On a condition of strong precompactness and the decay of periodic entropy solutions to scalar conservation laws

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Abstract

We propose a new sufficient non-degeneracy condition for the strong precompactness of bounded sequences satisfying the nonlinear first-order differential constraints. This result is applied to establish the decay property for periodic entropy solutions to multidimensional scalar conservation laws.

1 Introduction

Let \( \Omega \) be an open domain in \( \mathbb{R}^n \). We consider the sequence \( u_k(x), \ k \in \mathbb{N} \), bounded in \( L^\infty(\Omega) \), which converges weakly-* in \( L^\infty(\Omega) \) to some function \( u(x): u_k \rightarrow u \).

Now let \( \varphi(x,u) \in L^2_{\text{loc}}(\Omega,C(\mathbb{R},\mathbb{R}^n)) \) be a Caratheodory vector-function (i.e. it is continuous with respect to \( u \) and measurable with respect to \( x \)) such that the functions

\[
\alpha_M(x) = \max_{|u| \leq M} |\varphi(x,u)| \in L^2_{\text{loc}}(\Omega) \quad \forall M > 0
\]

(1.1)

(here and below \(| \cdot |\) stands for the Euclidean norm of a finite-dimensional vector).

By \( \theta(\lambda) \) we shall denote the Heaviside function:

\[
\theta(\lambda) = \begin{cases} 
1, & \lambda > 0, \\
0, & \lambda \leq 0.
\end{cases}
\]

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Suppose that for every \( p \in \mathbb{R} \) the sequence of distributions

\[
\text{div}_x [\theta(u_k - p)(\varphi(x,u_k) - \varphi(x,p))] \text{ is precompact in } W^{-1}_{d,\text{loc}}(\Omega)
\]

(1.2)

for some \( d > 1 \). Recall that \( W^{-1}_{d,\text{loc}}(\Omega) \) is a locally convex space of distributions \( u(x) \) such that \( uf(x) \) belongs to the Sobolev space \( W^{-1}_d \) for all \( f(x) \in C_0^\infty(\Omega) \).

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The topology in $W^{-1}_{d,\text{loc}}(\Omega)$ is generated by the family of semi-norms $u \rightarrow \|u f\|_{W^{-1}_d}$, $f(x) \in C_0^\infty(\Omega)$.

If the distributions $\text{div}_x \varphi(x, k)$ are locally finite measures on $\Omega$ for all $k \in \mathbb{R}$, then the notion of entropy solutions (in Kruzhkov’s sense) of the equation

$$\text{div} \varphi(x, u) + \psi(x, u) = 0 \quad (1.3)$$

(with a Caratheodory source function $\psi(x, u) \in L^1_{\text{loc}}(\Omega, C(\mathbb{R}))$) is defined, see [15] and [16] (in the latter paper the more general ultra-parabolic equations are studied). As was shown in [16], assumption (1.2) is always satisfied for bounded sequences of entropy solutions of (1.3).

Our first result is the following strong precompactness property.

**Theorem 1.1.** Suppose that for almost every $x \in \Omega$ and all $\xi \in \mathbb{R}^n$, $\xi \neq 0$ the function $\lambda \rightarrow \xi \cdot \varphi(x, \lambda)$ is not constant in any vicinity of the point $u(x)$ (here and in the sequel “·” denotes the inner product in $\mathbb{R}^n$). Then $u_k(x) \rightarrow u(x)$ in $L^1_{\text{loc}}(\Omega)$ (strongly).

Theorem 1.1 extends the results of [15], where the strong precompactness property was established under the more restrictive non-degeneracy condition: for almost every $x \in \Omega$ and all $\xi \in \mathbb{R}^n$, $\xi \neq 0$ the function $\lambda \rightarrow \xi \cdot \varphi(x, \lambda)$ is not constant on nonempty intervals.

The proof of Theorem 1.1 is based on a new localization principle for H-measure (with “continuous” indexes) corresponding to the sequence $u_k$, see Theorem 3.5 and its Corollary 3.6 below.

Using this theorem and results of [17], we will also derive the more precise criterion for the decay of periodic entropy solutions of scalar conservation laws

$$u_t + \text{div}_x \varphi(u) = 0, \quad (1.4)$$

where $u = u(t, x), (t, x) \in \Pi = (0, +\infty) \times \mathbb{R}^n$. The flux vector $\varphi(u) = (\varphi_1(u), \ldots, \varphi_n(u))$ is supposed to be merely continuous: $\varphi(u) \in C(\mathbb{R}, \mathbb{R}^n)$. Recall the definition of entropy solution to equation (1.4) in the Kruzhkov sense [7].

**Definition 1.2.** A bounded measurable function $u = u(t, x) \in L^\infty(\Pi)$ is called an entropy solution (e.s. for short) of (1.4) if for all $k \in \mathbb{R}$

$$|u - k|_t + \text{div}_x [\text{sign}(u - k)(\varphi(u) - \varphi(k))] \leq 0 \quad (1.5)$$

in the sense of distributions on $\Pi$ (in $\mathcal{D}'(\Pi)$).

As usual, condition (1.5) means that for all non-negative test functions $f = f(t, x) \in C_0^1(\Pi)$

$$\int_\Pi [u - k]f_t + \text{sign}(u - k)(\varphi(u) - \varphi(k)) \cdot \nabla_x f] dt dx \geq 0.$$
As was shown in [13] (see also [14]), an e.s. $u(t, x)$ always admits a strong trace $u_0 = u_0(x) \in L^\infty(\mathbb{R}^n)$ on the initial hyperspace $t = 0$ in the sense of relation

$$\text{ess lim}_{t \to 0} u(t, \cdot) = u_0 \quad \text{in} \quad L^1_{\text{loc}}(\mathbb{R}^n),$$

that is, $u(t, x)$ is an e.s. to the Cauchy problem for equation (1.4) with initial data

$$u(0, x) = u_0(x).$$

Remark 1.3. It was also established in [13, Corollary 7.1] that, after possible correction on a set of null measure, an e.s. $u(t, x)$ is continuous on $[0, +\infty)$ as a map $t \mapsto u(t, \cdot)$ into $L^1_{\text{loc}}(\mathbb{R}^n)$. In the sequel we will always assume that this property is satisfied.

Suppose that the initial function $u_0$ is periodic with a lattice of periods $L$, i.e., $u_0(x + e) = u_0(x)$ a.e. on $\mathbb{R}^n$ for every $e \in L$ (we will call such functions $L$-periodic). Denote by $\mathbb{T}^n = \mathbb{R}^n / L$ the corresponding $n$-dimensional torus, and by $L'$ the dual lattice $L' = \{ \xi \in \mathbb{R}^n \mid \xi \cdot x \in \mathbb{Z} \quad \forall x \in L \}$. In the case under consideration when the flux vector is merely continuous the property of finite speed of propagation for initial perturbation may be violated, which, in the multidimensional situation $n > 1$, may even lead to the nonuniqueness of e.s. to Cauchy problem (1.4), (1.7), see examples in [8, 9]. But for a periodic initial function $u_0(x)$, an e.s. $u(t, x)$ of (1.4), (1.7) is unique (in the class of all e.s., not necessarily periodic) and space-periodic, the proof can be found in [12]. It is also shown in [12] that the mean value of e.s. over the period does not depend on time:

$$\int_{\mathbb{T}^n} u(t, x) dx = I = \int_{\mathbb{T}^n} u_0(x) dx,$$

where $dx$ is the normalized Lebesgue measure on $\mathbb{T}^n$. The following theorem generalizes the previous results of [3, 17].

**Theorem 1.4.** Suppose that

$$\forall \xi \in L', \xi \neq 0 \text{ the function } u \to \xi \cdot \varphi(u)$$

is not affine on any vicinity of $I$. \hfill (1.9)

Then

$$\lim_{t \to +\infty} u(t, \cdot) = I = \int_{\mathbb{T}^n} u_0(x) dx \quad \text{in} \quad L^1(\mathbb{T}^n).$$

Moreover condition (1.9) is necessary and sufficient for the decay property (1.10).

In the case $\varphi(u) \in C^2(\mathbb{R}, \mathbb{R}^n)$ Theorem (1.4) was proved in [3]. As was noticed in [3, Remark 2.1], decay property (1.10) holds under the the weaker regularity requirement $\varphi(u) \in C^1(\mathbb{R}, \mathbb{R}^n)$ but under the more restrictive assumption that for each $\xi \in L'$ $I$ is not an interior point of the closure of the union of
all open intervals, over which the function $\xi \cdot \varphi'(u)$ is constant. Let us demonstrate that condition (1.9) is less restrictive than this assumption even in the case $\varphi(u) \in C^1(\mathbb{R}, \mathbb{R}^n)$. Suppose that $n = 1$, $\varphi(u) \in C^1(\mathbb{R})$ is a primitive of the Cantor function, so that $\varphi'(u)$ is increasing, continuous, and maximal intervals, over which it remains constant, are exactly the connected component of the complement $\mathbb{R} \setminus K$ of the Cantor set $K \subset [0,1]$. Since $K$ has the empty interior the assumption of [3] is never satisfied while (1.9) holds for each $I \in K$.

2 Preliminaries

We need the concept of measure valued functions (Young measures). Recall (see [4, 20]) that a measure-valued function on $\Omega$ is a weakly measurable map $x \mapsto \nu_x$ of $\Omega$ into the space Prob$_0(\mathbb{R})$ of probability Borel measures with compact support in $\mathbb{R}$.

The weak measurability of $\nu_x$ means that for each continuous function $g(\lambda)$ the function $x \mapsto \langle \nu_x, g(\lambda) \rangle = \int g(\lambda) d\nu_x(\lambda)$ is measurable on $\Omega$.

A measure-valued function $\nu_x$ is said to be bounded if there exists $M > 0$ such that $\text{supp} \nu_x \subset [-M, M]$ for almost all $x \in \Omega$.

Measure-valued functions of the kind $\nu_x(\lambda) = \delta(\lambda - u(x))$, where $u(x) \in L^\infty(\Omega)$ and $\delta(\lambda - u^*)$ is the Dirac measure at $u^* \in \mathbb{R}$, are called regular. We identify these measure-valued functions and the corresponding functions $u(x)$, so that there is a natural embedding of $L^\infty(\Omega)$ into the set MV(\Omega) of bounded measure-valued functions on $\Omega$.

Measure-valued functions naturally arise as weak limits of bounded sequences in $L^\infty(\Pi)$ in the sense of the following theorem by L. Tartar [20].

**Theorem 2.1.** Let $u_k(x) \in L^\infty(\Omega)$, $k \in \mathbb{N}$, be a bounded sequence. Then there exist a subsequence (we keep the notation $u_k(x)$ for this subsequence) and a bounded measure valued function $\nu_x \in \text{MV}(\Omega)$ such that

$$\forall g(\lambda) \in C(\mathbb{R}) \quad g(u_k) \to_{k \to \infty} \langle \nu_x, g(\lambda) \rangle \quad \text{weakly-\ast in } L^\infty(\Omega).$$  \hspace{1cm} (2.1)

Besides, $\nu_x$ is regular, i.e., $\nu_x(\lambda) = \delta(\lambda - u(x))$ if and only if $u_k(x) \to_{k \to \infty} u(x)$ in $L^1_{\text{loc}}(\Omega)$ (strongly).

We will essentially use in the sequel the variant of H-measures with “continuous indexes” introduced in [10]. This variant extends the original concept of H-measure invented by L. Tartar [21] and P. Gerard [5] and it appears to be a powerful tool in nonlinear analysis.

Suppose $u_k(x)$ is a bounded sequence in $L^\infty(\Omega)$. Passing to a subsequence if necessary, we can suppose that this sequence converges to a bounded measure valued function $\nu_x \in \text{MV}(\Omega)$ in the sense of relation (2.1). We introduce the
measures \( \gamma^k_\lambda = \delta(\lambda - u_k(x)) - \nu_x(\lambda) \) and the corresponding distribution functions \( U_k(x, p) = \gamma^k_\lambda(p, +\infty) \), \( u_0(x, p) = \nu_x(p, +\infty) \) on \( \Omega \times \mathbb{R} \). Observe that \( U_k(x, p), u_0(x, p) \in L^\infty(\Omega) \) for all \( p \in \mathbb{R} \), see [10] Lemma 2]. We define the set

\[
E = E(\nu_x) = \left\{ p_0 \in \mathbb{R} \mid u_0(x, p) \to u_0(x, p_0) \text{ in } L^1_{loc}(\Omega) \right\}.
\]

As was shown in [10] Lemma 4], the complement \( \mathbb{R} \setminus E \) is at most countable and if \( p \in E \) then \( U_k(x, p) \to 0 \) weakly-* in \( L^\infty(\Omega) \).

Let \( F(u)(\xi), \xi \in \mathbb{R}^n \), be the Fourier transform of a function \( u(x) \in L^2(\mathbb{R}^n) \), \( S = S^{n-1} = \{ \xi \in \mathbb{R}^n \mid |\xi| = 1 \} \) be the unit sphere in \( \mathbb{R}^n \). Denote by \( u \to \overline{u}, \ u \in \mathbb{C} \) the complex conjugation.

The next result was established in [10] Theorem 3], [11] Proposition 2, Lemma 2].

**Proposition 2.2.** (i) There exists a family of locally finite complex Borel measures \( \{\mu_{pq}\}_{p,q \in E} \) in \( \Omega \times S \) and a subsequence \( U_r(x, p) = U_{k_r}(x, p) \) such that for all \( \Phi_1(x), \Phi_2(x) \in C_0(\Omega) \) and \( \psi(\xi) \in C(S) \)

\[
\langle \mu_{pq}, \Phi_1(x)\Phi_2(x)\psi(\xi) \rangle = \lim_{r \to \infty} \int_{\mathbb{R}^n} F(\Phi_1 U_r(\cdot, p))(\xi)F(\Phi_2 U_r(\cdot, q))(\xi)\psi\left(\frac{\xi}{|\xi|}\right) \, d\xi;
\]

(2.2)

(ii) For any \( p_1, \ldots, p_l \in E \) the matrix \( \{\mu_{pq}\}_{i,j=1}^l \) is Hermitian and nonnegative definite, that is, for all \( \xi_1, \ldots, \xi_l \in \mathbb{C} \) the measure

\[
\sum_{i,j=1}^l \xi_i\overline{\xi_j} \mu_{pq} \geq 0.
\]

We call the family of measures \( \{\mu_{pq}\}_{p,q \in E} \) the H-measure corresponding to the subsequence \( u_r(x) = u_{k_r}(x) \).

As was demonstrated in [10], the H-measure \( \mu_{pq} = 0 \) for all \( p, q \in E \) if and only if the subsequence \( u_r(x) \) converges as \( r \to \infty \) strongly (in \( L^1_{loc}(\Omega) \)).

Since \( |U_k(x, p)| \leq 1 \), it readily follows from (2.2) and Plancherel’s equality that \( \text{pr}_\Omega|\mu_{pq}| \leq \text{meas} \) for \( p, q \in E \), where \( \text{meas} \) is the Lebesgue measure on \( \Omega \), and by \( |\mu| \) we denote the variation of a Borel measure \( \mu \) (this is the minimal of nonnegative Borel measures \( \nu \) such that \( |\mu(A)| \leq \nu(A) \) for all Borel sets \( A \)). This implies the representation \( \mu_{pq} = \mu_{pq}(dx) \) (the disintegration of H-measures). More exactly, choose a countable dense subset \( D \subset E \). The following statement was proved in [11] Proposition 3], see also [15] Proposition 3].

**Proposition 2.3.** There exists a family of complex finite Borel measures \( \mu^p_{x} \in M(S) \) in the sphere \( S \) with \( p, q \in D, x \in \Omega' \), where \( \Omega' \) is a subset of \( \Omega \) of full measure, such that \( \mu_{pq} = \mu^p_{x}(dx) \), that is, for all \( \Phi(x, \xi) \in C_0(\Omega \times S) \) the function

\[
x \to \langle \mu^p_{x}(\xi), \Phi(x, \xi) \rangle = \int_{S} \Phi(x, \xi) \, d\mu^p_{x}(\xi)
\]
is Lebesgue-measurable on $\Omega$, bounded, and

$$\langle \mu_{x}^{pq}, \Phi(x, \xi) \rangle = \int_{\Omega} \langle \mu_{x}^{pq}(\xi), \Phi(x, \xi) \rangle \, dx.$$  

Moreover, for $p, p', q \in D$, $p' > p$

$$\text{Var} \mu_{x}^{pq} = |\mu_{x}^{pq}|(S) \leq 1 \quad \text{and} \quad \text{Var} (\mu_{x}^{pq} - \mu_{x}^{pq}) \leq 2 (\nu_{x}((p, p')))^{1/2}. \quad (2.3)$$

We choose a non-negative function $K(x) \in C_{0}^{\infty}(\mathbb{R}^{n})$ with support in the unit ball such that $\int K(x) \, dx = 1$ and set $K_{m}(x) = m^{n}K(mx)$ for $m \in \mathbb{N}$. Clearly, the sequence of $K_{m}$ converges in $\mathcal{D}'(\mathbb{R}^{n})$ to the Dirac $\delta$-function (that is, this sequence is an approximate unity). We define $\Phi_{m}(x) = (K_{m}(x))^{1/2}$. As was shown in [11, Remark 4] (see also [15, Remark 2(b)]), the measures $\mu_{x}^{pq}$ can be explicitly represented by the relation

$$\Phi(x)\langle \mu_{x}^{pq}, \psi(\xi) \rangle = \lim_{m \to \infty} \langle \mu_{x}^{pq}(y, \xi), \Phi(y)K_{m}(x-y)\psi(\xi) \rangle =$$

$$\lim_{m \to \infty} \lim_{r \to \infty} \int_{\mathbb{R}^{n}} F(\Phi_{m}U_{r}(\cdot, p))(\xi)F(\Phi_{m}U_{r}(\cdot, q))(\xi)\psi \left( \frac{\xi}{|\xi|} \right) d\xi \quad (2.4)$$

for all $\psi(\xi) \in C(S)$, where $\Phi_{m}U_{r}(y) = \Phi(y)\Phi_{m}(x-y)U_{r}(y, p)$, $\Phi_{m}U_{r}(y) = \Phi_{m}(x-y)U_{r}(y, q)$, and $\Phi(y) \in L_{loc}^{2}(\Omega)$ be an arbitrary function such that $x$ is its Lebesgue point.

From this representation (with $\Phi \equiv 1$) and Proposition 2.2(ii) it follows that for all $p_{1}, \ldots, p_{l} \in D$, $x \in \Omega'$, $\zeta_{1}, \ldots, \zeta_{l} \in \mathbb{C}$ the measure

$$\mu = \sum_{i,j=1}^{l} \zeta_{i}\overline{\zeta}_{j}\mu_{x}^{p_{i}p_{j}} \geq 0. \quad (2.5)$$

Indeed, for every nonnegative $\psi(\xi) \in C(S)$

$$< \mu(\xi), \psi(\xi) > = \lim_{m \to \infty} \left( \sum_{i,j=1}^{l} \zeta_{i}\overline{\zeta}_{j}\mu_{x}^{p_{i}p_{j}}(y, \xi), K_{m}(x-y)\psi(\xi) \right) \geq 0.$$

This, in particular implies, that $\mu_{x}^{pp} \geq 0$, $\mu_{x}^{qq} = \overline{\mu_{x}^{pp}}$, and for every Borel set $A \subset S$

$$|\mu_{x}^{pq}|(A) \leq (\mu_{x}^{pp}(A)\mu_{x}^{qq}(A))^{1/2} \quad (2.6)$$

(see [11] [15]). For completeness we provide below the simple proof of (2.6). In view of (2.6) (with $l=2$) the matrix $M = \left( \begin{array}{cc} \mu_{x}^{pp}(A) & \mu_{x}^{pq}(A) \\
\mu_{x}^{qp}(A) & \mu_{x}^{qq}(A) \end{array} \right)$ is Hermitian and nonnegative definite. Therefore,

$$\mu_{x}^{pp}(A)\mu_{x}^{qq}(A) - |\mu_{x}^{pq}(A)|^{2} = \mu_{x}^{pp}(A)\mu_{x}^{qq}(A) - \mu_{x}^{pp}(A)\mu_{x}^{pp}(A) = \det M \geq 0.$$
By Young’s inequality for any positive constant $c$ and all Borel sets $A \subset S$
\[
|\mu_x^{pq}(A)| \leq (\mu_x^{pp}(A)\mu_x^{qq}(A))^{1/2} \leq \frac{c}{2} \mu_x^{pp}(A) + \frac{1}{2c} \mu_x^{qq}(A).
\]

Since $\mu = \frac{c}{2} \mu_x^{pp} + \frac{1}{2c} \mu_x^{qq}$ is nonnegative Borel measure, it follows from this inequality that the variation $|\mu_x^{pq}| \leq \mu$. This implies that
\[
|\mu_x^{pq}(A)| \leq \frac{c}{2} \mu_x^{pp}(A) + \frac{1}{2c} \mu_x^{qq}(A) \quad \forall c > 0. 
\]

It is easily computed that
\[
\inf_{c>0} \left( \frac{c}{2} \mu_x^{pp}(A) + \frac{1}{2c} \mu_x^{qq}(A) \right) = (\mu_x^{pp}(A)\mu_x^{qq}(A))^{1/2}
\]
and (2.6) follows from (2.7).

### 3 Localization principles and the strong pre-compactness property

**Lemma 3.1.** For each $p, q \in \mathbb{R}$, $x \in \Omega'$ there exist one-sided limits in the space $M(S)$ of finite Borel measures on $S$ (with the standard norm $\text{Var} \, \mu$):
\[
\mu_x^{p'q'} \rightarrow \mu_x^{p'q'}^+ \quad \text{as} \quad (p', q') \rightarrow (p, q), \quad p', q' \in D, \quad p' > p, \quad q' > q,
\]
\[
\mu_x^{p'q'} \rightarrow \mu_x^{p'q'}^- \quad \text{as} \quad (p', q') \rightarrow (p, q), \quad p', q' \in D, \quad p' < p, \quad q' < q.
\]

Moreover, $\text{Var} \, \mu_x^{pq} \leq 1$ and for every Borel set $A \subset S$ and each $p_i, q_i \in \mathbb{R}$, $i = 1, \ldots, l$ the matrices $\{\mu_x^{p_iq_i}(A)\}_{i=1}^l$ are Hermitian and nonnegative definite, that is, the measures
\[
\sum_{i,j=1}^l \zeta_i\zeta_j \mu_x^{p_ip_j} \geq 0 \quad (3.1)
\]
for all complex $\zeta_i \in \mathbb{C}$, $i = 1, \ldots, l$.

**Proof.** Let $x \in \Omega'$, $p, q \in \mathbb{R}$, $p_1, q_1, p_2, q_2 \in D$, $p_2 > p_1 > p$, $q_2 > q_1 > q$. Then, in view of (2.3) and the equality $\mu_x^{pq} = \mu_x^{qp}$,
\[
\text{Var} \, (\mu_x^{p_2q_2} - \mu_x^{p_1q_1}) \leq 2\nu_x((p_1, p_2)) + 2\nu_x((q_1, q_2)) \leq 2\nu_x((p, p_2)) + 2\nu_x((q, q_2)) \rightarrow (p_2, q_2) \rightarrow (p, q), 0.
\]

By the Cauchy criterion, this implies that there exists a limit $\mu_x^{pq}^+$ in $M(S)$ as $(p', q') \rightarrow (p, q)$, $p', q' \in D$, $p' > p$, $q' > q$. Similarly, for each $p_1, q_1, p_2, q_2 \in D$ such that $p_2 < p_1 < p$, $q_2 < q_1 < q$
\[
\text{Var} \, (\mu_x^{p_2q_2} - \mu_x^{p_1q_1}) \leq 2\nu_x((p_2, p_1)) + 2\nu_x((q_2, q_1)) \leq 2\nu_x((p_2, p)) + 2\nu_x((q_2, q)) \rightarrow (p_2, q_2) \rightarrow (p, q), 0.
\]
which implies existence of a left-sided limit $\mu^{pq}_x$ in $M(S)$ as $(p', q') \to (p, q)$, $p', q' \in D$, $p' < p$, $q' < q$. By Proposition 2.3 Var $\mu^{pq}_x \leq 1$, which implies in the limits as $p' \to p \pm$, $q' \to q \pm$ that Var $\mu^{pq}_x \leq 1$. Finally, for every $p_i' \in D$, $\zeta_i \in \mathbb{C}$, $i = 1, \ldots, l$ the measures

$$
\sum_{i,j=1}^l \zeta_i \zeta_j \mu^{pq}_x \geq 0.
$$

In the limits as $p_i' \to p_i \pm$ this implies \(\text{(3.1)}\). \qed

**Corollary 3.2.** Let $p, q \in \mathbb{R}$, $x \in \Omega'$. Then for every Borel set $A \subset S$

$$
|\mu^{pq}_x|(A) \leq \left( (\mu^{pp}_x(A) \mu^{qq}_x(A))^{1/2} \right),
$$

$$
(3.2)
$$

**Proof.** Relations (3.2) follow from (3.1) in the same way as in the proof of inequality (2.6) above. \qed

**Remark 3.3.** By continuity of $\mu^{pq}_x$ with respect to variables $p, q \in D$, we see that for $p \in D$

$$
\mu^{pq}_x = \lim_{q' \to q \pm} \lim_{p' \to p \pm} \mu^{p'q'}_x = \lim_{q' \to q \pm} \mu^{pq}_x' \text{ in } M(S).
$$

Analogously, if $q \in D$, then

$$
\mu^{pq}_x = \lim_{p' \to p \pm} \mu^{pq}_x \text{ in } M(S).
$$

If the both indices $p, q \in D$, then evidently $\mu^{pq}_x = \mu^{pq}_x'$.

Now we suppose that $f(y, \lambda) \in L^2_{\text{loc}}(\Omega, C(\mathbb{R}, \mathbb{R}^n))$ is a Caratheodory vector-function on $\Omega \times \mathbb{R}$. In particular,

$$
\forall M > 0 \quad \|f(x, \cdot)\|_{M, \infty} = \max_{|\lambda| \leq M} |f(x, \lambda)| = \alpha_M(x) \in L^2_{\text{loc}}(\Omega). \quad (3.3)
$$

Since the space $C(\mathbb{R}, \mathbb{R}^n)$ is separable with respect to the standard locally convex topology generated by seminorms $\|\cdot\|_{M, \infty}$, then, by the Pettis theorem (see \[6\], Chapter 3), the map $x \to F(x) = f(x, \cdot) \in C(\mathbb{R}, \mathbb{R}^n)$ is strongly measurable and in view of estimate (3.3) we see that $|F(x)|^2 \in L^1_{\text{loc}}(\Omega, C(\mathbb{R}))$. In particular (see \[6\], Chapter 3), the set $\Omega_f$ of common Lebesgue points of the maps $F(x), |F(x)|^2$ has full measure. As was demonstrated in \[15\], for $x \in \Omega_f$

$$
\lim_{m \to \infty} \int K_m(x - y)\|F(x) - F(y)\|_{M, \infty}^2 dy = 0 \quad \forall M > 0. \quad (3.4)
$$

Clearly, each $x \in \Omega_f$ is a common Lebesgue point of all functions $x \to f(x, \lambda)$, $\lambda \in \mathbb{R}$. Let $\Omega'' = \Omega' \cap \Omega_f$, $\gamma_x^\tau(\lambda) = \delta(\lambda - u_x(x)) - \nu_x(\lambda)$.

Suppose that $x \in \Omega'', p \in \mathbb{R}$, $H_+, H_-$ are the minimal linear subspaces of $\mathbb{R}^n$, containing supports of the measures $\mu^{pp}_x$, $\mu^{pp}_x$, respectively. We fix $q \in D$ and introduce for $p' \in D$ the function

$$
I_r(y, p') = \int f(y, \lambda)(\theta(\lambda - p') - \theta(\lambda - q))d\gamma_y^\tau(\lambda) \in L^2_{\text{loc}}(\Omega). \quad (3.5)
$$
Proposition 3.4. Assume that $q > p$ and $f(x, \lambda) \in H^+_\lambda$ for all $\lambda \in \mathbb{R}$. Then
\[
\lim_{p' \to p+} \lim_{m \to \infty} \lim_{r \to \infty} \int_{\mathbb{R}^n} \frac{\xi}{|\xi|} \cdot F(\Phi_m I_r(\cdot, p'))(\xi) \overline{F(\Phi_m U_r(\cdot, p'))(\xi)} \psi \left( \frac{\xi}{|\xi|} \right) d\xi = 0 \quad (3.6)
\]
for all $\psi(\xi) \in C(S)$. Analogously, if $q < p$ and $f(x, \lambda) \in H^-_\lambda \forall \lambda \in \mathbb{R}$, then $\forall \psi(\xi) \in C(S)$
\[
\lim_{p' \to p-} \lim_{m \to \infty} \lim_{r \to \infty} \int_{\mathbb{R}^n} \frac{\xi}{|\xi|} \cdot F(\Phi_m I_r(\cdot, p'))(\xi) \overline{F(\Phi_m U_r(\cdot, p'))(\xi)} \psi \left( \frac{\xi}{|\xi|} \right) d\xi = 0. \quad (3.7)
\]
Here $\Phi_m = \Phi_m(x - y) = \sqrt{K_m(x - y)}$ and $I_r(y, p'), U_r(y, p')$ are functions of the variable $y \in \Omega$.

Proof. Note that starting from some index $m$ the supports of the functions $\Phi_m(x - y)$ lie in some compact subset $B$ of $\Omega$. Without loss of generality we can assume that $\text{supp } \Phi_m \subset B$ for all $m \in \mathbb{N}$. Let
\[
I_r(y, p') = \int f(x, \lambda)(\theta(\lambda - p') - \theta(\lambda - q)) d\gamma_y^r(\lambda) \in L^2_{\text{loc}}(\Omega),
\]
\[
M = \sup_{r \in \mathbb{N}} \|u_r\|_{\infty}. \quad \text{Then } \text{supp } \gamma_y^r \subset [-M, M], \text{ and}
\]
\[
|I_r(y, p') - I_r(y, p')| \leq \int |f(y, \lambda) - f(x, \lambda)| d|\gamma_y^r(\lambda)| \leq 2\|F(y) - F(x)\|_{M, \infty}.
\]
By Plancherel’s identity
\[
\left| \int_{\mathbb{R}^n} \frac{\xi}{|\xi|} \cdot F(\Phi_m I_r(\cdot, p'))(\xi) \overline{F(\Phi_m U_r(\cdot, p'))(\xi)} \psi \left( \frac{\xi}{|\xi|} \right) d\xi \right| = \left| \int_{\mathbb{R}^n} \frac{\xi}{|\xi|} \cdot F(\Phi_m \tilde{I}_r(\cdot, p'))(\xi) \overline{F(\Phi_m U_r(\cdot, p'))(\xi)} \psi \left( \frac{\xi}{|\xi|} \right) d\xi \right| \leq \|\psi\|_{\infty}\|\Phi_m(I_r(\cdot, p') - \tilde{I}_r(\cdot, p'))\|_2 \|\Phi_m U_r(\cdot, p')\|_2 \leq 2\|\psi\|_{\infty} \left( \int K_m(x - y)\|F(y) - F(x)\|_{M, \infty}^2 dy \right)^{1/2}.
\]
Here we take account of the equality
\[
\|\Phi_m\|_2 = \left( \int_{\Omega} K_m(x - y) dy \right)^{1/2} = 1.
\]
Moreover, by the density of $D$ for all $\psi$ we obtain

$$g$$

valued function $p$ of the form $g(\lambda) = \sum_{i=1}^{k} v_i \theta(\lambda - p_i)$, where $v_i \in H_+^1$, $p = p_1 < p_2 < \cdots < p_k = q$ such that $\|\tilde{f}(\lambda) - g\|_{\infty} \leq \varepsilon$ on $\mathbb{R}$. Here $\chi(\lambda) = \theta(\lambda - p) - \theta(\lambda - q)$. Moreover, by the density of $D$, we may suppose that $p_i \in D$ for $i > 1$. We define for $p' \in D \cap (p, p_2)$

$$J_r(y, p') = \int g(\lambda) \theta(\lambda - p') d\gamma_y^r(\lambda).$$

Using again Plancherel’s identity and the fact that

$$|I_r(y, p') - J_r(y, p')| = \left| \int (\tilde{f} \cdot \chi - g)(\lambda) \theta(\lambda - p') d\gamma_y^r(\lambda) \right| \leq \int |(\tilde{f} \cdot \chi - g)(\lambda)| d|\gamma_y^r|(\lambda) \leq 2\varepsilon,$$

we obtain

$$\left| \int_{\mathbb{R}^n} \frac{\xi}{|\xi|} \cdot F(\Phi_m I_r(\cdot, p'))(\xi) \overline{F(\Phi_m U_r(\cdot, p'))(\xi)} \psi \left( \frac{\xi}{|\xi|} \right) d\xi \right| - \int_{\mathbb{R}^n} \frac{\xi}{|\xi|} \cdot F(\Phi_m J_r(\cdot, p'))(\xi) \overline{F(\Phi_m U_r(\cdot, p'))(\xi)} \psi \left( \frac{\xi}{|\xi|} \right) d\xi =$$

$$\left| \int_{\mathbb{R}^n} \frac{\xi}{|\xi|} \cdot F(\Phi_m (I_r(\cdot, p') - J_r(\cdot, p')))(\xi) \overline{F(\Phi_m U_r(\cdot, p'))(\xi)} \psi \left( \frac{\xi}{|\xi|} \right) d\xi \right| \leq \| \Phi_m (I_r(\cdot, p') - J_r(\cdot, p')) \|_{2} \cdot \| \Phi_m U_r(\cdot, p') \|_{2} \cdot \| \psi \|_{\infty} \leq 2 \| \psi \|_{\infty} \varepsilon \quad (3.9)$$

for all $\psi(\xi) \in C(S)$. Since

$$J_r(y, p') = \int \left( \sum_{i=1}^{k} v_i \theta(\lambda - p_i') \right) d\gamma_y^r(\lambda) = \sum_{i=1}^{k} v_i U_r(y, p'_i),$$

where $p'_i = \max(p_i, p'_i) \in D$, it follows from (2.4) with account of Remark 3.3 that

$$\lim_{p' \to p^+} \lim_{m \to \infty} \lim_{r \to \infty} \int_{\mathbb{R}^n} \frac{\xi}{|\xi|} \cdot F(\Phi_m J_r(\cdot, p'))(\xi) \overline{F(\Phi_m U_r(\cdot, p'))(\xi)} \psi \left( \frac{\xi}{|\xi|} \right) d\xi =$$

$$\lim_{p' \to p^+} \sum_{i=1}^{k} \langle \mu_x^{p'_i}, (v_i \cdot \xi) \psi(\xi) \rangle = \sum_{i=1}^{k} \langle \nu_x^{p'_i}, (v_i \cdot \xi) \psi(\xi) \rangle = 0. \quad (3.10)$$
The last equality is a consequence of the inclusion supp \( \mu_{x}^{p,q+} \subset \text{supp} \mu_{x}^{Pq+} \subset H_{+} \) (because of Corollary 3.2) combined with the relation \( v_{i} \perp H_{+} \). By (3.8), (3.9) and (3.10), we have

\[
\lim_{p' \to p+} \lim_{m \to \infty} \lim_{r \to \infty} \left| \int_{\mathbb{R}^{n}} \frac{\xi}{|\xi|} \cdot F(\Phi_{m}I_{r}(\cdot,p'))(\xi)F(\Phi_{m}U_{r}(\cdot,p'))(\xi) \psi \left( \frac{\xi}{|\xi|} \right) d\xi \right| \leq \text{const} \varepsilon,
\]

and it suffices to observe that \( \varepsilon > 0 \) can be arbitrary to complete the proof of (3.6). The proof of relation (3.7) is similar to the proof of (3.6) and is omitted. \( \square \)

Now we assume that the sequence \( u_{k} \) satisfies constraints (1.2). We choose a subsequence \( u_{r} \) and the corresponding H-measure \( \mu_{p} = \mu_{x}^{p} dx \). Assume that \( x \in \Omega'' = \Omega' \cap \Omega_{\varphi}, p_{0} \in \mathbb{R} \). As above, let \( H_{+}, H_{-} \) be the minimal linear subspaces of \( \mathbb{R}^{n} \) containing supp \( \mu_{x}^{q0p+} \), supp \( \mu_{x}^{p0q} \), respectively.

**Theorem 3.5** (localization principle). There exists a positive \( \delta \) such that \( (\varphi(x,\lambda) - \varphi(x,p)) \cdot \xi = 0 \) for all \( \xi \in H_{+} \), \( \lambda \in [p_{0}, p_{0} + \delta] \) and all \( \xi \in H_{-} \), \( \lambda \in [p_{0} - \delta, p_{0}] \).

**Proof.** The proof is analogous to the proof of [15, Theorem 4] (if \( d = 2 \)), for arbitrary \( d > 1 \) see the proof of [16, Theorem 4] (where the more general case of ultra-parabolic constraints was treated). For completeness we provide the details. Observe firstly that in view of (1.2) the sequence of distributions

\[
L_{p}^{x}(y) = \text{div}_{y} \left( \int_{\mathbb{R}^{n}} \theta(\lambda - p)(\varphi(y,\lambda) - \varphi(y,p))d\gamma_{y}^{r}(\lambda) \right) \to 0 \text{ in } W_{d,\text{loc}}^{-1}(\Omega). \quad (3.11)
\]

For \( p, q \in D, q > p > p_{0} \) we consider the sequence of distributions

\[
L_{q}^{r} - L_{p}^{r} = \text{div}_{y} \left( Q_{q}^{r}(y) \right), \quad r \in \mathbb{N},
\]

where the vector-valued functions \( Q_{p}^{r}(y) \) (for fixed \( q \in D \)) are as follows:

\[
Q_{q}^{r}(y) = \int (\varphi(y,\lambda) - \varphi(y,q))\theta(\lambda - q)d\gamma_{y}^{r}(\lambda) - \\
\int (\varphi(y,\lambda) - \varphi(y,p))\theta(\lambda - p)d\gamma_{y}^{r}(\lambda) = \\
\int (\varphi(y,q) - \varphi(y,\lambda))\chi(\lambda)d\gamma_{y}^{r}(\lambda) - \\
\int (\varphi(y,q) - \varphi(y,p))\theta(\lambda - p)d\gamma_{y}^{r}(\lambda) = \\
\int (\varphi(y,q) - \varphi(y,\lambda))\chi(\lambda)d\gamma_{y}^{r}(\lambda) - (\varphi(y,q) - \varphi(y,p))U_{r}(y,p);
\]

here \( \chi(\lambda) = \theta(\lambda - p) - \theta(\lambda - q) \) is the indicator function of the segment \( (p,q) \). As was already noted, \( \text{div}_{y} \left( Q_{q}^{r}(y) \right) \to 0 \) in \( W_{d,\text{loc}}^{-1}(\Omega) \) and if \( \Phi(y) \in C_{0}^{\infty}(\Omega) \) then

\[
\text{div}_{y} \left( Q_{q}^{r}\Phi(y) \right) \to 0 \text{ in } W_{d}^{-1}. \quad (3.13)
\]
Using the Fourier transformation, from (3.13) we obtain

\[ |\xi|^{-1} \cdot F(Q_r^p \Phi)(\xi) = F(g_r), \quad g_r \to 0 \text{ in } L^d(\mathbb{R}^n) \]  
(3.14)

(see [15, 16] for details).

Let \( \psi(\xi) \in C^\infty(S) \). By the known Marcinkiewicz multiplier theorem (cf. [19, Chapter 4]) \( \psi(\xi/|\xi|) \) is a Fourier multiplier in \( L^s \) for all \( s > 1 \). This implies that

\[ F(U_r(\cdot,p)\Phi)(\xi)\psi(\frac{\xi}{|\xi|}) = F(h_r)(\xi), \]  
(3.15)

where the sequence \( h_r \) is bounded in \( L^{d'} \), \( d' = d/(d - 1) \).

By (3.14), (3.15) we obtain

\[ \int_{\mathbb{R}^n} |\xi|^{-1} \cdot F(Q_r^p \Phi)(\xi)F(U_r(\cdot,p)\Phi)(\xi)\psi\left(\frac{\xi}{|\xi|}\right) = \int_{\mathbb{R}^n} g_r(x)h_r(x)dx \to 0 \]
as \( r \to \infty \), or in view of (3.12),

\[ \lim_{r \to \infty} \left\{ \int_{\mathbb{R}^n} |\xi|^{-1} \cdot F(U(\cdot,p)f \Phi)(\xi)F(U(\cdot,p)\Phi)(\xi)\psi\left(\frac{\xi}{|\xi|}\right) \right\} \to 0, \]  
(3.16)

where

\[ f(y) = \varphi(y, q) - \varphi(y, p) \quad \text{and} \quad V_r(y, p) = \int (\varphi(y, q) - \varphi(y, \lambda)) \chi(\lambda) d\gamma_\lambda^r(\lambda). \]

Obviously, (3.16) remains valid for merely continuous \( \psi(\xi) \). We set in (3.16) \( \Phi(y) = \Phi_m(x - y) \), where the functions \( \Phi_m \) were defined in section 2 and pass to the limit as \( m \to \infty \), \( p \to p_0^+ \). By (2.4) with \( \Phi(y) = \varphi(y, q) - \varphi(y, p) \) and Lemma 3.1 we obtain

\[ \lim_{p \to p_0^+} \lim_{m \to \infty} \lim_{r \to \infty} \int_{\mathbb{R}^n} |\xi|^{-1} \cdot F(U_r(\cdot,p)f \Phi_m)(\xi)F(U_r(\cdot,p)\Phi_m)(\xi)\psi\left(\frac{\xi}{|\xi|}\right) d\xi = \]

\[ \lim_{p \to p_0^+} (\varphi(x, q) - \varphi(x, p)) \cdot \langle \mu_{x^p}^{p_0}, \xi \psi(\xi) \rangle = (\varphi(x, q) - \varphi(x, p_0)) \cdot \langle \mu_{x^p}^{p_0p_0^+}, \xi \psi(\xi) \rangle, \]

therefore

\[ (\varphi(x, q) - \varphi(x, p_0)) \cdot \langle \mu_{x^p}^{p_0p_0^+}, \xi \psi(\xi) \rangle = \]

\[ \lim_{p \to p_0^+} \lim_{m \to \infty} \lim_{r \to \infty} \int_{\mathbb{R}^n} |\xi|^{-1} \cdot F(V_r(\cdot,p)\Phi_m)(\xi)F(U_r(\cdot,p)\Phi_m)(\xi)\psi\left(\frac{\xi}{|\xi|}\right) d\xi. \]  
(3.17)

Let \( \pi_1 \) and \( \pi_2 \) be the orthogonal projections of \( \mathbb{R}^n \) onto the subspaces \( H_+ \) and \( H_+^\perp \), respectively; let \( \tilde{\varphi}(x, \lambda) = \pi_1(\varphi(x, \lambda)), \tilde{\varphi}(x, \lambda) = \pi_2(\varphi(x, \lambda)) \). Recall that
Further, \( \langle \varphi(x, q) - \varphi(x, p_0) \rangle \cdot \langle \mu_{x_0}^{p_0}, \xi \psi(\xi) \rangle = (\tilde{\varphi}(x, q) - \tilde{\varphi}(x, p_0)) \cdot \langle \mu_{x_0}^{p_0}, \xi \psi(\xi) \rangle. \) (3.18)

Further, \( V_r(y, p) = \pi_1(V_r(y, p)) + \pi_2(V_r(y, p)) \) and

\[
\pi_1(V_r(y, p)) = \int (\tilde{\varphi}(y, q) - \tilde{\varphi}(y, \lambda)) \chi(\lambda)d\gamma_0^r(\lambda),
\]

\[
\pi_2(V_r(y, p)) = \int (\tilde{\varphi}(y, q) - \tilde{\varphi}(y, \lambda)) \chi(\lambda)d\gamma_0^r(\lambda).
\]

Observe that

\[
\pi_2(V_r(y, p)) = I_r(y, p),
\]

where the function \( I_r(y, p) \) is defined in (3.5) (with \( p' \) replaced by \( p \)) for a vector-function \( f(y, \lambda) = \tilde{\varphi}(y, q) - \tilde{\varphi}(y, \lambda) \in H_r^+. \) By Proposition 3.4 we obtain

\[
\lim_{p \to p_0} \lim_{m \to \infty} \lim_{r \to \infty} \int_{\mathbb{R}^n} |\xi|^{-1} \cdot F(\pi_2(V_r(y, p))\Phi_m(\xi)F(U_r(\cdot, p)\Phi_m(\xi))\psi \left( \frac{\xi}{|\xi|} \right) d\xi = 0.
\]

(3.19)

Let \( \tilde{V}_r(y, p) = \pi_1(V_r(y, p)) \). From (3.17), in view of (3.18) and (3.19), we see that

\[
(\tilde{\varphi}(x, q) - \tilde{\varphi}(x, p_0)) \cdot \langle \mu_{x_0}^{p_0}, \xi \psi(\xi) \rangle = \lim_{p \to p_0} \lim_{m \to \infty} \lim_{r \to \infty} \int_{\mathbb{R}^n} |\xi|^{-1} \cdot F(\tilde{V}_r(\cdot, p)\Phi_m(\xi)F(U_r(\cdot, p)\Phi_m(\xi))\psi \left( \frac{\xi}{|\xi|} \right) d\xi,
\]

which in turn, by Bunyakovskii inequality and Plancherel’s equality, gives us the estimate

\[
\lim_{p \to p_0} \lim_{m \to \infty} \lim_{r \to \infty} \|\tilde{V}_r(\cdot, p)\Phi_m\|_2 \cdot \|U_r(\cdot, p)\Phi_m\|_2 \cdot \|\psi\|_\infty \leq \frac{1}{\|\tilde{V}_r(\cdot, p)\Phi_m\|_2 \cdot \|\psi\|_\infty}.
\]

(3.20)

Next, for \( M_q(y) = \max_{\lambda \in [p_0, q]} |\tilde{\varphi}(y, q) - \tilde{\varphi}(y, \lambda)| \)

\[
|\tilde{V}_r(y, p)| \leq M_q(y) \left| \int \chi(\lambda)d\left( \nu_y^r(\lambda) + \nu_y^0(\lambda) \right) \right| = M_q(y)(u_r(y, p) - u_r(y, q) + u_0(y, p) - u_0(y, q)).
\]

In view of the elementary inequality \( (a + b)^2 \leq 2(a^2 + b^2) \) and the relation \( 0 \leq u_r(y, p) - u_r(y, q) \leq 1, r \in \mathbb{N} \cup \{0\} \), we have

\[
\|\tilde{V}_r(\cdot, p)\Phi_m\|_2^2 \leq 2 \int_{\Omega} (M_q(y))^2((u_r(y, p) - u_r(y, q))^2 + (u_0(y, p) - u_0(y, q))^2) K_m(x - y)dy \leq 2 \int_{\Omega} (M_q(y))^2(u_r(y, p) - u_r(y, q) + u_0(y, p) - u_0(y, q)) K_m(x - y)dy.
\]

(3.21)
Since \( p, q \in D \subseteq E \), then

\[
    u_r(y, p) - u_r(y, q) \to u_0(y, p) - u_0(y, q)
\]
as \( r \to \infty \) in the weak-* topology of \( L^\infty(\Omega) \) and from (3.21) we now obtain the estimate

\[
    \lim_{r \to \infty} \| \tilde{V}_r(\cdot, p) \Phi_m \|_2^2 \leq 4 \int_\Omega (M_q(y))^2(u_0(y, p) - u_0(y, q))K_m(x - y)dy,
\]
from which, passing to the limit as \( m \to \infty \), we obtain

\[
    \lim_{m \to \infty} \lim_{r \to \infty} \| \tilde{V}_r(\cdot, p) \Phi_m \|_2^2 \leq 4(M_q(x))^2(u_0(x, p_0) - u_0(x, p)).
\]

(3.22)

Here we bear in mind that by the definition of \( \Omega' \) (see, for instance, [35 Proposition 3]) \( x \) is a Lebesgue point of the functions \( u_0(y, p_0) \), \( u_0(y, p) \). It is also used that \( x \in \Omega_\varphi \) is a Lebesgue point of the function \( (M_q(y))^2 \) as well (this easily follows from the fact that \( x \) is a Lebesgue point of the maps \( y \to |\varphi(y, \cdot)|^2 \) into the spaces \( C(\mathbb{R}, \mathbb{R}^n) \), \( C(\mathbb{R}) \), respectively). From (3.22) in the limit as \( p \to p_0 \) it follows that

\[
    \lim_{p \to p_0} \lim_{m \to \infty} \lim_{r \to \infty} \| \tilde{V}_r(\cdot, p) \Phi_m \|_2^2 \leq 4(M_q(x))^2(u_0(x, p_0) - u_0(x, q)).
\]

(3.23)

In view of (3.20) and (3.23),

\[
    |(\tilde{\varphi}(x, q) - \tilde{\varphi}(x, p_0)) \cdot \langle \mu_{x_0}^{p_0}, \xi \psi(\xi) \rangle| \leq 2\|\psi\|_\infty M_q(x)\omega(q),
\]

\[
    \omega(q) = (u_0(x, p_0) - u_0(x, q))^{1/2} = (\nu_x(p_0, q))^{1/2} \to 0.
\]

(3.24)

It is clear that the set of vectors of the form \( \langle \mu_{x_0}^{p_0}, \xi \psi(\xi) \rangle \), with real \( \psi(\xi) \in C(S) \) spans the subspace \( H_+ \). Hence we can choose functions \( \psi_i(\xi) \in C(S), i = 1, \ldots, l \) such that the vectors \( v_i = \langle \mu_{x_0}^{p_0}, \xi \psi_i(\xi) \rangle \) make up an algebraic basis in \( H_+ \).

By (3.22), for \( \psi(\xi) = \psi_i(\xi), i = 1, \ldots, l \), we obtain

\[
    |(\tilde{\varphi}(x, q) - \tilde{\varphi}(x, p_0)) \cdot v_i| \leq c_i\omega(q)M_q(x), \quad c_i = \text{const},
\]

and since \( v_i, i = 1, \ldots, l \) is a basis in \( H_+ \), these estimates show that

\[
    |\tilde{\varphi}(x, q) - \tilde{\varphi}(x, p_0)| \leq c\omega(q)M_q(x) =
\]

\[
    c\omega(q) \max_{\lambda \in [p_0, q]} |\tilde{\varphi}(x, q) - \tilde{\varphi}(x, \lambda)|, \quad c = \text{const}.
\]

(3.25)

We take \( q = p_0 + \delta \), where \( \delta > 0 \) is so small that \( 2c\omega(q) = \varepsilon < 1 \). Then, in view of (3.25),

\[
    |\tilde{\varphi}(x, q) - \tilde{\varphi}(x, p_0)| \leq \frac{\varepsilon}{2} \max_{\lambda \in [p_0, p]} |\tilde{\varphi}(x, q) - \tilde{\varphi}(x, \lambda)|.
\]

(3.26)
and since $\varphi(x, q)$ is continuous with respect to $q$ and the set $D$ is dense, the estimate (3.26) holds for all $q \in [p_0, p_0 + \delta]$.

We claim that now $\tilde{\varphi}(x, p) = \tilde{\varphi}(x, p_0)$ for $p \in [p_0, p_0 + \delta]$. Indeed, assume that for $p' \in [p_0, p_0 + \delta]$

$$|\tilde{\varphi}(x, p') - \tilde{\varphi}(x, p_0)| = \max_{\lambda \in [p_0, p_0 + \delta]} |\tilde{\varphi}(x, \lambda) - \tilde{\varphi}(x, p_0)|.$$ 

Then for $\lambda \in [p_0, p']$ we have

$$|\tilde{\varphi}(x, p') - \tilde{\varphi}(x, \lambda)| \leq |\tilde{\varphi}(x, \lambda) - \tilde{\varphi}(x, p_0)| + |\tilde{\varphi}(x, p') - \tilde{\varphi}(x, p_0)|$$

and

$$\max_{\lambda \in [p_0, p']} |\tilde{\varphi}(x, p') - \tilde{\varphi}(x, \lambda)| \leq 2|\tilde{\varphi}(x, p') - \tilde{\varphi}(x, p_0)|.$$ 

We now derive from (3.26) with $p = p'$ that

$$|\tilde{\varphi}(x, p') - \tilde{\varphi}(x, p_0)| \leq \varepsilon|\tilde{\varphi}(x, p') - \tilde{\varphi}(x, p_0)|,$$

and since $\varepsilon < 1$, this implies that

$$|\tilde{\varphi}(x, p') - \tilde{\varphi}(x, p_0)| = \max_{\lambda \in [p_0, p_0 + \delta]} |\tilde{\varphi}(x, \lambda) - \tilde{\varphi}(x, p_0)| = 0.$$ 

We conclude that $\varphi(x, \lambda) - \varphi(x, p_0) \in H_+^\perp$ for all $\lambda \in [p_0, p_0 + \delta]$, i.e., $(\varphi(x, \lambda) - \varphi(x, p_0)) \cdot \xi = 0$ on the segment $[p_0, p_0 + \delta]$ for all $\xi \in H_+^\perp$.

To prove that for some sufficiently small $\delta > 0 (\varphi(x, \lambda) - \varphi(x, p_0)) \cdot \xi = 0$ on the segment $[p_0 - \delta, p_0]$ for all $\xi \in H_-^\perp$, we take $p, q \in D$, $q < p < p_0$ and repeat the reasonings used in the first part of the proof. As a result, we obtain the relation similar to (3.24)

$$|(\tilde{\varphi}(x, q) - \tilde{\varphi}(x, p_0)) \cdot \langle \mu_{x_0p_0}^+, \xi \psi(\xi) \rangle| \leq 2||\psi||\infty M_q(x)\omega(q),$$

where

$$M_q(x) = \max_{\lambda \in [q, p_0]} |\tilde{\varphi}(y, q) - \tilde{\varphi}(y, \lambda)|,$$

$$\omega(q) = \lim_{p \to p_0^+} (u_q(x, q) - u_0(x, p))^{1/2} = (\nu_x(q, p_0))^{1/2} \to 0.$$ 

This relation readily implies the desired statement $(\varphi(x, \lambda) - \varphi(x, p_0)) \cdot \xi = 0$ on the segment $[p_0 - \delta, p_0]$ for all $\xi \in H_-^\perp$, where $\delta$ is sufficiently small.

The proof is complete.

**Corollary 3.6.** Let $x \in \Omega''$, $[a, b]$ be the minimal segment, containing $\text{supp} \nu_x$ and $p_0 \in (a, b)$. Then, in the notations of Theorem 3.2, $\text{supp} \mu_{x_0p_0}^+ \cap \text{supp} \mu_{x_0p_0}^- \neq \emptyset$ and for all $\xi \in H_+ \cap H_-^\perp, \xi \neq 0$ the function $\xi \cdot \varphi(x, \lambda)$ is constant in a vicinity of $p_0$. 

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Proof. First, note that since \( x \in \Omega'' \subset \Omega' \) is a Lebesgue point of the functions \( u_0(\cdot, p) \) for all \( p \in D \) while \( D \) is dense, the distribution function \( u_0(x, \lambda) = \nu_x((\lambda, +\infty)) \) is uniquely defined by the relation \( u_0(x, \lambda) = \sup_{p \in D, p > \lambda} u_0(x, p) \). In particular, the measure \( \nu_x \) is well-defined at the point \( x \).

The statement that the function \( \lambda \to \xi \cdot \varphi(x, \lambda) \) is constant in a vicinity of \( p_0 \) for all \( \xi \in H_+ \cap H_- \), \( \xi \neq 0 \) readily follows from the assertion of Theorem 3.5. Hence, we only need to show that \( \text{supp} \mu_x^{p_0+} \cap \text{supp} \mu_x^{p_0-} \neq \emptyset \). We assume to the contrary that \( S_+ \cap S_- = \emptyset \), where \( S_\pm = \text{supp} \mu_x^{p_0\pm} \). Denote \( C_+ = S \setminus S_+ \), \( C_- = S \setminus S_- \), Then \( S = C_+ \cup C_- \), \( \mu_x^{p_0+}(C_+ \setminus C_-) = \mu_x^{p_0-}(C_- \setminus C_+) = 0 \). Therefore, by relation (2.6), for all \( p, q \in D \), \( p < p_0 < q \)

\[
\text{Var} \mu_x^{pq} = |\mu_x^{pq}|(S) \leq |\mu_x^{pq}|(C_+) + |\mu_x^{pq}|(C_-) \leq (\mu_x^{pq}(C_+))^{1/2} + (\mu_x^{pq}(C_-))^{1/2} \leq (\mu_x^{pq}(C_+))^{1/2} + (\mu_x^{pq}(C_-))^{1/2},
\]

where we use that \( \mu_x^{pq}(A) \leq \mu_x^{pq}(S) \leq 1 \) for all \( p \in D \) and every Borel set \( A \subset S \), see (2.3). It follows from the obtained estimate and Lemma 3.1 that

\[
\lim_{p \to p_0-} \lim_{q \to p_0+} \text{Var} \mu_x^{pq} \leq \left( \mu_x^{p_0+}(C_+) \right)^{1/2} + \left( \mu_x^{p_0-}(C_-) \right)^{1/2} = 0.
\]

Thus,

\[
\mu_x^{pq} \to 0 \quad \text{in } M(S) \quad \text{as } p \to p_0-, q \to p_0+.
\] (3.27)

On the other hand, by (2.4)

\[
\mu_x^{pq}(S) = \lim_{m \to \infty} \lim_{r \to \infty} \int_{\mathbb{R}^n} F(\Phi_m U_r(\cdot, p))((\xi)) F(\Phi_m U_r(\cdot, q))((\xi)) d\xi = \lim_{m \to \infty} \lim_{r \to \infty} \int_{\mathbb{R}^n} U_r(y, p) U_r(y, q) K_m(x - y) dy.
\] (3.28)

Observe that \( U_r(x, \lambda) = \theta(u_r(x) - \lambda) - u_0(x, \lambda) \). Since \( U_r(\cdot, p) \to 0 \) for all \( p \in D \) and \( (\theta(u_r(y) - p) - 1)\theta(u_r(y) - q) \equiv 0 \), we find

\[
\lim_{r \to \infty} \int_{\mathbb{R}^n} U_r(y, p) U_r(y, q) K_m(x - y) dy = 0,
\]

\[
\lim_{r \to \infty} \int_{\mathbb{R}^n} (U_r(y, p) - 1) U_r(y, q) K_m(x - y) dy = 0,
\]

\[
\lim_{r \to \infty} \int_{\mathbb{R}^n} (\theta(u_r(y) - p) - 1 - u_0(y, p)) (\theta(u_r(y) - q) - u_0(y, q)) K_m(x - y) dy = \int_{\mathbb{R}^n} (1 - \theta(u_r(y) - p)) u_0(y, q) - u_0(y, p) (\theta(u_r(y) - q) - u_0(y, q)) K_m(x - y) dy.
\]
In the limit as \( m \to \infty \) this yields
\[
\lim_{m \to \infty} \lim_{r \to \infty} \int_{\mathbb{R}^n} U_r(y, p) U_r(y, q) K_m(x - y) dy = \]
\[
\lim_{m \to \infty} \int_{\mathbb{R}^n} (1 - u_0(y, p)) u_0(y, q) K_m(x - y) dy = (1 - u_0(x, p)) u_0(x, q).
\]

Here we take into account that \( x \) is a Lebesgue point of the functions \( u_0(y, p) \), \( u_0(y, q) \). By (3.27), (3.28) we find
\[
0 = \lim_{p \to p_0 -} \lim_{q \to p_0 +} u_0(x, p) u_0(x, q) = (1 - u_0(x, p)) u_0(x, q) = \nu_x((a, p_0]) \nu_x((p_0, b)) > 0,
\]

since \( a < p_0 < b \) and \([a, b]\) is the minimal segment containing \( \text{supp} \nu_x \). The obtained contradiction implies that \( S_+ \cap S_- \neq \emptyset \) and completes the proof.

Now we are ready to prove Theorem 1.1

**Proof.** Let \( u_r = u_{kr} \) be a subsequence of \( u_k \) chosen in accordance with Proposition 2.2. In particular, this subsequence converges to a measure-valued function \( \nu_x \in \text{MV}(\Omega) \). In view of (2.1) for a.e. \( x \in \Omega \)
\[
u_x(\lambda) = \int \lambda d\nu_x(\lambda).\] (3.29)

We define the set of full measure \( \Omega'' \subset \Omega \) and the minimal segment \([a(x), b(x)]\), containing \( \text{supp} \nu_x \), \( x \in \Omega'' \). In view of (3.29) \( u(x) \in (a(x), b(x)) \) whenever \( a(x) < b(x) \). By Corollary 3.6 the function \( \xi \cdot \varphi(x, \cdot) \) is constant in a vicinity of \( u(x) \) for some vector \( \xi \neq 0 \). But this contradicts to the assumption of Theorem 1.1. Therefore, \( a(x) = b(x) = u(x) \) for a.e. \( x \in \Omega \). This means that \( \nu_x(\lambda) = \delta(\lambda - u(x)) \). By Theorem 2.1 the subsequence \( u_r \to u \) as \( r \to \infty \) in \( L^1_{\text{loc}}(\Omega) \). Finally, since the limit function \( u(x) \) does not depend of the choice of a subsequence \( u_r \), we conclude that the original sequence \( u_k \to u \) in \( L^1_{\text{loc}}(\Omega) \) as \( k \to \infty \). The proof is complete.

\[ \square \]

4 Decay property

This section is devoted to the proof of Theorem 1.4. Suppose that \( u(t, x) \) is a unique e.s. to problem (1.4), (1.7) with the periodic initial data \( u_0(x) \). By Remark 1.3 we can assume that \( u(t, x) \in C([0, +\infty), L^1(\mathbb{T}^n)) \) (after possible correction on a set of null measure). We consider the sequence \( u_k(t, x) = u(kt, kx) \), \( k \in \mathbb{N} \), consisting of e.s. of (1.4). As was firstly shown in [2], the decay property (1.10) is equivalent to the strong convergence \( u_r(t, x) \to I = \text{const} \in L^1_{\text{loc}}(\Pi) \) of a subsequence \( u_r = u_{kr}(t, x) \). As follows from [17] Lemma 3.2(i)], \( u_r \to u^* \), where
\( u^* = u^*(t) \) is a weak-* limit of the sequence \( a_0(k, t) \), where 
\( a_0(t) = \int_{T_0} u(t, x) dx. \)

Since \( u(t, x) \) is an e.s. of (1.4), this function is constant: 
\( a_0(t) \equiv I = \int_{T_0} u_0(x) dx, \) in view of (1.8). Therefore, \( u_r \to I \) as \( r \to \infty \) (actually, the original sequence \( u_k \to I \) as \( k \to \infty \)).

Let \( \mu^{pq} \), \( p, q \in E \), be the H-measure corresponding to a subsequence \( u_r = u_{k_r}(t, x) \). Recall that \( \mu^{pq} = \mu^{pq}(t, x, \tau, \xi) \in M_{loc}(\Pi \times S) \), where

\[
S = \{ \hat{\xi} = (\tau, \xi) \in \mathbb{R} \times \mathbb{R}^n \mid |\hat{\xi}|^2 = \tau^2 + |\xi|^2 = 1 \}
\]
is a unit sphere in the dual space \( \mathbb{R}^{n+1} \) (the variable \( \tau \) corresponds to the time variable \( t \)).

By [17] Theorem 3.1 the following localization principle holds

\[
\text{supp } \mu^{pq} \subset \Pi \times S_0,
\]
where

\[
S_0 = \{ \hat{\xi}/|\hat{\xi}| \mid \hat{\xi} = (\tau, \xi) \neq 0, \tau \in \mathbb{R}, \xi \in \Pi' \}.
\]

As was demonstrated in Proposition 2.3 \( \mu^{pq} = \mu^{pq}_{t,x} dtdx \) for all \( p, q \in D \), where \( D \subset E \) is a countable dense subset and measures \( \mu^{pq}_{t,x} \in M(S) \), are defined for all \( (t, x) \) belonging to a set of full measure \( \Pi' \subset \Pi \). Obviously, the identity

\[
\langle \mu^{pq}, \Phi(t, x, \hat{\xi}) \rangle = \int_{\Pi} \langle \mu^{pq}_{t,x}(\hat{\xi}), \Phi(t, x, \hat{\xi}) \rangle dtdx, \tag{4.1}
\]

\( \Phi(t, x, \hat{\xi}) \in C_0(\Pi \times S) \), remains valid also for compactly supported Borel functions \( \Phi \). Taking \( \Phi = \phi(t, x) h(\hat{\xi}) \), where \( \phi(t, x) \in C_0(\Pi) \), \( \phi(t, x) \geq 0 \) while \( h(\hat{\xi}) \) is an indicator function of the set \( S \setminus S_0 \), we derive from (4.1) that

\[
\int_{\Pi} \mu^{pq}_{t,x}(S \setminus S_0) \phi(t, x) dtdx = 0
\]

and since \( \mu^{pq}_{t,x} \geq 0 \) and \( \phi(t, x) \in C_0(\Pi) \) is arbitrary nonnegative function, it follows from this identity that \( \mu^{pq}_{t,x}(S \setminus S_0) = 0 \) for all \( p \in D \), \( (t, x) \in \Pi' \). By relation 2.6 we claim that, more generally, \( |\mu^{pq}_{t,x}(S \setminus S_0)| = 0 \) for all \( p, q \in D \), \( (t, x) \in \Pi' \). Finally, in view of Lemma 3.1 we find that \( |\mu^{pq}_{t,x}(S \setminus S_0)| = 0 \), that is,

\[
\text{supp } \mu^{pq}_{t,x} \subset S_0 \forall p, q \in \mathbb{R}, (t, x) \in \Pi'. \tag{4.2}
\]

Further, \( u_r(t, x) \) is a sequence of entropy solutions of (1.4). Therefore (see for instance [16]) the sequences

\[
\text{div } [\theta(u_r - p)(\hat{\varphi}(u_r) - \hat{\varphi}(p))] = ((u_r - p)^+) + \text{div}_x [\theta(u_r - p)(\varphi(u_r) - \varphi(p))]
\]

are compact in \( H_{d,loc}^{-1}(\Pi) \) for some \( d > 1 \) and all \( p \in \mathbb{R} \), where \( \hat{\varphi}(u) = (u, \varphi(u)) \in C(\mathbb{R}, \mathbb{R}^{n+1}) \), and we use the notation \( v^+ = \max(v, 0) \).
Denote by \( \nu_{t,x} \in \text{MV}(\Pi) \) the limit measure valued function for a sequence \( u_r \), and by \([a(t,x), b(t,x)]\) the minimal segment containing \( \text{supp } \nu_{t,x} \).

Suppose that \((t,x) \in \Pi', a(t,x) < b(t,x)\). Then \( I = \int \lambda d\nu_{t,x}(\lambda) \in (a(t,x), b(t,x)) \). By Corollary 3.9 we find that there exists \( \hat{\xi} = (\tau, \xi) \in \text{supp } \mu_{t,x}^{II+} \cap \text{supp } \mu_{t,x}^{II-} \) and \( \delta > 0 \) such that the function

\[
\lambda \to \hat{\xi} \cdot \hat{\varphi}(\lambda) = \tau u + \xi \cdot \varphi(u) = c = \text{const}
\]

on the interval \( V = \{ \lambda \mid |\lambda - I| < \delta \} \). By (4.2) \( \hat{\xi} \in S_0 \), which implies that we can assume that \( \xi \in L' \) in (4.3). Evidently, \( \xi \neq 0 \) (otherwise, \( \tau u \equiv c \) on \( V \) for \( \tau \neq 0 \)). Hence the function \( \xi \cdot \varphi(u) = c - \tau u \) is affine, which contradicts (1.9). Thus, \( a(t,x) = b(t,x) = I \) for a.e. \((t,x) \in \Pi\). We conclude that \( \nu_{t,x}(\lambda) = \delta(\lambda - I) \) an by Theorem 2.1 the sequence \( u_r \to I \) as \( r \to \infty \) strongly (in \( L^1_{\text{loc}}(\Pi) \)). As was mentioned above (one can simply repeat the conclusive part of the proof of Theorem 1.1 in [17]), this implies (1.10).

Conversely, if the assumption (1.9) fails, we can find \( \xi \in L', \xi \neq 0 \), and constants \( a, b \in \mathbb{R} \) such that \( \xi \cdot \varphi(\lambda) \equiv au + b \) on a segment \([I - \delta, I + \delta], \delta > 0\). Then, as is easily verified, the function

\[
u(t,x) = I + \delta \sin(2\pi(\xi \cdot x - at))
\]

is the e.s. of (1.4), (1.7) with initial data \( u_0(x) = I + \delta \sin(2\pi(\xi \cdot x)) \). It is clear that \( u_0(x) \) is \( L \)-periodic and \( \int_{\mathbb{T}^n} u_0(x)dx = I \), but the e.s. \( u(t,x) \) does not satisfy the decay property.

**Example.** Let \( n = 1 \), \( \varphi(u) = |u| \). Let \( u = u(t,x) \) be an e.s. of the problem

\[
u_t + (|u|)_x = 0, \quad u(0, x) = u_0(x), \tag{4.4}
\]

where \( u_0(x) \in L^\infty(\mathbb{R}) \) is a nonconstant periodic function with a period \( l \) (for a constant \( u_0 \equiv c \) the e.s. \( u \equiv c \) and the decay property is evident). Notice that no previous results [2, 3, 17] can help to answer the question whether the decay property is satisfied. However, as follows from Theorem 1.4 if \( I = \frac{1}{l} \int_0^l u_0(x)dx = 0 \), then the decay property holds: \( \int_0^l |u(t,x)|dx \to 0 \) as \( t \to \infty \). Actually, the condition \( \int_0^l u_0(x)dx \) is also necessary for the decay property (1.10). Indeed, \( u(t,x) = u_0(x \mp t) \) if \( \pm u_0(x) \geq 0 \) (then \( \pm l > 0 \)), and the decay property is evidently violated. In the remaining case when \( u_0 \) changes sign we define the functions \( v_+(t,x) = v_+(x - t) \), \( u_-(t,x) = v_-(x + t) \), where \( v_+(x) = \max(u_0(x), 0) \geq 0 \), \( v_-(x) = \min(u_0(x), 0) \leq 0 \). Note that this functions take zero values on sets of positive measures. By the construction, \( v_-(x) \leq u_0(x) \leq v_+(x) \) and \( u_\pm(t,x) \) are e.s. of (1.4) with initial data \( v_\pm(x) \). In view of the known property of monotone dependence of e.s. on initial data \( u_-(t,x) \leq u(t,x) \leq u_+(t,x) \) a.e. on \( \Pi \). These inequality can be written in the form

\[
u(t, x - t) \geq v_-(x), \quad u(t, x + t) \leq v_+(x). \tag{4.5}
\]
Assuming that \( u(t, x) \) satisfies the decay property, we find, with the help of \( x\)-periodicity of \( u(t, \cdot) \), that
\[
\int_0^l |u(t, x \pm t) - I| \, dx = \int_0^l |u(t, x) - I| \, dx \to 0 \quad \text{as} \quad t \to +\infty,
\]
that is, the functions \( u(t, x \pm t) \to I \) in \( L^1([0, l]) \). Passing to the limit as \( t \to +\infty \) in (4.5), we find that \( v_-(x) \leq I \leq v_+(x) \) for a.e. \( x \in \mathbb{R} \). The latter is possible only if \( I = 0 \). We conclude that the decay property holds only in the case \( I = 0 \).

**Remark 4.1.** Theorem 1.4 can be extended to more general case of almost periodic initial data (in the Besicovitch sense [1]). Repeating the arguments of [18], we arrive at the following analogue of Theorem 1.4.

**Theorem 4.2.** Let \( M_0 \) be the additive subgroup of \( \mathbb{R}^n \) generated by the spectrum of \( u_0 \). Assume that for all \( \xi \in M_0, \xi \neq 0 \) the function \( \xi \cdot \varphi(\lambda) \) is not affine in any vicinity of \( I = \int_{\mathbb{R}^n} u_0(x) \). Then the e.s. \( u(t, x) \) of (1.4), (1.7) satisfies the decay property
\[
\lim_{t \to +\infty} \int_{\mathbb{R}^n} |u(t, x) - I| \, dx = 0.
\]
Here \( \int_{\mathbb{R}^n} v(x) \, dx \) denotes the mean value of an almost periodic function \( v(x) \) (see [1]).

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