Spin-charge separation for the SU(3) gauge theory

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The idea of a spin-charge separation of the SU(2) gauge potential is extended to the SU(3) case. It is shown that in this case there exist different non-perturbative ground states characterized by different gauge condensate $A^a_\mu A^a_\nu \neq 0$.

I. INTRODUCTION

One of the main problems in quantum field theory is the quantization of strongly interacting fields. In quantum chromodynamics this problem leads to the fact that up to now we do not completely understand the confinement of quarks. Mathematically the problem is connected with quartic term $g^2 (f^{ABC} A^B_\mu A^C_\nu)^2$ in the SU(3) Lagrangian: we have no exact mathematical tools for the non-perturbative path integration of such non-quadratic Lagrangian.

In this case it is useful to have any analogy with other area of physics. In Ref. [1] the authors considered the similarity between High-$T_c$ cuprate superconductivity in condensed matter physics and the problem of a mass gap in the Yang-Mills theory. The authors suggest that in both cases the basic theoretical problems is the absence of a natural condensate to describe the symmetry breaking. The method which is applied in this investigation is a slave-boson decomposition [2] - [5].

In Ref. [6] the idea is presented that an analogy may exist between the SU(2) Yang-Mills theory in the low-temperature phase and a nematic liquid crystal. The idea is based on a spin-charge separation of the gluon field in the Landau gauge.

In Ref. [7] the idea is proposed that in High-$T_c$ superconductivity may exist an analog of a hypothesized flux tube between quarks in quantum chromodynamics where such flux tube essentially increases the interaction energy of two interacting quarks in comparison with the interaction energy for two electrons.

In this paper we would like to investigate such spin-charge separation for the SU(3) gauge field theory and additionally to show that the SU(2) gauge field theory may have another spin-charge separation.

II. SPIN-CHARGE SEPARATION

In the matrix theory [8] there exists the theorem that any real $(m \times n), m > n$ matrix $A$ can be decomposed as

$$A = QR$$

(1)

where $Q$ is an $(m \times n)$ orthogonal matrix ($Q^T Q = 1$) and $R$ is $(n \times n)$ upper triangular matrix. If $A$ is $(m \times n), m < n$ then $Q$ is an $(m \times m)$ orthogonal matrix and $R$ is $(m \times n)$ upper triangular matrix. Following to this theorem, we can decompose any SU(2) gauge component $A^a_\mu$ as

$$A^a_\mu = \tilde{e}^i_\mu \tilde{\phi}^{ia}$$

(2)

where $a = 1, 2, 3$ is the SU(2) color index and enumerates the columns; $\mu = 1, 2, 3, 4$ (we consider the Euclidean version of the theory) and enumerates the rows; $i = 1, 2, 3$ is an inner index which enumerates the columns. Let us introduce the unity

$$1 = \Lambda \Lambda^{-1}$$

(3)
where Λ is an SO(3) orthogonal matrix. The unity can be inserted in Eq. (2) by such a way that
\[ A_\mu^a = (\hat{e}_j^i \Lambda^{ij}) (\Lambda^{kj} \hat{\Phi}^{ka}) = e_\mu^i \Phi^{ia} \] (4)
where \( e_\mu^i \) = \( \hat{e}_j^i \Lambda^{ij} \), \( \Phi^{ia} = \Lambda^{ij} \hat{\Phi}^{ja} \). This decomposition is the subject of the investigation in Ref. [6]. The matrix \( A_\mu^a \) is a \((4 \times 3)\) matrix, \( \hat{e}_j^i \) is a \((4 \times 3)\) matrix and \( \hat{\Phi}^{ka} \) is a \((3 \times 3)\) matrix.

In Ref. [6] the idea is presented that the SU(2) Yang–Mills theory can be associated with a nematic crystal in which the “molecules” are directed in the internal SO(3) space. The adjoint “matter” field
\[ \chi^{ij} = \sum_a \Phi^{ai} \Phi^{aj} \] (5)
can be associated with the dielectric susceptibility
\[ \chi_{\alpha\beta} = \Delta \sum_s n_s(n_s) \] (6)
where \( n_s \) is the direction of the axis of the \( s^{th} \) molecule; \( \Delta \chi = \chi_{\|} - \chi_{\perp} \) is the anisotropy in the diamagnetic susceptibility along and perpendicular to the molecule axis.

III. ANOTHER DECOMPOSITION OF SU(2) GAUGE FIELDS

One can present the potential \( A_\mu^a \) also as a \((3 \times 4)\) matrix where \( a \) enumerates the rows and \( \mu \) – the columns. Then the corresponding decomposition will be
\[ A_\mu^a = \hat{\Phi}^{ai} \hat{e}_\mu^i \] (7)
where \( \hat{\Phi}^{ai} \) is the orthogonal matrix \( \hat{\Phi}^{ai} \hat{\Phi}^{aj} = \delta^{ij} \) and \( e_\mu^i \) is an upper triangular matrix. Again we can insert the unity 1 = ΛΛ⁻¹ between \( \hat{\Phi} \) and \( \hat{e} \) on the r.h.s. of Eq. (7). Finally we have
\[ A_\mu^a = \Phi^{ai} \hat{e}_\mu^i \] (8)
where \( \Phi = \hat{\Phi} \Lambda \) and \( e = \Lambda^{-1} \hat{e} \). Now we would like to rewrite the SU(2) Lagrangian in terms of the fields \( \Phi^{ai} \) and \( e_\mu^i \) similar to Ref. [6]. The field strength \( F_{\mu\nu}^a \) is
\[ F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g e^{abc} A_\mu^b A_\nu^c = \Phi^{ai} (\partial_\mu e_\nu^i - \partial_\nu e_\mu^i) + (e^i_\nu \partial_\mu \Phi^{ai} - e^i_\mu \partial_\nu \Phi^{ai}) + g e^{abc} \Phi^{bi} \Phi^{cj} e_\mu^i e_\nu^c \] (9)
where \( e^{abc} \) are the SU(2) structural constants. The terms without coupling constant \( g \) can be rewritten as
\[ \Phi^{ai} (\partial_\mu e_\nu^i - \partial_\nu e_\mu^i) + (e^i_\nu \partial_\mu \Phi^{ai} - e^i_\mu \partial_\nu \Phi^{ai}) = \Phi^{bi} \left[ \delta^{ab} \partial_\mu e_\nu^i + \frac{1}{2} (e^j_\nu \partial_\mu \Phi^{aj} - e^j_\mu \partial_\nu \Phi^{ai}) \Phi^{bj} \right] - \Phi^{bi} \left[ \delta^{ab} \partial_\nu e_\mu^i + \frac{1}{2} (e^j_\mu \partial_\nu \Phi^{aj} - e^j_\nu \partial_\mu \Phi^{ai}) \Phi^{bj} \right] = \Phi^{bi} [D^{ab}_{\mu\nu}(\Gamma) e_\nu^i - D^{ab}_{\nu\mu}(\Gamma) e_\mu^i] \] (10)
where \( D^{ab}_{\mu\nu}(\Gamma) \) is an analog of the covariant derivative with the “connection” \( \Gamma \)
\[ \Gamma^{ab,ij}_{\mu\nu} (e_\nu^i) = \frac{1}{2} (e^j_\nu \partial_\mu \Phi^{aj} - e^j_\mu \partial_\nu \Phi^{ai}) \] (11)
Then the SU(2) Lagrangian can be written as
\[ L = L_0 + L_1 + L_2 \] (12)
with
\[ L_0 = \frac{1}{4} \{ \Phi^{bi} [D^{ab}_{\mu\nu}(\Gamma) e_\nu^i - D^{ab}_{\nu\mu}(\Gamma) e_\mu^i] \}, \] (13)
\[ L_1 = \frac{g}{2} \Phi^{bi} [D^{ab}_{\mu\nu}(\Gamma) e_\nu^i - D^{ab}_{\nu\mu}(\Gamma) e_\mu^i] e^{abc} \Phi^{ck} e_\mu^l e_\nu^c, \] (14)
\[ L_2 = \frac{g^2}{4} \left[ (\text{Tr} \chi)^2 - \text{Tr} (\chi^2) \right] \] (15)
where

$$\chi^{ij} = e_i^\mu e_j^\mu. \quad (16)$$

Similar to Ref. 6, the quantity $\chi^{ij}$ can be associated with the nematic crystal with one difference: the "molecules" are directed in the Euclidean space-time with the coordinates $x^\mu$. Absolutely by the same way as in Ref. 6, one can calculate the ground state of the nematic associated with the Yang-Mills theory $[13]-[15]$. If we introduce the eigenvalues of the matrix $\chi = \text{diag} \{\chi_1, \chi_2, \chi_3\}$, the ground state $\chi = \chi_0$ is defined as

$$\sum_{i,j=1}^{4} \chi_i^{(0)} \chi_j^{(0)} = 0 \quad (17)$$

with the constraints

$$\sum_{i=1}^{4} \chi_i^{(0)} \geq 0, \quad \prod_{i=1}^{4} \chi_i^{(0)} \geq 0. \quad (18)$$

The solutions of (17) (18) are given as

$$\chi_1^{(0)} = \chi_2^{(0)} = 0, \quad \chi_3^{(0)} \geq 0 \quad (19)$$

The most interesting in this consideration is a non-perturbative vacuum which corresponds to $\chi_3^{(0)} \neq 0$. Clearly, this vacuum state is an $A^2$-condensate

$$\langle A_\mu^a A_\mu^a \rangle = \chi_3^{(0)} \neq 0. \quad (20)$$

### IV. SU(3) SPIN-CHARGE SEPARATION

In this section we would like to repeat the SU(2) matrix decomposition of the previous section for the SU(3) case.

#### A. $A_\mu^B$ gauge potential as a $(4 \times 8)$ matrix

This case is similar to the spin-charge separation used in Ref. 6

$$A_\mu^B = e_\mu^i \Phi_i^B \quad (21)$$

The matrix $e_\mu^i$ is orthogonal one $e_\mu^i e_\mu^j = \delta^i_j; i,j = 1,2,3,4$. The matrix $e_\mu^i$ is similar to the 4-bein but with one essential difference. Generally speaking, one has

$$e_\mu^i e_\nu^i \neq \delta_{\mu\nu} \quad (22)$$

Using this decomposition, one can write

$$L_{SU(3)} = \frac{1}{4} (F^a_{\mu\nu})^2 = L_0 + L_1 + L_2 \quad (23)$$

with

$$L_0 = \frac{1}{2} (D_\mu \phi_i^B)^2 + \frac{1}{8} \phi_i^B \phi_j^B \left[ (\partial_\mu e_\nu^i - \partial_\nu e_\mu^i) \right] \left[ (\partial_\mu e_\alpha^i - \partial_\alpha e_\mu^i) \right] - \frac{1}{2} \left[ e_\mu^i \partial_\nu \Phi_i^B + \frac{1}{2} (\partial_\mu e_\nu^i - \partial_\nu e_\mu^i) \right], \quad (24)$$

$$L_1 = \frac{g}{2} f^{BCD} \left[ e_\nu^i \partial_\mu \Phi_i^B + \frac{1}{2} (\partial_\mu e_\nu^i - \partial_\nu e_\mu^i) \phi_i^B \right] - \left[ \mu \leftrightarrow \nu \right] \phi_j^C \phi_k^D e_\nu^j e_\nu^k, \quad (25)$$

$$L_2 = \frac{g^2}{4} f^{BCD} f^{BMN} \chi^{CD} A_\mu^C A_\nu^D A_\rho^M A_\lambda^N = \frac{g^2}{4} \chi^{CM} f^{BCD} f^{BMN} \chi^{DN} = -\frac{g^2}{4} \text{Tr} (\Phi f^B \Phi^T)^2 \quad (26)$$
where \( f^B \) is the matrix \((f^B)^{MN}\) and
\[
\chi^{AB} = \Phi^i A \Phi^j B. \tag{27}
\]
The covariant derivative \( D_\mu^B \phi^j^B \) is defined in the following way
\[
D_\mu^B \phi^j^B = \partial_\mu \Phi^j^B + \Gamma(e)^{ij}_\mu \Phi^j^B \tag{28}
\]
and the connection \( \Gamma(e) \) as
\[
\Gamma^{ij}_\mu (e) = e^i_\nu (\partial_\mu e^j_\nu - \partial_\nu e^j_\mu) \tag{29}
\]
In order to find possible vacuum state we should to find the values of the condensate \( A^B_\mu A^B_\mu = \text{Tr} \chi \) for which the potential term \( L_2 \) is zero. Let the matrix \( \chi^{AB} \) is diagonalized
\[
\chi^{AB} = \text{diag} \{ \chi_1, \cdots, \chi_8 \}. \tag{30}
\]
In this case
\[
L_2 = \frac{g^2}{4} \left[ 2 (\chi_1 \chi_2 + \chi_1 \chi_3 + \chi_2 \chi_3) + \right. \\
\left. \frac{1}{2} (\chi_1 \chi_4 + \chi_1 \chi_5 + \chi_1 \chi_6 + \chi_1 \chi_7 + \chi_2 \chi_4 + \chi_2 \chi_5 + \chi_2 \chi_6 + \chi_2 \chi_7 + \right. \\
\chi_3 \chi_4 + \chi_3 \chi_5 + \chi_3 \chi_6 + \chi_3 \chi_7 + 4 \chi_4 \chi_5 + \chi_4 \chi_6 + \chi_4 \chi_7 + 3 \chi_4 \chi_8 + \\
\chi_5 \chi_6 + \chi_5 \chi_7 + 3 \chi_5 \chi_8 + 4 \chi_6 \chi_7 + \chi_6 \chi_8 + 3 \chi_7 \chi_8 \left. \right] \tag{31}
\]
The first term in eq. (31) correspons to the SU(2) subgroup [13]. For the perturbative vacuum the solution is
\[
\chi_i = 0, \quad i = 1, \cdots, 8 \tag{32}
\]
The possible non-perturbative vacuum is more complicated then in the SU(2) case. One can exist different vacuum states. The first vacuum state is similar to the SU(2) case and it is defined by the relation
\[
\chi_i = 0, \quad \chi_j \neq 0, \tag{33}
\]
\( j \) is a fixed number. In this case the vacuum condensate is given in the following manner
\[
\text{Tr} \chi = \langle A^B_\mu A^B_\mu \rangle = \chi_j \tag{34}
\]
From eq. (32) we see that not all \( \chi_i \) are equivalent that means that the corresponding vacuum states may be nonequivalent in the contrast with the SU(2) case.

The second possibility is the case when
\[
\chi_i = 0 \tag{35}
\]
but, for example, three \( \chi_{6,7,8} \neq 0 \). In this case we have the following relation between \( \chi_{6,7,8} \)
\[
\frac{4}{3} \chi_{6}\chi_{7} + \chi_{6}\chi_{8} + \chi_{7}\chi_{8} = 0 \tag{36}
\]
but \( \chi_{6,7,8} \) are independent degrees of freedom and they can be do not satisfy the relation (36). Thus in this case it will be a \textit{special} vacuum state and the vacuum \textit{special} condensate is
\[
\langle A^B_\mu A^B_\mu \rangle = \chi^{AA} = \sum \chi^i = \chi_6 + \chi_7 - \frac{4}{3} \frac{\chi_{6}\chi_{7}}{\chi_{6} + \chi_{7}} \tag{37}
\]
Other cases with four and more non-zero \( \chi_i \) can be considered analogously.
B. $A_\mu^B$ gauge potential as a $(8 \times 4)$ matrix

In this case

$$A_\mu^B = \Phi^B_i e^i_\mu$$  (38)

where $\Phi^B_i \Phi^B_j = \delta^i_j$. The same calculations as in the section III gives us

$$F^B_{\mu \nu} = \partial_\mu A^B_\nu - \partial_\nu A^B_\mu + g f^{BCD} A^C_\mu A^D_\nu =$$

$$\Phi^B_i \left( \partial_\mu e^i_\nu - \partial_\nu e^i_\mu \right) + (e^i_\mu \partial_\mu \Phi^B_i - e^i_\mu \partial_\nu \Phi^B_i) + g f^{BCD} \Phi^C_i \Phi^D_j e^i_\mu e^j_\nu$$  (39)

and the SU(3) Lagrangian

$$L_{SU(3)} = L_0 + L_1 + L_2$$  (40)

can be written as

$$L_0 = \frac{1}{4} \left\{ \Phi^B_i \left[ D^{AB}_{\mu\nu}(\Gamma) e^i_\nu - D^{AB}_{\nu\mu}(\Gamma) e^i_\mu \right] \right\},$$  (41)

$$L_1 = \frac{g}{2} \Phi^B_i \left[ D^{AB}_{\mu\nu}(\Gamma) e^i_\nu - D^{AB}_{\nu\mu}(\Gamma) e^i_\mu \right] f^{ABC} \Phi^B_k \Phi^C_l e^k_\mu e^l_\nu,$$  (42)

$$L_2 = \frac{g^2}{4} \left( f^{BCD} f^{BMN} \right) \left( \Phi^C_i \Phi^D_j \Phi^M_k \Phi^N_l \right) \left( e^i_\mu e^j_\nu e^k_\mu e^l_\nu \right)$$  (43)

Unfortunately in this case it is impossible to simplify the quartic term in the consequence of the specific form of the SU(3) structural constant $f^{ABC}$.

V. SUMMARY

In this paper we have applied the spin-charge separation for the SU(3) gauge field and have shown that the SU(2) gauge field may have two different spin-charge separations. We have shown that ground states in the SU(3) case can be divided into two branches: the first one is similar to the SU(2) case, but the second branch contains special vacuum states as there exist relations between eigenvalues of the matrix $A^B_\mu A^B_\mu$. The existence of these vacuum states shows that the perturbative vacuum state of the SU(3) gauge theory can be broken down to different vacuum states characterized by different gauge condensates $A^B_\mu A^B_\mu$.

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