Optimal stopping under $g_T$-expectation

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Abstract

In this paper, we solve the existence problem of optimal stopping problem under some kind of nonlinear expectation named $g_T$-expectation which was recently introduced in Peng, S.G. and Xu, M.Y. [8]. Our method based on our preceding work on the continuous property of $g_T$-solution. Generally, the strict comparison theorem does not hold under such nonlinear expectations any more, but we can still modify the classical method to find out an optimal stopping time via continuous property. The mainly used theory in our paper is the monotonic limit theorem of BSDE and nonlinear decomposition theorem of Doob-Meyer’s type developed by Peng S.G. [6]. With help of these useful theories, an RCLL modification of the value process can also be obtained by a new approach instead of down-crossing inequality.

Keywords: CBSDE, $g_T$-expectation, Optimal stopping

1 Introduction

Optimal stopping is a very classical and meaningful problem in pure stochastic analysis and applications, a well-known example is that the price of American claims under complete market without arbitrage can be represented by the value function of an optimal stopping under some linear expectation induced by a probability. In recent years, nonlinear expectation become more and more wildly studied by authors. Among all kinds of nonlinear expectations, $g$-expectation which was introduced by Peng S.G. [7] is a nice example for it enjoys many nice properties like linear expectation such as continuous property and strict comparison property as well as time-consistence property. In more general case, for a given family of $\sigma$-fields $\{\mathcal{F}_t\}_{t \in [0,T]}$, some kind of $\mathcal{F}$-expectation can be defined by axioms. An interesting problem that when $\mathcal{F}$-expectation can be represented by $g$-expectation was considered in Coquet, F., Hu, Y., Memin, J., and Peng, S.G. [2] where we refer to the definition of $\mathcal{F}$-expectation.

From an application point of view, many nonlinear expectations are inevitable just because of the world in our reality is not idea and perfect. For example, the pricing and hedging problem can be modeled by linear BSDE under complete market without arbitrage while BSDE driven by nonlinear generator function $g$ becomes reasonable when the market is incomplete or other kinds of constraints be considered.

In the framework of nonlinear expectation, Riedel, F. [9] studied the optimal stopping problem with multiple priors. The author developed a theory of the optimal stopping along the classical lines using and extending suitable results from martingale theory in finite discrete time model. This approach works as long as the set of priors is time consistent. Krätschmer, V. and Schoenmakers, J. [4] considered the optimal stopping for more general dynamic utility functionals satisfying nice properties such as time consistency an recursiveness but without strict comparison property in finite discrete time case. In their paper, the authors provided sufficient conditions for Bellman principle and the existence of optimal stopping. For continuous
time, an optimal stopping problem was considered under ambiguity by Cheng, X. and Riedel, F. [3]. In that paper, the author solve the optimal stopping problem using nonlinear Doob-Meyer-Peng decomposition of \( g \)-supermartingale. However, Bayraktar, E. and Yao, S [1] developed a theory for solving continuous time stopping problems for general non-linear expectations. Given a stable family of \( \mathcal{F} \)-expectations \( \{ \mathcal{E}_i \}_{i \in I} \) defined well on a common domain, the authors considered the optimal problems

\[
\sup_{(i, \tau) \in I \times S_{0,T}} \mathcal{E}_i(Y_\tau + H^i_\tau) \tag{1.1}
\]

and

\[
\sup_{\tau \in S_{0,T}} \inf_{i \in I} \mathcal{E}_i(Y_\tau + H^i_\tau). \tag{1.2}
\]

where \( S_{0,T} \) denotes the whole stopping times valued on \([0, T]\) and \((Y_i + H^i_t), i \in I\) are the model-dependent reward processes. Among all above papers except for Kr"atschmer, V. and Schoenmakers, J. [4], nonlinear expectations all satisfy the uncommon property of strict comparison and stable property.

In our paper, we consider the optimal stopping problem under \( gr \)-expectation which was introduced by Peng, S.G. and Xu, M.Y. [8] as follows:

\[
\sup_{\tau \in S_{0,T}} \mathcal{E}_0^{g,\phi}(X_\tau). \tag{1.3}
\]

where \( \mathcal{E}_0^{g,\phi}(\cdot) \) is the \( g \)-expectation under some constraints \( \phi(t, y, z) = 0 \) well defined on some suitable space and \((X_t)\) is a reward process satisfying some mild assumptions.

Although the \( gr \)-expectation can not easily be represented by a stable class of \( g \)-expectations, but it is still a increasing limit of a sequence of \( g_n \)-expectations via penalization method. A main difficulty is that the strict comparison theorem may not holds for \( gr \)-expectation any more. The method used to solve optimal stopping problem in above mentioned papers must be modified to work well in our case. Fortunately, with the help of some results about the continuous property of \( gr \)-expectation obtained in our preceding paper Wu, H.L. [10], we can still find out an optimal solution of this problem.

Our paper is organized as follows: In section 2, we give some necessary definitions such as \( gr \)-expectation and some useful properties of it. In section 3, we work out the optimal problem by a modified method of classical one.

## 2 CBSDE and \( gr \)-expectation

Given a probability space \((\Omega, \mathcal{F}, P)\) and \(R^d\)-valued Brownian motion \(W(t)\), we consider a sequence \(\{(\mathcal{F}_t); t \in [0, T]\}\) of filtrations generated by Brownian motion \(W(t)\) and augmented by \(P\)-null sets. \( \mathcal{P} \) is the \( \sigma \)-field of predictable sets of \( \Omega \times [0, T] \). We use \(L^2(\mathcal{F}_T)\) to denote the space of all \( \mathcal{F}_T \)-measurable random variables \( \xi : \Omega \to R^d \) for which

\[
\| \xi \|^2 = E[|\xi|^2] < +\infty.
\]

and use \(H^2_{T}(R^d)\) to denote the space of predictable process \( \varphi : \Omega \times [0, T] \to R^d \) for which

\[
\| \varphi \|^2 = E[\int_0^T |\varphi|^2] < +\infty.
\]

For a given probability \( P \), we denote the Banach space of all \( P \)-essentially bounded real functions on a probability space \((\Omega, \mathcal{F}, P)\) as \( L^\infty(\mathcal{F}_T)\).
Given a function \( \varphi : [0, T] \times R \times R^d \rightarrow R \), following assumptions always used in theory of BSDE.

\[
|\varphi(\omega, t, y_1, z_1) - \varphi(\omega, t, y_2, z_2)| \leq M(|y_1 - y_2| + |z_1 - z_2|), \quad \forall (y_1, z_1), (y_2, z_2)
\]  

(A1)

for some \( M > 0 \).

\[
(\cdot, y, z) \in H^2_T(R) \quad \forall y \in R, z \in R^d
\]  

(A2)

The backward stochastic differential equation (shortly BSDE) driven by \( g(t, y, z) \) is given by

\[
dy_t = g(t, y_t, z_t)dt + z^*_tdW(t)
\]  

(2.1)

where \( y_t \in R \) and \( W(t) \in R^d \). Suppose that \( \xi \in L^2(\mathcal{F}_T) \) and \( g \) satisfies (A1) and (A2), Pardoux, E., Peng, S.G. [5] proved that \( g(\cdot, \xi) \) standard parameters for the BSDE.

We call the pair \((y_t, z_t)\) satisfying (2.1) a \( g \)-solution, but when an increasing process is added in a BSDE, the notion of super-solution is introduced by researchers.

**Definition 2.1.** *(super-solution)* A super-solution of a BSDE associated with the standard parameters \((g, \xi)\) is a vector process \((y_t, z_t, C_t)\) satisfying

\[
dy_t = g(t, y_t, z_t)dt + dC_t - z^*_tdW(t), \quad y_T = \xi,
\]  

(2.2)

or being equivalent to

\[
y_t = \xi + \int_t^T g(s, y_s, z_s)ds - \int_t^T z^*_sdW_s + \int_t^T dC_s,
\]  

(2.2')

where \((C_t, t \in [0, T])\) is an increasing, adapted, right-continuous process with \( C_0 = 0 \) and \( z^*_t \) is the transpose of \( z_t \).

In many analysis and applications, constraints always put on \((y_t, z_t)\). We formulate the constraints like stated in Peng, S.G. [6]. For a given function \( \phi(t, y, z) : [0, T] \times R \times R^d \rightarrow R^+ \), we define a subset in \([0, T] \times R \times R^d\) as \( \Gamma_t = \{(t, y, z) : \phi(t, y, z) = 0\} \).

A super-solution \((y_t, z_t, A_t)\) is said to satisfies constraints if the following condition holds,

\[
(t, y_t, z_t) \in \Gamma_t.
\]  

(2.3)

Constraints like (2.3) is always considered in this paper. In such case, we give the following definition.

**Definition 2.2.** *(\( g_T \)-solution or the minimal solution)* A \( g \)-supersolution \((y_t, z_t, C_t)\) is said to be the minimal solution of a constrained backward differential stochastic equation (shortly CB-SDE), given \( y_T = \xi \), subject to the constraint (2.3) if for any other \( g \)-supersolution \((y'_t, z'_t, C'_t)\) satisfying (2.3) with \( y'_T = \xi \), we have \( y_t \leq y'_t \) a.e., a.s.. The minimal solution is denoted by \( \mathcal{E}^{\phi}_{g, T}(\xi) \) and for convenience called as \( g_T \)-solution. Sometimes, we also call \( g_T \)-expectation \( \mathcal{E}^{\phi}_{g, T}(\xi) \) the dynamic \( g_T \)-expectation with constraints (2.3).

For any \( \xi \in L^2(\mathcal{F}_T) \), we denote \( \mathcal{H}^d(\xi) \) as the set of \( g \)-supersolutions \((y_t, z_t, C_t)\) subjecting to (2.3) with \( y_T = \xi \). When \( \mathcal{H}^d(\xi) \) is not empty, Peng, S.G. [6] proved that \( g_T \)-solution exists.

In general case, unlike \( g \)-solution, the increasing part of \( g_T \)-solution is different with different terminal value and it is impossible to get a similar priori estimation. The continuous property seems hard to hold, however, we can prove it is still continuous from below similarly like Wu, H.L. [10].
Proposition 2.1. Suppose the generator function \( g(t, y, z) \) and the constraint function \( \phi(t, y, z) \) both satisfy conditions (A1) and (A2), \( \{\xi_n \in L^\infty(F_T), \; n = 1, 2, \cdots\} \) is a norm-bounded increasing sequence in \( L^\infty(F_T) \) and converges almost surely to \( \xi \in L^\infty(F_T) \), if \( E_t^{g,\phi}(\xi_n) \) exists for \( \xi = \xi_n, n = 1, 2, \cdots \), then
\[
\lim_{n \to \infty} E_t^{g,\phi}(\xi_n) = E_t^{g,\phi}(\xi).
\]
In order to obtain a whole continuity, we always assume that both \( g \) and \( \phi \) are convex functions.

The convexity of \( E_t^{g,\phi}(\xi) \) can be easily deduced from the same proposition of solution of BSDE with convex generator function, see also Peng, S.G. and Xu, M.Y. [8].

Proposition 2.2. Suppose that \( \phi(t, y, z) \) and \( g(t, y, z) \) are both convex in \((y, z)\) and satisfy (A1) and (A2), then
\[
E_t^{g,\phi}(a\xi + (1-a)\eta) \leq aE_t^{g,\phi}(\xi) + (1-a)E_t^{g,\phi}(\eta) \quad \forall t \in [0, T]
\]
holds for any \( \xi, \eta \) in the effective domain of CBSDE and \( a \in [0, 1] \).

Proof According to Peng, S.G. [6], the solutions \( y_n^t(\xi) \) of
\[
y_n^t(\xi) = \xi + \int_t^T g(y_n^s(\xi), z_n^s, s)ds + A_n^T - A_n^t - \int_t^T z_n^s dW_s.
\]
is an increasing sequence and converges to \( E_t^{g,\phi}(\xi) \), where
\[
A_n^t := n \int_0^t \phi(y_n^s, z_n^s, s)ds.
\]
For any fixed \( n \), by the convexity of \( g \) and \( \phi \), \( y_n^t(\xi) \) is a convex in \( \xi \), that is
\[
y_n^t(a\xi + (1-a)\eta) \leq a y_n^t(\xi) + (1-a)y_n^t(\eta),
\]
taking limit as \( n \to \infty \), we get the required result. \( \square \)

By the same method of penalization, we can get the comparison theorem of \( E_t^{g,\phi}(\xi) \).

Proposition 2.3. Under the same assumptions as above proposition, we have
\[
E_t^{g,\phi}(\xi) \leq E_t^{g,\phi}(\eta)
\]
for any \( \xi, \eta \in L^2_T(R) \) when \( P(\eta \geq \xi) = 1 \).

In order to make the domain of definition of CBSDE more explicitly for our use, we give another mild assumption below,
\[
\varphi(\cdot, y, 0) = 0 \quad \forall y \in R.
\]
(A3)
The following result can be easily obtained with the help of Peng, S.G. and Xu, M.Y. [8].

Proposition 2.4. Suppose the generator function \( g \) and the constraint function \( \phi \) satisfy assumptions \( A(i), i = 1, 2, 3 \), then the \( g_T \)-solution exists for any \( \xi \in L^\infty(F_T) \) with terminal condition \( y_T = \xi \).
Proof In the paper Peng, S.G. and Xu, M.Y. [8], the author define a new subspace $L^2_T(R)$:

$$L^2_{t,\infty}(\mathcal{F}_T) \triangleq \{\xi \in L^2(\mathcal{F}_T), \xi^+ \in L^\infty(\mathcal{F}_T)\}.$$

For any $\xi \in L^2_{t,\infty}(\mathcal{F}_T)$ with terminal condition $y_T = \xi$, the existence of $g_T$-solution was proved in that paper under the assumption

$$g(t, y, 0) \leq L_0 + M|y| \quad \text{and} \quad (y, 0) \in \Gamma_t$$

holds for a large constant $L_0$ and for any $y \geq L_0$.

It is obvious $L^\infty(\mathcal{F}_T) \subset L^2_{t,\infty}(\mathcal{F}_T)$ and under assumptions $A(i), i = 1, 2, 3, (2.4)$ holds for any $L_0 \geq 0$ and $M$ in (A1), thus $g_T$-solution is defined well on the whole space $L^\infty(\mathcal{F}_T)$. $\square$.

The following nice properties of $\mathcal{E}_t^{g,\phi}(\cdot)$ will be helpful in our study, their proofs can be found in Peng, S.G. and Xu, M.Y. [8].

**Proposition 2.5.** Suppose the generator function $g$ and the constraint function $\phi$ satisfy assumptions $A(i), i = 1, 2, 3$, then the $g_T$-expectation satisfies:

(i) Self-preserving: $\mathcal{E}_t^{g,\phi}(\xi_t) = \xi_t$ for any $\xi_t \in L^\infty(\mathcal{F}_t)$.

(ii) Time consistency: $\mathcal{E}_t^{g,\phi}(\mathcal{E}_s^{g,\phi}(\xi)) = \mathcal{E}_t^{g,\phi}(\xi), \quad 0 \leq s \leq t \leq T \quad \xi \in L^\infty(\mathcal{F}_T)$.

(iii) 1-0 law: $1_\Delta \mathcal{E}_t^{g,\phi}(\xi) = \mathcal{E}_t^{g,\phi}(1_\Delta \xi), \quad \forall A \in \mathcal{F}_t$.

When both $g(t, y, z)$ and $\phi(t, y, z)$ are convex in $(y, z)$, with the help of convex analysis, Wu, H.L. [10] has proved $g_T$-solution is continuous according to the norm of $L^\infty(\mathcal{F}_T)$ on the whole space $L^\infty(\mathcal{F}_T)$. The continuous property will play a crucial role in our analysis in next section.

## 3 Optimal stopping under $g_T$-expectation

In this section, we want to find an optimal stopping time which attains the supremum:

$$\sup_{\tau \in \mathcal{S}_{0,T}} \mathcal{E}_0^{g,\phi}(X_{\tau}).$$

For simplicity and making $\mathcal{E}_0^{g,\phi}(X_{\tau})$ meaningful, we assume that the model-depend reward process $X_{t, t} \in [0, T]$ is an adapted, nonnegative process with continuous sample paths. Furthermore, we still assume $X_{t, t} \in [0, T]$ is bounded in $L^\infty(\mathcal{F}_T)$. Similarly, like definition in Bayraktar, E. and Yao, S. [1], a process $X_{t, t} \in [0, T]$ is called uniformly-left-continuous if for any sequence $\{\tau_n\}_{n \in \mathcal{N}} \subset \mathcal{S}_{0,T}$ increasing a.s to $\tau$, we can find a subsequence $\{n_k\}_{k \in \mathcal{N}_4}$ of the set of positive nature numbers $\mathcal{N}$ such that the sequence of random variables $X_{\tau_{n_k}, k = 1, 2, \cdots}$ converges to $X_\tau$ in $L^\infty(\mathcal{F}_T)$ according to norm.

Under $g_T$-expectation, we define the value function of the optimal stopping problem as

$$V_t \triangleq \text{ess sup}_{\tau \in \mathcal{S}_{t,T}} \mathcal{E}_t^{g,\phi}(X_{\tau}).$$

As usual, we define supermartingale (respectively submartingale, martingale) under $g_T$-expectation as done in Peng, S.G. and Xu, M.Y. [8].

**Definition 3.1.** A process $(X_t)$ which is adapted and $X_t \in L^\infty(\mathcal{F}_t)$ for every $t \in [0, T]$ is called a $g_T$-supermartingale (respectively submartingale, martingale) on $[0, T]$, if for $0 \leq s \leq t \leq t \leq T$ we have

$$\mathcal{E}_t^{g,\phi}(X_t) \leq X_s, (\text{resp.}, \geq, = X_s).$$
Just as classical case, we show that \((V_t)\) defined by (3.2) is a \(g_t\)-supermartingale, it is based on the continuous property of \(g_t\)-solution and the following lemma.

**Lemma 3.1.** For all \(t \geq 0\), the family 
\[ \{ \mathcal{E}^{\mathbb{G}, \phi}_t(X_{\tau}) : \tau \geq t \} \]
is upwards directed.

**Proof** Thanks to the useful property of 1-0 law of \(g_t\)-expectation, we can prove this result by the usual way, for details, see for example Lemma B.1 in Cheng, X. and Riedel, F. [3].

With the help of this lemma, we have

**Proposition 3.1.** Under the assumptions on the reward process in our paper, the value function \((V_t)\) defined by (3.2) is a \(g_t\)-supermartingale.

**Proof** For every \(t \geq 0\), the lemma above allows us to choose a sequence \(\{ \tau_n(t), n = 1, 2, \cdot \} \) of stopping times greater or equal \(t\) with 
\[ \mathcal{E}^{\mathbb{G}, \phi}_t(X_{\tau_n(t)}) \uparrow V_t. \]
Since \(\mathcal{E}^{\mathbb{G}, \phi}_t(X_{\tau_n(t)})\) converges to \(V_t\) increasingly, by the continuous property from below of proposition (2.1) and time consistency property \((ii)\) in proposition 2.5, for \(0 \leq s \leq t \leq T\), we have
\[
\mathcal{E}^{\mathbb{G}, \phi}_s(V_t) = \mathcal{E}^{\mathbb{G}, \phi}_s(\sup_{\tau \in S_t, T} \mathcal{E}^{\mathbb{G}, \phi}_t(X_{\tau_n(t)})) = \mathcal{E}^{\mathbb{G}, \phi}_s(\lim_{n \to \infty} \mathcal{E}^{\mathbb{G}, \phi}_t(X_{\tau_n(t)})) \\
= \lim_{n \to \infty} \mathcal{E}^{\mathbb{G}, \phi}_n(X_{\tau_n(t)}) = \lim_{n \to \infty} \mathcal{E}^{\mathbb{G}, \phi}_t(X_{\tau_n(t)}) \\
\leq \text{ess sup}_{\tau \in S_t, T} \mathcal{E}^{\mathbb{G}, \phi}_s(X_{\tau}) = V_s.
\]

To obtain an optimal stopping time, we want to show that there is a right-continuous modification of \((V_t)\). However, this time, the strict comparison theorem does not hold for \(G_t\)-expectation anymore in general, so the usual way to find a right-continuous modification of the value function \((V_t)\) by downcrossing inequality may not work. Fortunately, with the help of important results obtained in Peng, S.G. [6], we can still have the following claim.

**Theorem 3.1.** Under the assumptions in our paper, the value process \((V_t)\) defined by (3.2) has a right-continuous modification.

**Proof** Let \(g_n = g + n\phi\) as in proposition (2.2), we define the value function \((V_n(t))\) under \(g_n\)-expectation
\[ V_n(t) \triangleq \text{ess sup}_{\tau \in S_t, T} \mathcal{E}^{\mathbb{G}, \phi}_t(X_{\tau}). \] (3.3)
According to Lemma F.1 in Cheng, X. and Riedel, F. [3] or Lemma 5.2 in Coquet, F., Hu, Y., Memin, J., and Peng, S.G. [2], \((V_n(t))\) is a \(g_n\)-supermartingale with a right-continuous modification for any \(n\). By the comparison theorem of BSDE, without lose of generality, we can say \((V_n(t))\) is also a RCLL \(g\)-supermartingale, hence by Theorem 3.3 in Peng, S.G. [6], it is a \(g\)-supersolution. At the same time, we can easily prove that 
\[ V_n(t) \uparrow V_t. \]
In fact, since \((g_n)\) is an increasing sequence of generator functions, we have

\[
V_t = \operatorname{ess} \sup_{\tau \in [S_t, T]} E^{g, \phi}(X_\tau) = \sup_{n \in \mathbb{N}} \sup_{\tau \in [S_t, T]} E^{g_n}(X_\tau)
\]

All the process in our paper are bounded in \(L^\infty(F_T)\), with the help of Theorem 3.6 of Peng, S.G. [6], \((V_t)\) is also a RCLL \(g\)-supersolution or \(g\)-supermartingale.

As usual, it is easy to see that \((V_t)\) is the smallest \(g\)-supermartingale with RCLL sample path which we state it as a proposition below.

**Proposition 3.2.** The value function process \((V_t)\) is the smallest \(g\)-supermartingale with RCLL sample path which dominates the reward process \(X_t\).

**Proof** Suppose \((S_t)\) is another RCLL \(g\)-supermartingale with \(S_t \geq X_t\) for all \(t \in [0, T]\) and \(S_t\) is meaningful for any stopping time.

Choose a sequence of stopping times \(\{\tau_n(t), n = 1, 2, \cdots\}\) in \([S_t, T]\) as in the proof of proposition 3.1, then we have

\[
V_t = \lim E^{g, \phi}_t(V_{\tau_n(t)}) \leq \lim inf E^{g, \phi}_t(S_{\tau_n(t)}) \leq S_t.
\]

Hence, \((V_t)\) is the smallest RCLL \(g\)-supermartingale dominating \(X\).

With these results in hand, we then go on to find an optimal stopping of problem (3.1) by a similar constructive way as usual.

For any \(0 < \lambda < 1, 0 \leq t \leq T\), we define the stopping times

\[
\tau^\lambda(t) \triangleq \inf\{u \geq t | X_u \geq \lambda V_u\}.
\]

The next lemma is a crucial step to construct an optimal stopping time for our problem.

**Lemma 3.2.** With the notation introduced above, then we have

\[
V_t = E^{g, \phi}_t(V_{\tau^\lambda(t)}).
\]

**Proof** Introduce the process

\[
W_t \triangleq E^{g, \phi}_t(V_{\tau^\lambda(t)}).
\]

and for each \(n\),

\[
W_n(t) \triangleq E^{g_n}_t(V_{\tau^\lambda(t)}), n = 1, 2, \cdots.
\]

Since \((V_t)\) is \(g_n\)-supermartingale for each \(n = 1, 2, \cdots\), by the same way similar with Cheng, X. and Riedel, F. [3], \(W_n\) is a \(g_n\)-supermartingale with RCLL sample paths. Furthermore, we can claim that \(W\) is a \(g\)-supermartingale. For \(0 \leq t \leq t + u \leq T\) we have

\[
E^{g, \phi}_t(W_{t+u}) = E^{g, \phi}_t(E^{g, \phi}_{t+u}(V_{\tau^\lambda(t+u)})) = E^{g, \phi}_t(E^{g, \phi}_{t+u}(V_{\tau^\lambda(t+u)})) \leq E^{g, \phi}_t(V_{\tau^\lambda(t+u)}) = W_{t+u}.
\]

It is obviously that \(W_n(t)\) converges increasingly to \(W_t\) and they are all \(g\)-supermartingales, hence we can use the same skill in the proof of Theorem (3.2) to prove that \(W\) admits a RCLL modification.

The following proof can go on similarly as the last part of the proof of Lemma B.3 in Cheng, X. and Riedel, F. [3]. For convenience, we state it still in our paper.

Let

\[
Y_t = \lambda V_t + (1 - \lambda)W_t.
\]
We claim that $W$ dominates $X$. For $X_t \geq V_t$, we have $\tau^\lambda(t) = t$, hence $W_t = V_t$ and $Y_t = V_t \geq X_t$. If $X_t < \lambda V_t$, we have $W_t \geq 0$ as $X_t \geq 0$, so

$$Y_t \geq \lambda V_t \geq X_t.$$ 

From Proposition 3.2, we get $Y \geq V$. This equivalent to $W \geq V$. On the other hand, by definition of $W$ and $g_\Gamma$-supermartingale property of $V$: $W \leq V$. So we conclude $W = V$. In other words, we finally get

$$V_t = W_t = E_{g,\phi_t}(V_{\tau^\lambda(t)}).$$

Now let us back to the definition of stopping times of $\tau^\lambda(t)$ at $t = 0$, which, for simplicity, we denote it as $\tau^\lambda$.

Noting that $\tau^\lambda$ is increasing with $\lambda$ and dominated by the stopping time $\tau^* \triangleq \inf\{t \geq 0 : X_t = V_t\}$, we can choose a sequence of real numbers $(\lambda_n) \subset (0, 1)$ such that $\tau^\lambda_n$ converges increasingly to some stopping time $\bar{\tau}$.

We state our last result in this paper.

**Theorem 3.2.** Under our assumptions in our paper about the generator function $g$ and constraint function $\phi$ as well as the model-dependent reward process $(X_t)$, if furthermore both $g$ and $\phi$ are convex, then with notations above, the stopping time $\bar{\tau}$ is an optimal stopping for problem (3.1).

**Proof** First, by Lemma 3.2, with $t = 0$, we have

$$V_0 = E_0^{\beta,\phi}(V_{\tau^\lambda_n}) \leq \frac{1}{\lambda_n} E_0^{\beta,\phi}(X_{\tau^\lambda_n}).$$

On the other hand, since $E_0^{\beta,\phi}(\cdot)$ is a convex functional defined well on $L^\infty(\mathcal{F}_T)$, it is then continuous on the space $L^\infty(\mathcal{F}_T)$ according to the norm, for details see Wu, H.L. [10].

Our assumption help us to obtain a subsequence of $(X_{\tau^\lambda_n})$, which we still denote as $(X_{\tau^\lambda_n})$, converges to $X_{\bar{\tau}}$ under norm of $L^\infty(\mathcal{F}_T)$, thus

$$V_0 \leq \lim_{\lambda_n \to 1} \frac{1}{\lambda_n} E_0^{\beta,\phi}(X_{\tau^\lambda_n}) = E_0^{\beta,\phi}(X_{\bar{\tau}}) \leq V_0.$$ 

Thus $\bar{\tau}$ is an optimal stopping time. 

**Remark 3.1.** According to Peng, S.G. and Xu, M.Y. [8], $V_t$ can be viewed as the solution of the Reflected Backward stochastic differential equation with constraint, and $\tau^*$ defined above is another optimal stopping time with the stopped process $(V_t\wedge \tau^*)$ be a $g_\Gamma$-martingale. However, different from classical case, $\bar{\tau}$ may not coincide with $\tau^*$, so whether the stopped process $(V_t\wedge \bar{\tau})$ is also a $g_\Gamma$-martingale is questioned.

**Remark 3.2.** By the penalization method to obtain the $g_\Gamma$-solution, $V_t$ can be represented by

$$V_t = \text{ess sup}_{\tau \in S_{t,T}} \sup_n E_0^{\beta_n}(X_{\tau})$$

which is a stopper and controller problem.
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