ERRATUM TO: DEFORMATION QUANTIZATION IN
ALGEBRAIC GEOMETRY

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Abstract. This note contains a correction of the proofs of the main results of the paper [A. Yekutieli, Deformation quantization in algebraic geometry, Adv. Math. 198 (2005), 383-432]. The results are correct as originally stated.

0. Introduction

This note contains a correction of the proofs of the main results of [Ye1], namely Theorems 0.1 and 0.2. The results are correct as originally stated.

The mistake in my original proofs was discovered Michel Van den Bergh, and I thank him for calling my attention to it. The way to fix the proofs is essentially contained in his paper [VdB].

Let me begin by explaining the mistake. As can be seen in Example 0.1 below, the mistake itself is of a rather elementary nature, but it was obscured by the complicated context.

Suppose \( K \) is a field of characteristic 0, and \( X \) is a smooth separated \( n \)-dimensional scheme over \( K \). Recall that the coordinate bundle \( \text{Coor}_X \) is an infinite dimensional bundle over \( X \), with free action by the group \( \text{GL}_{n,K} \). The quotient bundle is by definition \( \text{LCC}_X := \text{Coor}_X / \text{GL}_{n,K} \), and the projection \( \pi_{\text{gl}} : \text{Coor}_X \to \text{LCC}_X \) is a \( \text{GL}_{n,K} \)-torsor.

The erroneous (implicit) assertion in [Ye1] is that the de Rham complexes satisfy

\[
(\pi_{\text{gl}}^* \Omega_{\text{Coor}_X}^{\text{GL}_{n,K}}) = \Omega_{\text{LCC}_X}.
\]

From that I deduced (incorrectly, top of page 424) that the Maurer-Cartan form \( \omega_{MC} \) is a global section of the sheaf

\[
\Omega^1_{\text{LCC}_X} \otimes_{\mathcal{O}_{\text{LCC}_X}} \pi_{\text{LCC}}^*(\mathcal{P}_X \otimes_{\mathcal{O}_X} \mathcal{T}^0_{\text{poly}_X}).
\]

(This false, as can be seen from [VdB, Lemma 6.5.1]). This led to many incorrect formulas in [Ye1, Section 7].

The correct thing to do is to work with the infinitesimal action of the Lie algebra \( \mathfrak{g} := \mathfrak{gl}_n(K) \). For \( v \in \mathfrak{g} \) one has the contraction (inner derivative) \( \iota_v \), which is a degree \(-1\) derivation of the de Rham complex \( \pi_{\text{gl}}^* \Omega_{\text{Coor}_X} \). Recall that the Lie derivative is \( L_v := d \circ \iota_v + \iota_v \circ d \). A local section \( \omega \in \pi_{\text{gl}}^* \Omega_{\text{Coor}_X} \) is said to be \( \mathfrak{g} \)-invariant if \( \iota_v(\omega) = L_v(\omega) = 0 \) for all \( v \in \mathfrak{g} \). According to [VdB, Lemma 9.2.3] one has

\[
(\pi_{\text{gl}}^* \Omega_{\text{Coor}_X})^\mathfrak{g} = \Omega_{\text{LCC}_X}.
\]
It is worthwhile to note that in my incorrect proof there was no need to invoke Kontsevich’s property (P5) from [Ko]. The correct proof does require property (P5) – cf. [vdB3, Lemma 9.2.1].

**Example 0.1.** Here is a simplified example. Suppose $G$ is the affine algebraic group $\text{GL}_1, K = \text{Spec} K[t, t^{-1}]$, and $X$ is the variety $G$, with regular left action. The group of rational points is $G(K) = K^\times$. The action of $G$ on $X$ is free, the invariant ring is $\mathcal{O}(X)^{G(K)} = K$, and the quotient is $X/G = \text{Spec} K$. For the de Rham complex

$$\Omega(X) = \mathcal{O}(X) \oplus \Omega^1(X) = K[t, t^{-1}] \oplus K[t, t^{-1}] \cdot dt$$

we have $\Omega(X)^{G(K)} \neq K$, since it contains $t^{-1}dt$. But for the infinitesimal action of the Lie algebra $g := gl_1(K)$ it is easy to see that $\Omega(X)^g = K$.

After some deliberation I decided that the best way to present the erratum is by completely rewriting [Ye1, Section 7]. This is Section 1 below. Section 2 contains some additional minor corrections to [Ye1].

1. The Global $L_\infty$ Quasi-isomorphism

This is a revised version of [Ye1, Section 7]. In this section we prove the main results of the paper [Ye1], namely Theorem 0.1 (which is repeated here as Corollary 1.19), and Theorem 0.2 (which is repeated here, with more details, as Theorem 1.2). Throughout $K$ is a field containing $\mathbb{R}$, and $X$ is a smooth irreducible separated $n$-dimensional scheme over $K$. We use all notation, definitions and results of [Ye1, Sections 1-6] freely. However the bibliography references relate to the list at the end of this note.

Suppose $U = \{U_0, \ldots, U_m\}$ is an open covering of the scheme $X$, consisting of affine open sets, each admitting an étale coordinate system, namely an étale morphism $U_i \to \mathbb{A}^2_k$. For every $i$ let $\sigma_i : U_i \to \text{LCC}(X)$ be the corresponding section of $\pi_{lcc} : \text{LCC}(X) \to X$, and let $\sigma$ be the resulting simplicial section (see [Ye1, Theorem 6.5]).

Let $\mathcal{M}$ be a bounded below complex of quasi-coherent $\mathcal{O}_X$-modules. The mixed resolution $\text{Mix}_U(\mathcal{M})$ was defined in [Ye1, Section 6]. For any integer $i$ let

$$G^i \text{Mix}_U(\mathcal{M}) := \bigoplus_{j=i}^{\infty} \text{Mix}_U^j(\mathcal{M}),$$

so $\{G^i \text{Mix}_U(\mathcal{M})\}_{i \in \mathbb{Z}}$ is a descending filtration of $\text{Mix}_U(\mathcal{M})$ by subcomplexes, with $G^i \text{Mix}_U(\mathcal{M}) = \text{Mix}_U(\mathcal{M})$ for $i \leq 0$, and $\bigcap_i G^i \text{Mix}_U(\mathcal{M}) = 0$. Let

$$\text{gr}^i G \text{Mix}_U(\mathcal{M}) := G^i \text{Mix}_U(\mathcal{M}) / G^{i+1} \text{Mix}_U(\mathcal{M})$$

and $\text{gr}_G \text{Mix}_U(\mathcal{M}) := \bigoplus_i \text{gr}^i G \text{Mix}_U(\mathcal{M})$.

By [Ye1, Proposition 6.3], if $G_X$ is either $\mathcal{T}_{\text{poly}, X}$ or $\mathcal{D}_{\text{poly}, X}$, then $\text{Mix}_U(G_X)$ is a sheaf of DG Lie algebras on $X$, and the inclusion

$$\eta_G : G_X \to \text{Mix}_U(G_X)$$

is a DG Lie algebra quasi-isomorphism.

Note that if $\phi : \text{Mix}_U(\mathcal{M}) \to \text{Mix}_U(\mathcal{N})$ is a homomorphism of complexes that respects the filtration $\{G^i \text{Mix}_U\}$, then there exists an induced homomorphism of complexes

$$\text{gr}_G(\phi) : \text{gr}_G \text{Mix}_U(\mathcal{M}) \to \text{gr}_G \text{Mix}_U(\mathcal{N}).$$
Suppose \( \mathcal{G} \) and \( \mathcal{H} \) are sheaves of DG Lie algebras on a topological space \( Y \). An \( L_\infty \) morphism \( \Psi : \mathcal{G} \to \mathcal{H} \) is a sequence of sheaf morphisms \( \psi_j : \prod [\mathcal{G}] \to \mathcal{H} \), such that for every open set \( V \subset Y \) the sequence \( \{ \Gamma(V, \psi_j) \}_{j \geq 1} \) is an \( L_\infty \) morphism \( \Gamma(V, \mathcal{G}) \to \Gamma(V, \mathcal{H}) \). If \( \psi_1 : \mathcal{G} \to \mathcal{H} \) is a quasi-isomorphism then \( \Psi \) is called an \( L_\infty \) quasi-isomorphism.

Recall that there is a canonical quasi-isomorphism of complexes of \( O_X \)-modules
\[
U_1 : T_{\text{poly}, X} \to D_{\text{poly}, X}.
\]
According to [Ye2] Theorem 4.17, the induced homomorphism
\[
gr_G(\text{Mix}_U(U_1)) : gr_G \text{Mix}_U(T_{\text{poly}, X}) \to gr_G \text{Mix}_U(D_{\text{poly}, X})
\]
is a quasi-isomorphism.

**Theorem 1.2.** Let \( X \) be an irreducible smooth separated \( \mathbb{K} \)-scheme. Let \( U = \{U_0, \ldots, U_m\} \) be an open covering of \( X \) consisting of affine open sets, each admitting an étale coordinate system, and let \( \sigma \) be the associated simplicial section of the bundle \( \text{LCC}_X \to X \). Then there is an induced \( L_\infty \) quasi-isomorphism
\[
\Psi_{\sigma : 1} : \text{Mix}_U(T_{\text{poly}, X}) \to \text{Mix}_U(D_{\text{poly}, X}).
\]
The homomorphism \( \Psi_{\sigma : 1} \) respects the filtration \( \{G^i \text{Mix}_U\} \), and
\[
gr_G(\Psi_{\sigma : 1}) = gr_G(\text{Mix}_U(U_1)).
\]

**Proof.** Let \( Y \) be some \( \mathbb{K} \)-scheme, and denote by \( \mathbb{K}_Y \) the constant sheaf. For any \( p \) we view \( \Omega^p_Y \) as a discrete inv \( \mathbb{K}_Y \)-module, and we put on \( \Omega_Y = \bigoplus_{p \in \mathbb{N}} \Omega^p_Y \) direct sum dir-inv structure. So \( \Omega_Y \) is a discrete (and hence complete) DG algebra in \( \text{Dir Inv Mod} \mathbb{K}_Y \).

We shall abbreviate \( \mathcal{A} := \Omega_{\text{Coor}, X} \), so that \( \mathcal{A}^0 = O_{\text{Coor}, X} \) etc. As explained above, \( \mathcal{A} \) is a DG algebra in \( \text{Dir Inv Mod} \mathbb{K}_{\text{Coor}, X} \), with discrete (but not trivial) dir-inv module structure.

There are sheaves of DG Lie algebras \( \mathcal{A} \otimes T_{\text{poly}}(\mathbb{K}[[t]]) \) and \( \mathcal{A} \otimes D_{\text{poly}}(\mathbb{K}[[t]]) \) on the scheme \( \text{Coor} X \). The differentials are \( d_{\text{for}} = d \otimes 1 \) and \( d_{\text{for}} + 1 \otimes d_{\text{P}} \) respectively. As explained just prior to [Ye1] Theorem 3.16, \( \mathcal{U} \) extends to a continuous \( \mathcal{A} \)-multilinear \( L_\infty \) morphism
\[
\mathcal{U}_A = \{ \mathcal{U}_{A j} \}_{j \geq 1} : \mathcal{A} \otimes T_{\text{poly}}(\mathbb{K}[[t]]) \to \mathcal{A} \otimes D_{\text{poly}}(\mathbb{K}[[t]])
\]
of sheaves of DG Lie algebras on \( \text{Coor} X \).

The MC form \( \omega := \omega_{\text{MC}} \) is a global section of \( \mathcal{A}^1 \otimes T_{\text{poly}}^0(\mathbb{K}[[t]]) \) satisfying the MC equation in the DG Lie algebra \( \mathcal{A} \otimes T_{\text{poly}}(\mathbb{K}[[t]]) \). See [Ye1] Proposition 5.9. According to [Ye1] Theorem 3.16, the global section \( \omega' := \mathcal{U}_{A,1}(\omega) \in \mathcal{A}^1 \otimes D_{\text{poly}}^0(\mathbb{K}[[t]]) \) is a solution of the MC equation in the DG Lie algebra \( \mathcal{A} \otimes D_{\text{poly}}(\mathbb{K}[[t]]) \), and there is a continuous \( \mathcal{A} \)-multilinear \( L_\infty \) morphism
\[
\mathcal{U}_{A,\omega} = \{ \mathcal{U}_{A,\omega j} \}_{j \geq 1} : \left( \mathcal{A} \otimes T_{\text{poly}}(\mathbb{K}[[t]]) \right)_\omega \to \left( \mathcal{A} \otimes D_{\text{poly}}(\mathbb{K}[[t]]) \right)_\omega
\]
between the twisted DG Lie algebras. The formula is
\[
\mathcal{U}_{A,\omega j}(\gamma_1 \cdots \gamma_j) = \sum_{k \geq 0} \frac{1}{(j + k)!} \mathcal{U}_{A,j+k}(\omega^k \cdot \gamma_1 \cdots \gamma_j)
\]
for \( \gamma_1, \ldots, \gamma_j \in \mathcal{A} \otimes T_{\text{poly}}(\mathbb{K}[[t]]) \). The two twisted DG Lie algebras have differentials \( d_{\text{for}} + \text{ad}(\omega) \) and \( d_{\text{for}} + \text{ad}(\omega') + 1 \otimes d_{\text{P}} \) respectively.
This sum in (1.3) is actually finite, the number of nonzero terms in it depending on the bidegree of $\gamma_1 \cdots \gamma_j$. Indeed, if $\gamma_1 \cdots \gamma_j \in A^q \otimes T^p_{\text{poly}}(K[[t]])$, then

$$U_{A; j+k}(\omega^k \cdot \gamma_1 \cdots \gamma_j) \in A^{q+k} \otimes T^p_{\text{poly}}(K[[t]])$$

which is is zero for $k > p - j + 2$; see proof of [Ye2 Theorem 3.23].

By [Ye1 Theorem 5.6] (the universal Taylor expansions) there are canonical isomorphisms of graded Lie algebras in $\text{Dir Inv Mod} K\text{Coor}_X$

$$(1.5) \quad A \otimes T_{\text{poly}}(K[[t]]) \cong A \otimes \Lambda^0 \pi_{\text{coor}}(P_X \otimes_{O_X} T_{\text{poly}, X})$$

and

$$(1.6) \quad A \otimes D_{\text{poly}}(K[[t]]) \cong A \otimes \Lambda^0 \pi_{\text{coor}}(P_X \otimes_{O_X} D_{\text{poly}, X})$$

According to [VdB, Lemma 9.2.1], the $L\infty$ morphism $U_{A;1}$ gives a continuous $A$-multilinear $L\infty$ morphism between these DG Lie algebras, whose differentials are $\nabla_\nu$ and $\nabla_{\nu} + 1 \otimes_D$ respectively. As in the proof of [Ye1 Theorem 5.6], under the identifications (1.5) and (1.6) we have the equality

$$U_{A;1} = 1 \otimes \pi_{\text{coor}}(1 \otimes U_1),$$

i.e., it is the pullback of the map (1.1).

Let us filter the DG algebra $A$ by the descending filtration $\{G^j A\}_{j \in \mathbb{Z}}$, where $G^j A := \bigoplus_{i=j}^{\infty} A^i$. The DG Lie algebras appearing in equation (1.7) inherit this filtration. From formulas (1.3) and (1.4) we see that the homomorphism of complexes $U_{A; \omega:1}$ respects the filtration, and from (1.8) we see that

$$\text{gr}_G(U_{A; \omega:1}) = \text{gr}_G(U_{A;1}) = 1 \otimes \pi_{\text{coor}}(1 \otimes U_1).$$

Let $n := \dim X$. As noted earlier, the action of $g := gl_n(K)$ gives

$$(\pi_{gl_n}(\Lambda)^\theta = (\pi_{gl_n} \cdot \Omega_{\text{Coor}_X})^\theta = \Omega_{LCC}_X.$$

According to [VdB Lemma 9.2.1], the $L\infty$ morphism $U_{A; \omega}$ commutes with the action of the Lie algebra $g$. Therefore $U_{A; \omega}$ descends (i.e., restricts) to a continuous $\Omega_{LCC}_X$-multilinear $L\infty$ morphism

$$(1.9) \quad U_{A; \omega}^\theta : \Omega_{LCC}_X \otimes_{C_{\text{CC}X}} \pi_{\text{lecc}}(P_X \otimes_{O_X} T_{\text{poly}, X})$$

$$\rightarrow \Omega_{LCC}_X \otimes_{C_{\text{CC}X}} \pi_{\text{lecc}}(P_X \otimes_{O_X} D_{\text{poly}, X}).$$

The DG Lie algebras in formula (1.9) also have filtrations $\{G^j\}_{j \in \mathbb{Z}}$, the homomorphism $U_{A; \omega:1}^\theta$ respects this filtration, and we now have

$$\text{gr}_G(U_{A; \omega:1}^\theta) = \text{gr}_G(U_{A;1}^\theta) = 1 \otimes \pi_{\text{lecc}}(1 \otimes U_1).$$

According to [Ye1 Theorem 6.4] there are induced operators

$$\Psi_{\sigma; j} := \sigma^*(U_{A; \omega:1}^\theta) : \text{Mix}_U(T_{\text{poly}, X}) \rightarrow \text{Mix}_U(D_{\text{poly}, X})$$

for $j \geq 1$. The $L\infty$ identities in [Ye1 Definition 3.7], when applied to the $L\infty$ morphism $U_{A; \omega}^\theta$, are of the form considered in [Ye1 Theorem 6.4(iii)]. Therefore these
identities are preserved by $\sigma^*$, and we conclude that the sequence $\Psi_\sigma = \{\Psi_{\sigma,j}\}_{j=1}^\infty$ is an $L_\infty$ morphism. Furthermore, $\Psi_{\sigma,1}$ respects the filtration $\{G^i\text{Mix}_U\}$, and from (1.10) we get

$\text{gr}_G(\Psi_{\sigma,1}) = \text{gr}_G(\sigma^*(U_{i,1}^\theta)) = \text{gr}_G(M\text{ix}_U(U_1)).$

According to [Ye2, Theorem 4.17] the homomorphism $\text{gr}_G(M\text{ix}_U(U_1))$ is a quasi-isomorphism. Since the complexes $\text{Mix}_U(T_{\text{poly},X})$ and $\text{Mix}_U(D_{\text{poly},X})$ are bounded below, and the filtration is nonnegative and exhaustive, it follows that $\Psi_{\sigma,1}$ is also a quasi-isomorphism.

**Corollary 1.12.** Taking global sections in Theorem 1.2 we get an $L_\infty$ quasi-isomorphism

$\Gamma(X, \Psi_\sigma) = \{\Gamma(X, \Psi_{\sigma,j})\}_{j\geq 1} : \Gamma(X, \text{Mix}_U(T_{\text{poly},X})) \to \Gamma(X, \text{Mix}_U(D_{\text{poly},X})).$

**Proof.** Theorem 1.2 tells us that $\Psi_{\sigma,1}$ is a quasi-isomorphisms of complexes of sheaves. By [Ye1, Theorem 6.2] it follows that

$\Gamma(X, \Psi_{\sigma,1}) : \Gamma(X, \text{Mix}_U(T_{\text{poly},X})) \to \Gamma(X, \text{Mix}_U(D_{\text{poly},X}))$

is a quasi-isomorphism. □

**Corollary 1.13.** The data $(U, \sigma)$ induces a bijection

$\text{MC}(\Psi_\sigma) : \text{MC}(\Gamma(X, \text{Mix}_U(T_{\text{poly},X})[[h]^+]) \cong \text{MC}(\Gamma(X, \text{Mix}_U(D_{\text{poly},X})[[h]^+]).$

**Proof.** Use Corollary 1.12 and [Ye1 Corollary 3.10]. □

Recall that $T_{\text{poly},X} = \Gamma(X, T_{\text{poly},X})$ and $D_{\text{poly},X} = \Gamma(X, D_{\text{poly},X})$; and the latter is the DG Lie algebra of global poly differential operators that vanish if one of their arguments is 1.

Suppose $f : X' \to X$ is an étale morphism. According to [Ye2 Proposition 4.6] there are DG Lie algebra homomorphisms $f^* : T_{\text{poly},X} \to T_{\text{poly},X'}$ and $f^* : D_{\text{poly},X} \to D_{\text{poly},X'}$. These homomorphisms extend to formal coefficients, and we get functions

$\text{MC}(f^*) : \text{MC}(T_{\text{poly},X}[[h]^+]) \to \text{MC}(T_{\text{poly},X'}[[h]^+])$

etc.

One says that $X$ is a $D$-affine variety if $H^q(X, \mathcal{M}) = 0$ for every quasi-coherent left $D_X$-module $\mathcal{M}$ and every $q > 0$.

**Theorem 1.14.** Let $X$ be an irreducible smooth separated $\mathbb{K}$-scheme. Assume $X$ is $D$-affine. Then there is a canonical function

$Q : \text{MC}(T_{\text{poly},X}[[h]^+]) \to \text{MC}(D_{\text{poly},X}[[h]^+])$

called the quantization map. It has the following properties:

(i) The function $Q$ preserves first order terms.

(ii) The function $Q$ respects étale morphisms. Namely if $X'$ is another $D$-affine scheme, with quantization map $Q'$, and if $f : X' \to X$ is an étale morphism, then

$Q' \circ \text{MC}(f^*) = \text{MC}(f^*) \circ Q.$

(iii) If $X$ is affine, then $Q$ is bijective.
The quantization map $Q$ is a DG Lie algebra homomorphism. So by continuity we might as well assume that

$$\sigma \equiv \text{MC}(\mathcal{L}) \overset{Q}{\longrightarrow} \text{MC}(\mathcal{R})$$

in which the arrows $\text{MC}(\sigma)$ and $\text{MC}(\eta_D)$ are bijections. Here $\Psi_{\sigma}$ is the $L_\infty$ quasi-isomorphism from Theorem 1.2, and $\eta_D$ and $\eta$ are the inclusions of DG Lie algebras.

Let's elaborate a bit on the statement above. It says that to any MC solution $\alpha = \sum_{j=1}^{\infty} \alpha_j h^j \in \mathcal{T}_{\text{poly}}(X)[[h]]^+$ there corresponds an MC solution $\beta = \sum_{j=1}^{\infty} \beta_j h^j \in \mathcal{D}_{\text{poly}}(X)[[h]]^+$. The element $\beta = Q(\alpha)$ is uniquely determined up to gauge equivalence by the group $\exp(D_{\text{poly}}^0(X)[[h]]^+).$ Given any local sections $f, g \in \mathcal{O}_X$ one has

$$\frac{i}{\hbar}(\beta_1(f, g) - \beta_1(g, f)) = \alpha_1(f, g) \in \mathcal{O}_X.$$ 

The quantization map $Q$ can be calculated (at least in theory) using the collection of sections $\sigma$ and the universal formulas for deformation in Ye1, Theorem 3.13.

We'll need a lemma before proving the theorem.

**Lemma 1.16.** Let $f, g \in \mathcal{O}_X = \mathcal{D}_{\text{poly}, X}^{-1}$ be local sections.

1. For any $\beta \in \text{Mix}_U^0(\mathcal{D}_{\text{poly}, X}^1)$ one has

$$[[\beta, f], g] = \beta(g, f) - \beta(f, g) \in \text{Mix}_U^0(\mathcal{O}_X).$$

2. For any $\beta \in \text{Mix}_U^0(\mathcal{D}_{\text{poly}, X}^0) \oplus \text{Mix}_U^0(\mathcal{D}_{\text{poly}, X}^{-1})$ one has $[[\beta, f], g] = 0.$

3. Let $\gamma \in \text{Mix}_U(\mathcal{D}_{\text{poly}, X}^0)^0,$ and define $\beta := (d_{\text{mix}} + d_{\text{D}})(\gamma).$ Then $[[\beta, f], g] = 0.$

**Proof.** (1) [Ye1] Proposition 6.3 implies that the embedding ([Ye1] (6.1):

$$\text{Mix}_U(\mathcal{D}_{\text{poly}, X}) \subset \bigoplus_{p, q, r} \prod_{j \in \mathbb{N}} \prod_{i \in \Delta^n} g_{i*} g_{i}^{-1}(\Omega^p(\Delta^n)^{\otimes} (\Omega^q_X \otimes \mathcal{O}_X \circ \mathcal{O}_X \circ \mathcal{D}_{\text{poly}, X}^r))$$

is a DG Lie algebra homomorphism. So by continuity we might as well assume that

$$\beta = aD$$

with $a \in \Omega_X^0 = \mathcal{O}_X$ and $D \in \mathcal{D}_{\text{poly}, X}^1$. Moreover, since the Lie bracket of $\Omega_X \otimes \mathcal{O}_X \circ \mathcal{O}_X \circ \mathcal{D}_{\text{poly}, X}$ is $\mathcal{O}_X$-bilinear, we may assume that $a = 1,$ i.e., $\beta = D.$ Now the assertion is clear from the definition of the Gerstenhaber Lie bracket, see [Ko] Section 3.4.2).

(2) Applying the same reduction as above, but with $D \in \mathcal{D}_{\text{poly}, X}^r$ and $r \in \{0, -1\},$ we get $[[D, f], g] \in \mathcal{D}_{\text{poly}, X}^{r-2} \circ \mathcal{O}_X = 0.$

(3) By part (2) it suffices to show that $[[\beta, f], g] = 0$ for $\beta := d_{\text{D}}(\gamma)$ and $\gamma \in \text{Mix}_U^0(\mathcal{D}_{\text{poly}, X}^0).$ As explained above we may further assume that $\gamma = D \in \mathcal{D}_{\text{poly}, X}^0.$ Now the formulas for $d_{\text{D}}$ and $[-, -]$ in [Ko] Section 3.4.2 imply that $[[d_{\text{D}}(D), f], g] = 0.$
Proof of Theorem 1.13  Step 1. Take an open covering \( U \) as in property (iv). Since the sheaves \( \mathcal{D}_{\text{nor},X}^{\text{pol}} \) are quasi-coherent left \( \mathcal{D}_X \)-modules, it follows that \( H^q(X, \mathcal{D}_{\text{nor},X}^{\text{pol}}) = 0 \) for all \( p \) and all \( q > 0 \). Therefore \( \Gamma(X, \mathcal{D}_{\text{nor},X}^{\text{pol}}) \to R\Gamma(X, \mathcal{D}_{\text{poly},X}^{\text{pol}}) \) in the derived category \( D(\text{Mod} \mathbb{K}) \). Now by [Ye1] Theorem 3.12 the inclusion \( \mathcal{D}_{\text{nor},X}^{\text{pol}} \to \mathcal{D}_{\text{poly},X}^{\text{pol}} \) is a quasi-isomorphism, and by [Ye1] Theorem 6.2(1) the inclusion \( \mathcal{D}_{\text{poly},X}^{\text{pol}} \to \text{Mix}_U(\mathcal{D}_{\text{poly},X}) \) is a quasi-isomorphism. According to [Ye1] Theorem 6.2(2) we have \( \Gamma \left( \mathbb{X}, \text{Mix}_U(\mathcal{D}_{\text{poly},X}) \right) = R\Gamma \left( \mathbb{X}, \text{Mix}_U(\mathcal{D}_{\text{poly},X}) \right) \). The conclusion is that

\[
\mathcal{D}_{\text{nor}}^{\text{pol}}(X) = \Gamma(X, \mathcal{D}_{\text{nor},X}^{\text{pol}}) \to \Gamma(X, \text{Mix}_U(\mathcal{D}_{\text{poly},X}))
\]

is a quasi-isomorphism of complexes of \( \mathbb{K} \)-modules. But in view of [Ye1] Proposition 6.3, this is in fact a quasi-isomorphism of DG Lie algebras.

From (1.17) we deduce that

\[
\eta_{\mathcal{D}} : \mathcal{D}_{\text{nor}}^{\text{pol}}(X)[[\hbar]]^+ \to \Gamma(X, \text{Mix}_U(\mathcal{D}_{\text{poly},X}))[\hbar]^+
\]

is a quasi-isomorphism of DG Lie algebras. Using [Ye1] Corollary 3.10 we see that \( \text{MC}(\eta_{\mathcal{D}}) \) is bijective. Therefore the diagram in property (iv) defines \( Q \) uniquely.

According to Corollary 1.13 the arrow marked \( \text{MC}(\Psi_{\sigma}) \) is a bijection. So we have established property (iv), except for the independence of the open covering.

Step 2. The left vertical arrow comes from the DG Lie algebra homomorphism

\[
\eta_T : T_{\text{poly}}(X)[[\hbar]]^+ \to \Gamma(X, \text{Mix}_U(T_{\text{poly},X}))[\hbar]^+;
\]

which is a quasi-isomorphism when \( H^q(X, T_{\text{poly},X}) = 0 \) for all \( p \) and all \( q > 0 \). So in case \( X \) is affine, the quantization map \( Q \) is bijective. This establishes property (iii).

Step 3. Now suppose \( U' = \{ U'_0, \ldots, U'_{m'} \} \) is another such affine open covering of \( X \), with sections \( \sigma'_i : U'_i \to \text{LCC} \). Without loss of generality we may assume that \( m' \geq m \), and that \( U'_0 = U_0 \) and \( \sigma'_i = \sigma_i \) for all \( i \leq m \). There is a morphism of simplicial schemes \( f : U \to U' \), that is an open and closed embedding. Correspondingly there is a commutative diagram

\[
\begin{array}{ccc}
\text{MC}(T_{\text{poly}}(X)[[\hbar]]^+) & \xrightarrow{Q} & \text{MC}(\mathcal{D}_{\text{nor}}^{\text{pol}}(X)[[\hbar]]^+) \\
\downarrow \text{MC}(\eta_T) & & \downarrow \text{MC}(\eta_{\mathcal{D}})
\end{array}
\]

\[
\begin{array}{ccc}
\text{MC}(\Gamma(X, \text{Mix}_{U'}(T_{\text{poly},X}))[\hbar]^+) & \xrightarrow{\text{MC}(\Psi_{\sigma'})} & \text{MC}(\Gamma(X, \text{Mix}_U(\mathcal{D}_{\text{poly},X}))[\hbar]^+) \\
\downarrow \text{MC}(f^+) & & \downarrow \text{MC}(f^+)
\end{array}
\]

\[
\begin{array}{ccc}
\text{MC}(\Gamma(X, \text{Mix}_{U'}(T_{\text{poly},X}))[\hbar]^+) & \xrightarrow{\text{MC}(\Psi_{\sigma'})} & \text{MC}(\Gamma(X, \text{Mix}_U(\mathcal{D}_{\text{poly},X}))[\hbar]^+)
\end{array}
\]

where the vertical arrows on the right are bijections. We conclude that \( Q \) is independent of \( U \) and \( \sigma \). This concludes the proof of property (iv).

Step 4. Suppose \( f : X' \to X \) is an étale morphism. Then we can choose an affine open covering \( U' \) of \( X' \) that refines \( U \) in the obvious sense. Each of the open sets \( U'_i \) inherits an étale coordinate system, and hence a section \( \sigma'_i : U'_i \to \text{LCC} \). We
get a commutative diagram

\[
\begin{array}{cccc}
\text{MC}(\Gamma(X, \text{Mix}_U(\mathcal{T}_\text{poly}, X)))[[h]]^+ & \xrightarrow{\text{MC}(\Psi_{\sigma,1})} & \text{MC}(\Gamma(X, \text{Mix}_U(\mathcal{D}_\text{poly}, X)))[[h]]^+) \\
\downarrow \text{MC}(f^*) & & \downarrow \text{MC}(f^*) \\
\text{MC}(\Gamma(X', \text{Mix}_U(\mathcal{T}_\text{poly}, X')))[[h]]^+ & \xrightarrow{\text{MC}(\Psi_{\sigma',1})} & \text{MC}(\Gamma(X', \text{Mix}_U(\mathcal{D}_\text{poly}, X')))[[h]]^+
\end{array}
\]

This proves property (ii).

Step 5. Finally we must show that \(Q\) preserves first order terms, i.e. property (i). Let

\[\alpha = \sum_{j=1}^{\infty} \alpha_j h^j \in \mathcal{T}_\text{poly}(X)^1[[h]]^+\]

be an MC solution, and let

\[\beta = \sum_{j=1}^{\infty} \beta_j h^j \in \mathcal{D}_\text{poly}(X)^1[[h]]^+\]

be an MC solution such that \(\beta = Q(\alpha)\) modulo gauge equivalence. This means that there exists some

\[\gamma = \sum_{k \geq 1} \gamma_k h^k \in \Gamma(X, \text{Mix}_U(\mathcal{D}_\text{poly}, X))^0[[h]]^+\]

such that

\[\sum_{j \geq 1} \frac{1}{j} \Psi_{\sigma,j}(\alpha^j) \exp(af)(\exp(\gamma))(\beta),\]

with notation as in [Ye1, Lemma 3.2]. Cf. [Ye1, Theorem 3.8]. In the first order term (i.e. the coefficient of \(h^1\)) of this equation we have

(1.18) \(\Psi_{\sigma,1}(\alpha_1) = \beta_1 - (d_{\text{mix}} + d_{\mathcal{D}})(\gamma_1)\);

see [Ye1, equation (3.3)].

In order to apply Lemma 1.16(2), we are interested in the component of \(\Psi_{\sigma,1}(\alpha_1)\) living in the summand \(\text{Mix}_U(\mathcal{T}_1, X)\). But this is exactly

\[\text{gr}_G(\Psi_{\sigma,1})(\alpha_1) \in \text{gr}_G(\text{Mix}_U(\mathcal{T}_1, X)) = \text{Mix}_U^0(\mathcal{D}_1, X).\]

Since according to Theorem 1.2 we have

\[\text{gr}_G^0(\Psi_{\sigma,1}) = \text{gr}_G^0(\text{Mix}_U(U_1)),\]

it follows that the component we are interested in is

\[\text{gr}_G^0(\text{Mix}_U(U_1))(\alpha_1) = U_1(\alpha_1).\]

Now take any two local sections \(f, g \in O_X\). Using Lemma 1.16 we get

\[[[\Psi_{\sigma,1}(\alpha_1), f], g] = [[U_1(\alpha_1), f], g] = U_1(\alpha_1)(g, f) - U_1(\alpha_1)(f, g) = -2\alpha_1(f, g),\]

and

\[[[\beta_1, f], g] = \beta_1(g, f) - \beta_1(f, g)\]

and

\[[[(d_{\text{mix}} + d_{\mathcal{D}})(\gamma_1), f], g] = 0.\]

Combining these equations with equation (1.18) we see that equation (1.15) indeed holds. So the proof is done. \[\square\]
Corollary 1.19. Let $X$ be an irreducible smooth separated $\mathbb{K}$-scheme. Assume $X$ is $\mathcal{D}$-affine. Then the quantization map $Q$ of Theorem 1.14 may be interpreted as a canonical function

$$Q : \left\{ \text{formal Poisson structures on } X \right\}_{\text{gauge equivalence}} \rightarrow \left\{ \text{deformation quantizations of } \mathcal{O}_X \right\}_{\text{gauge equivalence}}.$$ 

The quantization map $Q$ preserves first order terms, and commutes with étale morphisms $f : X' \rightarrow X$. If $X$ is affine then $Q$ is bijective.

Proof. By definition the left side is $\text{MC}(\mathcal{T}_{\text{poly}}(X)[[\hbar]]^+)$. On the other hand, according to [Ye1, Theorem 1.13] every deformation quantization of $\mathcal{O}_X$ can be trivialized globally, and by [Ye1, Proposition 1.14] any gauge equivalence between globally trivialized deformation quantizations is a global gauge equivalence. Hence the right side is $\text{MC}(\mathcal{D}_{\text{nor}}_{\text{poly}}(X)[[\hbar]]^+)$. \qed

2. Miscellaneous Errors

Here is a list of minor errors in the paper [Ye1].

1. Section 3, bottom of page 395: the formula should be

$$\text{af}(\gamma)(\omega) := [\gamma, \omega] - d(\gamma) = \text{ad}(\gamma)(\omega) - d(\gamma) \in \mathfrak{m} \otimes \mathfrak{g}^1,$$

2. Definition 5.2, page 411: the formula should be

$$\nabla_P : \mathcal{P}_X \rightarrow \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{P}_X.$$ 

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