Performance Limits on the Classification of
Kronecker-structured Models

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Abstract

Kronecker-structured (K-S) models recently have been proposed for the efficient representation, processing, and classification of multidimensional signals such as images and video. Because they are tailored to the multi-dimensional structure of the target images, K-S models show improved performance in compression and reconstruction over more general (union of) subspace models. In this paper, we study the classification performance of Kronecker-structured models in two asymptotic regimes. First, we study the diversity order, the slope of the error probability as the signal noise power goes to zero. We derive an exact expression for the diversity order as a function of the signal and subspace dimensions of a K-S model. Next, we study the classification capacity, the maximum rate at which the number of classes can grow as the signal dimension goes to infinity. We derive upper and lower bounds on the prelog factor of the classification capacity. Finally, we evaluate the empirical classification performance of K-S models for both the synthetic and the real world data, showing that they agree with the diversity order analysis.

Index Terms

Machine learning, subspace models, Kronecker-structured models, Gaussian mixture models, matrix normal distribution, diversity order.

I. INTRODUCTION

The classification of high-dimensional signals arises in a variety of image processing settings: object and digit recognition [1], [2], speaker identification [3], [4], tumor classification [5], [6], and more. A standard technique is to find a low-dimensional representation of the signal, such as a subspace or union of subspaces on which the signal approximately lies. However, for many signals, such as dynamic scene videos [7] or tomographic images [8], the signal inherently is multi-dimensional, involving dimensions of space and/or time. To use standard techniques, one vectorizes the signal, which throws out the spatial structure of the data which could be leveraged to improve representation fidelity, reconstruction error, or classification performance.

In order to exploit multi-dimensional signal structure, researchers have proposed tensor-based dictionary learning techniques, in which the signal of interest is a matrix or a higher-order tensor and the dictionary defining the (union of) subspace model is a tensor. Recent work along these lines includes the K-CPD method [9], which is an extension of traditional K-SVD [10] to learning an overcomplete tensor dictionary. Similarly, [11], [12] employ
the TUCKER model to learn tensor dictionaries. Other techniques include the t-SVD method \cite{6} and \cite{13}, where a separable structure is imposed on the dictionaries. These methods boast improved performance over traditional methods on a variety of signal processing tasks, including image reconstruction, image denoising and inpainting, video denoising, and speaker classification.

A simple tensor-based model is the \textit{Kronecker-structured} (K-S) model, in which a two-dimensional signal is represented by a coefficient matrix and two matrix dictionaries that pre- and post-multiply the coefficient matrix, respectively. Vectorizing this model leads to a dictionary that is the Kronecker product of two smaller dictionaries; hence the K-S model is a specialization of subspace models. This model is applied to spatio-temporal data in \cite{14}, low-complexity methods for estimating K-S covariance matrices are developed in \cite{15}, and it is shown that the sample complexity of K-S models is smaller than standard union-of-subspace models in \cite{16}.

As standard union-of-subspace models have proven successful for classification tasks \cite{17}, \cite{18}, \cite{19}, a natural question is the classification performance of K-S subspace models. In this paper, we address this question from an information-theoretic perspective. We consider a signal model in which each signal class is associated with a subspace whose basis is the Kronecker product of two smaller dictionaries; equivalently, we suppose that each signal class has a matrix normal distribution, where the row and column covariances are approximately low rank. In this sense, signals are drawn from a matrix Gaussian mixture model (GMM), similar to \cite{20}, where each K-S subspace is associated with a mixture component.

Similar to \cite{21}, we study the \textit{diversity order} and \textit{classification capacity} of K-S models, characterizing performance in the limit of high SNR and large signal dimension, respectively. In Section \textbf{II} we describe the K-S classification model in detail. In Section \textbf{III} we derive the diversity order for K-S classification problems, showing the exponent of the probability of error as the SNR goes to infinity. This analysis depends on a novel expression, presented in Lemma \textbf{4} for the rank of sums of Kronecker products of tall matrices. In Section \textbf{IV} we provide high-SNR approximations to the classification capacity. In Section \textbf{V} we show that the empirical classification performance of K-S models agrees with the diversity analysis.

\section*{II. Problem Definition}

\subsection*{A. Kronecker-structured Signal Model}

To formalize the classification problem, let the signal of interest $Y \in \mathbb{R}^{m_1 \times m_2}$ be a matrix whose entries are distributed according to one of $L$ class-conditional densities $p_l(Y)$. Each class-conditional density corresponds to a Kronecker-structured model described by the pair of matrices $A_l \in \mathbb{R}^{m_1 \times n_1}$ and $B_l \in \mathbb{R}^{m_2 \times n_2}$. The matrix $A_l$ describes the subspace on which the columns of $Y$ approximately lie, and $B_l$ describes the subspace on which the rows of $Y$ approximately lie. More precisely, if $Y$ belongs to class $l$, it has the form

$$Y = A_l X B_l^T + Z,$$

where $Z \in \mathbb{R}^{m_1 \times m_2}$ has i.i.d. zero-mean Gaussian entries with variance $\sigma^2 > 0$, and $X \in \mathbb{R}^{n_1 \times n_2}$ has i.i.d. zero-mean Gaussian entries with unit variance. We can also express $Y$ in vectorized form:

$$y = (B_l \otimes A_l)x + z,$$
for coefficient vector $\mathbf{x} = \text{vec}(\mathbf{X}) \in \mathbb{R}^N$, and noise vector $\mathbf{z} \in \mathbb{R}^M$, where $N = n_1 n_2$, $M = m_1 m_2$, and where $\otimes$ is the usual Kronecker product. Then, the class-conditioned density of $\mathbf{y}$ is

$$p_l(\mathbf{y}) = \mathcal{N}(0, (\mathbf{B}_l \otimes \mathbf{A}_l)(\mathbf{B}_l \otimes \mathbf{A}_l)^T + \sigma^2 \cdot \mathbf{I}).$$

(3)

In other words, the vectorized signal $\mathbf{y}$ lies near a subspace with a Kronecker structure that encodes the row and column subspaces of $\mathbf{Y}$.

In the sequel, we will characterize the performance limits over ensembles of classification problems of this form. To this end, we parameterize the set of class-conditioned densities via

$$\mathcal{A}(m_1, m_2, n_1, n_2) = \mathbb{R}^{m_1 \times n_1} \times \mathbb{R}^{m_2 \times n_2},$$

(4)

which contains the set of matrices indicating the row and column subspaces given signal and subspace dimensions $m_1, m_2, n_1, n_2$. We can represent an $L$-ary classification problem by a tuple $\mathbf{a} = (a_1, \cdots, a_L) \in \mathcal{A}^L(m_1, m_2, n_1, n_2)$, where each $a_l \in \mathcal{A}(m_1, m_2, n_1, n_2)$ is the pair of matrices $a_l = (\mathbf{A}_l, \mathbf{B}_l)$. Let $p(y|a_l) = p(y|\mathbf{A}_l, \mathbf{B}_l) = p_l(y)$, for $1 \leq l \leq L$, denote the class conditional densities parametrized by $\mathbf{a} \in \mathcal{A}(m_1, m_2, n_1, n_2)$. For a classification problem defined by $\mathbf{a}$, we can define the average misclassification probability:

$$P_e(\mathbf{a}) = \frac{1}{L} \sum_{l=1}^{L} \text{Pr}(\hat{l} \neq l|y \sim p(y|a_l)), $$

(5)

where $\hat{l}$ is the output of the maximum-likelihood classifier over the class-conditioned densities described by $a_l$. In this paper, we provide two asymptotic analyses of $P_e(\mathbf{a})$. First, we consider the diversity order, which characterizes the slope of $P_e(\mathbf{a})$ for a particular $\mathbf{a}$ as $\sigma^2 \to 0$. Second, we consider the classification capacity, which characterizes the asymptotic error performance averaged over $\mathbf{a}$ as $n_1, m_1, n_2, m_2$ go to infinity. For the latter case, we define a prior distribution over the matrix pairs $(\mathbf{A}_l, \mathbf{B}_l)$ in each class:

$$p(\mathbf{a}) = \prod_{p=1}^{m_1} \prod_{q=1}^{n_1} \prod_{r=1}^{m_2} \prod_{s=1}^{n_2} \mathcal{N}(a_{pq}, 0, 1/n_1) \cdot \mathcal{N}(b_{rs}, 0, 1/n_2)$$

(6)

where $a_{pq}$ is the $(p, q)$th element of matrix $\mathbf{A}$ and $b_{rs}$ is the $(r, s)$th element of matrix $\mathbf{B}$. Note that the column and row subspaces described by $\mathbf{A}$ and $\mathbf{B}$ are uniformly distributed over the Grassmann manifold because the matrix elements are i.i.d. Gaussian; however, the resulting K-S subspaces are not uniformly distributed.

### B. Diversity order

For a fixed classification problem $\mathbf{a}$, the diversity order characterizes the decay of the misclassification probability as the noise power goes to zero. By analogy with the definition of the diversity order in wireless communications [22], we consider the asymptotic slope of $P_e(\mathbf{a})$ on a logarithmic scale as $\sigma^2 \to 0$. Formally, the diversity order is defined as

$$d(\mathbf{a}) = \lim_{\sigma^2 \to 0} \frac{\log P_e(\mathbf{a})}{\frac{1}{2} \log (1/\sigma^2)}. $$

(7)

In Section III we characterize exactly the diversity order for almost every $\mathbf{a}$. 
C. Classification capacity

The classification capacity characterizes the number of unique subspaces that can be discerned as \( n_1, n_2, m_1 \) and \( m_2 \) go to infinity. That is, we derive bounds on how fast the number of classes \( L \) can grow as a function of signal dimension while ensuring the misclassification probability decays to zero almost surely. Here, we define a variable \( m \) and let it go to infinity. As \( m \) goes to infinity we let the dimensions \( m_1, m_2, n_1 \) and \( n_2 \) scale linearly with \( m \) as follows:

\[
m_1(m) = \lceil \kappa_1 m \rceil, \quad m_2(m) = \lceil \kappa_2 m \rceil, \quad n_1(m) = \lceil v_1 m \rceil, \quad n_2(m) = \lceil v_2 m \rceil
\]  

(8)

for \( v_1, v_2 \geq 1 \) and \( 0 \leq \kappa_1, \kappa_2 \leq 1 \). We let the number of classes \( L \) grow exponentially in \( m \) as:

\[
L(m) = \lfloor 2^{m_1(m)m_2(m)} \rfloor,
\]

(9)

for some \( \rho \geq 0 \), which we call the classification rate. We say that the classification rate \( \rho \) is achievable if

\[
\lim_{m \to \infty} \frac{1}{E[P_e(a)]} = \lim_{m \to \infty} \frac{1}{\int_{A^L(m)}(m_1(m),m_2(m),n_1(m),n_2(m))} P_e(a) \prod_{i=1}^{L(m)} p(a_i) da = 0.
\]

(10)

For fixed signal dimension ratios \( v_1, v_2, \kappa_1 \) and \( \kappa_2 \), we define \( C(v_1, v_2, \kappa_1, \kappa_2) \) as the supremum over all achievable classification rates, and we call \( C(v_1, v_2, \kappa_1, \kappa_2) \) (sometimes abbreviated by \( C \)) the classification capacity.

We can bound the classification capacity by the mutual information between the signal vector \( y \) and the matrix pair \( (A, B) \) that characterizes each Kronecker-structured class.

**Lemma 1.** The classification capacity satisfies:

\[
C \leq \lim_{m \to \infty} \frac{I(y; A, B)}{m_1(m)m_2(m)}
\]

(11)

Where the mutual information is computed with respect to \( p(a) \).

To prove lower bounds on the diversity order and classification capacity, we will need the following lemma, which gives the well-known Bhattacharyya bound on the probability of error of a maximum-likelihood classifier that chooses between two Gaussian hypotheses.

**Lemma 2** ([23]). Consider a signal distributed according to \( \mathcal{N}(\mu_1, \Sigma_1) \) or \( \mathcal{N}(\mu_2, \Sigma_2) \) with equal priors. Then, define

\[
b = \frac{1}{2} \ln \left( \frac{\Sigma_1 + \Sigma_2}{\Sigma_1^{\frac{1}{2}} \Sigma_2^{\frac{1}{2}}} \right) + \frac{1}{8} (\mu_1 - \mu_2) \left[ \frac{\Sigma_1 + \Sigma_2}{2} \right]^{-1} (\mu_1 - \mu_2)
\]

(12)

Supposing maximum likelihood classification, the misclassification probability is bounded by

\[
P_e(\mu_1, \Sigma_1, \mu_2, \Sigma_2) \leq \frac{1}{2} \exp(-b).
\]

(13)

We will also need the following lemma, which characterizes the dimension of intersections of subspaces spanned by Kronecker products of matrices. To the best of our knowledge this result is not in the literature, although its statement is intuitive.

\(^1\)Note that \( m \) is different from \( M \), where \( m \) is the variable we let to go to infinity and \( M = m_1m_2 \).
Lemma 3. Suppose \( \dim[\mathcal{R}(A_i) \cap \mathcal{R}(A_j)] = x \) and \( \dim[\mathcal{R}(B_i) \cap \mathcal{R}(B_j)] = y \), where \( \mathcal{R}(\cdot) \) denotes the range space of a matrix. Then,

\[
\dim[\mathcal{R}(B_i \otimes A_i) \cap \mathcal{R}(B_j \otimes A_j)] = xy. \tag{14}
\]

Proof: From [24, p. 447] for \( p \in \mathbb{R}^{m_1 \times n_1} \) and \( q \in \mathbb{R}^{m_2 \times n_2} \), we have

\[
\mathcal{R}(p \otimes q) = \mathcal{R}(p \otimes I_{m_2 \times n_2}) \cap \mathcal{R}(I_{m_1 \times m_1} \otimes q). \tag{15}
\]

Therefore, we can write the dimension as

\[
\dim \left[ \mathcal{R}(B_i \otimes A_i) \cap \mathcal{R}(B_j \otimes A_j) \right] = \dim \left[ \mathcal{R}(B_i \otimes I_{m_1 \times m_1}) \cap \mathcal{R}(B_j \otimes I_{m_1 \times m_1}) \right] \cap \mathcal{R}(B_i \otimes I_{m_2 \times m_2} \otimes A_i) \cap \mathcal{R}(B_j \otimes I_{m_1 \times m_1} \otimes A_j) \cap \mathcal{R}(I_{m_2 \times m_2} \otimes A_j). \tag{16}
\]

Rearranging terms, we obtain

\[
\dim \left[ \mathcal{R}(B_i \otimes A_i) \cap \mathcal{R}(B_j \otimes A_j) \right] = \dim \left[ \mathcal{R}(B_i \otimes I_{m_1 \times m_1}) \cap \mathcal{R}(B_j \otimes I_{m_1 \times m_1}) \right] \cap \mathcal{R}(B_i \otimes I_{m_1 \times m_1} \otimes A_i) \cap \mathcal{R}(B_j \otimes I_{m_2 \times m_2} \otimes A_j) \cap \mathcal{R}(I_{m_2 \times m_2} \otimes A_j). \tag{17}
\]

Next, let \( A_{ij} \) and \( B_{ij} \) be matrices whose column spans are \( \mathcal{R}(A_i) \cap \mathcal{R}(A_j) \) and \( \mathcal{R}(B_i) \cap \mathcal{R}(B_j) \), respectively. It is straightforward to verify that

\[
\mathcal{R}(B_i \otimes I_{m_1 \times m_1}) \cap \mathcal{R}(B_j \otimes I_{m_1 \times m_1}) = \mathcal{R}(B_{ij} \otimes I_{m_1 \times m_1}), \tag{18}
\]

and

\[
\mathcal{R}(I_{m_2 \times m_2} \otimes A_i) \cap \mathcal{R}(I_{m_2 \times m_2} \otimes A_j) = \mathcal{R}(I_{m_2 \times m_2} \otimes A_{ij}). \tag{19}
\]

Therefore, we can rewrite the subspace dimension as

\[
\dim \left[ \mathcal{R}(B_i \otimes A_i) \cap \mathcal{R}(B_j \otimes A_j) \right] = \dim \left[ \mathcal{R}(B_{ij} \otimes I_{m_1 \times m_1}) \cap \mathcal{R}(I_{m_2 \times m_2} \otimes A_{ij}) \right]. \tag{20}
\]

Next, we can apply the lemma of [24, p. 447] in reverse, yielding

\[
\dim[\mathcal{R}(B_i \otimes A_i) \cap \mathcal{R}(B_j \otimes A_j)] = \dim[\mathcal{R}(B_{ij} \otimes A_{ij})] = r(B_{ij}) \cdot r(A_{ij}) = xy.
\]

III. Diversity Order

As mentioned in Section II, the diversity order measures how quickly misclassification probability decays with the noise power for a fixed number of discernible subspaces. By careful analysis using the Bhattacharyya bound, we derive an exact expression for the diversity order for almost every classification problem. First, we state an expression that holds in general.

\footnote{With respect to the Lebesgue measure over \( \mathcal{A}_L \).}
Theorem 1. For a classification problem described by the tuple \( a \in \mathcal{A} \) such that \( r(A_i) = n_1 \) and \( r(A_j) = n_2 \) for every \( l \), the diversity order is \( d(a) = r^* - n_1 n_2 \), where

\[
r^* = \min_{i,j} r \left( B_i \otimes A_i \quad B_j \otimes A_j \right),
\]

and where \( r(\cdot) \) denotes the matrix rank.

Proof: Applying the Bhattacharya bound, the probability of a pairwise error between two Kronecker-structured classes \( i \) and \( j \) is bounded by

\[
P_e(D_i, D_j) \leq \frac{1}{2} \left( \frac{|D_i D_i^T + D_j D_j^T + 2\sigma^2 I|}{|D_i D_i^T + \sigma^2 I|^{\frac{1}{2}} |D_j D_j^T + \sigma^2 I|^{\frac{1}{2}}} \right)^{-\frac{1}{2}}
\]

where

\[
D_i D_i^T = B_i B_i^T \otimes A_i A_i^T
\]

\[
D_j D_j^T = B_j B_j^T \otimes A_j A_j^T.
\]

Using the well-known Kronecker product identities \((p \otimes q) \cdot (r \otimes s) = (pq \otimes rs)\) and \((p \otimes r)^T = (p^T \otimes r^T)\) we can write the matrix \( D_i D_i^T + D_j D_j^T \) as

\[
D_i D_i^T + D_j D_j^T = \left[ B_i \otimes A_i \quad B_j \otimes A_j \right] \cdot \left[ B_i^T \otimes A_i^T \quad B_j^T \otimes A_j^T \right]
\]

It is trivial that \( r(A) = r(AA^T) \), thus

\[
r(D_i D_i^T + D_j D_j^T) = r \left( \left[ B_i \otimes A_i \quad B_j \otimes A_j \right] \right) = r^*
\]

Let \( \lambda_i \) and \( \lambda_j \) denote the nonzero eigenvalues of \( D_i D_i^T \) and \( D_j D_j^T \) respectively, and let \( \lambda_{ij} \) denote the nonzero eigenvalues of \( D_i D_i^T + D_j D_j^T \) and \( r_{ij}^* \) denote its rank. Then, we can write the pairwise bound as:

\[
P_e(D_i, D_j) \leq \frac{1}{2} \left( \frac{\left( \sigma^2 \right)^{m_1 m_2 - n_1 n_2} \prod_{l=1}^{n_1 n_2} \left( \lambda_{ijl} + \sigma^2 \right)}{\sqrt{\left( \sigma^2 \right)^{m_1 m_2 - n_1 n_2} \prod_{l=1}^{n_1 n_2} \left( \lambda_{il} + \sigma^2 \right) \cdot \left( \sigma^2 \right)^{m_1 m_2 - n_1 n_2} \prod_{l=1}^{n_1 n_2} \left( \lambda_{jl} + \sigma^2 \right)}} \right)^{-\frac{1}{2}}
\]

\[
= \frac{1}{2} \left( \frac{1}{\sigma^2} \right)^{\frac{r_{ij}^* - n_1 n_2}{2}} \cdot \left( \prod_{l=1}^{n_1 n_2} \left( \lambda_{ijl} + \sigma^2 \right) \right)^{-\frac{1}{2}}
\]

By construction,

\[
D_i D_i^T + D_j D_j^T \geq D_i D_i^T, D_j D_j^T
\]

Using Weyl’s monotonicity theorem \( 2\lambda_{ijl} \geq \lambda_{il} \) and \( 2\lambda_{ijl} \geq \lambda_{jl} \) for every \( 1 \leq l \leq n_1 n_2 \). Therefore,

\[
\prod_{l=1}^{n_1 n_2} 2(\lambda_{ijl} + \sigma^2) \geq \left[ \prod_{l=1}^{n_1 n_2} (\lambda_{il} + \sigma^2) \cdot \prod_{l=1}^{n_1 n_2} (\lambda_{jl} + \sigma^2) \right]^{\frac{1}{2}}
\]
From this we can write
\[
P_c(D_i, D_j) \leq \frac{1}{2} \left( \frac{1}{\sigma^2} \right)^{\frac{n_1 n_2}{2}} \cdot 2^{\frac{\delta n_2}{2}} \cdot \left( \prod_{l=n_1 n_2 + 1} r_{ij}^{l} \left( \lambda_{ij} + \sigma^2 \right)^{-\frac{1}{2}} \right) \tag{26}
\]

\[
\leq 2^{\frac{n_1 n_2 - 2}{2}} \left( \frac{1}{\sigma^2} \right)^{\frac{n_1 n_2}{2}} \cdot \left( \lambda_{ij} r_{ij}^{*} + \sigma^2 \right)^{-\frac{1}{2}} \left( \lambda_{ij} r_{ij}^{*} \right)^{-\frac{1}{2}} \tag{27}
\]

\[
= 2^{\frac{n_1 n_2 - 2}{2}} \left( 1 + \frac{\lambda_{ij} r_{ij}^{*}}{\sigma^2} \right)^{-\frac{1}{2}}. \tag{28}
\]

Next, we bound \( P_c(a) \leq \sum_{i \neq j} P_c(D_i, D_j) \) via the union bound. For all the \( L(m) \) subspaces, we obtain the pairwise error probability and by invoking the union bound over all the subspaces we obtain:
\[
E[P_c(a)] \leq \frac{1}{L(m)} \sum_{i \neq j} \sum_{i} E[P_c(D_i, D_j)]
\]
\[
= (L(m) - 1) E[P_c(D_i, D_j)]
\]
\[
\leq 2^{\rho m l m_2} E[P_c(D_i, D_j)]
\]

Taking logarithm on both sides we obtain:
\[
\log_2(E[P_c(a)]) \leq \rho m l m_2 + \frac{n_1 n_2 - 2}{2} - \frac{r_{ij}^{*} - n_1 n_2}{2} \log_2 \left( 1 + \frac{\lambda_{ij} r_{ij}^{*}}{\sigma^2} \right) \tag{29}
\]

Putting this and (28) into the definition of the diversity order from (7), we obtain
\[
d(a) \geq \min_{i,j} \lim_{\sigma^2 \to 0} \log_2 \left( 1 + \frac{\lambda_{ij} r_{ij}^{*}}{\sigma^2} \right)
\]
\[
= \min_{i,j} \frac{r_{ij}^{*} - n_1 n_2}{2} \log_2 \left( 1 + \frac{\lambda_{ij} r_{ij}^{*}}{\sigma^2} \right) \tag{30}
\]
\[
= r^{*} - n_1 n_2. \tag{31}
\]

Finally, (23) shows that the Bhattacharyya bound is exponentially tight as the pairwise error decays to zero. Furthermore, the union bound is exponentially tight. Therefore, the above inequality holds with equality, and
\[
d(a) = r^{*} - n_1 n_2. \tag{32}
\]

For almost every classification problem, the rank \( r^{*} \) has the same value, as we show in the next lemma.

**Lemma 4.** *For almost every classification problem \( a \), the matrices \( [B_i \otimes A_i, B_j \otimes A_j] \) have rank
\[
r_{ij}^{*} = 2n_1 n_2 - [2n_1 - m_1]^+[2n_2 - m_2]^+, \tag{33}
\]*

where \([\cdot]^{+}\) denotes the positive part of a number.

**Proof:** Using standard matrix properties (e.g., (25)), we can write
\[
r([B_i \otimes A_i, B_j \otimes A_j]) = r(B_i \otimes A_i) + r(B_j \otimes A_j) - \dim [\mathcal{R}(B_i \otimes A_i) \cap \mathcal{R}(B_j \otimes A_j)]. \tag{34}
\]

Applying Lemma (3), we obtain
\[
r([B_i \otimes A_i, B_j \otimes A_j]) = r(B_i \otimes A_i) + r(B_j \otimes A_j) - \dim [\mathcal{R}(A_i) \cap \mathcal{R}(A_j)] \cdot \dim [\mathcal{R}(B_i) \cap \mathcal{R}(B_j)]. \tag{35}
\]
Almost every matrix has full rank, so 
\[ r(B_i \otimes A_i) = r(B_j \otimes A_j) = n_1n_2 \] almost everywhere, so we can rewrite (35) as
\[ r(\begin{bmatrix} B_i \otimes A_i & B_j \otimes A_j \end{bmatrix}) = 2n_1n_2 - \dim[R(A_i) \cap R(A_j)] \cdot \dim[R(B_i) \cap R(B_j)]. \] (36)

Next, we study the three possible cases for (36).

Case 1: \( n_2 < m_2 < 2n_2 \) and \( n_1 \leq \frac{m_1}{2} \). Here,
\[ \dim[R(A_i) \cap R(A_j)] = 0 \]
\[ \dim[R(B_i) \cap R(B_j)] = (2n_2 - m_2) \]
\[ r(\begin{bmatrix} B_i \otimes A_i & B_j \otimes A_j \end{bmatrix}) = 2n_1n_2 \]

Case 2: \( n_2 \leq \frac{m_2}{2} \) and \( n_1 < m_1 < 2n_1 \). Here,
\[ \dim[R(B_i) \cap R(B_j)] = 0 \]
\[ \dim[R(A_i) \cap R(A_j)] = (2n_1 - m_1) \]
\[ r(\begin{bmatrix} B_i \otimes A_i & B_j \otimes A_j \end{bmatrix}) = 2n_1n_2 \]

Case 3: \( n_2 < m_2 < 2n_2 \) and \( n_1 < m_1 < 2n_1 \). Here,
\[ \dim[R(A_i) \cap R(A_j)] = (2n_1 - m_1) \]
\[ \dim[R(B_i) \cap R(B_j)] = (2n_2 - m_2) \]
\[ r(\begin{bmatrix} B_i \otimes A_i & B_j \otimes A_j \end{bmatrix}) = 2n_1n_2 - (2n_1 - m_1)(2n_2 - m_2), \]

where the first and second equalities for each case hold almost everywhere, and the third equality for each case follows from Lemma 3. Combining the three cases yields the claim.

Applying Lemma 4 to Theorem 1, an exact expression for the diversity order follows immediately.

**Corollary 1.** For almost every classification problem \( a \), the diversity order is
\[ d(a) = n_1n_2 - [2n_1 - m_1] + \lceil 2n_2 - m_2 \rceil. \] (37)

In the symmetric case \( m_1 = m_2 \) and \( n_1 = n_2 \), the diversity order is the same as that predicted for general subspaces in [21]. The high-SNR classification performance of K-S subspaces is the same as general subspaces, even though K-S subspaces are structured, involve fewer parameters, and are easier to train.

**IV. Classification Capacity**

In this section, we derive upper and lower bounds on the classification capacity that hold approximately for large \( \sigma^2 \). Detailed analysis can be found in the long version of the paper.

**Theorem 2.** The classification capacity is upper bounded by
\[ C \leq \frac{\min\{\nu_1, \nu_2\}(\kappa_1 - \nu_1 + \kappa_2 - \nu_2)}{2\kappa_1\kappa_2} \log_2(1/\sigma^2) + O(1), \]
and
\[ C \geq \frac{\nu_1 \nu_2 - [2\nu_1 - \nu_1]\sigma^2}{2\kappa_1 \kappa_2} \log_2(1/\sigma^2) + O(1). \]

**Upper Bound:** The upper bound follows from an upper bound on the mutual information \( I(y; A, B) = h(y) - h(y|A, B) \) between the dictionary pairs \((A, B)\) and the signal \(y\) and invoke Lemma 1. In particular,
\[ I(y; A, B) = h(y) - h(y|A, B). \] (38)
The conditional distribution of \( y \) is
\[ p(y|(B \otimes A) = N(0,(B \otimes A)(B \otimes A)^T + \sigma^2 \cdot I). \]
Let \( \lambda_i \) be the \( i \)th eigenvalue of \((B \otimes A)(B \otimes A)^T\); then the conditional entropy is
\[ h(y|A, B) = \sum_{i=1}^{n_1 n_2} \frac{1}{2} E[\log_2(2\pi e \lambda_i)] + \frac{m_1 m_2 - n_1 n_2}{2} \log_2(2\pi e \sigma^2), \] (39)
which is bounded by
\[ h(y|A, B) \geq \frac{n_1 n_2}{2} E[\log_2(\lambda_{n_1 n_2} + \sigma^2)] + \frac{m_1 m_2 - n_1 n_2}{2} \log_2(\sigma^2) + \frac{m_1 m_2}{2} \log_2(2\pi e), \] (40)
where \( \lambda_{n_1 n_2} \) is the smallest possible positive eigenvalue, the value of which we now bound. Both \( A \) and \( B \) have i.i.d. Gaussian entries with variance \( 1/n_1 \) and \( 1/n_2 \) respectively, so
\[ AA^T \overset{d}{=} \frac{1}{n_1} W_1 \]
\[ BB^T \overset{d}{=} \frac{1}{n_2} W_2, \]
Where \( W_1 \sim \mathcal{W}(I, m_1) \) and \( W_2 \sim \mathcal{W}(I, m_2) \), and \( \mathcal{W}(R, n) \) denotes the Wishart distribution with \( k \) degrees of freedom and shape matrix \( R \). As \( m \) goes to infinity, the smallest eigenvalue of \( 1/m_1 W_1 \) converges to \((1 - \sqrt{\nu_1/\kappa_1})^2\) and the smallest eigenvalue of \( 1/m_2 W_2 \) converges to \((1 - \sqrt{\nu_2/\kappa_2})^2\) almost surely [26 Theorem 1]. So the minimum eigenvalue of \( AA^T \), denoted \( \lambda_{n_1} \), converges on \((\sqrt{\kappa_1/\nu_1} - 1)^2\) almost surely. Along the same lines, the minimum eigenvalue of \( BB^T \), denoted \( \lambda_{n_2} \), converges on \((\sqrt{\kappa_2/\nu_2} - 1)^2\) almost surely. Since the eigenvalues of the Kronecker product of two matrices is equal to the product of the pairs of the individual eigenvalues, \( \lambda_{n_1 n_2} \) converges on \((\sqrt{\kappa_1/\nu_1} - 1)^2 \cdot (\sqrt{\kappa_2/\nu_2} - 1)^2\) almost surely. We therefore, bound the conditional entropy as:
\[ h(y|A, B) \geq \frac{n_1 n_2}{2} E[\log_2((\sqrt{\kappa_1/\nu_1} - 1)^2 \cdot (\sqrt{\kappa_2/\nu_2} - 1)^2 + \epsilon(m) + \sigma^2)] + \frac{m_1 m_2 - n_1 n_2}{2} \log_2(\sigma^2) + \frac{m_1 m_2}{2} \log_2(2\pi e), \] (41)
From the i.i.d. Gaussian outer bound on entropy, we can derive a naive bound on the marginal entropy:
\[ h(y) \leq \frac{m_1 m_2}{2} \log(1 + \sigma^2) \] (42)
Now consider the case when both \( A \) and \( B \) are tall i.e. \( m_1 > n_1 \) and \( m_2 > n_2 \). Further suppose that \( n_1 < m_2 \). Then, we can derive a tighter outer bound on \( h(y) \). Let \( y_p \) be the first \( n_1 \) columns of \( y \) and let \( y'_p \) be the rest...
From (41) and (50), we can write an expression for the mutual information as

$$h(y) = h(y_A) + h(y'_p|y_p)$$

(43)

and obtained a bound on the probability of a pairwise error between two kronecker-subspaces $i$ and $j$.

Combining (45) and (48), we obtain an expression for the differential entropy:

$$h(y) = m_1n_1 \frac{1}{2} \log(1 + \sigma^2) + \frac{[m_2 - n_1]^+(m_1 - n_1)}{2} \log(\sigma^2)$$

(45)

Now, let $y_q$ be the first $n_2$ columns of $y$ and let $y'_q$ denotes the rest $m_1 - n_2$ columns of $y$. Then, $y'_p \in \mathbb{R}^{(m_2 - n_2) \times m_2}$, and we can derive the following high-SNR approximation on $h(y)$:

$$h(y) = h(y_q) + h(y'_q|y_q)$$

(46)

$$h(y) \leq \min(h(y_A), h(y_B))$$

(49)

Combining (43) and (48), we obtain an expression for the differential entropy:

$$h(y) \leq \frac{\min\{(m_2 - n_1)(m_1 - n_1)(m_1 - n_2)(m_2 - n_2)\}}{2} \log(\sigma^2) + \frac{\min\{m_1n_1, m_2n_2\}}{2} \log(1 + \sigma^2)$$

(50)

From (41) and (50), we can write an expression for the mutual information as $m \to \infty$ as:

$$I(y; A, B) \leq h(y) - h(y|A, B)$$

$$I(y; A, B) \leq \frac{\min\{(m_2 - n_1)(m_1 - n_1)(m_1 - n_2)(m_2 - n_2)\}}{2} \log(\sigma^2) + \frac{\min\{m_1n_1, m_2n_2\}}{2} \log(1 + \sigma^2)$$

(51)

$$I(y; A, B) = \frac{\min\{n_1, n_2\}(m_1 - n_1 + m_2 - n_2)}{2} \log(\frac{1}{\sigma^2}) + \frac{\min\{m_1n_1, m_2n_2\}}{2} \log(1 + \sigma^2) - \frac{m_1m_2 - n_1n_2}{2} \log(2\pi e)$$

Lower Bound: In order to obtain the lower bound on classification capacity we apply the same Bhattacharya bound on the probability of a pairwise error between two kronecker-subspaces $i$ and $j$ as described in section $\text{III}$ and obtained $E[P_e(a)]$ from (29) as follows:

$$\log(2E[P_e(a)]) \leq \rho m_1m_2 + \frac{n_1n_2 - 2}{2} - \frac{2n_1 - m_1 + 2n_2 - m_2}{2} \log\left(1 + \frac{\lambda_{ij}[2m_2 - 2n_1 - m_1 + 2n_2 - m_2]}{\sigma^2}\right)$$

(53)
We conjecture that the value of $\lambda_{ij}\{2n_1n_2-[2n_1-m_1]^+[2n_2-m_2]^+\}$ is bounded away from zero as $m \to \infty$. Therefore, if

$$\rho < \frac{n_1n_2 - 2}{2m_1m_2} - \frac{[2n_1-m_1]^+[2n_2-m_2]^+}{2m_1m_2} \cdot \log_2 \left(1 + \frac{\lambda_{ij}\{2n_1n_2-[2n_1-m_1]^+[2n_2-m_2]^+\}}{\sigma^2}\right)$$

(54)

Then surely $\mathbb{E}[P_e(a)]$ goes to zero as $m \to \infty$ and hence $P_e(a)$ goes to zero.

To compare the upper and lower bounds, consider the symmetric case, i.e. $m_1 = m_2 = m$ and $n_1 = n_2 = n$ and $m > n$. The gap between the prelog factor of the upper and lower bounds is $(m-n)^2$. We believe that both the upper and lower bounds are loose in general, and future work will involve tighter bounds on the mutual information in order to improve the prelog estimates.

V. NUMERICAL RESULT

In this section we demonstrate the classification performance numerically on both synthetic data and images taken from the YaleB face dataset, the empirical performance agrees with the diversity order derived in Section III. We also explore the representation and classification performance of K-S dictionaries on real world face recognition dataset and compare the performance of K-S dictionary learning with the standard subspace dictionary learning.

A. Synthetic Data

We randomly choose two classes by drawing matrix pairs $A_i$ and $B_i$ independently from the distribution in (6). Then, we draw data samples i.i.d. from the class-conditional densities in (2). We classify each data sample by minimizing the Mahalanobis distance associated with the covariance of each class-conditional density. We consider five cases, in which we fix $m_1 = m_2 = m$ and vary $n_1$ and $n_2$. In Figure 1 we plot the misclassification probability $P_e$ against the SNR in dB, averaged over $10^5$ random draws from each class. We also plot the slope predicted by the diversity order for each case, which is as small as 1 and as large as 12. In each case, the empirical performance agrees with the diversity predictions. For larger values of $m$, the diversity order is sufficiently high that it is difficult to estimate $P_e$ reliably.

![Fig. 1: Misclassification probability $P_e$ Vs. SNR](image)
B. YaleB Faces

Now, we present the performance of our analysis on YaleB face dataset. We randomly choose 10 classes out of the 38 face classes in the set. For each class we learn the Kronecker-structured dictionary that best fits the dataset images. Then, we project the images from each class onto its learned Kronecker subspace. This enforces the Kronecker structure on the images, which makes it possible to evaluate the diversity performance. We calculate the misclassification probability via minimizing the residual error between the image and its noisy Kronecker subspace projection for 10,000 noisy instantiations of each image. In this problem, the ambient dimensions \( m = 32 \). We show results for \( n_1 = 4 \) and \( n_2 = 2 \), which leads to a diversity order sufficiently small that we can estimate \( P_e \) reliably. For this particular case when \( m = 32, n_1 = 4 \) and \( n_2 = 2 \) diversity order is 8 and the misclassification probability is plotted with respect to SNR in Figure 2. The empirical performance is similar to theoretical predictions.

Fig. 2: Misclassification probability \( P_e \) Vs. SNR

C. Face Recognition

To evaluate the representation and classification performance of K-S dictionaries on a real-world dataset, we learn K-S dictionary pair \((A, B)\) for each class. Where a signal is represented by a sparse matrix coefficients in a dictionary pair. A dictionary pair describes the training data well if there exists a coefficient matrix which minimizes the reconstruction error. For the signals from different class to live in K-S subspace that are far apart we add an incoherence term between dictionaries. We evaluate the performance on ten classes chosen from the extended YaleB face recognition dataset. From each class we use 10 images for training/dictionary learning and the remaining 54 images for testing. In Figure 3 we show the dictionaries learned by K-S model and standard subspace model, we observe that the standard model learns dictionary atoms that look similar to a few reference faces for each class, whereas the K-S model learns more abstract dictionary atoms. This is in part due to imposition of the Kronecker structure on the dictionary atoms, as well as the larger number of atoms possible in a K-S dictionary.

In Table II we show that the classification accuracy of K-S model is better than the standard subspace model. We also observe that K-S obtains similar performance compared to the standard subspace dictionary for SIFT features of faces [18], without any feature engineering.

For K-S model, we find \( n_1 = 7 \) and \( n_2 = 9 \) gives the best classification performance with diversity order 63 and having 63 dictionary atoms. While for standard subspace model, we obtain the best classification accuracy for
|               | Subspaces | SIFT [18] | K-S |
|---------------|-----------|-----------|-----|
| Test Accuracy (%) | 79.36     | 84.3      | 84  |
| Number of parameters | 10240    | 5040      | 512 |

**TABLE I: Comparison between different approaches**

![Figure 3](image1.png)

Fig. 3: (a) A subset of dictionary atoms learned by (a) K-S model, and (b) standard subspace model.

10 dictionary atoms. The K-S model uses more atoms overall, but Figure 3 shows that they are more abstract, whereas the standard subspace dictionary atoms are direct face images. In terms of scalar parameters K-S model needs to learn a few number of parameters than the other models. Overall, K-S dictionary provides a more compact representation of the signal.

![Figure 4](image2.png)

Fig. 4: (a) A subset of test samples; (b) Image reconstruction and classification using K-S dictionary; (c) Image reconstruction and classification using standard subspace dictionary; {White box indicates incorrect classification}

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