ZERO-CYCLES ON CANCIAN–FRAPPORTI SURFACES

ROBERT LATERVEER

ABSTRACT. An old conjecture of Voisin describes how 0-cycles on a surface $S$ should behave when pulled-back to the self-product $S^m$ for $m > p_g(S)$. We show that Voisin’s conjecture is true for a 3-dimensional family of surfaces of general type with $p_g = q = 2$ and $K^2 = 7$ constructed by Cancian and Frapporti, and revisited by Pignatelli–Pezzi.

1. INTRODUCTION

Let $X$ be a smooth projective variety over $\mathbb{C}$, and let $A^i(X)_\mathbb{Z} := CH^i(X)$ denote the Chow groups of $X$ (i.e. the groups of codimension $i$ algebraic cycles on $X$ with $\mathbb{Z}$-coefficients, modulo rational equivalence \cite{9}). Let $A^i_{hom}(X)_\mathbb{Z}$ (and $A^i_A(X)_\mathbb{Z}$) denote the subgroup of homologically trivial (resp. Abel–Jacobi trivial) cycles.

The Bloch–Beilinson–Murre conjectures describe an alluring kind of paradise, in which Chow groups are precisely determined by cohomology and the coniveau filtration \cite{11}, \cite{12}, \cite{23}, \cite{14}, \cite{24}, \cite{34}. The following particular glimpse of this paradise was first formulated by Voisin:

Conjecture 1.1 (Voisin 1993 \cite{33}). Let $S$ be a smooth projective surface. Let $m$ be an integer strictly larger than the geometric genus $p_g(S)$. Then for any 0-cycles $a_1, \ldots, a_m \in A^2_A(S)_\mathbb{Z}$, one has

$$\sum_{\sigma \in \mathfrak{S}_m} \text{sgn}(\sigma) a_{\sigma(1)} \times \cdots \times a_{\sigma(m)} = 0 \text{ in } A^{2m}(S^m)_\mathbb{Z}.$$  

(Here $\mathfrak{S}_m$ is the symmetric group on $m$ elements, and $\text{sgn}(\sigma)$ is the sign of the permutation $\sigma$. The notation $a_1 \times \cdots \times a_m$ is shorthand for the 0-cycle $(p_1)^*(a_1) \cdot (p_2)^*(a_2) \cdots (p_m)^*(a_m)$ on $S^m$, where the $p_j : S^m \to S$ are the various projections.)

For surfaces of geometric genus 0, conjecture 1.1 reduces to Bloch’s conjecture \cite{4}. As for geometric genus 1, Voisin’s conjecture is still open for a general K3 surface; examples of surfaces of geometric genus 1 verifying the conjecture are given in \cite{33}, \cite{15}, \cite{17}, \cite{18}. Examples of surfaces with geometric genus strictly larger than 1 verifying the conjecture are given in \cite{21}. One can also formulate versions of conjecture 1.1 for higher-dimensional varieties; this is studied in \cite{33}, \cite{16}, \cite{19}, \cite{20}, \cite{4}, \cite{22}, \cite{32}, \cite{6}.

The modest goal of this note is to add to the stock of surfaces verifying conjecture 1.1 by considering Cancian–Frapporti surfaces. These are minimal surfaces $S$ of general type with

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\( p_g(S) = q(S) = 2 \) and \( K^2_S = 7 \) constructed as semi-isogenous mixed surfaces in [7] and revisited in [26]. The main result of this note is:

**Theorem** (=theorem 5.1). Let \( S \) be a Cancian–Frapporti surface. Then conjecture 1.1 is true for \( S \).

This is proven by exploiting the facts that Cancian–Frapporti surfaces have (a) finite-dimensional motive (in the sense of [14]) and (b) surjective Albanese morphism [26]. A key ingredient of the argument is a strong form of the generalized Hodge conjecture for self-products of abelian surfaces [1], [32]. Because of the use of this key ingredient, I am not sure whether the argument can be adapted to other surfaces with \( p_g = q = 2 \) verifying (a) and (b) (cf. remark 5.7).

As a corollary, certain instances of the generalized Hodge conjecture are verified:

**Corollary** (=corollary 5.6). Let \( S \) be a Cancian–Frapporti surface, and let \( m > 2 \). Then the sub-Hodge structure

\[ \wedge^m H^2(S, \mathbb{Q}) \subset H^{2m}(S^m, \mathbb{Q}) \]

is supported on a divisor.

**Conventions.** In this note, the word *variety* will refer to a reduced irreducible scheme of finite type over \( \mathbb{C} \). A *subvariety* is a (possibly reducible) reduced subscheme which is equidimensional.

Unless indicated otherwise, all Chow groups will be with rational coefficients: we will denote by \( A_j(X) \) the Chow group of \( j \)-dimensional cycles on \( X \) with \( \mathbb{Q} \)-coefficients (and by \( A_j(X)_\mathbb{Z} \) the Chow groups with \( \mathbb{Z} \)-coefficients); for \( X \) smooth of dimension \( n \) the notations \( A_j(X) \) and \( A^{n-j}(X) \) are used interchangeably.

The notations \( A^j_{hom}(X), A^j_{AJ}(X) \) will be used to indicate the subgroups of homologically trivial, resp. Abel–Jacobi trivial cycles. The contravariant category of Chow motives (i.e., pure motives with respect to rational equivalence as in [29], [24]) will be denoted \( \mathcal{M}_{rat} \).

## 2. CANCIAN–FRAPPORTI SURFACES

**Theorem 2.1** (Cancian–Frapporti [7], Pignatelli–Polizzi [26]). There exist minimal surfaces \( S \) of general type with \( p_g(S) = q(S) = 2 \) and \( K^2_S = 7 \), and surjective Albanese map (of degree 3). These surfaces fill out a dense open subset of a 3-dimensional component of the Gieseker moduli space of general type minimal surfaces with these invariants.

**Proof.** We present a condensed outline of the construction, following [26].

Let \( C_4 \subset \mathbb{P}^3 \) be a genus 4 curve defined as a smooth complete intersection

\[ r(x_0, x_1) + x_2 x_3 = s(x_0, x_1) + x_2^3 + x_3^3 = 0, \]

As explained in loc. cit., only two families of minimal surfaces of general type with invariants \( p_g = q = 2 \) and \( K^2 = 7 \) are known: the 3-dimensional family of Cancian–Frapporti, and a 2-dimensional family (distinct from the first family) constructed as bidouble covers by Rito [27]. For Rito’s surfaces, proving conjecture 1.1 seems difficult as they are not known to have finite-dimensional motive.
where \( r(x_0, x_1), s(x_0, x_1) \) are homogeneous polynomials of degree 2 resp. 3. The curve \( C_4 \) admits a free action of an order 3 automorphism \( \xi \) defined as
\[
\xi[x_0, x_1, x_2, x_3] = [x_0, x_1, \nu x_2, \nu^2 x_3]
\]
where \( \nu \) is a primitive third root of unity. The quotient \( C_2 := C_4/\langle \xi \rangle \) is a smooth genus 2 curve.

The product \( C_4 \times C_4 \) admits an involution \( \sigma \) (switching the two factors) and an order 3 diagonal automorphism \( \xi_{xy} \) (acting as \( \xi \) on both factors). The surface \( S \) is now defined as a quotient
\[
S := (C_4 \times C_4)/\langle \xi, \xi_y \rangle \cong C_2 \times C_2
\]

(The surface \( S \) is smooth, because it is a \textit{semi-isogenous mixed surface} in the sense of [7, Definition 2.1], cf. [7, Corollary 1.11].)

The group \( G \) is a non-normal, abelian subgroup of the group
\[
H := \langle \xi_x, \xi_y, \sigma \rangle \subset \text{Aut}(C_4 \times C_4)
\]
where \( \xi_x, \xi_y \) act as \( \xi \) on the first, resp. second, factor. As shown in [26, (4)], there is a commutative diagram
\[
\begin{align*}
C_4 \times C_4 & \quad \downarrow \\
(C_4 \times C_4)/\langle \xi_{xy} \rangle & \quad \cong (C_4 \times C_4)/\langle \xi_x, \xi_y \rangle \cong C_2 \times C_2 \\
\downarrow & \quad \downarrow \\
S := (C_4 \times C_4)/\langle \xi_{xy}, \sigma \rangle & \quad \cong Y := (C_4 \times C_4)/H \cong \text{Sym}^2(C_2)
\end{align*}
\]

Here, the unnamed horizontal arrows are the natural quotient morphisms, the morphism \( \pi \) is the contraction of the unique rational curve contained in \( Y \), and the morphism \( \alpha \) is the Albanese map. The fact that the morphism \( \alpha \) making the diagram commute is the Albanese map (which is thus surjective) is contained in [26, Proposition 1.8].

The invariants of \( S \) and the minimality are justified in [26, Proposition 1.5]. Finally, the statement about the moduli space is [26, Theorem 2.7].

\[ \square \]

\textbf{Definition 2.2.} We will call surfaces as in theorem 2.1 \textit{Cancian–Frapporti surfaces}.

\section*{3. Transcendental Part of the Motive of a Surface}

\textbf{Theorem 3.1 (Kahn–Murre–Pedrini [13]).} Let \( S \) be a smooth projective surface. There exists a decomposition
\[
h(S) = h^0(S) \oplus h^1(S) \oplus h^2_{tr}(S) \oplus h^2_{alg}(S) \oplus h^3(S) \oplus h^4(S) \in \mathcal{M}_{\text{rat}},
\]
such that
\[ H^*(h^2_{tr}(S), \mathbb{Q}) = H^2_{tr}(S, \mathbb{Q}), \quad H^*(h^2_{alg}(S), \mathbb{Q}) = NS(S)_{\mathbb{Q}}, \]
(here \(H^2_{tr}(S)\) is defined as the orthogonal complement of the \(\text{Néron–severi group} NS(S)_{\mathbb{Q}}\) in \(H^2(S, \mathbb{Q})\)), and
\[ A^*(h^2_{tr}(S))_{\mathbb{Q}} = A^2_{AJ}(S). \]
(The motive \(h^2_{tr}(S)\) is called the transcendental part of the motive.)

4. A RESULT OF VIAL’S

This section contains a “Bloch conjecture” type of statement. As already shown in [32], this statement is very useful in dealing with Voisin’s conjecture on 0-cycles.

Definition 4.1. Let \(M \in \mathcal{M}_{\text{rat}}\) and let \(X\) be a smooth projective variety. We say that \(M\) is motivated by \(X\) if \(M\) is isomorphic to a direct summand of a sum of tensor powers of motives of the form \(h(X)(j), j \in \mathbb{Z}\).

Theorem 4.2 (Vial [32]). Let \(M \in \mathcal{M}_{\text{rat}}\) be motivated by an abelian variety of dimension \(\leq 2\). Assume that
\[ H^{i,j}(M) = 0 \quad \text{for all} \quad j < n. \]
Then also
\[ A_i(M) = 0 \quad \text{for all} \quad i < n. \]

Proof. This is not stated verbatim in [32], but the argument is the same as that of [32, Theorem 4.7]. In a nutshell, the point is that (as proven in [32, Corollary 3.13]) \(M\) satisfies a strong form of the generalized Hodge conjecture, i.e. there is equality
\[ N^r_H H^i(M) = \Gamma^* H^{i-2r}(A), \]
where \(A\) is a disjoint union of abelian varieties and \(\Gamma\) is a correspondence from \(A\) to \(M\). (Here, \(N^r_H\) denotes the Hodge coniveau filtration [32, Definition 1.4].)

Writing \(M = (X, p, m) \in \mathcal{M}_{\text{rat}}\), the cohomological assumption thus translates into the fact that the cohomology class of \(p\) factors as
\[ h(X) \xrightarrow{\Psi} h(A)(n - m) \xrightarrow{\Xi} h(X), \]
where \(A\) is a disjoint union of abelian varieties, and \(\Psi\) and \(\Xi\) are correspondences in \(A^*(A \times A)\) resp. in \(A^*(A \times X)\). Since \(M\) is Kimura finite-dimensional, one can apply the nilpotence theorem to \(p \circ \Xi' \circ \Psi \circ p\); the outcome is that the rational equivalence class of \(p\) factors as
\[ h(X) \xrightarrow{\bar{\Psi}} h(A)(n - m) \xrightarrow{\bar{\Xi}} h(X). \]
Taking Chow groups, this proves the theorem. \(\square\)
5. Main result

**Theorem 5.1.** Let \( S \) be a Cancian–Frapparti surface. For any \( a, b, c \in A^2_{AJ}(S) \), there is equality

\[
a \times b \times c - b \times a \times c - c \times b \times a - a \times c \times b + b \times c \times a + c \times a \times b = 0 \quad \text{in} \quad A^6(S^3) .
\]

**Proof.** A first reduction step is that thanks to Roitman [28], one may replace \( A^*() \) by Chow groups with \( \mathbb{Q} \)-coefficients \( A^*() \).

Next, let us consider the decomposition of the Chow motive of \( S \)

\[
h(S) = h^0(S) \oplus h^1(S) \oplus h^2_{tr}(S) \oplus h^2_{alg}(S) \oplus h^3(S) \oplus h^4(S) \quad \text{in} \quad M_{\text{rat}} ,
\]

where \( h^2_{tr}(S) \) is the transcendental part of the motive of \( S \) (theorem [3.1]).

The dominant morphism \( \beta : S \to Y \) (proof of theorem [2.1]) identifies the motive of \( Y \) with a submotive of the motive of \( S \), in particular this gives (non-canonical) splittings

\[
\begin{align*}
h^2_{tr}(S) &= h^2_{tr}(Y) \oplus M_{tr} = h^2_{tr}(Y) \oplus M_{tr} , \\
h^2_{alg}(S) &= h^2_{alg}(Y) \oplus M_{alg} \quad \text{in} \quad M_{\text{hom}} .
\end{align*}
\]

The surfaces \( S \) and \( Y \), being dominated by a product of curves, have finite-dimensional motive. This implies (using the nilpotence theorem [14]) that the splittings (1) also exist on the level of \( M_{\text{rat}} \).

We remark that the motive \( M := M_{tr} \oplus M_{alg} \) has

\[
\begin{align*}
dim_{\mathbb{C}} H^{2,0}(M) &= \dim_{\mathbb{C}} H^{2,0}(S) - \dim_{\mathbb{C}} H^{2,0}(A) = 2 - 1 = 1 , \\
dim_{\mathbb{C}} H^{1,1}(M) &= \dim_{\mathbb{C}} H^{1,1}(S) - \dim_{\mathbb{C}} H^{1,1}(Y) = 7 - 5 = 2 .
\end{align*}
\]

One has \( A^*(h^2_{tr}(S)) = A^2_{AJ}(S) \) and \( A^*(h^2_{tr}(Y)) = A^2_{2}(A) = A^2_{AJ}(A) \) (here and below, for any abelian variety \( A \), we write \( A^*_v(A) \) for the Fourier decomposition of \( \pi^* \), and \( \pi^* \) for the Chow–Künneth projectors inducing the Fourier decomposition as in [8]). This splitting of \( h^2_{tr}(S) \) induces a splitting

\[
A^2_{AJ}(S) = A^2_{2}(A) \oplus A^2(M_{tr}) .
\]

We make two claims, that deal with the two pieces of this splitting separately:

**Claim 5.2.** For any \( a_1, a_2 \in A^2_{2}(A) \), there is equality

\[
a_1 \times a_2 = a_2 \times a_1 \quad \text{in} \quad A^4(A \times A) .
\]

**Claim 5.3.** For any \( v_1, v_2 \in A^2(M_{tr}) \), there is equality

\[
v_1 \times v_2 = v_2 \times v_1 \quad \text{in} \quad A^4(S \times S) .
\]

Because of the equality

\[
\bigwedge^3 \left( A^2_{2}(A) \oplus A^2(M_{tr}) \right) = \bigoplus_{j=0}^3 \bigwedge^j A^2_{2}(A) \otimes \bigwedge^{3-j} A^2(M_{tr}) ,
\]

these two claims together suffice to prove theorem [5.1].
The first claim is easy, and directly follows from a more general result of Voisin’s (this is Example 4.40):

**Proposition 5.4** (Voisin [34]). *Let $A$ be an abelian variety of dimension $g$. Let $a_1, a_2 \in A^g(A)$. Then*

$$a_1 \times a_2 = (-1)^g a_2 \times a_1 \text{ in } A^g(A \times A).$$

*In order to prove the second claim, we first need to understand the motive $M_{tr}$ a bit better.*

**Proposition 5.5.** *There exist an abelian surface $B$, and a correspondence inducing a surjection*

$$H^2(B \times B, \mathbb{Q}) \to H^2(M_{tr}, \mathbb{Q}).$$

*Proof. This follows from the specific geometry of the construction of $S$. Reverting to the notation of the proof of theorem [2.1] the covering morphism $C_4 \times C_4 \to S$ induces a surjection*

$$H^2_{tr}(C_4 \times C_4, \mathbb{Q}) \to H^2_{tr}(S, \mathbb{Q}).$$

*An application of the Künneth formula gives a surjection*

$$H^1(C_4, \mathbb{Q}) \otimes H^1(C_4, \mathbb{Q}) \to H^2_{tr}(C_4 \times C_4, \mathbb{Q}).$$

*The Abel–Jacobi map of the curve $C_4$ into the 4-dimensional abelian variety $A_4 := \text{Jac}(C_4)$ induces an isomorphism*

$$H^1(A_4, \mathbb{Q}) \otimes H^1(A_4, \mathbb{Q}) \cong H^1(C_4, \mathbb{Q}) \otimes H^1(C_4, \mathbb{Q}).$$

*Choosing base points for the Abel–Jacobi maps in a compatible way, the triple covering of curves $C_4 \to C_2$ induces a surjective homomorphism $A_4 \to A := \text{Jac}(C_2)$. Using Poincaré’s complete reducibility theorem, this implies that $A_4$ is isogenous to $B \times A$, where $B$ is an abelian surface. This gives a decomposition*

$$H^1(A_4, \mathbb{Q}) = H^1(A \times B, \mathbb{Q}) = H^1(A, \mathbb{Q}) \oplus H^1(B, \mathbb{Q}).$$

*Combining all these maps, we obtain a surjection*

$$\left( H^1(A, \mathbb{Q}) \oplus H^1(B, \mathbb{Q}) \right)^{\otimes 2} \cong H^1(C_4, \mathbb{Q}) \otimes H^1(C_4, \mathbb{Q}) \to H^2_{tr}(C_4 \times C_4, \mathbb{Q}) \to H^2_{tr}(S, \mathbb{Q}) \cong H^2_{tr}(A, \mathbb{Q}) \oplus H^2(M_{tr}, \mathbb{Q}).$$

*It follows from the truth of the standard conjectures for surfaces and abelian varieties that all arrows in (3) are induced by correspondences. Let us now consider the summand $H^1(A, \mathbb{Q}) \otimes H^1(A, \mathbb{Q})$ of the left-hand side of (3). The triple covering $C_4 \to C_2$ induces a commutative diagram*

$$\begin{array}{ccc}
H^1(C_4, \mathbb{Q}) \otimes H^1(C_4, \mathbb{Q}) & \to & H^2_{tr}(\text{Sym}^2 C_4, \mathbb{Q}) \\
\downarrow & & \downarrow \\
H^1(C_2, \mathbb{Q}) \otimes H^1(C_2, \mathbb{Q}) & \to & H^2_{tr}(\text{Sym}^2 C_2, \mathbb{Q})
\end{array}$$

*where the composition of upper horizontal arrows is the same map $H^1(C_4, \mathbb{Q}) \otimes H^1(C_4, \mathbb{Q}) \to H^2_{tr}(S, \mathbb{Q})$ as in (3), and $\alpha$ and $\pi$ are as in the proof of theorem [2.1]. Because the summand*
It follows that this summand maps onto $H^2_{tr}(A, \mathbb{Q})$ in (3). More precisely, the map

$$H^1(A, \mathbb{Q}) \otimes H^1(A, \mathbb{Q}) \to H^2_{tr}(A, \mathbb{Q}) \oplus H^2(M_{tr}, \mathbb{Q})$$

deducted from diagram (3) induces a surjection onto $H^2_{tr}(A, \mathbb{Q})$ and the zero-map to $H^2(M_{tr}, \mathbb{Q})$, under both projections.

Let us now analyze the other summands of the left-hand side of (3). There is an induced action of $\xi \in \text{Aut}(C_4)$ on $A_4$, and an eigenspace decomposition

$$H^1(A_4, \mathbb{C}) = H^1(A_4, \mathbb{C})^{(1)} \oplus H^1(A_4, \mathbb{C})^{(\nu)} \oplus H^1(A_4, \mathbb{C})^{(\nu^2)}$$

(where $\nu$ is a primitive third root of unity). The first eigenspace (which is 2-dimensional) corresponds to $H^1(A, \mathbb{C}) \cong H^1(C_2, \mathbb{C})$, while the sum of the two other (1-dimensional) summands corresponds to $H^1(B, \mathbb{C})$. The covering morphism $C_4 \times C_4 \to S$ factors as

$$C_4 \times C_4 \to (C_4 \times C_4)/\langle \xi_{xy} \rangle \to S$$

(where $\xi_{xy} \in \text{Aut}(C_4 \times C_4)$ is the order 3 automorphism acting diagonally as in the proof of theorem [2.1]), and so there is a factorization

$$H^2(C_4 \times C_4, \mathbb{C}) \to H^2((C_4 \times C_4)/\langle \xi_{xy} \rangle, \mathbb{C}) \to H^2(S, \mathbb{C}).$$

It follows that the summands of type $H^1(A_4, \mathbb{C})^{(1)} \otimes H^1(A_4, \mathbb{C})^{(\nu)}$ and $H^1(A_4, \mathbb{C})^{(1)} \otimes H^1(A_4, \mathbb{C})^{(\nu^2)}$ (and their permutations) map to zero under the natural map. In other words, the natural map

$$H^1(A_4, \mathbb{C}) \otimes H^1(A_4, \mathbb{C}) \to H^2(S, \mathbb{C})$$

is the same as the composition

$$H^1(A_4, \mathbb{C})^{(1)} \otimes H^1(A_4, \mathbb{C})^{(1)}$$

$$\oplus \left( H^1(A_4, \mathbb{C})^{(\nu)} \otimes H^1(A_4, \mathbb{C})^{(\nu^2)} \oplus H^1(A_4, \mathbb{C})^{(\nu^2)} \otimes H^1(A_4, \mathbb{C})^{(\nu)} \right) \to H^2(S, \mathbb{C}).$$

The first summand corresponds to $H^1(A, \mathbb{C}) \otimes H^1(A, \mathbb{C})$, the second is contained in $H^1(B, \mathbb{C}) \otimes H^1(B, \mathbb{C})$. Thus, we see that “mixed terms” $H^1(A, \mathbb{Q}) \otimes H^1(B, \mathbb{Q})$ and $H^1(B, \mathbb{Q}) \otimes H^1(A, \mathbb{Q})$ in (3) map to zero. It follows that the summand $H^1(B, \mathbb{Q}) \otimes H^1(B, \mathbb{Q})$ in (3) maps onto $H^2(M_{tr}, \mathbb{Q})$.

Let us now prove claim 5.3 (and hence theorem 5.1). Proposition 5.5 in combination with the fact that the standard conjectures hold for surfaces and abelian varieties, shows that there is a map

$$M_{tr} \to h^2(B \times B) \text{ in } \mathcal{M}_{\text{hom}}$$

admitting a left-inverse. Using Kimura finite-dimensionality (cf. for instance [31 Section 3.3]), the same holds in $\mathcal{M}_{\text{rat}}$, i.e. the motive $M_{tr}$ is motivated by the abelian surface $B$. The motive $M := \wedge^2 M_{tr}$ (being a submotive of $M_{tr}^{\otimes 2}$) is also motivated by $B$. The motive $M$ has $H^j(M) = 0$ for all $j \neq 4$ and $H^{4,0}(M) = \wedge^2 H^{2,0}(M_{tr}) = 0$, since $\dim H^{2,0}(M_{tr}) = 1$ (cf. (2)). Applying theorem 4.2 to $M$ (with $n = 1$), we find that

$$\wedge^2 A_0(M_{tr}) = A_0(M) = 0,$$

proving claim 5.3.

□
**Corollary 5.6.** Let \( S \) be a Cancian–Frapporti surface, and let \( m > 2 \). Then the sub-Hodge structure
\[
\wedge^m H^2(S, \mathbb{Q}) \subset H^{2m}(S^m, \mathbb{Q})
\]
is supported on a divisor.

**Proof.** As Voisin had already remarked [33, Corollary 3.5.1], this is implied by the truth of conjecture [1.1] for \( S \) (as can be seen using the Bloch–Srinivas argument [5]).

**Remark 5.7.** The strong form of the generalized Hodge conjecture (as mentioned in the proof of theorem [4.2]) is a result specific to self-products of abelian surfaces, and seems out of reach for self-products of higher-dimensional abelian varieties. As such, the argument employed here crucially hinges on the fact that the Cancian–Frapporti surfaces \( S \) are constructed starting from a Galois cover \( C_m \to C_n \), where \( C_m, C_n \) are curves of genus \( m \) resp. \( n \) and \( m - n \leq 2 \). While the other surfaces with \( p_g = q = 2 \) constructed in [7] still have surjective Albanese map [25, Theorem 4], for all but one of them the difference \( m - n \) is larger than 2. As such, they do not enter in the set-up of the present note; some new argument is needed to prove conjecture [1.1] for them.

**Remark 5.8.** My initial hope was to establish that Cancian–Frapporti surfaces have a multiplicative Chow–K"unneth decomposition (in the sense of [30]), and satisfy the condition \((\ast)\) of [10]. This proved to be unfeasibly difficult, however.

(The problem was that I could not prove that the class of the curve \( C_4 \) in \( A^3(A_4) \) is symmetrically distinguished. This cannot possibly be true for a general genus 4 curve, but might perhaps be true for \( C_4 \) because it is a triple cover over \( C_2 \)?)

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INSTITUT DE RECHERCHE MathÉMATIQUE AVANCÉE, CNRS – UNIVERSITÉ DE STRASBOURG, 7 RUE RENÉ DESCARTES, 67084 STRASBOURG CEDEX, FRANCE. 

E-mail address: robert.laterveer@math.unistra.fr