Twisting tensor and spin squeezing

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A unified tensor description of quadratic spin squeezing interactions is proposed, covering the single- and two-axis twisting as two special cases of a general scheme. Equations of motion of the first moments and variances are derived and their solutions are discussed from the perspective of fastest squeezing generation. It turns out that the optimum rate of squeezing generation is governed by the difference between the largest and the smallest eigenvalues of the twisting tensor. A cascaded optical interferometer with Kerr nonlinear media is proposed as one of possible realizations of the general scheme.

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Introduction.— Suppressed noise in two-mode multiparticle systems known as “spin squeezing” is an essential tool in quantum metrology protocols [1–4]. The interferometric schemes utilizing this effect cover broad area of possible physical systems, ranging from collective spins of neutral atoms interacting by collisions [5, 6], atoms interacting with light by Faraday rotation and ac-Stark shift [7], atoms interacting by Rydberg blockade [8], polarized light [9], to Bose-Einstein condensates (BEC) in double-well potentials (bosonic Josephson junctions) [10–13]. Typically, the preparation of spin squeezed states is based on nonlinear inter-particle interactions. In terms of the collective “spin” operators \( \vec{J} \), the procedures have been classified as “one-axis twisting” with a term \( J_z^2 \), and “two-axis counter-twisting” with a term \( J_x^2 - J_y^2 \) [1].

The two-axis counter-twisting was shown to be more efficient to produce highly squeezed states. Since the two-axis counter twisting is more challenging to produce experimentally, a scheme has been proposed to combine a sequence of single-axis twisting and spin rotations to an effectively two-axis counter-twisting procedure [14]. Efficient preparation of spin-squeezed states has become an objective of various optimized procedures [15]. Here I show that any quadratic interaction in the collective spin can be described by means of a twisting tensor, encompassing the single-axis twisting and two-axis counter-twisting as special cases. The model is illustrated by an example with coupled optical resonators with Kerr media. Equations of motion for the first and second moments in the Gaussian approximation are used to show how squeezing is generated in various cases of the twisting tensor. At the initial stage, the maximum squeezing rate only depends on the difference between the maximum and minimum eigenvalues of the twisting tensor. For certain times, deviations from the optimum squeezing rate can be compensated by suitable rotations. The results are applicable for optimizing strategies of interferometric measurements with various nonlinear media.

System Hamiltonian and equations of motion.— Consider a two-mode bosonic system described by annihilation operators \( a \) and \( b \) with total number of particles \( N \) conserved. The dynamics can be expressed by operator \( \vec{J} \) defined as \( J_x = \frac{1}{2}(a^\dagger b + ab^\dagger) \), \( J_y = \frac{1}{2i}(a^\dagger b - ab^\dagger) \), \( J_z = \frac{1}{2}(a^\dagger a - b^\dagger b) \), with \( N = a^\dagger a + b^\dagger b \). The components of \( \vec{J} \) satisfy the angular momentum commutation relations \( [J_x, J_y] = iJ_z \), \( [J_y, J_z] = iJ_x \), and \( [J_x, J_z] = iJ_y \). Let the Hamiltonian be composed of \( a, b, a^\dagger \), and \( b^\dagger \) such that in each term the same number of creation and annihilation operators occurs (total number of particles is conserved), and the highest power of each operator is 2.

The Hamiltonian then can be written as

\[
H = \omega_k J_k + \chi_{kl} J_k J_l + f(N),
\]

where \( \omega_k \) and \( J_k \) transform as vectors and \( \chi_{kl} = \chi_{lk} \) transforms as a tensor under \( O(3) \) rotations. Here \( k, l \in \{x, y, z\} \), and the Einstein summation is used. In Eq. (1), \( f(N) \) is a linear or quadratic function of the total particle number, generating an unimportant overall phase. Let us call \( \chi \) the twisting tensor and note that the special case of \( \chi_{k,l} = 0 \) for \( k \neq l \neq z \) and \( \chi_{z,z} \neq 0 \) corresponds to the one-axis twisting scenario, and the case \( \chi_{xx} = -\chi_{yy} \neq 0 \), \( \chi_{xy} = 0 \) otherwise, corresponds to the two-axis counter-twisting scenario of [1]. Since \( J_x^2 + J_y^2 + J_z^2 = \frac{N}{2}(N+1) \), addition of an arbitrary multiple of unit matrix to \( \chi \) can be absorbed in the unimportant term \( f(N) \). Therefore, any diagonal \( \chi \) in which one element is exactly in the middle of the remaining two elements also corresponds to the two-axis counter-twisting.

Using the Heisenberg equations of motion, \( i\dot{A} = [A,H] \), and calculating the mean values of the operators, we arrive at the equations for \( \dot{J}_j \equiv \langle \dot{J}_j \rangle \)

\[
\dot{J}_j = \epsilon_{jkl} [\omega_k J_l + 2\chi_{kn}(J_n J_l + V_{nl})],
\]

where \( V \) is the variance tensor, \( V_{nl} \equiv \frac{1}{2} \langle (J_n - J_l)(J_l - J_n) \rangle + \langle J_l - J_l \rangle \langle J_n - J_n \rangle \). The equations of motion for \( V_{nl} \) can be obtained in a similar way, however, in this case mean values of cubic terms such as \( \langle J_k J_x J_p \rangle \) occur. Our approximation is based on the assumption that the distribution of the components \( J_k \) is close to Gaussian for which all higher moments are functions of the first and second moments. In particular, we express the third moments as \( \langle J_k J_x J_p \rangle \approx \langle J_k J_x J_p \rangle + \langle J_k V_{sp} + J_x V_{pk} + J_p V_{ks} \rangle \). In this
Thus, e.g., light passing through the medium in branch a experiences a phase shift proportional to the intensity of the field. For each of the four branches there is a Kerr medium inducing a phase shift proportional to $\gamma_a |a|^2$, etc. This setup leads to the Hamiltonian of Eq. (1).

A special case of $\gamma_a = \gamma_b = \gamma_z$, and $\gamma_c = \gamma_d = \gamma_x$ reduces the Hamiltonian to $H = \frac{1}{2} \left( \gamma_z J_z^2 + \gamma_x J_x^2 \right) + f(N)$. Note that already this simplest form of interaction covers all the main categories of spin squeezing: the one-axis twisting ($\gamma_x = 0$ or $\gamma_z = 0$), the two-axis counter-twisting ($\gamma_x = 2\gamma_z$ or $\gamma_z = 2\gamma_x$), or more general twistings (other relations between $\gamma_x$ and $\gamma_z$). Rotating terms of various $\omega_{x,y,z}$ of Hamiltonian (1) can be introduced by shifting the positions of the mirrors and/or by tuning the parameters of the beam splitter. Any more general form of the twisting tensor, including off-diagonal terms, can be achieved by chaining the beam splitters and nonlinear zones as in Fig. (b).

**Squeezing rate.**— Let us first assume that the coordinate system is chosen such that the state is centered at the pole of the Poincaré sphere with $J_x = J_y = 0$. From (3) we find

$$V_{xx} = \left[ -2\omega_x + 4(\chi_{yy} - \chi_{zz})J_z \right] V_{yy} + 4\chi_{xy} J_z V_{xx},$$

$$V_{yy} = \left[ 2\omega_y - 4(\chi_{xx} - \chi_{zz})J_z \right] V_{xy} - 4\chi_{xy} J_z V_{yy},$$

$$V_{xy} = \omega_z (V_{xx} - V_{yy}) + 2 J_z [(\chi_{zz} - \chi_{xx})V_{xx} - (\chi_{xx} - \chi_{yy})V_{yy}].$$

The variance matrix $V_{kl}$, $k, l = x, y$ can be expressed as the rotated diagonal matrix of principal variances $V_{\pm}$,

$$\begin{pmatrix} V_{xx} & V_{xy} \\ V_{xy} & V_{xx} \end{pmatrix} = U \begin{pmatrix} V_- & 0 \\ 0 & V_+ \end{pmatrix} U^\dagger,$$

where

$$U = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix},$$

and $V_{\pm} = \frac{1}{2} (V_{xx} + V_{yy}) \pm \sqrt{V_{xx}^2 + (V_{xx} - V_{yy})^2}/4$. On using Eqs. (3)-(8) one finds

$$V_{\pm} = \pm 2 J_z [\chi_{yy} - \chi_{xx}] \sin 2\alpha - 2 \chi_{xy} \cos 2\alpha] V_{\pm}.$$  

This equation shows how the rate of squeezing generation depends on the orientation angle of the squeezed state. The optimum rate occurs for $\alpha$ satisfying

$$\tan 2\alpha = \frac{\chi_{xx} - \chi_{yy}}{2\chi_{xy}}.$$  

FIG. 1: (color online). A possible scheme realizing the interaction. (a) Modes of two crossed resonators mix at a balanced beam splitter according to $c = (a+b)/\sqrt{2}$, and $d = (a-b)/\sqrt{2}$, and each of the four beams propagates through a nonlinear medium which changes its phase such that the phase of mode $a$ is proportional to $\gamma_a |a|^2$, etc. This setup leads to the Hamiltonian of Eq. (1). (b) Chaining of the mode mixing plus nonlinearities leads to the general form of Hamiltonian (1).
for which \( \dot{\chi}^{(\text{opt})} = \pm Q V_{\pm} \), where

\[
Q = 2|J_2| \sqrt{(\chi_{xx} - \chi_{yy})^2 + 4\chi_{xy}^2} \tag{12}
\]

is the optimum squeezing rate. Note that \( Q \) is invariant with respect to rotations of the coordinate system around the \( J_z \) axis.

So far, the coordinate system was chosen such that the state was centered at the pole of the Poincaré sphere. To find the maximum squeezing rate and optimum variance orientation for arbitrary location of the state, we have to transform the components of the twisting tensor. For simplicity, we now choose the coordinate system oriented in the main directions of the twisting tensor, i.e., such that \( \chi \) is diagonal. Two angles, \( \theta \) and \( \varphi \), determine the direction of the state as follows (see Fig. 2b)): the state starts centered at \( J_x = J_y = 0 \), the system is then rotated around \( J_z \) by \( \varphi \) and then around the new \( J_y \) by \( \theta \). Calculating the elements of \( \chi \) in the new coordinates

and using Eq. 12, one finds

\[
Q = 2|J_2| \left\{ \left[ \chi_x (\cos^2 \theta \cos^2 \varphi - \sin^2 \varphi) + \chi_y (\sin^2 \theta \sin^2 \varphi) + \chi_z \sin^2 \varphi \right]^2 + 4(\chi_x - \chi_y)^2 \cos^2 \varphi \sin^2 \varphi \right\}^{1/2}, \tag{13}
\]

where for the diagonal \( \chi \) we used \( \chi_x = \chi_{xx} \), etc. If \( \alpha \) denotes the orientation angle of optimum squeezing with respect to the parallels on the Poincaré sphere, one finds

\[
\tan 2\alpha = \frac{\chi_x (\cos^2 \theta \cos^2 \varphi - \sin^2 \varphi) + \chi_y (\sin^2 \theta \sin^2 \varphi) + \chi_z \sin^2 \varphi}{\chi_y - \chi_x \cos \theta \sin 2\varphi}^{-1}. \tag{14}
\]

From Eq. 13 one can find for which directions (\( \theta, \varphi \)) the squeezing rate is maximum and for which it is zero. Let us first consider the general case when the three eigenvalues of \( \chi \) are all different (Fig. 2f). Let us assume, e.g., \( \chi_y < \chi_z < \chi_x \). Then there are four points where \( Q = 0 \), all at the equator of the Poincaré sphere, \( \theta = \pi/2 \), and \( \sin^2 \varphi = (\chi_z - \chi_y)/(\chi_x - \chi_y) \). The maximum of \( Q \) is achieved at the poles at \( \theta = 0 \) and \( \varphi = \pm \pi/2 \) (Fig. 2d). The most important general result is that the maximum achievable squeezing rate only depends on the difference between the maximum and minimum eigenvalues of the twisting tensor.

Another example is degenerate \( \chi \), with, say \( \chi_y = \chi_z \), \( \chi_x > 0 \) (one-axis twisting, Fig. 2b). In this case there are two zeros of \( Q \) located at \( \varphi = \pm \pi/4 \) and \( \pm 3\pi/4 \). Another point of interest is the point where \( \varphi \) is kept at the meridian \( \varphi = \pm \pi/2 \) with \( Q = 2|J_2| \chi_x \).

Approximate solution of the equations of motion.—For choosing a good squeezing strategy for an initially spin coherent state, it is instructive to find approximate analytical solutions of the set of Eqs. 2, 3. For simplicity we choose the coordinate system such that the initial state is centered at the pole of the Poincaré sphere \( J_x = J_y = 0 \) oriented such that the off-diagonal term \( \chi_{xy} \) vanishes. Let us further assume that the frequency components \( \omega_x \) and \( \omega_y \) are chosen such that the system is centered at the pole (this is simply \( \omega_{x,y} = 0 \) if \( \chi_{xx} = \chi_{yy} = 0 \), but otherwise nontrivial expression for \( \omega_{x,y} \) has to be used to compensate for the Bogoliubov backreaction). It is convenient to introduce scaled variables \( v_{kl} \), \( j \) and \( \tau \) as \( v_{kl} \equiv (N/4)w_{kl} \), \( J_2 \equiv (N/2)j \), \( \omega_z \equiv N\omega \), and \( \tau \equiv \tau/N \). Thus we get a closed set of...
The results are illustrated in Fig. 3a. Two special cases longer times these results can only be used as long as counter-twisting case. One should keep in mind that for longer times it drops to zero much faster in the two-axis difference of the biggest and smallest eigenvalues of cases the squeezing evolves at the same rate given by the zero and one can take \( \chi \equiv \frac{\chi_y}{N} \) equations

\[
\frac{dv_{xx}}{d\tau} = 2 \left[ -\dot{\omega} + (\chi_y - \chi_x)j \right] v_{xy},
\]

\[
\frac{dv_{yy}}{d\tau} = 2 \left[ \dot{\omega} - (\chi_x - \chi_y)j \right] v_{xy},
\]

\[
\frac{dv_{xy}}{d\tau} = \dot{\omega} (v_{xx} - v_{yy})
\] + \( j \left[ (\chi_x - \chi_y)v_{xx} - (\chi_z - \chi_y)v_{yy} \right],
\]

\[
\frac{dj}{d\tau} = \frac{1}{N} (\chi_x - \chi_y)v_{xy}.
\]

In the limit of \( N \to \infty \), the derivative \( dj/d\tau \) approaches zero and one can take \( j = \pm 1 \) and solve the equations with the initial condition of the spin coherent state \( v_{xx}(0) = v_{yy}(0) = 1, \ v_{xy}(0) = 0 \). Let us assume \( \chi_x \geq \chi_z \geq \chi_y \). For convenience we define \( \chi_x - \chi_z \equiv \Delta \chi_z, \chi_z - \chi_y \equiv \Delta \chi_y \), and \( \Delta \chi = 2\sqrt{\Delta \chi_z \Delta \chi_y} \). In the special case of \( \dot{\omega} = 0 \) we find

\[
v_{xx} = \frac{1}{2\Delta \chi_x} \left[ (\Delta \chi_x + \Delta \chi_y) \cosh(\Delta \chi \tau) + \Delta \chi_x - \Delta \chi_y \right],
\]

\[
v_{yy} = \frac{1}{2\Delta \chi_y} \left[ (\Delta \chi_x + \Delta \chi_y) \cosh(\Delta \chi \tau) + \Delta \chi_y - \Delta \chi_x \right],
\]

\[
v_{xy} = \frac{\Delta \chi_x + \Delta \chi_y}{2\sqrt{\Delta \chi_x \Delta \chi_y}} \sinh(\Delta \chi \tau),
\]

from which we find the squeezing parameter \( \xi^2 = \frac{1}{2} (v_{xx} + v_{yy}) - v_{xy}^2 (v_{xx} - v_{yy})^2 / 4 \)

\[
\xi^2 = -\frac{1}{4\Delta \chi_x \Delta \chi_y} \left\{ (\Delta \chi_x + \Delta \chi_y)^2 \cosh(\Delta \chi \tau) - \frac{1}{2} \left[ (\Delta \chi_x - \Delta \chi_y)^2 - \left( [\Delta \chi_x + \Delta \chi_y]^2 \cosh^2(\Delta \chi \tau) - 2(\Delta \chi_x - \Delta \chi_y)^2(\Delta \chi_x + \Delta \chi_y)^2 \cosh(\Delta \chi \tau) + (\Delta \chi_x - \Delta \chi_y)^4 - 16\Delta \chi_x^2 \Delta \chi_y^2 \right]^{1/2} \right\}.
\]

The results are illustrated in Fig. 3a. Two special cases are worth mentioning, first, in the one-axis twisting with \( \Delta \chi_y = 0 \) (green broken line with symbol “∞” in Fig. 3b) the squeezing is \( \xi^2 = 1 - \Delta \chi_x \tau \sqrt{1 + (\Delta \chi_x \tau)^2} / 4 + \frac{1}{2} (\Delta \chi_x \tau)^2 \). For short times this expression drops linearly as \( 1 - \Delta \chi_x \tau \), and for long times it approaches zero as \( 1/(\Delta \chi_x \tau)^2 \). The second special case is two-axis counter-twisting with \( \Delta \chi = 2\Delta \chi_x = 2\Delta \chi_y = \chi_x - \chi_y \) (blue full line with symbol “∞” in Fig. 3b) when we get \( \xi^2 = \exp(-\Delta \chi \tau) \). Although at the beginning for the two cases the squeezing evolves at the same rate given by the difference of the biggest and smallest eigenvalues of \( \chi \), for longer times it drops to zero much faster in the two-axis counter-twisting case. One should keep in mind that for longer times these results can only be used as long as the used approximations are valid (the deviation of the exact values for various finite \( N \) from these approximate solutions can be seen in Fig 3b).

Optimum rotation.— So far the special case of \( \dot{\omega} = 0 \) has been considered. To generate squeezing at the maximum rate, one has to keep the state optimally oriented with respect to the main twisting axes, so that \( \alpha = \pi/4 \) (see Eq. 14 with \( \chi_{xy} = 0 \)). This leads to \( v_{xx} = v_{yy} = (v_+ + v_-)/2 \) and \( v_{xy} = (v_+ - v_-)/2 \). It follows that \( dv_{xx}/d\tau = dv_{yy}/d\tau \), and from Eqs. 13 and 16 that the optimum rotation frequency should satisfy

\[
\dot{\omega} = j \left( \frac{\chi_y + \chi_x}{2} - \chi_z \right)
\]

Thus, for the exact two-axis counter-twisting with \( \chi_z = (\chi_x + \chi_y)/2 \) no rotation is needed to achieve the optimum squeezing rate. For any other values of the twisting parameters one needs to keep the variance ellipse optimally oriented by means of suitable rotation frequency.

The evolution of the covariance matrix then follows from Eqs. 15–17 as \( dv_{xx}/d\tau = dv_{yy}/d\tau = j(\chi_y - \chi_x)v_{xy} \), and \( dv_{xy}/d\tau = j(\chi_y - \chi_x)v_{xx} \). If the sys-
tem starts in the spin coherent state with $v_{xx}(0) = v_{yy}(0) = 1$, and $v_{xy}(0) = 1$, one finds $v_{xx} = v_{yy} = \cosh[\chi_x - \chi_y \tau]$, $v_{xy} = \sinh[\chi_x - \chi_y \tau]$, and $\xi^2 = \exp\left[-(\chi_x - \chi_y \tau)\right]$. These results are illustrated in Fig. 3b. Note that the Gaussian approximation with finite $N$ works for relatively short times, after which the state undergoes an $S$-shape deformation and the squeezing is deteriorated (see inset of Fig. 3b). These results hint for which parameters it might be suitable to apply the additional rotation during the squeezing preparation stage, e.g., the data of $N \approx 400$, $\chi \tau \approx 3$ suggest a possible room for further optimization by this means.

**Conclusions.**— The twisting tensor approach is suitable for two-mode systems with quadratic nonlinearities, such as two-mode optical resonators with Kerr media, BEC in structured traps, collective atomic spins, etc. As special cases it covers the one-axis twisting and two-axis counter-twisting scenarios of spin squeezing. At the early stages of squeezing, the most relevant parameter is the difference between the maximum and minimum eigenvalues of the twisting tensor which determines the squeezing rate. For longer times, most efficient squeezing is achieved if the middle eigenvalue halves the interval between the extreme ones (two-axis counter-twisting). In other cases, for certain times the imbalance of the middle eigenvalue can be compensated by suitable rotation. The results, relevant mostly for quantum metrology and interferometry, can be further generalized to cover losses and decoherence and various squeezing optimization strategies.

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