Photon splitting cascades and a new statistics

L. Accardi, I.Ya. Aref’eva *and I.V. Volovich †

Centro Vito Volterra Universita di Roma Tor Vergata

March 26, 2022

Abstract

Photon splitting cascades in a magnetic field are considered. It is shown, in the stochastic limit approximation, that photons in cascades might form entangled states (“triphons”) and that they obey not Bose but a new type of statistics, the so called infinite or quantum Boltzmann statistics. These states belong to an interacting Fock space which is a generalization of the ordinary Fock space. The new photon statistics in principle can be detected in future astrophysical experiments such as the planned Integral mission and also in nonlinear quantum optics.

*Steklov Mathematical Institute, Gubkin St.8, GSP-1, 117966, Moscow, Russia
†Steklov Mathematical Institute, Gubkin St.8, GSP-1, 117966, Moscow, Russia
One of the most interesting manifestations of the non–linearity of Maxwell’s equations with radiative corrections is the splitting of a photon into two in an external magnetic field. In a constant uniform field, this process occurs with conservation of energy and momentum. The process was considered by Adler et al. in the early ’70s by using the Heisenberg–Euler effective Lagrangian [1, 2, 3]. Photon splitting was considered as a possible mechanism for the production of linearly polarized gamma–rays in a pulsar field. Recently the splitting of photons has found astrophysical applications in the study of annihilation line suppression in gamma ray pulsars and spectral formation of gamma ray bursts from neutron stars [4, 5]. Photon splitting cascades have also been used in models of soft gamma-ray repeaters, where they soften the photon spectrum [6, 7]. The process of photon splitting is potentially important in applications to a possible explanation of the origin of high energy cosmic rays from Active Galactic Nuclei [8]. A recalculation of the amplitude for photon splitting in a strong magnetic field has been performed recently in [9, 10, 11].

In this note we argue that photon splitting cascades in the magnetic field might create entangled states and that photons in cascades obey not Bose but a new type of statistics, the so called infinite or quantum Boltzmann statistics. These states are formed from triples of entangled photons and may be called triphons. They belong to a generalization of the ordinary Fock space which is called an interacting Fock space. Creation and annihilation operators for infinite (quantum Boltzmann) statistics satisfy neither Bose nor Fermi commutational relations but

\begin{equation}
 b(k)b^+(p) = \delta(k - p)
\end{equation}

The relations (1) have been considered in several recent works in quantum field theory. The second quantized example of infinite statistics has been discussed by Greenberg [12] and in the present note it is shown that this statistics has a physical meaning since it describes photons in cascades and more generally the dominating diagrams in the long time/week coupling limit in quantum field theory. The notion of interacting Fock space was introduced by Accardi and Lu [13] in nonrelativistic QED and it was related with the role of non–crossing diagrams in the stochastic limit. The master field describing the large \( N \) limit in QCD was obtained in [14], it is quantized by using the relations (1); see [15, 16] for a recent discussion of the large \( N \) limit.

We will start from a discussion of the theory of photon splitting cascades and show the emergence of infinite statistics in this theory and then discuss its connection with the stochastic limit of quantum field theory.

In the decay of a photon with momentum \( k \) into photons with momentum \( k_1 \) and \( k_2 \), we have conservation of momentum and energy \( k = k_1 + k_2 \), \( \omega(k) = \omega_1(k_1) + \omega_2(k_2) \). For photons in vacuum, in the absence of external fields, \( \omega = \omega_1 = \omega_2 = k \) and although these two equations have a solution the decay is forbidden by the invariance under charge conjugation (Furry’s theorem).

In a constant uniform magnetic field \( B_0 \) there are only two decay processes kinematically allowed, \( \gamma \| \rightarrow \gamma_\perp + \gamma_\perp \) and \( \gamma \| \rightarrow \gamma_\| + \gamma_\perp \) [2]. Here the subscripts \( \perp \) and \( \| \) will denote polarizations of the photon with respect to the vector \( B_0 \). More precisely, in presence of a magnetic field one has a distinctive plane, namely the \( kB_0 \) plane. One takes the linear polarization of the magnetic field of the photon parallel and orthogonal to this plane as the two independent polarizations of the photon, \( \| \) and \( \perp \), respectively.
The vacuum in the presence of the field \( \mathbf{B}_0 \) acquires an index of refraction \( n \), and the photon dispersion relation is modified from \( k/\omega = 1 \) to \( k/\omega = n \). The indices of refraction \( n_{\|,\perp} \) can be calculated from the Heisenberg–Euler effective lagrangian. Adler showed that for subcritical fields in the limit of weak vacuum dispersion only the splitting mode \( \parallel \rightarrow \perp + \perp \) operates below pair production threshold. For weak dispersion \( n_{\perp} = 1 + \frac{7}{90} \beta \) and \( n_{\|} = 1 + \frac{2}{3} \beta \), where \( \beta = \frac{e^4 \hbar}{m^4 c^2} B_0^2 \sin^2 \theta \) and \( \theta \) is the angle between \( \mathbf{k} \) and \( \mathbf{B}_0 \). It is mentioned by Harding et al. that in magnetar models of soft gamma repeaters [3], where supercritical fields are employed, moderate vacuum dispersion arises. In such a regime, it is not clear whether Adler’s selection rules still endure since in his analysis higher order contributions to the vacuum polarization are omitted. In [3] photon cascades are considered for the case where all three photon splittings modes allowed by CP invariance are operating. Baier et al. [10] have found that there is only one allowed transition (\( \parallel \rightarrow \perp + \perp \)) for any magnetic field. They suggested that a photon cascade could develop only if magnetic field changes its direction. It seems that the question on the validity of Adler’s rule for a non weak vacuum dispersion deserves a further study. In this work we consider photon cascades when both kinematically allowed modes (\( \parallel \rightarrow \perp + \perp \) and \( \parallel \rightarrow \parallel + \perp \)) operate.

The interaction operator for the decay \( \parallel \rightarrow \perp + \perp \) is known to be [3]

\[
V_1(t) = \lambda_1 \int (\mathbf{B}_0 \mathbf{E}_1)(\mathbf{B}_0 \mathbf{E}_2)(\mathbf{B}_0 \mathbf{B})d^3x,
\]

(2)

where the coupling constant \( \lambda_1 = 13e^6/315\pi^2m^8 \) and magnetic and electric parts of photon field are

\[
\mathbf{B} = i(4\pi)^{1/2} \mathbf{k} \times \mathbf{e}_{\parallel} e^{-i(kr-\omega t)}a_{\parallel}(\mathbf{k}_1), \quad \mathbf{E}_1 = -i(4\pi)^{1/2} \omega_1 \mathbf{e}_{\perp} e^{i(kr-\omega t)}a_{\perp}^+(\mathbf{k})
\]

(3)

and similarly for \( \mathbf{E}_2 \).

For the decay \( \parallel \rightarrow \parallel + \perp \) one has a similar interaction operator with the operator structure

\[
\mathcal{A}_1^+(t) = \lambda a_{\parallel}(\mathbf{k}_1)a_{\perp}(\mathbf{k}_2)a_{\parallel}(\mathbf{k})e^{-itE} \delta(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2)
\]

(4)

where \( E = \omega_{\parallel}(|\mathbf{k}|) - \omega_{\parallel}(\mathbf{k}_1) - \omega_{\perp}(\mathbf{k}_2) \). The coupling constant \( \lambda \) in this case can be estimated as \( \lambda/\lambda_1 = \alpha(B_0/B_{cr})^2 \), where \( \alpha \) is the fine structure constant, \( \alpha = e^2/\hbar c \) and \( B_{cr} = m^2c^3/\hbar k \approx 4.4 \times 10^{13} \text{ Gauss} \).

Let us consider a photon cascade created by a photon with momentum \( \mathbf{k} \) and polarization \( \parallel \). The photon splits as \( \gamma_{\parallel}(\mathbf{k}) \rightarrow \gamma_{\perp}(\mathbf{k}_1) + \gamma_{\parallel}(\mathbf{k} - \mathbf{k}_1) \). Then one has the next generation of splitting: \( \gamma_{\parallel}(\mathbf{k} - \mathbf{k}_1) \rightarrow \gamma_{\perp}(\mathbf{k}_2) + \gamma_{\parallel}(\mathbf{k} - \mathbf{k}_1 - \mathbf{k}_2) \) etc. After \( N \) generations of splitting one gets a cascade with \( N \) photons with \( \perp \) polarization and momenta \( \mathbf{k}_1, \ldots, \mathbf{k}_N \) and also one photon with \( \parallel \) polarization and momentum \( \mathbf{k} - \mathbf{k}_1 - \ldots - \mathbf{k}_N \). An example of a cascade with two generations is shown in Fig 1.

It is important to notice that we consider cascades with real photons and therefore the diagram in Fig.1 is not a Feynman one because all the lines (including an intermediate one) correspond to real particles on the mass shell and not to virtual states. From the point of view of the standard quantum field theory all the lines in the diagram are ”dressed” lines on the mass shell and moreover the initial photon \( \gamma_{\parallel}(\mathbf{k}) \) is prepared in a special way such that it undergoes the decay in a finite time. So we cannot use the standard S-matrix approach and the standard Feynman diagram technique to describe this process. The diagram is also not a diagram in the non–covariant diagram technique [17] because we
have the conservation of energy at every vertex. The cascade in Fig.1 may be intuitively
described by the following state

$$|\psi(k, k_1, k_2)\rangle = a_\parallel^+(k - k_1 - k_2)f(k - k_1, k_2)a_\parallel^+(k_2)f(k, k_1)a_\perp^+(k_1)|0\rangle \quad (5)$$

where momentum conservation is built in creation and annihilation operators and energy
conservation is accounted for by the factor $f(p, k) = f(\omega_\parallel(p) - \omega_\perp(k) - \omega_\parallel(p - k))$ where
$f(\omega)$ is a function with support at $\omega = 0$. As we shall see below this is not the $\delta$–
function but roughly speaking its square root. Indeed, the transition amplitude between
two cascade states is given by scalar product

$$\langle \psi(k', k'_1, k'_2)|\psi(k, k_1, k_2)\rangle = |f(k, k_1)|^2 |f(k - k_1, k_2)|^2 \delta(k - k')\delta(k_1 - k'_1)\delta(k_2 - k'_2) \quad (6)$$

Notice that in the scalar product (6) only the non–crossing diagram, (Fig.2a) contributes.
In fact the contribution from the crossing diagram in Fig.2b vanishes because of conserva-
tion of energy and momentum. This is a crucial point where the difference between
our diagrams describing real particles in intermediate states and the Feynman diagrams
having virtual particles in intermediate states is evidentiated. In the Feynman diagram
technique the amplitude of emission of the two photons is represented by a sum of two
diagrams differing by the order in which the two photons are emitted. Here we have only
one diagram, Fig.1.

Now let us observe that if in (6) we replace operators $a_\parallel^+(k_1)$ and $a_\parallel^+(k_2)$ by the quan-
tum Boltzmann operators $b_\perp(k_1)$ and $b_\perp(k_2)$ satisfying the relations (1), i.e. $b_\perp(k)b_\perp^+(p) =
\delta(k - p)$, then it will be automatically guaranteed that only the non-crossing diagrams
survive. Therefore it is natural to describe cascade wave functions in terms of these opera-
tors. It is well known that standard free photons are bosons. Therefore to see the quantum
Boltzmann statistics we have to prepare a special state depending on interaction. A nat-
ural method, leading to this result, is suggested by the stochastic limit technique. In fact
it is natural to expect that the cascades with physical intermediate states occur at a time
scale slower than the one occurring in the standard $S$-matrix approach to multiparticle production. Notice also that the coupling constant in the interaction term $\lambda$ are small. Therefore we are precisely in the situation in which one considers long time–commulative effects of weak interactions. The stochastic limit captures exactly these effects in the van Hove limit $\lambda \to 0$, $t \to \infty$ so that $\lambda^2 t \sim \tau$ (new time scale) which means that we measure time in units of $1/\lambda^2$ where $\lambda$ measures the strength of the self–interaction (proportional to the magnitude of the magnetic field in our case). It is remarkable that in this limit the triplets of photons ("triphons") behave like a single new quantum field obeying a new statistics.

Now let us consider the question how one can prepare a state with the new statistics for photons. If we would deal with the scattering at infinite time ($S$–matrix) we simply have to consider two Feynman diagrams to take into account the Bose statistics of photons. However in the cascade we deal with evolution in finite time and the states of photons $\gamma_\perp (k_1)$ and $\gamma_\perp (k_2)$ are prepared in a special way because they are emitted at time $t_1$ and $t_2$, respectively. Therefore, there is a reason not to add the second diagram. There exist a special procedure which is adequate to this situation. This is so-called stochastic limit technic. This limiting procedure is widely used in the consideration of the long time/weak coupling behaviour of quantum dynamical systems with dissipation, see for example [18, 19].

In this procedure one deals with states generated by products of rescaling interaction in different times in the interaction picture $\frac{1}{\lambda} V \left( \frac{t}{\lambda^2} \right)$ in the limit of $\lambda \to 0$. This is connected with anisotropic asymptotics, see [20], where one deals with correlators $< V(t_1/\lambda)...V(t_n/\lambda) >$ when $\lambda \to 0$. The equation for the evolution operator in interaction picture reads

$$ \frac{dU^{(\lambda)}(t)}{dt} = -i \lambda V(t) U^{(\lambda)}(t) $$

where $\lambda$ is the coupling constant. In the stochastic approximation one replaces $U^{(\lambda)}(t)$ to another operator $\mathcal{U}(t)$

$$ U^{(\lambda)}(t) \approx \mathcal{U}(\lambda^2 t) $$

where $\mathcal{U}(t)$ is obtained by performing the van Hove rescaling of time $t \to t/\lambda^2$ and taking the limit $\lambda \to 0$. Then for the limiting evolution operator $\mathcal{U}(t) = \lim_{\lambda \to 0} U^{(\lambda)}(t/\lambda^2)$ one gets the equation

$$ \frac{d\mathcal{U}(t)}{dt} = -i \mathcal{V}(t) \mathcal{U}(t) $$

where

$$ \mathcal{V}(t) = \lim_{\lambda \to 0} \frac{1}{\lambda} V \left( \frac{t}{\lambda^2} \right) $$

For the interaction Hamiltonian (4) we consider the asymptotic behaviour of the collective operator $A_\lambda (t) = \frac{1}{\lambda} A \left( \frac{t}{\lambda^2} \right)$ and its Hermitian conjugate. We obtain

$$ \lim_{\lambda \to 0} A_\lambda^\dagger (t) = b^\dagger_\parallel (k_1, k_2, k) \delta(k - k_1 - k_2) $$

(7)

where

$$ B^\dagger (k_1, k_2, k) = b^\dagger_\parallel (k_1) b^\dagger_\parallel (k_2) b_\parallel (k)(2\pi)^{1/2} \delta^{1/2}(E) $$

$$ E = \omega_\parallel (k) - \omega_\parallel (k_1) - \omega_\perp (k_2). $$
and the following commutation relations take place

\[ b_i b_i^\dagger = \delta(t - t') \]  
\[ b_\perp(p) b_\perp^\dagger(p') = \delta(p - p'), \quad b_\parallel(p) b_\parallel^\dagger(p') = \delta(p - p') \]  

and also the following relation, which explains our notation \( \delta^{1/2}(E) \)

\[ B(k_1, k_2, k) B^+(k'_1, k'_2, k') = 2\pi \delta(E) \delta(k_1 - k_1) \delta(k_2 - k'_2) \delta(k - k') b_\parallel^\dagger(k) b_\parallel(k) \]  

The relation (7) is understood in the sense of convergence of matrix elements,

\[ \langle 0| a_\parallel^\dagger(p_1) \mathcal{A}^\#(t_1) \ldots \mathcal{A}^\#(t_n) a_\parallel^\dagger(p_2)|0 \rangle \rightarrow_{\lambda \rightarrow 0} \langle 0| b_\parallel^\dagger(p_1) B^\#_1 \ldots B^\#_n b_\parallel^\dagger(p_2)|0 \rangle \]  

where \( \mathcal{A}^\# \) means \( \mathcal{A} \) or \( \mathcal{A}^\dagger \).

Relations (8) define the free or Boltzmann commutation relations. Notice the Boltzmannian white noise relation (9), which makes our model particularly suitable for Monte Carlo simulations. The origin for arising these new commutation relations lies in the fact that the crossing diagrams in the computation of the matrix element (10) are suppressed in the weak coupling /large time limit. The presence of the \( \delta(E) \)-factor has two important physical consequences. First, the commutation relations for the \( B^\# \) are not a consequence of the corresponding relations for \( b_\parallel^\# \) and \( b_\perp^\# \): the three photons are entangled into a single new object (triphon). Second, the triphon creation and annihilation operators \( B^\# \) operate not on the usual Fock space but in interacting Fock space.

Let us illustrate on the example of four triphons that only diagrams with non–crossing lines survive in the limit \( \lambda \rightarrow 0 \). An arbitrary diagram with four ”triphons” schematically can be written as

\[ \frac{1}{\lambda^4} \int \exp \left\{ \frac{i}{\lambda^2} \sum (\pm) E^{(i)} t_i \right\} \phi(t, p, k) dk \prod dt_i \]  

Here \( E^{(i)} \) are the same as in (11) with \( k^{(i)} \) being momenta of line coming in and \( k_1^{(i)}, k_2^{(i)} \) momenta of lines coming out from \( i \)-vertex; \( \pm \) correspond to vertex (12) and its complex conjugated; \( p \) are external momenta and \( k \) are two independent momenta; \( \phi \) accumulates all form-factors and test functions. In general, the sets of momenta corresponding to different vertices are different. But if there are two vertices such that momenta coming in the first vertex come out from the second one and via versa we call these vertices as conjugated ones (see Fig.2a where conjugated vertices are denoted by hat). Only diagrams consistent of pairs of conjugated vertices survive in the limit \( \lambda \rightarrow 0 \). Indeed, making a change of variables \( (t_1, t_2, t_1, t_2) \rightarrow (\tau_1, \tau_2, t_1, t_2) \), \( \tau_1 = \frac{t_1 - t_2}{\lambda^2}, \quad \tau_2 = \frac{t_2 - t_1}{\lambda^2} \) we see that diagram Fig.2a gives contributions containing the following factors

\[ \delta(E^{(1)}_{\parallel \rightarrow \parallel + \perp}) \delta(E^{(2)}_{\parallel \rightarrow \parallel + \perp}) \delta(t_1 - t_1) \delta(t_2 - t_2), \]  

where the energy factors \( E^{(1)}, E^{(2)} \) are not independent due to momentum conservation, that is typical for interacting Fock space (13). For all others diagrams a similar change of variables do not remove the dependence of exponent of \( \lambda \) that due to fast oscillations produces zero contributions.

A photon splits into two not only in a magnetic field but also in a nonlinear medium. In fact such processes are well known in nonlinear quantum optics, see for example (14). In the nonlinear process of parametric down conversion a high frequency photon splits into
two photons with frequencies such that their sum equals that of the high-energy photon. The two photons produced in this process possess quantum correlations and have identical intensity fluctuations. The photon pairs generated through parametric down-conversion carry quantum correlations of the Einstein-Podolsky-Rosen type. Experiments to test Bell inequalities were designed using a correlated pair of photons. A two-photon cascade was used in the initial experiment by A. Aspects to generate the correlated photons but more recent experiments have used parametric down-conversion. These experiments have consistently given predictions of quantum theory and in violation of "realistic" classical predictions. In our case the quantum correlations do not come from superpositions of spins or polarizations but have a deeper dynamical origin expressed by the $\delta(E)$-function in formula (11). It would be very interesting to extend these experiments to observe the new statistics considered in this paper.

In conclusion, in this note we have argued that photon cascades in a strong magnetic field might create entangled states (triphons) which obey non Bose but the quantum Boltzmann statistics. This prediction is based on the assumption that both kinematically allowed photon splitting modes operate. Another assumption is that intermediate photons in a cascade are physical particles (i.e. on the mass shell) but not virtual states. This is equivalent to the validity of the stochastic limit approximation. In fact both of these assumptions deserve a further study. A better theoretical understanding of the photon splitting with a non weak dispersion is required. From the experimental side new more precise devices such as the planned Integral mission [4] might significantly advance our understanding of the fundamental problem of photon statistics.

Acknowledgement. I.A. and I.V. are grateful to the Centro Vito Volterra Universita di Roma Tor Vergata for the kind hospitality.

References

[1] S.L. Adler, J.N. Bahcall, C.G. Callan and M.N. Rosenbluth, *Phys. Rev. Lett.*, 25(1970)1061.

[2] S.L. Adler, *Ann. Phys.* 67(1971)599

[3] V.B. Berestetskii, E.M. Lifshitz and L.P. Pitaevskii, *Quantum Electrodynamics*, Pergamon Press, 1982.

[4] M.G. Baring, A.K. Harding, and P.L. Gonthier, *The attenuation of gamma-ray emission in strongly-magnetized pulsars*, Astrop-ph/9704210.

[5] A.K. Harding, M.G. Baring and P.L. Gonthier, *Photon splitting cascades in gamma-ray pulsars and the spectrum of PSR1509-58*, astro-ph/9609167.

[6] M.G. Baring, *Astrophys. Journ. Lett.*, 440 (1995) L 69.

[7] A.K. Harding and M.G. Baring, *Photon Splitting in Soft Gamma Repeaters*, astro-ph/9603093.

[8] R.J. Protheroe, *Origin and propagation of the highest energy cosmic rays*, astro-ph/9610100.
[9] S.L. Adler and C. Schubert, *Photon splitting in a strong magnetic field: recalculation and comparison with previous calculations*, hep-th/9605035.

[10] V.N. Baier, A.I. Milstein and R.Zh. Shaisultanov, *Phys. Rev. Lett.*, 77(1996)1691.

[11] J.S. Heyl and L. Hernquist, *Birefringence and Dichroism of the QED Vacuum*, hep-ph/9705367.

[12] O. Greenberg, *Phys.Rev.Lett.* 64 (1990) 705.

[13] L. Accardi, Y.G. Lu, *Comm. Math. Phys.* 108 (1996) 605

[14] I.Ya. Aref’eva and I.V. Volovich, *Nucl.Phys.* B462(1996)600.

[15] D. Gross and R. Gopakoumar, *Nucl.Phys.* B451 (1995) 379

[16] M. Douglas, *Nucl.Phys.Proc.Suppl.* 41(1995)66

[17] C.Cohen-Tannoudji, J.Dupont-Roc and G.Grinberg, *Atom-Photon Interactions, Basic Processes and Applications*, John Wiley & Sons, Inc., 1992

[18] L. Accardi, Y.G. Lu and I.V.Volovich, *Quantum theory and its stochastic limit*, Oxford Univ.Press (in press)

[19] L. Accardi, S.V. Kozyrev and I.V. Volovich, *Dynamics of Dissipative Two-Level Systems in the Stochastic Approximation*, quant-ph/9706021, to be publ. in Phys. Rev. A

[20] I.Ya.Aref’eva, Phys. Lett. (1994); I.Ya.Aref’eva and I.V.Volovich, in: ”Quarks 94”, eds, V.Matveev and V.Rubakov, Worl Scientifis, 1995

[21] D.F. Walls and G.J. Milburn, *Quantum Optics*, Springer-Verlag, 1994