Approximate $(\sigma - \tau)$-Contractibility

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Abstract

In this paper, the generalized Hyers–Ulam–Rassias stability of $(\sigma - \tau)$-derivations on normed algebras into Banach bimodules is established. We introduce the notion of approximate $(\sigma - \tau)$-contractibility and prove that a Banach algebra is $(\sigma - \tau)$-contractible if and only if it is approximately $(\sigma - \tau)$-Contractible.

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1 Introduction.

The stability theory of functional equations was started in 1940 with a problem on approximate homomorphisms raised by S. M. Ulam; cf. [13]. Indeed, as noted by Z. Páles [8], the first stability theorem was discovered in 1925 anticipating the question of Ulam [11]. In 1941, D. H. Hyers gave a partial solution of Ulam’s problem for linear mappings. Since then many mathematicians have been working on this area of research; cf. [2] and [3]. The stability of derivations was studied by C.-G. Park [9], [10] and by the author [7].

Let $A$ be an algebra and $X$ be an $A$-bimodule. A linear mapping $d : A \to X$ is called a $(\sigma - \tau)$-derivation if there exists two linear operators $\sigma$ and $\tau$ on $A$ such that $d(ab) = d(a)\sigma(b) + \tau(a)d(b)$ for all $a, b \in A$. Familiar examples are

(i) the ordinary derivations from $A$ to $X$;
(ii) the so-called $(\sigma - \tau)$-inner derivations i.e. those are defined by $d_x(a) = x\sigma(a) - \tau(a)x$ for a fixed arbitrary element $x \in X$ and endomorphisms $\sigma$ and $\tau$ on $A$. Note that $d_x(ab) = d_x(a)\sigma(b) + \tau(a)d_x(b)$.
(iii) all endomorphisms $\phi$ on $A$. Notice that such $\phi$ is clearly a $(\frac{1}{2}\phi - \frac{1}{2}\phi)$-derivation. See [1].

The present paper is devoted to study of stability of $(\sigma - \tau)$-derivations and to generalize some results in [9]. We also introduce $(\sigma - \tau)$-contractibility ($(\sigma - \tau)$-amenability) and approximate $(\sigma - \tau)$-contractibility (approximate $(\sigma - \tau)$-amenability) and investigate their relationships which may seem to be interesting on their own right.

2 Stability of $(\sigma - \tau)$-Derivations.

Throughout the section, $A$ denotes a (not necessary unital) normed algebra and $X$ is a Banach $A$-bimodule. Our aim is to establish the generalized Hyers–Ulam–Rassias stability of $(\sigma - \tau)$-derivations. We indeed extend main results
of C.-G. Park [4] to \((\sigma - \tau)\)-derivations. We use the direct method which was first devised by D. H. Hyers to construct an additive function from an approximate one; cf. [5] and Găvruta’s technique [4]. Some ideas of [10] are also applied.

2.1. Theorem. Suppose \(f : A \to X\) is a mapping with \(f(0) = 0\) for which there exist maps \(g_1, g_2 : A \to A\) with \(g_1(0) = g_2(0) = 0\) and a function \(\varphi : A \times A \to [0, \infty)\) such that

\[
\tilde{\varphi}(a, b) := \frac{1}{2} \sum_{n=0}^{\infty} 2^{-n} \varphi(2^n a, 2^n b) < \infty \tag{1}
\]

\[
\|f(\lambda a + \lambda b) - \lambda f(a) - \lambda f(b)\| \leq \varphi(a, b) \tag{2}
\]

\[
\|g_k(\lambda a + \lambda b) - \lambda g_k(a) - \lambda g_k(b)\| \leq \varphi(a, b) \tag{3}
\]

\[
\|f(ab) - f(a)g_1(b) - g_2(a)f(b)\| \leq \varphi(a, b) \tag{4}
\]

for \(k = 1, 2\), for all \(\lambda \in T = \{\lambda \in \mathbb{C} : |\lambda| = 1\}\) and for all \(a, b \in A\). Then there exist unique linear maps \(\sigma\) and \(\tau\) from \(A\) to \(X\) satisfying \(\|g_1(a) - \sigma(a)\| \leq \tilde{\varphi}(a, a)\) and \(\|g_2(a) - \tau(a)\| \leq \tilde{\varphi}(a, a)\), and there exists a unique \((\sigma - \tau)\)-derivation \(d : A \to X\) such that

\[
\|f(a) - d(a)\| \leq \tilde{\varphi}(a, a) \tag{5}
\]

for all \(a \in A\). Moreover,

(i) if

\[
\|g_2(ab) - g_2(a)g_2(b)\| \leq \varphi(a, b) \tag{6}
\]

for all \(a, b \in A\), and either \(A\) has no nonzero divisor of zero or \(d\) is surjective and \(\text{ran}(A) = \{0\}\), then either \(d = 0\) or both linear mappings \(\sigma\) and \(\tau\) are endomorphisms on \(A\).
(ii) if $f$ and $\psi(a) = \tilde{\varphi}(a, a)$ are continuous at a point $a_0$ then $d$ is continuous on $A$.

**Proof.** Putting $\lambda = 1$ in (2), we have
\[
\|f(a + b) - f(a) - f(b)\| \leq \varphi(a, b) \quad a, b \in A
\] (7)

Now we use the Gâvruta method on inequality (4) (see [3] and [12]). One can use the induction to show that
\[
\left\| \frac{f(2^n a)}{2^n} - f(a) \right\| \leq \frac{1}{2} \sum_{k=0}^{n-1} 2^{-k} \varphi(2^k a, 2^k a)
\] (8)

and
\[
\left\| \frac{f(2^p a)}{2^p} - \frac{f(2^q a)}{2^q} \right\| \leq \frac{1}{2} \sum_{k=q}^{p-1} 2^{-k} \varphi(2^k a, 2^k a)
\]

for all $a \in A$, all $n$ and all $p > q$. It follows from the convergence (1) that the sequence $\{\frac{f(2^n a)}{2^n}\}$ is Cauchy. Due to the completeness of $X$, this sequence is convergent. Set
\[
 d(a) := \lim_{n \to \infty} \frac{f(2^n a)}{2^n}
\] (9)

Applying (2), we get $\|2^{-n} f(2^n (\lambda a + \lambda b)) - 2^{-n} \lambda f(2^n a) - 2^{-n} \lambda f(b)\| \leq 2^{-n} \varphi(2^n a, 2^n a)$. Passing to the limit as $n \to \infty$ we obtain $d(\lambda a + \lambda b) = \lambda d(a) + \lambda d(b)$. Note that the convergence of (1) implies that
\[
\lim_{n \to \infty} 2^{-n} \varphi(2^n a, 2^n a) = 0.
\]

Next, let $\gamma \in C(\gamma \neq 0)$ and let $M$ be a positive integer greater than $4|\gamma|$. Then $|\frac{\gamma}{M}| < \frac{1}{4} < 1 - \frac{2}{3} = 1/3$. By Theorem 1 of [3], there exist three elements $\theta_1, \theta_2, \theta_3 \in T$ such that $3 \frac{\gamma}{M} = \theta_1 + \theta_2 + \theta_3$. By the additivity of $d$ we get
\[
 d(\frac{1}{3} a) = \frac{1}{3} d(a)
\] for all $a \in A$. Therefore,
\[
 d(\gamma a) = d\left(\frac{M}{3} \cdot \frac{\gamma}{M} a\right) = M d\left(\frac{1}{3} \cdot \frac{\gamma}{M} a\right) = \frac{M}{3} d\left(\frac{\gamma}{M} a\right) = \frac{M}{3} d(\theta_1 a + \theta_2 a + \theta_3 a) = \frac{M}{3} (d(\theta_1 a) + d(\theta_2 a) + d(\theta_3 a)) = \frac{M}{3} (\theta_1 + \theta_2 + \theta_3) d(a) = \frac{M}{3} \cdot \frac{\gamma}{M} = \gamma d(a)
\]
for all $a \in A$.

Thus $d$ is ($C$-)linear.

Moreover, it follows from (8) and (9) that $\|f(a) - d(a)\| \leq \varphi(a, a)$ for all $a \in A$. It is known that additive mapping $d$ satisfying (2) is unique \[4\].

Similarly one can use (3) to show that there exist unique linear mappings $\sigma$ and $\tau$ defined by $\lim_{n \to \infty} 2^{-n}g_1(2^n b)$ and $\lim_{n \to \infty} 2^{-n}g_2(2^n a)$, respectively.

Replacing $a, b$ in (4) by $2^n a$ and $2^n b$ respectively, we have

$$\|f(2^{2n} ab) - f(2^n a)g_1(2^n b) - g_2(2^n a)f(2^n b)\| \leq \varphi(2^n a, 2^n b)$$

$$\|2^{-2n}f(2^{2n} ab) - 2^{-n}f(2^n a)2^{-n}g_1(2^n b) - 2^{-n}g_2(2^n a)2^{-n}f(2^n b)\| \leq 2^{-2n}\varphi(2^n a, 2^n b)$$

It follows that

$$d(ab) = d(a)\sigma(b) + \tau(a)d(b).$$

Therefore $d$ is a ($\sigma - \tau$)-derivation.

(i) Inequality (6) yields

$$\|2^{-2n}g_2(2^{2n} ab) - 2^{-n}g_2(2^n a)2^{-n}g_2(2^n b)\| \leq 2^{-n}\varphi(2^n a, 2^n b).$$

for all $a, b \in A$. So that $\tau(ab) = \tau(a)\tau(b)$ for all $a, b \in A$.

On the other hand

$$d(cab) = d(c)\sigma(ab) + \tau(c)d(ab)$$

$$d(c)\sigma(ab) = d(cab) - \tau(c)d(ab)$$

$$= (d(ca)\sigma(b) + \tau(ca)d(b)) - \tau(c)d(ab)$$

$$= (d(c)\sigma(a) + \tau(c)d(a))\sigma(b) + \tau(ca)d(b) - \tau(c)d(ab)$$

$$= d(c)\sigma(a)\sigma(b) + \tau(c)d(a)\sigma(b) + \tau(ca)d(b) - \tau(c)d(ab)$$

$$= d(c)\sigma(a)\sigma(b) + \tau(c)d(a)\sigma(b) + \tau(ca)d(b)$$

$$- \tau(c)(d(a)\sigma(b) + \tau(a)d(b))$$

$$d(c)(\sigma(ab) - \sigma(a)\sigma(b)) = (\tau(ca) - \tau(c)\tau(a))d(b) = 0$$
Hence \( d(c)(\sigma(ab) - \sigma(a)\sigma(b)) = 0 \). Therefore if \( d \neq 0 \), under any one of the assumptions that \( A \) has no nonzero divisor of zero, or \( d \) is surjective and \( \text{ran}(A) = \{0\} \) we conclude that \( \sigma \) is an endomorphism.

(ii) If \( d \) were not continuous at a point \( a \in A \) then there would be an integer \( m \) and a sequence \( \{a_n\} \) of \( A \) such that \( \lim_{n \to \infty} a_n = 0 \) and \( \|d(a_n)\| > \frac{1}{m} \) for all \( n \). Let \( k \) be an integer greater than \( m(2\psi(a_0)+1) \). Since \( \lim_{n \to \infty} f(ka_n + a_0) = f(a_0) \), there is an integer \( N \) such that for all \( n \geq N \), \( \|f(ka_n + a_0) = f(a_0)\| < 1 \). Hence

\[
2\psi(a_0) + 1 < \frac{k}{m} \leq 2\psi(a_0) + 1,
\]

for all \( n \geq N \). Tending \( n \) to \( \infty \) we conclude that \( 2\psi(a_0) + 1 < \frac{k}{m} \leq 2\psi(a_0) + 1 \), a contradiction (see [5]).

2.2. Remark. (i) The algebra \( \mathbb{C}[x] \) consisting of all complex polynomials in one variable has no zero-divisor.

(ii) \( \text{ran}A = \text{lan}A = 0 \) for each normed algebra \( A \) with a bounded approximate identity or a unital normed algebra \( A \) containing an invertible element.

(iii) A similar statement to Theorem 2.1 holds if

\[
\|g_1(ab) - g_1(a)g_1(b)\| \leq \varphi(a,b)
\]

for all \( a, b \in A \), and either \( A \) has no nonzero divisor of zero or \( d \) is surjective and \( \text{lan}(A) = \{0\} \).

2.3. Proposition. Suppose the normed algebra \( A \) is spanned by a set \( S \subseteq A \), \( f : A \to X \) is a mapping with \( f(0) = 0 \) for which there exist maps \( g_1, g_2 : A \to A \) with \( g_1(0) = g_2(0) = 0 \) and a function \( \varphi : A \times A \to [0, \infty) \) such
that
\[
\tilde{\varphi}(a, b) := \frac{1}{2} \sum_{n=0}^{\infty} 2^{-n} \varphi(2^n a, 2^n b) < \infty
\]

\[
\|f(\lambda a + \lambda b) - \lambda f(a) - \lambda f(b)\| \leq \varphi(a, b)
\] (10)

\[
\|g_k(\lambda a + \lambda b) - \lambda g_k(a) - \lambda g_k(b)\| \leq \varphi(a, b)
\]

for \(k = 1, 2\), for \(\lambda = 1, i\) and for all \(a, b \in A\); and

\[
\|f(ab) - f(a)g_1(b) - g_2(a)f(b)\| \leq \varphi(a, b)
\]

for all \(a, b \in A\). If for \(k = 1, 2\) and each fixed \(a \in A\) the functions \(t \mapsto f(ta), t \mapsto g_k(ta)\) are continuous on \(\mathbb{R}\) then there exist unique linear maps \(\sigma\) and \(\tau\) from \(A\) to \(X\) satisfying \(\|g_1(a) - \sigma(a)\| \leq \tilde{\varphi}(a, a)\) and \(\|g_2(a) - \tau(a)\| \leq \tilde{\varphi}(a, a)\), and a unique \((\sigma - \tau)\)-derivation \(d : A \to X\) such that

\[
\|f(a) - d(a)\| \leq \tilde{\varphi}(a, a)
\]

for all \(a \in A\).

**Proof.** Put \(\lambda = 1\) in (10). It follows from the proof of Theorem 2.1 that there exists a unique additive mapping \(d : A \to X\) given by

\[
d(a) = \lim_{n \to \infty} \frac{f(2^n a)}{2^n}, a \in A.
\]

By the same reasoning as in the proof of the theorem of [12], the mapping \(d\) is \(\mathbb{R}\)-linear.

Assuming \(b = 0\) and \(\lambda = i\) in (10) we have \(\|f(ia) - if(a)\| \leq \varphi(a, 0), a \in A\). Hence \(\frac{1}{2^n}\|f(2^n ia) - if(2^n a)\| \leq \varphi(2^n a, 0) a \in A\). The right hand side tends to zero as \(n \to \infty\), so that

\[
d(ia) = \lim_{n \to \infty} \frac{f(2^n ia)}{2^n} = \lim_{n \to \infty} \frac{if(2^n a)}{2^n} = id(a), a \in A.
\]

For every \(\lambda \in \mathbb{C}, \lambda = r_1 + ir_2\) in which \(r_1, r_2 \in \mathbb{R}\). Hence

\[
d(\lambda a) = d(r_1 a + ir_2 a) = r_1 d(a) + ir_2 d(a) = \lambda d(a).
\]

Thus \(d\) is \(\mathbb{C}\)-linear.

Similarly there exist linear maps \(\sigma\) and \(\tau\) from \(A\) to \(X\) satisfying the required inequalities. One can easily verify \(\|f(a) - d(a)\| \leq \tilde{\varphi}(a, a)\) and
\[ d(ab) = d(a)\sigma(b) + \tau(a)d(b). \] Since \( A \) is linearly generated by \( S \) we conclude that \( d \) is \( (\sigma - \tau) \)-derivation. \( \square \)

2.4. **Remark.** (i) Similar statements to Theorem 2.1 and Proposition 2.3 hold if \( \varphi(a, b) = \alpha + \beta(\|a\|^p + \|b\|^p) \). Note that

\[ \tilde{\varphi}(a, b) := \frac{1}{2}\sum_{n=0}^{\infty} 2^{-n} \varphi(2^n a, 2^n b) = \alpha + \beta \frac{\|a\|^p + \|b\|^p}{2 - 2^p}. \]

(ii) Conclusion of Proposition 2.3 holds if \( A \) is a \( C^* \)-algebra and \( S \) is the unitary group of \( A \) or the positive part of \( A \).

### 3 Approximately \((\sigma - \tau)\)-Contractible

Throughout this section \( A \) is a Banach algebra and \( X \) is a Banach \( A \)-bimodule. In addition, \( \sigma \) and \( \tau \) are assumed to be bounded endomorphisms on \( A \).

If every bounded \((\sigma - \tau)\)-derivation is \((\sigma - \tau)\)-inner, then \( A \) is said to be \((\sigma - \tau)\)-contractible. In particular, the ordinary contractibility is indeed \((\text{id} - \text{id})\)-contractibility where \( \text{id} \) denotes the identity map.

We say a mapping \( f : A \to X \) with \( f(0) = 0 \) is approximate \((\sigma - \tau)\)-derivation if it is continuous at a point \( a \in A \) and there is a positive number \( \alpha > 0 \) such that \( \|f(\lambda a + \lambda b) - \lambda f(a) - \lambda f(b)\| \leq \alpha \) and \( \|f(ab) - f(a)\sigma(b) - \tau(a)f(b)\| \leq \alpha \) for all \( \lambda \in T \) and for all \( a, b \in A \).

The Banach algebra \( A \) is called approximately \((\sigma - \tau)\)-contractible if for every approximate \((\sigma - \tau)\)-derivation there exist a positive number \( \beta \) and an \( x \in X \) such that \( \|x\sigma(a) - \tau(a)x - f(a)\| \leq \beta \).

3.1. **Theorem.** A Banach algebra \( A \) is approximately \((\sigma - \tau)\)-contractible if and only if \( A \) is \((\sigma - \tau)\)-contractible.

**Proof.** Let \( A \) be \((\sigma - \tau)\)-contractible, \( \alpha > 0 \) and \( f : A \to X \) is a mapping which is continuous at a point \( a \in A \) and \( f(0) = 0 \) such that

\[ \|f(\lambda a + \lambda b) - \lambda f(a) - \lambda f(b)\| \leq \alpha \]
\[ \| f(ab) - f(a)\sigma(b) - \tau(a)f(b) \| \leq \alpha \]

for all \( \lambda \in T \) and for all \( a, b \in A \).

By Theorem 2.1 there exists a bounded \((\sigma - \tau)\)-derivation \( d : A \rightarrow X \) defined by

\[
d(a) = \lim_{n \to \infty} \frac{f(2^na)}{2^n}, \quad a \in A \]

which satisfies \( \| d(a) - f(a) \| \leq \alpha \). Since \( A \) is \((\sigma - \tau)\)-contractible, there is some \( x \in X \) such that \( d(a) = x\sigma(a) - \tau(a)x \).

Hence

\[
\| x\sigma(a) - \tau(a)x - f(a) \| \leq \| x\sigma(a) - \tau(a)x - d(a) \| + \| d(a) - f(a) \| \leq \alpha
\]

Therefore \( A \) is approximately \((\sigma - \tau)\)-contractible.

Conversely, let \( A \) be approximately \((\sigma - \tau)\)-contractible and \( d : A \rightarrow X \) be a bounded \((\sigma - \tau)\)-derivation. Then \( d \) is trivially an approximate \((\sigma - \tau)\)-derivation. Due to the approximate \((\sigma - \tau)\)-contractibility of \( A \), there exist \( \beta > 0 \) and \( x \in X \) such that

\[
\| x\sigma(a) - \tau(a)x - d(a) \| \leq \beta.
\]

Replacing \( a \) by \( 2^na \) in the later inequality we get

\[
\| x\sigma(a) - \tau(a)x - d(a) \| \leq 2^{-n}\beta
\]

for all positive integer \( n \). Hence \( x\sigma(a) - \tau(a)x = d(a) \). It follows that \( A \) is \((\sigma - \tau)\)-contractible.\( \square \)

One can similarly define the notions of \((\sigma - \tau)\)-amenability and approximate \((\sigma - \tau)\)-amenability and establish the following theorem.

3.2. Theorem. A Banach algebra \( A \) is approximately \((\sigma - \tau)\)-amenable if and only if \( A \) is \((\sigma - \tau)\)-amenable.

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