Pricing and Hedging Long-Term Options

Hyungbin Park
Courant Institute of Mathematical Sciences
Department of Mathematics,
New York University
18 Jul 2014

Abstract

In this article, we investigate the behavior of long-term options. In many cases, option prices follow an exponential decay (or growth) rate for further maturity dates. We determine under what conditions option prices are characterized by this property. To see this, we use the martingale extraction method through which a pricing operator is transformed into a semigroup operator, which is easier to address.

We also explore notions of hedging long-term options. Hedging is an attempt to reduce market risks, and we investigate the price sensitivities (Greeks) with respect to such risks, which are typically represented by variations in the underlying process of an option. We combine the Malliavin calculus with the martingale extraction method to analyze Greeks. We see that the ratios between Greeks and the option price are expressed in a simple form in the long term.

Key Words: Long-term options, Greeks, Martingale extraction, Malliavin Calculus

1 Introduction

We study the behavior of long-term option prices and their associated price sensitivities (Greeks). In financial mathematics, the current price of an option whose payoff occurs at time $T$ is expressed by

$$ p_T := \mathbb{E}_Q \left[ e^{-\int_0^T r_s ds} \cdot (\text{payoff}) \right] $$

for some measure $Q$, where $r_t$ is the short interest rate. Measure $Q$ is defined below. This article investigates the price of a long-term option; thus, we examine the behavior of $p_T$ as $T$ goes to infinity. We also explore the Greeks of long-term options with respect to variations in the underlying process of an option. We will demonstrate that Greeks divided by the option price can be expressed in a simple form over the long run.

To clarify the meaning of a financial market, we formally define a financial market as a probability space $(\Omega, \mathcal{F}, L)$ having a Brownian motion and filtration $\mathcal{F} = (\mathcal{F}_t)_{t=0}^\infty$ that is generated by the Brownian motion. For purposes of simplicity, we assume that the Brownian motion is one-dimensional. It is straightforward to extend to the multi-dimensional case (except for section 2.2). The measure $L$ is referred to as the objective measure or the real-world measure of the market. All processes in this article are assumed to be adapted to the filtration $\mathcal{F}$. In this financial market, there are two processes. One is a short interest rate that is denoted by $r_t$ and is assumed to be nonnegative. The process defined by

$$ e^{\int_0^t r_s ds} $$

is called a money-market account. The other process is a risky asset, which is denoted by $S_t$ and is typically expressed in the following stochastic differential equation (SDE) form:

$$ dS_t = \mu_t S_t dt + \sigma_t S_t dW_t $$

*hyungbin@cims.nyu.edu, hyungbin2015@gmail.com
for some processes $\mu_t$ and $\sigma_t$. It is assumed that $\sigma_t$ is not equal to zero.

The fundamental theorem of asset pricing posits that if $F_t$ is a price of an asset or an option in a financial market and if the market has no-arbitrage, then there exists a measure $Q$ such that

$$e^{-\int_0^T r_s \, ds} F_t$$

is a martingale under $Q$. We say that the measure $Q$ is a risk-neutral measure of this market. Thus, the current price of an option with maturity $T$ is

$$p_T = \mathbb{E}_Q^T \left[ e^{-\int_0^T r_s \, ds} \cdot \text{payoff} \right].$$

For example, if we denote the price of a stock by $S_t$, then the current price of the call option of the stock is $\mathbb{E}[e^{-\int_0^T r_s \, ds} (S_T - K)_+]$ where $K$ is the strike price of the call option.

We assume in the financial market that there is a time-homogeneous Markov diffusion process that is denoted by $X_t$ and that determines the interest rate and the risky asset: The interest rate $r_t$ and the risky asset $S_t$ are expressed as a function of $X_t$. More precisely, $r_t$ is expressed by $r_t = r(X_t)$ for some function $r(\cdot)$, and $S_t$ is also expressed as $S_t = h(X_t)$ for some function $h(\cdot)$. We define the process $X_t$ as the driver process of this market and the function $r(\cdot)$ as the interest rate function. The payoff of an option whose underlying asset is $S_t$ with maturity $T$ is expressed by

$$\text{(payoff)} = f(X_T)$$

for some $f(\cdot)$. For example, the payoff function of the call option is $f(y) = (h(y) - K)_+$. This function $f$ is called the payoff function of the option. In conclusion, we obtain the following expression for the current price of an option with maturity $T$:

$$p_T = \mathbb{E}_Q^T \left[ e^{-\int_0^T r_s \, ds} f(X_T) \right]$$

where $x = X_0$ for the payoff function $f$. We restate these assumptions more precisely as follows.

**Assumption 1.** In the financial market, there is a time-homogeneous Markov diffusion process $X_t$ with the following SDE form:

$$dX_t = b(X_t) \, dt + \sigma(X_t) \, dW_t$$

$$X_0 = x$$

where $W_t$ is a Brownian motion under $Q$. Here, $b(\cdot)$ and $\sigma(\cdot)$ are Lipschitz continuous. This process $X_t$ is called a driver process. We assume that there are functions, $r(\cdot)$ and $f(\cdot)$, that are called the interest rate function and the payoff function, respectively, such that the interest rate $r_t$ is expressed by $r_t = r(X_t)$, and the payoff of an option with maturity $T$ is expressed by $f(X_T)$.

One purpose of this article is to investigate the behavior of price $p_T$ of long-term options. Given $X_t$ and $r(\cdot)$, the value $p_T$ for large $T$ is highly sensitive for payoff function $f$. For example, consider the Black-Scholes model, which describes the case in which $r(\cdot)$ is a constant function. i.e., $r(\cdot) = r$ for some nonnegative $r$ and a stock price $S_t$ that follows a geometric Brownian motion. Let the driver process be equal to the stock price, that is, let $X_t = S_t$. Then,

$$dX_t = rX_t \, dt + \sigma X_t \, dW_t$$

where $W_t$ is a Brownian motion under $Q$. In this model, if $r - \frac{1}{2} \sigma^2 > 0$, then the stock price $X_T$ will typically be found at very large values for large time $T$. Thus, if the payoff function $f$ has a compact support in $[0, \infty]$, then stock price $X_T$ will lie outside of the range of payoffs with increasing probability. Consequently, the option value $p_T$ will be very small for large time $T$. By contrast, consider the price of the call option. The payoff function is $f(y) = (y - K)_+$, which does not have compact support, and the payoff function retains significant value even when maturity $T$ is very large. Therefore, the decay rate of $p_T$ is zero; in fact, $p_T$ converges to $X_0$ as $T$ goes infinity. To further demonstrate the dependency of option prices on payoff function, consider $f(y) = (y^2 - K)_+$. In this case, the price grows exponentially and the rate is $e^{(r+\sigma^2)T}$. We will see more details in section 4.5.
As we have seen, for a given \( X_t \) and \( r(\cdot) \), the behavior of \( p_T \) is determined by \( f \). Given a payoff function \( f \) which is financially practical for many but not all cases, price \( p_T \) decays (or grows) at an exponential rate in time \( T \). When the price converges, we say for the sake of convenience that it decays (or grows) at an exponential rate of zero. More precisely,

\[
\beta := \lim_{T \to \infty} \left( -\frac{1}{T} \log p_T \right)
\]

exists, and the limit

\[
\lim_{T \to \infty} e^{\beta T} p_T
\]

also exists and is nonzero. Below, we will show that \( \lim_{T \to \infty} e^{\beta T} p_T \) depends on the initial value \( x \), that we denote by \( l(x) := \lim_{T \to \infty} e^{\beta T} p_T \).

This implies that price \( p_T \) of a long-term option decays (\( \beta \geq 0 \)) or grows (\( \beta \leq 0 \)) exponentially at rate

\[
e^{-\beta T} l(x).
\]

**Notation.** Let \( p_T \) and \( q_T \) be two nonzero functions of \( T \). We denote this by \( p_T \cong q_T \) if

\[
\lim_{T \to \infty} \frac{p_T}{q_T} = 1.
\]

Using the notation, we can write

\[
p_T \cong e^{-\beta T} l(x)
\]

in the shorter form.

Two important questions arise:

(i) Under what conditions on \( X_t, r(\cdot) \) and \( f \), does the price of an option decay (or grow) exponentially in time? (more precisely, when \( p_T \) satisfies \( p_T \cong e^{-\beta T} l(x) \) for some \( \beta \) and nonzero function \( l(x) \)).

(ii) If \( p_T \) satisfies \( p_T \cong e^{-\beta T} l(x) \), then how can we find the value \( \beta \) and the function \( l(x) \)?

We investigate these topics below in section 2. Hansen and Scheinkman in [16], [17] and [18] proposed a brilliant method for modeling the long run and applied it to many economic cases. This method is known as the *martingale extraction* method, and we explore it below.

The next topic in this article is a sensitivity analysis of the price of long-term options with respect to the perturbation of driver process \( X_t \), where price sensitivities are referred to as *Greeks*. Recall that

\[
dX_t = b(X_t) \, dt + \sigma(X_t) \, dW_t
\]

\[
X_0 = x.
\]

Let \( X'_t \) be a perturbed process of \( X_t \) (with the same initial value \( x = X_0 = X'_0 \)), then the perturbed option price is then given by

\[
\mathbb{E}_x^Q \left[ e^{-\int_0^T r(X'_s) \, ds} f(X'_T) \right]
\]

and we denote this by \( p'_T \). For the sensitivity analysis, we compute

\[
\left. \frac{\partial}{\partial \epsilon} \right|_{\epsilon=0} p'_T
\]

and investigate the behavior of this quantity for large \( T \). The sensitivity of the perturbation of the drift term \( b(X_t) \) is called the *rho*, and the sensitivity of the diffusion term \( \sigma(X_t) \) is called the *vega*. The sensitivity of the initial value \( x \) is given by

\[
\left. \frac{\partial p_T}{\partial x} \right|_{x=0} \left( = \left. \frac{\partial}{\partial x} \mathbb{E}_x^Q \left[ e^{-\int_0^T r(X_s) \, ds} f(X_T) \right] \right) \right.
\]
Suppose $p_T$ satisfies $p_T \cong e^{-\beta T} l(x)$. When $T$ is large, because $e^{-\beta T}$ dominates the price $p_T$, we can expect that the long-term behaviors of the rho and the vega are mainly determined by $e^{-\beta T}$. Thus, more precisely, assume that $p_T^\varepsilon$ also satisfies $p_T^\varepsilon \cong e^{-\beta(\varepsilon)T} l_\varepsilon(x)$ for some function $\beta(\varepsilon)$ and $l_\varepsilon(x)$. We may then expect

$$\frac{\partial}{\partial \varepsilon} \bigg|_{\varepsilon=0} p_T^\varepsilon \cong -\beta'(0) T \cdot e^{-\beta T} l(x) + e^{-\beta T} \frac{\partial}{\partial \varepsilon} \bigg|_{\varepsilon=0} l_\varepsilon(x)$$

and we thus obtain the following simple equation:

$$\frac{\partial}{\partial \varepsilon} \bigg|_{\varepsilon=0} p_T^\varepsilon \cong -\beta'(0).$$

For the delta, because $\beta$ is independent of the initial value of $X_t$ - as we will soon see - we have

$$\frac{\partial p_T}{\partial x} \cong e^{-\beta T} l'(x),$$

thus, we obtain

$$\frac{\partial p_T}{\partial x} \cong \frac{l'(x)}{l(x)}.$$

To justify this argument, we use the method of Fournie in [12], in which there is a remarkable technique for calculating Greeks using the Malliavin calculus. Unfortunately, this method cannot be applied to functionals of the following form:

$$E^Q_x \left[ e^{-\int_0^T r(X_s) ds} f(X_t) \right],$$

and this is the form that interests us for option pricing. This method (for calculating the delta and vega) is valid only for discretely monitored functionals of the following form:

$$E^Q_x[f(X_{t_1}, X_{t_2}, \cdots, X_{t_m})]$$

such that the perturbed process $X^\varepsilon_t$ is detected only for finite times up to maturity $T$. In our case, however, the expectation contains the term

$$e^{-\int_0^T r(X_s) ds}$$

which depends on the entire path of $X_t$ up to time $T$; therefore, the perturbed process $X^\varepsilon_t$ is also detected for the whole path up to time $T$.

The martingale extraction is useful in overcoming this problem. While applying the martingale extraction, the Fournie method is able to be successfully applied to our cases, and we see this in section 3. Without the martingale extraction method, it may be helpful to use the functional Ito calculus proposed by Dupire in [10], which is an extension of the Ito calculus to functions of whole paths. Cont in [8] achieved remarkable results with the functional Ito calculus. Jazaerli and Saporito in [21] develop an approach to compute the Greeks for path-dependent options using the functional Ito calculus. In this article, however, we do not employ this method.

The following provides an overview of this article. First, in section 2, we investigate under what conditions on $X_t$, $r(\cdot)$ and $f$, the option price decays or grows at an exponential rate. Under this circumstance, we explore Greeks over the long term in section 3. Section 4 presents several examples. The last section summarizes the paper.

## 2 Pricing long-term options

In this section, we develop answers to two important questions posed in the introduction. We discuss the following two approaches: spectral decomposition and martingale extraction.
2.1 Pricing operator and generator

It is occasionally useful to regard an option price $p_T$ as an operator of payoff functions. Given $X_t$ and $r(\cdot)$, we define the \textit{pricing operator} by

$$P_T f(x) := \mathbb{E}_x^Q \left[ e^{-\int_0^T r(X_s) \, ds} f(X_T) \right].$$

where $x = X_0$. We denote the \textit{infinitesimal operator} of this pricing operator by $\mathcal{L}$ and its domain by $D(\mathcal{L})$. It is recognized that $C^2_c(\mathbb{R}) \subseteq D(\mathcal{L})$ and the infinitesimal operator is

$$\mathcal{L}g = \frac{1}{2} \sigma^2 g'' + bg' - rg$$

for $g \in C^2_c(\mathbb{R})$.

2.2 Spectral decomposition

Consider the \textit{speed measure} $\mu$ of $X_t$ defined by $d\mu(y) := w(y) dy$ where

$$w(y) = \frac{1}{\sigma^2(y)} e^{\int_y^{\infty} \frac{2b(z)}{\sigma^2(z)} dz}.$$

The speed measure $\mu$ is assumed to be $\sigma$-finite with respect to the Lebesgue measure on $\mathbb{R}$. From this assumption, we know that $C^2_c(\mathbb{R}) \subseteq L^2(\mu)$ and it is easy to check that $C^2_c(\mathbb{R})$ is dense in $L^2(\mu)$. We denote the inner product in $L^2(\mu)$ by $\langle \cdot, \cdot \rangle$. In other words,

$$\langle f, g \rangle := \int fg \, d\mu.$$

We can directly compute to obtain the following:

$$\langle \mathcal{L}(f), g \rangle = \langle f, \mathcal{L}(g) \rangle \quad \text{for} \quad f, g \in C^2_c(\mathbb{R}).$$

and

$$\langle f, \mathcal{L}(f) \rangle \leq 0.$$

For more details, see [2] and [31]. Therefore, the infinitesimal operator $\mathcal{L}$ is a \textit{densely defined symmetric nonpositive} operator from $L^2(\mu)$ to itself.

For convenience, we denote $-\mathcal{L}$ by $A$ so operator $A$ is a densely defined symmetric \textit{nonnegative} operator from $L^2(\mu)$ to itself. Let $\overline{A}$ be a self-adjoint extension of $A$; so that $\overline{A}$ is a densely defined self-adjoint \textit{nonnegative} operator from $L^2(\mu)$ to itself. We shall denote $A = \overline{A}$ with no ambiguity. Because $A$ is self-adjoint, using the spectral theory, we write

$$A = \int_{\mathbb{R}^+} \lambda \, dE_\lambda$$

for some spectral measure $E$, where $\mathbb{R}^+ = \{ y \geq 0 \}$. We obtain

\textbf{Theorem 2.1. (Spectral Decomposition)}

$$P_T = e^{-AT} = \int_{\mathbb{R}^+} e^{-\lambda T} \, dE_\lambda.$$

The proof is by either \textit{Stone’s theorem} or the \textit{Hille-Yosida theorem}. We also used the fact that $\sigma(A) \subseteq \mathbb{R}^+$. Here, $\sigma(A)$ denotes the set of all the spectrums of $A$.

Because a self-adjoint operator has no residual spectrum, the spectrum of $A$ is expressed by $\sigma(A) = \sigma_p(A) \cup \sigma_c(A)$, where $\sigma_p(A)$ and $\sigma_c(A)$ denotes the set of eigenvalues (or point spectrum) and the set of continuous spectrum of $A$, respectively. We assume that $\sigma_p(A)$ is nonempty, i.e., that it has at least one
eigenvalue. Denote the eigenfunction corresponding to the eigenvalue \( \lambda \in \sigma_p(A) \) by \( \phi_\lambda \). By the theorem 2.1 for \( f \in L^2(\mu) \), we have
\[
P_T f = \int_{\mathbb{R}^+} e^{-\lambda T} d\langle E_\lambda, f \rangle \\
= \sum_{\lambda \in \sigma_p(A)} c_\lambda e^{-\lambda T} \phi_\lambda + \int_{\lambda \in \sigma_c(A)} e^{-\lambda T} d\langle E_\lambda, f \rangle .
\]

**Property 2.1.** The \( \sigma_p(A) \) point spectrum is located below the continuous spectrum \( \sigma_c(A) \) on \( \mathbb{R}^+ \). More precisely, for any \( \lambda \in \sigma_p(A) \) and for any \( \xi \in \sigma_c(A) \), we have \( \lambda < \xi \).

**Property 2.2.** The point spectrum \( \sigma_p(A) \) is countable, discrete and has a minimum value.

For details of the two above properties, see [14]. From these facts, we know the behavior of \( P_T f(x) \) for large \( T \) is determined by the minimum eigenvalue, or \( \beta \), when \( \langle f, \phi_\beta \rangle \neq 0 \).

**Property 2.3.** The eigenfunction corresponding to the minimum eigenvalue has no zeros, and we can thus make it positive. The other eigenfunctions always have zeros.

From this property, we always assume that the the eigenfunction corresponding to the minimum eigenvalue is positive.

The following theorem gives partial answers for the two questions mentioned in the introduction.

**Theorem 2.2.** Suppose the negative infinitesimal operator \( A \) has at least one eigenvalue. Let \( \beta \) be the minimum eigenvalue and denote its eigenfunction by \( \phi \). If \( f \in \text{Dom}(A) \subseteq L^2(\mu) \) and \( f \geq 0, f \neq 0, \mu\text{-almost surely} \), then \( \kappa := \langle f, \phi \rangle/ \langle \phi, \phi \rangle \) is positive and
\[
\lim_{T \to \infty} \left( -\frac{1}{T} \log P_T f(x) \right) = \beta \\
\lim_{T \to \infty} e^{\beta T} P_T f(x) = \phi(x) \kappa
\]
are obtained.

Note that the eigenfunction corresponding to the minimum eigenvalue is positive; thus, \( \langle f, \phi \rangle \) is positive for \( f \) with \( f \geq 0, f \neq 0 \). This theorem gives the decay rate of the price of a long-term option. When the payoff function \( f \) is in \( L^2(\mu) \), the decay rate is equal to the minimum eigenvalue of the negative infinitesimal operator. Also note that when \( f \) is not positive, this theorem remains valid if \( \langle f, \phi \rangle \neq 0 \).

The set \( L^2(\mu) \) is determined by \( X_t \). For certain \( X_t \), the set is large enough that most financially practical payoff functions are included in the set. Intuitively, if \( X_t \) goes to infinity (or minus infinity) with high probability as \( t \to \infty \), then \( f(x) \) should decay rapidly for large \( |x| \) to be in \( L^2(\mu) \). That is, \( f \) should be small in some sense. Conversely, if \( X_t \) lies on a compact set with high probability, then \( L^2(\mu) \) includes functions that do not decay as rapidly or occasionally grow as \( |x| \to \infty \); thus, the set \( L^2(\mu) \) becomes larger.

As we will see, when \( X_t \) is an interest rate, most financially practical payoff functions are included in the set \( L^2(\mu) \) because, in practice, an interest rate typically does not go to infinity with high probability. Most interest rate models are typically mean-reverting or have an invariant measure, and high probability of the invariant measure lies on a compact set. Therefore, the theorem above is useful in these cases.

By contrast, when \( X_t \) goes to infinity (or minus infinity) with high probability, then the payoff function should be small to apply the above theorem. For example, let \( X_t \) be the stock price in the Black-Scholes model, i.e., \( X_t \) is a geometric Brownian motion:
\[
dX_t = rX_t \, dt + \sigma X_t \, dW_t .
\]
Suppose \( r - \frac{1}{2}\sigma^2 > 0 \). It is easy to check that \( f_c(y) = (y - K)_+ \) is not in \( L^2(\mu) \), thus the theorem above cannot be applied to see the behavior of long-term call options.

How can we analyze the behavior of a long-term option if the payoff function is not in \( L^2(\mu) \)? For such a function \( f \), we use the martingale extraction in the section below. We will review this method and discuss its pros and cons.
2.3 Martingale extraction

Hansen and Scheinkman in [16], [17], [18] proposed a brilliant way to analyze the price of a long-term option. The method is known as the martingale extraction, which we review in this section. At this time, we must distinguish the meaning of a solution pair from an eigenvalue and its eigenfunction. Let \((\beta, \phi)\) be a solution pair of \(\mathcal{L}\phi = -\beta\phi\) with a positive function \(\phi\) (suppose it exists). Next, it is easily checked that

\[ M_t := e^{-\int_0^t r(X_s)ds} e^{\beta t} \phi(X_t) \phi^{-1}(x) \]

is a local martingale. When the local martingale \(M_t\) is a martingale, the expression

\[ e^{-\int_0^t r(X_s)ds} = M_t \phi^{-1}(X_t) \phi(x) e^{-\beta t} \]

is called the martingale extraction of \(e^{-\int_0^t r(X_s)ds}\) with respect to \((\beta, \phi)\).

Suppose \(M_t\) is a martingale; in this case, we can define a new measure in the following way. Define a measure \(P\) on \((\Omega, \mathcal{F})\) by

\[ P(A) := \int_A M_t \, dQ = \mathbb{E}^Q [\mathbb{I}_A M_t] \quad \text{for } A \in \mathcal{F}_t. \]

The measure \(P\) is called the transformed measure from \(Q\) with respect to \((\beta, \phi)\). The definition is well-defined: If \(A \in \mathcal{F}_t\), then for \(0 < t < s\), we have \(\mathbb{E}^Q [\mathbb{I}_A M_t] = \mathbb{E}^Q [\mathbb{I}_A M_s]\). Using this transformed measure \(P\), the pricing operator \(P_T f(x)\) can be expressed by

\[ P_T f(x) = \mathbb{E}_x^P \left[ e^{-\int_0^T r(X_s)ds} f(X_T) \right] = \phi(x) e^{-\beta T} \cdot \mathbb{E}_x^P \left[ (\phi^{-1} f)(X_T) \right]. \tag{2.1} \]

**Theorem 2.3.** Suppose there is a solution pair \((\beta, \phi)\) of

\[ \mathcal{L}\phi = -\beta\phi \]

with a positive function \(\phi\) and suppose \(e^{-\int_0^t r(X_s)ds} e^{\beta t} \phi(X_t) \phi^{-1}(x)\) is a martingale. Let \(P\) be the transformed measure with respect to \((\beta, \phi)\). If

\[ \mathbb{E}_x^P \left[ (\phi^{-1} f)(X_T) \right] \]

converges to a nonzero constant as \(T\) goes to infinity, say the limit \(\kappa(x)\), then

\[ \lim_{T \to \infty} \left( -\frac{1}{T} \log P_T f(x) \right) = \beta \]

or

\[ \lim_{T \to \infty} e^{\beta T} P_T f(x) = \phi(x) \kappa(x). \]

are obtained.

The theorem above explains two questions that are first discussed in the introduction. The behavior of the price of long-term option is obtained from the theorem represented above.

This theorem encompasses a much larger class of payoff functions \(f\) when compared with theorem 2.2. For example, as we have observed, theorem 2.2 cannot be applied to long-term call option prices in the Black-Scholes model, but this theorem can be so applied. We will examine more details in section 3.3.3.

By contrast, theorem 2.3 has certain flaws when put to practical use. To use this theorem, we must find a solution pair \((\beta, \phi)\) that satisfy the described conditions in the theorem above. However, this theorem does not even tell us about the existence of such a solution pair. This is important because, as a general matter, such a solution pair may not exist. We see an example of such non-existence in section 3.3.3. The existence of the solution pair is an essential part of theorem 2.2. Additionally, even if such a solution pair exists, it
may not be easy to find. Finally, it is no easier to compute the limit value \( \kappa \) when compared with theorem 2.2 in which \( \kappa \) was obtained by the inner product.

In conclusion, it is not easy to know whether the given \( X_t, r(\cdot) \) and \( f \) satisfy the hypothesis of theorem 2.3. To overcome this obstacle, Hansen and Scheinkman in [16], [17], [18] assumed additional special structure in \( X_t \) so we can determine it more easily. Additionally, under this assumption, the possibility of convergence of

\[
E^P_x \left[ (\phi^{-1} f)(X_T) \right]
\]

and the value \( \kappa \) are also relatively easily obtained, which we will see in section 2.5.

In the remainder of this section, we illustrate the relationship between theorem 2.2 and theorem 2.3.

**Proposition 2.1.** Suppose the negative infinitesimal operator \( A \) has at least one eigenvalue. Let \( \beta \) be the minimum eigenvalue and denote its eigenfunction by \( \phi \). Assume that \( e^{-\int_0^t r(X_s)ds} \ e^{\beta t} \ \phi(X_t) \ \phi^{-1}(x) \) is a martingale. If \( f \) is in \( \text{Dom}(A) \) and \( f \geq 0 \), \( f \neq 0 \), \( \mu \)-almost surely, then

\[
E^P_x \left[ (\phi^{-1} f)(X_T) \right]
\]

converges to a nonzero constant, say \( \kappa \), as \( T \) goes to infinity, where \( P \) is the transformed measure with respect to \( (\beta, \phi) \).

In fact, \( \kappa = \langle f, \phi \rangle / \langle \phi, \phi \rangle \). In other words, the pair \( (\beta, \phi) \) satisfies the hypothesis of theorem 2.2.

### 2.4 Pricing semigroup

Recall (2.1):

\[
P_T f(x) = E^Q_x \left[ e^{-\int_0^T r(X_s)ds} f(X_T) \right] = \phi(x) \ e^{-\beta T} \ E^P_x \left[ (\phi^{-1} f)(X_T) \right].
\]

Define a semigroup operator by

\[
U_T h(x) := E^P_x[h(X_T)].
\]

Next, we obtain the following expression:

\[
P_T f(x) = \phi(x) \ e^{-\beta t} \cdot U_T (\phi^{-1} f)(x).
\]

This relationship implies that the pricing operator can be expressed by using the semigroup operator, which is relatively more manageable than the pricing operator and will be useful for the sensitivity analysis in section 3.

Recall the process \( X_t \) in assumption [1]

\[
dX_t = b(X_t) \ dt + \sigma(X_t) \ dW_t, \quad X_0 = x
\]

where \( W_t \) is a Brownian motion under \( Q \). We observe how the dynamic of \( X_t \) is changed when the underlying measure is changed from the risk-neutral measure \( Q \) to the transformed measure \( P \). We know that the Radon-Nikodym derivative of \( Q \) with respect to \( P \) on \( F_t \) is

\[
M_t = e^{-\int_0^t r(X_s)ds} \ e^{\beta t} \ \phi(X_t) \ \phi^{-1}(x).
\]

Using the Ito formula, we have

\[
\frac{dM_t}{M_t} = \sigma(X_t) \phi'(X_t) \phi^{-1}(X_t) \ dW_t.
\]

For convenience, put

\[
\varphi(\cdot) := \sigma(\cdot) \phi'(\cdot) \phi^{-1}(\cdot).
\]

By the Girsanov theorem, we know that a process \( B_t \) defined by

\[
B_t := W_t - \int_0^t \varphi(X_s) \ ds
\]
is a Brownian motion under $P$. Therefore, $X_t$ follows
\[
dX_t = b(X_t) \, dt + \sigma(X_t) \, dW_t
\]
\[
= (b(X_t) + \sigma(X_t) \varphi(X_t)) \, dt + \sigma(X_t) \, dB_t.
\]
This equation gives us the dynamic of $X_t$ under $P$.

2.5 Invariant probability and ergodicity

In this section, we investigate the ergodic properties of an invariant distribution. Recall theorem 2.3. As a special case, when $X_t$ has an invariant distribution under $P$, ergodic theory is useful for checking the possibility of convergence of
\[
U_T(\phi^{-1}f)(x)
\]
and to compute the limit value $\kappa$.

**Theorem 2.4.** ($L^2$-ergodicity)
Assume that $X_t$ has an invariant distribution under $P$, say $\nu$. For $h \in L^2(\nu)$,
\[
\lim_{T \to \infty} U_T h = \int h \, d\nu
\]
pointwise and in $L^2(\nu)$.

For proof, see [3] and [9]. Therefore, in theorem 2.4 if $\phi^{-1}f \in L^2(\nu)$ then $E^P_x [(\phi^{-1}f)(X_T)]$ is convergent as $T$ goes to infinity and the limit is equal to $\int \phi^{-1}f \, d\nu$. Occasionally, the following proposition is useful.

**Proposition 2.2.** Assume that $X_t$ has an invariant distribution under $P$, say $\nu$. If $E^P_x [(\phi^{-1}f)^2(X_t)]$ is bounded for $t$, then $\phi^{-1}f \in L^2(\nu)$. Thus,
\[
\kappa = \int \phi^{-1}f \, d\nu
\]
is obtained.

Occasionally, we require a more delicate tool to check the convergence. We state the Lyapunov criteria for this purpose.

**Theorem 2.5.** (Lyapunov criteria)
Assume that $X_t$ has an invariant distribution under $P$, say $\nu$. Let $h \geq 0$. If these constants, $a > 0$ and $b < \infty$, exist such that
\[
\mathcal{L}^P h(x) \leq -ah(x) + b\varphi_K(x),
\]
where $\mathcal{L}^P$ is the infinitesimal operator of the semigroup $U_T$ and $K$ is a compact set, then
\[
\lim_{T \to \infty} U_T h(x) = \int h \, d\nu.
\]

For more details, see [3], [26] and [29].

3 Sensitivity analysis

Hedging options are used to try to reduce market risks, which are typically represented by variations in the underlying process of the option. The sensitivity of the option price to a change in the underlying process is called Greeks. We investigate the behavior of Greeks in long-term options when the underlying driver process $X_t$ is perturbed. In this section, we assume that $X_t$, $r(\cdot)$ and $f$ satisfy the hypothesis of theorem 2.3.
We see the sensitivity behavior of $P$.

In this section, we examine the derivative with respect to the initial value:

### 3.1 The delta

To analyze the Greeks, we use (2.1).

\[ \frac{\partial}{\partial x} P_T f(x) \]

This quantity is called the delta of the option price. We have

\[
\frac{\partial}{\partial x} P_T f(x) = \frac{\phi'(x)}{\phi(x)} + \frac{\partial}{\partial x} U_T(\phi^{-1}f)(x) \quad (3.1)
\]

To analyze the second term, the following fact from [12] is useful.

\[
\frac{\partial}{\partial x} U_T(\phi^{-1}f)(x) = E^P_x \left[ (\phi^{-1}f)'(X_T) \cdot Y_T \right]
\]

\[
= E^P_x \left[ (\phi^{-1}f)(X_T) \cdot \frac{1}{T} \int_0^T \sigma^{-1}(X_s) Y_s dB_s \right] \quad (3.2)
\]

Here, $Y_t$ is the first variation process of $X_t$ under $\mathbb{P}$, defined by

\[ dY_t = \left( b + \sigma \varphi \right)'(X_t) Y_t dt + \sigma'(X_t) Y_t dB_t, \quad Y_0 = 1. \]

This process $Y_t$ measures the derivative of $X_t$ with respect to the initial value $x$, that is, $Y_t = \frac{\partial X_t}{\partial x}$. Intuitively, we may expect that if $X_t$ has an invariant distribution, then $Y_t$ becomes small in some sense as $t$ goes to infinity because the distribution of $X_t$ converges to the invariant distribution, and the invariant distribution is independent of the initial value.

For convenience, we put

\[ \Delta_T := \frac{\partial}{\partial x} P_T f(x). \]

**Theorem 3.1.** Under assumption [1] and [2], if $E^P_x [(\phi^{-1}f)'(X_T) \cdot Y_T]$ goes to zero as $T$ goes to infinity, then

\[ \lim_{T \to \infty} \Delta_T = \phi'(x) \quad (3.3) \]

is obtained.

The proof is direct by (3.1) and (3.2). This theorem implies that the ratio between the delta and the option price in the long run can be expressed in a simple form: $\frac{\phi'(x)}{\phi(x)}$.

**Proposition 3.1.** Under assumptions [1] and [2], if both $E^P_x [(\phi^{-1}f)^2(X_t)]$ and $E^P_x [(\sigma^{-1}(X_t)Y_t)^2]$ are bounded for $t$, then

\[ \lim_{T \to \infty} \Delta_T = \phi'(x) \quad (3.4) \]

See appendix [A] for proof.
3.2 The rho

Consider the perturbed process $X_t^\epsilon$ defined by

$$dX_t^\epsilon = (b(X_t^\epsilon) + \epsilon \dot{b}(X_t^\epsilon)) dt + \sigma(X_t^\epsilon) dW_t.$$ 

Set the pricing operator with respect to this perturbed process by

$$P_T^\epsilon f(x) := \mathbb{E}_x^\mathbb{P} \left[ e^{-\int_0^T r(X_s^\epsilon) ds} f(X_T) \right].$$

In this section, we explore the sensitivity with respect to the perturbation $\dot{b}(\cdot)$ in the drift term. The quantity

$$\left. \frac{\partial}{\partial \epsilon} \right|_{\epsilon=0} P_T^\epsilon f(x)$$

is called the rho of the option price.

We assume that $X_t^\epsilon$, $r(\cdot)$ and $f$ satisfy assumption 2 and accordingly define $\beta(\epsilon)$, $\phi_\epsilon$, $\mathbb{P}_x$ and $\varphi_\epsilon$, the meanings of which are self-explanatory. Using the martingale extraction and the transformed measure $\mathbb{P}_x$, we have

$$P_T^\epsilon f(x) = \phi_\epsilon(x) e^{-\beta(\epsilon)T} \cdot \mathbb{E}_x^\mathbb{P}_x \left[ (\phi_\epsilon^{-1} f)(X_T^\epsilon) \right].$$

Differentiate with respect to $\epsilon$ and evaluate at $\epsilon = 0$, then

$$\left. \frac{\partial}{\partial \epsilon} \right|_{\epsilon=0} P_T^\epsilon f(x) = \left. \frac{\partial}{\partial \epsilon} \right|_{\epsilon=0} \phi_\epsilon(x) - \beta'(0) T$$

$$+ \left. \frac{\partial}{\partial \epsilon} \right|_{\epsilon=0} \mathbb{E}_x^\mathbb{P}_x \left[ (\phi_\epsilon^{-1} f)(X_T) \right] + \left. \frac{\partial}{\partial \epsilon} \right|_{\epsilon=0} \mathbb{E}_x^\mathbb{P}_x \left[ (\phi^{-1} f)(X_T) \right].$$

For the last term, the following proposition is useful.

**Proposition 3.2.** Assume that $\phi_\epsilon(x)$ is differentiable at $\epsilon = 0$. Let $\overline{\varphi}(x)$ be the first order term of $\varphi_\epsilon(x)$ with respect to $\epsilon$ at $\epsilon = 0$.

$$\overline{\varphi}(x) := \left. \frac{\partial}{\partial \epsilon} \right|_{\epsilon=0} \phi_\epsilon(x).$$

Then,

$$\left. \frac{\partial}{\partial \epsilon} \right|_{\epsilon=0} \mathbb{E}_x^\mathbb{P}_x \left[ (\phi^{-1} f)(X_T) \right] = \mathbb{E}_x^\mathbb{P}_x \left[ (\phi^{-1} f)(X_T) \int_0^T (\sigma^{-1} \dot{b} + \overline{\varphi})(X_s) dB_s \right].$$

See appendix A for proof. For convenience, we put

$$\rho_T := \left. \frac{\partial}{\partial \epsilon} \right|_{\epsilon=0} P_T^\epsilon f(x)$$

and denote $\beta'(0)$ by $\overline{\varphi}$.

**Theorem 3.2.** Under assumption 2 and 3, if $\mathbb{E}_x^\mathbb{P} \left[ \left. \frac{\partial}{\partial \epsilon} \right|_{\epsilon=0} \phi_\epsilon^{-1} f(X_t) \right]$, $\mathbb{E}_x^\mathbb{P} \left[ (\phi^{-1} f)^2(X_t) \right]$ and $\mathbb{E}_x^\mathbb{P} \left[ (\overline{\varphi} + \sigma^{-1} \dot{b})^2(X_t) \right]$ are bounded for $t$, then

$$\lim_{T \to \infty} \frac{\rho_T}{T \cdot \rho_T} = -\overline{\varphi}.$$

is obtained.

This gives the behavior of the rho of long-term options. Occasionally, the bounded conditions in the hypothesis for this theorem can be easily checked by the tools stated in section 2.5.
3.3 The vega

Consider the perturbed process \( X_t^\epsilon \) defined by

\[
dX_t^\epsilon = b(X_t^\epsilon) \, dt + (\sigma(X_t^\epsilon) + \epsilon \sigma(X_t^\epsilon)) \, dW_t .
\]

Set

\[
P_T f(x) := \mathbb{E}_x^\mathbb{P} \left[ e^{-\int_0^T r(X_s^\epsilon) \, ds} f(X_T^\epsilon) \right].
\]

We assume that \( X_t^\epsilon, r(\cdot) \) and \( f \) satisfy assumption 2 and accordingly define \( \beta(\epsilon), \phi_\epsilon, \mathbb{P}_\epsilon \) and \( \varphi_\epsilon \), the meanings of which are self-explanatory. To distinguish from the notations used in section 3.2, we use \( \gamma(\epsilon), \pi_\epsilon \) and \( \psi_\epsilon \), instead of \( \beta(\epsilon), \phi_\epsilon \) and \( \varphi_\epsilon \), respectively.

Similarly to section 3.2, we have

\[
\frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} P_T f(x) = \frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} \psi_\epsilon(x) - \gamma'(0) T
\]

\[
+ \frac{\mathbb{E}_x^\mathbb{P} \left[ (\varphi_\epsilon^{-1} f)(X_T^\epsilon) \right]}{\mathbb{E}_x^\mathbb{P} \left[ (\varphi_\epsilon^{-1} f)(X_T^\epsilon) \right]} + \frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} \mathbb{E}_x^\mathbb{P} \left[ (\varphi_\epsilon^{-1} f)(X_T^\epsilon) \right].
\]

For the last term, the following proposition is useful.

Proposition 3.3. Assume \( \psi_\epsilon(x) \) is differentiable at \( \epsilon = 0 \). Let \( \overline{\psi}(x) \) be the first order term of \( \psi_\epsilon(x) \) with respect to \( \epsilon \) at \( \epsilon = 0 \).

Then

\[
\frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} \mathbb{E}_x^\mathbb{P} \left[ (\varphi_\epsilon^{-1} f)(X_T^\epsilon) \right] = \mathbb{E}_x^\mathbb{P} \left[ (\varphi_\epsilon^{-1} f)(X_T^\epsilon) \int_0^T (\overline{\psi} + \sigma^{-1} \dot{\sigma})(X_s) \, dB_s \right]
\]

\[
+ \mathbb{E}_x^\mathbb{P} \left[ (\varphi_\epsilon^{-1} f)'(X_T^\epsilon) \cdot Z_T \right].
\]

where \( Z_t \) is defined by

\[
dZ_t = (b + \sigma \dot{\varphi})'(X_t) Z_t \, dt + \sigma'(X_t) Z_t \, dB_t + \dot{\sigma}(X_t) \, dB_t , \quad Z_0 = 0 .
\]

See appendix A for proof. We set

\[
v_T := \frac{\partial}{\partial \epsilon} \bigg|_{\epsilon=0} P_T f(x)
\]

and denote \( \gamma'(0) \) by \( \overline{v} \).

Theorem 3.3. Under assumption 2 and 3, if \( \mathbb{E}_x^\mathbb{P} \left[ \frac{\partial}{\partial \epsilon} \big|_{\epsilon=0} \pi_\epsilon^{-1} f(X_t^\epsilon) \right], \mathbb{E}_x^\mathbb{P} \left[ (\varphi_\epsilon^{-1} f)(X_t^\epsilon) \right], \mathbb{E}_x^\mathbb{P} \left[ (\overline{\psi} + \sigma^{-1} \dot{\sigma})^2(X_t^\epsilon) \right] \)

and \( \mathbb{E}_x^\mathbb{P} \left[ (\varphi_\epsilon^{-1} f)'(X_t^\epsilon) \cdot Z_t \right] \) are bounded for \( t \), then

\[
\lim_{T \to \infty} \frac{v_T}{T \cdot p_T} = -\overline{v}
\]

is obtained.

Occasionally, the following proposition is useful to estimate the quantity \( \mathbb{E}_x^\mathbb{P} \left[ (\varphi_\epsilon^{-1} f)'(X_t^\epsilon) \cdot Z_t \right] \).

Proposition 3.4. Let \( \delta \) be the divergence operator under \( \mathbb{P} \). Then we have

\[
\mathbb{E}_x^\mathbb{P} \left[ (\varphi_\epsilon^{-1} f)'(X_t^\epsilon) \cdot Z_t \right] = \frac{1}{t} \mathbb{E}_x^\mathbb{P} \left[ (\varphi_\epsilon^{-1} f)(X_t^\epsilon) \cdot \delta(\sigma^{-1}(X_s) Y_s Y_{t-s}^{-1} Z_t) \right].
\]

Here, \( s \) is the parameter for the divergence operator \( \delta \).

For more details for the divergence operator \( \delta \), see [28].
4 Applications

4.1 The CIR model

We explore the case in which the driver process is the interest rate process. Assume the interest rate \( r_t \) follows the CIR model:

\[
dr_t = a(\theta - r_t) \, dt + \sigma \sqrt{r_t} \, dW_t
\]

and we assume \( 2a\theta \geq \sigma^2 \) such that the interest rate of zero is precluded. We use theorem 2.2 to find the behavior of the price of long-term options. The associated infinitesimal operator is

\[
L g(r) = \frac{1}{2} \sigma^2 r g''(r) + a(\theta - r) g'(r) - r g(r)
\]

and the minimum eigenvalue of \( A = -L \) is

\[
\beta = ka\theta
\]

with its eigenfunction

\[
\phi(r) = e^{-kr}
\]

where \( k := \frac{\sqrt{a^2+2\sigma^2} - a}{\sigma^2} \). The speed measure \( \mu \) (or the invariant measure) of \( r_t \) is given by (up to constant multiples)

\[
d\mu(r) = \frac{r \frac{2a\theta}{\sigma^2} - 1 - e^{-\frac{2a\theta}{\sigma^2} r}}{(\frac{2a\theta}{\sigma^2})^{\frac{2a\theta}{\sigma^2}} \cdot \Gamma(\frac{2a\theta}{\sigma^2})} \, dr.
\]

Here, \( \Gamma(u) = \int_0^\infty y^{u-1} e^{-y} \, dy \) is the gamma function, and the denominator is a normalizing constant for the total measure of \( \mu \) to be equal to 1.

4.1.1 Interest rate options

Because the speed measure \( \mu \) decays exponentially, we know that any function whose growth rate is equal to or less than a polynomial function is in \( L^2(\mu) \). In particular, as concrete examples, we see functions such as \( f_b(r) \equiv 1 \), \( f_p(r) := (R - r)_+ \), \( f_c(r) := (r - R)_+ \).

By analyzing these functions, we obtain the behavior of the prices of long-term bonds, (fixed-rate) put options, and call options, denoted by \( p^b_T \), \( p^p_T \) and \( p^c_T \), respectively. We have

\[
\lim_{T \to \infty} \left( -\frac{1}{T} \log p_T \right) = k a\theta = \frac{(\sqrt{a^2+2\sigma^2} - a) a\theta}{\sigma^2}
\]

for \( p_T = p^b_T \), \( p^p_T \) and \( p^c_T \) and

\[
\lim_{T \to \infty} e^{(ka\theta)T} \cdot p_T = e^{-k\rho_o \kappa}
\]

for \( \kappa = \kappa_b , \kappa_p \) and \( \kappa_c \), the meanings of which are self-explanatory.

We compute \( \kappa_b , \kappa_p \) and \( \kappa_c \). By using theorem 2.2, we know those \( \kappa \)'s are computed by the inner product; thus, by direct calculation

\[
\kappa_b = \langle f_b, \phi \rangle / \langle \phi, \phi \rangle = \left( \frac{2\sqrt{a^2+2\sigma^2}}{a + \sqrt{a^2+2\sigma^2}} \right)^{\frac{2a\theta}{\sigma^2}}
\]

\[
\kappa_p = \frac{1}{\left( \frac{2\sqrt{a^2+2\sigma^2}}{\sigma^2} \right)^{\frac{2a\theta}{\sigma^2}} \cdot \Gamma(\frac{2a\theta}{\sigma^2})} \int_0^R (R - r)^{\frac{2a\theta}{\sigma^2} - 1} e^{-\left( \frac{\sqrt{a^2+2\sigma^2} - a}{\sigma^2} \right) r} \, dr
\]

\[
\kappa_c = \frac{1}{\left( \frac{2\sqrt{a^2+2\sigma^2}}{\sigma^2} \right)^{\frac{2a\theta}{\sigma^2}} \cdot \Gamma(\frac{2a\theta}{\sigma^2})} \int_R^\infty (r - R)^{\frac{2a\theta}{\sigma^2} - 1} e^{-\left( \frac{\sqrt{a^2+2\sigma^2} + a}{\sigma^2} \right) r} \, dr
\]
are obtained.

We investigate the price behavior of a long-term (fixed-rate) swap. Consider the payoff function

\[ f_s(r) := r - R. \]

This function has negative values for small \( r \), so we cannot apply theorem 2.2 directly; indeed, there is a possibility that \( \langle \phi, f_s \rangle = 0 \). See the following remark for more about this phenomenon. Denote the price of this swap by \( p_s^T \). Observe that \( p_s^T = p_c^T - p_p^T \). Therefore,

\[
\lim_{T \to \infty} e^{(ka\theta)T} p_s^T = \lim_{T \to \infty} e^{(ka\theta)T} (p_c^T - p_p^T) = e^{-kr_0} (\kappa_p - \kappa_c)
\]

\[
= e^{-kr_0} \left( \frac{2a\theta}{\alpha + \sqrt{\alpha^2 + 2\sigma^2}} - R \right) \left( \frac{2\sqrt{\alpha^2 + 2\sigma^2}}{\alpha + \sqrt{\alpha^2 + 2\sigma^2}} \right).
\]

For the last equation, the gamma function formulas

\[
\frac{\Gamma(u)}{\alpha^u} = \int_0^\infty y^{u-1}e^{-\alpha y} dy
\]

\[
\Gamma(u + 1) = u\Gamma(u)
\]

are useful.

**Remark 4.1.** If \( \frac{2a\theta}{\alpha + \sqrt{\alpha^2 + 2\sigma^2}} - R = 0 \), then \( \lim_{T \to \infty} e^{(ka\theta)T} p_s^T = 0 \); thus, the decay rate is not determined by the minimum eigenvalue. The decay rate is in fact determined by the second eigenvalue or by another eigenvalue; consequently, it decays more rapidly to zero than the decay rate of case \( \frac{2a\theta}{\alpha + \sqrt{\alpha^2 + 2\sigma^2}} - R \neq 0 \).

However even for case \( \frac{2a\theta}{\alpha + \sqrt{\alpha^2 + 2\sigma^2}} - R = 0 \), if we change the parameter \((\theta, a, \sigma)\) by a tiny amount, then \( \frac{2a\theta}{\alpha + \sqrt{\alpha^2 + 2\sigma^2}} - R \neq 0 \) is satisfied; thus, the decay rate is \( e^{-(ka\theta)T} \). This result indicates that the behavior of a long-term option is highly sensitive when parameters \((\theta, a, \sigma)\) are located near \( \frac{2a\theta}{\alpha + \sqrt{\alpha^2 + 2\sigma^2}} - R = 0 \).

**Remark 4.2.** The bond price \( p_T \) with the CIR interest rate model has a closed form of solution.

\[ p_T = A(T)e^{-C(T)r_0} \]

with

\[
C(T) = \frac{\sinh(\xi T)}{\xi \cosh(\xi T) + \frac{1}{2}a \sinh(\xi T)}
\]

\[
A(T) = \left( \frac{\xi e^{\frac{1}{2}a T}}{\xi \cosh(\xi T) + \frac{1}{2}a \sinh(\xi T)} \right)^{\frac{2a\theta}{\alpha^2}}
\]

where \( \xi = \frac{1}{2}\sqrt{a^2 + 2\sigma^2} \). We can use this formula to compute the long-term behavior of bond prices, and the results are the same as above.

### 4.1.2 Pricing semigroup

Recall section 2.4. Let \( \mathbb{P} \) be the transformed measure with respect to \((ka\theta, e^{-kr})\). The Radon-Nikodym derivative \( M_t \) satisfies

\[
dM_t = \varphi(r_t)M_t dW_t
\]

with

\[ \varphi(r) = -\sigma k \sqrt{r}. \]
(We can easily show that $M_t$ is a martingale by checking the Novikov condition.) We know that a process $B_t$ defined by

$$dB_t = dW_t + \sigma \sqrt{r_t} \, dt$$

is a Brownian motion under $\mathbb{P}$. The interest rate $r_t$ follows

$$dr_t = \sqrt{a^2 + 2\sigma^2} \left( \frac{a\theta}{\sqrt{a^2 + 2\sigma^2}} - r_t \right) \, dt + \sigma \sqrt{r_t} \, dB_t .$$

The interest rate $r_t$ has an invariant distribution under $\mathbb{P}$ and is given by

$$d\nu(r) = \frac{2a\theta}{\sigma^2} - 1 \, e^{-2\sqrt{a^2 + 2\sigma^2}r} \left( \frac{2a\theta}{\sigma^2} + 2 \right) \Gamma \left( \frac{2a\theta}{\sigma^2} \right) \, dr .$$

### 4.1.3 The delta

We investigate the behavior of the delta for long-term options. We use proposition 3.1. First, for a payoff function $f$ whose growth rate is equal to or less than a polynomial function, we show that the condition that $\lim_{t \to \infty} \mathbb{E}_x^P[(\phi^{-1} f)^2(r_t)]$ is bounded for $t$ is satisfied. Using theorem 2.5, it can be shown that $\mathbb{E}_x^P[e^{\nu r_t}]$ is convergent as $t$ goes to infinity for $m < \frac{2\sqrt{a^2 + 2\sigma^2}}{\sigma^2}$. Because $2k < \frac{2\sqrt{a^2 + 2\sigma^2}}{\sigma^2}$, we know that

$$\mathbb{E}_x^P[(\text{polynomial})^2 e^{2k r_t}]$$

is bounded for $t$. Thus, we arrive at the desired result. Second, we show that $\mathbb{E}_x^P[(r_t^{-\frac{1}{2}} Y_t)^2]$ is bounded for $t$. The process $Y_t$ is given by

$$dY_t = -\sqrt{a^2 + 2\sigma^2} Y_t \, dt + \frac{1}{2} \sigma r_t^{-\frac{1}{2}} Y_t \, dB_t .$$

Using the Ito formula, we have

$$d(r_t^{-\frac{1}{2}} Y_t) = \left( -\frac{1}{2} a\theta + \frac{1}{8} \sigma^2 \right) r_t^{-1} - \frac{\sqrt{a^2 + 2\sigma^2}}{2} r_t^{-\frac{1}{2}} Y_t \, dt$$

$$r_t^{-\frac{1}{2}} Y_t = r_0^{-\frac{1}{2}} e^{-\left( -\frac{1}{2} a\theta + \frac{1}{8} \sigma^2 \right) t_0} \int_0^t r_s^{-1} ds - \frac{\sqrt{a^2 + 2\sigma^2}}{2} t ,$$

and because we assumed $2a\theta \geq \sigma^2$, we obtain

$$r_t^{-\frac{1}{2}} Y_t \leq r_0^{-\frac{1}{2}} e^{-\frac{\sqrt{a^2 + 2\sigma^2}}{2} t} \text{ pathwise} .$$

Thus, $\mathbb{E}_x^P[(r_t^{-\frac{1}{2}} Y_t)^2]$ is bounded for $t$ (and in fact converges to zero).

In conclusion, for options whose payoff is equal to or less than a polynomial function, we can apply proposition 3.1. As concrete cases, functions can be defined by

$$f_b(r) = 1 , \quad f_p(r) = (R - r)_+ , \quad f_c(r) = (r - R)_+ ;$$

thus, we obtain the behavior of the long-term bond delta, the (fixed-rate) put option and the call option:

$$\lim_{t \to \infty} \frac{\partial}{\partial r} \frac{\mathbb{E}_x^P}{\mathbb{P}} = \phi'(r_0) = \phi(r_0) = -k = \frac{-\sqrt{a^2 + 2\sigma^2} + a}{\sigma^2} .$$

For the (fixed-rate) swap whose payoff function is $f_s(r) = r - R$, we have the same result when $\frac{2a\theta}{a + \sqrt{a^2 + 2\sigma^2}} - R \neq 0$. 

15
4.1.4 The rho

Now, we see the behavior of the rho for long-term options. We use theorem 3.2 to find the sensitivity of variable $a$ in the drift coefficient. The sensitivity of the $\theta$ variable can be similarly analyzed. Consider the perturbed process $r^\epsilon_t$ defined by

$$dr^\epsilon_t = (a + \epsilon)(\theta - r^\epsilon_t)\,dt + \sigma\sqrt{r^\epsilon_t}\,dW_t.$$ 

$\epsilon(a - r^\epsilon_t)$ in the drift term is the perturbed term. We know the minimum eigenvalue, say $\beta(\epsilon)$, and its eigenfunction $\phi_\epsilon$ corresponding to $r^\epsilon_t$:

$$\beta(\epsilon) = k(\epsilon)(a + \epsilon)\theta$$ 

$$\phi_\epsilon(r) = e^{-k(\epsilon)r}$$

where

$$k(\epsilon) = \frac{\sqrt{(a + \epsilon)^2 + 2\sigma^2} - (a + \epsilon)}{\sigma^2}.$$ 

We also know

$$\varphi_\epsilon(r) = -k(\epsilon)\sigma\sqrt{r}$$

and

$$\varphi'(r) = -k'(0)\sigma\sqrt{r} = \frac{\sqrt{a^2 + 2\sigma^2} - a}{\sigma\sqrt{a^2 + 2\sigma^2}}\sqrt{r}.$$ 

It is easy to confirm that one of the assumptions of theorem 3.2, $\varphi + a(\theta - r)\frac{\partial}{\partial a} \in L^2(\nu)$, is satisfied by checking the decay rate of $\nu$ near zero and at infinity.

By direct calculation,

$$\frac{\partial}{\partial \epsilon} \bigg{|}_{\epsilon=0} \phi^{-1}_\epsilon(r) = \frac{\partial}{\partial a} \phi^{-1}(r) = \frac{-\sqrt{a^2 + 2\sigma^2} - a}{\sigma^2\sqrt{a^2 + 2\sigma^2}} e^{kr}.$$ 

$\frac{\partial}{\partial \epsilon} \bigg{|}_{\epsilon=0} \phi^{-1}_\epsilon(r)$ is a constant multiple of $\phi^{-1}$, so for $f$ such that $\mathbb{E}_r^p \left[ (\phi^{-1} f)^2(r_t) \right]$ is bounded for $t$, the condition that $\mathbb{E}_r^p \left[ \frac{\partial}{\partial \epsilon} \bigg{|}_{\epsilon=0} \phi^{-1}_\epsilon f \right]$ is bounded for $t$ is automatically satisfied. We previously showed that when $f$ has a growth rate equal to or less than a polynomial function, then $\mathbb{E}_r^p \left[ (\phi^{-1} f)^2(r_t) \right]$ is bounded for $t$. Therefore, by theorem 3.2, we obtained

$$\lim_{T \to \infty} \frac{\partial}{\partial \epsilon} \bigg{|}_{\epsilon=0} \frac{p^T_T - p^T_T}{p^T_T} = \beta'(0) = k'(0)a\theta + k\theta = \frac{2\theta(a^2 + \sigma^2 - a\sqrt{a^2 + 2\sigma^2})}{\sigma^2\sqrt{a^2 + 2\sigma^2}}$$

for any $f$ such that $f \geq 0, f \neq 0$ $\mu$-almost surely, and the growth rate of $f$ is equal to or less than a polynomial function. We used the trivial equality:

$$\frac{\partial}{\partial \epsilon} \bigg{|}_{\epsilon=0} = \frac{\partial}{\partial a}.$$ 

As concrete examples, we obtained the behavior of the rho of the long-term bond, the (fixed-rate) put option and the call option. For the (fixed-rate) swap whose payoff function is $f_s(r) := r - R$, we have the same result when $\frac{\partial}{\partial \epsilon} \bigg{|}_{\epsilon=0} \frac{p^T_T - p^T_T}{p^T_T} = \frac{\partial}{\partial \epsilon} \bigg{|}_{\epsilon=0} \frac{(p^T_T - p^T_T)}{T \cdot (p^T_T - p^T_T)}$. 

16
4.1.5 The vega

We investigate the sensitivity of variable $\sigma$ in the diffusion coefficient. Consider the perturbed process $r'_t$, defined by

$$dr'_t = a(\theta - r'_t) \, dt + (\sigma + \epsilon) \sqrt{r'_t} \, dW_t.$$ 

Here, $\hat{\sigma}(r'_t) := \sqrt{r'_t}$ is the perturbed term. We know the minimum eigenvalue, $\gamma(\epsilon)$, and its eigenfunction, $\psi_\epsilon$, of the negative infinitesimal operator corresponding to the process $r'_t$ are given by

$$\gamma(\epsilon) = l(\epsilon)a\theta,$$

$$\psi_\epsilon(r) = e^{-l(\epsilon)r},$$

where

$$l(\epsilon) := \frac{\sqrt{a^2 + 2(\sigma + \epsilon)^2} - a}{(\sigma + \epsilon)^2}.$$ 

Furthermore,

$$\pi_\epsilon(r) = -l(\epsilon)(\sigma + \epsilon)\sqrt{r}$$

$$\nu(r) = \frac{(\sqrt{a^2 + 2\sigma^2} - a)^2}{\sigma^3 \sqrt{a^2 + 2\sigma^2}} e^{-kr}.$$ 

It is easy to confirm that one of the assumptions in theorem 3.3

$$\nu(r) + \sigma - \frac{2}{l} \hat{\sigma}(r) \phi(r) \in L^2(\nu)$$

is satisfied. Therefore

$$\lim_{T \to \infty} \frac{\partial}{\partial T} \frac{\partial pr}{\partial T} = \gamma'(0) = -\frac{(\sqrt{a^2 + 2\sigma^2} - a)^2 a\theta}{\sigma^3 \sqrt{a^2 + 2\sigma^2}}$$

for any $f$ such that $f \geq 0, f \neq 0$ $\mu$ almost surely, and $E_p^T[(\phi^{-1}f)^2(r_t)]$ and $E_p^T[(\phi^{-1}f)'(r_t) \cdot Z_t]$ are bounded for $t$. These conditions are satisfied if both $f$ and $f'$ have a growth rate equal to or less than a polynomial function. To see that $E_p^T[(\phi^{-1}f)'(r_t) \cdot Z_t]$ is bounded for $t$, we compute

$$E_p^T[(\phi^{-1}f)'(r_t) \cdot Z_t] \leq E_p^T[(\text{polynomial}) e^{2kr_t}]^{1/2} \cdot E_p^T[Z_t^2 r_t^{-1}]^{1/2}$$

We already proved that the first term is convergent. To see the second term, by using

$$dZ_t = -\sqrt{a^2 + 2\sigma^2} Z_t \, dt + \frac{\sigma}{\sqrt{r_t}} Z_t \, dB_t + \sqrt{r_t} dB_t, \quad Z_0 = 0,$$

we have

$$d \left( Z_t^2 r_t^{-1} \right) = \left( \left( \frac{\sigma^2}{4} - a\theta \right) Z_t^2 r_t^{-2} - \sqrt{a^2 + 2\sigma^2} Z_t^2 r_t^{-1} - \sigma Z_t r_t^{-1} + 1 \right) \, dt$$

$$+ \text{(something)} \, dB_t.$$ 

Because the drift term is

$$1 + \text{(negative)} - \sqrt{a^2 + 2\sigma^2} Z_t^2 r_t^{-1}$$

(we assumed $2a\theta \leq \sigma^2$), we know

$$E_p^T[e^{\sqrt{a^2 + 2\sigma^2} t} Z_t^2 r_t^{-1}] \leq \int_0^t e^{\sqrt{a^2 + 2\sigma^2} s} ds = \frac{1}{\sqrt{a^2 + 2\sigma^2}} \left( e^{\sqrt{a^2 + 2\sigma^2} t} - 1 \right)$$

and hence $E_p^T[Z_t^2 r_t^{-1}]$ is bounded for $t$. 

17
4.2 The Vasicek model

4.2.1 Interest rate options

In this section, we explore the case that the driver process is the interest rate process and that the interest rate is the Vasicek model defined by

$$dr_t = a(\theta - r_t) \, dt + \sigma \, dW_t.$$  

The process $r_t$ has an invariant distribution, denoted by $\mu$, and given by (up to constant multiples)

$$d\mu(r) = e^{-\frac{a}{2}r^2 + 2a \theta r} \, dr.$$  

Consider a payoff function $f$ with a growth rate equal to or less than a polynomial function. We assume $f \geq 0$, $f \neq 0$. It can be confirmed that $f$ is in $L^2(\mu)$. We apply theorem 2.2. By direct calculation, we know that the minimum eigenvalue of the negative infinitesimal operator is $\theta - \frac{\sigma^2}{2a}$, and that its eigenfunction is $\phi(r) = e^{-\frac{1}{a}r}$. Thus, we have

$$\lim_{T \to \infty} \left( -\frac{1}{T} \log p_T \right) = \theta - \frac{\sigma^2}{2a^2},$$

$$\lim_{T \to \infty} e^{\left( \theta - \frac{\sigma^2}{2a} \right) T} p_T = e^{-\frac{1}{a} \theta_0} \kappa$$

where $\kappa = \langle e^{-\frac{1}{a}r}, f \rangle / \langle e^{-\frac{1}{a}r}, e^{-\frac{1}{a}r} \rangle$. From these observations, we can determine that the Vasicek model is not appropriate to model long-term options if $\theta - \frac{\sigma^2}{2a} < 0$, in which case the longer the time to maturity, the higher the bond price. This phenomenon is not practical.

**Remark 4.3.** The Vasicek model can have negative values. In this case, as we have seen above, it is possible for the bond price to blow up to infinity as time to maturity increases. This is largely because the minimum eigenvalue of the negative infinitesimal generator can be negative. This fact implies that interest rate models such as the Vasicek model, the Ho-Lee model, and the Hull-White model, which have negative values, are occasionally not suitable for long-term option models when put to practical use.

4.2.2 The Greeks

Consider the transformed measure $\mathbb{P}$ with respect to $\left( \theta - \frac{\sigma^2}{2a}, e^{-\frac{1}{a}r} \right)$. We see the dynamics of $r_t$ under $\mathbb{P}$. The Radon-Nikodym derivative $M_t$ is given by

$$dM_t = \frac{\sigma}{a} M_t \, dW_t.$$  

A process $B_t$ defined by

$$B_t = W_t - \frac{\sigma}{a} t$$

is a Brownian motion under $\mathbb{P}$; therefore, $r_t$ follows

$$dr_t = a \left( \theta + \frac{\sigma^2}{a^2} - r_t \right) \, dt + \sigma \, dB_t.$$  

We investigate the sensitivity analysis for this model. We continue to assume that the payoff function $f$ has a growth rate equal to or less than a polynomial function and that $f \geq 0$, $f \neq 0$. For the delta, we use theorem 3.1. The hypothesis of the theorem can easily be confirmed, so we have

$$\lim_{T \to \infty} \frac{\partial}{\partial r_0} p_T = -\frac{1}{a}.$$  

We obtain the sensitivities of the $a$ and $\theta$ variables by using theorem 3.2. The hypothesis of the theorem is also easily checked, and we have

$$\lim_{T \to \infty} \frac{\partial}{\partial a} p_T = -\frac{\sigma^2}{a^3}, \quad \lim_{T \to \infty} \frac{\partial}{\partial \theta} p_T = 1.$$
For the vega, we obtain
\[
\lim_{T \to \infty} \frac{\partial_T p_T}{p_T} = \frac{\sigma}{a^2}.
\]
In theorem 3.3 we only check the condition that \(E^P_T[(\phi^{-1} f)'(r_T) \cdot Z_t] )\) is bounded for \(t\). The other conditions are easily checked. We assume that both \(f\) and \(f'\) have a growth rate equal to or less than a polynomial function. By the Cauchy-Schwarz inequality, it is enough to show that \(E^P_T[Z_t^2]\) is bounded for \(t\); this can easily be computed by using
\[
Z_t = e^{-at} \int_0^t e^{as} dB_s.
\]

4.3 The geometric Brownian motion

4.3.1 Call options

When analyzing long-term options, theorem 2.2 cannot be applied when the payoff function is not small; more precisely, we can apply the theorem when \(f\) is in \(L^2(\mu)\). For example, consider an ordinary call option in the Black-Scholes model. The interest is constant, \(r_t = r\) and the stock price, denoted by \(S_t\), follows a geometric Brownian motion:
\[
dS_t = rS_t \, dt + \sigma S_t \, dW_t.
\]
(In this case, the driver process is the stock price process \(S_t\).) We assume \(r - \frac{1}{2}\sigma^2 > 0\). Because \(S_t\) goes to infinity as \(t\) goes to infinity, we may expect that the payoff function of call option \(f_c(s) := (s - K)_+\) is not small, and in fact, \(f_c\) is not in \(L^2(\mu)\). More generally, the payoff function of the form \(f_{c,m}(y) = (y^m - K)_+\) for \(m > 0\) is not in \(L^2(\mu)\). Indeed, the speed measure \(\mu\) is given by (up to constant multiples)
\[
d\mu(s) = s^{2m-2} ds.
\]
To see the behavior of the long-term option with payoff function \(f_{c,m}\), we use theorem 2.3 instead of theorem 2.2. Find a solution pair \((\beta, \phi)\) of
\[
L\phi = -\beta \phi
\]
where
\[
(L\phi)(s) = \frac{1}{2} \sigma^2 s^2 \phi''(s) + rs \phi'(s) - r \phi(s).
\]
Let \(\phi(s) = s^m\) and let \(\beta = r - rm - \frac{1}{2} \sigma^2 m(m - 1)\). Let \(\mathbb{P}\) be the transformed measure with respect to \((s^m, r - rm - \frac{1}{2} \sigma^2 m(m - 1))\). The Radon-Nikodym derivative \(M_t\) satisfies
\[
dM_t = \varphi(S_t) M_t \, dW_t
\]
with
\[
\varphi(s) = m \sigma.
\]
(It can easily be shown that \(M_t\) is a martingale by checking the Novikov condition.) We know that a process \(B_t\), defined by
\[
 dB_t = dW_t - m \sigma \, dt,
\]
is a Brownian motion under \(\mathbb{P}\). The stock price \(S_t\) follows
\[
dS_t = (r + m \sigma^2) S_t \, dt + \sigma S_t \, dB_t.
\]
We can easily show that
\[
E^\mathbb{P}_t \left[ S_{tT}^{-m}(S_{tT}^m - K)_+ \right]
\]
converges to 1 as $T$ goes to infinity. Hence the pair $(s^m, r - rm - \frac{1}{2} \sigma^2 m(m - 1))$ satisfies the hypothesis of theorem 2.3. In conclusion,

$$\lim_{T \to \infty} e^{(r - rm - \frac{1}{2} \sigma^2 m(m - 1)) T} \cdot p^m_T = S_0^m$$

where $p^m_T$ is the price of the option with payoff function $f_{c,m}$ with maturity $T$. If $m = 1$, this gives the ordinary call option price, and we have

$$\lim_{T \to \infty} p_T^c = S_0$$

as is commonly understood, where $p_T^c := p_{T}^{c,1}$.

4.3.2 The Greeks

We investigate the sensitivity analysis for this option whose payoff function is $f_{c,m}$. For the sake of convenience, set $p_T = p^m_T$. First, we find the delta by applying proposition 3.1, and we have

$$\lim_{T \to \infty} \frac{\partial}{\partial S} p_T = \frac{\phi'(S_0)}{\phi(S_0)} = \frac{m}{S_0} .$$

For the rho,

$$\frac{\partial}{\partial r} p_T = \frac{\partial}{\partial r} (e^{-rT} \cdot \mathbb{E}_T^Q ((S_T^m - K)_+)) = -T p_T + e^{-rT} \cdot \frac{\partial}{\partial r} \left( \mathbb{E}_T^Q ((S_T^m - K)_+) \right) .$$

When we apply theorem 3.2 to the second part, then

$$\lim_{T \to \infty} \frac{e^{-rT} \cdot \frac{\partial}{\partial r} \left( \mathbb{E}_T^Q ((S_T^m - K)_+) \right)}{T \cdot p_T} = m .$$

Therefore, we obtain

$$\lim_{T \to \infty} \frac{\partial}{\partial r} p_T = m - 1 .$$

For the vega, by using theorem 3.3, we have

$$\lim_{T \to \infty} \frac{\partial}{\partial \sigma} p_T = -\frac{\partial}{\partial \sigma} = \sigma m(m - 1) .$$

To show that $\mathbb{E}_s^P ((\phi^{-1} f)'(S_t) \cdot Z_t)$ is bounded for $t$, we use proposition 3.4. By direct computation, we have

$$dY_t = (r + m \sigma^2) Y_t dt + \sigma Y_t dB_t$$
$$dZ_t = (r + m \sigma^2) Z_t dt + \sigma Z_t dB_t + S_t dB_t .$$

Solving these equations, we obtain

$$Y_t = s^{-1} S_t$$
$$Y_t^{-1} Z_t = B_t - \sigma t .$$

Thus, we obtain

$$\mathbb{E}_s^p [(\phi^{-1} f)'(S_t) \cdot Z_t] = \frac{1}{t} \mathbb{E}_s^p [(\phi^{-1} f)(S_t) \cdot \delta(\sigma^{-1}(S_u) Y_u Y_t^{-1} Z_t)]$$

$$= \mathbb{E}_s^p \left[ (S_t^{-m}(S_t^m - K)_+) \cdot \delta \left( \frac{B_t}{\sigma t} - 1 \right) \right] \cdot s^{-1}$$

$$= \mathbb{E}_s^p \left[ (S_t^{-m}(S_t^m - K)_+) \cdot \left( \frac{B_t^2}{\sigma t} - B_t - \frac{1}{\sigma} \right) \right] \cdot s^{-1} .$$

We see that

$$\mathbb{E}_s^p [(S_t^{-m}(S_t^m - K)_+) \cdot B_t]$$
is bounded for \( t \), because the other terms are clearly bounded for \( t \). For simplicity, let \( m = 1 \),

\[
\mathbb{E}_t^\pi \left[(S_t^{-1}(S_t - K)\cdot B_t)\right] = \mathbb{E}_t^\pi \left[B_t \cdot \mathbb{I}_{(S_t < K)}\right] - K \mathbb{E}_t^\pi \left[S_t^{-1}B_t \cdot \mathbb{I}_{(S_t > K)}\right] 
\]

We can directly compute the value of the first term. It is easy to confirm that it is bounded (in fact, it converges to zero). The second term is clearly bounded because the function \( g(y) := |y|e^{-\sigma y} \) is bounded for \( y \) and \( S_t^{-1}|B_t| = g(B_t) e^{-(r+\sigma^2/2)} \).

### 4.3.3 Put options

For the put option, the payoff function \( f_p(y) = (K - y)_+ \) is in \( L^2(\mu) \) because \( f_p \) has compact support. Unfortunately, however, neither theorem 2.2 nor 2.3 can be applied to see the option because the infinitesimal operator \( \mathcal{L} \) given by

\[
\mathcal{L}\phi(s) = \frac{1}{2} \sigma^2 s^2 \phi''(s) + rs\phi'(s) - r\phi(s)
\]

has no eigenvalues. Indeed, the ordinary put option in the Black-Scholes model decays faster than at an exponential rate. Note that we assumed \( r - \frac{1}{2}\sigma^2 > 0 \).

The existence of an eigenvalue depends not only on the driver process but also on the interest rate function. For example, suppose the interest rate function is

\[
\tilde{r}(s) := \frac{1}{2} \alpha^2 \sigma^2 \left( \alpha \ln(s) + \frac{1}{2} \right)^2
\]

where \( \alpha := 1 - \frac{2}{\tilde{r}} \). In this case, the infinitesimal operator

\[
\mathcal{L}\phi(s) = \frac{1}{2} \sigma^2 s^2 \phi''(s) + rs\phi'(s) - \tilde{r}(s)\phi(s)
\]

has an eigenvalue, and we can compute the closed forms of the eigenvalue and its eigenfunction. Indeed, \( \frac{5}{8} \alpha^2 \sigma^2 \) is the minimum eigenvalue of the negative infinitesimal operator \( -\mathcal{L} \), and its eigenfunction is \( e^{-\frac{1}{2} \alpha^2 (\ln s)^2} \). Therefore, the put-option price \( p_T \) has an exponential decay rate; more precisely, we obtain

\[
\lim_{T \to \infty} \left(-\frac{1}{T} \log p_T \right) = \frac{5}{8} \alpha^2 \sigma^2
\]

\[
\lim_{T \to \infty} e^{\left(\frac{5}{8} \alpha^2 \sigma^2\right)T} p_T = e^{-\frac{1}{2} \alpha^2 (\ln S_0)^2} \kappa
\]

where \( \kappa = \langle f_p, e^{-\frac{1}{2} \alpha^2 (\ln s)^2} \rangle / \langle e^{-\frac{1}{2} \alpha^2 (\ln s)^2}, e^{-\frac{1}{2} \alpha^2 (\ln s)^2} \rangle \).

### 5 Conclusion

This article consists of four sections. In the first section, we investigated the behavior of long-term options with two approaches: the spectral decomposition of the pricing operator and the martingale extraction of the inverse of the money-market account. Through these methods, we explored the conditions under which the option price decays or grows at an exponential rate as time to maturity increases. Under these circumstances, we investigated the Greeks in the long run in section 3. The Greeks divided by the option price was expressed in a simple form for the long run. In section 4, we applied these theories to several models: the CIR model, the Vasicek model, and a geometric Brownian motion.

We suggest following extensions for further research. First, it would be interesting to find more general conditions that guarantee the existence of the solution pair that satisfies the hypothesis of theorem 2.3. In this article, only one condition for the solution pair is offered by using the spectral decomposition in theorem 2.2. Second, it would be interesting to explore a more general decay or growth rate of option prices over time - we only explored the case that the decay or growth rate is exponential. Third, with a more general decay or growth rate, it would be interesting to see the behavior of the Greeks in the long run.
A Proofs of main theorems

In this section, we prove three propositions 3.1, 3.2 and 3.3.

Proof of proposition 3.1

Proof.

\[ \frac{\partial}{\partial x} U_T(\phi^{-1} f)(x) = \mathbb{E}_x^p \left[ (\phi^{-1} f)(X_T) \cdot \frac{1}{T} \int_0^T \sigma^{-1}(X_s) Y_s \, dB_s \right] \]

\[ \leq \frac{1}{\sqrt{T}} \left( \mathbb{E}_x^p [(\phi^{-1} f)^2(X_T)] \right)^{1/2} \left( \frac{1}{T} \int_0^T \mathbb{E}_x^p [(\sigma^{-1}(X_s) Y_s)^2] \, ds \right)^{1/2} \]

\[ \rightarrow 0 \quad \text{as} \quad T \to \infty . \]

The last convergence is by the assumption that both \( \mathbb{E}_x^p [(\phi^{-1} f)^2(X_T)] \) and \( \mathbb{E}_x^p [(\sigma^{-1}(X_s) Y_s)^2] \) are bounded for \( t \). This completes the proof.

Proof of proposition 3.2

Proof. A process \( B_t \) defined by \( dB_t = dW_t - \varphi_t(X_t) dt \) is a Brownian motion under \( \mathbb{P}_x \).

\[ dX_t^e = (b(X_t^e) + \epsilon \dot{b}(X_t^e)) \, dt + \sigma(X_t^e) \, dW_t \]

\[ = (b(X_t^e) + \epsilon \dot{b}(X_t^e) + \varphi(X_t^e) + \sigma(X_t^e)) \, dt + \sigma(X_t^e) \, dB_t . \]

Under \( \mathbb{P} \), a process \( \tilde{X}_t^e \) defined by

\[ d\tilde{X}_t^e = (b(\tilde{X}_t^e) + \epsilon \dot{b}(\tilde{X}_t^e) + \varphi(\tilde{X}_t^e)) \, dt + \sigma(\tilde{X}_t^e) \, dB_t \]

has the same distribution with \( X_t^e \) under \( \mathbb{P}_x \), thus

\[ \frac{\partial}{\partial \epsilon} \mathbb{E}_x^p [(\phi^{-1} f)(X_T^e)] = \frac{\partial}{\partial \epsilon} \mathbb{E}_x^p [(\phi^{-1} f)(\tilde{X}_T^e)] . \]

On the other hand, by using the Taylor expansion, write \( \varphi = \varphi + \epsilon \bar{\sigma} + \eta(\epsilon) \Phi \) for some \( \Phi \) with \( \eta'(0) = 0 \).

\[ d\tilde{X}_t^e = (b(\tilde{X}_t^e) + (\varphi(s)(\tilde{X}_t^e) + \epsilon(b + \varphi') \sigma(\tilde{X}_t^e) + (\eta(s) \sigma \Phi)(\tilde{X}_t^e)) \, dt + \sigma(\tilde{X}_t^e) \, dB_t . \]

We see the higher order term \( (\eta(s) \sigma \Phi)(\tilde{X}_t^e) \) is negligible. Using the chain rule, we have

\[ \frac{\partial}{\partial \epsilon} \mathbb{E}_x^p [(\phi^{-1} f)(\tilde{X}_T^e)] = \partial_{\epsilon} \mathbb{E}_x^p [(\phi^{-1} f)(\tilde{X}_T^e)] + \partial_{\eta} \mathbb{E}_x^p [(\phi^{-1} f)(\tilde{X}_T^e)] \cdot \eta'(\epsilon) . \]

With \( \eta'(0) = 0 \), it is obtained that

\[ \frac{\partial}{\partial \epsilon} \mathbb{E}_x^p [(\phi^{-1} f)(\tilde{X}_T^e)] = \partial_{\epsilon} \mathbb{E}_x^p [(\phi^{-1} f)(\tilde{X}_T^e)] . \]

By [12], we have

\[ \partial_{\epsilon} \mathbb{E}_x^p [(\phi^{-1} f)(\tilde{X}_T^e)] = \mathbb{E}_x^p \left[ (\phi^{-1} f)(X_T) \int_0^T (\sigma^{-1} \dot{b} + \varphi)(X_s) \, dB_s \right] . \]

This completes the proof.

Proof of proposition 3.3
Proof. A process $B_t^\epsilon$ defined by $dB_t^\epsilon = dW_t - \psi_t(X_t^\epsilon) dt$ is a Brownian motion under $\mathbb{P}_\epsilon$.

$$dX_t^\epsilon = b(X_t^\epsilon) dt + (\sigma + \epsilon \hat{\sigma})(X_t^\epsilon) dW_t$$

$$= (b + (\sigma + \epsilon \hat{\sigma})\psi_t)(X_t^\epsilon) dt + (\sigma + \epsilon \hat{\sigma})(X_t^\epsilon) dB_t^\epsilon .$$

Under $\mathbb{P}$, a process $\tilde{X}_t^\epsilon$ defined by

$$d\tilde{X}_t^\epsilon = (b + (\sigma + \epsilon \hat{\sigma})\psi_t)(\tilde{X}_t^\epsilon) dt + (\sigma + \epsilon \hat{\sigma})(\tilde{X}_t^\epsilon) dB_t$$

has the same distribution with $X_t^\epsilon$ under $\mathbb{P}_\epsilon$, thus

$$\frac{\partial}{\partial \epsilon} \left| \mathbb{E}_x^\mathbb{P}_\epsilon [(\phi^{-1} f)(X_T)] \right| = \frac{\partial}{\partial \epsilon} \left| \mathbb{E}_x^\mathbb{P} [(\phi^{-1} f)(\tilde{X}_T^\epsilon)] \right| .$$

On the other hand, by using the Taylor expansion, write $\psi_t = \varphi + \epsilon \tilde{\psi} + \eta(\epsilon) \Psi$ for some $\Psi$ with $\eta'(0) = 0$.

$$d\tilde{X}_t^\epsilon = (b + \sigma \varphi + \epsilon (\sigma \tilde{\psi} + \hat{\sigma} \varphi) + \eta(\epsilon) \sigma \Psi + \epsilon^2 \sigma \tilde{\psi} + \epsilon \eta(\epsilon) \hat{\sigma} \Psi)(\tilde{X}_t^\epsilon) dt$$

$$+ (\sigma + \epsilon \hat{\sigma})(\tilde{X}_t^\epsilon) dB_t .$$

Similar to the proof of proposition 3.2, by using the chain rule, the higher order term $(\eta(\epsilon) \sigma \Psi + \epsilon^2 \sigma \tilde{\psi} + \epsilon \eta(\epsilon) \hat{\sigma} \Psi)(\tilde{X}_t^\epsilon)$ is negligible. We may regard $\tilde{X}_t^\epsilon$ as a solution of the following SDE:

$$d\tilde{X}_t^\epsilon = (b + \sigma \varphi + \epsilon (\sigma \tilde{\psi} + \hat{\sigma} \varphi))(\tilde{X}_t^\epsilon) dt + (\sigma + \epsilon \hat{\sigma})(\tilde{X}_t^\epsilon) dB_t .$$

The first $dt$ part means a perturbation of the drift term and the second $dB_t$ part means a perturbation of the diffusion term. Hence,

$$\frac{\partial}{\partial \epsilon} \left| \mathbb{E}_x^\mathbb{P} [(\phi^{-1} f)(\tilde{X}_T)] \right| = \mathbb{E}_x^\mathbb{P} \left[ (\phi^{-1} f)(\tilde{X}_T) \int_0^T (\tilde{\psi} + \sigma^{-1} \hat{\sigma} \varphi)(\tilde{X}_s) dB_s \right]$$

$$+ \mathbb{E}_x^\mathbb{P} \left[ (\phi^{-1} f)'(\tilde{X}_T) \cdot \tilde{Z}_T \right] .$$

The last equality is by [12]. This completes the proof.

References

[1] Amrein, W.O., Hinz, A.M., Pearson, D.B.: Sturm-Liouville theory, past and present. Birkhauser-Verlag, Basel, Boston, Berlin (2005)

[2] Bakry, D.: Functional inequalities for Markov semigroups. Probability measures on groups, Mumbai: Inde 2004 (2009)

[3] Baudoin, F: Convergence of the semigroup, Poincare and log-Sobolev. Lecture note 15 for Curvature Dimension Inequalities (2013). [http://fabricebaudoin.wordpress.com/]

[4] Berkowitz, J.: On the discreteness of spectra of singular Sturm-Liouville problems. Commun. Pur. Appl. Math. 12, 523-542 (1959)

[5] Bjork, T.: Arbitrage theory in continuous time. Oxford University Press (2009)

[6] Cattiaux, P.: Long time behavior of Markov process. Lecture note at the workshop, Metastability and Stochastic Processes (2011). [http://cermics.enpc.fr/~lelievre/Journees_MAS/Pcat.pdf]

[7] Coddington, E.A., Levinson, N.: Theory of ordinary differential equations. McGraw-Hill, New York, Toronto, London (1955)

[8] Cont, R., Fournie, D.: Functional Ito calculus and stochastic integral representation of martingales. Ann. Probab. 41, 109-133 (2013)
[9] Davydov, D., Linetsky, V.: Pricing options on scalar diffusions: An eigenfunction expansion approach. Operations Research 51, 185-290 (2003)
[10] Dupire, B.: Functional Itô calculus. Portfolio Research Paper, Bloomberg (2009)
[11] Durrett, R.: Stochastic Calculus: A practical introduction. CRC Press (1996)
[12] Fournie, E., Lasry J., Lebuchoux, J., Lions P., Touzi, N.: Applications of Mallivin calculus to Monte Carlo methods in finance. Finance Stoch. 3, 391-412 (1999)
[13] Friederichs, K.O. Criteria for discrete spectra. Commun. Pur. Appl. Math. 3, 439-449 (1950)
[14] Fulton, C.T., Pruess, S., Xie, Y.: The automatic classification of Sturm-Liouville problems. mimeo (1993).
[15] Gorovoi, V., Linetsky, V.: Black’s model of interest rates as options, eigenfunction expansions and Japanese interest rates. Math. Financ 14, 49-78 (2004)
[16] Hansen, L.P.: Dynamic valuation decomposition with stochastic economies. Econometrica 80, 911-967 (2012)
[17] Hansen, L.P., Scheinkman, J.A.: Long-term risk: an operator approach. Econometrica 77, 177-234 (2009)
[18] Hansen, L.P., Scheinkman, J.A.: Pricing growth-rate risk. Finance Stoch. 16, 1-15 (2012)
[19] Hull, J.C.: Options, futures, and other derivatives. Pearson Prentice Hall (1997)
[20] Ito, K., McKean, H.P. Diffusion processes and their sample paths. Springer-Verlag, Berlin, Heidelberg, New York (1974)
[21] Jazaeri, S., Saporito, Y.: Functional Itô calculus, path-dependent and the computation of Greeks. arXiv:1311.3881v1[q-fin.CP]
[22] Jorgens, K.: Spectral theory of second-order ordinary differential operators. Arhus Universitet, Matematisk institut (1964)
[23] Karatzas, I., Shreve, S.E.: Brownian motion and stochastic calculus. Springer-Verlag, New York (1991)
[24] Lax, P.D.: Functional analysis. John Wiley & Sons, Inc (2002)
[25] Linetsky, V.: The spectral decomposition of the option value. Int. J. Theor. Appl. Finance 7 337-384 (2004)
[26] Meyn, S.P., Tweedie, R.L.: Stability of Markov processes III: Foster-Lyapunov criteria for continuous-time processes. Adv. Appl. Prob. 25, 487-517 (1993)
[27] McKeon, H.P.: Elementary solutions for certain parabolic partial differential equations. T. Am. Math. Soc. 82, 519-548. (1956)
[28] Nualart, D.: The Malliavin Calculus and Related Topics. Probability and Its Applications. Springer, New York (1995)
[29] Rey-Bellet, L.: Ergodic properties of Markov processes. Lecture note (2006).
[30] Rogers, L.C.G., Williams, D.: Diffusions, Markov processes, and martingales, volume 2: Ito calculus. John Wiley & Sons Ltd. (1987)
[31] Zettl, A.: Sturm-Liouville theory. AMS 121, (2005)