DETERMINATION OF VACUUM SPACE-TIMES FROM THE
EINSTEIN-MAXWELL EQUATIONS

MATTI LASSAS, GUNTHER UHLMANN, AND YIRAN WANG

Abstract. We study inverse problems for the Einstein-Maxwell equations. We prove that it is
possible to generate gravitational waves from the nonlinear interactions of electromagnetic waves.
By sending electromagnetic waves from a neighborhood of a freely falling observer and taking
measurements of the gravitational perturbations in the same neighborhood, one can determine the
vacuum space-time structure up to diffeomorphisms in the largest region where these waves can
travel to from the observer and return.

1. Introduction

This paper continues the study of inverse problems with sources for non-linear hyperbolic equa-
tions in 4 dimensional space-times initiated in [21], where Einstein equations are coupled with
matter fields. It is shown in [21] that one can recover the topology, differentiable structure and
conformal class of the metric in a larger set than where the observations are made. The article
[24] studies a general class of semilinear wave equations extending the previous work [20] that con-
sidered quadratic non-linearities. See also [34] for equations with quadratic derivative nonlinear
terms. In this paper we consider Einstein field equations in vacuum coupled with Maxwell equa-
tions. We show that just using electromagnetic sources we can generate gravitational waves and
use them to determine the metric, topology and differentiable structure in a larger set than where
the observations are made. The method of proof uses asymptotic expansions and the precise study
of the singularities of each term in the expansion. We use as in [21, 20] four plane waves interacting
to create new point singularities and use these to determine the earliest light observation set. Then
we appeal to the geometric result in [20] that states that from the earliest light observation set we
can determine the topology, differentiable structure and conformal class of the metric in a larger
set than where the observations are made. The asymptotics needed in the current paper are more
subtle than the previously mentioned works since the highest order term vanishes and one needs
to look at the lower order terms carefully. As in the previously mentioned papers we remark that
the linearized inverse problem is not known to be solvable. We use the non-linear interaction of
waves in a significant way to create new point sources.

In the following, we first introduce the mathematical model, then we formulate the inverse
problem and state the main result.

1.1. The Einstein-Maxwell equations. Let M be a 4 dimensional smooth manifold and g be
a Lorentzian metric on M satisfying the vacuum Einstein equations

\[ \text{Ein}(g) = 0. \]

Here Ein(g) denotes the Einstein tensor given by

\[ \text{Ein}(g) = \text{Ric}(g) - \frac{1}{2} R(g) g \]
where $\text{Ric}(g)$ denotes the Ricci curvature tensor and $R(g)$ the scalar curvature. In particular, $(M,g)$ is called a vacuum space-time. The propagation of electromagnetic waves on $(M,g)$ are governed by the Maxwell equations. In the covariant formulation, the electromagnetic field can be described as a two form $F$ and the (four) electric current $J$ is a vector field on $M$. The Maxwell equations for $F$ on $(M,g)$ with source $J$ are given by

$$dF = 0, \quad \delta_g F = J^b,$$

where the codifferential $\delta_g$ is the dual of the exterior differential $d$ with respect to $g$ and $J^b$ denotes the one form obtained from $J$ by lowering the index using $g$. In simply connected domains, one can find a one form $\phi$ by Poincaré lemma such that $F = d\phi$, and $\phi$ is called the electromagnetic potential. The Maxwell equations can be written as

$$\delta_g d\phi = J^b,$$

while the first set of Maxwell equations is automatically satisfied as $d^2 = 0$. The source $J$ is subject to the conservation law $\text{div}_g J = 0$, where $\text{div}_g$ denotes the divergence operator.

To understand the gravitational effects of incidenting electromagnetic waves on $(M,g)$ with electric current $J$ as a source, we need to couple the Einstein and Maxwell equations in a physically meaningful way. There are many models for the Einstein-Maxwell equations in the literature, for example equations with no electric current [7, Section 6.10] and [29], and equations for charged dust [7] and [33, Chap. 18]. In this work, we derive the equations with sources in Section 2.2 by ignoring the contribution from the mass of the dust in the charged dust model. Roughly speaking, the total Lagrangian consists of the Einstein-Hilbert Lagrangian, the electromagnetic Lagrangian and the interaction term. The Einstein-Maxwell equations are the Euler-Lagrange equations of the total Lagrangian and they can be written as

$$\text{Ein}(g) = T_{\text{sour}},$$

$$\delta_g d\phi = J^b,$$

where $T_{\text{sour}}$ is the source stress-energy tensor which can be written as

$$T_{\text{sour}} = T_{\text{em}} + T_{\text{inter}},$$

$$(1.2) \quad T_{\text{em},\alpha\beta} = F^\lambda_\alpha F^\lambda_\beta - \frac{1}{4} g_{\alpha\beta} F^{\lambda\mu} F^\mu_\lambda, \quad T_{\text{inter}} = -\frac{1}{2} (J^\mu \phi_\mu) g.$$
sources, and construct certain component of the source \( J \) given the others. We remark that a similar problem for the Einstein equations with matter fields is addressed by Girbau and Bruna [13] for the Cauchy problem.

Now we formulate the local problem of Einstein-Maxwell equations with sources to be considered in this work, starting with some notions of Lorentzian geometry. For \( p, q \) on a Lorentzian manifold \((M, g)\), we denote by \( p \ll q \) (\( p < q \)) if \( p \neq q \) and there is a future pointing time-like (causal) curve from \( p \) to \( q \). We denote by \( p \leq q \) if \( p = q \) or \( p < q \). The chronological (causal) future of \( p \in M \) is the set \( I_g^+(p) = \{ q \in M : p \ll q \} \) (\( J_g^+(p) = \{ q \in M : q \leq p \} \)). Similarly, we can define the chronological past and causal past, which are denoted by \( I_g^-(p) \) and \( J_g^-(p) \) respectively.

For any set \( A \subset M \), we denote the causal future by \( J_g^\pm(A) = \bigcup_{p \in A} J_g^\pm(p) \). Also, we denote \( J_g(p, q) = J_g^+(p) \cap J_g^-(q) \) and \( I_g(p, q) = I_g^+(p) \cap I_g^-(q) \).

We consider globally hyperbolic manifold \((M, g)\), which can be identified with the product manifold \( \mathbb{R} \times M \) with \( M \) a 3-dimensional manifold and metric \( g = -\beta(t, y)dt^2 + \kappa(t, y) \) where \( \beta > 0 \) is a smooth function and \( \kappa \) is a family of Riemannian metrics on \( M \) smooth in \( t \), see Section 2.1. For \( T_0 \in \mathbb{R} \), we denote \( M(T_0) = (-\infty, T_0) \times M \). For a vector field \( J \) on \( M \), we shall write \( J = (J^0, \bar{J}) \) where \( \bar{J} \in C^\infty(\mathbb{R}_t; TM) \). In particular, \( \bar{J} \) is a family of smooth vector fields on \( M \) and smooth in \( t \in \mathbb{R} \).

We consider \((M, \bar{g})\) as a background space-time where \( \bar{g} \) satisfies the vacuum Einstein equations. Let \( \bar{\mu}(t) \subset M \) be a time-like geodesic where \( t \in [-1, 1] \). In the language of general relativity, \( \bar{\mu} \) represents a freely falling observer. Let \( V \subset M \) be an open relatively compact neighborhood of \( \bar{\mu}(\bar{s}_-, \bar{s}_+) \) where \( -1 < \bar{s}_- < \bar{s}_+ < 1 \). Take \( T_0 > 0 \) such that \( V \subset M(T_0) \). For \( \bar{J} \) compactly supported in \( V \), the Einstein-Maxwell equations with sources we consider are following equations for the field \((g, \phi, J^0)\)

\[
\begin{cases}
\text{Ein}(g) = T_{\text{sour}}(g, \phi, J) \\
\delta_g d\phi = J^0 \\
div_g J = 0
\end{cases}
\text{in } M(T_0),
\]

\[ g = \bar{g}, \quad \phi = 0, \quad J^0 = 0, \quad \text{in } M(T_0) \backslash J_g^+(\text{supp}(\bar{J})). \]

As is well-known, the solution of the Einstein equation is unique only up to diffeomorphisms and the electric potential is unique up to a gauge choice. Hence one should think of \( g, \phi \) and \( J \) in the above equations as representatives of their equivalence classes.

In Section 2 we study the well-posedness of (1.3) in wave and Lorentz gauge. Moreover, in Section 3 we consider the gravitational perturbations in the weak field approximation and show that small electric current \( \bar{J} \) will generate non-trivial gravitational waves. We remark that the coupling of progressive electromagnetic waves and gravitational waves is known for the electro-vacuum space-time (with no source) using the WKB method, see for example Choquet-Bruhat [7, Chap. XI]. Also, we’d like to point to the interesting work by Füzfa [12], who demonstrated numerically that strong electromagnetic fields (current loops and solenoid) can generate gravitational field curving the space-time.

1.2. The inverse problem and the main results. Assume that we are given \( V \) a neighborhood of an observer as a differentiable manifold. Suppose we can control the electric current \( \bar{J} \) which are compactly supported in \( V \) and we take the measurements of the gravitational perturbations \( g \) in \( V \). Can we determine the background metric \( \bar{g} \) on \( I^{g}_{\bar{g}}(p_-, p_+) \) using these information? See Fig. 1. This problem becomes particularly interesting thanks to the breakthrough discovery by the LIGO project [22, 23], i.e. direct detection of gravitational waves. We remark that the region \( I^{g}_{\bar{g}}(p_-, p_+) \) is the maximal region where the electromagnetic waves can travel to from \( V \) and from
where the waves can travel back to $V$. Therefore, it is the maximal region that we can expect to determine the space-time structure in this setup.

To formulate the inverse problem precisely, one need to keep in mind that the solutions to the Einstein-Maxwell equations are unique up to diffeomorphisms of certain regularity. Here we follow [21] to formulate the observations in Fermi coordinates associated with freely falling observers $\hat{\mu}([-1, 1])$. Assume that $p = \hat{\mu}(-1) \in \{0\} \times M$. We shall take some basis $X_j(p) \in T_p M$, $j = 0, 1, 2, 3$ such that $X_0(p) = \dot{\hat{\mu}}(-1)$ is time-like. We let $Z_j, j = 0, 1, 2, 3$ be a frame along $\hat{\mu}$ obtained from parallel translating $X_j(p)$. Then we consider the following Fermi coordinates:

$$
\Phi_{g,p}(z^0, z^1, z^2, z^3) = \exp_{g,\hat{\mu}(z^0)}(\sum_{j=1}^3 z^j Z_j), \quad \Phi_{g,p} : \tilde{V}_p \to V^g \subset V,
$$

where $\tilde{V}_p = (-1, 1) \times B_p$ is a subset of $\mathbb{R}^4$, $B_p$ is a small neighborhood of $0 \in \mathbb{R}^3$ depending on $p$ (or $\hat{\mu}$) so that the exponential maps are well-defined and $V^g = \Phi_{g,p}(\tilde{V}_p)$. In particular, $(\tilde{V}_p, \Phi_{g,p})$ gives a coordinate system of $V^g$, the Fermi coordinates. We formulate the problem in this coordinate.

For the inverse problem, it is natural to think of the four current $J$ as the source and define the source-to-solution map of the Einstein-Maxwell system as $J \to (g, \phi)$. Moreover, it is convenient to take the graph of the source-to-solution map as the data set. On the manifold $M$, we introduce a complete Riemannian metric $\tilde{g}^+$ so we can define seminorms in $C^4(M)$. For $\delta > 0$ small, we define the data set as

$$
\mathcal{D}(\delta) = \{(\Phi_{g,p}^*, g) : \|J\|_{C^4(M)} < \delta, \supp(J) \subset V^g \}
$$

Roughly speaking, the data set $\mathcal{D}(\delta)$ corresponds to the following measurement settings: We control the electric current $\mathbf{J}$ that is supported in the neighborhood $V^g$ of the geodesic $\hat{\mu}$. This gives rise
to the moving electric charges \( J^0 \). Together \( J = (J^0, \mathcal{J}) \) cause a perturbation in the electric field \( \phi \) and the gravitational field \( g \) that start to propagate in space time. This perturbed fields can be considered as a non-linear wave that scatters from non-homogenous background metric \( \hat{g} \). Some of the scattered waves return back to \( V^g \) where we observe the fields \((g, \phi)\). In the data set \( D(\delta) \) the controlled source \( \mathcal{J} \) and the other fields \( J^0, g, \phi \) are given in the Fermi coordinates \( \Phi = (g, \phi) \) associated to the freely falling observer (time-like geodesic) \( \mu_q \). Our main result is the unique determination of two vacuum space-times up to isometries.

**Theorem 1.1.** Let \( M \) be a 4 dimensional simply connected smooth manifold and \( \hat{g}^{(i)}, i = 1, 2 \) be two globally hyperbolic Lorentzian metrics on \( M \) satisfying the vacuum Einstein equations. Let \( \tilde{\mu}^{(i)}([-1, 1]) \) be time like geodesics with respect to \( \hat{g}^{(i)} \). Assume that \( p^{(i)} = \tilde{\mu}^{(i)}(-1) \in \{0\} \times M \) and take \( p^{(i)}_{\pm} = \tilde{\mu}^{(i)}(s_{\pm}) \) with \(-1 < s_- < s_+ < 1\). Suppose that \( V^{(i)} \) are neighborhoods of \( \tilde{\mu}([s_-, s_+]) \) and \( V^{(i)} \subset M(T_0) \backslash M(0) \) for \( T_0 > 0 \). Consider the Einstein-Maxwell systems

\[
\begin{align*}
\text{Ein}(g^{(i)}) &= T_{\text{sour}}(g^{(i)}, \phi^{(i)}, J^{(i)}) \quad \text{in } M(T_0), \\
\delta_{g^{(i)}}d\phi^{(i)} &= J^{(i), b} \quad \text{in } M(T_0),
\end{align*}
\]

which is well-posed on data set \( D^{(i)}(\delta), i = 1, 2 \) defined as \( \{1.5\} \) for some small \( \delta \). If

\[
\mathcal{D}^{(1)}(\delta) = \mathcal{D}^{(2)}(\delta),
\]

then there is a diffeomorphism \( \Psi : I_{\hat{g}^{(1)}}(p^{(1)}_-, p^{(1)}_+) \rightarrow I_{\hat{g}^{(2)}}(p^{(2)}_-, p^{(2)}_+) \) such that \( \Psi^* \hat{g}^{(2)} = \hat{g}^{(1)} \) in \( I_{\hat{g}^{(1)}}(p^{(1)}_-, p^{(1)}_+) \).

A key ingredient in the proof of Theorem 1.1 is to produce artificial point sources in \( I_{\hat{g}(p_-, p_+)} \) from the nonlinear interaction of linear waves. For the Einstein-Maxwell equations we study in this paper, the linear waves are electromagnetic waves caused by small perturbations of the electromagnetic sources. In Section 5 we show that the nonlinear interaction of four electromagnetic waves could produce a point source of gravitational waves and this helps us to solve the inverse problem, see Fig. 2.

**Figure 2.** Interaction of singularities. Four electric current sources are placed on submanifolds \( Y_i \), which produce four electromagnetic waves (distorted plane waves) propagating along geodesics from \( Y_i \). Their nonlinear interactions at \( q_0 \) produce new gravitational waves propagating on the light-cone \( L_{q_0}^+ = \text{exp}_{q_0}(L_{q_0}^+ M) \).
We remark that the propagation and interaction of singularities for nonlinear hyperbolic equations was actively studied in the 80’s and 90’s, mainly for 2 + 1 dimension by Bony [6], Melrose-Ritter [27], Rauch-Reed [31], etc, see also Beals [2] for an overview. The Einstein-Maxwell equations reduces to a quasilinear hyperbolic system in 3 + 1 dimensions in wave and Lorentz gauge, for which the methods in the mentioned literatures do not apply directly. We use the method developed in [20] and [24] to analyze the singularities in the asymptotic expansion of solutions with sources depending on some small parameters. We point out that we only treat weak conormal type of singularities. For the propagation and interaction of strong gravitational waves, for example impulse waves, see [25, 26].

The paper is organized as following. In Section 2 we formulate the source problem for the Einstein-Maxwell equations using Lagrangians. Then we discuss the well-posedness of the Einstein-Maxwell equations, first for the reduced equations in wave and Lorentz gauge then the full equations. In Section 3, we consider the linearization of the Einstein-Maxwell equations and construct distorted plane waves. Before solving the inverse problem, we show in Section 4 the generation of gravitational waves using electromagnetic sources in wave gauge. In Section 5 we study the leading singularities generated from nonlinear interaction of linear waves and prove that they are not always vanishing. Finally, we solve the inverse problem in Section 6.

Acknowledgments

The authors would like to thank Peter Hintz for valuable comments on a preliminary version of the paper and Judith Arms for helpful discussions. Matti Lassas has been supported by Academy of Finland. Gunther Uhlmann is partially supported by NSF, a Si-Yuan Professorship of HKUST and FiDiPro Professorship of Academy of Finland.

2. Locally well-posedness of the Einstein-Maxwell equations

2.1. Notations. We review some facts from Lorentzian geometry and set up notations. Some of them are briefly discussed in the introduction and we collect them here for convenience. Our main references are [3], [7], [21].

Assume that \((M, g)\) is a 4 dimensional Lorentzian manifold which is time oriented and globally hyperbolic. From [4], we know that \((M, g)\) is globally hyperbolic if there is no closed causal paths in \(M\) and for any \(p, q \in M\) and \(p < q\), the set \(J_g(p, q)\) is compact. We take the signature of the metric \(g\) as \((-tt, +xx, +xx, +xx)\). It is proved in [5] that \((M, g)\) is isometric to a product manifold \(\mel{R} \times M\) with \(g = -\beta(t, y)dt^2 + \kappa(t, y)\), where \(M\) is a 3 dimensional manifold, \(\beta: \mel{R} \times M \to \mel{R}+\) is smooth and \(\kappa\) is a Riemannian metric on \(M\) and smooth in \(t\). In this work, we identify \((M, g)\) with this isometric image. We shall use \(x = (t, y) = (x^0, x^1, x^2, x^3)\) as the local coordinates on \(M\). For \(T_0 \in \mel{R}\), we set \(M(T_0) = (-\infty, T_0) \times M\). It is worth noting that for each \(t \in \mel{R}\), the submanifold \(\{t\} \times M\) is a Cauchy surface, i.e. every inextendible causal curve intersects the submanifold only once, see for example [3, Page 65].

For \(p \in M\), we denote the collection of light-like vectors at \(p\) by \(L_p^M = \{\theta \in T_pM \setminus \{0\} : g(\theta, \theta) = 0\}\) and the bundle by \(L^M = \bigsqcup_{p \in M} L_p^M\). The future (past) light-like vectors are denoted by \(L^+_p(M) = L^+_p)\), and the bundle \(L^{\pm} M = \bigcup_{p \in M} L^{\pm}_p M\). Also, the set of light-like covectors at \(p\) is denoted by \(L_p^*M\) and the bundles \(L^*M, L^{\pm, *}(M)\) are defined similarly. Since the metric \(g\) is nondegenerate, there is a natural isomorphism \(\iota_p: T_pM \to T^*_p M\). With this isomorphism, we sometimes use vectors and co-vectors interchangeably. Let \(\exp_p: T_pM \to M\) be the exponential map. The geodesic from \(p\) with initial direction \(\theta\) is denoted by \(\gamma_{p, \theta}(t) = \exp_p(t\theta), t \geq 0\). The
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forward light-cone at \( p \in M \)

\[ \mathcal{L}^+_p = \{ \gamma_{p, \theta}(t) : \theta \in L^+_p M, t > 0 \} \]

is a submanifold of \( M \) and we notice that \( p \notin L^+_p M \).

We recall some useful formulas. We use the standard Einstein summation notation, i.e. summation is over repeated indices. In local coordinates, the Christoffel symbols are given by

\[ \Gamma^\mu_{\alpha \beta} = \frac{1}{2} g^{\mu \lambda} \left( \partial_\beta g_{\lambda \alpha} + \partial_\alpha g_{\lambda \beta} - \partial_\lambda g_{\alpha \beta} \right), \quad \partial_i = \frac{\partial}{\partial x^i}, \quad i = 0, 1, 2, 3. \]

The contracted Christoffel symbol is

\[ \Gamma^\mu = g^{\alpha \beta} \Gamma^\mu_{\alpha \beta} = -\frac{1}{2} \det g \partial_\alpha \left( \frac{1}{2} g^{\beta \alpha} \partial_\beta \right), \]

see for example [10, Section 3] and [21]. The Laplace-Beltrami operator defined via \( \Box_g = \delta_g d + d \delta_g \) can be written as

\[ (2.1) \quad \Box_g = -\frac{1}{2} \det g \partial_\beta \left( \frac{1}{2} g^{\beta \alpha} \partial_\alpha \right) = -g^{\alpha \beta} \partial_\alpha \partial_\beta + \Gamma^\alpha \partial_\alpha. \]

Let \( J \) be a smooth vector field. The divergence of \( J \) is given in local coordinates by

\[ (2.2) \quad \text{div}_J = \sum_{i=0}^3 \frac{\partial}{\partial x^i} \left( (-\det g)^{\frac{1}{2}} J^i \right), \]

see for example [32] and [29, p. 222]. Let \( \text{Ric}(g) \) denote the Ricci curvature tensor. For any covariant 2-tensor \( T \), recall that the trace of \( T \) with respect to \( g \) is defined as \( \text{Tr}_g(T) = g^{\alpha \beta} T_{\alpha \beta} \).

In particular, the scalar curvature \( R(g) = \text{Tr}_g(\text{Ric}(g)) \).

Besides the Lorentzian metric, we will take a complete Riemannian metric \( \tilde{g}^+ \) on \( M \), whose existence is guaranteed by [30]. With this metric, we can introduce distances on \( M, TM, \) and Sobolev spaces on \( M \).

2.2. The Lagrangian formulation. We derive the coupled Einstein-Maxwell equations from the Lagrangian point of view. One can compare this with the equations for the charged dust, see Taylor [33, Chap. 18]. Recall that the (vacuum) Einstein equations are the Euler-Lagrange equations of the Einstein-Hilbert Lagrangian:

\[ \mathcal{L}_{\text{grav}}(g) = \int_M R(g) dg, \]

see for example [7, Theorem 7.1, Chap. III]. Here \( dg \) denotes the volume element and in local coordinates, \( dg = |\det g|^{\frac{1}{2}} dx \). Next, the electromagnetic field can be described as a two form \( F \) on \( (M, g) \) i.e. \( F = F_{\alpha \beta} dx^\alpha \wedge dx^\beta \). On a simply connected domain, we can write \( F = d\phi \) or equivalently \( F_{\alpha \beta} = \partial_\alpha \phi_\beta - \partial_\beta \phi_\alpha \) with a one form \( \phi \), the electromagnetic potential. The Lagrangian of the electromagnetic field is given by

\[ \mathcal{L}_{\text{em}} = \frac{1}{4} \int_M F^{\alpha \beta} F_{\alpha \beta} dg. \]

In case there are electric current \( J \) a vector field, we should add the interaction Lagrangian

\[ \mathcal{L}_{\text{inter}} = -\int_M J^\mu \phi_\mu dg, \]

so the source Lagrangian is \( \mathcal{L}_{\text{sour}} = \mathcal{L}_{\text{em}} + \mathcal{L}_{\text{inter}} \). Indeed, the Maxwell equations (1.1) with source \( J \) are the Euler-Lagrange equations of this Lagrangian.
Now we consider the coupled Lagrangian of gravitational field and electromagnetic field:

\[ \mathcal{L}_{total} = \mathcal{L}_{grav} + \mathcal{L}_{sour}. \]

According to [7, Theorem 7.3, Chap. III], we can write the first variation of \( \mathcal{L}_{sour} \) as

\[ \delta \mathcal{L}_{sour} = \int_{M} \Psi(g, \phi) \delta \phi dg - \int_{M} T_{sour} \delta g dg. \]

(Note here \( \delta \) denotes the variation and is different from the codifferential \( \delta g \).) Then we get the field equations i.e. Maxwell equations

\[ \delta g F = J^p, \]

from \( \Psi(g, \phi) = 0 \) and the Einstein equations

\[ \text{Ein}(g) = T_{sour} \]

if the Maxwell equations are satisfied. More importantly, since the Lagrangian \( \mathcal{L}_{sour} \) is invariant under diffeomorphisms, Theorem 7.3 of [7, Chap. III] guarantees that the stress-energy tensor \( T_{sour} \) satisfies the conservation law \( \text{div}_g T_{sour} = 0 \)

if the Maxwell equations are satisfied. It also follows from the Maxwell equations that the electric current satisfies the conservation law \( \text{div}_g J = 0 \). A more physical elaboration of electrodynamics on curved space-time, especially the validation of the conservation laws, can be found in [29, Section 22.4].

We find \( T_{sour} \) explicitly. We can write

\[ T_{sour} = T_{em} + T_{inter}, \]

where

\[ T_{em, \alpha \beta} = F^\lambda_\alpha F_{\beta \lambda} - \frac{1}{4} g_{\alpha \beta} F^{\lambda \mu} F_{\lambda \mu} \]

is the stress-energy tensor of the electromagnetic field \( F \). To find the stress-energy tensor \( T_{inter} \) of \( \mathcal{L}_{inter} \), we consider the first variation

\[ \delta \mathcal{L}_{inter} = - \int_{M} J^\mu \phi_\mu \delta (dg) = \int_{M} \frac{1}{2} J^\mu \phi_\mu g(\delta g) dg, \]

since \( \delta (dg) = -\frac{1}{2} g_{\alpha \beta} \delta g^{\alpha \beta} dg, \) see formula (7.4) of [7, Chap. III]. So the stress-energy tensor \( T_{inter} = -\frac{1}{2} (J^\mu \phi_\mu) g \). Finally, we can write the coupled Einstein-Maxwell equations with sources as

\[ \text{Ein}(g) = T_{em} - \frac{1}{2} (J^\mu \phi_\mu) g, \]

\[ \delta g d\phi = J^p. \]

We remark that in absence of the electric current i.e. \( J = 0 \), these equations reduce to the sourceless Einstein-Maxwell equations in literatures, see e.g. [7, 29].

For our analysis, it is convenient to use an equivalent form of the Einstein equations. We take the trace of the Einstein equation in (2.4) to obtain

\[ \text{Tr}_g(\text{Ein}(g)) = -R(g) = -2J^\mu \phi_\mu, \]

because \( T_{em} \) is trace-free in dimension 4. Hence the Einstein equations in (2.4) is equivalent to

\[ \text{Ric}(g) = T_{em} + \frac{1}{2} (J^\mu \phi_\mu) g. \]

2.3. The reduced Einstein-Maxwell equations. The Einstein equations form a second order system of PDEs. In harmonic gauge i.e. local coordinates such that \( \Gamma^\alpha = 0, \alpha = 0, 1, 2, 3, \) the Einstein equation (2.5) becomes a second order quasi-linear hyperbolic system for \( g \). Actually, in local coordinates, the Ricci tensor can be written as

\[ \text{Ric}_{\mu \nu}(g) = \text{Ric}^{(h)}_{\mu \nu}(g) + \frac{1}{2} (g_{\mu q} \partial_\nu \Gamma^q + g_{\nu q} \partial_\mu \Gamma^q), \]

\( \mu, \nu = 0, 1, 2, 3, \)
where $\text{Ric}_{\mu\nu}^{(h)}(g) = -\frac{1}{2}g^{pq}\partial_{p}\partial_{q}g_{\mu\nu} + P_{\mu\nu}$ is the harmonic Ricci tensor and

$$P_{\mu\nu} = g^{ab}g_{ba}\Gamma_{\mu\nu}^{\alpha}\Gamma_{\alpha}^{s} + \frac{1}{2}(\partial_{a}g_{\mu\nu}\Gamma^{a} + g_{\alpha\beta}\Gamma_{\alpha\beta}^{a}g^{bd}\partial_{d}g_{\mu\nu} + g_{\mu\nu}\Gamma_{ab}^{a}g^{bd}\partial_{d}g_{\mu\nu}),$$

see for example [21, Appendix A] and [7, Section VI.7]. Note that $P_{\mu\nu}$ is a polynomial of $g_{pq}, g^{pq}$ and the first derivatives of $g_{pq}$. When $\Gamma^{a} = 0$, $\text{Ric}(g)$ is reduced to $\text{Ric}_{\mu\nu}^{(h)}(g)$.

This reduction can be made coordinate invariant using wave map. Let $\hat{g}$ be a smooth Lorentzian metric and $g'$ be a Lorentzian metric close to $\hat{g}$ in $C^{n}(M)$ with $m$ to be specified later. Consider the wave map $f : (M, g') \rightarrow (M, \hat{g})$ satisfying the following equation

$$\Box_{\hat{g}, \hat{g}}f = 0$$

in $M(T_{1})$, $f = \text{Id}$ in $M(0)$,

where $T_{1} > T_{0}$ and $\Box_{\hat{g}, \hat{g}}$ is the wave map operator defined independent of local coordinates, see [7, Section VI.7] and [21, Appendix A] for more details. In local coordinates $x = (x^{0}, x^{1}, x^{2}, x^{3})$ for $(M, g')$ and $y = (y^{0}, y^{1}, y^{2}, y^{3})$ for $(M, \hat{g})$ so that $y^{A} = f^{A}(x)$, $A = 0, 1, 2, 3$, we have

$$\Box_{\hat{g}, \hat{g}}f = \left(\frac{\partial^{2}}{\partial x^{j}\partial x^{k}}f^{A}(x) - \Gamma_{jk}^{n}(x)\frac{\partial}{\partial x^{n}}f^{A}(x) + \hat{\Gamma}_{BC}^{A}(f(x))\frac{\partial}{\partial x^{j}}f^{B}(x)\frac{\partial}{\partial x^{k}}f^{C}(x),
$$

where $\Gamma$, $\hat{\Gamma}$ denote the Christoffel symbols on $(M, g')$ and $(M, \hat{g})$ respectively. In particular, (2.7) is a second order semilinear hyperbolic system for $f$. The well-posedness of (2.7) has been studied in [21, Appendix A.3], see also [7, Appendix III]. Actually, for $g'$ sufficiently close to $\hat{g}$ in $C^{n}(M(T_{1})), m \geq 5$, the wave map equation (2.7) has a unique solution

$$f \in C^{0}([0, T_{1}]; H^{m-1}(M)) \cap C^{1}([0, T_{1}]; H^{m-2}(M)).$$

Here we take $T_{1} > T_{0}$ such that $M(T_{0}) \subset f(M(T_{1}))$ and $g = f_{*}g'$ is defined on $M(T_{0})$. Let

$$E^{s}(M(T_{0})) = \cap_{p=0}^{s} C^{p}([0, T_{0}]; H^{s-p}(M)).$$

We also abbreviate the notation as $E^{s}$ when the manifold is clear. If $m$ is even, the wave map $f \in E^{m-1}$ and $f$ depends continuously on $g' \in C^{m}(M(T_{1})).$ In particular, for $s \geq 4$, we notice that $E^{s} \subset C^{p}([0, T_{0}] \times M)$ for $0 \leq p < s - 2$. Therefore $f \in C^{m-3}(M(T_{0}))$ if $m \geq 4$ is even. We remark that these regularity results may not be optimal but are sufficient for our analysis.

Suppose $f$ is a wave map with respect to $(g', \hat{g})$ and set $g = f_{*}g'$, then the identity map $\text{Id}$ is a wave map with respect to $(g, \hat{g})$ and the wave map equation for $\text{Id}$ is equivalent to the harmonicity condition $\Gamma^{n} = \hat{\Gamma}^{n}, n = 0, 1, 2, 3$, where $\hat{\Gamma}^{n} = g^{\alpha\beta}\hat{\Gamma}_{\alpha\beta}^{n}$. Let $\hat{F}^{n} = \Gamma^{n} - \hat{\Gamma}^{n}$, the Ricci curvature tensor can be written as

$$\text{Ric}_{pq}(g) = (\text{Ric}_{\hat{g}}(g))_{pq} + \frac{1}{2}(g_{pm}\hat{\nabla}_{q}\hat{F}^{n} + g_{qn}\hat{\nabla}_{p}\hat{F}^{n}),$$

where $\text{Ric}_{\hat{g}}(g)$ is called the reduced Ricci curvature tensor. Therefore, in wave gauge where $\hat{F}^{n}$ vanishes, the Ricci tensor is the reduced Ricci tensor. Using (2.6) and the harmonicity condition, we obtain that

$$R_{\mu\nu}(g) = \frac{1}{2}(\partial_{a}g_{\mu\nu}\Gamma^{a} + g_{\alpha\beta}\Gamma_{\alpha\beta}^{a}g^{bd}\partial_{d}g_{\mu\nu} + g_{\mu\nu}\Gamma_{ab}^{a}g^{bd}\partial_{d}g_{\mu\nu})
$$

$$+ \frac{1}{2}(g_{pq}\partial_{p}\hat{\Gamma}^{q} + g_{qp}\partial_{q}\hat{\Gamma}^{p}).$$

(2.9)
Suppose that \( g' \) is a solution to the Einstein equation \( \text{Ric}(g') = T'_{em} + \frac{1}{2}(J^\mu \phi_\mu)g' \) in \( M(T_0) \) where \( T'_{em} \) is defined using \( g' \). Then \( g = f^*g' \) satisfies the reduced Einstein equations

\[
\text{Ric}_g(g) = T_{em}(g) + \frac{1}{2}(J^\mu \phi_\mu)g
\]

in wave gauge, where \( T_{em} \) is defined with respect to \( g \).

Next we consider the Maxwell equations \( \delta\gamma d\phi = \mathcal{J}^\delta \) for electromagnetic potential \( \phi \). We impose the Lorentz gauge condition \( \delta\gamma \phi = 0 \) to fix the gauge choice of \( \phi \), see for example Section 10.2 of [7] Chap. VI. Then we have \( \Box g\phi = \delta\gamma d\phi + d\delta\gamma \phi = \mathcal{J}^\delta \). So we can write the Maxwell equation as

\[
\Box g\phi = -g^{\alpha\beta}\partial_\alpha \partial_\beta \phi + \hat{\Gamma}^\mu \partial_\mu \phi_\beta = \mathcal{J}_\beta, \quad \beta = 0, 1, 2, 3,
\]

where we have used the expression (2.1) and the harmonicity condition \( \Gamma^\mu = \hat{\Gamma}^\mu \).

Now we formulate the reduced Einstein-Maxwell equations in wave and Lorentz gauge. We consider small perturbations near a background field. Suppose that \( (\hat{g}, \hat{\phi}) \) is the background field where \( \hat{\phi} = 0 \) and \( \hat{g} \) satisfies \( \text{Ric}(\hat{g}) = 0 \). Let \( \bar{J} \in C^\infty(\mathbb{R}; TM) \) with \( \text{supp} (\bar{J}) \subset V \). In wave and Lorentz gauge, the Einstein-Maxwell equations (1.3) with unknowns \( (g, \phi, \mathcal{J}^\delta) \) are reduced to

\[
\begin{align*}
\text{Ric}_g(g) = T_{em}(g) + \frac{1}{2}(J^\mu \phi_\mu)g \\
\Box g\phi = \mathcal{J}^\delta \\
\text{div}_g \mathcal{J} = 0
\end{align*}
\]

(2.12)

Here we denote by \( T_{em}(\phi) \) the stress-energy tensor associated with \( F = d\phi \). Note that \( T_{em}(\phi) \) is a polynomial involving \( g_{ij}, g^{ij} \) and first derivatives of \( \phi \). To establish the well-posedness for this system, it is convenient to consider the perturbed fields

\[
\bar{w} \doteq (u, \phi) = (g, \phi) - (\hat{g}, 0),
\]

where \( \doteq \) means by definition. Using (2.2), we can express \( \mathcal{J}^\delta \) in terms of \( \bar{J} \) and \( g \) i.e.

\[
\mathcal{J}^\delta = F(x, g, \bar{J}, \partial g, \partial \bar{J}) \doteq -(-\text{det} g)^{-\frac{1}{2}} \sum_{i=1}^3 \int_0^t \partial_i(-\text{det} g)^{\frac{1}{2}} \bar{J}^i ds.
\]

(2.14)

In particular, \( F \) is a smooth function of its arguments when \( g \) is non-degenerate. Note that here we can replace the vector field \( \partial_i \) along which the divergence equation \( \text{div}_g \mathcal{J} = 0 \) is solved by any time-like vector field, but for clarity of notations we use just \( \partial_i \). Using (2.20) and the expression of \( \Box g\phi \), we see that equations (2.12) in terms of \( \bar{w} \) can be reduced to

\[
\begin{align*}
-g^{\alpha\beta}\partial_\beta \partial_\alpha u_{\mu\nu} + B^\mu_\nu(x, \bar{w}, \partial \bar{w}) &= G(x, \bar{w}, \bar{J}, \partial \bar{J}), \\
-g^{\alpha\beta}\partial_\beta \partial_\alpha \phi_\beta + B^\beta(x, \bar{w}, \partial \bar{w}) &= \sum_{\alpha=1}^3 g_{\beta\alpha} \bar{J}^\alpha + g_{\beta\alpha} F(x, g, \bar{J}, \partial g, \partial \bar{J}),
\end{align*}
\]

(2.15)

\( \bar{w} = 0 \), in \( M(T_0) \backslash J^+_g \text{(supp} (\bar{J})) \),

where \( \mu, \nu, \beta = 0, 1, 2, 3 \), \( B^\bullet \) are smooth functions of \( \bar{w}, \partial \bar{w} \) and \( G(\bullet) \) is a smooth function in its arguments. In particular, \( G(\bullet) \) comes from the term \( \frac{1}{2}(J^\mu \phi_\mu)g \). We shall establish first the well-posedness of system (2.15) and then the coupled system (2.12). The system (2.15) is a quasilinear hyperbolic system of PDEs for \( \bar{w} \). In Appendix B of [21], the authors studied the local solvability and stability for certain nonlinear hyperbolic system of PDEs arising from the Einstein-scalar field equations. The proof follows similar fixed point iteration arguments as for the Cauchy problem, see for example [16] and [7] Appendix III. Although our nonlinear function \( G(x, \bar{w}, \bar{J}, \partial \bar{J}) \) is non-local...
due to (2.14), the proofs in [21] can be slightly modified to apply to our system (2.15) since we consider local problem for \( t \) finite. We briefly recall the results in [21].

In (2.13), \( u \) is a symmetric \( 4 \times 4 \) matrix and \( \phi \) is a 4-vector. By renumbering, we can regard \( \bar{w} = (u, \phi) \) as a 14-vector. For integer \( \kappa \), we let \( B^\kappa \) be a vector bundle on \( M \) such that the fiber \( B^\kappa_x \) is a \( \kappa \) dimensional vector space. In the following, we use section-valued Sobolev space \( H^m(M; B^\kappa) \).

We denote
\[
E^m(M(T_0); B^\kappa) = \bigcap_{j=0}^m C^j([0, T_0]; H^{m-j}(M; B^\kappa)), \quad m \in \mathbb{N}.
\]

Also, we use the abbreviation \( E^m \) and denote \( E^m_0 \) the compactly supported functions in \( E^m \). Now we state the local well-posedness for (2.15), hence the reduced Einstein-Maxwell equations (2.12). As we already mentioned, the proof can be found in Appendix B of [21].

**Proposition 2.1.** Let \( m_0 \geq 4 \) be an even integer and \( T_0 > 0 \). If \( \bar{J} \) is compactly supported and \( \| \bar{J} \|_{E^{m_0}} < \epsilon_0 \) is small enough, there exists a unique solution \( \bar{w} \) satisfying the equation (2.15) on \( M(T_0) \) and
\[
(2.16) \quad \| \bar{w} \|_{E^{m_0}} \leq C \| \bar{J} \|_{E^{m_0+1}},
\]
where \( C \) denotes a generic constant.

We remark that these regularity requirements are not optimal but sufficient for our work. Also, the space for \( J \) is \( E^{m_0+1} \) because the functions \( F \) and \( G \) contains derivative of \( J \). We see that for \( \| \bar{J} \|_{E^{m_0+1}} \) sufficiently small so that \( g = \bar{g} + u \) is non-degenerate, we can solve \( J^0 \) from (2.14). Moreover, we have \( J^0 \in E^{m_0} \) and \( \| J^0 \|_{E^{m_0}} \leq C \| \bar{J} \|_{E^{m_0+1}} \) for \( m \geq 4 \) even. This proves the well-posedness of the system (2.12).

2.4. The full Einstein-Maxwell equations. We’ve found solutions to the reduced Einstein-Maxwell equations in wave and Lorentz gauge. To make sure that they give solutions to the full Einstein-Maxwell equations, we need to check that the wave gauge condition \( \Gamma^a = \Gamma^a, \, \forall a = 0, 1, 2, 3 \) and the Lorentz gauge condition \( \delta_g \phi = 0 \) are satisfied. For the sourceless Einstein-Maxwell equations, this can be found in e.g. Lemma 10.1 of [7, Section 10.3]. The same idea applies here. The key is that the stress-energy tensor \( T_{\text{source}} \) satisfies the conservation law when \( J \) does.

**Lemma 2.2.** If \( (g, \phi) \) and \( J \) satisfies (2.12) in wave and Lorentz gauge, the harmonic gauge condition \( \Gamma^a = \Gamma^a \) and Lorentz gauge condition \( \delta_g \phi = 0 \) are satisfied on \( M(T_0) \).

**Proof.** First consider the function \( \mathcal{G} = \delta_g \phi \). We know that
\[
(2.17) \quad \Box_g \phi = \delta_g F + d \mathcal{G}, \quad F = d \phi.
\]

Also, we know from (2.12) that \( \Box_g \phi = J^g \) and \( J \) satisfies the conservation law for Maxwell equations which implies \( \delta_g J^g = 0 \). Now we take the codifferential of equation (2.17) and use \( \delta^2_g = 0 \) to obtain \( \delta_g (d \mathcal{G}) = 0 \), which is \( \Box_g \mathcal{G} = 0 \) as \( \mathcal{G} \) is a zero form (function). Therefore, \( \mathcal{G} \) is a solution to the linear hyperbolic system
\[
\Box_g \mathcal{G} = 0 \text{ in } M(T_0),
\]
\[
\mathcal{G} = 0 \text{ in } M(T_0) \setminus J^g_+ (\text{supp } (\bar{J})).
\]

The system has a unique solution. Hence we conclude that \( \mathcal{G} = 0 \) i.e. \( \delta_g \phi = 0 \) in \( M(T_0) \).
Next, consider \( \hat{F} = (\hat{F}^n)_{n=0}^3 \) with \( \hat{F}^n = \Gamma^u - \hat{\Gamma}^n \). The proof of \( \hat{F} = 0 \) follows from standard arguments. We briefly repeat the proof for completeness. If we write the reduced Einstein equation \([2,12]\) as

\[
\text{Ein}^\phi(g) = \text{Ric}^\phi(g) - \frac{1}{2} \text{Tr}(\text{Ric}^\phi(g))g = T_{\text{sour}},
\]

it follows from the Maxwell equation that \( T_{\text{sour}} \) satisfies the conservation law \( \text{div}_\phi T_{\text{sour}} = 0 \). From standard calculation for example \([21\text{ Appendix A.4}]\), we know that

\[
\text{Ein}_{jk}(g) - T_{jk} = \frac{1}{2} (g_{jm} \hat{\nabla}_k \hat{F}^m + g_{kn} \hat{\nabla}_j \hat{F}^n - g_{jk} \hat{\nabla}_n \hat{F}^n), \quad j, k = 0, 1, 2, 3.
\]

Taking divergence and using Bianchi identity, we arrive at

\[
0 = g^{nm} \nabla_n \hat{\nabla}_m \hat{F}^q + W^q(\hat{F}), \quad q = 0, 1, 2, 3,
\]

where \( W \) is a first order linear differential operator whose coefficients are polynomials of \( \hat{g}, g \) and their derivatives. Thus, \( \hat{F} \) satisfies a second order linear hyperbolic system

\[
g^{nm} \nabla_n \hat{\nabla}_m \hat{F}^q + W^q(\hat{F}) = 0, \quad \text{in } M(T_0),
\]

\[
\hat{F} = 0, \quad \text{in } M(T_0) \setminus J^+_g \text{ (supp } (\hat{J})\).
\]

Since this system is uniquely solvable, we conclude that \( \hat{F} = 0 \) i.e. the harmonic gauge condition is satisfied. This finishes the proof. \( \square \)

Finally, we discuss the well-posedness of the full Einstein-Maxwell equations and we’d like to explain the regularity requirements in the set \( D(\delta) \). In wave gauge, we take \( J = (J^0, \vec{J}) \) a one form with \( J^0 \in E^{m_0}, \vec{J} \in E^{m_0+1}, m_0 \geq 4 \) even and by Prop. \([21]\) we know that the solution \( g = \hat{g} + u \) and \( \phi \) are in \( E^{m_0} \) as well. Since \( E^{m_0} \) is an algebra, we can take the physical field \( J \) as a vector field in \( E^{m_0} \) in wave gauge. For the wave map \( \vec{f} \), we know that \( f \in C^{p-3} \) if \( \vec{g}' \in C^p \) where \( \vec{g}' \) is the solution to the full Einstein-Maxwell equations with electric current \( J' = f^*J \). For solving the inverse problem, we’d like to take into account all \( J \) in \( E^{12} \subset C^8 \) in wave gauge. Then \( g \in C^8 \) and \( f \in C^4 \). Thus we should include \( J' \in C^4 \) and \( \vec{g}' \in C^4 \) in the data set.

3. Linearization of the Einstein-Maxwell Equations

We’ve found the set of \( J \) for which the Einstein-Maxwell equations are well-posed. In the rest of the paper, we shall work with the following system in wave and Lorentz gauge

\[
\begin{cases}
\text{Ric}^\phi(g) = T_{em}(\phi) + \frac{1}{2} (J^\mu \phi_\mu)g & \text{in } M(T_0), \\
\Box g = J^0 & \text{in } M(T_0) \setminus J^+_g \text{ (supp } (\vec{J})\).
\end{cases}
\]

The corresponding system for the perturbed fields \( \vec{w} = (u, \phi) = (g, \phi) - (\hat{g}, 0) \) are

\[
\begin{cases}
-g^{\mu\nu} \partial_\mu \partial_\nu u + B_{\mu\nu}(x, \vec{w}, \partial \vec{w}) = (J^\mu \phi_\mu)g, & \text{in } M(T_0), \\
-g^{\mu\nu} \partial_\mu \partial_\nu \phi + B_{\mu\nu}(x, \vec{w}, \partial \vec{w}) = J_{\beta}, & \text{in } M(T_0), \\
\vec{w} = 0, & \text{in } M(T_0) \setminus J^+_g \text{ (supp } (\vec{J})\).
\end{cases}
\]

This is a second order quasilinear hyperbolic system for \( \vec{w} \). We consider its linearization in this section and construct the so-called distorted plane wave solutions. The theory for linear hyperbolic PDEs, for example wave equations, is well-developed, see e.g. \([11, 14, 1]\). We review the causal...
inverse for the linearized Einstein equation as a paired Lagrangian distribution [28]. This is particularly convenient for the analysis of propagation and interaction of singularities.

3.1. The causal inverse. Consider the equation (3.2) with source \( J = \epsilon \mathcal{J} \) depending on a small parameter \( \epsilon > 0 \). For the moment, we can assume that \( \mathcal{J} \in C^0(M; \mathcal{B}^4) \) is compactly supported. We will return to the issue of constructing \( \mathcal{J} \) belonging to the data set in Section 3.3. By the stability result Prop. 2.1, we can write the solution \((g, \phi) = (\tilde{g}, 0) + \epsilon (\dot{g}, \dot{\phi}) \) modulo terms of order \( \epsilon^2 \). Then \((\dot{g}, \dot{\phi})\) satisfy the linearized Einstein-Maxwell equations (for \( \mu, \nu, \beta = 0, 1, 2, 3 \))

\[
\begin{align*}
-\tilde{g}^{pq} \partial_q \dot{g}_{\mu\nu} + A_{\mu\nu}(\dot{g}) &= 0, \quad \text{in } M(T_0), \\
-\tilde{g}^{pq} \partial_q \dot{\phi}_{\beta} + A_{\beta}^{\alpha}(\dot{\phi}) &= \tilde{g}_{\beta\alpha},
\end{align*}
\]

where \( A_\bullet \) are first order differential operators. From (3.3), we notice that using (2.9), (2.3) and (2.11), we emphasize that the system (3.3) is actually decoupled and we have \( \dot{g} = 0 \). This facilitates our analysis and is the reason why we consider the vacuum background.

It is convenient to write the system in matrix form. Let \( \mathcal{J} = (\dot{g}, \dot{\phi}) \) be section valued in \( \mathcal{B}^{14} \). We let \( \mathcal{J} = (0, (\tilde{g}_{\beta\alpha}^{\mathcal{J}})_{\beta=0}^{3}) \) where \( 0 \) is the zero section in \( \mathcal{B}^{10} \). Then equation (3.3) can be written as

\[
P \mathcal{J} = (-\tilde{g}^{pq} \partial_p \partial_q) \text{Id} \mathcal{J} + V(x, \partial) \mathcal{J} = \mathcal{J},
\]

where \( \text{Id} \) denotes the 14 \times 14 identity matrix and \( V \) is a 14 \times 14 block-diagonalized matrix whose elements are first order differential operators. From (3.3), we notice that \( P \) is hyperbolic with principal term the wave operator \( \Box \mathcal{J} \text{Id} \) for which we know there exists a causal inverse if \( (M, \tilde{g}) \) is globally hyperbolic, see for example [1]. Moreover, the Schwartz kernel of the causal inverse can be described as a paired Lagrangian distribution as shown in [28], see also [21]. We review the construction in the following.

For two Lagrangians \( \Lambda_0, \Lambda_1 \subset T^*X \) intersecting cleanly at a codimension \( k \) submanifold i.e.

\[
T_q \Lambda_0 \cap T_q \Lambda_1 = T_q(\Lambda_0 \cap \Lambda_1), \quad \forall q \in \Lambda_0 \cap \Lambda_1,
\]

the paired Lagrangian distribution associated with \((\Lambda_0, \Lambda_1)\) is denoted by \( I^P(\Lambda_0, \Lambda_1) \), see [8, 28, 18, 17] for details. Let \( \mathcal{P}(x, \xi) = (\xi_i)_{i=1}^{2} \) be the symbol of \(-\tilde{g}^{pq} \partial_p \partial_q \) with \( \tilde{g}^{*} = \tilde{g}^{-1} \) the dual metric. Let \( \Sigma_{\tilde{g}} = \{(x, \xi) \in T^*M : \mathcal{P}(x, \xi) = 0\} \) be the characteristic set which consists of light-like co-vectors. The Hamilton vector field of \( \mathcal{P} \) is denoted by \( H_\mathcal{P} \) and in local coordinates

\[
H_\mathcal{P} = \sum_{i=0}^{3} \frac{\partial \mathcal{P}}{\partial \xi_i} \frac{\partial}{\partial x_i} - \frac{\partial \mathcal{P}}{\partial x_i} \frac{\partial \mathcal{P}}{\partial \xi_i}.
\]

The integral curves of \( H_\mathcal{P} \) in \( \Sigma_{\tilde{g}} \) are called null bicharacteristics and there projections to \( M \) are geodesics. Let \( \text{Diag} = \{ (z, z') \in M \times M : z = z' \} \) be the diagonal and

\[
N^* \text{Diag} = \{ (z, \zeta, \zeta') \in T^*(M \times M) \setminus 0 : z = z', \zeta' = -\zeta \}
\]

be the conormal bundle of \( \text{Diag} \) minus the zero section. We let \( \Lambda_{\tilde{g}} \) be the Lagrangian obtained by flowing out \( N^* \text{Diag} \cap \Sigma_{\tilde{g}} \) under \( H_\mathcal{P} \). Here we regard \( \Sigma_{\tilde{g}}, H_\mathcal{P} \) as objects on product manifold \( T^*M \times T^*M \) by lifting from the left factor. It is proved in Lemma 3.1 of [21] that for the linear differential operator \( \mathcal{P} \), there exists a causal inverse \( \mathcal{Q} \in I^{-\frac{3}{2},-\frac{1}{2}}(N^* \text{Diag}, \Lambda_{\tilde{g}}; \mathcal{B}^{14}) \) such that \( \mathcal{P} \mathcal{Q} = \text{Id} \) on \( E'(M; \mathcal{B}^{14}) \). Later, we also use \( \mathcal{Q}_{\tilde{g}} \) to emphasis the dependence on the metric. Moreover, from [8 Prop. 5.6 ] or [17 Theorem 3.3], we know that \( \mathcal{Q} : H^m_{\text{comp}}(M; \mathcal{B}^{14}) \to H^{m+1}_{\text{loc}}(M; \mathcal{B}^{14}) \) is continuous for \( m \in \mathbb{R} \).
3.2. Distorted plane waves. We shall consider source \( J \) with singularities in the normal directions of a submanifold. For a submanifold \( Y \subset X \) of codimension \( k \), the conormal bundle \( N^*Y \) is a Lagrangian submanifold. We review Lagrangian distributions in the scalar case, see [14, 15].

Let \( X \) be a \( n \) dimensional smooth manifold and \( \Lambda \) be a smooth conic Lagrangian submanifold of \( T^*X\setminus 0 \). Following the standard notation, we denote by \( I^\mu(\Lambda) \) the Lagrangian distribution of order \( \mu \) associated with \( \Lambda \). In particular, for \( U \) open in \( X \), let \( \phi(x, \xi) : U \times \mathbb{R}^N \to \mathbb{R} \) be a smooth non-degenerate phase function that locally parametrizes \( \Lambda \) i.e. \( \{ (x, d_x \phi) : x \in U, d_\xi \phi = 0 \} \subset \Lambda \). Then \( u \in I^\mu(\Lambda) \) can be locally written as a finite sum of oscillatory integrals

\[
\int_{\mathbb{R}^N} e^{i\phi(x, \xi)} a(x, \xi) d\xi, \quad a \in S^{\mu+\frac{n}{2}-\frac{N}{2}}(U \times \mathbb{R}^N),
\]

where \( S^\bullet(\bullet) \) denotes the standard symbol class, see [14, Section 18.1]. For \( u \in I^\mu(\Lambda) \), we know that the wave front set \( WF(u) \subset \Lambda \) and \( u \in H^s(X) \) for any \( s < -\mu - \frac{n}{2} \). The principal symbol of \( u \) is well-defined as a half-density bundle tensored with the Maslov bundle on \( \Lambda \), see [15, Section 25.1].

For a submanifold \( Y \subset M \), the conormal distributions to \( Y \) are denoted by \( I^\mu(N^*Y) \). Occasionally, we also use the notation \( I^m(Y) = I^{m+\frac{4}{n-k}}(N^*Y) \) with \( d \) the codimension of \( Y \) so that \( m \) is the order of the symbols. Consider local representations of such distributions. We can find local coordinates \( x = (x', x'') \), \( x' \in \mathbb{R}^k \), \( x'' \in \mathbb{R}^{n-k} \) such that \( Y = \{ x' = 0 \} \). Let \( \xi = (\xi', \xi'') \) be the dual variable, then \( N^*Y = \{ x'' = 0 \} \). We can write \( u \in I^\mu(N^*Y) \) as

\[
u = \int_{\mathbb{R}^k} e^{ix'\xi'} a(x'', \xi') d\xi', \quad a \in S^{\mu+\frac{n-k}{2}}(\mathbb{R}^{n-k}; \mathbb{R}^k).
\]

In this case, the principal symbol is

\[
s(\nu) = (2\pi)^{\frac{n-k}{2}} a_0(x'', \xi') |d_2 x''|^{\frac{1}{2}} |d\xi'|^{\frac{1}{2}},
\]

where \( a_0 \in S^{\mu+\frac{n-k}{2}}(\mathbb{R}^{n-k}; \mathbb{R}^k) \) is such that \( a - a_0 \in S^{\mu+\frac{n-k}{2}-1}(\mathbb{R}^{n-k}; \mathbb{R}^k) \), see [14, Section 18.2].

Later, we also use the notation \( \sigma_{N^*Y}(u) \) to emphasize where the symbol is defined. For a Lorentzian manifold \( (M, g) \), there is a natural choice of the density bundle, the volume element \( dvol_\gamma \). Hence we can trivialize the distributional half densities and regard \( u \) as distributions. We emphasis that in our notation for principal symbols, we do not specify the order but refer to the distribution space for the orders.

We follow [21] to construct distorted plane waves. For \( (x_0, \theta_0) \in L^+ M \), recall that \( \gamma_{x_0, \theta_0}(t), t \geq 0 \) is the geodesic from \( x_0 \) with direction \( \theta_0 \). For \( s_0 > 0 \) a small parameter, we let

\[
K(x_0, \theta_0; t_0, s_0) = \{ \gamma_{x_0', \theta_0'}(t) \in M(T_0); \theta \in O(s_0), t \in (0, \infty) \},
\]

where \( (x', \theta') = (\gamma_{x_0, \theta_0}(t_0), \dot{\gamma}_{x_0, \theta_0}(t_0)) \) and \( O(s_0) \subset L^+_\gamma M \) is an open neighborhood of \( \theta' \) consisting of \( \zeta \in L^+_\gamma M \) such that \( \| \zeta - \theta' \|_{g^+} < s_0 \). We observe that as \( s_0 \to 0 \), \( K(x_0, \theta_0; t_0, s_0) \) tends to the geodesic \( \gamma_{x_0, \theta_0} \). Next, let

\[
Y(x_0, \theta_0; t_0, s_0) = K(x_0, \theta_0; t_0, s_0) \cap \{ t = 2t_0 \}
\]

be a 2-dimensional surface, which intersects the geodesic at \( \gamma_{x_0, \theta_0}(2t_0) \), see Fig. 3.2. We let \( \Lambda(x_0, \theta_0; t_0, s_0) \) be the Lagrangian submanifold obtained from flowing out \( N^*K(x_0, \theta_0; t_0, s_0) \cap N^*Y(x_0, \theta_0; t_0, s_0) \) under the Hamilton vector field of \( \mathcal{P} \) in \( \Sigma \). More precisely, for a smooth vector field \( V \) on \( M \) and \( (x, \xi) \in TM \), we denote the integral curve of \( V \) from \( (x, \xi) \) by \( exp(tV)(x, \xi), t \in \mathbb{R} \).
Then we have
\[
\Lambda(x_0, \theta_0; t_0, s_0) = \bigcup_{t \geq 0} \exp \left( tH \right) \left( N^* K(x_0, \theta_0; t_0, s_0) \cap N^* Y(x_0, \theta_0; t_0, s_0) \right) \cap \Sigma_{\hat{g}}.
\]

It is convenient to introduce a notation for the flow out under \( \Lambda_{\hat{g}} \). For any \( \Gamma \subset T^* M \), we denote the flow out of \( \Gamma \) by \( \Lambda'_{\hat{g}} = \Lambda_{\hat{g}} \circ (\Gamma \cap \Sigma_{\hat{g}}) \), where as usual in microlocal analysis
\[
\Lambda'_{\hat{g}} = \{(x, \xi, y, \eta) \in T^* M \times T^* M : (x, \xi, y, -\eta) \in \Lambda_{\hat{g}}\},
\]
and \( \circ \) denotes the composition of sets as relations. The proposition below is essentially proved in \[21\] and is stated for the current setting.

**Proposition 3.1.** Suppose \( Y(x_0, \theta_0; t_0, s_0) \) and \( \Lambda(x_0, \theta_0; t_0, s_0) \) are defined as in (3.5) and (3.6) respectively. For \( \vec{J} \in I^{\mu+1}(N^* Y; B^{14}) \) with \( \mu \) an integer, the solution to the linearized Einstein-Maxwell equations \( P\vec{v} = \vec{J} \) is
\[
\vec{v} = Q(\vec{J}) \in I^{\mu - \frac{1}{2}}(\Lambda; B^{14}) \text{ in } M(T_0) \setminus Y,
\]
and \( \vec{v} \) is called distorted plane waves. Moreover, for \( (p, \xi) \in N^* Y \cap \Sigma_{\hat{g}} \) and \( (q, \eta) \in \Lambda(x_0, \theta_0; t_0, s_0) \) which lie on the same bicharacteristics, the principal symbol of \( \vec{v} \) is given by
\[
\sigma(\vec{v})(q, \eta) = \sigma(Q)(q, \eta; p, \xi)\sigma(\vec{J})(p, \xi),
\]
where \( \sigma(Q) \) is a 14 \times 14 invertible matrix.

We remark that in general \( \vec{v} \) is a well-defined Lagrangian distribution and before the first conjugate point, it is a conormal distribution. As discussed in [20], it is complicated to analyze the interaction of singularities past the conjugate points and this difficulty will be overcome using another argument. Let \( \tau_0 > 0 \) be such that \( \gamma_{x_0, \theta_0}(\tau_0) \) is the first conjugate point of \( x_0 \) along \( \gamma_{x_0, \theta_0} \). Then the exponential map \( \exp_{x_0} \) is a local diffeomorphism from a neighborhood of \( t \theta_0 \in T_{x_0} M \) to a neighborhood of \( \gamma_{x_0, \theta_0}(t) \) for \( t < \tau_0 \). Therefore, \( K(x_0, \theta_0; t_0, s_0) \) is a co-dimension 1 submanifold near \( \gamma_{x_0, \theta_0}(t) \) and
\[
\Lambda(x_0, \theta_0; t_0, s_0) = N^* K(x_0, \theta_0; t_0, s_0) \text{ near } \gamma_{x_0, \theta_0}(t) \text{ for } t < \tau_0.
\]
In particular, before the first conjugate point of \( x_0 \) along \( \gamma_{x_0, \theta_0} \), \( \vec{v} \) is a conormal distribution to \( K(x_0, \theta_0; t_0, s_0) \). We mention that in [20], the null cut points was considered, see [20] Section 2.1.\[\text{Figure 3. Illustration of distorted plane waves.}\]
On a globally hyperbolic Lorentzian manifold, the first null cut point $\tilde{x}$ of $x_0$ along the geodesic $\gamma_{x_0,0}$ is either the first conjugate point or there are at least two light-like geodesics joining $x_0$ and $\tilde{x}$. So the first cut point appears on or before the first conjugate point.

3.3. Microlocal linearization conditions. Now we address the issue that for the source $J$ to be in the data set, it must satisfy the conservation law. Suppose $J = J$, depending on some small parameter $\varepsilon$ and we let $\mathcal{J} = \partial_{\varepsilon} J_{|\varepsilon = 0}$ be its linearization. From the conservation law $\text{div}_g J = 0$, we derive its linearization

$$\partial_i ((-\det \hat{g})^{\frac{1}{2}} J^i) = 0.$$  

Notice that this is one equation and the Einstein summation is over $i = 0, 1, 2, 3$. For $\mathcal{J} \in I^\mu(N^*Y; B^4)$ where $Y$ is defined in (3.5), the principal symbol at $(x, \xi) \in N^*Y$ should satisfy the microlocal linearized conservation law

$$\xi_i \sigma(J^i)(x, \xi) = 0.$$  

We shall denote such conormal distributions by $I^\mu(N^*Y; B^4)$. We emphasize that the space $X_{x, \xi}$ of principal symbols of $\mathcal{J} = (0, 0)$ at $(x, \xi)$ satisfying (3.8) is a 3-dimensional subspace of a 14-dimensional vector space.

Since we derived the linearized Einstein-Maxwell equations from the reduced equations in wave and Lorentz gauge, the linearized solutions must also satisfy the linearized gauge conditions. Their microlocalization imposes conditions on the symbols of the distorted plane waves. We start with the harmonicity condition $g^{mn} \Gamma^j_{nm} = g^{mn} \hat{\Gamma}^j_{nm}; j = 0, 1, 2, 3$. By linearization we obtain

$$-\hat{g}^{an} \partial_a \hat{g}_{nj} + \frac{1}{2} \hat{g}^{pq} \partial_j \hat{g}_{pq} = m_j \hat{g}_{pj}, \quad j = 0, 1, 2, 3,$$

where $m_j$ depends on $\hat{g}$ and its derivative, see [21 Sec. 3.2.3]. When the background metric $\hat{g}$ is Minkowski, we have $m_j = 0$. The Lorentz gauge condition is $g^{\alpha \lambda} \partial_\alpha \phi_\lambda = \hat{\Gamma}^{\mu} \phi_\mu$. The linearization gives $\hat{g}^{\alpha \lambda} \partial_\lambda \phi_\alpha = \hat{\Gamma}^{\mu} \phi_\mu$. Also, when $\hat{g}$ is Minkowski, the right hand side vanishes. Let $\hat{g}, \hat{\phi}$ be conormal distributions in $I^\mu(\Lambda)$ where $\Lambda$ is defined in (3.6). For any $(y, \eta) \in \Lambda$, we obtain that the principal symbols of $\hat{g}, \hat{\phi}$ should satisfy the microlocal linearized gauge conditions i.e.

$$-\hat{g}^{an} \xi_a \sigma(\hat{g}_{nj}) + \frac{1}{2} \hat{g}^{pq} \xi_j \sigma(\hat{g}_{pq}) = 0, \quad j = 0, 1, 2, 3,$$

$$\hat{g}^{\alpha \lambda} \xi_\lambda \sigma(\hat{\phi}_\alpha) = 0.$$  

We shall denote conormal distributions satisfying these conditions by $I^\mu_\Sigma(\Lambda)$. In particular, for our consideration, the linearized metric component $\hat{g}$ vanishes thus the microlocal gauge condition is reduced to (3.10). Then we notice that the space $\mathcal{Y}_{y, \eta}$ of principal symbols of $(\hat{g}, \hat{\phi})$ at $(y, \eta)$ is also a 3-dimensional vector space. Now if we consider the linear map $R = \sigma(Q)(y, \eta, x, \xi)$ in Proposition 3.1 we see that

$$R$$

is bijective from $X_{x, \xi}$ to $\mathcal{Y}_{y, \eta}$.

Now we make a connection with the so called microlocal linearization stability condition, introduced in [21 Assumption $\mu$-LS] for the Einstein-scalar field equations. The condition plays an important role because it allows one to construct distorted plane wave solutions by prescribing the principal symbols. Let’s consider the setup of the Einstein-Maxwell equations (1.3) in the introduction, see also Theorem 1.1. Let $Y$ be defined as in (3.5) and $Y \subset V$. For any $(x, \xi) \in N^*Y \cap L^*M$ and vector $A = (A_i)_{i=0}^3 \in \mathbb{R}^4$ satisfying $\xi_i A^i = 0$, one can find $\mathcal{J}^i \in I^{\mu+1}(N^*Y), i = 1, 2, 3$ compactly supported in $V$ such that $\sigma(\mathcal{J}^i)(x, \xi) = A^i, i = 1, 2, 3$. Then by the well-posedness of the
Einstein-Maxwell equations with sources in wave and Lorentz gauge, we know that there exists a family of solutions \((\gamma_t, \phi_t, J^0_t)\) for \(J = e^3\epsilon > 0\) such that \(J_t = (J^0_t, J_t)\) satisfies the conservation law. Let \(\mathcal{J}^0\) be the linearization of \(J^0_t\). From (2.14), we obtain that

\[
\mathcal{J}^0 = -(-\text{det}(\mathcal{g}))^{-\frac{1}{2}} \sum_{i=1}^{3} \int_0^t \partial_i((-\text{det}(\mathcal{g}))^{\frac{1}{2}} \mathcal{J}^i)ds.
\]

Lemma 3.2. Let \(Y\) be defined as in (3.5) and suppose that \(\mathcal{J}^i \in I^{u+1}(N^*Y)\), \(i = 1, 2, 3\). Then we have \(\mathcal{J}^0 \in I^{u+2}(N^*Y)\). Moreover, near \(L^*M \cap N^*Y\), we have \(\mathcal{J}^0 \in I^{u+1}(N^*Y)\).

Proof. We recall the equivalent definition of conormal distributions e.g. [14, Def. 18.2.6]. So \(u \in I^u(N^*Y)\) if and only if \(V_1 \cdots V_k u \in \infty H^{-\mu - \frac{n}{2}}(M)\), \(\forall k \geq 0\) where the \(V_j\) are smooth vector fields on \(M\) tangent to \(Y\) and \(\infty H^u_{loc}\) denotes Besov spaces, see e.g. [14, Appendix B]. By our construction, we know that \(Y \subset \{t = 2t_0\}\) so that \(TY \subset TM\). In particular, the vector fields tangent to \(Y\) are spanned by \(\frac{\partial}{\partial y_i}, i = 1, 2, 3\) with \((y^i)\) the local coordinates for \(M\). We observe that

\[
V_1 \cdots V_k \mathcal{J}^0 = -\sum_{i=1}^{3} V_1 \cdots V_k \int_0^t (-\text{det}(\mathcal{g}))^{-\frac{1}{2}} \partial_i((-\text{det}(\mathcal{g}))^{\frac{1}{2}} \mathcal{J}^i)ds = \sum_{i=1}^{N} \int_0^t f_i(s, y)ds,
\]

where \(f_i \in \infty H^{-\mu - 2 - \frac{n}{4}}(M)\). Using for example [14, Corollary B.1.6], we conclude that \(V_1 \cdots V_k \mathcal{J}^0 \in \infty H^{-\mu - 2 - \frac{n}{4}}(M)\) so that \(\mathcal{J}^0 \in I^{u+2}(N^*Y)\). Now we use that the principal symbols of \(J\) should satisfy the microlocal linearized conservation law. So for \((x, \xi) \in N^*Y\) and \(\xi\) close to \(L^*_M\) such that \(\xi_0 \neq 0\), we have \(\xi_0 \sigma(\mathcal{J}^0)(x, \xi) = 0\) which implies \(\sigma(\mathcal{J}^0) = 0\). Thus the principal symbol vanishes and \(\mathcal{J}^0 \in I^{u+1}(N^*Y)\) microlocally near \(L^*M \cap N^*Y\).

Notice that \(\mathcal{J}^0\) and \(\mathcal{J}\) satisfy the linearized conservation law, therefore, their principal symbols satisfy the microlocal linearized conservation law. Continue our argument, we have \(\sigma(\mathcal{J}^0)(x, \xi) = A^0\). To conclude, given the symbols \(A\) at \((x, \xi) \in N^*Y \cap L^*M\), we can find a one parameter family of sources \(J_\epsilon\) such that \(\mathcal{J} = \partial_\epsilon J_\epsilon\rvert_{\epsilon=0} \in I^u(N^*Y)\) with \(\sigma(\mathcal{J})(x, \xi) = A\). Moreover, for \(-\mu\) large enough, there exists a unique solution \((g_\epsilon, \phi_\epsilon) \in C^4(M)\) to the Einstein-Maxwell equations \([13]\) with the source \(J_\epsilon\). Notice that we can choose \(J_\epsilon = e^3\mathcal{J}\) compactly supported in \(V\) and from the formula (2.14) of \(J^0_t\), we can shrink the support of \(\mathcal{J}\) so that \(J^0_t\) is supported in \(V\) (but not compactly supported). This is actually the corresponding version of the microlocal linearization stability condition for the Einstein-Maxwell equations (compare with [21] Assumption \(\mu\)-LS]). Finally, combining with the observation (3.11), we conclude that for any given symbols of \((\mathcal{J}^0, \mathcal{J})\) in \(\mathcal{Y}_{y, \eta}\), one can find the corresponding source \(J_\epsilon\).

4. LINEARIZED ELECTROMAGNETIC AND GRAVITATIONAL WAVES

In this section, we demonstrate that due to the nonlinearity and coupling of the Einstein-Maxwell equations, it is possible to generate gravitational waves using electromagnetic sources. For simplicity, we consider in this section the linearization of Einstein-Maxwell equations, in wave gauge, when the background manifold is Minkowski.

Let \((M, \bar{g})\) be the Minkowski space-time \((\mathbb{R}^4, h)\) with \(h = -(dx^0)^2 + \sum_{i=1}^{3} (dx^i)^2\). In this case, the Einstein-Maxwell equations are simple because \(\bar{g}^{-1}_{ij}\) and the derivatives of \(\bar{g}_{ij}\) all vanish. We start with the expression of the nonlinear equations (3.2). Recall that \((u, \phi) = (g - \bar{g}, \phi - 0)\) is the
perturbed fields. From (2.9), we get the reduced Ricci tensor $(\mu, \nu = 0, 1, 2, 3)$
\[
(\text{Ric}_g)(\mu\nu) = -\frac{1}{2} g^{pq} \partial_p \partial_q u_{\mu\nu} + g^{ab} g_{pq} \Gamma^p_{\mu b} \Gamma^q_{\nu a} + \frac{1}{2} (g_{pq} \Gamma^l_{ab} g^{aa} g^{bd} \partial_{\mu} u_{qd} + g_{pq} \Gamma^l_{ab} g^{aa} g^{bd} \partial_{\nu} u_{qd}).
\]
The stress-energy tensor for the electromagnetic field is $(\alpha, \beta = 0, 1, 2, 3)$
\[
T_{\text{em,}\alpha\beta} = g^{\lambda\mu} F_{\alpha\mu} F_{\beta\lambda} - \frac{1}{4} g_{\alpha\beta} g^{\gamma\delta} F_{\gamma\alpha} F_{\delta\beta}, \quad F_{\alpha\beta} = \partial_{\alpha} \phi_{\beta} - \partial_{\beta} \phi_{\alpha}.
\]
Let $J = (J^0, \bar{J})$ be the source where $J^0$ is defined in (2.14). The reduced Einstein equation is of the form
\[
-g^{pq} \partial_p \partial_q u_{\mu\nu} + 2P_{\mu\nu} - 2T_{\text{em,}\mu\nu} = (J^i \phi_i) g, \quad \mu, \nu = 0, 1, 2, 3,
\]
where $P_{\mu\nu}$ is the semilinear term in $\text{Ric}_g$ and $J^0$ is regarded as a nonlinear function of $g, J^i, i = 1, 2, 3$. Finally, the reduced Maxwell equations (2.11) is
\[
-g^{\alpha\lambda} \partial_\lambda \partial_\alpha \phi_{\beta} = g_{\beta\alpha} J^\alpha, \quad \beta = 0, 1, 2, 3.
\]
In the above equations, $g^{\alpha\beta}$ can be computed as following
\[
g^{-1} = (h + u)^{-1} = (\text{Id} + h^{-1} u + (h^{-1} u)^2 + (h^{-1} u)^3 + \cdots) h^{-1},
\]
where the 2-tensors are treated as matrices in local coordinates. The components are
\[
g^{ab} = h^{ab} + h^{aa} u_{a} u_{b} + h^{bb} u_{ab} + h^{cc} u_{ac} u_{cb} + \cdots
\]
(4.1)
It is worth mentioning that the first line of the above formula is in Einstein summation but the second line is not and we made simplifications using properties of $h$. Using these formulas, we can find the nonlinear terms $B$ in (3.2) explicitly. Next, let $\hat{J} = \epsilon \hat{J}$ and $\bar{g}$ be the linearization of $J^0$ defined in (3.12). The linearized equations (3.3) are simply
\[
-h^{pq} \partial_p \partial_q \bar{g}_{\mu\nu} = 0, \quad \mu, \nu = 0, 1, 2, 3
\]
(4.2)
\[
-h^{\alpha\lambda} \partial_\lambda \partial_\alpha \hat{\phi}_{\beta} = h_{\beta\alpha} \hat{\phi}_{\alpha}, \quad \beta = 0, 1, 2, 3.
\]
We denote the causal inverse $Q_h = (-h^{pq} \partial_p \partial_q)^{-1}$. The main result of this section is

**Theorem 4.1.** Let $(M, h)$ be the Minkowski space-time with $M = \mathbb{R}^4$ and $M(T_0) = (-\infty, T_0) \times \mathbb{R}^3, T_0 > 0$. Let $Y$ be a 2-dimensional surface on $\{t = 0\}$ and $\bar{Y}$ be the null hyper-surface from $Y$ i.e. $\bar{Y} = \{ \exp_B(tV) \in M(T_0) : t > 0, V \in \mathbb{L}^+_Y M \}$. Let $\bar{J} \in I^{n+1}(N^*Y), i = 1, 2, 3, \mu \leq -10$ be compactly supported. Consider the Einstein-Maxwell equations in wave gauge

\[
\begin{cases}
\text{Ein}(g) = T_{\text{sour}} \\
\delta_g \phi = \epsilon \hat{J}^0 & \text{in } M(T_0), \\
\text{div}_g \bar{J} = 0
\end{cases}
\]
\[
g = h, \quad \phi = 0, \quad \hat{J}^0 = 0, \quad \text{in } M(T_0) \setminus J^+_g(\text{supp } \bar{J}).
\]
For $\epsilon > 0$ sufficiently small, the solution to (4.2) on $M \setminus Y$ satisfies
\[
g = h + \epsilon^2 g_1 + o(\epsilon^2), \quad \phi = \epsilon \phi_1 + o(\epsilon^2),
\]
where the term in $o(\epsilon^2)$ is small in $H^4(M)$, such that on $M \setminus Y$,
(1) \( g_1 \in H^9(M), \phi_1 \in H^8(M) \) with singsupp\((g_1)\), singsupp\((\phi_1)\) \( \subset \mathcal{Y} \).

(2) \( g_1, \phi_1 \) are non-vanishing if \( \mathcal{J} \) is non-vanishing.

**Proof.** According to the discussion in Section 2 of the Einstein-Maxwell equations, we have that there exists a unique solution \( \mathcal{J} \in C^{4}(M) \) such that the elements can be written as matrices. The term \( \mathcal{J} \) is smooth and \( \mathcal{J} = E \) with \( 0 \in B^{10} \). The solution to the linearized Einstein-Maxwell equations \( (3,3) \) is \( \mathcal{J} \in I^{u_{\mathcal{J}}} = (0, \mathcal{J}) \) with \( \mathcal{J} \) in \( \mathcal{Y} \). Also, from \( (3,3) \), we know that the metric components \( \mathcal{J} \) are in \( I^{u_{\mathcal{J}}} = (0, \mathcal{J}) \) and \( \mathcal{J} \) is a Lagrangian distribution but may not be conormal. The projection of \( \Lambda_1 \) \( \mathcal{Y} \) to \( \mathcal{Y} \) is exactly \( \mathcal{Y} \). Also, from \( (3,3) \), we know that the metric components \( \mathcal{J} \) in \( I^{u_{\mathcal{J}}} = (0, \mathcal{J}) \) are all zero.

From the equations \( (3.2) \) and \( (3.3) \), we find that

\[
P_2(x, \mathcal{J}) = \frac{\partial^2}{\partial x^p \partial x^q} \mathcal{J},
\]

where \( h \) and \( u \) are treated as matrices. The term \( H_2 \) is section valued in \( B^{14} \) and comes from the nonlinear terms of \( (3.2) \). More explicitly, the elements can be written as

\[
H_2 = \sum_{i,j=1}^{4} \sum_{a,b=1}^{14} H_{i,j,a,b} \mathcal{J}^a \mathcal{J}^b w_i w_j,
\]

where the coefficients are all smooth and \( \gamma = 1, 2, \ldots, 14 \). From \( (1.3) \), we obtain

\[
\mathcal{J} = \frac{\partial^2}{\partial x^p \partial x^q} \mathcal{J}.
\]

Substitute \( \mathcal{J} \) to the right hand side, we get

\[
\mathcal{J} = \frac{\partial^2}{\partial x^p \partial x^q} \mathcal{J}.
\]

Notice that since \( H^4(M) \) is an algebra and \( Q \) is continuous from \( H_{\text{comp}}(M) \) to \( H_{\text{loc}}^5(M) \), the remainder term \( o(e^2) \) is still in \( H^4(M) \). Since \( \mathcal{J} = 0 \), we have \( P_2(x, \mathcal{J}) = 0 \). The terms in \( H_2 \) with metric components vanish as well. So \( h \) suffices to consider terms in \( H_2 \) which only have the electric potentials. We shall denote them by \( \tilde{H}_2 \). Observe that these terms can only come from the stress-energy tensor \( T_{em}(\phi) \). Actually, we find using \( (2.3) \) and \( (1.1) \) that

\[
\tilde{H}_{2,\alpha\beta}(x, \mathcal{J}) = -2(h^{a\alpha}F_{a\alpha}F_{\beta\beta} - \frac{1}{4}h_{\alpha\beta}h^{a\alpha}h^{b\beta} F_{a\beta} F_{a\beta} F_{a\beta}), \quad \alpha, \beta = 0, 1, 2, 3,
\]

where the \( F_{a\beta} \) are defined in terms of \( \phi \). Here \( H_{2,\alpha\beta} \) are the terms of \( \tilde{H}_2 \) in the reduced Einstein equations and \( \tilde{H}_{2,\mu} \) are the terms from the Maxwell equations. Also, we renumbered the section valued \( \tilde{H}_2 \). Now we can simplify \( (1.4) \) to

\[
\mathcal{J} = \frac{\partial^2}{\partial x^p \partial x^q} \mathcal{J}.
\]

In particular, the metric components are

\[
u = \frac{\partial^2}{\partial x^p \partial x^q} \mathcal{J}, \quad g_{1,\alpha\beta} = -Q_h(\tilde{H}_{2,\alpha\beta} + (\mathcal{J}^i\mathcal{J}^i)h_{\alpha\beta}).
\]
Using the support that $H^8(M)$ is an algebra and the continuity of $Q_h$, we see that $g_1 \in H^9(M)$. The singular support properties follows from standard wave front analysis, see e.g. [9] Section 1.3.

Finally, we observe from [415] that if $\alpha = \beta = 0$, we have that

$$
\hat{H}_{2,00} = -2(h^{aa}F_{0a}F_{0a} - \frac{1}{4}h_{00}h^{aa}h^{bb}F_{ab}F_{a'b'}) = -2(\sum_{a=0}^{3} h^{aa}F_{0a}^2 + \sum_{a,b=0}^{3} \frac{1}{4}h^{aa}h^{bb}F_{ab}^2)
$$

$$
= -2(\sum_{a=1}^{3} F_{0a}^2 - \frac{1}{2}\sum_{a=1}^{3} F_{0a}^2 + \frac{1}{4}\sum_{a,b=1}^{3} F_{ab}^2) = -\sum_{a=1}^{3} F_{0a}^2 - \frac{1}{2}\sum_{a,b=1}^{3} F_{ab}^2.
$$

Here we used the fact that $F_{ab}$ is antisymmetric and $F_{00} = 0$. It is clear that $\hat{H}_{2,00}$ vanishes if and only if $F_{ab} = 0, a, b = 0, 1, 2, 3$. It follows from the Maxwell equation (2.11) and its linearization that $F = 0$ implies $\mathcal{J} = 0$. Thus if $\mathcal{J}$ is non-vanishing, we conclude that $\phi$ is non-vanishing from the linearized equations and hence $g_1$ is non-vanishing outside the support of $\mathcal{J}$. This ends the proof of the theorem.

5. INTERACTIONS OF DISTORTED PLANE WAVES

From this section, we return to the setting of a general globally hyperbolic vacuum spacetime $(\hat{M}, \hat{g})$ instead of the Minkowski spacetime in Section 4. When four distorted plane waves meet at a point, new singularities could be produced. In [21], the nature of these singularities are analyzed using Gaussian beam solutions and stationary phase type arguments. Recently in [24], the authors studied such interactions carefully for scalar waves using paired Lagrangians and symbol calculus. We shall apply these results to the Einstein-Maxwell equations.

Let $x^{(j)} \in V$ and $(x^{(j)}, \theta^{(j)}) \in L^+M, j = 1, 2, 3, 4$ be such that

$$
\gamma_{x^{(j)}, \theta^{(j)}}([0, t_0]) \subset V, \quad x^{(j)}(t_0) \notin J^+_g(x^{(k)}(t_0)), \quad j \neq k,
$$

which means that the points are causally independent. We define $K_j = K(x^{(j)}, \theta^{(j)}; t_0, s_0), j = 1, 2, 3, 4$ and $\Lambda_j(x^{(j)}, \theta^{(j)}; t_0, s_0)$ similar to (3.5) and (3.6). Let $\tau_j, j = 1, 2, 3, 4$ be such that $\gamma_{x^{(j)}, \theta^{(j)}}(\tau_j)$ is the first conjugate point of $x^{(j)}$ along the geodesics and $\tau_{\min} = \min_{j=1,2,3,4}(\tau_j)$. In the rest of this section, we shall study the interactions only in the following set

$$
\mathcal{N}((\vec{x}, \vec{\theta}), t_0) = M(T_0) \setminus \bigcup_{j=1}^{4} J^+_g(\gamma_{x^{(j)}, \theta^{(j)}}(\tau_j)),
$$

where $\vec{x} = (x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}), \vec{\theta} = (\theta^{(1)}, \theta^{(2)}, \theta^{(3)}, \theta^{(4)})$, i.e. away from the causal future of the conjugate points. In particular, $\Lambda_j = N^*K_j$ in $\mathcal{N}((\vec{x}, \vec{\theta}), t_0)$.

Recall that two submanifolds $X, Y$ of $M$ intersect transversally if

$$
T_qX + T_qY = T_qM, \forall q \in X \cap Y.
$$

For the codimension 1 submanifolds $K_i, i = 1, 2, 3, 4$ we consider, $\Lambda_i = N^*K_i \subset L^+M$ i.e. the co-vectors normal to $K_i$ are light-like. For the moment, we assume that they only intersect at $q_0$ transversally meaning

1) $K_i, K_j$ intersect transversally at a codimension 2 submanifold $K_{ij}, i < j$;
2) $K_i, K_j, K_k$ intersect at a codimension 3 submanifold $K_{ijk}, i < j < k$;
3) $K_i, i = 1, 2, 3, 4$ intersect at a point $q_0$. 

In particular, (2) implies that $K_{ij} \cap K_k$ transversally and (3) implies that the four submanifolds intersect at a point $q_0$ and the normal co-vectors $\zeta_i$ to $K_i$ at $q_0$ are linearly independent so they span the cotangent space $T^*_q M$. We remark that for any $q \in M$, we can find $K_i$ intersect transversally at $q$. For $i = 1, 2, 3, 4$, we shall denote

$$\Lambda_i = N^* K_i; \quad \Lambda_{ij} = N^* K_{ij}, i < j; \quad \Lambda_{ijk} = N^* K_{ijk}, i < j < k; \quad \Lambda_{q_0} = T^* q_0 M \setminus 0.$$  

These are Lagrangian submanifolds of $T^* M$. We introduce the following notations

$$\Lambda^{(1)} = \bigcup_{i=1}^4 \Lambda_i; \quad \Lambda^{(3)} = \bigcup_{i,j,k=1, i < j < k} \Lambda_{ijk}.$$  

It is obvious that $\Lambda^{(1)} \hat{\Theta} = \Lambda^{(1)}$, but $\Lambda^{(3)} \hat{\Theta}$ is not the same as $\Lambda^{(3)}$. We denote $\Theta = \Lambda^{(1)} \cup \Lambda^{(3)} \hat{\Theta}$ which is of particular importance below.

We begin with the interaction of scalar valued distorted plane waves. Let $v_i \in I^\mu (N^* K_i), i = 1, 2, 3, 4$ and $Q_{\hat{g}} \in I^{-\frac{2}{3} - \frac{1}{9}} (N^* \text{Diag}, \Lambda_{\hat{g}})$ be the causal inverse of $\Box_{\hat{g}}$ on $(M, \hat{g})$. For semilinear wave equations studied in [24], singularities of the following terms were analyzed

\begin{equation}
\begin{align*}
y_1 &= Q_{\hat{g}} (cv_1 v_2 v_3 v_4), \\
y_2 &= Q_{\hat{g}} (av_1 Q_{\hat{g}} (bv_2 v_3 v_4)), \\
y_3 &= Q_{\hat{g}} (bv_1 v_2 Q_{\hat{g}} (av_3 v_4)), \\
y_4 &= Q_{\hat{g}} (av_1 Q_{\hat{g}} (av_2 Q_{\hat{g}} (av_3 v_4))), \\
y_5 &= Q_{\hat{g}} (a Q_{\hat{g}} (av_1 v_2 Q_{\hat{g}} (av_3 v_4))),
\end{align*}
\end{equation}

where $a, b, c$ are smooth functions on $M$. The terms $y_i, i = 1, 2, 3, 4, 5$ involve multiplication of four conormal distributions whose singular support intersect at $q_0$. It is proved in Prop. 3.9 of [24] that $y_i$ have conormal singularities at $\Lambda_{q_0} \setminus \Theta$ i.e. the flow out of $\Lambda_{q_0}$ away from $\Theta$. We will see that such terms also appear in the asymptotic analysis of the Einstein-Maxwell equations, but they may contain derivatives. So we slightly generalize the result in [24] to include cases when the order of $v_i$ are different.

**Proposition 5.1.** Let $v_i \in I^\mu (\Lambda_i), i = 1, 2, 3, 4$ and $\bar{\mu} = \sum_{i=1}^4 \mu_i$. Let $\nabla$ be a smooth vector field. We have the following conclusions

1. $Q_{\hat{g}} (cv_1 v_2 v_3 v_4) \in I^{\bar{\mu} + \frac{4}{9}} (\Lambda_{q_0} \setminus \Theta)$;
2. $Q_{\hat{g}} (av_1 \nabla Q_{\hat{g}} (bv_2 v_3 v_4)), Q_{\hat{g}} (bv_1 v_2 \nabla Q_{\hat{g}} (av_3 v_4)) \in I^{\bar{\mu} + \frac{4}{9}} (\Lambda_{q_0} \setminus \Theta)$;
3. $Q_{\hat{g}} (av_1 \nabla Q_{\hat{g}} (av_2 \nabla Q_{\hat{g}} (av_3 v_4))), Q_{\hat{g}} (a \nabla Q_{\hat{g}} (av_1 v_2) \nabla Q_{\hat{g}} (av_3 v_4)) \in I^{\bar{\mu} + \frac{4}{9} - \frac{1}{2}} (\Lambda_{q_0} \setminus \Theta)$.

*Proof.* For a smooth vector field $\nabla$, $\nabla v_i \in I^{\mu + 1} (N^* Y_i)$ if $v_i \in I^\mu (N^* Y_i), i = 1, 2, 3, 4$. This is addressed for $\nabla = \nabla_{\hat{g}}$ in [31] Lemma 4.1] and the general case is the same. Also, that lemma tells that $\nabla Q_{\hat{g}} \in I^{-\frac{2}{3} + 1 - \frac{1}{2}} (N^* \text{Diag}, \Lambda_{\hat{g}})$ and the principal symbols can be found. Then the proof is that of [24 Prop. 3.9] by adjusting the orders. □

We emphasize that the set $\Theta$ is the union of the wave front set of $v_i$ and $\Lambda^{(3)} \hat{\Theta}$ is the wave front set of singularities generated by the triple wave interactions. So we only look at the new singularities in $\mathfrak{u}^{(4)}$ produced by the four wave interactions.

In Section 3.5 of [24], the principal symbols of the terms (5.1) are found explicitly. For our purpose, we just need the symbol of $y_3, y_4$ and $y_5$. Consider the symbols at $(q, \eta) \in \Lambda_{q_0} \setminus \Theta$, which is joined with $(q_0, \zeta) \in \Lambda_{q_0} \setminus \Theta$. Let
We assume that \( J ^ \flat \) derivatives is quite straightforward because "\( J ^ \flat \)" trivialized the density factors in the distributions in a local coordinate linearized metric component and \( \dot{J} \).

We remark that here we trivialized the density factors in the distributions in a local coordinate near \( q_0 \), so the symbols are functions. The generalization of the above formulas to the case with derivatives is quite straightforward because \( \sigma(\mathcal{V}v_i) = \sigma(\mathcal{V})(v_i) \), see also \cite{34} Lemma 4.1.

Now we are ready to study the singularities for the Einstein-Maxwell equations. As in Section 3.3 we assume that \( g^{(i)} \in H^{\mu+1}(N^*Y;B^4), i=1,2,3,4 \) are supported in \( V \) and \( \tilde{g}^{(i)} \) are compactly supported. In order that \( g^{(i)} \in E^4 \), we shall take \( \mu \leq -10 \) so that \( H^{\mu+1}(N^*Y) \subset H^7(M(T_0)) \subset C^4(M(T_0)) \subset E^4 \). We denote \( \tilde{g}^{(i)} = (0,\tilde{g}^{(i)}) \) and let \( \mathcal{P}^{(i)} = Q(\tilde{g}^{(i)}) \in H^{\mu-\frac{7}{2}}(N^*K;B^{14}) \) be distorted plane waves. Let \( \epsilon_i, i=1,2,3,4 \) be small parameters and \( \tilde{J} = \sum_{i=1}^4 \epsilon_i \tilde{g}^{(i)} \). Then \( \tilde{v} = \sum_{i=1}^4 \epsilon_i \tilde{v}^{(i)} = (\tilde{g},\tilde{\phi}) \) is the solution to the linearized equation \( \tilde{g}^{(i)} \mathcal{P}\tilde{v} = \tilde{J} \). We recall that \( \dot{g} \in H^{\mu-\frac{1}{2}}(\Lambda_2;B^{10}) \) is the linearized metric component and \( \tilde{\phi} \in H^{\mu-\frac{1}{2}}(\Lambda_2;B^{14}) \) is the linearized electromagnetic component.

By the microlocal linearization condition, we let \( J_\epsilon \) be such that \( \partial_{\epsilon_i} J_\epsilon |_{\epsilon_i = 0} = \tilde{g}^{(i)}, i = 1,2,3,4 \). Let \( \tilde{w} = (u,\phi) \) be the solution of the nonlinear equation \( (3.2) \) with source \( J_\epsilon \). From Prop. 2.1 we can write the asymptotic expansion of \( \tilde{w} \) as \( \epsilon_i \to 0 \)

\[
\tilde{w} = \tilde{v} + \sum_{1 \leq i < j \leq 4} \epsilon_i \epsilon_j \mathcal{U}^{(2)} + \sum_{1 \leq i < j < k \leq 4} \epsilon_i \epsilon_j \epsilon_k \mathcal{U}^{(3)} + \epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4 \mathcal{U}^{(4)} + \mathcal{R}_\epsilon,
\]

where \( \mathcal{R}_\epsilon \) denotes the collection of terms in \( H^4(M(T_0)) \) and \( \bigcup_{i=1}^4 O(\epsilon_i^2) \). In particular,

\[
\mathcal{U}^{(4)} = \partial_{\epsilon_1} \partial_{\epsilon_2} \partial_{\epsilon_3} \partial_{\epsilon_4} \tilde{w} |_{\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = 0}.
\]

Our goal is to analyze the singularities in \( \mathcal{U}^{(4)} \). It is not easy to find the terms in \( (5.3) \) explicitly because the nonlinear Einstein-Maxwell equations is complicated. We will simplify the computation by identifying the most singular terms in \( \mathcal{U}^{(4)} \).

From equations \( (3.2) \) and \( (3.3) \) which \( \tilde{w}, \tilde{v} \) satisfy, we derive

\[
\mathbf{P} (\tilde{w} - \tilde{v}) + \sum_{i=2}^4 P_i(x,\tilde{w}) + \sum_{i=2}^4 H_i(x,\tilde{w}) + \mathcal{R}_\epsilon = 0.
\]

We explain the terms in \( (5.5) \) in detail. For the moment, we shall ignore the asymptotic expansion terms involving \( \tilde{g} \) and we shall see later that they do not matter for our analysis.

First of all, the \( P_i, i = 2,3,4 \) terms come from the quasilinear term \( -g^{pq} \partial_p \partial_q \tilde{w} \), \( i = 1,2,\cdots,14 \) in \( (3.2) \). In a given local coordinates, we can express two tensors as \( 4 \times 4 \) matrices. It is easy to
see that

\[ g^{-1} = (\tilde{g} + u)^{-1} = (\text{Id} + \tilde{g}^{-1}u)^{-1}\tilde{g}^{-1} = \tilde{g}^{-1} - \tilde{g}^{-1}u\tilde{g}^{-1} + (\tilde{g}^{-1}u)^2\tilde{g}^{-1} + (\tilde{g}^{-1}u)^3\tilde{g}^{-1} + \cdots. \]

Then the \( P_i \) terms are

\[
P_2(x, \bar{w}) = (\tilde{g}^{-1}u\tilde{g}^{-1})^{pq} \frac{\partial^2 \bar{w}}{\partial x^p \partial x^q}, \quad P_3(x, \bar{w}) = -(\tilde{g}^{-1}u\tilde{g}^{-1})^{pq} \frac{\partial^2 \bar{w}}{\partial x^p \partial x^q},
\]

\[
P_4(x, \bar{w}) = (\tilde{g}^{-1}u\tilde{g}^{-1})^{pq} \frac{\partial^2 \bar{w}}{\partial x^p \partial x^q}, \quad i = 1, 2, \ldots, 14.
\]

More precisely, we write down the components of these terms as

\[
P_{2,i}(x, \bar{w}) = \tilde{g}^{pa} u_{ab} \tilde{g}^{bq} \frac{\partial^2 \bar{w}_i}{\partial x^p \partial x^q},
\]

\[
P_{3,i}(x, \bar{w}) = -\tilde{g}^{pa} u_{ab} \tilde{g}^{bc} u_{cd} \tilde{g}^{dq} \frac{\partial^2 \bar{w}_i}{\partial x^p \partial x^q},
\]

\[
P_{4,i}(x, \bar{w}) = \tilde{g}^{pa} u_{ab} \tilde{g}^{bc} u_{cd} \tilde{g}^{de} u_{ef} \tilde{g}^{fq} \frac{\partial^2 \bar{w}_i}{\partial x^p \partial x^q}, \quad i = 1, 2, \ldots, 14.
\]

Notice that each term has two derivatives and the coefficients of the derivatives are polynomials of the metric component. It is also convenient to regard them as multi-linear functions. For example,

\[
P_{2,i}(x, \bar{w}^{(1)}, \bar{w}^{(2)}) = \tilde{g}^{pa} u_{ab} \tilde{g}^{bq} \frac{\partial^2 \bar{w}_i^{(2)}}{\partial x^p \partial x^q}, \quad i = 1, 2, \ldots, 14.
\]

Next, the terms \( H_i, i = 2, 3, 4 \) in (5.5) come from the semilinear terms of (3.2). They are section valued in \( \mathbb{B}^{14} \). Each component of \( H_i \) is a sum of \( i \)-th order monomials of \( \bar{w}, \partial \bar{w} \), but at most quadratic in \( \partial \bar{w} \), where \( \bullet = 1, 2, \ldots, 14 \) denote a generic index. This follows from the expression of the reduced Ricci tensor (2.9) and the fact that the Christoffel symbols only have first derivatives of the metric \( g \). We can write the component of \( H_i, i = 2, 3, 4 \) as multilinear functions as

\[
H_{2,\theta}(x, \bar{w}^{(1)}, \bar{w}^{(2)}) = \sum_{i,j=1}^{1} \sum_{\alpha, \beta=1}^{4} \sum_{a,b=0}^{1} \tilde{g}^\theta_{i,j\alpha\beta\theta} \partial^{\alpha}_{\alpha} \bar{w}_i^{(1)} \partial^{\beta}_{\beta} \bar{w}_j^{(2)},
\]

\[
H_{3,\theta}(x, \bar{w}^{(1)}, \bar{w}^{(2)}, \bar{w}^{(3)}) = \sum_{i,j,k=1}^{4} \sum_{\alpha, \beta, \gamma=1}^{2} \sum_{a,b,c=0,1,a+b+c\leq 2} H^\theta_{3,i,j,k\alpha\beta\gamma\theta} \partial^{\alpha}_{\alpha} \bar{w}_i^{(1)} \partial^{\beta}_{\beta} \bar{w}_j^{(2)} \partial^{\gamma}_{\gamma} \bar{w}_k^{(3)},
\]

\[
H_{4,\theta}(x, \bar{w}^{(1)}, \bar{w}^{(2)}, \bar{w}^{(3)}, \bar{w}^{(4)}) = \sum_{i,j,k,l=1}^{4} \sum_{\alpha, \beta, \gamma, \delta=1}^{2} \sum_{a,b,c,d\in \mathbb{A}} H^\theta_{4,i,j,k,l\alpha\beta\gamma\delta\theta} \partial^{\alpha}_{\alpha} \bar{w}_i^{(1)} \partial^{\beta}_{\beta} \bar{w}_j^{(2)} \partial^{\gamma}_{\gamma} \bar{w}_k^{(3)} \partial^{\delta}_{\delta} \bar{w}_l^{(4)},
\]

where the set \( \mathbb{A} = \{(a, b, c, d) : a, b, c, d = 0, 1, a+b+c+d \leq 2\} \), \( \theta = 1, 2, \ldots, 14 \) and the coefficients are all smooth. We emphasize that the derivatives of \( \bar{w} \) in these terms appear at most twice and such terms are especially important for the analysis below. We denote such terms i.e. terms in \( H_i \) with two derivatives, by \( \tilde{H}_i, i = 2, 3, 4 \).

For convenience, we denote

\[ G_i(x, \bar{w}) = P_i(x, \bar{w}) + H_i(x, \bar{w}), \quad i = 2, 3, 4, \]
then $G_i$ are polynomials in $\vec{w}, \partial \vec{w}$ of order $i$. We also denote $\tilde{G}_i = P_i + \tilde{H}_i$. Then the asymptotic expansion (5.5) can be written as

$$
\vec{w} = \vec{v} - \mathcal{Q}(G_2 + G_3 + G_4) + \mathcal{R}_e.
$$

By the stability estimate, we have $\|\vec{w}\|_{E^4} \leq C \sum_{i=1}^4 \epsilon_i$. Note $E^4 \subset H^4(M)$ is an algebra. We have

$$
\|\vec{w}_i \vec{w}_j\|_{H^4(M)} \leq C(\sum_{i=1}^4 \epsilon_i)^2.
$$

Therefore, by the continuity of $\mathcal{Q}$, we obtained the first term in (5.5). Also, we notice that all the terms in $\mathcal{R}_e$ are in $H^4(M)$. To get other terms, we will iterate the formula (5.8) i.e. plug (5.8) to the right hand side of (5.8). This will generate many terms. However, we only need terms of the order $\epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4$ which can only be obtained from the multiplication of four terms of $\vec{v}$ and $\partial \vec{v}$.

**Proposition 5.2.** Consider the fourth order interaction term $\mathcal{U}^{(4)}$ defined in (5.4).

1. If $\bigcap_{j=1}^4 Q_{x_j, \theta_j}(t) = q_0$, for $s_0$ sufficiently small so that $K_i$ only intersect at $q_0$, then on $N((\vec{x}, \vec{0}), t_0)$, we can write $\mathcal{U}^{(4)} = \mathcal{Q}(\mathcal{H} + \tilde{\mathcal{H}})$ such that

$$
\mathcal{Q}(\mathcal{H}) \in \mathcal{I}^{4\mu + \frac{1}{2}}(\Lambda^2 \Theta \setminus \Theta; \mathbf{B}^{14}) \text{ and } \mathcal{Q}(\tilde{\mathcal{H}}) \in \mathcal{I}^{4\mu + \frac{1}{2}}(\Lambda^2 \Theta \setminus \Theta; \mathbf{B}^{14}).
$$

The term $\mathcal{H}$ is given by

$$
\mathcal{H} = \sum_{(i,j,k,l) \in \sigma(4)} \left( \tilde{H}_3(x, \vec{v}^{(i)}, \vec{v}^{(j)}, \mathcal{Q}(\tilde{H}_2(x, \vec{v}^{(k)}, \vec{v}^{(l)}))) + \tilde{H}_3(x, \vec{v}^{(i)}, \mathcal{Q}(\tilde{H}_2(x, \vec{v}^{(j)}, \vec{v}^{(k)})), \vec{v}^{(j)}) + \tilde{H}_3(x, \mathcal{Q}(\tilde{H}_2(x, \vec{v}^{(i)}, \vec{v}^{(j)})), \vec{v}^{(k)}, \vec{v}^{(l)}))

- \sum_{(i,j,k,l) \in \sigma(4)} \tilde{G}_2(x, \mathcal{Q}(\tilde{H}_2(x, \vec{v}^{(i)}, \vec{v}^{(j)})), \mathcal{Q}(\tilde{H}_2(x, \vec{v}^{(k)}), \vec{v}^{(l)})))

- \sum_{(i,j,k,l) \in \sigma(4)} \left( \tilde{H}_2(x, \vec{v}^{(i)}, \mathcal{Q}(P_2(x, \mathcal{Q}(\tilde{H}_2(x, \vec{v}^{(j)}, \vec{v}^{(k)})), \vec{v}^{(l)})))

+ \tilde{H}_2(x, \mathcal{Q}(P_2(x, \mathcal{Q}(\tilde{H}_2(x, \vec{v}^{(i)}, \vec{v}^{(j)})), \vec{v}^{(k)})), \vec{v}^{(l)}))

\right)

\text{ with } \sigma(4) \text{ denoting the set of permutations of } (1, 2, 3, 4).

2. If $\bigcap_{j=1}^4 Q_{x_j, \theta_j}(t) = \emptyset$, then $\mathcal{U}^{(4)}$ is smooth on $N((\vec{x}, \vec{0}), t_0)$ microlocally away from $\Theta$.

**Proof.** (1) We divide the proof into two steps.

**Step 1:** We first determine the most singular terms in $\mathcal{U}^{(4)}$. We claim that $\mathcal{U}^{(4)} = \mathcal{Q}(\mathcal{H} + \tilde{\mathcal{H}})$ such that

$$
\mathcal{Q}(\mathcal{H}) \in \mathcal{I}^{4\mu + \frac{1}{2}}(\Lambda^2 \Theta \setminus \Theta; \mathbf{B}^{14}), \quad \mathcal{Q}(\tilde{\mathcal{H}}) \in \mathcal{I}^{4\mu + \frac{1}{2}}(\Lambda^2 \Theta \setminus \Theta; \mathbf{B}^{14}).
$$
Here $\mathcal{H} = \sum_{i=1}^{3} \mathcal{H}_i$ in which

\begin{equation}
\mathcal{H}_1 = - \sum_{(i,j,k,l) \in \sigma(4)} \tilde{G}_4(x, \tilde{v}^{(i)}, \tilde{v}^{(j)}, \tilde{v}^{(k)}, \tilde{v}^{(l)})
\end{equation}

\begin{equation}
\mathcal{H}_2 = \sum_{(i,j,k,l) \in \sigma(4)} \left( \tilde{G}_3(x, \tilde{v}^{(i)}, \tilde{v}^{(j)}, Q(\tilde{G}_2(x, \tilde{v}^{(k)}, \tilde{v}^{(l)}))) + \tilde{G}_3(x, \tilde{v}^{(i)}, Q(\tilde{G}_2(x, \tilde{v}^{(j)}, \tilde{v}^{(k)})), \tilde{v}^{(j)}) \right. \\
+ \tilde{G}_3(x, Q(\tilde{G}_2(x, \tilde{v}^{(i)}, \tilde{v}^{(j)})), \tilde{v}^{(k)}, \tilde{v}^{(l)}) \\
+ \sum_{(i,j,k,l) \in \sigma(4)} \left( \tilde{G}_2(x, Q(\tilde{G}_3(x, \tilde{v}^{(i)}, \tilde{v}^{(j)}), \tilde{v}^{(l)})) + \tilde{G}_2(x, \tilde{v}^{(i)}, Q(\tilde{G}_3(x, \tilde{v}^{(j)}), \tilde{v}^{(k)})), \tilde{v}^{(l)})) \right)
\end{equation}

and

\begin{equation}
\mathcal{H}_3 = - \sum_{(i,j,k,l) \in \sigma(4)} \tilde{G}_2(x, Q(\tilde{G}_2(x, \tilde{v}^{(i)}), \tilde{v}^{(j)})), Q(\tilde{G}_2(x, \tilde{v}^{(k)}, \tilde{v}^{(l)}))) \\
+ \sum_{(i,j,k,l) \in \sigma(4)} \left( \tilde{G}_2(x, \tilde{v}^{(i)}, Q(\tilde{G}_2(x, \tilde{v}^{(j)}), \tilde{v}^{(k)})), \tilde{v}^{(l)})) \right)
\end{equation}

We use the formula (5.8) to iterate to get the asymptotic terms. First of all, we put (5.8) to the right hand side of (5.8) to get one term from $G_4(x, \vec{w})$:

\begin{equation}
G_4(x, \vec{w}) = G_4(x, \vec{v}) + \mathcal{R}_\epsilon = \sum_{(i,j,k,l) \in \sigma(4)} G_4(x, \tilde{v}^{(i)}, \tilde{v}^{(j)}, \tilde{v}^{(k)}, \tilde{v}^{(l)}) + \mathcal{R}_\epsilon.
\end{equation}

The summation terms consist of two types of terms because $G_4 = P_4 + H_4$. The terms from $P_4$ can be found in (5.7) and they all have two derivatives. Using Prop. 5.1 we conclude that $Q(P_4(x, \tilde{v}^{(i)}, \tilde{v}^{(j)}, \tilde{v}^{(k)}, \tilde{v}^{(l)})) \in I^{4\mu+2}(\Lambda_{\mathcal{D}_4}^{3} \setminus \Theta)$. The terms from $H_4$ are of the form

\begin{equation}
A_{ijkl\alpha\beta\gamma\delta} v_i^{(a)} v_j^{(b)} \partial^{\gamma\delta}_{\alpha\beta} v_k^{(c)} \partial^{\gamma\delta}_{\alpha\beta} v_l^{(d)} , \quad m, n \leq 1 ; \quad i, j, k, l = 1, \cdots, 14; \quad \alpha, \beta = 1, 2, 3, 4,
\end{equation}

and $a, b, c, d$ are permutations of $1, 2, 3, 4$ (Note this is not in Einstein summation.) We can apply Prop. 5.1 to conclude that when $m, n = 1$, the term after applying $Q$ is in $I^{4\mu+2}(\Lambda_{\mathcal{D}_4}^{3} \setminus \Theta)$ and otherwise in $I^{4\mu+\frac{1}{2}}(\Lambda_{\mathcal{D}_4}^{3} \setminus \Theta)$. When $m = n = 1$, the terms only come from $\tilde{H}_4$. Thus we obtain the leading term $\mathcal{H}_1$.

Next consider the other terms in the asymptotic expansion. To get order $\epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4$ terms, we need to iterate twice or three times using (5.8). From the term $G_3(x, \vec{w})$, we get

\begin{equation}
G_3(x, \vec{w}) = - \sum_{(i,j,k,l) \in \sigma(4)} \left( G_3(x, \tilde{v}^{(i)}, \tilde{v}^{(j)}, Q(G_2(x, \tilde{v}^{(k)})), \tilde{v}^{(l)})) \right)
\end{equation}

\begin{equation}
+ G_3(x, \tilde{v}^{(i)}, Q(G_2(x, \tilde{v}^{(j)}), \tilde{v}^{(k)})), \tilde{v}^{(j)}) + G_3(x, Q(G_2(x, \tilde{v}^{(i)}), \tilde{v}^{(j)})), \tilde{v}^{(k)}), \tilde{v}^{(l)}) \right) + \mathcal{R}_\epsilon.
\end{equation}

From Prop. 5.1, we know that the term after applying $Q$ is in $I^{4i+2} (\Lambda^q_y \setminus \Theta)$ if the $G_i$, $i = 2, 3$ terms involved have two derivatives i.e. they are $\hat{G}_i$, $i = 2, 3$. Otherwise, the terms are in $I^{4i+\frac{2}{3}} (\Lambda_y^q \setminus \Theta)$ which are less singular. So we get one piece in $\mathcal{H}_2$.

Finally, from the term $G_2(x, \bar{w})$, we get from the iteration using (5.8) that

$$G_2(x, \bar{w})$$

$$= G_2(x, \bar{v} - Q(G_2(x, \bar{w}) + G_3(x, \bar{w}))) + R_\epsilon$$

$$= G_2(x, Q(G_2(x, \bar{v})) - G_2(x, \bar{v}, Q(G_2(x, \bar{v}))) - G_2(x, Q(G_3(x, \bar{v})), \bar{v})$$

$$- G_2(x, Q(G_2(x, \bar{v}))) - G_2(x, Q(G_2(x, \bar{v}))) - G_2(x, Q(G_3(x, \bar{v})), \bar{v})$$

$$= G_2(x, Q(G_2(x, \bar{v})) - G_2(x, \bar{v}, Q(G_3(x, \bar{v}))) - G_2(x, Q(G_3(x, \bar{v})), \bar{v})$$

$$+ G_2(x, \bar{v}, Q(G_2(x, \bar{v}))) + G_2(x, Q(G_2(x, \bar{v}))) - G_2(x, Q(G_3(x, \bar{v})), \bar{v})$$

$$+ G_2(x, Q(G_2(x, \bar{v}))) + G_2(x, Q(G_2(x, \bar{v}))) + R_\epsilon.$$  

Using $\bar{v} = \sum_{i=1}^4 \bar{v}^{(i)}$, we find that

$$G_2(x, \bar{v}, Q(G_3(x, \bar{v}))) + G_2(x, Q(G_3(x, \bar{v})), \bar{v})$$

$$= \sum_{(i,j,k,l) \in \sigma(4)} \left( G_2(x, \bar{v}^{(i)}, Q(G_3(x, \bar{v}^{(j)}, \bar{v}^{(k)})), \bar{v}^{(l)}) + G_2(x, Q(G_3(x, \bar{v}^{(i)}), \bar{v}^{(j)}), \bar{v}^{(k)}), \bar{v}^{(l)}) \right)$$

Applying Prop. 5.1, we see that the leading order term is achieved when the $G_i$, $i = 2, 3$ are $\hat{G}_i$ and the leading terms are in $I^{4i+2} (\Lambda^q_y \setminus \Theta)$. So we get the other piece of $\mathcal{H}_3$. Using the same argument, we can obtain the terms in $\mathcal{H}_3$ from the rest of terms in (5.13). The details are omitted here.

Finally, let’s consider the asymptotic expansion terms involving $\bar{g}^{(i)}$ which we have dropped in (5.5). Notice that our source $J_\epsilon = (J_0^\epsilon, J^\epsilon)$ where $J^\epsilon_a = \sum_{i=1}^4 \epsilon_i \bar{g}^{(i,a)}$, $a = 1, 2, 3$ and according to (4.12) we have

$$J^\epsilon_a = - (\det g)^{-\frac{1}{2}} \sum_{a=1}^3 \int_0^t \partial_a ((\det g)^{\frac{1}{2}} \sum_{i=1}^4 \epsilon_i \bar{g}^{(i,a)})ds$$

So for example the order $\epsilon_i \epsilon_j, i \neq j$ terms involving $\bar{g}^{(i)}$ in (5.5) is a summation of the product of $\bar{w}_a$, $a = 1, 2, \cdots, 14$ and $\bar{g}^{(i)}_b$, $b = 0, 1, 2, 3$ and possibly their derivatives in $y$ and integrals in $t$. But WF($\bar{g}^{(i)}$) is close to $\Lambda_i \cap N^*Y_i$. By a wave front analysis and the argument in Lemma 3.2, we conclude that the wave front of these terms are included in $\Lambda^{(1)}$. For the order $\epsilon_i \epsilon_j \epsilon_k$, $i < j < k$ terms involving $\bar{g}^{(i)}$, a similar analysis tells that the wave front set of these terms are contained in $\Lambda^{(1)}$ as well, and finally the wave front set of order $\epsilon_1 \epsilon_2 \epsilon_3 \epsilon_4$ terms involving $\bar{g}^{(i)}$ are contained in $\Theta$. This finishes the proof of the claim.

**Step 2:** We recall the notation that $\bar{v}^{(i)} = (g^{(i)}, \dot{g}^{(i)})$ is the linearized wave with source $\bar{g}^{(i)}$, and $\bar{v} = (\dot{g}, \dot{\phi}) = \sum_{i=1}^4 \epsilon_i \bar{v}^{(i)}$. We already observed that the metric components $\dot{g}$ are all zero because of the choice of the source and the fact that the linearized equation (3.3) are decoupled. We use these to further identify the most singular terms in $\mathcal{H}$.
(i) We see that
\[ H_{4,\alpha\beta}(x, \vec{v}, \vec{v}) = 0 \quad \text{and} \quad P_{4,\alpha\beta}(x, \vec{v}, \vec{v}) = 0, \quad \alpha, \beta = 0, 1, 2, 3 \]
because they all have the metric component. This implies that \( \mathcal{H}_1 \) vanishes.

(ii) Notice that \( P_3(x, \vec{v}^{(i)}, \vec{v}^{(j)}, \vec{v}^{(k)}) \) is at least quadratic in the metric components and \( P_2(x, \vec{v}^{(i)}, \vec{v}^{(j)}) \) and \( \tilde{H}_3(x, \vec{v}^{(i)}, \vec{v}^{(j)}, \vec{v}^{(k)}) \) are at least linear in the metric components. As the metric components of the linearized waves \( \tilde{v}^{(i)} \) vanish, the non-vanishing term in \( \mathcal{H}_2 \) is
\[
\mathcal{H}_2 = \sum_{(i,j,k,l) \in \sigma(4)} \left( \tilde{H}_3(x, \vec{v}^{(i)}, \vec{v}^{(j)}, Q(\tilde{H}_2(x, \vec{v}^{(k)})), \vec{v}^{(i)})) \right.
\]
\[
+ \tilde{H}_3(x, Q(\tilde{H}_2(x, \vec{v}^{(j)}), \vec{v}^{(k)}), \vec{v}^{(i)})) \right).
\]

(iii) The components in \( \tilde{H}_2(x, \vec{v}, \vec{v}) \) involving \( \hat{g} \) components vanish. For analyzing terms in \( \mathcal{H}_3 \), we identify the \( \tilde{H}_2 \) terms which have \( \hat{\phi} \) components. In the reduced Einstein equations, such \( \tilde{H}_2 \) terms are quadratic in \( \hat{\phi} \) and they come from the stress-energy term \( T_{\text{em}}(\phi) \). For the Maxwell equations, \( \tilde{G}_2 \) terms are only linear in \( \hat{\phi} \) and this is when \( \tilde{G}_2 = P_2 \). As a result, the electromagnetic potential component of \( Q(\tilde{G}_2(x, \vec{v}^{(i)}, \vec{v}^{(j)})), i, j = 1, 2, 3, 4, i \neq j \) are zero but the metric components could be non-zero when we have \( Q(\tilde{H}_2(x, \vec{v}^{(i)}), \vec{v}^{(j)})) \). Then only the electromagnetic components of \( \tilde{G}_2(x, \vec{v}^{(k)}, Q(\tilde{H}_2(x, \vec{v}^{(i)}), \vec{v}^{(j)})) \) could be non-zero when \( \tilde{G}_2 = P_2 \). Thus, we can simplify \( \mathcal{H}_3 \) to
\[
\mathcal{H}_3 = - \sum_{(i,j,k,l) \in \sigma(4)} \tilde{G}_2(x, Q(\tilde{H}_2(x, \vec{v}^{(i)}), \vec{v}^{(j)})), Q(\tilde{H}_2(x, \vec{v}^{(k)}), \vec{v}^{(i)}))
\]
\[
= - \sum_{(i,j,k,l) \in \sigma(4)} \left( \tilde{H}_2(x, \vec{v}^{(i)}, Q(P_2(x, Q(\tilde{H}_2(x, \vec{v}^{(j)}), \vec{v}^{(k)})), \vec{v}^{(i)})) \right)
\]
\[
+ \tilde{H}_2(x, Q(P_2(x, Q(\tilde{H}_2(x, \vec{v}^{(i)}), \vec{v}^{(j)})), \vec{v}^{(k)}), \vec{v}^{(i)})) \right),
\]
in which all \( \tilde{H}_2 \) are quadratic in \( \hat{\phi} \) components. This completes the proof of part (1).

(2) The proof is the same as that of Prop. 4.1 of [24] for the scalar case. If \( K_i, i = 1, 2, 3, 4 \) do not intersect, the singularities of \( U^{(4)} \) are at most conic (when three of \( K_i \) intersect) and the wave front set is contained in \( \Theta \).

Finally, we show that by choosing distorted plane waves, the newly generated singularities of \( U^{(4)} \) are not always vanishing.

**Proposition 5.3.** Suppose that geodesics \( \gamma_{x(j), \theta(j)}(\mathbb{R}^+), j = 1, 2, 3, 4 \) intersect at \( \gamma_{x(j), \theta(j)}(t_j) = q_0 \) and \( \gamma_{x(j), \theta(j)}(t_j), j = 1, 2, 3, 4 \), are linearly independent. Let \( \tilde{v}^{(i)} = (\hat{g}^{(i)}, \hat{\phi}^{(i)}) \) be distorted plane waves propagating near geodesics \( \gamma_{x(j), \theta(j)}(\mathbb{R}^+) \), described in the beginning of Section 5.2, so that \( \tilde{v}^{(i)} \in I^{\mu-\frac{5}{2}}(N^* K_i; \mathbf{B}^{14}) \) in \( M(T_0) \backslash Y_0 \). For the fourth order interaction term \( U^{(4)} \) produced by the waves \( \tilde{v}^{(i)} \), we denote by \( U^{(4), \text{met}} \) the metric component and by \( U^{(4), \text{em}} \) the electromagnetic potential component. Let \( (q, \eta) \in A_{q_0}^g \backslash \Theta \) which is joined to \( (q_0, \zeta) \in A_{q_0} \) by bi-characteristics, and \( \zeta^{(i)} \in N^*_q K_i \) be such that \( \zeta = \sum_{i=1}^4 \zeta^{(i)} \).

We have the following conclusions for the principal symbol of \( U^{(4)} \) in \( I^4+\frac{3}{2}(\Lambda_{q_0}^g \backslash \Theta; \mathbf{B}^{14}) \).

(i) \( \sigma(U^{(4), \text{em}})(q, \eta) = 0 \).
terms \( \hat{0} \) come from the reduced Einstein equation. In particular, the electromagnetic component we see that
\[
\vec{\zeta} \vec{A} \tag{5.16}
\]
can do the considerations below in the case when the space \((q, \eta) \) near to show they are non-zero functions. Without loss of generality, we can use local coordinate \( q \) of the metric at \((q, \eta) \) such that \( \sigma = 0 \), \( \mu = 0, 1, 2, 3 \) are homogeneous polynomial of degree 4 in \( A^{(i)} \in (\mathbb{R}^4)^4, i = 1, 2, 3, 4 \) whose coefficients are real-analytic functions defined on
\[
\mathcal{X}(\zeta) = \{(\zeta^{(1)}, \zeta^{(2)}, \zeta^{(3)}, \zeta^{(4)}) \in (L^*_q M)^4 : \sum_{i=1}^{4} \zeta^{(i)} = 0 \}.
\]

(iii) Assume that \( q \) has a neighborhood \( B \) such that the intersection of \( B \) and the light cone \( L^+_{q_0} \) is a smooth 3-dimensional surface \( S = B \cap L^+_{q_0} \).

We denote below \( x^{(i)}, \theta^{(i)}, \), \( i = 1, 2, 3, 4 \) so that the geodesic \( \gamma_{x^{(i)}, \theta^{(i)}} \) intersect at \( q_0 \). For any \((q, \eta) \in \Lambda^q \setminus \Theta \), one can find \( (x^{(i)}, \theta^{(i)}) \) in an arbitrarily small neighborhood of \( (x^{(i)}, \theta^{(i)}) \) and distorted plane waves \( \tilde{\sigma}^{(i)} \) associated with \( \gamma_{x^{(i)}, \theta^{(i)}} \) such that the interaction term \( \mathcal{U}^{(4)} \) has non-vanishing singularities at \((q, \eta) \) i.e. \( \mathcal{U}^{(4)} \) is not smooth at \( q \). Moreover, some of these singularities have such polarization that they can be observed by taking inner products of the field \( \mathcal{U}^{(4)} \) and tensor products of vector fields that are tangent to the light cone \( L^+_{q_0} \).

**Proof of Prop. 5.3**  
**Claim (i).** From Prop. 5.2 we can write the symbol of \( \mathcal{U}^{(4)} \) at \((q, \eta) \) as
\[
\sigma(\mathcal{U}^{(4)})(q, \eta) = \sigma(\mathcal{Q})(q, \eta, q_0, \zeta)\sigma(\mathcal{H})(q_0, \zeta), \tag{5.16}
\]
where \( \mathcal{H} \) is given in Prop. 5.2. From the proof of Prop. 5.2 we’ve seen that it suffices to consider \( \mathcal{H} \) the terms \( \tilde{H}_2 \) that are quadratic in \( \phi \) the electromagnetic components, and such terms only come from the reduced Einstein equation. In particular, the electromagnetic component \( \tilde{H}_{2, \mu}, \mu = 0, 1, 2, 3 \) are all zero. However, from the Maxwell equation (3.2), we observe that the nonlinear terms \( \tilde{P}_{2, \mu}, \tilde{H}_{3, \mu}, \mu = 0, 1, 2, 3 \) are always linear in \( \phi \). Therefore, from the expression of \( \mathcal{H} \) above, we see that \( \mathcal{H}_\mu = 0, \mu = 0, 1, 2, 3 \). This implies that the principal symbol \( \sigma(\mathcal{U}^{(4)}_{\text{em}})(q, \eta) = 0 \).

**Claim (ii).** We denote below \( \tilde{\zeta} = (\zeta^{(1)}, \zeta^{(2)}, \zeta^{(3)}, \zeta^{(4)}) \) and \( \tilde{A} = (A^{(1)}, A^{(2)}, A^{(3)}, A^{(4)}) \). As we already discussed in part (i), in \( \mathcal{H} \) it suffices to consider \( \tilde{H}_2 \) which are quadratic in \( \phi \). The terms in \( B^{(i)} \) are similar to \( y_3, y_4, y_5 \) in discussion after Prop. 5.2 for which we have the principal symbol, see (5.2). Using these formulas for (5.9), we see that the principal symbol \( \sigma(\mathcal{H})(q_0, \zeta) = \mathcal{P}(\tilde{\zeta}, \tilde{A}) \) is a fourth order polynomial of \( \tilde{A}_a^{(i)}, i = 1, 2, 3, 4, a = 0, 1, 2, 3 \). Therefore, they are real analytic functions of \( \tilde{\zeta} \) and \( \tilde{A} \), see more details in the proof of (iii). This proves claim (ii).

**Claim (iii).** Now we compute the metric component of \( \sigma(\mathcal{H})(q_0, \zeta) \) in (5.16) in terms of \( \tilde{\zeta} \) and \( \tilde{A} \) to show they are non-zero functions. Without loss of generality, we can use local coordinate near \( q_0 \) such that \( \tilde{g} \) is the standard Minkowski metric \( h \) in \( \mathbb{R}^4 \) at \( q_0 \). Then we can use the Einstein-Maxwell equations in Minkowski space-time found in Section 4. We remark that the derivatives of the metric at \( q_0 \) in the local coordinates do not contribute in the principal symbols and thus we can do the considerations below in the case when the space \((M, \tilde{g})\) is the Minkowski space.
We assume that $\zeta$ is chosen to be $\zeta = (9/4, -7/4, 1, 1)$. Consider the following set of vectors
\[(5.17) \quad \zeta^{(1)} = (1, 0, 1, 0), \quad \zeta^{(2)} = (1, 0, 0, 1), \quad \zeta^{(3)} = (-3/4, -3/4, 0, 0), \quad \zeta^{(4)} = (1, -1, 0, 0).\]
These are light-like vectors and $\sum_{i=1}^{4} \zeta^{(i)} = (9/4, -7/4, 1, 1) = \zeta$. Let $i$ be the imaginary unit i.e. $i^2 = -1$. We shall take
\[(5.18) \quad A^{(1)} = -(0, 0, 0, i), \quad A^{(2)} = -(0, 0, i, 0), \quad A^{(3)} = -(0, 0, 0, i), \quad A^{(4)} = -(0, 0, 0, i)\]
as the symbols of $\phi^{(i)}$ at $\zeta^{(i)}$. We will also denote the symbols given in (5.18) by $A^{(i)} = A^{(i)}$ and $1\tilde{A} = (1A^{(i)})_{i=1}^{4}$. With these choices, we see that the symbols $A^{(i)}$ satisfy the microlocal linearized gauge condition i.e.
\[(5.19) \quad \hbar^{\alpha\lambda} \zeta^{(i)} A^{(i)} = 0, \quad i = 1, 2, 3, 4.\]
Next we compute the principal symbol of $F^{(i)}$ at $(q_{0}, \zeta^{(i)})$ (in $P^{\mu+1}$) using $\sigma(F^{(i)}_{\alpha\beta}) = i\zeta^{(i)}_{\beta} A^{(i)}_{\beta} - \alpha_{\beta} A^{(i)}_{\beta}$. Due to the choices of the vectors, many components of $F$ vanish. Actually, the non-zero ones are
\[
\begin{align*}
F_{03}^{(1)} &= 1, \quad F_{23}^{(1)} = 1; \quad F_{02}^{(2)} = 1, \quad F_{32}^{(2)} = 1;
F_{03}^{(3)} &= -3/4, \quad F_{13}^{(3)} = -3/4; \quad F_{03}^{(4)} = 1, \quad F_{13}^{(4)} = -1,
\end{align*}
\]
and their (anti-)symmetric terms. By straightforward computations carried out in Appendix A, we find that
\[(5.20) \quad \mathcal{P} = \sigma(\mathcal{H})(q_{0}, \zeta) = c_{\pi} \begin{pmatrix}
* & * & -5.3 & -0.7 \\
* & * & 2.1 & 4.9 \\
-5.3 & 2.1 & * & -4.5 \\
-0.7 & 4.9 & -4.5 & *
\end{pmatrix}
\]
where $*$ stands for the elements which we didn’t compute and $c_{\pi} = (2\pi)^{-3}$ is a non-zero constant. This proves that $\mathcal{P}$ hence the symbol of $\mathcal{U}^{(i),\text{met}}$ is non-vanishing for the choice of $\zeta^{(i)}$ and $A^{(i)} = 1\tilde{A}^{(i)}$.

We need to construct four other examples of choices of $A^{(i)}$ such that the symbol of $\mathcal{H}^{\alpha\beta}$ is non-vanishing. For simplicity, we define an operator $T$ from $\text{Sym}^{2}(T_{q_{0}}M \otimes T_{q_{0}}M) \simeq \mathbb{R}^{10}$ to $\mathbb{R}^{5}$ as follows:
\[T(B) = (B_{02}, B_{03}, B_{12}, B_{13}, B_{23}),\]
where $B \in \text{Sym}^{2}(T_{q_{0}}M \otimes T_{q_{0}}M)$. In this notation, we have that
\[T\mathcal{P}(\zeta, 1\tilde{A}) = c_{\pi}(-5.3, -0.7, 2.1, 4.9, -4.5).\]
In Appendix A we construct four set of vectors $a\tilde{A}, a = 2, 3, 4, 5$ that satisfy the microlocal linearized gauge conditions (recall microlocal linearized gauge condition) and that $T\mathcal{P}(\zeta, a\tilde{A}), a = 1, 2, 3, 4, 5$ are non-zero and linearly independent in $\mathbb{R}^{5}$. Actually, the proof is based on the same kind of computation as for $1\tilde{A}$. We emphasize that in these examples we keep the vectors $\zeta^{(i)}$ fixed and we construct different choices of $\tilde{A}$.

Next we consider the dependency of $\sigma(\mathcal{H})$, or more precisely, of $\mathcal{P}(\zeta, \tilde{A})$ on the variables $\zeta$ and $\tilde{A}$. We will consider the case when $\zeta^{(i)} = \alpha_{i}\xi^{(i)}$, with $\alpha_{i} \in \mathbb{R}$ and $\xi^{(i)} \in L_{q_{0}}^{*+}M$, and denote
\[ \tilde{\xi} = (\xi^{(1)}, \xi^{(2)}, \xi^{(3)}, \xi^{(4)}) \in (L^* q_0 \cdot M)^4. \] Also, let
\[
\mathcal{Y} = \{ (\xi^{(1)}, \xi^{(2)}, \xi^{(3)}, \xi^{(4)}, A^{(1)}, A^{(2)}, A^{(3)}, A^{(4)}) \in (L^* q_0 \cdot M)^4 \times (\mathbb{R}^4)^4 : \]
\[
A^{(i)} \text{ satisfy the linearized gauge condition } \tilde{g}^{\alpha \lambda} \xi^{(i)} \alpha = 0. \}
\]

When \( \xi^{(1)}, \xi^{(2)}, \xi^{(3)}, \xi^{(4)} \) are linearly independent, let \( \alpha_i(\xi, \tilde{\xi}) \in \mathbb{R} \) be such that \( \alpha_i = \alpha_i(\xi, \tilde{\xi}) \) satisfy \( \xi = \sum_{i=1}^{4} \alpha_i(\xi) \). Then \( \alpha_i(\xi, \tilde{\xi}) \) is a quotient of real-analytic functions, that is, \( \alpha_i(\xi, \tilde{\xi}) = p_i(\xi, \tilde{\xi})/q_i(\xi, \tilde{\xi}) \) where \( p_i(\xi, \tilde{\xi}) \) and \( q_i(\xi, \tilde{\xi}) \) are real-analytic functions of \( \xi, \tilde{\xi} \in (L^* q_0 \cdot M)^5 \) and \( q_i(\xi, \tilde{\xi}) \) is not identically vanishing. Next, let us consider \( \zeta = (9/4, -7/4, 1, 1) \). Here vectors \( \xi^{(i)} \) are given in \([5.17]\) and we use \( \xi^{(i)} = \tilde{\xi}^{(i)} / \alpha_i \) where \( (\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (1, 1, -3/4, 1) \). Let
\[
S(\tilde{\xi}, \tilde{A}) = \mathcal{P}(\alpha_1(\xi, \tilde{\xi}) \tilde{\xi}^{(1)}, \alpha_2(\xi, \tilde{\xi}) \tilde{\xi}^{(2)}, \alpha_3(\xi, \tilde{\xi}) \tilde{\xi}^{(3)}, \alpha_4(\xi, \tilde{\xi}) \tilde{\xi}^{(4)}, \tilde{A}), \quad \text{for } (\tilde{\xi}, \tilde{A}) \in \mathcal{Y}.
\]

Note we have chosen \( \zeta \) to a fixed light-like vector, function \( S(\tilde{\xi}, \tilde{A}) \), that gives the symbol \( \sigma(\mathcal{Y})(q_0, \zeta) \) of the fourth interaction source corresponding to the waves \( \sigma^{(i)} \) having directions \( \xi^{(i)} \) and symbols \( A^{(i)} \), can be considered as a quotient of real-analytic functions of \( \xi \) and \( \tilde{A} \).

Now we let \( e_a \in \text{Sym}^2(T_{q_0} M \otimes T_{q_0} M), a = 1, 2, 3, 4, 5 \) be constructed as follows.
\[
e_{1,02} = e_{1,20} = 1, \quad e_{2,03} = e_{2,30} = 1,
\]
\[
e_{3,12} = e_{3,21} = 1, \quad e_{4,13} = e_{4,31} = 1, \quad e_{5,23} = e_{5,32} = 1
\]
and all the rest of the components are zero. Let \( Z = \text{span}\{ e_a, a = 1, 2, 3, 4, 5 \} \) be a 5 dimensional subspace of \( \text{Sym}^2(T_{q_0} M \otimes T_{q_0} M) \). Then, \( T \) is a projection onto the space \( Z \).

Consider the map
\[
D(\tilde{\xi}, (a \tilde{A})_{a=1}^5) = \det \left( (S(\tilde{\xi}, a \tilde{A}), e_{ib})^5_{a,b=1} \right).
\]
By substituting in \( D(\tilde{\xi}, (a \tilde{A})_{a=1}^5) \) the vectors \( \tilde{\xi} \) and the symbols given in formula \([5.18]\) and the other four sets, for which \( TS(\tilde{\xi}, a \tilde{A}) \), \( a = 1, 2, 3, 4, 5 \), are linearly independent, we see that the function \( D : \mathcal{Y} \rightarrow \mathbb{C} \) obtains at some point \( (\tilde{\xi}, (a \tilde{A})_{a=1}^5) \) \( \mathcal{Y} \) a finite and non-zero value, and thus it is not identically vanishing. Since \( \mathcal{Y} \) is a real-analytic manifold and \( D : \mathcal{Y} \rightarrow \mathbb{C} \cup \{ \infty \} \) is a quotient of real-analytic functions that is not identically vanishing, we see that \( D : \mathcal{Y} \rightarrow \mathbb{C} \) does not vanish or is infinity in any open subset of \( \mathcal{Y} \). Thus we see that in any open set \( W \subset (L^* q_0 \cdot M)^4 \) there are \( \tilde{\xi} = (\xi^{(1)}, \xi^{(2)}, \xi^{(3)}, \xi^{(4)}) \in W \) and \( (\tilde{\xi}, (a \tilde{A})_{a=1}^5) \in \mathcal{Y} \) such that vectors \( \xi^{(1)}, \xi^{(2)}, \xi^{(3)}, \xi^{(4)} \) are linearly independent and matrices \( S(\tilde{\xi}, a \tilde{A}), a = 1, 2, \cdots, 5 \) are linearly independent. Let \( \xi^{(i)} = \alpha_i(\xi, \tilde{\xi}) \xi^{(i)}, i = 1, 2, 3, 4 \) and \( a \vartheta^{(i)} = (a \vartheta^{(i)}) \in I^{(\mu) \frac{1}{2}}(N^* K_i), a = 1, 2, 3, 4, 5 \) be distorted plane waves described in the beginning of this section, such that \( a \vartheta^{(i)} = 0 \) and \( \vartheta^{(i)}(q_0, \zeta) = a A^{(i)} \). When \( a \vartheta^{(i)} \) is the wave produced by the linearized waves \( a \vartheta^{(i)}(x) \) corresponding to the vectors \( \xi \) and matrices \( a A^{(i)} \), we see that
\[
(5.21) \quad \langle \sigma(a \vartheta^{(i)}, \text{met})(q, \eta), V \rangle = \langle S(\tilde{\xi}, a \tilde{A}), \mathcal{R} T V \rangle, \quad \mathcal{R} = \sigma(Q \vartheta)(q, \eta, q_0, \zeta)
\]
where \( V \in \text{Sym}^2(T_{q} M \otimes T_{q} M) \). Since the linear map \( \mathcal{R} \) is bijective, and the space \( \text{Sym}^2(T_{q} S \otimes T_{q} S) \) has co-dimension 4 in \( \text{Sym}^2(T_{q} M \otimes T_{q} M) \), also the space \( \mathcal{R} T (\text{Sym}^2(T_{q} S \otimes T_{q} S)) \) has co-dimension 4 in \( \text{Sym}^2(T_{q} M \otimes T_{q} M) \). If the dualities \([5.21]\) would vanish for all \( a = 1, 2, 3, 4, 5 \), and for all \( \mathcal{R} T V \in \mathcal{R} T (\text{Sym}^2(T_{q} S \otimes T_{q} S)) \), then all the vectors \( S(\xi, a \tilde{A}), a = 1, 2, 3, 4, 5 \) would be in the four dimensional space \( (\mathcal{R} T (\text{Sym}^2(T_{q} S \otimes T_{q} S)))^\perp \). However, since the vectors \( S(\xi, a \tilde{A}), a = 1, 2, 3, 4, 5 \) are linearly independent, this is not possible.
The above implies that for some $V \in \text{Sym}^2(T_q S \otimes T_q S)$ and for some $a \in \{1, 2, 3, 4, 5\}$, the linearized waves $\pi^{(i)}(x)$, propagating near geodesics $\gamma_{q_0, \zeta}$ and having symbols $A^{(i)} = \hat{a} A^{(i)}$ at $(q_0, \zeta^{(i)})$, are such that the wave $\Upsilon^{(4)}$ produced by the fourth order interaction satisfies $\langle \sigma(\Upsilon^{(4)}, \text{met})(q, \eta), V \rangle \neq 0$. This proves the claim (iii).

6. Determination of the space-time

We finish the proof of Theorem 6.1. Since the rest of the proof is essentially the same as that of [21, Theorem 1.1] and the argument is quite involved, we shall only repeat the key argument and refer the interested readers to [21] for more details. Basically, we need to consider two things. First, we analyzed the singularities in wave gauge and we need to show such singularities can be observed in the data set in Fermi coordinates, that is roughly speaking, by freely falling observers that move in the perturbed spacetime. This follows from [21, Section 4]. Second, from the observed singularities, one can determine the earliest light observation set and the space-time structure. This is contained in [20, Section 4], see also [21, Section 3.5 and 5].

We first show that an analogue of [20, Lemma 4.2] holds for our problem, which is the key lemma for changing observations to Fermi coordinate. We remark that Prop. 5.2 and Prop. 5.3 are the analogous of Prop. 3.3 and Prop. 3.4 of [21]. Following [21, Section 4], we say that the interaction condition (I) is satisfied for $y \in V$ with light-like vectors $(\bar{x}, \bar{\xi}) = ((x^{(i)}, \xi^{(i)}))^4_{i=1}$ and $t_0 > 0$, if

(I) There exist $q_0 \in \bigcap_{i=1}^4 \gamma_{x^{(i)}(t_0), \xi^{(i)}(t_0)}((0, \tau_i))$, $\zeta \in L_{q_0}^\infty M$ and $t \geq 0$ such that $y = \gamma_{q_0, \zeta}(t)$. Here we recall that $\tau_i$ are defined in the beginning of Section 5 i.e. $\gamma_{x^{(i)}(t_0), \xi^{(i)}(t_0)}((0, \tau_i))$ is the first conjugate point.

Also, we say that $y \in V$ satisfies the singularity detection condition (D) with light-like directions $(\bar{x}, \bar{\xi})$ and $t_0, \bar{s} > 0$ if

(D) For any $s, s_0 \in (0, \bar{s})$ and $i = 1, 2, 3, 4$, there are $(\bar{x}^{(i)}, \bar{\xi}^{(i)})$ in the $s$-neighborhood of $(x^{(i)}, \xi^{(i)})$, $(2s_0)$-neighborhood $B^{(i)}$ of $x^{(i)}$ (these neighborhoods are defined using the Riemannian metric $\bar{g}^{\bar{x}}$) satisfying

$$J^{-}_{\bar{g}}(B^{(j)}) \cap J^{+}_{\bar{g}}(B^{(j)}) \subset V_{\bar{g}} \text{ and } J^{+}_{\bar{g}}(B^{(j)}) \cap J^{-}_{\bar{g}}(B^{(k)}) = \emptyset \text{ for all } j \neq k.$$

Also, consider sources $\bar{g}^{(i)}$ constructed as in the beginning of Section 5. Let $(g_\varepsilon, \phi_\varepsilon)$ be the solutions to (1.3) with sources $J_\varepsilon$ such that $\partial_\varepsilon J_\varepsilon|_{\varepsilon=0} = \bar{g}^{(i)}$ and $\bar{J}^{(i)}_\varepsilon$ are supported in $B^{(i)}$. Let $\Phi_{g_\varepsilon}$ be the normal coordinates with respect to $g_\varepsilon$ centered at the point $y_\varepsilon = \Psi^{-1}_\varepsilon(\Psi_0(y))$, where $\Psi_\varepsilon$ are the Fermi coordinates, see (1.4). Then $\partial_\varepsilon \partial_{x_1} \partial_{x_2} \partial_{x_4} \Phi_{g_\varepsilon}^* g_{\varepsilon}|_{\varepsilon=0} = 0$ is not smooth near $\Phi_0(y)$. We remark that due to this condition we can determine the wave gauge coordinates in the sets $J^{+}_{\bar{g}}(B^{(j)}) \cap J^{-}_{\bar{g}}(B^{(j)})$.

Using [21, Lemma 4.1] and Prop. 5.2 and 5.3 we can adapt the proof of [21, Lemma 4.2] to our case and conclude that using the data set $D(\bar{s})$, we can determine whether the condition (D) is valid for the given point $y \in V$ or not. We remark that here the key to adapt the proof of [21, Lemma 4.2] is Prop. 5.3 part (3), which is required by [21, Lemma 4.1]. Then, [21, Lemma 4.2], parts (1)-(2), tell that the interaction condition (I) and the singularity detection condition (D) are equivalent when $y$ is in not the future of any cut point of geodesics $\gamma_{x^{(i)}(t_0), \xi^{(i)}(t_0)}$.
Next, the light observation set of \( q \in M \) in \( V \) is defined as \( \mathcal{P}_V(q) = \mathcal{L}_q^+ \cap V \). The earliest light observation set is defined as

\[
\mathcal{E}_V(q) = \{ x \in \mathcal{P}_V(q) : \text{there is no } y \in \mathcal{P}_V(q) \text{ and future-pointing time-like path } \alpha : [0,1] \to V \text{ such that } \alpha(0) = y \text{ and } \alpha(1) = x \} \subset V,
\]

see [21 Def. 1.1]. For \( W \subset M \) open, the collection of the earliest light observation sets with source points in \( W \) is \( \mathcal{E}_V(W) = \{ \mathcal{E}_V(q) : q \in W \} \). In particular, if \( q_0 \) is the interaction point as in Prop. 5.2, then \( \mathcal{E}_V(q_0) \subset N((\bar{x}, \bar{t}), t_0) \). From Section 2.2.1 of [21], we know that \( \mathcal{E}_V(q_0) \) contains a 3-dimensional submanifold hence is not empty.

The argument in [21 Section 5] can be applied to show that from the data set \( \mathcal{D}(\delta) \), we can determine the set \( \mathcal{E}_{V,\tilde{g}}(I_{\tilde{g}}(p_-, p_+)) \). This means that we can produce “artificial point source” at any point \( q \) of \( I_{\tilde{g}}(p_-, p_+) \) and determine the intersection of the observation set \( V_{\tilde{g}} \) and the light cone emanating from the point \( q \). Now for the set up of Theorem 1.1 we can apply [20 Theorem 1.2] to conclude that \( \tilde{g}^{(1)} \) is conformal to \( \tilde{g}^{(2)} \). Moreover, since \( \tilde{g}^{(1)}, \tilde{g}^{(2)} \) are Ricci flat, we can further apply [20 Corollary 1.3] to conclude that the conformal diffeomorphism is an isometry i.e. there exists a diffeomorphism \( \Psi : I_{\tilde{g}^{(1)}}(p_-, p_+) \to I_{\tilde{g}^{(2)}}(p_-, p_+) \) such that \( \Psi^* \tilde{g}^{(2)} = \tilde{g}^{(1)} \). This finishes the proof of Theorem 1.1.

**APPENDIX A. COMPUTATION OF PRINCIPAL SYMBOLS OF THE INTERACTION TERM**

We complete the calculation required in the proof of Prop. 5.3. Our goal is to compute the principal symbols of \( \mathcal{H} \) given in Prop. 5.2. For convenience, we split \( \mathcal{H} \) into three parts

\[
\mathcal{H}_1 = \sum_{(i,j,k,l) \in \sigma(4)} \left( \hat{H}_3(x, \bar{v}^{(i)}, \bar{v}^{(j)}), Q(H_2(x, \bar{v}^{(k)}, \bar{v}^{(l)})) + \hat{H}_3(x, Q(H_2(x, \bar{v}^{(i)}, \bar{v}^{(j)})), \bar{v}^{(k)}, \bar{v}^{(l)}) \right)
\]

\[
\mathcal{H}_2 = - \sum_{(i,j,k,l) \in \sigma(1)} \hat{G}_2(x, Q(H_2(x, \bar{v}^{(i)}, \bar{v}^{(j)})), Q(H_2(x, \bar{v}^{(k)}, \bar{v}^{(l)})))
\]

\[
\mathcal{H}_3 = - \sum_{(i,j,k,l) \in \sigma(4)} \left( \hat{H}_2(x, \bar{v}^{(i)}), Q(P_2(x, Q(H_2(x, \bar{v}^{(j)}, \bar{v}^{(k)})), \bar{v}^{(l)})) + \hat{H}_2(x, Q(P_2(x, Q(H_2(x, \bar{v}^{(i)}, \bar{v}^{(j)})), \bar{v}^{(k)})), \bar{v}^{(l)}) \right)
\]

For the computation, we recall that we use local coordinate near \( q_0 \) so that \( \tilde{g} \) is the standard Minkowski metric at \( q_0 \), and we found the Einstein-Maxwell equations in Minkowski space-time in Section 4. Let \( (u, \phi) = (g - \tilde{g}, \phi) \) and \( F = d\phi \). For our purpose, we need the nonlinear terms \( \hat{H}_2, \hat{H}_3 \) which are quadratic in \( \phi \), which can only come from the stress-energy tensor. For \( \alpha, \beta = 0, 1, 2, 3 \), we have

\[
(\hat{H}_2)_{\alpha\beta} = 2(-h^{\lambda\mu} F_{\alpha\lambda} F_{\beta\mu} + \frac{1}{4} h_{\alpha\beta} h^{{\lambda\gamma}} h^{{\mu\delta}} F_{\gamma\delta} F{_{\lambda\mu}}),
\]

\[
= 2\sum_{\lambda=0}^{3} -h^{\lambda\alpha} F_{\alpha\lambda} F_{\beta\mu} + \sum_{\lambda, \mu=0}^{3} \frac{1}{4} h_{\alpha\beta} h^{\lambda\nu} h^{\mu\phi} F{^2 _{\lambda\mu}}
\]
and for $\alpha \neq \beta$, we have
\[
(\hat{H}_3)_{\alpha\beta} = 2(-h^{aa'}h^{bb'}u_{ab}F_{aa'}F_{bb'} + \frac{1}{4}u_{a\beta}h^{aa'}h^{bb'}F_{ab}F_{a'b'})
\]
\[
= \sum_{a,b=0}^3 2(-h^{aa}h^{bb}u_{ab}F_{aa}F_{bb} + \frac{1}{4}u_{a\beta}h^{aa}h^{bb}F_{a}^2).
\]

Also, we need
\[
P_2(x,u,v) = (h^{-1}uh^{-1})^{pq} \frac{\partial^2}{\partial x^p \partial x^q} u.
\]

Sometimes, it is convenient to write $\hat{H}_2$ in matrix form. Let $H = (h_{ij})$ and $F^{(i)} = d\hat{\phi}^{(i)}$, $i = 1, 2, 3, 4$. Notice that $F^{(i)}$ are anti-symmetric so $F^{(i)} = -F^{(i)}$, $\mu, \beta = 0, 1, 2, 3$. Thus we obtain that
\[
\hat{H}_2(x,v^{(i)},v^{(j)}) = 2F^{(i)}HF^{(j)} + \frac{1}{2}H \text{Tr}(F^{(i)}F^{(j)}),
\]
where
\[
\text{Tr}(F^{(i)}F^{(j)}) = \sum_{\lambda,\mu=0}^3 h^{\lambda\mu}h^{\mu\lambda}F^{(i)}_{\lambda\mu}F^{(j)}_{\lambda\mu}, \quad 1 \leq i < j \leq 4.
\]

Also, we notice that $\hat{H}_2(x,v^{(i)},v^{(j)}) = \hat{H}_2^T(x,v^{(i)},v^{(j)})$, where $T$ denotes the transpose of a matrix. In view of Prop. [5.2] we can write the causal inverse $Q = Q_g\text{Id}$ at $q_0$ below. The computation of the symbol is straightforward but very lengthy, so we compute each term in a separate subsection.

A.1. Computation of $\sigma(\mathcal{H}_1)$. In this subsection, we write $\mathcal{H}_1 = \mathcal{H}$. For $\alpha \neq \beta$, we have
\[
\mathcal{H}_{\alpha\beta} = \sum_{(i,j,k,l)\in\sigma(4)} \hat{H}_{3,\alpha\beta}(Q\hat{H}_2(\bar{v}^{(i)},\bar{v}^{(j)}),\bar{v}^{(k)},\bar{v}^{(l)})
\]
\[
= 4 \sum_{(i,j,k,l)\in\sigma(4)} -h^{aa'}h^{bb'}Q_g(-h^{\lambda\mu}F^{(i)}_{aa}F^{(j)}_{bb} + \frac{1}{4}h_{ab}h^{\lambda\gamma}h^{\mu\delta}F^{(i)}_{\gamma\delta}F^{(j)}_{\lambda\mu}F^{(k)}_{ab}F^{(l)}_{a'b'}
\]
\[
+ 4 \sum_{(i,j,k,l)\in\sigma(4)} \frac{1}{4}Q_g(-h^{\lambda\mu}F^{(i)}_{aa}F^{(j)}_{bb})h^{aa'}h^{bb'}F^{(k)}_{ab}F^{(l)}_{a'b'}
\]
\[
= 4(\mathcal{J}_{1,\alpha\beta} + \mathcal{J}_{2,\alpha\beta} + \mathcal{J}_{3,\alpha\beta}),
\]
where $F^{(i)} = d\hat{\phi}^{(i)}$ and
\[
\mathcal{J}_{1,\alpha\beta} = \sum_{a,b,\lambda=0}^3 \sum_{(i,j,k,l)\in\sigma(4)} h^{aa}h^{bb}h^{\lambda\mu}Q_g[F^{(i)}_{aa}F^{(j)}_{bb}]F^{(k)}_{ab}F^{(l)}_{a'b'}
\]
\[
\mathcal{J}_{2,\alpha\beta} = -\sum_{a,\lambda,\mu=0}^3 \sum_{(i,j,k,l)\in\sigma(4)} \frac{1}{4}h^{\lambda\mu}h^{\lambda\mu}h^{aa}Q_g[F^{(i)}_{a\lambda}F^{(j)}_{b\mu}]F^{(k)}_{ab}F^{(l)}_{a'b'}
\]
\[
\mathcal{J}_{3,\alpha\beta} = -\sum_{a,b,\lambda=0}^3 \sum_{(i,j,k,l)\in\sigma(4)} \frac{1}{4}h^{\lambda\mu}h^{aa}h^{bb}Q_g(F^{(i)}_{a\lambda}F^{(j)}_{b\beta})F^{(k)}_{ab}F^{(l)}_{a'b'}.
\]
To see the structure of the terms, we find that

\[
\begin{align*}
J_1 &= \sum_{(i,j,k,l)\in\sigma(4)} F^{(k)}HQ_{\beta}(F^{(i)}HF^{(j)})HF^{(l)}, \\
J_2 &= \frac{1}{4} \sum_{(i,j,k,l)\in\sigma(4)} Q_{\beta}(\text{Tr}(F^{(i)}F^{(j)}))F^{(k)}HF^{(l)}, \\
J_3 &= \frac{1}{4} \sum_{(i,j,k,l)\in\sigma(4)} Q_{\beta}(F^{(i)}HF^{(j)})\text{Tr}(F^{(k)}F^{(l)}).
\end{align*}
\]  

\((A.3)\)

In these formulas, \(Q_{\beta}\) applies to each element of the matrix. Also, the sign of \(J_2, J_3\) are positive because we used the anti-symmetry of \(F\). We emphasize that we can only use these formula to express the off-diagonal elements. We know (see e.g. Lemma 4.1 [34]) that if \(\dot{\phi}^{(i)} \in I^\mu(\Lambda)\), then the principal symbol of \(F_{\alpha\beta}^{(i)}\) (in \(I^{\mu+1}(\Lambda)\)) are given by

\[
\sigma(F_{\alpha\beta}^{(i)})(x, \zeta) = i\zeta_\alpha \sigma(\dot{\phi}_\beta^{(i)})(x, \zeta) - i\zeta_\beta \sigma(\dot{\phi}_\alpha^{(i)})(x, \zeta), \quad (x, \zeta) \in \Lambda.
\]

So at the intersection point \(q_0\), we get \(\sigma(F_{\alpha\beta}^{(i)})(q_0, \zeta^{(i)}) = i\zeta_\alpha A_\beta^{(i)} - i\zeta_\beta A_\alpha^{(i)}\).

We start with the set of vectors considered in Prop. 5.3

\[
\zeta^{(1)} = (1, 0, 1, 0), \quad \zeta^{(2)} = (1, 0, 0, 1), \quad \zeta^{(3)} = (-3/4, -3/4, 0, 0), \quad \zeta^{(4)} = (1, -1, 0, 0)
\]

so that the sum \(\zeta = (9/4, -3/4, 1, 1)\) is light-like. The vectors \(\vec{A}\) are taken as

\[
\vec{A}^{(1)} = -(0, 0, 0, 1), \quad \vec{A}^{(2)} = -(0, 0, 1, 0), \quad \vec{A}^{(3)} = -(0, 0, 0, 1), \quad \vec{A}^{(4)} = -(0, 0, 0, 1).
\]

We calculate the principal symbol \(\sigma(H^{l}(\vec{A}))\) in detail for this set of vectors. We start with some preparations. For the set of vectors \(\zeta^{(i)}, i = 1, 2, 3, 4\), we first compute \(\sigma(i,j) = |\zeta^{(i)} + \zeta^{(j)}|^2_h = 2h(\zeta^{(i)}, \zeta^{(j)})\):

\[
\begin{align*}
\sigma(1, 2) &= -2, & \sigma(1, 3) &= 3/2, & \sigma(1, 4) &= -2, \\
\sigma(2, 3) &= 3/2, & \sigma(2, 4) &= -2, & \sigma(3, 4) &= 3.
\end{align*}
\]

Next, we compute the trace terms \(W(i,j) = \text{Tr}(F^{(i)}F^{(j)})\):

\[
\begin{align*}
W(1, 2) &= -2, & W(1, 3) &= 3/2, & W(1, 4) &= -2, \\
W(2, 3) &= 0, & W(2, 4) &= 0, & W(3, 4) &= 3.
\end{align*}
\]

Now we compute the terms \(H_{\alpha\beta}\) using the formulas. Although there are many summation terms due to the permutations, eventually the non-trivial terms are few thanks to the sparseness of \(F^*\). We remark that one can also compute using the matrix representation \((A.3)\). We split the rest into five parts, each dealing with one term.
(a) $H_{02}$: We begin with

\[
J_{1,02} = \sum_{(i,j,k,l)\in\sigma(4)} h^{aa}h^{bb}h^{\lambda\lambda}Q_{\bar{g}}[F_{a\lambda}^{(i)}F_{b\lambda}^{(j)}]F_{0a}^{(k)}F_{2b}^{(l)}
\]

\[
= \sum_{(i,j,k,l)\in\sigma(4)} \sum_{\lambda=0}^{3} h^{\lambda\lambda}Q_{\bar{g}}[F_{3\lambda}^{(i)}F_{0\lambda}^{(j)}]F_{03}^{(k)}F_{23}^{(l)} + \sum_{(i,j,k,l)\in\sigma(4)} \sum_{\lambda,b=0}^{3} h^{bb}h^{\lambda\lambda}Q_{\bar{g}}[F_{3\lambda}^{(i)}F_{b\lambda}^{(j)}]F_{0b}^{(k)}F_{2b}^{(l)}
\]

\[
= \sum_{(i,j,k,l)\in\sigma(4)} \sum_{\lambda=0}^{3} -h^{\lambda\lambda}Q_{\bar{g}}[F_{3\lambda}^{(i)}F_{0\lambda}^{(j)}]F_{03}^{(k)}F_{23}^{(l)} + \sum_{(i,j,k,l)\in\sigma(4)} \sum_{\lambda=0}^{3} h^{\lambda\lambda}Q_{\bar{g}}[F_{3\lambda}^{(i)}F_{0\lambda}^{(j)}]F_{03}^{(k)}F_{23}^{(l)}
\]

\[
= \sum_{(i,j,k,l)\in\sigma(4)} -Q_{\bar{g}}[F_{30}^{(i)}F_{03}^{(j)}]F_{03}^{(k)}F_{23}^{(l)} + \sum_{(i,j,k,l)\in\sigma(4)} Q_{\bar{g}}[F_{30}^{(i)}F_{03}^{(j)}]F_{03}^{(k)}F_{23}^{(l)}
\]

\[
= -2Q_{\bar{g}}[F_{30}^{(1)}F_{03}^{(3)}]F_{03}^{(2)}F_{23}^{(2)} - 2Q_{\bar{g}}[F_{30}^{(1)}F_{03}^{(4)}]F_{03}^{(2)}F_{23}^{(2)} - 2Q_{\bar{g}}[F_{30}^{(3)}F_{03}^{(4)}]F_{03}^{(1)}F_{23}^{(2)}
\]

\[
- 2Q_{\bar{g}}[F_{31}^{(4)}F_{31}^{(4)}]F_{03}^{(1)}F_{23}^{(2)}.
\]

The symbols can be calculated using the expressions in Section 5. For simplicity, we shall ignore the $c_\pi = (2\pi)^{-3}$ factor as well as the multiple of 4 in the computations. We add them in the final answer. For example, we shall compute using

\[
\sigma(Q_{\bar{g}}[F_{0}^{(i)}F_{0}^{(j)}]F_{0}^{(k)}F_{0}^{(l)}) = \frac{1}{|\zeta^{(i)} + \zeta^{(j)}|^2} \sigma(F_{0}^{(i)})\sigma(F_{0}^{(j)})\sigma(F_{0}^{(k)})\sigma(F_{0}^{(l)}),
\]

where $\bullet$ stands for a generic index. Hereafter the principal symbols $\sigma(F^{(i)})$ are always evaluated at $(q_0,\zeta^{(i)})$ which shall be omitted in the notations. Now we use the values of $F^\bullet$ and $\bar{G}(\cdot, \cdot)$ to get

\[
\sigma(J_{1,02}) = -2(2/3)(-1)(3/4)1(-1) - 2(-1/2)(-1)(-1)(-3/4)(-1) - 2(1/3)(3/4)(-1)(-1)
\]

\[
+ 2(1/3)(3/4)1\cdot1(-1) = -5/4.
\]

Next, consider

\[
J_{2,02} = -\sum_{(i,j,k,l)\in\sigma(4)} \sum_{\lambda=0}^{3} \frac{1}{4} h^{\lambda\lambda}h^{\mu\mu}Q_{\bar{g}}[F_{\lambda\mu}^{(i)}F_{\lambda\mu}^{(j)}]F_{03}^{(k)}F_{23}^{(l)}
\]

When we compute the symbol, we will have $W(i, j)$ which is zero for $(i, j) = (2, 3)$ and $(2, 4)$. Now we determine the non-trivial terms for all permutations of $(i, j, k, l)$. First, $l$ has to be 1 or 2. If $l = 1$, then $k$ must be 3 or 4 so that at least one of $i, j$ is 2. But then the corresponding $W$ term become zero. So it suffices to take $l = 2$. Then we have

\[
J_{2,02} = \sum_{\lambda=0}^{3} \left( -\frac{1}{2} h^{\lambda\lambda}h^{\mu\mu}Q_{\bar{g}}[F_{\lambda\mu}^{(3)}F_{\lambda\mu}^{(4)}]F_{03}^{(1)}F_{23}^{(2)} - \frac{1}{2} h^{\lambda\lambda}h^{\mu\mu}Q_{\bar{g}}[F_{\lambda\mu}^{(1)}F_{\lambda\mu}^{(4)}]F_{03}^{(3)}F_{23}^{(2)}
\]

\[
- \frac{1}{2} h^{\lambda\lambda}h^{\mu\mu}Q_{\bar{g}}[F_{\lambda\mu}^{(1)}F_{\lambda\mu}^{(3)}]F_{03}^{(4)}F_{23}^{(2)} \right).
\]

Then we compute the symbol and find that

\[
\sigma(J_{2,02}) = -\frac{3}{2} 1(-1) + \frac{2}{1} (-3/4)(-2) + \frac{3/2}{3/4}(1(-1)) = 5/8.
\]
Finally, consider
\[ J_{3,02} = - \sum_{(i,j,k,l) \in \sigma(4)} \sum_{a,b=0}^{3} \frac{1}{4} h^{aa} h^{bb} Q_{\bar{g}}(F_{03}^{(i)} F_{13}^{(j)} F_{ab}^{(k)} F_{ab}^{(l)}) \]

This is similar to \( J_{2,02} \). When \( j = 1 \), the symbol of the terms vanishes. For \( j = 2 \), we obtain
\[ J_{3,02} = \sum_{a,b=0}^{3} \left( -\frac{1}{2} h^{aa} h^{bb} Q_{\bar{g}}(F_{03}^{(1)} F_{13}^{(2)} F_{ab}^{(3)} F_{ab}^{(4)}) - \frac{1}{2} h^{aa} h^{bb} Q_{\bar{g}}(F_{03}^{(3)} F_{13}^{(2)} F_{ab}^{(1)} F_{ab}^{(4)}) \right) \]

Then we can find the symbol is \( \sigma(J_{3,02}) = -5/8 \). Finally, we have \( \sigma(J_{02}) = 4c(5/4) \).

(b) \( J_{12} \): We start with
\[ J_{1,12} = \sum_{(i,j,k,l) \in \sigma(4)} \sum_{b,l=0}^{3} h^{bb} h^{\lambda\lambda} Q_{\bar{g}}[F_{3\lambda}^{(i)} F_{0\lambda}^{(j)} F_{13}^{(k)} F_{2b}^{(l)}] \]
\[ = \sum_{(i,j,k,l) \in \sigma(4)} \sum_{l=0}^{3} h^{\lambda\lambda} Q_{\bar{g}}[F_{3\lambda}^{(i)} F_{0\lambda}^{(j)} F_{13}^{(k)} F_{20}^{(l)}] + \sum_{(i,j,k,l) \in \sigma(4)} \sum_{l=0}^{3} h^{\lambda\lambda} Q_{\bar{g}}[F_{3\lambda}^{(i)} F_{0\lambda}^{(j)} F_{13}^{(k)} F_{23}^{(l)}] \]
\[ = -Q_{\bar{g}}[F_{30}^{(i)} F_{30}^{(j)} F_{13}^{(k)} F_{23}^{(l)}]. \]

We know that \( l \) must be 1, 2 and \( k \) must be 3, 4. So we just need to consider these permutations.
\[ \sigma(J_{1,12}) = -2\sigma(Q_{\bar{g}}[F_{30}^{(1)} F_{30}^{(2)} F_{13}^{(3)} F_{23}^{(2)}]) - 2\sigma(Q_{\bar{g}}[F_{30}^{(3)} F_{30}^{(2)} F_{13}^{(1)} F_{23}^{(2)}]) \]
\[ = -2(-1/2)(-1)(-1)(-3/4)(-1) - 2(2/3)(-1)(3/4)(-1)(-1) = 7/4. \]

Next, we consider
\[ J_{2,12} = - \sum_{(i,j,k,l) \in \sigma(4)} \sum_{l=0}^{3} \frac{1}{4} h^{\lambda\lambda} h^{\mu\mu} Q_{\bar{g}}[F_{\lambda\mu}^{(i)} F_{\lambda\mu}^{(j)} F_{13}^{(k)} F_{23}^{(l)}] \]

We know that \( l \) must be 2 because if \( l = 1 \), we always have \( i, j \) are 2, 3 or 2, 4 and the corresponding \( W \) vanish. So we have
\[ \sigma(J_{2,12}) = 3 \sum_{\lambda,\mu=0}^{3} \left( -\frac{1}{2} \sigma(h^{\lambda\lambda} h^{\mu\mu} Q_{\bar{g}}[F_{\lambda\mu}^{(1)} F_{\lambda\mu}^{(4)} F_{13}^{(3)} F_{23}^{(2)}]) - \frac{1}{2} \sigma(h^{\lambda\lambda} h^{\mu\mu} Q_{\bar{g}}[F_{\lambda\mu}^{(1)} F_{\lambda\mu}^{(3)} F_{13}^{(4)} F_{23}^{(2)}]) \right) \]
\[ = \frac{1}{2} \cdot -2(-3/4)(-1) - \frac{1}{2} \cdot 3/2(1)(-1) = -7/2. \]

Finally, we find that
\[ J_{3,12} = - \sum_{(i,j,k,l) \in \sigma(4)} \sum_{a,b=0}^{3} \frac{1}{4} h^{aa} h^{bb} Q_{\bar{g}}(F_{13}^{(i)} F_{13}^{(j)} F_{a}^{(k)} F_{a}^{(l)}). \]
By similar consideration, we get
\[
\sigma(\mathcal{I}_{3,12}) = \sum_{a,b=0}^{3} \left( -\frac{1}{2} \sigma(h^{aa} h^{bb} Q_{\bar{g}}(F_{13}^{(3)} F_{23}^{(2)} F_{ab}^{(1)} F_{ab}^{(4)}) - \frac{1}{2} \sigma(h^{aa} h^{bb} Q_{\bar{g}}(F_{13}^{(3)} F_{23}^{(2)} F_{ab}^{(1)} F_{ab}^{(4)}) )
\right)
= -\frac{1}{2} \cdot \frac{2}{3/2}(-3/4)(-1) - \frac{1}{2} \cdot \frac{3/2}{3/2}(-1)(-1) = 7/2
\]

Summing up the symbols, we get \( \sigma(\mathcal{I}_{12}) = 4\epsilon_{\pi}(7/4) \).

(c) \( \mathcal{H}_{03} \): We start with
\[
\mathcal{J}_{2,03} = - \sum_{(i,j,k,l) \in \sigma(4)} \sum_{\lambda,\mu,a=0}^{3} \frac{1}{4} h^{\lambda\lambda} h^{\mu\mu} h^{aa} Q_{\bar{g}}(F_{\lambda\mu}^{(i)} F_{\lambda\mu}^{(j)}) F_{00}^{(k)} F_{30}^{(l)}
\]
\[
= - \sum_{(i,j,k,l) \in \sigma(4)} \sum_{\lambda,\mu,a=0}^{3} \frac{1}{4} h^{\lambda\lambda} h^{\mu\mu} Q_{\bar{g}}(F_{\lambda\mu}^{(i)} F_{\lambda\mu}^{(j)}) F_{02}^{(k)} F_{32}^{(l)} = \sum_{\lambda,\mu=0}^{3} \frac{1}{4} h^{\lambda\lambda} h^{\mu\mu} Q_{\bar{g}}(F_{\lambda\mu}^{(3)} F_{\lambda\mu}^{(4)}) F_{02}^{(2)} F_{32}^{(1)}.
\]

So the symbol is \( \sigma(\mathcal{J}_{2,03}) = -\frac{1}{2} \cdot \frac{3}{2}1(-1) = 1/2 \). Similarly, we have
\[
\mathcal{J}_{3,03} = \sum_{\lambda,\mu=0}^{3} \frac{1}{4} h^{\lambda\lambda} h^{\mu\mu} F_{\lambda\mu}^{(3)} F_{\lambda\mu}^{(4)} Q_{\bar{g}}(F_{02}^{(2)} F_{32}^{(1)}),
\]
and the symbol is \( \sigma(\mathcal{J}_{3,03}) = -\frac{1}{2} \cdot \frac{3}{2}1(-1) = -3/4 \). Finally we calculate
\[
\mathcal{J}_{1,03} = \sum_{(i,j,k,l) \in \sigma(4)} \sum_{a,b=0}^{3} h^{aa} h^{bb} h^{\lambda\lambda} Q_{\bar{g}}(F_{\lambda\lambda}^{(i)} F_{\lambda\lambda}^{(j)}) F_{00}^{(k)} F_{30}^{(l)}
\]
\[
= \sum_{(i,j,k,l) \in \sigma(4)} \sum_{b=0}^{3} h^{bb} h^{\lambda\lambda} Q_{\bar{g}}(F_{\lambda\lambda}^{(i)} F_{\lambda\lambda}^{(j)}) F_{02}^{(k)} F_{30}^{(l)} - \sum_{(i,j,k,l) \in \sigma(4)} \sum_{\lambda=0}^{3} h^{\lambda\lambda} Q_{\bar{g}}(F_{\lambda\lambda}^{(3)} F_{\lambda\lambda}^{(4)}) F_{03}^{(2)} F_{32}^{(1)}
\]
\[
= - Q_{\bar{g}}[F_{23}^{(1)} F_{00}^{(3)} F_{23}^{(4)} F_{03}^{(2)}] - Q_{\bar{g}}[F_{23}^{(1)} F_{03}^{(4)} F_{02}^{(2)} F_{30}^{(3)}] + Q_{\bar{g}}[F_{23}^{(1)} F_{13}^{(3)} F_{02}^{(2)} F_{31}^{(4)}] + Q_{\bar{g}}[F_{23}^{(1)} F_{13}^{(3)} F_{03}^{(4)} F_{32}^{(1)}]
\]
\[+ Q_{\bar{g}}[F_{13}^{(4)} F_{02}^{(2)} F_{23}^{(3)} F_{03}^{(4)} - 2Q_{\bar{g}}[F_{32}^{(1)} F_{02}^{(2)} F_{03}^{(4)} F_{30}^{(3)} - Q_{\bar{g}}[F_{30}^{(3)} F_{20}^{(4)} F_{04}^{(1)} F_{32}^{(3)} - Q_{\bar{g}}[F_{30}^{(4)} F_{20}^{(4)} F_{04}^{(1)} F_{32}^{(3)}]
\]

We compute the symbol as
\[
\sigma(\mathcal{J}_{1,03}) = -(2/3)1(-3/4)(-1) - (-1/2)1 \cdot 1 \cdot 1(3/4) + (2/3)1(-3/4)1 \cdot 1
\]
\[+ (-1/2)1(-1)1(3/4) - 2(-1/2)(-1)1(-3/4)(-1) - (2/3)(3/4)(-1)1 \cdot (-1)
\]
\[= -(1/2)(-1)(-1)(3/4)(-1) = -9/8.
\]
Summing up the symbols, we get $\sigma(\mathcal{H}_{03}) = 4c_\pi(-11/8)$.

**d) $\mathcal{H}_{13}$:** We compute

$$\mathcal{J}_{1,13} = \sum_{(i,j,k,l) \in \sigma(4)} \sum_{b,\lambda=0}^3 h^{bb} h^{\lambda \lambda} Q_{\tilde{g}}[F_{3\lambda}^{(i)} F_{b\lambda}^{(j)}] F_{13}^{(k)} F_{3b}^{(l)}$$

$$= \sum_{(i,j,k,l) \in \sigma(4)} \sum_{\lambda=0}^3 \left( -h^{\lambda \lambda} Q_{\tilde{g}}[F_{3\lambda}^{(i)} F_{1\lambda}^{(j)}] F_{13}^{(k)} F_{30}^{(l)} + h^{\lambda \lambda} Q_{\tilde{g}}[F_{3\lambda}^{(i)} F_{1\lambda}^{(j)}] F_{31}^{(k)} F_{31}^{(l)} + h^{\lambda \lambda} Q_{\tilde{g}}[F_{3\lambda}^{(i)} F_{2\lambda}^{(j)}] F_{13}^{(k)} F_{32}^{(l)} \right)$$

$$= -\sum_{(i,j,k,l) \in \sigma(4)} Q_{\tilde{g}}[F_{32}^{(i)} F_{02}^{(j)}] F_{13}^{(k)} F_{30}^{(l)} - \sum_{(i,j,k,l) \in \sigma(4)} Q_{\tilde{g}}[F_{30}^{(i)} F_{20}^{(j)}] F_{13}^{(k)} F_{32}^{(l)}$$

$$= -Q_{\tilde{g}}[F_{32}^{(1)} F_{02}^{(2)}] F_{13}^{(3)} F_{30}^{(4)} - Q_{\tilde{g}}[F_{30}^{(3)} F_{20}^{(2)}] F_{13}^{(4)} F_{32}^{(3)} - Q_{\tilde{g}}[F_{32}^{(1)} F_{02}^{(2)}] F_{13}^{(4)} F_{30}^{(3)} - Q_{\tilde{g}}[F_{30}^{(4)} F_{20}^{(2)}] F_{13}^{(3)} F_{32}^{(1)}.$$

We can compute the symbol as

$$\sigma(\mathcal{J}_{1,13}) = -(-1/2)(-1)1(-3/4)(-1) - (2/3)(3/4)(-1)(-1)(-1)$$

$$- (-1/2)(-1)1(-1)(3/4) - (-1/2)(-1)(-1)(-3/4)(-1) = 7/8.$$  

For $\mathcal{J}_{2,13}$, we notice that $\mathcal{J}_{2,13} = -\sum_{(i,j,k,l) \in \sigma(4)} \sum_{b,\mu=0}^3 \frac{1}{4} h^{\mu \mu} h^{\lambda \lambda} Q_{\tilde{g}}[F_{\lambda \mu}^{(i)} F_{b\mu}^{(j)}] F_{13}^{(k)} F_{33}^{(l)} = 0.$ Similarly, $\mathcal{J}_{3,13} = 0.$ Thus $\sigma(\mathcal{H}_{13}) = 4c_\pi \sigma(\mathcal{J}_{1,13}) = 4c_\pi(7/8)$.

**e) $\mathcal{H}_{23}$:** We begin with

$$\mathcal{J}_{1,23} = \sum_{(i,j,k,l) \in \sigma(4)} \sum_{a,b,\lambda=0}^3 h^{aa} h^{bb} h^{\lambda \lambda} Q_{\tilde{g}}[F_{a\lambda}^{(i)} F_{b\lambda}^{(j)}] F_{2a}^{(k)} F_{3b}^{(l)}$$

$$= \sum_{(i,j,k,l) \in \sigma(4)} \sum_{b=0}^3 (-h^{bb} Q_{\tilde{g}}[F_{03}^{(i)} F_{b3}^{(j)}] F_{20}^{(k)} F_{3b}^{(l)}) + \sum_{(i,j,k,l) \in \sigma(4)} \sum_{b,\lambda=0}^3 h^{bb} h^{\lambda \lambda} Q_{\tilde{g}}[F_{b\lambda}^{(i)} F_{b\lambda}^{(j)}] F_{23}^{(k)} F_{3b}^{(l)}.$$  

There are many terms in the summations so we deal the two summations separately.

$$\mathcal{A}_1 = \sum_{(i,j,k,l) \in \sigma(4)} \sum_{b=0}^3 (-h^{bb} Q_{\tilde{g}}[F_{03}^{(i)} F_{b3}^{(j)}] F_{20}^{(k)} F_{3b}^{(l)})$$

$$= \sum_{(i,j,k,l) \in \sigma(4)} \left( Q_{\tilde{g}}[F_{03}^{(i)} F_{03}^{(j)}] F_{20}^{(k)} F_{30}^{(l)} - Q_{\tilde{g}}[F_{03}^{(i)} F_{13}^{(j)}] F_{20}^{(k)} F_{31}^{(l)} - Q_{\tilde{g}}[F_{03}^{(i)} F_{23}^{(j)}] F_{20}^{(k)} F_{32}^{(l)} \right)$$

$$= 2 \left( Q_{\tilde{g}}[F_{03}^{(3)} F_{03}^{(2)}] F_{20}^{(1)} F_{30}^{(1)} + Q_{\tilde{g}}[F_{03}^{(1)} F_{03}^{(4)}] F_{20}^{(2)} F_{30}^{(3)} + Q_{\tilde{g}}[F_{03}^{(1)} F_{03}^{(3)}] F_{20}^{(2)} F_{30}^{(4)} \right)$$

$$+ \left( -Q_{\tilde{g}}[F_{03}^{(1)} F_{13}^{(3)}] F_{20}^{(2)} F_{34}^{(4)} - Q_{\tilde{g}}[F_{03}^{(1)} F_{13}^{(4)}] F_{20}^{(2)} F_{33}^{(3)} \right).$$

Then we find

$$\sigma(\mathcal{A}_1) = 2 \left( (1/3)(-3/4)1(-1)(-1) + (-1)1(-1)(-1)(3/4) + (2/3)1(-3/4)(-1)(-1) \right)$$

$$+ \left( -2(3/4)1(-3/4)(-1)1 - (-1)(-1)(3/4) \right) = -7/8.$$
Therefore, \( \sigma = \sum_{i,j,k,l} h h h h \). Summing up the symbols we get

\[
A_2 = \sum_{i,j,k,l} \sum_{\lambda,\mu = 0} 3 h h h h Q \left[ F_{\sigma(4)}^{(i)} F_{\sigma(4)}^{(j)} \right] F_{\sigma(4)}^{(k)} F_{\sigma(4)}^{(l)}
\]

\[
= \sum_{i,j,k,l} \sum_{\lambda,\mu = 0} 3 \left( -h h h h Q \left[ F_{\sigma(4)}^{(i)} F_{\sigma(4)}^{(j)} \right] F_{\sigma(4)}^{(k)} F_{\sigma(4)}^{(l)} + h h h h Q \left[ F_{\sigma(4)}^{(i)} F_{\sigma(4)}^{(j)} \right] F_{\sigma(4)}^{(k)} F_{\sigma(4)}^{(l)} + h h h h Q \left[ F_{\sigma(4)}^{(i)} F_{\sigma(4)}^{(j)} \right] F_{\sigma(4)}^{(k)} F_{\sigma(4)}^{(l)} \right)
\]

\[
= \sum_{i,j,k,l} \sum_{\lambda,\mu = 0} 3 \left( -Q \left[ F_{\sigma(4)}^{(i)} F_{\sigma(4)}^{(j)} \right] F_{\sigma(4)}^{(k)} F_{\sigma(4)}^{(l)} - Q \left[ F_{\sigma(4)}^{(i)} F_{\sigma(4)}^{(j)} \right] F_{\sigma(4)}^{(k)} F_{\sigma(4)}^{(l)} \right) = 0.
\]

Therefore, \( \sigma(3, 2) = 1/8 \). Next, we compute

\[
J_{2,23} = \sum_{i,j,k,l} \sum_{\lambda,\mu = 0} 3 \left( \frac{1}{4} h h h h Q \left[ F_{\lambda,\mu}^{(i)} F_{\lambda,\mu}^{(j)} \right] F_{\lambda,\mu}^{(k)} F_{\lambda,\mu}^{(l)} \right)
\]

\[
= \sum_{\lambda,\mu = 0} 3 \left( \frac{1}{2} h h h h Q \left[ F_{\lambda,\mu}^{(i)} F_{\lambda,\mu}^{(j)} \right] F_{\lambda,\mu}^{(k)} F_{\lambda,\mu}^{(l)} + \frac{1}{2} h h h h Q \left[ F_{\lambda,\mu}^{(i)} F_{\lambda,\mu}^{(j)} \right] F_{\lambda,\mu}^{(k)} F_{\lambda,\mu}^{(l)} + \frac{1}{2} h h h h Q \left[ F_{\lambda,\mu}^{(i)} F_{\lambda,\mu}^{(j)} \right] F_{\lambda,\mu}^{(k)} F_{\lambda,\mu}^{(l)} \right).
\]

Therefore, the symbol is

\[
\sigma(J_{2,23}) = \frac{1}{2} \cdot \frac{3}{3} (-1)(-1) + \frac{1}{2} \cdot \frac{3}{3} (-1)(3/4) + \frac{1}{2} \cdot \frac{3}{3} (-1)(-1) = 5/8.
\]

Finally, we consider

\[
J_{3,23} = \sum_{i,j,k,l} \sum_{\lambda,\mu = 0} 3 \left( \frac{1}{4} h h h h Q \left[ F_{\lambda,\mu}^{(i)} F_{\lambda,\mu}^{(j)} \right] F_{\lambda,\mu}^{(k)} F_{\lambda,\mu}^{(l)} \right)
\]

\[
= \sum_{\lambda,\mu = 0} 3 \left( \frac{1}{2} h h h h Q \left[ F_{\lambda,\mu}^{(i)} F_{\lambda,\mu}^{(j)} \right] F_{\lambda,\mu}^{(k)} F_{\lambda,\mu}^{(l)} + \frac{1}{2} h h h h Q \left[ F_{\lambda,\mu}^{(i)} F_{\lambda,\mu}^{(j)} \right] F_{\lambda,\mu}^{(k)} F_{\lambda,\mu}^{(l)} + \frac{1}{2} h h h h Q \left[ F_{\lambda,\mu}^{(i)} F_{\lambda,\mu}^{(j)} \right] F_{\lambda,\mu}^{(k)} F_{\lambda,\mu}^{(l)} \right).
\]

The symbol can be found as

\[
\sigma(J_{3,23}) = \frac{1}{2} \cdot \frac{3}{3} (-1)(-1) + \frac{1}{2} \cdot \frac{3}{3} (-1)(3/4) + \frac{1}{2} \cdot \frac{3}{3} (-1)(-1) = -5/8.
\]

Summing up the symbols we get \( \sigma(J_{2,3}) = 4c_\pi(-7/8) \).

To conclude, we obtained

\[
\sigma(J_{1, \hat{A}})(q_0, \zeta) = c_\pi \begin{pmatrix}
* & * & -5 & -5.5 \\
* & * & 7 & 3.5 \\
-5 & 7 & * & -3.5 \\
-5.5 & 3.5 & -3.5 & *
\end{pmatrix}
\]

For the proof of Prop. 5.3 part (3), we need to find another four set of vectors \( a, \hat{A}, a = 2, 3, 4, 5 \) so that the principal symbols \( \sigma(J_{1, a, \hat{A}}) \), \( a = 1, 2, 3, 4, 5 \) are linearly independent. We will fix the choices of \( \zeta(i), i = 1, 2, 3, 4 \) and vary the choices of \( \hat{A} \). This is reasonable because each \( A^{(i)} \), \( i = 1, 2, 3, 4 \) varies in a 3 dimensional vector space (we lose one dimension due to the gauge condition). In the following, we shall omit the details of the computation as they are the same as or \( 1, \hat{A} \).
result can be checked quite straightforwardly with some tedious computations. Also, one can check the result via symbolic computations using the formulas \([A.3]\).

We take \(A^1\) as
\[
A^{(1)} = -(0, 0, 0, 0), \quad A^{(2)} = -(0, 0, 0, 0), \quad A^{(3)} = -(0, 0, 0, 0), \quad A^{(4)} = -(0, 0, 0, 0).
\]
Notice that comparing to \(A^1\), we just changed \(A^{(1)}\). We find
\[
\sigma(\mathcal{H}_1(A^2)) = c_\pi \begin{pmatrix}
* & * & 3.5 & 3.5 \\
* & * & -3.5 & -5.5 \\
3.5 & -3.5 & * & -7 \\
3.5 & -5.5 & -7 & *
\end{pmatrix}
\]

We take \(A^3\) as
\[
A^{(1)} = -(0, 0, 0, 0), \quad A^{(2)} = -(0, 0, 0, 0), \quad A^{(3)} = -(0, 0, 0, 0), \quad A^{(4)} = -(0, 0, 0, 0).
\]
Notice that comparing to \(A^1\), we just changed \(A^{(2)}\). We find
\[
\sigma(\mathcal{H}_1(A^3)) = c_\pi \begin{pmatrix}
* & * & 7 & 7 \\
* & * & -5 & -7 \\
7 & -5 & * & 7 \\
7 & -7 & 7 & *
\end{pmatrix}
\]

We take \(A^4\) as
\[
A^{(1)} = -(0, 0, 0, 0), \quad A^{(2)} = -(0, 0, 0, 0), \quad A^{(3)} = -(0, 0, 0, 0), \quad A^{(4)} = -(0, 0, 0, 0).
\]
Here we just changed \(A^{(3)}\) compared to \(A^1\). We find
\[
\sigma(\mathcal{H}_1(A^4)) = c_\pi \begin{pmatrix}
* & * & 5 & -5.5 \\
* & * & 0 & -3.5 \\
5 & 0 & * & 1 \\
-5.5 & -3.5 & 1 & *
\end{pmatrix}
\]

We take \(A^5\) as
\[
A^{(1)} = -(0, 0, 0, 0), \quad A^{(2)} = -(0, 0, 0, 0), \quad A^{(3)} = -(0, 0, 0, 0), \quad A^{(4)} = -(0, 0, 0, 0).
\]
Here we changed \(A^{(4)}\) compared to \(A^1\). We find
\[
\sigma(\mathcal{H}_1(A^5)) = c_\pi \begin{pmatrix}
* & * & -5.5 & 5 \\
* & * & -3.5 & 0 \\
-5.5 & -3.5 & * & 1 \\
5 & 0 & 1 & *
\end{pmatrix}
\]

This completes the calculation for \(\sigma(\mathcal{H}_1)\) needed in the proof of Prop. 5.3

A.2. Computation of \(\sigma(\mathcal{H}_2)\). The computation of this term is relatively simple. We shall use the matrix form. For convenience, we let
\[
W^{(ij)} = Q \left( \hat{H}_2(x, \vec{v}^{(i)}, \vec{v}^{(j)}) + \tilde{H}_2(x, \vec{v}^{(i)}, \vec{v}^{(j)}) \right), \quad 1 < i < j \leq 4.
\]

Then we can write
\[
\sigma(\mathcal{H}_2)(q_0, \zeta) = -\sum \sigma \left( \hat{P}_2(x, W^{(ij)}, W^{(k\ell)}) \right)(q_0, \zeta).
\]
and the summation is over the pairs \((i, j) = (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\). So we reduce to summation of six terms. Using the expressions of principal symbols in Section 5, we find that

\[
-\sigma \left( \hat{P}_2(x, W^{(12)}, W^{(34)}) \right) (q_0, \zeta) = \left( H\sigma (W^{(12)}) H \right)_{pq} (c_p^{(3)} + \zeta_p^{(4)}) (c_q^{(3)} + \zeta_q^{(4)}) \sigma (W^{(34)})
\]

\[
= \frac{c_\pi}{2h(\xi^{(1)}, \xi^{(2)})} \left( H (2\sigma (F^{(1)}) H \sigma (F^{(2)}) + 2\sigma (F^{(2)}) H \sigma (F^{(1)}) + H Tr(\sigma (F^{(1)}) \sigma (F^{(2)}))) H \right)_{pq} \\
\quad \cdot (c_p^{(3)} + \zeta_p^{(4)}) (c_q^{(3)} + \zeta_q^{(4)}) \\
\quad \cdot \frac{1}{2h(\xi^{(3)}, \xi^{(4)})} (2\sigma (F^{(3)}) H \sigma (F^{(4)}) + 2\sigma (F^{(4)}) H \sigma (F^{(3)}) + H Tr(\sigma (F^{(3)}) \sigma (F^{(4)}))) .
\]

The other terms in (A.4) can be written in a similar way and are omitted here. So the computation is reduced to several matrix multiplications. Again, we do the calculation for the first set of vectors. For our choices of \(\hat{A}\), we have

\[
\sigma (F^{(1)}) = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
-1 & 0 & -1 & 0
\end{pmatrix}, \quad \sigma (F^{(2)}) = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix},
\]

\[
\sigma (F^{(3)}) = \begin{pmatrix}
0 & 0 & 0 & -\frac{3}{4} \\
0 & 0 & 0 & -\frac{3}{4} \\
0 & 0 & 0 & 0 \\
\frac{3}{4} & \frac{3}{4} & 0 & 0
\end{pmatrix}, \quad \sigma (F^{(4)}) = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0
\end{pmatrix}.
\]

By straightforward computations, we get that

\[
-\sigma \left( \hat{P}_2(x, W^{(12)}, W^{(34)}) \right) = c_\pi \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 \\
0 & 0 & 0 & -3
\end{pmatrix}, \quad -\sigma \left( \hat{P}_2(x, W^{(34)}, W^{(12)}) \right) (q_0, \zeta) = 0
\]

Similarly, we have

\[
-\sigma \left( \hat{P}_2(x, W^{(13)}, W^{(24)}) \right) = c_\pi \begin{pmatrix}
0 & 0 & -8 & 0 \\
0 & 0 & 8 & 0 \\
-8 & 8 & 0 & -8 \\
0 & 0 & -8 & 0
\end{pmatrix},
\]

\[
-\sigma \left( \hat{P}_2(x, W^{(24)}, W^{(13)}) \right) = c_\pi \begin{pmatrix}
-1 & -1 & -1 & 0 \\
-1 & -1 & -1 & 0 \\
-1 & -1 & -1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]
\[-\sigma \left( \tilde{P}_2(x, \mathcal{W}^{(14)}, \mathcal{W}^{(23)}) \right) = c_\pi \begin{pmatrix} 0 & 0 & 0.75 & 0 \\ 0 & 0 & 0.75 & 0 \\ 0.75 & 0.75 & 0 & 0.75 \\ 0 & 0 & 0.75 & 0 \end{pmatrix},\]

\[-\sigma \left( \tilde{P}_2(x, \mathcal{W}^{(23)}, \mathcal{W}^{(14)}) \right) = c_\pi \begin{pmatrix} 6 & -6 & 6 & 0 \\ -6 & 6 & -6 & 0 \\ 6 & -6 & 6 & 0 \\ 0 & 0 & 0 & -6 \end{pmatrix}.\]

So finally, for this set of vectors \(1\tilde{A}\), we denote the term \(\mathcal{H}_2\) by \(\mathcal{H}_2^{(1\tilde{A})}\) and we find that

\[\sigma(\mathcal{H}_2^{(1\tilde{A})})(q_0, \zeta) = c_\pi \begin{pmatrix} 5 & -7 & -2.25 & 0 \\ -7 & 5 & 1.75 & 0 \\ -2.25 & 1.75 & 8 & -7.25 \\ 0 & 0 & -7.25 & -8 \end{pmatrix}.\]

For the other sets of vectors, we get

\[\sigma(\mathcal{H}_2^{(2\tilde{A})})(q_0, \zeta) = c_\pi \begin{pmatrix} 0 & 0 & -7 & 7 \\ 0 & 0 & 5 & -5 \\ -7 & 5 & -0.875 & 0 \\ 7 & -5 & 0 & 0.875 \end{pmatrix},\]

\[\sigma(\mathcal{H}_2^{(3\tilde{A})})(q_0, \zeta) = c_\pi \begin{pmatrix} 0 & 0 & -8.75 & 8.75 \\ 0 & 0 & 7.25 & -7.25 \\ -8.75 & 7.25 & -18.375 & 0 \\ 8.75 & -7.25 & 0 & 18.375 \end{pmatrix},\]

\[\sigma(\mathcal{H}_2^{(4\tilde{A})})(q_0, \zeta) = c_\pi \begin{pmatrix} 9.25 & -8.75 & 4 & 2.25 \\ -8.75 & 5.25 & -2 & 0.25 \\ 4 & -2 & 10.75 & -6 \\ -2.25 & 0.25 & -6 & -6.75 \end{pmatrix},\]

\[\sigma(\mathcal{H}_2^{(5\tilde{A})})(q_0, \zeta) = c_\pi \begin{pmatrix} 9.25 & -8.75 & 2.25 & 4 \\ -8.75 & 5.25 & 0.25 & -2 \\ 2.25 & 0.25 & -6.75 & -6 \\ 4 & -2 & -6 & 10.75 \end{pmatrix}.\]

A.3. **Computation of \(\sigma(\mathcal{H}_3)\).** We simplify the symbols as following. We know that

\[-\sigma(\tilde{H}_2(x, \bar{v}^{(i)}), Q(\bar{P}_2(x, Q(\tilde{H}_2(x, \bar{v}^{(j)}, \bar{v}^{(k)})), \bar{v}^{(l)})))) = \frac{c_\pi}{|\zeta^{(j)}| + |\zeta^{(k)}| + |\zeta^{(l)}| + |\zeta^{(l)}|} \left( H\sigma(\tilde{H}_2(\bar{v}^{(j)}, \bar{v}^{(k)}))\right)_{pq} \zeta^{(l)}_{p} \cdot \sigma(\tilde{H}_2(\bar{v}^{(i)}, \bar{v}^{(l)})),\]

where

\[\sigma(\tilde{H}_2(\bar{v}^{(i)}, \bar{v}^{(l)})) \equiv 2\sigma(F^{(i)})H\sigma(\bar{F}^{(l)}) + \frac{1}{2} HTr(\sigma(F^{(i)})\sigma(\bar{F}^{(l)})),\]

\[\sigma(F^{(l)}) = \bar{v}^{(l)}_\alpha \sigma(\bar{v}_\beta^{(l)}) - \bar{v}_\beta^{(l)} \sigma(\bar{v}_\alpha^{(l)}), \quad \bar{v}^{(l)} = \zeta^{(j)} + \zeta^{(k)} + \zeta^{(l)}.\]
Recall that it suffices to show that the vectors $T_1$.

However, we need to compute twelve terms. The details will be omitted. For the first choice of $l, i$

where the summation is over $l, i$.

Notice that the principal symbols of $F$ is evaluated at $(q_0, \tilde{\zeta}(l))$. The other piece is similar

$$-\sigma(\tilde{H}_2(x, Q(P_2(x, Q(\tilde{H}_2(x, \tilde{v}(j), \tilde{v}(k))), \tilde{v}(l))), \tilde{v}(l)))$$

$$= \frac{c_\pi}{|\zeta(j) + \zeta(k)|^2 + |\zeta(j) + \zeta(l)|^2} \left( H \sigma(\tilde{H}_2(\tilde{v}(j), \tilde{v}(k))) H \right)^{pq} \zeta_p^{(l)} \zeta_q^{(l)} \cdot \sigma(\tilde{H}_2(\tilde{v}(l), \tilde{v}(l))).$$

To simplify the computation a bit, we set

$$\sigma(\varnothing^l) = \sigma(\tilde{H}_2(\tilde{v}(i), \tilde{v}(l)) + \tilde{H}_2(\tilde{v}(i), \tilde{v}(l)))$$

$$= 2\sigma(F(i)) H \sigma(\varnothing^l) + 2\sigma(F(i)) H \sigma(F(i)) + H \text{Tr}(\sigma(F(i)) \sigma(\varnothing^l)),$$

so that

$$\sigma(\mathcal{H}_3) = \sum \frac{c_\pi}{|\zeta(j) + \zeta(k)|^2 + |\zeta(j) + \zeta(l)|^2} \left( H \sigma(W^{jk}) H \right)^{pq} \zeta_p^{(l)} \zeta_q^{(l)} \cdot \sigma(\varnothing^l),$$

where the summation is over $l, i = 1, 2, 3, 4, l \neq i$ and $(j, k)$ are determined by the choice of $l, i$.

The computation is straightforward and one can use many terms $W^*$ already computed in $\mathcal{H}_2$. However, we need to compute twelve terms. The details will be omitted. For the first choice of the vectors $\vec{A}$, we get

$$\sigma(\mathcal{H}_3(\vec{A}))(q_0, \zeta) = c_\pi \begin{pmatrix} -4 & 7 & 1.95 & 4.8 \\ 7 & -14 & -6.65 & 1.4 \\ 1.95 & -6.65 & 1 & 6.25 \\ 4.8 & 1.4 & 6.25 & 9 \end{pmatrix}.$$
independent where \( T \) is defined in the proof of Prop. 5.3. So we have
\[
T(\sigma(3\tilde{A})) = c_\pi(-5.3, -0.7, 2.1, 4.9, -4.5), \\
T(\sigma(4\tilde{A})) = c_\pi(-0.35, 7, -0.25, -4, -10.5), \\
T(\sigma(5\tilde{A})) = c_\pi(-1.75, 5.6, 7.55, -8, 7), \\
T(\sigma(6\tilde{A})) = c_\pi(9, -5, -0.4, 0.4, -1), \\
T(\sigma(7\tilde{A})) = c_\pi(-5, 9, 0.4, -0.4, -1).
\]

We check that the rank of these five vectors is 5. This completes the proof of the claim in Prop. 5.3.

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Matti Lassas  
Department of Mathematics, University of Helsinki  
*E-mail address: matti.lassas@helsinki.fi*

Gunther Uhlmann  
Department of Mathematics, University of Washington,  
Institute for Advanced Study, the Hong Kong University of Science and Technology  
and Department of Mathematics, University of Helsinki  
*E-mail address: gunther@math.washington.edu*

Yiran Wang  
Department of Mathematics, University of Washington  
and Institute for Advanced Study, the Hong Kong University of Science and Technology  
*E-mail address: wangy257@math.washington.edu*