Efficient decoy-states for the reference-frame-independent measurement-device-independent quantum key distribution

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Reference-frame-independent measurement-device-independent quantum key distribution (RFI-MDI-QKD) is a novel protocol which eliminates all possible attacks on detector side and necessity of reference-frame alignment in source sides. However, its performance may degrade notably due to statistical fluctuations, since more parameters, e.g. yields and error rates for mismatched-basis events, must be accumulated to monitor the security. In this work, we find that the original decoy-states method estimates these yields over pessimistically since it ignores the potential relations between different bases. Through processing parameters of different bases jointly, the performance of RFI-MDI-QKD is greatly improved in terms of secret key rate and achievable distance when statistical fluctuations are considered. Our results pave an avenue towards practical RFI-MDI-QKD.

I. INTRODUCTION

Quantum key distribution (QKD) [1], based on information-theoretic security guaranteed by quantum mechanics [2–5], allows two remote users, Alice and Bob, to share secret keys. In recent decades, numerous efforts have been made to improve the practical security and applicability of QKD protocols and systems [6–19]. The measurement-device-independent quantum key distribution (MDI-QKD) [16] and reference-frame-independent quantum key distribution (RFI-QKD) [17] are two of the most successful efforts of them.

The MDI-QKD is proposed to eliminate all possible detector side channel attacks. In this protocol, the most fragile assumption [20, 24] that the measurement unit is ideal is removed. The protocol offers a great balance between practical security and usability. Up to now, several MDI-QKD experiment has been presented [10, 24, 27]. However, no matter which coding scheme, e.g. polarization coding or phase coding, was deployed, the coding reference frame of Alice and Bob have to be rigorous calibrated, which limits practical performance and may bring side channels to QKD systems.

On the other hand, the RFI-QKD allows Alice and Bob to share secret key bits without active alignment of reference-frame. It was pointed out that the RFI-QKD is particularly relevant for the earth to satellite QKD [28] and time-bin encoded QKD [4]. However, the RFI-QKD still suffers from the fatal detector side channel attacks [20, 23].

Fortunately, the combination of MDI-QKD and RFI-QKD solved above problems. The RFI-MDI-QKD protocol [24, 30] does not need active reference frame calibration and immune to all detector side channel attacks. The protocol simplifies the MDI-QKD, reduces the aligning expenses as well as security risk. Meanwhile, it closes all detector side loopholes of RFI-QKD, which makes the RFI-QKD more secure.

Nevertheless, there are still two gaps between practice and theory. One of the gaps is the absence of ideal single-photon source. The gap is solved by decoy-state method [6–8, 31–35]. By randomly sending between so-called signal, decoy or vacuum states (weak coherent pulses with different intensities), Alice and Bob can establish a secret key strictly from information conveyed by single-photon component in weak coherent state. Another gap is named statistic fluctuation [32] which occurs when users only have limited resources while the theory assumes unlimited. The Refs. [32, 34, 35] have proposed countermeasure by analyzing statistical fluctuation. However, the secret key rate of RFI-MDI-QKD are rather undesirable. Compared with general MDI-QKD, the RFI version prepares more bases and more states, which means it requires more data to reduce the impact of the statistical fluctuations. The Ref. [35] experimentally demonstrate the RFI-MDI-QKD protocol to verified its feasibility and The Ref. [39] further improves its the performance. However, to eliminate the impact of statistical fluctuations, the Ref. [38] accumulates $3.5 \times 10^{11}$ pulse pairs for about 100 hours and the Ref. [39] accumulates $3 \times 10^{12}$ pulse pairs pairs for about 16 hours. Since the RFI protocol assumes the misalignment of two unbiased bases are unknown but fixed, which may be spoiled in such a long system-running time and make the protocol not RFI any more.

In this work, a new decoy-state method for RFI-MDI-QKD is proposed. By considering the potential relations of different bases and applying the joint-study method...
our new method reaches much higher secret key rate and longer achievable distance in non-asymptotic scenarios compared with existing theoretical and experimental methods. Our new method can generate considerable key rate given a smaller number of trials, which means less time for data accumulation.

The remainder of this paper is organized as follows: In Sec. II A we briefly introduce the RFI-MDI-QKD and the previous decoy-state method [30, 33, 34]. Then in Sec. II B we present our improved protocol and introduce our theories and parameter estimation strategies for improving the performance in non-asymptotic cases. In Sec. III the simulation results are showed to verify the superiority of our new method. Finally, in Sec IV we summarize our new method and introduce a more intuitive explanation of the improvement.

II. EFFICIENT DECOY-STATE RFI-MDI-QKD

A. Original protocol

We use subscript A (B) to denote a variable or a parameter belongs to Alice (Bob). To simplicity, we will omit the subscript A or B if not causing any ambiguity. We define $\mu$, $\nu$, $\omega$ and $\sigma$ are intensities of weak coherent pulse in decoy-state method, the $\mu > \nu > \omega > \sigma$ and $\omega$ is the vacuum state. The probability of a weak coherent pulse with intensity $\lambda$ contains $k$ photons is $\lambda^k e^{-\lambda} / k!$, especially, we define $p^\mu_k = (\lambda A)^k e^{-\lambda_A} / k!$ and $p^\nu_k = (\lambda B)^k e^{-\lambda_B} / k!$ are photon number distribution of Alice and Bob’s weak coherent pulse respectively. The $X$, $Y$, $Z$ are three coding bases. We define set $B = \{X, Y, Z\}$, the $b \in B$ denotes the coding basis selected by Alice and Bob, especially, the $S = ZZ$ is defined as both of the users select $Z$ basis, and $D = \{X, Y, Y, Y\}$ is defined as a set which both of the users select $X$ or $Y$ basis. In original MDI-QKD, all bases should be carefully aligned, but in RFI-MDI-QKD, the misalignment of $X$ and $Y$ bases can be an unknown but fixed value, namely, the $X$ and $Y$ bases of Alice (Bob) can be changed as:

\begin{align}
X_A(B) &= X \cos \beta_A(B) + Y \sin \beta_A(B); \\
Y_A(B) &= Y \cos \beta_A(B) - X \sin \beta_A(B),
\end{align}

where the $\beta$ denotes the reference frame misalignment.

The original decoy-state RFI-MDI-QKD is described as follows:

**Step 1. Preparation:** On each trial, Alice (Bob) randomly codes her (his) phase randomized weak coherent pulse on $Z$ $X$ or $Y$ basis and randomly selects an intensity $l (r)$ from a pre-decided intensity set $\{\mu_A, \nu_A, o\}$. After that, they send their weak coherent pulse to the untrusted third party Charlie. When both of Alice and Bob select intensity $\mu$ and $Z$ basis, the pulse pair can be used to generate key bits.

**Step 2. Measurement and announcement:** When charlie receives a weak coherent pulse pair from Alice and Bob, he does the Bell-state measurement and publicly announces his measurement result.

**Step 3. Post-processing:** After the trial repeats $N_{tot}$ times, there are $N_{lr}^b$ pulse pairs with intensity $lr$ and basis $b$ in total. Alice and Bob can observe the overall gain and the overall error yield of each kind of pulse pair. We define $Q_{lr}^b$ and $T_{lr}^b$ as the overall gain and the overall error yield of the weak coherent pulse pairs with intensity $lr$ and basis $b$ respectively. Then by applying the decoy-state method [30, 33, 35, 40], they can estimate yield and quantum bit error rate of single-photon pulse pairs for each basis. We define $Y_{tot}^b$ and $e_{tot}^b$ respectively, the yield and qubit error rate of single-photon pulse pairs with basis $b$, where $b \in B$. The information leakage is described as:

\begin{equation}
I_{AE}(C, e_{11}^S) = (1 - e_{11}^S)H_2[(1 + u)/2] + e_{11}^S H_2[(1 + v)/2],
\end{equation}

where

\begin{equation}
C = \sum_{d \in D} (1 - 2e^d_{11})^2,
\end{equation}

and

\begin{equation}
u = \min \left(\sqrt{C/2} / (1 - e_{11}^S), 1\right), \quad u = \sqrt{C/2 - (1 - e_{11}^S)^2}w^2 / e_{11}^S.
\end{equation}

The $H_2(p) = -p \log_2 p$ is the Shannon entropy [14, 20]. The secret key rate can be estimated by the GLLP formula:

\begin{equation}
R \geq p_1^l p_1^\mu \left[ Y_{11}^S [1 - I_{AE}(C, e_{11}^S)] - Q_{\mu u} f_{e} T_{\mu u}^S Q_{\mu u}^{-1} \right],
\end{equation}

where the $Q_{\mu u}^S$ and $T_{\mu u}^S$ are overall gain and overall error rate when Alice and Bob both select intensity $\mu$ and $Z$ basis; the $f_e$ denotes the reconciliation efficiency in key reconciliation phase.

In the non-asymptotic scenario, the data size is finite, thus the statistical fluctuation of observed overall gains and overall error rates must be considered. The statistical fluctuation seriously harms the secret key rate and achievable distance of QKD systems. Especially, compared with other types of QKD protocol, the statistical fluctuation harms the RFI-MDI-QKD system particularly serious since the RFI-MDI-QKD has more kinds of pulse pair which scatters the accumulated data and makes the fluctuation more obvious. Meanwhile, the $\beta$ may drift while accumulating data, which may invalidate the RFI protocol.

In this work, we find that the previous works [31, 38–40] estimate the yield and quantum bit error rate of each basis individually and ignore the potential relations of different basis. By analyzing these relations, we propose an improved decoy method for RFI-MDI-QKD which performs much better in non-asymptotic scenario.
B. improved protocol

In our new decoy-state method, the step 1 is modified as follows:

New step 1: Alice (Bob) randomly selects $X$, $Y$, $Z$ basis or selects vacuum state $\circ$ with no basis. If $Z$ basis is selected, Alice (Bob) only select intensity $\mu_A$ ($\mu_B$); if $X$ or $Y$ basis is selected, Alice (Bob) randomly selects intensity $\nu_A$ or $\omega_A$ ($\nu_B$ or $\omega_B$). In the improved protocol, the weak coherent pulses in $Z$ basis are only employed to generate key bits, we define the $Z$ basis as the signal basis; the $X$ and $Y$ bases are employed to estimated parameters, we define them as decoy basis.

In non-asymptotic scenarios, the differences of observed values and expected values must be taken into consider. Similar to previous works, we estimate the upper and lower bound of $Q$ and $T$ by Chernoff bound $38, 39, 41$:

$$
N^b_{ir}(M^b_{ir}) \leq F^+(N^b_{ir}, M^b_{ir}) = N^b_{ir} + f\left(\frac{\epsilon}{2}\right) \sqrt{N^b_{ir} M^b_{ir}},
$$

$$
N^b_{ir}(M^b_{ir}) \geq F^-(N^b_{ir}, M^b_{ir}) = N^b_{ir} M^b_{ir} - f\left(\frac{\epsilon}{2}\right) \sqrt{N^b_{ir} M^b_{ir}},
$$

(6)

where $M^b_{ir}$ denotes $Q^b_{ir}$ or $T^b_{ir}$, the $\langle M^b_{ir} \rangle$ donates the expected value of $M^b_{ir}$, the $\epsilon$ is the failure probability and $f(x) = \sqrt{2 \ln(x^{-1})}$.

To simplification, we define:

$$
a_k = p_k^\mu, a_k^\nu = p_k^\nu,
$$

$$
b_k = p_k^\omega, b_k^\nu = p_k^\nu,
$$

$$
c_k = p_k^\omega, c_k^\nu = p_k^\nu.
$$

(7)

In the original decoy-state method, the yield and quantum bit error rate of single-photon pulse pairs for each basis are estimated individually:

$$
Y^d_{11} \geq \Sigma^d_{11} = \{ b_1 b_2 Q^b_{11} + c_1 c_2 c_0 Q^b_{00} + c_1 c_2 b_0 Q^d_{11} + b_1 b_2 c_0 c^\nu Q^d_{00} \}
$$

$$
- \{ c_1 c_2 Q^d_{11} + b_1 b_2 c_0 Q^d_{00} + b_1 b_2 c_0 c^\nu Q^d_{11} + b_0 b_1 c_1 c^\nu Q^d_{00} \}/[b_1 b_2 (c_1^\nu + b_1 c_2)] , \text{for } d \in B,
$$

(8)

$$
e^d_{11} \leq e^d_{11} = \left[ T^d_{11} + \alpha_0 a_0 Q^d_{00} - \alpha_0 T^d_{00} \right] / \alpha_1 a_1^\nu \Sigma^d_{11} , \text{for } d \in B,
$$

(9)

$$
Q^b_{ir} = F^+(N^b_{ir} Q^b_{ir})/N^b_{ir}, \quad Q^b_{ir} = F^-(N^b_{ir} Q^b_{ir})/N^b_{ir},
$$

$$
T^b_{ir} = F^+(N^b_{ir} T^b_{ir})/N^b_{ir}, \quad T^b_{ir} = F^-(N^b_{ir} T^b_{ir})/N^b_{ir}.
$$

(10)

Here we introduce our improved decoy-state method based on potentional relations of different basis for estimating secret key rate in non-asymptotic case.

Relation 1: Since the desity matrices of single-photon pulse pair in $XX$, $XY$, $YY$, $YY$ and $ZZ$ bases are quite the same, the yield $Y^{XX}_{11} = Y^{XY}_{11} = Y^{YY}_{11} = Y^{YY}_1 = Y^{ZZ}_{11} 38$.

Based on the Relation 1, we can estimate the $Y^{S}_{11}$ by $X$ or $Y$ basis, namely, the decoy basis. In signal basis $Z$, we only modulate intensity $\mu$ to generate key bits.

Relation 2: When estimating $Y^{S}_{11}$, all pulse pairs in decoy basis can be regarded as an entirety. The set $D$ include all decoy basis, define $Y^{D}_{11}$ as the average yield of single-photon pulse pairs in the joint decoy basis. It is obvious that, the $Y^{S}_{11} = Y^{D}_{11}$.

We define $Q^D_{ir} = \frac{\sum D N^b_{ir} Q^b_{ir}}{N^D_{ir}}$ and $T^D_{ir} = \frac{\sum D N^b_{ir} T^b_{ir}}{N^D_{ir}}$ as the gain and error yield of the weak coherent pulse pair in the joint decoy basis respectively, where the $N^D_{ir} = N^{XX}_{ir} + N^{XY}_{ir} + N^{YY}_{ir} + N^{YY}_{ir}$.

By applying the relation 1, relation 2 and the joint-study method proposed in Ref. 34, we can estimate a tight common lower bound of $Y^{S}_{11}$ for any of the basis by solving the linear programming.
\[
\min: \sum_{11} = \sum_1 = \left\{ [c_1 c'_2(Q_{o\nu}) + b_1 b_2 c_0(Q_{o\nu}) + b_1 b'_2 c'_0(Q_{o\nu}) + b b'_2 c_0 c'_2(Q_{o\nu})] \right. \\
\left. - [b b'_2(Q_{o\nu}) + c_1 c'_2 b_0(Q_{o\nu}) + c_1 c'_2 b'_0(Q_{o\nu}) + b_1 b'_2 c_0 c'_2(Q_{o\nu})] / [b b'_2 (b'_1 c'_2 - c'_1 b)] \right\};
\]

s.t.:
\[
F^+(N_{d5}^{\nu}Q_{o\nu}^{\nu}) \geq N_{d5}^{\nu}(Q_{o\nu}^{\nu}) \geq F^-(N_{d5}^{\nu}Q_{o\nu}^{\nu}); \text{ for any } l, r \in \{\nu, \omega, o\},
\]
\[
F^+(N_{d5}^{\nu}Q_{o\nu}^{\nu} + N_{d6}^{\nu}Q_{o\omega}^{\nu}) \geq N_{d5}^{\nu}(Q_{o\nu}^{\nu}) + N_{d6}^{\nu}(Q_{o\omega}^{\nu}) \geq F^{-}(N_{d5}^{\nu}Q_{o\nu}^{\nu} + N_{d6}^{\nu}Q_{o\omega}^{\nu}),
\]
(11)
\[
F^+(N_{d5}^{\nu}Q_{o\omega}^{\nu} + N_{d6}^{\nu}Q_{o\omega}^{\nu}) \geq N_{d5}^{\nu}(Q_{o\nu}^{\nu}) + N_{d6}^{\nu}(Q_{o\omega}^{\nu}) \geq F^{-}(N_{d5}^{\nu}Q_{o\nu}^{\nu} + N_{d6}^{\nu}Q_{o\omega}^{\nu}),
\]
\[
N_{d5}^{\nu}(Q_{o\nu}^{\nu}) + N_{d6}^{\nu}(Q_{o\omega}^{\nu}) + N_{d7}^{\nu}(Q_{o\omega}^{\nu}) \leq F^{+}(N_{d5}^{\nu}Q_{o\nu}^{\nu} + N_{d6}^{\nu}Q_{o\omega}^{\nu} + N_{d7}^{\nu}Q_{o\omega}^{\nu}),
\]
\[
N_{d5}^{\nu}(Q_{o\nu}^{\nu}) + N_{d6}^{\nu}(Q_{o\omega}^{\nu}) + N_{d7}^{\nu}(Q_{o\omega}^{\nu}) \geq F^{-}(N_{d5}^{\nu}Q_{o\nu}^{\nu} + N_{d6}^{\nu}Q_{o\omega}^{\nu} + N_{d7}^{\nu}Q_{o\omega}^{\nu}).
\]

Relation 3: Any combinations of $XX$, $XY$, $YX$ and $YY$ basis can be regarded as an entirety. Define $e_{11}^{d1d2}$ is the quantum bit error rate of the single-photon pulse pairs in the joint basis of $d_1$ and $d_2$, where $d_1, d_2 \in D$ and $d_1 \neq d_2$. Define $e_{11}^{d}$ is the quantum bit error rate of the single-photon pulse pairs the joint basis $D$. When Alice and Bob select $X$ and $Y$ basis with equal probability, the $e_{11}^{d1d2} = (e_{11}^{d1} + e_{11}^{d2})/2$, the $e_{11}^{d} = (\sum_{d \in D} e_{11}^{d})/4$.

By applying Relation 3, we can introduce more constraints for estimating a tighter lower bound of information leakage. We define:
\[
E_{\lambda, 11}^{d} = \frac{[T_{\lambda}^{d} + p_{\lambda}^{\nu} \beta_{\nu}^{\nu}(T_{o\nu}^{d})] - [p_{\lambda}^{\nu} \beta_{\nu}^{\nu}(T_{o\nu}^{d})]}{p_{\lambda}^{\nu} (\beta_{\nu}^{\nu})^{1}}.
\]
The $e_{11}^{d} \leq \frac{E_{\lambda, 11}^{d}}{Y_{11}}$. Similarly, we define $\frac{E_{\lambda, 11}^{d}}{Y_{11}}$ and $\frac{E_{\lambda}^{d}}{Y_{11}}$. It is obvious that $e_{11}^{d1d2} \leq \frac{E_{\lambda, 11}^{d}}{Y_{11}}$ and $e_{11}^{d} \leq \frac{E_{\lambda}^{d}}{Y_{11}}$.

The $E_{\lambda, 11}^{d}$, $E_{\lambda}^{d}$, and $E_{11}^{d}$ can be estimated by the joint study method 33 as follows:

\[
\text{max: } E_{\lambda, 11}^{d} = \left\{ [(T_{\lambda}^{d})], p_{\lambda}^{\nu} \beta_{\nu}^{\nu}(T_{o\nu}^{d})] - [p_{\lambda}^{\nu} \beta_{\nu}^{\nu}(T_{o\nu}^{d})] \right\}; \text{ for } \lambda \in \{\nu, \omega\}, d \in B,
\]

s.t.:
\[
F^{+}(N_{d5}^{\nu}T_{o\nu}^{\nu}) \geq N_{d5}^{\nu}(T_{o\nu}^{\nu}) \geq F^{-}(N_{d5}^{\nu}T_{o\nu}^{\nu}),
\]
\[
F^{+}(N_{d5}^{\nu}T_{o\nu}^{\nu} + N_{d6}^{\nu}T_{o\omega}^{\nu}) \geq N_{d5}^{\nu}(T_{o\nu}^{\nu}) + N_{d6}^{\nu}(T_{o\omega}^{\nu}) \geq F^{-}(N_{d5}^{\nu}T_{o\nu}^{\nu} + N_{d6}^{\nu}T_{o\omega}^{\nu}),
\]
\[
F^{+}(N_{d5}^{\nu}T_{o\nu}^{\nu} + N_{d6}^{\nu}T_{o\omega}^{\nu}) \geq N_{d5}^{\nu}(T_{o\nu}^{\nu}) + N_{d6}^{\nu}(T_{o\omega}^{\nu}) \geq F^{-}(N_{d5}^{\nu}T_{o\nu}^{\nu} + N_{d6}^{\nu}T_{o\omega}^{\nu}).
\]

max: $E_{\lambda, 11}^{d} = \left\{ [(T_{\lambda}^{d1d2})], p_{\lambda}^{\nu} \beta_{\nu}^{\nu}(T_{o\nu}^{d1d2})] - [p_{\lambda}^{\nu} \beta_{\nu}^{\nu}(T_{o\nu}^{d1d2})] \right\}; \text{ for } \lambda \in \{\nu, \omega\}, d_1, d_2 \in D$ and $d_1 \neq d_2$,

s.t.:
\[
F^{+}(N_{d5}^{d1d2}T_{o\nu}^{d1d2}) \geq N_{d5}^{d1d2}(T_{o\nu}^{d1d2}) \geq F^{-}(N_{d5}^{d1d2}T_{o\nu}^{d1d2}),
\]
\[
F^{+}(N_{d5}^{d1d2}T_{o\nu}^{d1d2} + N_{d6}^{d1d2}T_{o\omega}^{d1d2}) \geq N_{d5}^{d1d2}(T_{o\nu}^{d1d2}) + N_{d6}^{d1d2}(T_{o\omega}^{d1d2}) \geq F^{-}(N_{d5}^{d1d2}T_{o\nu}^{d1d2} + N_{d6}^{d1d2}T_{o\omega}^{d1d2}),
\]
\[
F^{+}(N_{d5}^{d1d2}T_{o\nu}^{d1d2} + N_{d6}^{d1d2}T_{o\omega}^{d1d2}) \geq N_{d5}^{d1d2}(T_{o\nu}^{d1d2}) + N_{d6}^{d1d2}(T_{o\omega}^{d1d2}) \geq F^{-}(N_{d5}^{d1d2}T_{o\nu}^{d1d2} + N_{d6}^{d1d2}T_{o\omega}^{d1d2}).
\]

max: $E_{\lambda, 11}^{d} = \left\{ [(T_{\lambda}^{d})], p_{\lambda}^{\nu} \beta_{\nu}^{\nu}(T_{o\nu}^{d})] - [p_{\lambda}^{\nu} \beta_{\nu}^{\nu}(T_{o\nu}^{d})] \right\}; \text{ for } \lambda \in \{\nu, \omega\},$
By applying the relation 3, a tighter lower bound for the intermediate variable $C$ can be estimated by optimization algorithms:

$$\min : \quad C = \sum_{d \in \mathcal{D}} (1 - 2e_{11}^d)^2,$$

s.t. :

$$e_{11}^d \geq 0,$$
$$e_{11}^d \leq \min(E_{\nu,11}^d, E_{\omega,11}^d)/Y_{11},$$
$$e_{11}^d + e_{11}^d \leq 2 \min(E_{\nu,11}^d, E_{\omega,11}^d)/Y_{11},$$
$$\sum_{d \in \mathcal{D}} e_{11}^d \leq 4 \min(E_{\nu,11}^D, E_{\omega,11}^D)/Y_{11}. \tag{16}$$

Finally, a tighter secret key rate in non-asymptotic scenario can be estimated as follows:

$$R \geq Pr_A^S Pr_B^S a_1 a_1' \{ Y_{11} [1 - I_{AE}(C, \tau_{11}^S)] - Q_{\mu \nu} f H_2 \left( \frac{T_{\mu \nu}}{Q_{\mu \nu}} \right) \}, \tag{17}$$

where $Pr_A^S$ ($Pr_B^S$) denotes the probability that Alice (Bob) sends code mode, the $\tau_{11}^S = \tau_{\mu,11}^S / Y_{11}$ and the intermediate values are:

$$v = \sqrt{C/2 - (1 - \tau_{11}^S)^2} u^2 / \tau_{11}^S,$$
$$u = \min \left( \sqrt{C/2} / (1 - \tau_{11}^S), 1 \right),$$
$$I_{AE}(C, \tau_{11}^S) = (1 - \tau_{11}^S) H_2 [(1 + u) / 2] + \tau_{11}^S H_2 [(1 + v) / 2].$$

III. SIMULATION

In this section, we simulate our new decoy-state method for RFI-MDI-QKD and compare the results with the best known prior article results (Refs. 39, 40) with device parameters which listed in Table I.

Firstly, we illustrate the improvement of estimating the $Y_{11}$ and $C$ in Fig. 1. By employing our relation 2 and relation 3, the $Y_{11}$ and $C$ are significantly estimated tighter, especially when $N_{tot}$ is less than 10$^{11}$.

Then we simulate the optimized secret key rate as a function of transmission distance. We focus on the symmetric case where the distance from Alice to Charlie and from Bob to Charlie are equal, due to the symmetric case, we treat the parameters of Alice Bob equivalently for simplicity. We define $\mu = \mu_A = \mu_B$, $\nu = \nu_A = \nu_B$.

| $\eta_d$ | $p_d$ | $e_d$ | $f_e$ | $\epsilon$ | $\alpha$ |
|---------|-------|------|------|----------|--------|
| 25%     | $10^{-10}$ | 0.5% | 1.16 | $10^{-1}$ | 0.02 dB/km |

TABLE I. The table lists the device parameters of our simulation. The $\eta_d$ denotes the detection efficiency; the $p_d$ denotes the dark count rate; the $e_d$ denotes the misalignment error rate; the $f_e$ denotes the reconciliation efficiency in key reconciliation phase; the $\epsilon$ denotes the failure probability of parameter estimation; the $\alpha$ denotes the fiber loss per km.

$\omega = \omega_A = \omega_B$ and define $Pr_A^S$, $Pr_B^S$, $Pr_\nu^D$, $Pr_\omega^D$ as the probability of Alice (Bob) selects intensity $\mu \nu \omega$ respectively. The probability of selecting vacuum state o is $1 - Pr_\mu^S - Pr_\nu^D - Pr_\omega^D$. Alice (Bob) selects X or Y basis with equal probability. By optimizing $\mu$, $\nu$, $\omega$, $Pr_\mu^S$, $Pr_\nu^D$, $Pr_\omega^D$ with particle swarm optimization algorithm (PSO) [42], we investigate our new decoy method at different data size $N_{tot}$ and compare our result with previous works [32, 41].

In Fig. 2 and Fig. 3 the comparison results at different reference frame misalignment are illustrated. The simulation results indicate that our new method and the previous method have the same performance in the asymptotic case, but when data size is finite, our method performs much better. Our method can generate considerable key rate with about only 10$^{10}$ data size, which is significantly less than previous works. When 50 MHz system [32] is employed, only several minutes are needed to accumulate enough data. Especially, if the GHz high-speed system [11, 14, 43] system is employed, only several seconds are needed, which can make the protocol practically reference frame independent.

The Fig. 4 illustrate the secret key rate as a function of data size $N_{tot}$. The simulation results indicate that our improved method can generate considerable key rate when $N_{tot} = 10^{10}$ while the original method need more than 10$^{11}$ or 10$^{12}$. When the GHz system is employed,
the $N_{\text{tot}} = 10^{10}$ is accepted data size to ensure the system practical RFI. In the scenario of $N_{\text{tot}} = 10^{10}$, $\beta = 0$ and $L = 10$-km, the secure key rate of our method is 130 times of previous method; when $L = 20$ km, our method is 3000 times of previous.

FIG. 1. The estimated $Y_{11}$ and $C$ as a function of data size. The blue and red line denote the estimated value of $Y_{11}$ and $C$, respectively; the solid line denotes our improved method and the dash line denotes the original method; the dash-dot line denotes the asymptotic cases. The $\nu$ and $\omega$ are fixed to 0.2 and 0.05 respectively, the probability of selecting $\mu$, $\nu$, $\omega$ and $\phi$ are equal; the device parameters are listed in Tab. I.

FIG. 2. The secret key rate as function of communication distance. In this figure, the reference frame of Alice and Bob is well aligned. The black dash-dot line denotes the asymptotic case. The solid line and dash line denote, respectively, our improved decoy method and original method; the blue, red and yellow denote, respectively, $N_{\text{tot}}$ is $10^{12}$, $10^{11}$ and $10^{10}$.

FIG. 3. The secret key rate as function of communication distance. In this figure, the reference frame misalignment of Alice and Bob is fixed to $25^\circ$. The black dash-dot line denotes the asymptotic case. The solid line and dash line denote, respectively, our improved decoy method and original method; the blue, red and yellow denote, respectively, $N_{\text{tot}}$ is $10^{12}$, $10^{11}$ and $10^{10}$.

FIG. 4. The secure key rate versus different data size. The reference-frame misalignment $\beta$ is fixed to 0 here. The blue and red line denote, respectively, transmission distance is 20 and 40 km. The solid line denotes our new method, the dash line denotes the previous method and the dash-dot lines are the asymptotic cases.

IV. DISCUSSION

In the practical RFI-MDI-QKD systems, the statistical fluctuation must be taken into consideration. The previous works [38–40] ignore the potential relations of each modes and each basis and only use the worst-case calculation for estimating $Y_{11}$ and $C$. To alleviate the impact of statistical fluctuation, a new decoy-state method for RFI-MDI-QKD is proposed. By employing three rela-
mulations introduced in Sec. [11] and the joint-study method proposed in Ref. [34]. Our method only modulates intensity $\mu$ in $\mathcal{Z}$ basis and estimates the yield by the combination of decoy basis. We also introduce more constraints for intermediate variable $C$, which makes the information leakage estimated much tighter. The simulation results indicate that our method can generate a considerable secret key rate with only $10^{10}$ data size, which is significantly less than the previous method. Our method significantly reduces the time for accumulating data, which mitigates the drift of the reference frame during accumulating data and makes the protocol practically reference frame independent.

Here, we introduce a more intuitive explanation for our improvement. Firstly, in previous works, the intensity $\mu$ is used to generate key rates and parameter estimation. However, in non-asymptotic cases, the optimized intensity for generating key rates is usually not equal to it for parameter estimation. In our new method, we introduce the fourth intensity $\omega$ for parameter estimation and the $\mu$ is not used to estimate $Y_{11}$ anymore, which leads to a higher secret key rate when all parameters are optimized. Secondly, the new method only modulates $\mu$ on $\mathcal{Z}$ basis, which means less observed values are needed. Finally, by considering the potential relations of different bases and different modes, more constraints can be introduced when estimating $\Sigma_1$ and $\Sigma_2$, which makes them estimated much tighter. By applying our improved decoy-stated method, the performance of RFI-MDI-QKD can be greatly improved and the time for data accumulation can be significantly reduced. Our results pave an avenue towards practical RFI-MDI-QKD.

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