Collision Thermalization of Nucleons in Relativistic Heavy-Ion Collisions

D. Anchishkin\textsuperscript{a}, A. Muskeyev\textsuperscript{b}, S. Yezhov\textsuperscript{b}

\textsuperscript{a} Bogolyubov Institute for Theoretical Physics, 03680 Kiev, Ukraine
\textsuperscript{b} Taras Shevchenko Kiev National University, 03022 Kiev, Ukraine

Abstract

We consider a possible mechanism of thermalization of nucleons in relativistic heavy-ion collisions. Our model belongs, to a certain degree, to the transport ones; we investigate the evolution of the system created in nucleus-nucleus collision, but we parametrize this development by the number of collisions of every particle during evolution rather than by the time variable. We based on the assumption that the nucleon momentum transfer after several nucleon-nucleon (-hadron) elastic and inelastic collisions becomes a random quantity driven by a proper distribution. This randomization results in a smearing of the nucleon momenta about their initial values and, as a consequence, in their partial isotropization and thermalization. The trial evaluation is made in the framework of a toy model. We show that the proposed scheme can be used for extraction of the physical information from experimental data on nucleon rapidity distribution.

1 Introduction

The problem of isotropization and thermalization in the course of collisions between heavy relativistic ions attracts much attention, because the application of thermodynamic models is one of the basic phenomenological approaches to the description of experimental data. Moreover, the assumption about a local thermodynamic equilibrium, along with other factors, is successfully used in various domains of high-energy physics. Meanwhile, many of the questions concerning this issue remain open for discussion \cite{1-8}.

The main goal of the investigations of the collisions of relativistic nuclei is the extraction of a physical information about nuclear matter and its constituents. It is a matter of fact that we can get know more about quarks and gluons (constituents) just under extreme conditions, i.e. at high densities and temperatures. During last two decades, one of the celebrated tools on the way of the theoretical investigations of this subject was relativistic hydrodynamics which started to be applied to elementary particle physics by the famous Landau’s paper \cite{9}.

Applying relativistic hydrodynamics, one can partially describe experimental data and get know that the matter created in relativistic nucleus-nucleus collisions can be regarded on some stage of evolution as a continuous one, i.e. as a liquid. Moreover, as was discovered in Brookheaven National Laboratory, it can be regarded even as a perfect fluid \cite{10, 11} which consistent with a description of the created quark-gluon plasma (QGP). Unfortunately, the important physical information is hidden in sophisticated numerical codes which solve the Euler hydrodynamic equations of motion.

In the present paper, we propose an approach to description of relativistic heavy-ion collisions which allows one to extract the physical information from experimental data on the basis of a transparent analytical model.

2 The Model

Our model is aimed at the extraction of the physical information from the nucleon spectra for such collision energies when the number of created nucleon-antinucleon pairs is much less than the number of net nucleons. This means that the model can be applied at AGS and low SPS energies.

1. We separate all nucleons in the final state, i.e. after freeze-out, into two groups in accordance with their origination: a) the first group consists of net nucleons that went through just hadron reactions; b) the second group includes nucleons which were created in the collective processes, for instance, during hadronization of the
QGP. In accordance with this, the nucleon momentum spectrum can be represented as a sum of two different contributions:

\[
\frac{dN}{d^3p} = \left(\frac{dN}{d^3p}\right)_{\text{hadron}} + \left(\frac{dN}{d^3p}\right)_{\text{QGP}}.
\]

In turn, the total number of registered nucleons equals \(N_{\text{total}} = N_{\text{hadron}} + N_{\text{QGP}}\). If this separation can be done, then we can define the “nucleon power” of the created QGP as \(P_{\text{QGP}} = N_{\text{QGP}}/N_{\text{total}}\). In our further investigation, we will deal mainly with the nucleons from the first group which come to the final state after a chain of sequential nucleon-nucleon (-hadron) elastic and inelastic collisions.

2. The collision number for every nucleon (hadron) is finite because the lifetime of the fireball is limited. To determine the maximal number of collisions, \(M_{\text{max}}\), in a particular experiment we use the results of UrQMD simulations \[12, 13\].

3. Because the colliding nuclei are the spatially restricted many-nucleon systems, the different nucleons experience different collision numbers: it is intuitively clear that the collision histories of the inner and surface nucleons will be different. That is why, we partition all amount of nucleons of the first group into different ensembles in accordance with a number of collisions before freeze-out. Then the nucleons from every ensemble give their own contribution to the total nucleon spectrum. If we denote the number of particles in a particular ensemble where the particles experienced \(M\) collisions by \(C(M)\), then, in correspondence to Eq. (1), we can write the total nucleon spectrum as

\[
\frac{dN}{d^3p} = \sum_{M=1}^{M_{\text{max}}} C(M) D_M(p) + C_{\text{therm}} D_{\text{therm}}(p),
\]

where \(D_M(p)\) is the spectrum (normalized to unity) of the particles in the \(M\)-th ensemble. The last term on the r.h.s. of (2) corresponds to the possible contribution from the totally thermalized source which we associate with QGP. Here, \(C_{\text{therm}}\) is the number of nucleons from the second group which are created during the hadronization, and \(D_{\text{therm}}(p)\) is the thermal distribution normalized to unity.

Consider successive variations of the momentum of the \(n\)-th nucleon from nucleus \(A\) which moves along the collision axis from left to right toward nucleus \(B\). Every \(m\)-th collision induces the momentum transfer, \(q_n^{(m)}\), for the \(n\)-th nucleon. So that, after \(M\) collisions, the nucleon acquires the momentum \(k_n\):

\[
k_0 \rightarrow k_0 + q_0^{(1)} \rightarrow k_0 + q_0^{(1)} + q_0^{(2)} \rightarrow \cdots \rightarrow k_0 + Q_n \equiv k_n,
\]

where \(Q_n = \sum_{m=1}^{M} q_n^{(m)}\) is the total momentum transfer finally obtained by the \(n\)-th nucleon (see Fig. 1).

Let us assume for the moment that the elastic scattering gives the main contribution to the two-nucleon collision amplitude. The initial momentum of every nucleon in nucleus \(A\) is \(k_a = k_0 = (0, 0, k_{0z})\), while the initial momentum of every nucleon in nucleus \(B\) is \(k_b = -k_0 = (0, 0, k_{0z})\). The energy and momentum are conserved in every separate collision of two particles, \(\omega(k_a) + \omega(k_b) = \omega(p_a) + \omega(p_b)\), \(k_a + k_b = p_a + p_b\), where

\[\begin{align*}
\omega(k) &= \sqrt{k^2 + m^2}, \\
\omega(p) &= \sqrt{p^2 + m^2}.
\end{align*}\]
\( p_a \) and \( p_b \) are the momenta of the particles after the collision. We assume that the particles are on the mass shell, so that \( \omega(k) = \sqrt{m^2 + k^2} \) (the system of units \( \hbar = c = 1 \) is adopted). Determining the six unknown quantities, \( p_a \) and \( p_b \), from four equations is straightforward, but two quantities, e.g. \( (p_a)_x \) and \( (p_b)_x \), remain uncertain and can be considered as such which accept random values driven by the scattering probability. So, after the first collision, one component of the particle momentum becomes random. After the third collision of the particle, each component of the momentum becomes random, hence the particle momentum, for instance \( p_n \), becomes completely random. As a result, we can regard the momentum transfer, \( q = p_n - k_a \), as a random quantity as well. If we follow the nucleon sequential elastic scattering from the first collision to the last one, we would see the full randomization of the particle momentum after each three sequential acts of scattering. So, we can look at the process of randomization from the equivalent point of view: starting from the definite initial momentum \( k_0 \), the \( n \)-th particle gains completely random momentum transfer \( q_n^{(1)}, q_n^{(2)}, \ldots, q_n^{(M)} \) after each three sequential collisions in the chain of \( 3M \) collisions, see \((3)\).

At the same time, the nucleon momentum transfer undergoes the even faster randomization in the inelastic collisions. Indeed, let us consider, for example, the process of creation of \( d \) pions in the nucleon-nucleon reaction: \( N + N \rightarrow N + N + N_\pi \). The process leads to randomization of \([3(2 + N_\pi) - 4]\) degrees of freedom, or \( d_\pi \) degrees of freedom per particle becomes random, where \( d_\pi = 3 - 4/(2 + N_\pi) \) and we assume that all particles are on the mass shell. For instance, if \( N_\pi = 2 \) (see the physical diagram of the collision depicted in Fig. 2), then two components of the momentum of every particle after a reaction becomes random, i.e. \( d_\pi = 2 \) in this process. If \( N_\pi \gg 1 \), the number of the random degrees of freedom per particle achieves its maximal value \( d_\pi = 3 \). Thus, we see that, in the inelastic collisions, the randomization of the nucleon momentum transfer attributed to one physical diagram goes faster than that in the elastic scattering.

To estimate the effective number of collisions, \( M \), we analyze all nucleon collisions (physical diagrams), \( N_{\text{coll}} \), and obtain a total sum of the random degrees of freedom, \( d_{\text{tot}} \), gained by the particle during all reactions before freeze-out, i.e. we need to know \( d_{\text{tot}} = \sum_{i=1}^{N_{\text{coll}}} d_i^{(3)} \). Then, the effective number of collisions is determined as \( M = d_{\text{tot}}/3 \). So, we assume a randomization of the momentum transfer, \( q \), after every effective collision; the number of effective collisions, \( M \), is determined by the conditions of a particular experiment.

### 2.1 Many-particle distribution function for \( M \)-th ensemble

We would like to determine the density distribution function in the momentum space, \( f_{2N}(E_{\text{tot}}; k_1, \ldots, k_{2N}) \), which describes \( 2N \) nucleons in the final state, after \( M \) effective collisions which are experienced by every nucleon before freeze-out (we name this group of nucleons as the \( M \)-th ensemble). All consideration is carried out in the c.m.s. of two identical colliding nuclei, hence the initial total momentum of the system \( A + B \) is equal to zero. Let us write down the density distribution function in the form

\[
f_{2N}(E_{\text{tot}}; k_1, \ldots, k_{2N}) = \frac{1}{\Omega_{2N}(E_{\text{tot}})} \tilde{f}_{2N}(E_{\text{tot}}; k_1, \ldots, k_{2N}),
\]

(4)

where we normalize the density distribution function in such a way that simultaneously determines the density of states in the system:

\[
\Omega_{2N}(E_{\text{tot}}) = \int d\tilde{k}_1 \cdots d\tilde{k}_{2N} \tilde{f}_{2N}(E_{\text{tot}}; k_1, \ldots, k_{2N}).
\]

(5)

The measure of integration in the single-particle phase space looks like (in units of \( \hbar \)) \( d\tilde{k}_n = V \frac{d^3k_n}{(2\pi)^3} \).

The unnormalized distribution function \( \tilde{f}_{2N}(E_{\text{tot}}; k_1, \ldots, k_{2N}) \) is determined in a two-fold way: first, we follow all collisions of a particular nucleon by integration with respect to all nucleon random momentum transfer, and, second, we fix the total energy of the \( 2N \)-nucleon system after freeze-out in a microcanonical-like way: \( E_{\text{tot}} = \sum_{n=1}^{2N} \omega(k_n) \). Then, it reads

\[
\tilde{f}_{2N}(E_{\text{tot}}; k_1, \ldots, k_{2N}) = \delta \left( E_{\text{tot}} - \sum_{n=1}^{2N} \omega(k_n) \right) \times \prod_{n=1}^{N} \frac{dP_1}{V} \prod_{n=1}^{N} \left[ (2\pi)^3 \delta^3 \left( k_n - k_0 - \sum_{m=1}^{M} q_n^{(m)} \right) \right] \prod_{n'=N+1}^{2N} \left[ (2\pi)^3 \delta^3 \left( k_{n'} + k_0 - \sum_{m=1}^{M} q_n^{(m)} \right) \right],
\]

(6)
where $V$ is the volume of the system in the coordinate space. Here, we assume the independence of the sequential scatterings, which results in the independence of the particular momentum transfer (see for details [14]). Hence, the element of the probability to accept a particular chain of the momentum transfer by the $n$-th particle in the series of $M$ collisions reads

$$dP_n = \prod_{m=1}^{M} J_m \left( q_n^{(m)} \right) \frac{d^3 q_n^{(m)}}{(2\pi)^3},$$

(7)

where the distribution of the momentum transfer in the $m$-th collision is characterized by the presence of the form-factor $J_m(q)$ with

$$\int \frac{d^3 q}{(2\pi)^3} J_m(q) = 1.$$  

(8)

In what follows, for the sake of simplicity, we assume the independence of $J_m(q)$ on the collision number, i.e. $J_m(q) \rightarrow \langle J_m(q) \rangle_{\text{collisions}} = J(q)$. Hence, we adopt an approximation where just one form-factor $J(q)$ characterizes a distribution of the momentum-transfer in a series of collisions which are experienced by a nucleon during its traveling through the fireball.

With making use of this approximation, we make a step in the evaluation of the $2N$-distribution function [15]. If one represents $\delta$-functions in (9) in terms of the Fourier integrals with respect to the auxiliary variables $a_n$, $b_n$ and then with making allowance for definition of the integration measure [7], the unnormalized density distribution (9) can be written down in the form

$$\tilde{f}_{2N}(E_{\text{tot}}; k_1, \ldots, k_{2N}) = \delta \left( E_{\text{tot}} - \sum_{n=1}^{2N} \omega(k_n) \right) \prod_{n=1}^{N} \left[ \int \frac{d^3 a_n}{V} e^{-i a_n \cdot (k_n - k_0)} [J(a_n)]^M \right] \times \prod_{n=N+1}^{2N} \left[ \int \frac{d^3 b_n}{V} e^{-i b_n \cdot (k_n + k_0)} [J(b_n)]^M \right],$$

(9)

where $J(r) = \int \frac{d^3 q}{(2\pi)^3} J(q) e^{i q \cdot r}$ is the Fourier component of the momentum-transfer form-factor which corresponds to the single nucleon scattering.

We introduce the “multi-scattering form-factor” ($M$-scattering form-factor)

$$I_M(Q) = \frac{1}{V} \int d^3 r e^{-i Q \cdot r} [J(r)]^M.$$  

(10)

Then we can rewrite the unnormalized distribution function (9) as

$$\tilde{f}_{2N}(E_{\text{tot}}; k_1, \ldots, k_{2N}) = \delta \left( E_{\text{tot}} - \sum_{n=1}^{2N} \omega(k_n) \right) \prod_{n=1}^{N} I_M(k_n - k_0) \prod_{n'=N+1}^{2N} I_M(k_{n'} + k_0).$$

(11)

To obtain the partition function, we make the Laplace transformation of the density-of-states function (5) with respect to the variable $E_{\text{tot}}$ (the total energy of the nucleon system after freeze-out). As a result, we obtain the partition function of the canonical ensemble

$$Z_{2N}(\beta) = \int_{E_{\text{min}}}^{\infty} dE_{\text{tot}} e^{-\beta E_{\text{tot}}} \Omega_{2N}(E_{\text{tot}}).$$

(12)

If we define the unnormalized $2N$-particle distribution function of the canonical ensemble as

$$\tilde{\Omega}_{2N}(\beta; k_1, \ldots, k_{2N}) = \int_{E_{\text{min}}}^{\infty} dE_{\text{tot}} e^{-\beta E_{\text{tot}}} \tilde{f}_{2N}(E_{\text{tot}}; k_1, \ldots, k_{2N}),$$

(13)

we can write the partition function as the integral

$$Z_{2N}(\beta) = \int d\tilde{k}_1 \ldots d\tilde{k}_{2N} \tilde{\Omega}_{2N}(\beta; k_1, \ldots, k_{2N}).$$

(14)
It is obvious that the partition function $Z_{2N}(\beta)$ plays the role of the normalization constant with respect to the $2N$-particle distribution function $F_{2N}$ defined in (13). Next, we determine the $2N$-particle distribution function in the canonical ensemble as

$$F_{2N}(\beta; k_1, \ldots, k_{2N}) = \frac{1}{Z_{2N}(\beta)} \overline{F}_{2N}(\beta; k_1, \ldots, k_{2N}) .$$

(15)

Taking Eq. (13) into account, for the distribution functions $\overline{F}_{2N}(\beta; k_1, \ldots, k_{2N})$ from (13), we obtain

$$\overline{F}_{2N}(\beta; k_1, \ldots, k_{2N}) = \prod_{n=1}^{N} \left[ e^{-\beta \omega(k_n)} I_M(k_n - k_0) \right] \prod_{n'=N+1}^{2N} \left[ e^{-\beta \omega(k_{n'})} I_M(k_{n'} + k_0) \right] .$$

(16)

So, we can write the distribution function which characterizes the $2N$-nucleon system where each particle experiences $M$ effective collisions ($M$-th ensemble) in a factorized form

$$F_{2N}(\beta; k_1, \ldots, k_{2N}) = \prod_{n=1}^{N} f_a(k_n) \prod_{n=N+1}^{2N} f_b(k_n) ,$$

(17)

where

$$f_a(b)(k) = \frac{1}{z_{a(b)}(\beta)} e^{-\beta \omega(k)} I_M(k \mp k_0) \quad \text{with} \quad z_{a(b)}(\beta) = V \int \frac{d^3k}{(2\pi)^3} e^{-\beta \omega(k)} I_M(k \mp k_0) ,$$

(18)

are the single-particle distribution function and the single-particle partition function, respectively, attributed to nucleus “A” for subindex $a$ or to nucleus “B” for subindex $b$.

### 2.1.1 Saddle-point approximation

For large enough collision numbers $M$, we can calculate the integral in (10) within the saddle-point method. To do this, we first represent the integral in the form $I_M(Q) = \frac{1}{V} \int d^3r \exp\left[-iQ \cdot r + M \ln J(r)\right]$. Then we expand the exponent in $J(r)$ into a series at the point $r = 0$ up to the second order. We take into account that $J(r)|_{r=0} = 1$ (see Eq. (5)). If the form-factor depends on the modulus of the momentum transfer $J(|q|)$, the expansion of the logarithm in $J(r)$ looks like

$$\ln J(r) \approx -\frac{1}{6} \langle q^2 \rangle r^2 , \quad \text{where} \quad \langle q^2 \rangle = \int \frac{d^3q}{(2\pi)^3} q^2 J(|q|)$$

(19)

with $q = |q|$ and $r = |r|$. Here, we take into account that $\partial_r J(r)|_{r=0} = 0$, which is valid when $J(-q) = J(q)$. Finally, we obtain integral (10) in the approximate form

$$I_M(Q) \approx \frac{1}{V} \left( \frac{6\pi}{M\langle q^2 \rangle} \right)^{3/2} \exp\left( -\frac{3Q^2}{2M\langle q^2 \rangle} \right) ,$$

(20)

where $Q = |Q|$.

So, the $M$-scattering form-factor $I_M(Q)$ for a large number of the effective collisions $M$ can be approximately represented as the normal distribution $\propto \exp\left(-Q^2/2\sigma^2\right)$ with the variance $\sigma = \sqrt{M\langle q^2 \rangle}/3$, which is obviously a reflection of the central limit theorem.

Following approximation (20), the single-particle distribution functions (18) take the form

$$f_{a(b)}(k) = \frac{1}{z_{a(b)}(\beta)} e^{-\beta \omega(k) - \frac{3(k \mp k_0)^2}{2M\langle q^2 \rangle}} \quad \text{with} \quad z_{a(b)}(\beta) = \int \frac{d^3k}{(2\pi)^3} e^{-\beta \omega(k) - \frac{4(k \mp k_0)^2}{2M\langle q^2 \rangle}} ,$$

(21)

where we skip the common factors in $f_{a(b)}(k)$ and in $z_{M}(\beta)$, respectively. In the limit case $M \to \infty$, the dependence on the initial momentum $\pm k_0$ is washed out, and both single-particle distributions $f_a(k)$ and $f_b(k)$ take the same “thermal” limit:

$$f_{a(b)}(k) \to f_{\text{therm}}(k) = \frac{1}{z_{\text{therm}}(\beta)} e^{-\beta \omega(k)} , \quad z_{\text{therm}}(\beta) = \int \frac{d^3k}{(2\pi)^3} e^{-\beta \omega(k)} .$$

(22)
In the non-relativistic case, i.e. when \( \omega(k) = k^2/2m + m \), one can rewrite distributions (21) as the Jüttner ones [15] (see [16])

\[
f_{a(b)}(k) = \frac{1}{z_M(\beta)} \exp \left[ -\frac{(k \pm m v_{h})^2}{2mT_{\text{eff}}} \right] \quad \text{with} \quad z_M(\beta) = \int \frac{d^3k}{(2\pi)^3} \exp \left[ -\frac{(k - m v_{h})^2}{2mT_{\text{eff}}} \right]. \tag{23}
\]

Here, we define the collective (hydrodynamical) velocity \( v_{h} \) as

\[
v_{h} = \left( \frac{1}{1 + \frac{\beta}{\beta_{\text{coll}}(M)}} \right) \frac{k_0}{m} \quad \text{with} \quad \beta_{\text{coll}}(M) = \frac{3m}{Mq^2}, \tag{24}
\]

and the effective temperature as

\[
T_{\text{eff}} = \frac{1}{\beta + \frac{\beta_{\text{coll}}(M)}{\beta}}. \tag{25}
\]

The quantity \( \beta_{\text{coll}} \) can be put in correspondence to the “collision” temperature \( T_{\text{coll}} = 1/\beta_{\text{coll}} = \frac{2}{3}M\langle \omega(q) \rangle \), where \( \langle \omega(q) \rangle = \langle q^2 \rangle/2m \) is the mean energy transfer in one nucleon collision. We see that every collision ensemble has its own effective temperature and own collective velocity. Indeed, with increase in the number of collisions \( M \), the effective temperature increases and the collective velocity is going down. These quantities have the following limits:

\[
(M = 0) \quad \frac{k_0}{\omega(k_0)} \geq v_{h} \geq 0 \quad \quad 0 \leq T_{\text{eff}} \leq T \quad (M \to \infty) \tag{26}
\]

where \( \omega(k_0) = \sqrt{m^2 + k_0^2}; \quad T = 1/\beta \), and we marked conventionally the nucleons which do not take part in any collisions (spectators) by \( M = 0 \). Here, the left limit corresponds to the case \( M = 0 \), and the right limit corresponds to \( M \to \infty \). Hence, the every step during the increase in the number of collisions results in the redistribution of the energy accumulated initially in the longitudinal movement, i.e. \( v_{h} = k_0/\omega(k_0) \) and \( T_{\text{eff}} = 0 \) before first collision. The energy passes partially to the transverse degrees of freedom, which increases the isotropization and the temperature of the system and decreases, in turn, the collective velocity. When the number of collisions is large enough, i.e. when \( M \to \infty \), we come to the limits: \( v_{h} \to 0 \) and \( T_{\text{eff}} \to T \). These conclusions are valid, of course, for relativistic energies as well.

Meanwhile, when the number of collisions is finite we see that, just for the non-relativistic dispersion law, the obtained distribution (21) can be regarded as a locally thermal one (the Jüttner distribution function).

### 2.1.2 Single-particle spectrum for \( M \)-th ensemble

We can construct a “two-source” single-particle distribution function \( f(k_a, k_b) = f_a(k_a)f_b(k_b) \). Then the averaging of the two-source random quantity \( W(k_a, k_b) \) gives

\[
\langle W \rangle = \int \frac{d^3k_a}{(2\pi)^3} \frac{d^3k_b}{(2\pi)^3} W(k_a, k_b) f(k_a, k_b). \tag{27}
\]

If we take \( W(k_a, k_b) = \frac{1}{2} [\delta^3(p - k_a) + \delta^3(p - k_b)] \), we obtain, after the averaging, the single-particle spectrum which is attributed to the system where all particles experienced \( M \) collisions before freeze-out:

\[
D_M(p) = \left( \frac{1}{2N} \frac{d^3N}{d^3p} \right)_M = \frac{1}{2} e^{-\beta \omega(p)} \left[ \frac{I_M(p - k_0)}{z_a(\beta)} + \frac{I_M(p + k_0)}{z_b(\beta)} \right], \tag{28}
\]

where \( I_M(Q) \) is defined in [18]. Hence, with accounting for notations [18], the distribution functions \( D_M(p) \) can be represented also as

\[
D_M(p) = \frac{1}{2} \left[ f_a(p) + f_b(p) \right]. \tag{29}
\]

For large collision numbers \( M \), the distribution function \( D_M(p) \) reads

\[
D_M(p) \approx e^{-\beta \omega(p)} \left[ e^{\frac{\beta p \cdot k_0^2}{2M(\pi^2)}} + e^{\frac{\beta p \cdot k_0^2}{2M(\pi^2)}} \right] \quad \text{with} \quad z_M(\beta) = \int \frac{d^3k}{(2\pi)^3} e^{-\beta \omega(k)} \frac{\beta k \cdot k_0^2}{2M(\pi^2)}. \tag{30}
\]
It is evidently seen that the spectrum has two items which can be attributed to the first and to the second colliding nuclei, respectively. In accordance with this structure, it can be named a “two-source single-particle spectrum”. It is clear also that this picture is just the effective one: after a nucleon-nucleon collision, we cannot say anything about their origination from a particular nucleus. In what follows, we use the distributions $D_M(p)$ from [28] to describe a nucleon spectrum arising in relativistic nucleus-nucleus collisions.

### 2.2 Nucleon Rapidity Distribution and Transverse Spectrum

To obtain the transverse mass and rapidity distributions, we pass to new variables $(p_x, p_y, p_z) \rightarrow (\phi, m_\perp, y)$:

$$m_\perp = (m^2 + p_\perp^2)^{1/2}, \quad p_\perp^2 = p_x^2 + p_y^2, \quad \tanh y = \frac{p_z}{\omega(p)},$$

then $d^3p = d\phi \omega_p m_\perp dm_\perp dy$, where $\phi$ is the azimuth angle. In accordance with [2], the double differential cross-section reads (for the sake of simplicity, we consider here the most central collisions which possess the azimuth symmetry)

$$\frac{d^2N}{m_\perp dm_\perp dy} = 2\pi m_\perp \cosh y \left[ \sum_{M=1}^{M_{\text{max}}} C(M) D_M(m_\perp, y) + C_{\text{therm}} D_{\text{therm}}(m_\perp, y) \right].$$

We can define the distribution functions in the rapidity space as

$$\Phi_M(y) = 2\pi \cosh y \int_m^\infty dm_\perp m_\perp^2 D_M(m_\perp, y) \quad \text{with} \quad \int dy \Phi_M(y) = 1.$$  \hspace{1cm} (32)

Then the rapidity distribution looks like

$$\frac{dN}{dy} = \sum_{M=1}^{M_{\text{max}}} C(M) \Phi_M(y) + C_{\text{therm}} \Phi_{\text{therm}}(y),$$

where

$$\Phi_{\text{therm}}(y) = \frac{2\pi \cosh y}{z_{\text{therm}}(\beta) \beta^3 \cosh^3 y} e^{-x} (x^2 + 2x + 2)$$

with $x = m\beta \cosh y$ and $z_{\text{therm}}(\beta) = 4\pi \frac{m^2}{\beta^2} K_2(m\beta)$.  \hspace{1cm} (34)

### 3 Toy model

We would like to obtain the explicit results in the framework of the proposed approach. We note that the crucial quantity of our approach is the form-factor $J(q)$ which can be considered as the probability density (see [3]) to find the nucleon momentum transfer $q$ in a particular nucleon collision. Then the mean value of the quantity which depends on the momentum transfer, for instance $f(q)$, is obtained as $\langle f(q) \rangle = \int d^3q/(2\pi)^3 f(q) J(q)$. To estimate the behavior of the form-factor $J(q)$, we made evaluation of the mean values of $q_\perp^2$, $q_{\perp z}$, and $q_z^2$ in the framework of the UrQMD [12, 13]. The results of the evaluation are depicted in Fig. 4. It is evident that, for the central collisions, the equality $\langle q_\perp^2 \rangle = \langle q_z^2 \rangle$ should be valid; we see this, indeed, in Fig. 4 (\langle q_2^2 \rangle \approx 2\langle q_z^2 \rangle). It is seen that, for AGS energies, the mean values $\langle q_\perp^2 \rangle$ and $\langle q_z^2 \rangle$ approximately equal one another or of the same order. We take this as a basis to formulate a model: let the form-factor $J(q)$ be chosen as a homogeneous distribution of the momentum transfer in the sphere of finite radius $q_{\text{max}}$ (see Fig. 4),

$$J(q) = \frac{(2\pi)^3}{V_q} \theta(q_{\text{max}} - |q|), \quad \text{where} \quad V_q = \frac{4}{3} \pi q_{\text{max}}^3 \quad \text{and} \quad \int \frac{d^3q}{(2\pi)^3} J(q) = 1.$$  \hspace{1cm} (35)

As a matter of fact, an application of this toy model can be regarded just as an approximate description of the nucleon distributions at AGS energies.

We calculated the Fourier transform of form-factor [35] in the explicit way and obtained

$$J(x) = \frac{3}{x^3} \left( \sin x - x \cos x \right),$$

\hspace{1cm} (36)
where \( x \equiv r q_{\text{max}} \) and \( r = |r| \). So, the multiscattering form-factor \( I_M(Q) \), see (11), reads now as

\[
I_M(Q) = \frac{4\pi}{V Q} \int_0^\infty dr \sin(Qr) \left[ \frac{3}{r^3} \left( \sin x - x \cos x \right) \right] M ,
\]
(37)

where \( Q = |Q| \), \( x = r q_{\text{max}} \).

With the help of these functions, we can evaluate the partial distribution \( D_M(p) \) from (28) as

\[
D_M(m_\perp, y) = \frac{1}{2z(\beta)} e^{-\beta m_\perp \cosh y} \left[ I_M(|p - k_0|) + I_M(|p + k_0|) \right] ,
\]
(38)

where \( \omega(p) = m_\perp \cosh y, p_z = m_\perp \sinh y, |p \mp k_0| = \sqrt{m_\perp^2 - m^2 + (m_\perp \sinh y \mp k_0 z)^2} \) and

\[
z(\beta) = V \int \frac{d^3p}{(2\pi)^3} e^{-\beta \omega(p)} I_M(|p - k_0|) .
\]
(39)

Next, we use these partial distribution functions directly in (31) to evaluate the double differential (transverse mass) spectrum and the functions \( \Phi_M(y) \) from (32) which will be used then as the partial rapidity distributions in (33).

In the framework of the toy model, we obtain that the parameter of the model, \( q_{\text{max}} \), can be related to the mean value of the momentum transfer squared

\[
\langle q^2 \rangle = \frac{3}{5} q_{\text{max}}^2 .
\]
(40)

Taking into account \( \langle q^2 \rangle \) evaluated in UrQMD \[12,13\] for AGS energies (see Fig. 3), we obtain the following estimation from (40): \( q_{\text{max}} \approx 800 \text{ MeV} \).

In the framework of the toy model for large collision numbers \( M \), the distribution functions \( D_M(m_\perp, y) \) reads

\[
D_M(m_\perp, y) \approx \frac{e^{-\beta m_\perp \cosh y}}{2z(\beta)} \left[ e^{-\frac{5(p - k_0)^2}{2M q_{\text{max}}^2}} + e^{-\frac{5(p + k_0)^2}{2M q_{\text{max}}^2}} \right] \text{ with } z(\beta) = \int \frac{d^3p}{(2\pi)^3} e^{-\beta \omega(p)} \frac{5(p - k_0)^2}{2M q_{\text{max}}^2} ,
\]
(41)

where \( |p - k_0| \) and \( |p + k_0| \) are defined in the same way as in (38).
3.1 Extraction of the physical information from experimental data

By making use of the rapidity distribution [39], we fit the experimental data on the rapidity distribution of net protons which were measured at the AGS (E802 Collaboration) [17]. The proton data is remarkable in that sense that we know exactly the initial momentum, \( k_{0z} \), of every nucleon.

We note that, in the case of a small number of experimental points \( N_{\text{exp}} \), the set of functions \( \Phi_M(y) \) is overcomplete (\( N_{\text{exp}} < M_{\text{max}} + 1 \)). To choose a unique configuration of the variable parameters \( C(M) \), we use the maximum entropy method [18, 19]. Namely, we define the information entropy

\[
\sigma = - \sum_{M=1}^{M_{\text{max}}+1} C(M) \ln C(M),
\]

where \( C(M) = C(M)/N_p \), \( N_p \) is the total number of protons, and, for the unification of notations, we adopt \( C(M_{\text{max}} + 1) = C_{\text{therm}} \). Our goal is to find the maximum of the information entropy \( \sigma \). Meanwhile, entropy (42) is supplemented by \( (N_{\text{exp}} + 1) \) constraints:

\[
\sum_{M=1}^{M_{\text{max}}+1} C(M) = 1, \quad I_i^{\text{(exp)}} = \sum_{M=1}^{M_{\text{max}}+1} C(M) \Phi_M(y_i),
\]

where \( I_i^{\text{(exp)}} \) is an experimental data value of the distribution \( dN/dy \) taken at the rapidity point \( y_i \), \( i = 1, 2, \ldots, N_{\text{exp}} \) and \( \Phi_{M_{\text{max}}+1}(y) \equiv \Phi_{\text{therm}}(y) \). Using the method of the Lagrangian multipliers, we reformulate the problem in the following way: it is necessary to find the maximum of the expression

\[
L = - \sum_{M=1}^{M_{\text{max}}+1} C(M) \ln C(M) + \mu \left[ \sum_{M=1}^{M_{\text{max}}+1} C(M) - 1 \right] + \sum_{i=1}^{N_{\text{exp}}} \lambda_i \left[ I_i^{\text{(exp)}} - N_p \sum_{M=1}^{M_{\text{max}}+1} C(M) \Phi_M(y_i) \right],
\]

where \( \mu \) and \( \lambda_i \) are the \( (N_{\text{exp}} + 1) \) Lagrangian multipliers. Derivatives of this expression with respect to all unknown coefficients give a set of \( (M_{\text{max}} + 1 + N_{\text{exp}} + 1) \) equations which we solve using the variation method: first, we exclude \( \mu \) and all \( C(M) \) from the system of equations and left with a reduced system of \( N_{\text{exp}} \) transcendental equations for the unknown Lagrangian multipliers \( \lambda_i \), where \( i = 1, 2, \ldots, N_{\text{exp}} \). The latter reduced system of equations is solved by variations of the \( \lambda_i \). So, we seek out the minimum of the expression

\[
\chi^2(\lambda_1, \lambda_2, \ldots, \lambda_{N_{\text{exp}}}) = \sum_{i=1}^{N_{\text{exp}}} \left[ I_i^{\text{(exp)}} - N_p \sum_{M=1}^{M_{\text{max}}+1} C(M) \Phi_M(y_i) \right]^2,
\]

where \( \bar{C}(M) = X_M/ \left( \sum_{M=1}^{M_{\text{max}}+1} X_M \right) \) with \( X_M = \exp \left[ N_p \sum_{i=1}^{N_{\text{exp}}} \lambda_i \Phi_M(y_i) \right] \). Actually, Eq. (43) is nothing more as a total \( \chi^2 \), with a help of which we find a theoretical curve of the closest fit to \( dN/dy \) experimental data. At the same time, it is easy to rewrite (43) with accounting for normalization with respect to the experimental error bars at every rapidity point \( y_i \).

The fit was carried out with a help of the program MINUIT, and the variable parameters are the Lagrangian multipliers \( \lambda_i \). The slope parameter \( \beta \) was first extracted from the double differential yield for protons with the use of the thermal distribution. All evaluations are carried out in the c.m.s. of colliding nuclei. The obtained theoretical curves are depicted in Fig. 5. The solid thick curve represents the result of the fit, total \( \chi^2 = 13.3 \). The broken curves marked by the numbers \( M \) represent the partial contributions from every ensemble, \( C(M) \phi_M(y) \). The thermal contribution is represented by the central bell-like dashed curve. The obtained parameters for \( T = 1/\beta = 280 \text{ MeV} \) are shown in Table 1. In the form of histograms, these coefficients are depicted in Fig. 6.

| \( C(1) \) | \( C(2) \) | \( C(3) \) | \( C(4) \) | \( C(5) \) | \( C(6) \) | \( C(7) \) | \( C(8) \) | \( C(9) \) | \( C(10) \) | \( C(11) \) | \( C(12) \) | \( C(13) \) | \( C_{\text{therm}} \) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1.3 | 23.2 | 4.7 | 5.3 | 6.6 | 8.2 | 9.5 | 10.6 | 11.6 | 12.3 | 12.9 | 13.5 | 13.9 | 16.6 |

So, each coefficient \( C(M) \) tells us how many protons undergo \( M \) effective collisions or which is the population of each collision ensemble. For instance, the ensemble of protons which participated just in two effective collisions, \( M = 2 \), consists of 23 protons, i.e. \( C(2) = 23.2 \). It is worth to note that this ensemble is maximally populated in comparison to others. What is very important, we learn from this expansion that \( C_{\text{therm}} = 16.6 \). This
means that approximately seventeen protons come from a thermal source, what makes up 11% (eleven percent) of all participated protons. The maximal number of collisions, \( M_{\text{max}} = 13 \) was determined from the UrQMD evaluation \([12, 13]\) as the maximum number of effective proton collisions which is obtained by averaging over the number of Monte-Carlo events. Actually, the histograms depicted in Fig. 6 represent a tomography picture with respect to the number of collisions.

The partial distribution functions in the rapidity space, \( \Phi_M(y) \) and \( \Phi_{\text{therm}}(y) \) (see \([32, 51]\)), are depicted in Fig. 6. One can say that the total rapidity distribution \( dN/dy \) from \([33]\) is nothing more as an expansion over the set of these \( M_{\text{max}} + 1 \) functions. In Fig. 7, we depicted also, as the scatter curves, the functions \( \Phi_M(y) \) evaluated in the framework of the saddle-point approximation (see \([21]\)). Starting from \( M = 4 \), these curves, as seen, differ very slightly from the distribution evaluated in a rigorous way.

So, we see that the top of partial rapidity distributions which corresponds to the rapidity point \( y_{\text{top}} \) is shifted to the central rapidity region with increase in the number of effective collisions \( M \) (in Fig. 7 the number \( M \) marks every particular distribution). This means that the collective velocity \( v_h \approx \tanh y_{\text{top}} \) of the distribution \( \Phi_M(y) \) decreases with increase in \( M \). Evidently, the energy which was initially accumulated in the longitudinal degrees of freedom is converted to the particle creation and to the excitation of the transverse degrees of freedom with decrease in the longitudinal velocity \( v_h \), i.e. after every collision. The latter results in the collision-by-collision increase of the isotropization and the effective temperature. These two conclusions are in full correspondence with the analysis made in Section 2.1.1.

By making use of the obtained coefficients \( C_M \) and \( C_{\text{therm}} \), which are listed in Table 1 (see histograms of the coefficients in Fig. 6), we evaluated the double differential cross-section in accordance with \([21]\). The results of this evaluation for different rapidity windows is depicted in Fig. 8. A very good description of the experimental data \([17]\) for all rapidity windows is seen. Hence, our model satisfactory describes the \( m_\perp \)-spectra.

We assume that the thermal source has absolutely different nature of origination, i.e. it cannot be created just due to the hadron reactions of nucleons which result in the randomization and the subsequent isotropization of the nucleon momentum. The thermal source can emerge as a result of the appearance of many other (not hadron) degrees of freedom. We know just one candidate to this role, it is the quark-gluon plasma, for instance, its creation can occur in collisions of nucleons at high energies in the presence of a dense medium. A many-parton system, which emerges in the collision, consists of a large number of gluons and quarks. All momenta of quarks and gluons can be regarded from the very beginning as random ones, and the thermalization of the system occurs during a time span \( \tau_{\text{therm}} = 0.6 \text{ fm/c} \) \([20]\). Hence, the protons which come from the thermal
source indicate the presence of the QGP in the fireball, and we can determine a nucleon power of the QGP, \( P_{\text{qgp}} \), by the ratio of the number of protons coming from the thermal source, \( C_{\text{therm}} \), and total number of participanted protons: \( P_{\text{qgp}} \equiv C_{\text{therm}} / N_p \). For instance, it turns out that the nucleon power of the QGP equals eleven percent, \( P_{\text{qgp}} \approx 11\% \), in \( Au + Au \) collisions \(^{[17]}\) (0-3% centrality).

Figure 7: The curves marked by numbers \( M \) represent the functions \( \Phi_M (y) \) (see \(^{[23]}\)), the thermal distribution \( \Phi_{\text{therm}} (y) \) is represented by the central Gaussian-like curve. The scatter curves represent the same \( \Phi_M (y) \) evaluated in the saddle-point approximation.

Figure 8: Solid curves represent the evaluation of the \( m_\perp \)-spectra obtained in accordance with Eq. \(^{\ref{eq:fit}}\), where we use the same values of the coefficients \( C(M) \) which were obtained as the result of a fit of the \( dN/dy \) data (see Fig. \(^{\ref{fig:fit}}\)). Experimental points are from \(^{[17]}\).

4 Summary and discussion

The starting assumption of our approach is given in Eq. \(^{[11]}\): we propose to separate the registered nucleons by their origination into two groups. The first group consists of the net nucleons after the multiple rescattering and the second group consists of the nucleons created by the QGP. We consider the latter as a source of the QGP. We consider the former as a source of the QGP.

The second assumption consists in full independence of the momentum transfer in the chain of sequential collisions of a nucleon:

\[
J (q^{(1)}, \ldots, q^{(M)}) = \prod_{m=1}^{M} J_m (q^{(m)})
\]

where \( J (q^{(1)}, \ldots, q^{(M)}) \) is the probability density to find the set \( q^{(1)}, \ldots, q^{(M)} \) of the momentum transfer during \( M \) collisions of the nucleon in the fireball; \( J_m (q) \) is the probability density to find the momentum transfer \( q \) in the \( m \)-th collision. Additionally, we assume that the scattering conditions do not depend on the collision number:

\[
J_m (q) \rightarrow \langle J_m (q) \rangle_{\text{collisions}} = J (q)
\]

The essence of the third assumption is a partition of all nucleons, which took part in rescattering processes, into collisions ensembles in accordance with the effective collision number \( M \), see Eq. \(^{[11]}\). The physical basis of this assumption is a spatial finiteness of the multi-nucleon system.

So, we parametrize the time axis by the number of particle’s effective collisions \( M = \langle \nu \rangle t \), where \( \langle \nu \rangle \) is the mean frequency of collisions and \( t \) is the time interval. If we considers the dependence of the distribution
function (28) or (30) on the variable $M$, we can regard this distribution as a nonequilibrium one ($M \propto$ time). As we see from (30), the increase of $M$ effectively mimics an approach to the Boltzmann distribution (for the sake of simplicity, we consider no special statistics). If the number of collisions $M$ is fixed, this means that the process of thermalization stopped at the time moment when the particles had experienced just $M$ collisions and, at this moment, they were frozen out; all this results in a partial thermalization of the subsystem which we name as the $M$-th collision ensemble. So, the level of thermalization and isotropization depends on the number of effective collisions, $M$, which is determined, first, by the lifetime of the system or, more precisely, by the number of physical collisions of every particle and, second, by the level of randomization of the momentum transfer in every physical collision.

Actually, in the framework of the proposed model, we obtain a physical picture which looks like a discrete “fireball model”, where every fireball can be associated with a particular collision ensemble. In accordance with the fireball model [21], the rapidity axis is populated with thermalized fireballs following a distribution $\rho(y_h)$ which is taken as the Gaussian one. (Here, the rapidity $y_h$ determines the collective (hydrodynamical) velocity $v_h = \tanh y_h$ of the particles attributed to the particular fireball.) The essential difference which occurs in our model, except the discretization of the fireball set, is that “our” fireballs (collision ensembles) are not yet the fully thermalized systems. The degree of thermalization of every collision ensemble is frozen on some stage of evolution toward a full thermalization (the collective velocity of the particles which belong to the particular ensemble can be evaluated approximately as $v_h \approx \tanh y_{top}$, where $y_{top}$ determines the top of the $\Phi_M(y)$ distribution in the rapidity space, see Fig. [7].

Properly, the results of the proposed model crucially depend on the single-particle form-factor $J(q)$ which is nothing more as the probability density to find a particular value $q$ of the momentum transfer in one collision. As a trial evaluation (toy model), we take the form-factor as a homogeneous restricted distribution of the momentum transfer, $J(q) = \frac{(2\pi)^3}{V_q} \theta(q_{\text{max}} - |q|)$, where $V_q = \frac{4\pi}{3} q_{\text{max}}^3$ (see Fig. [4]).

The maximum number of collisions (reactions) $M_{\text{max}}$ is assumed to be finite and determined by the nuclear number $A$, initial energy, and centrality. With the help of the UrQMD transport model [12, 13], it was found that, under AGS (Au+Au, 11.6A GeV/c, 0-3% centrality) conditions [17], $M_{\text{max}} = 13$. Using the thermal distribution, we extract the slope parameter from the experimental data on the proton $m_\perp$-spectra, $T = 280$ MeV.

We made fit of the experimental data [17] on the rapidity distribution of net protons and obtained the collection of coefficients $C(M)$ (see Table 1 and Fig. [9]) which are nothing more as the absolute number of protons in every collision ensemble, i.e. $N_p = \sum_{M=1}^{M_{\text{max}}} C(M) + C_{\text{therm}}$, where $N_p$ is the total number of protons.

The knowledge of the number of protons, $C_{\text{therm}}$, which come from the QGP gives us a possibility to evaluate the “nucleon power” of the QGP, $P_{\text{QGP}}^{(N)}$, created in a particular experiment on the nucleus-nucleus collision. We find that, under the AGS conditions [17] (a centrality of 0-3%), $P_{\text{QGP}}^{(N)} \approx 11\%$. So, in the framework of the proposed criterion, it could be claimed that the QGP (as a nucleon source) was created not only at SPS energies [22], but it was also created in the central collisions at AGS energies.

Meanwhile, if we do not use the thermal contribution in the partial expansion (33) for the description of the particular experimental data [17], we obtain as well a good description (the same $\chi^2$) of both the rapidity distribution and $m_\perp$-spectra. So, we cannot resolve unambiguously the problem about the presence of the thermal source. In fact, to overcome the problem, we need a more detailed experimental information for the central rapidity region.

Solid curves in Fig. [5] represent the results of calculations of the $m_\perp$-spectra obtained in accordance with Eq. (31), where we use the same values of the coefficients $C(M)$ (see Table 1) which were obtained as the result of a fit of the $dN/dy$ data. It is seen from the results depicted in Figs. [5] and [7] that the main contribution to the distribution for small rapidity values (in the laboratory system) is given by collision ensembles with small $M$. For instance, for the particular experiment [17], the partial distribution for $M = 2$ in Eq. (33) determines the distribution in the rapidity windows $0.4 \leq y \leq 0.6$. We note that, in this rapidity domain, the thermal contribution is practically zero. So, we can conclude that the proposed model satisfactorily describes the $m_\perp$-spectra even in those regions, where the contribution from a pure thermal source is absent and the spectrum is determined only by the rescattering of net nucleons.

All this leaves us with the continued challenge of applying the model to other experiments and problems.
Acknowledgements

D.A. is grateful to A. Rebhan, S. Mrówczyński, and K. Tuchin for useful and encouraging discussions. The authors are grateful to R. Lednicky for reading the manuscript and for reasonable remarks. D.A. was partially supported by the program “Fundamental properties of physical systems under extreme conditions” (Division of the Physics and Astronomy of the NAS of Ukraine).

References

[1] R. Baier, A.H. Mueller, D. Schiff and D.T. Son, Phys. Lett. B 539, 46 (2002); arXiv:hep-ph/0009237.
[2] D. Molnar and M. Gyulassy, Nucl. Phys. A 697, 495 (2002); arXiv:nucl-th/0104073.
[3] A. El, Z. Xu and C. Greiner, Nucl. Phys. A 806, 287 (2008); arXiv:0712.3734 [hep-ph].
[4] A. Rebhan, P. Romatschke, Phys. Rev. Lett. 94, 102303 (2005); arXiv:hep-ph/0412016.
[5] S. Mrowczynski, J. Phys. Conf. Ser. 27, 204 (2005); arXiv:hep-ph/0506179.
[6] P. Arnold, G.D. Moore, Phys. Rev. D 76, 045009 (2007); arXiv:0706.0490 [nucl-th].
[7] D. Kharzeev, Nucl. Phys. A774, 315 (2006); arXiv:hep-ph/0511354.
[8] P. Castorina, D. Kharzeev, H. Satz, Eur. Phys. J. C 52, 187 (2007); arXiv:0704.1426 [hep-ph].
[9] L. D. Landau, Izv. Akad. Nauk, Ser. Fiz., 17, 51 (1953).
[10] M. Gyulassy, arXiv:nucl-th/0403032; arXiv:nucl-th/0403032.
[11] J. Kapusta, J. Phys. G 34, S295 (2007); arXiv:0705.1277 [nucl-th].
[12] S. A. Bass, M. Belkacem, M. Bleicher et al., Prog. Part. Nucl. Phys. 41, 225 (1998);
[13] M. Bleicher, E. Zabrodin, C. Spieles et al., J. Phys. G: Nucl. Part. Phys. 25, 1859 (1999).
[14] D. Anchishkin, S. Yezhov, arXiv:0802.0259 [nucl-th].
[15] F. Jüttner, Ann. Physik u. Chemie, 34, 856 (1911).
[16] S.R. de Groot, W.A. van Leeuwen, Ch.G. van Weert, Relativistic Kinetic Theory, North-Holland, Amsterdam, 1980.
[17] L. Ahle et al. (E802 Collaboration), Phys. Rev. C 60, 064901 (1999).
[18] L.M. Soroko, Physics of Elementary Particles and Atomic Nuclei, V. 12, No. 3, p. 754-795 (1981) [in Russian].
[19] A. Papoulis, Probability, Random Variables and Stochastic Processes, 4th ed., McGraw-Hill, New York, 2002.
[20] A. Adil and M. Gyulassy, arXiv:0709.1716 [nucl-th].
[21] F. Becattini, J. Cleymans, J. Phys. G 34, S959 (2007); arXiv:hep-ph/0701029.
[22] U.W. Heinz, M. Jacob, arXiv:nucl-th/0002042.