A Summation of Series Involving Bessel Functions and Order Derivatives of Bessel Functions

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Abstract

In this note, we derive the closed-form expression for the summation of series $\sum_{n=0}^{\infty} n J_n(x) \partial J_n / \partial n$, which is found in the calculation of entanglement entropy in 2-d bosonic free field, in terms of $Y_0$, $J_0$ and an integral involving these two Bessel functions. Further, we point out the integral can be expressed as a Meijer G function.

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I. INTRODUCTION

In this note, we find that the sum of a series involving Bessel functions and the order derivatives of Bessel functions:

\[ P(x) \equiv \sum_{n=0}^{\infty} nJ_n(x) \frac{\partial J_n(x)}{\partial n} \]  

(1)
can be expressed in a closed-form.

The motivation of finding this sum rule arises from the study of entanglement entropy in two dimensional bosonic free field. In [1], the authors provide a way to compute such entanglement entropy. The key point is the Green’s function in the n-sheeted geometry \( G_n(r, r) \). It obeys the Helmholtz equation in polar coordinates and can be expressed as

\[ G_n(r, \theta; m^2) = \frac{1}{2\pi \nu} \sum_{k=0}^{\infty} \varepsilon_k \int_{0}^{\infty} \lambda d\lambda \frac{J_{k/\nu}(\lambda r)}{\lambda^2 + m^2}, \]

(2)

where \( \varepsilon_0 = 1 \) or \( \varepsilon_n = 2 \) otherwise. With the Green’s function in the n-sheeted geometry, the Rényi entropy can be expressed as

\[ \text{Tr} \rho^n = \exp \left\{ -\frac{1}{2} \int_{a^2}^{m^2} \int d^2r \left( G_n(r, \theta; m^2) - nG_1(r, \theta; m^2) \right) \right\}. \]

(3)

Then the von Neumann entropy

\[ S = -\text{Tr} \log \rho = -\frac{\partial}{\partial n} \text{Tr} \rho^n \bigg|_{n=1} = \frac{1}{2} \int_{a^2}^{m^2} \int d^2r \left( \frac{\partial}{\partial n} G_n(r, \theta; m^2) \bigg|_{n=1} - G_1(r, \theta; m^2) \right). \]

(4)

It is not hard to see the summation over the product of Bessel functions in (2) playing an important role. In the last step calculating the von Neumann entropy, the derivative with respect to \( n \) requires the order derivatives and the summation now becomes

\[ \frac{\partial}{\partial n} \sum_{k=0}^{\infty} \varepsilon_k J_{k/\nu}(\lambda r) \bigg|_{n=1} = 2 \sum_{k=0}^{\infty} \varepsilon_k \frac{k}{n^2} J_{k/\nu}(\lambda r) \frac{\partial J_{k/\nu}(\lambda r)}{\partial (k/\nu)} \bigg|_{n=1} = -4P(\lambda r). \]

(5)

II. DERIVATION OF \( P(x) \)

Let us start with this formula found in [2]

\[ \int_{x}^{\infty} \frac{J_\nu^2(t)}{t} dt = \frac{1}{2\nu} - \frac{1}{2\nu} \sum_{n=0}^{\infty} \varepsilon_n J_{\nu+n}(x), \]

(6)

where \( \varepsilon_0 = 1 \) and \( \varepsilon_n = 1 \) otherwise. By multiplying \( \nu \) on both sides and taking the derivative with respect to \( \nu \), we have

\[ \int_{x}^{\infty} \frac{J_\nu^2(t) + 2\nu J_\nu(t)J_\nu(t)}{t} dt = -\sum_{n=0}^{\infty} \varepsilon_n J_{\nu+n}(x) \frac{\partial J_{\nu+n}(x)}{\partial \nu} = -2 \sum_{n=1}^{\infty} J_{\nu+n}(x) \frac{\partial J_{\nu+n}(x)}{\partial \nu} - J_\nu(x) \frac{\partial J_\nu(x)}{\partial \nu}, \]

(7)

where \( \frac{\partial J_\nu}{\partial \nu} = \partial J_\nu/\partial \nu \). Then we set \( \nu \in \mathbb{N} \) and sum over \( \nu \) form 0 to \( \infty \). (7) becomes

\[ \int_{x}^{\infty} \sum_{\nu=0}^{\infty} J_\nu^2(t) + 2P(t) \frac{dt}{t} = -\sum_{\nu=0}^{\infty} J_\nu(x) \frac{\partial J_\nu(x)}{\partial \nu} = -2 \sum_{\nu=0}^{\infty} \sum_{n=0}^{\infty} J_{\nu+n}(x) \frac{\partial J_{\nu+n}(x)}{\partial \nu}. \]

(8)

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Notice that the RHS gives $-2P(x)$. From (3.2) in [2], the second term in the LHS of (8) gives

$$
\sum_{\nu=0}^{\infty} J_{\nu}(x) \dot{J}_{\nu}(x) = -\frac{1}{2} \int_{x}^{\infty} \frac{J_{0}^2(t)}{t} dt + \frac{1}{2} J_{0}(x) \dot{J}_{0}(x),
$$

and also

$$
\sum_{\nu=0}^{\infty} J_{\nu}^2(t) = \frac{1}{2} \left( \sum_{\nu=0}^{\infty} J_{\nu}(t) + J_{0}(t) \right) = \frac{J_{0}^2(t) + 1}{2}.
$$

Thus, (8) reduces to

$$
\frac{1}{2} \int_{x}^{\infty} \left( \frac{J_{0}^2(t) + 1}{2} + 2P(t) \right) dt - \frac{1}{2} \int_{x}^{\infty} \frac{J_{0}^2(t)}{t} dt + \frac{1}{2} J_{0}(x) \dot{J}_{0}(x) = -2P(x).
$$

After taking the derivative with respect to $x$ and substituting $\dot{J}_{0}(x) = \frac{8}{x} Y_{0}(x)$ (8.486(1) in [2]), we have the differential equation

$$
P'(x) - \frac{P(x)}{x} = \frac{1}{4x} - \frac{\pi}{8} (J_{0}(x) Y_{0}(x))'.
$$

This equation can be easily solved by the method of variation of parameters. The solution

$$
P(x) = -x \left( \int_{x}^{\infty} \left( \frac{1}{4t^2} - \frac{\pi}{8} J_{0}(t) Y_{0}(t) \right) dt + C \right)
$$

$$
= -x \left( \frac{1}{4} - \frac{\pi}{8} J_{0}(t) Y_{0}(t) \right)_{x} - \frac{\pi}{8} \int_{x}^{\infty} \frac{J_{0}(t) Y_{0}(t)}{t^2} dt + C
$$

$$
= -\frac{1}{8} \left( 2 + \pi J_{0}(x) Y_{0}(x) - \pi x \int_{x}^{\infty} \frac{J_{0}(t) Y_{0}(t)}{t^2} dt + Cx \right).
$$

To determine the integral constant $C$, we can analyse the asymptotic behavior of $P(x)$ at large $x$. When $x \to \infty$, we have $P(x) \sim -\frac{1}{8}(2 + Cx)$, which means when $x$ is sufficiently large, $P(x)$ is approximately a linear function and the slope is $C$. However, the asymptotic form of $\dot{J}_{n}$ can be written as [2]

$$
\dot{J}_{n}(x) \sim \sqrt{\frac{\pi}{2x}} \sin \left( x - \frac{n\pi}{2} - \frac{\pi}{4} \right) \quad (x \to \infty).
$$

Combining with the asymptotic approximation of $J_{n}$ in [21], it is easy to find that $P(x) \sim \frac{a}{x} + b \sim b$ when $x \to \infty$, where $a$ and $b$ are constants. More detailed discussion about the asymptotic behavior can be found in the next section. So the integral constant $C = 0$ and finally we obtain a closed-form

$$
P(x) = -\frac{1}{8} \left( 2 + \pi J_{0}(x) Y_{0}(x) - \pi x \int_{x}^{\infty} \frac{J_{0}(t) Y_{0}(t)}{t^2} dt \right).
$$

Also, we can evaluate the integral (see Appendix), which can be expressed by a Meijer G function

$$
\int_{x}^{\infty} \frac{J_{0}(t) Y_{0}(t)}{t^2} dt = \frac{1}{2\sqrt{\pi}} G_{20}^{13} \left( x^2 \left| \begin{array}{c} \frac{1}{2} \\ \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \end{array} \right. \right).
$$

Hence, we have an another expression

$$
P(x) = -\frac{1}{8} \left( 2 + \sqrt{\frac{\pi x}{2}} G_{20}^{13} \left( x^2 \left| \begin{array}{c} \frac{1}{2} \\ \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \end{array} \right. \right) \right).
$$
III. ASYMPTOTIC APPROXIMATION OF $P(x)$ AND REGULARIZATION OF GREEN’S FUNCTIONS

We now obtain the asymptotic approximation of $P(x)$ at large $x$. The asymptotic behavior can be helpful to analyze the behavior of the combination of Green’s functions $\partial G_n(r; \theta; m^2)/\partial n|_{n=1} - G_1(r; \theta; m^2)$, especially the divergence and the regularization of it.

Obviously, when $x \to \infty$, the last two terms in (15), involving $J_0$ and $Y_0$, go to zero fast (at the order of $O(x^{-1})$). So we have

$$\lim_{x \to \infty} P(x) = -\frac{1}{4}, \quad (18)$$

This is not a good news for us because $P(x)$ as a part of the integrand in $\partial G_n(r; \theta; m^2)/\partial n|_{n=1}$, the non-zero limits highly increases the probability that we have a divergent integral. Indeed, the integral is actually divergent.

However, things are not so bad. We define

$$Q(x) \equiv -4P(x) - \sum_{k=0}^{\infty} J_k^2(x) = -4P(x) - 1 \quad (19)$$

We can rewrite the combination of Green’s functions in terms of $Q(x)$:

$$\left. \frac{\partial}{\partial n} G_n(r; \theta; m^2) \right|_{n=1} - G_1(r; \theta; m^2) = \frac{1}{2\pi} \int_0^\infty \frac{Q(\lambda r)}{\lambda^2 + m^2} d\lambda \quad (20)$$

It is easy to check that $Q(x)$ vanishes when $x$ approaches to infinity. By introducing asymptotic approximations for $J_0(x)$ and $Y_0(x)$ at large $x$

$$J_0(x) \sim \sqrt{\frac{2}{\pi x}} \cos \left( x - \frac{n\pi}{2} - \frac{\pi}{4} \right),$$

$$Y_0(x) \sim \sqrt{\frac{2}{\pi x}} \sin \left( x - \frac{n\pi}{2} - \frac{\pi}{4} \right), \quad (21)$$

we have $\pi J_0(x)Y_0(x) \sim -\cos(2x)/x$. Substituting this approximation into the integral and integrating by part, we derive the asymptotic approximation for the second term in $Q(x)$, which gives

$$\int_{x}^{\infty} \frac{J_0(t)Y_0(t)}{t^2} dt \sim -\int_{x}^{\infty} \frac{\cos(2t)}{t^3} dt = \frac{\sin(x)}{2x^3} - \frac{3}{2} \int_{x}^{\infty} \frac{\sin(2t)}{t^4} dt \sim O(x^{-3}). \quad (22)$$

Comparing to the first term, we can neglect the second term when $x$ is sufficiently large. Thus, the asymptotic approximation of $Q(x)$ turns

$$Q(x) \sim -\frac{\cos(2x)}{x}. \quad (23)$$

Also, we can evaluate the value at $x = 0$. By using L’Hospital rule, we find

$$\lim_{x \to 0} \frac{1}{x} \left( \frac{J_0(x)Y_0(x)}{x} - \int_{x}^{\infty} \frac{J_0(t)Y_0(t)}{t^2} dt \right) = \lim_{x \to 0} \frac{-1}{x^2} \left( \frac{x(J_0(x)Y_0(x))' - J_0(x)Y_0(x)}{x^2} + \frac{J_0(x)Y_0(x)}{x^2} \right) = -\lim_{x \to 0} x(J_0(x)Y_0(x))'. \quad (24)$$
Using identities (8.471 and 8.477 in [3])

\[ Z'_n(x) = \frac{1}{2}(Z_{n-1}(x) - Z_{n+1}(x)), \]

\[ J_n(x)Y_{n+1}(x) - J_{n+1}(x)Y_n(x) = \frac{2}{\pi x}, \]

where \( Z \) denotes \( J \) or \( Y \), we have

\[ -\lim_{x \to 0} x(J_0(x)Y_0(x))' = \lim_{x \to 0} x(J_1(x)Y_0(x) + J_0(x)Y_1(x)) = \frac{2}{\pi x} \]

\[ = \frac{2}{\pi}. \quad (25) \]

Thus, \( Q(0) = -1 \). Moreover, no singularity at the origin (of course, no singularity at other place as well) and the simple asymptotic form of \( Q(x) \) tell us the combination in (20) is convergent. Also, by using the identity (7.811.5) in [4] and the relation (9.311) in [3] to deal with the Meijer G function, one can evaluate the combination exactly

\[ \partial_{\eta} G_n(r, \theta; m^2) \bigg|_{\eta=1} - G_1(r, \theta; m^2) = \frac{1}{16\pi} \left( \sqrt{\pi m^2} G^{(3)}_{13} \left( m^2 r^2 \left[ -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right] - 4K_0^2 (m^2) \right) \right). \]

Furthermore, we can also obtain

\[ \int d^2r \left( \partial_{\eta} G_n(r, \theta; m^2) \bigg|_{\eta=1} - G_1(r, \theta; m^2) \right) = -\frac{1}{6m^2}, \quad (27) \]

which agrees the result in [4]. Because the radial integral is definitely a Mellin transform so this result can be found in Mellin transform table in [3].

**IV. CONCLUSIONS AND REMARKS**

We study a summation of series involving Bessel functions and the order derivatives of Bessel functions, which arises in the study of entanglement entropy in two dimensional bosonic free field. We find a closed-form expression of such summation which can be expressed simply in terms of \( J_0, Y_0 \) and a Meijer G function. Further, we study the asymptotic behavior of this summation of series and finally we find a simple expression for the combination of Green’s functions in (26).

The expression in (26) and also \( P(x), Q(x) \) might be useful for calculating entanglement entropies in different fields. For instance, if the field is coupled with a potential such that the equation of motion is no longer a homogeneous Helmholtz equation and the Green’s equation becomes

\[ (\nabla_r^2 + m^2 - f(r))G_n(r, r') = \delta(r - r'). \quad (28) \]

Generally, the exact solution would not be found easily. However, the equation can be solved perturbatively by applying the resolvent formalism

\[ G_n(r, r') = \tilde{G}_0(r, r') + \tilde{G}_1(r, r') + .... \]

(29)
where
\[
\tilde{G}_0(r, r') = \tilde{G}(r, r'),
\]
\[
\tilde{G}_1(r, r') = (\tilde{G} \circ f \circ \tilde{G})(r, r'),
\]
\[
\vdots
\]
\[
\tilde{G}_i(r, r') = (\tilde{G} \circ f \circ \tilde{G} \circ \ldots \circ f \circ \tilde{G})(r, r'),
\]
(30)

where \( \tilde{G} \) is the Green's function with \( f = 0 \) and \( (G_n \circ f)(r) = \int G(r, r') f(r') d^2 r \). In this method, the exact Green's function can be also written as (see Appendix B)
\[
G_n(r, r') = \sum_{i=0}^{\infty} \sum_{j=0}^{l} B_{i,j} f(2\omega) \frac{\partial}{\partial \omega} \tilde{G}^{(1+i+j)}_n(r, r') \bigg|_{\omega = 0}.
\]
(31)

This method is well developed in [3, 4] for Sturm–Liouville problems. More details can be found in [5, 6]. Furthermore, we can apply the Green's function to calculate the entanglement entropy
\[
\frac{\partial}{\partial n}G_n(r, \theta; m^2) \bigg|_{n=1} - G_1(r, \theta; m^2)
\]
\[
= \sum_{i=0}^{\infty} \sum_{j=0}^{l} \left\{ B_{i,j} f(2\omega) \frac{\partial}{\partial \omega} \tilde{G}^{(1+i+j)}_n(r, r') \bigg|_{\omega = 0} - \tilde{G}^{(1+i+j)}_1(r, r') \right\}
\]
(32)

For those terms with \( i = j \) in the summation, it is easy to figure out by using \( Q(x) \), i.e.
\[
(4\partial_\omega \partial_\omega) \left[ \frac{\partial}{\partial n} \tilde{G}^{(1+i+i)}_n(r, r') \bigg|_{\omega = 0} - \tilde{G}^{(1+i+i)}_1(r, r') \right]_{r=r'} = \frac{1}{2\pi i} \int_{0}^{\infty} -(-\lambda)^{2i+1} Q(\lambda r) d\lambda.
\]
(33)

However, the remaining terms where \( i \neq j \) are still a problem that we can not solve easily. So how to deal with other terms can be a topic for further study.

**Appendix A: Evaluation of \( \int_{x}^{\infty} J_0(t)Y_0(t)/t^2 dt \)**

The definition of Meijer G function is following:
\[
G^{m,n}_{p,q} \left( \begin{array}{c}
\begin{array}{c}
a_1, \ldots, a_p \\
b_1, \ldots, b_q
\end{array}
\end{array} \bigg| z \right) = \frac{1}{2\pi i} \int_{L} \frac{\Gamma(1-s) \Gamma(1-a_j+s) \Gamma(1-b_j-s) \Gamma(1-a_j-s) \Gamma(1+b_j-s) \Gamma(1+s)}{\prod_{j=0}^{n} \Gamma(1-b_j-s) \prod_{j=0}^{n} \Gamma(1-a_j-s) \prod_{j=0}^{n} \Gamma(1-a_j+s) \prod_{j=0}^{n} \Gamma(1+b_j+s) \prod_{j=0}^{n} \Gamma(1+s)} z^s ds,
\]
(A1)

where the integral contour \( L \) can be found in (9.302) in [3]. This is a Mellin-Barnes type integral which can be viewed as an inverse Mellin transform. The Mellin transform of this function can be written as
\[
\mathcal{M} \left( G^{m,n}_{p,q} \left( \begin{array}{c}
\begin{array}{c}
a_1, \ldots, a_p \\
b_1, \ldots, b_q
\end{array}
\end{array} \bigg| z \right) \right)(s) = \frac{\Gamma(1-s) \Gamma(1-a_j+s) \Gamma(1+b_j-s) \Gamma(1+a_j+s)}{\prod_{j=0}^{n} \Gamma(1-b_j-s) \prod_{j=0}^{n} \Gamma(1-a_j-s) \prod_{j=0}^{n} \Gamma(1-a_j+s) \prod_{j=0}^{n} \Gamma(1+b_j+s) \prod_{j=0}^{n} \Gamma(1+s)}
\]
(A2)

First, we apply Mellin transform and use the multiplicative convolution formula of Mellin transform [3]
\[
\mathcal{M} \left( \int_{x}^{\infty} \frac{J_0(t)Y_0(t)}{t^2} dt \right)(s) = \mathcal{M} \left( \int_{0}^{x} u \left( 1 - \frac{x}{t} \right) \frac{J_0(t)Y_0(t)}{t^2} dt \right)(s)
\]
\[
= \mathcal{M} (u (1-x)) (s) \mathcal{M} \left( \frac{J_0(x)Y_0(x)}{x} \right)(s),
\]
(A3)

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where \( u(x) \) is Heaviside step function. From Eq. 5, we have
\[
\mathcal{M} (u(1-x))(s) = \frac{1}{s} = \frac{\Gamma(s/2)}{2\Gamma(s/2 + 1)},
\]
\[
\mathcal{M} \left( J_0(x)Y_0(x) \right)(s) = -2^{s-1}\pi^{-1}\frac{\Gamma^2(s/2)\Gamma(1 - s)}{\Gamma^2(1 - s/2)} \cos(s/2)\pi
\]
\[
= -\frac{1}{2\sqrt{\pi}} \frac{\Gamma(s/2)}{\Gamma((1 + s)/2)}. \quad (A4)
\]

If we set
\[
\mathcal{M} \left( J_0(x)Y_0(x) \right)(s) = F(s), \quad (A5)
\]
then according to the property of Mellin transform, we have
\[
\mathcal{M} \left( \frac{J_0(x)Y_0(x)}{x} \right)(s) = F(s - 1). \quad (A6)
\]
Thus, the Mellin transform of the original integral turns
\[
\mathcal{M} \left( \int_x^{\infty} \frac{J_0(t)Y_0(t)}{t^2} dt \right)(s) = -\frac{1}{4\sqrt{\pi}} \frac{\Gamma^2(s/2 - 1/2)\Gamma(s/2)}{\Gamma(3/2 - s/2)\Gamma(s/2 + 1)}. \quad (A7)
\]

By using the property of Mellin transform
\[
\mathcal{M} (f(t^a))(s) = \frac{1}{a} F \left( \frac{s}{a} \right), \quad (A8)
\]
and comparing to the Mellin transform of the Meijer G function in \( A2 \), it is easy to check
\[
\int_x^{\infty} \frac{J_0(t)Y_0(t)}{t^2} dt = G_{13}^{20} \left( x^2 \left| \begin{array}{c} \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \end{array} \right. \right). \quad (A9)
\]

**Appendix B: Derivation of Eq. 31**

We can derive this form in complex plane. By setting \( \omega = x + iy \), the Laplace operator becomes \( \nabla^2 = 4\partial_x\partial_x \). Now the potential \( f = f(\omega, \bar{\omega}) \) and it is real, i.e. \( f = \bar{f} \). The Green’s function \( G_n(\mathbf{r}, \mathbf{r}') \) is the integral kernel of the resolvent \( (-4\partial_x\partial_x + m^2 - f)^{-1} \).

The expansion of the resolvent operator in powers of \( (-4\partial_x\partial_x + m^2)^{-1} \) gives
\[
(-4\partial_x\partial_x + m^2 - f)^{-1} = (-4\partial_x\partial_x + m^2)^{-1} + (-4\partial_x\partial_x + m^2)^{-1} f (-4\partial_x\partial_x + m^2)^{-1} + \ldots
\]
\[
(B1)
\]
Then following the chapter 5 in the paper \( 7 \), the commutator of the resolvent \( (-4\partial_x\partial_x + m^2)^{-1} \) and \( f \) shows
\[
(-4\partial_x\partial_x + m^2)^{-1} f \quad = \quad f (-4\partial_x\partial_x + m^2)^{-1} + 4 (-4\partial_x\partial_x + m^2)^{-1} (\partial_x f \partial_x + \partial_x f \partial_x) (-4\partial_x\partial_x + m^2)^{-1} \quad (B2)
\]
By recurring the commutator, we can finally factor out all \( f \) and its derivatives and leave the resolvent \( (-4\partial_x\partial_x + m^2)^{-1} \) and differential operators \( \partial_x \) and \( \partial_x \). Hence, the full resolvent can be expressed as
\[
(-4\partial_x\partial_x + m^2 - f)^{-1} = \sum_{l=0}^{\infty} \sum_{i+j=0}^{l} B_{i,j}(f)[2\partial_x]^i[2\partial_x]^j (-4\partial_x\partial_x + m^2)^{-1-i-j}, \quad (B3)
\]
where $i + j + l$ is always even. By acting $(-4\partial_x \partial_2 + m^2 - f)$ on each side, we have the recurrence relations to determine the coefficients:

\[
\begin{cases}
B_{0,0,0} \equiv 1, & B_{i,j,l} = 0 \text{ for } i, j, l < 0, \\
B_{i,j,l} = 4\partial_x \partial_2 B_{i,j,l-2} + 2\partial_x B_{i,j-1,l-1} + 2\partial_2 B_{i-1,j,l-1} + f B_{i,j,l-2}.
\end{cases}
\]  

(B4)

Also, we need an additional constraint, $B_{i,j,l} = B_{l,j,i}$, to keep the resolvent real.

To connect the resolvents and the Green’s functions, we can consider function basis in the $n$-sheeted geometry $\{ e^{ik/n\theta} J_{k/n}(\lambda r) | k \in \mathbb{Z}, \lambda > 0, \lambda \in \mathbb{R} \}$. Hence, the Delta function can be constructed

\[
\delta(r-r')\delta(\theta-\theta') = \frac{1}{2\pi n} \sum_{k=-\infty}^{\infty} e^{ik/n(\theta-\theta')} J_{k/n}(\lambda r) J_{k/n}(\lambda r') \lambda d\lambda.
\]  

(B5)

Then we can plug this expression into (B3) and we have the form in (B3)

\[
G_n(r,r') = \sum_{l=0}^{\infty} \sum_{i+j=0} B_{i,j,l} [f(2\partial_x)^i(2\partial_2)^j \tilde{G}_n^{1/(i+j+1)}(r,r')],
\]  

(B6)

where

\[
\tilde{G}_n^{1/(i+j+1)}(r,r') = \left( \tilde{G}_n \circ \ldots \circ \tilde{G}_n \right)(r,r') = \frac{1}{2\pi n} \sum_{k=-\infty}^{\infty} e^{ik/n(\theta-\theta')} \int_0^{\infty} \frac{J_{k/n}(\lambda r) J_{k/n}(\lambda r')}{(\lambda^2 + m^2)^{1/(i+j+1)}} \lambda d\lambda.
\]  

(B7)

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