Abstract. Let \( H \) be a finite quasisimple classical group, i.e. \( H \) is perfect and \( S := H/Z(H) \) is a finite simple classical group. We prove in this paper that, excluding the cases when the simple group \( S \) has a very exceptional Schur multiplier such as \( \text{PSL}_3(4) \) or \( \text{PSU}_4(3) \), \( H \) is uniquely determined by the structure of its complex group algebra. The proofs make essential use of the classification of finite simple groups as well as the results on prime power character degrees and relatively small character degrees of quasisimple classical groups.

1. Introduction

An important question in representation theory of finite groups is the extent to which the complex group algebra of a finite group determines the group or its properties. In late 1980s, Isaacs proved that if \( CG \cong CH \) and \( p \) is a prime, then \( G \) has a normal \( p \)-complement if and only if \( H \) has a normal \( p \)-complement, and therefore the nilpotency of a group is determined by the complex group algebra of the group (cf. [Isa]). Later on, Hawkes gave a counterexample showing that the same statement does not hold for supersolvability (cf. [Haw]). It is still unknown nowadays whether the solvability of a finite group is preserved by its complex group algebra (cf. [MaK, Problem 11.8]).

While some properties of a solvable group \( G \) is determined by \( CG \), it is well known that \( G \) in general is not determined by \( CG \). For instances, \( CD_8 \cong CQ_8 \) or Dade even constructed two non-isomorphic metabelian groups with isomorphic group algebras over any field (cf. [Dade]).

In contrast to solvable groups, simple groups or more generally quasisimple groups are believed to have a stronger connection with their complex group algebras. Recently, Tong-Viet in a series of papers [Ton1, Ton2, Ton3] has succeeded in proving that, if \( G \) is a finite group and \( H \) is a finite simple group such that \( CG \cong CH \), then \( G \cong H \).

The main purpose of this paper is to go one step further and improve Tong-Viet’s result to quasisimple groups in the case of classical groups.

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Theorem 1.1. Let \( G \) be a finite group and let \( H \) be a quasisimple group such that \( H/Z(H) \) is a simple classical group different from \( \text{PSL}_3(4) \) and \( \text{PSU}_4(3) \). If \( CG \cong CH \) then \( G \cong H \). In other words, every quasisimple classical group \( H \) with \( H/Z(H) \neq \text{PSL}_3(4) \) and \( \text{PSU}_4(3) \) is uniquely determined (up to isomorphism) by the structure of its complex group algebra.

We should mention that, as the complex group algebra of a finite group determines and also is determined by the degrees (counting multiplicities) of irreducible characters of the group, the statement of the theorem is equivalent to say that, excluding those exceptions, every quasisimple classical group is determined by its multiset of character degrees or its character-degree-frequency function (cf. [Haw] for the definition).

If \( G \) is a finite group and \( H \) is a quasisimple group having the same complex group algebra then it is easy to see that they have the same order and multiset of character degrees (i.e., the set of character degrees counting multiplicities). Moreover, as \( H \) has a unique linear character, so does \( G \) and hence \( G \) is perfect. We will prove that, if \( M \) is a maximal normal subgroup of \( G \) then \( G/M \) must be isomorphic to \( S := H/Z(H) \). This basically eliminates the involvement of all nonabelian simple group other than \( S \) in the structure of \( G \). Let \( \text{cd}(G) \) denote the character degree set of \( G \) and let \( \text{Schur}(S) \) denote the Schur cover of \( S \). On the way to the proof of Theorem 1.1, we in fact prove the following:

Proposition 1.2. Let \( S \) be a simple classical group. Let \( G \) be a nontrivial perfect group and \( M \) be a maximal subgroup of \( G \) such that \( |S| \mid |G| \mid |\text{Schur}(S)| \) and \( \text{cd}(S) \subseteq \text{cd}(G) \subseteq \text{cd}(\text{Schur}(S)) \). Then \( G/M \cong S \).

This proposition will be proved in sections 3, 4, 5, and 6 for linear groups, unitary groups, symplectic groups, and orthogonal groups, respectively. Assume for now that it is true. Since \( |G| = |H| \) and \( G/M \cong H/Z(H) \), we deduce that \( |M| = |Z(H)| \). It follows that, if \( H \) is simple then \( M \) is trivial and we would have immediately that \( G \cong H \). However, since we are working on a quasisimple group \( H \), the problem becomes much more complicated. To illustrate the difficulty of the problem for quasisimple groups, let us mention that we are unable to answer whether the complex group algebra of a quasisimple group, whose quotient by its center is \( \text{PSL}_3(4) \) or \( \text{PSU}_4(3) \), determines the group uniquely up to isomorphism. This is due to the very exceptional Schur multiplier of these groups (cf. Lemmas 7.3 and 7.4).

In section 7 we will show that if \( H/Z(H) \) is different from \( \text{PSL}_3(4) \) and \( \text{PSU}_4(3) \), then \( H/Z(H) \) can not be imbedded into the automorphism group of an abelian group of order not larger than that of the Schur multiplier of \( H/Z(H) \). We then use this to show that \( G \) is isomorphic to a quotient of the Schur cover of \( H/Z(H) \) (cf. Lemma 7.5). From there, with a careful analysis of the Schur multipliers of finite simple groups, we are able to show that \( G \cong H \). We note that many results in section 7 are stated for every finite group of Lie type, not only classical groups.
With the techniques presented here, we hope to continue the problem for the Schur covers of symmetric and alternating groups (cf. [NgT]), quasisimple exceptional groups of Lie type as well as quasisimple sporadic groups. To end this introduction, we put a list of notation that will be used throughout the paper.

| Notation | Meaning |
|----------|---------|
| \(Z(G)\) | the center \(G\) |
| \(\text{Aut}(G)\) | the automorphism group of \(G\) |
| \(\text{Irr}(G)\) | the set of irreducible characters of \(G\) |
| \(\text{cd}(G)\) | the set of irreducible character degrees of \(G\) |
| \(d_i(G)\) | the \(i\)th smallest number in \(\text{cd}(G)\setminus\{1\}\) |
| \(G^{(i)}\) | the \(i\)th derived subgroup of \(G\) |
| \(\text{Mult}(G)\) | the Schur multiplier of \(G\) |
| \(\text{Schur}(G)\) | the Schur cover of a perfect group \(G\) |
| \(\text{St}_G\) | the Steinberg character of a group of Lie type \(G\) |
| \(n_p\) | the \(p\)-part of an integer number \(n\) |

### 2. Character degrees of quasisimple classical groups

In this section, we prove a part of Proposition 1.2. More explicitly, we show that \(G/M\) is a simple group of Lie type in characteristic \(p\). We start with a couple of known lemmas.

**Lemma 2.1** (Gallagher’s lemma). Let \(M \leq G\) and \(\chi \in \text{Irr}(G)\). If \(\chi_M \in \text{Irr}(M)\) then \(\chi \theta \in \text{Irr}(G)\) for every \(\theta \in \text{Irr}(G/M)\).

**Lemma 2.2** ([Hup], Lemma 3). Let \(M \leq G\) and let \(\theta \in \text{Irr}(M)\) be \(G\)-invariant. If \(\chi \in \text{Irr}(G)\) lying above \(\theta\), then \(\chi = \theta_0 \tau\), where \(\theta_0\) is a character of an irreducible projective representation of \(G\) of degree \(\theta(1)\) and \(\tau\) is a character of an irreducible projective representation of \(G/M\).

**Lemma 2.3** ([Ton1], Lemma 2.2). If \(n \geq 10\) then the maximal degree of \(A_n\) is at least \(2^{n-1}\).

The next lemma is a special property of the Steinberg character, denoted by \(\text{St}_S\), of a simple group of Lie type \(S\) in characteristic \(p\). It is well known that the degree of \(\text{St}_S\) is \(|S|_p\).

**Lemma 2.4.** Let \(S\) be simple group of Lie type. Then no proper multiple of \(\text{St}_S(1)\) is a degree of \(\text{Schur}(S)\).

**Proof.** If \(S\) is one of the following groups of exceptional Schur multiplier:

\[
\text{PSL}_2(4), \text{PSL}_2(9), \text{PSL}_3(2), \text{PSL}_3(4), \text{PSL}_4(2), \Omega_7(2), \Omega_7(3), P\Omega_8^+(2), P\Omega_8^+(2),
\]

\[
\text{PSU}_4(2), \text{PSU}_4(3), \text{PSU}_6(2), F_4(2), G_2(3), G_2(4)^2 E_6(2), \text{Suz}(8),
\]
the statement can be checked directly using [At1]. So we assume that \( S \) is none of these. Then \( \text{Schur}(S) \) is indeed the finite Lie-type group of simply-connected type corresponding to \( S \) and moreover \(|S|_p = |\text{Schur}(S)|_p \). Therefore, using [Cur], we have that \( \text{St}_S \) is the unique character of \( p \)-defect zero of \( \text{Schur}(S) \) where \( p \) is the characteristic of the underlying field of \( S \).

The next result is a slight improvement of a result of Tong-Viet in [Ton3, Proposition 3.1], which has a stronger hypothesis that \(|G| = |S| \) and \( \text{cd}(G) = \text{cd}(S) \).

**Proposition 2.5.** Let \( S \) be a simple classical group defined over a field of characteristic \( p \). Let \( G \) be a nontrivial perfect group and \( M \) be a maximal subgroup of \( G \) such that \(|S| \mid |G| \mid |\text{Schur}(S)| \) and \( \text{cd}(S) \subseteq \text{cd}(G) \subseteq \text{cd}(\text{Schur}(S)) \). Then \( L := G/M \) is a simple group of Lie type in characteristic \( p \).

**Proof.** The proof can be adapted from that of [Ton3, Proposition 3.1] with some minor modifications. Therefore, we only sketch the main ideas of the proof and leave some detailed check to the reader. From the hypothesis, we have \(|L| \mid |\text{Schur}(S)| \), \( \pi(L) \subseteq \pi(\text{Schur}(S)) \), and \( d_i(L) \geq d_i(\text{Schur}(S)) \) for every \( i \). As \( \text{cd}(S) \subseteq \text{cd}(G) \) and \( \text{St}_S(1) = |S|_p \), \( G \) has an irreducible character, say \( \chi \), of degree \(|S|_p \). Let \( \theta \in \text{Irr}(M) \) lying under \( \chi \) and \( I := I_G(\theta) \).

(i) Assume that \( L \) is a sporadic group or the Tits group. If \( \theta \) is not \( G \)-invariant, then \( I \) would be a proper subgroup of \( G \). By Clifford theory, \( \chi = \phi^G \) for some \( \phi \in \text{Irr}(I/\theta) \) and also \(|S|_p = \chi(1) = |G : I|\phi(1) \). Using the classification of subgroups of prime power index in a simple group by Guralnick (cf. [Gur]), we have that

\[
L = M_{11}, I/M = M_{10} \quad \text{and} \quad |G : I| = 11,
\]

or

\[
L = M_{23}, I/M = M_{22} \quad \text{and} \quad |G : I| = 23.
\]

In particular, the proper subgroup \( I/M \) of \( G/M \) has order coprime to \( p \). It then follows that \( (\phi(1), |I : M|) = 1 \) and hence \( \theta \) extends to \( \phi \). Together with Gallagher’s lemma, we obtain \( \phi \psi \in \text{Irr}(I/\theta) \) for every \( \phi \in \text{Irr}(I/M) \). Therefore \((\phi \psi)^G \) is an irreducible character of \( G \) of degree \( \chi(1)\psi(1) = |S|_p\psi(1) \). Taking \( \psi \) to be a nonlinear character of \( I/M \), we get a contradiction by Lemma 2.4.

So \( \theta \) is \( G \)-invariant and hence \( \chi_M = e\theta \) for some \( e \in \mathbb{N} \). If \( e = 1 \), \( \chi \) would be an extension of \( \theta \) and hence \( \chi \psi \in \text{Irr}(G) \) for every \( \psi \in \text{Irr}(L) \). Taking \( \psi \) to be nonlinear and note that \( \text{cd}(G) \subseteq \text{cd}(\text{Schur}(S)) \), we deduce that \( \text{Schur}(S) \) has a degree which is a proper multiple of \( \chi(1) = |S|_p \), violating Lemma 2.4 again. Thus \( e \) is a nontrivial \( p \)-power. Moreover, by Lemma 2.2, \( e \) is a degree of a projective irreducible representation of \( L \). By the classification of prime power degrees of irreducible characters of quasisimple groups by Malle and Zalesskii (cf. [MaZ]), we come up with the following cases:

\[
(L, e) = (M_{11}, 11), (M_{11}, 2^4), (M_{12}, 11), (M_{12}, 2^4), (M_{12}, 2^5), (M_{24}, 23),
\]
The cases $L = M_{11}$ and $M_{12}$ can not happen since no quasisimple classical groups in characteristic 2 or 11 have both degrees 11 and 2 by [MaZ Theorem 1.1]. Similarly, $L \neq 2F_4(2)'$ since no quasisimple classical groups in characteristic 2 or 3 have both degrees 3 and 2. If $(L, e) = (M_{24}, 23), (Co_2, 23), (Co_3, 23), (J_2, 2^6), (Ru, 2^{13}), (2F_4(2)', 3^3)$, or $(2F_4(2)', 2^{11})$, then $p = 23$ and moreover 23 must be the degree of the Steinberg character of $S$. However, one can check from [Atl] that, in these cases, $L$ has a degree 253, which is a proper multiple of 23 and this violates Lemma 2.4. For the remaining cases $(L, e) = (J_2, 2^6)$ or $(Ru, 2^{13})$, we can argue as follows. Suppose that $S = S_n(q)$ where $n$ is the dimension of $S$ and $q$ is the cardinality of the underlying field of $S$. Since $d_1(L) \geq d_1(Schur(S))$, by using the result on minimal degrees of quasisimple classical groups of Tiep and Zalesskii in [TZa], one can bound $d$ and come up with a short list of possibilities for $S$. At this stage, one just checks directly by using [Atl] or [Lub] or [GAP] to verify that $cd(L) \notin cd(Schur(S))$.

(ii) Assume that $L$ is an alternating group $A_m$ with $m = 7$ or $m \geq 9$. We note that $A_5, A_6$, and $A_8$ can be considered as Lie-type groups. Since the arguments for each family of classical groups are fairly similar, we only present here the linear group case.

Suppose that $S = PSL_n(q)$. Using [Atl], one can easily eliminate those groups with exceptional Schur multiplier such as: $PSL_2(4) = PSL_2(5), PSL_2(9), PSL_3(2), PSL_3(4)$, and $PSL_4(2)$ and also the group $PSL_2(7)$. So we assume $S$ is none of these and hence $Schur(S) = SL_n(q)$. Assume first that $m \leq 9$. Then we have $d_1(SL_n(q)) \leq d_1(A_m) \leq 8$. Using the table of minimal degrees of $SL_n(q)$ in [TZa Table IV] again, we deduce that $(n, q) = (2, q \text{ odd } \leq 17) \text{ or } (2, q \text{ even } \leq 8)$. It is now routine to check that $cd(A_m) \notin cd(SL_n(q))$ for these cases.

Thus we can assume that $m \geq 10$. It then follows by the hypothesis and Lemma 2.3 that

$$b(SL_n(q)) \geq b(L) = b(A_m) \geq 2^{m-1} \geq 2^{d_1(L)} \geq 2^{d_1(SL_n(q))}.$$ 

With the list of character degrees of Lie-type groups of low rank available in [Lnb], one can check that this inequality is violated for $n = 2, 3, 4$. For $n \geq 5$, as $d_1(SL_n(q)) = (q^n - q)/(q - 1)$ by [TZa Table IV] and $b(SL_n(q)) < \sqrt{|SL_n(q)|} < q^{n^2/2}$, the above inequality implies

$$2^{(q^n - q)/(q - 1)} < q^{n^2/2},$$

which in turn implies

$$2q^{n-1} < \log_2 q \cdot n^2.$$ 

This last inequality is impossible when $n \geq 5$.

(iii) Assume that $L$ is a simple group of Lie type in characteristic $r \neq p$. Then, as $cd(L) \in cd(Schur(S))$, $cd(Schur(S))$ has at least two prime power members: $|S|_p$ and
Again using [MaZ], we get a list of possibilities for $S$ and $|L|_L$. These possibilities can be eliminated similarly as in [Ton3, Proposition 3.1] except the following one:

$$S = \text{PSL}_2(q), \quad |L|_L = q \pm 1 \text{ or } |L|_L = (q \pm 1)/2 \text{ with odd } q.$$ 

In fact, Tong-Viet assumed in his proposition that the simple classical group $S$ has rank $\geq 3$ and therefore $\text{PSL}_2(q)$ was not in his consideration. We now present the arguments to eliminate this remaining possibility. Again, as it is routine to check the lemma for small $q$, we assume that $q \geq 11$. We then have that $\text{Schur}(S) = \text{SL}_2(q)$ and therefore $\text{cd}(L) \subseteq \text{cd}(\text{SL}_2(q))$. Since $\text{SL}_2(q)(q \geq 11)$ has at most 5 character degrees, we deduce that $L = \text{SL}_2(q_1)$ for some $q_1$ coprime to $p$. So we have

$$\{q_1, q_1 - 1, q_1 + 1\} \subseteq \text{cd}(L) \subseteq \text{cd}(\text{SL}_2(q)) \subseteq \{q, q - 1, q + 1, (q - 1)/2, (q + 1)/2\},$$

which is impossible as $q \geq 11$. □

3. Linear groups

We have seen from Proposition 2.5 that $L := G/M$ is a simple group of Lie type in characteristic $p$ and so we write $L = L(p^b)$ to indicate that $L$ is defined over a field of $p^b$ elements. Assuming the hypothesis of Proposition 1.2, we have $|L| | |\text{Schur}(S)|$ and $\text{cd}(L) \subseteq \text{cd}(\text{Schur}(S))$.

From now on to the end of section 6, we often use these conditions without notice. The following classical result of Zsigmondy is very useful in proving the non-divisibility between orders and also character degrees of classical groups.

**Lemma 3.1** (Zsigmondy’s theorem). If $x > y > 0$ are coprime integers, then for any natural number $n > 1$ there is a prime number denoted by $\ell(x, y, n)$ (called a primitive prime divisor) that divides $x^n - y^n$ and does not divide $x^i - y^i$ for any positive integer $i < n$, with the following exceptions:

(i) $x = 2, y = 1$, and $n = 6$; or
(ii) $x + y$ is a power of two, and $n = 2$.

**Table 1.** Possible nontrivial character degrees up to $q^n$ of $\text{SL}_n(q)$ (cf. [Lub], [TZa]).

| Value of $n$ | Degrees |
|--------------|----------|
| $n = 2$      | $(q \pm 1)/2, q, q \pm 1$ |
| $n = 3$      | $q(q + 1), q^2 + q + 1, (q + 1)(q - 1)^2, q^3 - 1, (q + 1)(q - 1)^2/3, (q + 1)(q^2 + q + 1)/3$ |
| $n = 4$      | $q(q^2 + q + 1), (q + 1)(q^2 + 1), (q - 1)(q^3 - 1), (q - 1)(q^3 - 1)/2, (q^2 + 1)(q^2 + q + 1)/2$ |
| $n \geq 5$   | $(q^n - q)/(q - 1), (q^n - 1)/(q - 1)$ |
Lemma 3.2. Proposition 1.2 is true when $S$ is a simple linear group.

Proof. One can verify the lemma easily for the following groups: $S = \text{PSL}_2(4) \cong \text{PSL}_2(5)$, $\text{PSL}_2(9)$, $\text{PSL}_3(2)$, $\text{PSL}_4(2)$, and $\text{PSL}_4(2)$ by using [Atl]. So we assume that $S = \text{PSL}_n(q) = \text{PSL}_n(p^a)$ is none of these. In that case, $\text{Schur}(S) = \text{SL}_n(p^a)$ and $\text{St}_S(1) = p^{an(n-1)/2}$ is the only $p$-power degree of $\text{SL}_n(q)$ by the classification of prime power degrees of quasisimple groups in [MaZ, Theorem 1.1]. Recall from Proposition 2.5 that $L := G/M$ is a simple group of Lie type in characteristic $p$ and we aim to show that $L \cong S$.

(i) We first outline the arguments to eliminate the simple Lie-type groups of exceptional type. Suppose that $|L| \mid p^{bm}$ and the largest factor of the form $p^{x} - 1$ in the formula of $|L|$ is $p^{bm'} - 1$. Then, as $\text{St}_L(1) = p^{bm}$ is a $p$-power degree of $\text{SL}_n(q)$, we get $p^{bm} = p^{an(n-1)/2}$ and therefore

$$(3.1) \quad bm = an(n-1)/2.$$ 

It follows that

$$bm' = \frac{an(n-1)}{2} \cdot \frac{m'}{m}. $$

As

$$(p^{bm'} - 1) \mid |L| \mid |\text{SL}_n(q)| = p^{an(n-1)/2} \prod_{i=2}^{n}(p^{ia} - 1),$$

using Zsigmondy’s theorem, one gets

$$\frac{an(n-1)}{2} \cdot \frac{m'}{m} \leq an \quad \text{or} \quad \frac{an(n-1)}{2} \cdot \frac{m'}{m} = 6 > an.$$ 

Therefore,

$$n \leq \max\{5, 2m/m' + 1\}. $$

Since $m/m'$ is known (the maximum value of $m/m'$ is 4, obtained when $L = E_8$), we can bound above the value of $n$ (indeed is at most $4 \cdot 2 + 1 = 9$ for all groups of exceptional type). For small $n$, we use Equation 1.1 to show that either $|L| \nmid |\text{SL}_n(q)|$ or $\text{cd}(L) \nsubseteq \text{cd}(\text{SL}_n(q))$ with the help of [Lub] and Zsigmondy’s theorem. To illustrate this, let us present the arguments for the most difficult case $L(p^b) = E_8(p^b)$.

Assume by contrary that $L(p^b) = E_8(p^b)$. Then $m = 120$ and $m' = 30$ by [Atl] p. xvi. Therefore

$$n \leq \max\{5, 2 \cdot 120/30 + 1\} = 9.$$ 

For every $n \leq 9$, we in fact obtain a contradiction by showing that $|E_8(p^b)| \nmid |\text{SL}_9(p^a)|$. For instance, if $n = 9$ then $120b = 36a$ by (1.1). Therefore $b = 9c$ and $a = 30c$ for some positive integer $c$. Now the Zsigmondy prime $\ell(p, 1, 216c)$ is a divisor of $|E_8(p^{bc})|$ but does not divide $|\text{SL}_9(p^{30c})|$. Other values of $n \leq 8$ are handled similarly.

(ii) Next, we eliminate the remaining simple classical groups in characteristic $p$ except $\text{PSL}_n(p^a)$. 

\[ L = \Omega_{2m+1}(p^b) \] or \( \text{PSp}_{2m}(p^b) \) with \( m \geq 2 \). As \( \text{St}_L(1) \in \text{cd}(\text{SL}_n(p^a)) \), we have \( p^{bm^2} = p^{an(n-1)/2} \) and hence

\[ (3.2) \quad 2bm^2 = an(n-1). \]

Moreover, since \( (p^{2bm} - 1) \mid |L| \mid |\text{SL}_n(p^a)| \), \( 2bm \leq an \) or \( 2bm = 6 > an \) by Zsigmondy’s theorem. The case \( 2bm = 6 > an \) indeed does not happen by \( (3.2) \) and the fact that \( m \geq 2 \). Thus we must have \( 2bm \leq an \). It follows by \( (3.2) \) that \( m \geq n-1 \) and hence \( 2b(n-1) \leq an \). In particular, we obtain \( b < a \) and \( a \geq 2 \).

From the description of unipotent characters of \( |L| \) (cf. 
\[ \text{Car}, \text{p. 466} \]), we see that \( L \) has a unipotent character \( \chi \) of degree \( \chi(1) = (p^{bm} - 1)(p^{bm} - b^2)/2(p^b + 1) \) and moreover

\[ \chi(1) < p^{2bm} \leq p^{an}. \]

We note that \( \chi(1) \) is indeed the smallest degree of unipotent characters of \( L \) excluding some exceptions (cf. \[ \text{Ng}, \text{Table 1} \]). Recall that \( \chi(1) \in \text{cd}(\text{SL}_n(p^a)) \) and we will get to a contradiction by showing that \( \chi(1) \) can not be equal to a degree smaller than \( p^{an} \) of \( \text{SL}_n(p^a) \).

When \( n \geq 5 \), by inspecting the list of three smallest nontrivial degrees of \( \text{SL}_n(q) \) in \[ \text{TZa}, \text{Table IV} \], we see that

\[ d_3(\text{SL}_n(p^a)) \geq \frac{(p^{an} - 1)(p^{an(n-1)} - p^{2a})}{(p^a - 1)(p^{2a} - 1)} > p^{an}. \]

We deduce that \( \chi(1) < d_3(\text{SL}_n(p^a)) \) and hence

\[ \chi(1) = d_1(\text{SL}_n(p^a)) = \frac{p^{an} - p^a}{p^a - 1} \text{ or } \chi(1) = d_2(\text{SL}_n(p^a)) = \frac{p^{an} - 1}{p^a - 1}. \]

In particular, \( |\chi(1)|_p = 1 \) or \( p^a \), which is impossible as \( \chi(1) = (p^{bm} - 1)(p^{bm} - p^b)/2(p^b + 1) \) and \( b < a \).

For \( n \leq 4 \), one can argue similarly by using the list of character degrees of \( \text{SL}_n(q) \) available in the website of Lubeck (cf. \[ \text{Lub} \]). In fact, from there, one observes that every degree smaller than \( q^a \) of \( \text{SL}_n(q) \) (\( n \leq 4 \)) has \( p \)-part either 1, or \( q \) (cf. Table [1]). Therefore, \( \chi(1) \) can not be one of these degrees.

\[ \bullet L = P\Omega_{2m}^\pm(p^b) \] with \( m \geq 4 \). Then we have \( p^{bm(m-1)} = p^{an(n-1)/2} \) and hence

\[ (3.3) \quad 2bm(m-1) = an(n-1). \]

Moreover, since \( (p^{2b(m-1)} - 1) \mid |L| \mid |\text{SL}_n(p^a)| \), it follows by Zsigmondy’s theorem that \( 2b(m-1) \leq an \). Therefore, \( m \geq n-1 \) and also \( 2b(n-2) \leq an \). We claim that \( b < a \). Assume the contrary, then \( b \geq a \) and hence \( (3.3) \) implies that \( n(n-1) \geq 2m(m-1) \). In particular, \( n \geq 5 \). Now the inequality \( 2b(n-2) \leq an \) is violated.

From the description of unipotent characters of \( |L| \) (cf. \[ \text{Car}, \text{p. 471} \]), we see that \( L \) has a unipotent character \( \chi \) of degree \( (p^{bm} \mp 1)(p^{bm} \mp p^b)/(p^{2b} - 1) \), where \( \chi(1) < p^{2b(m-1)} \leq p^{an} \). Now one just argues as in the previous case.
• \( L = \text{PSU}_m(p^b) \) with \( m \geq 3 \). As \(|L|_p = |\text{SL}_n(p^a)|_p\), we have \( bm(m-1) = an(n-1) \). Since \(|L| \mid |\text{SL}_n(p^a)|\), the Zsigmondy’s theorem implies that \( b(m-1) \leq an \). It then follows that \( m \geq n - 1 \) and hence \( an \geq b(n-2) \), which in turn implies that \( b \leq 2a \). Moreover, if \( b = 2a \) then we must have \( n = 4 \) and \( m = 3 \) and it is easy to check that \( \text{cd}(L) \not\subseteq \text{cd}(\text{SL}_n(q)) \) in this case. So we conclude that \( b < 2a \).

By [Car, p. 465], \( \text{PSU}_m(p^b) \) has a unipotent character \( \chi \) of degree \((p^{bm} + (-1)^m p^b)/(p^b + 1)\). Observe that \( \chi(1) < p^{b(m-1)} \) and therefore \( \chi(1) < p^{an} \). As before, since \( |\chi(1)|_p = p^b \) and the \( p \)-part of a degree smaller than \( p^{an} \) of \( \text{SL}_n(p^a) \) is either 1 or \( p^a \) (cf. Table I), we deduce that \( b = a \), which in turn implies that \( m = n \) as \( bm(m-1) = an(n-1) \). This leads to a contradiction as \( \text{cd}(L) = \text{cd}(\text{PSU}_n(p^a)) \not\subseteq \text{cd}(\text{SL}_n(p^a)) \).

• \( L = \text{PSL}_m(p^b) \) with \( m \geq 2 \). Arguing exactly as in the unitary group case, we obtain that \( a = b \) and \( m = n \). This means \( L = \text{PSL}_n(p^a) \), as wanted. \( \square \)

4. UNITARY GROUPS

If we want to prove Proposition [L2] for unitary groups similarly to the case of linear groups, we need to know all the character degrees up to \( q^{2n} \) of \( \text{SU}_n(q) \). The characters of relatively small degrees of unitary groups have been worked out in [LOST]. From there, one can obtain a list of those degrees smaller than \( q^{2n} \) when \( n \) is large enough, say \( n \geq 10 \). However, when \( n \leq 9 \), the list is fairly long and therefore the arguments would be quite complicated. In this section, we have found some new arguments to avoid the analysis of degrees of \( \text{SU}_n(q) \) in small dimensions.

**Table 2.** Possible nontrivial character degrees up to \( q^{2n} \) of \( \text{SU}_n(q) \) (cf. [LOST]).

| Value of \( n \) | Degrees |
|---|---|
| \( n \geq 10 \) | \( q^n + (-1)^v q, (q^n - (-1)^v) (q^{n-1} + (-1)^v q^2), (q^n + (-1)^v q^2 (q^{n-1} - (-1)^v q^2), (q^n - (-1)^v) (q^{n-1}) (q^2 - 1), (q^n - (-1)^v) (q^{n-1} + (-1)^v q^2), (q^n + (-1)^v q^2 (q^{n-1} - (-1)^v q^2), (q^n - (-1)^v) (q^{n-1}) (q^2 - 1) \) |

**Lemma 4.1.** Proposition [L2] is true when \( S \) is a simple unitary group.

**Proof.** Since we can verify the lemma for the groups with exceptional Schur multiplier such as \( \text{PSU}_4(2), \text{PSU}_4(3), \text{PSU}_6(2) \) by using [At], we assume that \( S = \text{PSU}_n(q) = \text{PSU}_n(p^a) \ (n \geq 3) \) is none of these. In that case, \( \text{Schur}(S) = \text{SU}_n(p^a) \) and \( \text{St}_S(1) = p^{an(n-1)/2} \) is the only \( p \)-power degree of \( \text{SU}_n(q) \) by [MaZ, Theorem 1.1]. Recall that \( L := G/M \) is a simple group of Lie type in characteristic \( p \) and we aim to show that \( L \cong S \).

(i) We first eliminate the simple Lie-type groups of exceptional type. Suppose that \( |L|_p = p^{bm} \) and the largest factor of the form \( p^x - 1 \) in the formula of \( |L| \) is \( p^{bm'} - 1 \). Then, as \( \text{St}_L(1) = p^{bm} \) is a \( p \)-power degree of \( \text{SU}_n(q) \), we get \( p^{bm} = p^{an(n-1)/2} \) and
therefore
\begin{equation}
(4.1) \quad bm = an(n - 1)/2.
\end{equation}
It follows that
\begin{equation}
bm' = \frac{an(n - 1)}{2} \cdot \frac{m'}{m}.
\end{equation}
As
\begin{equation}
(p^{bm'} - 1) \mid |SU_n(q)| = p^{an(n-1)/2} \prod_{i=2}^{n} (p^{ia} - (-1)^i),
\end{equation}
we have
\begin{equation}
(p^{bm'} - 1) \mid \prod_{i=2}^{n} (p^{2ia} - 1).
\end{equation}
Using Zsigmondy’s theorem and recalling that \( n \geq 3 \), we deduce
\begin{equation}
bm' = \frac{an(n - 1)}{2} \cdot \frac{m'}{m} \leq 2an.
\end{equation}
Therefore,
\begin{equation}
n \leq 4m/m' + 1.
\end{equation}
Now we just argue as in the proof of Lemma 3.2 to get a contradiction.

(ii) Next, we eliminate the simple classical groups in characteristic \( p \) except \( PSU_n(p^a) \).
\begin{itemize}
  \item \( L = \Omega_{2m+1}(p^b) \) or \( PSp_{2m}(p^b) \) with \( m \geq 2 \). As \( \text{St}_L(1) \in \text{cd}(SU_n(p^a)) \), we have \( p^{bm^2} = p^{an(n-1)/2} \) and hence
\begin{equation}
(4.2) \quad 2bm^2 = an(n - 1).
\end{equation}
This in particular implies that \( b \neq 2a \) since \( n(n-1) \) is never a square. Moreover, since 
\begin{equation}
(p^{2bm} - 1) \mid |L| \mid |SU_n(p^a)| ,
\end{equation}
it follows by Zsigmondy’s theorem that \( bm \leq an \), which in turn implies that \( 2m \geq n - 1 \) by (4.2). Therefore, \( bm \leq an \leq a(2m + 1) < 3am \) and hence \( b < 3a \).
Recall from the previous cases that \( L \) has a unipotent character \( \chi \) of degree \( \chi(1) = (p^{bm} - 1)(p^{bm} - p^b)/2(p^b + 1) \). As this degree has \( p \)-part \( p^b \) while the \( p \)-part of any degree of \( SU_n(p^a) \) is a power of \( p^a \), we deduce that \( b = a \) since \( 2a \neq b < 3a \). Now equation (4.2) implies
\begin{equation}
2m^2 = n(n - 1).
\end{equation}
This happens only if \( (n, m) = (9, 6) \) or \( n \geq 10 \). The former case leads to a contradiction since \( |\Omega_{13}(p^a)| = |PSp_{12}(p^a)| \) does not divide \( |SU_9(p^a)| \). Thus \( n \geq 10 \).
Recall that the degree of \( \chi \) is \( (p^{bm} - 1)(p^{bm} - p^b)/2(p^b + 1) \), which is smaller than \( p^{2bm} \leq p^{2am} \). By inspecting the list of degrees smaller than \( q^{2m} \) of \( SU_n(q) \) with \( n \geq 10 \).
Moreover, since Propositions 6.3 and 6.6, which has been reproduced in Table 2, we come up with three possibilities as follow:

\[
\chi(1) = \frac{q^n + (-1)^n q}{q + 1} \frac{(q^n - (-1)^n)(q^n + (-1)^n)}{(q + 1)(q^2 - 1)}, \quad \text{or} \quad \frac{(q^n - (-1)^n)(q^{n-1} + (-1)^n q)}{(q + 1)^2},
\]

where \( q = p^a \). However, with the conditions \( b = a \) and \( 2m^2 = n(n-1) \), it is easy to see that these equations have no solutions.

- \( L = \text{PSL}_m(p^b) \) with \( m \geq 2 \). As \( |L|_p = |\text{SU}_n(p^a)|_p \), we have \( bm(m-1) = an(n-1) \). Since \( |L| \mid |\text{SU}_n(p^a)| \), the Zsigmondy’s theorem implies that \( bm \leq 2an \). It then follows that \( 2(m-1) \geq n-1 \) and hence \( bm \leq 2an \leq 2a(m-1) < 4am \). Thus \( b < 4a \).

We know that \( \text{PSL}_m(p^b) \) has a unipotent character \( \chi \) of degree \( (p^{bm} - p^b)/(p^b - 1) \) and this degree belongs to \( \text{cd}(\text{SU}_n(p^a)) \). As \( |\chi(1)|_p = p^b \) and the \( p \)-part of any degree of \( \text{SU}_n(p^a) \) is a power of \( p^a \), the fact \( b < 4a \) implies that \( b = a, 2a, \) or \( 3a \). First, if \( b = a \) then \( m = n \) and this is impossible since \( |L| = |\text{PSL}_n(p^a)| \mid |\text{SU}_n(p^a)| \) for every \( n \geq 3 \). Next, if \( b = 2a \) then

\[
2m(m-1) = n(n-1),
\]

which implies that \( (n, m) = (4, 3) \) or \( n \geq 10 \). The former case does not happen since \( |\text{PSL}_m(p^b)| = |\text{PSL}_3(p^{2a})| \mid |\text{SU}_4(p^a)| \). Hence we must have \( n \geq 10 \). Recall that the degree of \( \chi \) is \( (p^{bm} - p^b)/(p^b - 1) \), which is smaller than \( p^{2an} \). Using Table 2 again, one sees that there is only one possibility

\[
\frac{p^{bm} - p^b}{p^b - 1} = \frac{(p^{an} - (-1)^n)(p^{a(n-1)} + (-1)^n p^{2a})}{(p^n + 1)(p^{2a} - 1)}.
\]

Again, it is easy to see that this equation has no solutions since \( b = 2a \) and \( 2m(m-1) = n(n-1) \).

Finally, we consider the remaining case \( b = 3a \). Then \( 3m(m-1) = n(n-1) \) and hence \( (n, m) = (3, 2) \) or \( n \geq 10 \). The case \( (n, m) = (3, 2) \) can not happen as \( |L| = |\text{PSL}_2(p^{3a})| > |\text{SU}_3(p^a)| \) and the case \( n \geq 10 \) is handled exactly as above.

- \( L = P\Omega_{2m}^\pm(p^b) \) with \( m \geq 4 \). Then we have \( p^{bm(m-1)} = p^{an(n-1)/2} \) and hence

\[
2bm(m-1) = an(n-1).
\]

Moreover, since \( (p^{2b(m-1) - 1}) \mid |L| \mid |\text{SU}_n(p^a)| \), it follows by Zsigmondy’s theorem that \( b(m-1) \leq an \). Therefore, \( 2m \geq n-1 \) and hence \( b(m-1) \leq a(2m+1) < 4a(m-1) \). This means \( b < 4a \).

Recall \( L \) has a unipotent character \( \chi \) of degree \( (p^{bm} + 1)(p^{b(m-1)} + p^b)/(p^{2b} - 1) \), where \( \chi(1) < p^{2b(m-1)} \leq p^{2an} \). As above, we deduce that \( b = a, 2a, \) or \( 3a \).

First, if \( b = 3a \) then the inequality \( b(m-1) \leq a(2m+1) \) implies that \( m = 4 \) and hence \( n = 9 \). Now one can check that \( |L| = |P\Omega_8^\pm(p^{3a})| \) does not divide \( |\text{SU}_9(p^a)| \), a contradiction. Second, if \( b = 2a \) then (4.3) implies that \( 4m(m-1) = n(n-1) \).
Also,
\[ 2(m - 1) \leq n \leq 2m + 1. \]
Now one see that two above equations violate each other. Finally we assume that \( b = a \). Then it follows by (1.3) that
\[ 2m(m - 1) = n(n - 1), \]
which again implies that \((n, m) = (4, 3)\) or \( n \geq 10 \). Arguing similarly as in the previous case, one gets to a contradiction.

\[ L = \text{PSU}_m(p^b) \text{ with } m \geq 3. \text{ Again we have } bm(m - 1) = an(n - 1) \text{ and moreover } b(m - 1) \leq 2an \text{ since } |L| \mid |\text{SU}_n(p^b)|. \text{ We deduce that } 2m \geq n - 1 \text{ and hence } b(m - 1) \leq 2an \leq 2a(2m + 1) \leq 7a(m - 1), \text{ whence } b \leq 7a. \]

Recall that \( \text{PSU}_m(p^b) \) has a unipotent character \( \chi \) of degree \((p^{bm} + (-1)^m)p^b)/(p^b + 1)\). Now one just argues as in the previous case to conclude that \( b = a \), which also implies \( m = n \). That means \( L = \text{PSU}_n(p^a) \), as we wanted to prove. \( \square \)

5. **Symplectic groups**

**Table 3.** Possible nontrivial degrees up to \( q^{2n} \) of \( \text{Sp}_{2n}(q) \), \( q \) odd (cf. [Lub, Ng]).

| Value of \( n \) | Degrees |
|------------------|---------|
| \( n = 2 \)      | \( \frac{q^2+1}{2}, \frac{q(q+1)}{2}, \frac{q(q+1)^2}{2}, (q \pm 1)(q^2+1), \frac{q(q+1)}{2}, \frac{q(q^2+1)}{2}, (q \pm 1)(q^2+1), \frac{q(q+1)}{2}, \frac{q(q^2+1)}{2}, \) |
| \( n = 3 \)      | \( q(q-1)(q^2+1), (q^2-1)^2q^4-1 \frac{q^2+1}{2}, q^3+1, q(q+1)(q^2+1), \frac{q^2+1}{2}, \frac{q^4+1}{2}, \frac{q^4+2}{2}, (q^2+1)(q^3+1) \) |
| \( n \geq 4 \)   | \( q^n+1, q^n(q^n+q), q^n+1, q^n(q^n+q), q^n+1, q^n(q^n+q), q^n+1, q^n(q^n+q), q^n+1 \) |

**Table 4.** Possible nontrivial degrees up to \( q^{2n} \) of \( \text{Sp}_{2n}(q) \), \( q \) even (cf. [GuT, Lub]).

| Value of \( n \) | Degrees |
|------------------|---------|
| \( n = 2 \)      | \( \frac{q(q+1)}{2}, (q \pm 1)(q^2+1), (q-1)(q^2+1), (q^2-1)^2q^4-1 \) |
| \( n = 3, q \neq 2 \) | \( \frac{(q^3+1)(q^3+q)}{2(q+1)}, \frac{q^6-1}{q+1}, (q-1)(q^2+1)(q^3-1) \) |
| \( n \geq 4, (n, q) \neq (4, 2) \) | \( \frac{q^n+1}{2(q+1)}, \frac{q^n(q^n+q)}{2(q+1)}, \frac{q^n+1}{q+1}, \frac{q^n(q^n+q)}{q+1} \) |

**Lemma 5.1.** Proposition [1.2] is true when \( S \) is a simple symplectic group.
Proof. Assume that $S = \text{PSp}_{2n}(q) = \text{PSp}_{2n}(p^a)$ where $n \geq 2$. As the case $\text{Sp}_{6}(2)$ can be checked directly by [Atl], we assume that $(n, q) \neq (3, 2)$ and hence $\text{Schur}(S) = \text{Sp}_{2n}(p^a)$ and $\text{St}_S(1) = p^{an^2}$ is the only $p$-power degree of $\text{Sp}_{2n}(q)$ by [MaZ] Theorem 1.1. By Proposition 2.5, $L := G/M$ is a simple group of Lie type in characteristic $p$ and we aim to show that $L \cong S$.

(i) The simple Lie-type groups of exceptional type can be eliminated as follows. Assume so and suppose that $|L|_p = p^{bm}$ and the largest factor of the form $p^x - 1$ in $|L|$ is $p^{bm'} - 1$. Then, as $\text{cd}(L) \subseteq \text{cd}(\text{Sp}_{2n}(q))$, we get $p^{bm} = p^{an^2}$ and therefore

\begin{equation}
(5.1) \quad bm = an^2 \text{ and } bm' = an^2 \cdot \frac{m'}{m}.
\end{equation}

As

\begin{equation}
(p^{bm'} - 1) \mid |L| \mid |\text{Sp}_{2n}(p^a)| = p^{an^2} \prod_{i=1}^{n}(p^{2ia} - 1),
\end{equation}

using Zsigmondy’s theorem, one gets

\[ an^2 \cdot \frac{m'}{m} \leq 2an \text{ or } an^2 \cdot \frac{m'}{m} = 6 > 2an. \]

Since the latter case can not happen, we must have $an^2m'/m \leq 2an$ and therefore

\[ n \leq 2m/m'. \]

Again, as $m/m'$ is known, one obtains an upper bound for $n$. For small values of $n$, we use Equation (5.1) to show that either $|L| \nmid |\text{SL}_n(q)|$ or $\text{cd}(L) \nsubseteq \text{cd}(\text{SL}_n(q))$ with the help of [Lub] and Zsigmondy’s theorem. Let us present the arguments for the case $L(p^b) = E_7(p^b)$ as an example.

Assume by contrary that $L(p^b) = E_7(p^b)$. Then $m = 63$ and $m' = 18$. Therefore $n \leq 7$. For every $n \leq 7$, we obtain a contradiction by showing that $|E_7(p^b)| \nmid |\text{Sp}_{2n}(p^a)|$. For instance, if $n = 7$ then $63b = 49a$ by (5.1). Therefore $9b = 7a$ and hence $a = 9c$ and $b = 7c$ for some positive integer $c$. Now the Zsigmondy prime $\ell(p, 1, 98c)$ is a divisor of $|E_7(p^{7c})|$ but does not divide $|\text{Sp}_{14}(p^{9c})|$.

(ii) We next eliminate the remaining simple classical groups in characteristic $p$ except $\text{PSp}_{2n}(p^a)$.

- $L = P\Omega_{2m}^\pm(p^b)$ with $m \geq 4$. Then we have $p^{bm(m-1)} = p^{an^2}$ and hence

\begin{equation}
(5.2) \quad bm(m - 1) = an^2.
\end{equation}

Since $(p^{2b(m-1)} - 1) \mid |\text{Sp}_{2n}(p^a)|$, it follows by Zsigmondy’s theorem that $b(m-1) \leq an$, whence $m \geq n$ by (5.2). This in particular implies that $b < 2a$. Moreover, as $m(m-1)$ can not be equal to $n^2$, we obtain that $b \neq a$.

As before, $L$ has a unipotent character $\chi$ of degree $(p^{bm \mp 1})(p^{b(m-1)} \pm p^b)/(p^{2b} - 1)$ and

\[ \chi(1) < p^{2b(m-1)} \leq p^{2an}. \]
First we consider the case when \( q \) is odd. Using the classification of low-dimensional irreducible characters of \( \text{Sp}_{2n}(q) \) of degree up to \( q^{2n} \) in [Ng2020, Corollary 4.2] for \( n \geq 6 \) and [Lub2019] for \( n \leq 5 \), we see that all degrees less than \( p^{2am} \) of \( \text{Sp}_{2n}(p^a) \) have \( p \)-parts \( 1, p^2, \) or \( p^{2a} \). Therefore, these degrees can not be \( \chi(1) \) since \( |\chi(1)|_p = p^b \) where \( b \neq a, 2a \). We have shown that \( \chi(1) \notin \text{cd}(\text{Sp}_{2n}(p^a)) \), a contradiction. The case when \( q \) even is handled exactly in the same way by using a result on low-dimensional characters of symplectic groups in even characteristic of Guralnick and Tiep [GuT2018, Theorem 6.1].

- \( L = \text{PSL}_m(p^b) \) or \( \text{PSU}_m(p^b) \) with \( m \geq 2 \). Then we have \( p^{bm(m-1)/2} = p^{an^2} \) and hence \( bm(m-1) = 2an^2 \). Since \( (p^{bm}-1) \mid |\text{Sp}_{2n}(p^a)| \), it follows by Zsigmondy’s theorem that \( bm \leq 2an \) since the case \( 6 = bm > 2an \) can not happen. It follows that \( m-1 \geq n \) and hence \( 2an \geq b(n+1) \). In particular, \( b < 2a \).

- \( L = \text{PSp}_{2n}(p^b) \) or \( \Omega_{2m+1}(p^b) \) with \( m \geq 2 \). Then we have \( p^{bm^2} = p^{an^2} \) and hence \( bm^2 = an^2 \) and in particular we have \( b \neq 2a \). Since \( (p^{bm^2}-1) \mid |\text{Sp}_{2n}(p^a)| \), it follows by Zsigmondy’s theorem that \( bm \leq 2an \) since the case \( 6 = bm > 2an \) can not happen.

Recall that \( L \) has a unipotent character \( \chi \) of degree \( (p^{bm^2} - 1)/2(p^b + 1) \), which is smaller than \( p^{2bm} \leq p^{2an} \). Arguing as in \( L = P\Omega_{2m}^{\pm}(p^b) \) case and recalling that \( b \neq 2a \), we obtain that \( b = a \) and therefore \( m = n \). We now just need to eliminate the possibility \( L = \Omega_{2n+1}(p^a) \) with \( n \geq 3 \) and \( p \geq 3 \) (note that \( \Omega_5(q) \simeq \text{PSp}_4(q) \) and \( \Omega_{2n+1}(2^a) \simeq \text{PSp}_{2n}(2^a) \)). As mentioned in [TZa2002, §6], \( \Omega_{2n+1}(p^a) \) with \( n \geq 3 \) and \( p \) an odd prime has an irreducible character of degree \( (p^{2an} - 1)/(p^{2a} - 1) \), which is also smaller than \( p^{2an} \). Again, the classification of irreducible characters of \( \text{Sp}_{2n}(p^a) \) of degrees smaller than \( p^{2an} \) shows that \( \text{Sp}_{2n}(p^a) \) has no such degree, as desired. \( \square \)

### 6. Orthogonal Groups

| \( n \) | Degrees |
|---|---|
| \( n = 3 \) | \( \frac{q^2-1}{2}, \frac{q(q^2+1)(q^3+1)}{2}, \frac{q^2-1}{q-1}, \frac{q(q+1)(q^2+1)}{q+1}, \frac{q^2-1}{q+1}, \frac{q(q+1)(q^2+1)}{q+1}, \frac{q^2-1}{q+1}, \frac{q(q+1)(q^2+1)}{q+1} \) |
| \( n = 4 \) | \( \frac{q^2-1}{2}, \frac{q(q^2+1)(q^3+1)}{2}, \frac{q^2-1}{q-1}, \frac{q(q+1)(q^2+1)}{q+1}, \frac{q^2-1}{q+1}, \frac{q(q+1)(q^2+1)}{q+1} \) |
| \( n \geq 5 \) | \( \frac{q^2-1}{2}, \frac{q(q^2+1)(q^3+1)}{2}, \frac{q^2-1}{q+1}, \frac{q(q+1)(q^2+1)}{q+1} \) |

Table 5. Possible nontrivial degrees up to \( q^{2n} \) of \( \operatorname{Spin}_{2n+1}(q) \), \( n \geq 3 \), \( q \) odd (cf. [Lub2019, Ng2020]).
Lemma 6.1. Proposition 1.2 is true when $S$ is a simple orthogonal group in odd dimension.

Proof. Assume that $S = \Omega_{2n+1}(q) = \Omega_{2n+1}(p^a)$ where $n \geq 3$ and $p$ an odd prime. As the case $\Omega_7(3)$ can be checked directly by [Atl], we assume that $(n, q) \neq (3, 3)$ and hence Schur($S$) = Spin$_{2n+1}(p^a)$ and $\text{St}_S(1) = p^{an^2}$ is the only $p$-power degree of Spin$_{2n+1}(q)$ by [MaZ, Theorem 1.1]. In view of Proposition 2.5, we have known that $L := G/M$ is a simple group of Lie type in characteristic $p$ and we want to show that $L = S$. The simple groups of exceptional Lie type is eliminated as in Lemma 5.1. For classical groups, we also follow the proof of Lemma 5.1 and make use of the classification of irreducible characters of low degrees of Spin$_{2n+1}(q)$ by Nguyen in [Ng, Theorem 1.2].

Table 6. Possible nontrivial degrees up to $q^{2n-2}$ of Spin$_{2n}^\pm(q)$, $n \geq 4$

| Value of $n, q$ | Degrees |
|-----------------|---------|
| $n \geq 4$, $q$ odd | $\frac{(q^a+1)(q^{a-1}+q)}{q^2-1}$, $\frac{(q^a+1)(q^{a-1}+1)}{2(q+1)}$, $\frac{(q^a+1)(q^{a-1}+1)}{q+1}$ |
| $n \geq 4$, $q$ even, $(n, q) \neq (4, 2), (5, 2)$ | $\frac{(q^a+1)(q^{a-1}+q)}{q^2-1}$, $\frac{(q^a+1)(q^{a-1}+1)}{q+1}$ |

Lemma 6.2. Proposition 1.2 is true when $S$ is a simple orthogonal group in even dimension.

Proof. We only present here the proof of the case $S = PO_{2n}^\pm(p^a)$ with $n \geq 4$. The minus type orthogonal groups are dealt similarly.

We assume that $S \neq PO_8^+(2)$ as this special case can be eliminated easily by using [Atl]. Then Schur($S$) = Spin$_{2n}^\pm(p^a)$ and $\text{St}_S(1) = p^{an(n-1)}$ is the only $p$-power degree of Spin$_{2n}(p^a)$ by [MaZ, Theorem 1.1]. We have already known that $L := G/M$ is a simple group of Lie type in characteristic $p$ and we aim to show that $L = S$.

(i) The simple Lie-type groups of exceptional type can be eliminated as follows. Assume that $L$ is of exceptional type. Suppose that $|L|^p = p^{bm}$ and the largest divisor of $|L|$ of the form $p^x - 1$ is $p^{bm'} - 1$. Then, as $\text{cd}(L) \subseteq \text{cd}(\text{Spin}_{2n}^+ (p^a))$, we get $p^{bm} = p^{an(n-1)}$ and therefore

\begin{equation}
(6.1) \quad bm = an(n-1) \text{ and } bm' = an(n-1) \cdot \frac{m'}{m}.
\end{equation}

As

$$p^{bm'} - 1 \mid |\text{Spin}_{2n}^+ (p^a)| = p^{an(n-1)}(p^{an} - 1) \prod_{i=1}^{n-1}(p^{2ia} - 1),$$
using Zsigmondy’s theorem and note that \( n \geq 4 \), one gets \( an(n-1) : \frac{m'}{m} \leq 2a(n-1) \), and therefore
\[
n \leq 2m/m'.
\]
Now one can argue as in Lemma 5.1 to get a contradiction.

(ii) Next, we eliminate the remaining simple classical groups in characteristic \( p \) except \( P\Omega_{2n}^+(p^a) \).
- \( L = \text{PSp}_{2m}(p^b) \) or \( \Omega_{2m+1}(p^b) \) with \( m \geq 2 \). Then we have \( p^{bn^2} = p^{an(n-1)} \) and hence \( bm^2 = an(n-1) \). Since \( (p^{bn^2} - 1) | \text{Spin}^+_2(p^a) \), it follows by Zsigmondy’s theorem that \( bm \leq a(n-1) \) as \( n \geq 4 \). We then obtain that \( m \geq n \), whence \( a(n-1) \geq bn \). In particular, \( b < a \).

Recall that \( L \) has a unipotent character \( \chi \) of degree \( (p^{bn} - 1)(p^{bn} - p^b)/2(p^b + 1) \), which is smaller than \( p^{2bn} \leq p^{2a(n-1)} \). Using the classification of irreducible characters of \( \text{Spin}^+_2(p^a) \) of degree smaller than \( q^{2(n-1)} \) in [Ng Theorems 1.3,1.4] for \( n \geq 5 \) and the list of degrees of \( \text{Spin}^+_2(p^a) \) for \( n \leq 4 \) in [Lub], we observe that the \( p \)-part of such a degree is either nothing or \( p^a \) (cf. Table 4), and hence any of them can not be equal to \( \chi(1) \) as \( |\chi(1)| = p^b \) and \( a > b \).

- \( L = \text{PSL}_m(p^b) \) with \( m \geq 2 \). Then we have \( p^{bm(m-1)/2} = p^{an(n-1)} \) and hence \( bm(m-1) = 2an(n-1) \). Since \( (p^{bm} - 1) | \text{Spin}^+_2(p^a) \), we deduce by Zsigmondy’s theorem that \( bm \leq 2a(n-1) \) and hence \( m-1 \geq n \). Thus \( 2a(n-1) \geq bm \geq b(n+1) \), which in turn implies that \( b < 2a \).

We know that \( L \) has a unipotent character \( \chi \) of degree \( (p^{bn} - p^b)/(p^b - 1) \), which is smaller than \( p^{bn} \leq p^{2a(n-1)} \). Now using the results in [Ng Theorems 1.3,1.4] and [Lub] again, we see that every degree of \( \text{Spin}^+_2(p^a) \) smaller than \( p^{2a(n-1)} \) has \( p \)-part 1, \( p^a \) or \( p^{2a} \). As, \( |\chi(1)| = p^b \) and \( b < 2a \), it follows that \( b = a \) and therefore we have
\[
m(m-1) = 2n(n-1).
\]
In particular, we obtain \( n \geq 6 \). Table 4 now says that \( \text{Spin}^+_2(p^a) \) has the only two following degrees that are smaller than \( p^2a(n-2) \) and divisible by \( p \):
\[
\frac{(p^{an} \pm 1)(p^{an(n-1)} \mp p^a)}{p^2a - 1}.
\]
However, one can check that
\[
\chi(1) = \frac{p^{bn} - p^b}{p^b - 1} = \frac{p^{an} - p^a}{p^a - 1}
\]
is not equal to neither of them as \( m(m-1) = 2n(n-1) \).
- \( L = P\Omega_{2m}^\pm(p^b) \) with \( m \geq 4 \). Then we have \( p^{bm(m-1)} = p^{an(n-1)} \) and hence
\[
(6.2) \quad bm(m-1) = an(n-1).
\]
Since \((p^{2b(m-1)} - 1) | \text{Spin}_{2n}^+(p^a)\), it follows by Zsigmondy’s theorem that \(b(m-1) \leq a(n-1)\), whence \(m \geq n\) by (6.2). Therefore, \(a(n-1) \geq b(m-1) \geq b(n-1)\) and so \(b < 2a\).

As before, \(L\) has a unipotent character \(\chi\) of degree \((p^{bn} + 1)(p^{b(m-1)} + p^b)/(p^{2b} - 1)\) and

\[
\chi(1) < p^{2b(m-1)} \leq p^{2a(n-1)}.
\]

As in the previous case, by using the results in [Ng, Theorems 1.3, 1.4] and [Lub], we obtain that \(\chi(1) \in \text{cd}(\text{Spin}_{2n}^+(p^a))\) only if \(a = b\), which also means \(m = n\).

We now just need to eliminate \(L = P\Omega_{2n}^-(p^a)\). In fact, even in this case, one can check that the degree \((p^{an} + 1)(p^{a(n-1)} - p^a)/(p^{2a} - 1)\) of \(P\Omega_{2n}^-(p^a)\) is not equal to any degree of \(\text{Spin}_{2n}^+(p^a)\) of degree smaller than \(p^{2a(n-1)}\).

\[\square\]

7. Proof of the main result

Proof of Proposition 1.2. This is an immediate consequence of Lemmas 3.2, 4.1, 5.1, 6.1, and 6.2

We establish a couple of important lemmas leading to the proof of the main theorem at the end of the section. The next two lemmas are well known. The first one is due to Bianchi, Chillag, Lewis, and Pacifici and the second one is due to Moretó.

Lemma 7.1 ([BCLP], Lemma 5). Let \(N = T \times \cdots \times T\), a direct product of \(k\) copies of a nonabelian simple group \(T\), be a minimal normal subgroup of \(G\). If \(\chi \in \text{Irr}(T)\) extends to \(\text{Aut}(T)\), then \(\chi^k\) extends to a character of \(G\).

Lemma 7.2 ([Mor], Lemma 4.2). Let \(T\) be a nonabelian simple group. Then there exists a non-principal irreducible character of \(T\) that extends to \(\text{Aut}(T)\).

The following lemma basically says that every nonabelian simple group different from \(\text{PSL}_3(4)\) and \(\text{PSU}_4(3)\) can not be embedded in to an automorphism group of an abelian group of order not bigger than than of \(\text{Mult}(S)\).

Lemma 7.3. Let \(S\) be a nonabelian simple group different from \(\text{PSL}_3(4)\) and \(\text{PSU}_4(3)\). Let \(A\) be an abelian group of order less than or equal to \(|\text{Mult}(S)|\). Then \(|S| > |\text{Aut}(A)|\).

Proof. If \(|\text{Mult}(S)| \leq 4\) then the lemma is obvious. Therefore, one only needs to consider nonabelian simple groups with Schur multiplier of order at least 5.

First we consider \(S = \text{PSL}_2(9)\) or \(S = \Omega_7(3)\). Then \(|\text{Mult}(S)| = 6\). If \(|A| \leq 5\) then \(|\text{Aut}(A)| \leq 5! = 120\) and we are done. On the other hand, if \(|A| = 6\) then \(A \cong \mathbb{Z}_6\) as \(A\) is abelian and we are done also. Next we consider \(S = \text{PSU}_6(2)\) or \(2E_6(2)\). Then \(|\text{Mult}(S)| = 12\) and therefore \(|S| > 12! > |\text{Aut}(A)|\), as desired.

Now we consider the linear groups \(S = \text{PSL}_n(q)\). Excluding all the exceptional cases already considered, we have \(|\text{Mult}(S)| = (n, q - 1) \leq n\) and so \(|\text{Aut}(A)| \leq n!\). On the other hand, it is easy to check that \(|S| > n!\). For the simple unitary groups
Recall that, for each nonnegative integer $i$, $M^{(i)}$ denotes the $i$th derived subgroup of $M$.

**Lemma 7.4.** Let $S$ be a nonabelian simple group different from $PSL_3(4)$ and $PSU_4(3)$. Let $G$ be a perfect group and $M \lhd G$ such that $G/M \cong S$ and $|M| \leq |\text{Mult}(S)|$. Then, for every nonnegative integer $i$, $G/M^{(i)}$ is isomorphic to a quotient of $\text{Schur}(S)$.

**Proof.** We prove by induction that $G/M^{(i)}$ is isomorphic to a quotient of $\text{Schur}(S)$ for every $i$. The induction base $i = 0$ is exactly the hypothesis. Now assuming that $G/M^{(i)} \cong \text{Schur}(S)/Z_i$ for some normal subgroup $Z_i$ of $\text{Schur}(S)$, we have to show $G/M^{(i+1)}$ is also a quotient of $\text{Schur}(S)$.

As $M^{(i)}/M^{(i+1)}$ is abelian and normal in $G/M^{(i+1)}$, we have

$$\frac{M^{(i)}}{M^{(i+1)}} \leq C_{G/M^{(i+1)}} \left( \frac{M^{(i)}}{M^{(i+1)}} \right) \leq \frac{G}{M^{(i+1)}}.$$ 

We first consider the case $C_{G/M^{(i+1)}} \left( \frac{M^{(i)}}{M^{(i+1)}} \right) = G/M^{(i+1)}$. Then $M^{(i)}/M^{(i+1)}$ is central in $G/M^{(i+1)}$. As $G$ is perfect, $G/M^{(i+1)}$ is a stem extension of $G/M^{(i)} \cong \text{Schur}(S)/Z_i$. As $\text{Schur}(S)/Z_i$ is a quasisimple group whose quotient by the center is $S$, we deduce from the main result of [Har] that $G/M^{(i+1)}$ is a quotient of the Schur cover of $\text{Schur}(S)/Z_i$. Therefore, $G/M^{(i+1)}$ is a quotient of $\text{Schur}(S)$, as wanted.

The lemma is completely proved if we can show that $C_{G/M^{(i+1)}} \left( \frac{M^{(i)}}{M^{(i+1)}} \right)$ can not be a proper normal subgroup of $G/M^{(i+1)}$. Assume so, then it follows by the induction hypothesis that

$$\frac{C_{G/M^{(i+1)}} \left( \frac{M^{(i)}}{M^{(i+1)}} \right)}{M^{(i)}/M^{(i+1)}} \leq \frac{G/M^{(i+1)}}{M^{(i)}/M^{(i+1)}} \cong \frac{G}{M^{(i)}} = \frac{\text{Schur}(S)}{Z_i}.$$ 

Therefore,

$$\left| \frac{C_{G/M^{(i+1)}} \left( \frac{M^{(i)}}{M^{(i+1)}} \right)}{M^{(i)}/M^{(i+1)}} \right| \leq \left| \frac{\text{Mult}(S)}{Z_i} \right| = \left| \frac{M}{M^{(i)}} \right|$$

and hence

$$\left| C_{G/M^{(i+1)}} \left( \frac{M^{(i)}}{M^{(i+1)}} \right) \right| \leq \left| M/M^{(i+1)} \right|.$$ 

Thus

$$\frac{G/M^{(i+1)}}{C_{G/M^{(i+1)}} \left( \frac{M^{(i)}}{M^{(i+1)}} \right)} \geq |G/M| = |S|.$$ 

Since the quotient group on the left side can be embedded in $\text{Aut}(M^{(i)}/M^{(i+1)})$ and $M^{(i)}/M^{(i+1)}$ is abelian of order less than or equal to $|M|$, this last inequality leads to a contradiction by Lemma 7.3. □
Lemma 7.5. Let $S$ be a simple group of Lie type different from $\text{PSL}_3(4)$ and $\text{PSU}_4(3)$. Let $G$ be a perfect group and $M < G$ such that $G/M \cong S$, $|M| \leq |\text{Mult}(S)|$, and $\text{cd}(G) \subseteq \text{cd}(\text{Schur}(S))$. Then $G$ is isomorphic to a quotient of $\text{Schur}(S)$.

**Proof.** By Lemma 7.2, we are done if $M$ is solvable. So it remains to consider the case when $M$ is nonsolvable. If $M$ is nonsolvable, there is an integer $i$ such that

$$M^{(i)} = M^{(i+1)} > 1.$$ 

Let $N \leq M^{(i)}$ be a normal subgroup of $G$ so that $M^{(i)}/N \cong T^k$ for some non-abelian simple group $T$. By Lemma 7.2, $T$ has a non-principal irreducible character $\varphi$ that extends to $\text{Aut}(T)$. Lemma 7.1 then implies that $\varphi^k$ extends to $G/N$. Therefore, by Gallagher’s lemma, $\varphi^k\chi \in \text{Irr}(G/N)$ for every $\chi \in \text{Irr}(G/M^{(i)})$. In particular,

$$\varphi(1)^k\chi(1) \in \text{cd}(G/N) \subseteq \text{cd}(G) \subseteq \text{cd}(\text{Schur}(S)).$$

Taking $\chi$ to be the Steinberg character of $S$. By Lemma 7.4, $S$ is a quotient of $G/M^{(i)}$ and hence $\chi$ can be considered as a character of of $G/M^{(i)}$. We now get a contradiction since $\varphi(1)^k\chi(1)$, which is larger than $\chi(1) = \text{St}_S(1)$, can not be degree of $\text{Schur}(S)$ by Lemma 2.4. \qed

**Lemma 7.6.** Assume the hypothesis of Lemma 7.5. Then $G$ is uniquely determined (up to isomorphism) by $S$ and the order of $G$.

**Remark 7.7.** In the case $S = \text{PSU}_4(3)$, this lemma is true if Lemma 7.4 is true. However, this lemma is wrong in the case $S = \text{PSL}_3(4)$ because of the following. Note that $\text{Mult}(S) = \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_3$. Let $Z_1$ and $Z_2$ be subgroups of $\text{Mult}(S)$ isomorphic respectively to $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$. The non-isomorphic groups $\text{Schur}(S)/Z_1$ and $\text{Schur}(S)/Z_2$ (cf. [At] where these groups are denoted by 12, 12, 12, 12, 12, 12, 12, 12) both satisfies the hypothesis of the lemma.

**Proof.** First we consider the case $S = \text{PO}_4^+(2), \text{Sp}_6(8)$, or $\text{PO}_{2n}^+(q)$ with $n$ even and $q$ odd. Then $\text{Mult}(S) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. By Lemma 7.5, $S$ is isomorphic to a quotient of $\text{Schur}(S)$ so that we can assume $G \cong \text{Schur}(S)/Z$ with $Z \leq \text{Mult}(S)$ (note that $Z$ can not be $\text{Schur}(S)$). If $|M| = 1$ or $4$ then $Z = \text{Mult}(S) or 1$, respectively, and so we are done. Thus it remains to consider $|M| = 2$. We then have $|Z| = 2$ and hence $Z$ is generated by an involution of $\text{Mult}(S)$. However, as these three involutions are permuted by an outer automorphism of $S$ of degree 3, the quotient groups of the form $\text{Schur}(S)/\langle t \rangle$ for any involution $t \in \text{Mult}(S)$ are isomorphic and we are done again. Next, we assume that $S = \text{PSU}_6(2)$ or $2E_6(2)$. Though the Schur multipliers of these groups are more complicated, these cases in fact can be argued similarly as above. We leave details to the reader.

If $S$ is none of the groups already considered and also $S \neq \text{PSL}_3(4)$ and $\text{PSU}_4(3)$, then $\text{Mult}(S)$ indeed is cyclic. Again, $S$ is isomorphic to a quotient of $\text{Schur}(S)$ and we can assume

$$G \cong \text{Schur}(S)/Z,$$
where $Z \leq \text{Mult}(S)$. As $G/M \cong S$, we then deduce that $|Z| = |\text{Mult}(S)|/|M| = |\text{Schur}(S)|/|G|$. Since the cyclic group $\text{Mult}(S)$ has a unique subgroup of order $|\text{Schur}(S)|/|G|$, $Z$ is uniquely determined by $S$ and $|G|$ and therefore the lemma follows.

We are now ready to prove the main result.

**Proof of Theorem 1.1.** Recall the hypothesis that $\mathbb{C}G \cong \mathbb{C}H$ where $H$ is a quasisimple group such that $S := H/Z(H)$ is a simple classical group different from $\text{PSL}_3(4)$ and $\text{PSU}_4(3)$. As $H$ has only one linear character, we deduce that so does $G$ and hence $G$ is perfect. It is obvious that $G$ is nontrivial. Therefore, if $M$ is a maximal normal subgroup of $G$, then $G/M$ is isomorphic to a nonabelian simple group. Moreover, as $\mathbb{C}G \cong \mathbb{C}H$,

$$\text{cd}(G/M) \subseteq \text{cd}(G) = \text{cd}(H) \subseteq \text{cd}(\text{Schur}(S)).$$

Applying Proposition 1.2 we obtain that

$$G/M \cong S.$$ 

It follows that $|M| = |Z(H)| \leq |\text{Mult}(S)|$ since $|G| = |H|$. Now we see that the two groups of the same order $G$ and $H$ both satisfy the hypothesis of Lemma 7.6. It then follows by Lemma 7.6 that $G \cong H$, as desired.

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DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF AKRON, AKRON, OHIO 44325, UNITED STATES

*E-mail address: hungnguyen@uakron.edu*