Prime Suspects in a Quantum Ladder

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In this paper we set up a suggestive number theory interpretation of a quantum ladder system made of $N$ coupled chains of spin 1/2. Using the hard-core boson representation, we associate to the spins $\sigma_a$ along the chains the prime numbers $p_a$ so that the chains become quantum registers for square-free integers. The Hamiltonian of the system consists of a hopping term and a magnetic field along the chains, together with a repulsion rung interaction and a permutation term between next neighborhood chains. The system has various phases, among which there is one whose ground state is a coherent superposition of the first $N$ prime numbers. We also discuss the realization of such a model in terms of an open quantum system with a dissipative Lindblad dynamics.

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Introduction. Prime numbers are the building block of arithmetic and, arguably, one of the pillars of the entire mathematics [1–4]. Their nature has two fascinating but opposite features [5]: if their appearance in the sequence of natural numbers are rather unpredictable, their coarse graining properties (e.g. their total number $\pi(x)$ less than $x$) can be captured instead rather efficiently by simple statistical considerations [6–11]. In particular, the scaling of the $k$-th prime is particularly plain

$$p_k \approx k \log k.$$  

(1)

Equally fascinating is the connection between prime numbers and quantum mechanics: prime numbers, for instance, was the main concern of Shor’s algorithm, one of the first quantum computing algorithm [12]. Moreover, the scaling behaviour (1) permits to show the existence of a single-particle one-dimensional quantum mechanical potential $V(x)$ with eigenvalues given just by the prime numbers and therefore to address the primality test of a natural number in terms of a quantum scattering [13]: such a potential $V(x)$ can be determined either semiclassically [13] or exactly using methods of supersymmetric quantum mechanics [14,15]. In experimental setups of cold atom systems, $V(x)$ could be realised using a holographic trap [16].

Turning now the attention to quantum many-body systems, we consider here for the first time a quantum spin ladder system which has a suggestive number theory interpretation [17]. Such a system has a rich spectrum of ground states, in particular there is one where the ground state wave-function is given in terms of highly coherent superposition of prime number occupations. Quantum ladder systems, which consist of coupled one-dimensional chains, have attracted considerable interest in recent years as truly interpolating one- and two-dimensional systems [18–24]. In our case we have $N$ coupled legs, made of half-infinite chain of spins 1/2 subjected to hopping interaction and a magnetic field which increases along the chains. In such a system, as we are going to see, there is a simple way to put in correspondence the spins with the prime numbers and to reformulate the spin-spin rung interaction in terms of coprimality conditions (two integers are coprime if they do not share common factors other than 1).

Degrees of freedom and Hamiltonian. Let’s use the hard-core boson representation for spin 1/2, given by $\sigma_z = f^\dagger f - 1/2; \sigma_+ \equiv \sigma_x + i\sigma_y = f^\dagger; \sigma_- \equiv \sigma_x - i\sigma_y = f$. Hence, instead of the spins, as our degrees of freedom we will take the annihilation and creation operators $f_i(a)$ and $f_i^\dagger(a)$ of hard-core bosons, where the index $i$ refers to the $i$-th leg ($i = 1, 2, \ldots, N$), while $a = 1, 2, \ldots$ to the position along the half-infinite chain (see Fig. 1). Since $(f_i(a))^2 = (f_i^\dagger(a))^2 = 0$, each level has occupation numbers $\{0, 1\}$. Let’s $|\text{vac}\rangle$ be the vacuum state, i.e. the state which is annihilated by all the $f_i(a)$’s. For each chain we can then define the state

$$|n_i\rangle = \left( \prod_{a=1}^{k} (f_i^\dagger(a))^{\alpha_a} \right) |\text{vac}\rangle, \quad \alpha_a = \{0,1\}. $$  

(2)

Now, associating to the $a$-th level the $a$-th prime number,
on each chain we can also define a set of integers whose general form is
\[ n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}, \quad \alpha_a = \{0, 1\}. \] (3)
These are the so-called square-free numbers, i.e. those integers whose prime factors do not divide them more than once. Their first representatives are \( n = 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, \ldots \) Remarkably, these numbers are a finite fraction (i.e. 6/\( \pi^2 \)) of all the integers [9]; indeed, assuming 1/p to be the probability that a generic integer is divisible by a prime \( p \), the probability that it is not divisible more than once by a prime is given by
\[ \Pi_p \left( 1 - \frac{1}{p} \right) = 1/\zeta(2) = \frac{6}{\pi^2}, \] where \( \zeta(x) \) is the Riemann zeta function. So, in short, each leg plays the role of a quantum register for the square-free numbers.

Below we use the notation \( |p_n\rangle_i = f_i^T(a)|\text{vac}\rangle \). Other useful definitions are \( F_i(a) = f_i^T(a) f_i(a) \), (the number operators of the \( a \)-th hard-core boson on the \( i \)-th chain), \( F_i = \sum_{a=1} F_i(a) \) (the total number of hard-core bosons of the \( i \)-th leg) and \( F^{(a)}(a) = \sum F_i(a) \) (the total number of the \( a \) hard-core bosons on the entire ladder lattice). It is also convenient to introduce the numbers operators \( \hat{N}_i \) for each leg, such that
\[ \hat{N}_i |n_j\rangle = \delta_{i,j} n_j |n_j\rangle. \] (4)
It is worth stressing that the \( n_i \)'s, however, are not true occupation numbers (the actual occupation numbers at each leg are given on the contrary by the the \( F_i \)'s). The \( n_i \)'s may be regarded just as useful labels of the hard-core boson degrees of freedom present at each leg. Of course, one can also adopt the inverse point of view, i.e. to consider the \( n_i \)'s as the basic quantities of our system and the hard-core bosons, on the contrary, as basis of their “spectroscopy” [3]: as a matter of fact, there is a one-to-one correspondence between configurations of hard-core bosons and the \( n_i \)'s since, after all, a decomposition as [3] not only exists but is also unique.

The Hamiltonian of our system consists of two terms, relative to leg (L) and rung (R) interactions
\[ H = \sum_{i=1}^{N} [H_i + H_{i,i+1}] = H_L + H_R, \] (5)
where
\[ H_L = \sum_{i=1}^{N} H_i(b, \nu) = \sum_{i=1}^{N} \left[ \frac{\nu}{2} \sum_{a \neq b} J_{ab} \left( f_i^T(a) f_i(b) + f_i^T(b) f_i(a) \right) \right] + \sum_{a=1}^{\Lambda} \hat{h}(a) F_i(a), \] (6)
with \( \Lambda \) being a cut-off in the length of the chain (with \( \Lambda \rightarrow \infty \) taken firstly and independently of \( N \)) and
\[ H_R = \sum_{i=1}^{N} H_{i,i+1} = \sum_{i=1}^{N} \left[ P_{i,i+1} + \lambda \sum_{a=1}^{\Lambda} C_{i,i+1}(a) \right], \] (7)

with \( \lambda \geq 0 \), where \( P_{i,i+1} \) is the permutation operator between the two n.n. “occupation numbers” \( n_i \) defined in eq. (4), while \( C_{i,i+1}(a) \equiv F_i(a) F_{i+1}(a) \) are the coprimality operators. Each of these operators, in fact, probes whether two n.n. sites share the same flavour \( a \) and, if so, takes value 1 otherwise value 0. Hence, \( C_{i,i+1} = \sum_{a=1} n_i(a) \) counts how many common prime factors are shared between two n.n. numbers \( n(i) \) and \( n(i+1) \). As shown below, such Hamiltonian has quite a rich phase structure.

Let’s discuss first \( H_L \), namely \( H_i \), since \( H_L \) is a sum thereof. The first term in \( H_i \), associated to the magnetic field \( \hbar \), tends to localise the hard-core bosons around one of the \( |p_n\rangle \)'s, while the second term tends to delocalise them along the whole chain. In particular, for \( \nu / \hbar \rightarrow \infty \), the ground state of \( H_i \) (with periodic b.c., when \( J_{ab} \) is a circulant matrix), is the Prime state [20]
\[ |P_0\rangle_\Lambda = \frac{1}{\sqrt{\Lambda}} \left( |p_1\rangle + |p_2\rangle + \cdots |p_\Lambda\rangle \right), \] (8)
which is completely delocalised in the space of the primes. Equally delocalised are also the excited states
\[ |P_k\rangle = \frac{1}{\sqrt{\Lambda}} \left( |p_1\rangle + \omega^{1k} |p_2\rangle + \cdots \omega^{(\Lambda-1)k} |p_\Lambda\rangle \right), \] (9)
where \( \omega = e^{2\pi i / \Lambda} \). As shown in the Supplementary Material, for any assigned monotonic increasing function \( \hat{h}(a) \) of the local magnetic field, there is an associated function \( \hat{f}(a) \) which gives rise to a true competition between these two terms. Notice that the magnetization of the state (9) is given by \( M_{n_i} = \hbar \sum_a \alpha_a \hat{h}(a) \) and, depending on which function \( \hat{h}(a) \) has been chosen, some of its values may be degenerate. However, we can avoid once and for all any degeneracy in \( M_{n_i} \) by choosing for \( \hat{h}(a) \) the function \( \hat{h}(a) = \log p_a \) because in this case, as before, we can rely on the unique decomposition in terms of primes of any number \( n_i \) (see eq. (3))
\[ M_{n_i} = \hbar \sum_{a=1}^{k} \alpha_a \log(p_a) = \hbar \log n_i. \] (10)
With this choice of \( \hat{h}(a) \), for the hopping term we need to take \( J_{ab} = J_{|a-b|} = 1/|a - b| \). The competition between

![FIG. 2: IPR(x) vs x, with x/(1-x) = \nu/h, for the Hamiltonian \( H_i \), with \( \hbar = \log p_a \) and \( J_{ab} = 1/|a - b| \).](image)
the two terms in $H_1$ is well captured by studying the Inverse Partecipation Ratio of the ground states wave function in one particle sector $|\phi_0\rangle = \sum_{\alpha} c_{\alpha} |p_{\alpha}\rangle$, defined as IPR$(x) = 1/(\lambda \sum |c_{\alpha}|^2)$, where $x/(1-x) = \nu/h$. The IPR is indeed a good diagnosis of the transition between a localised phase (IPR $\approx 0$, for $x \to 0$) and a delocalised phase (IPR $\approx 1$, for $x \to 1$): the corresponding plot is in Figure 2.

Let’s now consider the rung Hamiltonian $\mathcal{H}_R$, given in (7). It clearly conserves the total numbers of each $n_i$: the Hilbert space is then partitioned in sectors $S_{\omega_1,\ldots,\omega_k}(u_1,\ldots, u_k)$, identified by a set of square-free numbers $(u_1, u_2, \ldots, u_k)$, with $k \leq \mathcal{N}$ and multiplicities $(\omega_1,\ldots,\omega_k)$ such that $\sum_{i=1}^k \omega_i = \mathcal{N}$. The dimensions of these sectors are $d(\omega_1,\ldots,\omega_k) = \frac{\mathcal{N}!}{\omega_1!\omega_2!\ldots\omega_k!}$. Notice that even though the number $\mathcal{N}$ of legs may be finite, there are nevertheless infinite sectors, which are obtained by varying both the set of the numbers $u_i$ and their multiplicity $\omega_i$. The Hamiltonian (7) also conserves the $a$-th numbers $F^{(a)}$ separately.

On a general ground, an Hamiltonian as (7) can be decomposed in block-forms according to the Irreducible Representation (IR) of the Symmetric Group $S_N$ given by Young Tableaux, and then each block diagonalised separately [27].

While this diagonalization procedure is in general highly elaborate, on the contrary it is quite easy to identify the two IR’s which gives rise to the highest and lowest energy states $E = \pm \mathcal{N}$: these are given respectively by the first and the last Young Tableaux in Fig. 3 relative to the fully symmetric and antisymmetric 1d representations $IR_S$ and $IR_A$. Note that while $IR_S$ always appears in the decomposition of any sector, on the contrary $IR_A$ only appears in the decomposition of those sectors where all $n_i$ are different numbers.

**Manifold of the ground states of $\mathcal{H}_R$.** Let’s consider the ground states of $\mathcal{H}(\lambda)$ varying $\lambda$. For $\lambda = 0$, the minimum energy is $E^* = -\mathcal{N}$ and this level is infinitely degenerate, since the Slater determinant built in terms of any set of $\mathcal{N}$ different square-free numbers $|n_i\rangle_i$:

$$|n^{(1)}, \ldots, n^{(\mathcal{N})}\rangle = \frac{1}{\sqrt{\mathcal{N}!}} \begin{vmatrix} |n^{(1)}\rangle_1 & \cdots & |n^{(\mathcal{N})}\rangle_1 \\ |n^{(1)}\rangle_2 & \cdots & |n^{(\mathcal{N})}\rangle_2 \\ \vdots & \ddots & \vdots \\ |n^{(1)}\rangle_\mathcal{N} & \cdots & |n^{(\mathcal{N})}\rangle_\mathcal{N} \end{vmatrix}$$  

(11)

gives rise to a ground state. For any finite number of legs $\mathcal{N}$, the probability $P$ to get one of these ground states is equal to 1, in other words their density is as much as the density of all states of the Hilbert space. Indeed, let $\Delta$ be a cut-off for the number of square free integers: on a lattice of $\mathcal{N}$ sites, the dimension of the Hilbert space is $D = \Delta^{\mathcal{N}}$ while the ground states are given by $\mathcal{N}$ different square-free integers, whose number is then $d = \Delta(\Delta - 1)(\Delta - 2)\cdots(\Delta - \mathcal{N} + 1)$. Hence, $P = d/D$ and, taking the limit $\Delta \to \infty$, we see that $P \to 1$, independently on the number $\mathcal{N}$ of legs.

However, switching on the coupling constant $\lambda > 0$, the associate coprimality term lifts the degeneracy of many of the previous ground states but leaves nevertheless several of them untouched: the restricted new set of ground states (which have the same minimum energy $E = -\mathcal{N}$ as before) is associated to square-free numbers $n_i$ which have to be not only different but also coprime each other! It is indeed the only way to minimise the coprime operator, because in this case the matrix elements of this operator simply vanish. As shown in the Supplementary Material, for $\lambda > 0$ the probability to get a ground state of the Hamiltonian (7) is thus given by

$$P = \prod_{p} \left[ \frac{1 + \mathcal{N} - 1}{p_a + 1} \left( 1 - \frac{1}{p_a + 1} \right)^{\mathcal{N}-1} \right].$$  

(12)

This quantity, contrary to the $\lambda = 0$ case, depends on $\mathcal{N}$ and rapidly decreases to 0 by increasing the number $\mathcal{N}$ of legs: in this case the numbers of ground states of $\mathcal{H}_R$ for large $\mathcal{N}$ is infinitesimally small with respect to the dimension of the Hilbert space.

**Phases of the system.** Let’s now discuss some of the phases of the full Hamiltonian (6). It is worth reminding that the Hamiltonian (6) conserves in general the total number $F = \sum_n F_n$ of hard-core bosons while its individual terms conserve different quantities, such as: (a) the permutation term conserves only $F$ but breaks all other quantum numbers, such as $F_i(a)$ or $F_i$, while the coprimality term commutes with all $F_i(a)$ and therefore also with $F_i$; (b) the magnetic field term also commutes with all $F_i(a)$ and then also with $F_i$, while the hopping term, on the other hand, commutes only with $F_i$ but violates the local quantum numbers $F_i(a)$. Let’s now probe the Hamiltonian (6) in some limit of its coupling constants.

It is easy to see that, taking $\nu \to \infty$ (keeping all other coupling constant fixed), the system goes into a ”stripe phase” described by the factorised ground state made of the Prime states (8):

$$|\Psi_0\rangle \simeq \otimes_i |P_i\rangle_i. $$  

(13)

Expanding each $|P_i\rangle_i$ in the prime basis, one can see that in this limit the ground state is made of equally weighted vectors of all possible sectors of the theory. The degeneracy of many of these terms will be eventually solved by taking into account the coprimality interaction and the magnetic field ($\lambda$ and $h$ both small compared to $\nu$). In this phase, factorised expressions also hold similarly for
the excited states, $|Ψ⟩_k ≃ ⊗_i |P_k⟩_i$. Assuming periodic boundary conditions and a cut-off $Λ$ along each leg, for the ground and excited state energies of this phase we get $E_k ≃ −νN e_k$ where, for $Λ → ∞$

$$e_0 ≃ −2 \left( \frac{\log{2} + \gamma_E}{2} + 1 + \cdots \right),$$
$$e_k ≃ \log \left[ 1 - \cos \left( \frac{2πk}{Λ} \right) \right] + \log{2}.$$ 

One can make these expressions finite by subtracting the leading divergent term $\log{Λ}/2$.

Let’s analyse another situation: taking now $h → ∞$ (and neglecting for simplicity the hopping term in $H_L$), the system goes instead into its “ordered phase”, characterised by an occupation number at each leg given by the lowest prime $p_1 = 2$

$$|Ψ⟩_0 ≃ ⊗ |2⟩_i$$

and ground state energy $E_{0}^{(ord)} ≃ N(h \log{2} + 1 + λ)$. For small $h$, the ground state is however unstable with respect the proliferation of other numbers $n_i$ (which replace some of the 2’s present). Indeed, when such numbers $n_i ≠ 2$ exist in some of the legs, the ground state energy tends to decrease for: (i) the presence of other IR’s in the decomposition of the permutation term of the Hamiltonian, in addition to IRs, the only present in the ordered phase; these IR typically have lower energy than $E = N$ (the rule of thumb being, the longer the Young Tableau in the vertical direction, the lower the corresponding minimum energy in that IR); (ii) a lower contribution coming from the coprimality term, since there are less pairs of equal particles. Imagine, for instance, to replace one of the 2’s in the ordered phase with a 3: the new IR needed in this case is the second Young Tableau (from left) in Figure 3 which, with periodic boundary conditions, has dimension $N$ and spanned by the $N$ vectors $(m = 1, 2, \ldots, N)$

$$|m⟩ = |2, 2, 2, \ldots, 3⟩_{m-\text{leg}}.$$  \hspace{1cm} (15)

The number 3 plays a role of a defect w.r.t. the ordered ground state. On the space spanned by these $N$ vectors, the term $∑_{i+1} P_{i,i+1}$ in the Hamiltonian has $|v_k⟩ = 1/√N e^{i km} |m⟩$ as eigenvectors and spectrum given by $E_k = N - 2 + 2 \cos{2πk/N}$, whose minimum is $E_{min} = (N - 4)$. The expectation value of the coprimality operators on the $|v_k⟩$ eigenvectors is simply $(N - 2)λ$.

So, putting together the two terms, the minimum energy in the defect sector is $E_{min}^{(def)} = N - 4 + λ(N - 2) + N' h \log{2} + h \log{(3/2)}$ (neglecting for simplicity the hopping term in $H_L$). Comparing now $E_{0}^{(ord)}$ with $E_{min}^{(def)}$ we can determine the minimum value of $h$, i.e. $h_c$, for which the ordered phase is stable

$$E_{0}^{(ord)} ≤ E_{min}^{(def)} \text{ if } h_c \log{(3/2)} ≥ 4 + 2λ \hspace{1cm} (16)$$

Let’s finally considering the limit in which $λ → ∞$ and $h → 0$ (in a way determined below), also imposing the extra condition $ν ≤ h$. In this case, the system prefers to go into a “prime-number phase”, which consists in minimizing simultaneously the permutation and the coprimality operators, adjusting accordingly the magnetization operators. A state which satisfies all these requirements consists of a coherent ground state made of a Slater determinant of the first $N$ primes. This condition defines a Fermi energy given by filling the first $N$ levels and its value is

$$E_F = ∑_{a=1}^{N} \log{p_a} = \log \left( ∏_{a=1}^{N} p_a \right) = \log{P(N)}, \hspace{1cm} (17)$$

where $P(N)$ is the primorial, i.e. the product of the first $N$ prime numbers. Since this quantity goes asymptotically as $P(N) ≃ e^{N log(N)}$, we have $E_F ≃ N \log{N}$. Hence the ground state energy in the “prime phase” is given by

$$E_{0}^{(prime)} = −N + hN \log{N}.$$

So, letting $h$ vanishing as $h ≃ 1/\log{N}$ for $N → ∞$ we have a ground state energy of the “prime phase” which scales linearly with the number of legs $N$. Hence, this state gives rise to a Fermi surface in terms of the first $N$ prime numbers. These primes are simultaneously present on each leg, being spread on the entire ladder system, although quantum coherently assembled by a Slater determinant, see eq. (11). As for other Fermi surfaces, there are soft modes above this ground state it: indeed, if we replace one prime number $p_c = p_{N-\epsilon}$ (inside and close to the Fermi surface) with other one $p_c = p_{N+\delta}$ (placed outside and close to it), the variation of the energy of the corresponding wave functions is simply

$$ΔE = log \frac{p_c}{p_{c-\epsilon}} ≃ \frac{\delta + \epsilon}{N}, \hspace{1cm} (19)$$

where we have used the scaling law (1). So, for a finite $N$, the system has a gap which however scales to zero as $1/N$ if we send the number of legs to infinity.

Lindbladian dynamics. A natural question is how the system is able to reach one of its ground states, let’s say the “prime ground state” given by the coherent superposition of the first $N$ primes. One way is to set up a dissipative dynamics able to efficiently “filter” such a ground state starting from an initial configuration made of an arbitrary mixture of excited states. This procedure can be implemented by choosing a suitable and optimized set of Lindblad operators (see, e.g. [29]) which induce a dissipative dynamics for the density matrix $ρ$ of our system ruled by the master equation

$$\dot{ρ} = −i \hbar [H, ρ] + L[ρ], \hspace{1cm} (20)$$

where $H$ is the ladder Hamiltonian [5] while $L[ρ]$ is the Lindbladian term describing spontaneous emission pro-
cesses. Of course one has to specify the Lindbladian operator, a goal achieved by posing
\[ \mathcal{L}[\rho] = \sum_{i=1}^{N} \left( \gamma_i L_i \rho L_i^\dagger - (\gamma_i/2) \{ L_i^\dagger L_i, \rho \} \right), \]
and identifying a suitable set of the \( L_i \)’s operators.

For our purposes, notice that the quantum superposition is induced by the rung Hamiltonian \( \mathcal{H}_R \) while the intra-leg term \( \mathcal{H}_L \) permits the hopping, i.e. the reshuffling, of the particles among the \( |p_a\rangle \) levels inside each of the legs. Hence, if our aim is to target the ground state made of the first \( N \) primes, it is sufficient to choose for \( L_i \) the following operators
\[ L_i = \sum_{a>N} \sum_{b \leq N} f_1(a) f_i(b) \dagger, \quad (21) \]
(in principle, one could also consider level-dependent coefficients \( \gamma_i^{(ab)} \)). The rationale behind this choice is that the dissipative term does not act in the space of the first \( N \) levels, while at the same time does favour the occupation of such a subspace. We expect an interplay between the term \( \propto \nu \) present in \( \mathcal{H}_L \) and the Lindbladian term \( \mathcal{L} \), in the sense that a non-vanishing \( \nu \) tends to decrease the characteristic time in which the system reaches our target subspace.

**Conclusions.** Number Theory is the paradigmatic example of pure mathematics. Yet the theory of integers can appear totally unexpected in quantum mechanics systems, providing new perspectives on their dynamics. In this paper we have considered a many-body quantum ladder system, made of \( N \) coupled quantum chains, whose degrees of freedom and interactions have a very direct interpretation in terms of prime numbers and basic properties thereof. We have shown that such a system has many different phases. Among the major capabilities of this system there is the possibility of realising a ground state made of a coherent superposition of the first \( N \) primes.

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[1] G.H. Hardy and E.M. Wright, *An Introduction to Theory of Numbers* (Oxford University Press, 1979).
[2] T.M. Apostol, *Introduction to Analytic Number Theory*, 5th ed. (Springer, New York, 1998).
[3] K. Rosen, *Elementary Number Theory and its Applications* (Addison-Wesley, 1993).
[4] P. Ribenboim, *The New Book of Prime Number Records* (Springer-Verlag, Berlin-New York, 1996).
[5] T. Tao, *Structure and Randomness in the Prime Numbers* in *An Invitation to Mathematics*, eds. D. Schleicher and M. Lackmann (Springer 2011).
[6] M. Kac, *Statistical Independence in Probability, Analysis and Number Theory* (The Mathematical Association of America, New Jersey, 1959).
[7] H. Cramér, Acta. Arith. 2, 23 (1936).
[8] P. Erdős and M. Kac, Am. J. Math. 62, 738 (1940).
[9] M. Schroeder, *Number Theory in Science and Communication*, (Springer-Verlag, 2009).
[10] P. Billingsley, Amer. Math. Monthly 80, 1099 (1973).
[11] B. Julia, *Statistical Theory of Numbers*, in *Number Theory and Physics*, Proceedings of the Winter School, Les Houches 1989, (Springer-Verlag, 1990).
[12] P.W. Shor, SIAM J. Comput. 26, 1484 (1997).
[13] G. Mussardo, *The Quantum mechanical potential for the prime numbers*, arXiv:cond-mat/9712010
[14] A. Ramani, B. Grammaticos and E. Caurier, Phys. Rev. E 51, 6323 (1995); B.P. van Zyl and D.A. Hutchinson, Phys. Rev. E 67, 066211 (2003).
[15] F. Cooper, A. Khare, and U. Sukhatme, Phys. Rep. 251, 267 (1995).
[16] S. Bergamini, B. Darquie, M. Jones, L. Jacobowiez, A. Browaeys, and P. Granger, Opt. Soc. Am. B 21, 1889 (2004); M. Pasienski and B. DeMarco, Phys. Rev. X 4, 021034 (2014); F. Nogrette, H. Labuhn, S. Ravets, D. Barredo, L. Béquin, A. Vernier, T. Lahaye, and A. Browaeys, Opt. Express 16, 2176 (2008); D. Bowman, P. Ireland, G.D. Bruce, D. Cassettari, Opt. Express 23, 8365 (2015).
[17] The closest example to our present model is the one-dimensional coprime quantum chain, studied in [30], where there have been observed an exponentially large degenerate ground states, the existence of phase transitions in the class of universality of Ising model, 3-state Potts model, etc.
[18] E. Dagotto and T.M. Rice, Science 271, 618 (1996); T.M. Rice, Z. Phys. B103, 165 (1997).
[19] S.R. White and D.J. Scalapino, Phys. Rev. B 60, R753 (1999).
[20] F.D.M. Haldane, Phys. Rev. Lett. 50, 1153 (1983).
[21] T. Giamarchi and A.M. Tsvelik, Phys. Rev. B 59, 11398 (1999).
[22] M. Azuma, Z. Hiroi, M. Takano, K. Ishida, and Y. Kitaoka, Phys. Rev. Lett. 73, 3463 (1994).
[23] D.C. Cabra, A. Honecker, and P. Pujol, Phys. Rev. B 58, 6241 (1998).
[24] D. Poilblanc, A.W.W. Ludwig, S. Trebst, and M. Troyer, Phys. Rev. B 83, 134439 (2011).
[25] R. Friedberg, T.D. Lee, and H.C. Ren, Ann. Phys. 228, 52 (1993).
[26] J.I. Latorre and G. Sierra, Quant. Inf. Comput. 14, 577 (2014); Quant. Inf. Comput. 15, 622 (2015); D. Garcia-Martin, E. Ribas, S. Carrazza, J. I. Latorre, G. Sierra, *The Prime state and its quantum relatives.*
[27] Note that for \( \lambda = 0 \), the Hamiltonian was diagonalised by nested Bethe-ansatz in [31].
[28] S.M. Ruiz, Math. Gaz. 81, 269 (1997).
[29] H.P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, 2002).
[30] G. Mussardo, G. Giudici, and J. Viti, J. Stat. Mech. 033104 (2017).
[31] B. Sutherland, Phys. Rev. B 12, 3795 (1975).
Supplementary Material

The competition between the hopping and the magnetic field. Let’s consider the 1-d Hamiltonian

\[ H_i(h, \nu) = h \sum_{a=1}^{\Lambda} \hat{h}(a) F_i(a) + \left( -\nu \right) \sum_{a \neq b}^{\Lambda} J_{ab} \left( f_i^\dagger(a) f_i(b) + f_i^\dagger(b) f_i(a) \right) . \]  

We want to show that, fixed the behaviour of \( h(a) \) (which we assume constant or monotonically increasing), there is a simple condition of \( J_{ab} \) which ensures that the two terms in this Hamiltonian are truly competing. Assuming \( J_{ab} \) a circulant matrix, the eigenvectors of the hopping term alone is particularly simple and can be expressed in terms of the \( \Lambda \)-roots of the unity \( \omega = e^{2\pi i/\Lambda} \), with \( \omega^{kn} = e^{(2\pi i/\Lambda)kn} \). In fact, we have

\[ |\phi_k\rangle = \frac{1}{\sqrt{\Lambda}} \begin{pmatrix} \omega^{0k} \\ \omega^{1k} \\ \omega^{2k} \\ \vdots \\ \omega^{(\Lambda-1)k} \end{pmatrix} , \]

with \( k = 0, 1, \ldots, \Lambda - 1 \). In particular, the expression for the ground state, which corresponds to the non-degenerate value \( e_0 \), is particularly simple

\[ |\phi_0\rangle = \frac{1}{\sqrt{\Lambda}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} . \]

Expressing it as \( \phi_0 = \sum a c_a |v_a\rangle \) we can compute the ground state energy of the hopping term alone, given by

\[ E_0 = \frac{1}{\Lambda} \sum_{a=1}^{\Lambda} \sum_{b=1}^{\Lambda} J_{ab} , \]

Let’s now compute the first order correction given by the magnetic field on this ground state. This correction is given by

\[ \delta e_0 = \frac{1}{\Lambda} \sum_{a=1}^{\Lambda} \tilde{h}(a) . \]

Imposing that \( \delta_0 \simeq E_0 \), we arrive to the condition \( \sum_{a,b} J_{a,b} \simeq \sum \tilde{h}(a) \) which, assuming \( J_{a,b} = J|a-b| \), can be also expressed as

\[ J(x) \simeq \tilde{h}'(x) . \]  

**Probability of divisibility by \( p_a \) of a square-free number.** In order to determine the probability (12) given in the text, we have firstly to determine the probability that a randomly chosen square-free number is divisible by a prime factor \( p_a \). We will show that such a probability is given by \( 1/(p_a + 1) \) by using the inclusion-exclusion principle as follows. Let \( F(t) \) be the number of square-free numbers which are less than \( t \): this function asymptotically goes as

\[ F(t) \simeq \frac{6}{\pi^2} t . \]  

(S3)
Using \(F(t)\), we can give the first estimate of the number of square-free numbers which are less than \(t\) and multiples of the prime \(p_a\). This number is approximatively equal to \(F(t/p_a)\). If we now multiply a square-free number \(y\) (with \(y \leq t/p_a\)) by \(p_a\), this yields (for sure) a multiple of \(p_a\) which is \(\leq t\). This multiplication usually gives rise to a number which is also square-free, for the only perfect square that could possibly divide the number \(yp_a\), where \(y \leq t/p_a\) and \(y\) is square-free, is \(p_a^2\). This implies that \(F(t/p_a)\) overcounts the set of multiples of \(p_a\) that are \(\leq t\) and square-free. In order to correct this discrepancy, we must subtract approximately \(F(t/p_a^2)\), which almost counts how many numbers \(\leq t\) are divisible by \(p_a^2\) but which are otherwise square-free. But this time we have subtracted too much, since we have also subtracted the numbers \(\leq t\) which are divisible by \(p_a^3\) but which are otherwise square-free. So, we need to add back approximately \(F(t/p_a^3)\) and so on. In this way, we have to deal with the sum of the infinite series

\[
\frac{1}{p} - \frac{1}{p^2} + \frac{1}{p^3} - \frac{1}{p^4} + \cdots = \frac{1}{p+1}
\]  

(S4)

This yields the sought probability for a random chosen square-free number to be divisible by a prime \(p_a\). To see how accurate this prediction is, we have generated the first \(10^7\) square-free numbers and we have count how many of them were divisible by 3, 5, 7, \ldots. The outputs of this analysis is in Table I and, as one can see, the agreement between “theory” and “experiment” is pretty remarkable.

**Probability of getting a ground state in the Hamiltonian \(\mathcal{H}_R\).** Let’s now compute the probability of getting one of the ground states of the Hamiltonian \(\mathcal{H}_R\). As discussed in the text, this consists in estimating the probability of pairwise coprimality of \(N\) randomly selected square-free numbers. An important input of this computation is the probability that a square-free number is divisible by a prime factor \(p_a\) which, as shown above, is given by \(1/(p_a + 1)\). With this information, we can follow the argument given in Schroeder’s book [9]: the probability that none of \(N\) square-free integers has the prime factor \(p_a\) is

\[
\left(1 - \frac{1}{p_a + 1}\right)^N,
\]

(S5)

while the probability that exactly one has \(p_a\) as a factor is

\[
\frac{N}{p_a + 1} \left(1 - \frac{1}{p_a + 1}\right)^{N-1}.
\]

(S6)

The sum of these two probabilities is the probability that at most one of the \(N\) square-free numbers has \(p_a\) as a factor. Hence, taking the product over all primes, we get the probability that \(N\) square-free numbers are pairwise coprime, i.e. the probability to get one of the ground states of the Hamiltonian \([7]\)

\[
\text{Prob(ground states)} = \prod_{p_a} \left[\left(1 + \frac{N-1}{p_a + 1}\right) \left(1 - \frac{1}{p_a + 1}\right)^{N-1}\right].
\]

(S7)

This probability rapidly decreases by increasing the number \(N\) of legs of the ladder, as shown in Table II. In the limit \(N \to \infty\), the dimension of the ground state manifold is infinitesimally small with respect to the dimension of the Hilbert space.

| primes | numerical probability | theoretical probability |
|--------|-----------------------|-------------------------|
| 2      | 0.333331              | 0.333333                |
| 3      | 0.249998              | 0.250000                |
| 5      | 0.166670              | 0.166666                |
| 7      | 0.1250001             | 0.125000                |
| 11     | 0.0833331             | 0.083333                |
| 13     | 0.0714281             | 0.0714286               |
| 17     | 0.0555547             | 0.0555556               |

**TABLE I:** Numerical vs theoretical probability of divisibility of square-free numbers by a prime. The numerical data are given by the ratio \(N_a/N\), where \(N = 10^7\) is the number of square-free numbers considered and \(N_a\) is the number of them divisible by the prime \(p_a\). The theoretical probability is \(1/(p_a + 1)\).
| $\mathcal{N}$ | Prob     |
|------------|----------|
| 3          | 0.511335 |
| 4          | 0.299667 |
| 5          | 0.160472 |
| 6          | 0.0799262|
| 7          | 0.0374877|
| 8          | 0.0167083|
| 9          | 0.0071255|
| 10         | 0.00292332|
| 50         | $1.55 \times 10^{-24}$ |
| 100        | $7.74 \times 10^{-57}$ |

TABLE II: Probability to get a ground state of the Hamiltonian for different number of lattice sites $\mathcal{N}$. 