Isometry groups of three-dimensional Lie groups

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Abstract
We compute the full isometry group of any left-invariant metric on a simply connected, non-unimodular Lie group of dimension three. As an application, we determine the index of symmetry of such metrics and prove that the singularities of the moduli space of left-invariant metrics, up to isometric automorphism are contained in the subspace of classes of metrics with maximal index of symmetry.

Keywords Lie group · Left-invariant metric · Isometry group · Symmetric space · Index of symmetry

Mathematics Subject Classification 53C30 · 53C35

1 Introduction

Lie groups endowed with a left-invariant metric are objects of great importance among Riemannian homogeneous spaces. If \( G \) is a Lie group and \( g \) is a left-invariant metric on \( G \), then the geometry of \( (G, g) \) is locally determined, in a simple algebraic way, by choosing an inner product on the Lie algebra \( g \) of \( G \). An interesting problem in Riemannian geometry, which is also important from the point of view of theoretical physics, is the explicit computation of the full isometry group of a homogeneous space. If the Lie group \( G \) is nice enough, then the structure of the isometry group \( \text{Isom}(G, g) \) of \( g \) can be recovered, at least locally, from the underlying algebraic structure. For example, if \( G \) is a compact Lie group, it is proved in [1] that the connected component \( \text{Isom}_0(G, g) \) of the isometry group is a subgroup of \( L(G) \cdot R(G) \subset \text{Diff}(G) \), where \( L(G) \) (resp. \( R(G) \)) is the subgroup of left (resp. right) translations on \( G \). In a different context, it was proved by Wolf in [2] (see also [3]) that if \( G \) is nilpotent, then \( \text{Isom}(G, g) \cong L(G) \rtimes (\text{Aut}(g) \cap \text{O}(g, g)) \) is the semi-direct product of \( L(G) \).
and the isometric automorphisms of $g$, which are identified, via the isotropy representation, with the isotropy group of $\text{Isom}(G, g)$ of the identity element $e$. Notice that in the general case, if $K$ is the full isotropy group of $e$, then $\text{Isom}(G, g) = L(G) \cdot K$, but the Lie group structures of $G$ and $K$ do not determine the structure of $\text{Isom}(G, g)$. In fact, it is known (see, for instance, [4]) that $L(G)$ is a normal subgroup of $\text{Isom}(G, g)$ if and only if the connected component of $K$ is contained in $\text{Aut}(g)$. Thus, in general, the full isometry group is not a semidirect product of $L(G)$ and $K$, nor is contained in $L(G) \cdot R(G)$.

This paper is devoted to the case where $G$ has dimension 3. Recall that the unimodular case was already treated in [5]. Such groups are well behaved in the sense of the above paragraph. There, the authors take advantage of this in order to obtain the full isotropy subgroup as the isometric automorphisms of the Lie algebra. The classification of the isometry groups, in the unimodular case, relies on the previous classification of Ha–Lee [6] of the moduli spaces of left-invariant metrics, up to isometric automorphism, of 3-dimensional Lie groups. The explicit knowledge of these moduli spaces is a starting point for approaching the non-unimodular case, but the methods used in the unimodular case are of no use. Some works related to this topic can be mentioned. For instance, in the article by Lauret [7] (see also [8]) there is a classification of Lie groups with a unique left-invariant metric, up to isometric automorphism and scaling. In 3 dimensions, a solvable non-unimodular Lie algebra appears in Lauret’s classification, the so-called Lie algebra of the hyperbolic space, where all the left-invariant metrics are of constant negative curvature. In [9], the authors determine the left-invariant metrics on 3-dimensional Lie groups which are locally symmetric. This is done by solving a polynomial system on the structure coefficients of the underlying Lie algebras, associated with the parallel curvature condition. Hence, one can easily determine the full isometry groups from the eigenvalues of the Ricci tensor. We can also mention the article of Gordon and Wilson [10] on transitive isometry subgroups of Riemannian solvmanifolds, which are in, what they call, standard position inside the full isometry group. Their results are mainly useful in the case where $G$ is unimodular solvable.

In this article, we compute the full isometry group of any left invariant metric on a non-unimodular 3-dimensional Lie group. The main ideas involved can be summarized as follows. In order to compute the full isotropy Lie algebra, we use a theorem by Singer, which describes such Lie algebra as the skew-symmetric endomorphisms of the tangent space preserving all the derivatives of the curvature tensor, up to a certain order (see Theorem 2.1). Notice that this theorem is stated in the article [11] with no proof. A proof can be found in the later paper [12] by Nicolodi and Tricerri. In the case where the isotropy group has positive dimension, we determine the group structure of the full isometry group thinking of the Killing fields of $G$ as parallel sections of the associated vector bundle $TG \oplus \mathfrak{so}(TG)$, as used in [13]. We want to remark two interesting families of examples that follow from our classification. In the first place, we explicitly find a curve of simply connected Lie groups with a left-invariant metric which are isometric but not isomorphic (see Remark 3.9). Secondly, we find a curve of left-invariant metrics on the same Lie group whose isometry groups are not isomorphic to each other (see Remark 3.4). This example is somehow surprising since such behaviour occurs only for the Lie algebra given by the structure coefficients

$$[e_0, e_1] = 0, \quad [e_0, e_2] = -e_1, \quad [e_1, e_2] = -2e_1.$$ 

In the generic case, the full isometry group consists only of left translations, or it is the isometry group of a non-compact symmetric space.

As an application of our results, we compute the index of symmetry of all the metrics under study. Recall that the index of symmetry is a geometric invariant, introduced in [14], which measures how far is a homogeneous metric from being symmetric. The index of symmetry
was successfully computed for several distinguished families of homogeneous spaces, such as compact naturally reductive spaces [14] and naturally reductive nilpotent Lie groups [15], flag manifolds [16] and 3-dimensional unimodular Lie groups [17]. It follows from our results that every 3-dimensional Lie group admits a left-invariant metric with non-trivial index of symmetry. We have noticed that in a recent preprint [18], R. May also compute, independently and with other techniques, the index of symmetry for these metrics.

Finally, we obtain a result relating the topology of the moduli space of left-invariant metrics with the index of symmetry. More precisely, we prove that the singularities of such moduli space are contained in the subsets of (equivalence classes of) metrics with maximal index of symmetry (cfr. Theorem 4.1). We believe these topological considerations could be extended to other families of homogeneous spaces in the future.

2 Preliminaries

2.1 3-Dimensional, non-unimodular, metric Lie algebras

The classification of 3-dimensional real Lie algebras, up to isomorphism, has been known for many years (see, for instance, [19]). There are six isomorphism classes of unimodular Lie algebras and uncountably many isomorphism classes of non-unimodular Lie algebras, which can be parametrized in the following way. Let \( g \) be a non-unimodular real Lie algebra of dimension 3. Then, there exists a basis \( e_0, e_1, e_2 \) of \( g \) such that

\[
[e_0, e_1] = 0, \quad [e_2, e_0] = \alpha e_0 + \beta e_1, \quad [e_2, e_1] = \gamma e_0 + \delta e_1,
\]

where the matrix \( A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \) has \( \text{tr} A = 2 \). Recall that \( A \) is the transpose to the matrix of \( \text{ad} e_2 \) (when restricted to the subalgebra spanned by \( e_0 \) and \( e_1 \)). Moreover, except for the Lie algebra where \( A \) is the identity matrix, the number \( c = \det A \) is a complete isomorphism invariant. In the rest of the article, we will denote by

- \( g_I \), the Lie algebra with \( A = I \) and
- \( g_c \), the Lie algebra with \( A = \begin{pmatrix} 0 & 1 \\ -c & 2 \end{pmatrix} \).

In the article [6], Ha–Lee classifies all the inner products on 3-dimensional Lie algebras up to isometric automorphism, and hence, they determine the moduli space of left invariant metrics for 3-dimensional Lie groups. We summarize their results, for the cases of our interest, in Table 1, where such inner products are presented by means of the corresponding matrix in the basis \( e_0, e_1, e_2 \). Notice that the expression of the inner products when \( 0 < c < 1 \) is rather complicated as it involves the action of the matrix

\[
P = \begin{pmatrix}
-\frac{1+\sqrt{1-c}}{2\sqrt{1-c}} & -\frac{1}{2\sqrt{1-c}} & 0 \\
\frac{1-\sqrt{1-c}}{2\sqrt{1-c}} & \frac{1}{2\sqrt{1-c}} & 0 \\
0 & 0 & 1
\end{pmatrix} \in \text{GL}_3(\mathbb{R})
\]

on a non-diagonal, symmetric, positive-definite matrix.

We shall denote by \( G_I \) and \( G_c \) the simply connected Lie groups with Lie algebra \( g_I \) and \( g_c \), respectively. Since these Lie algebras are solvable, \( G_I \) and \( G_c \) are diffeomorphic to \( \mathbb{R}^3 \). The corresponding Lie group structures can be described in the following way (as
Table 1 Left-invariant metrics up to isometric automorphism on 3-dimensional, non-unimodular Lie groups

| Lie algebra | Left-invariant metrics | Parameters |
|-------------|------------------------|------------|
| \( \mathfrak{g}_I \) | \( g_\nu = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu \end{pmatrix} \) | \( 0 < \nu \) |
| \( \mathfrak{g}_c \) | \( g_{\mu,\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix} \) | \( 0 < \mu \leq |c| \) and \( 0 < \nu \) |
| \( c = 0 \) | \( g_{\mu,\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \nu \end{pmatrix} \) | \( 0 < \mu, \nu \) |
| \( g_\nu = \begin{pmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu \end{pmatrix} \) | \( 0 < \nu \) |
| \( 0 < c < 1 \) | \( g_{\mu,\nu} = P^T \begin{pmatrix} 1 & \mu & 0 \\ \mu & 1 & 0 \\ 0 & 0 & \nu \end{pmatrix} P \) | \( P \) as in (2.2), \( 0 \leq \mu < 1 \) and \( 0 < \nu \) |
| \( c = 1 \) | \( g_{\mu,\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix} \) | \( 0 < \mu \leq 1 \) and \( 0 < \nu \) |
| \( g'_{\lambda,\nu} = \begin{pmatrix} 1 & \lambda & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \nu \end{pmatrix} \) | \( 0 < \lambda < 1 \) and \( 0 < \nu \) |
| \( 1 < c \) | \( g_{\mu,\nu} = \begin{pmatrix} 1 & \mu & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \nu \end{pmatrix} \) | \( 1 < \mu \leq c \) and \( 0 < \nu \) |

presented in [6]). In all cases, the Lie group will be a semi-direct product \( \mathbb{R}^2 \rtimes_{\varphi} \mathbb{R} \), where the representation \( \varphi : \mathbb{R} \to \text{GL}_2(\mathbb{R}) \) is given by:

\[
\varphi(t) = \begin{cases} 
(e^t 0 \\
0 e^t 
\end{cases} \begin{pmatrix} \cosh(t\sqrt{1-c}) & 0 \\
0 & \sinh(t\sqrt{1-c}) \end{pmatrix} \begin{pmatrix} 1 & 0 \\
0 & 1 
\end{pmatrix} + \frac{\sinh(t\sqrt{1-c})}{\sqrt{1-c}} \begin{pmatrix} -1 & -c \\
1 & 1 
\end{pmatrix}
\text{ for } G_I \tag{2.3}
\]

and hence the product on \( G_c \) is given by

\[
(v, t)(w, s) = (v + \varphi(t)w, t + s). \tag{2.4}
\]

Note that the formula in (2.3) makes sense even if \( c > 1 \).

The left-invariant metrics on \( G_{\bullet} \), where \( \bullet \in \{ I, c \} \), associated with the inner products on \( \mathfrak{g}_{\bullet} \) listed in Table 1 will be denoted by the same symbol \( g \in \{ g_\nu, g_{\mu,\nu}, g'_{\lambda,\nu} \} \). This presents certain ambiguity as \( g_\nu \) and \( g_{\mu,\nu} \) have different meaning for different Lie algebras, but such ambiguity disappears when one specifies the value of \( \bullet \).

### 2.2 A theorem by Singer on the Lie algebra of the isometry group

Let \( V \) be an Euclidean space and let \( T \) be an algebraic tensor of type \( (s, t) \) on \( V \). The natural action of the orthogonal Lie algebra \( \text{so}(V) \) on \( T \) is defined by

\[
(A \cdot T)(\omega_1, \ldots, \omega_s, v_1, \ldots, v_t) = -\sum_{i=1}^{s} T(\omega_1, \ldots, \omega_{i-1}, A \circ \omega_i, \omega_{i+1}, \ldots, \omega_s, v_1, \ldots, v_t) \\
+ \sum_{j=1}^{t} T(\omega_1, \ldots, \omega_s, v_1, \ldots, v_{j-1}, Av_j, v_{j+1}, \ldots, v_t),
\]
where $A \in \mathfrak{so}(\mathbb{V})$, $\omega_1, \ldots, \omega_s \in \mathbb{V}^*$ and $v_1, \ldots, v_1 \in \mathbb{V}$. In the work [11], Singer discusses under what conditions a family of algebraic tensors $R^s (s \geq 0)$ of type $(1, s + 3)$ on $\mathbb{V} = T_p M$, the tangent space of a differentiable manifold $M$ coincides with the covariant derivatives of order $s$ at $p$ of the curvature tensor of some homogeneous Riemannian metric on $M$. In particular, if $s$ ranges between 0 and certain integer $k$, one can theoretically recover the Lie algebra of the isometry group of $M$ from the action of $\mathfrak{so}(T_p M)$ on the algebraic tensors $R_p, (\nabla R)_p, \ldots, (\nabla^{k+1} R)_p$. We find it useful to rephrase such a theorem in order to give a procedure to find the Lie algebra of the isotropy subgroup of the full isometry group of $M$.

**Theorem 2.1** (Singer [11]) Let $M$ be a homogeneous Riemannian space of dimension $n$. Let $p \in M$ and $A \in \mathfrak{so}(T_p M)$. Then, there exists a Killing vector field $X$ on $M$ such that $X_p = 0$ and $(\nabla X)_p = A$ if and only if $A \cdot (\nabla^s R)_p = 0$ for all $0 \leq s < \frac{n(n-1)}{2}$, where $R$ denotes the Riemannian curvature tensor of $M$.

One can improve the statement of the above theorem. In fact, it is enough to assume that $s$ takes values between 0 and the so-called Singer invariant of $M$ ([11], see also [12, 13]).

### 2.3 The index of symmetry of a homogeneous manifold

Let $(M, g)$ be a Riemannian homogeneous space and let $\text{Isom}(M, g)$ be its full isometry Lie group. The Lie algebra of $\text{Isom}(M, g)$ is canonically identified with the Lie algebra of Killing vector fields $\mathcal{K}(M, g)$. The distribution of symmetry $\mathcal{s}_g$ of $(M, g)$ is defined at the point $p \in M$ by

$$\mathcal{s}_g = \{ v \in T_p M : v = X_p \text{ for some } X \in \mathcal{K}(M, g) \text{ with } (\nabla X)_p = 0 \}.$$

One has that $\mathcal{s}_g$ is an autoparallel distribution of $M$, that is, $\mathcal{s}_g$ is integrable with totally geodesics leaves. Moreover, the distribution of symmetry is invariant under $\text{Isom}(M, g)$ and its integrable manifolds are extrinsically isometric to a globally symmetric space, which is called the leaf of symmetry of $(M, g)$ (see, for instance, [14, 20]). The rank of $\mathcal{s}_g$ is usually called the index of symmetry of $(M, g)$, and it is denoted by $i_g(M, g)$. The index of symmetry is a geometric invariant which measures how far is $(M, g)$ from being a symmetric space, in the sense that $i_g(M, g) = \dim M$ if and only if $(M, g)$ is a symmetric space. It is known that the distribution of symmetry can never have corank equal to 1 (see [17]). In particular, if $M$ has dimension 3 and $(M, g)$ is not a symmetric space, then the index of symmetry is equal to 0 or 1.

### 3 The full isometry groups

In this section, we compute the full isometry groups of $(G_\bullet, g)$ where $\bullet \in \{ I \} \cup \mathbb{R}$ and $g$ is a left-invariant metric on $G_\bullet$. Recall that we will be using the notations given in Sect. 2. We analyze several cases according to the structure of the Lie algebra of $G_\bullet$ and the moduli space of left-invariant metrics. Before getting into the case-by-case analysis, we outline the general idea for dealing with the generic case. Assume that the left-invariant metric $g$ on $G_\bullet$ is not symmetric, and let $G = \text{Isom}(G_\bullet, g)$. Notice that, $\dim G \leq 4$. In fact, the isotropy group of $G$ at any point is isomorphic, via the isotropy representation, to a compact subgroup of $O(3)$. Let $e \in G_\bullet$ be the identity element. If $\varphi \in G$ and $\varphi(e) = x$, then $\varphi = L_x \circ (L_{x^{-1}} \circ \varphi)$, where $L_x \in G$ is the left translation by $x$. So, the full isometry group decomposes as

$$G \simeq G_\bullet \cdot G_e,$$

(3.1)
where $G_e$ is the isotropy subgroup of $G$ at $e$ and we are identifying $G_*$ with $L(G_*)$. Notice that, in general, (3.1) is not a semi-direct product. In fact, according to [4,Lemma 1.1], $G_*$ is a normal subgroup of the connected component of $G$ if and only if the connected component of $G_e$ is contained in $\text{Aut}(g_*)$, under the usual identifications. However, one can recover the algebraic structure of $G$ from the group structure of $G_*$ and $G_e$ and the geometry of the metric $g$. In fact, identify the Lie algebra of $G$ with the Lie algebra of Killing fields $\mathcal{K}(G_*, g)$. We have from (3.1) that every element of $\mathcal{K}(G_*, g)$ can be written as $X + Y$ where $X$ is a right-invariant vector field on $G_*$ and $Y$ is a Killing field such that $Y_e = 0$. On the other hand, if $X, X'$ are two Killing vector fields with initial conditions $X_e = v, X'_e = v'$ and $(\nabla X)_e = B, (\nabla X')_e = B'$, then the initial conditions of the Lie bracket $[X, X']$ are

$$[X, X']_e = B'v - Bv', \quad (\nabla [X, X'])_e = R_{v, v'} - [B, B']. \tag{3.2}$$

In fact, the first condition follows from the fact that the Levi–Civita connection is torsion-free, and the second one can be derived from the so-called Killing affine equation (see [13] or [17] for more details). It follows that in order to compute the full isometry group one only needs to compute $G_e$. We identify $G_e$, via the isotropy representation, with a subgroup of $O(g_*, g) \simeq O(3)$. At the Lie algebra level, one can recover the connected component of $G_e$ from its Lie algebra $g_e \subset \mathfrak{so}(g_*, g) \simeq \mathfrak{so}(3)$. In order to simplify some long calculations, it is useful to note that an element of $G_e$ must preserve the Ricci tensor. In most cases, the Ricci tensor of $G_*$ is semi-definite, and therefore, if an element $A \in \mathfrak{so}(g_*, g)$ is induced by a one-parameter subgroup of $G_e$, then $A$ belongs to a Lie subalgebra $\mathfrak{h} \subset \mathfrak{gl}(g_*)$, which is isomorphic to $\mathfrak{so}(2, 1)$. This reduces the complexity of the problem, as we now only need to look for

$$A \in \mathfrak{so}(g_*, g) \cap \mathfrak{h},$$

which, in the generic case, has dimension at most 1. In the case that this dimension is positive, we then can check if $A$ is in fact induced by isometries by using Theorem 2.1.

We now proceed with the study of all the possible cases. We deal with the isolated cases first.

### 3.1 The case of $G_I$

From (2.1), the bracket relations for $g_I$ are

$$[e_0, e_1] = 0, \quad [e_2, e_0] = e_0, \quad [e_2, e_1] = e_1,$$

and so, $g_I$ is isomorphic to the so-called *Lie algebra of the hyperbolic space* $H^3$. That is, the Lie algebra of the solvable, transitive group of isometries of $H^3$. Notice that this Lie group admits a left-invariant metric which makes it isometric to $H^3$. Moreover, according to [7] (see also [8]), this is the only possible left-invariant metric for $G_I$ up to automorphism and scaling. This proves the following result.

**Theorem 3.1** Let $g$ be a left-invariant metric on $G_I$, then

$$\text{Isom}(G_I, g) \simeq \text{SO}(3, 1).$$

**Remark 3.2** It follows from routine calculations that if we take $g = g_v$ in the above theorem, then $(G_I, g_v)$ is isometric to the real hyperbolic space of curvature $-\frac{1}{v^2}$ in three dimensions.
3.2 The case of $G_0$

According to (2.4), the Lie group structure of $G_0 \simeq \mathbb{R}^2 \rtimes \mathbb{R}$ is given by

$$(x_0, x_1, x_2)(y_0, y_1, y_2) = (x_0 + y_0, x_1 + e^{2x_2} (y_0 + 2y_1) - \frac{y_0}{2}, x_2 + y_2)$$

and the canonical frame of left-invariant vector fields is given by

$$e_0 = \frac{\partial}{\partial x_0} + e^{2x_2} \frac{\partial}{\partial x_1}, \quad e_1 = e^{2x_2} \frac{\partial}{\partial x_1}, \quad e_2 = \frac{\partial}{\partial x_2}.$$ 

We also have the explicit expression for the corresponding right invariant vector fields:

$$r_0 = \frac{\partial}{\partial x_0}, \quad r_1 = \frac{\partial}{\partial x_1}, \quad r_2 = (x_0 + 2x_1) \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}.$$ 

Theorem 3.3 Keeping the notation of Table 1, we have that

1. $\text{Isom}(G_0, g_{\mu, \nu}) \simeq G_0 \cdot \text{SO}(2)$ where its Lie algebra has a basis $r_0, r_1, r_2, A$ such that

$$[r_0, r_1] = 0, \quad [r_0, r_2] = r_1, \quad [r_1, r_2] = 2r_1,$n

$$[r_0, A] = r_2, \quad [r_1, A] = 2r_2, \quad [r_2, A] = -\nu r_0 - \frac{2\nu}{\mu} r_1 + 2A.$$ 

2. $\text{Isom}(G_0, g_\nu) \simeq E(1) \times \text{SO}(2, 1)$, where $E(1)$ is the Euclidean group in one dimension.

**Proof** Let us consider first the case of $g = g_{\mu, \nu}$. It is not difficult to see that if $X$ is a left-invariant vector field on $G_0$, that is also a Killing vector field, then $X$ must be a scalar multiple of $e_0 = r_0 - \frac{1}{2} e_1 = r_0 - \frac{1}{2} r_1$, which is also right-invariant. So, there are no new Killing vector fields arising as the difference of a left- and a right-invariant vector fields taking the same value at the identity.

In order to compute the isometry group, we first compute the orthogonal Lie algebra

$$\mathfrak{so}(g_0, g_{\mu, \nu}) = \left\{ \begin{pmatrix} 0 & -a_{10}\mu & -a_{20}\nu \\ a_{10} & 0 & -\frac{a_{21}\nu}{\mu} \\ a_{20} & a_{21} & 0 \end{pmatrix} : a_{10}, a_{20}, a_{21} \in \mathbb{R} \right\}.$$ 

Now we compute the Ricci tensor of the metric, which in the left-invariant frame $e_0, e_1, e_2$ takes the form

$$\text{Ric} = \begin{pmatrix} -\frac{\mu}{2\nu} & -\frac{2\mu}{\mu(\nu-8)} & 0 \\ -\frac{2\mu}{\nu} & \frac{\nu(\nu-8)}{2\nu} & 0 \\ 0 & 0 & -\frac{\mu+8}{2} \end{pmatrix}$$

and it induces a non-degenerate, semi-definite, symmetric bilinear form on $g_0$. The connected component of the isotropy group is then identified with the intersection

$$\mathfrak{so}(g_0, g_{\mu, \nu}) \cap \mathfrak{so}(g_0, \text{Ric}) = \mathbb{R}A,$$

where $A$ is the matrix

$$A = \begin{pmatrix} 0 & 0 & -\nu \\ 0 & -\frac{2\nu}{\mu} & 1 \\ 1 & 2 & 0 \end{pmatrix}.$$ 


Let us denote by $R$ the Riemannian curvature tensor of $g_{\mu,\nu}$. It is relatively easy to see that $A \cdot R_e = 0$ and after some heavy calculations\(^1\) we can check that $A \cdot (\nabla R)_e = 0$ and $A \cdot (\nabla^2 R)_e = 0$. Hence, by Theorem 2.1, $A = (\nabla Z)_e$ for some Killing field such that $Z_e = 0$. This implies that the connected component of the isometry group is a subgroup of $SO(g_0, g_{\mu,\nu})$ isomorphic to $SO(2)$. Since $\dim G_0 = 3$ and the metric is not symmetric, we get that the isometry group is connected and so $\text{Isom}(G_0, g_{\mu,\nu}) \simeq G_0 \cdot SO(2)$. Otherwise, there would be non-trivial isometries in the two connected components of $O(g_0, g_{\mu,\nu})$, which would imply that the geodesic symmetries are isometries. In order to determine the Lie group structure of the isometry group, we identify $\mathcal{K}(G_0, g_{\mu,\nu}) \simeq g_0 \oplus \mathbb{R} Z$ (direct sum of vector spaces), where $g_0'$ is the Lie algebra of right-invariant vector fields on $G_0$. Let us $r_0, r_1, r_2$ be the basis of right-invariant vector fields defined above the theorem. In order to compute the brackets $[r_i, Z]$, we use identities (3.2) and the fact that

$$
(\nabla r_0)_e = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -\frac{1}{2} \\
0 & \frac{1}{2} & 0
\end{pmatrix}, \quad
(\nabla r_1)_e = \begin{pmatrix}
0 & 0 & -\frac{1}{2} \mu \\
0 & 0 & -2 \\
\frac{1}{2} \nu & 2 \mu & 0
\end{pmatrix}, \quad
(\nabla r_2)_e = \begin{pmatrix}
0 & -\frac{1}{2} \mu & 0 \\
\frac{1}{2} & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}.
$$

Now we direct our attention to the metric $g = g_{\nu}$. If we define a new left-invariant vector frame by

$$
\hat{e}_0 = 2e_0 - e_1, \quad \hat{e}_1 = \frac{1}{2} e_2, \quad \hat{e}_2 = e_1,
$$

then $\mathfrak{g}(g_0) = \mathbb{R} \hat{e}_0$ and $[\hat{e}_1, \hat{e}_2] = \hat{e}_2$. So $g_0 \simeq \mathbb{R} \oplus \mathfrak{g}_{\mathbb{R} H^2}$ (direct sum of Lie algebras), where $\mathfrak{g}_{\mathbb{R} H^2}$ is the Lie algebra of the real hyperbolic space $\mathbb{R} H^2$. Moreover, the left-invariant metric $g_{\nu}$ takes now the form

$$
g_{\nu} = 3 \hat{e}^0 \otimes e^0 + \frac{1}{4} \nu \hat{e}^1 \otimes \hat{e}^1 + \hat{e}^2 \otimes \hat{e}^2.
$$

Thus, $g_{\nu}$ splits off its center and $(G_0, g_{\nu})$ is isometric to (a rescaling of) the Riemannian product $\mathbb{R} \times \mathbb{R} H^2$. In fact, as we noticed in the proof of Theorem 3.1, the restriction of $g_{\nu}$ to $\mathfrak{g}_{\mathbb{R} H^2}$ is unique up to automorphism and scaling. Therefore, the isometry group of $(G_0, g_{\nu})$ is isomorphic to $E(1) \times SO(2, 1)$.

\[\square\]

**Remark 3.4** If two left-invariant metrics on $G_0$ are not equivalent up to isometric automorphism or scaling, then the corresponding isometry groups are not isomorphic. In fact, the case when one of the metrics is equivalent of some $g_{\nu}$, is trivial. So we only need to consider metrics of the form $g_{\mu,\nu}$, and we can assume further that $\nu = 1$. In these conditions, the Killing form distinguish the Lie algebras given in item 1 of Theorem 3.3. In fact, it is easy to see that if $\nu = 1$, the eigenvalues of the Killing form of the isometry Lie algebra of $g_{\mu,1}$ are

$$
-\frac{\mu + 4 \pm \sqrt{81 \mu^2 + 8 \mu + 16}}{\mu}, 0, 8.
$$

### 3.3 The case of $G_1$

Taking $c = 1$ in (2.4), we can compute explicit expressions for the left- and right-invariant vector frames, which will be used in further calculations. The left-invariant vector frame $e_0, e_1, e_2$ is given by

\[\text{Calculations for this paper were verified using the extension SageManifolds of the computer algebra system SageMath. The corresponding jupyter-notebooks are available at https://github.com/silvioreggiani/isom-dim-3.}\]
After standard computations, we get that
\[ A = \frac{\partial}{\partial x_0} + x_2 \frac{\partial}{\partial x_1} \],
\[ e_1 = -x_2 e^{x_2} \frac{\partial}{\partial x_0} + (1 + x_2) e^{x_2} \frac{\partial}{\partial x_1}, \]
\[ e_2 = \frac{\partial}{\partial x_2}, \]
and the corresponding right-invariant vector frame is
\[ r_0 = \frac{\partial}{\partial x_0}, \quad r_1 = \frac{\partial}{\partial x_1}, \quad r_2 = -x_1 \frac{\partial}{\partial x_0} + (x_0 + 2x_1) \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}. \]

**Theorem 3.5** Let \( g \) be a left-invariant metric on \( G_1 \), then
\[ \text{Isom}(G_1, g) \simeq G_1. \]

**Proof** Assume first that \( g = g_{\mu, v} \) in Table 1. Notice that in the frame \( e_0, e_1, e_2 \) the orthogonal Lie algebra has the same matrix representation as in (3.3), but with the constrain \( 0 < \mu \leq 1 \). The Ricci tensor of \( g_{\mu, v} \) is represented in our frame by the matrix
\[
\text{Ric} = \begin{pmatrix}
-\frac{\mu^2 - 1}{2\mu v} & -\frac{\mu}{v} & 0 \\
-\frac{\mu}{v} & \frac{\mu^2 - 8\mu - 1}{2v} & 0 \\
0 & 0 & -\frac{\mu^2 + 6\mu + 1}{2\mu}
\end{pmatrix}.
\]
After standard computations, we get that \( A \in \mathfrak{so}(g_1, g_{\mu, v}) \) is also an element of \( \mathfrak{so}(g_1, \text{Ric}) \) if and only if \( a_{10} = 0 \) and
\[ a_{20}(1 + 3\mu) - 2a_{21}\mu = 2a_{20}\mu + a_{21}(1 - \mu) = 0 \]
which trivially implies \( a_{20} = a_{21} = 0 \) (recall that we are assuming that \( A \) has the form given in (3.3)). Since the metric is not symmetric, we get that the isotropy group of \( \text{Isom}(G_1, g_{\mu, v}) \) is trivial.

Finally, for the case of \( g = g'_{\lambda, v} \) we have
\[
\mathfrak{so}(g_1, g'_{\lambda, v}) = \left\{ \begin{pmatrix}
-a_{11} - \frac{a_{11}}{\lambda} (2\lambda + 1) \nu \\
a_{11} - \frac{a_{20}}{\lambda} (2\lambda + 1) \nu \\
0 - \frac{4}{(2\lambda + 1) \nu}
\end{pmatrix} : a_{11}, a_{20}, a_{21} \in \mathbb{R} \right\}
\]
and
\[
\text{Ric} = \begin{pmatrix}
-\frac{\lambda^2 - 1}{(2\lambda + 1) \nu} & -\frac{2(\lambda^2 + 1)}{(2\lambda + 1)^2} & 0 \\
-\frac{2(\lambda^2 + 1)}{(2\lambda + 1)^2} & \frac{4\lambda^2 - 4}{(2\lambda + 1)^2} & 0 \\
0 & 0 & -\frac{4}{\lambda + 1}
\end{pmatrix}.
\]
(3.6)
Reasoning as in the above paragraph we prove that \( \mathfrak{so}(g_1, g'_{\lambda, v}) \cap \mathfrak{so}(g_1, \text{Ric}) = 0 \). Notice that we are making here some abuse of notation, since \( \text{Ric} \) in (3.6) is degenerate when \( \lambda = \sqrt{5} - 2 \). Using again that the metric is not symmetric, we conclude that the full isometry group is isomorphic to \( G_1 \).

**3.4 The case of \( G_c, c < 0 \)**

In order to simplify the notation and make some calculations easier, we write
\[ c = 1 - c_1^2, \quad \text{for } c_1 > 1. \]
This trick is very useful when checking our calculations with SageMath, since this software finds it very hard simplifying certain expressions where \( \sqrt{1-c} \) appears, which now are just
replaced by \( c_1 \). The left-invariant frame in which the metrics of Table 1 are represented is given by

\[
e_0 = e^{x_2} \left( \cosh(c_1 x_2) - \frac{\sinh(c_1 x_2)}{c_1} \right) \frac{\partial}{\partial x_0} + e^{x_2} \frac{\sinh(c_1 x_2)}{c_1} \frac{\partial}{\partial x_1},
\]

\[e_1 = -e^{x_2} \frac{\sinh(c_1 x_2)}{c_1} \frac{\partial}{\partial x_0} + e^{x_2} \left( \cosh(c_1 x_2) + \frac{\sinh(c_1 x_2)}{c_1} \right) \frac{\partial}{\partial x_1},
\]

\[
e_2 = \frac{\partial}{\partial x_2},
\]

and the associated right-invariant vector frame is

\[
r_0 = \frac{\partial}{\partial x_0}, \quad r_1 = \frac{\partial}{\partial x_1}, \quad r_2 = -cx_1 \frac{\partial}{\partial x_0} + (x_0 + 2x_1) \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}.
\] (3.8)

**Theorem 3.6** If \( c < 0 \), then

\[\text{Isom}(G_c, g) \simeq G_c.\]

**Proof** The proof is similar to the ones of previous cases. We may assume that \( g = g_{\mu, \nu} \) and hence the orthogonal Lie algebra can be presented as in (3.3). For brevity, we present here the Ricci tensor in terms of \( c \) (instead of \( c_1 \)):

\[
\text{Ric} = \begin{pmatrix}
\frac{e^2 - \mu^2}{2\mu^v} & -\frac{2\mu}{v} & 0 \\
-\frac{2\mu}{v} & \frac{e^2 + 8\mu(1-\mu)}{2v} & 0 \\
0 & 0 & \frac{(c-\mu)^2 - 8\mu}{2\mu}
\end{pmatrix}.
\]

Now it is not hard to see that \( \text{so}(g_c, g_{\mu, \nu}) \cap \text{so}(g_c, \text{Ric}) = 0 \). In particular, this shows that the metric is not symmetric and hence \( \text{Isom}(G_c, g_{\mu, \nu}) \simeq G_c. \)

\[\Box\]

**3.5 The case of \( G_c, 0 < c < 1 \)**

As in the previous case, we write \( c = 1 - c_1^2 \) and hence the left- and right-invariant vector frames have now the exact same form as in (3.7) and (3.8), respectively, but now \( 0 < c_1 < 1 \).

**Theorem 3.7** If \( 0 < c < 1 \) and \( g \) is a left-invariant metric on \( G_c \), then

\[\text{Isom}(G_c, g) \simeq G_c.\]

**Proof** We take \( g = g_{\mu, \nu} \). Recall that in this case the coefficients of the metric with respect to the frame \( e_0, e_1, e_2 \) are rather involved. In fact, it needs to be computed as in Table 1 using the matrix \( P \) as in (2.2). After some suitable simplifications we can express, in the usual frame, the metric as

\[
g_{\mu, \nu} = \begin{pmatrix}
\frac{e^2 + \mu - 2}{2(c-1)c} & \frac{\mu - 1}{2(c-1)c} & 0 \\
\frac{\mu - 1}{2(c-1)c} & \frac{\mu - 1}{2(c-1)c} & 0 \\
0 & 0 & v
\end{pmatrix}
\] (3.9)

and the Ricci tensor as

\[
\text{Ric} = \begin{pmatrix}
(\mu^2 + \mu)e^2 + 2\mu^2 - (\mu^3 + 2\mu^2 + 2)c^2 & \mu^2 - c & 0 \\
(\mu^2 + \mu)c^2 + (1-\mu)^2v(1-c) & (\mu+1)v(1-c)c & 0 \\
0 & (\mu+1)v(1-c)c & 2(\mu^2 + \mu - 2)/(1-\mu^2)
\end{pmatrix}.
\]
The same argument used previously gives us that $\mathfrak{so}(g_c, g_{\mu, \nu}) \cap \mathfrak{so}(g_c, \text{Ric}) = 0$ and since the metric is not symmetric, $\text{Isom}(G_c, g_{\mu, \nu}) \simeq G_c$. □

### 3.6 The case of $G_c$, $1 < c$

We write $c = 1 + c_1^2$ with $c_1 > 0$. We can perform some formal manipulation in (3.7) in order to compute the left-invariant frame for $g_c$ in an easy way. In fact, rewriting $c = 1 - (ic_1)^2$, we can present the left-invariant vector fields as in (3.7) replacing $c_1$ by $ic_1$. Now, from $\cosh(iz) = \cos z$ and $\sinh(iz) = i \sin z$ follows that

$$e_0 = e^{x^2} \left( \cos(c_1 x_2) - \frac{\sin(c_1 x_2)}{c_1} \right) \frac{\partial}{\partial x_0} + e^{x^2} \frac{\sin(c_1 x_2)}{c_1} \frac{\partial}{\partial x_1},$$

$$e_1 = -c e^{x^2} \sin(c_1 x_2) \frac{\partial}{\partial x_0} + e^{x^2} \frac{\cos(c_1 x_2) + \sin(c_1 x_2)}{c_1} \frac{\partial}{\partial x_1},$$

$$e_2 = \frac{\partial}{\partial x_2},$$

and the associated right-invariant vector frame is

$$r_0 = \frac{\partial}{\partial x_0}, \quad r_1 = \frac{\partial}{\partial x_1}, \quad r_2 = -cx_1 \frac{\partial}{\partial x_0} + (x_0 + 2x_1) \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}.$$ 

**Theorem 3.8** If $1 < c$ and $g_{\mu, \nu}$ is the left-invariant metric on $G_c$ described in Table 1, then

$$\text{Isom}(G_c, g_{\mu, \nu}) \simeq \begin{cases} G_c & \text{if } 1 < \mu < c, \\
\text{SO}(3, 1) & \text{if } \mu = c. \end{cases}$$

**Proof** The orthogonal Lie algebra of $g_{\mu, \nu}$ is given by

$$\mathfrak{so}(g_c, g_{\mu, \nu}) = \left\{ \left( \begin{array}{ccc} a_{11} & -a_{11} \mu & (a_{20} \mu - a_{21}) \mu \mu^{-1} \\ a_{11} & a_{11} & (a_{20} \mu - a_{21}) \mu^{-1} \\ a_{20} & a_{21} & 0 \end{array} \right) : a_{11}, a_{20}, a_{21} \in \mathbb{R} \right\} \quad (3.10)$$

in the frame $e_0, e_1, e_2$. Since the matrix representation of the Ricci tensor becomes too involved, we rather give its components with respect to the same frame:

$$\text{Ric}_{00} = \frac{c_1^4 - \mu^2 - 2 \mu + 3}{2(\mu - 1)^v} = \frac{(c - 1)^2 - (\mu + 1)^2 + 4}{2(\mu - 1)^v},$$

$$\text{Ric}_{01} = \frac{c_1^4 + 2c_1^2 \mu - 2c_1^2 - 3 \mu^2 + 2 \mu + 1}{2(\mu - 1)^v} = \frac{(c + \mu)^2 - 4(\mu^2 + 1 - 1)}{2(\mu - 1)^v},$$

$$\text{Ric}_{02} = 0,$$

$$\text{Ric}_{11} = -\frac{c_1^4 \mu - 2c_1^4 - 4c_1^2 \mu - \mu^3 + 4c_1^2 + 12 \mu^2 - 17 \mu + 6}{2(\mu - 1)^v} = \frac{(2 - \mu)c^2 + (6 \mu - 8)c + \mu(\mu^2 - 12 \mu + 12)}{2(\mu - 1)^v},$$

$$\text{Ric}_{12} = 0,$$

$$\text{Ric}_{22} = -\frac{c_1^4 - 2c_1^2 \mu + 2c_1^2 + \mu^2 + 2 \mu - 3}{2(\mu - 1)^v} = -\frac{(c - \mu)^2 + 4(\mu - 1)}{2(\mu - 1)^v}.$$
Let $A$ be defined as in (3.10). The condition $A \in \mathfrak{so}(g_e, \text{Ric})$, or equivalently

$$A^T \text{Ric} + \text{Ric} A = 0$$

is given by the equations

$$\frac{2a_{11}(c-\mu)}{\nu} = \frac{(a_{20}(c-2) + a_{21})(\mu - c)}{\mu - 1} = \frac{(a_{20}(2\mu + c - 4) + a_{21}(2 - \mu))(\mu - c)}{\mu - 1} = 0.$$

Notice that if $\mu \neq c$, then the only possible solution is $A = 0$. In this case, we use the same ideas from previous cases to prove $\text{Isom}(G_c, g_{\mu, v}) \simeq G_c$.

Finally, assume that $\mu = c$. In this case, one can see that $\text{Ric} = -\frac{2}{v} g_{c,v}$ and so the metric on $G_c$ is Einstein. Moreover, $(G_c, g_{c,v})$ is isometric to the hyperbolic space of curvature $-\frac{1}{v}$, and hence, its isometry group is isomorphic to $\text{SO}(3, 1)$.

\[\square\]

**Remark 3.9** Putting together Theorems 3.1 and 3.8, one can construct and infinite family of pairs of solvable Lie groups endowed with a left-invariant metric, say $(S, g)$ and $(S', g')$, such that $(S, g)$ is isometric to $(S', g')$ but $S$ is not isomorphic to $S'$. In fact, if $0 < v$ and $1 < c$, then $(G_1, g_v)$ and $(G_c, g_{c,v})$ are both isometric to the 3-dimensional hyperbolic space of curvature $-\frac{1}{v}$.

**4 Computation of the index of symmetry**

With the results obtained in Sect. 3, we can easily compute the index of symmetry for all the non-unimodular Lie groups of dimension 3. Let us explain the general procedure to compute this invariant. Let $(G, g)$ be a simply connected, non-unimodular 3-dimensional Lie group endowed with a left-invariant metric $g$. We may assume that $(G, g)$ is not a symmetric space, otherwise the index of symmetry is equal to 3. It follows from Sect. 2.3 that $i_s(G, g) \leq 1$ and we have that $i_s(G, g) = 1$ if and only if there exists a Killing vector field $X$ on $G$ such that $X_e \neq 0$ and $(\nabla X)_e = 0$, where $e$ is the identity element of $G$. Now, if $\dim \text{Isom}(G, g) = 3$ such $X$ will be a right-invariant field. In the case where $\dim \text{Isom}(G, g) = 4$, the Lie algebra of the isotropy group is generated by a single element, say $A \in \mathfrak{so}(T_e G) \simeq \mathfrak{so}(g, g)$ and so we need to find a right-invariant vector field, $X \neq 0$ such that $(\nabla X)_e = \alpha A$ for some $\alpha \in \mathbb{R}$. Recall that the second situation only occurs when $G$ is isomorphic to $G_0$ and $(G, g)$ is isometric to $(G_0, g_{\mu, v})$ (see Theorem 3.3) and in this case we can take

$$A = \begin{pmatrix} 0 & 0 & -\nu \\ 0 & 0 & -\frac{2\nu}{\mu} \\ 1 & 2 & 0 \end{pmatrix} \quad (4.1)$$

with respect to the basis $e_0, e_1, e_2$. After some standard calculations, we obtain the index of symmetry for every metric described in Table 1. We summarize the results in Table 2. It is worth mentioning that the fourth column of Table 2 collects the vector fields generating the distribution of symmetry $s_g$, and since $s_g$ is invariant by isometries, such fields can be taken left-invariant. For example, when $c = 0$ and $g = g_{\mu, v}$, the distribution of symmetry is generated by $e_0 - \frac{1}{2}e_1$ because there is a Killing vector field $X$ such that $X_e = (e_0 - \frac{1}{2}e_1)|_e$ and $(\nabla X)_e = 0$. Notice that $X$ is not right-invariant. In fact, in order to construct such an $X$ one shows that $(\nabla X)_e$ is a multiple of the skew-symmetric endomorphism given by (4.1).
where $X$ can be thought as the gluing of the perpendicular half-planes

\[ \{ (\mu, 0, v) : \mu \in (0, 1), v \in \mathbb{R}^+ \} \cup \{(1, \lambda, v) : \lambda \in [0, 1), v \in \mathbb{R}^+ \} \]

along the line $\mathcal{L} = \{(1, 0, v) : v \in \mathbb{R}^+ \}$. Notice that $g'_{\lambda, v}$ in Table 1 makes sense for $\lambda = 0$ if one defines $g'_{0, v} = g_{1, v}$. Recall that $\mathcal{X}$ is homeomorphic to an open set of $\mathbb{R}^2$, but the natural inclusion of $\mathcal{X} \subset \mathbb{R}^3$ into $\mathbb{R}^3$ is not differentiable along the gluing line.

Now let $\mathcal{S}(G) \subset \mathcal{M}(G)$ be the subset of (equivalence classes of) metrics with maximal index of symmetry and let us denote by $\mathcal{Z}(G)$ the set of singularities of $\mathcal{M}(G)$. Recall that $\mathcal{Z}(G_1) = \emptyset; \mathcal{Z}(G_c) \simeq [c] \times \mathbb{R}^+$ if $c < 0; \mathcal{Z}(G_0) = \mathbb{R}^+; \mathcal{Z}(G_c) = \{0\} \times \mathbb{R}^+ \cup \{\sqrt{c}\} \times \mathbb{R}^+$ if $0 < c < 1; \mathcal{Z}(G_1) = \mathcal{L};$ and $\mathcal{Z}(G_c) = \{c\} \times \mathbb{R}^+$ if $1 < c$. 

### 4.1 The geometric meaning of the subset of metrics with maximal index of symmetry inside the moduli space of left-invariant metrics

Let $G$ be a 3-dimensional non-unimodular Lie group and denote by $\mathcal{M}(G)$ the moduli space of left-invariant metrics up to isometric automorphism. Using the classification of Ha–Lee given in Table 1, we can identify $\mathcal{M}(G)$ with a topological subspace of the symmetric space $\text{Sym}_3^+ = \text{GL}_3(\mathbb{R})/\text{O}(3)$ of positive-definite inner products on $\mathbb{R}^3$, and thus, $\mathcal{M}(G)$ inherits this natural topology. More precisely, we have the following homeomorphisms

\[
\mathcal{M}(G) \simeq \begin{cases} \mathbb{R}^+, & G = G_1, \\
(0, [c]) \times \mathbb{R}^+, & G = G_c \text{ with } c < 0, \\
(\mathbb{R}^+ \times \mathbb{R}^+) \cup \mathbb{R}^+, & G = G_0, \\
(0, 1) \times \mathbb{R}^+, & G = G_c \text{ with } 0 < c < 1, \\
\mathcal{X}, & G = G_1, \\
(1, c) \times \mathbb{R}^+, & G = G_c \text{ with } 1 < c, 
\end{cases}
\]

where $\mathcal{X}$ can be thought as the gluing of the perpendicular half-planes

\[
\{(\mu, 0, v) : \mu \in (0, 1), v \in \mathbb{R}^+ \} \cup \{(1, \lambda, v) : \lambda \in [0, 1), v \in \mathbb{R}^+ \}
\]
We can prove the following result by direct inspection using Table 2.

**Theorem 4.1** Let $G$ be a non-unimodular 3-dimensional Lie group. Then

$$Z(G) \subset S(G).$$

Moreover, equality holds for every $G$ such that $G \not\cong G_I$ and $G \not\cong G_c$ with $0 < c < 1$.

**Remark 4.2** Let $0 < c < 1$ and let $g$ be a left-invariant metric on $G_c$ such that $(G_c, g)$ is isometric to $(G_c, g_{\mu,1})$. It is easy to see that

$$\text{scal } g = -\frac{2(4 - c - 3\mu^2)}{(1 - \mu^2)\nu}.$$ 

In particular, for all the metrics of the form $g = g_{\sqrt{c},\nu}$, which are precisely the ones in $S(G_c) - Z(G_c)$, the scalar curvature $\text{scal } g = -8/\nu$ does not depend on $c$.

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