Interaction of Finitely-Strained Viscoelastic Multipolar Solids and Fluids by an Eulerian Approach

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Abstract. A mechanical interaction of compressible viscoelastic fluids with viscoelastic solids in Kelvin–Voigt rheology using the concept of higher-order (so-called 2nd-grade multipolar) viscosity is investigated in a quasi-static variant. The no-slip contact between fluid and solid is considered and the Eulerian-frame return-mapping technique is used for both the fluid and the solid models, which allows for a “monolithic” formulation of this fluid–structure interaction problem. Existence and a certain regularity of weak solutions is proved by a Schauder fixed-point argument combined with a suitable regularization.

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1. Introduction

The mechanical interaction between fluids and solids, briefly a fluids-solid (or fluid–structure) interaction (FSI), is a largely scrutinized task in continuum mechanics. Numerically, there are more than dozen of methods to cope with this problem, cf. [34, Fig.1] for a survey. Here we focus on the so-called Eulerian-frame Reference/Return map Technique (RMT). Actually, it can be seen as philosophically similar to the popular arbitrary Lagrangian–Eulerian (ALE) formulation which maps Eulerian fluid onto a fixed reference framework to be coupled with the Lagrangian solid domain. Yet, RMT goes just opposite when mapping the solid into actual Eulerian framework to be compatible with fluid, cf. [55, Ch.6] or also [70] for a comparison.

Engineering literature exploiting the return mapping (sometimes confining on incompressible fluids or on a “mushy” interface by a level-set method) as [10,15,16,22–24,26,33,34,36,38,52,54,56,63,67,69,71] shows a high computational efficiency, although the supporting rigorous analysis or an attempt to pass to the desired limit in the diffuse mushy interface are missing. It is quite generally understood that the truly general FSI problem at large strains is troublesome for many reasons and some regularization or linearization is needed for a rigorous analysis, cf. the analysis for an incompressible linearized fully Eulerian variant in [39]. Also, the opposite fully Lagrangian alternative would not make the life easier: although such description would allow for evolving the outer boundary of the domain, it would need a higher gradient in the stored energy [37] which would not be conceptually consistent in the fluidic part where the shear elastic response should completely vanish. Moreover, in such Lagrangian setting, the frame-indifferent viscosity would have to be very nonlinear and also the interaction of spacial field (as gravitational) would be more complicated, cf. e.g. the incompressible fluid interacting with elastic solid but only in finite-dimensional approximation or small perturbation from rigid body locally in time until topology changes in [12,13]. For all these reasons, we will use higher gradients, involved here in viscosity instead of stored energy.

The main attributes (and in their combination also novelty) of the devised model are:
– Concept of hyperelastic solid materials (whose conservative-stress response comes from a stored energy) combined with viscosity in the Kelvin–Voigt viscoelastic rheology.
– The fully Eulerian rate formulation of the solid part in terms of velocity and deformation gradient is used while the deformation itself does not explicitly occur, although it can be reconstructed a-posteriori.
– Viscoelastic compressible fluid formulated compatibly with the solids but, of course, exhibiting zero shear elastic response.
– The frame indifference of the stored energy (which has to be nonconvex in terms of deformation gradient) and a physically desirable singularity under infinite compression in relation with local non-interpenetration allowed.
– The nonconservative part of the stress in the Kelvin–Voigt model containing a higher-order component reflecting the concept of nonsimple multipolar media is exploited.
– The model allows for rigorous mathematical analysis as far as global existence and certain regularity of weak solutions concerns.

To highlight the main concepts and phenomena, we confine ourselves on spatially homogeneous fluids and homogeneous solids (except Remark 3.2) and on quasistatic models (except Remark 3.1) which is legitimate in regimes not creating elastic waves or too fast vibrations, in particular for external loading varying only slowly in time and for initial conditions not far from steady states. We also confine ourselves on isothermal situations, although the energy dissipation equality as claimed in Proposition 4.3(ii) indicates that fully thermodynamical extension would be analytically amenable, too.

The plan of this article is: First, we specify the models for viscoelastic solids and fluids and their mechanical interaction in Sect. 2. Then we reformulate this model in a unified way (called “monolithic”) in Sect. 3 and show its energetics. Weak solutions are then defined and their existence are proved in Sect. 4 by the Schauder fixed point argument combined with a suitable cut-off regularization, when using also nontrivial arguments to cope with sharp interface between solid and fluidic domains.

The main notation used in this paper is summarized in the following Table 1:

2. The Mechanical Model

It is important to distinguish carefully the referential and the actual time-evolving coordinates. Our aim is to formulate the model eventually in actual configurations, i.e. the Eulerian formulation, reflecting also the reality in many (or even most) situations (and a certain general agreement) that a reference configuration is only an artificial construction and, even if relevant in some situations, becomes successively more and more irrelevant during evolution at truly large deformations (especially in fluids) and large (finite) strains. On the other hand, available experimental material data are typically referential—in particular it concerns the mass density and the stored energy per mass (in J/kg) or per referential volume (in J/m$^3$=Pa), as considered here.

2.1. Finite-strain Kinematics and Mass and Momentum Transport

We will present briefly the fundamental concepts and formulas which can mostly be found in many monographs, as e.g. [29, Part XI] or [41, Sect. 7.2]. In finite-strain continuum mechanics, the basic geometrical concept is the time-evolving deformation $y : \Omega \to \mathbb{R}^d$ as a mapping from a reference configuration of the body $\Omega \subset \mathbb{R}^d$ into a physical space $\mathbb{R}^d$. The “Lagrangian” space variable in the reference configuration will be denoted as $X \in \Omega$ while in the “Eulerian” physical-space variable by $x \in \mathbb{R}^d$. The basic geometrical object is the (referential) deformation gradient $F_R = \nabla_X y$.

We will be interested in deformations $x = y(t,X)$ evolving in time, which are sometimes called “motions”. The important quantity is the referential “Lagrangian” velocity $v_R = \frac{\partial}{\partial t} y$. As already said, our formulation will be purely Eulerian, i.e. in deforming configurations. To this aim, one defines the

\[ F_R = \nabla_X y \]

\[ v_R = \frac{\partial}{\partial t} y \]
Table 1. Summary of the basic notation used through Sects. 2–4

| Symbol | Description |
|--------|-------------|
| $\Omega$ | A fixed domain in $\mathbb{R}^d$, $d = 2, 3$ |
| $\Gamma$ | The boundary of $\Omega$ |
| $\Omega_F, \Omega_S$ | Open subsets of $\Omega$ |
| $x \in \Omega$ | Actual (Eulerian) coordinate |
| $F$ | Deformation gradient (Eulerian) |
| $T$ | Cauchy stress (symmetric—in Pa) |
| $v$ | Velocity (in m/s) |
| $\varrho$ | Mass density (in kg/m$^3$) |
| $\varrho_R = \varrho_R(X)$ | Referential mass density |
| $\xi$ | Return mapping (in m) |
| $A = F^{-1}$ | Distortion (Eulerian) |
| $\nu > 0$ | A boundary viscosity |
| $\mathbb{R}^{d \times d}_{\text{sym}} = \{ A \in \mathbb{R}^{d \times d}; A^T = A \}$ | The unit matrix |
| $\mathbb{I} \in \mathbb{R}^{d \times d}_{\text{sym}}$ | Cofactor matrix |
| $n$ | The unit outward normal to $\Gamma$ |
| $X \in \Omega$ | Referential (Lagrangian) coordinate |
| $J = \det F$ | Jacobian deformation gradient |
| $p$ | Pressure in fluid (in J/m$^3$ = Pa) |
| $\varepsilon(v) = \frac{1}{2} \nabla v^\top + \frac{1}{2} \nabla v$ | Small strain rate (in s$^{-1}$) |
| $D = D(\varepsilon(v))$ | Dissipative part of Cauchy stress |
| $\varphi_R = \varphi_R (X, F)$ | Referential stored energy (in Pa) |
| $(\cdot)' = \frac{\partial}{\partial t} (\cdot) + (v \cdot \nabla)(\cdot)$ | Convective time derivative |
| $\det(\cdot)$ | Determinant of a matrix |

inverse of a motion $\xi(t) : x \mapsto y^{-1}(t, X)$, mostly called return (alternatively called also a reference mapping or sometimes also as the inverse of a motion or inverse deformation map, etc. Then we consider the actual (i.e. Eulerian) deformation gradient $F = F_R \circ \xi$ and the actual velocity $v = v_R \circ \xi$. The return mapping $\xi$ satisfies the transport equation

$$\dot{\xi} = 0,$$

where (and thorough the whole article) we use the dot-notation $(\cdot)' := \frac{\partial}{\partial t} (\cdot) + (v \cdot \nabla)(\cdot)$ for the convective time derivative applied to scalars or, component-wise, to vectors or tensors.

Then the velocity gradient $\nabla v = \nabla_X v \nabla_x X = \dot{F} F^{-1}$, where we used the chain-rule calculus and $F^{-1} = (\nabla_X x)^{-1} = \nabla_x X$. This gives the transport-and-evolution equation for the deformation gradient as

$$\dot{F} = (\nabla v) F.$$  

From this, we also obtain the evolution-and-transport equation for the Jacobian $J = \det F$ as

$$\dot{J} = \text{Cof}(F) \dot{F} = J F^{-1} \dot{F} = \mathbb{I} : \dot{F} F^{-1} = J \mathbb{I} : \nabla v = J \text{div} v,$$

where $\mathbb{I}$ denotes the unit matrix and where “Cof” denotes the cofactor matrix, i.e. the matrix composed from signed $(d-1) \times (d-1)$ minors; actually Cof$F = (\det F) F^{-\top}$. Sometimes, the stored energy is expressed in terms of the left Cauchy-Green tensor $B = F F^\top$ which is then evolving/transported as $\dot{B} = (\nabla v)^\top B + B(\nabla v)$. 

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The reference-mapping gradient, called a distortion, is \( A = \nabla \xi = F^{-1} \) and together with its determinant \( \det A = 1/J \), likewise (2.2) and (2.3), satisfy the transport-and-evolution equations
\[
\dot{A} = -A(\nabla \nu) \quad \text{and} \quad \dot{\det A} = -(\det A)\text{div} \nu .
\]
(2.4)
To highlight the dependence on the Eulerian coordinate \( x \), one can express \( F = (\nabla \xi)^{-1} \) by the standard algebra through cofactors as
\[
F(x) = \frac{\text{Cof}(\nabla \xi(x))^{\top}}{\det(\nabla \xi(x))} .
\]
(2.5)

Basic ingredients of mechanical models are defined in the referential frame, which also corresponds to the standardly available experimental data. We consider a hyperelastic material in the viscoelastic rheology of the Kelvin–Voigt type. The basic ingredients are thus the referential mass density \( \rho_R = \rho_R(X) \), the elastic stored energy \( \varphi_R = \varphi_R(X, \cdot) \), and the non-conservative dissipative stress \( D_R = D_R(X, \cdot) \). Let us remind that the adjective "hyperelastic" refers to the (widely accepted) concept that the elastic response comes from the mentioned stored energy \( \varphi_R \). Wanting to formulate the model in Eulerian frame, we use the shorthand notation which however indicates dependence on \( t \) considered at a current time \( t \):
\[
\rho^\xi_R(t, x) = \rho_R(\xi(t, x)), \quad \varphi^\xi_R(t, t, \cdot) = \varphi_R(\xi(t, x), \cdot), \quad \text{and} \quad D^\xi_R(t, t, \cdot) = D_R(\xi(t, x), \cdot).
\]
(2.6)
This stored energy \( \varphi^\xi_R = \varphi^\xi_R(t, t, F) \) (still counted per referential volume) is to be consistently considered dependent on the Eulerian deformation gradient \( F \) from (2.5) while \( D^\xi_R = D^\xi_R(t, t, e(\nu)) \) is dependent on the symmetric gradient of the Eulerian velocity \( \nu \). The actual Eulerian stored energy would then be \( \varphi^\xi(t, x, F) = \varphi^\xi_R(t, x, F)/\det F \). Then also the conservative part of the Cauchy stress
\[
T^\xi(t, x, F) = \frac{[\varphi^\xi_R/t_F(t, x, F)]^{\top} F}{\det F(x)}
\]
(2.7)
depends on Eulerian variables \( x \) and \( F \). The formula (2.7) is actually obtained by the variation in the reference frame of the first Piola-Kirchhoff stress with respect to \( F \) when push forward to the actual Eulerian frame.

The mass density (in kg/m\(^3\)) is a so-called extensive variable, and its transport (expressing the conservation of mass) writes as the continuity equation \( \frac{\partial}{\partial t} \rho + \text{div}(\rho \nu) = 0 \), or, equivalently, as the mass evolution-and-transport equation
\[
\dot{\rho} = -\rho \text{div} \nu .
\]
(2.8)
Imposing the initial condition \( \rho|_{t=0} = \rho_R/F|_{t=0} \) for (2.8), one can alternatively determine the density \( \rho \) instead of the differential equation (2.8) from the algebraic relation
\[
\rho = \frac{\varphi^\xi_R}{\det F} = \varphi^\xi_R/\det(\nabla \xi) ;
\]
(2.9)
recall that \( \rho_R = \rho_R(X) \) is the mass density in the reference configuration.

All the equations (2.1)–(2.9) are understood for a.a. \( x \) and are thus truly Eulerian. For the Eulerian formulation of continuum mechanics see classical textbooks as e.g. [29,40,41].

2.2. Viscoelastic Multipolar Solids

For simplicity, let us consider spatially homogeneous material, cf. Remark 3.2 for an inhomogeneous case. Then the referential density and the referential stored energy are \( X \)-independent and thus \( \varphi^\xi_R \) and \( \varphi^\xi_R \) do not depend on \( \xi \) and we can write simply \( \rho_S \) and \( \varphi_S \), respectively, with the subscript "S" standing for "solid". Actually, we consider \( \varphi_S : \text{GL}^+(d) \to \mathbb{R} \) with the group of invertible matrices with positive determinant \( \text{GL}^+(d) = \{ F \in \mathbb{R}^{d\times d}; \det F > 0 \} \). The further mentioned ingredient is a (possibly nonlinear) monotone non-conservative stress, i.e. here \( D_S : \mathbb{R}_{\text{sym}}^{d\times d} \to \mathbb{R}_{\text{sym}}^{d\times d} \). Here, in addition, we consider
also the nonsimple (multipolar) viscosity, i.e. the non-conservative monotone, so-called hyperstress $\mathcal{H} : \mathbb{R}^{d \times d \times d} \to \mathbb{R}^{d \times d \times d} : G \to \nu|G|^{-2}G$ with some $\nu > 0$ and $s > d$; the preposition "hyper" means that it contributes to the stress through its divergence. Neglecting inertia, this sort of solids are then governed by the quasistatic system for $v$ and $\xi$:

$$\begin{align*}
\text{div} \left( \frac{\partial S}{\partial F} F^T \det F \right) + D_S(e(v)) - \text{div} \mathcal{H}(\nabla e(v)) + \frac{\rho_S}{\det F} \mathbf{g} &= 0 \\
\text{with } F &= \text{Cof}(\nabla \xi) \det(\nabla \xi), \quad \mathcal{H}(\nabla e(v)) = \nu|\nabla e(v)|^{s-2}\nabla e(v), \quad \text{and} \quad (2.10a) \\
\dot{\xi} &= 0. \quad (2.10b)
\end{align*}$$

The dynamic variant with an inertial force will be discussed in Remark 3.1 below. The boundary conditions for the 4th-order equation (2.10a) are rather delicate because of the higher-order viscosity and will be specified later.

The classical simple-material models involve basic strain/stress (and their rates), but it brings serious analytical troubles due to inevitable geometrical and material nonlinearities at large strains, as articulated in particular by Ball [5,6]. The mentioned concept of the so-called nonsimple materials represents a widely used generalization by considering strain or stress (rates) gradients (or even fractional gradients leading to integral-type truly nonlocal models). It opens a great menagerie of options and of microstructural interpretations, cf. [65]. It dates back to the general concepts of Green, Rivlin, Mindlin, and Toupin [28,42,45,68]. Mechanically, higher gradients can suit for fitting various phenomenological effects more properly than simple-material models, in particular flow profiles [8,11,27] or, in the dynamical variant as in Remark 3.1, attenuation and speed dispersion in propagation of elastic waves as discussed in [35,57]. More specifically, the higher gradients in strain rate, which is the option adopted here under the name multipolar materials, leads to so-called normal dispersion, i.e. the speed of wave decreases with their frequency.

The original concept from 60ieith was later developed by J. Nečas et al. as multipolar fluids [43,44,46,47] or solids [45,51,64], later also by E. Fried and M. Gurtin [27]. In the context of fluid with rigid solid interaction, one refers to [25].

More specifically, in (2.10a) we have used the concept of 2nd-grade nonsimple media, in fluidic variant also called bipolar (or dipolar) fluids [7,11]. Here we used it in a nonlinear variant with the “analytical” goal to ensure $v(t, \cdot)$ Lipschitz continuous integrably in time, i.e. $\|\nabla v(t, \cdot)\|_{L^\infty(\Omega; \mathbb{R}^{d \times d})} \in L^1(I)$. This avoids formation of singularities in the transported fields, i.e. in $\xi, g, F, A$, and $J$, cf. also Remark 4.9 below. For this, we will need $s > d$ to rely on the Sobolev embedding $W^{1, s}(\Omega) \subset L^\infty(\Omega)$. This qualification together with the “impenetrability” boundary condition $v \cdot n = 0$ seems “nearly” necessary, cf. [19] for $s \geq d$.

Reminding the mentioned difficulty of nonlinear viscoelastodynamics of simple materials [5,6] as far as mere existence of conventional weak-solutions, let us mention results for Eulerian nonsimple incompressible materials with linear viscosity of the 2nd-grade [7] and of the 4th-grade [45]; the former one not yielding enough regular velocity as needed here for eliminating singularities in the transport (2.1) and (2.2). For nonlinear 2nd-grade viscosity in the compressible media, we refer to [59–61]. For completeness, let us mention that there are some results [9] for multipolar variants involving also higher time derivatives [21].

It is physically reasonable to assume the frame indifference of the stored energy, i.e. $\varphi_S(F) = \varphi_S(QF)$ for any $Q \in \text{SO}(d) = \{A \in \mathbb{R}^{d \times d}; A^T A = A A^T = I, \text{det } A = 1\}$. In particular, this makes the conservative part of the Cauchy stress $T$ symmetric.

An example for the frame-indifferent energy $\varphi_S$ is the so-called neo-Hookean material whose elastic response is governed by the stored-energy

$$\varphi_S(F) = \frac{1}{2} K_E (\text{det } F - 1)^2 + \frac{1}{2} G_E \left( \frac{\text{tr}(F F^T)}{(\text{det } F)^2/d} - d \right) \quad (2.11)$$
for $\det F > 0$ otherwise $\varphi(F) = +\infty$. Alternatively, the second term in (2.11) is considered as $G_F (\text{tr} (FF^\top)/(\det F)^{2/d} - d - \ln(\det F))/2$ to comply with (4.2b) below. In (2.11), $K_F$ means the elastic bulk modulus and $G_F$ is the elastic shear modulus, also called the second Lamé coefficient. This ansatz has been used e.g. in [10,23,52,67,70], possibly in an incompressible variant as in Remark 3.5 below. Another popular stored energy describes a so-called St. Venant-Kirchhoff material, as used e.g. in [10,23,32,50,52]::

$$
\varphi_S(F) = \frac{1}{2} \left( K_F - \frac{2}{d} G_F \right) (\text{tr} E)^2 + G_F |E|^2 \quad \text{with} \quad E = (F^\top F - I)/2
$$

(2.12)

where $E$ is Green-Lagrange (sometimes also called Green-St. Venant) strain tensor and $K_F$ is the bulk modulus; actually $K_F - 2G_F/d$ is the first Lamé coefficient.

### 2.3. Viscoelastic Fluids

Again, we confine to homogeneous fluids where the referential density and the referential stored energy are $X$-independent and thus we can write simply $\varrho_F$ and $\varphi_F$ instead of $\varrho_F^X$ and $\varphi_F^X$, respectively, with the subscript “$F$” standing for “fluid”.

Fluids are characterized as continua whose shear elastic response vanishes and whose volumetric response does not bear negative pressures (or at least not big negative pressures). Thus the only elastic response is in the volumetric part, i.e. $\varphi_F$ depends only on the isochoric part $(\det F)I$ of $F$ and completely ignores the deviatoric part $F/\det F$ and we can write $\varphi_F(F) = \varphi_F(J)$ with $J = \det F$ and some $\phi_F : (0, +\infty) \to (0, +\infty)$, cf. e.g. [40, p.10]. As fluids do not withstand (too much big) tension, $\phi_F$ is not coercive. Actually fluids are primarily liquids or gases (disregarding some other materials as magma etc). From the mechanical viewpoint, the difference between liquids and gases is that gases have zero mass density (vacuum) if the pressure $p = -\varphi'_F(J)$ is zero, in contrast to liquids which sometimes bear even a slightly negative pressure without going to vacuum.

Using the calculus $\det'(\cdot) = \text{Cof}(\cdot)$ and the algebra $A^{-1} = \text{Cof}A^\top/\det A$ for a square matrix $A$, the conservative part of the Cauchy stress then reduces to

$$
\frac{\varphi'_F(F)F^\top}{\det F} = \frac{\varrho_F(\det F)(\det' F)F^\top}{\det F} = \frac{\varrho_F(\det F)(\text{Cof} F)F^\top}{\det F} = \phi'_F(J) I.
$$

(2.13)

Thus gases are characterized by $p \to 0+$ for $J \to +\infty$.

The system (2.10) of $d+d$ equations thus reduces to the system of $d+1$ equations:

$$
\text{div}(D_F(e(v)) - \text{div} \mathcal{H}(\nabla e(v))) + \frac{\varrho_F g}{J} = \nabla p
$$

with $p = -\varphi'_F(J)$, \quad $\mathcal{H}(\nabla e(v)) = \nu|\nabla e(v)|^{s-2}\nabla e(v)$, \quad and

(2.14a)

$$
\dot{J} = J \text{ div } v.
$$

(2.14b)

Actually, the evolution-and-transport equation $\dot{J} = J \text{ div } v$ in (2.14b) can be replaced by the algebraic relation $J = 1/\det(\nabla \xi)$ provided the transport equation (2.10b) would be added. This is actually used in what follows.

Realizing that the actual density $\varrho = \varrho_F / J$, we can express the pressure as a function of density as $p = -\varphi'_F(J) = -\phi'_F(\varrho_F / \varrho)$. This is to be understood as an isentropic state equation.

### 2.4. Fluid–solid Interaction

We consider a fixed bounded domain $\Omega \subset \mathbb{R}^d$ covered up to zero measure by two (not necessarily connected open sets $\Omega_S(t)$ and $\Omega_F(t)$ depending on time $t \in I = [0, T]$ with $T$ a fixed time horizon, containing the solid and the fluid, respectively. The interface between $\Omega_F(t)$ and $\Omega_S(t)$ is denoted by $\Gamma_{FS}(t) = \partial \overline{\Omega}_F(t) \cap \partial \overline{\Omega}_S(t)$ with the bar denoting the closure. The (fixed) boundary of $\Omega$ will be denoted by $\Gamma$, cf. Fig. 1.
Of course, we need to prescribe the boundary conditions on $\Gamma$ and also the coupling conditions on the (evolving) interface $\Gamma_{FS}(t)$. Let us note that both (2.10a) and (2.14a) are elliptic problems for $d$-dimensional vectorial fields $v$ respectively on $\Omega_S(t)$ and on $\Omega_F(t)$ parameterized by time. Therefore, they need $d+d$ conditions on the boundaries of each domains. On the fluid–solid interface $\Gamma_{FS}(t)$ where two domains merge, we thus need 4$d$ transition conditions.

On the boundary $\Gamma$ of $\Omega$ which is considered fixed, cf. Fig. 1, we denote by $n$ the unit outward normal to the boundary $\Gamma$ and prescribe $d+d$ boundary conditions:

$$ v = 0 \quad \text{and} \quad \nabla e(v) : (n \otimes n) = 0. $$

(2.15)

The first condition fixes not only the shape of the boundary (due to the normal velocity zero) as most frequently adopted in literature for Eulerian formulation, but even the entire velocity field also in the tangential direction. This is needed for global invertibility of $\xi(t, \cdot) : \Omega \to \Omega$ used for (4.16) below. The latter condition in (2.15) is one of simple variationally consistent options for the weak formulation of the 4th-order hyper-viscosity terms in (2.10a) and (2.14a). Another analytically well amenable option instead of Neumann-type conditions $\nabla e(v) : (n \otimes n) = 0$ would be the higher-order Dirichlet condition $(n \cdot \nabla)v = 0$ or a higher-order Neumann condition $n \cdot \text{div}(|\nabla e(v)|^{s-2} \nabla e(v)) = 0$. Some options for a relaxation of these (sometimes rather artificial) conditions are later outlined in Remarks 4.7 and 4.8.

The transition conditions on $\Gamma_{FS}(t)$ are more or less dictated by our (quite natural) intention to allow for a “monolithic” description of the overall fluid–solid interaction problem. More specifically, as quite standard, we prescribe continuity of velocity (i.e. no-slip condition) and traction equilibrium and, in addition due to the multipolar material, also the continuity of symmetric velocity gradient, i.e.

$$ [v]_{\Gamma_{FS}(t)} = 0, \quad t_S = t_F, \quad \text{and} \quad [e(v)]_{\Gamma_{FS}(t)} = 0 \quad \text{on} \quad \Gamma_{FS}(t), $$

(2.16)

where $[\cdot]_{\Gamma_{FS}(t)}$ denotes the jump of normal component of the indicated field across $\Gamma_{FS}(t)$. The tractions of the solid $t_S$ and the fluid $t_F$ on $\Gamma_{FS}(t)$ are respectively the vectors

$$ t_S = (\frac{\varphi_S(F)}{\det F} F^\top + D_S(e(v)) - \text{div} \mathcal{H}(\nabla e(v))) n_S $$

and

$$ t_F = (\frac{\varphi_F(\det F)}{\det F} \mathbb{I} + D_F(e(v)) - \text{div} \mathcal{H}(\nabla e(v))) n_F, $$

(2.17a)

(2.17b)

where, at a current time $t$, we denote the “outward” unit normal $n_S(x)$ to $\Gamma_{FS}(t)$ at $x \in \Gamma_{FS}(t)$ oriented from the solid towards the fluid domain, and $n_F(x) = -n_S(x)$. The fields $F$ and $e(v)$ in (2.17a) consider values on $\Omega_S(t)$ while in (2.17b) they consider values on $\Omega_F(t)$. Actually, the traction equilibrium $t_S = t_F$ means that there is no interface tension considered in the model.

The overall fluid–solid interaction problem then reads in its classical formulation as

the system (2.10) on $\Omega_S(t)$,

(2.18a)

the system (2.14) on $\Omega_F(t)$,

(2.18b)

the transient conditions (2.16) on $\Gamma_{FS}(t)$,

(2.18c)

the boundary conditions (2.15) on $\Gamma$.

(2.18d)
3. A monolithic Model of Fluid–Solid Interaction

The Eulerian formulation of the solid model and the relation of the fluidic model with the stored energy degenerated in the deviatoric part allows for a monolithic formulation to merge both models and to incorporate the transient conditions on the evolving fluid–solid interface $\Gamma_{FS}(t)$.

Let us first define a “monolithic” referential mass density $\varrho_R$, stored energy $\varphi_R$, conservative part of the Cauchy stress (let us denote it by $T_R$), and dissipative stress $D_R$ as

$$
\varphi_R(X, F) = \begin{cases} 
\varphi_S(F) & \text{if } X \in \Omega_S, \\
\phi_F(\det F) & \text{if } X \in \Omega_F,
\end{cases},
T_R(X, F) = \begin{cases} 
\frac{\varphi'_S(F)F^T}{\det F} & \text{if } X \in \Omega_S, \\
\phi'_F(\det F)\mathbb{I} & \text{if } X \in \Omega_F,
\end{cases} \tag{3.1a}
$$

$$
D_R(X, e) = \begin{cases} 
D_S(e) & \text{if } X \in \Omega_S, \\
D_F(e) & \text{if } X \in \Omega_F,
\end{cases}, \quad \varrho_R(X) = \begin{cases} 
\varrho_S & \text{if } X \in \Omega_S, \\
\varrho_F & \text{if } X \in \Omega_F.
\end{cases} \tag{3.1b}
$$

Actually, the reference (presumably “small”) 2nd-grade hyper-viscosity coefficient $\nu$ can also be different for the fluid and the solid.

As now $\varphi_R$, $T_R$, $D_R$, and $\varrho_R$ are inhomogeneous, their Eulerian representation will depend on $\xi$. Therefore, from now on, we will use the notation (2.6), i.e. the quantities $\varphi^\xi_R$, $T^\xi_R$, $D^\xi_R$ and $\varrho^\xi_R$ are thus defined for a.a. $x \in \Omega$.

In terms of (3.1), for all $t \in I$, the systems (2.10) and (2.14) can be written “monolithically” as a single system of $d+d$ equations for $v$ and $\xi$ of $\Omega_F(t) \cup \Omega_S(t)$:

$$
div \left( T^\xi_R \left( \frac{\text{Cof}(\nabla \xi)\nabla e(v)}{\det(\nabla \xi)} \right) + D^\xi_R(e(v)) - div \mathcal{H}(\nabla e(v)) \right) + \varphi^\xi_R g \det(\nabla \xi) = 0 
$$

with $\mathcal{H}(\nabla e(v)) = \nu |\nabla e(v)|^{s-2} \nabla e(v)$ and \( \frac{\partial \xi}{\partial t} = -(v \cdot \nabla) \xi; \) \hspace{1cm} (3.2a)

$$
\varrho^\xi_R(x) = -\frac{(v \cdot \nabla) \xi}{\rho_R(x)}
$$

here and in what follows, the $x$-dependence of $T^\xi_R$ and $D^\xi_R$ and of $\varrho^\xi_R$ is omitted for notational simplicity. This monolithic model is to be completed by the boundary (2.15), while it hides the fluid–solid interface $\Gamma_{FS}(t)$ as well as the transient conditions (2.16), cf. Remark 4.5 below.

The energetics behind (3.2) can be revealed by testing (3.2a) by $v$ and by using Green’s formula twice on $\Omega$ together with a surface Green formula on $\Gamma$ because of the multipolar viscosity, cf. (3.7) below.

Let us first analyze the conservative part $\text{div} T^\xi_R(F)$ with $F = \text{Cof}(\nabla \xi)/\det(\nabla \xi)$. Using the algebra $F^{-1} = \text{Cof} F^\top/\det F$ and the calculus $\text{det}(F) = C$, we can write the conservative part of the Cauchy stress as

$$
T^\xi_R(F) = \left[ \frac{\varphi^\xi_R(F)}{\det F} \right] F^\top = \left[ \frac{\varphi^\xi_R(F)}{\det F} - \frac{\varphi^\xi_R(F) F^{-1}}{\det F} \right] F^\top + \frac{\varphi^\xi_R(F)}{\det F} \mathbb{I}
$$

$$
= \left( \frac{\varphi^\xi_R(F)}{\det F} - \frac{\varphi^\xi_R(F) \text{Cof} F}{(\det F)^2} \right) F^\top + \frac{\varphi^\xi_R(F)}{\det F} \mathbb{I} = \left[ \frac{\varphi^\xi_R(F)}{\det F} \right]^\top F^\top + \frac{\varphi^\xi_R(F)}{\det F} \mathbb{I}.
$$

(3.3)

Using the calculus (3.3) and the matrix algebra $A:(BC) = (B^\top A):C = (AC^\top):B$ for any square matrices $A$, $B$, and $C$ and also the evolution-and-transport equation (2.2), we obtain

$$
T^\xi_R(F):e(v) = \left[ \frac{\varphi^\xi_R(F)}{\det F} \right]^\top e(v) = \left( \frac{\varphi^\xi_R(F)}{\det F} \right)^\top F^\top + \varphi^\xi_R(F) \det F \mathbb{I} = \left( \frac{\varphi^\xi_R(F)}{\det F} \right)^\top : \left( \frac{\partial F}{\partial t} + (v \cdot \nabla) F \right) + \varphi^\xi_R(F) \det F \mathbb{I} \div v.
$$

(3.4)
Let us recall that we agreed to omit the dependence on the variable \( x \) for notational simplicity. Further, we use the calculus
\[
\frac{\partial}{\partial t} \left( \frac{\varphi^\xi_R (F)}{\det F} \right) = \left[ \frac{[\varphi^\xi_R]_X (F)}{\det F} \right] \frac{\partial \xi}{\partial t} + \left[ \frac{\varphi^\xi_R (F)}{\det F} \right]' \frac{\partial F}{\partial t} \quad \text{and}
\]
\[
\nabla \left( \frac{\varphi^\xi_R (F)}{\det F} \right) \cdot v = \left[ \frac{[\varphi^\xi_R]_X (F)}{\det F} \right] (v \cdot \nabla) \xi + \left[ \frac{\varphi^\xi_R (F)}{\det F} \right]' (v \cdot \nabla) F,
\]
so that (3.4) turns into
\[
T^\xi_R (F) : e(v) = \frac{\partial}{\partial t} \left( \frac{\varphi^\xi_R (F)}{\det F} \right) + \nabla \left( \frac{\varphi^\xi_R (F)}{\det F} \right) \cdot v \\
+ \frac{\varphi^\xi_R (F)}{\det F} \text{div } v - \left[ \frac{[\varphi^\xi_R]_X (F)}{\det F} \right]' \left( \frac{\partial \xi}{\partial t} + (v \cdot \nabla) \xi \right),
\]
(3.5)
\[
= 0 \text{ due to (3.2b)}
\]

Analytically, it is rather nontrivial that the last term indeed vanishes, cf. Step 4 in the proof of Proposition 4.3 below. Anyhow, continuing formally at this moment and relying further on the integrability rigorously proved later in Sect. 4, we can integrate (3.5) over \( \Omega \) and use it for the following calculations (exploiting Green’s formula twice on \( \Omega \)) to obtain:
\[
\int_{\Omega} -\text{div} T^\xi_R (F) \cdot v \, dx = \int_{\Omega} T^\xi_R (F) : e(v) \, dx - \int_{\Gamma} v \cdot T^\xi_R (F) n \, dS \\
= \frac{d}{dt} \int_{\Omega} \frac{\varphi^\xi_R (F)}{\det F} \, dx + \int_{\Omega} \nabla \left( \frac{\varphi^\xi_R (F)}{\det F} \right) \cdot v + \frac{\varphi^\xi_R (F)}{\det F} \text{div } v \, dx - \int_{\Gamma} v \cdot T^\xi_R (F) n \, dS \\
= \frac{d}{dt} \int_{\Omega} \frac{\varphi^\xi_R (F)}{\det F} \, dx + \int_{\Gamma} \frac{\varphi^\xi_R (F)}{\det F} (v \cdot n) - [T^\xi_R (F) n]_T \cdot v \, dS.
\]
(3.6)

The further contribution is from the dissipative part of the Cauchy stress, let us abbreviate it as \( D^\xi = D^\xi_R (e(v)) - \text{div } \mathcal{H}(\nabla e(v)) \) with the hyperstress \( \mathcal{H}(\nabla e(v)) = \nu |\nabla e(v)|^{s-2} \nabla e(v) \). It uses Green’s formula over \( \Omega \) twice and the surface Green formula over the boundary \( \Gamma \) assumed smooth. Then
\[
\int_{\Omega} -\text{div } D^\xi \cdot v \, dx = \int_{\Omega} D^\xi : \nabla v \, dx - \int_{\Gamma} v \cdot D^\xi n \, dS \\
= \int_{\Omega} D^\xi_R (e(v)) : e(v) + \mathcal{H}(\nabla e(v)) |\nabla v|^{s} \, dx - \int_{\Gamma} v \cdot D^\xi n - \mathcal{H}(\nabla e(v)) : (n \otimes \nabla v) \, dS \\
= \int_{\Omega} D^\xi_R (e(v)) : e(v) + \nu |e(v)|^{s} \, dx - \int_{\Gamma} (D^\xi n + \text{div}_r (\mathcal{H}(\nabla e(v)) n) \cdot v \, dS.
\]
(3.7)

Here we used the surface Green formula and the boundary condition \( \mathcal{H}(\nabla e(v)) : (n \otimes n) = 0 \) for
\[
\int_{\Gamma} \mathcal{H}(\nabla e(v)) : (n \otimes \nabla v) \, dS = \int_{\Gamma} \mathcal{H}(\nabla e(v)) : (n \otimes n) \frac{\partial v}{\partial n} + \mathcal{H}(\nabla e(v)) : (n \otimes \nabla v) \, dS \\
= 0 \text{ due to (2.15)}
\]
\[
= \int_{\Gamma} (\text{div}_r n) (\mathcal{H}(\nabla e(v)) : (n \otimes n)) \cdot v - \text{div}_r (\mathcal{H}(\nabla e(v)) n) \cdot v \, dS - \int_{\Gamma} \text{div}_r (\mathcal{H}(\nabla e(v)) n) \cdot v \, dS,
\]
where we used the decomposition of \( \nabla v \) into its normal and tangential parts, i.e. written as \( \nabla v = (\partial v / \partial n) n + \nabla_r v \). Here, \( \text{div}_r = \text{tr}(\nabla_r) \) denotes the \((d-1)\)-dimensional surface divergence with \( \text{tr}(\cdot) \) being the trace of a \((d-1) \times (d-1)\)-matrix and \( \nabla_r v = \nabla v - (\partial v / \partial n) n \).
Now we sum (3.6) and (3.7). The last integral in (3.6) summed with the last integral in (3.7) allows us to use the latter boundary condition in (2.15). Altogether, we thus obtain (at least formally) the mechanical energy dissipation balance:

\[
\frac{d}{dt} \int_{\Omega} \phi_R(F) \, dx + \int_{\Omega} D_R(e(v)) \cdot (\nu \nabla e(v) + \nu \nabla e(v))^s \, dx = \int_{\Omega} \phi_R(\nu \cdot v) \, dx. \tag{3.8}
\]

Remark 3.1. (Dynamic problems: adding inertia) The dynamical modification of the presented quasistatic models by considering also inertia leads to expansion of the first equations in (2.10a) and (2.14a) respectively as

\[
\frac{\partial (\rho v)}{\partial t} = \left( \frac{\varphi_R^\prime(F)}{\det F} F^\top \right) + D_S(e(v)) - \text{div} \mathcal{H}(\nabla e(v)) - \rho \nu \nabla v + \rho g \quad \text{with} \quad \rho = \frac{\rho_S}{\det F} \quad \text{and}
\]

\[
\frac{\partial (\rho v)}{\partial t} = \text{div} (D_F(e(v)) - \text{div} \mathcal{H}(\nabla e(v)) - \rho \nu \nabla v + \nabla \phi_F(J) + \rho g \quad \text{with} \quad \rho = \frac{\rho_F}{J}.
\]

Naturally, this expansion needs an additional initial condition for \( v \). Such expanded model allows for propagation of elastic waves. In solid regions, there can be both longitudinal (pressure) and shear waves (denoted respectively as P- and S-waves) while in fluidic regions, only P-waves can propagate. On the fluid–solid interface, various reflections of S-waves in solids and refractions and transformations of P-waves can occur, as well as so-called Love, Rayleigh, or head (von Schmidt) waves. Adding inertia would help in a-priori estimation strategy (4.4) working for more general \( g \) and, in the modification of Remark 4.7 below, would allow for pure slip on \( \Gamma \) (i.e. \( \nu = 0 \)) even if \( f \neq 0 \). Also, the mentioned improved estimation strategy (4.4) by considering inertia would facilitate an anisothermal extension like in [60]. On the other hand, handling the inertial force \( \rho \dot{v} \), i.e. the additional terms \( \frac{\partial}{\partial t}(\rho v) + \text{div} (\rho v) \), brings various technicalities in convergence of the momentum \( \rho v \) and the velocity \( v \) itself and, mainly, the uniqueness of the response \( v \) for fixed \( \xi \) is troublesome and thus the Schauder fixed-point arguments used below do not work. Also e.g. (4.5d) is not at disposal. As a result, the analysis would need to use a semi-Galerkin approximation in the spirit of Remark 4.6 below like in [60,62].

Remark 3.2. (Spatially inhomogeneous media) The homogeneity of particular solid and fluid areas in their reference configuration was considered in Sect. 2.2 and 2.3 rather for notational simplicity and smoothly inhomogeneous media can be easily considered, as well. Actually, when admitting a “thin” mushy-like layer instead of a sharp interface \( \Gamma_{FS} \) between fluid and solid regions so that \( \varphi_R(\cdot,F) \in W^{1,1}(\Omega) \), the last term in (3.5) would be zero a.e. pointwise and we can even avoid the requirement of global invertibility of \( \xi \) used in the proof of Proposition 4.3 to justify that this last term in (3.5) vanishes at least when integrated over \( \Omega \) in (3.6). This would allow for considering a more general boundary conditions (2.15) on \( \Gamma \) in Remark 4.7 below. The inhomogeneous modification of the mass density \( \rho_R(X) \) and the dissipative stress \( D_R(X,e) \) is even simpler.

Remark 3.3. (Usage of distortion) In view of (2.5), we implicitly formulated the problem in terms of the distortion \( A = \nabla \xi \) instead of \( F = (\nabla \xi)^{-1} \) like e.g. in [49]. Let us realize that the continuity equation (2.9) can also be written as \( \rho = \rho_R(\det A) \) and the stress \( \varphi_R^\prime(F)F^\top/\det F \) in (2.10a) as \( (\det A)\varphi_R^\prime(A^{-1})A^{-\top} \).

Remark 3.4. (Level-set method) In literature as e.g. [33,39,66,69], one also plays with the tensorial evolution-and-transport (2.2) of \( F \) or of \( B = FF^\top \) and replace the vectorial transport (2.1) of \( \xi \) by a transport of a scalar \([-1,1]\)-valued “phase-field” variable \( \chi \), i.e. \( \dot{\chi} = 0 \). Its positive (resp. negative) value decides whether the medium is solid (resp. fluid), while \( \Gamma_{FS}(t) = \{ x \in \Omega \mid \chi(t,x) = 0 \} \). Then, instead of (3.1), one can then make \( \varphi, T, \) and \( D \) dependent on \( \chi \) in a discontinuous way instead of \( x \). The analysis seems however complicated due to this discontinuity, unless one admits some “mushy” interface by admitting the \( \chi \)-dependence of \( \varphi, T, \) and \( D \) continuous.
Remark 3.5. (Incompressible models) Often, the compressibility is small and is thus neglected, which is legitimate in particular in fluids. It means that \( \det F = 1 \). In solids, this is a nonlinear constraint while in fluids \( J = 1 \) is affine. Anyhow, monolithically this constraint can be ensured by the linear constraint \( \text{div} \mathbf{v} = 0 \) provided \( \det F |_{t=0} = 1 \), i.e. \( \det(\nabla \xi |_{t=0}) = 1 \) on \( \Omega \), which can be easily seen from (2.3). In particular, an incompressible neo-Hookean material, as considered e.g. in [23,24,30,52,71], simplifies by omitting the first term in (2.11); actually the incompressible variant is motivated in particular by the fluidic part where the elastic shear modulus \( G_F \) vanishes so it suggests putting \( K_F = +\infty \) in (2.11) as a legitimate approximation which clearly leads to \( \det F = 1 \).

4. Existence of Weak Solutions

We will be interested in an initial-value problem (3.2) and provide a proof of global-in-time existence and of a certain regularity of weak solutions. To this aim, the concept of multipolar viscosity is essential but, anyhow, still quite nontrivial and carefully ordered arguments will be needed.

We thus need to prescribe a suitable initial condition, i.e. here in our quasistatic problem only for the return mapping \( \xi \). Notably, in terms of (2.2), we would prescribe an initial condition \( \mathbf{F}_0 \) for \( \mathbf{F} \) while, in terms of (2.4), we would prescribe an initial condition \( \mathbf{A}_0 = \mathbf{F}_0^{-1} \). For the return-mapping transport equation (2.1), we prescribe the initial condition

\[
\xi(0) = \xi_0,
\]

assuming \( \nabla \xi_0 = \mathbf{A}_0 \). This implicitly determines an initial condition for the mass density \( \varrho_0 = \varrho_{\mathbf{R}} \det(\nabla \xi_0) \).

We will use the standard notation concerning the Lebesgue and the Sobolev spaces, namely \( L^p(\Omega; \mathbb{R}^n) \) for Lebesgue measurable functions \( \Omega \to \mathbb{R}^n \) whose Euclidean norm is integrable with \( p \)-power, and \( W^{k,p}(\Omega; \mathbb{R}^n) \) for functions from \( L^p(\Omega; \mathbb{R}^n) \) whose all derivatives up to the order \( k \) have their Euclidean norm integrable with \( p \)-power. We also write briefly \( H^k = W^{k,2} \). The notation \( p^* \) will denote the exponent from the embedding \( W^{1,p}(\Omega) \subset L^{p^*}(\Omega) \), i.e. \( p^* = \frac{dp}{d-p} \) for \( p < d \) while \( p^* \geq 1 \) arbitrary for \( p = d \) or \( p^* = +\infty \) for \( p > d \). Moreover, for a Banach space \( X \) and for \( I = [0,T] \), we will use the notation \( L^p(I; X) \) for the Bochner space of Bochner measurable functions \( I \to X \) whose norm is in \( L^p(I) \) while \( W^{1,p}(I; X) \) stands for functions \( I \to X \) whose distributional derivative is in \( L^p(I; X) \). Also, \( C(\cdot) \) and \( C^1(\cdot) \) will denote spaces of continuous and continuously differentiable functions, respectively. Eventually, \( C^w(I; X) \) for \( X \) reflexive will denote the space of weakly continuous functions \( I \to X \).

Moreover, as usual, we will use \( C \) for a generic constant which may vary from estimate to estimate. Let us first state a result exploiting spatial homogeneity of elastic response in the particular (solid and fluidic) subdomains and a bounded-distortion argument, relying on that the referential \((d-1)\)-dimensional fluid–solid interface \( I_{FS} \) cannot “inflate” during the evolution to some positive measure set:

**Lemma 4.1.** Let \( \xi \) be continuous and \( \xi(t, \cdot) \in C^1(\overline{\Omega}; \mathbb{R}^d) \) with \( \det(\nabla \xi) > 0 \) on \( \overline{\Omega} \). Then, for a.a. \((t, x)\), the value \( \xi(t, x) \) belongs to the open set \( \Omega_F \cup \Omega_S \).

**Proof.** Here we rely on a subtle analytical argument that \( \xi(t, \cdot) \) is of a so-called bounded distortion in the sense that \( |\nabla \xi|/\det(\nabla \xi) \) is bounded. Thus we have at our disposal the well-known property [31,53] that pre-images of zero-measure sets (here \( I_{FS}(t) \)) are sets of measure zero unless \( \xi(t, \cdot) \) is constant. Yet, \( \xi(t, \cdot) \) cannot be constant since \( \det(\nabla \xi) > 0 \) is assumed.

As \( \xi \) is assumed continuous and \( \Omega_F \cup \Omega_S \) is open and of a full measure, we can conclude that \( \xi(t, x) \in \Omega_F \cup \Omega_S \) for a.a. \((t, x) \in I \times \Omega \).

\(\varepsilon\) Birkhäuser
Let us summarize the assumptions (with some \( \kappa > 1, r > d, \) and \( \eta > 0 \)):

\[
\Omega \quad \text{a smooth bounded domain of } \mathbb{R}^d, \; d = 2, 3, \\
\Omega_F, \Omega_S \subset \Omega \quad \text{open,} \; \Omega_F \cap \Omega_S = \emptyset, \; \Omega_F \cup \Omega_S = \Omega, \\
\varphi_S \in C^1(\text{GL}^+(d)), \; \forall F \in \text{GL}^+(d) : \; \varphi_S(F) \geq \eta/(\text{det } F)^{\kappa-1} \quad \text{and} \\
\forall Q \in \text{SO}(d) : \; \varphi_S(QF), \\
\phi_F \in C^1((0, +\infty)), \; \forall J > 0 : \; \phi_F(J) \geq J^{\kappa-1}, \\
D_F, D_S \in C(\mathbb{R}^{d \times d}, \mathbb{R}^{d \times d}) \; \text{strictly monotone and} \; \forall e \in \mathbb{R}^{d \times d} : \\
\eta|e|^2 \leq D_F(e) : e \leq (1 + |e|^2)/\eta \quad \text{and} \; \eta|e|^2 \leq D_S(e) : e \leq (1 + |e|^2)/\eta, \\
g \in L^\infty(I; L^\kappa(\Omega; \mathbb{R}^d)), \\
\xi_0 \in W^{2,r}(\Omega; \mathbb{R}^d), \; \min \text{det}(\nabla \xi_0) > 0, \; \xi_0|_\Gamma \text{ a homeomorphism } \Gamma \to \Gamma.
\]

The standard choice in literature is \( \xi_0(x) = x \), which means \( F_0 = I \) and \( \varrho_0 = \rho_R \), i.e. the unstretched continuum at \( t = 0 \), and then \( \Omega_S(0) = \Omega_S \) and \( \Omega_F(0) = \Omega_F \) and also \( \Gamma_{FS}(0) = \Gamma_{FS} \). The condition \( (4.1) \) covers this situation as a special case, the qualification \( (4.2f) \) being then satisfied trivially.

As we consider only quasistatic problem to assume that the constant mass densities \( \rho_R \) and \( \rho_L \). Let us note that \( (4.2f) \) with \( r > d \) implies that \( F_0 = (\nabla \xi_0)^{-1} \in W^{1,r}(\Omega; \mathbb{R}^{d \times d}) \) because

\[
\nabla F_0 = \nabla(\nabla \xi_0)^{-1} = \nabla \frac{\text{Cof}(\nabla \xi_0)}{\text{det}(\nabla \xi_0)} = \left( \frac{\text{Cof}(\nabla \xi_0)}{\text{det}(\nabla \xi_0)} - \frac{\text{Cof}(\nabla \xi_0) \otimes \text{Cof}(\nabla \xi_0)}{\text{det}(\nabla \xi_0)^2} \right) : \nabla^2 \xi_0,
\]

from which one can see that \( \nabla F_0 \in L^r(\Omega; \mathbb{R}^{d \times d \times d}) \).

We will fit the definition of weak solutions to the exponents \( r \) and \( s \) used in \( (3.2a) \) and in \( (4.2f) \):

**Definition 4.2.** (Weak solutions to \( (3.2) \)) A triple \((v, F, \xi)\) with \( v \in L^\infty(I; W^{2,s}(\Omega; \mathbb{R}^d)) \) with \( v = 0 \) on \( I \times \Gamma \), and \( F \in L^\infty(I; W^{1,r}(\Omega; \mathbb{R}^{d \times d})) \cap W^{1,s}(I; L^\kappa(\Omega; \mathbb{R}^{d \times d})) \) with \( \min_{I \times \Gamma} \text{det } F > 0 \), and \( \xi \in L^\infty(I; W^{2,r}(\Omega; \mathbb{R}^d)) \cap W^{1,s}(I; W^{1,r}(\Omega; \mathbb{R}^d)) \) is called a weak solution to the initial-boundary-value problem for the system \( (3.2) \) with \( (3.1) \) and with the boundary/transient conditions \( (2.15)-(2.16) \) and the initial condition \( (4.1) \) if \( \frac{\partial}{\partial t} \xi = -(v \cdot \nabla) \xi \) holds a.e. in \( I \times \Omega \) and \( \xi(0) = \xi_0 \) a.e. in \( \Omega \), and if the integral identity

\[
\int_\Omega \left( T^\xi_R \left( \frac{\text{Cof}(\nabla \xi_0)^T}{\text{det}(\nabla \xi_0)} \right) + D^\xi_R(e(v)) \right) : e(\tilde{v}) + v|\nabla e(v)|^{s-2} \nabla e(v) : \nabla e(\tilde{v}) \; dx \\
= \int_\Omega \text{det}(\nabla \xi_0) g^\xi_R \tilde{v} \; dx
\]

holds for any \( \tilde{v} \) smooth with \( \tilde{v} = 0 \) on \( \Gamma \) and for a.a. time instants \( t \in I \), the argument \( t \) being omitted in \( (4.3) \) for notational simplicity.

**Proposition 4.3.** (Existence and regularity of weak solutions) Let the assumptions \( (4.2) \) hold for \( r > d \) and \( \kappa > 2 \) and \( s > d \). Then:

(i) there exist a weak solution \((v, F, \xi)\) according Definition 4.2.

(ii) Moreover, this solution complies with energetics in the sense that the energy dissipation balance \( (3.8) \) integrated over time interval \([0, t]\) with the initial conditions \( (4.1) \) hold.

**Proof.** For clarity, we will divide the proof into five steps.

**Step 1: formal a-priori estimates.** Let us first make formally the a-priori estimates which follow from the energetics \( (3.8) \) when one uses the assumptions \( (4.2) \) for \( \kappa > 1 \) with some \( r > d \) and \( s > d \).
The only difficult term is \( \varrho \mathbf{g} \cdot \mathbf{v} \) on the right-hand side of (3.8), which can be estimated by Hölder’s and Young’s inequalities for \( x > 2 \) as

\[
\int_{\Omega} \frac{\varrho}{\det \mathbf{F}} \frac{\varrho}{\det \mathbf{F}} \mathbf{v} \, dx \leq \left\| \frac{\varrho}{\det \mathbf{F}} \right\|_{L^\infty(\Omega)} \left\| \mathbf{v} \right\|_{L^\infty(\Omega; \mathbb{R}^d)} \left\| \mathbf{g} \right\|_{L^\infty(\Omega; \mathbb{R}^d)}
\leq C_{\nu, \eta, \kappa} \left\| \mathbf{g} \right\|_{L^\infty(\Omega; \mathbb{R}^d)} \left( 1 + \left\| \frac{\varrho}{\det \mathbf{F}} \right\|_{L^\infty(\Omega)} \right) + \frac{\eta}{2} \left\| \mathbf{e}(\mathbf{v}) \right\|_{L^2(\Omega; \mathbb{R}^{d \times d})}^2 + \frac{\nu}{2} \left\| \nabla \mathbf{e}(\mathbf{v}) \right\|_{L^1(\Omega; \mathbb{R}^{d \times d \times d})}^2,
\]

with some \( C_{\nu, \eta, \kappa} \) depending on \( (\nu, \eta, \kappa) \). In (4.4) we used the Korn inequality, i.e. surely \( \| \mathbf{v} \|_{L^\infty(\Omega; \mathbb{R}^d)} \leq C(\| \nabla \mathbf{e}(\mathbf{v}) \|_{L^\infty(\Omega; \mathbb{R}^d)} + \| \mathbf{e}(\mathbf{v}) \|_{L^2(\Omega; \mathbb{R}^{d \times d})} \). More in details, one is to compose the Sobolev embedding \( W_{0}^{1,\kappa}(\Omega; \mathbb{R}^{d \times d}) \subset L^\kappa(\Omega; \mathbb{R}^{d \times d}) \) for an inequality \( \| \mathbf{e}(\mathbf{v}) \|_{L^\kappa(\Omega; \mathbb{R}^{d \times d})} \leq C_{\kappa}(\| \nabla \mathbf{e}(\mathbf{v}) \|_{L^\infty(\Omega; \mathbb{R}^{d \times d \times d})} + \left\| \nabla \mathbf{e}(\mathbf{v}) \right\|_{L^1(\Omega; \mathbb{R}^{d \times d \times d})} \leq C_{\kappa}(1 + \| \mathbf{e}(\mathbf{v}) \|_{L^2(\Omega; \mathbb{R}^{d \times d})}^2 + \| \nabla \mathbf{e}(\mathbf{v}) \|_{L^1(\Omega; \mathbb{R}^{d \times d \times d})}^2) \) for \( s \geq 2 \) and, relying on Dirichlet boundary conditions, the Korn inequality \( \| \nabla \mathbf{v} \|_{L^\kappa(\Omega; \mathbb{R}^{d \times d})} \leq C_{\kappa}\| \mathbf{v} \|_{L^\kappa(\Omega; \mathbb{R}^d)} \) with some \( C_{\kappa} \), and eventually again the Sobolev embedding \( W_{0}^{1,\kappa}(\Omega; \mathbb{R}^d) \subset L^\infty(\Omega; \mathbb{R}^d) \). The term \( \| \varrho \|^2_{L^\kappa(\Omega; \mathbb{R}^d)} \) in (4.4) can thus be treated by the Gronwall inequality relying on the blow-up assumption \( \varphi_{R}(\mathbf{X}, \mathbf{F}) \geq \eta/(\det \mathbf{F})^{\kappa-1} \) in (4.2b,c); note that \( \int_{\Omega} \varphi_{R}(\mathbf{F})/\det \mathbf{F} \, dx \geq \eta^2/(\det \mathbf{F})^{\kappa+1} \) and \( \varphi_{R}(\mathbf{F})/\det \mathbf{F} \leq \eta^2/(\det \mathbf{F})^{\kappa+1} \) with \( \eta \). Of course, the last and the penultimate terms in (4.4) can be absorbed in the left-hand side of the energy balance. From (3.8), exploiting the boundary condition \( \mathbf{v} = 0 \) and Korn’s inequality, we thus obtain

\[
\| \mathbf{v} \|_{L^\infty(I; W^{1,\kappa}(\Omega; \mathbb{R}^d)) \cap L^2(I; H^1(\Omega; \mathbb{R}^d))} \leq C \quad \text{and} \quad \sup_{t \in I} \int_{\Omega} \varphi_{R}(\mathbf{F}(t))/\det \mathbf{F}(t) \, dx \leq C.
\]

The former estimate in (4.5a) with \( s > d \) in particular means Lipschitz continuity of \( \mathbf{v}(t, \cdot) \) with a time-integrable Lipschitz constant, which further implies that the transport equations copies regularity of the initial conditions, cf. \cite{60}, Appendix 5.3. Here, in particular, (2.1) with the qualification of \( \xi_0 \in W^{1,q}(\Omega; \mathbb{R}^d) \) gives \( \xi \in L^\infty(I; W^{1,q}(\Omega; \mathbb{R}^d)) \cap W^{1,s}(I; L^q(\Omega; \mathbb{R}^d)) \) with any \( 1 \leq q < \infty \). Conceptually, the \( L^\infty(I; W^{1,q}(\Omega; \mathbb{R}^d)) \)-estimate is obtained by testing (2.1) by \( \nabla(\varphi^2 \xi)^{1/2} \varphi \varphi \) and using Gronwall’s inequality, cf. \cite{60} for a lot of technicalities. For the \( W^{1,s}(I; L^q(\Omega; \mathbb{R}^d)) \)-estimate, we used \( \varphi_{R}(\mathbf{X}, \mathbf{F}) \). Similarly, from the evolution-and-transport (2.4) for the distortion \( \mathbf{A} = \nabla \xi \) together with the qualification \( \mathbf{A}_0 = \nabla \xi_0 \in W^{1,r}(\Omega; \mathbb{R}^{d \times d}) \) gives \( \mathbf{A} = \nabla \xi \in L^\infty(I; W^{1,r}(\Omega; \mathbb{R}^{d \times d})) \cap W^{1,s}(I; L^r(\Omega; \mathbb{R}^{d \times d})) \). For the \( L^\infty(I; W^{1,r}(\Omega; \mathbb{R}^{d \times d})) \)-estimate, (2.4) is to be tested by \( \nabla(\varphi^2 \xi)^{1/2} \varphi \varphi \) while using Gronwall’s inequality now additionally handling the right-hand side term \( -A(\nabla \mathbf{v}) \) in (2.4) relying on the regularity of \( \mathbf{v} \) in (4.5a), cf. again \cite{60} for a lot of technicalities. For the \( W^{1,s}(I; L^r(\Omega; \mathbb{R}^{d \times d})) \)-estimate, we used (4.5a) for a comparison to see that \( \nabla \varphi_{R}(\mathbf{F})/\det \mathbf{F} \, dx \geq \eta^2/(\det \mathbf{F})^{\kappa+1} \) with \( \eta \). Of course, the last and the penultimate terms in (4.4) can be absorbed in the left-hand side of the energy balance. From (3.8), exploiting the boundary condition \( \mathbf{v} = 0 \) and Korn’s inequality, we thus obtain

\[
\| \mathbf{v} \|_{L^\infty(I; W^{1,\kappa}(\Omega; \mathbb{R}^d)) \cap L^2(I; H^1(\Omega; \mathbb{R}^d))} \leq C \quad \text{and} \quad \sup_{t \in I} \int_{\Omega} \varphi_{R}(\mathbf{F}(t))/\det \mathbf{F}(t) \, dx \leq C.
\]

Similarly, from the evolution-and-transport equation (2.3) for \( 1/\det \mathbf{A} \) and from the qualification (4.2f) of the initial condition \( 1/\det \mathbf{A}_0 = 1/\det(\nabla \xi_0) \), we also obtain the boundedness of \( 1/\det \mathbf{A} \) in \( L^\infty(I; W^{1,r}(\Omega)) \cap W^{1,s}(I; L^r(\Omega)) \) so that \( 1/\det \mathbf{A} \in C(I \times \Omega) \) and also \( \min_{I \times \Omega} \det \mathbf{A} > 0 \). As \( r > d \), this implies also estimates for \( \mathbf{A}^{-1} = \mathbf{F} \), specifically

\[
\| \mathbf{F} \|_{L^\infty(I; W^{1,r}(\Omega; \mathbb{R}^{d \times d}))} \cap W^{1,s}(I; L^r(\Omega; \mathbb{R}^{d \times d})) \leq C \quad \text{with} \quad \min_{I \times \Omega} \det \mathbf{F} > 1/C;
\]

and also \cite{62}, Lemma 3.2. We can eventually use the quasistatic equation (3.2a) itself: from the \( L^\infty \)-estimate in (4.5b), we can eventually improve the former estimate in (4.5a) as

\[
\| \mathbf{v} \|_{C^{1,\kappa}(I; W^{2,\kappa}(\Omega; \mathbb{R}^d))} \leq C.
\]
Step 2: a cut-off regularization. Referring to the formal estimates (4.5c), we can choose $\varepsilon > 0$ so small that, for any possible sufficiently regular solution, it holds
\[
\det F > \varepsilon \quad \text{and} \quad |F| < \frac{1}{\varepsilon} \quad \text{a.e. on } I \times \Omega. \tag{4.6}
\]
We regularize the stress $T^\xi_R(F)$ in (3.2a) by considering a smooth cut-off $\pi_\varepsilon \in C^1(\mathbb{R}^{d \times d})$ as
\[
\pi_\varepsilon(F) = \begin{cases} 
1 & \text{for } \det F \geq \varepsilon \text{ and } |F| \leq 1/\varepsilon, \\
0 & \text{for } \det F \leq \varepsilon/2 \text{ or } |F| \geq 2/\varepsilon,
\end{cases} \tag{4.7}
\]
Here $| \cdot |$ stands for the Frobenius norm $|F| = (\sum_{i,j=1}^d F^2_{ij})^{1/2}$ for $F = [F_{ij}]$, which guarantees that $\pi_\varepsilon$ is frame indifferent. Furthermore, we also cut-off and regularize the singular nonlinearity $1/\det(\cdot)$ which is employed in the momentum equation and the stress $T^\xi_R$ as
\[
det_\varepsilon(F) := \min \left( \max \left( \det F, \frac{\varepsilon}{2} \right), \frac{\varepsilon}{2} \right) \quad \text{and} \quad T^\xi_{R,\varepsilon}(F) = \frac{\left[ \pi_\varepsilon \varphi^\xi_R \right]_F(F)F^t}{\det_\varepsilon(F)}.
\]
Note that also $[\pi_\varepsilon \varphi^\xi_R](x, \cdot) \in C^1(\mathbb{R}^{d \times d})$ if $\varphi_R(X, \cdot) \in C^1(\text{GL}^+(d))$ and that $[\pi_\varepsilon \varphi^\xi_R]_F$ together with the regularized Cauchy stress $T^\xi_{R,\varepsilon}$ are bounded, continuous, and vanish if the argument $F \in \mathbb{R}^{d \times d}$ “substantially” violates the constraints (4.6), specifically:
\[
\left( \det F \leq \frac{\varepsilon}{2} \text{ or } |F| \geq \frac{2}{\varepsilon} \right) \Rightarrow T^\xi_{R,\varepsilon} = 0.
\]
We then consider a regularized momentum equation (3.2a) and thus the overall monolithic system
\[
\text{div} \left( T^\xi_{R,\varepsilon} \left( \frac{\text{Cof}(\nabla \xi)^T}{\det_\varepsilon(\nabla \xi)} \right) + D^\xi_R(e(v)) - \text{div}(\nu |\nabla e(v)|^{s-2}\nabla e(v)) \right) + \det_\varepsilon(\nabla \xi) g^\xi_R g = 0, \tag{4.9a}
\]
\[
\frac{\partial \xi}{\partial t} = -(v \cdot \nabla) \xi. \tag{4.9b}
\]
Of course, for (4.9a) we consider the boundary conditions (2.15) while for (4.9b) we consider the initial condition (4.1).

Step 3: solving (4.9) by Schauder’s fixed point. We organize this step for a fixed point of the mapping composed from
\[
B \ni \xi \xrightarrow{(4.9b)} \xi \quad \text{and} \quad \xi \xrightarrow{(4.9a)} v \in B \tag{4.10}
\]
with a (sufficiently large) convex set $B \subset C_w(I; W^{2,s}(\Omega; \mathbb{R}^d))$ with $v = 0$ on $I \times \Gamma$. More specifically, we put
\[
B := \left\{ v \in C_w(I; W^{2,s}(\Omega; \mathbb{R}^d)); \forall a, t \in I: v(t) = 0 \text{ on } \Gamma \right\}
\]
and
\[
\|\nabla e(v(t))\|^s_{L^\infty(\Omega; \mathbb{R}^{d \times d})} + \|e(v(t))\|^2_{L^2(\Omega; \mathbb{R}^{d \times d})} \leq \max \left( \frac{s'}{\nu}, \frac{2}{\eta} \right) \left( \frac{2\text{meas}(\Omega)}{\eta} \right) L^2_{\nu} + \frac{N s' \max(g^s_R, g^s_F)}{s' \nu^{2/(s-1)}} \|g\|_{L^\infty(\Omega; L^1(\mathbb{R}^d))},
\]
where $\eta$ is from (4.2d), $N$ denotes the norm of the embedding of $\{v \in W^{2,s}(\Omega; \mathbb{R}^d); v|_f = 0\}$ normed by $\|\nabla e(\cdot)\|_{L^s(\Omega; \mathbb{R}^{d \times d})}$ into $L^\infty(\Omega; \mathbb{R}^d)$, and
\[
L_{\varepsilon} := \sup_{X \in \Omega, F \in \mathbb{R}^{d \times d}} \left| \frac{[\pi_\varepsilon \varphi_R(X, \cdot)]_F(F)F^t}{\det_\varepsilon(F)} \right| < +\infty. \tag{4.11}
\]
The motivation of this choice of $B$ will be seen later in the estimate (4.15).
First, let us consider a fixed \( \mathbf{v} \) from \( B \). Then the transport equation (4.9b) with \( \xi(0) = \xi_0 \in W^{2,r}(\Omega; \mathbb{R}^d) \) has a solution \( \xi \) in a bounded subset of \( L^\infty(I; W^{2,r}(\Omega; \mathbb{R}^d)) \cap W^{1,\sigma}(I; L^r(\Omega; \mathbb{R}^d)) \), cf. (4.5b) with some constant \( C \) depending on the already chosen ball \( B \) but not on the particular choice of \( \mathbf{v} \in B \), and the equation (4.9b) holds a.e. on \( I \times \Omega \) and \( \xi \in C(I \times \Omega; \mathbb{R}^d) \); cf. again [60, Appendix 5.3] as used already for the argumentation (4.5). For the mentioned uniqueness of \( \xi \), let us note that the transport equation (4.9b) is linear for a fixed \( \mathbf{v} \); more specifically, considering two solutions \( \xi_1 \) and \( \xi_2 \) satisfying
\[
\frac{\partial}{\partial t}(\xi_1 - \xi_2) = (\mathbf{v} \cdot \nabla)\xi_1 - (\mathbf{v} \cdot \nabla)\xi_2,
\]
gives, when tested by \( 2(\xi_1 - \xi_2) \), the following estimate for a.a. \( t \in I \):
\[
\frac{d}{dt} \int_\Omega |\xi_1 - \xi_2|^2 \, dx = 2 \int_\Omega ((\mathbf{v} \cdot \nabla)\xi_2 - (\mathbf{v} \cdot \nabla)\xi_1) \cdot (\xi_1 - \xi_2) \, dx
\]
\[
= \int_\Omega (\nabla \mathbf{v})\xi_1 \cdot \nabla \xi_2 \leq \| \nabla \mathbf{v} \|_{L^\infty(\Omega)} \|\xi_1 - \xi_2\|_{L^2(\Omega; \mathbb{R}^d)}^2,
\]
from which we conclude \( \xi_1 = \xi_2 \) by Gronwall’s inequality when taking \( \xi_1(0) = \xi_0 = \xi_2(0) \) into account.

Further, we use this also for the nonhomogeneous evolution-and-transport equations for \( \mathbf{A} = \nabla \xi \) and for \( \mathbf{F} = (\nabla \xi)^{-1} \) to obtain \( \mathbf{A} \) and \( \mathbf{F} \) a-priori bounded in \( L^\infty(I; W^{1,\sigma}(\Omega; \mathbb{R}^{d \times d})) \cap W^{1,\sigma}(I; L^r(\Omega; \mathbb{R}^{d \times d})) \), cf. (4.5b,c) with some constant \( C \) again depending on that bounded set \( B \) of \( \mathbf{v} \)'s. In particular, we can also see that \( \mathbf{A}, \mathbf{F} \in C(I \times \Omega; \mathbb{R}^d) \) and that the equations (2.2) and (2.4) are satisfied a.e. on \( I \times \Omega \).

Moreover, in view of these a-priori bounds and uniqueness of the response \( \xi \) for a current \( \mathbf{v} \), it is easy to see that the dependence of \( \xi \) on \( \mathbf{v} \) is (weak,weak*)-continuous as a mapping \( L^1(I; W^{2,s}(\Omega; \mathbb{R}^d)) \to L^\infty(I; W^{2,r}(\Omega; \mathbb{R}^d)) \). Actually, for any weakly convergent sequence of \( \mathbf{v} \)'s, the corresponding weak* sequence of \( \xi \)'s converges, for a moment in terms of subsequences in \( L^\infty(I; W^{2,r}(\Omega; \mathbb{R}^d)) \cap W^{1,\sigma}(I; W^{1,\sigma}(\Omega; \mathbb{R}^d)) \) and hence strongly in \( C(I \times \Omega; \mathbb{R}^d) \), which makes it easy to pass to the limit in the transport equation (3.2b). From the mentioned uniqueness for \( \xi \), we can see that eventually the whole sequence of \( \xi \)'s converges, cf. again [60].

Now we go on to the latter mapping in (4.10), considering a fixed \( \xi \). The quasistatic equation (4.9a) together with the mentioned boundary conditions represents, at each time instant \( t \) and for \( \xi = \xi(t) \) fixed, a static boundary-value problem for the 4th-order quasilinear equation
\[
\text{div}^2(\nu|\nabla \mathbf{e}(\mathbf{v})|^{s-2}\nabla \mathbf{e}(\mathbf{v})) - \text{div} \mathbf{D}_R^\xi(\mathbf{e}(\mathbf{v})) = f_\varepsilon(\xi, \nabla \xi)
\]
with
\[
f_\varepsilon(\xi, \mathbf{A}) := \text{div} \left( \mathbf{T}_R^\varepsilon \left( \frac{\text{Cof} \mathbf{A}^\top}{\det \varepsilon(\mathbf{A})} \right) \right) + \det \varepsilon(\mathbf{A}) \varepsilon \mathbf{g},
\]
which has a solution \( \mathbf{v} = \mathbf{v}(t) \in W^{2,s}(\Omega; \mathbb{R}^d) \). If \( \mathbf{D}_R(X, \cdot) : \mathbb{R}^{d \times d} \to \mathbb{R}^{d \times d} \) would have possessed some (dissipation) potential as often assumed in literature, the solution to (4.14) could be proved simply by the direct minimization method, while in our general case we can use the classical Browder-Minty theorem about surjectivity for a coercive radially-continuous monotone operator, here \( \mathbf{v} \mapsto \text{div}^2(\nu|\nabla \mathbf{e}(\mathbf{v})|^{s-2}\nabla \mathbf{e}(\mathbf{v})) - \text{div} \mathbf{D}_R^\xi(\mathbf{e}(\mathbf{v})) \) with the respective boundary conditions \( \mathbf{v} = 0 \) and \( \nabla \mathbf{e}(\mathbf{v}) \cdot (\mathbf{n} \otimes \mathbf{n}) = 0 \) on \( \Gamma \). Moreover, due to the strong monotonicity of this quasilinear operator, the mentioned solution is unique and the mapping \( \xi \mapsto \mathbf{v} \) is bounded as \( W^{1,1}(\Omega; \mathbb{R}^d) \to W^{2,s}(\Omega; \mathbb{R}^d) \); here it is trivial due to the \( \varepsilon \)-cut-off regularization (4.8) so that the right-hand side \( f_\varepsilon(\xi, \nabla \xi) \) of (4.14) is a-priori bounded in \( W^{2,s}(\Omega; \mathbb{R}^d)^* \). It is important to quantify this boundedness: testing (4.14) by \( \mathbf{v} \), at a current time instant \( t \in I \), we obtain the estimate
\[
\nu \| \nabla \mathbf{e}(\mathbf{v}) \|_{L^2(\Omega; \mathbb{R}^{d \times d})}^2 + \eta \| \mathbf{e}(\mathbf{v}) \|_{L^2(\Omega; \mathbb{R}^{d \times d})}^2 \leq \int_\Omega f_\varepsilon(\xi, \nabla \xi) \cdot \mathbf{v} \, dx
\]
\[
= \int_\Omega \det \varepsilon(\nabla \xi) \varepsilon \mathbf{g} \cdot \mathbf{v} - \mathbf{T}_R^\varepsilon \left( \frac{\text{Cof} \nabla \xi^\top}{\det \varepsilon(\nabla \xi)} \right) \mathbf{e}(\mathbf{v}) \, dx \leq \frac{N^s}{s' \nu^{1/(s-1)}} \max(g_0^s, g_0^s) \| \mathbf{g} \|_{L^1(\Omega; \mathbb{R}^d)}^s + \frac{2 \text{meas}(\Omega)}{\eta} L_\varepsilon^2 + \frac{\eta}{2} \| \mathbf{e}(\mathbf{v}) \|_{L^2(\Omega; \mathbb{R}^{d \times d})}^2
\]
with \( L_\varepsilon \) from (4.12) and with \( N \) as in (4.11). Thus \( t \mapsto \mathbf{v}(t) \) belongs to \( B \) defined in (4.11).
Next, we are to show that this mapping \( \xi \mapsto v \) is even continuous while taking into account that the Nemytskii operators in (4.9a) involve a discontinuity across \( I_{FS} \) and that we do not have any estimates on \( \frac{\partial}{\partial x} v \) at disposal. One ingredient is surely the mentioned strong monotonicity of the underlying quasilinear operator. Further, we use the Arzelà-Ascoli-type compact embedding \( C_w(I; W^{2,r}(\Omega; \mathbb{R}^d)) \cap W^{1,s}(I; L^r(\Omega; \mathbb{R}^d)) \subseteq C(I; C^1(\overline{\Omega}; \mathbb{R}^d)) \) due to the compact embedding \( W^{2,r}(\Omega) \subseteq C^1(\overline{\Omega}) \) for \( r > d \); cf. e.g. [58, Lemma 7.10]. We now need to show continuity of the mappings \( \xi \mapsto \text{det}_s(\nabla \xi) g_{R} : C(I; C^1(\overline{\Omega}; \mathbb{R}^d)) \to L^1(I \times \Omega; \mathbb{R}^d) \) and \( \xi \mapsto T_{R,\varepsilon}^\xi(\text{Cof}(\nabla \xi) \top / \text{det}_s(\nabla \xi)) : C(I; C^1(\overline{\Omega}; \mathbb{R}^d)) \to L^1(I \times \Omega; \mathbb{R}^{d \times d}) \). This is however a bit delicate issue because \( g_{R} \) and \( T_{R}^\varepsilon(\cdot, F) \) as well as \( D_{R}(\cdot, e) \) are discontinuous across the surface \( I_{FS} \) inside the reference domain \( \Omega \), cf. (3.1), so that the usual arguments of continuity of Nemytskii operators cannot be used. Instead, we can use Lemma 4.1. Realizing that, having a sequence \( \{\xi_k\}_{k \in \mathbb{N}} \) converging in \( C(I \times \overline{\Omega}; \mathbb{R}^d) \) to \( \xi \), due to Lemma 4.1, we can see that \( \xi(t, x) \) is valued (together with an neighbourhood) in the open set \( \Omega_t \cup \Omega_{\xi} \) for a.a. \((t, x)\) so that, \( g_{R}({\xi_k}(t, x)) \to g_{R}({\xi}(t, x)) \) for a.a. \((t, x)\). Also \( \nabla \xi_k \) converges in \( C(I \times \overline{\Omega}; \mathbb{R}^{d \times d}) \) to \( \nabla \xi \); here we again use the Arzelà-Ascoli-type compact embedding of \( C_w(I; W^{1,r}(\Omega; \mathbb{R}^{d \times d})) \cap W^{1,s}(I; L^r(\Omega; \mathbb{R}^{d \times d})) \) into \( C(I \times \overline{\Omega}; \mathbb{R}^{d \times d}) \), cf. (4.5b). Thus \( \text{det}_s(\nabla \xi_k) \to \text{det}_s(\nabla \xi) \) in \( C(I \times \overline{\Omega}) \) since \( \text{det}_s(\cdot) : \mathbb{R}^{d \times d} \to \mathbb{R} \) is continuous. Altogether, \( \text{det}_s(\nabla \xi_k) \to \text{det}_s(\nabla \xi) g_{R} \to \text{det}_s(\nabla \xi) g_{R} \) a.e., and thus also in \( L^1(I \times \Omega; \mathbb{R}^{d \times d}) \) by the Lebesgue dominated-convergence theorem. Also \( T_{R,\varepsilon}^\xi(\text{Cof}(\nabla \xi_k) \top / \text{det}_s(\nabla \xi_k)) \to T_{R,\varepsilon}^\xi(\text{Cof}(\nabla \xi) \top / \text{det}_s(\nabla \xi)) \) in \( L^1(I \times \Omega; \mathbb{R}^{d \times d}) \) by similar arguments. The discontinuity of \( D_{R}(\cdot, e(v)) \) occurring in the weak formulation (4.3) integrated over the time interval \( I \) can be treated by the same manner.

Then we can apply the Schauder fixed-point argument for the composed mapping \( v \mapsto \xi \) and \( \xi \mapsto v \), viz (4.10). The former mapping is considered on the convex bounded closed set \( B \) from (4.11) in the Banach space \( C_w(I; W^{2,s}(\Omega; \mathbb{R}^d)) \) endowed with the weak topology from \( L^s(I; W^{2,s}(\Omega; \mathbb{R}^d)) \) which makes \( B \) weakly compact. As to the mapping \( \xi \mapsto v \), it is important that, due to our cut-off \( \varepsilon \)-regularization, the mentioned a-priori bounds for \( v \) are entirely independent of \( \xi \) and hold even for \( \xi \)'s with \( \text{det}(\nabla \xi) \) not positive. Therefore, one does not need to consider the nonconvex constraint \( \text{det}(\nabla \xi) \geq \varepsilon \) at this stage and can work with the whole mentioned ball \( B \) which is, of course, convex. This gives a uniquely defined velocity field \( v \in C(I; W^{2,s}(\Omega; \mathbb{R}^d)) \) with \( v \cdot n = 0 \) on \( I \times \Gamma \). We then choose \( B \) so big that the mentioned bounded set of \( v \)'s is contained in it. The mentioned composed mapping is single-valued and (weak,weak)-continuous, and thus it has a fixed point \( \xi \in B \). Together with the corresponding \( v \) we thus get a weak solution to the regularized system (4.9) with the initial and boundary conditions (2.15) and (4.1).

**Step 4: energetics rigorously.** The estimates in Step 1 relied on the energy dissipation balance which was derived in Sect. 3 rather formally. We can prove it for (4.9) rigorously.

Due to the cut-off regularization of (4.9a), we can be sure that \( v \in L^\infty(I; W^{2,s}(\Omega; \mathbb{R}^d)) \). Thus, the transport of \( \xi \) by (4.9b) is done through a Lipschitz velocity field, and thus surely \( \text{det}(\nabla \xi) > 0 \) due to the qualification of the initial condition (4.2f). Therefore, \( F = \text{Cof}(\nabla \xi) \top / \text{det}(\nabla \xi) \) is well defined and \( T_{R,\varepsilon}^\xi(F) = [\pi_{\varepsilon} \varphi_{R}]_{F}(F) F \top / \text{det}(F) = [\pi_{\varepsilon} \varphi_{R}]_{F}(F) F \top / \text{det}(F) \) due to (4.8) because, if \( \text{det}(\nabla F) \leq \varepsilon/2 \), then \( \pi_{\varepsilon}(F) = 0 \) due to (4.7) and thus also \( [\pi_{\varepsilon} \varphi_{R}]_{F}(F) = 0 \). Also (2.2) is at our disposal. Therefore, we can use the calculus (3.3)–(3.4). Yet, (3.6) is more delicate because it is not trivial to show that the last term in (3.5), when integrated over \( \Omega \), is indeed zero—here we emphasize that \( \varphi_{R}(\cdot, F) \) is discontinuous across \( I_{FS} \) so that \([\varphi_{R}]_{x}(t)(F) \) is a measure supported on \( I_{FS}(t) \), in general.

At each time instant \( t \in I \), with omitting the argument \( t \) for notational simplicity and by the change-of-variable formula for \( x = \xi^{-1}(X) \) and by the Green formula, we have

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\[ \int_{\Omega} \left[ \frac{[\varphi R]^T}{\text{det } F(x)} \right] \xi^{(x)}(F(x)) \left( \frac{\partial \xi}{\partial t} + (v \cdot \nabla) \xi \right) \, dx = \int_{\Omega} [\varphi R]^{-1}(X, F_R(X)) \cdot r(\xi^{-1}(X)) \, dX \]
\[ =: r(x) \]
\[ = \int_{\Gamma} [\varphi R](X, F_R(X)) \left( n \cdot r(\xi^{-1}(X)) \right) \, dS \]
\[- \int_{\Omega} [\varphi R](X, F_R(X)) \text{div}(r(\xi^{-1}(X))) \, dX = 0, \quad (4.16)\]

where we used the notation \( F = F_R \circ \xi \) from Sect. 2.1. For the change of variable, we needed however the global invertibility of \( \xi \) for which the classical result of Ball [4, Thm.1(ii)] is exploited; here the local invertibility \( \text{det}(\nabla \xi(t)) > 0 \) in \( \Omega \) together with the invertibility of \( \xi(t) \) on \( I \) were used. The latter property is ensured by the boundary invertibility of \( \xi_0 \) assumed in (4.2f) together with the boundary condition \( v = 0 \) in (2.15) so that \( \xi|_\Gamma = \xi_0|_\Gamma \) stays constant during the evolution. Further, we needed that the residuum \( r \) vanishes on \( \Gamma \) and its divergence vanishes in \( \Omega \). This is indeed guaranteed by the transport equation (4.9b) together with the mentioned regularity of its solution \( \xi \in L^\infty(I; W^{2,r}(\Omega; \mathbb{R}^d)) \cap W^{1,s}(I; L^r(\Omega; \mathbb{R}^d)) \) so that \( r(t) \) has a well defined (zero) trace on \( \Gamma \) and

\[ \text{div } r = \frac{\partial}{\partial t} \text{div } \xi + \text{div}(v \cdot \nabla \xi) = \frac{\partial}{\partial t} \text{div } \xi + (\nabla v)^\top : \nabla \xi + v \cdot \nabla(\text{div } \xi) \]
\[ = \frac{\partial}{\partial t} (\text{tr } A) + (\nabla v)^\top : A + v \cdot \nabla(\text{tr } A) \in L^s(I; L^r(\Omega)), \quad (4.17)\]

due to the already mentioned regularity of the distortion \( A = \nabla \xi \in L^\infty(I; W^{1,r}(\Omega; \mathbb{R}^{d \times d})) \cap W^{1,s}(I; L^r(\Omega; \mathbb{R}^{d \times d})) \).

Due to the mentioned quality of \( v \), we can legitimately test also the dissipation part of the momentum equation in its weak formulation (4.3) by \( \tilde{v} = v \in L^\infty(I; W^{2,\text{s}}(\Omega; \mathbb{R}^d)) \). Thus we can rigorously obtain the energy dissipation balance (3.8) modified for the regularized system as

\[ \frac{d}{dt} \int_{\Omega} \phi_R(F(x)) \phi_x(F(x)) \, dx + \int_{\Omega} D_R(e(v)) : e(v) + (v \cdot \nabla e(v))^s \, dx = \int_{\Omega} \phi_x g \cdot v \, dx. \quad (4.18)\]

**Step 5: the original problem.** Let us remind that \( F = \text{Cof}(\nabla \xi)^\top / \det_\varepsilon(\nabla \xi) \) resulted as the fixed point in Step 3 lives in \( L^\infty(I; W^{1,r}(\Omega; \mathbb{R}^{d \times d})) \cap W^{1,s}(I; L^r(\Omega; \mathbb{R}^{d \times d})) \) and this space is embedded into \( C(I \times \overline{\Omega}; \mathbb{R}^{d \times d}) \) since \( r > d \). Similarly, \( \det_\varepsilon F = 1 / \det_\varepsilon(\nabla \xi) \) lives in \( L^\infty(I; W^{1,r}(\Omega)) \cap W^{1,s}(I; L^r(\Omega)) \). Therefore \( F \) together with \( \det F \) evolve continuously in time, being valued respectively in \( C(\overline{\Omega}; \mathbb{R}^{d \times d}) \) and \( C(\overline{\Omega}) \). Let us recall that the initial condition \( F_0 \) complies with the bounds (4.6) and we used this \( F_0 \) also for the \( \varepsilon \)-regularized system. Therefore \( F \) satisfies these bounds not only at \( t = 0 \) but also at least for small times \( t > 0 \). Yet, in view of the choice (4.6) of \( \varepsilon \), this means that the \( \varepsilon \)-regularization is nonactive and \( (v, F, \xi) \) solves, at least for a small time, the original nonregularized problem (3.2) with the initial/boundary conditions (2.15) and (4.1). For this solution, the a-priori \( L^\infty \)-bounds (4.6) hold. Eventually, by the continuation argument, we may see that the \( \varepsilon \)-regularization remains inactive within the whole evolution of \( (v, F, \xi) \) on the whole time interval \( I \).

**Remark 4.4. (Deformation \( y \))** The global invertibility of \( \xi(t) \) used in the above proof ensures also existence a deformation \( y \) such that \( F = \nabla y \), which was the original motivation of the model when defining the deformation gradient tensor \( F \).

**Remark 4.5. (Transient conditions across \( \Gamma_{FS}(t) \))** The conditions on the evolving fluid–solid interface are rather implicit in the monolithic formulation. Clearly, \( v(t) \in W^{1,s}(\Omega; \mathbb{R}^d) \subset C(\overline{\Omega}; \mathbb{R}^d) \) implies \( [v(t)]_{\Gamma_{FS}(t)} = 0 \) and \( e(v(t)) \in W^{1,s}(\Omega; \mathbb{R}^{d \times d}) \) for \( s > d \) implies \( [e(v(t))]_{\Gamma_{FS}(t)} = 0 \). The other condition comes from the momentum equilibrium, when assuming the weak solution to be sufficiently smooth. Indeed, when testing the monolithic momentum equation (3.2a) by \( \tilde{v} \) with a compact support in \( \Omega_F(t) \), by the Green formula we obtain \( \int_{\Omega_F(t)} \phi g \tilde{v} - (T_F(F) + D_F(e(v)) \cdot \text{div } \mathcal{H}(\nabla e(v))) \cdot e(v) \, dx = 0 \). Similarly
it holds for $\Omega_S(t)$. Then taking $\tilde{v}$ with a compact support in $\Omega$ and integrating (3.2a) and using Green formula separately over $\Omega_F(t)$ and $\Omega_S(t)$, we obtain the identity

$$\int_{\Gamma_{FS}(t)} t_F \cdot \tilde{v} + t_S \cdot \tilde{v} \, dS = 0,$$

cf. (2.17). As $\tilde{v}$ takes arbitrary values on $\Gamma_{FS}(t)$, it must hold $t_F = t_S$, as claimed in (2.16). Altogether, in the physically relevant case $d = 3$, we obtained the expected number $3 + 6 + 3 = 4d$ of the transient conditions on $\Gamma_{FS}(t)$.

Remark 4.6. (A constructive proof) Actually, we can use the Galerkin approximation separately of both the regularized momentum equation (4.9a) and the transport equation regularized by an $r$-Laplacian, and then make a successive limit passage first in the transport equation (3.2b) and then in the momentum equation as in [59]. Such a space discretization leading to an initial-value problem for a system of ordinary differential equations would give a conceptually implementable algorithm. On the other hand, the proof of Proposition 4.3 would then be much more technical, which is why the shorter but non-constructive proof based on Schauder’s theorem was here presented.

Remark 4.7. (Navier-type boundary conditions: local-in-time solutions) The normal deformation $y \cdot n$ is usually fixed by prescribing the normal velocity $v \cdot n = 0$ on $\Gamma$ to avoid serious troubles in Eulerian approach. Yet, one can think at least about making the tangential velocity not vanishing. Zero tangential velocity was actually needed only for the global invertibility of $\xi$ used in (4.16). If one has weaker (but in the FSI-context quite usual) ambitions to prove only local-in-time solution existence, one can admit non-zero tangential velocity and rely on that the return mapping $\xi(t, \cdot) : \Omega \to \Omega$ is originally globally invertible for $t = 0$ and, evolving continuously, it stays invertible at least for sufficiently small $t > 0$. Then one can think about more general boundary conditions not fixing the tangential velocity to zero, as illustrated in Fig. 2.

Specifically, we prescribe $1 + (d-1) + d = 2d$ boundary conditions:

$$v \cdot n = 0,$$  \hspace{1cm} (4.19a)

$$[ (T^\xi_{\mathbf{R}}(F) + D^\xi_{\mathbf{R}}(e(v))) - \text{div} \mathcal{H}(\nabla e(v)) ] n - \text{div}_r (\mathcal{H}(\nabla e(v)) n)]_T + \nu_y v = f,$$  \hspace{1cm} (4.19b)

$$\nabla e(v) \cdot (n \otimes n) = 0,$$  \hspace{1cm} (4.19c)

with $\nu_y > 0$ a boundary viscosity coefficient while $[\cdot]_T$ denotes the tangential part of a vector. The condition (4.19b) involves a boundary “friction” $\nu_y$ and, together with (4.19a), forms the Navier boundary
conditions largely used in fluid dynamics. The mechanical energy dissipation balance (3.8) extends to

\[
\frac{d}{dt} \int_{\Omega} \varphi R(\det F) \, dx + \int_{\Omega} D_R(\varepsilon(v)) : \varepsilon(v) + \nu |\nabla \varepsilon(v)|^2 \, dx + \int_{\Gamma} \nu \beta |v|^2 \, dS
\]

\[
= \int_{\Omega} \varphi g \cdot v \, dx + \int_{\Gamma} f \cdot v \, dS.
\]

Remark 4.8. (Making the boundary \( \Gamma \) free: a fictitious-domain or sticky-air approach) In some applications, an entirely free \( \Gamma \) is urgently desirable. This seems possible only by an “engineering” approach, embedding \( \Omega \) into a fictitious bigger domain containing a very soft material (fluid) with a very low viscosity and having with a fixed boundary to comply the usual non-penetration requirement of the outer boundary \( \Gamma \) and also to the analysis presented above, see e.g. [48]. In geophysical modelling, such rather rough (although numerical simple and efficient) trick is called a “sticky-air” approach, cf. e.g. [17].

Remark 4.9. (Transport by non-Lipschitz velocity field) The multipolar viscosity ensures the velocity field regular, namely \( v \in L^s(I; W^{2,s}(\Omega; \mathbb{R}^d)) \), so that, in particular, it is valued in \( W^{1,\infty}(\Omega; \mathbb{R}^d) \) when \( r > d \) is assumed, i.e. the velocity field is Lipschitz continuous in space integrably in time. Since the seminal paper by R. DiPerna and P.L. Lions [20], it is well understood that transport by non-Lipschitz velocity fields can (and even, in general, must [1]) lead to development of singularities and is very difficult, as documented in hundreds of subsequent articles, as in particular [2, 3, 14, 18, 19]. Relaxing such higher-gradient multipolar viscosity is therefore great but surely very difficult challenge.

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Declarations

Conflict of interest The author states that there is no conflict of interest (in the sense that the “author has no financial or personal relationship with any third party whose interests could be positively or negatively influenced by the article’s content”, without being sure about the precise meaning of it, because the author is a mere mathematician without any juristic education).

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