A VARIATIONAL APPROACH TO THREE-PHASE TRAVELING WAVES FOR A GRADIENT SYSTEM

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Abstract. In this paper, we use a variational approach to study traveling wave solutions of a gradient system in an infinite strip. As the even-symmetric potential of the system has three local minima, we prove the existence of a traveling wave that propagates from one phase to the other two phases, where these phases correspond to the three local minima of the potential. To control the asymptotic behavior of the wave at minus infinity, we successfully find a certain convexity condition on the potential, which guarantees the convergence of the wave to a constant state but not to a one-dimensional homoclinic solution or other equilibria. In addition, a non-trivial steady state in \( \mathbb{R}^2 \) is established by taking a limit of the traveling wave solutions in the strip as the width of the strip tends to infinity.

1. Introduction. This paper is concerned with traveling waves for the reaction-diffusion system of gradient type:

\[
\partial_t u = \Delta u - \nabla W(u),
\]

where \( u(x,t) : \Omega \times \mathbb{R} \to \mathbb{R}^m \) is a vector-valued function, \( \Omega \subset \mathbb{R}^n \) is a smooth cylindrical-like domain and \( W(u) : \mathbb{R}^m \to \mathbb{R} \) is a non-negative potential possessing several discrete minimum points \( a_1, a_2, \ldots, a_N \). In this paper, we focus on waves with a three-phase structure on a two dimensional infinite strip with \( m = 2 \). One important motivation to study gradient system (1) originates from the gradient theory of phase transitions proposed by Cahn and Hilliard [13] in 1950's. Their model, derived from the interfacial free energy, describes the mixtures of a noninteracting binary fluid. The variable \( u \) represents the density of the fluid or the thermodynamic state of the system near the phase transition points. In general, \( u \) can be a real-, complex- or vector-valued function. For instance, the Allen-Cahn equation [6] is a scalar case of (1). Another typical example, used to model superconductivity phenomena, is the Ginzburg-Landau equation with

\[
W(u) = (|u|^2 - 1)^2/4, \ u \in \mathbb{C}.
\]

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A good illustration of phase transitions comes from the study of a rescaling equation of (1), that is,

$$\partial_t u = \epsilon^2 \Delta u - \nabla W(u).$$

(2)

It is shown that under suitable conditions, for $\epsilon$ small enough, as $t \to \infty$ the domain of $u$ is divided into several regions on which $u$ is close to one of the phases. In addition, there exist internal transition layers in the midst of these phases. (See [16, 12, 20].) To study the steady state of (2), one usually investigates the corresponding variational energy functional:

$$\int_{\Omega} \left( \frac{\epsilon}{2} |\nabla u|^2 + \frac{1}{\epsilon} W(u) \right) \, dx$$

(3)

and its critical points. If $\Omega$ is bounded and $\epsilon$ is small, then the shape of the internal layers of the minimizer to (3) can be approximated by the minimal hyper-surfaces obtained via the theory of gamma convergence, which is developed by Gurtin et al. [24, 25] and Modica [32] for the scalar case and by Baldo [7] and Sternberg [39, 40] for the vector-valued case. When $\Omega$ is unbounded, a nontrivial steady state of (2) has an unbounded energy in general. Hence it is more difficult to apply a variational approach to obtain solutions with multiple phases in such a situation. However, when $W(u)$ is invariant under actions of some symmetry group, such a difficulty may be overcome. With the property of the symmetry group employed suitably, multi-phase saddle solutions or more general equivariant solutions on $\mathbb{R}^n$ were successfully constructed by Dang et al. [19], Bronsard et al. [11], Gui et al. [23], Alama et al. [1] and Alikakos et al. [3, 4].

In the study of the phase-transition dynamics described by (1), it is important to consider a traveling wave solution $(w, c)$ of (1) satisfying

$$u(x, t) = w(x - ct) \text{ in } \Omega,$$

where $\Omega$ is an infinite cylinder or $\mathbb{R}^n$, $k$ is a unit vector indicating the direction of wave propagation, and $c > 0$ is the wave speed. Without loss of generality, let $k$ be $(1, 0, \ldots, 0)$. Then, a traveling wave $(w, c)$ satisfies

$$\Delta w + c \partial_{x_1} w - \nabla W(w) = 0.$$

Typically a traveling wave $w$ connects two constant equilibria at $x_1 = \pm \infty$, namely,

$$\lim_{x_1 \to -\infty} w(x_1, \ldots, x_n) = a_2, \quad \lim_{x_1 \to \infty} w(x_1, \ldots, x_n) = a_1.$$

For the scalar case ($m = 1, n \geq 1$), the existence of planar traveling front solutions has been extensively studied for a very general potential. The stability of such waves were also successfully explored by Fife and McLeod [21], Levermore and Xin [29], and Matano et al. [31]. As $n > 1$, it is very interesting that there also exist non-planar traveling fronts such as V-shape fronts, pyramidal fronts and paraboloid-like fronts. (See [37, 27, 17, 33, 34, 41, 42, 43, 44].) For planar, almost-planar and non-planar transition fronts of a scalar equation, the reader can refer to [26] by Hamel.

For the vector-valued case ($m \geq 2$), unlike the scalar case, one can not apply the maximum principle as well as the method of super- and sub-solutions to (1) in general. However since (1) is a gradient system, it is natural to consider a variational approach to study traveling wave solutions. Such an approach was successfully developed to prove the existence of a traveling wave connecting two equilibria on $\mathbb{R}$ or a cylindrical domain by Heinze [28], Muratov and Novaga [36], Lucia, Muratov,
and Novaga [30], and Alikakos and Katzourakis [5]. Especially this variational method was also used to construct traveling waves between non-constant steady states. For example, in a pioneer paper by Bertsch, Muratov, and Primi [8], a traveling waves connecting two non-constant stationary heat flows was established. The trick to acquire such a traveling wave is subtracting the “sectional energy” of one of the steady states in the variational energy. Assume that the potential $W$ has three equal depth wells at the points $a$, $b_1$ and $b_2$ and that the wave in our consideration propagates along the $z$-axis on an infinite strip domain. The main goal in this paper is to use such a variational idea to find traveling waves connecting the constant equilibrium $a$ at $z = -\infty$ to an approximate heteroclinic solution $\phi_t$ between $b_1$ and $b_2$ at $z = \infty$. For this purpose, we use the techniques developed in [1] for the construction of approximate heteroclinic solutions and the methods developed in [8, 30, 35] for the construction of traveling waves. Although the variational approach can be widely used in many situations, one main difficulty often comes from how to determine the asymptotic behavior at the tail of the obtained traveling wave. In our case, for example, if there exists a low-energy one-dimensional homoclinic solution connecting the equilibrium $a$ to itself, it will be difficult to show that the wave tail tends to $a$ rather than to an approximation of the homoclinic solution when $z \to -\infty$. Hence we would like to point out that overcoming this difficulty is one of the main contributions of our study. More precisely, we find a convexity condition (H4) (see below), which requires the Hessian of $W$ to be positive in a region whose size is determined by the energy of $\phi_t$. With this condition, we successfully exclude the possibility that the constructed wave can approach to a profile other than $a$ as $z \to -\infty$.

More recently, variational methods for a gradient system with a term of non-local interaction, which includes the FitzHugh-Nagumo system as an example, were developed by two of the authors of this paper and their collaborator [14], and Chen et al. [15].

Now we state our main assumptions and results. In this paper, we focus on waves with a three-phase structure with $m = n = 2$ and $\Omega = D_\ell = \mathbb{R} \times (-\ell, \ell)$ for $\ell > 0$. We consider the equation (1) with Neumann boundary condition on $D_\ell$, that is,

$$\begin{align*}
\partial_t u(x, t) &= \Delta u(x, t) - \nabla W(u(x, t)), \quad x = (x, y) \in D_\ell, \; t \in \mathbb{R}; \\
\partial_y u(x, \pm \ell, t) &= 0,
\end{align*}$$

(4)

where $u(x, t) : D_\ell \times \mathbb{R} \to \mathbb{R}^2$ and $W : \mathbb{R}^2 \to \mathbb{R}$. We study the traveling wave solution of (4) by setting $z = x - ct$ and denote it by $w(z, y) = u(x, t)$. Then the wave profile $w(z, y) = (u(z, y), v(z, y))$ and the speed $c$ satisfy the following equation:

$$\begin{align*}
\partial_z^2 w + c \partial_z w + \partial_y^2 w - \nabla W(w) &= 0, \quad (z, y) \in D_\ell; \\
\partial_y w &= 0, \quad (z, y) \in \partial D_\ell.
\end{align*}$$

(5)

To solve (5), we assume the following four hypotheses (H1) to (H4).

**Hypothesis 1. (H1).** The potential $W : \mathbb{R}^2 \to \mathbb{R}$ is a $C^2$-function and satisfies the following conditions:

1. $W = 0$ is the only local and global minimum, attained on $\mathcal{S} = \{a = (-a, 0), b_1 = (1, b), b_2 = (1, -b)\}$ with $a, b > 0$ and $W > 0$ on $\mathbb{R}^2 \setminus \mathcal{S}$.
2. The Hessian matrix of $W$, denoted by $\text{Hess } W$, is strictly positive definite at $u = a, b_1$ and $b_2$. 
3. There exists an $R > 0$ such that
$$\nabla W(u) \cdot u \geq C_0 |u| \quad \forall |u| \geq R$$
for some $C_0 > 0$.

**Hypothesis 2. (H2)** Let $u = (u, v)$. We assume that $W$ is invariant under the reflection $\gamma u = \gamma(u, v) := (u, -v)$, i.e., $W(\gamma u) = W(u)$.

Also, we assume the uniqueness of the heteroclinic connection from $b_1$ to $b_2$.

**Hypothesis 3. (H3)** The energy functional
$$F_\infty[\psi] := \int_{-\infty}^{\infty} \left[ \frac{1}{2} |\psi'(y)|^2 + W(\psi(y)) \right] dy$$
has one unique global minimizer $\phi_\infty$ in the set
$$\tilde{A}_b := \{ \psi \in H^1_{loc}(\mathbb{R}; \mathbb{R}^2) \mid \psi(-y) = \gamma \psi(y), \ \psi(\infty) = b_1 \}.$$
In addition, assume $\phi_\infty(y) \neq a$ for all $y \in \mathbb{R}$.

Our purpose is to establish a traveling wave solution that connects the equilibrium $a$ at $z = -\infty$ and a profile $\phi_\ell$ close to $\phi_\infty$ at $z = \infty$, where $\phi_\ell \in H^1(-\ell, \ell)$ solves
$$\begin{cases}
\phi''_\ell(y) - \nabla W(\phi_\ell(y)) = 0, & y \in (-\ell, \ell);
\phi_\ell(-y) = \gamma \phi_\ell(y), & y \in [-\ell, \ell];
\phi'_\ell(\pm \ell) = 0.
\end{cases}$$

To state hypothesis (H4), we introduce a Riemannian metric $\text{dist}_W$ on $W$, defined by Sternberg [39]. Set
$$\text{dist}_W(p, q) := \inf_{\zeta \in C^1} \left\{ \int_0^1 \sqrt{2W(\zeta(t))} |\zeta'(t)| \ dt \mid \zeta(0) = p, \ \zeta(1) = q \right\},$$
which involves the profile of $W$. We can define the *locally convex balls* with centers $a, b_1$ and $b_2$ by
$$\mathfrak{B}_\rho(q) := \{ p \mid \text{dist}_W(q, p) < \rho \} \text{ for } q = a, b_1 \text{ or } b_2.$$ 
To ensure that $a$ is the unique stationary solution of (6), which achieves a lower energy than $\phi_\ell$, we make the following convexity assumption of $W$ on a geodesic ball centered at $a$. See Proposition 6.1 for the details.

**Hypothesis 4. (H4)** There exists a number $\rho_0 > \frac{1}{2} F_\infty[\phi_\infty]$ such that the Hessian matrix of $W$ is strict positive definite in $\mathfrak{B}_{\rho_0}(a)$.

We remark that the condition (H4) is not difficult to fulfill if the distance between $b_1$ and $b_2$ is short enough so that $F_\infty[\phi_\infty]$ is decreasing in $|b_1 - b_2|$. See Section 8. We will also give an example of $W$ which fulfills (H1) to (H4) in section 8.

The following proposition, proved in Section 2, ensures the existence of $\phi_\ell$.

**Proposition 1.1.** Assume $W$ satisfies (H1), (H2) and (H3). Then there exist $\epsilon_0 > 0$ and $\ell_\epsilon > 0$ for all $\epsilon \in (0, \epsilon_0]$ such that if $\ell > \ell_\epsilon$, we have an approximation $\phi_\ell$ which solves (6) and satisfies $||\phi_\ell - \phi_\infty||_{H^1(-\ell, \ell)} < \epsilon$.

Now we state the main result which describes a wave propagation among the three phases, i.e., the global minimum points of $W$, in an infinite strip.
Main Theorem. Assume $W$ satisfies (H1), (H2), (H3) and (H4). Then for any $\ell > \ell_0$, which is determined in Proposition 1.1, the system (5) has a $C^{2,\alpha}$ solution $(w^*, c^*)$ for some $a \in (0, 1)$ such that $c^* > 0$,

$$\lim_{z \to -\infty} w^*(z, y) = a \quad \text{and} \quad \lim_{z \to \infty} w^*(z, y) = \phi_\ell(y)$$

uniformly in $y$.

Remark. The wave speed $c^*$ can be characterized by a variational description and is unique in some sense as stated in Proposition 4.2. However, this property does not exclude the possibility that (5) has other solutions with different wave speeds due to a different variational setting in solving the system.

By reflecting $w^*(z, y)$, obtained in Main Theorem, with respect to $y$-coordinate, we can construct a traveling wave solution in $\mathbb{R}^2$ as follows. For any $n \in \mathbb{Z}$ let

$$w_\ell(z, y) = \begin{cases} w^*(z, y - 4n\ell) & \text{for } y \in [4n\ell - \ell, 4n\ell + \ell]; \\ w^*(z, y + 4n\ell + 2\ell) & \text{for } y \in [4n\ell + \ell, 4n\ell + 3\ell]; \end{cases}$$

and

$$\phi_\ell(y) = \begin{cases} \phi(y - 4n\ell) & \text{for } y \in [4n\ell - \ell, 4n\ell + \ell]; \\ \phi(y + 4n\ell + 2\ell) & \text{for } y \in [4n\ell + \ell, 4n\ell + 3\ell]. \end{cases}$$

A reflection argument shows that $w_\ell(z, y) \in C^2(\mathbb{R}^2)$ and the following corollary holds.

Corollary 1.2. Under hypotheses (H1), (H2), (H3) and (H4), $(w_\ell, c^*)$ solves

$$\Delta w_\ell + c^* \partial_z w_\ell - \nabla W(w_\ell) = 0 \quad \text{in } \mathbb{R}^2,$$

$$\lim_{z \to -\infty} w_\ell(z, y) = a \quad \text{and} \quad \lim_{z \to \infty} w_\ell(z, y) = \phi_\ell(y).$$

As $\ell \to \infty$, the domain $D_\ell$ of the traveling wave $w^*$ tends to $\mathbb{R}^2$. By extracting a subsequence of these $w^*$ and showing $c^* \to 0$ as $\ell \to \infty$, we are able to establish the existence of a standing wave in $\mathbb{R}^2$.

Corollary 1.3. Under the hypotheses (H1), (H2), (H3) and (H4), there exists a non-constant $w_\infty(z, y) \in C^2(\mathbb{R}^2)$ satisfying $w_\infty \neq \phi_\infty$ and

$$\Delta w_\infty - \nabla W(w_\infty) = 0 \quad \text{in } \mathbb{R}^2.$$

Remark. We give a non-rigorous remark about why $c^* \to 0$ as $\ell \to \infty$. It seems reasonable to say that the difference between $F_\infty(\phi_\infty)$ and $F_\infty(a)$ pushes the wave to move while the “inertia” of the wave retarding the propagation is roughly proportional to the width of the domain. Since $F_\infty(\phi_\infty) - F_\infty(a)$ remains bounded but the “inertia” tends to $\infty$ as the domain width $\ell$ tends to $\infty$, we expect $c^* \to 0$ as $\ell \to \infty$.

To construct the traveling wave in Main Theorem, we use the variational approach, developed by Heinzl [28], Muratov and Novaga [35], and Lucia, Muratov, and Novaga [30] in which a weighted minimizing problem with a constraint is considered. For our purpose, we use a modified variational setting so that the asymptotic limit of the traveling wave at $z = \infty$ is $\phi_\ell$. We prove the existence of such a traveling wave by seeking a minimizer under a suitable constraint. Due to the exponential weight in the variational functional, besides the task of proving existence it becomes another difficulty to determine the asymptotic behavior of the traveling wave at $z = -\infty$. In general, the limit behavior at $z = -\infty$ can be complicated. To
overcome this difficulty, we successfully find the convex condition, (H4), to ensure
the asymptotic behavior of the wave tending to \( a \) at \( z = -\infty \). We believe that a
condition like (H4) is necessary. Without it, the traveling wave may tends to a
solution of (6) other than \( a \) at \( z = -\infty \).

This paper is organized as follows. In Section 2, we first establish the existence
of the one-dimensional phase transitions on a bounded interval (Proposition 1.1)
and investigate the properties of them. In Section 3 and Section 4 we study the
variational formulation for traveling waves and demonstrate some of its important
properties. In Section 5, we prove the existence of a minimizer for the variational
problem and discuss its regularity. In Section 6 and Section 7, Main Theorem and
Corollary 1.3 are proved. In addition, we give an example satisfying (H1) to (H4)
in Section 8.

2. The one-dimensional phase transition. Throughout the paper, we assume
the potential \( W \) satisfies the hypotheses (H1) to (H4). Notice that (6) is the
Euler-Lagrange equation corresponding to the variational problem

\[
\min_{\psi \in \mathcal{A}} \{ F_\ell[\psi] \},
\]

where

\[
F_\ell[\psi] : = \int_{-\ell}^\ell \left[ \frac{1}{2} |\psi'(y)|^2 + W(\psi(y)) \right] \, dy
\]

and

\[
\mathcal{A}_\ell = \{ \psi \in H^1((-\ell, \ell); \mathbb{R}^2) \mid \psi(-y) = \gamma \psi(y) \},
\]

\[
\mathcal{A}_\infty = \{ \psi \in H^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^2) \mid \psi(-y) = \gamma \psi(y) \}. \tag{12}
\]

In order to construct \( \phi_\ell \) in (6) more explicitly, we consider the quantity

\[
e_{b_1, a, b_2} = \inf \left\{ F_\infty[\psi] \mid \psi \in \mathcal{A}_\infty, \psi(\infty) = b_1 \text{ and } \psi(0) = a \right\},
\]

which represents the energy of the transition connecting \( b_1 \) and \( b_2 \) and passing
through \( a \). By (H3), we have

\[
e_{b_1, a, b_2} > F_\infty[\phi_\infty].
\]

Since \( a, b_1 \) and \( b_2 \) are the only three minima of \( W \), for \( 0 < \epsilon \ll 1 \), we can
uniquely define three disjoint simply connected sets \( B_{\epsilon,a}, B_{\epsilon,b_1}, \) and \( B_{\epsilon,b_2} \) such that
for \( q = a, b_1, b_2, B_{\epsilon,q} \) contains \( q \) and

\[
B_{\epsilon,a} \cup B_{\epsilon,b_1} \cup B_{\epsilon,b_2} = \{ u \in \mathbb{R}^2 \mid W(u) \leq \epsilon \}. \tag{13}
\]

We can deduce directly from (H1) to (H4) the following local properties for
these \( B_{\epsilon,q} \)'s. See also Figure 1.

Lemma 2.1. Let \( \rho_0 \) be the number defined in (H4). Choose \( \rho > 0 \) and \( \delta > 0 \) such
that

\[
\rho < \min \left\{ \frac{1}{4} \left( e_{b_1, a, b_2} - F_\infty[\phi_\infty] \right), \rho_0 - \frac{1}{2} F_\infty[\phi_\infty] \right\}, \tag{14}
\]

\[
\delta < \inf_{y \in \mathbb{R}} \{ |\phi_\infty(y) - a| \} \quad \text{and} \quad \{ p \mid |p - a| \leq \delta \} \subset 2B_\rho(a) \cap \{ p \mid |p| < R \}, \tag{15}
\]

where \( R \) is given in (H1). Then there exists an \( \epsilon_0 > 0 \) such that for all \( 0 < \epsilon \leq \epsilon_0 \),
the following conditions hold for \( q = a, b_1 \) and \( b_2 \):

(a) \( B_{\epsilon,q} \) is compact and convex,
(b) \( \text{Hess } W \) is positive definite on \( B_{\epsilon,q} \).
The union of gray regions is \( \{ W(u) \leq \epsilon \} \). The green curves are geodesic balls. And the red circle is \( \{ p \mid |p - a| = \delta \} \).

(e) \( B_{\epsilon, q} \subset B_{\rho} (q) \), the geodesic ball defined in (8).

Furthermore \( \epsilon_0 \) can be chosen small enough such that

(d) \( B_{\epsilon, a} \subset \{ p \mid |p - a| < \delta \} \).

We remark that condition (14) for \( \rho \) will be used in Lemma 2.4 and Proposition 6.1 and condition (15) for \( \delta \) will be used to truncate the potential \( W \) in (16).

Now we use a variational approach to construct \( \phi_\ell \) by following the ideas for a scalar equation in [2] by Alessio, Calamai and Montecchiari. To exclude the trivial solution \( \phi_\ell \equiv a \) of (6), we replace \( W \) by

\[
\tilde{W}(u) = \begin{cases} 
W(u), & \text{if } |u - a| \geq \delta; \\
\infty, & \text{if } |u - a| < \delta.
\end{cases}
\]  

(16)

For \( \ell \in \mathbb{R}^+ \cup \{ \infty \} \) we set

\[
\bar{F}_\ell[\psi] := \int_{-\ell}^{\ell} \left[ \frac{1}{2} |\psi'(y)|^2 + \tilde{W}(\psi(y)) \right] \, dy
\]

and

\[
e_\ell := \inf_{\psi \in \mathcal{A}_\ell} \bar{F}_\ell[\psi], \quad e_\infty := \inf_{\psi \in \mathcal{A}_\infty} \bar{F}_\infty[\psi].
\]

It is easy to see that \( e_\ell \) increases with \( \ell \) and \( e_\ell \leq e_\infty \) for all \( \ell > 0 \). Since \( \delta < \inf_{\psi \in \mathbb{R}} \{ |\phi_\infty - a| \} \), we have \( \bar{F}_\infty[\phi_\infty] = F_\infty[\phi_\infty] \) and then conclude that

\[
e_\infty = \inf_{\phi \in \mathcal{A}_\infty} \bar{F}_\infty \leq \bar{F}_\infty[\phi_\infty] < \infty.
\]

The following proposition shows the existence of a minimizer for \( \bar{F}_\ell \) and indicates that the endpoints of \( \phi_\ell \) can be arbitrarily close to the minima \( b_1 \) and \( b_2 \) if \( \ell \) is sufficiently large. This shows the minimum of \( \bar{F}_\infty \) in \( \mathcal{A}_\infty \) will also be attained in the function space \( \mathcal{A}_b \) (\( \mathcal{A}_b \) is defined in (H3)), i.e. \( e_\infty = \inf_{\mathcal{A}_b} \bar{F}_\infty \). Since \( \bar{F}_\infty[\psi] \geq F_\infty[\psi] \) for any \( \psi \in \mathcal{A}_b \),

\[
e_\infty = \inf_{\mathcal{A}_b} \bar{F}_\infty \geq \inf_{\mathcal{A}_b} F_\infty = F_\infty[\phi_\infty] = \bar{F}_\infty[\phi_\infty] \geq e_\infty.
\]

Hence we have \( e_\infty = F_\infty[\phi_\infty] \).
Proposition 2.2. Let $\epsilon_0$ be defined in Lemma 2.1 and let $R$ be the number be defined in (H1). Then for $0 < \epsilon \leq \epsilon_0$ and $\ell > \frac{e_{\infty}}{2\epsilon}$, there exists $\phi_\ell \in \hat{A}_\ell$ satisfying $\|\phi_\ell\|_{L^\infty(-\ell,\ell)} \leq R$, $e_\ell = \tilde{F}_\ell[\phi_\ell]$ and
\[
\phi_\ell(y) \in B_{\epsilon,\ell}, \quad \text{for all } y \in (e_{\infty}/2\epsilon, \ell].
\]
Moreover, we have
\[
\phi_\infty(y) \in B_{\epsilon,\ell}, \quad \text{for } y > e_{\infty}/2\epsilon. \tag{18}
\]
Proof. Let $\{\psi_k\}_{k \in \mathbb{N}}$ be a minimizing sequence of $\tilde{F}_\ell$. Since $\delta$ in (15) is chosen such that $\{p \mid |p - a| < \delta \} \subset \{|p| \leq R\}$, without loss of generality, we can choose $|\psi_k(y)| \leq R$ for all $y \in [-\ell, \ell]$. (See Proposition 3.3 in [30]). By the Banach-Alaoglu theorem and the Sobolev embedding theorem, up to a subsequence, $\psi_k$ has a weak limit $\phi_\ell \in \hat{A}_\ell$, $\psi_k \to \phi_\ell$ strongly in $L^2$ and pointwise a.e. on $[-\ell, \ell]$. Then the weak lower-semi-continuity of the semi-norm $\|\psi'\|^2_{L^2(-\ell,\ell)}$ and continuity of $W$ imply that
\[
\tilde{F}_\ell[\phi_\ell] \leq \liminf_{k \to \infty} \tilde{F}_\ell[\psi_k] = e_\ell.
\]
Hence $\phi_\ell \in \hat{A}_\ell$ is a minimizer of $\tilde{F}_\ell$. Next, we prove (17). Note that for any $Y > e_{\infty}/2\epsilon$ there exists $\tau \in (-Y, Y)$ such that $\phi_\ell(\tau) \in B_{\epsilon, \ell} \cup B_{\epsilon, -\ell}$. Indeed, if $\phi_\ell(y) \notin B_{\epsilon, \ell} \cup B_{\epsilon, -\ell}$ for all $y \in (-Y, Y)$, then by Lemma 2.1(c) and the definition of $W$, we have
\[
W(\phi_\ell(y)) \geq \epsilon
\]
and
\[
e_\ell = \tilde{F}_\ell[\phi_\ell] \geq \int_{-Y}^{Y} W(\phi_\ell(y)) \, dy \geq 2Y\epsilon > e_{\infty},
\]
which contradicts to $e_\ell \leq e_{\infty}$. Now without loss of generality, we may assume that $\phi_\ell(\tau) \in B_{\epsilon, \ell}$ for $\tau \in (0, Y)$ by considering $\gamma \phi_\ell$ instead, if necessary. We may also assume $\phi_\ell(y) \in \{(u, v) \in \mathbb{R}^2 \mid u \geq 0\}$ for all $y \in [0, \ell]$ by replacing $\phi_\ell$ by
\[
\psi(y) = \begin{cases} \phi_\ell(y) & \text{if } y \in [0, \ell] \text{ satisfies } \phi_\ell(y) \in \{(u, v) \in \mathbb{R}^2 \mid u \geq 0\}; \\
\gamma \phi_\ell(y) & \text{if } y \in [0, \ell] \text{ satisfies } \phi_\ell(y) \in \{(u, v) \in \mathbb{R}^2 \mid u < 0\}; \end{cases}
\]
since $\tilde{F}_\ell[\psi] = \tilde{F}_\ell[\phi_\ell]$. Define $\Pi : \mathbb{R}^2 \to \mathbb{R}^2$ with $\Pi u$ being the closest point to $u$ in $B_{\epsilon, \ell}$. (See also Section 2 in [30].) To complete the proof of (17), we claim that $\phi_\ell(y) \in B_{\epsilon, \ell}$ for all $y \in (\tau, \ell]$. If there is $y_0 \in (\tau, \ell]$ such that $\phi_\ell(y_0) \notin B_{\epsilon, \ell}$, then
\[
\int_{\tau}^{\ell} \bar{W}(\phi_\ell(y)) \, dy > \int_{\tau}^{\ell} \bar{W}(\Pi \phi_\ell(y)) \, dy,
\]
due to Lemma 2.1(b). Let $\psi^* \in \hat{A}_\ell$ be
\[
\psi^*(y) := \begin{cases} \phi_\ell(y) & \text{if } y \in [0, \tau]; \\
\Pi \phi_\ell(y) & \text{if } y \in (\tau, \ell]. \end{cases}
\]
Then $\tilde{F}_\ell[\psi^*] < \tilde{F}_\ell[\phi_\ell]$, which leads to a contradiction. Finally, (18) follows from an argument similar to proof of (17). \hfill \Box

Before discussing the relation of $\tilde{F}_\ell$ and $F_\ell$, we first show the following result.

Lemma 2.3.
\[
e_{b_1, a, b_2} = 2 \text{dist}_W(a, b_1). \tag{19}
\]
From the definition of $\text{dist}_W$ (see (7)) in the introduction, it may not be so easy to assert the regularity of a path $\zeta$ which realizes the distance $\text{dist}_W(a, b_1)$. To overcome this difficulty, we verify (19) through a series of approximation of $\text{dist}_W(a, b_1)$ instead. See (20) below.

**Proof of Lemma 2.3.** We first claim $c_{b_1, a, b_2} \geq 2 \text{dist}_W(a, b_1)$. Given $\psi \in \hat{A}_b$, with $\psi(0) = a$, for all small $\tau > 0$ we can choose $y^* > 0$ and $b^* \in \mathcal{B}_\sigma(b_1)$ (the geodesic ball with radius $\tau$ as defined in (8)) such that $\psi(y^*) = b^*$. Therefore,

$$F_\infty[\psi] = 2 \int_0^\infty \left[ \frac{1}{2}|\psi'(y)|^2 + W(\psi(y)) \right] \, dy$$

$$\geq 2 \int_0^{y^*} \sqrt{2W(\psi(y))}|\psi'(y)| \, dy$$

$$\geq 2 \, \text{dist}_W(\psi(0), \psi(y^*)) \geq 2 \, \text{dist}_W(a, b_1) - 2\tau.$$

As $\tau \to 0$, we have $F_\infty[\psi] \geq 2 \text{dist}_W(a, b_1)$ for arbitrary $\psi \in \hat{A}_b$ with $\psi(0) = a$. Thus, $c_{b_1, a, b_2} \geq 2 \text{dist}_W(a, b_1)$.

Conversely, let $\sigma > 0$ and consider the geodesic balls $\mathcal{B}_\sigma(a)$, $\mathcal{B}_\sigma(b_1)$. For $\sigma$ small enough, $\mathcal{B}_\sigma(a)$ and $\mathcal{B}_\sigma(b_1)$ are disjoint convex compact set. We consider this infimum

$$\text{dist}_W(\mathcal{B}_\sigma(a), \mathcal{B}_\sigma(b_1)) := \inf \left\{ \text{dist}_W(p, q) \mid p \in \overline{\mathcal{B}_\sigma(a)}, \ q \in \overline{\mathcal{B}_\sigma(b_1)} \right\}.$$

By compactness, there exist $a_\sigma \in \partial(\mathcal{B}_\sigma(a))$ and $b_\sigma \in \partial(\mathcal{B}_\sigma(b_1))$ such that $\text{dist}_W(a_\sigma, b_\sigma) = \text{dist}_W(\mathcal{B}_\sigma(a), \mathcal{B}_\sigma(b_1))$. The path that realizes $\text{dist}_W(a_\sigma, b_\sigma)$ can be obtained by solving the minimization problem

$$\inf \left\{ \int_0^1 \sqrt{2W(\eta(s))} |\eta'(s)| \, ds \mid \eta \in C^1([0, 1]; \mathbb{R}^2), \ \eta(0) = a_\sigma, \ \eta(1) = b_\sigma \right\}.$$

However, instead of a minimal path, there exists some $\eta_\sigma \in C^1([0, 1]; \mathbb{R}^2)$ satisfying $\eta_\sigma(0) = a_\sigma$, $\eta_\sigma(1) = b_\sigma$, $\eta_\sigma(s) \neq 0$ for all $s \in [0, 1]$, and

$$\text{dist}_W(a_\sigma, b_\sigma) \leq \int_0^1 \sqrt{2W(\eta_\sigma(s))} |\eta_\sigma'(s)| \, ds < \text{dist}_W(a_\sigma, b_\sigma) + \sigma.$$

We may assume that $\eta_\sigma \notin \mathcal{B}_\sigma(a) \cup \mathcal{B}_\sigma(b_1)$ for $0 < s < 1$. Otherwise we let $s_1 = \sup \{ s \mid \eta(s) \in \partial \mathcal{B}_\sigma(a) \}$, $s_2 = \inf \{ s \mid \eta(s) \in \partial \mathcal{B}_\sigma(b_1) \}$, $p = \eta(s_1)$ and $q = \eta(s_2)$, and consider $\eta$ restricted on $[s_1, s_2]$ instead. (Note that we do not need that $(p, q)$ coincides with $(a_\sigma, b_\sigma)$ in the following argument.) Due to $\eta_\sigma \neq 0$, we can solve the equation for a new parametrization $\beta_\sigma$ on an interval $[1, T]$:

$$\begin{cases}
\beta_\sigma'(y) = \frac{\sqrt{2W(\eta_\sigma(\beta_\sigma(y)))}}{|\eta_\sigma'(\beta_\sigma(y))|}, & y \in (1, T) \\
\beta_\sigma(1) = 0, \quad \text{and} \quad \beta_\sigma(T) = 1.
\end{cases} \quad (20)$$

Since $\eta_\sigma \notin \mathcal{B}_\sigma(a) \cup \mathcal{B}_\sigma(b_1)$ for $0 < s < 1$ and $\eta_\sigma'$ is bounded, we can see that $\beta_\sigma$ is strictly increasing and there exists some $T$ such that $\beta_\sigma(T) = 1$.

Then we define a new function by $\zeta_\sigma(\cdot) := \eta_\sigma(\beta_\sigma(\cdot)), \ y \in [1, T]$. From (20), it is easy to see the *equi-partition* of the energy: for $y \in (1, T)$,

$$\frac{1}{2} |\zeta_\sigma'(y)|^2 = \frac{1}{2} |\beta_\sigma'(\beta_\sigma(y))\beta_\sigma'(y)|^2 = W(\eta_\sigma(\beta_\sigma(y))) = W(\zeta_\sigma(y)). \quad (21)$$
Hence the equality in the inequality of arithmetic and geometrical means holds, that is, for \( y \in (1, T) \),
\[
\frac{1}{2} |\zeta_\sigma'(y)|^2 + W(\zeta_\sigma(y)) = \sqrt{2W(\eta_\sigma(\beta_\sigma(y)))} |\eta_\sigma'(\beta_\sigma(y))| |\beta_\sigma'(y)|.
\]

Now we extend \( \zeta_\sigma \) to the domain \([0, 1) \cup (T, \infty)\) by
\[
\zeta_\sigma(y) = \begin{cases} 
1 - y & y \in [0, 1], \\
T + 1 - y & y \in [T, T + 1], \\
\eta_\sigma & y \in [T + 1, \infty).
\end{cases}
\]

And furthermore we define \( \eta_\sigma(y) = \gamma \eta_\sigma(-y) \) for \( y \in (-\infty, 0) \). Consequently, by the relation (21) and using the Taylor expansion of \( W \) near \( a \) and \( b_1 \), we obtain
\[
2 \text{dist}_W(a, b_\sigma) + 2\sigma > 2 \int_0^1 \sqrt{2W(\eta_\sigma(\tau))} |\eta_\sigma'(\tau)| \, d\tau
\]
\[
= 2 \int_0^T \sqrt{2W(\zeta_\sigma(y))} |\zeta_\sigma'(y)| \, dy = 2 \int_0^T \left[ \frac{1}{2} |\zeta_\sigma'(y)|^2 + W(\zeta_\sigma(y)) \right] \, dy
\]
\[
= 2 \int_0^\infty \left[ \frac{1}{2} |\zeta_\sigma'(y)|^2 + W(\zeta_\sigma(y)) \right] \, dy - 2 \int_0^1 \left[ \frac{1}{2} |a - a_\sigma|^2 + W(a + y(a_\sigma - a)) \right] \, dy
\]
\[
- 2 \int_T^{T + 1} \left[ \frac{1}{2} |b_1 - b_\sigma|^2 + W(b_1 + (T + 1 - y)(b_\sigma - b_1)) \right] \, dy
\]
\[
= \int_{-\infty}^\infty \left[ \frac{1}{2} |\zeta_\sigma'(y)|^2 + W(\zeta_\sigma(y)) \right] \, dy - O(|a - a_\sigma|^2 + |b_1 - b_\sigma|^2).
\]

Now for each \( \sigma > 0 \) small enough, we have that \( F_\infty(\zeta_\sigma) \) and \( \|\zeta_\sigma'\|_{L^\infty} \) are uniformly bounded from (21). Therefore there exists a subsequence \( \zeta_{\sigma_k} \) with \( \sigma_k \to 0 \) and a function \( \zeta_0 \in \tilde{A}_b \) such that \( \zeta_{\sigma_k} - \zeta_0 \to 0 \) weakly in \( H^1(R; \mathbb{R}^2) \) and \( \zeta_{\sigma_k} - \zeta_0 \to 0 \) strongly in \( L^\infty(R; \mathbb{R}^2) \) with \( \zeta(0) = \zeta_{\sigma_k}(0) = a \). Then by the weak lower-semi-continuity of semi-norms and the continuity and non-negativity of \( W \) together with Fatou’s lemma, we have
\[
2 \text{dist}_W(a, b_1) \geq \lim_{k \to \infty} [2 \text{dist}_W(a_{\sigma_k}, b_{\sigma_k}) - 4\sigma_k]
\]
\[
\geq \lim_{k \to \infty} \left[ \int_{-\infty}^\infty \left( \frac{1}{2} |\zeta_{\sigma_k}'(y)|^2 + W(\zeta_{\sigma_k}(y)) \right) \, dy - O(|a - a_{\sigma_k}|^2 + |b_1 - b_{\sigma_k}|^2) - 6\sigma_k \right]
\]
\[
\geq \int_{-\infty}^\infty \liminf_{k \to \infty} \left[ \frac{1}{2} |\zeta_{\sigma_k}'(y)|^2 + W(\zeta_{\sigma_k}(y)) \right] \, dy
\]
\[
= F_\infty(\zeta_0) \geq \epsilon_{b_1, a, b_2}.
\]

\[\square\]

**Lemma 2.4.** For any \( \psi \in \tilde{A}_\infty \), if there exists \( y_1, y_2 \in (0, \ell) \) such that
\[
\psi(y_1) \in \mathcal{B}_\rho(a) \quad \text{and} \quad \psi(y_2) \in \mathcal{B}_\rho(b_1),
\]
then \( F_\ell(\psi) > c_\infty \geq c_\ell \). Moreover, for \( \ell > \frac{c_\ell}{\epsilon_\ell} \), we have
\[
|\phi_\ell(y) - a| > \delta, \quad \forall \ y \in [-\ell, \ell] \quad \text{and} \quad c_\ell = F_\ell(\phi_\ell).
\]

**Proof.** The assumption of \( \psi \) and Lemma 2.3 lead to
As a sequence of the following argument can also be used to show that the limit of any sub-

Proof. By Proposition 2.2, for 0

\[ F_\ell[\psi] = 2 \int_0^\ell \left[ \frac{1}{2} |\psi'(y)|^2 + W(\psi(y)) \right] dy \]

\[ \geq 2 \int_0^\ell \sqrt{2W(\psi(y))} |\psi'(y)| dy \]

\[ \geq 2 \text{dist}_W(\psi(y_1), \psi(y_2)) \geq 2 \text{dist}_W(a, b_1) - 4\rho = e_{b_1, a, b_2} - 4\rho. \]  

It follows from (14) and (22) that

\[ F_\ell[\psi] \geq e_{b_1, a, b_2} - 4\rho > F_\infty[\phi_\infty] = e_\infty \geq e_\ell. \]  

(23)

Since for \( y \in (e_\infty/2\epsilon, \ell] \) we have \( \phi_\ell(y) \in B_{\epsilon, b_1} \subset \mathcal{B}_y(b_1) \), we can see that

\[ |\phi_\ell(y) - a| > \delta, \quad \forall y \in [-\ell, \ell] \quad \text{and} \quad e_\ell = \tilde{F}_\ell[\phi_\ell] = F_\ell[\phi_\ell]; \]

Otherwise, similar arguments in (22) and (23) imply \( \tilde{F}_\ell[\phi_\ell] > e_\infty \geq e_\ell \), which is a contradiction. \( \square \)

Define an extension \( \tilde{\phi}_\ell \in \tilde{A}_\infty \) of \( \phi_\ell \) by

\[ \tilde{\phi}_\ell(y) := \begin{cases} 
\phi_\ell(y) & \text{for } y \in [0, \ell] \\
(y - \ell)b_1 + (\ell + 1 - y)\phi_\ell(\ell) & \text{for } y \in (\ell, \ell + 1) \\
b_1 & \text{for } y \in [\ell + 1, \infty). 
\end{cases} \]

It is easy to see \( |\tilde{\phi}_\ell(y) - a| \geq \delta \) for all \( y \in \mathbb{R} \) if \( \ell \) is large. Now we investigate the convergence of \( \tilde{\phi}_\ell \) on \( \mathbb{R} \).

Lemma 2.5. As \( \ell \to \infty \), we have \( F_\infty[\tilde{\phi}_\ell] \to e_\infty \).

Proof. By Proposition 2.2, for \( 0 < \epsilon \leq \epsilon_0 \) and \( \ell > e_\infty/2\epsilon \), we have \( \phi_\ell(\ell) \in B_{\epsilon, b_1} \). For \( y \in (\ell, \ell + 1) \), the Taylor expansion of \( W \) implies

\[ W(\tilde{\phi}_\ell(y)) = W(\tilde{\phi}_\ell(y)) - W(b_1) \leq \frac{M}{2} |\tilde{\phi}_\ell(y) - b_1|^2 \]

where \( M \) is an upper bound for all eigenvalues of \( \text{Hess} \) \( W \) on \( B_{\epsilon_0, b_1} \). Therefore, we have for large \( \ell \)

\[ e_\infty \leq \tilde{F}_\infty[\tilde{\phi}_\ell] = e_\ell + 2 \int_\ell^{\ell + 1} \left[ \frac{1}{2} |\tilde{\phi}_\ell(y)|^2 + W(\tilde{\phi}_\ell(y)) \right] dy \]

\[ \leq e_\infty + (M + 1)|\phi_\ell(\ell) - b_1|^2. \]  

(24)

If \( \epsilon \to 0 \), then we obtain \( \phi_\ell(\ell) \to b_1 \) as \( \ell \to \infty \). Consequently, \( F_\infty[\tilde{\phi}_\ell] \to e_\infty \) as \( \ell \to \infty \). \( \square \)

Observing (24), we can also see that \( e_\ell \to e_\infty \) as \( \ell \to \infty \). As follows, we will show

\[ \|\phi_\ell - \phi_\infty\|_{H^1(-\ell, \ell)} \leq \|\tilde{\phi}_\ell - \phi_\infty\|_{H^1(\mathbb{R})} \to 0 \quad \text{as } \ell \to \infty, \]

which is the approximation in Proposition 1.1.

Lemma 2.6. As \( \ell \to \infty \), we have \( \|\tilde{\phi}_\ell - \phi_\infty\|_{H^1(\mathbb{R})} \to 0 \).

Proof. It is sufficient to show that there exists a subsequence \( \tilde{\phi}_{\ell_k} \) of \( \tilde{\phi}_\ell \) satisfying

\[ \|\tilde{\phi}_{\ell_k} - \phi_\infty\|_{H^1(\mathbb{R})} \to 0 \quad \text{as } k \to \infty \]

since the following argument can also be used to show that the limit of any subsequence of \( \tilde{\phi}_\ell \) exists and has to be \( \phi_\infty \). Proposition 2.2 and Lemma 2.5 imply that \( \|\tilde{\phi}_\ell\|_{L^\infty(\mathbb{R})} \leq R \) and \( \|\tilde{\phi}_\ell\|_{L^2(\mathbb{R})} \leq M \) for some \( M > 0 \). Therefore, there exists
a subsequence $\overline{\phi}_{\ell_k} \to \psi_\infty$ weakly in $H^1_{\text{loc}}(\mathbb{R})$, strongly in $L^2_{\text{loc}}(\mathbb{R})$ and pointwisely a.e. on $\mathbb{R}$ for some $\psi_\infty \in \widehat{A}_\infty$. Also, $\overline{\phi}_{\ell_k} \to \psi'_\infty$ weakly in $L^2(\mathbb{R})$. From the weak lower-semicontinuity of $F_\infty$, we derive
\[ e_\infty \leq F_\infty[\psi_\infty] \leq \liminf_{k \to \infty} F_\infty[\overline{\phi}_{\ell_k}] = e_\infty. \]

In view of Proposition 2.2, we can show $\psi_\infty(y) \to b_1$ as $y \to \infty$. So, the uniqueness of global minimizer of $F_\infty$ in (H3) yields $\psi_\infty \equiv \phi_\infty$.

To show the convergence of $\phi_{\ell_k}$ in $H^1$-norm, we first show that $\overline{\phi}_{\ell_k} \to \phi'_\infty$ in $L^2(\mathbb{R})$. It is sufficient to show that $\|\overline{\phi}_{\ell_k}\|_{L^2(\mathbb{R})} \to \|\phi'_\infty\|_{L^2(\mathbb{R})}$ due to the weak convergence of $\overline{\phi}_{\ell_k}$ in $L^2(\mathbb{R})$. Indeed, the weakly lower semicontinuity of the $L^2$-norm implies that
\[ \|\phi'_\infty\|_{L^2(\mathbb{R})} \leq \liminf_{k \to \infty} \|\overline{\phi}_{\ell_k}\|_{L^2(\mathbb{R})}. \]

On the other hand, from Lemma 2.5 and Fatou’s lemma, we have
\[ \limsup_{k \to \infty} \frac{1}{2\|\overline{\phi}_{\ell_k}\|_{L^2(\mathbb{R})}} = \limsup_{k \to \infty} \left( F_\infty[\overline{\phi}_{\ell_k}] - \int_\mathbb{R} W(\overline{\phi}_{\ell_k}(y)) \, dy \right) = e_\infty - \liminf_{k \to \infty} \int_\mathbb{R} W(\overline{\phi}_{\ell_k}(y)) \, dy \leq F_\infty[\phi_\infty] - \int_\mathbb{R} W(\phi_\infty(y)) \, dy = \frac{1}{2} \|\phi'_\infty\|_{L^2(\mathbb{R})}. \]

Therefore, we prove $\overline{\phi}_{\ell_k} \to \phi'_\infty$ in $L^2(\mathbb{R})$. Next, we prove that $\|\overline{\phi}_{\ell_k} - \phi_\infty\|_{L^2(\mathbb{R})} \to 0$. Fix some $\ell > e_\infty/2\epsilon$. It is easy to see that for $\ell > \ell$,
\[ \|\phi_\ell - \phi_\infty\|_{L^2(\mathbb{R})}^2 = \int_{-\ell}^\ell |\phi_\ell(y) - \phi_\infty(y)|^2 \, dy + 2 \int_{\ell}^{\infty} |\phi_\ell(y) - \phi_\infty(y)|^2 \, dy \leq \|\phi_\ell - \phi_\infty\|_{L^2(-\ell, \ell)}^2 + 4 \int_\ell^{\infty} |\phi_\ell(y) - b_1|^2 + |b_1 - \phi_\infty(y)|^2 \, dy. \]

By Lemma 2.1 and Proposition 2.2, we have for $y > e_\infty/2\epsilon$, $\overline{\phi}_\ell(y), \phi_\infty(y) \in B_{\ell_0} b_1$, on which Hess $W$ is positive definite. Using the Taylor expansion of $W$, for $y > e_\infty/2\epsilon$ we obtain
\[ W(\overline{\phi}_\ell(y)) = W(\overline{\phi}_\ell(y)) - W(b_1) \geq \frac{m}{2} |\overline{\phi}_\ell(y) - b_1|^2 \]
\[ W(\phi_\infty(y)) = W(\phi_\infty(y)) - W(b_1) \geq \frac{m}{2} |\phi_\infty(y) - b_1|^2 \]
for some fixed $m > 0$. Hence
\[ \|\overline{\phi}_\ell - \phi_\infty\|_{L^2(\mathbb{R})}^2 \leq \|\overline{\phi}_\ell - \phi_\infty\|_{L^2(-\ell, \ell)}^2 + \frac{8}{m} \int_\ell^{\infty} \left[ W(\overline{\phi}_\ell(y)) + W(\phi_\infty(y)) \right] \, dy. \]

A simple calculation gives
\[ \int_\ell^{\infty} W(\overline{\phi}_\ell(y)) \, dy = \frac{1}{2} \epsilon_\ell - \frac{1}{4} \|\phi'_\ell\|_{L^2(-\ell, \ell)}^2 - \frac{1}{2} \int_{-\ell}^{\ell} W(\phi_\ell(y)) \, dy + \int_{\ell}^{\ell+1} W(\overline{\phi}_\ell(y)) \, dy. \]

Notice that
\[ \|\phi'_\ell\|_{L^2(-\ell, \ell)}^2 = \|\overline{\phi}_\ell\|_{L^2(\mathbb{R})}^2 - 2|\phi_\ell(\ell) - b_1|^2 \to \|\phi'_\infty\|_{L^2(\mathbb{R})} as \ell \to \infty \]
and
\[ \int_{\ell}^{\ell+1} W(\overline{\phi}_\ell(y)) \, dy \leq \epsilon. \]
Therefore, using the convergence of \( \epsilon \ell \) and Fatou’s lemma, we have
\[ \limsup_{\ell \to \infty} \int_{\ell}^{\infty} W(\overline{\phi}_\ell(y)) \, dy \leq \frac{1}{2} \epsilon \ell \to \infty - \frac{1}{4} \| \phi_\infty' \|^2_{L^2(\mathbb{R})} - \frac{1}{2} \int_{-\ell}^{\ell} W(\phi_\infty(y)) \, dy + \epsilon \\
= \int_{-\ell}^{\ell} W(\phi_\infty(y)) \, dy + \epsilon. \]
Consequently, from (25) and by \( \overline{\phi}_{\ell_k} \to \phi_\infty \) strongly in \( L^2_{\text{loc}}(\mathbb{R}) \), we obtain
\[ \limsup_{k \to \infty} \| \overline{\phi}_{\ell_k} - \phi_\infty \|^2_{L^2(\mathbb{R})} \leq \limsup_{k \to \infty} \frac{8}{m} \int_{-\ell}^{\ell} W(\overline{\phi}_{\ell_k}(y)) + W(\phi_\infty(y)) \, dy \\
\leq \frac{8}{m} \int_{-\ell}^{\ell} 2W(\phi_\infty(y)) \, dy + \frac{8}{m} \epsilon. \]
By taking \( \epsilon \to 0 \) and \( \ell \to \infty \), we can conclude that \( \| \overline{\phi}_{\ell_k} - \phi_\infty \|^2_{L^2(\mathbb{R})} \to 0 \) as \( k \to \infty \).
Consequently, \( \overline{\phi}_{\ell_k} - \phi_\infty \to 0 \) in \( H^1(\mathbb{R}) \) as \( k \to \infty \).

At the end of this section, we establish the local minimality of \( \phi_\ell \) to \( F_\ell \) on \( \tilde{\mathcal{A}}_\ell \), which is used in the later section, and Proposition 1.1.

**Proposition 2.7** (local minimality of \( \phi_\ell \)). Let \( \epsilon_0 \) be defined in Lemma 2.1. For \( 0 < \epsilon \leq \epsilon_0 \) and \( \ell > \frac{\delta}{2\epsilon} \), there exists \( \alpha_0 = \alpha_0(\ell, \epsilon) > 0 \) such that if \( \psi \in \tilde{\mathcal{A}}_\ell \) with \( \| \psi - \phi_\ell \|^2_{L^2(-\ell, \ell)} \leq \alpha_0 \), then
\[ F_\ell[\psi] \geq F_\ell[\phi_\ell]. \]

**Proof.** Let \( \psi \in \tilde{\mathcal{A}}_\ell \) satisfy \( \| \psi - \phi_\ell \|^2_{L^2(-\ell, \ell)} \leq \alpha_0 \) with \( \alpha_0 \) to be determined later. Our strategy is to consider the regions where \( \psi \) is close to one of the minimum points,
\[ \Omega_1 := \{ y \in [-\ell, \ell] \mid \| \psi(y) - a \| \leq \delta \} \]
and
\[ \Omega_2 := \{ y \in [-\ell, \ell] \mid \| \psi(y) \in B_{\epsilon, b_1} \cup B_{\epsilon, b_2} \}. \]
We divide the argument into three cases.

**Case 1:** \( \Omega_1 \neq \emptyset \) and \( \Omega_2 \neq \emptyset \). In the situation, there exist \( y_1 \) and \( y_2 \) such that \( \psi(y_1) \) satisfies \( \| \psi(y_1) - a \| \leq \delta \) and \( \psi(y_2) \in B_{\epsilon, b_1} \cup B_{\epsilon, b_2} \). Without loss of generality, we assume \( \psi(y_2) \in B_{\epsilon, b_1} \). Then by Lemma 2.1, and Lemma 2.4, we have \( \psi(y_1) \in \mathfrak{B}_p(a), \phi(y_2) \in \mathfrak{B}_p(b_1) \) and \( F_\ell[\phi] > e_\ell = F_\ell[\phi_\ell] \).

**Case 2:** \( \Omega_1 = \emptyset \). In this case, it is easy to see that \( F_\ell[\psi] = F_\ell[\psi] \geq e_\ell = F_\ell[\phi_\ell] = F_\ell[\phi_\ell] \).

**Case 3:** \( \Omega_2 = \emptyset \). From Lemma 2.1(d), we have \( W(\psi(y)) \geq \epsilon \) for \( y \in [-\ell, \ell] \setminus \Omega_1 \) and
\[ F_\ell[\psi] \geq \int_{[-\ell, \ell] \setminus \Omega_1} W(\psi(y)) \, dy \geq (2\ell - |\Omega_1|)\epsilon. \]

Define
\[ \eta := \inf_{\ell \geq \frac{\delta}{2\epsilon}} \inf_{y \in [-\ell, \ell]} \{ |\phi_\ell(y) - a| \} \]
and
\[ \alpha_0 := (\eta - \delta) \sqrt{\frac{2}{\ell - \frac{\delta}{2\epsilon}}} \].
To ensure $\alpha_0 > 0$, we argue $\eta > \delta$ by a contradiction. Suppose there exists $\ell_k \to \infty$ and $y_k \in (0, \ell_k)$ such that $|\phi_{\ell_k}(y_k) - a| \to \delta$. From (15), we have $\phi_{\ell_k}(y_k) \in B_\rho(a)$ for $k$ large enough. In addition, for $y \in (\frac{x}{2\ell_k}, \ell_k]$ we have $y \in B_{\epsilon, b_1} \subset B_\rho(b_1)$. By Lemma 2.4, these properties imply

$$F_{\ell_k} [\phi_{\ell_k}] > e_\infty,$$

which is a contradiction. Consequently, $\eta > \delta$ and $\alpha_0 > 0$.

For any $y \in \Omega_1$, by the assumptions of $\psi$, we have $|\psi(y) - \phi_\ell(y)| \geq \eta - \delta$ and

$$|\Omega_1| \leq (\eta - \delta)^{-2} \|\psi - \phi_\ell\|_{L^2(-\ell, \ell)}^2 \leq 2\ell - \frac{e_\infty}{\epsilon}, \tag{29}$$

Combining (26) and (29) implies

$$F_{\ell}[\psi] \geq e_\infty \geq F_{\ell} [\phi_\ell].$$

Proof of Proposition 1.1. By Proposition 2.7, $\phi_\ell$ is a local minimizer of $F_{\ell}$ on $\hat{A}_\ell$. A standard argument in calculus of variation and the variational principle of symmetric criticality [38] imply that $\phi_\ell$ is a classical solution. Consequently, $\phi_\ell$ solves the system (6). By Lemma 2.6, there exists $\ell \geq \frac{x}{2\ell_k}$ such that for $\ell > \ell_c$

$$\|\phi_\ell - \phi_\infty\|_{H^1(-\ell, \ell)} \lessgtr \|\phi_\ell - \phi_\infty\|_{H^1(\mathbb{R})} < \epsilon.$$  

This finishes the proof of Proposition 1.1.  

3. Formulations of the variational structure. In this section we begin to study solutions of (5) in $D_\ell := \mathbb{R} \times (-\ell, \ell)$. Since we consider the traveling wave solution which tends to $\phi_\ell$ at $z = \infty$, the potential energy $\int_{-\ell}^{\ell} W(u(z, y)) \, dy$ of the solution at $z = -\infty$ is smaller than the one at $z = \infty$. (See Corollary 6.7 in [30].) Hence it is natural to focus on the solutions with wave speed $c > 0$ in the following. Throughout the following Section 3 - 6, we fix some $\ell > \ell_c$, determined in Proposition 1.1, and suppress the subscript of $D_\ell$ and $\phi_\ell$ to simplify the notation. Instead of $z = x - ct$ in (5), we make the change of variable $z = c(x - ct)$, introduced by Heinze in [28], such that our variation functional will be continuous in $c$ also. (See the form of the energy defined below.) Therefore the traveling wave $w$ and the speed $c$ of (4) satisfy

$$\begin{cases}
    c^2 (\partial_z^2 w + \partial_\gamma w) + \partial_y^2 w - \nabla W(w) = 0, & (z, y) \in D; \\
    \partial_\gamma w = 0, & (z, y) \in \partial D, 
\end{cases} \tag{30}$$

which is the Euler-Lagrange equation associated with the functional

$$\int_D c^2 \left[ \frac{\partial_z w}{2} |\partial_z w|^2 + \frac{1}{2} |\partial_y w|^2 + W(w) \right] \, dz \, dy.$$

To assure the profile of $w$ to converge to $\phi$ at $z = \infty$, we modify the above functional as follows:

$$E_c[w] := \int_D c^2 \left[ \frac{\partial_z w}{2} |\partial_z w|^2 + \frac{1}{2} |\partial_y w|^2 - \frac{1}{2} |\partial_y \phi|^2 + W(w) - W(\phi) \right] \, dz \, dy. \tag{31}$$

The admissible function space is chosen as

$$\mathcal{A} = \{ w \mid w(z, y) - \phi(y) \in H, w(z, -y) = \gamma w(z, y), \|w\|_{L^\infty} < \infty \},$$

where $H$ is the weighted Sobolev space of functions from $D$ to $\mathbb{R}^2$, which is the completion of $C_c^\infty(\mathbb{R}^2, \mathbb{R}^2)$ restricted on the domain $D$ under the norms
\[
\|v\|_H^2 := \|v\|_{L_\infty}^2 + \|\partial_z v\|_{L_2}^2 + \|\partial_y v\|_{L_2}^2, \quad \|v\|_{L_\infty}^2 := \int_D e^z |v|^2 \, dz dy.
\]

We will show that \(E_c\) is well-defined. As a remark, \((30)\) is still the Euler-Lagrange equation associated with the functional \((31)\) even we have \(\phi\) in \((31)\). In this setting, we will show that a minimizer of \((31)\) on \(A\) converges to \(\phi\) as \(z \to \infty\).

**Lemma 3.1.** The energy functional \(E_c\) defined in \((31)\) is well-defined on \(A\).

**Proof.** Let \(v \in A\). For any \(\alpha, \beta \in \mathbb{R}\) with \(\alpha < \beta\), we consider the integral in \(E_c\) restricted on the rectangle \((\alpha, \beta) \times (-\ell, \ell)\)

\[
E_c[v, (\alpha, \beta)] := \int_{\alpha}^{\beta} \int_{-\ell}^{\ell} e^z \left[ \frac{c^2}{2} |\partial_z v|^2 + \frac{1}{2} |\partial_y v|^2 - \frac{1}{2} |\partial_y \phi|^2 + W(v) - W(\phi) \right] \, dy dz
\]

\[
= \int_{\alpha}^{\beta} \int_{-\ell}^{\ell} e^z \left[ \frac{c^2}{2} |\partial_z v|^2 + \frac{1}{2} |\partial_y (v - \phi)|^2 + \partial_y (v - \phi) \cdot \partial_y \phi + W(v) - W(\phi) \right] \, dy dz.
\]

We divide the integral into two parts as we denote above. Clearly, the first part \((I)\) is absolutely bounded by \(\frac{(c^2+1)}{2} \|v - \phi\|_H^2\) independent of \(\alpha\) and \(\beta\). For the second part \((II)\), by integration by parts in \(y\) and the equation \((6)\) satisfied by \(\phi\),

\[
= \int_{\alpha}^{\beta} \int_{-\ell}^{\ell} e^z \left[ -(v - \phi) \cdot \partial_y \phi + W(v) - W(\phi) \right] \, dy dz.
\]

By the Taylor expansion on the line segment between \(v(z, y)\) and \(\phi(y)\), there exists a \(\xi \in (0, 1)\) such that

\[
W(v) - W(\phi) - (v - \phi) \cdot \nabla W(\phi) = \frac{1}{2} (v - \phi)^t \left[ \text{Hess} W \left( \xi v + (1 - \xi) \phi \right) \right] (v - \phi),
\]

where the superscript \(tr\) means the transpose of the vector. Since \(v \in A\), \(\|v\|_{L_\infty}\) is bounded and then the eigenvalues of \(\text{Hess} W(\xi (z, y))\) are bounded by some large \(M_1\). Hence \(|(II)| \leq M_1 \|v - \phi\|_H^2\) which is independent of \(\alpha, \beta\). And we can conclude that \(E_c\) is well-defined on \(A\). \(\square\)

As we study traveling waves in \(D\), one difficulty is to deal with the translation effect along \(z\)-axis. For the energy functional \((31)\), we can easily infer the following property.

**Proposition 3.2.** Set \(w^{(\tau)}(z, y) = w(z - \tau, y)\). For \(w - \phi \in H\), we have \(w^{(\tau)} - \phi \in H\) with \(\|w^{(\tau)} - \phi\|_H^2 = e^{\tau} \|w - \phi\|_H^2\) and

\[
E_c[w^{(\tau)}] = e^{\tau} E_c[w].
\]

**Proof.** It is easy to see \(\|w^{(\tau)} - \phi\|_H^2 = e^{\tau} \|w - \phi\|_H^2\) and \(w^{(\tau)} - \phi \in H\). Similarly a simple calculation yields

\[
E_c[w^{(\tau)}] = \int_D e^{\tau} \left[ \frac{c^2}{2} |\partial_z w^{(\tau)}|^2 + \frac{1}{2} |\partial_y w^{(\tau)}|^2 - \frac{1}{2} |\partial_y \phi(y)|^2 + W(w^{(\tau)}) - W(\phi) \right] \, dz dy
\]

\[
= \int_D e^{\tau + \tau} \left[ \frac{c^2}{2} |\partial_z w|^2 + \frac{1}{2} |\partial_y w|^2 - \frac{1}{2} |\partial_y \phi(y)|^2 + W(w) - W(\phi) \right] \, dz dy = e^{\tau} E_c[w].
\]

\(\square\)
To avoid getting the trivial solution \( w(z, y) \equiv \phi(y) \), we investigate the minimizing problem by proposing the constraint:

\[
K[w] := \frac{1}{2} \int_D e^z |\partial_z w|^2 \, dz \, dy = 1,
\]

first introduced by Heinze [28] and Lucia et al. [30]. Set the space of admissible functions with the constraint to be

\[
B := \{ u \in \mathcal{A} \mid K[u] = 1 \}.
\]

From Proposition 3.2, it is clear that for \( w \in \mathcal{A} \) with \( w \not\equiv \phi(y) \) there is a unique \( \tau_0 \in \mathbb{R} \) such that \( w(\tau_0) \in B \). We consider in this paper the following minimizing problem

\[
\mu_c := \inf_{B} E_c = \frac{1}{2} \int_D e^z |\partial_z u|^2 \, dz \, dy + \frac{1}{2} |\partial_y \phi|^2 + W(u) - W(\phi) \, dydz.
\]

To show \( \mu_c \) is bounded below, we quote the Poincare-type inequalities for functions in \( H \) as shown in [30]. For the convenience of the readers, we give a proof of them here.

**Lemma 3.3.** If \( u - \phi \in H \), then

\[
\int_{\tau}^{\infty} \int_{-\ell}^{\ell} e^z |u - \phi|^2 \, dy \, dz \leq 4 \int_{\tau}^{\infty} \int_{-\ell}^{\ell} e^z |\partial_z u|^2 \, dy \, dz,
\]

and therefore

\[
\int_{\tau}^{\infty} \int_{-\ell}^{\ell} e^z |u - \phi|^2 \, dy \, dz \leq \frac{1}{4} \int_{\tau}^{\infty} \int_{-\ell}^{\ell} e^{\tau} |u - \phi|^2 \, dy \, dz.
\]

**Proof.** Integration by parts implies

\[
\int_{\tau_1}^{\tau_2} \int_{-\ell}^{\ell} e^z |u - \phi|^2 \, dy \, dz
\]

\[
= \left[ \int_{-\ell}^{\ell} e^z |u - \phi|^2 \, dy \right]_{z=\tau_1}^{z=\tau_2} - \int_{\tau_1}^{\tau_2} \int_{-\ell}^{\ell} e^z 2(u - \phi) \cdot \partial_z u \, dy \, dz.
\]

Since \( u - \phi \in H \), there exists a suitable sequence \( \tau_2 \to \infty \) such that

\[
\int_{-\ell}^{\ell} e^{\tau_2} |u(\tau_2, y) - \phi(y)|^2 \, dy \to 0.
\]

Hence by taking limit along such a sequence \( \tau_2 \), we have

\[
\int_{\tau_1}^{\infty} \int_{-\ell}^{\ell} e^z |u - \phi|^2 \, dy \, dz
\]

\[
= \int_{\tau_1}^{\infty} \int_{-\ell}^{\ell} e^{\tau_1} |u - \phi|^2 \, dy \, dz - \int_{\tau_1}^{\infty} \int_{-\ell}^{\ell} e^{\tau_2} 2|u - \phi| \cdot [\partial_z u] \, dy \, dz
\]

\[
\leq \int_{\tau_1}^{\infty} \int_{-\ell}^{\ell} 2[e^{z/2}|u - \phi|] \cdot [e^{z/2}\partial_z u] \, dy \, dz
\]
\[ \leq 2 \left[ \int_{\tau_1}^{\infty} \int_{-\ell}^{\ell} e^z |u - \phi|^2 \, dy \, dz \right]^{1/2} \left[ \int_{\tau_1}^{\infty} \int_{-\ell}^{\ell} e^z |\partial_z u|^2 \, dy \, dz \right]^{1/2} \]

which gives (33) and (35). Now considering
\[ \int_{\tau}^{\infty} \int_{-\ell}^{\ell} e^z |(u - \phi) + \partial_z (u - \phi)|^2 \, dy \, dz \geq 0, \]

and observing that from integration by parts,
\[ \int_{\tau}^{\infty} \int_{-\ell}^{\ell} e^z \partial_z (u - \phi) \cdot (u - \phi) \, dy \, dz = \int_{-\ell}^{\ell} e^z |(u - \phi) + \partial_z (u - \phi) + |u - \phi|)^2 \, dy \, dz, \]

we have
\[ \int_{\tau}^{\infty} \int_{-\ell}^{\ell} e^z |\partial_z (u - \phi)|^2 \, dy \, dz \geq \int_{\tau}^{\infty} \int_{-\ell}^{\ell} e^z \left[ -2(u - \phi) \cdot \partial_z (u - \phi) - |u - \phi|^2 \right] \, dy \, dz = \int_{-\ell}^{\ell} e^z |u(\tau, y) - \phi(y)|^2 \, dy \, dz. \]

This proves the inequality (34).

For showing that \( \mu_c \) is bounded below and for the later use, we give an energy estimate for \( u(z, y) \in B \) when \( z \) is large by combining Proposition 2.7 and Lemma 3.3.

**Lemma 3.4.** Let \( z_0 \in \mathbb{R} \) satisfy \( 2e^{-z_0} < \alpha^2_0 \), where \( \alpha_0 \) is given in Proposition 2.7. Then
\[ F_{c}[u(z, \cdot)] \geq e\ell > 0 \]
for all \( z \geq z_0 \) and \( u \in B \).

**Proof.** Using (34) in Lemma 3.3 and \( u \in B, \) for all \( z \) we have
\[ \int_{-\ell}^{\ell} |u(z, y) - \phi(y)|^2 \, dy \leq 2e^{-z}. \]

Then the lemma easily follows from Proposition 2.7.

**Lemma 3.5.** \( \mu_c > -\infty \). And more explicitly,
\[ \mu_c \geq c^2 - \frac{e \cdot e\ell}{2(\eta - \delta)^2(\ell - \frac{c_{\infty}}{2})}. \]

**Proof.** It is sufficient to prove that \( E_c \) is bounded from below on \( B \). For all \( u \in B \), Lemma 3.4 implies
\[ E_c[u] \geq c^2 + \int_{-\infty}^{z_0} \int_{-\ell}^{\ell} e^z \left( -\frac{1}{2} |\partial_y \phi|^2 - W(\phi) \right) \, dy \, dz = c^2 - e\ell e^{-z_0}. \]

Here we choose
\[ z_0 = 1 - 2 \ln \alpha_0 = 1 - \ln \left[ 2 (\eta - \delta)^2 (\ell - \frac{c_{\infty}}{2\ell}) \right], \]
satisfying \( 2e^{-z_0} < \alpha_0^2 \) in Lemma 3.4. (The constants \( \alpha_0 \) and \( \eta \) are defined in (28) and (27).) Consequently, the lemma is proved.
4. **Traveling wave speed.** To show the existence of a traveling wave solution, determining the wave speed is important. We refer to [28], [36] and [30] for ingenious ideas to determine the wave via variational consideration which lead to an energy relation given in Lemma 4.1 below or related criteria. Here as in [14], we use a slightly different setting, in which $E_c$ remains a continuous function of $c$, to capture the wave speed. For this purpose, we first study the relation between the wave speed and the functional $E_c$. (See Proposition 3.5 in [30].)

**Lemma 4.1.** If $(w, c)$ is a smooth solution of (30) with $w - \phi \in H$ and $c > 0$, then $E_c[w] = 0$.

**Proof.** Let $\alpha, \beta \in \mathbb{R}$ and $\alpha < \beta$. Multiplying the equation in (30) by $e^z \partial_z w$ and integrating by parts, we obtain

$$0 = \int_{-\ell}^{\ell} e^z \partial_z w \cdot \left[ e^2 (\partial_z^2 w + \partial_z w) + \partial_y^2 w - \nabla W(w) \right] dydz$$

$$= \int_{-\ell}^{\ell} e^z \left[ e^2 \frac{1}{2} \partial_z \partial_z w + \frac{1}{2} \partial_z w \right] dydz$$

$$+ \int_{-\ell}^{\ell} \left[ e^z \partial_y w \right]_{y=\ell}^{y=-\ell} dz - \int_{-\ell}^{\ell} \int_{-\ell}^{\ell} e^z \frac{1}{2} \partial_z \left( |\partial_y w|^2 - |\partial_y \phi|^2 \right) dydz$$

$$+ \int_{-\ell}^{\ell} \int_{-\ell}^{\ell} e^z \left[ - \partial_z (W(w) - W(\phi)) \right] dydz$$

$$= \left[ \int_{-\ell}^{\ell} e^z \left( \frac{1}{2} |\partial_z w|^2 - \frac{1}{2} |\partial_y w|^2 + \frac{1}{2} |\partial_y \phi|^2 - W(w) + W(\phi) \right) dy \right]_{z=\alpha}^{z=\beta}$$

$$+ \int_{-\ell}^{\ell} \int_{-\ell}^{\ell} e^z \left[ \frac{1}{2} |\partial_z w|^2 + \frac{1}{2} |\partial_y w|^2 - \frac{1}{2} |\partial_y \phi|^2 + W(w) - W(\phi) \right] dydz.$$ 

Since $w - \phi \in H$, we can choose two sequences $\alpha_k$ and $\beta_k$ such that $\alpha_k \to -\infty$, $\beta_k \to \infty$ and the boundary terms at $z = \alpha_k$ and $\beta_k$ will converge to 0 as $k \to \infty$. Then we will obtain $0 = E_c[w]$.

In view of Lemma 4.1, if there exists a smooth global minimizer of $E_c$ on $B$, then we can conclude that $\mu_c = \inf_{u \in B} E_c[u] = 0$. Therefore, a root of $\mu_c$ is a possible candidate for the wave speed.

**Proposition 4.2.** There exists a unique $c^* > 0$ such that $\mu_{c^*} = 0$.

**Proof.** Note that in (32) that $\mu_c$ is a strictly increasing and continuous function of $c$. From Lemma 3.5, it is easy to see $\mu_c > 0$ if $c$ is large enough. The result follows if we can show that $\mu_c < 0$ as $0 < c \ll 1$. To do this, it suffices to ensure that there exists a function $\psi \in B$ such that $E_c[\psi] < 0$ as $c \ll 1$. Indeed, consider the function

$$\psi(z, y) = \begin{cases} a, & z < 0; \\
\frac{z}{c} \phi(y) + (1 - \frac{z}{c}) a, & 0 \leq z < c; \\
\phi(y), & z \geq c.
\end{cases}$$

Due to the boundedness of $\psi$, $\phi$ and $\partial_y \phi$, a simple calculation yields
lower-semi-continuity of $E$ and may cause some problems. To overcome the difficulty, we prove the weak cases, the energy $E$ and $\mu$.

Then it follows from the existence of a traveling wave solution by looking for a minimizer of $E$. Minimizer for the constrained problem.

5. Minimizer for the constrained problem. In this section, we obtain the existence of a traveling wave solution by looking for a minimizer of $E_c$ with $c = c^*$. To do this, we first show the weak lower-semi-continuity of $E_c$. Unlike the usual cases, the energy $E_c(u)$ contains the weight $e^z$, which goes to infinity for large $z$ and may cause some problems. To overcome the difficulty, we prove the weak lower-semi-continuity of $E_c$ by Lemma 3.4.

**Proposition 4.3.** For $\ell > \ell_c \geq \frac{e\epsilon}{2\epsilon}$, we have

$$c^* \leq \frac{C_1}{\sqrt{\ell - \frac{e\epsilon}{2\epsilon}}}$$

where $C_1 := \sqrt{e \cdot e_{\infty}/2 \cdot (\eta - \delta)^{-1}}$.

**Proof.** For all $u \in B$, by Lemma 3.5,

$$\mu_c \geq c^2 - \frac{e \cdot e_{\ell}}{2(\eta - \delta)^2(\ell - \frac{e\epsilon}{2\epsilon})}.$$ 

Then it follows from $\mu_{c^*} = 0$ and $e_{\ell} < e_{\infty}$ that

$$(c^*)^2 \leq \frac{C_1^2}{\ell - \frac{e\epsilon}{2\epsilon}},$$

which implies (36).

5. Minimizer for the constrained problem. In this section, we obtain the existence of a traveling wave solution by looking for a minimizer of $E_c$ with $c = c^*$. To do this, we first show the weak lower-semi-continuity of $E_c$. Unlike the usual cases, the energy $E_c[u]$ contains the weight $e^z$, which goes to infinity for large $z$ and may cause some problems. To overcome the difficulty, we prove the weak lower-semi-continuity of $E_c$ by Lemma 3.4.

**Proposition 5.1.** Suppose $\{u_k\} \subset B$, $(u_k - \phi) \to (u_{\infty} - \phi)$ weakly in $H$, and $u_k \to u_{\infty}$ pointwise a.e. on $D$. Then

$$\liminf_{k \to \infty} E_c[u_k] \geq E_c[u_{\infty}]$$

and

$$\liminf_{k \to \infty}(E_c[u_k] - c^2 K[u_k]) \geq E_c[u_{\infty}] - c^2 K[u_{\infty}].$$

**Proof.** For $k \in \mathbb{N} \cup \{\infty\}$, set $E_c[u_k, (\tau_1, \tau_2)] = \int_{\tau_1}^{\tau_2} g_k(z)dz$ where

$$g_k(z) = \int_{-\ell}^{\ell} e^z \left[ \frac{e^2}{2} |\partial_x u_k|^2 + \frac{1}{2} |\partial_y u_k|^2 - \frac{1}{2} |\partial_y \phi|^2 + W(u_k) - W(\phi) \right] dy.$$

By Lemma 3.4, for all $\tau \geq z_0$ we have $g_k(\tau) \geq 0$ and $E_c[u_k, (\tau, \infty)] \geq 0$. Since $e^z dydz$ is a finite measure on $(-\infty, \tau) \times (-\ell, \ell)$ for any fixed $\tau \in \mathbb{R}$, a standard weak lower-semi-continuity result and Fatou’s lemma imply

$$\liminf_{k \to \infty} E_c[u_k, (-\infty, \tau)] \geq E_c[u_{\infty}, (-\infty, \tau)].$$
(See Proposition 2.1 in [18].) Also note that for any \( u - \phi \in H \), we have
\[
\lim_{\tau \to \infty} E_c[u, (\tau, \infty)] = 0,
\]
due to (33). Consequently, we derive that for any \( \tau \geq z_0 \)
\[
\liminf_{k \to \infty} E_c[u_k] \geq \liminf_{k \to \infty} E_c[u_k, (-\infty, \tau)] + \liminf_{k \to \infty} E_c[u_k, (\tau, \infty)]
\]
\[
\geq E_c[u_\infty, (-\infty, \tau)] = E_c[u_\infty] - E_c[u_\infty, (\tau, \infty)].
\]
As \( \tau \to \infty \), we obtain (37). Lastly, (38) also follows from the above proof.

**Proposition 5.2.** There exists \( w \in B \) such that
\[
E_{c^*}[w] = \mu_{c^*} = \inf_{A} E_{c^*} = 0.
\]
In addition, \( w \in C^2_{\text{loc}}(D) \) solves (30) and \( |w| \leq R \).

**Proof.** Let \( u_k \in B \) be a minimizing sequence for \( \inf_B E_{c^*} = 0 \). Without loss of generality, we suppose \( E_{c^*}[u_k] \leq \frac{e^{2u}}{2} \) for all \( k \). Let
\[
\tilde{E}_{c^*}[u] = E_{c^*}[u] - (c^*)^2K[u]
\]
and
\[
\sigma(u) = \begin{cases} u & \text{for } |u| \leq R \\ \frac{R}{u} & \text{for } |u| \geq R \\ \end{cases}.
\]
Observing in (H1) that
\[
\nabla W(u) \cdot u \geq C_0|u| \quad \text{on } \{|u| \geq R\},
\]
we have \( \tilde{E}_{c^*}[\sigma(u_k)] \leq \tilde{E}_{c^*}[u_k] \leq 0 \). Since \( \int_D e^2|\partial_2 \sigma (u_k)|^2dydz \leq \int_D e^2|\partial_2 u_k|^2dydz \),
there is an \( \tau_k \geq 0 \) such that \( \sigma(u_k)(\tau_k) := \sigma(u_k)(z - \tau_k, y) \in B \). Moreover, by Proposition 3.2,
\[
0 \leq E_{c^*}[\sigma(u_k)(\tau_k)] = e^{2\tau_k} + \tilde{E}_{c^*}[\sigma(u_k)]
\]
\[
= e^{2\tau_k} + e^{\tau_k} \tilde{E}_{c^*}[\sigma(u_k)] \leq e^{2\tau_k} + \tilde{E}_{c^*}[u_k] = E_{c^*}[u_k].
\]
Consequently, we may also assume that \( |u_k| \leq R \) if we replace \( u_k \) by \( \sigma(u_k)(\tau_k) \).

From Lemma 3.3, it is easy to see
\[
\|u_k - \phi\|_{L^2_w}^2 \leq 8.
\]
Again we can use the Taylor expansion of \( W \) along the line segment between \( w(z, y) \) and \( \phi(y) \), we get \( \xi \in (0, 1) \) such that
\[
|W(w) - W(\phi) - \nabla W(\phi) \cdot (w - \phi)|
\]
\[
= \frac{1}{2} |(w - \phi)^{tr} \text{Hess } W(\xi w + (1 - \xi) \phi) (w - \phi) - \tilde{M}|w - \phi|^2,
\]
where the superscript \( tr \) means the transpose of the vector and \( \tilde{M} \) is the maximum of all the eigenvalues of \( \text{Hess } W(v) \) on \( \{v \mid |v| \leq R\} \). Then we have
Proposition 5.3.

Schauder's theory to show uniform bound of $w$. 

Proof. To reduce the influence of the boundary $0$ for any $\epsilon > 0$, let $\tau > 0$ be such that $|\tau| = 2\epsilon$. Since $w$ is a solution of (30) on $\mathcal{A}$, we have

$$
\int_D e^2 \left[ c^2 |\partial_x (u_k - \phi)|^2 + |\partial_y (u_k - \phi)|^2 \right] dy dz 
= 2E_{c^*}[u_k] + 2 \int_D e^2 \left[ W(\phi) - W(u_k) - (\partial_y u_k - \partial_y \phi) \cdot \partial_y \phi \right] dy dz 
= 2E_{c^*}[u_k] + 2 \int_D e^2 \left[ -W(u_k) + W(\phi) + (u_k - \phi) \cdot \partial_y^2 \phi \right] dy dz 
\leq c^2 + 2 \int_D e^2 M[u_k - \phi]^2 dy dz 
\leq c^2 + 16M.
$$

With $\|u_k - \phi\|_H$ being uniformly bounded, we apply the Banach-Alaoglu theorem to this sequence on the symmetry space $\mathcal{A}$. Up to a subsequence, we obtain $u_k - \phi \to u_\infty - \phi$ weakly in $\mathcal{A}$ and $u_k \to u_\infty$ pointwise a.e. in $D$. The latter leads to $|u_\infty| \leq R$ a.e. in $D$. We claim that $u_\infty(y)$ is not equal to $\phi(y)$ identically. Otherwise, $u_k - \phi \to 0$ weakly in $H$ and (38) implies

$$
0 = \liminf_{k \to \infty} E_{c^*}[u_k] = c^2 + \liminf_{k \to \infty} (E_{c^*}[u_k] - K[u_k]) \geq c^2,
$$

being a contradiction. Therefore, there is a $\tau \in \mathbb{R}$ such that $u^{(\tau)} \in \mathcal{B}$. Set $w = u^{(\tau)}$. By invoking (37), Proposition 3.2 gives

$$
0 \leq E_{c^*}[w] = e^\tau \liminf_{k \to \infty} E_{c^*}[u_k] = 0.
$$

Hence $E_{c^*}[u] = E_{c^*}[w] = 0$, and $w \in \mathcal{A}$ also solves (30) by the standard variational theory and by the principle of symmetric criticality (which is shown by Palais [38]). Since $W$ is $C^2$, we conclude that $w$ is a $C^{2,\alpha}$ solution of (30) on $D$ for any $0 < \alpha < 1$. \hfill \Box

As to further investigate the asymptotic behavior of $w$ at $z = \pm \infty$, we divide the domain $D$ into rectangles $R_k := (k - 1, k + 1) \times (-\ell, \ell)$ for $k \in \mathbb{Z}$ and apply Schauder’s theory to show uniform bound of $w$ on $R_k$.

**Proposition 5.3.** Let $w$ be a minimizer obtained by Proposition 5.2. Then we have

$$
\sup_{k \in \mathbb{Z}} \|w\|_{C^{2,\alpha}(R_k)} < \infty
$$

for any $0 < \alpha < 1$.

**Proof.** To reduce the influence of the boundary $y = \pm \ell$, we extend $R_k$ to $\tilde{R}_k := (k - 2, k + 2) \times (-\ell - 1, \ell + 1)$ by an even reflection. Set

$$
\tilde{w}(z, y) = \begin{cases} 
    w(z, 2\ell - y), & \text{for } y \in (\ell, \ell + 1); \\
    w(z, y), & \text{for } y \in [-\ell, \ell]; \\
    w(z, -2\ell - y), & \text{for } y \in [-\ell - 1, \ell].
\end{cases}
$$

Then $\tilde{w}$ solves the following Dirichlet boundary problem on $\tilde{R}_k$:

$$
\begin{cases}
    e^2 (\partial^2_w u + \partial_y u) + \partial_y^2 u - \nabla W(\tilde{w}(z, y)) = 0, & (z, y) \in \tilde{R}_k; \\
    u |_{\partial \tilde{R}_k} = \tilde{w} |_{\partial \tilde{R}_k}.
\end{cases}
$$

(39)
By using the mean value theorem on $\nabla W(p) - \nabla W(q)$ for any $p, q \in \{v \mid |v| \leq R\}$, there exist two points $\xi_1, \xi_2$ on the line segment between $p$ and $q$ such that

$$\partial_a W(p) - \partial_a W(q) = \nabla(\partial_a W)(\xi_1) \cdot (p - q);$$
$$\partial_c W(p) - \partial_c W(q) = \nabla(\partial_c W)(\xi_2) \cdot (p - q).$$

Let $\hat{M} > 0$ be a bound of all eigenvalues of the matrices

$$\begin{bmatrix}
\partial_{uu} W(v_1) & \partial_{uW} W(v_1) \\
\partial_{uW} W(v_2) & \partial_{ww} W(v_2)
\end{bmatrix}, \quad \forall v_1, v_2 \in \{v \mid |v| \leq R\}.$$

Then we have $|\nabla W(p) - \nabla W(q)| \leq \hat{M}|p - q|$ and

$$\|\nabla W(\hat{w}(\cdot))\|_{C^{0,\alpha}(\hat{R}_k)} = \|\nabla W(\hat{w}(\cdot))\|_{C(\hat{R}_k)} + \sup_{|w| \leq R} |\nabla W(\hat{w}(z_1, y_1)) - \nabla W(\hat{w}(z_2, y_2))|$$
$$\leq \|\nabla W(\hat{w}(\cdot))\|_{C(\hat{R}_k)} + \hat{M}\|(\partial_z \hat{w}, \partial_y \hat{w})\|_{C(\hat{R}_k)} (\text{diam} (\hat{R}_k))^{1-\alpha}.$$

So $\nabla W(\hat{w}(\cdot)) \in C^{0,\alpha}(\hat{R}_k)$. By solving elliptic equations in each component utilizing the elliptic interior estimates inside $R_k$, the system (39) has a unique $C^{2,\alpha}$ solution $\hat{w}$ satisfying

$$\|\hat{w}\|_{C^{2,\alpha}(R_k)} \leq C\left( \|\hat{w}\|_{C(\hat{R}_k)} + \sup_{|w| \leq R} |\nabla W(\hat{w})(z_1, y_1)| \right),$$

where $C > 0$ depends only on $\alpha, \ell$ and $\hat{M}$. (See Theorem 6.2, Theorem 6.13 and the interpolation inequality in Appendix 1 in § 6.8 in [22].) Therefore, we have

$$\|w\|_{C^{2,\alpha}(R_k)} \leq \|\hat{w}\|_{C^{2,\alpha}(R_k)} \leq C\left( R + \sup_{|w| \leq R} |\nabla W(\hat{w})| \right),$$

which is independent of $k \in \mathbb{Z}$. \qed

6. Proof of Main Theorem. Now let $w(z, y)$ be determined in Proposition 5.2 and set $w^*(z, y) = w(c^* z, y)$. Then $(w^*, c^*)$ is a solution of (5). To finish the proof of Main Theorem, we study the behavior of $w^*$ at $z = \pm \infty$. It follows from Proposition 5.3 that the family $\{w^*(z, y)\}_{z \in \mathbb{R}}$ considered as functions of $y$ is uniformly bounded and equi-continuous in $C^2(-\ell, \ell) \cap C^1[-\ell, \ell]$. By the Arzela-Ascoli theorem, there exist $z^\pm \rightarrow \pm \infty$ and $v^\pm(y)$ such that

$$w^*(z^\pm, y) \rightarrow v^\pm(y)$$

uniformly in $C^2(-\ell, \ell) \cap C^1[-\ell, \ell]$. In the case of $z \rightarrow \infty$, from (34), we have

$$||w^*(z, \cdot) - \phi(\cdot)||_{L^2(-\ell, \ell)} \rightarrow 0.$$

Hence $v^+ = \phi$ is the only possible sequential limit and $w^*(z, \cdot) \rightarrow \phi$ as $z \rightarrow \infty$.

Now, we investigate the asymptotic behavior of $\partial_z w^*$ at $z = -\infty$. Denote $\Omega_n := (z_n^-, z_n^+) \times (-\ell, \ell)$. Multiplying (5) by $\partial_z w^*$ and integrating by parts, for $c = c^*$ we have

$$0 = \int_{\Omega_n} \partial_z w^* \cdot [\partial_z^2 w^* + c^* \partial_z w^* + \partial_y^2 w^* - \nabla W(w^*)] \, dy \, dz$$
$$= \left[ \int_{-\ell}^\ell \frac{1}{2} |\partial_z w^*|^2 - \frac{1}{2} |\partial_y w^*|^2 - W(w^*) \, dy \right]_{z = z_n^+}^{z = z_n^-} + \int_{\Omega_n} c^* |\partial_z w^*|^2 \, dy \, dz. \quad (40)$$
Since $w^* \in C^1(D)$ and its derivatives are uniformly bounded, we conclude that $\partial_z w^* \in L^2(D)$ by letting $n \to \infty$ in (40). With this, after passing to a suitable subsequence, we have

$$\int_{-\ell}^\ell |\partial_z w^*(z_n^+ + y)|^2 dy \to 0 \text{ as } n \to \infty.$$  

Furthermore, as $n \to \infty$ we obtain

$$F_\ell[v^-] = F_\ell[\phi] - c^* \int_D |\partial_z w^*|^2 dydz < F_\ell[\phi]$$

due to $c^* > 0$ and $\partial_z w^* \not\equiv 0$.

Next, following the proof of Proposition 6.6 in [30] we show that $v^-(y)$ solves

$$\begin{align*}
\partial_y^2 v(y) - \nabla W(v(y)) &= 0, \quad y \in (\ell, 0), \\
\partial_y v(\pm \ell) &= 0
\end{align*}$$

(41)

Indeed, for any $\psi(y) \in H^1((\ell, 0); \mathbb{R}^2)$, multiplying (5), with $c = c^*$, by $\psi$, then we have by integration by parts

$$0 = \int_{z_n^-}^{z_n^+ + 1} \int_{-\ell}^\ell \psi \cdot (\partial_y^2 w^* + c^* \partial_z w^* + \partial_y^2 w^* - \nabla W(w^*)) dydz$$

(42)

$$= \int_{-\ell}^\ell \psi \cdot \partial_z w^* dy \bigg|_{z_n^-}^{z_n^+ + 1} + \int_{z_n^-}^{z_n^+ + 1} \int_{-\ell}^\ell \psi \cdot c^* \partial_z w^* dydz$$

$$- \int_{z_n^-}^{z_n^+ + 1} \int_{-\ell}^\ell [\partial_y \psi \cdot \partial_y w^* + \psi \cdot \nabla W(w^*)] dydz.$$  

(43)

Since $\partial_z w^* \in L^2(D)$ is uniformly continuous, we have

$$\lim_{z \to -\infty} \partial_z w^* = 0.$$  

Therefore, as $n \to \infty$ in (43) we obtain

$$\int_{-\ell}^\ell [\partial_y \psi \cdot \partial_y v^- + \psi \cdot \nabla W(v^-)] dy = 0.$$  

By a standard argument, we can conclude that $v^-$ solves (41).

Finally, we complete the proof of Main Theorem by showing that $v^-(y) \equiv a$ is the unique limit at $z = -\infty$ as follows.

**Proposition 6.1.** Let $\ell_\epsilon > e_\epsilon/(2\epsilon)$ be the number in Proposition 1.1 and $\tilde{A}_\ell$ be introduced in (12). If $\ell > \ell_\epsilon$, $v \in \tilde{A}_\ell \cap C^2(\ell, 0) \cap C^1(\ell, 0)$ satisfies (41) and

$$F_\ell[v] < F_\ell[\phi],$$

(44)

then $v(y) = a$ for all $y \in [-\ell, 0]$.

**Proof.** Multiplying (41) by $\partial_y v$ and using the boundary condition in (41) we have

$$\partial_y \left[ \frac{1}{2} |\partial_y v|^2 - W(v) \right] = 0$$

and for all $y \in [-\ell, \ell]$

$$W(v(y)) = \frac{1}{2} |\partial_y v(y)|^2 + W(v(y)).$$
Then in view of (44), we derive

\[ e_\ell = F_\ell(\phi_\ell) > F_\ell(v) = \int_{-\ell}^{\ell} |\partial_y v|^2 \, dy + 2\ell \cdot W(v(\ell)) \geq 2\ell \cdot W(v(\ell)). \]

Hence \( W(v(\ell)) < e_\ell/2\ell \). From the setting in Lemma 2.1, if \( \ell > \ell_* \geq [e_\infty/(2\epsilon)] \), then \( v(\ell) \in B_{\epsilon A} \) for \( q = a, b_1 \) or \( b_2 \). We argue that \( v(\ell) \in B_{\epsilon A} \) by contradiction. Without loss of generality, suppose \( v(\ell) \in B_{\epsilon A} \subset B_{\rho_0}(b_1) \), and then there are two cases: (i) \( |v(y) - a| \geq \delta, \forall y \in (-\ell, \ell) \); (ii) there exists \( Y \) such that \( |v(Y) - a| \leq \delta \).

In case (i), by the construction of \( \phi \), we have \( F_\ell[v] \geq F_\ell[\phi] \). In case (ii), by Lemma 2.4, we also have \( F_\ell[v] \geq F_\ell[\phi] \). These two cases lead to a contradiction. Thus \( v(\ell) \notin B_{\epsilon A} \). Now we are going to show that for all \( y \in [-\ell, \ell] \), \( v(y) \in B_{\rho_0}(a) \), being a region that Hess \( W \) is positive definite. Note that for all \( y \in [0, \ell] \),

\[ F_\infty[\phi_\infty] \geq F_\ell[\phi] > F_\ell[v] \geq 2\text{dist}_W(v(\ell), v(y)). \]

By Lemma 2.1, for all \( y \in [0, \ell] \) we have

\[ \text{dist}_W(v(y), a) \leq \text{dist}_W(v(y), v(\ell)) + \text{dist}_W(v(\ell), a) < \frac{1}{2}F_\infty[\phi_\infty] + \rho < \rho_0. \]

It follows from \( v \in \tilde{A}_\ell \) that \( v(y) \in B_{\rho_0}(a) \) for all \( y \in [-\ell, \ell] \). Now we express \( v \) by

\[ v(y) = a + r(y)(\cos \theta(y), \sin \theta(y)) \quad \text{for} \quad r(y) \geq 0. \]

A simple calculation gives

\[
\begin{align*}
\partial_y v(y) &= r'(\cos \theta, \sin \theta) + r\theta'(-\sin \theta, \cos \theta), \\
\partial_y^2 v(y) &= (r'' - r\theta'^2)(\cos \theta, \sin \theta) + (2r'\theta' + r\theta'')(\sin \theta, \cos \theta), \\
\partial_y v(y) \cdot (v - a) &= r' r \quad \text{for} \quad r > 0
\end{align*}
\]

and

\[ \partial_y^2 v(y) \cdot \frac{v - a}{r} = r'' - r(\theta')^2 \quad \text{for} \quad r > 0. \]

We argue that \( r(y) = 0 \) for all \( y \in [-\ell, \ell] \) by contradiction. If \( r(y) > 0 \) for some \( y \in [-\ell, \ell] \), then there exists \( y_0 \in [-\ell, \ell] \) such that \( r(y_0) = \max_{y \in [-\ell, \ell]} r(y) \). If \( y_0 \in (-\ell, \ell) \), then \( r''(y_0) \leq 0 \) and the right hand side of (46) is nonpositive at \( y = y_0 \). However, by using (41) and positive definiteness of Hess \( W \) on \( B_{\rho_0}(a) \), at \( y = y_0 \) we obtain

\[ \partial_y^2 v \cdot \frac{v - a}{r} = \nabla W(v) \cdot \frac{v - a}{r} = \partial_y W > 0, \]

which leads to a contradiction. In the case of \( y_0 = \pm \ell \), we have \( r'(y_0) = 0 \) due to \( \partial_y v(y_0) = 0 \) and (45). Since \( y_0 \) is a maximum point on the boundary \( \pm \ell \), there exists \( y_1 \in (-\ell, \ell) \), being near \( y_0 \), such that \( r''(y_1) \leq 0 \). A similar argument also leads to a contradiction. Consequently, \( v(y) = a \) for all \( y \in [-\ell, \ell] \). \( \square \)

7. Proof of Corollary 1.3. On the domain \( D_\ell := \mathbb{R} \times (-\ell, \ell) \) we denote \((w^*, c^*) \) in Main Theorem by \((w^*_\ell, c^*_\ell) \). We choose a \( \beta \) lying between \(-a \) and the first component of \( \phi_\infty(0) \) such that \( (\beta, 0) \) is not a critical point of \( W \) and \((\beta, 0) \neq \phi_\infty(z) \) for all \( z \in \mathbb{R} \). Since for any \( \ell > \ell_* \)

\[ \lim_{z \to -\infty} w^*_\ell(z, y) = a = (-a, 0) \quad \text{and} \quad \lim_{z \to \infty} w^*_\ell(z, y) = \phi_\ell(y), \]

there exists a point \((z_\ell, 0)\) such that \( w^*_\ell(z_\ell, 0) = (\beta, 0) \) when \( \ell \) is large. By translating \( w^*_\ell \) with respect to \( z \)-coordinate, we may assume \( w^*_\ell(0, 0) = (\beta, 0) \). In view of
Proposition 5.3, for any compact set \( V \subset \mathbb{R}^2 \) we may choose \( \ell \), being sufficiently large, such that \( V \subset D_\ell \) and
\[
\| w^*_k \|_{C^2,\alpha(V)} \leq \tilde{C}_1 \left( R + \sup_{|\nu| \leq R} |\nabla W| \right),
\]
where \( \tilde{C}_1 \) depends only on \( W \) and the size of the region \( V \).

By a diagonal process, there exist a subsequence \( w^*_k \rightarrow w_\infty \) in \( C^2(V) \) as \( k \rightarrow \infty \), i.e., \( w^*_k \rightarrow w_\infty \) in \( C^2_{loc}(\mathbb{R}^2) \) and \( c^*_k \rightarrow c_\infty \) as \( k \rightarrow \infty \). Applying Proposition 4.3, we have
\[
0 \leq c_\infty = \lim_{k \rightarrow \infty} c^*_k \leq \lim_{k \rightarrow \infty} \left( \frac{C_1}{\sqrt{\ell_k - \frac{c_\infty}{2\epsilon}}} \right) = 0.
\]
Consequently, \( (w_\infty(x, y), c_\infty) \) satisfies \( w_\infty(0, 0) = (\beta, 0) \) and solves (11). Since \( (\beta, 0) \) is not a critical point of \( W \), we conclude that \( w_\infty \) is not a constant.

8. An example satisfying (H1) - (H4). Now we give an example fulfilling (H1) to (H4) In particular, (H4) can be verified by calculating the distance between the minima of \( W \).

Example. We consider a quadruple well potential \( \widetilde{W} \) with minimum points at \( a = (-1, 0), (3, 0), b_1 = (1, b) \) and \( b_2 = (1, -b) \):
\[
\widetilde{W} = \frac{1}{2} [(u + 1)^2 + v^2] [(u - 1)^2 + (v - b)^2] \left[ (u - 1)^2 + (v + b)^2 \right] [(u - 3)^2 + v^2],
\]
where \( b > 0 \) is small enough and will be determined later. Define \( W \), a modification of \( \widetilde{W} \), by
\[
W(u, v) = \widetilde{W}(u, v) \text{ for } u \leq 2
\]
and extend \( W \) smoothly for \( u > 2 \) such that
\[
\nabla W(u, v) \cdot (u, v) \geq C \sqrt{u^2 + v^2} \text{ for some } C > 0 \text{ and } W(u, -v) = W(u, v).
\]

Verification. It is easy to see (H1) and (H2) hold for \( W \). By a numerical calculation, being not too difficult but complicated by hand, we have the following properties of \( W \). (See also Figure 2.)
\[
\begin{align*}
\partial_u W(u, v) &> 0, \quad \text{for } 1 < u \leq 2; \\
\partial_u W(u, v) &< 0, \quad \text{for } u = 1; \\
\partial_u W(u, v) &> 0, \quad \text{for } 0 \leq u < 1,
\end{align*}
\]
and
\[
\text{Hess } W \text{ is positive definite on } \{ p \mid |p - a| < 0.2 \}. \quad \text{(P2)}
\]

By the lemma in [39], there exists a path \( \phi_\infty \) connecting \( b_1 \) and \( b_2 \) such that \( \phi_\infty \) is a global minimizer of \( F_\infty \) on \( \bar{A}_b \) and \( F_\infty[\phi_\infty] = \text{dist}_W(b_1, b_2) \). We claim \( \text{dist}_W(b_1, b_2) \rightarrow 0 \) as \( b \rightarrow 0 \). Indeed, set
\[
\tau = \begin{cases} 
\begin{array}{ll}
b, & \text{for } y \geq \frac{b}{2}; \\
2y, & \text{for } -\frac{b}{2} \leq y \leq \frac{b}{2}; \\
-b, & \text{for } y \leq -\frac{b}{2};
\end{array}
\end{cases}
\]
A simple calculation implies
\[
\text{dist}_W(b_1, b_2) \leq F_\infty[(1, \tau)] = \int_{-\frac{b}{2}}^{\frac{b}{2}} (2 + W(1, \tau))dy = \mathcal{O}(b) \text{ as } b \rightarrow 0.
\]
Figure 2. The portrait of $\partial_u \tilde{W} = 0$, $\det(\text{Hess} \tilde{W}) = 0$, $\partial_{uu} W = 0$ and $|p - a| = 0.2$.

To show the uniqueness of $\phi_\infty$, we claim that as $b \ll 1$, $\phi_\infty := (u_\infty, v_\infty)$, where $u_\infty \equiv 1$ and $v_\infty$ is the unique global minimizer of $E[v] := F_\infty[(1, v)]$ for $(1, v) \in \hat{A}_b$. Set $A = \{(u, v) \in \mathbb{R}^2 \mid u \leq 0\}$. Choose $b$, being small, such that

$$\text{dist}_W(b_1, b_2) < \text{dist}_W(b_1, A).$$

We argue $u_\infty \equiv 1$ by contradiction. If $u_\infty \not\equiv 1$, then there exists $p \in \mathbb{R}$ such that $u_\infty(p) < 0$ otherwise the energy of $(1, v_\infty)$ will be smaller than the energy of $(u_\infty, v_\infty)$ due to (P1). Therefore, we have

$$\text{dist}_W(b_1, b_2) = F_\infty[\phi_\infty] \geq \text{dist}_W(b_1, \phi_\infty(p)) \geq \text{dist}_W(b_1, A),$$

being a contradiction. Consequently, we have $u_\infty \equiv 1$ and deduce the minimizer problem $F_\infty$ on $\hat{A}_b$ to the scalar problem $E[v]$ for $(1, v) \in \hat{A}_b$. Following the arguments in [2], we obtain the existence and uniqueness of $v_\infty$ and consequently the uniqueness of $\phi_\infty$.

Set

$$\tilde{W} = \frac{1}{2} \left[ (u + 1)^2 + v^2 \right] \left[ (u - 1)^2 + v^2 \right] \left[ (u - 3)^2 + v^2 \right],$$

being the limit of $\tilde{W}$ as $b \to 0$. To show $\phi_\infty$ does not pass through $a$, we take $b$ small enough such that

$$2\text{dist}_W((1, 0), b_1) + \text{dist}_W(b_1, b_2) < 2\text{dist}_W(a, (1, 0)).$$

(49)

Note that (49) can be realized for small $b$ since

$$\text{dist}_W(a, (1, 0)) \to \text{dist}_W(a, (1, 0)) > 0 \text{ as } b \to 0$$

and

$$\text{dist}_W(b_1, b_2), \text{dist}_W((1, 0), b_1) \to 0 \text{ as } b \to 0.$$
In view of Lemma 2.3, for any $\psi \in \mathcal{A}_a$ passing $a$, we have
\begin{align*}
F_\infty[\psi] \geq 2\text{dist}_W(a, b_1) \geq 2\text{dist}_W(a, (1, 0)) - 2\text{dist}_W((1, 0), b_1) \\
> \text{dist}_W(b_1, b_2) = F_\infty[\phi_\infty],
\end{align*}
which verifies (H3). For (H4), we claim that $\mathcal{B}_{0.19}(a) \subset \{p \mid |p - a| < 0.2\}$. Indeed, for all $p$ with $|p - a| = 0.2$, and for arbitrary $C^1$ path $P$ connecting $a$ and $p$ with parametrization $\zeta$, we can easily obtain a lower bound for $\sqrt{2W}^2$ and
\begin{align*}
\int_P \sqrt{2W(\zeta)} \ |d\zeta| \geq \int_P |\zeta - a| \left[\frac{(7/4)^2 + (1/3)^2}{3} \cdot 3 \right] |d\zeta|.
\end{align*}
Since the line integral on the right hand side reaches a lower bound when the $C^1$ path is just the line segment between $a$ and $p$, we have
\begin{align*}
\int_P |\zeta - a| \ |d\zeta| \geq \frac{1}{2}|p - a|^2 = \frac{1}{2}(0.2)^2.
\end{align*}
Hence for all $p$ such that $|p - a| = 0.2$,
\begin{align*}
\text{dist}_W(p, a) = \inf \left\{ \int_a^p \sqrt{2W(\zeta)} \ |d\zeta| \right\} \\
\geq \inf \left\{ \int_a^p |\zeta - a| \left[\frac{(7/4)^2 + (1/3)^2}{3} \cdot 3 \right] |d\zeta| \right\} \\
\geq \frac{1}{2}(0.2)^2 \cdot \frac{457}{144} \cdot 3 \approx 0.1904 \ldots.
\end{align*}
And by (P2), Hess $W$ is positive definite on $\mathcal{B}_{0.19}(a)$. Choosing $b$ further such that
\begin{align*}
\frac{1}{2}F_\infty[\phi_\infty] = \frac{1}{2}\text{dist}_W(b_1, b_2) < 0.19,
\tag{50}
\end{align*}
we obtain (H4). As a conclusion, if $b$ is sufficiently small such that (48), (49) and (50) hold, then $W$ satisfies (H1) to (H4).

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