Statistical interaction description of Pauli crystals in two-dimensional systems of harmonically confined fermions

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It has been conjectured that the Pauli exclusion principle alone may be responsible for a particular geometric arrangement of confined systems of identical fermions even when there is no interaction between them. These geometric structures, called Pauli crystals, are predicted for a two-dimensional system of free fermions under harmonic confinement. It is assumed that the system consists of neutral fermionic atoms with their spins frozen (spin-polarized) in order to avoid any form of electromagnetic interaction. These crystalline patterns emerge as the most frequent configurations seen in a large collection of single-shot pictures of the system. In this work, we pursue the possibility of this outcome and consider a theoretical model that may capture both qualitatively and quantitatively key features of the above mentioned setup. Our approach treats a quantum system of non-interacting fermions as an effective classical system of particles that interact with an effective statistical interaction potential that mimics the quantum statistics. For this model, we consider two-dimensional few-body systems of harmonically confined particles that interact with a statistical potential and calculate analytically the minimum energy configuration for specific values of relevant parameters. The results for $N = 3$ and 6 particles show that the minimum energy configuration corresponds to and is in good quantitative agreement with the reported values of Pauli crystals seen in single-shot imaging data obtained via the configuration density technique. Numerical results for larger systems of $N = 15$ and 30 particles show that the crystalline configurations observed are not the same as the classical Wigner crystal structures that emerge should the confined charged particles interact with a Coulomb potential. An important question floated is whether such crystalline structures do really exist in a quantum system or whether they are artifacts of the methods used to analyze them.

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I. INTRODUCTION

Pauli’s exclusion principle\cite{1} enforces high-order correlations in systems of identical fermions and its consequences are well known in quantum physics\cite{2,3}. This principle applies to fermions (such as electrons) and not bosons\cite{4}. Pauli’s exclusion principle basically states that no two fermions can occupy the same quantum state and, thus, anti-symmetrization of the full wave function is required\cite{4,5}. Recent work\cite{6,7} focused on confined systems of free fermions in a harmonic trap argues that quantum correlations imposed by Pauli’s exclusion principle may lead to the stabilization of crystalline structures, called Pauli crystals. These crystalline structures emerge even when there is no mutual interaction between fermions. It was argued that a crystal-like structure is observed when many identical fermions at very low temperature are confined within an external two-dimensional (2D) harmonic trap. The authors of this work\cite{6,7} showed how to extract these geometric structures from multiple single-shot pictures of the many-body system using a numerical approach known as the configuration density technique. The authors of Ref.\cite{6,7} refer to the results obtained by using the numerical configuration density approach as ”single-shot imaging experiments”. Although we will refer to these works with a similar terminology, it is our understanding that the works that we just mentioned are theoretical (they represent computational processing of large data sets). As far as we know, any experimental verification of the existence of the Pauli crystals does not exist at the moment.

It is important to remark that Pauli crystals are different from Wigner crystals of electrons\cite{11} or Coulomb crystals of ions since there is no interaction between fermions in the case of Pauli crystals. On the other hand, it is known that Wigner crystals of electrons or Coulomb crystals of ions are stabilized by interaction effects between particles\cite{12}. This means that Pauli crystals have a very different origin. Absence of any interaction between fermions is a pre-requisite to observe the effects of quantum statistics. Thus, one may detect these structures in few-body systems of non-interacting charge-neutral spin-polarized atomic fermions under 2D harmonic confinement at ultracold temperatures\cite{8,10}.

In this work we introduce a theoretical model for...
a Pauli crystal for the setup considered in Ref.\[8–10\]. The model allows us to treat a quantum system of free fermions as a classical ensemble of particles interacting with a statistical interaction potential [13, 14]. The statistical interaction potential mimics the quantum statistics of the particles. We consider small systems of $N = 3$ and $N = 6$ particles confined in a 2D harmonic trap and calculate exactly the minimum energy configuration corresponding to various parameters (such as temperature, etc.). The results obtained for systems with $N = 3$ and $N = 6$ particles show that the minimum energy configurations in this model correspond to the Pauli crystal configurations reported in Ref.\[8–10\] and are in good quantitative agreement with the equilibrium parameters reported in Ref.\[8–10\]. Numerical results for larger systems with $N = 15$ and $N = 30$ particles show that the crystalline configurations observed are not the same as the corresponding classical Wigner crystal structures of confined charges in a 2D harmonic trap.

The article is composed as follows. In Section II we briefly describe the methods used in single-shot imaging experiments. In Section III we introduce the theory and model for the case of particles under 2D harmonic confinement. Section IV contains the key results that apply to few-body systems of particles. A brief discussion and some concluding remarks are found in Section V.

II. SINGLE-SHOT IMAGING EXPERIMENTS OF FREE FERMIONS IN A HARMONIC TRAP

In this section we provide a brief description of recent single-shot imaging experiments which seem to provide hints for the existence of Pauli crystals. The discussions are based and rely on the work described in Ref.\[8–10\]. The system considered is a cloud of non-interacting fermions confined in a 2D isotropic harmonic trap with frequency, $\omega$. The fermions have their spins frozen (the system is spin-polarized). For such a case, the quantum one-particle wave functions would be written as:

$$\phi_{n_x n_y}(x, y) = N_{n_x n_y} \exp\left(-\frac{x^2 + y^2}{2l_0^2}\right) H_{n_x} \left(\frac{x}{l_0}\right) H_{n_y} \left(\frac{y}{l_0}\right),$$

(1)

where $N_{n_x n_y}$ is the appropriate normalization constant, $H_n(x)$ is a Hermite polynomial, $n_{x,y} = 0, 1, \ldots$ are the allowed quantum numbers and

$$l_0 = \sqrt{\frac{\hbar}{m \omega}},$$

(2)

is the so-called harmonic oscillator length. The number of quantum states with energy,

$$E_{n_x n_y} = \hbar \omega (n_x + n_y + 1),$$

(3)

is $(n_x + n_y + 1)$ and this represents the degeneracy of that energy value. An energy shell is said to be filled if all quantum states corresponding to that energy level are occupied. Systems with $N = 1, 3, 6, 10, 15, 21, \ldots$ particles correspond to filled shells. A Slater determinant wave function of occupied one-particle states,

$$\Psi(\vec{r}_1, \ldots, \vec{r}_N) = \frac{1}{\sqrt{N!}} \text{Det}\left\{ \phi_{\vec{n}_i}(\vec{r}_j) \right\},$$

(4)

represents the many-body ground state wave function consistent with Pauli’s exclusion principle. In the above notation $\vec{n}_i = (n_{ix}, n_{iy})$ and $\vec{r}_j$ is a 2D position vector for each of the $j = 1, \ldots, N$ particles.

In the single-shot measurements reported in Ref.\[8–10\], one attempts to determine the position configuration of $N$ fermions that maximizes the value of the probability distribution of the system, namely the configuration that maximizes the value of the modulus squared of the wave function, $|\Psi(\vec{r}_1, \ldots, \vec{r}_N)|^2$. A single-shot picture of the system allows one to obtain the particle’s positions. Since positions of particles in a quantum system are probabilistic variables one searches for the most probable ones. To this effect, the above-mentioned authors have found efficient ways to process a very large number of single-shots using a method known as the configuration density technique. A single-shot measurement leads to the corresponding sets of particle’s position configurations. The large amount of data for a multitude of single-shots are analyzed and from there one can extract the most probable configurations of the fermions. In a nutshell, after processing a multitude of single-shot imaging measurements, the results seem to suggest that the most probable configurations observed have a distinct crystalline nature and are universal if the number $N$ of fermions in a 2D harmonic trap corresponds to filled shells [8]. Specifically speaking, an equilateral triangle for $N = 3$ and a pentagon (with an additional atom at the center) for $N = 6$ fermions is observed.

The above-mentioned crystalline patterns are extracted from the measured noisy structure by using image processing techniques [10]. An instantaneous picture of the atoms gives a set of $N$ positions vectors. However, as mentioned earlier, a single picture of the many-body system in its ground state cannot reveal anything about the underlying configuration predicted by using the probability distribution [8]. In order to circumvent this problem and obtain the $N$-particle correlation function a multi-shot analysis procedure is employed in order to uncover the structure of the most probable configuration. The details of how to analyze the outcomes of single-shot measurements is explained in Ref.\[8–10\]. The number of single-shot pictures that leads to identifiable structures for the case of $N = 3$ and $N = 6$ fermions can be as small as $10^3$ but preferably should be as large as possible [8]. The results reported in Ref.\[8, 10\] for a system of $N = 6$ fermions are obtained after image processing is applied to an ensemble of $2 \times 10^6$ configurations at a temperature, $T = 1 \hbar \omega/k_B$. The crystalline structure in this case has one particle at the center of the harmonic trap and five particles in an outer pentagon structure each at a distance $r/l_0 = 1.265$ from the center.
III. MODEL

In quantum statistical mechanics, the mean thermal wavelength parameter, $\lambda$ is defined as:

$$\lambda = \sqrt{\frac{2\pi\hbar^2}{mk_B T}}, \quad (5)$$

where $m$ represents the mass of the particle, $k_B$ is Boltzmann’s constant $T$ is the absolute temperature of the system in Kelvin degrees and $\hbar$ is the reduced Planck’s constant. The ratio of the mean thermal wavelength to the mean interparticle distance can be related to the “indistinguishability” of quantum particles. If $\lambda$ is smaller than typical interparticle separations, the system of fermions will be nondegenerate and will be approximately classical satisfying Boltzmann statistics. This quantity is called the thermal wavelength because has the same order of magnitude as the de Broglie wavelength of a particle of mass $m$ with energy $k_B T$. The statistical correlations between quantum particles are expected to be irrelevant if temperature is so high that $\lambda$ is much smaller than the average interparticle distance. For such conditions, the system can be treated as an ideal gas. The first quantum correction to the classical partition function of an ideal gas can be rigorously calculated by following a recipe developed by Uhlenbeck et al. The central notion of the method is the mapping of non-interacting fermions into a classical interacting system via the so-called Uhlenbeck’s effective temperature-dependent statistical interaction potential. This correction has the same effect as endowing the particles with an inter-particle effective statistical interaction potential, $v_s(r)$ and treating the system classically. The treatment is general and applies to both fermions and bosons. The statistical potential between quantum particles (fermions or bosons) that arises from the symmetry properties of the $N$-particle wave function can be written as:

$$v_s(r) = -k_B T \ln \left[ 1 \pm \exp \left( -\frac{2\pi r^2}{\lambda^2} \right) \right], \quad (6)$$

where the “$\pm$” sign applies to, respectively, bosons/fermions. As seen from Fig. 1, the interaction of two bosons is effectively ”attractive” while that of two of fermions is highly ”repulsive” at short separation distances. In few words, the effective interaction mimics Pauli-repulsion effects for the case of fermions. Reduction of the temperature would lead to an increase of the range of the statistical potential. This implies that very low temperatures are desirable for any possible experimental observation of such purely statistical quantum effects.

The current model inspired by this approach consists of $N$ interacting classical particles under an isotropic 2D harmonic confinement:

$$E = \sum_{i<j} v(r_{ij}) + \frac{m}{2} \omega^2 \sum_{i=1}^{N} r_i^2, \quad (7)$$

where $v(r_{ij}) = -k_B T \ln \left[ 1 - \exp \left( -2\pi r_{ij}^2 / \lambda^2 \right) \right]$ is the effective fermionic statistical interaction potential, $r_{ij} = |\vec{r}_i - \vec{r}_j|$ represents the 2D inter-particle separation distance, $m$ is the mass, $\omega$ is the angular frequency of the harmonic potential and $\vec{r}_i$ are 2D position vectors. The energy of the system can be conveniently expressed in units of $\hbar\omega$ and can be written as:

$$\frac{E(\alpha)}{\hbar\omega} = -\alpha \sum_{i<j} \ln \left[ 1 - \exp \left( -\alpha \frac{r_{ij}^2}{l_0^2} \right) \right] + \frac{1}{2} \sum_{i=1}^{N} r_i^2, \quad (8)$$

where

$$\alpha = \frac{k_B T}{\hbar \omega}, \quad (9)$$

is a dimensionless temperature parameter and $l_0$ is the harmonic oscillator length defined in Eq. 2. Observe that we denoted energy as $E(\alpha)$ drawing attention to the fact that the value of this quantity depends on the parameter $\alpha$ as well as the spatial arrangement of the particle positions. It is easy to verify that $2\pi / \lambda^2 = \alpha / l_0^2$.

IV. FEW-BODY SYSTEMS

Earlier results obtained using the Monte Carlo simulated annealing method for finite 2D systems of harmonically confined particles that interact with a statistical
analytically at any arbitrary value of \( t \). The distance of each of the particles from the center of the trap for the case of the system with \( N = 3 \) particles was a crystalline structure consisting of an equilateral triangle with its center corresponding to the center of the harmonic trap. This result compares favorably to the previously reported results for systems with 6 particles.

Given the availability of quantitative results for \( N = 6 \) particles, we choose a system of \( N = 6 \) particles as our principal case study to verify analytically the suitability of the approach. The geometry of the \( N = 6 \) crystalline system under consideration is shown in Fig. 2. One can express all the distances of interest in terms of

\[
r = r_1 = r_2 = r_3 = r_4 = r_5 ,
\]

which represents the distance of each of the five particles located at the vertices of the pentagon relative to the center of the harmonic trap. The sixth particle is located at the center of the harmonic trap. It simple to verify that:

\[
\begin{align*}
 r_{16} &= r_{26} = r_{36} = r_{46} = r_{56} = r , \\
r_{12} &= r_{23} = r_{34} = r_{45} = r_{51} = 2 r \sin(36^\circ) , \\
r_{13} &= r_{24} = r_{35} = r_{41} = r_{52} = 2 r \sin(72^\circ) .
\end{align*}
\]

One can write the energy of the system in dimensionless units as:

\[
\frac{E(\alpha)}{\hbar \omega} = -5 \alpha \left\{ \ln \left[ 1 - \exp \left( -\alpha \frac{r_{16}^2}{l_0^2} \right) \right] + \ln \left[ 1 - \exp \left( -\alpha \frac{r_{12}^2}{l_0^2} \right) \right] + \ln \left[ 1 - \exp \left( -\alpha \frac{r_{13}^2}{l_0^2} \right) \right] \right\} + \frac{5 r^2}{2 l_0^2} ,
\]

where \( r_{16}, r_{12} \) and \( r_{13} \) depend on \( r \) and are given from Eq. (11). Since there are previously reported results for \( \alpha = 1 \) we choose \( \alpha = 1 \) and minimize the corresponding value of the energy, \( E(\alpha = 1)/\hbar \omega \) with respect to \( r \). The optimal value for \( r \) found is:

\[
\frac{r}{l_0} = 1.226 \quad ; \quad N = 6 \quad ; \quad \alpha = 1 . \quad (13)
\]

This result compares favorably to the previously reported value of \( r/l_0 = 1.265 \) in single-shot measurements. The relative percentage discrepancy is approximately 3%. The observed good quantitative agreement between the results is very rewarding considering the simplicity of the semi-classical model under consideration.

The case of the equilateral triangle configuration of \( N = 3 \) particles is simpler. One can calculate the distance of each of the particles from the center of the trap analytically at any arbitrary value of \( \alpha \) with the final result that reads:

\[
\frac{r}{l_0} = \sqrt{\frac{\ln(6 \alpha^2 + 1)}{3 \alpha}} \quad ; \quad N = 3 . \quad (14)
\]

As shown in Fig. 2, the optimal radial distance from the center of the trap was found to be

\[
\frac{r}{l_0} = 0.805 \quad ; \quad N = 3 \quad ; \quad \alpha = 1 . \quad (15)
\]

for the case of the system with \( N = 3 \) particles at \( \alpha = 1 \).

So far, the number of particles considered (\( N = 3 \) and 6 particles) is consistent with a closed shell numbering of the quantum system, i.e., the quantum many-body ground state is degenerate. It is also interesting to consider other numbers of particles, for example \( N = 4 \) and 5 particles that correspond to an open shell structure where the quantum ground state is degenerate. The

\[\text{FIG. 2: Equilibrium configuration for a system of } N = 6 \text{ particles under 2D harmonic confinement for a value of } \alpha = k_BT/(\hbar \omega) = 1.0. \text{ Dimensionless distances are used.}\]
In terms of the minimum energy configuration of the system is that of a pentagon where the center of the pentagon corresponds to the center of the trap. While there are no readily available quantitative results to whom we can compare our findings, the authors of Ref.\[9\] mention briefly that their single-shot imaging approach indicates that there are two equivalent configurations maximizing the 5-particle probability for the case of \( N = 5 \) fermions, i.e., the case of an open shell structure. For an open shell structure (case of \( N = 5 \) particles), one has the freedom to choose two occupied orbitals out of three quantum states leaving one empty. To lift the degeneracy of the ground state, the authors of Ref.\[9\] assumed that the harmonic frequencies, \( \omega_x \) and \( \omega_y \) are slightly different. The quantum energies in this case are \( E_{n_x, n_y} = \hbar \omega_x (n_x + 1/2) + \hbar \omega_y (n_y + 1/2) \). One can verify that \( E_{n_x=0, n_y=2} - E_{n_x=2, n_y=0} = 2 \hbar (\omega_y - \omega_x) \). This means that the degeneracy of the ground state is exactly lifted if one assumes that, let’s say \( \omega_x \) is a little bit smaller than \( \omega_y \), namely, \( \omega_y/\omega_x = 1 + \epsilon \) for a positive \( \epsilon \). This is the choice assumed in Ref.\[9\] for \( N = 5 \) particles and, for such a choice, the empty orbital (the one with higher energy) is \( n_x = 0 \) and \( n_y = 2 \). For such a case, the two quantum crystalline configurations observed were isosceles trapezoids differing by the reflection. For such a choice, the quantum ground state has no rotational symmetry. The only symmetry is reflection \( y \rightarrow -y \). The statistical interaction potential of our model in Eq.\[12\] leads to a minimum energy state with a circular shell structure of a pentagon for \( N = 5 \) particles. This means that in the case of \( N = 5 \) particles, the quantum configuration studied in Ref.\[9\] is not appropriately captured by the statistical interaction.

Similar arguments as above can be applied to the case of \( N = 4 \) particles. Obviously, one should be aware that the statistical interaction potential represents an approximation (at pair level) of the exact high-order quantum correlations between quantum particles and, thus, has disadvantages and limitations. While the main advantage of the method relies on its simplicity, the reader should be alerted to the limitations of using a classical approach to a quantum problem. For example, based on the results for \( N = 5 \) and \( N = 6 \) particles, it is probable that the description of the system in terms of the statistical interaction is limited to particular cases of very symmetric confinements and very symmetric quantum states. Alternatively, one also might argue that the choice of the initial quantum configuration in Ref.\[9\] may have influenced what sort of Pauli crystalline structure is observed at the end.

The reason why we studied small values of \( N \) in this work is to generate results that can be analytically checked up to a good degree. Obviously, for larger systems of particles, it is not possible to find the minimum energy configuration by using analytical tools. For such a scenario, the entire approach is considerably more difficult and should be fully numerical. In order to get glimpses of the behavior of the system at larger values of \( N \), we carried out additional numerical calculations using the simulated annealing method \[15\] in order to find the minimum energy configuration for some of the largest values of \( N \) that we could reliably simulate.
The minimum energy configurations for a system of $N = 15$ harmonically confined particles interacting with a statistical potential at various values of the parameter $\alpha = k_B T/(\hbar \omega)$. For $\alpha = 1.5$ (circles, red lines) the particles form a shell structure with a $(1,4,10)$ configuration. For $\alpha = 2.0$ (diamonds, green lines) we still have the same $(1,4,10)$ configuration as before. However, for $\alpha = 3.0$ (squares, blue lines) the configuration of the particles changes to a $(5,10)$ one. Distances are measured in arbitrary units.

FIG. 4: Plot of the minimum energy configuration for $N = 15$ particles under 2D harmonic confinement for different values of parameter $\alpha = k_B T/(\hbar \omega)$. For $\alpha = 1.5$ (circles, red lines) the particles form a shell structure with a $(1,4,10)$ configuration. For $\alpha = 2.0$ (diamonds, green lines) we still have the same $(1,4,10)$ configuration as before. However, for $\alpha = 3.0$ (squares, blue lines) the configuration of the particles changes to a $(5,10)$ one. Distances are measured in arbitrary units.

FIG. 5: Plot of the minimum energy configuration for a system of $N = 30$ particles under 2D harmonic confinement for a value of $\alpha = k_B T/(\hbar \omega) = 1.0$. Distances are measured in arbitrary units.

a classical approximation at the two-body level of the high order correlations originating from the Fermi statistics. Therefore, one should neither expect, nor suggest that the statistical potential can reproduce the high order correlations originating from quantum Fermi statistics. As we increase the temperature further to a larger value of $\alpha = 3.0$ the more stable configuration structure changes to $(5,10)$ that would be that of a 2D system of charged classical particles with a parabolic confinement $^{16}$, namely, a classical model of a Wigner crystal in a parabolic confinement potential. A larger value of $\alpha$ implies that the statistical potential is less relevant at typical separation distances between particles and, thus, the overall structure of the configuration tends to resemble that of repelling classical particles under a parabolic confinement potential.

In Fig. 5 we show the minimum energy configuration for the largest system that we were able to simulate. This system consists of $N = 30$ harmonically confined particles interacting with a statistical interaction potential for $\alpha = 1.0$. We noticed that the stabilized shell structure, $(4,10,16)$ differs from the Wigner crystal one $^{10}$ when no statistics is involved, that is, $(5,10,15)$. This result suggests that the most stable configurations for high number of particles (for instance, $N = 30$ particles) at sufficiently low temperatures is not that of a Wigner crystal of parabolically confined charges $^{10}$. However, a structure that has a geometry different from that of the
Wigner crystal configuration at \( N = 30 \) particles may be or may not be the same as the Pauli crystal counterpart (as seen earlier for the case of \( N = 15 \) particles). We are unsure if this the situation for the case of \( N = 30 \) particles since there are no available quantum mechanical single-shot data results to whom we can compare. In a nutshell, despite the fact that our approach is classical and has its limitations, some interesting insights do come from the numerical results that we obtained for systems with \( N = 15 \) and \( N = 30 \) particles.

V. DISCUSSION AND CONCLUSIONS

From the theoretical point of view, a good case for the existence of Pauli crystals should be based on showing that these structures provide an appropriate description of relevant aspects of a full quantum mechanical solution of the few-body problem under consideration. At zero temperature, the quantum state of a non-interacting system of \( N \) confined fermions is given by the Slater determinate wave function in Eq. (1). At a finite temperature, the quantum description of the system is provided by a thermal state, corresponding to the canonical density matrix. Does any of these two quantum solutions (for instance, the ground state for a small system of \( N = 3 \) fermions at \( T = 0 \)) resemble a Pauli crystal? The answer is not affirmative if one looks at the form of the one-particle density function, \( \rho(x, y) = |\phi_{00}(x, y)|^2 + |\phi_{10}(x, y)|^2 + |\phi_{01}(x, y)|^2 \) for \( N = 3 \) which is plotted in Fig. 6. However, it has been already noted in Ref. [8–10] that the one-particle density function cannot reveal any underlying geometric arrangement of the particles. Instead, one should look at the (conditional) \( N \)-particle correlation function if there are any underlying geometric arrangements (for instance, see Figure 2 of Ref. [8]). Based on these explanations, the answer to the question of whether such a geometric arrangement of fermions due to quantum statistics is firmly based on a theoretical model is still lacking.

With that said, we take the opportunity to iterate again that the objective of this work is simpler than a full-fledged theoretical investigation of the possible existence (or not) of the Pauli crystals. The focal point of this work was a practical one and had as its main objective to consider and investigate a simple model that may be able to capture both qualitatively and quantitatively all the elements of the physics of few-body Pauli crystals under the same conditions as recently reported in the literature [8–10]. The basic idea of the approach is the mapping of non-interacting fermions into a classical interacting system via the so-called effective statistical interaction potential method [13]. Such an effective interaction between the classical particles mimics the Pauli repulsion effects. We considered few-body systems of \( N = 3 \) and \( N = 6 \) particles under 2D harmonic confinement as well as some larger systems. The most energetically stable structure for \( N = 6 \) particles and the equilibrium shell radius of the pentagon structure for \( N = 6 \) particles calculated at a temperature that corresponds to \( \alpha = 1 \) was found to compare very favorably to the reported value obtained from single-shot experiments [8]. The current results for \( N = 3 \) and \( N = 6 \) particles are not in disagreement with the idea of Pauli crystals. However, results for open shell structures (\( N = 4 \) and \( N = 5 \) particles) suggest that the question of whether such crystalline structures do really exist in a quantum system or whether they are artifacts of the configuration density approach used is legitimate question open to discussion.

The nature of the statistical interaction potential adopted in this work is such that it is relevant for interparticle distances \( r_{ij} < \lambda \propto 1/\sqrt{T} \). Therefore, an experimental observation of Pauli crystals would be possible only for very low temperatures in systems of non-interacting fermions such as confined charge-neutral fermionic atoms. Such systems are known that can be cooled to very low temperatures. By the same token, confined electronic states that can be created in semiconductor quantum dots [17] or oxide quantum wells [18] would not be good choices since, most likely, the Coulomb interaction effects between electrons will dominate over quantum statistics at any given temperature. The effective statistical potential involves a temperature-dependent characteristic length scale known as the mean thermal wavelength, \( \lambda \) given in Eq. (4). At \( T = 0 \), the mean thermal wavelength becomes infinite. This means that we cannot reliably extend the use of the statistical potential to temperatures in the \( T \to 0 \) limit. The statistical potential has a thermodynamic origin and, thus, depends on the consideration of a nonzero temperature. Therefore, the reader should be aware of the limitations of the method. After all, the statistical interaction model represents a classical approximation to the real high-order quantum correlations between particles since it includes only pair-wise correlations.

FIG. 6: One-particle density function for a system of \( N = 3 \) fermions at their ground state. Brighter colors imply larger values of the function.
Note that the general expression for the energy of $N = 3$ and $N = 6$ particles is initially given for an arbitrary $\alpha$ and then numerically minimized for $\alpha = 1$. On the other hand, numerical simulations for larger systems with $N = 15$ and $N = 30$ particles involve different values of $\alpha$, respectively, $\alpha = 1.5, 2.0, 3.0$ ($N = 15$) and $\alpha = 1$ (the $N = 30$ case). This means that, in our calculations, we considered different choices of the temperature parameter $\alpha$ with no particular focus on any specific value. The choice $\alpha = 1$ encountered in few places of our study such as in Eq.(13) or Eq.(15) is because of the availability of results for $\alpha = 1$ in Ref.[8–10]. For example, we can immediately compare our analytical result for $\alpha = 1$ in Eq.(13), namely $r/l_0 = 1.226$ for $N = 6$ to the corresponding value of $r/l_0 = 1.265$ at $\alpha = 1$ as reported in Ref.[9]. In this sense, there no deep physical argument for the choice of $\alpha = 1$ except the opportunity for a direct comparison of the results of this work to corresponding results in the literature (reported at $\alpha = 1$). Overall, the results suggest that the use of this model has advantages due to its simplicity, but also it has limitations. It is probable that an accurate description of the system in terms of the statistical interaction may be limited to particular cases of symmetric quantum states. Furthermore, being a classical approximation at two-body level, the statistical interaction potential may not be able to fully reproduce all the exact high-order correlations between quantum particles arising from the Fermi statistics.

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Conflict of interest

The authors declare no conflict of interest.

[1] W. Pauli, Z. Physik 1925, 31, 765.
[2] P. A. M. Dirac, Proc. R. Soc. Lond. A 1929, 123, 714.
[3] D. Vvedensky, S. Crampin, M. E. Eberhart, J. M. Maclaren, Contemp. Phys. 1990, 31, 73.
[4] M. L. Cohen, MRS Bull. 2015, 40, 516.
[5] O. Ciftja, J. Phys. Chem. Sol. 2014, 75, 931.
[6] J. C. Slater, Phys. Rev. 1929, 34, 1293.
[7] O. Ciftja, AIP Adv. 2015, 5, 017148.
[8] D. Rakshit, J. Mostowski, T. T. Sowiński, M. Zaluska-Kotur, M. Gajda, Sci. Rep. 2017, 7, 15004.
[9] M. Gajda, J. Mostowski, T. Sowiński, M. Zaluska-Kotur, Europhys. Lett. 2016, 115, 20012.
[10] M. Gajda, J. Mostowski, T. Sowiński, M. Zaluska-Kotur, arXiv:1511.01036v3 2015.
[11] E. P. Wigner, Phys. Rev. B 1934, 46, 1002.
[12] C. C. Grimes, G. Adams, Phys. Rev. Lett. 1979, 42, 795.
[13] G. E. Uhlenbeck, L. Gropper, Phys. Rev. 1932, 41, 79.
[14] J. Batle, O. Ciftja, A. Farouk, M. Alkhambashi, S. Abdalla, Ann. Phys. 2017, 384, 11.
[15] S. Kirkpatrick, C. D. Gelatt, M. P. Vecchi, Science 1983, 220, 671.
[16] V. N. Bedanov, F. M. Peeters, Phys. Rev. B 1994, 49, 2667.
[17] O. Ciftja, Phys. Scr. 2013, 88, 058302.
[18] S. Stemmer, A. J. Millis, MRS Bull. 2013, 38, 1032.