ON THE ERGODIC PRINCIPLE FOR MARKOV AND QUADRATIC
STOCHASTIC PROCESSES AND ITS RELATIONS

NASIR GANIKHODJAEV, HASAN AKIN, AND FARRUKH MUKHAMEDOV

Abstract. In the paper we prove that a quadratic stochastic process satisfies the ergodic
principle if and only if the associated Markov process satisfies one.

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1. Introduction

It is known that Markov processes are well-developed field of mathematics which have various
applications in physics, biology and so on. But there are some physical models which cannot
be described by such processes. One of such models is a model related to population genetics.
Namely, this model is described by quadratic stochastic processes (see [7] for review). To define
it, we denote

\[ \ell_1 = \{ x = (x_n) : \|x\|_1 = \sum_{n=1}^{\infty} |x_n| < \infty; \ x_n \in \mathbb{R} \}, \]
\[ S = \{ x \in \ell_1 : x_n \geq 0; \ \|x\|_1 = 1 \}. \]

Hence this process is defined as follows (see [1],[9]): Consider a family of functions \( \{ P_{ij,k}^{[s,t]} : i,j,k \in \mathbb{N}, \ s,t \in \mathbb{R}^+, \ t-s \geq 1 \} \). This family is said to be quadratic stochastic process (q.s.p.)
if for fixed \( s,t \in \mathbb{R}_+ \) it satisfies the following conditions:

(i) \( P_{ij,k}^{[s,t]} = P_{ji,k}^{[s,t]} \) for any \( i,j,k \in \mathbb{N} \).

(ii) \( P_{ij,k}^{[s,t]} \geq 0 \) and \( \sum_{k=1}^{\infty} P_{ij,k}^{[s,t]} = 1 \) for any \( i,j,k \in \mathbb{N} \).

(iii) An analogue of Kolmogorov-Chapman equation; here there are two variants: for the
initial point \( x^{(0)} \in S, \ x^{(0)} = (x_1^{(0)}, x_2^{(0)}, \cdots) \) and \( s < r < t \) such that \( t-r \geq 1, r-s \geq 1 \)

\[ P_{ij,k}^{[s,t]} = \sum_{m,l=1}^{\infty} P_{ij,m}^{[s,r]} P_{ml,k}^{[r,t]} x_l^{(r)} \]

where \( x_k^{(r)} \) is defined as follows:

\[ x_k^{(r)} = \sum_{i,j=1}^{\infty} P_{ij,k}^{[0,r]} x_i^{(0)} x_j^{(0)} \]
\[ P^{[s,t]}_{ij,k} = \sum_{m,l,g,h=1}^{\infty} P^{[s,r]}_{im,l} P^{[s,r]}_{jg,h} P^{[r,t]}_{lh,k} x_{m}^{(s)} x_{g}^{(s)}. \]

We say that the q.s.p. \( \{P^{[s,t]}_{ij,k}\} \) is of type (A) or (B) if it satisfies the fundamental equations \((\text{iii}_A)\) or \((\text{iii}_B)\), respectively. In this definition the functions \( P^{[s,t]}_{ij,k} \) denotes the probability that under the interaction of the elements \( i \) and \( j \) at time \( s \) the element \( k \) comes into effect at time \( t \). Since for physical, chemical and biological phenomena a certain time is necessary for the realization of an interaction, we shall take the greatest such time to be equal to 1 (see the Boltzmann model [4] or the biological model [7]). Thus the probability \( P^{[s,t]}_{ij,k} \) is defined for \( t - s \geq 1 \). It should be noted that the quadratic stochastic processes are related to quadratic transformations (see [5],[7]) in the same way as Markov processes are related to linear transformations.

The equations \((\text{iii}_A)\) and \((\text{iii}_B)\) can be interpreted as different laws of behavior of the "off-spring".

Some examples of q.s.p. were given in [1],[3]. We note that quadratic processes of type (A) were considered in [1],[9]. One of the central problems in this theory is the study of limit behaviors of q.s.p. An ergodic principle is a such concept relating to limit behaviors one (see [6]). In [10] some conditions were given for q.s.p. to satisfy this principle. It is known [1],[2] that a certain Markov process can be defined by means of q.s.p., therefore, it is interesting to know the following question: if this Markov process satisfies the ergodic principle, then would q.s.p. satisfy that principle? The answer to this question gives us to find more conditions for fulfilling the ergodic principle for q.s.p., since the theory of Markov processes are well-developed filed. In the paper we are going to solve the formulated question for discrete time q.s.p.

We note that a part of the results were announced in [8],[10].

### 2. Ergodic Principle for Quadratic Stochastic Processes

In this section we will answer to the above formulated question for discrete time q.s.p. Before doing it, we prove some results concerning about Markov processes.

In the sequel we will consider discrete time q.s.p., i.e. for \( \{P^{[s,t]}_{ij,k}\} \) the numbers \( s, t \) belong to \( \mathbb{N} \).

Recall that a matrix \( (Q_{ij}) \) is called stochastic if

\[ Q_{ij} \geq 0; \sum_{j=1}^{\infty} Q_{ij} = 1. \]

First recall that a family of stochastic matrices \( \{(Q_{ij}^{m,n})_{i,j \in \mathbb{N}} : m, n \in \mathbb{N}, n - m \geq 1\} \) is called discrete time Markov process if the following condition holds: for every \( m < n < l \)

\[ Q_{ij}^{m,l} = \sum_{k=1}^{\infty} Q_{ik}^{m,n} Q_{kj}^{n,l}. \quad (2.1) \]

This equation is known as the Kolmogorov-Chapman equation.

A Markov process \( \{Q_{ij}^{m,n}\} \) is said to satisfy the ergodic principle if

\[ \lim_{n \to \infty} |Q_{ik}^{m,n} - Q_{jk}^{m,n}| = 0 \]
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is valid, for every $i, j, k, m \in \mathbb{N}$. Note that this notion firstly was introduced in [6]. Each $(Q_{ij}^{m,n})$-stochastic matrix defines a linear operator $Q^{m,n} : \ell^1 \to \ell^1$ as follows

$$(Q^{m,n}(x))_j = \sum_{i=1}^{\infty} Q_{ij}^{m,n} x_i, \quad x = (x_n) \in \ell^1. \quad (2.2)$$

Stochasticity of $(Q_{ij}^{m,n})$ implies that $Q^{m,n}(S) \subset S$ and $\|Q^{m,n}x\|_1 \leq \|x\|_1, \quad x \in \ell^1$; \quad (2.3)

By $\{e^{(n)}\}$ we denote standard basis of $\ell^1$, i.e.

$$e^{(n)} = (0, 0, \ldots, 1, \ldots), \quad n \in \mathbb{N}.$$ 

Now we formulate the following well-know fact (see, for example [11]).

**Lemma 2.1.** A sequence $\{x_n\} \subset \ell^1$ converges weakly if and only if it converges in norm of $\ell^1$.

We have following;

**Theorem 2.2.** Let $\{Q_{ij}^{m,n}\}$ be a Markov process. The following conditions are equivalent:

(i) $\{Q_{ij}^{m,n}\}$ satisfies the ergodic principle;

(ii) For every $i, j, m \in \mathbb{N}$ the following relation holds:

$$\lim_{n \to \infty} \|Q_{m,n}e^{(i)} - Q_{m,n}e^{(j)}\|_1 = 0.$$ 

(iii) For every $\varphi, \psi \in S$ and $m \in \mathbb{N}$ the following relation holds:

$$\lim_{n \to \infty} \|Q_{m,n}\varphi - Q_{m,n}\psi\|_1 = 0.$$ 

**Proof.** (i)$\Rightarrow$(ii). The ergodic principle means that the a sequence $x_{ij,m}^{(n)} = (Q_{ik}^{m,n} - Q_{jk}^{m,n})_{k \in \mathbb{N}}$ converges weakly in $\ell^1$, here $i, j, m \in \mathbb{N}$ are fixed numbers. According Lemma 2.1 we infer that $x_{ij,m}^{(n)}$ converges strongly in $\ell^1$, this means

$$\lim_{n \to \infty} \sum_{k=1}^{\infty} |Q_{ik}^{m,n} - Q_{jk}^{m,n}| \to 0.$$ 

On other hand, from (2.2) we find

$$\|Q_{m,n}e^{(i)} - Q_{m,n}e^{(j)}\|_1 = \sum_{k=1}^{\infty} |Q_{ik}^{m,n} - Q_{jk}^{m,n}|. \quad (2.4)$$

Hence, the considered implication is proved.
(ii)⇒(iii). First consider the following elements: 
\[ \xi = \sum_{i=1}^{M} \alpha_i e^{(i)}, \quad \eta = \sum_{j=1}^{N} \beta_j e^{(j)}, \] where \( \alpha_i, \beta_j \geq 0, \quad \sum_{i=1}^{M} \alpha_i = \sum_{j=1}^{N} \beta_j = 1. \) Using (ii) we have

\[
\| Q_{m,n}^{m,n} \xi - Q_{m,n}^{m,n} \eta \|_1 = \left\| \sum_{i=1}^{M} \alpha_i Q_{m,n}^{m,n} e^{(i)} - \sum_{j=1}^{N} \beta_j Q_{m,n}^{m,n} e^{(j)} \right\|
\]

\[
= \left\| \sum_{i=1}^{M} \sum_{j=1}^{N} \alpha_i \beta_j Q_{m,n}^{m,n} e^{(i)} - \sum_{i=1}^{M} \sum_{j=1}^{N} \alpha_i \beta_j Q_{m,n}^{m,n} e^{(j)} \right\|
\]

\[
\leq \sum_{i=1}^{M} \sum_{j=1}^{N} \alpha_i \beta_j \| Q_{m,n}^{m,n} e^{(i)} - Q_{m,n}^{m,n} e^{(j)} \|_1 \to 0 \quad \text{as} \quad n \to \infty. \quad (2.5)
\]

Now let \( \varphi, \psi \in S \) and \( \varepsilon > 0. \) Denote

\[ G = \{ \xi = \sum_{i=1}^{M} \alpha_i e^{(i)} : \alpha_i \geq 0; \sum_{i=1}^{M} \alpha_i = 1, \ M \in \mathbb{N} \}. \]

It is clear that \( G \) is dense in \( S. \) Therefore, there exist \( \xi, \eta \in G \) such that

\[ \| \varphi - \xi \|_1 < \varepsilon/3, \quad \| \psi - \eta \|_1 < \varepsilon/3. \]

According to (2.5) there is \( n_0 \in \mathbb{N} \) such that

\[ \| Q_{m,n}^{m,n} \xi - Q_{m,n}^{m,n} \eta \|_1 < \varepsilon/3, \quad \forall n \geq n_0. \]

Hence, by means of above relations and (2.3) we obtain

\[
\| Q_{m,n}^{m,n} \varphi - Q_{m,n}^{m,n} \psi \|_1 \leq \| Q_{m,n}^{m,n} (\varphi - \xi) \|_1 + \| Q_{m,n}^{m,n} (\psi - \eta) \|_1 + \| Q_{m,n}^{m,n} \xi - Q_{m,n}^{m,n} \eta \|_1
\]

\[
\leq \| \varphi - \xi \|_1 + \| \psi - \eta \|_1 + \varepsilon/3 < \varepsilon
\]

for all \( n \geq n_0. \) Thus the implication is proved. The implication (iii)⇒(i) is obvious. \( \square \)

Let \( \{ P_{ij,k}^{[m,n]} \} \) be a q.s.p. Define

\[
\mathbb{H}_{ij}^{m,n} = \sum_{l=1}^{\infty} P_{il,j}^{[m,n]} x_{l}^{(m)}, \quad i, j \in \mathbb{N}. \quad (2.6)
\]

It is clear \( \{ \mathbb{H}_{ij}^{m,n} \} \) is a stochastic matrix.

**Lemma 2.3.** Let \( \{ P_{ij,k}^{[m,n]} \} \) be a q.s.p. Then \( \{ \mathbb{H}_{ij}^{m,n} \} \) is a Markov process.

**Proof.** Consider two distinct cases with respect to types of q.s.p.
Case (a). Let \( P_{ij,k}^{[m,n]} \) be a q.s.p of type (A). Then we have
\[
\begin{align*}
\sum_{k=1}^{\infty} \mathbb{H}_{k}^{m,n} \mathbb{H}_{k}^{l} &= \sum_{k=1}^{\infty} \left( \sum_{u=1}^{\infty} P_{iu,k}^{[m,n]} x_{u}^{(m)} \right) \left( \sum_{v=1}^{\infty} P_{kv,j}^{[n,l]} x_{v}^{(n)} \right) \\
&= \sum_{u=1}^{\infty} \left( \sum_{k,v=1}^{\infty} P_{iu,k}^{[m,n]} P_{kv,j}^{[n,l]} x_{u}^{(m)} x_{v}^{(n)} \right) \\
&= \sum_{u=1}^{\infty} P_{iu,j}^{[m,l]} x_{u}^{(m)} = \mathbb{H}_{ij}^{m,l}.
\end{align*}
\]
So \( \{Q_{ij}^{m,n}\} \) is a Markov process.

Case (b). Let \( P_{ij,k}^{[m,n]} \) be a q.s.p of type (B). First consider
\[
\begin{align*}
\sum_{i,j=1}^{\infty} P_{ij,k}^{[m,n]} x_{i}^{(m)} x_{j}^{(m)} &= \sum_{i,j=1}^{\infty} P_{ij,k}^{[m,n]} \left( \sum_{e,f=1}^{\infty} P_{ef,i}^{[0,m]} x_{e}^{(0)} x_{f}^{(0)} \right) \left( \sum_{c,d=1}^{\infty} P_{cd,j}^{[0,n]} x_{c}^{(0)} x_{d}^{(0)} \right) \\
&= \sum_{c,e=1}^{\infty} \left( \sum_{j,d,i=1}^{\infty} P_{ef,i}^{[0,m]} P_{cd,j}^{[0,n]} P_{ij,k}^{[m,n]} x_{e}^{(0)} x_{c}^{(0)} \right) x_{c}^{(0)} x_{c}^{(0)} \\
&= \sum_{c,e=1}^{\infty} P_{ec,k} x_{e}^{(0)} x_{c}^{(0)} = x_{k}^{(n)}.
\end{align*}
\]
So
\[
x_{k}^{(n)} = \sum_{i,j=1}^{\infty} P_{ij,k}^{[m,n]} x_{i}^{(m)} x_{j}^{(m)}.
\]

Using this equality check Markovianity:
\[
\begin{align*}
\mathbb{H}_{ij}^{m,l} &= \sum_{k=1}^{\infty} P_{ik,j}^{[m,l]} x_{k}^{(m)} = \sum_{k=1}^{\infty} \left( \sum_{a,b,c,d=1}^{\infty} P_{ia,b}^{[m,n]} P_{kc,d}^{[m,n]} P_{bd,j}^{[n,l]} x_{a}^{(m)} x_{c}^{(m)} \right) x_{k}^{(m)} \\
&= \sum_{b,d=1}^{\infty} \left( \sum_{a=1}^{\infty} P_{ia,b}^{[m,n]} x_{a}^{(m)} \right) \left( \sum_{k,c=1}^{\infty} P_{kc,d}^{[m,n]} x_{k}^{(m)} x_{c}^{(m)} \right) P_{bd,j}^{[n,l]} \\
&= \sum_{b=1}^{\infty} \left( \sum_{a=1}^{\infty} P_{ia,b}^{[m,n]} x_{a}^{(m)} \right) \left( \sum_{d=1}^{\infty} P_{bd,j}^{[n,l]} x_{d}^{(n)} \right) = \sum_{b=1}^{\infty} \mathbb{H}_{lb}^{m,n} \mathbb{H}_{b}^{m,l}.
\end{align*}
\]

The defined process \( \{\mathbb{H}_{ij}^{m,n}\} \) is called associated Markov process with respect to q.s.p. By \( \mathbb{H}_{ij}^{m,n} \) we denote the linear operator associated with this Markov process (see (2.2)).

We say that the ergodic principle holds for the q.s.p. \( \{P_{ij,k}^{[m,n]}\} \) if
\[
\lim_{n \to \infty} |P_{ij,k}^{[m,n]} - P_{uv,k}^{[m,n]}| = 0
\]
is valid for any $i, j, u, v, k \in \mathbb{N}$ and arbitrary $m \in \mathbb{N}$.

Define

$$R_{ij}^{m,n}(x) = \sum_{l=1}^{\infty} P_{il,j}^{[m,n]} x_l,$$  \hspace{1cm} (2.7)

here $x = (x_n) \in S$. It is clear that for each $m, n \in \mathbb{N}$ and $x \in S$ the matrix \((R_{ij}^{m,n}(x))\) is stochastic.

**Proposition 2.4.** Let \(\{P_{il,j}^{[m,n]}\}\) be a q.s.p. Then the following conditions are equivalent:

(i) For every $i, j, k \in \mathbb{N}$ and $x \in S$ the following holds:

$$\lim_{n \to \infty} |R_{ik}^{m,n}(x) - R_{jk}^{m,n}(x)| = 0.$$

(ii) The Markov process \(\{H_{ij}^{m,n}\}\) satisfies the ergodic principle.

**Proof.** The implication (ii)⇒(i). Again divide the proof into two cases.

**Case (a).** Let \(\{P_{ij,k}^{[m,n]}\}\) be a q.s.p. of type (A). Then we have

$$R_{ik}^{m,n}(x) = \sum_{l=1}^{\infty} P_{il,k}^{[m,n]} x_l = \sum_{l=1}^{\infty} \left( \sum_{u,v=1}^{\infty} P_{il,u}^{[m,m+1]} P_{uv,k}^{[m+1,n]} x_v^{(m+1)} \right) x_l$$

$$= \sum_{l=1}^{\infty} \sum_{u=1}^{\infty} P_{il,k}^{[m,m+1]} H_{uk}^{m+1,n} x_l$$

$$= \sum_{u=1}^{\infty} Q_{uk}^{m+1,n} y_u(i) = (H_{m+1,n}^{m+1} y(i))_k$$

where $y_u(i) = \sum_{l=1}^{\infty} P_{il,u}^{[m,m+1]} x_l$. Similarly, one gets

$$R_{jk}^{m,n}(x) = \sum_{u=1}^{\infty} H_{u,k}^{m+1,n} y_u(j) = (H_{m+1,n}^{m+1} y(j))_k.$$

The ergodic principle for the Markov process with Theorem 2.2 implies that

$$\|H_{m+1,n}^{m+1} y(i) - H_{m+1,n}^{m+1} y(j)\|_1 \to 0 \text{ as } n \to \infty.$$

Therefore,

$$|R_{ik}^{m,n}(x) - R_{jk}^{m,n}(x)| = |(H_{m+1,n}^{m+1} y(i))_k - (H_{m+1,n}^{m+1} y(j))_k|$$

$$\leq \|H_{m+1,n}^{m+1} y(i) - H_{m+1,n}^{m+1} y(j)\|_1 \to 0 \text{ as } n \to \infty.$$

**Case (b).** Now suppose that \(\{P_{ij,k}^{[m,n]}\}\) is a q.s.p. of type (B). Given $x \in S$ define operator

$$R_{ij}^{m,n}(x) : \ell^1 \to \ell^1$$

as follows:

$$(R_{ij}^{m,n}(x))(y)_k = \sum_{i=1}^{\infty} P_{ik}^{m,n}(x) y_i,$$

$y = (y_i) \in \ell^1$. Using stochasticity of \((R_{ij}^{m,n}(x))\) we infer

$$\|(R_{ij}^{m,n}(x))(y)\|_1 \leq \|y\|_1, \text{ \forall } y \in \ell^1.$$  \hspace{1cm} (2.8)
Now using (iii) we find
\[ R_{nk}^{m,n+1}(x) = \sum_{l=1}^{\infty} \left( \sum_{a,b,c,d=1}^{\infty} P_{ika,b}^{[m,n]} P_{lc,d}^{[m,n]} P_{bd,k}^{[m,n+1]} x_{a,c}^{(m)} x_{c}^{(m)} \right) x_{l} \]
\[ = \sum_{l=1}^{\infty} \sum_{b,d=1}^{\infty} H_{ib}^{m,n} H_{id}^{m,n} P_{bd,k}^{[n,n+1]} x_{l} \]
\[ = (R^{n,n+1}(y))(H^{m,n}(e^{(i)}))_{k}, \tag{2.9} \]
here \( y = H^{m,n}(x) \). Similarly, one gets
\[ R_{jk}^{m,n+1}(x) = (R^{n,n+1}(y))(H^{m,n}(e^{(j)}))_{k}. \tag{2.10} \]
Therefore, it follows from (2.8)-(2.10) that
\[ |R_{nk}^{m,n+1}(x) - R_{jk}^{m,n+1}(x)| = |(R^{n,n+1}(y))(H^{m,n}(e^{(i)})) - H^{m,n}(e^{(j)}))_{k}| \]
\[ \leq \|(R^{n,n+1}(y))(H^{m,n}(e^{(i)})) - H^{m,n}(e^{(j)}))\|_{1} \]
\[ \leq \|H^{m,n}(e^{(i)}) - H^{m,n}(e^{(j)})\|_{1} \to 0 \] as \( n \to \infty \).

This completes the proof.

\[ \square \]

**Proposition 2.5.** Let \( \{P_{ij,k}^{[m,n]}\} \) be a q.s.p. Then the following conditions are equivalent:

(i) \( \{P_{ij,k}^{[m,n]}\} \) satisfies the ergodic principle;

(ii) For every \( x \in S \) and \( i, j, k, m \in \mathbb{N} \) the following holds:
\[ \lim_{n \to \infty} |R_{nk}^{m,n}(x) - R_{jk}^{m,n}(x)| = 0. \]

**Proof.** (i)\( \Rightarrow \) (ii). Define a bilinear operator \( P^{[m,n]} : \ell^{1} \times \ell^{1} \to \ell^{1} \) as follows
\[ (P^{[m,n]}(x,y))_{k} = \sum_{i,j=1}^{\infty} P_{ij,k}^{[m,n]} x_{i} y_{j}, \]
where \( x = (x_{n}) \), \( y = (y_{n}) \in \ell^{1} \). According to Lemma 2.1 from ergodic principle we find
\[ \lim_{n \to \infty} \|P^{[m,n]}(e^{(i)}, e^{(j)}) - P^{[m,n]}(e^{(u)}, e^{(v)})\|_{1} = 0 \]
for every \( i, j, u, v, m \in \mathbb{N} \). The same argument of the proof of Theorem 2.2 implies that
\[ \lim_{n \to \infty} \|P^{[m,n]}(e^{(i)}, x) - P^{[m,n]}(e^{(u)}, x)\|_{1} = 0, \]
for every \( x, y \in S \). Hence, we have
\[ |R_{nk}^{m,n}(x) - R_{jk}^{m,n}(x)| \leq \|P^{[m,n]}(e^{(i)}, x) - P^{[m,n]}(e^{(j)}, x)\|_{1} \to 0 \] as \( n \to \infty \).

Now consider the implication (ii)\( \Rightarrow \) (i). From (2.7) we have
\[ |P_{iu,k}^{[m,n]} - P_{iu,k}^{[m,n]}| = |R_{ik}^{m,n}(e^{(u)}) - R_{jk}^{m,n}(e^{(u)})| \to 0 \] as \( n \to \infty \);
for every \( i, j, k, u \in \mathbb{N} \). Whence one gets
\[ |P_{ij,k}^{[m,n]} - P_{ij,k}^{[m,n]}| \leq |P_{ij,k}^{[m,n]} - P_{ij,k}^{[m,n]}| + |P_{ij,k}^{[m,n]} - P_{ij,k}^{[m,n]}| \to 0 \] as \( n \to \infty \),
here we have used the equation (i) of definition q.s.p. \[ \square \]
Now we are ready to formulate our main result.

**Theorem 2.6.** Let \( \{P_{ij,k}^{[m,n]}\} \) be a q.s.p. The following conditions are equivalent:

(i) \( \{P_{ij,k}^{[m,n]}\} \) satisfies the ergodic principle;

(ii) The Markov process \( \{H_{ij,k}^{[m,n]}\} \) satisfies the ergodic principle.

The proof immediately follows from Propositions 2.4 and 2.5.

### 3. An application of the main result

In this section we give certain conditions for the Markov process which ensure fulfilling the ergodic principle for q.s.p.

Now we need some auxiliary facts.

**Lemma 3.1.** Let \( \{a_n\} \) be a nonnegative sequence which satisfies the following inequality

\[
a_n \leq (1 - \lambda_n)a_{n-1} + \prod_{k=1}^{n}(1 - \lambda_k),
\]

where \( \lambda_n \in (0, 1) \), \( \forall n \in \mathbb{N} \) and

\[
\sum_{n=1}^{\infty} \lambda_n = \infty,
\]

\[
\sum_{j=1}^{n} \frac{\prod_{k=1}^{n}(1 - \lambda_k)}{(1 - \lambda_j)} \to 0 \quad \text{as} \quad n \to \infty,
\]

then \( \lim_{n \to \infty} a_n = 0 \).

**Proof.** From (3.1) by iterating we get

\[
a_n \leq a_1 \prod_{i=2}^{n}(1 - \lambda_i) + \sum_{j=1}^{n} \frac{\prod_{k=1}^{n}(1 - \lambda_k)}{(1 - \lambda_j)}.
\]

The condition (3.2) with \( 0 < \lambda_n < 1 \) implies that \( \prod_{k=1}^{n}(1 - \lambda_k) \to 0 \) as \( n \to \infty \). Hence, by means of (3.3) we obtain \( a_n \to 0 \) as \( n \to \infty \). \( \square \)

**Corollary 3.2.** Let \( \{a_n\} \) be as above. If the sequence \( \{\lambda_n\} \), \( (0 < \lambda_n < 1, \forall n \in \mathbb{N}) \) satisfies (3.2) and the following relations

\[
n \prod_{k=1}^{n}(1 - \lambda_k) \to 0 \quad \text{as} \quad n \to \infty
\]

\[
\sum_{i=1}^{n} \frac{\lambda_i}{1 - \lambda_i} = O(n)
\]

then \( \lim_{n \to \infty} a_n = 0 \).
Proof. It is enough to show that the conditions (3.5), (3.6) imply (3.3). Indeed, since $0 < 1 - \lambda_j < 1$, we can write

$$\frac{1}{1 - \lambda_j} = 1 + \varepsilon_j,$$

here $\varepsilon_j$ is a some positive number. It then follows that

$$\sum_{j=1}^{n} \frac{1}{1 - \lambda_j} = n + \sum_{j=1}^{n} \varepsilon_j = n + \sum_{j=1}^{n} \frac{\lambda_j}{1 - \lambda_j}.$$

It follows from (3.6) that

$$\sum_{j=1}^{n} \frac{1}{1 - \lambda_j} \leq C \cdot n$$

for all $n \in \mathbb{N}$, here $C$ is a constant. Therefore, from (3.7), (3.4) and (3.5) we infer that

$$\sum_{j=1}^{n} \frac{\prod_{k=1}^{n} (1 - \lambda_k)}{1 - \lambda_j} \leq C \cdot n \prod_{k=1}^{n} (1 - \lambda_k) \to 0 \text{ as } n \to \infty$$

□

Theorem 3.3. Let $\{Q^{m,n}_{ij}\}$ be a Markov process. If there exists a number $k_0 \in \mathbb{N}$ and a sequence $\{\lambda_n\}$, $0 < \lambda_n < 1, \forall n \in \mathbb{N}$ satisfying (3.2), (3.3) such that

$$Q^{n-1,n}_{i,k_0} \geq \lambda_n \text{ for all } i, n \in \mathbb{N}.$$  

Then the Markov process satisfies the ergodic principle.

Proof. We set

$$\sup_{i \in \mathbb{N}} Q^{k,n}_{i,k_0} = M_{k,n}(k_0); \inf_{i \in \mathbb{N}} Q^{k,n}_{i,k_0} = m_{k,n}(k_0).$$

For $i < k < n$ we have

$$Q^{i,n}_{i,k_0} = \sum_{l=1}^{\infty} Q^{i,l}_{i,k_0} Q^{k,n}_{l,k_0} \leq M_{k,n}(k_0) \sum_{l=1}^{\infty} Q^{i,l}_{i,k_0} = M_{k,n}(k_0).$$

Similarly,

$$Q^{i,n}_{i,k_0} \geq m_{k,n}(k_0)$$

By means of (3.8) we infer

$$Q^{n-1,n}_{i,k_0} - \lambda_n Q^{n-1,n}_{j,k_0} \geq 0$$
for all $i, j \in \mathbb{N}$, because $0 \leq Q_{j k_0}^{-1,n} \leq 1$. It follows

$$Q_{i k_0}^{k-1,n} = \sum_{l=1}^{\infty} Q_{i l}^{k-1,l} Q_{l k_0}^{k,n}$$

$$= \sum_{l=1}^{\infty} [Q_{i l}^{k-1,l} - \lambda_k Q_{j l}^{k-1,k}] Q_{l k_0}^{k,n} + \lambda_k \sum_{l=1}^{\infty} Q_{j l}^{k-1,k} Q_{l k_0}^{k,n}$$

$$\geq m_{k,n}(k_0) \sum_{l=1}^{\infty} [Q_{i l}^{k-1,l} - \lambda_k Q_{j l}^{k-1,k}] + \lambda_k Q_{j k_0}^{k-1,n}$$

$$= (1 - \lambda_k)m_{k,n}(k_0) + \lambda_k Q_{j k_0}^{k-1,n},$$

whence

$$Q_{j k_0}^{k-1,n} - Q_{i k_0}^{k-1,n} \leq (1 - \lambda_k)(Q_{j k_0}^{k-1,n} - m_{k,n}(k_0)). \quad (3.12)$$

Since (3.12) holds for any $i, j \in \mathbb{N}$, from (3.9)-(3.10) we find

$$M_{k-1,n}(k_0) - m_{k-1,n}(k_0) \leq (1 - \lambda_k)(M_{k,n}(k_0) - m_{k,n}(k_0)). \quad (3.13)$$

So iterating the last inequality we get

$$M_{l,n}(k_0) - m_{l,n}(k_0) \leq \prod_{k=l+1}^{n-1} (1 - \lambda_k). \quad (3.14)$$

Using (2.1) we have

$$|Q_{i k}^{m,n} - Q_{j k}^{m,n}| = \sum_{l=1}^{\infty} |Q_{i l}^{m,n-1} - Q_{j l}^{m,n-1}| Q_{l k}^{n-1,n},$$

for every $i, j \in \mathbb{N}$. Hence by means of (3.8) it yields that

$$\sum_{k=1, k \neq k_0}^{\infty} |Q_{i k}^{m,n} - Q_{j k}^{m,n}| \leq \sum_{l,k=1}^{\infty} |Q_{i l}^{m,n-1} - Q_{j l}^{m,n-1}| Q_{l k}^{n-1,n}$$

$$- \sum_{l=1}^{\infty} |Q_{i l}^{m,n-1} - Q_{j l}^{m,n-1}| Q_{l k_0}^{n-1,n}$$

$$\leq (1 - \lambda_n) \sum_{l=1}^{\infty} |Q_{i l}^{m,n-1} - Q_{j l}^{m,n-1}|. \quad (3.15)$$

Now add $|Q_{i k_0}^{m,n} - Q_{j k_0}^{m,n}|$ to both sides of (3.15). Then

$$\sum_{k=1}^{\infty} |Q_{i k}^{m,n} - Q_{j k}^{m,n}| \leq (1 - \lambda_n) \sum_{l=1}^{\infty} |Q_{i l}^{m,n-1} - Q_{j l}^{m,n-1}| + |Q_{i k_0}^{m,n} - Q_{j k_0}^{m,n}|.$$

Now by means of (3.14) and (2.4) we infer

$$\|Q_{i}^{m,n}(e^{(i)}) - Q_{j}^{m,n}(e^{(j)})\|_1 \leq (1 - \lambda_n)\|Q_{i}^{m,n-1}(e^{(i)}) - Q_{j}^{m,n-1}(e^{(j)})\|_1 + \prod_{j=m+1}^{n-1} (1 - \lambda_j).$$
Denoting \( a_n = \| Q^{m,n}(e^{(i)}) - Q^{m,n}(e^{(j)}) \|_1 \) and applying Lemma 2.1 to \( a_n \) one gets that
\[
\| Q^{m,n}(e^{(i)}) - Q^{m,n}(e^{(j)}) \|_1 \to 0 \quad \text{as} \quad n \to \infty.
\]
So, according to Theorem 2.2 we obtain the required assertion. \( \square \)

Now we can formulate the following

**Theorem 3.4.** Let \( \{ P^{[m,n]}_{ij,k} \} \) be a q.s.p. If there exist a number \( k_0 \in \mathbb{N} \) and a sequence \( \{ \lambda_n \} \), \((0 < \lambda_n < 1)\) satisfying the conditions (3.2), (3.3) such that
\[
P^{[n-1,n]}_{ij,k_0} \geq \lambda_n \quad \text{for all} \quad i,j \in \mathbb{N}, \tag{3.16}
\]
then the q.s.p. satisfies the ergodic principle.

**Proof.** Consider the Markov process \( \{ M^{m,n}_{ij} \} \) associated with given q.s.p. (see (2.6)). Then (3.16) implies that
\[
M^{n-1,n}_{ik_0} = \sum_{l=1}^{\infty} P^{[n-1,n]}_{dl,k_0} x^{(n-1)}_l \geq \lambda_n \sum_{l=1}^{\infty} x^{(n-1)}_l = \lambda_n.
\]
Consequently, the Markov process satisfies the conditions of Theorem 3.3. Therefore the ergodic principle is valid for it. Now by means of Theorem 2.6 we infer that q.s.p. satisfies the ergodic principle. \( \square \)

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Nasir Ganikhodjaev, Department of Mechanics and Mathematics, National University of Uzbekistan, Vuzgorodok, 700174, Tashkent, Uzbekistan, and Faculty of Science, IIUM, 53100 Kuala Lumpur, Malaysia
E-mail address: nasirgani@yandex.ru

Hasan Akin, Department of Mathematics, Arts and Science Faculty, Harran University, Sanliurfa, 63200, Turkey
E-mail address: akinhasan@harran.edu.tr

Farruh Mukhamedov, Department of Mechanics and Mathematics, National University of Uzbekistan, Vuzgorodok, 700174, Tashkent, Uzbekistan
E-mail address: far75m@yandex.ru