On equation of geodesic deviation and its solutions

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Abstract

Equations of geodesic deviation for the 3-dimensional and 4-dimensional Riemann spaces are discussed. Availability of wide classes of exact solutions of such equations, due to recent results for the matrix Schrödinger equation, is demonstrated. Particular classes of exact solutions for the geodesic deviation equation as well as for the Raychaudhuri and generalized Raychaudhuri equation are presented. Solutions of geodesic deviation equation for the Schwarzschild and Kasner metrics are found.

1 Introduction

It is well-known that the equation of geodesic deviation is an important equation in general relativity. It relates the relative acceleration between two test particles to certain components of the Riemann-curvature tensor. A knowledge of geodesic deviations is also needed to evaluate or simplify the first and higher derivatives of two-point geometrical quantities (the world function or geodetic interval), the parallel propagator and others geometric quantities related to these [1-3].

The Raychaudhuri equation, which is closely connected with the equation of geodesic deviation plays a key role in an analysis of the focusing effects of gravity [3,4]. Evolution of deformations of a relativistic membranes and

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the focusing of families of surfaces are described by the generalisations of
the Raychaudhuri equation \[5,6\].

Equation of geodesic deviation is also an important in mathematics, in
the theory of Riemann spaces. For surfaces it coincides, in essence, with
the Gauss equation in geodesic coordinates while in the three and higher-
dimensional Riemann spaces it carries an essential information about these
spaces. Solutions of the geodesic deviations equation (Jacoby fields) and
their properties (e.g., an existence of conjugate points) are related to various
important characteristics of Riemann spaces \[7-8\].

The equations of geodesic deviation are the system of second order linear
equations. Their relevance to various problems in physics and mathematics
has became clear long time ago. But, suprisingly, still not many exact
solutions of these equations have been found. They basically correspond to
the particular case of reduction to a single second order equation \[4,6,9\].

In this paper we will present a wide variety of exact solutions of the
equation of geodesic deviations. These solutions became available due to
the recently discovered method of the inverse spectral transform \[10,11\].
We will consider equations of geodesic deviation in the two, three and four
dimensional Riemann spaces as well as the Raychaudhuri equation and its
generalisations.

The case of the diagonalized separated system of the geodesic deviation
equations which correspond to the diagonal metric is considered in details.
The nondiagonalizable case is studied with the use of the theory of the
matrix Schrödinger equation. General formulae and particular solutions are
presented.

Specification of the geodesic deviations equation via nonlinear integrable
(soliton) equations is discussed. Solutions of the geodesic deviations equa-
tion in general relativity for the Schwarzschild and Kasner metrics are pre-

2 Equation of geodesic deviations

Here we will present some known formulas, which will be used in what
follows. In the Riemann space with the metric

\[ ds^2 = g_{ij}dx^i dx^j \] (1)
the geodesics are defined via the equation
\[ \frac{d^2 x^i}{ds^2} + \Gamma^i_{kj} \frac{dx^i}{ds} \frac{dx^j}{ds} = 0 \] (2)
where \( \Gamma^i_{kj} \) are the Christoffel symbols of metric.

Let \( \eta^i \) be the components of deviation vector between two infinitesimally nearby geodesic lines. Then these components satisfy to the Jacobi equation [1]
\[ v^i \nabla_i (v^j \nabla_j \eta^l) = -v^i R^l_{ikm} v^m \eta^k \] (3)
where \( v^i \) are the components of tangent vector along geodesic line \( \gamma \), symbol \( \nabla_i \) means the covariant derivation along the vector field \( v^i \) with respect to the metric (1) and \( R^l_{ikm} \) are the curvature tensor of the metric (1).

In a special system of coordinates were axis \( x^i \) is a geodesic line equations (3) have the following form [1-3]
\[ \frac{d^2 \eta^i}{dx^2} + R^i_{jil} \eta^l = 0. \] (4)
This system of equations is a main object of our consideration.

The system of equations (4) has a form of the matrix Schrödinger operator:
\[ \left[ -\frac{d^2}{dx^2} + U(x) \right] \Psi = \lambda \Psi \] (5)
where \( U_{ij}(x) = (-R^l_{jil} + \lambda \delta_{jl}) \), \( x^i = x \) and \( \Psi \) is the fundamental matrix-solution.

In the one-dimensional case the equation of geodesic deviation is equivalent to the Gauss equation
\[ \frac{d^2 \Psi}{dx^2} + K(x, y) \Psi = 0 \] (6)
which connects the curvature \( K(x, y) \) of the surface with the metric
\[ ds^2 = dx^2 + \Psi^2(x, y)dy^2. \] (7)

The Raychaudhuri equation also is, in fact, the one-dimensional Schrödinger equation [4,3]. Indeed, introducing the variable \( \Theta = \frac{\partial}{\partial x} \log \det U \), one gets from equation (4) the following equation
\[ \frac{d\Theta}{dx} = -\frac{1}{3} \Theta^3 - \sigma^2 + \omega^2 - R, \]
where \( \sigma^2 = \sigma_{ab} \sigma_{ab}, \ \omega^2 = \omega_{ab} \omega_{ab} \text{ and } R \) are certain quantities depending on the metric. The change of variable

\[
\Theta = 3 \frac{\Psi_x}{\Psi}
\]

leads to the equation

\[
\Psi_{xx} = \frac{1}{3} (\omega^2 - \sigma^2 - R) \Psi.
\]

In particular, for the isotropic cosmological model with the Einstein equation

\[-8\pi T^\mu_\nu = R^\mu_\nu - \frac{1}{2} R \delta^\mu_\nu + \Lambda \delta^\mu_\nu\]

one has

\[
\frac{d^2}{dt^2} \Psi = \frac{1}{3} [\Lambda - 4\pi \rho - \phi^2] \Psi \quad (8)
\]

where \( \rho \) is a density of matter and \( \phi \) is a certain geometric characteristic.

The Raychaudhuri equation is the principal equation for an analysis of the singularity behaviour in general relativity. Few particular solutions of this equation have been found until now.

Wide classes of solutions of equation (6) have been presented in the paper [12]. Since for surfaces \( \eta = \Psi(x, y)dy \) then the results of [12] provide wide classes of the Jacoby field for the surfaces.

3 Geodesic deviation equation for three-dimensional space

In the case of three-dimensional space the use of the coordinate transformations allows us to reduce a general metric to the form

\[
ds^2 = dx^2 + A(x, y, z)dy^2 + 2B(x, y, z)dydz + C(x, y, z)dz^2. \quad (9)
\]

For such a metric the matrices of the Christoffel’s symbols are

\[
\Gamma_1 = \begin{pmatrix}
0 & 0 & 0 \\
0 & \frac{CA_x - BB_x}{2\Delta} & \frac{CB_x - BC_x}{2\Delta} \\
0 & \frac{AB_x - BA_x}{2\Delta} & \frac{AC_x - BB_x}{2\Delta}
\end{pmatrix},
\]
\[
\Gamma_2 = \begin{pmatrix}
0 & -A_x & -B_x \\
\frac{CA_x-BB_x}{2\Delta} & \frac{CA_y+BA_x-2BB_x}{2\Delta} & \frac{CA_z-BC_y}{2\Delta} \\
\frac{AB_x-BA_x}{2\Delta} & \frac{2AB_y-BA_y-AA_z}{2\Delta} & \frac{AC_y-BA_z}{2\Delta}
\end{pmatrix},
\]
\[
\Gamma_3 = \begin{pmatrix}
0 & -B_x & -C_x \\
\frac{CB_x-BC_x}{2\Delta} & \frac{CA_y-BC_x}{2\Delta} & \frac{2CB_y-CC_y-BC_z}{2\Delta} \\
\frac{AC_x-BA_x}{2\Delta} & \frac{AC_y-BA_z}{2\Delta} & \frac{AC_x+BC_z-2BB_x}{2\Delta}
\end{pmatrix}
\]
where \(\Delta = AC - B^2\).

The equations of geodesic deviations are of the form
\[
\frac{d^2\eta^2}{dx^2} + R_{121}^2 \eta^2 + R_{131}^2 \eta^3 = 0,
\]
\[
\frac{d^2\eta^3}{dx^2} + R_{121}^3 \eta^2 + R_{131}^3 \eta^3 = 0
\]
(10)

where the corresponding components of the curvature tensor are
\[
R_{121}^2 = -\frac{1}{2} \frac{\partial}{\partial x} \left( \frac{CA_x - BB_x}{\Delta} \right) - \frac{(CA_x - BB_x)^2}{4\Delta^2} - \frac{(CB_x - BC_x)(AB_x - BA_x)}{4\Delta^2},
\]
\[
R_{131}^2 = -\frac{1}{2} \frac{\partial}{\partial x} \left( \frac{CB_x - BC_x}{\Delta} \right) - \frac{(CB_x - BC_x)\Delta_x}{4\Delta^2},
\]
\[
R_{121}^3 = -\frac{1}{2} \frac{\partial}{\partial x} \left( \frac{AB_x - BA_x}{\Delta} \right) - \frac{(AB_x - BA_x)\Delta_x}{4\Delta^2},
\]
\[
R_{131}^3 = -\frac{1}{2} \frac{\partial}{\partial x} \left( \frac{AC_x - BB_x}{\Delta} \right) - \frac{(AC_x - BB_x)^2}{4\Delta^2} - \frac{(AB_x - BA_x)(CB_x - BC_x)}{4\Delta^2}.
\]

The system of equations (10) can be rewritten in the form of the matrix Schrödinger operator
\[
-\frac{d^2\eta^2}{dx^2} + (-R_{121}^2 + \lambda^2)\eta^2 + (-R_{131}^2)\eta^3 = \lambda^2 \eta^2,
\]
\[
-\frac{d^2\eta^3}{dx^2} + (-R_{121}^3)\eta^2 + (-R_{131}^3 + \lambda^2)\eta^3 = \lambda^2 \eta^3.
\]
4 Some examples of solutions of geodesic deviation equation

The simplest case of the deviation equations (7) corresponds to the diagonal metrics, i.e. to the case $B = 0$. In this case we have the system

\[- \frac{d^2 \eta^2}{dx^2} + \frac{d^2 A}{dx^2} \eta^2 = 0,\]  
\[- \frac{d^2 \eta^3}{dx^2} + \frac{d^2 B}{dx^2} \eta^3 = 0\]  

and its solutions correspond to the geodesic deviations for the 3-dimensional orthogonal metrics (9). We can construct wide classes of solutions of each of equations (12) using the results for the one-dimensional Schrödinger equation. In this case the components of deviations vector $\eta^2$ and $\eta^3$ are completely independent.

One can relates them by some special conditions. One possibility is to consider the system of equations (12) as related by the Darboux transformations, i.e.

\[\frac{d^2 B}{dx^2} = \frac{d^2 A}{dx^2} - 2 \frac{\partial^2}{\partial x^2} \log \eta_1,\]

\[\eta^3 = \eta_2^2 - \eta_2^2 \eta^2 \eta_1^2\]

where $\eta_1$ is some solution of the first equation (12). The set of metrics related by the conditions of such type remains always undetermined and for its further specification it is necessary to fix the dependence on the variables $y$ and $z$. We can fix this dependence with the help of the some equations or to consider the metrics as induced from the immersion in some Euclidean space or another type of spaces.

As example we can consider the class of 3-dimensional orthogonal metrics

\[ds^2 = A^2(x, y, z)dx^2 + B^2(x, y, z)dy^2 + C^2(x, y, z)dz^2\]

and will require that the functions $A$, $B$, $C$ obey the Darboux equations [13,14]

\[A_{zy} = \frac{C_y}{C} A_z + \frac{B_z}{B} A_y,\]

\[B_{zx} = \frac{C_z}{C} B_z + \frac{A_z}{A} B_x,\]
\[
C_{xy} = \frac{A}{A} C_x + \frac{B}{B} C_y.
\]

In the case \( A = 1 \) this system is reduced to the relations

\[
B_z = U(y, z) C, \quad C_y = V(y, z) B,
\]

which are equivalent to the Laplace-type equation for the functions \( A \) and \( B \)

\[
B_{zy} - \frac{U_y}{U} B_z - UVB = 0, \quad C_{zy} - \frac{V_z}{V} C_y - VUC = 0.
\]

These systems of equations are well known in theory of integrable equations and one can use different methods to construct their exact solutions.

In any case using the \( N \)-times iterated Darboux transformation and certain addition conditions one can get the discrete set of metrics

\[
ds_n^2 = dx^2 + B_n(x, y, z) dy^2 + C_n(x, y, z) dz^2,
\]

which may be of interest from the various points of view.

For analysis of nondiagonal metrics (9) we can use the results on the matrix Schrödinger equation and corresponding nonlinear integrable differential equations.

The simplest way consists in the use of the relations between the AKNS-system [10,11]

\[
\begin{align*}
\frac{\partial \psi_1}{\partial x} + i\lambda \psi_1 &= q(x, y, z) \psi_2, \\
\frac{\partial \psi_2}{\partial x} - i\lambda \psi_2 &= r(x, y, z) \psi_1
\end{align*}
\]

and the corresponding Schrödinger-like system of equations [15]:

\[
\left[ - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \frac{\partial^2}{\partial x^2} + \begin{pmatrix} rq & qx \\ rx & rq \end{pmatrix} \right] \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \lambda^2 \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.
\]

Comparing these equations with equations (8), one obtains the following conditions on curvature tensor

\[
\lambda^2 - R_{121}^2 = rq, \quad \lambda^2 - R_{131}^3 = rq,
\]

\[
R_{131}^2 = -qx, \quad R_{121}^3 = -rx.
\]

These conditions imply certain relations between components of metric.
In fact, from the condition 
\[ R_{121}^2 = R_{131}^1 \]
we have 
\[ CA_x - AC_x = \delta(y, z) \sqrt{AC - B^2}. \]

Other two conditions lead to the relations 
\[ AB_x - BA_x = 2 \sqrt{AC - B^2} \left( \int r_x \sqrt{AC - B^2} dx + \alpha(y, z) \right), \]
\[ CB_x - BC_x = 2 \sqrt{AC - B^2} \left( \int q_x \sqrt{AC - B^2} dx + \beta(y, z) \right) \]
where \( \alpha(y, z), \beta(y, z) \) and \( \delta(y, z) \) are arbitrary functions.

So, we have some relations between the functions and we can hope that the solutions of this system exist. As result, we can obtain the two parametrical family of solutions of the matrix Schrödinger equation and then we can fix dependence on variables \( y \) and \( z \) by nonlinear integrable equation which have the form of the Lax-equations
\[ L_y = [L, A] \quad \text{or} \quad L_z = [L, A] \]
where
\[ L = -\frac{d^2}{dx^2} + U(x, y, z) \]
and \( A \) is some matrix differential operator.

So this approach to the solutions of geodesic deviation equations allows us to calculate the examples of 3-dimensional metrics which could be useful in the various problems (for example, in general relativity).

5 Solutions of the matrix Schrödinger equation and geodesic deviation equation in the 4-dimensional space

We see that the construction of solutions of the matrix Schrödinger equation is a key step in the study of the geodesic deviation equation.

The matrix Schrödinger equation has been studied recently within the framework of the inverse spectral transform method [15,16]. We will present here some results from [16].
Solution of equation (5) is given by the formulas
\[ \Psi = \exp(ikx) + \int_x^\infty K(x, x') \exp(ikk'x')dx' \]
where \( k^2 = \lambda \) and
\[ U(x) = -2 \frac{d}{dx}K(x, x). \]
Here the matrix function \( K(x, x') \) is the solution of linear integral equation
\[ K(x, x') + M(x + x') + \int_x^\infty dx'' K(x, x'')M(x' + x'') = 0, \quad x < x' \]
with the kernel
\[ M(x) = \sum_{n=1}^{N} C_n \exp(-knx) + \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\lambda R(\lambda) \exp(i\lambda x) \]
where \( C_n \) and \( R(\lambda) \) are arbitrary matrices. The simplest solution of this integral equation corresponds to the case \( R = 0 \) and \( N = 1 \). The corresponding potential \( U \) and wave function \( \Psi \) are
\[ Q(x) = -2p^2 \cosh^2[p(x - \xi)]P, \quad (13) \]
\[ \Psi = (ik - p \tanh[p(x - \xi)]) \exp(i[x - \xi])P \]
where \( P \) is projector, i.e. the matrix which satisfies the condition
\[ P^2 = P. \]

More complicated solution corresponds to \( N = 2 \) and \( R = 0 \). It is of the form
\[ U(x, y, z) = -W_x, \]
\[ W(x, y, z) = -2(p_1 + p_2)[1 - \rho \tau_1 \tau_2]^{-1}[\tau_1 P_1 + \tau_2 P_2 - \tau_1 \tau_2 \{P_1, P_2\}], \]
\[ \tau_k = \frac{P_k}{(p_1 + p_2)} (1 - \tanh[p_k(x - \xi_k)]) \]
and the function \( \rho \) is defined from the relations
\[ \rho P_1 = P_1 P_2 P_1, \]
\[ \rho P_2 = P_2 P_1 P_2. \]
All solutions which correspond to the case \( R = 0 \) can be written in a closed form \([15,16]\). These exact solutions could be used to specify the components of the curvature tensor. For instance, for the solution (13) one has the following relations between components of curvature tensor

\[
R^2_{131} R^3_{131} = R^2_{121} R^3_{121}.
\]

In the case of 4-dimensional space with the geodesic coordinate system

\[
ds^2 = dt^2 + g_{ab} dx^a dx^b
\]

the geodesic deviations equation has the form

\[
\frac{d^2}{dt^2} \eta^1 + R^1_{010} \eta^1 + R^1_{020} \eta^2 + R^1_{030} \eta^3 = 0,
\]

\[
\frac{d^2}{dt^2} \eta^2 + R^2_{010} \eta^1 + R^2_{020} \eta^2 + R^2_{030} \eta^3 = 0,
\]

\[
\frac{d^2}{dt^2} \eta^3 + R^3_{010} \eta^1 + R^3_{020} \eta^2 + R^3_{030} \eta^3 = 0
\]

where \( R^i_{jkl} \) are the components of curvature tensor. In this case the conditions on \( R^i_{jkl} \) to be projector have more complicated form.

Using the generalizations of Wronskian relations between the various solutions \( \Psi_i \) of the matrix Schrödinger equation one can get the nonlinear relations for the potentials: \( Q(x, y, z) \) and \( Q(x, y + \Delta y, z + \Delta z) \) at the fixed value of variable \( x \) such that the corresponding matrix coefficients \( R(\lambda, y, z, x) \) and \( C(y, z, x) = SS^T \) will satisfy linear equations.

In a such a way the nonlinear ”boomeron” integrable equations of the form

\[
Q_y(x, y, z) = \Phi[Q(x, y, z), Q_z(x, y, z)]
\]

have been constructed \([16]\). Here \( \Phi \) is a certain function. Their solutions can be used for the construction of the solutions of geodesic deviation equation.

So we see that the soliton structures and equations may be useful in the study of the geodesic deviations equations and in the general relativity.

6 Some solutions of geodesic deviation equation in general relativity

Let us consider the case of 4-dimensional Riemann space with the metric \([17]\)

\[
ds^2 = dt^2 - A^2(t) dx^2 - B^2(t) dy^2 - C^2(t) dz^2.
\]
The corresponding geodesic deviation equations are of the form

\[
\frac{d^2}{dt^2} \eta^1 - \frac{A_{tt}}{A} \eta^1 = 0,
\]

\[
\frac{d^2}{dt^2} \eta^2 - \frac{B_{tt}}{B} \eta^2 = 0,
\]

\[
\frac{d^2}{dt^2} \eta^3 - \frac{C_{tt}}{C} \eta^3 = 0.
\]

If the metric (25) satisfies the Einstein equations, then:

\[A = t^{p_1}, \quad B = t^{p_2}, \quad C = t^{p_3}\]

where

\[p_1 + p_2 + p_3 = 1, \quad p_1^2 + p_2^2 + p_3^2 = 1.\]

It is the Kasner metric. The variables \(p_i\) admit the parametrisation:

\[p_1 = \frac{-s}{1 + s + s^2}, \quad p_2 = \frac{s(s + 1)}{1 + s + s^2}, \quad p_3 = \frac{1 + s}{1 + s + s^2}.\]

Hence, the corresponding equations are not independent, namely:

\[
\frac{d^2}{dt^2} \eta^1 - \frac{s(s + 1)^2}{(1 + s + s^2)t^2} \eta^1 = 0,
\]

\[
\frac{d^2}{dt^2} \eta^2 + \frac{s(s + 1)}{(1 + s + s^2)t^2} \eta^2 = 0,
\]

\[
\frac{d^2}{dt^2} \eta^3 + \frac{(1 + s)s^2}{(1 + s + s^2)t^2} \eta^3 = 0.
\]

Solutions of the equation of the type

\[y'' = cx^{-2} y\]

have the form:

\[y = c_1 x^{s + \frac{1}{2}} + c_2 x^{\frac{1}{2} - s},\]

\[c_1 \sqrt{x} + c_2 \sqrt{x} \log x \]

\[c_1 \sqrt{x} \cos(s \log x) + c_2 \sqrt{x} \sin(s \log x),\]

where \(2s = \sqrt{|4c + 1|}\) and for the first solution \(4c + 1 > 0\), for the second solution \(4c + 1 = 0\) and for the third \(4c + 1 < 0\). These solutions provide
us the solutions and properties of the geodesic deviations for the Kasner metric.

To study the properties of geodesic deviation for the Schwarzschild solution of Einstein equation it is convenient to use the metric in the Lemaître-form [17]

$$ds^2 = dt^2 - \left[ \frac{4}{2r^2} (r - t)^{2/3} \right] \left[ \frac{2}{3} \left( R - t \right) \right]^{4/3} r_g^{2/3} (d\theta^2 + \sin^2\theta d\phi^2).$$  (14)

The corresponding equations for geodesic deviations are

$$\frac{d^2}{dt^2} \eta^1 - \frac{4}{9(R - t)^2} \eta^1 = 0,$$

$$\frac{d^2}{dt^2} \eta^2 + \frac{2}{9(R - t)^2} \eta^2 = 0,$$

$$\frac{d^2}{dt^2} \eta^3 + \frac{2}{9(R - t)^2} \eta^3 = 0.$$

The solutions of these equations have the following simple form

$$\eta^1 = c_1 (R - t)^{4/3} + c_2 (R - t)^{-1/3},$$

$$\eta^2 = c_3 (R - t)^{1/2} + c_4 (R - t)^{1/3},$$

$$\eta^3 = c_5 (R - t)^{1/2} + c_6 (R - t)^{1/3}.$$

These solutions can be useful for the study the properties of Schwarzschild metric.

### 7 Solution of the Raychaudhuri equation

The Raychaudhuri equation (8) is the one-dimensional scalar Schrödinger equation with the potential

$$u = 4\pi \rho + \phi^2$$

and the cosmological constant $\lambda$ plays a role of energy ($\lambda = \frac{\Lambda}{3}$). So, any exact solution of the Schrödinger equation provides us the solution of the Raychaudhuri equation. Wide class of them is given by the so-called, Bargmann potentials and their eigenfunctions

$$U(x, y) = 2(\log \det A)_{xx}$$
and

$$\Psi = \text{Re}(\exp[-i\lambda_0 x] + \sum_{n=1}^{N} \frac{\det A_n \exp[-(\alpha_n(y) + i\lambda_0)x]}{\det A \alpha_n(y) + i\lambda_0})$$

where $y$ denotes all independent variables except $x$, $A$ is the $N \times N$ matrix with elements

$$A_{nk} = \delta_{nk} + \frac{\beta_n(y)}{\alpha_n(y) + \alpha_k(y)} \exp[-(\alpha_n(y) + \alpha_k(y))x].$$

The matrix elements of the matrix $A_n$ are given by (15) with the substitution of the last column by the column

$$-\beta_n(y) \exp(-\alpha_n(y)x)$$

$(n=1,2,3,...N)$.

In the simplest case $N = 1$ one has:

$$U = \frac{2\alpha^2(y)}{\cosh^2[\alpha(y)x - \gamma(y)]} - \frac{\Lambda}{3}$$

$$\Psi = \text{Re}(\frac{i\lambda_0 + \alpha(y) \tanh[\alpha(y)x - \gamma(y)]}{i\lambda_0 + \alpha(y)} \exp[-i\lambda_0 x])$$

where

$$\gamma(y) = \frac{1}{2} \log \frac{\beta(y)}{2\alpha(y)}$$

and $\alpha(y)$, $\beta(y)$ are arbitrary functions. These solutions have very special properties: they are transparent for all energies, the corresponding function $\Psi$ have in general $N - 1$ zeros.

The dependence of the functions $\alpha$, $\beta$ can be fixed by the requirement that $U$ and $\Psi$ obey additional equations. Theory of soliton equations provides us such equations. They are the famous, Korteweg-de Vries (KdV) equation and its higher partners. The KdV equation has the form

$$U_y + 6UU_x + U_{xxx} = 0$$

while $\Psi$ obeys the equation

$$\Psi_y + 4\Psi_{xxx} - 6U\Psi_x - 3U_x\Psi = 0.$$
8 Solutions of the generalized Raychaudhuri equation

A generalisation of Raychaudhuri equation in the case of the two-dimensional time-like surfaces embedded in a four-dimensional background has been derived in [5]. It has the form

$$\Delta \gamma + \frac{1}{2} \partial_a \gamma \partial^a \gamma + (M^2)^i_i = 0,$$

where $\Delta = \nabla^a \nabla_a$, $\nabla_a$ is the world-sheet covariant derivative and $\partial_a \gamma = \Theta_a$. The quantity $(M^2)^i_i$ is connected with the geometric characteristic of background and geometry of membrane embedded in this background.

In a simplest case this equation looks like [6]

$$-\frac{\partial^2 \Psi}{\partial \tau^2} + \frac{\partial^2 \Psi}{\partial \sigma^2} + \Omega^2(\sigma, \tau)(M^2)^i_i(\sigma, \tau)\Psi = 0$$

where $\Omega^2$ is the conformal factor of the induced metric written in isothermal coordinates.

It is is the second-order linear, hyperbolic, partial differential equation for $\Psi$ describing the deformation of the surfaces with respect to parameter $\tau$. It coincides with deviations equation in the case when $\Psi$ does not dependent on $\tau$.

We present this equation in form

$$- (\partial^2_\tau - \partial^2_\sigma)\Psi + U(\tau, \sigma)\Psi = E\Psi$$

(16)

where

$$U = \Omega^2(M^2)^i_i + E.$$

Particular solution of this equation has been obtained in [7] by separation of variables. A wide class of solutions of equation (16) can be constructed by the inverse spectral transform method. The simplest from them are of the form [18,19]. For $E \neq 0$

$$U(\tau, \sigma) = -2E + \frac{E(\alpha - \beta)}{\alpha \beta \cosh^2[(\alpha - \beta)(\tau + \sigma) + \frac{E(\tau - \sigma)}{\alpha \beta} + \gamma]/2},$$

$$\Psi(\tau, \sigma) = \frac{\cosh[(\alpha + \beta)(\sigma + \tau + \frac{E(\tau - \sigma)}{\alpha \beta} + \delta)]/2}{\cosh[(\alpha - \beta)(\sigma + \tau + \frac{E(\tau - \sigma)}{\alpha \beta} + \gamma)]/2}.$$
where $\alpha, \beta, \gamma, \delta$ are arbitrary parameters.

For the case $E = 0$ we have the solution

$$U(\tau, \sigma) = \frac{4\alpha\beta}{\cosh^2[\alpha(\tau + \sigma) + \beta(\tau - \sigma)]}$$

and

$$\Psi(\tau, \sigma) = A\frac{\cosh[\alpha(\tau + \sigma) - \beta(\tau - \sigma) + \delta]}{\cosh[\alpha(\tau + \sigma) + \beta(\tau - \sigma) + \gamma]}.$$

A very simple solution with $E = 0$ is of the form

$$U(\tau, \sigma) = \frac{6(\tau + \sigma - \alpha(\tau - \sigma))^2 - 2\beta^2}{(\tau + \sigma - \alpha(\tau - \sigma)^2 + \beta^2)^2}$$

and

$$\Psi(\tau, \sigma) = \frac{1}{(\tau + \sigma - \alpha(\tau - \sigma))^2 + \beta^2}$$

with arbitrary parameters $\alpha, \beta$. Such exact solutions of could be important for an analysis of dynamics of surfaces.

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