A 2-variable power series approach to the Riemann hypothesis

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Abstract: We consider the power series in two complex variables

\[ B_y(f^\flat)(x) = \sum_{n \geq 0} A^\flat_n x^n y^{n(n+1)/2}, \]

where \((-1)^n A^\flat_n\) are the non-zero coefficients of the Maclaurin series of the Riemann Xi function. The Riemann hypothesis is the assertion that all zeros of \(B_y(f^\flat)\) are real. We prove that every zero of \(B_y(f^\flat)\) is the inverse of a power series in \(y\) with real coefficients, which converges for \(|y| < 2078\). We show the existence of a constant \(\Theta\), similar to the de Bruijn-Newman constant, satisfying \(0 \leq y \leq \Theta\) if and only if all zeros of \(B_y(f^\flat)\) are real. We prove that \(1/4 \leq \Theta \leq 1\) and that \(\Theta = 1\) is equivalent to the Riemann hypothesis. We show that the Riemann hypothesis is equivalent to what the discriminant of each Jensen polynomial of \(B_y(f^\flat)\) does not vanish on the interval [1/4, 1]. We prove the Riemann hypothesis implies that the zeros of \(B_y(f^\flat)\) are simple for \(0 < y < \frac{1}{4}\), and we conjecture that the reciprocal implication is true.

keywords: Riemann hypothesis, two complex variables, Laguerre entire functions, de Bruijn-Newman constant, analytic continuation, simple zeros.

1 Introduction

Following standard practice, we define the Riemann Xi function as

\[ \Xi(z) = \frac{1}{2} s(s-1) \pi^{-s/2} \Gamma(s/2) \zeta(s); \quad s = \frac{1}{2} + iz \]

where \(\zeta\) is the Riemann zeta function. It is well known ([15] chap.1, 10) that \(\Xi\) is an entire real function of order 1 and that the Riemann hypothesis (abbreviated RH) is the assertion that all zeros of \(\Xi\) are real. We have

\[ \Xi(z) = \sum_{n=0}^{\infty} (-1)^n A^\flat_n z^{2n} \]

the coefficients \(A^\flat_n\) being some strictly positive numbers ([9], p.41). Introduce the auxiliary function

\[ f^\flat(x) = \Xi(-i\sqrt{x}) = \sum_{n=0}^{\infty} A^\flat_n x^n \]

\(f^\flat\) is an entire real function of order 1/2 and factorizes therefore, according to the Hadamard theorem, as

\[ f^\flat(x) = A^\flat_0 \prod_{p \geq 1} \left(1 - \frac{x}{x_p}\right) \]

Thus, RH is also the assertion that all the zeros \(x_p\) of \(f^\flat\) are negative numbers (in fact, it suffices to show that the zeros \(x_p\) are real since the coefficients \(A^\flat_n\) are positive).

In [3] a new method was introduced by the author, to determine the value and the reality of zeros of an entire function, whose all coefficients of its Maclaurin series are non-zero real numbers (this method has been generalized in [4] to the functions which may be represented by a Laurent series convergent in \(\mathbb{C}^*\)). It was therefore natural to apply this method to the function \(f^\flat\) that satisfies these characteristics (note that this is why we introduce the auxiliary function \(f^\flat\), rather than using the function \(\Xi\) that does not check these characteristics). This is the purpose of this article, whose here is the plan and the main results. In Section 2, we recall the principle of the method introduced in [3], in defining a functional transformation called \(y\)-Borel transform and denoted by \(B_y\). By direct application of Theorem 16 of [3], we show that the inverse of zeros

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of \( B_y(f^q) \) may be represented by power series in \( y \), convergent in a disk with center 0 and radius equal to \( \rho_0^2 = 0.2078 \ldots \) where \( \rho_0 \) is the positive root of the equation \( \sum_{k=1}^{\infty} \rho^k = 1/2 \) (Theorem 2.2). In the following sections, we limit ourselves to study \( B_y(f^q) \) for \( y \in \mathbb{R}_+ \). In Section 3, we recall some properties of the Laguerre entire functions, necessary for the rest of the article. In Section 4, we define from \( B_y(f^q) \) a constant \( \Theta \), whose Theorem 4.3 shows that it plays a role similar to the de Bruijn-Newman constant. In particular, the point iv) of Theorem 4.3 is a reformulation of RH. In Section 5, based on the results already established, we get another reformulation of RH by using the theory of analytic continuation along a path (Theorem 5.1). In Section 6, we prove (Corollary 6.2) that the Riemann hypothesis implies that the zeros of \( B_y(f^q) \) are simple for \( y \in [0,1] \) and we discuss the reciprocal implication.

**Remark 1** It is easy to see that RH is equivalent to what \( f^q \) is a Laguerre entire function (see Section 3 below for the definition of these functions). In fact, many of the necessary or sufficient conditions for the RH, that we exhibit in this article, follow from conditions of which we will demonstrate the necessity or the sufficiency, in order that an entire real function may be a Laguerre entire function. So, our article can also be considered from this point of view. We can say that our approach to the RH is an "external approach", in the words of Balazard in Section 3 of [1]. The theory of the Fourier transforms with real zeros (see [1] Section 3.2 with, in particular, Polya’s results), is a well known "external approach" of RH. These two "external approaches" present similarities as we will see below in Section 4.3.

## 2 Direct application of the method introduced in [3]

To present this method, it is convenient to introduce a functional transformation:

### 2.1 Formal \( y \)-Borel transform

**Definition 1** Let \( \mathbb{K} \) be a field and \( f(x) = \sum_{n \in \mathbb{N}} A_n x^n \in \mathbb{K}[[x]] \). We will call \( y \)-Borel transform of \( f \), the formal power series in \( x \) and \( y \)

\[
B_y(f)(x) = Q(x,y) = \sum_{n \in \mathbb{N}} A_n x^n y^{n(n+1)/2} \tag{5}
\]

This definition is still valid when \( y \) is a fixed element of \( \mathbb{K} \), \( B_y(f) \) is then a formal power series in \( x \) depending on the parameter \( y \). Note that we have in particular \( B_1(f) = f \). When \( \mathbb{K} = \mathbb{C} \), these definitions and notations extend naturally to cases of convergent power. In the following, we use the notation \( B_y(f) \) in general, but it may happen that we use the notation \( Q(x,y) \) when we consider the \( y \)-Borel transform as a function of two variables.

If we change variables \( y = 1/q, x = \xi q \) (here \( \xi \) is a simple variable) in the equation (5), \( B_y \) reduces to a \( q \)-analogue of the formal Borel transform, introduced by Ramis (see eg [16], Definition 4.4). The relationship is clearly between \( B_y \) and the \( q \)-Borel transform as in Ramis assume \( |q| \geq 1 \), whereas for us the interesting case is \( |y| \leq 1 \). Note however, that our results are independant of the theory of \( q \)-difference equations, for which the \( q \)-formal Borel transform was introduced by Ramis.

Note also that we have the following relation valid for \( y \in \mathbb{K} \) and \( y_0 \in \mathbb{K}^* \)

\[
B_y(f) = B_{y/y_0}(B_{y_0}(f)) \tag{6}
\]

and the following convergence properties valid for \( \mathbb{K} = \mathbb{C} \):

**Proposition 2.1** If for \( y_0 \in \mathbb{R}_+^* \), \( B_{y_0}(f) \) is an entire function, then \( B_y(f) \) is an entire function for \( |y| \leq y_0 \). Furthermore, when \( y \to y_0, 0 \leq y < y_0 \), \( B_y(f) \) converges uniformly to \( B_{y_0}(f) \) on every compact subsets of \( \mathbb{C} \), and the zeros of \( B_{y_0}(f) \) are the limits of zeros of \( B_y(f) \).
Proof. Suppose first that \( y_0 = 1 \). Then, it is a special case of Lemma 19 of [3]. So, we summarize the proof briefly. Let \( R \) be any positive number, for \( |x| \leq R \) and \( y \in \overline{D}(0,1) = \{ y \in \mathbb{C} : |y| \leq 1 \} \), we have \( |A_n x^n y^{n(n+1)/2}| \leq |A_n| R^n \), then the power series \( \sum A_n x^n y^{n(n+1)/2} \) converges normally with respect to \( |x| \leq R \) and \( |y| \leq 1 \). \( Q(x,y) \) is therefore a continuous function in \( \mathbb{C} \times \overline{D}(0,1) \). It follows that \( Q(x,y) \) is uniformly continuous on each \( K \times \overline{D}(0,1) \), where \( K \) is an arbitrary compact of \( \mathbb{C} \). We deduce that \( B_y(f) \) is an entire function for \( |y| \leq 1 \) and \( B_y(f) \) converges to \( B_1(f) = f \) uniformly with respect to \( x \) on each compact subsets of \( \mathbb{C} \), when \( y \to 1 \), \( y \) remaining in \( \overline{D}(0,1) \). Then we prove that, similarly to the Hurwitz theorem, the zeros of \( f \) are the limits of zeros of \( B_y(f) \) when \( y \to 1 \), \( y \) remaining in \( \overline{D}(0,1) \). This is a fortiori true when \( y \to 1 \) along the radius \([0,1]\).

For any \( y_0 \in \mathbb{R}^*_+ \), we reduce to the previous case using the relationship (6). ■

2.2 Summary of the method introduced in [3]

Let \( \mathbb{K} \) be a field of characteristic zero and \( f(x) = \sum_{n \geq 0} A_n x^n \in \mathbb{K}[[x]] \) with \( A_n \neq 0 \) for all \( n \). By considering \( B_y(f) \) as a formal power series in \( x \) over the field \( \mathbb{K}((y)) \), it was shown (Theorem 1 of [3]) that the zeros of \( B_y(f) \) are the elements of \( \mathbb{K}((y)) \), given for \( p \geq 1 \) by

\[
x_p(y) = -\frac{1}{\alpha_p(y)}
\]

where \( \alpha_p(y) \in \mathbb{K}[[y]] \) is given by

\[
\alpha_p(y) = \sum_{q \geq p} u_p(q) y^q
\]

and

\[
u_p(q) = v_p(q) - v_{p+1}(q)
\]

in particular

\[
u_p(p) = \frac{A_p}{A_{p-1}}.
\]

Further, Theorem 15 of [3] shows that for \( p \geq 1 \), \( q \geq p \)

\[
u_p(q) = \left[x^1 y^{q-(p-1)}\right] LogQ_p(x,y)
\]

with

\[Q_p(x,y) = \frac{y^{(p-2)(p-1)/2}}{A_{p-1} x^{p-1}} Q(x,y).
\]

In the equation (10), \( LogQ_p \) denotes the extended composition, as it was defined in Section 4 of [3], of the formal power series \( Q_p(x,y) \in \mathbb{K}((x)) [[y]] \) by \( Log(1+z) \in \mathbb{K}[[z]] \). And we use the notation \([x^i y^j]\) to denote the coefficient in \( x^i y^j \) in the 2-variable power series which follows this symbol. We thus get \( v_p(q) \) (Section 5 of [3]), as a product of \( A_1/A_0 \) by a polynomial expression in the "variables" \( \Omega_n \) defined for \( n \geq 1 \), by

\[
\Omega_n = \frac{A_{n-1} A_{n+1}}{A_n^2}.
\]

Suppose now that \( \mathbb{K} = \mathbb{C} \). When \( R > 0 \), we say that \( f \) satisfies the property \( A(R) \) if for all real number \( r \), \( 0 \leq r < R \), we have

\[
\sum_{p \geq 1, q \geq p} v_p(q) r^q < +\infty
\]

We then put down \( R^* = \sup \{ R > 0 / A(R) \) is true \} if this set is not empty, and \( R^* = 0 \) otherwise.

It has been shown (Theorem 3 of [3]) for \( y \in \mathbb{C} \) such that \( |y| < R^* \), that the power series \( \alpha_p(y) \) converge as well as the infinite product

\[
B_y(f)(x) = A_0 \prod_{p \geq 1} (1 + \alpha_p(y) x).
\]

The zeros of \( B_y(f) \) are therefore exactly given by Equation (7) for \( |y| < R^* \), \( p \) taking all values such that \( \alpha_p(y) \neq 0 \).
Using a method of majorant series, it was also shown (Theorem 16 of [3]) that if we put $\Omega = \sup_{n \geq 1} |\Omega_n|$, we have $R^* > \rho_o^2 \Omega^{-1}$ where $\rho_o$ is the positive root of the equation in $\rho$

$$\sum_{k=1}^{\infty} \rho^{k^2} = 1/2.$$  \hspace{1cm} (13)

In particular, if all coefficients $A_n$ are real and if $0 < \rho_o^2 = 0.2078...$, all the zeros of the entire function $f$ are real (corollary 18 of [3]).

Note that if $f$ is a polynomial of degree $n$ whose all coefficients are different from zero, the method outlined above is fully applicable for the $n$ zeros $x_p(y)$, $1 \leq p \leq n$, with the following convention: $\Omega_k = 0$ for $k \geq n$. In [4], these results were extended to the case of Laurent series $\sum_{n \in \mathbb{Z}} A_n x^n$, convergent on $\mathbb{C}^*$. Finally, note that the factorization of $Q(x, y)$ in infinite product and the formula (10) can be generalized to the case where:

$$Q(x, y) = \sum_n A_n(y) x^n y^{n(n+1)/2}$$

where $A_n(y)$ is now a series in $y$ with a valuation equal to zero, [5]. This latter result can be easily applied to factorize a large number of $q$-series in an infinite product.

2.3 Application to the function $f^\circ$

As noted in the introduction, the coefficients $A_n^\circ$ of the entire function $f^\circ$ are real and different from zero. We can therefore apply the method described above. This gives the following new theorem:

**Theorem 2.2** Let $\rho_o$ be the positive root of equation (13) and $f^\circ$ be the entire function defined by equation (3). Then the zeros of the $y$-Borel transform of $f^\circ$ can be calculated explicitly by the formulas (7), (8), (10) for $0 < |y| < \rho_o^2 = 0.2078...$ and they are all real for $0 < y \leq \rho_o^2$.

**Proof.** In [8] (Csordas and al, 1986), it was proved for $n \geq 1$, the double inequality

$$1 \leq \frac{b_{n-1}b_{n+1}}{b_n^2} \leq \frac{2n+1}{2n-1}$$

where the coefficients $b_n$ are related to our coefficients $A_n^\circ$ by $b_n/(2n)! = A_n^\circ/2^{n+3}$. It follows that

$$\left(\frac{2n-1}{2n+1}\right)^{n+1} \leq \Omega_n^\circ = \frac{A_n^\circ}{(A_n^\circ)^2} \leq \frac{n}{n+1}$$

(14)

(the inequality on the right side of (14) is sometime called the Turan inequality). Proof is now a direct application of Theorem 16 of [3] to the function $f^\circ$ with $\Omega = \sup_{n \geq 1} |\Omega_n^\circ| = 1$ and gives $R^* \geq \rho_o^2$. So, for $|y| < \rho_o^2$, the power series $a_p(y)$ converge for all $p \geq 1$, and we get all the zeros of $B_y(f^\circ)$ with (7) by taking all values of $p \geq 1$ such as $a_p(y) \neq 0$. Further, as the coefficients $A_n^\circ$ are real numbers, it is the same for $v_p(y)$ then for $a_p(y)$, and therefore for all the zeros of $B_y(f^\circ)$ when $y \in [0, \rho_o^2]$. It follows from proposition 2.1 that the zeros of $B_y(f^\circ)$ are also real numbers when $y = \rho_o^2$.

Assume now we can show that $R^* \geq 1$ for $f^\circ$. Then the results of Theorem 2.2 extend to the disk $|y| < 1$. Specifically, Theorem 16 of [3] shows that all series $a_p(y)$ converge and give by the equation (7), the zeros of $B_y(f^\circ)$ for $|y| < 1$. We then show as above that all zeros of $B_y(f^\circ)$ are real for $y \in [0, 1]$, and, in particular, that RH is true. Hence, we just prove the implication:

$$R^* \geq 1 \text{ for the function } f^\circ \implies \text{ RH is true}$$

(15)

Note that to prove the condition $R^* \geq 1$, we cannot use Theorem 16 of [3] for which the lower bound $R^* \geq \rho_o^2$ is optimum. So we have to seek a majoration of $v_p(y)$ different from that used in this theorem, and probably specific to the function $f^\circ$. This raises difficult technical problems. So we cannot, at present, increase the convergence radius of the power series $a_p(y)$. That’s why we have limited ourselves in the rest of the article, to study the reality of the zeros of $B_y(f^\circ)$ when $y$ increases along the real segment $[\rho_o^2, 1]$; in renouncing a priori to the convergence of the power series $a_p(y)$ in the disk $|y| < 1$.
3 Background on the Laguerre entire functions

For the reader’s convenience, we recall in this section the definitions and some known properties of these functions (see for example [12] or [7]).

Definition 2 i) We say that an entire function $f$ is of the first type of Laguerre (abbreviated $L_1$) if it can be written

$$f(x) = C x^m e^{a x} \prod_{p \geq 1} (1 + a_p x)$$

with $C, \sigma, a, p \geq 0$, $m \in \mathbb{N}$, and $\sum_{p \geq 1} a_p < +\infty$

ii) We say that an entire function $f$ is of the second type of Laguerre (abbreviated $L_2$) if it can be written

$$f(x) = C x^m e^{-ax^2+bx} \prod_{p \geq 1} [(1 + a_p x) e^{-a_p x}]$$

with $C, b, a_p \in \mathbb{R}$, $a \geq 0$, $m \in \mathbb{N}$, and $\sum_{p \geq 1} a_p^2 < +\infty$

Note that in [7], the Laguerre entire functions are called the Laguerre-Polya class.

Let us give two others definition that will be useful

Definition 3 Let $f(x) = \sum_{n \geq 0} a_n x^n$ be a formal power series on $\mathbb{C}$, finite or infinite. For $n \in \mathbb{N}$, the $n$-th Jensen polynomial of $f$ is the polynomial (we follow the definition given in [7] Section 3)

$$J_n(x) = \sum_{k=0}^{n} \frac{n!}{(n-k)!} A_k x^k.$$

Definition 4 We say that a formal power series $f(x) = \sum_{n \geq 0} A_n x^n$ is without gaps, if it can be written

$$f(x) = \sum_{n=n_1}^{n_2} A_n x^n$$

with $A_n \neq 0$ for $0 \leq n_1 \leq n \leq n_2 < +\infty$.

For any formal power series without gaps, we put $\Omega_n = A_{n-1} A_{n+1}/A_n^2$ for $n_1 \leq n \leq n_2$ and $\Omega_n = 0$ otherwise.

Here are the known properties of Laguerre entire functions we need:

Proposition 3.1 - If $f$ is $L_1$, then it is $L_2$.

- If $f$ is $L_2$ and if all the coefficients of its MacLaurin series are positive, then it is $L_1$.

- If $f(x) = \sum_{n \geq 0} A_n x^n$ is $L_1$, then it is without gaps and we have the inequality (of Turan)

$$\Omega_n \leq \frac{n}{n+1}$$

for all $n \geq 1$.

- The set of functions $L_1$ (respectively $L_2$) is closed for the topology of uniform convergence on compact subsets of $\mathbb{C}$, and stable by derivation.

Theorem 3.2 (Laguerre) let $f(x) = \sum_{n \geq 0} A_n x^n$ be $L_2$ and let $\varphi$ be $L_2$ without positive zeros, then the function $\sum_{n \geq 0} A_n \varphi(n) x^n$ is $L_2$.

Theorem 3.3 Let $f(x) = \sum_{n \geq 0} A_n x^n$ be an entire function with coefficient in $\mathbb{C}$, such that $f(0) \neq 0$, then (Jensen)

- $f$ is $L_1$ if and only if its Jensen polynomials have only negative zeros.

- $f$ is $L_2$ if and only if its Jensen polynomials have only real zeros.

Moreover (Polya, [14]) if $f$ is $L_2$ and not of the form $P(x) e^{ax}$, where $P$ is a polynomial and $a$ a real number different from zero, its Jensen polynomials have only simple real zeros.

Remark 2 We chek with (4) and Definition 2, that RH is equivalent to what $f^2$ is $L_1$, as announced in Remark 1. Conversely, if $f$ is $L_1$, Proposition 3.1 shows that $f$ has no gaps and satisfies the inequality (17). Hence, we can apply to $f$ the method introduced in [3] and summarized above. Further, the inequality (17) shows that $\Omega \leq 1$ for $f$, therefore Theorem 2.2 applies exactly to $f$. 


4 An analogue of the De Bruijn-Newman constant

4.1 Some additional properties of the $y$-Borel transform

We establish a principle of contraction, we need to prove the following theorems in the article:

**Theorem 4.1** Let $f(x) = \sum_{n \geq 0} A_n x^n$ be an entire function with real coefficients $A_n$.

i) If $f$ is $L_2$, then $B_y(f)$ is also $L_2$ for $y \in [0,1]$.

ii) If there is $y_0 \in [0,1]$ such that $B_{y_0}(f)$ has only real zeros, then $B_y(f)$ is $L_2$ for $y \in [0,y_0]$.

In [2] (see also [16] Definition 4.1) Bézivin introduced the following definition:

**Definition 5** Let $q \in \mathbb{C}^*$, we say that a formal power series $f(x) = \sum_{n \geq 0} A_n x^n \in \mathbb{C}[[x]]$ is $q$-Gevrey type 1, if the power series

$$B_{q^{-1}}(f)(xq) = \sum_{n \geq 0} A_n x^n q^{-n(n-1)/2}$$

has a convergence radius different from zero.

First, we prove

**Lemma 4.2** Let $q > 0$ be such that the formal power series $f(x) = \sum_{n \geq 0} A_n x^n \in \mathbb{C}[[x]]$ is $q$-Gevrey type 1. Then for $|y| < 1/q$, $B_y(f)$ is an entire function of order zero. In particular if $f$ is an entire function, $B_y(f)$ is an entire function of order zero for $|y| < 1$.

**Proof.** It is easy to see that $f$ is $q$-Gevrey type 1, is equivalent to the fact that there exist $C,A > 0$ such that for $n \in \mathbb{N}$

$$|A_n| < C q^{-n(n+1)/2} A^n. \quad (18)$$

Let $\chi$ be any positive number and $y$ a nonzero complex number such that $|y| < 1/q$. We have then for $n \in \mathbb{N}$

$$n^\chi |A_n y^{n(n+1)/2}|^{1/n} < n^\chi C^{1/n} (|y|q)^{n+1/2} A$$

with $|y|q < 1$. Hence the left-hand side of the above equation tends to zero as $n \rightarrow \infty$. There is therefor a naturel integer $n_1$ such that for $n \geq n_1$ we have

$$|A_n y^{n(n+1)/2}|^{1/n} \leq \frac{1}{n^\chi} \quad (19)$$

Let us write $B_y(f) = \sum_{n \geq 0} C_n x^n$, ie $C_n = A_n y^{n(n+1)/2}$. Taking $\chi = 1$, equation (19) gives $\lim_{n \rightarrow \infty} \sqrt[n]{|C_n|} = 0$, what already shows that $B_y(f)$ is an entire function. This implies in particular that there is a naturel integer $n_2$ such that for $n \geq n_2$

$$\log |C_n^{-1}| > 0 \quad \text{(20)}$$

We set $n_0 = \max (n_1, n_2)$. From (19) and (20) we get for $n \geq n_0$

$$0 \leq \frac{n \log n}{\log |C_n^{-1}|} \leq \frac{1}{\chi}$$

Since $\chi$ is arbitrary in $\mathbb{R}^*_+$, we deduce

$$\lim_{n \rightarrow \infty} \frac{n \log n}{\log |C_n^{-1}|} = 0 \quad (21)$$

It follows from Theorem 14.1.1 of [11] that the order of $B_y(f)$ is zero.

Now if $f$ is an entire function, then $\lim_{n \rightarrow \infty} \sqrt[n]{|A_n|} = 0$. There is thus a naturel integer $n'$ such that, for $n \geq n'$ we have $|A_n| < 1$. Hence, with $C = \max_{0 \leq n \leq n'} (|A_n| + 1)$ and $A = 1$, (18) shows that $f$ is 1-Gevrey type 1. $B_y(f)$ is therefore an entire function of order zero for $|y| < 1$. \[\blacksquare\]
Proof of Theorem 4.1.  i) If \( f \) is \( L_2 \), and \( y \in [0,1] \). We consider the function \( \varphi : \mathbb{C} \rightarrow \mathbb{C} \) defined by
\[
\varphi(x) = y \frac{x^{(x+1)/2}}{2} = \exp \left( \frac{\ln y}{2} x^2 + \frac{\ln y}{2} x \right)
\]
\( \varphi \) is an entire function and \( L_2 \) because \( \ln y \leq 0 \). And it has no positive zeros (because it has no zeros). The application of Theorem 3.2 to \( f \) and \( \varphi \), shows that the function
\[
\sum_{n \geq 0} A_n \varphi(n) x^n = B_y(f)(x)
\]
is \( L_2 \). The case \( y = 0 \) is obvious since \( B_0(f)(x) = A_0 \).

ii) If \( f \) is an entire function, and \( y_0 \in [0,1] \), we know from Lemma 4.2 that \( B_{y_0}(f) \) is an entire function of order zero. Thus, it can be written
\[
B_{y_0}(f)(x) = A_m y_0^m x^m \prod_{p \geq 1} (1 + a_p x)
\]
where \( A_m \) is the first non-zero coefficient of the Maclaurin series of \( f \), and where the product is normally convergent on each compact subset of \( \mathbb{C} \). It follows that \( B_{y_0}(f) \) is \( L_2 \) if \( f \) is real and \( B_{y_0}(f) \) has only real zeros.

For \( y \in [0,y_0] \), we get by (6)
\[
B_y(f) = B_{y_0}(B_{y_0}(f))
\]
with \( \frac{y}{y_0} \in [0,1] \). Thus, it follows from the previous cases that \( B_y(f) \) is \( L_2 \). ■

4.2 Constant \( \Theta \) associated with \( B_y(f^p) \)

Definition 6 Let \( f^p \) be given by (3). We define
\[
\Theta = \sup \left\{ y \in [0, +\infty[ \text{ such that } B_y(f^p) \text{ is an entire function and has only real zeros} \right\}
\]

We are going to show that \( \Theta \) satisfies the following equivalences:

\[
\left\{ \begin{array}{l}
* \text{ For } y \geq 0 \text{ we have: } y \leq \Theta \iff \{ B_y(f^p) \text{ is an entire function whose zeros are all real} \} \\
* 1 \leq \Theta \iff \text{RH}
\end{array} \right. \tag{21}
\]

In fact, we will prove the more accurate following result:

Theorem 4.3 We have

i) If \( 0 \leq y \leq \Theta \), \( B_y(f^p) \) is \( L_1 \) and has thus only negative zeros.

ii) If \( \Theta < 1 \) and \( \Theta < y \leq 1 \) \( B_y(f^p) \) is an entire function with at least two non-real zeros.

iii) If \( 1 < y \) the convergence radius of \( B_y(f^p) \) is zero.

iv) \( \Theta = 1 \) if and only if the Riemann hypothesis is true.

v) We have \( 0 < \rho^2 < 1/4 \leq \Theta \leq 1 \).

This theorem is one of the main results of the article. In particular, the point iv) gives a reformulation of RH. To prove this theorem, we need two Lemmas.

Lemma 4.4 Let \( f(x) = \sum_{n \geq 0} A_n x^n \) be an entire function with \( A_n \neq 0 \) for all \( n \) and satisfying \( \lim \inf_{n \geq 1} |\Omega_n| \geq 1 \). Then the radius of convergence of \( B_y(f) \) is zero for all \( y \in \mathbb{C} \) such that \( |y| > 1 \).
Proof. Let \( y \in \mathbb{C}, |y| > 1 \) we have \(|y| > \mu = \frac{|y| + 1}{2} > 1 \). By assumption, there exists an integer \( n_0 \geq 1 \) such that for \( n \geq n_0 \) we have \(|\Omega_n| > \mu^{-1} \). In addition, we have for all \( n \geq 2 \):

\[
\frac{A_n}{A_{n-1}} = \Omega_{n-1}\Omega_{n-2}...\Omega_1 \frac{A_1}{A_0}.
\]

If, as above \( C_n = A_n y^{n(n+1)/2} \), we have for \( n > n_0 \)

\[
\frac{C_n}{C_{n-1}} = \Omega_{n-1}...\Omega_n y^{n-n_0} M \quad \text{with} \quad M = \Omega_{n_0-1}...\Omega_1 y^{n_0} \frac{A_1}{A_0}
\]

thus

\[
\left| \frac{C_n}{C_{n-1}} \right| \geq \left( \mu^{-1} |y| \right)^{n-n_0} |M|.
\]

\( M \) is a constant different from zero and \( \mu^{-1} |y| > 1 \). It follows that the convergence radius of \( B_y(f) \) is zero. \( \blacksquare \)

Lemma 4.5 For any formal power without gaps \( f(x) = \sum_{n \geq 0} A_n x^n \in \mathbb{C}[[x]] \), satisfying \( A_0 \neq 0 \) and \( \Omega < +\infty \), the \( y \)-Borel transform of \( f \) is an entire function of order zero for \(|y| < 1/\Omega \). In addition, if the coefficients \( A_n \) are positive, \( B_y(f) \) is a function \( L_1 \) for \( y \in [0, 1/4\Omega] \).

Proof. With an easy recurrence, we have for \( n \geq 1 \)

\[
A_n = \Omega_{n-1} (\Omega_{n-2})^2 ...(\Omega_1)^{n-1} \frac{A_1^n}{A_0^n}
\]

then

\[
|A_n| \leq \Omega^{\frac{n(n+1)}{2}} \left| \frac{A_1}{A_0} \right|^n |A_0|
\]

By comparison with the inequality (18), it follows that \( f \) is \( q \)-Gevrey type 1 for \( q = \Omega \). Lemma 4.2 then shows that \( B_y(f) \) is an entire function of order zero when \(|y| < 1/\Omega \).

Assume now \( A_n > 0 \) and consider the Jensen polynomials of \( f \), \( J_n \). It is easy to see that the Jensen polynomials of the \( y \)-Borel transform of \( f \) are the \( y \)-Borel transform of the Jensen polynomials of \( f \), ie

\[
B_y(J_n)(x) = \sum_{k=0}^{n} \frac{n!}{(n-k)!} A_k x^k y^{k(k+1)/2} = \sum_{k=0}^{n} C_{n,k} x^k.
\]  

(22)

For \( y > 0 \), we have \( C_{n,k} > 0 \) and for \( 1 \leq k \leq n - 1 \)

\[
\frac{C_{n,k-1}C_{n,k+1}}{C_{n,k}^2} = \frac{A_{k-1}A_{k+1}}{A_k^2} \frac{(n-k)}{(n-k+1)} y
\]

and then, for \( y \in [0, 1/4\Omega] \)

\[
\frac{C_{n,k-1}C_{n,k+1}}{C_{n,k}^2} < 1/4
\]

It follows from a theorem of Kurtz (see section 4 of [7]), that all the zeros of \( B_y(J_n) \) are negative. This being true for any \( n \in \mathbb{N} \), we deduce from the theorem 3.3 that \( B_y(f) \) is \( L_1 \) for \( y \in [0, 1/4\Omega] \). The case \( y = 0 \) is obvious. \( \blacksquare \)

Proof of Theorem 4.3. According to the double inequality (14), we have \( \lim_{n \to \infty} \Omega_n^2 = 1 \). Thus Lemma 4.4 shows that for \( y > 1 \), \( B_y(f^2) \) has a convergence radius equal to zero, which proves \( iii \). This also shows that \( \Theta \leq 1 \), ie the last inequality of \( v \).
Now let $y$ be such that $0 \leq y < \Theta \leq 1$. According to the definition of $\Theta$, there is $y_0$ such that $y < y_0 < \Theta \leq 1$ and such that $B_{y_0} (f^p)$ has only real zeros. It follows from item ii) of Theorem 4.1 that $B_y (f^p)$ is $L_2$. In addition, it is clear that the coefficients of the Maclaurin series of $B_y (f^p)$ are positive. Therefore, $B_y (f^p)$ is actually $L_1$ with Proposition 3.1. Since $L_1$ is closed for the topology of uniform convergence on each compact subset of $\mathbb{C}$ (see Proposition 3.1), we deduce from Proposition 2.1 that $B_0 (f^p)$ is also $L_1$. This proves i).

We reason by contradiction to prove ii). Assume that $\Theta < 1$ and that there is $y_0 \in \left[\Theta, 1\right]$ such that $B_{y_0} (f^p)$ has only real zeros.

- If $y_0 \in \left[\Theta, 1\right]$ (the point ii) of Theorem 4.1 shows that for $y \in \left[0, y_0\right]$ the zeros of $B_y (f^p)$ are real numbers, in contradiction with the definition of $\Theta$.

- If $y_0 = 1$ the function $B_1 (f^p) = f^p$ is $L_2$ because we know that $f^p$ is written as $f^p (x) = A_0^p \prod_{p \geq 1} \left(1 - \frac{x}{x_p}\right)$ with $A_0^p > 0$. The point i) of Theorem 4.1, then shows that the zeros of $B_y (f^p)$ are real for $y \in [0, 1]$, in contradiction with $\Theta < 1$.

For the point iv), the implication: $\text{RH} \implies \Theta = 1$ is now clear with ii), since we already know that $\Theta \leq 1$. The reciprocal implication follows from i).

To complete the proof of v) we apply Lemma 4.5 to $f^p$ with $\Omega = 1$. Hence $B_y (f^p)$ is $L_1$ and has all its zeros real for $y \in [0, 1/4]$, which proves $\Theta \geq 1/4$.

**Remark 3** We can now prove that for the function $f^p$, we have $\rho^2 \leq R^* < 1$. The inequality $\rho^2 \leq R^*$ follows from Theorem 2.2. In addition, assume that $R^* > 1$, then the theorem 3 of [2] shows that we have for $|y| < R^*$

\[
B_y \left(f^p\right) (x) = A_0^p \prod_{p \geq 1} (1 + \alpha_p (y) x) ,
\]

where the product is normally convergent on any compact subset of $\mathbb{C}$. Thus, $B_y (f^p)$ would be an entire function for $y \in [1, R^*]$ contradicting iii) of Theorem 4.3.

### 4.3 Analogy of $\Theta$ with the de Bruijn-Newman constant

In [13] Newman, working after de Bruijn [6] on the Fourier transforms with real zeros (see Remark 1), defines for $\lambda \in \mathbb{R}$

\[
\Xi_{\lambda} (z) = 2 \int_0^\infty \exp (\lambda u^2) \Phi (u) \cos zu du .
\]

That is the Fourier transform of $\exp (\lambda u^2) \Phi (u)$ where

\[
\Phi (u) = \sum_{n=1}^\infty \left(4\pi^2 n^4 e^{3n/2} - 6\pi n^2 e^{5n/2}\right) \exp (-\pi n^2 e^{2u}) .
\]

We have $\Xi_0 (z) = \Xi (z)$ where $\Xi$ is the Riemann Xi function. The constant $\Lambda$ of de Bruijn-Newman, whose the existence was proved in [13], satisfies the properties (we follow the presentation given in [1] Section 3.2.2):

\[
\{ \begin{array}{l}
* \text{ For } \lambda \in \mathbb{R}, \text{ we have: } \Lambda \leq 4\lambda \iff \{ \Xi_{\lambda} \text{ is an entire function whose zeros are all real}\} \\
* \Lambda \leq 0 \iff \text{RH} \end{array} \}
\]

By comparison with the properties (21), we see that the constant $\Theta$ plays a role similar to that of the constant $\Lambda$. It would be interesting to find an equation relating $\Theta$ to $\Lambda$. We leave aside the issue in the following of the article.

### 5 Another reformulation of the Riemann hypothesis

Here is the reformulation:

**Theorem 5.1** Let $A_1^n$ be defined by the equation (2) where $\Xi$ is the Riemann Xi function given by (1). For $n \geq 0$, denote by $\Delta_n (y)$ the discriminant of the following polynomial

\[
B_y \left(J^n_{\Lambda}\right) (x) = \sum_{k=0}^n \frac{n!}{(n-k)!} A_k^n x^k y^{k(k+1)/2} .
\]
Then the Riemann hypothesis is true if and only if for all \( n \geq 2 \), \( \Delta_n (y) \) does not vanish on the interval \([1/4, 1] \).

To prove this theorem, we need two lemmas.

**Lemma 5.2** Let \( \mathbb{K} \) be a field of characteristic zero and \( P(x) = A_0 + A_1x + \ldots + A_n x^n \in \mathbb{K}[x] \) with \( A_k \neq 0 \) for \( 0 \leq k \leq n \). Then

i) The discriminant \( \Delta(y) \) of \( B_y (P) \) is a polynomial in \( y \) not identically zero.

ii) If \( \mathbb{K} = \mathbb{R} \), \( \Omega = \sup_{1 \leq k \leq n-1} |A_{k-1}A_{k+1}/A_k^2| \) and if we assume the roots of \( B_y (P) \) are simple for \( y \in [\rho_0^2 \Omega^{-1}, 1] \), \( P \) has only real roots.

**Proof.** i) By application of Theorem 1 of [3], we have in the ring \( \mathbb{K}[[x, y]] \):

\[
B_y (P) (x) = A_0 \prod_{p=1}^{n} (1 + \alpha_p(y) x)
\]

with (see equations (8) and (9) above)

\[
\alpha_p(y) = \frac{A_p}{A_{p-1}} y^p + \sum_{q=p+1}^{\infty} u_p(q) y^q.
\]  

(23)

Thus the zeros of \( B_y (P) \) considered as a polynomial in \( x \) on the field \( \mathbb{K}((y)) \), can be written as (see equation (7))

\[
x^p(y) = -(\alpha_p(y))^{-1} = -\frac{A_{p-1}}{A_p y^p} \left( 1 + \sum_{q=1}^{\infty} u_p(q) y^q \right)
\]

(24)

where \( u_p(q) \in \mathbb{K} \). It follows that the polynomial \( B_y (P) \) is separate over \( \mathbb{K}((y)) \) and that its zeros are all distinct because they have different valuations. This proves that the discriminant \( \Delta(y) \) of \( B_y (P) \) is a polynomial in \( y \) not identically zero.

ii) First, we assume \( \rho_0^2 \Omega^{-1} < 1 \), otherwise Corollary 18 of [3] shows that \( P \) has only real roots, without requirement of simplicity of the roots. We know by the theorem 16 of [3], that the series \( \alpha_p(y) \) are absolutely convergent in the disk \( D (0, \rho_0^2 \Omega^{-1}) = \{ y \in \mathbb{C} : |y| < \rho_0^2 \Omega^{-1} \} \).

Thus, they define holomorphic functions in this disk. Since \( A_n y^{n/2} \) does not vanish outside 0, it is the same for the functions \( \alpha_p(y) \). Therefore the \( n \) zeros \( x_p(y) = -1/\alpha_p(y) \) of \( B_y (P) \) are also holomorphic functions in the punctured disk \( D^*(0, \rho_0^2 \Omega^{-1}) = \{ y \in \mathbb{C} : 0 < |y| < \rho_0^2 \Omega^{-1} \} \). And their Laurent series at the point 0, given by (24), converge on this punctured disk. It is well known that the roots of \( Q(x, y) = B_y (P)(x) \) are the branches of one or several algebraic functions, that would be obtained by decomposing \( Q \) into irreducible factors in the factorial ring \( \mathbb{C}[x, y] \). The following decomposition shows that the case of several algebraic functions cannot be excluded.

\[
Q(x, y) = B_y (1 + x + x^2 + x^3) = 1 + xy + x^2 y^3 + x^3 y^6 = \left[ 1 + xy^2 \right] \left[ 1 + x (y - y^2) + x^2 y^4 \right]
\]

(25)

However, we will follow the reasoning of Section 12.1 of [11], which deals with the case of a single algebraic function. Indeed the reasoning of Section 12.1 of [11] does not depend, for the part that concern us, to the fact that \( Q \) is irreducible or not.

For a fixed value \( y \in \mathbb{C} \setminus \mathcal{S} \) (\( S \) is a set defined below), the equation

\[
Q(x, y) = \sum_{k=0}^{n} A_k x^k y^{k(k+1)/2} = 0
\]

has \( n \) distinct finite roots. The set \( S \) of exceptional values is composed of three subsets. The first of these is the subset of roots of \( A_{n/2} y^{n/2} = 0 \), ie \( \{ 0 \} \) (the Laurent series (24) shows that \( y = 0 \) is actually a pole of order \( p \) for \( x_p(y) \)). Secondly, we must exclude the values of \( y \) for which (25) has multiple roots. It is well known that these values of \( y \) are the roots of the discriminant \( \Delta(y) \). We know from the point i) of Lemma 5.2, that \( \Delta(y) \) is not identically null, so this second subset is finite. Third, we must exclude the point at infinity. \( S \) is thus finite and the hypothesis of the point ii) of Lemma 5.2 shows that \( S \cap [\rho_0^2 \Omega^{-1}, 1] = \emptyset \). Let \( d \) be the distance between \( S \setminus \mathbb{R} \) and \( \mathbb{R} \) (we set \( d = +\infty \) if \( S \setminus \mathbb{R} = +\infty \)). Put \( b = \max \{ y \in S \cap \mathbb{R} : y < \rho_0^2 \Omega^{-1} \} \); we have \( b \geq 0 \) since \( 0 \in S \).
And consider the real number \( a = \left( b + \rho_0^2 \Omega^{-1} \right) / 2 \), we have \( a \in D^* (0, \rho_0^2 \Omega^{-1}) \). Put \( \mu = 1/2 \min \{ d, a - b \} \), we have \( \mu > 0 \). For each \( p \in \{ 1, 2, \ldots, n \} \), the function \( x_p (y) \) holomorphic in \( D^* (0, \rho_0^2 \Omega^{-1}) \), can be expanded in a Taylor series around the point \( a \), which converges in a disk \( |y - a| < \nu_p \) \( (\nu_p > 0) \). Put \( \varepsilon = \min \{ \mu, \nu_1, \nu_2, \ldots, \nu_n \} \), we have \( \varepsilon > 0 \). For \( y_0 \in [\rho_0^2 \Omega^{-1}, 1] \), \( T(y_0) \) denotes the closed rectangular subset of the complex plane, whose vertices are the complex numbers \( a - \varepsilon + i \varepsilon, a - \varepsilon - i \varepsilon, y_0 + i \varepsilon, y_0 - i \varepsilon \). \( T(y_0) \) denotes the interior of \( T(y_0) \). Let \( y_0 \) be a fixed positive number in \( [\rho_0^2 \Omega^{-1}, 1] \), the distance between \( T(y_0) \) and \( S \) is a positive number, \( \delta (y_0) > 0 \). For \( p \in \{ 1, 2, \ldots, n \} \), the Taylor series of \( x_p (y) \) around the point \( a \) is convergent in the disk \( |y - a| < \varepsilon \), which is included in \( T(y_0) \) for all \( y_0 \in [\rho_0^2 \Omega^{-1}, 1] \). Following the reasoning of section 12.1 of [11] for the open subset \( T(y_0) \), the implicit function theorem shows that each of these Taylor series can be continued analytically along every path in \( T(y_0) \). Since \( T(y_0) \) is a simply-connected domain, the monodromy theorem (see [11] p.12) shows that these analytic continuations provide for each \( p \in \{ 1, 2, \ldots, n \} \), a holomorphic function in \( D^* (0, \rho_0^2 \Omega^{-1}) \cup T(y_0) \). If we do this for all \( y_0 \in [\rho_0^2 \Omega^{-1}, 1] \), we get for each \( p \in \{ 1, 2, \ldots, n \} \) an analytic continuation of \( x_p (y) \), which is now defined and holomorphic in \( D^* (0, \rho_0^2 \Omega^{-1}) \cup T(y_0) \).

If we denote \( \bar{x}_p (y) \), holomorphic in the open subset

\[
O = D^* (0, \rho_0^2 \Omega^{-1}) \cup T(1) = \bigcup_{y_0 \in [\rho_0^2 \Omega^{-1}, 1]} D^* (0, \rho_0^2 \Omega^{-1}) \cup T(y_0).
\]

Note that \([0,1] \subset O \). By construction, for \( p \in \{ 1, 2, \ldots, n \} \) and \( y \in O \), \( \bar{x}_p (y) \) is a zero of \( B_y (P) \). In addition, since \( Q(0, y) = A_0 \neq 0 \) we have \( \bar{x}_p (y) \neq 0 \) in \( O \) for each \( p \in \{ 1, 2, \ldots, n \} \). Hence, the law of permanence of functional equations (see section 10.7 of [11]) shows that the relation

\[
Q(x,y) = A_0 \prod_{p=1}^{n} \left( 1 - \frac{x}{x_p (y)} \right) = 0
\]

is still valid for \( x \in C \) and \( y \in O \). It follows that the \( n \) values \( \bar{x}_p (y) \) are exactly the roots of \( B_y (P) \) for \( y \in O \).

It is clear that the domain \( O \) is symmetrical about the real axis. By assumption, the coefficients \( A_k \) are real, thus \( \alpha_p (y) \) and \( x_p (y) = (\alpha_p (y))^{-1} \) are also real on the nonempty open interval \([0, \rho_0^2 \Omega^{-1}] \), where they coincide with, respectively, \( \alpha_p (y) \) and \( \bar{x}_p (y) \). Hence, applying the Schwarz reflection principle to \( O \) and each holomorphic function \( \bar{x}_p (y) \), we find that \( \bar{x}_p (y) \) is a real number for \( y \in [0,1] \) and \( 1 \leq p \leq n \). In fact, we have just shown that the zeros of \( B_y (P) \) are all real for \( y \in [0,1] \). The proposition 2.1 then allows to conclude.

**Lemma 5.3** Let \( f(x) = \sum_{n \geq 0} A_n x^n \) be an entire function and for \( n \in \mathbb{N} \), let \( J_n \) be the \( n \)-th Jensen polynomial of \( f \).

i) If the coefficients \( A_n \) are real, different from zero and satisfy the condition \( \Omega < +\infty \), we have

\[
f \text{ is } L_2 \text{ (and therefore its zeros are real) } \iff \{ \text{ the polynomials } B_y (J_n) \text{ have only simple zeros for } y \in [\rho_0^2 \Omega^{-1}, 1] \}.
\]

ii) If the coefficients \( A_n \) are strictly positive, we have

\[
f \text{ is } L_1 \text{ (and therefore its zeros are negative) } \iff \{ \Omega < +\infty \text{ and the polynomials } B_y (J_n) \text{ have only simple zeros for } y \in [1/4\Omega, 1] \}.
\]

**Proof.** i) Suppose first that for all \( n \in \mathbb{N} \), \( B_y (J_n) \) has only simple zeros for \( y \in [\rho_0^2 \Omega^{-1}, 1] \). As before, we assume \( \rho_0^2 \Omega^{-1} < 1 \), otherwise Corollary 18 of [3] and Lemma 4.2 above, show that \( B_y (f) \) is \( L_2 \) for \( y \in [0,1] \); and we deduce from Proposition 2.1 and the last point of Proposition 3.1 that \( f \) is \( L_2 \) without requirement of simplicity of zeros. We have

\[
J_n (x) = \sum_{k=0}^{n} C_{n,k} x^k \text{ with } C_{n,k} = \frac{n!}{(n-k)!} A_k \in \mathbb{R}^*
\]

and for \( 1 \leq k \leq n - 1 \)

\[
\left| \frac{C_{n,k-1} C_{n,k+1}}{C_k^2} \right| = \frac{n-k}{n-k+1} \left| \frac{A_{k-1} A_{k+1}}{A_k^2} \right| \leq \frac{n-1}{n} |\Omega_k|
\]
thus
\[
\sup_{1 \leq k \leq n-1} \left| \frac{C_{n,k-1} C_{n,k+1}}{C_{n,k}} \right| \leq \Omega = \sup_{k \geq 1} |\Omega_k|
\]

We then apply Lemma 5.2 to \(J_n\), which shows that this polynomial has only real zeros. Since this is true for any \(n \in \mathbb{N}\), it follows from Theorem 3.3 that \(f\) is \(L_2\).

Conversely, if we assume that \(f\) is \(L_2\), we know from Theorem 4.1 that \(B_y(f)\) is \(L_2\) for all \(y \in [0, 1]\). Furthermore, Lemma 4.2 shows that the order of \(B_y(f)\) is zero in this interval. Thus \(B_y(f)\) is not of the form \(P(x) e^{\alpha x}\) where \(P\) is a polynomial and \(\alpha\) a non-zero real number. It follows from the last point of Theorem 3.3 that the Jensen polynomials of \(B_y(f)\) have only simple zeros for \(y \in [0, 1]\) and in particular for \([\rho_y, \Omega^{-1}, 1]\).

\(\text{ii})\) Assume first that \(\Omega < +\infty\) (we assume \(1/4\Omega < 1\), otherwise Lemma 4.5 shows that \(f\) is \(L_1\) without requirement of simplicity of zeros) and that for all \(n \in \mathbb{N}\) the polynomial \(B_y(J_n)\) has only simple zeros for \(y \in [1/\Omega, 1]\). We know by Lemma 4.5 that \(B_y(f)\) is \(L_1\), thus also \(L_2\) for \(y \in [0, 1/4\Omega]\) and moreover it is zero order. It is thus not of the form \(P(x) e^{\alpha x}\) where \(P\) is a polynomial and \(\alpha\) a non-zero real number. And the zeros of \(B_y(J_n)\) are then also simple for \(y \in [0, 1/4\Omega]\) by Theorem 3.3. Hence, the zeros of \(B_y(f)\) are simple for \(y \in [0, 1]\) and we can apply the results of \(\text{ii})\), which shows that \(f\) is \(L_2\). It follows from Proposition 3.1 that \(f\) is also \(L_1\). We deduce the converse from \(\text{i})\) and the fact that \(\Omega \leq 1 < +\infty\) by inequality (17).

**Proof of Theorem 5.1.** Let \(f^\ast\) be the function defined by (3). We already know that The Riemann hypothesis is equivalent to that \(f^\ast\) is \(L_1\). We notice now that the polynomials
\[
B_y \left( J_n^\ast \right) (x) = \sum_{k=0}^{n} \frac{n!}{(n-k)!} A_k^x y^{k(k+1)/2}
\]
are the \(y\)-Borel transform of the Jensen polynomials of \(f^\ast\). We then use the \(\text{ii})\) of Lemma 5.3, taking into account the fact that we have \(\Omega = 1\) for \(f^\ast\) (see Section 2.3). Hence RH is true if and only if \(B_y(f^\ast)\) has only simple zeros for \(y \in [1/4, 1]\) and \(n \in \mathbb{N}\). This is obvious for \(n = 0\) and \(n = 1\), and for \(n \geq 2\) this amounts to saying that \(\Delta_n(y)\) does not vanish in this interval.

**Remark 4** It is well known that \(\Delta_n(y)\) can be expressed as a determinant. Theorem 5.1 can then be compared to a condition equivalent to RH, given in Section C.8 of [17], and using a different sequence of determinants.

### 6 Simplicity of the zeros of \(B_y(f^\ast)\)

A new necessary condition for the RH, follows from the following general theorem.

**Theorem 6.1** If \(f\) is \(L_2\) with \(f(0) \neq 0\), then the zeros of \(B_y(f)\) are simple (and real) for \(y \in [0, 1]\).

**Corollary 6.2** We can improve Theorem 4.3 by replacing the point \(\text{i})\) of this theorem by:

\(\text{i*})\) If \(0 \leq y \leq \Theta\), \(B_y(f^\ast)\) is \(L_1\) and thus, has only negative zeros. Furthermore, the zeros are simple if \(0 \leq y < \Theta\).

In particular, the Riemann hypothesis implies that the zeros of \(B_y(f^\ast)\) are simple for \(y \in [0, 1]\).

**Proof.** The first part of the point \(\text{i*})\) is the point \(\text{i})\) of Theorem 4.3, we have already proved. It then suffices to apply Theorem 6.1 to \(B_y(f^\ast)\), to deduce that the zeros of \(B_y(f^\ast)\) are simple for \(0 \leq y < \Theta\). Further, RH implies that \(\Theta = 1\) (see \(\text{iv})\) of Theorem 4.3), and the final implication follows from \(\text{i*})\).

We need the following lemma to prove Theorem 6.1.
Lemma 6.3 Let \( Q(x, y) = \sum_{n=0}^{N} A_n x^n y^{(n+1)/2} \) with \( A_n \in \mathbb{C} \) and \( N \) a natural number greater than 1 or \( N = +\infty \). If \( N = +\infty \), assume that the power series in two variables \( x \) and \( y \) converges in a neighborhood of \((a, b) \in \mathbb{C}^2 \) with \( a \) and \( b \) different from zero. Assume further that

\[
Q(a, b) = 0, \quad \frac{\partial Q}{\partial x}(a, b) = 0, \quad \frac{\partial^2 Q}{\partial x^2}(a, b) \neq 0.
\]

Then, there is a convergent Puiseux series, \( \psi(h) \), satisfying \( Q(a + \psi(h), b + h) = 0 \), and whose the first term is

\[
\left( -\frac{a^2}{b} h \right)^{1/2}
\] (26)

Proof. Let \( T(h, k) \) be the Taylor expansion of \( Q(x, y) \) about \((a, b)\), taken as

\[
T(h, k) = Q(a + k, b + h) = N(N+3)/2 \sum_{n=0}^{N} \sum_{i+j=n} a_{i,j} h^i k^j \text{ with } a_{i,j} = \frac{1}{i!j!} \frac{\partial^n Q}{\partial y^i \partial x^j}(a, b)
\]

By assumption, we have

\[
a_{0,0} = 0, \quad a_{0,1} = 0, \quad a_{0,2} \neq 0
\]

thus

\[
a_{0,1} = b \sum_{n=1}^{N} n A_n a^{n-1} b^{(n^2+n-2)/2} = 0
\]

and since \( b \neq 0 \)

\[
A_1 = -\sum_{n=2}^{N} n A_n a^{n-1} b^{(n^2+n-2)/2}
\] (27)

Then

\[
a_{1,0} = \frac{\partial Q}{\partial y}(a, b) = \sum_{n=1}^{N} \frac{n (n+1)}{2} A_n a^n b^{(n^2+n-2)/2}
\]

or

\[
a_{1,0} = a A_1 + \sum_{n=2}^{N} \frac{n (n+1)}{2} A_n a^n b^{(n^2+n-2)/2}
\]

and with (27)

\[
a_{1,0} = \sum_{n=2}^{N} \frac{n (n-1)}{2} A_n a^n b^{(n^2+n-2)/2}
\]

In addition, we have

\[
a_{0,2} = \frac{1}{2} \sum_{n=2}^{N} n (n-1) A_n a^{n-2} b^{(n+1)/2} \neq 0
\]

and since \( a \neq 0 \), we get

\[
a_{1,0} = \frac{a^2}{b} a_{0,2} \neq 0.
\]

The Newton polygon is thus reduced in a \((h, k)\)-plane, to the segment that connects the point \((0, 2)\) to point \((1, 0)\). The Puiseux theorem asserts the existence of a convergent Puiseux series \( \psi(h) \), satisfying \( T(h, \psi(h)) = 0 \). Let us determine the first term of this series with the method of Newton-Puiseux (see for exemple [10] Chap. 7). The quasi-homogeneous polynomial corresponding to the Newton polygon is

\[
\tilde{T}(h, k) = \left( \frac{a^2}{b} h + k^2 \right) a_{0,2}
\]

So, the parametrization is at the first approximation

\[
\begin{cases}
  h = t^2 \\
  k = \lambda t
\end{cases}
\]
where $\lambda$ is determined by the equation $\tilde{T}(h, k) = \left(\frac{a^2}{b} + \lambda^2\right) t^2 a_{0,2} = 0$. It follows $\lambda = (-a^2/b)^{1/2}$, giving the first term (26).

**Proof of Theorem 6.1.** We are going to show that there are no zeros of the function $B_y(f)$, which may have an order greater than one for $y \in [0, 1]$. Let $m$ be a natural number superior or equal to 2 and $f(x) = \sum_{n=0}^{\infty} A_n x^n$ be a $L_2$ function. Here, $N$ is an integer superior or equal to $m$ if $f$ is a polynomial of degree $N$, or $N = +\infty$ if $f$ is transcendental. Assume the existence of $b \in [0, 1]$ and $a \in \mathbb{R}$, such that $a$ is a zero of $B_y(f)$ with order $m$, i.e.

$$B_y(f)(a) = Q(a, b) = 0$$

$$\frac{\partial Q}{\partial x}(a, b) = 0$$

$$\frac{\partial^2 Q}{\partial x^2}(a, b) = \frac{\partial^m Q}{\partial x^m}(a, b) \neq 0$$

Note that we have $a \neq 0$, otherwise $B_y(f)(0) = A_0 = 0$ with $b \neq 0$, which implies $A_0 = f(0) = 0$ in contradiction with the hypothesis. An easy induction shows that we have formally for all $l \in \mathbb{N}$

$$B_y\left(f^{[l]}\right)(x) = y^{-(l+1)/2} (B_y(f))^{[l]} \left(\frac{x}{y}\right)$$

In particular, we have

$$B_y\left(f^{[m-2]}\right)(x) = b^{-(m-2)(m-1)/2} (B_y(f))^{[m-2]} \left(\frac{x}{b^{m-2}}\right)$$

where the power series in $x$ are convergent on $\mathbb{C}$ if $N = +\infty$. Therefore

$$\left(B_y\left(f^{[m-2]}\right)\right)'(x) = b^{-(m-2)(m+1)/2} (B_y(f))^{[m-1]} \left(\frac{x}{b^{m-2}}\right)$$

and

$$\left(B_y\left(f^{[m-2]}\right)\right)''(x) = b^{-(m-2)(m+3)/2} (B_y(f))^{[m]} \left(\frac{x}{b^{m-2}}\right)$$

We set $g = f^{[m-2]}$ and $c = ab^{m-2}$. The degree of $g$ is superior to 1 or $g$ is transcendent. $c$ is a real number different from zero and we have

$$B_y(g)(c) = b^{-(m-2)(m-1)/2} (B_y(f))^{[m-2]}(a) = 0$$

$$\left(B_y(g)\right)'(c) = b^{-(m-2)(m+1)/2} (B_y(f))^{[m-1]}(a) = 0$$

$$\left(B_y(g)\right)''(c) = b^{-(m-2)(m+3)/2} (B_y(f))^{[m]}(a) \neq 0$$

It follows that $B_y(g)$ satisfies the assumptions of Lemma 6.3 with $(c, b) \in \mathbb{C}^2$, where $c$ and $b$ are different from 0. Hence, there exists a convergent Puiseux series $k = \psi(h)$, such that $B_{b+h}(g)(c + \psi(h)) = 0$, and whose the first term is

$$\left(-\frac{c^2}{b}h\right)^{1/2}$$

It is possible to choose a determination of the power $\psi^{1/2}$, holomorphic in an open subset containing $\mathbb{R}_+$ and such that $(-1)^{1/2} = i$. So, we have for $h > 0$ and small enough

$$\psi(h) = \frac{|c|}{\sqrt{b}} \sqrt{h} + o \left(\sqrt{h}\right).$$

It follows that for positive small enough $h$, $\psi(h)$ will have a nonzero imaginary part. It is the same for $c + \psi(h)$ since $c$ is a real number. We just prove that there is $\varepsilon > 0$ such that $B_y(g)$ has at least one non-real zero, for $y \in [b, b + \varepsilon \subset [0, 1]$. But the last point of Proposition 3.1 shows that $g$ is $L_2$. Thus, we have a contradiction because the point $i)$ of Theorem 4.1 shows that $B_y(g)$ is also $L_2$, and has therefore only real zeros, for $y \in [0, 1]$.

$\blacksquare$
Remark 5 Theorem 6.1 can be used in the proof of Lemma 5.3 instead of the point 3 of Theorem 3.3. It can also be used to prove with Theorem 5.1, that RH implies actually that $\Delta_n(y)$ does not vanish on the interval $[0, 1]$ for all $n \geq 0$.

It is natural to think that the reciprocal of Theorem 6.1 is true. So, we formulate the following conjecture.

**Conjecture** If $f$ is an entire real function without gaps (in the sense of Definition 4) and $\Omega = \sup_{n \geq 1} |\Omega_n| < +\infty$, then $f$ is $L_2$ with $f(0) \neq 0$, is equivalent to that the zeros of $B_y(f)$ are simple for $y \in [0, 1]$.

First, note that with the function $f^p$, the conjecture gives the reciprocal of the final implication of Corollary 6.2, i.e the conjecture announced in the abstract.

We just prove the necessity of the condition of simplicity of the zeros. We now discuss the sufficiency of this condition.

For each $p \geq 1$ we can write the power series (see (8) et (9) above)

$$\alpha_p(y) = \frac{A_p}{A_p-1}y^p \left(1 + \frac{A_{p-1}}{A_p} \sum_{q=1}^{\infty} u_p(q) y^q\right)$$

where the power series in parentheses converges for $|y| < \rho_p^2 \Omega^{-1}$ (as before, we assume $\rho_p^2 \Omega^{-1} < 1$, otherwise we have already seen that $f$ is $L_2$ without requirement of simplicity of zeros). It follows that $\alpha_p(y)$ does not vanish in a punctured disk $0 < |y| < R_p \leq \rho_p^2 \Omega^{-1}$, where $R_p$ depends a priori on $p$. Each zeros of $B_y(f)$, $x_p(y) = - (\alpha_p(y))^{-1}$, is therefore a holomorphic function in this punctured disk. We can now try to follow the reasoning of the point (ii) of Lemma 5.2, by using the implicit function theorem. For each $p \geq 1$, this would lead to an analytic continuation $\tilde{x}_p(y)$ of the holomorphic function $x_p(y)$, where $\tilde{x}_p(y)$ is now defined in an open subsets containing $[0, 1]$. The rest of the reasoning would be the same as that of Lemma 5.2. However, there is now a challenge to make this reasoning rigorous, because we cannot exclude here the fact that some zeros of $B_y(f)$ tend to infinity for certain values of $y$ in the interval $[0, 1]$.

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**References**

[1] M. Balazard, Un siècle et demi de recherches sur l’hypothèse de Riemann, Gaz. Math.126 (2010), p. 7-24.

[2] J.P. Bezivin, Sur les équations fonctionnelles aux q-différences, Aequationes Math. 43 (2-3), (1992), p.159-176.

[3] V. Brugidou, A new method to determine the value or the reality of zeros for certain entire functions, J.Math.Pures Appl. 94, Nb.3 (2010), p. 244-276.

[4] V. Brugidou, Une généralisation de la formule du triple produit de Jacobi et quelques applications, C. R. A. S. 349 (2011), p. 357-484.

[5] V. Brugidou, private communication to Paul Malliavin (July 1, 2008) and others; in preparation for publication.

[6] N.G. de Bruijn, The roots of trigonometric integrals, Duke J. Math.17, (1950), p. 197-226.

[7] T. Craven, G. Csordas, Composition theorems, multiplier sequences and complex zero decreasing sequences, in: Value Distribution Theory and Its Related Topics, G. Barsegian, I. Laine, C.C. Yang (Eds.), Kluwer Academic, Dordrecht, 2004.

[8] G. Csordas, T.S. Norfolk, R.S. Varga, The Riemann hypothesis and the Turan inequalities, Trans.of the Amer.Math.Soc. 296, Nb.2 (1986), p. 521-541.

[9] H.M. Edwards, Riemann’s Zeta Function, Academic Press, 1974.

[10] G. Fischer, Plane algebraic curves, AMS, Student mathematical library, 2001.

[11] H. Hille, Analytic function theory, vol. II, Ginn and Co., Boston, 1962.

[12] L. Iliev, Laguerre entire functions, Publishing house of the Bulg. Acad. of Sciences, Sofia, 1987.

[13] C.M. Newman, Fourier Transforms with only real zeros, Proc. Amer. Math. Soc., 61 (1976), p. 245-251.

[14] G. Polya, Uber die algebraisch-funktionentheoretischen Untersuchungen von J.L.W.V. Jensen, Kgl. Danske Vid. Sel. Math-Fys. Medd. 7 (1927), p. 224-249.

[15] E.C. Titchmarsh, The Theory of the Riemann Zeta-Function, 2° ed., Oxford Press, 1986.

[16] L. Di.Vizio, J.P. Ramis, J. Sauloy, C. Zhang, Equations aux q-différences, Gaz. Math. 96 (2003), p. 20-49.

[17] http://www.aimath.org/pl/rhequivalences