The Four Hagge Circles

Christopher J Bradley

Abstract: The four Hagge circles of the triangles BCD, ACD, ABD, ABC of a cyclic quadrilateral ABCD with respect to an appropriate common axis of inverse similarity share many features which are analysed.

Fig. 1
The key points involved in describing the four triangles of a cyclic quadrilateral
1. **Preliminary analysis**

In later sections we state and prove a number of theorems about the four Hagge circles, with respect to an appropriately chosen centre of inverse similarity \( P \), of the four triangles \( BCD, ACD, ABD, ABC \) that comprise a cyclic quadrilateral \( ABCD \). It is desirable therefore to give a description of the basic configuration involved and to prove some straightforward facts that are required later.

We use vectors with origin \( O \), the centre of the circumscribing circle, and with \( A, B, C, D \) having vector positions \( a, b, c, d \) each of which has magnitude \( R \), the radius of the circle. The triangles \( BCD, ACD, ABD, ABC \) are denoted by \( \Delta_1, \Delta_2, \Delta_3, \Delta_4 \) respectively. Suffices are used for key points in these triangles. Thus \( H_k \) is the orthocentre of \( \Delta_k \), \( N_k \) is the nine-point centre of \( \Delta_k \) and \( G_k \) is the centroid of \( \Delta_k \), \( k = 1, 2, 3, 4 \). The vector positions corresponding to the points are

\[ h_1 = b + c + d, \quad h_2 = a + c + d, \quad h_3 = a + b + d, \quad h_4 = a + b + c \]

and, as in any triangle, \( n_k = \frac{1}{2} h_k \) and \( g_k = \frac{1}{3} h_k \).

The following facts are now easy to establish:

(i) \( AH_1, BH_2, CH_3, DH_4 \) are concurrent at the point \( N \) with vector position \( \frac{1}{2}(a + b + c + d) \);

(ii) \( AN_1, BN_2, CN_3, DN_4 \) are concurrent at the point \( G \) with vector position \( \frac{1}{3}(a + b + c + d) \);

(iii) \( AG_1, BG_2, CG_3, DG_4 \) are concurrent at a point \( M \) with vector position \( \frac{1}{4}(a + b + c + d) \);

(iv) \( H_1, H_2, H_3, H_4 \) lie on a circle of radius \( R \) with centre \( H \), which has position vector \( (a + b + c + d) \) and moreover \( H_1H_2H_3H_4 \) is the image of \( ABCD \) under a 180° rotation about \( N \);

(v) \( N_1, N_2, N_3, N_4 \) lie on a circle of radius \( \frac{1}{2}R \) with centre \( N \) and \( N_1N_2N_3N_4 \) is homothetic with \( H_1H_2H_3H_4 \), with centre \( O \) and enlargement factor \( \frac{1}{2} \);

(vi) The nine-point circles of triangles \( \Delta_1, \Delta_2, \Delta_3, \Delta_4 \), each having radius \( \frac{1}{2}R \) therefore all pass through \( N \);

(vii) A known result is that a rectangular hyperbola through the vertices of a triangle passes through its orthocentre. It follows that the rectangular hyperbola \( \Sigma \) through \( A, B, C, D \) passes through all of \( H_1, H_2, H_3, H_4 \);

(viii) Another known result is that the centre of a rectangular hyperbola passing through the vertices of a triangle lies on its nine-point circle. In consequence of (vii) the centre of \( \Sigma \) is the point \( N \);

(ix) \( G_1, G_2, G_3, G_4 \) lie on a circle of radius \( \frac{1}{2}R \) with centre \( G \), which has position vector \( \frac{1}{3} (a + b + c + d) \);

(x) The points \( O, M, G, N, H \) are collinear and if \( O \) and \( H \) are given co-ordinates 0, 1 on this line, then \( M, G, N \) have co-ordinates \( \frac{1}{4}, \frac{1}{3}, \frac{1}{2} \) respectively.
See Fig.1 for an illustration of these properties.

2. Setting the scene

It is clear from Speckman’s work on indirect similar perspective triangles that if we now choose any point $P$ on the rectangular hyperbola defined in Section 1 Result (vii), it is possible to find the Hagge circle of $P$ with respect to each of the four triangles $\Delta_1$, $\Delta_2$, $\Delta_3$, $\Delta_4$ and to draw them on the same diagram. These are the four Hagge circles $\Gamma_1$, $\Gamma_2$, $\Gamma_3$, $\Gamma_4$ of the article heading and they are shown in Fig. 2. Note that as the axes of inverse similarity through $P$ are parallel to the asymptotes of the rectangular hyperbola $\Sigma$, these axes coincide for each of the Hagge circles.

A consistent notation is essential and this we now describe. See Fig. 2. The centres of the circles are denoted by $Q_k$, $k = 1, 2, 3, 4$. For reasons that become clear shortly we re-label the four orthocentres $A_1$, $B_2$, $C_3$, $D_4$ so that, for example $D_4$ is the orthocentre of triangle $ABC$. The lines $AP$, $BP$, $CP$, $DP$ meet the circumcircle $\Gamma$ at points $A'$, $B'$, $C'$, $D'$. We now consider the labelling of points on the Hagge circle $\Gamma_4$, which is the Hagge circle of $P$ with respect to triangle $ABC$. They all carry the subscript 4. The reflection of $A'$ in $BC$ we denote by $A_4'$, the reflection of $B'$ in $CA$ we denote by $B_4'$ and the reflection of $C'$ in $AB$ we denote by $C_4'$. These replace the labels $U$, $V$, $W$ used in Article: CJB/2010/1 [1]. The points where $A_4'P$, $B_4'P$, $C_4'P$, $D_4P$ meet $\Gamma_4$ are denoted by $A_4$, $B_4$, $C_4$, $D_4'$ respectively. The first three of these replace the labels $X$, $Y$, $Z$ in Article 1. Points with suffices 1, 2, 3 are similarly defined on circles $\Gamma_1$, $\Gamma_2$, $\Gamma_3$. The labels $Pg_1$, $Pg_2$, $Pg_3$, $Pg_4$ are given to the isogonal conjugates of $P$ with respect to triangles $\Delta_1$, $\Delta_2$, $\Delta_3$, $\Delta_4$ respectively. The double lines of inverse symmetry through $P$ are denoted by $L$ and $L'$. The centre of the rectangular hyperbola $\Sigma$ is denoted by $M$.

In Section 3 we use Cartesian co-ordinates, origin $M$ and the asymptote parallel to $L$ is chosen as the $x$-axis. In this way $\Sigma$ has equation $xy = 1$ (by choice of scale), and we use a parameter $t$ on the hyperbola so that its points have co-ordinates $(t, 1/t)$. The points $A, B, C, D, P$ are given parameters $a, b, c, d, p$, where it is known, since $ABCD$ is cyclic, that $abcd = 1$ and the parameters of $A_1, B_2, C_3, D_4$, since they are the orthocentres, have parameters $-a, -b, -c, -d$ respectively.

The following theorems now hold:

Theorem 1
The centres, $Q_1, Q_2, Q_3, Q_4$, of the four Hagge circles are collinear.

Theorem 2
The points $A_1, A_2, A_3, A_4, A_1', A_2', A_3', A_4'$ are collinear and similarly for the points $B_k, B_k', k = 1, 2, 3, 4$ and $C_k, C_k', k = 1, 2, 3, 4$ and $D_k, D_k', k = 1, 2, 3, 4$. 

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Fig. 2

Illustrating the notation and results for the four Hagge circles

**Theorem 3**
The quadrilateral $Pg_1Pg_2Pg_3Pg_4$ is similar to the quadrilateral $A'B'C'D'$. 
3. The analysis

The co-ordinates of $O$

The chord $AB$ has equation
\[ aby + x = a + b. \] (3.1)

The midpoint of $AB$ has co-ordinates $(\frac{1}{2}(a + b), \frac{1}{2}(a + b)/(ab))$. It follows that the perpendicular bisector of $AB$ has equation
\[ 2ab(y - abx) = (a + b)(1 - a^2b^2). \] (3.2)

The perpendicular bisector of $BC$ has equation
\[ 2bc(y - bcx) = (b + c)(1 - b^2c^2). \] (3.3)

These meet at $O$, the centre of $\Gamma$, at the point with co-ordinates $(\frac{1}{2}(a + b + c + d), \frac{1}{2}(1/a + 1/b + 1/c + 1/d))$. Here we have used $abcd = 1$.

It can be shown that the radius of the circumcircle is
\[ \frac{1}{2}\sqrt{(a^2 + b^2 + c^2 + d^2 + 1/a^2 + 1/b^2 + 1/c^2 + 1/d^2)}. \] (3.4)

The equation of the circumcircle $\Gamma'$

From the above information this is easily proved to be
\[ 2x^2 - 2x(a + b + c + d) + 2y^2 - 2y(1/a + 1/b + 1/c + 1/d) + (ab + ac + ad + bc + bd + cd + 1/ab + 1/ac + 1/ae + 1/bc + 1/bd + 1/cd) = 0. \] (3.5)

Alternatively, in terms of $a$, $b$, $c$ only, it has the form
\[ abcx^2 - (abc(a + b + c) + 1)x + abcy^2 - (a^2b^2c^2 + ab + bc + ca)y + (abc)(ab + bc + ca) + a + b + c = 0. \] (3.6)

A digression

The reflection of the point with co-ordinates $(f, g)$ in the line with equation $lx + my = n$ is the point with co-ordinates $(h, k)$ where
\[ h = (ff(m^2 - l^2) + 2l(n - gm))/(l) \]
\[ k = (2mn + g(l^2 - m^2) - 2flm)/(l^2 + m^2). \]
This is straightforward and is left to the reader. This is used later to reflect \(ABCD\) in the line \(L\) and then dilate through \(P\) to obtain the Hagge circle \(A_4B_4C_4D_4\).

The indirect similarity

We now determine the equations governing the indirect similarity which maps the circumcircle into the Hagge circle of \(P\) with respect to \(\Delta_4\). This is done by following what happens to the point \(D(d, 1/d)\). We know the image of this point is the orthocentre \(D_4(-1/abc, -abc)\) and we know the mapping is effected by a reflection in the line \(L\), with equation \(y = 1/p\) followed by an enlargement (reduction) by a factor \(PD_4/PD\). As far as the reflection is concerned we may use the analysis of the last paragraph with \(l = 0, m = 1, n = 1/p\). The result of this reflection on \(D\) is to produce the point with coordinates \((d, 2/p - 1/d)\).

The enlargement (reduction) factor (comparing \(x\)-co-ordinates of the points concerned) is equal to \((p + d)/(p - d) = (abc + 1)/(abc - 1)\). It may now be checked that the reflection in \(L\) followed by the dilation with this enlargement (reduction) factor takes the point with coordinates \((h, k)\) to the point with coordinates \((x, y)\) where

\[
\begin{align*}
x &= (abchp - 2p + h)/(abc - 1), \quad (3.7) \\
y &= (abc(kp - 2) + k)/(1 - abc).
\end{align*}
\]

Using \(h = a, k = 1/a\) we deduce the coordinates of \(A_4\) to be

\[
((a^2bcp + a - 2p)/(abc - 1), (2a^2bc - abc - 1)/(a(abc - 1))),
\]

with similar expressions for the co-ordinates of \(B_4\) and \(C_4\) by using \(b\) and \(c\), in \(x\) and \(y\) above, instead of \(a\).

The equations of the Hagge circles

From here it is quite an elaborate calculation to obtain the equation of the Hagge circle \(A_4B_4C_4\) and check the fundamental theorem that \(D_4\) lies on this circle. The computer algebra package \(DERIVE\) was used to perform the calculations, and the result is that \(\Gamma_4\) has equation

\[
\begin{align*}
abc(abcp - 1)(x^2 + y^2) &- (a^2b^2c^2p(a + b + c) + abc(a + b + c) - 3abc + 1)x \\
+ (a^3b^3c^3p - 3a^2b^2c^2 + (abc + 1)(bc + ca + ab))y \\
- 2a^3b^3c^3 + a^2b^2c^2p(a + b + c) + abc(bc + ca + ab) - (abc + 1)(a + b + c) - 2p &= 0. 
\end{align*}
\] (3.9)

The equations of the other Hagge circles follow immediately by using other triplets of parameters instead of \(a, b, c\). The co-ordinates of the centres of these circles may now be written down and it has been checked, using \(DERIVE\), that any three of the four centres are collinear. Theorem 1 is now proved, but it also follows from a geometrical argument. The centre \(O\) of the...
circumcircle is the common circumcentre of all four triangles \( \Delta_1, \Delta_2, \Delta_3, \Delta_4 \). If we now reflect \( O \) in the two lines of inverse similarity, it maps into a pair of points collinear with \( P \). Since the centres of the four Hagge circles are now the images of these points by means of dilations through \( P \) with enlargements (reductions) using different factors, the resulting images all lie on this line, and this line also passes through \( P \). Theorem 2 follows by a similar argument, bearing in mind that we already know primed and unprimed pairs of points such as \( A_4, A_4' \) are collinear with \( P \).

**The quadrilateral \( A'B'C'D' \)**
The equation of the line \( AP \) is

\[
x + apy = a + p.
\] (3.10)

This meets the circumcircle again at the point \( A' \) with co-ordinates \((x,y)\) where

\[
x = \{a p^2 (a b(c + 1) + 1) - p(a^2 b^2 c^2 + a(b + c) - b c) + a b c\}/\{b c(a^2 p^2 + 1)\},
\] (3.11)

\[
y = \{a b c p^2 + p(a^2 b c - a b c(b + c) - 1) + a b^2 c^2 + b + c\}/\{b c(a^2 p^2 + 1)\}.
\] (3.12)

The point \( D' \) has co-ordinates similar to these, but with \( d \) replacing \( a \).

From these co-ordinates we can work out \((A'D')^2\) and the result is

\[
\{(a - d)^2(b - p)^2(c - p)^2(b^2 c^2 + 1)\}/\{b^2 c^2(a^2 p^2 + 1)(d^2 p^2 + 1)\}. \] (3.13)

**The quadrilateral \( P_g_1 P_g_2 P_g_3 P_g_4 \)**

The co-ordinates \((h,k)\) of \( P_{g_4} \), the isogonal conjugate of \( P \) with respect to triangle \( ABC \) are given by

\[
h = (a + b + c - p)/(1 - abcp),
\] (3.14)

\[
k = (p(b c + c a + a b) - a b c)\/(abcp - 1). \] (3.15)

This may be checked as follows: Let the line from \( A \) to \((h,k)\) meet \( \Gamma \) at \( A'' \), then it is easy to show that \( A'A'' \) is parallel to \( BC \). The symmetry of \( h, k \) with respect to \( a, b, c \) now proves that \((h,k)\) are the co-ordinates of \( P_{g_4} \), as this calculation reflects one of the standard constructions for an isogonal conjugate. Indeed the parallel \( A'A'' \) to \( BC \) indicates that \( AP \) and \( AP_{g_4} \) are reflections of each other in the internal bisector of angle \( A \). The co-ordinates of \( P_{g_1} \) now follow from \((h,k)\) by exchanging \( a \) and \( d \).

From these co-ordinates we can work out \((P_{g_1} P_{g_4})^2\) and the result is

\[
\{(a - d)^2(b^2 c^2 + 1)(b^2 p^2 + 1)(c^2 p^2 + 1)\}/\{(abcp - 1)^2(b c d p - 1)^2\}. \] (3.16)

**The similarity of the quadrangles**
The ratio of the squares of the side lengths and diagonals of the quadrangles \( P_{g_1} P_{g_2} P_{g_3} P_{g_4} \) and \( A'B'C'D' \) may now be calculated and is the totally symmetric expression

\[
\{(a^2 p^2 + 1)(b^2 p^2 + 1)(c^2 p^2 + 1)(d^2 p^2 + 1)\}/\{(a - p)^2(b - p)^2(c - p)^2(d - p)^2\}. \] (3.17)
Flat 4,
Terrill Court,
12-14 Apsley Road,
BRISTOL BS8 2SP