THE UNIVERSAL ADDITIVE DAHA OF TYPE \((C'_γ, C_1)\) AND LEONARD TRIPLES

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Abstract. Assume that \(\mathbb{F}\) is an algebraically closed field with characteristic zero. The universal Racah algebra \(\mathcal{R}\) is a unital associative \(\mathbb{F}\)-algebra generated by \(A, B, C, D\) and the relations state that \([A, B] = [B, C] = [C, A] = 2D\) and each of

\[
[A, D] + AC - BA, \quad [B, D] + BA - CB, \quad [C, D] + CB - AC
\]

is central in \(\mathcal{R}\). The universal additive DAHA (double affine Hecke algebra) \(\mathcal{H}\) of type \((C'_γ, C_1)\) is a unital associative \(\mathbb{F}\)-algebra generated by \(\{t_i\}_{i=0}^3\) and the relations state that

\[
t_0 + t_1 + t_2 + t_3 = -1,
\]

\(t_i^2\) is central for all \(i = 0, 1, 2, 3\).

Any \(\mathcal{H}\)-module can be considered as a \(\mathcal{R}\)-module via the \(\mathbb{F}\)-algebra homomorphism \(\mathcal{R} \to \mathcal{H}\) given by

\[
A \mapsto \frac{(t_0 + t_1 - 1)(t_0 + t_1 + 1)}{4},
\]

\[
B \mapsto \frac{(t_0 + t_2 - 1)(t_0 + t_2 + 1)}{4},
\]

\[
C \mapsto \frac{(t_0 + t_3 - 1)(t_0 + t_3 + 1)}{4}.
\]

Let \(V\) denote a finite-dimensional irreducible \(\mathcal{H}\)-module. In this paper we show that \(A, B, C\) are diagonalizable on \(V\) if and only if \(A, B, C\) act as Leonard triples on all composition factors of the \(\mathcal{R}\)-module \(V\).

Keywords: additive double affine Hecke algebras, Racah algebras, Leonard pairs, Leonard triples.

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1. Introduction

Throughout this paper, we adopt the following conventions: Assume that the ground field \(\mathbb{F}\) is algebraically closed with characteristic zero. An algebra is meant to be a unital associative algebra over \(\mathbb{F}\). An algebra homomorphism is meant to be a unital algebra homomorphism.

Definition 1.1 (Definition 2.1, \cite{9}). The universal Racah algebra \(\mathcal{R}\) is an algebra defined by generators and relations in the following way: The generators are \(A, B, C, D\) and the relations state that

\[
[A, B] = [B, C] = [C, A] = 2D
\]

and each of

\[
[A, D] + AC - BA, \quad [B, D] + BA - CB, \quad [C, D] + CB - AC
\]

commutes with \(A, B, C, D\).

The purpose of this unpublished work is only to make sure the existence of \(q \to 1\) version of \cite{8} and this work is based on the master thesis by the first author under the supervision of the second author.
Define
\[ \delta = A + B + C. \]  
Using (1) yields that \( \delta \) is central in \( \mathcal{R} \).

**Definition 1.2 (3[4]).** The universal additive DAHA (double affine Hecke algebra) \( \mathcal{H} \) of type \((C'_{\vee}, C_1)\) is an algebra defined by generators and relations. The generators are \( t_0, t_1, t_2, t_3 \) and the relations state that
\[ t_0 + t_1 + t_2 + t_3 = -1 \]  
and each of \( t_0^2, t_1^2, t_2^2, t_3^2 \) commutes with \( t_0, t_1, t_2, t_3 \).

**Theorem 1.3 (2[6]).** There exists a unique algebra homomorphism \( \zeta : \mathcal{R} \to \mathcal{H} \) that sends
\begin{align*}
A & \mapsto \frac{(t_0 + t_1 - 1)(t_0 + t_1 + 1)}{4}, \\
B & \mapsto \frac{(t_0 + t_2 - 1)(t_0 + t_2 + 1)}{4}, \\
C & \mapsto \frac{(t_0 + t_3 - 1)(t_0 + t_3 + 1)}{4}, \\
\delta & \mapsto \frac{t_0^2 + t_1^2 + t_2^2 + t_3^2}{4} - \frac{t_0}{2} - \frac{3}{4}.
\end{align*}

Note that \( \zeta \) is the Racah version of the algebra homomorphism from the universal Askey–Wilson algebra \( \Delta_q \) into the universal DAHA \( \mathcal{H}_q \). Let \( V \) denote a finite-dimensional irreducible \( \Delta_q \)-module. In [3][10] it was shown that the defining generators of \( \Delta_q \) are diagonalizable on \( V \) if and only if the defining generators of \( \Delta_q \) act as Leonard triples on all composition factors of the \( \Delta_q \)-module \( V \).

In this paper we are devoted to giving the Racah version of [3][10]. We begin with recalling the notion of Leonard pairs and Leonard triples. We will use the following terms: A square matrix is said to be tridiagonal if each nonzero entry lies on either the diagonal, the superdiagonal, or the subdiagonal. A tridiagonal matrix is said to be irreducible if each entry on the superdiagonal is nonzero and each entry on the subdiagonal is nonzero. Let \( V \) denote a vector space over \( \mathbb{F} \) with finite positive dimension. An eigenvalue \( \theta \) of a linear operator \( L : V \to V \) is said to be multiplicity-free if the algebraic multiplicity of \( \theta \) is equal to 1. A linear operator \( L : V \to V \) is said to be multiplicity-free if all eigenvalues of \( L \) are multiplicity-free.

**Definition 1.4 (Definition 1.1, [11]).** Let \( V \) denote a vector space over \( \mathbb{F} \) with finite positive dimension. By a Leonard pair on \( V \), we mean a pair of linear operators \( L : V \to V \) and \( L^* : V \to V \) that satisfy both (i), (ii) below.

(i) There exists a basis for \( V \) with respect to which the matrix representing \( L \) is diagonal and the matrix representing \( L^* \) is irreducible tridiagonal.
(ii) There exists a basis for \( V \) with respect to which the matrix representing \( L^* \) is diagonal and the matrix representing \( L \) is irreducible tridiagonal.

**Definition 1.5 (Definition 1.2, [1]).** Let \( V \) denote a vector space over \( \mathbb{F} \) with finite positive dimension. By a Leonard triple on \( V \), we mean a triple of linear operators \( L : V \to V, L^* : V \to V, L^\varepsilon : V \to V \) that satisfy the conditions (i)–(iii) below.
There exists a basis for \( V \) with respect to which the matrix representing \( L \) is diagonal and the matrices representing \( L^* \) and \( L^\varepsilon \) are irreducible tridiagonal.

(ii) There exists a basis for \( V \) with respect to which the matrix representing \( L^* \) is diagonal and the matrices representing \( L^\varepsilon \) and \( L \) are irreducible tridiagonal.

(iii) There exists a basis for \( V \) with respect to which the matrix representing \( L^\varepsilon \) is diagonal and the matrices representing \( L \) and \( L^* \) are irreducible tridiagonal.

The main results of this paper are as follows:

**Theorem 1.6.** Suppose that \( V \) is a finite-dimensional irreducible \( \mathfrak{H} \)-module. Then the following are equivalent:

(i) \( A \) (resp. \( B \)) (resp. \( C \)) is diagonalizable on \( V \).

(ii) \( A \) (resp. \( B \)) (resp. \( C \)) is diagonalizable on all composition factors of the \( \mathfrak{R} \)-module \( V \).

(iii) \( A \) (resp. \( B \)) (resp. \( C \)) is multiplicity-free on all composition factors of the \( \mathfrak{R} \)-module \( V \).

**Theorem 1.7.** Suppose that \( V \) is a finite-dimensional irreducible \( \mathfrak{H} \)-module. Then the following are equivalent:

(i) \( A, B \) (resp. \( B, C \)) (resp. \( C, A \)) are diagonalizable on \( V \).

(ii) \( A, B \) (resp. \( B, C \)) (resp. \( C, A \)) are diagonalizable on all composition factors of the \( \mathfrak{R} \)-module \( V \).

(iii) \( A, B \) (resp. \( B, C \)) (resp. \( C, A \)) are multiplicity-free on all composition factors of the \( \mathfrak{R} \)-module \( V \).

(iv) \( A, B \) (resp. \( B, C \)) (resp. \( C, A \)) act as Leonard pairs on all composition factors of the \( \mathfrak{R} \)-module \( V \).

**Theorem 1.8.** Suppose that \( V \) is a finite-dimensional irreducible \( \mathfrak{H} \)-module. Then the following are equivalent:

(i) \( A, B, C \) are diagonalizable on \( V \).

(ii) \( A, B, C \) are diagonalizable on all composition factors of the \( \mathfrak{R} \)-module \( V \).

(iii) \( A, B, C \) are multiplicity-free on all composition factors of the \( \mathfrak{R} \)-module \( V \).

(iv) \( A, B, C \) act as Leonard triples on all composition factors of the \( \mathfrak{R} \)-module \( V \).

The paper is organized as follows. In §2 we recall some preliminaries on the finite-dimensional irreducible \( \mathfrak{R} \)-modules. In §3 we give some sufficient and necessary conditions for \( A, B, C \) acting as a Leonard triple on finite-dimensional irreducible \( \mathfrak{R} \)-modules. In §4 we recall some facts concerning the even-dimensional irreducible \( \mathfrak{H} \)-modules. In §5 we prove Theorem 1.6 in the even-dimensional case. In §6 we recall some facts concerning the odd-dimensional irreducible \( \mathfrak{H} \)-modules. In §7 we prove Theorem 1.6 in the odd-dimensional case and we end this paper with the proofs for Theorems 1.7 and 1.8.

2. Preliminaries on the finite-dimensional irreducible \( \mathfrak{R} \)-modules

**Proposition 2.1** (Proposition 2.4, [9]). For any scalars \( a, b, c \in \mathbb{F} \) and any integer \( d \geq 0 \), there exists a \((d+1)\)-dimensional \( \mathfrak{R} \)-module \( R_d(a,b,c) \) satisfying the following conditions:
Lemma 3.1. There exists a basis for $R_d(a, b, c)$ with respect to which the matrices representing $A$ and $B$ are
\[
\begin{pmatrix}
\theta_0 & \theta_1 & 0 \\
1 & \theta_1 & \theta_2 \\
& 1 & \theta_3 \\
0 & & \ddots & \ddots \\
& & & 1 & \theta_d
\end{pmatrix}, \quad \begin{pmatrix}
\theta_0^* & \varphi_1 & 0 \\
\theta_1^* & \varphi_2 & \ddots \\
\theta_2^* & & \ddots & \ddots \\
0 & & & \ddots & \ddots \\
& & & \ddots & \ddots & \ddots \\
& & & & \ddots & \ddots & \ddots \\
& & & & & \ddots & \ddots \\
& & & & & & \ddots \\
& & & & & & \theta_d^*
\end{pmatrix}
\]
respectively, where
\[
\begin{align*}
\theta_i &= (a + \frac{d}{2} - i) (a + \frac{d}{2} - i + 1) & & \text{for } i = 0, 1, \ldots, d, \\
\theta_i^* &= (b + \frac{d}{2} - i) (b + \frac{d}{2} - i + 1) & & \text{for } i = 0, 1, \ldots, d, \\
\varphi_i &= i(i - d - 1) (a + b + c + \frac{d}{2} - i + 2) (a + b - c + \frac{d}{2} - i + 1) & & \text{for } i = 1, 2, \ldots, d.
\end{align*}
\]

Theorem 2.2 (Theorem 4.5, [9]). For any scalars $a, b, c \in \mathbb{F}$ and any integer $d \geq 0$, the $\mathbb{R}$-module $R_d(a, b, c)$ is irreducible if and only if
\[
a + b + c + 1, -a + b + c, a - b + c, a + b - c \not\in \left\{ \frac{d}{2} - i \mid i = 1, 2, \ldots, d \right\}.
\]

Theorem 2.3 (Theorem 6.3, [9]). Let $d \geq 0$ denote an integer. Suppose that $V$ is a $(d + 1)$-dimensional irreducible $\mathbb{R}$-module. Then there exist $a, b, c \in \mathbb{F}$ such that the $\mathbb{R}$-module $R_d(a, b, c)$ is isomorphic to $V$.

Lemma 2.4 (Theorem 6.6, [9]). Let $d \geq 0$ denote an integer and let $a, b, c$ denote scalars in $\mathbb{F}$. Suppose that the $\mathbb{R}$-module $R_d(a, b, c)$ is irreducible. Then the following are equivalent:

(i) $A$ (resp. $B$) (resp. $C$) is diagonalizable on $R_d(a, b, c)$.

(ii) $A$ (resp. $B$) (resp. $C$) is multiplicity-free on $R_d(a, b, c)$.

(iii) $2a$ (resp. $2b$) (resp. $2c$) is not in $\{i - d - 1 \mid i = 1, 2, \ldots, 2d - 1\}$.

3. The conditions for $A, B, C$ as a Leonard triple on finite-dimensional irreducible $\mathbb{R}$-modules

Define the following elements of $\mathbb{R}$:

\[
\begin{align*}
(4) & \quad \alpha = [A, D] + AC - BA, \\
(5) & \quad \beta = [B, D] + BA - CB, \\
(6) & \quad \gamma = [C, D] + CB - AC.
\end{align*}
\]

Lemma 3.1. The following relations hold in $\mathbb{R}$:

\[
\begin{align*}
(7) & \quad A^3 B - 2ABA + BA^2 - 2AB - 2BA = 2A^2 - 2A\delta + 2\alpha, \\
(8) & \quad AB^3 - 2BAB + B^2 A - 2AB - 2BA = 2B^2 - 2B\delta - 2\beta, \\
(9) & \quad A^2 C - 2ACA + CA^2 - 2AC - 2CA = 2A^2 - 2A\delta - 2\alpha, \\
(10) & \quad C^2 A - 2CAC + AC^2 - 2AC - 2CA = 2C^2 - 2C\delta + 2\gamma, \\
(11) & \quad B^2 C - 2BCB + CB^2 - 2BC - 2CB = 2B^2 - 2B\delta + 2\beta.
\end{align*}
\]
Proposition 3.3. \( C^2 B - 2C BC + BC^2 - 2BC - 2CB = 2C^2 - 2C\delta - 2\gamma. \)

Proof. From (11) and (2) we see that \( D = \frac{1}{2} [A, B] \) and \( C = \delta - A - B. \) The relations (7) and (8) can be obtained by using these two facts to eliminate \( C, D \) from (4) and (5) respectively. The relations (9) and (10) can be obtained by substituting \( B = \delta - A - C \) and \( D = \frac{1}{2} [C, A] \) into (4) and (6) respectively. The relations (11) and (12) can be obtained by substituting \( A = \delta - B - C \) and \( D = \frac{1}{2} [B, C] \) into (5) and (6) respectively. \(\Box\)

Recall that the Pochhammer symbol is defined by

\[(x)_n = \prod_{i=1}^{n}(x + i - 1)\]

for all \( x \in \mathbb{F} \) and all integers \( n \geq 0. \) Note that (13) is taken to be 1 when \( n = 0. \)

**Lemma 3.2.** Let \( d \geq 0 \) denote an integer and let \( a, b, c \in \mathbb{F}. \) Then there exists a basis for \( R_d(a, b, c) \) with respect to which the matrix representing \( C \) is

\[
\begin{pmatrix}
\theta_0^e & 0 \\
1 & \theta_1^e \\
& \ddots & \ddots \\
0 & 1 & \theta_d^e
\end{pmatrix},
\]

where \( \theta_i^e = (c + \frac{d}{2} - i)(c + \frac{d}{2} - i + 1) \) for \( i = 0, 1, \ldots, d. \)

Proof. Let \( \{v_i\}_{i=0}^{d} \) denote the basis for \( R_d(a, b, c) \) from Proposition 2.1(i). Let

\[w_i = (-1)^i \sum_{h=0}^{i} \binom{i}{h} (d - i + 1)h(a + b + c + \frac{d}{2} - i + 2)h v_{i-h}\]

for all \( i = 0, 1, \ldots, d. \) Note that \( w_i \) is a linear combination of \( v_0, v_1, \ldots, v_i \) and the coefficient of \( v_i \) in \( w_i \) is nonzero for all \( i = 0, 1, \ldots, d. \) Hence \( \{w_i\}_{i=0}^{d} \) is a basis for \( R_d(a, b, c). \) A routine but tedious calculation yields that

\[(C - \theta_i^e)w_i = \begin{cases} w_{i+1} & \text{for } i = 0, 1, \ldots, d - 1, \\ 0 & \text{for } i = d. \end{cases}\]

The lemma follows. \(\Box\)

The finite-dimensional irreducible \( \mathbb{R} \)-modules are related to Leonard pairs and Leonard triples in the following ways:

**Proposition 3.3.** Let \( d \geq 0 \) denote an integer and let \( a, b, c \in \mathbb{F}. \) Suppose that the \( \mathbb{R} \)-module \( R_d(a, b, c) \) is irreducible. Then the following are equivalent:

(i) \( A, B \) (resp. \( B, C \)) (resp. \( C, A \)) are diagonalizable on \( R_d(a, b, c). \)

(ii) \( A, B \) (resp. \( B, C \)) (resp. \( C, A \)) are multiplicity-free on \( R_d(a, b, c). \)

(iii) \( A, B \) (resp. \( B, C \)) (resp. \( C, A \)) act as a Leonard pair on \( R_d(a, b, c). \)

Proof. (i) \(\iff\) (ii): Immediate from Lemma 2.4

(iii) \(\implies\) (i): Immediate from Definition 1.4

(ii) \(\implies\) (iii): Without loss of generality we assume that \( A, B \) are multiplicity-free on \( R_d(a, b, c) \) and we show that \( A, B \) act as a Leonard pair on \( R_d(a, b, c). \)
Let \( \theta_i = (a + \frac{d}{2} - i)(a + \frac{d}{2} - i + 1) \) for every integer \( i \). By Proposition 2.1 the scalars \( \{\theta_i\}_{i=0}^d \) are the eigenvalues of \( A \) in \( R_d(a, b, c) \). By Lemma 2.4 the eigenvalues \( \{\theta_i\}_{i=0}^d \) of \( A \) are mutually distinct. Let \( v_i \) denote the \( \theta_i \)-eigenvector of \( A \) in \( R_d(a, b, c) \) for \( i = 0, 1, \ldots, d \). Note that \( \{v_i\}_{i=0}^d \) form a basis for \( R_d(a, b, c) \). Given any element \( X \in \mathbb{R} \) let \( \langle X \rangle \) denote the matrix representing \( X \) with respect to \( \{v_i\}_{i=0}^d \). By construction the matrix \( [A] \) is

\[
\begin{pmatrix}
\theta_0 & & & 0 \\
& \theta_1 & & \\
& & \ddots & \\
0 & & & \theta_d
\end{pmatrix}.
\]

Let \( i \) be an integer with \( 0 \leq i \leq d \). Applying \( v_i \) to either side of (7) yields \((A - \theta_{i-1})(A - \theta_{i+1})Bv_i \) is a scalar multiple of \( v_i \). Hence

\[
(A - \theta_{i-1})(A - \theta_i)(A - \theta_{i+1})Bv_i = 0.
\]

It follows that \( [B] \) is a tridiagonal matrix. Since the \( \mathbb{R} \)-module \( R_d(a, b, c) \) is irreducible, the tridiagonal matrix \( [B] \) is irreducible.

Let \( \theta_i^* = (b + \frac{d}{2} - i)(b + \frac{d}{2} - i + 1) \) for every integer \( i \). By Proposition 2.1 the scalars \( \{\theta_i^*\}_{i=0}^d \) are the eigenvalues of \( B \) in \( R_d(a, b, c) \). By Lemma 2.4 the eigenvalues \( \{\theta_i^*\}_{i=0}^d \) of \( B \) are mutually distinct. Let \( w_i \) denote the \( \theta_i^* \)-eigenvector of \( B \) in \( R_d(a, b, c) \) for \( i = 0, 1, \ldots, d \). Note that \( \{w_i\}_{i=0}^d \) form a basis for \( R_d(a, b, c) \). Given any element \( X \in \mathbb{R} \) let \( \langle X \rangle \) denote the matrix representing \( X \) with respect to \( \{w_i\}_{i=0}^d \). By construction the matrix \( \langle B \rangle \) is

\[
\begin{pmatrix}
\theta_0^* & & & 0 \\
& \theta_1^* & & \\
& & \ddots & \\
0 & & & \theta_d^*
\end{pmatrix}.
\]

Let \( i \) be an integer with \( 0 \leq i \leq d \). Applying \( w_i \) to either side of (8) yields \((B - \theta_{i-1}^*)(B - \theta_{i+1}^*)Aw_i \) is a scalar multiple of \( w_i \). Hence

\[
(B - \theta_{i-1}^*)(B - \theta_i^*)(B - \theta_{i+1}^*)Aw_i = 0.
\]

It follows that \( \langle A \rangle \) is a tridiagonal matrix. Since the \( \mathbb{R} \)-module \( R_d(a, b, c) \) is irreducible, the tridiagonal matrix \( \langle A \rangle \) is irreducible. Therefore (iii) follows. \( \square \)

**Proposition 3.4.** Let \( d \geq 0 \) denote an integer and let \( a, b, c \) denote scalars in \( \mathbb{F} \). Suppose that the \( \mathbb{R} \)-module \( R_d(a, b, c) \) is irreducible. Then the following are equivalent:

(i) \( A, B, C \) are diagonalizable on \( R_d(a, b, c) \).
(ii) \( A, B, C \) are multiplicity-free on \( R_d(a, b, c) \).
(iii) \( A, B, C \) act as a Leonard triple on \( R_d(a, b, c) \).

**Proof.** (i) \( \iff \) (ii): Immediate from Lemma 2.4

(iii) \( \Rightarrow \) (i): Immediate from Definition 1.5

(ii) \( \Rightarrow \) (iii): In view of Lemma 3.1 the part follows by an argument similar to the proof of Proposition 3.3. \( \square \)
Proposition 4.1 \((\mathbb{5})\). For any scalars \(a, b, c \in \mathbb{F}\) and any odd integer \(d \geq 1\), there exists a \((d + 1)\)-dimensional \(\mathfrak{H}\)-module \(E_d(a, b, c)\) satisfying the following conditions:

(i) There exists a basis \(\{v_i\}_{i=0}^{d}\) for \(E_d(a, b, c)\) such that

\[
\begin{align*}
t_0v_0 &= -\frac{d+1}{2}v_0, \quad t_0v_d = -\frac{d+1}{2}v_d, \\
t_1v_i &= \begin{cases} i(i-1)v_{i-1} + av_i + v_{i+1} & \text{for } i = 2, 4, \ldots, d-1, \\
-av_i & \text{for } i = 1, 3, \ldots, d,
\end{cases} \\
t_1v_0 &= av_0 + v_1, \\
t_2v_i &= \begin{cases} bv_i - (\sigma + i)(\tau + i)v_{i-1} - bv_i - v_{i+1} & \text{for } i = 0, 2, \ldots, d-1, \\
-\frac{\sigma + \tau + 2i + 2}{2}v_i - v_{i+1} & \text{for } i = 0, 2, \ldots, d-1, \\
(\sigma + i)(\tau + i)v_{i-1} + \frac{\sigma + \tau + 2i}{2}v_i & \text{for } i = 1, 3, \ldots, d,
\end{cases}
\]

where

\[
\sigma = a + b + c - \frac{d+1}{2}, \quad \tau = a + b - c - \frac{d+1}{2}.
\]

(ii) The elements \(t_0^2, t_1^2, t_2^2, t_3^2\) act on \(E_d(a, b, c)\) as scalar multiplication by \(\frac{(d+1)^2}{4}, a^2, b^2, c^2\) respectively.

Theorem 4.2 (Theorem 2.5, \(\mathbb{5}\)). Let \(a, b, c \in \mathbb{F}\) and let \(d \geq 1\) denote an odd integer. Then the \(\mathfrak{H}\)-module \(E_d(a, b, c)\) is irreducible if and only if

\[
a + b + c, -a + b + c, a - b + c, a + b - c \notin \left\{ \frac{d-1}{2} - i \mid i = 0, 2, \ldots, d-1 \right\}.
\]

Recall that \(\{\pm 1\}\) is a group under multiplication and the group \(\{\pm 1\}^2\) is isomorphic to the Klein 4-group. By Definition 12 there exists a unique \(\{\pm 1\}^2\)-action on \(\mathfrak{H}\) such that each \(\varepsilon \in \{\pm 1\}^2\) acts on \(\mathfrak{H}\) as an algebra automorphism in the following way:

| \(u\) | \(t_0\) | \(t_1\) | \(t_2\) | \(t_3\) |
|---|---|---|---|---|
| \(u^{(1,1)}\) | \(t_0\) | \(t_1\) | \(t_2\) | \(t_3\) |
| \(u^{(1,-1)}\) | \(t_1\) | \(t_0\) | \(t_3\) | \(t_2\) |
| \(u^{(-1,1)}\) | \(t_2\) | \(t_3\) | \(t_0\) | \(t_1\) |
| \(u^{(-1,-1)}\) | \(t_3\) | \(t_2\) | \(t_1\) | \(t_0\) |

Table 1. The \(\{\pm 1\}^2\)-action on \(\mathfrak{H}\)
Lemma 5.1. Let \( V \) denote an \( \mathfrak{H} \)-module. For any algebra automorphism \( \varepsilon \) of \( \mathfrak{H} \) the notation \( V^\varepsilon \) stands for an alternate \( \mathfrak{H} \)-module structure on \( V \) given by

\[
{xv := x^\varepsilon v ~ \text{ for all } x \in \mathfrak{H} \text{ and all } v \in V.}
\]

**Theorem 4.3** (Theorem 6.3, [3]). Let \( d \geq 1 \) denote an odd integer. Suppose that \( V \) is a \((d + 1)\)-dimensional irreducible \( \mathfrak{H} \)-module. Then there exist \( a, b, c \in \mathbb{F} \) and \( \varepsilon \in \{\pm 1\}^2 \) such that the \( \mathfrak{H} \)-module \( E_d(a, b, c)^\varepsilon \) is isomorphic to \( V \).

**Theorem 4.4** (Theorem 5.3, [5]). Let \( a, b, c \in \mathbb{F} \) and let \( d \geq 1 \) denote an odd integer. Suppose that the \( \mathfrak{H} \)-module \( E_d(a, b, c) \) is irreducible. Then the following hold:

(i) The \( \mathfrak{H} \)-module \( E_d(a, b, c) \) is isomorphic to \( E_d(-a, b, c) \).

(ii) The \( \mathfrak{H} \)-module \( E_d(a, b, c) \) is isomorphic to \( E_d(a, -b, c) \).

(iii) The \( \mathfrak{H} \)-module \( E_d(a, b, c) \) is isomorphic to \( E_d(a, b, -c) \).

**Lemma 4.5.** Let \( a, b, c \in \mathbb{F} \) and let \( d \geq 1 \) denote an odd integer. Then the traces of \( t_0, t_1, t_2, t_3 \) on \( E_d(a, b, c) \) are \(-d + 1, 0, 0, 0\) respectively.

**Proof.** It is routine to verify the lemma by using Proposition 4.1(i). \( \square \)

By means of Theorem 4.3 and Lemma 4.5 we develop the following discriminant to determine the scalars \( a, b, c \in \mathbb{F} \) and \( \varepsilon \in \{\pm 1\}^2 \) in Theorem 4.3.

**Theorem 4.6.** Let \( d \geq 1 \) denote an odd integer. Suppose that \( V \) is a \((d + 1)\)-dimensional irreducible \( \mathfrak{H} \)-module. For any scalars \( a, b, c \in \mathbb{F} \) and any \( \varepsilon \in \{\pm 1\}^2 \) the following are equivalent:

(i) The \( \mathfrak{H} \)-module \( E_d(a, b, c)^\varepsilon \) is isomorphic to \( V \).

(ii) The trace of \( t_0 \) on \( V^\varepsilon \) is \(-d + 1\) and \( t_1^2, t_2^2, t_3^2 \) act on \( V^\varepsilon \) as scalar multiplication by \( a^2, b^2, c^2 \) respectively.

**Proof.** (i) \( \Rightarrow \) (ii): By (i) the \( \mathfrak{H} \)-module \( E_d(a, b, c) \) is isomorphic to \( V^\varepsilon \). Hence (ii) follows by Proposition 4.1(ii) and Lemma 4.5.

(ii) \( \Rightarrow \) (i): By Theorem 4.3 there are an \( \varepsilon' \in \{\pm 1\}^2 \) and \( a', b', c' \in \mathbb{F} \) such that the \( \mathfrak{H} \)-module \( E_d(a', b', c')^{\varepsilon'} \) is isomorphic to \( V \). Hence the \( \mathfrak{H} \)-module \( E_d(a', b', c') \) is isomorphic to \( V^{\varepsilon'} \). By Lemma 4.5 the traces of \( t_0, t_1, t_2, t_3 \) on \( V^{\varepsilon'} \) are \(-d + 1, 0, 0, 0\) respectively. Since the trace of \( t_0 \) on \( V^{\varepsilon} \) is \(-d + 1\), it follows from Table C that \( \varepsilon = \varepsilon' \). Combined with Proposition 4.1(ii) this yields that \( a', b', c' \) are \( \pm a, \pm b, \pm c \) respectively. Now (i) follows from Theorem 4.3. \( \square \)

5. The conditions for \( A, B, C \) as diagonalizable on even-dimensional irreducible \( \mathfrak{H} \)-modules

For convenience we adopt the following conventions in this section: Let \( d \geq 1 \) denote an odd integer. Let \( a, b, c \) denote any scalars in \( \mathbb{F} \). Let \( \{v_i\}_{i=0}^d \) denote the basis for \( E_d(a, b, c) \) from Proposition 4.1(i).

**Lemma 5.1.** The action of \( t_0 + t_1 \) on \( E_d(a, b, c) \) is as follows:

\[
\left( t_0 + t_1 + (-1)^i \left( \frac{d}{2} - a - i \right) + \frac{1}{2} \right) v_i = \begin{cases} 
v_{i+1} & \text{for } i = 0, 1, \ldots, d - 1, \\
0 & \text{for } i = d.
\end{cases}
\]

**Proof.** It is routine to verify the lemma by using Proposition 4.1(i). \( \square \)
Lemma 5.2. Suppose that the $\mathfrak{h}$-module $E_d(a, b, c)$ is irreducible. Then the following hold:

(i) There exists a basis $\{w_i\}_{i=0}^d$ for $E_d(a, b, c)$ such that

\[ (t_0 + t_2 + (-1)^i \left( \frac{d}{2} - b - i \right) + \frac{1}{2}) w_i = \begin{cases} w_{i+1} & \text{for } i = 0, 1, \ldots, d - 1, \\ 0 & \text{for } i = d. \end{cases} \]

(ii) There exists a basis $\{w_i\}_{i=0}^d$ for $E_d(a, b, c)$ such that

\[ (t_0 + t_3 + (-1)^i \left( \frac{d}{2} - c - i \right) + \frac{1}{2}) w_i = \begin{cases} w_{i+1} & \text{for } i = 0, 1, \ldots, d - 1, \\ 0 & \text{for } i = d. \end{cases} \]

Proof. Let $V$ denote the $\mathfrak{h}$-module $E_d(a, b, c)$.

(i): By Definition 1.2 there exists a unique algebra automorphism $\rho : \mathfrak{h} \to \mathfrak{h}$ given by

\[(t_0, t_1, t_2, t_3) \mapsto (t_0, t_2, t_3, t_1)\]

whose inverse sends $(t_0, t_1, t_2, t_3)$ to $(t_0, t_3, t_1, t_2)$. By Proposition 4.1(ii) the elements $t_0^2$, $t_1^2$, $t_2^2$, $t_3^2$ act on $V^\rho$ as scalar multiplication by $\frac{(d+1)^2}{4}$, $b^2$, $c^2$, $a^2$ respectively. By Lemma 4.3 the trace of $t_0$ on $V^\rho$ is $-(d+1)$. Therefore the $\mathfrak{h}$-module $V^\rho$ is isomorphic to $E_d(b, c, a)$ by Theorem 4.6. It follows from Lemma 5.1 that there exists a basis $\{w_i\}_{i=0}^d$ for $V^\rho$ such that

\[ (t_0 + t_1 + (-1)^i \left( \frac{d}{2} - b - i \right) + \frac{1}{2}) w_i = \begin{cases} w_{i+1} & \text{for } i = 0, 1, \ldots, d - 1, \\ 0 & \text{for } i = d. \end{cases} \]

It follows from (14) that the action of $t_0 + t_1$ on $V^\rho$ is identical to the action of $t_0 + t_2$ on $V$. Hence (i) follows.

(ii): By Definition 1.2 there exists a unique algebra automorphism $\rho : \mathfrak{h} \to \mathfrak{h}$ given by

\[(t_0, t_1, t_2, t_3) \mapsto (t_0, t_3, t_2, t_1)\]

whose inverse sends $(t_0, t_1, t_2, t_3)$ to $(t_0, t_3, t_2, t_1)$. By Proposition 4.1(ii) the elements $t_0^2$, $t_1^2$, $t_2^2$, $t_3^2$ act on $V^\rho$ as scalar multiplication by $\frac{(d+1)^2}{4}$, $c^2$, $b^2$, $a^2$ respectively. By Lemma 4.3 the trace of $t_0$ on $V^\rho$ is $-(d+1)$. Therefore the $\mathfrak{h}$-module $V^\rho$ is isomorphic to $E_d(c, b, a)$ by Theorem 4.6. It follows from Lemma 5.1 that there exists a basis $\{w_i\}_{i=0}^d$ for $V^\rho$ such that

\[ (t_0 + t_1 + (-1)^i \left( \frac{d}{2} - c - i \right) + \frac{1}{2}) w_i = \begin{cases} w_{i+1} & \text{for } i = 0, 1, \ldots, d - 1, \\ 0 & \text{for } i = d. \end{cases} \]

It follows from (15) that the action of $t_0 + t_1$ on $V^\rho$ is identical to the action of $t_0 + t_3$ on $V$. Hence (ii) follows.

\[ \square \]

Lemma 5.3. Let $j$ denote an integer with $0 \leq j \leq d$. Then

\[ \prod_{i=0}^d \left( t_0 + t_1 + (-1)^i \left( \frac{d}{2} - a - i \right) + \frac{1}{2} \right) v_0 \neq 0. \]

Proof. It is immediate from Lemma 5.1 that the left–hand side of (16) is equal to

\[ \prod_{i=j+1}^d \left( t_0 + t_1 + (-1)^i \left( \frac{d}{2} - a - i \right) + \frac{1}{2} \right) v_j. \]

If $j = d$ then (17) is equal to $v_d$. Suppose that $j \neq d$. Applying Lemma 5.1 yields that (17) is equal to $v_d$ plus a linear combination of $v_0, v_1, \ldots, v_{d-1}$. Hence (17) is nonzero. The lemma follows.

\[ \square \]
Lemma 5.4. Suppose that the $\mathfrak{G}$-module $E_d(a, b, c)$ is irreducible. Then the following are equivalent:

(i) $t_0 + t_1$ is diagonalizable on $E_d(a, b, c)$.
(ii) $t_0 + t_1$ is multiplicity-free on $E_d(a, b, c)$.
(iii) $2a$ is not among $d - 1, d - 3, \ldots, 1 - d$.

Proof. By Lemma 5.1, this characteristic polynomial of $t_0 + t_1$ in $E_d(a, b, c)$ has the roots

\[ (-1)^{i-1} \left( \frac{d}{2} - a - i \right) - \frac{1}{2} \]

for $i = 0, 1, \ldots, d$.

(ii) $\Leftrightarrow$ (iii): Since $F$ is of characteristic zero, the scalars $\left[ \frac{1}{2} \right]$ for $i = 0, 2, \ldots, d - 1$ are mutually distinct and the scalars $\left[ \frac{1}{2} \right]$ for $i = 1, 3, \ldots, d$ are mutually distinct. Hence the scalars $\left[ \frac{1}{2} \right]$ for all $i = 0, 1, \ldots, d$ are mutually distinct if and only if (iii) holds. Therefore (ii) and (iii) are equivalent.

(ii), (iii) $\Rightarrow$ (i): Trivial.

(i) $\Rightarrow$ (ii), (iii): By Lemma 5.1, the product

\[ \prod_{i=0}^{d} \left( t_0 + t_1 + (-1)^i \left( \frac{d}{2} - a - i \right) + \frac{1}{2} \right) \]

vanishes at $v_0$. Combined with Lemma 5.3, the $(t_0 + t_1)$-annihilator of $v_0$ is equal to the characteristic polynomial of $(t_0 + t_1)$ in $E_d(a, b, c)$. Therefore (i) implies (ii) and (iii). \(\square\)

Lemma 5.5. Suppose that the $\mathfrak{G}$-module $E_d(a, b, c)$ is irreducible. Then the following are equivalent:

(i) $t_0 + t_2$ is diagonalizable on $E_d(a, b, c)$.
(ii) $t_0 + t_2$ is multiplicity-free on $E_d(a, b, c)$.
(iii) $2b$ is not among $d - 1, d - 3, \ldots, 1 - d$.

Proof. In view of Lemma 5.2(i), the lemma follows by an argument similar to the proof of Lemma 5.4. \(\square\)

Lemma 5.6. Suppose that the $\mathfrak{G}$-module $E_d(a, b, c)$ is irreducible. Then the following are equivalent:

(i) $t_0 + t_3$ is diagonalizable on $E_d(a, b, c)$.
(ii) $t_0 + t_3$ is multiplicity-free on $E_d(a, b, c)$.
(iii) $2c$ is not among $d - 1, d - 3, \ldots, 1 - d$.

Proof. In view of Lemma 5.2(ii), the lemma follows by an argument similar to the proof of Lemma 5.4. \(\square\)

Lemma 5.7. Suppose that the $\mathfrak{G}$-module $E_d(a, b, c)$ is irreducible. Then the following hold:

(i) If $2a$ is not among $d - 3, d - 5, \ldots, 3 - d$, then $(t_2 + t_3 - 1)(t_2 + t_3 + 1)$ is diagonalizable on $E_d(a, b, c)$.
(ii) If $2b$ is not among $d - 3, d - 5, \ldots, 3 - d$, then $(t_1 + t_3 - 1)(t_1 + t_3 + 1)$ is diagonalizable on $E_d(a, b, c)$.
(iii) If $2c$ is not among $d - 3, d - 5, \ldots, 3 - d$, then $(t_1 + t_2 - 1)(t_1 + t_2 + 1)$ is diagonalizable on $E_d(a, b, c)$.
Proof. (i): For convenience we write \( L = (t_2 + t_3 - 1)(t_2 + t_3 + 1) \). Given any element \( X \) of \( \mathfrak{H} \), let \([X]\) denote the matrix representing \( X \) with respect to \( \{v_i\}_{i=0}^d \). Using (3) yields that

\[
L = (t_0 + t_1)(t_0 + t_1 + 2).
\]

Combined with Lemma 5.1 yields that the matrix \([L]\) is a lower triangular matrix of the form

\[
\begin{pmatrix}
\theta_0 & 0 & \theta_0 & \theta_1 & \cdots & \theta_d \frac{d-1}{2} \\
0 & \theta_0 & \theta_1 & \cdots & \theta_d \frac{d-1}{2} & 0 \\
0 & \theta_0 & \theta_1 & \cdots & \theta_d \frac{d-1}{2} & \theta_d \\
0 & \theta_0 & \theta_1 & \cdots & \theta_d \frac{d-1}{2} & \theta_d \\
0 & \theta_0 & \theta_1 & \cdots & \theta_d \frac{d-1}{2} & \theta_d \frac{d-1}{2} \\
0 & \theta_0 & \theta_1 & \cdots & \theta_d \frac{d-1}{2} & \theta_d \frac{d-1}{2} \\
\end{pmatrix}
\]

(19)

where

\[
\theta_i = \left(a - \frac{d+1}{2} + 2i\right) \left(a - \frac{d+1}{2} + 2i + 2\right) \quad \text{for } i = 0, 1, \ldots, \frac{d-1}{2}.
\]

Since \( 2a \) is not among \( d - 3, d - 5, \ldots, 3 - d \), the scalars \( \{\theta_i\}_{i=0}^{\frac{d-1}{2}} \) are mutually distinct. Hence the \( \theta_i \)-eigenspace of \( L \) in \( E_d(a, b, c) \) has dimension less than or equal to two for all \( i = 0, 1, \ldots, \frac{d-1}{2} \). By (19) the first two rows of \( L - \theta_0 \) are zero and the last two columns of \( L - \theta_\frac{d-1}{2} \) are zero. Hence the \( \theta_i \)-eigenspace of \( L \) in \( E_d(a, b, c) \) has dimension two for all \( i = 0, 1, \ldots, \frac{d-1}{2} \).

By Lemma 5.1 the matrix \([t_0 + t_1]\) is a lower triangular matrix of the form

\[
\begin{pmatrix}
\vartheta_0 & -\vartheta_1 & 0 \\
\vartheta_1 & -\vartheta_2 & \cdots & 0 \\
\vartheta_2 & \cdots & \vartheta_d & 0 \\
0 & \cdots & 0 & -\vartheta_\frac{d+1}{2} \\
\end{pmatrix}
\]

(20)

where

\[
\vartheta_i = a - \frac{d+1}{2} + 2i \quad \text{for } i = 0, 1, \ldots, \frac{d+1}{2}.
\]

Let \( i \in \{1, 2, \ldots, \frac{d-3}{2}\} \) be given. Let \( u \) and \( w \) denote the eigenvectors of \( t_0 + t_1 \) in \( E_d(a, b, c) \) corresponding to the eigenvalues \( \vartheta_i \) and \( -\vartheta_{i+1} \), respectively. Since \( 2a \) is not among \( d - 5, d - 9, \ldots, 5 - d \), the scalars \( \vartheta_i \) and \( -\vartheta_{i+1} \) are distinct. It follows that \( u \) and \( w \) are linearly independent. Observe that \( u \) and \( w \) are the \( \theta_i \)-eigenvectors of \( L \). Hence the \( \theta_i \)-eigenspace of \( L \) in \( E_d(a, b, c) \) has dimension two. The statement (i) follows.

(ii), (iii): Using Lemma 5.2(i), (ii) the statements (ii), (iii) follow by the arguments similar to the proof for (i). \( \Box \)

Recall the finite-dimensional irreducible \( \mathbb{R} \)-modules from §2 and the even-dimensional irreducible \( \mathfrak{H} \)-modules from §4.
Theorem 5.8 ([§4.2–§4.5, 7]). Suppose that the $\mathfrak{H}$-module $E_d(a, b, c)$ is irreducible. Then the following hold:

(i) If $d = 1$ then the $\mathbb{R}$-module $E_d(a, b, c)$ is irreducible and it is isomorphic to

\[ R_1 \left( -\frac{a+1}{2}, -\frac{b+1}{2}, -\frac{c+1}{2} \right). \]

(ii) If $d \geq 3$ then the factors of any composition series for the $\mathbb{R}$-module $E_d(a, b, c)$ are isomorphic to

\[ R_{d+1} \left( -\frac{a+1}{2}, -\frac{b+1}{2}, -\frac{c+1}{2} \right), \]
\[ R_{d-3} \left( -\frac{a+1}{2}, -\frac{b+1}{2}, -\frac{c+1}{2} \right). \]

(iii) The factors of any composition series for the $\mathbb{R}$-module $E_d(a, b, c)^{(1,-1)}$ are isomorphic to

\[ R_{d+1} \left( -\frac{a}{2}, -\frac{b+1}{2}, -\frac{c+1}{2} \right), \]
\[ R_{d-2} \left( -\frac{a}{2}-1, -\frac{b+1}{2}, -\frac{c+1}{2} \right). \]

(iv) The factors of any composition series for the $\mathbb{R}$-module $E_d(a, b, c)^{(-1,1)}$ are isomorphic to

\[ R_{d+1} \left( -\frac{a+1}{2}, -\frac{b}{2}, -\frac{c+1}{2} \right), \]
\[ R_{d-4} \left( -\frac{a+1}{2}, -\frac{b-1}{2}, -\frac{c+1}{2} \right). \]

(v) The factors of any composition series for the $\mathbb{R}$-module $E_d(a, b, c)^{(-1,-1)}$ are isomorphic to

\[ R_{d+1} \left( -\frac{a+1}{2}, -\frac{b+1}{2}, -\frac{c}{2} \right), \]
\[ R_{d-2} \left( -\frac{a+1}{2}, -\frac{b+1}{2}, -\frac{c}{2}-1 \right). \]

We now give the necessary and sufficient conditions for $A, B, C$ to be multiplicity-free on the composition factors of even-dimensional irreducible $\mathfrak{H}$-modules.

Lemma 5.9. Suppose that the $\mathfrak{H}$-module $E_d(a, b, c)$ is irreducible. Then the following hold:

(i) $A$ is multiplicity-free on all composition factors of the $\mathbb{R}$-module $E_d(a, b, c)$ if and only if $2a$ is not among $d-1, d-3, \ldots, 1-d$.

(ii) $B$ is multiplicity-free on all composition factors of the $\mathbb{R}$-module $E_d(a, b, c)$ if and only if $2b$ is not among $d-1, d-3, \ldots, 1-d$.

(iii) $C$ is multiplicity-free on all composition factors of the $\mathbb{R}$-module $E_d(a, b, c)$ if and only if $2c$ is not among $d-1, d-3, \ldots, 1-d$.

Proof. Immediate from Lemma 5.4 and Theorem 5.8(i), (ii). \qed

Lemma 5.10. Suppose that the $\mathfrak{H}$-module $E_d(a, b, c)$ is irreducible. Then the following hold:
5.10. The actions of $A$, $B$, $C$

Suppose that the $\mathfrak{H}$-module $E_d(a, b, c)$ is irreducible. Then the following hold:

(i) $A$ is multiplicity-free on all composition factors of the $\mathcal{R}$-module $E_d(a, b, c)^{(1,-1)}$ if and only if $2a$ is not among $d - 1, d - 3, \ldots, 1 - d$.

(ii) $B$ is multiplicity-free on all composition factors of the $\mathcal{R}$-module $E_d(a, b, c)^{(1,-1)}$ if and only if $2b$ is not among $d - 3, d - 5, \ldots, 3 - d$.

(iii) $C$ is multiplicity-free on all composition factors of the $\mathcal{R}$-module $E_d(a, b, c)^{(1,-1)}$ if and only if $2c$ is not among $d - 3, d - 5, \ldots, 3 - d$.

Proof. Immediate from Lemma 2.4 and Theorem 5.8(iii). □

Lemma 5.11. Suppose that the $\mathfrak{H}$-module $E_d(a, b, c)$ is irreducible. Then the following hold:

(i) $A$ is multiplicity-free on all composition factors of the $\mathcal{R}$-module $E_d(a, b, c)^{(-1,1)}$ if and only if $2a$ is not among $d - 3, d - 5, \ldots, 3 - d$.

(ii) $B$ is multiplicity-free on all composition factors of the $\mathcal{R}$-module $E_d(a, b, c)^{(-1,1)}$ if and only if $2b$ is not among $d - 3, d - 5, \ldots, 3 - d$.

(iii) $C$ is multiplicity-free on all composition factors of the $\mathcal{R}$-module $E_d(a, b, c)^{(-1,1)}$ if and only if $2c$ is not among $d - 3, d - 5, \ldots, 3 - d$.

Proof. Immediate from Lemma 2.4 and Theorem 5.8(iv). □

Lemma 5.12. Suppose that the $\mathfrak{H}$-module $E_d(a, b, c)$ is irreducible. Then the following hold:

(i) $A$ is multiplicity-free on all composition factors of the $\mathcal{R}$-module $E_d(a, b, c)^{(-1,-1)}$ if and only if $2a$ is not among $d - 3, d - 5, \ldots, 3 - d$.

(ii) $B$ is multiplicity-free on all composition factors of the $\mathcal{R}$-module $E_d(a, b, c)^{(-1,-1)}$ if and only if $2b$ is not among $d - 3, d - 5, \ldots, 3 - d$.

(iii) $C$ is multiplicity-free on all composition factors of the $\mathcal{R}$-module $E_d(a, b, c)^{(-1,-1)}$ if and only if $2c$ is not among $d - 3, d - 5, \ldots, 3 - d$.

Proof. Immediate from Lemma 2.4 and Theorem 5.8(v). □

We are in the position to prove Theorem 1.6 in the even-dimensional case.

Theorem 5.13. Suppose that $V$ is an even-dimensional irreducible $\mathfrak{H}$-module. Then the following are equivalent:

(i) $A$ (resp. $B$) (resp. $C$) is diagonalizable on $V$.

(ii) $A$ (resp. $B$) (resp. $C$) is diagonalizable on all composition factors of the $\mathcal{R}$-module $V$.

(iii) $A$ (resp. $B$) (resp. $C$) is multiplicity-free on all composition factors of the $\mathcal{R}$-module $V$.

Proof. (i) $\Rightarrow$ (ii): Trivial.

(ii) $\Rightarrow$ (iii): Immediate from Lemma 2.4

(iii) $\Rightarrow$ (i): Suppose that (iii) holds. Let $d = \dim V - 1$. By Theorem 1.3 there exist $a, b, c \in \mathbb{F}$ and $\varepsilon \in \{\pm 1\}^2$ such that the $\mathfrak{H}$-module $E_d(a, b, c)^\varepsilon$ is isomorphic to $V$.

Using Lemmas 5.3, 5.6 and 5.10 yields that (i) holds for the case $\varepsilon = (1, 1)$. Note that the actions of $A, B, C$ on $E_d(a, b, c)^{(1,-1)}$ are identical to the actions of

$$(t_0 + t_1 - 1)(t_0 + t_1 + 1), \quad (t_1 + t_3 - 1)(t_1 + t_3 + 1), \quad (t_1 + t_2 - 1)(t_1 + t_2 + 1)$$

on $E_d(a, b, c)$ respectively. Hence (i) holds for the case $\varepsilon = (1, -1)$ by Lemmas 5.3, 5.7 and 5.10. The actions of $A, B, C$ on $E_d(a, b, c)^{(-1,1)}$ are identical to the actions of

$$(t_2 + t_3 - 1)(t_2 + t_3 + 1), \quad (t_0 + t_2 - 1)(t_0 + t_2 + 1), \quad (t_1 + t_2 - 1)(t_1 + t_2 + 1)$$

on $E_d(a, b, c)$ respectively. Hence (i) holds for the case $\varepsilon = (-1, 1)$ by Lemmas 5.3, 5.7 and 5.10. The actions of $A, B, C$ on $E_d(a, b, c)^{(-1,-1)}$ are identical to the actions of

$$(t_2 + t_3 - 1)(t_2 + t_3 + 1), \quad (t_0 + t_2 - 1)(t_0 + t_2 + 1), \quad (t_1 + t_2 - 1)(t_1 + t_2 + 1)$$

on $E_d(a, b, c)$ respectively. Hence (i) holds for the case $\varepsilon = (-1, -1)$ by Lemmas 5.3, 5.7 and 5.10.
on \( E_d(a, b, c) \) respectively. Hence (i) holds for the case \( \varepsilon = (-1, 1) \) by Lemmas 5.10, 5.11, and 5.12. The actions of \( A, B, C \) on \( E_d(a, b, c)^{(−1, −1)} \) are identical to the actions of

\[
\frac{(t_2 + t_3 - 1)(t_2 + t_3 + 1)}{4}, \quad \frac{(t_1 + t_3 - 1)(t_1 + t_3 + 1)}{4}, \quad \frac{(t_0 + t_3 - 1)(t_0 + t_3 + 1)}{4}
\]
on \( E_d(a, b, c) \) respectively. Hence (i) holds for the case \( \varepsilon = (-1, −1) \) by Lemmas 5.6, 5.7, and 5.12. We have shown that (i) holds for all elements \( \varepsilon \in \{±1\}^2 \). Therefore (i) follows. \( \Box \)

6. Preliminaries on the odd-dimensional irreducible \( \mathfrak{f} \)-modules

**Proposition 6.1** ([5]). For any scalars \( a, b, c \in \mathbb{F} \) and any even integer \( d \geq 0 \), there exists a \((d + 1)\)-dimensional irreducible \( \mathfrak{f} \)-module \( O_d(a, b, c) \) satisfying the following conditions:

(i) There exists a basis \( \{v_i\}_{i=0}^d \) for \( O_d(a, b, c) \) such that

\[
t_0 v_i = \begin{cases} 
-i(\sigma + i)v_{i-1} + \frac{\sigma + 2i}{2} v_i & \text{for } i = 2, 4, \ldots, d, \\
-\frac{\sigma + 2i}{2} v_i + v_{i+1} & \text{for } i = 1, 3, \ldots, d - 1,
\end{cases}
\]

\[
t_0 v_0 = \frac{\sigma}{2} v_0,
\]

\[
t_1 v_i = \begin{cases} 
i(\sigma + i)v_{i-1} + \frac{\lambda}{2} v_i + v_{i+1} & \text{for } i = 2, 4, \ldots, d - 2, \\
\frac{\lambda}{2} v_i & \text{for } i = 1, 3, \ldots, d - 1,
\end{cases}
\]

\[
t_1 v_0 = \frac{\lambda}{2} v_0 + v_1, \quad t_1 v_d = d(\sigma + d)v_{d-1} + \frac{\lambda}{2} v_d,
\]

\[
t_2 v_i = \begin{cases} 
\frac{\nu}{2} v_i & \text{for } i = 0, 2, \ldots, d, \\
(d - i - 1)(\tau + i)v_{i-1} - \frac{\nu}{2} v_i - v_{i+1} & \text{for } i = 1, 3, \ldots, d - 1,
\end{cases}
\]

\[
t_3 v_i = \begin{cases} 
\frac{2d + \mu - 2i}{2} v_i - v_{i+1} & \text{for } i = 0, 2, \ldots, d - 2, \\
(i - d - 1)(\tau + i)v_{i-1} - \frac{2d + \mu - 2i + 2}{2} v_i & \text{for } i = 1, 3, \ldots, d - 1,
\end{cases}
\]

\[
t_3 v_d = \frac{\mu}{2} v_d.
\]

where

\[
\begin{align*}
\sigma &= a + b + c - \frac{d + 1}{2}, & \tau &= a + b - c - \frac{d + 1}{2}, \\
\lambda &= a - b - c - \frac{d + 1}{2}, & \mu &= c - a - b - \frac{d + 1}{2}, \\
\nu &= b - a - c - \frac{d + 1}{2}.
\end{align*}
\]

(ii) The elements \( t_0^2, t_1^2, t_2^2, t_3^2 \) act on \( O_d(a, b, c) \) as scalar multiplication by

\[
\left( \frac{a + b + c}{2} - \frac{d + 1}{4} \right)^2, \quad \left( \frac{a - b - c}{2} - \frac{d + 1}{4} \right)^2,
\]

\[
\left( \frac{a - b - c}{2} - \frac{d + 1}{4} \right)^2.
\]
Lemma 6.3. Let \( t, a, b, c \in \mathbb{F} \) such that the \( \mathfrak{H} \)-module \( O_d(a, b, c) \) is irreducible if and only if
\[
\begin{align*}
& a + b + c, \ a - b - c, \ -a + b - c, \ -a - b + c, \\
& \left\{ \frac{d + 1}{2} - i \right\} 
\end{align*}
\]
for \( i = 2, 4, \ldots, d \).

Proof. It is routine to verify the lemma by using Proposition 6.1(i). \( \square \)

By means of Lemma 6.3 we develop the following discriminant to determine the scalars \( a, b, c \in \mathbb{F} \) in Theorem 6.2(ii):

Theorem 6.2 (Theorem 2.8, [5]). Let \( d \geq 0 \) denote an even integer. Then the following hold:
\begin{enumerate}
  \item For any \( a, b, c \in \mathbb{F} \) the \( \mathfrak{H} \)-module \( O_d(a, b, c) \) is irreducible if and only if
  \[
  a + b + c, \ a - b - c, \ -a + b - c, \ -a - b + c \not\in \left\{ \frac{d + 1}{2} - i \right\} 
  \]
  for \( i = 2, 4, \ldots, d \).
  \item Suppose that \( V \) is a \((d+1)\)-dimensional irreducible \( \mathfrak{H} \)-module. Then there exist \( a, b, c \in \mathbb{F} \) such that the \( \mathfrak{H} \)-module \( O_d(a, b, c) \) is isomorphic to \( V \).
\end{enumerate}

Proof. Immediate from Theorem 6.2(ii) and Lemma 6.3. \( \square \)

7. The Conditions for \( A, B, C \) as Diagonalizable on Odd-Dimensional Irreducible \( \mathfrak{H} \)-Modules

For convenience we adopt the following notations in this section: Let \( d \geq 0 \) denote an even integer. Let \( a, b, c \) denote any scalars in \( \mathbb{F} \). Let \( \{ v_i \}_{i=0}^d \) denote the basis for \( O_d(a, b, c) \) from Proposition 6.1(i).

Lemma 7.1. The action of \( t_0 + t_1 \) on \( O_d(a, b, c) \) is as follows:
\[
\begin{align*}
& \left( t_0 + t_1 + (-1)^i \left( \frac{d}{2} - a - i \right) + \frac{1}{2} \right) v_i = \left\{ \begin{array}{ll}
& v_{i+1} \quad \text{for } i = 0, 1, \ldots, d - 1, \\
& 0 \quad \text{for } i = d.
\end{array} \right.
\end{align*}
\]

Proof. It is routine to verify the lemma by using Proposition 6.1(i). \( \square \)

Lemma 7.2. Suppose that the \( \mathfrak{H} \)-module \( O_d(a, b, c) \) is irreducible. Then the following hold:
\begin{enumerate}
  \item There exists a basis \( \{ w_i \}_{i=0}^d \) for \( O_d(a, b, c) \) such that
  \[
  \left( t_0 + t_2 + (-1)^i \left( \frac{d}{2} - b - i \right) + \frac{1}{2} \right) w_i = \left\{ \begin{array}{ll}
& w_{i+1} \quad \text{for } i = 0, 1, \ldots, d - 1, \\
& 0 \quad \text{for } i = d.
\end{array} \right.
  \]
  \item There exists a basis \( \{ w_i \}_{i=0}^d \) for \( O_d(a, b, c) \) such that
  \[
  \left( t_0 + t_3 + (-1)^i \left( \frac{d}{2} - c - i \right) + \frac{1}{2} \right) w_i = \left\{ \begin{array}{ll}
& w_{i+1} \quad \text{for } i = 0, 1, \ldots, d - 1, \\
& 0 \quad \text{for } i = d.
\end{array} \right.
  \]
\end{enumerate}
Proof. Let $V$ denote the $\mathcal{H}$-module $O_d(a, b, c)$.

(i): By Definition 1.2 there exists a unique algebra automorphism $\rho : \mathcal{H} \rightarrow \mathcal{H}$ given by

$$(t_0, t_1, t_2, t_3) \mapsto (t_0, t_2, t_3, t_1)$$

whose inverse sends $(t_0, t_1, t_2, t_3)$ to $(t_0, t_3, t_1, t_2)$. By Lemma 6.3 the traces of $t_0 + t_1 + \frac{1}{2}, t_0 + t_2 + \frac{1}{2}, t_0 + t_3 + \frac{1}{2}$ act on $V^\rho$ are $b, c, a$ respectively. Therefore the $\mathcal{H}$-module $V^\rho$ is isomorphic to $O_d(b, c, a)$ by Theorem 6.4. It follows from Lemma 7.1 that there exists a basis $\{w_i\}_{i=0}^d$ for $V^\rho$ such that

$$\left(t_0 + t_1 + (-1)^i \left(\frac{d}{2} - b - i\right) + \frac{1}{2}\right) w_i = \begin{cases} w_{i+1} & \text{for } i = 0, 1, \ldots, d - 1, \\ 0 & \text{for } i = d. \end{cases}$$

Observe that the action of $t_0 + t_1$ on $V^\rho$ is identical to the action of $t_0 + t_2$ on $V$. Hence (i) follows.

(ii): By Definition 1.2 there exists a unique algebra automorphism $\rho : \mathcal{H} \rightarrow \mathcal{H}$ given by

$$(t_0, t_1, t_2, t_3) \mapsto (t_0, t_3, t_2, t_1)$$

whose inverse sends $(t_0, t_1, t_2, t_3)$ to $(t_0, t_3, t_2, t_1)$. By Lemma 6.3 the traces of $t_0 + t_1 + \frac{1}{2}, t_0 + t_2 + \frac{1}{2}, t_0 + t_3 + \frac{1}{2}$ on $V^\rho$ are $c, b, a$ respectively. Therefore the $\mathcal{H}$-module $V^\rho$ is isomorphic to $O_d(c, b, a)$ by Theorem 6.4. It follows from Lemma 7.1 that there exists a basis $\{w_i\}_{i=0}^d$ for $V^\rho$ such that

$$\left(t_0 + t_1 + (-1)^i \left(\frac{d}{2} - c - i\right) + \frac{1}{2}\right) w_i = \begin{cases} w_{i+1} & \text{for } i = 0, 1, \ldots, d - 1, \\ 0 & \text{for } i = d. \end{cases}$$

Observe that the action of $t_0 + t_1$ on $V^\rho$ is identical to the action of $t_0 + t_3$ on $V$. Hence (ii) follows.

Lemma 7.3. Suppose that the $\mathcal{H}$-module $O_d(a, b, c)$ is irreducible. Then the following are equivalent:

(i) $t_0 + t_1$ is diagonalizable on $O_d(a, b, c)$.
(ii) $t_0 + t_1$ is multiplicity-free on $O_d(a, b, c)$.
(iii) $2a$ is not among $d - 1, d - 3, \ldots, 1 - d$.

Proof. Similar to the proof for Lemma 6.4.

Lemma 7.4. Suppose that the $\mathcal{H}$-module $O_d(a, b, c)$ is irreducible. Then the following are equivalent:

(i) $t_0 + t_2$ is diagonalizable on $O_d(a, b, c)$.
(ii) $t_0 + t_2$ is multiplicity-free on $O_d(a, b, c)$.
(iii) $2b$ is not among $d - 1, d - 3, \ldots, 1 - d$.

Proof. In view of Lemma 7.2(i) the lemma follows by an argument similar to the proof for Lemma 7.3.

Lemma 7.5. Suppose that the $\mathcal{H}$-module $O_d(a, b, c)$ is irreducible. Then the following are equivalent:

(i) $t_0 + t_3$ is diagonalizable on $O_d(a, b, c)$.
(ii) $t_0 + t_3$ is multiplicity-free on $O_d(a, b, c)$.
(iii) $2c$ is not among $d - 1, d - 3, \ldots, 1 - d$. 

Proof. In view of Lemma 7.2(i) the lemma follows by an argument similar to the proof for Lemma 7.3.
Proof. In view of Lemma 7.2(ii) the lemma follows by an argument similar to the proof for Lemma 7.3.

Lemma 7.6. Suppose that the $\mathcal{H}$-module $O_d(a, b, c)$ is irreducible. Then the following hold:

(i) If $2a$ is not among $d - 1, d - 3, \ldots, 3 - d$ then $A$ is diagonalizable on $O_d(a, b, c)$.
(ii) If $2b$ is not among $d - 1, d - 3, \ldots, 3 - d$ then $B$ is diagonalizable on $O_d(a, b, c)$.
(iii) If $2c$ is not among $d - 1, d - 3, \ldots, 3 - d$ then $C$ is diagonalizable on $O_d(a, b, c)$.

Proof. (i): Given any element $X$ of $\mathcal{H}$, let $[X]$ denote the matrix representing $X$ with respect to $\{v_i\}_{i=0}^d$. Using Lemma 7.1 a direct calculation yields that $[A]$ is a lower triangular matrix of the form

$$
\begin{pmatrix}
\theta_0 & \theta_1 & \theta_2 & \cdots & \theta_{d/2} & 0 \\
0 & \theta_0 & \theta_1 & \cdots & \theta_{d/2} & 0 \\
0 & 0 & \theta_0 & \cdots & \theta_{d/2} & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \theta_0 & \theta_1 \\
0 & 0 & 0 & \cdots & 0 & \theta_0
\end{pmatrix}
$$

(21)

where

$$
\theta_i = \left(\frac{a}{2} - \frac{d + 3}{4} + i\right) \left(\frac{a}{2} - \frac{d + 3}{4} + i + 1\right) \quad \text{for } i = 0, 1, \ldots, d/2.
$$

Since $2a$ is not among $d - 1, d - 3, \ldots, 3 - d$, the scalars $\{\theta_i\}_{i=0}^{d/2}$ are mutually distinct. Hence the $\theta_0$-eigenspace of $A$ in $O_d(a, b, c)$ has dimension one and the $\theta_i$-eigenspace of $A$ in $O_d(a, b, c)$ has dimension less than or equal to two for all $i = 1, 2, \ldots, d/2$. By (21) the last two columns of $[A - \theta_{d/2}]$ are zero. Hence the $\theta_{d/2}$-eigenspace of $A$ in $O_d(a, b, c)$ has dimension two. To see the diagonalizability of $A$ it remains to show that the $\theta_i$-eigenspace of $A$ in $O_d(a, b, c)$ has dimension two for all $i = 1, 2, \ldots, d/2 - 1$.

By Lemma 7.1 the matrix $[t_0 + t_1]$ is a lower triangular matrix of the form

$$
\begin{pmatrix}
\vartheta_0 & \vartheta_1 & \vartheta_2 & \cdots & \vartheta_{d/2} & 0 \\
0 & \vartheta_0 & \vartheta_1 & \cdots & \vartheta_{d/2} & 0 \\
0 & 0 & \vartheta_0 & \cdots & \vartheta_{d/2} & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \vartheta_0 & \vartheta_1 \\
0 & 0 & 0 & \cdots & 0 & \vartheta_0
\end{pmatrix}
$$

where

$$
\vartheta_i = a - \frac{d + 1}{2} + 2i \quad \text{for } i = 0, 1, \ldots, d/2.
$$

Let $i \in \{1, 2, \ldots, d/2 - 1\}$ be given. Let $u$ and $w$ denote the eigenvectors of $t_0 + t_1$ in $O_d(a, b, c)$ corresponding to the eigenvalues $\vartheta_i$ and $-\vartheta_i$, respectively. Since $2a$ is not among $d - 3, d - 7, \ldots, 5 - d$, the scalars $\vartheta_i$ and $-\vartheta_i$ are distinct. It follows that $u$ and $w$ are linearly
Lemma 7.8. Suppose that the composition factors of odd-dimensional irreducible $H$ hold:

(ii), (iii): Using Lemma 7.2(i), (ii) the statements (ii), (iii) follow by the arguments similar to the proof for (i). □

Recall the finite-dimensional irreducible $\mathcal{R}$-modules from §2 and the odd-dimensional irreducible $J$-modules from §6.

Theorem 7.7 (§4.6, [7]). Suppose that the $J$-module $O_d(a, b, c)$ is irreducible. Then the following hold:

(i) If $d = 0$ then the $\mathcal{R}$-module $O_d(a, b, c)$ is irreducible and it is isomorphic to

$$R_0 \left( -\frac{a}{2} - \frac{1}{4}, -\frac{b}{2} - \frac{1}{4}, -\frac{c}{2} - \frac{1}{4} \right).$$

(ii) If $d \geq 2$ and $a + b + c = \frac{d + 1}{2}$ then the factors of any composition series for the $\mathcal{R}$-module $O_d(a, b, c)$ are isomorphic to

$$R_{\frac{d}{2} + 1} \left( -\frac{a}{2} - \frac{3}{4}, -\frac{b}{2} - \frac{3}{4}, -\frac{c}{2} - \frac{3}{4} \right),$$

$$R_0 \left( -\frac{b + c + 1}{2}, -\frac{a + c + 1}{2}, -\frac{a + b + 1}{2} \right).$$

(iii) If $d \geq 2$ and $a + b + c \neq \frac{d + 1}{2}$ then the factors of any composition series for the $\mathcal{R}$-module $O_d(a, b, c)$ are isomorphic to

$$R_{\frac{d}{2}} \left( -\frac{a}{2} - \frac{1}{4}, -\frac{b}{2} - \frac{1}{4}, -\frac{c}{2} - \frac{1}{4} \right),$$

$$R_{\frac{d}{2} - 1} \left( -\frac{a}{2} - \frac{3}{4}, -\frac{b}{2} - \frac{3}{4}, -\frac{c}{2} - \frac{3}{4} \right).$$

We now give the necessary and sufficient conditions for $A, B, C$ to be multiplicity-free on the composition factors of odd-dimensional irreducible $J$-modules.

Lemma 7.8. Suppose that the $J$-module $O_d(a, b, c)$ is irreducible. If $a + b + c = \frac{d + 1}{2}$ then the following hold:

(i) $A$ is multiplicity-free on all composition factors of the $\mathcal{R}$-module $O_d(a, b, c)$ if and only if $2a$ is not among $d - 5, d - 7, \ldots, 3 - d$.

(ii) $B$ is multiplicity-free on all composition factors of the $\mathcal{R}$-module $O_d(a, b, c)$ if and only if $2b$ is not among $d - 5, d - 7, \ldots, 3 - d$.

(iii) $C$ is multiplicity-free on all composition factors of the $\mathcal{R}$-module $O_d(a, b, c)$ if and only if $2c$ is not among $d - 5, d - 7, \ldots, 3 - d$.

Proof. Immediate from Lemma 2.4 and Theorem 7.7(i), (ii). □

Lemma 7.9. Suppose that the $J$-module $O_d(a, b, c)$ is irreducible. If $a + b + c \neq \frac{d + 1}{2}$ then the following hold:

(i) $A$ is multiplicity-free on all composition factors of the $\mathcal{R}$-module $O_d(a, b, c)$ if and only if $2a$ is not among $d - 1, d - 3, \ldots, 3 - d$.

(ii) $B$ is multiplicity-free on all composition factors of the $\mathcal{R}$-module $O_d(a, b, c)$ if and only if $2b$ is not among $d - 1, d - 3, \ldots, 3 - d$. 

(iii) \( C \) is multiplicity-free on all composition factors of the \( \mathcal{R} \)-module \( O_d(a, b, c) \) if and only
if \( 2c \) is not among \( d - 1, d - 3, \ldots, 3 - d \).

Proof. Immediate from Lemma 2.4 and Theorem 7.7(i), (iii). □

We are in the position to prove Theorem 1.6 in odd-dimensional case.

**Theorem 7.10.** Suppose that \( V \) is an odd-dimensional irreducible \( \mathcal{H} \)-module. Then the
following are equivalent:

(i) \( A \) (resp. \( B \)) (resp. \( C \)) is diagonalizable on \( V \).
(ii) \( A \) (resp. \( B \)) (resp. \( C \)) is diagonalizable on all composition factors of the \( \mathcal{R} \)-module \( V \).
(iii) \( A \) (resp. \( B \)) (resp. \( C \)) is multiplicity-free on all composition factors of the \( \mathcal{R} \)-module \( V \).

Proof. (i) \( \Rightarrow \) (ii): Trivial.

(ii) \( \Rightarrow \) (iii): Immediate from Lemma 2.4.

(iii) \( \Rightarrow \) (i): Suppose that (iii) holds. Let \( d = \dim V - 1 \). By Theorem 6.2(ii) there exist
\( a, b, c \in \mathbb{F} \) such that \( \mathcal{H} \)-module \( O_d(a, b, c) \) is isomorphic to \( V \).

Suppose that \( a + b + c = \frac{d+1}{2} \). Since the \( \mathcal{H} \)-module \( O_d(a, b, c) \) is irreducible it follows from
Theorem 6.2(i) that none of \( 2a, 2b, 2c \) is among \( 1, 3, \ldots, d - 1 \). Combined with Lemmas 7.6
and 7.8 this yields that (i) holds. Suppose that \( a + b + c \neq \frac{d+1}{2} \). By Lemmas 7.6 and 7.9 the statement (i) holds. Therefore (i) follows. □

Proof of Theorem 1.7. It is immediate from Theorem 1.6 and Proposition 3.3.

Proof of Theorem 1.8. It is immediate from Theorem 1.6 and Proposition 3.4.

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