Galois Groups in Rational Conformal Field Theory II.

The Discriminant.

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ABSTRACT

We express the discriminant of the polynomial relations of the fusion ring, in any conformal field theory, as the product of the rows of the modular matrix to the power $-2$. The discriminant is shown to be an integer, always, which is a product of primes which divide the level. Detailed formulas for the discriminant are given for all WZW conformal field theories.
The classification of rational conformal field theories is one of the intriguing problems in relation to string theory and to critical phenomena. The study of such theories were initiated in the seminal works [1, 2, 3, 4].

Here we continue the work of ref. [5], where some equations were derived for the fusion ring. Our aim here is to define the discriminant of the relations of the fusion ring. This is expressed as the product of the rows of the modular matrix to the power $-2$. It is shown to be an integer which has the striking property of being a product of primes dividing the level. We hope that these results will help in the classification of rational conformal field theories.

From the discussion in ref. [5], we recall the theorem proven originally in ref. [6], which states as follows:

**Theorem (1):** Any fusion ring is a ring of polynomials $\mathbb{Z}[x_1, x_2, \ldots, x_m]/I$ where $I$ is the ideal of all the polynomials vanishing on the points of the fusion variety, $x^i_\alpha = \frac{S^i_\alpha}{S_0^\alpha}$, where $\alpha = 1, 2, \ldots, m$, label the generators, and $i$ denotes the different points of the variety, whose number is the number of primary fields. Furthermore, the value of any primary field $[a]$ evaluated on any of the points of the fusion variety is given by $p_a(x^1_1, x^2_1, \ldots, x^m_1) = \frac{S^i_{a,i}}{S^i_{0,i}}$.

For further explanation and a proof see refs. [5, 6].

For simplicity, we assume that the fusion ring is generated by one primary field, which we label as $x = [1]$. It follows that the fusion ring is given by the quotient,

$$ R \approx \frac{\mathbb{Z}[x]}{(q(x))}, $$

where $\mathbb{Z}[x]$ is the ring of polynomials in $x$ with integer coefficients, and $q(x)$ is the relation in this ring. We can write the primary field in this ring as $p_r(x) = x^r + a_1 x^{r-1} + \ldots$, where by convention $p_0(x) = 1$ and $p_1(x) = x$ and $r$ goes from
0 to $n - 1$. Using the theorem we may write the following matrix equation,

$$M_{i\alpha} = \frac{S^\dagger_{i\alpha}}{S^\dagger_{0\alpha}} = \begin{pmatrix}
1 & 1 & \ldots & 1 \\
x_0 & x_1 & \ldots & x_{n-1} \\
p_2(x_0) & p_2(x_1) & \ldots & p_2(x_{n-1}) \\
\vdots & \vdots & \ddots & \vdots \\
p_{n-1}(x_0) & p_{n-1}(x_1) & \ldots & p_{n-1}(x_{n-1})
\end{pmatrix}, \quad (2)
$$

where $x_0, x_1, \ldots, x_{n-1}$ label the points of the fusion variety, i.e., the solutions of the equation $q(x) = 0$.

Now, take the determinant of both sides of eq. (2). On the right side, since we can eliminate the lower order terms by adding rows recursively, we have the Vandermonde determinant,

$$\det(M) = \prod_{i<j}(x_i - x_j), \quad (3)$$

As is well known, this determinant is given as the product,

$$\det(M) = \prod_{i<j}(x_i - x_j). \quad (4)$$

Squaring the Vandermonde determinant we get the discriminant,

$$D = \prod_{i<j}(x_i - x_j)^2, \quad (5)$$

which since it is a symmetric polynomial in $x_i$ can be expressed in terms of the coefficients of the polynomial $q(x)$. Say,

$$q(x) = x^m - S_1x^{m-1} + S_2x^{m-2} - \ldots + (-1)^mS_m, \quad (6)$$

where the $S_r$ are the symmetric polynomials of the roots,

$$S_r = \prod_{i_1 < i_2 < \ldots < i_r} x_{i_1}x_{i_2}\ldots x_{i_r}. \quad (7)$$

The discriminant may easily be computed for any equation. For example, see the appendix of ref. [6] for a Mathematica program. We quote the result for
equations of order 2 and 3,

\[ R_2 = S_1^2 - 4S_2, \]
\[ R_3 = S_1^2S_2^2 - 4S_2^3 - 4S_1^2S_3 + 18S_1S_2S_3 - 27S_3^2. \]  

(8)

Very importantly, since the coefficients \( S_i \) are integers, the discriminant is always an integer.

Now, consider the left hand side of eq. (2). Since \( S \) obeys, \( S^2 = C \), where \( C \) is the charge conjugation matrix, \( \det(S) = i^s \), where \( s \) is some integer. So we find for the determinant,

\[ \det(M) = i^s \prod_i S_{i0}^{-1}. \]  

(9)

Comparing eqs. (4,9), we find immediately an expression for the discriminant,

\[ |D| = \prod_i S_{i0}^{-2}. \]  

(10)

Since we are free to act with an element of the Galois group, \( G \), we find also that

\[ |D| = \prod_i S_{i,a}^{-2}, \]  

(11)

where \( a \) is any primary field in the path of the Galois element,

\[ a = t(0), \quad \text{where } t \in G. \]  

(12)

Quite strikingly, since the discriminant is an integer, we find that the product of the elements of the modular matrix is always the inverse square root of an integer,

\[ D = \prod_i S_{0,i}^{-2} = \text{integer}. \]  

(13)

Although our proof utilized the assumption of one generator, we believe that eq. (13), holds for any number of generators since it depends only on the product
of the rows of the modular matrix. In particular, since the generator is not used, the discriminant is independent of the choice of generators, and is always the same for any generator.

Actually, we may calculate the determinant with a generator which is not a primary field, thus proving that eq. (13) holds for any number of generators.

Of course, since the points \( x_i \) have to be distinct, which is a consequence of the semi–simplicity of the ring, the discriminant \( D \), cannot vanish.

For example, take the WZW model \( SU(2)_2 \). Here the first row of the modular matrix reads,

\[
S_{i0} = \left( 1/2, 1/\sqrt{2}, 1/2 \right).
\]

and so the product of the row to the power of \(-2\) is,

\[
D = 32.
\]

The relation for this ring is \( q(x) = x^3 - 2x \), and so from eq. (8), we find for the discriminant,

\[
R_3 = -4S_2^3 = 32,
\]

agreeing with eq. (15).

Before proceeding, it will be useful to work out some examples. For WZW models we find the following formula to calculate the modular matrix very useful,

\[
S_{0,\lambda} = M \prod_{\alpha > 0} \sin \left[ \frac{\pi(\lambda + \rho)\alpha}{k + g} \right],
\]

where \( \lambda \) is a highest weight at the central charge \( k \), \( \rho \) is half the sum of positive roots, \( g \) is the dual Coxeter number, and \( \alpha \) runs over all the positive roots. \( M \) is some normalization independent of \( \lambda \).
For $SU(2)_k$ WZW model we have for $S$,

$$S_{0,r} = \sqrt{\frac{2}{k+2}} \sin\left[\frac{\pi(r+1)}{k+2}\right],$$

(18)

where $r = 0, 1, \ldots, k$, is twice the isospin of the primary fields. Multiplying the $S$ matrix according to eq. (13), we find for the discriminant,

$$D = 2^{k+1}(k+2)^{k-1},$$

(19)

which is obviously an integer.

As a further example take the theory $SU(2)_{2n-1}/Z_2$. This theory is composed from the integer isospin representations of $SU(2)$ at an odd level [7]. Denoting the fields by $l = 1, 2, \ldots, n$, we have for the modular matrix,

$$S_{l,m} = \sqrt{\frac{4}{2n+1}} \sin\left[\frac{\pi(2l-1)(2m-1)}{2n+1}\right],$$

(20)

where $2l - 1$ is twice the isospin plus one. For a discussion of the Galois group of these theories see the appendix of ref. [5]. Multiplying the rows of the $S$ matrix according to eq. (13), we find for the discriminant,

$$D = \prod_l S_{l0}^{-2} = (2n+1)^{n-1}. $$

(21)

It will be useful to work out an example which is not related to current algebra. This is a theory with six primary fields, $\phi_1, \phi_2, \ldots, \phi_6$, where by convention $\phi_1 = 1$. 
The fusion rules of the model are given by,

\[ \begin{align*}
\phi_2^2 &= 1 + 2\phi_2 + \phi_3 + \phi_4 + 2\phi_5 + \phi_6 \\
\phi_2\phi_3 &= \phi_2 + \phi_4 + \phi_5 + \phi_6 \\
\phi_2\phi_4 &= \phi_2 + \phi_3 + \phi_5 + \phi_6 \\
\phi_2\phi_5 &= 2\phi_2 + \phi_3 + \phi_4 + \phi_5 + \phi_6 \\
\phi_2\phi_6 &= \phi_2 + \phi_3 + \phi_4 + \phi_5 \\
\phi_3^2 &= 1 + \phi_4 + \phi_5 \\
\phi_3\phi_4 &= \phi_2 + \phi_3 \\
\phi_3\phi_5 &= \phi_2 + \phi_3 + \phi_5 \\
\phi_3\phi_6 &= \phi_2 + \phi_6 \\
\phi_4^2 &= 1 + \phi_5 + \phi_6 \\
\phi_4\phi_5 &= \phi_2 + \phi_4 + \phi_5 \\
\phi_4\phi_6 &= \phi_2 + \phi_4 \\
\phi_5^2 &= 1 + \phi_2 + \phi_3 + \phi_4 + \phi_5 + \phi_6 \\
\phi_5\phi_6 &= \phi_2 + \phi_5 + \phi_6 \\
\phi_6^2 &= 1 + \phi_3 + \phi_5
\end{align*} \] (22)

The first row of the modular matrix corresponding to this theory can be seen to be, to eleven digits precision,

\[ S_{l,0} := \{1/3, 1/3, 0.62646174719, 0.51069629541, 1/3, 0.11576545177\}. \] (23)

Multiplying these numbers, we find the discriminant to be,

\[ D = \prod_l S_{l,0}^{-2} = 531441 = 3^{12}. \] (24)

From these examples alone we already see an interesting property of the discriminant. Denote by \( N \) the level of the conformal field theory. This is defined as
the least integer such that the dimensions of the theory all obey,

$$\Delta N = \text{integer},$$  \hspace{1cm} (25)

where $\Delta$ is any of the dimensions, and $N$ is the level, thus defined. We observe the following intriguing arithmetical result about the discriminant.

**Theorem (2):** The discriminant is always a product of primes which divide the level,

$$|D| = \prod_a p_a^{n_a}, \quad \text{where } p_a | N,$$  \hspace{1cm} (26)

and the $n_a$ are some non-negative integer powers.

In the case of $SU(2)_k$ the level is $2(k+2)$ and so eq. (26) is obeyed, using eq. (19). For $SU(2)_{2n-1}/Z_2$ the level is $2n+1$ so again eq. (26) holds, when compared with eq. (21). Finally for the non WZW example, using Vafa equations, [8], it can be shown that the level is 36, and so again the arithmetic properties, eq. (26) are obeyed.

Actually, it is not hard to prove directly this result. Consider the fusion ring $R$ over the field of integers modulo $p$, where $p$ is some prime which is strange to the level, $Z_p$. Since the fusion coefficients are integers, we are allowed to do that.

Now, the matrix $M$ is given here by the same expression, eq. (2), but now the coefficients are in the Galois Field of the equation $q(x) = 0$, now over $Z_p$. To get $S$ we must normalize $M$ such that each row is of length one. The only obstruction for doing so is the coefficient in front of $S$ which is a square root of the level. Thus, if $p$ is strange to the level, we obtain a unitary matrix $S$, $SS^\dagger = 1 \mod p$, and so

$$\det S \neq 0 \mod p.$$  \hspace{1cm} (27)

This implies that the points of the fusion variety are distinct modulo $p$ and so the discriminant cannot vanish,

$$D \neq 0 \mod p.$$  \hspace{1cm} (27)

Thus the discriminant has to be a product of primes dividing the level, proving
the theorem. This proof works for any number of generators assuming only that the discriminant is an integer.

We turn now to the calculation of the discriminant for WZW models based on any group $G$. Before proceeding, we have to establish the following result, or conjecture.

Assume some simple Lie algebra, $G$. Denote by $\bar{g}_i$, $i = 1, 2, \ldots, n + 1$, the dual Coxeter numbers of the algebra, and by $g_i$ the Coxeter numbers. Let $m_i$, $i = 1, 2, \ldots, n$, be the exponents of the algebra, where $n$ is the rank of the algebra. Let $M_k$ denote the number of primary fields in the Dual affine theory, at the level $k$. Explicitly, $M_k$ is the number of solutions to the equation,

$$
\sum_i n_i g_i = k,
$$

where $n_i$ are some non-negative integers. It is clear that a generating function for $M_k$ is

$$
\sum_{k=0}^{\infty} M_k z^k = \prod_{i=1}^{n+1} (1 - z^{\bar{g}_i})^{-1},
$$

proven by expanding the right hand side of this equation.

The result that we need is as follows.

**Conjecture (1):** For level $k$ which are not zero modulo any of the Coxeter numbers, $g_i$, the number of primary fields in the Dual algebra, $M_k$, is given by

$$
M_k = s \prod_{i=1}^{n} \frac{k + m_i}{m_i + 1},
$$

where $s$ is the number of primary fields at the level $k = 1$, $s = M_1$.

At the table the Coxeter numbers and the exponents of the different Lie algebras are listed. We can check this conjecture for all the algebras, which are of
Table.
Exponents and Coxeter Numbers of simple Lie algebras.

| Algebra | Coxeter Numbers | Dual Numbers | Exponents |
|---------|----------------|--------------|-----------|
| $A_l$   | $1, 1, 1, \ldots, 1$ | $1, 1, 1, \ldots, 1$ | $1, 2, 3, \ldots, l$ |
| $B_l$   | $1, 1, 2, 2, \ldots, 2$ | $1, 1, 1, 2, 2, \ldots, 2$ | $1, 3, 5, \ldots, 2l - 1$ |
| $C_l$   | $1, 1, 2, 2, \ldots, 2$ | $1, 1, \ldots, 1$ | $1, 3, 5, \ldots, 2l - 1$ |
| $D_l$   | $1, 1, 1, 2, 2, \ldots, 2$ | $1, 1, 1, 2, 2, \ldots, 2$ | $1, 3, 5, \ldots, 2l - 3, l - 1$ |
| $E_6$   | $1, 1, 1, 2, 2, 2, 3$ | $1, 1, 1, 2, 2, 2, 3$ | $1, 4, 5, 7, 8, 11$ |
| $E_7$   | $1, 1, 2, 2, 2, 3, 3, 4$ | $1, 1, 2, 2, 2, 3, 3, 4$ | $1, 5, 7, 9, 11, 13, 17$ |
| $E_8$   | $1, 2, 3, 4, 5, 6, 3, 4, 2$ | $1, 2, 3, 4, 5, 6, 3, 4, 2$ | $1, 7, 11, 13, 17, 19, 23, 29$ |
| $F_4$   | $1, 2, 3, 4, 2$ | $1, 2, 3, 2, 1$ | $1, 5, 7, 11$ |
| $G_2$   | $1, 2, 3$ | $1, 2, 1$ | $1, 5$ |

types $A_n$, $B_n$, $C_n$, $D_n$, $G_2$, $F_4$ and $E_r$, for $r = 6, 7, 8$, for small levels, and it is always obeyed.

As an example of conjecture (1) consider $A_l$ (or $SU(l+1)$). Substituting the exponents from the table we arrive at,

$$M_k = (k + 1)(k + 2)\ldots(k + l)/l!, \quad (31)$$

which can easily be verified directly by using the binomial theorem on eq. (29).

Before proceeding let us collect some data about the simple Lie algebras. We denote my $R$ the index of the algebra which is defined as the number of elements,

$$R = |M/M^*|, \quad (32)$$

where $M$ is the long root lattice and $M^*$ its dual, i.e., the weight lattice. The index of the various algebras are, $A_l$, $l + 1$. $B_l$, 2. $C_l$, 2. $D_l$, 4. $E_6$, 3. $E_7$, 2. $E_8$, 1. $F_4$, 2. $G_2$, 3.
We shall need also the total Coxeter and dual Coxeter numbers, which are the sums of the respective Coxeter numbers, which we denote by $h$ and $g$. For simply laced algebras the two numbers are the same, $g = h$, and we have, $A_l$, $l + 1$. $D_l$, $2l - 2$. $E_6$, 12. $E_7$, 18. $E_8$, 30. For the non-simply laced algebras we have: For $B_l$, $h = 2l$, $g = 2l - 1$. For $C_l$, $h = 2l$ and $g = l + 1$. For $F_4$ we have, $h = 12$ and $g = 9$. For $G_2$ we have, $h = 6$ and $g = 4$.

The level of the WZW model based on a group $G$ at the central charge $k$ is given by

$$N = (k + g)|M/M^*| = R(k + g).$$

(33)

Thus, we expect the discriminant to be a product of primes which divide $N$. Our result is actually stronger, giving the exact value of the discriminant, and we will state it in stages. First, we have the following,

**Conjecture (2a):** The discriminant, $D$ for any WZW based on the algebra $G$, and for the central charge $k$, such that $k + h - g \neq 0 \mod \tilde{g}_i$, for any $i$, is the integer

$$D = R^m(k + g)^r,$$

(34)

where $r$ and $m$ are integers which are expressed as polynomials in $k$ of the order rank$(G) = n$.

We shall give the exact expression for the powers $m$ and $r$. We find it convenient to separate the simply laced case, $A, D, E$, from the not simply laced cases, $B, C, G, F$. The result for the simply laced algebras is:

**Conjecture (2b):** For the simply laced algebras, and the level as in Conjecture (2a), the discriminant is given by eq. (34), where the power $m$ is equal to the number of primary fields at the central charge $k$, and the power $r$ is given by,

$$r = r_0m(k - 1)/(k + g - 1),$$

(35)

where $r_0$ is some rational coefficient to be listed below.
Using conjecture (1) we may express the number of primary fields through the exponents,

\[ m = c \prod \frac{k + m_i}{1 + m_i}, \]  

(36)

where \( c \) is the number of primary fields at the central charge one. Since \( g - 1 \) is always an exponent, we see that we can divide by \( k + g - 1 \), as we do in eq. (36).

We can list explicitly, the polynomial expressions for \( r \) and \( m \), for each of the simply laced algebras. These follow by substituting the exponents from the table.

For \( A_l \) at any central charge \( k \), we have,

\[ D = (l + 1)^m(k + l + 1)^r, \]  

(37)

where

\[ m = (k + 1)(k + 2) \ldots (k + l)/l!, \]  

(38)

and \( r \) is,

\[ r = m(k - 1)l/(k + l) = (k - 1)(k + 1)(k + 2) \ldots (k + l - 1)/(l - 1)! . \]  

(39)

For \( D_l \) we have, for \( k \neq 0 \mod 2 \), or for odd \( k \),

\[ D = 4^m(k + 2l - 2)^r, \]  

(40)

where

\[ m = (k + 1)(k + 3)(k + 5) \ldots (k + 2l - 3)(k + l - 1)/(2^{l-3}l!). \]  

(41)

For \( r \) we have,

\[ r = (k-1)lm/(k+2l-3) = (k-1)(k+1)(k+3) \ldots (k+2l-5)(k+l-1)/(2^{l-3}(l-1)!]. \]  

(42)
For $E_6$ we find for the central charge $k$, which is not divisible by 2 or 3,

$$D = 3^m(k + 12)^r,$$

where

$$m = (k + 1)(k + 4)(k + 5)(k + 7)(k + 8)(k + 11) \cdot 3/(2 \cdot 5 \cdot 6 \cdot 8 \cdot 9 \cdot 12),$$

is the number of primary fields, and the power $r$ is

$$r = (k - 1)(k + 1)(k + 4)(k + 5)(k + 7)(k + 8) \cdot 3/(4 \cdot 5 \cdot 6 \cdot 8 \cdot 9).$$

For $E_7$ we have, for any central charge not divisible by 2 or 3,

$$D = 2^m(k + 18)^r,$$

where $m$ is the number of primary fields,

$$m = (k + 1)(k + 5)(k + 7)(k + 9)(k + 11)(k + 13)(k + 17)/(6 \cdot 8 \cdot 10 \cdot 12 \cdot 14 \cdot 18),$$

and

$$r = (k - 1)(k + 1)(k + 5)(k + 7)(k + 9)(k + 11)(k + 13) \cdot 7/(9 \cdot 2 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot 14).$$

For $E_8$ we have, for any central charge, $k$, not divisible by 2, 3, 5,

$$D = (k + 30)^r.$$

Here, since $R = |M/M^*| = 1$ (it is a self dual lattice), we have only the power $r$, and $m$ is irrelevant. However, the same formula, eq. (35), holds for the power $r$,

$$r = (k-1)(k+1)(k+7)(k+11)(k+13)(k+17)(k+19)(k+23) \cdot 4/(15 \cdot 2 \cdot 8 \cdot 12 \cdot 14 \cdot 18 \cdot 20 \cdot 24).$$

We see that these are indeed polynomials of order rank($G$), in accordance with Conjecture (2a).
It will be useful at this stage to work out some cases. So, take $SU(4)$ or $A_3$, as an example. From conjecture (2b) we have

$$D = 4^m (k + 4)^r,$$

where

$$m = (k + 1)(k + 2)(k + 3)/6,$$

is the number of primary fields at the central charge $k$, and $r$ is

$$r = (k - 1)(k + 1)(k + 2)/2.$$

Substituting some low $k$'s we get, for the discriminant, $D(k)$ where $k$ is the central charge, $D(1) = 2^8$, $D(2) = 2^{26}3^6$, $D(3) = 2^{40}7^{20}$, $D(4) = 2^{205}$, $D(5) = 2^{112}3^{168}$, $D(6) = 2^{308}5^{140}$, $D(7) = 2^{240}11^{216}$, etc. These numbers can be calculated directly from eq. (17). A sample program is given in the appendix. Substituting into eqs. (52,53), we get the exact same numbers. We see that the numbers tend to grow very fast, for example, $D(7)$ has 297 digits. It is thus, rather striking, that these are integers which are product of small primes, i.e., the divisors of the level, $N$.

Let us turn now to the non-simply laced case. Here we have the following conjecture.

**Conjecture (2c):** For the non simply laced algebras $B_n$, $C_n$, $F_4$ and $G_2$, the discriminant is given by

$$R^m(k + g)^r,$$

where $m$ is proportional to the number of primary fields in the dual algebra, shifted by $h - g$, except for $C_n$ which will be listed separately. Using Conjecture (1) we may write,

$$m = m_0 \prod_i \frac{k + m_i + g - h}{1 + m_i},$$

where $m_0$ is the number of primary fields for $k = 1 + h - g$. The level $k$ is assumed
to obey,

\[ k + g - h \neq 0 \mod \tilde{g}_i, \]  \hspace{1cm} (56)

where \( \tilde{g}_i \) are any of the Dual Coxeter numbers. For \( r \) we have a formula similar from before,

\[ r = r_0 m(k + a)/(k + g - 1), \]  \hspace{1cm} (57)

where \( a \) is some number and \( r_0 \) some normalization. An exception for this formula is for \( G_2 \) which we will list explicitly below.

For \( B_n \) we have,

\[ D = 2^m(k + 2n - 1)^r, \]  \hspace{1cm} (58)

and in accordance with Conjecture (2c),

\[ m = k(k + 2)(k + 4) \ldots (k + 2n - 2)/(6 \cdot 8 \cdot 10 \cdot \ldots \cdot (2n)), \]  \hspace{1cm} (59)

for all even \( k \). For \( r \) we have,

\[ r = k(k + 2)(k + 4) \ldots (k + 2n - 4)[k + (n - 2)/2]/(4 \cdot 6 \cdot \ldots \cdot (2n - 2)). \]  \hspace{1cm} (60)

For \( C_n \) we have

\[ D = 2^m(k + n + 1)^r. \]  \hspace{1cm} (61)

For \( m \) we have the exceptional formula, where half of the exponents ‘mutate’:

\[ m = k(k + 1) \ldots (k + n - 2)(k + n)/(n - 1)!, \]  \hspace{1cm} (62)

and \( r \) is given by eq. (57),

\[ r = k(k + 1) \ldots (k + n - 3)(k + n - 2)^2/(n - 1)!. \]  \hspace{1cm} (63)

Since for \( C_n \) the dual Coxeter numbers are 1, these formulas hold for all the central charges, \( k \).
For $F_4$ we have,

$$D = 2^m (k+9)^r,$$

where $m$ is again the number of primary fields in the dual algebra, shifted by $h - g = 3$.

$$m = (k-2)(k+2)(k+4)(k+8)/(8\cdot18),$$

where $k - 3 \neq 0 \mod 2,3$, i.e., $k$ is an even number not divisible by 3. For $r$ we find, again as an exception to conjecture (2c),

$$r = (k+1)(k+2)^2(k+4)/72.$$

Here two exponents have been mutated.

For $G_2$ we find

$$D = 3^m (k+4)^r,$$

where $m$ is given by conjecture (2c),

$$m = (k-1)(k+3)/2,$$

and $k$ is any odd number. The formula for $r$ is exceptional,

$$r = k^2 + 1.$$

This formulae summarizes all the algebras, albeit, only for central charges $k$ obeying $k + g - h \neq 0 \mod \tilde{g}_i$. For other $k$, not covered by conjecture (2), the factorization, eq. (34), does not always hold. In some cases, it does hold with polynomial expressions for the powers, but, this is not always the case. We leave the determination of these cases to further work. Of course, in accordance with theorem (2), the discriminant is always an integer which is a product of primes dividing the level, and we checked this for all the algebras, with small $k$ not covered by conjecture (2).
We presented here the rather intriguing algebraic and arithmetical properties of the discriminant. We hope that this would lead to an arithmetic approach to the classification of RCFT problem. In particular, one may ask what values of the discriminant may appear in RCFT. Since the discriminant is an integer which is a product of primes dividing the level, this question is of a particular significance.

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APPENDIX

An example program for calculating the discriminant.

Below is a sample Mathematica program which calculates the discriminant for $SU(4)$ level $k$, with an example of $k = 6$. The program uses eq. (17) and the explicit form of the $A_3$ root system.

Here the weight is $mf_1 + nf_2 + lf_3$, where $f_i$ are the three fundamental weights. For $\rho$ we take, $\rho = f_1 + f_2 + f_3$. We take for the positive roots in eq. (17) their explicit form, $a_1, a_2, a_3, a_1 + a_2, a_2 + a_3$ and $a_1 + a_2 + a_3$, where $a_1, a_2, a_3$ are the three simple roots. To calculate eq. (17), we use the fact that $f_i a_j = \delta_{i,j}$.

Here $SA4$ calculates the modular matrix for $SU(4)$, and $FA4$ calculates the discriminant as a product of the row of the modular matrix to the power $-2$.

```mathematica
SA4[m_, n_, l_] := N[(Sin[(m + n + l + 3) Pi/(k + 4)] Sin[Pi (m + n + 2)/(k + 4)] Sin[Pi (n + l + 2)/(k + 4)] Sin[Pi (m + 1)/(k + 4)] Sin[Pi (l + 1)/(k + 4)] Sin[Pi (n + 1)/(k + 4)]), ss];
FA4 := Function[k, dx := 0; Do[If[n1 + n2 + n3 <= k, dx = N[dx + SA4[n1, n2, n3]^2, ss]], {n1, 0, k}, {n2, 0, k}, {n3, 0, k}]; dd := 1; Do[If[n1 + n2 + n3 <= k, dd = N[dd SA4[n1, n2, n3]/Sqrt[dx], ss]], {n1, 0, k}, {n2, 0, k}, {n3, 0, k}]; N[dd^(-2), ss]];
k := 6; ss := 280;
FactorInteger[Round[FA4[k]]]

{{2, 308}, {5, 140}}
```