The Rise-Contact involution on Tamari intervals

Viviane Pons
Paris-Saclay University
Motivations: Tamari intervals (they’re cool)

2005 Chapoton proves a nice formula counting intervals of the Tamari lattice (also counting triangular maps!)

\[
\frac{2}{n(n+1)} \binom{4n+1}{n-1}
\]

F. Chapoton. Sur le nombre d’intervalles dans les treillis de Tamari. Sém. Lothar. Combin., 2005.
Motivations: Tamari intervals (they’re cool)

2010 Bergeron and Prévillé-Ratelle give the definition of $m$-Tamari lattices and conjecture that the number of intervals is counted by

$$
\frac{m + 1}{n(mn + 1)} \binom{(m + 1)^2 n + m}{n - 1}
$$

F. Bergeron and L.-F. Prévillé-Ratelle. Higher trivariate diagonal harmonics via generalized Tamari posets. *J. Comb.*, 2012.
Motivations: Tamari intervals (they’re cool)

2011 Bousquet-Mélou, Fusy, and Préville-Ratelle prove the formula for $m$-Tamari intervals.

The generating function shows a symmetry between the *number of contacts* and the *initial rises* but they have no combinatorial explanation.

M. Bousquet-Mélou, E. Fusy, and L.-F. Préville-Ratelle. The number of intervals in the $m$-Tamari lattices. *Electron. J. Combin.*, 2011.
Motivations: Tamari intervals (they’re cool)

2012 Préville-Ratelle leaves a conjecture at this end of his thesis about the symmetry of two series of statistics “contacts” and “rises”

L.-F. Préville-Ratelle. *Combinatoire des espaces coinvariants trivariés du groupe symétrique*. Thèse de Doctorat, Université du Québec à Montréal, 2012.
Motivations: Tamari intervals (they’re cool)
Today, I give you the involution that proves this symmetry.

V. Pons. The Rise-Contact involution on Tamari intervals. *The Electronic Journal of Combinatorics*, 2019
The Rise-Contact involution on Tamari intervals
Contacts : 2
Contacts : 2
Contact vector: 2,
Contacts : 2
Contact vector: 2, 2,
Contacts: 2
Contact vector: 2, 2, 0,
Contacts : 2
Contact vector: 2, 2, 0, 1,
Contacts : 2
Contact vector: 2, 2, 0, 1, 0,
Contacts : 2
Contact vector: 2, 2, 0, 1, 0, 2,
Contacts : 2
Contact vector: 2, 2, 0, 1, 0, 2, 0,
Contacts : 2
Contact vector: 2, 2, 0, 1, 0, 2, 0, 2,
Contacts: 2
Contact vector: 2, 2, 0, 1, 0, 2, 0, 2, 0,
Contacts: 2
Contact vector: 2, 2, 0, 1, 0, 2, 0, 2, 0, 1,
Contacts : 2
Contact vector: 2, 2, 0, 1, 0, 2, 0, 2, 0, 1, 0
The Rise-Contact involution on Tamari intervals
The Rise-Contact involution on Tamari intervals
Rises : 2
Rises vector: 2,
Rises : 2
Rises vector: 2, 2,
Rises : 2
Rises vector: 2, 2, 0,
Motivations
Contacts and Rises
Involutions
Extra slides

Dyck paths
Tamari lattice

Rises : 2
Rises vector: 2, 2, 0, 0,
Rises : 2
Rises vector: 2, 2, 0, 0, 2,
Rises : 2
Rises vector: 2, 2, 0, 0, 2, 2,
Rises : 2
Rises vector: 2, 2, 0, 0, 2, 2, 2,
Rises : 2
Rises vector: 2, 2, 0, 0, 2, 2, 2, 0,
Rises : 2
Rises vector: 2, 2, 0, 0, 2, 2, 2, 0, 0,
Rises : 2
Rises vector: 2, 2, 0, 0, 2, 2, 2, 0, 0, 0,
Rises : 2
Rises vector: 2, 2, 0, 0, 2, 2, 2, 0, 0, 0, 0
The Rise-Contact involution on Tamari intervals
The Rise-Contact involution on Tamari intervals
Viviane Pons

The Rise-Contact involution on Tamari intervals

Contact vector: 3, 0, 2, 0, 0, 4, 0, 0, 1, 0
Rise vector: 3, 1, 0, 2, 3, 0, 1, 0, 0, 0
The Rise-Contact involution on Tamari intervals
The Rise-Contact involution on Tamari intervals

Contact vector: 2, 2, 0, 1, 0, 2, 0, 2, 0, 1
Rise vector: 2, 2, 0, 0, 2, 2, 2, 0, 0, 0
Motivations
Contacts and Rises
Involutions
Extra slides
Dyck paths
Intervals
Conclusion

Contact vector: 2, 2, 0, 1, 0, 2, 0, 2, 0, 1
Rise vector: 2, 2, 0, 0, 2, 2, 2, 0, 0, 0

↓ Φ
Contact vector: 2, 2, 0, 1, 0, 2, 0, 2, 0, 1
Rise vector: 2, 2, 0, 0, 2, 2, 2, 0, 0, 0

\[ \Phi \]

Contact vector: 2, 2, 2, 1, 0, 0, 0, 2, 1, 0
Rise vector: 2, 2, 0, 0, 2, 2, 2, 0, 0, 0

Viviane Pons
The Rise-Contact involution on Tamari intervals
Viviane Pons

The Rise-Contact involution on Tamari intervals

Contact vector: 2, 2, 2, 1, 0, 0, 0, 2, 1, 0
rotcev esiR: 2, 2, 0, 0, 2, 2, 2, 0, 0, 0
Contact vector: 2, 2, 2, 1, 0, 0, 0, 2, 1, 0

rotcev esiR: 2, 2, 0, 0, 2, 2, 2, 0, 0, 0

\[
\Psi
\]
Contact vector: 2, 2, 2, 1, 0, 0, 0, 2, 1, 0

rotcev esiR: 2, 2, 0, 0, 2, 2, 2, 0, 0, 0

ψ

Contact vector: 2, 2, 0, 0, 2, 2, 2, 0, 0, 0

rotcev esiR: 2, 2, 2, 1, 0, 0, 0, 2, 1, 0
Contact vector: 2, 2, 0, 0, 2, 2, 2, 0, 0, 0
rotcev esiR: 2, 2, 2, 1, 0, 0, 0, 2, 1, 0
The Rise-Contact involution on Tamari intervals
Contact vector: 2, 2, 0, 2, 0, 2, 0, 2, 0
Rise vector: 2, 2, 1, 0, 0, 0, 2, 1, 0

\[ \Phi \]
Viviane Pons

The Rise-Contact involution on Tamari intervals

Contact vector: 2, 2, 0, 1, 0, 2, 0, 2, 0, 1
Rise vector: 2, 2, 0, 0, 2, 2, 2, 0, 0, 0

↓ \( \Phi \circ \Psi \circ \Phi \)

Contact vector: 2, 2, 0, 2, 0, 2, 0, 0, 2, 0
Rise vector: 2, 2, 2, 1, 0, 0, 0, 2, 1, 0

Contact vector: 2, 2, 0, 2, 0, 2, 0, 0, 2, 0
Rise vector: 2, 2, 2, 1, 0, 0, 0, 2, 1, 0

\(1 - 2^4, 1^2, 0^4\)

\(1 - 2^5, 0^5\)
Contact vector: 4, 0, 1, 0, 2, 0, 0, 1
Rise vector: 1, 2, 0, 3, 2, 0, 0, 0

Viviane Pons
The Rise-Contact involution on Tamari intervals
Contact vector: 4, 0, 1, 0, 2, 0, 0, 1
Rise vector: 1, 2, 0, 3, 2, 0, 0, 0
Contact vector: 4, 0, 1, 0, 2, 0, 0, 1
Rise vector: 1, 2, 0, 3, 2, 0, 0, 0
Contact vector: 4, 0, 1, 0, 2, 0, 0, 1
Rise vector: 1, 2, 0, 3, 2, 0, 0, 0

\[ \Phi \]

Contact vector: 4, 2, 0, 1, 0, 0, 1, 0
Rise vector: 1, 2, 0, 3, 2, 0, 0, 0

Viviane Pons
The Rise-Contact involution on Tamari intervals
Contact vector: 4, 2, 0, 1, 0, 0, 1, 0
rotcev esiR: 1, 2, 0, 3, 2, 0, 0, 0

Viviane Pons
The Rise-Contact involution on Tamari intervals
Contact vector: 4, 2, 0, 1, 0, 0, 1, 0
rotcev esiR: 1, 2, 0, 3, 2, 0, 0, 0

\[ \Psi \]
Contact vector: 4, 2, 0, 1, 0, 0, 1, 0
rotcev esiR: 1, 2, 0, 3, 2, 0, 0, 0

↓ \( \Psi \)

Contact vector: 1, 2, 0, 3, 2, 0, 0, 0
rotcev esiR: 4, 2, 0, 1, 0, 0, 1, 0

Viviane Pons

The Rise-Contact involution on Tamari intervals
Contact vector: 1, 2, 0, 3, 2, 0, 0, 0

rotcev esiR: 4, 2, 0, 1, 0, 0, 1, 0
Contact vector: 1, 2, 0, 3, 2, 0, 0, 0
Rise vector: 4, 2, 0, 1, 0, 0, 1, 0
Contact vector: $4, 0, 1, 0, 2, 0, 0, 1 - 4, 2, 1^2, 0^4$
Rise vector: $1, 2, 0, 3, 2, 0, 0, 0 - 3, 2^2, 1, 0^4$

$\downarrow \Phi \circ \Psi \circ \Phi$

Contact vector: $1, 2, 3, 2, 0, 0, 0, 0 - 3, 2^2, 1, 0^4$
Rise vector: $4, 2, 0, 1, 0, 0, 1, 0 - 4, 2, 1^2, 0^4$
A combinatorial bijection which switches the contacts and the rises

Can be extended to the $m$-Tamari case

Preserve the “distance” of the interval

**The paper** : The Rise-Contact involution on Tamari intervals. *The Electronic Journal of Combinatorics, 26*(2):P2.32, 2019. arXiv:1701.07995

**SageMath Demo** :
github.com/VivianePons/public-notebooks/
“distance” preserving
$m$-Tamari

$m$-Contact vector: 5, 0, 0, 1, 0, 0, 0, 0, 2, 1, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0

$m$-Rise vector: 1, 0, 2, 0, 0, 1, 0, 2, 0, 1, 0, 0, 1, 1, 0, 0, 2, 0, 0, 0, 0, 0
Viviane Pons

The Rise-Contact involution on Tamari intervals
Motivations
Contacts and Rises
Involutions
Extra slides

The Rise-Contact involution on Tamari intervals

\[ \downarrow \Phi \circ \Psi \circ \Phi \]
Motivations
Contacts and Rises
Involutions
Extra slides

\[m\text{-Contact vector}: 5, 0, 0, 1, 0, 0, 0, 0, 2, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0 - 5, 2, 1^4, 0^{16}\]

\[m\text{-Rise vector}: 1, 0, 2, 0, 0, 1, 0, 2, 0, 1, 0, 0, 1, 0, 0, 2, 0, 0, 0, 0, 0, 0 - 2^3, 1^5, 0^{14}\]

Expand \(\Phi \circ \Psi \circ \Phi\) \(\rightarrow\) contract

\[m\text{-Contact vector}: 1, 2, 0, 0, 0, 1, 0, 2, 0, 1, 0, 1, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0 - 2^3, 1^5, 0^{14}\]

\[m\text{-Rise vector}: 5, 1, 2, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0 - 5, 2, 1^4, 0^{16}\]