Phantom without UV pathology

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Abstract

We present a simple model in which the weak energy condition is violated for spatially homogeneous, slowly evolving fields. The excitations about Lorentz-violating background in Minkowski space do not contain ghosts, tachyons or superluminal modes at spatial momenta ranging from some low scale $\epsilon$ to the ultraviolet cutoff scale, while tachyons and possibly ghosts do exist at $p^2 < \epsilon^2$. We show that in the absence of other matter, slow roll cosmological regime is possible; in this regime $p + \rho < 0$, and yet homogeneity and isotropy are not completely spoiled (at the expense of fine-tuning), since for given conformal momentum, the tachyon mode grows for short enough period of time.

1 Introduction

There is growing interest in possible exotic gravitational phenomena at cosmological distances, triggered in part by the observation of the accelerated expansion of the Universe and in part by theoretical considerations. Approaches in this direction include the introduction of new light fields and the modification of gravity at ultra-large scales. Among the former, phantom energy with equation of state $p + \rho < 0$, thus violating weak energy condition, is certainly of interest both from purely theoretical viewpoint and possibly also for cosmology \[1\]. If phantom were to dominate the late-time cosmological evolution, a potentially observable feature would be the accelerated acceleration of the Universe\[1\]; if phantom were to drive inflation, the spectrum of relic gravitational waves would be blue, which would be potentially observable as well. However, most field theories violating weak energy condition, constructed to date, are pathological in the ultraviolet domain (UV; hereafter by UV we mean sufficiently high energies, but still below a UV cutoff of an effective field theory):

\[1\)This effect may be mimicked \[2\] in scalar-tensor theories of gravity with matter obeying $p + \rho > 0$, see, however Ref. \[3\].
they have either ghosts or tachyons or superluminally propagating modes, or combinations thereof. Furthermore, there are general arguments implying that in four dimensions, any\(^2\) such theory has UV instabilities \([1, 5, 6]\). Yet, purely phenomenological analysis \([8]\) reveals that this situation may not be completely generic. Also, the brane-world DGP model \([7]\) which may be well-behaved in UV \([9]\), allows for a period of accelerated cosmological expansion with effective \(p + \rho < 0\) \([10, 11]\). So, it makes sense to try to construct an example of a four-dimensional field-theoretic phantom with non-pathological UV behavior.

In view of the arguments of Ref. \([6]\), it is rather unlikely that there exist field theories with \(p + \rho < 0\) and no problematic features at all. As a modest approach, one may begin with a field theory in Minkowski space, which is consistent at energy scales from zero all the way up to the UV cutoff scale \(\mathcal{M}\) of an effective theory. One then deforms this theory in IR in such a way that its high energy behavior remains healthy, while the weak energy condition is violated for spatially homogeneous configurations, and pathological states appear below a certain low scale \(\epsilon\) only\(^3\). When pursuing this approach, one has to worry about possible superluminal modes which may emerge at high momenta after the theory is deformed in IR: this effect, even of small magnitude, would signal an inconsistency of the whole theory \([13]\).

In cosmological setting, a theory of this sort may be acceptable provided that the energy scale \(\epsilon\) is below, or at least sufficiently close to the Hubble scale. One envisages that the latter property needs fine tuning over and beyond other adjustments of parameters required to make the theory consistent with observations.

One way to implement this approach is to begin with a more or less conventional two-derivative theory\(^4\) with healthy behavior below the scale \(\mathcal{M}\), at least in certain backgrounds, and then add one-derivative terms to the Lagrangian, the latter suppressed by the small parameter \(\epsilon\). Such a construction is difficult to realize in scalar theories, so one is naturally lead to consider theories involving vector field(s). By now it is understood that UV problems inherent in vector theories without gauge invariance may be avoided \([17]\), and that one-derivative terms indeed give rise to interesting IR dynamics \([18]\).

In this paper we construct a simple model along these lines. The model is described in Section 2. The background is assumed to break Lorentz-invariance, but leave three-dimensional rotational invariance unbroken. We will check in Section 3 that in flat space-time, there are no ghosts, tachyons or superluminal modes at three-momenta above the

\(^2\)A possible exception has to do with (spontaneous) violation of 3d rotational invariance \([6]\).

\(^3\)In theories with ghosts and unbroken Lorentz symmetry this would not be acceptable anyway, since instabilities would occur at all spatial momenta and frequencies. Once Lorentz-invariance is broken, this observation no longer applies, and a theory with ghosts may be viable even for rather high IR scale \([12]\).

\(^4\)Another interesting possibility may emerge in a ghost condensate model \([14]\) with negatively tilted potential \([15]\). In the expanding Universe, this model indeed has \(p + \rho < 0\). A potential problem with this model is wrong sign of the quadratic gradient term in the action for perturbations \([16]\); it remains to be understood how dangerous this feature is.
critical momentum $p_c \sim \epsilon$. At $p < p_c$ there are tachyons and possibly ghosts.

Turning to the cosmological evolution in Section 4 we will show that, in the absence of other matter, a slow roll regime is possible with appropriate choice of the potential, once the homogeneous fields filling the Universe take appropriate values. In this regime, the equation of state corresponds to $p + \rho < 0$, with $|p + \rho| \ll \rho$, and, indeed, the fields slowly roll up the potential. In the model we consider, the slow roll regime occurs at $H \ll \epsilon$, so that the tachyonic mode is sub-horizon for some period of time.

We will then study inhomogeneous perturbations of the fields about slowly-rolling background (Section 5). We will show that for given conformal momentum $k$, the dangerous tachyonic mode grows for finite period of time, until it becomes super-horizon. Super-horizon modes do not grow; some of them, including would-be tachyon, freeze in (like in the case of the minimal scalar field in accelerating Universe), some decay. This property is not entirely trivial for the would-be tachyonic mode, and has to do with the fact that this mode is gapless in Minkowski background. With extra fine-tuning, the period of growth of the tachyonic mode can be made sufficiently small, so that the model can be made viable. We conclude in Section 6.

2 Model

The model we wish to present has two-derivative kinetic terms similar to those in Ref. [17], one-derivative term reminiscent of that in Ref. [18], and also a potential term. It has one vector field $B_\mu$ and one scalar field $\Phi$. The Lagrangian is

$$ L = L^{(2)} + L^{(1)} + L^{(0)} $$

(1)

where superscripts indicate the number of derivatives, and we take

$$ L^{(2)} = -\frac{1}{2} \alpha(\xi) g^{\rho\lambda} D_\rho B_\mu D^\mu B_\lambda + \frac{1}{2} \beta(\xi) D_\mu B_\nu D^\nu B_\lambda \cdot \frac{B^\nu B^\lambda}{M^2} + \frac{1}{2} \partial_\mu \Phi \partial^\mu \Phi $$

(2)

$$ L^{(1)} = \epsilon \partial_\mu \Phi B^\mu $$

(3)

$$ L^{(0)} = -V(B, \Phi) $$

(4)

where

$$ \xi = \frac{B_\mu B^\mu}{M^2} $$

Here $\alpha$ and $\beta$ are functions roughly of order 1, and $\epsilon$ is a free positive parameter – the IR scale; $M$ can be viewed as the UV cutoff scale of the effective theory, so that $M \gg \epsilon$. We will be interested in Lorentz-violating backgrounds with $\xi \neq 0$.

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5Space-time signature is $(-,+,+,+)$. 

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The Lagrangian is Lorentz-invariant provided that the potential $V$ depends on a combination $X = \sqrt{B_\mu B^\mu}$. For simplicity we will mostly consider potentials of the form

$$V = U(X) + W(\phi)$$

We begin with Minkowski space, and consider static homogeneous background

$$B_0 = X = \text{const}, \quad B_i = 0, \quad \Phi = \phi = \text{const}$$

where $B_0$ and $B_i$ are time and space components of $B_\mu$. We will be interested in small inhomogeneous perturbations

$$B_0 = X + b_0, \quad B_i = b_i, \quad \Phi = \phi + \varphi$$

Their mass terms are

$$-\frac{m_0^2 b_0^2}{2} - \frac{m_i^2 b_i^2}{2} - \frac{m_\phi^2 \varphi^2}{2}$$

with

$$m_0^2 = U_{XX},$$

$$m_1^2 = -\frac{1}{X} U_X ,$$

$$m_\phi^2 = W_{\phi\phi}$$

Here we introduced convenient notation for derivatives,

$$\frac{\partial U}{\partial X} \equiv U_X, \quad \frac{\partial^2 U}{\partial X^2} \equiv U_{XX},$$

etc.

Let us make a point concerning the energy-momentum tensor. For a field configuration with $B_i = 0$, Noether’s energy-momentum tensor is

$$T_{\mu\nu} = \kappa \partial_\mu B_0 \partial_\nu B_0 + \partial_\mu \Phi \partial_\nu \Phi + \delta_{\mu0} \epsilon \partial_\nu \Phi B_0 - \eta_{\mu\nu} L$$

where

$$\kappa(X) = \frac{X^2}{M^2} \beta(X) - \alpha(X)$$

In what follows we assume

$$\kappa > 0$$

If $B_0$ and $\Phi$ are spatially homogeneous, $B_0 = X(t), \, \Phi = \phi(t)$, energy density and pressure are

$$\rho = \frac{\kappa}{2} \dot{X}^2 + \frac{1}{2} \dot{\phi}^2 + V$$

$$p = \frac{\kappa}{2} \dot{X}^2 + \frac{1}{2} \dot{\phi}^2 + \epsilon \dot{\phi} X - V$$
The relation \( p + \rho < 0 \) is satisfied provided the time derivatives of the fields are small, and
\[
\dot{\phi}X < 0 \tag{13}
\]
In the cosmological context, time derivatives will indeed be small if the fields slowly roll along the potential; we will see later how this requirement, as well as the relation (13), can be satisfied.

3 Minkowski spectrum

3.1 Massless case

Let us switch gravity off and consider the spectrum of this model in Minkowski space and in background (6).

To begin with, we neglect the mass terms and write the quadratic Lagrangian for perturbations,
\[
L_{b_i, b_0, \varphi} = \frac{\alpha}{2} \partial_\mu b_i \partial^\mu b_i + \frac{\kappa}{2} \partial_\mu b_0 \partial^\mu b_0 + \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi + \epsilon \partial_0 \varphi b_0 - \epsilon \partial_i \varphi b_i \tag{14}
\]
From the Lagrangian (14) it is straightforward to find the spectrum. The transverse components of \( B_i \) decouple and have the standard massless dispersion relation \( \omega^2 = p^2 \). Among the three remaining modes — linear combinations of \( b_i = (p_i/p)b_L, b_0 \) and \( \varphi \), there exists one mode with \( \varphi = 0, b_0 = b_L \) and the same dispersion relation, \( \omega^2 = p^2 \), and two modes with dispersion relations
\[
\omega^2 = p^2 + \frac{\epsilon^2}{2\kappa} \pm \sqrt{\left( \frac{1}{\alpha} + \frac{1}{\kappa} \right) \epsilon^2 p^2 + \frac{\epsilon^4}{4\kappa^2}} \tag{15}
\]
These modes are not ghosts at any \( p \). Let us consider their properties in two different ranges of momenta.

We begin with high momenta, \( p^2 \gg (\alpha^{-1} + \kappa^{-1})\epsilon^2 \). Expanding in inverse momentum, we obtain
\[
\omega = p \pm \frac{1}{2} \epsilon \sqrt{\frac{1}{\alpha} + \frac{1}{\kappa} + \frac{1}{8p} \left( \frac{1}{\kappa} - \frac{1}{\alpha} \right)} \tag{16}
\]
We see that the modes have group velocity \( \partial \omega/\partial p \) smaller than the speed of light provided that
\[
\alpha > \kappa \tag{17}
\]
We concentrate on this case in what follows.
Let us turn to low momenta. There is a critical momentum,

$$p^2_c = \frac{\epsilon^2}{\alpha}$$  \hspace{1cm} (18)

Above this momentum, no tachyonic modes exist. At $p^2 < p^2_c$ one of the modes (that with the negative sign in (15)) is tachyonic. The highest possible tachyonic $|\omega^2|$ is at

$$p^2 = \frac{\epsilon^2}{4\alpha} \frac{2\alpha + \kappa}{\alpha + \kappa}$$

and at this momentum one has

$$\omega^2_{\text{min}} = -\frac{\kappa}{4(\alpha + \kappa)} \frac{\epsilon^2}{\alpha}$$  \hspace{1cm} (19)

It is worth noting that this value is much smaller than the maximum tachyonic momentum squared (18) provided that

$$\alpha \gg \kappa ,$$  \hspace{1cm} (20)

which of course requires a sort of fine-tuning.

Thus, the massless model in Minkowski background does not have superluminal, ghost or tachyonic modes at high momenta, and have tachyons at low momenta.

### 3.2 Non-zero masses

Let us now turn on masses of the perturbations, i.e., add the terms (8) to the Lagrangian (14). In what follows we will be interested in the case

$$m_\varphi^2 = 0 , \quad m_0^2 \sim m_1^2 , \quad \epsilon^2 \gg m_0^2, m_1^2$$  \hspace{1cm} (21)

Furthermore, we assume for simplicity of the analysis that the relation (20) holds.

The transverse modes $b_i$ again decouple; they have the dispersion relation

$$\omega^2_{\text{transv}} = p^2 + \frac{m_i^2}{\alpha}$$  \hspace{1cm} (22)

Equations for $b_0, \varphi$ and $b_L$ are

$$\left( \omega^2 - p^2 - \frac{m_0^2}{\kappa} \right) b_0 + \frac{i \epsilon}{\kappa} \omega \varphi = 0$$

$$\left( \omega^2 - p^2 - \frac{m_1^2}{\alpha} \right) b_L - \frac{i \epsilon}{\alpha} p \varphi = 0$$

$$\left( \omega^2 - p^2 \right) \varphi - i \epsilon \omega b_0 + i \epsilon p b_L = 0$$  \hspace{1cm} (23)
The analysis of high momentum modes goes through almost as before. Expanding $\omega$ in inverse momentum, one obtains under our assumptions that the two modes (16) remain the same (modulo small corrections), while the third mode behaves as

$$\omega = p + \frac{m_0^2 + m_1^2}{\alpha + \kappa} \frac{1}{2p} \tag{24}$$

For positive $m_0^2$ and $m_1^2$ this mode is also subluminal. All modes have healthy kinetic terms at high momenta, so there are no ghosts in UV.

Let us consider now the low-momentum modes, and assume $m_0^2 > 0$, $m_1^2 > 0$ (we will comment on the case $m_0^2 < 0$, $m_1^2 > 0$ in the end of this section). It is straightforward to see that the tachyonic mode again exists at $p^2 < p_c^2$, and the lowest tachyonic $\omega^2$ is still given by (19). It is instructive to study the dispersion relations at very low momenta, $p^2 \ll m_1^2/\alpha$. By inspecting eqs. (23), one finds that one mode contains mostly the field $b_L$ and has

$$\omega_{b_L}^2 = \frac{m_1^2}{\alpha} \tag{25}$$

There are two more modes: one non-tachyonic mode with

$$\omega_{\text{normal}}^2 = \frac{\epsilon^2}{\kappa} \tag{26}$$

and one tachyonic mode with

$$\omega_{\text{tachyonic}}^2 = -\frac{m_0^2}{m_1^2} p^2 \tag{27}$$

Neither of the modes with positive $\omega^2$ is a ghost. It is important that $\omega^2$ of the tachyonic mode vanishes as $p^2 \to 0$; in fact, even though eq. (27) as it stands is valid for $\epsilon^2 \gg m_0^2, m_1^2$ only, the property that the tachyonic mode is gapless, $\omega_{\text{tachyonic}}^2 \propto -p^2$ as $p^2 \to 0$, is of general validity.

To end up this section, let us comment on the case

$$m_0^2 < 0, \quad m_1^2 > 0$$

still assuming that (21) holds for absolute values. It follows from (16) and (21) that in this case all modes are still subluminal at high momenta, provided that

$$|m_0^2| < m_1^2$$

The tachyonic mode again appears at $p^2 = p_c^2$. The special property of this case is that at even lower momenta, $p^2 < |m_0^2|/\kappa$, this mode again becomes non-tachyonic, and there appears a negative energy state — ghost in the spectrum. Consider, e.g., the special case

$$m_0^2 = -m_1^2 < 0 \tag{28}$$
In this case, the tachyon exists in a finite interval of momenta, \(m_1^2 < p^2 < \frac{\epsilon^2 - m_1^2}{\alpha}\). There is a mode with massless dispersion relation at all momenta,

\[
\omega^2 = p^2
\]  

(29)

together with the modes (16) which are subluminal at high momenta. At high \(p^2\) the mode (29) has positive energy, while precisely at that value of momentum for which the tachyon disappears, \(p^2 = m_1^2/\kappa\), this mode becomes a ghost. It remains to be the ghost at all momenta below \(m_1^2/\kappa\).

4 Cosmological evolution

4.1 Field equations

In the case of spatially homogeneous fields with \(B_i = 0\) in spatially flat FRW metric

\[
ds^2 = N^2(t)dt^2 - a^2(t)dx^2
\]

the Lagrangian (11) reads

\[
\sqrt{g}L = \frac{\kappa}{2N} \dot{X}^2 - \frac{3\alpha}{2} \frac{\dot{a}^2 a}{N} X^2 + \frac{1}{2N} \dot{\phi}^2 + \epsilon a^3 \dot{\phi} X - a^3 NV(X, \phi)
\]

(30)

where \(X = N^{-1}B_0\). It is worth noting that, up to an \(X\)-dependent factor, the second term here is proportional to the gravitational Lagrangian, \(\sqrt{-g}R \propto (\dot{a}^2 a)/N\) modulo total derivative. Therefore, the effective “cosmological” Planck mass, entering the Friedmann equation, is modified as follows (cf. Ref. [19]),

\[
M_{Pl}^2 \rightarrow M_{Pl}^2 + \frac{\alpha}{4\pi} X^2
\]

(31)

We will have to make sure that this change is small\(^7\).

\(^6\)The tachyon mode can be avoided completely at the expense of fine-tuning \(m_1^2/\kappa = \frac{\epsilon^2 - m_1^2}{\alpha}\). In that case the only potentially dangerous feature of the model is a ghost in IR, i.e., at \(p < p_c\). This ghost is phenomenologically acceptable even for relatively large \(p_c\), well exceeding the Hubble parameter [12].

\(^7\)Another possibility is that \(X\) varies in time sufficiently slowly. We will not need to resort to this option.
where $H = \dot{a}/a$ is the Hubble parameter, and we set $N = 1$ for the rest of this section. The quantity of interest is

$$\rho + p = \epsilon \dot{\phi} X + \alpha \ddot{X} X^2 + 2\alpha H \dot{X} X + \kappa \dot{X}^2 + \dot{\phi}^2$$  \hspace{1cm} (32)$$

The most important for our purposes is the first term here.

Equations of motion for homogeneous fields are

$$-\kappa \left( \ddot{X} + 3 H \dot{X} \right) - \frac{1}{2} \kappa X \dot{X}^2 - \frac{3}{2} \alpha X H^2 X^2 - 3 \alpha H^2 X + \epsilon \dot{\phi} = V_X$$  \hspace{1cm} (33)$$

$$-\left( \dot{\phi} + 3 H \dot{\phi} \right) - \epsilon (\dot{X} + 3 H X) = V_\phi$$  \hspace{1cm} (34)$$

We will assume in what follows that $\alpha(X)$ and $\kappa(X)$ are not rapidly changing functions, so that

$$|\alpha_X| \lesssim \frac{\alpha}{X}$$  \hspace{1cm} (35)$$

$$|\kappa_X| \lesssim \frac{\kappa}{X}$$  \hspace{1cm} (36)$$

We are going now to discuss slow roll regime.

### 4.2 Slow roll

Besides the usual requirement that the fields and the Hubble parameter evolve slowly, we define the particular slow roll regime of interest for our purposes as the regime at which terms proportional to $\epsilon$ dominate in the left hand sides of the field equations (33), (34). The slow roll conditions are therefore not exactly the same here as in the case of inflation driven by scalar inflaton (or acceleration driven by scalar quintessence). One point is that there are terms without time derivatives in the field equation (33). These terms are undesirable, so one of our conditions is

$$\alpha H^2 X \ll \epsilon \dot{\phi}$$  \hspace{1cm} (37)$$

In view of (35) this condition implies that both non-derivative terms in eq. (33) are small. Hereafter when writing inequalities, we will always mean absolute values of the quantities on the left and right hand sides.

Another non-trivial slow-roll condition that ensures that $\dot{\phi}$-term is subdominant in eq. (34) is

$$\dot{\phi} \ll \epsilon X$$  \hspace{1cm} (38)$$

Other conditions are fairly obvious:

$$\dot{X} \ll H X$$  \hspace{1cm} (39)$$

$$\dot{\phi} \ll H \dot{\phi}$$  \hspace{1cm} (40)$$

$$\ddot{H} \ll H^2$$  \hspace{1cm} (41)$$
Note that because of (35) and (36), $\epsilon \dot{\phi}$-term indeed dominates over all other terms in the left hand side of eq. (33), provided that (37) and (39) are satisfied. Finally, the potential term dominates the cosmological evolution provided that

$$V \gg \epsilon \dot{\phi}X$$

(42)

Once these slow-roll conditions are satisfied, the field equations take simple form,

$$\epsilon \dot{\phi} = V_X$$

(43)

$$3H\epsilon X = -V_\phi$$

(44)

In this slow roll regime, the first term in eq. (32) dominates, and

$$\rho + p = XV_X$$

which is negative for

$$V_X < 0$$

(45)

(without loss of generality we assume that $X > 0$).

Equation (44) is algebraic, so the whole system of the slow-roll equations may appear unusual. To understand it better, it is instructive to consider the potentials of the form (5) and view eq. (43) as an equation for $X$, whose solution is

$$X = F(\dot{\phi})$$

Keeping only the terms proportional to $\epsilon$ in eq. (34), we write eq. (34) as follows

$$\epsilon \left[ F'(\dot{\phi}) \dddot{\phi} + 3HF(\dot{\phi}) \right] = -W_\phi$$

(46)

This equation can be understood as the field equation in a scalar field theory with the action

$$\int d^4x \sqrt{g} \left[ K(\dot{\phi}) - W(\dot{\phi}) \right]$$

(47)

where

$$K(z) = \epsilon \int F(z) \, dz$$

Truncated equation (44) is then the slow-roll equation for this theory. As an example, for

$$U(X) = -\frac{M^2}{2}X^2$$

(48)

one has from eq. (43)

$$X = -\frac{\epsilon}{M^2} \ddot{\phi}$$

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and equation (46) reads
\[ \frac{\epsilon^2}{M^2} (\ddot{\phi} + 3H\dot{\phi}) = +W_\phi \]
which is the usual scalar field equation, but in upside-down potential. In other words, the kinetic term in the effective Lagrangian (47) has negative sign. This explains why \( \rho + p < 0 \) in the slow roll regime.

Let us come back to the general case. Making use of eqs. (43) and (44) one can rewrite the slow-roll conditions in terms of the potential \( V(X, \phi) \) and its derivatives, or, in the special case (4), in terms of \( U(X) \), \( W(\phi) \) and their derivatives. As an example, assuming that (41) is satisfied, one finds from (39)
\[ \frac{1}{H\epsilon} W_\phi U_X \ll W_\phi \] (49)

For potentials obeying
\[ U_X \sim U/X, \quad W_\phi \sim W/\phi, \quad W_\phi \sim W/\phi^2 \] (50)
(e.g., for power-law potentials) one makes use of eq. (43) again, and finds from the latter inequality that
\[ W \gg U \]

Thus, the potential \( W(\phi) \) dominates in the Friedmann equation, so that
\[ H^2 = \frac{8\pi}{3M^2_{\text{Pl}}} W \]

It is now clear that the energy density indeed increases in time in our slow roll regime. Assuming without loss of generality that \( X > 0 \), one finds from (44) that \( \dot{W}_\phi < 0 \), while from (43) and (45) it follows that \( \dot{\phi} < 0 \). This means that the field \( \phi \) climbs the potential up; the value of \( W(\phi) \), and hence the energy, increases.

The condition (40) gives
\[ \frac{1}{(\epsilon H)^2} U_{XX} W_\phi \ll 1 \] (51)
\[ \frac{\dot{H}}{H^2} \frac{1}{H\epsilon} U_{XX} W_\phi \ll U_X \] (52)

For potentials obeying (50), one makes use of eq. (43) and finds that the second relation here is equivalent to (41). The relations (49) and (51) suggest that the potentials are sufficiently flat.

The condition (47) gives the relation of somewhat different sort,
\[ W_\phi \ll \frac{\epsilon}{\alpha H} U_X \] (53)
Similarly, one finds from (38)
\[ U_X \ll \frac{\epsilon}{H} W_\phi \] (54)
These can be satisfied simultaneously only for
\[ \frac{\epsilon^2}{\alpha H^2} \gg 1 \] (55)
Finally, one finds from (41)
\[ \frac{M_{Pl}}{\epsilon W^{3/2}} W_\phi U_X \ll 1 \]
To see that all above conditions can indeed be satisfied, let us begin with the example (48). In this case the evolution of the field \( \phi \) and the scale factor corresponds to the action (47) with
\[ K = -\frac{\epsilon^2}{2M^2} \phi^2 \]
Hence, the slow roll conditions (39), (40) and (41), written in terms of the field
\[ \sigma = \frac{\epsilon}{M} \phi \]
are the same as in inflationary theory; for power-law potentials \( W \) they give \( \sigma \gg M_{Pl} \), i.e.
\[ \phi \gg \frac{M M_{Pl}}{\epsilon} \] (56)
The validity of (37) and (38) is not guaranteed, however, so these conditions are to be imposed in addition to (56). One obtains from (38)
\[ M^2 \ll \epsilon^2 \] (57)
while (37) gives
\[ \frac{M^2}{\alpha} \gg H^2 \] (58)
The two relations (56) and (58) are generally satisfied simultaneously only in a finite interval of the values of \( \phi \); for example, for \( W = (1/2)\mu^2 \phi^2 \) this interval is
\[ \frac{M M_{Pl}}{\epsilon} \ll |\phi| \ll \frac{M M_{Pl}}{\sqrt{\alpha \mu}} \]
(note that because of (44), slow-roll dynamics occurs at \( \phi < 0 \) for positive \( X \)) which requires
\[ \mu^2 \ll \frac{\epsilon^2}{\alpha} \]
To end up with this example, we note that (42) is equivalent to one of the standard slow-roll conditions

$$\dot{\sigma}^2 \ll W$$

and that one indeed has

$$\alpha X^2 \ll M_{Pl}^2$$

(see discussion after eq. (31)), due to the relations (58) and (59). Thus, our first example indeed provides the case in which all our requirements are satisfied.

The above example has the property that $U_{XX} < 0$, so that the mass (59) is negative. In fact, in this example the masses obey (28), so there are no superluminal modes and/or ghosts at high momenta. The existence of the ghost (29) at low momenta is in accord with the negative sign of $K(\dot{\phi})$ in (17). If one adds extra terms to the potential (48), the mode (29) becomes either subluminal or superluminal at high spatial momenta.

An example which does not necessarily have $U_{XX} < 0$ is determined, e.g., by potentials

$$U(X) = \zeta^2 X_0^4 u \left( \frac{X}{X_0} \right)$$
$$W(\phi) = \tau^2 \phi^4$$

(60)

where $X_0$ is some typical value of the field $X$, $u$ is a smooth function of order 1 whose derivatives are also of order 1 and $\zeta$ and $\tau$ are coupling constants (in this example we set $\alpha = 1$ by redefining the fields). The quartic potential (60) is chosen for concreteness only. Equation (44) tells that the slow roll occurs at $X \sim X_0$ if

$$\phi \sim \frac{\epsilon X_0}{\tau M_{Pl}}$$

It is straightforward to check that all slow-roll conditions are satisfied provided that the parameters obey

$$\frac{M_{Pl}^4 \tau^2 \zeta^2}{\epsilon^4} \ll 1$$

and the value of the field $X$ satisfies

$$X_0^2 \ll M_{Pl}^2 \frac{M_{Pl}^4 \tau^2 \zeta^2}{\epsilon^4}$$
$$X_0^2 \ll \frac{\epsilon^2}{\zeta^2}$$

Clearly, such a choice can indeed be made.
5 Perturbations about slowly rolling background

Let us consider perturbations of the fields $B_\mu$ and $\Phi$ in the slow-roll regime. Assuming that all masses are small compared to $M_{Pl}$, we neglect mixing with perturbations of gravitational field\(^8\). To simplify formulas, we take $\alpha$ and $\beta$ independent of $X$; our main conclusions remain valid in the general case.

Before proceeding, let us make a few comments on the relations between the mass terms, the value of $\epsilon$ and the Hubble parameter. First, we note that eq. (45), i.e., the negative sign of $XV_X$, ensures that the mass term for vector perturbations $b_i$, eq. (10), is positive. Second, from eqs. (53) and (54) and the slow-roll equation of motion (44) it follows that

$$\epsilon^2 \gg m_1^2 \gg \alpha H^2$$

(61)

where, as before, $m_1^2 = -\frac{1}{X} U_X$. Likewise, eq. (61) implies that $m_\phi^2 \equiv W_{\phi\phi}$ is small. Assuming that $U_{XX} \sim X^{-1} U_X$, we thus come to consider the case studied in Section 3.2 in Minkowski background.

Due to (55) and (61), the Hubble parameter is small compared to $\epsilon/\sqrt{\alpha}$, $\epsilon/\sqrt{\kappa}$ and $m_1/\sqrt{\alpha}$. Therefore, the expansion of the Universe has no effect on transverse modes which have the gap in the spectrum (22) as well as on the two modes with the gaps (25) and (26). For given conformal momentum $k$, one mode becomes tachyonic as the physical momentum redshifts to $p_c$, which is given by (18), and which is much larger than $H$. As the physical momentum redshifts further, this mode enters super-horizon regime. Thus, we have to understand the behavior of this mode in that regime\(^9\).

It is convenient to work with conformal metric,

$$ds^2 = a^2(\eta)(d\eta^2 - dx^2)$$

In this metric, the fields with perturbations are

$$B_0 = aX + b_0, \quad B_i = b_i, \quad \Phi = \phi + \frac{1}{a} \chi$$

where $X$ and $\phi$ are background fields. Equations for longitudinal $b_i = \frac{k_i}{k} b_L$, $b_0$ and $\chi$ in

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\(^8\)Mixing with perturbations of the gravitational field in Minkowski background leads to new effects at even lower momentum scales as compared to those considered here \cite{17, 18}. These scales are well below the Hubble scale in our context, so they are not of interest here.

\(^9\)In the case $m_0^2 = -m_1^2$ considered in the end of section 3.2, there are no tachyonic modes at $p^2 < m_1^2/\kappa$. The analysis below applies to the low-momentum ghost mode, which obeys $\omega^2 = p^2$ in Minkowski space-time.
conformal time are
d's

\begin{align}
- \kappa \left( b_0'' - \frac{a''}{a} b_0 + k^2 b_0 \right) - 3 \alpha \frac{a'^2}{a^2} b_0 - a^2 m_0^2 b_0 + 2 i k \alpha \frac{a'}{a} b_L + \epsilon a \left( \chi' - \frac{a'}{a} \chi \right) &= 0 \quad (62) \\
- \alpha \left( b_L'' - \frac{a''}{a} b_L - \frac{a'^2}{a^2} b_L + k^2 b_L \right) - a^2 m_1^2 b_L - 2 i k \alpha \frac{a'}{a} b_0 - i e k a \chi &= 0 \quad (63) \\
\chi'' + \frac{a''}{a} \chi - k^2 \chi - a^2 m_1^2 \chi - \epsilon a \left( b_0'' + 2 \frac{a''}{a} b_0 \right) + i e k a b_L &= 0 \quad (64)
\end{align}

where $k$ is conformal momentum and prime denotes $\partial / \partial \eta$. We are interested in slow modes whose physical momenta are small,

$$k / a \ll m_{0.1} / \sqrt{\alpha}$$  

and time-derivatives are small compared to $m_{0.1} / \sqrt{\alpha}$. Recalling that $m_{0.1}^2 \gg \alpha H^2$, we neglect the first two terms in eq. (62) and the first term in eq. (63). Then these two equations become algebraic equations for $b_0$ and $b_L$, giving

$$b_0 = \frac{\epsilon}{m_0^2} \left[ \frac{1}{a} \chi' - H \chi - \frac{2 \alpha H}{m_1^2} \left( \frac{k}{a} \right)^2 \chi \right]$$  

$$b_L = \frac{\epsilon}{m_1^2} \left[ -i \frac{k}{a} \chi + \frac{2 i \alpha H k}{m_0^2} \left( \frac{1}{a} \chi' - H \chi \right) \right]$$

where $H = a' / a^2$ is the Hubble parameter, and we made use of (61) and (65). Plugging these expressions into eq. (64), we obtain the equation for $\chi$,

$$- \chi'' + \frac{a''}{a} \chi - k^2 \chi + \frac{\epsilon^2}{m_0^2} \left[ - \chi'' + \frac{a''}{a} \chi + \frac{m_0^2}{m_1^2} k^2 \chi - a^2 \frac{m_0^2 m_1^2}{\epsilon^2} \chi + 2 \alpha \frac{k^2}{m_1^2} \left( H^2 - \frac{1}{a} H' \right) \chi \right] = 0$$

Recalling again the relations (61), we obtain finally the equation for the soft mode,

$$- \chi'' + \frac{a''}{a} \chi + \frac{m_0^2}{m_1^2} k^2 \chi - a^2 \frac{m_0^2 m_1^2}{\epsilon^2} \chi = 0$$  

(68)

We see that for $m_\varphi = 0$, the field $\chi$ obeys the equation for the scalar field with tachyonic dispersion relation (27), but now in the expanding Universe. For non-vanishing $m_\varphi$, the relation (61) implies that the term with $m_\varphi^2$ is small compared to the second term (the latter is of order $H^2 a^2 \chi$). Thus, the solutions of eq. (68) have tachyonic behavior at $k / a \gg H$ with the dispersion relation (27).

\footnote{We do not write here the terms proportional to $X'$. It can be shown that in the slow-roll regime, these terms are subdominant for both super-horizon and sub-horizon modes. We also neglect the time dependence of masses for the same reason.}
On the other hand, in super-horizon regime $k/a \ll H$, there are usual “constant” mode $\chi \propto a$ and decaying mode (for time-independent $H$ the latter is $\chi \propto a^{-2}$). More precisely, in de Sitter space-time with

$$a = \frac{1}{H\eta},$$

the “constant” mode is

$$\chi = -\frac{c}{H\eta^{1-\delta}},$$

where $c$ is the small amplitude and

$$\delta = \frac{m^2/\epsilon}{3\epsilon^2 H^2} \ll 1.$$ 

The perturbation of the field $\Phi$ is, therefore, almost time-independent,

$$\varphi \equiv \frac{1}{a} \chi = c\eta^\delta \quad (69)$$

It follows from (66) that for the “constant” mode one has

$$b_0 = \frac{\epsilon H}{m^2_\varphi} \delta \chi$$

This means that the physical temporal component of the vector field perturbation also slowly varies in time,

$$\frac{b_0}{a} \sim \frac{\epsilon m^2_\varphi}{\epsilon H \eta^\delta} \quad (70)$$

On the other hand, it follows from (67) that for the “constant” mode, $b_L$ does not depend on time, so the physical spatial component of the vector field, $B_i/a$, decays as $a^{-1}$.

We conclude that in the superhorison regime, the tachyonic mode is no longer dangerous, as it gets almost frozen in. This mode has finite time to develop in the expanding Universe.

### 6 Conclusions

Needless to say, our model needs a lot of fine tuning to be a viable candidate for describing inflation or present-time cosmological evolution. In particular, the ratio $\epsilon^2/(\alpha H^2)$ is to be large in the slow roll regime, but not exceedingly large. Indeed, from the time a mode becomes tachyonic (this occurs when $k/a = \epsilon/\sqrt{\alpha}$) to the time this mode exits the horizon (at that time $k/a \sim H$), the growth factor for this mode is

$$\exp (N_{growth}) \sim \exp \left( \frac{1}{H} \int_H^{k/a} \frac{\sqrt{\epsilon^2/\alpha}}{\omega(p)} \frac{dp}{p} \right)$$
The integral here saturates at \( p \sim p_c \), and using (19) one estimates

\[
N_{\text{growth}} = \text{const} \cdot \sqrt{\frac{\kappa}{\alpha}} \cdot \sqrt{\frac{\epsilon^2}{\alpha H^2}}
\]

For the growth factor not to be huge, one either has to fine tune \( \kappa \) to be much smaller than \( \alpha \), or fine tune \( \epsilon^2/(\alpha H^2) \) to be not too large, or both\(^\ast\). Crudely speaking, energy density in inhomogeneous modes makes a small fraction of the background energy density only if

\[
\frac{\sqrt{\kappa} \epsilon}{\alpha H} \ll \log \frac{M_{\text{Pl}}}{\epsilon}
\]

It would be interesting to see whether this and numerous other fine tunings can occur automatically.

The model presented here is almost certainly not the most appealing UV-safe phantom theory. Our attitude was rather to show that such a theory is possible at all. We therefore did not attempt to study a number of important issues: whether the slow roll regime can be a cosmological attractor, whether in inflationary context there is a natural way to exit from this regime, whether in the context of the recent Universe the tachyon instability at long wavelengths may have interesting consequences, etc. Even before addressing these issues it would be desirable to understand what properties of this model are generic, and what properties are model-dependent. In particular, it would be interesting to see whether tachyons, which exist in our model in a fairly wide range of spatial momenta above \( H \), may be avoided, leaving behind less dangerous IR ghosts.

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\(^\ast\)As we mentioned above, another possibility is to fine tune \( \frac{m^2}{\kappa} = \frac{\epsilon^2 - m^2}{\alpha} \). In that case the tachyon does not exist at all.
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