1. Introduction

1.1. Stable propagation. Let $M := U^* \mathbb{T}^n$ be the open unit cotangent bundle of the $n$-dimensional Euclidean torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. We express the elements of $M$ in terms of the canonical coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$, where $q_i \equiv q_i + 1$. Thus we identify $M$ with $\mathbb{T}^n \times \mathbb{D}^n$, where $\mathbb{D}^n = \{|p| < 1\}$ is the open unit ball in $\mathbb{R}^n$ and $|v|$ stands for the Euclidean norm of a vector $v \in \mathbb{R}^n$. Consider the space $\mathcal{H}$ of all smooth compactly supported functions on $[0, 1] \times M$. Every function $H \in \mathcal{H}$ gives rise to the Hamiltonian system on $M$

$$\dot{q} = \frac{\partial H}{\partial p}(t, q, p), \quad \dot{p} = -\frac{\partial H}{\partial q}(t, q, p).$$

The flow $h_t$ which sends any initial condition $x(0) = (q(0), p(0))$ to the solution $x(t) = (q(t), p(t))$ at time $t$ is called the Hamiltonian flow (or isotopy) generated by $H$, and the time-one-map $h_1$ is called the Hamiltonian diffeomorphism generated by $H$. The set of all Hamiltonian diffeomorphisms of $M$ form a group denoted by $\mathcal{D}$.

Let $f^* = \{f_k\}_{k=1,2,\ldots}$ be an infinite sequence of Hamiltonian diffeomorphisms. We consider $f^*$ as a dynamical system as follows. Put $f^{(k)} = f_k \cdots f_1$. This sequence is called the evolution of $f^*$. The orbit $\{x_k\}_{k \in \mathbb{N}}$ of a point $x \in M$ is defined by $x_k = f^{(k)} x$. Classical dynamical systems (iterations of a single map $f$) correspond to constant sequences $f_k \equiv f$. Sequential systems arise naturally as perturbations of the classical ones. In a number of interesting situations stability has been observed in the sense that these perturbations inherit dynamical properties of the original system (cf. [PR]). In this paper we present a new stability phenomenon of this type – stable propagating behaviour – which we are going to describe next.

Every Hamiltonian diffeomorphism $h$ has a canonical lift $\tilde{h}$ to the universal cover $\widetilde{M} = \mathbb{R}^n \times \mathbb{D}^n$ of $M$. Throughout we denote

$$Q := \left\{(q, p) \in \widetilde{M} \mid |p| < 1, \max_i |q_i| \leq 1/2\right\};$$

this is a fundamental domain of the covering.

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Definition. A sequential system $f_*$ propagates to infinity with speed (at least) $c$ if for every vector $v \in \mathbb{R}^n$ with $|v| \leq c$ there exists a sequence of points $\tilde{x}_k \in Q$ such that the sequence $(p_k, q_k) := \tilde{f}^{(k)}(\tilde{x}_k)$ satisfies

$$\lim_{k \to \infty} \frac{q_k}{k} = v.$$ 

In other words, the projection to $\mathbb{R}^n$ of $\tilde{f}^{(k)}(Q)$ covers the Euclidean ball of radius $kc$ up to an error which is small with respect to $k$.

Let us illustrate this notion in the following simple example. Consider a Hamiltonian function $H \in \mathcal{H}$ which depends on momenta variables only: $H = H(p)$. The lift of the corresponding Hamiltonian flow to the universal cover is given by $\tilde{h}^t(q, p) = (q + t\nabla H(p), p)$. Therefore the projection of $\tilde{h}^t(Q)$ to $\mathbb{R}^n$ coincides up to a bounded error with the set $I$, where $I$ denotes the image of the gradient map $p \mapsto \nabla H(p)$. Assume now that $H(0) > c$ for some $c > 0$. We claim that $I$ contains the Euclidean ball of radius $c$ centered at zero. Indeed, for every $v \in \mathbb{R}^n$ with $|v| \leq c$ the function $F(p) := H(p) - pv$ satisfies $F(0) > c$ and $F \leq c$ near $\partial \mathbb{D}^n = S^{n-1}$. Therefore $F$ attains its maximum at a point $p_0 \in \mathbb{D}^n$. Hence $\nabla H(p_0) = v$ and the claim follows. This shows that in our example we have propagation with speed $c$.

The group $\mathcal{D}$ carries a remarkable biinvariant metric $\rho$ called Hofer’s metric (see [H]). The corresponding geometry provides us with a suitable language for the study of various stable phenomena in Hamiltonian dynamics. Given a diffeomorphism $f \in \mathcal{D}$, write $\rho(id, f)$ for $\inf \max F - \min F$ where the infimum is taken over all Hamiltonians $F \in \mathcal{H}$ generating $f$. Define Hofer’s distance $\rho(f, g)$ between two elements $f, g \in \mathcal{D}$ as $\rho(id, fg^{-1})$.

**Theorem A (Stable propagation).** Let $0 < a < c$ and suppose that $h$ is a Hamiltonian diffeomorphism generated by a Hamiltonian $H \in \mathcal{H}$ such that $H(t, q, 0) \geq c$ for all $t$ and $q$. Let $f_*$ be any sequential system such that $\rho(f_*, h) < a$ for all $i \in \mathbb{N}$. Then $f_*$ propagates to infinity with speed $c - a$.

Theorem A will be proved in Section 2.3 below. Interestingly enough, such a propagating behaviour may be completely destroyed by an appropriate arbitrarily $C^\infty$-small dissipative perturbation even in the framework of classical dynamics. In Section 2.4 below we elaborate this in the case $n = 1$. We show that every Hamiltonian diffeomorphism $h$ generated by the Hamiltonian $H = H(p)$ admits an arbitrarily small smooth perturbation $f$ such that the images $\tilde{f}^k(Q)$, $k \in \mathbb{N}$, of the set $Q$ under the iterates of $\tilde{f}$ remain in a compact part of $\tilde{M}$.

Note that Theorem A does not provide any information about propagation of individual trajectories on the universal cover (and we doubt that such information is available at
all in this generality). The situation improves when one considers sequences \( f_* = \{f_i\} \) which roughly speaking are uniformly distributed with respect to some “nice” measure on \( D \) whose support is close to \( h \) in the sense of Hofer’s metric. It turns out that such sequential systems have trajectories which propagate to infinity with constant velocity. We refer the reader to Section 2.3 for the details.

1.2. Noncontractible closed orbits. The main tool for studying stable propagation as described in the previous section is an existence result for noncontractible periodic solutions of compactly supported Hamiltonian systems under quite robust assumptions on the Hamiltonian functions. This result is based on Floer homology filtered by the symplectic action and its proof is quite involved. A solution \( x(t) = (q(t), p(t)) \) of a Hamiltonian system generated by a function \( H \in \mathcal{H} \) is called periodic if \( p(1) = p(0) \) and \( q(1) = q(0) + e \) for some integer vector \( e \in \mathbb{Z}^n \). The lattice \( \mathbb{Z}^n \) is identified in a natural way with the fundamental group of \( M \), hence we refer to \( e \) as the homotopy class of the solution. An important quantity associated to a periodic solution is its action

\[
A_H(x) = \int_0^1 \left( H(t, q(t), p(t)) - \sum_{i=1}^n p_i(t) \dot{q}_i(t) \right) dt.
\]

Denote by \( Z \subset M \) the zero section \( \{p = 0\} \).

**Theorem B.** For every compactly supported smooth Hamiltonian function \( H \in \mathcal{H} \) and every \( e \in \mathbb{Z}^n \) such that

\[
|e| \leq c := \inf_{[0,1] \times \mathbb{Z}} H,
\]

the Hamiltonian system \( [4] \) has a periodic solution \( x(t) \) in the homotopy class \( e \) with action \( A_H(x) \geq c \).

The result above is sharp in the following sense. Firstly, the inequalities in Theorem B which guarantee the existence of periodic solutions in the class \( e \) cannot be improved. For instance, for every \( \varepsilon > 0 \) it is easy to produce a Hamiltonian of the form \( H = H(|p|) \) whose restriction to \([0,1] \times Z\) equals \( 1 - \varepsilon \) and such that all periodic solutions are contractible (such a function \( H(|p|) \) can be obtained from \( 1 - |p| \) by an appropriate smoothing). Secondly, the zero section \( Z \) in the inequality \( [2] \) cannot be replaced by an arbitrary smooth section. Consider, for instance an arbitrary \( C^\infty \)-small perturbation \( S = \{p = u(q)\} \) of \( Z \) such that the 1-form \( u(q) dq \) on \( \mathbb{T}^n \) is not closed (in symplectic terms this means that the section \( S \) is non-Lagrangian; this is only possible when \( n \geq 2 \)).
Theorem C. Given any $c > 0$, there exists a Hamiltonian function $H \in \mathcal{H}$ such that $H(t,x) \geq c$ for every $t \in [0,1]$ and every $x \in S$ and such that every 1-periodic solution of (2) is contractible.

If one takes $S$ as the graph of an exact 1-form on $\mathbb{T}^n$, the assertion of Theorem B remains valid with $Z$ replaced by $S$. Thus the existence mechanism for noncontractible periodic solutions described in Theorem B is quite sensitive to the choice of a subset where the Hamiltonian is large enough. This phenomenon will be studied below in terms of a relative symplectic capacity (see Section 3).

On the other hand, the statement of Theorem B is robust from the following viewpoint: if the restriction of a Hamiltonian $H$ to $[0,1] \times Z$ is bigger than a certain positive number, (most of) the periodic solutions guaranteed by the theorem persist under $C^0$-perturbations of the Hamiltonian. This robustness plays a crucial role in the study of stable propagation.

In Section 3 below we prove Theorem B and its generalization where the torus is replaced by a hyperbolic manifold. The proof uses Floer homology for the action functional on the space of noncontractible loops. The main difficulty we have to go round is as follows. Since we deal with compactly supported Hamiltonians, noncontractible solutions are “non-essential” from the viewpoint of Floer homology – they may not persist under deformations of the Hamiltonian. In brief, the idea of the proof can be described as follows. One can squeeze the Hamiltonian function $H$ between two more or less standard functions, $H_- \leq H \leq H_+$ where $H_-$ and $H_+$ depend only on $|p|$. The filtered Floer homologies of $H_-$ and $H_+$ as well as the natural morphism between them can be computed explicitly. This morphism turns out to be nontrivial, and since it factors through the Floer homology of $H$, we get nontrivial information about the noncontractible periodic solutions corresponding to $H$. The calculations are quite involved, hence we introduce a convenient algebraic tool – relative symplectic homology – which helps us perform them in a more organized way.

Noncontractible orbits of Hamiltonian systems were studied earlier in a number of interesting situations. In a beautiful paper [GL] Gatien and Lalonde considered the following setting. Let $L_0$ and $L_1$ be two disjoint closed Lagrangian submanifolds of a symplectic manifold. Assume that $H$ is an autonomous Hamiltonian function which is “small” on $L_0$ and “large” on $L_1$. It turns out that under certain additional assumptions of a topological nature one can prove the existence of noncontractible periodic orbits for the Hamiltonian flow generated by $H$. This result was the starting point for Theorem B. Another important idea of a symplectic capacity which feels the fundamental group is contained in Schwarz’s work [Sc]. We will develop it further in Section 3 below. Noncontractible orbits of autonomous Hamiltonians on cotangent bundles whose levels are starshaped were considered by a number of authors (see e.g. [C-1]). In contrast to our case, these closed orbits
are homologically essential in the sense of Floer homology. Let us mention finally that the interest in noncontractible periodic orbits on cotangent bundles comes from classical mechanics where one considers Hamiltonians \( H(t, q, p) \) which are convex with respect to the momenta variable \( p \). In this case the existence of closed orbits in given homotopy classes can be derived with methods from the classical calculus of variations.

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2. **Stable propagation in sequential Hamiltonian dynamics**

In this section we study stable propagation along the lines mentioned in the Introduction. The main tool is Theorem B.

2.1. **Preliminaries on Hamiltonian diffeomorphisms.** Let \( M \) be an open manifold (i.e. \( M \) is connected, noncompact and has no boundary) and denote by \( \pi : \widetilde{M} \to M \) its universal cover. Let \( \text{Diff}_0(M) \) denote the group of compactly supported diffeomorphisms of \( M \) that are isotopic to the identity by isotopies with compact support in \([0, 1] \times M\). Then every \( h \in \text{Diff}_0(M) \) has a canonical lift \( \widetilde{h} : \widetilde{M} \to \widetilde{M} \) to the universal cover. To see this, choose a compactly supported isotopy \([0, 1] \to \text{Diff}_0(M) : t \mapsto h_t\) from \( h_0 = \text{id} \) to \( h_1 = h \). Given \( \widetilde{x}_0 \in \widetilde{M} \), lift the path \([0, 1] \to M : t \mapsto h_t(\pi(\widetilde{x}_0))\) to a path \([0, 1] \to \widetilde{M} : t \mapsto \tilde{x}(t)\) such that \( \tilde{x}(0) = \widetilde{x}_0 \) and define \( \widetilde{h}(\widetilde{x}_0) := \tilde{x}(1) \). The following remark shows that this definition is independent of the choice of the isotopy \( t \mapsto h_t \).

**Remark 2.1.1.** Let \( t \mapsto \varphi_t \) and \( t \mapsto \psi_t \) be two compactly supported isotopies such that \( \varphi_1 = \psi_1 = h \in \text{Diff}_0(M) \). Given a point \( x \in M \), consider the paths \( t \mapsto \varphi_t(x) \) and \( t \mapsto \psi_t(x) \) connecting \( x \) to \( h(x) \). We claim that they are homotopic with fixed endpoints. To see this choose a path \([0, 1] \to M : s \mapsto \gamma(s) \) such that \( \gamma(0) = x \) and \( \gamma(1) \) lies outside the support of the isotopies. Looking at \( \varphi_t(\gamma(s)) \) and \( \psi_t(\gamma(s)) \) it is easy to see that both paths \( t \mapsto \varphi_t(x) \) and \( t \mapsto \psi_t(x) \) are homotopic with fixed endpoints to the path \( \Gamma \) obtained by going first from \( x \) to \( \gamma(1) \) along \( \gamma \) and then from \( \gamma(1) \) to \( h(x) \) along \( h(\gamma^{-1}) \). Here \( \gamma^{-1} \) stands for the reverse of \( \gamma \).

Now suppose that \( M \) is equipped with a symplectic form \( \omega \) and denote by \( \mathcal{D} \subset \text{Diff}_0(M) \) the group of Hamiltonian diffeomorphisms that are generated by compactly supported Hamiltonian functions on \([0, 1] \times M\). Given a compact subset \( A \subset M \) and a real number \( c \) let us denote by \( \mathcal{D}_c = \mathcal{D}_c(M, A) \subset \mathcal{D} \) the subset of Hamiltonian diffeomorphisms that are generated by compactly supported Hamiltonian functions \( H \in C_0^\infty([0, 1] \times M) \) with \( \inf_{[0,1] \times A} H \geq c \). As before denote by \( \rho \) the Hofer metric on \( \mathcal{D} \).
Proposition 2.1.2. Let \( c > a \) be positive numbers, and let \( f, g \in D \) be Hamiltonian diffeomorphisms with \( f \in D_c \) and \( \rho(f, g) < a \). Then \( g \in D_{c-a} \).

Proof. This is an immediate consequence of the following product formula. If the functions \( \Phi_t \) and \( \Psi_t \) generate Hamiltonian flows \( \varphi_t \) and \( \psi_t \), respectively, then the product flow \( \varphi_t \circ \psi_t \) is generated by the Hamiltonian \( \Phi_t + \Psi_t \circ \varphi_t^{-1} \). \( \square \)

In what follows we shall deal also with time periodic Hamiltonians, namely functions \( H \) satisfying \( H(t, \cdot) = H(t + 1, \cdot) \). It is useful to think of these Hamiltonians as smooth functions \( H : S^1 \times M \to \mathbb{R} \), where we identify \( S^1 \cong \mathbb{R}/\mathbb{Z} \). The following proposition shows that every \( h \in D_c \) can be generated by a periodic Hamiltonian bounded below by \( c \) on \( S^1 \times A \).

Proposition 2.1.3. (Periodic Hamiltonians) Let \( M \) be an open symplectic manifold and \( A \subset M \) be a compact subset. Let \( h \) be a Hamiltonian diffeomorphism of \( M \) generated by a Hamiltonian \( H \in C^\infty_{[0,1]}(S^1 \times M) \) with \( \inf_{S^1 \times A} H \geq c \). Then there exists a Hamiltonian \( \overline{H} \in C^\infty(S^1 \times M) \) with \( \inf_{S^1 \times A} \overline{H} \geq c \) that generates \( h \).

Proof. Let \( h_t \) denote the Hamiltonian isotopy generated by \( H \) and \( f_t \) denote the Hamiltonian flow generated by a time independent compactly supported function \( F : M \to \mathbb{R} \). Let \( \tau : [0, 1] \to [0, 1] \) be a smooth nondecreasing function which equals 0 near \( t = 0 \) and equals 1 near \( t = 1 \). Then the Hamiltonian isotopy

\[
\overline{h}_t := f_{t-\tau(t)} \circ h_{\tau(t)}
\]

is generated by the Hamiltonian functions

\[
\overline{H}_t := F + \tau'(t)(H_{\tau(t)} - F) \circ f_{\tau(t)-t}.
\]

The function \( \overline{H}_t \) equals \( F \) near \( t = 0 \) and \( t = 1 \) and hence defines a smooth Hamiltonian on \( S^1 \times M \). Moreover, \( \overline{h}_t = h_1 = h \). If \( F \) is chosen to be equal to \( c \) in a neighbourhood of \( A \), then \( f_t \) is equal to the identity on \( A \) and \( H_{\tau(t)} - F \) is nonnegative on \( A \), and hence \( \inf_A \overline{H}_t \geq c \) for every \( t \). \( \square \)

In the remainder of this section we assume that \( M = U^* \mathbb{T}^n \) is the open unit cotangent bundle and \( A = Z = \mathbb{T}^n \subset U^* \mathbb{T}^n \) is the zero section. Given \( e \in \mathbb{Z}^n \), we denote by \( T_e \) the deck transformation \((q, p) \to (q + e, p)\) of the covering. Note that 1-periodic solutions of the Hamiltonian system associated to \( H \) are in one-to-one correspondence with the fixed points of \( h \). The homotopy class of the periodic solution \( x(t) \) corresponding to the fixed point \( x \in M \) (or, in brief, the homotopy class of the fixed point \( x \)) can be determined as follows. Pick a lift \( y \) of \( x \). Then \( \tilde{h}(y) = T_{-e}(y) \) for some \( e \in \mathbb{Z}^n \) and this \( e \) is the homotopy class in question. In this terminology Theorem B asserts that every
diffeomorphism \( h \in \mathcal{D}_c = \mathcal{D}_c(U^* \mathbb{T}^n, \mathbb{T}^n) \) has a fixed point in the homotopy classes \( e \) for every \( e \) such that \( |e| \leq c \). It turns out that the same result allows us to get information on the fixed points of the iterates \( h^k \) for \( k \in \mathbb{N} \).

**Proposition 2.1.4.** If \( h \in \mathcal{D}_c \) then \( h^k \in \mathcal{D}_{kc} \) for every \( c > 0 \) and every \( k \in \mathbb{N} \). In particular, \( h^k \) has a fixed point in every homotopy class \( e \in \mathbb{Z}^n \) such that \( |e| \leq kc \).

**Proof.** We have seen in Proposition 2.1.3 that, for every \( h \in \mathcal{D}_c \), one can find a function \( H : \mathbb{R} \times M \to \mathbb{R} \) which is 1-periodic in time, satisfies \( \inf_{\mathbb{R} \times A} H \geq c \), and generates \( h \) as the time-1-map. It follows that the \( k \)th iterate \( h^k \) is generated by \( kH(kt, x) \), hence \( h^k \in \mathcal{D}_{kc} \), and hence the result follows from Theorem B.

2.2. **Preliminaries from ergodic theory.** Let \( X \) be a compact metrizable topological space and \( f : X \to X \) be a homeomorphism. Let \( \mathcal{M} \) denote the set of Borel probability measures on \( X \) and \( \mathcal{M}(f) \subset \mathcal{M} \) denote the subset of \( f \)-invariant Borel probability measures. Both sets are convex and compact with respect to the weak-\(*\) topology (\( \mu_i \to \mu \) iff \( \int u \, d\mu_i \to \int u \, d\mu \) for every continuous function \( u : X \to \mathbb{R} \)). An \( f \)-invariant measure \( \mu \in \mathcal{M}(f) \) is called **ergodic** if every Borel set \( \Lambda \subset X \) such that \( f(\Lambda) = \Lambda \) has measure \( \mu(\Lambda) \in \{0, 1\} \).

**Theorem 2.2.1.** (Birkhoff’s Ergodic Theorem \([CFS]\)) If \( \mu \in \mathcal{M}(f) \) is an ergodic \( f \)-invariant Borel probability measure and \( u \in C^0(X) \), then there exists a Borel set \( Y \subset X \) such that \( \mu(Y) = 1 \) and

\[
\int u \, d\mu = \lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} u(f^k(x))
\]

for every \( x \in Y \).

**Remark 2.2.2.** Let \( X \) be a Banach space and \( \mathcal{K} \subset X^* \) be a weak-\(*\) compact convex subset of its dual space. A point \( x^* \in \mathcal{K} \) is called **extremal** if, for all \( y^*, z^* \in \mathcal{K} \) and all real numbers \( t \),

\[
x^* = (1-t)y^* + tz^*, \quad 0 < t < 1 \quad \Rightarrow \quad y^* = z^*.
\]

The Krein–Milman theorem \([R]\, p. 242\) asserts that every weak-\(*\) compact convex subset of \( X^* \) is equal to the weak-\(*\) closure of the convex hull of its extremal points. In particular, every compact convex subset \( \mathcal{K} \subset \mathbb{R}^n \) with nonempty interior has at least \( n + 1 \) extremal points. Namely, if there were less than or equal to \( n \) extremal points then \( \mathcal{K} \) would be contained in an affine subspace of dimension less than \( n \) and so have empty interior.
Remark 2.2.3. Every extremal point of $\mathcal{M}(f)$ is ergodic. To see this suppose that $\mu \in \mathcal{M}(f)$ is not ergodic. Then there exists an invariant Borel set $\Lambda = f(\Lambda)$ such that $0 < \mu(\Lambda) < 1$. Let $t := \mu(\Lambda)$. Then $\mu = t\mu_\Lambda + (1-t)\mu_{X\setminus\Lambda}$ where $\mu_\Lambda \in \mathcal{M}(f)$ is defined by $\mu_\Lambda(A) := \mu(\Lambda \cap A) / \mu(\Lambda)$ and similarly for $\mu_{X\setminus\Lambda}$. Hence $\mu$ is not an extremal point of $\mathcal{M}(f)$.

Remark 2.2.4. A homeomorphism $f : X \to X$ is called \textbf{uniquely ergodic} if $\mathcal{M}(f)$ consists of a single point, i.e. there exists a unique $f$-invariant Borel probability measure $\mu$ on $X$. In this case, by Remark 2.2.3, the invariant measure $\mu$ is necessarily ergodic.

Example 2.2.5. Let $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$ be a vector with rationally independent components. Then the homeomorphism $\varphi : \mathbb{T}^d \to \mathbb{T}^d$ induced by the map $\mathbb{R}^d \to \mathbb{R}^d : y \mapsto y + \alpha$ is uniquely ergodic and the unique $\varphi$-invariant measure is the Lebesgue measure (cf. [CFS]).

Lemma 2.2.6. Let $f : X \to X$ be a homeomorphism of a compact metric space and $u : X \to \mathbb{R}^n$ be a continuous function. Consider the linear projection $\mathcal{M}(f) \to \mathbb{R}^n : \mu \mapsto R(\mu) := \int u \, d\mu$. Let $r$ be an extremal point of the compact convex set $K := R(\mathcal{M}(f)) \subset \mathbb{R}^n$. Then there exists an ergodic $f$-invariant Borel probability measure $\mu \in \mathcal{M}(f)$ such that $r = \int u \, d\mu$.

Proof. The set $\mathcal{M}_r(f) := \{ \mu \in \mathcal{M}(f) \mid R(\mu) = r \}$ is a nonempty weak-* compact convex subset of the dual space of $C^0(X)$. Hence, by the Krein–Milman theorem, it has an extremal point $\mu$ (Remark 2.2.3). We prove that $\mu$ is an extremal point of $\mathcal{M}(f)$. To see this, suppose that $\mu = (1-t)\mu_0 + t\mu_1$ such that $\mu_0, \mu_1 \in \mathcal{M}(f)$ and $0 < t < 1$. We must prove that $\mu_0 = \mu_1$. To see this, let $r_0 := R(\mu_0)$ and $r_1 := R(\mu_1)$. Then $r = (1-t)r_0 + tr_1$ and $r_0, r_1 \in K$. Since $r$ is an extremal point of $K$ it follows that $r_0 = r_1 = r$ and hence $\mu_0, \mu_1 \in \mathcal{M}_r(f)$. Since $\mu$ is an extremal point of $\mathcal{M}_r(f)$ this implies that $\mu_0 = \mu_1$. Thus we have proved that $\mu$ is an extremal point of $\mathcal{M}(f)$ and hence, by Remark 2.2.3, it is an ergodic $f$-invariant Borel probability measure.

2.3. Stable propagation revisited. Let us now return to the case where $M = U^*\mathbb{T}^n$ is the open unit cotangent bundle of the $n$-torus and $\mathcal{D}$ is the group of compactly supported Hamiltonian diffeomorphisms of $M$. For $a > 0$ and $h \in \mathcal{D}$ denote by

$$B(h, a) := \{ f \in \mathcal{D} \mid \rho(f, h) < a \}$$

the open ball of radius $a$ in Hofer’s metric. Recall that Theorem A states the following. Let $c > a > 0$ be real numbers, $h \in \mathcal{D}_c$ be a Hamiltonian diffeomorphism, and $f_*$ be a
sequential system such that \( f_i \in B(h, a) \) for every \( i \in \mathbb{N} \). Then \( f_* \) propagates to infinity with speed \( c - a \). We now prove Theorem A, assuming Theorem B.

**Proof of Theorem A.** Write \( f_i = h\psi_i \) with \( \rho(\text{id}, \psi_i) < a \). Then the evolution of \( f_* \) can be written as follows:

\[
 f^{(k)} = (h\psi_k h^{-1})(h^2 \psi_{k-1} h^{-2}) \cdots (h^k \psi_1 h^{-k}) h^k.
\]

Combining the fact that \( \rho \) is biinvariant with the triangle inequality we get

\[
 \rho(f^{(k)}, h^k) < ka.
\]

Now let \( w \in \mathbb{R}^n \) be any vector such that \( |w| \leq c - a \) and choose a sequence \( v_k \in \mathbb{Z}^n \) such that

\[
 w = \lim_{k \to \infty} \frac{v_k}{k}, \quad |v_k| < k(c - a).
\]

Since \( h^k \in D_{kc} \) (see Proposition 2.1.2) we obtain from Proposition 2.1.2 that \( f^{(k)} \in D_{k(c-a)} \). Hence, by Theorem B, there exists a point \( \tilde{x}_k = (q_k, p_k) \in Q \) so that \( \tilde{f}^{(k)}(\tilde{x}_k) = (q_k + v_k, p_k) \). Since the sequence \( q_k \) is bounded, we have \( \lim_{k \to \infty} (q_k + v_k)/k = w \), and hence \( f_* \) propagates to infinity with speed \( c - a \). \( \square \)

Now we turn to the study of the long time behaviour of individual trajectories. We start with the following general definition. Consider the trajectory \( x_k = f^{(k)}(x) \) of a point \( x \in M \) under a sequential system \( f_* \). Let \( \tilde{x} \in \tilde{M} \) be a lift of \( x \) and consider the lifted trajectory \( \tilde{x}_k = \tilde{f}^{(k)}(\tilde{x}) \) in the universal cover.

**Definition.** A trajectory \( x_k \) of a sequential system with a lift \( \tilde{x}_k = (q_k, p_k) \in \tilde{M} \) is said to **propagate with velocity vector** \( r \in \mathbb{R}^n \) if

\[
 \lim_{k \to \infty} \frac{q_k}{k} = r.
\]

The Euclidean norm of \( r \) is called the **speed** of the trajectory \( x_k \).

As an illustration, let us mention that every \( k \)-periodic orbit of a Hamiltonian diffeomorphism \( h \) in a nonzero homotopy class \( e \in \mathbb{Z}^n \) propagates with velocity vector \( e/k \). Note also that propagation in the universal cover \( \tilde{M} \) corresponds to “rotation” in \( M \). In classical dynamics the velocity vector of a propagating orbit is called the rotation vector.

Consider now the \( d \)-dimensional torus \( \mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d \). Fix a vector \( \alpha = (\alpha_1, \ldots, \alpha_d) \) whose coordinates are independent over the rationals. Let \( g : \mathbb{T}^d \to D \) be a map which is continuous with respect to the \( C^0 \)-topology on \( D \). For a point \( y \in \mathbb{T}^d \) define a sequential system

\[
 f_*(y) := \{g(y + k\alpha)\}_{k \in \mathbb{N}}.
\]
Roughly speaking, the sequence \( f^*(y) \) is “random” – it is uniformly distributed on the subset \( g(T^d) \). The next result is an improvement of Theorem A for (almost all) such sequences.

**Theorem 2.3.1.** Let \( 0 < a < c \) be real numbers and \( h \in D_c \) be a Hamiltonian diffeomorphism. Let \( g : T^d \to D \) be a continuous map whose image is contained in \( B(h,a) \). Then, for set of Lebesgue measure one of points \( y \in T^d \), the sequential system \( f^*(y) \) has at least \( n + 1 \) trajectories which propagate with speed at least \( c - a \).

In fact there exists a compact convex set \( K \subset \mathbb{R}^n \) containing the closed ball of radius \( c - a \) in \( \mathbb{R}^n \) centered at the origin so that the following holds: for every extremal point \( r \in K \) there exists a subset \( Y_r \subset T^d \) of Lebesgue measure one such that, for every \( y \in Y_r \), the system \( f^*(y) \) has a trajectory which propagates with velocity vector \( r \).

The basic feature of the irrational shift of the torus which is crucial for our purposes is unique ergodicity (see Example 2.2.5). The assertion of Theorem 2.3.1 continues to hold, and the proof remains the same, if one replaces the torus \( T^d \) by an arbitrary compact metric space \( Y \), the Lebesgue measure by any Borel probability measure \( \sigma \), and the irrational shift by any \( \sigma \)-preserving uniquely ergodic homeomorphism of \( Y \).

**Proof of Theorem 2.3.1.** Let \( \overline{M} \) denote the closed unit cotangent bundle and \( \widetilde{M} \) its universal cover. Write \( Y := T^d \) and \( \varphi(y) := y + \alpha \) and denote by \( \sigma \) the Lebesgue measure on \( Y \). Consider the skew-product map

\[
\overline{M} \times Y \to \overline{M} \times Y : (x,y) \mapsto S(x,y) := (g(y)(x), \varphi(y))
\]

and its canonical lift \( \widetilde{S} \) to \( \widetilde{M} \times Y \). There exist functions \( u, v : M \times Y \to \mathbb{R}^n \) so that

\[
\widetilde{S}(q,p,y) = (q + u(q,p,y), v(q,p,y), \varphi(y)).
\]

Here we slightly abuse notation and write \( u(q,p,y) \) instead of \( u(x,y) \), whenever \( \tilde{x} = (q,p) \). The function \( u \) which measures displacement along the \( q \)-plane on \( \widetilde{M} \) will play an important role below. Note that \( u \) has compact support in \( M \times Y \).

Let \( \mathcal{M}(S) \) denote the set of all \( S \)-invariant Borel probability measures on \( \overline{M} \times Y \). This space is convex and weak-* compact (see Section 2.2). For a measure \( \mu \in \mathcal{M} \) define its **rotation vector** \( R(\mu) \in \mathbb{R}^n \) by

\[
R(\mu) := \int u \, d\mu \in \mathbb{R}^n.
\]

The map \( R : \mathcal{M}(S) \to \mathbb{R}^n \) is affine and continuous. Hence its image \( K := R(\mathcal{M}(S)) \) is a compact convex subset of \( \mathbb{R}^n \). This set has all the properties formulated in the theorem.
We prove that $K$ contains a ball of radius $c - a$. Pick a vector $v \in \mathbb{R}^n$ with $|v| \leq c - a$ and a point $y \in Y$. Then, by Theorem A, there exists a sequence $x_k \in M$ such that

$$\lim_{k \to \infty} \frac{1}{k} \sum_{i=0}^{k-1} u(S^i(x_k, y)) = v.$$ 

Denote by $\delta_{i,k}$ the Dirac measure on $M \times Y$ concentrated at $S^i(x_k, y)$ and consider the sequence of measures $\mu_k$ on $M \times Y$ defined by

$$\mu_k := \frac{1}{k} \sum_{i=0}^{k-1} \delta_{i,k}.$$ 

By Alaoglu’s theorem, this sequence has a limit point $\mu = \lim_{\nu \to \infty} \mu_{k,\nu}$ with respect to the weak-$*$ topology. Since

$$\lim_{k \to \infty} \left( \int F \circ S \, d\mu_k - \int F \, d\mu_k \right) = \lim_{k \to \infty} \frac{F(S^k(x_k, y)) - F(x_k, y)}{k} = 0$$ 

for every continuous function $F : M \times Y \to \mathbb{R}$ the limit point $\mu$ is $S$-invariant. Moreover, it satisfies

$$R(\mu) = \int u \, d\mu = \lim_{\nu \to \infty} \int u \, d\mu_{k,\nu} = \lim_{\nu \to \infty} \frac{1}{k_{\nu}} \sum_{i=0}^{k_{\nu}-1} u(S^i(x_{k,\nu}, y)) = v.$$ 

Hence $v \in K$.

We prove that the extremal points of $K$ satisfy the requirements of the theorem. Let $r$ be an extremal point of $K$. Then, by Lemma 2.2.6, there exists an ergodic measure $\mu_r \in \mathcal{M}(S)$ such that $R(\mu_r) = r$. By Birkhoff’s Ergodic Theorem 2.2.1, there exists a Borel set $Z_r \subset M \times Y$ such that $\mu_r(Z_r) = 1$ and

$$\lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} u(S^i(z)) = \int u \, d\mu_r = r$$

for every $z \in Z_r$. Note that $Z_r \subset M \times Y$. Let $Y_r \subset Y$ be the image of $Z_r$ under the obvious projection $M \times Y \to Y$ and let $\sigma_r$ be the pushforward of the measure $\mu_r$ (i.e. $\sigma_r(B) := \mu_r(M \times B)$ for every Borel set $B \subset Y$). Then $\sigma_r$ is a $\varphi$-invariant Borel probability measure on $Y$ and so, by unique ergodicity, equals $\sigma$. Hence

$$\sigma(Y_r) = \sigma_r(Y_r) = \mu_r(M \times Y_r) \geq \mu_r(Z_r) = 1.$$ 

Moreover, for every $y \in Y_r$ there exists a point $x \in M$ such that $(x, y) \in Z_r$. By definition of the set $Z_r$, the trajectory of the point $x$ under the system $f_*(y)$ propagates with velocity vector $r$.  \[\square\]
It is instructive to compare the proof above with Aubry–Mather theory for Hamiltonian diffeomorphisms of the annulus generated by a Hamiltonian which is convex with respect to the momentum \( p \). Aubry and Mather constructed remarkable orbits which lie on special Cantor-type sets called cantori. These orbits propagate in our sense. Roughly speaking, these orbits are obtained as the limits of periodic ones. A crucial point for such a limiting procedure is the fact that, due to convexity, one can control the relative positions of closed orbits in phase space. In our setting the Hamiltonians are not assumed to be convex, and we have no information on the positions of the closed orbits. We went round this difficulty by using an idea which goes back to Mather [2] – to look at limits of invariant measures which sit on periodic orbits provided by Theorem B.

2.4. A dissipative counter-example. Assume \( n = 1 \) so \( M = S^1 \times (-1,1) \) is the annulus. Let \( H : (-1,1) \to \mathbb{R} \) be any compactly supported function (which depends on the momenta variable only and has an arbitrarily large value at \( p = 0 \)). Then the corresponding Hamiltonian diffeomorphism \( h \) has the form

\[
h(q,p) = (q + H'(p), p),
\]

where \( H' \) denotes the derivative of \( H \). Recall that \( Q \) denotes the fundamental domain \( Q := [-1/2, 1/2] \times (-1,1) \) in the universal cover \( \tilde{M} = \mathbb{R} \times (-1,1) \). We present an example of an arbitrarily small smooth perturbation \( f \) of \( h \) such that the images \( \tilde{f}^k(Q) \) under the iterates of \( \tilde{f} \) remain in a compact part of \( \tilde{M} \), thus the propagating behaviour disappears. Assume without loss of generality that the support of \( H = H(p) \) is contained in the open interval \((-1/2, 1/2)\) and let \( \gamma := \max_p |H'(p)| \).

Let \( u : [-1,1] \to [-1,1] \) be an orientation preserving diffeomorphism such that \( u(s) > s \) for \(-3/4 < s < 3/4 \) and \( u(s) = s \) for \( |s| \geq 3/4 \). Note that \( u \) can be chosen arbitrarily close to the identity. Choose \( N \in \mathbb{N} \) such that \( u^N(-2/3) > 2/3 \).

Define a map \( \varphi : M \to M \) by \( \varphi(q,p) := (q, u(p)) \) and let \( f := \varphi h \). For a point \( x = (q_0, p_0) \in Q \) consider its orbit \( (q_i, p_i) := \tilde{f}^i(x) \). We claim that \( |q_i - q_0| \leq (N + 1)\gamma \) for all \( x \in Q \) and \( i \in \mathbb{Z} \). This universal estimate means the absence of propagation.

To prove this, note that \( h \) preserves the \( p \)-coordinate, and \( \varphi \) preserves the \( q \)-coordinate of each point. Moreover, the sequence \( p_i \) is nondecreasing. If \( |p_0| \geq 3/4 \) then \( (p_i, q_i) = (p_0, q_0) \) for all \( i \in \mathbb{Z} \). If \( |p_0| < 3/4 \) then \( \lim_{i \to \pm \infty} p_i = \pm 3/4 \). In this case let \( j_0 \in \mathbb{Z} \) be the largest integer such that \( p_{j_0} < -2/3 \) and \( j_1 \geq j_0 \) be the smallest integer such that \( p_{j_1} > 2/3 \). Then \( |j_1 - j_0| \leq N + 2 \) and \( q_i = q_{j_0} \) for \( i \leq j_0 \) and \( q_i = q_{j_1} \) for \( i \geq j_1 \). For every \( i \in [j_0, j_1 - 1] \) we have \( |q_{i+1} - q_i| = |H'(p_i)| \leq \gamma \). Hence, for all \( j, j' \in \mathbb{Z} \) such that \( j' > j \),

\[
|q_{j'} - q_j| \leq \sum_{i=j}^{j'-1} |q_{i+1} - q_i| \leq \sum_{i=j_0}^{j_1-1} |q_{i+1} - q_i| \leq (N + 1)\gamma.
\]
3. Relative symplectic topology

Below we discuss existence and nonexistence results for noncontractible closed orbits in a more general topological framework. The main player in this section is a relative symplectic capacity — a symplectic invariant which provides a convenient language for thinking about these results. Using this language, we prove Theorems B and C as stated in the Introduction.

3.1. Symplectic action. This is a preparatory section in which we set notation. Let \((M, \omega)\) be an open symplectic manifold. We assume throughout that the symplectic form \(\omega\) is exact and fix a 1-form \(\lambda \in \Omega^1(M)\) such that
\[
d\lambda = \omega.
\]
Let \(S^1 := \mathbb{R}/\mathbb{Z}\) and denote the free loop space of \(M\) by \(LM := C^\infty(S^1, M)\). We denote the set of free homotopy classes of loops in \(M\) by \(\tilde{\pi}_1(M)\) and for \(x \in LM\) we write \([x] \in \tilde{\pi}_1(M)\) for its free homotopy class. Given a subset \(\alpha \subset \tilde{\pi}_1(M)\), we write
\[
L_\alpha M := \{x \in LM \mid [x] \in \alpha\}.
\]
We shall mostly consider single elements \(\alpha \in \tilde{\pi}_1(M)\), however, for some of our applications it is useful to consider more general subsets of \(\tilde{\pi}_1(M)\). We denote the space of compactly supported time dependent 1-periodic Hamiltonian functions on \(M\) by \(H(M) := C^\infty_0(S^1 \times M)\). We do not distinguish in notation between the function \(H \in H(M)\) and its lift \(H : \mathbb{R} \times M \to \mathbb{R}\) and, for \(H \in H(M)\) and \(t \in \mathbb{R}\), we define \(H_t : M \to \mathbb{R}\) by \(H_t(x) := H(t, x)\). Every Hamiltonian function \(H \in H(M)\) determines a 1-periodic family of Hamiltonian vector fields \(X_t = X_{t+1} \in \text{Vect}(M, \omega)\) via \(\iota(X_t)\omega = -dH_t\). The space of 1-periodic solutions of the corresponding Hamiltonian differential equation representing a class in the set \(\alpha \subset \tilde{\pi}_1(M)\) will be denoted by
\[
\mathcal{P}(H; \alpha) := \{x \in L_\alpha M \mid \dot{x}(t) = X_t(x(t))\}.
\]
The elements of \(\mathcal{P}(H; \alpha)\) are the critical points of the symplectic action \(A_H : L_\alpha M \to \mathbb{R}\), defined by
\[
A_H(x) := \int_0^1 (H_t(x(t)) - \lambda(\dot{x}(t))) \, dt
\]
for \(x \in L_\alpha M\). The sole purpose of the 1-form \(\lambda\) is to fix a normalization of the symplectic action (which otherwise is only well defined up to an additive constant). Some of the invariants discussed in this paper do not depend on this normalization and this will be pointed out at the appropriate places. However, we do not indicate the dependence of \(A_H\) on \(\lambda\) in the notation.
Remark 3.1.1. The notation in this section differs from the one used in Section 1.2, where the Hamiltonian functions are not required to be periodic in the $t$-variable. Proposition 2.1.3 shows that this does not effect the class of Hamiltonian diffeomorphisms to which the theory applies. It also does not effect the value of the symplectic action at the periodic orbits. To see this, fix a compactly supported, but not necessarily periodic, Hamiltonian function $H \in C^\infty_0([0, 1] \times M)$ and let $[0, 1] \times M \to M : (t, x) \mapsto h_t(x)$ be the Hamiltonian isotopy generated by $H$. Fix any point $x_0 \in M$ and consider the path $x : [0, 1] \to M$ given by
$$x(t) := h_t(x_0).$$
Let $\gamma : [0, 1] \to M$ be a smooth path such that $\gamma(0)$ lies outside of the support of $H$ and $\gamma(1) = x_0$. Differentiating the function $s \mapsto A_{H}(x_s)$, where $x_s(t) := h_t(\gamma(s))$, we find
$$A_{H}(x) = \int_0^1 (h^s \lambda - \lambda)(\dot{\gamma}(s)) \, ds.$$ 
Thus the action of $x$ depends only on $x_0$, $h$, and $\lambda$.

3.2. A relative symplectic capacity. Let $M$ be an open symplectic manifold with symplectic form $\omega = d\lambda$ and $A \subset M$ be a compact subset. In this section we define a relative symplectic capacity
$$C(M, A) : 2\widetilde{\pi}_1(M) \times [-\infty, \infty) \to [0, \infty]$$
for the pair $(M, A)$. This capacity has the following feature: given $\alpha \subset \widetilde{\pi}_1(M)$ and $a \in \mathbb{R} \cup \{-\infty\}$ such that $C(M, A; \alpha, a) < \infty$, every $H \in \mathcal{H}(M)$ with $\inf_{S^1 \times A} H > C(M, A; \alpha, a)$ must have a 1-periodic orbit representing one of the homotopy classes in $\alpha$ with symplectic action $A_{H}(x) \geq a$. Moreover, $C(M, A; \alpha, a)$ is the optimal bound for the existence of such an orbit. More precisely, for $c > 0$ let
$$\mathcal{H}_c(M, A) := \left\{ H \in \mathcal{H}(M) \mid \inf_{S^1 \times A} H \geq c \right\}.$$ 
For a subset $\alpha \subset \widetilde{\pi}_1(M)$ and a number $a \geq -\infty$ we define the relative symplectic capacity $C(M, A; \alpha, a) \geq 0$ by
$$C(M, A; \alpha, a) := \inf \left\{ c > 0 \left| \forall H \in \mathcal{H}_c(M, A) \exists x \in \mathcal{P}(H; \alpha) \text{ such that } A_{H}(x) \geq a \right. \right\}.$$ 
We use the convention that $\inf \emptyset = \infty$. The infimum in the definition of $C(M, A; \alpha, a)$ is always achieved (see Proposition 3.3.3 below).

In order to relax the notation we identify the single element subset $\{\alpha\} \subset \widetilde{\pi}_1(M)$ with $\alpha \in \widetilde{\pi}_1(M)$. Also, for $a = -\infty$ we abbreviate
$$C(M, A; \alpha) := C(M, A; \alpha, -\infty) = \inf \left\{ c > 0 \left| \mathcal{P}(H; \alpha) \neq \emptyset \text{ for every } H \in \mathcal{H}_c(M, A) \right. \right\}.$$
Note that the invariant \( C(M, A; \alpha) \) is independent of the choice of \( \lambda \). It turns out that \( C(M, A; \alpha) \) is \textit{finite} in some interesting cases. To begin with let us consider the case \( A = \text{pt} \). Then the invariant \( C(M, A; \alpha) \), with \( \alpha = \tilde{\pi}_1(M) \), is analogous to the Hofer-Zehnder capacity \([HZ]\). The difference is that Hofer and Zehnder considered nonnegative and time-independent Hamiltonian functions. Lalonde \([L]\) suggested to consider more general subsets \( A \).

Here are some examples in which our capacity can be computed. Let \( X \) be a closed (i.e. compact without boundary) connected Riemannian manifold. Denote by

\[
U^*X := \{ v \in T^*X \mid |v| < 1 \} \subset T^*X
\]

the open unit cotangent bundle. This manifold is equipped with the canonical symplectic form \( \omega_{\text{can}} = d\lambda_{\text{can}} \) of \( T^*X \). We identify \( X \) with the zero section of \( U^*X \) and \( \tilde{\pi}_1(U^*X) \). Moreover, in the case of the standard torus \( X = \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n \) there is a natural isomorphism

\[
\tilde{\pi}_1(U^*\mathbb{T}^n) \cong \tilde{\pi}_1(\mathbb{T}^n) \cong \mathbb{Z}^n.
\]

We denote the Euclidean norm of an integer vector \( k = (k_1, \ldots, k_n) \in \mathbb{Z}^n \) by

\[
|k| := \left( \sum_{j=1}^n k_j^2 \right)^{1/2}.
\]

\[\textbf{Theorem 3.2.1.} \ (i) \ Let \( \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n \) be the flat equilateral torus with the metric induced by the standard metric on \( \mathbb{R}^n \). Then, for every \( k \in \mathbb{Z}^n \cong \tilde{\pi}_1(U^*\mathbb{T}^n) \) and every \( a \in \mathbb{R} \),

\[
C(U^*\mathbb{T}^n, \mathbb{T}^n; k, a) = \max\{|k|, a\}.
\]

(ii) Let \( X \) be a closed Riemannian manifold with negative sectional curvature. Then, for every \( \alpha \in \tilde{\pi}_1(X) \cong \tilde{\pi}_1(U^*X) \) and every \( a \in \mathbb{R} \),

\[
C(U^*X, X; \alpha, a) = \max\{\text{length}(\gamma_\alpha), a\},
\]

where \( \gamma_\alpha : S^1 \to X \) is the (unique up to time shift) closed geodesic that represents the homotopy class \( \alpha \).

\[\textbf{3.3. Properties.}\]

In this section we establish some basic properties of the relative capacity and prove Theorems B and C, assuming Theorem 3.2.1 (i).

\[\textbf{Proposition 3.3.1. (Monotonicity)} \ Let \( A_2 \subset A_1 \subset M_1 \subset M_2 \), where \( M_1 \) is an open subset of \( M_2 \). Let \( \iota_* : \tilde{\pi}_1(M_1) \to \tilde{\pi}_1(M_2) \) denote the map induced by the inclusion \( \iota : M_1 \subset M_2 \). Then, for \( \alpha_i \in \tilde{\pi}_1(M_i) \) and \( a_i \geq -\infty \), we have

\[
\iota_*^{-1}(\alpha_2) \subset \alpha_1, \ a_1 \leq a_2 \quad \Rightarrow \quad C(M_1, A_1; \alpha_1, a_1) \leq C(M_2, A_2; \alpha_2, a_2).
\]
In particular, if \( \iota_* \) is injective then for every \( \alpha \subset \tilde{\pi}_1(M_1) \) and \( a_1 \leq a_2 \) we have

\[
C(M_1, A_1; \alpha, a_1) \leq C(M_2, A_2; \iota_*(\alpha), a_2).
\]

**Proof.** Let \( c > C(M_2, A_2; \alpha, a_2) \). Then every Hamiltonian function \( H \in \mathcal{H}_c(M_2, A_2) \) has a 1-periodic orbit representing a class in \( \alpha_2 \) with action at least \( a_2 \). In particular, for every \( H \in \mathcal{H}_c(M_1, A_1) \) there exists a 1-periodic orbit \( x : S^1 \to M_1 \) such that \( \iota_*[x] \in \alpha_2 \) (and hence \( [x] \in \alpha_1 \)) and \( \mathcal{A}_H(x) \geq a_2 \geq a_1 \). This shows that \( c \geq C(M_1, A_1; \alpha_1, a_1) \). \( \square \)

Our next result provides a useful condition which guarantees that the relative capacity is trivial.

**Proposition 3.3.2. (Displacement)** Suppose there exists a compactly supported Hamiltonian isotopy \( f_t : M \to M \), \( 0 \leq t \leq 1 \), such that \( f_1(A) \cap A = \emptyset \) and all 1-periodic orbits of \( f_t \) are contractible. Then, for every constant \( c > 0 \), there exists a Hamiltonian \( H \in \mathcal{H}_c(M, A) \) such that \( \mathcal{P}(H; \alpha) = \emptyset \) for every nontrivial class \( \alpha \in \tilde{\pi}_1(M) \). In particular, \( C(M, A; \alpha) = \infty \) for every nontrivial class \( \alpha \).

**Proof.** Let \( F \in \mathcal{H} \) be the Hamiltonian function generating \( f_t \) and suppose that \( U \) is a neighbourhood of \( A \) such that \( f_1(U) \cap U = \emptyset \). Choose a smooth function \( G : M \to \mathbb{R} \) such that

\[
\text{supp} G \subset U, \quad \inf_A G + \inf_{[0,1] \times M} F \geq c.
\]

Let \( g_t \) denote the Hamiltonian flow of \( G \) and consider the Hamiltonian flow \( h_t := f_t g_t \).

Since \( f_1(U) \cap U = \emptyset \), the Hamiltonian diffeomorphisms \( h_1 \) and \( f_1 \) have the same fixed points. Hence all 1-periodic orbits of the flow \( h_t \) are contractible: they have the form \( x(t) = x(t + 1) = h_t(x_0) = f_t(x_0) \) for some \( x_0 \in M \setminus U \). The same holds for the conjugate isotopy \( \varphi_t := f_t^{-1} h_t f_1 \). Moreover, \( \varphi_1 = g_1 f_1 = \psi_1 \), where \( \psi_t = g_t f_t \). Hence, by Remark 2.1.1, all 1-periodic orbits of the isotopy \( \psi_t \) are contractible as well. Observe now that the flow \( \psi_t \) is generated by the Hamiltonian functions \( \Psi_t := G + F_t \circ g_t^{-1} \) and they satisfy \( \inf_A \Psi_t \geq c \) for \( x \in A \). By Proposition 2.1.3, there is a periodic Hamiltonian in \( \mathcal{H}_c(M, A) \) with the same time-1-map as \( \Psi_t \) and, by Remark 2.1.1, the 1-periodic orbits of this periodic Hamiltonian are also contractible. \( \square \)

Let us now look at the relative capacity from the geometric viewpoint. The group \( \mathcal{D} \) of all compactly supported Hamiltonian diffeomorphisms of \( M \) carries Hofer’s metric (see Section 1.1 above). Denote by \( \mathcal{D}_c \subset \mathcal{D} \) the subset consisting of all Hamiltonian diffeomorphisms generated by functions from \( \mathcal{H}_c(M, A) \). Recall that every \( h \in \mathcal{D} \) has a canonical lift to the universal cover and hence the homotopy class (in \( \tilde{\pi}_1(M) \)) of a fixed point is independent of the choice of the compactly supported Hamiltonian generating \( h \) (see Remark 2.1.1).
Proposition 3.3.3. (Stability) (i) Suppose that \( C(M, A; \alpha) \leq c < \infty \) for some non-trivial class \( \alpha \in \varpi_1(M) \). Let \( f \in D_{c+a} \) be a Hamiltonian diffeomorphism, where \( a > 0 \). Then every compactly supported Hamiltonian diffeomorphism \( h \in D \) with \( \rho(f, h) < a \) has a fixed point in the class \( \alpha \). In particular, \( \rho(f, id) \geq a \).

(ii) Suppose that
\[
\lim_{a \to 0} C(M, A; 0, a) = 0
\]
and let \( c > 0 \). Then \( \rho(f, id) \geq c \) for every \( f \in D_c \).

Proof. We prove (i). Let \( f \in D_{c+a} \) and \( h \in D \) such that \( \rho(f, h) < a \). Then, by Proposition 2.1.2, \( h \in D_c \) and hence there exists a Hamiltonian isotopy \( h_t \) with \( h_1 = h \) whose Hamiltonian function belongs to \( \mathcal{H}_c(M, A) \). By definition of the relative capacity, this flow has a 1-periodic orbit in the class \( \alpha \). In particular \( h \neq id \). This proves (i).

We prove (ii). Suppose, by contradiction, that \( \rho(f, id) < c \) for some \( f \in D_c \). Then, by Propositions 2.1.2 and 2.1.3, there exists a Hamiltonian function \( H \in \mathcal{H}(M) \) that generates the identity and satisfies \( \inf_{S^1 \times A} H > 0 \). By assumption, there exists a constant \( a > 0 \) such that \( C(M, A; 0, a) < \inf_{S^1 \times A} H \). Hence there exists a contractible periodic orbit \( x \in \mathcal{P}(H; 0) \) with action \( \mathcal{A}_H(x) \geq a \). On the other hand, since \( H \) generates the identity, every orbit is a contractible periodic orbit with action zero. This contradiction proves (ii).

Proposition 3.3.4. Let \( \alpha \subset \varpi_1(M) \) and \( a \geq -\infty \). Then every Hamiltonian \( H \in \mathcal{H}(M) \) with \( \inf_{S^1 \times A} H \geq C(M, A; \alpha, a) \) must have a 1-periodic orbit representing one of the homotopy classes in \( \alpha \) with symplectic action \( \mathcal{A}_H(x) \geq a \). In other words, the set \( \{ c > 0 \mid \forall H \in \mathcal{H}_c(M, A) \exists x \in \mathcal{P}(H; \alpha) \text{ such that } \mathcal{A}_H(x) \geq a \} \) is either empty or has a minimum.

Proof. Without loss of generality we may assume that either \( 0 \notin \alpha \) or \( a > 0 \), since otherwise there is nothing to prove (all points outside the support of \( H \) are constant periodic orbits in the class 0 and with action 0). Choose a compact subset \( K \subset M \) such that \( A \subset \text{Int} K \) and \( S^1 \times K \supset \text{supp} H \). Next, let \( \sigma : M \to \mathbb{R} \) be a Hamiltonian function supported in \( K \) and such that \( \sigma|_A > 0 \). Consider the sequence of Hamiltonians \( H_n := H + \frac{1}{n} \sigma \). Clearly \( \inf_{S^1 \times A} H_n > C(M, A; \alpha, a) \), hence for every \( n \), \( H_n \) has a 1-periodic orbit \( x_n \) with \([x_n] \in \alpha \) and \( \mathcal{A}_{H_n}(x_n) \geq a \). Note that \( x_n \subset K \) for every \( n \) due to the assumption that either \( 0 \notin \alpha \) or \( a > 0 \).

Now \( H_n \) converges to \( H \) in the \( C^\infty \) topology and \( K \) is compact. Hence, replacing \( H_n \) by a suitable subsequence if necessary, we obtain a sequence of periodic orbits \( x_n \), that converges to a periodic orbit \( x \) of \( H \). If follows that, for \( n \) sufficiently large, the loops
\( x_n \) all represent the same homotopy class. Hence \( [x] = \lim_{n \to \infty} [x_n] \in \alpha \) and \( \mathcal{A}_H(x) = \lim_{n \to \infty} \mathcal{A}_{H_n}(x_n) \geq a. \)

**Proof of Theorem B.** Due to Proposition 2.1.3 and the discussion on homotopy classes following its proof we may assume our Hamiltonians to be defined on \( S^1 \times U^*T^n \) rather than on \([0,1] \times U^*T^n\). Let \( H : S^1 \times U^*T^n \to \mathbb{R} \) be a compactly supported Hamiltonian function and \( k \in \mathbb{Z}^n \) an integer vector such that

\[
|k| \leq c := \inf_{S^1 \times T^n} H.
\]

By Theorem 3.2.1 (i), \( C(U^*T^n, T^n; k, c) = c \). It follows from the definition of our capacity and Proposition 3.3.4 that there exists a periodic solution \( x \in P(H; k) \) of (1) such that \( x \) represents the homotopy class \( k \) and \( \mathcal{A}_H(x) \geq c \). This proves the assertion of Theorem B for periodic Hamiltonian functions. The assertion in the non-periodic case follows from the periodic case, Proposition 2.1.3, and Remark 3.1.1.

**Proof of Theorem C.** Let \( S \subset U^*T^n \) be a non-Lagrangian section and \( c > 0 \) be any real number. It is shown in [Po1, LS] that there exists a Hamiltonian function \( H : U^*T^n \to \mathbb{R} \) such that the vector field \( X_H \) is nowhere tangent to \( S \):

\[
x \in S \implies X_H(x) \notin T_xS.
\]

We may assume without loss of generality that \( H \) has compact support. Now let \( \varphi_t : U^*T^n \to U^*T^n \) denote the flow generated by \( X_H \). Then there exists an \( \varepsilon > 0 \) such that

\[
0 < t < \varepsilon \implies \varphi_t(S) \cap S = \emptyset.
\]

If \( \delta \) is sufficiently small then the only 1-periodic solutions of the Hamiltonian flow \( t \mapsto \varphi_{\delta t} \) are constant and \( \varphi_\delta(S) \cap S = \emptyset \). Hence, by Proposition 3.3.2, there exists, for every \( c > 0 \), a Hamiltonian function \( F \in \mathcal{H}_c(U^*T^n, S) \) such that \( P(F; k) = \emptyset \) for every nonzero integer vector \( k \in \mathbb{Z}^n \).

**3.4. Existence of closed orbits on hypersurfaces.** As a by-product of our study of the relative capacity we obtain existence of closed orbits on hypersurfaces in various situations.

**Theorem 3.4.1.** Let \( X \) be either \( T^n \) or a closed negatively curved manifold and \( H : TX \to \mathbb{R} \) a proper and bounded below Hamiltonian. Suppose that the sublevel set \( \{ H < c \} \) contains the zero section. Then for every nontrivial homotopy class \( \alpha \in \tilde{\pi}_1(T^*X) \) there exists a dense subset \( S_\alpha \subset (c, \infty) \) with the property that for every \( s \in S_\alpha \) the level set \( \{ H = s \} \) contains a closed orbit \( x_s \) in the class \( \alpha \) and with \( \int_{x_s} \lambda_{\mathrm{can}} > 0 \).
Proof. Fix a nontrivial homotopy class $\alpha \in \pi_1(T^*X)$. To prove the statement of the theorem we shall show that for every $b > a > c$ there exists $s \in (a, b)$ such that the level set $\{H = s\}$ carries a closed orbit $x$ in the class $\alpha$ and with $\int_x \lambda_{\text{can}} > 0$.

Pick a Riemannian metric $g$ on $X$ which in case $X = \mathbb{T}^n$ is the flat equilateral metric, or in the other case has negative curvature. Denote by $U^*X \subset T^*X$ the unit cotangent bundle of $(X, g)$ endowed with the canonical symplectic structure.

Let $b > a > c$. Due to a rescaling argument we may assume without loss of generality that the sublevel set $\{H \leq b\}$ is contained in $U^*X$. Next, put $C_\alpha = C(U^*X, X; \alpha^{-1})$. Note that by Theorem 3.2.1 the number $C_\alpha$ is finite (and in fact equals the length of a geodesic in the class $\alpha^{-1}$). Now choose a smooth function $\sigma : \mathbb{R} \to \mathbb{R}$ with the following properties:

- $\sigma(r) = C_\alpha$ for $r \leq a$.
- $\sigma(r) = 0$ for $r \geq b$.
- $\sigma'(r) < 0$ for $a < r < b$.

Consider the compactly supported Hamiltonian $F : U^*X \to \mathbb{R}$ defined by $F = \sigma \circ H$. As $F|_X = C_\alpha$ it follows from the definition of $C_\alpha$ that $F$ has a 1-periodic orbit, say $y$, in the class $\alpha^{-1}$. Note that $y$ is nonconstant (because $\alpha$ is nontrivial). Therefore $y$ must lie on one of the level sets $\{F = \rho\}$ where $0 < \rho < C_\alpha$. Since $\sigma$ takes the interval $(a, b)$ injectively onto $(0, C_\alpha)$ there exists $s \in (a, b)$ such that $\{F = \rho\} = \{H = s\}$, hence $y$ lies on the level set $\{H = s\}$. Applying to $y$ a suitable orientation reversing reparametrization will give us a closed orbit $x_s$ of $H$ in the class $\alpha$. This follows easily from the fact that $X_F = (\sigma' \circ H)X_H$ and $\sigma'(s) < 0$.

Finally note that by Theorem 3.2.1, the orbit $y$ has action $A_F(y) \geq C_\alpha$, hence

$$C_\alpha \leq \int_0^1 (F(y(t)) - \lambda_{\text{can}}(\dot{y}(t))) \, dt = \rho + \int_{x_s} \lambda_{\text{can}} < C_\alpha + \int_{x_s} \lambda_{\text{can}}.$$ 

Consequently $\int_{x_s} \lambda_{\text{can}} > 0$. \hfill $\square$

3.4.1. Symplectically convex boundaries. Let $(M, \omega)$ be a symplectic manifold and $U \subset M$ a relatively compact domain with smooth convex boundary $Q = \partial \overline{U}$. Recall that this means, by definition, that there exists a Liouville vector field $Y$ (namely $\mathcal{L}_Y \omega = \omega$), defined on a neighbourhood of $Q$ in $M$, such that $Y$ points outside of $U$ along $Q$. Note that in this case $Q$ is a hypersurface of contact type since the vector field $Y$ gives rise to a contact form $\lambda_Q = (\iota(Y)\omega)|_{\partial Q}$ on $Q$ with $d\lambda_Q = \omega|_{\partial Q}$.

Denote by $\mathcal{L}_Q = \ker(\omega|_{\partial Q}) \subset TQ$ the characteristic line field of $Q$. The Reeb vector field $R$ of $\lambda_Q$ is a nonvanishing section of $\mathcal{L}_Q$ and so defines an orientation on $\mathcal{L}_Q$. We
call this the **canonical orientation** of $\mathcal{L}_Q$. It induces an orientation on each leaf of the characteristic foliation of $Q$ (namely the foliation corresponding to $\mathcal{L}_Q$).

**Corollary 3.4.2.** Let $X$ be either $\mathbb{T}^n$ or a closed negatively curved manifold. Let $U \subset T^*X$ be a relatively compact domain containing the zero section, and with smooth convex boundary $Q = \partial \overline{U}$. Let $\mathcal{L}_Q$ be equipped with its canonical orientation. Then for every nontrivial homotopy class $\alpha \in \tilde{\pi}_1(X)$ the characteristic foliation of $Q$ has a closed leaf $x \subset Q$ with $j_*[x] = \alpha$, where $j_* : \tilde{\pi}_1(Q) \to \tilde{\pi}_1(X)$ is the map induced by the composition $j : Q \subset T^*X \xrightarrow{\pi} X$.

**Remark.** We were informed by C. Viterbo that the above corollary should follow also from the variational techniques of [HV].

**Proof of Corollary 3.4.2.** Denote by $\omega$ the (canonical) symplectic form on $T^*X$ and let $\lambda_Q$ be the contact form on $Q$ induced by a local Liouville vector field $Y$ as described above. Since $U$ has convex boundary there exist a neighbourhood $W$ of $Q = \partial \overline{U}$ in $M$ and a diffeomorphism $\Phi : (-\varepsilon, \varepsilon) \times Q \to W$ such that (see [MS]):

- $\Phi(0, q) = q$ for every $q \in Q$.
- $\Phi((-\varepsilon, 0) \times Q) \subset U$, and $\Phi((0, \varepsilon) \times Q) \subset W \setminus \overline{U}$.
- $\Phi^*\omega = d(e^\tau \lambda_Q)$, where $\tau$ is the coordinate on $(-\varepsilon, \varepsilon)$.

Choose a smooth function $h : (-\varepsilon, \varepsilon) \to \mathbb{R}$ with the following properties:

- $h(\tau) = 0$ for $\tau \leq -\varepsilon/2$, and $h(\tau) = 1$ for $\tau \geq \varepsilon/2$.
- $h'(\tau) > 0$ for $-\varepsilon/2 < \tau < \varepsilon/2$.

Define now $H : W \to \mathbb{R}$ by $H(\Phi(\tau, q)) = h(\tau)$. Next, extend $H$ to a smooth function $H : T^*X \to \mathbb{R}$ in such a way that:

- $H \equiv 0$ on $U \setminus W$.
- $H \geq 1$ on $T^*X \setminus (U \cup W)$.
- $H$ is proper.

Pick $a, b$ with $0 < a < b < 1$. By Theorem 3.4.1, there exists an $s_0 \in (a, b)$ such that the level set $\{H = s_0\}$ carries a closed orbit of $H$, say $y$, with $\pi_* [y] = \alpha$.

Since $0 < s_0 < 1$, there exists $\tau_0 \in (-\varepsilon/2, \varepsilon/2)$ such that $\Phi^{-1}(\{H = s_0\}) = \tau_0 \times Q$. Thus $\Phi^{-1}(y) = (\tau_0, x)$ where $x \subset Q$ is a closed curve. On the other hand $\Phi^{-1}(y)$ is a closed orbit for the Hamiltonian $h(\tau, q) = h(\tau)$ on $\left( (-\varepsilon, \varepsilon) \times Q, d(e^\tau \lambda_Q) \right)$. A simple computation shows that the Hamiltonian vector field $X_h$ of $h$ is given along $\tau_0 \times Q$ by $X_h|_{\tau_0 \times Q} = h'(\tau_0)e^{-\tau_0}R$, where $R$ is the Reeb vector field of $\lambda_Q$. This proves that $x \subset Q$ is a closed leaf of the characteristic foliation of $Q$. The orientation of $x$ agrees with the one of the characteristic foliation because $h'(\tau_0) > 0$. 
Finally, the map $\Phi(\tau_0, \cdot) : Q \to T^*X$ is homotopic to the inclusion $\varrho \subset X$, hence $j_*[x] = \pi_*[y] = \alpha$. 

The main point in the proofs of both Theorem 3.4.1 and Corollary 3.4.2 is that when $X$ is either $\mathbb{T}^n$ or a negatively curved manifold we have $C(U^*X, X; \alpha) < \infty$ for every $\alpha \in \tilde{\pi}_1(X)$. This suggests the following definition.

**Definition 3.4.3.** Let $X$ be a smooth closed manifold. We say that a nontrivial free homotopy class $\alpha \in \tilde{\pi}_1(X)$ is **symplectically essential** if there exists a domain $U \subset T^*X$ containing the zero-section such that $C(U^*X, X; \iota_* \alpha) < \infty$. Here $\iota_* : \tilde{\pi}_1(X) \to \tilde{\pi}_1(U)$ is the map induced by the inclusion $\iota : X \to U$ of the zero section into $U$.

**Example.** Let $X$ be either $\mathbb{T}^n$ or a closed negatively curved manifold. It follows from Theorem 3.2.1 that every nontrivial homotopy class $\alpha \in \tilde{\pi}_1(X)$ is symplectically essential.

In this language, Theorem 3.4.1 has the following obvious generalization.

**Theorem 3.4.4.** Let $X$ be a smooth closed manifold and $H : T^*X \to \mathbb{R}$ be a proper and bounded below Hamiltonian. Suppose that the sublevel set $\{ H < c \}$ contains the zero section. Then for every nontrivial symplectically essential class $\alpha \in \tilde{\pi}_1(X)$ there exists a dense subset $S_\alpha \subset (c, \infty)$ with the property that for every $s \in S_\alpha$ the level set $\{ H = s \}$ contains a closed orbit whose projection to the zero section is in the class $\alpha^{-1}$.

The proof is an obvious modification of the one of Theorem 3.4.1. Similarly, the proof of the next result is analogous to that of Corollary 3.4.2.

**Corollary 3.4.5.** Let $X$ be a closed manifold and $U \subset T^*X$ a relatively compact domain containing the zero section, and with smooth convex boundary $Q = \partial U$. Let $\mathcal{L}_Q$ be equipped with its canonical orientation. Then for every nontrivial symplectically essential homotopy class $\alpha \in \tilde{\pi}_1(X)$ the characteristic foliation of $Q$ has a closed leaf $x \subset Q$ with $j_*[x] = \alpha$, where $j_* : \tilde{\pi}_1(Q) \to \tilde{\pi}_1(X)$ is the map induced by the composition $j : Q \subset T^*X \to X$.

### 3.5. Topological applications.

**Definition (Manifolds of type $\mathcal{F}$).** Let $X$ be a smooth closed manifold. We say that $X$ is of **type $\mathcal{F}$** if there exist two nontrivial free homotopy classes $\alpha_1, \alpha_2 \in \tilde{\pi}_1(X)$ such that:

(i) $\alpha_1$ and $\alpha_2$ are not positively proportional, namely there exist no $k_1, k_2 \in \mathbb{N}$ with $\alpha_1^{k_1} = \alpha_2^{k_2}$.

(ii) $\alpha_1$ and $\alpha_2$ are both symplectically essential. It is easy to see that this is equivalent to existence of one domain $U \subset T^*X$ containing the zero section and such that both $C(U^*X, X; \iota_* \alpha_1)$ and $C(U^*X, X; \iota_* \alpha_2)$ are finite. Here $\iota_*$ is the map induced by the inclusion of zero section $X$ into $U$. 

Examples. (i) Let $X$ be a closed negatively curved manifold. Then $X$ is of type $\mathcal{F}$. To see this note first that $X$ is not simply connected because the universal cover is diffeomorphic to Euclidean space. Now let $\alpha \in \tilde{\pi}_1(X)$ be a nontrivial free homotopy class and denote by $\gamma$ the unique (up to parametrization) closed geodesic representing the class $\alpha$. We claim that $\alpha$ and $\alpha^{-1}$ are not positively proportional. Suppose otherwise that there exist positive integers $k$ and $\ell$ such that $\alpha^k = \alpha^{-\ell}$. Then the $k$'th and $\ell$'th iterates $\gamma^k$ and $\gamma^{-\ell}$ of $\gamma$ and $\gamma^{-1}$, respectively, are two different geodesics representing the same free homotopy class. This is impossible for negatively curved manifolds because every geodesic has Morse index zero.

Now let $\alpha_1, \alpha_2 \in \tilde{\pi}_1(X)$ be two nontrivial positively non-proportional classes. By Theorem 3.2.1 (ii), $C(U^*X, X; \alpha_1)$ and $C(U^*X, X; \alpha_2)$ are finite, hence $X$ is of type $\mathcal{F}$.

(ii) It follows from Theorem 3.2.1(i) that $\mathbb{T}^n$ is also of type $\mathcal{F}$.

3.5.1. Applications to Hamiltonian circle actions.

**Theorem 3.5.1.** Let $(M, \omega)$ be a compact symplectic manifold (possibly with boundary) with $\dim \mathbb{R} M \geq 4$ and $L \subset \text{Int}(M)$ a compact connected Lagrangian submanifold. Suppose that $M \setminus L$ admits a Hamiltonian circle action with a surjective moment map $\mu : M \setminus L \to [r, R]$ that is proper onto its image ($R$ may possibly be $\infty$). Then $L$ cannot be of type $\mathcal{F}$. In particular, $L$ cannot be diffeomorphic to $\mathbb{T}^n$, $n \geq 2$, or to any negatively curved manifold.

Note that the Theorem, as stated, is not true when $\dim \mathbb{R} M = 2$. For example take $M = U^*S^1$ and let $L = S^1$ be the zero section. Then $U^*S^1 \setminus S^1$ is diffeomorphic to $S^1 \times ((-1, 0) \cup (0, 1))$ and has a circle action with moment map $\mu(q, p) = -|p|$ which satisfies all the assumptions of the theorem.

The crucial difference between dimension two and higher ones is that in dimension two, $M \setminus L$ might be disconnected whereas in higher dimension this never happens. Indeed the following 2-dimensional version of Theorem 3.5.1 holds:

**Theorem 3.5.2.** Let $(M, \omega)$ be a compact symplectic 2-manifold (possibly with boundary) and $S^1 \approx L \subset \text{Int}(M)$ an embedded circle. Suppose that $M \setminus L$ admits a Hamiltonian circle action with a surjective moment map $\mu : M \setminus L \to [r, R]$ that is proper onto its image ($R$ may possibly be $\infty$). Then $M \setminus L$ is disconnected.

Before we turn to the proofs of Theorems 3.5.1 and 3.5.2 let us remark that in both Theorems it is impossible to drop the assumption that $\mu$ is proper onto its image and that its image is a half open interval. Indeed take $M = \mathbb{T}^2 \approx S^1 \times S^1$ and $L = \text{pt} \times S^1$. Then $M \setminus L$ is diffeomorphic to $S^1 \times (0, 1)$ and so is connected. It has an obvious circle
action whose moment map \( \mu : S^1 \times (0,1) \to (0,1) \) is the projection onto the second factor. However, there is no Hamiltonian circle action on \( S^1 \times (0,1) \) such that the image of the moment map is a half open interval and the moment map is proper. One can easily produce higher dimension examples as well, e.g. multiplying \( M \) by another \( \mathbb{T}^2 \) factor and \( L \) by \( S^1 \).

**Proof of Theorem 3.5.4.** By Darboux’ theorem there exist an open relatively compact neighbourhood \( U_0 \) of \( L \) in \( T^*L \), an open neighbourhood \( W_0 \) of \( L \) in \( M \) and a symplectomorphism \( f : (U_0, \omega_{\text{can}}) \to (W_0, \omega) \) taking \( L \subset U_0 \) identically to \( L \subset W_0 \).

Note that \( M \setminus U_0 \) is compact and let
\[
R_0 := \max_{M \setminus W_0} \mu, \quad U := f^{-1}(M \setminus \{ \mu \leq R_0 \}) \subset U_0.
\]
Then \( U \setminus L \) carries a circle action as well, with moment map \( \mu \circ f : U \setminus L \to (R_0, R) \) which is proper onto its image. By reducing \( U_0 \) if necessary we may assume that \( U \) is connected. Note that \( U \setminus L \) is also connected because the codimension of \( L \) in \( U \) is at least two. It follows that any two \( S^1 \)-orbits in \( U \setminus L \) represent positively proportional homotopy classes in \( \tilde{\pi}_1(U) \).

Now let \( R_0 < R_1 < R_2 < R \) and choose a smooth function \( \sigma : \mathbb{R} \to \mathbb{R} \) with the following properties:

- \( \sigma(r) = 0 \) for every \( r \leq R_1 \).
- \( \sigma(r) = c \) for every \( r \geq R_2 \), where \( c > 0 \) is a constant that will be determined later.
- \( \sigma'(r) > 0 \) for every \( R_1 < r < R_2 \).

Define now a compactly supported Hamiltonian \( H : U \to \mathbb{R} \) by putting \( H := \sigma \circ \mu \circ f \) on \( U \setminus L \), and extending \( H \) to be \( c \) on \( L \). Note that the vector field \( X_H \) is everywhere tangent to the orbits of the circle action on \( U \setminus L \) and moreover along \( \mu^{-1}(R_1, R_2) \), \( X_H \) points in the direction induced by the circle action (because \( \sigma' > 0 \) there). Since any two orbits of the circle action represent positively proportional homotopy classes in \( \tilde{\pi}_1(U) \) it follows that any two nonconstant closed orbits of \( X_H \) must also be positively proportional in \( \tilde{\pi}_1(U) \).

Suppose now that \( L \) is of type \( \mathcal{F} \). We shall get a contradiction by showing that once the constant \( c \) from the definition of \( \sigma \) is chosen to be large enough, \( X_H \) must carry two closed orbits whose classes are positively non-proportional. Indeed, if \( L \) is of type \( \mathcal{F} \) then there exist two positively non-proportional classes \( \alpha_1, \alpha_2 \in \pi_1(L) \) and a domain \( U' \) containing the zero section such that both capacities \( C(U', L; \iota'_s(\alpha_1)) \) and \( C(U', L; \iota'_s(\alpha_2)) \) are finite, where \( \iota'_s \) is the map induced by the inclusion of the zero section \( L \) into \( U' \). By reducing the size of the Darboux neighbourhood \( W_0 \) if necessary we may assume that \( U \subset U' \). Denote by \( \iota : L \to U \) and \( j : U \to U' \) the obvious inclusions and by \( \pi : U \to L \) the
obvious projection. Then \( \pi_* \iota_* = \text{id} \) and hence \( j_*^{-1}(\iota'_*(\alpha_i)) \subset \pi_*^{-1}(\alpha_i) \) for \( i = 1, 2 \). Hence, by monotonicity (Proposition 3.3.1), we have
\[
C(U, L; \pi_*^{-1}(\alpha_1)) \leq C(U_1, L; \iota'_*(\alpha_1)) < \infty \\
C(U, L; \pi_*^{-1}(\alpha_2)) \leq C(U_2, L; \iota'_*(\alpha_2)) < \infty.
\]
Now choose the constant \( c \) in the definition of \( \sigma \) to be larger than both of \( C(U, L; \pi_*^{-1}(\alpha_1)) \) and \( C(U, L; \pi_*^{-1}(\alpha_2)) \). Since \( H|_L = c \), the Hamiltonian \( H \) must have two periodic orbits \( x_1, x_2 \subset U \) with \( \pi_*[x_1] = \alpha_1 \) and \( \pi_*[x_2] = \alpha_2 \). As \( \alpha_1 \) and \( \alpha_2 \) are not positively proportional, neither are the classes \([x_1]\) and \([x_2]\). This contradicts the fact, established above, that any two periodic orbits of \( X_H \) represent proportional homotopy classes in \( \tilde{\pi}_1(U) \).

**Proof of Theorem 3.5.2.** Let \( W_0 \) be a Darboux neighbourhood of \( L \) in \( M \) as in the proof of Theorem 3.5.1. Choose an orientation on \( L \approx S^1 \) and denote by \( \gamma_0 \in \tilde{\pi}_1(W_0) \) the homotopy class represented by the oriented circle \( L \). By Theorem 3.2.1 both \( C(W_0, L; \gamma_0) \) and \( C(W_0, L; \gamma_0^{-1}) \) are finite.

Let \( \sigma : \mathbb{R} \to \mathbb{R} \) be the function defined in the proof of Theorem 3.5.1, where the constant \( c \) used in its definition is now taken to be larger than both of \( C(W_0, L; \gamma_0) \) and \( C(W_0, L; \gamma_0^{-1}) \). Define now a compactly supported Hamiltonian \( H : W_0 \to \mathbb{R} \) by putting \( H := \sigma \circ \mu \) on \( W_0 \setminus L \) and extending \( H \) to be \( c \) on \( L \). By the choice of the constant \( c \), \( H \) must have two periodic orbits \( x_1, x_2 \subset W_0 \setminus L \) with \([x_1] = \gamma_0\) and \([x_2] = \gamma_0^{-1} \).

As in the proof of Theorem 3.5.1, every nonconstant periodic orbit of \( H \) represents a class that is positively proportional in \( \tilde{\pi}_1(W_0) \) to a class represented by an orbit of the \( S^1 \) action. Thus both \( \gamma_0 \) and \( \gamma_0^{-1} \) are positively proportional to classes represented by orbits of the \( S^1 \) action.

Suppose, by contradiction, that \( M \setminus L \) is connected. Then all the classes represented by orbits of the \( S^1 \) action are positively proportional in \( \tilde{\pi}_1(M \setminus L) \), in particular also in \( \tilde{\pi}_1(M) \). It follows that \( \gamma_0 = [x_1] \) and \( \gamma_0^{-1} = [x_2] \) become positively proportional when viewed as classes in \( \tilde{\pi}_1(M) \), namely the classes \( \iota_*(\gamma_0), (\iota_*(\gamma_0))^{-1} \in \tilde{\pi}_1(M) \) are positively proportional, where \( \iota : W_0 \to M \) is the inclusion. Hence there exists \( k > 0 \) such that \( (\iota_*(\gamma_0))^k \) is the trivial class in \( \tilde{\pi}_1(M) \). Passing to homology we get that \( k[L] = 0 \in H_1(M; \mathbb{Z}) \), where \([L] \in H_1(M; \mathbb{Z}) \) is the homology class represented by \( L \). But \( H_1(M; \mathbb{Z}) \) has no torsion, hence \([L] = 0 \). Finally, note that if an *embedded* circle \( L \) in an orientable surface \( M \) is zero in homology then \( M \setminus L \) must be disconnected. Contradiction.

### 3.5.2. Applications to Stein manifolds

Let \((W, J)\) be a Stein manifold. Recall that a smooth function \( \varphi : W \to \mathbb{R} \) is called **plurisubharmonic** if the 2-form
\[
\omega_\varphi := -d(d\varphi \circ J)
\]
is a $J$-positive symplectic form, i.e. $\omega_\varphi(v, Jv) > 0$ for every nonzero tangent vector $v \in TW$. We denote by
\[ g_\varphi(\cdot, \cdot) := \omega_\varphi(\cdot, J\cdot) \]
the associated Kähler metric. Let $\varphi : W \to \mathbb{R}$ be an **exhausting plurisubharmonic function**, namely in addition to being plurisubharmonic $\varphi$ is also proper, bounded from below and has no critical points outside some compact subset of $W$. Let
\[ X_\varphi := \text{grad}_{g_\varphi} \varphi \]
be the gradient vector field of $\varphi$ with respect to the metric $g_\varphi$. Then
\[ L_{X_\varphi} \omega_\varphi = d\iota_{X_\varphi} \omega_\varphi = -d(g_\varphi(X_\varphi, J\cdot)) = -d(d\varphi \circ J) = \omega_\varphi, \]
and hence the flow $X_\varphi^t : W \to W$ of $X_\varphi$ satisfies $(X_\varphi^t)^* \omega_\varphi = e^t \omega_\varphi$. Denote by $\Delta_\varphi$ the union of all the stable submanifold of the flow of $X_\varphi$:
\[ \Delta_\varphi := \bigcup_{p \in \text{Crit}(\varphi)} W_p^s(X_\varphi). \]
Note that $\Delta_\varphi \subset W$ is the maximal compact invariant subset for the flow of $X_\varphi$. We call $\Delta_\varphi$ the **associated skeleton** of $\varphi$. When $\varphi$ is Morse, each stable submanifold in the union (5) is isotropic with respect to $\omega_\varphi$ (see [EG]), in particular $\dim \Delta_\varphi \leq \frac{1}{2} \dim \mathbb{R} W$. We remark that, even if $\varphi$ is Morse, $\Delta_\varphi$ may have a quite “wild” structure. However, for a generic exhausting plurisubharmonic function $\varphi$, $\Delta_\varphi$ has the structure of an isotropic CW-complex (see [Bi]). In rare situations it may even happen that some $\varphi$’s (Morse or not) have smooth skeletons.

We now turn to a special class of Stein manifolds, namely those obtained from removing an ample divisor from a smooth algebraic variety. More precisely, let $(M, J)$ be a closed algebraic manifold and $\Sigma \subset M$ a smooth ample divisor. It is well known that $(M \setminus \Sigma, J)$ is an affine variety, in particular Stein. The following theorem deals with topological restrictions on the possible smooth manifolds that may arise as skeletons of Stein manifolds of the type just mentioned.

**Theorem 3.5.3.** Let $(M, J)$ be a closed algebraic manifold and $\Sigma \subset M$ a smooth and reduced ample divisor. If the Stein manifold $(M \setminus \Sigma, J)$ admits an exhausting plurisubharmonic function $\varphi : M \setminus \Sigma \to \mathbb{R}$ whose skeleton $\Delta_\varphi$ is a smooth connected Lagrangian submanifold, then $\Delta_\varphi$ cannot be of type $\mathcal{F}$. In particular, $\Delta_\varphi$ cannot be diffeomorphic to $\mathbb{T}^n$, $n \geq 2$, or to any closed negatively curved manifold.

**Proof.** Put $W := M \setminus \Sigma$ and endow $W$ with the complex structure $J$. The proof has two steps.
Step 1. For every exhausting plurisubharmonic function $\varphi : W \to \mathbb{R}$ there exists an open relatively compact domain $W_0 \subset W$ with the following properties:

(i) $W_0 \supset \Delta \varphi$.
(ii) The boundary $P := \partial W_0$ is smooth, connected, and convex with respect to $\omega_\varphi$.
(iii) The leaves of the characteristic foliation (with respect to $\omega_\varphi$) on $P$ are all orbits of a free circle action.

The idea of the proof is the following. Since $W$ is the complement of smooth ample divisor it is possible to endow $W$ with an exhausting plurisubharmonic function $\varphi_0$ for which the above statement holds. In order to pass from $\varphi_0$ to any given plurisubharmonic function $\varphi$ we modify $\varphi_0$ and $\varphi$ at infinity to obtain new plurisubharmonic functions $\overline{\varphi}_0$ and $\overline{\varphi}$ for which the vector fields $X_{\overline{\varphi}_0}$ and $X_{\overline{\varphi}}$ are complete. Then by the theory of Eliashberg-Gromov \[EG\] the symplectic forms $\omega_{\overline{\varphi}_0}$ and $\omega_{\overline{\varphi}}$ are diffeomorphic. Hence the statement being true for $\varphi_0$ is true also for $\varphi$.

Here are the precise details. We first adjust $\varphi$ at infinity so that the vector field $X_\varphi$ becomes complete. This is a standard procedure (\[EG\], see also \[Bi-Ci\] Lemma 3.1). More precisely, let $\varphi : W \to \mathbb{R}$ be an exhausting plurisubharmonic function with the following properties:

- $\varphi = \varphi$ on a relatively compact domain $W' \subset W$ that contains $\Delta \varphi$.
- $\Delta \varphi = \Delta \varphi$.
- The vector field $X_\varphi$ is complete.

Next we endow $W$ with another plurisubharmonic function $\varphi_0$ that arises from $W$ being the complement of an ample divisor. For this purpose, denote by $\mathcal{L} = \mathcal{O}_M(\Sigma) \to M$ the holomorphic line bundle defined by $\Sigma$, and let $s : M \to \mathcal{L}$ be a holomorphic section with $\Sigma = \{s = 0\}$. Since $\Sigma$ is ample there exists a hermitian metric $\| \cdot \|$ on $\mathcal{L}$ for which the associated metric connection $\nabla$ has positive curvature $R^\nabla$. Then the (real) 2-form $\omega = \frac{1}{2\pi i} R^\nabla$ is a $J$-compatible symplectic form. Let $\varphi_0 : W \to \mathbb{R}$ be the function defined by $\varphi_0 = -\frac{1}{4\pi} \log \|s\|^2$. Then on $W$ we have $\omega = -d(d\varphi_0 \circ J) = \omega_{\varphi_0}$, hence $\varphi_0$ is plurisubharmonic. Note that $\varphi_0$ is exhausting. Indeed $\varphi_0$ is proper, bounded below, and, since $\Sigma$ is smooth and reduced, $\varphi_0$ has no critical points near $\Sigma$.

Let $E(\rho) := \{v \in \mathcal{L}|_{\Sigma} \mid \|v\| < \rho\}$ and $P(\rho) := \{v \in \mathcal{L}|_{\Sigma} \mid \|v\| = \rho\}$ be the radius-$\rho$ disc and circle subbundles of $\mathcal{L}|_{\Sigma}$ and denote by $\pi : E(\rho) \to \Sigma$ the projection. Pick a connection 1-form $\gamma$ on $P(1)$ such that $d\gamma = -\pi^*(\omega|_{\Sigma})$ and consider the symplectic form $\omega_0 = \pi^*(\omega|_{\Sigma}) + d(r^2 \gamma)$ where $r$ is the radial coordinate on the fibres induced by the hermitian metric. A simple computation shows that the vector field $Z := \frac{r^2 - 1}{2r} \frac{\partial}{\partial r}$ defined on $E(1) \setminus \Sigma$ satisfies $\mathcal{L}_Z \omega_0 = \omega_0$. Moreover for every $0 < \rho < 1$, $Z$ is transverse to $P(\rho)$ and points towards the inside of $E(\rho)$. Finally note that the leaves of characteristic foliation
on $P(\rho)$ are all orbits of a free circle action, in fact they all coincide with the fibres of the principal circle bundle $P(\rho) \to \Sigma$. Since $\Sigma$ is connected so is $P(\rho)$.

By the symplectic tubular neighbourhood theorem there exist a neighbourhood $B$ of $\Sigma$ in $M$, an $\varepsilon > 0$, and a symplectomorphism $f : (E(\varepsilon), \omega_0) \to (B, \omega)$ that sends $\Sigma \subset M$ identically onto $\Sigma \subset E(\varepsilon)$. Pick any $0 < \varepsilon_0 < \varepsilon$ and put $B_0 := f(E(\varepsilon_0))$ and $U_0 := M \setminus B_0$. Then $U_0 \subset W$ is a relatively compact domain with convex boundary $\partial U_0$ on which the characteristic foliation coincides with the orbits of a free circle action. Taking $\varepsilon_0$ to be smaller if necessary we may assume that $U_0$ contains $\Delta_{\varphi_0}$.

Similarly to $\varphi$ we adjust $\varphi_0$ at infinity so that the vector field $X_{\varphi_0}$ becomes complete. More precisely, let $\overline{\varphi}_0 : W \to \mathbb{R}$ be an exhausting plurisubharmonic function with the following properties:

- $\overline{\varphi}_0 = \varphi_0$ on an open subset containing $\overline{U}_0$.
- $\Delta_{\overline{\varphi}_0} = \Delta_{\varphi_0}$.
- The vector field $X_{\overline{\varphi}_0}$ is complete.

By [EC] there exists a symplectomorphism $F : (W, \omega_{\overline{\varphi}_0}) \to (W, \omega_\varphi)$. Denote by $X^t_{\overline{\varphi}_0}$ the flow of $X_{\overline{\varphi}_0}$. Then we have $\bigcup_{t \geq 0} X^t_{\overline{\varphi}_0}(U_0) = W$, hence for $t_0 > 0$ large enough $F(X^t_{\overline{\varphi}_0}(U_0)) \supset \Delta_\varphi$. Clearly, the domain $F(X^t_{\overline{\varphi}_0}(U_0)) \subset (W, \omega_\varphi)$ has a smooth convex connected boundary whose characteristic foliation has leaves which are orbits of a free circle action. Since $(X^{-t}_{\overline{\varphi}})^* \omega_\varphi = e^{-t} \omega_\varphi$ the same holds also for each of the domains $X^{-t}_\varphi \circ F \circ X^t_{\overline{\varphi}_0}(U_0)$. Pick now $t_1 > 0$ large enough so that $W_0 := X^{-t_1}_\varphi \circ F \circ X^t_{\overline{\varphi}_0}(U_0) \subset W'$. Recall that on $W'$ we have $\omega_\varphi = \omega_\varphi$. Hence the domain $W_0$ satisfies all three conditions claimed in Step 1.

**Step 2. We prove the theorem.**

Let $\varphi : W \to \mathbb{R}$ be an exhausting plurisubharmonic function with skeleton $\Delta_\varphi$ which is a connected smooth manifold of type $\mathcal{F}$.

By Darboux' theorem there exist a neighbourhood $V(\Delta_\varphi)$ of $\Delta_\varphi$ and a symplectic embedding $g : (V(\Delta_\varphi), \omega_\varphi) \to (T^*\Delta_\varphi, \omega_\text{can})$ which takes $\Delta_\varphi \subset V(\Delta_\varphi)$ identically onto the zero section $\Delta_\varphi \subset T^*\Delta_\varphi$.

Let $X_\varphi := \text{grad}_{g^* \omega_\varphi} \varphi$, denote by $X^t_\varphi$ the flow of $X_\varphi$, and recall that $(X^t_\varphi)^* \omega_\varphi = e^t \omega_\varphi$. Let $W_0 \subset W$ be the domain defined by Step 1 and let $W_1 \subset W$ be a relatively compact domain containing $\overline{W}_0$. From the definition of $\Delta_\varphi$ it follows that for $T > 0$ large enough we have $\Delta_\varphi \subset X^{-T}_\varphi(W_1) \subset V(\Delta_\varphi)$.

Denote by $Y = \frac{\partial}{\partial y}$ the standard Liouville vector field on $T^*\Delta_\varphi$. Then its flow $Y^t$ satisfies $(Y^t)^* \omega_\text{can} = e^t \omega_\text{can}$. Hence $Y^T \circ g \circ X^{-T}_\varphi : (W_1, \omega_\varphi) \to (T^*\Delta_\varphi, \omega_\text{can})$ is a symplectic embedding. It follows that $U := Y^T \circ g \circ X^{-T}_\varphi(W_0) \subset T^*\Delta_\varphi$ is an open relatively compact domain containing the zero section and with convex boundary $Q := Y^T \circ g \circ X^{-T}_\varphi(P)$. 

Moreover, by condition (iii) in Step 1, the leaves of the characteristic foliation on $Q$ coincide with the orbits of a free circle action on $Q$. Since $Q$ is connected, it follows that any two leaves of the characteristic foliation on $Q$ represent positively proportional homotopy classes. On the other hand, $\Delta_{\varphi}$ is of type $F$ and so, by Corollary 3.4.3, the characteristic foliation of $Q$ contains two closed leaves whose homotopy classes are positively non-proportional. Contradiction. 

4. A HOMOLOGICAL CAPACITY

In order to study and compute the relative capacity $C(M, A; \alpha)$ we shall define another quantity $\widehat{C}(M, A; \alpha) \geq C(M, A; \alpha)$ which captures the existence of homologically essential periodic orbits in given homotopy classes. Here the term “homologically essential” refers to Floer homology. The homological capacity $\widehat{C}(M, A; \alpha)$ will be defined in purely Floer-homological terms. It is easier to compute and enjoys some nice functorial properties. We begin with a brief discussion of convex boundaries.

4.1. Convex boundaries. Let $(\overline{M}, \omega)$ be a compact connected symplectic manifold with convex boundary and denote $M := \overline{M} \setminus \partial \overline{M}$. Recall (\cite{EG}, see also Section 3.4 above) that the boundary is called convex if there exist a vector field $X \in \text{Vect}(\overline{M})$ and a neighbourhood $U$ of $\partial \overline{M}$ such that $X$ points out on the boundary and is dilating on $U$, namely $\mathcal{L}_X \omega = \omega$ on $U$. Let $\varphi_t$ denote the flow of $X$, suppose that $U = \{ \varphi_t(x) \mid x \in \partial M, -\varepsilon < t \leq 0 \}$, and denote by $\xi := \ker(\iota(X) \omega|_{\partial \overline{M}})$ the contact structure on the boundary determined by $X$ and $\omega$. Under these hypotheses (the existence of $X$ and $U$) there is an $\omega$-compatible almost complex structure $J$ on $\overline{M}$ such that

$$J \xi = \xi, \quad \omega(X(x), J(x)X(x)) = 1, \quad D\varphi_t(x)J(x) = J(\varphi_t(x))D\varphi_t(x)$$

for all $x \in \partial \overline{M}$ and $t \in (-\varepsilon, 0]$. Such an almost complex structure is called convex near the boundary.

Consider the function $f : U \to \mathbb{R}$ given by

$$f(\varphi_t(x)) := e^t$$

for $x \in \partial \overline{M}$ and $-\varepsilon < t \leq 0$. A simple computation shows that its gradient with respect to the metric $\omega(\cdot, J \cdot)$ is $X$ and hence $-d(df \circ J) = \omega$. This means that $f$ is plurisubharmonic on $U$, or in other words the 2-form $-d(df \circ J)$ is positive on every $J$-complex tangent line in $TU$. Let $u : \Omega \to U$ be a nonconstant $J$-holomorphic curve, defined on a connected open subset $\Omega \subset \mathbb{C}$. Then the function $f \circ u : \Omega \to \mathbb{R}$ is subharmonic and, by the mean value inequality, cannot have a strict interior maximum. Hence a nonconstant $J$-holomorphic curve in $\overline{M}$ cannot intersect $\partial \overline{M}$ at an interior point of its domain $\Omega$. 

Remark 4.1.1. Since $J$ is invariant under the flow of $X$ (near $\partial \overline{M}$) we have $0 = (L_X J)v = (\nabla_X J)v + J\nabla_v X - \nabla_{Jv} X$, where $\nabla$ denotes the Levi-Civita connection of the metric $\omega(\cdot, J\cdot)$. Now consider the 1-form $\alpha := -df \circ J = \iota(X)\omega = \langle JX, \cdot \rangle$. A simple calculation, using $L_X J = 0$, shows that $d\alpha(v, w) = \langle \nabla_v (JX), w \rangle - \langle \nabla_w (JX), v \rangle = 2\langle J\nabla_v X, w \rangle$. Hence the identity $d\alpha = \omega$ is equivalent to $\nabla_v X = v/2$. Since $X$ is the gradient of $f$ it follows that the Laplacian of $f \circ u$ is given by $\Delta(f \circ u) = |du|^2/2$ for every $J$-holomorphic curve $u : \Omega \to \overline{M}$.

Remark 4.1.2. The space of almost complex structures on $\overline{M}$ that are convex near the boundary is connected. To see this fix first the dilating vector field $X$ on a neighbourhood of $\partial M$. Then the space of $\omega$-compatible almost complex structures on the symplectic bundle $\xi = \ker(\iota(X)\omega|_{\partial M})$ is connected (see [MS]), hence the space of $\omega$-compatible almost complex structures $J$ satisfying (4.1) is also connected. Finally, we may allow $X$ to vary since the space of vector fields that are dilating near $\partial M$ and point out on $\partial M$ is convex hence connected.

4.2. The setting. From now on our standing hypotheses are that $(\overline{M}, \omega)$ is a compact connected symplectic manifold with convex boundary $\partial \overline{M}$ and $A \subset M := \overline{M} \setminus \partial \overline{M}$ is a compact subset. We assume that the symplectic form is exact and fix a 1-form $\lambda \in \Omega^1(\overline{M})$ such that $d\lambda = \omega$. We call $\lambda$ an $\omega$-primitive.

As in section 2.1, we denote by $\mathcal{H} := \mathcal{H}(M) := C_0^\infty(S^1 \times M)$ the space of smooth compactly supported smooth Hamiltonian functions on $S^1 \times M$ and by $\mathcal{D} \subset \text{Diff}_0(M)$ the group of Hamiltonian diffeomorphisms of $M$ generated by functions from $\mathcal{H}$. For $c > 0$ we denote by $\mathcal{H}_c = \mathcal{H}_c(M, A)$ the subspace of all Hamiltonian functions $H \in \mathcal{H}$ that satisfy $\inf_{S^1 \times A} h \geq c$ and by $\mathcal{D}_c = \mathcal{D}_c(M, A)$ the set of all Hamiltonian diffeomorphisms that are generated by functions from $\mathcal{H}_c$.

4.3. Floer homology. Floer homology is an essential ingredient in the definition of our invariants. The purpose of this section is to summarize the main building blocks of this theory needed for our applications. The reader is referred to [F4, FH2, CFH, V] for a detailed foundation of the subject (see also [Sa] for a general exposition).

Fix a nontrivial free homotopy class $\alpha \in \tilde{\pi}_1(M)$ and recall from Section 2.1 that $\mathcal{P}(H; \alpha) \subset L_\alpha M$ denotes the set of periodic solutions of the Hamiltonian system associated to $H \in \mathcal{H}$ and that these periodic solutions are the critical points of the symplectic action functional $\mathcal{A}_H : L_\alpha M \to \mathbb{R}$ defined by (4). The set of critical values of $\mathcal{A}_H$ is called
the action spectrum and will be denoted by

$$S(H;\alpha) := A_H(\mathcal{P}(H;\alpha)) = \{A_H(x) \mid x \in L_\alpha M, \dot{x}(t) = X_{H_t}(x(t))\}.$$ 

Here $X_{H_t} \in \text{Vect}(M)$ is given by $\iota(X_{H_t})\omega = -dH_t$. Now let $-\infty \leq a < b \leq \infty$ and denote by $\mathcal{P}^{[a,b]}(H;\alpha)$ the set of 1-periodic solutions of the Hamiltonian system associated to $H$ that represent the class $\alpha$ and whose action lies in the interval $[a,b)$:

$$\mathcal{P}^{[a,b]}(H;\alpha) := \mathcal{P}^b(H;\alpha) \setminus \mathcal{P}^a(H;\alpha), \quad \mathcal{P}^a(H;\alpha) := \{x \in \mathcal{P}(H;\alpha) \mid A_H(x) < a\}.$$ 

Suppose that $H \in \mathcal{H}$ is a Hamiltonian function that satisfies the following hypothesis:

(H) $a, b \notin S(H;\alpha)$ and every 1-periodic orbit $x \in \mathcal{P}(H;\alpha)$ is nondegenerate.

Then the Floer homology group $HF^{[a,b]}(H;\alpha)$ is defined as the homology of a chain complex over $\mathbb{Z}_2$ generated by the 1-periodic orbits in $\mathcal{P}^{[a,b]}(H;\alpha)$. It is useful to think of this chain complex as the quotient

$$\text{CF}^{[a,b]}(H;\alpha) := \text{CF}^b(H;\alpha) / \text{CF}^a(H;\alpha), \quad \text{CF}^a(H;\alpha) := \bigoplus_{x \in \mathcal{P}^a(H;\alpha)} \mathbb{Z}_2 x.$$ 

The Floer boundary operator is defined as follows. Let $J_t = J_{t+1} \in \mathcal{J}(M,\omega)$ be a $t$-dependent smooth family of $\omega$-compatible almost complex structures on $\overline{M}$ such that $J_t$ is convex and independent of $t$ near the boundary $\partial\overline{M}$ (see Section 4[4]). Consider the Floer differential equation

$$\partial_s u + J_t(u) (\partial_t u - X_{H_t}(u)) = 0. \quad (6)$$

For a smooth solution $u : \mathbb{R} \times S^1 \to M$ of (6) define its energy to be

$$E(u) := \int_0^1 \int_{-\infty}^\infty |\partial_s u|^2 ds dt.$$ 

Then if $u : \mathbb{R} \times S^1 \to M$ is a smooth solution of (6) with finite energy then the limits

$$\lim_{s \to \pm\infty} u(s,t) = x^\pm(t), \quad \lim_{s \to \pm\infty} \partial_s u(s,t) = 0 \quad (7)$$

exist and are uniform in the $t$-variable. Moreover, $x^\pm \in \mathcal{P}(H)$ and we have

$$E(u) = A_H(x^-) - A_H(x^+).$$

The following observations allow us to define Floer homology groups in the present situation.

(i) Since every periodic solution $x \in \mathcal{P}(M;\alpha)$ is nonconstant (the class $\alpha$ is nontrivial) and $J$ is convex near the boundary, there exists an open set $U \subset M$ such that $M \setminus U$ is compact and $u(\mathbb{R} \times S^1) \cap U = \emptyset$ for every finite energy solution of (6).

1We use the convention that the complex generated by the empty set is 0.
(ii) By (i) and the energy identity, the space of finite energy solutions of (9) is compact with respect to $C^\infty$-convergence on compact sets, i.e. only the splitting into a finite sequence of adjacent Floer connecting orbits can occur in the limit.

(iii) For a generic family of almost complex structures $J = \{J_t\}$ (that are convex and independent of $t$ on $U$) the linearized operator for equation (9) is surjective for every finite energy solution of (9) in the homotopy class $\alpha$ (see [FHS]). Such a family of almost complex structures is called regular and the space of regular families of almost complex structures will be denoted by $J_{\text{reg}}(H; \alpha)$.

For every $J \in J_{\text{reg}}(H; \alpha)$ and every pair $x^\pm \in \mathcal{P}(H; \alpha)$ the space $M(x^-, x^+; H, J)$ of solutions of (6) and (7) is a smooth manifold whose dimension near a solution $u$ of (6) and (7) is given by the difference of the Conley–Zehnder indices (see [SZ]) of $x^-$ and $x^+$ (relative to $u$). The subspace of solutions of index one will be denoted by $M_1(x^-, x^+; H, J)$. For $J \in J_{\text{reg}}(H; \alpha)$ it follows from (i) and (ii) that the quotient $M_1(x^-, x^+; H, J)/\mathbb{R}$ (modulo time shift) is a finite set for every pair $x^\pm \in \mathcal{P}(H; \alpha)$. The Floer boundary operator $\partial^{H,J}$ on $\text{CF}^b(H; \alpha)$ is defined by

$$\partial^{H,J}x := \sum_{y \in \mathcal{P}^b(H; \alpha)} \#(M_1(x, y; H, J)/\mathbb{R}) y$$

for every $x \in \mathcal{P}^b(H; \alpha)$. That this is indeed a boundary operator is proved as in Floer’s original work [F4]. The energy identity shows that $\text{CF}^a(H; \alpha)$ is a subcomplex, namely it is invariant under the Floer boundary operator. We thus get an induced boundary operator $[\partial^{H,J}]$ on the quotient $\text{CF}^{[a,b]}(H; \alpha)$. We denote the homology of the quotient complex by

$$\text{HF}^{[a,b]}(H, J; \alpha) := \frac{\ker([\partial^{H,J}] : \text{CF}^{[a,b]}(H; \alpha) \to \text{CF}^{[a,b]}(H; \alpha))}{\text{im}([\partial^{H,J}] : \text{CF}^{[a,b]}(H; \alpha) \to \text{CF}^{[a,b]}(H; \alpha))}.$$

These Floer homology groups are independent of the choice of the almost complex structure $J = \{J_t\}_{t \in S^1}$ in the sense that for any two almost complex structures $J_0, J_1 \in J_{\text{reg}}(H; \alpha)$ there is a natural isomorphism

$$\tau_{J_1, J_0} : \text{HF}^{[a,b]}(H, J_0; \alpha) \to \text{HF}^{[a,b]}(H, J_1; \alpha).$$

If the two almost complex structure agree near the boundary then this follows from the standard arguments as in Floer’s original paper [F4] (choose a homotopy of almost complex structures $\{J_{s,t}\}$ from $J_0$ to $J_1$, independent of $s$ and $t$ near the boundary, and use the solutions of equation (8) below with $H_{s,t} = H_t$ to construct the isomorphism between the two Floer homology groups; see also [Sa, SZ]). To show that the Floer homology groups are also independent of the choice of the convex almost complex structure near the boundary one can use the fact that the space of convex almost complex structures near
the boundary is connected (Remark 4.1.2) and that the Floer chain complex associated
to a regular almost complex structure remains unchanged under sufficiently small pertur-
bations of \( J \). The upshot is that the Floer homology groups are independent of \( J \) up to
natural isomorphisms. For this reason we shall sometimes drop the argument \( J \) and refer
to \( HF^{[a,b]}(H; \alpha) := HF^{[a,b]}(H, J; \alpha) \) as the Floer homology associated to \( H \).

### 4.4. Homotopy invariance.

Following the work of Floer–Hofer [FH2], Cieliebak-Floer-Hofer [CFH], and Viterbo [V] we describe the local isomorphisms of Floer homology in a
given interval of the action spectrum. Consider the space

\[
\mathcal{H}^{a,b}(M; \alpha) := \{ H \in \mathcal{H}(M) \mid a, b \notin S(H; \alpha) \}
\]

of all Hamiltonians \( H \in \mathcal{H} \) that do not contain \( a \) and \( b \) in their action spectrum. We
consider the space \( \mathcal{H} \) with the strong Whitney \( C^\infty \)-topology. Note that the action spec-
trum \( S(H; \alpha) \) is compact for every \( H \) and is a lower semicontinuous function of \( H \)
(i.e. for every open neighbourhood \( V \subset \mathbb{R} \) of \( S(H; \alpha) \) there exists a neighbourhood \( U \subset \mathcal{H} \) of \( H \) such that \( S(H'; \alpha) \subset V \) for every \( H' \in U \)). Hence the set \( \mathcal{H}^{a,b}(M; \alpha) \) is open in \( \mathcal{H} \). We
now explain why the Floer homology groups \( HF^{[a,b]}(H; \alpha) \) are independent of \( H \) in every
component of \( \mathcal{H}^{a,b}(M; \alpha) \).

Fix a Hamiltonian function \( H \in \mathcal{H}^{a,b}(M; \alpha) \) and choose a (convex) neighbourhood \( U \) of
\( H \) such that \( U \subset \mathcal{H}^{a,b}(M; \alpha) \). Now suppose that \( H^+, H^- \in U \) satisfy \( (H) \), i.e. all periodic
solutions \( x \in \mathcal{P}(H^\pm; \alpha) \) are nondegenerate. Connect \( H^- \) and \( H^+ \) by a smooth homotopy
\( \mathbb{R} \mapsto U : s \mapsto H_s = \{ H_{s,t} \} \) such that \( H_{s,t} = H^-_t \) for \( s \leq -T \) and \( H_{s,t} = H^+_t \) for \( s \geq T \).
Consider the equation

\[
\partial_s u + J_{s,t}(u)(\partial_t u - X_{H_{s,t}}(u)) = 0,
\]
where \( s \mapsto \{ J_{s,t} \} \) is a regular homotopy of families of almost complex structures. This
means that \( J_{s,t} \) satisfies the following conditions.

- \( J_{s,t} \) is convex and independent of \( s \) and \( t \) near the boundary of \( \overline{M} \).
- \( J_{s,t} = J_t^- \) is regular for \( H^-_t \) for \( s \leq -T \).
- \( J_{s,t} = J_t^+ \) is regular for \( H^+_t \) for \( s \geq T \).
- The finite energy solutions of \( (8) \) are transverse (i.e. the associated Fredholm oper-
ators are surjective) and hence form finite dimensional moduli spaces.

The key observation is the energy identity

\[
E(u) = A_{H^-}(x^-) - A_{H^+}(x^+) + \int_0^1 \int_{-\infty}^\infty (\partial_s H)(s, t, u(s, t)) \, ds \, dt
\]
for every solution of (8) and (7). It follows from (9) that

$$A_{H^+}(x^+) \leq A_{H^-}(x^-) + \int_{-\infty}^{\infty} \max_{S^1 \times M} \partial_s H_s \, ds.$$  

In particular, if the homotopy has the form $H_{s,t} := H_{0,t} + \beta(s)(H_1^+ - H_1^-)$ for a non-decreasing function $\beta : \mathbb{R} \to [0, 1]$ we obtain $\partial_s H_s = \beta(s)(H_1^+ - H_1^-)$ and hence

$$(10) \quad A_{H^+}(x^+) \leq A_{H^-}(x^-) + \max_{S^1 \times M} (H_1^+ - H_1^-).$$

Now choose $\varepsilon > 0$ such that

$$\mathcal{S}(H^\pm; \alpha) \cap [a - 4\varepsilon, a + 4\varepsilon] = \emptyset, \quad \mathcal{S}(H^\pm; \alpha) \cap [b - 4\varepsilon, b + 4\varepsilon] = \emptyset,$$

and suppose that $\sup_{S^1 \times M} |H^\pm - H| \leq \varepsilon$. Then $\sup_{S^1 \times M} |H_1^+ - H_1^-| \leq 2\varepsilon$ and hence, by (10), the Floer chain map (see [F4, FH2, CFH, S4, S2, S3]) from $\text{CF}(H^-; \alpha)$ to $\text{CF}(H_1^+; \alpha)$ defined by the solutions of (8) preserves the subcomplexes $\text{CF}^a$ and $\text{CF}^b$. The same applies to the Floer chain map from $\text{CF}(H_1^+; \alpha)$ to $\text{CF}(H^-; \alpha)$ and to the chain homotopy equivalence associated to a suitable homotopy of homotopies. Hence the solutions of (8) define a homomorphism $\text{CF}^{[a,b]}(H^-; \alpha) \to \text{CF}^{[a,b]}(H_1^+; \alpha)$ which induces an isomorphism of Floer homology, whenever $H^\pm$ are sufficiently close to a given Hamiltonian function $H \in \mathcal{H}^{a,b}(M; \alpha)$.

Remark 4.4.1. (Local isomorphisms) The above discussion shows that every Hamiltonian function $H \in \mathcal{H}^{a,b}(M; \alpha)$ has a neighbourhood $\mathcal{U}$ such that the Floer homology groups $\text{HF}^{[a,b]}(H'; J'; \alpha)$, for every $H' \in \mathcal{U}$ that satisfies (H) and every regular almost complex structure $J' \in \mathcal{J}_{\text{reg}}(H'; \alpha)$, are naturally isomorphic. We can use these local isomorphisms to define the Floer homology groups $\text{HF}^{[a,b]}(H; \alpha)$ for every Hamiltonian $H \in \mathcal{H}^{a,b}(M; \alpha)$, whether or not it satisfies (H).

Remark 4.4.2. (Contractible loops) When $\alpha \in \tilde{\pi}_1(M)$ is the homotopy class of the constant loops we are not allowed to work with intervals $[a, b]$ that contain zero, since the Hamiltonians we work with always have degenerate periodic orbits with action zero as they vanish at infinity. In this case we are forced to work with either $0 < a < b \leq \infty$ or $-\infty \leq a < b < 0$.

Remark 4.4.3. (Composition) We emphasize that the canonical isomorphism

$$\text{HF}^{[a,b]}(H^-, J^-; \alpha) \to \text{HF}^{[a,b]}(H_1^+, J_1^+; \alpha)$$

only exists locally, when $H^\pm$ are sufficiently close to a given Hamiltonian function $H \in \mathcal{H}^{a,b}(M; \alpha)$. It is easy to construct Hamiltonian functions $H_0, H_1 \in \mathcal{H}^{a,b}(M; \alpha)$ such that $\text{HF}^{[a,b]}(H_0; \alpha)$ is not isomorphic to $\text{HF}^{[a,b]}(H_1; \alpha)$. If $H_0$ and $H_1$ belong to the same
component of $\mathcal{H}^{a,b}(M; \alpha)$ then there is a smooth path $[0, 1] \to \mathcal{H}^{a,b}(M; \alpha) : s \mapsto H_s$
connecting $H_0$ to $H_1$. Hence in this case $\text{HF}^{[a,b]}(H_0; \alpha)$ is isomorphic to $\text{HF}^{[a,b]}(H_1; \alpha)$. However, in general the isomorphism cannot be defined directly in terms of the solutions of (8). It can only be constructed as a composition of isomorphisms

$$\text{HF}^{[a,b]}(H_{s_i}; \alpha) \to \text{HF}^{[a,b]}(H_{s_{i+1}}; \alpha)$$

for a regular homotopy, where each of these isomorphisms is defined in terms of the solutions of (8). Moreover, it is an open question if this composition along a loop $s \mapsto H_s$ with $H_0 = H_1$ is always the identity.

4.5. **Monotone homotopies.** Suppose that $H_0, H_1 \in \mathcal{H}^{a,b}(M; \alpha)$ satisfy

$$H_0(t, x) \geq H_1(t, x)$$

for all $(t, x) \in \mathbb{R} \times M$ as well as $(H)$. Then there exists a homotopy $s \mapsto H_s$ from $H_0$ to $H_1$ such that $\partial_s H_s \leq 0$. We call such a homotopy of Hamiltonian functions **monotone**. In the monotone case it follows from (9) that the Floer chain map $\text{CF}(H_0; \alpha) \to \text{CF}(H_1; \alpha)$, defined in terms of the solutions of (8) preserves the subcomplexes $\text{CF}^a$ and $\text{CF}^b$. Hence every monotone homotopy $s \mapsto H_s$ induces a natural homomorphism

$$\sigma_{H_1 H_0} : \text{HF}^{[a,b]}(H_0; \alpha) \to \text{HF}^{[a,b]}(H_1; \alpha).$$

We call such a homomorphism **monotone**. The standard arguments in Floer homology [F2, FH2, CFH, V, Sa, SZ] show that this homomorphism is independent of the choice of the monotone homotopy of Hamiltonians, used to define it, and that

$$\sigma_{H_2 H_1} \circ \sigma_{H_1 H_0} = \sigma_{H_2 H_0},$$

whenever $H_0, H_1, H_2 \in \mathcal{H}^{a,b}(M; \alpha)$ satisfy $H_0 \geq H_1 \geq H_2$, and $\sigma_{HH} = \text{id}$ for every $H \in \mathcal{H}^{a,b}(M; \alpha)$.

The homomorphism $\sigma_{H_1 H_0}$ is in general neither injective nor surjective. For example, it may happen that during the homotopy the action of some periodic orbit of $H_s$ leaves or enters the interval $[a, b)$. It turns out that this is the only possible reason for $\sigma_{H_1 H_0}$ not to be an isomorphism. More precisely, we have the following proposition which is an easy consequence of the theory developed in [FH1, CFH] (as outlined above) and appears in an explicit form in [V].

**Proposition 4.5.1.** Let $-\infty \leq a < b \leq \infty$, $\alpha \in \pi_1(M)$ be a nontrivial homotopy class, and $H_0, H_1 \in \mathcal{H}^{a,b}(M; \alpha)$ be such that $H_0 \geq H_1$. Suppose that there exists a monotone homotopy $\{H_s\}_{0 \leq s \leq 1}$ from $H_0$ to $H_1$ such that $H_s \in \mathcal{H}^{a,b}(M; \alpha)$ for every $s \in [0, 1]$. Then $\sigma_{H_1 H_0} : \text{HF}^{[a,b]}(H_0; \alpha) \to \text{HF}^{[a,b]}(H_1; \alpha)$ is an isomorphism. This continues to hold for the trivial homotopy class $\alpha = 0$ provided that $0 \notin [a, b]$. 
Proof. The monotone homomorphism \( \sigma_{H_{s_1}H_0} \) agrees with the local isomorphism of Section 4.4 whenever \( s_0 \) and \( s_1 \) are both sufficiently close to a number \( s \in [0, 1] \) such that \( H_s \in \mathcal{H}^{a,b}(H_s; \alpha) \). By assumption, we have \( H_s \in \mathcal{H}^{a,b}(H_s; \alpha) \) for every \( s \in [0, 1] \). Hence we can write \( \sigma_{H_{s_1}H_0} \) as a composition of finitely many isomorphisms of the form \( \sigma_{H_{s_i+1}H_{s_i}} \), where \( 0 = s_0 < s_1 < \cdots < s_{N-1} < s_N = 1 \).

4.6. Direct and inverse limits. The next step towards defining the relative capacity is to define two kinds of symplectic homologies especially suited for our purposes. The definitions of these invariants require the algebraic notions of direct and inverse limits. In this subsection we recall the basic definitions (for more details see [GM], but note that below we use somewhat different conventions than theirs).

Let \((I, \preceq)\) be a partially ordered set. Think of \( I \) as a category with precisely one morphism from \( i \) to \( j \) whenever \( i \preceq j \). Let \( R \) be a commutative ring. A partially ordered system of \( R \)-modules over \( I \) is a functor from \((I, \preceq)\) into the category of \( R \)-modules. We write this functor as a pair \((G, \sigma)\) where \( G \) assigns to each \( i \in I \) an \( R \)-module \( G_i \) and \( \sigma \) assigns to each pair \( i, j \in I \) with \( i \preceq j \) a homomorphism \( \sigma_{ji} : G_i \to G_j \) such that \( \sigma_{kj} \circ \sigma_{ji} = \sigma_{ki} \) for \( i \preceq j \preceq k \) and \( \sigma_{ii} = \text{id} \) is the identity map on \( G_i \).

The partially ordered set \((I, \preceq)\) is called upward directed if for every pair \( i, j \in I \) there exists an \( \ell \in I \) such that \( i \preceq \ell \) and \( j \preceq \ell \). In this case the functor \((G, \sigma)\) is called a directed system of \( R \)-modules. The direct limit of such a directed system is defined as the quotient

\[
\varinjlim_{i \in I} G := \varinjlim_{i \in I} G_i := \{(i, x) \mid i \in I, x \in G_i\} / \sim
\]

where \((i, x) \sim (j, y)\) iff there exists an \( \ell \in I \) such that \( i \preceq \ell \), \( j \preceq \ell \) and \( \sigma_{\ell i}(x) = \sigma_{\ell j}(y) \). Since \( I \) is upward directed, this is an equivalence relation. The direct limit is an \( R \)-module with the operations \([i, x] + [j, y] := [\ell, \sigma_{\ell i}(x) + \sigma_{\ell j}(y)]\) for \( \ell \in I \) such that \( i \preceq \ell \) and \( j \preceq \ell \) and \( r[i, x] := [i, rx] \) for every \( r \in R \). For \( i \in I \) we denote by \( \iota_i : G_i \to \varinjlim G \) the homomorphism given by \( \iota_i(x) := [i, x] \). Then \( \iota_i = \iota_j \circ \sigma_{ji} \) for \( i \preceq j \). Despite the notation, \( \iota_i \) need not be injective. Up to isomorphism the direct limit is characterized by the following universal property. If \( H \) is any \( R \)-module and \( \tau_i : G_i \to H \) is a family of homomorphisms, indexed by \( i \in I \), such that \( \tau_i = \tau_j \circ \sigma_{ji} \) whenever \( i \preceq j \), then there exists a unique homomorphism \( \tau : \varinjlim G \to H \) such that \( \tau_i = \tau \circ \iota_i \) for every \( i \in I \). (The homomorphism \( \tau \) is given by \([i, x] \mapsto \tau_i(x)\).)

The partially ordered set \((I, \preceq)\) is called downward directed if for every pair \( i, j \in I \) there exists a \( k \in I \) such that \( k \preceq i \) and \( k \preceq j \). In this case the functor \((G, \sigma)\) is called an inverse system of \( R \)-modules. The inverse limit of such an inverse system is defined
as
\[
\lim G := \lim_{i \in I} G_i := \left\{ \{x_i\}_{i \in I} \in \prod_{i \in I} G_i \mid i \not\preceq j \implies \sigma_{ji}(x_i) = x_j \right\}.
\]
For \( j \in I \) we denote by \( \pi_j : \lim_{i \in I} G_i \to G_j \) the obvious projection to the \( j \)th component. Then \( \pi_j = \sigma_{ji} \circ \pi_i \) for \( i \not\preceq j \). Despite the notation, \( \pi_j \) need not be surjective. Up to isomorphism the inverse limit is characterized by the following universal property. If \( H \) is any \( R \)-module and \( \tau_j : H \to G_j \) is a family of homomorphisms, indexed by \( j \in I \), such that \( \tau_j = \sigma_{ji} \circ \tau_i \) whenever \( i \not\preceq j \), then there exists a unique homomorphism \( \tau : H \to \lim_{i \in I} G_i \) such that \( \tau_j = \pi_j \circ \tau \) for every \( j \in I \). (The homomorphism \( \tau \) is given by \( y \mapsto \{\tau_i(y)\}_{i \in I} \).)

Remark. Note that inverse and direct limits are related via the following duality. Let \( (G, \sigma) \) be a directed system of \( R \)-modules and \( H \) be any \( R \)-module. Denote by \( (I^*, \preceq^*) \) the oppositely partially ordered set, namely \( I^* := I \) and \( i \preceq^* j \) iff \( i \succeq j \). Then there exists a canonical isomorphism
\[
\text{Hom}_R(\lim_{i \in I} G_i, H) \cong \lim_{i \in I^*} \text{Hom}_R(G_i, H).
\]
In particular, if \( R \) is a field and the \( G_i \) are vector spaces over \( R \) then \( (\lim_{i \in I} G_i)^* \cong \lim_{i \in I^*} G_i^* \).

In most of our applications the partially ordered set \( (I, \preceq) \) will be bidirected, i.e. both upward and downward directed. In this case we call the functor \( (G, \sigma) \) a bidirected system of \( R \)-modules. The next lemma follows directly from the definitions.

**Lemma 4.6.1.** Let \( (I, \preceq) \) be a downward directed partially ordered set and \( I' \subset I \) be an upward directed subset (with respect to the restriction of the partial order \( \preceq \) to \( I' \)). Let \( (G, \sigma) \) be a partially ordered system of \( R \)-modules over \( I \). Then there exists a unique homomorphism
\[
T : \lim_{i \in I} G_i \to \lim_{i' \in I'} G_{i'}
\]
such that the following diagram commutes for all \( j', k' \in I' \) with \( j' \preceq k' \):
\[
\begin{array}{ccc}
\lim_{i \in I} G_i & \xrightarrow{T} & \lim_{i' \in I'} G_{i'} \\
\pi_{j'} \downarrow & & \sigma_{k',j'} \downarrow \\
G_{j'} & \xrightarrow{\iota_{k'}} & G_{k'}
\end{array}
\]

**Proof.** The map \( T \) is given by \( \{x_i\}_{i \in I} \mapsto [i', x_{i'}] \) for \( i' \in I' \). By definition of direct and inverse limits, this map is independent of the choice of \( i' \).
\[\Box\]
In Lemma 4.6.1 we do not assume that for every \( i \in I \) there exists an \( i' \in I' \) such that \( i \preceq i' \) (and indeed this condition is not satisfied in our application). If this holds then the map \( T \) factors through \( \pi_i \) for every \( i \in I \) and not just for \( i \in I' \). However, there are examples where \( G_i = \{ 0 \} \) for some \( i \in I \) and \( T \neq 0 \).

### 4.7. Exhausting sequences

To compute direct and inverse limits we introduce the notion of exhausting sequences. Let \((G, \sigma)\) be a partially ordered system of \( R \)-modules over \((I, \preceq)\) and denote \( \mathbb{Z}^\pm := \{ \nu \in \mathbb{Z} \mid \pm \nu > 0 \} \). A sequence \( \{i_\nu\}_{\nu \in \mathbb{Z}^+} \) is called **upward exhausting** for \((G, \sigma)\) iff the following holds

- For every \( \nu \in \mathbb{Z}^+ \) we have \( i_\nu \preceq i_{\nu + 1} \) and \( \sigma_{i_{\nu + 1}i_\nu} : G_{i_\nu} \to G_{i_{\nu + 1}} \) is an isomorphism.
- For every \( i \in I \) there exists a \( \nu \in \mathbb{Z}^+ \) such that \( i \preceq i_\nu \).

A sequence \( \{i_\nu\}_{\nu \in \mathbb{Z}^-} \) is called **downward exhausting** for \((G, \sigma)\) iff the following holds

- For every \( \nu \in \mathbb{Z}^- \) we have \( i_{\nu - 1} \preceq i_\nu \) and \( \sigma_{i_{\nu - 1}i_\nu} : G_{i_{\nu - 1}} \to G_{i_\nu} \) is an isomorphism.
- For every \( i \in I \) there exists a \( \nu \in \mathbb{Z}^- \) such that \( i_\nu \preceq i \).

The importance of such sequences is that they can be used to simplify computations of direct and inverse limits.

#### Lemma 4.7.1

Let \((G, \sigma)\) be a partially ordered system of \( R \)-modules over \((I, \preceq)\).

(i) If \( \{i_\nu\}_{\nu \in \mathbb{Z}^+} \) is an upward exhausting sequence for \((G, \sigma)\) then \((I, \preceq)\) is upward directed and the homomorphism \( \iota_{i_\nu} : G_{i_\nu} \to \varinjlim G \) is an isomorphism for every \( \nu \in \mathbb{Z}^+ \).

(ii) If \( \{i_\nu\}_{\nu \in \mathbb{Z}^-} \) is a downward exhausting sequence for \((G, \sigma)\) then \((I, \preceq)\) is downward directed and the homomorphism \( \pi_{i_\nu} : \varprojlim G \to G_{i_\nu} \) is an isomorphism for every \( \nu \in \mathbb{Z}^- \).

**Proof.** To prove (i) we fix an integer \( \nu \in \mathbb{Z}^+ \). Let \( x \in G_{i_\nu} \) and suppose that \( \iota_{i_\nu}(x) = 0 \). Then there exists an \( i \in I \) such that \( i_\nu \preceq i \) and \( \sigma_{i_{i_\nu}i_\nu}(x) = 0 \). Choose an integer \( \nu' \geq \nu \) such that \( i \preceq i_{\nu'} \). Then \( \sigma_{i_{\nu'}i_\nu}(x) = \sigma_{i_{\nu'}i_\nu}(\sigma_{i_{i_\nu}i_\nu}(x)) = 0 \) and hence \( x = 0 \). Hence \( \iota_{i_\nu} \) is injective. Now let \( y \in G_j \) and choose an integer \( \nu' \geq \nu \) such that \( j \preceq i_{\nu'} \). Since \( \sigma_{i_{\nu'}i_\nu} \) is surjective there exists an \( x \in G_{i_\nu} \) such that \( \sigma_{i_{\nu'}i_\nu}(x) = \sigma_{i_{\nu'}i_\nu}(y) \). Hence \( (j, y) \sim (i_\nu, x) \). This shows that \( \iota_{i_\nu} \) is surjective.

To prove (ii) we fix an integer \( \nu \in \mathbb{Z}^- \). Let \( \{x_i\}_{i \in I} \in \varprojlim G \) such that \( x_{i_\nu} = 0 \). Given \( i \in I \) choose an integer \( \nu' \leq \nu \) such that \( i_{\nu'} \preceq i \). Then \( \sigma_{i_{\nu'}i_\nu}(x_i) = x_{i_{\nu'}} = 0 \), hence \( x_i = \sigma_{i_{\nu'}i_\nu}(x_i) = 0 \). This shows that \( \pi_{i_\nu} \) is injective. Now let \( x \in G_{i_\nu} \).

Given \( i \in I \), choose an integer \( \nu' \leq \nu \) such that \( i_{\nu'} \preceq i \) and define \( x_i \in G_i \) by

\[
  x_i := \sigma_{i_{\nu'}i_\nu}(x_{i_{\nu'}}), \quad \sigma_{i_{\nu'}i_\nu}(x_{i_{\nu'}}) := x.
\]

Since \( \sigma_{i_{\nu'}i_\nu} \) is surjective the element \( x_i \in G_i \) exists. Since \( \sigma_{i_{\nu'}i_\nu} \) is injective, the element \( x_i \) is unique, and it is independent of the choice of \( \nu' \). We prove that \( \sigma_{ji}(x_i) = x_j \) whenever
i ≤ j. To see this choose \( \nu' \leq \nu \) such that \( i_{\nu'} \leq i \leq j \). Then

\[
x_j = \sigma_{ji,\nu'}(x_{i,\nu'}) = \sigma_{ji} \circ \sigma_{i,\nu'}(x_{i,\nu'}) = \sigma_{ji}(x_i).
\]

Hence \( \{x_i\}_{i \in I} \in \lim\limits_{\leftarrow} G \) and \( \pi_{i,\nu}(\{x_i\}_{i \in I}) = x_{i,\nu} = x \). This shows that \( \pi_{i,\nu} \) is surjective.  

4.8. **Symplectic homology.** The set \( \mathcal{H}(M) \) of Hamiltonian functions on \( S^1 \times M \) with compact support is partially ordered by

\[
H_0 \preceq H_1 \iff H_0(t, x) \geq H_1(t, x) \quad \forall (t, x) \in S^1 \times M.
\]

This defines a bidirected partial order on \( \mathcal{H}(M) \). Let \( \alpha \in \pi_1(M) \) be a nontrivial homotopy class and \( a, b \in \mathbb{R} \cup \{\pm \infty\} \) such that \( a < b \). As in Section 4.4 we denote by \( \mathcal{H}^{a,b}(M; \alpha) \) the subset of all Hamiltonian functions \( H \in \mathcal{H}(M) \) such that \( a, b \notin S(H; \alpha) \). In Subsection 4.5 we have seen that there is a natural homomorphism

\[
\sigma_{H_1H_0} : \text{HF}^{[a,b]}(H_0; \alpha) \to \text{HF}^{[a,b]}(H_1; \alpha)
\]

whenever \( H_0, H_1 \in \mathcal{H}^{a,b}(M; \alpha) \) satisfy \( H_0 \preceq H_1 \). These homomorphisms define an inverse (in fact bidirected) system of Floer homology groups over \( (\mathcal{H}^{a,b}(M; \alpha), \preceq) \). The inverse limit of this system is called the *symplectic homology* of \( M \) in the homotopy class \( \alpha \) for the action interval \([a, b]\). A version of this homology group was introduced in \([FH2, CFH]\) for the homotopy class of contractible loops and later on for general homotopy classes in \([C-2]\). We denote it by

\[
\text{SH}^{[a,b]}(M; \alpha) := \lim\limits_{\mathcal{H}^{a,b}(M; \alpha)} \text{HF}^{[a,b]}(H; \alpha).
\]

Now fix a compact subset \( A \subset M \) and a constant \( c \in \mathbb{R} \). Consider the set \( \mathcal{H}^{a,b}_c(M, A; \alpha) \) of all Hamiltonian functions \( H \in \mathcal{H}^{a,b}(M; \alpha) \) that satisfy \( H \geq c \) on \( S^1 \times A \), namely

\[
\mathcal{H}^{a,b}_c(M, A; \alpha) := \left\{ H \in \mathcal{H}^{a,b}(M; \alpha) \mid \inf_{S^1 \times A} H > c \right\}.
\]

This gives rise to a directed (in fact bidirected) system of Floer homology groups over \( \mathcal{H}^{a,b}_c(M, A; \alpha) \). The direct limit of this system is called the *relative symplectic homology* of the pair \((M, A)\) at the level \( c \) in the homotopy class \( \alpha \) for the action interval \([a, b]\). We denote it by

\[
\text{SH}^{[a,b]:c}(M, A; \alpha) := \lim\limits_{\mathcal{H}^{a,b}_c(M, A; \alpha)} \text{HF}^{[a,b]}(H; \alpha).
\]

**Remark.** Since we have chosen to work with \( \mathbb{Z}_2 \)-coefficients all the Floer homology groups \( \text{HF}^{[a,b]}(H; \alpha) \) are in fact \( \mathbb{Z}_2 \)-vector spaces. Consequently also the symplectic homologies \( \text{SH}^{[a,b]}(M; \alpha) \) and \( \text{SH}^{[a,b]:c}(M, A; \alpha) \) have the structure of \( \mathbb{Z}_2 \)-vector spaces.
An important feature of absolute and relative symplectic homologies is the existence of a homomorphism between them which factors through Floer homology.

**Proposition 4.8.1.** Let \( \alpha \in \tilde{\pi}_1(M) \) be a nontrivial homotopy class and suppose that \(-\infty \leq a < b \leq \infty \). Then, for every \( c \in \mathbb{R} \), there exists a unique homomorphism

\[
T^{(a,b);c}_\alpha: \text{SH}^{(a,b);c}(M; \alpha) \rightarrow \text{SH}^{(a,b);c}(M, A; \alpha)
\]

such that for any two Hamiltonian functions \( H_0, H_1 \in \mathcal{H}^{a,b}_c(M, A; \alpha) \) with \( H_0 \geq H_1 \) the following diagram commutes:

\[
\begin{array}{ccc}
\text{SH}^{(a,b);c}(M; \alpha) & \xrightarrow{T^{(a,b);c}_\alpha} & \text{SH}^{(a,b);c}(M, A; \alpha) \\
\pi_{H_0} \downarrow & & \sigma_{H_1} \downarrow \\
\text{HF}^{(a,b);c}(H_0; \alpha) & \xrightarrow{\iota_{H_1}} & \text{HF}^{(a,b);c}(H_1; \alpha)
\end{array}
\]

Here \( \pi_{H_0} : \text{SH}^{(a,b);c}(M; \alpha) \rightarrow \text{HF}^{(a,b);c}(H_0; \alpha) \) and \( \iota_{H_1} : \text{HF}^{(a,b);c}(H_1; \alpha) \rightarrow \text{SH}^{(a,b);c}(M, A; \alpha) \) are the canonical homomorphisms introduced in Section 4.6. In particular, since \( \sigma_{HH} = \text{id} \) for every \( H \in \mathcal{H}^{a,b}_c(M, A; \alpha) \), we have

\[
\begin{array}{ccc}
\text{SH}^{(a,b);c}(M; \alpha) & \xrightarrow{T^{(a,b);c}_\alpha} & \text{SH}^{(a,b);c}(M, A; \alpha) \\
\pi_H \downarrow & & \iota_H \\
\text{HF}^{(a,b);c}(H; \alpha)
\end{array}
\]

The statements above continue to hold also for the trivial class \( \alpha = 0 \), provided that \( 0 \notin [a, b] \).

**Proof.** The proof follows at once from Lemma 4.6.1. \(\square\)

4.9. **The homological relative capacity.** For every nontrivial homotopy class \( \alpha \in \tilde{\pi}_1(M) \) and every real number \( c > 0 \) we define the set

\[
\mathcal{A}^c_c(M, A; \alpha) := \left\{ a \in \mathbb{R} \mid \text{The homomorphism } T^{[a, \infty);c}_\alpha \text{ does not vanish} \right\},
\]

where \( T^{[a, \infty);c}_\alpha: \text{SH}^{[a, \infty);c}(M; \alpha) \rightarrow \text{SH}^{[a, \infty);c}(M, A; \alpha) \) is the homomorphism from Proposition 4.8.1.

For the trivial homotopy class \( \alpha = 0 \in \tilde{\pi}_1(M) \) we define \( \mathcal{A}^c_0(M, A; 0) \) by the same formula except that we only consider real numbers \( a > 0 \) (for which \( T^{[0, \infty);c}_0 \neq 0 \)). The **homological relative capacity** of the pair \((M, A)\) is the function

\[
\tilde{C}(M, A): \tilde{\pi}_1(M) \times [-\infty, \infty) \rightarrow [0, \infty]
\]
which assigns to the class $\alpha \in \tilde{\pi}_1(M)$ and the number $a \geq -\infty$ the number

$$\tilde{C}(M, A; \alpha, a) := \inf \left\{ c > 0 \mid \sup_{i \in H} \mathcal{A}_c(M, A; \alpha) > a \right\}. $$

Here we use the convention that $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$. For $a = -\infty$ we abbreviate

$$\tilde{C}(M, A; \alpha) := \tilde{C}(M, A; \alpha, -\infty) = \inf \left\{ c > 0 \mid \mathcal{A}_c(M, A; \alpha) \neq \emptyset \right\}. $$

The latter quantity is independent of the $\omega$-primitive $\lambda$ while $\tilde{C}(M, A; \alpha, a)$ does depend on this choice: the set $\mathcal{A}_c(M, A; \alpha) =: \mathcal{A}_c^{\lambda}(M, A; \alpha)$ depends on $\lambda$, but for two $\omega$-primitives $\lambda, \lambda'$ we have $\mathcal{A}_c^{\lambda'}(M, A; \alpha) = \mathcal{A}_c^{\lambda}(M, A; \alpha) - \int_\alpha (\lambda' - \lambda)$.

**Proposition 4.9.1.** Let $\alpha \in \tilde{\pi}_1(M)$ and $a \in \mathbb{R}$. If $\tilde{C}(M, A; \alpha, a) < \infty$ then every compactly supported Hamiltonian $H$ on $S^1 \times M$ with $H|_{S^1 \times A} \geq \tilde{C}(M, A; \alpha, a)$ has a $1$-periodic orbit in the homotopy class $\alpha$ with action $\mathcal{H}_1(x) \geq a$. In particular,

$$\tilde{C}(M, A; \alpha, a) \geq C(M, A; \alpha, a).$$

**Proof.** Assume first that $\inf_{S^1 \times A} H > \tilde{C}(M, A; \alpha, a)$. Then, by definition of $\tilde{C}(M, A; \alpha, a)$, there exist two real numbers $b$ and $c$ such that

$$0 < c < \inf_{S^1 \times A} H, \quad a < b, \quad b \in \mathcal{A}_c(M, A; \alpha).$$

Hence, by definition of the set $\mathcal{A}_c(M, A; \alpha)$, the homomorphism

$$T_{\alpha}^{(b, \infty); c} : \text{SH}^{(b, \infty)}(M; \alpha) \to \text{SH}^{(b, \infty); c}(M; A; \alpha)$$

is nonzero. Now choose a sequence of Hamiltonian functions $H_i \in \mathcal{H}(M)$ such that $H_i$ converges to $H$ in the $C^\infty$-topology, $b \notin S(H_i; \alpha)$, and $\inf_{S^1 \times A} H_i > c$ for every $i$. Then $H_i \in \mathcal{H}_{c, \infty}(M, A; \alpha)$ and so, by Proposition 4.8.1, the nonzero homomorphism $T_{\alpha}^{(b, \infty); c}$ factors through the Floer homology group $\text{HF}^{(b, \infty); c}(H_i; \alpha)$ for every $i$. Hence there exits a sequence of periodic orbits $x_i \in \mathcal{P}(H_i; \alpha)$ such that $\mathcal{A}_{H_i}(x_i) > b$. Passing to a converging subsequence we get a periodic orbit $x \in \mathcal{P}(H; \alpha)$ with $\mathcal{A}_{H}(x) \geq b > a$. This proves the assertion in the case $\inf_{S^1 \times A} H > \tilde{C}(M, A; \alpha, a)$. If $\inf_{S^1 \times A} H = \tilde{C}(M, A; \alpha, a)$ the result follows by another approximation argument. \hfill $\Box$

**Remark 4.9.2.** Both relative capacities have the following rescaling property. Let $c > 0$ and replace the symplectic form $\omega$ by $c\omega$ and the $\omega$-primitive $\lambda$ by $c\lambda$. Then

$$\tilde{C}(M, A, c\lambda; \alpha, a) = c\tilde{C}(M, A, \lambda; \alpha, a), \quad C(M, A, c\lambda; \alpha, a) = cC(M, A, \lambda; \alpha, a).$$

To see this, note that the Hamiltonian function $\tilde{H} := cH$ has the same Hamiltonian vector field with respect to $\tilde{\omega} := c\omega$ as the Hamiltonian function $H$ with respect to $\omega$, and that the symplectic action of a periodic orbit $x \in \mathcal{P}(\tilde{H}, \tilde{\omega}; \alpha) = \mathcal{P}(H, \omega; \alpha)$ with respect to $\tilde{H}$ and $\tilde{\lambda} := c\lambda$ is equal to $c$ times the action with respect to $H$ and $\lambda$. 


5. Computation of the capacities

We are now in a position to compute in certain cases the relative symplectic homology of the unit cotangent bundle $U^*X$ of a compact connected Riemannian manifold $X$ without boundary. We shall always work with the Liouville form $\lambda_{can}$ as a primitive of the canonical symplectic form $\omega_{can}$. We consider the following two cases.

(T) $X = \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is the flat torus.
(N) $X$ has negative sectional curvature.

In either case we identify $\tilde{\pi}_1(U^*X)$ with $\tilde{\pi}_1(X)$ and in the case of the torus we identify $\tilde{\pi}_1(\mathbb{T}^n)$ with $\mathbb{Z}^n$. More precisely, we identify $k \in \mathbb{Z}^n$ with the homotopy class of the loop $[0, 1] \to \mathbb{T}^n : t \mapsto tk + \mathbb{Z}^n$.

5.1. The main results. In this section we state the main results about the (relative) symplectic homology of open subsets of cotangent bundles and show how they can be used to establish Theorem 3.2.1 (i) and (ii). The subsequent sections are devoted to their proofs. The first result concerns the symplectic homology in the trivial homotopy class.

Theorem 5.1.1. Assume (T) or (N) and consider the trivial class $\alpha = 0 \in \tilde{\pi}_1(X)$. Then, for $a, c > 0$, we have

$$\text{SH}^{[a, \infty)}(U^*X; 0) \cong H_*(X; \mathbb{Z}_2),$$

and

$$\text{SH}^{[a, \infty):c}(U^*X, X; 0) \cong \begin{cases} H_*(X; \mathbb{Z}_2), & \text{if } a \leq c, \\ 0, & \text{if } c < a. \end{cases}$$

Moreover, the homomorphism $T^{[a, \infty):c}_0 : \text{SH}^{[a, \infty)}(U^*X; 0) \to \text{SH}^{[a, \infty):c}(U^*X, X; 0)$ is an isomorphism for $0 < a \leq c$. In particular, for every $a \in \mathbb{R}$

$$\widehat{C}(U^*X, X; 0, a) = \max\{0, a\}.$$
Moreover, the homomorphism $T^\alpha_{a,\infty} : \text{SH}^{[a,\infty)}(U^*X;\alpha) \to \text{SH}^{[a,\infty)c)(U^*X,X;\alpha)$ is an isomorphism for $\ell \leq a \leq c$. In particular, for every $a \in \mathbb{R}$

$$\tilde{C}(U^*X,X;\alpha,a) = \max\{\ell,a\}.$$ 

We are now in position to prove Theorem 3.2.1.

Proof of Theorem 3.2.1 (i) and (ii). Assume that $X$ satisfies (T) or (N). If $\alpha = 0$ we must prove that $C(U^*X,X;0,a) = \max\{0,a\}$ To see this, note that every compactly supported Hamiltonian function $H \in H(U^*X)$ has a contractible periodic orbit $x$ with action $A_H(x) = 0$ and hence $C(U^*X,X;0,a) = 0$ whenever $a \leq 0$. If $a > 0$ then Theorem 5.1.1 asserts that $\hat{C}(U^*X,X;0,a) = a$ and hence

$$a = \hat{C}(U^*X,X;0,a) \geq C(U^*X,X;0,a) \geq a.$$ 

Here the middle inequality follows from Proposition 4.9.1. To prove the right-hand inequality let $0 < \delta < a$ and choose any Hamiltonian function $H = H(p)$ that depends only on the momenta variables and satisfies $\max H = a - \delta$. Then every contractible periodic orbit $x \in \mathcal{P}(H,0)$ is (up to parametrization) a contractible geodesic and hence, since $X$ satisfies (T) or (N), is constant and has action $A_H(x) = H(x) = a - \delta$. This shows that $C(U^*X,X;0,a) \geq a - \delta$ for every $\delta > 0$. Thus we have proved that $C(U^*X,X;0,a) = \max\{0,a\}$ as claimed.

Now assume $\alpha \neq 0$ and abbreviate $\ell := \ell(\gamma_\alpha)$ in the case (N) and $\ell := |k|$ in the case (T) with $\alpha = k \in \mathbb{Z}^n$. Then Theorem 5.1.1 asserts that $\hat{C}(U^*X,X;\alpha,a) = \max\{\ell,a\}$ and hence

$$\max\{\ell,a\} = \hat{C}(U^*X,X;\alpha,a) \geq C(U^*X,X;\alpha,a) \geq \max\{\ell,a\}$$

for every real number $a$. Again the middle inequality follows from Proposition 4.9.1 and the rightmost inequality from an explicit construction of a Hamiltonian function. Namely, for any $\delta > 0$ choose a compactly supported function $f : [0,1) \to \mathbb{R}$ such that

$$f(r) = \begin{cases} m - \delta, & \text{for } r \text{ near } 0, \\ 0, & \text{for } r \text{ near } 1, \end{cases}$$

where $m := \max\{\ell,a\}$, and

$$f(r) < (1-r)m, \quad -m < f'(r) \leq 0$$

for every $r$. Now consider the compactly supported Hamiltonian function $H := f(|p|)$ on $U^*X$. Its 1-periodic solutions are reparametrized closed geodesics. The sphere bundle
$|p| = r$ contains a periodic orbit $x$ in the class $\alpha$ if and only if $f'(r) = -\ell$ and the action of this periodic orbit is

$$A_H(x) = f(r) - rf'(r) < f(r) + rm < m.$$  

(See Lemma 5.3.2 below.) If $a \leq \ell$ then $f'(r) > -\ell$ for all $r$, hence there is no 1-periodic solution of length $\ell$, and hence none in the class $\alpha$. If $\ell \leq a$ then every 1-periodic solution has action $A_H(x) < a$. In either case there is no 1-periodic orbit in the class $\alpha$ with action at least $a$, and hence $C(U^*X, X; 0, a) \geq m - \delta$. Since this holds for every $\delta > 0$ we obtain $C(U^*X, X; 0, a) \geq m$ as claimed.

5.2. **Morse–Bott theory in Floer homology.** Let us return to the general setting of Section 4.2 where $(M, \omega)$ is a compact connected symplectic manifold with convex boundary, $\omega = d\lambda$ is an exact symplectic form, $M = \overline{M} \setminus \partial \overline{M}$, $\mathcal{H} = \mathcal{H}(M)$ denotes the space of compactly supported functions on $S^1 \times M$, and $\mathcal{J}$ denotes the space of 1-periodic $\omega$-compatible almost complex structures $J_t = J_{t+1}$ on $M$.

A subset $P \subset \mathcal{P}(H)$ is called a **Morse–Bott manifold of periodic orbits** if the set $C_0 := \{x(0) \mid x \in P\}$ is a compact submanifold of $M$ and $T_{x_0}C_0 = \ker (D\psi_1(x_0) - \mathbb{I})$ for every $x_0 \in C_0$.

**Remark 5.2.1.** The Morse–Bott condition can be reformulated as follows. Firstly, a subset $P \subset \mathcal{P}(H)$ is a compact submanifold of the loop space $LM$ if and only if the set $C_0 = \{x(0) \mid x \in P\}$ is a compact submanifold of $M$. Secondly, for every $x \in P$ the kernel of the linear map $D\psi_1(x(0)) - \mathbb{I}$ on $T_{x(0)}M$ is isomorphic to the space of periodic solutions of the following **linearized Hamiltonian differential equation** for vector fields $\xi(t) \in T_{x(t)}M$ along $x$:

$$\nabla_\xi \xi = \nabla_\xi X_H(x), \quad \xi(t + 1) = \xi(t),$$

where $\nabla$ stands for the Levi-Civita connection of the metric $\omega(\cdot, J\cdot)$. To see this just note that $\xi$ satisfies (11) if and only if $\xi(t) = D\psi_1(x(0))\xi(0)$ for all $t$. Note that every tangent vector of $P$ is a solution of (11). The Morse–Bott condition can now be expressed in the form that $P$ is a compact submanifold of $LM$ and

$$T_x P = \{\xi \in C^\infty(S^1, x^*TM) \mid \xi \text{ satisfies (11)}\}.$$  

We emphasize that the Hessian of the symplectic action functional $A_H : LM \to \mathbb{R}$ at a critical point $x$ is the linear operator $\xi \mapsto \nabla_\xi (\text{grad}H) - (\nabla_\xi J)\dot{x} - J\nabla_\xi \xi = J(\nabla_\xi X_H - \nabla_\xi \xi)$ on $C^\infty(S^1, x^*TM)$ (equipped with the $L^2$-inner product). Hence the space of solutions of (11) is the kernel of the Hessian of $A_H$ at $x$, and the Morse–Bott condition asserts that the kernel of the Hessian agrees with the tangent space of the critical manifold $P$.  


Theorem 5.2.2. Let \(-\infty \leq a < b \leq \infty\), \(\alpha \in \mathcal{P}_1(M)\), and \(H \in \mathcal{H}^{a,b}(M;\alpha)\). Suppose that the set \(P := \{x \in \mathcal{P}(H;\alpha) \mid a < A_H(x) < b\}\) is a connected Morse–Bott manifold of periodic orbits. Then \(HF^{a,b}(H;\alpha) \cong H_s(P;\mathbb{Z}_2)\).

This is a version of Pozniak’s theorem \([\text{P}]\) which was originally proved in the context of Floer homology for Lagrangian intersections. In this section we explain the reduction of this theorem to Pozniak’s original one.

In order to reformulate Floer homology in the Lagrangian setting let us consider the symplectic manifold

\[\tilde{M} := M \times M, \quad \tilde{\omega} := \omega \oplus (-\omega) = d\tilde{\lambda}, \quad \tilde{\lambda} := \lambda \oplus (-\lambda).\]

Since \(\tilde{\omega}\) is exact there are no nonconstant holomorphic spheres in \(\tilde{M}\), for any \(\tilde{\omega}\)-compatible almost complex structure \(\tilde{J} \in \mathcal{J}(\tilde{M}, \tilde{\omega})\). Since \(\tilde{\lambda}\) vanishes on the diagonal \(\Delta \subset M \times M\), there are also no nonconstant holomorphic disks with boundary in \(\Delta\). Hence the standard theory of Floer homology for Lagrangian intersections applies as in Floer’s original work \([\text{F1, F2, F3}]\). Given \(H \in \mathcal{H}(M)\) define \(\tilde{H}_t : \tilde{M} \to \mathbb{R}\) by

\[\tilde{H}_t(x_0, x_1) := H_t(x_0) + H_{1-t}(x_1).\]

The Hamiltonian isotopy generated by \(\tilde{H}_t\) with respect to \(\tilde{\omega}\) is given by \(\tilde{\psi}_t(x_0, x_1) = (\psi_t(x_0), \psi_{1-t} \circ \psi_{1}^{-1}(x_1))\). Given \(J \in \mathcal{J}(M)\) define \(\tilde{J}_t \in \mathcal{J}(\tilde{M}, \tilde{\omega})\) by

\[\tilde{J}_t := \psi_t^*J_t \times (- (\psi_{1-t} \circ \psi_{1}^{-1})^*J_{1-t}) = \left(\tilde{\psi}_t^{-1}\right)^*(J_t \times (-J_{1-t}))\]

for \(0 \leq t \leq 1/2\). Given \(u : \mathbb{R} \times S^1 \to M\) define \(\tilde{u} : \mathbb{R} \times [0, 1/2] \to \tilde{M}\) by

\[\tilde{u}(s, t) := (\psi_t^{-1}(u(s, t)), \psi_1 \circ \psi_{1-t}^{-1}(u(s, 1-t))) = \left(\tilde{\psi}_t^{-1}\right)^{(s, 1-t)}(u(s, t), u(s, 1-t)).\]

Then \(u\) satisfies (\[\text{[\text{I}]}\]) if and only if \(\tilde{u}\) satisfies the Lagrangian boundary value problem

\[\partial_s \tilde{u} + \tilde{J}_t(\tilde{u})\partial_t \tilde{u} = 0, \quad \tilde{u}(s, 0) \in \Delta, \quad \tilde{u}(s, 1/2) \in \text{graph}(\psi_1).\]

It satisfies (\[\text{[\text{I}]}\]) if and only if \(\tilde{u}\) satisfies

\[\lim_{s \to \pm\infty} \tilde{u}(s, t) = \tilde{x}^\pm, \quad \lim_{s \to \pm\infty} \partial_s \tilde{u}(s, t) = 0,\]

where \(\tilde{x}^\pm := (x^\pm(0), x^\pm(0)) \in \Delta \cap \text{graph}(\psi_1)\). The solutions of (\[\text{[\text{I}]}\]) can be interpreted as the gradient flow lines of the action functional

\[\tilde{A}(\tilde{x}) := -\int_0^{1/2} \tilde{\lambda}(\tilde{x}(t)) \, dt\]

on the space \(\tilde{P}\) of paths \(\tilde{x} : [0, 1/2] \to \tilde{M}\) with endpoints \(\tilde{x}(0) \in \Delta\) and \(\tilde{x}(1/2) \in \text{graph}(\psi_1)\) (with respect to the \(L^2\) metric determined by \(\tilde{J}\)). Note that the composition of \(\tilde{A}\) with
the map \( LM \to \tilde{P} : x \mapsto \tilde{x} \), given by \( \tilde{x}(t) := (\psi_t^{-1}(x(t)), \psi_t \circ \psi_t^{-1}(x(1-t))) \), agrees with \( \mathcal{A}_H \). Note also that this map induces a bijection

\[
\pi_1(M) = \pi_0(LM) \to \pi_0(\tilde{P}) : \alpha \mapsto \tilde{\alpha}.
\]

Hence the solutions of (13) can be used to define the Floer homology groups of the pair \((\Delta, \text{graph}(\psi_1))\) of Lagrangian submanifolds of \((\tilde{M}, \tilde{\omega})\). Moreover, the Floer homology groups defined by the solutions of (13) are independent of the choice of the (regular) almost complex structure \( \tilde{J}_t \) used to define them. More precisely, denote by \( \tilde{J} \) the space of smooth functions \([0, 1/2] \to J(\tilde{M}, \tilde{\omega}) : t \mapsto \tilde{J}_t \) such that \( \tilde{J}_t = J \times (-J) \) near the boundary of \( \tilde{M} \), where \( J \in J(M, \omega) \) is convex. Given a Hamiltonian \( H \in \mathcal{H} \) that satisfies \((H)\) denote by \( \tilde{J}_{\text{reg}}(H) \) the set of all almost complex structures \( \tilde{J} \in \tilde{J} \) such that every finite energy solution \( \tilde{u} \) of (13) is regular in the sense that the linearized operator along \( \tilde{u} \) is surjective. Then the solutions of (13) give rise to Lagrangian Floer homology groups \( HF^{[a, b]}(\Delta, \text{graph}(\psi_1); \tilde{J}, \tilde{\alpha}) \). Moreover, it follows as in [F1, F2, F3] (and as outlined above) that these Floer homology groups are independent of the almost complex structure \( \tilde{J} \in \tilde{J}_{\text{reg}}(H) \) used to define them. Note that if \( J \in J_{\text{reg}}(H) \) and \( \tilde{J} \) is given by (12) then \( \tilde{J} \in \tilde{J}_{\text{reg}}(H) \). Hence there is a natural isomorphism

\[
HF^{[a, b]}(H; \alpha) \cong HF^{[a, b]}(\Delta, \text{graph}(\psi_1); \tilde{\alpha})
\]

for every \( \alpha \in \pi_1(M) \) and every Hamiltonian \( H \in \mathcal{H}^{[a, b]}(M; \alpha) \), where \( \tilde{\alpha} \) is the image of \( \alpha \) under the above homomorphism \( \pi_1(M) = \pi_0(LM) \to \pi_0(\tilde{P}) \). The advantage of the Lagrangian approach in the present context is that we can use any (regular) family of \( \tilde{\omega} \)-compatible almost complex structures \( \{\tilde{J}_t\}_{0 \leq t \leq 1/2} \) to define the Floer homology groups, and are not restricted to those arising from periodic families of almost complex structures on \( M \) via (12). Hence we can apply the results of Pozniak.

Let \( \tilde{J} \in \tilde{J} \) and \( H \in \mathcal{H} \) and denote by \( \mathbb{R} \to \text{Diff}(M, \omega) : t \mapsto \psi_t \) the Hamiltonian isotopy generated by \( H \). The \textbf{graph} of a loop \( x \in LM \) is the set

\[
\Gamma(x) := \{ (t, \psi_t^{-1}(x(t)), \psi_t \circ \psi_t^{-1}(x(1-t))) \mid 0 \leq t \leq 1/2 \} \subset [0, 1/2] \times \tilde{M}.
\]

For a subset \( P \subset LM \) we write \( \Gamma(P) := \bigcup_{x \in P} \Gamma(x) \), and for a map \( \tilde{u} : \mathbb{R} \times [0, 1/2] \to \tilde{M} \) we write

\[
\Gamma(\tilde{u}) := \{ (t, \tilde{u}(s, t)) \mid s \in \mathbb{R}, 0 \leq t \leq 1/2 \}.
\]

A subset \( P \subset \mathcal{P}(H) \) is called a \textbf{\( \tilde{J} \)-isolated periodic set} if there exists an open neighbourhood \( U \subset [0, 1/2] \times \tilde{M} \) of \( \Gamma(P) \) such that the following holds:

(P1) The closure \( \overline{\mathcal{J}} \) is a compact subset of \([0, 1/2] \times \tilde{M} \).

(P2) If \( \tilde{u} : \mathbb{R} \times [0, 1/2] \to \tilde{M} \) is a finite energy solution of (13) with \( \Gamma(\tilde{u}) \subset \overline{\mathcal{J}} \) then there exists an \( x \in P \) such that \( u(s, t) = x(t) \) for every \( (s, t) \in \mathbb{R}^2 \).
An open neighbourhood \( U \subset [0, 1/2] \times \tilde{M} \) of \( \Gamma(P) \) that satisfies \((P1)\) and \((P2)\) is called \( \tilde{J} \)-isolating. Note that every \( \tilde{J} \)-isolated periodic set is compact.

**Lemma 5.2.3.** Let \( H \in \mathcal{H}(M) \). Then every Morse–Bott manifold \( P \subset \mathcal{P}(H) \) of periodic orbits is a \( \tilde{J} \)-isolated periodic set for every almost complex structure \( \tilde{J} \in \mathcal{J} \).

**Proof.** We may assume without loss of generality that \( P \) is connected. Let \( \tilde{d}_t \) denote the distance function of the metric \( \langle \cdot, \cdot \rangle_t := \tilde{\omega}(\cdot, \tilde{J}_t \cdot) \) and consider the open set
\[
U := \left\{ (t, \tilde{x}) \mid 0 \leq t \leq 1/2, \tilde{x} \in \tilde{M}, \sup_{y \in P} \tilde{d}_t(\tilde{x}, (y(1/2 - t), y(1/2 + t))) < \varepsilon \right\} \subset [0, 1/2] \times \tilde{M}.
\]
Let \( \varepsilon > 0 \) be sufficiently small. Then since \( \mathcal{C}_0 \) is an isolated fixed point set for \( \psi_1 \), it follows that every \( x \in \mathcal{P}(H) \) with \( \Gamma(x) \subset \overline{U} \) is an element of \( P \). Now the set \( U \) satisfies \((P2)\) because every finite energy solution \( \tilde{u} : \mathbb{R} \times [0, 1/2] \to \tilde{M} \) of \((13)\) with \( \Gamma(\tilde{u}) \subset \overline{U} \) is asymptotic to the set \( P \) as \( s \to \pm \infty \). Since \( \mathcal{A}_H = \mathcal{A}_H \) is constant along \( P \) it follows that every such solution \( \tilde{u} \) has energy \( E(\tilde{u}) = 0 \) and hence has the form \( \tilde{u}(s, t) = \tilde{x}(t) = (x(1/2 - t), x(1/2 + t)) \) for some \( x \in \mathcal{P}(H) \).

**Lemma 5.2.4.** Let \( H \in \mathcal{H} \) and \( J \in \mathcal{J} \). Suppose that \( P \subset \mathcal{P}(H) \) is a \( \tilde{J} \)-isolated periodic set and \( U \subset [0, 1/2] \times \tilde{M} \) is a \( \tilde{J} \)-isolating neighbourhood of \( \Gamma(P) \). Then there exist a compact neighbourhood \( V \subset U \) of \( \Gamma(P) \) and a constant \( \delta > 0 \) such that the following holds. If \( \mathbb{R} \to \mathcal{H}(M) : s \mapsto H_s \) and \( \mathbb{R} \to \mathcal{J}(\tilde{M}) : s \mapsto \tilde{J}_s \) are smooth homotopies such that
\[
\|H_s - H\|_{C^2} + \|\tilde{J}_s - \tilde{J}\|_{C^1} + \|\partial_s H_s\|_{C^2} + \|\partial_s \tilde{J}_s\|_{C^1} < \delta
\]
and \( \partial_s H_s = 0 \) and \( \partial_s \tilde{J}_s = 0 \) for \( |s| \geq 1 \) then every finite energy solution \( \tilde{u} : \mathbb{R} \times [0, 1/2] \to \tilde{M} \) of \((13)\) with \((\tilde{H}, \tilde{J})\) replaced by \((\tilde{H}_s, \tilde{J}_s)\) satisfies
\[
\Gamma(\tilde{u}) \subset \overline{U} \implies \Gamma(\tilde{u}) \subset V.
\]

**Proof.** Suppose, by contradiction, that there exist sequences
\[
\mathbb{R} \to \mathcal{H} : s \mapsto H_s^\nu, \quad \mathbb{R} \to \mathcal{J} : s \mapsto \tilde{J}_s^\nu, \quad \tilde{u}^\nu : \mathbb{R} \times [0, 1/2] \to \tilde{M},
\]
and \((s^\nu, t^\nu) \in \mathbb{R} \times [0, 1/2] \) such that the following holds:

(i) \( \lim_{\nu \to \infty} \sup_{s \in \mathbb{R}} \left( \|H_s^\nu - H\|_{C^2} + \|\partial_s H_s^\nu\|_{C^2} + \|\tilde{J}_s^\nu - \tilde{J}\|_{C^1} + \|\partial_s \tilde{J}_s^\nu\|_{C^1} \right) = 0. \)

(ii) \( \partial_s H_s^\nu = 0 \) and \( \partial_s \tilde{J}_s^\nu = 0 \) for \( |s| \geq 1 \).

(iii) \( \tilde{u}^\nu \) is a finite energy solution of \((13)\) with \((\tilde{H}, \tilde{J})\) replaced by \((\tilde{H}_s^\nu, \tilde{J}_s^\nu)\).

(iv) \( \Gamma(\tilde{u}^\nu) \subset \overline{U} \) and \( \lim_{\nu \to \infty} \tilde{u}^\nu(s^\nu, t^\nu) \in \partial U. \)

Since there are no nonconstant \( \tilde{J}_r \)-holomorphic spheres in \( \tilde{M} \) and no nonconstant \( \tilde{J}_r \)-holomorphic disks with boundary in \( \Delta \), the first derivatives of the functions \( \tilde{u}^\nu \) are uniformly bounded. Hence, by Floer–Gromov compactness \([F4, G, MS, Sa]\), there exists a
subsequence, still denoted by $\tilde{u}^\nu$, such that the shifted sequence $\tilde{u}^\nu(s^\nu + s, t)$ converges in the $C^1$-topology on compact sets to a finite energy solution $\tilde{u} : \mathbb{R} \times [0,1/2] \to \tilde{M}$ of (13) such that $\Gamma(\tilde{u}) \subset \tilde{U}$. By taking a further subsequence we may assume that $t^\nu \to t$ and hence $\tilde{u}(0,t) = \lim_{\nu \to \infty} \tilde{u}^\nu(s^\nu, t^\nu)$ satisfies $(t, \tilde{u}(0,t)) \in \partial U \subset [0,1/2] \times (M \setminus U)$. This contradicts (P2). \hfill \Box

Lemma 5.2.4 enables us to define the local Floer homology $HF^{loc}(H; P)$ of a $J$-isolated periodic set $P \subset \mathcal{P}(H)$ as follows. Choose a $\tilde{J}$-isolating neighbourhood $U \subset S^1 \times M$ of $\Gamma(P)$, let $\delta > 0$ be as in Lemma 5.2.4, choose a Hamiltonian function $H'$ such that all periodic solutions $x \in \mathcal{P}(H')$ are nondegenerate and $\|H' - H\|_{C^2} < \delta/4$, and choose a regular almost complex structure $\tilde{J}' \in \tilde{J}_{reg}(H')$ such that $\|\tilde{J}' - \tilde{J}\|_{C^2} < \delta/4$. Then, by Lemma 5.2.4, all the Floer connecting orbits of $(\tilde{H}', \tilde{J}')$ (i.e. solutions $\tilde{u}'$ of (13)) with $(\tilde{H}, \tilde{J})$ replaced by $(\tilde{H}', \tilde{J}')$ in $\tilde{U}$ are actually contained in $V$. Denote the set of local periodic orbits of $H'$ near $P$ by

$$\mathcal{P}(H'; U) := \{x' \in \mathcal{P}(H') \mid \Gamma(x') \subset U\}$$

and consider the local Floer chain complex

$$CF^{loc}(H'; U) := \bigoplus_{x' \in \mathcal{P}(H'; U)} \mathbb{Z}_2 x'.$$

The boundary operator $\partial^{H', \tilde{J}'; U} : CF^{loc}(H'; U) \to CF^{loc}(H'; U)$ is defined by counting the index-1 solutions $\tilde{u}'$ of (13), with $(\tilde{H}, \tilde{J})$ replaced by $(\tilde{H}', \tilde{J}')$, such that $\Gamma(\tilde{u}') \subset U$. Since these solutions can never converge to the boundary of $U$ it follows that $\partial^{H', \tilde{J}'; U}$ is indeed a boundary operator and the local Floer homology is defined by

$$HF^{loc}(H', \tilde{J}'; U) := H_*(CF^{loc}(H'; U), \partial^{H', \tilde{J}'; U}).$$

The same arguments as in Floer’s original theory [F1, F2, F3] now show that this local Floer homology is independent (up to natural isomorphisms) of the isolating neighbourhood $U$, and of the perturbations $H'$ and $\tilde{J}'$ used to define it. We write

$$HF^{loc}(H; P) := HF^{loc}(H', \tilde{J}'; U)$$

for the local Floer homology in a $\tilde{J}$-isolating neighbourhood $U$ of $\Gamma(P)$. Strictly speaking, this is a connected simple system in the sense of Conley, namely a small category whose objects are the triples $(H', \tilde{J}'; U)$ of local perturbations and whose morphisms are the canonical (unique) isomorphisms between the local Floer homologies $HF^{loc}(H'_0, \tilde{J}'_0, U_0)$ and $HF^{loc}(H'_1, \tilde{J}'_1, U_1)$. The details of this construction were carried out by Pozniak [P] in the context of Lagrangian intersections.
Theorem 5.2.5. (Pozniak [P]) Let $H \in \mathcal{H}(M)$ and suppose that $P \subset \mathcal{P}(H)$ is a connected Morse–Bott manifold of periodic orbits. Then $HF^{\text{loc}}(H; P) \cong H_*(P; \mathbb{Z}_2)$.

Proof. The local Floer homology of $H$ near $P$ is isomorphic to the local Floer homology of the pair of Lagrangian submanifolds $L_0 := \Delta \subset M \times M$ and $L_1 := \text{graph}(\psi_1) \subset M \times M$ near their clean intersection $\Lambda := \{(x(0), x(0)) | x \in P\}$. Hence, by [P, Theorem 3.4.11], it is isomorphic to $H_*(\Lambda; \mathbb{Z}_2) \cong H_*(P; \mathbb{Z}_2)$.

Proof of Theorem 5.2.2. Fix a 1-periodic almost complex structure $J \in \mathcal{J}$ and let $\tilde{J} \in \mathcal{J}$ be given by (12). Then, by Lemma 5.2.3, $P$ is a $\tilde{J}$-isolated periodic set. Let $U$ be a $\tilde{J}$-isolating neighbourhood of $\Gamma(P)$ and choose a sequence of regular perturbations $(H_\nu, J_\nu)$ that agree with $(H, J)$ in some neighbourhood of $\partial M$ and converge to $(H, J)$ in the $C^2$-norm. We claim that, for $\nu$ sufficiently large, all the Floer connecting orbits (i.e solutions of (6)) for the pair $(H_\nu, J_\nu)$ in the homotopy class $\alpha$ and the action interval $[a, b]$ are contained in $U$. Otherwise, there has to be a sequence $u_\nu$ of such connecting orbits passing through $M \setminus U$ and we can argue as in the proof of Lemma 5.2.4 that, in the limit $\nu \to \infty$, there must be a finite energy solution of (11) for the pair $(H, J)$ in the homotopy class $\alpha$ and the action interval $[a, b]$ that passes through $M \setminus U$. However, every such connecting orbit has the form $u(s, t) = x(t)$ for some $x \in P$ and so is contained in $U$. This contradiction proves the claim. Hence $HF^{[a, b)}(H; \alpha) \cong HF^{\text{loc}}(H; P)$, and hence the result follows from Theorem 5.2.5. \hfill \Box

5.3. The main example. In this section we consider the case where $M = U^*X$ is the open unit cotangent bundle of a compact connected Riemannian $n$-manifold $X$ without boundary that satisfies either (T) (i.e. $X$ is a flat torus) or (N) (i.e. $X$ has negative sectional curvature). We shall use the metric to identify the tangent bundle with the cotangent bundle and denote a point in $U^*X$ by $x = (q, p)$ where $q \in X$ and $p \in T_qX$. Let $H : U^*X \to \mathbb{R}$ be a compactly supported Hamiltonian function of the form

$$H(q, p) = f(|p|),$$

where $f : \mathbb{R} \to \mathbb{R}$ is a smooth function such that $f(r) = f(-r)$. The corresponding Hamiltonian differential equation has the form

$$\dot{q} = \frac{f'(|p|)}{|p|} p, \quad \nabla_q p = 0.$$  \hfill (15)

Here $\nabla$ denotes the Levi-Civita connection. Since $|p|$ is constant it follows that $q$ is a geodesic for every solution $(q, p)$ of (15). Moreover, since $f'(0) = 0$, the zero section $\{p = 0\}$ consists of constant solutions. There are other constant solutions $x(t) \equiv (q, p)$ whenever $f'(|p|) = 0$ but these will not be relevant in the context of the present paper.
**Lemma 5.3.1.** The set \( P_0 := \{ x = (p, q) : S^1 \to T^*X | \dot{q} \equiv 0, p \equiv 0 \} \) is a Morse–Bott manifold of periodic orbits for \( H \) if and only if \( f''(0) \neq 0 \).

**Proof.** Since \( f \) is even there exists a smooth function \( h : \mathbb{R} \to \mathbb{R} \) such that \( f(r) = h(r^2)/2 \). Then \( h'(r^2) = f'(r)/r \) and \( h'(0) = f''(0) \). So equation (13) reads

\[
\dot{q} = h'(|p|^2)p, \quad \nabla_q p = 0.
\]

By Remark 5.2.1, \( P_0 \) is a Morse–Bott manifold if and only if the space of periodic solutions of the linearized equation is equal to the tangent space of \( P_0 \) for every \( x = (q, p) \in P_0 \). This means that the space of periodic solutions of the linearized equation has the same dimension as \( P_0 \). Now the linearized equation at a constant solution with \( p \equiv 0 \) has the form

\[
\frac{d}{dt} \dot{q} = h'(0)\dot{p} = f''(0)\dot{p}, \quad \frac{d}{dt} \dot{p} = 0.
\]

If \( f''(0) \neq 0 \) then the space of periodic solutions of this equation has dimension \( n = \dim P_0 \) and if \( f''(0) = 0 \) it has dimension \( 2n \). \( \square \)

Lemma 5.3.1 continues to hold for any compact Riemannian manifold \( X \). However, in the case \((T)\) or \((N)\), every nonconstant closed geodesic is not contractible. Let us now consider a nonzero homotopy class \( 0 \neq \alpha \in \pi_1(X) \) and denote by \( \ell \) the (unique) length of the closed geodesics in the class \( \alpha \). The space of solutions to equation (13) that represent the class \( \alpha \) consists of \( \{ (q(t), p(t)) \} \), where

\[
(16) \quad \left\{ \begin{array}{l}
q(t) \text{ is a geodesic in the class } \alpha, \text{ parametrized so that } |\dot{q}| \equiv \ell. \\
p(t) = \pm \frac{r}{2\ell} \dot{q}(t), \text{ where } r > 0 \text{ is such that } f'(r) = \pm \ell.
\end{array} \right.
\]

Given \( r > 0 \) with \( f'(r) = \pm \ell \) we denote

\[ P^\pm(r, \alpha) := \{ x = (q, p) : S^1 \to U^*X | p(t), q(t) \text{ satisfy (16)} \}. \]

In the case \((T)\) the space \( P^\pm(r, \alpha) \) is diffeomorphic to \( X \) and in the case \((N)\) it is diffeomorphic to \( S^1 \).

**Lemma 5.3.2.** Assume \( X \) satisfies \((T)\) or \((N)\) and let \( \alpha \neq 0, \ell, r > 0 \) with \( f'(r) = \pm \ell \), and \( P^\pm(r, \alpha) \) be as above. Then \( P^\pm(r, \alpha) \) is a Morse–Bott manifold of periodic orbits for \( H \) if and only if \( f''(r) \neq 0 \). Moreover, \( A_H(x) = f(r) - rf'(r) = f(r) \mp r \ell \) for every \( x \in P^\pm(r, \alpha) \).

**Proof.** As in the proof of Lemma 5.3.1, it follows from Remark 5.2.1 that \( P^\pm(r, \alpha) \) is a Morse–Bott manifold if and only if the space of periodic solutions of the linearized equation has the same dimension as \( P^\pm(r, \alpha) \), namely \( n \) in the case \((T)\) and 1 in the case \((N)\). We begin by linearizing equation (13). Given a path \( x : \mathbb{R} \to L\!U^*X : s \mapsto x_s = (q_s, p_s) \), we
represent a variation of $x$ by a pair $\hat{x} = (\hat{q}, \hat{p})$ of periodic vector fields along $q$ via $\hat{q} := \partial_s q_s$ and $\hat{p} := \nabla_s p_s$. Since $\partial_s |p| = |p|^{-1} \langle p, \nabla_s p \rangle$, the linearized equation has the form

$$\nabla_t \hat{q} = \frac{f'(r)}{r} \left( \hat{p} - \frac{\langle p, \hat{p} \rangle}{r} \right) + f''(r) \left( \frac{\langle p, \hat{p} \rangle}{r} \right) \frac{p}{r}, \quad \nabla_t \hat{p} + \frac{f'(r)}{r} R(\hat{q}, p) p = 0.$$  

(17)

Here $R \in \Omega^2(X, \text{End}(TX))$ denotes the Riemann curvature tensor. Note that in the case $f(r) = r^2/2$ we have $\nabla_t \hat{q} = \hat{p}$ and so (17) is equivalent to the standard Jacobi equation $\nabla \nabla \hat{q} + R(\hat{q}, p) q = 0$.

The periodic solutions of equation (17) form the kernel of the Hessian of the symplectic action (see Remark 5.2.1). Taking the pointwise inner product of the second equation in (17) with $p$ and using $\nabla_t p = 0$ we find that $\langle p, \hat{p} \rangle$ is constant. Hence, taking the $L^2$-inner product of the first equation in (17) with $p$ and using the fact that $|p| = r$, we find that every periodic solution of (17) satisfies

$$f''(r) \langle p, \hat{p} \rangle = 0.$$  

Moreover, taking the $L^2$-inner product of the second equation in (17) with $\hat{q}$ and using the first equation we find that every periodic solution of (17) satisfies

$$\int_0^1 \left( |\hat{p}|^2 - \langle \frac{p}{r}, \hat{p} \rangle^2 - \langle R(\hat{q}, p)p, \hat{q} \rangle \right) dt = 0.$$  

Now suppose that $X$ has nonpositive sectional curvature and that $f''(r) \neq 0$. Then $\hat{p} = 0$ and $\nabla_t \hat{q} = 0$ for every periodic solution of (17). Hence the space of periodic solutions of (17) has dimension $n$ in the torus case (namely $\hat{q}$ is uniquely determined by $\hat{q}(0) \in T_{q(0)}X$) and has dimension one in the negative curvature case (namely, $\hat{q}$ is a scalar multiple of $p$). In both cases it follows that the kernel of the Hessian of the symplectic action at every point $x \in P^\pm(r, \alpha)$ has the same dimension as $P^\pm(r, \alpha)$ and hence is equal to the tangent space of $P^\pm(r, \alpha)$ at $x$. This is equivalent to the Morse–Bott nondegeneracy condition (Remark 5.2.1). If, on the other hand, $f''(0) = 0$ then the dimension of the space of periodic solutions of (17) is $n + 1$ in the torus case and is $2$ in the negative curvature case.

5.4. Proof of Theorem 5.1.1. Fix a real number $c > 0$ and choose a smooth family of real functions $\{f_s(r)\}_{s \in \mathbb{R}}$, defined for $r \in \mathbb{R}$, with the following properties (see Figure 4):

(i) $f_s(-r) = f_s(r)$ for all $s$ and $r$.

(ii) For every $s \in \mathbb{R}$

$$f_s(0) > c, \quad f_s''(0) < 0, \quad \lim_{s \to -\infty} f_s(0) = c, \quad \lim_{s \to \infty} f_s(0) = \infty.$$  

(iii) For all $s$ and $r$ we have $\partial_s f_s(r) \geq 0$.  

(iv) $f'_s(r) \leq 0$ for $r \geq 0$ and, for $s \geq 1$,
\[
f_s(r) = \begin{cases} 
  f_s(0)(1 - r^2), & \text{if } 0 \leq r \leq 1 - 1/4s, \\
  0, & \text{if } r \geq 1 - 1/8s.
\end{cases}
\]

(v) $f'_s(r) \leq 0$ for $r \leq 1/2$, $f'_s(r) \geq 0$ for $r \geq 1/2$, and, for $s \leq -1$,
\[
f_s(r) = \begin{cases} 
  f_s(0)(1 - r^2), & \text{if } 0 \leq r \leq 1/8|s|, \\
  s, & \text{if } 1/4|s| \leq r \leq 1 - 1/4|s|, \\
  0, & \text{if } r \geq 1 - 1/8|s|.
\end{cases}
\]

(vi) For every $s$ the only critical point $r$ of $f_s$ with $f_s(r) > 0$ is $r = 0$.

It is not hard to prove that such a family $\{f_s(r)\}_{s \in \mathbb{R}}$ indeed exists.

Now define $H_s : U^*X \to \mathbb{R}$ by
\[
H_s(q, p) := f_s(|p|).
\]
Since $X$ satisfies $(T)$ or $(N)$, every contractible closed geodesic is constant. Moreover,
the constant periodic orbits $x \in \mathcal{P}(H_s)$ have the form $x \equiv (q, p)$ where $f'_s(|p|) = 0$, and
the symplectic action of such a constant solution is $\mathcal{A}_{H_s}(x) = f_s(|p|)$. Hence, by (vi), the
contractible periodic solutions of $H_s$ with positive action have the form $x = (q, 0)$ with
\[
\mathcal{A}_{H_s}(x) = f_s(0) > c.
\]

By Lemma 5.3.1, these solutions form a Morse–Bott manifold of periodic orbits for $H_s$.
Hence, by Theorem 5.2.2, we have
\[
HF^{[a, \infty]}(H_s; 0) \cong \begin{cases} 
  H_s(X; \mathbb{Z}_2), & \text{if } 0 < a < f_s(0), \\
  0, & \text{if } 0 < f_s(0) < a,
\end{cases}
\]
for every $s \in \mathbb{R}$. By Proposition 4.5.1, the monotone homomorphism
\[
\sigma_{H_{s_1}H_{s_0}} : HF^{[a, \infty]}(H_{s_0}; \alpha) \to HF^{[a, \infty]}(H_{s_1}; \alpha)
\]
is an isomorphism whenever $s_1 \leq s_0$ and $a \notin [f_{s_1}(0), f_{s_0}(0)]$. Now, for every $H \in \mathcal{H}(U^*X)$
there exists an $s \in \mathbb{R}$ such that $H \leq H_s$. Hence, by Lemma 4.7.1 (ii), the homomorphism
\[
\pi_s : SH^{[a, \infty]}(U^*X; 0) \to HF^{[a, \infty]}(H_s; 0)
\]
is an isomorphism for every $s \in \mathbb{R}$ such that $f_s(0) > a$. Hence
\[
SH^{[a, \infty]}(U^*X; 0) \cong H_s(X; \mathbb{Z}_2).
\]
Moreover, for every $H \in \mathcal{H}_{c_a}^{a,\infty}(U^*X, X; 0)$ there exists an $s \in \mathbb{R}$ such that $H_s \leq H$. Hence, by Lemma 4.7.1 (i), the homomorphism

$$\iota_s : \text{HF}^{[a,\infty]}(H_s, 0) \to \text{SH}^{[a,\infty);c}(U^*X, X; 0)$$

**Figure 1.** A family of profile functions.
is an isomorphism for every $s \in \mathbb{R}$ in the case $a \leq c$, and for every $s \in \mathbb{R}$ with $f_s(0) < a$ in the case $a > c$. Hence

$$\text{SH}^{(a, \infty); c}(U^*X, X; 0) \cong \begin{cases} H_*(X; \mathbb{Z}_2), & \text{if } a \leq c, \\ 0, & \text{if } a > c. \end{cases}$$

In the case $a > c$ it follows that $T_0^{(a, \infty); c} = 0$. In the case $a \leq c$ it follows from Proposition 4.8.1 that the map $T_0^{(a, \infty); c}$ can be expressed as the composition

$$T_0^{(a, \infty); c} = \iota_s \circ \pi_s : \text{SH}^{(a, \infty)}(U^*X; 0) \to \text{SH}^{(a, \infty); c}(U^*X, X; 0)$$

for every $s \in \mathbb{R}$. Hence, in this case, $T_0^{(a, \infty); c}$ is an isomorphism.

It remains to prove the statement on $\hat{C}(U^*X, X; 0, a)$. Indeed, it follows from what we have proved above that $\mathcal{A}_c(U^*X, X; 0) = (0, c]$ for every $c > 0$. Therefore, for every $a \in \mathbb{R}$ we have:

$$\hat{C}(U^*X, X; 0, a) = \inf \{c > 0 \mid \sup \mathcal{A}_c(U^*X, X; \alpha) > a\} = \max\{0, a\}.$$

The proof of Theorem 5.1.1 is complete.

5.5. Proof of Theorem 5.1.2. Fix a nontrivial homotopy class $\alpha \in \tilde{\pi}_1(X)$ and let $\ell$ denote the length of the geodesics in this class. Moreover, fix a real number $c > 0$ and choose a smooth family of real functions $\{f_s(r)\}_{s \in \mathbb{R}}$, defined for $r \in \mathbb{R}$, with the following properties (see Figure 2):

(i) $f_s(-r) = f_s(r)$ for all $s$ and $r$.

(ii) For every $s \in \mathbb{R}$

$$f_s(0) > c, \quad \lim_{s \to -\infty} f_s(0) = c, \quad \lim_{s \to \infty} f_s(0) = \infty.$$

(iii) For all $s$ and $r$ we have $\partial_s f_s(r) \geq 0$.

(iv) If $s \geq 1$ then

$$f_s(r) = \begin{cases} f_s(0), & \text{if } 0 \leq r \leq 1 - 3/8s, \\ 0, & \text{if } r \geq 1 - 1/8s, \end{cases}$$

$$f'_s(r) \leq 0 \text{ for all } r \geq 0, \text{ and }$$

$$f''_s(r) = \begin{cases} < 0, & \text{if } 1 - 3/8s < r < 1 - 2/8s, \\ > 0, & \text{if } 1 - 2/8s < r < 1 - 1/8s, \end{cases}$$

$$f'_s(1) = \begin{cases} < 0, & \text{if } 1 - 1/8s < r < 1 - 1/4s, \\ > 0, & \text{if } 1 - 1/4s < r < 1. \end{cases}$$
(v) If $s \leq -1$ then

$$f_s(r) = \begin{cases} f_s(0), & \text{if } 0 \leq r \leq 1/8|s|, \\ s, & \text{if } 3/8|s| \leq r \leq 1 - 3/8|s|, \\ 0, & \text{if } r \geq 1 - 1/8|s|, \end{cases}$$

$$f'_s(r) \leq 0 \text{ for } r \leq 1/2, \quad f'_s(r) \geq 0 \text{ for } r \geq 1/2,$$

and

$$f''_s(r) = \begin{cases} < 0, & \text{if } 1/8|s| < r < 2/8|s|, \\ > 0, & \text{if } 2/8|s| < r < 3/8|s|. \end{cases}$$

(vi) For every $s \in \mathbb{R}$ such that $f_s(0) > \ell$ there exist real numbers $r'_s > r_s > 0$ such that

$$f'_s(r_s) = f'_s(r'_s) = -\ell, \quad f''_s(r_s) < 0, \quad f''_s(r'_s) > 0,$$

and $f'_s(r) \neq -\ell$ for every $r \in [0, \infty) \setminus \{r_s, r'_s\}$.

(vii) For every $s \in \mathbb{R}$, the only possible points $r > 0$ with $f'_s(r) = \ell$ must satisfy $f_s(r) < 0$. It is not hard to show that such a family of functions $\{f_s(r)\}_{s \in \mathbb{R}}$ indeed exists. Now define $H_s : U^*X \to \mathbb{R}$ by

$$H_s(q, p) := f_s(|p|).$$

Consider first periodic orbits of $H_s$ that belong to one of the sets $P^+ (r, \alpha)$, $(r > 0)$, as defined by (16). We claim that the corresponding action is negative. Indeed, at such a value of $r > 0$ we have $f'_s(r) = \ell$, and the action is $f_s(r) - \ell r$ which is negative due to (vii).

Next, denote by $P_s := P^-(r_s, \alpha)$ and $P'_s := P^-(r'_s, \alpha)$ the other two components of the set of periodic solutions in the class $\alpha$ as defined by (16). Then $P_s$ and $P'_s$ are both diffeomorphic to $\mathbb{T}^n$ in the case $(T)$ and to $S^1$ in the case $(N)$. Moreover, by (vi) and Lemma 5.3.2, they are Morse–Bott manifolds of periodic orbits for $H_s$ for every $s \in \mathbb{R}$ and the values of the symplectic action functional on these two critical manifolds are

$$\mathcal{A}_{H_s}(P_s) = f_s(r_s) + r_s\ell, \quad \mathcal{A}_{H_s}(P'_s) = f_s(r'_s) + r'_s\ell.$$  

Fix a real number $a$ and denote $P := \mathbb{T}^n$ in the case $(T)$ and $P := S^1$ in the case $(N)$. We prove Theorem 5.1.2 in five steps.

**Step 1.** If $a < \ell$ then $\underline{SH}^{[a, \infty)}(U^*X; \alpha) = 0$.

By (iv), $f'_s(r) \leq 0$ for every $s \geq 1, r > 0$ hence for $s \geq 1$ there are no periodic orbits of the type $P^+(r, \alpha)$. Thus for $s \geq 1$ the only families of periodic orbits are $P_s$ and $P'_s$. Since both $r_s$ and $r'_s$ converge to $1$ as $s \to \infty$ it follows that the sets $P_s$ and $P'_s$ both have action bigger than $a$ for $s$ sufficiently large. Hence

$$HF^{[a, \infty)}(H_s; \alpha) \cong HF^{[a, \infty)}(H_s; \alpha) = 0$$
for $s$ sufficiently large. The last equation holds because $\alpha \neq 0$, so $HF^{[-\infty, \infty)}(H; \alpha)$ is independent of $H$, and there is a Hamiltonian function with only contractible 1-periodic orbits. Now Step 1 follows from Lemma [1.7.1] (ii).

**Step 2.** If $a \geq \ell$ then $\mathcal{SH}^{[a, \infty)}(U^*X; \alpha) \cong H_s(P; \mathbb{Z}_2)$. Moreover, the homomorphism

$$\pi_a : \mathcal{SH}^{[a, \infty)}(U^*X; \alpha) \to HF^{[a, \infty)}(H_s; \alpha)$$

is an isomorphism whenever $f_s(0) > a$. 

---

**Figure 2.** Another family of profile functions.
As \( a > \ell > 0 \) we may ignore all periodic orbits of the type \( P^+(r, \alpha) \) and consider only the families \( P_s, P'_s \). The numbers \( r_s \) and \( r'_s \) are the critical points of the function \( f_{s, \ell} : [0, 1] \to \mathbb{R} \) given by

\[
(18) \quad f_{s, \ell}(r) := f_s(r) + r \ell.
\]

By (vi), the point \( r_s \) is a strict local maximum and the point \( r'_s > r_s \) is a strict local minimum. Suppose that \( f_s(0) > a \). Then \( f_{s, \ell}(0) = f_s(0) > a \) and \( f_{s, \ell}(1) = \ell \leq a \), hence it follows that \( f_{s, \ell}(r_s) > a \) and \( f_{s, \ell}(r'_s) < a \). This means that

\[
A_{H_s}(P_s) > a, \quad A_{H_s}(P'_s) < a.
\]

Hence, by Theorem 5.2.2, \( \text{HF}^{[a, \infty]}(H_s; \alpha) \cong H_*(P; \mathbb{Z}_2) \) and, by Proposition 4.5.1, the monotone homomorphism \( \sigma_{H_s, H_{s_0}} : \text{HF}^{[a, \infty]}(H_{s_0}; \alpha) \to \text{HF}^{[a, \infty]}(H_{s_1}; \alpha) \) is an isomorphism whenever \( f_s(0) > a \) for \( i = 0, 1 \) and \( s_1 \leq s_0 \) and Step 2 follows from Lemma 4.7.1 (ii).

**Step 3.** If \( a > c > 0 \) then \( \text{SH}^{[a, \infty); c}(U^*X, X; \alpha) = 0 \).

As \( a > 0 \) we can again ignore all orbits of type \( P^+(r, \alpha) \). Since both \( r_s \) and \( r'_s \) converge to 0 as \( s \to -\infty \) it follows that the sets \( P_s \) and \( P'_s \) both have action less than \( a \) for \( -s \) sufficiently large. Hence \( \text{HF}^{[a, \infty]}(H_s; \alpha) = 0 \) for \( -s \) sufficiently large. Hence Step 3 follows from Lemma 4.7.1 (i).

**Step 4.** If \( 0 < a \leq c \) then \( \text{SH}^{[a, \infty); c}(U^*X, X; \alpha) \cong H_*(P; \mathbb{Z}_2) \). Moreover, the homomorphism

\[
\iota_s : \text{HF}^{[a, \infty]}(H_s; \alpha) \to \text{SH}^{[a, \infty); c}(U^*X, X; \alpha)
\]

is an isomorphism for \( s \ll -1 \).

Since \( a > 0 \), we may ignore as in previous steps orbits of type \( P^+(r, \alpha) \). Let \( f_{s, \ell} : [0, 1] \to \mathbb{R} \) be given by (18). Then, by (ii), \( f_{s, \ell}(0) = f_s(0) > c \geq a \) and hence

\[
A_{H_s}(P_s) = f_{s, \ell}(r_s) > f_{s, \ell}(0) > a.
\]

If \( s < \min\{-1, a - \ell/2\} \) then \( f_{s, \ell}(1/2) = s + \ell/2 < a \) and hence

\[
A_{H_s}(P'_s) = f_{s, \ell}(r'_s) < a.
\]

By Theorem 5.2.2, \( \text{HF}^{[a, \infty]}(H_s; \alpha) \cong H_*(P_s, \mathbb{Z}_2) \) for \( s < \min\{-1, a - \ell/2\} \). By Proposition 4.5.1, the monotone homomorphism \( \sigma_{H_s, H_{s_0}} : \text{HF}^{[a, \infty]}(H_{s_0}; 0) \to \text{HF}^{[a, \infty]}(H_{s_1}; 0) \) is an isomorphism for \( s_1 < s_0 < \min\{-1, a - \ell/2\} \). Step 4 follows now from Lemma 4.7.1 (i).

**Step 5.** If \( \ell \leq a \leq c \) then the homomorphism

\[
T^{[a, \infty); c}_\alpha : \text{SH}^{[a, \infty]}(U^*X; \alpha) \to \text{SH}^{[a, \infty); c}(U^*X, X; \alpha)
\]

is an isomorphism.
By (ii), \( f_s(0) > c \geq a \) for every \( s \). Hence, by Step 2, \( \pi_s \) is an isomorphism for every \( s \in \mathbb{R} \). Moreover, by Step 4, \( \iota_s \) is an isomorphism for \( s \ll -1 \). By Proposition 4.8.1, \( T_{[a,\infty);c} = \iota_s \circ \pi_s \) for every \( s \). Hence \( T_{[a,\infty);c} \) is an isomorphism.

It remains to prove the statement on \( \widehat{\mathcal{C}}(U^*X, X; \alpha, a) \). Indeed, it follows from what we have proved above that \( \mathcal{A}_c(U^*X, X; \alpha) = [\ell, c] \) for every \( c > 0 \). Therefore, for every \( a \in \mathbb{R} \) we have:

\[
\widehat{\mathcal{C}}(U^*X, X; \alpha, a) = \inf \{ c > 0 \mid \sup \mathcal{A}_c(U^*X, X; \alpha) > a \} = \max \{ \ell, a \}.
\]

The proof of Theorem 5.1.2 is complete.

\[ \square \]

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