LINEAR QUADRATIC MEAN FIELD TYPE CONTROL AND MEAN FIELD GAMES WITH COMMON NOISE, WITH APPLICATION TO PRODUCTION OF AN EXHAUSTIBLE RESOURCE

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ABSTRACT. We study a general linear quadratic mean field type control problem and connect it to mean field games of a similar type. The solution is given both in terms of a forward/backward system of stochastic differential equations and by a pair of Riccati equations. In certain cases, the solution to the mean field type control is also the equilibrium strategy for a class of mean field games. We use this fact to study an economic model of production of exhaustible resources.

Keywords: mean field type control, mean field games, linear-quadratic, optimal control, riccati equations, exhaustible resource production
MSC: 93E20

1. Introduction

The purpose of this work is to study mean field games and mean field type control problems of linear quadratic type, primarily those motivated by a certain kind of application to economics, the quintessential example being the production of an exhaustible resource. Let us recall that mean field game theory was introduced by the parallel works of Caines, Huang and Malhamé [40] and of Lasry and Lions [45, 46, 47], with a general aim to study the interactions of large populations of rational actors. A mean field game refers essentially to an equilibrium which occurs when the strategy employed by a representative agent of a given crowd is optimal given the costs imposed by that crowd. A useful overview of the topic can be found in the notes of Cardaliaguet [14] based on the lectures of Lions at the Collège de France [51]. For an introduction to both theory and applications of mean field games, see especially the Paris-Princeton Lectures of Guéant, Lasry and Lions [35]. See also the survey by Gomes [29]. For a probabilistic analysis of mean field games, see Carmona and Delarue [20]. We also mention that numerical methods have been important in the development of mean field game theory; see especially Achdou and Cappuzzo-Dolcetta [3] and Achdou, et al. [2].

A related but distinct concept is that of mean field type control. In this case, the goal is to assign a strategy to all agents at once, such that the resulting crowd behavior is optimal with respect to costs imposed on a central planner. For a comparison of mean field games and mean field type control, see the book of Bensoussan, Frehse, and Yam [5] as well as the article by Carmona, Delarue, and Lachapelle [23]. A key reference is the work of Carmona and Delarue [22], which characterizes solutions to the mean field type control problem in terms of a stochastic maximum principle for McKean-Vlasov type dynamics.

While mean field type control is conceptually distinct from mean field games, and although in general an optimal control on the one hand is not an equilibrium strategy on the other, nevertheless

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in many cases a mean field Nash equilibrium is also the solution to an optimal control problem, as pointed out at least as early as [47]. Many researchers have used this insight to generate results concerning existence and uniqueness [15, 17, 18, 53, 19] as well as computation of mean field Nash equilibria [7]. It must be understood that the overall minimized cost is smaller than the total cost to all individual players; the difference between the two is called the price of anarchy (see the discussion in [7]). The present work also highlights this point of view. Our motivating example from economics, while conceptually construed as a Nash equilibrium, can be solved via a reformulation as an optimal control problem. For this reason our results are more heavily inclined toward the study of mean field type control, even though, a priori, we are interested in mean field games.

One of the most natural ways to apply mean field game theory is to such fields as economics and systemic risk, since here the critical questions concern the behavior of large numbers of individuals motivated by similar incentives. See, for instance, the thesis of Guéant [32] on mean field games and economics, as well as related work by Guéant et al. [48] and by Lachapelle, Salomon, and Torinici [42]; the influential paper of Lucas and Moll [52]; the work of Carmona, Fouque and Sun on systemic risk [25]; and many other references, many of which can be found in the survey articles [1, 13, 28].

In this paper we are particularly motivated by a model of the production of an exhaustible resource, such as oil. We draw our inspiration from a model found in [35] and later adapted by Chan and Sircar in [26, 27]. (See also the work of Bauso, Tembine, and Basar [6].) A well-posedness result for a related system of partial differential equations appears in a paper by Bensoussan and the present author [31]. The basic structure of the model is as follows. Let $X$ represent the amount of remaining reserves held by a firm, $v$ the level of production, and the dynamics governed by a linear stochastic differential equation:

$$dX(s) = -v(s)ds + \nu X(s)dW(s) + \nu_0 X(s)dW_0(s), \quad X(t) = x.$$ 

A (mean field) Nash equilibrium is obtained whenever each firm has solved the profit maximization problem

$$\sup_v \mathbb{E} \left[ \int_t^T e^{-\mu(s-t)} v(s)(2\alpha - 2\beta \hat{\psi}(s) - v(s))ds - e^{-\mu(T-t)}|X(T)|^2 \right]$$

where $\mu$ is the discount rate and $\hat{\psi}(s) = \mathbb{E}[\hat{\psi}(s)|\mathcal{F}_s]$ is the conditional expectation (given the common noise) of the equilibrium strategy $\hat{\psi}(s)$. The reason for the appearance of the conditional expectation of the equilibrium strategy in the objective functional is that the market price is determined by taking an average. In this model, we have simplified the calculation of the price considerably by taking a linear demand schedule (cf. Chan and Sircar [26]). Hence the mean field game is of linear-quadratic form.

Linear quadratic models were among the first to receive full mathematical treatment by researchers studying mean field games. For the infinite time horizon case, we mention the work of Guéant [33], of Caines, Huang, and Malhame [36, 37, 38, 39], and of Li and Zhang [50]. For the finite time horizon case, we mention in particular the work of Bensoussan et al. [10], which deals both with mean field games and mean field type control problems. See also [5, 34, 41, 42, 43, 58, 59]. For the discrete time case, see the recent work of Ni, Zhang, and Li [54].

On the mean field type control side, the general linear quadratic case without common noise has been dealt with in the work of Yong [60]. Unlike most other references, his result allows the cost functional to depend on the expected value of the control as well as of the state variable. In that work the motivation for this dependence comes from problems involving the minimization of the variance, both of the control and state variables. In this paper, by contrast, we are interested in economic applications, and in fact we will see that the mean field game described above can
be written as a mean field type control problem, in that the equilibrium strategy is the optimal
decision for a central planner trying to minimize the following objective functional:
\[
E \left[ \int_t^T e^{-\mu(s-t)} v(s) \left( v(s) + \beta \mathbb{E}[v(s)|\mathcal{F}_s^0] - 2\alpha \right) ds + e^{-\mu(T-t)} |X(T)|^2 \right].
\]

Three other recent works of particular importance in the context of mean field type control have
recently been published by Pham and Wei \cite{56, 57} and by Pham \cite{55}. Pham and Wei develop a
dynamic programming technique (see also \cite{49}), with corresponding Hamilton-Jacobi equations on
an infinite dimensional space of probability measures, for solving mean field type optimal control
problems, first without \cite{56} and then with common noise \cite{57}. Both of these citations include brief
applications to linear-quadratic problems. We note that \cite{56} reproduces and slightly extends the
results of \cite{60} using dynamic programming. Meanwhile \cite{57} treats the case of a common noise;
however, in that case the control is already adapted to the common noise, which allows the authors
to prove that the distribution of the state variable is Markovian (a crucial step in proving the
dynamic programming principle). In \cite{55} Pham further develops the theory of linear quadratic
problems with common noise by considering random coefficients; again in this reference the control
is adapted to the common noise. The present work is complementary to \cite{56, 57, 55} in that (a) we
investigate the case where the control is not necessarily adapted to the common noise, and both the
control and its conditional expectation (given the common noise) are variables in the dynamics and
the cost; (b) we explicitly consider the connection with mean field games; and (c) we are inspired
by a particular application to economics, in contrast to the financial applications of these other
works. However, we only consider deterministic coefficients.

The distinguishing features of the present work are as follows. First, we will deal with the case of
a common noise, which is taken to represent an inherent uncertainty in nature affecting simultane-
ously all the agents participating in the game (or being controlled by a central planner). Second, as
in \cite{60, 56} but unlike most of the previous references, we will consider the conditional expectation
of the control variable (or the equilibrium strategy in the case of mean field games) as a factor
in the dynamics and quadratic cost. Finally, we use an economic model from recent literature to
illustrate the applicability of our general framework.

Mean field games with common noise have been analyzed recently using several different approaches.
First we mention the master equation, first introduced by Lions in his lectures at the Collège de
France \cite{51} (see also \cite{14, 21, 9}). This is a partial differential equation on the Wasserstein space
of probability measures, where the common noise is encoded in a second-order derivative with respect
to the measure variable. A well-posedness result for the master equation can be found in \cite{16},
including in the case of a common noise. A different approach, from a probabilistic point of view,
can be found in the work of Ahuja \cite{41} and Carmona, Delarue, and Lacker \cite{24}. The former starts
with the stochastic maximum principle for a representative player and uses a fixed point argument
to find a mean field Nash equilibrium. The latter also develops a theory of weak solutions for which
there is a quite general existence result.

A question of particular interest in the case of common noise is whether the mean field equilibrium
can act as an approximate solution of an N-player game for large N. Such an approximation is
known as “ε-Nash equilibrium” \cite{14, 36, 37, 40}. The first largely comprehensive account of this
problem for mean field games with common noise was given by Lacker in \cite{44}. Note, however, that
this general result does not cover all linear quadratic models. In the present work we will prove
that mean field game solutions serve as ε-Nash equilibria for large N-player games, but only in a
special case which most directly motivates our results, namely when the game is in fact solved by
an optimal control problem.
To complete this introduction, we give an outline of the rest of the paper. In Section 2 we completely solve the linear-quadratic mean field type control problem. We characterize the solution both in terms of a stochastic maximum principle (forward-backward system of stochastic differential equations) and Riccati equations. In Section 3 we discuss the question of Nash equilibrium. Rather than seek the most general possible solution, our main goal will be to provide criteria that allow an equilibrium to be interpreted as a global optimal solution for a mean field control problem, in which case the results of Section 2 can be applied to show that there is a unique solution to the mean field game. Finally, in Section 4 we describe and solve a linear-quadratic version of an economic production model.

2. The Mean Field Type Control Problem

Fix an initial time $t$ and a final time $T$. Let $(\Omega, (\mathcal{F}_s)_{s \in [t,T]}, \mathbb{P})$ be a complete filtered probability space. We suppose that $W(s), W_0(s)$ are independent $(\mathcal{F}_s)_{s \in [t,T]}$-Wiener processes, and that $x$ is a random variable independent of $W(s), W_0(s)$. (Here $W_0(s)$ is considered as the common noise.) Throughout we will denote by $(\mathcal{F}_s^0)_{s \in [t,T]}$ the filtration generated by $W_0(s), s \in [t,T]$. If $(X(s)$ is any stochastic process adapted to $(\mathcal{F}_s)_{s \in [t,T]}$, we denote $X(s) = \mathbb{E}[X(s)|\mathcal{F}_s^0]$, the conditional expectation of $X(s)$ given $W_0(s)$.

The linear-quadratic mean field type control problem is formulated as follows. An admissible control is defined to be a square integrable $(\mathcal{F}_s)_{s \in [t,T]}$-adapted process with values in $\mathbb{R}^m$. The corresponding state variable $X(s)$ is an $\mathbb{R}^d$-valued adapted process satisfying the dynamics

$$
(2.1) \quad dX(s) = \{A(s)X(s) + \dot{A}(s)\dot{X}(s) + B(s)v(s) + \ddot{B}(s)\ddot{v}(s)\} \, ds \\
+ \{C(s)X(s) + \dot{C}(s)\dot{X}(s) + D(s)v(s) + \ddot{D}(s)\ddot{v}(s)\} \, dW(s) \\
+ \{F(s)X(s) + \dot{F}(s)\dot{X}(s) + G(s)v(s) + \ddot{G}(s)\ddot{v}(s)\} \, dW_0(s), \quad X(t) = x.
$$

Let $\langle \cdot, \cdot \rangle$ be the inner product on Euclidean space. The objective functional is given by

$$
(2.2) \quad J^{LQ}_{x,t}(v) = \mathbb{E}\left\{ \int_t^T \left[ \langle Q(s)X(s), X(s) \rangle + \langle Q(s)\dot{X}(s), \dot{X}(s) \rangle + \langle R(s)v(s), v(s) \rangle + \langle \ddot{R}(s)\ddot{v}(s), \ddot{v}(s) \rangle \\
+ 2\langle S(s)X(s), v(s) \rangle + 2\langle \dot{S}(s)\dot{X}(s), \ddot{v}(s) \rangle + 2\langle q(s), X(s) \rangle + 2\langle \dot{q}(s), \dot{X}(s) \rangle \\
+ 2\langle r(s), v(s) \rangle + 2\langle \ddot{r}(s), \ddot{v}(s) \rangle \right] ds + \langle HX(T), X(T) \rangle + \langle H\dot{X}(T), \dot{X}(T) \rangle \right\}.
$$

We seek an optimal control $\hat{v}$ such that

$$
(2.3) \quad J^{LQ}_{x,t}(\hat{v}) = \inf_{v} J^{LQ}_{x,t}(v).
$$

Let us now give some standing assumptions on the coefficients. First, we define $S^n$ to be the set of all $n \times n$ symmetric matrices with real entries. Now we state the following:

**Assumption 2.1.** The coefficient matrices satisfy

1. $A, \dot{A}, C, \dot{C}, F, \ddot{F} \in L^{\infty}(0, T; \mathbb{R}^{d \times d})$
2. $B, \dot{B}, D, \dot{D}, G, \ddot{G} \in L^{\infty}(0, T; \mathbb{R}^{d \times m})$
3. $Q, \dot{Q} \in L^{\infty}(0, T; \mathbb{R}^{d})$, $R, \dot{R} \in L^{\infty}(0, T; S^n)$, $H, \dot{H} \in S^d$
4. $H, \dot{H} + \ddot{H} \geq 0$, and for some $\delta_1 \geq 0, \delta_2 > 0$, $Q, Q + \ddot{Q} \geq \delta_1 I$ and $R, R + \ddot{R} \geq \delta_2 I$
(5) \( S, \bar{S} \in L^\infty(0,T;\mathbb{R}^{m \times d}); q, \bar{q} \in L^\infty(0,T;\mathbb{R}^d); r, \bar{r} \in L^\infty(0,T;\mathbb{R}^m) \)

(6) \( \|S\|_\infty^2, \|S + \bar{S}\|_\infty^2 < \delta_1 \delta_2 \) if \( \delta_1 > 0 \), \( S = \bar{S} = 0 \) otherwise.

Under these assumptions, the dynamics (2.1) are well-posed in the following sense:

**Lemma 2.2.** Let \( v \) be an admissible control process and \( x \) an \( L^2(\Omega, \mathcal{F}_t, \mathbb{P}) \) random variable. Then there exists a unique \((\mathcal{F}_s)_{t \leq s \leq T}\)-adapted state process \( X(s) \) satisfying (2.1) with a continuous version such that

\[
\mathbb{E} \int_t^T |X(s)|^2 \, ds < \infty.
\]

**Proof.** This is proved in a straightforward manner using a fixed point argument, following standard theory for McKean-Vlasov dynamics. (See [60] for details.) \( \square \)

**Lemma 2.3.** The functional \( J_{x,t}^{LQ} \) is uniformly convex and has a unique minimizer.

**Proof.** Observe that

\[
\mathbb{E}[\langle Q(s)X(s), \bar{X}(s) \rangle] = \mathbb{E}[\mathbb{E}[\langle Q(s)X(s), \bar{X}(s) \rangle | \mathcal{F}_s]] = \mathbb{E}[\mathbb{E}[\langle Q(s)X(s), \bar{X}(s) \rangle | \mathcal{F}_s]] = \mathbb{E}[Q(s)\bar{X}(s), \bar{X}(s)],
\]

so that

\[
\mathbb{E}[\langle Q(s)X(s), X(s) \rangle + \langle \bar{Q}(s)\bar{X}(s), \bar{X}(s) \rangle] = \mathbb{E}[\langle Q(s)(X(s) - \bar{X}(s)), X(s) - \bar{X}(s) \rangle + \langle (Q(s) + \bar{Q}(s))\bar{X}(s), \bar{X}(s) \rangle].
\]

In like manner, we have

\[
\mathbb{E}[\langle R(s)v(s), v(s) \rangle + \langle \bar{R}(s)\bar{v}(s), \bar{v}(s) \rangle] = \mathbb{E}[\langle R(s)(v(s) - \bar{v}(s)), v(s) - \bar{v}(s) \rangle + \langle (R(s) + \bar{R}(s))\bar{v}(s), \bar{v}(s) \rangle].
\]

and

\[
\mathbb{E}[\langle S(s)X(s), v(s) \rangle + \langle \bar{S}(s)\bar{X}(s), \bar{v}(s) \rangle] = \mathbb{E}[\langle S(s)(X(s) - \bar{X}(s)), v(s) - \bar{v}(s) \rangle + \langle (S(s) + \bar{S}(s))\bar{X}(s), \bar{v}(s) \rangle].
\]

Strict convexity now follows from Assumption 2.1. The existence and uniqueness of the minimizer follows from the weak lower semicontinuity of the functional. \( \square \)

### 2.1. Optimality conditions.

**Proposition 2.4.** Suppose \( v \) is an optimal control minimizing the functional \( J_{x,t}(v) \), with corresponding trajectory \( \bar{X}(s) \) (the solution of (2.1)). Then there exists a unique adapted solution \((Y,Z,Z_0)\) of the BSDE

\[
\begin{aligned}
\frac{dY(s)}{ds} &= -\left( A^T(s)Y(s) + \bar{A}^T(s)\bar{Y}(s) + C^T(s)Z(s) + \bar{C}^T(s)\bar{Z}(s) + F^T(s)Z_0(s) + \bar{F}^T(s)\bar{Z}_0(s) \right. \\
&\quad \quad \quad + Q(s)X(s) + \bar{Q}(s)\bar{X}(s) + S^T(s)v(s) + \bar{S}^T(s)\bar{v}(s) + q(s) + \bar{q}(s) \bigg) \, ds \\
&\quad \quad \quad + Z(s)dW(s) + Z_0(s)dW_0(s), \quad s \in [0,T], \\
Y(T) &= HX(T) + H\bar{X}(T)
\end{aligned}
\]
satisfying the coupling condition

\[ (2.5) \quad B_T(s)Y(s) + \tilde{B}_T(s)\tilde{Y}(s) + D_T(s)Z(s) + \tilde{D}_T(s)\tilde{Z}(s) + G_T(s)Z_0(s) + \tilde{G}_T(s)\tilde{Z}_0(s) \]
\[ + R(s)v(s) + \tilde{R}(s)\tilde{v}(s) + S(s)X(s) + \tilde{S}(s)\tilde{X}(s) + r(s) + \tilde{r}(s) = 0, \quad s \in [0, T], \quad a.s. \]

Here, as usual, \( \tilde{Y}(s) := \mathbb{E}[Y(s)|\mathcal{F}_s^0], \tilde{Z}(s) := \mathbb{E}[Z(s)|\mathcal{F}_s^0], \) and \( \tilde{Z}_0(s) := \mathbb{E}[Z_0(s)|\mathcal{F}_s^0]. \)

Conversely, suppose \((X, v, Y, Z)\) is an adapted solution to the forward-backward system \( (2.1), (2.4) \). Then \( v \) is the optimal control minimizing \( J_{x,t}^{LQ}(v) \), and \( X(s) \) is the optimal trajectory.

**Proof.** The Gâteaux derivative of \( J_{x,t}^{LQ}(v) \) is

\[ (2.6) \quad \frac{d}{dh} J_{x,t}^{LQ}(v + hv) \bigg|_{h=0} = 2\mathbb{E} \left\{ \int_t^T \left[ (Q(s)X(s), \tilde{X}(s)) + (\tilde{Q}(s)\tilde{X}(s), \tilde{X}(s)) + (R(s)v(s), \tilde{v}(s)) \right. \right. \]
\[ + (\tilde{R}(s)\tilde{v}(s), (S(s)X(s), \tilde{v}(s)) + (S(s)\tilde{X}(s), v(s)) + (S(s)\tilde{X}(s), \tilde{v}(s)) \right. \right. \]
\[ + (q(s), \tilde{X}(s)) + (\tilde{q}(s), \tilde{X}(s)) + (r(s), \tilde{v}(s)) + (\tilde{r}(s), \tilde{v}(s)) \bigg] ds + (H(T), \tilde{X}(T)) + (\tilde{H}(T), \tilde{X}(T)) \right\} \]

where \( \tilde{X} \) is the solution of \( (2.4) \) with \( v \) replaced by \( \tilde{v} \) and \( X \) replaced by 0. If \((X, v)\) is optimal, then we get the optimality condition

\[ (2.7) \quad \mathbb{E} \left\{ \int_t^T \left[ (Q(s)X(s) + \tilde{Q}(s)\tilde{X}(s) + S_T(s)v(s) + \tilde{S}_T(s)\tilde{v}(s) + q(s) + \tilde{q}(s), \tilde{X}(s)) \right. \right. \]
\[ + (R(s)v(s) + \tilde{R}(s)\tilde{v}(s) + S(s)X(s) + \tilde{S}(s)\tilde{X}(s) + r(s) + \tilde{r}(s), \tilde{v}(s)) \bigg] ds \]
\[ + (H(T), \tilde{X}(T) + \tilde{H}(T)) \right\} = 0 \]

Now by \( [11], [12] \) we have a solution to the McKean-Vlasov type BSDE \( (2.4) \). By the Itô formula,

\[ (2.8) \quad \mathbb{E}(Y(T), \tilde{X}(T)) \]
\[ = \mathbb{E} \int_t^T \langle B_T(s)Y(s) + \tilde{B}_T(s)\tilde{Y}(s) + D_T(s)Y(s) + \tilde{D}_T(s)\tilde{Y}(s) + G_T(s)Z_0(s) + \tilde{G}_T(s)\tilde{Z}_0(s), \tilde{v}(s) \rangle ds \]
\[ - \mathbb{E} \int_t^T \langle \tilde{X}(s), Q(s)X(s) + \tilde{Q}(s)\tilde{X}(s) + S_T(s)v(s) + \tilde{S}_T(s)\tilde{v}(s) + q(s) + \tilde{q}(s) \rangle ds, \]

which by using the optimality condition \( (2.7) \) and the fact that \( Y(T) = HX(T) + \tilde{H}\tilde{X}(T) \) becomes

\[ (2.9) \quad 0 = \mathbb{E} \int_t^T \langle B_T(s)Y(s) + \tilde{B}_T(s)\tilde{Y}(s) + D_T(s)Y(s) + \tilde{D}_T(s)\tilde{Y}(s) + G_T(s)Z_0(s) + \tilde{G}_T(s)\tilde{Z}_0(s), \tilde{v}(s) \rangle ds \]
\[ + \mathbb{E} \int_t^T \langle R(s)v(s) + \tilde{R}(s)\tilde{v}(s) + S(s)X(s) + \tilde{S}(s)\tilde{X}(s) + r(s) + \tilde{r}(s), \tilde{v}(s) \rangle ds. \]

Since \( \tilde{v} \) is arbitrary, we obtain the coupling condition \( (2.5) \), as desired.

To prove the converse, it suffices to note that our assumptions imply that \( J_{x,t}^{LQ}(\cdot) \) is strictly convex. Then, given a solution \((X(s), v(s), Y(s), Z(s), Z_0(s))\) to the system \( (2.1), (2.4) \), we know from \( (2.6) \) that the Gâteaux derivative of \( J_{x,t}^{LQ} \) at \( v \) is zero, which implies \( v \) is the minimizer, as desired. \( \square \)
2.2. **Riccati equations.** In order to find explicit solutions of the mean field type control problem, we derive a system of Riccati equations. We use a technique developed by Yong in [60]. We suppose

\[ Y(s) = P(s)(X(s) - \bar{X}(s)) + \Pi(s)\bar{X}(s) + \phi(s) \]

where \( P \) and \( \Pi \) are \( S^d \)-valued processes such that

\[ P(T) = H, \quad \Pi(T) = H + \bar{H}. \]

and \( \phi(s) \) is an \( \mathbb{R}^d \)-valued process. Note that \( P, \Pi, \) and \( \phi \) are deterministic. Our goal is to derive a system of ordinary differential equations governing their evolution (backwards) in time.

By taking conditional expectation we have

\[ \bar{Y}(s) = \Pi(s)\bar{X}(s) + \phi(s) \quad \text{and} \quad Y(s) - \bar{Y}(s) = P(s)(X(s) - \bar{X}(s)). \]

Now recall that

(2.10) \[ d\bar{X} = \{(A + \bar{A})\bar{X} + (B + \bar{B})\bar{v}\} ds + \{(F + \bar{F})\bar{X} + (G + \bar{G})\bar{v}\} dW_0, \]

which implies

(2.11) \[ d(X - \bar{X}) = \{A(X - \bar{X}) + B(v - \bar{v})\} ds \]
\[ + \{C(X - \bar{X}) + (C + \bar{C})X + D(v - \bar{v}) + (D + \bar{D})\bar{v}\} dW + \{F(X - \bar{X}) + G(v - \bar{v})\} dW_0. \]

Now recall that

(2.12) \[ dY = -\left( A^T Y + \bar{A}^T \bar{Y} + C^T Z + \bar{C}^T \bar{Z} + F^T \bar{Z}_0 + \bar{F}^T \bar{Z}_0 + QX + \bar{Q}\bar{X} \right. \]
\[ + S^T v + \bar{S}^T \bar{v} + q + \bar{q} \right) ds + ZdW + Z_0dW_0 \]
\[ = -\left( A^T (Y - \bar{Y}) + (A^T + \bar{A}^T)\bar{Y} + C^T (Z - \bar{Z}) + (C^T + \bar{C}^T)\bar{Z} + F^T (Z_0 - \bar{Z}_0) + (F^T + \bar{F}^T)\bar{Z}_0 \right. \]
\[ + Q(X - \bar{X}) + (Q + \bar{Q})\bar{X} + S^T v + \bar{S}^T \bar{v} + q + \bar{q} \right) ds + ZdW + Z_0dW_0. \]

On the other hand,

(2.13) \[ d(Y - \bar{Y}) = \dot{P}(X - \bar{X})ds + Pd(X - \bar{X}) \]
\[ = \left\{ \dot{P}(X - \bar{X}) + PA(X - \bar{X}) + PB(v - \bar{v}) \right\} ds \]
\[ + P\left\{ C(X - \bar{X}) + (C + \bar{C})\bar{X} + D(v - \bar{v}) + (D + \bar{D})\bar{v} \right\} dW + P \{ F(X - \bar{X}) + G(v - \bar{v}) \} dW_0. \]

while

(2.14) \[ d\bar{Y} = (\phi + \bar{\Pi}\bar{X})ds + \Pi d\bar{X} \]
\[ = \left\{ \phi + \bar{\Pi}\bar{X} + \Pi(A + \bar{A})\bar{X} + \Pi(B + \bar{B})\bar{v} \right\} ds + \Pi \left\{ (F + \bar{F})\bar{X} + (G + \bar{G})\bar{v} \right\} dW_0. \]

Note that \( dY = d(Y - \bar{Y}) + d\bar{Y} \). By comparing the diffusion terms, we get

(2.15) \[ Z = P \left\{ C(X - \bar{X}) + (C + \bar{C})\bar{X} + D(v - \bar{v}) + (D + \bar{D})\bar{v} \right\} \]

and

(2.16) \[ Z_0 = P \left\{ F(X - \bar{X}) + G(v - \bar{v}) \right\} + \Pi \left\{ (F + \bar{F})\bar{X} + (G + \bar{G})\bar{v} \right\} \]
which imply
\begin{align}
(2.17) & \quad \bar{Z} = P \{(C + \bar{C})\bar{X} + (D + \bar{D})\bar{v}\}, \\
(2.18) & \quad Z - \bar{Z} = P \{C(X - \bar{X}) + D(v - \bar{v})\}, \\
(2.19) & \quad \bar{Z}_0 = \Pi \{(F + \bar{F})\bar{X} + (G + \bar{G})\bar{v}\},
\end{align}
and
\begin{align}
(2.20) & \quad Z_0 - \bar{Z}_0 = P \{F(X - \bar{X}) + G(v - \bar{v})\}.
\end{align}

Next we use the coupling condition (2.5) to find a formula for \(v(s)\). We have
\begin{align}
(2.21) & \quad 0 = B^T(Y - \bar{Y}) + (B^T + \bar{B}^T)\bar{Y} + D^T(Z - \bar{Z}) + (D^T + \bar{D}^T)\bar{Z} + G^T(Z_0 - \bar{Z}_0) + (G^T + \bar{G}^T)\bar{Z}_0 \\
& \quad \quad + R(v - \bar{v}) + (R + \bar{R})\bar{v} + S(X - \bar{X}) + (S + \bar{S})\bar{X} + r + \bar{r} \\
& \quad = B^T P(X - \bar{X}) + (B^T + \bar{B}^T)\Pi \bar{X} + (B^T + \bar{B}^T)\phi + D^T P \{C(X - \bar{X}) + D(v - \bar{v})\} \\
& \quad \quad + (D^T + \bar{D}^T) P \{(C + \bar{C})\bar{X} + (D + \bar{D})\bar{v}\} + G^T P \{F(X - \bar{X}) + G(v - \bar{v})\} \\
& \quad = \Lambda_0(X - \bar{X}) + \Lambda_1 \bar{X} + \Sigma_0(v - \bar{v}) + \Sigma_1 \bar{v} + (B^T + \bar{B}^T)\phi + r + \bar{r}
\end{align}
with
\begin{align*}
\Lambda_0 & = B^T P + D^T PC + G^T PF + S, \\
\Lambda_1 & = (B^T + \bar{B}^T)\Pi + (D^T + \bar{D}^T) P(C + \bar{C}) + (G^T + \bar{G}^T)\Pi(F + \bar{F}) + S + \bar{S}, \\
\Sigma_0 & = D^T PD + R, \\
\Sigma_1 & = (D^T + \bar{D}^T) P(D + \bar{D}) + (G^T + \bar{G}^T)\Pi(G + \bar{G}) + (R + \bar{R}).
\end{align*}

Taking conditional expectation, we deduce
\begin{align}
(2.22) & \quad \Sigma_1(s)\bar{v}(s) + \Lambda_1(s)\bar{X}(s) + r(s) + \bar{r}(s) + (B^T(s) + \bar{B}^T(s))\phi(s) = 0
\end{align}
so that, assuming \(\Sigma_1(s)\) is invertible,
\begin{align}
(2.23) & \quad \bar{v}(s) = -\Sigma_1(s)^{-1}(\Lambda_1(s)\bar{X}(s) + r(s) + \bar{r}(s) + (B^T(s) + \bar{B}^T(s))\phi(s)).
\end{align}

Assuming \(\Sigma_0(s)\) is also invertible, we therefore have
\begin{align}
(2.24) & \quad v(s) = v(s) - \bar{v}(s) + \bar{v}(s) \\
& \quad = -\Sigma_0(s)^{-1}(\Lambda_0(s)(X(s) - \bar{X}(s)) + \Lambda_1(s)\bar{X}(s) + \Sigma_1(s)\bar{v}(s) + r(s) + \bar{r}(s) + (B^T(s) + \bar{B}^T(s))\phi(s)) + \bar{v}(s) \\
& \quad = -\Sigma_0(s)^{-1}\Lambda_0(s)(X(s) - \bar{X}(s)) - \Sigma_1(s)^{-1}(\Lambda_1(s)\bar{X}(s) + r(s) + \bar{r}(s) + (B^T(s) + \bar{B}^T(s))\phi(s)).
\end{align}
Now we compare the drift terms from (2.12) to those of (2.13) and (2.14). Using the relations (2.17), (2.18), (2.19), (2.20), (2.23), and (2.24) proved above, we get

\begin{equation}
0 = \left( \dot{P} + A^T P + PA + C^T PC + F^T PF + Q - (PB + C^T PD + F^T PG + S^T)\Sigma_0^{-1} \Lambda_0 \right)(X - \bar{X}) \\
+ \left( \dot{\Pi} + (A^T + \bar{A})\Pi + \Pi(A + \bar{A}) + (C^T + \bar{C}^T)\Pi(C + \bar{C}) + (F^T + \bar{F}^T)\Pi(F + \bar{F}) + Q + \bar{Q} \\
- (\Pi(B + \bar{B}) + (C^T + \bar{C}^T)P(D + \bar{D}) + (F^T + \bar{F}^T)\Pi(G + \bar{G}) + S^T + \bar{S}^T)\Sigma_1^{-1} \Lambda_1 \right)\dot{X} \\
+ \dot{\phi} - (\Pi(B + \bar{B}) + (C^T + \bar{C}^T)P(D + \bar{D}) + (F^T + \bar{F}^T)\Pi(G + \bar{G}) + S^T + \bar{S}^T)\Sigma_1^{-1}(r + \bar{r}) + q + \bar{q}.
\end{equation}

We deduce that \( P \) and \( \Pi \) should satisfy the following Riccati equations:

\begin{align}
\dot{P} + A^T P + PA + C^T PC + F^T PF + Q \\
- (PB + C^T PD + F^T PG + S^T)(D^T PD + R)^{-1}(B^T P + D^T PC + G^T PF + S) = 0,
\end{align}

and

\begin{align}
\dot{\Pi} + (A^T + \bar{A})\Pi + \Pi(A + \bar{A}) + (C^T + \bar{C}^T)\Pi(C + \bar{C}) + (F^T + \bar{F}^T)\Pi(F + \bar{F}) + (Q + \bar{Q}) \\
- (\Pi(B + \bar{B}) + (C^T + \bar{C}^T)P(D + \bar{D}) + (F^T + \bar{F}^T)\Pi(G + \bar{G}) + S^T + \bar{S}^T)\Sigma_1^{-1} \\
\cdot \left( (B + \bar{B})^T \Pi + (D + \bar{D})^T P(C + \bar{C}) + (G + \bar{G})^T \Pi(F + \bar{F}) + S + \bar{S} \right) = 0,
\end{align}

\begin{align}
\Pi(T) &= H + \bar{H}.
\end{align}

Once we have \( P, \Pi \) solutions to (2.26), (2.27), respectively, we set

\begin{align}
\phi(s) &= \int_t^s \left\{ (\Pi(B + \bar{B}) + (C^T + \bar{C}^T)P(D + \bar{D}) + (F^T + \bar{F}^T)\Pi(G + \bar{G}) + S^T + \bar{S}^T)\Sigma_1^{-1}(r + \bar{r}) + q + \bar{q} \right\} d\tau.
\end{align}

A standard reference on optimal control, e.g. [61], suffices to show that under the given assumptions (2.26) has a unique solution, which in addition is symmetric. To see that (2.27) has a unique solution, we note that \((C^T + \bar{C}^T)P(C + \bar{C})\) is also a symmetric matrix, and therefore by the same reference we can deduce there exists a unique solution of (2.27), which is also symmetric.
We can deduce the dynamics of the optimal trajectory using (2.24).

\[ dX = \left\{ A(X - \bar{X}) + (A + \bar{A})\bar{X} + B(v - \bar{v}) + (B + \bar{B})\bar{v} \right\} ds \]
\[ + \left\{ C(X - \bar{X}) + (C + \bar{C})\bar{X} + D(v - \bar{v}) + (D + \bar{D})\bar{v} \right\} dW \]
\[ + \left\{ F(X - \bar{X}) + (F + \bar{F})\bar{X} + G(v - \bar{v}) + (G + \bar{G})\bar{v} \right\} dW_0 \]
\[ = \left\{ (A - B\Sigma_0^{-1}\Lambda_0)(X - \bar{X}) + (A + \bar{A} - (B + \bar{B})\Sigma_1^{-1}\Lambda_1)\bar{X} - (B + \bar{B})\Sigma_1^{-1}(r + \bar{r}) \right\} ds \]
\[ + \left\{ (C - D\Sigma_0^{-1}\Lambda_0)(X - \bar{X}) + (C + \bar{C} - (D + \bar{D})\Sigma_1^{-1}\Lambda_1)\bar{X} - (D + \bar{D})\Sigma_1^{-1}(r + \bar{r}) \right\} dW \]
\[ + \left\{ (F - G\Sigma_0^{-1}\Lambda_0)(X - \bar{X}) + (F + \bar{F} - (G + \bar{G})\Sigma_1^{-1}\Lambda_1)\bar{X} - (G + \bar{G})\Sigma_1^{-1}(r + \bar{r}) \right\} dW_0. \]

A formula for the process \( Z \) can also be deduced from (2.18), (2.17), and (2.24):

\[ Z = Z - \bar{Z} + \bar{Z} = P \left\{ C - D\Sigma_0^{-1}\Lambda_0 \right\} (X - \bar{X}) \]
\[ + P \left\{ (C + \bar{C} - (D + \bar{D})\Sigma_1^{-1}\Lambda_1)\bar{X} - (D + \bar{D})\Sigma_1^{-1}(r + \bar{r}) \right\}, \]
and for \( Z_0 \) we get the following by using (2.20) and (2.19):

\[ Z_0 = Z - \bar{Z} + \bar{Z}_0 = P \left\{ F - G\Sigma_0^{-1}\Lambda_0 \right\} (X - \bar{X}) \]
\[ + \Pi \left\{ (F + \bar{F} - (G + \bar{G})\Sigma_1^{-1}\Lambda_1)\bar{X} - \Pi(G + \bar{G})\Sigma_1^{-1}(r + \bar{r}) \right\}. \]

We summarize our results here:

**Theorem 2.5.** There exists a unique solution \( P, \Pi \) to the pair of Riccati equations (2.26) and (2.27), where \( P, \Pi \) are both \( S^d \)-valued deterministic processes. Moreover, the unique optimal trajectory for Problem (2.3) is given by the solution to the SDE

\[ dX = \left\{ (A - B\Sigma_0^{-1}\Lambda_0)(X - \bar{X}) + (A + \bar{A} - (B + \bar{B})\Sigma_1^{-1}\Lambda_1)\bar{X} - (B + \bar{B})\Sigma_1^{-1}(r + \bar{r}) \right\} ds \]
\[ + \left\{ (C - D\Sigma_0^{-1}\Lambda_0)(X - \bar{X}) + (C + \bar{C} - (D + \bar{D})\Sigma_1^{-1}\Lambda_1)\bar{X} - (D + \bar{D})\Sigma_1^{-1}(r + \bar{r}) \right\} dW \]
\[ + \left\{ (F - G\Sigma_0^{-1}\Lambda_0)(X - \bar{X}) + (F + \bar{F} - (G + \bar{G})\Sigma_1^{-1}\Lambda_1)\bar{X} - (G + \bar{G})\Sigma_1^{-1}(r + \bar{r}) \right\} dW_0. \]

with

\[ \Lambda_0 = B^T P + D^T PC + G^T PF + S, \]
\[ \Lambda_1 = (B^T + \bar{B})\Pi + (D^T + \bar{D})P(C + \bar{C}) + (G^T + \bar{G})\Pi(F + \bar{F}) + S + \bar{S}, \]
\[ \Sigma_0 = D^T PD + R, \]
\[ \Sigma_1 = (D^T + \bar{D})P(D + \bar{D}) + (G^T + \bar{G})\Pi(G + \bar{G}) + (R + \bar{R}). \]

The SDE (2.31) has a unique solution. The optimal control is given by

\[ v(s) = -\Sigma_0(s)^{-1}\Lambda_0(s)(X(s) - \bar{X}(s)) - \Sigma_1(s)^{-1}(\Lambda_1(s)\bar{X}(s)) + r(s) + \bar{r}(s) + (B^T(s) + \bar{B}^T(s))\phi(s), \]
If we define the adjoint processes

\[
Y(s) = P(s)(X(s) - \hat{X}(s)) + \Pi(s)\hat{X}(s) + \phi(s),
\]

\[
Z(s) = P\left\{ C - DS_0^{-1}\Lambda_0 \right\} (X - \hat{X}) + P\left\{ (C + \hat{C}) - (D + \hat{D})\Sigma_1^{-1}\Lambda_1 \right\} \hat{X} - P(D + \hat{D})\Sigma_1^{-1}(r + \hat{r}),
\]

\[
Z_0(s) = P\left\{ F - G\Sigma_0^{-1}\Lambda_0 \right\} (X - \hat{X}) + \Pi\left\{ (F + \hat{F}) - (G + \hat{G})\Sigma_1^{-1}\Lambda_1 \right\} \hat{X} - \Pi(G + \hat{G})\Sigma_1^{-1}(r + \hat{r})
\]

where

\[
\phi(s) = \int_{t}^{s}\left\{ (\Pi(B + \hat{B}) + (C^T + \hat{C}^T)P(D + \hat{D}) + (F^T + \hat{F}^T)\Pi(G + \hat{G}) + S^T + \hat{S}^T)\Sigma_1^{-1}(r + \hat{r}) + q + q' \right\} dt,
\]

then the quintuple \((X, v, Y, Z, Z_0)\) is an adapted solution to the mean field FBSDE (3.1), (2.4).

3. The Mean Field Game

In this section, we consider the problem of Nash equilibrium rather than optimal control. We modify the dynamics (2.1) as follows. Let \(\xi(s)\) and \(\psi(s)\) be given processes adapted to the filtration \(\mathcal{F}^0_s\). Consider

\[
(3.1) \quad dX(s) = \left\{ A(s)X(s) + \hat{A}(s)\xi(s) + B(s)v(s) + \hat{B}(s)\hat{\psi}(s) \right\} ds
\]

\[
+ \left\{ C(s)X(s) + \hat{C}(s)\xi(s) + D(s)v(s) + \hat{D}(s)\hat{\psi}(s) \right\} dW(s)
\]

\[
+ \left\{ F(s)X(s) + \hat{F}(s)\xi(s) + G(s)v(s) + G(s)\hat{\psi}(s) \right\} dW_0(s), \quad X(t) = x
\]

and the objective functional

\[
(3.2) \quad J^{mf}_{x,t}(v; \xi, \psi) = \mathbb{E}\left\{ \int_{t}^{T}\left[ (Q(s)X(s), X(s)) + 2\langle \hat{Q}(s)\xi(s), X(s) \rangle + \langle R(s)v(s), v(s) \rangle + 2\langle \hat{R}(s)\hat{\psi}(s), v(s) \rangle
\right.
\]

\[
+ 2\langle S(s)X(s), v(s) \rangle + 2\langle \hat{S}_1(s)\xi(s), v(s) \rangle + 2\langle \hat{S}_2(s)X(s), \psi(s) \rangle
\]

\[
+ 2\langle q(s), X(s) \rangle + 2\langle \hat{q}(s), \xi(s) \rangle + 2\langle r(s), v(s) \rangle + 2\langle \hat{r}(s), \hat{\psi}(s) \rangle \right] ds + \langle HX(T), X(T) \rangle + 2\langle \hat{H}\xi(T), X(T) \rangle \}
\]

The goal is to find a process \(\hat{\psi}(s)\) such that, given the process \(\hat{X}(s)\) generated by \(\hat{\psi}\), we have

\[
(3.3) \quad J^{NF}_{x,t}(\hat{v}; \xi, \psi) = \inf_{v} J^{NF}_{x,t}(v; \xi, \psi) \quad \text{and} \quad \mathbb{E}[\hat{X}(s)|\mathcal{F}^0_s] = \hat{\xi}, \quad \mathbb{E}[\hat{\psi}(s)|\mathcal{F}^0_s] = \hat{\psi}.
\]

Such a process \(\hat{\psi}\) is called a mean field Nash equilibrium. Note that, rather than an optimizer, we are seeking a fixed point of the map

\[
v \mapsto (X, v) \mapsto (\xi, \psi) \mapsto \hat{\psi},
\]

where any given control \(v\) generates a state process \(X, (X, v)\) generate processes \((\xi, \psi)\) by taking conditional expectation with respect to the common noise, and \(\hat{v}\) is an optimal control with respect to these given processes.

We make similar assumptions as before:

**Assumption 3.1.** The following are the assumption on the coefficient matrices for the mean field game:

1. \(A, \hat{A}, C, \hat{C}, F, \hat{F} \in L^{\infty}(0, T; \mathbb{R}^{d\times d})\)
2. \(B, \hat{B}, D, \hat{D}, G, \hat{G} \in L^{\infty}(0, T; \mathbb{R}^{d\times m})\)
3. \(Q, \hat{Q} \in L^{\infty}(0, T; \mathbb{S}^d), R, \hat{R} \in L^{\infty}(0, T; \mathbb{S}^m), H, \hat{H} \in \mathbb{S}^d\)
Let us first see that, for given processes \( \bar{\xi}, \bar{\psi} \), the Nash equilibrium can be computed by finding an optimizer to a mean field type control problem. Following subsections, we will focus more on the case which is of most interest to us, namely when the first part of this section, we will give a brief discussion of where the difficulty lies. In the following subsections, we will focus more on the case which is of most interest to us, namely when the Nash equilibrium can be computed by finding an optimizer to a mean field control problem. Let us first see that, for given processes \( \bar{\xi}, \bar{\psi} \), there is indeed an optimal control \( \bar{\nu} \). Using the same arguments as in the previous section, we obtain the following characterization.

**Proposition 3.2.** For a given pair \((\bar{\xi}, \bar{\psi})\), there exists a unique optimal control \( v \) minimizing the functional \( J_{XE}^{NE}(v; \bar{\xi}, \bar{\psi}) \). Furthermore, let \( X(s) \) be the corresponding trajectory (the solution of \( (3.1) \) with control \( v(s) \)). Then there exists a unique adapted solution \( (Y, Z, Z_0) \) of the BSDE

\[
\begin{align*}
 dY(s) &= -\left( A^T(s)Y(s) + C^T(s)Z(s) + F^T(s)Z_0(s) + Q(s)X(s) + \bar{Q}(s)\bar{\xi}(s) + S^T(s)v(s) + \bar{S}_2^T(s)\bar{\psi}(s) + q(s) \right)ds \\
 &\quad + Z(s)dW(s) + Z_0(s)dW_0(s), \quad s \in [0, T], \\
 Y(T) &= HX(T) + \bar{H}\bar{\xi}(T)
\end{align*}
\]

satisfying the coupling condition

\[
B^T(s)Y(s) + D^T(s)Z(s) + G^T(s)Z_0(s) + R(s)v(s) + \bar{R}(s)\bar{\psi}(s) + S(s)X(s) + \bar{S}_1(s)\bar{\xi}(s) + r(s) = 0, \quad s \in [0, T], \text{ a.s.}
\]

Conversely, suppose \((X, v, Y, Z)\) is an adapted solution to the forward-backward system \( (3.1), (3.4) \) and coupling condition \( (3.5) \). Then \( v \) is the optimal control minimizing \( J_{XE}^{NE}(v; \bar{\xi}, \bar{\psi}) \), and \( X(s) \) is the optimal trajectory. If, in addition, we have \( \mathbb{E}[X(s)|\mathcal{F}_s^0] = \bar{\xi}(s) \) and \( \mathbb{E}[v(s)|\mathcal{F}_s^0] = \bar{\psi}(s) \) then \( v(s) \) is a mean field Nash equilibrium.

Proposition 3.2 can be seen as an abstract condition for the solvability of the fixed point problem. However, one would like to have a more concrete criterion giving existence of a mean field Nash equilibrium. Let us attempt to follow the discussion in [10] and find out where the difficulty lies.

Suppose there exists a mean field Nash equilibrium \( v(s) \). Then \( \bar{X}(s) = \mathbb{E}[X(s)|\mathcal{F}_s^0] \) satisfies the dynamics

\[
(3.6) \quad d\bar{X} = \{(A + A)\bar{X} + (B + B)^{\bar{\nu}}\}ds + \{(F + F)\bar{X} + (G + G)^{\bar{\nu}}\}dW_0, \quad \bar{X}(t) = X
\]

while \( Y \) satisfies

\[
(3.7) \quad d\bar{Y} = -\left( A^T\bar{Y} + C^T\bar{Z} + F^T\bar{Z}_0 + (Q + Q)\bar{X} + (S^T + \bar{S}_2^T)\bar{\psi} + q \right)ds + \bar{Z}_0dW_0, \quad \bar{Y}(T) = (H + \bar{H})\bar{\xi}(T)
\]

and we have the coupling condition

\[
(3.8) \quad B^T\bar{Y} + D^T\bar{Z} + G^T\bar{Z}_0 + (R + \bar{R})\bar{\psi} + (S + \bar{S}_1)\bar{X} + r = 0.
\]
Conversely, suppose that the system \((3.6), (3.7), (3.8)\) has a solution which we denote \((\hat{\xi}, \hat{\eta}, \hat{\psi})\) (corresponding to \((\hat{X}, \hat{Y}, \hat{v})\)). Then we let \(v\) be the optimal control minimizing \(J_{X,t}^{NE}(v; \hat{\xi}, \hat{\psi})\), and let \(X(s)\) be the state solving the dynamics \((3.1)\) and \(Y(s)\) the adjoint state solving \((3.4)\). Note that, by Proposition 3.4 the coupling condition \((3.5)\) is satisfied. Now, if we knew that \((X, Y, v)\) is also a solution to the system \((3.6), (3.7), (3.8)\), and it would not be difficult to see that therefore \(X = \xi\) and \(Y = \eta\) as well. Thus we would have that \(v\) is a Nash equilibrium. However, this is a nontrivial criterion on \(v\), since there is nothing which obviously connects the control problem of minimizing \((3.2)\) with the system \((3.6), (3.7), (3.8)\). So this fails to be an appropriate criteria for determining the existence of Nash equilibrium.

On the other hand, if we take \(\tilde{v}\) out of the problem, i.e. if we set \(\tilde{B} = \tilde{D} = \tilde{G} = \tilde{R} = \tilde{S}_2 = 0\), then we obtain, as in [10], a necessary and sufficient condition for the existence of Nash equilibrium. This is summarized in the following proposition, whose proof is essentially the same as that of [10 Theorem III.4].

**Proposition 3.3.** Let \(\tilde{B} = \tilde{D} = \tilde{G} = \tilde{R} = \tilde{S}_2 = 0\). Then there exists a mean-field Nash equilibrium for the objective functional \(J_{X,t}^{NE}\), given in \((3.2)\), if and only if there exists a solution to the forward-backward system of stochastic differential equations given by \((3.6), (3.7), (3.8)\).

There is another condition on the coefficients, given in Section 3.1 below, which permits us to assert that there always exists a unique mean field Nash equilibrium, namely when the mean field game corresponds to an optimal control problem. While this is not the most general case, it is directly applicable to the economics example which motivated this work. Moreover, as we will see, we need not give up all dependence on the conditional expectation of the control variable.

### 3.1. When is a Mean Field Game equivalent to a Mean Field Type Control Problem?

It is now well known that, as pointed out in [47], mean field Nash equilibria can be characterized—at least formally—as optimality conditions for mean field type control problems. Often this is exploited to obtain results on existence and uniqueness of solutions to mean field games. See, for instance, [15, 30, 17, 18, 7]. Here we point out a condition under which the Mean Field Type Control Problem and the Mean Field Game are equivalent for the general linear-quadratic case.

**Proposition 3.4.** Let \(\hat{A}, \hat{B}, \hat{C}, \hat{D}, \hat{F}, \hat{G}, \hat{q}, \) and \(\hat{r}\) all be zero. Additionally, assume \(\hat{S} = \hat{S}_1 = \hat{S}_2\).

Suppose \(\hat{v}(s)\) is the optimal control for the linear-quadratic functional \(J_{X,t}^{LQ}(v)\) defined by \((2.1)\), with corresponding optimal trajectory \(\hat{X}(s)\) defined by \((2.1)\). Define \(\hat{\xi}(s) := \mathbb{E}[X(s)|\mathcal{F}_s^0]\) and \(\hat{\psi}(s) := \mathbb{E}[\hat{v}(s)|\mathcal{F}_s^0]\). Then \(\hat{v}(s)\) is a mean field Nash equilibrium for \(J_{X,t}^{NE}(v; \hat{\xi}, \hat{\psi})\) defined by \((3.2)\).

Conversely, if \(\hat{v}(s)\) is a mean field Nash equilibrium for \(J_{X,t}^{NE}(v; \hat{\xi}, \hat{\psi})\), then it is also an optimal control for \(J_{X,t}^{LQ}(v)\).

**Proof.** It suffices to note that, under the given assumptions, the (forward-backward) systems of stochastic differential equations and their coupling conditions given by Propositions 2.1 and 3.2 are equivalent, once we have taken into account the equilibrium condition \(\hat{\xi}(s) = \mathbb{E}[\hat{X}(s)|\mathcal{F}_s^0]\) and \(\hat{\psi}(s) = \mathbb{E}[\hat{v}(s)|\mathcal{F}_s^0]\). \(\square\)

We note the following simple corollary of Proposition 3.4.

**Corollary 3.5.** Let \(\hat{A}, \hat{B}, \hat{C}, \hat{D}, \hat{F}, \hat{G}, \hat{q}, \) and \(\hat{r}\) all be zero. Additionally, assume \(\hat{S} = \hat{S}_1 = \hat{S}_2\).
Let $\xi, \bar{\psi}$ be given $\mathcal{F}_s^0$-adapted processes with values in $\mathbb{R}^d, \mathbb{R}^m$ respectively. Then the linear-quadratic functional $J^NE_{X,t}(v; \xi, \bar{\psi})$ defined by (3.2) has a mean field Nash equilibrium if and only if $\xi(s) = \mathbb{E}[\bar{X}(s)|\mathcal{F}_s^0]$ and $\bar{\psi}(s) = \mathbb{E}[\bar{v}(s)|\mathcal{F}_s^0]$, where $\bar{v}$ is the optimal control of $J^{LQ}_{X,t}(v)$ with corresponding optimal trajectory $\bar{X}(s)$ defined by (2.1). Moreover, this equilibrium is unique.

Note that the objective functionals $J^{LQ}_{X,t}$ and $J^NE_{X,t}$ given in (2.2) and (3.2), respectively, are not precisely the same under the assumptions of Proposition 3.4. In other words, even though $\hat{v}$ solves both a fixed point Nash equilibrium and an mean field type control problem, the costs are different. Indeed, suppose the assumptions of Proposition 3.4 hold, and take $\hat{v}$ to be a mean field Nash equilibrium with corresponding trajectory $\hat{X}$. Define $\bar{\xi}(s) := \mathbb{E}[\hat{X}(s)|\mathcal{F}_s^0]$ and $\bar{\psi}(s) := \mathbb{E}[\hat{v}(s)|\mathcal{F}_s^0]$. By the proposition, $\hat{v}$ is the minimizer of both $J^NE_{X,t}(v; \xi, \bar{\psi})$ and $J^{LQ}_{X,t}(v)$. Now observe that

\begin{equation}
J^NE_{X,t}(\hat{v}; \xi, \bar{\psi}) - J^{LQ}_{x,t}(\hat{v}) = \mathbb{E} \left\{ \int_0^T \left[ \langle Q(s)\hat{X}(s), \hat{X}(s) \rangle + \langle R(s)\hat{v}(s), \hat{v}(s) \rangle + \langle S(s)\hat{X}(s), \hat{v}(s) \rangle \right] ds + \langle H\hat{X}(T), \hat{X}(T) \rangle \right\}.
\end{equation}

We call this difference the “price of anarchy,” since it is the added aggregate cost of allowing all players to independently choose their optimal strategy.

3.2. $\epsilon$-Nash equilibrium of an $N$-player game. In this section we discuss the relationship between the mean field game given in the previous section and the analogous $N$-player game, which can be formulated as follows. First, we specify that $W_1, \ldots, W_N$ are $N$ independent Wiener processes, and $X_1, \ldots, X_N$ are $N$ i.i.d. random variables. The state of player $i \in \{1, 2, \ldots, N\}$ is given by the dynamics

\begin{equation}
\begin{aligned}
dX_i(s) &= \left\{ A(s)X_i(s) + \bar{A}(s)\frac{1}{N-1} \sum_{j \neq i} X_j(s) + B(s)v_i(s) + \bar{B}(s)\frac{1}{N-1} \sum_{j \neq i} v_j(s) \right\} ds \\
&\quad + \left\{ C(s)X_i(s) + \bar{C}(s)\frac{1}{N-1} \sum_{j \neq i} X_j(s) + D(s)v_i(s) + \bar{D}(s)\frac{1}{N-1} \sum_{j \neq i} v_j(s) \right\} dW_i(s) \\
&\quad + \left\{ F(s)X_i(s) + \bar{F}(s)\frac{1}{N-1} \sum_{j \neq i} X_j(s) + G(s)v_i(s) + \bar{G}(s)\frac{1}{N-1} \sum_{j \neq i} v_j(s) \right\} dW_0(s), \quad X_i(t) = X_i
\end{aligned}
\end{equation}
where $v_i$ is the control chosen by player $i$. The cost functional for player $i$ is given by

$$J_{X_{i}, t}^{N,i}(v_1, \ldots, v_N) = J_{X_{i}, t}^{N,i}(v_i; \{v_j\}_{j \neq i})$$

$$= E \left\{ \int_t^T \left[ \langle Q(s)X_i(s), X_i(s) \rangle + 2 \frac{1}{N-1} \sum_{j \neq i} \langle Q(s)X_j(s), X_i(s) \rangle \right. \\
+ \langle R(s)v_i(s), v_i(s) \rangle + 2 \frac{1}{N-1} \sum_{j \neq i} \langle R(s)v_j(s), v_i(s) \rangle + 2 \langle S(s)X_i(s), v_i(s) \rangle \\
+ 2 \frac{1}{N-1} \sum_{j \neq i} \langle S_1(s)X_j(s), v_i(s) \rangle + 2 \frac{1}{N-1} \sum_{j \neq i} \langle S_2(s)X_i(s), v_j(s) \rangle \\
+ 2 \langle q(s), X_i(s) \rangle + 2 \frac{1}{N-1} \sum_{j \neq i} \langle q(s), X_j(s) \rangle + 2 \langle r(s), v_i(s) \rangle + 2 \frac{1}{N-1} \sum_{j \neq i} \langle r(s), v_j(s) \rangle \rangle \right] ds \\
+ \langle H X_i(T), X_i(T) \rangle + 2 \frac{1}{N-1} \sum_{j \neq i} \langle H X_j(T), X_i(T) \rangle \right\}.$$

We seek to prove that the mean field equilibrium $\hat{v}$ can be used as an approximate Nash equilibrium for the $N$-player game, in a way that is stated precisely below.

**Definition 3.6.** We say that $\{\hat{v}_i\}_{i=1}^N$ is an “$\epsilon$-Nash equilibrium” for the $N$-player game provided that, for all $i \in \{1, \ldots, N\}$,

$$J_{X_{i}, t}^{N,i}(v_i; \{\hat{v}_j\}_{j \neq i}) \geq J_{X_{i}, t}^{N,i}(\hat{v}_i; \{\hat{v}_j\}_{j \neq i}) - \epsilon$$

for any set of controls $\{v_i\}_{i=1}^N$.

See, for instance, [14, 36, 37, 40]. The following theorem states that the mean field Nash equilibrium is an $\epsilon$-Nash equilibrium for the $N$-player game. However, we are unable to prove it here in the general case. Instead, we restrict our attention to the case where the mean field game is equivalent to an optimal control problem, as in Proposition 3.3.

**Theorem 3.7.** Let $\bar{A}, \bar{B}, \bar{C}, \bar{D}, \bar{F}, \bar{G}, \bar{q},$ and $\bar{r}$ all be zero. Additionally, assume $\bar{S} = \bar{S}_1 = \bar{S}_2$.

Let $v_i^*$ be a mean field Nash equilibrium for (3.12) with $X = X_i$ and $W = W_i$. Then for any $\epsilon > 0$ there exists $N_\epsilon$ large enough such that if $N \geq N_\epsilon$, then $\{v_i^*\}_{i=1}^N$ is an $\epsilon$-Nash equilibrium for the $N$-player game.

**Proof.** To begin with, we write down the dynamics which, given the hypotheses of the theorem, are much simpler than (3.10):

$$dX_i = (AX_i + BV_i)ds + (CX_i + Dw_i)dW_i + (FX_i + Gv_i)dW_0, \quad X_i(t) = x_i$$
Recall that these formulas, we can similarly assert that

Next, we recall that, by Proposition 3.4 and Theorem 2.5, we have a formula for $t_i$, namely (2.32) where $P$ and $\Pi$ are the solutions to the Riccati equations (2.26) (2.27). Based on these formulas, we can similarly assert that

$$\bar{v}_i = \frac{1}{N-1} \sum_{j \neq i} v_j$$

Now, let us establish some notation. Fix $i \in \{1, \ldots, N\}$. We will let $v_i$ be an arbitrary given control. Recall that $v_j^*, j \in \{1, \ldots, N\}$ is the mean field Nash equilibrium for (3.2) with initial condition $X_j$.

- Let $X_j^*(s)$, $j \in \{1, \ldots, N\}$ be the corresponding state as given by the mean field dynamics (3.1) with initial condition $X_j$ and $W = W_j$. Note that under our assumptions, $X_j^*(s)$ is also the solution to the $N$-player game dynamics given by (3.10) with $v_j = v_j^*$.

- Let $X_i(s)$ refer to the solution of the system given by (3.10) with an arbitrary given control $v_i$.

Now, as a first step towards the necessary estimates to prove the theorem, we compare the difference between the mean field cost and that of the $N$-player game.

$$\bar{v}_i = \frac{1}{N-1} \sum_{j \neq i} v_j^*$$

Observe that the processes $X_i^*$ are conditionally i.i.d. Since, in addition, they are continuous in $L^2(\Omega)$, we have, as $N \to \infty$,

$$\bar{X}_i^*(s) - \frac{1}{N-1} \sum_{j \neq i} X_j^*(s) \to 0 \text{ in } L^2(\Omega), \text{ uniformly in } s \in [t, T].$$

Next, we recall that, by Proposition 3.4 and Theorem 2.5, we have a formula for $v_i^*$ in terms of $X_i^*$, namely (2.32) where $P$ and $\Pi$ are the solutions to the Riccati equations (2.26) (2.27). Based on these formulas, we can similarly assert that

$$\bar{v}_i = \frac{1}{N-1} \sum_{j \neq i} v_j^* \to 0 \text{ in } L^2.$$
Combining these with the a priori bounds on \( X_i^* \) and \( v_i^* \) in \( L^2 \) as well as the \( L^\infty \) bounds on the coefficients, we see that, as \( N \to \infty \),
\[
J^{mf}_{x,i,t}(v_i^*; \tilde{X}_i^*, \tilde{v}_i^*) - J^{N,i}_{x,i,t}(v_i^*; \{v_j^*\}_j \neq i) = o(1).
\]
By analogous reasoning, we see that
\[
J^{mf}_{x,i,t}(v_i; \tilde{X}_i^*, \tilde{v}_i^*) - J^{N,i}_{x,i,t}(v_i; \{v_j^*\}_j \neq i) = o(1)
\]
as well. Now by definition of mean field Nash equilibrium, \( v_i^* \) is the optimal control for \( J^{mf}_{x,i,t}(\cdot; \tilde{X}_i^*, \tilde{v}_i^*) \). Therefore, we have
\[
J^{N,i}_{x,i,t}(v_i; \{v_j^*\}_j \neq i) = J^{mf}_{x,i,t}(v_i; \tilde{X}_i^*, \tilde{v}_i^*) + o(1)
\]
\[
\geq J^{mf}_{x,i,t}(v_i^*; \tilde{X}_i^*, \tilde{v}_i^*) + o(1)
\]
\[
= J^{N,i}_{x,i,t}(v_i^*; \{v_j^*\}_j \neq i) + o(1),
\]
which is what we wanted to show. \( \square \)

4. Example from economics: production of an exhaustible resource

In this section we develop and analyze a model of exhaustible resource production, following the work of Guéant, Lasry, and Lions in [35] and of Chan and Sirca r in [26, 27]. As in the general mean field setting, we assume the number of producers of a given resource (oil, for example) is very large. Consider an arbitrary producer. Let \( v(s) \) represent the quantity produced at time \( s \), while \( X(s) \) is the producer’s current level of reserves. Following [35], we assume the dynamics are stochastic, with the noise proportional to the current number of reserves. In addition, we deal with a common noise, modeling uncertainty inherent in nature itself, rather than in the measurements of individual producers. We have

\[
dX(s) = -v(s)ds + \nu X(s)dW(s) + \nu_0 X(s)dW_0(s), \quad X(t) = x.
\]

The goal of each individual producer is the maximization of profit. We model the market competition as a Nash equilibrium. Define \( k(s) \) to be the price at which a producer can sell, and define \( \bar{k}(s) \) to be the market price. To simplify the analysis and allow the model to fall under our linear-quadratic framework, we follow [26] and consider a linear demand schedule

\[
v(s) = \gamma + \delta \bar{k}(s) - k(s)
\]
for given parameters \( \gamma, \delta \). In [26] they are given by

\[
\gamma = \frac{1}{1 + \epsilon}, \quad \delta = \frac{\epsilon}{1 + \epsilon}
\]

for a parameter \( \epsilon \geq 0 \) which measures the degree of competition (\( \epsilon = 0 \) corresponds to monopoly, as the market price is unseen by consumers, whereas \( \epsilon = +\infty \) corresponds to perfect competition, as the market price has exactly the same weight as the price offered by each individual firm). The revenue maximization problem can now be stated as

\[
\sup_v \mathbb{E} \left[ \int_t^T e^{-\mu(s-t)} v(s)k(s)ds - e^{-\mu(T-t)}|X(T)|^2 \right].
\]
To make this into a linear-quadratic functional of the form [3.2], we must first compute \( k(s) \) in terms of \( v(s) \) and \( \psi(s) \), where we recall that \( \psi(s) = \mathbb{E}[\psi(s)|\mathcal{F}_s^0] \) is the conditional expectation of the optimal control. To do this, it will first be necessary to find a formula for the market price.
Let \( \hat{k}(s) \) be the price corresponding to the optimal quantity \( \hat{v}(s) \). In equilibrium, the market price is precisely the (conditional) expected value of \( \hat{k}(s) \), so that
\[
\hat{k}(s) = E[\hat{k}(s)|\mathcal{F}_s^0] = E[\gamma + \delta \hat{k}(s) - \hat{v}(s)|\mathcal{F}_s^0] = \gamma + \delta \hat{k}(s) - \hat{v}(s) \quad \Rightarrow \quad \hat{k}(s) = \frac{\gamma}{1-\delta} - \frac{1}{1-\delta} \hat{v}(s).
\]
Hence from (4.2) we get
\[
k(s) = \frac{\gamma}{1-\delta} - \frac{\delta}{1-\delta} \hat{v}(s) - v(s).
\]
Then (4.4) becomes
\[
\inf_v \mathbb{E}\left[ \int_t^T e^{-\mu(s-t)} (v^2(s) + 2\beta v(s) \hat{v}(s) - 2\alpha v(s)) ds + e^{-\mu(T-t)} |X(T)|^2 \right]
\]
where we have set
\[
\alpha := \frac{\gamma}{2(1-\delta)}, \quad \beta := \frac{\delta}{2(1-\delta)}.
\]
The dynamics and objective functional above are a much simplified form of (3.1), with
\[
A = \bar{A} = \bar{B} = \bar{C} = D = \bar{D} = F = G = \bar{G} = 0, B = -1, C = \nu, F = \nu_0.
\]
Likewise in the cost functional (3.2) we have
\[
R(s) = e^{-\mu(s-t)}, \quad \tilde{R}(s) = \beta e^{-\mu(s-t)}, \quad r(s) = -\alpha e^{-\mu(s-t)}, \quad H = e^{-\mu(T-t)}.
\]
Moreover, this setup satisfies the assumptions of Proposition 3.4, so the Nash equilibrium \( \hat{v} \) quantity is in fact an optimizer for a mean field type control problem. The corresponding objective functional is
\[
J^{LQ}_{x,t}(v(\cdot)) = \mathbb{E}\left\{ \int_t^T e^{-\mu(s-t)} \left( v^2(s) + \beta (\mathbb{E}[v(s)|\mathcal{F}_s^0])^2 - 2\alpha v(s) \right) ds + e^{-\mu(T-t)} \mathbb{E}[|X(T)|^2] \right\},
\]
and the price of anarchy is given by
\[
J^{mfg}_{X,t}(\hat{v}(\cdot)) - J^{LQ}_{X,t}(\hat{v}(\cdot)) = \beta \mathbb{E}\int_t^T e^{-\mu(s-t)} (\mathbb{E}[v(s)|\mathcal{F}_s^0])^2 ds.
\]
If we take, as in (26), the formula \( \delta = \epsilon/(1+\epsilon) \), we get simply \( \alpha = 1/2 \) and \( \beta = \epsilon/2 \); so we see that the price of anarchy is directly proportional to the competition coefficient \( \epsilon \):
\[
J^{mfg}_{X,t}(\hat{v}(\cdot)) - J^{LQ}_{X,t}(\hat{v}(\cdot)) = \frac{\epsilon}{2} \mathbb{E}\int_t^T e^{-\mu(s-t)} (\mathbb{E}[v(s)|\mathcal{F}_s^0])^2 ds.
\]

4.1. Computation of the equilibrium strategy. By the theory developed in the previous section, we can compute the market equilibrium quite explicitly. The Riccati equations are
\[
\begin{align*}
\dot{p} + (\nu^2 + \nu_0^2)p - e^{\mu(s-t)}p^2 &= 0, \\
p(T) &= e^{-\mu(T-t)}
\end{align*}
\]
and
\[
\begin{align*}
\dot{\pi} + \nu^2 \pi + \nu_0^2 \pi - (1 + \beta)^{-1} e^{\mu(s-t)} \pi^2 &= 0, \\
\pi(T) &= e^{-\mu(T-t)}.
\end{align*}
\]
We can explicitly solve for $p$. If $\lambda := \mu - (\nu^2 + \nu_0^2) \neq 0$, we get
\begin{equation}
(4.10) \quad p(s) = \frac{\lambda e^{-\mu(s-t)}}{(\lambda + 1)e^{\lambda(T-s)} - 1}
\end{equation}
while if $\lambda = 0$ we find
\begin{equation}
(4.11) \quad p(s) = \frac{e^{-\mu(s-t)}}{1 + T - s}.
\end{equation}
Let us also remark that in the special case $\nu = 0$ we can even compute $\pi$ explicitly:
\begin{equation}
(4.12) \quad \pi(s) = \frac{(1 + \beta)\lambda e^{-\mu(s-t)}}{((1 + \beta)\lambda + 1)e^{\lambda(T-s)} - 1} \quad \text{if } \lambda \neq 0, \quad \frac{(1 + \beta)e^{-\mu(s-t)}}{1 + \beta + T - s} \quad \text{if } \lambda = 0.
\end{equation}
For the optimal trajectory, we have, by Equation (2.28),
\begin{equation}
(4.13) \quad d\bar{X}(s) = \left( -\frac{1}{1 + \beta} e^{\mu(s-t)}\pi(s)\bar{X}(s) - \frac{\alpha}{1 + \beta} \right) ds + \nu_0 \bar{X}(s)dW_0(s)
\end{equation}
and
\begin{equation}
(4.14) \quad d(X(s) - \bar{X}(s)) = -e^{\mu(s-t)}p(s)(X - \bar{X}(s))ds + \nu X(s)dW(s) + \nu_0(X - \bar{X}(s))dW_0(s).
\end{equation}
These can be solved explicitly in terms of $\pi$ and $p$. We have
\begin{equation}
(4.15) \quad \bar{X}(s) = e^{\Psi(s)} \left( \mathbb{E}[x | \mathcal{F}_t] - \frac{\alpha}{1 + \beta} \int_t^s e^{-\Psi(\tau)} d\tau \right)
\end{equation}
where
\begin{equation}
(4.16) \quad \Psi(s) := \frac{1}{1 + \beta} \int_t^s e^{\mu(x-\bar{X})}(x - \mathbb{E}[x | \mathcal{F}_t]) - \nu^2 \int_t^s e^{-\Phi(\tau)} \bar{X}(\tau)d\tau + \nu \int_t^s e^{-\Phi(\tau)} \bar{X}(\tau)dW(\tau)
\end{equation}
where
\begin{equation}
(4.17) \quad \bar{X}(s) = e^{\Phi(s)} \left( x - \mathbb{E}[x | \mathcal{F}_t] - \nu \int_t^s e^{-\Phi(\tau)} \bar{X}(\tau)d\tau + \nu \int_t^s e^{-\Phi(\tau)} \bar{X}(\tau)dW(\tau) \right)
\end{equation}
where
\begin{equation}
(4.18) \quad \Phi(s) := \int_t^s e^{\mu(x-\bar{X})}(x - \mathbb{E}[x | \mathcal{F}_t]) + \frac{1}{2}(\nu^2 + \nu_0^2)(s - t) + \nu(W(s) - W(t)) + \nu_0(W_0(s) - W_0(t)).
\end{equation}
Then we have the following for the optimal control, by Equation (2.23):
\begin{equation}
(4.19) \quad v(s) = e^{\mu(s-t)} \left( p(s)(X(s) - \bar{X}(s)) + \frac{1}{1 + \beta} \pi(s)\bar{X}(s) + \frac{\alpha}{(1 + \beta)^2} \int_t^s \pi(\tau)d\tau + \frac{\alpha}{1 + \beta} \right).
\end{equation}

4.2 Market price. From Equation (4.19) and formula (4.12), we see that the market price is given by
\begin{equation}
(4.20) \quad \bar{k}(s) = 2\alpha - (1 + 2\beta)\bar{v}(s) = \frac{\alpha}{1 + \beta} - e^{\mu(s-t)} \left( \frac{1 + 2\beta}{1 + \beta} \pi(s)\bar{X}(s) + \frac{\alpha(1 + 2\beta)}{(1 + \beta)^2} \int_t^s \pi(\tau)d\tau \right)
\end{equation}
which, when using the model of Chan and Sircar [26] so that $\alpha = 1/2$ and $\beta = \epsilon/2$, becomes
\begin{equation}
(4.21) \quad \bar{k}(s) = \frac{1}{2 + \epsilon} - \frac{2 + 2\epsilon}{2 + \epsilon} e^{\mu(s-t)} \left( \pi(s)\bar{X}(s) + \int_t^s \pi(\tau)d\tau \right).
\end{equation}
An interesting question is the behavior of the market price as the competition parameter $\epsilon$ increases. Let us focus on the expected market price:
\begin{equation}
(4.22) \quad \mathbb{E}[\bar{k}(s)] = \frac{1}{2 + \epsilon} - \frac{2 + 2\epsilon}{2 + \epsilon} e^{\mu(s-t)} \left( \pi(s)\bar{X}(s) + \int_t^s \pi(\tau)d\tau \right).
\end{equation}
where we define \( \chi(s) = \mathbb{E}[X(s)] = \mathbb{E}[\bar{X}(s)] \). It is possible to give conditions on the initial data such that the expected value of the state variable remains positive up to time \( T \). To see this, note that

\[
\chi'(s) = -\frac{1}{1 + \beta} e^{\mu(s-t)} \pi(s) \chi(s) - \frac{\alpha}{1 + \beta}, \quad \chi(t) = \mathbb{E}[x].
\]

We solve to get

\[
\chi(s) = \left( \mathbb{E}[x] - \frac{\alpha}{1 + \beta} \int_t^s \exp \left\{ \frac{1}{1 + \beta} \int_t^r \tilde{\pi}(\tau) d\tau \right\} \right) \exp \left\{ -\frac{1}{1 + \beta} \int_t^s \tilde{\pi}(\tau) d\tau \right\}
\]

where \( \tilde{\pi}(s) = e^{\mu(s-t)} \pi(s) \). By analyzing \( \tilde{\pi} \), we can find conditions under which \( \chi(s) \) will be positive. Note that

\[
\tilde{\pi}' + \nu^2 \tilde{\rho} + (\nu_0^2 - \mu) \tilde{\pi} - \frac{1}{1 + \beta} \tilde{\pi}^2 = 0, \quad \tilde{\pi}(T) = 1
\]

where \( \tilde{\rho}(s) = e^{\mu(s-t)} \rho(s) \). We use the inequality \( ab \leq \frac{1}{\alpha} a^2 + b^2 \) and the fact that \( \tilde{\rho} \leq 1 \) to get

\[
-\tilde{\pi}' \leq \nu^2 + \frac{1}{4} (1 + \beta)(\nu_0^2 - \mu)^2
\]

which yields

\[
\tilde{\pi}(s) \leq 1 + \frac{1}{4} (1 + \beta)(\nu_0^2 - \mu)^2 + 4\nu^2 (T - s) \leq \kappa := 1 + \frac{1}{4} ((1 + \beta)(\nu_0^2 - \mu)^2 + 4\nu^2) T.
\]

Using this estimate we deduce

\[
\chi(s) \geq \left( \mathbb{E}[x] - \frac{\alpha}{\kappa} \left( e^{\kappa(s-t)/(1+\beta)} - 1 \right) \right) \exp \left\{ -\frac{1}{1 + \beta} \int_t^s \tilde{\pi}(\tau) d\tau \right\}
\]

Therefore if we have

\[
(4.23) \quad \frac{\alpha}{\kappa} \left( e^{\kappa(T-t)/(1+\beta)} - 1 \right) \leq \mathbb{E}[x] \iff T - t < \frac{1 + \beta}{\kappa} \ln \left( 1 + \frac{\kappa}{\alpha} \mathbb{E}[x] \right),
\]

that is, if \( T - t \) is small enough, we have \( \chi(s) = \mathbb{E}[X(s)] \geq 0 \) for all \( s \in [t, T] \). We can interpret this smallness condition as saying that, on average, the initial reserves are not used up by time \( T \). Recalling that \( \beta = \epsilon/2 \) and \( \alpha = 1/2 \), we notice that condition (4.23) is equivalent to

\[
T - t < \frac{4 + 2\epsilon}{4 + ((1 + \epsilon/2)(\nu_0^2 - \mu)^2 + 4\nu^2) T} \left[ 1 + \frac{1}{2} (4 + (1 + \epsilon/2)(\nu_0^2 - \mu)^2 + 4\nu^2) T \mathbb{E}[x] \right]
\]

where the right-hand side is large when \( \epsilon \) is large.

Additionally, we observe that

\[
-\tilde{\pi}' \geq (\nu_0^2 - \mu) \tilde{\pi} - \frac{1}{1 + \beta} \tilde{\pi}^2
\]

using the fact that \( \tilde{\rho} \geq 0 \). Using the substitution \( u = \tilde{\pi}^{-1} \) we have

\[
uu' \geq (\nu_0^2 - \mu) u - \frac{1}{1 + \beta} \Rightarrow u(s) \leq \left( 1 + \frac{T - s}{1 + \beta} \right) e^{\nu_0^2 - \mu(T-s)} \Rightarrow \tilde{\pi}(s) > 0 \forall s \in [t, T].
\]

It follows that \( \pi \) is positive. Therefore, under condition (4.23) (in particular, for \( \epsilon \) large enough with respect to \( T \)) we have that the expected market price

\[
\mathbb{E}[ar{k}(s)] = \frac{1}{2 + \epsilon} - \frac{2 + 2\epsilon}{2 + \epsilon} e^{\mu(s-t)} \left( \pi(s) \chi(s) + \int_t^s \tilde{\pi}(\tau) d\tau \right)
\]

is decreasing in \( \epsilon \) and goes to zero as \( \epsilon \to \infty \).
5. Conclusion

In this paper we have discussed the solution of a linear-quadratic mean field type control problem with a common noise and a dependence on the conditional expectation of both state and control variables. We then compared this to mean field games, where it is seen that in certain cases, the two problems are the same, with a difference in the objective functionals which is called the price of anarchy. We then applied this to an economic model of production of exhaustible resources. Since it is natural for such aggregate quantities as the expected value of the control to appear in economic models, it is useful to note that variational methods can be used to study the Nash equilibrium in this case. It would be interesting to pursue this approach in future work on more general models than the linear-quadratic setting.

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