SMALL NOISE PERTURBATIONS IN MULTIDIMENSIONAL CASE

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Abstract. In this paper we study zero-noise limits of $\alpha$-stable noise perturbed ODE’s which are driven by an irregular vector field $A$ with asymptotics $A(x) \sim \pi(x|x|^{\beta-1}) x$ at zero, where $\pi > 0$ is a continuous function and $\beta \in (0, 1)$. The results established in this article can be considered a generalization of those in the seminal works of Bafico [5] and Bafico, Baldi [6] to the multi-dimensional case. Our approach for proving these results is inspired by techniques in [38] and based on the analysis of an SDE for $t \rightarrow \infty$, which is obtained through a transformation of the perturbed ODE.

1. Introduction

In order to illustrate small noise analysis for singular ordinary differential equations (ODE’s) from an application’s point of view, let us consider the stochastic gradient method in non-convex optimization:

In deep learning the problem of training neural networks with a training set can be translated into a problem of the minimization of a non-convex loss function $F : \mathbb{R}^d \rightarrow \mathbb{R}$. A popular approach for solving this minimization problem is the stochastic gradient algorithm (SGD), which is numerically more tractable than the classical gradient descent method in the case of high dimensional data; see e.g. [11], [9] and the references therein.

To give more details about this approach, let $f : \mathbb{R}^d \times \mathbb{R}^l \rightarrow \mathbb{R}$ and $\xi$ be a random variable in $\mathbb{R}^l$ such that $f(\cdot, y)$ is twice differentiable for all $y \in \mathbb{R}^l$ and $F(x) = \mathbb{E}[f(x, \xi)]$, $x \in \mathbb{R}^d$. Then a sequence $x_k, k \geq 0$, which converges to a local minimizer of $F$, is constructed recursively via

$$x_k^\varepsilon = x_{k-1}^\varepsilon - \varepsilon \nabla f(x, \xi_k), x_0 = \xi_0,$$

where $\varepsilon > 0$ is the learning rate, $\xi_k, k \geq 0$ an i.i.d—sequence of random variables and $\nabla = \nabla_x$ the gradient.

Here, a central issue is the study of the problem of how fast the iterations $x_k^\varepsilon, k \geq 0$ can escape from unstable stationary points, that is from points $x$ with $\nabla F(x) = 0$ such that the least eigenvalue of the Hessian $\nabla^2 F(x)$ is strictly negative. More recently (see [28], [24]), it could be shown that the speed of escape from non-stable stationary points can be analyzed by means of solutions $X_t, 0 \leq t \leq T$ to stochastic differential equations (SDE’s) which approximate $x_k^\varepsilon, k \geq 0$ in a certain weak sense. More precisely, it was proven in [28] for sufficiently smooth $f$ (in e.g. $C^7_b$) and positive semi-definite functions $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$

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that the solution $X$ to the SDE
\[ dX_t^\varepsilon = -\nabla F(X_t^\varepsilon)dt + \sqrt{\varepsilon}\sigma(X_t^\varepsilon)dW_t, 0 \leq t \leq T, X_0 = x_0, \]
for a Wiener process $W$, approximates $x_k^\varepsilon, k \geq 0$ in the following sense: For all $\varphi \in C^6_b$ there exist constants $C, \varepsilon_0 > 0$ (depending on $T$ and $\varphi$) such that
\[ |E[\varphi(x_k^\varepsilon)] - E[\varphi(X_k^\varepsilon)]| \leq Ce^k, k = 0, \ldots, \left\lceil \frac{T}{\varepsilon} \right\rceil, \varepsilon < \varepsilon_0. \]

On the other hand, one may encounter in applications the situation that the coefficients $A := -\nabla F$ and $\sigma$ in (1) do not meet the above smoothness requirements; see e.g. [35]. For example it may happen that the vector field $A$ is irregular, that is non-Lipschitz or discontinuous, while $\sigma$ is constant. From a practical and theoretical perspective, it would therefore important to extend those results to the case of irregular vector fields $A$. However, in pursuing such an objective one has to cope with the possibility of the occurrence of an interesting effect in connection with the small noise perturbed ODE (1), namely the stochasticity of a solution $X$ to (1) for $\varepsilon = 0$ as the limit of $X^\varepsilon$ for $\varepsilon \to 0$ in distribution.

To explain this phenomenon, consider the ODE
\[ X_t = x + \int_0^t A(X_s)ds, x \in \mathbb{R}^d, t \geq 0 \]
for $x \in \mathbb{R}^d$, where $A: \mathbb{R}^d \to \mathbb{R}^d$ is a Borel measurable vector field.

If we assume that $A$ is Lipschitzian, it is well-known that one can construct a global unique solution $X \in C([0, \infty); \mathbb{R}^d)$ to (2) by using Picard iteration.

On the other hand, if $A$ is not Lipschitzian, uniqueness or even existence of solutions of the ODE (2) are not any longer guaranteed. An example is the ODE (2) driven by the discontinuous vector field $A$ given by
\[ A(x) = \text{sgn}(x) \]
for $X_0 = 0$, which possesses infinitely many solutions, where $X_t = +t, -t, t \geq 0$ are extremal solutions.

Other examples of non-well-posedness of the ODE (2) in the above sense can be also observed in the case of continuous non-Lipschitz vector fields $A$ satisfying the growth condition $\langle A(x), x \rangle \leq K(|x|^2 + 1), x \in \mathbb{R}^d$ for a constant $K$. Then, using e.g. Peano’s theorem and the theorem of Arzelà-Ascoli one finds that the set $C(x)$ of solutions $X \in C([0, \infty); \mathbb{R}^d)$ to (2) is non-empty, compact (in $C([0, \infty); \mathbb{R}^d)$) and connected; see [42].

The case, when $C(x)$ is a singleton, that is the case of uniqueness of solutions to (2) was e.g. examined [17], [1], [2] by means of the concept of renormalized solutions in connection with the associated continuity equation.

However, when $C(x)$ is not a singleton, which corresponds to the situation of non-uniqueness, one may be faced with the problem of the selection of the ”right” or ”most appropriate” solution to (2).

An important approach for studying such a selection problem based on so-called Markov selections was developed by Krylov [32].
An alternative method to the latter one, which we aim at applying in this paper, is based on the zero-noise selection principle, that is on the selection of a solution \( X \) to (2) obtained as a limit in law of solutions \( X^\varepsilon \) to a small noise perturbed ODE (2) of the form

\[
X^\varepsilon_t = x + \int_0^t A(X_s)ds + \varepsilon W_t
\]

for \( \varepsilon \downarrow 0 \). We remark that in this case well-posedness of the equation (2) is restored by adding a regularizing noise \( \varepsilon W_t, \varepsilon > 0 \) to (2), since the SDE (4) has a unique strong solution or even a path-by-path unique solution in the space of continuous functions for each \( \varepsilon > 0 \), when e.g. \( A \) is bounded and measurable (see [16] and [16]). The purpose of this method, which was originally proposed by A. N. Kolmogorov, is to select solutions to (2), which exhibit stability under random perturbations.

When \( \text{Card}(C(x)) > 1 \), that is the case which is also referred to as Peano phenomenon in the literature, Bafico [3] and Bafico, Baldi [6] were the first authors, who studied the zero-noise problem in the case of one-dimensional time-homogeneous vector fields \( A \). They could show, by using estimates of mean exit times of \( X^\varepsilon \) with respect to small neighbourhoods of isolated singular points of \( A \), that \( X^\varepsilon \) weakly converges to a process \( X \), whose law is given by a linear convex combination of two Dirac measures. Other results in the one-dimensional case, which rest on large deviation techniques and which cover the example (3), can be found in [26], [27]. We also mention the paper [10], where the authors use viscosity solutions of perturbed backward Kolmogorov equations to study the small noise problem [11]. As for local time techniques we refer to [15]. Another more recent paper, which deals with vector fields \( A \) of the form \( c_+ |x|^\alpha, x > 0 \) and \( -c_- |x|^\alpha, x < 0 \) and self-similar driving noise \( B_\beta \) with self-similarity index \( \beta > 0 \), is [38]. Using a new qualitative technique, which doesn’t rely on quantitative tools based e.g. on Kolmogorov type equations, the authors characterize zero-noise limits by analyzing solutions to a “canonical equation”, that is solutions to a transformation of the small noise perturbed equation for \( t \to \infty \). Based on the latter time-space rescaling technique the authors in [36] extend the results in [38] to the case of SDE’s with multiplicative noise driven by a strictly \( \alpha \)-stable process and vector fields \( A \), which are given by the following rather large class of coefficients

\[
A(x) = \begin{cases} 
    x^\beta L_+(x), & x > 0 \\
    -|x|^\beta L_-(x), & x > 0
\end{cases}
\]

for continuous functions \( L_+: (0, \infty) \to (0, \infty) \) with \( L_+(x) \sim A_+l(\frac{1}{x}) \) as \( x \to +0 \), where \( l > 0 \) is a slowly varying function at infinity and \( A_+, A_- > 0 \) are constants.

Further, we also point out the works of [3], [4], which are concerned with the analysis of small noise limits in the case of one-dimensional linear transport equations. See also [21], [34] in the case of non-linear PDE’s.

Compared to the above results in the one-dimensional case, however, the literature on small noise analysis for multi-dimensional vector fields \( A \) is rather sparse.

Using viscosity solutions of Hamilton-Jacobi-Bellman equations, Zhang [47] e.g. discusses a characterization of weak limits of \( X^\varepsilon \) of [11], when \( A : \mathbb{R}^d \to \mathbb{R}^d \) is continuous and bounded, and generalizes results in [11] in certain cases.
As for the case of discontinuous vector fields $A$ in the multi-dimensional setting we mention the articles Delarue, Flandoli, Vincenzi [19] and Buckdahn, Ouknine, Quincampoix [12]. Based on a concept of solutions in the sense of Filippov, which differs from the classical one in the case of discontinuous vector fields, the work [12] is concerned with study of ODE’s when $A$ is merely measurable. In [19] the authors derive probability estimates of exit times with respect zero-noise limits of certain ODE’s in $\mathbb{R}^4$ and apply those to the analysis of small noise perturbations of the Vlasov-Poisson equation. Another result in this direction, whose proof compared to [6] does not presume knowledge of the explicit distribution of $X^\varepsilon$, is the paper [37]. Here the authors examine small noise limits of time-inhomogeneous vector fields $A$, which allow for discontinuity points in a halfplane. In [33] Pilipenko, Kulik further develop the methods of [38, 37] and apply new ideas based on an averaging principle, which is closely related to that for Markov processes, to describe zero-noise limits, when the Wiener noise driven SDE admits for multiplicative noise and vector fields $A$, that are locally Lipschitz continuous outside a fixed hyperplane $H$ and have Hölder asymptotics at $H$.

Finally, we point out the recent work [20]. Here the authors establish results on the limiting behaviour of $X^\varepsilon$ in (4) for vector fields $A = \nabla V$, which are not assumed to be Lipschitz continuous in the neighbourhood of the origin and where the potential $V$ is $C^{1,1}$ on compact subsets in $\mathbb{R}^d\setminus\{0\}$ given by functions of the form

$$V(x) = g(x) |x|^{1+\alpha} + h(x) |x|^{1+\beta}, x \neq 0, V(0) = 0$$

for functions $g, h$ such that $\{g \neq 0\} \cap \{h \neq 0\} = \emptyset$. There the analysis of zero-noise limits, which requires rather complicated conditions on $V$, rests on probability estimates with respect to the polynomial growth of $V(X^\varepsilon_t)$ for $\varepsilon \searrow 0$ and the submartingale dynamics of $V(X^\varepsilon_t)$, $t \geq 0$.

1.1. Goals of the paper. The main objective of our paper is characterization of limits of $X^\varepsilon$ of small $\alpha$–stable noise $B_\alpha$ perturbed ODE’s for $\alpha \in (1, 2]$, when $A$ is locally Lipschitz continuous on $\mathbb{R}^d\setminus\{0\}$ and has the asymptotics $A(x) \sim \overline{\alpha}(\frac{x}{|x|^{\beta-1}}) x$ for $x \to 0$, where $\overline{\alpha}$ is a positive continuous function on the unit sphere and $\beta \in (0, 1)$.

Let us set the problem more precisely.

Let $\{B_\alpha(t), t \geq 0\}$ be a $d$-dimensional symmetric Levy $\alpha$-stable process with $\alpha \in (1, 2]$, i.e., $\{B_\alpha(t), t \geq 0\}$ is a cadlag process with independent and homogeneous increments such that the characteristic function of the increment equals

$$E \exp\{i(z, B_\alpha(t))\} = \exp\{-|z|^\alpha t\}, z \in \mathbb{R}^d, t \geq 0,$$

where $c > 0$ is a constant.

For example, if $\alpha = 2$, then $\{B_\alpha(t), t \geq 0\}$ is a Brownian motion.

Let $A : \mathbb{R}^d \to \mathbb{R}^d$. Consider an ODE

$$dX_0(t) = A(X_0(t))dt, t \geq 0,$$

$$X_0(0) = 0$$

and a sequence of its perturbations by a small noise

$$dX^\varepsilon(t) = A(X^\varepsilon(t))dt + \varepsilon dB_\alpha(t), t \geq 0,$$

$$X^\varepsilon(0) = 0.$$
The problem is to study existence and identification of a limit \( \lim_{\varepsilon \to 0} X_\varepsilon \).

If the function \( A \) satisfies the Lipschitz condition, then it is easy to show the convergence \( X_\varepsilon \to X_0, \varepsilon \to 0 \) for a.a. \( \omega \) uniformly on the compact sets. However, the problem is much harder if there are multiple solutions to (5). This may happen, as mentioned before, if e.g. the Lipschitz property for \( A \) fails.

We will use the following notations

\[
\begin{align*}
  r &= |x| \geq 0, \\
  \varphi &= \frac{x}{|x|} \in S^{d-1}, x \neq 0,
\end{align*}
\]

\( x^\beta := |x|^{-1} x = r^\beta \varphi \). If \( x = 0 \), then the corresponding \( \varphi \) may be arbitrary; by definition we set \( 0^\beta := 0 \) for arbitrary \( \beta \).

A problem solved in this paper is to identify a limit of \( \{X_\varepsilon\} \) as \( \varepsilon \to 0 \) if

\[
A(x) \sim \bar{a}(\varphi)x^\beta, \ x \to 0, \tag{8}
\]

where \( \beta \in (0, 1), \bar{a} : S^{d-1} \to (0, \infty) \) is a positive continuous function on the sphere, and \( A \) is locally Lipschitz function in \( \mathbb{R}^d \setminus \{0\} \).

If \( \alpha + \beta > 1 \), then the existence of a weak solution follows from \([39, 8]\), uniqueness was proved in \([14]\). If \( d = 1 \) and \( \alpha + \beta < 1 \), then uniqueness fails, see the arguments of \([43, \text{Theorem 3.2}]\). We will always assume in this paper that \( \alpha + \beta > 1 \) to have uniqueness and other nice properties.

It can be easily seen that any limit point \( X_0 \) of \( \{X_\varepsilon\} \) (if it exists) satisfies ODE \([5]\) outside of 0. Since \( \bar{a}(x) > 0 \), equation \([5]\) yields that \( X_0 \) does not return to 0 after the exit from 0. Hence, to identify \( X_0 \) we have to study two problems: find the distribution of time spent at 0 and characterize “direction of exit” from 0.

We will show that \( X_0 \) spends zero time at 0 and the distribution of “direction of exit” coincides with the distribution of “direction of exit” of the limit for the sequence of the following model equations

\[
\begin{align*}
  d\bar{X}_\varepsilon(t) &= \bar{a}(\bar{\varphi}_\varepsilon(t))\bar{X}_\varepsilon^\beta(t)dt + \varepsilon dB_\alpha(t), \ t \geq 0, \\
  \bar{X}_\varepsilon(0) &= 0.
\end{align*}
\]

This is quite natural because a selection of exit should be done in a neighborhood of 0 and coefficients of the initial equation \([7]\) and the model equation \([9]\) are equivalent there.

Moreover, we will show that the exit distribution equals the distribution of the limit as \( t \to \infty \) of angle \( \frac{\bar{X}_\varepsilon(t)}{|\bar{X}_\varepsilon(t)|} \) for the model equation. It will also be shown that this limit exists a.s. and its distribution is independent of \( \varepsilon > 0 \).

To understand what happens with the limit of \([7]\), let us firstly explain what happens with the limit of \([9]\).

The formal limit equation for \([9]\) is

\[
\begin{align*}
  dX_0(t) &= \bar{a}(\bar{\varphi}_0(t))X_0^\beta(t)dt, \ t \geq 0, \\
  X_0(0) &= 0.
\end{align*}
\]

\[
\begin{align*}
  d\bar{X}_\varepsilon(t) &= \bar{a}(\bar{\varphi}_\varepsilon(t))\bar{X}_\varepsilon^\beta(t)dt + \varepsilon dB_\alpha(t), \ t \geq 0, \\
  \bar{X}_\varepsilon(0) &= 0.
\end{align*}
\]
Any solution to (10) is of the form:
\[ \bar{X}_0(t) = \bar{X}_0(t_0; \varphi) := \begin{cases} \frac{\bar{a}(\varphi)(1 - \beta)(t - t_0)}{t - t_0}, & t \geq t_0, \\ 0, & t \in [0, t_0], \end{cases} \tag{11} \]
where \( \varphi \in S^{d-1} \), \( t_0 \geq 0 \) are constants.

That is,
\[ \bar{\varphi}_0(t) \equiv \varphi, \quad \text{and} \quad \bar{r}_0(t) = \begin{cases} \frac{\bar{a}(\varphi)(1 - \beta)(t - t_0)}{t - t_0}, & t \geq t_0, \\ 0, & t \in [0, t_0]. \end{cases} \tag{12} \]

It can be seen from the change of variables formula and the self-similarity of \( B_\alpha \) that
\[ \{ \bar{X}_x(t), \ t \geq 0 \} \overset{d}{=} \{ \varepsilon^{1 + \beta} \bar{X}_1(\varepsilon^{\frac{\alpha(\beta - 1)}{\alpha + \beta - 1}} t), \ t \geq 0 \}. \tag{13} \]

Hence the distribution of the limit angle \( \bar{\varphi}_\varepsilon(t) = \frac{\bar{X}_x(t)}{|X_x(t)|\bar{r}_0(t)} \) as \( \varepsilon \to 0 \) should coincide with the distribution of
\[ \bar{\varphi}(+\infty) := \lim_{t \to \infty} \bar{\varphi}_0(t) \tag{14} \]
for any fixed \( \varepsilon_0 > 0 \). A.s. existence of the corresponding limit is given in §3.1. As we mentioned above, we will show that any limit process spends zero time at 0. Hence
\[ \bar{X}_x(\cdot) \Rightarrow \bar{X}_0(\cdot; 0, \bar{\varphi}(+\infty)), \quad \varepsilon \to 0. \]

The paper is organized as follows. In §2 we study a general problem of finding the asymptotic behavior as \( t \to \infty \) of a solution to the SDE
\[ dX(t) = A(X(t))dt + \text{general noise}. \tag{15} \]

This question is interesting by itself, see for example [25, 30, 13, 38]. We give sufficient conditions that ensure existence of a.s. limit of the angle \( \Phi(\infty) := \lim_{t \to \infty} \frac{X(t)}{|X(t)|} \) if \( A(x) \sim \bar{a}(\varphi)x^\beta, \ |x| \to \infty \). In particular, our result will imply existence of the limit in (14). It will also be shown that the absolute value \( R(t) = |X(t)| \) is equivalent to \( (\bar{a}(\Phi(\infty))(1 - \beta)t)^{\frac{1}{\beta}} \), compare with formula (11).

The main result of the paper is formulated and proved in section 3. It states that under some natural assumptions we have the following convergence:
\[ X_\varepsilon(\cdot) \Rightarrow X_0(\cdot, \bar{\varphi}(+\infty)), \quad \varepsilon \to 0, \tag{16} \]

where \( \bar{\varphi}(+\infty) \) is defined for the model equation in (14) and \( X_0(t, \varphi) \) is a solution of the limit equation (5) such that \( X_0(t, \varphi) \neq 0, t > 0 \) and \( \lim_{t \to 0^+} \frac{X_0(t, \varphi)}{|X_0(t, \varphi)|} = \varphi \).

Note that there are no reasons to expect that the distribution of the angle \( \bar{\varphi}(+\infty) \) can be calculated explicitly except of some trivial cases. It is doubtful that it has a discrete distribution or the uniform distribution. Moreover, if we take \( X_\varepsilon(0) = x\varepsilon^{\frac{\beta}{\alpha + \beta - 1}} \), instead of \( X_\varepsilon(0) = 0 \), then the answer may be different, see equality (13) for the model equation.

Proofs of auxiliary results are given in the Appendix. We also give there sufficient conditions that ensure existence of a family of solutions \( X_0(t, \varphi) \) to the limit equation (5) such that \( \lim_{t \to 0^+} \frac{X_0(t, \varphi)}{|X_0(t, \varphi)|} = \varphi \).
2. An asymptotic behavior of a solution to an SDE for a large time

Let $X(t)$ be a solution to the following SDE in $\mathbb{R}^d$:

$$dX(t) = A(X(t))dt + B(X(t))dW(t) + \int_U C_1(X(t), u)N_1(du, dt) + \int_U C_2(X(t), u)\hat{N}_2(du, dt),$$

(17)

where $W$ is a multi-dimensional Wiener process, $N_1$ is a Poissonian point measure, $\hat{N}_2$ is a compensated Poissonian point measure.

In this section we will always assume that any equation has a (weak) solution for any starting point, any integral is well-defined, etc.

Together with (17), let us consider an ODE

$$d\bar{X}(t) = \bar{A}(\bar{X}(t))dt,$$

(18)

where $\bar{A}$ is such that $\bar{A}(x) \sim A(x)$ as $|x| \to \infty$, i.e. $|\bar{A}(x) - A(x)| = o(|A(x)|)$ as $|x| \to \infty$.

The aim of this section is to find conditions that ensure equivalence

$$X(t) \sim \bar{X}(t), \ t \to \infty, \ a.s.$$

Note that the study of asymptotics for a solution to the ODE (18) may be much simpler than for the SDE (17).

Naturally we should have some assumptions that guarantee that the rate of growth of the noise terms in (17) is $o(|\bar{X}(t)|)$ as $t \to \infty$ a.s. We also have to discuss an initial condition or another parametrization for $\bar{X}(t)$.

Consider an integral equation

$$dZ(t) = A(Z(t))dt + d\xi(t),$$

(19)

where $\xi$ is locally bounded, measurable function, $\xi(0) = 0$.

At first let us obtain a result on the asymptotics for equation (19) with non-random $\xi$ and then we apply the results to equation (17) with

$$d\xi(t) = B(X(t))dW(t) + \int_U C_1(X(t), u)N_1(du, dt) + \int_U C_2(X(t), u)\hat{N}_2(du, dt).$$

We will consider the case when $A$ has power asymptotics at the infinity with an index that is less than 1.

Remark 1. If the coefficients have a power growth with an index that is less than 1, then solution will have the power asymptotic. However if coefficients has a linear growth, then the growth of a solution may be exponential. This case demands another technique and we do not consider this case in the paper.

Recall notations

$$r = |x|, \ \varphi = \frac{x}{|x|}, \ r(t) = r_X(t) = |X(t)|, \ \varphi(t) = \varphi_X(t) = \frac{X(t)}{|X(t)|}.$$ We define $\varphi$ arbitrary if denominator equals 0.

If we specify an initial condition $X(0) = x$, then we denote $X(t)$ by $X_x(t)$.
Represent the vector field $A$ as a sum of the radial and tangential components:
\[ A(x) = A_{\text{rad}}(x) + A_{\text{tan}}(x), \tag{20} \]
i.e.,
\[ A_{\text{rad}}(x) = \langle A(x), \varphi \rangle \varphi, \quad A_{\text{tan}}(x) = A(x) - \langle A(x), \varphi \rangle \varphi. \]

**Theorem 1.** Assume that
- \[ A_{\text{rad}}(x) = a(x)r^{\beta} \varphi \tag{21} \]
and
- \[ \forall \varphi_0 \in S^{d-1} \quad \exists \lim_{r \to +\infty} a(x) =: \bar{a}(\varphi_0), \tag{22} \]
where $\beta < 1$ and $\bar{a} : S^{d-1} \to \mathbb{R}$ is a positive and continuous function;
- \[ \exists \gamma > 0 \quad \exists C_{\text{tan}} > 0 \quad \exists R_{\text{tan}} > 0 \quad \forall r > R_{\text{tan}} : \sup_{|y|=r} |A_{\text{tan}}(y)| \leq C_{\text{tan}}r^{\beta - \gamma}; \tag{23} \]
- \[ \exists \delta > 0 \quad \exists C_{\xi} > 0 \quad \forall t \geq 0 : \left| \xi(t) \right| \leq C_{\xi} \left( 1 + t^{\frac{1}{2} - \beta - \delta} \right), \tag{24} \]

Then there is $R_0 = R_0(A, \delta, \beta, C_{\xi}) > 0$ such that for any $x \in \mathbb{R}^d$, $|x| \geq R_0$ and any solution $Z(t) = Z_x(t)$, $Z_x(0) = x$ we have for any solution of (19):
- \[ \lim_{t \to \infty} |Z_x(t)| = +\infty; \]
- there exists a limit
  \[ \varphi_{Z_x(+\infty)} := \lim_{t \to \infty} \varphi_{Z_x}(t) := \lim_{t \to \infty} \frac{Z_x(t)}{|Z_x(t)|}; \]
  \[ Z_x(t) \sim \left( (1 - \beta)\bar{a}(\varphi_{Z_x(+\infty)})t \right)^{-\beta} \varphi_{Z_x(+\infty)} \text{ as } t \to \infty. \]

We postpone the proof to the Appendix.

**Remark 2.** We don’t assume uniqueness of a solution. The limit angle $\varphi_{Z_x(+\infty)}$ may depend on a choice of a solution $Z_x(t)$.

**Remark 3.** Recall that the function
\[ \bar{X}(t) = \bar{X}(t, \varphi) := ((1 - \beta)\bar{a}(\varphi)t)^{-\beta} \varphi \tag{25} \]
is a solution to (10). The statement of Theorem 1 means that $Z_x(t) \sim \bar{X}(t, \varphi_{Z_x(+\infty)})$ as $t \to \infty$. Note that (24) yields $|\xi(t)| = o(|\bar{X}(t)|)$, $t \to \infty$, i.e., the perturbation is negligible w.r.t. the solution of (10). If the noise is large in the sense:
\[ \limsup_{t \to \infty} \frac{|\xi(t)|}{t^{1/2 - \beta + \delta}} > 0 \]
for some $\delta > 0$, then there are no reasons to expect equivalence $Z_x(t) \sim \bar{X}(t, \varphi)$ as $t \to \infty$ for any $x$ and $\varphi$. 
Remark 4. Assumptions (24), (22), (21) without (23) don’t imply convergence $\lim_{t \to \infty} |Z_x(t)| = +\infty$. For any starting point $x_0$ we give an example below of $A_{tan}$ and $\xi$ such that $|\xi(t)| \leq 1$, $t \geq 0$ and the function $Z_{x_0}$ is bounded.

Consider the case $d = 2$ and assume that $A_{rad}(x) = r^\beta \varphi, |x| \geq R_{rad}$. Let $r(t) = r_n(t)$ be the solution to the ODE
\[
\frac{dr(t)}{dt} = r^\beta(t), \quad r(0) = n,
\]
i.e., $r(t) = \left(n^{1-\beta} + (1-\beta)t\right)^{\frac{1}{1-\beta}}$.

Denote $\sigma := \frac{(n+1)^{1-\beta} - n^{1-\beta}}{1-\beta}$. Then $r(\sigma) = n + 1$.

Let $\xi(t) = (0, 0), t \in [0, \sigma]$. If $|x_0| = n$, then $|Z_{x_0}(t)| = r(t), t \in [0, \sigma]$, and in particular $|Z_{x_0}(\sigma-)| = n + 1$. It is possible to select $A_{rad}$ such that $Z_{(n,0)}(\sigma-): = (-n - 1, 0)$ and $Z_{(-n,0)}(\sigma-) = (n + 1, 0)$. Set
\[
\xi(t) := \begin{cases} 
(0, 0), & t \in [2k\sigma, (2k + 1)\sigma-); \\
(1, 0), & t \in [(2k + 1)\sigma, (2k + 2)\sigma-).
\end{cases}
\]

Then $Z_{(n,0)}(2k\sigma) = (n, 0), Z_{(n,0)}((2k + 1)\sigma) = (-n, 0)$ for any $k \geq 0$. Hence the function $Z_{(n,0)}(t), t \geq 0$ is bounded.

Remark 5. Notice that in Theorem 1 we assume some asymptotics for coefficients at infinity but for equation (7) we assume similar asymptotics at zero, see (8). We will see that the limit as $\varepsilon \to 0$ of time-space transformation $\tilde{X}_\varepsilon(t) := \varepsilon^{\frac{1+\alpha}{1+\beta}} \mathcal{X}_\varepsilon \left(\varepsilon^{\frac{1+\beta}{\alpha+\beta}} t\right)$, c.f. (13), equals a solution to the model equation (9) with $\varepsilon = 1$. This is the reason that the study of the limit angle at 0 for equation (7) with the small noise reduces to the study of the limit angle of the model equation for large $t$.

The main result of this section is the following.

Theorem 2. Suppose that for any initial condition $x \in \mathbb{R}^d$ there is a unique solution to (17), which is a strong Markov process.

Assume that (21), (23) are satisfied, and
\[
\forall x \quad P_x \left( \sup_{t \geq 0} |X(t)| = +\infty \right) = 1, \tag{26}
\]
\[
\exists \delta > 0, \forall \varepsilon > 0, \exists c_{\delta, \varepsilon} \forall x \quad P_x \left( |\xi_X(t)| \leq c(1 + t^{\frac{1-\beta}{1-\delta}}), \ t \geq 0 \right) \geq 1 - \varepsilon, \tag{27}
\]
where
\[
\xi_X(t) = \int_0^t B(X(s))dW(s) + \int_0^t \int_U C_1(X(s-), u) N_1(du, ds) + \int_0^t \int_U C_2(X(s-), u) N_2(du, ds).
\]

Then for any initial condition $X(0)$ we have
\[
\lim_{t \to \infty} |X(t)| = +\infty \ a.s.; \tag{28}
\]
there exists a (random) limit
\[ \varphi_X(+\infty) := \lim_{t \to \infty} \varphi_X(t) = \lim_{t \to \infty} \frac{X(t)}{X(0)} \] a.s.;

- \( X(t) \sim \bar{X}(t, \varphi_X(+\infty)) \) as \( t \to \infty \) a.s., where \( \bar{X} \) is defined in (25).

**Proof.** Set
\[ \tau_R := \inf \{ t \geq 0 : |X(t)| \geq R \}. \]

It follows from the assumptions of the Theorem that \( \tau_R < \infty \) a.s. for any \( R > 0 \) and any initial starting point \( X(0) \).

Let \( \varepsilon > 0 \) be arbitrary, constants \( \delta > 0 \) and \( c = c_{\delta, \varepsilon} \) be from (27), \( R_0 = R_0(\delta, c, A) \) is selected from Theorem 1.

Theorem 1 and the strong Markov property yield
\[
P(\lim_{t \to \infty} |X(t)| = +\infty) \geq P\left( \sup_{t \geq \tau_{R_0}} |\xi_X(t) - \xi_X(\tau_{R_0})| \leq c\left(1 + (t - \tau_{R_0})^{-\beta - \delta}\right) \right) = \\
\int_{|x| \geq R_0} P_x\left( \sup_{s \geq 0} |\xi_X(s)| \leq c(1 + s^{1/\beta - \delta}) \right) P_X(\tau_{R_0})(dx) \geq 1 - \varepsilon. \tag{29} \]

Since \( \varepsilon > 0 \) is arbitrary, the last inequality implies
\[ \lim_{t \to \infty} |X(t)| = +\infty \] a.s.

Proofs of all other items are similar. \( \square \)

**Remark 6.** We need the strong Markov property only for the justification of the equality in (29). Generally, we may replace assumption on Markovianity, (26) and (27) with
\[ \exists \delta > 0, c > 0 \forall R > 0 : P\left( \exists t_0 \forall t \geq 0 : |\xi_X(t_0 + t) - \xi_X(t_0)| \leq c(1 + t^{1/\beta - \delta}) \text{ and } |X(t_0)| \geq R \right) = 1. \]

**Example 1.** Condition (26) is satisfied if, for example, all coefficients are locally bounded and \( B \) is non-degenerate or, for example, if the jump part is non-degenerate at infinity and \( X \) can exit from any fixed ball by one large jump with positive probability that is independent of the initial point (but may depend on a ball).

Below we give three examples when apriory bound (27) of the noise growth is satisfied.

**Example 2.** Let \( X(t), t \geq 0 \) be a solution of the SDE
\[ dX(t) = A(X(t))dt + dB_\alpha(t), t \geq 0, \] where \( \alpha > 1 \), \( A \) is locally bounded and satisfies (21), (23).

Existence and uniqueness of the solution follows from [39, 8] and [14] if \( \alpha + \beta > 1 \). In this case the statement of Theorem 2 holds true because
\[ \forall \alpha' < \alpha : \lim_{t \to \infty} \frac{B_\alpha(t)}{t^{1/\alpha'}} = 0 \text{ a.s.,} \] see [31].
Example 3. Let $X(t)$ be a solution to (17), where $C_1 = C_2 \equiv 0$ and $B$ is bounded. Observe that
\[
\sum_j \int_0^t B_{kj}(X(s))dW_j(s) = \tilde{W}_k \left( \sum_j B_{kj}^2(X(s))ds, \right)
\]
where $\tilde{W}_k$ is a Brownian motion.

Hence $|\xi_X(t)|$ is dominated by $\sup_{s \in [0,t]} (\sum_k |\tilde{W}_k(cs)|^2)^{1/2}$, where
\[
c = \sup_x \sum_{k,j} |B_{k,j}(x)|^2 = \text{const}.
\]

It is well known that
\[
\forall \gamma > 1/2 : \lim_{t \to \infty} \frac{\tilde{W}_k(t)}{t^\gamma} = 0 \quad \text{a.s.}
\]
So, condition (27) is satisfied if $\beta > -1$.

It should be noted that the case $\beta = -1$ is really critical. Indeed, if $d = 1$, then the Bessel process
\[
d\zeta(t) = \frac{a}{\zeta(t)} dt + dW(t)
\]
does not converge to $\infty$ as $t \to \infty$ for $a \leq 1/2$.

Example 4. Let $\tilde{Z}$ be a zero mean Lévy process without a Gaussian component. Assume that its jump measure $\nu$ satisfies conditions
\[
\int_{|z| > x} \nu(dz) \leq \frac{K}{x^\gamma}, \quad x \geq 1,
\]
and
\[
\int_{|z| \leq 1} z^2 \nu(dz) \leq K
\]
for some $K > 0$ and $\gamma \in (1, 2)$.

It can be proved, see [36, Lemma 3.1] that for any $\varepsilon > 0$ and $\gamma' < \gamma$ there exists a generic constant $C = C(K, \gamma, \gamma', \theta)$ such that for any predictable process $\sigma(t)$, $|\sigma(t)| \leq 1$ a.s., we have
\[
P \left( \left| \int_0^t \sigma(s) d\tilde{Z}(s) \right| \leq C(1 + t^{1/\gamma'}), \quad t \geq 0 \right) \geq 1 - \varepsilon.
\]

3. Small-noise perturbations of ODEs

Let $\{X_\varepsilon(t)\}$ be a solution to the SDE (7) with with the initial condition $X_\varepsilon(0) = 0$.

In this section we formulate and prove the main result of the paper on a limit of $\{X_\varepsilon\}$ as $\varepsilon \to 0$, see Theorem 3 below.

Assumption 0. $A$ is such that there exists a unique weak solution of (7), and the solution is a strong Markov process.

This is true, for example, if $A \in L_p,loc$, where $p > \frac{d}{\alpha - 1}$ and $\alpha > 1$, see existence [39] and uniqueness in [14].

Recall that radial and tangential components $A_{rad}$ and $A_{tan}$ of the vector field $A$ are defined in (20).
Assumption 1.  
(1) the vector field $A$ is Lipschitz continuous function in any compact set $G \subset \mathbb{R}^d \setminus \{0\}$;  
(2) we have representation in a neighborhood of zero:  
$$A_{\text{rad}}(x) = a(x) r^\beta \varphi,$$
where $|\beta| < 1$ and for some positive continuous function $\tilde{a} : S^{d-1} \to \mathbb{R}$:  
$$\forall \varphi_0 \in S^{d-1} \quad \exists \lim_{r \to 0^+} a(r, \varphi) =: \tilde{a}(\varphi_0);$$
(3)  
$$\exists \gamma > 0 : \sup_{|y| = r} |A_{\text{tan}}(y)| = o(r^{\beta + \gamma}), \ r \to 0^+; \quad (31)$$

We need the following assumption on existence of a collection of solutions to (5), (6), which are parametrized by the angle $\varphi$ of exit from 0.

Assumption 2. For any $\varphi \in S^{d-1}$ there is a unique solution $X_0(t) = X_0(t, \varphi)$ to (5), (6) such that $X_0(t) \neq 0, t > 0$ and  
$$\lim_{t \to 0^+} \frac{X_0(t, \varphi)}{|X_0(t, \varphi)|} = \varphi.$$  
Moreover, for any $T > 0$ the pair $(R_0(t, \varphi), \Phi_0(t, \varphi)) := (|X_0(t, \varphi)|, \frac{X_0(t, \varphi)}{|X_0(t, \varphi)|})$ is uniformly continuous for $(t, \varphi) \in (0, T] \times S^{d-1}$.

Examples of vector fields that satisfy Assumption 2 are given in Appendix.

Theorem 3. Suppose that Assumptions 0, 1, 2 are satisfied and $\alpha + \beta > 1$, $|\beta| < 1$. Let $\{\tilde{X}(t)\}$ be a solution to the SDE  
$$d\tilde{X}(t) = \tilde{a}(\frac{\tilde{X}(t)}{|\tilde{X}(t)|}) \tilde{X}^\beta(t)dt + dB_\alpha(t), \ t \geq 0, \quad (32)$$
and  
$$\tilde{X}(0) = 0,$$  
(existence of a.s. limit follows from Theorem 2, see Example 3).

Then solutions of (7) converge in distribution  
$$X_\varepsilon(\cdot) \overset{D((0,\infty))}{\Rightarrow} X_0(\cdot, \tilde{\Phi}(+\infty)), \ \varepsilon \to 0,$$
where $X_0$ is from Assumption 2, $\tilde{\Phi}(+\infty)$ is defined in (34).

The course of the proof is the following. At first we show the weak relative compactness of $\{X_\varepsilon\}$ and that any limit point is continuous a.s. Then we will show that $X_\varepsilon$ spends a small time in a small neighborhood of zero. So any limit point spends zero time at 0. The noise disappears in the limit as $\varepsilon \to 0$, so any limit process $X_0$ must satisfy (5) for $t$ such that $X_0(t) \neq 0$. Since $\tilde{a}(x) > 0$ in Assumption 1 for any $t_0 > 0$ such that $X_0(t_0) \neq 0$, we have $X_0(t) \neq 0, t \geq t_0$ a.s. Since $X_0$ spends zero time at 0, we get $X_0(t) \neq 0$ for all $t > 0$.  


a.s. Hence, any limit point \( X_0(t) \) must be of the form \( X_0(t, \Phi) \), where \( X_0(t, \varphi) \) is defined in Assumption 2 and \( \Phi = \lim_{t \to 0} \frac{X_0(t)}{|X_0(t)|} \).

Let \( \tau_\delta^{(e)} := \inf \{ t \geq 0 : |X_e(t)| \geq \delta \} \).

We will show that if \( \varepsilon > 0 \) and \( \delta > 0 \) are small, \( \varepsilon \ll \delta \), then

\[
\frac{X_e(\tau_\delta^{(e)})}{|X_e(\tau_\delta^{(e)})|} \xrightarrow{d} \bar{\Phi}(+\infty) \quad \text{and} \quad |X_e(\tau_\delta^{(e)})| \approx \delta, \quad \text{so} \quad X_e(\tau_\delta^{(e)}) \approx \delta \bar{\Phi}(+\infty), \tag{35}
\]

where \( \bar{\Phi}(+\infty) \) is defined in (33).

Here signs \( \approx \) (or \( \approx \)) mean that processes are "close" ("close" in distribution). The rigorous statements will be given later in subsection 3.2.

This is the most difficult part of the proof. To do this, we apply a space-time transform, see (39) below, and analyze behavior of \( X_\varepsilon \) in a microscopic neighborhood of 0.

By \( X_{\varepsilon, y}, \varepsilon \geq 0 \) we denote a solution to an equation with initial starting point \( X_{\varepsilon, y}(0) = y \).

If we have (35) and we know that \( \tau_\delta^{(e)} \) is small, then the rest of the proof follows from the next reasoning, where we use Assumption 2 combined with some continuity arguments

\[
X_\varepsilon(t) \approx X_\varepsilon(\tau_\delta^{(e)} + t) \equiv X_{\varepsilon, y}(t) \bigg|_{y = X_\varepsilon(\tau_\delta^{(e)})} \equiv X_{\varepsilon, y}(t) \bigg|_{y = \delta \bar{\Phi}(+\infty)} \approx \approx X_{0, \delta \bar{\Phi}(+\infty)}(t) \approx X_0(t, \bar{\Phi}(+\infty)).
\]

3.1. Proof of Theorem 3. First steps.

**Lemma 1.** \( \{X_\varepsilon\} \) is weakly relatively compact in \( D([0, \infty)) \). Any limit process of \( \{X_\varepsilon\} \) as \( \varepsilon \to 0 \) is continuous with probability 1.

The proof is standard, see Appendix.

Let \( f \) be a cadlag function, \( f(0) = 0 \). By \( X_{f, x}(t) \) denote a solution to the integral equation

\[
X_{f, x}(t) = x + \int_0^t A(X_{f, x}(s))ds + f(t).
\]

**Remark 7.** Function \( X_{f, x}(t) \) is well-defined until it hits 0. Since \( A(x) \sim \bar{a}(\frac{x}{|x|})x^\beta \) as \( x \to 0 \), where \( \bar{a} \) is positive and continuous, the function \( X_0,x \) never hits 0 if \( x \neq 0 \).

**Lemma 2.** Let Assumption 7 be satisfied. Then for any \( x_0 \neq 0 \) and \( T > 0 \) we have the uniform convergence on the compact sets:

\[
\lim_{\|f\|_{[0,T]} \to 0} \|X_{f, x} - X_{0, x_0}\|_{[0,T]} = 0,
\]

where \( \|g\|_{[0,T]} := \sup_{[0,T]} |g(t)| \).

Moreover,

\[
\forall T > 0 \, \forall \mu > 0 \, \forall \delta \in (0, 1) \, \exists \varepsilon > 0 \, \forall f, \|f\|_{[0,T]} \leq \varepsilon \, \forall x, |x| \in [\delta, \delta^{-1}] \, \forall y, \, |y - x| \leq \varepsilon : \sup_{t \in [0,T]} |X_{f, y}(t) - X_{0, x}(t)| \leq \mu.
\]

The proof of this Lemma is standard. Indeed, observe that Assumption 8 ensures that \( X_{0, x}(t), t \in [0, T] \) is separated from 0. Since \( A \) satisfies the local Lipschitz condition outside of 0 and linear growth condition, the proof follows from the application of Gronwall’s lemma.
Let us denote by $X_{\varepsilon,x}, \varepsilon \geq 0$ the solution to (3) with initial condition $X_{\varepsilon,x}(0) = x \neq 0$ (i.e., $X_{\varepsilon,x}(t)$ is $X_{\varepsilon B_{\alpha},x}(t)$ from Lemma [2]).

**Corollary 1.** Let the assumptions of Theorem [3] be satisfied. Then for any $x_0 \neq 0$ we have a.s. convergence of stochastic processes:

$$\lim_{\varepsilon \to 0, x \to x_0} X_{\varepsilon,x}(\cdot) = X_{0,x_0}(\cdot) \text{ in } D([0, \infty)).$$

Moreover,

$$\forall T > 0 \ \forall \mu > 0 \ \forall \delta \in (0, 1) \ \exists \nu > 0 \ \exists \varepsilon_0 = \varepsilon_0(\mu, \delta) > 0 \ \forall \varepsilon \in [0, \varepsilon_0]$$

$$\forall x, |x| \in [\delta, \frac{1}{\delta}] \ \forall y, |y - x| \leq \nu : \ \sup_{t \in [0,T]} |X_{\varepsilon,y}(t) - X_{0,x}(t)| \geq \mu \leq \mu.$$  \hspace{1cm} (36)

In particular, if random variables $\{\zeta_{\varepsilon}, \varepsilon \geq 0\}$ are independent of the noise and $\zeta_{\varepsilon} \Rightarrow \zeta_0$ as $\varepsilon \to 0$, where $\zeta_0 \neq 0$ a.s., then

$$X_{\varepsilon,\zeta} \Rightarrow X_{0,\zeta_0}, \text{ as } \varepsilon \to 0 \text{ in } D([0, \infty)).$$ \hspace{1cm} (37)

**Remark 8.** Equation (39) may be convenient to write in terms of absolute value of vectors $x, y$ and their polar angles (with possibly another $\nu$):

$$\forall x, |x| \in [\delta, \frac{1}{\delta}] \ \forall y \text{ such that } \left|\frac{|x|}{|y|} - 1\right| \leq \nu \text{ and } \left|\frac{x}{|x|} - \frac{y}{|y|}\right| \leq \nu : \ \sup_{t \in [0,T]} |X_{\varepsilon,y}(t) - X_{0,x}(t)| \geq \mu \leq \mu.$$ \hspace{1cm} (38)

Consider the following time-space transformation. Set

$$\tilde{X}_{\varepsilon}(t) := \varepsilon^{\frac{\alpha}{\alpha + \beta - 1}} X_{\varepsilon}(\varepsilon^{\frac{\alpha(1-\beta)}{\alpha + \beta - 1}} t).$$ \hspace{1cm} (39)

Then

$$\tilde{X}_{\varepsilon}(t) = \varepsilon^{\frac{\alpha}{\alpha + \beta - 1}} X_{\varepsilon}(\varepsilon^{\frac{\alpha(1-\beta)}{\alpha + \beta - 1}}) =$$

$$\varepsilon^{-\frac{\alpha}{\alpha + \beta - 1}} \int_0^{\varepsilon^{\frac{\alpha(1-\beta)}{\alpha + \beta - 1}}} A(X_{\varepsilon}(s)) ds + \varepsilon^{1 + \frac{\alpha}{\alpha + \beta - 1}} B_{\alpha}(\varepsilon^{\frac{\alpha(1-\beta)}{\alpha + \beta - 1}} t) =$$

$$\varepsilon^{-\frac{\alpha}{\alpha + \beta - 1}} \int_0^{\varepsilon^{\frac{\alpha(1-\beta)}{\alpha + \beta - 1}}} A(X_{\varepsilon}(\varepsilon^{\frac{\alpha(1-\beta)}{\alpha + \beta - 1}} z)) dz + \varepsilon^{\frac{\beta}{\alpha + \beta - 1}} B_{\alpha}(\varepsilon^{\frac{\alpha(1-\beta)}{\alpha + \beta - 1}} t) =$$

$$\varepsilon^{-\frac{\alpha}{\alpha + \beta - 1}} \int_0^{t} A(\varepsilon^{\frac{\alpha}{\alpha + \beta - 1}} \tilde{X}_{\varepsilon}(z)) dz + B_{\alpha}^{(\varepsilon)}(t),$$

where $\{B_{\alpha}^{(\varepsilon)}(t)\} := \{\varepsilon^{\frac{\beta}{\alpha + \beta - 1}} B_{\alpha}(\varepsilon^{\frac{\alpha(1-\beta)}{\alpha + \beta - 1}} t)\}$.

We have, see (20),

$$d\tilde{X}_{\varepsilon}(t) = \varepsilon^{-\frac{\alpha}{\alpha + \beta - 1}} \left(A_{\text{rad}}(\varepsilon^{\frac{\alpha}{\alpha + \beta - 1}} \tilde{X}_{\varepsilon}(t)) + A_{\text{tan}}(\varepsilon^{\frac{\alpha}{\alpha + \beta - 1}} \tilde{X}_{\varepsilon}(t))\right) dt + dB_{\alpha}^{(\varepsilon)}(t).$$ \hspace{1cm} (40)

Further we will skip $\varepsilon$ in notation $\{B_{\alpha}^{(\varepsilon)}(t)\}$. 
It follows from Assumption 1 that for any \( x \neq 0 \):
\[
\varepsilon \frac{a}{|x|} A_{rad}(\varepsilon \frac{a}{|x|} x) = a(\varepsilon \frac{a}{|x|} x) x^\alpha = \bar{a}(x) x^\beta, \ \varepsilon \to 0;
\]
\[
\varepsilon \frac{a}{|x|} A_{tan}(\varepsilon \frac{a}{|x|} x) = \varepsilon \frac{a}{|x|} A_{tan}(\varepsilon \frac{a}{|x|} x) x|x|^{\beta + \gamma} \to 0, \ \varepsilon \to 0.
\]

Lemma 3. Let \( \alpha + \beta > 1, \alpha \in (1,2), |\beta| < 1 \), and \( Y_\varepsilon(t), t \geq 0, \varepsilon \geq 0 \) be solutions to the SDEs:
\[
dY_\varepsilon(t) = b_\varepsilon(Y_\varepsilon(t)) dt + dB_\alpha(t), \ t \geq 0, \ Y_\varepsilon(0) = 0,
\]
where \( b_\varepsilon \) are continuous functions everywhere except of 0.

Assume that
1) \( \lim_{\varepsilon \to 0} b_c(x) = b_0(x) \), uniformly on every compact that does not contain 0,
2) \( \exists C > 0 \forall \varepsilon \geq 0 \forall x \neq 0 : |b_\varepsilon(x)| \leq C(1 + |x|^\beta) \).

Then
\[
Y_\varepsilon \Rightarrow Y_0, \varepsilon \to 0 \ in \ D([0,\infty)).
\]
In particular, we have the weak convergence of stochastic processes:
\[
\tilde{X}_\varepsilon \Rightarrow \tilde{X}, \ as \ \varepsilon \to 0, \ in \ D([0,\infty)),
\]
where \( \tilde{X} \) is defined in \([32], [33]\).

Proof of Lemma 3. Using localization technique and sub-linear growth of coefficients, without loss of generality we may assume that supports of all \( \{b_\varepsilon\} \) belongs to the same compact.

It follows from \([39]\) that \( \{Y_\varepsilon(t)\} \) generates Feller semigroup and for any bounded continuous function \( f \) we have the uniform convergence:
\[
\sup_x |E_x f(Y_\varepsilon(t)) - E_x f(Y_0(t))| \to 0, \ \varepsilon \to 0,
\]
see \([39]\) Lemma 2.

The application of \([23]\) Theorem 2.5, Chapter 4] yields \([43]\).

Let us sketch a proof of Theorem 3 again in more details.

Recall that \( \tilde{X}_\varepsilon(t) := \varepsilon \frac{a}{|x|} X_\varepsilon(\frac{a}{|x|} t) \).

By \( \tilde{X}_{\varepsilon,x} \) denote a solution to \([40]\) with initial condition \( \tilde{X}_{\varepsilon,x}(0) = x \) (so \( \tilde{X}_\varepsilon = \tilde{X}_{\varepsilon,0} \)).

For \( M \geq 0 \) set
\[
\tau^\varepsilon_M := \inf\{t \geq 0 : |X_\varepsilon(t)| \geq M\},
\]
\[
\tau^\varepsilon_{M,x} := \inf\{t \geq 0 : |X_{\varepsilon,x}(t)| \geq M\},
\]
\[
\tilde{\tau}^\varepsilon_M := \inf\{t \geq 0 : |\tilde{X}_\varepsilon(t)| \geq M\},
\]
\[
\tilde{\tau}^\varepsilon_{M,x} := \inf\{t \geq 0 : |\tilde{X}_{\varepsilon,x}(t)| \geq M\}.
\]

Note that
\[
\tilde{\tau}^\varepsilon_M = \frac{\varepsilon^{(1-\beta)}}{\alpha + \beta - \gamma} \tau^\varepsilon_M, \ \tilde{\tau}^\varepsilon_{M,x} = \frac{\varepsilon^{(1-\beta)}}{\alpha + \beta - \gamma} \tau^\varepsilon_{M,x}.
\]

\[
\tilde{X}_\varepsilon(\tilde{\tau}^\varepsilon_M) = \varepsilon \frac{a}{|x|} X_\varepsilon(\frac{a}{|x|} \tilde{\tau}^\varepsilon_M).
\]
Step 1. Let \( R > 0 \) be a large number, \( \varepsilon > 0 \) be a small number. We apply Lemma 3 and Theorem 1 and show that if \( R \) is large and \( \varepsilon > 0 \) is small, then

\[
|\tilde{X}_\varepsilon(\tau_R^\varepsilon)| \approx R \quad \text{and} \quad \frac{\tilde{X}_\varepsilon(\tau_R^\varepsilon)}{|\tilde{X}_\varepsilon(\tau_R^\varepsilon)|} \approx \Phi(+\infty),
\]

where \( \Phi(+\infty) \) is defined in (33).

So

\[
|X_\varepsilon(\tau^\varepsilon_{\varepsilon + \beta - 1} R)| \approx \varepsilon^{\alpha - \beta + 1} R \quad (45)
\]

and

\[
\frac{X_\varepsilon(\tau^\varepsilon_{\varepsilon + \beta - 1} R)}{|X_\varepsilon(\tau^\varepsilon_{\varepsilon + \beta - 1} R)|} \approx \frac{\tilde{X}_\varepsilon(\tau_R^\varepsilon)}{|\tilde{X}_\varepsilon(\tau_R^\varepsilon)|} \approx \Phi(+\infty). \quad (46)
\]

Step 2. Let \( R > 0 \) be a large number, \( \delta > 0 \) be a small number, \( \varepsilon \ll \delta \). We show that if \( |x| \in [\varepsilon^{\alpha - \beta + 1} R, \delta] \), then

\[
|X_\varepsilon,x(\tau^\varepsilon_{\varepsilon + \beta - 1} R)| \approx \delta \quad \text{and} \quad \frac{X_\varepsilon,x(\tau^\varepsilon_{\varepsilon + \beta - 1} R)}{|X_\varepsilon,x(\tau^\varepsilon_{\varepsilon + \beta - 1} R)|} \approx \frac{x}{|x|}. \quad (47)
\]

The first approximate equality in (46) means that \( X \) cannot exit from the ball \( \{y : \|y\| \leq \delta\} \) by a large jump. The second equality means that the process \( X \) slightly rotates in a set \( \{y : \varepsilon^{\alpha - \beta + 1} R \leq \|y\| \leq \delta\} \). This happens because the effect of drift there dominates the effect of the noise if \( R \) is large enough. It will be seen from the proof that the effect of the drift and the noise comparable in the set \( \{y : \|y\| < \varepsilon^{\alpha - \beta + 1} R\} \).

It follows from (46), (47), and Corollary 1 that

\[
|X_\varepsilon(\tau^\varepsilon_{\delta})| \approx \delta, \quad \frac{X_\varepsilon(\tau^\varepsilon_{\delta})}{|X_\varepsilon(\tau^\varepsilon_{\delta})|} \approx \Phi(+\infty). \quad (48)
\]

Step 3. We deduce from Corollary 1 (see (37)) and (48) that

\[
X_\varepsilon(\tau^\varepsilon_{\delta} + t) \approx X_{0,\delta \Phi(+\infty)}(t). \quad (49)
\]

Step 4. We prove that \( \tau^\varepsilon_{\delta} \) is small if \( \varepsilon \) and \( \delta \) are small. So

\[
X_\varepsilon(\tau^\varepsilon_{\delta} + t) \approx X_{0}(t). \quad (50)
\]

Step 5. It follows from Assumption 2 that

\[
X_{0,\delta \Phi(+\infty)}(t) \approx X_0(t, \Phi(+\infty)). \quad (51)
\]

So, (49), (50), and (51) yield the approximate equality

\[
X_\varepsilon(t) \approx X_0(t, \Phi(+\infty)).
\]
3.2. End of the proof of Theorem 3. Let $d(\xi, \eta) = d(P_\xi, P_\eta)$ be the Levy-Prokhorov metric between distributions of $\xi$ and $\eta$.

**Lemma 4.** For any $\mu > 0$ there exists $R_1 > 0$ such that for all $R \geq R_1$:

\[
d\left(\frac{X(\bar{\tau}_R)}{|X(\bar{\tau}_R)|}, \Phi(+\infty)\right) < \mu; \\
d\left(\frac{X(\bar{\tau}_R)}{R}, 1\right) < \mu; \\
d\left(\frac{R^{1-\beta}}{\bar{\tau}_R}, (1-\beta)\bar{\alpha}(\Phi(+\infty))\right) < \mu; \tag{52}
\]

and there is $\epsilon_1 > 0$ such that for all $\epsilon \in (0, \epsilon_1]$:

\[
d\left(\frac{\tilde{X}_\epsilon(\bar{\tau}_R)}{|\tilde{X}_\epsilon(\bar{\tau}_R)|}, \Phi(+\infty)\right) < \mu; \\
d\left(\frac{\tilde{X}_\epsilon(\bar{\tau}_R)}{R}, 1\right) < \mu; \\
d\left(\frac{R^{1-\beta}}{\bar{\tau}_R}, (1-\beta)\bar{\alpha}(\Phi(+\infty))\right) < \mu. \tag{53}
\]

**Proof.** It follows from Theorems 1 and 2 (see Example 2) that $$\bar{\tau}_R(\bar{\tau}_R) \rightarrow \Phi(+\infty), t \rightarrow \infty \text{ a.s.}$$ and $|\bar{X}(t)| \sim ((1-\beta)\bar{\alpha}(\Phi(+\infty)))t^{\frac{1}{1-\beta}}$, $t \rightarrow \infty \text{ a.s.}$

This yields (52).

Equations (53) follow from Lemma 3. \qed

The next lemma is the most important part of the proof. Informally, the statement means that if $\epsilon$, $\delta$ are small and $|x| \geq \epsilon^{\frac{\alpha}{\alpha + \beta - 1}}R$ (see Step 2 above), then (47) holds true, i.e., the process can’t exit from the ball $\{y : |y| \leq \delta\}$ by a large jump and the radius at the instant of exit approximately equals $\delta$; moreover, the polar angle at the instant of exit approximately equals initial polar angle $\frac{x}{|x|}$. We also will prove that the exit time $\tau_\delta^x$ is small if $\epsilon$ and $\delta$ are small (see Step 4 above).

**Lemma 5.**

\[
\forall \mu > 0 \exists \delta_2 = \delta_2(\mu) > 0 \exists R_2 = R_2(\mu) > 0 \exists \epsilon_2 = \epsilon_2(\mu) > 0 \\
\forall R > R_2, \delta \in (0, \delta_2), \forall \epsilon \in (0, \epsilon_2) \text{ such that } R^{\frac{\alpha}{\alpha + \beta - 1}} < \delta, \forall x \in \mathbb{R}^d, \ R^{\frac{\alpha}{\alpha + \beta - 1}} < |x| < \delta:
\]

\[
P\left(\left|\frac{X_{\epsilon,x}(\tau_\delta^x)}{|X_{\epsilon,x}(\tau_\delta^x)|} - \frac{x}{|x|}\right| > \mu\right) < \mu; \tag{54}
\]

\[
P\left(\left|\frac{X_{\epsilon,x}(\tau_\delta^x)}{\delta} - 1\right| > \mu\right) < \mu; \tag{55}
\]

\[
P\left(|\tau_\delta^x| > \mu\right) < \mu. \tag{56}
\]
As for the proof see the Appendix.

It follows from equality (44), Lemmas 1, 2, and the strong Markov property that
\[ \forall \mu > 0 \exists \delta_3 = \delta_3(\mu) > 0 \forall \delta \in (0, \delta_3) \exists \epsilon_3 = \epsilon_3(\mu, \delta) > 0 \forall \varepsilon \in (0, \epsilon_3) : \]
\[ d \left( \frac{X_\varepsilon(\tau_\delta^\varepsilon)}{|X_\varepsilon(\tau_\delta^\varepsilon)|}, \hat{\Phi}(+\infty) \right) < \mu; \]  
(57)
\[ P \left( \left| X_\varepsilon(\tau_\delta^\varepsilon) \right| < 1, 1 + \mu \right) < \mu; \]  
(58)
\[ P(\tau_\delta^\varepsilon > \mu) < \mu. \]  
(59)

To prove Theorem 3 it is sufficient to verify that any sequence \( \{ X_{\varepsilon_n} \} \) contains a subsequence \( \{ X_{\varepsilon_{n_k}} \} \) such that
\[ X_{\varepsilon_{n_k}}(\cdot) \Rightarrow X_0(\cdot, \hat{\Phi}(+\infty)), \quad k \to \infty. \]

It follows from Lemma 1 that \( \{ X_{\varepsilon} \} \) is relatively compact. So without loss of generality we will assume that \( \{ X_{\varepsilon_n} \} \) is already convergent.

Let \( \mu > 0 \) be fixed. Select \( \delta_3 = \delta_3(\mu) \) from (57) – (59), and for fixed \( \delta \in (0, \delta_3) \) select \( \epsilon_3 = \epsilon_3(\mu, \delta) \).

Let \( n_3 = n_3(\mu) = n_3(\mu, \delta) \) be such that \( \varepsilon_n < \epsilon_3(\mu) \) for any \( n \geq n_3 \). It follows from Corollary 11.6.4 [22] that for any \( n \geq n_3 \) there are copies \( \hat{\Phi}(+\infty) \) of \( \hat{\Phi}(+\infty) \) and \( \tilde{\zeta}_n \) defined on some probability space such that for all \( n \geq n_3 \)
\[ P \left( \left| \tilde{\zeta}_n - \hat{\Phi}(+\infty) \right| \geq 2\mu \right) \leq 2\mu. \]

By conditioning, see for example [44] Chapter 4, we can construct all these random variables \( \hat{\Phi}(+\infty), \tilde{\zeta}_n, n \geq n_3 \) on the same probability space (nontrivial statement here is that \( \hat{\Phi}(+\infty) \) is independent of \( n \)). Moreover, by conditioning and extension of the probability space we can construct processes \( \hat{B}_\alpha \) and \( \hat{X}_{\varepsilon_n} \) on this extension of the probability space such that \( \tilde{\zeta}_n = \frac{\hat{X}_{\varepsilon_n}(\tilde{\tau}_n^{\varepsilon_n})}{|\hat{X}_{\varepsilon_n}(\tilde{\tau}_n^{\varepsilon_n})|} \) a.s.

Therefore, (57) – (59) yield
\[ P \left( \frac{\hat{X}_{\varepsilon_n}(\tilde{\tau}_n^{\varepsilon_n})}{|\hat{X}_{\varepsilon_n}(\tilde{\tau}_n^{\varepsilon_n})|} - \hat{\Phi}(+\infty) \geq 2\mu \right) \leq 2\mu, \]  
(60)
\[ P \left( \left| \frac{\hat{X}_{\varepsilon_n}(\tilde{\tau}_n^{\varepsilon_n})}{\delta_n} \right| \in [1, 1 + \mu] \right) \leq \mu, \]  
(61)
\[ P(\tilde{\tau}_n^{\varepsilon_n} \geq \mu) \leq \mu. \]  
(62)

We have
\[ \sup_{t \in [0, T]} \left| \hat{X}_{\varepsilon_n}(t) - X_0(t, \hat{\Phi}(+\infty)) \right| \leq \sup_{t \in [0, \tilde{\tau}_n^{\varepsilon_n}]} \left| X_{\varepsilon_n}(t) \right| + \sup_{t \in [0, \tilde{\tau}_n^{\varepsilon_n}]} \left| X_0(t, \hat{\Phi}(+\infty)) \right| + \]  
(63)
\[ \sup_{t \in [0, T]} \left| \hat{X}_{\varepsilon_n}(\tilde{\tau}_n^{\varepsilon_n} + t) - X_0(\tilde{\tau}_n^{\varepsilon_n} + t, \hat{\Phi}(+\infty)) \right| = \]
\[ |\hat{X}_{\varepsilon_n}(\hat{t}_\delta^{\varepsilon_n})| + X_0(\hat{t}_\delta^{\varepsilon_n}, \hat{\Phi}(+\infty)) + \sup_{t \in [0,T]} |\hat{X}_{\varepsilon_n}(\hat{t}_\delta^{\varepsilon_n} + t) - X_0(\hat{t}_\delta^{\varepsilon_n} + t, \hat{\Phi}(+\infty))| = I_1^{n,\delta} + I_2^{n,\delta} + I_3^{n,\delta}. \]

It follows from (68) that
\[ \forall n \geq n_3 : \quad P(I_1^{n,\delta} \geq \delta + \delta \mu) \leq \mu. \] (64)

It is easy to see that there exists \( K > 0 \) such that \( |X_0(t, \varphi)| \leq K t^{1-\beta} \) for small \( t \geq 0 \), where \( K \) is independent of \( \varphi \). So,
\[ P(I_2^{n,\delta} \geq K \mu^{1-\beta}) = \frac{I_2^{n,\delta}}{\mu} \leq P(\hat{t}_\delta^{\varepsilon_n} \geq \mu) \leq \mu. \] (65)

The last inequality follows from (62).

Consider \( I_3^{n,\delta} \). By the strong Markov property we have
\[ I_3^{n,\delta} = \sup_{t \in [0,T]} |X_{\varepsilon_n}(t)_{y=x_n(\hat{t}_\delta^{\varepsilon_n})} - X_0(t)_{x=x_n(\hat{t}_\delta^{\varepsilon_n}, \hat{\Phi}(+\infty))}|, \]
where \( \{X_{\varepsilon_n}(t)\} \) is independent of \( \hat{X}_{\varepsilon_n}(\hat{t}_\delta^{\varepsilon_n}) \).

It follows from Corollary 1, Remark 8 and Lemma 5 that
\[ P(I_3^{n,\delta} \geq \mu) \leq \mu \] (66)

if \( \delta \) is sufficiently small and \( n \) is sufficiently large. Let \( \mu > 0 \) be fixed. Combining estimates (63), (64), (66) we can select sufficiently small \( \delta = \delta(\mu) > 0 \) and \( N = N(\delta, \mu) \) such that
\[ \forall n \geq N : P\left( \sup_{t \in [0,T]} |\hat{X}_{\varepsilon_n}(t) - X_0(t, \hat{\Phi}(+\infty))| \geq 100 \mu + K \mu^{1-\beta} \right) \leq 100 \mu. \]

The last display yields the uniform convergence in probability
\[ \sup_{t \in [0,T]} |\hat{X}_{\varepsilon_n}(t) - X_0(t, \hat{\Phi}(+\infty))| \xrightarrow{P} 0, n \to \infty. \]

Since \( \hat{X}_{\varepsilon_n} \xrightarrow{d} X_{\varepsilon_n}, X_0(t, \hat{\Phi}(+\infty)) \xrightarrow{d} X_0(t, \Phi(\infty)) \) this concludes the proof of Theorem 3.

4. Appendix

**Proof of Theorem 7** Let \( \{Z(t)\} \) be a solution to (19). Set \( \tilde{Z}(t) := Z(t) - \xi(t) \). Then
\[ d\tilde{Z}(t) = A(\tilde{Z}(t) + \xi(t))dt = A(Z(t))dt, \quad t \geq 0. \] (67)

Assume for a while that \( \tilde{Z}(t) \neq 0, t \geq 0 \). Then for \( \tilde{r}(t) = |\tilde{Z}(t)|, r(t) = |Z(t)| \) we have:
\[ d\tilde{r}^{1-\beta}(t) = (1-\beta)\tilde{r}^{-\beta}(t)\frac{\langle \tilde{Z}(t), A(Z(t)) \rangle}{|Z(t)|}dt = (1-\beta)\tilde{r}^{-\beta}(t)\frac{\langle \tilde{Z}(t), A_{\text{rad}}(Z(t)) + A_{\text{tan}}(Z(t)) \rangle}{|Z(t)|}dt \]
\[ = (1-\beta)\tilde{r}^{-\beta}(t)\langle \tilde{Z}(t), a(Z(t))\frac{Z(t)}{|Z(t)|} \rangle dt + (1-\beta)\tilde{r}^{-\beta}(t)\langle \tilde{Z}(t), A_{\text{tan}}(Z(t)) \rangle dt. \] (68)

It follows from Assumption 1 of the Theorem that
\[ \exists R_{\text{rad}} > 0 \exists C_{\text{rad}} > 0 \forall x, |x| \geq R_{\text{rad}} : a(x) \geq C_{\text{rad}} |x|^{\beta}. \]
Suppose that for some $T > 0$ we have
\[ |\xi(t)| \leq \frac{1}{2} \frac{|\tilde{Z}(t)|}{2} \quad \text{and} \quad |\tilde{Z}(t)| \geq 2(R_{rad} \vee R_{tan}) \text{ for } t \in [0, T], \tag{69} \]
where $R_{tan}$ is from (23). Then
\[ |Z(t)|/2 \leq |\tilde{Z}(t)| \leq 3|Z(t)|/2, \quad (Z(t), \tilde{Z}(t)) \geq |\tilde{Z}(t)|/2, \quad t \in [0, T], \]
and assumptions of Theorem 1 yield the inequality
\[ K \]

Note that $t$ and assumptions of Theorem 1 are satisfied. Then the re is Lemma 6.

Remark 9. In this lemma we assume only existence of a solution but not a uniqueness.

Proof. Let $|\tilde{Z}(0)| > 2R_{rad} \vee 2R_{tan}$. It follows from (70) that (69) and (71) may fail only if there is $t$ such that $|\xi(t)| > \frac{|\tilde{Z}(0)|}{2}$.

It follows from (24) that
\[ \exists R_2 \forall t \geq 0 : 2|\xi(t)| \leq (R_2^{1-\beta} + Kt)\frac{1}{1-\beta} - 1, \tag{72} \]
where $K$ is from (71).

Assume that $|\tilde{Z}(0)| > R_0 := R_2 \vee 2R_{rad} \vee 2R_{tan}$. Set
\[ t_0 := \inf \left\{ t \geq 0 : |\xi(t)| > \frac{|\tilde{Z}(t)|}{2} \right\} = \inf \{ t \geq 0 : 2|\xi(t)| > |\tilde{r}(t)| \}. \tag{73} \]

Note that $t_0 \neq 0$ because $\tilde{Z}$ is continuous.

To prove the Lemma it suffices to verify that $t_0 = \infty$. Assume the converse, i.e., $t_0 \in (0, \infty)$.

It follows from (71) that $\tilde{r}(t) = |\tilde{Z}(t)| \geq 2(R_{rad} \vee R_{tan}), t \in [0, t_0]$ and for any $t \in [0, t_0] :$
\[ \tilde{r}(t) \geq (\tilde{r}^{1-\beta}(0) + Kt)\frac{1}{1-\beta} \geq (R_0^{1-\beta} + Kt)\frac{1}{1-\beta}. \]

Since $\tilde{r}$ is continuous,
\[ \exists \varepsilon > 0 \forall t \in [t_0, t_0 + \varepsilon] : \tilde{r}(t) \geq (R_0^{1-\beta} + Kt)\frac{1}{1-\beta} - 1. \]

This and (72) imply
\[ \tilde{r}(t) \geq 2|\xi(t)|, \quad t \in [t_0, t_0 + \varepsilon], \]
and we get a contradiction with the definition of $t_0$.

This proves the Lemma. \qed
Lemma 7. Let $Z(0) = x$, $|x| > R_0$, where $R_0$ is from Lemma 6. Then there exists a limit

$$
\varphi_{Z_x}(+\infty) := \lim_{t \to \infty} \frac{Z_x(t)}{|Z(t)|}.
$$

Proof. It follows from Lemma 6 that $t^{1-\beta} = O(|Z(t)|) = O(|\tilde{Z}(t)|)$, $t \to \infty$. So

$$
|Z(t) - \tilde{Z}(t)| = |\xi(t)| = o(|Z(t)|) = o(|\tilde{Z}(t)|), \quad t \to \infty. \tag{74}
$$

Hence, it is sufficient to verify existence of the limit $\tilde{\varphi}(t) := \frac{\tilde{Z}(t)}{|\tilde{Z}(t)|}$ as $t \to \infty$. Moreover, this limit will coincide with $\lim_{t \to \infty} \frac{\tilde{Z}(t)}{|\tilde{Z}(t)|}$.

It follows from the proof of Lemma 6 that $\tilde{Z}(t) \neq 0$ for all $t \geq 0$.

Since $\tilde{\varphi}(t)$ is absolutely continuous function, in order to prove existence of $\lim_{t \to \infty} \tilde{\varphi}(t)$ it suffices to show that

$$
\int_1^\infty \left| \frac{d \tilde{\varphi}(t)}{dt} \right| dt < \infty. \tag{75}
$$

We have

$$
\frac{d \tilde{\varphi}(t)}{dt} = \frac{d}{dt} \left( \frac{\tilde{Z}(t)}{|Z(t)|} \right) = \frac{|\tilde{Z}(t)|^2 I_d - \tilde{Z}(t)\tilde{Z}(t)^T}{|Z(t)|^3}. \tag{76}
$$

where $I_d$ is $d \times d$ identity matrix.

So, (67), (74), (76), and Lemma 6 yield that for $t \geq 1$:

$$
\left| \frac{d \tilde{\varphi}(t)}{dt} \right| \leq \left| \frac{\tilde{Z}(t)|^2 I_d - \tilde{Z}(t)\tilde{Z}(t)^T}{|Z(t)|^3} \cdot \left( A_{rad}(Z(t)) + A_{tan}(Z(t)) \right) \right| \leq \tag{77}
$$

$$
\left| \frac{\tilde{Z}(t)|^2 I_d - \tilde{Z}(t)\tilde{Z}(t)^T}{|Z(t)|^3} \cdot a(Z(t))|Z(t)|^{\beta-1}Z(t) \right| + K_1 \left| \frac{A_{tan}(Z(t))}{|Z(t)|} \right| \leq \left| \frac{\tilde{Z}(t)|^2 I_d - \tilde{Z}(t)\tilde{Z}(t)^T}{|Z(t)|^3} \cdot a(Z(t))|Z(t)|^{\beta-1}(\tilde{Z}(t) + \xi(t)) \right| + K_2 \left| \frac{Z(t)|^{\beta-\gamma}}{|Z(t)|} \right|.
$$

Notice that for any $z \in \mathbb{R}^d$:

$$
(\|z|^2 I_d - zz^T)z = \|z|^2 z - zz^T z = \|z|^2 z - z|z|^2 = 0.
$$
So, the right hand side in (77) equals
\[ \left| \frac{\ddot{Z}(t)^2 - \dddot{Z}(t)}{|\dot{Z}(t)|^3} \cdot a(Z(t))|Z(t)|^{\beta-1}\xi(t) \right| + K_2 \frac{|Z(t)|^{\beta-\gamma}}{|\dot{Z}(t)|} \leq \]
\[ K_3 \left( \frac{1}{|\dot{Z}(t)|} |Z(t)|^{\beta-1}|\xi(t)| + \frac{|Z(t)|^{\beta-\gamma}}{|\dot{Z}(t)|} \right) \leq \]
\[ K_4 \left( \frac{|\xi(t)|}{|Z(t)|^{2-\beta}} + \frac{1}{|\dot{Z}(t)|^{1-\beta+\gamma}} \right) \leq \]
\[ K_5 \left( \frac{1}{t^{1-\beta}} + \frac{1}{t^{1-\beta+\gamma}} + \frac{1}{t^{1+\gamma}} \right). \tag{78} \]
This implies (75).

Lemma 7 is proved. \( \square \)

Let us complete the proof of Theorem 1. It follows from (74) that it suffices to verify that
\[ \dot{r}(t) \sim ((1-\beta)\bar{a}(\varphi_Z(+\infty))t)^{1-\beta} \quad \text{as } t \to \infty, \]
where \( \varphi_Z(+\infty) \) is from Lemma 7.

It follows from Lemmas 5, 7, equations (68), (74), and (22) that
\[ \dot{r}^{1-\beta}(t) = \dot{r}^{1-\beta}(0) + \int_0^t [...] ds, \]
where expression in brackets converges to \( (1-\beta)\bar{a}(\varphi_Z(+\infty)) \) as \( s \to \infty \). Therefore
\[ \dot{r}^{1-\beta}(t) \sim (1-\beta)\bar{a}(\varphi_Z(+\infty))t, \quad t \to \infty. \]

Theorem 1 is proved. \( \square \)

Let us now give some an example of coefficients that satisfy Assumption 2 of §4.

Consider a system of ODEs in \( \mathbb{R} \times \mathbb{R}^d \):
\begin{align*}
&dR(t) = a(R(t), \Phi(t))R^\beta(t)dt, \\
&d\Phi(t) = b(R(t), \Phi(t))R^{3+\delta-1}(t)dt,
\end{align*} \( \tag{79} \)
where \( \delta, \beta \in (0, 1), a : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}, b : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d \) are continuous functions.

If \( (R(0), \Phi(0)) \in [0, \infty) \times \mathbb{R}^d \), then we will denote a solution by \( (R_{r,\varphi}(t), \Phi_{r,\varphi}(t)) \).

The following general result implies Assumption 2 if we suppose that the vector field \( A \) is of the form
\[ A(x) = f(r, \varphi)r^\beta \varphi + g(r, \varphi)r^{\beta+\delta}, \]
where \( f \) is a real-valued, positive function, \( g \) is a vector-valued function, \( f, g \) are continuous in \( r, \varphi \) and Lipschitz in \( \varphi \), and \( \beta + \delta < 1 \).

**Theorem 4.** Assume that
1) the function \( a \) is positive, bounded and separated from zero
\[ \exists a_+ > 0 \quad \forall r \geq 0, \quad \varphi \in \mathbb{R}^d : \quad a_- \leq a(r, \varphi) \leq a_+; \tag{80} \]
2) the function $b$ is bounded and Lipschitz continuous in $\varphi$ uniformly in $r$:

$$\exists L \forall r \geq 0 \; \varphi_1, \varphi_2 \in \mathbb{R}^d : |b(r, \varphi_1) - b(r, \varphi_2)| \leq L|\varphi_1 - \varphi_2|.$$ 

Then for any initial starting point $(r, \varphi) \in (0, \infty) \times \mathbb{R}^d$ there is a unique solution $(R_{r,\varphi}(t), \Phi_{r,\varphi}(t))$ to equation (79). Moreover, the map

$$[0, \infty) \times (0, \infty) \times \mathbb{R}^d \ni (t, r, \varphi) \rightarrow (R_{r,\varphi}(t), \Phi_{r,\varphi}(t))$$

can be extended by continuity to $[0, \infty) \times [0, \infty) \times \mathbb{R}^d$, and its extension $(R_{0,\varphi}(t), \Phi_{0,\varphi}(t))$ is a unique solution to (79) among all solutions satisfying $R_{0,\varphi}(t) > 0$ for $t > 0$.

**Proof.** Existence of solutions for $r > 0, \varphi \in \mathbb{R}^d$ follows from the Peano theorem. It follows from (80) and the comparison with the equation $d\rho(t) = a(\pm \rho(t))dt$ that

$$(a_-(1-\beta)t)^{\frac{1}{1-\beta}} \leq R_{r,\varphi}(t) \leq (r^{1-\beta} + a_+(1-\beta)t)^{\frac{1}{1-\beta}}$$

(81)

for any $r > 0, \varphi \in \mathbb{R}^d, t \geq 0$. Moreover, any solution $R_{0,\varphi}(t)$ such that $R_{0,\varphi}(t) > 0, t > 0$ also satisfies (81).

It follows from the compactness arguments that there is a sequence $\{r_n\}, \lim_{n \to \infty} r_n = 0$ such that the sequence $\{(R_{r_n,\varphi}, \Phi_{r_n,\varphi})\}$ converges uniformly on compact sets. It easy to see that its limit $(R_{0,\varphi}(t), \Phi_{0,\varphi}(t))$ is a solution to (79) and satisfies (81) with $r = 0$.

Let us prove uniqueness. Apply transformation of time arguments, e.g. [29]. Set

$$A_{r,\varphi}(t) := \int_0^t a(R_{r,\varphi}(z), \Phi_{r,\varphi}(z))R_{r,\varphi}^2(z)dz,$$

$$\hat{R}_{r,\varphi}(t) := R_{r,\varphi}(A_{r,\varphi}^{-1}(t)),$$

$$\hat{\Phi}_{r,\varphi}(t) := \Phi_{r,\varphi}(A_{r,\varphi}^{-1}(t)),$$

where $A_{r,\varphi}^{-1}$ is the inverse function. The function $A_{r,\varphi}^{-1}$ is well defined because $A_{r,\varphi}$ is continuous and increasing function.

Then

$$d\hat{R}_{r,\varphi}(t) = dt,$$

$$d\hat{\Phi}_{r,\varphi}(t) = \frac{b(\hat{R}_{r,\varphi}(t), \hat{\Phi}_{r,\varphi}(t))\hat{R}_{r,\varphi}^{\beta+\delta-1}(t)}{a(\hat{R}_{r,\varphi}(t), \hat{\Phi}_{r,\varphi}(t))\hat{R}_{r,\varphi}(t)}dt = \frac{b(\hat{R}_{r,\varphi}(t), \hat{\Phi}_{r,\varphi}(t))}{a(\hat{R}_{r,\varphi}(t), \hat{\Phi}_{r,\varphi}(t))}\hat{R}_{r,\varphi}^{\delta-1}(t)dt.$$ 

Hence

$$\hat{R}_{r,\varphi}(t) = r + t,$$

and

$$\hat{\Phi}_{r,\varphi}(t) = r + \int_0^t \frac{b(r+z, \hat{\Phi}_{r,\varphi}(z))}{a(r+z, \hat{\Phi}_{r,\varphi}(z))}(r+z)^{\delta-1}dz.$$ 

(82)

It follows from the assumption that the function $(r, t, \varphi) \rightarrow \frac{b(r+z, \varphi)}{a(r+z, \varphi)}$ is bounded and Lipschitz in $\varphi$ uniformly in $r, t$. Therefore, the local integrability of $t \rightarrow (r+t)^{\delta-1}$ yields existence and uniqueness of a solution to (82) for $r \geq 0, \varphi \in \mathbb{R}^d$, and continuity of $\hat{\Phi}_{r,\varphi}(t)$ in $(r, \varphi, t) \in [0, \infty) \times [0, \infty) \times \mathbb{R}^d$. 

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To get \((R(t), \Phi(t))\) we have to make the inverse transformation of time. Set
\[ \tilde{A}_{r, \phi}(t) := \int_0^t a^{-1}(\tilde{R}_{r, \phi}(z), \tilde{\Phi}_{r, \phi}(z)) \tilde{R}_{r, \phi}^{-\beta}(z) dz. \]

Notice that \(\tilde{A}_{r, \phi}(t)\) is well defined and increasing in \(t\).

The functions
\[ R_{r, \phi}(t) = \tilde{R}_{r, \phi}(\tilde{A}_{r, \phi}^{-1}(t)), \]
\[ \Phi_{r, \phi}(t) = \tilde{\Phi}_{r, \phi}(\tilde{A}_{r, \phi}^{-1}(t)) \]
satisfy (79). Uniqueness follows from uniqueness for (82) and the fact that the correspondence \((R_{r, \phi}, \phi_{r, \phi}) \leftrightarrow (\tilde{R}_{r, \phi}, \tilde{\Phi}_{r, \phi})\) is one-to-one.

The function \(A_{r, \phi}(t)\) is continuous in \((r, \phi, t) \in [0, \infty) \times [0, \infty) \times \mathbb{R}^d\) due to the Lebesgue dominated convergence theorem and continuity of \(\Phi_{r, \phi}(t)\). So, the inverse function \(A_{r, \phi}^{-1}(t)\) is also continuous in \((r, \phi, t)\).

This and continuity of \((\tilde{R}_{r, \phi}, \tilde{\Phi}_{r, \phi})\) in \((r, \phi, t)\) yields the continuity of \((R_{r, \phi}, \Phi_{r, \phi})\).

\[\square\]

Proof of Lemma [7] It is sufficient to check the following condition on modulus of continuity, see [7]:
\[
\forall T > 0 \quad \forall \mu > 0 \quad \exists \delta > 0 \quad \limsup_{\varepsilon \to 0} P\left( \exists s, t \in [0, T], |s - t| \leq \delta : |X_\varepsilon(s) - X_\varepsilon(t)| \geq \mu \right) < \mu.
\]

Since the vector field \(A\) is of linear growth at the infinity, we have for any \(T > 0\):
\[
\lim_{M \to +\infty} \sup_{\varepsilon \in (0,1]} P\left( \sup_{t \in [0,T]} |X_\varepsilon(t)| \geq M \right) = 0.
\]

Denote
\[ K(\mu, M) := \sup_{\mu \leq |x| \leq M} |A(x)|. \]

We have
\[
P\left( \exists s, t \in [0, T], |s - t| \leq \delta : |X_\varepsilon(s) - X_\varepsilon(t)| \geq \mu \right) \leq \]
\[
P\left( \exists s, t \in [0, T], |s - t| \leq \delta : \left| \int_s^t A(X_\varepsilon(z)) dz \right| \geq \mu/4 \right. \text{ and } \mu/2 \leq |X_\varepsilon(z)| \leq M, z \in [s, t] \bigg) + \]
\[
P\left( \varepsilon \sup_{s \in [0,T]} |B_\alpha(s)| \geq \mu/4 \right) + \sup_{\varepsilon \in (0,1]} P\left( \sup_{t \in [0,T]} |X_\varepsilon(t)| \geq M \right) \leq \]
\[
P\left( \delta K(\mu/2, M) \geq \mu/4 \right) + P\left( \sup_{s \in [0,T]} |B_\alpha(s)| \geq \mu/4 \varepsilon \right) + \sup_{\varepsilon \in (0,1]} P\left( \sup_{t \in [0,T]} |X_\varepsilon(t)| \geq M \right) = \]
\[
P\left( \sup_{s \in [0,T]} |B_\alpha(s)| \geq \mu/4 \varepsilon \right) + \sup_{\varepsilon \in (0,1]} P\left( \sup_{t \in [0,T]} |X_\varepsilon(t)| \geq M \right) \tag{83}
\]
if \(\delta < \frac{\mu}{4K(\mu/2, M)}\).

The right hand side of (83) can be done arbitrary small if \(M\) is large and \(\varepsilon\) is small for any fixed \(\mu\). This proves the Lemma. \[\square\]
Proof of Lemma 2. Introduce the following notations (see (89))
\[ \tilde{x} := x \varepsilon^{\frac{\alpha}{n+\beta-1}}, \quad \tilde{X}(t) := \tilde{X}_\varepsilon(t) := \varepsilon^{\frac{\alpha}{n+\beta-1}} X_{\varepsilon,x} \left( \varepsilon^{\frac{\alpha}{n+\beta-1}} t \right), \quad B_\alpha^{(e)}(t) := \varepsilon^{\frac{\alpha}{n+\beta-1}} B_\alpha \left( \varepsilon^{\frac{\alpha}{n+\beta-1}} t \right) \]
Then \( \tilde{X}(t) \) satisfies (10).

It follows from (11) that
\[ \frac{X_{\varepsilon,x}(\tau_{\delta}^{e,x})}{|X_{\varepsilon,x}(\tau_{\delta}^{e,x})|} = \frac{\tilde{X}_{\varepsilon,\tilde{x}}(\tilde{\tau}_{\varepsilon,\tilde{x}}^{e,\tilde{x}})}{|\tilde{X}_{\varepsilon,\tilde{x}}(\tilde{\tau}_{\varepsilon,\tilde{x}}^{e,\tilde{x}})|}; \quad (84) \]
\[ \frac{|X_{\varepsilon,x}(\tau_{\delta}^{e,x})|}{\delta} = \frac{|\tilde{X}_{\varepsilon,\tilde{x}}(\tilde{\tau}_{\varepsilon,\tilde{x}}^{e,\tilde{x}})|}{\varepsilon^{\frac{\alpha}{n+\beta-1} \delta}}, \quad \tau_{\delta}^{e,x} = \frac{\varepsilon^{\frac{\alpha}{n+\beta-1}} \tilde{\tau}_{\varepsilon,\tilde{x}}^{e,\tilde{x}}}{\varepsilon^{\frac{\alpha}{n+\beta-1} \delta}}; \quad (85) \]
Further the arguments are similar to the proof of Theorem 1. Set
\[ \hat{X}_{\varepsilon,\tilde{x}}(t) := \tilde{X}_{\varepsilon,\tilde{x}}(t) - B_\alpha^{(e)}(t), \quad \hat{\tau}_t := |\hat{X}_{\varepsilon,\tilde{x}}(t)|. \]
Assume that \( \omega \) is such that \( |\tilde{X}_{\varepsilon,\tilde{x}}(t)| \leq \hat{\tau}_t/2, t \in [0, \varepsilon^{\frac{\alpha}{n+\beta-1}} \delta], \) cf. (89).

If \( R_0 \) is sufficiently large, \( \delta_2 \) is sufficiently small and \( R \geq R_0, \delta \in (0, \delta_2) \), then applying (31), (11), (12), and (68) we get for this \( \omega \):
\[ (\hat{\tau}(t))^{1-\beta} \geq K(\delta_2) t, t \in [0, \varepsilon^{\frac{\alpha}{n+\beta-1}} \delta], \]
and moreover
\[ \varepsilon^{\frac{\alpha}{n+\beta-1} \delta} \leq \frac{\varepsilon^{ \frac{\alpha}{(1-\beta)\alpha+\beta-1} \delta } \varepsilon^{\frac{\alpha}{(1-\beta)\alpha+\beta-1} \delta \delta^{\frac{1}{\beta}}}}{K}; \quad (86) \]
We get (54), (56) if we use (30), (85), and reasoning of Lemma 1.
To prove (54) we have to apply (31), (42), (84), (68), and the arguments of Lemma 7 (see (77), (78)).

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