X_0(\Omega')$ there is then an inclusion

$$\tilde{\tilde{C}}^N(\Omega') \to \tilde{C}^N(\Omega) \quad \text{if} \quad \Omega' \subseteq \Omega.$$

By the homotopy equivalence $\Omega \sim \mathbb{R}^2$ we have

$$\pi_1(C^N(\Omega), X_0(\Omega)) \cong \pi_1(C^N(\mathbb{R}^2), X_0) = B_N,$$

$$[\gamma] \cdot \tilde{X}_0(\Omega) = [\gamma \gamma_0] \cdot \tilde{X}_0 = [\gamma_0 \gamma^{-1} \gamma_0] \cdot \tilde{X}_0 = \tilde{X}_0 \cdot \text{Ad}_{\gamma_0}[\gamma], \quad [\gamma] \in \pi_1(C^N(\Omega), X_0(\Omega)),$$

$$\tilde{X}_0(\Omega)[\eta] = [\gamma_0 \eta] \cdot \tilde{X}_0 = [\gamma_0 \eta \gamma^{-1} \gamma_0] \cdot \tilde{X}_0 = \tilde{X}_0 \cdot \text{Ad}_{\gamma_0}[\eta], \quad [\eta] \in \pi_1(C^N(\mathbb{R}^2), X_0).$$

The actions of paths and loops in $\tilde{C}^N$ thus carry over to $\tilde{\tilde{C}}^N(\Omega)$ straightforwardly, considering there only paths contained in $\tilde{C}^N(\Omega)$.

**Definition 4.15** (Function space). Given an $N$-anyon model $\rho: B_N \to U(\mathcal{F})$ and a simply connected open domain $\Omega \subseteq \mathbb{R}^2$, the space $L^2_\rho(\tilde{C}^N(\Omega); \mathcal{F})$ of ideal $N$-anyon wave functions on $\Omega$ is the closure of the space of $\rho$-equivariant smooth functions $\Psi \in C^\infty(\tilde{C}^N(\mathbb{R}^2); \mathcal{F})$ with $\tilde{\rho}(\text{supp} \Psi) \subseteq C^N(\mathbb{R}^2)$ compact,

$$\Psi(X, b^{-1}) = \rho(b) \Psi(X), \quad b \in B_N,$$

w.r.t. the norm $|| \cdot ||_{L^2_\rho}$ induced by the inner product on $C^N(\Omega)$

$$\langle \Phi, \Psi \rangle_{L^2_\rho} := \int_{C^N(\Omega)} \langle \Phi(X), \Psi(X) \rangle_{\mathcal{F}} dX = \frac{1}{N!} \int_{\Omega^N} \langle \Phi(X), \Psi(X) \rangle_{\mathcal{F}} dX.$$

**Definition 4.16** (Kinetic energy). Given an $N$-anyon model $\rho: B_N \to U(\mathcal{F})$ and a simply connected domain $\Omega \subseteq \mathbb{R}^2$, we define a Sobolev space $H^1_\rho(\tilde{C}^N(\Omega); \mathcal{F})$ of wave functions $\Psi$ with finite expected kinetic energy on $\Omega$

$$\mathcal{T}^\Omega_{\rho}[\Psi] := \int_{C^N(\Omega)} |\nabla \Psi(X)|^2_{\mathcal{F}^2 N} dX.$$

This is the Hilbert space with inner product

$$\langle \Phi, \Psi \rangle_{H^1_\rho} := \int_{C^N(\Omega)} \left( \langle \Phi(X), \Psi(X) \rangle_{\mathcal{F}} + \langle \nabla \Phi(X), \nabla \Psi(X) \rangle_{\mathcal{F}^2 N} \right) dX.$$
obtained by taking the closure of smooth \( \rho \)-equivariant functions on \( \hat{C}^N(\mathbb{R}^2) \) w.r.t. the corresponding norm,

\[
H^1_\rho(\hat{C}^N(\Omega); F) := \overline{C^\infty_{\rho,c}(\hat{C}^N; F)}^{H^1_\rho}.
\]

We may then define the \( N \)-anyon ground-state energy on \( \Omega, N \geq 2 \),

\[
E_N(\Omega) := \inf \left\{ \int_{\hat{C}^N(\Omega)} |\nabla \Psi(\hat{X})|_{F^2}^2 dX : \Psi \in H^1_\rho(\hat{C}^N(\Omega); F), \int_{\hat{C}^N(\Omega)} |\Psi|_{F}^2 dX = 1 \right\}.
\]

For convenience we also define \( E_0(\Omega) := 0 \) and the one-particle energy

\[
E_1(\Omega) := \inf \left\{ \int_{\Omega} |\nabla \Psi|_{F^2}^2 : \Psi \in H^1(\Omega; F), \int_{\Omega} |\Psi|_{F}^2 = 1 \right\} = 0.
\]

If we need to indicate the anyon model \( \rho \) we write \( E^{\rho}_N(\Omega) \).

Again, the \( \rho \)-equivariance (4.14) of functions assures that the integrands in (4.15), (4.16) and (4.17) are well defined. We will also need the corresponding one-body density:

**Definition 4.17** (One-body density). Let \( \rho: B_N \to U(\mathcal{F}) \) be an \( N \)-anyon model and \( \Omega \subseteq \mathbb{R}^2 \) a simply connected open subset. For \( \Psi \in L^2_\rho(\hat{C}^N(\Omega); F) \) a normalized \( N \)-body wave function on \( \Omega \) we define its one-body density on \( \Omega \) as the function \( \varrho_\Psi \in L^1(\Omega) \), where \( \varrho_\Psi(x) \) at \( x \in \Omega \) is the marginal of the probability distribution \( |\Psi|^2_F \) at \( X = \{x\} \cup X' \),

\[
\varrho_\Psi(x) := \int_{\hat{C}^{N-1}(\Omega \setminus \{x\})} |\Psi(\{x\} \cup X')|^2_F dX' = \frac{1}{(N-1)!} \int_{\Omega^{N-1}} |\Psi(x, x')|^2_{F^2} dx'.
\]

It may equivalently be defined via its integral over a subset \( \Omega' \subseteq \Omega \)

\[
\int_{\Omega'} \varrho_\Psi = \int_{\hat{C}^N(\Omega)} \sum_{j=1}^{N} I_{\{x_j \in \Omega'\}} |\Psi|_{F}^2 dX,
\]

(the expected number of particles on \( \Omega' \)) and the Lebesgue differentiation theorem, taking \( \Omega' = B_\varepsilon(x), \varepsilon \to 0 \).

**Definition 4.18** (Energy density). Similarly, we define for \( \Psi \in H^1_\rho(\hat{C}^N(\Omega); F) \) a local expected kinetic energy on \( \Omega' \subseteq \Omega \)

\[
T^{\Omega'_\subseteq \Omega}[\Psi] := \int_{\Omega'} T_{\rho, \Psi}(x) dx = \int_{\hat{C}^N(\Omega)} \sum_{j=1}^{N} I_{\{x_j \in \Omega'\}} |\nabla_j \Psi|_{F^2}^2 dX,
\]

and its corresponding kinetic energy density \( T_{\rho, \Psi} \in L^1(\Omega) \),

\[
T_{\rho, \Psi}(x) := \int_{\hat{C}^{N-1}(\Omega \setminus \{x\})} |\nabla_x \Psi(\{x\} \cup X')|^2_{F^2} dX'.
\]

Occasionally we shall suppress the \( F \) on the norms to make the notation less heavy.
5. Statistical repulsion

Given an $N$-anyon model $\rho: B_N \rightarrow U(\mathcal{F})$ with associated exchange operators $\{U_p\}_{p=0}^{N-2}$, we may define an exchange parameter for $p$ enclosed particles

$$\beta_p := \min\{\beta \in [0, 1] : e^{i\beta\pi} \text{ or } e^{-i\beta\pi} \text{ is an eigenvalue of } U_p\},$$

and for $n \in \{2, 3, \ldots, N\}$ involved particles (i.e. worst case of up to $n-2$ enclosed)

$$\alpha_n := \min_{p \in \{0, 1, 2, \ldots, n-2\}} \beta_p.$$

To a simple two-particle exchange $U_0 = \rho(\sigma_1)$ is then associated the parameter $\alpha_2 = \beta_0$. In the case of irreducible abelian anyons, with exchange phases $U_p = e^{i(2p+1)\alpha\pi}$, we have

$$\alpha_n = \min_{p \in \{0, 1, \ldots, n-2\}} \min_{q \in \mathbb{Z}} |(2p+1)\alpha - 2q|.$$

The goal of this section is to derive explicit lower bounds for the local kinetic energy in terms of the above model-dependent parameters which quantify the strength of repulsion among the particles due to their statistics (exchange phases).

5.1. Relative configuration space. Let $\Omega \subseteq \mathbb{R}^2$ be a convex open set. Consider the symmetric exchange of two particles inside $\Omega$ with the remaining $N-2$ particles fixed on $\Omega$. It is convenient to switch to relative and center-of-mass coordinates for the pair $(x_1, x_2)$:

$$r = x_1 - x_2, \quad R = \frac{1}{2}(x_1 + x_2) \in \Omega, \quad x' = (x_3, \ldots, x_N) \in \Omega^{N-2},$$

$$x(r; R; x') := (R + \frac{1}{2}r, R - \frac{1}{2}r, x_3, \ldots, x_N) \in \mathbb{R}^{2N}.$$

To keep all points in $\Omega$ we must restrict the values of $r \in \mathbb{R}^2$ to the subset

$$\Omega_{rel}(R; x') = \{r \in \mathbb{R}^2 : x(r; R; x') \in \Omega^N \setminus \Delta^N\}.$$

For these we have a surjection

$$X(r; R; x') := \text{pr}(x(r; R; x')) = \{R + \frac{1}{2}r, R - \frac{1}{2}r, x_3, \ldots, x_N\} \in \mathcal{C}^N(\Omega).$$

However, $\Omega_{rel}$ is not simply connected. For example, in the case $\Omega = \mathbb{R}^2$ we have

$$\Omega_{rel} = \mathbb{R}^2 \setminus \{0, \pm 2(x_3 - R), \ldots, \pm 2(x_N - R)\}$$

and

$$X(\pm e_1; \frac{3}{2}e_1; 3e_1, \ldots, Ne_1) = X_0 = X_0(\mathbb{R}^2),$$

while if $\bar{\Omega} = Q_0 = [0, 1]^2$ is the unit square then $\Omega_{rel} \subseteq [-1, 1]^2$ is a rectangle centered at 0 with up to $2N + 1$ points removed symmetrically around the origin (one point being the origin).

Note that several configurations $(r; R; x')$ map to the same point $X(r; R; x')$, and we have for example the antipodal symmetry

$$X(-r; R; x') = X(r; R; x'), \quad r \in \Omega_{rel}(R; x').$$

Upon fixing $R$ and $x'$ and a base point $r_0 \in \Omega_{rel}$ we may extend $r \mapsto X(r; R; x')$ to a smooth map on the covering space $\tilde{\Omega}_{rel}(R; x') \to \Omega_{rel}(R; x')$

$$\tilde{r} \mapsto \tilde{X}(\tilde{r}; R; x') \mapsto X(r; R; x').$$
such that $\tilde{X}(\tilde{r}; \mathbf{R}; x', \sigma) = \tilde{X}(\tilde{r}; \mathbf{R}; x') \cdot b$ for $\sigma \in S_{N-2}$ and some $b \in B_N$. By composing with $\Psi \in C^\infty_{\rho, \epsilon}$ we then have a smooth map

$$\psi : \tilde{\Omega}_{rel} \to \mathcal{F}, \quad \psi(\tilde{r}) := \Psi(\tilde{X}(\tilde{r}; \mathbf{R}, x'))$$

(5.1)
such that

$$\partial_k \psi(\tilde{r}) = \sum_{j,q} \frac{\partial \Psi}{\partial x_{j,q}} \frac{\partial x_j}{\partial r_k}(\tilde{r}) = \pm \frac{1}{2} \left( \frac{\partial \Psi}{\partial x_{1k}} - \frac{\partial \Psi}{\partial x_{2k}} \right)(\tilde{X}(\tilde{r})),$$

where the sign depends on the order of exchange of particles 1 and 2 done in $\tilde{r}$. Namely, the action of any $\beta \in \pi_1(\Omega_{rel}, \mathbf{r}_0)$ carries over to a corresponding one on $\tilde{X}$,

$$\tilde{X}(\tilde{r}; \beta; \mathbf{R}; x') = \tilde{X}(\tilde{r}; \mathbf{R}; x') \cdot b, \quad \text{some } b = b(\beta) \in B_N,$$

so that, with the transformation rule (4.8),

$$\partial_k \psi(\tilde{r}; \beta) = \pm \frac{1}{2} \rho(b^{-1}) \left( \frac{\partial \Psi}{\partial x_{1k}} - \frac{\partial \Psi}{\partial x_{2k}} \right)(\tilde{X}(\tilde{r})), \quad j' = \partial r(p)[j],$$

and hence

$$|\nabla \psi(\tilde{r}; \beta)|^2 = \frac{1}{4} |\nabla_\Psi - \nabla_\beta|^2$$

for any $\beta \mapsto b \mapsto \partial \mathbf{r}(b)$ which leaves invariant the partition $\{1, 2\}, \{3, 4, \ldots, N\}$.

Furthermore, given any $r = |\mathbf{r}| > 0$ such that the full circle $rS^1$ is contained in $\Omega_{rel}$, we may consider the corresponding lift of this circle to a subset in $\tilde{\Omega}_{rel}$ isomorphic to $\mathbb{R}$. By the convexity of $\Omega$, this will be possible on any of the annuli $A_p \subseteq \Omega_{rel}$ defined by $r \in I_p$, $I_0 = (r_0 := 0, r_1), I_1 = (r_1, r_2), \ldots, I_{M-1} = (r_{M-1}, r_M), I_M = (r_M, r_{\text{max}}), 0 \leq M \leq N-2$, with $r_{\text{max}} := \text{dist}(0, \partial \Omega_{rel}) = 2 \text{dist}(\mathbf{R}, \partial \Omega)$, and having written the relative distances to the other particles in increasing order

$$0 \leq r_1 = 2|x_{2+\sigma(1)} - \mathbf{R}| \leq r_2 = 2|x_{2+\sigma(2)} - \mathbf{R}| \leq \ldots \leq r_{N-2} = 2|x_{2+\sigma(N-2)} - \mathbf{R}|$$

for a suitable permutation $\sigma \in S_{N-2}$.

Comparing now to the two-particle relative configuration space $\tilde{C}_0 \to C_0$ in Example 4.2, we have the curve $\mathbb{R} \to S^1 \to S^1/\sim = C_0$ (with $x \sim -x$) of relative angles

$$\varphi \mapsto \mathbf{e}(\varphi) : = (\cos \varphi, \sin \varphi) \mapsto \{ \pm (\cos \varphi, \sin \varphi) \},$$

which after fixing a base point $\mathbf{e}(0) \in C_0$ lifts to a unique curve $\mathbb{R} \to \tilde{C}_0$

$$\varphi \mapsto \tilde{\mathbf{e}}(\varphi) : = [\gamma_\varphi, \tilde{\mathbf{e}}(0), \gamma_\varphi(t) : = \{ \pm \mathbf{e}(\varphi t) \}, \ t \in [0, 1].$$

In the same way there is for $r \in I_p$ a corresponding smooth curve $\mathbb{R} \to \tilde{C}^N(\Omega) \to C^N(\Omega)$ obtained by taking $\tilde{r} = r\tilde{\mathbf{e}}(\varphi)$ with the remaining coordinates fixed, i.e.

$$\varphi \mapsto \tilde{X}(r\tilde{\mathbf{e}}(\varphi); \mathbf{R}; x') \to X(r\mathbf{e}(\varphi); \mathbf{R}; x').$$

Given a base point on the curve, for example

$$\tilde{X}(r\tilde{\mathbf{e}}(0); \mathbf{R}; x') = [\eta].\tilde{X}_0(\Omega) = [\eta \gamma_{\Omega}].\tilde{X}_0$$

for some fixed path $\eta$ from $X_0(\Omega)$ to $X(r\mathbf{e}(0); x')$ (possibly via $X(r_0; \mathbf{R}; x')$), this curve can also be represented as

$$\varphi \mapsto \tilde{X}(r\tilde{\mathbf{e}}(\varphi); \mathbf{R}; x') = [\Gamma_\varphi].\tilde{X}(r\tilde{\mathbf{e}}(0); \mathbf{R}; x').$$
using the path in $C^N(\Omega)$
\[ \Gamma_\varphi(t) := \{ R + \frac{1}{2}r\gamma_\varphi(t), R - \frac{1}{2}r\gamma_\varphi(t), x_3, \ldots, x_N \}, \ t \in [0,1]. \]

We may then define a smooth function $v : \mathbb{R} \rightarrow \mathcal{F}$ as the restriction of $\Psi$ to this particular curve,
\[ v(\varphi) := \psi(\bar{e}(\varphi)) = \Psi(\bar{X}(\bar{e}(\varphi); R; x')), \quad (5.2) \]

with
\[ v'(\varphi) = \nabla \psi \cdot \bar{e}'(\varphi) = \pm \left[ \frac{1}{2} \left( \frac{\partial \Psi}{\partial x_{11}} - \frac{\partial \Psi}{\partial x_{21}} \right)(-r\sin \varphi) + \frac{1}{2} \left( \frac{\partial \Psi}{\partial x_{12}} - \frac{\partial \Psi}{\partial x_{22}} \right)(r\cos \varphi) \right]. \]

We also compute
\[ \partial_\psi(r\bar{e}(\varphi)) = \nabla \psi \cdot \mathbf{e}(\varphi) = \pm \left[ \frac{1}{2} \left( \frac{\partial \Psi}{\partial x_{11}} - \frac{\partial \Psi}{\partial x_{21}} \right) \cos \varphi + \frac{1}{2} \left( \frac{\partial \Psi}{\partial x_{12}} - \frac{\partial \Psi}{\partial x_{22}} \right) \sin \varphi \right], \]

which combines with the previous expression to yield
\[ |\nabla \psi|^2 = |\partial_\psi(r\bar{e}(\varphi))|^2 + \frac{1}{r^2} |v'(\varphi)|^2 = \frac{1}{4} |\nabla_1 \Psi - \nabla_2 \Psi|^2. \quad (5.3) \]

A two-particle exchange on $C_0$ with winding number $n \in \mathbb{Z}$ yields $\mathbf{e}(n\pi) = (-1)^n\mathbf{e}(0)$ and
\[ \bar{e}(n\pi) = [\gamma_{n\pi}]\mathbf{e}(0) = [\gamma_{\pi}]^n \mathbf{e}(0) = \bar{e}(0).[\gamma_{\pi}]^n, \]

with $[\gamma_{\pi}]$ the generator of the respective braid group $\pi_1(C_0) = \mathbb{Z}$ (base point $\{ \pm \mathbf{e}(0) \}$). With our corresponding curve on $C^N(\Omega)$
\[ \bar{X}(\bar{e}(n\pi); R; x') = [\Gamma_{n\pi}]\bar{X}(\bar{e}(0); R; x') = \bar{X}(\bar{e}(0); R; x').b^n, \]

with $b = \text{Ad}_{[\gamma_{\pi}]}^{-1}([\Gamma_{\pi}]) \in B_N$ the generator of exchange. In this case we perform continuous two-particle exchanges which are not necessarily simple but may, depending on $r$, involve other particles. Topologically we have a situation similar to (3.2), i.e. $[\Gamma_{\pi}] \sim b \sim \Sigma_p$, where there are $p$ enclosed particles if $r \in I_p$. Thus the $p$-equivariance of $\Psi \in C^\infty(\Omega)$ implies
\[ v(n\pi) = \Psi(\bar{X}(\bar{e}(n\pi); R; x')) = \Psi(\bar{X}(\bar{e}(0); R; x').b^n) = \rho(b^{-n})v(0) \]

where $\rho(b) \sim \rho(\Sigma_p) = U_p$, the corresponding two-anyon exchange operator.

5.2. Poincaré inequality. The statistical repulsion between a pair of anyons boils down to the following simple version of the Poincaré inequality.

**Lemma 5.1** (Poincaré inequality for semi-periodic functions). Let $\psi \in H^1([0, \pi]; \mathcal{F})$ satisfy the semi-periodic boundary condition
\[ \psi(\pi) = U\psi(0), \quad (5.4) \]

for some $U \in \mathcal{U}(\mathcal{F})$, then
\[ \int_0^\pi |\psi'(\varphi)|^2 d\varphi \geq \lambda_0(U)^2 \int_0^\pi |\psi(\varphi)|^2 d\varphi, \quad (5.5) \]

where
\[ \lambda_0(U) := \inf\{ \lambda \in [0,1] : e^{i\lambda\pi} \text{ or } e^{-i\lambda\pi} \text{ is an eigenvalue of } U \}. \quad (5.6) \]
Proof. We consider the operator $D$ on the space $L^2([0, \pi]; \mathcal{F})$ with $D\psi(\varphi) = -i\psi'(\varphi)$ and domain given by functions $\psi \in H^1([0, \pi]; \mathcal{F})$ satisfying the b.c. (5.4). This is a self-adjoint operator and its spectrum is given explicitly by $\lambda_{n,k} = \mu_n + 2k \in \mathbb{R}$, $n \in \{1, \ldots, \dim \mathcal{F}\}$, $k \in \mathbb{Z}$, with corresponding orthonormal eigenfunctions $u_{n,k}(\varphi) = e^{i\lambda_{n,k}\varphi}v_n$, where $Uv_n = e^{i\mu_n\varphi}v_n$, $\mu_n \in [-1, 1]$, a basis in $\mathcal{F}$ of eigenvectors of $U$. By the spectral theorem, the l.h.s. of (5.5) is then

$$\|D\psi\|^2 = \langle \psi, D^2\psi \rangle \geq \inf_{n,k} \lambda_{n,k}^2 \|\psi\|^2 = \lambda_0(U)^2 \|\psi\|,$$

according to the definition (5.6).

For completeness we also state a version suitable for magnetic anyon models modeled using bosons:

**Lemma 5.2** (Gauge-transformed Poincaré inequality). Let $\psi \in H^1([0, \pi]; \mathcal{F})$ satisfy the periodic boundary condition $\psi(\pi) = \psi(0)$, and assume $A \in C([0, \pi]; u(\mathcal{F}))$, where $u(\mathcal{F})$ are the anti-hermitian operators on $\mathcal{F}$. Then

$$\int_0^\pi \left| \left( \frac{d}{d\varphi} + A \right) \psi(\varphi) \right|^2 d\varphi \geq \lambda_0(U)^2 \int_0^\pi |\psi(\varphi)|^2 \frac{d\varphi}{\mathcal{F}},$$

(5.7)

where $U = \exp_p(\int_0^\pi A) \in U(\mathcal{F})$ is the path-ordered exponential of $A$.

Proof. Define the function $v(\varphi) = U(\varphi)\psi(\varphi)$, where $U(\varphi) := \exp_p(\int_0^\varphi A(t)dt)$, i.e. (by definition) $U(0) = I$ and $U'(\varphi) = A(\varphi)U(\varphi)$ for all $\varphi \in [0, \pi]$. Then $v'(\varphi) = \psi'(\varphi) + A(\varphi)\psi(\varphi)$, so the l.h.s. of (5.7) is $\int_0^\pi |v'|^2$, and furthermore $v$ satisfies the b.c.

$v(\pi) = U(\pi)\psi(\pi) = U\psi(0) = Uv(0)$.

Now we may apply Lemma 5.1 to $v$ and use $|v| = |\psi|$ to conclude the lemma.

\[ \square \]

5.3. Diamagnetic inequality. Consider estimates due to a polar decomposition of $\mathcal{F}$:

**Lemma 5.3** (Angular diamagnetic inequality). Let $\psi \in H^1([0, \pi]; \mathcal{F})$, $|\psi|_{\mathcal{F}} = (\sum_k |\psi_k|^2)^{1/2}$ where $\psi_k$ are the components in an ON basis of $\mathcal{F}$. Then $|\psi|_{\mathcal{F}} \in H^1([0, \pi]; \mathbb{R}^+)$,

$$|\psi'(\varphi)|^2 \geq |\psi'|^2_{\mathcal{F}}(\varphi)^2$$

(5.8)

pointwise a.e. on $[0, \pi]$, and thus

$$\int_0^\pi |\psi'(\varphi)|^2 d\varphi \geq \int_0^\pi |\psi'|^2_{\mathcal{F}}(\varphi)^2 d\varphi.$$  

(5.9)

Proof. This is a direct consequence of the usual diamagnetic inequality for functions with values in $\mathbb{C} \cong \mathbb{R}^2$ (see e.g. [LL01, Theorem 6.17]), i.e. pointwise a.e.

$$|(u_1, u_2)|^2 = |u'_1|^2 + |u'_2|^2 \geq \sqrt{|u_1^2| + |u_2^2|^2},$$

where the r.h.s. is zero if $u_1^2 + u_2^2 = 0$. Applied inductively,

$$|u_1|^2 + \ldots + |u_{n-1}|^2 + |u_n|^2 \geq \sqrt{|u_1^2 + \ldots + u_{n-1}^2|^2 + |u_n|^2} \geq \sqrt{|u_1^2 + \ldots + u_n^2|^2},$$
etc., extends to $\mathcal{F} \cong \mathbb{C}^D \cong \mathbb{R}^{2D}$. Thus, pointwise a.e.

$$|\psi'|^2_F = \sum_k |\psi_k'|^2 \geq ||\psi'||^2_F,$$

and since $|\psi|_F \in L^2$ and $\psi' \in L^2$, we also have $|\psi|_F \in H^1$ and the bound (5.9).

**Lemma 5.4 (Spatial diamagnetic inequality).** Let $\Omega \subseteq \mathbb{R}^2$ be a simply connected Lipschitz domain. Let $\Psi \in H^1_\rho(\tilde{\mathcal{C}}^N(\Omega); \mathcal{F})$, $|\Psi|_F = (\sum_k |\psi_k|^2)^{1/2}$, then $|\Psi|_F \in H^1(\tilde{\mathcal{C}}^N(\Omega); \mathbb{R}_+)$ extends to a function $|\Psi|_F \in H^1_s(\tilde{\mathcal{C}}^N(\Omega; \mathbb{R}_+))$,

$$||\nabla \Psi(X)\|^2_{L^2} \geq ||\nabla |\Psi|_F(X)\|^2_{L^2}$$

pointwise a.e. on $\mathcal{C}^N(\Omega)$, and

$$\int_{\mathcal{C}^N(\Omega)} |\nabla \Psi(X)|^2_{L^2} \, dX \geq \int_{\mathcal{C}^N(\Omega)} |\nabla |\Psi|_F(X)|^2_{L^2} \, dX = \frac{1}{N!} \int_{\Omega^N} |\nabla |\Psi|_F(x)|^2_{L^2} \, dx. \quad (5.10)$$

**Proof.** The inequality (5.8) may be replaced by a derivative in any direction,

$$\sum_{j,k} |\partial_{j,k} \Psi|^2 \geq \sum_{j,k} |\partial_{j,k} |\Psi|_F|^2.$$

We therefore have a pointwise a.e. inequality on $\tilde{\mathcal{C}}^N(\Omega)$ as above,

$$||\nabla \Psi|^2_{L^2} = \sum_{j,k} |\partial_{j,k} \Psi|^2 \geq \sum_{j,k} |\partial_{j,k} |\Psi|_F|^2 = ||\nabla |\Psi|_F|^2_{L^2}.$$

Furthermore both sides of this inequality are invariant under the action of $B_\Psi$ due to the $\rho$-equivariance of $\Psi$, and thus descend to $\mathcal{C}^N(\Omega)$. Hence we have a function $|\Psi|_F \in H^1(\mathcal{C}^N(\Omega; \mathbb{R}_+)$ satisfying the bound (5.10). This extends to a function $|\Psi|_F \in H^1_s(\mathcal{C}^N(\Omega^N \setminus \Delta^N; \mathbb{R}_+)$). We may also consider $|\Psi|_F$ as a function in $L^2_s(\mathcal{C}^N(\Omega)) = L^2(\mathcal{C}^N(\Omega^N \setminus \Delta^N)$ and its distributional derivative $\nabla |\Psi|_F \in H^{-1}(\Omega^N)$. Since $H^1(\mathbb{R}^{2N} \setminus \Delta^N) = H^1(\mathbb{R}^{2N} \setminus \Delta^N)$ this function extends to $H^1_s(\mathcal{C}^N(\Omega))$; cf. [LS14, Lemmas 3 and 4] and more generally [LLN19, Appendix B]. In fact $\Psi \in H^1_\rho$ is the limit of $\Psi_n \in C^\infty_\rho$ and the inequality (5.10) for such functions implies for the limit $|\Psi|_F \in H^1(\Omega^N) \cap H^1(\mathbb{R}^{2N} \setminus \Delta^N) = H^1(\Omega^N)$.}

The diamagnetic inequality shows that $\inf \text{spec } \tilde{T}_\rho \geq \inf \text{spec } \tilde{T}_{\rho_+}$, i.e. bosons always provide a lower bound to the energy, since given any $\Psi \in H^1_\rho(\tilde{\mathcal{C}}^N(\Omega); \mathcal{F})$ or sequence $\Psi_n \in C^\infty_\rho$ converging to $\Psi$ we obtain

$$T^\Omega_{\rho_+}[\Psi_n] \geq T^\Omega_{\rho_+}[||\Psi_n||_F] \geq E^\rho_+(\Omega),$$

and thus $E^\rho_+(\Omega) \geq E^\rho_+(\Omega)$. Furthermore, we obtain by finiteness of the integral

$$T^\Omega_{\rho_+}[||\Psi||_F] = T_0[\Phi] = \frac{1}{(N-1)!} \int_{\Omega^{N-1}} \int_{\Omega} |\nabla x \Phi(x, x')|^2 dx' dx, \quad \Phi(x) = ||\Psi(X)||_F,$$

the Sobolev embedding $||\Psi(\cdot, x')||_F \in L^p(\Omega), 2 \leq p < \infty$, for a.e. $x' \in \Omega^{N-1}$ [LL01, Theorems 8.5 and 8.8].
5.4. Hardy inequality. We are now ready to prove our first main result concerning the dynamics of the anyon gas, which extends the many-body Hardy inequality [LL18, Theorem 1.3] for irreducible abelian anyons in the magnetic representation to arbitrary geometric anyon models.

**Theorem 5.5 (Hardy inequality for non-abelian anyons).**
Let \( \rho: B_N \to \mathcal{U}(\mathcal{F}) \) be an \( N \)-anyon model with exchange parameters \( (\beta_p)_{p \in \{0,\ldots,N-2\}} \) and \( \alpha_N = \min_{p \in \{0,\ldots,N-2\}} \beta_p \), and let \( \Omega \subseteq \mathbb{R}^2 \) be open and convex. Then, for any \( \Psi \in H^1_p(\tilde{\mathcal{G}}^N(\Omega); \mathcal{F}) \),

\[
\int_{\mathcal{F}^N(\Omega)} \| \nabla \Psi \|_{\ell^2}^2 dX \geq \frac{1}{N} \int_{\mathcal{F}^N(\Omega)} \left| \sum_{j=1}^{N} \nabla_j \Psi \right|^2 dX
\]

\[+ \frac{4}{N} \int_{\mathcal{F}^N(\Omega)} \sum_{j<k} \left( \left| \partial_{r_{jk}} \Psi \right|^2 + \sum_{p=0}^{M} \beta_p^2 \mathbb{1}_{A_p}(r_{jk}) \frac{\left| \Psi \right|^2}{r_{jk}} \right) dX \]

\[\geq \pi \left( j'_{\alpha_N} \right)^2 \frac{4}{N} \int_{\mathcal{F}^N(\Omega)} \sum_{j<k} \frac{\mathbb{1}_{\Omega \cap \Omega}(x_j, x_k)}{4\pi \delta(x_j, x_k)^2} \left| \Psi \right|^2 dX,
\]

where \( 0 \leq M \leq N - 2 \) and the relative annuli \( A_p \) (defined in Section 5.1) depend on the relative positions of all particles, the reduced support

\[\mathbb{1}_{\Omega \cap \Omega}(x_j, x_k) := \mathbb{1}_{B_{2\delta}(x_{jk})}(0) = \sum_{p=0}^{M} \mathbb{1}_{A_p}(r_{jk})\]

defined in terms of pairwise coordinates and distances

\[r_{jk} := x_j - x_k, \quad X_{jk} := (x_j + x_k)/2, \quad r_{jk} := |r_{jk}|, \quad \delta(x) := \text{dist}(x, \partial \Omega),\]

while \( j'_{\nu} \) denotes the first positive zero of the derivative of the Bessel function \( J_\nu \), satisfying

\[\sqrt{2\nu} \leq j'_{\nu} \leq \sqrt{2\nu(1+\nu)}, \quad j'_0 := 0. \quad (5.11)\]

**Proof.** We adapt the strategy of the proof of [LL18, Theorem 1.3] to functions living on the covering space. By definition of the space \( H^1_p \), w.l.o.g. \( \Psi \in C^\infty_{\rho,c} \) where \( \rho(\text{supp} \Psi) \) is compact in \( \mathcal{G}^N \), i.e. \( \Psi \) is zero close to \( \Delta^N \). We may also use invariance of \( |\nabla \Psi|^2 = \sum_j |\nabla_j \Psi|^2 \) under the action of \( B_N \) and thus of \( S_N \) to replace \( N! \) copies of the integral \( \int_{\mathcal{F}^N(\Omega)} \) by \( \int_{\Omega^N} \).

Now use the parallelogram identity

\[\sum_{j=1}^{n} |z_j|^2 = \frac{1}{n} \sum_{1 \leq j < k \leq n} |z_j - z_k|^2 + \frac{1}{n} \sum_{j=1}^{n} |z_j|^2,
\]

valid for any \( z = (z_j)_{j} \in \mathcal{F}^n, n \in \mathbb{N} \), and Hilbert space \( \mathcal{F} \), to rewrite the kinetic energy

\[
\int_{\Omega^N} \sum_{j=1}^{N} |\nabla_j \Psi|^2 dX = \frac{1}{N} \sum_{1 \leq j < k \leq N} \int_{\Omega^{N-2}} \int_{\Omega^2} |\nabla_j \Psi - \nabla_k \Psi|^2 dX_j dX_k \prod_{l \neq j, k} dX_l
\]

\[+ \frac{1}{N} \int_{\Omega^N} \sum_{j=1}^{N} \left| \sum_{j=1}^{N} \nabla_j \Psi \right|^2 dX. \quad (5.12)\]
We consider the term
\[ \int_{\Omega^{N-2}} \int_{\Omega^2} |\nabla_1 \Psi - \nabla_2 \Psi|^2 dx_1 dx_2 dx' = \int_{\Omega^{N-2}} \int_{\Omega_{reel}(\mathbb{R}, x')} |\nabla_1 \Psi - \nabla_2 \Psi|^2 dr dR dx' \]
where, following the strategy of Section 5.1, we make the 1-to-1 coordinate transformation to relative coordinates \( R \in \Omega, r \in \Omega_{reel} \) with measure \( dx_1 dx_2 = dr dR \). Given a configuration \( x' \in \Omega^N \) and \( R \in \Omega \), by (5.3) it remains then to study the integral
\[ \int_{\Omega_{reel}(\mathbb{R}, x')} |\nabla_1 \Psi - \nabla_2 \Psi|^2 dr = 4 \int_{\Omega_{reel}(\mathbb{R}, x')} |\nabla \psi(\tilde{r})|^2 dr = 4 \sum_{p=0}^{M} \int_{A_p} |\nabla \psi(\tilde{r})|^2 dr + 4 \int_{A^c} |\nabla \psi(\tilde{r})|^2 dr \]
with the smooth function \( \psi \) in (5.1) defined on the covering space \( \Omega_{reel} \) and \( A^c := \Omega_{reel} \setminus \bigcup_{p} A_p \).

On each annulus \( A_p \) we write in terms of polar coordinates
\[ \int_{A_p} |\nabla \psi(\tilde{r})|^2 dr = \int_{r=r_p}^{r_{p+1}} \int_{\varphi=0}^{2\pi} \left( |\partial_\varphi \psi|^2 + \frac{1}{r^2} |\partial_r \psi|^2 \right) d\varphi rdr. \]

For the first term we use the pointwise diamagnetic inequality (5.8)
\[ |\partial_\varphi \psi| \geq |\partial_r \psi| \]
while in the second we may at fixed \( r \in I_p \) identify \( |\partial_\varphi \psi|^2 = |\partial r \psi|^2 \) for the smooth function (5.2) of the relative angle \( \varphi \in \mathbb{R} \) which obeys the semi-periodicity
\[ v(n\pi) = \rho(b^{-1})^n v(0), \quad n \in \mathbb{Z}, \]
with \( \rho(b^{-1}) \sim U_p^{-1} \). We stress that the choice of a base point \( \tilde{X}(r \tilde{e}(0); R; x') = [\eta], \tilde{X}_0(\Omega) = [\eta_\gamma],[.] \tilde{X}_0 \) for the curve used in the definition of \( v \) (respectively \( \psi \)) is arbitrary since if we instead took \([\eta_\gamma],[.] \tilde{X}_0(\Omega) = (\eta), \tilde{X}_0(\Omega)) \) they differ by the braid \( B = [\gamma_\Omega^{-1} \eta^{-1} \eta' \gamma_\Omega] \) and thus, as long as it preserves our division of the particles, we have
\[ \partial_\varphi [\psi(\tilde{X}(r \tilde{e}(\varphi); R; x')).B] = \rho(B^{-1}) \partial_\varphi [\psi(\tilde{X}(r \tilde{e}(\varphi); R; x'))], \]
leaving \( |\partial r \psi|^2 \) unchanged. In the same way we have \( v(\varphi + \pi) = \rho(b^{-1}) v(\varphi) \) and thus
\[ \int_0^{2\pi} |\partial r \psi|^2 = 2 \int_0^\pi |\partial r \psi|^2 \geq 2 \lambda_0(U_p^{-1})^2 \int_0^{2\pi} |v|^2 = \lambda_0(U_p)^2 \int_0^{2\pi} |v|^2, \]
by the Poincaré inequality, Lemma 5.1. Therefore
\[ \int_{A_p} |\nabla \psi|^2 dr \geq \int_{r=r_p}^{r_{p+1}} \int_0^{2\pi} \left( |\partial_r \psi|^2 + \frac{\lambda_0(U_p)^2}{r^2} |u|^2 \right) d\varphi rdr. \]

The remaining term can be bounded using the diamagnetic inequality,
\[ \int_{A^c} |\nabla \psi|^2 \geq \int_{A^c} |\partial_r \psi|^2 \geq \int_{A^c} |\partial_r \psi|^2. \]

Summarizing,
\[ \int_{\Omega^N} |\nabla_1 \Psi - \nabla_2 \Psi|^2 dx \geq 4 \int_{\Omega^N} \left( |\partial_r |\Psi|^2 \right) + \sum_{p=0}^{M} 1_{A_p} \beta^2_p |\Psi|^2 \]
which, after bounding uniformly \( \lambda_0(U_p) = \beta_p \geq \alpha_N \) in terms of the exchange parameters of the anyon model, simplifies further with the remaining support \( \sum_{p=0}^{M} 1_{A_p}(r) = 1_{B_{r_{\max}}}(0)(r) = 1_{\Omega_0}(x_1, x_2) \).

The other terms in (5.12) involving the pairs \((x_j, x_k)\) are exactly analogous in terms of corresponding relative coordinates \((r_{jk}, R_{jk})\). After collecting these we may finally use again the \( B_N \)-invariance of the collection of all the terms to write the integrals on \( C^N(\Omega) \). This proves the first bound of the theorem.

The second bound of the theorem follows exactly as in [LL18], passing again to polar coordinates on \( \Omega_{rel} \) and considering \( u(r) = |\Psi(X)|_2 \) in

\[
\int_0^{r_{\max}} \left( |u'|^2 + \frac{\alpha_N^2}{r^2} |u|^2 \right) r^2 dr \geq \lambda \int_0^{r_{\max}} |u|^2 r^2 dr.
\]

The minimizer of the Rayleigh quotient satisfies the Bessel equation

\[-u''(r) - u'(r)/r + \nu^2 u(r)/r^2 = \lambda u(r), \quad u(0) = 0, \quad u'(r_{\max}) = 0,
\]

i.e. \( u(r) = j_{\nu}(j_{\nu} r/r_{\max}) \), with the eigenvalue \( \lambda = (j_{\nu}^2)/r_{\max}^2 \), \( \nu = \alpha_N \in [0, 1] \). The bounds (5.11) for \( j_{\nu}' \) are given in [LL18, Appendix A].

Note that Theorem 5.5 implies the simpler inequality on \( \Omega = \mathbb{R}^2 \), \( \Psi \in H^1_\rho \),

\[
T_\rho[\Psi] \geq C_N \sum_{j<k} \frac{1}{N!} \int_{\mathbb{R}^{2N}} \frac{|\Psi|^2}{|x_j - x_k|^2} dx, \quad C_N = \frac{4\alpha_N^2}{N}.
\]  

**Proposition 5.6** (Counterexamples for \( \alpha_2 = 0 \), global case). A Hardy inequality of the type (5.13) cannot hold with any positive constant \( C_N \) in the case that \( \rho : B_N \rightarrow U(2) \) is abelian with \( \alpha_2 = 0 \) or \( \rho : B_3 \rightarrow U(2) \) is the non-abelian Burau representation given in Section 3.7 with \( w = 1 \) (\( \alpha_2 = \alpha_3 = 0 \)).

**Proof.** For \( \rho : B_N \rightarrow U(2) \) abelian with \( \alpha_2 = 0 \) we may take a joint unit eigenvector \( v_0 \in C^D \) such that \( \rho(\sigma_j)v_0 = v_0 \forall j \) and consider the state \( \Psi = \Phi_\varepsilon v_0 \in H^4_{\text{sym}} \), where \( \Phi_\varepsilon \in H^4_{\text{sym}} \cap C^{\infty}_c(\mathbb{R}^{2N} \setminus \Delta^N) \) approximates in \( H^1 \) the product state \( \Phi_0(x) = e^{-|x|^2} = \prod_{j=1}^N e^{-|x_j|^2} \) as \( \varepsilon \to 0 \). Thus \( \Psi \) is \( \rho \)-equivariant,

\[
\Psi(\tilde{X}, \sigma_j^{-1}) = \Phi_\varepsilon(X)v_0 = \rho(\sigma_j)\Psi(\tilde{X}),
\]

\[
\int_C |\Psi|^2 = (N!)^{-1} \int_{\mathbb{R}^{2N}} |\Phi_\varepsilon|^2 \rightarrow C, \quad and
\]

\[
T_\rho[\Psi] = \int_C |\nabla \Psi(\tilde{X})|^2 dX = \frac{1}{N!} \int_{\mathbb{R}^{2N}} |\nabla \Phi_\varepsilon|^2 \rightarrow \frac{1}{N!} \int_{\mathbb{R}^{2N}} |\nabla \Phi_0|^2 < \infty,
\]

while, due to the non-integrability of the inverse-square potential,

\[
\int_{\mathbb{R}^{2N}} \frac{|\Psi|^2}{|x_1 - x_2|^2} dx = \int_{\mathbb{R}^{2N}} \frac{|\Phi_\varepsilon|^2}{|x_1 - x_2|^2} dx \rightarrow \infty \quad \text{as} \quad \varepsilon \to 0.
\]

For \( N = 2 \) any representation is abelian and thus the above counterexample applies if and only if \( \alpha_2 = \beta_0 = 0 \).

For \( N = 3 \) we consider the reduced unitarized Burau representation with \( \mathcal{F} = \mathbb{C}^2 \) and \( w = 1 \):

\[
\rho(\sigma_1) = \frac{1}{2} \begin{bmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{bmatrix}, \quad \rho(\sigma_2) = \frac{1}{2} \begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}, \quad \rho(\sigma_1 \sigma_2 \sigma_1) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.
\]
Note that $\rho(\sigma)^2 = 1$ and thus it descends to a non-abelian representation of $S_3 = \{1, \sigma_1, \sigma_2, \sigma_1\sigma_2, \sigma_2\sigma_1, \sigma_1\sigma_2\sigma_1\}$. The eigenvectors corresponding to eigenvalue $1$ are

$$v_1 := \frac{1}{2} \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}, \quad v_2 := \frac{1}{2} \begin{bmatrix} \sqrt{3} \\ -1 \end{bmatrix},$$

and therefore we cannot simply take a constant eigenvector as in the abelian case. Instead, take two disjoint domains $\Omega_1$ and $\Omega_2$ in $\mathbb{R}^2$ and define $\Psi(\tilde{X}) = \Psi(\tilde{\rho}(\tilde{X}))$ by

$$\Psi(x_1, x_2) = \sum_{\sigma \in S_3} f(x_{\sigma(1)}, x_{\sigma(2)})g(x_{\sigma(3)})\rho(\sigma)v_1 + 2f(x_1, x_2)g(x_1)(-v_2) + f(x_3, x_1)g(x_2)(v_2 - v_1),$$

where $f \in H^1_{\text{sym}}(\Omega_1, \mathbb{C})$ supported in $\Omega_1^2$ and $g \in H^1(\Omega_2; \mathbb{C})$ supported in $\Omega_2$. We also used that $\rho(\sigma_1)v_2 = v_1 - v_2$ and $\rho(\sigma_2)v_1 = v_2 - v_1$. It then follows that the three terms above are pairwise orthogonal in $L^2_{\rho}$ and

$$\int_{C^3} |\Psi|^2 = 2\int_{\Omega_1^2} |f(x_1, x_2)|^2dx_1dx_2\int_{\Omega_2} |g(x_3)|^2dx_3.$$  

Furthermore,

$$\int_{C^3} |\nabla \Psi|^2 = 2\int_{\Omega_1^2} (|\nabla f(x_1, x_2)|^2 + |\nabla g(x_1)|^2)dx_1dx_2\int_{\Omega_2} |g(x_3)|^2dx_3$$

and $\Psi$ is $\rho$-equivariant by definition,

$$\Psi(\tilde{X}, \sigma^{-1}_j) = \sum_{\sigma \in S_3} f(x_{\sigma\sigma(1)}, x_{\sigma\sigma(2)})g(x_{\sigma\sigma(3)})\rho(\sigma_j^2\sigma)v_1 = \rho(\sigma_j)\Psi(\tilde{X}).$$

Again, by fixing $g$ and taking $f \in C_c^\infty(\Omega \setminus \Delta^2)$ to approximate a product state $u(x_1)u(x_2)$ where, e.g., $u$ is the g.s. of the Dirichlet Laplacian on $\Omega_1$, we find that $T_\rho[\Psi]$ stays uniformly bounded while

$$\int_{\mathbb{R}^6} |\Psi|^2 \geq \int_{\Omega_1^2} |f(x_1, x_2)|^2dx_1dx_2\int_{\Omega_2} |g|^2$$

grows unboundedly. This contradicts the validity of the Hardy inequality (5.13) for any positive value of the constant $C_N$. \hfill \Box

**Proposition 5.7** (Counterexamples for $\alpha = 0$, local case). In the case that $\rho : B_N \to U(D)$ is abelian with $\rho(\sigma_j^{-1}) \sim \text{diag}(e^{i\gamma_n\pi})_{n=1}^{D}$ we have

$$E_N^\alpha(\Omega) \leq \min_{\gamma \in \{1, \ldots, D\}} E_N^{\gamma\alpha}(\Omega),$$

with

$$E_N^{\gamma\alpha}([0, 1]^2) \leq 2\pi N(N - 1)\alpha \frac{1 + 2\pi\pi N - 2\alpha}{(1 - 2\pi\alpha N)^2}, \quad 0 \leq \alpha < (2\pi N)^{-1},$$

(5.14)

and thus for $\Omega = [0, 1]^2$ (and in fact for any bounded convex $\Omega$) the Hardy inequality of Theorem 5.5 necessarily trivializes for $\alpha_2 = \beta_0 = 0$. 

Proof. Use the abelian magnetic representation \((4.13)\) to estimate for any \(1 \leq n \leq D\)

\[
E_N^\rho(\Omega) \leq \inf \left\{ \int_{\mathbb{R}^{2N} \setminus \Delta^N} |\Phi_n|^2 = 1 \right\} = E_N^{\gamma_n}(\Omega).
\]

The bound \((5.14)\) for the abelian energy on \(\Omega = [0, 1]^2\) was proved in [LS18, Lemma 3.2] using a Dyson-type ansatz [Dys57]. In fact, if \(\gamma_n = 0\) and \(\Omega \subseteq \mathbb{R}^2\) bounded we may take a sequence of \(\Phi_n \in C^\infty_c(\mathbb{R}^{2N} \setminus \Delta^N)\)

\[
\int_{\mathbb{R}^{2N}} |\Phi_n|^2 = 1, \quad \int_{\mathbb{R}^{2N}} |\nabla \Phi_n|^2 \to 0,
\]

yielding \(E_N^\rho(\Omega) = 0\). \(\square\)

\textbf{Remark 5.8.} The question concerning the behavior of the optimal constant in \((5.13)\) for \(N \to \infty\) and \(\alpha^* = 0\) is an interesting but difficult open problem (in fact open even for fermions with \(\alpha^* = 1\) [FHOLS06]). It is a consequence of the uncertainty principle that in the ground state we must also consider exchanges \(U_p\) with \(p > 0\) enclosed particles (this was ignored in some earlier scenarios [FM88], [FM89, p.212-213] resp. [GM04] for anyonic exclusion). One may note that Theorem 5.5 seems to premiere clustering states where some of the annuli \(A_p\) can have a higher weight if \(\beta_p < \beta_0\). This has been discussed to some extent in [LS13b, Lun17, LL18, Lun19], and it would be even more relevant for point-attractive anyons (cf. Remark 4.7). Actually only the latter form of the Hardy inequality relies on the regularity assumptions on \(\Psi\) at \(\Delta^N\) implied by the Friedrichs extension. See also the remarks following Lemma 5.12 below.

\section*{5.5. Scale-covariant energy bounds.}

The goal of this subsection is to use the positivity of the anyonic energy due to the repulsion from the Hardy inequality for just a few particles to derive positivity and in fact a quadratic growth for large numbers of particles. We are here guided by the scale-covariant method introduced in [LS18] and formulated quite generally in [LLN19] (see also [Lun19, Sec. 5.5]):

\textbf{Lemma 5.9 (Covariant energy bound: [LLN19, Lemma 4.1]).} Assume that to any \(n \in \mathbb{N}_0\) and any cube \(Q \subset \mathbb{R}^d\) there is associated a non-negative number ('energy') \(e_n(Q)\) satisfying the following properties, for some constant \(s > 0\):

- (scale-covariance) \(e_n(\lambda Q) = \lambda^{-2s} e_n(Q)\) for all \(\lambda > 0\);
- (translation-invariance) \(e_n(Q + x) = e_n(Q)\) for all \(x \in \mathbb{R}^d\);
- (superadditivity) For any collection of disjoint cubes \(\{Q_j\}_{j=1}^J\) such that their union is a cube,

\[
e_n\left(\bigcup_{j=1}^J Q_j\right) \geq \min_{\{n_j\} \in \mathbb{N}_0^J \text{ s.t. } \sum_{j=1}^J n_j = n} \sum_{j=1}^J e_{n_j}(Q_j);
\]

- (a priori positivity) There exists \(q \geq 0\) such that \(e_n(Q) > 0\) for all \(n \geq q\).

There then exists a constant \(C > 0\) independent of \(n\) and \(Q\) such that

\[
e_n(Q) \geq C|Q|^{-2s/d} n^{1+2s/d}, \quad \forall n \geq q.
\]
Note that scale-covariance with \( s = 1 \) as well as translation-invariance follows directly from the definition of the anyonic energy \( E_N(\Omega) \):

**Lemma 5.10** (Covariance). Given \( \Omega \subseteq \mathbb{R}^2 \) open and simply connected, \( x \in \mathbb{R}^2 \) and \( \lambda > 0 \), we have
\[
E_N(\lambda \Omega + x) = \lambda^{-2} E_N(\Omega).
\]  

**Proof.** Denote \( \Omega_\lambda := \lambda \Omega + x \). For \( X = \{x_1, \ldots, x_N\} \in C^N(\Omega) \) we have \( X_\lambda := \lambda X + x \in C^N(\Omega_\lambda) \), and for \( \tilde{X} \in \tilde{C}^N(\Omega) \) an equivalence class of paths from \( X_0(\Omega) \) to \( X = \tilde{pr}(\tilde{X}) \) we have \( \tilde{X}_\lambda := \lambda \tilde{X} + x \in \tilde{C}^N(\Omega_\lambda) \) a corresponding translated and scaled equivalence class of paths from \( X_0(\Omega_\lambda) \) to \( X_\lambda = \tilde{pr}(\tilde{X}_\lambda) \). Given \( \Psi_\lambda \in H^1_{\rho}(\Omega_\lambda) \) we define \( \Psi(\tilde{X}) := \lambda^N \Psi_\lambda(\tilde{X}_\lambda) \).

Then, using \( dX_\lambda = \lambda^{2N} dX \), we have an \( L^2 \)-isomorphism
\[
\int_{\tilde{C}^N(\Omega_\lambda)} |\Psi(\tilde{X})|^2_{\tilde{\mathcal{F}}} dX = \int_{\tilde{C}^N(\Omega_\lambda)} |\Psi_\lambda(\tilde{X}_\lambda)|^2_{\tilde{\mathcal{F}}} dX_\lambda,
\]
and, with the corresponding rescaling in (4.5),
\[
\int_{\tilde{C}^N(\Omega_\lambda)} |\nabla \Psi(\tilde{X})|^2_{\tilde{\mathcal{F}}^{2N}} dX = \lambda^2 \int_{\tilde{C}^N(\Omega_\lambda)} |\nabla \Psi_\lambda(\tilde{X}_\lambda)|^2_{\tilde{\mathcal{F}}^{2N}} dX_\lambda,
\]
and vice versa by inversion \((\lambda, x) \mapsto (\lambda^{-1}, -x)\). Applied to a sequence of minimizers yields (5.15).

**Lemma 5.11** (Superadditivity). For \( K \geq 2 \), let \( \{\Omega_k\}_{k=1}^K \) be a collection of disjoint, convex subsets of \( \mathbb{R}^2 \). For any \( \bar{n} \in \mathbb{N}_0^K \) with \( \sum_k n_k = N \), let \( \mathbbm{1}_{\bar{n}} \) denote the characteristic function of the subset of \( C^N(\mathbb{R}^2) \) where exactly \( n_k \) of the points \( X = \{x_1, \ldots, x_N\} \) are in \( \Omega_k \), for all \( 1 \leq k \leq K \). Let
\[
W(X) := \sum_{\bar{n}} \sum_{k=1}^K E_{n_k}(\Omega_k) \mathbbm{1}_{\bar{n}}(X),
\]
and assume \( \Omega := \bigcup_k \Omega_k \) is also convex. We then have
\[
\int_{\tilde{C}^N(\Omega)} |\nabla \Psi|^2 dX \geq \int_{\tilde{C}^N(\Omega)} W|\Psi|^2
\]
for any \( \Psi \in H^1_{\rho}(\tilde{C}^N(\Omega)) \), and in particular
\[
E_N(\Omega) \geq \min_{\bar{n}} \sum_{k=1}^K E_{n_k}(\Omega_k).
\]

**Proof.** Using that \( \mathbbm{1} = \sum_{\bar{n}} \mathbbm{1}_{\bar{n}} \) on \( C^N(\Omega) \), we have for any \( \Psi \in C^{\infty}_{\rho,c}(\tilde{C}^N) \)
\[
\int_{\tilde{C}^N(\Omega)} |\nabla \Psi|^2 dX = \sum_{\bar{n}} \int_{\tilde{C}^N(\Omega)} |\nabla \Psi|^2 \mathbbm{1}_{\bar{n}} dX,
\]
where for any \( \tilde{X} = [\gamma].X_0(\Omega) \in \tilde{C}^N(\Omega) \)
\[
|\nabla \Psi(\tilde{X})|^2 \mathbbm{1}_{\bar{n}} = \sum_{k=1}^K \sum_{j:x_j \in \Omega_k} |\nabla_j \Psi(\tilde{X})|^2 \mathbbm{1}_{\bar{n}}.
\]
Hence for each $\vec{n}$, the integral to be considered is
\begin{equation}
\sum_{k=1}^{K} \int_{C^{N-n_k}(\Omega \setminus \Omega_k)} \int_{C^{n_k}(\Omega_k)} \sum_{j:X_j \in \Omega_k} |\nabla_j \Psi(\vec{X})|^2 dX_k \mathcal{I}_\beta dX_k^c. \tag{5.18}
\end{equation}

Fixing the set of points $X_k^c \in \mathcal{C}^{N-n_k}(\Omega \setminus \Omega_k)$ and considering the set $X_k \in \mathcal{C}^{n_k}(\Omega_k)$ s.t. $X = X_k \cup X_k^c$, we have by $\vec{X} = [\gamma].\vec{X}_0(\Omega)$ a path $\gamma$ in $\Omega$ where $n_k$ of the points at $X_0^N(\Omega)$ move into $\Omega_k$ and $N - n_k$ into $\Omega \setminus \Omega_k$. Any such path can equivalently be taken via first acting on $\vec{X}_0^N(\Omega)$ with a braid $b \in B_N$, then selecting a partition of particles at $X_0^N(\Omega)$, say $\{1, \ldots, n_k\}, \{n_k+1, \ldots, N\}$, then moving the former set of points via $X_0^N(\Omega_k)$ to $X_k$ and the latter via $X_0^N(\Omega_{k'})$ to $X_k^c$ for some $k' \neq k$. Note that any additional encircling of the points in $X_k$ around the fixed set $X_k^c$ in this process may be taken care of by writing the corresponding action of $[\gamma_k] \in \pi_1(\mathcal{C}^{n_k}(\Omega \setminus X_k^c), X_k)$ on $\vec{X}_k \mapsto X_k$ as $\vec{X}_k^N(\Omega).b$ for some $b \in B_N$ (actually in the subgroup of the pure braid group which keeps the indices fixed). Furthermore, any $\vec{X}_k \in \mathcal{C}^{n_k}(\Omega_k)$ can be represented as $\vec{X}_k = [\gamma_k].\vec{X}_0^N(\Omega_k)$ for a path $\gamma_k$ of $n_k$ points in $\Omega_k$ and using $X_0^N(\Omega_{k'}).\beta$ if it is a loop $\beta \in B_n \hookrightarrow B_N$. In this way we construct a surjective map

$$\mathcal{C}^{n_k}(\Omega_k) \times \mathcal{C}^{N-n_k}(\Omega \setminus \Omega_k) \times B_N \ni (\vec{X}_k, \vec{X}_k^c, b) \mapsto \vec{X} \in \mathcal{I}^{-1}(\text{supp } \mathcal{I}_n \subseteq \mathcal{C}^N(\Omega)).$$

Hence, if we define for a fixed $X_0^N(\Omega_k)$ and $X_k^c$ with fixed lift to $\vec{X} \in \mathcal{C}^N(\Omega)$ the function $\Psi_k(\vec{X}_k) := \Psi(\vec{X})$ we have
\begin{equation}
\Psi_k(\vec{X}_k, \beta) = \Psi(\vec{X}, \beta) = \rho(\beta^{-1})\Psi_k(\vec{X}_k). \tag{5.19}
\end{equation}

Note that it is also defined in a neighborhood of $\mathcal{C}^{n_k}(\Omega_k)$. By making a smooth cut-off on $\Omega_k^c$ away from the points $X_k^c$ we may consider this a smooth function on $\mathcal{C}^{n_k}(\mathbb{R}^2)$ which is $\rho$-equivariant and with compact support $\mathcal{I}^{-1}(\text{supp } \Psi_k)$. It is therefore in $H^1_\rho(\mathcal{C}^{n_k}(\Omega_k); \mathcal{F})$. Further,
\begin{equation}
\sum_{j=1}^{n_k} |\nabla_j \Psi_k(\vec{X}_k)|^2 = \sum_{j:X_j \in \Omega_k} |\nabla_j \Psi(\vec{X})|^2,
\end{equation}
for $\vec{X} \in \text{supp } \mathcal{I}_\beta$. Then
\begin{equation}
\int_{\mathcal{C}^{n_k}(\Omega_k)} \sum_{j=1}^{n_k} |\nabla_j \Psi_k(\vec{X}_k)|^2 dX_k \geq E_{n_k}(\Omega_k) \int_{\mathcal{C}^{n_k}(\Omega_k)} |\Psi_k|^2 dX_k
\end{equation}
and after putting this back into the original integral (5.18) and (5.17), and using that $|\Psi_k|^2 = |\Psi|^2$, we obtain the bound (5.16) in terms of the scalar and $B_N$-symmetric potential $W$. \hfill \Box

In the sequel we denote simply by $E_N := E_N([0, 1]^2)$ the $N$-anyon energy on the unit square with a given anyon model. Our starting point for positivity is the following bound originally derived for abelian anyons in [LL18]:

**Lemma 5.12** (Bound for $E_N$ directly from Hardy; [LL18, Lemma 5.3]). For $\nu > 0$ let $j_\nu'$ denote the first positive zero of the derivative of the Bessel function $J_\nu$, satisfying (5.11). There exists a function $f: [0, (j_\nu')^2] \rightarrow \mathbb{R}_+$ satisfying
\begin{equation}
t/6 \leq f(t) \leq 2\pi t \quad \text{and} \quad f(t) = 2\pi t (1 - O(t^{1/3})) \quad \text{as } t \to 0,
\end{equation}
for some positive constant $C$, and for each $0 < \nu < \nu_0$ and $0 < t < 1$ we have
\begin{equation}
E_N \leq C \int_{\mathcal{C}(\nu)} |\nabla \Psi|^2 dX \leq C \int_{\mathcal{C}(\nu)} |\Psi|^2 dX,
\end{equation}
with $\mathcal{C}(\nu) = \{x \in \mathbb{R}^2: |x_j| < \nu \text{ for all } j \}$.
such that

$$E_N \geq f(j_{\alpha N}^2)(N - 1)_+.$$  \hspace{1cm} (5.19)

**Proof.** After splitting the energy $T_\rho = \kappa T_\rho + (1 - \kappa) T_\rho$ and applying the diamagnetic inequality Lemma 5.4 to the first part, and the Hardy inequality of Theorem 5.5 in the later form to the second part, the proof is identical to that of [LL18, Lemma 5.3], where one expresses the bosonic $N$-particle energy as $(N - 1)/2$ copies of the two-particle energy, and defines (where subsequently $t = j_{\alpha N}^2$)

$$f(t) := \frac{1}{2} \sup_{\kappa \in (0, 1)} \int_{Q_0^2} \left( \kappa |\nabla_1| \psi | + \kappa |\nabla_2| \psi | + (1 - \kappa) t \frac{1}{\delta(X_{12})^2} |\psi|^2 \right) dx_1 dx_2.$$

The energy of the corresponding Schrödinger operator (with singular potential) on $L^2(Q_0^2)$ is then estimated in a standard way using projection on the constant function, yielding the bounds for $f(t)$ stated in the lemma (see also Figure 1.2). \hfill \square

**Remark 5.13.** We could replace the above bound by one which takes more of the particle distribution into account. Namely, after making a cut-off around the diagonals on the otherwise singular Hardy term we can take the average (i.e. expectation in the bosonic g.s.)

$$\sum_{j<k} \int_{Q_0^2} \sum_{p=0}^M \beta_p^2 \mathbbm{1}_{A_p \cap B_p}(0) \frac{1}{r^2_{jk}} dx,$$

which is finite and greater than the one used in obtaining the above bounds for $f$. However, we will be using the lemma mainly for $N = 2$, where nothing is gained by this, and use a different route to derive a bound for $E_N$ depending on $E_2$ and thus $\alpha_2 = \beta_0$ only.

**Remark 5.14.** In fact, it was pointed out in [LS18, Proposition 4.6] that Lemma 5.12 gives the correct behavior to leading order for $N = 2$ and $\alpha_2 \to 0$ as it is matched by the upper bound (5.14) of Proposition 5.7 (for $N = 2$ it is effectively abelian) to yield

$$E_2 = 4\pi \alpha_2 \left(1 + O(\alpha_2^{1/3})\right) \quad \text{as } \alpha_2 \to 0.$$

In other words, the energy per particle is asymptotically $2\pi \alpha_2$.

Given that $\alpha_N \leq \alpha_2$ for all $N \geq 2$, the bound (5.19) will be trivial if $\alpha_2 = 0$, i.e. if $E_2 = 0$. On the other hand, if $\alpha_2 > 0$ and thus $E_2 > 0$ then it is in fact sufficient to yield positivity $E_N > 0$ for all $N \geq 2$:

**Lemma 5.15** (A priori bounds in terms of $E_2$; [LS18, Lemma 4.3]). For any $N \geq 3$ we have

$$E_N \geq \frac{\pi^2 \binom{N}{2} \left(\frac{3}{4}\right)^{N-2} E_2}{(\pi + 4\sqrt{E_2})^2 + \left(\frac{\pi}{2}\right)^{N-2} E_2},$$

and thus $E_N > 0$ if $E_2 > 0$.

The proof is exactly analogous to the abelian case, [Lun13, Proposition 2] and [LS18, Lemma 4.3], where we split the unit square $Q_0$ in $K = 4$ smaller squares and use covariance (5.15) and superadditivity (5.16), with

$$W \geq W_2 := 4E_2 \sum_{k=1}^4 \sum_{\vec{n}_k=2} 1_{\vec{n}}.$$
(for each $k$, we keep only the terms in $W$ for which $n_k = 2$), and
\[
\int_{Q^N} W_2 = E_2 \left( \frac{N}{2} \right) \left( \frac{3}{4} \right)^{N-2}.
\]

Then we use the diamagnetic inequality of Lemma 5.4 to estimate the energy $E_N$ in terms of a standard bosonic or distinguishable particle problem, yielding the claimed bound (see [LS18, Lemma 4.3] for details).

**Lemma 5.16 (Covariant energy bound).** Given a sequence of anyon models $\rho_N : B_N \to U(\mathcal{F}_N)$ with two-anyon exchange parameters $\alpha_2(N) \geq \alpha_2 > 0$ (it could depend on the total number $N$ of particles but must then be uniformly bounded from below), the local $N$-anyon energy $E(N)(Q)$ on squares $Q$ satisfies the criteria of Lemma 5.9 with $s = 1$ and $q = 1$ and therefore there exists a constant $C = C(\alpha_2) > 0$ such that
\[
E_N \geq CN^2, \quad N \geq 2.
\]

For an explicit bound on this constant we will use the original method of [LS18, Lemma 4.8-4.9] and the local exclusion principle.

5.6. **Local exclusion principle.** Given $\Psi \in H^1(\mathcal{C}^N(\Omega))$, recall that it defines a corresponding one-body density $\rho_\Psi \in L^1(\Omega)$ and a kinetic energy density $T_{\rho_\Psi} \in L^1(\Omega)$ (Definitions 4.17 and 4.18).

**Lemma 5.17 (Local exclusion principle for non-abelian anyons).** Given a sequence of anyon models $\rho_N : B_N \to U(\mathcal{F}_N)$ with exchange parameters $\alpha_n = \min_p \beta_p$ (they may all depend on $N$), the local $n$-anyon energy $E_n = E^\rho_N(Q_0)$ on the unit square satisfies
\[
E_n \geq C_4(\rho_N)n, \quad 4 \leq n \leq N,
\] (5.20)

where
\[
C_4(\rho_N) := \frac{1}{4} \min \{E_2, E_3, E_4\} \geq \frac{1}{4} \min \{E_2, 0.147\} \geq c(\alpha_2) = \frac{1}{4} \min \{f(j_{\alpha_2}^2), 0.147\}.
\]

Furthermore, for any simply connected open domain $\Omega \subseteq \mathbb{R}^2$ and any square $Q \subseteq \Omega$, any $N \geq 1$ and $L^2$-normalized $\Psi \in H^1(\mathcal{C}^N(\Omega); \mathcal{F}_N)$ with one-particle density $\rho_\Psi$ on $\Omega$, we have
\[
T_{\rho_\Psi}^{Q \subseteq \Omega}[\Psi] \geq \frac{C(\rho_N)}{|Q|} \left( \int_Q \rho_\Psi(x) \, dx - 1 \right),
\] (5.21)

where
\[
C(\rho_N) := \max \left\{ C_4(\rho_N), f(j_{\alpha_2}^2), f(j_{\alpha_3}^2) \right\} \geq \max \left\{ \frac{1}{4} \min \{f(j_{\alpha_2}^2), 0.147\}, f(j_{\alpha_3}^2) \right\}.
\]

**Proof.** By splitting the unit square $Q_0$ into four equally large squares $Q_k$, $|Q_k| = |Q_0|/4$, one may derive from scale-covariance and superadditivity the recursive bound
\[
E_n \geq 4 \min_{m \in \{n/4, n/4+1, \ldots, n\}} E_m.
\]

From this and a priori positivity follows the linear bound (5.20) (see [LS18, Lemma 4.8] for details). The bound on $C_4(\rho_N)$ in terms of $E_2 \geq f(j_{\alpha_2}^2)$ follows from Lemma 5.15, where the positive root of $(\pi + 4\sqrt{x})^2 + \frac{9}{4}x = \frac{9}{4}\pi^2$ is $x \geq 0.147$. Furthermore one may bound $c(\alpha_2) \geq \frac{1}{4} \min \{\alpha_2/3, 0.147\}$ using (5.11) and the bounds on $f$ in Lemma 5.12.
For (5.21) we use the decomposition of the energy into particles in \( Q \) and in \( Q^c = \Omega \setminus Q \),

\[
T_{\rho_N}^{Q \subseteq \Omega} [\Psi] = \int_{C^N(\Omega)} \sum_{j=1}^N |\nabla_j \Psi(\tilde{X})|^2 \mathbb{1}_{\{x_j \in Q\}} \, dX
\]

\[
= \sum_n \int_{C^{N-n_1}(Q^c)} \int_{C^{n_1}(Q)} \sum_{j,x_j \in Q} |\nabla_j \Psi(\tilde{X})|^2 dX_1 \mathbb{1}_{_{\mathbb{R}^d}} dX_1^c
\]

and proceed similarly to the proof of superadditivity (Lemma 5.11), with \( \Omega = Q \) and \( \Omega_{k>1} \subseteq Q^c \) (e.g. splitting \( Q^c \) along the coordinate axes). Thus,

\[
T_{\rho_N}^{Q \subseteq \Omega} [\Psi] \geq \sum_n \int_{C^{N-n_1}(Q^c)} \int_{C^{n_1}(Q)} |\Psi(\tilde{X})|^2 dX_1 \mathbb{1}_{_{\mathbb{R}^d}} dX_1^c.
\]

Now use that from scale-covariance and (5.20)

\[
E_n(Q) \geq |Q|^{-1} C_4(\rho_N)(n-1)_+ \quad \text{for all } 0 \leq n \leq N,
\]

while from our earlier Lemma 5.12

\[
E_n(Q) \geq |Q|^{-1} f(j_{\alpha_n}^2)(n-1)_+ \geq |Q|^{-1} f(j_{\alpha_N}^2)(n-1)_+ \quad \text{for all } 0 \leq n \leq N.
\]

Defining the induced \( n \)-particle probability distribution on \( Q, n \in \{0,1,\ldots,N\} \),

\[
p_n(\Psi; Q) := \sum_{r,m_1=r} \int_{C^{N-n_1}(Q^c)} \int_{C^{n_1}(Q)} |\Psi(\tilde{X})|^2 dX_1 \mathbb{1}_{_{\mathbb{R}^d}} dX_1^c,
\]

with \( \sum_{n=0}^N p_n = 1 \) and \( \sum_{n=0}^N np_n = \int_Q \theta_{\Psi} \), we may use the convexity of \( x \mapsto (x-1)_+ \) to obtain

\[
T_{\rho_N}^{Q \subseteq Q_0} [\Psi] \geq \sum_{n=0}^N |Q|^{-1} C(\rho_N)(n-1)_+ p_n \geq |Q|^{-1} C(\rho_N)(n \rho_n - 1)_+,
\]

which proves (5.21). \( \square \)

6. The ideal non-abelian anyon gas

6.1. The homogeneous gas. The (zero-temperature) homogeneous anyon gas on a domain \( \Omega \) is defined by taking a ground state \( \Psi \) of the kinetic energy \( T_\rho^\Omega \), either with Dirichlet or Neumann boundary conditions. As \( N \to \infty \) and \( |\Omega| \to \infty \) while keeping the mean density \( \bar{\rho} := N/|\Omega| \) fixed (the thermodynamic limit) one expects on general grounds the one-body density \( \theta_{\Psi} \) to tend to the constant \( \bar{\rho} \), regardless of the shape of (reasonable) \( \Omega \) and the choice of b.c. (at least on average over large enough scales; cf. the almost-bosonic limit [CDLR19]).

**Definition 6.1** (Dirichlet energy). Given an \( N \)-anyon model \( \rho : B_N \to U(\mathcal{F}) \) and a simply connected domain \( \Omega \subseteq \mathbb{R}^2 \), we define the **Dirichlet Sobolev space** \( H_{\rho,0}^{1}(\tilde{C}^N(\Omega); \mathcal{F}) \), of wave functions \( \Psi \in L_{\rho}^2 \) vanishing on the boundary \( \partial \Omega \) and with finite expected local kinetic energy on \( \Omega \), as the closure of the space of functions \( \Psi \in C_{\rho,c}^\infty(\tilde{C}^N(\Omega)) \) (with \( \rho \text{r(supp } \Psi) \)}
Thus, normalizing $k$ ground states on $\Omega$ and $E$ where, to distinguish the two, we refer to

\[ \text{Define for } \tilde{x} \text{ a lift } \tilde{\Omega} \text{ a path } x \text{ where } x \in \tilde{\Omega} \text{ and } x \text{ is localized on } \Omega. \]

This point is in 1-to-1 correspondence to $X \in \Omega$. This follows by decomposing $|\nabla \Psi|^2 = \sum_{n=1}^D |\nabla \Psi_n|^2$ w.r.t. a basis in $F$, i.e. $T_0|H^1(\Omega; F) = \bigoplus D T_0|H^1(\Omega; C)$, and using the standard diamagnetic inequality, $|\nabla \Psi| \geq |\nabla |\Psi||$.

**Lemma 6.2** (Subadditivity). Let $\Omega_k, k = 1, \ldots, N$ be pairwise disjoint and simply connected subsets of $\mathbb{R}^2$, and assume $\Omega := \bigcup_k \Omega_k$ is also simply connected. If $\rho: B_N \to U(F)$ is an arbitrary geometric $N$-anyon model then

\[ \tilde{E}_N(\Omega) \leq \sum_{k=1}^N \tilde{E}_1(\Omega_k). \]

**Proof.** Let $u_k \in C^\infty_c(\Omega_k; C)$ and $v \in F, |v| = 1$, and consider the subset $\Omega = \prod_{k=1}^N \Omega_k \subseteq \mathbb{R}^{2N}$ where $x_j$ is localized on $\Omega_j$ (thus making all particles distinguishable). Fix a representative $x_0 \in \Omega$. This point is in 1-to-1 correspondence to $X_0$ (and $X_0(\Omega)$) by means of a simple path $\gamma_0$ in $C^N$, and any action of $b \in B_N$ on $X_0$ induces a permutation $\Omega \rho(b)$ in $\mathbb{R}^{2N}$ and a lift $\tilde{\Omega}, b$ in $\tilde{C}^N$. Thus, by the disjointness of $\Omega_k$ we can write the set of pre-images in $\tilde{C}^N$

\[ \tilde{\rho}^{-1} \rho(b) = \bigcup_{b \in B_N} \tilde{\Omega}, b. \]

Define for $\tilde{X} \in \tilde{\Omega}$ ($b = 1$)

\[ \Psi(\tilde{X}) := u_1(x_1)u_2(x_2) \ldots u_N(x_N)v \]

and on $\tilde{\Omega}, b$ for $b \neq 1$

\[ \Psi(\tilde{X}, b) := \rho(b^{-1})\Psi(\tilde{X}). \]

Then we have a $\rho$-equivariant function such that for any $\tilde{X} \mapsto X = \{x_1, \ldots, x_N\}, x_j \in \Omega_j$,

\[ |\Psi(\tilde{X})|^2 = |u_1(x_1)|^2 \ldots |u_N(x_N)|^2, \]

and

\[ |\nabla \Psi(\tilde{X})|^2 = |\nabla u_1(x_1)|^2 |u_2(x_2)|^2 \ldots |u_N(x_N)|^2 + \ldots + |u_1(x_1)|^2 \ldots |u_{N-1}(x_{N-1})|^2 |\nabla u_N(x_N)|^2. \]

Thus, normalizing $\int_{\mathbb{R}^2} |u_k|^2 = 1$ and taking sequences of $u_k$ converging to the respective ground states on $\Omega_k$ realizing $E_1(\Omega_k)$, we obtain the upper bound of the lemma. \qed
Theorem 6.3 (Uniform bounds for the homogeneous anyon gas). For any sequence of $N$-anyon models $\rho_N : B_N \to U(F_N)$ with $n$-anyon exchange parameters $\alpha_n = \alpha_n(N) \in [0,1]$ we have the uniform bounds
\[ \frac{1}{4} C(\rho_N) N^2 (1 - O(N^{-1})) \leq E_N(Q_0) \leq \bar{E}_N(Q_0) \leq 2\pi^2 N^2 (1 + O(N^{-1/2})), \] (6.1)
where
\[ C(\rho_N) = \max \{ C_4(\rho_N), f(j_{\alpha_N}^2) \} \geq \max \left\{ \frac{1}{4} \min \{ f(j_{\alpha_2}^2), 0.147 \}, f(j_{\alpha_N}^2) \right\}. \]

Remark 6.4. By scale-covariance, the g.s. energy per particle and unit density in the thermodynamic limit $N \to \infty$, $\Omega = [-L/2, L/2]^2$, $L \to \infty$ at fixed density $\bar{\rho} = N/|\Omega|$ is then
\[ e(\{\rho_N\}) := \liminf_{N \to \infty} E_N/N^2, \quad \max \left\{ \frac{1}{4} \min \{ f(j_{\alpha_2}^2), 0.147 \}, f(j_{\alpha_N}^2) \right\} \leq e(\{\rho_N\}) \leq 2\pi^2, \]
where $\alpha_n$ are the lim inf of $\alpha_n(\rho_N)$ as $N \to \infty$.

Remark 6.5. One can improve the upper bound in the case that the anyon model is transmutable and with a small statistics parameter, by treating the magnetic potential as a weak interaction and using techniques for weakly interacting Bose gases [Dys57, LSSY05]. This was done for almost-bosonic abelian anyons in [LS18, Proposition 3.3 and Lemma 4.9].

Proof. Given sub- and superadditivity and the local exclusion principle, the method to derive these bounds via splitting into smaller boxes is quite standard; we follow [LS18, Proposition 3.3 and Lemma 4.9].

For the lower bound, let $K \in \mathbb{N}$ and split $Q_0$ into $K^2$ smaller squares $Q_k$, $k = 1, \ldots, K^2$, of equal size $|Q_k| = K^{-2}$. By superadditivity Lemma 5.11 and scale-covariance Lemma 5.10 we have for any state $\Psi \in H^1_0(\mathcal{C}^N(Q_0); F_N)$
\[ \int_{\mathcal{C}^N(Q_0)} |\nabla \Psi|^2 \geq \sum_{\bar{n}} \sum_{k=1}^{K^2} K^2 E_{n_k} \int_{\mathcal{C}^N(Q_0)} 1_{\bar{n}} |\Psi|^2 = K^2 \sum_{k=1}^{K^2} \sum_{n=0}^{N} E_{n} p_n(\Psi; Q_k), \] (6.2)
where $p_n$ is the induced $n$-particle probability distribution on $Q_k$, defined in (5.22). Define also the average distribution of particle numbers
\[ \gamma_n := K^{-2} \sum_{k=1}^{K^2} p_n(\Psi; Q_k), \]

normalized
\[ \sum_{n=0}^{N} \gamma_n = 1, \quad \sum_{n=0}^{N} n \gamma_n = N/K^2 =: \rho_Q, \]
which is the expected number of particles on any one of the smaller squares. Hence the r.h.s. of (6.2) is, by Lemma 5.17 and convexity, bounded from below by
\[ K^2 \sum_{n=0}^{N} C(\rho_N)(n - 1)_+ \gamma_n \geq C(\rho_N) K^4 \left( \sum_{n=0}^{N} n \gamma_n - 1 \right)_+ = C(\rho_N) N^2 \rho_Q^2 (\rho_Q - 1)_+. \]
Optimizing then $\rho_Q \sim 2$ by our choice of $K$, we take $K := \lfloor \sqrt{N/2} \rfloor$ and obtain $2(1 + \sqrt{2/N})^{-2} \leq \rho_Q \leq 2$, and thus

$$E_N \geq C(\rho_N) N^{\frac{1}{4}} \left(2(1 + \sqrt{2/N})^2 - (1 + \sqrt{2/N})^4\right),$$

which proves the claimed lower bound.

For the upper bound let $K$ be the smallest integer such that $N \leq K^2$, and again split the unit square $Q_0 = [0,1]^2$ into $K^2$ smaller squares $Q_k$ of equal size. By subadditivity Lemma 6.2 and the well-known one-body energy $\bar{E}(Q_0) = 2\pi^2$ of the Dirichlet ground state $u(x,y) = \sin(\pi x) \sin(\pi y)$, we obtain

$$\bar{E}_N(Q_0) \leq \sum_{k=1}^{N} E_1(Q_k) + \sum_{k=N+1}^{K^2} E_0(Q_k) = NK^22\pi^2 \leq N(1 + \sqrt{N})^22\pi^2$$

by scale-covariance. This proves the claimed upper bound. \hfill \Box

With our results of Corollaries 3.11-3.12 respectively Section 3.6, and bounds for $f$ and $j'_\nu$, we have then

**Corollary 6.6.** For Fibonacci anyons the exchange parameters are $\alpha_2 = \beta_0 = 3/5$ and $\alpha_N = \beta_1 = 1/5$ for $N \geq 3$, and hence (6.1) holds with $C(\rho_N) \geq 1/15$ for $N \geq 3$.

**Corollary 6.7.** For Ising anyons the exchange parameters are $\alpha_N = \beta_0 = 1/8$ for all $N \geq 2$, and hence (6.1) holds with $C(\rho_N) \geq 1/24$ for $N \geq 2$.

**Corollary 6.8.** For Clifford anyons the exchange parameters are $\alpha_N = \beta_0 = 1/4$ for $2 \leq N \leq 4$ and $\alpha_N = \beta_3 = 0$ for $N \geq 5$, and hence (6.1) holds with $C(\rho_N) \geq 1/48$ for $N \geq 5$.

**Remark 6.9.** Note that the numerical estimate of $f(j_{\alpha}^2)$ in Figure 1.2 gives significant improvements to these analytical lower bounds for $C(\rho_N)$:

$$C(\rho_N^{\text{Fibonacci}}) \gtrsim 0.35, \quad C(\rho_N^{\text{Ising}}) \gtrsim 0.25, \quad \text{respectively} \quad C(\rho_N^{\text{Clifford}}) \gtrsim 0.147/4.$$

Let us make some additional remarks concerning the sharper first form of the Hardy inequality of Theorem 5.5 and Remark 5.13. While Ising anyons exhibit a uniform statistical repulsion with $\beta_p$ independent of $p$, Fibonacci and in particular Clifford anyons could in principle prefer to cluster to minimize their repulsion, just like we consider this also a possible preferred behavior (balanced by the uncertainty principle) for abelian anyons with $\alpha_s < \alpha$ [LS13b, Lun17]. We also note in a similar way that the statistical repulsion of Majorana fermions $\psi$ in the Ising model is potentially weakened to some probability, $U_{\psi,\sigma,\psi} = +1$ by (3.8), thus allowing for pairing to happen on larger scales.

### 6.2. Lieb–Thirring inequality

In the case that the gas is not homogeneous, such as if an additional external scalar potential is added to the Hamiltonian (4.1), a very powerful bound is given by the Lieb–Thirring inequality, which estimates the kinetic energy of $\Psi$ locally at $x \in \mathbb{R}^2$ in terms of the homogeneous gas energy at the corresponding density $\rho_{\Psi}(x)$ (for fermions that would be the famous Thomas–Fermi approximation (1.12)). The bound combines the uncertainty principle and the exclusion principle into a uniform bound for arbitrary $N$-anyon states $\Psi \in H^1_N$. 
Theorem 6.10 (Lieb–Thirring inequality for ideal anyons). There exists a constant $C > 0$ such that for any number of particles $N \geq 1$, for any $N$-anyon model $\rho : B_N \to U(F_N)$ with 2-particle exchange parameter $\alpha_2 \in [0, 1]$ (which may depend on $N$), and for any $L^2$-normalized $N$-anyon state $\Psi \in H^1_0(\mathbb{C}^N; F_N)$, we have
\[ T_\rho[\Psi] \geq C \alpha_2 \int_{\mathbb{R}^2} \varrho \Psi(x)^2 \, dx. \] (6.3)

Furthermore, given a one-body potential $V : \mathbb{R}^2 \to \mathbb{R}$ and $\hat{V}(X) = \sum_{j=1}^N V(x_j)$,
\[ \inf \text{spec } (\hat{T}_\rho + \hat{V}) \geq \frac{1}{4C \alpha_2} \int_{\mathbb{R}^2} V_-(x)^2 \, dx, \] (6.4)
where $V_- := \max\{-V, 0\}$.

Remark 6.11. For bosons, with $\alpha_2 = 0$, the inequality (6.3) cannot hold with any better constant since for product states (1.2) the l.h.s. is $\int |\nabla u|^2$ while $\varrho \Psi(x) = \int |u(x)|^2$ in the r.h.s., so the ratio cannot be better than $C_{\text{GNS}}/N \to 0$, where $C_{\text{GNS}} := \inf_{u \in H^1(\mathbb{R}^2); \int_{\mathbb{R}^2} |u|^2 = 1} \frac{\int_{\mathbb{R}^2} |\nabla u|^2}{\int_{\mathbb{R}^2} |u|^4}$ is the optimal constant of the corresponding Gagliardo-Nirenberg-Sobolev inequality for $N = 1$.

Remark 6.12. We expect that the optimal constant $C$ in (6.3) is on the order of $2\pi$, which is the Thomas-Fermi constant in 2D and is obtained in Weyl’s law (1.6) for the sum of the eigenvalues of the Laplacian on a bounded domain (although recall also our remarks after (1.12)). The standing conjecture for fermions [LT76] is that the optimal constant is slightly smaller and equal to $C_{\text{GNS}}$ (which is only known numerically). However, proving this even for fermions is a difficult open problem; see e.g. [Fra20] for a recent review.

We prove Theorem 6.10 as an application of the local exclusion principle, Lemma 5.17, combined with a local version of the uncertainty principle for bosons or distinguishable particles. The method was introduced for abelian anyons in [LS13a] and goes back to Dyson and Lenard’s approach to the proof of the stability of fermionic matter [DL67, Lemma 5]. We could simply replace [LS13a, Lemma 8] by our Lemma 5.17, however we give for completeness a more immediate proof from [LNP16] formulated using a covering lemma (see [Lun19] for a more detailed exposition of this local approach to Lieb-Thirring inequalities).

Lemma 6.13 (Local uncertainty principle [LS13a, Lemma 9-10], [LS14, Lemma 14]). For arbitrary $d \geq 1$ and $N \geq 1$ let $\Psi$ be a wave function in $H^1(\mathbb{R}^d)$, normalized $\int_{\mathbb{R}^dN} |\Psi|^2 = 1$, and let $Q$ be an arbitrary cube in $\mathbb{R}^d$. Then there exists a constant $C_1 > 0$ depending only on $d$, such that
\[ T_0^{Q \subseteq \mathbb{R}^d}[\Psi] = \int_{\mathbb{R}^dN} \sum_{j=1}^N |\nabla_j \Psi|^2 1_{\{x_j \in Q\}} \, dx \geq C_1 \left( \int_Q \varrho \Psi \right)^{1+2/d} - \frac{1}{C_1} \frac{1}{|Q|^{2/d}} \int_Q \varrho \Psi, \] (6.5)
with the one-body density
\[ \varrho_\Psi(x) := \sum_{j=1}^N \int_{\mathbb{R}^{d(N-1)}} |\Psi(x_1, \ldots, x_{j-1}, x, x_{j+1}, \ldots, x_N)|^2 \prod_{k \neq j} dx_k. \]

**Lemma 6.14** (Covering lemma [LNP16]). Let \( 0 \leq f \in L^1(\mathbb{R}^d) \) be a function with compact support such that \( \int_{\mathbb{R}^d} f \geq \Lambda > 0 \). Then the support of \( f \) can be covered by a collection of disjoint cubes \( \{Q\} \) in \( \mathbb{R}^d \) such that
\[ \int_{Q} f \leq \Lambda, \quad \forall Q \]
and
\[ \sum_{Q} \left( \frac{1}{|Q|^a} \left( \left[ \int_{Q} f - q \right]_+ - b \int_{Q} f \right) \right) \geq 0 \]
for all \( a > 0 \) and \( 0 \leq q < \Lambda 2^{-d} \), where
\[ b := \left( 1 - \frac{2^d q}{\Lambda} \right) \frac{2^{da} - 1}{2^{da} + 2^d - 2} > 0. \]

**Proof of Theorem 6.10.** If \( \alpha_2 = 0 \) the theorem is trivially true, and otherwise we have \( E_2(Q) > 0 \) and thus a non-trivial local exclusion principle on squares \( Q \) given by Lemma 5.17 with \( C_\rho \geq c(\alpha_2) > 0 \). By definition of the space \( H^1_\rho \) we may w.l.o.g. assume \( \Psi \) is smooth and the projection by \( \rho \) of its support to \( C^N \) contained in the projection by \( \rho \) of some large cube \( Q^N_L, Q^L \in [-L,L]^2 \).

Let \( q = 1 \) and \( \Lambda = 5 \). If \( N \leq \Lambda \), then (6.3) follows immediately from (6.5) with \( Q = Q_L, L \rightarrow \infty \). If \( N > \Lambda \), then we can apply Lemma 6.14 with \( f = \varrho_\Psi \) and \( a = 1 \), and obtain a collection of disjoint cubes \( \{Q\} \) covering \( Q_L \). Using that \( T_\rho[\Psi] = \sum_{Q} T_{\rho}^{Q \subseteq Q_L}[\Psi] \)
and the diamagnetic inequality of Lemma 5.4
\[ T_\rho[\Psi] \geq T_0[|\Psi|_F], \quad \varrho_\Psi|_F = \varrho_\Psi, \]
and finally combining the bounds (6.5) and (5.21), we obtain
\[ (\varepsilon + 1)T_\rho[\Psi] \geq \varepsilon \sum_{Q} \left[ C_1 \int_{Q} \varrho_\Psi^2 - \frac{1}{C_1} \frac{1}{|Q|} \int_{Q} \varrho_\Psi \right] + \sum_{Q} \left( C(\rho) \frac{1}{|Q|} \int_{Q} \varrho_\Psi dx - 1 \right) \]
\[ \geq \varepsilon C_1 \int_{\mathbb{R}^d} \varrho_\Psi^2 \]
for any fixed constant \( \varepsilon > 0 \) satisfying \( \varepsilon \leq C_1 C(\rho) b \). Taking \( \varepsilon = C_1 b \min\{\alpha_2/12, 0.147/4, (C_1 b)^{-1}\} \)
proves the first inequality (6.3) with \( C = C_1 \min\{C_1 b, 1\}/120 > 0 \).
For the second inequality (6.4) we use a well-known equivalence of the bounds, namely by Cauchy-Schwarz and simple optimization

\[
\langle \Psi, (\hat{T}_\rho + \hat{V}) \Psi \rangle_{L^2} = T_\rho[\Psi] - \int_{\mathbb{R}^2} V \varrho \Psi \geq T_\rho[\Psi] - \int_{\mathbb{R}^2} |V_-| \varrho \\
\geq C \alpha_2 \int_{\mathbb{R}^2} \varrho^2 - \left( \int_{\mathbb{R}^2} |V_-|^2 \right)^{1/2} \left( \int_{\mathbb{R}^2} \varrho^2 \right)^{1/2} \geq -\frac{1}{4C\alpha_2} \int_{\mathbb{R}^2} |V_-|^2,
\]

which is a uniform bound in \( N \) provided that \( \alpha_2 = \alpha_2(N) \) stays uniformly bounded away from zero.

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