Quantum Criticality at the Metal Insulator Transition

D. Schmeltzer

Physics Department, The City College, CUNY
Convent Ave. at 138 ST, New York, NY 10031, USA

Abstract

We introduce a new method to analysis the many-body problem with disorder. The method is an extension of the real space renormalization group based on the operator product expansion. We consider the problem in the presence of interaction, large elastic mean free path, and finite temperatures. As a result scaling is stopped either by temperature or the length scale set by the diverging many-body length scale (superconductivity). Due to disorder a superconducting instability might take place at \( T_{SC} \rightarrow 0 \) giving rise to a metallic phase or \( T > T_{SC} \). For repulsive interactions at \( T \rightarrow 0 \) we flow towards the localized phase which is analyzed within the diffusive Finkelstein theory.

For finite temperatures with strong repulsive backward interactions and non-spherical Fermi surfaces characterized by \( |\frac{d \ln N(b)}{\ln b}| \ll 1 \) one finds a fixed point \( (D^*, \Gamma_2^s) \) in the plane \( (D, \Gamma_2^s) \). \( (D \propto (K_F \ell)^{-1} \) is the disorder coupling constant, \( \Gamma_2^s \) is the particle-hole triplet interaction, \( b \) is the length scale and \( N(b) \) is the number of channels.) For weak disorder, \( D < D^* \), one obtains a metallic behavior with the resistance \( \rho(D, \Gamma_2^s, T) = \rho(D, \Gamma_2^s, T) \simeq \rho^* f \left( \frac{D-D^*}{D^*} \right) \frac{1}{\nu_1} \) \( (\rho^* = \rho(D^*, \Gamma_2^s, 1), z = 1, \text{ and } \nu_1 > 1) \) in good agreement with the experiments.
I. INTRODUCTION

The Metal-Insulator (M-I) transition has been understood within the seminal paper [1] in 1979. Focusing on noninteracting electrons the authors demonstrated that in two dimension (2D) even weak disorder is sufficient to localize the electrons at $T = 0$. Few years later [2] it has been realized by Finkelstein that the particle-hole interaction in the triplet channel might enhance the conductivity. However a detailed analysis revealed that at long scale the interaction term diverges making difficult to determine what will happen at long scales. Recently a remarkable experiment [3] has been performed on a 2D electron gas in zero magnetic field strongly points towards a M-I transition in two dimensions. The characteristic of this experiment performed on a 2DES silicon ($n_s \sim 10^{11} \text{cm}^{-2}$) the mean free path “$\ell$” is large, the electron-electron interaction was $\sim 5\text{mev}$, while the Fermi energy is only $0.6\text{mev}$. The lowest temperature in the experiment was $0.2K$. These experimental condition might suggest that the non-linear sigma model introduced in ref. [2] might not be applicable since it ignores the interaction effects at length scales shorter than the mean free path. Since the mean free path is large quantum effects in the momentum range $2\pi/\ell \leq |q| \leq \Lambda$ ($\Lambda^{-1} \sim a \sim$ particle separation) might be important for weak disorder, $\ell \longrightarrow \infty$. This suggests that a phase transition due to a collective many body interaction might occur before the diffusive limit is reached. One might have a phase transition from a superconductor to insulator [4], Wigner crystal [5,6], or quantum Hall-insulator transition [7]. In one dimension it is known that attractive interaction or ferromagnetic spin fluctuations can suppress the $2k_F$ backscattering leading to a delocalization transition [8]. We investigate the problem in the presence of interaction and large mean free paths. In order to clarify the situation in 2D we propose to use the Renormalization Group (RG) analysis. Motivated by the fact that the mean free path “$\ell$” can be large with respect to the particle separation $a \sim \Lambda^{-1}$ (standard transport theories start at the scale “$\ell$” and investigate only processes at larger scales governed by diffusion) we investigate at finite temperatures the competition between localization and interaction. The competition between multiple scattering (due to disorder)
and the interactions is investigated within a RG theory. The method used here is different from the procedure used in ref. [2]. In ref. [2] one emphasizes the disorder by replacing the multiple elastic scattering by a diffusion theory and in the second step the interactions are treated perturbatively. We consider a situation where the elastic mean free path is much larger than any microscopic length. Therefore we might have a situation that before entering the diffusive region we have to stop scaling. This can happen if the thermal wave length is shorter than the elastic mean free path or that the Cooper channel diverges giving rise to superconductivity. In the quantum region the single particle excitations are well-defined and the Fermi surface is parametrized in terms of \( N_0 = \frac{\pi k_F}{\Lambda} \) channels. When the cutoff \( \Lambda \) is reduced \( \Lambda \to \Lambda/b \), one finds that the interactions scale like \( \Gamma \to \Gamma b^{1-d} \) and the number of channels, increases like \( N = N_0 b \) [9]. The disorder scales like \( D \to D b^{2-d} \). Due to the fact that the number of channels increase under scaling, we find that the interaction is marginal and the disorder is relevant. The quantum region gives rise to a set of scaling equations for the interaction term \( \Gamma \): \( \Gamma^{(c)}_2 \)-particle-hole singlet, \( \Gamma^{(s)}_2 \)-particle-hole triplet, \( \Gamma^{(s)}_3 \)-particle-particle singlet and disorder \( D \) (\( d^{(s)}_3 \)-the Cooperon). Our results show that due to disorder \( \Gamma^{(s)}_3 \) might becomes negative resulting in a superconducting instability at \( T \to 0 \). This might give rise to an Insulator-Superconductor transition similar to what one has for superconducting films where a phase transition is expected [4]. In the absence of an instability the standard method at length scale \( b > b_{Dif} \), \( b_{Dif} = \frac{\Lambda}{2\pi/\ell} \) is the diffusion theory developed by Finkelstein. Here we consider the situation where the system is in the clean limit such that the microscopic mean free path \( \ell_o = \ell(b = 1) \) is large. Due to interaction we obtain that the mean free path \( \ell(b > 1) \) increases, \( \ell(b) > \ell_o \).

In this paper we will work at finite temperatures such the the thermal wavelength is shorter than the mean free path \( \ell \). We introduce a thermal length scale \( b_T = \frac{\pi k_F}{T} \) and consider the situation where \( b_{Dif} > b_T \). Since we have to stop the scaling scaling at \( b = b_T \) we are allowed to ignore the diffusive region. In the recent transport experiment \( E_F/T \sim 5 \) and \( K_F \ell \gg 5 \), therefore the condition \( b_{Dif} > b_T \) is realized. The presence of the cutoff \( b_T \) prevent
the number of channels to scale to infinity, instead we have \( N_o < N(b) \leq N(b_T) = \bar{N} = \frac{E_p}{T} \).

We solve the model under the condition \( b_{Dif} > b_T \) and find that the physics is controlled by the disorder “\( D \)” and the particle-hole triplet \( \Gamma_2^{(s)} \). We find that when the number of channels does not scale (This might be the case at finite temperature or for non-spherical Fermi surfaces, which obeys \( N(b) \simeq \text{Const.} \)), a fixed point in the plane \( \Gamma_2^{(s)} \) and \( D \) is obtained. This fixed point separates a metallic phase from a localized one. The metallic phase is caused by the fact that the particle-hole triplet flows to a stable fixed point causing a shift in the critical dimension from \( d = 2 \) to \( d < 2 \). The presence of the stable fixed point in the triplet channel causes power law behavior of the spin-spin correlations. The resistivity is expected to obey the scaling behavior: 

\[
\rho(D, \Gamma_2^{(s)}(b), T) = \rho(D(b), \Gamma_2^{(s)}(b), Tb^z); \quad z \simeq 1 \\
\Gamma_2^{(s)}(b) = \Gamma_2^* + (\Gamma_2^{(s)} - \Gamma_2^*)b^{-1/\nu_2} \quad \text{and} \quad D(b) = D^* + (D - D^*)b^{1/\nu_1}.
\]

Choosing \( Tb^z = T_o \) we obtain:

\[
\rho(D, \Gamma_2^{(s)}(b), T) \simeq \rho(D^*, \Gamma_2^*, T_o) + \text{const.}(\frac{D - D^*}{D^*})(\frac{T_o}{T_T})^{1/\nu_1}.
\]

In agreement with the experimental results given in ref. \[1\] the resistivity increases for \( D > D^* \) and decreases for \( D < D^* \). In the literature alternative theories have been proposed already: ref. \[14\] (phenomenological), ref. \[10\] (within the Finkelstein theory), as well as models which focus on the insulating side ref. \[15,6\].

The plan of this paper is: We introduce in Chapter \[1\] our microscopic model. We consider a two dimensional gas in the presence of a screened two-body potential and a static random potential. We follow a standard method for treating disorder. We use the “replica” method and perform the statistical average over the disorder. In the second step we parametrize the Fermi Surface (FS) in terms of \( N \) channels. Using this parametrization we identify in Appendix \[A\] all the possible interaction and disorder terms. We find that the interaction and disorder is best described in terms of chiral currents carrying indices of charge, spin, replica, and channel. In Chapter \[11\] the method of the Renormalization Group (RG) based on the Operator Product Expansion (OPE) is introduced. We compute the OPE rules for the different interaction terms, particle-hole (p-h) singlet, p-h triplet, particle-particle (p-p)
and the Cooperon (the effective interaction induced by the disorder). Chapter [V] is devoted to the derivation of the RG equations based on the OPE results obtained in Chapter [III]. In Chapter [V] we consider the scaling equations in the quantum limit. Chapter [VI] is devoted to the possible superconducting instability which might occur in the quantum region. In Chapter [VII] we investigate the scaling equations at finite temperatures. Here we observe that the physics is determined by the effective number of channels $\bar{N}$. In Chapter [VIII] we solve the RG equations and compute the resistivity. Chapter [IX] is limited to discussions and conclusions.

II. THE MICROSCOPIC MODEL

We introduce the screened two-body potential and perform a statistical average over the disorder using the replica method. We parametrize the FS in terms of $N$ Fermions. Using these Fermions we replace the interaction terms and the Cooperon by chiral currents. The starting point of our investigation is the averaged disorder partition function, $\bar{Z}^\alpha$, $\alpha = 1, ..., \alpha \to 0$,

$$\bar{Z}^\alpha = \int D[\bar{\psi}, \psi] e^{-S}, \quad \alpha = 1, ..., \alpha \to 0$$  \hspace{1cm} (1)

$$S_o = \int d^d x \int dt \left\{ \sum_\sigma \sum_\alpha [\bar{\psi}_{\sigma,\alpha} \partial_t \psi_{\sigma,\alpha} - \bar{\psi}_{\sigma,\alpha} (\nabla^2 \frac{m}{2} + E_F) \psi_{\sigma,\alpha}] \right\}$$  \hspace{1cm} (2)

$$S_{int} = \int d^d x \int d^d y \int dt \sum_\sigma \sum_\alpha \{ \bar{\psi}_{\sigma,\alpha}(x) \bar{\psi}_{\sigma',\alpha}(y) v(x - y) \psi_{\sigma',\alpha}(y) \psi_{\sigma,\alpha}(x) \}$$  \hspace{1cm} (3)

$$S_D = - \int dt_1 \int dt_2 \int d^d x \int d^d y \sum_{\sigma,\sigma', \alpha, \beta} \left\{ \bar{V}(x) \bar{V}(y) \bar{\psi}_{\sigma,\alpha}(x, t_1) \bar{\psi}_{\sigma',\alpha}(y, t_2) \psi_{\sigma',\beta}(y, t_2) \psi_{\sigma,\beta}(x, t_1) \right\}$$  \hspace{1cm} (4)

"$v(x - y)$" is the two body screened potential and $\bar{V}(x) \bar{V}(y) = D\delta(x - y)$ where $D = \frac{v_F^2}{\hbar v_F}$ is the disorder parameter controlled by the elastic scattering time $\tau = \ell/v_F$. Next we
parametrize the Fermi surface (FS) in terms of $N$ Fermions or $N/2$ pairs of right and left movers (see ref. [9]):

$$\psi_{\sigma,\alpha}(\vec{x}) = \sum_{n=1}^{N/2} (e^{i k_F \hat{n} \cdot \vec{x}} R_{n,\sigma,\alpha}(\vec{x}) + e^{-i k_F \hat{n} \cdot \vec{x}} L_{n,\sigma,\alpha}(\vec{x}))$$ (5)

$R_{n,\sigma,\alpha}(\vec{x})$ and $L_{n,\sigma,\alpha}(\vec{x})$ are right and left movers defined by momenta $|q||<\Lambda$, $|q_\perp|<\Lambda$ around each Fermi point $k_F = k_F \hat{n}$. The Fermi momentum is determined by the renormalized Fermi energy $\bar{E}_F$ which is related to the non-interacting Fermi energy $E_F$ by the relation $\bar{E}_F = E_F + \delta \mu_F$, such that $\bar{E}_F = k_F^2 2 m$. The value of $\delta \mu_F$ is obtained from the interaction. The two dimensional Fermions are expressed in terms of the one dimensional Fermions $\hat{R}_{n,\sigma,\alpha}(x_\parallel)$ and $\hat{L}_{n,\sigma,\alpha}(x_\parallel)$:

$$R_{n,\sigma,\alpha}(\vec{x}) = \hat{R}_{n,\sigma,\alpha}(x_\parallel) Z_n(x_\perp), \quad L_{n,\sigma,\alpha}(\vec{x}) = \hat{L}_{n,\sigma,\alpha}(x_\parallel) Z_n(x_\perp)$$

$Z_n(x_\perp)$ is scalar function which ensures the conservation of momentum in the transversal direction. The number of channels (Fermions) is related to $k_F$ and cutoff $\Lambda < k_F$, $N_o = \frac{\pi k_F}{\Lambda}$. Using the representation given in Eq.5 we introduce the normal order currents $J^R_{n,\alpha,\sigma}(Z)$ (right mover) and $J^L_{n,\alpha,\sigma}(\bar{Z})$ (left mover) with $Z$ and $\bar{Z}$ given by $Z = (Z_\parallel, Z_\perp)$, $\bar{Z} = (\bar{Z}_\parallel, \bar{Z}_\perp)$, $Z_\parallel = v_F t - i x_\parallel$, $\bar{Z}_\parallel = v_F t + i x_\parallel$, and $Z_\perp = \bar{Z}_\perp = x_\perp$,

$$J^R_{n,\alpha,\sigma}(Z) =: R_{n,\alpha,\sigma}^\dagger(Z) R_{n,\alpha,\sigma}(Z) := R_{n,\alpha,\sigma}^\dagger(Z + \epsilon) R_{n,\alpha,\sigma}(Z) - \langle R_{n,\alpha,\sigma}^\dagger(Z + \epsilon) R_{n,\alpha,\sigma}(Z) \rangle_o$$ (6)

with $\epsilon = \varepsilon - i \delta$, $\epsilon \to 0$ and the expectation value:

$$\langle R_{n,\alpha,\sigma_1}^\dagger(\vec{x}, t_1) R_{m,\beta,\sigma_2}(\vec{y}, t_2) \rangle_o \sim \delta_{n,m} \delta_{\alpha,\beta} \delta_{\sigma_1,\sigma_2} \delta_A^{-1} (x_\perp - y_\perp) [v_F(t_1 - t_2) - i(x_\parallel - y_\parallel)]^{-1}$$ (7)

Similarly we introduce for the left movers:

$$J^L_{n,\alpha,\sigma}(\bar{Z}) =: L_{n,\alpha,\sigma}^\dagger(\bar{Z}) L_{n,\alpha,\sigma}(\bar{Z}) := L_{n,\alpha,\sigma}^\dagger(\bar{Z} + \bar{\epsilon}) L_{n,\alpha,\sigma}(\bar{Z}) - \langle L_{n,\alpha,\sigma}^\dagger(\bar{Z} + \bar{\epsilon}) L_{n,\alpha,\sigma}(\bar{Z}) \rangle_o$$ (8)

We write the interaction and the disorder parts in the normal order form. From the disorder part we obtain the elastic scattering term $\frac{1}{27} \propto D$ (see ref. [12]). From the disorder part (Eq.4) we obtain the normal order form $\tilde{S}_D$. 6
From the screened two-body potential we find the normal order representation \( \tilde{S}_{\text{int}} \) plus a shift of the Fermi energy: \( \delta \mu_{\text{int}}(J^R_{n,\alpha,\sigma}(x, t) + J^L_{n,\alpha,\sigma}(x, t)) \). We choose \( \delta \mu_F \) such that it cancels the interaction shift, \( \delta \mu_F + \delta \mu_{\text{int}} = 0 \). As a result \( S_0 \) becomes:

\[
\tilde{S}_o = \frac{N/2}{\Lambda^d} \sum_{n=1}^{N/2} \sum_{\alpha} \int dt \int d^d x \sum_{\alpha} \left\{ \partial_t - v_F \vec{n} \cdot \vec{\partial} \right\} R_{n,\alpha,\sigma} + \tilde{L}_{n,\alpha,\sigma} \partial_t + v_F \vec{n} \cdot \vec{\partial} \right\} \{ \partial_t + v_F \vec{n} \cdot \vec{\partial} \} L_{n,\alpha,\sigma} \right\} \tag{9}
\]

Using the representation given in Eq. 8 we replace the interaction term and disorder in terms of the currents (see appendix A). The interaction part is decomposed in terms of forward scattering \( Q^{(F)}_{n,m}(t, \vec{x}, \vec{y}) \) (charge part), \( H^{(F)}_{n,m}(t, \vec{x}, \vec{y}) \) (spin part), \( Q^{(B)}_{n,m}(t, \vec{x}, \vec{y}) \) (particle-hole in the singlet channel), \( H^{(B)}_{n,m}(t, \vec{x}, \vec{y}) \) (particle-hole in the triplet channel), \( O^{(s)}_{n,m}(t, \vec{x}, \vec{y}) \) (particle-particle in the singlet channel), \( O^{(t)}_{n,m}(t, \vec{x}, \vec{y}) \) (particle-particle in the triplet channel).

From the screened two-body potential takes the form:

\[
\begin{align*}
\Gamma^{(c)}(\vec{n}, \vec{m}), \Gamma^{(s)}(\vec{n}, \vec{m}), & \Gamma^{(c)}_2(\vec{n}, \vec{m}), \Gamma^{(s)}_2(\vec{n}, \vec{m}), \Gamma^{(s)}_3(\vec{n}, \vec{m}), \Gamma^{(t)}_3(\vec{n}, \vec{m}). \\
\end{align*}
\]

For the screened case the matrix elements \( \Gamma(\vec{n}, \vec{m}) \) depend only on the angles “\( \theta \)” on the FS. For example, if \( \kappa \) is the inverse of the screening length we have \( \Gamma^{(s)}_2(\vec{n}, \vec{m}) = 2\kappa[1 + 2k_F \cos \theta/2]^{-1}, 0 \leq \theta \leq \pi \) (“\( \theta \)” is the angle between the unit vectors \( \vec{n} \) and \( \vec{m} \)). The particle-particle matrix is, \( \Gamma^{(s)}_3(\vec{n}, \vec{m}) = \frac{2}{\kappa}[(1 + 2k_F \sin \theta/2)^{-1} + (1 + 4k_F \cos \theta/2)^{-1}], 0 \leq \theta \leq \pi. \)

We introduce the left and right currents and obtain the representations for the interaction and disorder terms:

\[
\begin{align*}
J^R_{n,\alpha,\sigma_1; m,\beta,\sigma_2}(\vec{x}, t_1, t_2) =: & R^\dagger_{n,\alpha,\sigma_1}(\vec{x}, t_1) R_{m,\beta,\sigma_2}(\vec{x}, t_2) : \\
J^L_{n,\alpha,\sigma_1; m,\beta,\sigma_2}(\vec{x}, t_1, t_2) =: & L^\dagger_{n,\alpha,\sigma_1}(\vec{x}, t_1) L_{m,\beta,\sigma_2}(\vec{x}, t_2) : \tag{10}
\end{align*}
\]

For the interaction term we have \( t_1 = t_2 \) and \( \alpha = \beta \). We obtain that the interaction part for a screened two-body potential takes the form:

\[
\begin{align*}
\tilde{S}_{\text{int}} = & \frac{1}{2N_o} \sum_n \sum_m \int d^d x \int dt \sum_{\alpha} \left\{ \Gamma^{(c)}(\vec{n}, \vec{m}) Q^{(F)}_{n,m;\alpha}(\vec{x}, t) - \Gamma^{(s)}(\vec{n}, \vec{m}) H^{(F)}_{n,m;\alpha}(\vec{x}, t) \right. \\
+ & \left. \Gamma^{(c)}_2(\vec{n}, \vec{m}) Q^{(B)}_{n,m;\alpha}(\vec{x}, t) - \Gamma^{(s)}_2(\vec{n}, \vec{m}) H^{(B)}_{n,m;\alpha}(\vec{x}, t) + \Gamma^{(s)}_3(\vec{n}, \vec{m}) O^{(s)}_{n,m;\alpha}(\vec{x}, t) + \Gamma^{(t)}_3(\vec{n}, \vec{m}) O^{(t)}_{n,m;\alpha}(\vec{x}, t) \right\} \\
\end{align*}
\]

(11)
In Eq. [11] we have to restrict $\Gamma_3^{(s)}(\vec{n}, \vec{m})$ and $\Gamma_3^{(t)}(\vec{n}, \vec{m})$ to $\vec{n} \neq \vec{m}$ in order to avoid double counting. If we ignore the angle dependence of $\Gamma_3^{(t)}$ we have $\Gamma_3^{(t)} \approx 0$. For $\vec{n} = \vec{m}$ we have the relation $\Gamma_3^{(s)}(\vec{n}, \vec{n}) = \frac{1}{2} \Gamma_2^{(s)}(\vec{n}, \vec{n})$. Based on dimensional analysis we obtain that the interaction term has the dimension of $\Lambda^{1-d}$. Due to the fact that the interaction is defined at the scale $\Lambda \ll k_F$ we have the relation $k_F^{1-d} \Gamma(k_F) = \Lambda^{1-d} \frac{(k_F)}{N_o} \Gamma(\Lambda)$ where $N_o = \pi \frac{(k_F)}{X} d - 1$ is the number of channels. The operators $Q_{n,m;\alpha}^{(F)}$, $H_{n,m;\alpha}^{(F)}$, $Q_{n,m;\alpha}^{(B)}$, $H_{n,m;\alpha}^{(B)}$, $O_{n,m;\alpha}^{(s)}$, and $O_{n,m;\alpha}^{(t)}$ are given by:

$$Q_{n,m;\alpha}(\vec{x}, t) = J_{n,\alpha}^{R}(\vec{x}, t) J_{m,\alpha}^{R}(\vec{x}, t) + J_{n,\alpha}^{L}(\vec{x}, t) J_{m,\alpha}^{L}(\vec{x}, t),$$

$$H_{n,m;\alpha}(\vec{x}, t) = \vec{J}_{n,\alpha}^{R}(\vec{x}, t) \cdot \vec{J}_{m,\alpha}^{R}(\vec{x}, t) + \vec{J}_{n,\alpha}^{L}(\vec{x}, t) \cdot \vec{J}_{m,\alpha}^{L}(\vec{x}, t),$$

$$Q_{n,m;\alpha}(\vec{x}, t) = J_{n,\alpha}^{R}(\vec{x}, t) J_{m,\alpha}^{L}(\vec{x}, t) + J_{n,\alpha}^{L}(\vec{x}, t) J_{m,\alpha}^{R}(\vec{x}, t),$$

$$H_{n,m;\alpha}(\vec{x}, t) = \vec{J}_{n,\alpha}^{R}(\vec{x}, t) \cdot \vec{J}_{m,\alpha}^{L}(\vec{x}, t) + \vec{J}_{n,\alpha}^{L}(\vec{x}, t) \cdot \vec{J}_{m,\alpha}^{R}(\vec{x}, t),$$

$$O_{n,m;\alpha}^{(s)}(\vec{x}, t) = O_{n,m;\alpha}^{(||)}(\vec{x}, t) - O_{n,m;\alpha}^{(\perp)}(\vec{x}, t), \quad O_{n,m;\alpha}^{(t)}(\vec{x}, t) = O_{n,m;\alpha}^{(||)}(\vec{x}, t) + O_{n,m;\alpha}^{(\perp)}(\vec{x}, t),$$

$$O_{n,m;\alpha}^{(\perp)}(\vec{x}, t) = \sum_{\sigma} : R_{n,\alpha,\sigma}^{\dagger}(\vec{x}, t) R_{m,\alpha,-\sigma}(\vec{x}, t) : L_{n,\alpha,-\sigma}^{\dagger}(\vec{x}, t) L_{m,\alpha,\sigma}(\vec{x}, t) :$$

$$+ : L_{n,\alpha,\sigma}^{\dagger}(\vec{x}, t) L_{m,\alpha,-\sigma}(\vec{x}, t) : R_{n,\alpha,-\sigma}^{\dagger}(\vec{x}, t) R_{m,\alpha,\sigma}(\vec{x}, t) :,$$

$$O_{n,m;\alpha}^{(||)}(\vec{x}, t) = \sum_{\sigma} : R_{n,\alpha,\sigma}^{\dagger}(\vec{x}, t) R_{m,\alpha,\sigma}(\vec{x}, t) : L_{n,\alpha,\sigma}^{\dagger}(\vec{x}, t) L_{m,\alpha,\sigma}(\vec{x}, t) :$$

$$+ : L_{n,\alpha,\sigma}^{\dagger}(\vec{x}, t) L_{m,\alpha,\sigma}(\vec{x}, t) : R_{n,\alpha,\sigma}^{\dagger}(\vec{x}, t) R_{m,\alpha,\sigma}(\vec{x}, t) :,$$

where

$$J_{n,\alpha}^{R}(\vec{x}, t) = \sum_{\sigma} : R_{n,\alpha,\sigma}^{\dagger}(\vec{x}, t) R_{n,\alpha,\sigma}(\vec{x}, t) :,$$

$$J_{n,\alpha}^{R}(\vec{x}, t) = \frac{1}{2} : R_{n,\alpha,\sigma_{1}}^{\dagger}(\vec{x}, t) \sigma_{\sigma_{1},\sigma_{2}} R_{n,\alpha,\sigma_{2}}(\vec{x}, t) :.$$
with similar expressions for the left movers. \( \Lambda < k_F \) is the cutoff of the theory and we find that the naive dimension of the interaction field is \( \Lambda^{1-d} \) \( (d = 2) \). This follows from the fact that Eq. [9] is invariant under the scaling \( \Lambda \rightarrow \Lambda/b, \ x = x'b, \ t = t'b, \ R_n(\vec{x}, t) = b^{-d/2}R_n(\vec{x}', t'), \ L_n(\vec{x}, t) = b^{-d/2}L_n(\vec{x}', t'), \) and \( N(b) = bN_o \). Following the same procedure as for the interaction we express the disorder part using again the respective part:

\[
\tilde{S}_D = -\frac{\Lambda^{2-d}}{N_o} \sum_n \sum_m \sum_{\alpha,\beta} \int dt_1 \int dt_2 \int d^d x \left\{ d_2^{(d)} \rho_{n,m,\alpha,\beta}(\vec{x}; t_1, t_2) - d_2^{(c)} q_{n,m,\alpha,\beta}(\vec{x}; t_1, t_2) \right. \\
- \left. d_2^{(s)} h_{n,m,\alpha,\beta}(\vec{x}; t_1, t_2) + d_2^{(s)} c_{n,m,\alpha,\beta}(\vec{x}; t_1, t_2) + d_2^{(t)} c_{n,m,\alpha,\beta}(\vec{x}; t_1, t_2) \right\} 
\tag{14}
\]

The operators in Eq. [14] are in complete analogy with the ones in Eq. [11], except that they are at different times and have double replica index:

\[
\rho_{n,m,\alpha,\beta}(\vec{x}; t_1, t_2) \leftrightarrow Q^{(F)}_{n,m,\alpha}(\vec{x}, t);
\]

\[
q_{n,m,\alpha,\beta}(\vec{x}; t_1, t_2) \leftrightarrow Q^{(B)}_{n,m,\alpha}(\vec{x}, t);
\]

\[
h^{(B)}_{n,m,\alpha}(\vec{x}; t_1, t_2) \leftrightarrow H^{(B)}_{n,m,\alpha}(\vec{x}, t);
\]

\[
c^{(s)}_{n,m,\alpha,\beta}(\vec{x}; t_1, t_2) \leftrightarrow O^{(s)}_{n,m,\alpha}(\vec{x}, t);
\]

\[
c^{(t)}_{n,m,\alpha,\beta}(\vec{x}; t_1, t_2) \leftrightarrow O^{(t)}_{n,m,\alpha}(\vec{x}, t)
\]

The corresponding constants in Eq. [14] have the initial values: \( d_3^{(s)} = d_2^{(c)} = \frac{1}{2}d_2^{(s)} \equiv D, \)

\( d_3^{(t)} = 0 \). We will find that only the Cooperon term, \( d_3^{(s)} c_{n,m,\alpha,\beta}(\vec{x}; t_1, t_2) \) is important. For the rest part of this paper we will ignore the rest of the terms and consider only the Cooperon part.
III. THE RENORMALIZATION GROUP METHOD

In the first part of this chapter we will introduce the RG method based on the OPE. This method is needed in order to analyze the possible phase diagram of our problem. The real space method based on the Operator Product Expansion (OPE) introduced in ref. [13] is in particular advantageous. In order to explain how this works we express the action in Eq.11 by a formal expression
\[ S \sim \sum \Gamma_i \mathcal{A}_i \]
where \( \mathcal{A}_i \) are the operators and \( \Gamma_i \) are the coupling constants. Using the fact that the time ordered product of the single particle operator is given by,
\[ R_{n,\alpha,\sigma}(\vec{x},t_1)R_{m,\beta,\sigma}(\vec{y},t_2) \sim \frac{1}{2\pi} \delta_{n,m} \delta_{\alpha,\beta} \delta_{\sigma,\sigma} \delta_\Lambda^{d-1}(x_\perp - y_\perp) \theta(t_1 - t_2) \]
where \( x_{\parallel} = \hat{n} \cdot \vec{x} \), \( x_{\perp} = \vec{x} - \hat{n} \cdot \vec{x} \). We find for any two operators given in Eq.11 the OPE:
\[ A_i(\vec{x},t_1)A_j(\vec{x} + a,t_2) \sim \sum_K C_{ij}^{K} F_K(|t_1 - t_2|) A_K(\vec{x},t_1 + t_2) \]
with \( C_{ij}^{K} \) the structure constant and \( F_K(|t_1 - t_2|) \sim 1 \). As a result the product of any number of operators can be reduced to a sum of operators. This implies that once the cutoff \( \Lambda \) is reduced to \( \Lambda/b \), one can obtain the scaling equations for coupling constants \( \Gamma_i \). For \( \Gamma_i \) with the scaling dimension \( \Gamma_i \rightarrow \Gamma_i b^{(x_i - d)} \), one obtains:
\[ \frac{d\Gamma_K}{d \ln b} = -(d - x_K)\Gamma_K - \frac{1}{2} \sum_{i,j} \tilde{C}_{ij}^{K} \Gamma_i \Gamma_j + \frac{1}{3!} \sum_{i,j} \sum_{p,q} \tilde{C}_{ij}^{p} \tilde{C}_{ij}^{q} \Gamma_i \Gamma_j \Gamma_q \]
where the \( \tilde{C}_{i,j}^{K} \) are proportional to the structure constants \( C_{i,j}^{K} \). In order to be able to complete the RG equation given in Eq.17 we have to compute the operator product expansion of the operators which appear in Eqs.11 and 14. The second part of this chapter will be devoted to the calculation of the OPE for the interaction and disorder operators. Using current algebra of the chiral currents given in ref. [16] we will establish the OPE rules for our problem. The calculation is based on the Wick theorem which replaces the time order product by the
normal ordered form plus all the possible ways of contracting pairs of Fermion fields. This
calculation is standard and lengthy therefore we will present only the results. We start with
the results for the p-p singlet:

\[
O^{(s)}_{n,m,\alpha}(\vec{x}, t)O^{(s)}_{k,l,\beta}(\vec{x} + \vec{a}, t + \tau) = \frac{1}{(2\pi)^2} \left( \frac{\Lambda}{2\pi} \right)^{d-1} \delta_{\alpha,\beta} [a^2 + (v_F \tau)^2]^{-1} \{ O^{(s)}_{k,m,\alpha}(\vec{x}, t) \delta_{n,l} \\
+ O^{(s)}_{n,l,\alpha}(\vec{x}, t) \delta_{k,m} (Q^{(B)}_{n,m,\alpha}(\vec{x}, t) + Q^{(B)}_{m,n,\alpha}(\vec{x}, t)) \} + \text{"c" number}. \tag{18}
\]

From Eq.\ref{18} we learn that the OPE generates p-p and p-h singlets.

For the p-h in the triplet channel no new terms are generated:

\[
H^{(B)}_{n,m,\alpha}(\vec{x}, t)H^{(B)}_{k,l,\beta}(\vec{x} + \vec{a}, t + \tau) = \frac{1}{(2\pi)^2} \left( \frac{\Lambda}{2\pi} \right)^{d-1} \delta_{\alpha,\beta} [a^2 + (v_F \tau)^2]^{-1} \{ -2H^{(B)}_{n,m,\alpha}(\vec{x}, t) [\delta_{n,l}\delta_{k,m} + \delta_{n,k}\delta_{l,m}] \} + \text{"c" number} \tag{19}
\]

The p-h singlet generates only a "c" number:

\[
Q^{(B)}_{n,m,\alpha}(\vec{x}, t)Q^{(B)}_{k,l,\beta}(\vec{x} + \vec{a}, t + \tau) = \text{"c" number} \tag{20}
\]

The OPE for the Cooperon do not generate new terms:

\[
C^{(s)}_{n,m;\alpha,\beta}(\vec{x}; t_1, t_2)C^{(s)}_{k,l;\alpha',\beta'}(\vec{x} + \vec{a}; t_1 + \tau_1, t_2 + \tau_2) = \frac{1}{(2\pi)^2} \left( \frac{\Lambda}{2\pi} \right)^{d-1} \left[ \frac{1}{(v_F \tau_1 - ia)(v_F \tau_2 + ia)} \right]^{1/2} \\\n+ \frac{1}{(v_F \tau_1 + ia)(v_F \tau_2 - ia)} \right] \{ 2\delta_{\alpha,\beta} \delta_{\alpha',\beta'} [C^{(s)}_{n,m;\alpha,\beta}(\vec{x}; t_1, t_2) + \delta_{n,l} C^{(s)}_{k,m;\alpha,\beta}(\vec{x}; t_1, t_2)] \\
- \frac{1}{2} \delta_{\alpha,\alpha'} \delta_{\beta,\beta'} \delta_{n,l} [C^{(s)}_{n,m;\alpha,\beta}(\vec{x}; t_1, t_2) + C^{(s)}_{m,n;\alpha,\beta}(\vec{x}; t_1, t_2)] \} + \text{"c" number} \tag{21}
\]

The OPE between the p-p and p-h triplet generates the p-p operator and the p-h singlet:

\[
O^{(s)}_{n,m,\alpha}(\vec{x}, t)H^{(B)}_{k,l,\beta}(\vec{x} + \vec{a}, t + \tau) = \frac{1}{(2\pi)^2} \left( \frac{\Lambda}{2\pi} \right)^{d-1} \delta_{\alpha,\beta} [a^2 + (v_F \tau)^2]^{-1} \{ \frac{3}{4} \delta_{l,k}\delta_{n,l}\delta_{m,k} Q^{(B)}_{n,m}(\vec{x}, t) \} \\
- \frac{1}{8} \delta_{l,k} \{ \delta_{n,l} O^{(s)}_{l,m,\alpha}(\vec{x}, t) + \delta_{m,l} O^{(s)}_{n,l,\alpha}(\vec{x}, t) \} \} + \text{"c" number} \tag{22}
\]
The OPE between the p-p term and the p-h singlet generates a “c” number:

\[ O_{n,m,\alpha}^{(B)}(\vec{x}+\vec{a},t+\tau_1,t_2+\tau_2) = \frac{1}{(2\pi)^2} \frac{\Lambda}{2\pi} \frac{1}{(v_F \tau_1 - ia)(v_F \tau_2 + ia)} \]

(23)

The product for the product p-h triplet and p-h singlet gives a “c” number:

\[ H_{n,m,\alpha}^{(B)}(\vec{x}+\vec{a},t+\tau_1,t+\tau_2) = \{“c” \text{ number}\} \]

(24)

In the remaining part we present the OPE between the Cooperon and the interaction operators. For the p-p case we generate the Cooperon and p-p operator:

\[ O_{n,m,\gamma}^{(s)}(\vec{x},t) C_{k,l,\alpha,\beta}^{(s)}(\vec{x}+\vec{a},t_1+\tau_1,t_2+\tau_2) = \frac{1}{(2\pi)^2} \frac{\Lambda}{2\pi} \frac{1}{(v_F \tau_1 - ia)(v_F \tau_2 + ia)} \]

(25)

For the p-h triplet one obtains the p-p and Cooperon terms:

\[ H_{n,m,\gamma}^{(B)}(\vec{x},t) C_{k,l,\alpha,\beta}^{(s)}(\vec{x}+\vec{a},t_1+\tau_1,t_2+\tau_2) = \frac{1}{(2\pi)^2} \frac{\Lambda}{2\pi} \frac{1}{(v_F \tau_1 - ia)(v_F \tau_2 + ia)} \]

(26)

When we consider the p-h singlet we generate the p-p and Cooperon terms.

\[ Q_{n,m,\gamma}^{(s)}(\vec{x},t) C_{k,l,\alpha,\beta}^{(s)}(\vec{x}+\vec{a},t_1+\tau_1,t_2+\tau_2) = \frac{1}{(2\pi)^2} \frac{\Lambda}{2\pi} \frac{1}{(v_F \tau_1 - ia)(v_F \tau_2 + ia)} \]

(27)
\[
\frac{1}{2} (\delta_{n,k} \delta_{m,l} + \delta_{m,k} \delta_{n,l})(\delta \gamma, \alpha + \delta \gamma, \beta) + \left[ \frac{1}{2} \frac{1}{(v_F \tau_1 - ia)(v_F \tau_2 + ia)} + \frac{1}{2} \frac{1}{(v_F \tau_1 + ia)(v_F \tau_2 - ia)} \right] \\
\delta \gamma, \alpha \delta \gamma, \beta (\delta_{m,k} \delta_{m,l} + \delta_{n,k} \delta_{n,l}) (O^{(s)}_{m,n,\gamma}(\vec{x}, t) - O^{(t)}_{m,n,\gamma}(\vec{x}, t)) + \text{"c" number} \quad (27)
\]

The Eqs. 18-27 have been obtained using the free Fermion action given in Eq. 9. We will work at a finite temperature, therefore Eq. 13 is an approximation of the exact propagator \( \{ \frac{v_F \beta}{\pi} \sin \left( \frac{\pi}{v_F \beta} (v_F (t_1 - t_2) - i(x_\parallel - y_\parallel)) \right) \}^{-1} \). This means that the "t" range of integration in Eqs. 17 and 27 is restricted to \( t < \beta \).

**IV. DERIVATION OF THE RG EQUATIONS**

This chapter is designated to the computation of the RG equations. This will be done by expanding the partition function \( Z \) in terms of the interaction and disorder operators. Using the OPE rules derived in Eqs. 18-27 will allow to replace the product of operators in terms of a sum of operators. When rescaling the minimal distance "a" to "ba" will allow to find the scaling equations. It is important to remark that the method used here is different from the standard method used for problems with disorder. The traditional method [2] starts from the diffusion theory and includes the interaction terms as a perturbation. Here we start from the Fermion theory and include simultaneously on equal footing the effects of interaction and disorder. In the standard approach the quantum diffusion theory ignores completely the effects of interactions at short distances (distances shorter than the mean free path). We will see that considering the disorder and interaction on equal footing new terms will appear in the RG equations. The scaling equations will contain terms which are controlled by the number of channels.

Following the analysis given in section 4 we have:

\[
\tilde{S}_o = \frac{N}{2} \sum \sum_{\sigma} \int d^d x \int dt \{ \tilde{R}_{n,a,\sigma} [\partial_t - v_F \nabla \cdot \vec{\partial}] R_{n,a,\sigma} + \tilde{L}_{n,a,\sigma} [\partial_t + v_F \nabla \cdot \vec{\partial}] L_{n,a,\sigma} \}. \quad (28)
\]
The action $\tilde{S}_o$ determines the partition function $Z_o$,
\[
Z_o = \int D[\tilde{\psi}, \psi] e^{-\tilde{S}_o}.
\]

We perturb the partition function $Z_o$ by the interaction $\tilde{S}_{int}$ and disorder $\tilde{S}_D$:
\[
\tilde{S}_{int} = \frac{\Lambda^{1-d}}{2N_o} \sum_n \sum_m \sum_\alpha \int d^d x \int d\{\Gamma^{(c)}(\vec{n}, \vec{\bar{m}})Q^{(F)}_{n,m,\alpha}(\vec{x},t) - \Gamma^{(s)}(\vec{n}, \vec{\bar{m}})H^{(F)}_{n,m,\alpha}(\vec{x},t) + \Gamma^{(c)}(\vec{n}, \vec{\bar{m}})Q^{(B)}_{n,m,\alpha}(\vec{x},t)

- \Gamma^{(s)}(\vec{n}, \vec{\bar{m}})H^{(B)}_{n,m,\alpha}(\vec{x},t) + (1 - \delta_{n,m}) \Gamma^{(s)}_{3}(\vec{n}, \vec{\bar{m}})O_{n,m,\alpha}^{(s)}(\vec{x},t)\}
\]
\[
= \frac{\Lambda^{1-d}}{2N_o} \sum_n \sum_m \sum_\alpha \int d^d x \int d\{\Gamma^{(c)}(\vec{n}, \vec{\bar{m}})Q^{(F)}_{n,m,\alpha}(\vec{x},t) - \Gamma^{(s)}(\vec{n}, \vec{\bar{m}})H^{(F)}_{n,m,\alpha}(\vec{x},t) + e_2^{(s)}(\vec{n}, \vec{\bar{m}})Q^{(B)}_{n,m,\alpha}(\vec{x},t)

- e_2^{(s)}(\vec{n}, \vec{\bar{m}})H^{(B)}_{n,m,\alpha}(\vec{x},t) + e_3^{(s)}(\vec{n}, \vec{\bar{m}})O_{n,m,\alpha}^{(s)}(\vec{x},t)\}
\]
\]

In Eq.29 we have ignored for simplicity the particle-particle triplet and consider only the particle-particle singlet.

Due to the relation between the particle-particle and the particle-hole triplets we remove the term $(1 - \delta_{n,m})$ by defining new coupling constants:
\[
e^{(c)}_2(\vec{n}, \vec{\bar{m}}) = \Gamma^{(c)}_2(\vec{n}, \vec{\bar{m}}) - \frac{1}{2} \delta_{n,m} \Gamma^{(s)}_3(\vec{n}, \vec{\bar{m}}),
\]
\[
e^{(s)}_2(\vec{n}, \vec{\bar{m}}) = \Gamma^{(s)}_2(\vec{n}, \vec{\bar{m}}) - 2\delta_{n,m} \Gamma^{(s)}_3(\vec{n}, \vec{\bar{m}}),
\]
\[
e^{(s)}_3(\vec{n}, \vec{\bar{m}}) = \Gamma^{(s)}_3(\vec{n}, \vec{\bar{m}}).
\]

The results in Eqs.30 follows from the operator identity
\[
O^{(s)}_{n,m,\alpha}(\vec{x},t) = (1 - \delta_{n,m})O^{(s)}_{n,m,\alpha}(\vec{x},t) + \delta_{n,m}[\frac{1}{2} Q^{(B)}_{n,m,\alpha}(\vec{x},t) - 2H^{(B)}_{n,m,\alpha}(\vec{x},t)]
\]
\]

Eq.31 follows directly from the definitions of the particle-particle singlet for $\vec{n} = \vec{\bar{m}}$ in terms of the currents ( see Eqs.12-13 ). For $\vec{n} = \vec{\bar{m}}$ the particle-hole triplet $\Gamma^{(s)}_2(\vec{n}, \vec{\bar{m}})$ is related to the particle-particle singlet $\Gamma^{(s)}_3(\vec{n}, \vec{\bar{m}})$ ( see Eq.A4 )
\[
\frac{1}{2} \Gamma_2^{(s)}(\vec{n}, \vec{n}) = \Gamma_3^{(s)}(\vec{n}, \vec{n}).
\]  

(32)

For the disorder part we will consider only the dominant Cooperon term \(d_3^{(s)} C_{n,m;\alpha,\beta}^{(s)}(\vec{x}; t_1, t_2)\) and we will ignore the effect of forward disorder

\[
\bar{S}_D = -\frac{\Lambda^{2-d}}{N_o} \sum_n \sum_m \sum_\alpha \sum_\beta \int d^d x \int dt_1 \int dt_2 \{d_3^{(s)} C_{n,m;\alpha,\beta}^{(s)}(\vec{x}; t_1, t_2)\}.
\]  

(33)

Following ref. [13] we compute the partition function \(Z\) of the action \(\bar{S}_o + \bar{S}_{int} + \bar{S}_D\) by expanding up to the third order in \(\bar{S}_{int} + \bar{S}_D\). Using \(Z_o\) we obtain:

\[
Z = Z_o \{1 - [\langle \bar{S}_{int} \rangle_a + \langle \bar{S}_D \rangle_a - \frac{1}{2} \langle \bar{S}_{int}^2 \rangle_a - \langle \bar{S}_{int} \bar{S}_D \rangle_a - \frac{1}{2} \langle \bar{S}_D^2 \rangle_a
\]

\[
+ \frac{1}{3!} \langle \bar{S}_{int}^3 \rangle_a + \frac{1}{3!} \langle \bar{S}_D^3 \rangle_a + \frac{1}{2} \langle \bar{S}_{int}^2 \bar{S}_D \rangle_a + \frac{1}{2} \langle \bar{S}_{int} \bar{S}_D^2 \rangle_a]\}. 
\]  

(34)

The meaning of \(\langle \cdots \rangle_a\) is to take the expectation value with respect to \(\bar{S}_o\) defined in Eq.28.

Since we want to perform a RG analysis we will take the expectation value only in the interval \((\Lambda, \Lambda/b), b \geq 1\). In real space this means to integrate from the microscopic distance \(a\) to \(ba\).

Next we will compute the first term in Eq.34

\[
\langle \bar{S}_{int} \rangle_{ba} = b^{2-d} \frac{N_o}{N(b)} \langle \bar{S}_{int} \rangle_a, \quad N(b) = N_o b
\]  

(35)

where \(\langle \cdots \rangle_{ba}\) represents the expectation value with respect to Eq.28 with the new cutoff \(\Lambda/b = 2\pi/ba\). The expectation value of \(\langle \bar{S}_D \rangle_a\) is different from Eq.34. The difference is due to the two times \(t_1\) and \(t_2\). For times \(|t_1 - t_2| \leq a/v_F\), \(C_{n,m;\alpha,\beta}^{(s)}(\vec{x}; t_1, t_2)\) is replaced by the singlet particle-particle interaction.

\[
\langle \bar{S}_D \rangle_{ba} = b^{2-d} \frac{N_o}{N(b)} \langle \bar{S}_D \rangle_a
\]  

(36)

\[
\Delta \langle \bar{S}_{int} \rangle_{ba} = -\frac{2a}{v_F} b^{2-d} \frac{N_o}{N(b)} \langle \bar{S}_D(t_1 = t_2) \rangle_a. 
\]  

(37)
Eq. 37 represents the contribution from the disorder Cooperon to the singlet particle-particle term when $|t_1 - t_2| \leq a/v_F$.

In order to compute the higher order term we have to use the rule of the operator product expansion defined in Eqs. 18-27, and have to perform the time integration. We introduce the notation $\langle \cdots \rangle_{da}$ which stands for the expectation value in the domain $(a, ba) - \frac{1}{2}$

$$-\frac{1}{2} \langle \tilde{S}_{int}^2 \rangle_{da} = \frac{da}{a} \left\{ \frac{\Lambda^{1-d}}{2N} \frac{A^{-1}}{4N} (-1) \sum_n \sum_m \sum_{\alpha} \int d^4 x \int dt_1 \int dt_2 \right \} \left\{ T(\bar{n}, \bar{m}) O^{(B)}_{n,m,\alpha}(\bar{x}, t) - S(\bar{n}, \bar{m}) H^{(B)}_{n,m,\alpha}(\bar{x}, t) + R(\bar{n}, \bar{m}) O^{(B)}_{n,m,\alpha}(\bar{x}, t) \right \}$$

(38)

where $\frac{da}{a} \sim d \ln b$. $R(\bar{n}, \bar{m})$, $S(\bar{n}, \bar{m})$, and $T(\bar{n}, \bar{m})$ are a set of polynomials defined by the rules of the OPE given by Eqs. 18-27.

$$T(\bar{n}, \bar{m}) = -[2(e_3^{(s)}(\bar{n}, \bar{m}))^2 + \frac{3}{4} \delta_{n,m} e_2^{(s)}(\bar{n}, \bar{m}) e_3^{(s)}(\bar{n}, \bar{m})],$$

$$R(\bar{n}, \bar{m}) = 2 \sum_l e_3^{(s)}(\bar{n}, \bar{l}) e_3^{(s)}(\bar{l}, \bar{m}) + \frac{1}{4} e_3^{(s)}(\bar{n}, \bar{m}) e_2^{(s)}(0),$$

$$S(\bar{n}, \bar{m}) = 4(e_2^{(s)}(\bar{n}, \bar{m}))^2$$

(39)

where $e_2^{(s)}(0) \equiv e_2^{(s)}(\bar{n}, \bar{n})$. $A^{-1}$ is determined by the time integration

$$A^{-1} = \frac{I_1(\hat{\beta})}{(2\pi)^{d-1} 2\pi v_F}, \quad \hat{\beta} \equiv \frac{\beta}{\pi a}$$

(40)

$$I_1(\hat{\beta}) = \frac{2}{\pi} \int_0^\infty dx \frac{\cos x/\hat{\beta}}{x^2 + 1}$$

(41)

where $\hat{\beta}$ is the dimensionless inverse temperature. The function $I_1(\hat{\beta})$ originates at $T \neq 0$.

In the limit $\beta \gg 1$, $I_1(\hat{\beta}) \rightarrow 1$. In the limit $\beta \sim 1$, $I_1(\hat{\beta}) \ll 1$ and the time integration can be neglected.

$$-\frac{1}{2} \langle \tilde{S}_{B}^2 \rangle_{da} = \frac{da}{a} \left\{ \frac{\Lambda^{2-d}}{N} \frac{B^{-1}}{2N} (-1) \sum_n \sum_m \sum_{\alpha} \sum_{\beta} \int d^4 x \int dt_1 \int dt_2 [2(d_3^{(s)})^2(1 - \frac{1}{8N}) NC_{n,m,\alpha,\beta}(\bar{x}; t_1, t_2)] \right \}$$

(42)
\[ -(\tilde{S}_{int}\tilde{S}_D)_{da} = \frac{da}{a} \left\{ -\Lambda^{1-d} B^{-1} \frac{A^{-1}}{2N^2} \sum_n \sum_m \sum_{\alpha} \int d^d x \int dt [-d_3^{(s)}(\bar{n}, \bar{m})O_{n,m;\alpha}(\bar{x}, t)] \right. \]

\[ -\frac{\Lambda^{2-d} A^{-1}}{2N^2} \sum_n \sum_m \sum_{\alpha} \sum_{\beta} \int d^d x \int dt \int dt_1 \int dt_2 [-d_3^{(s)}(\bar{n}, \bar{m})O_{n,m;\alpha,\beta}(\bar{x}; t_1, t_2)] \right\} \]

where

\[ B^{-1} = \frac{I_2(2\hat{\beta})}{(2\pi)^{d-1}2v_F} \]

\[ I_2(\hat{\beta}) = [I_1(\hat{\beta})]^2 \]

The term \( \hat{L}(\bar{n}, \bar{m}) \) and \( \hat{M}(\bar{n}, \bar{m}) \) are given by:

\[ \hat{L}(\bar{n}, \bar{m}) = 2 \sum_l e_3^{(s)}(\bar{l}, \bar{m}) + \frac{1}{2} e_2^{(s)}(\bar{n}, \bar{m}) - 2e_2^{(c)}(\bar{n}, \bar{m}) \] (46)

and

\[ \hat{M}(\bar{n}, \bar{m}) = 3 \frac{1}{2} e_2^{(s)}(\bar{n}, \bar{m}) + 2\gamma_3^{(s)}(\bar{n}, \bar{m})\delta_{n,m} - 2e_2^{(c)}(\bar{n}, \bar{m}) \] (47)

The set of Eqs. (43-47) concludes the RG calculation to second order.

The presence of the elastic mean free path introduces a cutoff in the time domain and allows us to apply the method of OPE to higher order. To third order in the interaction parameters \( e_2^{(s)}, e_2^{(c)}, e_3^{(s)} \), and disorder \( d_3^{(s)} \) we obtain:

\[ \langle \tilde{S}_{int}\tilde{S}_D \rangle_{da} = \frac{da}{a} \left\{ \Lambda^{1-d} \frac{A^{-1}}{2N^2} \sum_n \sum_m \sum_{\alpha} \sum_{\beta} \int d^d x \int dt \int dt_1 \int dt_2 \frac{A^{-1} J_1(\hat{\beta})}{I_1(\hat{\beta})} G_1(\bar{n}, \bar{m}) \right. \]

\[ + B^{-1} \frac{J_2(2\hat{\beta})}{I_2(2\hat{\beta})} \hat{G}_2(\bar{n}, \bar{m}) |C^{(s)}_{n,m;\alpha,\beta}(\bar{x}; t_1, t_2) | + \frac{da}{a} \left\{ \Lambda^{1-d} \frac{A^{-1} B^{-1}}{N^2} \sum_n \sum_m \sum_{\alpha} \int d^d x \int dt \right. \]

\[ \left. \left[ \frac{J_2(\hat{\beta})}{2I_2(\hat{\beta})} K_3(\bar{n}, \bar{m}) + \frac{J_1(\hat{\beta})}{I_1(\hat{\beta})} K_2(\bar{n}, \bar{m}) + \frac{B J_1(\hat{\beta})}{A I_1(\hat{\beta})} K_3(\bar{n}, \bar{m}) |O^{(s)}_{n,m;\alpha}(\bar{x}, t) | \right \} \]
Next we compute:

\[
+ \frac{da}{a} \left\{ \frac{\Lambda^{1-d} A^{-1} B^{-1}}{2N} \sum \sum \sum \int d^d x \int dt \frac{J_1(\tilde{\beta})}{I_1(\tilde{\beta})} \right\} \sum \sum \sum \int d^d x \int dt F(\tilde{n}, \tilde{m}) Q^{(R)}_{n,m,\alpha}(\tilde{x}, t) \}
\]

(48)

In Eq.48 the time integration introduces:

\[
J_1(\tilde{\beta}) \simeq I_1(\tilde{\beta}), \quad J_2(\tilde{\beta}) \simeq I_2(\tilde{\beta}).
\]

(49)

The integral in Eq.49 depends explicitly on the dimensionless $\tilde{\beta}$. At the scale $b = 1$ we have $\tilde{\beta}(b = 1) = \tilde{\beta} \gg 1$ and for $b = b_T = \beta/a$ we have $\tilde{\beta}(b) = 1$ and have to stop scaling.

By Using the OPE rules we generate the polynomials $G_1, G_2, K_1, K_2, K_3,$ and $F$. These polynomials are obtained from the microscopic couplings and the OPE results obtained at second order (the polynomials $R, S, T, L,$ and $M$).

\[
G_1(\tilde{n}, \tilde{m}) = -d_3^{(s)} \left[ \frac{3}{2} S(\tilde{n}, \tilde{m}) + 2R(0)\delta_{n,m} - T(\tilde{n}, \tilde{m}) \right],
\]

(50)

\[
G_2(\tilde{n}, \tilde{m}) = -d_3^{(s)} \left[ 2 M(0) e_3^{(s)}(0) \delta_{n,m} + \frac{3}{2} M(\tilde{n}, \tilde{m}) e_2^{(s)}(\tilde{n}, \tilde{m}) - 2 M(\tilde{n}, \tilde{m}) e_2^{(s)}(\tilde{n}, \tilde{m}) \right],
\]

(51)

\[
K_1(\tilde{n}, \tilde{m}) = -d_3^{(s)} \left[ 2 \sum_{\tilde{l}} R(\tilde{l}, \tilde{m}) + \frac{1}{2} S(\tilde{n}, \tilde{m}) - T(\tilde{n}, \tilde{m}) \right],
\]

(52)

\[
K_2(\tilde{n}, \tilde{m}) = -d_3^{(s)} \left[ 2 \sum_{\tilde{l}} \tilde{M}(\tilde{n}, \tilde{l}) e_3^{(s)}(\tilde{l}, \tilde{m}) + \frac{1}{4} \tilde{L}(\tilde{n}, \tilde{m}) e_2^{(s)}(0) \right],
\]

(53)

\[
K_3(\tilde{n}, \tilde{m}) = -d_3^{(s)} \left[ 2 \sum_{\tilde{l}} \tilde{M}(\tilde{n}, \tilde{l}) e_3^{(s)}(\tilde{l}, \tilde{m}) + \frac{1}{2} \tilde{M}(\tilde{n}, \tilde{m}) e_2^{(s)}(\tilde{n}, \tilde{m}) + 2 \tilde{M}(\tilde{n}, \tilde{m}) e_2^{(s)}(\tilde{n}, \tilde{m}) \right],
\]

(54)

\[
F(\tilde{n}, \tilde{m}) = -d_3^{(s)} \left[ 2 \tilde{L}(\tilde{n}, \tilde{m}) e_3^{(s)}(\tilde{n}, \tilde{m}) - \frac{3}{4} \delta_{n,m} e_2^{(s)}(0) \tilde{L}(\tilde{n}, \tilde{m}) \right].
\]

(55)

Next we compute:

\[
\langle \tilde{S}_{int} S_D^2 \rangle_{da} = \frac{da}{a} \left\{ \frac{\Lambda^{2-d} A^{-1} B^{-1}}{N} \sum \sum \sum \int d^d x \int dt_1 \int dt_2 \frac{J_1(\tilde{\beta})}{I_1(\tilde{\beta})} (d_3^{(s)})^2 \tilde{M}(\tilde{n}, \tilde{m})(4N-1)
\]

\[+ \frac{J_2(\tilde{\beta})}{I_2(\tilde{\beta})} (d_3^{(s)})^2 (2 \tilde{L}(0) \delta_{n,m} + 4 \sum \tilde{M}(\tilde{l}, \tilde{m}) - \tilde{M}(\tilde{n}, \tilde{m})) C_{n,m,\alpha,\beta}^{(s)}(\tilde{x}; t_1, t_2) \}
\]

(56)
The OPE in Eq.56 determines the behavior of the Cooperon as a function of the polynomials \( \hat{M} \) (see Eq.47) and Cooperon coupling \( d_3^{(s)} \).

\[
\frac{1}{3!}\langle \hat{S}^3 \rangle_{da} = \frac{da}{a} \left\{ \frac{\Lambda^{1-d}}{2N} \frac{(A-1)^2}{3!4N^2} J_1(\hat{\beta}) \right\} \sum_n \sum_m \sum_a \int d^d x \int dt
\]

\[
[W(\bar{n}, \bar{m})Q^{(B)}_{n,m,\alpha}(\bar{x}, t) - V(\bar{n}, \bar{m})H^{(B)}_{n,m,\alpha}(\bar{x}, t) + U(\bar{n}, \bar{m})O^{(s)}_{n,m,\alpha}(\bar{x}, t)]
\]

where the functions \( U, V, \) and \( W \) are defined in terms of the microscopic couplings and the second order functions \( R \) and \( S \) defined in Eq.39.

\[
W(\bar{n}, \bar{m}) = -[2R(\bar{n}, \bar{m})e_3^{(s)}(\bar{n}, \bar{m}) + \frac{3}{4} \delta_n_m (R(0)e_2^{(s)}(0) + S(0)e_3^{(s)}(0))]
\]

\[
V(\bar{n}, \bar{m}) = 4S(\bar{n}, \bar{m})e_2^{(s)}(\bar{n}, \bar{m})
\]

\[
U(\bar{n}, \bar{m}) = 2 \sum_i R(\bar{n}, \bar{l})e_3^{(s)}(\bar{l}, \bar{m}) + \frac{1}{4} (R(\bar{n}, \bar{m})e_2^{(s)}(0) + S(0)e_3^{(s)}(\bar{n}, \bar{m}))
\]

and

\[
\frac{1}{3!}\langle \hat{S}^3 \rangle_{da} = \frac{da}{a} \left\{ -\frac{\Lambda^{2-d}}{N} \left( B^{-1} \right) J_2(2\hat{\beta}) \right\} \frac{16}{\beta^2} \sum_n \sum_m \sum_a \sum_{t_1} \int d^d x \int dt_1 \int dt_2
\]

\[
(d_3^{(s)})^3(1 - \frac{1}{4N})^3 C^{(s)}_{n,m,\alpha,\beta}(\bar{x}; t_1, t_2)
\]

Using the results given in Eqs.29-59 we will obtain the RG equations.

V. THE RG EQUATIONS IN THE QUANTUM LIMIT

The quantum region is defined by \( \Lambda/b_T < |q| < \Lambda \) where \( b_T = \frac{v_F \Lambda}{T} \). In principle it is possible that before the scale \( b_T \) has been reached, one of the coupling constants has reached values of order one. If this happens at a scale \( \Lambda \) we have to stop at \( b_o \) and for the interval \( \frac{\Lambda}{b_T} \leq |q| \leq \frac{\Lambda}{b_o} \) we have a different theory. If the Cooperon coupling constant \( \frac{\hat{t}}{N} \propto (k_F \ell)^{-1} \)
reaches values of order one at $b_o < b_T$ we must crossover to the Finkelstein diffusion theory. From the other hand if one of the two-body interactions reaches large values we have to construct a new theory. If the two-body interaction which grows under scaling is the Cooper coupling constant we have to construct a theory based on a superconductivity with disorder. We will consider here the situation where the effects of interactions are such that the value of $b_o \equiv b_{Dif}$ obeys $b_{Dif} > b_T$ or $b_o \equiv b_{SC}$, $b_{SC} < b_T$ ($b_{SC}$ is the length scale where the Cooper coupling constant diverges.). Therefore we will ignore the diffusive region.

We introduce the following rescaled coupling constants:

\[
d_{3}^{(s)} = \hat{t}B; \quad e_{2}^{(c)}(\vec{n}, \vec{m}) = \hat{e}_{2}^{(c)}(\vec{n}, \vec{m})A;
\]

\[
\Gamma_{2}^{(c)}(\vec{n}, \vec{m}) = \hat{\gamma}_{2}^{(c)}(\vec{n}, \vec{m})A; \quad \hat{e}_{2}^{(s)}(\vec{n}, \vec{m}) = \hat{e}_{2}^{(s)}(\vec{n}, \vec{m})A;
\]

\[
\Gamma_{2}^{(s)}(\vec{n}, \vec{m}) = \hat{\gamma}_{2}^{(s)}(\vec{n}, \vec{m})A; \quad \hat{e}_{3}^{(s)}(\vec{n}, \vec{m}) = \hat{e}_{3}^{(s)}(\vec{n}, \vec{m})A;
\]

\[
\Gamma_{3}^{(s)}(\vec{n}, \vec{m}) = \hat{\gamma}_{3}^{(s)}(\vec{n}, \vec{m})A.
\]

(60)

where the constants $A$ and $B$ are defined in Eqs.40 and 44. In the quantum regime the number of channels obeys $N_0 \to N(b) = \pi(k_F\Lambda/b) = N_0b$. Due to the fact that when the cutoff $\Lambda$ is reduced to $\Lambda/b$ the number of channels scales like $N(b) = N_0b$, it follows that the naive scaling dimension of the interaction and disorder will be

\[
\frac{\gamma}{N} \xrightarrow{\Lambda/b} \frac{\gamma}{N} b^{2-d} \quad \text{and} \quad \frac{\hat{t}}{N} \xrightarrow{\Lambda/b} \frac{\hat{t}}{N} b^{3-d}.
\]

We observe that the interaction becomes marginal while the disorder is relevant. In the opposite situation where the number of channels does not scale, we have: $\gamma \to \gamma b^{1-d}$ and $\hat{t} \to \hat{t} b^{2-d}$.

For the disorder Cooperon coupling constant $\hat{t}$ we have the scaling equation:
scaling law, "Fock" exchange term ($\gamma$ that Eq.61 must be linear in $\hat{\gamma}$.

In Eq.61 we use the notation:

$$\gamma_{n,m}^2 \equiv \frac{1}{N} \sum_{l} \gamma_{n,m}^{(s)}(\vec{n}, \vec{l}) \gamma_{n,m}^{(s)}(\vec{l}, \vec{n})$$

(62)

$$\langle \gamma_2^{(s)} \rangle = \frac{1}{N} \sum_{l} \gamma_{l,n}^{(s)}(\vec{l}, \vec{n}) \quad \langle \gamma_2^{(c)} \rangle = \frac{1}{N} \sum_{l} \gamma_{l,n}^{(c)}(\vec{l}, \vec{n})$$

(63)

From Eq.61 we see that we can have a M-I transition in two dimensions when the p-p interaction $\gamma_{3}^{(s)}$ and the p-h $\gamma_{2}^{(s)}$ increases such that the linear term in $\hat{\gamma}$ becomes negative (see Eq.61). We observe in Eq.61 that the effect of the p-h singlet $\gamma_2^{(c)}$ is opposite to the p-h triplet $\gamma_2^{(s)}$. $\gamma_2^{(c)}$ enhances the localization while $\gamma_2^{(s)}$ drives the system metallic. This is consistent with the known fact that a "Hartree" term ($\gamma_2^{(c)}$) favors localization while the "Fock" exchange term ($\gamma_2^{(s)}$) drives the system metallic. From dimensional analysis it follows that Eq.61 must be linear in $\hat{\gamma}$. In addition we have that the number of channels obey the scaling law, $N = N(b) = N_o b$.

The scaling equation for the particle-hole singlet is:

$$\frac{d\gamma_2^{(c)}(\vec{n}, \vec{m})}{d \ln b} = \frac{1}{N} \left\{ \langle \gamma_3^{(s)}(\vec{n}, \vec{m}) \rangle^2 + \frac{J_1(\hat{\beta})}{I_1(\hat{\beta})} \hat{\gamma}_{3}^{(c)}(\vec{n}, \vec{m})(1 - 3\delta_{n,m}) + \frac{3}{2} \delta_{n,m} \gamma_{2}^{(s)}(0) \right\}$$

$$- \frac{1}{24} \gamma_{n,m}^{2}(\vec{n}, \vec{m})(4 - 3\delta_{n,m}) + \frac{3}{2} \delta_{n,m} \gamma_{3}^{(s)}(0)) \} + \frac{1}{2} \delta_{n,m} \frac{d\gamma_3^{(s)}(\vec{n}, \vec{m})}{d \ln b}$$

(64)

and the particle-hole triplet $\gamma_2^{(s)}(\vec{n}, \vec{m})$ is given by:

$$\frac{d\gamma_2^{(s)}(\vec{n}, \vec{m})}{d \ln b} = \frac{1}{N} \left\{ -\langle \gamma_2^{(s)}(\vec{n}, \vec{m}) \rangle^2 + \frac{1}{6N} \frac{J_1(\hat{\beta})}{I_1(\hat{\beta})} \langle \gamma_2^{(s)}(\vec{n}, \vec{m}) \rangle^3 \right\} + 2\delta_{n,m} \frac{d\gamma_3^{(s)}(\vec{n}, \vec{m})}{d \ln b}$$

(65)

From Eqs.64 and 65 we see that the particle-particle channel affects the particle-hole singlet. In addition for $\vec{n} = \vec{m}$ the particle-particle channel $\gamma_3^{(s)}(\vec{n}, \vec{n})$ is identical to the particle-hole
the particle-particle singlet $\frac{1}{2} \hat{\gamma}_2^{(s)}(\vec{n}, \vec{n})$, $\hat{\gamma}_3^{(s)}(\vec{n}, \vec{n}) = \frac{1}{2} \hat{\gamma}_2^{(s)}(\vec{n}, \vec{n})$.

The particle-particle singlet term obeys the scaling equation:

$$
\frac{d\hat{\gamma}_3^{(s)}(\vec{n}, \vec{m})}{d \ln b} = -\frac{1}{2} [\hat{\gamma}_3^{(s)}]_{n,m}^2 + \hat{t} \langle \hat{\gamma}_3^{(s)} \rangle + \frac{1}{3!} \frac{J_1(\hat{\beta})}{I_1(\hat{\beta})} [\hat{\gamma}_3^{(s)}]_{n,m}^3 \\
-2 \frac{J_2(\hat{\beta})}{I_2(\hat{\beta})} \hat{t} \langle (\hat{\gamma}_3^{(s)})^2 \rangle - 4 \frac{J_1(\hat{\beta})}{I_1(\hat{\beta})} \hat{t} \langle (\hat{\gamma}_3^{(s)})^2 \rangle + 8 \frac{J_2(\hat{\beta})}{I_2(\hat{\beta})} \hat{t}^2 \langle (\hat{\gamma}_3^{(s)}) \rangle + 8 \frac{J_1(\hat{\beta})}{I_1(\hat{\beta})} \hat{t}^2 \langle (\hat{\gamma}_3^{(s)}) \rangle
$$

$$
+ \frac{1}{N} \{ [\hat{t} + 4\hat{t}^2 \frac{J_1(\hat{\beta})}{I_1(\hat{\beta})} + \frac{A}{B} \frac{J_2(\hat{\beta})}{I_2(\hat{\beta})}] \frac{1}{2} \hat{\gamma}_2^{(s)}(\vec{n}, \vec{m}) - 2 \hat{\gamma}_2^{(c)}(\vec{n}, \vec{m}) \} + \frac{1}{N} \left\{ \frac{1}{8} \hat{\gamma}_3^{(s)}(\vec{n}, \vec{m}) \hat{\gamma}_3^{(s)}(0) - \frac{1}{16} \hat{\gamma}_3^{(s)}(\vec{n}, \vec{m}) \hat{\gamma}_2^{(s)}(0) \right\}
$$

$$
- \frac{1}{3!} \frac{J_1(\hat{\beta})}{I_1(\hat{\beta})} \delta_{n,m} [\hat{\gamma}_3^{(s)}]_{n,m}^2 \hat{\gamma}_3^{(s)}(0) + \frac{1}{2 \cdot 3!} \frac{J_1(\hat{\beta})}{I_1(\hat{\beta})} [\hat{\gamma}_3^{(s)}]_{n,m}^2 \hat{\gamma}_2^{(s)}(0) - \frac{J_2(2\hat{\beta})}{I_2(2\hat{\beta})} \langle (\hat{\gamma}_3^{(s)}) \rangle \langle \frac{1}{2} \hat{\gamma}_2^{(s)}(0) - \hat{\gamma}_2^{(s)}(0) \delta_{n,m} \rangle
$$

where

$$
[\hat{\gamma}_3^{(s)}]_{n,m}^2 = \frac{1}{N} \sum_{\vec{l}} \hat{\gamma}_3^{(s)}(\vec{n}, \vec{l}) \hat{\gamma}_3^{(s)}(\vec{l}, \vec{m})
$$

$$
\langle (\hat{\gamma}_3^{(s)})^2 \rangle = \frac{1}{N} \sum_{\vec{l}} (\hat{\gamma}_3^{(s)}(\vec{n}, \vec{l}))^2
$$

$$
\langle (\hat{\gamma}_3^{(s)}) \rangle = \frac{1}{N} \sum_{\vec{l}} \hat{\gamma}_3^{(s)}(\vec{n}, \vec{l})
$$

$$
[\hat{\gamma}_3^{(s)}]_{n,m}^3 = \frac{1}{N^2} \sum_{\vec{l}} \sum_{\vec{l}'} \hat{\gamma}_3^{(s)}(\vec{n}, \vec{l}) \hat{\gamma}_3^{(s)}(\vec{l}, \vec{l}') \hat{\gamma}_3^{(s)}(\vec{l}', \vec{m}).
$$

The scaling relation for the forward part are trivial:

$$
\frac{d\Gamma^{(c)}}{d \ln b} = \frac{d\Gamma^{(s)}}{d \ln b} = 0
$$

The set of Eqs. 64-66 show that in the limit of $N \to \infty$ the interaction is controlled only by the particle-particle singlet $\hat{\gamma}_3^{(s)}(\vec{n}, \vec{m})$. In addition we observe that the disorder renormalizes
the \( \hat{\gamma}_3^{(s)}(\vec{n},\vec{m}) \). We observe that the scaling equation for \( \hat{\gamma}_3^{(s)}(\vec{n},\vec{m}) \) can be negative at \( b = 1 \).

The origin of the negative term is given by Eq.37, where it has been shown that at short times the Cooperon behaves like a Cooper p-p singlet. As a result the initial values of the particle-particle singlet \( \hat{\gamma}_3^{(s)}(\vec{n},\vec{m}; b = 1) \) are replaced by \( \hat{\gamma}_3^{(s)}(\vec{n},\vec{m}; b = 1) - 2v_F(\frac{B}{A})\hat{t} \). In Eqs.61, 64, 65, and 66 the scaling of the number of the channels is stopped when diffusive region is reached. At finite temperature we stop scaling at the scale \( b = b_T = E_F/T \). This will fix the number of channels to \( \bar{N} \equiv N_T = E_F/T \) (see ref. [9]). It might be possible that in two dimensions the decoherency introduced by the temperature might be stronger than \( T \). This might be the case if we have in mind dephasing effects in two dimensions which can define an effective temperature \( T_{eff}(T) > T \) replacing \( \bar{N} \) by \( E_F/T_{eff} \).

VI. THE CONDUCTING PHASE DUE TO THE SUPERCONDUCTING INSTABILITY IN THE QUANTUM REGION

In the low temperature limit we can ignore all the many body effect except the particle-particle singlet \( \hat{\gamma}_3^{(s)}(\vec{n},\vec{m}) \). The reason being the \( 1/N \) factor which appears in Eqs.54 and 55 and is missing for the particle-particle singlet in Eq.60. The growth of the number of channels \( N(b) \) is determined by the topology of the Fermi surface. In particular this is the case for spherical Fermi surface where \( N(b) = N_0 b \). For \( T \neq 0 \) we obtain \( N(b = b_T) = \frac{E_F}{T} \). (In chapter VIII we will consider non-spherical Fermi surface with repulsive interaction which might lead to a Ferromagnetic instability.)

Due to the fact that the \( 1/N \) factor is only absent for the particle-particle singlet, we will investigate the problem in the parameter space \( (\hat{\gamma}_3^{(s)}, \hat{t}) \) using the angular momentum representation:

\[
\gamma_3^{(s)}(r) \equiv \gamma_r = \int_0^\pi \frac{d\theta}{\pi} \gamma_3^{(s)}(\theta) \cos(r\theta), \quad r = 0, 2, 4, \ldots.
\]

For the singlet case \( r = 0 \) we have \( \gamma_{r=0} = \gamma_0 \):
\[ \gamma_0 \simeq \frac{1}{N} \sum_l \tilde{\gamma}_3^{(s)}(\vec{l}, \vec{n}) \]  

(69)

From Eq.66 we obtain to leading order in \(1/N\) the following equation for particle-particle singlet:

\[ \frac{d\gamma_0}{d \ln b} = -\frac{1}{2} \gamma_0^2 + \gamma_0 \dot{\gamma}_0 - 6 \gamma_0^2 \dot{\gamma}_0 + 16 \gamma_0 \ddot{\gamma}_0 + \frac{1}{3!} \gamma_0^3, \]  

(70)

with

\[ \gamma_0(b = 1) \rightarrow \gamma_0(b = 1) - 2 \tilde{v}_F \dot{\gamma}_0 \]

In Eq.70 we have used \(I_1 \sim I_2 \sim J_1 \sim 1\) and \(\tilde{v}_F = v_FB/A\) where the constants \(A\) and \(B\) have been defined in Eqs.40 and 44.

We investigate Eq.70 in the limit of weak disorder \(\dot{\gamma} \rightarrow 0\). We find that even for positive value of \(\gamma_0\) the effect of disorder is to drive \(\gamma_0(b)\) to negative values. The reason for this is the fact that the negative linear term in \(\dot{\gamma}_0\) can cause an initial negative value for \(\gamma_0(b = 1)\). As a result the term \(-\frac{1}{2} \gamma_0^2\) (for negative value of \(\gamma_0\), \(\gamma_0(b = 1) < 0\)) might drive the particle-particle interaction towards a superconducting instability. This behavior can be seen in the following way. In the limit of \(\dot{\gamma} \rightarrow 0\) we keep in Eq.70 only the two first order terms and obtain the solution for \(\gamma_0(b)\):

\[ \gamma_0(b = e^\delta) = \gamma_0(b = 1) \exp(\int_0^\delta \dot{\gamma}(x) \, dx)[1 + \frac{1}{2} \gamma_0(b = 1) \int_0^\delta dy \exp(\int_y^\delta \dot{\gamma}(x) \, dx)]^{-1} \]  

(71)

For \(\gamma_0(b = 1) < 0\), \(\gamma_0(b)\) diverges at a length scale \(b \equiv b_{SC} \equiv \frac{v_F \Lambda}{T_{SC}}\) where \(T_{SC}\) represents the superconducting instability temperature,

\[ T_{SC} = v_F \Lambda(1 + \frac{2\dot{\gamma}_0}{|\gamma_0(b = 1)|})^{-\frac{1}{4}} \tilde{\gamma}_0 \rightarrow \exp(-\frac{2}{|\gamma_0(b = 1)|}). \]

Next we consider the RG equation for the Cooperon (see Eq.61) with \(J_1(\tilde{\beta}) \sim I_1(\tilde{\beta}) \sim I_2(\tilde{\beta}) \sim 1\). From Eq.61 we observe that in the limit of vanishing interactions the Cooperon coupling constant scales like \(\frac{i_{N}(b)}{N(b)} \sim \frac{i_{N}(b = 1)}{N_{\omega}}[1 - 2\dot{i}(b = 1) \log b]^{-1}\) and diverges at \(b \equiv b_{Loc} \equiv \frac{v_F \Lambda}{T_{Loc}}\)

24
$b_{\text{Loc}} \geq b_{\text{Diff}}, \ T_{\text{Diff}} \geq T_{\text{Loc}}$, $T_{\text{Loc}} \approx v_F A \exp[-\frac{1}{2(t_b-1)}]$. In order to understand the physics of the system we have to compare the physical temperature $T$ with the other two, $T_{SC}$ and $T_{Loc}$. We have to consider separately the cases: a) $T < T_{SC} < T_{Loc}$; b) $T < T_{Loc} < T_{SC}$; c) $T_{SC} < T_{Loc} < T$; d) $T_{SC} < T < T_{Loc}$; e) $T_{Loc} < T < T_{SC}$; f) $T_{Loc} < T_{SC} < T$.

a) $T < T_{SC} < T_{Loc}$

This is the localized case where the mean free path “$\ell$” is the shortest length scale in the problem. This case will not be analyzed here. Most of the work in the past has been concentrated towards this case, in particular the Finkelstein theory which has investigated the interactions within the diffusion theory.

b) $T < T_{Loc} < T_{SC}$

Here the shortest length scale is the Cooper coherence length. Physically one can describe this region by a system of disorder bosons (the bosons describe the pairs). The critical theory might correspond to a disorder X-Y model.

c) $T_{SC} < T_{Loc} < T$

This is a region where interactions are not important. The physics is controlled by classical hopping transport.

d) $T_{SC} < T < T_{Loc}$

As in case a) here the system is localized. This case will not be considered here. (See the Finkelstein theory.)

e) $T_{Loc} < T < T_{SC}$

Again a bosonic X-Y theory with disorder is applicable here as in case b).

f) $T_{Loc} < T_{SC} < T$
In this region we will have transport controlled by pair breaking.

In the rest part of this section we will investigate the RG equation for the negative particle-particle singlet \( \hat{\gamma}^{(s)}(\vec{n}, \vec{m}) \) and the Cooperon coupling constant \( \hat{t} \). In agreement with Eq.39 we introduce the angular momentum representation for the Cooper and Cooperon channels: \( \gamma_o = \frac{1}{N} \sum_l \hat{\gamma}^{(s)}(\vec{l}, \vec{n}) \), \( t_o = \frac{1}{N} \sum_l \hat{t}(\vec{l}, \vec{n}) \). We obtain from Eq.51 and Eq.70 the following RG equations:

\[
\frac{d\lambda}{d\ln b} = \frac{1}{2} \lambda^2 + \lambda t_o, \quad \lambda \equiv -\gamma_o
\]

\[
\frac{dt_o}{d\ln b} = t_o[1 - (\frac{\lambda}{N})^2] + 2t_o^2
\]

\[
\rho(b) \propto \frac{t_o(b)}{N(b)} \equiv \bar{t}_o(b), \quad N(b) = N_o b
\]

(72)

\( \rho(b) \) is the resistance with \( b \) restricted to \( 1 < b \leq \frac{e\Delta}{T} \). From Eq.72 we observe that in the limit \( b \to \infty \) (\( T \to 0 \)) the parameter \( \lambda \) diverges. In particular we observe that the ratio \( \frac{\lambda(b)}{N(b)} \) \( b \to \infty \) \( \to \infty \). As a result the RG equation behaves like \( \frac{dt_o}{d\ln b} = -t_o(\frac{\lambda}{N})^2 \). Due to the large value of \( (\frac{\lambda}{N})^2 \) it follows that \( t_o(b) \) \( b \to \infty \) \( \to 0 \). As a result we obtain a superconducting ground state. At finite temperature we consider the case \( b_T < b_{SC} < b_{Loc} \). We substitute the solution of \( \lambda(b) \) into \( t_o(b) \) and obtain

\[
\bar{t}_o(b_T) = \frac{t_o}{N_o} exp\left\{ -\int_0^{\log b_T} (\frac{\lambda(x)}{N(x)})^2 dx \right\} \sim \frac{t_o}{N_o} exp\left\{ -\frac{4N_o^2 T_{SC}}{|T - T_{SC}|} \right\}, \quad T > T_{SC}
\]

(73)

\( N_o \simeq \frac{\pi k_F}{\Lambda} \sim 1 \) and \( T_{SC} \) is given by Eq.71. As a result we obtain that the resistance obeys \( \rho(T) \to 0 \) as \( T \to T_{SC} \), \( \rho(T) \to Const. \ exp\left\{ -\frac{4N_o^2 T_{SC}}{|T - T_{SC}|} \right\} \). To conclude this section (\( \gamma_o < 0 \)) we remark that the transport data \[3\] show some similarity with the one reported for disorder bosons in ref. \[4\]. This might suggest that the correct starting point might be a disordered bosonic system instead of a diffusion theory \[2\].

26
VII. THE RG EQUATION AT FINITE TEMPERATURE

At a temperature $T$ the scaling is restricted to $\frac{\Lambda}{b_T} < |q| < \Lambda$ where $b_T = \frac{vF \Lambda}{T}$. In this interval the number of channels is restricted to $\bar{N} = N(b_T) = \frac{E_c}{b}$, with $N(b)$ obeying the condition $N_o < N(b) \leq \bar{N}$. We replace in Eqs.61-68 $J_1(\beta) \sim J_2(\beta) \sim I_1(\beta) \sim I_2(\beta) \sim 1$ and find a simplified form

$$\frac{d\hat{t}}{d\ln b} = \epsilon(b)\hat{t} - \frac{\hat{t}}{N} \frac{3}{4} \hat{\gamma}_2(s)(\bar{n}, \bar{m}) - \hat{\gamma}_2^{(c)}(\bar{n}, \bar{m}) + \delta_{n,m}[\hat{\gamma}_3^{(s)}]^2_{n,m} + 2\hat{t}^2[1 - \frac{1}{N}(\frac{3}{2} \hat{\gamma}_2^{(s)}(\bar{n}, \bar{m}) - 2\hat{\gamma}_2^{(c)}(\bar{n}, \bar{m}))), \frac{1}{N}(3\hat{\gamma}_2^{(s)} - 2\hat{\gamma}_2^{(c)})] (74)$$

The parameter $\epsilon(b)$ controls the crossover at finite temperatures. $\epsilon(b)$ is given by, $\epsilon(b) = 1$ for $b < b_T$ and $\epsilon(b) \approx 0$ for $b > b_T$. Eq.74 replaces the scaling Eq.61 for the disorder coupling constant $\hat{t}$. In Eq.74 we observe that the interaction has produced a shift in the critical dimensionality. The disorder parameter $\hat{t}$ has accumulated a finite anomalous dimension, $\frac{1}{N}(\frac{3}{4} \hat{\gamma}_2^{(s)} \cdots)$, which will control the M-I transition. (In the limit $T \to 0$, $N \to \infty$ causing this term to disappear.)

At finite temperatures the scaling Eqs.64 and 65 for the interactions $\hat{\gamma}_2^{(s)}$ and $\hat{\gamma}_2^{(c)}$ are the same except that linear terms of the form $[e(b) - 1]\hat{\gamma}_2^{(s)}$ and $[e(b) - 1]\hat{\gamma}_2^{(c)}$ are added to the Eqs.64 and 65, respectively. For the particle-particle singlet $\hat{\gamma}_3^{(s)}$ we have

$$\frac{d\hat{\gamma}_3^{(s)}}{d\ln b} = [\epsilon(b) - 1]\hat{\gamma}_3^{(s)}(\bar{n}, \bar{m}) - \frac{1}{2}[\hat{\gamma}_3^{(s)}]_{n,m}^2 + \frac{1}{3}[\hat{\gamma}_3^{(s)}]_{n,m}^3 - 6\hat{t}(\hat{\gamma}_3^{(s)})^2 + 16\hat{t}\hat{\gamma}_3^{(s)}$$

$$+ \frac{1}{N}\{(\hat{t} + 8\hat{t}^2)(\frac{1}{2} \hat{\gamma}_2^{(s)}(\bar{n}, \bar{m}) - 2\hat{\gamma}_2^{(c)}(\bar{n}, \bar{m}))), \frac{1}{N}\{\frac{1}{8} \hat{\gamma}_3^{(s)}(\bar{n}, \bar{m})\hat{\gamma}_3^{(s)}(0) - \frac{1}{16} \hat{\gamma}_3^{(s)}(\bar{n}, \bar{m})\hat{\gamma}_2^{(s)}(0)$$

$$- \frac{1}{3!}\delta_{n,m}[\hat{\gamma}_3^{(s)}]_{n,m}^2 \hat{\gamma}_3^{(s)}(\bar{n}, \bar{m}) + \frac{1}{2!3!}[\hat{\gamma}_3^{(s)}]_{n,m}^2 \hat{\gamma}_2^{(s)}(0) - \hat{\gamma}_3^{(s)}(0)\hat{\gamma}_2^{(s)}(0)\delta_{n,m}]\} (75)$$

In Eq.73 we use the same definitions as given in Eq.61. Eq.75 must be supplemented by the condition $\frac{1}{2} \hat{\gamma}_2^{(s)}(\bar{n}, \bar{n}) = \hat{\gamma}_3^{(s)}(\bar{n}, \bar{n})$ plus Eqs.64 and 65.
VIII. THE SCALING EQUATIONS FOR THE RESISTIVITY AT FINITE TEMPERATURES AND STRONG REPULSIVE INTERACTIONS

We restrict ourselves to finite temperatures or/and cases where the scaling of the number of channels is different from \( N(b) = N_\circ b \) (spherical Fermi surface). For flat Fermi surface the number of the channels does not scale. We have \( N(b) \sim N(b = 1) \sim N_\circ \). At finite temperature for spherical Fermi surface the number of channels is finite and is restricted by the temperature \( N_\circ < N(b) < N(b_T) \sim E_F T \). Since the coupling constants depend on the number of channels (finite), we will normalize the coupling constant by \( N_\circ \), the number of channels

\[
\tilde{\gamma}_2^{(c)} \equiv \frac{\hat{\gamma}_2^{(c)}}{N}, \quad \tilde{\gamma}_2^{(s)} \equiv \frac{\hat{\gamma}_2^{(s)}}{N}, \quad \tilde{\gamma}_3^{(s)} \equiv \frac{\hat{\gamma}_3^{(s)}}{N}, \quad \tilde{\ell} \equiv \frac{\hat{\ell}}{N}.
\]

As a result the new RG equations are given in terms of the original Eqs. \(75, 65, \) and \(66\):

\[
\frac{d\tilde{\gamma}_2^{(c)}}{d\ln b} = \frac{1}{N} \left( \frac{d\hat{\gamma}_2^{(c)}}{d\ln b} \right) - \epsilon_T \tilde{\gamma}_2^{(c)};
\]

\[
\frac{d\tilde{\gamma}_2^{(s)}}{d\ln b} = \frac{1}{N} \left( \frac{d\hat{\gamma}_2^{(s)}}{d\ln b} \right) - \epsilon_T \tilde{\gamma}_2^{(s)};
\]

\[
\frac{d\tilde{\gamma}_3^{(s)}}{d\ln b} = \frac{1}{N} \left( \frac{d\hat{\gamma}_3^{(s)}}{d\ln b} \right) - \epsilon_T \tilde{\gamma}_3^{(s)}.
\]

(77)

The parameter \( \epsilon_T \) depends on the topology of the Fermi surface and temperature

\[
\epsilon_T \equiv \left| \frac{d\ln N(b)}{d\ln b} \right|, \quad N_\circ \leq N(b) \leq N(b_T).
\]

(78)

The parameter \( \epsilon_T \) takes values of \( 0 \leq \epsilon_T \leq 1 \). The value of \( \epsilon_T = 1 \) is obtained for spherical Fermi surface \( N(b) = N_\circ b \) and \( \epsilon_T = 0 \) is obtained for flat Fermi surface or high temperatures, \( N(b) \sim \bar{N} \sim \frac{E_F}{T} \).

Here we consider a special case of repulsive interactions such that the particle-particle singlet and particle-hole triplet are strong in the backward direction. This means that the
most relevant interactions are those with \( \bar{n} = \bar{m} \). In order to be specific we will consider a special model for which the terms \( \gamma_2^{(c)}(\bar{n}, \bar{m}), \gamma_2^{(s)}(\bar{n}, \bar{m}), \) and \( \gamma_3^{(s)}(\bar{n}, \bar{m}) \) are zero for \( \bar{n} \neq \bar{m} \).

We keep only terms with \( \bar{n} = \bar{m} \) and introduce the definition:

\[
\hat{\gamma}_2^{(c)} \equiv \gamma_2^{(c)}(\bar{n}, \bar{n}), \quad \hat{\gamma}_2^{(s)} \equiv \gamma_2^{(s)}(\bar{n}, \bar{n}) = 2\hat{\gamma}_3^{(s)}(\bar{n}, \bar{n})
\]

\[
\hat{\gamma}_2^{(c)}(\bar{n} \neq \bar{m}) \simeq \hat{\gamma}_2^{(s)}(\bar{n} \neq \bar{m}) \simeq \hat{\gamma}_3^{(s)}(\bar{n} \neq \bar{m}) \simeq 0 \quad (79)
\]

Using Eqs. (76), (77), and (78) we obtain:

\[
\frac{d\hat{\gamma}_2^{(c)}}{d\ln b} = -\hat{\gamma}_2^{(c)}(\epsilon_T + \hat{t}) + \frac{1}{2}\hat{\gamma}_2^{(s)} - \frac{3}{4}(\hat{\gamma}_2^{(s)})^2(\hat{t} - \frac{1}{4}) \quad (80)
\]

\[
\frac{d\hat{\gamma}_2^{(s)}}{d\ln b} = \hat{\gamma}_2^{(s)}(2\hat{t} + 8\hat{t}^2 - \epsilon_T) - (\hat{\gamma}_2^{(s)})^2\left(\frac{5}{4} + 3\hat{t}\right) + \frac{5}{24}(\hat{\gamma}_2^{(s)})^3 - 4\hat{\gamma}_2^{(c)}(\hat{t} + 8\hat{t}^2) \quad (81)
\]

\[
\frac{d\hat{t}}{d\ln b} = \hat{t}
\left[1 - \frac{3}{4}\hat{\gamma}_2^{(s)} + \hat{\gamma}_2^{(c)} - \frac{1}{4}(\hat{\gamma}_2^{(s)})^2\right] + 2\hat{t}^2
\left[1 - \frac{3}{2}\hat{\gamma}_2^{(s)} + 2\hat{\gamma}_2^{(c)}\right] \quad (82)
\]

\[
\hat{t} = \frac{\hat{t}}{N}, \quad N = N(b), \quad 1 \leq b \leq b_T = \frac{E_F}{T} \quad (83)
\]

From Eq. (80) we conclude that the particle-hole singlet \( \gamma_c \) is irrelevant. Therefore we will take \( \gamma_c = 0 \) and ignore Eq. (80). We will solve the RG equation in the space of \( \gamma_2^{(s)} \) and \( \hat{t} \) (Eqs. (81) and (82)). In the parameter space \( (\gamma_2^{(s)}, \hat{t}) \) we find a non-trivial fixed point. In the limit \( \epsilon_T \to 0 \) we find \( \langle \hat{\gamma}_2^{(s)} \rangle^* = \frac{8}{5}(\hat{t}^*) \simeq \frac{4}{25} \).

We linearize the equations around this fixed point and find: \( \hat{t}(b) = \hat{t}^* + (\hat{t} - \hat{t}^*)b^{1/\nu_1} \), \( \nu_1 \simeq 1 + \frac{2}{25} \) and \( \gamma_2^{(s)}(b) = \gamma_2^* + (\gamma_2^* - \gamma_2^*)b^{-1/\nu_2} \). These equations show that for \( \hat{t} < \hat{t}^* \) the disorder decreases and in the same time \( \hat{\gamma}_2^{(s)} \) flows to \( \gamma_2^* \). For large value of disorder we obtain that \( \hat{t} \) increase and \( \hat{\gamma}_2^{(s)} \) flows to \( \gamma_2^* \). Experimentally the presence of the stable fixed point \( \gamma_2^* \) might be identified by a power law behavior in the spin-spin correlation. This is similar to what one has in one dimension and might corresponds to a spin-liquid phase. For the transport properties, we believe that our predictions are in a qualitative agreement ...

29
with the experiments [1], we find for the resistivity $\rho(\bar{t}, \bar{\gamma}_2^{(s)}, T)$ at a finite temperature and the dynamical exponent $z \simeq 1$: $\rho(\bar{t}, \bar{\gamma}_2^{(s)}, T) = \rho(\bar{t}(b), \bar{\gamma}_2^{(s)}(b), T b^z) = \rho(\bar{t}^* + (\bar{t} - \bar{t}^*) b^{1/\nu_1}, 
\bar{\gamma}_2^* + (\bar{\gamma}_2^{(s)} - \bar{\gamma}_2^*) b^{-1/\nu_2}, T b^z)$. We introduce $T b^z \simeq T_o \Rightarrow b \simeq (\frac{T_o}{T})^{1/z}$ and use the definitions $\bar{t} \equiv \frac{\bar{t}}{N}$. As a result we find:

$$\rho(\bar{t}, \bar{\gamma}_2^{(s)}, T) \simeq \rho(\bar{t}^*, \bar{\gamma}_2^*, T_o) + \text{const.}(\frac{\bar{t} - \bar{t}^*}{\bar{t}^*})(\frac{T_o}{T})^{1/z\nu_1}$$

(84)

Eq. 84 shows that for $\bar{t} < \bar{t}^*$ the resistivity $\rho$ decreases as we lower the temperature and increases when $\bar{t} > \bar{t}^*$. In order to make contact with the experiments we replace: $\bar{t} \propto (k_F \ell)^{-1}, k_F \propto n_c^{1/2}, \bar{t}^* \propto (n_c^*)^{-1/2}$ ($n_c^*$ is the critical density) and identify $\frac{\bar{t} - \bar{t}^*}{\bar{t}^*} \propto \frac{n_c^* - n_c}{n_c^*} \equiv \delta$. As a result we find: $\rho(n_c, \bar{\gamma}_2^{(s)}, T) \simeq \rho^*(f(\frac{\delta}{T o^{1/z\nu_1}}), \rho^* \equiv \rho(n_c^*, \bar{\gamma}_2^*, T_o)$ which is the result observed in ref. [3]. We hope that more accurate experiments will confirm the existence of the suggested fixed point.

**IX. CONCLUSION**

A new method for studying many-body systems and disorder has been introduced. The method is based on the extension of the OPE to two dimensional systems. Using a real space version of RG we have derived a set of RG equations for disorder and interaction. We have constructed an alternative theory to the one constructed by Finkelstein [2]. The basic assumption in ref. [2] is that the elastic mean free path is the shortest length in the problem. As a result the multiple elastic scatterings are replaced by a diffusion theory (the non-linear $\sigma$-model) and the interactions are considered as a perturbation of the diffusion theory. The method used here is based on a RG analysis which studies the competitions between the multiple elastic scattering and the interaction. We identified the following regions:

1) The multiple elastic scattering is the shortest length scale and diverges first. For this case we agree with the results given in ref. [2] and do not have anything to add.

2) The particle-particle singlet is negative and a superconducting instability occurs for
$T \leq T_{SC}$ where $T_{SC} > T_{Loc}$. As a result one has to treat first the interaction within an effective Ginzburg-Landau theory. We reproduced a bosonic model (X-Y) which is perturbed by disorder.

3) The interactions are positive and the particle-hole is dominant in the backward direction. At finite temperature and non-spherical Fermi surfaces which obey $|\frac{d\ln N(b)}{d\ln b}| \ll 1$ one obtains a non-trivial fixed point in the plane $(\bar{\gamma}^{(s)}_2, \hat{t})$ which separates the conducting from the insulating phase. This fixed point is characterized by a stable fixed point in the $\bar{\gamma}^{(s)}_2$ direction. No divergence in the particle-hole triplet occurs expect the infinite correlation length for the spin-spin ferromagnetic correlations when $\bar{\gamma}^{(s)}_2 \rightarrow \bar{\gamma}^{*}_2$.

ACKNOWLEDGMENTS

D. Schmeltzer would like to thank professor A.M. Finkelstein for his valuable comment concerning the differences between his theory and the one presented here.
APPENDIX A:

The Fermion field $\psi_{\sigma,\alpha}(\vec{x})$ is decomposed into $N$ Fermions, $\psi_{\sigma,\alpha}(\vec{x}) = \sum_{\vec{\omega}}^{N} e^{i k_F \vec{\omega} \cdot \vec{x}} \psi_{\vec{\omega},\sigma,\alpha}$. Using this representation we obtain from Eq.3 the result:

$$S_{int} \simeq \int d^d x \int dt \sum_{\sigma,\sigma'} \sum_{\alpha} \sum_{\vec{\omega}_1} \sum_{\vec{\omega}_2} \sum_{\vec{\omega}_3} \sum_{\vec{\omega}_4} \delta_{\vec{\omega}_1 + \vec{\omega}_2 = \vec{\omega}_3 + \vec{\omega}_4} 
 v(\vec{\omega}_1, \vec{\omega}_2, \vec{\omega}_3, \vec{\omega}_4) \psi_{\vec{\omega}_1,\sigma,\alpha}^{\dagger}(\vec{x}) \psi_{\vec{\omega}_2,\sigma',\alpha}(\vec{x}) \psi_{\vec{\omega}_3,\sigma',\alpha}(\vec{x}) \psi_{\vec{\omega}_4,\sigma,\alpha}(\vec{x})$$

(A1)

where $v(\vec{\omega}_1, \vec{\omega}_2, \vec{\omega}_3, \vec{\omega}_4)$ represents the projection of the screened two-body potential on the Fermi surface. The presence of the Kronecker-delta function imposes the condition $\vec{\omega}_1 + \vec{\omega}_2 = \vec{\omega}_3 + \vec{\omega}_4$. As a result we separate the interaction term into three processes: 1) direct, 2) exchange, and 3) Cooperon channel:

1. The direct process is realized when $\vec{\omega}_1 = \vec{\omega}_4$, $\vec{\omega}_2 = \vec{\omega}_3$.

2. The exchange process: $\vec{\omega}_1 = \vec{\omega}_3 \equiv \vec{\omega}$, $\vec{\omega}_2 = \vec{\omega}_4 \equiv \vec{\omega}'$

3. The Cooperon channel: $\vec{\omega} \equiv \vec{\omega}_1 = -\vec{\omega}_2$, $\vec{\omega}' \equiv \vec{\omega}_3 = -\vec{\omega}_4$

As a result Eq.[A1] becomes

$$S_{int} \simeq \int d^d x \int dt \sum_{\sigma,\sigma'} \sum_{\alpha} \sum_{\vec{\omega}_1} \sum_{\vec{\omega}_2} \{ v(0) \psi_{\vec{\omega},\sigma,\alpha}^{\dagger}(\vec{x}) \psi_{\vec{\omega},\sigma,\alpha}(\vec{x}) \psi_{\vec{\omega}',\sigma',\alpha}(\vec{x}) \psi_{\vec{\omega}',\sigma',\alpha}(\vec{x}) 
 - v(\vec{\omega}, \vec{\omega}', \vec{\omega}, \vec{\omega}') \psi_{\vec{\omega},\sigma,\alpha}^{\dagger}(\vec{x}) \psi_{\vec{\omega},\sigma,\alpha}(\vec{x}) \psi_{\vec{\omega}',\sigma',\alpha}(\vec{x}) \psi_{\vec{\omega}',\sigma,\alpha}(\vec{x}) 
 + v(\vec{\omega}, -\vec{\omega}, \vec{\omega}', -\vec{\omega}') \psi_{\vec{\omega},\sigma,\alpha}^{\dagger}(\vec{x}) \psi_{-\vec{\omega},\sigma,\alpha}(\vec{x}) \psi_{\vec{\omega}',\sigma',\alpha}(\vec{x}) \psi_{-\vec{\omega}',\sigma,\alpha}(\vec{x}) \}$$

(A2)

In Eq.[A2] we observe that the Cooperon channel is identical to the exchange one if we substitute in the exchange term $\vec{\omega}' = -\vec{\omega}$. This means that we have to take into consideration this identity in order to avoid double counting.
We replace the “$N$” fermions by $N/2$ pairs of chiral fermions (see Eq.\.). In the second step we replace Eq.\. by the current representation:

$$S_{int} \simeq \int d^d x \int dt \sum_\alpha \sum_{n, \alpha} \{(v(0) - \frac{1}{2} v(\bar{n}, \bar{m})) (J^R_{n, \alpha}(\bar{x}) J^R_{m, \alpha}(\bar{x}) + J^L_{n, \alpha}(\bar{x}) J^L_{m, \alpha}(\bar{x}))$$

$$- 2v(\bar{n}, \bar{m})(J^R_{n, \alpha}(\bar{x}) J^R_{m, \alpha}(\bar{x}) + J^L_{n, \alpha}(\bar{x}) J^L_{m, \alpha}(\bar{x}))(v(0) - \frac{1}{2} v(\bar{n}, \bar{m} + \pi))(J^R_{n, \alpha}(\bar{x}) J^L_{m, \alpha}(\bar{x}))$$

$$+ J^L_{n, \alpha}(\bar{x}) J^R_{m, \alpha}(\bar{x})) - 2v(\bar{n}, \bar{m} + \pi)(J^R_{n, \alpha}(\bar{x}) J^L_{m, \alpha}(\bar{x}) + J^L_{n, \alpha}(\bar{x}) J^R_{m, \alpha}(\bar{x}))$$

$$(1 - \delta_{n,m}) v(\bar{n}, \bar{m}) \sum_{\sigma = \uparrow, \downarrow} \left(J^R_{n, \sigma; \alpha; \sigma, \alpha}(\bar{x}) J^L_{n, -\sigma; \alpha; m, -\sigma, \alpha}(\bar{x}) + J^L_{n, \sigma; \alpha; m, \sigma, \alpha}(\bar{x}) J^R_{n, -\sigma; \alpha; m, -\sigma, \alpha}(\bar{x})\right)$$

$$- v(\bar{n}, \bar{m} + \pi) \sum_{\sigma = \uparrow, \downarrow} \left(J^R_{n, \sigma; \alpha; m, -\sigma, \alpha}(\bar{x}) J^L_{n, -\sigma; \alpha; m, \sigma, \alpha}(\bar{x}) + J^L_{n, \sigma; \alpha; m, -\sigma, \alpha}(\bar{x}) J^R_{n, -\sigma; \alpha; m, -\sigma, \alpha}(\bar{x})\right)\} \quad (A3)$$

We introduce the following definitions:

$$\tilde{\Gamma}^{(c)}(\bar{n}, \bar{m}) \equiv v(0) - \frac{1}{2} v(\bar{n}, \bar{m}), \quad \tilde{\Gamma}^{(s)}(\bar{n}, \bar{m}) \equiv 2v(\bar{n}, \bar{m}),$$

$$\Gamma_2^{(c)}(\bar{n}, \bar{m}) \equiv v(0) - \frac{1}{2} v(\bar{n}, \bar{m} + \pi), \quad \Gamma_2^{(s)}(\bar{n}, \bar{m}) \equiv 2v(\bar{n}, \bar{m} + \pi),$$

$$\Gamma_3^{(s)}(\bar{n}, \bar{m}) \equiv \frac{1}{2}(v(\bar{n}, \bar{m}) + v(\bar{n}, \bar{m} + \pi)), \quad \Gamma_3^{(t)}(\bar{n}, \bar{m}) \equiv \frac{1}{2}(v(\bar{n}, \bar{m}) - v(\bar{n}, \bar{m} + \pi)). \quad (A4)$$

Using the definitions of the interaction operators given in Eqs. and we obtain the result given in Eq.\.
REFERENCES

[1] E. Abrahams, P.W. Anderson, D.C. Licciardelo, and T.V. Ramakrishnan, Phys. Rev. Lett. 42, 673 (1979).

[2] A.M. Finkelstein, Z. Phys. B56, 189 (1984); C. Castelani, et al, Phys. Rev B30, 527 (1984).

[3] S.V. Kravchenko, D. Simonian, M.P. Sarachik, W. Mason, and J.E. Funeou, Phys. Rev. Lett. 77, 4938 (1996); D. Popovic, A.B. Fowler, and S. Washburn, Phys. Rev. Lett. 79, 1543 (1997).

[4] A.M. Goldman and Nina Markovic, Phys. Today, 39 Nov. (1998).

[5] Y. Hanein, U. Meirav, D. Shahar, C.C. Lin, D.C. Tsui, and H. Shtrikman, Phys. Rev. Lett. 80, 1288 (1998).

[6] S. Chakraverty, S. Kivelson, C. Nayak, and K. Voelker, Phys. Rev. B58, R559 (1998).

[7] M. Hilke, D. Shahar, S.H. Strong, D.C. Tsui, Y.H. Xie, and D. Monroe, Nature, 395, 675 (1998).

[8] T. Giamarchi and H.J. Schultz, Phys. Rev. B37, 325 (1988).

[9] D. Schmeltzer, R. Berkovits, M. Kaveh, and E. Kogan, Letter to the editor, Cond. Matt. 10 L651 (1998).

[10] C. Castellani, C.Dicastro, and P.A. Lee, Phys. Rev. B57, R9381 (1998).

[11] D. Belitz and T.R. Kirkpatrick, Phys. Rev. B58, 9710 (1998); D. Belitz and T.R. Kirkpatrick, Rev. Mod. Phys. 66, 261 (1994).

[12] S. Hikami, Phys. Rev. B24, 2671 (1981).

[13] J. Cardy, “Scaling and Renormalization in Statistical Physics”, Chapter 5, Cambridge Univ. Press (1996)
[14] V. Dobrosavljevic, E. Abrahams, E. Miranda, and S. Chakraverty, Phys. Rev. Lett. 79, 455 (1997).

[15] Qimiao Si and C.M. Varma, Phys. Rev. Lett. 81, 4951 (1998).

[16] P. Di Francesco, P. Mathieu, and D. Senechal, “Conformal Field Theories”, Chapters 6 and 15, Springer-Verlag, New York (1997).