Adaptivity in Adaptive Submodularity

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Abstract

Adaptive sequential decision making is one of the central challenges in machine learning and artificial intelligence. In such problems, the goal is to design an interactive policy that plans for an action to take, from a finite set of $n$ actions, given some partial observations. It has been shown that in many applications such as active learning, robotics, sequential experimental design, and active detection, the utility function satisfies adaptive submodularity, a notion that generalizes the notion of diminishing returns to policies. In this paper, we revisit the power of adaptivity in maximizing an adaptive monotone submodular function. We propose an efficient batch policy that with $O(\log n \times \log k)$ adaptive rounds of observations can achieve an almost tight $(1 - 1/e - \varepsilon)$ approximation guarantee with respect to an optimal policy that carries out $k$ actions in a fully sequential setting. To complement our results, we also show that it is impossible to achieve a constant factor approximation with $o(\log n)$ adaptive rounds. We also extend our result to the case of adaptive stochastic minimum cost coverage where the goal is to reach a desired utility $Q$ with the cheapest policy. We first prove the conjecture by [16] that the greedy policy achieves the asymptotically tight logarithmic approximation guarantee without resorting to stronger notions of adaptivity. We then propose a batch policy that provides the same guarantee in polylogarithmic adaptive rounds through a similar information-parallelism scheme. Our results shrink the adaptivity gap in adaptive submodular maximization by an exponential factor.

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1. We also refer to an adaptive round as a batch query and use these two terms interchangeably throughout the paper.
1 Introduction

Adaptive stochastic optimization under partial observability is one of the fundamental challenges in artificial intelligence and machine learning with a wide range of applications, including active learning \cite{11}, optimal experimental design \cite{30}, interactive recommendations \cite{22}, viral marketing \cite{31}, and perception in robotics \cite{21}, to name a few. In such problems, one needs to adaptively make a sequence of decisions while taking into account the stochastic observations collected in previous rounds. For instance, in active learning, the goal is to learn a classifier by carefully requesting as few labels as possible from a set of unlabeled data points. Similarly, in experimental design, a practitioner may conduct a series of tests in order to reach a conclusion. Even though it is possible to determine all the selections ahead of time before any observations take place (e.g., select all the data points at once or conduct all the medical tests simultaneously), so called a priori selection, it is more efficient to consider a fully adaptive procedure that exploits the information obtained from past selections in order to make a new selection. Indeed, a priori and fully sequential selections are simply two ends of a spectrum. In this paper, we develop a semi-adaptive policy that enjoys the power of a fully sequential procedure while performing exponentially fewer adaptive rounds compared to previous work. In particular, we only need poly-logarithmic number of rounds for both adaptive stochastic submodular maximization and adaptive stochastic minimum cost coverage problems. We start by presenting the problems more formally, and then present the results in more details and compare to the previous work.

More formally, let $E = \{e_1, \ldots, e_n\}$ be a finite set of elements (e.g., tests in medical diagnostics, data points in active learning). Each element $e \in E$ is associated with a random variable $\Phi(e) \in \Omega$ where $\Omega$ is the set of all possible outcomes. A realization of the random variable $\Phi(e)$ is denoted by $\phi(e) \in \Omega$. For instance, in medical diagnosis, an element $e$ may represent a test such as blood pressure and $\Phi(e)$ its outcome, e.g., high or low. Or in active learning, an item $e$ may represent an unlabeled data point and $\Phi(e)$ its label. By overloading the notation, we use $\Phi$ to denote the set of random variables indexed by the elements of $E$ and $\phi$ their realizations. We refer to $\phi$ as a realization. Similarly for a subset of elements $A \subseteq E$ we use $\Psi_A$ to denote the set of random variables indexed by the elements of $A$ and $\psi_A$ their realizations. We refer to $\psi_A$ as a partial realization. We define $\text{dom}(\psi_A) = A$ and drop $A$ from the index when it is clear from the context.

We assume that there is a joint probability distribution $p(\Phi)$ over the set of random variables $\Phi$. This probability distribution encodes our uncertainty about the outcomes as well as their dependencies. In its simplest form, the outcomes maybe independent and the distribution $p$ completely factorizes, i.e., a product distribution over the set of random variables $\Phi$. The product distribution might be a valid model in sensor placement scenarios where sensors may fail to work independent of one another. However, in many practical scenarios, such as medical diagnosis and active learning, the underlying distribution may not factorize and the outcomes may depend on each other. A fully sequential policy $\pi : 2^E \times \Omega \rightarrow E$ is a partial mapping from partial observations $\psi_A$ to elements, stating which element to select next. The utility of a set of observations $\psi_A$ is specified through a utility function $f : 2^E \times \Omega \rightarrow \mathbb{R}_+$ where clearly $f(\psi_A)$ depends on the realization of the random variable $\psi_A$ and the chosen set of elements $A$. The expected utility of a policy is then defined as

$$f_{avg}(\pi) = \mathbb{E}_p[f(S(\pi, \Phi))] = \sum_{\phi} p(\phi) f(S(\pi, \phi)),$$

where $S(\pi, \phi)$ denotes the set of elements taken by policy $\pi$ under realization $\phi$. Without any structural assumptions, it is known that finding an optimal policy, the one that maximizes the
expected utility, is notoriously hard as in many cases the utility functions are computationally intractable [29].

Adaptive submodularity [16], a generalization of diminishing returns from sets to policies, is a sufficient condition under which a partially observable stochastic optimization problem admits (approximate) tractability. This condition ensures that the expected marginal benefit associated with any particular selection never increases as we make more observations. More formally, we define the marginal benefit \( \Delta(e|\psi_A) \) of an action \( e \) conditioned on the observations \( \psi_A \) as follows:

\[
\Delta(e|\psi_A) = \sum_{\phi(e)} [f(\psi_A \cup \{e\}) - f(\psi_A)] p(\Phi(e) = \phi(e)|\psi_A).
\]

We say that \( \psi_A \) is a subrealization of \( \psi_B \), denoted by \( \psi_B \succeq \psi_A \), if \( A \subseteq B \) and for all \( e \in A \), \( \psi_A(e) = \psi_B(e) \). The utility function \( f \) is adaptive submodular if for all subrealizations \( \psi_B \succeq \psi_A \), and all \( e \notin B \), we have \( \Delta(e|\psi_A) \geq \Delta(e|\psi_B) \). Moreover, we say that the utility function \( f \) is adaptive monotone if for all subrealizations \( \psi_A \), and all \( e \notin A \) we have \( \Delta(e|\psi_A) \geq 0 \).

The general goal in adaptive stochastic optimization is to develop policies that can maximize the expected utility while minimizing the cost of running the policy. One way to formalize it is through the adaptive stochastic submodular maximization problem where we aim to maximize the expected utility subject to a cardinality constraint, i.e.,

\[
\pi^* = \arg\max_{\pi} f_{\text{avg}}(\pi) \quad \text{s.t.} \quad |S(\pi,\phi)| \leq k \quad \text{whenever } p(\phi) > 0.
\]

It is known that when the utility is adaptive submodular and adaptive monotone, the greedy policy achieves the tight \((1 - 1/e)\) approximation ratio with respect to the optimal policy [16]. This result has lead to a surge of applications in decision making problems that are amenable to myopic optimization such as active learning [17], interactive recommender systems [23], value of information [8], and active object detection [10], to name a few. An alternative formalization is through adaptive stochastic minimum cost coverage where we prespecify a quota \( Q \) of utility to achieve, and aim to find a policy that achieves it with the cheapest policy, i.e.,

\[
\pi^* = \arg\min_{\pi} c_{\text{avg}}(\pi) \quad \text{s.t.} \quad f_{\text{avg}}(\pi) \geq Q \quad \text{whenever } p(\phi) > 0,
\]

where \( c_{\text{avg}}(\pi) = \mathbb{E}_p[|S(\pi,\phi)|] \) is the expected number of actions a policy \( \pi \) takes. Unlike the adaptive stochastic submodular maximization problem, the performance of the greedy policy is unknown for the above problem unless one makes strong assumptions about the distribution or the utility function. One of the contributions of this paper is to resolve this issue.

Fully sequential policies benefit from previous observations in order to make informed decisions. In many scenarios, however, it is more effective (and sometimes the only way) to select multiple elements in parallel and observe their realizations together. Examples include crowdsourcing (where a single task consists of a collection of unlabeled data to be labeled altogether), multi-stage viral marketing (where in each stage a subset of nodes are chosen as seed nodes), batch-mode pool-based active learning (where the label of a set of data points are requested simultaneously), or medical diagnosis (where there is a shared cost among experiments). A batch-mode, semi-adaptive policy is a mix of a priori and fully sequential selections. The focus of this paper is to answer the following question in the context of adaptive stochastic optimization:

How many adaptive rounds of observations are needed to be competitive to an optimal and fully sequential policy?
We answer the above question in the context of adaptive submodularity.

**Our contributions.** In this paper, we consider two adaptive problems, namely, adaptive stochastic maximization and adaptive stochastic minimum cost cover. We re-exam the required amount of adaptivity in order to be competitive to the optimal and fully sequential policy. In particular, we show the following results in the information-parallel stochastic optimization when the utility function is adaptive submodular and adaptive monotone.

- For the adaptive stochastic submodular maximization problem, we develop a semi adaptive policy that with $O(\log(n) \log(k))$ adaptive rounds (a.k.a. batch queries) achieves the tight $(1 - 1/e - \varepsilon)$ approximation guarantee with respect to the optimum policy $\pi^*$ that selects $k$ items fully sequentially, i.e., $f_{avg}(\pi) \geq (1 - 1/e - \varepsilon)f_{avg}(\pi^*)$.

- We complement the above result by showing that no policy can achieve a constant factor approximation guarantee with fewer than $o(\log(n))$ adaptive rounds. Moreover, the approximation guarantee of any semi adaptive policy that chooses batches of fixed size $r$ will degrade with a factor of $O(r/\log^2(r))$.

- For the adaptive stochastic minimum cost coverage problem, we show that the greedy policy achieves an asymptotically tight logarithmic approximation guarantee, effectively proving [16]'s conjecture. More precisely, we show that $c_{avg}(\pi_{greedy}) \leq (c_{avg}(\pi^*) + 1) \log \left( \frac{2Q}{\eta} \right) + 1$ where we make the common assumption that there is a value $\eta$ such that $f(\psi_A) > Q - \eta$ implies that $f(\psi_A) = Q$ for all realizations $x_A$.

- We also develop a semi adaptive policy for the the adaptive stochastic minimum cost coverage problem that achieves the same logarithmic approximation guarantee with $O(\log n \log(Qn/\eta))$ adaptive rounds.

2 **Comparison to the Related Work**

Submodularity captures an intuitive diminishing returns property where the gain of adding an element to a set decreases as the set gets larger. More formally, a non-negative set function $f : 2^V \rightarrow \mathbb{R}_+$ is **submodular** if for all sets $A \subseteq B \subset V$ and every element $e \in V \setminus B$, we have $f(A \cup \{e\}) - f(A) \geq f(B \cup \{e\}) - f(B)$. Submodular maximization has found numerous applications in machine learning and artificial intelligence, including neural network interpretation [12], data summarization [26], crowd teaching [32], privacy [27], fairness [5], and adversarial attacks in deep neural nets [25]. Moreover, in many information gathering and sensing scenarios, the objective functions satisfy submodularity [24, 33, 20]. However, the classic notion of submodularity falls short in interactive information acquisition settings as it requires the decision maker to commit to all of her selections ahead of time, in an open-loop fashion [19].

To circumvent this issue, [16] proposed adaptive submodularity, a generalization of submodularity from sets to policies. Like submodularity, adaptive submodularity is a sufficient condition that ensures tractability in adaptive settings. More precisely, in the adaptive stochastic submodular maximization problem, when the objective function is adaptive monotone and adaptive submodular, then the greedy policy achieves the tight $(1 - 1/e)$ approximation guarantee with respect to an optimum policy [16]. More generally, [18] proposed a random greedy policy that not only retains the aforementioned $(1 - 1/e)$ approximation ratio in the monotone setting, but also provides a
(1/e) approximation ratio for the non-monotone adaptive submodular functions. The results for adaptive stochastic minimum cost coverage problem are much weaker. Originally, [16] claimed that the greedy policy also achieves a logarithmic approximation factor but as pointed out by [28] the proof was flawed. Instead, under stronger conditions, namely, strong adaptive submodularity and strong adaptive monotonicity, Golovin and Krause proposed a new proof, with a the squared-logarithmic factor approximation, \( c_{\text{avg}}(\pi_{\text{greedy}}) \leq c_{\text{avg}}(\pi^*) \left( \log \left( \frac{nQ}{\eta} \right) + 1 \right)^2 \). In this paper, we prove the original conjecture and show that under adaptive submodularity (without resorting to stronger conditions), the greedy policy achieves a logarithmic approximation factor, namely, \( c_{\text{avg}}(\pi_{\text{greedy}}) \leq (c_{\text{avg}}(\pi^*) + 1) \log \left( \frac{nQ}{\eta} \right) + 1 \).

The main focus of this paper is to explore the information parallelism, a.k.a., batch-mode, stochastic optimization. Many active learning problems naturally fall into this setting when it is more cost-effective to request labels in large batches, rather than one-at-a-time (for detailed discussions, we refer the interested reader to [7]). Note that the two extremes of batch-mode stochastic optimization are full batch setting (i.e., all selections are done in a single batch, and hence the batch-mode setting reduces to the non-adaptive, open-loop optimization problem) and full sequential setting (i.e., elements are selected one-by-one in a closed-loop manner where each selection is based on the results of all previous selections). In this paper, we lay out a rigorous foundation for the semi-adaptive setting where elements are selected in a sequential and closed-loop way but with multiple selections at each round.

There are a few partial results regarding the semi-adaptive policy for the adaptive stochastic minimum cost coverage problem. In particular, [9] proposed a policy that selects batches of fixed size \( r \) and proved that under strong adaptive submodularity and strong adaptive monotonicity, this policy achieves a poly-logarithmic approximation to an optimal policy that is also constrained to picking up batches of size \( r \). Note that this result does not provide any guarantees with respect to the actual baseline, namely, the optimal and fully sequential policy. Moreover, Chen [7] showed that this policy has a sublinear-approximation\(^2\) guarantee against the fully sequential policy. In fact, we show that for the adaptive stochastic submodular maximization problem, the approximation factor of a fixed batch-policy suffers by at least a factor of \( \log^2(r)/r \) in the worst case, so unless \( r \) is a fixed constant, no constant factor approximation guarantee is possible.

Back to the adaptive stochastic minimum cost coverage, when the distribution \( p \) is fully factorized (i.e., the outcomes are independent) and for the special case of stochastic set cover, [11] showed very recently that there exists a policy that with \( O(\log(Q)/\log \log(Q)) \) rounds of adaptivity achieves a poly-logarithmic approximation to the optimal sequential policy. In this paper, we propose a batch policy that using only polylogarithmic adaptive rounds achieves an asymptotically tight logarithmic approximation to the fully sequential policy for general adaptive monotone submodular functions (we do not need to resort to stronger notions of adaptivity and monotonicity). To the best of our knowledge, no results are known for semi-adaptive policies for the adaptive stochastic submodular maximization problem. We develop a batch-mode policy that achieves an almost tight \( 1 - 1/e - \varepsilon \) approximation guarantee with only polylogarithmic adaptive rounds. We also complement our result by showing that no semi-adaptive policy can achieve a constant factor approximation to the optimal policy by fewer than \( o(\log(n)) \) adaptive rounds.

Our work is also related to the adaptivity complexity of submodular maximization, which refers

\(^2\)Unfortunately, the approximation factor grows polynomially in \( r \). Moreover, note that this result assumes strong adaptive submodularity and strong adaptive monotonicity.
to the number of parallel rounds required to achieve a constant factor approximation guarantee in the offline, open-loop setting. [4] developed a parallel algorithm that $O(\log n)$ rounds finds a solution with an approximation arbitrarily close to $\frac{1}{e}$ which was soon improved to $(1 - \frac{1}{e} - \epsilon)$-approximation [15, 3, 13]. The adaptivity complexity was also studied in the non-monotone setting [2, 14, 6]. We lift the notion of adaptivity complexity from the offline to the interactive setting where instead of parallelizing the optimization steps we parallelize the information acquisition.

3 Adaptive Stochastic Minimum Cost Coverage

In this section we prove that the greedy policy achieves a logarithmic approximation guarantee for adaptive stochastic minimum cost coverage. Let us start with some definitions. We define $\pi^*$ to be the policy that runs greedy and stops when the expected marginal gain of all of the remaining elements is less than or equal to $\tau$. We define $\tau_i$ to be a threshold such that the expected number of elements selected by $\pi^*$ is $i^3$. In this section as well as the rest of the paper we assume that our function of interest, $f$, is an adaptive monotone and adaptive submodular function with respect to the distribution $p(\phi)$. For two policies $\pi$ and $\pi'$ we define $\pi \oplus \pi'$ to be a policy that runs $\pi$ and after that runs $\pi'$ from a fresh start (i.e., ignoring the information gathered by $\pi$).

Whenever we use expectation notation $E$ for a random variable $\chi$, i.e., $E[\chi]$, the expectation is over all randomness of $\chi$, unless otherwise specified. Moreover, note that we always use capital letters for random variables, and small letters for realizations. For example $\Psi$ refers to a random variable, and $\psi$ refers to a realization of $\Psi$, and hence $\psi$ is a deterministic value.

Next we present the key lemma of this section.

**Lemma 1** For any policy $\pi^*$ and any positive integer $\ell$ we have

$$f_{\text{avg}}(\pi^\tau_\ell) > (1 - e^{-\frac{\tau}{\max \pi^*}}) f_{\text{avg}}(\pi^*),$$

where $k$ is a random variable that indicates the number of items picked by $\pi^*$.

**Proof**: First we provide an upper bound on $f_{\text{avg}}(\pi^*)$. Pick an arbitrary $i \in \{0, \ldots, \ell\}$. Note that by adaptive monotonicity, we have $f_{\text{avg}}(\pi^*) \leq f_{\text{avg}}(\pi^\tau_i \oplus \pi^*)$. Next we show that $f_{\text{avg}}(\pi^\tau_i \oplus \pi^*) \leq f_{\text{avg}}(\pi^\tau_i) + E[k] \Delta^i(\pi)$, where $\Delta^i(\pi) = f_{\text{avg}}(\pi^\tau_i) - f_{\text{avg}}(\pi^\tau_{i-1})$. This implies that

$$f_{\text{avg}}(\pi^*) \leq f_{\text{avg}}(\pi^\tau_i \oplus \pi^*) \leq f_{\text{avg}}(\pi^\tau_i) + E[k] \Delta^i(\pi).$$

(1)

Let $\Psi_i$ be a random variable that indicates the partial realization of $\pi^\tau_i$. We use $\psi_i$ to indicate a realization of $\Psi_i$. Note that by definition of $\pi^\tau_i$ we have $\tau_i \geq \max_{e \in E \setminus \text{dom}(\psi_i)} \Delta(e|\psi_i)$. Moreover, by adaptive submodularity for any consistent partial realization $\psi'_i \supseteq \psi_i$ and for all $e \in E \setminus \text{dom}(\psi'_i)$ we have $\Delta(e|\psi'_i) \geq \Delta(e|\psi_i)$. Therefore, we have

$$\forall \psi'_i \supseteq \psi_i, \forall e \in E \setminus \text{dom}(\psi'_i) \quad \tau_i \geq \Delta(e|\psi'_i).$$

(2)

We define a random variable $\Gamma(e|\psi'_i) = f(\psi'_i \cup \Phi(e)) - f(\psi'_i)$. Note that by definition $\Delta(e|\psi'_i) = E[\Gamma(e|\psi'_i)]$. Let $X_{e,\psi'_i}$ be a binary random variable that is 1 if and only if policy $\pi^\tau_i \oplus \pi^*$ observes subrealization $\psi'_i$ and picks item $e$ right after. Note that when a policy is deciding whether to pick $e$
or not, it is not aware of the actual outcome of $\Gamma(e|\psi_i')$. Hence $X_{e,\psi_i'}$ and $\Gamma(e|\psi_i')$ are independent. \footnote{Recall that by definition of $X_{e,\psi_i'}$, the randomness in $X_{e,\psi_i'}$ is from only two sources, 1) the randomness of the algorithm, 2) probability of observing $\psi_i'$. However, $\Gamma(e|\psi_i') = f(\psi_i' \cup \Phi(e)) - f(\psi_i')$ is defined given $\psi_i'$ (which is deterministic).} We have

$$f_{avg}(\pi^{\tau_i} @ \pi^*) - f_{avg}(\pi^{\tau_i}) = E \left[ \sum_{\psi_i} \sum_{\psi_i' \ni \psi_i} \sum_{e \notin \text{dom} (\psi_i')} X_{e,\psi_i'} \Gamma(e|\psi_i') \right]$$

By Definition

$$= \sum_{\psi_i} \sum_{\psi_i' \ni \psi_i} \sum_{e \notin \text{dom} (\psi_i')} E \left[ X_{e,\psi_i'} \Gamma(e|\psi_i') \right]$$

Linearity of expectation

$$= \sum_{\psi_i} \sum_{\psi_i' \ni \psi_i} \sum_{e \notin \text{dom} (\psi_i')} E \left[ X_{e,\psi_i'} \right] E \left[ \Gamma(e|\psi_i') \right]$$

Independency

$$= \sum_{\psi_i} \sum_{\psi_i' \ni \psi_i} \sum_{e \notin \text{dom} (\psi_i')} E \left[ X_{e,\psi_i'} \right] \Delta(e|\psi_i') \Delta(e|\psi_i') = E \left[ \Gamma(e|\psi_i') \right]$$

Equality \footnote{One can think of this as an optimal policy that minimizes the expected number of selected items and guarantees that every realization is covered.}

$$\leq \sum_{\psi_i} \sum_{\psi_i' \ni \psi_i} \sum_{e \notin \text{dom} (\psi_i')} E \left[ X_{e,\psi_i'} \right] \tau_i$$

Inequality \footnote{Recall that by definition of $X_{e,\psi_i'}$, the randomness in $X_{e,\psi_i'}$ is from only two sources, 1) the randomness of the algorithm, 2) probability of observing $\psi_i'$. However, $\Gamma(e|\psi_i') = f(\psi_i' \cup \Phi(e)) - f(\psi_i')$ is defined given $\psi_i'$ (which is deterministic).}

$$\leq \sum_{\psi_i} \sum_{\psi_i' \ni \psi_i} \sum_{e \notin \text{dom} (\psi_i')} E \left[ X_{e,\psi_i'} \right] \Delta^{\tau_i}(\pi)$$

$$= E \left[ \sum_{\psi_i} \sum_{\psi_i' \ni \psi_i} \sum_{e \notin \text{dom} (\psi_i')} X_{e,\psi_i'} \right] \Delta^{\tau_i}(\pi)$$

Linearity of expectation

$$\leq E \left[ k \right] \Delta^{\tau_i}(\pi).$$

This proves Inequality \footnote{Recall that by definition of $X_{e,\psi_i'}$, the randomness in $X_{e,\psi_i'}$ is from only two sources, 1) the randomness of the algorithm, 2) probability of observing $\psi_i'$. However, $\Gamma(e|\psi_i') = f(\psi_i' \cup \Phi(e)) - f(\psi_i')$ is defined given $\psi_i'$ (which is deterministic).} as promised.

Let us define $\Delta_i^* = f_{avg}(\pi^*) - f_{avg}(\pi^{\tau_i})$. Inequality \footnote{Recall that by definition of $X_{e,\psi_i'}$, the randomness in $X_{e,\psi_i'}$ is from only two sources, 1) the randomness of the algorithm, 2) probability of observing $\psi_i'$. However, $\Gamma(e|\psi_i') = f(\psi_i' \cup \Phi(e)) - f(\psi_i')$ is defined given $\psi_i'$ (which is deterministic).} implies that $\Delta_i^* \leq E \left[ k \right] (\Delta_{i-1}^* - \Delta_i^*).$ By a simple rearrangement we have $\Delta_i^* \leq (1 - \frac{1}{E[k]+1}) \Delta_{i-1}^*$. By iteratively applying this inequality we have $\Delta_i^* \leq (1 - \frac{1}{E[k]+1})^i \Delta_0^* \leq e^{-\frac{i}{E[k]+1}} \Delta_0^*$. By applying the definition of $\Delta_i^*$ and some rearrangements we have $f_{avg}(\pi^{\tau_i}) > (1 - e^{-\frac{\eta}{E[k]+1}}) f_{avg}(\pi^*)$ as desired. \qedsymbol

Next we present the main theorem of this section. To prove this theorem we use Lemma \footnote{Recall that by definition of $X_{e,\psi_i'}$, the randomness in $X_{e,\psi_i'}$ is from only two sources, 1) the randomness of the algorithm, 2) probability of observing $\psi_i'$. However, $\Gamma(e|\psi_i') = f(\psi_i' \cup \Phi(e)) - f(\psi_i')$ is defined given $\psi_i'$ (which is deterministic).} and follow the usual proof for set cover. We present the proof of this theorem in Appendix A.

**Theorem 2** Assume that there is a value $\eta \in (0, Q]$ such that $f(\psi) > Q - \eta$ implies $f(\psi) = Q$ for all $\psi$. Let $\pi^*$ be an arbitrary policy that covers everything i.e. $f(\pi^*) = Q$ for all $\phi$. Let $\pi$ be the greedy policy. We have

$$c_{avg}(\pi) \leq (c_{avg}(\pi^*) + 1) \log \left( \frac{nQ}{\eta} \right) + 1,$$

where $n = |E|$.
4 Adaptive Stochastic Submodular Maximization via Batch Queries

In this section we provide a policy for adaptive stochastic submodular maximization that makes only $O(\log n \log k)$ batch queries (a.k.a. adaptive rounds). We show that our policy provides a $(1 - \frac{1}{e} - \varepsilon)$ approximate solution compared to that of the best fully sequential policy. Our policy is based on two notions, semi-adaptive values, and, information gap. We first provide some intuition and notations, and then explicitly define these two notions. At any stage of the algorithm, semi-adaptive value of an item $e$ is our estimate of the expected value of selecting item $e$. We provide these estimations based on the information of the last batch query that we made (from the adaptive submodular function) and the set of the items that we decide to select but not queried yet. Information gap is our estimate of the accuracy of the maximum semi-adaptive value. We use the information gap to balance between the loss on the performance and the number of batch queries that we make.

We iteratively and greedily select elements based on their semi-adaptive values. We continue this selection non-adaptively until the information gap reaches $(1 - \varepsilon)$. When the information gap drops below $(1 - \varepsilon)$ we query the selected elements. We call this algorithm semi-adaptive greedy. In this section we use $\pi$ to refer to this policy. We use $\psi'$ to refer the last partial realization that is queried. We also use $\Psi^\pi_i$ to refer to the random partial realization at step $i$ of policy $\pi$. Notice that given $\psi'$ the domain of $\Psi^\pi_i$ is deterministically determined by the policy $\pi$. Now we are ready to define the semi-adaptive values and the information gap.

**Definition 3 (Semi-Adaptive Value)** At any point of the policy semi-adaptive value of an item $e$ is defined as follows.

$$E_{\Psi^\pi_i \sim \psi'} [\Delta(e | \Psi^\pi_i)].$$

Note that the semi-adaptive value of an item $e$ is equal to the expected marginal impact of $e$ condition of all unknown random variables (i.e. not on $\psi'$).

**Definition 4 (Information Gap)** At any point of the policy the information gap is defined as follows.

$$\frac{\max_{e \notin \text{dom}(\Psi^\pi_i)} E_{\Psi^\pi_i \sim \psi'} [\Delta(e | \Psi^\pi_i)]}{E_{\Psi^\pi_i \sim \psi'} [\max_{e \notin \text{dom}(\Psi^\pi_i)} \Delta(e | \Psi^\pi_i)]}.$$

**4.1 Performance**

The next lemma bounds the performance of our policy. We use the notions of semi-adaptive values and the information gap to prove this lemma. The proof of this lemma is partly similar to that of Lemma 1. In the following lemma, $\pi$ is the semi-adaptive greedy policy and $\pi[i]$ is a policy that runs $\pi$ and stops if it selects $i$ items. This is one of the main technical contributions of the paper, but due to the space constraint we provide the proof of this lemma in Appendix A.

**Lemma 5** Let $\pi$ be the semi-adaptive greedy policy. For any policy $\pi^*$ and positive integer $\ell$ we have

$$f_{avg}(\pi[i]) > (1 - e^{-\frac{1}{e}} - \varepsilon) f_{avg}(\pi^*).$$
4.2 Query Complexity

In this subsection we bound the number of batch queries of the semi-adaptive greedy policy. We define random variable $\Psi'_t$ to be the partial realization obtained by the $t$-th batch query. We use $\psi'_t$ to indicate a realization of random variable $\Psi'_t$.

The next lemma shows that after any $\log \frac{1}{1-\epsilon/2} \left( \frac{1}{\delta} \right)$ batch queries the maximum marginal increase drops by a factor $1 - \frac{\epsilon}{2}$, with high probability. We later apply this lemma iteratively for $O(\log k)$ times to show that after $O(\epsilon (\log n \log k)$ batch queries, the maximum marginal increase is vanishingly small.

Lemma 6 Pick an arbitrary $t$, and fix partial realization $\psi'_t$. Let $\Delta'_i = \max_{e \notin \text{dom}(\psi'_t)} \Delta(e|\psi'_t)$, and let $t^+ = t + \log \frac{1}{1-\epsilon/2} \left( \frac{1}{\delta} \right)$. We have $\max_{e \notin \text{dom}(\psi'_{t^+})} \Delta(e|\psi'_{t^+}) \leq (1 - \frac{\epsilon}{2})\Delta'_i$ with probability at least $1 - \delta$.

Proof: For any $t' \geq t$ we use random variable $S_{t'}$ to indicate the set of elements such that $\Delta(e|\Psi'_{t'}) \geq (1 - \frac{\epsilon}{2})\Delta'_i$. To prove this lemma we show that $E[|S_{t'}|] \leq (1 - \frac{\epsilon}{2}) E[|S_{t'+1}|]$. This together with $|S_t| \leq n$ implies that $E[|S_{t'+1}|] \leq \delta$ for $t^+ = t + \log \frac{1}{1-\epsilon/2} \left( \frac{1}{\delta} \right)$. Note that $|S_{t'+1}|$ is a non negative integer, and hence we have $S_{t'+1} = \emptyset$ with probability at least $1 - \delta$.

Next we show that $E[|S_{t'}|] \leq (1 - \frac{\epsilon}{2}) E[|S_{t'+1}|]$. First note that by adaptive monotonicity $e \in S_{t'+1}$ implies $e \in S_{t'}$, and hence we have $S_{t'+1} \subseteq S_{t'}$. Next we use the notion of information gap and show that for any element $e \in S_{t'}$ we have $e \notin S_{t'+1}$ with probability at least $\frac{\epsilon}{2}$. This directly implies $E[|S_{t'}|] \leq (1 - \frac{\epsilon}{2}) E [|S_{t'+1}|]$ as desired. Note that when we query $\Psi'_{t'+1}$ the information gap is at most $1 - \epsilon$. Hence for some $\Psi'_T$ which corresponds to $\Psi'_{t'+1}$ we have

$$\max_{e \notin \text{dom}(\Psi'_T)} E_{\Psi'_T \sim \psi'_T} \left[ \Delta(e|\Psi'_T) \right] \leq (1 - \epsilon) \max_{e \notin \text{dom}(\Psi'_T)} \left[ \max_{e \notin \text{dom}(\Psi'_T)} \Delta(e|\Psi'_T) \right]$$

by adaptive monotonicity.

This implies that for all $e \notin \text{dom}(\Psi'_T)$ with probability at least $\frac{\epsilon}{2}$ we have $\Delta(e|\Psi'_T) \leq (1 - \frac{\epsilon}{2})\Delta'_i$. Therefore, for any element $e \in S_{t'}$ we have $e \notin S_{t'+1}$ with probability at least $\frac{\epsilon}{2}$, as promised.

Now we are ready to prove the main theorem of this section. In the following theorem $\pi$ is the semi-adaptive greedy policy, $\pi_{[\ell]}$ is a policy that runs $\pi$ and stops if it selects $\ell$ items and $\pi^T_{[\ell]}$ is a policy that runs $\pi_{[\ell]}$ and stops if it makes $T$ batch queries.

Theorem 7 Let $\pi$ be the semi-adaptive greedy policy and let $\pi^T_{[\ell]}$ be a policy that runs $\pi_{[\ell]}$ and stops if it makes $T$ batch queries. For any policy $\pi^*$ and positive integer $\ell$ we have

$$f_{avg}(\pi^T_{[\ell]}) > (1 - e^{-\frac{\ell}{2}} - 3\epsilon) f_{avg}(\pi^*),$$

for some $T \in O(\epsilon (\log n \log \ell))$.

Proof: Let us set $\delta = \frac{\epsilon}{\log \frac{1}{1-\epsilon/2}}$ and let $\Delta_1(\pi)$ be the marginal increase of the first item selected.

By applying Lemma 6 iteratively $\log \frac{1}{1-\epsilon/2} \frac{\ell}{\epsilon}$ times we have

$$\max_{e \notin \text{dom}(\Psi'_T)} \Delta(e|\Psi'_T) \leq (1 - \frac{\epsilon}{2}) \frac{\ell}{\epsilon} \frac{\log \frac{1}{1-\epsilon/2}}{\epsilon} \Delta_1(\pi) = \frac{\epsilon}{\ell} \Delta_1(\pi),$$

9
with probability $1 - \delta \times \log \frac{1}{1 - \epsilon} = 1 - \epsilon$. This means that with probability $1 - \epsilon$ the total marginal increase of the elements added after the $T$-th batch query is at most $\ell \times \frac{\epsilon}{\log \frac{1}{1 - \epsilon} - \frac{\epsilon}{2}} \leq \mathcal{O}(\log n \log \ell)$. This together with Lemma 5 implies that if we stop policy $\pi$ after $T = \mathcal{O}(\log n \log \ell)$ batch queries for any policy $\pi^*$ we have

$$f_{avg}(\pi) > \left(1 - e^{-\frac{\epsilon}{\ell}} - 3\epsilon\right) f_{avg}(\pi^*),$$

as desired. \qed

5 Adaptive Stochastic Minimum Cost Coverage via Batch Queries

In this section we bound the efficiency and round complexity of the semi-adaptive greedy policy. To simplify the proofs, in this section we use a more restricted notion for information gap. It is easy to observe that the same proofs in the previous section hold using this version of information gap as well.

Definition 8 (Restricted Information Gap) At any point of the policy the restricted information gap is defined as follows.

$$\max_{e \in \text{dom}(\psi')} \Delta(e|\psi') \frac{E_{\psi \neq \psi'} \left[ \max_{e \in \text{dom}(\psi_i)} \Delta(e|\psi_i) \right]}{E_{\psi \neq \psi'} \left[ \max_{e \in \text{dom}(\psi_i)} \Delta(e|\psi_i) \right]},$$

5.1 Performance

Next theorem bounds the performance of our policy. The proof of this theorem is a combination of the ideas in Lemma 1, Lemma 5 and Theorem 2 and is presented in Appendix A.

Theorem 9 Assume that there is a value $\eta \in (0, Q]$ such that $f(\psi) > Q - \eta$ implies $f(\psi) = Q$ for all $\psi$. Let $\pi^*$ be an arbitrary policy that covers everything, i.e., $f(\pi^*) = Q$ for all $\phi$. Let $\pi$ be the semi-adaptive greedy policy. We have

$$c_{avg}(\pi) \leq \frac{(c_{avg}(\pi^*) + 1)}{1 - \epsilon} \log \left(\frac{nQ}{\eta}\right) + 1.$$

5.2 Query Complexity

Next we bound the number of batch queries of the semi-adaptive greedy policy. We use Lemma 6 presented in the previous section together with Theorem 9 to prove the following theorem. The proof of this theorem follows the proof of Theorem 7 and is presented in Appendix A.

Theorem 10 Assume that there is a value $\eta \in (0, Q]$ such that $f(\psi) > Q - \eta$ implies $f(\psi) = Q$ for all $\psi$. Let $\pi^*$ be an arbitrary policy that covers everything, i.e., $f(\pi^*) = Q$ for all $\phi$. Let $\pi$ be the semi-adaptive greedy policy and let $\pi^T$ be a policy that runs $\pi$ and stops if it makes $T$ batch queries. For some $T = \mathcal{O}(\log n \log (Qn/\eta))$ we have $f(\pi^T) = Q$ with probability $1 - 1/n$.

\footnote{We use this notion in this section for simplicity. However, since the previous notion of information gap is more intuitive, we keep the previous notion as well.}
6 Hardness Results

Theorem 11 Any constant approximation algorithm for adaptive stochastic submodular maximization requires $\Omega(\log n)$ batch queries.

Proof: Consider the following example. We have $n = 2^{k-1} - 1$ elements, and we want to select $k$ elements. The elements are decomposed into $k$ bags of sizes $1, 2, \ldots, 2^k$, where the decomposition is chosen uniformly at random. The objective function for a set $S$ is the number of distinct bags that elements in $S$ belong to. Whenever we select an element $e$ we see all of the elements that are in the same bag as $e$. It is easy to see that this function is adaptive monotone and adaptive submodular.

Note that, one can iteratively select $k$ elements each with marginal increase of 1 and hence the value of the optimum solution of this instance is $k$. Next we upper-bound the value of the solution of a policy with $T \in o(\log n)$ batch-queries.

Let $B_i$ be the $i$-th batch query and let $b_i = |B_i|$. Note that always the marginal increase of each element is either 0 or 1. Moreover all of the elements with marginal increase 1 are symmetric. Hence, without loss of generality we assume that $B_i$ is random subset of elements with marginal increase 1. Hence, with probability at least $1 - \frac{1}{b_i}$ all of the elements in $B_i$ belong to the $\log^2 b_i$ largest bags with marginal 1. Hence the expected marginal increase of batch $B_i$ is at most $(1 - \frac{1}{b_i}) \log^2 b_i + \frac{1}{b_i} b_i \leq \log^2 b_i + 1$. Hence the expected value of the solution of this policy is at most

$$\sum_{i=1}^{T} (\log^2 b_i + 1) \leq \sum_{i=1}^{T} (\log^2 \frac{k}{T} + 1) = T \log^2 \frac{k}{T} + T \in o(k),$$

where the last inequality is due to $T \in o(\log n) = o(k)$. □

Notice that in the hard example provided in the theorem above, we upper bound the marginal increase of each batch of size $r$ by $\log^2 r + 1$. Hence if we force each batch to query exactly $r$ elements, the expected value of the final solution is at most $\frac{k}{r} (\log^2 r + 1) \in O(\frac{k \log^2 r}{r})$.

Corollary 12 Let $\pi$ be a policy for adaptive stochastic submodular maximization that queries batches of size $r$. The approximation factor of $\pi$ is upper bounded by $O(\frac{\log^2 r}{r})$.

7 Conclusion

In this paper, we re-examined the required rounds of adaptive observations in order to maximize an adaptive submodular function. We proposed an efficient batch policy that with $O(\log n \times \log k)$ adaptive rounds of observations can achieve a $(1 - 1/e - \varepsilon)$ approximation guarantee with respect to an optimal policy that carries out $k$ actions, from a set of $n$ actions, in a fully sequential setting. We also extended our result to the case of adaptive stochastic minimum cost coverage and proposed a batch policy that provides the same guarantee in polylogarithmic adaptive rounds through a similar information-parallelism scheme.
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A Omitted proofs

A.1 Proof of Theorem 2

Proof of Theorem 2: Set \( \ell = (E[k] + 1) \log(nQ/\eta) \), and let \( k \) be a random variable that indicates the number of items picked by \( \pi^* \), i.e. \( k = S(\pi^*, \Phi) \). Note that by definition of \( \pi^* \) we have \( f(\pi^*) = Q \) for all \( \phi \), hence we have \( f_{\text{avg}}(\pi^*) = Q \). By Lemma 4 we have

\[
 f_{\text{avg}}(\pi^*) > \left( 1 - e^{-\frac{k}{E[k] + 1}} \right) f_{\text{avg}}(\pi^*) \quad \text{By Lemma 4}
\]

\[
 = \left( 1 - e^{-\log(nQ/\eta)} \right) f_{\text{avg}}(\pi^*) \quad \text{Since } \ell = (E[k] + 1) \log(nQ/\eta)
\]

\[
 = (1 - \frac{\eta}{nQ}) f_{\text{avg}}(\pi^*)
\]

\[
 = Q - \frac{ \eta }{ n } \quad \text{Since } f_{\text{avg}}(\pi^*) = Q
\]

Recall that, by definition we have \( E[f(\pi^\tau)] = f_{\text{avg}}(\pi^\tau) = Q - \frac{\eta}{n} \). Moreover, by adaptive monotonicity we have \( f(\pi^\tau) \leq f(\phi) = Q \). Hence by Markov inequality with probability \( 1 - 1/n \) we have \( f(\pi^\tau) > Q - n \). By definition of \( \eta \) this implies that with probability \( 1 - \frac{1}{n} \) we have \( f(\pi^\tau) = Q \). Therefore, with probability \( 1 - 1/n \), \( \pi^\tau \) reaches \( Q \) after selecting \( \ell = (E[k] + 1) \log(nQ/\eta) \) items in expectations. Otherwise, \( \pi \) picks at most \( n \) items. Hence the expected number of items that \( \pi \) picks is upper bounded by

\[
(1 - \frac{1}{n}) \times (E[k] + 1) \log \left( \frac{nQ}{\eta} \right) + \frac{1}{n} \times n \leq (c_{\text{avg}}(\pi^*) + 1) \log \left( \frac{nQ}{\eta} \right) + 1.
\]

\[ \square \]

A.2 Proof of Lemma 5

Proof of Lemma 5: First we provide an upper bound on \( f_{\text{avg}}(\pi^*) \). Pick an arbitrary \( i \in \{0, \ldots, \ell\} \). Note that by adaptive monotonicity, we have \( f_{\text{avg}}(\pi^*) \leq f_{\text{avg}}(\pi^{[i]}@\pi^*) \). Next we show that

\[
f_{\text{avg}}(\pi^{[i]}@\pi^*) \leq f_{\text{avg}}(\pi^{[i]}) + \frac{k}{1-\varepsilon} \Delta_i(\pi), \quad \Delta_i(\pi) = f_{\text{avg}}(\pi^{[i+1]}) - f_{\text{avg}}(\pi^{[i]})\]

This implies that

\[
f_{\text{avg}}(\pi^*) \leq f_{\text{avg}}(\pi^{[i]}@\pi^*) \leq f_{\text{avg}}(\pi^{[i]}) + \frac{k}{1-\varepsilon} \Delta_i(\pi).
\]

(3)

Let \( \bar{\Psi} \) be a random variable that indicates the last partial realization that is queried in \( \pi^{[i]} \). For \( j \geq i \) let \( \Psi_j \) be a random variable that indicates the partial realization of \( \pi^{[i]}@\pi^{[j-i]} \). We use \( \psi_j \) to indicate a realization of \( \Psi_j \). We define a random variable \( \Gamma(\epsilon|\psi_j) = f(\psi_j \cup \Phi(\epsilon)) - f(\psi_j) \). Note that by definition \( \Delta(\epsilon|\psi_j) = E[\Gamma(\epsilon|\psi_j)] \). Let \( X_{\epsilon,\psi_j} \) be a binary random variable that is 1 if and only if policy \( \pi^{[i]}@\pi^* \) observes subrealization \( \psi_j \) and picks item \( \epsilon \) right after. Note that when the policy

\[ \text{Note that if we let } \delta = \min_{\phi} p(\phi), \text{ for a deterministic policy having } f(\pi^\tau) > Q - \eta \text{ with probability greater than } 1 - \delta \text{ implies that we always have } f(\pi^\tau) > Q - \eta. \text{ This replaces the factor } n \text{ in the approximation factor with } \frac{1}{\delta}. \] However, since \( \delta \) might be exponentially smaller than \( \frac{1}{n} \), we find the current statement of the theorem more interesting.
is deciding whether to pick $e$ or not, it is not aware of the actual outcome of $\Gamma(e|\psi_j)$. Hence $X_{e,\psi_j}$ and $\Gamma(e|\psi_j)$ are independent. We define $p(\psi_j)$ to be the probability that policy $\pi_{[i]} \otimes \pi^*$ observes subrealization $\psi_j$. We have

\[
f_{\text{avg}}(\pi_{[i]} \otimes \pi^*) - f_{\text{avg}}(\pi_{[i]}) = \mathbb{E} \left[ \sum_{\psi_i} \sum_{\psi_j \triangleright \psi_i \in \text{dom}(\psi_j)} \sum_{e \notin \text{dom}(\psi_j)} X_{e,\psi_j} \Gamma(e|\psi_j) \right] \tag{By Definition}
\]
\[
= \sum_{\psi_i} \sum_{\psi_j \triangleright \psi_i \in \text{dom}(\psi_j)} \sum_{e \notin \text{dom}(\psi_j)} \mathbb{E} \left[ X_{e,\psi_j} \Gamma(e|\psi_j) \right] \tag{Linearity of expectation}
\]
\[
= \sum_{\psi_i} \sum_{\psi_j \triangleright \psi_i \in \text{dom}(\psi_j)} \sum_{e \notin \text{dom}(\psi_j)} \mathbb{E} \left[ X_{e,\psi_j} \right] \mathbb{E} \left[ \Gamma(e|\psi_j) \right] \tag{Independency}
\]
\[
= \sum_{\psi_i} \sum_{\psi_j \triangleright \psi_i \in \text{dom}(\psi_j)} \sum_{e \notin \text{dom}(\psi_j)} \mathbb{E} \left[ X_{e,\psi_j} \right] \Delta(e|\psi_j) \tag{$\Delta(e|\psi_j) = \mathbb{E} \left[ \Gamma(e|\psi_j) \right]$}
\]
\[
\leq \sum_{\psi_i} \sum_{\psi_j \triangleright \psi_i \in \text{dom}(\psi_j)} \sum_{e \notin \text{dom}(\psi_j)} \mathbb{E} \left[ X_{e,\psi_j} \right] \max_{e' \notin \text{dom}(\psi_i)} \Delta(e'|\psi_i) \tag{Adaptive Submodularity}
\]
\[
= \sum_{\psi_i} \left( \sum_{\psi_j \triangleright \psi_i \in \text{dom}(\psi_j)} \mathbb{E} \left[ X_{e,\psi_j} \right] \right) \max_{e' \notin \text{dom}(\psi_i)} \Delta(e'|\psi_i) \tag{definition of $p(\psi_i)$}
\]
\[
= k \sum_{\psi_i} p(\psi_i) \max_{e' \notin \text{dom}(\psi_i)} \Delta(e'|\psi_i) \tag{definition of $p(\psi_i)$}
\]
\[
= k \mathbb{E}_{\psi_i} \left[ \max_{e' \notin \text{dom}(\psi_i)} \Delta(e'|\psi_i) \right] \tag{by information gap bound}
\]
\[
= k \mathbb{E}_{\psi_i} \left[ \frac{1}{1-\varepsilon} \max_{e' \notin \text{dom}(\psi_i^\pi)} \mathbb{E}_{\psi_i^\pi} \left[ \Delta(e'|\psi_i^\pi) \right] \right]
\]
\[
= k \frac{1}{1-\varepsilon} \mathbb{E}_{\psi_i} \left[ \max_{e' \notin \text{dom}(\psi_i^\pi)} \mathbb{E}_{\psi_i^\pi} \left[ \Delta(e'|\psi_i^\pi) \right] \right]
\]
\[
= k \frac{1}{1-\varepsilon} \mathbb{E}_{\psi_i} \left[ \Delta_i(\pi) \right]
\]

This proves Inequality 3 as promised. Let us define $\Delta^*_i = f_{\text{avg}}(\pi^*) - f_{\text{avg}}(\pi_{[i]})$. Inequality 3 implies that $\Delta^*_i \leq k \mathbb{E}_{\psi_i} (\Delta^*_\ell - \Delta^*_{\ell+1})$. By a simple rearrangement we have $\Delta^*_i \leq (1 - \frac{1-k}{k}) \Delta^*_i$. By iteratively applying this inequality we have $\Delta^*_i \leq \left( 1 - \frac{1-k}{k} \right)^{\ell} \Delta^*_0 \leq e^{-\frac{1-k}{k}} \Delta^*_0$. By applying the definition of $\Delta^*_i$ and some rearrangements we have $f_{\text{avg}}(\pi_{[i]}) \geq (1 - e^{-\frac{1}{k}} - \varepsilon) f_{\text{avg}}(\pi^*)$ as desired.

$\square$
Let us start with some definitions. Let $\pi$ be the semi-adaptive greedy policy as defined in section 4, using restricted information gap (Definition 8). For an arbitrary number $\tau$ let $\pi^\tau$ be a policy that selects elements according to $\pi$ and stops when the semi-adaptive value of all of the remaining elements is less than or equal to $\tau$. We define $\tau_i$ to be a number such that the expected number of elements selected by $\pi^\tau_i$ is $i$. Now we are ready to prove Lemma 13.

**Lemma 13** For any policy $\pi^*$ and any positive integer $\ell$ we have

$$f_{\text{avg}}(\pi^\tau_\ell) > (1 - e^{-(1-\epsilon)\ell})f_{\text{avg}}(\pi^*),$$

where $k$ is a random variable that indicates the number of items picked by $\pi^*$.

**Proof:** Let us define $\Delta^\tau_i(\pi) = f_{\text{avg}}(\pi^\tau_i) - f_{\text{avg}}(\pi^\tau_{i-1})$. Recall that in expectation $\pi^\tau_i$ picks one item more than $\pi^\tau_{i-1}$. Moreover note that by definition of $\pi^\tau_i$ the semi-adaptive value of all of the items selected by $\pi^\tau_i$ is at least $\tau_i$. Hence

$$f_{\text{avg}}(\pi^\tau_i) - f_{\text{avg}}(\pi^\tau_{i-1}) = \Delta^\tau_i(\pi) \geq \tau_i. \quad (4)$$

Let $\Psi_i$ be a random variable that indicates the partial realization of $\pi^\tau_i$. We use $\psi_i$ to indicate a realization of $\Psi_i$. Let $\bar{\Psi}$ be a random variable that indicates the last partial realization that is queried in $\pi^\tau_i$.

We define a random variable $\Gamma(e|\psi'_i) = f(\psi'_i \cup \Phi(e)) - f(\psi'_i)$. Note that by definition $\Delta(e|\psi'_i) = E[\Gamma(e|\psi'_i)]$. Now consider the policy $\pi^\tau_i @ \pi^*$. Let $X_{e,\psi'_i}$ be a binary random variable that is 1 if and only if policy $\pi^\tau_i @ \pi^*$ observes subrealization $\psi'_i$ and picks item $e$ right after.

Next we show that

$$\Delta(e|\psi'_i) \leq \frac{\tau_i}{1 - \epsilon}. \quad (5)$$

We have two cases based on the time that the policy $\pi^\tau_i$ stops.

- The policy $\pi^\tau_i$ queries a batch and then observe that the semi-adaptive value of all items drop below $\tau_i$ and then $\pi^\tau_i$ stops.

- While adding items to a batch (and before performing the query), the semi-adaptive value of all items drop below $\tau_i$ and then $\pi^\tau_i$ stops.

Note that in the first case the semi-adaptive values of all of the items are equal to their actual expected marginal impact (i.e., $\Delta(e|\psi'_i)$). Hence, we have $\Delta(e|\psi'_i) \leq \Delta(e|\psi_i) \leq \tau_i$. In the second case, by the definition of the algorithm, the restricted information gap is at least $1 - \epsilon$. This together with the fact that the semi-adaptive values of all items are below $\tau_i$ implies that the expected marginal benefit of the first item that was added to the last batch is at most $\frac{\tau_i}{1 - \epsilon}$. This together with adaptive monotonicity implies $\Delta(e|\psi'_i) \leq \frac{\tau_i}{1 - \epsilon}$ as desired.
We have

\[
\begin{align*}
 f_{avg}(\pi_{[i]}@\pi^*) - f_{avg}(\pi_{[i]}) &= \mathbb{E}\left[ \sum_{\psi_i} \sum_{\psi'_i > \psi_i \in \text{dom}(\psi'_i)} \sum_{e \in \text{dom}(\psi'_i)} X_{e,\psi'_i} \Gamma(e|\psi'_i) \right] \\
 &= \sum_{\psi_i} \sum_{\psi'_i > \psi_i \in \text{dom}(\psi'_i)} \mathbb{E}\left[ X_{e,\psi'_i} \Gamma(e|\psi'_i) \right] \\
 &= \sum_{\psi_i} \sum_{\psi'_i > \psi_i \in \text{dom}(\psi'_i)} \mathbb{E}\left[ X_{e,\psi'_i} \right] \mathbb{E}\left[ \Gamma(e|\psi'_i) \right] \\
 &= \sum_{\psi_i} \sum_{\psi'_i > \psi_i \in \text{dom}(\psi'_i)} \mathbb{E}\left[ X_{e,\psi'_i} \Delta(e|\psi'_i) \right] \\
 &\leq \sum_{\psi_i} \sum_{\psi'_i > \psi_i \in \text{dom}(\psi'_i)} \mathbb{E}\left[ X_{e,\psi'_i} \right] \frac{\tau_i}{1 - \varepsilon} \\
 &= \mathbb{E}[k] \sum_{\psi_i} \sum_{\psi'_i > \psi_i \in \text{dom}(\psi'_i)} X_{e,\psi'_i} \frac{\tau_i}{1 - \varepsilon} \\
 &\leq \frac{\mathbb{E}[k]}{1 - \varepsilon} \Delta_i(\pi). \\
\end{align*}
\]

Now define \( \Delta_i^* = f_{avg}(\pi^*) - f_{avg}(\pi^*_i) \). The above inequality implies that \( \Delta_i^* \leq \frac{\mathbb{E}[k]}{1 - \varepsilon} (\Delta_i^* - \Delta_i^*) \). By a simple rearrangement we have \( \Delta_i^* \leq (1 - \frac{1}{\mathbb{E}[k] + 1}) \Delta_{i-1}^* \leq (1 - \frac{1}{1 - \varepsilon}) \Delta_{i-1}^* \). By iteratively applying this inequality we have \( \Delta_i^* \leq (1 - \frac{1}{\mathbb{E}[k] + 1})^\ell \Delta_{0}^* \leq e^{-(1-\varepsilon)\ell} \Delta_{0}^* \). By applying the definition of \( \Delta_i^* \) and some rearrangement we have \( f_{avg}(\pi^*_{\ell}) > (1 - e^{-(1-\varepsilon)\ell}) f_{avg}(\pi^*) \) as desired. \( \square \)

A.4 Proof of Theorem 9

Proof of Theorem 9: Set \( \ell = \frac{\mathbb{E}[k]+1}{1-\varepsilon} \log(nQ/\eta) \), and let \( k \) be a random variable that indicates the number of items picked by \( \pi^* \). Note that by definition of \( \pi^* \) we have \( f(\pi^*) = Q \) for all \( \phi \), hence we have \( f_{avg}(\pi^*) = Q \). By Lemma 13 we have

\[
\begin{align*}
 f_{avg}(\pi^*) &> (1 - e^{-(1-\varepsilon)\ell}) f_{avg}(\pi^*) \\
 &= (1 - e^{-\log(nQ/\eta)}) f_{avg}(\pi^*) \\
 &= (1 - \frac{\eta}{nQ}) f_{avg}(\pi^*) \\
 &= Q - \frac{\eta}{n}. \\
\end{align*}
\]

Since \( f_{avg}(\pi^*) = Q \)
Recall that, by definition $f_{avg}(\pi^T) = E[f(\pi^t)]$. Moreover, note that by adaptive monotonicity we have $f(\pi^t) \leq f(\phi) = Q$. Hence by Markov inequality with probability $1 - 1/n$ we have $f(\pi^t) > Q - \eta$. By definition of $\eta$ this implies that with probability $1 - \frac{1}{n}$ we have $f(\pi^t) = Q$. Therefore, with probability $1 - 1/n$, $\pi^t$ reaches $Q$ after selecting $\ell = (E[k] + 1) \log(nQ/\eta)$ items in expectations. Otherwise, $\pi$ picks at most all $n$ items. Hence the expected number of items that $\pi$ picks is upper bounded by

$$
(1 - \frac{1}{n}) \times \frac{E[k] + 1}{1 - \varepsilon} \log\left(\frac{nQ}{\eta}\right) + \frac{1}{n} \times n \leq \left(\frac{c_{avg}(\pi^*) + 1}{1 - \varepsilon}\right) \log\left(\frac{nQ}{\eta}\right) + 1.
$$

\[\square\]

A.5 Proof of Theorem $\textbf{[10]}$

First let us start with a couple of definitions. We define random variable $\bar{\Psi}_t$ to be the partial realization obtained by the $t$-th query. Next we prove Theorem $\textbf{[10]}$.

**Proof of Theorem $\textbf{[10]}$:** In order to prove this theorem we show that $f_{avg}(\pi^T) > Q - \frac{\eta}{n}$. Note that $f(\pi^T) \leq Q$. This together with a Markov bound imply $f(\pi^T) > Q - \eta$, and hence $f(\pi^T) = Q$, with probability $1 - 1/n$. In this theorem we simply set $\varepsilon = 0$.

Let us set $\delta = \frac{2Qn^2}{\eta}$. By applying Lemma $\textbf{[5]}$ log $\frac{1}{1 - \varepsilon/2}$ times iteratively we have

$$
\max_{e \in \text{dom}(\bar{\Psi}_T)} \Delta(e|\bar{\Psi}_T) \leq (1 - \frac{\varepsilon}{2})^{\log \frac{1}{1 - \varepsilon/2}} \frac{2Qn^2}{\eta} \Delta_1(\pi) = \frac{\eta}{2Qn^2} \Delta_1(\pi),
$$

with probability $1 - \delta \times \log \frac{1}{1 - \varepsilon/2} \frac{2Qn^2}{\eta} = 1 - \frac{n}{2Qn^2}$. This means that with probability $1 - \frac{n}{2Qn^2}$ the total marginal increase of the elements added after the $T$-th query is at most $n \times \frac{\eta}{2Qn^2} \Delta_1(\pi) = \frac{n}{2Qn^2} \Delta_1(\pi) \leq \frac{n}{2n}$, where

$$
T = \log \frac{1}{1 - \varepsilon/2} \left(\frac{n}{\delta}\right) \times \log \frac{1}{1 - \varepsilon/2} \frac{2Qn^2}{\eta}
\in O\left(\log n + \log \log(Qn/\eta)nlog(Qn/\eta)\right)
\in O\left(\log nlog(Qn/\eta)\right).
$$

This implies that $f_{avg}(\pi^T) \geq (1 - \frac{1}{2Qn^2})(Q - \frac{n}{2n}) = Q - \frac{\eta}{n}$. This implies that $f(\pi^T) = Q$ with probability at least $1 - \frac{1}{n}$, as desired.

---

*We can assume $\log n > \log \log(Qn/\eta)$, since otherwise $n \leq \log(Qn/\eta)$ and hence trivially $T \in O(\log(Qn/\eta))$ as desired.*

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