FLAT SML MODULES AND REFLEXIVE FUNCTORS

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Abstract. We give some functorial characterizations of flat strict Mittag-Leffler modules. We characterize reflexive functors of modules with similar tools, definitions and theorems.

1. Introduction

Let $R$ be a commutative (associative with unit) ring. Let $\mathcal{R}$ be the covariant functor from the category of commutative $R$-algebras to the rings defined by $\mathcal{R}(S) := S$ for any commutative $R$-algebra $S$. Let $M$ be an $R$-module. Consider the functor of $\mathcal{R}$-modules, $\mathcal{M}$, defined by $\mathcal{M}(S) := M \otimes_R S$, for any commutative $R$-algebra $S$. $\mathcal{M}$ is said to be the quasi-coherent $R$-module associated with $M$. The functors

$$\text{Category of } R\text{-modules} \to \text{Category of quasi-coherent } R\text{-modules}$$

$$\mathcal{M}(R) \leftrightarrow M$$

establish an equivalence of categories. Consider the dual functor $\mathcal{M}^* := \text{Hom}_R(\mathcal{M}, \mathcal{R})$ defined by $\mathcal{M}^*(S) := \text{Hom}_S(M \otimes_R S, S)$. In general, the canonical morphism $M \to M^{**}$ is not an isomorphism, but surprisingly $M = M^{**}$ (see [2,13]), that is, $\mathcal{M}$ is a reflexive functor of $\mathcal{R}$-module. This result has many applications in Algebraic Geometry (see [8]), for example the Cartier duality of commutative affine groups and commutative formal groups.

Given an $R$-module $N$ we shall say that $N^*$ is an $R$-module scheme. In [2], we proved that an $R$-module $M$ is a finitely generated projective module iff $\mathcal{M}$ is an $\mathcal{R}$-module scheme. In [10], we proved that $M$ is a flat $R$-module iff $\mathcal{M}$ is a direct limit of $\mathcal{R}$-module schemes. We proved too that the following statements are equivalent:

1. $M$ is a flat Mittag-Leffler module
2. $\mathcal{M}$ is the direct limit of its $\mathcal{R}$-submodule schemes.
3. The kernel of any morphism $N^* \to \mathcal{M}$ is an $\mathcal{R}$-module scheme.
4. The kernel of any morphism $R^n \to \mathcal{M}$ is an $\mathcal{R}$-module scheme.

In this paper we shall give some functorial characterizations of flat strict Mittag-Leffler modules. Mittag-Leffler conditions were first introduced by Grothendieck in [5], and deeply studied by some authors, for example, Raynaud and Gruson in [6]. Flat strict Mittag-Leffler modules have also been studied by Ohm and Rush under the name of "trace modules" in [9], by Garfinkel, who calls them "universally torsionless" in [4] and by Zimmermann-Huisgen, under the name of "locally projective modules" in [12]. We prove the following theorem.

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Theorem 1.1. Let $M$ be an $R$-module. The following statements are equivalent.

1. $M$ is a flat strict Mittag-Leffler module (see [3, II 2.3.2]). That is, $M$ is flat and it is isomorphic to a direct limit of finitely presented modules $F_i$, so that for every $R$-module $N$ and every $i$ there exists a $j \geq i$ such that
   \[ \text{Im}(\text{Hom}_R(M, N) \to \text{Hom}(F_i, N)) = \text{Im}(\text{Hom}_R(F_j, N) \to \text{Hom}(F_i, N)). \]

2. $M = \lim \rightarrow_i N_i^*$, where $\{N_i^*\}$ is the set of the $R$-submodule schemes of $M$, and the natural morphisms $M^* \to N_i$ are surjective.

3. $M^*$ is dually separated, that is, the natural morphism $M^* \otimes_R R/N \to \text{Hom}_R(M^*, R/N)$ is injective, for every $R$-module $N$.

4. The natural morphism $M^* \otimes_R N \to \text{Hom}_R(M^*, N)$ is injective, for every $R$-module $N$ (that is, $M$ is universally torsionless, see [4]).

5. There exists a monomorphism $M \to \prod I R$.

6. $M$ is a flat Mittag-Leffler module and the morphism
   \[ M \otimes_R R/\mathfrak{m} \to \text{Hom}_R(M^*, R/\mathfrak{m}) \]
   is injective, for every maximal ideal $\mathfrak{m} \subset R$.

7. The cokernel of every morphism $M^* \to N$ is quasi-coherent, for every $R$-module $N$.

8. The cokernel of every morphism $M^* \to R$ is quasi-coherent (which is equivalent to saying that $M$ is a trace module, see [6,13]).

More generally we shall give some characterizations of dually separated functors of $R$-modules.

Theorem 1.2. Let $\mathcal{M}$ be a functor of $\mathcal{R}$-modules. The following statements are equivalent

1. $\mathcal{M}$ is dually separated: The natural morphism $\mathcal{M}^*(S) \to \text{Hom}_S(\mathcal{M}(S), S)$ is injective, for any commutative $R$-algebra $S$.

2. The natural morphism $\text{Hom}_R(\mathcal{M}, N) \to \text{Hom}_R(\mathcal{M}(R), N)$ is injective, for any $R$-module $N$.

3. The natural morphism $\text{Hom}_R(\mathcal{M}, N) \to \text{Hom}_R(\mathcal{M}(R), N(R))$ is injective, for any dual functor $N$.

4. The cokernel of every morphism $\mathcal{M} \to N$ is quasi-coherent, for any $R$-module $N$.

Assume that $\mathcal{M}$ is reflexive.

5. There exists a monomorphism $\mathcal{M}^* \to \prod I \mathcal{R}$.

Now assume that $R$ is a field.

6. $\mathcal{M}^* = \lim \rightarrow_i N_i^*$, where $\{N_i\}$ is the set of the quasi-coherent quotient $\mathcal{R}$-modules of $\mathcal{M}$.

If $R$ is a field and $\mathcal{M}$ is a reflexive functor of $\mathcal{R}$-modules, we prove that $\mathcal{M}$ is dually separated and we obtain the following theorem.

Theorem 1.3. Let $R = K$ be a field. A functor of $K$-modules is reflexive iff it is equal to the inverse limit of its quasi-coherent quotient $\mathcal{R}$-modules.
If $I$ is a totally ordered set and $\{f_{ij} : M_i \to M_j\}_{i,j \in I}$ is an inverse system of $K$-vector spaces, we prove that $\lim_{i \in I} M_i$ is a reflexive functor of $K$-modules. Unfortunately, we do not know if arbitrary inverse limits of quasi-coherent $K$-modules are reflexive.

## 2. Preliminaries

Let $R$ be a commutative ring (associative with a unit). All the functors considered in this paper are covariant functors from the category of commutative $R$-algebras (always assumed to be associative with a unit) to the category of sets. A functor $X$ is said to be a functor of sets (resp. groups, rings, etc.) if $X$ is a functor from the category of commutative $R$-algebras to the category of sets (resp. groups, rings, etc.).

**Notation 2.1.** For simplicity, given a (covariant) functor $X$ (from the category of commutative $R$-algebras to the category of sets), we shall sometimes use $x \in X$ to denote $x \in X(S)$. Given $x \in X(S)$ and a morphism of commutative $R$-algebras $S \to S'$, we shall still denote by $x$ its image by the morphism $X(S) \to X(S')$.

Let $M$ and $M'$ be two $R$-modules. A morphism of $R$-modules $f : M \to M'$ is a morphism of functors such that the morphism $f_S : M(S) \to M'(S)$ defined by $f$ is a morphism of $S$-modules, for any commutative $R$-algebra $S$. We shall denote by $\text{Hom}_R(M, M')$ the family of all the morphisms of $R$-modules from $M$ to $M'$.

**Remark 2.2.** Direct limits, inverse limits of $R$-modules and kernels, cokernels, images, etc., of morphisms of $R$-modules are regarded in the category of $R$-modules.

One has

$$
\begin{align*}
(\text{Ker} f)(S) &= \text{Ker} f_S, \\
(\text{Coker} f)(S) &= \text{Coker} f_S, \\
(\text{Im} f)(S) &= \text{Im} f_S, \\
(\lim_{i \in I} M_i)(S) &= \lim_{i \in I} (M_i(S)), \\
(\lim_{j \in J} M_j)(S) &= \lim_{j \in J} (M_j(S)),
\end{align*}
$$

(where $I$ is an upward directed set and $J$ a downward directed set). $M \otimes_R M'$ is defined by $(M \otimes_R M')(S) := M(S) \otimes_S M'(S)$, for any commutative $R$-algebra $S$.

**Definition 2.3.** Given an $R$-module $M$ and a commutative $R$-algebra $S$, we shall denote by $M|_S$ the restriction of $M$ to the category of commutative $S$-algebras, i.e.,

$$
M|_S(S') := M(S'),
$$

for any commutative $S$-algebra $S'$.

We shall denote by $\mathcal{H}om_R(M, M')$ the $R$-module defined by

$$(\mathcal{H}om_R(M, M'))(S) := \text{Hom}_S(M|_S, M'|_S).$$

Obviously,

$$
(\mathcal{H}om_R(M, M'))|_S = \text{Hom}_S(M|_S, M'|_S).
$$

**Notation 2.4.** Let $M$ be an $R$-module. We shall denote $M^* = \mathcal{H}om_R(M, R)$.
Proposition 2.5. Let $\mathbb{M}$ and $\mathbb{N}$ be two $\mathcal{R}$-modules. Then,
\[ \text{Hom}_\mathcal{R}(\mathbb{M}, \mathbb{N}^*) = \text{Hom}_\mathcal{R}(\mathbb{N}, \mathbb{M}^*), \]  
where $\tilde{f}$ is defined as follows: $\tilde{f}(n)(m) := f(m)(n)$, for any $m \in \mathbb{M}$ and $n \in \mathbb{N}$.

Proof. $\text{Hom}_\mathcal{R}(\mathbb{M}, \mathbb{N}^*) = \text{Hom}_\mathcal{R}(\mathbb{M} \otimes_\mathcal{R} \mathbb{N}, \mathcal{R}) = \text{Hom}_\mathcal{R}(\mathbb{N}, \mathbb{M}^*)$. \hfill $\square$

Proposition 2.6.\cite{1} 1.15] Let $\mathbb{M}$ be an $\mathcal{R}$-module, $S$ a commutative $\mathcal{R}$-algebra and $\mathbb{N}$ an $S$-module. Then,
\[ \text{Hom}_\mathcal{R}(\mathbb{M}, \mathbb{N}) = \text{Hom}_S(\mathbb{M}|_S, \mathbb{N}), \]
where $\pi: N|_S \to \mathbb{N}$ is defined by $\pi_T(n \otimes_R t) := n \otimes_S t \in N \otimes_S T$, for any commutative $S$-algebra $T$ and any $n \otimes_R t \in N \otimes_R T = N|_S(T)$. In particular,
\[ \text{Hom}_\mathcal{R}(\mathbb{M}, S) = \mathbb{M}^*(S). \]

2.1. Quasi-coherent modules.

Definition 2.7. Let $\mathbb{M}$ (resp. $\mathbb{N}$, $\mathbb{V}$, etc.) be an $\mathcal{R}$-module. We shall denote by $\mathbb{M}$ (resp. $\mathbb{N}$, $\mathbb{V}$, etc.) the $\mathcal{R}$-module defined by $\mathbb{M}(S) := \mathbb{M} \otimes_\mathcal{R} S$ (resp. $\mathbb{N}(S) := N \otimes_\mathcal{R} S$, etc.). $\mathbb{M}$ will be called the quasi-coherent $\mathcal{R}$-module associated with $\mathbb{M}$.

$\mathbb{M}|_S$ is the quasi-coherent $S$-module associated with $\mathbb{M} \otimes_\mathcal{R} S$. For any pair of $\mathcal{R}$-modules $\mathbb{M}$ and $\mathbb{N}$, the quasi-coherent module associated with $\mathbb{M} \otimes_\mathcal{R} \mathbb{N}$ is $\mathbb{M} \otimes_\mathcal{R} \mathbb{N}$.

Proposition 2.8.\cite{1} 1.12] The functors
\[ \text{Category of } \mathcal{R}\text{-modules } \to \text{Category of quasi-coherent } \mathcal{R}\text{-modules} \]
\[ \mathbb{M} \mapsto \mathbb{M} \]
\[ \mathcal{M}(R) \leftrightarrow \mathbb{M} \]

establish an equivalence of categories. In particular,
\[ \text{Hom}_\mathcal{R}(\mathcal{M}, \mathcal{M}') = \text{Hom}_R(M, M'). \]

Let $f: M \to N$ be a morphism of $\mathcal{R}$-modules and $\tilde{f}: \mathcal{M} \to \mathcal{N}$ the associated morphism of $\mathcal{R}$-modules. Let $C = \text{Coker } f$, then $\text{Coker } \tilde{f} = C$, which is a quasi-coherent module.

Proposition 2.9.\cite{1} 1.3] For every $\mathcal{R}$-module $\mathbb{M}$ and every $\mathcal{R}$-module $M$, it is satisfied that
\[ \text{Hom}_\mathcal{R}(\mathcal{M}, \mathbb{M}) = \text{Hom}_R(M, M), \]
\[ f \mapsto f_R. \]

Notation 2.10. Let $\mathbb{M}$ be an $\mathcal{R}$-module. We shall denote by $\mathbb{M}_{qc}$ the quasi-coherent module associated with the $\mathcal{R}$-module $\mathbb{M}(R)$, that is,
\[ \mathbb{M}_{qc}(S) := \mathbb{M}(R) \otimes_\mathcal{R} S. \]

Proposition 2.11. For each $\mathcal{R}$-module $\mathbb{M}$ one has the natural morphism
\[ \mathbb{M}_{qc} \to \mathbb{M}, \quad m \otimes s \mapsto s \cdot m, \]
for any $m \otimes s \in \mathbb{M}_{qc}(S) = \mathbb{M}(R) \otimes_\mathcal{R} S$, and a functorial equality
\[ \text{Hom}_\mathcal{R}(\mathcal{N}, \mathbb{M}_{qc}) = \text{Hom}_R(N, \mathbb{M}), \]
for any quasi-coherent $\mathcal{R}$-module $\mathcal{N}$. 

Proof. Observe that \( \text{Hom}_R(N, M) \cong \text{Hom}_R(N, M(R)) \cong \text{Hom}_R(N, M_{qc}) \).

Obviously, an \( \mathcal{R} \)-module \( M \) is a quasi-coherent module iff the natural morphism \( M_{qc} \rightarrow M \) is an isomorphism.

**Theorem 2.12.** [1, 1.8] Let \( M \) and \( M' \) be \( \mathcal{R} \)-modules. Then,
\[
\mathcal{M} \otimes_\mathcal{R} \mathcal{M}' = \text{Hom}_\mathcal{R}(\mathcal{M}^*, \mathcal{M}'), \ m \otimes m' \mapsto \tilde{m} \otimes m',
\]
where \( \tilde{m} \otimes m'(w) := w(m) \cdot m' \), for any \( w \in \mathcal{M}^* \).

If we make \( \mathcal{M}' = \mathcal{R} \) in the previous theorem, we obtain the following theorem.

**Theorem 2.13.** [3, II, §1.2.5] [1, 1.10] Let \( M \) be an \( \mathcal{R} \)-module. Then
\[
M = M^{**}.
\]

**Definition 2.14.** Let \( M \) be an \( \mathcal{R} \)-module. We shall say that \( M^* \) is a dual functor. We shall say that an \( \mathcal{R} \)-module \( M \) is reflexive if \( M = M^{**} \).

**Example 2.15.** Quasi-coherent modules are reflexive.

2.2. \( \mathcal{R} \)-module schemes.

**Definition 2.16.** Let \( M \) be an \( \mathcal{R} \)-module. \( M^* \) will be called the \( \mathcal{R} \)-module scheme associated with \( M \).

**Definition 2.17.** Let \( N \) be an \( \mathcal{R} \)-module. We shall denote by \( N_{sch} \) the \( \mathcal{R} \)-module scheme defined by
\[
N_{sch} := ((N^*)_{qc})^*.
\]

**Proposition 2.18.** Let \( N \) be an \( \mathcal{R} \)-module. Then,
1. \( N_{sch}(S) = \text{Hom}_\mathcal{R}((N^*)_{qc}(R), S) \).
2. \( \text{Hom}_\mathcal{R}(N_{sch}, \mathcal{M}) = \text{Hom}_\mathcal{R}(N^*(R) \otimes_\mathcal{R} M, \mathcal{M}) \), for any quasi-coherent module \( \mathcal{M} \).

**Proof.** 1. \( N_{sch}(S) = \text{Hom}_\mathcal{R}((N^*)_{qc}(R), S) = \text{Hom}_\mathcal{R}(N^*(R), S) \).
2. \( \text{Hom}_\mathcal{R}(N_{sch}, \mathcal{M}) \cong ((N^*)_{qc}(R) \otimes_\mathcal{R} M = N^*(R) \otimes_\mathcal{R} M \).

The natural morphism \( (N^*)_{qc} \rightarrow N^* \) corresponds by Proposition 2.5 with a morphism
\[
N \rightarrow N_{sch}.
\]
Specifically, one has the natural morphism
\[
N(S) \rightarrow \text{Hom}_\mathcal{R}(N^*(R), S) = N_{sch}(S) \quad n \mapsto \tilde{n}, \text{ where } \tilde{n}(w) := w_S(n)
\]

**Proposition 2.19.** Let \( N \) be an \( \mathcal{R} \)-module and \( M \) an \( \mathcal{R} \)-module. Then, the natural morphism
\[
\text{Hom}_\mathcal{R}(N, \mathcal{M}^*) \rightarrow \text{Hom}_\mathcal{R}(N_{sch}, \mathcal{M}^*),
\]

is an isomorphism.

**Proof.** \( \text{Hom}_\mathcal{R}(N, \mathcal{M}^*) \cong \text{Hom}_\mathcal{R}(\mathcal{M}^*, N^*), \text{Hom}_\mathcal{R}(\mathcal{M}^*(N^*)_{qc}) = \text{Hom}_\mathcal{R}(N_{sch}, \mathcal{M}^*) \).
3. Dually separated $\mathcal{R}$-modules

**Definition 3.1.** We shall say that an $\mathcal{R}$-module $\mathcal{M}$ is dually separated if the natural morphism $\mathcal{M}^* \to \mathcal{M}_{qc}^*$ is a monomorphism.

**Example 3.2.** Quasi-coherent modules, $\mathcal{M}$, are dually separated, because $\mathcal{M}^* = \mathcal{M}_{qc}^*$.

**Example 3.3.** If $\mathcal{M} = \bigoplus_i R$ is a free $\mathcal{R}$-module, then $\mathcal{M}^*$ is dually separated: The obvious monomorphism $\mathcal{M} = \bigoplus_i \mathcal{R} \to \prod_i \mathcal{R}$, factors through $\mathcal{M} \to \mathcal{M}_{sch}$, by Proposition 2.19. Hence, the morphism $\mathcal{M} \to \mathcal{M}_{sch}$ is a monomorphism. That is, $\mathcal{M}^{**} = \mathcal{M} \to \mathcal{M}^*_{qc}^*$ is a monomorphism and $\mathcal{M}^*$ is dually separated.

**Proposition 3.4.** The direct limit of a direct system of dually separated $\mathcal{R}$-modules is dually separated. Every quotient of a dually separated $\mathcal{R}$-module is dually separated.

**Proof.** Let $\mathcal{M} = \lim_i \mathcal{M}_i$ be a direct limit of dually separated $\mathcal{R}$-modules. Then, the morphism

$$\mathcal{M}^* = \lim_i \mathcal{M}_i^* \hookrightarrow \lim_i \mathcal{M}_i,qc^* = (\lim_i \mathcal{M}_i,qc)^* = \mathcal{M}_{qc}^*$$

is a monomorphism. Then, $\mathcal{M}$ is dually separated.

Let $\mathcal{M}$ be dually separated and $\mathcal{M} \to \mathcal{N}$ an epimorphism. The morphism $\mathcal{N}^* \to \mathcal{N}_{qc}^*$ is a monomorphism because the diagram

$$\begin{array}{c}
\mathcal{N}^* \\
\downarrow \\
\mathcal{M}^* \\
\downarrow \\
\mathcal{M}_{qc}^*
\end{array}$$

is commutative. Then, $\mathcal{N}$ is dually separated. \[\square\]

**Proposition 3.5.** If $\mathcal{M}$ is a dually separated $\mathcal{R}$-module and $S$ is a commutative $\mathcal{R}$-algebra, then the $S$-module $\mathcal{M}|_S$ is dually separated.

**Proof.** Let $S$ be a commutative $\mathcal{R}$-algebra and let $T$ be a commutative $T$-algebra. The diagram

$$\begin{array}{c}
\mathcal{M}|_S^*(T) = \text{Hom}_T(\mathcal{M}|_T, T) \\
\downarrow \\
\mathcal{M}|_{S,qc}^*(T) = \text{Hom}_S(\mathcal{M}(S), T) \\
\downarrow \\
\text{Hom}_R(\mathcal{M}(R), T) = \mathcal{M}_{qc}^*(T)
\end{array}$$

is commutative, then the morphism $\mathcal{M}|_S^* \to \mathcal{M}|_{S,qc}^*$ is a monomorphism. \[\square\]

**Theorem 3.6.** An $\mathcal{R}$-module $\mathcal{M}$ is dually separated iff the map

$$\text{Hom}_R(\mathcal{M}, \mathcal{N}) \to \text{Hom}_R(\mathcal{M}(R), \mathcal{N}), \quad f \mapsto f_R$$

is injective, for any $\mathcal{R}$-module $\mathcal{N}$.

**Proof.** If the natural morphism $\mathcal{M}^* \to \mathcal{M}_{qc}^*$ is a monomorphism, then

$$\text{Hom}_R(\mathcal{M}, S) \hookrightarrow \text{Hom}_R(\mathcal{M}(R), S),$$

is an injection.
is injective for any commutative $R$-algebra $S$. Given an $R$-module $N$, consider the $R$-algebra $S := R \oplus N$, with the multiplication operation $(r, n) \cdot (r', n') := (rr', rn' + r'n)$. The composite morphism

$$\text{Hom}_R(M, \mathcal{R} \oplus N) = \text{Hom}_R(M, S) \hookrightarrow \text{Hom}_R(M(R), S) = \text{Hom}_R(M(R), R \oplus N)$$

is injective. Hence, $\text{Hom}_R(M, N) \rightarrow \text{Hom}_R(M(R), N)$ is injective.

Reciprocally, $M^*(S) = \text{Hom}_R(M, S) \hookrightarrow \text{Hom}_R(M(R), S) = M_{qc^*}(S)$ is injective for any commutative $R$-algebra $S$, hence the morphism $M^* \rightarrow M_{qc^*}$ is a monomorphism. □

**Theorem 3.7.** Let $M$ be an $\mathcal{R}$-module. $M$ is dually separated iff the morphism

$$\text{Hom}_R(M, M') \rightarrow \text{Hom}_R(M(R), M'(R)), \quad f \mapsto f_R$$

is injective, for every dual $\mathcal{R}$-module $M' = N^*$.

**Proof.** ⇒ From the commutative diagram

\[
\begin{array}{ccc}
\text{Hom}_R(M, M') & \xrightarrow{\text{qc}} & \text{Hom}_R(N, M^*) \\
\downarrow & & \downarrow \\
\text{Hom}_R(M(R), M'(R)) & \xrightarrow{\text{qc}} & \text{Hom}_R(N, M_{qc^*})
\end{array}
\]

one deduces that the morphism $\text{Hom}_R(M, M') \rightarrow \text{Hom}_R(M(R), M'(R))$ is injective.

⇐ It is an immediate consequence of Theorem 3.6. □

**Proposition 3.8.** Let $\mathbb{A}$ be an $\mathcal{R}$-algebra and dually separated, let $\mathcal{M}$ and $\mathcal{N}$ be $\mathbb{A}$-modules and let $M'$ be a direct summand of $M$. Then,

1. $\mathcal{M}'$ is a quasi-coherent $\mathbb{A}$-submodule of $\mathcal{M}$ iff $M'$ is an $\mathbb{A}(R)$-submodule of $M$.
2. A morphism $f: \mathcal{M} \rightarrow \mathcal{N}$ of $\mathcal{R}$-modules is a morphism of $\mathbb{A}$-modules iff $f_R: M \rightarrow N$ is a morphism of $\mathbb{A}(R)$-modules.

**Proof.** (1) Obviously, if $\mathcal{M}'$ is an $\mathbb{A}$-submodule of $\mathcal{M}$ then $M'$ is an $\mathbb{A}(R)$-submodule of $M$. Inversely, assume $M = M' \oplus M''$ and assume $M'$ is an $\mathbb{A}(R)$-submodule of $M$. Let us consider the morphism $h: \mathbb{A} \rightarrow \text{Hom}_R(\mathcal{M}', \mathcal{M}), h(a) := a$. Write

$$\text{Hom}_R(M', \mathcal{M}) = \text{Hom}_R(M', \mathcal{M}') \times \text{Hom}_R(M', \mathcal{M}'')$$

and write $h = (h_1, h_2)$. As $h_R = (h_{1R}, 0)$, then $h_2 = 0$ and $\mathcal{M}'$ is an $\mathbb{A}$-submodule of $\mathcal{M}$.

(2) The morphism $f$ is a morphism of $\mathbb{A}$-modules iff $F: \mathbb{A} \otimes \mathcal{M} \rightarrow \mathcal{N}$, $F(a \otimes m) := f(am) - af(m)$ is the zero morphism. Likewise, $f_R$ is a morphism of $\mathbb{A}(R)$-modules iff $F_R: \mathbb{A}(R) \otimes M \rightarrow N$, $F_R(a \otimes m) = f_R(am) - af_R(m)$ is the zero morphism. Now, it easy to conclude the proof because the composite morphism

$$\text{Hom}_R(\mathbb{A} \otimes \mathcal{M}, \mathcal{N}) = \text{Hom}_R(\mathbb{A}, \text{Hom}_R(\mathcal{M}, \mathcal{N})) \rightarrow \text{Hom}_R(\mathbb{A}(R), \text{Hom}_R(\mathcal{M}, \mathcal{N}))$$

$$= \text{Hom}_R(\mathbb{A}(R), \text{Hom}_R(M, N)) = \text{Hom}_R(\mathbb{A}(R) \otimes M, N)$$

is injective. □
Example 3.9. Let $G = \text{Spec} A$ be an affine group $R$-scheme. The category of comodules over $A$ is equivalent to the category of quasi-coherent $G$-modules ($G$ is the functor defined by $G (S) = \text{Hom}_{R\text{-alg}} (A, S)$). The category of quasi-coherent $G$-modules is equal to the category of quasi-coherent $A^\ast$-modules (see [1] 5.5). Let $M$ and $N$ be $A$-comodules and $f : M \to N$ a morphism of $R$-modules. Then, $f$ is a morphism of $A$-comodules iff $f$ is a morphism of $A^\ast$-modules. A direct summand $M' \subseteq M$ is a $A$-subcomodule iff $M'$ is an $A^\ast$-submodule of $M$.

Proposition 3.10. If $M$ and $M'$ are dually separated, $M \otimes_R M'$ is dually separated.

Proof. Let $N$ be a dual $R$-module. Then, the composite morphism

$$
\text{Hom}_R (M \otimes_R M', N) = \text{Hom}_R (M, \text{Hom}_R (M', N))
$$

is injective. Hence, $M \otimes_R M'$ is dually separated, by Theorem 3.5. □

Lemma 3.11. An $R$-module $M$ is dually separated iff the cokernel of every $R$-module morphism from $M$ to a quasi-coherent module is quasi-coherent, that is, the cokernel of any morphism $f : M \to N$ is the quasi-coherent module associated with $\text{Coker} f_R$.

Proof. $\Rightarrow$ Let $f : M \to N$ be a morphism of $R$-modules. Let $N' := \text{Coker} f_R$. Coker $f$ is a quotient $R$-module of $N'$. Let $\pi : N \to N'$ be the natural epimorphism. As $(\pi \circ f)_R = 0$, $\pi \circ f = 0$ by Theorem 3.6. Then, $\text{Coker} f = N'$.

$\Leftarrow$ Let $f : M \to N$ be a morphism of $R$-modules. If $f_R = 0$ then $\text{Coker} f = N$ and $f = 0$. Therefore, $M$ is dually separated, by Theorem 3.6. □

Theorem 3.12. Let $M$ be an $R$-module. $M$ is dually separated iff the natural morphism

$$
M^\ast (S) \to \text{Hom}_S (M (S), S),
$$

is injective, for any commutative $R$-algebra $S$.

Proof. $\Rightarrow$ $M^\ast (R) \to \text{Hom}_R (M (R), R)$ is injective because $M$ is dually separated. $M^\ast |_S$ is dually separated, by Proposition 3.5. Then, the morphism

$$
M^\ast (S) = M^\ast |_S (S) \to \text{Hom}_S (M^\ast |_S (S), S) = \text{Hom}_S (M (S), S),
$$

is injective.

$\Leftarrow$ Let $N$ be an $R$-module. Consider the commutative $R$-algebra $S = R \oplus N$ ($(r, n) \cdot (r', n') := (rr', rn' + r'n)$), the morphism $\pi_1 : S \to R$, $\pi_1 (r, n) = r$, the obvious morphism $\pi_{1, *} : \text{Hom} (S) \to \text{Hom} (R)$ and the induced morphism

$$
\pi^\ast_{1, N} : \text{Hom}_R (M (R), N) \to \text{Hom}_S (M (S), N),\quad \pi^\ast_{1, N} (v) = v \circ \pi_{1, *}.
$$

Let $\pi : N |_S \to N$ be defined by $\pi_T (n \otimes_R t) := n \otimes_S t$, for any commutative $S$-algebra $T$ and $n \otimes_R t \in N \otimes_R T$. The diagram

\[\begin{array}{ccc}
\text{Hom}_R (M, N) & \to & \text{Hom}_R (M (R), N) \\
\text{Hom}_S (M^\ast |_S, N) & \to & \text{Hom}_S (M (S), N) \\
\pi^\ast_1 & & \pi^\ast_1 \\
\pi_1 & & \pi_1 \\
\text{Hom}_S (M^\ast |_S, N) & \to & \text{Hom}_S (M (S), N) \\
\pi \circ w |_S & & \pi \circ w |_S \\
\end{array}\]


is commutative, because the diagram

\[
\begin{array}{ccc}
M(R) & \xrightarrow{w_R} & N \\
\downarrow & & \downarrow \\
M(S) & \xrightarrow{w_S} & N \otimes_R S \\
\downarrow & \leftarrow & \downarrow \\
\pi_1, S & \xleftarrow{\pi_1^*, w_R} & N \otimes_S S = N
\end{array}
\]

is commutative, therefore \( \pi_1^*, N \circ w_R = w_R \circ \pi_1, S \). The diagram

\[
\begin{array}{ccc}
\text{Hom}_R(M, N) & \xrightarrow{\pi_1, N} & \text{Hom}_R(M(R), N) \\
\downarrow & & \downarrow \\
\text{Hom}_S(M, S \otimes_R S) & \xrightarrow{\pi_1^*, w_R} & \text{Hom}_S(M(S), S)
\end{array}
\]

is commutative, then the morphism \( \text{Hom}_R(M, N) \to \text{Hom}_R(M(R), N) \) is injective. By Theorem 6.1, \( M \) is dually separated.

\[\square\]

**Theorem 3.13.** Let \( R = K \) be a field. A \( K \)-module, \( M \), is dually separated iff for every quasi-coherent \( K \)-module \( N \), the image of every morphism \( f: M \to N \) is a quasi-coherent \( K \)-module.

**Proof.** The kernel of every morphism between quasi-coherent \( K \)-modules is quasi-coherent. Then, the cokernel of a morphism \( f: M \to N \) is quasi-coherent iff \( \text{Im} f \) is quasi-coherent. This theorem is a consequence of Lemma 3.11.

\[\square\]

**Lemma 3.14.** [10, 1.28] It holds that

\[ \text{Hom}_R(N^*, \lim_{i \in I} M_i^*) = \lim_{i \in I} \text{Hom}_R(N^*, M_i^*) \].

**Theorem 3.15.** Let \( R = K \) be a field. Let \( M \) be a \( K \)-module and let \( \{N_i\}_{i \in I} \) be the family of all the quasi-coherent quotient modules of \( M \). Then, \( M \) is dually separated iff \( I \) is a downward directed set (in the obvious way) and \( M^* = \lim_{i \in I} N_i^* \).

**Proof.** \( \Rightarrow \) \( I \) is a set because it is a subset of the set of quotient \( K \)-modules of \( M(K) \), by 3.6. Given two quotient \( K \)-modules \( M \to N_1, N_2 \), the image, \( N_3 \), of the obvious morphism \( M \to N_1 \times N_2 \) is a quotient \( K \)-module of \( M \) and \( N_3 \leq N_1, N_2 \). Therefore, \( I \) is a downward directed set. Let \( S \) be a commutative \( K \)-algebra, the morphism

\[ \lim_{i \in I} N_i^*(S) \to \text{Hom}_K(M, S) \]

is obviously injective, and it is surjective by Theorem 3.13. Hence, \( M^* = \lim_{i \in I} N_i^* \).
\[ \forall \text{ Observe that} \]
\[ \text{Hom}_{K}(M, N) \cong \text{Hom}_{K}(N^*, M^*) = \text{Hom}_{K}(N^*, \lim_{\to i} N_i^*) \cong \lim_{\to i} \text{Hom}_{K}(N_i^*, N) \]

Then, every morphism \( M \to N \) factors through some \( N_i \) and then its cokernel is a quasi-coherent module. By Lemma 3.11 \( M \) is dually separated.

**Corollary 3.16.** Let \( R = K \) be a field. If \( M \) is dually separated, then \( M^* \) is dually separated.

**Proof.** It is a consequence of Theorem 3.15 Example 3.3 and Proposition 3.4. \( \Box \)

#### 4. Reflexive \( R \)-modules

**Proposition 4.1.** Let \( M \) be a reflexive \( R \)-module. \( M \) is dually separated iff there exist a subset \( I \) and a monomorphism \( M^* \to \prod^{I} R \).

**Proof.** Let \( M \) be dually separated. Consider an morphism \( \oplus^{I} R \to M(R) \). The composite morphism \( M^* \to M_{qc} \to \prod^{I} R \) is a monomorphism.

Now, let \( M^* \to \prod^{I} R \) be a monomorphism. The dual morphism \( \oplus^{I} R \to M \), factors as follows: \( \oplus^{I} R \to M_{qc} \to M \). Dually, we have \( M^* \to M_{qc} \to \prod^{I} R \). Therefore, the morphism \( M^* \to M_{qc} \) is a monomorphism and \( M \) is dually separated.

**Definition 4.2.** An \( R \)-module \( M \) is said to be (linearly) separated if for each commutative \( R \)-algebra \( S \) and \( m \in M(S) \) there exist a commutative \( S \)-algebra \( T \) and a \( w: T \to S \) such that \( w(m) \neq 0 \) (that is, the natural morphism \( M \to M^{**} \), \( m \mapsto \hat{m} \), where \( \hat{m}(w) := w(m) \) for any \( w \in M^{*} \), is a monomorphism).

Every \( R \)-submodule of a separated \( R \)-module is separated.

**Example 4.3.** If \( M \) is a dual \( R \)-module, then it is separated: Given \( 0 \neq w \in M = N^* \), there exists an \( n \in \mathbb{N}^* \) such that \( w(n) \neq 0 \). Let \( \hat{n} \in M^* \) be defined by \( \hat{n}(w') := w'(n) \), for any \( w' \in M \). Then, \( \hat{n}(w) \neq 0 \).

**Proposition 4.4.** Let \( R = K \) be a field and let \( M \) be a \( K \)-module such that \( M^* \) is well defined. \( M \) is separated iff the natural morphism \( M \to M_{sch} \) is a monomorphism. Therefore, \( M \) is separated iff it is a \( K \)-submodule of a \( K \)-module scheme.

**Proof.** Assume \( M \) is separated. Let \( m \in M(S) \) be such that \( m = 0 \in M_{sch}(S) \). \( M_{sch}(S) \overset{\text{1.15}}{\to} \text{Hom}_{K}(M_{sch}^*(K), S) \), then \( m(w) := w(m) = 0 \) for any \( w \in M^*(K) \).

Let \( T \) be a commutative \( S \)-algebra, and let \( \{ e_i \}_{i \in I} \) be a \( K \)-basis of \( T \). Consider the composite morphism

\[ M^*(T) \overset{\text{2.10}}{\to} \text{Hom}_{K}(M, T) = \text{Hom}_{K}(M, \oplus_{I} K) \subset \prod_{I} \text{Hom}_{K}(M, K), \]

which assigns to every \( w_T \in M^*(T) \) a \( (w_i) \in \prod_{I} M^*(K) \). Specifically, \( w_T(m') = \sum_{i} w_i(m') \cdot e_i \), for any \( m' \in M(T) \). Therefore, \( w_T(m) = 0 \) for any \( w_T \in M^*(T) \).
As $M$ is separated, this means that $m = 0$, i.e., the morphism $M \to M_{\text{sch}}$ is a monomorphism.

Now, assume $M \to M_{\text{sch}}$ is a monomorphism. Observe that $M_{\text{sch}}$ is separated because it is reflexive. Then, $M$ is separated.

Finally, the second statement of the proposition is obvious. □

**Theorem 4.5.** Let $R = K$ be a field. $M$ is a reflexive $K$-module iff $M$ is equal to the inverse limit of its quasi-coherent quotient $\mathcal{R}$-modules.

**Proof.** Suppose that $M$ is reflexive. $M^*$ is separated, because it is a dual $R$-module. By Proposition 4.4, the morphism $M^* \to M^*_{\text{sch}} = M_{\text{qc}}^*$ is a monomorphism. Then, $M$ is dually separated. Let $\{M_i\}_{i \in I}$ be the set of all quasi-coherent quotient modules of $M$. Then, $M^* = \lim_{i \in I} M_i^*$, by Theorem 3.15. Therefore,

$$M = M^{**} = \lim_{i \in I} M_i.$$

Suppose now that $M$ is equal to the inverse limit of its quasi-coherent quotient $K$-modules, $M = \lim_{i \in I} N_i$. Then, $M = (\lim_{i \in I} N_i^*)^*$ is dually separated, by Theorem 3.15 and Proposition 3.4. By Theorem 3.15, $M^* = \lim_{i \in I} N_i^*$ and $M = \lim_{i \in I} N_i = M^{**}$. □

Let $R = \mathbb{Z}$ and $M = \mathbb{Z}/2\mathbb{Z}$. Then, $M := M^*$ is reflexive but it is not dually separated, because $M_{\text{qc}}^* = 0$, because $M(R) = 0$.

### 5. Proquasi-coherent modules

**Definition 5.1.** An $R$-module is said to be a proquasi-coherent module if it is an inverse limit of quasi-coherent $R$-modules.

In this section, $K$ will be a field.

**Example 5.2.** Reflexive $K$-modules are proquasi-coherent, by Theorem 4.5.

**Proposition 5.3.** If $M$ is a proquasi-coherent $K$-module, then it is a dual $K$-module and is a direct limit of $K$-module schemes. In particular, proquasi-coherent $K$-modules are dually separated.

**Proof.** $M = \lim_{i \in I} M_i = (\lim_{i \in I} M_i^*)^*$. $\lim_{i \in I} M_i^*$ is dually separated by Example 3.3 and Proposition 3.4. Then, its dual, which is $M$, is a direct limit of $K$-module schemes, by Theorem 3.15 and it is dually separated by Corollary 3.16. □

**Proposition 5.4.** Let $P$ be a proquasi-coherent $K$-module and $M$ a separated $K$-module. Let $f : P \to M$ be a morphism of $K$-modules. Then, $\text{Ker} f$ is proquasi-coherent.

**Proof.** By Theorem 4.4, there exist a $K$-vector space $V$ and a monomorphism $M \to V^*$. We can assume $M = V^* = \prod_I K$. Given $I' \subset I$, let $f_{I'}$ be the composition of $f$ with the obvious projection $\prod_I K \to \prod_{I'} K$. Then,

$$\text{Ker} f = \lim_{I' \subset I, \# I' < \infty} \text{Ker} f_{I'}.$$
It is sufficient to prove that $\ker f_{i'}$ is proquasi-coherent, since the inverse limit of proquasi-coherent modules is proquasi-coherent. Let us write $I' = I'' \coprod \{i\}$. $\ker f_{i'}$ is the kernel of the composite morphism $\ker f_i \hookrightarrow \prod_{I'} K$. By induction on $\# I'$, it is sufficient to prove that $\ker f_i$ is proquasi-coherent. Let us write $I' = I'' \coprod \{i\}$. $\ker f_i$ is the kernel of the composite morphism $\ker f_i \hookrightarrow \prod_{I''} K$. By induction on $\# I''$, it is sufficient to prove that $\ker f_i$ is proquasi-coherent. Let us write $f = f_{i'}$. If $f : P \to K$ is the zero morphism the proposition is obvious. Assume $f \neq 0$. Then, $f$ is an epimorphism (because $P$ is dually separated). Let us write $P = \varprojlim_{i} V_i$ and let $v = (v_i) \in \varprojlim_{i} V_i = P(K)$ be a vector such that $f_K((v_i)) \neq 0$. Then, $P = \ker f \oplus K \cdot v$. Let $\bar{V}_i := V_i/(v_i)$. Let us prove that $\ker f \simeq \varinjlim_{i} \bar{V}_i$: Let $i'$ be such that $v_{i'} \neq 0$. Consider the exact sequences

$$0 \to K \cdot v_i \to V_i \to \bar{V}_i \to 0, \quad (i > i')$$

Dually, we have the exact sequences

$$0 \to \bar{V}^*_i \to V^*_i \to K \to 0$$

Taking the direct limit we have the exact sequence

$$0 \to \varinjlim_{i} (\bar{V}^*_i) \to \varinjlim_{i} (V^*_i) \to K \to 0$$

Dually, we have the exact sequence

$$0 \to K \cdot v \to P \to \varprojlim_{i} \bar{V}_i \to 0$$

Then, $\ker f \to \varprojlim_{i} \bar{V}_i, (v_i)_i \mapsto (\bar{v}_i)_i$ is an isomorphism.

**Proposition 5.5.** Every direct summand of a proquasi-coherent module is proquasi-coherent.

**Theorem 5.6.** Let $M$ be a $K$-module. $M$ is proquasi-coherent iff $M$ is a dual $K$-module and it is dually separated.

**Proof.** By Proposition 5.5, we only have to prove the sufficiency. Let us write $M = N^*$. The dual morphism of the natural morphism $N \to N^{**}$ is a retraction of the natural morphism $M \to M^{**}$. Then, $M^{**} = M \oplus M'$. By Proposition 5.5, $M$ is proquasi-coherent, because $M^{**}$ is proquasi-coherent by Theorem 3.15.

**Theorem 5.7.** A $K$-module is proquasi-coherent iff it is the dual $K$-module of a dually separated $K$-module.

**Proof.** If $M = \varprojlim_{i} M_i$ is proquasi-coherent, then $M = (\varprojlim_{i} M_i^*)^*$. $\varprojlim_{i} M_i^*$ is dually separated and $M = (\varprojlim_{i} M_i^*)^*$.

If $M'$ is dually separated, then $M'^*$ is dually separated, by Corollary 3.16. By Theorem 5.6, $M'^*$ is proquasi-coherent.

**Proposition 5.8.** If $P, P'$ are proquasi-coherent $K$-modules, then $\hom_K(P, P')$ is proquasi-coherent. In particular, $P^*$ is proquasi-coherent.
Proof. Let us write $\mathbb{P} = \varprojlim_i \mathbb{V}_i^*$ and $\mathbb{P}' = \varprojlim_j \mathbb{V}'_j^*$. Then,

$$\mathbb{H}om_K(\mathbb{P}, \mathbb{P}') = \mathbb{H}om_K(\varprojlim_i \mathbb{V}_i^*, \varprojlim_j \mathbb{V}'_j^*) = \varprojlim_i \mathbb{H}om_K(\mathbb{V}_i^*, \mathbb{V}'_j^*) = \varprojlim_i (\mathbb{V}_i \otimes \mathbb{V}'_j)$$

Hence, $\mathbb{H}om(\mathbb{P}, \mathbb{P}')$ is proquasi-coherent. \hfill $\square$

**Proposition 5.9.** Let $\mathbb{K}$ be an $\mathbb{R}$-algebra and dually separated, and let $\mathbb{P}, \mathbb{P}'$ be proquasi-coherent $\mathbb{K}$-modules and $\mathbb{R}$-modules. Then, a morphism of $\mathbb{K}$-modules, $f: \mathbb{P} \to \mathbb{P}'$, is a morphism of $\mathbb{R}$-modules iff $f_K: \mathbb{P}(\mathbb{K}) \to \mathbb{P}'(\mathbb{K})$ is a morphism of $\mathbb{K}(\mathbb{K})$-modules.

**Proof.** Proceed as in the proof of Proposition 3.8 (2).

**Lemma 5.10.** Let $\mathbb{M}$ be an $\mathbb{R}$-module. Then,

$$\mathbb{H}om_\mathbb{R}(\prod_i \mathbb{R}, \mathbb{M}) = \bigoplus_i \mathbb{H}om_\mathbb{R}(\mathbb{R}, \mathbb{M}) = \bigoplus_i \mathbb{M}$$

**Proof.** $\mathbb{H}om_\mathbb{R}(\prod_i \mathbb{R}, \mathbb{M}) = \mathbb{H}om_\mathbb{R}(\bigoplus_i \mathbb{R}, \mathbb{M}) = \bigoplus_i \mathbb{M}$. \hfill $\square$

**Lemma 5.11.** Let $\{\mathbb{M}_i\}_{i \in I}$ be a set of dual $\mathbb{R}$-modules and let $\mathbb{N}$ be an $\mathbb{R}$-module. Then,

$$\mathbb{H}om_\mathbb{R}(\prod_{i \in I} \mathbb{M}_i, \mathbb{N}) = \bigoplus_{i \in I} \mathbb{H}om_\mathbb{R}(\mathbb{M}_i, \mathbb{N})$$

In particular, $(\prod_{i \in I} \mathbb{M}_i)^* = \bigoplus_{i \in I} \mathbb{M}_i^*$ and if $\mathbb{M}_i$ is reflexive, for every $i$, then $\prod_{i \in I} \mathbb{M}_i$ is reflexive.

**Proof.** Let $f \in \mathbb{H}om_\mathbb{R}(\prod_{i \in I} \mathbb{M}_i, \mathbb{N})$ and $f_i := f|_{\mathbb{M}_i}$. If $f|_{\bigoplus_{i \in I} \mathbb{M}_i} = 0$, then $f = 0$. Given $m = (m_i)_{i \in I} \in \prod_{i \in I} \mathbb{M}_i(S)$, let $g: \prod_{i \in I} \mathbb{S} \to N|S, g(t_i) := f_T((t_i, m_i))$, for every commutative $T$-algebra $T$. Since $g|_{\bigoplus_{i \in I} \mathbb{M}_i} = 0$, then $g = 0$, by Proposition 5.10. Therefore, $f = 0$.

Consider the obvious inclusion morphism

$$\bigoplus_{i \in I} \mathbb{H}om_\mathbb{R}(\mathbb{M}_i, \mathbb{N}) \subseteq \mathbb{H}om_\mathbb{R}(\prod_{i \in I} \mathbb{M}_i, \mathbb{N})$$

Let $J := \{i \in I: f_i := f|_{\mathbb{M}_i} \neq 0\}$. For each $j \in J$, let $R_j$ be a commutative $\mathbb{R}$-algebra and $m_j \in \mathbb{M}_j(R_j)$ such that $0 \neq f_j(m_j) \in N \otimes_\mathbb{R} R_j$. Let $\mathbb{S} := \prod_{j \in J} R_j$. The obvious morphism of $\mathbb{R}$-algebras $\mathbb{S} \to \mathbb{R}_i$ is surjective, and this morphism of $\mathbb{R}$-modules has a section. Write $\mathbb{M}_i = N_i^*$. The natural morphism $\pi_i: \mathbb{M}_i(S) = \mathbb{H}om_\mathbb{R}(N_i, \mathbb{S}) \to \mathbb{H}om_\mathbb{R}(N_i, \mathbb{R}_i) = \mathbb{M}_i(\mathbb{R}_i)$ has a section of $\mathbb{R}$-modules. Let $m'_i \in \mathbb{M}_i(S)$ be such that $\pi_i(m'_i) = m_i$. The morphism of $\mathbb{S}$-modules $g: \prod_j \mathbb{S} \to \mathbb{N}|S, g((s_j)) := f((s_j, m'_j))$ satisfies that $g|_\mathbb{S} \neq 0$, for every factor $S \subseteq \prod_j \mathbb{S}$. Then, by Proposition 5.10, $\#J < \infty$.

Finally, define $h := \sum_{j \in J} f_j \in \bigoplus_{i \in I} \mathbb{H}om_\mathbb{R}(\mathbb{M}_i, \mathbb{N})$, then $f = h$. \hfill $\square$

**Proposition 5.12.** Let $I$ be a totally ordered set and $\{f_{ij}: M_i \to M_j\}_{i \geq j \in I}$ an inverse system of $K$-vector spaces. Then, $\varprojlim_i M_i$ is reflexive.
Proof. \( \lim \mathcal{M}_i \) is a direct limit of submodule schemes \( V_j^* \), by [5.13] and [5.7] If all the vector spaces \( V_j \) are finite dimensional then \( \lim \mathcal{M}_i \) is quasi-coherent, then it is reflexive. In other case, there exists an injective morphism \( f: \prod_i \mathcal{K} \to \lim \mathcal{M}_i \).

Let \( \pi_j: \lim \mathcal{M}_i \to \mathcal{M}_j \) be the natural morphisms. Let \( g_r: \mathcal{K} \to \prod_i \mathcal{K} \) be defined by \( g_r(\lambda_1, \cdots, \lambda_r) := (\lambda_1, \cdots, \lambda_r, 0, \cdots, 0, \cdots) \). Let \( i_1 \in I \) be such that \( \pi_{i_1} \circ f \circ g_1 \) is a monomorphism. Recursively, let \( i_n > i_{n-1} \) be such that \( \pi_{i_n} \circ f \circ g_n \) is a monomorphism. If there exists a \( j > i_n \) for any \( n \), the composite morphism \( \oplus_n \mathcal{K} \subset \prod_i \mathcal{K} \to \mathcal{M}_j \) is a monomorphism, and by Proposition [5.10] the morphism \( \prod_i \mathcal{K} \to \mathcal{M}_j \) factors through the projection onto some \( \mathcal{K}^r \), which is contradictory. Therefore, \( \lim \mathcal{M}_i = \lim \mathcal{M}_{i_n} \).

Let \( \mathcal{M}'_{i_n} \) be the image of \( \lim \mathcal{M}_{i_n} \) in \( \mathcal{M}_{i_n} \). Then, \( \lim \mathcal{M}'_{i_n} = \lim \mathcal{M}_{i_n} \). Let \( H_n := \text{Ker}[\mathcal{M}'_{i_n} \to \mathcal{M}'_{i_{n-1}}] \). Then, \( \lim \mathcal{M}_{i_n} \simeq \prod_n H_n \). By Lemma [5.11] \( \lim \mathcal{M}_{i_n} \) is reflexive. \( \square \)

6. Flat SML \( R \)-modules and dually separated \( R \)-modules

**Theorem 6.1.** \( \mathcal{M}^* \) is dually separated iff the morphism

\[ M \otimes_R N \to \text{Hom}_R(M^*, N) \]

is injective, for any \( R \)-module \( N \).

**Proof.** The morphism \( M \otimes_R N \to \text{Hom}_R(M^*, N) \) is injective, for any \( R \)-module \( N \) iff \( \mathcal{M}^* \) is dually separated, by Theorem 3.6. \( \square \)

**Corollary 6.2.** If \( \mathcal{M}^* \) is dually separated, then \( M \) is a flat \( R \)-module and the morphism \( M \to M^{**} \) is universally injective, that is, \( M \otimes_R S \to M^{**} \otimes_R S \) is injective for every commutative \( R \)-algebra \( S \).

**Proof.** \( M \otimes - \) is a left exact functor because \( \text{Hom}_R(M^*, -) \) is a left exact functor. Hence, \( M \) is flat. Finally, the composite morphism,

\[ M \otimes_R S \to M^{**} \otimes_R S \to \text{Hom}_R(M^*, S) \]

is injective, then \( M \otimes_R S \to M^{**} \otimes_R S \) is injective. \( \square \)

Let \( R = \mathbb{Z} \) and \( M = \mathbb{Q} \), which is a flat \( \mathbb{Z} \)-module. \( \mathcal{M}^* \) is not dually separated, because \( M \to M^{**} \) is the zero morphism, because \( M^* = 0 \).

**Corollary 6.3.** Let \( \mathcal{M}^* \) be dually separated. Then, the morphism

\[ M \otimes N^* \to \text{Hom}_R(N, M) \]

is injective, for any \( R \)-module \( N \).

**Proof.** The composite morphism

\[ M \otimes N^* \to \text{Hom}_R(N, M) \to \text{Hom}_R(M^*, N^*) \]

is injective, then \( M \otimes N^* \to \text{Hom}_R(N, M) \) is injective. \( \square \)
Theorem 6.4. \( M^* \) is dually separated iff the natural morphism
\[ M \otimes_R S \to (M \otimes_R S)^{**} := \text{Hom}_S(\text{Hom}_S(M \otimes_R S, S), S) \]
is injective, for any commutative \( R \)-algebra \( S \).

Proof. It is an immediate consequence of Theorem 3.12. □

Proposition 6.5. [4, Prop. 5.3] \( M^* \) is dually separated iff there exists a monomorphism \( M \hookrightarrow \prod I R \).

Proof. It is an immediate consequence of Proposition 4.1. □

Example 6.6. Let \( P \) be a projective module, then \( P^* \) is dually separated: \( P \) is a direct summand of a free module \( \oplus I R \). Then, \( P \subseteq \oplus I R \subseteq \prod I R \) and \( P^* \) is dually separated.

Corollary 6.7. Let \( N \hookrightarrow M \) be a universally injective morphism of \( R \)-modules. If \( M^* \) is dually separated, \( N^* \) is dually separated.

Proof. \( N \hookrightarrow M \) is a universally injective morphism of \( R \)-modules iff \( N \to M \) is a monomorphism. The corollary is an immediate consequence of Proposition 6.5. □

Noetherian rings are coherent rings (see [7, I 6-7]) for definition and properties).

Theorem 6.8. Let \( R \) be a coherent ring and \( M \) an \( R \)-module. \( M^* \) is dually separated iff there exists an inclusion \( M \subseteq \prod I R \) such that the cokernel is flat.

Proof. Observe that \( \text{Hom}_R(M, \prod I R) = \text{Hom}_R(M, \prod I R) \). \( \prod I R \) is a flat \( R \)-module and for every \( R \)-module \( S \) the natural morphism \( \prod I R \otimes_R S \to \prod I S \) is injective, because \( R \) is a coherent ring. Then, a morphism \( M \to \prod I R \) is injective and the cokernel is a flat module iff \( M \to \prod I R \) is a monomorphism.

Then, this theorem is a immediate consequence of Proposition 6.5. □

Lemma 6.9. Let \( f : M^* \to N \) be a morphism of \( R \)-modules. Then, \( \text{Coker} f \) is quasi-coherent iff \( f \) factors through the quasi-coherent module associated with \( \text{Im} f_R \).

Proof. Let \( N_1 = \text{Im} f_R \) and let \( N_2 = N/N_1 \). Observe that \( \text{Coker} f \) is quasi-coherent iff \( \text{Coker} f = N_2 \), and \( \text{Coker} f = N_2 \) iff the composite morphism \( M^* \to N \to N_2 \) is zero. Consider the diagram

\[
\begin{array}{ccc}
\text{Hom}_R(M^*, N_1) & \longrightarrow & \text{Hom}_R(M^*, N) \\
\text{Hom}_R(M^*, N) & \longrightarrow & \text{Hom}_R(M^*, N_2) \\
M \otimes_R N_1 & \longrightarrow & M \otimes_R N & \longrightarrow & M \otimes_R N_2 & \longrightarrow & 0 \\
\end{array}
\]

Then, the composite morphism \( M^* \to N \to N_2 \) is zero iff \( f \) factors through \( N_1 \), which is the quasi-coherent module associated with \( \text{Im} f_R \). We are done. □

Remarks 6.10. If \( f : M^* \to N \) is an epimorphism, \( N \) is a finitely generated module: \( f = \sum_{i=1}^r m_i \otimes n_i \in \text{Hom}_R(M^*, N) = M \otimes N \), therefore \( f \) factors through the coherent module associated with \( \langle n_1, \ldots, n_r \rangle \), then \( N = \langle n_1, \ldots, n_r \rangle \).
If \( N_1 \hookrightarrow N_2 \) is an injective morphism of \( R \)-modules and \( M \) is flat, the map 
\[
\text{Hom}_R(M^*,N_1) = M \otimes R N_1 \to M \otimes R N_2 = \text{Hom}_R(M^*,N_1)
\]
is injective.

**Theorem 6.11.** \( M^* \) is dually separated iff every morphism \( f: M^* \to N \) (uniquely) factors through the coherent module associated with \( \text{Im} f_R \).

**Proof.** It is an immediate consequence of 3.11 and 6.9.

**Theorem 6.12.** \( M^* \) is dually separated iff any morphism \( f: M^* \to R \) factors through the quasi-coherent module associated with \( \text{Im} f_R \).

**Proof.** \( \Rightarrow \) It is an immediate consequence of 6.11. 
\( \Leftarrow \) We have to prove that a morphism \( f: M^* \to N \) is zero if \( f_R = 0 \), by 3.6.

Any morphism \( f: M^* \to N \) factors through the quasi-coherent module associated with a finitely generated submodule of \( N \). Then, we can suppose that \( N \) is finitely generated, that is, \( N = \langle n_1, \ldots, n_r \rangle \).

Let us proceed by induction on \( r \). If \( r = 1 \), \( N \cong R/I \), for some ideal \( I \subset R \). Let \( \pi: R \to N \) be the quotient morphism. There exists a morphism \( g: M^* \to R \) such that the diagram

\[
\begin{array}{ccc}
M^* & \xrightarrow{\pi} & R \\
\downarrow{f} & & \\
N & \xrightarrow{} & \pi
\end{array}
\]

is commutative (recall \( \text{Hom}_R(M^*,N') \cong M \otimes_R N' \)). Then, \( \text{Im} g_R \subseteq I \), because \( \text{Im}(\pi_R \circ g_R) = \text{Im}(\pi \circ g)_R = \text{Im} f_R = 0 \). Then, \( g \) factors through \( I \) and \( f = 0 \).

Assume the statement is true for \( 1, \ldots, r-1 \) and \( N = \langle n_1, \ldots, n_r \rangle \). Let \( N' = N/\langle n_1 \rangle \) and let \( \pi: N \to N' \) be the quotient morphism. Observe that \( (\pi \circ f)_R = \pi_R \circ f_R = 0 \), then \( \pi \circ f = 0 \), by the induction hypothesis. Let \( N_1 \) be the quasi-coherent module associated with \( \langle n_1 \rangle \). Consider the diagram

\[
\begin{array}{cccccc}
M \otimes_R \langle n_1 \rangle & \xrightarrow{f} & M \otimes_R N & \xrightarrow{} & M \otimes_R N' & \xrightarrow{} & 0 \\
\downarrow{\text{Hom}_R(M^*,N_1)} & & \downarrow{\text{Hom}_R(M^*,N)} & & \downarrow{\text{Hom}_R(M^*,N')} & & \\
\text{Hom}_R(M^*,N_1) & \xrightarrow{\pi_*} & \text{Hom}_R(M^*,N) & \xrightarrow{} & \text{Hom}_R(M^*,N')
\end{array}
\]

Since \( \pi_*(f) = \pi \circ f = 0 \), \( f \) factors through a morphism \( g: M^* \to N_1 \). Observe that \( g_R = 0 \), because \( f_R = 0 \), then \( g = 0 \) and \( f = 0 \).

A module \( M \) is a trace module if every \( m \in M \) holds \( m \in M^*(m) \cdot M \), where \( M^*(m) := \{ w(m) \in R: w \in M^* \} \) (see [3]).

**Proposition 6.13.** \( M \) is a trace module iff any morphism \( f: M^* \to R \) factors through the quasi-coherent module associated with \( \text{Im} f_R \).

**Proof.** \( \text{Hom}_R(M^*,R) = M \), then \( f = m \in M \) and \( \text{Im} f_R = M^*(m) \). Let \( I \subseteq R \) be an ideal, then \( f = m \) factors through \( \mathcal{I} \) iff \( m \in I \cdot M \), as it is easy to see taking
into account the following diagram

\[ \begin{array}{ccc}
\text{Hom}_R(M^*, I) & \longrightarrow & \text{Hom}_R(M^*, R) \\
I \otimes_R M & \longrightarrow & M
\end{array} \]

We are done. \[ \square \]

**Corollary 6.14.** \( M^* \) is dually separated iff \( M \) is a trace module.

**Proof.** It is an immediate consequence of 6.12 and 6.13. \[ \square \]

**Lemma 6.15.** Let \( M \) be a flat \( R \)-module and \( P \) a finitely presented \( R \)-module. Then,

\[ \text{Hom}_R(M^*, P^*_{qc}) = \text{Hom}_R(M^*, P^*). \]

**Proof.** Consider an exact sequence of morphisms \( R^n \rightarrow R^m \rightarrow P \rightarrow 0 \). Dually, \( 0 \rightarrow P^* \rightarrow R^m \rightarrow R^n \) is exact. From the commutative diagram of exact rows

\[ \begin{array}{ccc}
0 & \longrightarrow & \text{Hom}_R(M^*, P^*) \\
& & \longrightarrow \text{Hom}_R(M^*, R^m) \\
& & \longrightarrow \text{Hom}_R(M^*, R^n) \\
0 & \longrightarrow & P^* \otimes_R M \\
& & \longrightarrow R^m \otimes_R M \\
& & \longrightarrow R^n \otimes_R M \\
& & \text{Hom}_R(M^*, P^*_{qc}) \\
\end{array} \]

one has that \( \text{Hom}_R(M^*, P^*_{qc}) = \text{Hom}_R(M^*, P^*) \).

\[ \square \]

**Proposition 6.16.** \( M^* \) is dually separated iff \( M \) is a flat strict Mittag-Leffler module.

**Proof.** Let \( \{ P_i \} \) be a direct system of finitely presented modules such that \( M = \lim_i P_i \). Then, \( M^* = \lim_i P_i^* \). Observe that

\[ \text{Hom}_R(M^*, N) = \lim_i \text{Hom}_R(P_i^*, N) = \text{Hom}_R(P_i^*, N). \]

\( \Rightarrow \) \( M \) is flat, by 6.2. The natural morphism \( M^* \rightarrow P_i^* \) factors through \( M^* \rightarrow P_i^*_{qc} \), by 6.15. The morphism \( M^* \rightarrow P_i^*_{qc} \) factors through an epimorphism \( M^* \rightarrow N \), by 6.11. \( M^* \rightarrow N \) factors through the natural morphism \( M^* \rightarrow P_j^* \), for some \( j \). We have the morphisms

\[ M^* \rightarrow P_j^* \rightarrow N \rightarrow P_i^* \]

(recall \( M^* \rightarrow N \) is an epimorphism). Then, \( \text{Im}(M^*(S) \rightarrow P_j^*(S)) = \text{Im}(P_j^*(S) \rightarrow P_i^*(S)) \), for any commutative \( R \)-algebra \( S \). Taking \( S = \mathbb{R} \oplus Q \) (for any \( R \)-module \( Q \)), we obtain

\[ \text{Im}(\text{Hom}_R(M, Q) \rightarrow \text{Hom}_R(P, Q)) = \text{Im}(\text{Hom}_R(P, Q) \rightarrow \text{Hom}_R(P, Q)). \]
Hence, $M$ is a flat strict Mittag-Leffler module.

$\Leftarrow$ Let $\{P_i\}$ be a direct system of finitely presented modules so that $M = \varinjlim P_i$ and for every $i$ there exists a $j \geq i$ such that

\[ \text{Im}(M^* \to P_i^*) = \text{Im}(P_j^* \to P_i^*). \]

Let $M^* \to N$ be a morphism of $R$-modules. $M^* \to N$ factors through the natural morphism $M^* \to P_i^*$, for some $i$. There exists $j \geq i$ such that $\text{Im}(M^* \to P_i^*) = \text{Im}(P_j^* \to P_i^*)$. Then,

\[ \text{Im}(M^* \to N) = \text{Im}(P_j^* \to N) =: N_j. \]

The natural morphism $M^* \to P_j^*$ factors through a morphism $M^* \to P_j^{qc}$, by 6.13. We have the morphisms

\[ M^* \to P_j^{qc} \to P_j^* \to N \]

The composite morphism $N_j \to \text{Im}(M^* \to N) \subseteq \text{Im}(P_j^{qc} \to N)$ is an epimorphism. Hence, $\text{Im}(M^* \to N) = \text{Im}(P_j^{qc} \to N)$. Therefore, $\text{Coker}(M^* \to N) = \text{Coker}(P_j^{qc} \to N)$, which is quasi-coherent. $M^*$ is dually separated by 3.11.

It is well known that a module is a flat strict Mittag-Leffler module iff it is a trace module (see [6, II. 2.3.4] and [4, Th. 3.2]).

**Proposition 6.17.** [4, Cor. 3] Let $M$ be a finitely generated module. Then, $M^*$ is dually separated iff $M$ is a projective module.

**Proof.** $\Rightarrow$ Let $R^n \to M$ be an epimorphism. The dual morphism $M^* \to R^n$ is a monomorphism and it factors through an epimorphism $M^* \to N$. Then, $M^* \cong N$ and by [2] $M$ is a projective module.

$\Leftarrow$ See Example 6.6.

**Theorem 6.18.** Let $M^*$ be dually separated and $\{N_i\}$ the set of the coherent quotient $R$-modules of $M^*$. Then, $M = \varprojlim N_i^*$.

**Proof.** Proceed as in the proof of Theorem 3.15 to prove that $M = \varprojlim N_i^*$.

**Theorem 6.19.** $M^*$ is dually separated iff $M$ is a flat Mittag-Leffler module and the morphism

\[ M \otimes_R R/m \to \text{Hom}_R(M^*, R/m) \]

is injective, for every maximal ideal $m \subset R$.

**Proof.** $\Rightarrow$ By Theorem 6.18 and [10, 4.5], $M$ is a flat Mittag-Leffler module. Now, the direct part of this proposition is a consequence of Theorem 6.1.

$\Leftarrow$ Let $f : M^* \to N'$ be a morphism of $R$-modules. By [10, 4.5.4.1], there exists a finitely generated submodule $N' \subset N$ such that $f$ factors through a morphism $f' : M^* \to N'$ and the dual morphism $f'^* : N'^* \to M$ is a monomorphism. If we prove that $f'^* : M^* \to N'$ is an epimorphism, we are done by 6.11. Assume $f'^* \not\text{ is an epimorphism.}$ By Nakayama’s Lemma, there exists a maximal ideal $m \subset R$ such that the composite morphism $M^* \to N' \to N'/mN'$ is not an epimorphism. Then
there exists an epimorphism $N'/mN' \to R/m$ such that the composite morphism $M^* \to R/m$ is zero. Let $R/m$ be the quasi-coherent module associated with $R/m$. We have a morphism $\mathcal{M}^* \to R/m$ which is not zero (because the dual morphism is a monomorphism) and $M^* \to R/m$ is zero. This is contradictory because the composite morphism
\[
\text{Hom}_R(\mathcal{M}^*, R/m) = M \otimes_R R/m \to \text{Hom}_R(M^*, R/m)
\]
is injective, by Theorem 6.1.

□

Theorem 6.20. Let $R$ be a noetherian ring. Let $M$ be a flat $R$-module such that there exists a set of finitely generated submodules of $M$, $\{M_i\}$, so that $M = \bigcup_{i \in I} M_i$ and the morphisms $M^* \to M^*_i$ are surjective. Then, $M^*$ is dually separated.

Proof. Consider a morphism $f: \mathcal{M}^* \to \mathcal{N}$. Then, $f = \sum_i m_i \otimes n_i \in M \otimes N = \text{Hom}_R(\mathcal{M}^*, \mathcal{N})$. Let $M_j$ be such that $m_i \in M_j$, for any $i$. Then, $f$ factors through $\mathcal{M}^* \to M^*_j$. By 6.15 $\text{Hom}_R(\mathcal{M}^*, M^*_j) = \text{Hom}_R(\mathcal{M}^*, M^*_j)$. Then, $f$ (uniquely) factors through a morphism $\mathcal{M}^* \to M^*_jqc$. By the hypothesis, this morphism is an epimorphism. By Lemma 6.11, $M^*$ is dually separated.

□

Corollary 6.21. Let $R$ be a Dedekind domain. An $R$-module $M^*$ is dually separated iff $M$ is the direct limit of its finitely generated projective submodules that are direct summands.

Proof. $\Rightarrow$) Let $\pi: \mathcal{M}^* \to \mathcal{N}$ be an epimorphism. Let $L = R^n \to N$ be an epimorphism and $g: \mathcal{L} \to \mathcal{N}$ the induced morphism. There exists a morphism $f: \mathcal{M}^* \to \mathcal{L}$ such that the diagram
\[
\begin{array}{ccc}
\mathcal{M}^* & \xrightarrow{f} & \mathcal{N} \\
\downarrow & & \downarrow \ x \\
\mathcal{L} & \xrightarrow{g} & \mathcal{N}
\end{array}
\]
is commutative, because the morphism $\text{Hom}_R(\mathcal{M}^*, \mathcal{L}) = M \otimes_R L \to M \otimes_R N = \text{Hom}_R(\mathcal{M}^*, \mathcal{N})$ is surjective. Let $L' = \text{Im} f \subseteq L$. Then, $L'$ is a finitely generated projective module, the obvious morphism $\mathcal{M}^* \to \mathcal{L}'$ is an epimorphism and we have the commutative diagram
\[
\begin{array}{ccc}
\mathcal{M}^* & \xrightarrow{f} & \mathcal{N} \\
\downarrow & & \downarrow \ x \\
\mathcal{L} & \xrightarrow{g} & \mathcal{N}
\end{array}
\]
Then, $\mathcal{M}^*$ is the inverse limit of its coherent quotient $R$-modules $\mathcal{L}'$, such that $L'$ are finitely generated projective modules. Equivalently, $M$ is the direct limit of its finitely generated projective submodules that are direct summands.

$\Leftarrow$) It is a consequence of Theorem 6.20.

□

Corollary 6.22. Let $R$ be a local ring. $\mathcal{M}^*$ is dually separated iff $M$ is the direct limit of its finite free submodules that are direct summands.

Proof. $\Rightarrow$) $\mathcal{M}^*$ is the inverse limit of its coherent quotient $R$-modules. We only have to prove that every epimorphism $f: \mathcal{M}^* \to \mathcal{N}$ onto a coherent module factors
through an epimorphism onto a free coherent module. Let \( m \) be the maximal ideal of \( R \). Let \( R^n \to N \) be an epimorphism such that \( R^n \otimes_R R/m \to N \otimes_R R/m \) is an isomorphism. Let \( \pi: R^n \to N \) be the induced epimorphism. There exists a morphism \( g: M^* \to R^n \) such that \( \pi \circ g = f \), because the map

\[
\text{Hom}_R(M^*, R^n) = M \otimes_R R^n \to M \otimes_R N = \text{Hom}_R(M^*, N)
\]
is surjective. As \( f: M^* \to N \) is an epimorphism, then

\[
(\text{Im } g_R) \otimes_R R/m \to R^n \otimes_R R/m = N \otimes_R R/m
\]
is an epimorphism. By Nakayama’s lemma \( \text{Im } g_R = R^n \). Then, \( f \) factors through the epimorphism \( g: M^* \to R^n \).

\[
\iff M = \lim_i L_i, \text{ where } \{L_i\} \text{ is the set of finite free modules that are direct summands. Then, } M^* = \lim_i L_i^* \text{. Let } f: M^* \to N \text{ be a morphism. Then, } f \in M \otimes N = (\lim_i L_i) \otimes N = \lim_i (L_i \otimes N) \text{ and } f \text{ factors through an epimorphism } g: M^* \to L_i^*, \text{ for some } i. \text{ Let } \pi: L_i^* \to N \text{ be a morphism such that } f = \pi \circ g. \text{ Coker } f = \text{Coker } \pi \text{ is a quasi-coherent module. Then, } M^* \text{ is dually separated, by Theorem 6.11.}
\]

\[\square\]

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