A LOWER BOUND ON BLOWUP RATES FOR THE 3D INCOMPRESSIBLE EULER EQUATION AND A SINGLE EXPONENTIAL BEALE-KATO-MAJDA TYPE ESTIMATE

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ABSTRACT. We prove a Beale-Kato-Majda type criterion for the loss of regularity for solutions of the incompressible Euler equations in $H^s(\mathbb{R}^3)$, for $s > \frac{5}{2}$. Instead of double exponential estimates of Beale-Kato-Majda type, we obtain a single exponential bound on $\|u(t)\|_{H^s}$ involving the length parameter introduced by P. Constantin in [3]. In particular, we derive lower bounds on the blowup rate of such solutions.

1. Introduction

In this paper, we revisit the Beale-Kato-Majda criterion for the breakdown of smooth solutions to the 3D Euler equations.

More precisely, we consider the incompressible Euler equations

\begin{align*}
\partial_t u + (u \cdot \nabla) u + \nabla p &= 0 \quad (1.1) \\
\nabla \cdot u &= 0 \quad (1.2) \\
u(x, 0) &= u_0 \quad (1.3)
\end{align*}

for an unknown velocity vector $u(x, t) = (u_i(x, t))_{1 \leq i \leq 3} \in \mathbb{R}^3$ and pressure $p = p(x, t) \in \mathbb{R}$, for position $x \in \mathbb{R}^3$ and time $t \in [0, \infty)$.

Existence and uniqueness of local in time solutions to (1.1) – (1.3) in the space

$$C([0, T], H^s) \cap C^1([0, T]; H^{s-1}),$$

has long been known for $s > \frac{5}{2}$, see for instance [7]. However, it is an open problem to determine whether such solutions can lose their regularity in finite time. An important result that addresses the question of a possible loss of regularity of solutions to Euler equations (1.1) – (1.3) is the criterion formulated by Beale-Kato-Majda [1] in terms of the $L^\infty$ norm of the vorticity $\omega = \nabla \wedge u$. More precisely, Beale-Kato-Majda in [1] proved the following theorem:

**Theorem 1.1.** Let $u$ be a solution to (1.1) – (1.3) in the class (1.3) for $s \geq 3$ integer. Suppose that there exists a time $T^*$ such that the solution cannot be continued in the class (1.4) to $T = T^*$. If $T^*$ is the first such time, then

$$\int_0^{T^*} \|\omega(\cdot, t)\|_{L^\infty} dt = \infty.$$  

(1.5)
The theorem is proved with a contradiction argument. Under the assumption
\[ \int_0^{T^*} \| \omega(\cdot, t) \|_{L^\infty} \, dt < \infty, \]
the authors of \[1\] show that \( \| u(\cdot, t) \|_{H^s} \leq C_0 \) for all \( t < T^* \), contradicting the hypothesis that \( T^* \) is the first time such that the solution cannot be continued to \( T = T^* \). In particular, Beale-Kato-Majda obtain a double exponential bound for \( \| u(\cdot, t) \|_{H^s} \), which follows from the following estimates:

**Step 1** An energy-type bound on \( \| u \|_{H^s} \) in terms of \( \| Du \|_{L^\infty} \), where \( Du = [\partial_i u_j]_{i,j} \) is a 3 × 3-matrix valued function. More specifically, one applies the operator \( D^\alpha \) to equations \((1.1)-(1.2)\), where \( \alpha \) is an integer-valued multi-index with \( |\alpha| \leq s \) and uses a certain commutator estimate to derive

\[ \frac{d}{dt} \| u(\cdot, t) \|_{H^s}^2 \leq 2C \| Du \|_{L^\infty} \| u(\cdot, t) \|_{H^s}^2, \]

which via Gronwall’s inequality gives the bound:

\[ \| u(\cdot, t) \|_{H^s} \leq \| u_0 \|_{H^s} \exp \left( C \int_0^t \| Du(\cdot, \tau) \|_{L^\infty} \, d\tau \right). \]

**Step 2** An estimate on \( \| Du(\cdot, t) \|_{L^\infty} \) based on the quantities \( \| \omega(\cdot, t) \|_{L^\infty} \), \( \| \omega(\cdot, t) \|_{L^2} \), and \( \log^+ \| u(\cdot, t) \|_{H^3} \), given by

\[ \| Du(\cdot, t) \|_{L^\infty} \leq C \left\{ 1 + \left( 1 + \log^+ \| u(\cdot, t) \|_{H^3} \right) \| \omega(\cdot, t) \|_{L^\infty} + \| \omega(\cdot, t) \|_{L^2} \right\}, \]

where \( C \) is a universal constant.

**Step 3** The bound on \( \| \omega(\cdot, t) \|_{L^2} \) in terms of \( \| \omega(\cdot, t) \|_{L^\infty} \) given by

\[ \frac{d}{dt} \| \omega(\cdot, t) \|_{L^2}^2 \leq 2 \tilde{C} \| \omega(\cdot, t) \|_{L^\infty} \| \omega(\cdot, t) \|_{L^2}^2, \]

which follows from taking the \( L^2(\mathbb{R}^3) \)-inner product of \( \omega \) with the equation for vorticity. Then, Gronwall’s inequality yields

\[ \| \omega(\cdot, t) \|_{L^2} \leq \| \omega(\cdot, 0) \|_{L^2} \exp \left( \tilde{C} \int_0^t \| \omega(\cdot, \tau) \|_{L^\infty} \, d\tau \right). \]

Consequently, one obtains the double exponential bound

\[ \| u(\cdot, t) \|_{H^s} \leq \| u_0 \|_{H^s} \exp \left( \exp \left( \tilde{C} \int_0^t \| \omega(\cdot, \tau) \|_{L^\infty} \, d\tau \right) \right). \]

from combining \((1.7), (1.8)\) and \((1.9)\).

It is an open question whether \((1.10)\) is sharp. While we do not attempt to answer that question itself in this paper, we obtain a single exponential bound on the \( H^s \)-norm of solution to Euler equations \((1.1)-(1.2)\) in terms of the quantity

\[ \ell_\delta(t) = \min \left\{ L, \left( \frac{\| \omega(t) \|_{C^\delta}}{\| u_0 \|_{L^2}} \right)^{-\frac{1}{\delta \delta}} \right\}, \]

where

\[ \| \omega \|_{C^\delta} = \sup_{|x-y| < L} \frac{|\omega(x) - \omega(y)|}{|x-y|^\delta} \]
denotes the $\delta$-Holder seminorm, for $L > 0$ fixed, and $\delta > 0$. More precisely, we prove the following theorem:

**Theorem 1.2.** Let $u$ be a solution to (1.1) - (1.3) in the class (1.4), for $s = \frac{5}{2} + \delta$. Assume that $\ell_\delta(t)$ is defined as above, and that

$$
\int_0^T (\ell_\delta(\tau))^{-\frac{5}{2}} \, d\tau < \infty.
$$

Then, there exists a finite positive constant $C_\delta = O(\delta^{-1})$ independent of $u$ and $t$ such that

$$
\|u(\cdot, t)\|_{H^s} \leq \|u_0\|_{H^s} \exp \left( C_\delta \|u_0\|_{L^2} \int_0^t (\ell_\delta(\tau))^{-\frac{5}{2}} \, d\tau \right)
$$

holds for $0 \leq t \leq T$.

The quantity $\ell_\delta(t)$ has the dimension of length, and was introduced by Constantin in [3] (see also the work of Constantin, Fefferman and Majda [5] where a criterion for loss of regularity in terms of the direction of vorticity was obtained), where it was observed that

$$
\int_0^T (\ell_\delta(t))^{-\frac{5}{2}} \, dt = \infty
$$

is a necessary and sufficient condition for blow-up of Euler equations. In particular, the necessity of the condition follows from the inequality obtained in [3]

$$
\|\omega(\cdot, t)\|_{L^\infty} \leq \|u(\cdot, t)\|_{L^2} (\ell_\delta(t))^{-\frac{5}{2}},
$$

and Theorem 1.1 of Beale-Kato-Majda. This is so because Theorem 1.1 implies that if the solution cannot be continued to some time $T$, then $\int_0^T \|\omega(\cdot, t)\|_{L^\infty} \, dt = \infty$. As a consequence of (1.16), and conservation of energy

$$
\|u(\cdot, t)\|_{L^2} = \|u_0\|_{L^2},
$$

this in turn implies (1.15). However, by invoking the result of Beale-Kato-Majda in this argument, one again obtains a double exponential bound on $\|u(\cdot, t)\|_{H^s}$ in terms of $\int_0^T (\ell_\delta(t))^{-\frac{5}{2}} \, dt$. We refer to [4, 6] for recent developments in this and related areas.

In this paper, we observe that one can actually obtain a single exponential bound on the $H^s$-norm of the solution $u(t)$ in terms of $\int_0^T (\ell_\delta(t))^{-\frac{5}{2}} \, dt$, as stated in Theorem 1.2. This is achieved by avoiding the use of the logarithmic inequality (1.8) from [1]. More precisely, we combine the energy bound (1.6) with a Calderon-Zygmund type bound on the symmetric and antisymmetric parts of $Du$.

We note that for $s \in \mathbb{N}$, the estimate (1.14) follows directly from combining Theorem 1 in G. Ponce’s paper [7] with our Lemma 2.2 below, which established the link between $\|Du^+\|_{L^\infty}$ and the length scale $\ell_\delta$ (as stated in Corollary 2.3).

As a second main result in this paper, we obtain a lower bound on the blowup rate of solutions in $H^{\frac{5}{2} + \delta}$, for an arbitrary, real-valued $\delta > 0$. Specifically, we prove:

**Theorem 1.3.** Let $u$ be a solution to (1.1) - (1.3) in the class

$$
C([0, T]; H^{\frac{5}{2}+\delta}) \cap C^1([0, T]; H^{\frac{5}{2}+\delta}).
$$

(1.18)
Suppose that there exists a time $T^*$ such that the solution cannot be continued in the class (1.18) to $T = T^*$. If $T^*$ is the first such time then there exists a finite, positive constant $C(\delta, \|u_0\|_{L^2})$ such that
\[
\|u(\cdot, t)\|_{H^{\frac{5}{2}+\delta}} \geq C(\delta, \|u_0\|_{L^2}) \left(\frac{1}{T^*-t}\right)^{1+\frac{5}{2}\delta},
\] (1.19)
for all $t$ sufficiently close to $T^*$ (see the conditions (3.22) and (3.23) below, with $t_0 = t$).

The proof of Theorem 1.3 can be outlined as follows. We assume that $u$ is a solution in the class (1.18) that cannot be continued to $T = T^*$, and that $T^*$ is the first such time. Invoking the local in time existence result, we derive a lower bound $T_{loc,t_1} > 0$ on the time of existence of solutions to Euler equations in (1.18) for initial data $u(t_1) \in H^{\frac{5}{2}+\delta}$ at an arbitrary time $t_1 < T^*$. By definition of $T^*$, we thus have
\[
t_1 + T_{loc,t_1} < T^*.
\] (1.20)

Based on an energy bound on the $H^{\frac{5}{2}+\delta}$-norm of the solution, we obtain in Section 3 an expression for $T_{loc,t_1}$ of the form
\[
\frac{1}{C(T^*-t_1)},
\] (1.21)
for all $t_1 < T^*$. This is an “a priori” lower bound on the blowup rate. Subsequently, we improve (1.21) by a recursion argument in Theorem 1.3 for times $t$ close to $T^*$, to yield the stronger bound (1.19).

After completing this work, V. Vicol called to our attention that in a recent work, D. Chae proved in [2] (see Theorem 1.1 part (i) of [2]) that for integer values of $s \in \mathbb{N}$ with $s > 1 + \frac{d}{2}$, and in dimensions $d \geq 2$,
\[
\liminf_{t \to T^*} (T^*-t) \|D^s u(t)\|_{L^2(\mathbb{R}^d)} \geq \frac{K}{\|u_0\|_{L^2}}
\] (1.22)
is a necessary and sufficient condition for blowup at time $T^*$, where $K = K(d, s)$ is an absolute constant. In our estimate (1.19), we allow for real values of $s = \frac{5}{2} + \delta$, $\delta > 0$, and provide a pointwise lower bound instead of an infimum limit.

2. PROOF OF THEOREM 1.2

First we recall that the full gradient of velocity $Du$ can be decomposed into symmetric and antisymmetric parts,
\[
Du = Du^+ + Du^- \tag{2.1}
\]
where
\[
Du^\pm = \frac{1}{2} (Du \pm Du^T). \tag{2.2}
\]
$Du^+$ is called the deformation tensor.
In the following lemma, we recall some important properties of \(Du^+\) and \(Du^-\). For the convenience of the reader, we give detailed proofs of those properties, although they are in part available in the literature, see e.g. [3].

**Lemma 2.1.** For both the symmetric and antisymmetric parts \(Du^+\), \(Du^-\) of \(Du\), the \(L^2\) bound

\[
\|Du^\pm\|_{L^2} \leq C\|\omega\|_{L^2}.
\]

holds.

The antisymmetric part \(Du^-\) satisfies

\[
Du^- v = \frac{1}{2} \omega \wedge v
\]

for any vector \(v \in \mathbb{R}^3\). The vorticity \(\omega\) satisfies the identity

\[
\omega(x) = \frac{1}{4\pi} \text{P.V.} \int \sigma(\hat{y}) \omega(x + y) \frac{dy}{|y|^1},
\]

("P.V." denotes principal value) where \(\sigma(\hat{y}) = 3 \hat{y} \otimes \hat{y} - \mathbf{1}\), with \(\hat{y} = \frac{y}{|y|}\). Notably,

\[
\int_{S^2} \sigma(\hat{y}) \, d\mu_{S^2}(y) = 0,
\]

where \(d\mu_{S^2}\) denotes the standard measure on the sphere \(S^2\).

The matrix components of the symmetric part have the form

\[
Du^+_{ij} = \sum_{\ell} T_{ij}^\ell(\omega \ell) = \sum_{\ell} K_{ij}^\ell \ast \omega \ell,
\]

where \(\omega \ell\) are the vector components of \(\omega\), and where the integral kernels \(K_{ij}^\ell\) have the properties

\[
K_{ij}^\ell(y) = \sigma_{ij}^\ell(\hat{y}) |y|^{-3}
\]

(2.8)

\[
\|\sigma_{ij}^\ell\|_{C_1(S^2)} \leq C
\]

(2.9)

\[
\int_{S^2} \sigma_{ij}^\ell(\hat{y}) \, d\mu_{S^2}(y) = 0.
\]

(2.10)

Thus in particular, \(T_{ij}^\ell\) is a Calderon-Zygmund operator, for every \(i, j, \ell \in \{1, 2, 3\}\).

**Proof.** An explicit calculation shows that the Fourier transform of \(Du\) as a function of \(\hat{\omega}\) is given by

\[
\hat{Du}(\xi) = -[(\partial_i (\Delta^{-1} \nabla \wedge \omega) j)]_{i,j} = \hat{G}(\xi) + \hat{H}(\xi)
\]

(2.11)

where

\[
\hat{G}(\xi) := \frac{1}{2|\xi|^2} \begin{bmatrix}
\xi_1 \xi_2 \hat{\omega}_3 - \xi_1 \xi_3 \hat{\omega}_2 & -\xi_2 \xi_3 \hat{\omega}_2 & \xi_2 \xi_3 \hat{\omega}_3 \\
-\xi_1 \xi_3 \hat{\omega}_1 & \xi_2 \xi_3 \hat{\omega}_1 - \xi_1 \xi_2 \hat{\omega}_3 & -\xi_1 \xi_3 \hat{\omega}_3 \\
-\xi_1 \xi_2 \hat{\omega}_1 & \xi_1 \xi_2 \hat{\omega}_2 & \xi_1 \xi_2 \hat{\omega}_2 - \xi_2 \xi_3 \hat{\omega}_1
\end{bmatrix}
\]

(2.12)

and

\[
\hat{H}(\xi) := \frac{1}{2|\xi|^2} \begin{bmatrix}
0 & \xi_2^2 \hat{\omega}_3 - \xi_3^2 \hat{\omega}_2 & -\xi_3^2 \hat{\omega}_2 \\
-\xi_1^2 \hat{\omega}_3 & 0 & \xi_1^2 \hat{\omega}_1 \\
\xi_2^2 \hat{\omega}_2 & -\xi_3^2 \hat{\omega}_2 & 0
\end{bmatrix},
\]

(2.13)

using the notation \(\hat{\omega}_j \equiv \hat{\omega}_j(\xi)\) for brevity.
Clearly, every component of $G$ is given by a sum of Fourier multiplication operators with symbols of the form $\frac{\xi_i}{|\xi|^r}$, $i \neq j$, applied to a component of $\omega$. For instance,

$$G_{21}(x) = \text{const.} \cdot \text{P.V.} \int \frac{y_1 \hat{y}_3}{|y|^3} \omega_1(x + y) \frac{dy}{|y|^3}$$  \hspace{1cm} (2.14)$$
corresponds to the component $G_{21}$. It is easy to see that every component $G_{i,j}$ is a sum of Calderon-Zygmund operators applied to components of $\omega$, with kernel satisfying the asserted properties (2.8) $\sim$ (2.10). The same is true for the symmetric part, $G^+ = \frac{1}{2}(G + G^T)$.

The symmetric part of $\hat{H}(\xi)$ is given by

$$\hat{H}^+(\xi) = \frac{1}{2|\xi|^2} \begin{bmatrix} 0 & (\xi_2^2 - \xi_1^2)\hat{\omega}_3 & (\xi_2^2 - \xi_1^2)\hat{\omega}_2 \\ (\xi_1^2 - \xi_3^2)\hat{\omega}_3 & 0 & (\xi_3^2 - \xi_2^2)\hat{\omega}_1 \\ (\xi_1^2 - \xi_3^2)\hat{\omega}_2 & (\xi_3^2 - \xi_2^2)\hat{\omega}_1 & 0 \end{bmatrix}$$  \hspace{1cm} (2.15)$$
so that each component defines a Fourier multiplication operator with symbol of the form $\frac{\xi_i^2 - \xi_j^2}{|\xi|^2}$, $i \neq j$, acting on a component of $\omega$ (with associated kernel of the form $\frac{x_i^2 - x_j^2}{|x|^{n+2}}$). That is, for instance,

$$H^+_{12}(x) = \text{const.} \cdot \text{P.V.} \int (\hat{y}_2^2 - \hat{y}_1^2) \omega_3(x + y) \frac{dy}{|y|^3}.$$  \hspace{1cm} (2.16)$$
The properties (2.8) $\sim$ (2.10) follow immediately.

The Fourier transforms of the integral kernels $K_{i,j}^t$ can be read off from the components $G_{i,j}^t + H_{i,j}^t$. In position space, one finds that $\sigma^t_i(\hat{y})$ is obtained from a sum of terms proportional to terms of the form $\hat{y}_i \hat{y}_j$, and $(\hat{y}_i^2 - \hat{y}_j^2)$.

For the antisymmetric part $Du^-$, one generally has $Du^- v = \frac{1}{2}(\nabla \wedge u) \wedge v$ for any $v \in \mathbb{R}^3$, and from $u = -\Delta^{-1} \nabla \wedge \omega$, we get $Du^- v = \frac{1}{2} \omega \wedge v$, using that $\nabla \cdot u = 0$.

As a side remark, we note that while $H^-$ does not by itself exhibit the properties (2.8) $\sim$ (2.10), it combines with $G^-$ in a suitable manner to yield the stated properties of $Du^-$, thanks to the condition $\nabla \cdot \omega = 0$. \hfill $\square$

Next, Lemma 2.2 below provides an upper bound in terms of the quantity $\ell_3(t)$ on singular integral operators applied to $\omega$ of the type appearing in (2.7). We note that similar bounds were used in [3] and [5] for the antisymmetric part $Du^-$. Here, we observe that they also hold for the symmetric part $Du^+$. As shown in [5] for $Du^-$, the proof of such a bound invokes standard arguments based on decomposing the singular integral into an inner and outer contribution. The inner contribution can be bounded based on a certain mean zero property, while the outer part is controlled via integration by parts.

**Lemma 2.2.** For $L > 0$ fixed, and $\delta > 0$, let $\ell_3(t)$ be defined as above. Moreover, let $\omega_\ell, \ell = 1, 2, 3$, denote the components of the vorticity vector $\omega(t)$. Then, any singular integral operator

$$T \omega_\ell(x) = \frac{1}{4\pi} \text{P.V.} \int \sigma_T(\hat{y}) \omega_\ell(x + y) \frac{dy}{|y|^3}.$$  \hspace{1cm} (2.17)$$
with
\[
\int_{S^2} \sigma_T(y) d\mu_{S^2}(y) = 0 \quad , \quad \|\sigma_T\|_{C^1(S^2)} < C ,
\] (2.18)
satisfies
\[
\|T\omega_t\|_{L^\infty} \leq C(\delta) \|u_0\|_{L^2} \ell_\delta(t)^{-\frac{2}{5}}
\] (2.19)
for \(\ell \in \{1, 2, 3\}\), for a constant \(C(\delta) = O(\delta^{-1})\) independent of \(u\) and \(t\).

**Proof.** Let \(\chi_1(x)\) be a smooth cutoff function which is identical to 1 on \([0, 1]\), and identically 0 for \(x > 2\). Moreover, let \(\chi_R(x) = \chi_1(x/R)\), and \(\chi_0^c = 1 - \chi_R\).

We consider
\[
\int_{|y| > \epsilon} \sigma_T(y) \omega_t(x + y) \frac{dy}{|y|^3} = (I) + (II)
\] (2.20)
for \(\epsilon > 0\) arbitrary, where
\[
(I) := \int_{|y| > \epsilon} \sigma_T(y) \omega_t(x + y) \chi_{\ell_\delta(t)}(|y|) \frac{dy}{|y|^3}
\] (2.21)
and
\[
(II) := \int \sigma_T(y) \omega_t(x + y) \chi_{\ell_\delta(t)}^c(|y|) \frac{dy}{|y|^3}.
\] (2.22)

From the zero average property (2.18), we find
\[
\|(I)\|_{L^\infty} = \left| \int_{|y| > \epsilon} \sigma_T(y) (\omega_t(x + y) - \omega_t(x)) \chi_{\ell_\delta(t)}(|y|) \frac{dy}{|y|^3} \right|
\]
\[
\leq \|\omega_t\|_{C^s} \int_{|y| < 2\ell_\delta(t)} \frac{dy}{|y|^{3-\delta}}
\]
\[
\leq \frac{C}{\delta} (\ell_\delta(t))^\delta \|\omega_t\|_{C^s}
\]
\[
\leq C \delta^{-1} \|u_0\|_{L^2} (\ell_\delta(t))^{-\frac{2}{5}}
\] (2.23)
since from the definition of \(\ell_\delta(t)\),
\[
\|\omega_t\|_{C^s} \leq \|u_0\|_{L^2} (\ell_\delta(t))^{-\delta - \frac{2}{5}}
\] (2.24)
follows straightforwardly. We can send \(\epsilon \searrow 0\), since the estimates are uniform in \(\epsilon\).

On the other hand,
\[
(II) \quad = \quad \int \sigma_T(y) \partial_y u_j(x + y) \chi_{\ell_\delta(t)}^c(|y|) \frac{dy}{|y|^3}.
\] (2.25)
It suffices to consider one of the terms in the difference,
\[
\left| \int \sigma_T(y) \partial_y u_j(x + y) \chi_{\ell_\delta(t)}^c(|y|) \frac{dy}{|y|^3} \right|
\]
\[
= \left| \int dy u_j(x + y) \partial_y \left( \sigma_T(y) \chi_{\ell_\delta(t)}^c(|y|) \frac{1}{|y|^3} \right) \right|
\]
\[
\leq C \|u_j\|_{L^2} \left\| \partial_y \left( \sigma_T(y) \chi_{\ell_\delta(t)}^c(|y|) \frac{1}{|y|^3} \right) \right\|_{L^2}
\]
\[
\leq C \|u_0\|_{L^2} (\ell_\delta(t))^{-\frac{2}{5}}
\] (2.26)
where to obtain the last line we used the conservation of energy (1.17) and the following three bounds:

(i)  
\[
\left\| \left( \partial_y, \chi_R(|y|) \right) \frac{\sigma_T(\hat{\gamma})}{|y|^3} \right\|_{L^2}^2 \leq C \frac{1}{R^2} \int_{R<|y|<2R} \frac{dy}{|y|^6} \leq CR^{-5},
\]

for \( R = \ell_\delta(t) \).

(ii)  
\[
\left\| \sigma_T(\hat{\gamma}) \chi_R(|y|) \frac{1}{|y|^3} \right\|_{L^2}^2 \leq C \int_{|y|>R} \frac{dy}{|y|^6} \leq CR^{-5}.
\]

(iii)  
\[
\left\| \chi_R(|y|) \frac{1}{|y|^3} \right\|_{L^2}^2 \leq C \int_{|y|>R} \frac{1}{|y|^6} \frac{1}{y^2} dy \leq CR^{-5},
\]

where we used that
\[
\left| \nabla_y \sigma_T(\hat{\gamma}) \right| = \left| \frac{1}{|y|} \left( \nabla_z \sigma_T(z_1, z_2, z_3) \right) \bigg|_{\hat{\gamma}} \right| \leq \frac{1}{|y|} \| \sigma_T \|_{C^1(S^2)}
\]

holds.

Summarizing, we arrive at
\[
\| T \omega_\ell \|_{L^\infty} \leq C(\delta) \| u_0 \|_{L^2} \ell_\delta(t)^{-\frac{5}{2}}
\]

for \( C(\delta) = O(\delta^{-1}) \), which is the asserted bound.

The form of the singular integral operator that appears in the statement of Lemma 2.2 is suitable for application to \( Du^+ \) and \( Du^- \), as we shall see in the following corollary.

Corollary 2.3. There exists a finite, positive constant \( C_\delta = O(\frac{1}{\delta}) \) independent of \( u \) and \( t \) such that the estimate
\[
\| Du^+ \|_{L^\infty} + \| Du^- \|_{L^\infty} \leq C_\delta \| u_0 \|_{L^2} \ell_\delta(t)^{-\frac{5}{2}}
\]

holds.

Proof. According to Lemma 2.1, the matrix components of both \( Du^+ \) and \( Du^- \) have the form (2.17).

Accordingly, Lemma 2.2 immediately implies the assertion. \( \Box \)
Now we are ready to give a proof of Theorem 1.2, which is based on combining an energy estimate for Euler equations with Corollary 2.3.

For $s \geq 3$ integer-valued, the energy bound (1.6)
\[
\frac{1}{2} \partial_t \|u(t)\|^2_{H^s} \leq \|Du(t)\|_{L^\infty} \|u(t)\|^2_{H^s},
\]
was proven in [1]. For fractional $s > \frac{5}{2}$, we recall the definitions of the homogenous and inhomogenous Besov norms for $1 \leq p, q \leq \infty$,
\[
\|u\|_{B^s_{p,q}} = \left( \sum_{j \in \mathbb{Z}} 2^{jqs} \|u_j\|^q_{L^p} \right)^{\frac{1}{q}},
\]
respectively,
\[
\|u\|_{B^s_{p,q}} = \left( \|u\|^q_{L^p} + \|u\|^q_{B^s_{p,q}} \right)^{\frac{1}{q}},
\]
where $u_j = P_j u$ is the Paley-Littlewood projection of $u$ of scale $j$. In analogy to (2.34), we obtain the bound on the $B^s_{2,2}$ Besov norm of $u(t)$ given by
\[
\frac{1}{2} \partial_t \|u(t)\|^2_{B^s_{2,2}} \leq \|Du(t)\|_{L^\infty} \|u(t)\|^2_{B^s_{2,2}},
\]
from a straightforward application of estimates obtained in [8]; details are given in the Appendix. Accordingly, since the left hand side yields
\[
\partial_t \|u(t)\|_{B^s_{2,2}} = 2\|u(t)\|_{B^s_{2,2}} \partial_t \|u(t)\|_{B^s_{2,2}},
\]
we get
\[
\partial_t \|u(t)\|_{B^s_{2,2}} \leq \|Du(t)\|_{L^\infty} \|u(t)\|_{B^s_{2,2}}.
\]
However, Corollary 2.3 implies that
\[
\|Du(t)\|_{L^\infty} \leq \|Du^+(t)\|_{L^\infty} + \|Du^-(t)\|_{L^\infty} \leq C_\delta \|u_0\|_{L^2} \left( \ell_\delta(t) \right)^{-\frac{s}{2}},
\]
Therefore, by combining (2.38) and (2.39) we obtain
\[
\partial_t \|u(t)\|_{B^s_{2,2}} \leq C_\delta \|u_0\|_{L^2} \left( \ell_\delta(t) \right)^{-\frac{s}{2}} \|u(t)\|_{B^s_{2,2}},
\]
which implies that
\[
\|u(t)\|_{H^s} \sim \|u(t)\|_{B^s_{2,2}} \leq \|u_0\|_{B^s_{2,2}} \exp \left[ C_\delta \|u_0\|_{L^2} \int_0^t \ell_\delta(s)^{-\frac{s}{2}} \, ds \right] \sim \|u_0\|_{H^s} \exp \left[ C_\delta \|u_0\|_{L^2} \int_0^t \ell_\delta(s)^{-\frac{s}{2}} \, ds \right],
\]
for $s \geq 0$, where we recall from (2.23) that $C_\delta = O(\delta^{-1})$ (see also [9] for a related bound, but without (2.39)).

This completes the proof of Theorem 1.2. \qed

\footnote{We note that for integer $s \geq 3$, G. Ponce obtained in [9] the following improvement of (2.33),
\[
\frac{1}{2} \partial_t \|u(t)\|^2_{H^s} \leq \|Du^+(t)\|_{L^\infty} \|u(t)\|^2_{H^s}.
\]
Our proof of (2.36) for fractional $s > \frac{5}{2}$ does not yield the analogous improved bound.
3. Lower bounds on the blowup rate

In this section, we prove Theorem 1.3.

Recalling the energy bound (2.38),
\[
\partial_t \|u(t)\|_{B^s_{2,2}} \leq \|Du(t)\|_{L^\infty} \|u(t)\|_{B^s_{2,2}},
\]
we invoke the Sobolev embedding
\[
\|Du\|_{L^\infty} \leq \|\hat{D}u\|_{L^1} \leq \left( \int d\xi \langle \xi \rangle^{-3-2\delta} \right)^{\frac{1}{2}} \|Du\|_{H^{\frac{1}{2}+\delta}} \leq C_\delta \|u\|_{H^{\frac{1}{2}+\delta}} \sim C_\delta \|u\|_{B^s_{2,2}},
\]
with \(C_\delta = O(\delta^{-\frac{1}{2}})\), to get, for \(s = \frac{5}{2} + \delta\),
\[
\partial_t \|u(t)\|_{B^s_{2,2}} \leq C_\delta (\|u(t)\|_{B^s_{2,2}})^2.
\]
Straightforward integration implies
\[
- \left( \frac{1}{\|u(t)\|_{B^s_{2,2}}} - \frac{1}{\|u(t_0)\|_{B^s_{2,2}}} \right) \leq C_\delta (t - t_0).
\]
Hence,
\[
\|u(t)\|_{H^s} \sim \|u(t)\|_{B^s_{2,2}} \leq \frac{\|u(t_0)\|_{B^s_{2,2}}}{1 - (t - t_0)C_\delta \|u(t_0)\|_{H^s}} \sim \frac{\|u(t_0)\|_{H^s}}{1 - (t - t_0)C_\delta \|u(t_0)\|_{H^s}},
\]
where a possible trivial modification of \(C_\delta\) is implicit in passing to the last line. This implies that the solution \(u(t)\) is locally well-posed in \(H^s\), with \(s = \frac{5}{2} + \delta\), for
\[
t_0 \leq t < t_0 + \frac{1}{C_\delta \|u(t_0)\|_{H^s}}.
\]
In particular, this infers that if \(T^*\) is the first time beyond which the solution \(u\) cannot be continued, one necessarily has that
\[
T^* > t_0 + \frac{1}{C_\delta \|u(t_0)\|_{H^s}}.
\]
This in turn implies an a priori lower bound on the blowup rate given by
\[
\|u(t)\|_{H^s} > \frac{1}{C_\delta (T^* - t)}
\]
for all \(0 \leq t < T^*\).
Next, we derive the lower bound on the blowup rate stated in Theorem 1.3 which is stronger than (3.8). To begin with, we note that
\[
\| \omega(t) \|_{C^\delta} \leq C_\delta \| u(t) \|_{H^{\frac{5}{2} + \delta}} \\
\leq \frac{C_\delta \| u(t_0) \|_{H^{\frac{5}{2} + \delta}}}{1 - (t - t_0)C_\delta \| u(t_0) \|_{H^{\frac{5}{2} + \delta}}}. \tag{3.9}
\]
That is, local well-posedness of \( u \) in \( H^{\frac{5}{2} + \delta} \) implies \( \delta \)-Holder continuity of the vorticity.

The parameter \( L \) in the definition (1.11) of \( \ell_\delta(t) \) is arbitrary. Thus, in view of (3.9), we may now let \( L \to \infty \) for convenience. Then,
\[
\ell_\delta(t)^{-\frac{\delta}{2}} = \left( \frac{\| \omega(t) \|_{C^\delta}}{\| u_0 \|_{L^2}} \right)^{\frac{\delta}{2}} \\
\leq \left( \frac{C_\delta \| u(t) \|_{H^{\frac{5}{2} + \delta}}}{\| u_0 \|_{L^2}} \right)^{1-\delta} \\
\leq \left( \frac{C_\delta}{\| u_0 \|_{L^2}} \right)^{1-\delta} \left( \frac{\| u(t_0) \|_{H^s}}{1 - (t - t_0)C_\delta \| u(t_0) \|_{H^s}} \right)^{1-\delta}, \tag{3.10}
\]
where
\[
\tilde{\delta} := \frac{2\delta}{5 + 2\delta} \quad \text{and} \quad s = \frac{5}{2} + \delta. \tag{3.11}
\]
We note that while the right hand side of (3.10) diverges as \( t \) approaches \( t_1 := t_0 + \frac{1}{C_\delta \| u(t_0) \|_{H^s}} \),
\[
\int_{t_0}^{t_1} \ell_\delta(t)^{-\frac{\delta}{2}} dt \leq \left( \frac{C_\delta}{\| u_0 \|_{L^2}} \right)^{1-\delta} \int_{t_0}^{t_1} \left( \frac{\| u(t_0) \|_{H^s}}{1 - (t - t_0)C_\delta \| u(t_0) \|_{H^s}} \right)^{1-\delta} dt \\
=: B_0(\delta) \tag{3.13}
\]
converges for \( \delta > 0 \) (\( \Leftrightarrow \tilde{\delta} > 0 \)). This implies that the solution \( u(t) \) for \( t \in [t_0, t_1) \) can be extended to \( t > t_1 \).

In particular, we obtain that
\[
\| u(t_1) \|_{H^{\frac{5}{2} + \delta}} \leq \| u(t_0) \|_{H^{\frac{5}{2} + \delta}} \exp \left( C_\delta \| u_0 \|_{L^2} \int_{t_0}^{t_1} (\ell_\delta(t))^{-\frac{\delta}{2}} dt \right) \\
\leq \| u(t_0) \|_{H^{\frac{5}{2} + \delta}} \exp \left( C_\delta \| u_0 \|_{L^2} B_0(\delta) \right) \tag{3.14}
\]
from Theorem 1.2.

We may now repeat the above estimates with initial data \( u(t_1) \) in \( H^{\frac{5}{2} + \delta} \), thus obtaining a local well-posedness interval \([t_1, t_2] \). Accordingly, we may set \( t_2 \) to be given by
\[
t_2 := t_1 + \frac{1}{C_\delta \| u(t_1) \|_{H^s}}. \tag{3.15}
\]
More generally, we define the discrete times $t_j$ by

$$t_{j+1} := t_j + \frac{1}{C_δ\|u(t_j)\|_{H^s}}.$$  \hspace{1cm} (3.16)

We then have

$$\|u(t_{j+1})\|_{H^s} \leq \exp\left(C_δ\|u_0\|_{L^2} B_j(δ)\right) \|u(t_j)\|_{H^s},$$  \hspace{1cm} (3.17)

where $B_j(δ)$ is defined by

$$C_δ\|u_0\|_{L^2} B_j(δ) := C_δ\|u_0\|_{L^2} \left(\frac{C_δ}{\|u_0\|_{L^2}}\right)^{1-δ} \int_{t_j}^{t_{j+1}} \left(\frac{\|u(t_j)\|_{H^s}}{1 - (t - t_j)C_δ\|u(t_j)\|_{H^s}}\right)^{1-δ} dt$$

$$= \frac{1}{δ} C_δ^{1-δ} \left(\frac{\|u_0\|_{L^2}}{\|u(t_j)\|_{H^s}}\right)^{δ}$$

$$=: b_δ \left(\frac{\|u_0\|_{L^2}}{\|u(t_j)\|_{H^s}}\right)^{δ}. \hspace{1cm} (3.18)$$

Letting

$$ρ_j := \exp\left(b_δ \left(\frac{\|u_0\|_{L^2}}{\|u(t_j)\|_{H^s}}\right)^{δ}\right),$$  \hspace{1cm} (3.19)

we have

$$\|u(t_j)\|_{H^s} \leq ρ_{j-1} \|u(t_{j-1})\|_{H^s}, \hspace{1cm} (3.20)$$

and we remark that $(ρ_j)_j$ satisfy the recursive estimates

$$ρ_j \geq \exp\left(b_δ \left(\frac{\|u_0\|_{L^2}}{ρ_{j-1}\|u(t_{j-1})\|_{H^s}}\right)^{δ}\right)$$

$$= (ρ_{j-1})^{b_δ^{δ-1}}$$

$$= \exp\left(ρ_{j-1}^{δ-1} \ln ρ_{j-1}\right). \hspace{1cm} (3.21)$$

We note that from its definition, $ρ_j > 1$ for all $j$.

We shall now assume that $T^* > 0$ is the first time beyond which the solution $u(t)$ cannot be continued. Thus, by choosing $t_0$ close enough to $T^*$, \hspace{2cm} (3.8) implies that $\|u(t_0)\|_{H^s}$ can be made sufficiently large that the following hold:

1. The quantity

$$b_δ \left(\frac{\|u_0\|_{L^2}}{\|u(t_0)\|_{H^s}}\right)^{δ} \ll 1$$  \hspace{1cm} (3.22)

is small.

2. There is a positive, finite constant $\tilde{C}$ independent of $j$ such that

$$\|u(t_j)\|_{H^s} \geq \tilde{C} \|u(t_0)\|_{H^s}$$  \hspace{1cm} (3.23)

holds for all $j \in \mathbb{N}$. Without any loss of generality (by a redefinition of the constant $b_δ$ if necessary), we can assume that $\tilde{C} = 1$. 


We note that in principle, there might be strong oscillations close to blowup so that (3.23) is not obviously true. The fact that (3.23) holds follows from the a priori bound (3.8).

Accordingly, (3.23) with $\tilde{C} = 1$ implies that $\rho_j \leq \rho_0$ for all $j$. Then, for any $N \in \mathbb{N}$,

$$T^* - t_0 \geq \sum_{j=0}^{N} (t_{j+1} - t_j)$$

$$= \frac{1}{C_\delta} \left( \frac{1}{\|u(t_0)\|_{H^s}} + \cdots + \frac{1}{\|u(t_N)\|_{H^s}} \right)$$

$$= \frac{1}{C_\delta \|u(t_0)\|_{H^s}} \left( 1 + \frac{\|u(t_0)\|_{H^s}}{\|u(t_1)\|_{H^s}} + \cdots + \frac{\|u(t_0)\|_{H^s}}{\|u(t_N)\|_{H^s}} \right)$$

$$\geq \frac{1}{C_\delta \|u(t_0)\|_{H^s}} \left( 1 + \frac{1}{\rho_0} + \cdots + \frac{1}{\rho_0 \cdots \rho_N} \right)$$

$$\geq \frac{1}{C_\delta \|u(t_0)\|_{H^s}} \left( 1 + \frac{1}{\rho_0} + \cdots + \frac{1}{\rho_0^N} \right) \quad (3.24)$$

from $\frac{1}{\rho_j} \geq \frac{1}{\rho_0}$ for all $j$, and the fact that $\rho_0 > 1$ since the argument in the exponent (3.19) is positive.

Then, letting $N \to \infty$, we obtain

$$\frac{1}{T^* - t_0} \leq C_\delta \|u(t_0)\|_{H^s} \left( 1 - \frac{1}{\rho_0} \right)$$

$$= C_\delta \|u(t_0)\|_{H^s} \left( 1 - \exp \left( - b_\delta \left( \frac{\|u_0\|_{L^2}}{\|u(t_0)\|_{H^s}} \right)^{\frac{1}{\delta}} \right) \right). \quad (3.25)$$

Next, we deduce a lower bound on the blowup rate.

Invoking (3.22), we obtain

$$\frac{1}{T^* - t_0} \leq C_\delta \|u(t_0)\|_{H^s} \left( 1 - \exp \left( - b_\delta \left( \frac{\|u_0\|_{L^2}}{\|u(t_0)\|_{H^s}} \right)^{\frac{1}{\delta}} \right) \right)$$

$$\approx C_\delta \|u(t_0)\|_{H^s} \|u_0\|_{L^2}^\delta \\|u(t_0)\|_{H^s}^{-\delta}$$

$$= C_\delta \|u_0\|_{L^2}^{\delta} \|u(t_0)\|_{H^s}^{1-\delta}. \quad (3.26)$$

This implies a lower bound on the blowup rate of the form

$$\|u(t_0)\|_{H^{s+\delta}} \geq C(\delta, \|u_0\|_{L^2}) \left( \frac{1}{T^* - t_0} \right)^{\frac{1}{1-\delta}}$$

$$= C(\delta, \|u_0\|_{L^2}) \left( \frac{1}{T^* - t_0} \right)^{\frac{2\delta + 5}{5}}, \quad (3.27)$$

under the condition that (3.22) and (3.19) hold.

This concludes our proof of Theorem 1.3. \qed
APPENDIX A. PROOF OF INEQUALITY (2.38) FOR $s > \frac{5}{2}$

In this Appendix, we prove (2.38) which follows from (2.36),

$$\frac{1}{2} \partial_t \|u(t)\|_{L^2_{B_{s,2}^z}}^2 \lesssim \|Du(t)\|_{L^\infty} \|u(t)\|_{L^2_{B_{s,2}^z}}^2,$$  \hspace{1cm} (A.1)

for $s > \frac{5}{2}$. We invoke Eq. (26) in the work [5] of F. Planchon, which is valid for $s > 1 + \frac{2}{n}$ in $n$ dimensions (thus, $s > \frac{5}{2}$ in our case of $n = 3$), for parameter values $p = q = 2$ in the notation of that paper. It yields

$$\frac{1}{2} \partial_t 2^{2js} \|u_j\|_{L^2}^2 \lesssim 2^{2js} \sum_{k \sim j} \|S_{j+1} Du\|_{L^\infty} \|u_k\|_{L^2} \|u_j\|_{L^2}$$

$$+ 2^{2js} \sum_{j \leq k \sim k'} \|u_k\|_{L^2} \|u_k\|_{L^2} \|Du_j\|_{L^\infty}$$  \hspace{1cm} (A.2)

where $u_k = P_k u$ is the Paley-Littlewood projection of $u$ at scale $k$, and $S_j = \sum_{j' \leq j} P_{j'}$ is the Paley-Littlewood projection to scales $\leq j$.

Summing over $j$,

$$\frac{1}{2} \partial_t \sum_j 2^{2js} \|u_j\|_{L^2}^2 \lesssim \sup_j \|S_{j+1} Du\|_{L^\infty} \left( \sum_j \|S_{j+1} Du\|_{L^\infty} \left( \sum_{k \sim j} 2^{2js} \|u_k\|_{L^2} \|u_j\|_{L^2} 

+ \sum_k \sum_{j \leq k \sim k' \geq j} 2^{2s(j-k)} 2^{k} \|u_k\|_{L^2} 2^{k'} \|u_k\|_{L^2} \right) \right)$$

$$\lesssim \|Du\|_{L^\infty} \left( \sum_j \|u_j\|_{L^2}^2 

+ \sum_k \left( \sum_{j \leq k} 2^{2s(j-k)} \right) \|u_k\|_{L^2}^2 \right) \right)$$

$$\lesssim \|Du\|_{L^\infty} \sum_j 2^{2js} \|u_j\|_{L^2}^2.$$  \hspace{1cm} (A.3)

To pass to the second inequality, we used that

$$\|S_{j+1} Du\|_{L^\infty} = \|m_{j+1} * Du\|_{L^\infty} \lesssim \|Du\|_{L^\infty} \|m_{j+1}\|_{L^1},$$  \hspace{1cm} (A.4)

where $m_j$ is the symbol of the Fourier multiplication operator $S_j$, and the fact that $\|m_j\|_{L^1} \sim 1$ uniformly in $j$. Accordingly, we get

$$\frac{1}{2} \partial_t \|u(t)\|_{B^z_{s,2}}^2 \lesssim \|Du(t)\|_{L^\infty} \|u(t)\|_{B^z_{s,2}}^2.$$  \hspace{1cm} (A.5)

From

$$\|u(t)\|_{B^z_{s,2}}^2 = \|u(t)\|_{L^2}^2 + \|u(t)\|_{B^z_{s,2}}^2,$$  \hspace{1cm} (A.6)

and energy conservation, $\partial_t \|u(t)\|_{B^z_{s,2}}^2 = 0$, we obtain

$$\frac{1}{2} \partial_t \|u(t)\|_{B^z_{s,2}}^2 = \frac{1}{2} \partial_t \|u(t)\|_{B^z_{s,2}}^2 \lesssim \|Du(t)\|_{L^\infty} \|u(t)\|_{B^z_{s,2}}^2$$

$$\lesssim \|Du(t)\|_{L^\infty} \|u(t)\|_{B^z_{s,2}}^2 \lesssim \|Du(t)\|_{L^\infty} \|u(t)\|_{B^z_{s,2}}^2.$$  \hspace{1cm} (A.7)

This proves (A.1). □
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