Entropy operates in Non-linear Semifields

Francisco J. Valverde-Albacete, Member, IEEE, and Carmen Peláez-Moreno, Member, IEEE

Abstract

We set out to demonstrate that the Rényi entropies with parameter $\alpha$ are better thought of as operating in a type of non-linear semiring called a positive semifield. We show how the Rényi’s postulates lead to Pap’s $g$-calculus where the functions carrying out the domain transformation are Renyi’s information function and its inverse. In its turn, Pap’s $g$-calculus under Rényi’s information function transforms the set of positive reals into a family of semifields where “standard” product has been transformed into sum and “standard” sum into a power-deformed sum. Consequently, the transformed product has an inverse whence the structure is actually that of a positive semifield. Instances of this construction lead into idempotent analysis and tropical algebra as well as to less exotic structures. Furthermore, shifting the definition of the $\alpha$ parameter shows in full the intimate relation of the Rényi entropies to the weighted generalized power means. We conjecture that this is one of the reasons why tropical algebra procedures, like the Viterbi algorithm of dynamic programming, morphological processing, or neural networks are so successful in computational intelligence applications. But also, why there seem to exist so many procedures to deal with “information” at large.

Index Terms

Rényi entropy. Pap’s $g$-calculus. Generalized means. Positive semifields. Idempotent semifields.

I. INTRODUCTION AND MOTIVATION

Some non-linear algebras like the max-plus semiring or, in general, positive semirings have wide application in Machine Learning (ML), Computational Intelligence and Artificial Intelligence (AI). In this paper we propose the following explanation for their ubiquity: that they are in fact the natural algebras in which each of the infinite varieties of the Renyi $\alpha$-informations operate so that applications are “matched” to specific values of this parameter. The quasi-genetic process of social scientific and technological advance would then select which value for $\alpha$ is most suited to make a technique work in a particular application.

We will prove in this paper that these non-linear algebras are the positive semifields, a special type of semiring. Recall that a semiring is an algebra $S = (S, \oplus, \otimes, \epsilon, e)$ whose additive structure, $(S, \oplus, \epsilon)$, is a commutative monoid and whose multiplicative structure, $(S \setminus \{\epsilon\}, \otimes, e)$, is a monoid with multiplication distributing over addition from right and left and an additive neutral element absorbing for $\otimes$, i.e. $\forall a \in S, \epsilon \otimes a = \epsilon$.

A semiring is commutative if its product is commutative, and all semirings considered in this paper are commutative, whence we will drop the qualification. A semiring is zerosumfree if whenever a sum is null all summands are
also null, \( \bigoplus_i a_i = \epsilon \Rightarrow \forall i, a_i = \epsilon \), and it is \textit{entire} if it has no non-null factors of zero, \( a \otimes b = \epsilon \Rightarrow (a = \epsilon) \lor (b = \epsilon) \).

A semiring is \textit{positive} if it is \textit{zerosumfree} and \textit{entire} \cite{2}. Finally, a semiring is a \textit{semifield} if there exists a multiplicative inverse for every element \( a \in S \), except the null, notated as \( a^{-1} \). All semifields are entire. In this paper we concern ourselves with positive semifields, e.g. entire, zerosumfree semirings with a multiplication inverse.

Positive semifield applications abound in many areas of research. To cite but a few examples:

- Machine learning (ML) \cite{3} makes heavy use of Probability Theory, which is built around the positive semifield of the non-negative reals with their standard algebra \( \mathbb{R}_{\geq 0} \) and negative logarithms thereof, called \textit{log-probabilities}, both of which are positive semifields, as shown in Sections II-C and III-A.
- Computational Intelligence (CI) makes heavy use of positive semirings in the guise of \textit{fuzzy semirings}. Although semifields cannot be considered “fuzzy” for several technical reasons, the term is sometimes an umbrella under which to include \textit{non-standard algebras}, many of which are semifields, e.g. the morphological semifield of morphological processing and memories, a special case of the semifields in Section III-D.
- Artificial Intelligence (AI) \cite{4} is also a wide field under which applications dealing with minimizing costs or maximizing utilities abound. Semifields and their dual-orderings (cfr. Sections II-C and II-D) provide a perspective to mix these two kinds of valuations.

Beyond rigid disciplinary boundaries, the Viterbi algorithm \cite{5} is the paramount example of a discovery that leads to non-linear, positive semifield algebra in an algebraically-generic and application-independent setting. Initially devised as a teaching aid for convolutional codes, it soon was proven an optimal algorithm for shortest-path decoding in a specific type of network \cite{6}. But also, when this network comes from the unfolding over time of a Markov chain, it can be used to recover an “optimal” sequence of states and transition over a generative model for a given sequence of observations. In this natural generalization, it has been applied to text and speech recognition, among other cognitively-relevant application, that used to be considered part of classical AI but are modernly better tackled with ML or CI techniques.

A crucial issue in this application was to realise that the “optimality” of the decoding strategy is brought about by the nature of the operations being used in the decoding—in the language of this paper, it is \( \mathbb{R}_{\min,+} \)-optimal. It follows that several other algorithms can be built with the template of the Viterbi algorithm by changing the underlying semifield, but all require that the addition be \textit{idempotent}, that is \( \forall a \in S, a \oplus a = a \). A semiring with idempotent addition is simply called an \textit{idempotent semiring} and it is always positive.

Other applications of positive semifields include Electrical Network analysis and design (see the example in Section II-C), queuing theory \cite{7} and flowshop processing \cite{8}.

To build a basis for argumentation, we first revisit a number of seemingly unrelated topics: the discrete random variables and their probability distributions, the classical theory of weighted means, and positive semifields. We then introduce Pap’s g-calculus as applied to semifield construction and end the Preliminaries section with an introduction to the standard Rényi \( \alpha \)-entropies:

\[
H_{\alpha}(P_X) = \frac{1}{1-\alpha} \log \left( \sum_{i=1}^{n} p_i^\alpha \right)
\]
In our results we briefly introduce an “entropic” semifield and then make the case for shifting the Rényi entropy to \( r = \alpha - 1 \) prior to proving how this shifted Rényi entropy takes values in positive semifields. This allows us to unfold the argumentation for our conjecture that AI, ML and CI applications are mostly dealing with Rényi entropies of different order. We end the paper with a discussion of the issues touched upon it and some conclusions.

II. Preliminaries

A. Probability spaces and random variables

Let \((\Omega, \Sigma, P)\) be a measure space, with \(\Omega = \{\omega_1, \ldots, \omega_n\}\) the set of outcomes of a random experiment, \(\Sigma\) the sigma-algebra of this set and probability measure \(P: \Omega \rightarrow \mathbb{R}_{\geq 0}, P(\omega_i) = p_i, 1 \leq k \leq n\). For our purposes, we do not need to distinguish whether \(P\) is a probability measure in the \(\Omega\) simplex \(P \in \mathcal{S}(\Omega) \iff \sum_i P(\omega_i) = 1\). We define the support of \(P\), as the set of outcomes with positive probability \(\text{supp} (P_X) = \{\omega \in \Omega \mid P(\omega) > 0\}\).

Shannon and Renyi set out to find how much information can be gained by a single performance of the experiment \(\Omega\) under different suppositions. For that purpose we will sometimes need another probability space sharing the same measurable space \((\Omega, \Sigma)\) but different probability measure, \((\Omega, \Sigma, Q)\) with \(Q(\omega_i) = q_i\).

Let \((\mathcal{X}, \Sigma_{\mathcal{X}})\) be a measurable space with \(\mathcal{X}\) a domain and \(\Sigma_{\mathcal{X}}\) its sigma algebra and consider the random variable \(X: \Omega \rightarrow \mathcal{X}\), that is a measurable function so that for each set of \(B \in \Sigma_{\mathcal{X}}\) we have \(X^{-1}(B) \in \Sigma\). Then \(P\) induces a measure \(P_X\) on \((\mathcal{X}, \Sigma_{\mathcal{X}})\) with \(\forall x \in \Sigma_{\mathcal{X}}, P_X(x) = P(X = x) = P(X^{-1}(x))\), where \(x\) is an event in \(\Sigma_{\mathcal{X}}\), and \(P_X(x) = \sum_{\omega \subseteq X^{-1}(x)} P(\omega)\) whereby \((\mathcal{X}, \Sigma_{\mathcal{X}}, P_X)\) becomes a measure space.

Sometimes we will use \((X, P_X)\) to denote a random variable, instead of its measurable space. The reason for this is that since information measures are defined on distributions, this is the more fundamental notion for us. This is what the more usual \(X \sim P_X\) means. We make some remarks in passing:

- Sometimes co-occurring random variables are defined on the same sample space and sometimes on different ones.
- For all purposes, discrete distributions are nothing but sets of non-negative numbers adding up to 1, but see [9] on using incomplete distributions, that is with \(\sum_i p_i < 1\). Indeed nothing substantial changes in what follows by allowing \(\sum_i p_i \neq 1\), that is, measures in general, while it provides a measure of generalization.

B. Generalized means

Recall that given an invertible real function \(f: \mathbb{R} \rightarrow \mathbb{R}\) the Kolmogorov mean of a set of non-negative numbers \(\bar{x} = [x_i]_{i=1}^n \in (\mathbb{R}_{\geq 0})^n\) is:

\[
K_f(\bar{x}) = f^{-1}\left(\frac{1}{n}\sum_{i=1}^n f(x_i)\right) \tag{2}
\]

From now on, let \(\bar{w} = [w_i]_{i=1}^n \in (\mathbb{R}_{\geq 0})^n\) and \(\bar{x} = [x_i]_{i=1}^n \in (\mathbb{R}_{\geq 0})^n\) be equally long, co-indexed sequences of non-negative numbers. Formula (2) is an instance of the Kolmogorov-Nagumo formula to work out the mean with a set of weights.

\[
KN_f(\bar{w}, \bar{x}) = f^{-1}\left(\sum_{i=1}^n w_i f(x_i)\right) \tag{3}
\]
When a non-unitary $\sum w_i = W \neq 1$ is used, we can normalize $p_i = \frac{w_i}{W}$ and this is the condition implied when we use $KN_f(\vec{w}, \vec{x})$. But sometimes the normalization condition cannot be met and we will be using expressions that apply to this more general situation when possible.

Our interest in (3) lies in the fact that Shannon’s and Renyi’s entropies can be seen as special cases for it, which makes its properties specially interesting.

**Proposition 1** (Properties of the Nagumo-Kolmogorov means). Let $\vec{p}, \vec{x} \in (\mathbb{R}_0^+)^n$ and $r, s \in \mathbb{R} \setminus 0$. The following conditions hold if and only if there is a strictly monotonic and continuous function $f$ such that (2) holds.

1) **Continuity and strict monotonicity in all coordinates.**
2) **Symmetry or permutation invariance.** Let $\sigma$ be a permutation, then $M_r(p, x) = M_r(\sigma(p), \sigma(x))$.
3) **Reflexivity.** The mean of a series of constants is the constant itself:
   $$M_r(p, \{k\}_{i=1}^n) = k$$
4) **Blocking** The computation of the mean can be split into computations of equal size sub-blocks.
5) **Associativity:** replacing a $k$-subset of the $x$ with their partial mean in the same multiplicity does not change the overall mean.

For a minimal axiomatization, Blocking and Associativity are redundant. A review of the axiomatization of these and other properties can be found in [10].

1) **The weighted power means:** Recall that the weighted power or Hölder mean of order $r$ is defined as
   $$M_r(\vec{w}, \vec{x}) = \left(\frac{\sum_{i=1}^n w_i \cdot x_i^r}{\sum_{i=1}^n w_i}\right)^\frac{1}{r} \quad (4)$$

By formal identification, the power mean is nothing but the Kolmogorov-Nagumo mean with $f(x) = x^r$, and for $w_i = \frac{1}{n}$ we get the Kolmogorov mean. Reference [11] provides proof that this functional mean also has the properties [11][13] of Proposition 1 and Associativity.

Important cases of this mean for historical and practical reasons are:

- When $r = 0$ the (weighted) geometric mean results as the limit
  $$M_0(p, x) = \lim_{r \to 0} M_r(p, x) = \prod_{i=1}^n x_i^{p_i}$$

- When $r = 1$ the weighted arithmetic mean results:
  $$M_1(p, x) = \sum_{i=1}^n p_i \cdot x_i$$

- When $r = -1$ the weighted harmonic mean is produced:
  $$M_{-1}(p, x) = \left(\sum_{i=1}^n p_i \cdot x_i^{-1}\right)^{-1} = \frac{1}{\sum_{i=1}^n \frac{1}{p_i} \cdot \frac{1}{x_i}}$$

- When $r = 2$ the quadratic mean appears:
  $$M_2(p, x) = \left(\sum_{i=1}^n w_i \cdot x_i^2\right)^\frac{1}{2}$$
Finally, the max- and min-means appear as limits:

\[
M_\infty(p, x) = \lim_{r \to \infty} M_r(p, x) = \max_{i=1}^n x_i \\
M_{-\infty}(p, x) = \lim_{r \to -\infty} M_r(p, x) = \min_{i=1}^n x_i
\]

Furthermore, the weighted means all show the following properties:

**Proposition 2** (Properties of the weighted power means). Let \( \vec{w}, \vec{x} \in (\mathbb{R}_{\geq 0})^n \) and \( r, s \in \mathbb{R} \setminus 0 \). Then, the following formal identities hold:

1) **0- and 1-order homogeneity in weights and values.** Let \( k_1, k_2 \in \mathbb{R}_{\geq 0} \) then \( M_r(k_1\vec{w}, k_2\vec{x}) = k_1^k k_2^s M_r(\vec{w}, \vec{x}) \).

2) **Order factorization.** \( M_{rs}(\vec{w}, \vec{x}) = (M_s(\vec{w}, (\vec{x})^r))^{1/r} \)

3) **Reduction to the arithmetic mean.** \( M_r(\vec{w}, \vec{x}) = [M_1(\vec{w}, (\vec{x})^r)]^{1/r} \) where \( M_1(\cdot, \cdot) \) is the weighted arithmetic mean.

4) **Reduction to the harmonic mean** \( M_{-r}(\vec{w}, \vec{x}) = [M_{-1}(\vec{w}, (\vec{x})^r)]^{1/r} = [M_r(\vec{w}, 1)]^{-1} = [M_1(\vec{w}, \frac{1}{(\vec{x})^r})]^{-1/r} \) where \( M_{-1}(\cdot, \cdot) \) is the weighted harmonic mean and \( \frac{1}{r} \) has to be understood entry-wise.

5) **Monotonicity in r** Let \( \vec{w}, \vec{x} \in (\mathbb{R}_{\geq 0})^n \) and \( r, s \in [-\infty, \infty] \). Then

\[
\min_i x_i = M_{-\infty}(\vec{w}, \vec{x}) \leq M_r(\vec{w}, \vec{x}) \leq M_\infty(\vec{w}, \vec{x}) = \max_i x_i
\]

and the mean is a strictly monotonic function of \( r \), that is \( r < s \) implies \( M_r(\vec{w}, \vec{x}) < M_s(\vec{w}, \vec{x}) \), unless:

- \( x_i = k \) is constant, in which case \( M_r(\vec{w}, \vec{x}) = M_s(\vec{w}, \vec{x}) = k \).
- \( s \leq 0 \) and some \( x_i = 0 \), in which case \( 0 = M_r(\vec{w}, \vec{x}) \leq M_s(\vec{w}, \vec{x}) \).
- \( 0 \leq r \) and some \( x_i = \infty \), in which case \( M_r(\vec{w}, \vec{x}) \leq M_s(\vec{w}, \vec{x}) = \infty \).

6) **Non-null derivative.** Call \( q_r(\vec{w}, \vec{x}) = \left\{ \frac{w_k x_i^r}{\sum_i w_i x_i^r} \right\}_{k=1}^n \). Then

\[
\frac{\delta}{\delta r} M_r(\vec{w}, \vec{x}) = \frac{1}{r} \cdot M_r(\vec{w}, \vec{x}) \ln \frac{M_0(q_r(\vec{w}, \vec{x}), \vec{x})}{M_r(\vec{w}, \vec{x})}
\]

**Proof.** Property 1 follows from the commutativity, associativity and cancellation of sums and products in \( \mathbb{R}_{\geq 0} \).

Property 2 follows from identification in the definition, then 3 and 4 follow from it with \( s = 1 \) and \( s = -1 \) respectively. Property 5 and the special cases in it are well known and studied extensively in [12]. We will next prove Property 6

\[
\frac{d}{dr} M_r(\vec{w}, \vec{x}) = \frac{d}{dr} e^{\frac{1}{r} \ln \left( \sum_i \frac{w_k x_i^r}{\sum_i w_i x_i^r} \right)}
\]

\[
= M_r(\vec{w}, \vec{x}) \left( \frac{-1}{r^2} \ln \left( \sum_k \frac{w_k x_i^r}{\sum_i w_i x_i^r} \right) + \frac{1}{r} \cdot \frac{\sum_k w_k x_i^r \ln x_k}{\sum_i w_i x_i^r} \right)
\]

Note that if we call \( q_r(\vec{w}, \vec{x}) = \left\{ w_k x_i^r \right\}_{k=1}^n \), this again is a positive weight and we may rewrite:

\[
\sum_k \frac{w_k x_i^r}{\sum_i w_i x_i^r} \cdot \ln x_k = \sum_k w_k \ln x_k = \ln \left( \prod_k x_k^{w_k} \right) = \ln M_0(q_r(\vec{w}, \vec{x}), x)
\]

whence

\[
\frac{d}{dr} M_r(\vec{w}, \vec{x}) = M_r(\vec{w}, \vec{x}) \left( \frac{1}{r} \cdot \ln M_0(q_r(\vec{w}, \vec{x}), x) - \frac{1}{r} \cdot \ln M_r(\vec{w}, \vec{x}) \right) = \frac{1}{r} \cdot M_r(\vec{w}, \vec{x}) \ln \frac{M_0(q_r(\vec{w}, \vec{x}), \vec{x})}{M_r(\vec{w}, \vec{x})}.
\]
The distribution $q_r(\vec{w}, \vec{x})$ when $\vec{w} = \vec{x}$ is extremely important in the theory of generalized entropy functions, where it is called an **escort distribution** (of $\vec{w}$) \cite{13}, and we will prove below that its importance stems, at least partially, from this property. Also, due to Property 5, the evolution of $M_r(\vec{w}, \vec{x})$ with $r$ is also called the **H"older path** (of an $\vec{x}$).

An important application is that the weighted power means directly allow us to define the moments of a Random Variable. Let $X \sim P_X$ be a discrete random variable. Then the $r$-th moment of $X$ is:

$$E_X\{X^r\} = \sum_i P_X x_i^r = (M_r(P_X, X))^r$$

\hspace{1cm} (6)

### C. Positive Semifields and Semimodules

From the material in Sections II-A and II-B it seems evident that non-negative quantities are important for our purposes. This is the concept of **zerosumfree semiring** mentioned below, but we focus in the slightly less general notion of **dioid** (for double dioid) where there is a nice order available that “goes together” well with the operations of the algebra.

1) **Complete and positive dioids:** Recall that a **dioid** is a commutative semiring $D$ where the canonical preorder relation, $a \preceq b$ if and only if there exists $c \in D$ with $a \oplus c = b$ is actually an order $\langle D, \preceq \rangle$. For this order, the additive zero is always the bottom $\bot = \wedge D = \epsilon$.

In a dioid, the canonical order relation is compatible with both $\oplus$ and $\otimes$ \cite[Chap. 1, Prop. 6.1.7]{2}. Dioids are all zero-sum free, that is, they have no non-null additive factors of zero: if $a, b \in D, a \oplus b = \epsilon$ then $a = \epsilon$ and $b = \epsilon$.

A dioid is **complete** if it is complete as an ordered set for the canonical order relation, and the following distributivity properties hold, for all $A \subseteq D, b \in D$,

$$\left( \bigoplus_{a \in A} a \right) \otimes b = \bigoplus_{a \in A} (a \otimes b)$$

$$b \otimes \left( \bigoplus_{a \in A} a \right) = \bigoplus_{a \in A} (b \otimes a)$$

\hspace{1cm} (7)

In complete dioids, there is already a top element $\top = \oplus_{a \in D} a$.

A semiring is **entire or zero-divisor free** if $a \otimes b = \epsilon$ implies $a = \epsilon$ or $b = \epsilon$. If the dioid is entire, its order properties justifies calling it a **positive dioid or information algebra** \cite{2}.

2) **Positive semifields:** A semifield, as mentioned in the introduction, is a semiring whose multiplicative structure $\langle K \setminus \{\epsilon\}, \otimes, e, \cdot^{-1} \rangle$ is a group, where $\cdot^{-1} : K \rightarrow K$ is the function to calculate the inverse such that $\forall u \in K, u \otimes u^{-1} = e$. Since all semifields are entire, dioids that are at the same time semifields are called **positive semifields**, of which the positive reals or rationals are a paragon.

**Example** (Semifield of non-negative reals). The nonnegative reals

$$\mathbb{R}_{\geq 0} = \langle [0, \infty), +, \times, \cdot^{-1}, \bot = 0, e = 1 \rangle$$

are the basis for the computations in Probability Theory and other multiplicative costs. The bottom element has no inverse hence it is incomplete. Also, it is somewhat directed “away” from the bottom element hence the
underlying order is $([0, \infty), \leq)$. But its dual order has no bottom (see below), hence applications using, for instance, multiplicative costs and utilities at the same time, will be difficult to carry out in this algebra and notation.

In incomplete semifields like the one above, the inverse of the bottom element is the “elephant in the room” to be avoided in computations. Fortunately, semiring theory provides a construction to supply this missing element [14]. However, the problem with the dual order in the semifield above suggests that we introduce the completions appearing in the following theorem.

**Theorem 3.** For every (incomplete) semifield $\mathbb{K}$ there is a pair of completed semifields with dual order structures $\overline{\mathbb{K}} = \langle K, \ll \rangle$ and $(\overline{\mathbb{K}})^{-1} = \langle K, \gg \equiv \delta \rangle$, and dual algebraic structures. Define $\overline{\mathbb{K}} = K \cup \{\top\}$ where $\top = \bot^{-1}$ by definition. Then

$$\overline{\mathbb{K}} = \langle K, \oplus, \otimes, \cdot, -1, \bot, e, \top \rangle \quad \text{and} \quad (\overline{\mathbb{K}})^{-1} = \langle K, \hat{\oplus}, \hat{\otimes}, \cdot, -1, \top, e, \bot \rangle$$

On top of the individual laws as positive semifields, we have the modular laws:

$$(u \oplus v) \otimes (u \hat{\oplus} v) = u \otimes v$$

the analogues of the De Morgan laws:

$$u \hat{\oplus} v = (u^{-1} \oplus v^{-1})^{-1}$$
$$u \otimes v = (u^{-1} \otimes v^{-1})^{-1}$$

and the self-dual inequality

$$(u \otimes v) \hat{\otimes} w \geq u \otimes (v \hat{\otimes} w)$$

This is a well-known result we do not prove. Note that:

- the notation to “speak” about these semirings tries to follow a convention reminiscent of that of boolean algebra, where the inversion is complement.
- the dot notation is a mnemonic for where do the multiplication of the bottom and top go:

$$\bot \otimes \top = \bot \quad \text{and} \quad \top \otimes \bot = \top$$

implying that the “lower” addition and multiplication are aligned with the usual order in the semiring while the “upper” addition and multiplication are aligned with its dual. All other cases remain as defined for $\oplus$ in the incomplete semifield.

Regarding the intrinsic usefulness of completed positive semifields that are not fields—apart from the very obvious but degenerate case of $\mathbb{B}$ the booleans—we have the following example used, for instance, in Convex Analysis and Electrical Network theory.
Example (Dual semifields for the Non-negative Reals). The previous procedure shows that there are some problems with the notation of Example II-C2 and this led to the definition of the following signatures for this semifield and its inverse in convex analysis [15]:

$$\mathbb{R}_{\geq 0} = \langle [0, \infty], +, \times, \cdot^{-1}, 0, 1, \infty \rangle \quad \mathbb{R}_{\geq 0}^{-1} = \langle [0, \infty], +, \times, \cdot^{-1}, \infty, 1, 0 \rangle$$ (9)

Both of these algebras are used, for instance, in Electrical Engineering (EE), the algebra of complete positive reals to carry out the series sum of resistances, and its dual semifield to carry out parallel summation of conductances. With the convention that \(\mathbb{R}_{\geq 0}\) semiring models resistances, it is easy to see that the bottom element, \(\bot = 0\) models a shortcircuit, that the top element \(\top = \infty\) models an open circuit (infinite resistance) and these conventions are swapped in the dually-ordered semifield of conductances. Interestingly, the required formulae for the multiplication of the extremes:

$$0 \otimes \infty = 0 \quad 0 \lhd \infty = \infty$$ (10)

are a no-go for circuit analysis, which suggests that what is actually being operated with are the incomplete versions of these semifields, and the many problems that EE students have in learning how to properly deal with these values may stem from this fact. \(\square\)

3) Semimodules over positive semifields: Let \(\mathcal{D} = (D, +, \times, \epsilon_D, e_D)\) be a commutative semiring. A \(\mathcal{D}\)-semimodule \(\mathcal{X} = (X, \oplus, \odot, \epsilon_X)\) is a commutative monoid \((X, \oplus, \epsilon_X)\) endowed with a scalar action \(\lambda, x \mapsto \lambda \odot x\) satisfying the following conditions for all \(\lambda, \mu \in D\), \(x, x' \in X\):

\[
\begin{align*}
(\lambda \times \mu) \odot x &= \lambda \odot (\mu \odot x) \\
(\lambda + \mu) \odot x &= \lambda \odot x \oplus \mu \odot x \\
\epsilon_D \odot x &= x
\end{align*}
\]

Matrices form a \(\mathcal{D}\)-semimodule \(D^{g \times m}\) for given \(g, m\). In this paper, we only use finite-dimensional semimodules where we can identify semimodules with column vectors, e.g. \(\mathcal{X} \equiv D^g\). If \(\mathcal{D}\) is commutative, naturally-ordered or complete, then \(\mathcal{X}\) is also commutative, naturally-ordered or complete [11]. If \(\mathcal{K}\) is a semifield, we may also define an inverse for the semimodule by the coordinate-wise inversion, \((x^{-1})_i = (x_i)^{-1}\).

Similarly, the may define a matrix conjugate \((A^\circ)_{ij} = A_{ji}^{-1}\). For complete idempotent semifields, the following matrix algebra equations are proven in [16, Ch.8]:

Proposition 4. Let \(\mathcal{K}\) be an idempotent semifield, and \(A \in \mathcal{K}^{m \times n}\). Then:

1) \(A \hat{\odot} (A^\circ \hat{\odot} A) = A^\circ \hat{\odot} (A \hat{\odot} A^\circ) = (A \hat{\odot} A^\circ) \hat{\odot} A = A \hat{\odot} A^\circ = A \hat{\odot} (A \hat{\odot} A^\circ) = A^\circ \hat{\odot} (A \hat{\odot} A^\circ)\)

2) Alternating \(A - A^\circ\) products of 4 matrices can be shortened as in:

\[A^\circ \hat{\odot} (A \odot (A^\circ \hat{\odot} A)) = A^\circ \hat{\odot} A = (A^\circ \hat{\odot} A) \odot (A^\circ \hat{\odot} A)\]
3) Alternating $A \ast A$ products of 3 matrices and another terminal, arbitrary matrix can be shortened as in:

$$A \ast (A \ast (A \ast M)) = A \ast M = (A \ast A) \ast (A \ast M)$$

4) The following inequalities apply:

$$A \ast (A \ast M) \geq M \quad A \ast (A \ast M) \leq M$$

D. A construction for positive semifields

There is a non-countable number of semifields obtainable from $\mathbb{R}_{\geq 0}$. Their discovery is probably due to Maslov, but we present here the generalized procedure introduced by Pap and collaborators that includes Maslov’s results.

Construction 1 (Pap’s dioids and semifields). Let $\mathbb{R}_{\geq 0}$ be the semiring of non-negative reals, and consider a strictly monotone generator function $g$ on an interval $[a, b] \subseteq [-\infty, \infty]$ with values in $[0, \infty]$. Since $g$ is strictly increasing it admits an inverse $g^{-1}$, so set

1) the pseudo-addition, $u \oplus v = g^{-1}(g(u) + (g(v))$

2) the pseudo-multiplication, $u \otimes v = g^{-1}(g(u) \times (g(v))$

3) neutral element, $e = g^{-1}(1)$

4) inverse, $x^{\ast} = g^{-1}\left(\frac{1}{g(x)}\right)$.

Then,

1) if $g$ is strictly increasing such that $g(a) = 0$ and $g(b) = \infty$, then a complete positive semifield whose order is aligned with that of $\mathbb{R}_{\geq 0}$ is:

$$\mathcal{K}_g = ([a, b], \oplus, \otimes, \ast, \perp = a, \ominus = b) .$$

2) if $g$ is strictly decreasing such that $g(a) = \infty$ and $g(b) = 0$, then a complete positive semifield whose order is aligned with that of $(\mathbb{R}_{\geq 0})^{-1}$ is

$$(\mathcal{K}_g)^{-1} = (\langle a, b], \oplus, \otimes, \ast, \perp^{-1} = b, \ominus^{-1} = a) .$$

Proof. See [17] [18] for the basic dioid, and [2] p. 44] for the inverse operation and the fact that it is a semifield, hence a positive semifield.

Our use of Construction 1 is to generate different kind of semifields by providing different generator functions.

Construction 2 (Multiplicative-product real semifields [19]). Consider a free parameter $r \in (-\infty, 0] \cup (0, \infty]$ and the function $g(x) = x^r$ in $[a, b] = [0, \infty]$ in Construction 1. For the operations we obtain:

$$u \oplus_r v = \left(u^r + v^r\right)^{\frac{1}{r}} \quad u \otimes_r v = \left(u^r \times v^r\right)^{\frac{1}{r}} = u \otimes v \quad u^{\ast} = \left(\frac{1}{u^r}\right)^{\frac{1}{r}} = u^{-1} \quad (12)$$

where the basic operations are to be interpreted in $\mathbb{R}_{\geq 0}$. Now,

- if $r \in (0, \infty]$ then $g(x) = x^r$ is strictly monotone increasing whence $\perp_r = 0$, $e_r = 1$, and $\ominus_r = \infty$, and the complete positive semifield generated, order-aligned with $\mathbb{R}_{\geq 0}$, is:

$$\langle \mathbb{R}_{\geq 0} \rangle_r = ([0, \infty], \oplus_r, \otimes, \perp, \ominus^{-1}_r = 0, \ominus = 0, \ominus_r = \infty) \quad (13)$$
• if \( r \in [-\infty, 0) \) then \( g(x) = x^r \) is strictly monotone decreasing whence \( \perp_r = \infty, e_r = 1, \) and \( \top_r = 0, \) and the complete positive semifield generated, order-aligned with \((\mathbb{R}_{\geq 0})^{-1}\), or dually aligned with \(\mathbb{R}_{\geq 0}\), is:

\[
(\mathbb{R}_{\geq 0})_r = (\mathbb{R}_{\geq 0})_r^{-1} = ([0, \infty], \delta_r, \times, \cdot, \perp_r^{-1} = \infty, e, \top_r^{-1} = 0)
\]  

Proof. By instantiation of the basic case.

In particular, consider the cases:

**Proposition 5.** In the previous Construction 2, if \( r \in \{\pm 1\} \) then

\[
(\mathbb{R}_{\geq 0})_1 = \mathbb{R}_{\geq 0} \quad \quad (\mathbb{R}_{\geq 0})_{-1} = \mathbb{R}_{\geq 0}^{-1}
\]

and

\[
\lim_{r \to \infty} (\mathbb{R}_{\geq 0})_r = \mathbb{R}_{\text{max}, x} \quad \quad \lim_{r \to -\infty} (\mathbb{R}_{\geq 0})_r^{-1} = \mathbb{R}_{\text{min}, x}
\]

Proof. The proof of (15) by inspection. For (16) see [19].

This suggests the following corollary:

**Corollary 6.** \((\mathbb{R}_{\geq 0})_r\) and \((\mathbb{R}_{\geq 0})_r^{-1}\) are inverse, completed positive semifields.

**E. Renyi’s entropy**

It is important to recall that Shannon set out to define the *amount of information*, discarding any notions of information itself. Both notions should be distinguished clearly for methodological reasons, but can be ignored for applications that deal only with quantifying information.

1) The approach to Rényi’s information functions based in postulates: Recall the Faddeev postulates for the generalization of Shannon’s entropy [9, Chap. IX. §2]:

1) The (amount of) information \( H(\cdot) \) of a sequence of \( n \) numbers \( P = [p_k]_{k=1}^n \) is a symmetric function of this set of values \( H(P) = H(\sigma(P)) = H(\{p_k\}_{k=1}^n) \), where \( \sigma \) is any permutation of \( n \)-elements.

2) \( H(\{p, 1-p\}) \) is a continuous function of \( p, 0 \leq p \leq 1 \).

3) \( H(\{\frac{1}{2}, \frac{1}{2}\}) = 1 \).

4) The following relation holds:

\[
H(\{p_1, p_2, \ldots, p_n\}) = H(\{p_1 + p_2, \ldots, p_n\}) + (p_1 + p_2)H(\{\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2}\})
\]

These postulates lead to Shannon’s entropy for \( X \sim P_X \) with binary logarithm [9]

\[
H(P_X) = - \sum_k p_k \log p_k
\]

Similar postulates lead to Rényi’s entropy function, and we use the exposition in [20] to propose them:

1) The amount of information provided by a single random event \( x_k \) should be a function of its probability \( P_X(x_k) = p_k \), not its value \( x_k = X(\omega_k) \).

\[
\mathcal{I}(x_k) = \mathcal{I}(p_k), p_k \in [0, 1]
\]
2) This amount of information should be additive on independent events.

\[ I(pq) = I(p) + I(q) \]  \hfill (20)

3) The amount of information of a binary equiprobable decision is one bit.

\[ I(1/2) = 1 \]  \hfill (21)

4) If different amounts of information occur with different probabilities the total amount of information \( I \) is an average of the individual information amounts weighted by the probability of occurrence.

Note how the last postulate describes aggregate amounts of information, not atomic ones.

These postulates may lead to the following consequences:

- Postulates [1] and [2] fix Hartley’s function as the single possible amount of information of a basic event \( I(p) = -k \log p \).
- Postulates [3] fixes the base of the logarithm in Hartley’s formula to 2 by fixing \( k = 1 \). Any other value \( k = 1/\log b \) fixes b as the base for the logarithm and changes the unit.
- Postulate [4] defines an average amount of information, or entropy proper. Its basic formula is a form of the Kolmogorov-Nagumo applied to information

\[ H(P_X, \varphi, I) = \varphi^{-1} \left( \sum_{k=1}^{n} p_k \varphi(I(p_k)) \right) . \]  \hfill (22)

It has repeatedly been proven that only two forms of the function \( \varphi \) can actually be used in the Kolmogorov-Nagumo formula that respect the previous postulates [9, 20, 21]:

- The one generating Shannon’s entropy:

\[ \varphi(h) = ah + b \text{ with } a \neq 0, \]  \hfill (23)

- That for Rényi’s generalized entropy

\[ \varphi(h) = 2^{(1-\alpha)h}, \text{ with } \alpha \in [-\infty, \infty] \setminus \{1\} . \]  \hfill (24)

These decisions lead to the following formulae for the generalized entropy for a random variable \( X \sim P_X \).

Taking the first form (23) and plugging it into (22) leads to Shannon’s measure of information,

\[ H(P_X) = -\frac{1}{a} \left( \sum_{i} p_i \left( a \log \frac{1}{p_i} + b \right) \right) = \frac{1}{a} \left( \sum_{i} p_i a \log \frac{1}{p_i} \right) = -\sum_{i} p_i \log p_i , \]

and taking the second form leads to Rényi’s measure of information [1], so in fact we have:

\[ H_\alpha(P_X) = \frac{1}{1-\alpha} \log \left( \sum_{i=1}^{n} p_i^\alpha \right) \text{ } \alpha \neq 1 \quad \lim_{\alpha \to 1} H_\alpha(P_X) = H(P_X) = -\sum_{i} p_i \log p_i , \]  \hfill (25)

where the fact that Shannon’s entropy is the Rényi entropy when \( \alpha \to 1 \) in (25) is found by a continuity argument. Note that nothing precludes using \( m_X \) a sort of “mass” function with \( \sum_k m_k = M \neq 1 \) as a basis for entropy: corrections for the formulae are easy to develop.
Actually, Rényi used the postulate approach to define the gain of information or divergence (between) distributions when \( Y \sim P_Y \) is substituted by \( X \sim P_X \) being continuous wrt the latter \( \text{supp}(Y) \subseteq \text{supp}(X) \) [9] [21], as

\[
D_\alpha(P_X \| P_Y) = \frac{1}{\alpha - 1} \log \sum_{i=1}^{n} p_i^\alpha q_i^{1-\alpha}, \quad \alpha \neq 1 \quad \text{and} \quad \lim_{\alpha \to 1} D_\alpha(P_X \| P_Y) = D_{KL}(P_X \| P_Y)
\]

and the fact that Kullback-Leibler’s divergence emerges as the limit when \( \alpha \to 1 \) follows from the same procedure as before, and such special cases will not be stated again, as motivated in Section III-B.

As in Shannon’s case, the rest of the quantities arising in Information Theory can be defined in terms of the generalized entropy and its divergence, but we will not follow this road in this paper. See, for instance [20].

### III. Results

**A. The basic entropic semifield**

The effect of Hartley’s information function is to induce a semifield from the set of positive numbers (restricted to the \([0, 1]\) interval) to a semiring whose carrying set if \([0, \infty]\). To see this, we actually consider it acting on the whole of the non-negative reals \( \mathbb{R}_{\geq 0} \) of \([0, \infty]\) onto the algebra of entropies denoted by \( \mathbb{H} \).

**Theorem 7** (Hartley’s semifields). The algebra \( \mathbb{H} = ([-\infty, \infty], \oplus, \otimes, \ominus, -\infty, 0) \) with

\[
\begin{align*}
    h_1 \oplus h_2 &= h_1 + h_2 - \ln(e^{h_1} + e^{h_2}) \\
    h_1 \otimes h_2 &= h_1 + h_2 \\
    h_\ominus &= -h
\end{align*}
\]

obtained from that of positive numbers by Hartley’s information function is a positive semifield that can be completed in two different ways to two mutually dual semifields:

\[
\mathbb{H} = ([-\infty, \infty], \oplus, \otimes, \ominus, -\infty, e = 0, T = \infty) \quad \mathbb{H}^{-1} = ([-\infty, \infty], \hat{\oplus}, \hat{\otimes}, \hat{\ominus}, \bot = \infty, e = 0, \top = -\infty)
\]

(27)

whose elements can be considered as entropic values and operated accordingly.

**Proof.** Let \( p \in \mathbb{R}_{\geq 0} \) be a positive number, and define the extension to Hartley’s information function \( \mathcal{H}_*(\cdot) : [0, \infty] \to [-\infty, \infty] \) as \( \mathcal{H}_*(p) = -\ln(p) \). This is one-to-one from \([0, \infty]\) and total onto \([-\infty, \infty]\), with inverse

\[
(\mathcal{H}_*)^{-1}(h) = e^{-h} \quad \text{for} \quad h \in [-\infty, \infty].
\]

Since \( (\mathcal{H}_*)^{-1}(h) = e^{-h} \) is monotone, it is a generator (function) for Construction [1] with the following addition, multiplication and inversions.

\[
\begin{align*}
    h_1 \oplus h_2 &= \mathcal{H}_*((\mathcal{H}_*)^{-1}(h_1) + (\mathcal{H}_*)^{-1}(h_2)) = -\ln(e^{-h_1} + e^{-h_2}) = \ln \frac{e^{h_1} + e^{h_2}}{e^{h_1} + e^{h_2}} = h_1 + h_2 - \ln(e^{h_1} + e^{h_2}) \\
    h_1 \otimes h_2 &= \mathcal{H}_*((\mathcal{H}_*)^{-1}(h_1) \times (\mathcal{H}_*)^{-1}(h_2)) = -\ln(e^{-h_1} \cdot e^{-h_2}) = -\ln(e^{-(h_1 + h_2)}) = h_1 + h_2 \\
    h_\ominus &= \mathcal{H}_*(\frac{1}{(\mathcal{H}_*)^{-1}(h)}) = -\ln \frac{1}{e^{-h}} = -h
\end{align*}
\]

(28)

(29)

(30)

1In the \( g \)-calculus these are always named with a “pseudo-” prefix, but it does not agree with Semiring Theory practice, hence we drop it.
The “interesting” points \( \{0, e, \infty\} \) are transformed as:

\[
I^*(0) = \infty \quad I^*(1) = 0 \quad I^*(\infty) = -\infty
\] (31)

The rest follows by Construction 1 and Theorem 3.

This is the proof of the following corollary.

**Corollary 8.** Hartley’s information function extended to the non-negative reals \( I^*(m) = -\log_b m = h \leftrightarrow (I^*)^{-1}(h) = b^{-h} = m \) is a dual-order isomorphism of completed positive semifields between \( \mathbb{R}_{\geq 0} \) and \( \mathbb{H}^{-1} \) and their inverses.

**Proof.** In the previous theorem, note that \( I^*(\cdot) \) is monotonic decreasing, entailing that the construction inverts orders in semifields, that is \( I^*(\mathbb{R}_{\geq 0}) = \mathbb{H}^{-1} \) and \( I^*(\mathbb{R}_{\geq 0}^{-1}) = \mathbb{H} \).

On purpose, we have not restricted this dual-order isomorphism to the sub-semiring of probabilities. Note that on \( I^*((0,1]) = [0, \infty] \subseteq [\infty, \infty] \), our intuitions about amounts of informations hold, whence we still believe that “the information in \( I^*(0) = \infty \) is the highest, whereas the probability of \((I^*)^{-1}(\infty) = 0 \) is the smallest.

**B. The shifted Rényi entropy**

To leverage the theory of generalized means to our advantage, we start with a correction to Rényi’s entropy definition: The investigation into the form of the transformation function for the Rényi entropy is arbitrary in the parameter \( \alpha \) that it chooses. In fact, we may substitute in \( r = \alpha - 1 \) to obtain the pair of formulas:

\[
\varphi'(h) = 2^{-rh} \quad \varphi'^{-1}(p) = \frac{-1}{r} \log p
\] (32)

**Definition 1.** The shifted Rényi entropy of order \( r \neq 0 \) for a discrete random variable \( X \sim P_X \), is the Kolmogorov-Nagumo \( \varphi' \)-mean of the information function \( I^*(p) = -\ln p \) over the probability values.

\[
\tilde{H}_r(P_X) = \frac{-1}{r} \log \left( \sum_i p_i \varphi'^{-1}(p_i) \right).
\] (33)

For \( r \neq 0 \) this comes from

\[
\tilde{H}_r(P_X) = \frac{-1}{r} \log \left( \sum_i p_i 2^{r \log p_i} \right) = \frac{-1}{r} \log \left( \sum_i p_i 2^{\log \varphi'^{-1}(p_i)} \right) = \frac{-1}{r} \log \left( \sum_i p_i \varphi'^{-1}(p_i) \right).
\]

Note that for \( r = 0 \) we could use the linear mean \( \varphi(h) = ah + b \) with inverse \( \varphi^{-1}(p) = \frac{1}{a}(p-b) \) as per the standard definition. Also, note that the base of the logarithm is not important as long as it is maintained in \( \varphi'(\cdot), I^*(\cdot) \) and their inverses. The domain diagram in Fig. D.1 summarizes the actions of these functions to obtain the shifted Rényi entropy. It would seem we first transform the probabilities into entropies using Hartley’s function and then we use the \( \varphi' \) function to work out an average of these using the Kolmogorov-Nagumo formula. A similar diagram is, of course, available for the standard entropy, using \( \varphi \) with the \( \alpha \) parameter.

The shifted divergence can be obtained in the same manner—the way that Rényi followed himself [9].
\[ R \geq 0 \xrightarrow{\varphi'^{-1}} \mathbb{H} \xrightarrow{\varphi'} R_{\geq 0} \xrightarrow{(\varphi')^{-1}} (\cdot) P_X \]

Fig. 1. Domain diagram to interpret the Rényi transformation in the context of the expectations of probabilities and informations.

**Definition 2.** The shifted Rényi divergence between two distributions \( P_X(x_i) = p_i \) and \( Q_Y(y_i) = q_i \) with compatible support is the following quantity.

\[
\tilde{D}_r(P_X \parallel Q_Y) = \frac{1}{r} \log \sum_i p_i \left( \frac{p_i}{q_i} \right)^r
\]

(34)

It is important to realize that the values of the Rényi entropy and divergence are not modified by this shifting.

**Proposition 9.** The Rényi entropy and the shifted Rényi entropy produce the same value, and similarly for their respective divergences.

**Proof.** We consider a new parameter \( r = \alpha - 1 \) in which case:

\[
H_\alpha(P_X) = \frac{1}{1 - \alpha} \log \left( \sum_{i=1}^{n} p_i^\alpha \right) = -\frac{1}{r} \log \left( \sum_{i=1}^{n} p_i^{r+1} \right) = -\frac{1}{r} \log \left( \sum_{i=1}^{n} p_i^{r} q_i \right) = \tilde{H}_r(P_X).
\]

and similarly for the divergence:

\[
D_\alpha(P_X \parallel Q_X) = \frac{1}{\alpha - 1} \log \sum_{i=1}^{n} p_i^\alpha q_i^{1-\alpha} = \frac{1}{r} \log \sum_{i=1}^{n} p_i^{r+1} q_i^{-r} = \frac{1}{r} \log \sum_{i=1}^{n} p_i \left( \frac{p_i}{q_i} \right)^r = \tilde{D}_r(P_X \parallel Q_X).
\]

So what could be the reason for the shifting? First of all, it is a re-alignment with the more basic concept of generalized mean.

**Lemma 1.** (Rényi information spectrum) The Shifted Rényi Entropy and Divergence are logarithmic transformations of the power means:

\[
\tilde{H}_r(P_X) = \log \frac{1}{M_r(P_X, P_X)}
\]

(35)

\[
\tilde{D}_r(P_X \parallel Q_X) = \log M_r(P_X, \frac{P_X}{Q_X})
\]

(36)

**Proof.** Simple identification in the definition of power mean definitions of Property 2.

This means that the properties of the Rényi entropy and divergence stem from those of the means, inversion and logarithm, a great simplification. For instance, it is no longer necessary to make the distinction between the case \( r \to 0 \) and the rest, since the means are already defined with this caveat. This emphasizes the peculiar feature of Shannon’s entropy, arising from the geometric mean:

\[
\tilde{H}_0(p_x) = \log \frac{1}{M_0(P_X, P_X)} = -\log \left( \prod_i p_i^{p_i} \right) = -\sum_i p_i \log p_i
\]
Table I lists the shifting of these entropies and their relation both to the means and to the original Rényi definition in the parameter $\alpha$.

Also, since $M_r(P_X, P_X)$ is a peculiar kind of Hölder mean, for fixed $P_X$ we will refer to $\tilde{H}_r(P_X)$ as its Rényi information spectrum over parameter $r$. This shows the notorious following properties:

**Proposition 10. (Properties of the Rényi entropy in shifted formulation)** Let $r, s \in \mathbb{R} \cup \{\pm \infty\}, P_X, Q_X \in \mathcal{S}(\mathcal{X})$ where $\mathcal{S}(\mathcal{X})$ is the simplex over the support $\mathcal{X}$, with cardinal $|\mathcal{X}| = n$. Then,

1) (Boundedness, Monotonicity) The Rényi entropy is a non-increasing function of the order $r$.

$$s < r \Rightarrow \tilde{H}_{-\infty}(P_X) \geq \tilde{H}_s(P_X) \geq \tilde{H}_r(P_X) \geq \tilde{H}_\infty(P_X)$$  \hspace{1cm} (37)

2) The entropy of the uniform pmf $U_X$ is constant over $r$.

$$\forall r \in \mathbb{R} \cup \{\pm \infty\}, \tilde{H}_r(U_X) = \log |\mathcal{X}|$$  \hspace{1cm} (38)

3) The Hartley entropy $(r = -1)$ is constant over the distribution simplex.

$$\forall P_X \in \mathcal{S}(\mathcal{X}), \tilde{H}_{-1}(P_X) = \log |\mathcal{X}|$$  \hspace{1cm} (39)

4) (Divergence from uniform.) The divergence of any distribution $P_X$ from the uniform $U_X$ can be written in terms of the entropies as:

$$\tilde{D}_r(P_X \parallel U_X) = \tilde{H}_r(U_X) - \tilde{H}_r(P_X)$$  \hspace{1cm} (40)

5) (Derivative of the shifted entropy) The derivative in $r$ of Rényi’s $r$-th order entropy is

$$\frac{d}{dr} \tilde{H}_r(P_X) = \frac{-1}{r^2} \tilde{H}_0(\tilde{q}_r(P_X) \parallel P_X) = \frac{1}{r} \log \frac{M_0(\tilde{q}_r(P_X), P_X)}{M_r(P_X, P_X)}$$  \hspace{1cm} (41)

where \( \tilde{q}_r(P_X) = \left( \frac{p_i^r}{\sum_k p_k^r} \right)_{i=1}^n \).

6) (Relationship with the moments of $P_X$) The shifted Rényi Entropy of order $r$ is the logarithm of the inverse $r$-th root of the $r$-th moment of $P_X$.

$$\tilde{H}_r(P_X) = -\frac{1}{r} \log E_{P_X}\{P_X^r\} = \log \frac{1}{\sqrt[r]{E_{P_X}\{P_X^r\}}}$$  \hspace{1cm} (42)
Proof. Properties used in the following are referred to the Proposition they are stated in. Property 1 issues from property 2 and Hartley’s information function being a dual-order isomorphism from Corollary 8. With respect to property 2 we have, from $U_X = 1/|X| = 1/n$ and property 11

$$\hat{H}_r\left(\frac{1}{n}\right) = -\log M_r\left(\frac{1}{n}, \frac{1}{n}\right) = -\log \frac{1}{n} = \log n.$$ 

For property 3 we have:

$$\hat{H}_{-1}(P_X) = -\log\left(\sum_i p_i \cdot p_i^{-1}\right)^{-1} = -\log(n)^{-1} = \log n$$

For property 4

$$\hat{D}_r(P_X || U_X) = \frac{1}{r} \log \left[ \sum_i p_i \left( \frac{p_i}{\bar{u}_i} \right)^r \right] = \frac{1}{r} \log \left[ \sum_i p_i \left( \frac{p_i}{1/n} \right)^r \right] = \frac{1}{r} \log \left[ n^r \left( \sum_i p_i \tilde{p}_i^r \right) \right]$$

$$= \log n + \log \left( \sum_i p_i \tilde{p}_i^r \right)^{1/r} = \hat{H}_r(U_X) - \hat{H}_r(P_X).$$

For the third term of property 5 we have from (35) with natural logarithm, with $P_X$ in the role both of $\tilde{w}$ and $\tilde{\bar{x}}$

$$\frac{d}{dr} \hat{H}_r(P_X) = -\frac{d}{dr} \frac{M_r(P_X, P_X)}{M_r(P_X, P_X)},$$

whence the property follows directly from (5). For the first identity, though, we have:

$$\frac{d\hat{H}_r(P_X)}{dr} = -\frac{d}{dr} \left[ \frac{1}{r} \log \sum_i p_i \tilde{p}_i^r \right] = -\left[ -\frac{1}{r^2} \log \sum_i p_i \tilde{p}_i^r + \frac{1}{r} \sum_i \tilde{p}_i \log(p_i) \right]$$

we introduce $\tilde{q}_r(P_X) = \{\tilde{q}_r(P_X)_i\}_{i=1}^n = \{\frac{p_i \tilde{p}_i^r}{\sum_k p_k \tilde{p}_k^r}\}_{i=1}^n$, notice that $\log \sum_k p_k \tilde{p}_k^r = \sum_i \tilde{q}_r(P_X)_i \log(\sum_k p_k \tilde{p}_k^r)$, since $\tilde{q}_r(P_X)$ is a distribution, and extract $-1/r^2$:

$$\frac{d\hat{H}_r(P_X)}{dr} = -\frac{1}{r^2} \left[ -\sum_i \tilde{q}_r(P_X)_i \log(\sum_k p_k \tilde{p}_k^r) + r \left( \sum_i \tilde{q}_r(P_X)_i \log(p_i) \right) \pm \sum_i \tilde{q}_r(P_X)_i \log(p_i) \right]$$

$$= -\frac{1}{r^2} \left[ -\sum_i \tilde{q}_r(P_X)_i \log(\sum_k p_k \tilde{p}_k^r) + (r + 1) \sum_i \tilde{q}_r(P_X)_i \log(p_i) - \sum_i \tilde{q}_r(P_X)_i \log(p_i) \right]$$

$$= -\frac{1}{r^2} \left[ \sum_i \tilde{q}_r(P_X)_i \log(\sum_k p_k \tilde{p}_k^r) - \sum_i \tilde{q}_r(P_X)_i \log(p_i) \right] = -\frac{1}{r^2} \hat{D}_0(\tilde{q}_r(P_X)||P_X).$$

For property 6 in particular, the probability of any event is a function of the random variable $P_X(x_i) = p_i$ wherefore the r-th moment of $P_X$ is

$$E_X\{P_X^r\} = \sum_i p_i \tilde{p}_i^r = (M_r(P_X, P_X))^r$$

(43)

The result follows by applying the definition of the shifted entropy in terms of the means. □

In this proof we have also introduced the notion of shifted escort probabilities $\tilde{q}_r(P_X)$ acting in the shifted Rényi entropies as the analogues of the escort probabilities in the standard definition. Note that for $P_X \in \mathbb{S}(\mathcal{X})$ we have $\tilde{q}_0(P_X) = P_X$ and $\tilde{q}_{-1}(P_X) = |\text{supp}(P_X)|$.
C. The equivalent probability function

On the one hand, the existence of Hartley’s information function ties up all information values to probabilities and vice versa. In particular, from every average measure of information, an equivalent average probability emerges:

Definition 3. Let $X \sim P_X$ with Rényi spectrum $\tilde{H}_r(P_X)$. Then the equivalent probability function of $\tilde{P}_r(P_X)$ is the Hartley inverse of $\tilde{H}_r(P_X)$ over all values of $r \in [-\infty, \infty]$:

$$\tilde{P}_r(P_X) = (I^*)^{-1}(\tilde{H}_r(P_X))$$  \hspace{1cm} (44)

Note that the equivalent probability function, as the Rényi entropy, for a fixed $P_X$ is a function of parameter $r$ whose values are probabilities, but it is not a probability function.

Lemma 2. Let $X \sim P_X$. The equivalent probability function $\tilde{P}_r(P_X)$ is the Hölder path of the probability function $P_X$ (as a set of numbers) using the same probability function as weights.

$$\tilde{P}_r(P_X) = M_r(P_X, P_X)$$  \hspace{1cm} (45)

Proof. From the definition, using $b$ as the basis chosen for the logarithm in the information function.

$$\tilde{P}_r(P_X) = (I^*)^{-1}(\tilde{H}_r(P_X)) = b^{-\tilde{H}_r(P_X)} = b^{\log_b M_r(P_X, P_X)} = M_r(P_X, P_X)$$

Note that these means apply, in general, to sets of non-negative numbers and not only to the probabilities in a distribution, given the homogeneity properties of the means. In the light of Lemma 2 the properties of the equivalent probability function are a corollary of those of the weighted generalized power means of Proposition 2 in Section (II-B).

Corollary 11. Let $X \sim P_X$ be a random variable with equivalent probability function $\tilde{P}_{-\infty}(P_X)$. Then:

1) For all $r \in [-\infty, \infty]$, there holds that

$$\min_k p_k = \tilde{P}_{-\infty}(P_X) \leq \tilde{P}_r(P_X) \leq \max_k p_k = \tilde{P}_{\infty}(P_X)$$  \hspace{1cm} (46)

2) If $P_X \equiv U_X$ the uniform over the same domain, then $\forall k, \forall r \in [-\infty, \infty], p_k = \tilde{P}_r(U_X) = \frac{1}{|X|}$.

3) if $P_X \equiv \delta_{X,k}$ the Kroneker delta centered on $x_k = X(\omega_k)$, then $\tilde{P}_{\infty}(\delta_{X,k}) = 1$

And thus, in their turn, the properties of Rényi entropy can be proven from those of the equivalent probability function and Hartley’s generalized information function, being a dual isomorphim of positive semifields.

An interesting property might help transforming the equivalent probability function into a probability distribution:

Lemma 3. Let $X \sim P_X$ be a random variable with equivalent probability function $\tilde{P}_r(P_X)$. Then: for every $p_k$ in $P_X$ there exists an $r_k \in [-\infty, \infty]$ such that $p_k = \tilde{P}_{r_k}(P_X)$.

Proof. This is implied by the continuity of the means with respect to its parameters $\bar{w}$ and $\bar{x}$. \hfill \square
So if we could actually find those values $r_k, 1 \leq k \leq n$ which return $p_k = \tilde{P}_r(P_X)$ we could actually retrieve $P_X$ by sampling $\tilde{P}_{r_k}(P_X)$ in the appropriate values $P_X = \{\tilde{P}_{r_k}(P_X)\}_{k=1}^n$. Since $n \geq 2$ we know that at least two of these values are $r = \pm \infty$ retrieving the value of the highest and lowest probabilities for $k = 1$ and $k = n$ when they are sorting by increasing probability value.

D. Entropic semifields

The shifted Rényi entropy of [35] and the fact that Hartley’s information function, embedded within it, is a dual order isomorphism of semifields entails that every Rényi measure of entropy is working in a semifield. To prove this we first investigate the form of those semifields.

Theorem 12 (Additive-product real semifields or Entropy semifields). Let $r \in [-\infty, \infty] \setminus \{0\}$ and $b \in (1, \infty)$. Then the algebra

$$ \mathbb{H}_r = \langle [-\infty, \infty], \oplus_r, \otimes_r, \cdot^{-1}, \bot = -\infty, e = 0 \rangle $$

whose basic operations are:

$$ u \oplus_r v = u + v - \log_b (b^ru + b^rv)^\frac{1}{r} \quad u \otimes_r v = u + v \quad u^{-1} = -\frac{1}{r} \log_b \left( \frac{1}{b^{-ru}} \right) = -u $$

(48)

can be completed with $\top = \infty$ to two dually-ordered positive semifields

(49)

whose elements can be considered as emphasized, entropic values and operated accordingly.

Proof. We build these semifields with a composition of results. The first one is the well known result from the theory of functional means we choose to cast into the framework of Pap’s $g$-calculus: the power mean of order $r$ is the pseudo arithmetic-mean with generator $\varphi_r(x) = x^r$ and inverse $\varphi_r^{-1}(y) = y^{1/r}$. This was used in Construction 2 to build the semifields of [13] and [14]. The second result is Corollary 8 where $\mathcal{J}_s(\cdot)$ is proven a dual order isomorphism of semirings.

We now use the composition of functions $\mathcal{J}_s \circ \varphi_r$ and its inverse $\varphi_r^{-1} \circ \mathcal{J}_s^{-1}$. The latter exists, since it is a composition of isomorphisms and it is a dual order isomorphism, since $\mathcal{J}_s(\cdot)$ is order-inverting while $\varphi_r$ is not. That composition is the function $\varphi'_{r}(h) = b^{-rh}$ with inverse $\varphi'^{-1}_{r}(p) = \frac{1}{r} \log_b p$, hence the operations are:

$$ u \oplus_r v = \varphi'^{-1}_{r}(\varphi_r'(u) + \varphi_r'(v)) = -\frac{1}{r} \log_b (b^{-ru} + b^{-rv}) = -\frac{1}{r} \log_b \left( \frac{b^r(u+v)}{b^ru + b^rv} \right) = u + v - \log_b (b^ru + b^rv)^\frac{1}{r} $$

$$ u \otimes_r v = \varphi'^{-1}_{r}(\varphi_r'(u) \times \varphi_r'(v)) = -\frac{1}{r} \log_b (b^{-ru} \times b^{-rv}) = -\frac{1}{r} \log_b \left( \frac{b^{-r(u+v)}}{b^{-ru} \times b^{-rv}} \right) = u + v $$

$$ u^{-1} = \varphi'^{-1}_{r}(\frac{1}{\varphi_r'(u)}) = -\frac{1}{r} \log_b \left( \frac{1}{b^{-ru}} \right) = -u $$

This function is strictly increasing when $r \in [-\infty, 0)$ and strictly decreasing when $r \in (0, \infty]$, hence, when applying Construction 1 with it:

- For $r \in [-\infty, 0)$ we obtain $\mathbb{H}_r = \mathcal{J}_s((\mathbb{R}_{\geq 0})^{-1}) = \langle [-\infty, \infty], \oplus_r, \otimes_r, \cdot^{-1}, \bot = -\infty, e = 0, \top = \infty \rangle$
For \( r \in (0, \infty] \) we obtain
\[
\mathbb{H}_r^{-1} = \mathcal{J}_*(\mathbb{R}_{\geq 0})_r = \langle [-\infty, \infty], \oplus_r, \odot_r, \cdot^-1, \perp^-1 = \infty, e = 0, T^{-1} = -\infty \rangle
\]
with the extended operations:
\[
\begin{align*}
\perp^-1 &= -\infty^-1 = \infty = T \\
\cdot^-1 &= -\infty^-1 = \infty = \top^-1 \\
\mathcal{J}_*(\mathbb{R}_{\geq 0})_r &= \langle [-\infty, \infty], \oplus_r, \odot_r, \cdot^-1, \perp^-1 = \infty, e = 0, \top^-1 = -\infty \rangle
\end{align*}
\]
\begin{align*}
\begin{cases}
T = \infty & \text{if } u = T \text{ or } v = T \\
u \oplus v & \text{otherwise}
\end{cases}
\begin{cases}
\top = -\infty & \text{if } u = \top \text{ or } v = \top \\
u \odot v & \text{otherwise}
\end{cases}
\end{align*}
\begin{align*}
\begin{cases}
\top = \infty & \text{if } u = \top \text{ or } v = \top \\
u \odot v & \text{otherwise}
\end{cases}
\begin{cases}
\top = -\infty & \text{if } u = \top \text{ or } v = \top \\
u \odot v & \text{otherwise}
\end{cases}
\end{align*}

Further details for compositions of generating functions and other averaging constructions can be found in [22]. Note that since conjugation \( -1 \) is a dual order isomorphism and an involution we, and \( \phi' \) is a dual order isomorphism may write \( (\mathbb{H}_r)^{-1} = (\mathbb{H}_r)^{-1} \).

The following corollary is the result we announced at the beginning of this section.

**Corollary 13.** The Rényi entropies in the Rényi spectrum \( \tilde{H}_r(\mathbb{P}_X) \) are members of the semifields \( \mathbb{H}_r \).

**Proof.** We notice that the generating function to activate Construction 1 in Theorem 12 is none other than the function to calculate the Rényi non-linear average \( \phi' \) of (32). Hence the values resulting from Rényi’s entropies belong in that semifield of Theorem 12 with the respective \( r \) parameter.

We would like to clarify the meaning of these many semifields in relation to the entropy. The following sections try to do so.

**E. A conjecture on the abundance of semifields in Machine Learning and Computational Intelligence applications**

We are now in a position to argue our conjecture about the abundance of semifields in knowledge domains that model intelligent behaviour.

1) First, by shifting the definition of the Rényi entropy by \( r = \alpha - 1 \) in (1), we found a straightforward relation (35) between the power means of the probability distribution and the shifted Rényi entropy.

2) The function used by Rényi to define the generalized entropy, when shifted, is the composition of two functions: Hartley’s information function and the power function of order \( r \), which are monotone and invertible in the extended non-negative reals \([0, \infty]\). They are also bijections:
   - The power function is a bijection over the extended non-negative reals, and
   - Hartley’s is a bijection between the extended reals and the extended non-negative reals.

3) In Construction 1 where both the power function and Hartley’s prove to be isomorphisms of positive semifields, which are semirings whose multiplicative structure is that of a group, while the additive structure lacks additive inverses. Positive semifields are all naturally ordered and the power function respects this order within the non-negative reals, being an order isomorphism for generic power \( r \). Importantly, positive semifields come in
dually-ordered pairs and the expressions mixing operations from both members in the pair are reminiscent of boolean algebras.

a) The power function \( g(x) = x^r \) actually generates with \( r \in [-\infty, \infty] \setminus \{0\} \), a whole family of semifields \((\mathbb{R}_{\geq 0})_r\), related to emphasizing smaller (with small \( r \)) or bigger values (with small \( r \)) in the non-negative reals \( \mathbb{R}_{\geq 0} \). Indeed, the traditional weighted means are explained by the Construction 2 as being power-deformed arithmetic means, also known as Kolmogorov-Nagumo means with the power function as generators. These, semirings come in dually-ordered pairs for orders \( r \) and \(-r\) whose orders are aligned or inverted with respect to that of \( \mathbb{R}_{\geq 0} \). Indeed, \( \mathbb{R}_{\geq 0} \cong (\mathbb{R}_{\geq 0})_1 \).

b) However, Hartley’s function is a dual-order isomorphism, entailing that the new order in the extended reals is the opposite of that on the non-negative reals. It actually mediates between the (extended) probability semifield \( \mathbb{R}_{\geq 0} \) and the semifield of informations, notated as a homage to Hartley as \( \mathbb{H}^r \).

4) Since the composition of the power mean and Hartley’s information function produces the function that Rényi used for defining his information measures, and this is a dual-order semifield isomorphism, being the composition of one dual-order isomorphism—Hartley’s function—and an order isomorphism—the power function—we can see that entropies are actually operated in modified versions of Hartley’s semifields \( \mathbb{H}^r \) and \( \mathbb{H}^{-r} \), which come in pairs, as all completed positive semifields do.

5) For a given probability function or measure \( P_X \) the evolution of entropy with \( r \in [-\infty, \infty] \) resembles an information spectrum \( \tilde{H}_r(P_X) \). In a procedure reminiscent of defining an inverse transform, we may consider an equivalent probability \( \tilde{P}_r(P_X) = b^{-\tilde{H}_r(P_X)} \).

6) Many of the \((\mathbb{R}_{\geq 0})_r\) and \( \mathbb{H}^r \) semifields appear domains that model intelligent behaviour:

- In ML \( \mathbb{R}_{\geq 0} \) itself is used to model uncertainty as probabilities and \( \mathbb{H} \) to model log-probabilities. A new branch of ML is solely based upon the (standar) Rényi entropy with \( \alpha = 2 H_2(P_X) = \tilde{H}_1(P_X) \) and we know that the arithmetic mean has some special representational capabilities.

- In AI, maximizing utilities and minimizing costs is used by many applications and algorithms, e.g. heuristic search, to mimic “informed” behaviour. This points to using either \((\mathbb{R}_{\geq 0})_r\) when \( r \to \pm \infty \) for multiplicatively-aggregated costs and utilities, or using \( \mathbb{H}^r \) when \( r \to \pm \infty \) for additively-accumulated costs and utilities, Both a semifield and its order-dual are needed to express mixed utility-cost expressions, as in Electrical Network Analysis.

- In CI—apart from the Boolean semifield, which is a sub-semifield of every complete semifield by restricting the carrier set to \( \{\bot, \top\} \)—the sub-semifield obtained by the restriction of the operations to \( \{\bot, e, \top\} \) appears as a ternary logic. Spohn’s logical Rank theory \[23\] essentially spells out the isomorphisms of semifields between \( \mathbb{R}_{\geq 0} \) and the \( \mathbb{R}_{\min,+} \equiv \mathbb{H}_{-\infty} \) semifield in logical applications. Mathematical morphology and morphological processing need to operate in the dual pair \( (\mathbb{R}_{\max,+}, \mathbb{R}_{\min,+}) \equiv (\mathbb{H}_{\infty}, \mathbb{H}_{-\infty}) \) for image processing applications \[24\].

These hints lead us to our main conjecture, namely that applications in machine learning and machine intelligence operate with information, equivalent probability or a proxy thereof. And that those calculations are, therefore, better conceptualized and carried out in the framework of the adequate pairs of positive semifields.
F. Discussion

A number of decisions taken in the paper might seem arbitrary. In the following, we try to discuss these issues as well as alternatives not taken.

1) Rényi entropies on non-probabilistic measures: Although so far we conceived the origin of information to be a probability function, nothing precludes applying the same procedure to non-negative, non-normalized quantities with \( \sum_{x \in (X)} I_X(x) = K \neq 1 \), e.g. masses, sums, amounts of energy, etc. It is well-understood that in this situation Rényi’s entropy has to be slightly modified to accept this procedure. The reason for this is the one of the axioms of the means: generalized means are \( 1 \)-homogeneous in the numbers being averaged, but \( 0 \)-homogeneous in the weights. This entails that when using a mass-distributed variable \( X \sim M_X \) with \( M_X(x_i) = m_i \) such that \( \sum_i m_i = M \neq 1 \), then the normalized probability distribution \( q_1(M_X) = \{ m_i/M \}_{i=1}^{s^n} \) provides a Rényi spectrum that is displaced relative to that of the mass function:

\[
\hat{H}_r(q_1(M_X)) = -\log M r(q_1(M_X), q_1(M_X)) = -\frac{1}{r} \log \sum_i \frac{m_i}{M} \left( \frac{m_i}{M} \right)^r = \log M - \frac{1}{r} \log \sum_i \frac{m_i}{M} \left( \frac{m_i}{M} \right)^r = \log M + \hat{H}_r(q_1(M_X))
\]

so the entropy spectrum of the mass function is displaced by an amount \( -\log M \) with respect to that of its probability distribution:

\[
\hat{H}_r(M_X) = \hat{H}_r(q_1(M_X)) - \log M \quad (51)
\]

Notice that when \( M \geq 1, -\log M \leq 0 \) with equality for \( M = 1 \) and that if \( M < 1 \) then \( -\log M > 0 \).

2) Pervasiveness of Rényi entropies: Rényi’s entropy is a measure of variety in several disciplines (Economics, Ecology, etc.) it is not unconceivable that its applicability comes from the same properties that we have expounded in this paper as applied to positive distributions of wealth in a population or energy in a community.

3) The case for shifting the Rényi entropy: The shifting of the Rényi entropies is motivated by a number of results:

- The shifting of the Rényi entropy and divergence aligns them with the power means.
- It explains entropies as an application in semifield algebra, perhaps opening new insights and avenues of research.
- The shifting of the Rényi entropy aligns it with the moments of the distribution.
- It highlights the “information spectrum” quality of the measure for fixed \( P_X \).

This might or might not be justified by applications. We believe that for our purposes in explaining its relation to computations in ML, CI and AI, it is the best.

Of course, the insights obtained by the shifting might become more or less natural to the original proposal so that it may be skipped over. If manipulation of the formulae are needed to highlight the relation to the means, we recommend the shifted formulation.

4) Yet an alternate way of defining a generalization of Rényi’s entropy: Not only the parameter, but also de sign of the parameter is somewhat arbitray in the form of \( (24) \). If we choose \( r' = 1 - \alpha \) another generalization evolves
that is, in a sense, symmetrical to the shifted Rényi entropy we have presented above, since \( r' = -r \). This may be better or worse for the general formulae describing entropy, etc., but presents the problem that it no longer aligns with Shannon’s original choice of sign. The \( r = 0 \) order Rényi entropy would actually be Boltzmann’s, and perhaps more suitable for applications in Thermodynamics \([13]\).

Yet another formulations suggest the use of \( \alpha = 1/2 \), equivalently \( r = -1/2 \) as the origin of the parameter \([25]\). In our perspective, this only suggests that the origin of the Rényi entropy can be chosen adequately for applications.

IV. CONCLUSION

In the context of information measures, we have reviewed the notion of positive semifield—a semiring with a multiplicative group structure—distinct from that of the more usual fields with an additive group structure: in positive semirings there are no additive inverses, but there is a “natural order” compatible with addition and multiplication.

Through Pap’s \( g \)-calculus and Mesiar and Pap’s semifield Construction, we have related the Hölder means to the shifted Rényi measures of information, which appear as just the Kolmogorov-Nagumo means in different semifields obtained by ranging the \( r \) parameter in \([-\infty, \infty]\).

Our avowed intention with this exploration was to provide a conjecture about the abundance of semifields in a variety of machine learning and computational intelligence tasks from an information theoretic point of view. Namely, that such semifields are being used either as Rényi information measures or as proxies of such.

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