Let $U'_{L}(n, d)$ be the moduli space of stable vector bundles of rank $n$ with determinant $L$ where $L$ is a fixed line bundle of degree $d$ over a nodal curve $Y$. We prove that the projective Poincaré bundle on $Y \times U'_{L}(n, d)$ and the projective Picard bundle on $U'_{L}(n, d)$ are stable for suitable polarisation. For a nonsingular point $x \in Y$, we show that the restriction of the projective Poincaré bundle to $x \times U'_{L}(n, d)$ is stable for any polarisation. We prove that for the arithmetic genus $g \geq 3$ and for $g = n = 2, d$ odd, the Picard group of the moduli space $U_{L}(n, d)$ of semistable vector bundles of rank $n$ with determinant $L$ of degree $d$ is isomorphic to $\mathbb{Z}$.

1. Introduction

Let $X$ be an irreducible smooth projective curve of genus $g \geq 3$. Let $M_{X}(n, d)$ denote the moduli space of stable vector bundles of rank $n$ and degree $d$ on $X$. It is well-known that there is a Poincaré bundle (universal vector bundle) over $X \times M_{X}(n, d)$ if and only if $d$ is coprime to $n$ (see [19] for existence in the coprime case, [23] for non-existence in the non-coprime case and [20] for a topological version of non-existence in the case $d = 0$). If $n$ and $d$ have a common divisor, then there exists a projective Poincaré bundle over $X \times M_{X}(n, d)$. For a fixed line bundle $\xi$ on $X$, let $M_{\xi} \subset M_{X}(n, d)$ be the subvariety corresponding to vector bundles with determinant $\xi$. The existence and stability of a projective Poincaré bundle on $X \times M_{\xi}$ was proved in [11]. Our aim in this paper is to prove the existence and stability of a projective Poincaré bundle when $X$ is a nodal curve (for $n$ and $d$ not necessarily coprime).

Henceforth, we assume that $Y$ is an integral projective nodal curve of arithmetic genus $g \geq 2$ defined over an algebraically closed field. Let $p : X \to Y$ be a normalisation map. Let $U_{Y}(n, d)$ be the moduli space of semistable torsionfree sheaves of rank $n$ and degree $d$ on the nodal curve $Y$ and $U'_{Y}(n, d)$ its open dense subvariety corresponding to vector bundles. Denote by $U'_{L}(n, d)$ the subvariety of $U'_{Y}(n, d)$ corresponding to vector bundles with determinant $L$ where $L$ is a fixed line bundle. Let $U_{L}(n, d)$ be its closure in $U_{Y}(n, d)$. The superfix $'$ (respectively $''$) will denote the subset corresponding to stable bundles.
If \((n, d) = 1\), then there is a Poincaré sheaf on \(Y \times U_Y(n, d)\) [18, Theorem 5.12]. We show that if \(Y\) has geometric genus \(g(X) \geq 2\), then for \(n\) and \(d\) not coprime, there does not exist a Poincaré sheaf on \(Y \times V\) for any Zariski open subset of \(V \subset U_L(n, d)\) (see Theorem 7.14, see also Corollary 2.11). However, there exists a projective Poincaré bundle \(PU\) over \(Y \times U_L'(n, d)\) whose restriction to \(Y \times \{E\}\) is isomorphic to \(P(E)\) for all \(E \in U_L'(n, d)\). Let \(x \in Y\) be a nonsingular point. Define \(PU_x = PU|_{x \times U_L'(n, d)}\). Our first main results are summed up in the following theorem.

**Theorem 1.1. (Theorem 4.3)**

Let \(Y\) be an integral projective nodal curve of arithmetic genus \(g \geq 3\) defined over an algebraically closed field. Let \(n \geq 2\) be an integer and let \(L\) be a line bundle of degree \(d\) on \(Y\). Let \(U_L'(n, d)\) denote the moduli space of stable vector bundles on \(Y\) of rank \(n\) and determinant \(L\). Denote by \(PU\) the projective Poincaré bundle on \(Y \times U_L'(n, d)\) and define \(PU_x = PU|_{x \times U_L'(n, d)}\), where \(x \in Y\) is a nonsingular point.

1. The projective Poincaré bundle \(PU_x\) on \(U_L'(n, d)\) is stable for any polarisation.
2. Let \(\eta\) and \(\theta_L\) be divisors defining the polarisation on \(Y\) and \(U_L'(n, d)\) respectively. Then the projective Poincaré bundle \(PU\) is stable with respect to \(\eta + b\theta_L\), \(a, b > 0\).

If \((n, d) = 1\), then for \(d > n(2g - 1)\), the direct image of a Poincaré bundle on \(U_Y'(n, d)\) is a vector bundle, called a Picard bundle. For \(n\) and \(d\) non-coprime, there is no Poincaré bundle and hence no Picard bundle. However, one can construct a projective Picard bundle on \(U_L'(n, d)\) (following [11]).

**Theorem 1.2. (Theorem 5.2)**

Let the assumptions be as in Theorem 4.3. Let \(d > n(2g - 1)\). Then the projective Picard bundle \(PW\) on \(U_L'(n, d)\) is stable.

For our definitions and theorems to be valid for \(g \geq 3\), we needed to prove a few results on codimensions of complements of some open subsets of \(U_Y(n, d)\) and \(U_L(n, d)\). These results are of independent interest. They have many applications, we give here a few of them. Some of these results were proved by one of the authors under the assumption that the geometric genus \(g(X)\) of \(Y\) is at least two, this restriction is removed now.

**Theorem 1.3. (Theorem 6.1)**

Let \(Y\) be an integral nodal curve of arithmetic genus \(g\). Assume that \(n\) and \(d\) are not coprime.

1. For \(g \geq 2\) and \(n \geq 3\) (resp. \(n = 2\)),
\[
\text{codim}_{U_Y'(n, d)}(U_Y'(n, d) - U_Y'(n, d)) \geq 2(g - 1)(\text{ resp. } \geq g - 1) .
\]
2. For \(g \geq 2\) and \(n \geq 3\) (resp. \(n = 2\)),
\[
\text{codim}_{U_L'(n, d)}(U_L'(n, d) - U_L'(n, d)) \geq 2(g - 1)(\text{ resp. } \geq g - 1) .
\]

As an application, we have the following corollary.

**Corollary 1.4. (Corollary 6.2)** Let \(Y\) be an integral nodal curve of arithmetic genus \(g \geq 2\). Assume that if \(n = 2\) and \(g = 2\) then \(d\) is odd. Then
(1) \( \text{Pic } U_L'(n, d) \cong \mathbb{Z} \).
(2) \( \text{Pic } U_L'(n, d) \cong \mathbb{Z} \).
(3) The class group \( Cl(U_L(n, d)) \cong \mathbb{Z} \). The class group \( Cl(U_L'(n, d)) \cong \mathbb{Z} \).

Let \( U_Y^{ss}(n, d) \) (respectively \( U_Y^s(n, d) \)) be the subset of \( U_Y(n, d) \) corresponding to vector bundles \( F \) such that \( p^* F \) is semistable (respectively stable). Similarly, \( U_L^{ss}(n, d) \) (respectively \( U_L^s(n, d) \)) denotes the subset of \( U_L(n, d) \) corresponding to vector bundles \( F \) such that \( p^* F \) is semistable (respectively stable).

**Theorem 1.5.** (Theorem 7.6) For \( n \geq 3 \) (resp. \( n = 2 \)) and \( U_Y^{ss}(n, d) \neq U_Y^s(n, d) \) one has:

1. \( \text{codim}_{U_Y(n, d)}(U_Y(n, d) - U_Y^{ss}(n, d)) \geq 2g(X) - 1 \) (resp. \( g(X) \)).
2. \( \text{codim}_{U_L(n, d)}(U_L(n, d) - U_L^{ss}(n, d)) \geq 2g(X) - 1 \) (resp. \( g(X) \)).
3. \( \text{codim}_{U_Y(n, d)}(U_Y(n, d) - U_Y^{ss}(n, d)) \geq 2g(X) - 2 \) (resp. \( g(X) - 1 \)).
4. \( \text{codim}_{U_L(n, d)}(U_L(n, d) - U_L^{ss}(n, d)) \geq 2g(X) - 2 \) (resp. \( g(X) - 1 \)).
5. \( \text{codim}_{U_Y(n, d)}(U_Y(n, d) - U_Y^{ss}(n, d)) \geq 2g(X) - 2 \) (resp. \( g(X) - 1 \)).
6. \( \text{codim}_{U_L(n, d)}(U_L(n, d) - U_L^{ss}(n, d)) \geq 2g(X) - 2 \) (resp. \( g(X) - 1 \)).

From this, we deduce the following result.

**Corollary 1.6.** Let \( Y \) be a complex nodal curve (with at least one node). For \( g(X) \geq 2 \), except possibly when \( g(X) = n = 2 \) and \( d \) even, the moduli space \( U_Y'(n, 0) \) (respectively \( U_L'(n, 0) \)) has a big open dense subset (i.e. this open subset has complement of codimension at least 2) whose elements correspond to vector bundles which come from representations of the fundamental group of \( Y \).

For similar results for \( U_Y'(n, d) \) and \( U_L'(n, d) \), for any \( d \), see Corollary 7.7.

The codimension computations are done in Sections 6 and 7. These sections are independent and can be read directly without going through the previous sections. The results on projective Poincaré bundles are proved in Section 4, those on projective Picard bundles are proved in Section 5. Constructions and results needed for the proofs of the results in Sections 4 and 5 are contained in the rest of the sections.

## 2. Preliminaries

We start with some definitions and general results needed in the paper.

### 2.1. Degree of a sheaf on a big open set.

Let \( W \) be a variety of dimension \( m \) with an ample line bundle \( H \). Let \( \mathcal{E} \) denote a torsion free sheaf on the projective variety \( W \). Let \( C \) denote a general complete intersection curve rationally equivalent to \( H^{m-1} \). If the singular set of \( W \) has codimension at least 2, then the general complete intersection curve can be chosen to lie on the set of nonsingular points of \( W \).

We define the degree of \( \mathcal{E} \) with respect to the polarisation \( H \) as,

\[
\text{deg } \mathcal{E} = \text{deg } \mathcal{E}|_C.
\]
Remark 2.1. Let $U$ be an open subset of $W$ such that codimension of $S = W - U$ in $W$ is at least 2. Since $\dim S \leq \dim W - 2$, for any torsion-free coherent sheaf $\mathcal{F}$ on the open subset $U$, the direct image $i_* (\mathcal{F})$ is a torsion-free coherent sheaf on $W$.

Proof: Note that the restriction of any nonzero section of $\mathcal{F}$ over $V \subset W$ on $U \cap V$ is non zero and $\mathcal{F}$ is torsion free on $U$.

Definition 2.2. Let $U$ be an open subset of $W$ such that codimension of $W - U$ in $W$ is at least 2. Let $\mathcal{F}$ be a torsion free sheaf on $U$. We define $\deg(\mathcal{F}) = \deg(i_*(\mathcal{F}))$, where $i: U \to W$ is the inclusion map and $i_*(\mathcal{F})$ is the direct image sheaf of $\mathcal{F}$ on $W$.

2.2. Stability of Projective Bundles.

Let $U$ be an open subset of a projective variety $W$ such that the codimension of the complement of $U$ is at least 2. Let $P$ be a projective bundle on $U$ and let $P'$ be a projective subbundle of the restriction of $P$ to a Zariski open subset $Z$ of $U$ whose complement has codimension at least 2.

Let $p$ and $p'$ denote the projections of $P$ and $P'$ to $U$ and $Z$ respectively. As $P'$ and $P'|_Z$ are smooth over $Z$ and $i: P' \to P|_Z$ is a closed embedding with ideal sheaf, say $\mathcal{I}$, by [24] Theorem C.15, D.2.7 we get an exact sequence of vector bundles on $P'$

$$0 \to \mathcal{I}/\mathcal{I}^2 \to \Omega_{P'/Z} \otimes \mathcal{O}_{P'} \to \Omega_{P'/Z} \to 0.$$ 

Dualising it we get,

$$0 \to \mathcal{T}_{P'/Z} \to \mathcal{T}_{P|Z} \to N_{P'/P} \to 0$$

where $\mathcal{T}_{P'/Z}, \mathcal{T}_{P|Z}$ are the relative tangent bundles (with respect to the maps $p': P' \to Z$ and $p: P|_Z \to Z$ respectively) and $N_{P'/P}$ is the normal bundle of $P'$ in $P|_Z$.

Let $N_1 = N_{P'/P}$ and $N = p'_*(N_1)$. Using a theorem of Grauert ([23, Corollary 12.9]) and the fact that $H^i(\mathbb{P}^n, T\mathbb{P}^n) = 0$ for all $i \geq 1$, we get $N$ is a vector bundle on $Z$.

Definition 2.3. The projective bundle $P$ is stable (semistable) if for every subbundle $P'$,

$$\deg N > 0 \quad (\deg N \geq 0)$$

Remark 2.4. (1) Definition [23] is equivalent to the standard definition of stability for a principal $PGL(n)$-bundle (Remark 2.2, [11]). Also note that if $P = P(E)$ then $P' = P(F)$, where $F$ is a subbundle of $E|_Z$ such that $E|_Z/F$ is locally free on $Z$. In this case it is easy to verify that the stability of $E$ is same as the stability of $P(E)$. Note that the above definition is for a variety of dimension at least 2.

(2) There is also a notion of stability for a projective bundle on a curve which is similar to the above definition. For any integral curve $Y$, all projective bundles are of the form $P(E)$ for some vector bundle $E$ (since the Brauer Group $Br(X) = 0$). We remark that the stability of a projective bundle $P(E)$ and the Mumford stability of $E$ are same on a smooth curve but it is not the case for a singular curve.

2.3. Torsionfree sheaves on nodal curves.

Let $Y$ be an integral projective curve with only $m$ nodes (ordinary double points) as singularities defined over an algebraically closed field $k$. Let $p: X \to Y$ be the normalisation map. Let $g = h^1(Y, \mathcal{O}_Y)$ be the arithmetic genus of $Y$, we assume that $g \geq 2$. Let $g(X) = h^1(X, \mathcal{O}_X)$ be the geometric genus of $Y$. 


For a torsionfree sheaf $E$, let $r(E)$ denote the generic rank of $E$ and $d(E) = \chi(E) - r(E)\chi(O_Y)$ denote the degree of $E$. Let $\mu(E) = d(E)/r(E)$ denote the slope of $E$.

We say that $E$ is stable (respectively semistable) if for every coherent subsheaf $F$ of $E$ on $Y$,

$$\mu(F) < (\leq) \mu(E).$$

This is equivalent to the following: for every torsion free quotient $G$ of $E$ on $Y$,

$$\mu(E) < (\leq) \mu(G).$$

The stalk $E_{y_j}$ of a torsionfree sheaf $E$ at a node $y_j$ is isomorphic to $O_{y_j}^{a_j} \oplus m_{y_j}^{b_j}$, where $O_{y_j}$ is the local ring at $y_j$ and $m_{y_j}$ is its maximum ideal, $a_j, b_j$ are integers with $a_j + b_j = r(E)$. Then $b_j$ is called the local type of $E$ at $y_j$.

**Remark 2.5.** Let $U_Y(n, d)$ denote the moduli space of $S$-equivalence classes of semistable torsionfree sheaves of rank $n$ and degree $d$ on $Y$ and $U_L'(n, d)$ its open dense subvariety corresponding to locally free sheaves (vector bundles). For a line bundle $L$ over $Y$, denote by $U_L'(n, d)$ the closed subset of $U_L'(n, d)$ corresponding to vector bundles with fixed determinant $L$, and $U_L(n, d)$ the closure of $U_L'(n, d)$ inside $U_Y(n, d)$ with a reduced structure.

The moduli space $U_Y(n, d)$ is an irreducible seminormal projective variety ([26, 25 Theorem 4.2]). $U_L'(n, d)$ and $U_L'(n, d)$ are normal quasi-projective varieties ([18]). They are known to be locally factorial for $g(X) \geq 2$ ([3, Theorem 3A]. We shall soon see that $U_L'(n, d)$ is locally factorial for $g \geq 2$ (except possibly for $g = 2 = n, d$ even). The open subset $U_L'(n, d) \subset U_L'(n, d)$ corresponding to stable vector bundles is nonsingular.

There is a canonically defined ample line bundle $\theta_U$ on $U_Y(n, d)$, the determinant of cohomology line bundle. Let $\theta_L \rightarrow U_L(n, d)$ be its restriction. Assume that $g \geq 2$ and if $g = 2 = n$ then $d$ is odd. Then we shall prove that Pic $U_L'(n, d) \cong \mathbb{Z}$ (Corollary 6.2 [2 Theorem 1], for $g(X) \geq 2$). The restriction of $\theta_L$ to $U_L'(n, d)$ is the generator of Pic $U_L'(n, d)$.

We start with some results on codimensions of subsets of moduli spaces needed in the nodal case.

### 2.4. Codimensions of subsets of moduli spaces.

**Theorem 2.6.** ([27, Theorem 2.5])

Let $Y$ be an integral nodal curve of arithmetic genus $g \geq 2$ with $m$ nodes, $m \geq 1$. For $n \geq 2$, the codimension of $U_L(n, d) - U_L'(n, d)$ in $U_L(n, d)$ is more than 2.

**Theorem 2.7.** (Theorem 6.1) Let $Y$ be an integral nodal curve of arithmetic genus $g$.

1. For $g \geq 2$ and $n \geq 3$ (resp. $n = 2$),

$$\text{codim}_{U_L'(n, d)}(U_L'(n, d) - U_L'(n, d)) \geq 2(g - 1)(\text{resp. } \geq g - 1).$$

2. For $g \geq 2$ and $n \geq 3$ (resp. $n = 2$),

$$\text{codim}_{U_L(n, d)}(U_L'(n, d) - U_L'(n, d)) \geq 2(g - 1)(\text{resp. } \geq g - 1).$$

This theorem and its corollary will be proved in Section 6. The proofs in Section 6 are independent of the results in the sections before it.
Corollary 2.8. Let $Y$ be an integral nodal curve of arithmetic genus $g \geq 2$ with $m$ nodes. Assume that $g \geq 2$ and if $n = 2$ and $g = 2$ then $d$ is odd. Then

1. Pic $U'_L(n, d) \cong \mathbb{Z}$.
2. Pic $\tilde{U}'_L(n, d) \cong \mathbb{Z}$.
3. The class group $\text{Cl}(U_L(n, d)) \cong \mathbb{Z}$. The class group $\text{Cl}(U'_{L}(n, d)) \cong \mathbb{Z}$.

Remark 2.9. (1) Let the assumptions be as in Corollary 2.8. Then as in [8, Lemma 2.3], we can show that $U'_L(n, d)$ and $\tilde{U}_L(n, d)$ are locally factorial and

$$\text{Pic } \tilde{U}_L(n, d) \cong \mathbb{Z}.$$ (2) For $g = 2, n = 2$ and $d$ even, codim$U'_L(2, d)(U'_L(2, d) - U'_L(2, d)) = 1$. The variety $U_L(2, d) = U'_L(2, d) \cong \mathbb{P}^3$ has Picard group isomorphic to $\mathbb{Z}$. One has Pic $U'_L(2, d) \cong \mathbb{Z}/4\mathbb{Z}$ [5, Proposition 4.3].

2.5. Non-existence of a universal vector bundle on $Y \times U'_L(n, d)$.

From the exact sequence of group schemes

$$1 \to G_m \to GL(n) \to PGL(n) \to 1,$$

we have the following exact sequence of cohomology (non-abelian) groups

$$H^1(X_{\text{et}}, \mathbb{G}(L(n))) \xrightarrow{\mathcal{P}} H^1(X_{\text{et}}, \mathbb{P}GL(n)) \xrightarrow{\delta} H^2(X_{\text{et}}, \mathbb{G}_m).$$

The natural map $H^1(X_{\text{et}}, \mathbb{G}(L(n))) \xrightarrow{\mathcal{P}} H^1(X_{\text{et}}, \mathbb{P}GL(n))$ corresponds to projectivisation. The Brauer group captures the projective bundles that do not arise from vector bundles. For (quasi) projective varieties, the Brauer group is the same as the cohomological Brauer group $H^2(X_{\text{et}}, \mathbb{G}_m)$ (\cite{15}).

One way to see that there is no Poincaré bundle on $U'_L(n, d)$ is to exhibit a Brauer obstruction to the existence of such a bundle. Let $Br(U'_L(n, d))$ denote the cohomological Brauer group $H^2_\text{et}(U'_L(n, d), \mathbb{G}_m)$. We have the following result from [9] where Brauer group and other properties of this moduli space has been studied.

Theorem 2.10. Assume that $g(X) \geq 2$; also, if $g(X) = 2 = n$, assume that $d$ is odd. Then the Brauer group $Br(U'_L(n, d)) \cong \mathbb{Z}/h\mathbb{Z}$ where $h = \gcd(n, d)$ and the Brauer group is generated by the Brauer class of $\mathcal{P}U_x$, the restriction of the projective Poincaré bundle to $U'_L(n, d) \times x$, where $x \in Y$ is a nonsingular point.

Corollary 2.11. Assume that $g(X) \geq 2$; also, if $g(X) = 2 = n$, assume that $d$ is odd. Then for $n$ and $d$ non-coprime, there is no Poincaré bundle on $Y \times U'_L(n, d)$.

Proof. If there exists a Poincaré bundle $E$ on $U'_L(n, d) \times Y$, then by uniqueness of projective Poincaré bundles, we have $\mathcal{P}U \cong P(E)$ and $\mathcal{P}U_x \cong P(E_x)$ for a nonsingular point $x \in Y$. This will imply that the Brauer class of $\mathcal{P}U_x$ is trivial as it is the projectivisation of a vector bundle. As a consequence we get $Br(U'_L(n, d)) = \{0\}$ contradicting Theorem 2.10. \qed

In Section 7 we shall prove the more general result that for $n$ and $d$ noncoprime, there is no Poincaré bundle on $Y \times V$ where $V \subset U'_L(n, d)$ is any Zariski open subset.
3. The Morphism $\psi_{F,x}$.

3.1. Elementary Transformations. For a vector bundle $F$ of rank $n$ on a projective algebraic curve $X$, the elementary operation of degree $k$ is defined as follows: Let $x \in X$ be a regular point of $X$. Consider a $k$-dimensional subspace $w \subset F_x$ in the fibre of $F$ over the point $x$. This uniquely determines a surjective homomorphism

$$F \xrightarrow{\alpha(w)} k_x^{n-k} \longrightarrow 0,$$

where $k_x^{n-k}$ is the skyscraper sheaf of dimension $n-k$ supported at $x \in X$ by requiring the condition that $\ker \alpha(w)_x = w$. Here $\alpha(w)_x$ denotes the restriction of $\alpha(w)$ to the fibre over $x$. Then the kernel of $\alpha(w)$ is a locally free sheaf and is usually denoted by $\text{elm}_x^k(w)(F)$. Thus we obtain the following exact sequence:

$$0 \longrightarrow \text{elm}_x^k(w)(F) \longrightarrow F \xrightarrow{\alpha(w)} k_x^{n-k} \longrightarrow 0,$$

and the operation $F \mapsto \text{elm}_x^k(w)(F)$ is called an elementary operation of degree $k$ (17).

Let $x \in Y$ be a nonsingular point of $Y$, and $E$ a vector bundle over $Y$ of rank $n$ and degree $d$ with $\det E = L$. Let $\ell \subset E_x$ be a line in the fiber of $E$ at $x$. We obtain a vector bundle $F$ defined by the short exact sequence

$$0 \rightarrow F(-x) \rightarrow E \rightarrow E_x/\ell \rightarrow 0. \quad (3.1)$$

Now let $F$ be a vector bundle of rank $n$ on the nodal curve $Y$ such that $\det F \cong L(x)$, $L$ being a line bundle of degree $d$. Let $P := P(F^*_x)$ be the projective space parametrising the hyperplanes in the fibre $F_x$. Denote by $p : Y \times P(F^*_x) \rightarrow Y$ the projection onto $Y$ and by $i : P(F^*_x) \rightarrow Y \times P(F^*_x)$ the inclusion map defined by $\phi \mapsto (x, \phi)$.

Consider a nontrivial $\phi \in P(F^*_x)$, i.e., a nontrivial homomorphism $\phi : F \rightarrow k_x$. We have an exact sequence

$$0 \longrightarrow E \longrightarrow F \xrightarrow{\phi} k_x \longrightarrow 0, \quad (3.2)$$

where $E = E_{\phi} := \ker \phi$. Then $E$ is a vector bundle since $x$ is a nonsingular point.

We can reconstruct $E$ and $F$ from each other from the following commutative diagram.

$$\begin{array}{c}
0 & \longrightarrow & F(-x) \quad \Downarrow \\
\downarrow & & \downarrow \\
E_x/\ell & \longrightarrow & F_x \quad \Downarrow \\
\downarrow & & \downarrow \\
0 & \longrightarrow & k_x \quad \longrightarrow & 0
\end{array} \quad (3.3)$$
For every
\[ \phi \in \text{Hom}(F, k_x) = \text{Hom}(F_x, k_x) = F_x^* \]
we get a vector bundle \( E_\phi \) and as \( \phi \) varies over \( P(F_x^*) \) we obtain a family of vector bundles parametrised by \( P(F_x^*) \), say \( \mathcal{E} \) and we have the following exact sequence on \( Y \times P \):
\[ 0 \to \mathcal{E} \to p^*F \to i_*\mathcal{O}_{\{x\} \times P}(1) \to 0. \]
where \( \mathcal{E} \) is a vector bundle.

### 3.2. \((l,m)\)-stability of a coherent sheaf on \( Y \) and the morphism \( \psi_{F,x} \).

**Definition 3.1.** Let \( l \) and \( m \) be integers. A torsionfree sheaf \( \mathcal{E} \) on \( Y \) is said to be \((l,m)\)-stable if for every proper subsheaf \( \mathcal{G} \) of \( \mathcal{E} \) with a torsionfree quotient
\[ \frac{d(\mathcal{G}) + l}{r(\mathcal{G})} < \frac{d(\mathcal{E}) + l - m}{r(\mathcal{E})}. \]

**Remark 3.2.**
1. A torsionfree sheaf \( \mathcal{E} \) is stable if and only if it is \((0,0)\)-stable.
2. A \((0,1)\)-stable torsionfree sheaf is stable.

**Lemma 3.3.** If the bundle \( F \) in the sequence (3.2) is \((0,1)\)-stable, then \( E_\phi \) is stable.

**Proof.** The proof is similar to the case when \( Y \) is smooth [21, Lemma 5.5]. If \( E_\phi \) is not stable, then there exists a proper subsheaf \( G \) of \( E_\phi \) with a torsionfree quotient and satisfying \( \mu(G) > \mu(E_\phi) \). If we denote by \( G' \) the maximal torsionfree subsheaf generated by the image of \( G \) in \( F \), then
\[ \frac{\deg G' + 0}{r(G')} \geq \mu(G) > \mu(E_\phi) = \frac{\deg F - 1}{r(F)}, \]
contradicting the \((0,1)\)-stability of \( F \).

By the above lemma, for a \((0,1)\)-stable vector bundle \( F \in U'_L(n, d + 1) \) we get the vector bundle \( \mathcal{E} \) in (3.4), i.e., a family of stable vector bundles of rank \( n \) and determinant \( L \) on \( Y \). And using the universal property of \( U'_L(n, d) \), we get a morphism,
\[ \psi_{F,x} : P(F_x^*) \to U'_L(n, d). \]

**Lemma 3.4.** \( \psi_{F,x} \) is an isomorphism onto its image.

**Proof.** This can be proved exactly as [21] Lemma 5.9 (see [21] lemma 5.6] and [12] Lemma 3] for injectivity). We note that \( \psi_{F,x} \) maps into \( U'_L(n, d) \).

The diagrams (3.3) obtained by varying \( \phi \in P(F_x^*) \) combine to form the following diagram:
Lemma 3.5 ([12], Lemma 3.1). Let $E_x := \mathcal{E}|_{x \times P(F^*)}$. There is an exact sequence of vector bundles

\[
\begin{align*}
0 & \rightarrow \mathcal{E} \rightarrow p_1^*F \rightarrow i_*\mathcal{O}_P(1) \rightarrow 0 \\
0 & \rightarrow i_*\Omega^1_P(1) \rightarrow F_x \otimes_C i_*\mathcal{O}_P \rightarrow i_*\mathcal{O}_P(1) \rightarrow 0
\end{align*}
\]

(3.6) on $P(F^*)$.

Lemma 3.6. Let $W \subset E_x$ be a non-zero coherent subsheaf of the vector bundle $E_x$ in (3.6) such that:

- the quotient $E_x/W$ is torsion-free, and
- $\deg W/\text{rk } W \geq \deg(E_x)/\text{rk } (E_x)$.

Then $W$ contains the line subbundle $\mathcal{O}_P(1)$ of $E_x$ defined in (3.6).

Proof. By considering Harder-Narasimhan filtration of $W$, we can choose a sub-sheaf $W_1$ of $W$ such that $W_1$ is stable, $W/W_1$ is torsion free, and

\[
\frac{\deg W_1}{\text{rk } W_1} \geq \frac{\deg W}{\text{rk } W}.
\]

It follows from the exact sequence (3.6) that $\deg E_x = 0$ and we have the following inequality:

\[
\frac{\deg W_1}{\text{rk } W_1} \geq \frac{\deg E_x}{\text{rk } E_x} = 0 > \frac{-1}{n - 1} = \frac{\deg \Omega^1_P(1)}{\text{rk } \Omega^1_P(1)}.
\]

(Note that $\dim \mathbf{P} = \dim P(F^*) = n - 1$.)

The vector bundle $\Omega^1_P(1)$ is stable (see [22], Chapter II, Theorem 1.3.2). Since $W_1$ is semistable and $\frac{\deg W_1}{\text{rk } W_1} > \frac{\deg \Omega^1_P(1)}{\text{rk } \Omega^1_P(1)}$, there is no nonzero homomorphism from $W_1 \rightarrow \Omega^1_P(1)$ (The image of such a morphism will contradict the stability of $\Omega^1_P(1)$). This implies that $W_1 \subset \mathcal{O}_P(1)$ so that $\text{rk}(W_1) = 1$ and $\mathcal{O}_P(1)/W_1$ is a torsion sheaf if non-zero. Also we have the exact sequence

\[
0 \rightarrow W/W_1 \rightarrow E_x/W_1 \rightarrow E_x/W \rightarrow 0.
\]

Since $W/W_1$ and $E_x/W$ are torsion free so is $E_x/W_1$. Then $\mathcal{O}_P(1)/W_1$ is a subsheaf of $E_x/W_1$ implies that $\mathcal{O}_P(1)/W_1$ is torsionfree. It follows that $W_1 = \mathcal{O}_P(1)$. \qed
Lemma 3.7. The (0,1)-stable vector bundles on $Y$, of rank $n$ and determinant $L'$ of degree $d'$ form a non-empty open subset of the moduli space of stable torsionfree sheaves denoted by $U_{L'}^s(n,d')$.

Proof. We briefly sketch the proof given in [7]. Let $C$ be the complement in $U_{L'}^s(n,d')$ of the set of (0,1)-stable vector bundles. A vector bundle $F \in C$ if and only if it has a torsionfree subsheaf $G$ of rank $r$ and degree $e$ satisfying $rd' > ne \geq r(d' - 1)$. This implies that the ranks and degrees of quotients of $F \in C$ are bounded. The closedness of $C$ follows from the properness of quot schemes.

Next we estimate the dimension of $C$ and show that it is strictly less than the dimension of $U_{L'}^s(n,d')$. One can see that to estimate the dimension of the space of extensions of $H$ by $G$, where $G$ is as above and $H$ is of rank $n - r$ and degree $d' - e$.

We may assume that $G$ and $H$ are stable, so that $h^0(\text{Hom}(H,G)) = 0$. Using the Riemann-Roch Theorem we have

$$\chi(F) = d(F) + (1 - g)r(F).$$

From [4, Lemma 2.5(B)], we have

$$\dim \text{Ext}^1(H,G) = \dim H^1(Y,\text{Hom}(H,G)) + 2 \sum_j b_j(G)b_j(H)$$

$$= (g - 1)r(\text{Hom}(H,G)) - d(\text{Hom}(H,G)) + 2 \sum_j b_j(G)b_j(H)$$

$$= rd' - ne + r(n - r)(g - 1) + \sum_j b_j(G)b_j(H).$$

Hence the dimension $\delta$ of the space of bundles of rank $n$ and degree $d'$ with a fixed determinant $L'$ obtained as an extension of $H$ by $G$ is

$$\delta \leq (r^2(g - 1) + 1 - \sum_j b_j(G)^2 + ((n - r)^2(g - 1) + 1 - \sum_j b_j(H)^2) - g + \dim \text{Ext}^1(H,G) - 1$$

$$= (g - 1)(r^2 + (n - r)^2 + r(n - r) - 1) + rd' - ne - \sum_j (b_j(G) - b_j(H))^2 - \sum_j b_j(G)b_j(H)$$

$$\leq (g - 1)(r^2 + n^2 - rn - 1) + rd' - ne.$$

For $C$ to be a proper closed subset, it suffices that the last expression is strictly less than $\dim U_{L'}^s(n,d') = (n^2 - 1)(g - 1)$ which is true when $rd' - ne < r(n - r)(g - 1)$ and that indeed is the case here since $g \geq 3$. \hfill \Box

We remark that the lemma holds for any $n$ and $d'$ and the condition in [7] that $(n, d' - 1) = 1$ is not required since we are considering curves with genus $g \geq 3$.

4. Projective Poincaré Bundles

If $(n, d) = 1$, there is a Poincaré bundle on $Y \times U_{L'}^s(n, d)$, unique up to tensoring by a line bundle on $U_{L'}^s(n, d)$. If $(n, d) \neq 1$, there is no Poincaré bundle on $Y \times U_{L'}^s(n, d)$ (Corollary
Theorem [7.14]. However we can define a projective Poincaré Bundle on $U_Y^s(n, d) \times Y$ for any $n$ and $d$ with $n \geq 2$. Our construction is very similar to the construction in [11].

Grothendieck has proved that, for a positive integer $p$, the torsion free quotients of $\mathcal{O}_Y$ which have a fixed Hilbert polynomial $P$ can be parametrised by the points of a projective algebraic scheme $Q$ (Quot Scheme). Moreover there exists a universal quotient coherent sheaf $U$ over $Y \times Q$. The group $GL(p)$ acts on $Q$ as the group of automorphisms of $\mathcal{O}_Y$ and also on the sheaf $U$. Moreover the action of $GL(p)$ on $Q$ goes down to an action of $PGL(p)$ on $Q$. However, this is not true for the action on $U$, the scalar matrix $\lambda(\text{Id})$ acts on $U$ by scalar multiplication by $\lambda$.

$U_Y(n, d)$ can be seen as the geometric invariant theoretic (GIT) quotient of a Zariski open set $R_Y$ of $Q$ by the action of $PGL(p)$. In general, $R_Y$ need not be irreducible or non-singular, but the subset $R'$ of $R_Y$ consisting of those $q$ for which $U_q$ is locally free is irreducible, non-singular, $PGL(p)$ invariant and open in $Q$ ([13, Chapter-5]). Thus $U_Y^s(n, d)$ can be realized as a quotient $\pi : R^s \to U_Y^s(n, d)$. The universal quotient $U$ restricts to a vector bundle $U_{R^s}$ on $Y \times R^s$, such that $U_{R^s} | _{Y \times \{r\}}$ is the stable bundle corresponding to $\pi(r)$ for all $r \in R^s$. The isotropy group of $GL(p)$ at $r \in R^s$ is $\text{Aut } U_r \cong \text{scalar matrices}$. It acts on $U_r$ with $\lambda \text{Id}$ acting by multiplication by $\lambda$. Hence $PGL(p)$ acts on the associated projective bundle $P(U_{R^s})$. The quotient $\mathcal{P}U := P(U_{R^s})/PGL(p)$ is then a projective bundle on $Y \times U^s_Y(n, d)$ whose restriction to $Y \times \{E\}$ is isomorphic to $P(E)$ for all $E \in U^s_Y(n, d)$.

**Uniqueness of Projective Poincare Bundle.**

**Lemma 4.1.** Let $E$ be a vector bundle on $Y \times Z$ such that the restriction of $E$ to $Y \times \{z\}$ is stable of rank $n$ and determinant $L$ for all $z \in Z$, and let $\psi_E : Z \to U_Y^s(n, d)$ be the corresponding morphism. Then the projective bundles $P(E)$ and $(\text{id}_Y \times \psi_E)^*(\mathcal{P}U)$ are isomorphic.

**Proof.** This can be proved exactly as [11] Proposition 2.3].

**Corollary 4.2.** Suppose that $\pi' : R^s \to U_Y^s(n, d)$ defines $U_Y^s(n, d)$ as a quotient of $R^s$ by a free action of $PGL(M')$ and that:

1. $E_{R^s}$ is a vector bundle on $Y \times R^s$ such that $E_{R^s} | _{X \times \{r\}}$ is the stable bundle $\pi'(r)$ for all $r' \in R^s$;
2. the action of $PGL(M')$ lifts to $P(E_{R^s})$.

Then $P(E_{R^s})/PGL(M') \cong \mathcal{P}U$.

**Proof.** Apply Proposition 4.1 to by taking $Z = R^s$ and $E = E_{R^s}$.

**Definition 4.3.** $\mathcal{P}U$ is a projective bundle whose restriction to $Y \times \{E\}$ is isomorphic to $P(E)$ for all $E \in U_Y^s(n, d)$ and we call it the projective Poincaré bundle.

We call the restriction of $\mathcal{P}U$ on $Y \times U^s_L(n, d)$ the projective Poincaré bundle over $Y \times U^s_L(n, d)$ and denote it again by $\mathcal{P}U$.

Note that by Theorem [2.6] and Theorem [6.1] the codimension of $U_L(n, d) - U^s_L(n, d)$ is at least 2. So we can define the notion of degree on $U^s_L(n, d)$ and we can talk about the stability of any projective bundle on $U^s_L(n, d)$. Similarly $Y \times U^s_L(n, d) \hookrightarrow Y \times U_L(n, d)$.
via $Id \times i$ (where $i : U_L'(n, d) \hookrightarrow U_L(n, d)$ is the inclusion) as an open set of projective variety $Y \times U_L(n, d)$ whose complement has codimension at least 2. So we can talk about the degree and stability of a projective bundle on $Y \times U_L'(n, d)$.

In the next section we will study the stability of $\mathcal{P}U$ and $\mathcal{P}U_x := \mathcal{P}U|_{x \times U_L'(n, d)}$ for $x \in Y_{\text{reg}}$.

4.1. Stability of the restriction of projective Poincaré bundle to regular points on $Y$.

Each point of $\mathcal{P}U_x$ corresponds to a pair $(E, \ell)$ where $E \in U_L'(n, d)$ and $\ell \in P(E_x)$. Define,

$$H_x = \{(E, \ell) \in \mathcal{P}U_x : F \text{ defined in (3.1) is (0,1)-stable}\}.$$

Let $p : H_x \to U_L'(n, d)$ and $q : H_x \to U_{L(x)}'(n, d + 1)$ be defined by $p(E, \ell) = E$ and $q(E, \ell) = F$ (defined in (3.1)) respectively.

**Lemma 4.4.**

(1) The set $H_x$ is nonempty and Zariski open in $\mathcal{P}U_x$ and $q$ is a morphism.

(2) $p(H_x)$ is non-empty and Zariski open in $U_L'(n, d)$.

**Proof.** The proof is essentially the same as in [11] Lemma 3.5, we give an outline of it. We can construct $U_L'(n, d)$ as a quotient $\pi : R^s \to U_L'(n, d)$ and for a fixed regular point $x \in Y$ we have the bundle $P(U_{R^s})_x$ on $R^s$. A point in $P(U_{R^s})_x$ corresponds to a pair $(E, \ell)$, where $E \in R^s$ and $\ell \in P(E_x)$. Now corresponding to each such pair with $E$ in a Zariski open subset of $R^s$, we obtain an element $F \in U_{L(x)}'(n, d + 1)$ from the exact sequence (3.1). Thus a $\text{PGL}(p)$-invariant open subset $V^s$ of $P(U_{R^s})_x$ parametrises a family of elements of $U_{L(x)}'(n, d + 1)$ and by the universal property of moduli space we get a morphism $q' : V^s \to U_{L(x)}'(n, d + 1)$ which goes down to a morphism $q : V \to U_{L(x)}'(n, d + 1)$. Let $H_x$ be the subset of $\mathcal{P}U_x$ for which $F$ corresponding to $(E, \ell)$ is (0,1)-stable. Then by Lemma 3.7 $H_x$ is nonempty and Zariski open in $\mathcal{P}U_x$. □

**Theorem 4.5.** Let $Y$ be an integral projective nodal curve of arithmetic genus $g \geq 3$ defined over an algebraically closed field. Let $n \geq 2$ be an integer and let $L$ be a line bundle of degree $d$ on $Y$. Let $U_L'(n, d)$ denote the moduli space of stable vector bundles on $Y$ of rank $n$ and determinant $L$ and let $\mathcal{P}U$ be the projective Poincaré bundle on $Y \times U_L'(n, d)$. Then $\mathcal{P}U_x$ is stable for all $x \in Y_{\text{reg}}$ where $Y_{\text{reg}}$ is the set of all nonsingular points of $Y$.

**Proof.** Let $P'$ be a projective subbundle of the restriction of $\mathcal{P}U_x$ to a Zariski open subset $Z$ of $U_L'(n, d)$ with complement of codimension at least 2. Then as in the definition of stability for projective bundles, we have the following exact sequence

$$(4.1) \quad 0 \to T_{P'/Z} \to T_{\mathcal{P}U|_{x \times U_L'(n, d)}} \to N_{P'/\mathcal{P}U|_{x \times U_L'(n, d)}} \to 0$$

and we take $N = q_s(N_{P'/\mathcal{P}U|_{x \times U_L'(n, d)}})$. Our aim is to conclude $\deg N > 0$.

Claim 1: $p^{-1}(Z)$ is a Zariski open subset of $H_x$ and its complement $S$ has codimension at least 2.

Proof of Claim 1: From Lemma 4.4 we have $p(H_x)$ is non-empty and open in $U_L'(n, d)$ which implies $\dim p(H_x) = \dim U_L'(n, d)$. Since $Z$ is an open subset of $U_L'(n, d)$ with
its complement $Z^c$ of codimension $\geq 2$, $p(H_x) \cap Z$ is a non-empty open subset of $p(H_x)$ with $\text{codim}_{p(H_x)}(Z^c \cap p(H_x)) \geq 2$. Let $p_P : \mathcal{PU}_x \to U_L^n(n,d)$ be the projection. Then $p_F^1(Z^c) = p_F^1(Z)^c$ has codimension $\geq 2$ in $\mathcal{PU}_x$ and hence its intersection $S := p_F^{-1}(Z)^c$ with the open set $H_x$ has codimension $\geq 2$ in $H_x$.

Claim 2: For a general $F$, $\psi_{F,x}^{-1}(Z)$ has complement of codimension $\geq 2$ in $P(F_x^*)$.

Proof of Claim 2: Note first that, since codim $S \geq 2$, we get $\dim S \leq \dim H_x - 2 = \dim U_L^n(n,d) + n-1-2 = \dim U_L^n(n,d) + n-3$. The image of $q$ is the subset of $U_L(x)^*(n, d+1)$ consisting of $(0, 1)$-stable vector bundles. Using Lemma 3.7, we know that $q(H_x)$ is a nonempty open subset and hence has dim $q(H_x) = \dim U_L'(x)(n, d + 1) = \dim U_L^n(n, d)$. For a general $F$, we have $\dim(S \cap q^{-1}(F)) = \dim q^{-1}(F) + \dim S - \dim H_x \leq \dim q^{-1}(F) - 2 = n - 1 - 2 = n - 3$ since the intersection is proper.

The fibre of $q$ over $F$ is $P(F_x^*)$. The restriction of $p$ to this fibre is $\psi_{F,x}$. Thus $q$ maps the fibre over $F$ isomorphically onto $\psi_{F,x}(P(F_x^*))$. For a general $F$, we have $S \cap q^{-1}F = S \cap P(F_x^*) = P(F_x^*) - \psi_{F,x}^{-1}(Z)$ has dimension $\leq n - 3$, i.e., codimension $\geq 2$ in $P(F_x^*)$.

The complement of $P'$ in $\mathcal{PU}_x|_{Z}$ is open in $\mathcal{PU}_x$ as $\mathcal{PU}_x|_{Z} = p^{-1}(Z)$ and $Z$ is open. Since $H_x$ is open in $\mathcal{PU}_x$ and $\mathcal{PU}_x$ is irreducible, $H_x \cap (P')^c \neq \emptyset$. For $(E, \ell) \in H_x \cap (P')^c$, $\ell \subset E_x$ is not in the fibre of $P'$ at $x$.

It follows from Lemma 4.1 that $(id_Y \times \psi_{F,x})^*(\mathcal{PU}) \cong P(\mathcal{E})$, where $\mathcal{E}$ is as defined in (3.4). As remarked in [11, Remark 2.2], there exists a vector subbundle $V'$ of $\mathcal{E}'_x := \mathcal{E}_x|_{\psi_{F,x}^{-1}(Z)}$ such that $\psi_{F,x}^*(P') \cong P(V')$. It follows that, if $N$ is defined as in (2.3) we have $\psi_{F,x}^*N \cong V'' \otimes (\mathcal{E}'_x/V')$.

Claim 3: The condition on $\ell$ gives us that the line subbundle $\mathcal{O}_P(1)|_{\psi_{F,x}^{-1}(Z)}$ of $\mathcal{E}_x'$ is such that $\mathcal{O}_P(1)|_{\psi_{F,x}^{-1}(Z)} \not\subset V'$.

Proof of Claim 3: Let $\phi_E$ be the inverse image of $E$ under $\psi_{F,x}$. i.e, we have

\[
\begin{array}{ccccccccc}
0 & & 0 & & | & & | & & |
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
F(-x) & \longrightarrow & F(-x) & \longrightarrow & F & \longrightarrow & k_x & \longrightarrow & 0 \\
0 & \longrightarrow & E = \ker \phi_E & \longrightarrow & F & \phi_E & \phi_E & \phi_E & \phi_E \\
0 & \longrightarrow & E_x/\ell & \longrightarrow & F_x & \phi_E & k_x & \longrightarrow & 0 \\
0 & & 0 & & | & & | & & |
\end{array}
\]

(4.2)

It is enough to show that $\ell$ is the fibre of $\mathcal{O}_P(F_x^*)(1)$ at $\phi_E$.

$(\mathcal{O}_P(F_x^*)(1))_{\phi_E} = V_{\phi_E}^*$ where $V_\phi$ denote the 1 dimensional subspace generated by $\phi$ in $F_x^*$. We know that, for $v \in F_x$, $v^{**}(\phi_E) = \phi_E(v)$. Let $v^{**} \neq 0$ be in the fibre of $(\mathcal{O}_P(F_x^*)(1))$.
at $\phi_E$. This implies $\phi_E(v) \neq 0$, i.e., $v \notin \text{Im}(E_x/\ell \to F_x)$ and hence $v$ corresponds to the line $\ell$.

It follows from Lemma 3.6 that

$$\frac{\deg V'}{\text{rk} V'} < \frac{\deg \mathcal{E}_x}{\text{rk} \mathcal{E}_x},$$

and we get $\deg \psi_{F,x}^* N > 0$. Let $\bar{N} = i_*(N)$ where $i : U'_L(n,d) \hookrightarrow U_L(n,d)$ is the inclusion, then $\deg \bar{N} = \deg N$ so that

$$\deg \psi_{F,x}^* \bar{N} > 0.$$ Since $\theta_L$ is an ample line bundle, by Lemma 3.4, $\psi_{F,x}^* \theta_L$ is an ample line bundle on $P(F_x^*)$.

As Pic $P(F_x^*) \cong \mathbb{Z}$ and it is generated by $\mathcal{O}_{P(F_x^*)}(1)$, it follows that $\psi_{F,x}^* \theta_L$ is isomorphic to $\mathcal{O}_{P(F_x^*)}(r)$ for some $r > 0, r \in \mathbb{Z}$.

We have $c_1(\bar{N}) = \lambda_N \theta_L, \lambda_N \in \mathbb{Z}$. Let $u$ denote the dimension of $U_L(n,d)$. Then

$$\deg \bar{N} = [c_1(\bar{N}), \theta_L^{u-1}](U_L(n,d)) = \lambda_N \theta_L^u(U_L(n,d)) = \lambda_N C_U,$$

$C_U$ being a positive constant. We have $c_1(\psi_{F,x}^* \bar{N}) = \lambda_N\psi_{F,x}^* \mathcal{O}_{P(F_x^*)}(1)$ so that

$$\deg(\psi_{F,x}^* \bar{N}) = \lambda_N\psi_{F,x}^* \mathcal{O}_{P(F_x^*)}(1) > 0.$$ Since $\psi_{F,x}$ has degree 1, one has $c_1(\psi_{F,x}^* \bar{N}) = \psi_{F,x}^*(c_1(\bar{N}))$. Now

$$\psi_{F,x}^*(c_1(\bar{N})) = \psi_{F,x}^*(\lambda_N \theta_L) = \lambda_N\psi_{F,x}^*(\theta_L) = \lambda_N r \mathcal{O}_{P(F_x^*)}(1)$$

so that $(\deg \psi_{F,x})^*(c_1(\bar{N})) = r\lambda_N$. Hence $r\lambda_N = \lambda_N\psi_{F,x}^* \mathcal{O}_{P(F_x^*)}(1) > 0$. Therefore $\deg \bar{N} > 0 \quad \square$

**Theorem 4.6.** Let $Y$ be an integral projective nodal curve of arithmetic genus $g \geq 3$ defined over an algebraically closed field. Let $n \geq 2$ be an integer and let $L$ be a line bundle of degree $d$ on $Y$. Let $U_L^\text{ss}(n,d)$ denote the moduli space of stable vector bundles on $Y$ of rank $n$ and determinant $L$ and let $PU$ be the projective Poincaré bundle on $Y \times U_L^\text{ss}(n,d)$. Let $\eta$ and $\theta_L$ be divisors defining the polarisation on $Y$ and $U_L^\text{ss}(n,d)$ respectively. Then $PU$ is stable with respect to $a\eta + b\theta_L, a, b > 0$.

**Proof.** Suppose $Z \subset Y \times U_L^\text{ss}(n,d)$ is a Zariski open subset with complement of codimension at least 2 and $P'$ a projective subbundle of the restriction of $PU$ to $Z$.

**Claim:** For general $E \in U_L^\text{ss}(n,d), Y \times \{E\} \subset Z$.

The codimension of $Z$ in $Y \times U_L^\text{ss}(n,d)$ is at least 2, implies that

$$\dim(Z^c) \leq \dim(U_L^\text{ss}(n,d)) - 1. \quad (*)$$

Consider the projection map $\rho : Y \times U_L^\text{ss}(n,d) \to U_L^\text{ss}(n,d)$.

For all $E \in p(Z^c) \subset U_L^\text{ss}(n,d)$, we have $Y \times \{E\} \subset Z$. To show this holds for general $E \in U_L^\text{ss}(n,d)$, it is enough to show that $\dim p(Z^c) < \dim U_L^\text{ss}(n,d)$ which follows from $\ast$.

It is also easy to see that, for general $x \in Y$, the codimension of the complement of $\{x\} \times (U_L^\text{ss}(n,d) \cap Z)$ in $\{x\} \times U_L^\text{ss}(n,d)$ is at least 2.

Let $N$ be as in Definition 2.3. By applying Theorem 4.5, we have $\deg N_x > 0$ for $x \in Y_{\text{reg}}$. Also stability of $E$ implies that $\deg N_{|Y \times \{E\}} > 0$. 
Let $\tilde{N} = (Id \times i)_* N$. So by definition of degree we get, $\deg N = \deg \tilde{N}$. Since the codimension of the complement of $\{x\} \times (U_L^{ts}(n, d) \cap Z)$ in $\{x\} \times U_L^{ts}(n, d)$ is at least 2, $\deg N_x = \deg \tilde{N}_x$ for a general $x$.

Now, if $u$ is the dimension of $U_L(n, d)$ then
\[
\deg N = \deg \tilde{N} = [c_1(\tilde{N}) \cdot (a\alpha + b\theta_L)u](Y \times U_L(n, d)) \\
= [c_1(\tilde{N}) \cdot (\lambda\alpha, \theta_L^{n-1} + \mu\theta_L^u)](Y \times U_L(n, d)) \quad \text{for some } \lambda, \mu > 0 \\
= \lambda\alpha(Y) \cdot \deg \tilde{N}_x + \mu \deg(N|_{Y \times \{E\}}) \cdot \theta_L^u(U_L(n, d)) \\
> 0
\]

\[\square\]

5. **Projective Picard bundle**

The construction of the projective Picard bundle follows exactly as in [11]. We outline it here for the sake of completeness.

Recall that $U_L^{ts}(n, d)$ is a GIT quotient of $R^s$ by $PGL(M)$, let $\pi : R^s \to U_L^{ts}(n, d)$ be the quotient map. There is a (locally universal) vector bundle $E_R^s \to Y \times R^s$. Let $p_{R^s} : Y \times R^s \to R^s$ denote the projection to the second factor. Then $p_{R^s}, E_R^s$ is a torsionfree sheaf on $R^s$ and is non-zero if and only if $d > n(g - 1)$ [7, Theorem 1.3]. For $d \geq 2n(g - 1), n \geq 2$, the set
\[
\{ r \in R^s | H^1(Y, E_R^s|_{Y \times \{r\}}) = 0 \}
\]
equals $R^s$ and $p_{R^s}, E_R^s$ is a vector bundle on $R^s$. The projective Picard bundle on $U_L^{ts}(n, d)$ is defined to be the quotient of $P(p_{R^s}(E_R^s))$ by the action of $PGL(M)$ on it and is denoted by $\mathcal{PW}$.

**Lemma 5.1.** Let $E$ be a vector bundle on $Y \times Z$ such that the restriction of $E$ to $Y \times \{z\}$ is stable of rank $n$ and determinant $L$ for all $z \in Z$, and let $\psi_E : Z \to U_L^{ts}(n, d)$ be the corresponding morphism. Also suppose that $H^1(Y, E|_{Y \times \{z\}}) = 0$ for all $z \in Z$. Then the projective bundles $P(p_{Z\times E})$ and $\psi_E^*(\mathcal{PW})$ are isomorphic.

**Proof.** See [11] Proposition 4.2] for a proof. \[\square\]

An analogue of Corollary [12] shows that $\mathcal{PW}$ is independent of the choice of $R'$ and $\pi$. Its restriction $\mathcal{PW}|_{U_L^{ts}(n, d)}$ will be called the projective Picard bundle on $U_L^{ts}(n, d)$.

5.1. **Stability of the projective Picard bundle.**

**Theorem 5.2.** Let $Y$ be an integral projective nodal curve of arithmetic genus $g \geq 3$ defined over an algebraically closed field. Let $n \geq 2$ be an integer and let $L$ be a line bundle of degree $d$ on $Y$. Let $U_L^{ts}(n, d)$ denote the moduli space of stable vector bundles on $Y$ of rank $n$ and determinant $L$, suppose further that $d > 2n(g - 1)$. Then the projective Picard bundle $\mathcal{PW}|_{U_L^{ts}(n, d)}$ is stable.

**Proof.** The proof is exactly on the same lines as in the case when $Y$ is smooth [11]. Let $P'$ be a projective subbundle of the restriction of $\mathcal{PW}$ to some Zariski-open subset $Z'$ of $U_L^{ts}(n, d)$ with complement of codimension at least 2. For a nonsingular point $x \in Y$ and a $(0, 1)$-stable vector bundle $F$ of rank $n$ and determinant $L(x)$, we have a morphism,
\[ \psi_{F,x} : \mathbb{P}(F_x^*_p) \to U^*_L(n, d) \] (see Section 3). For a general \( F \), the complement of \( Z := \psi_{F,x}^{-1}(Z') \) has codimension at least 2. By (3.1), \( F \) corresponds to a pair \((E, \ell)\) with \( E \in U^*_L(n, d) \) and \( \ell \subset E_x \) is a line.

We restrict Diagram 3.5 to \( Y \times Z \). We want to take the direct image by the projection \( p_2 : Y \times Z \to Z \). We first note the following observations.

- \( p_2^*(p_1^*F(-x))_z = H^0(p_1^*F(-x)|_{Y \times \{z\}}) = H^0(Y, F(-x)) \) for all \( z \in Z \),
- Similarly \( p_2^*(p_1^*F)_z = H^0(Y, F) \) for all \( z \in Z \).
- One has \( i_*O_P(1)|_Z = (i_Z)_*O_Z(1) \), \( i_Z \) being the inclusion map restricted to \( Z \). Since \( p_2 \circ i_Z = id_Z \), \( p_2((i_Z)_*O_Z(1)) = O_Z(1) \).
- Similarly \( p_2^*i_*(\Omega^1_P(1)|_{Y \times Z}) = \Omega^1_Z(1) \)

Taking the direct image by the projection \( p_2 \) and using these observations, we have the following diagram of exact sequences on \( Z \):

\[
\begin{array}{ccccccccc}
0 & 0 & \downarrow & \downarrow & \downarrow \\
\downarrow & & & & & \downarrow & & & \\
H^0(Y, F(-x)) \otimes O_Z & \longrightarrow & H^0(Y, F(-x)) \otimes O_Z & \longrightarrow & \Omega^1_Z(1) & \longrightarrow & F_x \otimes O_Z & \longrightarrow & O_Z(1) & \longrightarrow & 0
\end{array}
\]

Now \( \mathcal{E}|_{Y \times Z} \) satisfies the hypothesis of Lemma 5.1 and hence we have \( P(p_2^*(\mathcal{E}|_{Y \times Z})) \cong (\psi_{F,x}^*PW)|_Z \). Now \( (\psi_{F,x}^*P')|_Z \) being a subbundle of \( P(p_2^*(\mathcal{E}|_{Y \times Z})) \) can be realized as \( P(V) \) for some vector subbundle \( V \) of \( p_2^*(\mathcal{E}|_{Y \times Z}) \).

We claim that the image of \( V \) in \( \Omega^1_Z(1) \) is non-zero, we prove the claim. Any \( 0 \neq v \in V_E \) corresponds to a section \( s \in H^0(Y, E) \). As in the beginning of [7, Subsection (4.1)], one can find a general line \( \ell \subset E_x \) such that \( s(x) \notin \ell \). Note that \( \ell \) is the image of \( F(-x)_x \) in \( E_x \) and since \( s(x) \notin \ell \), \( s \notin H^0(Y, F(-x)) \subset H^0(Y, E) \). The claim follows.

From the last row of Diagram 5.1, one sees that \( \deg(\Omega^1_Z(1)) = -1 \). Since \( \Omega^1_Z \) is a stable bundle, we have degree of \( \text{Im}(V \to \Omega^1_Z(1)) \leq -1 \), since the \( \ker(V \to \Omega^1_Z(1)) \) is a subsheaf of the trivial sheaf, its degree is at most 0 and hence we have \( \deg V \leq -1 \). i.e.

\[ \deg V^* \geq 1. \]

Also from the second row of Diagram 5.1, we get \( \deg p_2^*(\mathcal{E}|_{Y \times Z}) = -1 \). Hence

\[ \deg(p_2^*(\mathcal{E}|_{Y \times Z})/V) \geq 0. \]

Now as in beginning of the proof of Theorem 4.5, we have \( \psi_{F,x}^*N = V^* \otimes p_2^*(\mathcal{E}|_{Y \times Z})/V \).

\[ \deg \psi_{F,x}^*N = \deg(V^*) \deg(p_2^*(\mathcal{E}|_{Y \times Z})/V) + \deg(p_2^*(\mathcal{E}|_{Y \times Z})/V) \deg V^* \geq \deg(p_2^*(\mathcal{E}|_{Y \times Z})/V) > 0 \]

Similar arguments as at the end of the proof of Theorem 4.5 show that \( \deg \psi_{F,x}^*N > 0 \) implies that \( \deg N > 0 \). This proves the stability of \( PW|_{U^*_L(n, d)} \). \( \square \)
Remark 5.3. In case \((n, d) = 1\), there is a Poincaré bundle \(\mathcal{U}\) on \(Y \times U'_L(n, d)\) (respectively a Picard bundle \(\mathcal{W}\) on \(U'_L(n, d)\)) with the associated projective Poincaré bundle \(\mathcal{PU}\) (respectively the projective Picard bundle \(\mathcal{PW}\)). In view of remark 2.4(1), the stability of the projective Poincaré bundles \(\mathcal{PU}\), \(\mathcal{PU}_x\) is equivalent to the stability of the Poincaré bundles \(\mathcal{U}, \mathcal{U}_x\) and the stability of the projective Picard bundle \(\mathcal{PW}\) is equivalent to the stability of the Picard bundle \(\mathcal{W}\).

6. Codimension of the stable moduli space

In this section we assume that \(n\) and \(d\) are non-coprime. Recall that \(U'_Y(n, d)\) denotes the moduli space of vector bundles of rank \(n\), degree \(d\) on the curve \(Y\) and \(U'_{Ys}(n, d)\) its open subset consisting of stable vector bundles. For a fixed line bundle \(L\) of degree \(d\) on \(Y\), \(U'_L(n, d)\) denotes the moduli space of vector bundles of rank \(n\) with determinant \(L\) and \(U'_{Ls}(n, d)\) its open subset corresponding to stable vector bundles. We estimate the codimension (in \(U'_L(n, d)\)) of the complement \(U'_L(n, d) - U'_{Ls}(n, d)\).

Theorem 6.1. Let \(Y\) be an integral nodal curve of arithmetic genus \(g\).

(1) For \(g \geq 2\) and \(n \geq 3\) (resp. \(n = 2\)),
\[
\text{codim}_{U'_Y(n, d)}(U'_Y(n, d) - U'_{Ys}(n, d)) \geq 2(g - 1)\text{ (resp. } \geq g - 1)\,.
\]

(2) For \(g \geq 2\) and \(n \geq 3\) (resp. \(n = 2\)),
\[
\text{codim}_{U'_L(n, d)}(U'_L(n, d) - U'_{Ls}(n, d)) \geq 2(g - 1)\text{ (resp. } \geq g - 1)\,.
\]

Proof. (1) A semistable vector bundle that is not stable has a filtration of length \(r\) of the form
\[
(6.1) \quad 0 = E_0 \subset E_1 \subset \ldots \subset E_r = E
\]
where

(1) \(E_i\) are semistable torsionfree sheaves with \(\mu(E_i) = \mu(E)\) for all \(i = 1, \ldots, r\).

(2) \(F_i = E_i/E_{i-1} \in U_Y(n_i, d_i)\) which are stable torsionfree sheaves.

Note that \(\mu(E_i) = \mu(E)\) for all \(i\), implies \(\mu(F_i) = \mu(E)\) for all \(i = 1, 2, \ldots, r\). Hence \(n_id_j - n_jd_i = 0\) for all \(1 \leq i, j \leq r\).

For \(r \geq 2\), let \(S_r\) denote the subset of \(U'_Y(n, d)\) consisting of vector bundles \(E = E_r\) having a filtration \((6.1)\) with \(F_i \in U_Y(n_i, d_i)\) and let \(e_r\) denote the dimension of the subset \(S_r\) in \(U'_Y(n, d)\).

We first consider the case when length of the filtration \(r\) is 2.

(6.2) \[
(F_2) \quad 0 = E_0 \subset E_1 \subset E_2 = E
\]
such that \(F_1 = E_1\) and \(F_2 = E/E_1\).

We have the following exact sequence
\[
(6.3) \quad 0 \rightarrow F_1 \rightarrow E \rightarrow F_2 \rightarrow 0
\]
and to estimate the number of parameters determining such extensions, we compute \(\text{dim } \text{Ext}^1(F_2, F_1)\).
Note that \( \text{deg}(\text{Hom}(F_2, F_1)) = n_2d_1 - n_1d_2 + \sum_k b_k(F_1)b_k(F_2) \) and \( \mu(F_1) = \mu(F_2) \) implies
\[
\text{deg}(\text{Hom}(F_2, F_1)) = \sum_k b_k(F_1)b_k(F_2).
\]

Therefore,
\[
\dim \text{Ext}^1(F_2, F_1) = h^1(\text{Hom}(F_2, F_1)) + 2 \sum_k b_k(F_1)b_k(F_2)
\]
\[
= h^0(\text{Hom}(F_2, F_1)) - \chi(\text{Hom}(F_2, F_1)) + 2 \sum_k b_k(F_1)b_k(F_2)
\]
\[
= h^0(\text{Hom}(F_2, F_1)) - [\text{deg}(\text{Hom}(F_2, F_1)) + n_1n_2(1-g)] + 2 \sum_k b_k(F_1)b_k(F_2)
\]
\[
= n_1n_2(g-1) + \sum_k b_k(F_1)b_k(F_2) + h^0(\text{Hom}(F_2, F_1)).
\]

Since \( F_1 \) and \( F_2 \) are stable sheaves with same slope, \( h^0(\text{Hom}(F_2, F_1)) = 1 \) (resp. \( 0 \)) when \( F_2 \cong F_1 \) (resp. \( F_2 \not\cong F_1 \)).

Recall that \( S_2 \) denotes the subset of \( U_Y'(n,d) \) consisting of vector bundles \( E \) having a filtration \( 0 \rightarrow E_1 \rightarrow E_2 \rightarrow \cdots \rightarrow E_n \rightarrow 0 \) with \( E_i \in U_Y(n_i,d_i) \) and \( e_2 \) denotes the dimension of the subset \( S_2 \). We have
\[
e_2 \leq n_1^2(g-1) + 1 - \sum_k b_k(F_1)^2 + n_2^2(g-1) + 1 - \sum_k b_k(F_2)^2 + \dim \mathbb{P}(\text{Ext}^1(F_2, F_1))
\]
\[
= (n_1^2 + n_2^2)(g-1) + 2 - \sum_k b_k(F_1)^2 - \sum_k b_k(F_2)^2 + n_1n_2(g-1)
\]
\[+ \sum_k b_k(F_1)b_k(F_2) + h^0(\text{Hom}(F_2, F_1)) - 1
\]
\[
= (n_1^2 + n_2^2)(g-1) + 2 + n_1n_2(g-1) - \sum_k [b_k(F_1) - b_k(F_2)]^2 - \sum_k b_k(F_1)b_k(F_2)
\]
\[+ h^0(\text{Hom}(F_2, F_1)) - 1
\]

Here \( k \) varies over the nodes. If \( F_2 \not\cong F_1 \), \( h^0(\text{Hom}(F_2, F_1))=0 \) and
\[
(6.4) \quad e_2 \leq (n_1^2 + n_2^2)(g-1) + 2 + n_1n_2(g-1) - \sum_k [b_k(F_1) - b_k(F_2)]^2 - \sum_k b_k(F_1)b_k(F_2) - 1
\]

Since \( b_k(F_i) \geq 0 \) and terms with \( b_k(F_i) \) appear with a negative sign, we note that \( e_2 \) is maximal if \( b_k(F_i) = 0 \). i.e., \( F_i \)'s are vector bundles. Hence we have
\[
(6.5) \quad e_2 \leq (n_1^2 + n_2^2)(g-1) + 1 + n_1n_2(g-1)
\]

Let \( S_2^{12} \) denote the subset of \( U_Y'(n,d) \) corresponding to vector bundles given by the short exact sequence \( 0 \rightarrow F_1 \rightarrow F_2 \rightarrow 0 \) with \( F_1 \cong F_2 \). Then \( (F_1, F_2) \) belongs to the diagonal \( \Delta \) of \( U_Y(n_1,d_1) \times U_Y(n_2,d_2) \). Let \( e_2^{12} \) denote the dimension of \( S_2^{12} \). Then
$e_2^{12} = \dim \Delta + \dim \mathbb{P}(\text{Ext}^1(F_2, F_1))$

$$= n_1^2(g - 1) + 1 + n_1n_2(g - 1) + 1 - 1$$

Note that $e_2 - e_2^{12} = n_2^2(g - 1) \geq 1$ for $g \geq 2$.

Hence the codimension in $U_r' \times (n, d)$ of the subset corresponding to vector bundles $E$ given by 6.3 is

$$\geq n_2^2(g - 1) + 1 - e_2$$

$$= n_1n_2(g - 1)$$

In particular, for $n = 2$, $n_1 = n_2 = 1$ and

$$(6.6) \quad \text{codim}_{U_r' \times (n, d)}(U_r'(n, d) - U_r'(n, d)) \geq g - 1$$

$$(6.7) \quad \text{For } n \geq 3, \text{ codim } S_2 \geq 2(g - 1).$$

We can now assume that $r \geq 3, n \geq 3$. A vector bundle $E = E_r$ having a filtration $6.1$ comes in the following exact sequence

$$0 \to E_{r-1} \to E_r \to F_r \to 0$$

with $E_{r-1}$ varying on the space $S_{r-1}$.

As seen in case $r = 2$, to estimate the dimension of space of extensions, we may assume that $E_i$ are locally free. The group $G_r = \text{Aut}E_{r-1} \times \text{Aut}F_r$ acts on these extensions and $E_r$ given by the extensions in an orbit of $G_r$ are all isomorphic. One has dim $\text{Aut}F_r = 1$. Since $\text{End}E_{r-1} \supset \mathbb{C}^* \text{Id} \oplus \text{Hom}(F_{r-1}, E_{r-2})$, it follows that dim $\text{Aut}E_{r-1} \geq 1 + h^0(F_{r-1} \otimes E_{r-2})$ so that dim $G_r \geq 2 + h^0(F_{r-1} \otimes E_{r-2})$.

Hence $e_r \leq e_{r-1} + n_r^2(g - 1) + 1 + h^1(F_r^* \otimes E_{r-1}) - \dim G_r$. We have

$$h^1(F_r^* \otimes E_{r-1}) = -\chi(F_r^* \otimes E_{r-1}) + h^0(F_r^* \otimes E_{r-1}) = (\sum_{i=1}^{r-1} n_i)n_r(g - 1) + h^0(F_r^* \otimes E_{r-1}).$$

Hence for any $r$ ($r \geq 3$),

$$(6.9) \quad e_r \leq e_{r-1} + n_r^2(g - 1) - 1 + (\sum_{i=1}^{r-1} n_i)n_r(g - 1) + h^0(F_r^* \otimes E_{r-1}) - h^0(F_r^* \otimes E_{r-2}).$$

We replace $r$ by $r - 1$ in the inequality 6.9 and substitute it for $e_{r-1}$ in the inequality 6.9 to get

$$e_r \leq e_{r-1} + n_r^2(g - 1) - 1 + (\sum_{i=1}^{r-1} n_i)n_r(g - 1) + h^0(F_r^* \otimes E_{r-1}) - h^0(F_r^* \otimes E_{r-2})$$

$$\leq n_r^2(g - 1) - 1 + (\sum_{i=1}^{r-1} n_i)n_r(g - 1) + h^0(F_r^* \otimes E_{r-1}) - h^0(F_r^* \otimes E_{r-2})$$

$$+ e_{r-2} + n_{r-1}^2(g - 1) - 1 + (\sum_{i=1}^{r-2} n_i)n_{r-1}(g - 1) + h^0(F_{r-1}^* \otimes E_{r-2}) - h^0(F_{r-1}^* \otimes E_{r-3}).$$
We now substitute for \( e_{r-2} \) (using inequality (6.9)) in the resulting inequality and go on like this till we get \( e_2 \), then we substitute for \( e_2 \) from equation (6.5). We finally get

\[
e_r \leq \sum_{i=1}^{r} n_i^2 (g - 1) - (r - 3) + \sum_{1 \leq i < j \leq r} n_i n_j (g - 1) + (r - 1)
\]

where

\[
\delta_r = h^0(F_*^{r} \otimes E_{r-1}) - h^0(F_*^{r-1} \otimes E_{r-2}) + h^0(F_*^{r-2} \otimes E_{r-3}) + \ldots \nonumber
\]

Hence

\[
e_r \leq \sum_{i=1}^{r} n_i^2 (g - 1) + \sum_{1 \leq i < j \leq r} n_i n_j (g - 1) + 2.
\]

Therefore,

\[
\text{codim } S_r \geq \sum_{i=1}^{r} n_i^2 (g - 1) + 1 - \sum_{i=1}^{r} n_i (g - 1) - \sum_{1 \leq i < j \leq r} n_i n_j (g - 1) - 2
\]

\[
= \sum_{1 \leq i < j \leq r} n_i n_j (g - 1) - 1
\]

\[
\geq 3(g - 1) - 1 \text{ for } r \geq 3.
\]

Thus

\[
\text{codim } S_r \geq 3(g - 1) - 1 \geq 2(g - 1) \text{ for } r \geq 3, g \geq 2.
\]

This, together with equations (6.6) and (6.7), proves the first part of the theorem.

(2) There is a determinant homomorphism \( \text{det} : U'_L(n,d) \to J(Y) \) defined by \( E \mapsto \wedge^n E \) with all its fibres isomorphic. Hence Part 2 follows from Part 1.

\(\square\)

**Corollary 6.2.** Let \( Y \) be an integral nodal curve of arithmetic genus \( g \geq 2 \). Assume that if \( n = 2 \) and \( g = 2 \) then \( d \) is odd. Then

1. \( \text{Pic } U'_L(n,d) \cong \mathbb{Z} \).
2. \( \text{Pic } U'_L(n,d) \cong \mathbb{Z} \).
3. The class group \( \text{Cl}(U_L(n,d)) \cong \mathbb{Z} \). The class group \( \text{Cl}(U'_L(n,d)) \cong \mathbb{Z} \).

**Proof.** (1) Let \( \tilde{U}_L(n,d) \) denote a normalisation of \( U_L(n,d) \) and \( \pi : \tilde{U}_L(n,d) \to U_L(n,d) \) the normalisation map. Since \( \pi \) is a finite map, codim\( \text{codim} \tilde{U}_L(n,d) \tilde{U}_L(n,d) - \pi^{-1}U'_L(n,d) = \text{codim} U_L(n,d) - U'_L(n,d) \geq 2 \) by Theorems 2.6 and 6.1. Since \( \tilde{U}_L(n,d) \) is normal,
this implies that the restriction map $\text{res}_p : \text{Pic} \, \bar{U}_L(n, d) \to \text{Pic} \, \pi^{-1}U'_L(n, d)$ is injective. Note that $\pi$ is an isomorphism over $U'_L(n, d)$ so that $\text{Pic} \, \pi^{-1}U'_L(n, d) = \text{Pic} \, U'_L(n, d)$. Hence $\text{Pic} \, U'_L(n, d)$ has rank $\geq 1$. By [2, Proposition 2.3], for $g \geq 2$, we have $\text{Pic} \, U'_L(n, d) \cong \mathbb{Z}$ or $\mathbb{Z}/q\mathbb{Z}$ for some integer $q$. It follows that $\text{Pic} \, U'_L(n, d) \cong \mathbb{Z}$.

(2) The proof of Part 1 also implies that $\text{Pic} \, \bar{U}_L(n, d) \cong \mathbb{Z}$. The restriction map $\text{res}_p$ in the proof of Part (1) factors through the injective restriction map $\text{Pic} \, U'_L(n, d) \hookrightarrow \text{Pic} \, U'_L(n, d) \cong \mathbb{Z}$ (by Theorem 6.1). It follows that $\text{Pic} \, U'_L(n, d) \cong \mathbb{Z}$.

(3) Since $U_L(n, d)$ is nonsingular in codimension 1 by Theorems 2.6 and 6.1 by [10, Proposition 6.5, p.133], we have an isomorphism of class groups

$$\text{Cl}(U_L(n, d)) \cong \text{Cl}(U'_L(n, d)).$$

Since $U'_L(n, d)$ is nonsingular, one has $\text{Cl}(U'_L(n, d)) \cong \text{Pic} \, U'_L(n, d) \cong \mathbb{Z}$ by Part (1).

7. Non-existence of Poincaré bundle in the non-coprime case

If $(n, d) = 1$, one has $U'_L(n, d) = U'_L(n, d)$ and there is a Poincaré bundle on $U'_L(n, d) \times Y$ [18, Section 7, Theorem 5.2]. In this section we show that if $n$ and $d$ are not coprime, then there is no Poincaré bundle on $V \times Y$ for any Zariski open subset of $V \subset U'_L(n, d)$. We follow the proof of Ramanan for the non-existence of Poincaré bundle in non-coprime case over smooth curves [23, Theorem 2]. Hence we retain some of his notations and the proofs of lemmas whose proofs mostly go through in the nodal case are only sketched. There are some additional results and many modifications needed in the nodal case which we give in greater detail.

Let $\mathcal{O}_Y(1)$ denote an ample line bundle on $Y$, we can assume that its degree is 1. For a vector bundle $E$, we write $E(m) = E \otimes \mathcal{O}_Y(m)$.

We start with some results on codimensions needed in the nodal case.

7.1. Codimensions of subsets of moduli spaces. The moduli space $U_X = U_X(n, d)$ of semistable vector bundles of rank $n$ and degree $d$ on $X$ is the GIT quotient of the moduli space $\text{Quot}$ of coherent quotients $\mathcal{O}_X^m \to E \to 0$ with fixed Hilbert polynomial $P(m) = mn + d + n(1 - g), n = P(0)$, assume $d$ sufficiently large. Let $R$ be the open smooth subscheme of $\text{Quot}$ corresponding to vector bundles $E$ with $H^1(E) = 0$ and $H^0(X, E) \otimes \mathcal{O}_X \cong \mathcal{O}_X^m$. Let $R^s$ and $R^{ss}$ denote the subsets of $R$ consisting of stable and semistable points respectively for the action of $\text{PGL}(N)$. Let $U_X^s$ be the open subset of $U_X$ corresponding to stable vector bundles. Then $U_X$ (resp. $U_X^s$) is the GIT quotient of $R^{ss}$ (resp. $R^s$) by $\text{PGL}(N)$.

**Proposition 7.1.** ([2, Proposition 1.2]).

Let $g(X)$ be the genus of $X$. For $g(X) \geq 2$ and $n \geq 3$ (resp. $n = 2$) one has

1. $\text{codim}_{R^s}(R - R^s) \geq 2g(X) - 2$ (resp. $g(X) - 1$)
2. $\text{codim}_{R^{ss}}(R - R^{ss}) \geq 2g(X) - 1$ (resp. $g(X)$)
3. $\text{codim}_{R^{ss}}(R^{ss} - R^s) \geq 2g(X) - 2$ (resp. $g(X) - 1$).

**Corollary 7.2.** ([2, Corollary 1.3]). Fix a line bundle $L$ on $X$. Let $R_L \subset R, R_L^s \subset R^s, R_L^{ss} \subset R^{ss}$ be the closed subvarieties corresponding to bundles $E$ with fixed determinant
Proposition 7.3. For \( n \geq 3 \) (resp. \( n = 2 \)) one has:

1. \( \text{codim}_{R} (R_{L} - R_{n}^{ss}) \geq 2g(X) - 1 \) (resp. \( \geq g(X) \))
2. \( \text{codim}_{R} (R_{L} - R_{ss}^{s}) \geq 2g(X) - 2 \) (resp. \( \geq g(X) - 1 \))
3. \( \text{codim}_{R} (R_{L} - R_{ss}^{s}) \geq 2g(X) - 2 \) (resp. \( \geq g(X) - 1 \)).

Proof. (1) Since \( pr : \tilde{R}' \to R \) is a fibre bundle with fibres isomorphic to \( \prod_{j} GL(n) \), from [L4] it follows that

\[
\text{codim}_{\tilde{R}} (\tilde{R}' - \tilde{R}^{ss}) = \text{codim}_{\tilde{R}} (\tilde{R}' - pr^{-1}(R^{ss})) \geq 2g(X) - 1, \text{ for } n \geq 3 (\geq g(X) \text{ for } n = 2).
\]

(2) Part (2) can be proved similarly.

(3) One has \( \tilde{T}^{s}s - \tilde{R}^{s} = pr^{-1}(R^{ss} - R^{s}) \). Hence \( \text{codim}_{\tilde{R}} (\tilde{T}^{ss} - \tilde{T}^{s}) \geq 2g(X) - 2 \) for \( n \geq 3 \) (and \( \geq g(X) - 1 \) for \( n = 2 \)). \( \square \)

Corollary 7.4. For \( n \geq 3 \) (resp. \( n = 2 \)) and for a general \( L \), one has:

1. \( \text{codim}_{H'} (H' - \tilde{T}^{ss}) \geq 2g(X) - 1 \) (resp. \( g(X) \))
2. \( \text{codim}_{H'} (H' - \tilde{T}^{s}) \geq 2g(X) - 2 \) (resp. \( g(X) - 1 \))
3. \( \text{codim}_{H'} (\tilde{T}^{ss} - \tilde{T}^{s}) \geq 2g(X) - 2 \) (resp. \( g(X) - 1 \))

Taking GIT quotients by \( PGL(N) \), we get the following corollary from Proposition 7.3 and Corollary 7.4.
Corollary 7.5. For \( n \geq 3 \) (resp. \( n = 2 \)) and \( \bar{M}^{ss} \neq \bar{M}^s \) we have:

1. \( \text{codim}_{M^s}(M' - \bar{M}^{ss}) \geq 2g(\mathcal{X}) - 1 \) (resp. \( g(\mathcal{X}) \))
2. \( \text{codim}_{M^s}(M' - \bar{M}^s) \geq 2g(\mathcal{X}) - 2 \) (resp. \( g(\mathcal{X}) - 1 \))
3. \( \text{codim}_{M^{ss}}(\bar{M}^{ss} - \bar{M}^s) \geq 2g(\mathcal{X}) - 2 \) (resp. \( g(\mathcal{X}) - 1 \))
4. \( \text{codim}_{M^s}((M' - \bar{M}^{ss}) \geq 2g(\mathcal{X}) - 1 \) (resp. \( g(\mathcal{X}) \)).
5. \( \text{codim}_{M^s}((M' - \bar{M}^s) \geq 2g(\mathcal{X}) - 2 \) (resp. \( g(\mathcal{X}) - 1 \)).
6. \( \text{codim}_{M^{ss}}((\bar{M}^{ss} - \bar{M}^s) \geq 2g(\mathcal{X}) - 2 \) (resp. \( g(\mathcal{X}) - 1 \)).

Let \( \bar{U}^{ss}_Y(n, d) \) (respectively \( \bar{U}^{ss}_Y(n, d) \)) be the subset of \( \bar{U}_Y(n, d) \) corresponding to vector bundles \( F \) such that \( p^*F \) is semistable (respectively stable).

Theorem 7.6. For \( n \geq 3 \) (resp. \( n = 2 \)) and \( \bar{U}^{ss}_Y(n, d) \neq \bar{U}^{ss}_Y(n, d) \) one has:

1. \( \text{codim}_{U'_Y(n, d)}((U'_Y(n, d) - \bar{U}^{ss}_Y(n, d)) \geq 2g(\mathcal{X}) - 1 \) (resp. \( g(\mathcal{X}) \))
2. \( \text{codim}_{U'_Y(n, d)}((U'_Y(n, d) - \bar{U}^s(n, d)) \geq 2g(\mathcal{X}) - 1 \) (resp. \( g(\mathcal{X}) \))
3. \( \text{codim}_{U'_Y(n, d)}((U'_Y(n, d) - \bar{U}^{ss}_Y(n, d)) \geq 2g(\mathcal{X}) - 2 \) (resp. \( g(\mathcal{X}) - 1 \))
4. \( \text{codim}_{U'_Y(n, d)}((U'_Y(n, d) - \bar{U}^s(n, d)) \geq 2g(\mathcal{X}) - 2 \) (resp. \( g(\mathcal{X}) - 1 \))
5. \( \text{codim}_{U^{ss}_Y(n, d)}((U^{ss}_Y(n, d) - \bar{U}^{ss}_Y(n, d)) \geq 2g(\mathcal{X}) - 2 \) (resp. \( g(\mathcal{X}) - 1 \))
6. \( \text{codim}_{U^{ss}_Y(n, d)}((U^{ss}_Y(n, d) - \bar{U}^s(n, d)) \geq 2g(\mathcal{X}) - 2 \) (resp. \( g(\mathcal{X}) - 1 \)).

Proof. Since \( M' \) (respectively \( M'_L \)) and \( U'_Y(n, d) \) (respectively \( U'_L(n, d) \)) are isomorphic, the theorem follows from Corollary 7.5. \( \square \)

7.2. The group \( \pi_Y \) and bundles associated to its representations.

Let \( Y \) be a complex nodal curve with \( g(\mathcal{X}) \geq 2 \). By tensoring by a line bundle, we normalise \( d \) by the condition \( -n < d \leq 0 \). Let \( \pi_X \) denote the group (a Fuchian group) generated by \( 2g(\mathcal{X}) + 1 \) generators \( a_1, b_1, a_2, b_2, \ldots, a_g, b_g, c \) with only relations

\[
(\Pi_i a_i b_i a_i^{-1} b_i^{-1}) c = 1, \ c^n = 1.
\]

Define

\[
\pi_Y := \pi_X \ast \mathbb{Z} \ast \cdots \ast \mathbb{Z},
\]

with \( \mathbb{Z} \) repeated as many times as the number of nodes and \( \ast \) denoting the free product of groups. Let \( 1_j \) denote the generator of the \( j \)-th factor \( \mathbb{Z} \).

Let \( \zeta \) be a primitive \( n \)-th root of unity. Let \( \tau \) be the character on the cyclic group generated by \( \zeta \) defined by \( \tau(\zeta) = \zeta^{-d} \). We say that a representation \( \rho \) of \( \pi_Y \) in \( GL(n, \mathbb{C}) \) is of type \( \tau \) if \( \rho(c) = \tau(\zeta) I_n \), where \( I_n \) is the identity matrix of rank \( n \). Let \( \rho_X = \rho_{|\pi_X} \).

To a representation \( \rho \) of \( \pi_Y \) in \( GL(n, \mathbb{C}) \) of type \( \tau \), we can associate a generalised parabolic vector bundle (GPB) on \( X \) and hence a vector bundle \( E_\rho \) on \( Y \) of rank \( n \) and degree \( d \) (see [8 Subsection 2.3] for details).

Corollary 7.7. Let \( Y \) be a complex nodal curve with \( g(\mathcal{X}) \geq 2 \). The subset of \( U'_Y(n, d) \) (respectively of \( U'_L(n, d) \)) consisting of vector bundles which come from representations of
the group $\pi(Y)$ has complement of codimension at least 2 for $g(X) \geq 2$ except possibly when $n = g(X) = 2, d$ even.

Proof. By [6, Theorem 2.8], there is a bijective correspondence between $Rep := \{\text{Equivalence classes of representations } \rho : \pi_Y \to \text{GL}(n, \mathbb{C}) \text{ of type } \tau \text{ such that } \rho_X \text{ is irreducible and unitary} \}$ and elements of the open dense subset $U_Y^\tau(n, d)$. This bijection is in fact a homeomorphism as can be seen using the fact that, over a smooth curve the Narasimhan-Seshadri correspondence is a homeomorphism. Hence the corollary follows from [7,6]. In case of $U_L(n, d)$ we take $\rho$ to have values in $\text{SL}(n, \mathbb{C})$. □

We note that for $d = 0$, the Fuchsian group $\pi_X$ is replaced by the fundamental group $\pi_1(Y)$ of $Y$ [1].

7.3. Proof of the non-existence theorem. We start with a few lemmas needed for the proof.

Lemma 7.8. Let $E$ (respectively $F$) be a vector bundle of rank $n$, degree $d$ and determinant $L$ (respectively of rank $n + 1$, degree $d'$ and determinant $L'$) on $Y$. Then there exists an integer $m_0 = m_0(E,F)$ and an injective homomorphism of vector bundles $i : E \to F(m_0)$. One has $F(m_0)/iE \cong \mathcal{O}_Y(n + 1) \otimes \text{det}F \otimes (\text{det}E)^{-1} = L' \otimes L^{-1}(n + 1)$.

Proof. This is proved exactly as [23, Lemma 3.1]. □

The tensor product of two semistable vector bundles of rank $\geq 2$ on $Y$ may not be semistable [1]. So we may not be able to choose $m_0$ dependent only on the integers $n, d, d'$. If $E$ is (semi)stable, $E \otimes I_y$ may not be (semi)stable, where $I_y$ is the ideal sheaf at a node $y$. However, if a vector bundle $E$ is (semi)stable, then it is easy to see that $E \otimes N$ is (semi)stable for any line bundle $N$ on $Y$. If $E$ is (semi)stable, then the dual bundle $E^*$ is (semi)stable [4, Lemma 2.6(2)].

Lemma 7.9. Assume that $E$ and $F$ are as in Lemma 7.8 and both are stable vector bundles. Let $\delta_Y$ be the number of nodes of $Y$. (1) $H^1(Y, \text{Hom}(E, F(m))) = 0$ for $m \geq m_1$ where $m_1 = d/n - d'/((n + 1) + 2g - 2$.

(2) $H^1(Y, \text{Hom}(F(m), L^{-1} \otimes L'(m(n + 1)))) = 0$ for $m \geq m_2$ where $m_2 = \frac{d'/((n + 1) + d' + d + 2g - 2}{n}$.

(3) The vector bundle $\text{Hom}(F(m), L^{-1} \otimes L'(m(n + 1)))$ is generated by sections for $m \geq m_3$, where $m_3 = (1/n)((1 + \delta_Y)(n + 1) + 2g - 2 - d - d' + d'/((n + 1)).$

Proof. (1) By Serre duality $h^1(Y, \text{Hom}(E, F(m))) = h^1(Y, E^* \otimes F(m)) = h^0(Y, \text{Hom}(F(m), E \otimes \omega_Y))$. Since $F(m)$ and $E \otimes \omega_Y$ are stable, one has $h^0(Y, \text{Hom}(F(m), E \otimes \omega_Y)) = 0$ if $\mu(F(m)) \geq \mu(E \otimes \omega_Y)$ i.e., $\mu(F) + m \geq \mu(E) + 2g - 2$ or equivalently, $m \geq \mu(E) - \mu(F) + 2g - 2$.

(2) As in the proof of part (1), $h^1(Y, \text{Hom}(F(m), L^{-1} \otimes L'(m(n + 1)))) = 0$ if $dL^{-1} \otimes L'(m(n + 1)) \geq \mu(F) + m + 2g - 2$ i.e., $mn \geq d'/((n + 1) - d' + d + 2g - 2$.

(3) The vector bundle $\text{Hom}(F(m), L^{-1} \otimes L'(m(n + 1))) = F^* \otimes L^{-1} \otimes L'(mn)$ is stable [4, Lemma 2.6(2)]. Hence as in [13, Lemma 5.2], one sees that $\text{Hom}(F(m), L^{-1} \otimes L'(m(n + 1)))$ is generated by global sections if $-\mu(F) + mn + d' - d \geq 2g - 2 + (n + 1)(1 + \delta_Y)$ i.e. if $m \geq (1/n)((1 + \delta_Y)(n + 1) + 2g - 2 - d - d' + d'/((n + 1)).$ □
Lemma 7.10. Let $E$ and $F$ be as in Lemma 7.8 and assume that both $p^*(E)$ and $p^*(F)$ are stable vector bundles on $X$. Then:

1. $E^* \otimes F$ is semistable.
2. $H^0(Y, E^* \otimes F(m))$ generates $E^* \otimes F(m)$ for $m > m_4$ where $m_4 = n(n+1)(1+\delta_Y) + 2g - 2 + d/n - d'(n+1)$.

Proof. (1) If $p^*(E)$ and $p^*(F)$ are stable, then $p^*(E)^*$ is stable and $p^*(E) \otimes p^*(F)$ is semistable as $X$ is a smooth curve. Then $p^*(E^* \otimes F) = p^*(E)^* \otimes p^*(F)$ is semistable and therefore $E^* \otimes F$ is semistable [10, Proposition 3.6].

(2) By [13, Lemma 5.2], the conclusion of Part (2) holds if one has $\mu(E^* \otimes F(m)) > r(E^* \otimes F)(1+\delta_Y) + 2g - 2$ i.e. if $\mu(F) + m - \mu(E) > n(n+1)(1+\delta_Y) + 2g - 2$, hence the result. \hfill \Box

Assumptions Henceforth, we fix an $E \in U'_L(n,d)$ such that $p^*E$ is stable. We choose an $m > \max\{m_1, m_2, m_3, m_4\}$. We assume that $(n+1, d') = 1$. Then there is a Poincare bundle $U'$ on $U'_L(n+1, d') \times Y$. Let $p_1, p_2$ be the projections from $U'_L(n+1, d') \times Y$ to $U'_L(n+1, d')$ and $Y$ respectively. We denote by $\overline{U}'_L(n+1, d') \subset U_L(n+1, d')$ the open subvariety of $U_L(n+1, d')$ consisting of vector bundles $F$ such that $p^*F$ is stable.

7.4. The projective space $P$.

Define the projective space $P$ by

$$P := P(H^1(Y, \text{Hom}(L' \otimes L^{-1}(m(n+1)), E))).$$

Let $H$ denote the hyperplane bundle on $P$. Let $p_\mathbb{P}$ and $p_Y$ denote the projections from $P \times Y$ to $P$ and $Y$ respectively. On $P \times Y$, there is a family of vector bundles $W$ given by the exact sequence

$$0 \to p_Y^* E \otimes p_\mathbb{P}^* H \to W \to p_Y^* (L^{-1} \otimes L'(m(n+1))) \to 0.$$

Let $P^*$ denote the open subset of $P$ parametrising stable vector bundles. For $F \in \overline{U}'_L(n+1, d')$, $H^0(Y, \text{Hom}(E, F(m)))$ generates $\text{Hom}(E, F(m))$ at every point $t \in Y$ by Lemma 7.10. For every $t \in Y$, the set of homomorphisms in $\text{Hom}((E_t, F(m)_t))$ which are not injective is an irreducible set of codimension at least 2. The inverse image of this set, under the surjective evaluation map $H^0(Y, \text{Hom}(E, F(m))) \to \text{Hom}(E_t, F(m)_t)$ is the set $S_t$ of homomorphisms $E \to F(m)$ which are not injective at $t$. Hence there is a nonempty open subset of $\text{Hom}(E, F(m))$ consisting of injective homomorphisms. Hence $F(m)$ belongs to the family $W$ parametrised by $P^*$. In particular, $P^*$ is nonempty. By the universal property of moduli spaces, there is a morphism

$$\lambda : P^* \to U'_L(n+1, d') \ .$$

One has

$$0 \to (\lambda \times 1_Y)^* \mathcal{U} \cong W \otimes (p_Y)^* (\mathcal{O}_Y(-m)) \otimes p_\mathbb{P}^* (N)$$

for some line bundle $N$ on $P^*$ [23, Lemma 2.5]. The restriction of $H$ to $P^*$ will be denoted by $\mathcal{H}$ again.
7.5. The projective bundle $P_1$.

Define
\[ V_1 := (p_1)_*(\text{Hom}(U \otimes \mathcal{O}_Y(m), p_2^*(L^{-1} \otimes L'(m(n + 1))))) , \]
By Lemma 7.9(2), $V_1$ is a vector bundle on $U'_L(n + 1, d')$. Define $P_1 := P(V_1)$, a projective bundle on $U'_L(n + 1, d')$. Let $S_1 \subset P_1$ be the subset corresponding to surjections $F(m) \to L^{-1} \otimes L'(m(n + 1))$. Let $\pi_1 : S_1 \to U'_L(n + 1, d')$ be the projection. To give a morphism $\phi : \mathbb{P}^s \to S_1$ such that $\pi_1 \circ \phi = \lambda$, it suffices to give a line subbundle of $\lambda^*V_1$. By equation (7.2),
\[ \lambda^*V_1 \cong p_{p^*}(\text{Hom}(W, p_Y^*(L^{-1} \otimes L'(m(n + 1)))))) \otimes N^{-1}. \]
The exact sequence (7.1) gives a nowhere vanishing section of $p_{p^*}(\text{Hom}(W, p_Y^*(L^{-1} \otimes L'(m(n + 1))))))$ giving a trivial line subbundle of it. Hence $\lambda^*V_1$ contains a line subbundle isomorphic to $N^{-1}$. It follows that
\[ \phi^*\tau_1 = N \]
where $\tau_1$ is the relative hyperplane bundle on $P_1$.

7.6. The projective bundle $P_2$.

By Lemma 7.9(1),
\[ V_2 := (p_1)_*(\text{Hom}(p_2^*E, U \otimes p_2^*(\mathcal{O}_Y(m)))) \]
is a vector bundle on $U'_L(n + 1, d')$. Let $P_2 = \mathbb{P}(V_2)$. Let $S_2 \subset P_2$ be the subset corresponding to injections $E \to F(m)$. Let $\pi_2 : S_2 \to U'_L(n + 1, d')$ be the projection. We have
\[ \lambda^*V_2 \cong p_{p^*}(\text{Hom}(p_Y^*E, W)^\ast) \otimes N. \]
Hence $\lambda^*V_2$ has a line subbundle isomorphic to $H \otimes N$. This gives a morphism
\[ \psi : \mathbb{P}^s \to S_2 \]
such that $\pi_2 \circ \psi = \lambda$. Then one has
\[ \psi^*\tau_2 = N^{-1} \otimes H^{-1} \]
where $\tau_2$ is the relative hyperplane bundle on $P_2$.

**Lemma 7.11.** Assume that $g(X) \geq 2$ for $n \geq 3$ and $g(X) \geq 3$ for $n = 2$. Then the subset $D_2 = P_2 - S_2$ has codimension $\geq 2$ in $P_2$ or it is the union of an irreducible divisor and (possibly) a closed subset of codimension $\geq 2$.

**Proof.** For $F \in U'_L(n + 1, d')$, let $P_{2,F}$ be fibre of $P_2 \to U'_L(n + 1, d')$ and $D_{2,F}$ the fibre of $D_2 = P_2 - S_2$ over $F$. Let $d_F$ denote the constant dimension of $P_{2,F}$. Since $n + 1 \geq 3$ and $(n + 1, d') = 1$, codim$_{U'_L(n + 1, d')} (U'_L(n + 1, d') - \overline{U}^{ss}_L(n + 1, d')) \geq 2g(X) - 1$ by Theorem 7.6. It follows that codim$_{U'_L(n + 1, d')} (U'_L(n + 1, d') - \overline{U}^{ss}_L(n + 1, d')) \geq 3$ for $g(X) \geq 2$. Hence $pr^{-1}(U'_L(n + 1, d') - \overline{U}^{ss}_L(n + 1, d'))$ has codimension $\geq 3$ in $P_2$ and therefore its intersection with $D_2$ has codimension $\geq 2$ in $D_2$. For $F \in \overline{U}^{ss}_L(n + 1, d')$, by Lemma 7.10 Hom($E, F(m)$) generates Hom($E_t, F(m)_t$) for every $t \in Y$. Since the subset of Hom($E_t, F(m)_t$) corresponding to non-injective homomorphisms is irreducible subset of codimension $\geq 2$ (independent of $t$), it follows that the subset $D_{2,t}$ of Hom($E, F(m)$)
corresponding to homomorphisms which are not injective at $t$ is irreducible subset of codimension $\geq 2$. Hence $D_{2,F} = \bigcup_{t \in Y} D_{2,t}$ is an irreducible subset of codimension $\geq 1$. Since $U'_L(n + 1, d')$ is irreducible, so is its open subset $U'_{L'}(n + 1, d')$, hence $\bigcup_{F \in \mathcal{W}_{s}(n + 1, d')} D_{2,F}$ is an irreducible subset of $D_2$ of codimension $\geq 1$. This proves the lemma.

**Lemma 7.12.** The restriction map $\text{Pic } P \to \text{Pic } P^s$ is an isomorphism so that $\text{Pic } P^s \cong \mathbb{Z}$.

**Proof.** Since $P$ is smooth, the restriction map $\text{Pic } P \to \text{Pic } P^s$ is surjective. As $\text{Pic } P \cong \mathbb{Z}$, it follows that $\text{Pic } P^s$ is isomorphic to $\mathbb{Z}$ or $\mathbb{Z}/c\mathbb{Z}$ for some integer $c \geq 2$, in particular rank $\text{Pic } P^s \leq 1$. For $g(X) \geq 2$, we have $\text{Pic } U'_L(n + 1, d') \cong \mathbb{Z}$ [2, Theorem 1]. Hence rank $\text{Pic } P_2 = 2$. By Lemma 7.11, $\text{rank } \text{Pic } S_2 \geq \text{rank } \text{Pic } P_2 - 1 = 1$. Since $\psi : P^s \to P_2$ maps $P^s$ isomorphically onto $S_2$, now it follows that rank $\text{Pic } P^s = 1$ and $\text{Pic } P^s = \mathbb{Z}$ proving that the restriction map is an isomorphism. □

**Lemma 7.13.** The image of $\phi^* : \text{Pic } P_1 \to \text{Pic } P^s$ is the subgroup of $\mathbb{Z}$ generated by $n$ and $d$.

**Proof.** This can be proved similarly as [23, Lemma 3.6]. We briefly sketch it, see [23, Lemma 3.6] for details. Pic $P_1$ is generated by $\tau_1$ and $\pi^*\theta_{L'}$. Hence the required image is generated by $\phi^*\pi^*\theta_{L'} = \lambda^*\theta_{L'}$ and $\phi^*\tau_1 = N$. Let $T_{U'_L}$ denote the tangent bundle of $U'_L(n + 1, d')$. By [3, Theorem 4], $\det(T_{U'_L}) = \theta_{L'}^2$. By [23, Remark 2.11], $\lambda^*\theta_{L'} = H^{2r}$, where $H$ is the restriction of the hyperplane bundle on $P^s$ and $r = (n + 1)(mn - d) + nd'$. Since $(n + 1, d') = 1$, there exists integers $l, e$ with $ld' - e(n + 1) = 1$. Taking $N = H^{n'}$, one shows that $n' = l(d - mn) - en$, where $m$ is as chosen in assumptions. It follows that image of $\phi^* : \text{Pic } P_1 \to \text{Pic } P^s$ is the subgroup of $\mathbb{Z}$ generated by $r$ and $n'$, which is the same as the subgroup generated by $n$ and $d$. □

**Theorem 7.14.** Let $Y$ be an integral nodal curve of geometric genus $g(X) \geq 2$. If $n$ and $d$ are not coprime, then there does not exist a Poincaré family on any Zariski open subset of $U_L(n, d)$.

**Proof.** Since $U_L(n, d)$ is irreducible and $U'_L(n, d)$ is an open subset of $U_L(n, d)$, any Zariski open subset of $U_L(n, d)$ intersects $U'_L(n, d)$ in a nonempty Zariski open subset $V$. Hence it suffices to show that there does not exist a Poincaré family on any Zariski open subset $V$ of $U'_L(n, d)$.

Suppose that there exists a Poincaré family $E \to V \times Y$. Since $\text{Hom } (L^{-1} \otimes L'(m(n + 1))), E_v) = 0$ for all $v \in V$,

$$V := R^1(p_V)_*(\text{Hom}(p_Y^*(L^{-1} \otimes L'(m(n + 1))) \otimes E))$$

is a vector bundle on $V$. Let $P := P(V)$. The projective space $P$ defined in subsection 7.3.1 is the fibre of $P$ over $E \in V$. The projective bundle $P$ parametrises a family $W$ of vector bundles of rank $n + 1$ and degree $d'$. Let $P^s$ be the open subset of $P$ parametrising stable vector bundles. Clearly, $P^s \subset P^s$ and the morphism $\phi : P^s \to S_1$ extends to $\phi : P^s \to S_1$ which is an isomorphism onto an open subset of $S_1$.

Since $P = P(V)_E$ is a fibre of $P = P(V)$, the restriction map $\text{Pic } P(V) \to \text{Pic } P$ is a surjective (with the relative hyperplane bundle $H_V$ on $P$ mapping to $H$). Using
Lemma 7.12, one gets a surjection \( \text{Pic} \mathcal{P} \rightarrow \text{Pic} \mathcal{P}^s \) which factors through a surjection \( \text{Pic} \mathcal{P}^s \rightarrow \text{Pic} \mathcal{P}^s \).

Since \( \mathcal{P}^s \) is an open subset of \( S_1 \) and \( S_1 \) an open subset of \( P_1 \), one has a composite of surjections

\[
\text{Pic} P_1 \rightarrow \text{Pic} S_1 \rightarrow \text{Pic} \mathcal{P}^s \rightarrow \text{Pic} \mathcal{P}^s = \mathbb{Z}.
\]

By Lemma 7.13, this composite is a surjection if and only if \( n \) and \( d \) are coprime. Hence the family \( \mathcal{E} \) on \( V \) exists if and only if \((n,d) = 1\). \( \square \)

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