Maximally nonlocal subspaces

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A nonlocal subspace $\mathcal{H}_{NS}$ is a subspace within the Hilbert space $\mathcal{H}_n$ of a multi-particle system such that every state $\psi \in \mathcal{H}_{NS}$ violates a given Bell inequality $\mathcal{B}$. Subspace $\mathcal{H}_{NS}$ is maximally nonlocal if each such state $\psi$ violates $\mathcal{B}$ to its algebraic maximum. We propose ways by which states with a stabilizer structure of graph states can be used to construct maximally nonlocal subspaces, essentially as a degenerate eigenspace of Bell operators derived from the stabilizer generators. Two cryptographic applications—quantum information splitting and quantum subspace certification—are discussed.

I. INTRODUCTION

Quantum nonlocality is a fundamental quantum feature that demonstrates that quantum mechanics can’t be explained by a local theory [1]. It forms the basis for nonclassical tasks, such as secure key distribution with uncharacterized devices [2] and device-independent random number generation [3]. For a recent review, see [4]. It has been extensively studied over the last 50 years, with its various aspects studied in the bipartite and multipartite scenario, involving dichotomic or many-valued measurements.

In particular, it is known that the set $\mathcal{Q}$ of bipartite correlations is contained in the set of bipartite correlations obtained based on the assumption that the local operators of two observers commute [5, 6], and strictly contained in the set of no-signaling correlations [7]. As the set $\mathcal{L}$ of local correlations for a given finite number of inputs and outputs is convex, it follows from the hyperplane separation theorem that given element $p' \notin \mathcal{L}$, there is a correlation inequality, linear in the inputs and outputs, that is satisfied by all elements $p \in \mathcal{L}$ but violated by $p'$—i.e., a witness of the nonlocality of $p'$. These inequalities are called Bell inequalities. (The corresponding inequalities for the quantum set $\mathcal{Q}$ are called Tsirelson inequalities.) Bell inequalities that are facet inequalities—i.e., tight witnesses—characterize $\mathcal{L}$ minimally. The facets of $\mathcal{L}$ can be determined by computer codes, but in general the problem of determining whether a correlation is local in the Bell scenario with dichotomic, multiple inputs, is hard (in fact, NP-complete).

Bell inequalities for the bipartite case have been extended to the $n$-particle situation [8, 9], which can form the basis for witnessing multipartite entanglement without assumptions about measurement devices or underlying dimension [10]. Bell inequalities to witness $k$-partite nonlocality, and thus also $k$-partite entanglement, are known [11].

Here, we shall be concerned with another, quite distinct aspect of quantum multipartite nonlocality: namely, finding Bell inequalities that are, for certain given measurements settings, violated equally by any pure state in a subspace, called the nonlocal subspace. Consequently, any superpositions or mixtures of these pure states also violate the inequality to the same extent. In a sense, the concept of a nonlocal subspace generalizes the idea of a nonlocal state to a subspace.

To the best of our knowledge, the problem of characterizing or identifying nonlocal subspaces hasn’t been studied before. Here, we will show how the stabilizing properties of graph states naturally conduce to the construction of quantum nonlocal subspaces. The subspaces we construct are maximally nonlocal, in the sense that the Bell-type inequality is violated to its algebraic maximum. In addition to their theoretical interest, nonlocal subspaces are experimentally interesting because they can be demonstrated readily using practically the same setup used for tests of violation of Bell-type inequalities.

Furthermore, their structure makes them amenable to application in certain quantum cryptographic tasks, among them, quantum information splitting (QIS) and quantum subspace certification. QIS requires a quantum state to be teleported over an entangled state distributed among various parties. The nonlocal subspace of $n$ particles provides a natural subspace in which to encode the quantum secret (not unlike the encoding of an unknown particle), such that the nonlocality serves as the basis to test security. We discuss later below illustrative examples that underscore these cryptographic applications.

The plan of article is as follows. In Section II we formally define nonlocal subspaces and their maximal kind. In Section III we briefly review graph states and how Mermin-Bell type inequalities can be constructed for them. In Section IV we point out how to construct maximally nonlocal subspaces using graph states, which essentially reduces to the problem of finding such an inequality corresponding to a degenerate Bell operator. Examples where the degeneracy can be easily identified are pointed out in Sections V A (case of a linear cluster state) and

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II. NONLOCAL SUBSPACES

A measurement setting in a multipartite Bell scenario is a set of given input choices of various observers (say, Alice, Bob, Charlie et al.) in an experiment to test whether a state violates a Bell inequality. For example, in an experiment to test the violation of the CHSH inequality, a measurement setting could be \{\{X, Z\}, \{\sqrt{2}X \pm Z\}\}, i.e., that Alice chooses one of the Pauli observables X and Z, and Bob chooses one of \(\frac{1}{\sqrt{2}}(X \pm Z)\).

Definition 1 (Nonlocal subspace) Suppose \(\mathcal{H}_n\) is the Hilbert space of \(n\)-qubits, and subspace \(\mathcal{G} \subset \mathcal{H}_n\), such that any every state \(\psi \in \mathcal{G}\) violates a given Bell inequality \(\langle B \rangle \leq L\) to the same degree for the given measurement setting. Then, \(\mathcal{G}\) is a nonlocal subspace.

Here, \(\mathcal{B}\) is the Bell operator and \(L\) the local-realism bound. It is important to stress that this definition requires that the measurement setting, degree and Bell inequality should be the same. Otherwise, the set of all pure entangled states would form a nonlocal set in the sense that any such state will violate a Bell inequality to some degree for a suitable choice of measurement setting [12].

The basic idea here is that if two distinct states \(|\psi_a\rangle\) and \(|\psi_b\rangle\) span a nonlocal subspace, such that \(\langle \psi_a | B | \psi_b \rangle = \langle \psi_b | B | \psi_a \rangle \equiv C > L\), then without any further calculation, we know that any superposition \(\alpha |\psi_a\rangle + \beta |\psi_b\rangle\) will also violate inequality \(\langle B \rangle \leq L\) to the same degree \(C\) for a certain fixed measurement setting.

The singlet state \(|\psi^-\rangle \equiv \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)\) violates the CHSH inequality \(\langle \mathcal{B}_{\text{CHSH}} \rangle \equiv |\langle A_1 B_1 \rangle + \langle A_1 B_2 \rangle - \langle A_2 B_1 \rangle + \langle A_2 B_2 \rangle| \leq 2\) by reaching the Tsirelson bound of \(2\sqrt{2}\). That this is not a member of a nonlocal subspace can be shown by seeing that any neighboring state to \(|\Psi^-\rangle\) violates the inequality to a lesser degree, that directly depends on the fidelity.

Specifically, let \(|\psi(\theta, \phi)\rangle \equiv \cos(\theta/2) |01\rangle - e^{i\phi} \sin(\theta/2) |10\rangle\). Then, one finds that \(\langle \mathcal{B}_{\text{CHSH}} \rangle = 2\sqrt{2}F(\theta, \phi)\), where \(F(\theta, \phi)\) is the fidelity between the states \(|\Psi^-\rangle\) and \(|\psi(\theta, \phi)\rangle\). Here, the fidelity between states \(\rho_1\) and \(\rho_2\) is defined by \(\frac{1}{2} \text{Tr} \left( \sqrt{\sqrt{\rho_1} \rho_2 \sqrt{\rho_1} \rho_2} \right)\).

In general, the no-signaling bound on a Bell inequality exceeds the quantum (or, Tsirelson) bound. For example, for the CHSH inequality, the Tsirelson bound is \(2\sqrt{2}\), whereas the no-signaling bound is 4 [11]. However, for “all-or-nothing” type of Bell inequalities (discussed below), based on a logical contradiction à la GHZ [13], the quantum and no-signaling bounds can be the same, being equal to the algebraic maximum (the number of terms in the Bell expression). In such a case, one can consider a strengthened form of a nonlocal subspace, as defined below.

Definition 2 (Maximally nonlocal subspace (MNS).) Suppose \(\mathcal{G}(\subset \mathcal{H}_n)\) is a nonlocal subspace associated with the Bell inequality \(\langle B \rangle \leq L\), such that any every state \(\psi \in \mathcal{G}\) violates the Bell inequality to its algebraic maximum for the same measurement setting. Then, \(\mathcal{G}\) is a maximally nonlocal subspace.

Following an introduction to graph states, we shall discuss how maximally nonlocal subspaces can be naturally constructed for graph states, essentially by exploiting their stabilizer structure.

III. GRAPH STATES AND BELL INEQUALITIES: A ROUNDUP

Specifically, graph states are a class of highly entangled multi-qubit states, representable by a graph [14]. Given graph \(G = (n, E)\), with \(n\) and \(E\) being the number of vertices and the set of edges, respectively, the graph state \(|G\rangle\) is defined as:

\[
|G\rangle = \Pi_{(i,j) \in E} C(i,j) |+\rangle^\otimes n,
\]

where vertices represent spin systems and edges \(C(i,j)\) represent the controlled-phase gate between qubits \(i\) and \(j\), which can be realized using Ising interactions. Figure 1 depicts various graph types with \(n = 4\).

For \(1 \leq j \leq V\), define mutually commuting local observables (stabilizers):

\[
g_j = X_j \bigotimes_{k \in N(j)} Z_k.
\]

where \(N(j)\) denotes the neighborhood of vertex \(j\), i.e., the set of vertices having an edge with vertex \(j\). The graph state \(|G\rangle\) is simultaneously the +1 eigenstate of the \(n\) stabilizers \(g_i\):

\[
\forall i \ g_i |G\rangle = |G\rangle.
\]

Any graph state is equivalent a stabilizer state, up to local rotations [13]. The set of all \(2^n\) possible products...
(denoted $h_k$) of the generators $g_j$ forms a group, $S$, called the stabilizer. Obviously, the graph state is stabilized by all elements $h_k \in S$.

A complete basis for the Hilbert space $\mathcal{H}_n$ of $n$ qubits can be derived from $|G\rangle$ by all possible local applications of Pauli $Z$ to the $n$ vertices. This is the graph state basis, which consists of $2^n$ simultaneous eigenstates of stabilizer generators $g_j$:

$$|G_k\rangle \equiv |G_{x_1x_2\cdots x_n}\rangle = \bigotimes_j (Z_j)^{x_j} |G_{000\cdots 0}\rangle, \quad (4)$$

where $x_j \in \{0, 1\}$ and $|G_{000\cdots 0}\rangle \equiv |G\rangle$. It can be shown that

$$g_j |G_{x_1x_2\cdots x_n}\rangle = (-1)^{x_j} |G_{x_1x_2\cdots x_n}\rangle. \quad (5)$$

We define the syndrome of a graph basis state by the string $((-1)^{x_1}, (-1)^{x_2}, \ldots, (-1)^{x_n}) \in \{\pm 1\}^\otimes n$, which uniquely fixes the graph basis state in the graph basis.

Among various applications of graph states we may count the use of cluster states in measurement-based quantum computing (MBQC) \cite{14, 18, 17} and verifying MBQC \cite{18, 19}. Brickwork states, which are graph states with the underlying graph being a “brickwork” and which require only $X, Y$-plane measurements (rather than arbitrary $SU(2)$ measurements) constitute a basic resource for delegated quantum computation, specifically universal blind quantum computation \cite{20}. Graph states can be used for quantum secret sharing or quantum information splitting \cite{21, 20}, quantum error correction \cite{27} and quantum metrology \cite{19}.

Graph states have been realized experimentally \cite{28–31}. Their robustness in the presence of decoherence \cite{32} enhances their practical value.

As highly entangled states, not surprisingly, graph states show a rich variety of nonlocal correlations through the violation of Mermin-type inequalities \cite{8} based on stabilizer measurements \cite{33, 35} generating perfect correlations of GHZ type \cite{12}, and also through violations of Bell-Ardehali inequalities \cite{8} based on non-stabilizer measurements \cite{36, 37}.

From any subset of the stabilizer $S$, we can construct the operator:

$$\mathcal{B} = \sum_{k=1}^{m} h_k, \quad (6)$$

where $m \leq 2^n$, and the $h_k$’s are any $m$ distinct stabilizer elements. Since the $g_j$’s are tensor products of local (in fact, Pauli) operators, therefore $h_k$’s are also tensor products of Pauli operators. In view of Eq. (3):

$$\mathcal{B}|G\rangle = m|G\rangle. \quad (7)$$

Let $q$ denote the largest number of $h_k$’s in Eq. (6) that can assume a positive value (+1) under a local-realistic value assignment to the individual Pauli operators.

In the classical world, each property of each particle can be assigned values independently of the settings of other particles, and we would expect $q = m$. However, as we shall illustrate by a simple example below, in quantum theory on account of the non-commutativity and the intransitivity of commutativity of observables, for certain choices of operators $h_k$ in Eq. (1) one may encounter a Greenberger-Horne-Zeilinger (GHZ) type \cite{13} logical contradiction, making $q$ strictly smaller than $m$. Therefore, if for a given choice of $m$ operators $h_k$, $q < m$, then the operator defined by Eq. (6) constitutes a Bell operator, for which we can write down a Bell inequality (BI) of the type:

$$(\mathcal{B}) \leq \mathcal{L} = 2q - m, \quad (8)$$

which is violated to its algebraic maximum (of $m$) by the relevant graph state. We note that the set of $m$ stabilizer elements which form the GHZ-type contradiction is not unique. There can be several such contradictions derived from other subsets of $S$, leading to different Bell operators. Indeed, the full set $S$ will lead to a Bell operator, as noted below. The degree of violation of BI may be quantified by $D = \frac{m}{2^{q - m}}$, which would be the relevant figure of merit that determines resistance of the violation to noise and detection loophole. In Eq. (8), there may not be an obvious pattern that allows us to compute $q$. However, it can be determined by straightforward computer search, by assigning values ±1 to the (at most) three variables $X, Y$ and $Z$ for each of the $n$ qubits, i.e., by searching through (at most) $3^2n$ possibilities.

Any graph state violates a BI, which can be shown using an inductive argument \cite{34}. Extending this argument, the sum of all stabilizer elements $h_k$ is a Bell operator, though not a maximal one. In fact:

$$2^{-n} \sum_{k=1}^{2^n} h_k = |G\rangle \langle G|, \quad (9)$$

which is easily verified. There are $2^{2n}$ potential Bell operators of the type (6). For $3 \leq n \leq 6$, they are fully characterized into 14 equivalent classes (up to local rotations). Among them are the multiqubit GHZ states, which correspond to the star graph \cite{35}.

Let us consider a simple example of a Mermin inequality for a graph state, in the case of the linear cluster state LC$_4$:

$$|G\rangle = \frac{1}{2} (|+0+0\rangle + |+0-1\rangle + |-1-0\rangle + |-1+1\rangle), \quad (10)$$

stabilized by generators: $g_1 \equiv X_1Z_2$, $g_2 \equiv Z_1X_2Z_3$, $g_3 \equiv Z_2X_3Z_4$ and $g_4 \equiv Z_3X_4$.

One constructs a contradiction in a manner analogous to the GHZ argument \cite{12}, which is based on perfect (“all-or-nothing”) correlations. Consider the 4 stabilizing operators:

$$g_1g_3 = +X I X Z \rightarrow +1, \quad g_2g_3 = +Z Y Y Z \rightarrow +1,$$

$$g_1g_3g_4 = +X I Y Y \rightarrow +1, \quad -g_2g_3g_4 = +Z Y X Y \rightarrow -1, \quad (11)$$
Each column has two copies of a Pauli operator, meaning that under a local-realistic assignment of value +1 or −1 to the individual Pauli operators, the column product is 1. But, the product on the RHS is −1, leading to a contradiction. Therefore, the sum

\[ B = IXIZ + ZYYZ + XIYY - ZXYX \]  \hspace{1cm} (12)

provides a Bell operator of the Mermin type.

By design, \( \langle G \parallel B \rangle (G) = 4 \) for \( B \) in Eq. (12), the number of summands \( m \) in the Bell operator. On the other hand, the above contradiction argument shows that only 3 terms in Eq. (12) can be local-realistically made positive, so that \( q = 3 \). From Eq. (5), the local bound \( \mathcal{L} = 2q - m = 2 \). We thus have the Bell-type inequality

\[ \langle B \rangle \leq 2, \]  \hspace{1cm} (13)

for the Bell operator in (12).

For a large graph state, \( q \) can be derived by computer search. A helpful tip here is that the local-realistic value assignment scheme may be assumed to assign \( Z = +1 \) [4]. Another tip is that the value of \( q \) is invariant under local complementation. Thus, in the case of completely connected graphs and star graphs, we have \( \mathcal{B} (\text{ST}_n) = \mathcal{B} (\text{FC}_n) \) for a given \( \mathcal{B} \) (see Figure 1).

IV. BELL-DEGENERACY

Given any graph basis \( |G'\rangle \), let \( \hat{g}_j \) denote the eigenvalue of generator \( g_j \), i.e., \( g_j |G'\rangle = \hat{g}_j |G'\rangle \). Since all the generators \( g_j \) commute with each other, and \( |G'\rangle \) is a joint eigenstate of theirs, it is readily seen by direct substitution that any sum of products of \( \hat{g}_j \)'s acting on \( |G'\rangle \) equals the corresponding sum of products of \( \hat{g}_j \)'s multiplying \( |G'\rangle \). That is,

\[ (g_1, g_2, \ldots, g_n, + g_3, g_4, \ldots, g_m, + \cdots) |G'\rangle = (g_1, g_2, \ldots, g_n, + g_3, g_4, \ldots, g_m, + \cdots) |G'\rangle. \]  \hspace{1cm} (14)

In particular, this means that the Bell operator can be replaced by the corresponding function of the respective eigenvalues:

\[ \mathcal{B}(g_1, g_2, \ldots, g_n) |G'\rangle = \mathcal{B}(g_1, g_2, \ldots, g_n) |G'\rangle \]

In this light, Eq. (7) can, in view of the form Eq. (6), be considered as a set of \( m \) constraints (“Bell conditions”) on the graph syndrome:

\[ \forall_{k=1}^m h_k (\hat{g}_1, \hat{g}_2, \ldots, \hat{g}_n) = 1. \]  \hspace{1cm} (15)

That is, the solution to these conditions would represent the graph syndrome(s) of graph basis state(s) that satisfy Eq. (7).

If these constraints don’t uniquely fix the graph basis state to the graph state \( |G\rangle \), then there will be multiple syndrome solutions to Eq. (7), making the Bell operator \( \mathcal{B} \) degenerate.

Let \( |G_j\rangle \) denote these multiple graph basis state solutions to Eq. (7). By virtue of linearity, any normalized state \( \sum_j \alpha_j |G_j\rangle \) also violates the BI \( \langle B \rangle \leq \mathcal{L} \) by reaching its algebraic maximum. Thus, the span of these \( |G_j\rangle \)'s defines a subspace associated with maximal violation. Accordingly, this degenerate +1-eigenspace of \( \mathcal{B} \) constitutes a maximally nonlocal subspace (MNS), denoted \( \mathcal{H}_{\text{MNS}} \). Various ways to produce Bell degeneracy are exemplified below.

A. Bell degeneracy with LC state

The straightforward method is to solve the equation \( \mathcal{B}(g_1, g_2, \ldots, g_n) = m \). Multiplicity of solutions leads to Bell degeneracy, which may be determined by computer search for large-\( n \) graph states. For the state \( \text{LC}_4 \), characterized by Eq. (12), solving \( \hat{g}_1 \hat{g}_3 = \hat{g}_2 \hat{g}_4 = \hat{g}_1 \hat{g}_3 \hat{g}_4 = \hat{g}_2 \hat{g}_3 \hat{g}_4 = 1 \), we find solutions given by syndromes \( (\hat{g}_1, \hat{g}_2, \hat{g}_3, \hat{g}_4) \to (\pm 1, \pm 1, \pm 1) \). Thus, the corresponding graph basis states span \( \mathcal{H}_{\text{MNS}} \).

The first of these syndromes correspond to graph state \( |G\rangle \) given by Eq. (10), while the other state to

\[ |G'\rangle = Z_1 Z_2 Z_3 |G\rangle \]

\[ = \frac{1}{2} (|000\rangle + |001\rangle - |100\rangle - |101\rangle). \]  \hspace{1cm} (16)

Any superposition in subspace \( \mathcal{H}_{\text{MNS}} \), namely, \( \alpha |G\rangle + \beta |G'\rangle \) also violates BI (13) to its algebraic maximum of 4.

B. Bell degeneracy via Common generators

In a Bell operator \( \mathcal{B} \), suppose \( l (>1) \) stabilizer generators \( g_1, g_2, \ldots, g_l \) appear in all the summands \( h_k \) (1 ≤ \( j \) ≤ \( m \)). Then, \( \dim(\mathcal{H}_{\text{MNS}}) \geq 2^{l-1} \), which is the number of value assignments to \( (\hat{g}_1, \hat{g}_2, \ldots, \hat{g}_l) \) consistent with \( \hat{g}_1 \hat{g}_2 \cdots \hat{g}_l = 1 \).

As an example, consider the 6-qubit linear cluster state \( \text{LC}_6 \) [25]:

\[ \mathcal{B} = g_2 g_5 (I + g_1) (I + g_3) (I + g_4) (I + g_6) \leq 4, \]  \hspace{1cm} (17)

where \( g_1 = X_1 Z_2, g_6 = Z_5 X_6 \) and \( g_j = Z_{j-1} X_j Z_{j+1} \) for \( j = 2, 3, 4, 5 \). In this case, \( l = 2 \) and the two graph basis states spanning \( \mathcal{H}_{\text{MNS}} \) are \( (\hat{g}_1, \hat{g}_2, \hat{g}_3, \hat{g}_4, \hat{g}_5, \hat{g}_6) \to (\pm 1, \pm 1, \pm 1, \pm 1) \), with

\[ |G'\rangle = Z_2 Z_3 |\text{LC}_6\rangle \]

being the second state in addition to \( |G\rangle \) that violates BI (17) to its algebraic maximum.
V. BELL DEGENERACY FOR QEC CODES

Quantum error correcting (QEC) codes have a natural association with MNS. A $[[n,k]]$ QEC code encodes $k$ qubits in $n$ qubits, such that the code space is stabilized by $n-k$ commuting syndrome operators $g_j$ [38]. Any Bell operator $B$ formed from these $(n-k)$ generators will obviously have a $2^{k}$-fold degeneracy, since all states in the code space will produce maximal violation, by construction.

A. 5-qubit QEC code

As an example, let $|G_0\rangle$ and $|G_1\rangle$ be the code words for the 5-qubit code [39], which corrects one arbitrary qubit error.

$$|G_0\rangle = \frac{1}{4}(-|00000\rangle - |11000\rangle - |01100\rangle - |00110\rangle$$
$$- |00011\rangle - |10011\rangle + |10100\rangle + |01010\rangle + |01001\rangle$$
$$+ |01101\rangle + |11110\rangle + |11101\rangle + |11011\rangle)$$

$$|G_1\rangle = \cdots$$

where $\cdots \equiv X^\otimes 5$. The stabilizers are $g_1 = XYYYI$, $g_2 = IXYXY$, $g_3 = ZYIYZ$ and $g_4 = XYZYX$.

It may be checked that

$$B = g_4g_1(1 + g_3) + g_2(g_1 + g_3) + g_1 \leq 3$$

constitutes a Mermin inequality with $m = 5$. Our previous observation entails that any encoded state in this QEC code will violate BI (19) maximally. It follows from Eq. (19) that $\tilde{g}_2\tilde{g}_3 = \tilde{g}_2\tilde{g}_1 = 1$, and therefore that $\tilde{g}_i$ ($i = 1,2,3$) have the same sign. Because of the first summand in Eq. (19), $g_1 = 1$ and thus $g_4 = 1$. In other words, the “Bell conditions” fully fix the code space, and there is no further degeneracy. But this is not necessary, as we discuss with the Steane code.

B. Steane QEC code

A BI that can be constructed for the 7-qubit Steane QEC code [40], given by:

$$B = g_2g_1(1 + g_4 + g_5g_4) + g_5g_3(g_1 + g_2) + g_5 \leq 4(20)$$

where the stabilizer generators for the Steane code are $g_1 = IIIXXXX, g_2 = IXXIXXX, g_3 = XIXIXXX, g_4 = IIIZZZZ, g_5 = ZZIIZZZ$ and $g_6 = ZZIIZIZ$.

Note that the generator $g_6$ doesn’t appear in the BI (20), meaning that the value assignment $g_6$ is unrestricted. Solving the “Bell conditions” for $\tilde{g}_j$ ($1 \leq j \leq 5$) gives two solutions: $(\tilde{g}_1,\tilde{g}_2,\tilde{g}_3,\tilde{g}_4,\tilde{g}_5) \rightarrow (\pm 1,\pm 1,\pm 1, 1, 1)$. For BI (20), we thus find

$$\dim(H_{MNS}) = 4 \times \dim(\text{code space}) = 8.$$
the secret, which may ideally be an arbitrary qubit state. For perfect secrecy, graphs that correspond to QEC codes [42] are appropriate, while the properties of MNS may be used for testing the code space.

In the first step of the protocol, Alice measures her qubit in the computational basis. The result is given in Table I. In step 2, Bob measures in the computational basis, too, the results of which are depicted in Table II for the case where Alice obtained $|0\rangle$ in the first step. In step 3, Charlie measures his qubit in the computational basis. These steps leave the secret with Rex’s qubit, up to a Pauli operation. Clearly, Rex can find out this Pauli operator, and thereby recover the quantum secret, based on classical communication from Alice, Bob, and Charlie, and can’t recover it without input from even one of them.

| Outcome of Alice | State with Bob, Charlie and Rex |
|------------------|---------------------------------|
| $|0\rangle$      | $\mu\left( -|0000\rangle - |0011\rangle - |0110\rangle - |1100\rangle + |0101\rangle + |0010\rangle + |1010\rangle + |1111\rangle \right) + \nu\left( |1101\rangle - |0111\rangle - |1110\rangle + |1011\rangle + |0010\rangle + |1000\rangle + |0100\rangle \right) - |1100\rangle$ |

| $|1\rangle$      | $\mu\left( -|1000\rangle - |0001\rangle + |0100\rangle + |0100\rangle + |1101\rangle + |1101\rangle + |0111\rangle + |1111\rangle - |1001\rangle - |0000\rangle + |1010\rangle + |0101\rangle \right) + \nu\left( |0110\rangle \right) - |0011\rangle - |1111\rangle - |1001\rangle - |0000\rangle + |1010\rangle + |0101\rangle \right) - |1100\rangle$ |

TABLE I. QIS using the 5-qubit code (18): Outcome of Alice’s measurement with corresponding state left with Bob, Charlie and Rex.

Let us consider a simple security scenario of this QIS scheme. Suppose Eve, as part of eavesdropping, attacks the 4th qubit of an encoded state of above 5-qubit QECC, as part of which she employs the two-qubit controlled-qubit interaction:

$$U(\theta) = |0\rangle \langle 0| \otimes \mathbb{I} + |1\rangle \langle 1| \otimes \left( \begin{array}{cc} \cos \eta & \sin \eta \\ \sin \eta & -\cos \eta \end{array} \right),$$

where $0 \leq \eta \leq \pi/2$. By straightforward calculation, one finds that under this interaction, the expectation values for the stabilizing elements are

$$\langle h_m \rangle = \begin{cases} \cos(\eta) & (m = 1, 5) \\ 1 & (m = 2, 3, 4) \end{cases},$$

from which it follows that for BI (19)

$$\langle B \rangle = 2\cos(\eta) + 3,$$

which reaches the local bound 3 when $\eta = \pi/2$. Thus, the basic idea is that any intervention by Eve diminishes the level of violation away from maximality. Quite generally, this behavior is related to the monogamy of quantum entanglement and of nonlocal no-signaling correlations.

B. Quantum subspace certification

Given an unknown system and uncharacterized measurement devices, some system features, such as its dimension or entanglement, may be inferable from the observed measurement statistics. Thus, such features admit self-testing [43, 44], wherein one makes no assumptions about preparations, channels and measurements.

State tomography or entanglement witnesses also test states, but under the assumption of trusted preparation and measurement procedures. The problem of certifying states requires a higher level of trust, where measurements are trusted, but sources and channels aren’t. This can be extended to a self-test, essentially by showing robustness against errors in the measuring instruments. Typically, a self-test requires the violation of a suitable Bell-type inequality against specific local measurements.

Here, we shall briefly discuss how our approach to identify an MNS can be used to construct a produce to certify the subspace. This generalizes the problem of certifying a given graph state [45]. Because graph basis states form a complete basis, stabilizer tests which admit an MNS such that $\dim(H_{\text{MNS}}) > 1$ can be used to certify that the state belongs to the subspace $H_{\text{MNS}}$ in question by verifying that it maximally (or, to sufficiently high degree) violates the associated Bell inequality $B$. Two security criteria here are [19]: (Completeness) that the test accepts an ideal preparation; (Soundness) that acceptance indicates sufficient closeness to the ideal state preparation.

We note that the state certified in this way, while guaranteed to be an element of $H_{\text{MNS}}$, may be a pure or mixed. If, further, a guarantee of purity is needed, a further component to self-test purity must be added. In this framework, we obtain the certification of a given graph state as a special case of subspace certification, where one seeks an MNS of unit dimension. That is, suppose $|G\rangle$ uniquely violates BI $B$ maximally, but no other graph basis state does (an example is discussed below). Stabilizers $g_j$ associated with $B$ obviously accept $|G\rangle$, which guarantees completeness. By virtue of the assumed uniqueness, any deviation of the prepared state from $|G\rangle$ will increase chances of rejection, leading to soundness.
By way of an example: Ref. [32] lists BI’s for graph states of various families with up to 6 qubits. Three 4-qubit inequalities listed for $|LC_4\rangle$ are:

\[ B_1 = (I + g_1)g_2(I + g_3) \leq 2 \quad (24a) \]
\[ B_2 = (I + g_1)g_2(I + g_3g_4) \leq 2 \quad (24b) \]
\[ B_3 = (I + g_1)g_2(g_3 + g_4) \leq 2, \quad (24c) \]

where $g_1 = X_1Z_2, g_4 = Z_3X_4$ and $g_j = Z_{j-1}X_jZ_{j+1}$ ($j = 2, 3$). The maximal algebraic and quantum bound are 4 in each case.

By inspection, for each of these three inequalities, we find that the dimension of the corresponding MNS is 2, since (corresponding) $(g_1, g_2, g_3, g_4)_1 \rightarrow (1, 1, 1, \pm 1)$ and $(g_1g_2, g_3, g_4)_2 \rightarrow (1, 1, \pm 1, \pm 1)$ and $(g_1, g_2, g_3, g_4)_1 \rightarrow (1, \pm 1, \pm 1, \pm 1)$ all violate the corresponding BI maximally. Therefore, maximal (or close to maximal) violation of one of these inequalities can be used to certify the corresponding graph subspace. For example, a (near) maximal violation of inequality (24a) would indicate the state is a superposition of $|LC_4\rangle$ and $Z_4|LC_4\rangle$.

The 5-qubit state $|GHZ_5\rangle$ [33], which is stabilized by the five operators $g_1 = X_1Z_2Z_3Z_4$ and $g_j = Z_1X_j$ ($j = 2, 3, 4, 5$) is the unique state that maximally violates

\[ g_1(I + g_2)(I + g_3)(I + g_4)(I + g_5) \leq 4 \quad (25) \]

to its algebraic maximum of 16. Therefore, maximal (or close to maximal) violation of inequality Eq. (25) can be used to certify the state $|GHZ_5\rangle$.

This method of certification can be employed in the context of verifiable MBQC, allowing this idea to be extended to fault tolerance by having client (Alice) ask server (Bob) for a suitable resource graph state (cf. [46]), such as the 3D cluster state used in a fault-tolerant topological scheme [47].

VII. CONCLUSIONS AND DISCUSSIONS

We proposed various ways by which graph states can be used to construct maximally nonlocal subspaces, essentially as the degenerate eigenspaces of Bell operators derived from the stabilizer generators. Applications to quantum cryptography were discussed, in particular, quantum information splitting and quantum subspace certification.

A future direction would be to extend our approach to develop a method for creating nonlocal subspaces for Bell-Ardehali-type inequalities, which aren’t based on stabilizer measurements but may lead to stronger violations of the relevant BI. Another direction would be to derive Svetlichny-type inequalities for graph states leading to absolutely nonlocal subspaces for graph states.

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