Effective log Iitaka fibrations for surfaces and threefolds.

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Abstract

We prove an analogue of Fujino and Mori’s “bounding the denominators” [7, Theorem 3.1] in the log canonical bundle formula (see also [19, Theorem 8.1]) for Kawamata log terminal pairs of relative dimension one. As an application we prove that for a klt pair \((X, \Delta)\) of Kodaira codimension one and dimension at most three such that the coefficients of \(\Delta\) are in a DCC set \(A\), there is a natural number \(N\) that depends only on \(A\) for which \(\lfloor N(K_X + \Delta) \rfloor\) induces the Iitaka fibration. We also prove a birational boundedness result for klt surfaces of general type.

1 Introduction.

Let us start by recalling Kodaira’s canonical bundle formula for a minimal elliptic surface \(f: S \rightarrow C\) defined over the complex number field:

\[ K_S = f^*(K_C + B_C + M_C). \]

The moduli part \(M_C\) is a \(\mathbb{Q}\)-divisor such that \(12M_C\) is integral and \(\mathcal{O}_C(12M_C) \cong J^*\mathcal{O}_{\mathbb{P}^1}(1)\), where \(J: C \rightarrow \mathbb{P}^1\) is the \(J\)-invariant function. The discriminant \(B_C = \sum_P b_PP\), supported by the singular locus of \(f\), is computed in terms of the local monodromies around the singular fibers \(S_P\). Kawamata [10, 11] proposed an equivalent definition, which does not require classification of the fibers: \(1 - b_P\) is the log canonical threshold of the log pair \((S, S_P)\) in a neighborhood of the fiber \(S_P\).

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A higher dimensional analogue consists of a log klt pair \((X, \Delta)\) and surjective morphism such that the Kodaira dimension of \(K_X + \Delta\) restricted to the general fibre is zero. For now let us assume that \(K_X + \Delta = f^*D\) for some \(\mathbb{Q}\)-divisor \(D\) on \(Y\). Then we can define the discriminant or divisorial part on \(Y\) for \(K_X + \Delta\) to be the \(\mathbb{Q}\)-Weil divisor \(B_Y := \sum_P b_P P\), where \(1 - b_P\) is the maximal real number \(t\) such that the log pair \((X, \Delta + tf^*(P))\) has log canonical singularities over the generic point of \(P\). The sum runs over all codimension one points of \(Y\), but it has finite support. The moduli part or \(J\)-part is the unique \(\mathbb{Q}\)-Weil divisor \(M_Y\) on \(Y\) satisfying

\[ K_X + \Delta = f^*(K_Y + B_Y + M_Y). \]

According to Kawamata [9, Theorem 2](see also Ambro [3, Theorem 0.2 (ii)] and Fujino [6]) we know that on some birational model \(\mu : Y' \to Y\) the moduli divisor \(M_{Y'}\) is nef.

Some of the main questions concerning the moduli part are the following.

**Conjecture 1.1.** ([19, Conjecture 7.12]) Let \((X, \Delta)\) and \(f : X \to Y\) be as above and let us write as before

\[ K_X + \Delta = f^*(K_Y + B_Y + M_Y). \]

Then we have the following

1. **(Log Canonical Adjunction)** There exists a birational contraction \(\mu : Y' \to Y\) such that after base change the induced moduli divisor \(M_{Y'}\) on \(Y'\) is semiample.

2. **(Particular Case of Effective Log Abundance Conjecture).** Let \(X_\eta\) be the generic fibre of \(f\). Then \(I(K_{X_\eta} + \Delta_{X_\eta}) \sim 0\), where \(I\) depends only on \(\dim X_\eta\) and the horizontal multiplicities of \(\Delta\).

3. **(Effective Adjunction)** There exist a positive integer depending only on the dimension of \(X\) and the horizontal multiplicities of \(\Delta\) such that \(IM_{Y'}\) is base point free on some model \(Y'/Y\).

There is a proof of the above conjecture by Shokurov and Prokhorov in the case in which the relative dimension of \(f\) is one (Theorem 8.1 of [19]). For results towards (1) see Ambro [5]. Here we prove that there exist a positive integer \(I\) depending only on the dimension of \(X\) and the horizontal multiplicities of \(\Delta\) such that \(IM\) is integral when the relative dimension is one using ideas of Mori and Fujino [7] (see also [12]). The main advantage of our proof is that the number \(I\) that we produce is explicitly computable.
Our main interest in Conjecture 1.1 is because of its applications towards boundedness results for Iitaka fibrations. When \( X \) is of general type the existence of a natural number \( N \) such that \(|NK_X|\) induces the Iitaka fibration is known by results of C. Hacon and J. M'Kernan (cf. [8]) and Takayama (cf. [20]) following ideas by Tsuji. Similar results in low dimension when \( X \) is not of general type appear in the recent preprints [22, 18, 17]. Here we address the boundedness of Iitaka fibrations in the log case.

**Theorem 1.2.** Let \((X, \Delta)\) be a klt log pair of Kodaira codimension one and dimension at most three. Then there is a natural number \( N \) depending only on the coefficients of \( \Delta \) such that \(|N(K_X + \Delta)|\) induces the Iitaka fibration.

The proof of the above Theorem in dimension two relies on the existence of \( I \) as in the Conjecture and follows the strategy in Section 6 of [7]. For the proof of the Theorem in dimension three we need to bound the smallest positive number \( N \) such that \(|N(K_X + \Delta)|\) induces a birational map for any log surface of general type with the coefficients of \( \Delta \) in a DCC set \( \mathcal{A} \) as a function of the DCC set only (i.e. \( N = N(\mathcal{A}) \)). This is an interesting question in its own right (cf. [22]) and we address it in the last section. We can show that:

**Theorem 1.3.** Let \((X, \Delta)\) be a klt surface and assume that the coefficients of \( \Delta \) are in a DCC set \( \mathcal{A} \). Then there is a number \( N \) depending only on \( \mathcal{A} \) such that \(|m(K_X + \Delta)|\) (and \(|m(K_X + \Delta)|\)) defines a birational map for \( m \geq N \).

The above two Theorems complete the boundedness of Iitaka fibrations of klt pairs of dimension two (for the case of Kodaira dimension zero see [11]). The proof is based on the fact that by a result of [11] (see also [2]) for these surfaces we have a lower bound of the volume which allows us to produce centres of log canonical singularities of a controlled multiple of \( K_X + \Delta \). Using standard techniques we reduce to the case where the centres are isolated points. In order to achieve this, using ideas of M'Kernan [15] and Tsuji, we produce a morphism to a curve and we use this morphism to produce the required sections (cf. [21]).

2 Preliminaries.

2.1 Notations and Conventions.

We will work over the field of complex numbers \( \mathbb{C} \). A \( \mathbb{Q} \)-Cartier divisor \( D \) is nef if \( D \cdot C \geq 0 \) for any curve \( C \) on \( X \). We call two \( \mathbb{Q} \)-divisors \( D_1, D_2 \)
Q-linearly equivalent $D_1 \sim_Q D_2$ if there exists an integer $m > 0$ such that $mD_i$ are integral and linearly equivalent. We call two $\mathbb{Q}$-Cartier divisors $D_1, D_2$ numerically equivalent $D_1 \equiv D_2$ if $(D_1 - D_2) \cdot C = 0$ for any curve $C$ on $X$. A log pair $(X, \Delta)$ is a normal variety $X$ and an effective $\mathbb{Q}$-Weil divisor $\Delta$ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. A projective morphism $\mu : Y \rightarrow X$ is a log resolution of the pair $(X, \Delta)$ if $Y$ is smooth and $\mu^{-1}(\Delta) \cup \{\text{exceptional set of } \mu\}$ is a divisor with simple normal crossing support. For such $\mu$ we write $\mu^*(K_X + \Delta) = K_Y + \Gamma$, and $\Gamma = \sum a_i \Gamma_i$ where $\Gamma_i$ are distinct integral divisors. A pair is called klt (resp. lc) if there is a log resolution $\mu : Y \rightarrow X$ such that in the above notation we have $a_i < 1$ (resp. $a_i \leq 1$). The number $1 - a_i$ is called log discrepancy of $\Gamma_i$ with respect to the pair $(X, \Delta)$. We say that a subvariety $V \subset X$ is a log canonical centre if it is the image of a divisor of log discrepancy at most zero. A log canonical place is a valuation corresponding to a divisor of log discrepancy at most zero. A log canonical centre is pure if $K_X + \Delta$ is log canonical at the generic point of $V$. If moreover there is a unique log canonical place lying over the generic point of $V$, then we say that $V$ is exceptional. $	ext{LCS}(X, \Delta, x)$ is the union of all log canonical centres of $(X, \Delta)$ through the point $x$. We will denote by $\text{LLC}(X, \Delta, x)$ the set of all log canonical centres containing a point $x \in X$.

2.2 Generalities on cyclic covers.

Definition 2.1.

Let $X$ be a smooth variety and $L$ a line bundle on $X$ and $D$ an integral divisor. Assume that $L^m \sim \mathcal{O}_X(D)$. Let $s$ be any rational section and $1_D$ the constant section of $\mathcal{O}_X(D)$. Then $1_D/s^m$ is a rational function which gives a well defined element of the quotient group $k(X)^*/(k(X)^*)^m$, thus a well defined degree $m$ field extension $k(X)^{(m\sqrt{1_D/s^m})}$. Let $\pi : X' \rightarrow X$ denote the normalization of $X$ in the field $k(X)^{(m\sqrt{1_D/s^m})}$. Then

1. $\pi_*\mathcal{O}_{X'} = \sum_{i=0}^{m-1} L^{-i}([iD/m])$, and
2. $\pi_*\omega_{X'} = \sum_{i=0}^{m-1} \omega_X \otimes L^i(-[iD/m])$.

In particular, if $E$ is any integral divisor then the normalized cyclic cover obtained from $L^m \sim \mathcal{O}_X(D)$ is the same as the normalized cyclic cover obtained from $(L(E))^m \sim \mathcal{O}_X(D + mE)$. If $D$ has simple normal crossing support then $X'$ has only rational singularities.
2.3 DCC sets

Definition 2.2. A subset \( A \) of \( \mathbb{R} \) is said to satisfy the descending chain condition if any strictly decreasing subsequence of elements of \( A \) is finite. In this case we also say that \( A \) is a DCC set.

For the general properties of DCC sets we refer to Section 2 of [2].

Definition 2.3. A sum of \( n \) sets \( A_1, A_2, \ldots, A_n \) is defined as

\[
\sum_{i=1}^{n} A_i = \{a_1 + a_2 + \ldots + a_n | a_i \in A_i\}.
\]

Define also

\[
A_\infty = \{0\} \cup \bigcup_{n=1}^{\infty} \sum_{i=1}^{n} A_i.
\]

If \( A \) is a DCC set and it contains only non-negative numbers then it is easy to see that \( A_\infty \) is also a DCC set.

Definition 2.4. For \( A \subset [0, 1] \) we define the derivative set

\[
A' = \left\{ \frac{n-1 + a_\infty}{n} | n \in \mathbb{N}, a_\infty \in A_\infty \cap [0, 1] \right\} \cup \{1\}.
\]

It is easy to verify that if \( A \) is a DCC set then so is \( A' \).

3 Bounding the moduli part.

We start by describing the moduli part as it appears in [12]. Let \( f : (X, R) \to Y \) be a proper morphism of normal varieties with generic fibre \( F \) and \( R \) a \( \mathbb{Q} \)-divisor such that \( K_X + R \) is \( \mathbb{Q} \)-Cartier and assume that \( (F, R|_F) \) is lc and that \( K_F + R_i|_F \sim_{\mathbb{Q}} 0 \). Let \( Y^0 \subset Y \) and \( X^0 = f^{-1}(Y^0) \) be open subsets such that \( K_{X^0} + R^0 \sim_{\mathbb{Q}} 0 \) where \( R^0 := R|_{X^0} \) (cf. [12, Lemma 8.3.4]). Write \( R^0 = D^0 + \Delta^0 \) with \( D \) integral and \( |\Delta| = 0 \).

Assume that \( X^0, Y^0 \) are smooth and \( R^0 \) is relative simple normal crossing over \( Y^0 \).

Define \( V = \mathcal{O}_{X^0}(-K_{X^0} - D^0) \). Let \( m \) be (the smallest) positive integer such that \( m\Delta^0 \) is an integral divisor. Then we have an isomorphism

\[
V^\otimes m \cong \mathcal{O}_{X^0}(m\Delta^0),
\]

which defines a local system \( V \) on \( X^0 \setminus R^0 \) (cf. [12, Definition 8.4.6]).
Assume also that $Y$ is smooth, $Y \setminus Y^0$ is a simple normal crossing divisor and that $R^{\dim F} f_* \mathcal{V}$ has only unipotent monodromies. Then the bottom piece of the Hodge filtration of $R^{\dim F} f_* \mathcal{V}$ has a natural extension to a line bundle $J$. Set $J(X/Y, R)$ to be the divisor class corresponding to $J(X/Y, R)$.

If the smoothness, normal crossing, and unipotency assumptions above are not satisfied, take a generically finite morphism $\pi : Y' \longrightarrow Y$ and a resolution of the main component $f' : X' \longrightarrow X \times_Y Y' \longrightarrow Y'$ for which the assumptions hold and $R'$ the corresponding divisor. Then define

$$J(X/Y, R) = \frac{1}{\deg \pi} \pi_* J(X'/Y', R').$$

We need the following definition.

**Definition 3.1.** ([12, Definition 8.4.2]) Assume that $(X, R)$ is lc and $K_X + R \sim_\mathbb{Q} 0$ and write $R = R_{\geq 0} - R_{\leq 0}$ as the difference of its positive and negative parts. Define

$$p^+_g(X, R) := h^0(X, \mathcal{O}_X([R_{\leq 0}])).$$

**Theorem 3.1.** ([12, Theorem 8.5.1]) Let $X, Y$ be normal projective varieties and let $f : X \longrightarrow Y$ a dominant morphism with generic fibre $F$. Let $R$ be a $\mathbb{Q}$-divisor on $X$ such that $K_X + R$ is $\mathbb{Q}$-Cartier and $B$ a reduced divisor on $Y$. Assume that

(1) $K_X + R \sim f^*(\text{some } \mathbb{Q}\text{-Cartier divisor on } Y)$,

(2) $p^+_g(F, R|_F) = 1$, and

(3) $f$ has slc fibres in codimension 1 over $Y \setminus B$ (cf. [12]).

Then one can write

$$K_X + R \sim_\mathbb{Q} f^*(K_Y + J(X/Y, R) + B_R),$$

where

(i) $J(X/Y, R)$ is the moduli part defined above,

(ii) $B_R$ is the unique $\mathbb{Q}$-divisor supported on $B$ for which there is a codimension $\geq 2$ closed subset $Z \subset Y$ such that $(X \setminus f^{-1}(Z), R + f^*(B - B_R))$ is lc and every irreducible component of $B$ is dominated by a log canonical centre of $(X, R + f^*(B - B_R))$.

Let $(X_1, R_1), f_1 : X_1 \longrightarrow Y_1$ and $B_1$ be a pair satisfying the assumptions of Theorem 3.1 and $R_1$ effective on the general fibre. Assume furthermore that the relative dimension of $f_1$ is one and that $(X_1, R_1)$ is klt. Then the following holds.

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**Theorem 3.2.** There exist an integer $N$ depending only on the horizontal multiplicities of $R_1$ such that the divisor $NJ(X_1/Y_1, R_1)$ is integral.

We are going to prove the theorem in the case when the restriction of $\Delta$ to the general fibre of $f$ is non-trivial. When the restriction is trivial the Theorem follows as in [7, Theorem 3.1].

**Step I.** We start with some harmless reductions. Cutting by hyperplanes we can reduce to the case when $Y_1$ is a curve. From Step 1 of the proof of [12, Theorem 8.5.1], we can reduce to the case with normal crossing assumptions, that is we can assume that $X_1, Y_1$ are smooth, $R_1 + f_1^*B_1$ and $B_1$ are snc divisors, $f_1$ is smooth over $Y_1 \setminus B_1$ and $R_1$ is relative snc divisor over $Y_1 \setminus B_1$.

By Step 2 of the same proof we can assume also that $B_1 = B_{R_1}$.

**Step II - Galois cover of $Y_1$.** By [16, (4.6) and (4.7)] there is a finite Galois cover $\pi : Y \to Y_1$ with Galois group $G$, such that for the induced morphism $f : X \to Y$ every possible local system $R_{\dim F}f_*\mathcal{V}_j$ (the difference being given by a choice of isomorphism between two line bundles, compare [12, Remark 8.4.7]) has unipotent monodromies around every irreducible component of $B$. Note that we can also arrange that $G$ acts on $X$.

Thus by [12] Theorem 8.5.1] we have that the moduli part $L := J(X/Y, R)$ is integral. Here $\pi_X^*(K_{X_1} + R_1) = K_X + R$ and $R = (\pi_X)_*R_1$.

**Step III - Constructing the right cyclic cover.** There is a unique way to write

\[ R - f^*B = \Delta - G + E, \]

where $\Delta$ is effective, $[\Delta] = 0$, the divisors $E$ and $G$ are integral and vertical.

Let $M = f^*(K_Y + L + B) - (K_X + E - G + f^*B)$. Notice that $M$ is integral and that $\Delta \sim_{\mathbb{Q}} M$. Pick $m > 0$ such that $m\Delta$ is integral on the general fibre $F$. Such $m$ depends only on the horizontal multiplicities of $\Delta$. Since $m\Delta_{|F} \sim mM_{|F}$ there is an integral divisor $D$ on $X$ such that $D \sim mM$ and $D_{|F} = m\Delta_{|F}$.

Construct the cyclic covering $h : Z \to X$ corresponding to $D \sim mM$ and let $X^0$ be as in Definition [2.43]. After possibly changing the birational model of $X$ we can assume that $D$ is simple normal crossing and $Z$ has
rational singularities. We have the following diagram.

\begin{center}
\begin{tikzpicture}

\node (X) at (0,0) {$X$};
\node (Y) at (0,-2) {$Y$};
\node (Z) at (2,1) {$Z$};

\draw[->] (X) -- node[above] {$\pi_X$} (Z);
\draw[->] (X) -- node[below] {$f_1$} (Y);
\draw[->] (Y) -- node[below] {$f$} (X);
\end{tikzpicture}
\end{center}

The restriction of $\pi_X$ to $X^0$ gives one of the cyclic covers used in the construction of the local systems $R^\dim F_\ast \mathcal{V}_j$.

We have that

$$h_\ast \omega_Z = \sum_{i=0}^{m-1} O_X(K_X + iM - \lfloor iD/m \rfloor).$$

**Step IV - The $G$-action on $h'_\ast(\omega_{Z/Y})$.**

We now proceed as [7, 3.8]. By the pull-back property [12, Proposition 8.4.9 (3)] we have that $L = \pi^\ast J(X_1/Y_1, R_1)$. Let $P \in Y$ and localize everything in a neighborhood of $P_0 = \pi(P)$ and $P$, and let $e$ be the ramification index at $P$. Let $z_1$ be a local coordinate for the germ $(Y_1, P_1)$ and $z = (z_1)^{1/e}$ for $(Y, P)$. Since the divisor $D$ is $\mu_e$-equivariant over an open set $Y_0 \subset Y$ there is a group $G_0$ acting on $Z|_{Y_0}$ which fits in the sequence $0 \to \mu_m \to G_0 \to \mu_e \to 0$. In fact if locally $X$ is Spec$A$ then $Z$ is Spec$A[\phi^{1/m}]$ where $\phi$ is a local equation of $D$. Since locally $D$ is $\mu_e$-equivariant $\mu_e$ acts on $\phi$ by multiplication by $e$-th root of unity $\epsilon$ and $\mu_m$ acts on $\phi^{1/m}$ by a multiplication by an $m$-th root of unity $\epsilon$ there is $\mu_m \times \mu_e$ action on $Z$. Thus we can define a $\mu_{er}$-action on the local systems $R^1 h'_\ast \mathcal{V}_j$ where $r = m/(m, e)$ and hence on the canonical extension $h'_\ast(\omega_{Z/Y} \otimes \mathbb{C}(P))$. The action on the summand $L \otimes \mathbb{C}(P) \subset h'_\ast(\omega_{Z/Y} \otimes \mathbb{C}(P))$ is by a character $\chi_P$.

Let $E$ be the general fibre of $h'$. Then by [2, 3] we have that

$$h^0(E, \omega_E) = h^0(F, \sum_{i=0}^{m-1} \omega_F^{1-i}(-\lfloor i\Delta_1F \rfloor)) \leq (m-1)^2.$$

Reasoning as in [7, 3.8] if $l$ is the order of $\chi_P$, then $\varphi(l) \leq (m-1)^2$, where $\varphi(l)$ is the Euler function. Set $N(x) = \text{lcm}\{l|\varphi(l) \leq x\}$. Then for $N_1 = N((m-1)^2)$, the divisor $N_1 J(X_1/Y_1, R_1)$ is integral. \hfill $\square$

**Remark.** Note that the number above is easy to compute explicitly. This is then main advantage of our approach.
3.1 Auxiliary Lemma.

Let $Y$ be a smooth curve and let $h : Y' \rightarrow Y$ be a finite Galois cover with group $G$. Let $D$ be a $\mathbb{Q}$-divisor on $Y$ such that $D' = h^*D$ is Cartier. For $p' \in Y'$ let $G_{p'}$ be the stabilizer. We have that $G_{p'}$ acts on $(O_Y(D'))\otimes O_{P'}$ via a character $\chi_{p'} : G_{p'} \rightarrow \mathbb{C}$. In this setting we have the following lemma due to Fujino and Mori [7].

Lemma 3.3. (cf. [7]) For an integer $N$ the divisor $ND$ is integral if and only if for each $p' \in Y'$ the character $\chi_{p'}^N$ is trivial.

4 Iitaka fibrations for surfaces of log Kodaira dimension one.

In this section we prove Theorem 1.2 in dimension two. We start with the following lemma.

Lemma 4.1. Let $(X, \Delta)$ be a klt pair of dimension $n$ where the coefficients of $\Delta$ are in a DCC set $A \subset [0,1]$. Let $f : X \rightarrow Y$ be a surjective projective morphism such that for the general fibre $F \cong \mathbb{P}^1$ we have that $(K_X + \Delta)|_F \sim_{\mathbb{Q}} 0$. Then the set $B$ of coefficients of the horizontal components of $\Delta$ is finite. In particular there is an integer $m = m(B)$ that clears all the denominators of the horizontal components.

Proof. We can describe $B$ as the set $\{b \in A | b + a = 2, \text{for some} \ a \in A_\infty\}$. $B$ is a subset of a bounded DCC set, so it is itself a bounded DCC set. If $B$ is infinite, then there is an increasing infinite sequence. But this would give a decreasing infinite sequence in $A_\infty$, which is impossible since $A_\infty$ is a DCC set.

Theorem 4.2. Let $(X, \Delta)$ be a klt pair of dimension two and assume that the coefficients of $\Delta$ are in a DCC set of rational numbers $A \subset [0,1]$. Assume that $\kappa(K_X + \Delta) = 1$. Then there is an explicitly computable constant $N$ depending only on the set $A$ such that $\lceil N(K_X + \Delta) \rceil$ induces the Iitaka fibration.

Proof. To prove the theorem we are free to change the birational model of $(X, \Delta)$ (without changing the coefficients of $\Delta$). So after running the Log Minimal Model Program we can assume that $K_X + \Delta$ is nef. Log abundance for surfaces implies that $K_X + \Delta$ is semiample. Therefore there exists a positive integer $k$ such that $\lfloor k(K_X + \Delta) \rfloor$ defines the Iitaka fibration.
The morphism \( f : X \to Y \) for \( K_X + \Delta \) satisfies the hypothesis of Theorem 3.1, and hence we can write

\[
K_X + \Delta \sim_Q f^*(K_Y + B + J).
\]

By replacing the morphism \( f : X \to Y \) by an appropriate model we can assume that we have an isomorphism

\[
H^0(X, [n(K_X + \Delta)]) \cong H^0(Y, [n(K_Y + B + J)]),
\]

for every natural number \( n \) divisible by \( m \) as of Lemma 4.1 and \( \Delta \) is simple normal crossing over the generic point of \( Y \) (cf. [7, Theorem 4.5]). Here \( Y \) is a smooth curve. The coefficients of \( B \) are in a DCC set depending only on \( A \) (cf. [4, Remark 3.1.4]). We follow the argument in Section 6 of [7] to compute an integer \( N \) depending only on \( A \) for which \( [N(K_Y + B + J)] \) is an ample divisor. By Theorem 3.2 there is an integer \( m \), depending only on the DCC set \( A \) by Lemma 4.1, for which \( mJ \) is integral. Also note that \( [B] \geq 0 \).

We treat three cases.

**Case 1** \((g \geq 2)\). For \( N = 3m \) we obtain that \( \deg [N(K_Y + B + J)] \geq 2g + 1 \) and so the divisor in question is ample.

**Case 2** \((g = 1)\). We have that \( \deg (J + B) > 0 \) and the coefficients of \( m(J + B) \) are of the form integer plus an element in a fixed DCC set. Hence there is a positive constant \( c = c(A) \) such that the multiplicity at of \( m(J + B) \) at some point is greater than \( c \). Then for \( N > \frac{3}{c} \) we have that \( \deg [N(J + B)] \geq 3 \).

**Case 3** \((g = 0)\). In this case we have to find an integer \( N \) such that \( \deg [N(J + B)] - 2N > 0 \). This follows immediately from Lemma 4.3.

**Lemma 4.3.** For any set of elements \( a_i \) in a DCC set \( A \subset (0,1) \) such that \(-2 + \sum_{i=1}^{n} a_i > 0 \) there is an integer \( N = N(A) \) such that \(-2N + \sum_{i=1}^{n} \lfloor Na_i \rfloor \geq 0 \).

**Proof.** We proceed by induction on \( n \). Let \( c \) be any number \( 0 < c < \min \mathcal{A} \) and \( k \) such that \( 0 < k < \min \{ \mathcal{A}_\infty \cap (2,\infty) \} - 2 \). The base case is \( n = 3 \) and then it is enough to take \( N > \frac{4}{k} \). In fact

\[
\lfloor Na_1 \rfloor + \lfloor Na_2 \rfloor + \lfloor Na_3 \rfloor \geq \lfloor Na_1 \rfloor + \lfloor Na_2 \rfloor + \lfloor 2N \rfloor - \lfloor N(2 - a_3) \rfloor - 1.
\]
But $N(a_1 + a_2 + a_3 - 2) > 4$ hence $[Na_1] + [Na_2] - [N(2 - a_3)] > 2$ and so the desired inequality follows.

For the inductive step suppose that $\sum_{i=1}^n a_i \geq 3$ and order the $a_i$ so that $a_i \leq a_{i+1}$. Then $\sum_{i=1}^{n-1} a_i > 2$ and the assertion follows by induction. If not we have that $\sum_{i=1}^n a_i < 3$ and hence $n < \frac{3}{c}$. It suffices to take $N > \frac{3+c}{ck} > \frac{n+1}{k}$ since then

$$\sum_{i=1}^n [Na_i] - 2N \geq \sum_{i=1}^n Na_i - 2N - n + 1 \geq 2.$$

\[\square\]

5 Iitaka fibration for threefolds of log Kodaira dimension two.

In this section we complete the proof of Theorem 1.2 by proving it in dimension three.

Theorem 5.1. Let $(X, \Delta)$ be a klt pair of dimension three and assume that the coefficients of $\Delta$ are in a DCC set of rational number $A \subset [0, 1]$. Assume that $\kappa(K_X + \Delta) = 2$. Then there is a constant $N$ depending only on the set $A$ such that $[N(K_X + \Delta)]$ induces the Iitaka fibration.

Proof. Performing the same type of reductions in the proof of Theorem 4.2 we assume that we are in the case when we have a morphism $f : X \to Y$ where $Y$ is a surface, $\Delta|_F$ is non-trivial and we have an isomorphism $H^0(X, [n(K_X + \Delta)]) \cong H^0(Y, [n(K_Y + B + M)])$ for every $n$ sufficiently divisible. Here the divisor $K_Y + B + M$ is big, the coefficients of $B$ are in a DCC set depending only on $A$ (cf. [4, Remark 3.1.4]), $M$ is nef. Now take $n$ also divisible by by $l$ where $l$ is an integer such that $lM$ is integral and $|lM|$ is base point free. The integer $l$ depends only on the DCC set $A$. Such $l$ exists by the case of Conjecture 1.1 that is proven in [19, Theorem 8.1].

Notice that $(Y, B)$ is klt by [3, Theorem 3.1] and also by [19, Corollary 7.17]. The divisor $lM$ is base point free so we can replace it with a linearly equivalent divisor in $M'$, such that the the pair $(Y, B + \frac{1}{l}M')$ is klt and $H^0(Y, [n(K_Y + B + \frac{1}{l}M')]) = H^0(Y, [n(K_Y + B + M)])$ for every natural number $n$ divisible by $l$.

Now define the DCC set $B = A' \cup \{\frac{1}{l}\}$. Observe that $B$ depends only on $A$. Define $B_1 = B + \frac{1}{l}M' = \sum_i b_iB_i$ where $B_i$ are distinct irreducible divisors. By [2, Theorem 4.6] there is a computable constant $\beta$ that depends
only on $B$ such that $K_Y + (1 - \beta)B_1$ is a big divisor. Let $b$ be the minimum of the set $B$ and let $k = \left\lceil \frac{1}{m}\right\rceil$. Then define $B' = \sum_i b_iB_i$ where $b_i = \frac{kb_i}{k}$. We have that the divisor $K_Y + B'$ is big with coefficients in the DCC set $C = \{\frac{1}{k}i | i = 1, \ldots, k-1\}$. Also we have the inclusion $H^0(Y, [m(K_Y + B')]) \subset H^0(Y, [m(K_Y + B + \frac{1}{k}M')])$ for every $m$.

Now Theorem 6.1 implies that there is a number $N'$ depending only on $A$ such that $[m(K_Y + B')]$ defines a birational map for $m \geq N'$. Define $N = kN'$. Then we have that $H^0(Y, [N(K_Y + B')]) = H^0(Y, [N(K_Y + B + \frac{1}{k}M')])$ and hence the theorem follows.

\[\square\]

6 Birational boundedness for log surfaces of general type.

In this section we prove that for a surface pair $(X, \Delta)$ of log general type with the coefficients of $\Delta$ in a DCC set $A$ there is a number $N$ depending only on $A$ such that the linear system $[m(K_X + \Delta)]$ gives a birational map. Again by [1] or [2, Theorem 4.8] we have that $\text{vol}(K_X + \Delta) > \alpha^2$ for some $\alpha$ depending only on the DCC set $A$. We are going to use this lower bound of the volume to create a log canonical centre. The good case is when the volume of the restriction of $K_X + \Delta$ to the log canonical centre is large. Then we can proceed by cutting down the log canonical centre to a point and we generate a section of an appropriate multiple of $K_X + \Delta$. If the volume of the restriction is smaller then we are going to proceed as in [21].

**Theorem 6.1.** Let $(X, \Delta)$ be a klt surface of log general type and assume that the coefficients of $\Delta$ are in a DCC set $A \subset \mathbb{Q}$. Then there is a number $N$ depending only on $A$ such that $[m(K_X + \Delta)]$ defines a birational map for $m \geq N$.

**Proof.** Consider a log resolution $f : X' \to X$ of $(X, \Delta)$ and write $f^*(K_X + \Delta) = K_{X'} + (f^{-1})_*\Delta + \sum_i e_iE_i$ with $E_i$ exceptional. There is a natural number $n$ such that $e_i < 1 - \frac{1}{n}$ for every $i$. Define $\Delta' = (f^{-1})_*\Delta + \sum(1 - \frac{1}{n})E_i$. Since we have the inclusion $H^0(X', [m(K_{X'} + \Delta')]) \subset H^0(X, [m(K_X + \Delta)])$ by replacing the $A$ with the DCC set $A \cup \{1 - \frac{1}{n} | n \in \mathbb{N}\}$ we can assume that $X$ is smooth.

By [1] or [2, Theorem 4.8] we have that $\text{vol}(K_X + \Delta) > \alpha^2$ for some $\alpha$ depending only on the DCC set $A$. Take a Zariski decomposition $K_X + \Delta \sim_{\mathbb{Q}}$
\(A + E\) with \(A\) nef and \(E\) effective and \(A\) orthogonal to each component of \(E\). We have that \(\text{vol}(K_X + D) = \text{vol}(A) > a^2\).

Choose two general points \(x_1, x_2 \in X\). Arguing as in [20, Lemma 5.4 and Lemma 5.5] we can produce a divisor \(D_1 \sim a_1 A\), with \(a_1 < \sqrt{2}\alpha\) such that there is a non-empty subset \(I_1\) of \(\{1, 2\}\) with the following property:

\[\star (X, D_1)\] is lc but not klt at \(x_i\) for \(i \in I_1\) and not lc at \(x_i\) for \(i \notin I_1\).

With this choice of \(a_1\) we can furthermore assume that either \(\text{codim Nklt}(X, D_1) = 2\) at \(x_i\) for \(i \in I_1\) or \(\text{Nklt}(X, D_1) = Z \cup Z_+\) such that \(Z\) is irreducible curve and \(x_i\) is in \(Z\) but not in \(Z_+\) for \(i \in I_1\).

Assuming that \(Z \cdot A > c\) for some constant \(c\) and still following [20, Lemma 5.8] we can produce a divisor \(D_2 \sim a_2 A\) with \(a_2 < c + \epsilon + a_1\) such that there is a subset \(I_2\) of \(\{1, 2\}\) with the property that \((X, D_2)\) is lc but not klt at \(x_i\) for \(i \in I_2\) and not lc at \(x_i\) for \(i \notin I_2\) and codim \(\text{Nklt}(X, D_2) = 2\) at \(x_i\) for \(i \in I_2\).

Now if we set \(G = D_2 + (m - 1 - a_2 - \epsilon)A + (m - 1)E + F\) where \(0 < \epsilon \ll 1\) and \(F = [(m - 1)K_X + m\Delta] - (m - 1)K_X - (m - 1)\Delta\) we observe that \([(m - 1)K_X + m\Delta] - G \sim Q \epsilon A\). Since \(A\) is an ample divisor Kawamata-Viehweg vanishing implies that \(H^1(X, [m(K_X + \Delta)] \otimes \mathcal{I}(G)) = 0\) for \(m > a_2 + 1\) and hence the linear system \(\vert [m(K_X + \Delta)] \vert\) gives a birational map onto its image (cf. [14, Chapter 9]).

Thus we can now assume that for every general point \(x \in X\) we have a pair \((D_x, V_x)\), such that \(D_x \sim a_1 A\), \(V_x\) is a pure log canonical centre of \(D_x\), and \(\text{dim } V_x = 1\). By [15, Lemma 3.2] we have a diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{\pi} & X \\
\downarrow f & & \\
B & & \\
\end{array}
\]

where \(\pi\) is dominant and generically finite morphism of normal projective varieties, and the image of the general fibre of \(f\) is \(V_x\) for some \(x \in X\).

Arguing as in section 3 of [21] we can assume that the map \(\pi\) is birational. In fact if \(\pi\) is not birational we have at least two centres of log canonical singularities through a general point. Replacing each such pair of centres with a minimal centre we may assume that the dimension of the centres is zero and this way \([m(K_X + \Delta)]\) gives a birational map onto its image for \(m > 3a_1 + 1\) (compare [21, page 11]).

Thus we consider the case when \(\pi\) is birational. We replace \(X\) with a model on which \(K_X + \Delta\) is nef and big. To complete the proof we will
show that the degree of the restriction of $K_X + \Delta$ to a log canonical centre through a general point on an appropriate model is bounded from below by a constant that depends only on the DCC set $\mathcal{A}$. This is enough since we can apply Kawamata-Viehweg vanishing as before to produce sections with the desired properties and hence a birational map.

If $X \rightarrow B$ is not a morphism (over a general point $b \in B$) then there is a point $x \in X$ such that we have at least two pairs $(D_1, V_1)$ and $(D_2, V_2)$, such that $D_i \sim a_1(K_X + \Delta)$, $x \in V_i$ a pure log canonical centre of $K_X + \Delta + D_i$ of dimension 1 and $V_1 \neq V_2$ corresponding to two general fibres of $f$. If $x$ is a smooth point, we have that

$$(K_X + \Delta) \cdot V_1 = \frac{1}{a_1}D_2 \cdot V_1 \geq \frac{1}{a_1}V_2 \cdot V_1 \geq \frac{1}{a_1}$$

since $V_1^2 \geq 0$.

If $x$ is not smooth then $(X, \Delta)$ is not terminal at $x$ and so there is a projective birational morphism $\pi : X' \rightarrow X$ extracting a divisor of discrepancy less than or equal to zero. Therefore $\pi^*(K_X + \Delta) = K_{X'} + \Delta'$ where $\Delta' \geq 0$ and $K_{X'} + \Delta'$ is still nef and big. Since there are only finitely many divisors of non-positive discrepancy after finitely many extractions as above we may assume that there is a morphism $f : X' \rightarrow B$. Thus we may write $\pi^*(K_X + \Delta) = K_{X'} + \Delta'$. Here $\Delta'$ is effective and $K_{X'} + \Delta'$ is nef and big.

Now let $\beta = \beta(A)$ be as defined in 3.5 of [2]. We can assume that every $\pi$-exceptional divisor that dominates $B$ appears with a coefficient grater than $1 - \beta$ in $K_{X'} + \Delta'$. In fact suppose that this is not the case for an exceptional divisor $E$. Away from the intersection of $E$ with the other components of $\Delta'$ the divisor $E$ intersects two general fibres $F_1$ and $F_2$ corresponding to two log canonical centres as before. With this choice there are no other log canonical places of $K_X + \Delta + D_1 + D_2$ lying over $\pi(E)$ connecting the intersection of $E$ with $F_1$ and $F_2$. Then by the Connectedness Principle $E$ is a log canonical place for $K_X + \Delta + D_1 + D_2$. In particular $mult_E \pi^*D_i \geq \frac{\beta}{a_1}$ for at least one of the $D_i$, say $D_1$, and hence $(K_X + \Delta) \cdot V_1 > \frac{\beta}{a_1}$.

Now take a log resolution $g : X'' \rightarrow X'$ of $(X', \Delta')$ and let $f' = f \circ g$ and write $K_{X''} + \Delta'' + \sum e_i E_i + N_1 = g^*(K_{X'} + \Delta') + N_2$ where $\Delta'' + \sum e_i E_i + N_1$ and $N_2$ are effective with no common components, $\Delta''$ is the strict transform of $\Delta$ and the $E_i$ are the strict transforms of the $\pi$-exceptional divisors that dominate $B$ with $g_*(\Delta'' + \sum e_i E_i + N_1) = \Delta' (N_1$ and $N_2$ do not intersect the general fibre of $f'$). The divisor $K_{X''} + \Delta'' + \sum E_i + [N_1]$ is lc with the coefficients in a DCC set that depends only on $\mathcal{A}$ and so by [2] Theorem 4.6 the divisor $K_{X''} + \Delta'' + (1 - \beta) \sum E_i + [N_1]$ is still big. Hence for the general fibre $F'$ of $f'$ we have that $deg(K_{X''} + \Delta'' + \sum E_i + [N_1])$
$(1 - \beta) \sum E_i + [N_1]_{|F'} = \deg(K_{X''} + \Delta'' + (1 - \beta) \sum E_i)_{|F'} \geq c > 0$ where $c$ depends only on $A$. Now $g_*(\Delta'' + (1 - \beta) \sum E_i) \leq \Delta'$ and so $(K_{X'} + \Delta')_{|F} \geq c$. Since $K_{X'} + \Delta' = \pi_*(K_X + \Delta)$ it follows that $(K_X + \Delta) \cdot V_1 \geq c$.

**Corollary 6.1.** Let $(X, \Delta)$ be a surface klt pair of log general type and assume that the coefficients of $\Delta$ are in a DCC set $A$. Then there is a number $N$ depending only on $A$ such that $\lfloor N(K_X + \Delta) \rfloor$ defines a birational map.

**Proof.** Change the coefficients as in the last part of the proof of Theorem 5.1 and reduce to the case in which all the denominators of the coefficients of $\Delta$ are the same. Then by taking an appropriate multiple proceed with integral divisors only.

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