Abstract

We give an explicit simplicial model for the Hopf map $S^3 \to S^2$. For this purpose, we construct a model of $S^3$ as a principal twisted cartesian product $K \times_{\eta} S^2$, where $K$ is a simplicial model for $S^1$ acting by left multiplication on itself, $S^2$ is given the simplest simplicial model and the twisting map is $\eta : (S^2)_n \to (K)_{n-1}$. We construct a Kan complex for the simplicial model $K$ of $S^1$. The simplicial model for the Hopf map is then the projection $K \times_{\eta} S^2 \to S^2$.

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1 Introduction

The motivation for finding a simplicial model for the Hopf map arose when trying to find a simple test to decide whether the stabilisation of a certain interesting model category $M$ is different from that of the category of chain complexes of abelian groups. As detailed in [2, chapter 6], consider the following situation. Let $M$ be a symmetric monoidal model category whose stabilisation exists and suppose there is a monoidal Quillen adjunction $F : sS \leftrightarrow M : G$ between the category of simplicial sets and $M$. In the stable category of chain complexes, the Hopf map vanishes. Therefore, if we have a good simplicial model for the Hopf map that allows us to show that the
multiply suspended images under the functor $F$ of this simplicial model never vanish, then the stabilisation of $M$ is different from that of chain complexes.

The main result of this paper is that a very good simplicial model of the Hopf map is the projection $p : K \times \eta S^2 \rightarrow S^2$ of a principal twisted cartesian product of a simplicial model $K$ of $S^1$ with the simplest simplicial model for $S^2$. The proof of this result shows that we are able to model simplicially any $S^1$-bundle of base $S^2$.

This paper is structured in the following manner. Section 2 recalls the notions and results related to principal twisted cartesian products. In section 3 we construct a Kan model for $S^1$, which is required to carry enough structure. Section 4 gives explicit computations of the Kan model as well as of the twisting map. Finally, we prove the main result in section 5.

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2 Principal twisted cartesian products

This section is devoted to explaining the tools for building a simplicial model for $S^3$ with enough structure to capture the Hopf map.

**Definition 2.1.** Let $F$ and $B$ be two simplicial sets. Let $H$ be a simplicial group acting on the left on $F$. Let $\zeta : B \rightarrow H$ be a map of graded sets of degree $-1$ such that $\zeta_n : B_n \rightarrow H_{n-1}$ satisfies the following identities:

\[
\partial_0 \zeta(b) = (\zeta(\partial_0 b))^{-1} \zeta(\partial_1 b) \\
\partial_i \zeta(b) = \zeta(\partial_{i+1} b) \text{ for } i > 0 \\
s_i \zeta(b) = \zeta(s_{i+1} b) \text{ for } i \geq 0 \\
\zeta(s_0 b) = \text{id}_n \text{ for } b \in B_n.
\]

The map $\zeta$ is the twisting map. A twisted cartesian product of fibre $F$, base $B$ and group $H$ is a simplicial set denoted $F \times_\zeta B$ satisfying

\[(F \times_\zeta B)_n = F_n \times B_n\]

with faces and degeneracies as follows:

1. $\partial_i (f, b) = (\partial_i f, \partial_i b)$ for $i > 0$
2. \( \partial_0(f, b) = (\zeta(b)\partial_0 f, \partial_0 b) \)

3. \( s_i(f, b) = (s_i f, s_i b) \) for \( i \geq 0 \).

Furthermore, if \( F = H \) acting on itself by left multiplication, then \( F \times_\zeta B \) is a principal twisted cartesian product (PTCP).

We will also use the terminology “twisted cartesian product” for the projection \( p : F \times_\zeta B \to B \).

The following proposition is a classical result whose proof can be found in [1, proposition 18.4].

**Proposition 2.2.** Let \( p : F \times_\zeta B \to B \) be a twisted cartesian product with group \( H \). If the fiber \( F \) is a Kan complex, then:

1. the projection \( p \) is a Kan fibration, and
2. if \( F = H \), \( p \) is a principal fibration.

**Remark 2.3.** Let \( S : \text{TOP} \to \text{sS} \) be the singular functor from the category \( \text{TOP} \) of topological spaces to the category \( \text{sS} \) of simplicial sets. A map \( f \) is a (Serre) fibration if and only if \( S(f) \) is a Kan fibration. Thus, a principal fibration (or fibre bundle) in \( \text{TOP} \) passes via the functor \( S \) to a principal fibration in \( \text{sS} \). Since the Hopf map \( S^3 \to S^2 \) is a fibration in \( \text{TOP} \), the corresponding simplicial model has to be a Kan fibration. As a consequence, if we want to model \( S^3 \) as a PTCP \( K \times_\eta S^2 \to S^2 \), this has to be a Kan fibration, which it is when \( K \) is a Kan complex, by proposition 2.2. We construct such a PTCP in the following sections.

## 3 The simplicial model for \( S^1 \)

In short, to build a Kan model of \( S^1 \) we let \( \mathbb{Z}(2) \) denote a chain complex concentrated in degree two, and we apply a functor \( \Gamma \) to obtain a simplicial abelian group \( \Gamma \mathbb{Z}(2) \). By applying the loop group functor \( G \), the model of \( S^1 \) is given by \( G\Gamma \mathbb{Z}(2) \). The latter is always a Kan complex, since every simplicial group is a Kan complex. More precisely we give the following definitions.

**Definition 3.1.** Let \( \text{sAb} \) be the category of simplicial abelian groups and let \( \text{CC} \) be the category of chain complexes of abelian groups. We define the functor \( \Gamma : \text{CC} \to \text{sAb} \) as follows. For any \( (X, \partial) \in \text{CC} \), the simplicial abelian group \( \Gamma(X) \) is given by:
1. \[ \Gamma_n(X) = X_n \bigoplus_{r=0}^{n-1} \sum_{k=n-r} \sigma_{jk} \ldots \sigma_{j1} X_r \] (1)

where \( \sigma_{jk} \ldots \sigma_{j1} X_r \) is the abelian group whose elements are the symbols \( \sigma_{jk} \ldots \sigma_{j1} x \) with \( x \in X_r \). The sum \( \sum_{k=n-r} \) is taken over all sequences of indices \( \{j_i\} \) such that \( 0 \leq j_1 < j_2 < \cdots < j_k < n \).

The addition of symbols is defined by

\[ \sigma_{jk} \ldots \sigma_{j1} x + \sigma_{jk} \ldots \sigma_{j1} y = \sigma_{jk} \ldots \sigma_{j1} (x + y). \]

Degeneracies and faces are given by:

2. \( s_i : \Gamma_n(X) \to \Gamma_{n+1}(X) \) is defined by

(a) \( s_i x = \sigma_i x \) for \( x \in X_n \)

(b) if \( k = n - r \) and \( x \in X_r \) then

\[ s_i \sigma_{jk} \ldots \sigma_{j1} x = \sigma_{h_{k+1}} \ldots \sigma_{h_1} x \]

when \( s_i s_{jk} \ldots s_{j1} = s_{h_{k+1}} \ldots s_{h_1} \) and where \( s_{h_{k+1}} \ldots s_{h_1} \) is written in the canonical form\(^1\), i.e. \( h_{k+1} > h_k > \ldots > h_1 \).

3. \( \partial_i : \Gamma_n(X) \to \Gamma_{n-1}(X) \) is defined by

(a) \( \partial_n x = \partial(x) \) and \( \partial_i x = 0 \) if \( i < n \) and \( x \in X_n \).

(b) if \( k = n - r \) and \( x \in X_r \) then

\[ \partial_i \sigma_{jk} \ldots \sigma_{j1} x = \begin{cases} \sigma_{h_{k-1}} \ldots \sigma_{h_1} x & \\
\sigma_{h_k} \ldots \sigma_{h_1} \partial(x) & \\
0 & 
\end{cases} \]

if respectively

\[ \partial_i s_{jk} \ldots s_{j1} = \begin{cases} s_{h_{k-1}} \ldots s_{h_1} & \\
s_{h_k} \ldots s_{h_1} \partial_r & \\
s_{h_k} \ldots s_{h_1} \partial_j & j < r \end{cases} \]

where the right hand side is written in the canonical form.

\(^1\)Every composition of degeneracies and/or faces can be written in the canonical form with the aid of the simplicial identities.
We now define the functor $G$.

**Definition 3.2.** Let $sGr$ the category of simplicial groups and let $K$ be a simplicial set. We define the functor $G : sS \rightarrow sGr$ as follows. The group $G_n(K) = G(K)_n$ is the free group generated by the elements of $K_{n+1}$ modulo the relations $s_0 x = \text{id}_n$ for all $x \in K_n$.

If $x \in K_{n+1}$, let $\zeta(x)$ be the class of $x$ in $G_n(K)$. Faces and degeneracies of $G(K)$ are defined on generators by the relations:

1. $\zeta(\partial_0 x) \partial_0 \zeta(x) = \zeta(\partial_1 x)$
2. $\partial_i \zeta(x) = \zeta(\partial_{i+1} x)$ if $i > 0$
3. $s_i \zeta(x) = \zeta(s_{i+1} x)$ if $i \geq 0$.

By extension we have homomorphisms $\partial_i : G_n(K) \rightarrow G_{n-1}(K)$ and $s_i : G_n(K) \rightarrow G_{n+1}(K)$. Clearly, $G(K)$ is a simplicial group.

**Remark 3.3.** The morphism $\zeta$ of definition 3.2 is clearly a twisting map. Hence, for every simplicial abelian group $K$ we have a twisted cartesian product $G(K) \times_\zeta K$, which is acyclic. The reader may refer to [1, pp. 118–123] for details.

By [1, Remarks 23.7], $\Gamma Z(2)$ is a $K(Z, 2)$, hence a simplicial model for $BS^1$. $G \Gamma Z(2)$ is then a model for $\Omega BS^1$, hence for $S^1$.

## 4 Some computations

This section is devoted to clarifying the previous construction by giving explicit computations of $G \Gamma Z(2)$ and the map $\eta$. For this we will choose a simplicial model for $S^2$ consisting in one non degenerate simplex in degree two and only degeneracies above.

To compute $\Gamma_nZ(2)$ we use formula (1). Since $Z(2)$ is concentrated in degree two, we obtain

$$\Gamma_n Z(2) = \bigoplus_{0 \leq j_1 < \cdots < j_{n-2} < n} \sigma_{j_{n-2}} \cdots \sigma_{j_1} \mathbb{Z},$$  

As an example, in degree three, the faces and degeneracies are given for all $z \in \mathbb{Z}$ by

- $\partial_0 (\sigma_0 z) = z$  
- $\partial_1 (\sigma_0 z) = z$  
- $\partial_2 (\sigma_0 z) = 0$  
- $\partial_3 (\sigma_0 z) = 0$
- $\partial_0 (\sigma_1 z) = 0$  
- $\partial_1 (\sigma_1 z) = z$  
- $\partial_2 (\sigma_1 z) = z$  
- $\partial_3 (\sigma_1 z) = 0$
- $\partial_0 (\sigma_2 z) = 0$  
- $\partial_1 (\sigma_2 z) = 0$  
- $\partial_2 (\sigma_2 z) = z$  
- $\partial_3 (\sigma_2 z) = z$
For $G\Gamma Z(2)$, we have

$$(G\Gamma Z(2))_0 = \{e\}, \quad (G\Gamma Z(2))_1 = \mathcal{F}\{Z\setminus\{0\}\}$$

$$(G\Gamma Z(2))_2 = \mathcal{F}\{\sigma_2 Z \pm \sigma_1 Z\}$$

$$(G\Gamma Z(2))_3 = \mathcal{F}\{\sigma_2 \sigma_1 Z \pm \sigma_3 \sigma_2 Z \pm \sigma_4 \sigma_3 Z \pm \sigma_4 \sigma_2 \sigma_1 Z\}$$

$$\vdots$$

$$(G\Gamma Z(2))_n = \mathcal{F}\left\{\bigoplus_{0 < j_1 < \cdots < j_{n-1} < n+1} \sigma_{j_{n-1}} \cdots \sigma_{j_1} Z\right\}. \quad (6)$$

where $\mathcal{F}\{\}$ stands for the free group generated by elements inside $\{}$. Notice that $s_0(\sigma_{j_{n-1}} \cdots \sigma_{j_1} z)$ can always be expressed in a form ending by $\sigma_0 z$. Hence each term containing $\sigma_0 Z$ is trivial and gives the first strict inequality in $0 < j_i < \cdots < j_{n-1} < n + 1$. Faces and degeneracies are given by the formulae (2)–(4).

Let $x$ be the class of $x \in \Gamma_{n+1}Z(2)$ in $(G\Gamma Z(2))_n$. For $S^2$ we consider the simplicial model consisting in one generator $y$ in degree two and only degeneracies above. The twisting morphism $\eta : S^2 \to G\Gamma Z(2)$ is defined by the relations:

$$\eta_0(*) = e, \quad \eta_1(*) = e$$

$$\eta_2(y) = 1$$

$$\eta_3(s_1y) = \sigma_1 1$$

$$\eta_3(s_2y) = \sigma_2 1$$

$$\eta_3(s_0y) = e$$

$$\eta_4(s_2s_1y) = \sigma_2 \sigma_1 1$$

$$\eta_4(s_3s_2y) = \sigma_3 \sigma_2 1$$

$$\eta_4(s_3s_1y) = \sigma_3 \sigma_1 1$$

$$\eta_4(s_is_0y) = e \quad \text{for } 0 \leq i < 4$$

$$\vdots$$
where \( \bar{1} \) is a generator of \( F \{ \mathbb{Z} \setminus \{0\} \} \). In general, for \( n \geq 2 \)

\[
\eta_n(s_{j_{n-2}} \ldots s_{j_1} y) = \begin{cases} 
  e & \text{if } s_{j_1} = s_0 \\
  \sigma_{j_{n-2}} \ldots \sigma_{j_1} \bar{1} & \text{otherwise}
\end{cases}
\]

where \( s_{j_{n-2}} \ldots s_{j_1} y \) is written in the canonical form.

The map \( \eta \) is then determined by its value on the generator \( y \) of the model of \( S^2 \), as is clear from the formula (\ref{eq:eta}).

5 The simplicial model for the Hopf map

We now have all the tools to build our simplicial model for \( S^3 \). Denote by \( \mathbb{Z} \) the set of integers and by \( \mathbb{Z}(2) \) the chain complex of abelian groups consisting in one copy of \( \mathbb{Z} \) in degree two and 0 elsewhere. We apply the functor \( \Gamma \) to get a simplicial abelian group \( \Gamma \mathbb{Z}(2) \). Therefore, by remark \[3.3\],

\[
p : \Gamma \mathbb{Z}(2) \times_{\eta} S^2 \to S^2
\]

is a principal twisted cartesian product whose fiber is \( \Gamma \mathbb{Z}(2) \) acting on itself by left multiplication. The map \( \eta : S^2 \to \Gamma \mathbb{Z}(2) \) is explained in the previous section.

**Theorem 5.1.** Let \( S^2 \) be endowed with the above simplicial model. A simplicial model for the Hopf map \( S^3 \to S^2 \) is then given by the principal twisted cartesian product

\[
p : \Gamma \mathbb{Z}(2) \times_{\eta} S^2 \to S^2.
\]

**Proof.** The fibration \( p : \Gamma \mathbb{Z}(2) \times_{\eta} S^2 \to S^2 \) is a model for an element of the set of \( S^1 \)-bundles of base \( S^2 \), which contains the Hopf map. Now, \( S^1 \)-bundles of base \( S^2 \) are classified by \( \mathbb{Z} \), and the Hopf map corresponds to the class \( 1 \in \mathbb{Z} \). All we have to show is that our model \( \Gamma \mathbb{Z}(2) \times_{\eta} S^2 \to S^2 \) corresponds indeed to the class \( 1 \in \mathbb{Z} \). Consider the diagramm

\[
\begin{array}{c}
\Gamma \mathbb{Z}(2) \\
\downarrow \\
\Gamma \mathbb{Z}(2) \times_{\eta} S^2 \\
\downarrow \\
S^2 \\
\end{array} \quad \begin{array}{c}
\rightarrow \\
\Gamma \mathbb{Z}(2) \times_{\eta} \Gamma \mathbb{Z}(2) \\
\downarrow \\
\Gamma \mathbb{Z}(2) \quad \begin{array}{c}
\leftarrow \\
\Gamma \mathbb{Z}(2) \times_{\eta} \Gamma \mathbb{Z}(2) \\
\downarrow \\
\Gamma \mathbb{Z}(2) \end{array}
\end{array}
\]

\[
\beta
\]


where the two columns are fibrations and the composition $\beta \alpha$ is the twisting map $\eta$. Recall from last section that the bottom composition $\eta$ sends the generator $y$ of $S^2$ to the class of the generator $1 \in \mathbb{Z}$. Note that $G\Gamma\mathbb{Z}(2) \times_{\zeta} \Gamma\mathbb{Z}(2)$ is acyclic and that the first vertical fibration is classified by the map $\beta \alpha = \eta$. By choosing $\eta$ to send a generator of $S^2$ to the generator $1 \in \mathbb{Z}$ we guaranty that our fibration $G\Gamma\mathbb{Z}(2) \times_{\eta} S^2 \to S^2$ lies in the same class as the Hopf map does and hence is a model of the later.

Remark 5.2. In the previous proof, if we choose to send $y \in S^2$ to $m1 \in \mathbb{Z}$ via the map $\alpha$, our fibration can model any $S^1$-bundle of base $S^2$ by letting $m$ vary over $\mathbb{Z}$.

References

[1] J. P. May, *Simplicial objects in algebraic topology*, Mathematical Studies 11, Van Nostrand, 1967.

[2] O. R. Sauvageot, *Stabilisation des complexes croisés*, PhD. thesis 2692, Ecole Polytechnique fédérale de Lausanne, 2003. Available at [http://hopf.math.purdue.edu/](http://hopf.math.purdue.edu/)