On the onset of cosmological backreaction

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Cosmological backreaction has been suggested as an explanation of dark energy and is heavily disputed since. We combine cosmological perturbation theory with Buchert’s non-perturbative framework, calculate the relevant averaged observables up to second order in the comoving synchronous gauge, and discuss their gauge dependence. With the help of an integrability condition, the leading second order contributions follow from the first order calculation.

We focus on the onset of cosmological backreaction, as a perturbative treatment is necessarily restricted to the era when the effect is still small. We demonstrate that the leading contributions to all averaged physical observables are completely specified on the boundary of the averaged domain. For any finite domain these surface terms are nonzero in general and thus backreaction is for real.

We map the backreaction effect on an effectively homogeneous and isotropic fluid. The generic effective equation of state is not only time dependent, but also depends on scale.

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I. INTRODUCTION

The accelerated expansion of the Universe has now been confirmed by various observations [1] [2] [3]. To understand this mysterious phenomenon, different explanations have been suggested, e.g., dark energy in the form of a quintessence field or a modification of gravity on large scales. Among the many proposals, a cosmological constant (or vacuum energy) in the context of the inflationary Λ cold dark matter (ΛCDM) model seems to be the most attractive one, as it is simple and provides a good fit to all cosmological data currently available.

However, the observational evidence is based on the assumption of a homogeneous and isotropic universe (the Friedmann-Lemaître-Robertson-Walker (FLRW) model). But, spatial homogeneity and isotropy are rather rough approximations for the Universe, only valid on scales larger than \( \sim 100 \, h^{-1} \) Mpc [4, 5]. Thus, before assuming dark energy to be a component of the Universe, it is worthy to investigate the effects of local inhomogeneities and anisotropies. On the largest scales, the deviations from homogeneity and isotropy are tiny and a description in terms of linear cosmological perturbations is very well justified. However, when the matter and metric fluctuations enter the nonlinear regime (currently the largest observed non-linear objects extend to the 100 \( h^{-1} \) Mpc scale, among the most prominent the Shapely supercluster and the Sloan great wall), the problem of averages [6, 10] arises, and as a consequence the observed cosmological parameters might actually differ from the actual parameters of the underlying cosmology.

Let us pick the most important cosmological parameter, the Hubble expansion rate \( \dot{a} / a \) to discuss this issue in some detail. The idealized measurement of the Hubble expansion rate proceeds as follows. Take a set of \( N \) standard candles (in reality supernovae of type Ia) that sample a local (i.e., objects at redshift \( z \ll 1 \)) physical volume \( V \) homogeneously. Measure their luminosity distances \( d_i \) (via magnitudes) and recession velocities \( v_i = cz_i \) and take the average

\[
H_0 \equiv \frac{1}{N} \sum_{i=1}^{N} \frac{v_i}{d_i}.
\]

In the limit of a very big sample \( (N \to \infty) \), this turns into a volume average

\[
H_0 = \frac{1}{V} \int \frac{v}{d} dV.
\]

In the second step, we neglect the effect of the light cone, but for \( z \ll 1 \) the spatial average is a good approximation for an average over the past light cone, because the expansion rate of the Universe is not changing significantly at time scales much shorter than the Hubble time.

On the other hand, we have a theoretical object that we call the expansion rate, defined as \( H_0^{th} \equiv \dot{a} / a \), where a denotes the scale factor of the background model. The issue in the averaging problem now is to establish the connection between \( H_0 \) and \( H_0^{th} \). In linear theory, by construction, the average \( H_0 \) and the background \( H_0^{th} \) agree if the volume \( V \) becomes large enough. However, due to the nonlinearity of the Einstein equations, cosmological perturbations can affect the evolution of the average, which we often identify with the “background” Universe. This is the so-called backreaction mechanism [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29].

As in the example of the Hubble constant, we typically measure the value of a physical observable by taking an average of the observable in a domain of space-time. Other examples are number counts, correlation functions, power spectra, etc. Consequently, a comparison of the theory with observation needs to utilize averaged quantities.

The influence of cosmological perturbations on the expansion of the Universe shows up in many aspects, e.g.,

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for an underdense patch of the Universe the local expansion rate is naturally larger than its average. The task is to find the evolution equations for the averaged observables and thus the effective equation of state of the Universe.

If this effective equation of state $w_{\text{eff}} \equiv p_{\text{eff}}/\rho_{\text{eff}} < -1/3$, where $\rho_{\text{eff}}$ and $p_{\text{eff}}$ are the effective energy density and pressure, the expansion of the averaged Universe would accelerate. So, if backreaction (i.e., averaging) would give rise to a negative effective pressure, we might be able to explain the observed acceleration of the Hubble expansion.

At the same time such a mechanism might be able to resolve the coincidence problem: why does the onset of acceleration happen around the present time? The answer of cosmological backreaction could be that originally tiny perturbations grow with time, and lead to the formation of (weakly) nonlinear structures at the scales that we use to fix the cosmological parameters. Thus, any observation that is based on physics in the (weakly or strongly) nonlinear regime might be influenced by the backreaction effect, especially the determination of $H_0$ (based on local measurements), the Hubble diagrams from supernovae Type Ia (need data at low redshift and thus local information), the integrated Sachs-Wolfe effect, which can be mistaken for the nonlinear Rees-Sciama effect, and others.

The discussion above does not imply that we have solved the puzzle of dark energy by the backreaction mechanism. It could well be that the backreaction effect is tiny and the nature of dark energy is indeed a constant vacuum energy density. Nevertheless, the backreaction effect is of interest, because with increasing experimental precision, e.g., for the cosmic microwave background (CMB) effects of order $10^{-7}$ are measured [31], and we must take the backreaction effect into account seriously.

Recent research on the backreaction mechanism explored two directions. One is to study the properties of the averaged physical quantities in the perturbed Universe. In Buchert’s work [12, 13, 14] (see [28, 29] for a recent review), the averaged Einstein equations were derived in the synchronous coordinates with two fluctuation terms, the kinematical backreaction term derived in the synchronous coordinates with two fluctuation terms. Following the averaging procedure of Buchert, we rewrite the Einstein equations in terms of these quantities $\langle Q \rangle_D$ and the averaged spatial curvature $\langle R \rangle_D$. The behavior of the perturbed Universe thus depends on the properties of these averaged terms. In Buchert’s approach, the average is taken over a physically comoving spatial volume. This seems to be the appropriate procedure for observations in our local neighborhood, such as for galaxy redshift surveys, which are currently limited to small redshifts ($z < 1$). However, for $z > 1$, spatial averaging does not reflect actual observations, as we observe the past light cone and not a spatial hypersurface. We therefore consider spatial averaging as a first reasonable approach to the cosmological backreaction problem.

The second direction is to use cosmological perturbation theory to study the evolution of the perturbed Universe, such as [24, 22, 31, 32, 33, 34, 35, 36, 37, 38] and references therein. All these works discussed the possibility to explain the accelerated expansion of the Universe as the result of structure formation, without introducing dark energy into the Einstein equations. For example, in [20], the Hubble expansion rate was calculated to second order in a dust (i.e., matter-dominated) Universe.

Recent review and criticism on the backreaction mechanism can be found in [24, 29] and references therein.

Here, we synthesize these two lines of research. Because doing perturbative calculations without averaging, we cannot obtain the global property of the Universe. Whereas, averaging without perturbative calculations, we cannot get quantitative information of the Universe. Therefore, in this paper, we calculate the averaged physical quantities in the cosmological perturbation theory to second order, but without the need to use the metric perturbations of second order. In contrast to previous studies in the literature, we do not aim at calculating ensemble means or variances of spatial averages. It seems to us that the ensemble means of spatially averaged quantities are of limited interest for the interpretation of actual observations. Our interest must be to quantify the backreaction effect in the Milky Way’s neighborhood (the local spatial domain of $\sim (100 \ h^{-1} \ \text{Mpc})^3$, i.e., the domain used to measure $H_0$). We aim at predicting the amount of backreaction based on a measurement of the distribution of matter density within that domain. As we show in this work, the knowledge of the peculiar gravitational potential on the boundary of a physically comoving domain at some initial time allows us to predict the time evolution of the spatially averaged quantities (as long as the effect is small).

Our paper is organized as follows. In Section 2, we introduce the concepts of the expansion, shear and rotation of the Universe, and use the ADM decomposition to rewrite the Einstein equations in terms of these quantities. Following the averaging procedure of Buchert, we arrive at the averaged Einstein equations (Buchert equations [12]) for an irrotational dust universe and an integrability condition in Section 3. The integrability condition provides a consistency relation for the two backreaction terms $\langle Q \rangle_D$ and $\langle R \rangle_D$. With Section 4 we turn to cosmological perturbation theory, and solve for the first order metric perturbations $\Psi$ and $\chi$ in the comoving synchronous gauge. In Sections 5 and 6, we calculate the backreaction terms $\langle Q \rangle_D$ and $\langle R \rangle_D$, the averaged expansion rate $\langle \theta \rangle_D$, the averaged energy density $\langle \rho \rangle_D$, the effective speed of sound $c_{\text{eff}}^2$, and the square of the effective equation of state $w_{\text{eff}}$ and $c_{\text{eff}}^2$ are gauge independent. Conclusions and discussions are provided in Section 8.

In the following, the Greek indices run from 0 to 3 and the Latin indices from 1 to 3, and we use units with $c = 1$. 
II. KINEMATICS AND DYNAMICS OF THE EXPANDING UNIVERSE

The standard FLRW model is based on the assumption of spatial homogeneity and isotropy of the Universe. However, these assumptions are not valid (even approximately) at the scales on which structure formation happens, i.e., the scales smaller with respect to the Hubble radius and sufficiently long after the matter-radiation equality. So, necessarily one must consider not only the expansion, but also the shear and rotation of the Universe in order to understand its kinematics thoroughly.

A. Expansion, shear and rotation

To describe the kinematics of the Universe, we need to calculate the gradient field of the 4-velocity $u^\mu \equiv dx^\mu/d\tau$ of comoving observers, where $\tau$ is their proper time. We introduce the projection onto the spatial hypersurface defined by the comoving observers $h^\mu_\nu \equiv \delta^\mu_\nu + u^\mu u_\nu$. Thus, the components of the gradient field of the 4-velocity define the expansion tensor

$$\theta_{\mu\nu} \equiv u_{\mu,\nu} = h^\alpha_\mu h^\beta_\nu u_{\alpha;\beta} = \frac{1}{3} h_{\mu\nu} \theta + \sigma_{\mu\nu} + \omega_{\mu\nu},$$

where $\theta \equiv u^\mu_{;\mu}$, $\sigma_{\mu\nu} \equiv h^\alpha_\mu h^\beta_\nu (u_{(\alpha;\beta)} - \frac{1}{4} h_{\alpha\beta} u^\lambda_\lambda)$, and $\omega_{\mu\nu} \equiv h^\alpha_\mu h^\beta_\nu u_{(\alpha\beta)}$ are the expansion scalar, shear tensor and rotation tensor, respectively.

In the following, we restrict our attention to an irrotational universe, i.e., $\omega_{\mu\nu} = 0$. Neglecting rotations seems to be a reasonable assumption in the context of inflationary cosmology, as there exist no seeds for vector perturbations and the conservation of angular momentum also implies that only nonlinear effects could lead to a generation of rotation.

The metric of the inhomogeneous and anisotropic Universe may be expressed in terms of synchronous coordinates

$$ds^2 = -d\tau^2 + g_{ij}(t, x) dx^i dx^j,$$

where $\tau$ is the cosmic time, and $x$ denotes the spatial coordinates. The corresponding nontrivial Christoffel symbols are

$$\Gamma^0_{ij} = \frac{1}{2} g_{ij,0}, \quad \Gamma^i_{0j} = \frac{1}{2} g^{ik} g_{kj,0},$$

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} (g_{jl,k} + g_{kl,j} - g_{jk,l}).$$

For an irrotational universe it is possible to use comoving synchronous coordinates (the observer is at rest with respect to the cosmic medium), i.e., $u^\mu = (1, 0)$ and $h^i_j = \delta^i_j$. Thus, the nontrivial components of the shear tensor are

$$\sigma_{ij} = \theta_{ij} - \frac{1}{3} g_{ij} \theta,$$

and we define the shear scalar as

$$\sigma^2 \equiv \frac{1}{2} \sigma_\mu \sigma^\mu = \frac{1}{2} \left( \theta_j^j \frac{1}{3} \theta^2 \right).$$

Furthermore, the nontrivial components of the expansion tensor become

$$\theta_{ij} = \Gamma^0_{ij} = \frac{1}{2} g_{ij,0}, \quad \theta_i^i = \frac{1}{2} g^{ij} g_{ij,0}.$$  

From (3), we have $\theta = \dot{J}/J$, where $\dot{}$ is the derivative with respect to the cosmic time, $J \equiv \sqrt{|\text{det} g_{ij}|}$ and $\text{det} g_{ij}$ denotes the determinant of the metric. Thus,

$$\dot{J} = \theta J.$$  

B. ADM decomposition

Having obtained all the quantities to describe the kinematics of the expanding Universe, we turn to its dynamics. For the dust Universe (in the comoving synchronous coordinates), the only nontrivial component of the energy-momentum tensor is $T^0_0 = -\rho$, the energy density of dust.

According to Arnowitt, Deser, and Misner [39], the Einstein equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ in the present situation can be decomposed into:

the energy constraint

$$R + \theta^2 - \theta^i \theta_i^j = 16\pi G \rho,$$

the momentum constraint

$$\theta^i_{;j} = \theta_i^j,$$

and the evolution equation

$$\dot{\theta}^i_j = -\theta^k_{;j} - R^i_j + 4\pi G \rho \delta^i_j,$$

where $R^i_j$ denotes the spatial Ricci tensor and $R \equiv R^i_i$ is the spatial Ricci scalar. Combining the trace of (7) with (5) and (2) leads to the Raychaudhuri equation [40], which links the expansion and shear scalars together,

$$\dot{\theta} = -\frac{1}{3} \sigma^2 - 2\sigma^2 - 4\pi G \rho.$$

So far, we have not made any approximations apart from neglecting rotation and restricting the matter to dust. These equations are satisfied at any point in spacetime. However, our observations do not allow us to measure all the data that would be necessary to put a well-posed Cauchy problem, but realistic observations deliver averaged quantities. In the next section we discuss the averaged properties of these equations, and in Sections 5 and 6 we use cosmological perturbation theory to evaluate the averaged observables to first and second orders, respectively.
III. DYNAMICS OF FINITE DOMAINS

In the last section, we set up the local dynamical equations for a general irrotational dust universe, but realistic observations provide us with averaged quantities. We follow the averaging procedure by Buchert [12], and obtain the averaged equations of motion for the irrotational dust universe.

A particular choice in Buchert’s formalism is the focus on comoving domains. As long as we neglect the difference between cold dark matter and baryons, the comoving synchronous coordinate system is uniquely defined. From the theoretical point of view, this appears to be the most natural set-up, given that we work in synchronous comoving coordinates. We argue in Sec. 7, that there are convincing reasons to pick this particular coordinate system, to be the one that is best adapted to the actual observational situation. However, we should keep in mind that for observational purposes it is highly nontrivial to identify comoving domains. Nevertheless, we think that comoving domains are a reasonable approximation as the effect from peculiar motion is small compared to the Hubble expansion at scales beyond 100 h⁻¹ Mpc.

A. Averaging procedure

The spatial average of an observable $O(t, x)$ in a physically comoving domain $D$ (with the dust particles) at a fixed time $t$ is defined as [12]

$$
\langle O \rangle_D \equiv \frac{1}{V_D(t)} \int_D J(t, x)O(t, x)dx,
$$

where $V_D(t) \equiv \int_D J(t, x)dx$ is the volume of the domain, and the boundary of the domain is assumed to be moving. Following this spatial averaging procedure, we calculate the averaged expansion rate $\langle \theta \rangle_D$ as an example.

From the definition of $V_D(t)$, we may introduce an effective scale factor $a_D$

$$
a_D \equiv \left( \frac{V_D(t)}{V_{D_0}} \right)^{1/3},
$$

where $a_{D_0}$ and $V_{D_0}$ are the values of $a_D$ and $V_D$ at the present time. With the help of (11) and (10) we find the averaged expansion rate $\langle \theta \rangle_D$

$$
\langle \theta \rangle_D = \frac{1}{V_D} \int_D J(t)dx = \frac{1}{V_D} \int_D J(t)dx = \frac{\dot{V}_D}{V_D} = 3\dot{a}_D/a_D.
$$

The effective Hubble expansion rate can thus be defined as

$$
H_D \equiv \frac{\dot{a}_D}{a_D} = \frac{1}{3} \langle \theta \rangle_D.
$$

An important consequence of the definition (9) is that the spatial average and the time derivative do not commute with each other. It is straightforward to prove a corresponding Lemma (commutation rule) [12]

$$
\langle O \rangle_D - \langle \dot{O} \rangle_D = \langle O\theta \rangle_D - \langle \dot{O} \rangle_D \langle \theta \rangle_D.
$$

This Lemma is used to calculate the second order term of the averaged expansion rate $\langle \theta \rangle_D$ in Section 6.

B. Buchert equations

With the definition of the spatial average (9) and the Lemma (13), we yield the Buchert equations [12] from averaging the Einstein equations (5) – (7) and the Raychaudhuri equation [8],

$$
\left( \frac{\dot{a}_D}{a_D} \right)^2 + \frac{k_D}{a_D^2} = \frac{8\pi G}{3} \rho_{\text{eff}},
$$

$$
- \frac{\dot{a}_D}{a_D} = \frac{4\pi G}{3} (\rho_{\text{eff}} + 3p_{\text{eff}}),
$$

where $\rho_{\text{eff}}$ and $p_{\text{eff}}$ are the effective energy density and effective pressure of an isotropic fluid, which read

$$
\rho_{\text{eff}} = \langle \rho \rangle_D - \frac{1}{16\pi G} \left[ \langle Q \rangle_D + \left( \langle R \rangle_D - \frac{6k_D}{a_D^2} \right) \right],
$$

$$
p_{\text{eff}} = -\frac{1}{16\pi G} \left[ \langle Q \rangle_D - \frac{1}{3} \left( \langle R \rangle_D - \frac{6k_D}{a_D^2} \right) \right].
$$

The expression $\langle Q \rangle_D$ is the kinematical backreaction term,

$$
\langle Q \rangle_D = \frac{2}{3} \langle (\theta - \langle \theta \rangle_D)^2 \rangle_D - 2\langle \sigma^2 \rangle_D
$$

$$
= \frac{2}{3} \left( \langle \theta^2 \rangle_D - \langle \theta \rangle_D^2 \right) - 2\langle \sigma^2 \rangle_D,
$$

which consists of the variance of the averaged expansion rate and the averaged shear scalar.

Equations (16) and (17) express a highly nontrivial result! They closely resemble the Friedmann equations, but have been obtained without the assumption of homogeneity and isotropy. What has been shown is that any irrotational dust Universe, averaged over comoving (spatial) domains appears to the observers to be a FLRW-like Universe. The “curvature term” $k_D/a_D^2$ has been introduced to show that actually any FLRW geometry might be picked. Two geometries differ by their expressions for the effective fluid (16) and (17). Without loss of generality, we can thus take $k_D = 0$ in the following calculations.

This formulation of backreaction provides the link to arguments in favor of cosmological backreaction that have been put forward by one of the authors in [16]. It has been argued that on the largest scales (where the cosmic principle applies) we can view the Universe as being described by a FLRW model filled with a single...
isotropic, but imperfect fluid, i.e., we can then understand structure formation as a dissipative process that creates entropy and it has been shown in [16] that the second law of thermodynamics implies for an expanding dust Universe that \( p_{\text{eff}} < 0 \).

From the Buchert equations, we see that the evolution of the inhomogeneous and anisotropic Universe depends not only on the energy density, but also the back-reaction term \( \langle Q \rangle_D \) and the averaged spatial curvature \( \langle R \rangle_D \). So it is quite important to know the values of \( \langle Q \rangle_D \) and \( \langle R \rangle_D \). For instance, we find from [15] that if \( p_{\text{eff}} + 3p_{\text{eff}} < 0 \), i.e., \( \langle Q \rangle_D > 4\pi G \langle p \rangle_D \), the averaged expansion accelerates. In other words, the averaged Universe can expand in an accelerating way in the dust era, even if the local expansion is decelerating everywhere in the Universe. Accelerated expansion of the averaged expansion rate does not violate the strong energy condition!

We calculate \( \langle Q \rangle_D \) and \( \langle R \rangle_D \) in cosmological perturbation theory to both first and second orders in the next two sections. Furthermore, we can define the effective equation of state as

\[
w_{\text{eff}} \equiv \frac{\rho_{\text{eff}}}{\rho_{\text{eff}}} = \frac{\langle R \rangle_D - 3\langle Q \rangle_D}{2\langle \theta \rangle_D^2},
\]

and the square of an effective speed of sound as

\[
c_{\text{eff}}^2 = \frac{\dot{\rho}_{\text{eff}}}{\rho_{\text{eff}}}
\]

This effective speed of sound is the characteristic speed at which a small perturbation propagates through the effective fluid. An example would be a deformation of the boundary of the domain, or a perturbation caused by the introduction of some extra mass into the domain. Our effective speed of sound is different from the isentropic speed of sound. We calculate \( w_{\text{eff}} \) and \( c_{\text{eff}}^2 \) in Section 6.

C. Integrability condition

The Buchert equations contain two averaged quantities, \( \langle Q \rangle_D \) and \( \langle R \rangle_D \), which influence the evolution of the inhomogeneous and anisotropic Universe. However, these two terms are not independent, but can be linked by an integrability condition.

In the irrotational dust universe, pressure is negligible, so the energy-momentum tensor is given by \( T_{\mu\nu} = \rho u^\mu u_\nu \). From the covariant conservation of its time-like part we find the continuity equation

\[
\dot{\rho} = -\theta \rho.
\]

The space-like conservation is the Euler equation, which is trivial for an irrotational dust Universe, if expressed in the comoving synchronous coordinates. Taking the average of [21] and applying the Lemma [13] we have

\[
\langle \rho \rangle_D = -\langle \theta \rangle_D \langle \rho \rangle_D = -3H_D \langle \rho \rangle_D.
\]

From [14], [15] and [22], we find the relation between \( \langle Q \rangle_D \) and \( \langle R \rangle_D \) (the integrability condition) [12]

\[
(a_D^6 \langle Q \rangle_D) + a_D^4 (a_D^2 \langle R \rangle_D) = 0.
\]

The integrability condition is an essential equation for the following calculations. Its advantage is that it can be used to any order in perturbative calculations, as it is an exact result. This is shown in Section 6, where we make use of the integrability condition to derive the second order terms of \( \langle R \rangle_D \), \( \langle \theta \rangle_D \) and \( \langle \rho \rangle_D \) without using the metric perturbations of second order. If we had not made the choice \( k_D = 0 \), we would need to replace \( \langle R \rangle_D \) in [23] by \( \langle R \rangle_D - 6k_D/a_D^2 \), which would not change the solution of the integrability condition, as can be seen easily.

So far, all our results are exact for an irrotational dust universe. In order to get quantitative information on the observed Universe, we turn to cosmological perturbation theory. We use the comoving synchronous gauge below.

IV. LINEARIZED EINSTEIN EQUATIONS IN THE COMOVING SYNCHRONOUS GAUGE

In this section, we first introduce the metric perturbations \( \Psi \) and \( \chi \) and find the linearized Einstein equations. Solving these equations, we find the time dependence of \( \Psi \) and \( \chi \). With the help of these solutions, we calculate \( \langle Q \rangle_D \), \( \langle R \rangle_D \), \( \langle \theta \rangle_D \) and \( \langle \rho \rangle_D \) to both first and second orders in the next two sections.

A. Einstein equations for the perturbed Universe

We start now from a spatially flat FLRW dust model with the scale factor \( a(t) \). In the synchronous gauge we write the first order linearly perturbed metric as [20]

\[
ds^2 = -dt^2 + a^2(t)([1 - 2\Psi] \delta_{ij} + D_{ij} \chi)dx^i dx^j,
\]

where \( \Psi \) and \( \chi \) are the scalar metric perturbations at first order, \( D_{ij} \equiv \partial_i \partial_j - \frac{1}{a} \delta_{ij} \Delta \) and \( \Delta \) denotes the Laplace operator in a three-dimensional Euclidean space. The scale factor \( a \) in [24] is not the same as the effective scale factor \( a_D \) defined in [10], and their relation is shown in Section 6.

From the line element [24], we straightforwardly obtain the nontrivial Christoffel symbols,

\[
\Gamma^0_{ij} = a \dot{a} \delta_{ij} - 2a \dot{a} \Psi \delta_{ij} - a^2 \ddot{\Psi} \delta_{ij} + a \ddot{a} D_{ij} + \frac{a^2}{2} D_{ij} \dot{\chi},
\]

\[
\Gamma^i_{0j} = \frac{\dot{a}}{a} \delta^i_j - \dot{\Psi} \delta^i_j - \frac{1}{2} D^i_j \dot{\chi},
\]

\[
\Gamma^i_{jk} = -\partial_k \Psi \delta^i_j - \partial_j \Psi \delta^i_k + \partial^i \delta_{jk} + \frac{1}{2} D^i_j \partial_k \chi + \frac{1}{2} D^i_k \partial_j \chi - \frac{1}{2} D_{jk} \partial^i \chi,
\]

\[\text{in } t^2 \]
and the components of the Einstein tensor
\[
G^0_0 = -3 \left( \frac{\dot{a}}{a} \right)^2 + 6 \frac{\dot{a}}{a} \dot{\Psi} - \frac{2}{a^2} \left( \Psi + \frac{1}{6} \Delta \chi \right),
\]
\[
G^0_i = -2 \partial_i \left( \Psi + \frac{1}{6} \Delta \chi \right),
\]
\[
G^j_i = - \left[ \left( \frac{\dot{a}}{a} \right)^2 + 2 \frac{\ddot{a}}{a} \right] \delta^i_j + \frac{2}{3} \left[ 3 \ddot{\Psi} + 9 \frac{\dot{a}}{a} \dot{\Psi} - \frac{1}{a^2} \Delta \left( \Psi + \frac{1}{6} \Delta \chi \right) \right] \delta^i_j + D^j_i \left[ \frac{1}{2} \ddot{\chi} + 3 \frac{\dot{a}}{2a} \dot{\chi} + \frac{1}{a^2} \left( \Psi + \frac{1}{6} \Delta \chi \right) \right].
\]

The energy-momentum tensor of dust becomes
\[
T^0_0 = -\rho = -\rho^{(0)} - \rho^{(1)},
\]
where \(\rho^{(0)}\) and \(\rho^{(1)}\) are the energy density of the background and at first order, respectively.

We are now ready to obtain the linearized equations of motion in the ADM decomposition. The different components at different orders are

- the energy constraint at zeroth order
  \[
  \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho^{(0)},
  \]

- and at first order
  \[
  -3 \frac{\dot{a}}{a} \dot{\Psi} + \frac{1}{a^2} \Delta \left( \Psi + \frac{1}{6} \Delta \chi \right) = 4\pi G \rho^{(1)};
  \]

- the momentum constraint
  \[
  \partial_i \left( \Psi + \frac{1}{6} \Delta \chi \right) = 0;
  \]

- the evolution equation at zeroth order
  \[
  \left( \frac{\dot{a}}{a} \right)^2 + 2 \frac{\ddot{a}}{a} = 0,
  \]

- its diagonal \((i = j)\) piece at first order
  \[
  3 \ddot{\Psi} + 9 \frac{\dot{a}}{a} \dot{\Psi} - \frac{1}{a^2} \Delta \left( \Psi + \frac{1}{6} \Delta \chi \right) = 0,
  \]

- and its off-diagonal \((i \neq j)\) piece at first order
  \[
  D^j_i \left[ \frac{1}{2} \ddot{\chi} + 3 \frac{\dot{a}}{2a} \dot{\chi} + \frac{1}{a^2} \left( \Psi + \frac{1}{6} \Delta \chi \right) \right] = 0.
  \]

From the covariant energy-momentum conservation, we find at zeroth order
\[
\rho^{(0)} + 3 \frac{\dot{a}}{a} \rho^{(0)} = 0,
\]
and at first order
\[
\rho^{(1)} + 3 \frac{\dot{a}}{a} \rho^{(1)} - 3 \dot{\Psi} \rho^{(0)} = 0.
\]
Equation \((34)\) has a first integral,
\[
\ddot{\zeta}(x) = \frac{\rho^{(0)}}{3\rho^{(0)}} - \Psi,
\]
which resembles the famous hypersurface-invariant variable (here for dust, expressed in the synchronous coordinates)
\[
\zeta(t, x) \equiv \ddot{\zeta} - \frac{1}{6} \Delta \chi,
\]
commonly used to characterize the primordial power spectrum \([41]\).

**B. Solutions for \(a, \Psi\) and \(\chi\)**

**a. Solution for \(a\).** From \([38]\), we have \(\rho^{(0)} a^3 = \rho_0^{(0)} a_0^3\), where \(\rho_0^{(0)}\) and \(a_0\) are the values of \(\rho^{(0)}\) and \(a\) at the present time. By means of \((27)\), we find
\[
\frac{a}{a_0} = \left( \frac{t}{t_0} \right)^{2/3}.
\]

We can see that \(a\) grows as \(t^{2/3}\), which is the result of the spatially flat FLRW dust cosmology. But this does not mean that the perturbed Universe expands in the same way as the unperturbed one, because in the perturbed Universe, the meaningful scale factor is the effective scale factor \(a_D\) defined in \([11]\), which, however, is not equivalent to \(a\), and their relation is obtained in Section 6. So we cannot know the behavior of the expansion of the perturbed Universe in terms of the scale factor \(a\).

**b. Solution for \(\Psi\).** We first eliminate \(\rho^{(1)}\) from the equations of motion with the help of the first integral \(\dot{\zeta}\) from \([35]\),
\[
\rho^{(1)} = \frac{3\rho_0^{(0)} a_0^3}{a^3} \left( \Psi + \ddot{\zeta}(x) \right).
\]

This allows us to obtain an equation for \(\Psi\). Namely, from \([28], [31]\) and \([38]\), we have
\[
3 \ddot{\Psi} + 6 \frac{\dot{a}}{a} \dot{\Psi} = 4\pi G \rho^{(1)} = \frac{12\pi G \rho_0^{(0)} a_0^3}{a^3} \left( \Psi + \ddot{\zeta}(x) \right),
\]
and using \((37)\), we obtain
\[
3 \ddot{\Psi} + \frac{4}{t} \dot{\Psi} - \frac{2}{t^2} \Psi = \frac{2}{t^2} \ddot{\zeta}(x).
\]

So we find the solution for \(\Psi\)
\[
\Psi(x, t) = A(x) t^{2/3} + B(x) t^{-1} - \ddot{\zeta}(x),
\]
where \(A(x)\) and \(B(x)\) are constants of integration, i.e., functions of the spatial coordinates only. We can see from \([40]\) that \(\Psi\) consists of one growing mode \(A(x) t^{2/3}\), one decaying mode \(B(x) t^{-1}\), and one constant mode \(-\ddot{\zeta}(x)\),
in which the free spatial functions must be fixed by the initial conditions.

In the next sections, we see that only the time derivatives \( \dot{\Psi} \) and \( \ddot{\Psi} \) show up in the observables that are of interest to us. Also, we are only concerned with the evolutions of perturbations at late times, so we can neglect the decaying mode \( B(x)t^{-1} \). Hence, only the growing mode \( A(x)t^{2/3} \) is of importance for the following calculations. Thus, we get the time derivatives of \( \Psi \)

\[
\dot{\Psi} = \frac{2}{3} A(x)t^{-1/3}, \quad \ddot{\Psi} = \frac{2}{9} A(x)t^{-4/3}.
\] (41)

c. Solution for \( \chi \). From \( G^j_i = 0 \), we have

\[
\frac{2}{3} \left[ 3 \ddot{\Psi} + 9 \frac{\dot{a}}{a} \dot{\Psi} - \frac{1}{a^2} \Delta \left( \Psi + \frac{1}{6} \Delta \chi \right) \right] \delta^j_i
+ D^j_i \left[ \frac{1}{2} \dot{\chi} + \frac{3 \dot{a}}{2a} \chi + \frac{1}{a^2} \left( \Psi + \frac{1}{6} \Delta \chi \right) \right] = 0.
\] (42)

Multiplying \( a^2 \) on \( (12) \), taking the time derivative and inserting \( (29) \), we have

\[
\left( \frac{a^2}{2} D^j_i \ddot{\chi} + \frac{3a \dot{a}}{2} D^j_i \dot{\chi} \right) + \left[ a^2 \left( 2 \ddot{\Psi} + 6 \frac{\dot{a}}{a} \dot{\Psi} \right) \right] \delta^j_i = 0.
\] (43)

From \( (37) \) and \( (41) \), the second part in \( (43) \) is 0, so we get

\[
D^j_i \ddot{\chi} = C^j_i(x)t^{-1/3} + E^j_i(x)t^{-2},
\]

where \( C^j_i(x) \) and \( E^j_i(x) \) are functions of spatial coordinates only. In the following calculation we neglect the decaying mode \( E^j_i(x)t^{-2} \), so

\[
D^j_i \ddot{\chi} = C^j_i(x)t^{-1/3}.
\] (44)

From \( (44) \), we obtain the solution for \( \chi \)

\[
\chi(x,t) = C(x)t^{2/3} + g(x),
\] (45)

where \( C(x) \) and \( g(x) \) are also the functions of spatial coordinates only. The function \( g(x) \) does not carry physical information, as it can be fixed by the residual spatial gauge transformation of the comoving synchronous gauge, and we utilize this freedom to set \( g(x) = 0 \). Substituting \( (45) \) into \( (44) \), we can get the relation between \( C(x) \) and \( C^j_i(x) \)

\[
C^j_i(x) = \frac{2}{3} D^j_i C(x).
\]

Finally, the time derivatives of \( \chi \) are

\[
\dot{\chi} = \frac{2}{3} C(x)t^{-1/3}, \quad \ddot{\chi} = -\frac{2}{9} C(x)t^{-4/3}.
\] (46)

We see from \( (10) \) and \( (15) \) that both \( \Psi \) and \( \chi \) grow as \( t^{2/3} \) at late times. Because \( a \propto t^{2/3} \), both \( \Psi \) and \( \chi \) grow linearly as the scale factor \( a \) in the perturbed dust Universe. So if cosmological perturbation theory is valid, i.e., the perturbative terms \( \Psi \) and \( \chi \) are small, the scale factor should not be too large. In other words, a perturbative analysis (at any order) is restricted to the “linear” regime.

C. Relation between \( A(x) \), \( C(x) \) and \( \zeta(x) \)

We have obtained the solutions for \( \Psi \) and \( \chi \), but these two solutions are not independent due to \( (29) \). From \( (29) \), \( (41) \) and \( (46) \), we have

\[
\partial_i \left( A + \frac{1}{6} \Delta C \right) = 0.
\]

Because both \( A \) and \( C \) are functions of the spatial coordinates only, we have

\[
A + \frac{1}{6} \Delta C = K,
\] (47)

where \( K \) is a constant. In the spatially flat universe \( K = 0 \), which is shown in the next section, and we find the relation between \( A \) and \( C \), i.e., the relation between \( \Psi \) and \( \chi \)

\[
A = -\frac{1}{6} \Delta C.
\] (48)

Finally, with the help of the evolution equations \( (31) \) and \( (42) \), we obtain

\[
\zeta(x) = \frac{5}{4} \frac{a^2}{t_0^3} C(x).
\] (49)

Let us note that at superhorizon scales \( \zeta \approx \zeta \). As the amplitude of \( \zeta \) at superhorizon scales is measured by cosmic microwave background experiments, the magnitudes of the time independent functions \( \zeta \), \( C \) and \( A \) are thus fixed.

D. Relation between \( C(x) \), \( \zeta(x) \) and the peculiar gravitational potential \( \varphi(x) \)

The peculiar gravitational potential \( \varphi(x) \) is defined from the Poisson equation as \( (20) \)

\[
\Delta \varphi(x) \equiv 4 \pi G \rho^{(1)} a^2.
\] (50)

From \( (50) \), using \( (11) \) and \( (18) \), we have

\[
\Delta C(x) = -12 \pi G \rho^{(1)} t^{4/3}.
\] (51)

So, with the help of \( (50) \), \( (51) \) and \( (37) \), we obtain the relation between \( C(x) \), \( \zeta(x) \) and \( \varphi(x) \)

\[
\varphi(x) = -\frac{1}{3} \frac{a^2}{t_0^{4/3}} C(x) = -\frac{3}{5} \zeta(x).
\] (52)
We see that the peculiar gravitational potential \( \varphi(x) \) is just a linear function of the metric perturbation \( \chi \).

Therefore, we know the solutions for \( \alpha, \Psi \) and \( \chi \), which we use in the next two sections to calculate the first and second order contributions to the averaged physical quantities, and we focus on the investigation of the time dependence of the averaged observables.

**V. FIRST ORDER PERTURBATIONS**

In this section, we calculate \( \langle \theta \rangle_D \), \( \langle R \rangle_D \) and \( \langle \rho \rangle_D \) in the perturbed Universe, and these results are the first step to the derivation of the second order contributions. We do not calculate \( \langle Q \rangle_D \), because it is a pure second order term, which is proven in the next section.

For the first order calculations of averaged quantities, the integration measure \( J \) must be expanded to the first order as well,

\[
J = a^3(1 - 3\Psi) = a^3 \left( 1 - 3At^{2/3} \right) = a^3 \left( 1 + \frac{1}{2} \Delta C t^{2/3} \right),
\]

as at late times, the decaying and constant modes are negligible. In the following, let us denote

\[
\langle O \rangle_{D1} = \frac{\int_D O J d\mathbf{x}}{\int_D J d\mathbf{x}},
\]

which is defined to be the average on the background (i.e., \( J = a^3 \)). Watch out that the average is still over a physically comoving domain, which might have a distorted geometry, even on the background. Thus, for the first order calculations, the averages of the zeroth and first order quantities are

\[
\langle O^{(0)} \rangle_D = \frac{\int_D O^{(0)} J d\mathbf{x}}{\int_D J d\mathbf{x}} = O^{(0)},
\]

\[
\langle O^{(1)} \rangle_D = \frac{\int_D O^{(1)} J d\mathbf{x}}{\int_D J d\mathbf{x}} = \frac{\int_D O^{(1)} d\mathbf{x}}{\int_D d\mathbf{x}} = \langle O^{(1)} \rangle_{D1}.
\]

*Therefore, the perturbation in \( J \) does not affect the first order calculations.*

**A. Averaged expansion rate \( \langle \theta \rangle_D \)**

Substituting the perturbative connections in (50) into (31), we have

\[
\theta^j = \frac{\dot{a}}{a} \delta^j_0 - \Psi \delta^j_0 + \frac{1}{2} D^j_0 \dot{\chi},
\]

and taking the trace, we get the perturbative expansion rate to first order,

\[
\theta = 3 \frac{\dot{a}}{a} - 3 \Psi,
\]

in which we have used the property \( D^j_0 \dot{\chi} = 0 \). Using (67) and (11), we obtain the averaged expansion rate \( \langle \theta \rangle_D \) as a function of cosmic time \( t \) and \( A \),

\[
\langle \theta \rangle_D = 3 \frac{\dot{a}}{a} - 3 \langle \Psi \rangle_D = \frac{2}{t} - \frac{2}{t^{4/3}} A_{D1}^1.
\]

From (57), the first order perturbation decays as \( t^{-1/3} \), which is slower than that of the zeroth order term \( 2/t \). Therefore, the perturbation becomes more and more important as the Universe evolves. However, this does not mean that this perturbation dominates at late times, as in the perturbative approach, we must restrict our analysis to \( \langle \Psi \rangle_D \ll 1 \).

**B. Averaged spatial curvature \( \langle R \rangle_D \)**

From (31) and the trace of (7), we have

\[
R = 16\pi G \rho - \theta^2 + \theta_j \theta^j,
\]

\[
R = 12\pi G \rho - \dot{\theta} - \theta^2,
\]

and thus

\[
R = -\dot{\theta}^2 - 4\dot{\theta} - 3\theta_j \theta^j.
\]

By means of (55), (56), (37) and (41), we find to first order

\[
\langle R \rangle_D = \frac{40\langle A \rangle_{D1}}{3t^{4/3}}.
\]

Closer inspection of (60) shows that

1. \( \langle R \rangle_D \) has only a first order term while the zeroth order term vanishes, as the background metric is spatially flat.
2. \( \langle R \rangle_D \) decays as \( t^{-4/3} \). From (37), \( a \propto t^{2/3} \), so \( \langle R \rangle_D \propto 1/a^2 \) as the Universe expands (24).
3. we may, with the help of (47), rewrite \( \langle R \rangle_D \) as

\[
\langle R \rangle_D = \frac{40}{3^{1/3}} \left( -\frac{1}{6} \langle \Delta C \rangle_{D1}^2 + \langle K \rangle \right).
\]

We can see that the constant \( K \) in (47) contributes to the averaged spatial curvature a term \( 40K/3t^{4/3} \propto 1/a^2 \). We know that in the unperturbed \( k \neq 0 \) universe, \( R = 6k/a^2 \). Thus, \( 40K/3t^{4/3} \) is just the background spatial curvature term, and as we discuss the perturbations in the spatially flat Universe, this term vanish naturally. This is the reason for \( A = -\Delta C/6 \) in (48).

**C. Averaged energy density \( \langle \rho \rangle_D \)**

Similarly, from (58), (55) and (50), we have

\[
\langle \rho \rangle_D = -\frac{\langle \dot{\theta} \rangle_D + \langle \theta_j \theta^j \rangle_D}{4\pi G} = \frac{1}{6\pi G t^2} + \langle A \rangle_{D1} \frac{1}{2\pi G t^{4/3}}.
\]
So we find for a domain overdense in average that \( \langle A \rangle_{D_1} \) is positive. At the same time, from (57) the averaged expansion rate is reduced. This is consistent with the intuition of the gravitational collapse, which decreases the expansion rate of the Universe. Also, from (60) we have a positive averaged spatial curvature for the overdense regions.

We find in the first order perturbative calculations that only \( \Psi \) enters the expressions of \( \langle \theta \rangle_D \), \( \langle R \rangle_D \) and \( \langle \rho \rangle_D \), and the metric perturbation \( \chi \) does not show up. We show in the next section that \( \sigma^2 = 2D_i^j\chi D_j^i \chi \), so \( \chi \) is related to the shear of the perturbed Universe. This means that only the expansion influences the evolution of the averaged spatial curvature term and the averaged energy density in the perturbed Universe at first order.

Let us finally note that the first order contributions to \( \langle \theta \rangle_D \), \( \langle R \rangle_D \) and \( \langle \rho \rangle_D \) are all surface terms actually, as we may write them as integrals of total derivatives

\[
\langle A \rangle_{D_1} = -\frac{\int_D \delta (C)dx}{\int_D dx}.
\]

More surface terms show up below, when we turn to the second order perturbations.

VI. SECOND ORDER PERTURBATIONS

We move on to the second order perturbations of physical quantities. Second order cosmological perturbation theory has been discussed widely in the literature, such as [20, 42, 43, 44]. However, in these papers, the metric perturbations of second order are always needed for calculations, and these calculations are always rather complicated and tedious. In this paper, we show how to obtain the leading terms of second order contributions to \( \langle Q \rangle_D \), \( \langle R \rangle_D \), \( \langle \theta \rangle_D \) and \( \langle \rho \rangle_D \) from the metric perturbations of first order only.

We first prove that the kinematical backreaction term \( \langle Q \rangle_D \) is a second order term, and then use the integrability condition, which is a crucial new input, to find the second order terms of \( \langle R \rangle_D \), \( \langle \theta \rangle_D \) and \( \langle \rho \rangle_D \). In these calculations, the shear \( \sigma^2 \) and thus \( \chi \) show up in the expressions. The effective equation of state \( w_{\text{eff}} \) and the square of the effective speed of sound \( c_{\text{eff}}^2 \) are also given to second order.

Different from the first order contributions, at second order we have to consider the perturbation of the integration measure \( J \). Therefore, the averaged quantities of physical observables of different orders become

\[
\langle O^{(0)} \rangle_D = O^{(0)},
\langle O^{(1)} \rangle_D = \langle O^{(1)} \rangle_{D_1} - 3\langle O^{(1)} \rangle_{D_1} + 3\langle O^{(1)} \rangle_{D_1}\langle \Psi \rangle_{D_1},
\langle O^{(2)} \rangle_D = \langle O^{(2)} \rangle_{D_1}.
\]

(62)

We can see that at second order, the average of a first order quantity \( \langle O^{(1)} \rangle_D \) picks up two second order modifications \(-3\langle O^{(1)} \rangle_{D_1} + 3\langle O^{(1)} \rangle_{D_1}\langle \Psi \rangle_{D_1} \). In the following, we show that these modifications show up in the second order calculation of \( \langle \theta \rangle_D \).

---

A. Averaged kinematical backreaction term \( \langle Q \rangle_D \)

Let us recall the kinematical backreaction term \( \langle Q \rangle_D \) defined in (18)

\[
\langle Q \rangle_D = \frac{2}{3} \left( \langle \theta^2 \rangle_D - \langle \theta \rangle_{D_1}^2 \right) - 2\langle \sigma^2 \rangle_D.
\]

Now we prove that \( \langle Q \rangle_D \) is a pure second order term. We show that the first part \( \langle \theta^2 \rangle_D - \langle \theta \rangle_{D_1}^2 \) and the second part \( \langle \sigma^2 \rangle_D \) are both of second order.

To calculate the variance \( \langle \theta^2 \rangle_D - \langle \theta \rangle_{D_1}^2 \) to second order, we write

\[
\theta = \theta^{(0)} + \theta^{(1)} + \theta^{(2)},
\]

(63)

where \( \theta^{(0)} \), \( \theta^{(1)} \) and \( \theta^{(2)} \) are the zeroth, first and second order contributions to \( \theta \), respectively. \( \theta^{(0)} \) and \( \theta^{(1)} \) have been calculated in (56). Using (62), to second order we have

\[
\langle \theta^2 \rangle_D - \langle \theta \rangle_{D_1}^2 = \left( \langle \theta^{(0)} \rangle_D + \langle \theta^{(1)} \rangle_D + \langle \theta^{(2)} \rangle_D \right)^2 - \langle \theta^{(1)} \rangle_{D_1}^2 = \langle \theta^{(1)} \rangle_{D_1}^2 \langle \theta^{(1)} \rangle_{D_1} - \langle \theta^{(1)} \rangle_{D_1}^2.
\]

(64)

This means that the first part of \( \langle Q \rangle_D \) is of second order, however, for calculating it we do not need to know the detailed form of \( \theta^{(2)} \). All we need is \( \theta \) up to first order (see (56)).

Similarly, we calculate the average of the shear scalar \( \langle \sigma^2 \rangle_D \). From (1) and (55) we find at first order that

\[
\sigma_j^i = \theta_j^i - \frac{1}{3} \theta \delta_j^i = \theta_j^{(0)} + \theta_j^{(1)} - \frac{1}{3} \langle \theta^{(0)} \rangle_D + \langle \theta^{(1)} \rangle_{D_1} \delta_j^i = \frac{1}{2} D_j^i \chi,
\]

(65)

so \( \sigma_j^i \) has no zeroth order contribution. Hence, using (46), we have

\[
\sigma^2 = \frac{1}{2} \sigma_j^i \sigma^i_j = \frac{1}{8} D_j^i \chi D^i_j \chi = \frac{1}{18t^{2/3}} \left[ \partial^i \partial_j C \partial^j \partial_i C - \frac{1}{3} \langle \Delta C \rangle^2 \right].
\]

(66)
This means that \( \langle r^2 \rangle_D \), the second part of \( \langle Q \rangle_D \), is also of second order, and can be calculated by using the expression of \( \chi \) only at first order.

So far, we have proved that both two parts of \( \langle Q \rangle_D \) are of second order, and consequently \( \langle Q \rangle_D \) is a second order term, but nevertheless can be calculated from the first order contributions to \( \Psi \) and \( \chi \). Using \( (56) \), \( (59) \), \( (41) \) and \( (40) \), we get \( \langle Q \rangle_D \) at second order

\[
\langle Q \rangle_D = 6 \left( \langle \dot{\Psi}^2 \rangle_{D1} - \langle \dot{\Psi} \rangle_{D1}^2 \right) - \frac{1}{4} \langle D_i \dot{\chi} D_i \dot{\chi} \rangle_{D1} = \frac{1}{27} \left[ 3 \left( \langle (\Delta C)^2 \rangle_{D1} - \langle \partial_i \partial_j C \partial^i \partial_j C \rangle_{D1} \right) - 2 \langle \Delta C \rangle_{D1}^3 \right] = \frac{F}{27} \left[ 3 \left( \langle (\partial^i \partial^j \Delta C) \rangle_{D1} - \langle \partial_i \partial_j C \partial^i \partial_j C \rangle_{D1} \right) - 2 \langle \Delta C \rangle_{D1}^3 \right].
\]

where

\[
F \equiv \frac{1}{27} \left[ 3 \left( \langle (\Delta C)^2 \rangle_{D1} - \langle \partial^i \partial_j C \partial^i \partial_j C \rangle_{D1} \right) - 2 \langle \Delta C \rangle_{D1}^3 \right] = \frac{1}{27} \left[ 3 \left( \langle \partial^i \partial_j C \partial^i \partial_j C \rangle \right)_{D1} - 2 \langle \Delta C \rangle_{D1}^3 \right] - 2 \langle \Delta C \rangle_{D1}^3. \]

\( F \) has only second order terms and is a function of spatial coordinates, and here we use \( (18) \) to express the final result in terms of the variable \( C \) only.

Some remarks on this result for \( \langle Q \rangle_D \) are in order. From \( (67) \) we find that

1. \( \langle Q \rangle_D \), written in the form abbreviated \( F \), contains two second order terms, which are total derivatives and become surface terms when averaging. Meanwhile, the third term \( \langle \Delta C \rangle_{D1}^3 \) is the square of a first order surface term, and thus its second order modifications in \( (62) \) do not show up in \( F \). Therefore, the kinematical backreaction is a function of the derivatives of \( C \) on the boundary of the averaged domain only.

2. Because \( \Delta C \) is a fluctuating term, and can be stochastically positive in some regions and negative in others, its average is expected to be negligible, if the averaged domains become larger (but are still on subhorizon scales). However, \( \partial^i \partial_j C \partial^i \partial_j C \) and \( (\Delta C)^2 \) are positive definite, and therefore give nontrivial surface terms when averaging. Thus, \( \langle Q \rangle_D \) consists of these two surface terms on large scales. If they cancel, we can say that there is no kinematical backreaction at second order. In Newtonian limit, this cancelation was discussed in \( (18) \) for periodic boundary conditions, and in \( (16, 17) \) for spherically symmetric spaces. In relativistic cosmological perturbation theory, this problem was treated in \( (18, 18) \). However, in general case, there is no reason for this cancelation. A review of this cancelation problem can be found in \( (24) \).

3. We can see from \( (67) \) that \( \langle Q \rangle_D \) decreases as \( t^{-2/3} \), which means that \( \langle Q \rangle_D \propto 1/a \). And we already know that \( \langle \rho^{(0)} \rangle_D \propto 1/a^3 \) and \( \langle R \rangle_D \propto 1/a^2 \). So the kinematical backreaction term \( \langle Q \rangle_D \) decays slower than \( \langle \rho \rangle_D \) and \( \langle R \rangle_D \). Therefore, in the course of the expansion of the Universe the kinematical backreaction becomes more and more important in the effective energy density \( \rho_{\text{eff}} \) and effective pressure \( p_{\text{eff}} \). Of course, we should pay attention that \( \langle Q \rangle_D \) is a pure second order term, but \( \langle R \rangle_D \) has got a first order term, and \( \langle \rho \rangle_D \) even contains a zeroth order term, so we cannot conclude that \( \langle Q \rangle_D \) dominates the late time evolution of the Universe. The effect of the kinematical backreaction on the expansion of the perturbed Universe depends on its evolution with time \( t \) (or scale factor \( a \)) in its denominator, and also on the value of the surface terms in the numerator.

### B. Averaged spatial curvature \( \langle R \rangle_D \)

In the last section, we have calculated the averaged spatial curvature \( \langle R \rangle_D \) to first order in \( (60) \), and in this subsection, we use the integrability condition \( (23) \) to get its second order part. But to do so, we first need to find the relation between the effective scale factor \( a_D \), which is defined as the cubic root of the volume of integral in the averaging procedure in \( (10) \), and the scale factor \( a \), which is defined to describe the evolution of the Universe in the perturbative metric \( (24) \), and then express \( a_D \) as the function of the perturbation \( A \). Since the integrability condition is an exact relation to any order, and we have already got \( \langle Q \rangle_D \) to second order in \( (67) \), solving the integrability condition \( (a_D^3 \langle Q \rangle_D )' + a_D^3 (a_D^3 \langle R \rangle_D )' = 0 \), it is straightforward to obtain \( \langle R \rangle_D \) to second order. However, in the following, we show that we do not need to calculate \( a_D \) to second order, but only first order is sufficient for our purpose.

We may rewrite the integrability condition as

\[
6 \frac{\dot{a}_D}{a_D} \langle Q \rangle_D + \langle Q \rangle_D' + 2 \frac{\dot{a}_D}{a_D} \langle R \rangle_D + \langle R \rangle_D' = 0. \tag{68}
\]

Because \( \langle Q \rangle_D \) is already of second order, in the first term of \( (68) \), we only need the zeroth order term of \( \dot{a}_D/a_D \). In the third term, since \( \langle R \rangle_D \) has no zeroth order term, we need the zeroth and first order terms of \( \dot{a}_D/a_D \).
From (11) and (56), to first order we have
\[ \frac{\dot{a}_D}{a_D} = \frac{1}{3} \langle \dot{\theta}_D \rangle D_1 = \frac{\dot{a}}{a} - \langle \Psi \rangle D_1 = \frac{2}{3t} - \frac{2\langle A \rangle D_1}{3t^{1/3}}, \]
so at first order
\[ \frac{a_D(t)}{a_D(t_0)} = \left( \frac{t}{t_0} \right)^{2/3} \left[ 1 - \langle A \rangle D_1 \left( t^{2/3} - t_0^{2/3} \right) \right]. \]  
(70)

Thus, if \( \langle A \rangle D_1 \) is negative, the effective scale factor \( a_D \) grows faster than the ordinary result \( t^{2/3} \) in the unperturbed dust model. We can also see this from (61) that if \( \langle A \rangle D_1 < 0 \), \( \langle \rho \rangle D_1 \) is reduced with respect to the background and the expansion rate gets positive modification, which is consistent with the intuition that the underdense regions expand faster than the overdense ones.

Substituting (69) and (67) into (68), to second order we have
\[ \langle R \rangle_D = \frac{4}{9} \left( \frac{1}{t} - \frac{\langle A \rangle D_1}{t^{1/3}} \right) \langle R \rangle_D + \frac{10}{3} \frac{F}{t^{5/3}} = 0. \]
Solving this differential equation, we find
\[ \langle R \rangle_D = D \left( \frac{t_0}{t} \right)^{4/3} \exp \left( 2\langle A \rangle D_1 \left( t^{2/3} - t_0^{2/3} \right) \right) + \frac{5}{2} \frac{F}{\langle A \rangle D_1 t^{4/3}} \left[ 1 - \exp \left( 2\langle A \rangle D_1 \left( t^{2/3} - t_0^{2/3} \right) \right) \right], \]  
(71)
where \( D \) is the constant of integration, which is a function of spatial coordinates. For consistency, we must expand the solution up to second order
\[ \langle R \rangle_D = \frac{1}{t^{4/3}} \left( D(1) t_0^{4/3} - 2\langle A \rangle D_1 t_0^2 + 5F t_0^{2/3} \right) + \frac{1}{t^{2/3}} \left( 2\langle A \rangle D_1 D_1 t_0^{4/3} - 5F \right). \]
(72)

There is only one undetermined constant of integration \( D \) in the above expression. From (60), we know that \( \langle R \rangle_D \) has no zeroth order term, so \( D \) has only the first and second order terms, otherwise, the terms in the first bracket would give rise to a zeroth order contribution. We may write \( D = D^{(1)} + D^{(2)} \), where \( D^{(1)} \) and \( D^{(2)} \) are the first and second order terms of \( D \), respectively. Because \( \langle A \rangle D_1 \) is a first order term and \( F \) is a second order one, \( \langle R \rangle_D \) can be written as
\[ \langle R \rangle_D = \frac{1}{t^{4/3}} \left( D^{(1)} t_0^{4/3} + D^{(2)} t_0^{4/3} - 2\langle A \rangle D_1 D^{(1)} t_0^2 + 5F t_0^{2/3} \right) + \frac{1}{t^{2/3}} \left( 2\langle A \rangle D_1 D^{(1)} t_0^{4/3} - 5F \right), \]  
(73)
where the first term \( D^{(1)} t_0^{4/3} / t^{4/3} \) represents the first order term of \( \langle R \rangle_D \). It is matched with the first order expression (60) to fix
\[ D^{(1)} = \frac{40\langle A \rangle D_1}{3t_0^{1/3}}. \]  
(74)

Substituting (73) into (72), and using (48), we find the averaged spatial curvature \( \langle R \rangle_D \) to second order,
\[ \langle R \rangle_D = -\frac{20}{9t^{4/3}} \langle \Delta C \rangle D_1 + \frac{G^{(2)}}{t^{4/3}} + \frac{5}{9} \frac{G^{(2)}}{t^{2/3}} \left[ \langle (\partial^i \partial_j C \partial^i \partial_j C) \rangle D_1 - \langle (\partial^i \partial_j C \Delta C) \rangle D_1 \right] + 2\langle \Delta C \rangle^2 D_1, \]  
where \( G^{(2)} \equiv D^{(2)} t_0^{4/3} - 2\langle A \rangle D_1 D^{(1)} t_0^2 + 5F t_0^{2/3} \). Neither \( D^{(2)} \) nor \( G^{(2)} \) can be fixed by matching to some known coefficients. However, the term \( G^{(2)} / t^{4/3} \) is unimportant at any time. Early on, \( -\frac{20}{9t^{4/3}} \langle \Delta C \rangle D_1 \) is a first order term, while \( G^{(2)} \) is a second order one, so it is negligible compared to the first term in (74). Similarly, at late times, \( G^{(2)} / t^{4/3} \) decays faster than \( \frac{5}{9} \frac{G^{(2)}}{t^{2/3}} \left[ \langle (\partial^i \partial_j C \partial^i \partial_j C) \rangle D_1 - \langle (\partial^i \partial_j C \Delta C) \rangle D_1 \right] + 2\langle \Delta C \rangle^2 D_1 \), because both numerators are of second order, but the exponent of the denominator in \( G^{(2)} / t^{4/3} \) is the larger one. Thus, \( -\frac{20}{9t^{4/3}} \langle \Delta C \rangle D_1 \) is the first order term of \( \langle R \rangle_D \), which is the same as the result in (60), and \( \frac{5}{9} \frac{G^{(2)}}{t^{2/3}} \left[ \langle (\partial^i \partial_j C \partial^i \partial_j C) \rangle D_1 - \langle (\partial^i \partial_j C \Delta C) \rangle D_1 \right] + 2\langle \Delta C \rangle^2 D_1 \] is the leading second order part at late times. Therefore, in the following calculations, we write \( \langle R \rangle_D \) as
\[ \langle R \rangle_D = -\frac{20}{9t^{4/3}} \langle \Delta C \rangle D_1 + \frac{5}{9} \frac{G^{(2)}}{t^{2/3}} \left[ \langle (\partial^i \partial_j C \partial^i \partial_j C) \rangle D_1 - \langle (\partial^i \partial_j C \Delta C) \rangle D_1 \right] + 2\langle \Delta C \rangle^2 D_1. \]  
(75)
Thus, at second order, \( \langle R \rangle_D \) is again the function of surface terms. So with (60), we find that \( \langle R \rangle_D \) is the function of surface terms at both first and second orders.

In this subsection, we have extended the calculation of the averaged spatial curvature \( \langle R \rangle_D \) to second order by using the integrability condition. Its advantage is that we can do the second order calculation, without knowing the metric perturbations of second order. This is because the integrability condition is an exact result to any order, and we have got \( \langle Q \rangle_D \) to second order with only the first order perturbation theory.
C. Averaged expansion rate $\langle \theta \rangle_D$

The second order perturbation of the expansion rate has been discussed in the literature. For instance, in [20], Kolb et al. used the metric perturbations of second order to calculate the averaged Hubble expansion rate and its variance. Here, the expansion rate $\theta$ is defined in the same way as that in [20], namely $\theta = u^i_\chi$. However, the Hubble expansion rate in [20] is defined as $\sqrt{8\pi G \rho} / D / 3$, in contrast to the one in our work, which is defined in [12] as $H_D \equiv \frac{2 \dot{a}}{a} = \frac{1}{3} \langle \theta \rangle_D$. Below, the second order perturbation of the expansion rate is calculated, but without using the metric perturbations of second order again. We also show that our calculation is consistent with the result in [20].

From (63), we have

$$\langle R \rangle_D = -\langle \theta^2 \rangle_D - 4\langle \dot{\theta} \rangle_D - 3\langle \theta \dot{\theta} \rangle_D = -2\langle \theta^2 \rangle_D - 4\langle \dot{\theta} \rangle_D - 6\langle \sigma^2 \rangle_D. \quad (76)$$

Since $\langle R \rangle_D$ has already been calculated to second order in (75), $\langle \sigma^2 \rangle_D$ is a pure second order term, and we know the zeroth and first order terms of $\langle \theta \rangle_D$ from (67), we can obtain the second order perturbation of $\langle \theta \rangle_D$ from (76).

Using (56), we expand $\theta$ as

$$\theta = \theta^{(0)} + \theta^{(1)} + \theta^{(2)} = 3 \frac{\dot{a}}{a} - 3 \Psi + \theta^{(2)} = \frac{2}{t} \frac{2A}{t^{1/3}} + \theta^{(2)}. \quad (77)$$

so to second order

$$\theta^2 = \frac{4}{t^2} - \frac{8A}{t^{4/3}} + \frac{4A^2}{t^{2/3}} + \frac{4\theta^{(2)}}{t}, \quad \dot{\theta} = \frac{2}{t^2} + \frac{2A}{3 t^{4/3}} + \dot{\theta}^{(2)}. \quad (78)$$

Substituting (78) into (70), and using (65) and (43), we find

$$R = \frac{40A}{3t^{4/3}} - 4\dot{\theta}^{(2)} - \frac{8}{t} \dot{\theta}^{(2)} - \frac{8A^2}{t^{2/3}} - \frac{3}{4} D^j_i \chi D^i_j \dot{\chi}. \quad (79)$$

We see from (79) that $R$ has both the first and second order terms, so at second order, the first order term $\frac{40A}{3t^{4/3}}$ gives two additional second order modifications when averaging as shown in (62). Therefore, the average of the spatial curvature $R$ to second order is

$$\langle R \rangle_D = \frac{40\langle A \rangle_D}{3t^{4/3}} - 4\langle \dot{\theta}^{(2)} \rangle_D - \frac{8}{t} \langle \theta^{(2)} \rangle_D - \frac{8\langle A^2 \rangle_D}{t^{2/3}} - \frac{3}{4} \langle D^j_i \chi D^i_j \dot{\chi} \rangle_D$$

$$= -\frac{20}{9t^{4/3}} \langle \Delta C \rangle_{D1} - 4\langle \theta^{(2)} \rangle_{D1} - \frac{8}{t} \langle \theta^{(2)} \rangle_{D1} - \frac{1}{9t^{2/3}} \left[ 11\langle (\Delta C)^2 \rangle_{D1} - 10\langle (\Delta C)^2 \rangle_{D1} + 3\langle \theta \partial_i C \partial^i \partial_i C \rangle_{D1} \right]. \quad (80)$$

Above, the Lemma (13) allows us to write $\langle \dot{\theta}^{(2)} \rangle_{D1} = \langle \dot{\theta}^{(2)} \rangle_{D1}$ at second order. Matching (80) with (74) yields

$$\langle \theta^{(2)} \rangle_{D1} = \frac{2}{t} \langle \theta^{(2)} \rangle_{D1} + \frac{1}{18t^{2/3}} \left[ 3\langle (\Delta C)^2 \rangle_{D1} + 4\langle \theta \partial_i C \partial^i \partial_i C \rangle_{D1} \right] = 0. \quad (81)$$

Solving this differential equation provides us with the second order contribution to the averaged expansion rate $\langle \theta \rangle_D$

$$\langle \theta^{(2)} \rangle_{D1} = -\frac{t^{1/3}}{42} \left[ 3\langle (\Delta C)^2 \rangle_{D1} + 4\langle \theta \partial_i C \partial^i \partial_i C \rangle_{D1} \right] + \frac{I}{t^2}, \quad (81)$$

where $I$ is the constant of integration, and at late times the term $I/t^2$ is negligible without doubt. Therefore, we find the averaged expansion rate $\langle \theta \rangle_D$ to second order,

$$\langle \theta \rangle_D = \langle \theta^{(0)} \rangle_D + \langle \theta^{(1)} \rangle_D + \langle \theta^{(2)} \rangle_D$$

$$= \frac{2}{t} + \frac{1}{3t^{1/3}} \langle \Delta C \rangle_{D1} - \frac{t^{1/3}}{42} \left[ 4 \langle (\theta^i \partial_i C \partial^j \partial_i C) \rangle_{D1} - \langle (\dot{\theta}^i \partial_i C \partial^i C) \rangle_{D1} \right] + 7\langle (\Delta C)^2 \rangle_{D1}, \quad (82)$$

where the first order term $\langle \theta^{(1)} \rangle$ also contributes via averaging to the second order result (see (92)). Straightforwardly, from (22), the averaged Hubble expansion rate is

$$H_D = \frac{2}{3t} + \frac{1}{9t^{4/3}} \langle \Delta C \rangle_{D1} - \frac{t^{1/3}}{126} \left[ 4 \langle (\theta^i \partial_i C \partial^j \partial_i C) \rangle_{D1} - \langle (\dot{\theta}^i \partial_i C \partial^i C) \rangle_{D1} \right] + 7\langle (\Delta C)^2 \rangle_{D1}. \quad (82)$$
We can see that \( \langle \theta \rangle_D \) and \( \langle H \rangle_D \) are also functions of surface terms at both first and second orders.

Finally, we show that the result in (82) can also be obtained by using the metric perturbations of second order in [20],

\[
d s^2 = a^2(\eta) \left[ -d\eta^2 + \left( 1 - 2\Psi^{(1)} - \Psi^{(2)} \right) \delta_{ij} + D_{ij} \left( \chi^{(1)} + \frac{1}{2} \chi^{(2)} \right) + \frac{1}{2} \left( \partial_i \chi^{(2)} + \partial_j \chi^{(2)} + \chi^{(2)} \right) \right] dx^i dx^j, \tag{83}
\]

where \( \eta \) is the conformal time, \( \Psi^{(1)} \) and \( \chi^{(1)} \) are the first order scalar perturbations (the same as in this paper), \( \Psi^{(2)} \) and \( \chi^{(2)} \) are the second order scalar perturbations, \( \chi_i^{(2)} \) is the second order transverse vector perturbation, and \( \chi^{(2)}_j \) is the second order transverse and traceless tensor perturbation. Ignoring decaying modes, for a dust Universe, \( \Psi \) and \( \chi \) only have to use the first order perturbation terms \( \Psi \) and \( \chi \), subleading terms as they can never (in the perturbative regime) overcome the first order ones.

Second order contributions show the same time dependence as the first order term. Thus, it is justified to neglect the second order contributions. These simplifications are based on the integrability condition and the fact that \( \Psi^{(1)} \), \( \chi^{(1)} \) and \( \Psi^{(2)} \) are also given in [20].

From (83) and (84), in the comoving synchronous gauge, we have

\[
\Psi^{(1)}(\eta, x) = \frac{3}{5} \varphi(x) + \frac{\eta^2}{18} \Delta \varphi(x), \quad \chi^{(1)}(\eta, x) = -\frac{\eta^2}{3} \varphi(x),
\]

\[
\Psi^{(2)}(\eta, x) = -\frac{5}{9} \varphi^2(x) - \frac{5\eta^2}{54} \partial^i \varphi(x) \partial_i \varphi(x) + \frac{\eta^4}{252} \left[ (\Delta \varphi(x))^2 - \frac{10}{3} \partial^i \partial_j \varphi(x) \partial^i \partial_j \varphi(x) \right]. \tag{84}
\]

From (83) and (84), in the comoving synchronous gauge, we have

\[\theta = u^i \chi = \frac{1}{a} \Gamma^i_{0j} = \frac{3a'}{a^2} \varphi^{(1)'} - \frac{3}{2a} \Psi^{(1)'} - \frac{6}{a} \Psi^{(1)} \Psi^{(1)'} - \frac{1}{2a} D^{ij} \chi^{(1)} D_{ij} \chi^{(1)'},\]

where \( ' \) denotes the derivative with respect to the conformal time \( \eta \), and in the dust Universe, \( t = (a0\eta)^3/(27t_0^2) \). So using (82), we can straightforwardly get the average of the expansion rate \( \theta \),

\[
\langle \theta \rangle_D = \frac{2}{t} + \frac{1}{3t^{4/3}} \langle \Delta C \rangle_D - \frac{5t_0^2}{54t_0^{4/3}t^{1/3}} \left[ \langle \partial^j (C \partial_i C) \rangle_D + \langle C \Delta C \rangle_D - 6\langle C \rangle_D \langle \Delta C \rangle_D \right]
\]

\[\quad - \frac{t^{1/3}}{42} \left[ 4 \langle \partial^j (\partial_j C \partial^i \partial_i C) \rangle_D - \langle \partial^j (\partial_j C \Delta C) \rangle_D + 7\langle \Delta C \rangle_D^2 \right]. \tag{85}
\]

Thus, we find that the leading second order term in (83), which we get by using the metric perturbations of second order in [20], is the same as that in (82). One can see as already argued for the case of \( \langle R \rangle_D \) that the subleading second order contributions show the same time dependence as the first order term. Thus, it is justified to neglect the subleading terms as they can never (in the perturbative regime) overcome the first order ones.

### D. Averaged energy density \( \langle \rho \rangle_D \)

Similarly, from (11) and (14), we have

\[
\left( \frac{1}{3} \langle \theta \rangle_D \right)^2 = \frac{8G}{3} \left( \langle \rho \rangle_D - \frac{\langle Q \rangle_D + \langle R \rangle_D}{16G} \right). \tag{86}
\]

Using (67), (75) and (82), we get the averaged energy density to second order,

\[
\langle \rho \rangle_D = \frac{1}{6\pi G t^2} \left[ 1 - \frac{t^{2/3}}{2} \langle \Delta C \rangle_D + \frac{t^{4/3}}{28} \left( \langle \partial^j (\partial_j C \partial^i \partial_i C) \rangle_D - \langle \partial^j (\partial_j C \Delta C) \rangle_D + 7\langle \Delta C \rangle_D^2 \right) \right]. \tag{87}
\]

and \( \langle \rho \rangle_D \) is a function of surface terms at both first and second orders too.

Up to this point, we have obtained all the averaged quantities \( \langle Q \rangle_D \), \( \langle R \rangle_D \), \( \langle \theta \rangle_D \) and \( \langle \rho \rangle_D \) to second order, and we only have to use the first order perturbation terms \( \Psi \) and \( \chi \) (without the necessity of knowing the metric perturbations of second order). These simplifications are based on the integrability condition and the fact that \( \langle Q \rangle_D \) is a pure second order term. Unfortunately, we cannot extend this method to higher orders. For example, if we go to third order, to calculate \( \sigma^2 \), we would need \( \theta'' = \Gamma^i_{0j} \) to second order, and thus we must know the metric perturbations of second order. An exception is the irrotational and shearless universe, which is merely the FLRW model.
E. Effective equation of state and square of effective speed of sound

\( w_{\text{eff}} = w^{(1)}_{D} t^{1/3} + w^{(2)}_{D} t^{4/3} = -\frac{5}{18} \langle \Delta C \rangle_{D1} t^{2/3} + \left[ \frac{1}{9} \left( \langle \partial^i (\partial_j C \partial^j \partial_i C) \rangle_{D1} - \langle \partial^i (\partial_i C \Delta C) \rangle_{D1} \right) + \frac{7}{27} \langle \Delta C \rangle_{D1}^2 \right] t^{4/3}. \) (88)

Equation (88) is the perturbative expansion of the effective equation of state, which is both time and domain dependent. We can find its time dependence at different orders, and domain dependence in different coefficients. From (69), it is easy to get \( t \) as the function of the effective scale factor \( a_D \), and at late times (i.e., \( t \gg t_0 \)) \( w_{\text{eff}} \) can be rewritten as

\[
\begin{align*}
w_{\text{eff}} &= w^{(1)}_{D} a_D + w^{(2)}_{D} a_D^3 \\
&= -\frac{5}{18} \langle \Delta C \rangle_{D1} \frac{t^{2/3}}{a_D} a_D + \left[ \frac{1}{9} \left( \langle \partial^i (\partial_j C \partial^j \partial_i C) \rangle_{D1} - \langle \partial^i (\partial_i C \Delta C) \rangle_{D1} \right) + \frac{11}{36} \langle \Delta C \rangle_{D1}^2 \right] \frac{t^{4/3}}{a_D^2}. \end{align*}
\] (89)

Equation (89) is also a perturbative expansion of the effective equation of state, but in terms of \( a_D \), which is of more interest than \( a \). We can see that the second order coefficients in (88) and (89) are different, this is because \( a_D \) is not proportional to \( t^{2/3} \), and thus the second order coefficient picks up nontrivial contributions from the first order one.

Therefore, \( w_{\text{eff}} \) vanishes at zeroth order. This is different from the cosmological constant, for which \( w_{\Lambda} = -1 \). Consequently, in a perturbative framework, the backreaction mechanism cannot induce accelerated expansion of the Universe as that would imply \( w_{\text{eff}} < -1/3 \). Nevertheless, the cosmological perturbations allow us to investigate a possible change of the expansion rate of the averaged Universe that might in the later nonlinear stage lead to the accelerated expansion of the Universe. We discuss this on both small and large scales.

Using (51), we may rewrite \( w_{\text{eff}} \) as

\[
w_{\text{eff}} = \frac{10 \pi G}{3} \langle \rho^{(1)} \rangle_{D1} t^2 + \left[ \frac{1}{9} \left( \langle \partial^i (\partial_j C \partial^j \partial_i C) \rangle_{D1} - \langle \partial^i (\partial_i C \Delta C) \rangle_{D1} \right) + \frac{7}{27} \langle \Delta C \rangle_{D1}^2 \right] t^{4/3}. \] (90)

Firstly, on small scales, \( \langle \rho^{(1)} \rangle_{D1} \) may significantly deviate from 0, so the first order term dominates the value of \( w_{\text{eff}} \), i.e., \( w_{\text{eff}} = \frac{10 \pi G}{3} \langle \rho^{(1)} \rangle_{D1} t^2 \). We can see from (90) that if \( \langle \rho^{(1)} \rangle_{D1} < 0 \), which means that the energy density is underdense locally, \( w_{\text{eff}} \) is negative, and since \( w_{\text{eff}} \propto t^2 \), this effect will be more and more influential as time goes on, and might cause the accelerated (averaged) expansion of the inhomogeneous and anisotropic Universe. Of course, with the above expression we can trace the evolution only for small perturbations. Once they are in the nonlinear regime, our approach fails.

Secondly, for large averaged domains (but are still on subhorizon scales), like the discussion on \( F \), the average of \( \Delta C \) is expected to become negligible, since it is a fluctuating term, and only the two surface terms give nontrivial contributions. Therefore the value of \( w_{\text{eff}} \) is dominated by these two second order terms on large scales,

\[
w_{\text{eff}} = \frac{1}{9} \left( \langle \partial^i (\partial_j C \partial^j \partial_i C) \rangle_{D1} - \langle \partial^i (\partial_i C \Delta C) \rangle_{D1} \right) t^{4/3}. \] (91)

We can see from (91) that the sign of \( w_{\text{eff}} \) depends on the contrast of the two surface terms. It vanishes for certain boundary conditions, see (18, 24, 43, 46, 47, 48). However, we think that these boundary conditions are not natural and that the generic case for a finite domain in the Universe is that the effective equation of state is given by a finite surface term, that might be positive or negative, depending on the details of the fluctuations on the boundaries.

An important lesson that we learn here is that the cosmological backreaction introduces an effective equation of state, which is not only time dependent, but also scale dependent.

b. Square of the effective speed of sound \( c^2_{\text{eff}} \). Similarly, for the square of the effective speed of sound, we have

\[
c^2_{\text{eff}} = -\frac{5}{27} \langle \Delta C \rangle_{D1} t^{2/3} + \frac{1}{27} \left[ \langle \partial^i (\partial_j C \partial^j \partial_i C) \rangle_{D1} - \langle \partial^i (\partial_i C \Delta C) \rangle_{D1} \right] + \frac{47}{18} \langle \Delta C \rangle_{D1}^2 \right] t^{4/3}. \] (92)

Like \( w_{\text{eff}} \), on small scales,

\[
c^2_{\text{eff}} = -\frac{5}{27} \langle \Delta C \rangle_{D1} t^{2/3} = \frac{20 \pi G}{9} \langle \rho^{(1)} \rangle_{D1} t^2. \]
So if the cosmic medium is overdense locally, $c_{\text{eff}}^2 > 0$. But we also see that $c_{\text{eff}}^2$ can be negative in underdense regions. Usually this suggest that some damping is going on, which is related to dissipative phenomena and the increase of entropy. These aspects will be investigated in more detail elsewhere.

On large scales, the second order terms dominate and we find

$$c_{\text{eff}}^2 = \frac{1}{27} \langle (\partial_j C \partial_i C) \rangle_{D1} - \langle (\partial_j (C \Delta C)) \rangle_{D1} t^{4/3}.$$  

Also the sign of the square of the effective speed of sound depends on the contrast of the two surface terms.

To summarize this section, we can see that all studied physical quantities, $\langle Q \rangle_D$, $\langle R \rangle_D$, $\langle \theta \rangle_D$, $H_D$, $\rho_{\text{eff}}$, $w_{\text{eff}}$ and $c_{\text{eff}}^2$, can be expressed as functions of surface terms at both first and second orders. Thus, to know the values of these physical quantities, we do not need to know anything about the interior of the averaged domain. Only the physical information (i.e., the values of $C$, namely the peculiar gravitational potential $\varphi$, and its derivatives) encoded on the boundary of the domain matters.

**VII. GAUGE DEPENDENCE OF THE AVERAGED QUANTITIES**

We should finally discuss the gauge dependence of the averaged physical observables. In [42, 49], the gauge invariance of physical observables at different orders are discussed in detail: to second order, a quantity is gauge dependent unless its zeroth and first order terms vanish, apart from the trivial cases that it is a constant scalar field, or a linear combination of products of Kronecker deltas with constant coefficients on the background. Thus, we know that $\langle Q \rangle_D$ is a gauge invariant quantity, since it has only the second order term. Secondly, $\langle R \rangle_D$, $w_{\text{eff}}$ and $c_{\text{eff}}^2$, which have the first order terms, and $\langle \theta \rangle_D$, $H_D$ and $\rho_{\text{eff}}$, which have both the zeroth and first order terms depend on the gauge choice. However, the first order terms of $\langle R \rangle_D$, $w_{\text{eff}}$ and $c_{\text{eff}}^2$ are gauge independent as well. So in summary, we conclude that all leading terms of all physical observables are gauge invariant, while the higher order ones are not.

This raises the question of the coordinate system. It seems to us that the comoving synchronous gauge is very close to the coordinate system of a real observer. Real astronomers and their telescopes are comoving with matter (we neglect the difference between baryonic and dark matter here), they use their own proper time in all of their observations and regard space to be time-orthogonal, which defines precisely the slicing and gauge that we use throughout this work.

**VIII. CONCLUSIONS AND DISCUSSIONS**

In this paper, we use both the Buchert equations and cosmological perturbation theory to study the evolution of the perturbed dust Universe in the comoving synchronous gauge. We investigate the possibility to explain the accelerated expansion of the Universe without dark energy. We calculate the averaged kinematical backreaction term $\langle Q \rangle_D$ and the averaged spatial curvature $\langle R \rangle_D$, and find that $\langle Q \rangle_D$ is a pure second order term, and $\langle R \rangle_D$ has both the first and second order terms. As we use a perturbative approach, these terms can only affect the evolution of the Universe perturbatively, and thus we can only hope to find an onset of the cosmological backreaction mechanism in this work. In some circumstances, for example, on small scales, on which the effects of fluctuations of the energy density $\langle \rho \rangle_D$ are significantly nonzero, the backreaction mechanism should not be neglected carelessly.

We conclude that cosmological backreaction is for real and that it can both increase or decrease the expansion of the averaged Universe, depending on the averaged domain under consideration. Thus we argue that the effective equation of state of the Universe is time and scale dependent and so is the square of the effective speed of sound.

We find in our perturbative approach, that all physical quantities are surface terms or squares of surface terms. This suggests the conjecture that also a nonlinear treatment would find only functions of surface terms.

Another point of our paper is that we show in Section 6 how to calculate the averaged quantities to second order for the leading growing mode, but use only the metric perturbations of first order. This is a consequence of the integrability condition, which is valid to any order. And this greatly simplifies the perturbative calculations, which are usually done with help of the metric perturbations of second order in the previous papers.

Finally, we discuss some observational aspects of the perturbative calculations in our paper. The problem is how to measure the perturbations, such as $\Psi$ or $A$.

The fluctuation amplitude $\sigma_D^2$ is defined as

$$\sigma_D^2 \equiv \left\langle \left( \frac{\delta \rho}{\rho} \right)^2 \right\rangle_D = \left\langle \left( \frac{\rho^{(1)} + \rho^{(2)}}{\rho} \right)^2 \right\rangle_D = \left\langle \left( \frac{\rho^{(1)}}{\rho} \right)^2 \right\rangle_{D1}.$$  

Here we only calculate $\sigma_D^2$ to second order, so the terms containing $\rho^{(2)}$ are negligible. (Usually this amplitude is defined at the distance of 8 $h^{-1}$ Mpc, but we can also define it for any domain $D$.) From (61), we have

$$\sigma_D^2 = \frac{t^{4/3}}{4} \langle (\Delta C)^2 \rangle_{D1}. \quad (93)$$
So by measuring $\sigma^2_D$, we can know the value of $(\langle \Delta C^2 \rangle_{D1})$.

Secondly, a measurement of the averaged energy density $\langle \rho \rangle_D$ in $\mathbb{R}^3$ and the averaged expansion rate $\langle \theta \rangle_D$ in $\mathbb{R}^2$ and the averaged spatial curvature $\langle R \rangle_D$ in $\mathbb{R}^3$, together with the measurement of $\sigma^2_D$, would allow us to find $\langle \theta^2 \Delta C^2 \partial_i \partial_j C \rangle_D$. Because all of them are physical observables (at least in principle), from these measurements we can know the values of $(\langle \Delta C^2 \rangle_{D1})$ and $\langle \theta^2 \Delta C^2 \partial_i \partial_j C \rangle_D$. And since the other averaged physical quantities $Q_D$, $w_{\text{eff}}$ and $c_{\text{eff}}$ are the functions of the former, we can obtain the quantitative information of all these averaged terms. This will help us to understand the evolution of the inhomogeneous and anisotropic Universe and its relation to what we call the background model on a much deeper level. It also demonstrates that we do not need to retreat to a statistical treatment of the back-reaction effect, but we can try to design an experiment to directly measure its sign and magnitude in the Milky Way’s neighborhood (e.g., the local $\sim (100 \, h^{-1} \, \text{Mpc})^3$ domain).

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