Spatial distribution of superfluidity and superfluid distillation of Bose liquids

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Under the assumption of two fluid kinematics of a nonrelativistic Bose liquid in the presence of a local relative velocity field \( v(x) = v_s(x) - v_L(x) \), local Galilei transformations are used to derive formulae for the spatial distribution of superfluidity. The local formulation is shown to subsume several descriptions of superfluidity, from Landau’s free quasiparticle picture of the normal fluid to the fully microscopic winding number formula for superfluid density. We derive the superfluid distribution of generic pure states of 1-d bosonic systems by using the continuum analog of matrix product states. With a view toward spatially structured superfluid-based quantum devices, we consider the limits to local distillation of superfluidity within the framework of localized resource theories of quantum coherence.

The dissipationless flow of He II and its connection to the structure of the bosonic many-body quantum state remain central problems in the study of strongly correlated bosonic systems. Among the celebrated achievements for zero temperature systems are the London wavefunction describing superfluid motion via a local phase field \( \psi \), Feynman’s modifications to incorporate density fluctuations and vortices \( \phi \), \( \phi_{\phi} \), and Feynman and Cohen’s inclusion of particle pair correlations leading to agreement with Landau’s proposed excitation spectrum \( \Omega \). In He II in equilibrium, \( \text{ad hoc} \) quantification of the local kinematic superfluidity based on the winding number or projected area of imaginary time polymers has led to a microscopic understanding of nanoscale superfluidity at adsorptive surfaces \( \Omega \), \( \Omega_{\phi} \). However, despite these major theoretical advances, the local distribution of superfluidity in general classes of bosonic variational wavefunctions has not been analyzed and, furthermore, the estimators of local superfluid density based on properties of imaginary time polymers are not applicable to generic bosonic states \( \Omega \). Therefore, the development and interpretation of a widely applicable, quantum mechanical framework describing local superfluidity in systems that exhibit two-fluid behavior is desirable. Recent experiments demonstrating optomechanical control of excitations in He II film \( \Omega \), and He II-based interferometry \( \Omega \), which indicate the possibility of local control of superfluid dynamics on a range of length scales, highlight the necessity of a local theory of superfluidity for superfluid-based devices.

In the present work, we derive a general framework for calculating the local normal fluid distribution for bosonic systems that empirically exhibit local two fluid behavior. Because the framework is based on local Galilei transformations, it can be used to extend Landau’s quasiparticle picture of the normal fluid or to analyze local superfluidity in the microscopic theory. We do not aim to derive the two fluid picture from quantum mechanical principles, which is the subject of various monographs \( \Omega \), but rather determine how the superfluid kinematics (e.g., the distribution of superfluidity) depends on a given spatially varying velocity field in the system and on the structure of the bosonic quantum state. As examples, we generalize Landau’s formula for the phonon contribution to the superfluid density in equilibrium and extend this method to general Gaussian states of a quasiparticle field, calculate the local superfluid distribution of a \( k = 0 \) Bose-Einstein condensate in the presence of a generic smooth velocity field, and derive a general formula based on continuum matrix product states for the local superfluid density in one spatial dimension. Finally, we formulate a resource theory of quantum coherence with the aim of deriving fundamental limits for forming a perfect superfluid state in a spatial subregion of a bosonic system under a physically relevant class of quantum operations. Our results greatly extend the possibilities for numerical simulation of superfluid systems in the presence of flow, and can be applied to the design of gratings and external potentials in bosonic matter wave interferometers \( \Omega \), \( \Omega \) and local heat flux control protocols for interacting bosonic liquids and gases \( \Omega \).

To derive a formula for the local normal fluid distribution, we first generalize Landau’s quasiparticle picture of the normal fluid. In a given Bose liquid, we consider a subset of identical bosons that follow integral curves of a velocity field \( v(x) \), which defines a steady flow. In Landau’s treatment, this subset is idealized as a gas of noninteracting bosonic quasiparticles, but we do not insist on this interpretation. According to the local two fluid model (see below), the local normal density is the part of the fluid that is moving with respect to a local reference frame that has velocity \( -v(x) \) with respect to the subsystem. In particular, the aforementioned subset of bosons contributes to the normal density, which may have further contributions from other degrees of freedom. To derive the form of the local Galilei transformation (LGT) that maps between these local reference frames in the case of a single bosonic species of mass \( m \), we first use the canonical bosonic commutation relation \( [\psi(x),\psi(x')^\dagger] = \delta(x-x') \) \( \Omega \) to show that for any differentiable vector field \( f: \mathbb{R}^d \rightarrow \mathbb{R}^d \) and for \( U[f(x)] := \exp(i \int d^dx f(x)^T x \psi(x)^\dagger \psi(x)) \) a unitary

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operator-valued functional,
\[ U[f(x)]\psi(x)U[f(x)]^\dagger = \psi(x)e^{-i\int f(x)T_x} \]
\[ U[f(x)]\psi(x)U[f(x)]^\dagger = \psi(x)e^{i\int f(x)T_x}. \tag{1} \]
Taking \( P := \int d^dx g(x) \) to be the total momentum, where
\[ g(x) := (1/2\hbar)(\nabla - \nabla)\psi(x) \]
is the momentum density (we take \( \hbar = k_B = 1 \) throughout), it then follows that
\[ U[f(x)]PU[f(x)]^\dagger = P - \int d^dx (\nabla f(x)T_x)\psi(x)^\dagger\psi(x). \]
The connection to the usual Galilei transformation \([21\text{-}24]\) is made by choosing \( f(x) = mv \) to be a constant momentum in Eq.\((2)\). However, in the present work we do not make the assumption of homogeneity, and instead allow \( f(x) = mv(x) \), where \( v(x) \) is a smooth vector field on \( \mathbb{R}^d \). Note that the function \( \sum_{j=1}^N mv(R_j)T_j \), where \( R_j \) is the position of the \( j \)-th atom, was proposed by Feynman to give the phase of a superfluid state with slowly-varying velocity (see Eq.\((16)\) of \([3]\)).

**Local superfluidity:** Given a system of \( N \) bosons of mass \( m \) at thermal equilibrium at temperature \( \beta^\dagger \) and density \( \rho = \frac{mN}{\mathcal{V}} \), where \( \Omega \subset \mathbb{R}^d \) is a connected subset with volume \( \Omega > 0 \), a standard derivation of the global normal fluid tensor consists of the following recipe \([25\text{-}28]\):
1. The two fluid assumption \((\rho_{n})_{i,j} = \rho - (\rho_{g})_{i,j} \)
2. Calculation of the expectation of \( P_i \) in the Galilei boosted Gibbs state \( \sigma(\beta)_{n} := U[mv_j]e^{-\beta H}U[mv_j]^\dagger/\text{tr}e^{-\beta H} \), use of the two fluid assumption and the correspondence principle to equate the observed momentum density to the expectation value of the momentum
\[ (\rho_{n})_{i,j}v_i|\Omega| = \text{tr}P_i\sigma(\beta)_{v_j}. \tag{3} \]
In the limit of zero relative momentum between superfluid and normal fluid fractions, one has \( (\rho_{n})_{i,j}|v_i| = 0 = \lim_{v_i \to 0} (|v_i|)_{v_i} = \text{tr}P_i\sigma(\beta)_{v_j}. \)

A local, equilibrium version of the normal fluid tensor in the presence of a constant relative velocity between the normal and superfluid components can be formulated in terms of the momentum density \( g(x) \) \([29]\). We presently generalize this approach to include a spatially varying relative velocity field \( v(x) \) and an arbitrary bosonic quantum state associated to positive, trace class operator \( \sigma \). The analogous framework is as follows:
1. The local two-fluid assumption is given by \( \rho_{n}(x)_{i,j} := \text{tr}v(x)^\dagger\psi(x)\sigma - \rho_{g}(x)_{i,j} \) for all \( x \in \Omega \),
2. Calculation of the expectation of \( g(x) \), in the locally Galilei transformed state \( \sigma(\omega)_{v_j} := U[mv_j]\sigma U[mv_j]^\dagger/\text{tr}e^{-\beta H} \), use of the local two fluid assumption and the correspondence principle to equate the observed momentum density to the expectation value of the momentum density
\[ (\rho_{n}(x))_{i,j}v(x)_{j} = \text{trg}(x)_{i}\sigma_{v(x)_{j}}. \tag{4} \]
In particular, the prescription \( 3' \) leads to the following formula for the local normal fluid tensor in the limit \( v(x) \to 0 \):
\[ \rho_{n}(x)_{i,j}|v(x)| = \lim_{v(x) \to 0} v(x)_{j} \text{trg}(x)_{i}\sigma_{v(x)_{j}}. \tag{5} \]
We emphasize that the localized two fluid assumptions \( 1'\text{-}4' \) are kinematic in nature, i.e., they are well-defined without any assumption of the quantum dynamics of \( \sigma \) and, therefore, without the assumption of the existence of a Landau critical velocity \([20]\) in the fluid. Calculations of time-dependent superfluid response, which have been carried out in certain cases for both noiseless and dissipative quantum evolution \([31\text{-}33]\), can also be suitably localized.

As an immediate application of Eq.\((5)\), and to demonstrate that the above generalization subsumes well known results from the case of noninteracting excitations, we note that for a gas of free phonons with dispersion \( \epsilon_k = \epsilon_s ||k|| \) described by the Gibbs state \( \sigma(\beta) := Z^{-1}\epsilon_s \sum_{k} \epsilon_s ||k||a_k^\dagger a_k \), the momentum density in the locally Galilei transformed system is given by
\[ \text{trg}(x)_{i}\sigma(\beta)_{v(x)_{j}} = \frac{mT^4\pi^4S(d)}{15\epsilon_s^3} \nabla_i (v(x)_{j}x_j) \]
\[ - mc_s T \int d^dq \mathcal{G}(q;\beta)\sin(q^\dagger x)\hat{h}(q) \]
where \( \mathcal{G}(q;\beta) \) is a temperature-dependent form factor and \( S(d) \) is the area of the \((d-1)\) sphere. In \([20]\), we show that Eq.\((6)\) leads to Landau’s formula \([20, 32]\) for the phonon contribution to the normal fluid density when \( v(x) \) is a constant velocity field.

The present analysis can also be used to determine the conditions for which a local two-fluid description is valid for a state that exhibits Bose-Einstein condensation (BEC) of excitations. For example, whereas several previous calculations of kinematic superfluid density making use of global Galilei transformations assign \( \rho_{n} = 0 \) to the zero momentum BEC state \( |\psi_k=0\rangle := \frac{1}{\sqrt{N}}\sum_{k=0}^N|\text{VAC}\rangle \)
\([29, 33]\), we obtain the following more descriptive result \([20]\):
\[ \rho_{n}(x)_{i,j}|v(x)| = 0 = \frac{N}{|\Omega|} \delta_{i,j} + \frac{N}{|\Omega|} \lim_{v(x) \to 0} \frac{x_j \nabla_i v(x)_{j}}{v(x)_{j}}. \tag{7} \]
In the case of \( v(x) = \omega(-x_2, x_1) \) on \( \mathbb{R}^2 \), i.e., a rotating film with angular frequency \( \omega \), Eq.\((7)\) implies a complete longitudinal response \( \rho_{n}(x)_{i,j} = mN/|\Omega| \). By contrast, the vanishing transverse response implies complete superfluidity \( \rho_{n}(x)_{i,j} = mN/|\Omega| \), in agreement with the analysis of the rotating bucket experiment by linear response methods \([23]\).

In ab initio numerical calculations of He II in equilibrium, no assumption is made regarding the marginal quantum state of the quasiparticles. Instead, the superfluid density is calculated from a Gibbs state of the entire system from the second moment of a classical random
where $\mathbf{H}$ is the system Hamiltonian (e.g., defined by the sum of kinetic energy and a suitable two-body interaction). Under the change in the Helmholtz free energy satisfies $e^{-\beta \Delta F} = \mathbb{E} e^{im \sum_j=1 \{v(x_j)^T \nu^{x_j} - v(x_j^{(i)})^T \nu^{x_j^{(i)}}\}} \langle R | e^{-\beta \mathbf{H}} | R' \rangle$ on $\mathbb{R}^N$. By assuming small values of $v(x)$ and also assuming that the function $v(x)^T \nu$ varies slowly throughout the system, one can write $\beta \Delta F \approx m^2 w^2 \mathcal{E} W(\{x^{(i)}\})^2$, where $W(\{x^{(i)}\}) := \sum_{x=1}^m \nu(x)^T \nu(x^{(i)})$, and $\Delta F$ follows immediately from the commutation relation Eq. 11 [20]. Therefore, by Eq. 11, the following formula holds:

$$\rho_{\nu}(x,\beta) \approx m \partial_v (v(x)^T \nu \psi(x)) \rho_{\nu}(x,\beta) \frac{\delta}{2\beta} W(\{x^{(i)}\})^2$$

where $\hbar$ has been reintroduced for ease of comparison to the original winding number estimator of superfluid density [34], which is exactly obtained from Eq. 10 for constant $v(x) = v$. The expectation value in Eq. 11 can be calculated using, e.g., standard path integral Monte Carlo techniques.

Superfluidity of continuum matrix product states: Recent path integral Monte Carlo calculations of R"enyi entanglement entropy have shown that the superfluid behavior of liquid He II arises from quantum states that satisfy the area law for entanglement entropy [37]. Therefore, one expects that quantum states that describe superfluidity in the zero temperature limit can be expressed as continuum matrix product states (cMPS) in the appropriate dimension, although this expectation is rigorous only in one spatial dimension (1-d). A cMPS $|\psi(R,Q)\rangle$ on the circle $[-\frac{L}{2}, \frac{L}{2}]$ (with periodic boundary condition assumed) is defined by two auxiliary $D \times D$ matrix-valued functions $Q(x)$ and $R(x)$ via $|\psi(R,Q)\rangle := \text{Tr} \mathcal{P} \exp \left( \int_{-\frac{L}{2}}^{\frac{L}{2}} dx \mathcal{Q}(x) \otimes 1_D + R(x) \otimes |\psi(x)\rangle \right) | \text{VAC} \rangle$, where $\text{Tr}$ is the trace in the $D^2$-dimensional auxiliary matrix space and $\mathcal{P}$ is the path ordering operator. In order to calculate $\rho_{\nu}(x)$ for a cMPS $|\psi(R,Q)\rangle$, it suffices to utilize the formula for the one-body density $\rho_{Q,R}(x,y) := \langle \psi(x)^{\dagger} \psi(y) \rangle_{\psi(R,Q)}$ and note that $\langle g(x)^{\dagger} \mu_{\text{mV}(x)} \psi(Q,R) \rangle = \frac{\beta}{\beta} \lim_{x \rightarrow y} \left( \frac{\rho_{Q,R}(x,y)}{\rho_{Q,R}(x,y)} \right)$.

Working with the gauge $Q(x) = 0$ and assuming that the ground state $|\psi(R,Q)\rangle$ of the quasiparticles has no momentum at any point, the result is given by

$$\langle g(x)^{\dagger} \mu_{\text{mV}(x)} \psi(Q,R) \rangle = \text{Tr} T \left( -\frac{L}{2} \frac{L}{2} \right) A(x)$$

where $T(y,x) := \mathcal{P} \exp \left( \int_{y}^{x} dz R(z) \otimes R(z) \right)$ and $A(x) := m (x \frac{dv}{dx} + v(x)) R(x) \otimes R(x)$. Referring to Eq. 11 and noting that the local density $\rho(x) := \langle \psi(x)^{\dagger} \psi(x) \rangle$ in cMPS is given by $\rho(x) = \text{Tr} T \left( -\frac{L}{2}, \frac{L}{2} \right)$, we find the following general formula for the velocity-dependent normal fluid density in 1-d:

$$\rho_{\nu}(x) = m \rho(x) + mx \frac{dv}{dx} \rho(x).$$

The similarity of Eq. 11 to Eq. 111 belies the fully quantum nature of the latter result. Because the set of MPS is dense in the Fock space, Eq. 11 is an exact equation which can be utilized in calculations of the local superfluid density in non-translation invariant, quasi-one-dimensional systems of He II, which have been shown to exhibit Luttinger liquid behavior [39, 40].

Although systems of definite particle number are not described by $D = 1$ cMPS, the right hand side of Eq. 11 is still useful as a generating function for $\rho_{\nu}(x)$. As an example, note that given a sub-Hilbert space $K$ of $L^2([-\frac{L}{2}, \frac{L}{2}])$, any state of the form [41]

$$|\psi_{nk}\rangle := \frac{1}{\sqrt{n_k!}} \prod_{k=1}^{M} a^{\dagger}_{\nu_k} | \text{VAC} \rangle$$

where $\sum_{k=1}^{M} n_k = N$ and where the canonical boson operators $a_{\nu_k}^{\dagger}, a_{\nu_k}$ satisfy $[a_{\nu_k}, a_{\nu_k}^{\dagger}] = |\psi_k\rangle \langle \psi_k|$, can be generated from a cMPS of bond dimension $D = 1$ with $R(x) = \sum_{j=1}^M \xi_j \psi_j(x)$, where $|\psi_j(x)\rangle := \langle x | \psi_j \rangle$. Using the gauge freedom of cMPS to take $Q(x) = 0$, the normal fluid distribution is given by

$$\rho_{\nu}(x) = \langle \psi(x)^{-1} \left( \prod_{j=1}^{M} \delta^{\nu_{\epsilon_j}}_{\xi_j} \delta^{\nu_{\epsilon_j}}_{\xi_j} \right) \langle g(x)^{\dagger} \mu_{\text{mV}(x)} \psi(Q,R) \rangle \rangle | \xi_j, \bar{\xi}_j = 0, \rangle = \left( m + mx \frac{dv}{dx} \right) \left( \prod_{j=1}^{M} \delta^{\nu_{\epsilon_j}}_{\xi_j} \delta^{\nu_{\epsilon_j}}_{\xi_j} \right) \rho(x) | \xi_j, \bar{\xi}_j = 0. \rangle$$

Despite the fact that cMPS are dense in the bosonic Fock space generated by $L^2([-\frac{L}{2}, \frac{L}{2}])$, they do not provide an economical (i.e., low bond dimension) ex-
pression for general quantum phases of systems of pair-wise interacting bosons. In particular, a more general class of states than Eq. (12) that incorporates pair correlations can be written as

$$
\prod_{\ell=1}^{N/2} U^{(\ell)} \left( \sum_{j=1}^{r_{\ell}} \lambda_j^{(\ell)} |e_j^{(\ell)}\rangle \right)^{r_{\ell}} U^{(\ell)\dagger} |\text{VAC}\rangle
$$

where $U^{(\ell)}$ is a particle number conserving unitary generated by a quadratic Hamiltonian, $\sum_{\ell=1}^{N/2} r_{\ell} = N/2$, $r_{\ell} \in \mathbb{Z}_{\geq 0}$, and the use of a set of orthonormal bases $\{|e_j^{(\ell)}\rangle\}_{j=1,2,\ldots}$ for $\mathcal{K}$ is justified by the Schmidt decomposition for arbitrary two particle pure states. The states in Eq. (14) are not in general approximable by eMPS of small $D$. However, they appear as ground states of a class of exactly solvable bosonic models [44], as probe states for quantum estimation of bosonic Hamiltonians [45], and as variational ansätze for ground states of interacting Bose liquids and gases of a definite number of particles. In the case of a single nonzero $r_{\ell}$, the correlation functions and large-$N$ asymptotics of Eq. (11) have been calculated [46]. Although calculation of the local superfluid density of the class of states defined by Eq. (14) is beyond the scope of the present work, we note that the methods used to prove Eq. (8), (7) are applicable to any Gaussian state of the quasiparticle field and, therefore, to any projection of a Gaussian state to a finite particle number sector. For example, for $r_{\ell} = \frac{N}{2} \delta_{\ell,1}$, $|e_j^{(1)}\rangle := |\frac{2\pi j}{N}\rangle$ a single particle momentum eigenstate (the sum over $j$ is now taken from $-\infty$ to $\infty$), and $U^{(1)} := \prod_{k>0} e^{-i\frac{\pi}{4}(a_k^\dagger a_{-k} + h.c.)} e^{i\frac{\pi}{4}(a_k + a_{-k})}$ with $k \in \frac{2\pi}{N} \mathbb{Z}$, Eq. (13) is simply the projection to fixed depletion number $N$ of the $k \neq 0$ Bogoliubov ground state $e^{-\sum_{k>0} \sigma_k a_k^\dagger a_k}|\text{VAC}\rangle$ (which is Gaussian, with $\sigma_k := \lambda_k^{(1)}$ determined by Bogoliubov transformation) [47]. The normal fluid distribution of the Bogoliubov ground state can then be straightforwardly computed by applying a LGT and subsequently using the same methods that we have applied to calculate Eq. (1) for other Gaussian states.

**Limits on distillation of superfluidity:** Phenomena in He II such as the fountain effect and dissipationless flow through nanoporous Vycor packing provide examples of quantum dynamics that result in the conversion of partially superfluid system of $N$ atoms to a system of $N' < N$ atoms with greater superfluid fraction. As superfluid devices such as matter wave interferometers become controllable by locally modifying quasiparticle states and dynamics, the ultimate limits to such distillation protocols, taking into account potentially very many system copies, becomes relevant. It is known from early studies that the flow of He II through capillaries of diameter $O(1 \text{ mm})$ constitutes an “entropy filter” [28], but a quantum information-based analysis of this phenomenon has not been carried out. However, recent axiomatic formulation of a resource theory of quantum coherence (RTQC) allows to calculate asymptotic concentration and dilution rates between quantum states in terms of coherence measures [48]. An RTQC is defined by a 4-tuple $(\mathcal{B}, \mathcal{D}, T, C)$, where $\mathcal{B} = \{|f_j\rangle\}_{j \in J}$ is an orthonormal set in Hilbert space $\mathcal{H}$, $\mathcal{D}$ is the set of probability measures over the set of projections $\{|f_j\rangle\langle f_j|\}_{j \in J}$ (i.e., $\mathcal{D}$ is the set of incoherent quantum states), $T$ is the set of completely positive, trace preserving maps $\Phi$ such that $\Phi(\mathcal{D}) \subseteq \mathcal{D}$, and $C$ is a coherence monotone that satisfies the RTQC axioms.

To apply RTQC with the aim of deriving the rates for distillation of fully superfluid states in a subdomain $\Omega$ of the system, we first define for a region $\Omega$ and bosonic quantum state $\sigma$ the quantity $N_0 := \langle f_0 | d^4x |\psi(x)\rangle |\psi(x)\rangle_x$. For any value of $N_0$, we define $B_0 := \{\langle \prod_{j=1}^{N_0} |f_j\rangle |\psi(x)\rangle |\psi(x)\rangle_x : x \in \Omega\}$ and use an index $I$ to label elements of Eq. (19). One can clearly see that any $|\phi_i\rangle \in B_0$, $i \in I$, has $\rho_i(x)_{j,j} = 0$ for all $x \in \Omega$. For any bosonic state $\sigma$ of $N$ particles, one defines $\sigma_0 := \sum_{i,j \in I} \langle \phi_i | \sigma | \phi_i\rangle \| \phi_i \| \sigma | \phi_i\rangle$ to be the marginal state corresponding to $N_0$ particles in $\Omega$, where $|\Omega\rangle$ is the volume of $\Omega$. By defining the decohering map $\Delta_\Omega(\sigma_0) := \sum_{i,j \in I} \langle \phi_i | \sigma_0 | \phi_i\rangle \| \phi_i \| \sigma_0 | \phi_i\rangle$, one forms the coherence measure $C_\Omega(\sigma) := S(\Delta(\sigma_0)) - S(\sigma_0)$, where $S$ is the von Neumann entropy of a quantum state. By defining the Hilbert spaces $\mathcal{H}_\Omega := \text{span}_B \mathcal{B}_\Omega$, and associating to any open set $\Omega' \subset \Omega$ the product of linear maps $P_{N_0,\Omega'}: \mathcal{H}_\Omega \rightarrow \mathcal{H}_{\Omega'}$, where $P_{\Omega',\Omega} := \int_{\Omega'} d^4x \int_{N_0} |x_j\rangle |x_j\rangle_j$ is the domain projection to $\Omega'$ and $P_N$ is the projection to the $N$ particle sector, one obtains a sheaf of RTQCs over the system domain.

With these definitions in place, RTQC theorems can now be applied to calculate the asymptotic rate of distillation of superfluid states in $\Omega$. In the following, we use the term strongly superfluid state to mean a pure state such that the action of $U^{(\text{mev}(x))}$ on that state gives a zero eigenvector of $g(x)$. Given a velocity field $v(x)$, a rectangular subregion $\Omega$, and the state $\sigma$, Eq. (1), (1) imply that the state $|\text{SF}_\Omega\rangle \propto \left( \frac{1}{\sqrt{|\Omega|}} \int_{\Omega} d^4x e^{-\text{mev}(x)^T x \psi(x)^T} \right)^{N_0} |\text{VAC}\rangle$ is strongly superfluid in $\Omega$ (viz., because it is defined so that the action of $U^{(\text{mev}(x))}$ produces a $k = 0$ Bose-Einstein condensate of $N_0$ particles in $\Omega$, and it is readily verified that $C_\Omega(|\text{SF}_\Omega\rangle) = N_0 \log_2 |\Omega|$). The state $|\text{SF}_\Omega\rangle$ is also a maximally coherent state of the RTQC. It now follows from the basic distillation theorem of RTQC that for any $\epsilon > 0$, there is an incoherent operation $T$ and an $m \in \mathbb{N}$ such that $\| T(\sigma^\otimes n) - |\text{SF}_\Omega\rangle \langle \text{SF}_\Omega| \otimes n R \|_1 < \epsilon$ for all $n > m$ if and only if $R < \langle C_\Omega(\sigma) / N_0 \log_2 |\Omega| \rangle$, where $\| \cdot \|_1$ is the trace norm [49, 51]. The rate $R$ gives the ultimate limit for distillation of a perfect superfluid in independent copies of the domain $\Omega$ from independent copies of a marginal state of a Bose liquid, under potentially noisy quantum operations that, roughly speaking, map solid phases to solid phases.
Conclusions: In this work, a framework for the analysis of local kinematic superfluidity has been introduced and applied to quantum states of a wide variety of bosonic systems. The results set the stage for further analytical and numerical calculations of local superfluidity and provide a way to consider superfluidity in the context of modern quantum information via RTQCs. Extension of the present approach to spin-dependent flows accessible in systems of spinor bosons, and the quantum mechanical treatment of localized superfluidity for relativistic two-fluid systems, constitute two of several potential avenues of future research.

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In practice, $\mathcal{B}_0$ can be made to span a finite dimensional Hilbert space by using a fine spatial grid on $\Omega$.

I. APPENDIX

A. Proof of Eq. (2)

The proof utilizes the commutation relations $[\psi(x), \nabla \psi(x')] = \nabla \delta(x - x')$ and $[\psi(x)^\dagger, \nabla' \psi(x')] = -\nabla' \delta(x - x').$ In detail, by defining $A := \int d^d x f(x) \psi(x)^\dagger \psi(x)$, and using $P = -i \int d^d x \psi(x)^\dagger \nabla \psi(x)$, it follows that

$$AP - PA = -i \int d^d x \int d^d x' F(x) \left( \delta(x - x') \psi(x)^\dagger + \psi(x)^\dagger \psi(x) \right) \nabla' \psi(x') - PA$$

$$= -i \int d^d x f(x) \psi(x)^\dagger \nabla \psi(x) - i \int d^d x \int d^d x' F(x) \psi(x)^\dagger \left( -\nabla' \delta(x - x') \psi(x) + (\nabla' \psi(x')) \psi(x)^\dagger \right) - PA$$

$$= -i \int d^d x f(x) \psi(x)^\dagger \nabla \psi(x) + i \int d^d x \int d^d x' F(x) \psi(x')^\dagger \left( \nabla' \delta(x - x') \right) \psi(x)$$

$$= i \int d^d x \left( \nabla F(x) \right) \psi(x)^\dagger \psi(x)$$

where the last line follows from integration by parts. Then, since $[\int d^d x f_1(x) \psi(x)^\dagger \psi(x), \int d^d x f_2(x) \psi(x)^\dagger \psi(x)] = 0$ for any functions $f_1$, $f_2$, it is clear that $e^{iA}P e^{-iA} = P - \int d^d x \left( \nabla F(x) \right) \psi(x)^\dagger \psi(x). \blacksquare$

B. Proof of Eq. (6) and Eq. (7)

To demonstrate Eq. (6), note that Eq. (11) implies that

$$U[mv(x)] \sigma(\beta) U[mv(x)]^\dagger = Z^{-1} e^{-\frac{Z}{T} \sum_k \int d^d x d^d x' e^{i(k + m v(x))T x} e^{-i(k + m v(x'))T x'} \psi(x)^\dagger \psi(x').}$$

By using the expansion $F[v(x)] = F[0] + \int d^d x v(x) \frac{\delta F}{\delta v(x)}|_{v(x)=0} + \mathcal{O}(v(x)^2)$ for any functional $F$, one can write

$$\text{tr} g(x) \sigma(\beta) v(x) = \frac{-\beta \epsilon_m}{L^d} \text{tr} \left[ \sigma(\beta) g(x) \int d^d x' \sum_k i |k| (v(x)_j x_k - v(x')_j x'_k) e^{i k T(x-x')} \psi(x)^\dagger \psi(x') \right] + \mathcal{O}(v(x)^2)$$

because $\text{tr} g(x) \sigma(\beta) = 0$ by parity. Before considering the general case of nonconstant $v(x)$, we note that Landau’s formula for the phonon contribution to the normal fluid density is obtained by taking $v(x)_j = v_j = \text{const.}, \ d = 3$, and

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the $L \to \infty$ limit of Eq. (17). This fact is verified as follows:

\[
\text{trg}(x) \sigma(\beta)_{ij} = \frac{-\beta c_s m v_i}{L^d} \text{Re} \left[ i \sigma(\beta) \left( \sum_{k, k'} (k_i + k_{i'}) e^{-i(k - k')^T x} a_{k'}^i a_k \right) \right] + \mathcal{O}(v^2)
\]

where we have used $g(x) = \frac{1}{2\pi^d} \sum_{k, k' \in 2\pi^d / L} (k + k') e^{-i(k - k')^T x} a_{k'}^i a_k$ for the momentum density in terms of single particle momentum eigenstates. In the second line, a generating function has been temporarily introduced, with parameter $\lambda \in \mathbb{R}^3$, for ease of calculation. Eq. (18) can be used to extract $(\rho_n)_{ij}$ from Eq. (18). (Landau’s formula is usually reported as $\frac{1}{2} \sum_{j=1}^d (\rho_n)_{ij}$ and in the $L \to \infty$ limit). To consider Eq. (17) for periodic, nonconstant velocity fields $v(x)$, the Fourier transform $\tilde{h}(q) := \frac{1}{\pi} \int d^d x e^{-iq^T x} v(x)^T x$ is used, where we assume that $v(x) = v(-x)$ so that $\tilde{h}(q) \in \mathbb{R}$. Eq. (17) becomes

\[
\text{trg}(x) \sigma(\beta)_{ij} = \frac{-\beta c_s m v_i}{L^d} \text{Re} \left[ i \sigma(\beta) \left( \sum_{k, k'} (k_i + k_{i'}) e^{-i(k - k')^T x} a_{k'}^i a_k \right) \right] + \mathcal{O}(v^2)
\]

where, in the second line, we have noted that $\sigma(\beta)$ is diagonal in the number occupation basis, and, therefore, only the terms $k = k''$ and $k' = q + k''$ survive in the above sums (the $q = 0$ contribution is easily seen to vanish). In the second line of Eq. (19), we first analyse the contribution of the term that is quadratic in creation and annihilation operators:

\[
\frac{-\beta c_s m}{L^d} \text{Re} \left[ i \sigma(\beta) \sum_{k, k', q, q', q \neq 0} (2k_i'' + q_i) e^{-iq^T x} \|k''\|^2 \tilde{h}(q) a_{k'}^i a_{k''} \right] = \frac{c_s m}{L^d} \text{Re} \left[ i \sum_{k', q, q' \neq 0} \frac{(2k_i'' + q_i) e^{-iq^T x} \|k''\|^2 \tilde{h}(q)}{e^{\beta c_s \|k''\|} - 1} \right]
\]

where the second line follows due to the odd parity of $k_i''$. Taking $L \to \infty$ in the last line and evaluating $\int_0^\infty dx \frac{x^3}{e^x - 1} = \frac{\pi^4}{15}$ gives the first term in Eq. (19). The term in Eq. (19) that is quartic in creation and annihilation operators is evaluated
as follows:

\[-\beta c m \frac{\text{Re tr}}{L^{2d}} \left[ i\sigma(\beta) \sum_{k^{\prime},q} (2k^{\prime} + q_i) e^{-i q_i x} \|k^{\prime}\| \hat{h}(q) a_{k^{\prime}+q} \hat{a}_{k^{\prime}+q} \right] = \frac{-\beta^{-1} c m}{L^{2d}} \sum_{k^{\prime},q} (2k^{\prime} + q_i) \sin(q^{T} x) \|k^{\prime}\| \hat{h}(q) \]

By defining the form factor \( \mathcal{G}(q; \beta) := \lim_{L \to \infty} \frac{1}{L^{2d}} \sum_{k^{\prime}} \frac{(2k^{\prime} + q_i) \|k^{\prime}\|}{(e^{\beta c} \|k^{\prime}\| - 1)(e^{\beta c} \|k^{\prime}\| + 1)} \), the result is seen to be the second term of Eq.(9). \[\square\]

We now proceed to the proof of Eq.(7), for which we specialize to the box \( \Omega = [-L/2, L/2]^d \). The action of an LGT on the creation operator \( a_k^{\dagger} \) is given by:

\[ U[mv(x)] a_k^{\dagger} U[mv(x)]^\dagger = \frac{1}{\sqrt{N}} \left( \sum_k c_k a_k^{\dagger} \right)^N \langle \text{VAC} \rangle, \]

where the second line follows from integration by parts. Taking the limit in Eq.(24) as prescribed in Eq.(5) gives Eq.(7).

\[ \langle \text{VAC} \rangle \]

C. Proof of Eq.(9) and relation to the winding number formula for \( \rho_s \)

From the unitary action in Eq.(11), an LGT in the \( j \)-th direction transforms a normal ordered Hamiltonian describing the dynamics of bosons of mass \( m \) interacting pairwise with potential \( V(x - y) \) to the following Hamiltonian

\[ U[mv(x)] H U[mv(x)]^\dagger \approx \frac{1}{2m} \int d^d x \nabla \psi^\dagger(x) \cdot \nabla \psi(x) \]

\[- \int d^d x g(x) \cdot \nabla (v(x) j x_j) \]

\[ + \frac{m}{2} \int d^d x \|\nabla (v(x) j x_j)\|^2 \psi^\dagger(x) \psi(x) \]

\[ + \int d^d x d^d x' V(x - y) e^{2im(v(x) j x_j - v(x') j x'_j)} \psi^\dagger(x') \psi(x)^2 \]

where we have neglected terms in \( O(v^2), O(v \partial \psi) \). It follows from this LGT that

\[ \frac{\delta \Delta F}{\delta \partial_i (v(x) j x_j)} \approx \langle g(x) \rangle_{\rho(\beta)(v(x))} - m \partial_i (v(x) j x_j) \langle \psi^\dagger(x) \psi(x) \rangle_{\rho(\beta)(v(x))}, \]

Eq.(11) is now used to relate \( \langle g(x) \rangle_{\rho(\beta)(v(x))} \) to \( \rho_n(x), j \). Taking the appropriate functional derivative of the equation \( \Delta F \approx \frac{m^2}{25} \text{E} W(\{x^{(j)}\})^2 \) derived in the main text, and setting it equal to the right hand side of Eq.(26) results in Eq.(11). \[\square\]

Let us consider \( \rho_n(x), j \), which is relevant for isotropic systems. In this case, the local two fluid relation \( \rho_s(x), j +\]
\[ \rho_n(x)_{j,j} = m\langle \psi(x)^\dagger \psi(x) \rangle \] implies that Eq. (26) becomes

\[ -\frac{\delta \Delta F}{\delta \partial_i (v(x)_j x_j)} \approx -\rho_s(x)_{j,j} v(x)_j - mx_j \partial_j v(x)_j \langle \psi(x)^\dagger \psi(x) \rangle \rho(\beta) v(x)_j, \] (27)

The second term in the above equation vanishes if \( v(x)_j = v_j = \text{const} \) for all \( x \) and, therefore, Eq. (9) becomes

\[ \rho_s(x)_{j,j} \approx \frac{m^2}{2 \beta \hbar^2 v_j} \frac{d}{dv_j} E W(\{x^{(\ell)}\})^2 \] (28)

where, for \( v(x)_j = v_j \), \( W(\{x^{(\ell)}\}) = v_j \sum_{\ell=1}^N (x^{(\ell)}_j - x^{(s(\ell))}_j) \). Taking \( \frac{1}{3} \sum_{j=1}^3 \rho_s(x)_{j,j} \) gives the well known result Eq. (22) of Ref. [34], independent of \( x \).

D. cMPS for states of the form Eq. (12)

We define the displacement operator for cMPS by \( D(Q, R) = \mathcal{P} \exp \left[ \int_{-L/2}^{L/2} dx Q(x) \otimes I + R(x) \otimes \psi(x)^\dagger \right] \), where \( Q(x) \) and \( R(x) \) are \( D \times D \) matrix-valued distributions, \( D \) is the bond dimension, and \( \mathcal{P} \) is the path-ordering operator. For any \( n \in \mathbb{Z}_{\geq 0} \), the state \( a^{(n)}_{\psi} |\text{VAC}\rangle \) can be written as \( \partial_n \xi \text{Tr} D(\xi, \{ R_j \}|\text{VAC}\rangle \)\( |\xi = 0 \rangle \), where \( R(x) = \psi(x) \) is the wavefunction corresponding to \( |\psi\rangle \) and \( Q(x) = \lambda \) for any \( \lambda \in \mathbb{R} \) (i.e., the bond dimension of the generating cMPS is \( D = 1 \)), and \( \xi \in \mathbb{C} \). For the purposes of the present section, \( Q(x) \) can be gauged away [38]. Analogously, given a collection \( \{ |\psi_j\rangle \}_{j=1, \ldots, M} \), and defining \( R_j(x) = \psi_j(x) \in L^2([-L/2, L/2]) \), one has the following generating functional form of \( \prod_{j=1}^M a_{|\psi_j\rangle}^{(n)}|\text{VAC}\rangle \) in terms of a non-translationally invariant cMPS, again with \( D = 1 \):

\[
\prod_{j=1}^M a_{|\psi_j\rangle}^{(n)}|\text{VAC}\rangle = \prod_{j=1}^M \partial_{\xi_j}^{n_j} \left( \prod_{j=1}^M \text{Tr} D(\xi_j R_j)|\text{VAC}\rangle \right) \bigg|_{\{\xi_j\} = 0} = \prod_{j=1}^M \partial_{\xi_j}^{n_j} \text{Tr} D(\hat{R}_{\{\xi_j\}})|\text{VAC}\rangle \bigg|_{\{\xi_j\} = 0} \] (29)

where \( \hat{R}_{\{\xi_j\}}(x) = \sum_{j=1}^M \xi_j \psi_j(x) \) and \( \hat{Q}(x) = \sum_{j=1}^M \lambda_j, \lambda_j \in \mathbb{R} \).