We introduce a new approach to analyzing the interaction between classical and quantum systems that is based on a limiting procedure applied to multi-particle Schrödinger equations. The limit equations obtained by this procedure, which we refer to as the classical-quantum limit, govern the interaction between classical and quantum systems, and they possess many desirable properties that are inherited in the limit from the multi-particle quantum system. As an application, we use the classical-quantum limit equations to identify the source of the non-local signalling that is known to occur in the classical-quantum hybrid scheme of Hall and Reginatto. We also derive the first order correction to the classical-quantum limit equation to obtain a fully consistent first order approximation to the Schrödinger equation that should be accurate for modeling the interaction between particles of disparate mass in the regime where the particles with the larger masses are effectively classical.
I. INTRODUCTION

Consistent schemes for the interaction of classical and quantum systems are of great interest from both theoretical and practical points of view. For example, in the Copenhagen interpretation of quantum mechanics, measuring devices are treated as classical objects that interact with quantum systems. Consequently, from a foundational perspective, having a consistent and effective method for analyzing the interaction between classical and quantum systems is highly desirable. From the perspective of computational chemistry, the ability to treat heavy particles classically and light particles quantum mechanically results in a significant reduction in computational costs over fully quantum mechanical treatments. For systems with large particle numbers, this separation is essential to making computations feasible.

There has been much work done on the subject of the interaction of classical and quantum systems, and various schemes have been developed to investigate these types of interactions. In the literature these are known as classical-quantum hybrid schemes, and a wide range of distinct schemes have been developed for various purposes including modeling of chemical reactions, analyzing decoherence, and studying measurement theory [2, 4, 5, 7, 8, 13–25]. Additionally, a number of studies of the general structure and properties of classical-quantum hybrid schemes have been carried out [1, 6, 9, 11, 26, 28].

Although many classical-quantum hybrid schemes have been introduced, none are regarded as being the definitive solution to the problem of describing the interaction between classical and quantum systems. In this article, we introduce a new method to address this problem that is based on taking a singular limit of a fully quantum system in complete analogy with the more well-known classical limit. More specifically, we introduce a small dimensionless parameter $\epsilon$ into a multi-particle Schrödinger equation in such a way that the parameter only appears in a subset of the single particle Hamiltonian operators. Using a non-local polar representation for the wave function, we show that the multi-particle Schrödinger equation, when expressed in terms of our non-local polar decomposition, admits a formal $\epsilon \searrow 0$ limit. The resulting limit equations define our notion of a classical-quantum limit, which as far as we are aware is new. The derivation of these equations is carried out in Section II and we also justify our interpretation of these limit equations as describing the interaction between a classical and quantum system in that section.

The limit equations derived here define a completely natural classical-quantum hybrid scheme that has many attractive properties. We single out two properties. First, since our limit equations arise from the taking the singular limit of a fully quantum mechanical system, the desirable properties that hold true for the quantum system remain true for limit equations for any quantities with a well defined limit. In particular, this implies that the limit equations that arise from a non-interacting quantum system will satisfy the non-signalling property that is known to be satisfied by non-interacting quantum systems. This fact forms the basis of our first application where in Section III we use our classical-quantum limit equations to clarify the origin of the non-local signalling [16] that was observed to occur in the classical-quantum hybrid scheme of Hall and Reginatto [14, 15, 23]. Second, again due to the limiting procedure, it can be expected that solutions of the limit equations will approximate corresponding fully quantum mechanical solutions, at least on suitable time scales as is known to be true in the classical limit [5]. From this perspective, our limit equations capture the interactions between classical and quantum systems in a manner that is fully consistent with quantum theory. Moreover, we can view our limit equations as the zeroth order equations satisfied by an $\epsilon$ expansion of $\epsilon$-dependent solutions to the fully quantum mechanical equations. Using a standard perturbation expansion method, we derive in Section IV the equations satisfied by the first order expansion term. In this way, it is possible to derive a fully consistent perturbative approximation scheme for multi-particle quantum systems where the base approximation consists of an interacting classical-quantum system. We expect that this scheme will be useful in computational chemistry for modelling the interaction of heavy particles that are treated in the first approximation classically with light particles that are treated quantum mechanically.

II. A CLASSICAL-QUANTUM LIMIT

Our derivation of the classical-quantum limit begins with a 2-particle Schrödinger equation of the form

$$i\hbar \partial_t \psi = -\frac{\hbar^2}{2M_1} \partial_x^2 \psi + U \psi - \frac{\hbar^2}{2M_2} \partial_y^2 \psi + V \psi,$$

where $U = U(x)$ and $V = V(x, y)$ are time-independent potentials and we are using $\partial_x$ and $\partial_y$ to denote partial derivatives, i.e.

$$\partial_x \phi = \frac{\partial \phi}{\partial x} \quad \text{and} \quad \partial_y \phi = \frac{\partial \phi}{\partial y},$$

for any function $\phi = \phi(x, y)$.
Remark II.1. In the following, the partial derivatives will act on functions that directly follow them. For example,
\[ \partial_x \phi \psi = \frac{\partial \phi}{\partial x} \psi. \]
If we want the partial derivative to act on both functions, then we will employ brackets and write
\[ \partial_x (\phi \psi) = \frac{\partial (\phi \psi)}{\partial x}. \]

Next, we introduce time and length scales given by \( T \), \( L_1 \) and \( L_2 \), and define the dimensionless variables
\[
\tilde{t} = \frac{t}{T}, \quad \tilde{x} = \frac{x}{L_1}, \quad \tilde{y} = \frac{y}{L_2}, \\
\tilde{\psi}(\tilde{t}, \tilde{x}, \tilde{y}) = \psi(T \tilde{t}, L_1 \tilde{x}, L_2 \tilde{y}), \\
\tilde{U}(\tilde{x}) = \frac{T M_2}{h M_1} U(L_1 \tilde{x}) \\
\text{and} \\
\tilde{V}(\tilde{x}, \tilde{y}) = \frac{T}{h} V(L_1 \tilde{x}, L_2 \tilde{y}).
\]

Writing the Schrödinger equation (II.1) in terms of these dimensionless variables, we see that \( \tilde{\psi} \) satisfies
\[
i \partial_{\tilde{t}} \tilde{\psi} = -\frac{\hbar T}{2 M_1 L_1^2} \frac{M_1}{M_2} \tilde{U} \tilde{\psi} - \frac{\hbar T}{2 M_2 L_2^2} \partial^2_{\tilde{y}} \tilde{\psi} + \tilde{V} \tilde{\psi}.
\] (II.2)

We introduce a small dimensionless parameter \( \epsilon \) via
\[
\epsilon = \frac{M_2}{M_1}.
\] (II.3)

Defining the dimensionless masses
\[
m_1 = \frac{M_2 L_2^2}{\hbar T} \quad \text{and} \quad m_2 = \frac{M_2 L_2^2}{\hbar T},
\]
and dropping the tildes, we can write the dimensionless two-particle Schrödinger equation (II.2) as
\[
i \partial_t \psi^{\epsilon} = -\frac{\epsilon}{2 m_1} \frac{\partial^2}{\partial x^2} \psi^{\epsilon} + \frac{1}{\epsilon} U \psi^{\epsilon} - \frac{1}{2 m_2} \frac{\partial^2}{\partial y^2} \psi^{\epsilon} + V \psi^{\epsilon},
\] (II.4)

where we have written \( \psi = \psi^{\epsilon} \) to emphasize that the solution depends on the small parameter \( \epsilon \). We remark that we restrict ourselves to a pair of 1-dimensional particles and time-independent potentials for simplicity. All of our results generalize to higher dimensions, more particles and time-dependent potentials in an obvious manner.

The equation (II.4) governs the evolution of a pair of 1-dimensional quantum particles of (dimensionless) mass \( m_1/\epsilon \) and \( m_2 \), respectively. Assuming the length scales \( L_1 \) and \( L_2 \) are comparable, the ratio \( (m_1/\epsilon)/m_2 \approx 1/\epsilon \) is unbounded as \( \epsilon \searrow 0 \) showing that the limit \( \epsilon \searrow 0 \) is an extreme mass ratio limit that represents the interaction of a heavy particle with a light one. Furthermore, due to the positioning of the parameter \( \epsilon \), it is clear that the \( \epsilon \searrow 0 \) limit acts like a classical limit, but only for the first system with the configuration coordinate \( x \). From these considerations, we expect that a consistent \( \epsilon \searrow 0 \) limit represents the interaction of a classical particle with a quantum one. We justify this expectation in the remainder of this article.

\[ \text{A. A non-local polar representation} \]

The classical-quantum limit considered here is based on a non-local polar representation for the wave function where the non-locality is introduced by way of the averaging operator
\[
\hat{A}(\phi)(x, y) := \frac{1}{b - a} \int_a^b \phi(x, z) \, dz
\] (II.5)
with \( a, b \in \mathbb{R} \) fixed constants satisfying \( a < b \). We observe that \( \hat{A} \) satisfies
\[
\hat{A}^2 = \hat{A},
\]
and hence, defines a projection operator. It is also easy to see that \( \hat{A} \) satisfies the properties
\[
\partial_x \hat{A} = \hat{A} \partial_x \quad \text{and} \quad \partial_y \hat{A} = 0,
\]
and that
\[
\hat{A} \phi = \phi \quad \text{and} \quad \hat{A} (\phi \xi) = \phi \hat{A} \xi \quad \text{for} \quad \phi = \phi(x) \quad \text{and} \quad \xi = \xi(x, y).
\]

**Remark II.2.** The properties \((\text{II.6})-\text{II.8}\) are the essential properties of the operator \( \hat{A} \) that are needed to define our classical-quantum limit. Given any operator \( \hat{A} \) satisfying these properties, all of the arguments used in this article remain valid, and hence, there exists a classical-quantum limit associated with every operator \( \hat{A} \) satisfying \((\text{II.6})-\text{II.8}\), and in particular, for each choice of \( a, b \in \mathbb{R}, \ a < b \). These different classical-quantum limits are all related via a bijective transformation, and consequently, different choices of \( \hat{A} \) lead to different representations of the *same classical-quantum limit*; see Appendix A for the details. Thus there is no physical content in the particular choice of operator \( \hat{A} \) satisfying \((\text{II.6})-\text{II.8}\), and throughout this article, we will employ the operator defined by \((\text{II.5})\) for a fixed choice of constants \( a, b \in \mathbb{R}, \ a < b \).

Exponentiating \( \hat{A} \) yields
\[
\exp(\tau \hat{A}) = e^\tau \hat{A} + \hat{B}
\]
where
\[
\hat{B} := \mathbb{I} - \hat{A}
\]
is the complementary projection operator, which, by \((\text{II.6})-\text{II.7}\) and \((\text{II.8})\), satisfies
\[
\hat{B}^2 = \hat{B}, \quad \hat{B} \hat{A} + \hat{A} \hat{B} = 0, \quad \hat{A} + \hat{B} = \mathbb{I},
\]
\[
\partial_x \hat{B} = \hat{B} \partial_x, \quad \partial_y \hat{B} = \partial_y,
\]
and
\[
\hat{B} \phi = 0 \quad \text{and} \quad \hat{B} (\phi \xi) = \phi \hat{B} \xi \quad \text{for} \quad \phi = \phi(x) \quad \text{and} \quad \xi = \xi(x, y).
\]

Letting
\[
\psi^\epsilon(x, y) = e^{i\epsilon \theta^\epsilon(x, y)} R^\epsilon(x, y) \quad (R^\epsilon(x, y) := |\psi^\epsilon(x, y)|)
\]
denote the standard polar representation, we introduce a non-local phase variable \( \theta^\epsilon \) via the definition
\[
\theta^\epsilon := \exp(\ln(\epsilon) \hat{A}) \Theta^\epsilon \quad \overset{\text{(II.13)}}{=} \quad \epsilon \hat{A} \Theta^\epsilon + \hat{B} \Theta^\epsilon,
\]
or equivalently
\[
\Theta^\epsilon = \exp(- \ln(\epsilon) \hat{A}) \theta^\epsilon \quad \overset{\text{(II.14)}}{=} \quad \frac{1}{\epsilon} \hat{A} \theta^\epsilon + \hat{B} \theta^\epsilon.
\]

Substituting \((\text{II.13})\) into \((\text{II.4})\), we find using \((\text{II.15})\) that the pair \((R^\epsilon, \theta^\epsilon)\) satisfies
\[
\partial_t R^\epsilon = -\frac{\epsilon}{m_1} \partial_x (\exp(- \ln(\epsilon) \hat{A}) \theta^\epsilon) \partial_x R^\epsilon - \frac{\epsilon}{2m_1} \partial_x^2 (\exp(- \ln(\epsilon) \hat{A}) \theta^\epsilon) R^\epsilon
\]
\[
- \frac{1}{m_2} \partial_y (\exp(- \ln(\epsilon) \hat{A}) \theta^\epsilon) \partial_y R^\epsilon - \frac{1}{2m_2} \partial_y^2 (\exp(- \ln(\epsilon) \hat{A}) \theta^\epsilon) R^\epsilon,
\]
\[
\partial_t \theta^\epsilon = \exp(\ln(\epsilon) \hat{A}) \left\{ -\frac{\epsilon}{2m_1} \left[ \partial_x (\exp(- \ln(\epsilon) \hat{A}) \theta^\epsilon) \right]^2 - \frac{\epsilon}{2m_1} \frac{\partial_x^2 R^\epsilon}{R^\epsilon} \right. \\
\]
\[
- \frac{1}{2m_2} \left[ \partial_y (\exp(- \ln(\epsilon) \hat{A}) \theta^\epsilon) \right]^2 + \frac{1}{2m_2} \frac{\partial_y^2 R^\epsilon}{R^\epsilon} - V \right\},
\]
\[
\text{(II.17)}
\]
which is completely equivalent to the 2-particle Schrödinger equation (II.4) for any $\epsilon > 0$. We claim that this system has a regular $\epsilon \searrow 0$ limit. To see this, we observe, from repeated use of the relations (II.6)-(II.12), that the following terms that appear to be singular are in fact regular:

$$
\partial_\theta (\exp(-\ln(\epsilon) \hat{A}) \theta') = \partial_\theta \left( \frac{1}{\epsilon} \hat{A} \theta' \right) = \partial_\theta \left( \frac{1}{\epsilon} \hat{A} \theta' \right) = \partial_\theta \left( \frac{1}{\epsilon} (\epsilon \hat{A} + \hat{B}) \theta' \right) = U,
$$

$$
\exp(\ln(\epsilon) \hat{A}) \left( \partial_\theta (\exp(-\ln(\epsilon) \hat{A}) \theta') \right)^2 = \exp(\ln(\epsilon) \hat{A}) \left( \frac{1}{\epsilon} \hat{A} \theta' \right)^2 + \frac{2}{\epsilon} \partial_\theta \hat{A} \theta' \partial_\theta \theta' + \left( \partial_\theta \theta' \right)^2 = \exp(\ln(\epsilon) \hat{A}) \left( \frac{1}{\epsilon} \hat{A} \theta' \right)^2 + \frac{2}{\epsilon} \partial_\theta \hat{A} \theta' \partial_\theta \theta' + \left( \partial_\theta \theta' \right)^2.
$$

Using (II.18), (II.19) and (II.20) along with the relations (II.6)-(II.12), it is not difficult to verify that (II.16)-(II.17) can be written as

$$
\partial_\theta R' = -\frac{1}{m_1} (\partial_\theta \hat{A} \theta' + \epsilon \partial_\theta \hat{B} \theta') \partial_\theta R' - \frac{1}{2m_1} (\partial_\theta \hat{A} \theta' + \epsilon \partial_\theta \hat{B} \theta') R' - \frac{1}{2m_1} \partial_\theta \theta' \partial_\theta \theta' - \frac{1}{2m_1} \partial_\theta \theta' \partial_\theta \theta' - \frac{1}{2m_1} \partial_\theta \theta' \partial_\theta \theta',
$$

$$
\partial_\theta \theta' = \frac{1}{2m_1} \left[ (\partial_\theta \hat{A} \theta')^2 + \epsilon^2 \hat{A} \left( \partial_\theta \hat{B} \theta' \right)^2 + 2 \partial_\theta \hat{A} \theta' \partial_\theta \hat{B} \theta' + \epsilon \hat{B} \left( \partial_\theta \hat{B} \theta' \right)^2 \right] - U + \epsilon \left[ \frac{1}{2m_1} \frac{\partial_\theta^2 R'}{R'} - \frac{1}{2m_1} \left( \partial_\theta \theta' \right)^2 + \frac{1}{2m_1} \frac{\partial_\theta^2 R'}{R'} - V \right].
$$

Setting

$$
\theta_A' := \hat{A} \theta' \quad \text{and} \quad \theta_B' := \hat{B} \theta',
$$

we see after applying the projection operators $\hat{A}$ and $\hat{B}$ to (II.22), and again using the relations (II.6)-(II.12) that (II.21)-(II.22) are equivalent to

$$
\partial_\theta R' = -\frac{1}{m_1} (\partial_\theta \theta_A' + \epsilon \partial_\theta \theta_B') \partial_\theta R' - \frac{1}{2m_1} (\partial_\theta \theta_A' + \epsilon \partial_\theta \theta_B') R' - \frac{1}{2m_1} \partial_\theta \theta_A' \partial_\theta \theta' - \frac{1}{2m_1} \partial_\theta \theta_A' \partial_\theta \theta' - \frac{1}{2m_1} \partial_\theta \theta_A' \partial_\theta \theta',
$$

$$
\partial_\theta \theta_A' = \frac{1}{2m_1} \left[ (\partial_\theta \theta_A')^2 + \epsilon^2 \hat{A} \left( \partial_\theta \theta_B' \right)^2 \right] - U + \epsilon \left[ \frac{1}{2m_1} \frac{\partial_\theta^2 R'}{R'} - \frac{1}{2m_1} \left( \partial_\theta \theta_A' \right)^2 + \frac{1}{2m_1} \frac{\partial_\theta^2 R'}{R'} - V \right],
$$

$$
\partial_\theta \theta_B' = \hat{B} \left[ -\frac{1}{2m_1} \left( 2 \partial_\theta \theta_A' \partial_\theta \theta_B' + \epsilon \partial_\theta \theta_B' \right)^2 + \frac{1}{2m_1} \frac{\partial_\theta^2 R'}{R'} - \frac{1}{2m_1} \left( \partial_\theta \theta_B' \right)^2 + \frac{1}{2m_1} \frac{\partial_\theta^2 R'}{R'} - V \right].
$$

It is worth noting that we can treat $\theta_A' = \theta_A'(t, x)$ and $\theta_B' = \theta_B'(t, x, y)$ as independent variables provided that the initial data for $\theta_B$ at $t = 0$ is chosen to satisfy the constraint $\hat{A} \theta_B' = 0$. This is possible because it follows easily from (II.26) that the constraint $\hat{A} \theta_B' = 0$ propagates, i.e. satisfies $\partial_t \hat{A} \theta_B' = 0$, from which it follows that the function $\theta'$ defined by $\theta'(t, x, y) = \theta_A'(t, x) + \theta_B'(t, x, y)$ verifies the relations $\theta_A' = \hat{A} \theta'$ and $\theta_B' = \hat{B} \theta'$ in accordance with (II.23).

To summarize, the two systems (II.21)-(II.22) and (II.24)-(II.26) represent equivalent formulations of the 2-particle Schrödinger equation (II.4) under the transformation

$$
\psi' = e^{i \left( \frac{1}{\epsilon} \theta_A' + \theta_B' \right) R'},
$$

where

$$
\theta' = \theta_A' + \theta_B',
$$

and $\theta_A'$ and $\theta_B'$ are given in terms of $\theta'$ by (II.23). Moreover, it is clear from (II.27), that if $\chi_\Omega$ is the characteristic function of a subset $\Omega \subset \mathbb{R}^2$, i.e. $\chi_\Omega(x, y) = 1$ if $(x, y) \in \Omega$ and 0 otherwise, then the action of the operator $\chi_\Omega(\hat{x}, \hat{y})$ on the wave function $\psi'$ is equivalent to the following action on the pair $(\theta', R')$:

$$
\chi_\Omega(\hat{x}, \hat{y})(\theta', R')(x, y) = (\theta'(x, y), \chi_\Omega(x, y)R'(x, y)).
$$

From this, we conclude that position measurements are represented by $\epsilon$-independent maps in our non-local polar representation.
that is common to both quantum and classical systems. This can be seen by introducing the velocity field equations given by the conservation law for the probability density coupling term. It is also apparent that equation (II.33) is closely related to the quantum mechanical equation satisfied by the phase as a classical configuration coordinate. Further, we observe that the first equation (II.31) is a quantum mechanical solution of the 2-particle Schrödinger equation (II.4). From this starting point, higher order expansion terms in $\epsilon$ can be derived in a standard fashion. We derive the equations satisfied by the first order correction in Section IV.

B. The limit equations

Taking the formal $\epsilon \downarrow 0$ limit (II.21)-(II.22) and (II.24)-(II.26), we obtain the two equivalent forms of the limit equations given by

$$\partial_t R = -\frac{1}{m_1} \partial_x(A\theta)\partial_x R - \frac{1}{2m_1} \partial_x^2 (A\theta) R - \frac{1}{m_2} \partial_y \theta \partial_y R - \frac{1}{2m_2} \partial_y^2 \theta R,$$

$$\partial_t \theta = -\frac{1}{2m_1}\left[\left(\partial_x \hat{A}\theta\right)^2 + 2\partial_x \hat{A}\partial_x \partial_y \theta\right] - U + \hat{B} \left[-\frac{1}{2m_2} \left(\partial_y \theta\right)^2 + \frac{1}{2m_2} \frac{\partial^2 R}{R} - V\right],$$

and

$$\partial_t R = -\frac{1}{m_1} \partial_x \theta A \partial_x R - \frac{1}{2m_1} \partial_x^2 \theta A R - \frac{1}{m_2} \partial_y \theta B \partial_y R - \frac{1}{2m_2} \partial_y^2 \theta B R,$$

$$\partial_t \theta_A = -\frac{1}{2m_1} \left(\partial_x \theta_A\right)^2 - U,$$

$$\partial_t \theta_B = -\frac{1}{m_1} \partial_x \theta_A \partial_x \theta_B + \hat{B} \left[-\frac{1}{2m_2} \left(\partial_y \theta_B\right)^2 + \frac{1}{2m_2} \frac{\partial^2 R}{R} - V\right],$$

respectively. In the second formulation given by (II.31)-(II.33), we can, as above, treat $\theta_A = \theta_A(t, x)$ and $\theta_B = \theta_B(t, x, y)$ as independent variables provided that we choose initial data for $\theta_B$ satisfying $\hat{A}\theta_B|_{t=0} = 0$.

These limit equations define our notion of a classical-quantum limit. The classical nature of the limit system is clear from equation (II.32), which is of the form of a Hamilton-Jacobi equation from classical mechanics. We note that it only depends on the configuration coordinate $x$ of the first system, which allows us to interpret the first system as being classical and $x$ as a classical configuration coordinate. Further, we observe that the first equation (II.31) is the conservation law for the probability density

$$\rho = R^2$$

that is common to both quantum and classical systems. This can be seen by introducing the velocity field

$$v = (v_1, v_2) := \left(\frac{1}{m_1} \partial_x \theta_A, \frac{1}{m_2} \partial_y \theta_B\right),$$

which allows us to write (II.31) as the conservation law

$$\partial_t \rho + \text{div}(v\rho) = 0.$$
From similar results for the classical limit (see \(3\)), we can expect that time interval \([0, T]\) on which solutions to the limit equations will approximate the fully quantum system will be determined by the time interval for which the limit equations admit smooth solutions. As we establish below in Section IV this time interval is determined by the time of existence for the classical Hamiltonian flow associated to the Hamilton-Jacobi equation (II.32) and the time of existence of smooth solutions to (II.32), which is well known to be governed by the formation of caustics. Presumably, in line with known results for the classical limit, see again \(3\), it should be possible using a more complicated multi-phase construction to extend the time of the validity of the approximation to arbitrary time intervals. We will not consider this aspect further here and instead leave it for future consideration.

C. No quantum backreaction

In the setting of quantum-classical hybrid schemes, the notion of quantum backreaction refers to the effect of the quantum subsystem on the classical one. Here, we show that the classical-quantum limit equations derived above do not allow for quantum backreaction. This is in concordance with the theoretical studies carried out in \(6, 26–28\). Indeed, given a solution

\[ \rho_1(t, x) = \int_R \rho(t, x, y) dy, \]

which contains all of the statistical information relating to the classical system. It follows from integrating (II.34) together with the fact that \(v_1\) is independent of \(y\) that \(\rho_1\) satisfies

\[ \partial_t \rho_1 + \partial_x (v_1 \rho_1) = 0. \]

Noting that this equation and (II.32) form a closed system of evolution equations for \((\theta_A, \rho_1)\), it follows immediately that the quantum subsystem does not have any effect on the classical one and establishes the absence of a quantum backreaction.

D. A local formulation

Due to the appearance of the averaging operator in the limit equations, they are manifestly nonlocal, and consequently, difficult to interpret, and from a computational point of view, are more expensive to solve compared to local systems. Fortunately, solutions to the limit equations can be obtained from solutions of the following local system:

\[
\begin{align*}
\partial_t \theta_A &= -\frac{1}{m_1} \partial_x \theta_A \partial_x R - \frac{1}{2m_1} \partial^2_x \theta_A R - \frac{1}{m_2} \partial_y \tilde{\theta}_B \partial_y R - \frac{1}{2m_2} \partial^2_y \tilde{\theta}_B R, \quad \text{(II.35)} \\
\partial_t \tilde{\theta}_B &= -\frac{1}{m_1} \partial_x \theta_A \partial_x \tilde{\theta}_B - \frac{1}{2m_2} (\partial_y \tilde{\theta}_B)^2 - V.
\end{align*}
\]

Indeed, given a solution

\[ (\theta_A = \theta_A(t, x), \tilde{\theta}_B = \tilde{\theta}_B(t, x, y), R = R(t, x, y)) \]
of (II.35)-(II.37), a straightforward computation using the relations (II.10)-(II.12) establishes that

\[
(\theta_A, \theta_B := \tilde{\theta}_B, R)
\]

is a solution of (II.31)-(II.33).

We can further simplify the local system (II.35)-(II.37) by introducing a Lagrangian coordinate for the classical subsystem that is adapted to the vector field

\[
v_1(t, x) = \frac{1}{m_1} \partial_x \theta_A(t, x),
\]

which determines the motion of the classical particle. The Lagrangian transformation is defined using the flow \(F(t, x)\) of \(v_1(t, x)\), which is uniquely determined by the initial value problem

\[
\partial_tF(t, x) = v_1(t, F(t, x)),
\]

\[
F(0, x) = x.
\]

We note that the initial condition (II.41) implies that

\[
\partial_x F(0, x) = 1,
\]

and this, in turn, guarantees that the condition

\[
\partial_x F(t, x) > 0
\]

holds on some time interval \([0, T]\). The condition (II.42) is precisely the necessary condition for the Lagrangian coordinate transformation to be invertible; for the remainder of this section, we will assume that (II.42) is satisfied.

Defining

\[
\alpha_A(t, x) := \theta_A(t, F(t, x)),
\]

\[
\alpha_B(t, x, y) := \tilde{\theta}_B(t, F(t, x), y)
\]

and

\[
\mathcal{R}(t, x, y) := R(t, F(t, x), y) \sqrt{\partial_x F(t, x)},
\]

it follows directly from (II.39), (II.40) and the chain rule that

\[
\partial_t \alpha_A(t, x) = \partial_t \theta_A(t, F(t, x)) + \frac{1}{m_1} (\partial_x \theta_A(t, F(t, x)))^2,
\]

\[
\partial_t \alpha_B(t, x, y) = \partial_t \tilde{\theta}_B(t, F(t, x), y) + \frac{1}{m_1} \partial_x \tilde{\theta}_B(t, F(t, x), y) \partial_x \theta_A(t, F(t, x))
\]

and

\[
\partial_t \mathcal{R}(t, x, y) = \left[ \partial_t \mathcal{R}(t, F(t, x), y) + \frac{1}{m_1} \partial_x \mathcal{R}(t, F(t, x), y) \partial_x \theta_A(t, F(t, x)) \
\right.
\]

\[
\left. + \frac{1}{2m_1} \partial_x \mathcal{R}(t, F(t, x), y) \partial_x^2 \theta_A(t, F(t, x)) \right] \sqrt{\partial_x F(t, x)}.
\]

Given a solution

\[
(\theta_A = \theta_A(t, x), \tilde{\theta}_B = \tilde{\theta}_B(t, x, y), R = R(t, x, y))
\]

of (II.35)-(II.37), we see from (II.46)-(II.48) that the pair

\[
(\alpha_B = \alpha_B(t, x, y), \mathcal{R} = \mathcal{R}(t, x, y))
\]

satisfies the equations

\[
\partial_t \mathcal{R} = -\frac{1}{m_2} \partial_y \alpha_B \partial_y \mathcal{R} - \frac{1}{m_2} \partial_y^2 \alpha_B \mathcal{R},
\]

\[
\partial_t \alpha_B = -\frac{1}{m_2} (\partial_y \alpha_B)^2 + \frac{1}{2m_2} \frac{\partial_y^2 \mathcal{R}}{\mathcal{R}} - \tilde{V}
\]
where

\[ \tilde{V}(t, x, y) := V(F(t, x), y), \]

which we recognize as the polar representation of Schrödinger’s equation. This establishes that the wave function \( \tilde{\psi} \) defined by

\[ \tilde{\psi}(t, x, y) := e^{i\alpha B(t, x, y)}R(t, x, y) \quad (\text{II.49}) \]

satisfies the time-dependent Schrödinger equation

\[ i\partial_t \tilde{\psi} = -\frac{1}{2m_2} \partial_y^2 \tilde{\psi} + \tilde{V}. \quad (\text{II.50}) \]

The above equation shows that the \( y \) variable is best interpreted as the configuration variable of a 1-particle quantum subsystem in which the motion of the classical particle appears as a time dependent parameter in the potential \( \tilde{V}(t, x, y) = V(F(t, x), y) \). If \( V \) is chosen independent of \( x \), then \( \tilde{V} = V(y) \) and (II.50) reduces to precisely a 1-particle Schrödinger equation.

We can further simplify the equation (II.50) by replacing the flow \( F(t, x) \) with the Hamiltonian flow associated with the Hamilton-Jacobi equation (II.36). To see this, consider the Hamiltonian

\[ H_1(x, p) = \frac{1}{2m_1} p^2 + U(x) \]

and let

\[ \mathcal{F}(t, x, p) = (X(t, x, p), P(t, x, p)) \]

denote the corresponding Hamiltonian flow, i.e. \( \mathcal{F} \) satisfies

\[ \partial_t X(t, x, p) = \frac{\partial H_1}{\partial p}(\mathcal{F}(t, x, p)), \]
\[ \partial_t P(t, x, p) = -\frac{\partial H_1}{\partial x}(\mathcal{F}(t, x, p)), \]
\[ \mathcal{F}(0, x, p) = (x, p). \]

Then from the method of characteristics, see [12, §3.2 & 3.3], it follows that

\[ F(t, x) = X(t, x, \partial_x \theta_A(0, x)), \]

and that the solution \( \theta_B \) to the Hamilton-Jacobi equation (II.36) is given in terms of the Hamiltonian flow by the formula

\[ \theta_A(t, x) - \theta_A(0, G(t, x)) + \int_0^t \left( \frac{1}{2m_1} |P(\tau, \tilde{x}, \partial_x \theta_A(0, \tilde{x}))|^2 - U(F(\tau, \tilde{x})) \right) d\tau \bigg|_{\tilde{x} = G(t, x)} \quad (\text{II.55}) \]

where \( G(t, x) \) is for constant \( t \) the inverse of \( F(t, x) \), i.e. \( G(t, F(t, x)) = x \).

From the above calculations, it is clear that solutions to the classical-quantum limit system can be generated from solutions to the time-dependent 1-particle Schrödinger equation (II.50) and the classical Hamiltonian flow equations (II.51)–(II.53) via the transformations (II.38), (II.44), (II.45), (II.49), and (II.54), and the formula (II.55). In this way, we can view equations (II.50) and (II.51)–(II.53) as an equivalent formulation of the classical-quantum limit equations at least on time intervals for which \( F(t, x) \) is invertible, that is before the onset of caustics. Moreover, since (II.50) is linear, it is the time of existence for the classical Hamiltonian flow and the formation time for caustics that limit our ability to solve the classical-quantum limit equations.

III. THE ORIGIN OF NONLOCAL SIGNALLING IN THE HALL-REGINATTO HYBRID SCHEME

To understand the origin of the nonlocal signalling in the Hall-Reginatto hybrid scheme, we consider the case where the classical-quantum limit equations (II.29)–(II.30) arise from the limit of two commuting 1-particle Hamiltonians by assuming that the potential \( V \) is a function of \( y \) only, that is

\[ V = V(y). \]
We then write the classical-quantum limit equations as
\[
\partial_t \left( \frac{\rho}{\theta} \right) = X_C(\rho, \theta) + X_Q(\theta, \theta) \tag{III.1}
\]
where
\[
X_C(\rho, \theta) = \left( -\partial_x \left( \frac{1}{m_1} (\partial_x \hat{A} \rho) \right) - \frac{1}{2m_1} \left( (\partial_x \hat{A} \theta)^2 + 2\partial_x \hat{A} \theta \partial_x \theta \right) - U \right)
\]
and
\[
X_Q(\rho, \theta) = \left( -\partial_y \left( \frac{1}{m_2} (\partial_y \theta) \rho \right) \right)
\]
\[
\left( B - \frac{1}{2m_2} (\partial_y \theta)^2 + \frac{1}{2m_2} \frac{\partial^2 \theta}{\sqrt{\theta}} \right) - V \right).
\]

Since equation (III.1) arises from the limit of a 2-particle quantum system whose evolution is governed by the sum of the two commuting 1-particle Hamiltonian operators
\[
\hat{H}_1 = -\frac{\epsilon}{2m_1} \partial_x^2 + \frac{1}{\epsilon} U \quad \text{and} \quad \hat{H}_2 = -\frac{1}{2m_2} \partial_y^2 + V,
\]
and the \(\epsilon \to 0\) limit of the 1-particle Hamiltonian operators \(\hat{H}_1\) and \(\hat{H}_2\), in our variables, are given by the two vector fields (III.2) and (III.3), respectively, it follows that the flows of these vector fields must commute, and moreover, from the discussion above, that \(X_C\) and \(X_Q\) generate the evolution of independent classical and quantum subsystems, respectively. Thus, if we let \(F_{t}^{X_C}\), \(F_{t}^{X_Q}\), and \(F_{t}^{X_C+X_Q}\) denote the flows of the vector fields \(X_C\), \(X_Q\), and \(X_C + X_Q\), respectively, then \(F_{t}^{X_C}\) and \(F_{t}^{X_Q}\) must commute, i.e.
\[
F_{t}^{X_C} \circ F_{s}^{X_Q} = F_{s}^{X_Q} \circ F_{t}^{X_C}, \tag{III.4}
\]
and \(F_{t}^{X_C+X_Q}\) is given by
\[
F_{t}^{X_C+X_Q} = F_{t}^{X_C} \circ F_{t}^{X_Q}, \tag{III.5}
\]
which allows us to write solutions of (III.1) as
\[
\begin{pmatrix} R(t) \\ \theta(t) \end{pmatrix} = F_{t}^{X_C} \circ F_{t}^{X_Q} (R(0), \theta(0)).
\]
The two properties (III.4) and (III.5) reflect nothing more than the independence of the evolution of the classical and quantum subsystems.

Next, since the quantum system satisfies the no non-local signalling property with respect to position measurements of the 1-particle subsystems by virtue of the evolution being generate by the sum \(\hat{H}_1 + \hat{H}_2\) of commuting Hamiltonian operators, it follows automatically that the limit system must also satisfy this property since the position measurements are represented by \(\epsilon\) independent maps in our formalism. In contrast to our equations (III.1) for non-interacting classical-quantum systems, those of Hall and Reginatto are given by (II.13)
\[
\partial_t \left( \frac{\rho}{\theta} \right) = X_{HR}(\rho, \theta) \tag{III.6}
\]
where
\[
X_{HR}(\rho, \theta) = \left( -\partial_x \left( \frac{1}{m_1} (\partial_x \theta) \rho \right) \right)
\]
\[
\left( -\partial_y \left( \frac{1}{m_2} (\partial_y \theta) \rho \right) \right)
\]
\[
\left( -\frac{1}{2m_1} (\partial_x \theta)^2 - \frac{1}{2m_2} (\partial_y \theta)^2 + \frac{1}{2m_2} \frac{\partial^2 \theta}{\sqrt{\theta}} - (U + V) \right).
\]
These equations are known (II.16) to suffer from a non-local signalling effect with respect to position measurements.
Writing the Hall-Reginatto equations as

$$\partial_t \left( \frac{\theta}{\theta} \right) = X_C(\rho, \theta) + X_Q(\rho, \theta) + X_I(\rho, \theta)$$

where

$$X_I(\rho, \theta) := X_{HR} - X_C - X_Q \equiv \left( -\frac{1}{2m_1} \left( \frac{1}{m_1} \left( \partial_x (\hat{B} \theta) \rho \right) \right) \right) + \frac{1}{2m_2} \left( \epsilon \partial_y \left( \frac{\partial_x^2 \theta + \partial_y^2 \omega^s} {R + \epsilon \mu^s} \right) \right),$$

we see that $X_I$ represents a non-local interaction term that alters the evolution generated by the vector field $X_C + X_Q$. From this, we conclude that the non-local interaction term $X_I$ is responsible for the non-local signaling effect in the Hall-Reginatto equations \[14,15,25\]. It is also clear from the above calculations that we can obtain any of Hall-Reginatto equations in an $\epsilon$-expansion. For the purpose of improving the approximation and also determining the backreaction of the second subsystem on the first, it is necessary to include the effects of higher order corrections. In this section, we derive the equations satisfied by the first order correction.

IV. THE FIRST ORDER CORRECTION

As discussed above, the classical-quantum limit equations derived in Section II.B may be viewed as the zeroth order equations in an $\epsilon$-expansion. For the purpose of improving the approximation and also determining the backreaction of the second subsystem on the first, it is necessary to include the effects of higher order corrections. In this section, we derive the equations satisfied by the first order correction.

We begin the derivation by decomposing $R^\epsilon$, $\theta_A^\epsilon$, and $\theta_B^\epsilon$ as

$$R^\epsilon = R + \epsilon \mu^s, \quad \theta_A^\epsilon = \theta_A + \epsilon \nu^s$$

and

$$\theta_B^\epsilon = \theta_B + \epsilon \omega^s$$

where $\nu^s$ and $\omega^s$ satisfy $\hat{A} \nu^s = \nu^s$ and $\hat{A} \omega^s = 0$, respectively, and, as above, \[(R, \theta_A, \theta_B)\] satisfy the classical-quantum limit equations \[11.31-11.33\]. Substituting \[(IV.1)\] into the equations \[11.23-11.26\], we find, with the help of \[11.31-11.33\], that the triple \[(mu^s, \nu^s, \omega^s)\] satisfies the following equations:

$$\partial_t \mu^s = \frac{1}{m_1} \left( \partial_x \theta_A \partial_x \mu^s + \epsilon \partial_x \nu^s \left( \partial_x \nu^s + \partial_x \theta_B + \epsilon \partial_x \omega^s \right) \left( \partial_x R + \epsilon \partial_x \mu^s \right) \right) - \frac{1}{2m_1} \left( \partial_x \theta_A \mu^s \right) + \epsilon \partial_x \nu^s \left( \partial_x \nu^s + \partial_x \theta_B + \epsilon \partial_x \omega^s \right) \left( \partial_x R + \epsilon \partial_x \mu^s \right) \right)

+ \frac{1}{m_2} \left( \partial_x \theta_B \left( \partial_x \nu^s + \partial_x \theta_B + \epsilon \partial_x \omega^s \right) \left( \partial_x R + \epsilon \partial_x \mu^s \right) \right) - \frac{1}{2m_2} \left( \partial_x \theta_B \left( \partial_x \nu^s + \partial_x \theta_B + \epsilon \partial_x \omega^s \right) \left( \partial_x R + \epsilon \partial_x \mu^s \right) \right),$$

$$\partial_t \nu^s = \frac{1}{m_1} \left( 2 \partial_x \theta_A \partial_x \nu^s + \epsilon \left( \partial_x \nu^s + \partial_x \theta_B + \epsilon \partial_x \omega^s \right) \left( \partial_x R + \epsilon \partial_x \mu^s \right) \right)

+ \frac{1}{2m_1} \left( \epsilon \partial_x \theta_B \left( \partial_x \nu^s + \partial_x \theta_B + \epsilon \partial_x \omega^s \right) \left( \partial_x R + \epsilon \partial_x \mu^s \right) \right),$$

$$\partial_t \omega^s = \hat{B} \left[ \frac{1}{m_1} \left( 2 \partial_x \theta_A \partial_x \omega^s + \partial_x \nu^s \partial_x \theta_B + \epsilon \partial_x \nu^s \partial_x \nu^s + \left( \partial_x \theta_B + \epsilon \partial_x \omega^s \right)^2 \right) + \frac{1}{2m_1} \left( \partial_x \theta_B \left( \partial_x \nu^s + \partial_x \theta_B + \epsilon \partial_x \omega^s \right) \left( \partial_x R + \epsilon \partial_x \mu^s \right) \right) \right)

- \frac{1}{2m_2} \left( \partial_x \theta_B \partial_x \omega^s + \epsilon \left( \partial_x \nu^s + \partial_x \theta_B + \epsilon \partial_x \omega^s \right) \left( \partial_x R + \epsilon \partial_x \mu^s \right) \right),$$

We observe that the equations \[11.31-11.33\] and \[(IV.2)-(IV.4)\] for the variables \[(R, \theta_A, \theta_B, \mu^s, \nu^s, \omega^s)\] are equivalent to the Schrödinger equation \[11.4\] via the transformation

$$\psi = e^{i \left( \frac{\theta_A + \nu^s + \theta_B + \epsilon \omega^s}{R + \epsilon \mu^s} \right)} (R + \epsilon \mu^s).$$
Taking the formal $\epsilon \downarrow 0$ limit of the equations (IV.2)-(IV.3), we obtain the equations
\[
\partial_t \mu = -\frac{1}{m_1} \left( \partial_x \theta_A \partial_x \mu + (\partial_x \nu + \partial_x \theta_B) \partial_x R \right) - \frac{1}{2m_1} \left( \partial_{\mu}^2 \theta_A \mu + \left( \partial_{\nu}^2 \nu + \partial_{\theta_B}^2 \theta_B \right) R \right) - \frac{1}{2m_1} \partial_x \theta_B \partial_y \mu - \frac{1}{2m_1} \partial_{\nu}^2 \theta_B \mu, \tag{IV.5}
\]
\[
\partial_t \nu = -\frac{1}{m_1} \partial_x \theta_A \partial_x \nu + \hat{A} \left[ -\frac{1}{2m_2} (\partial_y \theta_B)^2 + \frac{1}{2m_2} \frac{\partial_{\nu}^2 R}{R} \right], \tag{IV.6}
\]
\[
\partial_t \omega = \hat{B} \left[ -\frac{1}{2m_1} \left( 2\partial_x \theta_A \partial_x \omega + 2\partial_x \nu \partial_x \theta_B + (\partial_x \theta_B)^2 \right) \right. \nonumber
\]
\[
\left. + \frac{1}{2m_1} \frac{\partial_{\nu}^2 R}{R} - \frac{1}{m_2} \partial_x \theta_B \partial_y \omega + \frac{1}{2m_2} \left( \frac{\partial_{\nu}^2 \mu}{R} - \frac{\mu \partial_{\nu}^2 R}{R^2} \right) \right]. \tag{IV.7}
\]
These equations define the first order correction to the limit equations, and we can expect, if $(R^c, \theta_A^c, \theta_B^c)$ is a solution of the full quantum mechanical system, then
\[
R^c = R + \epsilon \mu + O(\epsilon^2), \quad \theta_A^c = \theta_A + \epsilon \nu + O(\epsilon^2) \quad \text{and} \quad \theta_B^c = \theta_B + \epsilon \omega + O(\epsilon^2)
\]
on some fixed time interval $[0,T]$ where $(R, \theta_A, \theta_B)$ solves (II.31)-(II.33) and $(\mu, \nu, \omega)$ solves (IV.5)-(IV.7). Higher order corrections can be determined in the usual manner.

We emphasis that the system consisting of the equations (II.31)-(II.33) and (IV.5)-(IV.7) represents a fully consistent first order approximation to the Schrödinger equation that should be accurate for modeling the interaction between particles of disparate masses in the regime where the large mass particles are effectively classical. We expect these equations should be of use in computational chemistry and also the theory of measurement where interactions between quantum and classical systems play a distinguished role.

V. DISCUSSION AND OUTLOOK

In this article, we have introduced a new approach to analyzing the interaction between classical and quantum systems that is based on taking the classical-quantum limit of multi-particle Schrödinger equations. The classical-quantum limit equations obtained by this procedure describe the interaction between classical and quantum systems and possess many desirable properties that are inherited from the multi-particle quantum systems.

We considered two applications of the classical-quantum limit equations with the first being to identify the source of the non-local signalling that is present in the “non-interacting” classical-quantum hybrid equations of Hall and Reginatto. More specifically, we showed that the non-local signalling in the Hall-Reginatto scheme is due to a non-local correction to our classical-quantum limit equations, which we contend should be taken as the “true” form for the equations that govern non-interacting classical-quantum systems in the Hall-Reginatto hybrid scheme.

In the second application, we viewed the classical-quantum limit equations as the equations satisfied by the zeroth order expansion in $\epsilon$ of a fully quantum solution. We then, in the standard fashion, derived the system of equations satisfied by the first order correction. Together, the limit equations and their first order correction determine a fully consistent first order approximation to the multi-particle Schrödinger equation. We expect that these equations will be useful for modeling the interaction between particles of disparate masses in the regime where the large mass particles are effectively classical.

The results derived in this article can be taken in many directions. From the point of view of concrete applications, the most important is to develop analytical and numerical techniques to solve the limit equations (II.31)-(II.33) along with the first order corrections (IV.5)-(IV.7) in order to determine their suitability for modelling systems that consist of nearly classical subsystems interacting with fully quantum ones. Important examples include molecular reactions and the interaction of measuring devices with quantum systems. We are currently working on this and hope to have progress to report on in the near future.

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Appendix A: Equivalence of the classical-quantum limits

As discussed in Remark II.2, there is a classical-quantum limit associated to each operator satisfying (II.6)-(II.8). In this appendix, we show that these classical-quantum limits are equivalent by showing that there is an invertible transformation between the limits associated to different operators. An immediate consequence of this is that all choices of the operator $\hat{A}$ lead to physically equivalent classical-quantum limits.

1. Transformation formula

To establish the transformation formula and its invertibility, we suppose that $\hat{A}$ and $\hat{A'}$ are two operators that satisfy the properties (II.6)-(II.8), and we let $\hat{B} = \mathbb{I} - \hat{A}$ and $\hat{B'} = \mathbb{I} - \hat{A}'$ denote the complementary projection operators. Following the prescription (II.14), we define

\[
\theta := \epsilon \hat{A} \Theta + \hat{B} \Theta',
\]

\[
\theta' := \epsilon \hat{A}' \Theta + \hat{B} \Theta',
\]

where $\Theta$ and $\Theta'$ are as defined previously by the polar decomposition (II.13). These two definitions lead to two classical-quantum limits defined by

\[
(\theta, R) := \lim_{\epsilon \to 0} (\theta', R'),
\]

\[
(\theta, R') := \lim_{\epsilon \to 0} (\theta', R').
\]

From the relation (II.15), we observe that

\[
\frac{1}{\epsilon} \hat{A} \Theta + \hat{B} \Theta' = \Theta = \frac{1}{\epsilon} \hat{A} \Theta' + \hat{B} \Theta'.
\]

Applying $\hat{A}$ to this expression yields

\[
\frac{1}{\epsilon} \hat{A} \hat{A} \Theta + (\hat{A} - \hat{A}) \Theta' = \frac{1}{\epsilon} \hat{A} \Theta',
\]

where in deriving this we used the property $\hat{A}^2 = \hat{A}$. By assumption, $\hat{A}$ satisfies $\partial_y \hat{A} = 0$, and so, for any function $\xi = \xi(x, y)$, $\hat{A} \xi = \phi$ for some function $\phi = \phi(x)$. But since $\hat{A}(\phi) = \phi$ for all functions $\phi = \phi(x)$ by assumption, it follows that

\[
\hat{A} \hat{A} = \hat{A},
\]

and from similar considerations, that

\[
\hat{A} \hat{A} = \hat{A}.
\]

Using (A.5), we obtain from (A.4) that

\[
\hat{A} \theta' + \epsilon (\hat{A} - \hat{A}) \theta' = \hat{A} \theta'.
\]

Taking the limit $\epsilon \to 0$ gives

\[
\hat{A} \theta = \hat{A} \theta.
\]

Substituting (A.7) into (A.3) gives

\[
\hat{B} \theta' = \hat{B} \theta + (\hat{A} - \hat{A}) \theta',
\]

which, in turn, yields

\[
\hat{B} \theta = \hat{B} \theta + (\hat{A} - \hat{A}) \theta.
\]
by taking the limit $\epsilon \searrow 0$. Adding this to (A.8) then yields the relation
\[ \hat{\theta} = \hat{T} \theta \quad (A.9) \]
where the operator $\hat{T}$ is defined by
\[ \hat{T} = 1 + \hat{\Lambda} - \hat{\Lambda} . \quad (A.10) \]

The operator $\hat{T}$ is invertible with inverse given by
\[ \hat{T}^{-1} = 1 + \hat{\Lambda} - \hat{\Lambda} . \quad (A.11) \]
To see this, we observe that
\[ \hat{T}^{-1} \hat{T} = (1 + \hat{\Lambda} - \hat{\Lambda}) (1 + \hat{\Lambda} - \hat{\Lambda}) , \]
\[ = 1 + \hat{\Lambda} - \hat{\Lambda} + \hat{\Lambda} \hat{\Lambda} - \hat{\Lambda}^2 + \hat{\Lambda} \hat{\Lambda} , \quad (A.12) \]
where in deriving the last equality we used the identities $\hat{\Lambda}^2 = \hat{\Lambda}$, $\hat{\Lambda}^2 = \hat{\Lambda}$, $\hat{\Lambda} \hat{\Lambda} = \hat{\Lambda}$ and $\hat{\Lambda} \hat{\Lambda} = \hat{\Lambda}$. By a similar calculation, we also find that
\[ \hat{T} \hat{T}^{-1} = 1 . \quad (A.13) \]
Together, (A.12) and (A.13) show that $\hat{T}$ is invertible with inverse given by (A.11).

Equation (A.9) with $\hat{T}$ given by (A.10) defines the relationship between the two classical-quantum limits (A.1) and (A.2), and establishes the existence of an invertible transformation relating the two limits. As discussed above, this establishes the physical equivalence of the limits.

2. Examples of other operators satisfying (II.6)-(II.8)

There are many operators $\hat{\Lambda}$ that satisfy the properties (II.6)-(II.8) beyond those defined by (II.5). A more general class is defined by
\[ \hat{\Lambda}(\phi)(x, y) = \int \alpha(y) \phi(x, z) dz , \quad (A.14) \]
where $\alpha(y)$ is any function satisfying
\[ \int \alpha(y) dy = 1 . \quad (A.15) \]
We observe that we can recover the operators defined by (II.5) by setting
\[ \alpha(y) = \begin{cases} (b - a)^{-1} & \text{if } y \in (a, b) \\ 0 & \text{otherwise} \end{cases} . \quad (A.16) \]
It is also clear that by choosing functions $\alpha(y)$ satisfying (A.15) that are distinct from the step-functions (A.16) will yield operators that are not of the type (II.5), but nonetheless satisfy properties (II.6)-(II.8).

Another class of operators distinct from (A.14) is defined by
\[ \hat{\Lambda}(\phi)(x, y) = \phi(x, a) , \quad (A.17) \]
where, here, $a \in \mathbb{R}$ is an arbitrary, but fixed constant. From the Fundamental Theorem of Calculus, which holds for regular enough functions $\phi$, we see that
\[ \phi(x, a) = \frac{d}{dy} \int_a^y \phi(x, z) dz \bigg|_{y=a} = \lim_{b \searrow a} \frac{1}{b-a} \left( \int_a^b \phi(x, z) dz - \int_a^a \phi(x, z) dz \right) = \lim_{b \searrow a} \frac{1}{b-a} \int_a^b \phi(x, z) dz . \]
From this it is clear that the class (A.17) of operators is a limiting case of the operators defined by (II.15) or equivalently (A.14) with \( \alpha \) defined by (A.16).

It is worth noting at this point that although all operators \( \hat{A} \) satisfying the properties (II.6)-(II.8) are equivalent from the point of view of formal limits, the mathematical properties of the operators can be quite different. For example, the operators \( \hat{A} \) defined using averaging procedure (A.13) are well-defined, assuming appropriate integrability properties for the function \( \alpha \), on functions that are much rougher compared to those on which the operators defined via (A.17) can act. Considerations like this will likely become important in any attempt to go beyond formal considerations and rigorously establish the existence of the classical-quantum limit introduced in this article. We expect that only particular choices of the operators \( \hat{A} \) will be suitable for rigorous arguments.

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