Dimension-free convergence rates for gradient Langevin dynamics in RKHS

Boris Muzellec 1  Kanji Sato 2  Mathurin Massias 3  Taiji Suzuki 24

Abstract
Gradient Langevin dynamics (GLD) and stochastic GLD (SGLD) have attracted considerable attention lately, as a way to provide convergence guarantees in a non-convex setting. However, the known rates grow exponentially with the dimension of the space. In this work, we provide a convergence analysis of GLD and SGLD when the optimization space is an infinite dimensional Hilbert space. More precisely, we derive non-asymptotic, dimension-free convergence rates for GLD/SGLD when performing regularized non-convex optimization in a reproducing kernel Hilbert space. Amongst others, the convergence analysis relies on the properties of a stochastic differential equation, its discrete time Galerkin approximation and the geometric ergodicity of the associated Markov chains.

Introduction
Convex, finite-dimensional optimization problems have been studied at length, and there exists a variety of well-understood algorithms to solve them efficiently (Nesterov, 1983; 2004; Hiriart-Urruty and Lemaréchal, 1993; Boyd and Vandenberghe, 2004; Nocedal and Wright, 2006). In the non-convex case however, these methods are only guaranteed to converge to stationary points of the objective function. This is to be contrasted with the ubiquity of the non-convex case, largely due to the successes of deep learning methods, for which optimization methods with good empirical behavior are widely used (Robbins and Monro, 1951; Duchi et al., 2011; Zeiler, 2012; Kingma and Ba, 2014). In a different perspective, stochastic gradient Langevin dynamics (SGLD), which can be seen as stochastic gradient descent methods with additive Gaussian noise injection at each iteration, was introduced by Welling and Teh (2011). In the case of a strongly convex objective function $L$, recent studies (Dalalyan, 2017b) highlighted the connections between sampling from log-concave densities $f(x) \propto \exp(-\beta L(x))$ concentrated around the minimum of $L$, and minimizing $L$. Such distributions can be obtained as the stationary distributions of first order Langevin dynamics

$$dX(t) = -\nabla L(X(t)) dt + \sqrt{2\beta^{-1}} dB(t),$$

where $\{B(t)\}_{t \geq 0}$ is the standard Brownian motion in $\mathbb{R}^d$, and $\beta > 0$ is the inverse temperature. Chiang et al. (1987); Gelfand and Mitter (1991); Roberts and Tweedie (1996) studied the convergence of $X(t)$ to the stationary Gibbs distribution $\pi(dx) \propto \exp(-\beta L(x))$, and the concentration of the samples around the global minimum, while more recently Dalalyan (2017a); Durmus and Moulines (2016; 2017) analyzed the convergence rates of discrete-time Langevin updates for sampling from log-concave densities.

Recent studies have shown that Langevin dynamics based algorithms converge near a global minimum of $L$, even when $L$ is not convex, provided $L$ is dissipative and has Lipschitz gradient. The analysis relies on the connection between the iterates of Langevin dynamics based algorithms and the Markov chain solution of the continuous time Langevin equation, which admits the Gibbs measure as invariant distribution. Raginsky et al. (2017) provided a non asymptotic convergence rate in expectation to an almost minimizer for stochastic gradient Langevin dynamics (SGLD), which Xu et al. (2018) improved while also providing an extension to variance-reduced algorithms. In an alternative approach, Zhang et al. (2017) provided bounds on the hitting time of SGLD to neighborhoods of local minima.

However, these results only apply to finite-dimensional optimization, with rates growing polynomially or even exponentially with the dimension. In this paper, we study the rate of convergence when applying Langevin dynamics algorithms in infinite dimension. More precisely, we bound the probability of GLD of reaching prescribed level sets of the objective functional $L$ at iteration $m$. To our knowledge, this is the first application of GLD for infinite-dimensional

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non-convex optimization.

Our results rely on assumptions which are classical in the GLD/SGLD literature, and in the literature of approximation of invariant laws of stochastic partial differential equations (SPDE) in infinite dimension. In particular, we leverage the weak approximation error of the discrete time scheme of SPDEs analyzed by Bréhier and Kopec (2014); Bréhier and Kopec (2016) for general inverse parameter $\beta > 1$, where Debussche (2011); Wang and Gan (2013); Andersson and Larsson (2016) gave discretization error non-uniformly over the time horizon, and utilize the geometric ergodicity of continuous time dynamics (Jacquot and Royer, 1995; Goldys and Maslowski, 2006). Compared with Eq. (1), results in the infinite-dimensional setting usually involve a linear operator acting as a regularizer and whose spectrum “replaces” dimension in the convergence rates. More specifically, our contribution can be summarized as follows:

- We give a non-asymptotic error bound of the infinite dimensional GLD/SGLD implemented with a spectral Galerkin method, which has an explicit dependency on $\beta$ and is uniform over all time horizons.

- For that purpose, the geometric ergodicity of the time discretized dynamics is proven, which is known to be non-trivial.

- We give an upper bound of the distance between the expected objective value under the invariant measure and the global optimal solution in the infinite dimensional setting.

1. Notation and Framework

1.1. Notation and background on RKHS

Let $\mathcal{H}, \langle \cdot, \cdot \rangle$ be a Hilbert space. We will also use the notation $\| \cdot \|_\mathcal{H}$ to explicitly indicate the norm $\| \cdot \|$ is of $\mathcal{H}$. If $\phi$ and $V$ are two functions from $\mathcal{H}$ to $\mathbb{R}$ such that for all $x \in \mathcal{H}$, $|\phi(x)| \leq |V(x)|$, we write $\|\phi\|_V \leq 1$. $C^2_\phi$ is the set of bounded, twice continuously Fréchet differentiable functions with bounded first and second derivatives. We denote by $B(\mathcal{H})$ the set of bounded linear operators from $\mathcal{H}$ to $\mathcal{H}$ and $\| \cdot \|_{B(\mathcal{H})}$ denotes the operator norm. For a discrete or continuous Markov chain $\{X_t\}$, note $E_x[\cdot] \triangleq E[\cdot | X_0 = x]$.

Let $\mathcal{H}_K \subset \mathcal{H}$ be a Reproducing Kernel Hilbert Space (RKHS), with reproducing kernel $K$. By Mercer’s theorem (Mercer, 1909; Steinwart and Scovel, 2012), $\mathcal{H}_K$ can be described as:

$$\mathcal{H}_K = \left\{ \sum_{k=0}^\infty \alpha_k f_k : \sum_{k=0}^\infty \alpha_k^2 / \mu_k < \infty \right\},$$

(2)

where the $\mu_k$’s and $f_k$’s are the eigenvalues (in decreasing order) and corresponding eigenfunctions of $T_K$, the integral operator with kernel $K$ for a measure $\rho$:

$$T_K f_k(x) \triangleq \int K(x, y) f_k(y) \, d\rho(y) = \mu_k f_k(x),$$

(3)

and the $f_k$’s form an orthonormal system in $\mathcal{H}$. Therefore, in general $\mathcal{H}_K \subset \mathcal{H} = \{ \sum_{k=0}^\infty \alpha_k f_k : \sum_{k=0}^\infty \alpha_k^2 < \infty \}$.

Hence, we are working in two different geometries: if $f = \sum_{k \geq 0} \alpha_k f_k$ and $g = \sum_{k \geq 0} \beta_k f_k$, then $\mathcal{H}$ is equipped with the inner product $\langle f, g \rangle = \sum_{k \geq 0} \alpha_k \beta_k$, and $\mathcal{H}_K$ is equipped with the inner product $\langle f, g \rangle_{\mathcal{H}_K} = \sum_{k=0}^\infty \alpha_k \beta_k$. The norm in $\mathcal{H}_K$ induced by the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}_K}$ is denoted by $\| \cdot \|_{\mathcal{H}_K}$. Unless denoted by the $\mathcal{H}_K$ subscript, we will work in the geometry of $\mathcal{H}$.

In the following, for $L : \mathcal{H} \rightarrow \mathbb{R}$, the gradient $\nabla L(x)$ is defined as the Riesz representer of the Fréchet derivative of $L$, $DL(x)$ (i.e., the unique vector satisfying $\forall h, L(x+h) = L(x) + \langle \nabla L(x), h \rangle + O(\|h\|^2)$). We will identify $n$-order derivatives with $n$-th linear forms, and with vectors when there is no ambiguity (e.g., we write $D^3 L(x) \cdot (h, k)$ for the Riesz representer of $l \in \mathcal{H} \rightarrow D^3 L(x) \cdot (h, k, l)$).

1.2. Algorithm: gradient Langevin dynamics

We consider the following optimization problem:

$$\min_{x \in \mathcal{H}} L(x) = L(x) + \frac{\lambda}{2} \|x\|^2_{\mathcal{H}_K},$$

(4)

where $\lambda > 0$ and $L$ is potentially non convex. Assuming $L$ admits at least one global minimizer, we note

$$x^* \triangleq \arg \min_{x \in \mathcal{H}} L(x),$$

(5)

$$\hat{x} \triangleq \arg \min_{x \in \mathcal{H}} L(x) + \frac{\lambda}{2} \|x\|^2_{\mathcal{H}_K}. $$

(6)

In this study, we treat $\lambda > 0$ as a constant and assume that $L(x^*)$ and $L(\hat{x})$ are sufficiently close. The difference between these two quantities is extensively studied, for example, in the least squares estimation problem in RKHS (Caponnetto and De Vito, 2007).

We study the gradient Langevin dynamics (GLD) iterations to solve Problem (4). To define GLD, we need to make a heavy use of the infinite dimensional Brownian motion.

Definition 1 (Cylindrical Brownian motion/Wiener process). Given

- a complete orthonormal system of $\mathcal{H}$, $(e_i)_{i \in \mathbb{N}}$, where $I \subset \mathbb{N}$,
- a family $(\{W^i(t)\}_{t \geq 0})_{i \in I}$ of independent real Brownian motions, then $(\{W(t)\}_{t \geq 0} \triangleq \{\sum_{i \in I} W^i(t)e_i\}_{t \geq 0}$ is called a cylindrical Brownian motion.
Then, GLD updates are defined as follows:

\[
\begin{aligned}
X_0 &= x^0 \in \mathcal{H}, \\
X_{n+1} &= S_\eta X_n - \eta S_\eta \nabla L(X_n) + \sqrt{2\eta} S_\eta \epsilon_n,
\end{aligned}
\]  
(7)

where \( \eta > 0 \) is the stepsize, \( \beta \geq \eta \) is the inverse temperature parameter, the variables \( \epsilon_k \) are i.i.d. cylindrical standard Gaussian and \( S_\eta \equiv (\text{Id} + \eta \frac{1}{2} \nabla \| \cdot \|^2_{\mathcal{H}_K})^{-1} \). A crucial analysis tool is to see Eq. (7) as a time discretization of the following SPDE (Da Prato and Zabczyk, 1996):

\[
\begin{aligned}
X(0) &= x_0, \\
\text{d}X(t) &= -\nabla L(X(t)) + \sqrt{\frac{2}{\beta}} \text{d}W(t),
\end{aligned}
\]  
(8)

where \( \{W(t)\}_{t \geq 0} \) is a cylindrical Brownian motion (Definition 1). We refer to Da Prato and Zabczyk (1996) for the existence of solutions, its regularity conditions and related mathematical details. Note that the scheme Eq. (7) is semi-implicit: applying \((S_\eta)^{-1}\) reads

\[
X_{n+1} = X_n - \eta (\nabla L(X_n) + \sqrt{\frac{2}{\beta}} \| X_{n+1} \|^2_{\mathcal{H}_K}) + \sqrt{2\eta} \epsilon_n.
\]

**Approximated computation** Strictly speaking, the infinite dimensional GLD scheme presented above is computationally intractable. The *Galerkin approximation method* projects the dynamics to a finite dimensional subspace to make them computationally feasible. Let \( \mathcal{H}_N \) be an \( N + 1 \)-dimensional subspace of \( \mathcal{H} \) that is spanned by \( \{f_k\}_{k=0}^{N-1} : \mathcal{H}_N \equiv \text{Span}\{f_k \mid k = 0, \ldots, N\} \). Let \( P_N : \mathcal{H} \rightarrow \mathcal{H}_N \) be the orthogonal projection operator onto \( \mathcal{H}_N : P_N \sum_{k=0}^{\infty} \alpha_k f_k = \sum_{k=0}^{N} \alpha_k f_k \). Then, the GLD with Galerkin approximation can be formulated as

\[
X_{n+1}^N = S_\eta \left( X_n^N - \eta \nabla L_N(X_n^N) + \sqrt{2\eta} P_N \epsilon_n \right),
\]  
(9)

where \( X_0^N = P_N x_0 \in \mathcal{H}_N \) and \( \nabla L_N(x) \equiv P_N \nabla L(P_N x) \). Since this scheme is essentially finite dimensional, it can be implemented in practice.

Next, we consider a stochastic gradient variant of GLD (stochastic GLD; SGLD). Let us consider a finite sum risk minimization setting where

\[
\begin{aligned}
L(x) &= \frac{1}{n_b} \sum_{i=1}^{n_b} \ell_i(x),
\end{aligned}
\]

for \( \ell_i : \mathcal{H} \rightarrow \mathbb{R} \) which is Fréchet differentiable. SGLD makes use of a mini-batch of stochastic gradients (Welling and Teh, 2011) instead of the full gradient \( \nabla L(x) \): \( g_n(x) = \frac{1}{n_b} \sum_{i \in I_n} \nabla \ell_i(x) \) where \( I_n \) is a random subset of \( \{1, \ldots, N\} \) chosen uniformly at random and \( n_b = |I_n| \).

Then, its update rule is given by

\[
Y_{n+1}^N = S_\eta \left( Y_n^N - \eta g_n(x_n^N) + \sqrt{2\eta} P_N \epsilon_n \right),
\]  
(10)

where \( g_n(x) \equiv P_N(g_n P_N x) \) and \( Y_0^N = P_N x_0 \in \mathcal{H}_N \). These approximation techniques significantly reduce the computational cost.

### 1.3. Assumptions

Our goal is to study the convergence of the iterations Eq. (7), i.e., to bound \( L(x_n) - L(x^*) \) with high probability. For this, we need to make assumptions on the RKHS \( \mathcal{H}_K \) and on \( L \). We first make the following assumption on \( \mathcal{H}_K \), independently of the objective \( L \):

**Assumption 2.** It holds that:

\[
\mu_k \sim \frac{1}{k^2}.
\]  
(11)

We note that a finite dimensional situation is also allowed, i.e., \( \mu_k = 0 \) (\( \forall k \geq k_0 \)) for some \( k_0 \in \mathbb{N} \), as long as Eq. (11) is satisfied for \( k \leq k_0 \). The weaker assumption \( \mu_k \sim k^{-p} \) with \( p > 1 \) is sometimes made in the literature (Caponnetto and De Vito 2007, Definition 1.iii), Steinwart and Christmann 2008), but the numerical approximation result used in Section 3 requires the more restrictive \( p = 2 \) assumption (Bréhier and Kopec 2016, Assumption 2.2 (2)). As an example, one can consider the case where \( \mathcal{H} \) itself is an RKHS for a kernel \( K^* \), with Mercer decomposition \( K^*(x,y) = \sum_k \nu_k g_k(x) g_k(y) \). Then, the “rescaled” kernel \( K(x,y) = \sum_k \mu_k \nu_k g_k(x) g_k(y) \) with \( \mu_k \sim \frac{1}{k^2} \) satisfies Assumption 2. In fact, the role of Assumption 2 is to ensure that the trajectories (7), (8) will remain in the support of the Gaussian process corresponding to the kernel \( K \).

Next, we put assumptions on the objective function \( L \). The first one is classical for gradient-based optimization (Nesterov, 2004).

**Assumption 3** (Smoothness). \( L \) is \( M \)-smooth:

\[
\forall x,y \in \mathcal{H}, \quad \| \nabla L(x) - \nabla L(y) \| \leq M \| x - y \|.
\]  
(12)

In view of Eq. (2), we have that \( A \equiv -\frac{1}{\beta} \nabla \| \cdot \|^2_{\mathcal{H}_K} \) is a diagonal operator, characterized by \( A f_k = -\frac{1}{\mu_k} f_k \). The following assumptions enforce more smoothness on \( L \) w.r.t. a norm induced by \( A \) through its second and third order derivatives.

**Assumption 4.** There exists \( \alpha \in (1/4,1) \) and \( \lambda_0, C_{\alpha,2} \in (0,\infty) \) such that \( \forall x,h,k \in \mathcal{H}, \)

\[
|D^2 L(x) \cdot (h,k)| \leq C_{\alpha,2} \| h \|_{\mathcal{H}} \| k \|_{\alpha},
\]

where \( \| x \|_{\alpha} \equiv \left( \sum_{k \geq 0} (\mu_k)^{2\alpha} |(x,f_k)|^2 \right)^{1/2} \).
This assumption is not standard in the previous works. However, we put this assumption so that the time discretized dynamics satisfies geometric ergodicity. Fortunately, this assumption is not restrictive in machine learning applications (see Section 1.4 for details).

The next one is common in the SPDE discretization literature (Bréhier and Kopec (2016, Assumption 2.7), Debussche (2011, Assumption (2.3))). It is used in Section 4 to obtain the convergence of the stationary distribution \( \mu_0 \) of Eq. (7) to that of Eq. (8) as \( \eta \) goes to zero.

**Assumption 5** (Bréhier and Kopec (2016, Assumption 2.7, \( M \to \infty \))). Let \( L_N : \mathcal{H}_N \to \mathbb{R} \) be \( L_N = L(P_N x) \). \( L \) is three times differentiable, and there exists \( \alpha' \in (0, 1), C_{\alpha'} \in (0, \infty) \) such that for all \( N \in \mathbb{N} \) and \( \forall x, h, k \in \mathcal{H}_N \),

\[
\begin{align*}
\left\| D^3 L_N(x) \cdot (h, k) \right\|_{\alpha'} & \leq C_{\alpha'} \| h \|_0 \| k \|_0, \\
\left\| D^3 L_N(x) \cdot (h, k) \right\|_0 & \leq C_{\alpha'} \| h \|_{-\alpha'} \| k \|_0.
\end{align*}
\]

As an example, **Assumption 5** is satisfied with \( \alpha = 0 \) when \( L \) is \( C^3 \) with bounded second and third-order derivatives. Next, we assume the following condition to ensure the dissipativity (Proposition 7) which is essential to show geometric ergodicity.

**Assumption 6.** It either holds that

i) \( \lambda > M \mu_0 \) (Strict Dissipativity), or

ii) \( \| \nabla L(\cdot) \| \leq B, \quad B > 0 \) (Bounded gradients).

The \( C_0 \)-semigroup \( (S_t)_{t \geq 0} \) generated by \( A \) is the one of diagonal operators determined by \( S_t f_k = e^{-\lambda t/\mu} f_k \). It is easy to check that this semigroup is strongly continuous. Therefore, the Langevin SDE (8) is an instance of the more general semilinear SDE:

\[
dX(t) = \left( AX(t) + F(X(t)) \right) dt + \sqrt{Q} dW(t),
\]

where \( F \) is globally \( M \)-Lipschitz, \( Q \) is bounded and symmetrical and \( A \) is a linear unbounded operator on \( \mathcal{H} \) generating a strongly continuous semigroup. For the SDE Eq. (8), we have \( F = -\nabla L, Q = 2 \beta^{-1} \text{Id} \) and \( A = -\dot{\phi} \| \cdot \|^2_{L,K} \). The SDE (13) has been extensively studied in finite dimension (Khasminskii, 2011); in the infinite dimensional case, several results have been shown such as the existence and uniqueness of its invariant measure (Da Prato and Zabczyk, 1992; Maslowski, 1989; Sowers, 1992), the exponential convergence of the time \( t \) distribution to this invariant measure (Jacquet and Royer, 1995; Shardlow, 1999; Hairer, 2002) and its explicit convergence rate evaluation (Goldys and Maslowski, 2006); the invariant measure \( \pi \) is given by

\[
\frac{d\pi}{d\nu_0}(x) \propto \exp(-\beta L(x)),
\]

where \( \nu_0 \) is the Gaussian measure in \( \mathcal{H} \) with mean 0 and covariance \( (-\beta A)^{-1} \) (see Da Prato and Zabczyk (1996) for the precise definition of infinite dimensional Gaussian measures). If these assumptions are verified, we have a weaker condition than strong convexity: dissipativity.

**Proposition 7** (Dissipativity). Under Assumptions 2 and 3 and Assumption 6 (i) or Assumption 6 (ii), there exists constants \( m, c > 0 \) verifying

\[
\forall x \in \mathcal{H}, \langle Ax - \nabla L(x), x \rangle \leq -m \| x \|^2 + c.
\]

The dissipative condition proved in this proposition is quite standard to show the existence of the invariant law. For example, Raginsky et al. (2017); Xu et al. (2018) showed the convergence to the invariant law under the dissipative condition in the finite dimensional situation. This condition intuitively indicates that the dynamics stays inside a bounded domain in high probability. If \( X_n \) (or \( X(t) \)) is far away from the origin, then the dynamics are forced to get back around the origin. Thanks to this condition, the dynamics can possess finite moments, which is important to ensure the existence of an invariant law.

In fact, a result of **Assumption 6** is that there exits at least one invariant law.

**Proposition 8.** Under Assumption 6, the processes \( \{X(t)\}_{t \geq 0} \) and \( \{X_n\}_{n \in \mathbb{N}} \) admit (at least) an invariant law.

The proof can be found for example Proposition 4.1 of Bréhier and Kopec (2016), which utilizes the Krylov-Bogoliubov criterion (Da Prato and Zabczyk, 1996, Section 3.1). This proposition does not indicates the uniqueness of an invariant law. It is shown that the continuous time dynamics \( X(t) \) has a unique invariant law and is geometrically ergodic. As for the discrete time dynamics \( X_n \), the uniqueness of the invariant law is already well-known under the strict dissipative condition (Assumption 6 (i)) (see Bréhier and Kopec (2016) for example). However, the uniqueness has not been shown under the bounded gradient condition (Assumption 6 (ii)). In Section 3, we will show that the uniqueness also holds under Assumption 6 (ii) if we assume Assumption 4, which has not been assumed in previous work.

Finally, in the SGLD setting we put the following stronger assumption on each \( \ell_i \).

**Assumption 9.** Each \( \ell_i \) satisfies Assumptions 3 to 5 and Assumption 6 (ii) instead of \( L \), where the constants in each assumption are uniform over all \( \ell_i \) (\( i = 1, \ldots, n_x \)).

1.4. Motivation of problem settings

As examples, **Assumptions 3 and 5** encompass classification cases (e.g., logistic regression) and ordinary least
squares regression, among others. In the non-convex setting, examples include deep learning, tensor factorization (Signorett et al., 2013; Suzuki et al., 2016) and robust classification using non-convex losses such as Savage (Masnadi-Shirazi and Vasconcelos, 2009).

For the sake of instructive exposition, let us consider a situation where we observe n input-output pairs \((z_i, y_i)\) of \(i = 1\), where \(z_i \in \mathcal{Z}\) is an input and \(y_i \in \mathcal{Y}\) is the corresponding label. Here, we let \(\mathcal{H}\) be a Hilbert space of functions on \(\mathcal{Z}\) (which could be a RKHS) with complete orthonormal system \((f_k)_{k=0}^{\infty}\). Accordingly, we define a loss function \(\ell_i(\cdot, y_i) = \ell_i(\cdot) : \mathbb{R} \to \mathbb{R}\) for the \(i\)-th observation, and consider an empirical risk: 

\[ \hat{L}(f) = \frac{1}{n} \sum_{i=1}^{n} \ell_i(f(z_i)) \]

for a function \(f : \mathcal{Z} \to \mathbb{R}\). From the expression (2), the (sub-)RKHS \(\mathcal{H}_k\) can be expressed as an image of \(T_k^2\), i.e., \(\mathcal{H}_k = \{ f = T_k^\gamma h \mid h \in \mathcal{H}\} \) and \(\| f \|_{\mathcal{H}_k} = \inf_{h \in \mathcal{H}} \| f - T_k^\gamma h \|_H\). More generally, we define an RKHS \(\mathcal{H}_k^*\) for \(0 < \gamma < \infty\) as an image of \(T_k^\gamma: \mathcal{H}_k^* = \{ f = T_k^\gamma h \mid h \in \mathcal{H}\}\). We see that \(\gamma = 1\) corresponds to \(\mathcal{H}_k\). We employ \(\mathcal{H}_k^*\) as a model for \(f\) and let the corresponding empirical risk be \(L(x) = \hat{L}(T_k^\gamma x)\) (if needed, we may add a smoothing regularization term). Note that, for 

\[ x = \sum_{k=0}^{\infty} \sqrt{\mu_k} f_k \in \mathcal{H}, \quad T_k^\gamma x(z) = \sum_{k=0}^{\infty} \mu_k^{\gamma/2} \sqrt{\mu_k} f_k(z), \]

and thus we can obtain a reproducing formula \(T_k^\gamma x = \langle x, \psi_\gamma(z) \rangle_{\mathcal{H}}\) where \(\psi_\gamma(z) \triangleq \sum_{k=0}^{\infty} \mu_k^{\gamma/2} \mu_k^{\gamma/2} f_k(z) f_k(z')\). Using this, we see that \(\| \psi_\gamma(z) \|_2^2 = \sum_{k=0}^{\infty} \mu_k^{\gamma} f_k^2(z) = K_\gamma(z, z)\) and \(\| \psi_\gamma(z) \|_2^2 = \sum_{k=0}^{\infty} \mu_k^{\gamma} f_k^2(z) = K_\gamma(z, z)\). In this situation, if we have \(\max \sup_{z \in \mathcal{Z}} \| \psi_\gamma(z) \|_2 \leq G\) and \(\sup_{z \in \mathcal{Z}} K_\gamma(z, z) \leq R\), then

\[\| \nabla L(x) - \nabla L(x^*) \| \leq G R \| x - x^* \|_{\mathcal{H}}, \quad (15)\]

\[|D^2 L(x) \cdot (h, k) | \leq G \sqrt{R} \sum_{k=0}^{\infty} \mu_k^{\gamma-2\alpha}, \quad (16)\]

for \(x, h, k \in \mathcal{H}\) with \(\| h \| = 1\) and \(\| k \|_{\alpha} = 1\). The proof of these inequalities is given in Appendix A. Not that, Assumptions 3 and 4 are satisfied as long as \(R_\gamma < \infty\) for \(\gamma > 1\) because the condition \(\mu_k \lesssim 1/k^2\) makes the right hand of Eq. (16) finite by setting \(\alpha = (\gamma - 1)/2 + 1/4 > 1/4\). Assumption 5 is also verified in the same manner.

Finally, if we let \(f = T_k^\gamma x\), then \(\| f \|_{\mathcal{H}_k} = \| f \|_{\mathcal{H}_k^{\gamma+\gamma}}\) holds. Then, it follows that

\[L(x) + \lambda \| x \|_{\mathcal{H}_k}^2 = \hat{L}(f) + \lambda \| f \|_{\mathcal{H}_k}^2\]

Therefore, we see that our formulation covers a wide range of kernel regularization learning by adjusting \(\gamma\) appropriately.

We would like to remark that we may deal with a situation where \(L(x)\) contains a regularization term \(1/2 \mu_k \| x \|^2\) like \(L(x) = \hat{L}(x) + \lambda \mu_k \| x \|^2\). To deal with this situation, we should change the algorithm and analyze a little bit because it could violate Assumption 6 (especially, the bounded gradient condition). See Appendix A.1 for more details about how to deal with this setting.

2. Main Result

Here, we give our main result on the the non-asymptotic error bound of the GLD algorithm. Define a constant \(\hat{c}_\beta\) as

\[\hat{c}_\beta = \begin{cases} 1 & (\text{strict dissipativity: Assumption 6 (ii)}), \\ \sqrt{\beta} & (\text{bounded gradient: Assumption 6 (iii)}). \end{cases}\]

Theorem 10 (Main Result, GLD convergence rate). Let Assumptions 2, 3, 5 and 6 hold. We also assume Assumption 4 under the bounded gradient condition (Assumption 6 (ii)). Suppose the initial solution satisfies \(\| x_0 \| \leq 1\). Then, there exits \(\Lambda_{\eta}^\pi > 0\) for \(\eta \geq 0\) such that for any \(0 < \kappa < 1/4\) and \(\delta \in (0, 1)\), it holds that

\[\mathbb{P}(L(X_n) - L(x^*) > \delta) \leq \frac{1}{\delta} \left\{ L(\hat{x}) - L(x^*) + \exp(-\Lambda_{\eta}^\pi(\eta n - 1)) + \frac{\hat{c}_\beta}{\Lambda_0^\pi} \eta^{1/2 - \kappa} \right. \]

\[\left. + \left( \frac{2\pi}{\sqrt{\eta}} + 1 \right) \lambda \left( \| \hat{\pi} \| \mu_k^\gamma \right) \| K_\gamma \|_{\mathcal{H}} \right\}. \quad (17)\]

The proof is in Appendix C. A precise description of the spectral gap \(\Lambda_{\eta}^\pi\) is given in Proposition 14. \(\Lambda_{\eta}^\pi\) could be dependent on \(\beta\) and \(\eta\), but is uniformly lower bounded with respect to \(\eta > 0\). As can be seen in Eq. (17), there is a competing effect between the regularization \(\Lambda_{\eta}^\pi\) (ensuring faster convergence of the discrete chain) and the inverse temperature \(\beta\) (ensuring better concentration of the Langevin stationary distribution \(\pi\)). We can see that, for fixed \(\lambda\) and \(\beta\), by setting \(\eta \leq \frac{1}{\Lambda_{\eta}^\pi}\), Eq. (17) excluding the optimization unrelated term \(L(\hat{x}) - L(x^*)\) is of order

\[O \left( \frac{1}{n} + \frac{\hat{c}_\beta}{\Lambda_0^\pi} \left( \log \frac{n}{\Lambda_{\eta}^\pi} \right)^{1/2 - \kappa} + \frac{1}{\beta} + \lambda \right). \quad (18)\]

Note also that contrary to the finite dimensional setting where 1 order weak convergence is possible, the 1/2 rate in \(\eta\) is optimal (Brehier, 2014) – see Section 4.

Next, the convergence rate of SGLD is given as follows.

Theorem 11 (Main Result, SGLD convergence rate). Under Assumptions 2 and 9 and \(\| x_0 \| \leq 1\), SGLD has the following convergence rate:

\[\mathbb{P}(L(Y^N_n) - L(x^*) > \delta) \leq \frac{1}{\delta} \left\{ \Theta_n + \frac{\hat{c}_\beta}{\Lambda_0^\pi} N + \frac{1}{\beta} \left( \sqrt{T_n} + \sqrt{T_n} \right) \right\}. \quad (19)\]
where \( r_n = \frac{n^2 \eta (n \epsilon_n - n_h)}{n_h (n \epsilon_n - 1)} \) and \( \Theta_n = \exp(-\Lambda_n^2 (\eta n - 1)) + \frac{\tilde{\mu}_n}{N} n^{1/2 - \kappa} + \frac{\tilde{\mu}_n}{N} \left( \sqrt{\frac{2 \pi}{K}} + 1 \right) + \lambda \left( \frac{\|\tilde{x}\|^2}{\mathcal{H}_K} + 1 \right) + L(\tilde{x}) - L(x^*) \) which is the convergence rate of GLD shown in Theorem 10.

The approximation error induced by the Galerkin approximation corresponds to \( \frac{\tilde{\mu}_n}{N} n^{1/2 - \kappa} \). Since \( \mu_{N+1} \lesssim N^{-2} \), the approximation error decreases in a quadratic order as the dimension \( N \) is increased. The error induced by the stochastic gradient corresponds to \( \sqrt{\frac{2 \pi}{K}} + 1 \). As the minibatch size \( n_h \) increases, the stochastic gradient error converges to 0. This rate is slightly better than finite dimensional counter part (Raginsky et al., 2017; Xu et al., 2018), by a factor of \( \sqrt{\eta n} \). This is due to the regularization term \( \lambda \|x\|^2_{\mathcal{H}_K} \).

**Proof scheme** Applying GLD and SGLD for non-convex optimization in a finite dimensional space has been investigated extensively recently in Raginsky et al. (2017; Xu et al. 2018; Erdogdu et al. 2018) to name a few. Our analysis could be an infinite dimensional extension of Raginsky et al. (2017; Xu et al. 2018). However, unlike in the proof of such existing analyses for the finite dimensional case, \( E[L(X_n) - L(x^*)] \) cannot be directly bounded as the results used in an infinite-dimensional setting only apply to bounded test functions, Corollary 1.2 in Bréhier (2014) in particular. This is to be contrasted with Xu et al. (2018) where the finite-dimensional assumption allows to derive results for test functions bounded by a Lyapunov function of the type \( C(\|x\|^k + 1) \). Instead, sigmoid functions of the form \( \phi(x) = \sigma(L(x) - L(x^*)) \) with \( \sigma(x) = 1/(1 + e^{-x}) \) are used to bound the probability of the n-th iterate \( X_n \) of Eq. (7) being in a certain level set of \( L(x) - L(x^*) \), by bounding \( E[\phi(X_n)] \) and applying Markov’s inequality.

The seminal paper Raginsky et al. (2017) derived the finite time error bound of SGLD for non-convex learning problem utilizing the decomposition
\[
E[\phi(X_n) - \phi(x^*)] = E[\phi(X_n) - \phi(\nabla (X_n))] + E[\phi(X_n) - \phi(\nabla (X_n))] + E[\phi(\nabla (X_n)) - \phi(\nabla (x^*))],
\]
where \( \pi \) is the stationary distributions of the continuous Markov chain \( \{X(t)\}_{t \geq 0} \) and we denote by \( X^n \) a random variable obeying a probability distribution \( \mu_n \). On the other hand, Xu et al. (2018) observed that this decomposition could be improved by utilizing the geometric ergodicity of discrete time dynamics and proposed to use the following decomposition:
\[
E[\phi(X_n) - \phi(x^*)] = E[\phi(X_n) - \phi(X^n)] + E[\phi(X^n) - \phi(X^*)] + E[\phi(X^*) - \phi(x^*)],
\]
where \( \mu_n \) is the stationary distribution of the discrete Markov chain \( \{X_n\}_{n \in \mathbb{N}} \) (the existence of which is not trivial). By using this, it is shown that some polynomial order term with respect to \( n \) can be dropped to obtain a faster rate.\(^3\) Our analysis employs this strategy. That is, we control (i) the convergence of the discrete chain to its stationary distribution (whose existence we prove), (ii) the convergence of the GLD stationary distribution to that of the Langevin diffusion, and (iii) the concentration of the Langevin diffusion around the global minimum of \( L \).

The extension to an infinite dimensional setting is not trivial. For example, the boundedness of the noise \( \epsilon_n \) does no longer hold, and thus we need an additional regularization term \( AX(t) \) to make the solution bounded in \( \mathcal{H} \) and hit a compact set with high probability. The time discretization of the infinite dimensional Langevin dynamics (Da Prato and Zabczyk, 1996) has been studied especially as a numerical scheme of stochastic partial differential equation (Kuksin and Shirikyan, 2001; Debussche, 2011; Bréhier, 2014; Bréhier and Kopec, 2016; Andersson et al., 2016; Chen et al., 2017; 2018). Bréhier (2014) and Bréhier and Kopec (2016) derived a weak approximation error of the time discretization scheme (7) from the stationary distribution \( \pi \). However, their proof strategy utilizes the decomposition Eq. (19) as in Raginsky et al. (2017). As we have pointed out above, the error bound could be improved by using the decomposition Eq. (20) instead. Unfortunately, the geometric ergodicity of the discrete time dynamics has not been established so far. Therefore, we have introduced Assumption 4 so that the geometric ergodicity holds. Thanks to this, the decomposition (20) analogous to Xu et al. (2018) can be employed to yield a better rate.

### 3. Bounding the First Term: Geometric Ergodicity of the Discrete Chain

The proof from this section is adapted from Goldys and Maslowski (2006) for the discrete chain, i.e. it is shown that the hypothesis of Theorem 2.3 in Meyn and Tweedie (1994) are satisfied. Namely, we prove the existence of a Lyapunov function of the form \( V(x) = \|x\|^2 + 1 \), and that a minorization condition is satisfied on a ball \( B_\epsilon \subset \mathcal{H} \). These two properties act jointly in ensuring the geometric ergodicity (Meyn and Tweedie (1993, Ch.15)) of Eq. (7). Indeed, the Lyapunov condition ensures the attractiveness of \( B_\epsilon \) for the chain \( \{X_n\}_{n \in \mathbb{N}} \), while the minorization condition lower bounds the probability of staying in \( B_\epsilon \).

The following proposition controls the chain in the case when \( \nabla L = 0 \) and is used as an auxiliary result.

\(^3\)We would like to point out that we have found some incorrect analysis of the error bound in Xu et al. (2018). In particular, there are several wrong evaluations about dependency of constants (including the spectral gap) on the inverse temperature parameter \( \beta \).
Proposition 12. Let \( \{Z_n\}_{n \in \mathbb{N}} \) solve: \( Z_0 = 0 \) and
\[
Z_{n+1} = S_\eta Z_n + \sqrt{\frac{2\eta}{\beta}} S_\eta x_n,
\]
with \( \beta > \eta \). Then, \( \forall p > 0, k(p) \triangleq \sup_{n \geq 0} \mathbb{E}(\|Z_n\|^p) < \infty \).

The proof is rather straightforward and is deferred to Appendix D. Proposition 13 controls the decrease of \( \|X_n\| \) in expectation and is the key result towards proving the existence of a Lyapunov function. It relies on the regularization effect of \( A \), through \( S_\eta \); indeed, it holds that \( \forall \eta, S_\eta f_k = (\text{Id} - \eta A)^{-1} f_k = \frac{1}{1 + \lambda \eta/\mu} f_k \), hence \( S_\eta \) is a bounded linear operator of norm \( \|S_\eta\|_{\text{op}} = \frac{1}{1 + \lambda \eta/\mu_0} < 1 \).

Proposition 13. Let Assumptions 3 and 6 hold. We have
\[
\mathbb{E}_{x_0} \|X_n\| \leq \rho^n \|x_0\| + b, \quad \forall n \in \mathbb{N},
\]
with (i) (for Strict Dissipativity) \( \rho = \frac{1 + \eta M_1}{1 + \lambda \eta/\mu_0} < 1 \), \( b = \|x^*\| + 2k(1) \), or (ii) (for Bounded gradients) \( \rho = \frac{1}{1 + \lambda \eta/\mu_0} < 1 \), \( b = \frac{M_2}{2} B + k(1) \).

The proof is given in Appendix E. Combining this Lyapunov condition with a “minorization condition”, we can show the geometric ergodicity in the following proposition.

Proposition 14 (Geometric ergodicity). Let Assumptions 2, 3, 5 and 6 hold. We also assume Assumption 4 under the bounded gradient condition (Assumption 6 (ii)). Let \( \eta > 0, \beta > \eta \) and \( V(x) = \|x\| + 1 \). Then, there exists a unique invariant measure \( \mu_\eta \) and \( \Lambda^*_\eta > 0 \) such that for all \( \phi : \mathcal{H} \to \mathbb{R} \) with \( |\phi(\cdot)| \leq V(\cdot) \) and \( |\phi(x) - \phi(y)| \leq M'' \|x - y\| (x, y) \in \mathcal{H} \), it holds that
\[
|\mathbb{E}_{x_0}[\phi(X_n)] - \mathbb{E}[\phi(X_{t_n})]| \leq C_{x_0} \exp(-\Lambda^*_\eta (\eta n - 1)), \quad n \geq 0,
\]
where \( C_{x_0} \) and \( \Lambda^*_\eta > 0 \) are given by
i) (Strict dissipativity, Assumption 6 (i))
\[
\Lambda^*_\eta = \frac{\Lambda}{\mu_0} - M \frac{1}{1 + \eta \mu_0}, \quad C_{x_0} = M'(\|x_0\| \mathcal{H} + b),
\]
ii) (Bounded Gradient, Assumption 6 (ii))
\[
\Lambda^*_\eta = \min\left\{ \frac{\Lambda}{4\log(k(V + 1)/(1 - \delta))}, \frac{c}{k(V + 1)} \right\}, \quad C_{x_0} = \kappa [\hat{V} + 1] + \sqrt{\frac{2cV(x_0) + b}{\delta}},
\]
for \( 0 < \delta < 1 \), satisfying \( \delta = \Omega(\exp(-O(\delta))) \), \( \hat{b} = \max\{b, 1\} \), \( \kappa = \hat{b} + 1 \) and \( \hat{V} = \frac{4b}{\sqrt{(1 + \mu_0/\eta) - \mu_0/\eta}} \).

The proof is given in Appendix F. Unlike existing work, this theorem asserts the geometric ergodicity of the discrete time dynamics, whilst the geometric ergodicity for “continuous time” dynamics (Eq. (8)) has been well known, see as an example Debussche (2011; 2013). The proof follows a standard argument that utilizes the Lyapunov condition (Proposition 13) and the minorization condition. Here, the minorization condition asserts that the transition kernel with respect to the discrete time Markov process shares a common probability mass on a bounded region uniformly over initial state \( x_0 \) with some bounded norm. Once this condition is shown then the recurrence probability can be lower bounded combined with the Lyapunov condition, which yields the coupling argument. To show a faster convergence, we employed the coupling technique of Mattingly et al. (2002) and adopted it to the proof technique of Goldys and Maslowski (2006) developed for a continuous time dynamics. Hence, we obtained faster rates than Goldys and Maslowski (2006). In particular, the dependency on \( \beta \) is improved.

Transforming the continuous time argument to the discrete time setting is far from trivial because there appears a “integrability” problem in showing the minorization condition, which makes it difficult to show the geometric ergodicity. Indeed, Bréhier (2014); Bréhier and Kopec (2016) pointed out there has been no work that showed the geometric ergodicity of the time discretized dynamics. This difficulty does not occur in the finite dimensional setting. We resolved this problem by imposing Assumption 4. Thanks to this, we have exponential convergence \( \exp(-\Lambda^*_\eta (\eta n)) \) improving the polynomial order rate \( \frac{1}{\lambda_0} (\eta n)^{-1} \) of existing work.

4. Second Term: Weak Convergence of the Discrete Scheme

The second term is linked to the weak convergence of the numerical scheme, i.e., in our case the convergence of \( \phi(X_n) \) to \( \phi(X(\eta n)) \) for any admissible test function \( \phi \in C^2_b \). We rely directly on the results of Bréhier and Kopec (2016), who prove 1/2 order weak convergence in time and 1 order weak convergence in space for numerical schemes that have a semi-implicit discretization in time with \( \beta = 1 \), and a finite elements discretization in space; that is, they showed
\[
\left| \int \phi \mu_{\eta} - \int \phi \mu \right| \leq C \|\phi\|_{l^2} \eta^{1/2 - \kappa}, \quad \kappa < \frac{1}{2},
\]
where \( \|\phi\|_{l^2} \triangleq \max\{\|\phi\|_{l^\infty}, \sup_{x \in \mathcal{H}} \|\nabla \phi(x)\|_{l^1}, \sup_{x \in \mathcal{H}} \|D^2 \phi(x)\|_{l^1} \} \) for \( \phi \in C^2_b \).

In the general setting, \( \beta \neq 1 \), we need to evaluate the effect of \( \beta \). To that purpose, we essentially consider a re-scaling argument, that is, we observe that if we replace \( L \) with \( L' \triangleq \)
$\beta L$, $\lambda$ with $X' \triangleq \beta X$ and $\eta$ with $\eta' \triangleq \eta/\beta$ in Eq. (8) and Eq. (7), then it holds that
\[ S_n = (Id + \eta' \beta/\lambda \nabla ||)^{-1} (Id + \eta' \beta/\lambda \nabla ||)^{2n} = S_{\eta'}, \]
and thus
\[ X_{n+1} = S_{\eta'} X_n - \eta' S_{\eta'} \nabla L' (X_n) + \sqrt{2\eta' S_{\eta'}} \varepsilon_n, \]
i.e., $\{X_n\}_{n \in \mathbb{N}}$ is the numerical approximation of
\[ dX(t) = -\nabla L' (X(t)) + \sqrt{2} dW(t), \]
with time step $\eta'$. We carefully evaluate how the constant $C$ is Eq. (23) will be changed after rescaling. We can see that $\beta$ affects the rate through the spectral gap $\Lambda_0$, which corresponds to the continuous dynamics ($\eta = 0$). Eventually, we get the following result:

**Proposition 15** (Case $\beta \neq 1$). Under the same setting as Proposition 14, for any $0 < \kappa < 1/2, 0 < \eta_0$, there exists a constant $C$ such that for any bounded test function $\phi \in C_0^2$ and $0 < \eta < \eta_0$, it holds that
\[ \left| \int \phi d\mu - \int \phi d\pi \right| \leq C \frac{\|\phi\|_0}{\Lambda_0^{1/2-\kappa}} \beta^{1/2-\kappa}. \]  

The proof is given in Appendix H. Note that due to the infinite dimensional setting, the $1/2$ rate w.r.t the time discretization $\eta$ is optimal (Bréhier, 2014). This is to be contrasted with the finite-dimensional case, where 1 order weak convergence is attainable.

**5. Third Term: Concentration of the Gibbs Distribution Around the Global Minimum**

The last term corresponds to the concentration of the stationary Gibbs distribution around the global minimum of $L$. In this infinite-dimensional setting, the regularizing effect of operator $A$ is necessary to ensure good convergence properties of the discrete and continuous chains. Hence, even in the limit case $\beta \to 0$ one cannot expect to have arbitrary tight concentration around the global minimum. This is to be contrasted with the finite dimensional case (Chiang et al., 1987; Gelfand and Mitter (1991); Roberts and Tweedie (1996)). In fact, $A$ constrains the chain to remain within the support of a Gaussian process which is compactly embedded in $\mathcal{H}$.

**Proposition 16.** Under Assumptions 2 and 3, it holds that
\[ \int L d\pi - L(\bar{x}) \lesssim \frac{1}{\beta} \left( \frac{2M}{\lambda} + 1 \right) + \lambda \left( \frac{||\bar{x}||^2}{\mathcal{H}_K} + \frac{||\bar{x}||^2}{\mathcal{H}_K} \right). \]

The proof can be found in Appendix G. The proposition can be shown by utilizing an analogous technique to the convergence rate analysis of Gaussian process regression (van der Vaart and van Zanten, 2011). Along with this technique, the Gaussian correlation inequality (Royen, 2014; Latala, 2017) is used. This inequality gives a powerful tool to lower-bound the Gaussian probability measure of the intersection of two centered convex sets.

**6. Error Bound for the Galerkin Approximation and Stochastic Gradient**

The error induced by the Galerkin approximation can be evaluated as in the following proposition.

**Proposition 17.** Let Assumptions 2, 3, 5 and 6 hold and suppose $||x|| \leq 1$. Then, there exists an invariant measure $\mu_{(N,\eta)}$ for the discrete time Galerkin approximation scheme (Eq. (9)), and for any $0 < \kappa < 1/2, 0 < \eta_0$, there exists a constant $C > 0$ such that, for any $N \in \mathbb{N}$ and $0 < \eta < \eta_0$,
\[ \mathbb{E} \left[ \phi(X_{(N,\eta)}) - \phi(x) \right] \leq C \frac{\|\phi\|_0}{\Lambda_0} \beta \left( \mu_{N+1} + \eta^{1/2-\kappa} \right). \]

The proof is in Appendix H. We see that, by taking $N \to \infty$, we can replicate Proposition 15. Moreover, the geometric ergodicity of the time discretized dynamics with the Galerkin approximation holds completely in the same manner as Proposition 14. The discrepancy between GLD and SGLD with the Galerkin approximation can be bounded as follows.

**Proposition 18.** Suppose $||x_0|| \leq 1$. There exists a constant $C > 0$ such that, for any $n, N \in \mathbb{N}, any \beta > 1$ and sufficiently small $\eta > 0$,
\[ \mathbb{E} \left[ \phi(X_{(n,N)}), - \phi(Y_{(N)}^n) \right] \leq C \left( \sqrt{r_n} + \sqrt{\tau_n} \right). \]

The proof is given in Appendix I. From these propositions, we can see that the SGLD with the Galerkin approximation also gives a reasonably good solution for sufficiently large $N \in \mathbb{N}$, sufficiently small $\eta > 0$ and sufficiently large mini-batch size. Proposition 18 is analogous to those given for finite dimensional situations (Raginsky et al., 2017; Xu et al., 2018). However, thanks to the regularization term (appearing as $S^0$), our rate is better by a factor of $\sqrt{k}$. 

**7. Other Related Work**

In this section, we mention other related work that have not been exposed above. An analogous assumption to Assumption 4 has already been introduced in the analysis of infinite dimensional dynamics with nonlinear diffusion term, that is, $dW(t)$ is replaced by a nonlinear quantity $\sigma(X(t))dW(t)$ for $\sigma(X(t)) \in \mathcal{B}(\mathcal{H})$ (Conus et al., 2019;
Debussche, 2011; Bréhier and Debussche, 2018). These papers analyzed the existence of stationary distribution for continuous dynamics and discrete time approximation for finite time horizon. Chen et al. (2017; 2018) analyzed linear/nonlinear Schrödinger equations and derived geometric ergodicity, but they analyzed much more specific situations or stronger assumptions (e.g. the strong dissipativity condition).

The geometric ergodicity of infinite dimensional Markov processes for discrete time settings has been investigated by Kuksin and Shirikyan (2001) and infinite dimensional MCMC such as preconditioned Crank–Nicolson (pCN) (Hairer et al., 2014; Eberle, 2014; Vollmer, 2015; Rudolf and Sprungk, 2018), and in particular the Metropolis-Adjusted Langevin Algorithm (MALA) (Durmus and Moulines, 2015; Beskos et al., 2017). Among them, MALA is the most related to our setting (discrete time Langevin dynamics). The biggest difference the presence of a rejection step. Since the purpose of our work is rather optimization than sampling, and since the rejection step is not compatible with stochastic gradient descent, we do not pursue this direction.

## Conclusion and Future Work

In this paper, we have presented a non-asymptotic analysis of the convergence of GLD and SGLD in a RKHS and for a non-convex objective function. The bounds obtained in this infinite-dimensional setting involve the spectrum of the associated integral operator and a regularization factor instead of the dimension $d$, which to the best of our knowledge is the first result on applying GLD in RKHS to infinite-dimensional nonconvex optimization. In future work, we hope to alleviate the somewhat strict Assumption 2 linked to current results from the numerical approximation literature. Drawing inspiration from (Xu et al., 2018), we also plan to extend our analysis to variance-reduced SGLD algorithms.

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Y. Zhang, P. Liang, and M. Charikar. A hitting time analysis of stochastic gradient Langevin dynamics. *arXiv preprint arXiv:1702.05575*, 2017.
A. Proof of Eq. (15) and Eq. (16)

If $\|\epsilon''_i\|_{\infty} \leq G$, then it is $G$-Lipschitz continuous. Therefore, it holds that

$$
\|\nabla L(x) - \nabla L(x')\| \\
\leq \frac{1}{n_{tr}} \sum_{i=1}^{n_{tr}} |\epsilon''_i((x, \psi_\gamma(z))_H) - \epsilon''_i((x', \psi_\gamma(z'))_H)| \|\psi_\gamma(z_i)\|_H + \lambda_0 \|x - x'\|_H
$$

$$
\leq \frac{1}{n_{tr}} \sum_{i=1}^{n_{tr}} |\epsilon''_i((x, \psi_\gamma(z))_H) - \epsilon''_i((x', \psi_\gamma(z'))_H)| \sqrt{K_\gamma(z_i, z_i)} + \lambda_0 \|x - x'\|_H
$$

$$
\leq \sup_z \sqrt{K_\gamma(z, z)} G \frac{1}{n_{tr}} \sum_{i=1}^{n_{tr}} \|x - x', \psi_\gamma(z_i)\|_H + \lambda_0 \|x - x'\|_H
$$

$$
\leq G \sup_z K_\gamma(z, z) \|x - x'\|_H + \lambda_0 \|x - x'\|_H \leq (G\gamma + \lambda_0) \|x - x'\|_H.
$$

This yields Eq. (15). As for the second order derivative (Eq. (16)), first note that

$$
D^2L(x) \cdot (h, k) = \frac{1}{n_{tr}} \sum_{i=1}^{n_{tr}} \epsilon''((x, \psi_\gamma(z_i))_H) \langle \psi_\gamma(z_i), h \rangle_H \langle \psi_\gamma(z_i), k \rangle_H + \lambda_0 \langle h, k \rangle_H
$$

for $h, k \in H$. Therefore, we have that

$$
|D^2L(x) \cdot (h, k) - \lambda_0 \langle h, k \rangle_H |
$$

$$
\leq \frac{1}{n_{tr}} \sum_{i=1}^{n_{tr}} |\epsilon''((x, \psi_\gamma(z_i))_H)\|\langle \psi_\gamma(z_i), h \rangle_H \langle \psi_\gamma(z_i), k \rangle_H - \lambda_0 \|h\|_H \|k\|_H |
$$

$$
\leq G \max_i \|\psi_\gamma(z_i)\|_H \|h\|_H \frac{1}{n_{tr}} \sum_{i=1}^{n_{tr}} \|\psi_\gamma(z_i)\|_H - \lambda_0 \|h\|_H \|k\|_H |
$$

$$
= G \max_i \sqrt{K_\gamma(z_i, z_i)} \|h\|_H \frac{1}{n_{tr}} \sum_{i=1}^{n_{tr}} \sqrt{K_\gamma(z_i, z_i)} \|k\|_H |
$$

$$
\leq G \max_i \sqrt{K_\gamma(z_i, z_i)} \|h\|_H \sqrt{\frac{1}{n_{tr}} \sum_{i=1}^{n_{tr}} K_\gamma(z_i, z_i)} \|k\|_H |
$$

$$
\leq G \max_i \sqrt{K_\gamma(z_i, z_i)} \|h\|_H \sum_{k=0}^{\infty} \mu_k^{-2\alpha} \|k\|_H |
$$

$$
\leq G \sqrt{K_\gamma} \|h\|_H \sqrt{\sum_{k=0}^{\infty} \mu_k^{-2\alpha} \|k\|_H}.
$$

$\square$

### A.1. Remark on existence of regularization term

As an example of $L(x)$, it is useful to consider a setting where $L(x)$ can be expressed as $L(x) = \tilde{L}(x) + \frac{\beta}{2} \|x\|^2$ for $\tilde{L}(x)$ that satisfies the assumptions listed in the main text and $\lambda_0 \geq 0$. In this case, $L(x)$ does not satisfy the bounded gradient condition Assumption 6 (ii). However, by considering the following update rule, we can show the same error bound for $L(x)$:

$$
\begin{align*}
X_0 &= x_0 \in H, \\
X_{n+1} &= S'_{\eta}(X_n - \nabla \tilde{L}(X_n) + \sqrt{2\eta} \varepsilon_n),
\end{align*}
$$

where $S'_{\eta} = \left[ \text{Id} + \eta \left( \frac{\nabla}{\|\cdot\|_H} \right) + \frac{\lambda_0}{\|\cdot\|_H} \right]^{-1}$.
B. Proof of Proposition 7

Proof. Let us assume \( \lambda > M \mu_0 \) (Strict Dissipativity). Assumption 2 implies, for \( x = \sum_{k=0}^{\infty} \alpha_k f_k \),

\[
\langle Ax, x \rangle = -\lambda \left( \sum_{k=0}^{\infty} \frac{\alpha_k^2}{\mu_k} \right) \sum_{k=0}^{\infty} \alpha_k f_k
\]

\[
\leq -\frac{\lambda}{\mu_0} \sum_{k=0}^{\infty} \alpha_k^2 = -\frac{\lambda}{\mu_0} \|x\|^2,
\]

(26)

and Assumption 3 implies

\[
\langle -\nabla L(x), x \rangle \leq M \|x - x^*\| \|x\|
\]

\[
\leq M \|x\|^2 + M \|x\| \|x^*\|.
\]

(27)

Hence,

\[
\langle Ax - \nabla L(x), x \rangle \leq -(\frac{\lambda}{\mu_0} - M) \|x\|^2 + M \|x\| \|x^*\|.
\]

Therefore, if \( M < \frac{\lambda}{\mu_0} \), there exists \( m, c > 0 \) such that Eq. (14) holds. The proof when Assumption 6 (ii) holds is similar.

C. Proof of main result: Theorem 10 and Theorem 11

In light of Sections 3 to 5, we can now state our final result. We introduce the following bounded test function:

\[
\phi(x) = \sigma(L(x) - L(x^*)) \quad (x \in \mathcal{H}),
\]

(28)

where \( \sigma(u) = \frac{1}{1+e^{-u}} - \frac{1}{2} \) (\( u \in [0, \infty) \)) is concave and takes values in \([0, 1]\). In particular, \( \|\phi(\cdot)\|_V \leq 1 \) for \( V(x) = M\|x\| + 1 \) and \( \phi \in C_b^2(\mathcal{H}) \), hence \( \phi \) falls within the scope of Propositions 14 and 15.

First, we note that there exists a unique invariant measure \( \mu_{\eta} \) for the discrete time dynamics \( \{X_n\}_n \) and there also exists a unique invariant measure \( \mu_{(N, \eta)} \) for the discrete time Garelnik approximated dynamics \( \{X_n^N\}_n \) by Proposition 16. To obtain the result, we make use of Markov’s inequality: for any \( 0 < \delta < 1 \),

\[
P(L(X_n) - L(x^*) > \delta)
\leq P(\phi(X_n) > \sigma(\delta))
\leq \frac{\mathbb{E}[\phi(X_n)]}{\sigma(\delta)} \quad (\because \text{Markov’s inequality})
\]

\[
= \frac{1}{\sigma(\delta)} (\mathbb{E}[\phi(X_n) - \phi(X^N)] + \mathbb{E}[\phi(X^N) - \phi(X^\pi)] + \mathbb{E}[\phi(X^\pi)]).
\]

The first term \( (\mathbb{E}[\phi(X_n) - \phi(X^N)]) \) can be bounded by Proposition 14. The second term \( (\mathbb{E}[\phi(X^N) - \phi(X^\pi)]) \) can be bounded by Proposition 15. Next, we bound the third term. Since \( \sigma(u) \leq u \) for all \( u \in [0, \infty) \) and \( L(x) - L(x^*) \geq 0 \) for all \( x \in \mathcal{H} \), it holds that

\[
\mathbb{E}[\phi(X^\pi)] \leq \mathbb{E}[L(X^\pi)] - L(x^*)]
\]

(29)

Then, the first term \( (\mathbb{E}[L(X^\pi)] - L(x^*)) \) in the right hand side is bounded by Proposition 16. Finally, we observe that \( 1/\sigma(\delta) \leq 5/\delta \) for all \( \delta \in (0, 1) \). Combining all results, we obtain Theorem 10.

As for the Theorem 11, we use the following decomposition

\[
\mathbb{E}[\phi(X_n)] = \mathbb{E}[\phi(Y_n^N) - \phi(X_n^N)] + \mathbb{E}[\phi(X_n^N) - \phi(X_{(N,n)}^n)] + \mathbb{E}[\phi(X_{(N,n)}^n) - \phi(X^\pi)] + \mathbb{E}[\phi(X^\pi)].
\]

We apply Proposition 18 to the first term \( (\mathbb{E}[\phi(Y_n^N) - \phi(X_n^N)]) \) and apply Proposition 17 to the second term \( (\mathbb{E}[\phi(X_{(N,n)}^n) - \phi(X^\pi)]) \). As for the remaining terms, the same bound as Proposition 14 can be applied to the second term \( (\mathbb{E}[\phi(X_{(N,n)}^n) - \phi(X^\pi)]) \), and the last term \( \mathbb{E}[\phi(X^\pi)] \) can be bounded by Eq. (29) with Proposition 16. This yields Theorem 11.
D. Proof of Proposition 12

**Proof.** This is proved in Bréhier (2014) for $\beta = 1$. The $\beta > \eta$ assumption is necessary to ensure that $k(p)$ can be treated as a constant w.r.t $\beta$ and $\eta$ in the following. We recall the main arguments of the proof. \{$Z_n\}_{n \in \mathbb{N}}$ is the semi-implicit approximation of the continuous Markov chain defined by:

\[
\begin{cases}
    dZ(t) = AZ(t) \, dt + \sqrt{2} \, dW(t), \\
    Z(0) = 0.
\end{cases}
\]

Under Assumption 2, it can be shown that $\sup_{t \geq 0} \mathbb{E}(\|Z(t)\|^p) < \infty$, $\forall p \geq 1$. Finally, \{$Z_n\}$ is a numerical scheme with strong order $\frac{1}{2}$ (Printems (2001, Theorem 3.2)), which implies the result.

E. Proof of Proposition 13

**Proof.** The discrete chain $Y_n \triangleq X_n - Z_n$, $n \geq 0$ satisfies

\[ Y_{n+1} = S_\eta Y_n - \eta \nabla L(X_n). \]

Hence, using Assumption 3 and the fact that $X_n = Y_n + Z_n$, we get

\[
\|Y_{n+1}\| \leq \|S_\eta\|_{op} \|Y_n - \eta \nabla L(X_n)\| \\
\leq \frac{1}{1 + \lambda \eta / \mu_0} ((1 + \eta M) \|Y_n\| + \eta M(\|x^*\| + \|Z_n\|)).
\]

Taking the expectation and using $\mathbb{E}\|Z_n\| \leq k(1)$ (Proposition 12), this yields

\[ \mathbb{E}\|Y_{n+1}\| \leq \frac{1}{1 + \lambda \eta / \mu_0} ((1 + \eta M) \mathbb{E}\|Y_n\| + \eta M(\|x^*\| + k(1))), \]

from which we deduce

\[ \mathbb{E}\|Y_n\| \leq \rho^n \|x_0\| + \frac{\eta (1 - \rho^n) M}{(1 - \rho)(1 + \lambda \eta / \mu_0)} (\|x^*\| + k(1)). \]

Therefore,

\[ \mathbb{E}_{x_0}\|X_n\| \leq \rho^n \|x_0\| + \frac{\eta M(\|x^*\| + k(1))}{(1 - \rho)(1 + \lambda \eta / \mu_0)} + k(1). \]

Finally, we conclude by observing that $\frac{\eta}{1 - \rho} M = 1$.

The proof with bounded gradients is similar. Since

\[
\|Y_{n+1}\| \leq \|S_\eta\|_{op} \|Y_n - \eta \nabla L(X_n)\| \\
\leq \frac{1}{1 + \lambda \eta / \mu_0} (\|Y_n\| + \eta B),
\]

we have

\[
\|Y_n\| \leq \rho^n \|x_0\| + \frac{(1 - \rho^n) \eta B}{1 - \rho} \leq \rho^n \|x_0\| + \frac{\mu_0 B}{\lambda} \leq \rho^n \|x_0\| + \frac{\mu_0 B}{\lambda} + k(1),
\]

where $\rho = \frac{1}{1 + \lambda \eta / \mu_0}$. Hence, noting $\|X_n\| \leq \|Y_n\| + \|Z_n\|$, we have that $\mathbb{E}(\|X_n\|) \leq \rho^n \|x_0\| + \frac{\mu_0}{\lambda} B + k(1)$.  

F. Proof of Proposition 14

**Proof under the Strict Dissipativity Condition (Assumption 6 (i))** First we prove the geometric ergodicity under Assumption 6 (i). To show that we first prove the exponential contraction:

\[ \|X_n - Y_n\|_H \leq \left(1 - \frac{\lambda \mu_0 - M}{1 + \eta \mu_0} \right)^n \|X_0 - Y_0\|_H. \]

Once we have shown this inequality, it is easy to show the geometric ergodicity.
According to the update rule, we have that
\[
X_{n+1} = S_\eta \left( X_n - \eta \nabla L(X_n) + \sqrt{\frac{2\eta}{\beta}} \epsilon_n \right),
\]
\[
Y_{n+1} = S_\eta \left( Y_n - \eta \nabla L(Y_n) + \sqrt{\frac{2\eta}{\beta}} \epsilon_n \right) .
\]
Therefore, by taking difference, we obtain
\[
X_{n+1} - Y_{n+1} = S_\eta \left[ (X_n - Y_n) - \eta (L(x_n) - L(Y_n)) \right] .
\]
Then, by the triangular inequality, this yields
\[
\|X_{n+1} - Y_{n+1}\|_H \leq \frac{1}{1 + \frac{\eta}{\mu_0}} (\|X_n - Y_n\|_H + \eta \|L(X_n) - L(Y_n)\|_H)
\]
\[
\leq \frac{1}{1 + \frac{\eta}{\mu_0}} (\|X_n - Y_n\|_H + \eta M \|X_n - Y_n\|_H)
\]
\[
\leq \frac{1 + \eta M}{1 + \frac{\eta}{\mu_0}} \|X_n - Y_n\|_H
\]
\[
\leq \left( \frac{1 - \frac{\lambda}{\mu_0} - M}{1 + \frac{\eta}{\mu_0}} \right)^n \|X_0 - Y_0\|_H.
\]
Now, we already know that there exists an invariant low \( \mu_0 \) under the strong dissipativity condition. By assuming \( Y_0 \sim \mu_\eta \) and \( X_0 = x_0 \in H \), we can show the following geometric convergence:
\[
\mathbb{E}[\phi(X_n)] - \mathbb{E}[\phi(X)] = \mathbb{E}[\phi(X_n)] - \mathbb{E}[\phi(Y_n)] \leq M' \mathbb{E}[\|X_n - Y_n\|_H] \leq M' \left( 1 - \frac{\lambda}{\mu_0} - M \right)^n \mathbb{E}[\|X_0 - Y_0\|_H].
\]
Now, we see that
\[
\mathbb{E}[\|X_0 - Y_0\|_H] \leq \|x_0\|_H + \mathbb{E}[\|Y_0\|_H] \leq \|x_0\|_H + b.
\]
In the last inequality, we used that \( \mathbb{E}[\|Y_0\|_H] = \mathbb{E}[\|Y_0\|_H] \leq \mu_0 \mathbb{E}[\|Y_0\|_H] + b \) for all \( n = 1, 2, \ldots \) by Proposition 13 and we took \( n \to \infty \). As a consequence, we obtain
\[
\mathbb{E}[\phi(X_n)] - \mathbb{E}[\phi(X)] \leq M' \exp \left( -n \frac{\lambda}{\mu_0} - M \right) (\|x_0\|_H + b),
\]
where we used the relation \( 1 - a \leq \exp(-a) \) for \( a > 0 \). This yields the assertion.

**Proof under the Bounded Gradient Condition (Assumption 6 (ii))** Next, we prove the theorem under the bounded gradient case (Assumption 6 (ii)). Under the strict dissipative condition, the statement can be immediately shown and thus we omit the proof.

We adopt the technique of Theorems 5.2 & 5.3 from (Goldys and Maslowski, 2006), and show the geometric ergodicity via Theorem 2.5 of (Mattingly et al., 2002). We note that Theorem 2.5 of (Mattingly et al., 2002) is shown for a finite dimensional setting, but it can be adopted for an infinite dimensional setting if the “minorization condition” (Lemma 2.3 of (Mattingly et al., 2002)) and “Lyapunov condition” (Assumption 2.2 of (Mattingly et al., 2002)) are satisfied.

Since the Lyapunov condition is already shown by Proposition 13, we only need to show the minorization condition. Let \( \mu_{k,\eta}^n \) be the law of
\[
Z_{k,\eta}^n = S_{\eta}^k x + \sqrt{\frac{2\eta}{\beta}} \sum_{l=0}^{k-1} S_{\eta}^{k-l} \xi_l,
\]
and \( \mu_{k,\eta} \) be the law of
\[
Z_{k,\eta} = \sqrt{\frac{2\eta}{\beta}} \sum_{l=0}^{k-1} S_{\eta}^{k-l} \xi_l.
\]
Let $Q \triangleq \frac{2\eta}{\beta} \text{Id}$, and

$$Q_k \triangleq \sum_{l=0}^{k-1} Q S^{2(k-l)}_\eta,$$

for $k = 1, 2, \ldots$, and $Q_0 = 0$. Then, $\mu_k^\pm$ is the Gaussian process on $\mathcal{H}$ with mean $S^k_\eta x$ and covariance operator $Q_k$, and $\mu_k$ is the centered Gaussian process on $\mathcal{H}$ with the same covariance operator. By the Cameron-Martin formula, $\mu_k^\pm$ and $\mu_k$ are equivalent with density given by

$$\frac{d\mu_k^\pm}{d\mu_k}(y) = \exp \left\{ \langle Q_k^{-1} S^k_\eta x, y \rangle - \frac{1}{2} \| Q_k^{-1/2} S^k_\eta x \|^2 \right\},$$

(see Da Prato and Zabczyk (1996) for example). We can easily check that $Q_k \succeq k Q_\eta^\circ$. Then, we have that

$$\langle x, S^k_\eta Q_k^{-1} y \rangle - \frac{1}{2} \| Q_k^{-1/2} S^k_\eta x \|^2 \geq -\frac{\beta}{2} \| x \|^2 - \frac{1}{2\beta} \| S^k_\eta Q_k^{-1} y \|^2 - \frac{\beta}{4\eta k} \| x \|^2,$$

and thus we have the following lower bound of the density:

$$\frac{d\mu_k^\pm}{d\mu_k}(y) \geq \exp \left\{ -\frac{\beta}{2} \left( 1 + \frac{1}{2k\eta} \right) \| x \|^2 - \frac{1}{2\beta} \| S^k_\eta Q_k^{-1} y \|^2 \right\},$$

(37)

For a given $N$ (where $N$ will be determined later on), let

$$K_k \triangleq Q_k S^{N-k}_\eta Q_N^{-1/2},$$

for $k = 0, \ldots, N$. Here, we define

$$\hat{Z}_{k,\eta} \triangleq Z_{k,\eta} - K_k Q_N^{-1/2}(Z_{N,\eta} - y),$$

for $x, y \in \mathcal{H}$, and denote $\hat{Z}_{k,\eta} \triangleq \hat{Z}_{k,\eta}^{0,0}$. In particular, we notice that

$$\hat{Z}_{k,\eta} = Z_{k,\eta} - K_k Q_N^{-1/2} Z_{N,\eta},$$

by definition. Let

$$Y_k \triangleq \sum_{i=k}^{N-1} S^{N-i}_\eta Q^{1/2} \epsilon_i,$$

$$H_k \triangleq Q^{1/2}_N S^{N-k}_\eta Q^{1/2}_N.$$

By a simple calculation, we can show that

$$Y_k = Z_{N,\eta} - S^{N-k}_\eta Z_{k,\eta} = Q_{N-k} Q_N^{-1} Z_{N,\eta} - S^{N-k}_\eta \hat{Z}_{k,\eta}.$$

Finally, let

$$\alpha_k \triangleq Q_{N-k}^{-1/2} H_k Y_k = \frac{Q^{1/2}_N H_k Q^{1/2}_N}{\triangleq B_1(k)} \triangleq B_2(k) Z_{N,\eta} - \frac{Q^{1/2}_N H_k S^{N-k}_\eta}{\triangleq B_2(k)} \hat{Z}_{k,\eta},$$

and accordingly, define

$$\zeta_k \triangleq \epsilon_k - \alpha_k.$$

Then, we can show that $(\hat{Z}_{k,\eta})_k$ and $(\zeta_k)_k$ are independent of $Z_{N,\eta}$ by the same reasoning as (Goldys and Maslowski, 2006). To see this, we only have to show that their correlation is 0 because they are Gaussian process. First, we can show that

$$\mathbb{E} \left[ \epsilon_k \hat{Z}_{k,\eta} \right] = Q_{N-k}^{-1/2} H_k \mathbb{E} \left[ \epsilon_k Y_k \right] = \begin{cases} Q^{1/2}_N H_k \left( Q^{1/2}_N S^{N-k}_\eta - S^{N-k}_\eta' Q^{1/2}_N S^{N-k}_\eta' \right) & (k' < k) \\ Q^{1/2}_N H_k \left( Q^{1/2}_N S^{N-k}_\eta \right) & (k' \geq k) \end{cases}$$

Here, for $x, y \in \mathcal{H}$, the bounded linear operator $z \mapsto x\langle y, z \rangle$ is denoted by $xyz$ for simplicity.
For $k \leq k'$,

$$
\begin{align*}
\mathbb{E}[\alpha_k\alpha_{k'}^*] &= Q_{N-k}^{-1/2}H_k\mathbb{E}[Y_k Y_{k'}^*] H_{k'}^* Q_{N-k}^{-1/2} = Q_{N-k}^{-1/2}H_k \left(\sum_{i=k'}^{N-1} S_{\eta}^{2(N-i-1)} Q\right) H_{k'}^* Q_{N-k'}^{-1/2} \\
&= Q_{N-k}^{-1/2}H_k \left(\sum_{i=0}^{N-k'-1} S_{\eta}^{2(N-k'-i-1)} Q\right) H_{k'}^* Q_{N-k'}^{-1/2} \\
&= Q_{N-k}^{-1/2}H_k Q_{N-k'} H_{k'}^* Q_{N-k}^{-1/2} = Q_{N-k}^{-1/2}H_k Q_{N-k'}^2 H_{k'}.
\end{align*}
$$

Hence, when $k < k'$, it holds that

$$
\mathbb{E}[(\epsilon_k - \alpha_k)(\epsilon_{k'} - \alpha_{k'})^*] = 0,
$$

and when $k = k'$, we have that

$$
\mathbb{E}[(\epsilon_k - \alpha_k)(\epsilon_k - \alpha_k)^*] = \text{Id} - H_k^2.
$$

Finally, we can see that

$$
\begin{align*}
\mathbb{E}[(\epsilon_k - \alpha_k)Z_{N,\eta}^*] &= Q_{N-k}^{1/2}S_{\eta}^{N-k} - \left\{ Q_{N-k}^{1/2}H_k Q_N^{-1/2} Q_N - Q_{N-k}^{-1/2}H_k S_{\eta}^{N-k} (Q_k S_{\eta}^{-k} - K_k Q_N^{-1/2} Q_N) \right\} \\
&= Q_{N-k}^{1/2}S_{\eta}^{N-k} - Q_{N-k}^{1/2}S_{\eta}^{N-k} = 0,
\end{align*}
$$

which indicates $\zeta_k = \epsilon_k - \alpha_k$ is independent of $Z_{N,\eta}$. Furthermore, we have that

$$
\begin{align*}
\mathbb{E}[Z_{N,\eta}(\hat{Z}_{k,\eta}^{x,y} - \mathbb{E}[\hat{Z}_{k,\eta}^{x,y}])^*] &= \mathbb{E}[Z_{N,\eta}(\hat{Z}_{k,\eta}^{x,y})^*] - \mathbb{E}[Z_{N,\eta}Z_{N,\eta} Q_N^{-1/2} K_k] \\
&= Q \sum_{l=0}^{k-1} S_{\eta}^{k-l} S_{\eta}^{-l} - Q_N Q_N^{-1/2} K_k = Q_k S_{\eta}^{N-k} - Q_k S_{\eta}^{N-k} = 0.
\end{align*}
$$

This also yields that $Z_{N,\eta}$ and $\hat{Z}_{k,\eta}^{x,y}$ ($k = 1, \ldots, N - 1$) are independent.

As we have stated, we now show the minorization condition. Let $P_{x,\cdot}^y$ be the probability measure of the law of $X_{\eta}$ with $X_0 = x$, then by the Girsanov’s theorem, $P_{x,\cdot}^y$ is absolutely continuous with respect to $\mu_{x,\cdot}^\eta$, and the Radon-Nikodym density is given by

$$
\frac{dP_{x,\cdot}^y}{d\mu_{x,\cdot}^\eta}(y) = \mathbb{E} \left[ \exp \left\{ \frac{\beta}{2\eta} \sum_{k=0}^{N-1} \left( -\eta \nabla L(Z_{k,\eta}^x), \epsilon_k \right) \sqrt{2\eta/\beta} - \frac{\eta^2}{2} \left\| \nabla L(Z_{k,\eta}^x) \right\|^2 \right\} \bigg| Z_{N,\eta} = y \right].
$$

The right hand side can be evaluated as

$$
\begin{align*}
\mathbb{E} \left[ \exp \left\{ \frac{\beta}{2\eta} \sum_{k=0}^{N-1} \left( -\eta \nabla L(Z_{k,\eta}^x), \epsilon_k \right) \sqrt{2\eta/\beta} - \frac{\eta^2}{2} \left\| \nabla L(Z_{k,\eta}^x) \right\|^2 \right\} \bigg| Z_{N,\eta} = y - S_{\eta}^N x \right]
&= \mathbb{E} \left[ \exp \left\{ \frac{\beta}{2\eta} \sum_{k=0}^{N-1} \left( -\eta \nabla L(Z_{k,\eta}^x), \epsilon_k \right) \sqrt{2\eta/\beta} + \left( -\eta \nabla L(Z_{k,\eta}^x), (B_1(k)Z_{N,\eta} - B_2(k)\hat{Z}_{k,\eta}) \right) \sqrt{2\eta/\beta} \\
&\quad - \frac{\eta^2}{2} \left\| \nabla L(Z_{k,\eta}^x) \right\|^2 \right\} \bigg| Z_{N,\eta} = y - S_{\eta}^N x \right]
\end{align*}
$$

$$
\begin{align*}
&= \mathbb{E} \left[ \exp \left\{ \frac{\beta}{2\eta} \sum_{k=0}^{N-1} \left( -\eta \nabla L(Z_{k,\eta}^x), \epsilon_k \right) \sqrt{2\eta/\beta} \\
&\quad + \left( -\eta \nabla L(Z_{k,\eta}^x), B_1(k)(y - S_{\eta}^N x) - B_2(k)\hat{Z}_{k,\eta} \right) \sqrt{2\eta/\beta} - \frac{\eta^2}{2} \left\| \nabla L(Z_{k,\eta}^x) \right\|^2 \right\} \right],
\end{align*}
$$
Dimension-free convergence rates for gradient Langevin dynamics in RKHS

where we used the fact that $(\hat{Z}_k)_k$ and $(\zeta_k)_k$ are independent of $Z_{N,\eta}$. Therefore, by Jensen’s inequality, the right hand side is lower bounded by

$$
\exp \left\{ \frac{\beta}{2\eta} \sum_{k=0}^{N-1} \left[ \mathbb{E} \left[ (-\eta \nabla L(\hat{Z}^{x,y}_{k,\eta}), B_1(k)(y - S^N_{\eta} x) - B_2(k)\hat{Z}_{k,\eta}) \right] \right] \mathbf{1}_{n} \left( \frac{\eta^2}{\beta} + \mathbb{E} \left[ \|\nabla L(\hat{Z}_{k,\eta}^{x,y})\|^2 \right] \right) \right\}.
$$

Thus, by the assumption that $\|\nabla L(\cdot)\| \leq B$, the right hand side is lower bounded by

$$
\exp \left\{ -\sqrt{\frac{\beta}{2\eta}} \sum_{k=0}^{N-1} \mathbb{E} \left[ (-\eta \nabla L(\hat{Z}^{x,y}_{k,\eta}), B_1(k)(y - S^N_{\eta} x)) \right] + \frac{\beta}{2} \mathbb{E} \left[ \|B_2(k)\hat{Z}_{k,\eta}\| \right] \right\} \geq \exp \left\{ \frac{\beta}{2} \sum_{k=0}^{N-1} \left[ \mathbb{E} \left[ \left( \nabla L(\hat{Z}^{x,y}_{k,\eta}), B_1(k)(y - S^N_{\eta} x) \right) \right] - \frac{\beta}{2} \mathbb{E} \left[ \|B_2(k)\hat{Z}_{k,\eta}\|^2 \right] \right] \right\}.
$$

For $z \in \mathcal{H}$, we have

$$
\sqrt{\frac{\beta}{2\eta}} \sum_{k=0}^{N-1} \mathbb{E} \left[ (\eta \nabla L(\hat{Z}^{x,y}_{k,\eta}), B_1(k)z) \right] = \sqrt{\frac{\beta}{2\eta}} \sum_{k=0}^{N-1} \mathbb{E} \left[ (\eta \nabla L(\hat{Z}^{x,y}_{k,\eta}) - \nabla L(0)), B_1(k)z) \right] = \sqrt{\frac{3\beta}{2}} \left\langle \left( \sum_{k=0}^{N-1} B_1(k) \right) \nabla L(0), z \right\rangle + \sqrt{\frac{\beta}{2}} \sum_{k=0}^{N-1} \mathbb{E} \left[ (\nabla L(\hat{Z}^{x,y}_{k,\eta}) - \nabla L(0)), B_1(k)z) \right].
$$

The first term of the right hand side can be lower bounded by

$$
-\frac{\beta N}{4} - \frac{1}{2} \sum_{k=0}^{N-1} \mathbb{E} \left[ \langle B_1(k) \nabla L(0), z \rangle^2 \right] = \frac{\beta N}{4} - \frac{1}{2} \sum_{k=0}^{N-1} \mathbb{E} \left[ (QS^N_{\eta} - k)Q^{-1}_{N} \nabla L(0), z) \right]^2.
$$

The second term can be evaluated as

$$
\sqrt{\frac{\beta}{2}} \sum_{k=0}^{N-1} \mathbb{E} \left[ (\nabla L(\hat{Z}^{x,y}_{k,\eta})), B_1(k)z) \right] = \sqrt{\frac{\beta}{2}} \sum_{k=0}^{N-1} \mathbb{E} \left[ (\nabla L(\hat{Z}^{x,y}_{k,\eta})) \cdot \hat{Z}^{x,y}_{k,\eta}, B_1(k)z) \right],
$$

where $\hat{Z}^{x,y}_{k,\eta}$ is an intermediate point between $\hat{Z}^{x,y}_{k,\eta}$ and $0$, i.e., there exists $\theta \in [0, 1]$ such that $\hat{Z}^{x,y}_{k,\eta} = \theta \hat{Z}^{x,y}_{k,\eta}$. By Assumption 4, this can be further evaluated as

$$
\sqrt{\frac{\beta}{2}} \sum_{k=0}^{N-1} \mathbb{E} \left[ (\nabla L(\hat{Z}^{x,y}_{k,\eta})) \cdot \hat{Z}^{x,y}_{k,\eta}, B_1(k)z) \right] \geq \sqrt{\frac{\beta}{2}} \sum_{k=0}^{N-1} C_{\alpha,2} \mathbb{E} \left[ \|\hat{Z}^{x,y}_{k,\eta}\|_{\mathcal{H}} \|B_1(k)z\|_{\alpha} \right] - \frac{\beta}{4} \sum_{k=0}^{N-1} \mathbb{E} \left[ \|\hat{Z}^{x,y}_{k,\eta}\|_{\mathcal{H}}^2 \|B_1(k)z\|_{\alpha}^2 \right] - \frac{\beta}{4} \sum_{k=0}^{N-1} \mathbb{E} \left[ \|\hat{Z}^{x,y}_{k,\eta}\|_{\mathcal{H}} \|B_1(k)z\|_{\alpha}^2 \right] - \frac{\beta}{4} \sum_{k=0}^{N-1} \mathbb{E} \left[ \|\hat{Z}^{x,y}_{k,\eta}\|_{\mathcal{H}}^2 \|B_1(k)z\|_{\alpha}^2 \right].
$$

Here, we have

$$
\mathbb{E} \left[ \|\hat{Z}^{x,y}_{k,\eta}\|_{\mathcal{H}}^2 \right] = \mathbb{E} \left[ (Q_kQ^{-1}_{N-k}Q^{-1}_{N}) \right] + \mathbb{E} \left[ (S^k_{\eta} - K_kQ^{-1/2}_{N}S_{\eta}^N)x + K_kQ^{-1/2}_{N}y \right]_{\mathcal{H}}.
$$
Therefore, we obtain
\[
\sum_{k=0}^{N-1} \mathbb{E} \left[ (\eta \nabla L(\hat{Z}_{k;\eta}^y), B_{1}(k)z) \right] \geq -\frac{\beta \eta^2}{4} - \frac{\beta \eta^2}{4} C_{\alpha,2}^2 \sum_{k=0}^{N-1} \left( \text{Tr}[Q S^2_{\eta} (\text{Id} - S^{2N}_{\eta})^{-1}] + 2\|x\|_{\mathcal{H}}^2 + 2\|y\|_{\mathcal{H}}^2 \right)
\]
\[
- \frac{1}{2} \sum_{k=0}^{N-1} ((Q S^{N-k}_{\eta} Q^{-1} N \nabla L(0), z)^2 + \|Q S^{N-k}_{\eta} Q^{-1} N z\|_{A}^2).
\]

Next we give another bound for \( z = S^N_{\eta} x \). In this situation, thanks to the factor \( S^{N}_{\eta} \), we have a simpler bound:
\[
\sum_{k=0}^{N-1} \mathbb{E} \left[ (\eta \nabla L(\hat{Z}_{k;\eta}^y), B_{1}(k)S^N_{\eta} x) \right] \geq -\sqrt{\frac{\beta \eta^2}{2}} B \mathbb{E} \left[ B(k)S^N_{\eta} x \right]\geq -\frac{\beta \eta N}{4} B^2 + \frac{1}{2} \sum_{k=0}^{N-1} \|B_{1}(k)S^N_{\eta} x\|^2.
\]

Notice that
\[
\sum_{k=0}^{N-1} B_{1}(k)^2 = \sum_{k=0}^{N-1} (Q^{1/2}_{N-k} H_k Q^{-1}_{N})^2 = \sum_{k=1}^{N-1} Q_{N-k} H_k^2 Q^{-1}_{N} = \sum_{k=0}^{N-1} Q_{N-k} Q_{N-k}^2 S^{2(N-k)}_{\eta} Q^{-2}_{N} = \sum_{k=0}^{N-1} S^{2(N-k)}_{\eta} Q^{-2}_{N} = Q^{-1}_{N}.
\]

Therefore, \( \sum_{k=0}^{N-1} \|B_{1}(k)S^N_{\eta} x\|^2 \) can be bounded as
\[
\sum_{k=0}^{N-1} \|B_{1}(k)S^N_{\eta} x\|^2 = \|Q^{-1/2}_{N} S^N_{\eta} x\|^2 \leq \frac{1}{N} Q^{-1} \|x\|^2 = \frac{\beta}{2N \eta} \|x\|^2,
\]

where we used \( Q_N \geq N Q^{2N}_{\eta} \) and \( Q = \frac{2 \eta}{\beta} \text{Id} \). Therefore, we have
\[
\sqrt{\frac{\beta}{2N \eta}} \sum_{k=0}^{N-1} \mathbb{E} \left[ (\eta \nabla L(\hat{Z}_{k;\eta}^y), B_{1}(k)S^N_{\eta} x) \right] \geq \frac{\beta \eta N}{4} B^2 - \frac{\beta}{4N \eta} \|x\|^2.
\]
Therefore, we obtain, for all $y \in \text{Im}(Q_N^{1/2})$,

$$\frac{dP^\eta_N(x,\cdot)}{d\mu_{\mathcal{N},\eta}^\lambda}(y) \geq \exp \left\{ -\frac{\beta \eta N}{4} - \frac{\beta \eta N}{4} C_{a,2}^2 \sum_{k=0}^{N-1} (\text{Tr}[QS_N^{2}(\text{Id} - S_N^{2N})^{-1}] + 2\|x\|^2 + 2\|y\|^2) 
- \left( \frac{\beta \eta N}{4} B^2 - \frac{\beta}{4N\eta} \|x\|^2 \right) \right\} $$

Combining the inequalities (37) and (38), we finally obtain that

$$\frac{dP^\eta_N(x,\cdot)}{d\mu_{\mathcal{N},\eta}^\lambda}(y) \geq \exp \left\{ -\frac{\beta}{2} \left( 1 + \frac{1}{2\eta N} \right) \|x\|^2 - \frac{1}{2 \beta} \sum_{k=0}^{N-1} \left( \langle QS_N^{2N-k} Q_N^{-1} \nabla L(0), y \rangle^2 + \|QS_N^{2N-k} Q_N^{-1} y\|_2^2 \right) \right\} $$

From now on, we give a lower bound of the right hand side. To do so, we set $N = 1/\eta$. Under this setting, let $\Lambda_\alpha(x) := \frac{1}{2} \|x\|^2 + \Lambda_\alpha(x)$ and $\Lambda_\gamma(y) := \frac{4\beta}{\eta} \|S_N Q_N^{-1} y\|_2^2 + \Lambda_\gamma(y)$, i.e.,

$$\frac{dP^\eta_N(x,\cdot)}{d\mu_{\mathcal{N},\eta}^\lambda}(y) \geq \exp \left\{ -C_{\eta,N,\beta} - \Lambda_\alpha(x) - \Lambda_\gamma(y) \right\} \quad (39)$$

We evaluate the terms in the exponent in the right hand side one by one.

(i) (Bound of $C_{\eta,N,\beta}$): Note that

$$\|\text{Id} - S_N^{2N}\|_2 \leq \left[ 1 - (1 + \eta \lambda / \mu_0)^{-2N} \right]^{-1} \leq \left( 1 + \eta \lambda / \mu_0 \right)^{2N} \left( 1 + \eta \lambda / \mu_0 \right)^{-2N} - 1^{-1}$$

$$\leq \frac{\exp(2N\eta \lambda / \mu_0)}{2N\eta \lambda / \mu_0} = \frac{\exp(2\lambda / \mu_0)}{2\lambda / \mu_0}, \quad (41)$$

and thus

$$\text{Tr}[QS_N^{2}(\text{Id} - S_N^{2N})^{-1}] \leq \frac{2\eta}{\beta} \text{Tr}[S_N^{2}(\text{Id} - S_N^{2N})^{-1}] \leq \frac{2\eta}{\beta} \text{Tr}[S_N^{2}]\|S_N^{2}(\text{Id} - S_N^{2N})^{-1}\|_2 \leq \frac{2\eta}{\beta} \exp(2\lambda / \mu_0) \sum_{k=0}^{\infty} (1 + \eta \lambda / \mu_k)^{-2} \leq C_\mu \frac{2\eta}{\beta} \exp(2\lambda / \mu_0) \sqrt{\frac{1}{\eta \lambda}}$$

$$= C_\mu \frac{\sqrt{\eta} \mu_0 \exp(2\lambda / \mu_0)}{\lambda^{3/2}}.$$
where $C_\mu$ is a constant depending on $(\mu_k)_k$ and we used $\mu_k \lesssim 1/k^2$ in the last inequality. This converges to 0 as $\eta \to 0$ and $\beta \to \infty$, thus $\text{Tr}(Q S_\eta^2 (\text{Id} - S_\eta^{2N} )^{-1}) = O(1)$. Consequently, we have

$$C_{\eta, \lambda, \beta} = \frac{\beta(1 + B^2)}{4} + \frac{1}{4} C_{\alpha, 2}^2 \sqrt{\eta} \frac{\exp(2\lambda/\mu_0)}{\lambda^{\beta/2}} + \text{Tr} [(\text{Id} + 2A)^{-1}] = O(\beta).$$

(ii) (Bound of $\Lambda_\beta(x)$): By the definition of $\Lambda_\beta(x)$, it holds that

$$\Lambda_\beta(x) = \left\{ \begin{array}{ll} \beta \left( \frac{1}{2} + \frac{1}{2\eta N} \right) + \frac{\beta N}{2} C_{\alpha, 2}^2 + \frac{\beta}{4 N \eta} \right\} \|x\|^2 = \left( \beta + \frac{\beta}{2} C_{\alpha, 2}^2 \right) \|x\|^2 = O(\beta \|x\|^2).$$

(ii) (Bound of $\Lambda_\beta(y)$): Finally, we evaluate $\Lambda_\beta(y)$. When $\eta = 1/N$,

$$\Lambda_\beta(y) = \frac{1}{4\beta} \|S_\eta^N Q_\eta^{-1} y\|^2 + \frac{\beta}{2} C_{\alpha, 2}^2 \|y\|^2 + \sum_{k=0}^{N-1} \left( \langle Q S_\eta^{N-k} Q_\eta^{-1} \nabla L(0), y \rangle^2 + \|QS_\eta^{N-k} Q_\eta^{-1} y\|^2 \right).$$

We can show that $\Lambda_\beta(Z) \leq \infty$ for $Z \sim \mu_{N, \eta}$ almost surely, as follows. Since $0 \leq \Lambda_\beta(y)$, we only have to evaluate $E Z \sim \mu_{N, \eta}[\Lambda_\beta(Z)]$. To do so, we note that $\mu_{N, \eta}$ is a Gaussian process in $\mathcal{H}$ with mean 0 and covariance $Q_N$, which can be easily checked by its definition. By using this, we evaluate the expectation of each term as follows.

$$E Z \sim \mu_{N, \eta} \left[ \frac{1}{4\beta} \|S_\eta^N Q_\eta^{-1} y\|^2 \right] = \frac{1}{4\beta} \text{Tr}(S_\eta^N Q_\eta^{-2} Q_N) = \frac{1}{4\beta} \text{Tr}(S_\eta^N (S_\eta^2 + \ldots + S_\eta^{2N})^{-1} Q^{-1})$$

$$= \frac{1}{4\beta} \text{Tr}(Q^{-1} S_\eta^{2N} (\text{Id} - S_\eta^2 (S_\eta^2 - S_\eta^{2N})^{-1})) \leq \frac{1}{8\beta} \text{Tr}(Q^{-1} S_\eta^{2N} (2\eta A + \eta^2 A^2)) \frac{\exp(2\lambda/\mu_0)}{2\lambda/\mu_0} \quad (\because \text{Eq. (41)})$$

$$= \frac{1}{8\beta} \text{Tr}(Q^{-1} S_\eta^{2N} (S_\eta^{-2} - \text{Id})) \frac{\exp(2\lambda/\mu_0)}{2\lambda/\mu_0} \leq \frac{1}{8\beta} \text{Tr}(S_\eta^2 (2\lambda A + \lambda^2 A^2)) \frac{\exp(2\lambda/\mu_0)}{2\lambda/\mu_0} \leq 1 + 2\eta \exp(2\lambda/\mu_0) C_\mu' \leq O(1),$$

where $C_\mu'$ is a constant depending only on $(\mu_k)_k$ and we again used $\mu_k \lesssim 1/k^2$ in the last inequality.

$$E Z \sim \mu_{N, \eta} \left[ \frac{\beta}{2} C_{\alpha, 2}^2 \|y\|^2 \right] = \frac{\beta}{2} C_{\alpha, 2}^2 \text{Tr}(Q_N) = \frac{\beta}{2} C_{\alpha, 2}^2 \text{Tr}(Q (S_\eta^2 - S_\eta^{2(N+1)} (\text{Id} - S_\eta^{2}))^{-1})$$

$$= \eta C_{\alpha, 2}^2 \text{Tr}((\text{Id} - S_\eta^{2N}) (S_\eta^{-2} - \text{Id})) \leq \eta C_{\alpha, 2}^2 \sum_{k=0}^\infty (1 - \lambda^2/\mu_k)^{-2N} \leq \eta C_{\alpha, 2}^2 \sum_{k=0}^\infty (1 - \lambda^2/\mu_k)^{-2N} \leq \eta C_{\alpha, 2}^2 \sum_{k=0}^\infty (1 - \lambda^2/\mu_k)^{-2N} = O(1),$$

where $C_\mu''$ is a constant depending only on $(\mu_k)_k$ and we again used $\mu_k \lesssim 1/k^2$ in the last inequality.
and its summation becomes
\[ \sum_{k=0}^{\infty} \left\langle Q^2 S_{\eta}^{2(N-k)} Q_N^{-1} \nabla L(0), \nabla L(0) \right\rangle = \left\langle QQ_N Q_N^{-1} \nabla L(0), \nabla L(0) \right\rangle = \left\langle Q \nabla L(0), \nabla L(0) \right\rangle = \frac{2\eta}{\beta} \|L(0)\|^2 = O(\eta/\beta). \]

The second term can be evaluated as
\[ \sum_{k=0}^{\infty} \text{Tr}[Q S_{\eta}^{N-k} Q_N^{-1} \sqrt{Q N} \nabla L(0) \nabla L(0)], \]
where \( \beta > 0 \) and we used the assumption \( \alpha > 1/4 \) and \( \mu_k \leq 1/k^2 \). Summarizing the above arguments, we obtain that
\[ \mathbb{E}_{Z \sim \mu_{\eta,N}} [\Lambda_N(Z)] \leq O(1). \quad (42) \]

(iv) (Combining all bounds (i), (ii), (iii)). Combining these bounds for \( C_{\eta,N,\beta}, \Lambda_N(x), \Lambda_N(y) \), we may give a lower bound of \( P_N^\eta(x, \Gamma) \) for a measurable set \( \Gamma \subset \mathcal{H} \) uniformly for all \( x \) with norm smaller than a given \( R \), which is required to show the minorization condition. Let
\[ c_R \triangleq \exp \left( -C_{\eta,N,\beta} - \frac{\beta}{2} (2 + C_{\alpha,2}^2) R^2 \right) \]
for \( R \geq \frac{3}{2} k(1) \) which will be determined later, then we have shown that for all \( x \in \mathcal{H} \) with \( \|x\| \leq R \),
\[ \exp(-C_{\eta,N,\beta} - \Lambda_N(x)) \geq c_R. \]

By Eq. (39), this gives that
\[ P_N^\eta(x, \Gamma) \geq c_R \int_{\Gamma} e^{-\Lambda_N(z)} \mu_{N,\eta}(dz), \]
for all \( x \in B_R \) and a measurable set \( \Gamma \subset \mathcal{H} \). In particular, if we define
\[ \bar{\mu}(\Gamma) \triangleq \frac{1}{Z} \int_{\Gamma \cap B_R} e^{-\Lambda_N(z)} \mu_{N,\eta}(dz) \]
where \( Z = \int_{B_R} e^{-\Lambda_N(z)} \mu_{N,\eta}(dz) \) so that \( \bar{\mu} \) is a probability measure, then
\[ P_N^\eta(x, \Gamma) \geq c_R \int_{\Gamma \cap B_R} e^{-\Lambda_N(z)} \mu_{N,\eta}(dz) \geq \delta \bar{\mu}(\Gamma), \quad (43) \]
where
\[ \delta \triangleq c_R Z. \]

Here, we give a lower bound of \( \delta \). By Proposition 12,
\[ \mu_{N,\eta}(B_R) \geq 1 - \frac{\mathbb{E}_{Z \sim \mu_{N,\eta}} [\|Z\|]}{R} \geq 1 - \frac{1}{R} k(1) \geq \frac{1}{3}, \]
where we used \( R \geq \frac{3}{2} k(1) \) and thus \( \delta \) can be lower bounded as
\[ \delta = c_R \int_{B_R} e^{-\Lambda_N(z)} \mu_{N,\eta}(dz) = c_R \mu_{N,\delta}(B_R) \frac{1}{\mu_{N,\delta}(B_R)} \int_{B_R} e^{-\Lambda_N(z)} \mu_{N,\eta}(dz) \]
\[ \geq c_R \mu_{N,\delta}(B_R) \exp \left( - \frac{1}{\mu_{N,\delta}(B_R)} \int_{B_R} \Lambda_N(z) \mu_{N,\eta}(dz) \right) \]
\[ \geq \frac{1}{3} c_R \exp \left( -2 \int_{\mathcal{H}} \Lambda_N(z) \mu_{N,\eta}(dz) \right) \]
where we used Eq. (42) in the final inequality. Therefore, we have shown that there exists a probability measure $\bar{\mu}$, with $\bar{\mu}(B_R) = 1$ and $\bar{\mu}(B_R^c) = 0$, such that Eq. (43) is satisfied for any $x \in B_R$ and a measurable set $\Gamma \in \mathcal{B}(\mathcal{H})$, where $\delta \geq \frac{1}{3} c_R \exp(-O(1)) \geq \frac{1}{3} \exp(-C_{\eta,N,\beta} - \Lambda(x) - O(1)) \geq \exp(-O(\delta))$.

By Proposition 13, the following contraction condition holds for $\alpha_N = \rho^N = (\frac{1}{1 + \lambda^N/\mu_0})^N \leq \exp(-\lambda/\mu_0) < 1$, $\bar{b} = \max\{\frac{4}{\lambda N} B + k(1), 1\}$ under the bounded gradient condition:

$$E_{x_0} \|X_N\| \leq \alpha_N \|x_0\| + \bar{b} \quad (\forall n \in \mathbb{N}).$$

Set $V(x) = \|x\| + 1$ and $C = \left\{ x \in \mathcal{H} \mid V(x) \leq \frac{2b}{\sqrt{(1 + \alpha_N)/2 - \alpha_N}} \right\}$, then we have that $C = B_R$ for $R = \frac{2b}{\sqrt{(1 + \alpha_N)/2 - \alpha_N}} - 1$.

Here, we give lower and upper bounds of $R$. As for the lower bound, we can easily see that $R \geq \frac{2b}{\sqrt{a} - 1} \geq \frac{2b}{\sqrt{a}} \geq \frac{2b}{\sqrt{N}}$. Next, we give an upper bound. Jensen’s inequality and the fact $0 < \alpha_N < 1$ yield $\sqrt{(1 + \alpha_N)/2 - \alpha_N} \geq \frac{1 + \sqrt{\alpha_N}}{2} - \sqrt{\alpha_N} = \frac{1 - \sqrt{\alpha_N}}{2}$. Here for $a > 0$, it is easy to see $(1 + a)^{N/2} \geq 1 + aN/2$ and thus we have $1 - (1 + a)^{-N/2} \geq 1 - (1 + aN/2)^{-1} = \frac{N \alpha N/2}{1 + \alpha N/2}$. Substituting $a = \lambda N/\mu_0$, it is easy to check $\alpha_N \leq \frac{N \lambda N/2}{1 + \lambda N/2} < 1$. Then, by using $\eta = 1/N$, we obtain that

$$\frac{2b}{\sqrt{(1 + \alpha_N)/2 - \alpha_N}} \leq \frac{4b \mu_0 (1 + \alpha N/2 \lambda)}{1 + \lambda N/2} = \frac{2b(1 + 2N)}{\sqrt{\alpha_N}}.$$

Then, Theorem 2.5 of (Mattingly et al., 2002) asserts that there exists an invariant measure $\mu^0$ for the Markov chain $(X_{IN})_t$ and the chain satisfies the geometric ergodicity: for $\phi : \mathcal{H} \to \mathbb{R}$ such that $|\phi(\cdot)| \leq V(\cdot)$,

$$E[\phi(X_{IN})] - E_{X \sim \mu^0}[\phi(X)] \leq \kappa[\bar{V} + 1](1 - \delta)^a + \sqrt{2} V(x_0) \gamma^t(\kappa[\bar{V} + 1])^a \frac{1}{\sqrt{\delta}},$$

where $\kappa = \bar{b} + 1$, $\bar{V} = 2 \sup_{x \in C} V(x) = \frac{4b}{\sqrt{(1 + \alpha_N)/2 - \alpha_N}}$, $\gamma = \sqrt{(\alpha N + 1)/2}$ and $a \in (0, 1)$ so that $\gamma(\kappa[\bar{V} + 1])^a \leq (1 - \delta)^a$. In particular, we may choose $a \in (0, 1)$ as

$$a = \frac{\log(1/\gamma)}{\log(\kappa(\bar{V} + 1)/(1 - \delta))}.$$

Here, by noting that

$$\log(1/\gamma) = -\frac{1}{2} \log \left( \frac{1 + \alpha N}{2} \right) = \frac{1}{2} \log \left( \frac{1 - \alpha N}{2} \right) \geq \frac{1}{2} \left( \frac{1 - \alpha N}{2} \right) \geq \frac{1}{4} \min \left( \frac{\lambda}{2\mu_0}, \frac{1}{2} \right) = \Omega(\lambda/\mu_0),$$

we may assume

$$a \geq \frac{\min \left( \frac{\lambda}{2\mu_0}, \frac{1}{2} \right)}{4 \log(\kappa(\bar{V} + 1)/(1 - \delta))}.$$

Then Eq. (44) is simplified to

$$E[\phi(X_{IN})] - E_{X \sim \mu^0}[\phi(X)] \leq \left( \kappa[\bar{V} + 1] + \frac{\sqrt{2} V(x_0)}{\sqrt{\delta}} \right) (1 - \delta)^a.$$

This shows the geometric ergodicity of the sequence $(X_{IN})_{n=1}^\infty$. To extend this result to “unsampled” sequence $(X_n)_{n=1}^\infty$, we may apply the same argument to the sequence $(X_{IN + n})_{n=1}^\infty$ for each $n = 1, \ldots, N - 1$. Applying Eq. (44) where $x_0$ is replaced with $X_n$ and taking expectation with respect to $X_n$, we have

$$E[\phi(X_{IN + n})] - E_{X \sim \mu^0}[\phi(X)] \leq \left( \kappa[\bar{V} + 1] + \frac{\sqrt{2} E[V(X_n)]}{\sqrt{\delta}} \right) (1 - \delta)^a$$

$$\leq \left( \kappa[\bar{V} + 1] + \frac{\sqrt{2}(\rho^N \|x_0\| + b + 1)}{\sqrt{\delta}} \right) (1 - \delta)^a \quad (\therefore \text{Proposition 13})$$
Finally, we note that for $0 \leq n < N$,
\[
(1 - \delta)^{\alpha l} \leq (1 - \delta)^n [(N + n - 1)/N] \leq (1 - \delta)^n [(N + n)/N - 1]} \leq \exp \left( -\delta a[(N + n)/N - 1] \right),
\]
where we set
\[
\Lambda_n^* \triangleq a \delta \geq \min \left( \frac{1}{2 \rho x a} \right) \geq 4 \log \left( \frac{\kappa}{V + 1}/(1 - \delta) \right) \delta = \Omega(\exp(-O(\beta))).
\]
This yields the assertion. □

**G. Proof of Proposition 16**

**Lemma 19** (Gaussian correlation inequality). Let $\nu_\infty$ be the Gaussian measure in $\mathcal{H}$ given by a random variable $\sum_{i=0}^{\infty} \xi_i f_i$ where $(\xi_i)_{i=0}^{\infty}$ is a sequence of i.i.d. standard normal variables and $(\gamma_i)_{i=0}^{\infty}$ is a sequence of real variables with $0 < \sum_{i=0}^{\infty} \gamma_i^2 < \infty$. For two sets $C^1 = \{ X = \sum_{i=0}^{\infty} \alpha_i f_i \in \mathcal{H} \mid \sum_{i=0}^{\infty} \alpha_i^2 \mu_i^{(1)} \leq 1 \}$ and $C^2 = \{ X = \sum_{i=0}^{\infty} \alpha_i f_i \in \mathcal{H} \mid \sum_{i=0}^{\infty} \alpha_i^2 \mu_i^{(2)} \leq 1 \}$ where $(\mu_i^{(1)})_{i=1}^{\infty}$ is a fixed non-negative sequence and $(\mu_i^{(2)})_{i=1}^{\infty}$ is a fixed sequence of real numbers satisfying $\sum_{i=0}^{\infty} (\mu_i^{(2)})^2 < \infty$, we have
\[
\nu_\infty(C^1 \cap C^2) \geq \nu_\infty(C^1) \nu_\infty(C^2).
\]

**Proof.** Let $C_n^1$ and $C_n^2$ be the cylinder set that “truncates” $C^1$ and $C^2$ up to index $n$: $C_n^1 = \{ X = \sum_{i=0}^{\infty} \alpha_i f_i \in \mathcal{H} \mid \sum_{i=0}^{n} \alpha_i^2 \mu_i^{(1)} \leq 1 \}$ and $C_n^2 = \{ X = \sum_{i=0}^{\infty} \alpha_i f_i \in \mathcal{H} \mid \sum_{i=0}^{n} \alpha_i^2 \mu_i^{(2)} \leq 1 \}$. By the Gaussian correlation inequality (Royen, 2014; Talata, 2017), it holds that
\[
\nu_\infty(C_n^1 \cap C_n^2) \geq \nu_\infty(C_n^1) \nu_\infty(C_n^2).
\]

We note that $(C_n^1)_n$ is a monotonically decreasing sequence, i.e., $C_n^1 \subseteq C_m^1$ for $m < n$, and we see that $\cap_{n=1}^{\infty} C_n^1 = C^1$. By the continuity of probability measure, this yields that $\lim_{n \to \infty} \nu_\infty(C_n^1 \setminus C^1) = 0$ and $\lim_{n \to \infty} \nu_\infty(C_n^1) = \nu(C^1)$. On the other hand, for any $\epsilon > 0$, there exists $N$ such that $\sum_{i=N}^{\infty} (\gamma_i^2 \mu_i^{(2)})^2 \leq \epsilon$ by the assumption $(\sum_{i=0}^{\infty} \gamma_i^2 < \infty$ and $\sum_{i=0}^{\infty} (\mu_i^{(2)})^2 < \infty$). Hence, it holds that $\mathbb{E}[(\sum_{i=N}^{\infty} \gamma_i \xi_i \mu_i^{(2)})^2] = \sum_{i=N}^{\infty} (\gamma_i \mu_i^{(2)})^2 \leq \epsilon$, which indicates that, by Markov’s inequality,
\[
\nu_\infty(\{ \sum_{i=0}^{\infty} \alpha_i f_i \mid \sum_{i=N}^{\infty} \alpha_i^2 \mu_i^{(2)} > \delta \}) \leq \epsilon/\delta^2
\]
for any $\delta > 0$. If we set $C^2_\epsilon = \{ \sum_{i=0}^{\infty} \alpha_i f_i \in \mathcal{H} \mid \sum_{i=0}^{\infty} \alpha_i^2 \mu_i^{(2)} \leq 1 + \epsilon \}$, then this and the continuity of Gaussian measures (note that $\sum_{i=0}^{\infty} \xi_i \mu_i^{(2)}$ is a one dimensional Gaussian measure and has density with respect to the Lebesgue measure) yield that, for any $\epsilon > 0$, there exists $N$ such that for all $n \geq N$, it holds that
\[
\nu_\infty(C_n^1 \setminus C_n^2) - \epsilon \leq \nu_\infty(C_n^1) - \nu_\infty(C_n^2) + \epsilon,
\]
\[
\nu_\infty(C^1 \cap C_n^2 - \epsilon) - \epsilon \leq \nu_\infty(C^1 \cap C_n^2) - \nu_\infty(C_n^1 \cap C_n^2) + \epsilon.
\]

Since $\lim_{n \to \infty} \nu_\infty(C_n^1 \setminus C^1) = 0$, the second inequality also gives
\[
\nu_\infty(C_n^1 \cap C_n^2 - \epsilon) - 2\epsilon \leq \nu_\infty(C_n^1 \cap C_n^2) - \nu_\infty(C_n^1 \cap C_n^2) + 2\epsilon.
\]
for any $n \geq N'$ with sufficiently large $N'$. Therefore, since $\lim_{n \to \infty} \nu_\infty((C_n^2 \setminus C^2) \cup (C^2 \setminus C_n^2)) = 0$ by the continuity of Gaussian measures and $\lim_{n \to \infty} \nu_\infty(C_n^1 \setminus C^1)$, by taking the limit of $\epsilon$ and $n$ of this inequality, we have
\[
\nu_\infty(C^1 \cap C^2) = \lim_{n \to \infty} \nu_\infty(C_n^1 \cap C_n^2).
\]

Hence, applying the Gaussian correlation inequality to the right hand side yields
\[
\nu_\infty(C^1 \cap C^2) = \lim_{n \to \infty} \nu_\infty(C_n^1 \cap C_n^2).
\]
Proposition 16: The proof relies on comparing the stationary distribution $\pi$ of Eq. (8) to the Gaussian stationary distribution $\nu^{(\beta)}$ of Eq. (30) (case $F = 0$). We then conclude by using the small ball probability theorem (Kuelbs and Li, 1993; Li and Shao, 2001) and Lemma 19 on $\nu_{\infty}$. First note that

$$
\int L(x) \, d\pi(x) - L(\bar{x}) = -\frac{1}{\beta} \int \log \{e^{-\beta(L(x) - L(\bar{x}))}\} \, d\pi(x) = -\frac{1}{\beta} \int \log \{\frac{1}{\Lambda} e^{-\beta(L(x) - L(\bar{x}))}\} \, d\pi(x) - \frac{1}{\beta} \log \Lambda
$$

$$= -\frac{1}{\beta} \text{KL}(\pi||\nu_{\infty}^{(\beta)}) - \frac{1}{\beta} \log(\Lambda),
$$

(47)

where $\nu_{\infty}^{(\beta)}$ is the invariant distribution of Eq. (30), i.e., the centered Gaussian on $\mathcal{H}$ with covariance operator $(-\beta A)^{-1}$, $\Lambda \triangleq \int \exp[-\beta (L(x) - L(\bar{x}))] \, d\nu_{\infty}^{(\beta)}(x)$, and $\text{KL}(\mu||\nu) \triangleq \int \log(\mu/d\nu) \, d\mu$ for probability measures $\mu$ and $\nu$ that are mutually absolutely continuous. Since the KL-divergence $\text{KL}(\pi||\nu_{\infty}^{(\beta)})$ is non-negative, the right hand side is upper bounded by $-\frac{1}{\beta} \log(\Lambda)$. By definition of $\bar{x}$ (Eq. (6)), it holds that

$$\nabla L(\bar{x}) = -\frac{\lambda}{2} \nabla \|\bar{x}\|_{\mathcal{H}_K}^2 = -\lambda \sum_{k \geq 0} \langle \frac{f_k}{\mu_k}, f_k \rangle \mu_k.
$$

Hence, using the $M$-smoothness of $L$, we obtain

$$L(x) - L(\bar{x}) \leq \frac{1}{2} M \|x - \bar{x}\|^2 + \lambda \langle \bar{x}, x - \bar{x} \rangle_{\mathcal{H}_K}
$$

$$\leq \frac{1}{2} M \|x - \bar{x}\|^2 + \lambda \|\bar{x}\|_{\mathcal{H}_K} \langle \frac{\bar{x}}{\|\bar{x}\|_{\mathcal{H}_K}}, x - \bar{x} \rangle_{\mathcal{H}_K}.
$$

Therefore, $\log(\Lambda)$ can be lower bounded by

$$\log(\Lambda) \geq \log \int \exp\left\{-\beta \left[\frac{1}{2} M \|x - \bar{x}\|^2 + \lambda \|\bar{x}\|_{\mathcal{H}_K} \left\langle \frac{\bar{x}}{\|\bar{x}\|_{\mathcal{H}_K}}, x - \bar{x} \right\rangle_{\mathcal{H}_K} \right]\right\} \, d\nu_{\infty}^{(\beta)}(x)
$$

$$\geq -\beta \left[\frac{1}{2} M \|x - \bar{x}\|^2 + \lambda \|\bar{x}\|_{\mathcal{H}_K} \|x - \bar{x}\|_{\mathcal{H}_K} \right] + \log(\nu_{\infty}^{(\beta)}(\{x \in \bar{x} + C_{U}\}))
$$

where $C_{U} \triangleq \{x \in \mathcal{H} \, | \, \|x\| \leq \varepsilon, \|x + C_{U}\|_{\mathcal{H}_K} \leq U\}$ for arbitrary $\varepsilon > 0$ and $U > 0$ (if $\|\bar{x}\|_{\mathcal{H}_K} = 0$, then we treat $\frac{1}{\|\bar{x}\|_{\mathcal{H}_K}} = 0$). Then, by Borell’s inequality (Borell (1975), van der Vaart and van Zanten (2008, Lemma 5.2)), we have

$$\log(\nu_{\infty}^{(\beta)}(\{x \in \bar{x} + C_{U}\})) \geq \log(\nu_{\infty}^{(\beta)}(C_{U})) - \frac{\beta \varepsilon^2}{2} \|\bar{x}\|_{\mathcal{H}_K}^2.
$$

Finally, we lower bound $\log(\nu_{\infty}^{(\beta)}(C_{U}))$. Let $C_{U}^{(1)} \triangleq \{x \in \mathcal{H} \, | \, \|x\| \leq \varepsilon\}$ and $C_{U}^{(2)} \triangleq \{x \in \mathcal{H} \, | \, \|x + C_{U}\|_{\mathcal{H}_K} \leq U\}$ (that is, $C_{U} = C_{U}^{(1)} \cap C_{U}^{(2)}$), then by Lemma 19, it holds that

$$\log(\nu_{\infty}^{(\beta)}(C_{U})) \geq \log(\nu_{\infty}^{(\beta)}(C_{U}^{(1)})) + \log(\nu_{\infty}^{(\beta)}(C_{U}^{(2)})).$$
By the small ball probability theorem (Kuelbs and Li, 1993; Li and Shao, 2001), we can lower bound the first term of the left hand side as

$$- \log(\nu_\infty^{(\beta)}(C_1')) \lesssim (\sqrt{\beta \varepsilon})^{-2}.$$ 

To evaluate \(\nu_\infty^{(\beta)}(C_U^{(2)})\), we note that

$$\mathbb{E}_{x \sim \nu_\infty^{(\beta)}} \left[ \frac{\tilde{x}}{\|\tilde{x}\|_{\mathcal{H}_K}} , x \right]^2_{\mathcal{H}_K} \leq \beta^{-1}.$$ 

Therefore, by the Markov’s inequality,

$$\nu_\infty^{(\beta)}(C_U^{(2)}) \geq 1 - \frac{1}{\beta U^2}.$$ 

By setting \(U = \sqrt{2/\beta}\), we also have

$$- \log(\nu_\infty^{(\beta)}(C_U^{(2)})) \leq \log(1/2).$$

Combining these inequalities, we finally arrive at

$$\int L d\pi - L(\tilde{x}) \leq -\frac{1}{\beta} \log(\Lambda) \leq \frac{1}{4} M \epsilon^2 + \lambda \|\tilde{x}\|_{\mathcal{H}_K} \sqrt{\frac{2}{\beta}} + \frac{\lambda}{2} \|\tilde{x}\|_{\mathcal{H}_K}^2$$

$$+ \beta^{-1} [C(\sqrt{\beta \lambda \epsilon})^{-2} + \log(1/2)]$$

$$\lesssim \frac{M}{2} \epsilon^2 + \lambda \left(\|\tilde{x}\|_{\mathcal{H}_K} \beta^{-1/2} + \|\tilde{x}\|_{\mathcal{H}_K}^2\right)$$

$$+ (\sqrt{\beta \lambda \epsilon})^{-2} + \beta^{-1}. \quad (48)$$

Finally, differentiating the above w.r.t. \(\epsilon\), we get that the optimal bound is attained for \(\epsilon = \left(\frac{2M\lambda}{\beta}\right)^{1/4}\), and is then equal to

$$\frac{1}{\beta} \left(\sqrt{\frac{2M}{\lambda}} + 1\right) + \lambda \left(\|\tilde{x}\|_{\mathcal{H}_K} \beta^{-1/2} + \|\tilde{x}\|_{\mathcal{H}_K}^2\right).$$

\(\square\)

**H. Proof of time and space approximation error (Proposition 15 and Proposition 17)**

In this section, we prove Proposition 15 and Proposition 17. As we have noted, Proposition 15 is obtained as a corollary of Proposition 17 by taking the limit of \(N \to \infty\). More strongly, we can show the following lemma. Let

$$\hat{c}_\beta = \begin{cases} 
1 & \text{(strict dissipativity condition: Assumption 6 (i))}, \\
\sqrt{\beta} & \text{(bounded gradient condition: Assumption 6 (ii)).}
\end{cases} \quad (49)$$

**Lemma 20.** Suppose \(\|x_0\| \leq 1\). Under the same assumptions and notations as in Propositions 15 and 17, it holds that:

$$\frac{1}{n} \sum_{k=0}^{n-1} \mathbb{E} \left[ \phi \left( X_k^N \right) - \phi \left( X^N \right) \right]$$

$$\leq C_1 \hat{c}_\beta \left(1 + (n\eta')^{-1+\kappa} + (n\eta')^{-1}\right) \left(\eta^12^{-\kappa} + m^{1/2-\kappa} + (n\eta')^{-1}\right). \quad (50)$$

**Proof of Proposition 15 and Proposition 17.** Once we obtain this lemma (Lemma 20), then it is easy to show both propositions by taking into account that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{E} \left[ \phi \left( X_k^N \right) \right] = \mathbb{E}[\phi(X^N)] \quad (51)$$
where \( \mu_{(N, \eta)} \) is the invariant measure of \( (X^N_k)_k \) whose existence and uniqueness can be shown in the same manner as Proposition 14 (see also Bréhier and Kopec (2016) for this argument). This gives Proposition 17 under the condition \( \|x_0\| \leq 1 \). Since the invariant measure \( \mu_{(N, \eta)} \) is independent of the initial solution \( x_0 \), we may drop the condition \( \|x_0\| \leq 1 \). Then, we obtain Proposition 17.

We can see that the proof of Proposition 14 is valid to show the convergence of Eq. (51) uniformly over all \( N \) and its convergence is uniform over all \( N \). Moreover, it has been already shown (see Bréhier (2014) for example) that

\[
\lim_{N \to \infty} \mathbb{E}[\phi(X^N_k)] = \mathbb{E}[\phi(X_k)] \quad (\forall k \in \mathbb{N}).
\]

Consequently, we can exchange the order of limit, and by applying the geometric ergodicity \( \lim_{k \to \infty} \mathbb{E}[\phi(X_k)] = \mathbb{E}[\phi(X^\eta)] \) (Proposition 14) again, we also have

\[
\lim_{N \to \infty} \mathbb{E}[\phi(X^\mu_{(N, \eta)})] = \lim_{n \to \infty} \lim_{N \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{E}[\phi(X^N_k)] = \lim_{n \to \infty} \lim_{N \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{E}[\phi(X^N_k)] \quad (\because \text{uniformity of convergence})
\]

\[
= \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{E}[\phi(X_k)] = \mathbb{E}[\phi(X^\mu_{\eta})].
\]

Therefore, Lemma 20 gives the proof of Proposition 15 by taking the limit of \( n \to \infty \) and \( N \to \infty \). Here again, we would like to note that the assumption \( \|x\| \leq 1 \) can be dropped in the limit because the invariant measure \( \mu_{\eta} \) is independent of the initial solution.

In the following, we prove Lemma 20. Our proof follows the line of Bréhier and Kopec (2016). For lighter notation, our constants may differ from line to line.

### H.1. Preliminaries

In this subsection, we prepare some notations and state lemmas necessary to prove the statement. Here, we introduce the continuous time dynamics with the Galerkin approximation as

\[
\begin{aligned}
X^N(0) &= P_N x_0 \in \mathcal{H}_N, \\
\text{d}X^N(t) &= (AX^N(t) - \nabla L_N(X^N(t)))\text{d}t + \sqrt{\frac{2}{\beta}}P_N \text{d}W(t).
\end{aligned}
\]

Here, we denote by \( X(t, x) \) to represent \( X(t) \) with \( X_0 = x \) and similarly we write \( Y(t, x) \) for a continuous time process \( \{Y(t)\} \), to indicate \( Y(t) \) with \( Y(0) = x \). Notice that our constants should not depend on \( \beta \) while Bréhier and Kopec (2016) sets \( \beta = 1 \). Our technical contribution is to extend the work of Bréhier and Kopec (2016) to general case. To this end, we apply a change of variables with \( t' \triangleq 2t/\beta \). Accordingly, Eq. (8) transforms into

\[
\begin{aligned}
X(0) &= x_0 \in \mathcal{H}, \\
\text{d}X(t') &= -\nabla L'(X(t'))\text{d}t' + \text{d}W(t') \\
&= (A'X(t') - \nabla L'(X(t')))\text{d}t' + \text{d}W(t'),
\end{aligned}
\]

where \( A' \triangleq (\beta/2)A, \ L' \triangleq (\beta/2)L, \ L' \triangleq (\beta/2)L \). Accordingly, let the “time re-scaled version” of the process \( X(t) \) be

\[
\hat{X}(t') \triangleq X \left( \frac{\beta t'}{2} \right), \quad \hat{X}^N(t') \triangleq X^N \left( \frac{\beta t'}{2} \right) \quad (t' \geq 0).
\]

Similarly, the change of variables with \( \eta' = \eta/\beta \) translates the time discretized Galerkin approximation scheme (9) into

\[
\begin{aligned}
X^N_0 &= P_N x_0 \in \mathcal{H}_N, \\
X^N_{n+1} &= X^N_n - \eta'(A'X^N_n + \nabla L'_{N}(X^N_n)) + \sqrt{\eta'}\epsilon_n \\
&\Leftrightarrow X^N_{n+1} = \hat{S}_{\eta'}(X^N_n - \eta'\nabla L'_{N}(X^N_n) + \sqrt{\eta'}\epsilon_n),
\end{aligned}
\]

(54)
where \( L_N' \triangleq (\beta/2)L_N, \tilde{S}_N' = S_N'. \) Here we used the abuse of notation to let \( A', \tilde{S}_N' \) indicate the map from \( \mathcal{H}_N \) to \( \mathcal{H}_N \) which is naturally defined by \( A', \tilde{S}_N' : \mathcal{H} \rightarrow \mathcal{H} \) through the canonical imbedding \( \iota : x \in \mathcal{H}_N \mapsto \mathcal{H} : Ax \triangleq (A \circ \iota)x \) for \( x \in \mathcal{H}_N \) (the same argument is also applied to \( \tilde{S}_N' \)). Note that we may set \( \tilde{S}_N' = S_N' \) because \( \eta' A' = \eta A \). We write the (rescaled) continuous time corresponding to the \( k \)-th step as

\[
t_k \triangleq k\eta'.
\]

Our approach is to follow the proofs of Bréhier and Kopec (2016); Kopec (2014) and uncover the dependency of \( \beta \) step by step. For completeness, we restate the results of Bréhier and Kopec (2016) in our notations.

**Proposition 21.** 1. We have for any \( N \in \mathbb{N}, \gamma \in [-1/2, 1/2], \) and \( x \in \mathcal{H} \),

\[
\|(-A')^\gamma P_N x\| \leq \|(-A')^\gamma x\|.
\]

2. For \( P_N \), we have the following error estimate:

\[
\left\|(-A')^{s/2}(I - P_N)(-A')^{-r/2}\right\|_{\mathcal{B}(\mathcal{H})} = \left(\frac{\mu_{N+1}'}{\beta/2}\right)^{(r-s)/2} \quad \forall 0 \leq s \leq 1, \ s \leq r \leq 2,
\]

where

\[
\mu_i' \triangleq \mu_i/\lambda.
\]

The corresponding result in Bréhier and Kopec (2016) is on finite element approximations; see Andersson and Larsson (2016) for more details. However, it can be naturally extended to spectral Galerkin projection as pointed out in Bréhier and Kopec (2016). In fact, these two approximations are essentially the same; see Kruse (2013). As a consequence, we get the following result.

**Proposition 22.** For any \( \kappa > 0 \), the linear operator on \( \mathcal{H} \), \( P_N(-A')^{-1/2-\kappa}P_N \) is continuous, self-adjoint and positive semi-definite. Moreover, there exists \( C_\kappa > 0 \) such that for any \( \beta > 0 \)

\[
\sup_{N \in \mathbb{N}} \text{Tr} \left( P_N(-A')^{-1/2-\kappa}P_N \right) < \frac{C_\kappa}{\beta^{1/2+\kappa}}.
\]

This is an extension of Proposition 3.4 in Kopec (2014), where \( \beta \) is fixed to 1. The following fundamental inequality is important for the proof of Proposition 22.

**Proposition 23.** For \( M, N \in \mathcal{B}(\mathcal{H}) \) such that \( M \) is symmetric and positive semi-definite,

\[
|\text{Tr}(MN)| \leq \|M\|_{\mathcal{B}(\mathcal{H})} |\text{Tr}(N)|.
\]

Our next step is to extend Lemma 3.7 of Kopec (2014) to our case.

**Lemma 24.** For any \( 0 \leq \kappa \leq 1, N \in \mathbb{N}, \beta \geq \eta_0, \) and \( j \geq 1 \),

\[
\left\|(-A)^{1-\kappa}S_N'P_N\right\|_{\mathcal{B}(\mathcal{H})} \leq \frac{(\beta/2)^{1-\kappa}}{(j\eta)^{1-\kappa}(1+\eta/\mu_0)^j\kappa} = \frac{1}{l_j^1}(1+\eta/\mu_0)^j\kappa.
\]

Moreover,

\[
\forall \gamma \geq 1 \quad \exists C_\gamma > 0 \quad \forall j \geq \gamma \quad \left\|(-A')^\gamma S_N'P_N\right\|_{\mathcal{B}(\mathcal{H})} \leq \frac{C_\gamma}{(j\eta)^\gamma}(\beta/2)^\gamma = \frac{C_\gamma}{l_j^\gamma},
\]

and for any \( 0 \leq \gamma \leq 1 \),

\[
\left\|(-A')^{-\gamma}(\tilde{S}_N' - I)P_N\right\|_{\mathcal{B}(\mathcal{H})} \leq 2 \frac{\eta^{\gamma}}{(\beta/2)^\gamma} = 2\eta^{\gamma}.
\]
The proof is almost the same as in Lemma 3.7 of Kopec (2014), and thus we omit the proof. The relationship that \( \eta' = \eta/(\beta/2) \), \( A' = (\beta/2)A \) specifies the dependence on \( \beta \).

As in Bréhier and Kopec (2016), we have the following expression of \( X_N^N \):

\[
X_N^k = \tilde{S}_k \eta P_N x - \eta' \sum_{l=0}^{k-1} \tilde{S}_l \eta P_N \nabla L(X_N^l) + \sqrt{\eta' \sum_{l=0}^{k-l} \tilde{S}_l \eta} P_N \epsilon_{t+l},
\]

\[
\sqrt{\eta} \sum_{l=0}^{k-l} \tilde{S}_l \eta P_N \epsilon_{t+l} = \int_0^t \tilde{S}_l \eta P_N dW(s),
\]

where \( l_s \equiv \lfloor \frac{s}{\eta} \rfloor \) with the notation \( \lfloor \cdot \rfloor \) is the floor function. The advantage of this expression is that we can handle each term by simple estimates.

We introduce the following interpolation processes: for \( 0 \leq k \leq m - 1 \) and \( t_k \leq t \leq t_{k+1} \), it holds that:

\[
\tilde{X}(t) = X_N^k + \int_{t_k}^t \tilde{S}_l \eta' [A'X_N^l - P_N \nabla L(X_N^l)] ds + \int_{t_k}^t \tilde{S}_l P_N dW(s).
\]

The process \( \{\tilde{X}(t)\}_{t \geq 0} \) is a natural interpolation of the discrete scheme \( \{X_N^k\}_{k \in N} \): \( \{\tilde{X}(t_k)\}_{k \in N} \) and \( \{X_N^k\}_{k \in N} \) have the same joint distribution.

**H.2. Bounds on Moments**

In this subsection, we give a few bounds on moments of \( \{X(t)\}_{t \geq 0} \), \( \{X_N(t)\}_{t \geq 0} \), \( \{X_N^k\}_{k \in N} \). Note that the constants are uniform with respect to \( N \in \mathbb{N} \), \( 0 < \eta \leq \eta_0 \) and \( \beta \geq \eta_0 \).

**Lemma 25.** For any \( p \geq 1 \), there exists a constant \( C_p > 0 \) such that for every \( N \in \mathbb{N} \), \( t \geq 0 \), \( \beta \geq \eta_0 \) and \( x \in \mathcal{H} \),

\[
\mathbb{E} \left[ \|X(t,x)\|^p \right] < C_p (1 + \|x_0\|^p).
\]

**Lemma 26.** For any \( p \geq 1 \), there exists a constant \( C_p \) such that for every \( N \in \mathbb{N} \), \( 0 < \eta \leq \eta_0 \), \( \beta \geq \eta_0 \), \( k \in \mathbb{N} \), \( t \geq 0 \) and \( x \in \mathcal{H} \),

\[
\mathbb{E} \left[ \|X_N^k\|^p \right] < C_p (1 + \|x_0\|^p).
\]

Intuitively, these lemmas hold thanks to dissipativity, a kind of boundedness of a global optimum.

**Proof of Lemma 25 and Lemma 26.** The proof is very similar to that of Proposition 13. We only prove the statement for the bounded gradient condition. For the strict dissipativity condition, see Proposition 3.2 of Bréhier and Vilmart (2016). We prove the statement following the same line as Lemma 4.1 and 4.2 of Bréhier (2014). There is no essentially new ingredient, but we need to take care of the effect of \( \beta \). We define \( Z(t) = X(t) - W_{A'}(t) \) where \( W_{A'}(t) = \int_0^t e^{(t-s)A'} dW(s) \). It holds that \( W_{A'}(t/(\beta/2)) = W_{A}(t) / \sqrt{\beta/2} \). (2.6) in Kopec (2014) implies:

\[
\mathbb{E} \sup_{t' \geq 0} \left\| W_{A'}(t') \right\|^p = \mathbb{E} \sup_{t' \geq 0} \left\| W_{A'}(\frac{t}{\beta/2}) \right\|^p < \left( \frac{C_p}{(\beta/2)^{p/2}} \right) \left( \frac{C_p}{\beta/2} \right)^{p/2} < C_p',
\]

where \( C_p', C_p'' > 0 \) are constants independent from \( \beta \).

Then, we study \( \|Z(t)\| \). We have \( Z(0) = X_0 = x_0 \),

\[
\frac{dZ(t)}{dt} = \frac{\beta}{2} (AZ(t) - \nabla L(X(t))),
\]

and by Proposition 7,

\[
\frac{1}{2} \frac{d}{dt} \left\| Z(t) \right\|^2 = \frac{\beta}{2} (AZ(t) - \nabla L(X(t)), Z(t))
\]
Lemma 25

Remark 29

Bréhier and Kopec

Debussche et al.

As pointed out in Raginsky et al. Goldys and Maslowski

Assumptions 2 3 4 5 6.

since the estimates do not depend on the dimension

This also implies

The same argument yields



\[ \frac{\beta}{2} \langle AZ(t) - \nabla L(Z(t)), Z(t) \rangle + \frac{\beta}{2} \langle \nabla L(Z(t)) - \nabla L(\hat{X}(t)), Z(t) \rangle \]

\[ \leq \frac{\beta}{2} \langle -m \| Z(t) \|^2 + c + \| \nabla L \|_\infty \| Z(t) \| \rangle \]

\[ \leq \frac{\beta}{2} \langle -m' \| Z(t) \|^2 + C' \rangle, \]

where \( m' \) and \( C' \) are positive constants depending only on \( m, c, B \). Thus, we have for any \( t \geq 0 \)

\[ \| Z(t) \|^2 - C' / m' \leq \exp(-\beta m' t) \| x_0 \|^2 - C' / m' \]

\[ \implies \| Z(t) \|^2 \leq \exp(-\beta m' t) \| x_0 \|^2 - C' / m' + C' / m' \leq C(\| x_0 \|^2 + 1), \]

for a constant \( C > 0 \), which concludes the proof of Lemma 25, since the estimates do not depend on the dimension parameter \( N \).

Similarly, we introduce \( Z_k = X_k - w_k \), where \( \{ w_k \}_k \) is the numerical approximation of \( W^\lambda \) defined by

\[ w_{k+1} = \tilde{S}_\eta w_k + \sqrt{\eta} S_{\eta} \xi_{k+1}. \]

The same argument yields

\[ \mathbb{E} \| w_k \|^2 \leq \frac{C}{\beta} \leq C'. \] (67)

Now we have \( Z_0 = X_0 = x_0 \),

\[ Z_{k+1} = \tilde{S}_\eta Z_k - \eta' \tilde{S}_\eta \nabla L'(X_k), \]

since \( \| \tilde{S}_\eta \|_{\mathcal{B}(\mathcal{H})} \leq 1 / (1 + \eta / \mu_0) \), we obtain the almost sure estimates

\[ \| Z_{k+1} \| \leq \frac{1}{1 + \eta / \mu_0} \| Z_k \| + C \eta', \]

and therefore for \( \beta \geq 1 \)

\[ \| Z_k \| \leq C(1 + \| x_0 \|), \]

which concludes the proof of Lemma 26.

H.3. The Rate of Convergence to the Invariant Measure

Our focus in this subsection is just to state the convergence result to an invariant measure. For the existence and uniqueness of the invariant measure of the continuous time dynamics, see Debussche et al. (2011), Goldys and Maslowski (2006) and Bréhier and Kopec (2016). We have the following result thanks to a coupling argument presented in Debussche et al. (2011).

Proposition 27. Under Assumptions 2, 3, 5 and 6. There exist the “spectral gap” \( \lambda^* \) and a constant \( C > 0 \) such that for any bounded test function \( \phi : \mathcal{H} \to \mathbb{R}, t \geq 0, N \in \mathbb{N}, \beta \geq \eta_0 \) and \( x_1, x_2 \in \mathcal{H}_N \),

\[ \| \mathbb{E}[\phi(X^N(t, x_1))] - \mathbb{E}[\phi(X^N(t, x_2))] \| \leq C \| \phi \|_\infty (1 + \| x_1 \|^2 + \| x_2 \|^2) e^{-\lambda^* t}. \] (68)

This also implies

\[ \| \mathbb{E}[\phi(\hat{X}^N(t, x_1))] - \mathbb{E}[\phi(\hat{X}^N(t, x_2))] \| \leq C \| \phi \|_\infty (1 + \| x_1 \|^2 + \| x_2 \|^2) e^{-\beta \lambda^* t}. \] (69)

A proof of this result in case \( \beta = 1 \) can be found in Debussche et al. (2011). We can easily see the statement holds if \( \beta \) is arbitrary but we have to notice the convergence rate \( \lambda^* \) can be varied depending on \( \beta \). More concrete characterization of \( \lambda^* \) will be given in Remark 29. As pointed out in Raginsky et al. (2017), this spectral gap is supposed to decrease exponentially with respect to \( \beta \).
Corollary 28. For any $N \in \mathbb{N}$, the process $X^N$ admits a unique invariant probability measure $\pi^N$ and satisfies the following bound:

$$\exists c, C, \lambda^* > 0, \forall \phi : \mathcal{H} \to \mathbb{R}, t \geq 0, x \in \mathcal{H}_N,$$

$$\left| \mathbb{E}[\phi(X^N(t, x))] - \int_{\mathcal{H}_N} \phi d\pi^N \right| \leq C \|\phi\|_{\infty} (1 + \|x\|^2)e^{-\lambda^*t}. \quad (70)$$

These results naturally extend to an infinite dimensional scheme by similar arguments.

Remark 29 (Characterization of $\lambda^*$). Bréhier (2014, Theorem 1.1) showed that

$$\lim_{\eta \to 0} |\mathbb{E}[\phi(X_{t/\eta})] - \mathbb{E}[\phi(X(t))]| = 0.$$

In addition to that we have shown in Proposition 14 that the discrete time dynamics satisfies the geometric ergodicity:

$$|\mathbb{E}[\phi(X_n)] - \mathbb{E}[\phi(X^{\mu_n})]| \leq C(1 + \|x\|) \exp(-\Lambda^*_n(n\eta - 1)) \leq C'(1 + \|x\|^2) \exp(-\Lambda^*_n(n\eta)),$$

where we used a fact that we may set $\Lambda^*_n \leq 1$ (if this is not satisfied, we may set $\Lambda^*_n \leftarrow \min\{\Lambda^*_n, 1\}$). Moreover, Bréhier (2014, Corollary 1.2) gives that

$$\lim_{\eta \to 0} |\mathbb{E}[\phi(X^{\mu_n})] - \mathbb{E}[\phi(X^*)]| = 0.$$

Combining these arguments, we see that

$$|\mathbb{E}[\phi(X(t))] - \mathbb{E}[\phi(X^*)]| \leq \lim_{\eta \to 0} |\mathbb{E}[\phi(X_{t/\eta})] - \mathbb{E}[\phi(X^{\mu_n})]| \leq \lim_{\eta \to 0} C(1 + \|x\|^2) \exp(-\Lambda^*_n(n\eta)).$$

Finally, we note that Proposition 27 and Corollary 28 are used only for $\phi : \mathcal{H} \to \mathbb{R}$ satisfying $\|\phi\|_{\infty} \geq c$ for a positive constant $c > 0$. Hence, we may set $\lambda^* = \lim_{\eta \to 0} \Lambda^*_n = \Lambda^*_0$.

The same argument is also applied to $\{X^N_k\}_k, \{X^N(t)\}_t$, and $\{\hat{X}^N(t)\}_t$ with the same value of $\Lambda^*_n$. In the following, we use the notation $\lambda^*$ to indicate $\Lambda^*_0$.

Lemma 30. For any bounded test function $\phi : \mathcal{H} \to \mathbb{R}$, we have

$$\lim_{N \to \infty} \hat{\phi}_N := \lim_{N \to \infty} \int_{\mathcal{H}_N} \phi d\pi^N = \int_{\mathcal{H}} \phi d\pi =: \bar{\phi}. \quad (71)$$

Proof. For any $t \geq 0$ and any fixed initial condition $x \in \mathcal{H}$, we have

$$\bar{\phi}_N - \bar{\phi} = \phi_N - \mathbb{E}[\phi(X^N(t))]
+ \mathbb{E}[\phi(X^N(t)) - \mathbb{E}[\phi(X(t))]
+ \mathbb{E}[\phi(X(t)) - \bar{\phi}].$$

Since $\lim_{N \to \infty} E[\phi(X^N(t)) - \mathbb{E}[\phi(X(t))] = 0$ (Bréhier, 2014), we get that for any $t \geq 0$

$$\limsup_{N \to \infty} |\bar{\phi}_N - \bar{\phi}| \leq C e^{-\lambda^*t},$$

and then we may take $t \to \infty$. Notice that the constants are independent of the dimensionality $N$.

H.4. Proof of Lemma 20

In this subsection, we present a technical proof procedure of Lemma 20. As in Bréhier and Kopec (2016), we will use the following decomposition:

$$\frac{1}{n} \sum_{k=0}^{n-1} \mathbb{E}[\phi(X^N_k)] - \bar{\phi} = \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{E}[\phi(X^N_k)] - \bar{\phi}_N + \bar{\phi}_N - \bar{\phi} + \frac{1}{n} \sum_{k=0}^{n-1} \left( \mathbb{E}[\phi(X^N_k)] - \mathbb{E}[\phi(P_N^k X^N_k)] \right).$$
Our aim is to derive a $N'$-free bound of each term of this decomposition and to take $N' \to \infty$. It is obvious the last two terms converges to 0 as $N' \to +\infty$ thanks to Lemma 30 and $P_N \cdot X_k^N = X_k^N$ if $N' \geq N$.

It remains to bound the first term. We decompose the term by the solution of the Poisson equation defined in the following. Let $N' \in \mathbb{N}$, $\phi \in C^2_0(\mathcal{H})$. We define $\Psi^{N'}$ as the unique solution of the Poisson equation
\begin{equation}
\mathcal{L}^{N'} \Psi^{N'} = \phi \circ P_{N'} - \tilde{\phi}_{N'} \text{ and } \int_{\mathcal{H}_{N'}} \Psi^{N'} d\tau^{N'} = 0,
\end{equation}
where $\mathcal{L}^{N'}$ is the infinitesimal generator of the SPDE\footnote{Note that from here we also use the notation $t$ to indicate $t'$ for notational simplicity.}:
\begin{equation*}
\begin{cases}
\dot{X}^{N'}(0) = P_{N'} x_0 \in \mathcal{H}_{N'}, \\
dX^{N'}(t) = (A' \dot{X}^{N'}(t) - \nabla L_{N'}(\dot{X}^{N'}(t))) dt + P_{N'} dW(t),
\end{cases}
\end{equation*}
defined for $C^2$ functions $\psi : \mathcal{H} \to \mathbb{R}$ and $x \in \mathcal{H}$ by
\begin{equation*}
\mathcal{L}^{N'} \psi(x) = \langle A' P_{N'} x - P_{N'} \nabla L'(x), D\psi(x) \rangle + \frac{1}{2} \text{Tr}(P_{N'} D^2 \psi(x)).
\end{equation*}

The following proposition is essential for our result. This is an extension of Proposition 6.1 in Bréhier and Kopec (2016) in that dependence on $\beta$ is specified.

**Proposition 31.** Let $N' \in \mathbb{N}$ and $\phi \in C^2_0(\mathcal{H})$. The function $\Psi^{N'}$ defined for any $x \in \mathcal{H}_{N'}$ by
\begin{equation*}
\Psi^{N'}(x) = \int_0^\infty \mathbb{E} \left[ \phi(\hat{X}^{N'}(t), x)) - \tilde{\phi}_{N'} \right] dt,
\end{equation*}
is of class $C^2_0$ and the unique solution of Eq. (72). Moreover, we have the following estimates: for any $0 \leq \epsilon, \gamma < 1/2$ there exist $C, C_\epsilon, C_\gamma$, which are independent of $\gamma$ and $\beta$, such that for any $x \in \mathcal{H}_{N'}$,
\begin{align*}
\left\| \Psi^{N'}(x) \right\| &\leq \frac{C}{\lambda^{\beta}} (1 + \|x\|^2) \|\phi\|_{\infty}, \\
\left\| (-A')^\epsilon D\Psi^{N'}(x) \right\| &\leq \frac{C_\epsilon}{\lambda^{\beta}} (1 + \|x\|^2) \|\phi\|_{0,1}, \\
\left\| (-A')^\gamma D^2 \Psi^{N'}(x) (-A')^\gamma \right\|_{\mathcal{B}(\mathcal{H}_M)} &\leq \frac{C_\gamma}{\lambda^{\beta}} (1 + \|x\|^2) \|\phi\|_{0,2},
\end{align*}
where $\left\| \phi \right\|_{0,i} \equiv \max \left\{ \max_0 < j \leq 1, \|\phi\|_j, \|\phi\|_{\infty} \right\}$ for $\left\| \phi \right\|_{(1)} := \sup_{x \in \mathcal{H}} \|\nabla \phi(x)\|$ and $\left\| \phi \right\|_{(2)} := \sup_{x \in \mathcal{H}} \|D^2 \phi(x)\|$.

We give the proof of this proposition in Appendix H.6.

To show the proof, we prepare more theoretical tools. We define the function $\hat{\Psi}^{N'}$ for $x \in \mathcal{H}$ by
\begin{equation*}
\hat{\Psi}^{N'}(x) = \Psi^{N'}(P_{N'} x).
\end{equation*}
It can be interpreted as an extension of $\Psi^{N'}$ to the entire domain $\mathcal{H}$. Then we have for any $x \in \mathcal{H}$ and $h, k \in \mathcal{H}$,
\begin{align*}
\langle D \hat{\Psi}^{N'}(x), h \rangle &\equiv \langle D \Psi^{N'}(P_{N'} x), h P_{N'} \rangle, \\
D^2 \hat{\Psi}^{N'}(x) \cdot (h, k) &\equiv D^2 \Psi^{N'}(P_{N'} x) \cdot (P_{N'} h, P_{N'} k).
\end{align*}

**Proposition 31** can be also applied to $\hat{\Psi}^{N'}$ by these equations.

Then we define the generator $\mathcal{L}^{\epsilon, k, N'}$, discrete-time version of $\mathcal{L}^{N'}$, for all $k \in \mathbb{N}$ as
for $x_0 \in \mathcal{H}_{N'}$, $\phi \in \mathcal{B}(\mathcal{H})$,
\begin{equation*}
\mathcal{L}^{\epsilon, k, N'} \phi(x) = \langle \tilde{S}_{\epsilon'} (A' X_k^N - P_N \nabla L'(X_k^N)), D\phi(x) \rangle + \frac{1}{2} \text{Tr}(\tilde{S}_{\epsilon'} S_{\epsilon'}^* P_N D^2 \phi(x)).
\end{equation*}
Thanks to the Itô formula and Proposition 31, we have
\[
\mathbb{E} \tilde{\Psi}^{N'}(X_{k+1}^N) - \mathbb{E} \tilde{\Psi}^{N'}(X_k^N) = \int_{t_k}^{t_{k+1}} \mathbb{E} L^{n',k,N} \tilde{\Psi}^{N'}(\tilde{X}^N(s))ds.
\]
Similarly, we define the generator \(L^N\) of \(X^N\) by
\[
L^N \phi(x) = \langle A'x - P_N \nabla L'(x), D\phi(x) \rangle + \frac{1}{2} \text{Tr}(P_N D^2 \phi(x)).
\]
Putting all of the operators defined above, we have the following decomposition:
\[
\mathbb{E} \tilde{\Psi}^{N'}(X_{k+1}^N) - \mathbb{E} \tilde{\Psi}^{N'}(X_k^N) = \int_{t_k}^{t_{k+1}} \mathbb{E} \left( L^{n',k,N} - L^N \right) \tilde{\Psi}^{N'}(\tilde{X}^N(s))ds
+ \int_{t_k}^{t_{k+1}} \mathbb{E} \left( L^N - L^{N'} \right) \tilde{\Psi}^{N'}(\tilde{X}^N(s))ds
+ \int_{t_k}^{t_{k+1}} \mathbb{E} L^{N'} \tilde{\Psi}^{N'}(\tilde{X}^N(s))ds.
\]
Furthermore, the following equality for \(x \in \mathcal{H}\)
\[
L^{N'} \tilde{\Psi}^{N'}(x) = L^{N'} \Psi^{N'}(x) + \langle -P_N \nabla L'(x) + P_N \nabla L'(P_Nx), D\Psi^{N'}(P_Nx) \rangle,
\]
and the definition of \(\tilde{\Psi}^{N'}\) yields
\[
\mathbb{E} \tilde{\Psi}^{N'}(X_{k+1}^N) - \mathbb{E} \tilde{\Psi}^{N'}(X_k^N)
= \int_{t_k}^{t_{k+1}} \mathbb{E} \left( L^{n',k,N} - L^N \right) \tilde{\Psi}^{N'}(\tilde{X}^N(s))ds
+ \int_{t_k}^{t_{k+1}} \mathbb{E} \left( L^N - L^{N'} \right) \tilde{\Psi}^{N'}(\tilde{X}^N(s))ds
+ \eta' \left( \mathbb{E} \phi(P_NX_k^N) - \tilde{\phi}_{N'} \right)
+ \int_{t_k}^{t_{k+1}} \mathbb{E} \left[ \phi(P_N\tilde{X}^N(s)) - \phi(P_NX_k^N) \right] ds
+ \int_{t_k}^{t_{k+1}} \mathbb{E} \left( P_N \left( -\nabla L'(\tilde{X}^N(s)) + \nabla L'(P_N\tilde{X}^N(s)) \right), D\Psi^{N'}(P_N\tilde{X}^N(s)) \right) ds,
\]
and therefore
\[
\frac{1}{n} \sum_{k=0}^{n-1} \mathbb{E} \phi(P_NX_k^N) - \tilde{\phi}_{N'}
= \frac{1}{m'n'} \mathbb{E} \left[ \Psi^{N'}(P_NX_k^N) - \Psi^{N'}(P_NX_1^N) \right]
+ \phi(P_Nx) - \tilde{\phi}_{N'}
+ \frac{1}{m'n'} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \mathbb{E} \left( L^{N'} - L^N \right) \tilde{\Psi}^{N'}(\tilde{X}^N(s))ds
+ \frac{1}{m'n'} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \mathbb{E} \left( L^N - L^{n',k,N} \right) \tilde{\Psi}^{N'}(\tilde{X}^N(s))ds
- \frac{1}{m'n'} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \mathbb{E} \left[ \phi(P_N\tilde{X}^N(s)) - \phi(P_NX_k^N) \right] ds
+ \frac{1}{m'n'} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \mathbb{E} \left( P_N \left( -\nabla L'(\tilde{X}^N(s)) + \nabla L'(P_N\tilde{X}^N(s)) \right), D\Psi^{N'}(P_N\tilde{X}^N(s)) \right) ds
=: I_1 + I_2 + I_3 + I_4 + I_5 + I_6.
As in Bréhier and Kopec (2016), the fact that $\nabla L'$ is Lipschitz, Proposition 31 and Lemma 26 yield
\[
\lim_{N' \to \infty} I_6 = 0,
\]
and Proposition 31 and Lemma 26 yield for $0 < \eta \leq \eta_0$ and $\beta \geq \eta_0$,
\[
|I_1 + I_2| \leq \frac{C}{\lambda^* \beta \eta}(1 + \|x_0\|^2).
\]
The remaining three terms are controlled by the following lemmas, whose proofs we omit for the sake of conciseness. However, they can be shown by carefully tracing the proof lines of Bréhier and Kopec (2016); Kopec (2014) with the estimates in Proposition 31, Lemma 24 and Appendix H.5.

**Lemma 32** (The control of $I_3$; space discretization). For any $0 < \kappa < 1/2$ and $\eta_0$, there exists a constant $C > 0$ such that for any $\phi \in C_b^2(\mathcal{H})$, $x \in \mathcal{H}$, $\beta \geq \eta_0$ and $0 < \eta \leq \eta_0$
\[
\limsup_{N' \to \infty} \frac{1}{n \eta^2} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \mathbb{E} \left( L_{N'} - L_N \right) \Psi_{N'}(\tilde{X}_N(s)) ds \leq \frac{C}{\lambda^*} (1 + \|x_0\|^2) \|\phi\|_{0,2} \hat{c}_\beta \mu_{N+1}^{1/2-\kappa} (1 + (n \eta')^{-1}).
\]

**Lemma 33** (The control of $I_4$; time discretization). For any $0 < \kappa < 1/2$ and $\eta_0$, there exists a constant $C > 0$ such that for any $\phi \in C_b^2(\mathcal{H})$, $N' \in \mathbb{N}$, $x \in \mathcal{H}$, $\beta \geq \eta_0$ and $0 < \eta \leq \eta_0$
\[
\left| \frac{1}{n \eta^2} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \mathbb{E} \left( L_{N'} - L_{\eta',k,N} \right) \Psi_{N'}(\tilde{X}_N(s)) ds \right| \leq \frac{C}{\lambda^*} \|\phi\|_{0,2} (1 + \|x_0\|^2) \hat{c}_\beta \eta^{1/2-\kappa} (1 + (n \eta')^{-1} + (n \eta')^{-1}).
\]

**Lemma 34** (The control of $I_5$; more time discretization). For any $0 < \kappa < 1/4$ and $\eta_0$, there exists a constant $C, c' > 0$ such that for any $\phi \in C_b^2(\mathcal{H})$, $N' \in \mathbb{N}$, $x \in \mathcal{H}$, $\beta \geq \eta_0$ and $0 < \eta \leq \eta_0$
\[
\left| \frac{1}{n \eta^2} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \mathbb{E} \left[ \phi(P_{N',\tilde{X}_N(t)}) - \phi(P_{N',X_k}) \right] dt \right| \leq C \|\phi\|_{0,2} \hat{c}_\beta \eta^{1/2-2\kappa} \left( 1 + \frac{\|x_0\|}{(n \eta')^{1/2}} \right).
\]

Putting them together, we get the main result (Lemma 20).

**H.5. A Malliavin Integration by Parts Formula**

In the proofs of Lemma 32, 33 and 34, an integration by parts formula issued from Malliavin calculus is necessary to transform irregular stochastic integral terms into controllable ones; see Nualart (2006); Sanz-Solé (2005). Therefore, we restate the statement in this subsection. The notations are the same as in Bréhier and Kopec (2016); Debussche (2011).

**Lemma 35.** Let $N' \in \mathbb{N}$. For any $G \in \mathbb{D}^{1,2}(\mathcal{H}_{N'})$, $u \in C_b^2(\mathcal{H}_{N'})$ and $\Psi \in L^2(\Omega \times [0, T], \mathcal{L}_2(\mathcal{H}_{N'}))$, an adapted process,
\[
\mathbb{E} \left[ Du(G), \int_0^T \Psi(s) dW_N(s) \right] = \mathbb{E} \left[ \int_0^T \text{Tr}(\Psi(s)^* D^2 u(G) D_s G) ds \right],
\]
where $D_s G : x \in \mathcal{H} \mapsto D^s x G \in \mathcal{H}_{N'}$ stands for the Malliavin derivative of $G$, and $\mathbb{D}^{1,2}(\mathcal{H}_{N'})$ is the set of $\mathcal{H}_{N'}$-valued random variables $G = \sum_{i \leq N'} G_i f_i$, with $G_i \in \mathbb{D}^{1,2}$ the domain of the Malliavin derivative for $\mathbb{R}$-valued random variables for any $i$.

In the proof of Lemma 32, 33 and 34, we use the following estimates; see (Bréhier and Kopec, 2016; Bréhier, 2014; Kopec, 2014) for details.
Lemma 36. For any $0 \leq \gamma < 1$ and $\eta_0 > 0$, there exists a constant $C > 0$ such that for every $h \in (0, 1)$, $k \geq 1$, $0 < \eta \leq \eta_0$, $\beta > \eta_0$ and $s \in [t_k - 1/\beta, t_k]$,

$$
\left\|(-A')^\gamma D_s^\gamma X_k^N\right\|_{\mathcal{H}_N} \leq C(1 + M\eta)^{k-l_s} \left(\beta^\gamma + \frac{1}{(1 + \eta/\mu_0)(1-\gamma)(k-l_s) t_{k-l_s}}\right) \left\|x\right\|_{\mathcal{H}_N},
$$

for all $x \in \mathcal{H}_N$. Moreover, if $t_k \leq t < t_{k+1}$, we have

$$
\left\|(-A')^\gamma D_s^\gamma \tilde{X}^N(t)\right\|_{\mathcal{H}_N} \leq C\left\|(-A')^\gamma D_s^\gamma X_k^N\right\|_{\mathcal{H}_N},
$$

for $x \in \mathcal{H}_N$.

Note that the constant $C > 0$ is uniform with respect to $N' \in \mathbb{N}$, $\beta > \eta_0$.

Proof. The proof is almost the same as that of Lemma 6.5 in Kopec (2014).

The second inequality is a consequence of the following equality for $s \leq t_k \leq t < t_{k+1}$, thanks to (64):

$$
D_s^\gamma \tilde{X}^N(t) = D_s^\gamma X_k^N + (t - t_k)(A'\tilde{S}_\eta \eta L D_s^\gamma X_k^N - \tilde{S}_\eta L D(P_N \cdot \nabla L')(X_k^N) : D_s^\gamma X_k^N),
$$

and the conclusion follows since

$$
\sup_{N' \in \mathbb{N}} \left\|\eta' A' \tilde{S}_\eta\right\|_{\mathcal{B}(\mathcal{H}_N)} \leq C,
$$

where $C$ is a constant that does not depend on $\beta$ and the norm $\|\cdot\|_{\mathcal{B}(\mathcal{H}_N)}$ is taken as a linear map from $\mathcal{H}_N$ to $\mathcal{H}_N$.

Then we prove the first estimate. For any $k \geq 1$, $x \in \mathcal{H}_N$, and $s \in [t_k - 1/\beta, t_k]$, we have

$$
D_s^\gamma X_k^N = \tilde{S}_\eta^{k-l_s} x - \eta' \sum_{i=l_s+1}^{k-1} \tilde{S}_\eta^{k-i} D(P_N \cdot \nabla L')(X_k^N) : D_s^\gamma X_k^N.
$$

We recall that $l_s = [s/\eta']$, so that when $i \leq l_s$ we have $D_s^\gamma X_k^N = 0$.

As a consequence, the discrete Gronwall’s inequality ensures that for $k \geq l_s + 1$ and a constant $C > 0$,

$$
\left\|D_s^\gamma X_k^N\right\|_{\mathcal{H}_N} \leq (1 + M\eta)^{k-l_s} \left\|x\right\|_{\mathcal{H}_N},
$$

where we used $\eta' L' = \eta L$ and the Lipchitz continuity of $\nabla L$. Now using Lemma 24, we have

$$
\left\|(-A')^\gamma D_s^\gamma X_k^N\right\|_{\mathcal{H}_N} \leq \frac{1}{(1 + \eta/\mu_0)(1-\gamma)(k-l_s) t_{k-l_s}} \left\|x\right\|_{\mathcal{H}_N} + M\eta \sum_{i=l_s+1}^{k-1} \frac{(1 + M\eta)^{i-l_s}}{(1 + \eta/\mu_0)(1-\gamma)(k-i) t_{k-i}} \left\|x\right\|_{\mathcal{H}_N}.
$$

Note that $k - l_s \leq 1/(\eta' \beta) \leq 1/\eta$ yields $(1 + M\eta)^{k-l_s} \leq C$. To conclude, we see that when $0 < \eta \leq \eta_0$, it holds that for a constant $c_0$ (could be dependent on $\eta_0, \mu_0$),

$$
\eta \sum_{i=l_s+1}^{k-1} \frac{1}{(1 + \eta/\mu_0)(1-\gamma)(k-i) t_{k-i}} \leq \beta C \int_0^\infty \frac{t^{-\gamma}}{(1 + \eta/\mu_0)(1-\gamma)t/\eta} dt \\
\leq \beta C \int_0^\infty t^{-\gamma} \exp\left[-c_0(1-\gamma)(t/\eta')(\eta/\mu_0)\right] dt \\
\leq \beta C \int_0^\infty t^{-\gamma} \exp\left[-\frac{\beta}{2} c_0(1-\gamma)t/\mu_0\right] dt \\
\leq C \beta^\gamma.
$$
H.6. Proof of Proposition 31

In this subsection, we prove Proposition 31. Our argument follows the same line as Bréhier and Kopec (2016). Let $\phi \in C^2_0(\mathcal{H})$. For lighter notation, we assume $\tilde{\phi} = 0$ in this section. We define the function $u$ for any $t > 0$ and $x \in \mathcal{H}_{N'}$ by

$$u(t, x) = \mathbb{E}\left[\phi(\hat{X}^{N'}(t, x))\right], \quad (76)$$

which is the solution of a finite dimensional Kolmogorov equation associated with (52) where $N = N'$:

$$\frac{du}{dt}(t, x) = Lu(t, x) = \frac{1}{2} \text{Tr}(D^2 u(t, x)) + \langle A' x - \nabla L'_{N'}(x), Du(t, x) \rangle.$$  

To prove Proposition 31, we only need to show that $u \in C^2$ and that $u$ and its two first derivatives have estimates which are integrable with respect to $t$. Specifically we prove the following proposition.

**Proposition 37.** Let $\phi \in C^2$ such that $\tilde{\phi} = 0$ and $u$ defined by (76). Remember that $\hat{c}_\beta$ is defined in Eq. (49) as

$$\hat{c}_\beta = \begin{cases} 1 & \text{(strict dissipativity condition: Assumption 6 (ii))}, \\ \sqrt{\beta} & \text{(bounded gradient condition: Assumption 6 (iii))}. \end{cases}$$

There exist constant $c, C > 0$ such that for any $0 \leq \epsilon, \gamma < 1/2$ there exist constants $C_{\epsilon}$ and $C_{\epsilon, \gamma}$, which is independent of $\beta$, such that for any $t > 0$ and $x \in \mathcal{H}_{N'}$,

$$\|u(t, x)\| \leq C e^{-\beta \lambda^* t}(1 + \|x\|^2) \|\phi\|_\infty, \quad (77)$$

$$\|(-A')^\epsilon Du(t, x)\| \leq C_{\epsilon} \hat{c}_\beta \beta^\epsilon(1 + \frac{1}{(\beta t)^\epsilon}) e^{-\beta \lambda^* t}(1 + \|x\|^2) \|\phi\|_{0,1}, \quad (78)$$

$$\|(-A')^\gamma D^2 u(t, x)(-A')^\gamma\|_{\mathcal{B}(\mathcal{H})} \leq C_{\epsilon, \gamma} \hat{c}_\beta^2 \beta^\epsilon(1 + \frac{1}{(\beta t)^\epsilon} + \frac{1}{(\beta t)^\epsilon+\gamma}) e^{-\beta \lambda^* t}(1 + \|x\|^2) \|\phi\|_{0,2}, \quad (79)$$

where $\lambda^* > 0$ is the spectral gap introduced in Remark 29 (see also Proposition 27) and $\alpha' \in [0, 1]$ is the constant introduced in Assumption 5.

In fact the estimation (78) is true for $\alpha < 1$. The proof is a slight modification of the proof of Proposition 8.1 in (Kopec, 2014). Since $\phi \in C^2$, bounded and with bounded derivatives, $u \in C^2$ and the derivatives can be calculated in the following way:

- For any $h \in \mathcal{H}_{N'}$, we have

$$D_u(t, x).h = \mathbb{E}\left[D\phi(\hat{X}^{N'}(t, x)).\eta^{h,x}(t)\right], \quad (80)$$

where $\eta^{h,x}(t)$ is the solution of

$$\frac{d\eta^{h,x}(t)}{dt} = A'\eta^{h,x}(t) - D^2 L'_{N'}(\hat{X}^{N'}(t, x)).\eta^{h,x}(t),$$

$$\eta^{h,x}(0) = h.$$  

- For any $h, k \in \mathcal{H}_{N'}$, we have

$$D^2 u(t, x).(h, k) = \mathbb{E}\left[D^2 \phi(\hat{X}^{N'}(t, x)).(\eta^{h,x}(t), \eta^{k,x}(t)) + D\phi(\hat{X}^{N'}(t, x)).\zeta^{h,k,x}(t)\right],$$

where $\zeta^{h,k,x}$ is the solution of

$$\frac{d\zeta^{h,k,x}(t)}{dt} = A'\zeta^{h,k,x}(t) - D^2 L'_{N'}(\hat{X}^{N'}(t, x)).\zeta^{h,k,x}(t) - D^3 L'_{N'}(\hat{X}^{N'}(t, x)).(\eta^{h,x}(t), \eta^{k,x}(t)),$$

$$\zeta^{h,k,x}(0) = 0.$$  

Moreover, we already have the inequality (77) thanks to Corollary 28.
The proof requires several steps. First in Lemma 38 below we prove estimates for 0 < t ≤ 1/β and general 0 ≤ α, γ < 1/2; then in Lemma 39 we study the long-time behavior in case α = γ = 0; we finally conclude with the proofs of Proposition 37.

**Lemma 38.** Assume Assumption 6 (ii) (bounded gradient condition). For any 0 ≤ ε, γ < 1/2, there exist constants \( C_{\epsilon, \gamma} \) such that for any \( x \in \mathcal{H}^{N'} \), and any 0 < t ≤ 1/β,

\[
\|(-A')' Du(t, x)\| \leq \frac{C_{\epsilon}}{t^\epsilon} \|D\phi\|_\infty,
\]

\[
\|(-A')' D^2 u(t, x)(-A')^{-\gamma} \|_{B(\mathcal{H}^{N'})} \leq C_{\epsilon, \gamma} t^{-\beta + \gamma} \left( \frac{1}{(\beta t)^{\alpha'}} + \frac{1}{(\beta t)^{\epsilon + \gamma}} \right) \left( \|D\phi\|_\infty + \|D^2\phi\|_\infty \right),
\]

where \( \alpha' \) is defined in Assumption 5.

**Proof.** Owing to (80) and (81), we only need to prove the following almost sure estimates for some constants - which may vary from line to line below: for any 0 < t ≤ 1/β

\[
\|\eta^{h,x}(t)\| \leq \frac{C_{\epsilon}}{(\beta t)^{\epsilon}} \|h\|_\epsilon,
\]

\[
\|\zeta^{h,k,x}(t)\| \leq C_{\epsilon, \gamma} t^{-\beta + \gamma} \left( \frac{1}{(\beta t)^{\alpha'}} + \frac{1}{(\beta t)^{\epsilon + \gamma}} \right) \|h\|_\epsilon \|k\|_\gamma.
\]

To show these inequalities, first note that

\[
\left\| (e^{tA} h) \right\| = \left\| t^{-\epsilon} \left( -tA' \right)^{-\epsilon} e^{tA} \left( -A' \right)^{-\gamma} h \right\| = t^{-\epsilon} \left\| \left( -tA' \right)^{-\epsilon} e^{tA} \right\| \left\| \left( -A' \right)^{-\gamma} h \right\|
\]

\[
\leq t^{-\epsilon} \sup_{x \geq 0} \{x^\epsilon e^{-x}\} \left\| \left( -A' \right)^{-\epsilon} h \right\| = \frac{C_{\epsilon}}{t^\epsilon} \left\| \left( -A' \right)^{-\epsilon} h \right\|
\]

(82)

where \( C_{\epsilon} = \sup_{x \geq 0} \{x^\epsilon e^{-x}\} \). From this, we deduce that

\[
\|\eta^{h,x}(t)\| = \left\| e^{tA} h - \int_0^t e^{(t-s)A'} D^2 L'(\hat{X}(s, x)).\eta^{h,x}(s)ds \right\|
\]

\[
\leq \frac{C_{\epsilon}}{t^\epsilon} \left\| \left( -A' \right)^{-\epsilon} h \right\| + C \int_0^t \|\eta^{h,x}(s)\| ds.
\]

and by the Gronwall’s inequality and \( t \leq 1/\beta \), we get the result.

For the second-order derivative, we moreover use the properties of \( L \) to get

\[
\|\zeta^{h,k,x}(t)\| = \left| \int_0^t e^{(t-s)A'} D^2 L'(\hat{X}(s, x)).\zeta^{h,k,x}(s)ds \right|
\]

\[
+ \int_0^t e^{(t-s)A'} D^3 L'(\hat{X}(s, x)).(\eta^{h,x}(s), \eta^{k,x}(s))ds \right| 
\]

\[
\leq C_{\alpha'} \|\xi^{h,k,x}(s)\| ds + \int_0^t C_{\epsilon, \gamma} \|\eta^{h,x}(s)\| \|\eta^{k,x}(s)\| ds \leq C_{\alpha'} \|\xi^{h,k,x}(s)\| ds + C_{\alpha', \epsilon, \gamma} \|\left( -A' \right)^{-\epsilon} h \| \|\left( -A' \right)^{-\gamma} k \| \beta^{\epsilon + \gamma} \frac{1}{(1 - s)^{\alpha'}} ds.
\]

The Gronwall’s inequality yields the conclusion since for any 0 < βt ≤ 1 we have \( (\beta t)^{1-\alpha'-\epsilon-\gamma} < (\beta t)^{-\alpha'} \) due to the assumption \( \epsilon + \gamma < 1 \).

**Lemma 39.** Assume Assumption 6 (ii) (bounded gradient condition). There exist constants \( C, c > 0 \) such that for any \( t \geq 0 \), and any \( x \in \mathcal{H} \),

\[
\|Du(t, x)\| \leq C \sqrt{\beta} e^{-\beta x^2 t}(1 + \|x\|^2) \|\phi\|_\infty,
\]
and
\[ \|D^2 u(t,x)\|_{\mathcal{B}(\mathcal{H})} \leq C\beta e^{-\beta \lambda^* t} \left(1 + \frac{1}{(\beta t)^\alpha} \right) (1 + \|x\|^2) \|\phi\|_{\infty}. \]

**Proof of Lemma 39.** As in Kopec (2014), we use the Bismut-Elworthy-Li formula to get for \( \Phi : \mathcal{H}_N \rightarrow \mathbb{R} \) which belongs to class \( C^2 \) with bounded derivative and with at most quadratic growth, i.e.,
\[ \exists M(\Phi) > 0, \forall x \in \mathcal{H}_N, \|\Phi(x)\| \leq M(\Phi)(1 + \|x\|^2), \]
and \( v(t,x) \triangleq \mathbb{E}\Phi(\hat{X}^N(t,x)) \), we have two following formula:
\[ Dv(t,x).h = \frac{1}{t} \mathbb{E} \left[ \int_0^t \langle \eta^{h,x}(s), dW(s) \rangle \Phi(\hat{X}^N(t,x)) \right]. \]
Moreover, by the Markov property \( v(t,x) = \mathbb{E}v(t/2, \hat{X}^N(t/2,x)) \), we obtain
\[ Dv(t,x).h = \frac{2}{t} \mathbb{E} \left[ \int_0^{t/2} \langle \eta^{h,x}(s), dW(s) \rangle v(t/2, \hat{X}^N(t/2,x)) \right]. \]
and thus
\[ D^2 v(t,x).(h,k) = \frac{2}{t} \mathbb{E} \left[ \int_0^{t/2} \langle \zeta^{h,k,x}(s), dW(s) \rangle v(t/2, \hat{X}^N(t/2,x)) \right] \]
\[ + \frac{2}{t} \mathbb{E} \left[ \int_0^{t/2} \langle \eta^{h,x}(s), dW(s) \rangle Dv(t/2, \hat{X}^N(t/2,x)) \eta^{k,x}(s/2) \right]. \]
We then see, using Lemma 25 and Lemma 38 with \( \epsilon = \gamma = 0 \) that there exists \( C > 0 \) such that for any \( 0 < t \leq 1/\beta, x,h,k \in \mathcal{H}_N \),
\[ \|Dv(t,x).h\| \leq \frac{C}{\sqrt{t}} M(\Phi)(1 + \|x\|^2) \|h\|, \]
\[ \|D^2 v(t,x).(h,k)\| \leq \frac{C}{t} M(\Phi)(1 + \|x\|^2) \|h\| \|k\|. \]
Indeed, to see the first inequality, the Cauchy-Schwartz inequality gives
\[ Dv(t,x).h = \frac{1}{t} \mathbb{E} \left[ \int_0^t \langle \eta^{h,x}(s), dW(s) \rangle \Phi(\hat{X}^N(t,x)) \right] \]
\[ \leq \frac{1}{t} \sqrt{\mathbb{E} \left[ \left( \int_0^t \langle \eta^{h,x}(s), dW(s) \rangle \right)^2 \right] \mathbb{E}[\Phi(\hat{X}^N(t,x))^2]}, \]
and the isometry property of Ito integral and Lemma 38 give a bound of the first term as
\[ \sqrt{\mathbb{E} \left[ \left( \int_0^t \langle \eta^{h,x}(s), dW(s) \rangle \right)^2 \right]} \leq \sqrt{\int_0^t \|\eta^{h,x}(s)\|^2} ds \leq C\sqrt{t}\|h\|, \]
for \( t \leq 1/\beta \) and Lemma 25 gives a bound of the second term as
\[ \sqrt{\mathbb{E}[\Phi(\hat{X}^N(t,x))^2]} \leq CM(\Phi)(1 + \|x\|^2). \]
Now when \( \beta t \geq 1 \) the Markov property implies that \( u(t,x) = \mathbb{E}[u(t - 1/\beta, \hat{X}^N(1/\beta,x))] \) and by Corollary 28, we have
\[ \left\| u(t - 1/\beta, x) - \int_{\mathcal{H}_N} \phi d\mu \right\| \leq C e^{-\beta \lambda^*(t - 1/\beta)} (1 + \|x\|^2) \|\phi\|_{\infty}. \]
If we choose $\Phi_t(x) = u(t - 1/\beta, x) - \int_{\mathcal{H}} \phi d\bar{\mu}$, we have $u(t, x) = \mathbb{E} \Phi_t(\hat{X}'(1/\beta, x)) + \int_{\mathcal{H}} \phi d\mu$, with $M(\Phi_t) \leq Ce^{-\beta \lambda^*(t-1/\beta)} \|\phi\|_\infty$. With (83) at $t = 1/\beta$, we obtain for $t \geq 1/\beta$,
\[
\|Du(t, x)h\| \leq C\sqrt{\beta} \|\phi\|_\infty e^{-\beta \lambda^*(t-1/\beta)} (1 + \|x\|^2) \|h\|,
\|D^2u(t, x)(h, k)\| \leq C\beta \|\phi\|_\infty e^{-\beta \lambda^*(t-1/\beta)} (1 + \|x\|^2) \|h\| \|k\|.
\]
We have a control when $0 \leq t \leq 1/\beta$ in Lemma 38, so with a change of constants we get the result.

Next we show a corresponding lemma for the strict dissipativity condition in the following lemma.

**Lemma 40.** Assume Assumption 6 (i) (strict dissipativity condition). For any $0 \leq \epsilon, \gamma < 1/2$, there exist constants $C_{\epsilon}, C_{\epsilon, \gamma}$ such that for any $x \in \mathcal{H}_{N'}$, and any $0 < t$,
\[
\|(-A')^s Du(t, x)\| \leq C_{\epsilon, \beta^s} \left(1 + \frac{1}{(\beta t)\epsilon}\right) e^{-t\beta \lambda^*} \|D\phi\|_\infty,
\|(-A')^s D^2u(t, x)(-A')^\gamma\|_{B(\mathcal{H}_{N'})} \leq C_{\epsilon, \gamma, \beta^s} \left(1 + \frac{1}{(\beta t)\epsilon}\right) e^{-t\beta \lambda^*} \left(\|D\phi\|_\infty + \|D^2\phi\|_\infty\right),
\]
where $\alpha'$ is defined in Assumption 5.

**Proof.** From the definition of $\eta^{h, x}$, we have that
\[
\|\eta^{h, x}(t)\| = \left\|e^{tA'} h - \int_0^t e^{(t-s)A'} D^2L'(\hat{X}'(s, x)) \eta^{h, x}(s) ds\right\|
\leq \left\|e^{tA'} h\right\| + \int_0^t e^{-(t-s)\lambda/\mu_0} M\beta \left\|\eta^{h, x}(s)\right\| ds.
\]
As in Eq. (82), for any $0 \leq c_0 < 1$, the first term can be bounded by
\[
\|e^{tA'} h\| = \left\|t^{-\epsilon}(-tA')^s e^{c_0 tA'} (-A')^{-\epsilon} e^{(1-c_0)tA'} h\right\| = t^{-\epsilon} \left\|(-A')^{c_0} e^{tA'}\right\|_{B(\mathcal{H})} \left\|(-A')^{-\epsilon} e^{(1-c_0)tA'} h\right\|
\leq t^{-\epsilon} \sup_{x \geq 0} \left\{|x| e^{-c_0 x}\right\} \left\|(-A')^{-\epsilon} h\right\| = C_{\epsilon, c_0} \left\|(-A')^{-\epsilon} e^{(1-c_0)tA'} h\right\|
\]
where $C_{\epsilon, c_0} \triangleq \sup_{x \geq 0} \left\{|x| e^{-c_0 x}\right\}$. Then, Gronwall’s inequality gives
\[
e^{t\beta \lambda/\mu_0} \|\eta^{h, x}(t)\| \leq C_{\epsilon, c_0} e^{c_0 t\beta \lambda/\mu_0} \left\|(-A')^{-\epsilon} h\right\| + \int_0^t \beta M e^{s\beta \lambda/\mu_0} \left\|\eta^{h, x}(s)\right\| ds
\Rightarrow e^{t\beta \lambda/\mu_0} \|\eta^{h, x}(t)\| \leq C_{\epsilon, c_0} e^{c_0 t\beta \lambda/\mu_0} \left\|(-A')^{-\epsilon} h\right\| + \int_0^t \beta M e^{s\beta \lambda/\mu_0} \left\|\eta^{h, x}(s)\right\| ds
\Rightarrow e^{t\beta \lambda/\mu_0} \|\eta^{h, x}(t)\| \leq C_{\epsilon, c_0} e^{c_0 t\beta \lambda/\mu_0} \left\|(-A')^{-\epsilon} h\right\| + C_{\epsilon, c_0} \int_0^t e^{c_0 s\beta \lambda/\mu_0} s^{-\epsilon} \beta M \exp((t - s)\beta M) ds \left\|(-A')^{-\epsilon} h\right\|
\leq C_{\epsilon, c_0} \left\|(-A')^{-\epsilon} h\right\| \left[1 - \epsilon e^{c_0 t\beta \lambda/\mu_0} + \beta^\epsilon M e^{\beta M} \int_0^\infty e^{r\lambda(Ma_0 - M)} \frac{1}{r^{\epsilon}} dr\right] M \exp(-t\beta(Ma_0 - M)) \left\|(-A')^{-\epsilon} h\right\|
\Rightarrow \|\eta^{h, x}(t)\| \leq C_{\epsilon, c_0} \left\|(-A')^{-\epsilon} h\right\| \left[1 - \epsilon e^{c_0 t\beta \lambda/\mu_0} + \beta^\epsilon M e^{\beta M} \int_0^\infty e^{r\lambda(Ma_0 - M)} \frac{1}{r^{\epsilon}} dr\right] M \exp(-t\beta(Ma_0 - M)) \left\|(-A')^{-\epsilon} h\right\|
\]
Therefore, if we choose $c_0$ as $c_0 = (\lambda/\mu_0)^{-1} M/2$, then $0 \leq c_0 < 1$ by the strict dissipativity assumption and we obtain
\[
\|\eta^{h, x}(t)\| \leq C_{\epsilon} \left[1 + \frac{(t\beta M)^\epsilon}{t^\epsilon} \exp[-t\beta(Ma_0 - M)]\right] \left\|(-A')^{-\epsilon} h\right\|
= C_{\epsilon} \left[1 + (t\beta M)^\epsilon \exp[-t\beta\lambda^*]\right] \left\|(-A')^{-\epsilon} h\right\|
\]
(84)
where we used $\lambda^* = \lambda/\mu_0 - M > 0$. Applying this to Eq. (80), we have the first inequality.

The second inequality is also shown in the same way as Lemma 38. Notice that by the Lipschitz continuity of $\nabla L$, we have

$$
\|\zeta(t)\| \leq \int_0^t e^{-(t-s)\beta\lambda/\mu_0} M \|\zeta(s)\| \, ds + \int_0^t C_{\alpha',\epsilon,\gamma} \beta^{1-\alpha'} e^{-(1-c_0)(t-s)} \|\eta(t)\| \|\eta'(t)\| \, ds
$$

where we used

$$
\|\zeta(t)\| \leq \int_0^t e^{-(t-s)\beta\lambda/\mu_0} M \|\zeta(s)\| \, ds + C_{\alpha',\epsilon,\gamma} \beta^{1-\alpha'} \int_0^t \frac{(1 + (M\beta s)^{\epsilon+\gamma})}{(t-s)^{\alpha+\gamma}} e^{-2s\beta(\lambda/\mu_0-M)} e^{-(1-c_0)(t-s)} \beta\lambda/\mu_0 \, ds.
$$

From this inequality, we have

$$
e^{t\beta\lambda/\mu_0} \|\zeta(t)\| \leq \int_0^t \beta Me^{s\beta\lambda/\mu_0} \|\zeta(s)\| \, ds + C_{\alpha',\epsilon,\gamma} \beta^{1-\alpha'} \int_0^t \frac{(1 + (M\beta s)^{\epsilon+\gamma})}{(t-s)^{\alpha+\gamma}} e^{-2s\beta(\lambda/\mu_0-M)} e^{-(1-c_0)(t-s)} \beta\lambda/\mu_0 \, ds.
$$

Here, we set $c_0 = (\lambda/\mu_0)^{-1} M/2$, then we further obtain

$$
e^{t\beta\lambda/\mu_0} \|\zeta(t)\| \leq C_{\alpha',\epsilon,\gamma} \beta^{1-\alpha'} \int_0^t \beta Me^{s\beta\lambda/\mu_0} \|\zeta(s)\| \, ds + (1 + (M\beta t)^{\epsilon+\gamma}) t^{1-\alpha'-\epsilon-\gamma} e^{t\beta\lambda/\mu_0-M} \beta^{\epsilon+\gamma} \min(\lambda/\mu_0-M,M/2)
$$

By multiplying both terms by $e^{-t\beta\lambda/\mu_0}$, we obtain

$$
\|\zeta(t)\| \leq C_{\alpha',\epsilon,\gamma} \beta^{1-\alpha'} \beta^{\epsilon+\gamma} \min(\lambda/\mu_0-M,M/2)
$$

where we used that $\sup_{t>0} (1 + (M\beta t)^{\epsilon+\gamma}) e^{-t\beta\min(\lambda/\mu_0-M,M/2)} < C$ (bounded by a constant independent of $\beta$). Since $1 - \epsilon - \gamma > 0$ and $1 - \alpha' > 0$, it holds that $(\beta t)^{1-\alpha'-\epsilon-\gamma} \leq (\beta t)^{-\alpha'} + (\beta t)^{-\epsilon-\gamma}$. Then, we finally obtain

$$
\|\zeta(t)\| \leq C_{\alpha',\epsilon,\gamma} \beta^{1-\alpha'} \beta^{\epsilon+\gamma} \min(\lambda/\mu_0-M,M/2)
$$

where we used $\lambda^* = \lambda/\mu_0 - M > 0$. Applying this inequality and Eq. (84) to Eq. (81), we obtain the second inequality.
**Remark 41.** Note that Lemma 40 for the strict dissipativity condition does not require the restriction \( t \leq 1/\beta \) while Lemma 38 is for the bounded gradient condition. This is advantageous to show better dependency on \( \beta \) under the strict dissipativity condition than the bounded gradient condition.

We can finally prove Proposition 37. The proof is again in line with Kopec (2014).

**Proof of Proposition 37.** First, we show the assertion for the bounded gradient condition. By the Markov property and Lemma 39, for any \( t \geq 1/\beta \), we have

\[
\| Du(t, x)h \| \leq C \sqrt{\beta} \| \phi \|_\infty e^{-\beta \lambda^x(t-1/\beta)} \mathbb{E} \left[ (1 + \| \hat{X}^N(1/\beta, x) \|^2) \| \eta A^x(1/\beta) \| \right] \\
\leq C \sqrt{\beta} \| \phi \|_\infty e^{-\beta \lambda^x(t-1/\beta)} (1 + \| x \|^2) \beta e \| (-A')^{-\epsilon} h \| ,
\]

where the last estimate comes from Lemma 25 and Lemma 38. Combining this estimate and Eq. (78), we obtain Eq. (79). We can easily see Eq. (79) follows from the similar argument.

As for the strict dissipativity condition, Lemma 40 directly gives the assertion. \( \square \)

**I. Proof of SGLD convergence rate (Proposition 18)**

In this chapter, we prove Proposition 18. Before that, we need to prepare the following lemmas to bound \( \mathbb{E}[L(Y_k^N) - L(X_k^N)] \). For lighter notation, our constants may differ from line to line.

**Lemma 42.** For any \( x \in \mathcal{H}_N \), it holds that

\[
\mathbb{E} \| \nabla L(x) - g_k(x) \|^2 \leq \frac{C(n_{tr} - n_b)}{n_b(n_{tr} - 1)},
\]

where \( n_b \) is the mini-batch size and \( C > 0 \) is some constant.

We can prove the following bound similarly to Lemma 25 and 26 thanks to Assumption 9.

**Lemma 43.** For any \( p \geq 1 \), there exists a constant \( C_p \) such that for every \( N \in \mathbb{N} \), \( \beta \geq \eta_0 \), and \( x \in \mathcal{H}_N \),

\[
\mathbb{E} \| Y_k^N \|^p \leq C_p (1 + \| x_0 \|^p).
\]

**Lemma 44.** It holds that:

\[
\exists C_1, C_2 > 0, \forall \beta > \frac{2\mu_0}{2 + \eta/\mu_0}, \ log \mathbb{E} \left[ \exp(\| X_k^N \|^2) \right] \leq \| x_0 \|^2 + C_1/\beta + C_2,
\]

where \( C_1, C_2 > 0 \) is an constant.

**Remark 45.** Note that our estimate is not subject to “the curse of dimensionality” which explicitly appears in Lemma C.7 in Xu et al. (2018).

**Proof.** The proof is similar to that of Lemma C.7 in Xu et al. (2018). The main difference lies in the existence of regularizer in our scheme and the absence of dissipativity assumption of \( L_N \). Instead, we assume Assumption 9.

Let \( Q = \frac{2P}{\eta} \) and \( p_j = \frac{1}{(1 + \eta/\mu_j)^p} \). Let \( S' := \text{diag}(q_j)_{j=0}^N \) for \( q_j > 0 \ (j = 0, \ldots, N) \) and \( 1 > q_0 \geq q_1 \geq \cdots \geq q_N \), then we have

\[
\mathbb{E} \left[ \exp \| X_{k+1}^N \|^2_{S'} \right] \\
= \mathbb{E} \left[ \exp \left( S'_{q} (X_k^N - \eta \nabla L_N(X_k^N) + \sqrt{\frac{2\eta}{\beta} X_k^N} \right) \right],
\]

where the last estimate comes from Lemma 25 and Lemma 38. Combining this estimate and Eq. (78), we obtain Eq. (79). We can easily see Eq. (79) follows from the similar argument.

As for the strict dissipativity condition, Lemma 40 directly gives the assertion. \( \square \)
where \( \epsilon_k \sim \mathcal{N}(0, I_N) \). Let \( x_i, \epsilon_i \) denote the \( i \)-th component of \( X_k^N - \eta \nabla L_N(X_k^N) \), and \( \epsilon_k \) respectively, which corresponds to the coefficient of \( f_i \), the \( i \)-th eigenfunction of \( T_K \) defined in Eq. (3). Under this notation, we have the following estimate:

\[
E \left[ \exp \left\| S_\eta (X_k^N - \eta \nabla L_N(X_k^N) + \sqrt{\frac{2\eta}{\beta}} \epsilon_k) \right\|_{S^r}^2 \right] 
\]

\[
= E \left[ \exp \left\| S_\eta (X_k^N - \eta \nabla L_N(X_k^N) + \sqrt{\frac{2\eta}{\beta}} \epsilon_k) \right\|_{S^r}^2 \right] X_k^N 
\]

\[
= E \left[ \prod_{i=0}^{N} \int \exp \left( p_i q_i \left( x_i^2 + 2 \frac{2\eta}{\beta} x_i \epsilon_i + \frac{2\eta}{\beta} \epsilon_i^2 \right) \right) \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{\epsilon_i^2}{2} \right) \, d\epsilon_i \right] 
\]

\[
= E \left[ \prod_{i=0}^{N} \frac{1}{\sqrt{1 - p_i q_i}} \exp \left( \frac{x_i^2}{p_i q_i - 2Q} \right) \right] 
\]

\[
\leq \exp \left( \sum_{j=0}^{\sum \frac{Q p_j q_j}{1 - 2Q p_0 q_0}} \right) E \left[ \exp \left( \sum_{i=0}^{N} \frac{1}{p_i q_i - 2Q} \left( X_k^N - 2\eta X_k^N \nabla L_N,i(X_k^N) + \eta^2 \nabla L_N,i(X_k^N)^2 \right) \right) \right] 
\]

thanks to the formula of Gaussian integral, \( \mu_0' \geq \mu_i' \) and \( \log(1 - x) \geq -x/(1 - x) \).

Then, we have

\[
\exp \left( \sum_{j=0}^{\sum \frac{Q p_j q_j}{1 - 2Q p_0 q_0}} \right) E \left[ \exp \left( \sum_{i=0}^{N} \frac{x_i^2}{p_i q_i - 2Q} \right) \right] 
\]

\[
\leq \exp \left( \sum_{j=0}^{\sum \frac{Q p_j q_j}{1 - 2Q p_0 q_0}} \right) E \left[ \exp \left( \sum_{i=0}^{N} \frac{1}{p_i q_i - 2Q} \left( X_k^N - 2\eta X_k^N \nabla L_N,i(X_k^N) + \eta^2 \nabla L_N,i(X_k^N)^2 \right) \right) \right] 
\]

where \( X_k^N, L_N,i(X_k^N) \) denotes the \( i \)-th component of \( X_k^N, L_N(X_k^N) \) respectively.

Then Assumption 9 implies

\[
\exp \left( \sum_{j=0}^{\sum \frac{Q p_j q_j}{1 - 2Q p_0 q_0}} \right) E \left[ \exp \left( \sum_{i=0}^{N} \frac{1}{p_i q_i - 2Q} \left( X_k^N - 2\eta X_k^N \nabla L_N,i(X_k^N) + \eta^2 \nabla L_N,i(X_k^N)^2 \right) \right) \right] 
\]

\[
\leq \exp \left( \sum_{j=0}^{\sum \frac{Q p_j q_j}{1 - 2Q p_0 q_0}} \right) E \left[ \exp \left( \sum_{i=0}^{N} \frac{1}{p_i q_i - 2Q} \left( X_k^N - 2\eta X_k^N \nabla L_N,i(X_k^N) + \eta^2 \nabla L_N,i(X_k^N)^2 \right) \right) \right] 
\]

\[
\leq \exp \left( \sum_{j=0}^{\sum \frac{Q p_j q_j}{1 - 2Q p_0 q_0}} \right) E \left[ \exp \left( \sum_{i=0}^{N} \frac{1}{p_i q_i - 2Q} \left( X_k^N - 2\eta X_k^N \nabla L_N,i(X_k^N) + \eta^2 \nabla L_N,i(X_k^N)^2 \right) \right) \right] 
\]

\[
= E \left\{ \exp \left[ \| X_k^N \|_{S^r(i)}^2 + \sum_{j=0}^{\sum \frac{Q p_j q_j}{1 - 2Q p_0 q_0}} \left( \frac{C p_j q_j (1 + \frac{1}{\kappa})}{1 - 2Q p_0 q_0} \eta^2 \right) \right] \right\} 
\]

where \( S^{(k)} := \text{diag} \left( \frac{1 + \kappa}{\eta \sqrt{2Q}} \right) \).

Since \( q_0 \leq 1 \), it holds that

\[
\frac{Q p_j q_j}{1 - 2Q p_0 q_0} \leq \frac{Q p_j q_j}{1 - 2Q p_0}.
\]
If we have chosen $\kappa$ so that $\frac{1+\kappa}{p_j^{q_j}} - 2Q < 1$, then
\[
q_j^{(k)} := \frac{1 + \kappa}{p_j^{q_j}} - 2Q \leq \frac{1 + \kappa}{p_0^{q_0}} - 2Q < 1,
\]
and we also have $q_0^{(k)} \geq q_1^{(k)} \geq \cdots \geq q_N^{(k)}$. Here again, since $q_j \leq 1$, it holds that
\[
q_j^{(k)} = \frac{1 + \kappa}{p_j^{q_j}} - 2Q \leq \frac{1 + \kappa}{(p_j^{q_j} - 2Q)q_j^{(k)}}.
\]
Let $\kappa = \frac{1}{2} (2\eta / \mu_0 + (\eta / \mu_0)^2 - 2\eta / \beta)$, then it holds that
\[
\frac{1 + \kappa}{p_j^{q_j} - 2Q} = \frac{1 + \frac{1}{2} [2\eta / \mu_0 + (\eta / \mu_0)^2 - 2\eta / \beta]}{1 + \frac{1}{2} [2\eta / \mu_j + (\eta / \mu_j)^2 - 2\eta / \beta]} \leq \frac{1}{1 + \alpha_j} < 1.
\]
Therefore, we obtain the following evaluation for $q_j^{(k)}$:
\[
q_j^{(k)} \leq \frac{q_j}{1 + \alpha_j},
\]
which implies $\|\cdot\|_{S^{(k)}} \leq \|\cdot\|_{S^0}$. Hence, by noticing $p_j \leq (1 + \alpha_j)^{-1}$, a recursive argument yields
\[
E \left[ \exp \left( \|X_{k+1}^N\|^2 \right) \right] \leq \exp \left[ \|x_0\|^2 + \frac{Q + C (1 + 1/\kappa) \eta^2}{1 - 2Qp_0^{q_0}} \sum_{j=0}^{N} \sum_{i=0}^{k} \frac{q_j}{(1 + \alpha_j)^{k+1-i}} \right].
\]
The second term in the right hand side can be evaluated as
\[
\sum_{i=0}^{k} \frac{1}{(1 + \alpha_j)^{k+1-i}} = \frac{(1 + \alpha_j)^{-1} - (1 + \alpha_j)^{-(k+2)}}{1 - (1 + \alpha_j)^{-1}} \leq \frac{1}{\alpha_j}.
\]
Finally, if we set $q_j = 1$, then by observing that
\[
\sum_{j=0}^{N} \frac{1}{\alpha_j} \lesssim \int_{1}^{\infty} \frac{1}{\eta x^2} \, dx \lesssim \frac{1}{\eta},
\]
we have
\[
\frac{Q + C (1 + 1/\kappa) \eta^2}{1 - 2Qp_0^{q_0}} \sum_{j=0}^{N} \sum_{i=0}^{k} \frac{1}{(1 + \alpha_j)^{k+1-i}} q_j \lesssim \frac{Q + (1 + 1/\kappa) \eta^2}{\eta} = \frac{1}{\beta} + \left( 1 + \frac{1}{\alpha_0} \right) \eta.
\]
Since $\alpha_0 = O(\eta)$, the second term in the right hand side can be evaluated
\[
\left( 1 + \frac{1}{\alpha_0} \right) \eta \lesssim 1.
\]
Combining all arguments, we obtain that
\[
E \left[ \exp \left( \|X_{k+1}^N\|^2 \right) \right] \leq \exp \left( \|x_0\|^2 + \frac{C_1}{\beta} + C_2 \right).
\]

The following two lemmas are used to prove Theorem 3.6 in (Xu et al., 2018). These results can only be applied to finite dimensional spaces. However, our schemes $Y_k^N, X_k^N$ are no longer infinite dimensional, which means we can follow the same argument in (Xu et al., 2018).
Lemma 46 (Polyanskiy and Wu (2016); Raginsky et al. (2017); Xu et al. (2018)). For any two probability density functions $\mu, \nu$ with bounded second moments, let $g: \mathbb{R}^d \to \mathbb{R}$ be a $C^1$ function such that

$$\|\nabla g(x)\|_2 \leq C_1 \|x\|_2 + C_2, \forall x \in \mathbb{R}^d,$$

for some constants $C_1, C_2 \geq 0$. Then

$$\left|\int_{\mathbb{R}^d} g \mu - \int_{\mathbb{R}^d} g \nu\right| \leq (C_1 \sigma + C_2) W_2(\mu, \nu),$$

where $W_2$ is the 2-Wasserstein distance and $\sigma^2 = \max \left\{ \int_{\mathbb{R}^d} \|x\|_2^2 \mu(dx), \int_{\mathbb{R}^d} \|x\|_2^2 \nu(dx) \right\}$.

Lemma 47. (Corollary 2.3 in Bolley and Villani (2005)) Let $\nu$ be a probability measure on $\mathbb{R}^d$. Assume that there exist $x_0$ and a constant $\alpha > 0$ such that $\int \exp(\alpha \|x - x_0\|_2) \nu(dx) < \infty$. Then for any probability measure $\mu$ on $\mathbb{R}^d$, it satisfies

$$W_2(\mu, \nu) \leq C_\nu(D(\mu\|\nu)^{1/2} + D(\mu\|\nu)^{1/4}),$$

where $C_\nu$ is defined as

$$C_\nu = \inf_{x_0 \in \mathbb{R}^d, \alpha > 0} \frac{1}{\alpha} \left\{ \frac{3}{2} + \log \int \exp(\alpha \|x - x_0\|_2^2) \nu(dx) \right\}.$$

Proof of Proposition 18. Let $P_k, Q_k$ denote the probability measures for GLD scheme $X_k^N$ and SGLD scheme $Y_k^N$ respectively. Applying Lemma 46, Lemma 43 and Lemma 26 yields

$$|\mathbb{E} \left[ L(Y_k^N) \right] - \mathbb{E} \left[ L(X_k^N) \right]| \leq C(1 + \|x_0\|) W_2(Q_k, P_k), \quad (85)$$

where $C > 0$ are absolute constants. We further apply Lemma 47 to bound Wasserstein distance and get the following bound:

$$|\mathbb{E} \left[ L(Y_k^N) \right] - \mathbb{E} \left[ L(X_k^N) \right]| \leq C(1 + \|x_0\|) \Lambda(D(Q_k\|P_k)^{1/2} + D(Q_k\|P_k)^{1/4}), \quad (86)$$

where $\Lambda = \sqrt{3/2 + \log \mathbb{E} \left[ \exp \|X_k^N\|^2 \right]}$. Moreover, Lemma 44 yields

$$\Lambda \leq \sqrt{\frac{3}{2} + \|x_0\|^2 + \frac{C_1}{\beta} + C_2}, \quad (87)$$

where $C_1, C_2 > 0$ is some constants. To bound KL-divergence between $P_k$ and $Q_k$, we use the following decomposition:

$$D(Q_k\|P_k) \leq D(Q_k\|P_{k-1}) + D(Q_{1:k-1}\|P_{1:k-1}|P_k) = D(Q_{1:k}\|P_{1:k})$$

$$= D(Q_{1:1}\|P_1) + \sum_{i=2}^{k} D(Q_{1:i-1}\|P_{1:i-1}|P_{1:i})$$

$$= \sum_{i=1}^{k} D(Q_{1:i-1}\|P_{1:i-1}),$$

where $P_{1:k}, Q_{1:k}$ denotes joint distribution of $(X_1^N, \cdots, X_k^N)$ and $(Y_1^N, \cdots, Y_k^N)$ respectively and $Q_{1:i-1}$ denotes the conditional distribution of $X_i^N$ given $X_{i-1}^N$. The first inequality is based on non-negativity of KL-divergence and the final equality comes from the fact that $Q_0, P_0$ are deterministic and that $X_1^N$ and $(X_1^N, \cdots, X_i^N)$ are conditionally independent given $X_{i-1}^N$. For clarity, we write down the definition of conditional KL-divergence in the following line:

$$D(F_2\|F_1|G_2|G_1) = \int f(x_1, x_2) \log \frac{f(x_2|x_1)}{g(x_2|x_1)} dx_1 dx_2.$$
Now that $Q_i|Q_{i-1}$ and $P_i|P_{i-1}$ are both gaussian, that is,
\[ X_i^N|X_{i-1}^N = x \sim \mathcal{N}(S_\eta (x - \eta \nabla L_N (x)), \frac{\eta}{\beta} S_\eta S_\eta^T), \]
\[ Y_i^N|Y_{i-1}^N = x \sim \mathcal{N}(S_\eta (x - \eta g_{i-1} (x)), \frac{\eta}{\beta} S_\eta S_\eta^T), \]
we can calculate each conditional KL-divergence as below:
\[
D(Q_i|Q_{i-1}||P_i|P_{i-1}) = \mathbb{E}_Q \left[ \log \frac{dQ_i|Q_{i-1}}{dP_i|P_{i-1}} \right] \\
= \frac{\beta}{2\eta} \mathbb{E}_{(x,y) \sim Q_{i-1}} \left[ \left\| S_\eta^{-1} y - (x - \eta \nabla L_N (x)) \right\|^2 - \left\| S_\eta^{-1} y - (x - \eta g_{i-1} (x)) \right\|^2 \right] \\
= \frac{\beta}{2\eta} \mathbb{E}_{(x,y) \sim Q_{i-1}} \left[ 2\eta (S_\eta^{-1} y - x, \nabla L_N (x) - g_{i-1} (x)) + \eta^2 (\| \nabla L_N (x) \|^2 - \| g_{i-1} (x) \|^2) \right] \\
= \frac{\beta \eta}{2} \mathbb{E}_{x \sim Q_{i-1}} \left[ \| \nabla L_N (x) - g_{i-1} (x) \|^2 \right] \\
\leq \frac{\beta \eta}{2} \mathbb{E}_{x \sim Q_{i-1}} \left[ \frac{C(n_{\text{tr}} - n_{\text{rb}})}{n_{\text{rb}}(n_{\text{tr}} - 1)} \right] \\
\leq C \frac{\beta \eta (n_{\text{tr}} - n_{\text{rb}})}{n_{\text{rb}}(n_{\text{tr}} - 1)},
\]
thanks to Lemma 42 and 43. Therefore, we finally get the following bound:
\[
D(Q_k|P_k) \leq C \frac{\beta \eta k(n_{\text{tr}} - n_{\text{rb}})}{n_{\text{rb}}(n_{\text{tr}} - 1)}. \tag{88}
\]
Combining all of the above yields the claim. \qed