Half Spectral, an Another General Method for Linear Plasma Simulation

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There are two usual computational methods for linear (waves and instabilities) problem: eigen-value (dispersion relation) solver and initial value solver. In fact, we can introduce an idea of the combination of them, i.e., we keep time derivative $\partial/\partial t$ term (and other term if have, e.g., kinetic $\partial/\partial v$ term), but transform the linear spatial derivatives $\partial/\partial x$ term to $i\mathbf{k}$, which then can reduce the computational dimensions. For example, most (fluid and kinetic) normal mode problems can be reduced from treating cumbersome PDEs to treating simple ODEs. Examples for MHD waves, cold plasma waves and kinetic Landau damping are given, which show to be extremely simple or even may be the simplest method for simulating them. [I don’t know whether this idea is new, but it seems very interesting and useful. So, I choose making it public.]

I. INTRODUCTION AND BASIC IDEA

It’s well known in plasma physics community that for linear (waves and instabilities) problems we have two usual computational methods: eigenvalue (dispersion relation) solver and initial value solver. For the former, we transform the linear time derivative $\partial/\partial t$ and spatial derivatives $\nabla$ to spectral space using $-i\omega$ and $i\mathbf{k}$; for the latter, we solve the original equations directly. In some simple case, the eigenvalue method can be reduced to analytic tractable form, e.g., many well known dispersion relations are this type. However, numerical solutions are always OK (except some singularity cases).

Usually, the eigenvalue method is not intuitive and one needs be good at theoretical derivations; the conventional initial value method is complicated in computation and cumbersome in data analysis. Typically, the eigenvalue method can give all solutions of the system, while the initial value method can only give the $\gamma_{\text{max}}$ (most unstable) solution.

Can we combine these two methods? The answer is yes. We can keep time derivative $\partial/\partial t$ term (and other term if have, e.g., kinetic $\partial/\partial v$ term), but transform the linear spatial derivatives $\partial/\partial x$ term to $i\mathbf{k}$, which is still an initial value method, but is solved in half (only spatial not temporal) spectral space, then also has characteristics of eigenvalue method. For example, we can highlight the non-$\gamma_{\text{max}}$ solutions.

We will show how to do it with examples for normal mode problems in Sec.II. Since this method has been used for linear inhomogeneous eigenvalue and nonlinear problem by previous researchers, we will just give some necessary descriptions with citations in Sec.III.

II. NORMAL MODE PARADIGMS

In most literatures, normal mode and eigenmode are treated as a same concept since they are very similar. But, here, we distinguish them

**Normal mode**: homogeneous, without boundary conditions;

**Eigen mode**: inhomogeneous or with boundary conditions, all possible solutions of the system can be expressed by proper sum of eigenmodes.

Using Half Spectral (HS) method, many normal mode problems can be reduced from PDEs to ODEs. Then can be solved extreme easily. While, unfortunately, for eigen mode problems, the equations are in lower-dimensions but still PDEs.

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A. Simple example

We construct a simple example to show how to use this method. The original equations

\[
\begin{align*}
\frac{\partial f_1}{\partial t} + u_a \frac{\partial f_1}{\partial x} + u_b \frac{\partial f_2}{\partial x} &= 0, \\
\frac{\partial f_2}{\partial t} + u_b \frac{\partial f_1}{\partial x} + u_a \frac{\partial f_2}{\partial x} &= 0.
\end{align*}
\]

(1)

Half spectral solves

\[
\begin{align*}
\frac{\partial f_1}{\partial t} &= -(iku_a f_1 + iku_b f_2), \\
\frac{\partial f_2}{\partial t} &= -(iku_a f_1 + iku_b f_2).
\end{align*}
\]

(2)

The dispersion relation

\[
\begin{align*}
(\omega - ku_a) f_1 &= ku_b f_2, \\
(\omega - ku_a) f_2 &= ku_b f_1.
\end{align*}
\]

(3)

gives, \( \omega = k(u_a \pm u_b) \), \( f_1 = \pm f_2 \).

If we assume the initial values \( f_1 = y f_2 \), the ratio of \( \omega_\pm \) is \( x \) and \( 1 - x \) respectively, then \( x - (1 - x) = y \Rightarrow x = (y + 1)/2 \), then the ratio of the amplitudes for \( \omega_\pm \) is

\[
\frac{A_+}{A_-} = \frac{|x|}{|1 - x|} = \frac{|y + 1|}{|y - 1|},
\]

(4)

which means we can control the amplitudes of each modes of the system exactly by set the proper initial values.

A 4th order Runge-Kutta simulation of eq. 2 is shown in Fig. 1. We can see the frequencies and amplitudes of each modes (\( \omega_\pm = 0.24, 0.16, A_\pm = 0.2, 0.1 \)) are exact as predict. A small mismatch should be caused by the numerical discrete.

B. ES1D kinetic problem

Usually, we have two initial value method to simulate kinetic problem (e.g., Landau damping), i.e., Vlasov continuity solver and PIC method.
To show that half spectral method is not only for fluid problem, we give a kinetic example. At this subsection, the electrostatic 1D Landau damping and bump-on-tail simulations are given.

The linearized equations (ion immobile) are

\[
\begin{align*}
\frac{\partial \delta f}{\partial t} &= -ikv \delta f + \frac{e}{m} E \frac{\partial f_0}{\partial v}, \\
ike \delta E &= -e \int \delta f dv.
\end{align*}
\]

which gives

\[
\delta f = \frac{ieE}{m} \frac{\partial_v f_0}{\omega - kv}.
\]

contains both normal mode (which is independent with initial value) and ballistic mode (which is brought by initial value) or phase mixing (see e.g., [6]).

The initial distribution function \(f_0\) can be any form, e.g., Maxwellian gives Landau damping, bump-on-tail gives beam-plasma instabilities.
Fig. 2 shows the simulation of Landau damping. One can find very similar results (especially the fourth panel) from Vlasov continuity simulation. However, we should notice an unfavorable recurrence effect [1] caused by discrete $\Delta \nu$, which is also found in half spectral simulation. The recurrence time $T_R = 2\pi/k\Delta\nu$.

Fig. 3 is the simulation of bump-on-tail problem. Since we treat linear problem, the unit of the amplitude can be arbitrary large.

One can find, comparing with continuity solver and PIC method, using half spectral method for Landau damping simulation is extreme simple and can be more accurate. Comparing with numerical or analytical dispersion relation solver, half spectral method does not need treat troublesome or confusing integral contours. Especially, when the initial distribution function is not standard and the dispersion relation is hard to solve, the half spectral simulation can gives an reasonable solution for benchmark more complicated codes.

C. MHD waves

For MHD waves, we solve

$$\begin{align*}
\frac{\partial \delta \rho}{\partial t} &= -i\rho_0 \mathbf{k} \cdot \delta \mathbf{u}, \\
\frac{\partial \delta \mathbf{u}}{\partial t} &= \frac{i}{\mu_0} (\mathbf{k} \times \mathbf{B}) \times \mathbf{B}_0 - i k v_s^2 \delta \rho, \\
\frac{\partial \delta \mathbf{B}}{\partial t} &= i \mathbf{k} \times (\delta \mathbf{u} \times \mathbf{B}_0),
\end{align*}$$

(7)

where $\mathbf{B}_0 = (0, 0, B_0)$, $\mathbf{k} = (k \sin \theta, 0, k \cos \theta)$, $v_s^2 = \gamma p_0/\rho_0 = \gamma k T_0/m$, $v_A^2 = B_0^2/\mu_0 \rho_0$, $v_p = \omega/k$.

Three solutions are fast mode, slow mode and shear Alfvén wave (one can find introductions of them in textbooks, e.g., [4])

$$\begin{align*}
v_f^2 &= \frac{1}{2} (v_A^2 + v_s^2) + \frac{1}{2} \left( (v_s^2 - v_A^2)^2 + 4v_A^2 v_s^2 \sin^2 \theta \right)^{1/2}, \\
v_s^2 &= \frac{1}{2} (v_A^2 + v_s^2) - \frac{1}{2} \left( (v_s^2 - v_A^2)^2 + 4v_A^2 v_s^2 \sin^2 \theta \right)^{1/2}, \\
v_p^2 &= v_A^2 \cos^2 \theta.
\end{align*}$$

(8)

A simulation result is shown in Fig. 4. Again, we can find the simulation result exactly matches the theoretical solutions. The simulation is intuitive. The frequency signal in the second panel is taken from $\delta \rho$ and $\delta u_y$. If we only use $\delta \rho$ signal, the shear Alfvén wave solution will vanish.

If one want to go non-ideal MHD, e.g., including resistivity or using anisotropic pressure $\delta p \parallel \neq \delta p \perp$, but wouldn’t like to do analytical derivations, then half spectral method is an useful choice: it is very simple, intuitive and can give solutions exact enough.

Since all information for linear perturbation variables is kept in the simulation, we can use them for many more deeply analysis, e.g., the polarization and so on.

D. EM cold plasma waves

Equations are

$$\begin{align*}
\frac{\partial \delta \mathbf{v}_s}{\partial t} &= \frac{e_s}{m_s} \left[ \delta \mathbf{E} + \delta \mathbf{v}_s \times \delta \mathbf{B} \right], \\
\frac{\partial \delta \mathbf{E}}{\partial t} &= i c^2 \mathbf{k} \times \delta \mathbf{B} - \delta \mathbf{J}/\epsilon_0, \\
\frac{\partial \delta \mathbf{B}}{\partial t} &= -i \mathbf{k} \times \delta \mathbf{E}.
\end{align*}$$

(9)

where $\delta \mathbf{J} = \sum_s n_s e_s \delta \mathbf{v}_s$. And, $\mathbf{B}_0 = (0, 0, B_0)$, $\mathbf{k} = (k \sin \theta, 0, k \cos \theta)$, $\omega_{cs} = e_s B_0/m_s$ and $\omega_{ps} = n_s q_s^2/\epsilon_0 m_s$.

For one ion species, the final dispersion relation can be reduced (with heavy calculations) to a fifth order equation for $\omega^2$ (see [10] for details). While, using half spectral simulation, this is very easy. A result is shown in Fig. 5.

If we have more than one ion species or with beams, the dispersion relation can be very headache even though the problem seems not that complicated. While, using half spectral method, the problem is indeed still very easy.
FIG. 4: Solve eq. (7) for ideal MHD waves, red line is dispersion relation solutions

FIG. 5: Solve eq. (9) for EM cold plasma waves, red line is dispersion relation solutions

E. Summary

For normal mode problem, half spectral method may be the simplest method for simulating them, which is intuitive, simple and also exact enough. For analytical difficult problems, this method can not only be an intuitive tool but also has practical usages.
III. EIGENMODE AND NONLINEAR PROBLEMS

As claimed at the title, half spectral method can be a general (linear) plasma simulation method. So, we also need
discuss the eigenmode problem and some nonlinear treatments. While, it is found in the literatures that previous
researchers have given many examples of this. So, here we just give a short description and mention some citations.

A. Tearing mode

Eigenmode problems are similar. We take collisional tearing mode as example here.
A simulation matches the half spectral idea is given by Lee and Fu\[7\] (see also citations of that paper). For
simulation, we solve

\[
\begin{align*}
\frac{\partial \delta \rho}{\partial t} &= -\frac{\partial \rho}{\partial x} \delta u_x - \rho \frac{\partial \delta u_x}{\partial x} - i\alpha \rho \delta u_z, \\
\frac{\partial \delta u_x}{\partial t} &= -\frac{\beta}{2\rho} \frac{\partial \delta p}{\partial x} - \frac{B_z}{\rho} \frac{\partial \delta B_z}{\partial x} - \frac{1}{\rho} \frac{\partial B_z}{\partial x} \delta B_z + i\alpha \frac{B_z}{\rho} \delta B_z, \\
\frac{\partial \delta u_z}{\partial t} &= -i\alpha \frac{\beta}{2\rho} + \frac{1}{\rho} \frac{\partial B_z}{\partial x} \delta B_z, \\
\frac{\partial \delta B_z}{\partial t} &= i\alpha B_z \delta u_x + \frac{1}{R_m} \frac{\partial^2 \delta B_z}{\partial x^2} - \frac{\alpha^2}{R_m} \delta B_x, \\
\frac{\partial \delta B_x}{\partial t} &= -\frac{\partial B_z}{\partial x} \delta u_x - B_z \frac{\partial \delta u_x}{\partial x} + \frac{1}{R_m} \frac{\partial^2 B_z}{\partial x^2} - \frac{\alpha^2}{R_m} \delta B_z, \\
\frac{\partial \delta \rho}{\partial t} &= -\frac{\partial \rho}{\partial x} \delta u_z - \gamma \rho \frac{\partial \delta u_z}{\partial x} - i\alpha \gamma \delta u_z.
\end{align*}
\]

(10)

where parameter are \( \alpha = kl \), \( \beta = 2\mu_0 \rho_\infty / B_\infty^2 \), \( R_m = v_A / \eta, \gamma \). The normalization unit are \( B_\infty, \rho_\infty, v_A = B_\infty^2 / \mu_0 \rho_\infty, \rho_\infty, l, t_0 = l / v_A \).

The results are very well in that paper\[3\]. So, we can trust that half spectral method is also good for eigenmode
problem.

A bad thing is that, for eigenmode problem, we still need solve PDEs.

B. Hasegawa-Mima equation

In fact, an usual way for nonlinear drift wave turbulence by Hasegawa-Mima equation\[5\] simulation is doing in
spectral space, which is exact the half spectral idea of this manuscript. A very detailed introduction can be found in
Waltz’s lecture notes\[9\]. To give a rough impression, one of the equations is shown below

\[
\frac{d}{dt} \phi_k(t) = (-i\omega_k + \gamma_k) \phi_k(t) + \frac{1}{2} \sum_{k_1, k_2} \delta(k - k_1 - k_2) V_{k_1 k_2} \phi_{k_1}(t) \phi_{k_2}(t).
\]

(11)

We should comment here, for nonlinear problem, we need sum all \( k \) modes and the complex conjugate should also
be kept.

IV. SUMMARY AND COMMENTS

In this manuscript, we discussed the idea of half spectral method, and showed that it can be a general simulation
method. However, this idea maybe not new, especially for eigenmode and nonlinear problem, many previous
researchers used it. I don’t know whether this method is new for normal mode problems yet. And the name half
spectral using here is just for convenient.

For tokamak (or other problems with strong guide field) simulation, a similar method is called flux tube (e.g., \[8\]),
which using the idea of symmetry to reduce dimensions because the physics is mainly along magnetic field line then
we can take it as one coordinate. Poloidal and toroidal mode numbers \( m \) and \( n \) are often used in flux tube simulation.
I haven’t checked whether the equations for flux tube simulation are the same for half spectral simulation yet. They
may be exact the same or at least very similar.
But, we can tell that spectral or pseudo spectral method are different from this half spectral method, because that they need transform back to real space. For a same simulation, spectral or pseudo spectral method can be alternative by other discrete methods (e.g., finite difference) and won’t influent the simulation results. While, half spectral method is independent on discrete methods.

This idea is partly inspired by “A 2D Hasegawa-Mima Model of Electrostatic Drift Wave Turbulence” simulation by Deng ZHAO (PKU, 2011) and P239C course project (UCI, 2010) “Cold plasma-warm beam interaction” given by Prof. Liu CHEN. The basic idea is the same. The new here is that we find this method can be generalized.

If we do not care that whether this idea is new or not, we can find half spectral method is simple, interesting and useful. MATLAB codes for this manuscript are given in attached files.

[1] Cheng, C. and Knorr, G., The integration of the vlasov equation in configuration space, Journal of Computational Physics, 1976, 22, 330 - 351.
[2] For drift wave, if the gradient parameter $\partial \ln(f)/\partial x$ is constant, we also treat it as normal mode.
[3] There is also a Chinese book for space plasma simulation written by Zhu-feng FU and You-qiu HU in 1995, which has given more details of paper [7].
[4] Gurnett, D. A. and Bhattacharjee, A., Introduction to plasma physics: with space and laboratory applications, Cambridge, 2005.
[5] Hasegawa, A. and Mima, K. Pseudo-three-dimensional turbulence in magnetized nonuniform plasma, Physics of Fluids, 1978, 21, 87-92.
[6] Krall, N. and Trivelpiece, A., Principles of Plasma Physics, McGraw-Hill, 1973.
[7] Lee, L. C. and Fu, Z. F., Collisional Tearing Instability in the Current Sheet With a Low Magnetic Lundquist Number, J. Geophys. Res., AGU, 1986, 91, 3311-3313.
[8] A.G. Peeters, Y. Camenen, F.J. Casson, W.A. Hornsby, A.P. Snodin, D. Strintzi, G. Szepesi, The nonlinear gyro-kinetic flux tube code GKW, Computer Physics Communications, 180, 2650, 2009.
[9] Ronald Waltz, Lecture Series on Turbulent Transport in Tokamak, 1986.
[10] Swanson, D. G., Plasma Waves, IOP, 2nd (Ed.), 2003.