ON AN EXTENSION
OF KORN’S FIRST INEQUALITY
TO INCOMPATIBLE TENSOR FIELDS
ON DOMAINS OF ARBITRARY DIMENSIONS

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Abstract

For a bounded domain Ω in \( \mathbb{R}^N \) with Lipschitz boundary \( \Gamma = \partial \Omega \) and a relatively open and non-empty ‘admissible’ subset \( \Gamma_0 \) of \( \Gamma \), we prove the existence of a positive constant \( c \) such that inequality

\[
  c \| T \|_{L^2(\Omega, \mathbb{R}^{N \times N})} \leq \| \text{sym} \, T \|_{L^2(\Omega, \mathbb{R}^{N \times N})} + \| \text{Curl} \, T \|_{L^2(\Omega, \mathbb{R}^{N \times N}((N-1)/2))}
\]

(0.1)

holds for all tensor fields \( T \in \mathcal{H}(\text{Curl}; \Omega, \mathbb{R}^{N \times N}) \), this is, for all square-integrable tensor fields \( T : \Omega \to \mathbb{R}^{N \times N} \) having a row-wise square-integrable rotation tensor field \( \text{Curl} \, T : \Omega \to \mathbb{R}^{N \times N((N-1)/2)} \) and vanishing row-wise tangential trace on \( \Gamma_0 \).

For compatible tensor fields \( T = \nabla v \) with \( v \in H^1(\Omega, \mathbb{R}^N) \) having vanishing tangential Neumann trace on \( \Gamma_0 \) the inequality (0.1) reduces to a non-standard variant of Korn’s first inequality since \( \text{Curl} \, T = 0 \), while for skew-symmetric tensor fields \( T \) Poincaré’s inequality is recovered.

If \( \Gamma_0 = \emptyset \), our estimate (0.1) still holds at least for simply connected \( \Omega \) and for all tensor fields \( T \in \mathcal{H}(\text{Curl}; \Omega, \mathbb{R}^{N \times N}) \) which are \( L^2(\Omega, \mathbb{R}^{N \times N}) \)-perpendicular to \( \mathfrak{so}(N) \), i.e., to all skew-symmetric constant tensors.

Key Words Korn’s inequality, Poincaré’s inequality, Maxwell’s equations, Helmholtz’ decomposition, gradient plasticity, incompatible tensor fields, differential forms, mixed boundary conditions

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1 Introduction and Main Results

We extend the Korn-type inequalities from [16] presented earlier in less general settings in [13, 12, 15, 14] to the \( N \)-dimensional case. For this, let \( N \in \mathbb{N} \) and \( \Omega \) be a bounded domain in \( \mathbb{R}^N \) as well as \( \Gamma_t \) be an open subset of its boundary \( \Gamma := \partial \Omega \). Our main result reads:

**Theorem 1 (Main Theorem)** Let the pair \((\Omega, \Gamma_t)\) be admissible\(^*\). There exist constants \( 0 < c_1 \leq c_2 \) such that the following estimates hold:

(i) If \( \Gamma_t \neq \emptyset \), then the inequality
\[
\|T\|_{L^2(\Omega)} \leq c_1 \left( \|\text{sym} \, T\|_{L^2(\Omega)} + \|\text{Curl} \, T\|_{L^2(\Omega)} \right)
\]
holds for all tensor fields \( T \in \overset{\circ}{\mathcal{H}}(\text{Curl}; \Gamma_t, \Omega) \). In other words, on \( \overset{\circ}{\mathcal{H}}(\text{Curl}; \Gamma_t, \Omega) \) the right hand side defines a norm equivalent to the standard norm in \( \mathcal{H}(\text{Curl}; \Omega) \).

(ii) If \( \Gamma_t = \emptyset \), then for all tensor fields \( T \in \mathcal{H}(\text{Curl}; \Omega) \)\(^\dagger\) there exists a piece-wise constant skew-symmetric tensor field \( A \), such that
\[
\|T - A\|_{L^2(\Omega)} \leq c_2 \left( \|\text{sym} \, T\|_{L^2(\Omega)} + \|\text{Curl} \, T\|_{L^2(\Omega)} \right).
\]
Note that in general \( A \notin \mathcal{H}(\text{Curl}; \Omega) \).

(ii') If \( \Gamma_t = \emptyset \) and \( \Omega \) is additionally simply connected, then for all tensor fields \( T \in \mathcal{H}(\text{Curl}; \Omega) \) there exists a uniquely determined constant skew-symmetric tensor field \( A = A_T \in \mathfrak{so}(N) \)\(^\ddagger\) such that
\[
\|T - A_T\|_{L^2(\Omega)} \leq c_1 \left( \|\text{sym} \, T\|_{L^2(\Omega)} + \|\text{Curl} \, T\|_{L^2(\Omega)} \right).
\]

\(^*\)The precise meaning of ‘admissible’ will be given in Definition 25.

\(^\dagger\)If \( \Gamma_t = \emptyset \), then \( \overset{\circ}{\mathcal{H}}(\text{Curl}; \Gamma_t, \Omega) = \mathcal{H}(\text{Curl}; \Omega) \).

\(^\ddagger\)\( \mathfrak{so}(N) \) denotes the set of all constant skew-symmetric tensors, i.e., \( (N \times N) \)-matrices.
Since $A_T \in H(\text{Curl};\Omega)$ one can easily estimate $\|T - A_T\|_{H(\text{Curl};\Omega)}$ as well. Moreover, $T - A_T \in H(\text{Curl};\Omega) \cap \mathfrak{so}(N)^\perp$ and $A_T = 0$ if and only if $T \perp \mathfrak{so}(N)$. Thus, the inequality in (i) holds for all $T \in H(\text{Curl};\Omega) \cap \mathfrak{so}(N)^\perp$ as well. Therefore, also on $H(\text{Curl};\Omega) \cap \mathfrak{so}(N)^\perp$ the right hand side defines a norm equivalent to the standard norm in $H(\text{Curl};\Omega)$.

Remark 2

(i) Here, the differential operator $\text{Curl}$ denotes the row-wise application of the standard curl in $\mathbb{R}^N$ and a tensor field $T$ belongs to the Hilbert Sobolev-type space $H(\text{Curl};\Gamma_t,\Omega)$ if $T$ and its distributional $\text{Curl}T$ belong both to the standard Lebesgue spaces $L^2(\Omega)$ and the row-wise weak tangential trace of $T$ vanishes at the boundary part $\Gamma_t$. Exact definitions of all spaces and operators used will be given in section 2.

(ii) In (ii') the special constant skew-symmetric tensor field $A_T$ is given explicitly by $A_T = \pi_{\mathfrak{so}(N)}T \in \mathfrak{so}(N)$, where $\pi_{\mathfrak{so}(N)} : L^2(\Omega) \to \mathfrak{so}(N)$ denotes the $L^2(\Omega)$-orthogonal projection onto $\mathfrak{so}(N)$ and can be represented by

$$\pi_{\mathfrak{so}(N)}T = \text{skew} \oint_{\Omega} T \ d\lambda \in \mathfrak{so}(N).$$

Furthermore, $A_T$ can also be computed by

$$A_T = A_R := \pi_{\mathfrak{so}(N)}R = \text{skew} \oint_{\Omega} R \ d\lambda \in \mathfrak{so}(N),$$

where $R$ denotes the Helmholtz projection of $T$ onto $H(\text{Curl}_0;\Omega)$ according to Corollary 18.

(iii) The constants $c_1$ and $c_2$ are given by (3.3) and (3.4) and these depend in a simply algebraic way only on the constants $c_k, c_m$ in Korn’s first and the Maxwell inequality.

For the proof of Theorem 1 we follow in close lines the proofs from [16]. Therefore, again we need to combine three crucial tools, namely

- a Maxwell estimate, Corollary 17;
- a Helmholtz decomposition, Corollary 18;
- a generalized version of Korn’s first inequality, Lemma 29.

Our assumptions on the domain $\Omega$ and the part of the boundary $\Gamma_t$, i.e., on the pair $(\Omega, \Gamma_t)$, are precisely made for this three major tools to hold. We will present these assumptions in section 2 and a pair $(\Omega, \Gamma_t)$ satisfying those will be called admissible.

Theorem 1 can be looked at as a common generalization and formulation of two well known and very important classical inequalities, namely Korn’s first and Poincaré’s
inequality. This is, taking irrotational tensor fields $T$, i.e., $\text{Curl} \, T = 0$, then a non-standard version of Korn’s first inequality

$$c \|T - A_T\|_{L^2(\Omega)} \leq \|\text{sym} \, T\|_{L^2(\Omega)}$$

holds for all $T \in \mathcal{H}(\text{Curl}_0; \Gamma_t, \Omega)$, where $A_T = 0$ if $\Gamma_t \neq \emptyset$. Another, less general choice, is $T = \nabla v$ yielding

$$c \|\nabla v - A_v\|_{L^2(\Omega)} \leq \|\text{sym} \, \nabla v\|_{L^2(\Omega)}$$

with some vector field $v$ belonging to $\mathcal{H}(\Gamma_t; \Omega)$ or just to $H^1(\Omega)$ with $\nabla v_n$, $n = 1, \ldots, N$, normal at $\Gamma_t$. Note that

$$\nabla \mathcal{H}^1(\Gamma_t; \Omega), \nabla \{v \in H^1(\Omega) : \nabla v_n \text{ normal at } \Gamma_t \, \forall \, n = 1, \ldots, N \} \subset \mathcal{H}(\text{Curl}_0; \Gamma_t, \Omega).$$

On the other hand, taking a skew-symmetric tensor field $T$, i.e., $\text{sym} \, T = 0$, then Poincaré’s inequality in disguise

$$c \|T - A_T\|_{L^2(\Omega)} \leq \|\text{Curl} \, T\|_{L^2(\Omega)}$$

appears, where again $A_T = 0$ if $\Gamma_t \neq \emptyset$. We note that since $T$ can be identified with a vector field $v$ and the Curl is as good as the gradient $\nabla$ on $v$ we have

$$c \|v - c_v\|_{L^2(\Omega)} \leq \|\nabla v\|_{L^2(\Omega)}$$

with $c_v \in \mathbb{R}^N$ and $c_v = 0$ if $\Gamma_t \neq \emptyset$. These connections between Korn’s first and Poincaré’s inequalities and also to the Maxwell inequalities and the more general Poincaré-type inequalities are illustrated in Figure 1.

Figure 1: The three fundamental inequalities are implied by two. For the constants we have $c_p = c_{p,0}$, $c_m = c_{p,1}$ and $c_k, c_p \leq c_1$. 

\[ \begin{array}{ccc}
1. \text{Maxwell} & |v| \leq c_m(|\nabla \times v| + |\nabla \cdot v|) \\
2. \text{Poincaré} & |u| \leq c_p |\nabla u| \\
3. \text{First Korn} & |\nabla v| \leq c_k |\text{sym} \, \nabla v| \\
I. \text{Poincaré-type} & |E| \leq c_{p,q} (|d \, E| + |\delta E|) \\
II. \text{our new inequality} & |T| \leq c_1 (|\text{sym} \, T| + |\text{Curl} \, T|) \\
\end{array} \]
2 Definitions and Preliminaries

As mentioned before, let generally $N \in \mathbb{N}$ and $\Omega$ be a bounded domain in $\mathbb{R}^N$ as well as $\Gamma_t$ be an open subset of the boundary $\Gamma = \partial \Omega$. We will use the notations from our earlier papers \[14\] and \[13, 12, 15, 16\].

2.1 Differential Forms

In particular, we denote the Lebesgue spaces of square-integrable $q$-forms by $L^2,q(\Omega)$. Moreover, we have the standard Sobolev-type spaces for the exterior derivative $d$ and co-derivative $\delta := (-1)^{(q-1)N} * d *$ (acting on $q$-forms)

$$D^q(\Omega) := \{ E \in L^2,q(\Omega) : d E \in L^2,q+1(\Omega) \},$$
$$\Delta^q(\Omega) := \{ H \in L^2,q(\Omega) : \delta H \in L^2,q-1(\Omega) \},$$

where as usual * denotes Hodge’s star isomorphism. $C^\infty(\Omega)$ is the space of smooth and compactly supported $q$-forms on $\Omega$, often called test space. Due to the more complex geometry and topology of the domain $\Omega$ and its boundary parts $\Gamma, \Gamma_t$ we need some more test spaces

$$\mathring{C}^\infty,q(\Gamma_t, \Omega) := \{ E \in C^\infty,q(\Omega) : \text{dist}(\text{supp} E, \Gamma_t) > 0 \},$$
$$C^\infty,q(\overline{\Omega}) := \{ E|_\Omega : E \in \mathring{C}^\infty,q(\mathbb{R}^N) \}.$$

Then, we define

$$\mathring{D}^q(\Gamma_t, \Omega) := \overline{\mathring{C}^\infty,q(\Gamma_t, \Omega)}$$

taking the closure in $D^q(\Omega)$ and note that a $q$-form in $\mathring{D}^q(\Gamma_t, \Omega)$ has generalized vanishing tangential trace\footnote{alternating differential forms of rank $q \in \{0, \ldots, N\}$} on $\Gamma_t$. If $\Gamma_t = \Gamma$ we can identify $\mathring{C}^\infty,q(\Gamma_t, \Omega)$ with $\mathring{C}^\infty,q(\Omega)$ and write

$$\mathring{D}^q(\Gamma_t, \Omega) = \overline{\mathring{C}^\infty,q(\Gamma_t, \Omega)} = \mathring{C}^\infty,q(\Omega) =: \mathring{D}^q(\Omega).$$

If $\Gamma_t = \emptyset$ we have $\mathring{C}^\infty,q(\Gamma_t, \Omega) = C^\infty,q(\overline{\Omega})$ and thus

$$\mathring{D}^q(\Gamma_t, \Omega) = \overline{\mathring{C}^\infty,q(\Gamma_t, \Omega)} = \mathring{C}^\infty,q(\overline{\Omega}) \subset D^q(\Omega).$$

\footnote{This can be seen easily by Stokes’ theorem.}
Equality in the last relation means the density result $C^{\infty,q}(\Omega) = D^q(\Omega)$, which holds, e.g., if $\Omega$ has the segment property\footnote{See, e.g., \cite{1, 33, 7}.}. The latter is valid, e.g., for domains with Lipschitz boundary. An index 0 at the lower right corner indicates vanishing derivatives, e.g.,

$$D^q_0(\Gamma_t, \Omega) := \{ E \in D^q(\Gamma_t, \Omega) : dE = 0 \}.$$  

Analogously, we introduce the corresponding Sobolev-type spaces for the co-derivative $\delta$ which are usually assigned to the boundary complement $\Gamma_n := \Gamma \setminus \Gamma_t$ of $\Gamma_t$. We have, e.g.,

$$\Delta^q_0(\Omega) = \{ H \in \Delta^q(\Omega) : \delta H = 0 \}, \quad \overset{\circ}{\Delta}^q(\Gamma_n, \Omega), \quad \overset{\circ}{\Delta}^q_0(\Gamma_n, \Omega),$$

where in the latter spaces a vanishing normal trace on $\Gamma_n$ is generalized. Moreover, we define the spaces of so called ‘harmonic Dirichlet-Neumann forms’

$$\mathcal{H}^q(\Omega) := \overset{\circ}{D}^q_0(\Gamma_t, \Omega) \cap \overset{\circ}{\Delta}^q_0(\Gamma_n, \Omega).$$

(2.1)

We note that in classical terms a harmonic Dirichlet-Neumann $q$-form $E$ satisfies

$$dE = 0, \quad \delta E = 0, \quad \iota^* E|_{\Gamma_t} = 0, \quad \iota^* E|_{\Gamma_n} = 0,$$

where $\iota^*$ denotes the pullback of the canonical embedding $\iota : \Gamma \hookrightarrow \overline{\Omega}$ and the restrictions to $\Gamma_t$ and $\Gamma_n$ should be understood as pullbacks as well. Equipped with their natural graph norms all these spaces are Hilbert spaces.

Now, we can begin to introduce our regularity assumptions on the boundary $\Gamma$ and the interface $\gamma := \Gamma_t \cap \Gamma_n$. We start with the following:

**Definition 3** The pair $(\Omega, \Gamma_t)$ has the ‘Maxwell compactness property’ (MCP), if for all $q$ the embeddings

$$\overset{\circ}{D}^q(\Gamma_t, \Omega) \cap \overset{\circ}{\Delta}^q(\Gamma_n, \Omega) \hookrightarrow L^2(\Omega)$$

are compact.

**Remark 4**

(i) There exists a substantial amount of literature and different proofs for the MCP. See for example the papers and books of Costabel, Kuhn, Leis, Pauly, Picard, Saranen, Weber, Weck, Witsch\cite{3, 8, 9, 10, 11, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 28, 29, 30, 31, 32}. All these papers are concerned with the special cases $\Gamma_t = \Gamma$ resp. $\Gamma_t = \emptyset$. For the case $N = 3$, $q = 1$, i.e., $\Omega \subset \mathbb{R}^3$, we refer to\cite{3, 9, 10, 11, 22, 24, 26, 28, 29, 30, 32}, whereas for the general case, i.e., $\Omega \subset \mathbb{R}^N$ or even $\Omega$ a Riemannian manifold, we correspond to\cite{8, 17, 18, 19, 20, 21, 23, 25, 26, 28, 29, 31, 32}. We note that even weaker regularity of $\Gamma$ than Lipschitz is sufficient for the MCP to hold. The first proof of the MCP for non-smooth domains and even for smooth Riemannian
manifolds with non-smooth boundaries (cone property) was given in 1974 by Weck in [31]. To the best of our knowledge, the strongest result so far can be found in the paper of Picard, Weck and Witsch [26]. See also our discussion in [16]. An interesting proof has been given by Costabel in [3]. He made the detour of showing more fractional Sobolev regularity for the vector fields. More precisely, he was able to prove that for Lipschitz domains $\Omega \subset \mathbb{R}^3$ and $q = 1$ the embedding

$$\mathring{\mathcal{D}}^q(\Omega) \cap \Delta^q(\Omega) \hookrightarrow H^{1/2}(\Omega) \quad (2.2)$$

is continuous. Then, for all $0 \leq k < 1/2$ the embeddings

$$\mathring{\mathcal{D}}^q(\Omega) \cap \Delta^q(\Omega) \hookrightarrow H^k(\Omega)$$

are compact, especially for $k = 0$, where $H^k(\Omega) = L^2(\Omega)$ holds.

(ii) For the general case $\emptyset \subset \Gamma_t \subset \Gamma$ with possibly $\emptyset \subsetneq \Gamma_t \subsetneq \Gamma$, Jochmann gave a proof for the MCP in [6], where he considered the special case of a bounded domain $\Omega \subset \mathbb{R}^3$. He can admit $\Omega$ to be Lipschitz and $\gamma$ to be a Lipschitz interface. Generalizing the ideas of Weck in [31], Kuhn showed in his dissertation [7] that the MCP holds for domains $\Omega \subset \mathbb{R}^N$ or even for smooth Riemannian manifolds $\Omega$ with ‘smooth’ boundary and admissible interface $\gamma$. See also our discussion in [16]. A result, which meets our needs, has been proved quite recently by M. Mitrea and his collaborators. More precisely, we will use results by Gol'dshtein and Mitrea (I. & M.) from [4]. In the language of this paper we assume $\Omega$ to be a weakly Lipschitz domain, this is, $\Omega$ is a Lipschitz manifold with boundary, see [4, Definition 3.6], and $\Gamma_t \subset \Gamma$ to be an admissible patch (yielding $\gamma$ to be an admissible path), i.e., $\Gamma_t$ is a Lipschitz submanifold with boundary, see [4, Definition 3.7]. Roughly speaking, $\Omega$ and $\Gamma_t$ are defined by Lipschitz functions. Here, the main point in proving the MCP, i.e., [4, Proposition 4.4, (4.21)], is that then $\Omega$ is locally Lipschitz diffeomorphic to a ‘creased domain’ in $\mathbb{R}^N$, first introduced by Brown in [2]. See [4, Section 3.6] for more details and to find the statement ‘Informally speaking, the pieces in which the boundary is partitioned are admissible patches which meet at an angle $< \pi$. In particular, creased domains are inherently non-smooth’. Whereas in [4] everything is defined in the more general framework of manifolds, in [5] the MCP is proved by Jakab and Mitrea (I. & M.) for creased domains $\Omega \subset \mathbb{R}^N$. By the Lipschitz diffeomorphisms, the MCP holds then for general manifolds/domains $\Omega$ as well. In [5] the authors follow and generalize the idea (2.2) of Costabel from [3]. Particularly, in [5, (1.2), Theorem 1.1, (1.9)] the following regularity result has been proved: For all $q$ the embeddings

$$\mathring{\mathcal{D}}^q(\Gamma_t, \Omega) \cap \Delta^q(\Gamma_n, \Omega) \hookrightarrow H^{1/2}(\Omega)$$

are continuous. Therefore, as before, for all $q$ and for all $0 \leq k < 1/2$ the embeddings

$$\mathring{\mathcal{D}}^q(\Gamma_t, \Omega) \cap \Delta^q(\Gamma_n, \Omega) \hookrightarrow H^k(\Omega)$$

are continuous.
are compact, giving the MCP for \( k = 0 \).

By [4, Proposition 4.4, (4.21)] and the latter remark we have:

**Theorem 5**  Let \( \Omega \) be a weakly Lipschitz domain and \( \Gamma_t \) be an admissible patch, i.e., let \( \Omega \) be a (weakly) Lipschitz domain and \( \Gamma_t \) be an Lipschitz patch of \( \Gamma \). Then the pair \((\Omega, \Gamma_t)\) has the MCP.

**Corollary 6**  Let the pair \((\Omega, \Gamma_t)\) have the MCP. Then, for all \( q \) the spaces of harmonic Dirichlet-Neumann forms \( \mathcal{H}^q(\Omega) \) are finite dimensional.

**Proof**  The MCP implies immediately that the unit ball in \( \mathcal{H}^q(\Omega) \) is compact. \( \Box \)

For details about the particular dimensions see [22] or [4]. We note that the dimensions of \( \mathcal{H}^q(\Omega) \) depend only on topological properties of the pair \((\Omega, \Gamma_t)\).

**Lemma 7** (Poincaré-type Estimate for Differential Forms)  Let the pair \((\Omega, \Gamma_t)\) have the MCP. Then, for all \( q \) there exist positive constants \( c_{p,q} \), such that

\[
\|E\|_{L^2,q(\Omega)} \leq c_{p,q} \left( \|dE\|_{L^2,q+1(\Omega)}^2 + \|\delta E\|_{L^2,q-1(\Omega)}^2 \right)^{1/2}
\]

holds for all \( E \in \overset{\circ}{D}^q(\Gamma_t, \Omega) \cap \overset{\circ}{\Delta}^q(\Gamma_n, \Omega) \cap \mathcal{H}^q(\Omega) \perp \). Moreover,

\[
\|(id - \pi_q)E\|_{L^2,q(\Omega)} \leq c_{p,q} \left( \|dE\|_{L^2,q+1(\Omega)}^2 + \|\delta E\|_{L^2,q-1(\Omega)}^2 \right)^{1/2}
\]

holds for all \( E \in \overset{\circ}{D}^q(\Gamma_t, \Omega) \cap \overset{\circ}{\Delta}^q(\Gamma_n, \Omega) \), where \( \pi_q : L^2,q(\Omega) \to \mathcal{H}^q(\Omega) \) denotes the \( L^2,q(\Omega) \)-orthogonal projection onto the Dirichlet-Neumann forms \( \mathcal{H}^q(\Omega) \).

Here and throughout the paper, \( \perp \) denotes orthogonality in \( L^2,q(\Omega) \).

**Proof**  A standard indirect argument utilizing the MCP yields the desired estimates. \( \Box \)

By Stokes’ theorem and approximation always

\[
\overset{\circ}{D}^q_0(\Gamma_t, \Omega) \subset (\delta \overset{\circ}{\Delta}^{q+1}(\Gamma_n, \Omega)) \perp, \quad \overset{\circ}{\Delta}^q_0(\Gamma_n, \Omega) \subset (d \overset{\circ}{D}^{q-1}(\Gamma_t, \Omega)) \perp
\]

hold. Equality in the latter relations is not clear and needs another assumption on the pair \((\Omega, \Gamma_t)\).

**Definition 8**  The pair \((\Omega, \Gamma_t)\) has the ‘Maxwell approximation property’ (MAP), if for all \( q \)

\[
\overset{\circ}{D}^q_0(\Gamma_t, \Omega) = (\delta \overset{\circ}{\Delta}^{q+1}(\Gamma_n, \Omega)) \perp, \quad \overset{\circ}{\Delta}^q_0(\Gamma_n, \Omega) = (d \overset{\circ}{D}^{q-1}(\Gamma_t, \Omega)) \perp.
\]
Remark 9  By \( \ast \)-duality the pair \((\Omega, \Gamma_t)\) has the MAP, if and only if the pair \((\Omega, \Gamma_n)\) has the MAP, i.e., if and only if for all \(q\)

\[
\mathcal{D}_0^q(\Omega, \Omega) = (\delta \Delta^{q+1}(\Gamma_t, \Omega))^\perp, \quad \Delta_0^q(\Gamma_t, \Omega) = (d \mathcal{D}^{q-1}(\Gamma_n, \Omega))^\perp.
\]

Remark 10  If \(\Gamma_t = \Gamma\) or \(\Gamma_t = \emptyset\), the MAP is simply given by the projection theorem in Hilbert spaces and by the definitions of the respective Sobolev spaces. For the general case \(\emptyset \subset \Gamma_t \subset \Gamma\) with possibly \(\emptyset \subsetneq \Gamma_t \subsetneq \Gamma\), Jochmann proved the MAP in [4] considering the special case of a bounded domain \(\Omega \subset \mathbb{R}^3\). As in Remark 4 he needs \(\Omega\) to be Lipschitz and \(\gamma\) to be a Lipschitz interface. Kuhn showed the MAP in [7] for smooth domains \(\Omega \subset \mathbb{R}^N\) or even for smooth Riemannian manifolds \(\Omega\) with smooth boundary and admissible interface \(\gamma\). Again, a sufficient result for us has been given recently by Gol’dshtein and Mitrea (I. & M.) in [4, Theorem 4.3, (4.16)]. Like in Remark 4, for this \(\Omega\) has to be a weakly Lipschitz domain and \(\Gamma_t \subset \Gamma\) to be an admissible patch.

By [4, Theorem 4.3, (4.16)] and the latter remark we have:

**Theorem 11**  Let \(\Omega\) be a weakly Lipschitz domain and \(\Gamma_t\) be an admissible patch, i.e., let \(\Omega\) be a (weakly) Lipschitz domain and \(\Gamma_t\) be a Lipschitz patch of \(\Gamma\). Then the pair \((\Omega, \Gamma_t)\) has the MAP.

**Lemma 12**  (Hodge-Helmholtz Decomposition for Differential Forms) Let the pair \((\Omega, \Gamma_t)\) have the MAP. Then, the orthogonal decompositions

\[
L^2,q(\Omega) = \mathcal{D}^{q-1}(\Gamma_t, \Omega) \oplus \Delta_0^q(\Gamma_n, \Omega)
\]

hold. If the pair \((\Omega, \Gamma_t)\) has additionally the MCP, then

\[
d \mathcal{D}^{q-1}(\Gamma_t, \Omega) = d (\mathcal{D}^{q-1}(\Gamma_t, \Omega) \cap \delta \Delta^q(\Gamma_n, \Omega)) = \mathcal{D}_0^q(\Gamma_t, \Omega) \cap \mathcal{H}^q(\Omega)^\perp,
\]

\[
\delta \Delta^{q+1}(\Gamma_n, \Omega) = \delta (\Delta^{q+1}(\Gamma_n, \Omega) \cap d \mathcal{D}^q(\Gamma_t, \Omega)) = \Delta_0^q(\Gamma_n, \Omega) \cap \mathcal{H}^q(\Omega)^\perp
\]

and these are closed subspaces of \(L^2,q(\Omega)\). Moreover, then the orthogonal decompositions

\[
L^2,q(\Omega) = d \mathcal{D}^{q-1}(\Gamma_t, \Omega) \oplus \Delta_0^q(\Gamma_n, \Omega)
\]

hold.
Here, $\oplus$ denotes the $L^{2,q}(\Omega)$-orthogonal sum and all closures are taken in $L^{2,q}(\Omega)$.

**Proof** By the projection theorem in Hilbert space and the MAP we obtain immediately the two $L^{2,q}(\Omega)$-orthogonal decompositions

\[
d \overset{\circ}{D}^{q-1}(\Gamma_t, \Omega) \oplus \overset{\circ}{\Delta}^q \delta_\Omega(\Gamma_n, \Omega) = L^{2,q}(\Omega) = \overset{\circ}{D}^q \delta_\Omega(\Gamma_t, \Omega) \oplus \overset{\circ}{\Delta}^{q+1} \delta_\Omega(\Gamma_n, \Omega),
\]

where the closures are taken in $L^{2,q}(\Omega)$. Since

\[
d \overset{\circ}{D}^{q-1}(\Gamma_t, \Omega) \subset \overset{\circ}{D}^q \delta_\Omega(\Gamma_t, \Omega), \quad \overset{\circ}{\Delta}^{q+1} \delta_\Omega(\Gamma_n, \Omega) \subset \overset{\circ}{\Delta}^q \delta_\Omega(\Gamma_n, \Omega)
\]

and applying the latter decompositions separately to $\overset{\circ}{\Delta}^q \delta_\Omega(\Gamma_n, \Omega)$ or $\overset{\circ}{D}^q \delta_\Omega(\Gamma_t, \Omega)$ we get a refined decomposition, namely

\[
L^{2,q}(\Omega) = d \overset{\circ}{D}^{q-1}(\Gamma_t, \Omega) \oplus H^q(\Omega) \oplus \overset{\circ}{\Delta}^{q+1} \delta_\Omega(\Gamma_n, \Omega).
\]

Applying this decomposition to $d \overset{\circ}{D}^{q-1}(\Gamma_t, \Omega)$ and $\overset{\circ}{\Delta}^{q+1} \delta_\Omega(\Gamma_n, \Omega)$ yields also

\[
d d \overset{\circ}{D}^{q-1}(\Gamma_t, \Omega) = d \left( d \overset{\circ}{D}^{q-1}(\Gamma_t, \Omega) \cap \overset{\circ}{\Delta}^q \delta_\Omega(\Gamma_n, \Omega) \right),
\]

\[
\delta \overset{\circ}{\Delta}^{q+1} \delta_\Omega(\Gamma_n, \Omega) = \delta \left( \overset{\circ}{\Delta}^{q+1} \delta_\Omega(\Gamma_n, \Omega) \cap d \overset{\circ}{D}^q \delta_\Omega(\Gamma_t, \Omega) \right).
\]

Now, Lemma 7 shows that $d \overset{\circ}{D}^{q-1}(\Gamma_t, \Omega)$ and $\overset{\circ}{\Delta}^{q+1} \delta_\Omega(\Gamma_n, \Omega)$ are even closed subspaces of $L^{2,q}(\Omega)$. Hence, we obtain the asserted Hodge-Helmholtz decompositions of $L^{2,q}(\Omega)$. $\square$

### 2.2 Functions and Vector Fields

We turn to the special case $q = 1$, the case of vector fields, and use the notations and identifications from [14] and [12, 15, 16]. Especially, $L^{2,q}(\Omega)$ can be identified with the usual Lebesgue spaces of square integrable functions or vector fields on $\Omega$ with values in $\mathbb{R}^n$, $n := n_{N,q} := \binom{N}{q}$, and will be denoted by $L^2(\Omega) := L^2(\Omega, \mathbb{R}^n)$. We have the standard Sobolev spaces

\[
H(\text{grad}; \Omega) := \{ u \in L^2(\Omega, \mathbb{R}) : \text{grad} \, u \in L^2(\Omega, \mathbb{R}^N) \},
\]

\[
H(\text{div}; \Omega) := \{ v \in L^2(\Omega, \mathbb{R}^N) : \text{div} \, v \in L^2(\Omega, \mathbb{R}) \},
\]

\[
H(\text{curl}; \Omega) := \{ v \in L^2(\Omega, \mathbb{R}^N) : \text{curl} \, v \in L^2(\Omega, \mathbb{R}^{N(N-1)/2}) \}
\]

and by natural isomorphic identification

\[
D^0(\Omega) \cong H(\text{grad}; \Omega), \quad \Delta^1(\Omega) \cong H(\text{div}; \Omega), \quad D^1(\Omega) \cong H(\text{curl}; \Omega).
\]
On an Extension of Korn’s First Inequality to Incompatible Tensor Fields

| q | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| d | grad | curl | div | 0 |
| δ | 0 | div | - curl | grad |
| $\mathcal{D}^q(\Gamma_t, \Omega)$ | $\mathcal{H}(\text{grad}; \Gamma_t, \Omega)$ | $\mathcal{H}(\text{curl}; \Gamma_t, \Omega)$ | $\mathcal{H}(\text{div}; \Gamma_t, \Omega)$ | $L^2(\Omega)$ |
| $\mathcal{D}^q(\Gamma_n, \Omega)$ | $L^2(\Omega)$ | $\mathcal{H}(\text{div}; \Gamma_n, \Omega)$ | $\mathcal{H}(\text{curl}; \Gamma_n, \Omega)$ | $\mathcal{H}(\text{grad}; \Gamma_n, \Omega)$ |
| $\iota_\Gamma^* E$ | $E|_{\Gamma}$ | $\nu \times E|_{\Gamma}$ | $\nu \cdot E|_{\Gamma}$ | 0 |
| $\iota_\Gamma^* \Gamma_n \ast E$ | 0 | $\nu \cdot E|_{\Gamma_n}$ | $- \nu \times (\nu \times E)|_{\Gamma_n}$ | $E|_{\Gamma_n}$ |

Figure 2: identification table for $q$-forms and vector proxies in $\mathbb{R}^3$

General $\mathcal{D}^q(\Omega) \cong \Delta^{N-q}(\Omega)$ holds by Hodge star duality. For $v \in C^\infty(\Omega)$ and $N = 3, 4$

$$\text{curl } v = \begin{bmatrix} \partial_2 v_3 - \partial_3 v_2 \\ \partial_3 v_1 - \partial_1 v_3 \\ \partial_1 v_2 - \partial_2 v_1 \end{bmatrix} \in \mathbb{R}^3,$$

$$\text{curl } v = \begin{bmatrix} \partial_1 v_2 - \partial_2 v_1 \\ \partial_1 v_3 - \partial_3 v_1 \\ \partial_1 v_4 - \partial_4 v_1 \\ \partial_2 v_3 - \partial_3 v_2 \\ \partial_2 v_4 - \partial_4 v_2 \\ \partial_3 v_4 - \partial_4 v_3 \end{bmatrix} \in \mathbb{R}^6$$

hold, whereas $\text{curl } v = \partial_1 v_2 - \partial_2 v_1 \in \mathbb{R}$ or $\text{curl } v \in \mathbb{R}^{10}$ for $N = 2$ or $N = 5$, respectively.

Moreover, we have the closed subspaces

$$\mathcal{H}(\text{grad}; \Gamma_t, \Omega), \quad \mathcal{H}(\text{curl}; \Gamma_t, \Omega), \quad \mathcal{H}(\text{div}; \Gamma_n, \Omega),$$

in which the homogeneous scalar, tangential and normal boundary conditions

$$u|_{\Gamma_t} = 0, \quad \nu \times v|_{\Gamma_t} = 0, \quad \nu \cdot v|_{\Gamma_n} = 0$$

are generalized, as reincarnations of $\mathcal{D}^0(\Gamma_t, \Omega), \mathcal{D}^1(\Gamma_t, \Omega)$ and $\mathcal{D}^1(\Gamma_n, \Omega)$, respectively. Here $\nu$ denotes the outer unit normal at $\Gamma$. If $\Gamma_t = \Gamma$ (and $\Gamma_n = \emptyset$) resp. $\Gamma_t = \emptyset$ (and $\Gamma_n = \Gamma$) we obtain the usual Sobolev spaces

$$\mathcal{H}(\text{grad}; \Omega), \quad \mathcal{H}(\text{curl}; \Omega), \quad \mathcal{H}(\text{div}; \Omega)$$

resp.

$$\mathcal{H}(\text{grad}; \Omega), \quad \mathcal{H}(\text{curl}; \Omega), \quad \mathcal{H}(\text{div}; \Omega).$$

We note that $\mathcal{H}(\text{grad}; \Omega)$ and $\mathcal{H}(\text{grad}; \Omega)$ coincide with the usual standard Sobolev spaces $\mathcal{H}^1(\Omega)$ and $\mathcal{H}^1(\Omega)$, respectively. As before, the index 0, now attached to the symbols curl or div, indicates vanishing curl or div, e.g.,

$$\mathcal{H}(\text{curl}_0; \Gamma_t, \Omega) = \{ v \in \mathcal{H}(\text{curl}; \Gamma_t, \Omega) : \text{curl } v = 0 \},$$

$$\mathcal{H}(\text{div}_0; \Omega) = \{ v \in \mathcal{H}(\text{div}; \Omega) : \text{div } v = 0 \}.$$
Finally, we denote the ‘harmonic Dirichlet-Neumann fields’ by

\[ \mathcal{H}^1(\Omega) \cong \mathcal{H}(\Omega) := H(\text{curl}_0; \Gamma_t, \Omega) \cap H(\text{div}_0; \Gamma_n, \Omega). \]

Assuming the MCP for the pair \((\Omega, \Gamma_t)\), then \(\mathcal{H}(\Omega)\) is finite dimensional by Corollary 6 and we have the two (out of four) compact embeddings

\[ \mathcal{H}(\text{grad}; \Gamma_t, \Omega) \hookrightarrow L^2(\Omega), \quad (2.3) \]

\[ \mathcal{H}(\text{curl}; \Gamma_t, \Omega) \cap H(\text{div}; \Gamma_n, \Omega) \hookrightarrow L^2(\Omega), \quad (2.4) \]

i.e., Rellich’s selection theorem \((q = 0)\) and the vectorial Maxwell’s compactness property \((q = 1)\). Moreover, by Lemma 7 we get the following Poincaré and Maxwell estimates:

**Corollary 13 (Poincaré Estimate for Functions)** Let the pair \((\Omega, \Gamma_t)\) have the MCP and \(c_p := c_{p,0}\). Then

\[ \|u\|_{L^2(\Omega)} \leq c_p \|\text{grad} u\|_{L^2(\Omega)} \]

holds for all \(u \in \mathcal{H}(\text{grad}; \Gamma_t, \Omega)\) if \(\Gamma_t \neq \emptyset\) and for all \(u \in \mathcal{H}(\text{grad}; \Omega) \cap \mathbb{R}^d\) if \(\Gamma_t = \emptyset\). Moreover, for all \(u \in \mathcal{H}(\text{grad}; \Omega)\)

\[ \|(\text{id} - \pi_0)u\|_{L^2(\Omega)} \leq c_p \|\text{grad} u\|_{L^2(\Omega)} \]

holds, where \(\pi_0 : L^2(\Omega) \to \mathbb{R}\) denotes the \(L^2(\Omega)\)-orthogonal projection onto the constants.

We note that if \(\Gamma_t \neq \emptyset\) we have \(\mathcal{H}^0(\Omega) = \{0\}\). Furthermore, \(\mathcal{H}^0(\Omega) = \mathbb{R}\) and \(\mathcal{H}(\text{grad}; \Gamma_t, \Omega) = \mathcal{H}(\text{grad}; \Omega)\) hold if \(\Gamma_t = \emptyset\).

**Corollary 14 (Maxwell Estimate for Vector Fields)** Let the pair \((\Omega, \Gamma_t)\) have the MCP and \(c_m := c_{p,1}\). Then

\[ \|v\|_{L^2(\Omega)} \leq c_m (\|\text{curl} v\|_{L^2(\Omega)}^2 + \|\text{div} v\|_{L^2(\Omega)}^2) \]

holds for all \(v \in \mathcal{H}(\text{curl}; \Gamma_t, \Omega) \cap \mathcal{H}(\text{div}; \Gamma_n, \Omega) \cap \mathcal{H}(\Omega)\) as well as

\[ \|(\text{id} - \pi_1)v\|_{L^2(\Omega)} \leq c_m (\|\text{curl} v\|_{L^2(\Omega)}^2 + \|\text{div} v\|_{L^2(\Omega)}^2) \]

holds for all \(v \in \mathcal{H}(\text{curl}; \Gamma_t, \Omega) \cap \mathcal{H}(\text{div}; \Gamma_n, \Omega)\), where again \(\pi_1 : L^2(\Omega) \to \mathcal{H}(\Omega)\) denotes the \(L^2(\Omega)\)-orthogonal projection onto the Dirichlet-Neumann fields \(\mathcal{H}(\Omega)\).

Lemma 12 yields:

**Corollary 15 (Helmholtz Decompositions for Vector Fields)** Let the pair \((\Omega, \Gamma_t)\) have the MCP and the MAP. Then, the orthogonal decompositions

\[ L^2(\Omega) = \text{grad} \mathcal{H}(\text{grad}; \Gamma_t, \Omega) \oplus \mathcal{H}(\text{div}_0; \Gamma_n, \Omega) \]

\[ = \mathcal{H}(\text{curl}_0; \Gamma_t, \Omega) \oplus (\mathcal{H}(\text{div}_0; \Gamma_n, \Omega) \cap \mathcal{H}(\Omega)\)\]

hold.
2.3 Tensor Fields

Next, we extend our calculus to tensor fields, i.e., matrix fields. For vector fields \( v \) with components in \( H(\text{grad}; \Omega) \) and tensor fields \( T \) with rows in \( H(\text{curl}; \Omega) \) resp. \( H(\text{div}; \Omega) \), i.e.,

\[
v = \begin{bmatrix} v_1 \\ \vdots \\ v_N \end{bmatrix}, \quad v_n \in H(\text{grad}; \Omega), \quad T = \begin{bmatrix} T_1^\top \\ \vdots \\ T_N^\top \end{bmatrix}, \quad T_n \in H(\text{curl}; \Omega) \text{ resp. } H(\text{div}; \Omega)
\]

for \( n = 1, \ldots, N \) we define (in the weak sense)

\[
\text{Grad} v := \begin{bmatrix} \text{grad}^\top v_1 \\ \vdots \\ \text{grad}^\top v_N \end{bmatrix} = J v, \quad \text{Curl} T := \begin{bmatrix} \text{curl}^\top T_1 \\ \vdots \\ \text{curl}^\top T_N \end{bmatrix}, \quad \text{Div} T := \begin{bmatrix} \text{div} T_1 \\ \vdots \\ \text{div} T_N \end{bmatrix},
\]

where \( J v \) denotes the Jacobian of \( v \) and \( ^\top \) the transpose. We note that \( v \) and \( \text{Div} T \) are \( N \)-vector fields, \( T \) and \( \text{Grad} v \) are \( (N \times N) \)-tensor fields, whereas \( \text{Curl} T \) is a \( (N \times N(N-1)/2) \)-tensor field. The corresponding Sobolev spaces will be denoted by

\[
H(\text{Grad}; \Omega), \quad H(\text{Curl}; \Omega), \quad H(\text{Curl}_0; \Omega), \quad H(\text{Div}; \Omega), \quad H(\text{Div}_0; \Omega)
\]

and

\[
\dot{H}(\text{Grad}; \Gamma_t; \Omega), \quad \dot{H}(\text{Curl}; \Gamma_t; \Omega), \quad \dot{H}(\text{Curl}_0; \Gamma_t; \Omega), \quad \dot{H}(\text{Div}; \Gamma_t; \Omega), \quad \dot{H}(\text{Div}_0; \Gamma_t; \Omega),
\]

again with the usual notations if \( \Gamma_t \in \{\emptyset, \Gamma\} \).

From Corollaries 13, 14 and 15 we obtain immediately:

**Corollary 16 (Poincaré Estimate for Vector Fields)** Let the pair \((\Omega, \Gamma_t)\) have the MCP. Then

\[
\|v\|_{L^2(\Omega)} \leq c_p \|\text{Grad} v\|_{L^2(\Omega)}
\]

holds for all \( v \in \dot{H}(\text{Grad}; \Gamma_t; \Omega) \) if \( \Gamma_t \neq \emptyset \) and for all \( v \in H(\text{Grad}; \Omega) \cap (\mathbb{R}^N)^\perp \) if \( \Gamma_t = \emptyset \). Moreover, for all \( v \in H(\text{Grad}; \Omega) \)

\[
\| (\text{id} - \pi_0^N) v \|_{L^2(\Omega)} \leq c_p \|\text{Grad} v\|_{L^2(\Omega)}
\]

holds, where \( \pi_0^N : L^2(\Omega) \to \mathbb{R}^N \) denotes the \( L^2(\Omega) \)-orthogonal projection onto \( \mathbb{R}^N \).

**Sometimes, the Jacobian \( J_v \) is also denoted by \( \nabla v \).**
Corollary 17 (Maxwell Estimate for Tensor Fields) Let the pair \((\Omega, \Gamma_t)\) have the MCP. Then
\[
\|T\|_{L^2(\Omega)} \leq c_m \left( \|\operatorname{Curl} T\|_{L^2(\Omega)}^2 + \|\operatorname{Div} T\|_{L^2(\Omega)}^2 \right)^{1/2}
\]
holds for all \(T \in \mathring{\mathcal{H}}(\text{Curl}; \Gamma_t, \Omega) \cap \mathring{\mathcal{H}}(\text{Div}; \Gamma_n, \Omega) \cap (\mathcal{H}(\Omega)^N)^\perp\) as well as
\[
\|(\operatorname{id} - \pi_N^1)T\|_{L^2(\Omega)} \leq c_m \left( \|\operatorname{Curl} T\|_{L^2(\Omega)}^2 + \|\operatorname{Div} T\|_{L^2(\Omega)}^2 \right)^{1/2}
\]
holds for all \(T \in \mathring{\mathcal{H}}(\text{Curl}; \Gamma_t, \Omega) \cap \mathring{\mathcal{H}}(\text{Div}; \Gamma_n, \Omega)\), where \(\pi_N^1 : L^2(\Omega) \to \mathcal{H}(\Omega)^N\) denotes the \(L^2(\Omega)\)-orthogonal projection onto the \((N\text{-times})\)-Dirichlet-Neumann fields \(\mathcal{H}(\Omega)^N\).

Corollary 18 (Helmholtz Decompositions for Tensor Fields) Let the pair \((\Omega, \Gamma_t)\) have the MCP and the MAP. Then, the orthogonal decompositions
\[
L^2(\Omega) = \text{Grad} \mathring{\mathcal{H}}(\text{Grad}; \Gamma_t, \Omega) \oplus \mathring{\mathcal{H}}(\text{Div}_0; \Gamma_n, \Omega)
\]
\[
= \mathring{\mathcal{H}}(\text{Curl}_0; \Gamma_t, \Omega) \oplus (\mathring{\mathcal{H}}(\text{Div}_0; \Gamma_n, \Omega) \cap (\mathcal{H}(\Omega)^N)^\perp)
\]
hold.

We also need Korn’s First Inequality.

Definition 19 (Korn’s Second Inequality) The domain \(\Omega\) has the ‘Korn property’ (KP), if

(i) Korn’s second inequality holds, this is, there exists a constant \(c > 0\), such that for all vector fields \(v \in \mathcal{H}(\text{Grad}; \Omega)\)
\[
c \|\text{Grad} v\|_{L^2(\Omega)} \leq \|v\|_{L^2(\Omega)} + \|\text{sym Grad} v\|_{L^2(\Omega)},
\]

(ii) and Rellich’s selection theorem holds for \(\mathcal{H}(\text{grad}; \Omega)\), this is, the natural embedding \(\mathcal{H}(\text{grad}; \Omega) \hookrightarrow L^2(\Omega)\) is compact.

Here, we introduce the symmetric and skew-symmetric parts
\[
\text{sym} T := \frac{1}{2}(T + T^\top), \quad \text{skew} T := T - \text{sym} T = \frac{1}{2}(T - T^\top)
\]
of a tensor field \(T = \text{skew} T + \text{sym} T^\dagger\).

Remark 20 There exists a rich amount of literature for the KP, which we do not intend to list here. We refer to our overview on Korn’s inequalities in [16].

\dagger Note that \(\text{sym} T\) and \(\text{skew} T\) are point-wise orthogonal with respect to the standard inner product in \(\mathbb{R}^{N \times N}\).
Theorem 21  Korn’s second inequality holds for domains $\Omega$ having the strict cone property. For domains $\Omega$ with the segment property, Rellich’s selection theorem for $H(\text{grad}; \Omega)$ is valid. Thus, e.g., Lipschitz domains $\Omega$ possess the KP.

Proof  Book of Leis [11]. □

By a standard indirect argument we immediately obtain:

Corollary 22  (Korn’s First Inequality: Standard Version) Let $\Omega$ have the KP. Then, there exists a constant $c_{k,s} > 0$ such that the following holds:

(i) If $\Gamma_t \neq \emptyset$ then

$$
(1 + c_p^2)^{-1/2} \|v\|_{H^1(\Omega)} \leq \|\text{Grad} v\|_{L^2(\Omega)} \leq c_{k,s} \|\text{sym Grad} v\|_{L^2(\Omega)}
$$

holds for all vector fields $v \in H(\text{Grad}; \Gamma_t, \Omega)$.

(ii) If $\Gamma_t = \emptyset$, then the inequalities (2.5) hold for all vector fields $v \in H(\text{Grad}; \Omega)$ with $\text{Grad} v \perp \mathfrak{so}(N)$ and $v \perp \mathbb{R}^N$. Moreover, the second inequality of (2.5) holds for all vector fields $v \in H(\text{Grad}; \Omega)$ with $\text{Grad} v \perp \mathfrak{so}(N)$. For all $v \in H(\text{Grad}; \Omega)$

$$
(1 + c_p^2)^{-1/2} \|v - r_v\|_{H^1(\Omega)} \leq \|\text{Grad} v - A_{\text{Grad} v}\|_{L^2(\Omega)} \leq c_{k,s} \|\text{sym Grad} v\|_{L^2(\Omega)}
$$

holds, where the rigid motion $r_v$ and the skew-symmetric tensor $A_{\text{Grad} v} = \text{Grad} r_v$ are given by $r_v(x) := A_{\text{Grad} v} x + b_v$ and

$$
A_{\text{Grad} v} := \text{skew} \int_{\Omega} \text{Grad} v \, d\lambda \in \mathfrak{so}(N), \quad b_v := \int_{\Omega} v \, d\lambda - A_{\text{Grad} v} \int_{\Omega} x \, d\lambda \in \mathbb{R}^N.
$$

We note $v - r_v \perp \mathbb{R}^N$ and $\text{Grad}(v - r_v) = \text{Grad} v - A_{\text{Grad} v} \perp \mathfrak{so}(N)$.

Here, we generally define

$$
\int_{\Omega} u \, d\lambda := \lambda(\Omega)^{-1} \int_{\Omega} u \, d\lambda, \quad \lambda \text{ Lebesgue’s measure}.
$$

We note that $A_{\text{Grad} v} = \pi_{\mathfrak{so}(N)} \text{Grad} v \in \mathfrak{so}(N)$, where $\pi_{\mathfrak{so}(N)} : L^2(\Omega) \to \mathfrak{so}(N)$ denotes the $L^2(\Omega)$-orthogonal projection onto the constant skew-symmetric tensor fields $\mathfrak{so}(N)$. Moreover, we have generally for square integrable $(N \times N)$-tensor fields $T$

$$
\pi_{\mathfrak{so}(N)} T := A_T := \text{skew} \int_{\Omega} T \, d\lambda \in \mathfrak{so}(N).
$$
2.4 Sliceable and Admissible Domains

The essential tools to prove our main result Theorem 1 are

- the Maxwell estimate for tensor fields (Corollary 17),
- the Helmholtz decomposition for tensor fields (Corollary 18),
- and a generalized version of Korn’s first inequality (Corollary 22).

For the first two tools the pair \((\Omega, \Gamma_t)\) needs to have the MCP and the MAP. The third tool will be provided in Lemma 29 and needs at least the KP. As already pointed out, these three properties hold, e.g., for Lipschitz domains \(\Omega\) and admissible boundary patches \(\Gamma_t\). Moreover, we will make use of the fact that any irrotational vector field is already a gradient if the underlying domain is simply connected. For this, we present a trick, the concept of sliceable domains, which we have used already in [16].

**Definition 23** The pair \((\Omega, \Gamma_t)\) is called ‘sliceable’, if there exist \(J \in \mathbb{N}\) and \(\Omega_j \subset \Omega, j = 1, \ldots, J\), such that \(\Omega \setminus (\Omega_1 \cup \ldots \cup \Omega_J)\) has zero Lebesgue-measure and for \(j = 1, \ldots, J\)

(i) \(\Omega_j\) are open, disjoint and simply connected subdomains of \(\Omega\) having the KP,

(ii) \(\Gamma_{t,j} := \text{int}_{\text{rel}}(\Omega_j \cap \Gamma_t) \neq \emptyset\), if \(\Gamma_t \neq \emptyset\).

Here, \(\text{int}_{\text{rel}}\) denotes the interior with respect to the topology on \(\Gamma\).

**Remark 24** From a practical point of view, all domains considered in applications are sliceable, but it is unclear whether every Lipschitz pair \((\Omega, \Gamma_t)\) is already sliceable.

Now, we can introduce our general assumptions on the domain and its boundary parts.

**Definition 25** The pair \((\Omega, \Gamma_t)\) is called ‘admissible’, if

- the pair \((\Omega, \Gamma_t)\) possesses the MCP and the MAP,
- and the pair \((\Omega, \Gamma_t)\) is sliceable.

**Remark 26** In particular, the pair \((\Omega, \Gamma_t)\) is admissible if

- \(\Omega\) has a Lipschitz boundary \(\Gamma\),
- \(\Gamma_t\) is a Lipschitz patch,
- \((\Omega, \Gamma_t)\) is sliceable.
Figure 3: Some ways to ‘cut’ sliceable domains $\Omega$ in $\mathbb{R}^3$ and $\mathbb{R}^2$ into two ($J = 2$) or more ($J = 3, 4$) ‘pieces’. The boundary part $\Gamma_t$ is colored in light gray. Roughly speaking, a domain is sliceable if it can be cut into finitely many simply connected Lipschitz pieces $\Omega_j$, i.e., any closed curve inside some piece $\Omega_j$ is homotop to a point, this is, one has to cut all ‘handles’. In three and higher dimensions, holes inside $\Omega$ are permitted, but this is forbidden in the two-dimensional case. Note that, in these examples it is always possible to slice $\Omega$ into two ($J = 2$) pieces.
3 Proofs

Let the pair \((\Omega, \Gamma_t)\) be admissible. On our way to prove our main result we follow in close lines the arguments of [16, section 3]. First we prove a non-standard version of Korn’s first inequality Corollary 22, which will be presented as Lemma 29. Then, we prove our main result. Although, all subsequent proofs are very similar to the ones given in [16, Lemmas 8, 9, 12, Theorem 14], we will repeat them here for the convenience of the reader.

**Lemma 27** Let \(\Gamma_t \neq \emptyset\) and \(u \in H(\text{grad}; \Omega)\) with \(\text{grad} u \in \overset{\circ}{H}(\text{curl}_0; \Gamma_t, \Omega)\). Then, \(u\) is constant on any connected component of \(\Gamma_t\).

**Proof** Let \(x \in \Gamma_t\) and \(B_{2r} := B_{2r}(x)\) be the open ball of radius \(2r > 0\) around \(x\) such that \(B_{2r}\) is covered by a Lipschitz-chart domain and \(\Gamma \cap B_{2r} \subseteq \Gamma_t\). Moreover, we pick a cut-off function \(\varphi \in C^\infty(B_{2r})\) with \(\varphi|_{B_r} = 1\). Then, \(\varphi \text{grad} u \in H(\text{curl}; \Omega \cap B_{2r})\). Thus, the extension by zero \(v\) of \(\varphi \text{grad} u\) to \(B_{2r}\) belongs to \(H(\text{curl}; B_{2r})\). Hence, \(v|_{B_r} \in H(\text{curl}_0; B_r)\). Since \(B_r\) is simply connected, there exists a \(\tilde{u} \in H(\text{grad}; B_r)\) with \(\text{grad} \tilde{u} = v\) in \(B_r\). In \(B_r \setminus \Omega\) we have \(v = 0\). Therefore, \(\tilde{u}|_{B_r \setminus \Omega} = \tilde{c}\) with some \(\tilde{c} \in \mathbb{R}\). Moreover, \(\text{grad} u = v = \text{grad} \tilde{u}\) holds in \(B_r \cap \Omega\), which yields \(u = \tilde{u} + c\) in \(B_r \cap \Omega\) with some \(c \in \mathbb{R}\). Finally, \(u|_{B_r \cap \Gamma_t} = \tilde{c} + c\) is constant. Therefore, \(u\) is locally constant and hence the assertion follows. \(\square\)

**Lemma 28** (Korn’s First Inequality: Tangential Version) Let \(\Gamma_t \neq \emptyset\). Then, there exists a constant \(c_{k,t} \geq c_{k,s} > 0\), such that

\[
\|\text{Grad} v\|_{L^2(\Omega)} \leq c_{k,t} \|\text{sym Grad} v\|_{L^2(\Omega)}
\]

holds for all \(v \in H(\text{Grad}; \Omega)\) with \(\text{Grad} v \in \overset{\circ}{H}(\text{Curl}_0; \Gamma_t, \Omega)\).

In classical terms, \(\text{Grad} v \in \overset{\circ}{H}(\text{Curl}_0; \Gamma_t, \Omega)\) means that \(\text{grad} v_n = \nabla v_n, n = 1, \ldots, N\), are normal at \(\Gamma_t\).

**Proof** We pick a relatively open connected component \(\tilde{\Gamma} \neq \emptyset\) of \(\Gamma_t\). Then, there exists a constant vector \(c_v \in \mathbb{R}^3\) such that \(v - c_v\) belongs to \(H(\text{Grad}; \tilde{\Gamma}, \Omega)\) by Lemma 27 applied to each component of \(v\). Corollary 22 (i) (with \(\Gamma_t = \tilde{\Gamma}\) and a possibly larger \(c_{k,t}\)) completes the proof. \(\square\)

Now, we extend Korn’s first inequality from gradient to merely irrotational tensor fields.

**Lemma 29** (Korn’s First Inequality: Irrotational Version) There exists \(c_k \geq c_{k,t} > 0\), such that the following inequalities hold:
We define the piece-wise constant skew-symmetric tensor field $c \text{Grad} v \mid _{\Gamma}$ proving (i). Now, we assume $\Gamma \neq \emptyset$ so $\Omega$ be decomposed into $\Omega_j \in H(\text{Grad}; \Omega)$. This can be done for each $j$. Again we note that in classical terms a tensor $T \in H(\text{Curl}; \Omega)$ is irrotational and the vector field $T|_{\Gamma}$ vanishes for all tangential vector fields $\tau$ at $\Gamma$. Moreover, the slice-ability of $(\Omega, \Gamma)$ is precisely needed for Lemma 29 to hold.

**Proof** We start with proving (i). Let $\Gamma \neq \emptyset$ and $T \in H(\text{Curl}; \Omega)$. We choose a sequence $(T^k) \subset C^\infty(\Gamma; \Omega)$ converging to $T$ in $H(\text{Curl}; \Omega)$. According to Definition 23 we decompose $\Omega$ into $\Omega_1 \cup \ldots \cup \Omega_J$ and pick some $1 \leq j \leq J$. Then, the restriction $T_j := T|_{\Omega_j}$ belongs to $H(\text{Curl}; \Omega_j)$ and $(T^k|_{\Omega_j}) \subset C^\infty(\Gamma_{1,j}; \Omega)$ converges to $T_j$ in $H(\text{Curl}; \Omega_j)$. Thus, $T_j \in H(\text{Curl}; \Gamma_{1,j}, \Omega_j)$. Since $\Omega_j$ is simply connected, there exists a potential vector field $v_j \in H(\text{Grad}; \Omega_j)$ with $\text{Grad} v_j = T_j$. By Korn’s first inequality, Corollary (22) (ii), there exist positive $c_{k,t,j}$ and $A_T \in \mathfrak{so}(N)$ with

$$\|T_j\|_{L^2(\Omega_j)} \leq c_{k,t,j} \|\text{sym} T_j\|_{L^2(\Omega_j)},$$

proving (i). Now, we assume $\Gamma = \emptyset$. To show (ii), let $T \in H(\text{Curl}; \Omega)$ and, as before, let $\Omega$ be decomposed into $\Omega_1 \cup \ldots \cup \Omega_J$. Again, since every $\Omega_j$ is simply connected and $T_j \in H(\text{Curl}; \Omega_j)$, there exist vector fields $v_j \in H(\text{Grad}; \Omega_j)$ with $\text{Grad} v_j = T_j$ in $\Omega_j$. By Korn’s first inequality, Corollary (22) (ii), there exist positive $c_{k,s,j}$ and $A_T \in \mathfrak{so}(N)$ with

$$\|T_j - A_T\|_{L^2(\Omega_j)} \leq c_{k,s,j} \|\text{sym} T_j\|_{L^2(\Omega_j)},$$

proving (ii). We have also proved the first assertion.
of (ii’), since we do not have to slice if $\Omega$ is simply connected. The remaining assertion of (ii’) are trivial, since $\pi_{so(N)} : L^2(\Omega) \to so(N)$ is a $L^2(\Omega)$-orthogonal projector. We note that this can be seen also by direct calculations: To show that $T - A_T$ belongs to $H(Curl_0; \Omega) \cap so(N)^\perp$ we note $A_T \in H(Curl_0; \Omega)$ and compute for all $A \in so(N)$

$$\langle A_T, A \rangle_{L^2(\Omega)} = \int_\Omega T d\lambda, A \rangle_{R^{N \times N}} = \int_\Omega \langle T, A \rangle_{R^{N \times N}} d\lambda = \langle T, A \rangle_{L^2(\Omega)}.$$

Hence, $A_T = 0$ implies $T \perp so(N)$. On the other hand, setting $A := A_T$ shows that $T \perp so(N)$ also implies $A_T = 0$. □

We are ready to prove our main theorem.

**Proof of Theorem 1** Let $\Gamma_t \neq \emptyset$ and $T \in H(Curl; \Gamma_t, \Omega)$. By Corollary 18 we have

$$T = R + S \in H(Curl_0; \Gamma_t, \Omega) \oplus (\hat{H}(Div_0; \Gamma_n, \Omega) \cap (H(\Omega)^N)^\perp).$$

Moreover, by Corollary 17 we obtain

$$\|S\|_{L^2(\Omega)} \leq c_1 \|Curl T\|_{L^2(\Omega)}$$

(3.2)

since $Curl S = Curl T$ and $S \in H(Curl_0; \Gamma_t, \Omega) \cap \hat{H}(Div_0; \Gamma_n, \Omega) \cap (H(\Omega)^N)^\perp$. Then, by orthogonality, Lemma 29 (i) for $R$ and (3.2)

$$\|T\|_{L^2(\Omega)} = \|R\|_{L^2(\Omega)} + \|S\|_{L^2(\Omega)} \leq c_2 \|\text{sym } R\|_{L^2(\Omega)} + \|S\|_{L^2(\Omega)}$$

$$\leq 2c_2 \|\text{sym } T\|_{L^2(\Omega)} + (1 + 2c_2^2) \|S\|_{L^2(\Omega)}$$

$$\leq c_1^2 (\|\text{sym } T\|_{L^2(\Omega)} + \|\text{Curl } T\|_{L^2(\Omega)})$$

with

$$c_1 := \max\{\sqrt{2c_k}, c_\pi \sqrt{1 + 2c_k^2}\}$$

(3.3)

follows, which proves (i). Now, let $\Gamma_t = \emptyset$ and $T \in H(Curl; \Omega)$. First, we show (ii’). We follow in close lines the first part of the proof. For the convenience of the reader, we repeat the previous arguments in this special case. According to Corollary 18 we orthogonally decompose

$$T = R + S \in H(Curl_0; \Omega) \oplus (\hat{H}(Div_0; \Omega) \cap (H(\Omega)^N)^\perp).$$

Then, $Curl S = Curl T$ and $S \in H(Curl_0; \Omega) \cap \hat{H}(Div_0; \Omega) \cap (H(\Omega)^N)^\perp$. Again, by Corollary 17 we have (3.2). Note that

$$A_R = \pi_{so(N)} R = \text{skew} \int_\Omega R d\lambda \in so(N) \subset H(Curl_0; \Omega).$$
As before, by orthogonality, Lemma \[29\] (ii') applied to \(R\) and (3.2)

\[
\|T - A_R\|_{L^2(\Omega)}^2 = \|R - A_R\|_{L^2(\Omega)}^2 + \|S\|_{L^2(\Omega)}^2 \\
\leq c_k^2 \|\text{sym} R\|_{L^2(\Omega)}^2 + \|S\|_{L^2(\Omega)}^2 \\
\leq 2c_k^2 \|\text{sym} T\|_{L^2(\Omega)}^2 + (1 + 2c_k^2) \|S\|_{L^2(\Omega)}^2 \\
\leq c_1^2 \left( \|\text{sym} T\|_{L^2(\Omega)}^2 + \|\text{Curl} T\|_{L^2(\Omega)}^2 \right).
\]

For \(S = \text{Curl}^* X\) with \(X \in \text{H}(\text{Curl}^*; \Omega) = \text{H}(\text{Div}_0; \Omega) \cap (\text{H}(\Omega)^N)^\perp\), where \(\text{Curl}^* \approx -\delta_2\) denotes the formal adjoint of \(\text{Curl}\), we get a piece-wise constant skew-symmetric tensor \(A := A_R\). We note that in general \(A\) does not belong to \(\text{H}(\text{Curl}; \Omega)\) anymore. Hence, we loose the \(L^2(\Omega)\)-orthogonality \(R - A \perp S\). But again, by Lemma \[29\] (ii) and (3.2)

\[
\|T - A\|_{L^2(\Omega)} \leq \|R - A\|_{L^2(\Omega)} + \|S\|_{L^2(\Omega)} \\
\leq c_k \|\text{sym} R\|_{L^2(\Omega)} + (1 + c_k) \|S\|_{L^2(\Omega)} \\
\leq c_k \|\text{sym} T\|_{L^2(\Omega)} + (1 + c_k) c_a \|\text{Curl} T\|_{L^2(\Omega)} \\
\leq c_2 \left( \|\text{sym} T\|_{L^2(\Omega)}^2 + \|\text{Curl} T\|_{L^2(\Omega)}^2 \right)^{1/2}
\]

with

\[
c_2 := \sqrt{2} \max\{c_k, c_a(1 + c_k)\}, \quad (3.4)
\]

which proves (ii). \(\square\)

4 One Additional Result

As in \[16\] sections 3.4] we can prove a generalization for media with structural changes. To apply the main result from \[27\], let \(\mu \in C^0(\Omega)\) be a \((N \times N)\)-matrix field satisfying \(\det \mu \geq \bar{\mu} > 0\).

**Corollary 30** Let \(\Gamma_t \neq \emptyset\) and let the pair \((\Omega, \Gamma_t)\) be admissible. Then there exists \(c > 0\) such that

\[
c \|T\|_{L^2(\Omega)} \leq \|\text{sym} (\mu T)\|_{L^2(\Omega)} + \|\text{Curl} T\|_{L^2(\Omega)}
\]

holds for all tensor fields \(T \in \text{H}(\text{Curl}; \Gamma_t, \Omega)\). In other words, on \(\text{H}(\text{Curl}; \Gamma_t, \Omega)\) the right hand side defines a norm equivalent to the standard norm in \(\text{H}(\text{Curl}; \Omega)\).
A Construction of Hodge-Helmholtz Projections

We want to point out how to compute the projections in the Hodge-Helmholtz decompositions in Lemma 12. Recalling from Lemma 12 the orthogonal decompositions

\[ L^2,q(\Omega) = d^0 q - 1(\Gamma_t, \Omega) \oplus \delta \hat{\Delta} q + 1(\Gamma_n, \Omega) \]

we denote the corresponding \( L^2,q(\Omega) \)-orthogonal projections by \( \pi_d \), \( \pi_\delta \) and \( \pi_\mathcal{H} \). Then, we have

\[ \pi_\mathcal{H} = id - \pi_d - \pi_\delta \]

and

\[ \pi_d L^2,q(\Omega) = d^0 q - 1(\Gamma_t, \Omega) = d X^q - 1(\Omega), \quad X^q - 1(\Omega) := d^0 q - 1(\Gamma_t, \Omega) \cap \delta \hat{\Delta} q(\Gamma_n, \Omega), \]

\[ \pi_\delta L^2,q(\Omega) = \delta \hat{\Delta} q + 1(\Gamma_n, \Omega) = \delta Y^q + 1(\Omega), \quad Y^q + 1(\Omega) := \hat{\Delta} q + 1(\Gamma_n, \Omega) \cap d \hat{\Delta} q(\Gamma_t, \Omega), \]

\[ \pi_\mathcal{H} L^2,q(\Omega) = \mathcal{H}^q(\Omega). \]

By Poincaré’s estimate, i.e., Lemma 7, we have

\[ \forall E \in X^q - 1(\Omega) \quad \| E \|_{L^2,q - 1(\Omega)} \leq c_{p,q - 1} \| d E \|_{L^2,q(\Omega)}, \quad (A.1) \]

\[ \forall H \in Y^q + 1(\Omega) \quad \| H \|_{L^2,q + 1(\Omega)} \leq c_{p,q + 1} \| \delta H \|_{L^2,q(\Omega)}. \quad (A.2) \]

Hence, the bilinear forms

\[ (\tilde{E}, E) \mapsto \left\langle d \tilde{E}, d E \right\rangle_{L^2,q(\Omega)}, \quad (\tilde{H}, H) \mapsto \left\langle \delta \tilde{H}, \delta H \right\rangle_{L^2,q(\Omega)} \]

are continuous and coercive over \( X^q - 1(\Omega) \) and \( Y^q + 1(\Omega) \), respectively. Moreover, for any \( F \in L^2,q(\Omega) \) the linear functionals

\[ E \mapsto \left\langle F, d E \right\rangle_{L^2,q(\Omega)}, \quad H \mapsto \left\langle F, \delta H \right\rangle_{L^2,q(\Omega)} \]

are continuous over \( X^q - 1(\Omega) \) respectively \( Y^q + 1(\Omega) \). Thus, by Lax-Milgram’s theorem we get unique solutions \( E_d \in X^q - 1(\Omega) \) and \( H_\delta \in Y^q + 1(\Omega) \) of the two variational problems

\[ \langle d E_d, d E \rangle_{L^2,q(\Omega)} = \langle F, d E \rangle_{L^2,q(\Omega)} \quad \forall E \in X^q - 1(\Omega), \quad (A.3) \]

\[ \langle \delta H_\delta, \delta H \rangle_{L^2,q(\Omega)} = \langle F, \delta H \rangle_{L^2,q(\Omega)} \quad \forall H \in Y^q + 1(\Omega) \quad (A.4) \]

and the corresponding solution operators, mapping \( F \) to \( E_d \) and \( H_\delta \), respectively, are continuous. In fact, we have as usual

\[ \| d E_d \|_{L^2,q(\Omega)} \leq \| F \|_{L^2,q(\Omega)}, \quad \| \delta H_\delta \|_{L^2,q(\Omega)} \leq \| F \|_{L^2,q(\Omega)}. \]
respectively, and therefore together with (A.1) and (A.2)

\[ |E_d|_{\mathcal{V}_{q-\gamma}^1(\Omega)} = |E_d|_{\mathcal{D}_{q-\gamma}^{-1}(\Omega)} \leq \sqrt{1 + c_{p,q-\gamma}^2} \|F\|_{L^2(\Omega)}, \]

\[ |H_\delta|_{\mathcal{V}_{q+\gamma}^1(\Omega)} = |H_\delta|_{\mathcal{A}_{q+\gamma}^1(\Omega)} \leq \sqrt{1 + c_{p,q+\gamma}^2} \|F\|_{L^2(\Omega)}. \]

Since \( d\tilde{\mathcal{D}}_{q-1}(\Gamma_t, \Omega) = d\mathcal{X}_{q-1}(\Omega) \) and \( \delta\tilde{\mathcal{A}}_{q+1}(\Gamma_n, \Omega) = \delta\mathcal{Y}_{q+1}(\Omega) \) we see that (A.3) and (A.4) hold also for \( E \in \tilde{\mathcal{D}}_{q-1}(\Gamma_t, \Omega) \) and \( H \in \tilde{\mathcal{A}}_{q+1}(\Gamma_n, \Omega) \), respectively, and that

\[ F - dE_d \in (d\mathcal{X}_{q-1}(\Omega))^\perp = (d\tilde{\mathcal{D}}_{q-1}(\Gamma_t, \Omega))^\perp = \tilde{\mathcal{A}}_0^q(\Gamma_n, \Omega), \]

\[ F - \delta H_\delta \in (\delta\mathcal{Y}_{q+1}(\Omega))^\perp = (\delta\tilde{\mathcal{A}}_{q+1}(\Gamma_n, \Omega))^\perp = \tilde{\mathcal{D}}_0^q(\Gamma_t, \Omega). \]

Hence, we have found our projections since

\[ \pi_d F := dE_d \in d\mathcal{X}_{q-1}(\Omega) \subset \tilde{\mathcal{D}}_0^q(\Gamma_t, \Omega), \]

\[ \pi_\delta F := \delta H_\delta \in \delta\mathcal{Y}_{q+1}(\Omega) \subset \tilde{\mathcal{A}}_0^q(\Gamma_n, \Omega) \]

and

\[ \pi_H F := F - dE_d - \delta H_\delta \in \tilde{\mathcal{D}}_0^q(\Gamma_t, \Omega) \cap \tilde{\mathcal{A}}_0^q(\Gamma_n, \Omega) = \mathcal{H}^q(\Omega). \]

Explicit formulas for the dimensions of \( \mathcal{H}^q(\Omega) \) or explicit constructions of bases of \( \mathcal{H}^q(\Omega) \) depending on the topology of the pair \((\Omega, \Gamma_t)\) can be found, e.g., in [22] for the case \( \Gamma_t = \Gamma \) or \( \Gamma_t = \emptyset \), or in [41] for the general case.

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