Revisiting two well studied galaxy samples using alternative gravity

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Abstract. In the last few years, alternative gravity theories have seen increased interest due to the lack of observational evidence of dark matter. Further, new empirical patterns found in rotation curve data such as the Radial Acceleration Rule (RAR) have given new testable features for gravitational theories. In this paper, we revisit two popular surveys of galaxies (Randriamampandry et al 2013 and Bottema et al 2015) which when published were shown to be problematic for alternative gravity. Here, we apply the most recent observational parameters to the surveys and provide fits of Conformal Gravity, MOND as well as the RAR rotation curve formalism and show how these theories can apply to the new findings. We also provide the fits to the RAR and Tully-Fisher relation for each theory and discuss how the RAR may allow for some confining of parameters in the fitting procedure.

1. Introduction
Galactic rotation curves have been a fundamental testing ground for standard and alternative gravity for many years [1]. The missing matter in galaxies shown by the rotation curve problem is typically explained by the presence of massive dark matter halos [2]. Dark matter formalism such as the work of Navarro, Frenk and White has lead to some very accurate representations of galactic rotation curves. Although the missing mass problem in spiral galaxies has produced a puzzle for standard gravity and resulted in the inclusion of dark matter, some interesting empirical results have emerged relating to baryonic matter alone. The Tully-Fisher relation [3] (more recently known as the Baryonic Tully-Fisher relation) or BTF [4] has shown strong correlation between flattening rotation velocities and masses of galaxies. The Radial Acceleration Rule (RAR) of McGaugh, Lelli and Schombert (MLS) [5] is another empirical observation that highlights trends in the centripetal accelerations of spiral galaxies. Further, the MLS formalism focuses on sets of galaxies as opposed to individual galaxy fits. In the last two years, RAR has been explored and expanded in scope by many authors [6] to expand on the correlation between the predicted baryonic accelerations in spiral galaxies with observation. Further, one of the suggestions of the RAR, is that it is too be explained by new physics rather than dark matter.

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The overall lack of direct observational evidence of dark matter has lead many authors to pursue other possible explanations of the relevant physics. Even the RAR concludes that perhaps the empirical phenomena can be explained by new physics without dark matter. Since its inception, Modified Newtonian Gravity (MOND), originally formulated by Milgrom [7] has been the most prevalent alternative gravity formulation. Conformal gravity has been successfully fitting rotation curves for many years and has been rapidly extending its applications beyond rotation curves alone [8]. With recent advances such as the RAR and more reliable distance measurements to well studied galaxies, it is instructive to revisit some galaxy samples that proved problematic for alternative gravity. Further, direct comparisons of the treatments of such galaxies between MOND and CG can be explored in a systematic way.

2. The two selected samples
This work focuses around two heavily referenced galaxy surveys. The first survey used in this paper was published by Bottema et al. [9] and consists of twelve galaxies that have been studied by many other authors. The data in the Bottema survey was developed using strict criteria for the determination of reliable rotation curves such as inclination above 50 degrees and data points measuring at least five scale lengths. The sample consists of a diverse set of galaxies spanning a range of morphologies, gas masses, distances and luminosities. The second survey to be discussed comes from Randriamampandry and Carignan [10]. This sample of fifteen galaxies are homogeneous in terms of method used to acquire the distance, sampling of rotation curves, and the mass to light ratios (M/L). It should be noted that there is some overlap present between the two surveys. However, as will be seen in the fits, the two surveys used substantially different data for the overlapping galaxies and hence all repeats are kept for consistency and to avoid bias over a particular data set.

These two samples combined provide a collection of 27 well-referenced rotation curves. In each of the original works, Carignan and Bottema performed dark matter fits as well as various MOND fits with MOND serving as the only alternative to dark matter tested. Each paper made comparisons and analyses to how the single alternative gravity theory performed when compared against various dark matter models. The original conclusion from both works was that MOND as studied, had more trouble fitting the galaxies in question than standard gravity with dark matter. Since MOND was the most widely recognized alternative to dark matter, the two works presented a strong challenge for alternative gravity. However, in this work we show that after further systematic analysis of the two surveys, both MOND and CG can provide strong fits to the galaxies that are consistent with empirical data such as BTF and RAR.

3. Formalism
3.1. Conformal gravity
Conformal gravity, originally referred to as Weyl Gravity is a fourth-order renormalizable metric theory of gravity. Although the roots date back to Weyl, it has been developed and expanded by many authors [11] in recent years and has enjoyed much success in fitting galactic rotation curves [12]. As opposed to standard Einstein gravity being derived from the Einstein-Hilbert action, conformal gravity begins by variation of the Weyl Action, \( I_W = -2\alpha_g \int d^4x (-g)^{1/2} \left( R_{\mu \nu} R^{\mu \nu} - \frac{1}{3} R^2 \right) \). Since this action shares some similarities with the Einstein-Hilbert action, the theory retains some features, but solving the resulting fourth order field equations yields testable differences in the solutions. In order to apply the theory to rotation curves, one can follow a similar procedure to applying standard gravity to rotation curves, starting with a CG equivalent of an external spherical mass solution to the CG field equations [13]. The exterior spherical mass solution can then be extrapolated over the geometry
of the galaxy with total number of stars $N^*$ and galactic scale length $R_0$ yielding,

$$v_{CG}(R) = \sqrt{v_{NEW}^2 + \frac{N^* \gamma^* c^2 R^2}{2R_0} I_1 \left( \frac{R}{2R_0} \right) K_1 \left( \frac{R}{2R_0} \right) + \frac{\gamma_0 c^2 R}{2} - \kappa c^2 R}, \quad (1)$$

with $\gamma^* = 5.42 \times 10^{-41} \text{cm}^{-1}$, $\gamma_0 = 3.06 \times 10^{-30}\text{cm}^{-1}$, and $\kappa = 9.54 \times 10^{-54} \text{cm}^{-2}$ being pure constants set by the solution to the field equations [14] and $v_{NEW}$ being the Freeman Formula, viz,

$$v_{NEW}(R) = \sqrt{\frac{N^* \beta^* c^2 R^2}{2R_0^3} \left[ I_0 \left( \frac{R}{2R_0} \right) K_0 \left( \frac{R}{2R_0} \right) - I_1 \left( \frac{R}{2R_0} \right) K_1 \left( \frac{R}{2R_0} \right) \right]}.$$ \quad (2)

Using eq. (1), one can fit the rotation curve of the galaxy with the only free parameter being the total number of stars in the galaxy such that $N^* = M M^{-1}$.

### 3.2. MOND

Milgrom’s Modified Newtonian Dynamics was the first non-dark matter theory successful at fitting many rotation curves and describing empirical phenomena such as the BTF [15]. As stated in the name, MOND can solve the missing mass problem by modifying the Newtonian gravitational acceleration through the use of an interpolation function such as $\mu_{\text{orig}}(x) = \frac{x}{\sqrt{1 + x^2}}$.

We note the interpolation function with the subscript “orig” to signify that it was the original function proposed by Milgrom, although other authors have used similar functions with similar successful results [16]. The interpolation function then modifies the gravitational acceleration as

$$g = \frac{g_N}{\mu(g/a_0)}, \quad (3)$$

with the Newtonian gravitational potential $g_N$ and critical acceleration $a_0 \approx 1.21 \times 10^{-8} \text{cm s}^{-2}$.

Some authors have later extended MOND to a relativistic theory and other authors have let the critical acceleration be a free parameter [16]. For this work here, the interpolation function and critical acceleration $a_0$ will remain fixed. When using these values, one obtains the rotational velocity in a spiral galaxy [16] using MOND as:

$$v_{MOND}(R) = \sqrt{v_{NEW}^2(R) + \left( v_{NEW}^2(R) \left( 1 + \frac{4a_0 R}{v_{NEW}^2} - 1 \right) \right)}.$$ \quad (4)

We see the presence of $v_{NEW}$ appearing in both the MOND and CG equations, showing some overlap between the two theories and serving as the “standard” gravitational contribution to the theory. Just as in conformal gravity, this leaves only a single free parameter in $v_{NEW}$, namely $N^*$.

### 3.3. MLS for individual rotation curves

Although the work of McGaugh, Lelli and Schombert was used to extrapolate trends across entire samples of rotation curves, one can use their formalism to produce individual rotation curve fits, provided a mass (or $N^*$) prediction is already made. The MLS fits can be obtained from the empirical formula,

$$g_{OBS} = \frac{g_{NEW}}{\left[ 1 - \exp\left(-\frac{g_{NEW}}{g^1/2}\right) \right]^1}, \quad (5)$$
Table 1. Properties of the Bottema 2015 Galaxy Sample

| Galaxy  | Dist. (Mpc) | Lumin. ($10^9 L_\odot$) | $R_0$ (kpc) | $M_{gas}$ ($10^9 M_\odot$) | $(M/L)_C$ | $(M/L)_C^{-1}$ | $(M/L)_M$ | $(M/L)_M^{-1}$ |
|---------|-------------|--------------------------|-------------|-----------------------------|-----------|----------------|-----------|----------------|
| DDO 154 | 4.3         | 0.04                     | 0.8         | 0.45                        | 0.03      | 0.72           | 0.02      | 0.36           |
| NGC 1560| 3.2         | 0.70                     | 1.4         | 0.95                        | 1.78      | 2.55           | 0.64      | 0.91           |
| NGC 2403| 3.3         | 10.12                    | 2.1         | 4.67                        | 19.29     | 1.91           | 9.47      | 0.94           |
| NGC 2841| 16.2        | 101.61                   | 4.0         | 37.69                       | 152.76    | 1.50           | 218.39    | 2.15           |
| NGC 2903| 7.9         | 38.68                    | 2.6         | 4.87                        | 65.66     | 1.72           | 44.60     | 1.15           |
| NGC 2998| 59.6        | 75.52                    | 4.8         | 23.91                       | 91.13     | 1.21           | 68.27     | 0.90           |
| NGC 3109| 1.3         | 0.28                     | 1.2         | 0.43                        | 0.87      | 3.09           | 0.36      | 1.29           |
| NGC 3198| 12.8        | 24.53                    | 3.6         | 12.86                       | 36.64     | 0.64           | 1.17      | 0.40           |
| NGC 3992| 20.6        | 72.53                    | 4.6         | 5.52                        | 156.22    | 2.15           | 134.80    | 1.86           |
| NGC 5585| 7.4         | 2.48                     | 1.6         | 1.22                        | 2.90      | 1.17           | 1.00      | 0.40           |
| NGC 6930| 4.8         | 4.83                     | 1.4         | 1.42                        | 14.35     | 2.97           | 6.76      | 1.40           |
| NGC 7331| 13.4        | 49.80                    | 3.0         | 141.62                      | 1198.45   | 2.41           | 97.58     | 1.96           |

Table 1 Entries left to right are: Galaxy name, NED distance, total luminosity, disk scale length, mass of the gas (including helium), disk mass, mass to light ratio, disk mass for MOND, mass to light ratio for MOND.

Table 2. Properties of the Carignan 2013 Galaxy Sample

| Galaxy  | Dist. (Mpc) | Lumin. ($10^9 L_\odot$) | $R_0$ (kpc) | $M_{gas}$ ($10^9 M_\odot$) | $(M/L)_C$ | $(M/L)_C^{-1}$ | $(M/L)_M$ | $(M/L)_M^{-1}$ |
|---------|-------------|--------------------------|-------------|-----------------------------|-----------|----------------|-----------|----------------|
| DDO 154 | 4.3         | 0.08                     | 0.8         | 0.44                        | 0.03      | 0.38           | 0.02      | 0.23           |
| IC 2574 | 3.3         | 0.80                     | 3.1         | 1.42                        | 2.29      | 2.87           | 0.94      | 1.17           |
| NGC 55  | 1.9         | 0.81                     | 1.9         | 1.81                        | 2.93      | 3.63           | 1.06      | 1.32           |
| NGC 247 | 3.3         | 2.17                     | 3.8         | 1.85                        | 12.28     | 5.66           | 5.14      | 2.37           |
| NGC 300 | 1.9         | 1.74                     | 2.0         | 1.06                        | 7.00      | 4.01           | 2.88      | 1.65           |
| NGC 925 | 7.8         | 5.12                     | 3.5         | 4.67                        | 11.44     | 2.23           | 4.62      | 0.90           |
| NGC 2366| 3.2         | 0.38                     | 1.5         | 0.97                        | 0.83      | 2.17           | 0.28      | 0.74           |
| NGC 2403| 3.4         | 4.77                     | 2.1         | 4.02                        | 21.56     | 4.52           | 10.37     | 2.18           |
| NGC 2841| 16.2        | 37.93                    | 4.0         | 15.98                       | 204.58    | 5.39           | 201.41    | 5.31           |
| NGC 3031| 3.7         | 14.40                    | 2.6         | 5.25                        | 85.45     | 5.93           | 69.38     | 4.82           |
| NGC 3109| 1.3         | 0.19                     | 1.2         | 0.66                        | 0.14      | 0.71           | 0.07      | 0.38           |
| NGC 3198| 12.8        | 11.53                    | 3.6         | 12.31                       | 36.63     | 3.18           | 18.13     | 1.57           |
| NGC 3621| 6.7         | 5.37                     | 2.6         | 10.26                       | 28.05     | 5.22           | 14.08     | 2.62           |
| NGC 7331| 13.4        | 20.38                    | 3.0         | 10.67                       | 120.47    | 5.91           | 97.19     | 4.77           |
| NGC 7793| 3.9         | 6.51                     | 1.3         | 1.24                        | 5.69      | 0.87           | 3.10      | 0.48           |

Table 2 Entries left to right are: Galaxy name, NED Distance, total blue luminosity, disk scale length, mass of the gas (including helium), disk mass, mass to light ratios, disk mass for MOND, mass to light ratio for MOND.

as presented in the MLS framework which correlates the predicted newtonian (NEW) velocity $g_{NEW} = v_N^2/R$ to that velocity which should be observed $g_{OBS} = v_O^2/R$ using a fitting parameter, $g = a_0$. Thus for any predicted mass found in the fitting described above, $M_C$ or $M_M$ the fit from eq. [5] can be deduced by using the respective mass found using eqs. [1] and [3] to compute the respective $g_{NEW}$ for both CG and MOND. To this end, the MLS single rotation curve prediction for the given masses are included in Figs 1,2,3,4 as the solid green line. Although this does not make a claim on a particular rotation curve fit, it is illustrative to see the difference in sensitivity between the rotation curve problem and the universal trends in the acceleration scale to be discussed later.
Figure 1. Fitting to the rotational velocities (km/s) of the Bottema sample with quoted errors as a function of radial distance (kpc) using $M_C$. Blue dashed curve=Newtonian, full blue curve=conformal gravity, dashed red curve=MOND, full green curve=MLS. No dark matter is assumed.

Figure 2. Fitting to the rotational velocities (km/s) of the Bottema sample with quoted errors as a function of radial distance (kpc) using $M_M$. Blue dashed curve=Newtonian, full blue curve=conformal gravity, dashed red curve=MOND, full green curve=MLS. No dark matter is assumed.
3.4. Input data

In revisiting two surveys that were posited as troublesome for alternative gravity, the choice of input parameters must be systematic and standardized and cannot allow for any inputs to be treated as free parameters. The most important input quantity for rotation curve physics is the distance to the galaxy, as many of the other inputs (such as scale length, gas mass and luminosity) scale in some relation to the distance. A quick glance through the literature shows that for well-studied galaxies there are many distance estimates, sometimes varying by up to a factor of four. To this end, we use the NASA Extragalactic Database (NED), and take the average distance obtained by cepheid measurements. This method typically then results in the use of the most recent data available, and reduces any input bias towards a particular theory. Further, this method does not rely on using distances that were only inferred by using the Tully-Fisher relation [3]. More on the Tully-Fisher relation will be discussed in a later section.

Other inputs such as gas mass, disk scale length and luminosity are taken from the original sources [9] [10] and scaled appropriately to the NED distance. All 27 of the galaxies in the sample have non trivial gas contributions, and the gas masses are assumed to be the HI mass multiplied by a factor of 1.4 to account for the presence of other elements. A few of the galaxies in the samples are galaxies that are noted to have bulges and are thus included in the fits [13]. Thus, for either MOND or CG, the only free parameter per galaxy is the estimated disk mass.
Figure 4. Fitting to the rotational velocities (km/s) of the Carignan sample with quoted errors as a function of radial distance (kpc) using $M_M$. Blue dashed curve=Newtonian, full blue curve= conformal gravity, dashed red curve=MOND, full green curve=MLS. No dark matter is assumed.

and can then be tested against realistic mass to light ratios.

4. Analysis
In order to see some of the differences in the theories in question, we allow the disk mass to be a free parameter and determine the resulting mass value computationally for each theory. The result then, is two separate fits for each galaxy in the two samples using eqs. (1) and (4). To this end, Table 1 and Table 2 show the value for the best fit mass for both CG and MOND, labelled as $M_C$ and $M_M$ respectively (and thus two separate mass to light ratios). Each mass is then plotted for the two theories, showing how each responds to the input masses. Thus in each of Figs. 1, 2, 3, 4 there are numerous plots presented. In each of Figs. 1, 2, 3, 4, the blue dashed curve shows the standard gravity (no dark matter assumed) alone, the solid blue line shows the conformal gravity prediction for the given mass and the red dashed curve shows the MOND prediction for the given mass. In all plots, gas and applicable bulge contributions are included [13]. No dark matter fits are presented here since those fits would require at least a minimum of three free parameters per galaxy (disk mass, halo radius and halo density) and would be out of the scope of this work.

It is clear from the fits shown in Figs. 1, 2, 3, 4 that when the CG mass is used, CG outper-
forms MOND and vice versa. We also see from Tables 1 and 2 that the mass to light ratios are of order unity in the preferred mass fit, showing that with the stated input parameters and no other free parameters, both theories provide respectable fits without the need for dark matter. Since we are only using a single disk, some of the more exotic features in galaxies such as bars are not going to be captured (see for example NGC 7793). The fits presented here for both MOND and CG outperform the original alternative gravity fits from the two surveys, but do so in a more systematic approach with consistent input parameters.

4.1. Radial accelerations
In 2017, McGaugh, Lelli and Schombert used 153 galactic rotation curves, consisting of over 3000 individual data points and showed an empirical correlation in the centripetal accelerations of the baryonic matter ($g_{\text{NEW}}$) and the observed centripetal accelerations ($g_{\text{OBS}}$) for spiral galaxies [5]. The empirical result showed that the fitting function represented in eq. (5) with $g^t = a_0$ highlighted a new way of viewing the missing mass problem. Following the standard set by McGaugh, Lelli and Schombert, we use the 27 galaxy sample, consisting of 1108 total data points and plot the $g_{\text{NEW}}$ vs $g_{\text{OBS}}$ using $g_{\text{NEW}}$ for both $M_C$ and $M_M$. Although this sample is considerably smaller in both number of galaxies and individual data points to the study of McGaugh et al., it can be seen from Fig. 5 that the same trend originally observed in McGuagh et al. is preserved. Recently, two of the authors here have extended the work of McGaugh et al. to over 6000 data points and found similar trends when more galaxies and data points are used, but also found there is sensitivity to the mass used to generate the RAR plot [17]. In Fig. 6, we include the fitting function of eq. (5) in green, and it can be seen that the value of $g^t$ is not optimal in the case when the $M_C$ is used to generate $g_{\text{NEW}}$. However, even with the fixed value of $g^t = a_0$ for this work, we see that there is (as expected) departure from the line of unity (shown in red). It is this departure from unity that McGaugh et al. attributes to either dark matter or new physics.

Having shown that the RAR is preserved, it is instructive for alternative gravity to make similar plots to highlight the extent of the missing matter across the sample. Instead of plotting ($g_{\text{NEW}}$ vs $g_{\text{OBS}}$), one can plot ($g_{\text{CG}}$ vs $g_{\text{OBS}}$) and ($g_{\text{MOND}}$ vs $g_{\text{OBS}}$) to incorporate all of the influences of eqs. (1) and (4) instead of just the contributions to CG and MOND of $g_{\text{NEW}}$. Figure 6 shows...
these plots, with the line of unity shown in red. As expected, departure from the line of unity for these plots is minimal, showing that in both CG and MOND there is no need for dark matter and suggests that these alternative gravitational models may serve as the new physics called for in McGaugh et al.

4.2. The Baryonic Tully-Fisher relation

One of the remarkable features of the RAR, is that it is an empirical finding that shows trends across an entire data sample. The RAR thus functions as a more generalized version of the Baryonic Tully-Fisher relation (BTF) which highlights another empirical relationship between the velocity of the galaxy (in the flattening region of the rotation curve) and the mass of the galaxy as

\[ v_{OBS}^4 \propto M. \] (6)

This relation has long been observed in rotation curves [3] and constantly revisited since its discovery. In this work, since two different mass values \( M_C \) and \( M_M \) are derived, plots of the BTF relation are shown in Fig. 7. In order to keep the natural error in the velocity present, the plots shown are \((M_C^4 \text{ vs. } v_{OBS})\) and \((M_M^4 \text{ vs. } v_{OBS})\) where the last velocity point is used for each galaxy. Recently, two of the authors of this work showed that CG uniquely predicts a departure from pure BTF for the most massive of galaxies [17]. To this end, Fig. 7 shows two functions along with the plotted BTF points. The green function is the pure BTF relation of eq. (6), and the orange is the one predicted by CG theory [17]. We see that at the lower mass scale, the predictions are identical, but at larger masses, there is a drastic departure from pure BTF. Figure 7 highlights that although on an individual galaxy scale, CG and MOND predict different masses, each of those masses remain consistent with BTF. Moreover, both the CG and MOND masses show the departure predicted by conformal gravity. Although this predicted departure represents a testable and falsifiable feature unique to CG, its ramifications are a bit more severe. As mentioned earlier, rotation curve physics (for any theory) are extremely sensitive to distance measurements. Since so many galaxies (as seen in Fig. 7) directly align with BTF, when other distance estimates cannot be obtained, BTF is sometimes used to infer a distance to a galaxy (see NED for example). If a departure from BTF exists, then using BTF to infer a distance (which is then used to infer a mass) can become an incorrect and circular argument. Further work on large spirals and departures from BTF represents future extensions of this work.
5. Conclusion

This work revisits two problematic surveys for alternative gravity. Using a systematic approach for input parameters as well as utilizing tools such as BTF and RAR, we are able to complete a thorough analysis of the two surveys using CG and MOND. The rotation curves predicted by each theory provide reliable fits with mass to light ratios on the order of unity. The 1108 data points generate a RAR plot consistent with MLS formalism and Fig. 6 shows that alternative gravity could provide the new physics sought in McGaugh et al. Further, both MOND and CG yield galactic masses consistent with BTF and some of the galaxies in the sample show a departure predicted only by CG. Future work using the largest available spiral galaxies with reliable rotation curves will provide a more suitable testing ground for departures from pure BTF, and analysis of using the CG departures as a foil will provide some insight into the reliability of using BTF as a tool for galactic distance inference.

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