Decomposing equivariant indices of Spin\textsuperscript{c}-Dirac operators for cocompact actions

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Abstract

We generalise Paradan and Vergne’s result that Spin\textsuperscript{c}-quantisation commutes with reduction to cocompact actions. We consider equivariant indices of Spin\textsuperscript{c}-Dirac operators in terms of distributional characters of the group acting. If this group is semisimple with discrete series, these indices decompose into discrete series characters. Other results are a Spin\textsuperscript{c}-version of a conjecture by Landsman, and a generalisation of Atiyah and Hirzebruch’s vanishing theorem for actions on Spin-manifolds. The results are proved via induction formulas from compact to noncompact groups (which to a large extent also apply to non-cocompact actions), by applying them to the result in the compact case.

Contents

1 Indices of Spin\textsuperscript{c}-Dirac operators
  1.1 The distributional index ................................................. 6
  1.2 The invariant, transversally $L^2$-index .......................... 8
  1.3 Spin\textsuperscript{c}-Dirac operators .................................. 9

2 Quantisation and reduction: the main results .................. 10
  2.1 Reduced spaces ...................................................... 11
  2.2 Quantisation commutes with reduction in the compact case .. 11
  2.3 Quantisation commutes with reduction for cocompact actions 13
  2.4 Other results .......................................................... 16
3 Quantisation and induction
  3.1 Slices .................................................. 18
  3.2 Quantisation commutes with induction .................... 19
  3.3 The discrete series and $K$-theory ...................... 21

4 Decomposing the Dirac operator
  4.1 Spinors .................................................. 22
  4.2 Dirac operators on $N$ and $G/K$ ........................ 23
  4.3 The Levi–Civita connection ............................. 23
  4.4 Spinor connections ...................................... 25
  4.5 Proof of Proposition 4.1 ................................. 27
  4.6 Decomposing the kernel of $D$ ........................... 29

5 Proofs of Propositions 3.3 and 3.4 ........................ 30
  5.1 Densities .................................................. 30
  5.2 Cocompact actions ....................................... 32
  5.3 Non-cocompact actions .................................. 34

Introduction

Paradan and Vergne [22, 23] recently generalised Guillemin and Sternberg’s [10] *quantisation commutes with reduction* principle from symplectic manifolds to Spin$^c$-manifolds. This yields a decomposition into irreducible representations of the equivariant index of a Spin$^c$-Dirac operator on a compact, connected Spin$^c$-manifold, acted on by a compact, connected Lie group. In this paper, we generalise their result to cocompact actions, i.e. actions with compact orbit spaces. We consider the equivariant index of a Spin$^c$-Dirac operator in terms of distributional characters on the (possibly noncompact) group acting. To obtain this generalisation, we show that *quantisation commutes with induction* in a suitable sense. (A version of this also holds for non-cocompact actions.) This allows us to deduce results for cocompact actions from Paradan and Vergne’s result.

The compact case

For compact groups and manifolds, the study of Spin$^c$-quantisation goes back to [9], where it was shown to commute with reduction for circle ac-
tions. Paradan and Vergne’s result applies to an arbitrary compact, connected Lie group $K$ (with Lie algebra $\mathfrak{k}$) acting on an even-dimensional, compact, connected manifold $M$ with a $K$-equivariant Spin$^c$-structure. The Spin$^c$-quantisation of these data is the equivariant index of the associated Spin$^c$-Dirac operator $D$:

$$Q_K^{\text{Spin}^c}(M) := \text{index}_K(D) \in R(K),$$

where $R(K)$ is the representation ring of $K$.

If $L \to M$ is the determinant line bundle of the Spin$^c$-structure on $M$, and $\nabla^L$ is a connection on $L$ used in the definition of $D$, one can define a Spin$^c$-momentum map $\mu : M \to \mathfrak{k}^*$ by

$$2i \langle \mu, X \rangle = \mathcal{L}_X^L - \nabla^L_{X_M},$$

where $X \in \mathfrak{k}$, and $\mathcal{L}_X^L$ denotes the Lie derivative on sections of $L$. The reduced space at an element $\xi \in \mathfrak{k}^*$ is then defined as

$$M_{\xi} := \mu^{-1}(K \cdot \xi)/K.$$

Paradan and Vergne define $Q^{\text{Spin}^c}(M_{\xi}) \in \mathbb{Z}$, even for singular values $\xi$ of $\mu$. Their main result in [23] is an expression for the multiplicities $m_\pi \in \mathbb{Z}$ in

$$Q_K^{\text{Spin}^c}(M) = \bigoplus_{\pi \in \hat{K}} m_\pi \pi,$$

where $\hat{K}$ is the unitary dual of $K$. If the generic stabiliser of the action is Abelian, then

$$m_\pi = Q(M_{\lambda+\rho}),$$

where $\rho$ is half the sum of a choice of positive roots, and $\lambda$ is the highest weight of $\pi$ with respect to this choice. In general, the expression for $m_\pi$ is a sum of quantisations of reduced spaces (see (2.1) for details).

This result is a far-reaching generalisation of the symplectic case. For example, it includes Atiyah and Hirzebruch’s vanishing theorem for actions on Spin-manifolds [3] as a special case.

Indices for noncompact groups and manifolds

Consider a connected Lie group $G$, acting properly on a connected, even-dimensional manifold $M$. We suppose $M$ has a $G$-equivariant Spin$^c$-structure.
If \( M/G \) is compact, we define the \( \text{Spin}^c \)-quantisation of the action as the equivariant \( L^2 \)-index of the \( \text{Spin}^c \)-Dirac operator \( D \), as a distributional character on \( G \). This is shown to be well-defined in [25], and yields

\[
Q^\text{Spin}^c_G(M) := \text{index}_{L^2_G}(D) \in \mathcal{D}'(G).
\]

We will only prove quantisation commutes with reduction results for cocompact actions, but develop our techniques without the cocompactness assumption as much as possible. If \( M/G \) is possibly noncompact, it is still possible to define the \( G \)-invariant part of \( \text{Spin}^c \)-quantisation, as in [13, 14]. This definition starts with a deformation \( D_v \) of the Dirac operator \( D \) by a vector field \( v \) induced by the momentum map \( \mu \):

\[
D_v := D + c(v),
\]

where \( c \) denotes the Clifford action. If the set of zeros of \( v \) is cocompact, the invariant part of \( \text{Spin}^c \)-quantisation is defined as the index of \( D_v \) on \( G \)-invariant, transversally \( L^2 \) sections. Here transversally \( L^2 \) means \( L^2 \) after multiplication be a suitable cutoff function in the orbit directions. This yields

\[
Q^\text{Spin}^c(G)^G(M) := \text{index}_{L^2_G}(D_v) \in \mathbb{Z}.
\]

**Quantisation commutes with reduction**

Suppose \( M/G \) is compact. Let \( K < G \) be a maximal compact subgroup, and suppose \( G/K \) is even-dimensional and Spin. The Spin-condition holds for a double cover of \( G \). Under a mild regularity condition (see Subsection 2.3 for details) on the Riemannian metric on \( M \) induced by the Spin\(^c\)-structure, we have the following quantisation commutes with reduction results.

**Theorem 0.1.** One has

\[
Q^\text{Spin}^c_G(M) = \bigoplus_{\pi \in \hat{K}} m_\pi \left( \chi_\pi \otimes \text{index}_{L^2_G}(D_{G/K}) \right)^K \in \mathcal{D}'(G),
\]

where \( m_\pi \) is given be the same expression as in the compact case, and \( \chi_\pi \) denotes the character of \( \pi \).

Theorem 0.1 takes an explicit form if \( G \) is semisimple with discrete series. In the author’s opinion, this is the most interesting result in this paper.
Theorem 0.2. If $G$ is semisimple with discrete series, then

$$Q^{Spin_c}_G(M) = \bigoplus_{\pi \in \hat{G}_{ds}} m_{\pi} \Theta_{\pi} \in \mathcal{D}'(G),$$

where the sum runs over the discrete series of $G$, $\Theta_{\pi}$ is the character of $\pi$, and $m_{\pi}$ is given as in the compact case, where $\lambda$ is the Harish–Chandra parameter of $\pi$.

This result generalises the realisation of the discrete series by Atiyah and Schmid [4] and Parthasarathy [24] in the same way that the compact version of quantisation commutes with reduction generalises the Borel–Weil(–Bott) theorem. In particular, it implies that only the discrete series contributes to the equivariant $L^2$-index of $D$.

We also have a result for reduction at the trivial representation. Let $\Delta_p$ be the standard representation of $\text{Spin}(p)$. Since $G/K$ is even-dimensional, it decomposes as $\Delta_p = \Delta_p^+ \oplus \Delta_p^-$. Consider the representations

$$\pi_{p}^\pm : K \xrightarrow{\tilde{\text{Ad}}} \text{Spin}(p) \rightarrow \text{GL}(\Delta_p^\pm).$$

Here $\tilde{\text{Ad}}$ is a lift of the adjoint action, which exists if $G/K$ is Spin. Write $\pi_p := [\pi_p^+] - [\pi_p^-] \in R(K)$.

Theorem 0.3. One has

$$Q^{Spin_c}_G(M)^G = \bigoplus_{\pi \in \hat{K}} m_{\pi} n_{\pi} \in \mathbb{Z},$$

where $m_{\pi}$ is given be the same expression as in the compact case, and $n_{\pi}$ is the multiplicity of $\pi$ in $\pi_p$.

In addition to the above results, we obtain a generalisation of Atiyah and Hirzebruch’s vanishing theorem [3] for actions by compact groups on compact Spin-manifolds to the noncompact case.

Theorem 0.4. If $M/G$ is compact, $M$ is $G$-equivariantly Spin, and not all stabilisers of the action are maximal compact subgroups of $G$, then

$$Q^{Spin_c}_G(M) = 0 \in \mathcal{D}'(G).$$

Finally, we discuss links with the $K$-theoretic definition of quantisation studied in [11, 16, 18, 20]. In particular, Theorem 0.3 solves a $Spin^c$-version of Landsman’s conjecture [16, 18], which was not completely solved even in the symplectic setting.
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1 Indices of Spin$^c$-Dirac operators

The results in this paper are stated in terms of

1. the equivariant $L^2$-index in terms of distributional characters for co-compact actions, discussed in [25];

2. the invariant, transversally $L^2$-index for possibly non-cocompact actions, discussed in [8, 13, 14].

We will discuss induction from compact groups to noncopact groups for these indices. Using Paradan and Vergne’s result in [23], this allows us to obtain quantisation commutes with reduction results for cocompact actions.

We start by recalling the definitions of these indices in Subsections 1.1 and 1.2. In those subsections, we consider a Riemannian manifold $M$, on which a Lie group $G$ acts properly and isometrically. Fix a left Haar measure $dg$ on $G$. In addition, $E = E^+ \oplus E^- \to M$ will be a $G$-equivariant, $\mathbb{Z}_2$-graded, Hermitian vector bundle. We will consider the the $L^2$-inner product on sections of $E$ induced by the Riemannian density on $M$ and the Hermitian metric on $E$. We suppose a $G$-equivariant, first order, odd, essentially self-adjoint differential operator $D$ on $E$ to be given. We will write $D^\pm$ for the restriction of $D$ to $\Gamma^\infty(E^\pm)$. Then $D^- = (D^+)^*$. From Subsection 1.3 onwards, we will restrict our attention to cases where $D$ is a Spin$^c$-Dirac operator.

1.1 The distributional index

Suppose that $M/G$ is compact, and that $D$ is transversally elliptic. In the theorem and remark on page 29 of [25], Singer shows that the $L^2$-kernel of $D^\pm$ has a well-defined distributional index $\Theta^\pm$ in the space $\mathcal{D}'(G)$ of distributions on $G$. It is defined by

$$\langle \Theta^\pm, \varphi \rangle = \text{trace} \left( \int_G \varphi(g) \pi^\pm(g) \, dg \right),$$

where $\pi^\pm(g)$ is the projection of the $G$-equivariant bundle $E^\pm$ onto its $D^\pm$-eigensubbundle. The trace is taken with respect to a left Haar measure on $G$.
where $\varphi \in C_\infty^c(G)$, and $\pi^\pm$ is the unitary representation of $G$ in $\ker L^2 D^\pm$. Here $\overline{D^\pm}$ is the closure of $D^\pm$, whose graph is

\[(1.1) \quad \{(s^\pm, s^\mp) \in L^2(E^\pm) \oplus L^2(E^\mp); Ds^\pm = s^\mp\},\]

where $Ds^\pm$ should be interpreted in the distributional sense.

Singer attributes the above result to Hörmander. In the remark on page 29, he notes that this character decomposes discretely into irreducibles. In Theorem 2.5, we will see an explicit form of this decomposition if $D$ is a Spin$^c$-Dirac operator, and $G$ is semisimple with discrete series.

**Definition 1.1.** The equivariant $L^2$-index of $D^+$ is the distribution

$$\text{index}_{L^2}^G(D^+) = \Theta^+ - \Theta^- \in \mathcal{D}'(G).$$

For later use, we record the following basic fact about distributional characters.

**Lemma 1.2.** Let $G$ and $K$ be Lie groups, with $K$ compact. Let $\mathcal{H}$ be a Hilbert space carrying a unitary representation $\pi_{G \times K}$ of $G \times K$. Suppose this representation has a distributional character $\Theta(\mathcal{H}) \in \mathcal{D}'(G \times K)$. Then the representation $\pi_G$ of $G$ in the subspace $\mathcal{H}^K \subset \mathcal{H}$ of vectors fixed by $K$ has a distributional character $\Theta(\mathcal{H}^K) \in \mathcal{D}'(G)$, and for all $\varphi \in C_\infty^c(G)$,

$$\langle \Theta(\mathcal{H}^K), \varphi \rangle = \langle \Theta(\mathcal{H}), \varphi \otimes 1_K \rangle.$$

Here $1_K$ is the constant function 1 on $K$.

**Proof.** The averaging map

$$x \mapsto \int_K k \cdot x \, dk$$

(with $dk$ the Haar measure on $K$ giving it unit volume) is the orthogonal projection $P : \mathcal{H} \to \mathcal{H}^K$. For all $g \in G$ and $k \in K$, one has $\pi_G(g) = P \circ \pi_{G \times K}(g, k)|_{\mathcal{H}^K}$. Therefore, for all $\varphi \in C_\infty^c(G)$,

$$\int_G \varphi(g) \pi_G(g) \, dg = \int_G \varphi(g) \pi_{G \times K}(g, k) \, dg \, dk,$$

and the claim follows.  

\[7\]
In the setting of Lemma 1.2, let consider any distribution $\Theta \in \mathcal{D}'(G \times K)$. We will write $\Theta^K \in \mathcal{D}'(G)$ for the distribution defined by $\langle \Theta^K, \varphi \rangle = \langle \Theta, \varphi \otimes 1_K \rangle$, for $\varphi \in C_c^\infty(G)$. Then Lemma 1.2 states that

\begin{equation}
\Theta(H^K) = \Theta(H)^K.
\end{equation}

We also note that for $\Theta_1, \Theta_2 \in \mathcal{D}'(G \times K)$, we trivially have

\begin{equation}
\Theta^K_1 + \Theta^K_2 = (\Theta_1 + \Theta_2)^K.
\end{equation}

### 1.2 The invariant, transversally $L^2$-index

We now drop the assumption that $M/G$ is compact, but suppose that $M$ is complete, $G$ is unimodular, and that $D$ is elliptic. Let $f \in C^\infty(M)$ be a cutoff function, by which we mean that for all $m \in M$, the intersection of $\text{supp}(f) \cap G \cdot m$ is compact, and

$$\int_G f(gm)^2 \, dg = 1.$$ 

Cutoff functions always exist, see e.g. [6], Chapter 8, Section 2.4, Proposition 8. A $G$-invariant section $s$ of $E$ is transversally $L^2$ if $fs \in L^2(E)$. The $L^2$-norm of $fs$ is then independent of $f$. The $G$-invariant, transversally $L^2$-kernel of $D^\pm$ is

$$\ker_{L^2}(D^\pm)^G := \{ s \in \Gamma^\infty(E^\pm)^G ; s \text{ is transversally } L^2 \text{ and } Ds = 0 \}.$$ 

Instead of assuming $M/G$ to be compact, we now suppose there is a cocompact subset $X \subset M$, and a constant $C > 0$, such that for all $G$-invariant sections $s \in \Gamma^\infty(E)^G$, with $fs$ compactly supported outside $X$, we have

$$\| fDs \|_{L^2(E)} \geq C \| fs \|_{L^2(E)}.$$ 

It was shown in Proposition 4.8 of [13] that the spaces $\ker_{L^2}(D^\pm)^G$ are then finite-dimensional.

**Definition 1.3.** The invariant, transversally $L^2$-index of $D^+$ is

$$\text{index}^G_{L^2}(D^+) = \dim \ker_{L^2}(D^+)^G - \dim \ker_{L^2}(D^-)^G \in \mathbb{Z}.$$
1.3 Spin\textsuperscript{c}-Dirac operators

We now specialise to the case where \( M \) is even-dimensional, and has a \( G \)-equivariant Spin\textsuperscript{c}-structure. The vector bundle \( E = \mathcal{S} \) is taken to be the spinor bundle associated to the Spin\textsuperscript{c}-structure. We denote the determinant line bundle of the Spin\textsuperscript{c}-structure by \( L \to M \), and choose a \( G \)-invariant Hermitian connection \( \nabla_L \) on \( L \). Together with the Levi-Civita connection on \( TM \), this induces a connection \( \nabla^\mathcal{S} \) on \( \mathcal{S} \) (see e.g. Proposition D.11 in [19]). This in turn defines a Spin\textsuperscript{c}-Dirac operator \( D \) on \( \mathcal{S} \), by

\[
D : \Gamma^\infty(\mathcal{S}) \xrightarrow{\nabla^\mathcal{S}} \Omega^1(M;\mathcal{S}) \xrightarrow{\iota_c} \Gamma^\infty(\mathcal{S}).
\]

Here \( c \) denotes the Clifford action by \( T^\ast M \approx T^\ast_\mathbb{C}^\ast M \) on \( \mathcal{S} \).

We will study the decomposition of the equivariant \( L^2 \)-index of \( D \), and the value of its invariant, transversally \( L^2 \)-index, by interpreting these indices in terms of geometric quantisation. Then we can apply the quantisation commutes with reduction principle to obtain explicit multiplicities of irreducible representations.

**Definition 1.4.** If \( M/G \) is compact, then the Spin\textsuperscript{c}-quantisation\footnote{This Spin\textsuperscript{c}-quantisation depends on the Spin\textsuperscript{c}-structure, although it is suppressed in the notation.} of the action by \( G \) on \( M \) (and \( \mathcal{S} \)) is the equivariant \( L^2 \)-index of \( D^+ \):

\[
Q_{G}^{\text{Spin\textsuperscript{c}}} (M) = \text{index}_{L}^{L^2} (D^+) \in \mathcal{D}'(G).
\]

If \( M/G \) is not necessarily compact, then we can apply the invariant, transversally \( L^2 \)-index in the following way. Let

\[\mu : M \to \mathfrak{g}^\ast\]

be the Spin\textsuperscript{c}-momentum map, defined by

\[
2i\mu_X = \mathcal{L}^L_X - \nabla^L_{X_M} \in \text{End}(L) = C^\infty(M,\mathbb{C}).
\]

Here \( \mu_X \) is the pairing of \( \mu \) with an element \( X \in \mathfrak{g} \), \( \mathcal{L}^L \) denotes the Lie derivative of sections of \( L \), and \( X_M \) is the vector field induced by \( X \).

In addition, consider a metric \( \{(\cdot,\cdot)_m\}_{m \in M} \) on the trivial vector bundle \( M \times \mathfrak{g}^\ast \to M \), invariant under the action by \( G \) given by

\[g \cdot (m, \xi) = (gm, \text{Ad}^*(g)\xi),\]

where \( \text{Ad}^* \) is the coadjoint action of \( G \).
for $g \in G$, $m \in M$ and $\xi \in \mathfrak{g}^*$. Such a metric always exists (see e.g. Lemma 2.1 in [13], but one can also use a nonnegative cutoff function to average a given metric). Let $\mu^* : M \to \mathfrak{g}$ be the map defined by

$$\langle \xi, \mu^*(m) \rangle = (\xi, \mu(m))_m,$$

for $\xi \in \mathfrak{g}^*$ and $m \in M$. Consider the $G$-invariant vector field $v$ on $M$ defined by

$$v_m = 2\left(\mu^*(m)\right)^M_m,$$

where $m \in M$, $\left(\mu^*(m)\right)^M$ is the vector field induced by $\mu^*(m) \in \mathfrak{g}$, and the factor 2 was included for consistency with [13, 14, 26]. The deformed Dirac operator is the operator

$$D_v := D + c(v)$$

on smooth sections of $\mathcal{S}$, where $c$ denotes the Clifford action by $TM$ on $\mathcal{S}$. The main assumption we make is that the zeroes of $v$ form a cocompact subset of $M$.

It was shown in [14] that, for a well-chosen metric on $M \times \mathfrak{g}^*$, the operator $D_v$ has the properties of the operator $D$ in Subsection 1.2. This was generalised by Braverman in [8]. It therefore has a well-defined invariant, transversally $L^2$-index.

**Definition 1.5.** The invariant Spin$^c$-quantisation of the action by $G$ on $M$ is

$$Q^{\text{Spin}^c}(M)^G = \text{index}^G_{L^2_v}(D_v^+) \in \mathbb{Z}.$$

## 2 Quantisation and reduction: the main results

This section contains the main results of this paper. After stating Paradan and Vergne’s result that Spin$^c$-quantisation commutes with reduction in the compact case, we state our results for cocompact actions: Theorem 2.3 for the equivariant $L^2$-index, which specialises to Theorem 2.5 for discrete series representations, and Theorem 2.7 for reduction at the trivial representations. Finally, we discuss relations with a $K$-theoretic version of quantisation.
2.1 Reduced spaces

Consider the setting of Subsection 1.3. For any $\xi \in g^*$, the reduced space $M_\xi$ at $\xi$ is

$$M_\xi = \mu^{-1}(\xi)/G_\xi = \mu^{-1}(G \cdot \xi)/G.$$ 

Here $G_\xi$ is the stabiliser of $\xi$ with respect to the coadjoint action. It was shown in Lemma 3.3 of [14] that $M_\xi$ is a Spin$^c$-orbifold if $\xi$ is a Spin$^c$-regular value of $\mu$. This means that $\mu^{-1}(\xi)$ is smooth, $G_\xi$ acts locally freely on $\mu^{-1}(\xi)$, and the vector bundle $TM|_{\mu^{-1}(\xi)}/q^*TM_\xi$ (with $q : \mu^{-1}(\xi) \to M_\xi$ the quotient map) is $G_\xi$-equivariantly Spin. If $\xi$ is a regular value of $\mu$, then it is a Spin$^c$-regular value for example if

- $G$ is Abelian;
- $G$ is semisimple and $\xi = 0$;
- $G$ is unimodular and $G_\xi$ is compact.

(See Examples 3.6 and 3.7 in [14].) For our current purposes, the most important fact about Spin$^c$-regular values is that regular values of a momentum map for a certain kind of slice $N \subset M$ are Spin$^c$-regular values of $\mu$, as explained in Subsection 3.1.

If $\xi$ is a Spin$^c$-regular value of $\mu$, then the quantisation $Q^{Spin^c}(M_\xi) \in \mathbb{Z}$ is defined as the index of the Spin$^c$-Dirac operator on $M_\xi$ associated to the Spin$^c$-structure induced by the one on $M$. This is well-defined if $M_\xi$ is compact, which is the case if the set of zeros of the vector field $v$ in (1.5) is cocompact.

If $\xi$ is not a Spin$^c$-regular value of $\mu$, then $Q^{Spin^c}(M_\xi)$ can be defined by realising $M_\xi$ as a reduced space for an action by a compact group. Then Paradan and Vergne’s definition of quantisation of singular reduced spaces applies. We postpone the details of this procedure to Definition 3.2.

2.2 Quantisation commutes with reduction in the compact case

In this subsection, we state Paradan and Vergne’s result. (This subsection is adapted from Subsection 4.1 in [14].) Suppose for now that $G = K$ and $M$ are compact, as well as connected. Let $T < K$ be a maximal torus, with Lie algebra $t \subset k$. Let $t^*_+ \subset t^*$ be a choice of (closed) positive Weyl chamber.
Let $R$ be the set of roots of $(\mathfrak{t}_C, \mathfrak{t}_C)$, and let $R^+$ be the set of positive roots with respect to $\mathfrak{t}^*_+$. Set

$$\rho_K := \frac{1}{2} \sum_{\alpha \in R^+} \alpha.$$ 

Let $\mathcal{F}$ be the set of relative interiors of faces of $\mathfrak{t}^*_+$. Then

$$\mathfrak{t}^*_+ = \bigcup_{\sigma \in \mathcal{F}} \sigma,$$

a disjoint union. For $\sigma \in \mathcal{F}$, let $\mathfrak{k}_\sigma$ be the infinitesimal stabiliser of a point in $\sigma$. Let $R_\sigma$ be the set of roots of $((\mathfrak{k}_\sigma)_C, \mathfrak{t}_C)$, and let $R^+_\sigma := R_\sigma \cap R^+$. Set

$$\rho_\sigma := \frac{1}{2} \sum_{\alpha \in R^+_\sigma} \alpha.$$ 

Note that, if $\sigma$ is the interior of $\mathfrak{t}^*_+$, then $\rho_\sigma = 0$.

For any subalgebra $\mathfrak{h} \subset \mathfrak{k}$, let $(\mathfrak{h})$ be its conjugacy class. Set

$$\mathcal{H}_\mathfrak{k} := \{(\mathfrak{k}_\xi); \xi \in \mathfrak{h}\}.$$ 

For $(\mathfrak{h}) \in \mathcal{H}_\mathfrak{k}$, write

$$\mathcal{F}(\mathfrak{h}) := \{\sigma \in \mathcal{F}; (\mathfrak{k}_\sigma) = (\mathfrak{h})\}.$$ 

Let $(\mathfrak{f}^M)$ be the conjugacy class of the generic (i.e. minimal) infinitesimal stabiliser $\mathfrak{f}^M$ of the action by $K$ on $M$. If $\mu$ has regular values, then $(\mathfrak{f}^M) = 0$ by Lemma 2.4 in [14].

Let $\Lambda_+ \subset \mathfrak{i}^*$ be the set of dominant integral weights with respect to $\mathfrak{t}^*_+$. In the Spin$^c$-setting, it is natural to parametrise the irreducible representations of $K$ by their infinitesimal characters, rather than by their highest weights. For $\lambda \in \Lambda_+ + \rho_K$, let $\pi^K_\lambda$ be the irreducible representation of $K$ with infinitesimal character $\lambda$, i.e. with highest weight $\lambda - \rho_K$. Then one has, for such $\lambda$,

$$Q^{\text{Spin}^c}(K \cdot \lambda) = \pi^K_\lambda,$$

see Lemma 2.1 in [23].

Write

$$Q^K_{\text{Spin}^c}(M) = \bigoplus_{\lambda \in \Lambda_+ + \rho_K} m_\lambda [\pi^K_\lambda],$$

with $m_\lambda \in \mathbb{Z}$. Theorem 1.3 in Paradan and Vegne’s paper [23] is the following expression for $m_\lambda$ in terms of reduced spaces.

12
Theorem 2.1 ([Q^{Spin^c}, R] = 0 in the compact case). Suppose ([\mathfrak{t}^M, \mathfrak{t}^M]) = ([\mathfrak{h}, \mathfrak{h}]), for (\mathfrak{h}) \in H_\mathfrak{t}. Then

\[ m_\lambda = \sum_{\sigma \in F(\mathfrak{h}) \text{ s.t. } \lambda - \rho_\sigma \in \sigma} Q^{Spin^c}(M_{\lambda - \rho_\sigma}). \]

If the generic stabiliser \mathfrak{t}^M is Abelian, Theorem 2.1 simplifies considerably. As noted above, this occurs in particular if \mu has a regular value.

Corollary 2.2. If \mathfrak{t}^M is Abelian, then

\[ m_\lambda = Q^{Spin^c}(M_\lambda). \]

Proof. If one takes \mathfrak{h} = \mathfrak{t} in Theorem 2.1, then \mathcal{F}(\mathfrak{h}) only contains the interior of \mathfrak{t}^\ast_{\mathfrak{t}}. Hence \rho_\sigma = 0, for the single element \sigma \in \mathcal{F}(\mathfrak{h}). \]

In particular, if \rho_K is a regular value of \mu, then the invariant part of the Spin^c-quantisation of M is

\[ Q^{Spin^c}_K(M)^K = Q^{Spin^c}(M_{\rho_K}), \]

since \pi^K_{\rho_K} is the trivial representation.

2.3 Quantisation commutes with reduction for cocompact actions

Suppose that

- M/G is compact;
- M and G are connected;
- M and G/K are even-dimensional;
- G acts properly on M;
- The Riemannian metric on M induced by the Spin^c-structure is sufficiently regular, in the following sense. There is a K-invariant inner product on \mathfrak{g}, such that, if \mathfrak{p} \subset \mathfrak{g} is the orthogonal complement to \mathfrak{t}, then for all X, Y \in \mathfrak{p} and m \in M, the Riemannian inner product of the tangent vectors at m induced by X and Y equals their inner product in \mathfrak{g}.
In addition, suppose $\text{Ad} : K \to \text{SO}(p)$ lifts to

$$\tilde{\text{Ad}} : K \to \text{Spin}(p).$$

This is always possible if one replaces $G$ by a double cover. Indeed, consider the diagram

$$\begin{array}{ccc}
\tilde{K} & \xrightarrow{\tilde{\text{Ad}}} & \text{Spin}(p) \\
\pi_K & \downarrow & \downarrow \pi_{2:1} \\
K & \xrightarrow{\text{Ad}} & \text{SO}(p),
\end{array}$$

where

$$\tilde{K} := \{(k, a) \in K \times \text{Spin}(p); \text{Ad}(k) = \pi(a)\};$$

$$\pi_K(k, a) := k;$$

$$\tilde{\text{Ad}}(k, a) := a,$$

for $k \in K$ and $a \in \text{Spin}(p)$. Then for all $k \in K$,

$$\pi_{\tilde{K}}^{-1}(k) \cong \pi^{-1}(\text{Ad}(k)) \cong \mathbb{Z}_2,$$

so $\pi_{\tilde{K}}$ is a double covering map. Since $G/K$ is contractible, $\tilde{K}$ is the maximal compact subgroup of a double cover of $G$.

Under the above conditions, we have the quantisation commutes with reduction results stated below, in terms of the expression (2.1) for the multiplicities of irreducible representations. Recall from Corollary 2.2 that if the generic stabiliser of the action is Abelian, then this expression simplifies to

$$m_{\lambda} = Q_{\text{Spin}}(M_{\lambda}).$$

**Theorem 2.3** ($[Q_{\text{Spin}}, R] = 0$ for cocompact actions). One has

$$Q_{G}^{\text{Spin}}(M) = \sum_{\lambda \in \Lambda_{+} + \rho_G} m_{\lambda} \left(\chi_{\lambda}^{K} \otimes \text{index}_{L^2(D_{G/K})}^{G} \right)^{K}_{K},$$

with $m_{\lambda}$ as in (2.1), and where $\chi_{\lambda}^{K}$ is the character of $\pi_{\lambda}^{K}$.

This becomes more explicit if $G$ is semisimple with discrete series. Then the discrete series of $G$ was realised in Theorem 9.3 in [4] and Theorem 1 of [2.4] in terms of the characters $\left(\chi_{\lambda}^{K} \otimes \text{index}_{L^2(D_{G/K})}^{G}\right)^{K}$ appearing in Theorem 2.3.
Theorem 2.4 (Atiyah–Schmid, Parthasarathy). Let $\lambda \in \Lambda_+ + \rho_K$. If $\lambda$ is singular, then the character

\[(2.4) \quad (\chi^K_\lambda \otimes \text{index}_{L^2}^G(D_{G/K}))^K \in \mathcal{D}'(G)\]

is zero. If $\lambda$ is nonsingular, then (2.4) equals the character of the discrete series representation $\pi^\text{ds}_\lambda$ with Harish–Chandra parameter $\lambda$.

As a consequence, one obtains a complete decomposition of $Q^\text{Spin}^c_G(M)$ into discrete series representations. For regular $\lambda \in \Lambda_+ + \rho_K$, let $\Theta^\text{ds}_\lambda$ be the character of the discrete series representation $\pi^\text{ds}_\lambda$ with Harish–Chandra parameter $\lambda$.

Theorem 2.5 ($[Q^\text{Spin}^c, R] = 0$ at discrete series representations). One has

\[Q^\text{Spin}^c_G(M) = \sum_{\lambda \in \Lambda_+ + \rho_K \text{ regular}} m_\lambda \Theta^\text{ds}_\lambda.\]

with $m_\lambda$ as in (2.1).

In particular, the only irreducible representations occurring in the equivariant $L^2$-index of a Spin$^c$-Dirac operator for a cocompact action belong to the discrete series.

Example 2.6. Theorem 2.5 implies that the character of the discrete series representation $\pi^\text{ds}_\lambda$ with Harish–Chandra parameter $\lambda$ equals

\[\Theta^\text{ds}_\lambda = Q^\text{Spin}^c_G(G \cdot \lambda).\]

This also follows directly from Theorem 2.4 and a Spin$^c$-version of the Borel–Weil theorem (which are both used in the proof of Theorem 2.5). Indeed, since $\lambda$ is strongly elliptic, one has

\[G \cdot \lambda = G/G_\lambda = G/K_\lambda = G \times_K (K \cdot \lambda).\]

So now $N = K \cdot \lambda$, and $\text{index}_K(D_N)$ is the irreducible representation of $K$ with highest weight $\lambda - \rho_K$ (see Lemma 3.11 in [23]). Hence, by Theorem 2.4 the equivariant $L^2$-index $\text{index}_{L^2}^G(D_{G, \lambda})$ of the Spin$^c$-Dirac operator $D_{G, \lambda}$ on $G \cdot \lambda$ equals $\Theta^\text{ds}_\lambda$. 

15
Next, suppose $G$ is unimodular. To state a quantisation commutes with reduction formula for reduction at the trivial representation, consider the multiplicities $n_{\lambda} \in \mathbb{Z}$ in

$$\pi_p = \sum_{\lambda \in \Lambda_+ + \rho_K} n_{\lambda}[\pi^K_{\lambda}].$$

The weights of this virtual representation are described explicitly in Remark 2.1 in [24], if $G$ and $K$ have equal rank. (See also Proposition 8.10 in [21].) Let $\rho_G$ be half the sum of a choice of positive roots for $G$, compatible with the positive roots for $K$. Then, in particular, one has $n_{\rho_G - \rho_K} = 1$ in the equal rank case.

**Theorem 2.7** ([$Q^{Spin^c}, R] = 0$ at the trivial representation). One has

$$Q^{Spin^c}(M)^G = \sum_{\lambda \in \Lambda_+ + \rho_K} n_{\lambda}m_{\lambda},$$

with $m_{\lambda}$ as in (2.1).

### 2.4 Other results

Landsman [18] defined quantisation of cocompact Hamiltonian actions on symplectic manifolds. He defined quantisation as an element

$$Q_G(M) \in K_0(C^*G),$$

the $K$-theory of the group $C^*$-algebra of the group $G$ acting. He defined $Q_G(M)$ as the equivariant index of a Dirac operator in the sense of the analytic assembly map that is used in the Baum–Connes conjecture [5]. Applying the reduction map $R_0 : K_0(C^*G) \to K_0(\mathbb{C}) = \mathbb{Z}$, induced by integration over $G$, he conjectured that

$$R_0(Q_G(M)) = Q(M_0).$$

In [20], Mathai and Zhang solve Landsman’s quantisation commutes with reduction conjecture [16, 18] for large powers of the prequantum line bundle. In the appendix to [20], Bunke shows that the left hand side of this equality equals the invariant quantisation as defined here. (In the cocompact case, all sections are transversally $L^2$, and one does not need to deform the Dirac operator.)
In Corollary 9.1 [14], Mathai and Zhang’s result was generalised from symplectic manifolds to Spin$^c$-manifolds. Theorem 2.7 implies a generalisation of this result where one does not need high powers of the line bundle $L$. Indeed, as in [14], let
\[ \tilde{Q}_G^{\text{Spin}^c}(M) \in K_0(C^*G) \]
be the image of the $K$-homology class of $D$ under the analytic assembly map. By Bunke’s arguments in the appendix to [20], Theorem 2.7 is equivalent to the following statement.

**Corollary 2.8.** One has
\[ R_0(\tilde{Q}_G^{\text{Spin}^c}(M)) = \sum_{\lambda \in \Lambda_+ + \rho_K} n_\lambda m_\lambda, \]
with $n_\lambda$ and $m_\lambda$ as in Theorem 2.7.

As noted in Subsection 6.2 of [14], using large powers of $L$ leads to a simplified, ‘asymptotic’ statement that Spin$^c$-quantisation commutes with reduction. Corollary 2.8 is the appropriate refinement in terms of the exact multiplicities. This is the first non-asymptotic result stating that Landsman’s $K$-theoretic version of quantisation commutes with reduction.

In addition, the techniques we develop allow us to generalise Atiyah and Hirzebruch’s vanishing theorem [3] for actions by compact groups on compact Spin-manifolds to cocompact actions.

**Theorem 2.9.** If the Spin$^c$-structure on $M$ is a Spin-structure, and if not all stabilisers of the action by $G$ are maximal compact subgroups of $G$, then
\[ Q_G^{\text{Spin}}(M) = 0. \]

Theorem 2 in [15] is a $K$-theoretic version of this result. Note that all stabilisers being maximal compact subgroups is the closest a proper action can get to being trivial.

### 3 Quantisation and induction

We discuss a Spin$^c$-slice theorem, and state two quantisation commutes with induction results: Proposition 3.3 for cocompact actions, and Proposition 3.4 for possibly non-cocompact actions. These are the key ingredients of the proofs of the results in Subsection 2.3
3.1 Slices

To define $Q_{\text{Spin}^c}(M_\xi)$ if $\xi$ is not a Spin$^c$-regular value of $\mu$, we use a Spin$^c$-slice theorem based on Abels’ theorem. This will also play an important role in the proofs of our results. The smooth version of Abels’ theorem, as on page 2 of [1], is the following statement.

**Theorem 3.1.** Let $M$ be a smooth manifold, and let $G$ be a connected Lie group acting properly on $M$. Let $K < G$ be maximal compact. Then there is a smooth, $K$-invariant submanifold $N \subset M$, such that the map $[g, n] \mapsto gn$ is a $G$-equivariant diffeomorphism

\[(3.1)\quad G \times_K N \cong M\]

Here the left hand side is the quotient of $G \times N$ by the action by $K$ given by

$$k \cdot (g, n) = (gk^{-1}, kn),$$

for $k \in K$, $g \in G$ and $n \in N$.

Fix a slice $N \subset M$ as in Theorem 3.1.

Consider a $K$-invariant inner product on the Lie algebra $\mathfrak{g}$ of $G$, and let $\mathfrak{p} \subset \mathfrak{g}$ be the orthogonal complement to the Lie algebra $\mathfrak{k}$ of $K$. Then one has

\[(3.2)\quad TM \cong G \times_K (TN \oplus \mathfrak{p}_N).\]

Here $\mathfrak{p}_N$ is the trivial vector bundle $N \times \mathfrak{p} \to N$, on which $K$ acts diagonally via the adjoint action. A $K$-invariant Riemannian metric on $TN$, together with the restriction of the given inner product on $\mathfrak{g}$ to $\mathfrak{p}$, induces a Riemannian metric on $TM$ via the isomorphism (3.2). We will assume that the Riemannian metric on $TM$ induced by the Spin$^c$-structure is of this form.

In Section 3.2 of [11] and Section 3.2 of [13], an induction procedure of equivariant Spin$^c$-structures from $N$ to $M$ is described. Proposition 3.10 of [14] is a Spin$^c$-slice theorem, which states that there is a $K$-equivariant Spin$^c$-structure on $N$, such that the induced Spin$^c$-structure on $M$ equals the Spin$^c$-structure originally given. The connection $\nabla^k$ on $L \to M$ restricts to a connection on the determinant line bundle $L_N = L|_N$ of this Spin$^c$-structure on $N$. This defines a Spin$^c$-momentum map $\mu_N : N \to \mathfrak{k}^*$, analogously to
In Lemma 5.3 of [14], it was shown that the connection $\nabla^L$ can be chosen such that $\mu(N) \subset k^*$, and

\begin{equation}
\mu_N = \mu|_N.
\end{equation}

Here we embed $\mathfrak{k}^* \hookrightarrow \mathfrak{g}^*$ via the $K$-invariant inner product chosen earlier.

Then for all $\xi \in k^*$, the inclusion map $\mu_N^{-1}(\xi) \hookrightarrow \mu^{-1}(\xi)$ induces a homeomorphism

$$N_\xi \cong M_\xi.$$ 

If $\xi$ is a regular value of $\mu_N$, it is a Spin$^c$-regular value of $\mu$ by Proposition 3.12 in [14] (though not necessarily a regular value in the usual sense). Then Proposition 3.14 in [14] states that $N_\xi$ and $M_\xi$ are equal as Spin$^c$-orbifolds, so that

\begin{equation}
Q^{\text{Spin}^c}(M_\xi) = Q^{\text{Spin}^c}(N_\xi) \in \mathbb{Z}.
\end{equation}

If $\xi$ is not a regular value of $N$, then $Q^{\text{Spin}^c}(N_\xi)$ can be defined as in Definition 5.5 of [23].

**Definition 3.2.** If $\xi$ is not a regular value of $N$, then $Q^{\text{Spin}^c}(M_\xi) \in \mathbb{Z}$ is defined by (3.4).

This definition is independent of $\nabla^L$, by Proposition 4.25 of [23].

### 3.2 Quantisation commutes with induction

The quantisation commutes with induction results we will state now relate the Spin$^c$-quantisation of the action by $G$ on $M$ to the Spin$^c$-quantisation of the action by $K$ on $N$. For cocompact actions, this allows us to deduce the quantisation commutes with reduction results in Subsection 2.3 from the result in the compact case.

Let $\Delta_p$ be the standard representation of Spin($p$). Since $G/K$ is even-dimensional, $\Delta_p$ splits as $\Delta_p = \Delta_p^+ \oplus \Delta_p^-$. Let $D_{G/K}$ be the operator

\begin{equation}
D_{G/K} = \sum_{j=1}^k X_j \otimes c(X_j)
\end{equation}

on $C^\infty(G) \otimes \Delta_p$, where $\{X_1, \ldots, X_k\}$ is an orthonormal basis of $p$. By Proposition 1.1 in [24], this is the Spin-Dirac operator on $G/K$.

First, suppose $M/G$ is compact.
Proposition 3.3 ([Q^{Spin^c}, Ind] = 0 for cocompact actions). One has

\[ Q^{Spin^c}_G(M) = (Q^{Spin^c}_K(N) \otimes \text{index}^{L^2(D_{G/K})}_G)_K \in \mathcal{D}'(G). \]

Here we identify \( Q^{Spin^c}_K(N) \in R(K) \) with its character.

Theorem 2.1 follows directly from Proposition 3.3, Theorem 2.1, and the equality (3.4).

In addition, Proposition 3.3 allows us to deduce Theorem 2.9 from Atiyah and Hirzebruch’s result [3]. Indeed, by Lemma 4 in [15], the condition in Theorem 2.9 on the stabilisers of the action is equivalent to the action by \( K \) on \( N \) being nontrivial. Also, Lemma 5 in [15] implies that the Spin^c-structure on the slice \( N \) is a Spin-structure. Therefore, Atiyah and Hirzebruch’s theorem [3] implies that

\[ Q^{Spin^c}_K(N) = 0. \]

By Proposition 3.3, we also have \( Q^{Spin^c}_G(M) = 0 \), so Theorem 2.9 is proved.

We can also state a quantisation commutes with induction result for possibly non-cocompact actions. Recall that we assumed the connection \( \nabla^L \) on \( L \to M \) was chosen so that \( \mu_N = \mu|_N \). Using a metric on \( N \times \mathfrak{g}^* \to N \) compatible with the metric on \( M \times \mathfrak{g}^* \to M \), we then have \( v_N = v|_N \), where \( v_N \) is the vector field on \( N \) induced by \( \mu_N \) analogously to (1.5). Braverman showed in Theorem 2.9 of [7] that the equivariant index of the Dirac operator on \( N \), deformed by \( v_N \), is well-defined as an element of the generalised representation ring of \( K \). I.e. all irreducible representations occur with only finite multiplicity. We denote this index by \( Q^{Spin^c}_K(N) \).

Consider the representations

\[ \pi_p^\pm : K \xrightarrow{\tilde{\text{Ad}}} \text{Spin}(p) \to \text{GL}(\Delta^\pm_p) \]

of \( K \). We denote the formal difference of their equivalence classes by

\[ \pi_p = [\pi_p^+] - [\pi_p^-] \in R(K). \]

Proposition 3.4 ([Q^{Spin^c}, Ind] = 0 for non-cocompact actions). If \( G \) is unimodular, and the set of zeros of \( v_N \) is compact, then

\[ Q^{Spin^c}_G(M)^G = [Q^{Spin^c}_K(N) : \pi_p] \in \mathbb{Z}, \]

the multiplicity of \( \pi_p \) in \( Q^{Spin^c}_K(N) \).
Using this proposition in the cocompact case allows us to deduce Theorem 2.7 from Theorem 2.1. Indeed, if $M/G$, and hence $N$, is compact, then $Q^K_{\text{Spin}^c}(N)$ as defined above Proposition 3.4 is simply the equivariant index of the Spin$^c$-Dirac operator on $N$.

### 3.3 The discrete series and $K$-theory

Suppose $G$ is semisimple with discrete series. A version of Landman’s conjecture for classes of discrete series representations was studied in [11]. The arguments given there yield a relation between quantisation as defined here, and quantisation via the analytic assembly map. Indeed, let

$$\tilde{Q}^{\text{Spin}^c}_G(M)_r \in K_0(C^*_rG)$$

be the image of the $K$-homology class of $D$ under the analytic assembly map, this time defined with respect to the reduced group $C^*$-algebra $C^*_rG$. For every discrete series representation $\pi^\text{ds}$ there is a corresponding $K$-theory class $j(\pi^\text{ds})$, as in Section 2.2 of [17]. Identifying discrete series representations with their characters, we can extend $j$ to finite sums and differences of discrete series characters. Let $p : K_0(C^*_rG) \to K_0(C^*_rG)$ be the projection onto the part of $K_0(C^*_rG)$ generated by the discrete series, i.e. the image of $j$.

**Proposition 3.5.** We have

$$p(\tilde{Q}^{\text{Spin}^c}_G(M)_r) = (-1)^{\dim(G/K)/2} j(\tilde{Q}^{\text{Spin}^c}_G(M)).$$

**Proof.** By (5.3) in [12] we have the following $K$-theoretic analogue of Theorem 2.4:

$$(3.6) \quad j(\pi^\text{ds}_\lambda) = (-1)^{\dim(G/K)/2} \text{D-Ind}^G_K[\pi^K_\lambda],$$

where $\text{D-Ind}^G_K : R(K) \to K_*(C^*_rG)$ is the Dirac induction map. This follows from Lemma 1.3 in [11], which in turn is based on Lemma 2.1.1 in [17].

Write

$$\text{index}_K D^+_N = \bigoplus_{\lambda \in \Lambda_+ + \rho_K} m_\lambda[\pi^K_\lambda].$$
Then by Proposition 3.3, Theorem 2.4 and (3.6),

\[ j(Q^\text{Spin}_e^c_G(M)) = (-1)^{\text{dim}(G/K)/2} \bigoplus_{\lambda \in \Lambda_+ + \rho_K} m_\lambda \text{D-Ind}_K^G[\pi^K_\lambda] \]

\[ = (-1)^{\text{dim}(G/K)/2} p \circ \text{D-Ind}_K^G \left( \bigoplus_{\lambda \in \Lambda_+ + \rho_K} m_\lambda [\pi^K_\lambda] \right) \]

\[ = (-1)^{\text{dim}(G/K)/2} p \circ \text{D-Ind}_K^G(\text{index}_K(D_N^+)). \]

Since Spin$^c$-quantisation commutes with induction in the $K$-theory setting, as in Theorem 5.7 in [14], the latter expression equals $(-1)^{\text{dim}(G/K)/2} p(Q^\text{Spin}_e^c_G(M))$. □

4 Decomposing the Dirac operator

The proofs of Propositions 3.3 and 3.4 are based on the decomposition of the Spin$^c$-Dirac operator $D$ on $M$ that we will discuss in this section.

4.1 Spinors

As before, suppose that $M$ and $G$ are connected, $K < G$ is a maximal compact subgroup, and that $M$ and $G/K$ are even-dimensional. Let $P \to M$ be the $G$-equivariant Spin$^c$-structure used before. Let a $K$-invariant submanifold $N \subset M$ and a Spin$^c$-structure $P_N \to N$ be as in Subsection 3.1.

Since $M$ and $G/K$ are even-dimensional, so is $N$. Let $L_N = L|_N \to N$ be the determinant line bundle of $P_N$. Then $L = G \times_K L_N$. Let $S_N \to N$ be the spinor bundle associated to $P_N$. By Lemma 6.2 in [11], one has an equivariant vector bundle isomorphism

\[ S \cong G \times_K (S_N \otimes \Delta_p). \] (4.1)

With respect to the decompositions (3.2) and (4.1) The Clifford action $c$ by $TM$ on $S$ is given by

\[ c[g, v, X][g, s_N, \delta] = [g, c_N(v)s_N, \delta] + [g, \varepsilon s_N, c_p \delta]. \] (4.2)

Here $g \in G$, $n \in N$, $v \in T_n N$, $X \in \mathfrak{p}$, $s_N \in (S_N)_n$, and we used the Clifford actions $c_N : TN \to \text{End}(S_N)$ and $c_p : \mathfrak{p} \to \text{End}(\Lambda_p)$. Finally, $\varepsilon$ is the grading.
operator on $S_N$, equal to $\pm 1$ on $S_N^\pm$. At the level of smooth sections, we get

\begin{equation}
\Gamma^\infty(S) \cong (\Gamma^\infty(G \times N, p_N^* S_N) \otimes \Delta_p)^K,
\end{equation}

where $p_N : G \times N \to N$ is the natural projection map.

### 4.2 Dirac operators on $N$ and $G/K$

Let $D_N$ be the Spin$^c$-Dirac operator on $S_N$ induced by the restriction of the connection $\nabla^L$ to $L_N = L|_N$. Let $p_N^* D_N$ be the operator on $\Gamma^\infty(p_N^* S_N)$ given by

$$(p_N^* D_N s)(g, n) = D_N (s(g, -))(n),$$

for $s \in \Gamma^\infty(p_N^* S_N)$, $g \in G$ and $n \in N$.

Fix an orthonormal basis $\{X_1, \ldots, X_k\}$ of $\mathfrak{p}$, and consider the operator

\begin{equation}
D_p := \sum_{j=1}^k X_j \otimes c_p(X_j)
\end{equation}

on $\Gamma^\infty(S)$, where $X_j$ acts on $\Gamma^\infty(p_N^* S_N)$ by differentiation on the $G$-direction, and $c$ is the Clifford action. Then we have the following decomposition of $D$.

**Proposition 4.1.** Under the identification (4.3), one has

$$D = p_N^* D_N + \varepsilon D_p,$$

restricted to $K$-invariant sections. Here $\varepsilon$ is the grading operator on $S_N$.

**Remark 4.2.** If $N$ is a point, then Proposition 4.1 reduces to Proposition 1.1 in [24] (where one takes $V$ to be the trivial representation). If $G = K$ is compact, then one gets the trivial identity $D_N = D_N$. In Proposition 6.7 of [11], it was shown that Proposition 4.1 holds at the level of principal symbols.

### 4.3 The Levi–Civita connection

To prove Proposition 4.1, we start by decomposing the Levi–Civita connection on $M$. Let $\nabla^N$ be the Levi–Civita connection on $TN$, and let
\( \nabla^{G/K} \) be the Levi–Civita connection on \( T(G/K) \), for the Riemannian metric defined by the given inner product on \( \mathfrak{p} \). Consider the projection map \( p_{G/K} : G \times N \to G/K \). Using

\[
p_{G/K}^{*} T(G/K) = p_{G/K}^{*} (G \times_{K} \mathfrak{p}) = G \times N \times \mathfrak{p},
\]
we rewrite (3.2) as

\[
TM = (p_{N}^{*} TN \oplus p_{G/K}^{*} T(G/K))/K.
\]

In terms of the action map \( p_{M} : G \times N \to M \), this can be rephrased as

\[
p_{M}^{*} TM = p_{N}^{*} TN \oplus p_{G/K}^{*} T(G/K).
\]

We find that the space \( \mathfrak{X}(M) \) of vector fields on \( M \) decomposes as

\[
\mathfrak{X}(M) = \Gamma^{\infty} (G \times N, p_{N}^{*} TN \oplus p_{G/K}^{*} T(G/K))^{K}.
\]

Consider the connection

\[
\tilde{\nabla}^{M} := p_{N}^{*} \nabla^{N} \oplus p_{G/K}^{*} \nabla^{G/K}
\]
on \( p_{N}^{*} TN \oplus p_{G/K}^{*} T(G/K) \). Let \( \nabla^{M} \) be the connection on \( TM \) equal to the restriction of \( \tilde{\nabla}^{M} \) to \( K \)-invariant sections. In other words, \( p_{M}^{*} \nabla^{M} = \tilde{\nabla}^{M} \).

**Lemma 4.3.** The connection \( \nabla^{M} \) is the Levi–Civita connection on \( TM \).

**Proof.** The fact that \( \nabla^{N} \) and \( \nabla^{G/K} \) preserve the Riemannian metrics on \( N \) and \( G/K \), respectively, implies that \( \nabla^{M} \) preserves the Riemannian metric on \( M \).

To show that \( \nabla^{M} \) is torsion-free, we note that the torsion \( \text{Tor}^{\nabla^{M}} \) of \( \nabla^{M} \) is a tensor, so it is enough to show it vanishes on a set of vector fields spanning \( TM \). Therefore, we only need to show it vanishes on \( (K\text{-invariant}) \) vector fields of the forms \( p_{N}^{*} v_{N} \) and \( p_{G/K}^{*} v_{G/K} \); for \( v_{N} \in \mathfrak{X}(N) \) and \( v_{G/K} \in \mathfrak{X}(G/K) \cong (C^{\infty}(G), \mathfrak{p})^{K} \).

Now for \( v_{N}, w_{N} \in \mathfrak{X}(N) \), we have

\[
\nabla_{p_{N}^{*} v_{N}}^{M} (p_{N}^{*} w_{N}) = (p_{N}^{*} \nabla^{N})_{p_{N}^{*} v_{N}} (p_{N}^{*} w_{N}) = p_{N}^{*} (\nabla^{N}_{v_{N}} w_{N}).
\]
Hence, because $\nabla^N$ is torsion-free,
\[
\nabla^M_{p_N^*v_N} (p_N^*w_N) - \nabla^M_{p_N^*w_N} (p_N^*v_N) = p_N^* \left( \nabla^N_{v_N} w_N - \nabla^N_{w_N} v_N \right) \\
= p_N^* [v_N, w_N] \\
= [p_N^*v_N, p_N^*w_N].
\]

So
\[
\text{Tor}^M_{\nabla} (p_N^*v_N, p_N^*w_N) = 0.
\]

One similarly shows that for all $v_{G/K}, w_{G/K} \in \mathfrak{X}(G/K)$,
\[
\text{Tor}^M_{\nabla} (p_{G/K}^*v_{G/K}, p_{G/K}^*w_{G/K}) = 0.
\]

It therefore remains to show that
\[
(4.5) \quad \text{Tor}^M_{\nabla} (p_N^*v_N, p_{G/K}^*v_{G/K}) = 0,
\]
with $v_N$ and $v_{G/K}$ as above.

Since each of the vector fields $p_N^*v_N$ and $p_{G/K}^*v_{G/K}$ is tangent to the directions the other vector field is constant in, their Lie bracket vanishes. Also,
\[
\nabla^M_{p_N^*v_N} (p_{G/K}^*v_{G/K}) = (p_{G/K}^* \nabla^G_{v_{G/K}})_{p_N^*v_N} (p_{G/K}^*v_{G/K}) = 0,
\]
since the tangent map of $p_{G/K}$ is zero on the image of $p_N^*v_N$. Similarly, one has $\nabla^M_{p_{G/K}^*v_{G/K}} (p_N^*v_N) = 0$. So in particular\footnote{Note that the Lie bracket of sections of $p_N^*TN$ and $p_{G/K}^*T(G/K)$, and the analogous expression to (4.6), do not vanish in general. This is only the case for the pulled-back sections considered here, which is enough.}
\[
(4.6) \quad \nabla^M_{p_N^*v_N} (p_{G/K}^*v_{G/K}) - \nabla^M_{p_{G/K}^*v_{G/K}} (p_N^*v_N) = 0.
\]

We conclude that (4.5) holds. So $\nabla^M$ is torsion-free, and hence indeed the Levi–Civita connection on $TM$. \hfill \qed

### 4.4 Spinor connections

The decomposition of the Levi–Civita connection in Lemma 4.3 implies an analogous decomposition of the spinor connection $\nabla^S$ on $S$, associated to the connection $\nabla^L$ on $L \to M$. 
Lemma 4.4. In terms of the decomposition \[\text{(4.3)}\], one has for all \(X \in \mathfrak{p}\) and \(v \in TN\),

\[
\nabla_{p_N^*v + X}^S = (p_N^* \nabla^{S_N})_{p_N^*v} + X
\]

As before, \(X\) is considered as a left-invariant vector field on \(G\), acting by differentiation in the \(\mathcal{G}\)-direction.

Proof. Let \(U \subset N\) be a \(K\)-invariant open subset such that

\[S_N|_U = S_U^0 \otimes (L_N|_U)^{1/2},\]

where \(S_U^0 \to U\) is the spinor bundle for a local Spin-structure on \(U\). Then

\[S|_{G \times_K U} = (G \times_K (S_U^0 \otimes \Delta_p)) \otimes (G \times_K (L_N|_U))^{1/2}.\]

We have

\[
\nabla^S|_U = \nabla^{S_{G \times_K U}}_0 \otimes 1 + 1 \otimes \nabla^{(L|_{G \times_K U})^{1/2}},
\]

where \(\nabla^{S_{G \times_K U}}_0\) is the connection on the spinor bundle \(S_{0 \times_K U}^G \to G \times_K U\) induced by the Levi–Civita connection on \(G \times_K U \hookrightarrow M\).

First note that for all \(\sigma \in \Gamma^\infty(L_N)^K\), we have \(p_N^* \sigma \in \Gamma^\infty(p_N^* L_N)^K = \Gamma^\infty(L)\), and for all \(X \in \mathfrak{p}\) and \(v \in TN\),

\[
\nabla_{p_N^*v + X}^L(p_N^* \sigma) = (p_N^* (\nabla_v^{L_N} \sigma))_{p_N^*v} + p_N^* \sigma.
\]

This follows from the definition of \(\nabla^M\) in (22) in [11]. Furthermore, let \(\nabla^{S_U^0}\) be the connection on \(S_U^0\) induced by \(\nabla^N\), and let \(\nabla^{S_{0 \times_K U}^G}\) be the connection on the spinor bundle \(S_{0 \times_K U}^G = G \times_K \Delta_p \to G/K\) induced by \(\nabla^{G/K}\). Then Lemma 4.3 implies that one has

\[
\nabla^{S_{0 \times_K U}^G} = p_N^* \nabla^{S_U^0} + p_{G/K}^* \nabla^{S_{0 \times_K U}^G},
\]

restricted to \(K\)-invariant sections. The connection \(\nabla^{S_{0 \times_K U}^G}\) on \(\Gamma^\infty(S_{0 \times_K U}^G) = (C^\infty(G) \otimes \Delta_p)^K\) is simply given by

\[
\nabla^{S_{0 \times_K U}^G}_{X} s_{G/K} = X(s_{G/K}),
\]

for \(X \in \mathfrak{p}\) and \(s_{G/K} \in (C^\infty(G) \otimes \Delta_p)^K\). (As noted on page 7 of [24], the connection \(\nabla^{G/K}\) is induced by the canonical connection on the principal fibre bundle \(G \to G/K\).)
Since both sides of (4.7) satisfy the Leibniz rule, it is enough to check this equality on a set of sections spanning \( S|_U \). Hence it is enough to consider a section

\[
 s := p_N^*(s_N \otimes \sigma) \otimes p_{G/K}^* s_{G/K} \in \Gamma^\infty(S|_{G \times K U}),
\]

for

\[
 s_N \in \Gamma^\infty(S_0^U)^K; \\
 s_{G/K} \in (C^\infty(G) \otimes \Delta_g)^K; \\
 \sigma \in \Gamma^\infty(L_N^{1/2})^K.
\]

For such a section, and for all \( X \in \mathfrak{p} \) and \( v \in TU \), the preceding arguments allow us to compute

\[
 \nabla^S_{p_N^* v + X} s = \nabla^S_{p_N^* v + X} \left( p_N^* s_N \otimes p_{G/K}^* s_{G/K} \right) \otimes p_N^* \sigma + p_N^* s_N \otimes p_{G/K}^* s_{G/K} \otimes \nabla^L_{p_N^* v + X} p_N^* \sigma \\
 = (p_N^* \nabla^S_{p_N^* v})(p_N^* s_N) \otimes p_{G/K}^* s_{G/K} \otimes p_N^* \sigma + p_N^* s_N \otimes X(p_{G/K}^* s_{G/K}) \otimes p_N^* \sigma \\
 + p_N^* s_N \otimes p_{G/K}^* s_{G/K} \otimes p_N^* \left( \nabla^L_{v} p_N^* \right) \sigma \\
 = (p_N^* \nabla^S_{p_N^* v})(p_N^* s_N \otimes \sigma) \otimes p_{G/K}^* s_{G/K} + p_N^* (s_N \otimes \sigma) \otimes X(p_{G/K}^* s_{G/K}) \\
 = ((p_N^* \nabla^S_{p_N^* v}) p_N^* v + X) s,
\]

since \((p_N^* \nabla^S_{p_N^* v}) p_N^* v\) vanishes on sections pulled back from \( G/K \), while \( X \) vanishes on sections pulled back from \( N \).

\[\Box\]

4.5 **Proof of Proposition 4.1**

Using Lemma 4.4, we can prove Proposition 4.1. One ingredient of the proof is the following expression for the operator \( p_N^* D_N \).

**Lemma 4.5.** If \( \{e_1, \ldots, e_l\} \) is a local orthonormal frame for \( TN \), then locally,

\[
 (4.8) \quad p_N^* D_N = \sum_{s=1}^{l} c(p_N^* e_s)(p_N^* \nabla^S_{p_N^* v}) p_N^* e_s.
\]

**Proof.** Note that any section of \( \Gamma^\infty(p_N^* S_N) \) is a sum of sections of the form \( \varphi p_N^* s_N \), for \( \varphi \in C^\infty(G \times N) \) and \( s_N \in \Gamma^\infty(S_N) \). On such a section, one has

\[
 (4.9) \quad (p_N^* \nabla^S_{p_N^* v}) p_N^* e_s(\varphi p_N^* s_N) = \varphi p_N^* (\nabla^S_{p_N^* v} p_N^* s_N) + (p_N^* e_s)(\varphi) p_N^* s_N.
\]
At a point \((g, n) \in G \times N\), one has
\[(p^*_N e_s)(\varphi)(g, n) = e_s(\varphi(g, -))(n).\]

Therefore, at such a point, we find that \((4.9)\) equals
\[
\left(\nabla^{S_N} e_s(\varphi(g, -)s_N)\right)(n).
\]

We conclude that, at \((g, n)\), the right hand side of \((4.8)\) applied to \(\varphi p^*_N s_N\) yields
\[
\left(\sum_{s=1}^{l} c(p^*_N e_s)\nabla^S_{p^*_N e_s}(\varphi p^*_N s_N)\right)(g, n) = \sum_{s=1}^{l} \left( c(e_s) \left(\nabla^{S_N} e_s(\varphi(g, -)s_N)\right)\right)(n)
= \left( (p^*_N D_N)(\varphi p^*_N s_N)\right)(g, n).
\]

\[\square\]

\textbf{Proof of Proposition 4.1.} Let \(\{X_1, \ldots, X_k\}\) be an orthonormal basis of \(p\), and let \(\{e_1, \ldots, e_l\}\) be a local orthonormal frame for \(TN\). Then
\[
(4.10) \quad D = \sum_{r=1}^{k} c(X_r)\nabla^S_{X_r} + \sum_{s=1}^{l} c(p^*_N e_s)\nabla^S_{p^*_N e_s}.
\]

Note that for each \(r\) and \(s\), \(c(X_r)\) acts on \(\Delta_p\), and \(c(p^*_N e_s)\) acts on \(S_N\) in \(S = G \times_K (S_N \otimes \Delta_p)\), via \((4.2)\).

By Lemma 4.1 and \((4.2)\), the first term on the right hand side of \((4.10)\) equals
\[
\sum_{r=1}^{k} c(X_r)X_r = \varepsilon D_p.
\]

The same lemma implies that the second term equals
\[
\sum_{s=1}^{l} c(p^*_N e_s)(p^*_N \nabla^{S_N})_{p^*_N e_s},
\]
which by Lemma \(4.5\) equals \(p^*_N D_N\). \[\square\]
4.6 Decomposing the kernel of $D$

The decomposition of $D$ in Proposition 4.1 can be reformulated as follows. For $s \in \Gamma^\infty(S_N)$ and $\varphi \in C^\infty(G, \Delta_p)$, define $\sigma(s \otimes \varphi) \in \Gamma^\infty(p_N^*S) \otimes \Delta_p$ by

$$(\sigma(s \otimes \varphi))(g, n) = s(n) \otimes \varphi(g),$$

for $n \in N$ and $g \in G$.

**Corollary 4.6.** The map $\sigma$ is a $G$-equivariant isomorphism

$$\Gamma^\infty(S) \cong (\Gamma^\infty(S_N) \hat{\otimes} C^\infty(G, \Delta_p))^K,$$

where $\hat{\otimes}$ denotes the tensor product completed in the Fréchet topology on $\Gamma^\infty(S)$. Under this isomorphism, the Dirac operator $D$ corresponds to

$$(4.11) \quad D_N \otimes 1 + \varepsilon 1 \otimes D_{G/K},$$

where $D_{G/K}$ was defined in (3.5).

**Proof.** The map $\sigma$ maps $K$-invariant sections to $K$-invariant sections, and its image is dense in $\Gamma^\infty(p_N^*S) \otimes \Delta_p$. Furthermore, with notation as above,

$$(\sigma(DNs \otimes \varphi + s \otimes \varepsilon D_{G/K}\varphi))(g, n) = (DNs)(n) \otimes \varphi(g) + s(n) \otimes (\varepsilon D_{G/K}\varphi)(g) = ((p_N^*DN + \varepsilon D_p)\sigma(s \otimes \varphi))(g, n).$$

Proposition 4.1 states that $p_N^*DN + \varepsilon D_p$, restricted to $K$-invariant sections, is the Dirac operator $D$.

As a consequence of Corollary 4.6, we obtain a relation between the kernels of $D$ and $D_N$.

**Corollary 4.7.** One has

$$\ker(D) \cong (\ker(D_N) \hat{\otimes} \ker(D_{G/K}))^K.$$

**Proof.** As in the proof of Theorem 3.5 in [2], the presence of the grading operator $\varepsilon$ in (4.11) implies that

$$(D_N \otimes 1 + \varepsilon 1 \otimes D_{G/K})^2 = D_N^2 \otimes 1 + 1 \otimes D_{G/K}^2.$$

Since the operators $D_N$ and $D_{G/K}$ are essentially self-adjoint, and continuous in the Fréchet topology on $\Gamma^\infty(S_N)$ and $C^\infty(G, \Delta_p)$ (so that their kernels are closed) the claim follows.
In terms of the grading on $S$, we find that
\begin{equation}
\ker(D^+) \cong (\ker(D^+_{N}) \otimes \ker(D^+_{G/K}))^K \oplus (\ker(D^-_{N}) \otimes \ker(D^-_{G/K}))^K;
\end{equation}
\begin{equation}
\ker(D^-) \cong (\ker(D^-_{N}) \otimes \ker(D^+_{G/K}))^K \oplus (\ker(D^+_{N}) \otimes \ker(D^-_{G/K}))^K.
\end{equation}

5 Proofs of Propositions 3.3 and 3.4

We prove Propositions 3.3 and 3.4 by using Corollary 4.6 and keeping track of the $L^2$-norms on the various spaces involved. To compare $L^2$-norms, we use a relation between the Riemannian densities on $M$, $N$ and $G$.

5.1 Densities

Recall that the Riemannian metric on $M = G \times_K N$ is induced by the given inner product on $\mathfrak{p}$ and a $K$-invariant Riemannian metric on $N$. Let $dm$ and $dn$ be the densities on $M$ and $N$ defined by these Riemannian metrics. Recall that we fixed a Haar measure $dg$ on $G$. Let $dk$ be the Haar measure on $dk$ giving $K$ unit volume. We will use the fact that $dm$ equals the measure $d[g, n]$ on $G \times_K N$ induced by the product density $dg \otimes dn$ on $G \times N$. This is defined by
\[
\int_{G \times N} \varphi(g, n) \, dg \, dn = \int_{G \times_K N} \int_K \varphi(k \cdot \sigma[g, n]) \, dk \, d[g, n]
\]
for any $\varphi \in C_c(G \times N)$ and any Borel section $\sigma : G \times_K N \to G \times N$. (See e.g. [6], Chapter 7, Section 2, Proposition 4b.)

Lemma 5.1. Under the isomorphism
\[
\alpha : G \times_K N \to M
\]
given by $\alpha[g, n] = gn$, for $g \in G$ and $n \in N$, and for a suitable scaling of the Haar measure $dg$, we have $\alpha^* dm = d[g, n]$.

Proof. Consider the non-equivariant diffeomorphisms
\[
\Psi_M : \mathfrak{p} \times N \to M;
\Psi_{G \times N} : \mathfrak{p} \times K \times N \to G \times N,
\]

defined by

\[ \Psi_M(X, n) = \exp(X)n; \]
\[ \Psi_{G \times N}(X, k, n) = (\exp(X)k^{-1}, kn), \]

for \( X \in \mathfrak{p}, n \in N \) and \( k \in K \).

Let \( dX \) be the Riemannian density on \( \mathfrak{p} \). Then, since \( \Psi_M \) is an isometry,

(5.1) \[ \Psi_M^* dm = dX \otimes dn. \]

Now let the Haar measure \( dg \) be given by the Riemannian metric induced by the inner product on \( \mathfrak{g} \). Let \( dk \) be the Haar measure on \( K \) defined in the same way. By rescaling the inner product on \( \mathfrak{g} \), we can make sure that \( dk \) gives \( K \) unit volume. By Lemma 5.2 below, we have

(5.2) \[ \Psi_{G \times N}^*(dg \otimes dn) = dX \otimes dk \otimes dn. \]

The equalities (5.1) and (5.2) imply that for all \( \varphi \in C_c(M) \),

\[
\int_M \varphi(m)dm = \int_{\mathfrak{p} \times N} \varphi(\exp(X)n)dX \otimes dn
= \int_{\mathfrak{p} \times K \times N} \varphi(\exp(X)n)dX \otimes dk \otimes dn
= \int_{G \times N} \varphi(gn)dg \otimes dn
= \int_{G \times K \times N} \varphi(gn)d[g, n],
\]

where we used the fact that the map \( (g, n) \mapsto gn \) is invariant under the \( K \)-action given by \( k \cdot (g, n) = (gk^{-1}, kn) \). \( \square \)

Lemma 5.2. In the notation of the proof of Lemma 5.1, we have

\[ \Psi_{G \times N}^*(dg \otimes dn) = dX \otimes dk \otimes dn. \]

Proof. One can compute that for all \( X, Y \in \mathfrak{p}, Z \in \mathfrak{k}, k \in K, n \in N \) and \( v \in T_nN \),

\[
T_{(X, K, n)} \Psi_{G \times N}(Y, T_e l_k(Z), v) = \left( T_el_{\exp(X)k^{-1}}(\Ad(k)(Y+Z)), T_nk(\alpha_n(Z)+v) \right).
\]
Here the letter $l$ denotes left multiplication, and for $m \in M$, the map $\alpha_m : g \to T_m M$ is given by the infinitesimal action. Now the maps $T_e l_{\exp(X)k^{-1}}$, $\text{Ad}(k)$ and $T_n k$ preserve the Riemannian metrics on $TG$ and $TN$. So

$$T_{(X,K,n)} \Psi_{G \times N} = A \circ B,$$

where

$$A : T_{(X,K,n)}(p \times K \times N) \to T_{(\exp(X)k^{-1},kn)}(G \times N),$$

given by

$$A(Y, Te l_k(Z), v) = (T_e l_{\exp(X)k^{-1}}(\text{Ad}(k)(Y + Z)), T_n k(v))$$

is an isometry, and the automorphism $B$ of

$$T_{(\exp(X)k^{-1},kn)}(G \times N) \cong p \oplus \mathfrak{t} \oplus T_{kn} N$$

is given by the matrix

$$\text{mat}(B) = \begin{pmatrix}
I_p & 0 & 0 \\
0 & I_t & 0 \\
0 & T_n k \circ \alpha_n & I_{T_{kn} N}
\end{pmatrix},$$

where $I_t$, $I_p$ and $I_{T_{kn} N}$ are the identity maps on the respective spaces, so that $B$ has determinant one.

Since the map $A$ is an isometry, it relates the Riemannian density $dX \otimes dk \otimes dn$ on $p \times K \times N$ to the Riemannian density $dg \otimes dn$ on $G \times N$, at the point $(X, k, n)$. Since the map $B$ has unit determinant, it does not change densities, so the claim follows.

5.2 Cocompact actions

To apply the decomposition (4.12) to the $L^2$-kernel of $D$, we apply Lemma 5.1 in the following way. Note that we do not yet assume $M/G$ to be compact here.

**Lemma 5.3.** In the notation of Corollary 4.6, we have

$$\|\sigma(s \otimes \varphi)\|_{L^2(S)} = \|s\|_{L^2(S_N)} \|\varphi\|_{L^2(G, \Delta_p)},$$

for all $\sigma \in \Gamma_c(S_N)$ and $\varphi \in C_c^\infty(G, \Delta_p)$ such that $s \otimes \varphi$ is $K$-invariant.
Proof. By Lemma 5.1 and $K$-invariance of $s \otimes \varphi$ and of the norm on $S$, and implicitly using a Borel section $G \times K \times N \to G \times N$, one has
\[
\|\sigma(s \otimes \varphi)\|_{L^2(S)}^2 = \int_{G \times K \times N} \|s(n) \otimes \varphi(g)\|_S^2 \, d[g,n]
= \int_{G \times K \times N} \|s(n) \otimes \varphi(g)\|_S^2 \, dk \, d[g,n]
= \int_{G \times N} \|s(n)\|_{S_N}^2 \|\varphi(g)\|_{\Delta_p}^2 \, dg \, dn
= \|s\|^2_{L^2(S_N)} \|\varphi\|^2_{L^2(G,\Delta_p)}.
\]

Corollary 5.4. The map $\sigma$ induces a unitary isomorphism
\[
\ker_{L^2(D)} \cong \ker_{L^2(D_N) \hat{\otimes} L^2(D_{G/K})},
\]
where $\hat{\otimes}$ denotes the completion of the tensor product in the $L^2$-norm, and a bar over an operator denotes its closure in the sense of (1.1).

Proof. Since the algebraic tensor product
\[
\Gamma_c^\infty(S_N) \otimes C_c^\infty(G, \Delta_p)
\]
is dense in
\[
L^2(S_N) \hat{\otimes} L^2(G, \Delta_p),
\]
Corollary 4.6 and Lemma 5.3 imply that $\sigma$ induces a unitary isomorphism
\[
L^2(S) \cong L^2(S_N) \hat{\otimes} L^2(G, \Delta_p).
\]
Since $\sigma$ is unitary, Corollary 4.6 also implies that $\sigma$ intertwines the closures $\overline{D}$ and $\overline{D_N \otimes 1 + \varepsilon 1 \otimes D_{G/K}}$ (including their domains). Therefore, it induces a unitary isomorphism
\[
\ker_{L^2(D)} \cong \ker_{L^2(D_N \otimes 1 + \varepsilon 1 \otimes D_{G/K})}.
\]
Applying the argument in the proof of Corollary 4.7, and taking closures, we find that the right hand side of this equality equals
\[
\ker_{L^2(D_N)} \hat{\otimes} \ker_{L^2(D_{G/K})}.
\]
Corollary 5.4 allows us to prove Proposition 3.3. Indeed, if $M/G$ is compact, then
\[
Q^{\text{Spin}^c}_G(M) = \Theta(\ker L^2(D^+)) - \Theta(\ker L^2(D^-)) \\
= \Theta((\ker(D^+_N) \otimes \ker L^2(D^+_G/K))^K) + \Theta((\ker(D^-_N) \otimes \ker L^2(D^-_G/K))^K) \\
- \Theta((\ker(D^+_N) \otimes \ker L^2(D^+_G/K))^K) - \Theta((\ker(D^-_N) \otimes \ker L^2(D^-_G/K))^K) \\
= (Q^{\text{Spin}^c}_K(N) \otimes \text{index}^G_{L^2}(D_{G/K}))^K.
\]

Here we have used (1.2) and (1.3). Now, by compactness of $N$ and elliptic regularity, the kernel $\ker L^2(D^\pm_N) = \ker(D^\pm_N)$ is finite-dimensional, so it was not necessary to complete the tensor products that appear.

5.3 Non-cocompact actions

We now consider $G$-invariant, transversally $L^2$ sections of $S$, to prove Proposition 3.4.

Lemma 5.5. Restriction to $N$ is a linear isomorphism
\[
\Gamma^\infty(S)^G \cong (\Gamma^\infty(S_N) \otimes \Delta_p)^K.
\]

Proof. Note that for all $s \in \Gamma^\infty(S)^G$ and $n \in N$, we have $s(n) \in (S_N)_n \otimes \Delta_p \cong S_n$. Every $K$-invariant section in $(\Gamma^\infty(S_N) \otimes \Delta_p)^K$ has a unique $G$-invariant extension to a section in $\Gamma^\infty(S)^G$. This is the inverse to the restriction map. \qed

Fix $s \in \Gamma^\infty(S)^G$, and write $s^N := s|_N \in (\Gamma^\infty(S_N) \otimes \Delta_p)^K$. Let $f_G \in C^\infty(G)^K$ be such that
\[
\int_G f_G(g)^2 \, dg = 1
\]
for a Haar measure $dg$. Here the superscript $K$ denotes invariance under right multiplication by $K$. Define the cutoff function $f \in C^\infty(M)$ by
\[
f(gn) = f_G(g),
\]
for $g \in G$ and $n \in N$.

The characterisation of the density $dm$ in Lemma 5.1 allows us to relate transversally $L^2$ sections on $M$ to $L^2$-sections on $N$.  

34
Lemma 5.6. If $s^N \in \Gamma_c^\infty(S_N) \otimes \Delta_p$, then for the Riemannian metric on $M$ and the Haar measure on $G$ as in Lemma 5.1,
\[ \|fs\|_{L^2(S)} = \|s^N\|_{L^2(S_N) \otimes \Delta_p}. \]

Proof. By Lemma 5.1 we have
\[
\|fs\|_{L^2(S)}^2 = \int_M f(m)^2 \|s(m)\|^2 \, dm
= \int_{G \times K_N} f_G(g)^2 \|g^{-1}s^N(n)\|_S^2 \, dg \, dn
= \int_{G \times N} f_G(g)^2 \|s^N(n)\|_{S_N \otimes \Delta_p}^2 \, dg \, dn
= \|s^N\|_{L^2(S_N) \otimes \Delta_p}^2,
\]
where we have used $G$-invariance of the metric $\| \cdot \|_S$ and $K$-invariance of $s^N$.

Corollary 5.7. We have
\[ L^2_T(S)^G \cong (L^2(S_N) \otimes \Delta_p)^K, \]
via the restriction map.

In Proposition 4.1, the operator $D_p$ is zero on $G$-invariant sections. It therefore has the following consequence.

Corollary 5.8. One has
\[ (Ds)|_N = (D_N \otimes 1_{\Delta_p})s^N. \]

Now consider the deformed Dirac operator
\[ D_{v_N} := D_N + c_N(v_N). \]
Because of (3.3), we have $v_N = v|_N$. Therefore,
\[ (c(v)s)|_N = c_N(v_N)s^N \tag{5.3} \]

Corollary 5.7, Proposition 4.1 and (5.3) yield the following conclusion.
Corollary 5.9. We have
\[
\ker_{L^2}(D_v) \cong (\ker_{L^2}(D_{vN}) \otimes \Delta_p)^K.
\]

To deduce equalities between indices rather than kernels, we consider the gradings on the various bundles. Note that
\[
\begin{align*}
\Gamma^\infty(S^+)^G &\cong (\Gamma^\infty(S^+_N) \otimes \Delta^+_p)^K \oplus (\Gamma^\infty(S^-_N) \otimes \Delta^-_p)^K; \\
\Gamma^\infty(S^-)^G &\cong (\Gamma^\infty(S^+_N) \otimes \Delta^-_p)^K \oplus (\Gamma^\infty(S^-_N) \otimes \Delta^+)^K.
\end{align*}
\]

This allows us to prove Proposition 3.4. Indeed, by Corollary 5.9 and (5.4), we have
\[
\begin{align*}
\text{index}_{L^2}(D_v^+) &= \dim(\ker_{L^2}(D_{vN}^+) \otimes \Delta^+_p)^K + \dim(\ker_{L^2}(D_{vN}^-) \otimes \Delta^-_p)^K \\
&\quad - \dim(\ker_{L^2}(D_{vN}^+) \otimes \Delta^-_p)^K - \dim(\ker_{L^2}(D_{vN}^-) \otimes \Delta^+_p)^K \\
&= (\text{index}_{L^2}(D_{vN}) \otimes \pi_p)^K \\
&= [\text{index}_{L^2}(D_{vN}^+) : \pi_p].
\end{align*}
\]
In the last step, we used that \(\pi_p \cong \pi^*_p\), since the characters of these two representations coincide (see Remark 2.2 in [Parthasarathy]).

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