CVA and FVA to Derivatives Trades Collateralized by Cash

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Abstract

In this article, we consider replication pricing of derivatives that are partially collateralized by cash. We let issuer replicate the derivatives payout using shares and cash, and let buyer replicate the loss given the counterparty default using credit default swaps. The costs of funding for replication and collateral posting are accounted for in the pricing process. A partial differential equation (PDE) for the derivatives price is established, and its solution is provided in a Feynman-Kac formula, which decomposes the derivatives value into the risk-free value of the derivative plus credit valuation adjustment (CVA) and funding valuation adjustment (FVA). For most derivatives, we show that CVAs can be evaluated analytically or semi-analytically, while FVAs as well as the derivatives values can be solved recursively through numerical procedures due to their interdependence. In numerical demonstrations, continuous and discrete margin revisions are considered, respectively, for an equity call option and a vanilla interest-rate swap.
1 Introduction

One of the major consequences of the 2007-08 financial tsunami is the disappearance of the boundary between market risk and credit risk. After the tsunami, counterparty default risks have generally been priced into derivatives trades, while the pricing and managing of funding risks have arisen as a central issue for study. To mitigate potential losses due to counterparty defaults, financial institutions have increasingly been adopting the practice of collateralization. As a result, most trades today belong to CSA trades\textsuperscript{\textsuperscript{1}}. Meanwhile, the use of credit valuation adjustment (CVA) has become the market standard to account for the counterparty default risks for un-collateralized and partially collateralized trades, yet discussions and research are still ongoing regarding the so-called funding valuation adjustment (FVA) for funding risks. Since credit and funding risks are intimately related, a recent trend is to evaluate these two kinds of risks consistently under a unified framework. In this article, we present such a framework based on replication pricing and formulate bilateral CVA and FVA for trades collateralized through, in particular, margining.

Early literatures on counterparty risks may be traced back to Sorensen and Bollier (1994), where the pricing of interest-rate swaps subject to counterparty default risks is considered. The same problem is also studied by Duffie and Huang (1996), where any payout is discounted using the discount rate of the paying party. Their model was later extended by Huge and Lando (1999) to explicitly account for the credit ratings of the counterparties. Over the years there has been more research, including Bielecki and Rutkowski (2001), Canabarro and Duffie (2004), and Picoult (2005). During the same period of time, the notion of credit valuation adjustment was gradually established in industry. The effects of other features of trades on CVA have also been studied, including “netting agreements”, which allow multiple obligations to be consolidated into a single one upon a default, by Brigo and Masetti (2005) and also by Pyktin and Zhu (2007), and wrong-way risk by Brigo and Pallavicini (2007). As an important feature, collaterals was considered, among others, by Cherubini (2005) in the evaluation of CVA for basic products, and then in Li and Tang (2007) for general derivatives. Alavian et al. (2008) discuss minimum transfer amounts and collateral thresholds, and

\textsuperscript{1}CSA is for Credit Support Annex, a legal document of International Swap and Derivatives Association (ISDA) that regulates collateral posting. Non-CSA trades are also regulated by the ISDA master agreement.
provide model independent formulas for the counterparty exposure. Assefa et al. (2009) introduce a model for the collateral process without accounting for minimum transfer amounts and collateral thresholds. Not limited to the usual equity and interest-rate derivatives, the notion of CVA has also been extended to other asset classes, including commodity derivatives by Brigo and Bakkar (2009) and credit derivatives by Brigo and Chourdakis (2009).

Funding valuation adjustment became a focus a few years ago. In a classical Black-Scholes framework, Piterbarg (2010) derives replication pricing of derivatives under collateralization but without default risk. Funded replication pricing is also considered by Fries (2010) using the technique for pricing quantos, for which cash flows are indexed in one currency but paid in another. Fujii et al. (2010) analyze implications of currency risk for collateral modeling. For simple products like zero-coupon bonds or loans, Morini and Prampolini (2011) and Castagna (2011) investigate essential features of funding costs in the presence of default risk.

The primary cause for making CVA and FVA is credit risk, so CVA and FVA are interrelated. In the industry, the boundary between them has not been so clear cut, as debit valuation adjustment (DVA), which is the valuation adjustment for a trader’s own default risk, is considered a funding benefit by some researchers and included into FVA (Burgard and Kjaer, 2012; Hull and White, 2012). Besides, the adoption of collaterals in derivatives trades alters risk profile and makes the proper identification of CVA and FVA more challenging. A recent trend, seen for example in Burgard and Kjaer (2011), Pallavicini et al. (2011), Lu and Juan (2011) and Castagna (2011), is to study CVA and FVA in a consistent framework. In that regard, Pallavicini et al. (2011) derive a nonlinear and recursive equation for derivatives value subject to counterparty and funding risks, with or without collaterals, and resort to an iterative method for its solution. A similar equation is also obtained by Crépey (2011). For un-collateralized trades with default payments governed by the 2002 ISDA master agreement, Burgard and Kjaer (2011) present a framework of replication pricing using underlying share and zero-coupon bonds. Wu (2012) replaces the zero-coupon bonds by credit default swaps in the replication arguments and derives explicit formulae for CVA and FVA under stochastic interest rates.

Currently at the center of debates is the viewpoint of Hull and White (2012), who are against making FVA in asset pricing. Hull and White’s arguments are based on two principles: the risk-neutral valuation paradigm and the separation of pricing and funding. It is well understood that the
risk-neutral pricing measure should have taken into account all market risks, and the risk-neutral expected return of any investment is the risk-free rate. There is however no clear indication in most debates whether funding risks are treated as market risks. The correct answer to this question, in our opinion, is the key to argue for or against Hull and White’s viewpoint.

In this article, we will price derivatives trades that are partially collateralized by cash through unsecured borrowing, so-called margining, with the collateral posting subject to a threshold value and a minimum transfer amount. We operate in a market completed by shares and credit default swaps, so the risk-neutral measure is unique. Yet we show that the arbitrage-free price is no unique in the presence of funding costs or risks. As a matter of fact, if both parties want the risk-free rate as the “risk-neutral return”, then there will be a spread between the bid and ask prices, which is nothing else but the aggregated funding costs, or FVAs in another word, from both parties. The formulations we obtain for CVA and FVA remain valid under a general context, including general asset price dynamics, stochastic interest rates, stochastic hazard rates and stochastic recovery rates. We demonstrate that, while CVAs can often be evaluated semi-analytically or even in closed-form, FVA and the derivatives value will in general be evaluated through recursive numerical procedures due to their interdependence.

Our model has made clear that funding risks, which are idiosyncratic and non-diversifiable, cannot be taken into account by the risk-neutral measure. Whether and how much to accept an FVA charge depends on the risk appetite or tolerance of trading parties. If funding costs are not charged properly, then a trader may end up in red after all other risks have been hedged off and thus may become unwilling to enter a trade. On the other hand, if funding costs are taken into account, traders with different funding costs will come up with different prices, and the ones with lower funding costs will have competitive advantages in the market.

This paper has the following contributions to the literature. First, we recognize that the risk-neutral measure cannot take into account the funding risks, which is idiosyncratic, yet non-diversifiable, and leads to price non-uniqueness. Second, we identify and evaluate all adjustment terms in generality, and the formulae for CVA and FVA offer insights to the active management of counterparty risks and funding risks, allowing them to be actively hedged or marked to market. Finally, our approach of modeling and pricing can naturally price wrong-way risk or right-way risk, when exposure to a counterparty is adversely correlated with the credit quality of that
This article is organized as follows. In section 2 we start with replication pricing of an equity derivative. In section 3 we study the actual evaluation of the adjustment terms, delivering closed-form formulae for CVA and describing numerical methods for FVA. With an equity call option, we will demonstrate the pricing of valuation adjustments under continuous margin revision. In section 4 we consider the pricing of interest-rate swaps under discrete margin revision, and deliver closed-form solution to CVA. Finally, we conclude in section 5. Some technical details are placed in the appendix.

2 Pricing theory

2.1 Default payments

Consider a collateralized derivative trade between two defaultable parties, $B$ a bank (or issuer) and $C$ a counterparty (or buyer). If there is no default until the maturity of the derivative, say $T$, the buyer $C$ will receive or make a contractual payment, $Y_T$. In case of a premature default, the party of exposure will seize the collateral and, if insufficient, further claim the recovered marking-to-market (MtM) value of the derivative. To be more specific, we let $V(t)$ (or $V_t$) denote the transaction value of the derivative (which is a càdlàg function: right continuous with left limit), $M(t)$ be the MtM value of the derivative, and $c(t)$ be the value of the collateral, all seen from the viewpoint of party $C$. When $M(t) > 0$, party $B$ posts a collateral of value $c(t) \geq 0$ to party $C$. Conversely, when $M(t) < 0$, party $C$ posts a collateral of value $-c(t) \geq 0$ to $B$. Let $\tau_B$ and $\tau_C$ be the respective default times of $B$ and $C$, and $\tau = \tau_B \wedge \tau_C$ be the time of the first default, then the post-default value of the derivative can be written as

$$ V(\tau = \tau_B) = R_B [M(\tau_B) - c(\tau_B)]^+ + [M(\tau_B) - c(\tau_B)]^- + c(\tau_B), $$

$$ V(\tau = \tau_C) = R_C [M(\tau_C) - c(\tau_C)]^- + [M(\tau_C) - c(\tau_C)]^+ + c(\tau_C), $$

(1)

where $R_B$ and $R_C$ are the recovery rates for the losses due to respective defaults, and the sup-indexes “$+$” and “$-$” mean a cap and a floor of a function at level zero: $f^+(\tau) = \max\{f(\tau), 0\}$ and $f^-(\tau) = \min\{f(\tau), 0\}$. For notational simplicity, we write $V(\tau_B)$ for $V(\tau = \tau_B)$ and $V(\tau_C)$ for $V(\tau = \tau_C)$ hereafter. When $c(t) = 0$, (1) reduces to

$$ V(\tau_B) = R_B M^+(\tau_B) + M^-(\tau_B), $$

$$ V(\tau_C) = R_C M^-(\tau_C) + M^+(\tau_C), $$

(2)
which is the formalism of default settlement stipulated in the 2002 ISDA master agreement for un-collateralized trades. The MtM value of the derivative is usually obtained through a dealer poll mechanism. In financial modeling, this MtM value is often taken to be either the “risk-free value” or pre-default value of the derivative.

In this paper we consider a rather general formalism of cash collaterals, $c(t) = c^+(t) + c^-(t)$, such that

$$c^+(t) = (M(t) - H + X) 1_{\{M(t) \geq H\}},$$

$$c^-(t) = (M(t) + H - X) 1_{\{M(t) \leq -H\}},$$

where $H$ is the threshold for collateral posting and $X$ is the minimum transfer amount (Brigo et al., 2011), both can be time dependent, with $H \geq X \geq 0$.

### 2.2 The setup

We will model the market by the probability space $(\Omega, \mathcal{G}, \mathcal{G}_t, \mathbb{P})$, where $\mathbb{P}$ is a real-world measure, $\mathcal{G}_t$ is the filtration that represents all market information up to time $t$, such that $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$, where $\mathcal{F}_t$ contains all market information except defaults, while $\mathcal{H}_t$ carries only the information of default status. Our theory needs the following

**Assumption:** The derivative payoff $Y_T$ is $\mathcal{F}_T$-adapted.

This assumption excludes derivatives trades whose payoffs depend on the default status of the trading parties, like a CDS contract on the default of either trading party. In reality, two firms are unlikely to trade a CDS on either firm’s default because, once a default occurs, the defaulted firm can neither honor the contract nor benefit from the contract, while the surviving firm will either loss the protection or carry on with its liability.

For mathematical or notational simplicity, we make three more assumptions: 1) when the party of exposure defaults, there is no jump in the derivatives prices, 2) the CSA rate, CDS rates and recovery rates are deterministic and 3) there is no default by the current moment $t = 0$, i.e., $\tau > 0$. These assumptions are non-essential and can be easily removed.

### 2.3 Pricing by hedging

In a market completed by shares and CDS, market risks can be hedged off using shares, and the risk of loss given default (LGD) can be hedged off
using CDS by the party of exposure. So essentially, the value of a derivatives should be the replication cost for derivatives payoff minus the replication cost for LGD, and the arguments will be detailed below.

For both conceptual and notational simplicity, we begin with the pricing of a European equity derivative which is an asset to the buyer $C$, such that $Y_T \geq 0$. Once the bank sells the derivative to the counterparty at $t = 0$ for a premium $V_0 > 0$, it starts replicating the payoff using shares and cash until default or maturity. The value of the issuer’s replicating portfolio can be written as

$$\Pi_B(t) = \delta_B S_t + \beta_B(t),$$

where $S_t$ and $\delta_B$ are the unit price of the underlying shares and the hedge ratio, respectively, and $\beta_B(t)$ is the total value of the margin account and the collateral account. Cash for collateral is withdrawn from the margin account and then posted to the collateral account, the latter is under the custody of the counterparty or a third party. The margin account also takes the P&L from hedge and collateral revisions. In case of a default by the issuer, the buyer will seize the collateral and, if insufficient, be further compensated by the recovered value of the derivative. In case of a default by the buyer, the seller will pay the pre-default value and the derivative contract will be terminated.

The returns of the involving securities and cash accounts are described below. The share price is assumed to have a lognormal dynamics:

$$dS_t = S_t \left[ \mu_t dt + \sigma_S(t) dW_t^{(P)} \right],$$

where $W_t^{(P)}$ is a one-dimensional Brownian motion under $\mathbb{P}$, $\mu_t$ is the expected return, and $\sigma_S$ is the percentage volatility. The share pays continuous dividends with yield $q_t$. In this article, we assume that the share is acquired through repurchasing agreement (repo), and thus costs a repo spread $\lambda_S$ (i.e., the funding spread for borrowing cash using shares as collateral).

The cash of the issuer is separated in his margin account and in the collateral account, and the latter is under the custody of the buyer. The returns offered by these accounts are different, as described below.

1. The margin account earns the CSA rate for positive balance or costs the CSA rate plus a spread for negative balance, respectively. The balance starts from $V_0 - c(0) \geq 0$, the derivatives premium minus the initial collateral value. Besides, the profit and loss (P&L) from the repo
transactions and the surplus or deficit from collateral revisions will be credited to or debited from the margin account.

2. The collateral account offers only the CSA rate (and the cash is not re-hypothecated).

Let $\beta_{m,B}(t)$ denote the balance of the margin account, then $\beta_B(t) = \beta_{m,B}(t) + c(t)$. According to item 1 above, the return of the margin account over $[t, t + dt)$ yet before hedge rebalancing and collateral revision at $t + dt$ is

$$d\beta_{m,B}(t) = r_t \beta_{m,B}(t) dt + x_B \beta_{m,B}^{-}(t) dt,$$

where $r_t$ is the CSA rate and $x_B(t) \geq 0$ is the total funding spread of $B$ for unsecured borrowing. According to both item 1 and 2 above, there is

$$d\beta_B(t) = r_t \beta_{m,B}(t) dt + x_B \beta_{m,B}^{-}(t) dt + r_tC(t) dt.$$  \hspace{1cm} (5)

The LGD to the buyer can be perceived as his potential payout upon the issuer’s default, which can be replicated by holding CDS against the default of $B$. By doing so, the buyer also manages a portfolio, consisting of CDS and cash, of value

$$\Pi_C(t) = \alpha_B U_B(t) + \beta_C(t),$$

where $U_B(t) = 0$ and $\alpha_B$ are the value and number of units of a par CDS against the default of $B$, and $\beta_C(t)$ is the balance of a margin account that finances the CDS premium payments, so there will be $\beta_C(t) \leq 0$.

The par CDS contract is idealized, for the sake of simplicity, to have one dollar notional value and continuous premium payment according to a constant annualized CDS rate. The instantaneous return of the par CDS on the default of $B$ is

$$dU_B(t) = -s_B dt + L_B dJ_B^{(P)},$$

where $s_B$ is the annualized CDS rate, $J_B^{(P)}$ is a Poisson process under the physical measure that jumps from 0 to 1 upon the default of $B$, and $L_B$ is the corresponding loss rate, assumed to be a constant. In theory, the above CDS can be dynamically created by rolling over fixed-rate CDS of a short maturity, say, three months.
The margin account of the buyer is for funding the CDS premium payments, which awards the CSA rate for positive balance and charges the CSA rate plus a spread for negative balance:

\[ d\beta_{m,C}(t) = r_t\beta_{m,C}(t)dt + x_C\beta^-_{m,C}(t)dt, \]  

(6)

Our approach of pricing will be based on the so-called bilateral replication which, in the infinitesimal time scale, is represented by

\[ dV_t = d\Pi_B(t) - d\Pi_C(t). \]  

(7)

The equation above simply says that the change in the derivatives value is hedged jointly by the portfolios of the issuer and the buyer. Initially, there is \( V_0 = \Pi_B(0) \) and \( \Pi_C(0) = 0 \).

We want to make two comments before continuing. First, to the issuer, the replicating strategy is self-financing, in the sense of

\[ \Pi_B(t + dt) - \Pi_B(t) = d\Pi_B(t) = \delta_B dS_t + d\beta_B(t), \]

which holds even in the presence of asymmetric funding spreads. Note that, in addition, both hedge revision and collateral revision at time \( t + dt \) are zero-net transactions. Second, the unilateral self-financing trading strategy may not perfectly replicate the derivative. To see this, we suppose there is no default until \( t \) and integrate (7) over \( (0, t) \), yielding

\[ V_t = \Pi_B(t) - \Pi_C(t) = \Pi_B(t) - \beta_C(t) \geq \Pi_B(t), \]

which means that, once the insurance cost against the issuer’s default risk is accounted for, the derivatives premium may be insufficient to replicate the derivatives payouts (upon a premature default by the buyer or at the contract maturity).

We now proceed to pricing the derivative. To highlight funding costs, we will take the money market account,

\[ B_t = e^{\int_0^t r_u du}, \]  

(8)

as our numeraire asset and work with discount prices,

\[ \hat{A}_t = \frac{A_t}{B_t}, \]

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where $A_t$ represents the spot price of any cum-dividend tradeable assets. In particular, we define

$$\hat{S}_t = \frac{S_t e^{\int_0^t (q_s - \lambda_S)ds}}{B_t},$$

so as to include repo basis to the cost of carry. In terms of discount values, the dynamics of share price, CDS price and cash accounts become

$$d\hat{S}_t = \hat{S}_t \left[ (\mu_t - r_t + \lambda_S)dt + \sigma_t (t) dW_t^{(P)} \right],$$
$$d\hat{U}_B(t) = -\hat{s}_B dt + \hat{L}_B dJ_t^{(P)},$$
$$d\hat{\beta}_B(t) = x_B \hat{\beta}_{m,B}(t) dt,$$
$$d\hat{\beta}_C(t) = x_C \hat{\beta}_{m,C}(t) dt.$$

According to (7) and by the Ito’s lemma, we have the equality

$$\left( \partial_t \hat{V}_i + \frac{1}{2} \hat{S}_t^2 \sigma_t^2 \partial^2_S \hat{V}_i + x_B \hat{\beta}_{m,B}(t) \partial_{\beta_B} \hat{V}_i + x_C \hat{\beta}_{m,C}(t) \partial_{\beta_C} \hat{V}_i \right) dt + \partial_S \hat{V}_i d\hat{S}_t + \Delta \hat{V}_B dJ_t^{(P)} + \Delta \hat{V}_C dJ_t^{(P)}$$

$$= \delta_B d\hat{S}_t + x_B \hat{\beta}_{m,B}(t) dt - \alpha_B \left( -\hat{s}_B dt + \hat{L}_B dJ_t^{(P)} \right) - x_C \hat{\beta}_{m,C}(t) dt,$$

where $\Delta \hat{V}_i = \hat{V}(\tau_i) - \hat{V}(\tau_i^-)$, and $J_i^{(P)}$ jumps from 0 to 1 upon the default of party $i$, for $i = B$ and $C$. Since the derivative is an asset to $C$, there is $\Delta \hat{V}_C = 0$, according to the no-jump assumption made earlier. To eliminate the diffusion risk, the issuer will take

$$\delta_B = \partial_S \hat{V}_i,$$

while to hedge off the LGD upon the default of $B$, the buyer is long

$$\alpha_B = -\frac{\Delta \hat{V}_B}{L_B}$$

units the par CDS. Then, from (9) we obtain the governing PDE for the value function:

$$\partial_t \hat{V}_i + \frac{1}{2} \sigma_t^2 \hat{S}_t^2 \partial^2_S \hat{V}_i + \lambda_t^{(Q)} \Delta \hat{V}_B + x_B \hat{\beta}_{m,B}(t) \partial_{\beta_B} \hat{V}_i + x_C \hat{\beta}_{m,C}(t) \partial_{\beta_C} \hat{V}_i$$

$$= x_B \hat{\beta}_{m,B}(t) - x_C \hat{\beta}_{m,C}(t),$$

(10)
where $\lambda_B^{(Q)} = s_B/L_B$. Equation (10) is subject to either $\hat{V}_T = \hat{Y}_T$, the usual terminal condition of derivatives payoff at maturity, or (1), the payment upon a premature default.

Based on the governing PDE, we now can define the risk-neutral measure, $Q$, or, alternatively, the risk-neutral dynamics of asset prices:

\[
d\hat{S}_t = \hat{S}_t \sigma_S(t) dW^{(Q)}_t, \\
d\hat{U}_B(t) = -\lambda_B^{(Q)} \hat{L}_B dt + \hat{L}_B dJ^{(Q)}_B.
\] (11)

Here, $W^{(Q)}_t$ is a $Q$-Brownian motion and $J^{(Q)}_B$ is a jump process with risk-neutral intensity $\lambda_B^{(Q)}$. Note that the risk-neutral processes or measure has nothing to do with funding costs or risks.

We make a comment here. Equation (10) and (11) imply

\[
E^Q \left[ d \left( -\hat{V}_t + \hat{\Pi}_B(t) \right) \bigg| G_t \right] = x_B \hat{\beta}_{m,B}(t) dt \leq 0,
\]
or

\[
E^Q \left[ d \left( \hat{V}_t + \hat{\Pi}_C(t) \right) \bigg| G_t \right] = x_B \hat{\beta}_{m,C}(t) dt \leq 0.
\]

These two equations can be interpreted as follows: a party will have his expected aggregated return lower than the CSA rate if he bears any funding cost of the counterparty.

With $Q$ we can now specify the discounted risk-free value of the derivative:

\[
\hat{V}_e(t) := E^Q[Y_T | G_t] = E^Q[\hat{Y}_T | F_t],
\] (12)

where the second equality is due to the fact that both $B_T$ and $Y_T$ are $F_T$-adaptive. For notational simplicity, hereafter we will write $E^Q_t[X]$ for $E^Q[X|G_t]$.

Next, we will derive a financially intuitive solution to (10). Combining the diffusion and a jump of the value function, we have

\[
[1_{\tau > T} \hat{Y}_T + 1_{\tau \leq T} \hat{V}(\tau)] - \hat{V}_0 = \int_0^{T\wedge \tau} \left( \partial_u \hat{V}_u + \frac{1}{2} \hat{S}_u^2 \sigma_S^2 \partial^2_S \hat{V}_u + x_B \hat{\beta}_{m,B} \partial_{\beta_B} \hat{V}_u + x_C \hat{\beta}_{m,C} \partial_{\beta_C} \hat{V}_u \right) du \]
\[
+ \sigma_S \hat{S}_u \partial_S \hat{V}_u dW^{(Q)}_u + \Delta \hat{V}_B dJ^{(Q)}_B.
\]
Conditional on $G_0 = \mathcal{F}_0 \vee \{\tau > 0\}$, we take $\mathbb{Q}$-expectation on both sides of the above equation and then make use of equation (10), thus obtaining

$$E_Q^0[1_{\tau > T} \hat{Y}_T] + E_Q^0[1_{\tau \leq T} \hat{V}(\tau)] - \hat{V}_0 = E_Q^0 \left[ \int_0^{T^\wedge \tau} (x_B \hat{\beta}_{m,B}(u) - x_C \hat{\beta}_{m,C}(u)) du \right].$$

Rearranging, we then have the following expression for the derivatives value:

$$\hat{V}_0 = E_Q^0[1_{\{\tau > T\}} \hat{Y}_T] + E_Q^0[1_{\{\tau \leq T\}} \hat{V}(\tau)] - E_Q^0 \left[ \int_0^{T^\wedge \tau} (x_B \hat{\beta}_{m,B}(u) - x_C \hat{\beta}_{m,C}(u)) du \right]. \quad (13)$$

Next, we relate $\hat{V}_0$ to $\hat{V}_e(0)$, its risk-free counterpart defined in (12). In fact, there is

$$E_Q^0[1_{\{\tau > T\}} \hat{Y}_T] = \hat{V}_e(0) - E_Q^0[1_{\{\tau \leq T\}} \hat{Y}_T]. \quad (14)$$

Since $\hat{Y}_T$ is $\mathcal{F}_T$-adaptive and thus independent of $\mathcal{H}_t$, we have, by using the tower law,

$$E_Q^0[1_{\{\tau \leq T\}} \hat{Y}_T] = E_Q^0[1_{\{\tau \leq T\}} E_Q^0[\hat{Y}_T|\mathcal{G}_\tau]] = E_Q^0[1_{\{\tau \leq T\}} E_Q^0[\hat{Y}_T|\mathcal{F}_\tau]] = E_Q^0[1_{\{\tau \leq T\}} \hat{V}_e(\tau)]. \quad (15)$$

By substituting (14) and (15) back to (13), distinguishing between $\tau = \tau_B$ and $\tau = \tau_C$, noticing $\hat{V}_0 = \hat{V}_0; \hat{V}_e(0) = \hat{V}_e(0)$ and $\hat{V}(\tau_C) = \hat{V}_e(\tau_C)$, we finally arrive at the solution in the following form:

$$V_0 = V_e(0) + E_Q^0[1_{\{\tau = \tau_B \leq T\}} (\hat{V}(\tau_B) - \hat{V}_e(\tau_B))] - E_Q^0 \left[ \int_0^{T^\wedge \tau} (x_B \hat{\beta}_{m,B}(u) - x_C \hat{\beta}_{m,C}(u)) du \right].$$

Here, the evolution of margin account balance should take into account the P&L from hedge rebalancing and/or surplus or deficit after collateral revision:

$$d\hat{\beta}_{m,B}(t) = x_B \hat{\beta}_{m,B}(t) dt - d\hat{c}(t) + \partial_S \hat{V}_t d\hat{S}_t, \quad \hat{\beta}_{m,B}(0) = V_0^{(a)} - c(0),$$

$$d\hat{\beta}_{m,C}(t) = x_C \hat{\beta}_{m,C}(t) dt + \lambda_B^Q \Delta \hat{V}_B dt, \quad \hat{\beta}_{m,C}(0) = 0.$$
For a derivative that can switch between asset and liability during its life, both parties may need to hedge their exposures to the diffusion risk of the shares and the default risk of the counterparty, and both may need to post collaterals. In such generality, the derivatives price is subject to bilateral credit valuation and funding valuation adjustments, yet the counterparties may no longer share the unique risk-neutral valuation.

**Proposition 1:** Consider the pricing of a derivative with payoff \( Y_T \) at maturity \( T \). Define

\[
CV A_B = E_Q^0[1_{\{\tau_B \leq T\}}(\hat{V}(\tau_B) - \hat{V}_e(\tau_B))],
\]
\[
CV A_C = E_Q^0[1_{\{\tau_C \leq T\}}(\hat{V}(\tau_C) - \hat{V}_e(\tau_C))],
\]
\[
FV A_B = - E_Q^0 \left[ \int_0^{T \wedge \tau} x_B \hat{\beta}_{m,B}(u)du \right],
\]
\[
FV A_C = E_Q^0 \left[ \int_0^{T \wedge \tau} x_C \hat{\beta}_{m,C}(u)du \right].
\]

Here, the margin accounts \( \hat{\beta}_{m,B}(t) \) and \( \hat{\beta}_{m,C}(t) \) evolve according to

\[
d\hat{\beta}_{m,B}(t) = x_B \hat{\beta}_{m,B}(t)dt - d\hat{c}(t) + \delta_B d\hat{S}_t + \lambda^{(Q)}_C \Delta \hat{V}_C dt,
\]
\[
\hat{\beta}_{m,B}(0) = [V_0^{(a)} - c(0)]^+,
\]
\[
d\hat{\beta}_{m,C}(t) = x_C \hat{\beta}_{m,C}(t)dt + d\hat{c}(t) + \delta_C d\hat{S}_t + \lambda^{(Q)}_B \Delta \hat{V}_B dt,
\]
\[
\hat{\beta}_{m,C}(0) = -[V_0^{(b)} + c(0)]^-,
\]

where \( \delta_B \) and \( \delta_C \) represent necessary delta hedging of market risks by the two parties, and \( \lambda^{(Q)}_C = s_C/L_C \). Suppose both parties take only their own funding costs into account, then the ask price to be offered by the issuer is

\[
V_0^{(a)} = V_e(0) + CV A_B + CV A_C + FV A_B,
\]

while the bid price to be offered by the buyer is

\[
V_0^{(b)} = V_e(0) + CV A_B + CV A_C + FV A_C.
\]

Note that \( V_0^{(a)} \) and \( V_0^{(b)} \) appear in the initial conditions of \( \hat{\beta}_{m,B}(t) \) and \( \hat{\beta}_{m,C}(t) \), so (18) and (19) may define the ask and bid prices implicitly.\(^2\)

\(^2\)In the industry, \( CV A_B + CV A_C \) is called bilateral CVA, and \( CV A_B \) (or \( CV A_C \)) is also called the debit valuation adjustment (DVA) to B (or C).
We need to make a clarification here. Our notion of FVA is regarded as funding cost adjustment (FCA) by some researchers, including Burgard and Kjaer (2012), who instead define $FVA = FCA + DVA$. The rational behind this definition is that the DVA is the funding benefit realized from the yield spread of the bank’s own bond which is purchased to hedge against the bank’s own default. We however do not see such a funding benefit realizable because, even if the long bond position were kept outstanding, the benefit would still be in the expense of the bank itself.

We want to make a few more comments. First, since $FVA_B \geq 0 \geq FVA_C$, there is a spread between the two prices as long as the funding cost of either party is non-zero. To strike a deal, the two parties may take a price within $[V_0^{(b)}, V_0^{(a)}]$, then one or both parties’ expected returns under $Q$ will fall below the CSA rate. Second, the prices, bid or ask, and their corresponding FVAs may be interdependent. Similar interdependence has been obtained in, e.g. Pallavicini et al. (2011), yet the results here are more intuitive and transparent. Third, our CVA formulae give a clear-cut message: the CVA is the expected value of LGD, and the loss is relative to the risk-free value. A different yet equivalent result for CVA is also obtained by Burgard and Kjaer (2012), through the PDE approach. Finally, the price formula (18-17) in fact can hold in a more general context, including stochastic default intensities and stochastic funding spreads. It can also be generalized trivially to pricing derivatives with multiple cash flows, derivatives on multiple underlying assets, and portfolio of derivatives with a single counterparty under close-out netting agreements.

We may decompose a firm’s funding spread for unsecured borrowing into two components: firm-specific funding spread and market-wide funding spread, the latter may arise during a credit crunch. Arguably, the firm-specific funding spread is approximately the credit default swap rate on the firm. Thus we may write

$$x_B = s_B + \lambda_M = \lambda_B^{(Q)} L_B + \lambda_M,$$

$$x_C = s_C + \lambda_M = \lambda_C^{(Q)} L_C + \lambda_M,$$

where $\lambda_M \geq 0$ is market-wide funding spread. We may treat $\lambda_M$ as an indicator of market-wide funding liquidity. Equation (20) can be a starting point to study the issue of liquidity valuation adjustment (LVA).
3 Evaluation of the Adjustment Values

For a collateralized trade, the payment upon a default is described in (1), with the values of the collaterals given by (3), where the MtM value of the derivative can take either

1. the risk-free value, \( M(\tau) = V_e(\tau) \), or
2. the pre-default value, \( M(\tau) = V(\tau-) \).

For simplicity or analytical tractability in the valuation of CVA and FVA, we assume hereafter that the hazard rates for defaults are deterministic. When numerical methods are adopted for the valuation, these assumptions are not necessary.

3.1 When \( M(\tau) = V_e(\tau) \)

3.1.1 CVA

For \( M(t) = V_e(t) \), the loss upon the default of \( B \) is

\[
V(\tau_B) - V_e(\tau_B) = -(1 - R_B)[V_e(\tau_B) - c(\tau_B)]^+ = -(1 - R_B)[V_e^+(\tau_B) - c(\tau_B)].
\] (21)

Similarly, the loss upon the default of \( C \) is

\[
V(\tau_C) - V_e(\tau_C) = -(1 - R_C)[V_e^-(\tau_C) - c(\tau_C)].
\] (22)

Plugging (21) and (22) to the CVA terms in (16), we obtain

\[
CV_A_B = -(1 - R_B)E^Q_0 \left[ 1_{\{\tau_B \leq T\}} \left( CC(\hat{S}_0, \tau_B, 0, 0) - CC(\hat{S}_0, \tau_B, \hat{H}, \hat{X}) \right) \right],
\]

\[
CV_A_C = -(1 - R_C)E^Q_0 \left[ 1_{\{\tau_C \leq T\}} \left( CP(\hat{S}_0, \tau_C, 0, 0) - CP(\hat{S}_0, \tau_C, \hat{H}, \hat{X}) \right) \right],
\]

where

\[
CC(\hat{S}_0, \tau_B, \hat{H}, \hat{X}) = E^Q_0 \left[ (\hat{V}_e(\tau_B) - \hat{H})^+ + \hat{X}1_{\{\hat{V}_e(\tau_B) > \hat{H}\}} \bigg\vert \tau_B \right],
\]

\[
CP(\hat{S}_0, \tau_C, \hat{H}, \hat{X}) = E^Q_0 \left[ (\hat{V}_e(\tau_C) + \hat{H})^- - \hat{X}1_{\{\hat{V}_e(\tau_C) \leq -\hat{H}\}} \bigg\vert \tau_C \right].
\]
Note that \( CC \) and \( CP \) represent the values of a usual compound option plus or minus a digital option. While the latter can be evaluated analytically, the former can be valued through numerical means, including finite difference methods and Monte Carlo simulations. For some vanilla cases, the compound options can also be evaluated analytically (Geske, 1979).

Since the first default of either party follows a Poisson process, conditional on \( G_0 = F_0 \lor \{ \tau > 0 \} \), the risk-neutral probability density function of the first default by either party is

\[
h_i(u) = \lambda_i^{(Q)}(u) e^{-\int_0^u (\lambda_B^{(Q)}(v) + \lambda_C^{(Q)}(v)) dv}, \quad i = B \text{ or } C,
\]

so the evaluation of CVAs is just a matter of integrations:

\[
CV_{AB} = -\int_0^T h_B(u) \left( CC(\hat{S}_0, u, 0, 0) - CC(\hat{S}_0, u, \hat{H}, \hat{X}) \right) du,
\]

\[
CV_{AC} = -\int_0^T h_C(u) \left( CP(\hat{S}_0, u, 0, 0) - CP(\hat{S}_0, u, \hat{H}, \hat{X}) \right) du.
\]

When a derivative is either assets (\( Y_T \geq 0 \)) or liabilities (\( Y_T \leq 0 \)), there are

\[
CC(S_0, \tau_B, 0, 0) = V_e^+(0),
\]

\[
CP(S_0, \tau_C, 0, 0) = V_e^-(0).
\]

For non-collateralized trades, with \( c(t) = 0 \), the evaluation of the \( CV_{AB} \) and \( CV_{AC} \) can be much simplified:

\[
CV_{AB} = -E_0^Q \left[ 1_{\{ \tau_B \leq T \}} \right] (1 - R_B) V_e^+(0)
\]

\[
CV_{AC} = -E_0^Q \left[ 1_{\{ \tau_C \leq T \}} \right] (1 - R_C) V_e^-(0).
\]

By working out the default probabilities we arrive at

**Proposition 3.1** When the derivative is either an asset or a liability and both default intensities and loss rates are constants, there are

\[
CV_{AB} = \frac{-\lambda_B}{\lambda_B + \lambda_C} \left[ 1 - e^{- (\lambda_B + \lambda_C) T} \right] (1 - R_B) V_e^+(0),
\]

\[
CV_{AC} = \frac{-\lambda_C}{\lambda_B + \lambda_C} \left[ 1 - e^{- (\lambda_B + \lambda_C) T} \right] (1 - R_C) V_e^-(0). \quad \square
\]
The above results simply show that a CVA is equal to the product of first-default probability of the party of liability and the lost value to the surviving party, which generalize the existing result of CVA under unilateral default risk of the counterparty (see e.g. Pyktin and Zhu (2007) or Gregory (2009)):

\[ CV_A = -[1 - e^{-\lambda C T}] (1 - R_C)V_e^-(0). \]

### 3.1.2 FVA

In general, \( FVA_B \) and \( FVA_C \) are nonlinear functions of the ask and bid price, respectively, and thus have to be evaluated together with the corresponding prices. Take \( FVA_B \) for example, once a value of \( V_0^{(a)} \) is taken, \( FVA_B \) can be evaluated by either Monte Carlo simulation methods or lattice tree methods. As such, \( V_0^{(a)} \) is implied by

\[ V_0^{(a)} - FVA_B(V_0^{(a)}) = V_e(0) + CV_A + CV_C. \]  

(23)

Note that for any \( 0 \leq u \leq T \wedge \tau \), \( \beta_{m,B}(u) \) is a monotonically increasing function of \( V_0^{(a)} \), while

\[-FVA_B = E_0^Q \left[ \int_0^{T \wedge \tau} x_B \min(\beta_{m,B}(u), 0) du \right].\]

is monotonically increasing in \( \{\beta_{m,B}(u), 0 \leq u \leq T \wedge \tau\} \). As a result, \( V_0^{(a)} - FVA_B \) is monotonically increasing in \( V_0^{(a)} \). The above equation can be easily solved by a root-finding method like the bisection method or the fixed-point iteration method, and consequently we will also obtain \( FVA_B \). When the derivative stays as an asset or a liability, the party of exposure is not required to post collateral or hedge, then either \( FVA_B \) or \( FVA_C \) vanishes.

**Example 1.** For demonstration, we price an at-the-money (ATM) European call option for a range of issuer’s hazard rates. Let the stock price be \( S_t = 100 \), with volatility \( \sigma = 20\% \), dividend yield \( q_t = 0 \), repo basis \( \lambda_S = 0.75\% \), and let the interest rate be \( r_t = 3\% \). When there is no credit and funding risks, the price of one-year ATM call option is \( V_e = 9.8696 \), according to the Black-Scholes formula. In the presence of counterparty default and funding risks, we need to make valuation adjustments to the risk-free value. Since the call option is a liability to the issuer, there is \( CV_A = 0 \), so that

\[ V_0^{(a)} = V_e(0) + CV_B + FVA_B. \]
We will calculate the option value and its valuation adjustments using a binomial tree model for the underlying share price, with the time-step size of $\Delta t = 1/52$.

We take the following parameters for funding or credit risks. The funding spreads for margining are given by (20), where we take market-wide funding spread $\lambda_M = 0$, the loss rates $L_B = L_C = 0.6$, the default intensity for $C$ is $\lambda_C = 1.5\%$, and let $\lambda_B$, the default intensity for $B$, vary from 0 to 300 basis points. We consider both no-CSA and CSA trades. For the no-CSA trade, we apply a 40% recovery rate upon default to the risk-free value. For the CSA trade with cash collateral, we take the threshold value and the minimum transfer amount to be $H = 4$ and $X = 2$, respectively.

Figure 1 and 2 show the option values together with its valuation adjustments, without and with collaterals. In both figures, the plot on the right gives an enlarged view of the adjustment values. For both cases, the FVAs (due to margining) are negligibly small (and is under $10^{-5}$). This can be explained as follows. Under our diffusion model for the underlying share, the call option can be perfectly replicated until the first default or maturity, so the value of the margin account, $\beta_{m,B}(t) = V_t - c(t)$, stays positive due to partial collateralization and incurs no funding cost. The credit valuation adjustment, meanwhile, is sensitive to the hazard rate of $B$, with its magnitude depending on the amount of collateral being put down. In both cases, the CVAs are insignificant when compared with the option values.

3.2 When $M(\tau) = V(\tau)$

In reality, when the risk of counterparty default is looming, the derivative price will suffer quite substantially so that any default settlement based on the pre-default value will mean very little residual value to the holder. For this reason, we do not consider such default settlements equally important. Yet mathematically, it is still interesting.

For illustration, we model the ask price of a derivative asset, with $\hat{Y}(T) \geq 0$, in the absence of collateral and with proportional recovery upon default. It can be argued that there is $\hat{\beta}_{m,B}(t) \geq 0$ due to perfect replication. So, the cost for margining reduces to zero and the unconventional terminal-value
problem of the PDE can be simplified into
\[
\partial_t \hat{V}_t + \frac{1}{2} \sigma_S^2 \hat{S}^2 \partial^2_{S} \hat{V}_t = \lambda_B L_B \hat{V}_t, \quad 0 \leq t \leq \tau_B \land T,
\]
\[
\hat{V}(\tau_B) = R_B \hat{V}(\tau_B-), \quad \text{or} \quad \hat{V}(T) = \hat{f}(S),
\]
where \( f(S) \) is the payoff at maturity and \( L_B = 1 - R_B \). The solution to (24) is
\[
\hat{V}_0 = E^Q_0 \left[ e^{-\int_0^{\tau_B} \lambda_B L_B \text{d}u} R_B \hat{V}(\tau_B-)1_{\tau_B \leq T} \right] + E^Q_0 \left[ e^{-\int_0^{T} \lambda_B L_B \text{d}u} \hat{f}(S_T)1_{\tau_B > T} \right],
\]
(25)

Figure 1. Derivatives value and the adjustments without collateral

Figure 2. Derivatives value and the adjustments with cash collateral
which is an integral equation for $\hat{V}_0$, and it may only be solved recursively through a time-stepping discretization method for PDEs. Note that conditional on $\tau_B > T$, the first term of (25) vanishes and solution actually reduces to Equation (3) of Piterbarg (2010), when there is funding cost but no default risk.

4 CVA and FVA for Interest-rate Swaps

Next, we consider valuation adjustments to vanilla interest-rate swaps, the most liquid interest-rate derivatives traded over the counter. There will be two major differences from the theory for equity derivatives pricing established in section 2. First, there is no delta hedging for the swaps. Second, the cash collateral is revised in discrete time, which is the reality. In fact, with or without collateral, we can always evaluate the CVA terms analytically, using the swap market model. When there is cash collateral, the costs of margining are path-dependent, and their valuations should resort to Monte Carlo simulations. To avoid getting into the discussion of choosing term structure models, we in this article limit ourselves to the evaluation of the credit valuation adjustment in the absence of collateral, when there is no margining cost. For simplicity, we take the same payment frequency for both fixed and floating legs.

Our swap pricing will be based on the swap market model for swap and swaption pricing, which is briefly introduced below. Let $P(t, T)$ be the OIS discount curve, $f_j(T_j)$ be the LIBOR rate for the period $(T_j, T_{j+1})$, $Q_{j+1}$ be the $T_{j+1}$-forward measure corresponding to $P(t, T_{j+1})$ as numeraire. Define

$$A_{m,n}(t) = \sum_{j=m}^{n-1} \Delta T_j P(t, T_{j+1}),$$

$$s_{m,n}(t) = \sum_{j=m}^{n-1} \frac{\Delta T_j P(t, T_{j+1})}{A_{m,n}(t)} E_t^{Q_{j+1}}[f_j(T_j)],$$

for $t \leq T_m$. Note that after the 2007-08 financial tsunami, LIBOR rates are no longer considered riskless and they are no longer martingales under their cash-flow measures. When credit and funding risks are ignored, the values of a vanilla fixed-for-floating payer’s swap of tenor $(T_m, T_n)$ is given by

$$V_e(t) = A_{m,n}(t)(s_{m,n}(t) - s),$$
where $s$ is the fixed rate for the swap. Under the swap market model, the values of call and put options on fixed-for-floating swap are given by

$$BC(t, T_m, s_{m,n}(t), s, m, n, \sigma) = A_{m,n}(t)(s_{m,n}(t)\Phi(d_1) - s\Phi(d_2)), \quad BP(t, T_m, s_{m,n}(t), s, m, n, \sigma) = A_{m,n}(t)(s\Phi(-d_2) - s_{m,n}(t)\Phi(-d_1)),$$

where $\Phi(\cdot)$ is the standard normal accumulative function, and

$$d_{1,2} = \frac{\ln \frac{s_{m,n}(t)}{s} \pm \frac{1}{2} \sigma^2(T_m - t)}{\sigma \sqrt{T_m - t}},$$

where $\sigma$ is the volatility of the swap rate. For derivation of the swap and swaption formulae we refer to, e.g., Wu (2009).

We now proceed to evaluate $CVA_B$ (or $CVA$). According to the rules of default settlement, there is

$$CVA_B = E_0 \left[ 1_{\{\tau = \tau_B < T\}}(R_B - 1)\hat{V}_e^+(\tau_B) \right] = (R_B - 1) \sum_{j=1}^{n} E_0 \left[ \hat{V}_e^+(T_j) \right] E_0 \left[ 1_{T_{j-1} \leq \tau = \tau_B \leq T_j} \right],$$

where

$$E_0[1_{T_{j-1} \leq \tau = \tau_B \leq T_j}] = e^{-\int_{0}^{T_j-1}(\lambda_B + \lambda_C)du}(1 - e^{-\int_{T_{j-1}}^{T_j} \lambda_B du}).$$

Since

$$\hat{V}_e^+(T_j) = \hat{A}_{j,n}(T_j)(s_{j,n}(T_j) - s)^+$$

is the payoff of a swaption, we have

$$E_0 \left[ \hat{V}_e^+(T_j) \right] = BC(0, T_j, s_{j,n}(0), s, j, n, \sigma).$$

Putting (28) and (29) back to (27), we obtain $CVA_B$.

The formulae for $CVA_C$ (or $DVA$) can be derived similarly, and it is

$$CVA_C = (R_C - 1) \sum_{j=1}^{n} E_0 \left[ \hat{V}_e^-(T_j) \right] E_0[1_{T_{j-1} \leq \tau = \tau_C \leq T_j}],$$

with

$$E_0[1_{T_{j-1} \leq \tau = \tau_C \leq T_j}] = e^{-\int_{0}^{T_j-1}(\lambda_B + \lambda_C)du}(1 - e^{-\int_{T_{j-1}}^{T_j} \lambda_C du}).$$
and

$$E_0 \left[ \hat{V}_e^-(T_j) \right] = -BP(0, T_j, s_{j,n}(0), s, j, n, \sigma).$$

According to Proposition 1, the value of the swaps exposed to counterparty default risk and funding risk is

$$\hat{V}_0 = \hat{V}_e(0) + CVA_B + CVA_C.$$

The fair swap rate after adjusted for counterparty default risks can be solved from $\hat{V}_0 = 0$.

**Example 2.** We consider the pricing of a ten-year vanilla swap with semi-annual payment frequency, such that $\Delta T_j = 0.5$. The OIS discount curve and Euribor forward-rate curve are constructed using data of August 24, 2012, and they are shown in Figure 3. We take $s = s_{0,20}(0) = 1.45\%$, the risk-free ATM swap rate so that $\hat{V}_e(0) = 0$. For the adjustment terms, we take

$$R_B = R_C = 0.4,$$
$$\lambda_M = 0, \quad \lambda_C = 0.015\%, \quad \text{and} \quad \lambda_B = 0.000 : 0.005 : 0.03.$$

Figure 4 displays the swap value together with the adjustment terms. Without surprise, it shows that $CVA_B$ decreases with an increasing $\lambda_B$, while $CVA_C$ remains close to a constant due to the constant hazard rate for $C$.

## 5 Conclusions

Through replication using shares and CDS, we price derivative securities subject to bilateral default and funding risks, and identify CVA and FVA as price components. For derivatives trades collateralized by cash, we obtain CVA either analytically or semi-analytically. The evaluation of FVA, in general, resorts to numerical methods like Monte Carlo simulations. The results can be directly generalized to portfolios of derivatives with collaterals and/or netting agreements, and the framework can be applied to deal with re-hypothecation. The combined approach can be easily generalized to accommodate more general asset-price dynamics like dynamics like jump-diffusion processes.

There is room to enhance the results of this article. By considering jump-diffusion processes for the underlying assets, we can evaluate gap risk, the
effect of jumps on funding costs. By modeling the returns of collaterals, we can study re-hypothecation effect when the collaterals are used for funding purposes and thus exposed to default risks. Finally, the adjustment terms derived in the article are for European options. For application purposes, we need to generalize the results of this article to American or Bermudan options.

Figure 3 OIS discount curve and Euribor forward rates

Figure 4 Credit valuation adjustments for B and C
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A Price formulae for compound plus digital options

The compound-call plus digital option we encounter in the article can be expressed as

\[ CC(S_0, u, H, X) = E_0^Q \left[ \left( \hat{V}_e(u) - \hat{H} + X \right) 1_{\{\hat{V}_e(u) > \hat{H}\}} \right] = E_0^Q \left[ \left( \hat{V}_e(u) - \hat{H} \right)^+ \right] + E_0^Q \left[ \hat{X} 1_{\{\hat{V}_e(u) > \hat{H}\}} \right] , \]  

(30)

where the first term is a standard compound call option. Under stochastic interest rate, the compound call option can be treated as follows

\[ E_0^Q \left[ \left( \hat{V}_e(u) - \hat{H} \right)^+ \right] = P(0, u) E_0^Q \left[ \frac{P(u, u)}{P(0, u) B_u} (V_e(u) - H)^+ \right] = P(0, u) E_0^{Q_u} \left[ (V_e(u) - H)^+ \right] . \]  

(31)

Here, \( Q_u \) is the \( u \)-forward measure corresponding to numeraire \( P(t, u) \), the \( u \)-maturity risk-free discount bond. Compound options like (31) can be evaluated numerically or, in some circumstance, analytically.

When the derivatives underlying the compound option is a usual call or put option, we know from Geske (1979) that the value of compound options can be obtained in closed forms. Let \( C(S_u, u; K, T) \) denote the time-\( u \) price of a call option with maturity \( T \) and strike \( K \), then, assuming constant dividend yield, the European call-on-call option is

\[ E_0^{Q_u} \left[ (C(S_u, u; K, T) - H)^+ \right] = \frac{S_t e^{-q T}}{P(0, u)} \Phi_2(a_+, b_+, \sqrt{u/T}) - \frac{K P(0, T)}{P(0, u)} \Phi_2(a_-, b_-, \sqrt{u/T}) - H \Phi(a_-) , \]  

(32)

where \( \Phi_2(x, y; \rho) \) is the two-dimensional cumulative normal distribution functions,

\[ a_+ = \frac{\ln \frac{S_0 e^{-q u}}{S_u} + \frac{1}{2} \sigma^2 u}{\sigma \sqrt{u}} , \quad a_- = a_+ - \sigma \sqrt{u} , \]

\[ b_+ = \frac{\ln \frac{S_0 e^{-q T}}{K} + \frac{1}{2} \sigma^2 T}{\sigma \sqrt{T}} , \quad b_- = b_+ - \sigma \sqrt{T} , \]
$S_u$ solves $C(S_u, u; K, T) = H$, and $\sigma$ is the volatility of $\frac{S_u e^{-qt}}{P(0,u)}$, the $u$-forward price of the underlying. There is also

$$E_0^Q \left[ \hat{X} 1_{\{\hat{C}(u) > \hat{H} \}} \right] = P(0, u) E_0^Q u \left[ X 1_{\{C(u) > H \}} \right] = P(0, u) X \Phi(a_-). \quad (33)$$

Combining (30) to (33), we obtain the price formulae for the compound-call-plus-digital option:

$$CC(S_0, u, H, X) = S_0 e^{-qT} \Phi_2(a_+, b_+, \sqrt{u/T}) - KP(0, T) \Phi_2(a_-, b_-, \sqrt{u/T}) - (H - X) P(0, u) \Phi(a_-),$$

Function $CP(S_0, \tau_C, H, X)$ may be nonzero only if $\hat{V}_e(\tau_C) \leq 0$, when the derivative is a liability to $C$. We have

$$CP(S_0, \tau_C, H, X) = -E_0^Q \left[ \left( -\hat{V}_e(\tau_C) - \hat{H} + \hat{X} \right) 1_{\{-\hat{V}_e(\tau_C) \geq \hat{H} \}} \right],$$

which can be evaluated in the same way as for $CC(S_0, u, H, X)$.