Constant-Factor Approximation Algorithms for Socially Fair $k$-Clustering

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Abstract

We study approximation algorithms for the socially fair $((\ell_p, k))$-clustering problem with $m$ groups, whose special cases include the socially fair $k$-median ($p = 1$) and socially fair $k$-means ($p = 2$) problems. We present (1) a polynomial-time $(5 + 2\sqrt{6})^p$-approximation with at most $k + m$ centers (2) a $(5 + 2\sqrt{6} + \epsilon)^p$-approximation with $k$ centers in time $n^{O(p)} m^2$, and (3) a $(15 + 6\sqrt{6})^p$ approximation with $k$ centers in time $k^m \cdot \text{poly}(n)$. The first result is obtained via a refinement of the iterative rounding method using a sequence of linear programs. The latter two results are obtained by converting a solution with up to $k + m$ centers to one with $k$ centers using sparsification methods for (2) and via an exhaustive search for (3). We also compare the performance of our algorithms with existing bicriteria algorithms as well as exactly $k$ center approximation algorithms on benchmark datasets, and find that our algorithms also outperform existing methods in practice.

1 Introduction

Automated decision making using machine learning algorithms is being adopted in many aspects of society. The examples are innumerable and include applications with substantial societal effects such as automated content moderation and recidivism prediction. This necessitates designing new machine learning algorithms that incorporate societal considerations, especially fairness.

The facility location problem is a basic and well-studied problem in combinatorial optimization. Famous instances include the $k$-means, $k$-median and $k$-center clustering problems, where the input is a finite metric and the goal is to find $k$ points ("centers" or "facilities") such that a function of the distances of each given point to its nearest center is minimized. For $k$-means, the objective is the average squared distance to the nearest center; for $k$-median, it is the average distance; and for $k$-center, it is the maximum distance. These problems can all be captured by the $((\ell_p, k))$-clustering problem, defined as follows: given a set of clients $A$ of size $n$, a set of candidate facility locations $F$, and a metric $d$, find a subset $F \subset F$ of size $k$ that minimizes $\sum_{i \in A} d(i, F)^p$, where $d(i, F) = \min_{j \in F} d(i, j)$. This problem is NP-hard for any $p$, and also hard to approximate. A $2^{O(p)}$-approximation algorithm was given by [13].

Here we consider socially fair extensions of the $((\ell_p, k))$-clustering problem in which $m$ different (not necessarily disjoint) subgroups, $A = A_1 \cup \cdots \cup A_m$, among the data are given, and the goal is to minimize the maximum cost over the groups, so that a common solution is not too expensive for any one of them. Each group can be a subset of the data or simply any nonnegative weighting. The goal

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1 In some other works, the $p$'th root of the objective is considered and therefore the approximation factors look different in such works.
is to minimize the maximum weighted cost among the groups, i.e.,
\[
\min_{F \subseteq \mathcal{F} : |F| = k} \max_{s \in [m]} \sum_{i \in A_s} w_s(i) d(i, F)^p.
\] (1)

A weighting of \(w_s(i) = \frac{1}{|A_s|}\), for all \(i \in A_s\), corresponds to the average for each group. The groups usually arise from sensitive attributes such as race and gender (that are protected against discrimination under the Civil Rights Act of 1968 \[26\] \[8\]). The cases of \(p = 1\) and \(p = 2\) are the socially fair \(k\)-median and \(k\)-means, respectively, introduced in \[18\] \[1\]. As discussed in \[18\], the objective of the socially fair \(k\)-means promotes a more equitable average clustering cost among different groups.

The objective function of socially fair \(k\)-median problem was first studied by \[6\] who gave an \(O(\log m + \log n)\)-approximation algorithm. Moreover, the existing constant-factor approximation algorithms for the vanilla \(k\)-means and \(k\)-median problems can be used to find \(O(m)\)-approximate solutions for the socially fair \(k\)-means and \(k\)-median problems \[18\] \[1\]. The proof technique also directly yields a \(m \cdot 2^{O(p)}\)-approximation algorithm for the socially fair \((\ell_p, k)\)-clustering problem. The natural LP relaxation of the socially fair \(k\)-median problem has an integrality gap of \(\Omega(m)\) \[1\].

More recently, Makarychev and Vakilian strengthened the LP relaxation of the socially fair \((\ell_p, k)\)-clustering problem by a sparsification technique \[39\]. Their stronger LP has an integrality gap of \(\Omega(\log m \log \log m)\) and their rounding algorithm (similar to that of \[13\]) finds a \(2^{O(p) \log \log m}\)-approximation algorithm for the socially fair \((\ell_p, k)\)-clustering problem. For the socially fair \(k\)-median problem, this is asymptotically the best possible in polynomial time under the assumption \(\text{NP} \not\subseteq \bigcap_{\epsilon > 0} \text{DTIME}(2^{n^{o(\epsilon)}})\) \[10\]. This hardness result is for the \(k\)-clustering problem, where the algorithm must produce a solution with at most \(k\) centers. It is natural to consider a bicriteria approximation, which allows for more centers whose total cost is close to the optimal cost for \(k\) centers. For the socially fair \(k\)-median and \(0 < \epsilon < 1\), \[1\] presents an algorithm that gives at most \(k/(1 - \epsilon)\) centers with objective value at most \(2^{O(p)/\epsilon}\) times the optimum for \(k\) centers.

Our first result is an improved bicriteria approximation algorithm for the socially fair \(\ell_p\) clustering problem with only \(m\) additional centers (\(m\) is usually a small constant).

**Theorem 1.1.** For any \(\epsilon > 0\), there is a polynomial-time bicriteria approximation algorithm for the socially fair \((\ell_p, k)\)-clustering problem with \(m\) groups that finds a solution with at most \(k + m\) centers of cost at most \((5 + 2\sqrt{6})^p \approx 9.9^p\) times the optimal cost for a solution with \(k\) centers.

Goyal and Jaiswal \[20\] show that a solution to the socially fair \((\ell_p, k)\)-clustering problem with \(k' > k\) centers and cost \(C\) can be converted to a solution with \(k\) centers and cost at most \(3^{p-1}(C + 20\epsilon t)\) by simply taking the \(k\)-subset of the \(k'\) centers of lowest cost. A proof of this is included in the appendix for completeness. We improve this factor using a sparsification technique.

**Theorem 1.2.** For any \(\epsilon > 0\), there is a \((5 + 2\sqrt{6} + \epsilon)^p\)-approximation algorithm for the socially fair \((\ell_p, k)\)-clustering problem that runs in time \(n^{2O(p/m^2)}\); there is a \((15 + 6\sqrt{6})^p\)-approximation algorithm that runs in time \(k^{m \cdot \text{poly}(n)}\).

This raises the question of whether a faster constant-factor approximation is possible. Goyal and Jaiswal \[20\] show that assuming Gap-Exponential Time Hypothesis (Gap-ETH\(^2\)), it hard to approximate socially fair \(k\)-median and \(k\)-means within factors of \(1 + 2/e - \epsilon\) and \(1 + 8/e - \epsilon\) respectively, in time \(g(k) \cdot n^{f(m)} \cdot o(k)\), for \(g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}\) and \(f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}\): socially fair \((\ell_p, k)\)-clustering is hard to approximate within a factor of \(3^p - \epsilon\) in time \(g(k) \cdot n^{o(k)}\). They also give a \((3 + \epsilon)^p\)-approximation with time complexity \((k^2)^{O(k)} \cdot \text{poly}(\frac{n}{k})\). This leaves open the possibility of a constant-factor approximation algorithm that runs in time \(f(m) \cdot \text{poly}(n, k)\).

For the case of \(p \to \infty\), the problem reduces to fair \(k\)-center problem if we take \(p\)th root of the objective. The problem is much better understood and widely studied along with many generalization \[27\] \[4\] \[39\]. Makarychev et al.’s result \[39\] implies an \(O(1)\)-approximation in this case.

We compare the performance of our bicriteria algorithm against \[1\] and our algorithm with exactly \(k\) centers against \[39\] on three different benchmark datasets that are widely used in fair clustering.

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\(^2\)Informally Gap-ETH states that there is no \(2^{o(n)}\)-time algorithm to distinguish between a satisfiable formula and a formula that is not even \((1 - \epsilon)\) satisfiable.
literature. Our experiments show that our algorithms consistently outperform these in practice (see Section 3 and the number of centers that our algorithm selects is often less than that of selected by algorithm of [31] (see Section 3.2).

1.1 Approach and Techniques

Our starting point is a linear programming (LP) relaxation of the problem. The integrality gap of the natural LP relaxation of the problem is $m$ [1]. For our bicriteria approximation, we use an iterative rounding procedure, inspired by [13]. In each iteration, we solve an LP whose constraints change from one iteration to the next. We show that the feasible region of the final LP is the intersection of a matroid polytope and $m$ affine spaces. This implies that the size of the support of an optimal extreme solution of this LP is at most $k + m$ — see Lemma 1.3. Rounding up all of these fractional variables results in a solution with $k + m$ centers.

There are two approaches to convert a solution with up to $k + m$ centers to a solution with $k$ centers. The first is to take the best $k$-subset of the $k + m$ centers which results in a $(15 + 6\sqrt{6})\alpha$-approximation for an additional cost of $O(k^m n(k + m))$ in the running time. This follows from the work of [20]. For completeness, we include it as Lemma A.1 in the Appendix.

The second approach is to “sparsify” the given instance of the problem. We show if the instance is “sparse,” then the integrality gap of the LP is small. A similar idea was used by [36] for the classic work of [20]. For completeness, we include it as Lemma A.1 in the Appendix.

For an $O(\text{opt}/m)$-sparse instance of the socially fair $k$-median problem, a solution with $k + m$ centers can be converted to a solution with $k$ centers in time $n^{O(m^2)}$ while increasing the objective value only by a constant factor. Our conversion algorithm is based on the fact that there are at most $O(m^2)$ facilities that are far from the facilities in the optimal solution. We enumerate candidates for these facilities and then solve an optimization problem for the facilities that are close to the facilities in the optimal solution. This optimization step is again over the intersection of the polytope of a matroid with $m$ half-spaces. In summary, our algorithm consists of three main steps.

1. We produce $n^{O(m^2)}$ instances of the problem such that at least one is $O(\text{opt}/m)$-sparse and its optimal objective value is equal to that of the original instance (Section 3 Lemma 3.2).
2. For each of the instances produced in the previous step, we find a pseudo-solution with at most $k + m$ centers by an iterative rounding procedure (Section 2 Lemma 2.1).
3. We convert each pseudo-solution with $k + m$ centers to a solution with $k$ centers (Section 4 Lemma 4.2) and return the solution with the minimum cost.

1.2 Preliminaries

We use terms centers and facilities interchangeably. For sets $S_1, \ldots, S_k$, we denote their Cartesian product by $\bigotimes_{j \in [k]} S_j$, i.e., $(s_1, \ldots, s_k) \in \bigotimes_{j \in [k]} S_j$ if and only if $s_1 \in S_1, \ldots, s_k \in S_k$. For an instance $\mathcal{I}$ of the problem, we denote an optimal solution of $\mathcal{I}$ and its objective value by $\text{OPT}_\mathcal{I}$ and $\text{opt}_\mathcal{I}$, respectively.

A pair $\mathcal{M} = (E, \mathcal{I})$, where $\mathcal{I}$ is a non-empty family of subsets of $E$, is a matroid if: 1) for any $S \subseteq T \subseteq E$, if $T \in \mathcal{I}$ then $S \in \mathcal{I}$ (hereditary property); and 2) for any $S, T \in \mathcal{I}$, if $|S| < |T|$, then there exists $i \in T \setminus S$ such that $S + i \in \mathcal{I}$ (exchange property). We call $\mathcal{I}$ the set of independent sets of the matroid $\mathcal{M}$. The basis of $\mathcal{M}$ are all the independent sets of $\mathcal{M}$ of maximal size. We use the following lemma in the analysis of both our bicriteria algorithm and the algorithm with exactly $k$ centers.
Lemma 1.3. Let \( \mathcal{M} = (E, \mathcal{I}) \) be a matroid with rank \( k \) and \( P(\mathcal{M}) \) denote the convex hull of all basis of \( \mathcal{M} \). Let \( Q \) be the intersection of \( P(\mathcal{M}) \) with \( m \) additional affine constraints. Then any extreme point of \( Q \) has a support of size at most \( k + m \).

Related work. Unsupervised learning under fairness constraints has received significant attention over the past decade. Social fairness (i.e., equitable cost for different demographic groups) has been considered for problems such as PCA [41, 43]. There are other notions of fairness that has been considered for clustering problem. The most notable ones are balance in clusters (i.e., equitable representation of demographic groups in clusters) [15, 2, 9, 3], balance in representation (i.e., equitable representation of demographic groups in selected centers) [25, 33, 30], and individual fairness [31, 44, 12]. However as been discussed in the literature [24], different notions of fairness are incompatible with each other in the sense that they cannot be satisfied simultaneously. For example, see the discussion and experimental result regarding incompatibility of social fairness and equitable representation in [18]. In addition to these, several other notions of fairness for clustering has been considered in the literature [14, 28, 38, 40, 11].

2 Bicriteria Approximation

In this section, we prove Theorem 1.1. Our method relies on solving a series of linear programs and utilizing the iterative rounding framework for the \( k \)-median problem as developed in [34, 23]. We aim for a cleaner exposition over smaller constants below. We use the following standard linear programming (LP) relaxation (LP1).

\[
\begin{align*}
\min & \quad z \\
\text{s.t.} & \quad z \geq \sum_{i \in A, j \in \mathcal{F}} w_s(i) d(i, j)^p x_{ij}, & (LP1) \\
& \quad x_{ij} \leq y_j, \quad \forall i \in A, j \in \mathcal{F}, & (1) \\
& \quad \sum_{j \in \mathcal{F}} y_j = k, & (2) \\
& \quad \sum_{j \in \mathcal{F}} x_{ij} = 1, \quad \forall i \in A, & (3) \\
& \quad x, y \geq 0. & (4)
\end{align*}
\]

Theorem 1.1 follows as a corollary to Lemma 2.1 as we can pick all the fractional centers integrally. Observe that, once the centers have been fixed, the optimal allocation of clients to facilities is straightforward: every client connects to the nearest opened facility.

Lemma 2.1. Let \( 0 < \lambda \leq 1 \). There is a polynomial time algorithm that given a feasible solution \((\tilde{x}, \tilde{y}, \tilde{z})\) to the linear program LP1 returns a feasible solution \((x', y', z')\) where \( z' \leq \left( \left( 1 + \frac{2(1 + \lambda)}{\lambda} \right) (1 + \lambda) \right)^p \tilde{z} \) and \( y' \) has support \( k + m \) fractional variables. The running time of the algorithm is polynomial in \( n \) and the logarithm of the diameter of the metric divided by \( \lambda \).

Proof. We describe the iterative rounding argument to round the solution \((\tilde{x}, \tilde{y}, \tilde{z})\). As a first step, we work with an equivalent linear program LP2, where we have removed the assignment variables \( x \).

This can be achieved by splitting each facility \( j \) to the number of unique nonzero \( \tilde{x}_{ij} \)'s and setting the corresponding variable for these facilities accordingly, e.g., if the unique weights are \( \tilde{x}_{1j} < \tilde{x}_{2j} < \tilde{x}_{3j} \), then the corresponding weights for the facilities are \( \tilde{x}_{1j}, \tilde{x}_{2j} - \tilde{x}_{1j}, \) and \( \tilde{x}_{3j} - \tilde{x}_{2j} \) and the weights of the connections between these new facilities and clients are determined accordingly as either zero or the weight of the facility.

Let \( \mathcal{F} \) be the set of all (splitted) copies of facilities. Then we can assume \( \tilde{x}_{ij} \in \{0, \tilde{y}_j\} \) for each \( i, j \) (where \( j \in \mathcal{F} \)). We set \( F_i = \{ j \in \mathcal{F} : \tilde{x}_{ij} > 0 \} \). Note that \( F_i \) could contain multiple copies of original facilities. Observe that LP2 has a feasible solution \((\tilde{y}, \tilde{z})\) for this choice of \( F_i \) for each \( i \in A \). Moreover, any feasible solution to LP2 can be converted to a solution of LP1 of same cost while ensuring that each client \( i \) gets connected to the original copy of the facilities in \( F_i \).

The iterative argument is based on the following principle. We group nearby clients and pick only one representative for each group such that if each client is served by the facility that serves its representative, the cost is at most \( \left( \left( 1 + \frac{2(1 + \lambda)}{\lambda} \right) (1 + \lambda) \right)^p \tilde{z} \). Moreover, we ensure that candidate
facilities \( F_i \) for representative clients are disjoint. In this case, one observes that the constraints (5)-(4) in LP2 define the convex hull of a partition matroid and must be integral. Indeed, this already gives an integral solution to the basic \( k \)-median problem. But, in the socially fair clustering problem, there are \( m \) additional constraints, one for each of the \( m \) groups. Nevertheless, by Lemma [13] any extreme point solution to the matroid polytope intersected with at most \( m \) linear constraints has a support of size at most \( k + m \) (see also [35] Chap. 11).

We now formalize the argument and specify how one iteratively groups the clients. We need to iteratively remove/change constraints in LP2 as we do the grouping while ensuring that the cost of the linear program does not increase. We initialize \( D_i = \max\{d(i,j) : \hat{x}_{ij} > 0\} = \max\{d(i,j) : j \in F_i\} \) for each client \( i \). We maintain a set of representative clients \( U^* \). We say a client \( i \in U^* \) represents a client \( i' \) if they share a facility, i.e., \( F_i \cap F_{i'} \neq \emptyset \), and \( D_i \leq D_{i'} \). The representative clients do not share any facility with each other. The non-representative clients are put in the set \( \tilde{U} \).

We initialize \( U^* \) as follows. Consider all clients in increasing order of \( D_i \). Greedily add clients to \( U^* \) while maintaining \( F_i \) and \( F_{i'} \) are disjoint for any \( i, i' \in U^* \). Observe that \( U^* \) is maximal with above property, i.e., for every \( i' \not\in U^* \), there is \( i \in U^* \) such that \( F_i \cap F_{i'} \neq \emptyset \) and \( D_i \leq D_{i'} \). We will maintain this invariant in the algorithm. For clients \( i \in \tilde{U} \), we set \( B_i \) to be the facilities in \( F_i \) that are within a distance of \( \frac{D_i}{\lambda} \). In each iteration we solve the following linear program and update \( U^* \), \( \tilde{U} \), \( D_i \)'s, \( B_i \)'s, and \( F_i \)'s.

\[
\begin{align*}
\min z \\
\text{s.t. } & \quad z \geq \sum_{i \in A, i' \in \tilde{U}} w_i(s) \sum_{j \in F_i} d(i,j)^p y_j \\
& \quad + \sum_{i \in A, i' \in \tilde{U}} w_i(s) \left( \sum_{j \in B_i} d(i,j)^p y_j + (1 - y(B_i)) D_i^p \right), \forall 1 \leq s \leq m, \quad (5) \\
& \quad \sum_{j \in F_i} y_j = k, \quad (6) \\
& \quad \sum_{j \in B_i} y_j = 1, \quad \forall i \in \tilde{U}, \quad (7) \\
& \quad \sum_{j \in B_i} y_j \leq 1, \quad \forall i \in \tilde{U}, \quad (8) \\
& \quad y \geq 0. \quad (9)
\end{align*}
\]

For clients in \( i \in \tilde{U} \), we only insist that we pick at most one facility from \( B_i \) (see constraint (5)). The objective is modified to pay \( D_i \) for any fractional shortfall of facilities in this smaller ball (see constraint (5)). Observe that if this additional constraint (5) becomes tight for some \( j \in \tilde{U} \), we can decrease \( D_i \) by a factor of \( (1 + \lambda) \) for this client and then update \( U^* \) accordingly to see if \( i \) can be included in it. Also, we round each \( d(i,j) \) to the nearest power of \( (1 + \lambda) \). This only changes the objective by a factor of \( (1 + \lambda)^p \) and we abuse notation to assume that \( d \) satisfies this constraint (it might no longer be a metric but in the final assignment, we will work with its metric completion). The iterative algorithm will run as follows.

It is possible that a client moves between \( \tilde{U} \) and \( U^* \) but any time that a point is processed in \( \tilde{U} \) (Step 3(a) above), \( D_i \) is divided by \( (1 + \lambda) \). Therefore, the above algorithm takes at most \( O(n \log \frac{\text{diam}}{\lambda}) \) iterations, where \( \text{diam} \) is the distance between the two farthest points. Finally the result is implied by the following claims.

**Claim 2.2.** The cost of the LP is non-increasing over iterations. Moreover, when the algorithm ends, there are at most \( k + m \) facilities in the support.

**Claim 2.3.** For any client \( i' \in \tilde{U} \), there is always one total facility at a distance of at most \( \left(1 + \frac{2(1+\lambda)}{\lambda}\right) D_{i'} \), i.e., \( \sum_{j: d(i',j) \leq (1+2(1+\lambda)/\lambda) D_{i'}} y_j \geq 1 \).

**Claim 2.4.** Let \( \hat{y} \) be an integral solution to linear program LP(\( U^*, \tilde{U}, D \)) after the last iteration. Then, there is a solution \( (\hat{x}, \hat{y}) \) to the linear program LP1 such that objective is at most \( \left(1 + \frac{2(1+\lambda)}{\lambda}\right) (1+\lambda)^p \) times the objective of the linear program LP(\( U^*, \tilde{U}, D \)).

We prove the above claims in the following to finish the proof of Lemma 2.1.
Algorithm 1: Iterative Rounding

1. **Input:** $A = A_1 \cup \cdots \cup A_m, \mathcal{F}, d, \lambda$
2. **Output:** A set of centers of size at most $k + m$.
3. Solve LP1 to get optimal solution $(x^{*}, y^{*}, z^{*})$ and reate set $\mathcal{F}$ by splitting facilities.
4. Set $F_{i} = \{ j \in \mathcal{F} : x_{ij}^{*} > 0 \}, D_{i} = \max \{ d_{ij} : x_{ij}^{*} > 0 \}$ for each $i \in U$.
5. Sort clients in increasing order of $D_{i}$ and greedily include clients in $U' \subset U$ while maintaining that
   \[ \{ F_{i} : i \in U' \} \text{ remain disjoint.} \]
6. Set $U' = A \setminus U^*.$
7. **while** there is some tight constraint from (8) **do**
8. \quad **if** there exists $i \in U' \setminus D_{i}$ such that $y(B_{i}) = 1 \text{ (i.e., (8) is tight for i)}$ **then**
9. \quad \quad $F_{i} \leftarrow B_{i}, D_{i} \leftarrow D_{i} + \frac{d_{ij}}{1 + \lambda}, B_{i} \leftarrow \{ j \in F_{i} : d(i, j) \leq \frac{D_{i}}{1 + \lambda} \}, \text{ and Update-}\nu^{*}(i).$
10. **Find** an extreme point solution $y$ to the linear program $LP(U^*, U', D).$
11. return the support of $y$ in the solution of $LP(U^*, U', D)$.

**Procedure Update-$U^*(i)$**

1. if for every $i' \in U^*$ that $F_{i} \cap F_{i'} \neq \emptyset, D_{i'} > D_{i}$ then
2. \quad Remove all $i'$ that represent $i$ from $U^*$ and add them to $U'$.
3. $U^* \leftarrow U^* \cup \{ i \}.$

Figure 1: (a) Distance of $i'$ from the facilities of its representative $i$. (b) Solid circles are the balls corresponding to representative clients ($U^*$) and dashed circles are the balls corresponding to non-representative clients ($U'$).

**Proof of Claim 2.2** In each iteration, we put at most one client in $U^*$. For this client, (8) is tight, i.e., $\sum_{j \in B_{i}} y_{j} = 1$. Note that we update $F_{i}$ to $B_{i}$. Therefore the new point in $U^*$ satisfies (7). Moreover, for a point $i'$ that is removed from $U^*$, we have

\[ \sum_{j \in B_{i'}} y_{j} \leq \sum_{j \in F_{i'}} y_{j} = 1. \]

Therefore such a point satisfies (8). Hence a feasible solution to the LP of iteration $t$ is also feasible for iteration $t + 1$. Therefore the cost of the LP is non-increasing over iterations.

The second statement follows since if no constraint from (8) is tight, then the linear program is the intersection of a matroid polytope with $m$ linear constraints and the result follows from Lemma 1.3.

**Proof of Claim 2.3** Let $t$ be the iteration where $D_{i'}$ is updated for the last time. If $D_{i'}$ is only set once at Line 4 of Algorithm 1 and it is never updated, then $t = 0.$ We first show that immediately after iteration $t$, there is one total facility at a distance of at most $3D_{i'}$ from $i'$. If $t = 0$, then there existed $i \in U^*$ such that $F_{i} \cap F_{i'} \neq \emptyset$ and $D_{i} \leq D_{i'}$. Therefore by triangle inequality, all the facilities in $F_{i}$ are within a distance of at most $3D_{i'}$ from $i'$, see Figure 1 (a). Hence because (7) enforces one total facility in $F_{i}$, there exists one total facility at a distance of at most $3D_{i'}$ from $i'$. If $t > 0$, then $i'$ is moved from $U^*$ to $U'$ because at iteration $t$, a facility $i'$ is added to $U^*$ such that $D_{i} < D_{i'}$ and $F_{i} \cap F_{i'} \neq \emptyset$ — see the condition of Procedure Update-$U^*(i)$ in Algorithm 1. Again because of enforcement of (7) and triangle inequality, there exists one total facility at a distance of at most $3D_{i'}$ from $i'$ immediately after iteration $t$.

Now note that after iteration $t$, the facility $i \in U^*$ with $F_{i} \cap F_{i'} \neq \emptyset$ and $D_{i} \leq D_{i'}$ might get removed from $U^*$. In which case, we do not have the guarantee of (7) any longer. Let $i_{0} := t$. We define $i_{t+1}$
to be the client that has caused the removal of client \( i_p \) (through Procedure \( \text{Update-} \mathcal{U}^*(i) \)) from \( \mathcal{U}^* \) after iteration \( t \). Note that by the condition of \( \text{Update-} \mathcal{U}^*(i) \) from \( \mathcal{U}^*, D_{t_p+1} < D_{t_p} \). Therefore because we have rounded the distances to multiples of \((1 + \lambda)\), we have \( D_{t_p+1} \leq \frac{D_{t_p}}{1 + \lambda} \). Let \( i_r \) be the last point in this chain, i.e., \( i_r \) has caused the removal of \( i_{r-1} \) and \( i_r \) has stayed in \( \mathcal{U}^* \) until termination of the algorithm. Then by guarantee of (7) and triangle inequality, there is one total facility for \( i' \) within a distance of

\[
D_{i'} + \sum_{j=0}^{r} 2D_{ij} \leq D_{i'} + 2\sum_{j=0}^{r} \frac{D_{i'}(1 + \lambda)^j}{(1 + \lambda)^j} \leq \left( 1 + \frac{2(1 + \lambda)}{\lambda} \right) D_{i'}.
\]

\( \square \)

**Proof of Claim 2.4** By Claim 2.2, at every iteration, the cost of the linear program only decreases since a feasible solution to previous iteration remains feasible for the next iteration. Thus the objective value of \( \hat{y} \) in LP(\( \mathcal{U}^*, \mathcal{U}^t, D \)) is at most \((1 + \lambda)^p \) the optimal cost of LP1 (where we lost the factor of \((1 + \lambda)^p \) by rounding all distances to powers of \((1 + \lambda)\)).

We now construct \( \bar{x} \) such that \((\bar{x}, \hat{y})\) is feasible to LP1. First note that the above procedure always terminates. We construct \( \bar{x} \) by processing clients one by one. We process the clients in \( \mathcal{U}^t \) and \( \mathcal{U}^* \) as follows. For any \( i \in \mathcal{U}^* \), we define \( \hat{x}_{ij} = \hat{y}_j \) for each \( j \in F_i \). Observe that we have \( \sum_{j \in F_i} x_{ij} = 1 \) for such \( i \in \mathcal{U}^* \) and we obtain feasibility for this client. For any \( i \in \mathcal{U}^t \), we define \( \hat{x}_{ij} = \hat{y}_j \) for each \( j \in B_i \). Observe that we only insisted \( \sum_{j \in B_i} \hat{y}_j \leq 1 \) and therefore we still need to find \( 1 - \sum_{j \in B_i} \hat{x}_{ij} = 1 - \sum_{j \in B_i} \hat{y}_j \) facilities to assign to client \( i \). For this remaining amount \( 1 - \hat{y}(B_i) \), we notice by Claim 2.3, there is at least one facility within distance \( \left( 1 + \frac{2(1 + \lambda)}{\lambda} \right) D_i \) of this client. Thus we can assign the remaining \( 1 - \hat{y}(B_i) \) facility to client \( i \) at a distance of no more than \( \left( 1 + \frac{2(1 + \lambda)}{\lambda} \right) D_i \). Note that the cost is only increased by a factor of \( \left( 1 + \frac{2(1 + \lambda)}{\lambda} \right)^p \).

\( \square \)

Now we prove Theorem 1.1 by substituting the best \( \lambda \) in Lemma 2.1.

**Proof of Theorem 1.1** By Lemma 2.1, the output vector of Algorithm 1 corresponding to the centers has a support of size at most \( k + m \). Rounding up all the fractional centers, we get a solution with at most \( k + m \) centers and a cost of at most \( \left( 1 + \frac{2(1 + \lambda)}{\lambda} \right)^p \) of the optimal. We optimize over \( \lambda \) by taking the gradient of \( \left( 1 + \frac{2(1 + \lambda)}{\lambda} \right)^p \) \((1 + \lambda)\) and setting it to zero. This gives the optimum value of \( \lambda = \sqrt{\frac{2}{\sqrt{2} - 1}} \). Substituting \( \lambda \) in the approximation factor, gives a total approximation factor of \((5 + 2\sqrt{6})^p \).

\( \square \)

### 3 Approximation Algorithms for Fair \( k \)-Clustering

Here we first show how to generate a set of instances such that at least one of them is sparse and has the same optimal objective value as the original instance. Then we present our algorithm to find a solution with \( k \) facilities from a pseudo-solution with \( k + m \) facilities for a sparse instance, inspired by [36]. We need to address some new difficulties: the sparsity with respect to all groups \( s \in [m] \); and as our pseudo-solution has \( m \) additional centers (instead of \( O(1) \) additional centers), we need a sparser instance compared to [36]. One new technique is solving the optimization problem given in Step 11 of Algorithm 3 (see Lemma 4.5). This is trivial for the vanilla \( k \)-median but in the fair setting, we use results about the number of fractional variables in extreme points of intersection of a matroid polytope with half-spaces, and combine this with a careful enumeration.

For an instance \( \mathcal{I} \), we denote the cost of a set of facilities \( F \) by \( \text{cost}_\mathcal{I}(F) \). For a point \( q \) and \( r > 0 \), we denote the facilities in the ball of radius \( r \) at \( q \) by \( \text{Ball}_\mathcal{I}(q, r) \). This does not contain facilities at distance exactly \( r \) from \( q \). For a group \( s \in [m] \), the set of clients of \( s \) in the ball of radius \( r \) at \( q \) is \( \text{Ball}_\mathcal{I},s(q, r) \). Note that because we consider the clients as weights on points, \( \text{Ball}_\mathcal{I},s(q, r) \) is actually a set of (point, weight) pairs. We let \( |\text{Ball}_\mathcal{I},s(q, r)| = \sum_{i \in \text{Ball}_\mathcal{I},s(q, r)} w_s(i) \).
Algorithm 2: Sparsify

1. Input: $A = A_1 \cup \ldots \cup A_m$, $F, k, d, t \in \mathbb{N}$
2. Output: A set of fair $k$-median instances.
3. for $t' = 1, \ldots, m^2 t$ and $t'$ facility pairs $(j_1, j_1'), \ldots, (j_t', j_t')$
   do
   4. Output $I' = (F', A, k, d)$, where $F' = F \setminus \bigcup_{j=1}^{t'} \text{FBALL}(j, d(j, j'))$.

Definition 3.1. [Sparse Instance] For $\alpha > 0$, an instance of the fair $\ell_p$ clustering problem $I = (k, F, A, d)$ is $\alpha$-sparse if for each facility $j \in F$ and group $s \in [m]$, 

$$\left(\frac{\alpha}{3} d(j, \text{OPT}_I)\right)^p \cdot |\text{FBALL}_{I,s}(j, \frac{1}{2} d(j, \text{OPT}_I))| \leq \alpha.$$ 

We say that a facility $j$ is $\alpha$-dense if it violates the above for some group $s \in [m]$.

To motivate the definition, let $I$ be an $\alpha$-sparse instance, $\text{OPT}_I$ be an optimal solution of $I$, $j$ be a facility not in $\text{OPT}_I$ and $j^*$ be the closest facility in $\text{OPT}_I$ to $j$. Let $F$ be a solution that contains $j$ and $\eta_{j,s}$ be the total cost of the group $s \in [m]$ that are connected to $j$ in solution $F$. Then, 

$$(\text{cost of group } s \text{ solution } F \cup j \setminus j^*) \leq (\text{cost of group } s \text{ solution } F) + 2^{O(\rho)} \cdot (\alpha + \eta_{j,s}).$$

This property implies that if $\alpha \leq \text{opt}_I/m$, then replacing $m$ different facilities can increase the cost by a factor of $2^{O(\rho)}$ plus $2^{O(\rho)} \cdot \text{opt}_I$, and the integrality gap of the LP relaxation is $2^{O(\rho)}$. The next algorithm generates a set of instances such that at least one of them has objective value equal to $\text{opt}_I$ and is $(\text{opt}_I/mt)$-sparse for a fixed integer $t$.

Lemma 3.2. Algorithm 2 runs in $n^{O(m^2 t)}$ time and produces instances of the socially fair $\ell_p$ clustering problem such that at least one of them satisfies the following: (1) The optimal value of the original instance $I$ is equal to the optimal value of the produced instance $I'$; (2) $I'$ is $\text{opt}_I/mt$-sparse.

Proof: First note that a facility $i$ in $\text{OPT}_I$ cannot be $\alpha$-dense because $d(i, \text{OPT}_I) = 0$. Let $(j_1, j_1'), \ldots, (j_t, j_t')$ be a sequence of pairs of facilities such that for every $b = 1, \ldots, \ell,$

- $j_b \in F \setminus \bigcup_{z=1}^{b-1} \text{FBALL}_{I,s}(j_z, d(j_z, j'_z))$ is an $\text{opt}_I/m$-dense facility; and
- $j_b'$ is the closest facility to $j_b$ in $\text{OPT}_I$.

We show that $\ell \leq m^2 t$. For $b \in [\ell]$ and $s \in [m]$, let $B_{b,s} := \text{FBALL}_{I,s}(j_b, \frac{1}{2} d(j_b, j_b'))$. First we show that for any group $s \in [m]$, the client balls $B_{1,s}, \ldots, B_{\ell,s}$ are disjoint. Let $1 \leq z < w \leq \ell$. By triangle inequality $d(j_w, j_w') \leq d(j_w, j_z) + d(j_z, j_w')$. Moreover by definition $j_w \notin \text{FBALL}_{I}(j_z, d(j_z, j_w'))$. Thus $d(j_w, j_w') \leq 2 d(j_w, j_z)$. Since $j_w'$ is the closest facility to $j_w$ in $\text{OPT}_I$, $d(j_w, j_w') \leq d(j_w, j_z)$. Therefore $d(j_w, j_w') \leq 2d(j_w, j_z)$. Combining this with $d(j_z, j_z') \leq d(j_w, j_z)$ implies $\frac{1}{2} (d(j_z, j_z') + d(j_w, j_w')) \leq d(j_z, j_w)$. If $B_{s,w}$ and $B_{s,w}$ overlap then there exists $u \in B_{s,z} \cap B_{s,w}$, and by triangle inequality, $d(j_z, j_w) \leq d(j_z, u) + d(j_w, u) < \frac{1}{2} d(j_z, j_z') + \frac{1}{2} d(j_w, j_w')$, which is a contradiction.

Therefore for $s \in [m]$, $B_{1,j_1}, \ldots, B_{\ell,j}$ are disjoint. Also since $A_1, \ldots, A_s$ are disjoint, all of $B_{b,s}$’s are disjoint for $b \in [\ell]$ and $s \in [m]$. By definition, for any $b \in [\ell]$, there exists $s_b \in [m]$ such that $(\frac{\alpha}{3} d(j_b, \text{OPT}_I))^p |B_{b,s_b}| > \text{opt}_I/m$. Therefore, if $\ell > m^2 t$, $\sum_{b=1}^{\ell} (\frac{\alpha}{3} d(j_b, \text{OPT}_I))^p |B_{b,s_b}| > m \text{opt}_I$. Thus 

$$m \cdot \max_{s \in [m]} \sum_{b=1}^{\ell} \left(\frac{\alpha}{3} d(j_b, \text{OPT}_I)\right)^p |B_{b,s}| \geq \sum_{s \in [m]} \sum_{b=1}^{\ell} \left(\frac{\alpha}{3} d(j_b, \text{OPT}_I)\right)^p |B_{b,s}| > m \text{opt}_I.$$ 

Note that the connection cost of a client in $B_{b,s}$ in the optimal solution is at least $(\frac{\alpha}{3} d(j_b, \text{OPT}_I))^p = (\frac{\alpha}{3} d(j_b, j_b'))^p$. Therefore, as the $B_{b,s}$’s are disjoint, $\text{opt}_I \geq \max_{s \in [m]} \sum_{b=1}^{\ell} (\frac{\alpha}{3} d(j_b, \text{OPT}_I))^p |B_{b,s}|$. This is a contradiction. Therefore $\ell \leq m^2 t$. Thus Algorithm 2 returns an instance with the desired properties. \qed
4 Converting a Solution with \( k + m \) centers to a Solution with \( k \) centers

We first analyze the special case when the set of facilities is partitioned to \( k \) disjoint sets and we are constrained to pick exactly one from each set. This will be a subroutine in our algorithm.

**Lemma 4.1.** Let \( S_1, \ldots, S_k \) be disjoint sets such that \( S_1 \cup \cdots \cup S_k = \{n\} \). For \( g \in [m] \), \( v \in S_j \), let \( \alpha^{(g,j)} \geq 0 \). Then there is an \((nk)^{O(n^2/\epsilon)}\)-time algorithm that finds a \((1+\epsilon)\)-approximate solution to

\[
\min_{v_i \in S_i : i \in [k]} \max_{g \in [m]} \sum_{j \in [k]} \alpha^{(g,j)}.
\]

**Proof.** The LP relaxation of the above problem is

\[
\begin{align*}
\min \theta \\
\text{s.t.} \quad & \sum_{j \in [k]} \sum_{v \in S_j} \alpha^{(g,j)} x^{(j)}_{v} \leq \theta, \quad \forall g \in [m], \\
& \sum_{v \in S_j} x^{(j)}_{v} = 1, \quad \forall j \in [k], \\
& x^{(j)} \geq 0, \quad \forall j \in [k].
\end{align*}
\]

Note that this is equivalent to optimizing over a partition matroid with \( m \) extra linear constraints. Therefore by Lemma 3.3, an extreme point solution has a support of size at most \( k + m \). Now suppose \( \theta^* \) is the optimal integral objective value, and \( v^{*}_{j} \in S_1, \ldots, v^{*}_{k} \in S_k \) are the points that achieve this optimal objective. For each \( g \in [m] \), at most \( m/\epsilon \) many \( \alpha^{(g,j)} \), \( j \in [k] \), can be more than \( \frac{\epsilon}{m} \theta^* \) because \( \sum_{j \in [k]} \alpha^{(g,j)} \leq \theta^* \). Suppose, for each \( g \in [m] \), we guess the set of indices

\[
T_g = \{j \in [k] : \alpha^{(g,j)} \geq \frac{\epsilon}{m} \theta^*\}.
\]

This can be done in \( k^{O(m^2/\epsilon)} \) time by enumerating over set \([k]\). Let

\[
T = T_1 \cup \cdots \cup T_m.
\]

For \( j \in T \), we also guess \( v^{*}_{j} \) in the optimum solution by enumerating over \( S_j \)'s, \( j \in T \). This increases the running time by a multiplicative factor of \( n^{O(m^2/\epsilon)} \) since \( v^{*}_{j} \in S_j \) and \( |S_j| \leq n \). Based on these guesses, we can set the corresponding variables in the LP, i.e., for each \( S_j \) such that \( j \in T \), we add the following constraints for \( v \in S_j \):

\[
x^{(j)}_{v} = 1 \text{ if } v = v^{*}_{j}, \quad \text{and } x^{(j)}_{v} = 0, \text{ otherwise}.
\]

The number of LPs generated by this enumeration is \((nk)^{O(m^2/\epsilon)}\).

We solve all such LPs to get optimum extreme points. Let \((\overline{\theta}, \overline{x}^{(1)}, \ldots, \overline{x}^{(k)})\) be an optimum extreme point for the LP corresponding to the right guess (i.e., the guess in which we have identified all indices \( j \in [k] \) along with their corresponding \( v^{*}_{j} \) such that there exists \( g \in [m] \) where \( \alpha^{(g,j)} \geq \frac{\epsilon}{m} \theta^* \). Therefore \( \overline{\theta} \leq \theta^* \). Let \( R = \{j \in [k] : \overline{x}^{(j)} \notin \{0, 1\}^{\{S_i\}}\} \). Since the feasible region of the LP corresponds to the intersection of a matroid polytope and \( m \) half-spaces, by Lemma 1.3 the size of the support of an extreme solution is \( k + m \). Moreover any cluster with \( j \in [k] \) that have fractional centers in the extreme point solution, contributes at least 2 to the size of the support because of the equality constraint in the LP. Therefore \( 2|R| + |k - |R|| \leq k + m \) which implies \(|R| \leq m \).

Now we guess the \( v^{*}_{j} \) for all \( j \in R \). By construction, \( R \cap T = \emptyset \). Therefore for all \( j \in R \) and \( g \in [m] \), \( \alpha^{(g,j)} \leq \frac{\epsilon}{m} \theta^* \). Therefore for all \( g \in [m] \), \( \sum_{j \in R} \alpha^{(g,j)} \leq m \cdot \frac{\epsilon}{m} \theta^* = \epsilon \theta^* \). Thus for the right guess of \( v^{*}_{j} \), \( j \in R \), we get an integral solution with a cost less than or equal to \( \overline{\theta} + \epsilon \theta^* \leq (1 + \epsilon) \theta^* \). \( \square \)

Algorithm 3 is our main procedure to convert a solution with \( k + m \) centers to a solution with \( k \) centers. We need \( \beta \) to be in the interval mentioned in Lemma 1.2. To achieve this we guess \( \text{opt}_{x} \) as different powers of two and try the corresponding \( \beta \)'s. The main idea behind the algorithm is that in a pseudo-solution of a sparse instance, there are only a few (< \( m^2 t \)) facilities that are far from facilities in the optimal solution. So the algorithm tries to guess those facilities and replace them by facilities in the optimal solution. For the rest of the facilities in the pseudo-solution (which are close to facilities in the optimal solution), the algorithm solves an optimization problem (Lemma 4.1) to find a set of facilities with a cost comparable to the optimal solution.

Finally combining the following lemma with Lemma 3.2 (sparsification) and Theorem 1.1 (bicriteria algorithm) implies Theorem 1.2.

**Lemma 4.2.** Let \( I = (k, F, A, d) \) be an \( \frac{\text{opt}_{+}}{mt} \)-sparse instance of the \((\ell_p, k)\)-clustering problem, \( T \) be a pseudo-solution with at most \( k + m \) centers, \( \epsilon' > 0, \delta \in (0, \min\{\frac{1}{8}, \frac{\log(1+\epsilon')}{12}\}) \), \( t \geq 4(1 + \frac{3}{4})p \)
Algorithm 3: Obtaining a solution from a pseudo-solution

1. **Input:** Instance $\mathcal{I}$, $\beta$, a pseudo-solution $\mathcal{T}$, $\epsilon'>0$, $\delta \in (0, \min\{\frac{1}{8}, \frac{\log(1+\epsilon')}{\log^2(1+\epsilon')}\})$, and integer $t \geq 4 \cdot (1+\frac{3}{\delta})^p$.
2. **Output:** A solution with at most $k$ centers.

3. $T' \leftarrow T$
4. **while** $|T'| > k$ and there is $j \in T'$ s.t. $\text{cost}_T(T' \setminus \{j\}) \leq \text{cost}_T(T') + \beta$ **do** $T' \leftarrow T' \setminus \{j\}$
5. **if** $|T'| \leq k$ **then** return $T'$
6. **forall** $D \subseteq T'$ and $V \subseteq F$ such that $|D| + |V| = k$ and $|V| < m^2t$
7. **for** $j \in D$, set $L_j = d(j, T' \setminus \{j\})$
8. **for all** $s \in [m], j \in D$, $f_j \in \text{Fball}_T(j, \delta L_j)$, set $a_{j,s} = \sum_{i \in \text{Cball}_T(s, \frac{\delta L_j}{4})} \min\{d(i, f_j), d(i, V)^p\}$.
9. Let $(\tilde{f}_j : j \in D) \in \bigotimes_{j \in D} \text{Fball}_T(j, \delta L_j)$ be $(1+\epsilon)$-approximate solution to (see Lemma 4.1)

$$S_{D,V} \leftarrow V \cup \{\tilde{f}_j : j \in D\}$$
10. **return** $S := \arg \min_{S_{D,V}} \text{cost}_T(S_{D,V})$

be an integer, and

$$\frac{2}{mt} (opt_x + (1 + \frac{1}{3})^p \text{cost}_T(T)) \leq \frac{2}{mt} (2\text{opt}_x + (1 + \frac{3}{3})^p \text{cost}_T(T))$$

Then Algorithm 3 finds a set $S \in F$ in time $n^{m+2}2^{O(p)}$ such that $|S| \leq k$ and $\text{cost}_S(S) \leq (O(1) + (1+\epsilon'))(\text{cost}_T(T) + \text{opt}_T)$.

**Proof.** If Algorithm 3 ends in Step 7, then $\text{cost}_T(T) + m\beta$ is at most $\text{cost}_T(T) + \frac{2}{mt} (2\text{opt}_x + (1 + \frac{3}{3})^p \text{cost}_T(T)) = O(\text{opt}_x + \text{cost}_T(T))$. Otherwise, we run the loop. Now we show that there exist sets $D_0 \subseteq T'$ and $V_0 \subseteq F$ such that $|V_0| < m^2t$, $|D_0| + |V_0| = k$, and $S_{D_0,V_0}$ satisfies the desired properties.

For a facility $j \in T'$, let $L_j = d(j, T' \setminus \{j\})$ and $\ell_j = d(j, OPT_T)$. We say $j \in I$ is determined if $\ell_j \leq \delta L_j$. Otherwise, we say $j$ is undetermined. Let $D_0 = \{j \in T' : \ell_j \leq \delta L_j\}$. For each $j \in D_0$, let $f_j$ be the closest facility to $j$ in $OPT_T$. Let $V_0 = OPT_T \setminus \{f_j : j \in D_0\}$. First note that for any two distinct facilities in $j, j' \in D_0$, $d(j, j') \geq \max\{L_j, L_{j'}\}$. Moreover by definition, $d(j, f_j) \leq \delta L_j \leq \delta \max\{L_j, L_{j'}\}$. Therefore by triangle inequality, $d(j, f_j) \geq (1-\delta)\max\{L_j, L_{j'}\}$. Moreover by definition and because $\delta \in (0, \frac{1}{2}), (1-\delta)\max\{L_j, L_{j'}\} > \delta L_j \geq d(j, f_j)$. Therefore $d(j, f_j) > d(j, f_{j'})$. Thus for any two distinct facilities, $j, j' \in D_0, f_j \neq f_{j'}$.

Therefore $(f_j : j \in D_0) = |D_0|$. Thus $|V_0| = |OPT_T| - |D_0| = k - |D_0|$. Let $U_0 = T' \setminus D_0$ be the set of undetermined facilities. Since $|T'| > k$, $|V_0| = k - |D_0| = k - |T'| + |U_0| < |U_0|$. We show $|U_0| < m^2t$. For every $j \in T'$ and $s \in [m]$, let $A_{s,j}$ be the set of clients of group $s$ that are connected to $j$ in solution $T'$ and let $C_{s,j}$ be the total connection cost of these clients. Therefore $\text{cost}_T(T') = \max_{s \in [m]} \sum_{j \in T'} C_{s,j}$. Let $j^* := \arg \min_{j \in \Theta} \sum_{s \in [m]} C_{s,j}$. Let $\tilde{j}$ be the closest facility to $j^*$ in $T' \setminus \{j^*\}$, i.e., $d(\tilde{j}, j^*) = L_{j^*}$. Then $\text{cost}_T(T' \setminus \{j^*\}) - \text{cost}_T(T') \leq \max_{s \in [m]} \sum_{i \in A_{s,j^*}} d(i, j)^p$. For $s \in [m]$, let $A_{s,j^*} := A_{s,j^*} \cap \text{Cball}_T(s, \frac{1}{3} \delta L_{j^*})$. Similarly, $A_{s,j^*}$ is a $\frac{1}{m^2}$-sparse instance, $\sum_{i \in A_{s,j^*}} d(i, j)^p \leq \frac{\text{opt}_x}{mt}$. For $i \in A_{s,j^*}$, $d(i, j) \geq \frac{1}{3} \delta L_{j^*}$. Thus $\frac{1}{3} \delta d(i, j) \geq \text{opt}_x / mt$. And by triangle inequality, $d(i, j) \leq d(i, j^*) + d(j^*, \tilde{j}) = d(i, j^*) + L_{j^*} \leq (1 + \frac{3}{\delta}) d(i, j^*)$. Therefore

$$\text{cost}_T(T' \setminus \{j^*\}) - \text{cost}_T(T') \leq \max_{s \in [m]} \left( \frac{\text{opt}_x}{mt} + (1 + \frac{3}{\delta})^p C_{s,j^*} \right)$$

$$\leq \frac{\text{opt}_x}{mt} + (1 + \frac{3}{\delta})^p \sum_{s \in [m]} C_{s,j^*}.$$ (10)
By definition, \( \sum_{s \in [m]} C_{s,j^*} = \min_{t \in T_0} \sum_{s \in [m]} C_{s,j} \leq \frac{m \cost_{T}(T')}{|T_0|} \). So if \( |U_0| \geq m^2t \), then 
\( \sum_{s \in [m]} C_{s,j^*} \leq \frac{\cost_{T}(T')}{{m^2}} \). Moreover, since \(|T \setminus T'| < m\), 
\[ \cost_{T}(T') < \cost_{T}(T) + m\beta \leq \cost_{T}(T) + \frac{2}{t} \left( 2\opt_T + (1 + \frac{3}{\delta})p \cost_{T}(T) \right) \]
\[ \leq \left( \frac{1}{1 + \frac{3}{\delta}} \right)^p \opt_T + \frac{3}{2} \cost_{T}(T). \]

Combining with (10), \( \cost_{T}(T' \setminus \{j^*\}) - \cost_{T}(T') \leq 2\opt_T + \frac{3}{2} \left( 1 + \frac{3}{\delta} \right) p \cost_{T}(T) \leq \beta \). This is a contradiction because \( j^* \) should be removed in Step 4 of Algorithm 3. Therefore \(|U_0| < m^2 t\).

Now, we need to bound the cost of \( S_{D_0,V_0} \). For \( j \in D_0 \) and \( s \in [m], \) let \( i \in CBall_{T,s}(j, \frac{1}{3} L_j) \). By triangle inequality the distance of \( i \) to any facility in \( FBall_{T}(j, \delta L_j) \) is at most \( \left( \frac{1}{3} + \delta \right) L_j \). For a facility \( j' \in D_0 \), \( j' \neq j \), by triangle inequality and because \( d(j,j') \geq \max\{L_j,L_j'\} \), the distance of \( i \) to any facility in \( FBall_{T}(j', \delta L_j') \) is at least
\[ d(j,j') - \frac{1}{3} - \delta L_j \geq d(j,j') - \left( \frac{1}{3} + \delta \right) d(j,j') = \left( \frac{1}{3} - \delta \right) d(j,j') \geq \left( \frac{2}{3} - \delta \right) L_j. \]

For \( \delta < \frac{1}{3} \), we have \( \frac{1}{3} + \delta < \frac{2}{3} - \delta \). Therefore, \( i \) is either connected to \( f_j \) or to a facility in \( V_0 \). Let \( e_{f_j}^{(s,j)} \)'s be as defined in Algorithm 3 for \( D_0 \) and \( V_0 \). Let \( (f_j : j \in D_0) \) be a \( (1 + \epsilon) \)-approximate solution for the following, obtained by Lemma 4.4. For \( s \in [m], \) let \( T_s = \bigcup_{j \in D_0} CBall_{T,s}(j, L_j/3) \). Since for \( j \in D_0, f_j^{*} \in OPT_T \)'s are also in balls \( FBall_{T}(j, \delta L_j) \),
\[ \max_{s \in [m]} \sum_{i \in T_s} d(i, S_{D_0,V_0})^p \leq (1 + \epsilon) \max_{s \in [m]} \sum_{i \in T_s} d(i, OPT_T)^p. \]

Now consider a client \( i \in A_s \setminus T_s \). If in the optimal solution, \( i \) is connected to a facility in \( V_0 \), then by definition, \( d(i, OPT_T) \geq d(i, S_{D_0,V_0}) \). Otherwise, in the optimal solution, \( i \) is connected to \( f_j^* \in FBall_{T}(j, \delta L_j) \) for some \( j \in D_0 \). We compare \( d(i, f_j) \) to \( d(i, f_j^*) \). Since \( f_j \), \( f_j^{*} \in FBall_{T}(j, \delta L_j) \), by triangle inequality and because \( d(i,j) \geq \frac{L_j}{3} \),
\[ \frac{d(i, f_j)}{d(i, f_j^{*})} \leq \left( \frac{L_j/3 + \delta L_j}{L_j/3 - \delta L_j} \right)^p \leq \left( 1 + \frac{3 \delta}{L_j/3} \right)^p \leq \left( 1 + \frac{3}{\delta} \right)^p \leq 1 + 12 \delta. \]
Thus because \( \delta \leq \frac{1}{8} \), \( 1 + \frac{3}{\delta} \leq 1 + 12 \delta \). Moreover since \( \delta < \frac{\log(1+\epsilon)}{12} \),
\[ \cost_{T}(S_{D_0,V_0}) \leq (1 + \epsilon)^p \cdot \opt_T. \]
Finally note that the loop runs for \( n^{O(m^2 t)} \) iterations because \( |V| < m^2 t \) and \( |T \setminus D| \leq m^2 t + m \). Moreover by Lemma 4.4, each iteration runs in \( (nk)^{O(m^2 t/c)} \) time.

5 Empirical Study

In this section, we compare the performance of our algorithm against the previously best algorithms in the literature on benchmark datasets for socially fair \( k \)-median problem. More specifically, we compare our bicriteria algorithm with that of Abbasi-Bhaskara-Venkatasubramanian (ABV), and our exact algorithm (i.e., the algorithm that outputs exactly \( k \) centers) with that of Makarychev-Vakilian (MV). Since our bicriteria algorithm produces only a small number of extra centers (e.g., for the case of two groups, our algorithm only produces one extra center in practice — see Section B.2), we use the exhaustive search approach for our exact algorithm. Our code is written in MATLAB. We use IBM ILOG CPLEX 12.10 to solve the linear programs.

Datasets. We use three benchmark datasets that have been extensively used in the fairness literature. Similar to other works in fair clustering, we subsample the points in the datasets. More specifically, we consider the first 500 examples in each dataset. A quick overview of the used datasets is in the following. (1) Credit dataset contains records of 30000 individuals with 21 features. We divided the multi-categorical education attribute to “higher educated” and “lower educated”, and used these as the demographic groups. (2) COMPAS dataset is gathered by ProPublica and contains the recidivism rates for 9898 defendants. The data is divided to two racial groups of African-Americans (AA) and Caucasians (C). (3) Adult dataset contains records of 48842 individuals collected from census data, with 103 features. We consider five racial groups of “Amer-Indian-Eskim”, “AsianPac-Islander”, “Black”, “White”, and “Other” for this dataset.
Bicriteria approximation. The ABV algorithm, first solves the natural linear programming relaxation of the problem and then uses the “filtering” technique [37,13] to round the fractional solution to an integral one. Given a parameter $0 < \epsilon < 1$, the algorithm outputs at most $k/(1-\epsilon)$ centers and guarantees a $2/\epsilon$ approximation. In our comparison, we consider $\epsilon$ that gives almost the same number of centers as our algorithm. Tables in Section B.2 summarise the number of selected centers in our experiments for different $k$’s and different parameters $\epsilon$. The $\lambda$ parameter in our algorithm (see Algorithm 1 and Lemma 2.1) determines the factor of decrease in the radii of client balls in the iterative rounding algorithm. As illustrated in plots of Section B.1, the results of our algorithm do not change significantly by changing $\lambda$. So in our comparisons with other methods we fix $\lambda = 0.3$. Figure 2 illustrates that our algorithm outperforms ABV on different benchmark datasets.

Exactly $k$ centers. The MV algorithm, first sparsifies the linear programming relaxation by setting the connection variables of points that are far from each other to zero. It then adopts a randomized rounding algorithm similar to [13] based on consolidating centers and points. In the process of rounding, it produces a $(1-\gamma)$-restricted solution which is a solution where each center is either open by a fraction of at least $(1-\gamma)$ or it is zero. The algorithm needs $\gamma < 0.5$. The results of MV for different values of $\gamma$ are presented in Section B.1. It appears that MV performs better for larger values of $\gamma$, so below we use $\gamma = 0.1$ and $\gamma = 0.4$ for our comparisons. Figure 3 illustrates that our algorithms outperforms MV on different benchmark datasets.

6 Acknowledgements

This research is supported in part by NSF awards CCF-1909756, CCF-2007443 and CCF-2134105.

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A Exhaustive Search

Lemma A.1 ([20]). Let \( k' > k \) and \( S \) be a set of centers of size \( k' \) and cost \( C \) for the socially fair \((\ell_p, k)\)-clustering problem with \( m \) groups. Let \( T \subset S \) be a set of size \( k \) with minimum cost among all subsets of size \( k \) of \( S \). Then the cost of \( T \) is less than or equal to \( 3^p - 1 \) \((C + 2\text{opt})\) where \( \text{opt} \) is the cost of the optimal solution.

Proof. Let \( \text{OPT} \) be an optimal set of centers. For each center \( o \in \text{OPT} \), let \( s_o \) be the closest center in \( S \) to \( o \), i.e., \( s_o := \arg \min_{s \in S} d(s, o) \). Let \( T' := \{ s_o : o \in \text{OPT} \} \). Because the size of \( \text{OPT} \) is \( k \), \(|T'| \leq k\). We show that the cost of \( T' \) is less than or equal to \( 3^p - 1 \) \((C + 2\text{opt})\). The result follows from this because \( T' \subset S \) and \(|T'| \leq k\).

Let \( i \) be a client and \( o_i \) be the closest facility in \( \text{OPT} \) to \( i \). Let \( t'_i \) be the closest facility to \( o_i \) in \( T' \) which means \( t'_i \) is also the closest facility to \( o_i \) in \( S \). Moreover let \( s_i \) be the closest facility to \( i \) in \( S \). By triangle inequality, \( d(i, t'_i) \leq d(i, o_i) + d(o_i, t'_i) \). By definition of \( t'_i \), \( d(o_i, t'_i) \leq d(o_i, s_i) \). Therefore \( d(i, t'_i) \leq d(i, o_i) + d(o_i, s_i) \). Moreover by triangle inequality \( d(o_i, s_i) \leq d(i, o_i) + d(i, s_i) \). Therefore \( d(i, t'_i) \leq 2d(i, o_i) + d(i, s_i) \). Taking both sides to the power of \( p \) and using the power mean inequality, i.e., \((x + y + z)^p \leq 3^p - 1 \((x^p + y^p + z^p)\), we conclude \( d(i, t'_i)^p \leq 3^p - 1 \((2d(i, o_i)^p + d(i, s_i)^p)\). The result follows from summing such inequality for each group and taking the maximum over groups. \( \square \)
B  Omitted Empirical Results

B.1  Results of Different Algorithms for Different Parameters

Figure 4: Performance of our bicriteria algorithm for different values of $\lambda$. The max and min on Subfigure (c) are across the demographic groups.

Figure 5: Performance of our bicriteria algorithm of ABV [1] for different values of $\epsilon$. The max and min on Subfigure (c) are across the demographic groups.

Figure 6: Performance of our algorithm with exactly $k$ centers for different values of $\lambda$. The max and min on Subfigure (c) are across the demographic groups.

Figure 7: Performance of our MV algorithm [39] for different values of $\gamma$. The max and min on Subfigure (c) are across the demographic groups.
### B.2 Number of Selected Centers in The Bicriteria Algorithms

Table 1: The number of selected centers for our bicriteria algorithm on the Credit dataset. $\lambda$ is a parameter of the algorithm and denotes the amount of decrease in radii of balls around the clients in the iterative rounding algorithm.

| $\lambda$ | $\lambda = 0.1$ | $\lambda = 0.2$ | $\lambda = 0.3$ | $\lambda = 0.4$ | $\lambda = 0.5$ |
|----------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $k = 5$  | 6               | 6               | 6               | 6               | 6               |
| $k = 10$ | 11              | 11              | 11              | 10              | 11              |
| $k = 15$ | 16              | 16              | 16              | 15              | 15              |
| $k = 20$ | 21              | 21              | 21              | 20              | 20              |
| $k = 25$ | 26              | 26              | 26              | 25              | 25              |
| $k = 30$ | 31              | 31              | 31              | 30              | 30              |
| $k = 35$ | 36              | 36              | 36              | 35              | 35              |
| $k = 40$ | 41              | 40              | 40              | 40              | 40              |
| $k = 45$ | 46              | 45              | 45              | 46              | 46              |
| $k = 50$ | 50              | 50              | 50              | 51              | 51              |

Table 2: The number of selected centers for bicriteria algorithm of Abbasi-Bhaskara-Venkatasubramanian [1] on the Credit dataset. $\epsilon$ is a parameter of the algorithm. The maximum number of selected centers is $k/(1 - \epsilon)$ which achieves a $2/\epsilon$ approximation factor.

| $\epsilon$ | $\epsilon = 0.1$ | $\epsilon = 0.2$ | $\epsilon = 0.3$ | $\epsilon = 0.4$ | $\epsilon = 0.5$ |
|------------|------------------|------------------|------------------|------------------|------------------|
| $k = 5$    | 5                | 6                | 7                | 8                | 10               |
| $k = 10$   | 11               | 12               | 14               | 16               | 20               |
| $k = 15$   | 16               | 18               | 21               | 23               | 29               |
| $k = 20$   | 20               | 23               | 26               | 29               | 34               |
| $k = 25$   | 22               | 27               | 31               | 35               | 39               |
| $k = 30$   | 29               | 34               | 37               | 41               | 46               |
| $k = 35$   | 25               | 29               | 36               | 40               | 45               |
| $k = 40$   | 26               | 30               | 38               | 43               | 48               |
| $k = 45$   | 36               | 40               | 46               | 53               | 59               |
| $k = 50$   | 38               | 45               | 51               | 58               | 64               |

Table 3: The number of selected centers for our bicriteria algorithm on the COMPAS dataset. $\lambda$ is a parameter of the algorithm and denotes the amount of decrease in radii of balls around the clients in the iterative rounding algorithm.

| $\lambda$ | $\lambda = 0.1$ | $\lambda = 0.2$ | $\lambda = 0.3$ | $\lambda = 0.4$ | $\lambda = 0.5$ |
|----------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $k = 5$  | 6               | 6               | 6               | 6               | 6               |
| $k = 10$ | 11              | 10              | 11              | 11              | 10              |
| $k = 15$ | 16              | 16              | 16              | 16              | 16              |
| $k = 20$ | 21              | 21              | 21              | 20              | 20              |
| $k = 25$ | 26              | 26              | 25              | 26              | 26              |
| $k = 30$ | 31              | 31              | 31              | 31              | 31              |
| $k = 35$ | 36              | 36              | 36              | 36              | 36              |
| $k = 40$ | 40              | 40              | 41              | 41              | 40              |
| $k = 45$ | 46              | 46              | 46              | 46              | 46              |
| $k = 50$ | 51              | 51              | 51              | 51              | 50              |
Table 4: The number of selected centers for bicriteria algorithm of Abbasi-Bhaskara-Venkatasubramanian [1] on the COMPAS dataset. $\epsilon$ is a parameter of the algorithm. The maximum number of selected centers is $k/(1 - \epsilon)$ which achieves a $2/\epsilon$ approximation factor.

| $k$  | $\epsilon = 0.1$ | $\epsilon = 0.2$ | $\epsilon = 0.3$ | $\epsilon = 0.4$ | $\epsilon = 0.5$ |
|------|------------------|------------------|------------------|------------------|------------------|
| 5    | 5                | 6                | 7                | 8                | 10               |
| 10   | 11               | 12               | 14               | 15               | 16               |
| 15   | 16               | 18               | 19               | 20               | 21               |
| 20   | 22               | 24               | 24               | 25               | 27               |
| 25   | 27               | 29               | 30               | 31               | 32               |
| 30   | 30               | 34               | 35               | 36               | 38               |
| 35   | 35               | 40               | 40               | 42               | 43               |
| 40   | 42               | 46               | 48               | 49               | 51               |
| 45   | 45               | 50               | 51               | 52               | 53               |
| 50   | 50               | 56               | 58               | 59               | 61               |

Table 5: The number of selected centers for our bicriteria algorithm on the Adult dataset. $\lambda$ is a parameter of the algorithm and denotes the amount of decrease in radii of balls around the clients in the iterative rounding algorithm.

| $k$  | $\lambda = 0.1$ | $\lambda = 0.2$ | $\lambda = 0.3$ | $\lambda = 0.4$ | $\lambda = 0.5$ |
|------|------------------|------------------|------------------|------------------|------------------|
| 5    | 7                | 7                | 7                | 5                | 7                |
| 10   | 15               | 14               | 12               | 14               | 12               |
| 15   | 18               | 15               | 18               | 14               | 19               |
| 20   | 24               | 20               | 22               | 18               | 20               |
| 25   | 30               | 27               | 27               | 25               | 28               |
| 30   | 34               | 34               | 33               | 30               | 33               |
| 35   | 38               | 38               | 38               | 35               | 38               |
| 40   | 43               | 43               | 43               | 40               | 44               |
| 45   | 45               | 47               | 47               | 45               | 46               |
| 50   | 54               | 54               | 54               | 54               | 50               |

Table 6: The number of selected centers for bicriteria algorithm of Abbasi-Bhaskara-Venkatasubramanian [1] on the Adult dataset. $\epsilon$ is a parameter of the algorithm. The maximum number of selected centers is $k/(1 - \epsilon)$ which achieves a $2/\epsilon$ approximation factor.

| $k$  | $\epsilon = 0.1$ | $\epsilon = 0.2$ | $\epsilon = 0.3$ | $\epsilon = 0.4$ | $\epsilon = 0.5$ |
|------|------------------|------------------|------------------|------------------|------------------|
| 5    | 4                | 5                | 6                | 7                | 7                |
| 10   | 6                | 7                | 10               | 11               | 12               |
| 15   | 9                | 10               | 11               | 12               | 13               |
| 20   | 13               | 13               | 14               | 15               | 17               |
| 25   | 15               | 17               | 19               | 20               | 22               |
| 30   | 18               | 19               | 21               | 23               | 28               |
| 35   | 22               | 25               | 29               | 32               | 37               |
| 40   | 30               | 37               | 39               | 45               | 50               |
| 45   | 45               | 51               | 57               | 61               | 68               |
| 50   | 45               | 50               | 56               | 61               | 68               |