Universal fault-tolerant quantum computation using fault-tolerant conversion schemes

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Abstract

In this paper, we present the fault-tolerant conversion between quantum Reed–Muller (QRM)\((2, 5)\) and \((2, 7)\), and also the conversion between QBCH\((15, 7)\) and \((2, 7)\). Either of the two schemes provides a method to realize universal fault-tolerant quantum computation. In particular, the gate overhead and logical error rate of a logical \(T\) gate are provided, as well as the comparison with magic state distillation scheme. In addition, we propose two other fault-tolerant conversion schemes based on \((a|u + v\rangle\) and \((a + x|b + x|a + b – x\rangle\) constructions.

1. Introduction

A fault-tolerant computation architecture can remarkably allow arbitrarily accurate quantum computation against decoherence, provided that the error introduced by noise is below a certain threshold value \([1, 2]\). The simplest fault-tolerant operation is the use of transversal gates \([3]\). Various stabilizer codes can provide transversal implementations for various gates \([2]\), for example, the seven-qubit Steane code can realize the entire Clifford group transversally, while the 15-qubit quantum Reed–Muller (QRM) code admits a transversal \(T = \text{diag}(1, \sqrt{\varepsilon})\) gate. An important universal gate set is the Clifford + \(T\) set. Unfortunately, it has been proved that no quantum code can be constructed to allow a universal set of transversal gates \([4, 5]\). Therefore, other fault-tolerant methods are required.

Magic state distillation \([6–8]\) is now a leading proposal to complete a universal gate set. The key idea is to accept many low-fidelity magic states prepared by state injection or the previous round of distillation, and then to filter them into high-fidelity ones. In that case, magic state distillation can be orders of magnitude more costly than the transversal implementation \([9, 10]\). Another idea for universal fault-tolerant quantum computation is to combine the transversal gates in two different codes using fault-tolerant conversion scheme \([11, 12]\). Under the subsystem stabilizer formalism \([13]\), this conversion scheme can be interpreted as different gauge fixing procedures \([14–16]\) on the same subsystem code.

In this work, we first present the fault-tolerant conversion between QRM\((2, 5)\) and QRM\((2, 7)\) and the fault-tolerant conversion between QBCH\((15, 7)\) and QRM\((2, 7)\). Both schemes enable us to circumvent the no-go on transversal gates in universal quantum computation. In particular, we give the gate overhead and logical error rate of a logical \(T\) gate using conversion scheme. For comparison, the results of using magic state distillation scheme are also provided. In addition, we present two other fault-tolerant conversion schemes based on \((a|u + v\rangle\) construction and \((a + x|b + x|a + b – x\rangle\) construction. With a wide range of applications, these schemes are helpful in exploiting the advantages of different codes for specific scenarios \([17, 18]\).

The rest of this paper is organized as follows. In section 2, we describe the constructions and transversal properties of QRM codes and QBCH codes. In sections 3.1 and 3.2, we present the conversion between QRM\((2, 5)\) and QRM\((2, 7)\) and the conversion between QBCH\((15, 7)\) and QRM\((2, 7)\). The fault-tolerant logical \(T\) implementations are provided accordingly. In section 3.3, we illustrate two other conversion schemes based on the \((a|u + v\rangle\) construction and \((a + x|b + x|a + b – x\rangle\) construction respectively. In section 4, we compare...
the gate overhead and logical error rate of a logical $T$ gate using fault-tolerant conversion scheme and magicstate distillation scheme. Finally, we give concluding remarks in section 5.

2. Preliminaries

This section describes the classical and quantum error-correcting codes that are involved in the fault-tolerant code conversion procedures which we will discuss later. For the basic facts and notions in classical coding theories, the readers are referred to [2, 19].

Shortened-RM codes and BCH codes. — Suppose that $\gamma$ is a primitive $(2^m - 1)$th-root of unity for $m \geq 3$. Let $S_{(\gamma,i)}$ be the set of representatives of all the binary cyclotomic cosets modulo $2^m - 1$, and denote by $M_i(x)$ the minimal polynomial of $\gamma^i$ over $\mathbb{F}_2$. For $1 \leq r \leq m$, a shortened RM code $\mathbb{RM}(r, m)$ is a

$$
\bigg[2^m - 1, \sum_{i<r} \binom{m}{i}, 2^{m-r}\bigg]
$$

cyclic code with a generator polynomial

$$
g_{r,m}(x) = \prod_{0 \leq w(s) < m-r-1, s \in S_{(\gamma,i)}} M_i(x),
$$

(1)

where $w(s)$ indicates the number of 1s in the binary expansion of $s$. The dual code $\mathbb{RM}(r, m)^\perp$ is a

$$
\bigg[2^m - 1, \sum_{r=0}^{m-r-1} \binom{m}{i}, 2^{r+1} - 1\bigg]
$$

cyclic code with a generator polynomial

$$
g_{r,m}^\perp(x) = \prod_{1 \leq w(s) < m-r, s \in S_{(\gamma,i)}} M_i(x).
$$

(2)

Clearly, $\mathbb{RM}(m - r - 1, m)$ is not the dual of $\mathbb{RM}(r, m)$. Furthermore, the generator matrix of $\mathbb{RM}(r, m + 1)$ has a recursive definition

$$
\mathcal{G}_{r,m+1} = \begin{pmatrix}
\mathcal{G}_{r,m} & \mathcal{G}_{r,m} & 0 \\
0 & \mathcal{G}_{r-1,m} & 0 \\
0 & 1 & 1
\end{pmatrix},
$$

(3)

where $\mathbf{1}$ is a vector of all 1s (the columns are permuted for later convenience). In particular, the generator matrix of $\mathbb{RM}(1, m + 1)$ has the form

$$
\mathcal{G}_{1,m+1} = \begin{pmatrix}
\mathcal{G}_{1,m} & \mathcal{G}_{1,m} & 0 \\
0 & 1 & 1
\end{pmatrix},
$$

(4)

e.g. $\mathcal{G}_{1,2} = \begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1
\end{pmatrix}$.

A BCH code $\mathbb{BCH}(d, m)$ of length $2^m - 1$ and designed distance $d$ is a cyclic code of minimum distance at least $d$ with a generator polynomial

$$
g_{d,m}(x) = \prod_{i=1}^{d-1} M_i(x).
$$

(5)

When $m \geq 5$ and $3 \leq d \leq 2^\left\lfloor \frac{d}{2} \right\rfloor - 1$, $\mathbb{BCH}(d, m)$ has dimension $2^m - 1 - m\left\lceil \frac{d-1}{2} \right\rceil$ and contains its dual [20]. Moreover, if $r \leq \frac{m-1}{2}$, then $w(s) \leq m - r - 1$ holds for $1 \leq s < d \leq 2^\left\lceil \frac{d}{2} \right\rceil - 1 \leq 2^{r+1} - 1$, indicating that $\mathbb{RM}(r, m) \subset \mathbb{BCH}(d, m)$.

QRM codes. — QRM codes are defined as the quantum CSS codes derived from the dual codes of a pair of shortened-RM codes. Suppose that $C_1$ and $C_2$ are $[n, k_1, d_1]$ and $[n, k_2, d_2]$ codes with parity-check matrices $H_1$ and $H_2$, respectively. If $C_1^\perp \subset C_1$, the quantum CSS code $\mathbb{CSS}(C_1, C_2)$ is an $[n, k_1 + k_2 - n, \min\{d_1, d_2\}]$ stabilizer code [2]. The stabilizer $S$ has generators $H^X_1$ and $H^Z_1$, where $H^X_1$ ($H^Z_1$) denotes a set of operators obtained from rows of $H_1$ ($H_2$) by replacing 1s with Pauli operators $\mathbf{Z}$ ($\mathbf{X}$) and 0s with $\mathbf{I}$. The logical operators can transform code states but preserve the code space, and hence are included in $\mathbb{N}(S) \setminus S$, where $\mathbb{N}(S)$ denotes the normalizer of $S$ in the $n$-qubit Pauli group $\mathbb{P}_n$.

For $1 \leq r \leq \left\lfloor \frac{d}{2} \right\rfloor$, $\mathbb{QM}(r, m)$ is a $[2^m - 1, 1, 2^{r+1} - 1]$ CSS code derived from $C_1 = \mathbb{RM}(m - r - 1, m)$ and $C_2 = \mathbb{RM}(r, m)^\perp$, $\mathbb{QM}(r, m)$ has stabilizer generators $\mathcal{G}_{r,m}, \mathcal{G}_{r,m}^\perp$, and logical operators $\mathcal{X} = X^\otimes m - 1$, $\mathcal{Z} = Z^\otimes m - 1$. Specifically when $m = 2r + 1$, the stabilizer generators are $\mathcal{G}_{r,2r+1} = \mathcal{G}_{r,2r+1}$ and $\mathcal{G}_{r,2r+1}$. Hence, $\mathbb{QM}(r, 2r + 1)$ allows transversal implementation of the logical Hadamard gate and logical CNOT gate. For $m \geq 2r + 1$, $\mathbb{QM}(r, m)$ possesses a transversal phase gate $S = \text{diag}(1, e^{i\theta})$. This is evident since for any codeword $c \in \mathbb{QM}(r, m)$, the weight $|c|$ of $c$ satisfies $|c| \equiv 0 \pmod{2^{r+1} - 1}$ [21]. Especially when $m \geq 3r + 1$, $\mathbb{QM}(r, m)$ has a transversal $T$ gate [11, 22, 23].

QBCH codes. — For $m \geq 5$ and $3 \leq d \leq 2^\left\lfloor \frac{d}{2} \right\rceil - 1$, $\mathbb{QBCH}(d, m)$ is a $[2^m - 1, 2^m - 2m \left\lceil \frac{d-1}{2} \right\rceil - 1, \geq d]$ code $\mathbb{CSS}(\mathbb{BCH}(d, m), \mathbb{BCH}(d, m))$. Denote by $G_{d,m}$ and $H_{d,m}$ the generator matrix and parity-check matrix for
BCH \((d, m)\), respectively. The stabilizer generators are \(H^{\pm}_{d,m}\) and \(H^{\pm}_{d,m}\). Pick a basis for the code space such that the first logical X and Z operators are transversal X and Z, respectively. Suppose that all the other logical qubits are prepared as encoded \(|0\rangle\), then transversal Hadamard gates \(H^\otimes m\) can preserve the code space and effect a logical Hadamard gate on the first encoded qubit. Also, transversal CNOT gates effect a logical CNOT gate on the first encoded qubit. Since \(g_{d,t}(x)\) is divisible by \(g_{d,d}(x)\) for \(3 \leq d \leq 15\), we can obtain that \(\overline{\text{RM}(3, 7)^t} \subset \text{BCH}(d, 7)\). In other words, given an arbitrary codeword \(c \in \text{BCH}(d, 7)^t\), \(c\) is also contained in \(\overline{\text{RM}(3, 7)}\), indicating that \(c\) has weight divisible by 4. In that case, QBCH\((d,7)\) allows the transversal implementation of a logical \(S\) gate on the first encoded qubit.

### 3. Fault-tolerant conversion schemes

In this section, we first present the conversion between QRM codes, and the conversion between QRM codes and QBCH codes. Based on the \((a|b + y)\) construction and the \((a + x|b + x|a + b - x)\) construction, we then provide two other conversion schemes which are applicable to a wide range of quantum codes.

#### 3.1. Conversion between QRM codes

For \(1 \leq r \leq m\), denote by \(|\overline{\phi}\rangle_m\) an encoded state of QRM\((r, m)\). Let \(|\phi\rangle_m = \frac{1}{\sqrt{2}} (|\overline{\phi}\rangle_m + |1\rangle_m|1\rangle_m)\) be a maximally entangled state between a qubit encoded in QRM\((r, m)\) and a bare qubit. Then the combined state \(|\overline{\phi}\rangle_m \otimes |\phi\rangle_m\) can be viewed as an encoded state of a \((2^m + 1 - 1)\)-qubit code. We write stabilizer generators for this code and QRM\((r, m + 1)\) in table 1. Given two ordered sets \(A = \{A_1, A_2, ..., A_N\}\) and \(B = \{B_1, B_2, ..., B_N\}\), we use \(A \times B\) to represent the set \(\{A_1 \otimes B_1, A_2 \otimes B_2, ..., A_N \otimes B_N\}\), and use \(A \otimes I\) to represent the set \(\{A_1 \otimes I, A_2 \otimes I, ..., A_N \otimes I\}\).

Note that the generators of \(|\overline{\phi}\rangle_m \otimes |\phi\rangle_m\) that are listed in the first six rows can be rewritten as those of \(|\overline{\phi}\rangle_{m+1}\) without changing the code state. Thus we only need to substitute the remaining generators \(I^\otimes 2^m - 1 \otimes \tilde{G}_{r,m}^Z \otimes I\) by \(\tilde{G}_{r,m}^Z \times \tilde{G}_{r,m}^Z \otimes I\) when converting from \(|\overline{\phi}\rangle_m \otimes |\phi\rangle_m\) to \(|\overline{\phi}\rangle_{m+1}\). Furthermore, the common stabilizer generators contain \(G_{r,m+1}^X\) and \(G_{r,m+1}^Z\), that is

\[
\begin{align*}
\tilde{G}_{r,m}^X \times \tilde{G}_{r,m}^X \otimes I,
I^\otimes 2^m - 1 \otimes X^\otimes 2^m - 1 \otimes X,
\tilde{G}_{r,m}^Z \times \tilde{G}_{r,m}^Z \otimes I,
I^\otimes 2^m - 1 \otimes Z^\otimes 2^m - 1 \otimes Z,
\end{align*}
\]

the measurements of which give all the syndrome bits needed for diagnosing any single-qubit error on \(2^m + 1 - 1\) qubits. Hence by using the fault-tolerant implementations of state preparations and stabilizer measurements described in [24], we can complete the fault-tolerant conversion between QRM\((r, m)\) and QRM\((r, m + 1)\) as follows.

To convert from QRM\((r, m)\) to QRM\((r, m + 1)\), we first fault-tolerantly prepare the \(2^m\)-qubit entangled state \(|\phi\rangle_m\) and append it to the system. After fault-tolerantly measuring the stabilizer generators \(G_{r,m}^Z \times G_{r,m}^Z \otimes I\) of QRM\((r, m + 1)\), we remove any \(-1\) syndrome bit revealed by a generator \(A\) using the so-called ‘pure error’ operator [11] \(U \in \{I^\otimes 2^m - 1 \otimes \tilde{G}_{r,m}^Z \otimes I\}\), that is, \(U\) commutes with all the generators \(G_{r,m}^Z \times G_{r,m}^Z \otimes I\) except \(A\). Finally, we perform a fault-tolerant error-correction operation through fault-tolerant measurements of \(G_{r,m+1}^X\) and \(G_{r,m+1}^Z\) and a subsequent recovery procedure.

### Table 1. Stabilizer generators for \(|\overline{\phi}\rangle_{m+1}\) and \(|\overline{\phi}\rangle_m \otimes |\phi\rangle_m\)

| Stabilizer generators for \(|\overline{\phi}\rangle_{m+1}\) | Stabilizer generators for \(|\overline{\phi}\rangle_m \otimes |\phi\rangle_m\) |
|--------------------------------------------------|--------------------------------------------------|
| \(G_{r,m}^X \times G_{r,m}^X \otimes I\) | \(G_{r,m}^X \times G_{r,m}^X \otimes I\) |
| \(I^\otimes 2^m - 1 \otimes X^\otimes 2^m - 1 \otimes X\) | \(I^\otimes 2^m - 1 \otimes X^\otimes 2^m - 1 \otimes X\) |
| \(G_{r,m}^Z \times G_{r,m}^Z \otimes I\) | \(G_{r,m}^Z \times G_{r,m}^Z \otimes I\) |
| \(I^\otimes 2^m - 1 \otimes Z^\otimes 2^m - 1 \otimes Z\) | \(I^\otimes 2^m - 1 \otimes Z^\otimes 2^m - 1 \otimes Z\) |

\(a\) \(\tilde{G}_{r,m}^Z\) denotes the set \(\tilde{G}_{r,m}^Z \times \tilde{G}_{r,m}^Z \otimes \tilde{G}_{r,m}^Z\). 

\(b\) \(\tilde{G}_{r,m}^X\) denotes the set \(\tilde{G}_{r,m}^X \otimes \tilde{G}_{r,m}^X\).
fault-tolerantly, we apply a recovery procedure to correct any single-qubit error. By discarding the additional 2\(^m\) stabilizer generators and minimum distance d, we can implement the associated gauge qubit with the associated gauge operator if the outcome is Z \(^r\). The second scenario (shown in figure 1(b)) can be realized by measuring all X gauge operators and fixing the gauge qubit with corresponding Z gauge operator if the outcome is 1.

Recall that QRM\((r, 2r + 1)\) and QRM\((r, 3r + 1)\), jointly, possess a universal set of transversal gates. Hence using intermediate codes (QRM\((r, m): 2r + 1 < m < 3r + 1\)) for transition, the conversion between QRM\((r, 2r + 1)\) and QRM\((r, 3r + 1)\) provides a method for universal fault-tolerant quantum computation without the use of magic state distillation scheme. In particular, we give the fault-tolerant implementation of a logical T gate on QRM\((2, 5)\) in figure 2.
3.2. Conversion between QBCH codes and QRM codes

Now we present the fault-tolerant conversion between QBCH(\(d\), \(m\)) and QRM(\(r\), \(m\)) for \(m \geq 5\), \(2 \leq r \leq \frac{m-1}{3}\) and \(3 \leq d \leq \lfloor \frac{2m}{3} \rfloor - 1\). As an example, we consider the conversion between QBCH(15, 7) and QRM(2, 7).

Let \(|\tilde{\psi}\rangle\) be an encoded state of QBCH(15, 7) stabilized by \(H^X_{15,7} \text{ and } G^Z_{15,7} \notin \{Z^{127}\}\). Choose the generator matrix \(G_{4,7}\) of RM(4, 7) such that \(G_{15,7} - \{1\} \subseteq G_{4,7}\). Then we can write the stabilizer generators for \(|\tilde{\phi}\rangle\) in QRM(2, 7) in table 3. Note that the generators \(H^X_{7,3} \text{ and } H^X_{11,7} = H^X_{5,7}\) can be substituted by \(G^X_{127}\) without changing the code state |\(\tilde{\psi}\rangle\). Hence QBCH(15, 7) and QRM(2, 7) can correspond to the same [[127, 1, 21, 7]] subsystem code as described in table 4. In that case, the conversion between QRM(15, 7) and QRM(2, 7) consists in fixing all the 21 gauge qubits in encoded |0\rangle or |+\rangle.

Similar with the process in figure 1 (b), the fault-tolerant conversion from QBCH(15, 7) to QRM(2, 7) can be realized by first fault-tolerantly measuring the Z gauge operators and fixing the gauge qubit with corresponding X gauge operator if the outcome is \(-1\). After fault-tolerant measurements of stabilizer generators \(H^X_{5,7} \text{ and } H^X_{11,7}\), any single-qubit errors can be corrected with a recovery operation. By reversing these steps, the fault-tolerant conversion from QRM(2, 7) to QBCH(15, 7) can be accomplished.

Furthermore, a fault-tolerant logical \(T\) gate on QBCH(15, 7) can be implemented by first fault-tolerantly converting QBCH(15, 7) to QRM(2, 7), applying transversal \(T\) gates \(T^{127}\), and then converting the code back to QBCH(15, 7).

3.3. Other conversion schemes

We can tell that the key to a fault-tolerant conversion between two quantum codes is to correspond them to the same subsystem code. For example, the correspondence between QBCH codes relies mainly on the recursive definition of shortened-RM codes (equation (3)). Following that idea, techniques of generating new codes from the old ones may also help to build the required correspondence. In this subsection, we consider the use of two well-known construction methods—the \((u|u + v)\) construction and the \((a + x|b + x|a + b - x)\) construction.
Table 5. Stabilizer generators for $|\pi|_C$ and $|\pi|_1 \otimes |\pi|_2$.

| Stabilizer generators for $|\pi|_C$ | Stabilizer generators for $|\pi|_1 \otimes |\pi|_2$ |
|----------------------------------|----------------------------------|
| $G_1^X \otimes G_2^X$            | $G_1^X \otimes I^{2n}$          |
| $I^{2n} \otimes G_2^X$           | $I^{2n} \otimes G_1^X$          |
| $G_1^Z \times G_2^Z$             | $G_1^Z \otimes I^{2n}$          |
| $I^{2n} \otimes G_2^Z$           | $I^{2n} \otimes G_1^Z$          |
| $H_1^Z \times H_2^Z - G_1^Z \times G_2^Z - \{Z^{2n}\}$ | $H_1^Z \otimes H_2^Z - G_1^Z \otimes G_2^Z - \{Z^{2n}\}$ |
| $I^{2n} \otimes H_2^Z - I^{2n} \times G_2^Z$ | $I^{2n} \otimes H_2^Z - I^{2n} \times G_2^Z$ |

Table 6. Stabilizer generators, logical operators and gauge operators for a $[[2n, 1, 2k_1 - 2k_2, d_2^+]_2]$ subsystem code.

| Stabilizer generators | Logical operators | Gauge operators |
|-----------------------|-------------------|-----------------|
| $G_1^X \times G_2^X$  | $X^{2n}$          | $G_1^X \times G_2^X$ |
| $I^{2n} \otimes G_2^X$ | $Z^{2n}$          | $G_1^Z \times G_2^Z$ |
| $H_1^Z \times H_2^Z - G_1^Z \times G_2^Z - \{Z^{2n}\}$ | $I^{2n} \otimes H_2^Z - I^{2n} \times G_2^Z$ | $G_1^Z \otimes I^{2n} - G_1^Z \times I^{2n}$ |

3.3.1. The $(u|u + v)$ construction

Let $G_1$ be an $[n, k_1, d_1]$ code and $G_2$ be an $[n, k_2, d_2]$ code with generator matrices $G_1, G_2$ and parity check matrices $H_1, H_2$, respectively. Supposing that $G_2 \subset G_1$, we can then obtain a $[2n, k_1 + k_2, \min\{2d_1, d_2\}]$ code $C$ consisting of all vectors $(u|u + v): u \in G_1, v \in G_2$ [19]. The generator matrix $G_C$ of $C$ is

$$G_C = \begin{pmatrix} G_1 & G_1 \\ 0 & G_2 \end{pmatrix}.$$  \hspace{1cm} (7)

The dual code $C^\perp$ is $[2n, 2n - k_1 - k_2, \min\{2d_1^+, d_2^+\}]$ code with a generator matrix

$$H_C = \begin{pmatrix} H_1 & H_1 \\ 0 & H_2 \end{pmatrix}.$$  \hspace{1cm} (8)

where $d_1^+$ and $d_2^+$ denote the minimum distances of $C_1^+$ and $C_2^+$, respectively. Clearly, $C$ is self-orthogonal if $G_1$ and $G_2$ are both self-orthogonal.

Let $Q_C$ be a $[[2n, 2n - 2k_1 - 2k_2, \min\{2d_1^+, d_2^+\}]]$ code CSS$(C_1^+, C_2^+)$. Assume that $1 \in C_1^+$ and $1 \notin C_1$, then we can choose a basis for the code space of $Q_C$ such that logical $X$ and $Z$ operators on the first encoded qubit are transversal $X$ and $Z$, respectively. Provided that all the other logical qubits are prepared in $|0\rangle$, the stabilizer generators for this encoded state $|\pi|_C \in Q_C$ are listed in table 5. On the other hand, let $Q_{C_1}$ be an $[[n, n - 2k_2, d_2^+]]$ code CSS$(C_1^+, C_1^+)$ and let $Q_2$ be an $[[n, n - 2k_1, d_1^+]]$ code CSS$(C_2^+, C_2^+)$. Suppose that logical $X$ and $Z$ operators on the first encoded qubit of $Q_1$ ($Q_2$) are transversal $X$ and $Z$ respectively, and all other logical qubits are prepared as $|0\rangle$. Then the combined state $|\pi|_1 \otimes |\pi|_2$ for $|\pi|_1 \in Q_1$ and $|\pi|_2 \in Q_2$ has stabilizer generators as listed in table 5.

We can see that the stabilizer generators of $|\pi|_C$ from the first six rows can be substituted by those of $|\pi|_1 \otimes |\pi|_2$ without changing the code state. Therefore, $|\pi|_C$ and $|\pi|_1 \otimes |\pi|_2$ can be associated with the $[[2n, 1, 2k_1 - 2k_2, d_2^+]_2]$ subsystem code described in table 6. In that case, the conversion between $Q_C$ and $Q_C$ consists in fixing one half of the gauge qubits in encoded $|0\rangle$ and the other half in encoded $|+\rangle$. To be concrete, if $d_2^+ \geq 3$, the fault-tolerant conversion between $Q_1$ to $Q_C$ can be realized by first fault-tolerantly preparing an encoded state $|\pi|_2 \in Q_2$, appending it to the system, fault-tolerantly measuring the gauge operators $G_1^X \times G_1^X \times G_2^X \times G_1^Z \times G_2^Z \times G_2^Z$ and $G_1^Z \times G_1^Z \times G_2^Z \times G_2^Z \times G_2^Z$ and fixing the gauge qubit with the corresponding gauge operator in $(G_1^Z \otimes I^{2n} - G_2^Z \otimes I^{2n}, G_1^Z \otimes I^{2n} - G_2^Z \otimes I^{2n})$ if the outcome is $-1$. After fault-tolerant measurements of stabilizer generators.
we can apply a recovery procedure to correct any single-qubit errors. To convert from \( Q_c \) to \( Q_0 \), we simply fault-tolerantly measure the gauge operators \( G_2 \times I_{\text{comp}} - G_2 \times I_{\text{comp}} \) and \( G_1 \times I_{\text{comp}} - G_2 \times I_{\text{comp}} \), fix the gauge qubit with corresponding gauge operator in \( \{ G_1 \times G_2 - G_2 \times G_1 \} \), if the outcome is \(-1\), and perform a fault-tolerant error-correction operation. By discarding the additional \( n \) qubits, we can then obtain the required state of \( Q_0 \).

3.3.2. The \( (a + x)|b + x(a + b - x) \) construction
Let \( C_1, C_2 \) be defined as above, then we can construct a \([3n, 2k_1 + k_2, \min\{2d_1, d_2\}]\) code
\[
D = \{(a + x)|b + x(a + b - x)\} : a, b \in C_1, x \in C_2 \}
\]
with a generator matrix
\[
G_D = \begin{bmatrix}
G_1 & 0 & G_1 \\
0 & G_1 & G_1 \\
G_2 & G_2 & G_2
\end{bmatrix}
\] (10)
The dual code \( D^\perp \) is a \([3n, 3n - 2k_1 - k_2, \min\{2d_1^\perp, d_2^\perp\}]\) code with a generator matrix
\[
H_D = \begin{bmatrix}
H_1 & 0 & H_1 \\
0 & H_1 & H_1 \\
H_2 & H_2 & H_2
\end{bmatrix}
\] (11)

It is certain that if both \( C_1 \) and \( C_2 \) are self-orthogonal, \( D \) is also self-orthogonal.

Let \( Q_D \) be a \([3n, 3n - 4k_1 - 2k_2, \min\{2d_1^\perp, d_2^\perp\}]\) code CSS(\( D^\perp, D^\perp \)). Suppose that logical \( X \) and \( Z \) operators on the first encoded qubit are transversal and all other logical qubits are encoded in \([0]\). Then the stabilizer generators for \([\tilde{J}_3D] \in Q_D \) and \([\tilde{J}_3]_1 \otimes [\tilde{J}_3]_2 \otimes [\tilde{J}_3]_3 \) with \([\tilde{J}_3]_1, [\tilde{J}_3]_2 \in Q_1 \) and \([\tilde{J}_3]_3 \in Q_2 \), are listed in table 7. Clearly, the generators of \([\tilde{J}_3D] \) from the first 9 rows can be changed into those of \([\tilde{J}_3]_1 \otimes [\tilde{J}_3]_2 \otimes [\tilde{J}_3]_3 \) without altering the code state. Hence we can correspond \([\tilde{J}_3]_3 \) and \([\tilde{J}_3]_1 \otimes [\tilde{J}_3]_2 \) to the \([3n, 1, 2k_1 - 2k_2, d_1^\perp] \) subsystem code in table 8. In that case, the conversion between \( Q_3 \) and \( Q_D \) consists in fixing one half of the gauge qubits in encoded \([0]\) and the other half in encoded \([+\])

Note that the fault-tolerant conversion schemes based on the \((a|a + v)\) construction and the \((a + x)|b + x(a + b - x) \) construction only require that \( C_1 \) and \( C_2 \) are self-orthogonal codes satisfying \( C_2 \subset C_1 \) and \( I \not\subset C_i \). Hence, a wide range of codes could be applied to these schemes. Specifically by picking \( C_1 \) and \( C_2 \) to be Hamming codes, the Golay code derived from a variant of the \((a + x)|b + x(a + b - x) \) construction [25], has shown advantages in constructing quantum codes with higher threshold [26]. In that case, the fault-tolerant conversion from quantum Hamming code to quantum Golay code can help us exploit the particular advantage of the latter code.

4. Performance analysis

In this section, we analyze the gate overhead and logical error rate of a logical \( T \) gate using fault-tolerant conversion scheme. For simplicity, we assume that the error probability of any individual component in the

\[
G_2^X \times G_2^Z \\
I_{\text{comp}} \otimes G_2^X \\
G_2^Z \times G_2^Z \\
I_{\text{comp}} \otimes G_2^Z
\] (9)
Table 8. Stabilizer generators, logical operators and gauge operators for a $[[3n, 1, 2k_1 - 2k_2, d_z]]$ subsystem code.

| Stabilizer generators | Gates |
|-----------------------|-------|
| $G_1^Z \times G_2^Z \otimes I^{2n}$ | $G_1^Z \times G_2^Z \otimes I^{2n}$ |
| $I^{2n} \otimes G_2^Z \times G_2^Z$ | $I^{2n} \otimes G_2^Z \times G_2^Z$ |
| $G_1^Z \times G_2^Z \times G_2^Z$ | $G_1^Z \times G_2^Z \times G_2^Z$ |
| $H_1^Z \times H_2^Z \otimes I^{2n} - I^{2n} \times G_1^Z \times G_2^Z$ | $H_1^Z \times H_2^Z \otimes I^{2n} - I^{2n} \times G_1^Z \times G_2^Z$ |
| $I^{2n} \otimes H_2^Z \times H_2^Z - I^{2n} \otimes G_1^Z \times G_2^Z$ | $I^{2n} \otimes H_2^Z \times H_2^Z - I^{2n} \otimes G_1^Z \times G_2^Z$ |

Table 9. Gate overhead of several fault-tolerant circuits.

| Operation/Preparation | Gate overhead |
|------------------------|---------------|
| CAT$_i$ preparation$^a$ | $(3i - 1)\text{ CNOTs, 1 H, 1M}_{2i}/(1 - p)^i$ |
| $M_i$ measurement$^b$ | $3(1 \text{ CAT}_i, t \text{ CNOTs and/or controlled—Zs, } t - 1 \text{ CNOTs, } 1H, 1\text{M}_t)$ |
| $[\Phi]_m$ preparation$^c$ | $\left(\sum_{i=1}^{m} \frac{1}{m} (2^{2m+1-i} - 1) + 2^{m+1} - 5 \text{ CNOTs, } 2\sum_{i=1}^{m} \left(\frac{1}{m} \right) H_5, 1\text{M}_{2m-1}\right)/((t - p)^i)$, where $a = \sum_{i=1}^{m} \frac{1}{m} 2^{2m+1-i} + 2^{m+2} - 7$. |
| $[\Phi]_m$ preparation$^d$ | $1|\Phi\rangle, 1\text{M}_e, 0.5 X$ |

$^a$ A $t$-qubit CAT state requires 1 H gate, $t - 1$ CNOT gates for encoding and 2$t$ CNOT gates, $t$ Z measurements for verification. The final state is accepted if all the outcomes are +1s. With the use of $2t$ qubits, the probability of acceptance is $(1 - p)^i$.

$^b$ $M_i$ denotes a $t$-weight operator, e.g., a stabilizer generator, a logical operator or a gauge operator. The measurement is repeated by three times and the final result is decided by the majority value.

$^c$ An encoded state $[\Phi]_m \in$ QRM $(2, m)$ requires $\sum_{i=1}^{m} \left(\frac{1}{m} \right) (2^{m-i} - 1) + 2^{m+1} - 5 \text{ CNOTs, } 2\sum_{i=1}^{m} \left(\frac{1}{m} \right) H_5, 1\text{M}_{2m-1}$) H gates for encoding, along with $2^m - 1$ CNOT gates and a logical Z measurement for verification. The final state is accepted if the outcome is +1. With $2^{m+1} - 2$ qubits involved, the probability of acceptance is $(1 - p)^i$, where $a = \sum_{i=1}^{m} \frac{1}{m} 2^{2m+1-i} + 2^{m+2} - 2 = \sum_{i=1}^{m} \frac{1}{m} 2^{2m+1-i} + 2^{m+2} - 7$.

$^d$ The entangled state $[\Phi]_m = \frac{1}{\sqrt{2}} (|\Phi\rangle_0 + |\Phi\rangle_1)$ can be obtained by measuring Z on the combined state $|\Phi\rangle_m \otimes |+\rangle$ and flip the last qubit if the outcome is −1.

conversion circuit are the same $p$, including qubits, gates and measurements. In addition, all physical gates are assumed to have equal costs.

4.1. Gate overhead

4.1.1. Fault-tolerant conversion scheme

As shown in figure 1, the fault-tolerant conversion between QRM(2, 5) and QRM(2, 7) requires fault-tolerant state preparations and fault-tolerant measurements of gauge operators and stabilizer generators. Following the implementations demonstrated in [24], we list the gate overhead of these fault-tolerant circuits in table 9. In addition, we give the gate overhead of fault-tolerant conversion between QRM(2, 5) and QRM(2, 7) in tables 10 and 11.

By adding 127 T gates, we can then obtain the gate overhead of a fault-tolerant logical $T$ on QRM(2, 5) in table 12. It is clear that the gate overhead declines with a decreasing physical error probability $p$, especially when $p$ falls from $5 \times 10^{-3}$ to $3 \times 10^{-3}$. In addition, the fault-tolerant conversion from QRM(2, 5) to QRM(2, 7) consumes much more physical gates than its reverse when $p = 5 \times 10^{-3}$. The main reason is that fault-tolerant $[\Phi]_5$ and $[\Phi]_6$ preparations in the former case require intensive resources to maintain a reasonable probability of acceptance.

We give the gate overhead of fault-tolerant conversion between QBCH(15, 7) and QRM(2, 7) in tables 13 and 14. Combined with 127 T gates, the gate overhead of a fault-tolerant logical $T$ gate on QBCH(15, 7) is illustrated in table 15. By comparison with table 12 we can tell that, a fault-tolerant logical $T$ on QBCH(15, 7) consumes less physical gates than on QRM(2, 5). This is natural since the ancilla state preparations are not required for QBCH (15, 7) before the measurement of gauge operators. Also, the gate overhead drops sharply with $p$ descending from $5 \times 10^{-3}$ to $3 \times 10^{-3}$.
For comparison, we investigate the gate overhead of using magic state distillation scheme \([24, 27]\). A magic state \(|\text{+}\rangle\) can first be prepared by state injection and then be processed with several rounds of distillation procedures.

### Table 10. Gate overhead of the fault-tolerant conversion from QRM(2, 5) to QRM(2, 7).

| Operation | Gate overhead |
|-----------|---------------|
| \(|\Phi\rangle\) preparation | 21 Gate per \(|\text{+}\rangle\) |
| Measurement of Z gauge operators \(G_{a,b}^z\times G_{a,b}^z\otimes I\) | 11 Gates per \(|\text{+}\rangle\) |
| Fixing with X gauge operator in \((I^{10}\otimes G_{a,b}^x\otimes I)\) | at most 20 Gates per \(|\text{+}\rangle\) |
| Measurement of stabilizer generators \(\mathcal{C}_{\text{a},a}, \mathcal{C}_{\text{b},b}\) | 11 Gates per \(|\text{+}\rangle\) |
| \(|\Phi\rangle\) preparation | 21 Gate per \(|\text{+}\rangle\) |
| Measurement of gauge operators \(G_{a,b}^y\times G_{a,b}^y\otimes I\) | 11 Gates per \(|\text{+}\rangle\) |
| Fixing with X gauge operator in \((I^{10}\otimes G_{a,b}^x\otimes I)\) | at most 20 Gates per \(|\text{+}\rangle\) |
| Measurement of stabilizer generators \(\mathcal{C}_{\text{a},a}, \mathcal{C}_{\text{b},b}\) | 11 Gates per \(|\text{+}\rangle\) |

*An operator in \((I^{10}\otimes G_{a,b}^x\otimes I)\) has weight at most 20.

### Table 11. Gate overhead of the fault-tolerant conversion from QRM(2, 7) to QRM(2, 5).

| Operation | Gate overhead |
|-----------|---------------|
| Measurement of X gauge operators \(I^{10}\otimes G_{a,b}^x\otimes I\) | 15 Gates per \(|\text{+}\rangle\) |
| Fixing with Z gauge operator in \((G_{a,b}^z\times G_{a,b}^z\otimes I)\) | at most 44 Gates per \(|\text{+}\rangle\) |
| Measurement of stabilizer generators \(\mathcal{C}_{\text{a},a}, \mathcal{C}_{\text{b},b}\) | 14 Gates per \(|\text{+}\rangle\) |
| Measurement of X gauge operators \(I^{10}\otimes G_{a,b}^x\otimes I\) | 10 Gates per \(|\text{+}\rangle\) |
| Fixing with Z gauge operator in \((G_{a,b}^z\times G_{a,b}^z\otimes I)\) | at most 20 Gates per \(|\text{+}\rangle\) |
| Measurement of stabilizer generators \(\mathcal{C}_{\text{a},a}, \mathcal{C}_{\text{b},b}\) | 12 Gates per \(|\text{+}\rangle\) |

### Table 12. Gate overhead of a fault-tolerant logical \(T\) gate on QRM(2, 5) using fault-tolerant code conversion scheme.

| \(p\) | QRM(2, 5) to QRM(2, 7) \(T^{0\leftrightarrow 27}\) QRM(2, 7) to QRM(2, 5) Total |
|-------|-------------------------------|-------------------------------|-------------------|
| \(5 \times 10^{-3}\) | 1805 278 127 101 480 & 1906 885 |
| \(10^{-3}\) | 40 317 127 35 391 & 75 835 |
| \(10^{-4}\) | 29 054 127 29 587 & 58 768 |
| \(10^{-5}\) | 28 283 127 28 996 & 57 506 |

### Table 13. Gate overhead of the fault-tolerant conversion from QBCH(15, 7) to QRM(2, 7).

| Operation | Gate overhead |
|-----------|---------------|
| Measurement of Z gauge operators \(G_{a,b}^z\times (G_{a,b}^z - (Z^{0\leftrightarrow 127}))\) | 21 Gates |
| Fixing with X gauge operator in \((H_{a,b}^x + H_{a,b}^x - H_{n,1}^x)^2\) | At most 44 Gates |
| Measurement of stabilizer generators \(H_{a,b}^x, H_{a,b}^y\) | 14 Gates |

### Table 14. Gate overhead of the fault-tolerant conversion from QRM(2, 7) to QBCH(15, 7).

| Operation | Gate overhead |
|-----------|---------------|
| Measurement of X gauge operators \(H_{a,b}^x - H_{a,b}^x - H_{n,1}^x - H_{n,1}^y\) | 21 Gates |
| Fixing with Z gauge operator in \((G_{a,b}^z - (G_{a,b}^z - (Z^{0\leftrightarrow 127})))\) | At most 58 Gates |
| Measurement of stabilizer generators \(H_{a,b}^x, H_{a,b}^y\) | 14 Gates |

4.1.2. Comparison with magic state distillation scheme
For comparison, we investigate the gate overhead of using magic state distillation scheme \([24, 27]\). A magic state \(|\text{+}\rangle\) can first be prepared by state injection and then be processed with several rounds of distillation procedures.
Table 15. Gate overhead of a fault-tolerant logical $T$ gate on QBCH(15, 7) using fault-tolerant code conversion scheme.

| $p$               | QBCH(15, 7) to QRM(2, 7) | $T^{127}$ | QRM(2, 7) to QBCH(15, 7) | Total  |
|------------------|--------------------------|-----------|--------------------------|--------|
| $5 \times 10^{-3}$ | 82 812                   | 127       | 82 826                   | 165 765|
| $10^{-3}$        | 24 446                   | 127       | 24 460                   | 49 033 |
| $10^{-4}$        | 19 756                   | 127       | 19 770                   | 39 653 |
| $10^{-5}$        | 19 364                   | 127       | 19 378                   | 38 869 |

Table 16. Gate overhead of the teleportation circuit.

| Base code       | Gate overhead                                                                 |
|-----------------|-------------------------------------------------------------------------------|
| QRM(2, 5)       | 1 high-fidelity $T[\overline{\tau}]$, 31 CNOTs, 1 $M_{1s}$, 15.5 $X_s$, 15.5 $S_s$ |
| QBCH(15, 7)     | 1 high-fidelity $T[\overline{\tau}]$, 127 CNOTs, 1 $M_{1s}$, 63.5 $X_s$, 63.5 $S_s$ |

* The circuit consists of transversal CNOT gates from $T[\overline{\tau}]$ to $[\overline{\tau}]$ $\in$ QRM(2, 5) and transversal $X$ and $S$ gates on $T[\overline{\tau}]$ if the logical $Z$ measurement on $[\overline{\tau}]$ gives $-1$.

Table 17. Gate overhead of the state injection circuit.

| Base code       | Encoded entanglement and decoding (EED)                                      | Total  |
|-----------------|-------------------------------------------------------------------------------|--------|
| QRM(2, 5)       | (1 $|\overline{\tau}\rangle$, 1 $|\overline{\tau}\rangle$, 31 + 159 CNOTs, 15 $H_s$, 15 $M_{1s}$, 15 $M_{2s}$)/(1 - $p$)$^{298}$ | $1 - (1 - p)^{298}$ |
| QBCH(15, 7)     | (1 $|\overline{\tau}\rangle$, 1 $|\overline{\tau}\rangle$, 127 + 1469 CNOTs, 49 $H_s$, 49 $M_{1s}$, 77 $M_{2s}$)/(1 - $p$)$^{2026}$ | $1 - (1 - p)^{2026}$ |

* $p$ denotes the infidelity of the output state, which depends on the number of qubits, physical gates and measurements involved in physical Bell measurement. For example, the exponent 68 indicates 33 qubits, 33 physical gates and 2 measurements.

* The state injection circuit uses encoded entanglement and physical Bell measurement to inject a physical state into an encoded state. The decoding procedure involved depends on the base code. For example, a decoding circuit for QBCH(15, 7) consists of $21 \times 15 + 21 \times 31 + 7 \times 63 + 62 = 1469$ CNOT gates, 49 $H$ gates, 49 $X$ measurements and 77 $Z$ measurements. The decoding succeeds if all the outcomes are $+1$s. With $1 + 127 \times 2 = 255$ qubits involved, the probability is $(1 - p)^{127 - 1469 - 49 - 49 - 77 - 255} = (1 - p)^{2026}$.

Table 18. Gate overhead of one-round distillation circuit.

| Base code       | Twirl operation                                                             | Total  |
|-----------------|-------------------------------------------------------------------------------|--------|
| QRM(2, 5)       | 1 low-fidelity $T[\overline{\tau}]$, 1 $H_s$, 1 $M_{1s}$, 15.5 $X_s$, 15.5 $S_s$ | $(15$ twirl operations, 34 + 31 CNOTs, 14 $M_{1s}$)/$(1 - 15p)(1 - p)^{2014}$ |
| QBCH(15, 7)     | 1 low-fidelity $T[\overline{\tau}]$, 1 $H_s$, 1 $M_{1s}$, 63.5 $X_s$, 63.5 $S_s$ | $(15$ twirl operations, 34 + 127 CNOTs, 14 $M_{1s}$)/$(1 - 15p)(1 - p)^{1854}$ |

* After applying a twist operation on each low-fidelity magic state $T[\overline{\tau}]$ obtained from the injection procedure or a previous round of distillation procedure, the circuit decodes the distillation code QRM(1, 4). The output state is accepted provided that all the logical $Z$ (or $X$) measurements on the base code give $+1$. If the circuit is operated flawlessly, a high-fidelity $T[\overline{\tau}]$ will be successfully distilled with probability about $1 - 15p$, [6, 27]. The exponent 2014 indicates 465 qubits and 1549 physical gates and measurements that are involved in the circuit.

* The exponent 8158 indicates 1905 qubits and 6253 physical gates and measurements.

To increase the fidelity, given a high-fidelity $T[\overline{\tau}]$, a fault-tolerant logical $T$ gate can be implemented by the teleportation method. We give the gate overhead of these circuits in tables 16–18. To sum up, the gate overhead of a fault-tolerant logical $T$ gate using magic state distillation scheme is illustrated in table 19.

We can tell that a fault-tolerant logical $T$ gate on QBCH(15, 7) using magic state distillation scheme requires much more physical gates than using fault-tolerant conversion scheme under the same physical error probability $p$. For QRM(2, 5), the gate overhead of magic state distillation circuit is also larger than the conversion circuit provided that $p \geq 10^{-4}$. However, when $p$ descends to $10^{-5}$, the gate overhead of using distillation scheme can be reduced very fast, and becomes smaller than using conversion scheme.
4.2. Logical error rate

Now we proceed with the logical error rate of a logical T gate using fault-tolerant conversion scheme. From figure 2 we can see that, the output state $T\otimes\overline{0}$ contains more than 1 errors if two or more errors are introduced into the transversal circuit $T^{\otimes127}$ or the fault-tolerant conversion circuits. The first scenario occurs with probability about $(\binom{127}{2} p^2)$. The other scenario, however, has a few possibilities. As an example, we list all the possible ways of two errors entering the conversion circuit from QRM(2,5) to QRM(2,6) in table 20, along with the corresponding probabilities. With similar results of other conversion circuits, the logical error rate of a logical T gate on QRM(2,5) can be upper bounded by their sum. The logical T gate on QBCH(15,7) can be analyzed in the same manner. Given different $p$, we give the results in tables 21 and 22.

For comparison, we also analyze the logical error rate of a logical T gate using magic state distillation scheme. Given a magic state $T\otimes\overline{0}$ with an error rate $p$, the output state of the distillation circuit has an error rate about $35p^3$, provided that the circuit is error-free [6, 27]. The distillation circuits can be operated flawlessly with probabilities $(1 - p)^{914}[1 - 3 \times (31p)^2]^{14}$ and $(1 - p)^{158}[1 - 3 \times (127p)^2]^{14}$ based on QRM(2,5) and QBCH(15,7), respectively. Hence the error rate of the magic state after one-round distillation procedure can then be determined. Since the teleportation circuits on QRM(2,5) and QBCH(15,7) can be perfectly performed.

| Table 19. Gate overhead of a fault-tolerant logical T gate using magic-state distillation scheme. |
|---------------------------------------------------------------|
| Base code | Injection | Distillation | Teleportation |
| QRM(2,5) | 8 $\times 10^{-4}$ | 3946 | 1692432 | 1694084 |
| QRM(2,6) | 10$^{-8}$ | 2318 | 60277 | 60876 |
| QRM(2,7) | 10$^{-5}$ | 2167 | 43212 | 43805 |
| QBCH(15,7) | 10$^{-4}$ | 21507 | 1335076 | 1335740 |
| QBCH(16,7) | 10$^{-5}$ | 13478 | 271668 | 274223 |

| Table 20. Possible ways of two errors entering the conversion from QRM(2,5) to QRM(2,6). |
|---------------------------------------------------------------|
| Event | Probability |
| Failure of $|pI\rangle$, preparation | 3 $\times (32p)\otimes$ |
| Failure of a $\mathcal{Z}$ gauge operator measurement or a stabilizer measurement | $10^3 \times (8p)^2 + 6 \times (32p)^2$ |
| 2 errors in gauge operator measurements, or fixing operation, or stabilizer measurements | $\left(\frac{80}{2}\right)p^2 + \left(\frac{20}{2}\right)p^2 + \left(\frac{192}{2}\right)p^2$ |
| 2 errors in separate procedures | $80^3 \times p^2 + 80 \times 192p^2 + 20 \times 192p^2 + 80p^2 + 20p^2 + 192p^2$ |

| Table 21. Logical error rate of a logical T gate on QRM(2,5) using fault-tolerant conversion scheme. |
|---------------------------------------------------------------|
| $p$ | QRM(2,5) to QRM(2,6) | QRM(2,6) to QRM(2,7) | $T^{\otimes127}$ | QRM(2,7) to QRM(2,6) | QRM(2,6) to QRM(2,5) | Logical error rate |
| 10$^{-4}$ | 6.7$\times10^{-4}$ | 8.9$\times10^{-9}$ | 8.8$\times10^{-3}$ | 1.6$\times10^{-5}$ | 1.8$\times10^{-2}$ |
| 10$^{-5}$ | 7.1$\times10^{-6}$ | 8.9$\times10^{-7}$ | 8.8$\times10^{-1}$ | 1.6$\times10^{-5}$ | 1.8$\times10^{-4}$ |
| 10$^{-6}$ | 7.1$\times10^{-8}$ | 8.9$\times10^{-9}$ | 8.8$\times10^{-1}$ | 1.6$\times10^{-5}$ | 1.8$\times10^{-6}$ |
| 10$^{-7}$ | 7.1$\times10^{-10}$ | 8.9$\times10^{-11}$ | 8.8$\times10^{-1}$ | 1.6$\times10^{-5}$ | 1.8$\times10^{-8}$ |
| 10$^{-8}$ | 7.1$\times10^{-12}$ | 8.9$\times10^{-13}$ | 8.8$\times10^{-1}$ | 1.6$\times10^{-5}$ | 1.8$\times10^{-10}$ |

| Table 22. Logical error rate of a logical T gate on QBCH(15,7) using fault-tolerant conversion scheme. |
|---------------------------------------------------------------|
| $p$ | QBCH(15,7) to QBCH(2,7) | $T^{\otimes127}$ | QBCH(2,7) to QBCH(15,7) | Logical error rate |
| 10$^{-4}$ | 7.9$\times10^{-5}$ | 8.1$\times10^{-3}$ | 1.6$\times10^{-2}$ |
| 10$^{-5}$ | 7.9$\times10^{-5}$ | 8.1$\times10^{-3}$ | 1.6$\times10^{-4}$ |
| 10$^{-6}$ | 7.9$\times10^{-7}$ | 8.1$\times10^{-7}$ | 1.6$\times10^{-6}$ |
| 10$^{-7}$ | 7.9$\times10^{-9}$ | 8.1$\times10^{-9}$ | 1.6$\times10^{-8}$ |
| 10$^{-8}$ | 7.9$\times10^{-11}$ | 8.1$\times10^{-11}$ | 1.6$\times10^{-10}$ |
with respective probability \( (1 - p)^{22}[1 - 3 \times (31p)^2] \) and \( (1 - p)^{23}[1 - 3 \times (127p)^2] \), we can finally obtain the logical error rate in Table 23.

From the comparison between tables 21–23 we can tell that, given the same physical error probability \( p \), the logical error rate of a logical \( T \) gate using magic state distillation scheme is in general much smaller than that of using conversion scheme. In fact, the logical \( T \) gate using conversion scheme performs better than the individual physical gates only when \( p < 10^{-6} \), indicating a hard requirement on physical gate fidelity. The magic state distillation scheme, however, has a significant advantage in this aspect.

### 5. Conclusion

In this paper, we present the fault-tolerant conversion scheme between QRM(2, 5) and QRM(2, 7), as well as the conversion between QBCH(15, 7) and QRM(2, 7). Either of the two schemes enables us to fault-tolerantly perform a logical \( T \) gate without magic state distillation. The gate overhead is provided, along with the comparison with magic state distillation scheme. For QBCH(15, 7), the conversion scheme can achieve a smaller overhead due to the lack of fault-tolerant state preparations. For QRM(2, 5), the gate overhead of using magic state distillation scheme declines fast with a descending \( p \) and becomes smaller than using the conversion scheme when \( p \lesssim 10^{-5} \). In terms of the logical error rate, the distillation scheme has a significant advantage and enables a logical \( T \) gate to outperform the physical gate when \( p \lesssim 10^{-4} \).

On the other hand, the key to converting between two different codes is to correspond them to the same subsystem code with different gauge qubits. In other words, classical codes that construct these two quantum codes need to be closely related and have common codewords. Under such circumstances, we consider the use of the \( \alpha u \alpha + v \) construction and the \( \alpha + x|b + x|a + b - x \) construction, and present two fault-tolerant conversion schemes between quantum codes derived from the new codes and old codes. These schemes have a wide range of applications and can thus contribute to exploiting the advantages of different codes for specific scenarios. We should note that all the proposed fault-tolerant conversion schemes could be well-generalized to higher-dimensional systems, at least to the \( p \)-ary systems. In that case, more methods of constructing new codes from old ones [28] could be investigated.

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**Table 23. The logical error rate of a logical \( T \) gate using magic state distillation scheme.**

| Base code  | \( p \)   | Injected magic state | Distilled magic state | Logical error rate |
|------------|-----------|----------------------|-----------------------|-------------------|
| QRM(2, 5)  | \( 10^{-4} \) | \( 6.8 \times 10^{-3} \) | \( 1.3 \times 10^{-3} \) | \( 1.3 \times 10^{-3} \) |
|           | \( 10^{-3} \) | \( 6.8 \times 10^{-4} \) | \( 1.1 \times 10^{-3} \) | \( 1.1 \times 10^{-3} \) |
|           | \( 10^{-6} \) | \( 6.8 \times 10^{-5} \) | \( 1.1 \times 10^{-11} \) | \( 1.1 \times 10^{-11} \) |
|           | \( 10^{-7} \) | \( 6.8 \times 10^{-6} \) | \( 1.1 \times 10^{-14} \) | \( 1.1 \times 10^{-14} \) |
|           | \( 10^{-8} \) | \( 6.8 \times 10^{-7} \) | \( 1.1 \times 10^{-17} \) | \( 1.1 \times 10^{-17} \) |
| QBCH(15, 7)| \( 10^{-4} \) | \( 2.6 \times 10^{-2} \) | \( 1.3 \times 10^{-3} \) | \( 1.4 \times 10^{-3} \) |
|           | \( 10^{-5} \) | \( 2.6 \times 10^{-3} \) | \( 6.6 \times 10^{-7} \) | \( 6.7 \times 10^{-7} \) |
|           | \( 10^{-6} \) | \( 2.6 \times 10^{-4} \) | \( 6.2 \times 10^{-10} \) | \( 6.2 \times 10^{-10} \) |
|           | \( 10^{-7} \) | \( 2.6 \times 10^{-5} \) | \( 6.2 \times 10^{-13} \) | \( 6.2 \times 10^{-13} \) |
|           | \( 10^{-8} \) | \( 2.6 \times 10^{-6} \) | \( 6.2 \times 10^{-16} \) | \( 6.2 \times 10^{-16} \) |
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