A NON-LOCAL AREA PRESERVING CURVE FLOW

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ABSTRACT. In this paper, we consider a kind of area preserving non-local flow for convex curves in the plane. We show that the flow exists globally, the length of evolving curve is non-increasing, and the curve converges to a circle in $C^\infty$ sense as time goes into infinity.

1. INTRODUCTION

It is an interesting problem to study non-local flow for curves in the plane. The purpose of this paper is to introduce a new non-local flow which preserves the area enclosed by the evolving curve. Our research is motivated by the famous works of Gage and Hamilton [9] and [4] (see also [3] for background and more results). The curve shortening flow in a Riemannian manifold has been studied extensively in the last few decades (see [14]). The curve shortening flow in the plane is the family of evolving curves $\gamma(t)$ such that $\frac{\partial}{\partial t}\gamma(t) = kN$, where $k$ and $N$ are the curvature of curve $\gamma$ and the (inward pointing) unit normal vector to the curve. For this flow, deep results are obtained in [9], [7] and [11]. They have proved that a simple closed initial curve remains so along the flow, and the evolving curve becomes more and more circular during the curve shortening process, and it converges to a point in a finite time. Then another natural question arises for expanding evolution flow for curves. B.Chow and D.H. Tsai have studied the expanding flow such as $\frac{\partial}{\partial t}\gamma(t) = -G\left(\frac{1}{k}\right)N$, where $G$ is a positive smooth function with $G'>0$ everywhere. B.Andrews [1] has studied more general expanding flows, especially flows with anisotropic speeds. They have obtained deep results too. People then like to study curve flow problems preserving some geometric quantities. M.Gage [8] has considered an area-preserving flow $\frac{\partial}{\partial t}\gamma(t) = (k - \frac{2\pi}{L})N$, where $L$ are the length of the curve $\gamma$, and have proved that the length of the curve is non-increasing and finally converges to a circle. Based on this, it is interesting study a non-local curve flow which preserves the length of the evolving curve. For this, one may see [13] for a recent study. In a very recent paper [20], S.L.Pan and J.N.Yang consider a very interesting length preserving curve flow for convex curves in the plane of the form

$$\frac{\partial}{\partial t}\gamma(t) = \left(\frac{L}{2\pi} - k^{-1}\right)N,$$

where $L$, $N$, and $k$ are the length, unit normal vector, and the curvature of the curve $\gamma(t)$ respectively. They have proved that the convex plane curve will become more and more circular and converges to circle in the $C^\infty$ sense.

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We now consider the following non-local area preserving curve flow \( \frac{\partial}{\partial t} \gamma(t) = (\alpha(t) - \frac{1}{k})N \), where \( \alpha(t) = \frac{1}{L} \int_0^L \frac{1}{k} ds \), and obtain the following result.

**Theorem 1.** Suppose \( \gamma(u, 0) \) is a strictly convex curve (i.e. \( k(0) > 0 \)) in the plane \( \mathbb{R}^2 \). Assume \( \gamma(t) := \gamma(u, t) \) satisfies the following evolving equation

\[
\frac{\partial}{\partial t} \gamma(t) = (\alpha(t) - \frac{1}{k})N,
\]

where \( k \) is the curvature of the curve \( \gamma(t) \), \( N \) is inward pointing unit normal vector to the curve and \( \alpha(t) = \frac{1}{L} \int_0^L \frac{1}{k} ds \). Then the curve flow problem (1.1) has the global solution \( \gamma(t) \), for all \( t \in [0, \infty) \). Furthermore, the non-local curve flow (1.1) preserves the area enclosed by the evolving curve and keeps the strictly convexity under the evolution process. More over, \( \gamma(t) \) converges to a circle in the \( C^\infty \) sense as time \( t \) goes into infinity.

The difficulty in the study the non-local flow (1.1) lies in treating the possible collapsing point where \( k = \infty \) of the evolving curve at any finite time. To overcome this, we use the support function trick, which has also been used by B.Chow and B.Andrews in the Gauss curvature flow and in the curve shortening flow. We show the global existence of the support functions is equivalent to the globally existence of the non-local flow (1.1). However, this part is new in the research of the non-local flows. It is not clear to us how to get global existence of the area-preserving non-local flow for curves in non-flat surfaces. The convexity of the evolving flow is proved by the use of maximum principle to the curvature evolution equation. To prove the convergence of the global flow, we need the new isoperimetric inequality obtained Pan and Yang in [20], where they used a trick of Gage in [6] and [7].

The paper is organized as follows. In section 2, we calculate some evolution equations related to this curve flow. In section 3, we prove a long time existence for the curve flow (1.1) and show the strictly convexity of the flow is preserved. In section 4, we show that isoperimetric deficit decays to zero under the non-local curve flow (1.1) and the evolving curve converges to a circle in \( C^\infty \) sense. In the last section, we give a comment for more general non-local flows.

### 2. Preparation

In this section, we calculate some formulae for more general non-local flows than the non-local flow (1.1). Consider the evolving curve \( \gamma(t) \) defined by the map \( \gamma(u, t): S^1 \times I \to \mathbb{R}^2 \) satisfying the equation:

\[
\frac{\partial}{\partial t} \gamma(t) = (\alpha(t) - \frac{1}{k})N,
\]

where \( \alpha(t) \) is a \( C^\infty \) function only depends on the time \( t \). Since \( u \) and \( t \) are independent variables, \( \frac{\partial}{\partial u} \) and \( \frac{\partial}{\partial t} \) commute when applied to functions on \( \mathbb{R}^2 \). Let \( s \) denote the arc-length of the curve \( \gamma \). Then the operator \( \frac{\partial}{\partial s} \) is given in terms of \( u \) by

\[
\frac{\partial}{\partial s} = \frac{1}{v \frac{\partial}{\partial u}},
\]

where \( v = |\frac{\partial}{\partial u}| \).

The arc-length parameter is \( ds = v du \). Let \( T \) and \( N \) be the unit tangent vector and the (inward pointing) unit normal vectors to the curve respectively. Then the
Frenet equations can be written as
\[
\frac{\partial T}{\partial u} = vkN, \quad \frac{\partial N}{\partial u} = -vkT.
\]

We now introduce some formulas according to (2.1). First we have the following evolution equation for \( v \).

**Lemma 2.** \( \frac{\partial v}{\partial t} = (1 - k\alpha)v \).

**Proof.**
\[
\frac{\partial}{\partial t} (v^2) = \frac{\partial}{\partial t} \langle \frac{\partial \gamma}{\partial u}, \frac{\partial \gamma}{\partial u} \rangle = 2 \langle \frac{\partial \gamma}{\partial u}, \frac{\partial^2 \gamma}{\partial u \partial t} \rangle = 2 \langle vT, \frac{\partial}{\partial u}(\alpha - \frac{1}{k})N \rangle = 2(1 - k\alpha)v^2.
\]

Then the lemma follows immediately. \( \square \)

We also have the following useful relation for the operators \( \frac{\partial}{\partial t} \) and \( \frac{\partial}{\partial s} \).

**Lemma 3.**
\[
\frac{\partial}{\partial t} \frac{\partial}{\partial s} - \frac{\partial}{\partial s} \frac{\partial}{\partial t} = (k\alpha - 1) \frac{\partial}{\partial s}.
\]

**Proof.**
\[
\frac{\partial}{\partial t} \frac{\partial}{\partial s} = \frac{\partial}{\partial t} \left( \frac{1}{v} \frac{\partial}{\partial u} \right) = -\frac{v_t}{v} \frac{\partial}{\partial u} + \frac{1}{v} \frac{\partial}{\partial t} \frac{\partial}{\partial u} = -\frac{v_t}{v} \frac{\partial}{\partial u} + \frac{1}{v} \frac{\partial}{\partial u} \frac{\partial}{\partial t} = (k\alpha - 1) \frac{\partial}{\partial s} + \frac{\partial}{\partial s} \frac{\partial}{\partial t}.
\]

The derivatives of \( T \) and \( N \) are given by

**Lemma 4.**
\[
\frac{\partial}{\partial t} T = k_s k^2 N \quad \text{and} \quad \frac{\partial}{\partial t} N = -\frac{k_s}{k^2} T.
\]

**Proof.**
\[
\frac{\partial}{\partial t} T = \frac{\partial}{\partial t} \frac{\partial}{\partial s} \gamma = \frac{\partial}{\partial s} \frac{\partial}{\partial t} \gamma + (k\alpha - 1) \frac{\partial}{\partial s} \gamma = \frac{\partial}{\partial s}((\alpha - \frac{1}{k})N + (k\alpha - 1)T = \frac{\partial}{\partial s} \frac{\partial}{\partial s}((\alpha - \frac{1}{k})N + (k\alpha - 1)T = \frac{k_s}{k^2} N.
\]

The second equation follows from
\[
0 = \frac{\partial}{\partial t} < T, N > = < k_s \frac{k_s}{k^2} N, N > + < T, \frac{\partial}{\partial t} N >,
\]
and \( \frac{\partial N}{\partial t} \) must be perpendicular to \( N \). \( \square \)

We denote the angle between the tangent and the X-axis by \( \theta \). Then we have

**Lemma 5.**
\[
\frac{\partial \theta}{\partial t} = \frac{k_s}{k^2}.
\]
Proof. Since \( T = (\cos \theta, \sin \theta) \), we use the formula in lemma \( \text{[3]} \) to calculate
\[
\frac{\partial T}{\partial t} = \frac{k_s}{k^2} N = \frac{k_s}{k^2} (\cos \theta, \sin \theta).
\]
Comparing components on both sides we get the conclusion of this lemma. \( \square \)

The curvature for the evolving curve evolves according to

**Lemma 6.**
\[
(2.2) \quad \frac{\partial k}{\partial t} = \frac{1}{k^2} \frac{\partial^2 k}{\partial s^2} - \frac{2}{k^3} \left( \frac{\partial k}{\partial s} \right)^2 + (k \alpha - 1) k.
\]

Proof. By lemma \( \text{[3]} \) we have
\[
\frac{\partial k}{\partial t} = \frac{\partial}{\partial t} \frac{\partial \theta}{\partial s} = \frac{\partial}{\partial s} \frac{\partial \theta}{\partial t} + \left( k \alpha - 1 \right) \frac{\partial \theta}{\partial s} = \frac{\partial}{\partial s} \left( \frac{k_s}{k^2} \right) + \left( k \alpha - 1 \right) k = \frac{1}{k^2} \frac{\partial^2 k}{\partial s^2} k - \frac{2}{k^3} \left( \frac{\partial k}{\partial s} \right)^2 + (k \alpha - 1) k.
\]
\( \square \)

We denote \( A(t) \) be the area enclosed by the evolving curve. We have

**Lemma 7.** \( A(t) \) satisfies the equation
\[
\frac{d}{dt} A(t) = \int_0^L \frac{1}{k} ds - \alpha L.
\]

Hence \( A(t) \) remains constant when \( \alpha(t) = \frac{1}{L} \int_0^L \frac{1}{k} ds. \)

Proof. Since
\[
-2A(t) = \int_0^L < \gamma, N > ds = \int_0^{2\pi} < \gamma, vN > du,
\]
we have
\[
-2 \frac{d}{dt} A(t) = \int_0^{2\pi} < \gamma_t, vN > + \int_0^{2\pi} < \gamma, v_t N > + \int_0^{2\pi} < \gamma, vN_t > du.
\]

\[
= \int_0^{2\pi} < (\alpha - \frac{1}{k}) N, vN > du + \int_0^{2\pi} < \gamma, (1 - k \alpha) vN > du + \int_0^{2\pi} < \gamma, (\frac{k_s}{k^2}) vT > du
\]

\[
= \int_0^{2\pi} (\alpha - \frac{1}{k}) vdu + \int_0^{2\pi} < \gamma, (1 - k \alpha) vN > du + \int_0^{2\pi} \frac{\partial}{\partial u} \left( \frac{1}{k - \alpha} \right) < \gamma, T > du
\]

\[
= \int_0^{2\pi} (\alpha - \frac{1}{k}) ds + \int_0^{2\pi} < \gamma, (1 - k \alpha) vN > du + \int_0^{2\pi} \frac{\partial}{\partial u} \left( \frac{1}{k - \alpha} \right) < \gamma, T > du.
\]
By the use of integration by parts, we have
\[
-2 \frac{d}{dt} A(t) = \int_0^L (\alpha - \frac{1}{k}) ds + \int_0^{2\pi} <\gamma, (1-k\alpha)vN > du \\
+ \int_0^{2\pi} (\alpha - \frac{1}{k})(<\gamma_u, T> + <\gamma, T_u>) du \\
= \int_0^L (\alpha - \frac{1}{k}) ds + \int_0^{2\pi} <\gamma, (1-k\alpha)vN > du \\
+ \int_0^{2\pi} (\alpha - \frac{1}{k})(<vT, T> + <\gamma, vkN>) du \\
= 2 \int_0^L (\alpha - \frac{1}{k}) ds = -2(\int_0^L \frac{1}{k} ds - \alpha L).
\]

□

The evolution equation for \(\alpha(t)\) in the flow (1.1) is below.

Lemma 8. If \(\alpha(t) = \frac{1}{L} \int_0^L \frac{1}{k} ds\), we have
\[
\alpha \geq \frac{L}{2\pi}.
\]
The equality holds if and only if the curve \(\gamma\) has the constant curvature.

Proof. Since
\[
\int_0^L kds = 2\pi,
\]
by the Cauchy-Schwartz inequality we have
\[
\int_0^L kds \cdot \int_0^L \frac{1}{k} ds \geq (\int_0^L ds)^2 = L^2.
\]
Then we have the result. □

Lemma 9. The length of the evolving curve evolves by
\[
\frac{d}{dt} L = L - 2\pi \alpha(t),
\]
Moreover, \(\frac{d}{dt} L \leq 0\) when \(\alpha(t) = \frac{1}{L} \int_0^L \frac{1}{k} ds\).

Proof.
\[
\frac{d}{dt} L = \int_0^{2\pi} v_t du = \int_0^{2\pi} (1-k\alpha)vdu = \int_0^L (1-k\alpha) ds = L - 2\pi \alpha.
\]
We have \(\frac{d}{dt} L \leq 0\) if \(\alpha(t) = \frac{1}{L} \int_0^L \frac{1}{k} ds\) by lemma □

So much for the general flow (2.1).
3. Long Time Existence

Since the changing of the tangential components of the velocity vector of $\gamma_t$ affects only the parametrization, not the geometric shapes of the evolving curve, we can choose a suitable tangent component $\eta$ to simplify the analysis of the nolocal flow (1.1). This trick has been used by many authors, see, for example, [9] or [20]. So we consider the following evolution problem, which is equivalent to (1.1):

$$\gamma_t = (\alpha(t) - \frac{1}{k})N + \eta T.$$  (3.1)

Similar to the calculations in section 2, we have

**Lemma 10.**

$$\frac{\partial v}{\partial t} = \frac{\partial \eta}{\partial u} + (1 - k\alpha)v,$$

$$\frac{\partial T}{\partial t} = (\eta k + \frac{k_s}{k^2})N, \quad \frac{\partial N}{\partial t} = -\eta k + \frac{k_s}{k^2},$$

$$\frac{\partial \theta}{\partial t} = \eta k + \frac{k_s}{k},$$

$$\frac{\partial k}{\partial t} = \frac{1}{k^2} \frac{\partial^2 k}{\partial s^2} - \frac{2}{k^3} (\frac{\partial k}{\partial s})^2 + (k\alpha - 1)k + \eta \frac{\partial k}{\partial s},$$

$$\frac{d}{dt} A(t) = \int_0^L \frac{1}{k} ds - \alpha L,$$

$$\frac{d}{dt} L = L - 2\pi \alpha.$$  (3.3)

Note that $L$ and $A$ are both independent of $\eta$. In order to make $\theta$ independent of time $t$, we can choose suitable $\eta$ such that $\frac{\partial \eta}{\partial t} = 0$, i.e.

$$\eta = \frac{1}{k^3} k_s = \frac{1}{k^3} \frac{\partial k}{\partial \theta}.$$  (3.4)

So we consider the following equivalent problem instead from now on:

$$\gamma_t = (\alpha(t) - \frac{1}{k})N - \frac{1}{k^3} \frac{\partial k}{\partial s} T.$$  (3.2)

Then by lemma 10, we have the following result.

**Lemma 11.**

$$\frac{\partial T}{\partial t} = 0, \quad \frac{\partial N}{\partial t} = 0, \quad \frac{\partial \theta}{\partial t} = 0,$$

$$\frac{\partial k}{\partial t} = \frac{1}{k^2} \frac{\partial^2 k}{\partial s^2} - \frac{3}{k^3} (\frac{\partial k}{\partial s})^2 + (k\alpha - 1)k,$$

$$\frac{d}{dt} A(t) = \int_0^L \frac{1}{k} ds - \alpha L,$$

$$\frac{d}{dt} L = L - 2\pi \alpha.$$  (3.5)

By theorem 14, we can use the angle variable $\theta$ of the tangent line as a parameter for convex curves. To determine the evolution equation for curvature of the evolving when using $\theta$ as a parameter, we take $\tau = t$ as the time parameter; thus we change
variables from \((u, t)\) to \((\theta, \tau)\). We obtain the following equation for \(k\) in terms of \(\theta\) and \(\tau\).

**Lemma 12.**

\[
\frac{\partial k}{\partial \tau} = \frac{\partial^2 k}{\partial \theta^2} - \frac{2}{k} \left( \frac{\partial k}{\partial \theta} \right)^2 + (k\alpha - 1)k.
\]  

**Proof.** By the chain rule and lemma 11,

\[
\frac{\partial k}{\partial t} = \frac{\partial k}{\partial \tau} + \frac{\partial k}{\partial \theta} \frac{\partial \theta}{\partial t} = \frac{\partial k}{\partial \tau},
\]

and

\[
\frac{\partial^2 k}{\partial s^2} = \left( \frac{\partial \theta}{\partial s} \frac{\partial}{\partial \theta} \right) \left( \frac{\partial \theta}{\partial s} \frac{\partial}{\partial \theta} \right) = k^2 \frac{\partial^2 k}{\partial \theta^2} + k \left( \frac{\partial k}{\partial \theta} \right)^2.
\]

Substituting these expressions into the formula (3.4) in lemma 11 we get the result. \(\square\)

By direct calculation, we can derive a standard heat equation (see (3.9)) from formula (3.7).

**Lemma 13.**

\[
\frac{\partial}{\partial \tau} \left( \frac{1}{k} \right) = \frac{\partial^2}{\partial \theta^2} \left( \frac{1}{k} \right) + \frac{1}{k} - \alpha,
\]

Denote \(v = \frac{1}{k} - \frac{L}{2\pi}\) and \(w = ve^{-\tau}\), we have

\[
w_{\tau} = w_{\theta\theta}.
\]

Then \(w\) can be solved for time interval \([0, +\infty)\) as

\[
w(\theta, \tau) = \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi\tau}} e^{-\frac{\left(\theta - \xi\right)^2}{4\tau}} w(\theta, 0)d\xi.
\]

**Proof.** Since

\[
\frac{\partial}{\partial \tau} \left( \frac{1}{k} \right) = -\frac{k\tau}{k^2} = -\frac{k\theta}{k^2} + \frac{2}{k^3} k^2 \frac{\partial^2 k}{\partial \theta^2} - \frac{k^2}{k^2},
\]

and

\[
\frac{\partial^2}{\partial \theta^2} \left( \frac{1}{k} \right) = -\frac{\partial}{\partial \theta} \left( \frac{k}{k^2} \right) = -\frac{k\theta}{k^2} + \frac{2}{k^3} k^2,
\]

(3.8) follows immediately.

By lemma 9, we have

\[
v_{\tau} = \frac{\partial}{\partial \tau} \left( \frac{1}{k} \right) = \frac{L}{2\pi} - \frac{\partial^2}{\partial \theta^2} \left( \frac{1}{k} \right) + \frac{1}{k} - \alpha - \frac{L - 2\pi \alpha}{2\pi}
\]

\[
= \frac{\partial^2}{\partial \theta^2} \left( \frac{1}{k} \right) + \left( \frac{1}{k} - \frac{L}{2\pi} \right)
\]

\[
= v_{\theta\theta} + v.
\]

Then (3.9) follows immediately. \(\square\)

**Theorem 14.** Under the assumptions of theorem 1, the curve flow keeps convexity under the evolution process.
Proof. By lemma 13 there exists a constant $M > 0$ such that for $(\theta, \tau) \in [0, 2\pi] \times (0, \infty)$,

$$|w(\theta, \tau)| \leq M.$$ 

Then for any finite $T, \tau \in [0, T^*),$

$$\frac{1}{k} - \frac{L}{2\pi} \leq Me^{T^*}.$$ 

By lemma 11 $L$ is bounded above. Also by the isoperimetric inequality, $L$ has a lower bound $\sqrt{4\pi A}$. So we get

$$k(\theta, \tau) \neq 0, \text{ for } (\theta, \tau) \in [0, 2\pi] \times [0, T^*).$$

Now, from the continuity of $k(\theta, \tau)$ and the positivity of $k(\theta, 0)$, we know that

$$k(\theta, \tau) > 0, \text{ for } (\theta, \tau) \in [0, 2\pi] \times [0, T^*).$$

Then the theorem follows from the arbitrariness of $T^*$. □

We denote $S$ be the support function of the curve $\gamma$, i.e. $S = -\langle \gamma, N \rangle$. We then have the equation

$$\frac{1}{k} = \frac{\partial^2 S}{\partial \theta^2} + S. \tag{3.10}$$

We have the following evolution equation of support function.

**Lemma 15.**

$$\frac{\partial S}{\partial \tau} = \frac{\partial^2 S}{\partial \theta^2} + S - \alpha. \tag{3.11}$$

**Proof.** By lemma 11 we have

$$\frac{\partial S}{\partial \tau} = -\frac{\partial}{\partial \tau} \langle \gamma, N \rangle = -\frac{\partial}{\partial \tau} \langle \gamma, N \rangle$$

$$= -\langle (\alpha(\tau) - \frac{1}{k^2})N - \frac{1}{k} \frac{\partial k}{\partial \theta} T, N \rangle$$

$$= \frac{1}{k} - \alpha$$

$$= \frac{\partial^2 S}{\partial \theta^2} + S - \alpha. \quad \square$$

Similar to lemma 13 we have

**Theorem 16.** The support function $S$ can be solved for time interval $[0, +\infty)$ as

$$(S(\theta, \tau) - \frac{L(\tau)}{2\pi} e^{-\tau}) = \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi \tau}} e^{-\frac{\xi^2}{2\tau}} (S(\theta, 0) - \frac{L(0)}{2\pi}) d\xi.$$ 

Now we can follow the methods of B. Andrews, see theorem II.2 in [1]. It is convenient to choose the normal vector for parameter of the curve. We denote $n : \gamma \to S^1$ be the Gauss map. Let $z$ be the normal vector and $\gamma$ parametrized by $z$. So we have $S(z, t) = -\langle \gamma(n^{-1}(z)), z \rangle$. Let $r[S](z)$ be the radius of curvature at the point with normal $z$ is given by $r[S](z) = \frac{\partial^2 S}{\partial \theta^2}(z) + S(z)$. Then we obtain the following result.
Theorem 17. Assume $S : S^1 \times [0, \infty) \to \mathbb{R}$ is a smooth function of equation (3.11) with radius $r[S] > 0$, then there exists a solution $\gamma : \zeta \times [0, \infty) \to \mathbb{R}$ satisfies the equation (3.11) which has the initial data $\gamma_0 = \{ -S_0(z) z - \frac{\partial S_0}{\partial \theta}(z) \frac{\partial z}{\partial \theta} : z \in S^1 \}$ and such that the curve $\gamma(t)$ has the support function $S(t)$ for each $t \in [0, \infty)$.

Proof. We define the evolving curve by $\bar{\gamma} : S^1 \times [0, \infty) \to \mathbb{R}$ by

$$
(3.12) \quad \bar{\gamma}(z, t) = -S(z, t) z - \frac{\partial S}{\partial \theta}(z, t) \frac{\partial z}{\partial \theta}.
$$

Then we have a set of curves $\gamma(t)$ which have the support function $S$, and have the curvature $k$ by (3.10). By the assumptions of this theorem,

$$
\frac{\partial \gamma}{\partial t}(z, t) = -\frac{\partial S}{\partial t}(z, t) z - \frac{\partial^2 S}{\partial t \partial \theta} \frac{\partial z}{\partial \theta} = -(\frac{\partial^2 S}{\partial \theta^2} + S - \alpha) z - \frac{\partial}{\partial \theta} (\frac{\partial^2 S}{\partial \theta^2} + S - \alpha) \frac{\partial z}{\partial \theta} = (\alpha - r[S](z)) z - \frac{\partial}{\partial \theta} (r[S](z)) \frac{\partial z}{\partial \theta} = (\alpha - \frac{1}{k_\gamma}) N_\gamma(z) - T\bar{\gamma}(V),
$$

where $N_\gamma$ and $k_\gamma$ are the normal and curvature corresponding to $\bar{\gamma}$, and $V \in TS^1 \times [0, \infty)$ is the vector field on $S^1$ given by $k_\gamma \left( \frac{1}{\bar{\gamma}} \right) \frac{\partial}{\partial \theta}$. Here we used the fact $T\bar{\gamma}(V) = k_\gamma^{-1} V$ for any $V \in TS^1$. Next we define a family of diffeomorphisms $\phi$ such that $\gamma(p, t) = \bar{\gamma}(\phi(p, t), t)$ gives the solution of equation (1.1). Now take $\phi(p, t)$ solve the following ordinary differential equation for each $p$:

$$
(3.13) \quad \frac{d}{dt} \phi(p, t) = V(\phi(p, t), t).
$$

This equation has a unique solution for each $p$ as long as $S$ exists and remains smooth. Then we have

$$
\frac{\partial}{\partial t} \gamma(p, t) = \frac{\partial}{\partial t} \bar{\gamma}(\phi(p, t), t) = (\frac{\partial}{\partial t} \bar{\gamma})(\phi(p, t), t) + T\bar{\gamma}(\frac{\partial}{\partial t} \phi(p, t), t) = (\alpha - \frac{1}{k_\gamma(\phi(p, t), t)}) N_\gamma(\phi(p, t), t) - T\bar{\gamma}(V) + T\bar{\gamma}(V) = (\alpha - \frac{1}{k_\gamma(p, t)}) N_\gamma(p, t),
$$

where we have used $k_\gamma(p, t) = k_\gamma(\phi(p, t), t)$ and $N_\gamma(p, t) = N_\gamma(\phi(p, t), t)$. Hence the theorem holds. \hfill \Box

Hence we get the following result immediately by the use of theorem 16 and theorem 17.

Theorem 18. Under the assumptions of theorem 17, the curve flow (1.1) has the global solution in time interval $[0, \infty)$. 

4. Convergence

In order to get the results of the section, we need the following isoperimetric inequality due to S.L. Pan and J.N. Yang.

**Theorem 19.** [20] For the closed, convex $C^2$ curves in the plane, we have

$$\frac{L^2 - 2\pi A}{\pi} \leq \int_0^L \frac{1}{k} ds,$$

where $L, A$ and $k$ are the length of the curve, the area enclosed by the evolving curve, and its curvature.

Then we have the following theorem, which shows that the curve flow becomes more and more circular under the evolution process.

**Theorem 20.** If a convex curve evolves according to (2.1), then the isoperimetric deficit $L^2 - 4\pi A$ is non-increasing during the evolution process and converges to zero as the time $t$ goes to infinity.

**Proof.** By lemma 7 and lemma 9, we have

$$\frac{d}{dt}(L^2 - 4\pi A) = 2LL_t - 4\pi A_t$$

$$= 2L(L - 2\pi \alpha) - 4\pi \left(\int_0^L \frac{1}{k} ds - \alpha L\right)$$

$$= 2L^2 - 4\pi \int_0^L \frac{1}{k} ds.$$  

By the theorem 19, we have

$$\frac{d}{dt}(L^2 - 4\pi A) \leq 2L^2 - 4\pi \frac{L^2 - 2\pi A}{\pi} \leq -2(L^2 - 4\pi A),$$

We always have $L^2 - 4\pi A \geq 0$, so

$$\frac{d}{dt}(L^2 - 4\pi A) \leq 0.$$

Moreover, we have

$$0 \leq L^2 - 4\pi A(t) \leq C \exp(-2t).$$

As $t \to \infty$, we have the decay of the isoperimetric defect,

$$L^2 - 4\pi A \to 0.$$

□

We denote $r_{in}$ radii of the largest inscribed circle of the curve $\gamma$. Now we can use the same methods of M.Gage and R.S.Hamilton to show curvature $k$ converges to a constant as time goes into infinity, see Section 5 in [9]. First, we need a result in [9].

**Theorem 21.** [9] $k(\theta, t)r_{in}(t)$ converges uniformly to 1, when the isoperimetric deficit $L^2 - 4\pi A \to 0$.

**Theorem 22.** Under the assumptions of theorem [9], we have $k \to \frac{2\pi}{L}$ as $t \to \infty$. 
Proof. By the Bonnesen inequality \[16\]
\[
\frac{L^2}{A} - 4\pi \geq \frac{(L - 2\pi r_{in})^2}{A}
\]
and theorem 20, theorem 7, we have \( r_{in} \rightarrow \frac{L}{2\pi} \) as \( t \rightarrow \infty \). Hence the theorem follows immediately from theorem 21. \( \Box \)

Then we obtain the \( C^\infty \) convergence part in theorem 1.

**Theorem 23.** Under the assumptions of theorem 1, the curve flow (1.1) converges to a circle in \( C^\infty \) sense as time goes into infinity.

**Proof.** By lemma 13, the curvature \( k(t) \) is \( C^\infty \) differentiable. Then the theorem follows immediately from theorem 22. \( \Box \)

5. Appendix

Under the calculations and analysis in above sections, we observe that most of the theorems may hold with the assumption that \( \alpha(t) \) is a \( C^\infty \) differentiable function only depends on time \( t \) and bounded on compact interval. Note that the equations in Section 2 is calculated under the assumption \( \alpha(t) \) is a \( C^\infty \) differentiable function only depends on time \( t \). If we assume the curve flow (2.1) has the global solution in time interval \([0, \infty)\), i.e. the curve flow do not converge to a point in finite time, lemma 20 holds obviously. By lemma 9, the length of the evolving curve is bounded up in finite time. Then by the same arguments in lemma 13, the convexity is preserved under the evolution process. If \( \lim_{t \to \infty} L = \hat{L} \) exists and \( \hat{L} > 0 \), then \( \gamma(t) \) converge to a circle in \( C^\infty \) sense by the same arguments in theorem 21. If \( L \to \infty \) as \( t \to \infty \), then \( A \to \infty \) as \( t \to \infty \) by theorem 20. Then \( k \to 0 \) as \( t \to 0 \) by inequality (4.1). Hence \( \gamma(t) \) converge to a straight line in \( C^\infty \) sense. Note that the \( C^\infty \) convergence in last two cases is from same arguments in lemma 13. In fact we have the following theorem.

**Theorem 24.** Suppose \( \gamma(u, 0) \) is a strictly convex curve (i.e. \( k(0) > 0 \)) in the plane \( \mathbb{R}^2 \). Assume \( \gamma(t) := \gamma(u, t) \) satisfied the following evolving equation

\[
\frac{\partial}{\partial t} \gamma(t) = (\alpha(t) - \frac{1}{k})N,
\]

where \( \alpha(t) \) is a \( C^\infty \) differentiable function only depends on time \( t \) and bounded on compact time interval. Assume the curve \( \gamma(t) \) do not converge to a point in finite time. Then we have

1. Convexity of the evolving curve is preserved, that is, \( k(t) > 0 \) for \( t \in [0, \infty) \).
2. The isoperimetric deficit decays, \( L^2 - 4\pi A \to 0 \) and \( k - \frac{2\pi}{L} \to 0 \) as \( t \to \infty \).
3. If \( L \to \infty \) as \( t \to \infty \), then \( \gamma(t) \) converges to a straight line in \( C^\infty \) sense; if \( \lim_{t \to \infty} L = \hat{L} \) exists and \( \hat{L} > 0 \), then \( \gamma(t) \) converge to a circle in \( C^\infty \) sense.

The proof is similar to Theorem 1, so we omit the proof.

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