ACTION INTEGRALS ALONG CLOSED ISOTopies IN COADJOINT ORBITS

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Abstract. Let $O$ be the orbit of $\eta \in g^*$ under the coadjoint action of the compact Lie group $G$. We give two formulae for calculating the action integral along a closed Hamiltonian isotopy on $O$. The first one expresses this action in terms of a particular character of the isotropy subgroup of $\eta$. In the second one is involved the character of an irreducible representation of $G$.

1. Introduction

Let $(M, \omega)$ be a quantizable symplectic manifold. We will denote by $\text{Ham}(M)$ the group of Hamiltonian symplectomorphisms of $(M, \omega)$. In this note we will consider loops $\{\psi_t\}_{t \in [0,1]}$ in $\text{Ham}(M)$ at id.

Given $q \in M$, the loop $\psi$ generates the closed curve $\{\psi_t(q) \mid t \in [0,1]\}$ in $M$ which is homologous to zero [3, page 334]. As $(M, \omega)$ is quantizable, it makes sense to define the action integral $A_{\psi}(q)$ along such a curve as the element of $\mathbb{R}/\mathbb{Z}$ given by the formula [11]

$$A_{\psi}(q) = \int_S \omega - \int_0^1 f_t(\psi_t(q)) dt + \mathbb{Z},$$

(1.1)

where $S$ is any 2-surface whose boundary is the curve $\{\psi_t(q)\}$, and where $f_t$ a fixed time dependent Hamiltonian associated to $\{\psi_t\}$.

Since $(M, \omega)$ is quantizable, one can choose a prequantum bundle $L$ on $M$, endowed with a connection $D$ [13]. On the other hand, let $X_t$ be the corresponding Hamiltonian vector fields determined by $f_t$, then one can construct the operator $\mathcal{P}_t := -D_{X_t} - 2\pi i f_t$, which acts on the sections of $L$. The equation $\dot{\tau}_t = \mathcal{P}_t(\tau_t)$ defines a “transport” of the section $\tau_0 \in C^\infty(L)$ along $\psi_t$. This transport enjoys the following nice property: If $D_Y \tau_0 = 0$, with $Y$ a vector field on $M$, then $D_Y \tau_t = 0$, for

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\( Y_t = \psi_t(Y) \) (see [1]). From this fact one can prove that \( \tau_1 \) and \( \tau_0 \) differ in a constant factor \( \kappa(\psi) \); that is, \( \tau_1 = \kappa(\psi)\tau_0 \). A direct calculation shows that \( \kappa(\psi) = \exp(2\pi i A_\psi(q)) \), where \( q \) is an arbitrary point of \( M \) [10]. Consequently the action integral along \( \psi \) by (1.1).

The purpose of this note is to calculate the value of the invariant \( \kappa(\psi) \) when the manifold \( M \) is a coadjoint orbit [3] of a compact Lie group. However in Section 2 we study a more general situation. If a Lie group \( G \) acts on the manifold \( M \) by symplectomorphisms and there is a moment map for this action, then each \( A \in \mathfrak{g} \) determines a vector field \( X_A \) on \( M \) and the corresponding Hamiltonian \( f_A \). We can construct the respective operator \( P_A \) on \( C^\infty(L) \), so one has a representation \( P \) of the Lie algebra \( \mathfrak{g} \) on \( C^\infty(L) \). When this representation extends to an action \( \rho \) of the group \( G \), the prequantization is said to be \( G \)-invariant.

In this case we will prove that the value of \( \kappa(\psi) \) can be expressed in terms of \( \rho \). More precisely, if the isotopy \( \psi_t \) is determined by vector fields of type \( X_A_t \) we show that \( \tau_t = \rho(h_t)\tau_0 \), where \( h_t \) is the solution to Lax equation \( h_t h_t^{-1} = A_t \).

Section 3 is concerned with the invariant \( \kappa(\psi) \) for closed isotopies \( \psi \) in a coadjoint orbit of a compact Lie group \( G \). We study the value of \( \kappa(\psi) \), when the isotopy is defined by vector fields of type \( X_A \). Given \( \eta \in \mathfrak{g}^* \), the orbit \( O_\eta \) of \( \eta \) admits a \( G \)-invariant prequantization if the prequantum bundle is defined by a character \( \Lambda \) of \( G_\eta \), the subgroup of isotropy of \( \eta \). In this case we prove that \( \kappa(\psi) = \Lambda(h_1) \), with \( h_t \) the solution to the corresponding Lax equation (Theorem 3).

If \( G \) is a semisimple group, the choice of a maximal torus \( T \) contained in \( G_\eta \) permits us to define a \( G \)-invariant complex structure on \( G/G_\eta = O_\eta \). This complex structure, in turn, determines a holomorphic structure on \( L \). When the prequantization is \( G \)-invariant, \( P \) defines also a representation \( \rho \) of \( G \) on the space \( H^0(L) \) of holomorphic sections of \( L \). When \( G_\eta \) itself is a maximal torus, the Borel-Weil theorem allows us to characterize \( \rho \) in terms of its highest weight. We prove that the invariant \( \kappa(\psi) \) for the closed isotopy considered above is equal to \( \chi(\rho)(h_1)/\dim \rho \). This fact permits us to calculate \( \kappa(\psi) \) using the Weyl’s character formula. This stuff is considered in Section 4.

In Section 5 we check the results of Sections 3 and 4 in two particular cases. In the first one we calculate directly the value of \( \kappa(\psi) \) for a closed isotopy \( \psi \) in \( \mathbb{C}P^1 \); Theorem 3 and Weyl’s character formula applied to this example give the same result as the direct calculation. In [10] we determined the value \( \kappa(\psi) \) for a closed Hamiltonian flow \( \psi \) in \( S^2 \); here we recover this number by applying Theorem 3 to this isotopy.
2. G-invariant prequantum data.

Let $G$ be a compact Lie group which acts on the left on the symplectic manifold $(M, \omega)$ by symplectomorphisms. We assume that this action is Hamiltonian, and that $\Phi : M \to g^*$ is a map moment for this action.

Given $A \in g^*$, we denote by $X_A$ the vector field on $M$ generated by $A$. Then $(d\Phi(Y)) \cdot A = \omega(Y, X_A)$, for any vector field $Y$ on $M$. The $\mathbb{R}$-valued map $\Phi \cdot A$ will be denoted by $f_A$; so

$$i_{X_A} \omega = -df_A \quad \text{and} \quad \{f_A, f_B\} = \omega(X_B, X_A) = f_{[A,B]}.$$

As we said one assumes that $(M, \omega)$ is quantizable. Let $L$ be a prequantum bundle, i.e. $L$ is a Hermitian line bundle over $M$ with a connection $D$, whose curvature is $-2\pi i \omega$, then one can define the prequantization map [9]

$$A \in g \mapsto P_A = -D_{X_A} - 2\pi i f_A \in \text{End}(C^\infty(L)).$$

Proposition 1. The map $P$ is a Lie algebra homomorphism.

Proof. Since the action of $G$ is on the left, the map $A \in g \mapsto X_A \in \Xi(M)$, where $\Xi(M)$ denotes the set of vector fields on $M$, is a Lie Algebra antihomomorphism (see [8] p.42); that is,

$$X_{[A,B]} = -[X_A, X_B].$$

On the other hand, if $\tau$ is a section of $L$

$$[P_A, P_B] \tau = [D_{X_A}, D_{X_B}] \tau + 4\pi i \omega(X_A, X_B) \tau.$$

Since the curvature of $D$ is $-2\pi i \omega$

$$-2\pi i \omega(X_A, X_B) \tau = [D_{X_A}, D_{X_B}] \tau + D_{[X_A, X_B]} \tau.$$

Using (2.3), (2.1) and (2.4) one obtains

$$[P_A, P_B] \tau = P_{[A,B]} \tau.$$

The prequantum data $(L, D)$ are said to be $G$-invariant, if there is an action $\rho$ of $G$ on $C^\infty(L)$ which generates $P$ [1]. Henceforth in this Section we assume that the prequantum data are $G$-invariant.

Let $\{A_t\}$, be a curve in $g$ with $A_0 = 0$. Given $\tau \in C^\infty(L)$ we consider the equation for the section $\tau_t$ of $L$

$$\frac{d\tau_t}{dt} = P_{A_t} (\tau_t), \quad \tau_0 = \tau$$

This is the equation of the “transport” of the section $\tau$ along the isotopy determined by the vector fields $X_{A_t}$ (see [10]). We will try to find a curve $h_t$ in $G$, such that $h_0 = e$ and $\rho(h_t)(\tau) = \tau_t$, where $\tau_t$ is
solution to (2.5). As \( \rho : G \to \text{Diff}(C^\infty(L)) \) is a group homomorphism, 
\( \rho \circ \mathcal{L}_g = \mathcal{L}_{\rho(g)} \circ \rho \), where \( \mathcal{L}_a \) is the left multiplication by \( a \) in the respective group. The corresponding tangent maps satisfy 
\[ \rho_* \circ \mathcal{L}_{g_*} = \mathcal{L}_{\rho(g)_*} \circ \rho_* . \quad (2.6) \]
If we put \( F_t \) for diffeomorphism \( \rho(h_t) =: F_t \), and we define \( Y_t \in \mathfrak{g} \) by 
\[ \dot{h}(t) = \mathcal{L}_h(Y_t)_t(Y_t) , \]
then by (2.6) 
\[ \frac{dF_t}{dt} = \rho_*(\dot{h}_t) = \mathcal{L}_{F_t}(\mathcal{P}(Y_t)) . \quad (2.7) \]
As \( \mathcal{L}_{F_t}(C) = F_t \circ C \), if \( C \in \text{End}(C^\infty(L)) \subset \Xi(C^\infty(L)) \), then (2.7) can be written 
\[ \frac{dF_t}{dt} = F_t \circ \mathcal{P}(Y_t) . \]
If we introduce this formula in (2.5), we obtain 
\[ \frac{d\tau_t}{dt} = (F_t \circ \mathcal{P}(Y_t))\tau = (\mathcal{P}_{A_t} \circ F_t)\tau . \]
Hence 
\[ F_t \circ \mathcal{P}_{Y_t} \circ F_t^{-1} = \mathcal{P}_{A_t} \quad (2.8) \]
Let \( \{ m(u) \}_u \) a curve in \( G \) which defines \( Y_t \in \mathfrak{g} \), then 
\[ F_t \circ \mathcal{P}_{Y_t} \circ F_t^{-1} = \frac{d}{du} \bigg|_{u=0} \rho(h_t m(u) h_t^{-1}) . \]
By (2.8) one can take \( Y_t = \text{Ad}_{h_t^{-1}}A_t \); so \( h_t \) is the solution to the Lax equation 
\[ \dot{h}_t h_t^{-1} = A_t \quad h_0 = e. \quad (2.9) \]
We have proved

**Theorem 2.** The solution \( \tau_t \) to (2.3) is given by \( \rho(h_t)\tau \), where \( h_t \) satisfies equation (2.3).

Let \( \{ A_t \mid t \in [0,1] \} \) be a curve in \( \mathfrak{g} \) such that the Hamiltonian isotopy \( \{ \psi_t \}_{t \in [0,1]} \) generated by the vector fields \( X_{A_t} \) is closed; i.e. \( \psi_0 = \psi_1 = \text{id} \). We have proved in [10] that if \( \tau_t \) is the solution of (2.3), then 
\[ \tau_1 = \kappa(\psi)\tau, \quad (2.10) \]
for every \( \tau \in C^\infty(L) \), where \( \kappa(\psi) = \exp(2\pi i A_\psi(q)) \), and \( A_\psi(q) \) is the action integral along the curve \( \{ \psi_t(q) \}_t \), for \( q \) arbitrary in \( M \). On the other hand, if \( h_t \) is a curve in \( G \) solution to (2.3), by Theorem 2 \( \tau_1 = \rho(h_1)(\tau) \). It follows from (2.10) that \( \rho(h_1) = \kappa(\psi)\text{Id} \). Thus we have
Corollary 3. If $W$ is a finite dimensional $\rho$-invariant subspace of $C^\infty(L)$, and $\rho_W$ is the restriction of $\rho$ to this subspace, then for the character of $\rho_W$ holds the following formula
\[ \chi(\rho_W)(h_1) = \kappa(\psi) \dim(W). \]

3. The invariant $\kappa(\psi)$ in a coadjoint orbit

Let $G$ be a compact Lie group, and we consider the coadjoint action of $G$ on $\mathfrak{g}^*$ defined by
\[ (g \cdot \eta)(A) = \eta(g^{-1} \cdot A), \]
for $g \in G$, $\eta \in \mathfrak{g}^*$, $A \in \mathfrak{g}$ and $g \cdot A = \text{Ad}_g A$ (see [5] [13]).

If $X_A$ is the vector field on $\mathfrak{g}^*$ determined by $A$, the map $l_g : \mu \in \mathfrak{g}^* \mapsto g \cdot \mu \in \mathfrak{g}^*$ satisfies
\[ (l_g)_*(X_A(\mu)) = X_{g \cdot A}(g \cdot \mu). \quad (3.1) \]

Given $\eta \in \mathfrak{g}^*$, by $O_\eta := O$ will be denoted the orbit of $\eta$ under the coadjoint action of $G$. On $O$ one can consider the 2-form $\omega$ determined by
\[ \omega(\nu)(X_A(\nu), X_B(\nu)) = \nu([A, B]). \quad (3.2) \]
This 2-form defines a symplectic structure on $O$, and the action of $G$ preserves $\omega$. For each $A \in \mathfrak{g}$ one defines the function $h_A \in C^\infty(O)$ by
\[ h_A(\nu) = \nu(A), \]
and for this function holds the formula
\[ \iota_{X_A} \omega = dh_A. \quad (3.3) \]

The orbit $O$ can be identified with $G/G_\eta$, where $G_\eta$ is the subgroup of isotropy of $\eta$. The Lie algebra of this subgroup is
\[ \mathfrak{g}_\eta = \{ A \in \mathfrak{g} \mid \eta([A, B]) = 0, \text{ for every } B \in \mathfrak{g} \} \]

The orbit $O$ possesses a $G$-invariant prequantization iff the linear functional
\[ \lambda : \mathfrak{g} \ni C \mapsto 2\pi i \eta(C) \in i\mathbb{R} \quad (3.4) \]
is integral; i.e., iff there is a character $\Lambda : G_\eta \to U(1)$ whose derivative is the functional (3.4) (see [7]). Henceforth we assume the existence of such a character $\Lambda$. The corresponding prequantum bundle $L$ over $O = G/G_\eta$ is defined by $L = G \times_\Lambda \mathbb{C} = (G \times \mathbb{C})/\sim$, with $(g, z) \sim (gb^{-1}, \Lambda(b)z)$, for $b \in G_\eta$.

Each section $\sigma$ of $L$ determines a $\Lambda$-equivariant function $s : G \to \mathbb{C}$ by the relation
\[ \sigma(gG_\eta) = [g, s(g)]. \quad (3.5) \]
Theorem 4. The action in (2.2) to a representation of $g$ the form $\gamma$ value at $R$.

Let $v$ denote the element $[e, 1] \in L^\times$, then $T_v(L^\times) \cong (g \oplus \mathbb{C})/f_v$, with

$$f_v = \{(B, -2\pi i\eta(B) | B \in g_u\}.$$

The connection form $\Omega$ on $L^\times$ is constructed in [4] p.198. The form $\Omega$ can be written $\Omega = (\theta, d)$, where $\theta$ is the left invariant form on $G$ whose value at $e$ is $\eta$, and $d \in \text{Hom}_C(\mathbb{C}, \mathbb{C})$ is defined by $d(z) = (2\pi i)^{-1}z$. It is clear that $\Omega_v$ vanishes on $f_v$ and that it defines an element of $T^*_v(L^\times)$.

On the other hand the section $\sigma$ determines a lift $\sigma^\sharp : L^\times \to \mathbb{C}$ by the formula

$$\sigma(\pi(y)) = \sigma^\sharp(y) y,$$

here $\pi : L \to O$ is the projection map. It follows from (3.5)

$$s(g) = \sigma^\sharp([g, z]) z. \tag{3.7}$$

We denote by $E_\Lambda$ the space of $\Lambda$-equivariant functions on $G$. The identification $C^\infty(L) \sim E_\Lambda$ allows us to translate the action $P$ defined in (2.2) to a representation of $g$ on $E_\Lambda$.

**Theorem 4.** The action $P$ on $E_\Lambda$ is given by $P_A(s) = -R_A(s)$, where $R_A$ is the right invariant vector field on $G$ determined by $A$.

**Proof.** Let $\sigma$ be a section of $L$, by (3.3) $P_A(\sigma) = -X_A \sigma + 2\pi i h_A \sigma$. We will determine the lift $(P_A(\sigma))^\sharp$.

The vector $X_A(g \cdot \eta) \in T_{g \cdot \eta}(O)$ is defined by the curve $u \mapsto e^{uA} g \cdot \eta$ in $O$. A lift of this curve at the point $[g, z] \in L^\times$ will be a curve of the form $\gamma(u) = [e^{uA} g, z_u]$, with $z_u = z e^{ux}$. The vector tangent to $\gamma$ at $[g, z]$ is $\dot{\gamma}(0) = ([R_A(g), x], [R_A(g), x])$, where $R_A(g)$ is the value at $g$ of the right invariant vector field in $G$ defined by $A$.

The condition $\Omega(\dot{\gamma}(0)) = 0$ implies

$$x = -2\pi i\eta(g^{-1} \cdot A). \tag{3.8}$$

Therefore the horizontal lift of $X_A(g \cdot \eta)$ is

$$X^\sharp_A([g, z]) = [R_A(g), -2\pi i\eta(g^{-1} \cdot A)],$$

and by (1.7) the action of $X^\sharp_A([g, z])$ on the function $\sigma^\sharp$ can expressed in terms of $s$

$$X^\sharp_A([g, z])(\sigma^\sharp) = \frac{d}{du} \bigg|_{u=0} \left( s(e^{uA} g) \right) \frac{z e^{ux}}{z} - \frac{x s(g)}{z}. \tag{3.9}$$

Since $X^\sharp_A(\sigma^\sharp) = (D_{X_A} \sigma)^\sharp$ [1] page 115], from (3.8) and (3.7) it turns out that the equivariant function associated to $D_{X_A} \sigma$ is

$$g \in G \mapsto R_A(g)(s) + 2\pi i\eta(g^{-1} \cdot A)s(g) \in \mathbb{C}. \tag{3.9}$$
Obviously the equivariant function defined by the section $h_A \sigma$ is the function $\lambda_A \sigma$, where $\lambda_A(g) = h_A(gG_\eta) = (g \cdot \eta)(A) = \eta(g^{-1} \cdot A)$. It follows from (3.9) that the equivariant function which corresponds to 

$$-D_{X_A} \sigma + 2\pi i h_A \sigma = -R_A(s).$$

**Corollary 5.** The action $\mathcal{P}$ on $E_\Lambda$ is induced by the action

$$\rho : (b, s) \in G \times E_\Lambda \mapsto s \circ L_b^{-1} \in E_\Lambda,$$

where $L_c$ is left multiplication by $c$ in the group $G$.

**Proof.** If $g_t = e^{tA} \in G$, then

$$\frac{d}{dt} \rho_{g_t}(s) \bigg|_{t=0} = \frac{d}{dt} \bigg|_{t=0} s(e^{-tA} g) = -R_A(g)(s) = \mathcal{P}_A(s)(g).$$

From Corollary 5 it follows that the prequantum data $(L, D)$ are $G$-invariant.

Let $\{\psi_t | t \in [0, 1]\}$ be a closed Hamiltonian isotopy on $\mathcal{O}$; that is, a Hamiltonian isotopy such that $\psi_1 = \text{id}$. We also assume that the corresponding Hamiltonian vector fields are invariant; that is,

$$\frac{d\psi_t(q)}{dt} = X_{A_t}(\psi_t(q)), \quad \text{with} \ A_t \in \mathfrak{g}.$$

If $\sigma$ is a section of $L$, $\sigma_t$ will denote the solution to the equation

$$\frac{d\sigma_t}{dt} = \mathcal{P}_{A_t}(\sigma_t), \quad \sigma_0 = \sigma. \quad (3.10)$$

By Theorem 4, equation (3.10) on the points $\{g_t\}_{t \in [0,1]}$ of a curve in $G$ gives rise to

$$\dot{s}_t(g_t) = -R_{A_t}(g_t)(s_t), \quad (3.11)$$

for the corresponding equivariant functions. In particular, if $g_t$ is the curve such that $g_0 = e$ and $\dot{g}_t = R_{A_t}(g_t) \in T_{g_t}(G)$; in other words, $g_t$ satisfies the Lax equation $\dot{g}_t g_t^{-1} = A_t$, then

$$R_{A_t}(g_t)(s_t) = \frac{d}{du} \bigg|_{u=t} s_t(g_u).$$

Using (3.11) one deduces

$$\dot{s}_t(g_t) + \dot{g}_t(s_t) = 0 \quad (3.12)$$

If we consider the function $w : [0, 1] \to \mathbb{C}$ defined by $w_t = s_t(g_t)$; by (3.12) $w$ is constant. So $s_1(g_1) = s_0(e)$. If $g_1 \in G_\eta$, as $s_1$ is $\Lambda$-equivariant $s_1(g_1) = \Lambda(g_1^{-1}) s_1(e)$; so

$$\sigma_1(eG_\eta) = \Lambda(g_1) \sigma_0(eG_\eta). \quad (3.13)$$
The following Theorem is consequence of (2.10) and (3.13)

**Theorem 6.** If \( \{ \psi_t \} \) is the closed Hamiltonian isotopy in \( \mathcal{O} \) generated by the vector fields \( \{ X_{\lambda_\alpha} \} \), then \( \kappa(\psi) = \Lambda(g_1) \), where \( g_t \in G \) is the solution to \( \dot{g}_t g_t^{-1} = A_t \), with \( g_0 = e \) and \( g_1 \in G_\eta \).

**Remark.** Theorem 6 can also be deduced as a consequence of Theorem 2 and Corollary 5. In fact, if \( h_t \) is the solution to \( \dot{h}_t h_t^{-1} = A_t \), with the introduced notations

\[
\sigma_1(a) = (\rho(h_1)\sigma)(a) = [a, s(h_1^{-1}a)] = \Lambda(h_1)[a, s(a)] = \Lambda(h_1)\sigma(a).
\]

4. Relation with Weyl's character formula.

Let us assume that \( G \) is semisimple Lie group \([2]\), and let \( T \) a maximal torus with \( T \subset G_\eta \) (see \([4]\) p.166). One has the corresponding decomposition of \( g_C = g \otimes \mathbb{R} \mathbb{C} \) in direct sum of root spaces

\[ g_C = \mathfrak{h} \oplus \sum g_\alpha, \]

where \( \mathfrak{h} = t_C \), and \( \alpha \) ranges over the set of roots. This decomposition gives the real counterpart

\[ g = t \oplus \sum_{\alpha \in P} (g_\alpha \oplus g_{-\alpha}) \cap g, \]

where \( P \) is a set of positive roots. We denote by \( \alpha^\vee \) the element of \([g_\alpha, g_{-\alpha}]\) such that \( \alpha(\alpha^\vee) = 2 \), while \( \beta(\alpha^\vee) \in \mathbb{Z} \) for every root \( \beta \).

\( \eta \) extends in a natural way to \( g_C \). If \( Y \in g_\alpha \), then as \( \alpha^\vee \in g_\eta \)

\[ 0 = \eta([\alpha^\vee, Y]) = 2\eta(Y). \]

Hence \( \eta \) vanishes on \( \sum g_\alpha \). If \( \eta(\alpha^\vee) \neq 0 \), for all root \( \alpha \), then \( g_\eta = t \); in this case \( \eta \) is said to be regular. Henceforth we assume that \( \eta \) is regular.

We define \( \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n} \), where

\[ \mathfrak{n} = \sum_{\alpha \in P} g_\alpha. \]

Then \( \mathfrak{b} \) is a Borel subalgebra of \( g_C \), which corresponds to a Borel subgroup \( B \) of \( G \).

We have

\[ T_\eta(\mathcal{O}) = g/\mathfrak{g}_\eta = \sum_{\alpha \in P} (g_\alpha \oplus g_{-\alpha}) \cap g. \]

Hence

\[ T^C_\eta(\mathcal{O}) = \sum_{\alpha \in P} (g_\alpha \oplus g_{-\alpha}). \]
One defines
\[ T^{0,1}_\eta \mathcal{O} := \mathfrak{n}, \]
and
\[ T^{0,1}_{\eta g} \mathcal{O} := \{ X_{g \cdot A}(g \cdot \eta) \mid A \in \mathfrak{n} \}. \]
If \( g_1 \cdot \eta = g_2 \cdot \eta \), then \( g_1^{-1}g_2 \in T \). As \( \mathfrak{g}_\alpha \) is an eigenspace for the action of \( T \), then \( g_1^{-1}g_2 \cdot A \in \mathfrak{n} \), if \( A \in \mathfrak{n} \). Therefore the spaces \( T^{0,1}_{\eta g} \) are well-defined.

For \( A \in \mathfrak{n} \), one can define the vector field \( \mathcal{A} \) on \( \mathcal{O} \) by \( \mathcal{A}(g \cdot \eta) = X_{g \cdot A}(g \cdot \eta) \). By \((3.1)\) \((l_g)_* \mathcal{A} = \mathcal{A} \), hence the above complex foliation defined on \( \mathcal{O} \) is \( G \)-invariant. Since the vector \( X_{g \cdot A}(g \cdot \eta) \) is defined by the curve \( e^{t\theta g} g \cdot \eta = g e^{tA} \cdot \eta \), then the left invariant vector field \( L_A \) on \( G/T \) is the field which corresponds to \( \mathcal{A} \), in the identification of \( G/T \) with \( \mathcal{O} \).

The vector spaces \( T^{1,0} \) are defined in the obvious way. As \( \mathfrak{n} \) is a subalgebra of \( \mathfrak{g}_C \), the decomposition \( T^C(\mathcal{O}) = T^{1,0} \oplus T^{0,1} \) define a complex structure on \( \mathcal{O} \). This complex manifold can be identified with \( G_C/B \).

We assume that the integral functional \( \lambda \) in \((3.4)\) satisfies \( \lambda(\alpha^\vee) \leq 0 \) for every \( \alpha \in P \); this means that \( -\lambda \) is a dominant weight for \( T \) \([4]\). Using the complex structure on \( \mathcal{O} = G/T \) and the covariant derivative \( D \) on the prequantum bundle \( L = G \times_A \mathbb{C} \), it is possible to define a holomorphic structure in \( L \). The section \( \tau \) of \( L \) is said to be holomorphic iff \( D_Z \tau = 0 \) for any vector field \( Z \) of type \((0,1)\). In this way \( L \) can be regarded as a holomorphic line bundle over \( G_C/B \). The homomorphism \( \Lambda : T \to U(1) \) extends trivially to \( B \), since \( B \) is a semidirect product of \( H = T_C \) and the nilpotent subgroup whose Lie algebra is \( \mathfrak{n} \); and each section \( \sigma \) of \( L \) determines a function \( s : G_C \to \mathbb{C} \) which is \( \Lambda \)-equivariant.

Given \( A \in \mathfrak{n} \), the Proof of Theorem \([4]\) shows that the equivariant function associated to \( D_A \sigma \) is the map
\[ g \in G_C \mapsto R_{g \cdot A}(g)(s) + 2\pi i \eta(g^{-1}g \cdot A)s(g) \in \mathbb{C}. \]
As \( \eta \) vanishes on \( \mathfrak{n} \) and \( R_{g \cdot A}(g) = L_A(g) \), the function associated to \( D_A \sigma \) is \( L_A(s) \). The section \( \sigma \) is holomorphic if \( D_A \sigma = 0 \), for every \( A \in \mathfrak{n} \); in this case \( L_A(s) = 0 \) for \( A \in \mathfrak{n} \), that is, \( s \) is a holomorphic function on \( G_C \). So the space \( H^0(G_C/B, L) \) is isomorphic to the space
\[ \mathcal{E}_{\Lambda, P} := \{ s : G_C \to \mathbb{C} \mid s \text{ is holomorphic and } \Lambda - \text{equivariant} \}. \]

The Borel-Weil Theorem asserts that the action of \( G \) on the space \( \mathcal{E}_{\Lambda, P} \) given by \( g \ast s = s \circ L_{g^{-1}} \) is an irreducible representation of \( G \); more precisely the contragredient representation of that one whose highest weight is \( -\lambda \) (see \([4]\) pages 290, 300).
Lemma 7. If $A \in \mathfrak{n}$, then $[A, X_B] = 0$ for any $B \in \mathfrak{g}_C$.

Proof. The flow $\varphi_t$ determined by $X_B$ is given $\varphi_t(g \cdot \eta) = e^{tB}g \cdot \eta$. And the flow $\phi_t$ of $A$ is $\phi_t(g \cdot \eta) = e^{tA}g \cdot \eta$. Hence

$$(\varphi_t \circ \phi_t)(g \cdot \eta) = e^{tB}ge^{tA} \cdot \eta = (\phi_t \circ \varphi_t)(g \cdot \eta).$$

Proposition 8. If $D_A \sigma = 0$ for any $A \in \mathfrak{n}$, then $D_A P_B \sigma = 0$ for any $B \in \mathfrak{g}$.

Proof. Since $D_A \sigma = 0$, it follows from (2.2)

$$D_A(P_B \sigma) = -D_A D_{X_B} \sigma + 2\pi i A(h_B) \sigma.$$  \hfill (4.1)

As

$$[D_A, D_{X_B}] \sigma = D_{[A, X_B]} \sigma - 2\pi i \omega(A, X_B) \sigma,$$

from (4.1) and (3.3) we deduce

$$D_A(P_B \sigma) = -D_{[A, X_B]} \sigma.$$  \hfill (4.2)

Now the proposition is consequence of Lemma 7.

A direct consequence of Proposition 8 is

Corollary 9. $P$ defines a representation of $\mathfrak{g}$ on $H^0(G_C/B, L)$.

Denoting by $\pi$ the irreducible representation of $G$ whose highest weight is $-2\pi i \eta$, and by $\pi^*$ its dual, we have

Corollary 10. The representation $P$ on $H^0(G_C/B, L)$ is the derivative of $\pi^*$.

Proof. It is a consequence of Corollary 8 and Borel-Weil theorem

The subspace $E_{A, P} \subset E_A$ is invariant under the representation $\rho$ defined in Corollary 8, and the restriction of $\rho$ to $E_{A, P}$ is precisely the representation $\pi^*$. From by Corollary 8 it follows

Theorem 11. Let $\eta$ be an element of $\mathfrak{g}^*$, such that the orbit $O_\eta$ is quantizable and $-2\pi i \eta$ is a dominant weight for the maximal torus $G_\eta$. If $\{\psi_t\}$ is the closed Hamiltonian isotopy in $O_\eta$ generated by the vector fields $\{X_{A_i}\}$, then

$$\kappa(\psi) = \frac{\chi(\pi^*)(h_1)}{\dim \pi},$$  \hfill (4.2)

where $h_t \in G$ is the solution to $\dot{h}_t h_t^{-1} = A_t$, $h_0 = e$, and $\pi$ is the representation of $G$ whose highest weight is $-2\pi i \eta$.

Now the character $\chi(\pi^*)$ and the dimension $\dim \pi$ can be determined by Weyl’s character formula [1], and so $\kappa(\psi)$. 

5. Examples

The invariant $\kappa(\psi)$ in $\mathbb{C}P^1$. Let $G$ be the group $SU(2)$ and $\eta$ the element of $\mathfrak{su}(2)^*$ defined by

$$\eta \begin{pmatrix} ci & w \\ -\bar{w} & -ci \end{pmatrix} = -\frac{c}{2\pi}. \quad (5.1)$$

The subgroup of isotropy $G_\eta$ is $U(1) \subset SU(2)$, so the coadjoint orbit $O_\eta$ can be identified with $SU(2)/U(1) = \mathbb{C}P^1$. The element

$$g = \begin{pmatrix} z_0 & -\bar{z}_1 \\ z_1 & \bar{z}_0 \end{pmatrix} \in SU(2)$$

determines the point $(z_0 : z_1) \in \mathbb{C}P^1$. Hence to $\eta \in O_\eta$ corresponds $p = (1 : 0) \in \mathbb{C}P^1$. For $z_0 \neq 0$ we put $(z_0 : z_1) = (1 : x + iy)$.

Denoting by $A$ and $B$ the following matrices of $\mathfrak{su}(2)$

$$A := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad B := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (5.2)$$

by (3.2) $\omega_\eta(X_A, X_B) = \eta([A, B]) = \frac{1}{\pi}$. \quad (5.3)

As $e^{tA}$,

$$e^{tA} = \begin{pmatrix} \cos t & i \sin t \\ i \sin t & \cos t \end{pmatrix},$$

defines the curve $\{e^{tA} \eta\}$, which defines $X_A(p)$, is $(\cos t : i \sin t)$. Hence $X_A(p)$, expressed in the real coordinates $(x, y)$, is equal to $(\frac{\partial}{\partial y})_p$. Similarly $X_B(p) = -\left(\frac{\partial}{\partial x}\right)_p$. Hence it follows from (5.3)

$$\omega_p = \frac{1}{\pi} dx \wedge dy = \frac{i}{2\pi} dz \wedge d\bar{z},$$

where $z = x + iy$. Therefore $(\mathcal{O}_\eta, \omega)$ can be identified with $\mathbb{C}P^1$ endowed with the Fubini-Study form

$$\omega = \frac{i}{2\pi (1 + z\bar{z})^2} dx \wedge dy = \frac{1}{\pi (x^2 + y^2 + 1)^2} dx \wedge dy. \quad (5.4)$$

Let us consider the symplectomorphism $\psi_t$ on $\mathbb{C}P^1$ defined by

$$(z_0 : z_1) \in \mathbb{C}P^1 \mapsto (e^{-iat} z_0 : e^{iat} z_1) \in \mathbb{C}P^1,$$

where $a_t \in \mathbb{R}$. If we assume that $a_0 = 0$ and $a_1 = k\pi$, with $k \in \mathbb{Z}$, then $\{\psi_t \mid t \in [0, 1]\}$ is a closed Hamiltonian isotopy on $\mathbb{C}P^1$. We will
determine $\kappa(\psi)$ by direct calculation. In real coordinates
\[ \psi_t(x, y) = (x \cos 2a_t - y \sin 2a_t, x \sin 2a_t + y \cos 2a_t) . \] (5.5)

A straightforward calculation shows that the Hamiltonian vector field $X_t$ defined by
\[ \frac{d\psi_t(q)}{dt} = X_t(\psi_t(q)) \]
is $X_t(x, y) = 2\dot{a}_t (-y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y})$. It follows from (5.4)
\[ \iota_{X_t} \omega = -2\dot{a}_t \frac{\pi}{\pi (x^2 + y^2 + 1)^2} (dx + dy) . \]

A Hamiltonian function $f_t$ associated to $X_t$ is
\[ f_t(x, y) = -\frac{\dot{a}_t}{\pi (x^2 + y^2 + 1)} + c_t , \]
c_t being a constant. If we impose $\int f_t \omega = 0$, then
\[ c_t = c_t \int_{\mathbb{CP}^1} \omega = \frac{\dot{a}_t}{\pi^2} \int_{\mathbb{CP}^1} \frac{1}{(x^2 + y^2 + 1)^3} dx \wedge dy = \frac{\dot{a}_t}{2\pi} . \]

Thus the normalized Hamiltonian function is
\[ f_t(x, y) = -\frac{\dot{a}_t}{2\pi} \left( \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1} \right) . \]

Given $q = (x_0, y_0) \in \mathbb{CP}^1$, from (5.3) it follows that the set
\[ \{ \psi_t(x_0, y_0) | t \in [0, 1] \} \]
is a circle in the plane $(x, y)$ with centre at $(0, 0)$; therefore
\[ \int_0^1 f_t(\psi_t(q)) dt = k \frac{\dot{a}_t}{2} \left( \frac{x_0^2 + y_0^2 - 1}{x_0^2 + y_0^2 + 1} \right) . \] (5.6)

On the other hand the 1-form $\theta = (-x + iy)(x^2 + y^2 + 1)^{-1}(dx + idy)$ satisfies $d\theta = -2\pi i \omega$. And
\[ \int_0^1 \theta(X_t) dt = -2k \pi i \frac{x_0^2 + y_0^2}{x_0^2 + y_0^2 + 1} . \] (5.7)

From (1.1), (5.6) and (5.7) it follows $\mathfrak{A}_\psi(q) = k/2 + \mathbb{Z}$ and
\[ \kappa(\psi) = e^{ik\pi} . \] (5.8)

Next we determine the value of $\kappa(\psi)$ by using the results of Section 3. First of all the prequantum bundle for $(\mathbb{CP}^1, \omega)$ is the hyperplane bundle $[3]$ on $\mathbb{CP}^1$. On the other hand the functional
\[ ci \in \mathfrak{u}(1) \subset \mathfrak{su}(2) \mapsto 2\pi i \eta(\text{diag}(ci, -ci)) = -ic \]
is the derivative of $\Lambda : g \in U(1) \mapsto g^{-1} \in U(1)$. Therefore the respective prequantum data are $SU(2)$-invariant. The isotopy $\left\{ \psi_t \right\}$ of $\mathbb{CP}^1$ determines the vector fields $X_{\Lambda_t}$, where $A_t = \text{diag}(-ia_t, ia_t)$. In this case the solution to $\dot{h_t}h_t = A_t$ is $h_t = \text{diag}(e^{-ia_t}, e^{ia_t})$. Hence, by Theorem 6

$$\kappa(\psi) = \Lambda(h_1) = h_1^{-1} = e^{ik\pi}.$$ 

This result agrees with (5.8).

The invariant $\kappa$ of a Hamiltonian flow in $S^2$. For $G = SU(2)$, if $\eta : (ai w - \bar{w} - ai) \in su(2) \mapsto na^2 \pi \in \mathbb{R}$, with $n \in \mathbb{Z}$, then the orbit $O_\eta = SU(2)/U(1) = S^2$ admits and $SU(2)$-invariant quantization and the corresponding character $\Lambda$ of $U(1)$ is $\Lambda(z) = z^n$.

Let $\tilde{\psi}_t$ be the symplectomorphism of $S^2$ given by

$$\tilde{\psi}_t(q) = \exp(t(aA + bB)) \cdot q,$$

where $a, b \in \mathbb{R}$ and $A, B$ are the matrices introduced in (5.2). For $t_1 = (a^2 + b^2)^{-1/2} \pi$, $\tilde{\psi}_{t_1} = \text{id}$; in fact

$$\exp(t(aA + bB)) = \left( \begin{array}{cc} \cos |c| & \epsilon \sin |c| \\ -\bar{\epsilon} \cos |c| & \epsilon |c| \end{array} \right),$$

with $c = t(b + ai)$ and $\epsilon = c/|c|$ (see [10]). If we set

$$E := \pi(a^2 + b^2)^{-1/2}(aA + bB),$$

by (5.9) $\exp(E) = -\text{Id}$. So the family $\left\{ \psi_t \right\}_{t \in [0, 1]}$, defined by $\psi_t(q) = \exp(tE)q$, is a closed Hamiltonian flow on the orbit $O_\eta$. By Theorem 4

$$\kappa(\psi) = \Lambda(e^E) = \Lambda(-\text{Id}) = (-1)^n.$$ 

This result agrees with that one obtained in [10, Theorem 21] by direct calculation.

This result can be deduced from (1.2), when $n < 0$. Here Lax equation $\dot{h_t}h_t^{-1} = E$ has the solution $h_t = \exp(tE)$. The Weyl’s character formula [11] is very simple for the group $SU(2)$; in this case, there is only one positive root $\alpha$ and the Weyl group has only two elements. We take for $\alpha$ the linear map defined by

$$\alpha(\text{diag}(ai, -ai)) = 2ai;$$

so $\alpha^\vee = \text{diag}(1, -1)$. As we assume that $n < 0$, then $-\lambda := -2\pi i\eta$ is the highest weight of a representation $\pi$ of $SU(2)$. For $t \in U(1)$,
\[ t^\lambda = t^{-n} \text{ and } t^\alpha = t^2. \] Therefore (see [1])

\[ \dim \pi = -n + 1 \text{ and } \chi_{\pi}(t) = \sum_{k=0}^{-n} t^{-n-2k}. \]

Hence

\[ \chi_{\pi^*}(h_1) = \chi_{\pi}(-1) = (-n + 1)(-1)^n, \]

and from Corollary [3] we again obtain the value \((-1)^n\) for \(\kappa(\psi)\).

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