New Proofs for Several Properties of Capacities

Guangyan Jia
Qilu Securities Institute for Financial Studies, Shandong University
250199 Jinan, People’s Republic of China
E-mail: jiagy@sdu.edu.cn
Na Zhang*
College of Management and Economics, Tianjin University
China Center for Social Computing, Tianjin University
300072 Tianjin, People’s Republic of China
E-mail: znna1225@163.com

Abstract In this note, we find a new way to prove several properties of 2-alternating capacities.

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1 Introduction

Let \( \Omega \) denote the basic set and \( \mathcal{B} \) the \( \sigma \)-algebra on \( \Omega \). A set function \( c : \mathcal{B} \to [0, 1] \) is called a capacity if it satisfies:

(C1). \( c(\Omega) = 1, c(\emptyset) = 0 \);
(C2)(monotonicity). \( c(A) \leq c(B) \) for any \( A \subseteq B, A, B \in \mathcal{B} \).

A capacity \( c \) is called 2-alternating, if \( c(A \cup B) + c(A \cap B) \leq c(A) + c(B) \). It is called a probability measure if \( c(A \cup B) + c(A \cap B) = c(A) + c(B) \). We usually denote a probability measure by \( P \).

*corresponding author
For any expectation $E$, we can define a capacity $c$ by $c(A) = E[I_A], \forall A \in \mathcal{B}$; on the other hand, for any capacity $c$, we can define expectation through Choquet integral, i.e., $E[X] = \int X \, dc$. Choquet integral was first introduced by Choquet in 1953. The readers can refer to [1] or [2] for more details. In [2], Denneberg proved the following result.

**Lemma 1.1 ([2, Chapter 6])** If the integral with respect to a capacity $c$ is subadditive, i.e.,

$$\int (X + Y) \, dc \leq \int X \, dc + \int Y \, dc,$$

then $c$ is 2-alternating. Conversely, let $c$ be a 2-alternating capacity, then for any $\mathcal{B}$-measurable square integrable functions $X, Y$,

$$\int (X + Y) \, dc \leq \int X \, dc + \int Y \, dc.$$ 

In order to prove the above result, Denneberg proved the following result.

**Lemma 1.2 ([2, Lemma 6])** Suppose that $A_1, A_2, \ldots, A_n$ is a partition of $\Omega$, $\mathcal{B}$ is a $\sigma$-algebra generated by $A_1, A_2, \ldots, A_n$ and $c: \mathcal{B} \to [0,1]$ is a capacity. For any permutation $\pi$ of $(1, \ldots, n)$, we define

$$S^\pi_i := \bigcup_{j=1}^i A_{\pi_j}, \quad i = 1, \ldots, n, \quad S^\pi_0 := \emptyset.$$ 

We define a probability measure $P^\pi$ on $\mathcal{B}$ by

$$P^\pi(A_{\pi_i}) := \mu(S^\pi_i) - \mu(S^\pi_{i-1}), \quad i = 1, \ldots, n.$$ 

Suppose $X$ is a $\mathcal{B}$-measurable real valued function $X$ defined on $\Omega$. If $\mu$ is 2-alternating, then

$$\int X \, d\mu \geq \int X \, dP^\pi.$$ 

If $X(A_{\pi_1}) \geq X(A_{\pi_2}) \geq \ldots \geq X(A_{\pi_n})$, the above equality holds.

Since Choquet integral is positive homogeneous, any Choquet expectation generated by a 2-alternating capacity is sublinear expectation. Jia [3] defined a partial order "$\leq$" on the set of expectations as follows:

for any two expectations $E_1$ and $E_2$, $E_1 \leq E_2$ if for any $\mathcal{B}$-measurable square integrable random variable $X$, $E_1[X] \leq E_2[X],$

and proved the following results.

**Lemma 1.3 ([3, Theorem 2.7 and Theorem 3.1])** $E$ is a minimal member of the set of all the sublinear expectations if and only if $E$ is a linear expectation. Suppose $E_1$ is a subadditive expectation, $E_2$ is a superadditive expectation and $E_1 \geq E_2$, then there exists a linear expectation $E_0$ such that $E_1 \geq E_0 \geq E_2$. 

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Therefore we may wonder if the minimal members of all the 2-alternating capacities are exactly the probability measures? In fact, the answer is positive, since the following result holds:

Suppose $Q$ is a set, $\prec$ is a semiorder defined on $Q$ and $U$ denotes the set of all the minimal members in $Q$. Thus for any set $Z$ satisfying $U \subset Z \subset Q$, the set of all the minimal members of $Z$ is still $U$.

In this note, we’ll give another proof of the above results by means of capacity only. The method of constructing a probability measure step by step from a 2-alternating capacity is also given.

2 Main results

First we list the following definitions which will be used below. A capacity defined on $(\Omega, \mathcal{B})$ is said to be:

- **2-monotone** if $c(A \cup B) + c(A \cap B) \geq c(A) + c(B)$;
- **n-alternating** if $c(\bigcap_{i=1}^{n} A_i) \leq \sum_{\emptyset \neq I \subset \{1, \ldots, n\}} (-1)^{|I|+1} v(\bigcup_{i \in I} A_i), \forall A_1, \ldots, A_n \in \mathcal{B}$;
- **n-monotone** if $c(\bigcup_{i=1}^{n} A_i) \geq \sum_{\emptyset \neq I \subset \{1, \ldots, n\}} (-1)^{|I|+1} v(\bigcap_{i \in I} A_i), \forall A_1, \ldots, A_n \in \mathcal{B}$;
- **$\infty$-alternating** if $c$ is n-alternating, for all $n$;
- **$\infty$-monotone** if $c$ is n-monotone, for all $n$.

Furthermore, we have the following notations.

- $\mathcal{A}_n$ denotes the set of $n$-alternating capacities, for any $n \geq 2$;
- $\mathcal{M}_n$ denotes the set of $n$-monotone capacities, for any $n \geq 2$;
- $\mathcal{P}$ denotes the set of probability measures;
- $\mathcal{A}_\infty$ denotes the set of $\infty$-alternating capacities;
- $\mathcal{M}_\infty$ the set of $\infty$-monotone capacities.

It is known that $\mathcal{P} \subseteq \mathcal{A}_\infty \subseteq \mathcal{A}_{n+1} \subseteq \mathcal{A}_n$, $\mathcal{P} \subseteq \mathcal{M}_\infty \subseteq \mathcal{M}_{n+1} \subseteq \mathcal{M}_n$ and $\mathcal{A}_n \cap \mathcal{M}_m = \mathcal{P}$ for any $n \geq 2$ and $m \geq 2$.

Now let us define the partial order "$\leq$": for any two capacities $c_1$ and $c_2$, $c_1 \leq c_2$ means that $c_1(A) \leq c_2(A)$, for all $A \in \mathcal{B}$. If $c_1 \leq c_2$, we can also denote by $c_2 \geq c_1$. If $c_1 \leq c_2$ and $c_1 \geq c_2$, we have $c_1 = c_2$.

The following lemma holds.
Lemma 2.1 Let $\mathcal{T} \subset \mathcal{A}_2$ be a nonempty set and totally ordered (for each pair $c_1, c_2 \in \mathcal{T}$, one has either $c_1 \leq c_2$ or $c_2 \leq c_1$). Then the set function

$$\nu(A) \triangleq \inf_{c \in \mathcal{T}} c(A), \quad A \in \mathcal{B},$$

is a 2-alternating capacity, that is $\nu \in \mathcal{A}_2$.

Proof. It is obvious that $\nu(\Omega) = 1, \nu(\emptyset) = 0$ and $\nu$ is monotone. We now prove that it is 2-alternating.

$$\nu(A \cap B) = \inf_{c \in \mathcal{T}} c(A \cap B)$$

$$\leq \inf_{c \in \mathcal{T}} \{c(A) + c(B) - c(A \cup B)\}$$

$$\leq \inf_{c \in \mathcal{T}} \{c(A) + c(B)\} - \nu(A \cup B).$$

Since $\mathcal{T}$ is totally ordered, for $c_1, c_2 \in \mathcal{T}$, we suppose, without lost of generality, that $c_1 \leq c_2$, so $c_1(A) + c_2(B) \geq c_1(A) + c_1(B)$. Therefore,

$$\nu(A \cap B) \leq \inf_{c_1, c_2 \in \mathcal{T}} \{c_1(A) + c_2(B)\} - \nu(A \cup B)$$

$$= \inf_{c \in \mathcal{T}} \{c(A)\} + \inf_{c \in \mathcal{T}} \{c(B)\} - \nu(A \cup B)$$

$$= \nu(A) + \nu(B) - \nu(A \cup B).$$

Thus the result follows. \qed

Theorem 2.1 Any $P \in \mathcal{P}$ is a minimal member of $\mathcal{A}_2$. Conversely, if $c$ is a minimal member of $\mathcal{A}_2$, then $c \in \mathcal{P}$.

Proof. Suppose $c \in \mathcal{A}_2, c \leq P$. Then we have

$$\forall A \in \mathcal{B}, 1 - c(A^c) \leq c(A) \leq P(A) = 1 - P(A^c).$$

Since $c(A^c) \leq P(A^c)$, we have $c(A) = P(A)$, which means that there is no 2-alternating capacity $c$ satisfying $c \leq P$, i.e., $P$ is a minimal member of $\mathcal{A}_2$.

If $c$ is a minimal member of $\mathcal{A}_2$, for a fixed $A \in \mathcal{B}$, we define

$$c^A(B) := c(A \cup B) + c(A \cap B) - c(A), \forall B \in \mathcal{B}.$$

Obviously, $c^A \leq c$, $c^A(\Omega) = c(\Omega) + c(A) - c(A) = 1, c^A(\emptyset) = c(A) + 0 - c(A) = 0$. The monotonicity of $c^A$ can be easily deduced by the monotonicity of $c$. For any $B \in \mathcal{B}, F \in \mathcal{B}$,

$$c^A(B \cup F) + c^A(B \cap F) = c(A \cup (B \cup F)) + c(A \cap (B \cap F)) - c(A)$$

$$+ c(A \cup (B \cap F)) + c(A \cap (B \cup F)) - c(A)$$

$$= c((A \cup B) \cup (A \cup F)) + c((A \cap B) \cup (A \cap F))$$

$$+ c((A \cup B) \cap (A \cap F)) + c((A \cap B) \cap (A \cap F)) - 2c(A)$$

$$\leq c(A \cup B) + c(A \cup F) + c(A \cap B) + c(A \cap F) - 2c(A)$$

$$= c^A(B) + c^A(F),$$
i.e., $c^A \in A_2$. Note that $c$ is the minimal member of $A_2$, thus $c^A = c$, which means that, for any $B \in \mathcal{B}$, we have $c(A \cup B) + c(A \cap B) = c(A) + c(B)$. Since $A$ can be any set in $\mathcal{B}$, $c$ is a probability measure. □

**Remark 2.1** 1. By similar proof we can deduce that any minimal member of $A_n$ ($n \geq 2$) (resp. $A_\infty$) can only be probability measure and any probability measure is its minimal member.

2. The maximal member of $\mathcal{M}_n$ ($n \geq 2$) (resp. $\mathcal{M}_\infty$) can only be probability measure and any probability measure is its maximal member.

**Definition 2.1** For a capacity $c$, we define the invariant subfield $\mathcal{B}^c$ of $c$ as follows:

$$\mathcal{B}^c \triangleq \{ A \in \mathcal{B} : \forall B \in \mathcal{B}, c(A \cup B) + c(A \cap B) = c(A) + c(B) \}$$

It is obvious that $\mathcal{B}^c$ is nonempty, since $\Omega \in \mathcal{B}^c$ and $\emptyset \in \mathcal{B}^c$. A capacity $c$ is a probability measure if and only if $\mathcal{B}^c = \mathcal{B}$. Note that if $c$ is a 2-alternating capacity, then for all $A \in \mathcal{B}$, such that $c(A) = 0$, we have $A \in \mathcal{B}^c$. If $c$ is a 2-monotone capacity, then for all $A \in \mathcal{B}$, such that $c(A) = 1$, we have $A \in \mathcal{B}^c$.

∀ $c \in A_2$, it has been proved that $c^F \in A_2$. Thus we can define the following mapping.

**Definition 2.2** For all $F \in \mathcal{B}$, we define mapping $\Pi^F : A_2 \to A_2$ as follows:

$$\Pi^F c = c^F.$$  

**Proposition 2.1** The following properties about invariant subfield and the above mapping hold.

(i) $\forall c \in A_2$, $\Pi^F(c) \leq c$;

(ii) $\forall A \in \mathcal{B}$, if $A \subset F$ or $F \subset A$, one has $c^F(A) = c(A)$;

(iii) $\forall A \in \mathcal{B}^c$, $c^F(A) = c(A)$;

(iv) $F \in \mathcal{B}^c$;

(v) $\mathcal{B}^c \subset \mathcal{B}^{c^F}$;

(vi) If $F \in \mathcal{B}^c$, $c^F = c$.

**Proof.** (i) and (vi) are obvious.

(ii). Without lost of generality, suppose that $A \subset F$, thus $c^F(A) = c(A \cup F) + c(A \cap F) - c(F) = c(F) + c(A) - c(F) = c(A)$.

(iii). For all $A \in \mathcal{B}^c$, $c^F(A) = c(F \cup A) + c(F \cap A) - c(F) = c(A)$. 

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(iv). According to (ii), for all \( A \in \mathcal{B} \),

\[
\begin{align*}
& c^F(F \cup A) + c^F(F \cap A) - c^F(F) - c^F(A) \\
& = c(F \cup A) + c(A \cap F) - c(F) - (c(A \cup F) + c(A \cap F) - c(F)) = 0.
\end{align*}
\]

(v). Suppose \( A \in \mathcal{B}^c \), \( B \in \mathcal{B} \),

\[
\begin{align*}
& c^F(A \cup B) + c^F(A \cap B) - c^F(A) - c^F(B) \\
& = c(F \cup (A \cup B)) + c(F \cap (A \cup B)) - c(F) + c(F \cup (A \cap B)) \\
& + c(F \cap (A \cap B)) - c(F) - c(A) - c(F \cup B) - c(F \cap B) + c(F) \\
& = [c(A) + c(F \cup B) - c(A \cap (F \cup B))] + c(F \cap (A \cup B)) - c(F) \\
& + c(F \cup (A \cap B)) + [c(A) + c(F \cap B) - c(A \cup (F \cap B))] \\
& - c(F) - c(A) - c(F \cup B) - c(F \cap B) + c(F) \\
& = c(A) - c(A \cap (F \cup B)) + c(F \cap (A \cup B)) - c(F) + c(F \cup (A \cap B)) \\
& - c(A \cup (F \cap B)) \\
& = [c(F \cup (A \cap B)) - c(A \cap (F \cup B))] \\
& + [c(F \cap (A \cup B)) - c(A \cup (F \cap B))] + c(A) - c(F) \\
& = [c((F \cup (A \cap B)) \cup A) - c(A)] + [c((F \cap (A \cup B)) \cap A) - c(A)] \\
& + c(A) - c(F) \\
& = [c(F \cup A) - c(A)] + [c(F \cap A) - c(A)] + c(A) - c(F) = 0.
\end{align*}
\]

With the help of this mapping, we can prove Lemma 1.2, i.e., the following theorem, by way of capacity.

**Theorem 2.2** Consider \((\Omega, \mathcal{B})\). Suppose that \( \mathcal{B} \) is finite, \( c \) is a 2-alternating capacity defined on \( \mathcal{B} \). Take \( F_1, ..., F_n \in \mathcal{B} \) such that \( F_1 \subset F_2 \subset ... \subset F_n \). Thus there exists a probability measure \( P \), such that \( P(F_i) = c(F_i) \), for all \( i = 1, ..., n \) and \( P \leq c \).

**Proof.** First, we design a cyclic program as follows.

Set \( \mu = c \).

**Step I:** Check \( F_i, i = 1, ..., n \). If all the sets \( F_i \) belong to the invariant subfield of \( \mu \), go straight to Step III. Otherwise, suppose that \( F_i \) does not belong to the subfield of \( \mu \). By Proposition 2.1, the following result holds:

\[
\mu^{F_i}(F_j) = \mu(F_j), \forall j = 1, ..., n,
\]
i.e., $\mu^F_i$ and $\mu$ are equal on $F_j, j = 1, \ldots, n$;

$$F_i \in B^{\mu^F_i}, \quad B^\mu \subset B^{\mu^F_i},$$

i.e., from $\mu$ to $\mu^F_i$, the invariant subfield is enlarged and $F_i$ is also included.

**Step II**: Update $\mu$ by $c^F_i$. The invariant subfield of $\mu$ is enlarged by Step I. Repeating the procedures in Step I.

**Step III**: We get the final $\mu$, which satisfies $\mu \in \mathcal{A}_2$, $\mu \leq c$, and for all $i = 1, \ldots, n$, $\mu(F_i) \equiv c(F_i)$, $F_i \in B^\mu$, $B \subset B^\mu$.

Next, we consider $\mu$, and design another cyclic program.

**Step 1**: Check $B^\mu$ and $B$. If they are the same, go straight to Step 3. Otherwise, suppose $A \in B/B^\mu c^F_i$. Consider the transformation of $\mu$ induced by $A$. By Proposition 2.1, we have $\mu^A(F_i) \equiv \mu(F_i)$, $i = 1, \ldots, n$.

**Step 2**: Update $\mu$ by $\mu^A$. The invariant subfield of $\mu$ is enlarged. Repeat the procedures in Step 1.

**Step 3**: $\mu$ satisfies the following conditions: for all $i = 1, \ldots, n$, $\mu(F_i) = c(F_i)$, $\mu \leq c$. Furthermore, $B^\mu = B$, thus $\mu$ is just the probability measure satisfying the conditions needed. The proof is complete.  

**Theorem 2.3** Consider space $(\Omega, \mathcal{B})$. Suppose that $\mathcal{B}$ is finite, $\mu$ is a 2-alternating capacity defined on $\mathcal{B}$, $\nu$ is a 2-monotone capacity defined on $\mathcal{B}$. If $\mu \geq \nu$, there exists a probability measure $P$ such that $\mu \geq P \geq \nu$.

**Proof.** Since $\mathcal{B}$ is finite, we can take $A \in \mathcal{B}/B^\mu$ such that

$$\mu(A) - \nu(A) = \min_{B \in \mathcal{B}/B^\mu} \{\mu(B) - \nu(B)\}.$$  

Make transformation $\Pi^A$ on $\mu$, thus

$$\mu^A(B) = \mu(A \cup B) + \mu(A \cap B) - \mu(A) \geq \nu(A \cup B) + \nu(A \cap B) - \nu(A) \geq \nu(B),$$

i.e., $\mu \geq \mu^A \geq \nu$. By Proposition 2.1,

$$B^\mu \subset B^{\mu^A},$$

and $B^\mu \neq B^{\mu^A}$.

For $\mu^A$, repeat the above procedure, until we get a capacity $P$, such that $\mathcal{B}^P = \mathcal{B}$. $P$ satisfying that $\mu \geq P \geq \nu$. The proof is complete.  

**Remark 2.2** According the above theorem, we may get different probability measures if we make transformation by different sets or in a different order.
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