The strong renewal theorem with infinite mean via local large deviations.

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Abstract

A necessary and sufficient condition is established for an asymptotically stable renewal process to satisfy the strong renewal theorem. This result is valid for all $\alpha \in (0, 1)$, thus completing a result for $\alpha \in (1/2, 1)$ which was proved in the 1963 paper of Garsia and Lamperti [7]. This paper is superseded by arXiv:1612.07635.

1 Introduction

This paper contains new results about asymptotically stable random walk in two different, but related areas. The first result, which applies to any random walk $S = \{S_n, n \geq 0\}$ which is in the domain of attraction of a stable law of index $\alpha \in (0, 1) \cup (1, 2)$ without centering, is a local large deviation bound which improves the error term in Gnedenko’s local limit theorem without making any further assumptions. This bound may have other uses, but here we use it to give a complete answer to a question which has remained open since the 1963 paper of Garsia and Lamperti [7]: viz which renewal processes in the domain of attraction of a stable law of index $\alpha \in (0, 1)$ verify the strong renewal theorem (SRT). We also give an answer to the same question for certain generalized renewal functions, and indicate how the renewal process proof can be modified to get the corresponding result for asymptotically stable subordinators. In the main part of the paper, dealing with renewal processes, we will restrict attention to the case of an aperiodic distribution on the integer lattice, but it is easy to see that the non-lattice case can be treated by the same techniques. This paper is superseded by arXiv:1612.07635.

2 Results

We write $S_0 = 0, S_n = \sum_1^n X_r$, the $X$'s being i.i.d. with mass function $p$ and distribution function $F$. We also put $P(X > x) = F(x)$, and $X_{n,\ast}^\ast = \max_{1 \leq r \leq n} |X_r|$.

**Theorem 1** Suppose $S_n/a_n$ converges in distribution to a stable $(\alpha, \rho)$ law, where $\alpha \in (0, 1) \cup (1, 2)$ and the positivity parameter $\rho$ is positive. Then, given
any $\gamma > 0$, $\exists C_0, n_0, \theta_0$ such that, for all $n \geq n_0$ and $x \geq n\theta_0$,
\[ P\{S_n = x, X_n^* \leq \gamma x\} \leq \frac{C_0 (nF(x))^{1/\gamma}}{a_n}. \]  
(1)

and
\[ P\{S_n = x\} \leq C_0 \frac{nF(x)}{a_n}. \]  
(2)

Remark 2 A result similar to this for the case $\alpha = 2, EX^2 < \infty$ can be found in Lemma 4 of [8].

Remark 3 Gnedenko’s local limit theorem implies that $a_nP\{S_n = x\} \to 0$ whenever $\theta := x/a_n \to \infty$, but gives no information about the rate. This is given by (2), since we have
\[ nF(x) \sim \frac{cF(ax)}{F(a_n)}. \]
so that if we write $\Lambda = (nF(x))^{-1}$ it follows from Potter’s bounds that for any $\varepsilon > 0$ we have
\[ c\theta^{\alpha - \varepsilon} \leq \Lambda \leq c\theta^{\alpha + \varepsilon} \]  
for all sufficiently large $n$ and $\theta$. (Here, and in what follows, $c$ denotes a generic positive constant whose value can change from line to line.)

Our main application of this is to prove the second part of the following, in which we write $g(x) = \sum_{n}^\infty P(S_n = x)$ for the renewal mass function.

Theorem 4 (i) Assume that $F$ is aperiodic, $S \in D(\alpha, \rho)$ with $\alpha \in (1/2, 1)$ and $\rho > 0$. Then
\[ \lim_{x \to \infty} xF(x)g(x) = g(\alpha, \rho) := \alpha E(Y^{-\alpha} : Y > 0), \]  
(4)
where $Y$ denotes a random variable having the limiting stable law.

(ii) Assume that $F$ is aperiodic, $P(X \geq 0) = 1$, and $S \in D(\alpha, 1)$ with $\alpha \in (0, 1/2]$. Then (4) holds with $\rho = 1$ if and only if
\[ \lim_{x \to \infty} xF(x)p(x) = 0, \]  
(5)
and
\[ \lim_{\delta \to 0} \sup_{x \to \infty} xF(x) \sum_{1}^{\delta x} \frac{p(x-w)}{wF(w)^2} = 0. \]  
(6)

Remark 5 The condition (5) is easily seen to be equivalent to
\[ \lim_{x \to \infty} xF(x) \sum_{1}^{n_0} P(S_n = x) = 0 \text{ for any fixed } n_0, \]  
(7)
and we will use this repeatedly.
Remark 6 The statement \((4)\) is called the SRT and is the obvious analogue of the Renewal Theorem when the mean is infinite. The question as to which asymptotically stable renewal processes satisfy it has been extensively studied since the pioneering paper of Garsia and Lamperti \([7]\), who first established (i) above. They only considered the "lattice renewal case" i.e. they assumed \(P(X \in \mathbb{Z}^+) = 1\), but their results were extended to the case of a general random walk in \([6]\) and \([10]\). In the case \(\alpha \leq 1/2\) it is easy to show that (7) is necessary for the SRT to hold, and the papers \([10]\), \([6]\), \([9]\), \([3]\) and \([4]\) contain a succession of sufficient conditions for the SRT to hold, based on restrictions on the asymptotic behaviour of the ratio \(xp(x)/F(x)\), and its non-lattice counterpart.

Remark 7 When \(\alpha \in (1/2, 1)\) the fact that for fixed \(n\) we have \(P(S_n = x) \leq P(S_n > x - 1) \sim nF(x)\) shows that (6) holds and the fact that \(1/(xF(x)^2)\) is asymptotically increasing shows that

\[
\lim_{x \to \infty} xF(x)\sum_{|z| > r} p(x-w) \leq \lim_{x \to \infty} \frac{cxF(x)}{\delta xF(\delta x)^2} \left( F((1-\delta)x) - F(x) \right) = \delta^{2(1-\alpha)},
\]

so (4) also holds. So (ii) is technically also correct for \(\alpha \in (1/2, 1)\).

Remark 8 In the case \(\alpha = 1/2\), it is easy to check that if we put \(F(x) = (F(x))^{-1/2}\) then both (4) and (6) hold if \(\lim_{x \to \infty} L(x) > 0\), so only in case \(\lim_{x \to \infty} L(x) = 0\) is the NASC required. Thus the case \(F(x) \sim (cx)^{-1/2}\) represents the boundary of the situation where the SRT holds without further conditions.

3 Proof of Theorem 1

Proof. If we write \(P\{S_n = x\} = P_1 + P_2\), where \(P_2 = P\{S_n = x, X_n^* > \gamma x\}\), the simple estimate

\[
P_2 \leq n \sum_{|z| > r} p(z)P(S_{n-1} = x - z) \leq \frac{cnF(\gamma x) + F(-\gamma x)}{a_n} \leq \frac{cnF(x)}{a_n},
\]

which is a consequence of Gnedenko’s local limit theorem, shows that (2) follows from (1), which we now prove.

We introduce an associated distribution \(\tilde{P}\), by setting

\[
\tilde{p}(z) = \tilde{P}(X_1 = z) = e^{\mu z}p(z)1_{|z| \leq \gamma x}/m_0,
\]

(8)

where \(m_0 = \sum_{|z| \leq \gamma x} e^{\mu z}p(z)\) and we set \(\mu := \frac{\log \Lambda}{\gamma x}\). Note that \(\Lambda \to \infty\) as \(n, \theta \to \infty\), and, since \(e^{\gamma \mu x} = \Lambda\), iteration of (8) gives

\[
P_1 = m_0 \Lambda^{-1} \tilde{P}(S_n = x).
\]

(9)
We start by showing that \( m_0^n \leq c \), for all sufficiently large \( n \) and \( \theta \). When \( \alpha \in (0, 1) \), we write

\[
1 - m_0 = \sum_{|z| < 1/\mu} (1 - e^{\mu z}) p(z) + P(|X| \geq 1/\mu) - \sum_{1/\mu \leq |z| \leq \gamma x} e^{\mu z} p(z),
\]

so that

\[
|1 - m_0| \leq c\mu \sum_{|z| < 1/\mu} |z| p(z) + P(|X| \geq 1/\mu) + \sum_{1/\mu \leq |z| \leq \gamma x} |z| e^{\mu |z| - \log |z|} p(z)
\]

\[
\leq c\mathcal{F}(1/\mu) + \frac{e^{\gamma \mu x}}{\gamma x} \sum_{|z| \leq \gamma x} |z| p(z),
\]

where we have used the observation that \( \mu z - \log z \) is monotone increasing on \([1/\mu, \infty)\), and standard properties of regularly varying functions. The second term above is bounded by

\[
\frac{ce^{\gamma \mu x}}{x} e^{\gamma \mu x} \mathcal{F}(x) = c\Lambda \mathcal{F}(x) = \frac{c}{n},
\]

so that, again by Potter’s bounds, for any \( \tilde{\alpha} \in (0, \alpha) \)

\[
|1 - m_0| \leq \frac{c}{n} \left( 1 + n \mathcal{F}(1/\mu) \right)
\]

\[
\leq \frac{c}{n} \left( 1 + \frac{\mathcal{F}(1/\mu)}{\mathcal{F}(a_n)} \right) \leq \frac{c}{n} (1 + (a_n \mu)^{-\tilde{\alpha}}) \leq \frac{c}{n},
\]

the last step relying on the fact, which follows from (9), that

\[
\frac{1}{a_n \mu} = \frac{\gamma \theta}{\log \Lambda} \to \infty.
\]

When \( \alpha \in (1, 2) \) we have \( EX_1 = 0 \), so that

\[
\sum_{|z| < 1/\mu} (1 - e^{\mu z}) p(z) = \sum_{|z| < 1/\mu} (1 - e^{\mu z} + \mu z) p(z) + \mu \sum_{|z| \geq 1/\mu} z p(z)
\]

\[
\leq c \left\{ \sum_{-|z| < 1/\mu} \mu^2 z^2 p(z) + F(-1/\mu) + \mathcal{F}(1/\mu) \right\}
\]

\[
\leq c\mathcal{F}(1/\mu),
\]

and we can also write

\[
\sum_{1/\mu \leq |z| \leq \gamma x} e^{\mu |z|} p(z) = \sum_{1/\mu \leq |z| \leq 2/\mu} e^{\mu |z|} p(z) + \sum_{2/\mu \leq |z| \leq \gamma x} |z|^2 e^{\mu |z| - 2 \log |z|} p(z)
\]

\[
\leq c \left\{ P(|X| > 1/\mu) + \frac{e^{\gamma \mu x}}{(\gamma x)^2} E(X^2 : |X| \leq \gamma x) \right\}
\]

\[
\leq c \left\{ P(|X| > 1/\mu) + \Lambda \mathcal{F}(x) \right\}
\]
so again (11) holds. Thus we have 

\[ P_1 \leq c \Lambda^{-\frac{1}{\gamma}} \tilde{P}(S_n = x), \]

and we are left to prove that \( a_n \tilde{P}(S_n = x) \leq c \). We do this by applying a suitable Normal approximation, for which we need to estimate

\[ m_k := \tilde{E}X_k^k = \frac{1}{m_0} \sum_{|z| \leq \gamma x} z^k e^{\mu z} p(z) = \frac{\tilde{m}_k}{m_0}, \]

for \( k = 1, 2, 3 \). Since \( m_0 \geq \sum_{0 \leq z \leq \gamma x} p(z) \rightarrow \rho > 0 \), and (10) gives \( m_0 \leq c \), it suffices to estimate \( \tilde{m}_k \); first, when \( \alpha < 1 \) we have

\[ |\tilde{m}_k| \leq e^{\mu \gamma x} E\{ |X|^k : 0 \leq |X| \leq \gamma x \} \]

\[ \sim c \Lambda x^k \mathcal{F}(x) = cx^k/n \text{ for } k = 1, 2, 3. \]

When \( \alpha \in (1, 2) \) this is also valid for \( k = 2, 3 \), but for \( k = 1 \) a little more work is required. Specifically we write

\[ \left| \sum_{|z| < 1/\mu} e^{\mu z} p(z) \right| \leq \sum_{|z| < 1/\mu} |z(e^{\mu z} - 1)| p(z) + \sum_{|z| < 1/\mu} z p(z) \]

\[ \leq c \sum_{|z| < 1/\mu} \mu z^2 p(z) + \sum_{|z| \geq 1/\mu} z p(z) \leq c \mathcal{F}(1/\mu)/\mu \leq \frac{x}{n \log \Lambda}, \]

and

\[ \left| \sum_{1/\mu \leq |z| < \gamma x} e^{\mu z} z p(z) \right| \leq \sum_{1/\mu \leq |z| < \gamma x} e^{\mu |z| - \log |z|} z^2 p(z) \]

\[ \leq \frac{e^{\mu \gamma x}}{\gamma x} E\{X^2 : |X| \leq \gamma x \} \leq c \Lambda x \mathcal{F}(x) = cx/n, \]

so (12) also holds in this case. We also need a lower bound for \( \tilde{\sigma}^2 := \tilde{E}|X - \tilde{m}_1|^2 \). Since, for all \( x, n \) large enough and for any \( d \in (0, 1) \), \( (y - \tilde{m}_1)^2 \geq cy^2 \) for \( y \in (x(1 - d) \leq y \leq x) \) we get,

\[ \tilde{\sigma}^2 \geq e^{\mu x(1-d)} E\{(X - \tilde{m}_1)^2 : \gamma x(1-d) \leq X \leq \gamma x \} \]

\[ \geq c \Lambda^{1-d} x \mathcal{F}(x) = cx^2/n \Lambda^d \]

so that

\[ cx^2 \geq n \tilde{\sigma}^2 \geq \frac{cx^2}{\Lambda^d}. \]  

(13)

If \( \tilde{\nu} := \tilde{E}|X - \tilde{m}_1|^3 \), a similar calculation leads to the bounds

\[ cx^3 \geq n \tilde{\nu} \geq \frac{cx^3}{\Lambda^d}. \]  

(14)
With these results in hand, we can apply Lemma 3 of \[5\] to deduce that \( P_1 \) is bounded above by

\[
c \left( \frac{1}{\sqrt{n \sigma^2}} + \frac{n \nu}{(n \sigma^2)^2} + \int_{\sigma^2/4 \nu}^{\pi} e^{-n(1-|\phi(t)|)} dt \right),
\]

where \( \phi(t) = \hat{E}(e^{itX}) \). By \[13\] and \[14\], the first two terms are bounded above by \( c x^{-1} A^2 \), and this in turn is bounded asymptotically, for suitably chosen \( d \), by \( c/a_n \). We also have the bound

\[
1 - |\phi(t)| \geq 1 - R \phi(t) \geq c E(e^{i\nu X} (1 - \cos t X) : |X| \leq \gamma x)
\]

\[
\geq ct^2 E(X^2 : 0 \leq X \leq \gamma x \land t^{-1})
\]

\[
\geq c F(t^{-1}) \text{ for all } t \geq 1/\gamma x.
\]

For all \( n \) such that \( a_n \leq \gamma x \) we can therefore bound the integral term by

\[
\int_{1/a_n}^{1/a_n} dt + \int_{\sigma^2/4 \nu}^{\pi} e^{-cnF(a_n z^{-1})} dz \leq \frac{1}{a_n} (1 + \int_{1}^{a_n} e^{-cnF(a_n z^{-1})} dz).
\]

But by Potter’s bounds we have

\[
nF(a_n z^{-1}) = \frac{F(a z^{-1})}{F(a_n)} \geq c(z a_n)^{\alpha_1}
\]

for any \( \alpha_1 \in (0, \alpha) \), and we deduce that \( a_n \hat{P}(S_n = x) \) is bounded above, and \[15\] follows. ||

4 Proof of Theorem 4

**Proof.** We will introduce a quantity \( A(x) = x^\alpha L_0(x) \) where \( L_0(x) \) is a normalised slowly varying function, (see \[1\], p15) which satisfies \( A(x) \sim 1/F(x) \). Then \( A \) is differentiable and \( x A'(x)/A(x) \rightarrow \alpha \). We can and will take the normalising sequence \( a_n \) to be the value at \( x = n \) of \( a(x) \), where \( A(a(x)) = 1, x \geq 1 \).

Our first step is to establish that \[4\] holds iff for some fixed \( n_0 \)

\[
\lim \sup_{x \rightarrow \infty} \frac{x}{A(x)} \sum_{n \in (n_0, \delta A(x))]} P(S_n = x) = 0. \quad (15)
\]

To see this, fix any \( \delta > 0 \), note that \( n > \delta A(x) \iff x < a(n/\delta) \sim \delta^{-\alpha} a(n) \), so given \( \varepsilon > 0 \) Gnedenko’s local limit theorem allows us to choose \( x_0(\varepsilon) \) large enough to ensure that, for all \( x \geq x_0(\varepsilon) \), on this range we have

\[
a_n |P(S_n = x) - f(x/a_n)| \leq \varepsilon
\]
where \( f \) is the density of \( Y \). Then

\[
\left| \sum_{n > \delta A(x)} P(S_n = x) - \sum_{n > \delta A(x)} \frac{f(x/a_n)}{a_n} \right| \leq \varepsilon \sum_{n > \delta A(x)} \frac{1}{a_n} \leq \varepsilon c \delta A(x) \delta^* x,
\]

where \( a(\delta A(x)) = \delta^* x \), so that \( \delta^* x \sim \delta^n \) as \( x \to \infty \). Putting \( x/a_n = y \) so that \( n = A(x/y) \), we see that the second sum on the left is a Riemann approximation to

\[
\int_0^{x/a(\delta A(x))} \frac{f(y)A'(x/y)ydy}{x^2} \approx \frac{\alpha A(x)}{x} \int_0^{1/\delta^*} \frac{f(y)A(x/y)}{A(x)} y^{\delta - \eta} f(y)dy,
\]

where in the last step we used Potter’s bounds and dominated convergence. Since we can choose \( \delta \) as small as we like and \( \varepsilon \) is arbitrary, we see that

\[
\lim_{\delta \to 0} \lim_{x \to \infty} \frac{x}{A(x)} \sum_{n > \delta A(x)} P(S_n = x) = \alpha E(Y^{-\alpha} : Y > 0) = g(\alpha, \rho), \quad (16)
\]

and we are left to ascertain when (15) and (7) are valid.

When \( \alpha \in (1/2, 1) \); we have already noted that (7) holds. Also in this case we have \( n^2/a_n \in RV(2 - \eta) \), i.e. is regularly varying with index \( 2 - \eta > 0 \), so from (2)

\[
\frac{x}{A(x)} \sum_{n_0} P(S_n = x) \leq \frac{C_0 x}{A(x)} \sum_{n_0} \frac{nF(x)}{a_n} \leq \frac{cx}{A(x)^2} \frac{\delta^2 A(x)^2}{a(\delta A(x))} = \frac{c\delta^2}{\delta_x}.
\]

Since \( \delta^* x \sim \delta^n \) as \( x \to \infty \) with \( 1 < \eta < 2 \), (15) follows, and we have proved (i).

So from now on we take \( \alpha \in (0, 1/2] \) and consider only the renewal case, i.e. we assume \( P(X \geq 0) = 1 \). Since (7) is obviously necessary, from now on we will assume that it holds, and show that (15) holds iff (6) holds.

Clearly (6) is equivalent to

\[
\lim_{\delta \to 0} \lim_{x \to \infty} \sup \frac{x}{A(x)} \frac{\delta x}{A(x)} = 0, \quad (17)
\]

where

\[
\tilde{I}(\delta, x) = \sum_{1}^{\delta x} p(x - w) \frac{A(w)^2}{w}.
\]

We also note that if \( B \) denotes any non-negative asymptotically increasing function then an immediate consequence of (17) is that

\[
\lim_{\delta \to 0} \lim_{x \to \infty} \frac{x}{B(\delta x)A(x)} \sum_{1}^{\delta x} p(x - w) \frac{B(w)A(w)^2}{w} = 0. \quad (19)
\]
In connection with this we will often use the fact that for any fixed \( \delta_0 > 0 \),

\[
\limsup_{x \to \infty} \frac{x}{B(x)A(x)} \sum_{\delta_0 x < w < (1 - \delta_0)x} p(x - w) B(w)A(w)^2 \frac{\delta A(x)}{\delta_0 x} F(\delta_0 x) < \infty,
\]

and combining this with (19) we see that, whenever (17) holds,

\[
\limsup_{x \to \infty} \frac{x}{B(x)A(x)} \sum_{0 < w < (1 - \delta_0)x} p(x - w) B(w)A(w)^2 \frac{\delta A(x)}{\delta_0 x} F(\delta_0 x) < \infty,
\]

and

\[
\limsup_{x \to \infty} \frac{x}{B(x)A(x)} \sum_{\delta_0 x < w < (1 - \delta_0)x} p(x - w) B(w)A(w)^2 \frac{\delta A(x)}{\delta_0 x} F(\delta_0 x) < \infty.
\]

(20)

We start by showing the necessity of (6). If \( f(n, x) \) is any non-negative function we will write "\( f(n, x) \) is asymptotically negligble" to mean that for some fixed \( n_0 \)

\[
\lim_{\delta \downarrow 0} \limsup_{x \to \infty} \frac{x}{A(x)} \sum_{n_0} f(n, x) = 0.
\]

We can and will assume henceforth that \( n_0 \) and \( \delta \) are chosen so that when \( x \) is large enough the bounds (1). and (2) in Theorem 1 are operative. First, we consider possible values of \( Z_n^{(1)} := \max_{1 \leq r \leq n} X_r \), and for any fixed \( 0 < \lambda < C \)

we deduce the bound

\[
P^* := P(S_n = x, x - Ca_n \leq Z_n^{(1)} \leq x - \lambda a_n) = \sum_{\lambda a_n} P(S_n = x, Z_n^{(1)} = x - y)
\]

\[
= n \sum_{\lambda a_n} P(S_{n-1} = y, Z_{n-1}^{(1)} < x - y, X_n = x - y)
\]

\[
= n \sum_{\lambda a_n} p(x - y)P(S_{n-1} = y, Z_{n-1}^{(1)} < x - y).
\]

By Gnedenko’s Local Limit Theorem we see that \( a_n P(S_{n-1} = y) \) is bounded above and below by positive constants for \( y \in [\lambda a_n, Ca_n] \) when \( n \) and \( \theta = y/a_n \) are sufficiently large. We deduce the bound

\[
P(S_{n-1} = y, Z_{n-1}^{(1)} \geq x - y) \leq n \sum_{x-y} P(S_{n-2} = y - z) \leq \frac{cn F(x - y)}{a_n}.
\]

Since \( n F(x - y) \) can be made small for all \( y \leq Ca_n \) by making \( \theta \) large, we see that

\[
P^* \geq \frac{cn}{a_n} \sum_{\lambda a_n} p(x - z).
\]
Thus when $\alpha < 1/2$ so that $n/a_n \in RV(1-\eta)$ with $1-\eta < -1$, $\sum_{n \in (n_0, \delta A(x))] P^*$ is bounded below by a multiple of
\[
\sum_{n \in (n_0, \delta A(x))] \frac{C a_n}{\lambda a_n} \sum_{\lambda a_n} p(x - z) = \frac{C \delta^*_x}{C a(n_0)} \sum_{\delta A(x) \wedge A(z/\lambda)} p(x - z) \sum_{A(z/C)} \frac{n}{a_n} \geq \frac{\lambda \delta^*_x}{C a(n_0)} \sum_{A(z/C)} p(x - z) \sum_{A(z/C)} \frac{n}{a_n} \geq c \sum_{C a(n_0)} \frac{\lambda \delta^*_x}{C a(n_0)} p(x - z) \left( \frac{A(z/C)^2}{z/C} - \frac{A(z/\lambda)^2}{z/\lambda} \right) = c \tilde{I}(\lambda \delta^*_x, x) - c \sum_{1} p(x - z) \frac{A(z)^2}{z}.
\]

Since it follows from (7) that the final term is asymptotically negligible, we see that (17) is necessary for the SRT to hold when $\alpha < 1/2$. In the case $\alpha = 1/2$ we write $A(x) = \sqrt{x L(x)}$ and note that
\[
\sum_{A(z/C)} \frac{A(z/C)}{A(z/C)} \frac{n}{a_n} \sim c \int_{z/C}^{z/\lambda} \frac{A(y) A'(y)}{y} dy \sim c \int_{z/C}^{z/\lambda} \frac{A(y)^2}{y^2} dy = c \int_{z/C}^{z/\lambda} \frac{L(y)}{y} dy \sim c \int_{1/C}^{1/\lambda} \frac{L(z w)}{w} dw \sim c L(z) = c \frac{A(z)^2}{z}.
\]

Thus (17) is a necessary condition in all cases. To show that it is also sufficient, we write
\[
P(S_n = x) = \sum_{1} P_r^{(1)}, \text{ where } P_r^{(1)} = P(S_n = x, Z_r^{(1)} \leq \gamma x),
\]
\[
P_2^{(1)} = P(S_n = x, Z_2^{(1)} \in (\gamma x, x - \lambda a_n)), \text{ and } P_3^{(1)} = P(S_n = x, Z_3^{(1)} \in [x - \lambda a_n, x]).
\]

Note first that there is an upper bound for $P^*$ corresponding to the lower bound we established earlier, so the sufficiency for $P_3^{(1)}$ to be asymptotically negligible will follow if we can, given arbitrary $\varepsilon > 0$, find a $\lambda > 0$ such that
\[
\sum_{n \in (n_0, \delta A(x))] \sum_{0} \lambda \delta^*_x n p(x - z) P(S_{n-1} = z) \leq c \tilde{I}(C \delta^*_x, x) + o(A(x)/x).
\]

To see this, we use the following facts, which are contained in Lemma 4 and the
argument leading to (3.16) in [5]:  \( \exists n_0, \lambda > 0 \) such that

for \( z \geq n \geq n_0 \) and \( z/a_n \leq \lambda \) we have \( zP(S_{n-1} = z) \leq ce^{-c(n/A(z))} \); 

for \( z \leq n \) we have \( P(S_{n-1} = z) \leq ce^{-cn} \).

Splitting the LHS of (21) in the obvious way we see that it is bounded above by a multiple of

\[
\sum_{n \in (n_0, \delta A(x)])} n e^{-cn} \sup_{1 \leq z \leq n} p(x - z) + \sum_{n \in (n_0, \delta A(x)])} \sum_{z = n_0}^{n} e^{-c(n/A(z))} p(x - z)
\]

The first term here is \( o(A(x)/x) \) by condition (5). Writing \( n = A(z)\gamma \) in the second term we see that

\[
\sum_{n = A(z/\gamma)}^{z} n \sum_{n \in (n_0, \delta A(x)])} e^{-c(n/A(z))} p(x - z) 
\leq c \int_{\lambda - \delta}^{z/A(z)} e^{-cy} dy \leq c \frac{A(z)^2}{z},
\]

and then (21) follows.

Next, we choose \( \gamma \in (0, \alpha/(1 - \alpha)) \) so that by (1) we have \( P_1^{(1)} \leq \frac{c(nF(x))^{1/\gamma}}{a_n} \) and since \( \frac{a_n^{1+1/\gamma}}{a_n} \in RV(1 + 1/\gamma - \eta) \) with \( 1 + 1/\gamma - \eta > 0 \) it follows that

\[
\sum_{n \in (n_0, \delta A(x)])} P_1^{(1)} \leq c F(x)^{1/\gamma} \sum_{n \in (n_0, \delta A(x)])} \frac{n^{1/\gamma}}{a_n}
\leq c F(x)^{1/\gamma} \frac{(\delta A(x))^{1+1/\gamma}}{\delta_x^x} \frac{\delta^{1+1/\gamma} A(x)}{\delta_x^x}.
\]

Recalling that \( \delta_x^x \sim \delta^\eta \) as \( x \to \infty \), we see that \( P_1^{(1)} \) is also asymptotically negligible. This leaves us only to deal with \( P_2^{(1)} \) and this is more complicated.

First assume that \( \alpha \in (1/3, 1/2] \), i.e., \( \eta \in \[2, 3) \), so that \( \sum_{m} \frac{a_m^2}{a_m} \sim c \frac{a_m^3}{a_m} \) and we can assume, WLOG, that \( z^{-1}A(z)^3 \) is increasing. By the same argument used to get the upper bound for \( P_3^{(1)} \), but now using the bound (2), we get that
for any \( \delta_0 \in (C\delta^*_z, 1-\gamma) \)
\[
\sum_{n \in (n_0, \delta A(x))} P_2^{(1)}(n) \leq c \sum_{n \in (n_0, \delta A(x))} \frac{n^2}{a_n} \sum_{\alpha \in C_{\alpha n}} p(x - z) F(z)\]
\[
= c \sum_{\alpha \in C_{\alpha n}} p(x - z) F(z) \frac{\delta A(x) \wedge A(z/C)}{\delta^*_x \wedge z/C} \sum_{n_0} \frac{n^2}{a_n}\]
\[
\leq c \sum_{\alpha \in C_{\alpha n}} p(x - z) F(z) \frac{\frac{\delta A(x)}{A(x)^{\gamma}} \wedge z/C}{\delta^*_x \wedge z/C} \sum_{\alpha \in C_{\alpha n}} p(x - z) F(z). \quad \text{(22)}
\]

Now, given arbitrary \( \varepsilon > 0 \), we fix \( \delta_0 \) so that \( \limsup_{x \to \infty} \frac{\delta A(x)}{A(x)} \leq \varepsilon \). Then the second term in (22) is bounded above by
\[
\frac{\delta^3 A(x)^3}{\delta^*_x x} \cdot F(\gamma x) F(\delta_0 x) \cdot \frac{\delta^{\eta} A(x)}{(\gamma \delta_0)^{\eta} x} \text{ as } x \to \infty,
\]
and it follows that \( P_2^{(1)} \) is asymptotically negligible. This proves the theorem for \( \alpha \in (1/3, 1/2) \), so now we consider other values of \( \alpha \).

We let \( Z_n^{(i)}, i = 1, 2, \ldots, n \) denote the steps \( X_r, r = 1, 2, \ldots, n \) arranged in decreasing order, and put \( Y_k = \sum_{i=1}^k Z_n^{(i)} \) for \( k \geq 1 \), and \( Y_0 = 0 \), where we suppress the dependence on \( n \). Fix \( C_1 > C_2 > \cdots > C_k > 0 \). Then, if \( P_2^{(k)} = P(S_n = x, B_k) \) with
\[
B_k = \wedge \left( Z_n^{(r)} \in (\gamma(x - Y_{r-1}), x - Y_{r-1} - C_r a_n) \right),
\]
we claim first that \( P_2^{(k)} \) is asymptotically negligible for all \( 1/(k + 2) < \alpha \leq 1/(k + 1) \), where \( k \geq 2 \). We have
\[
P_2^{(k)} = \sum P(Z_n^{(1)} = z_1, \ldots, Z_n^{(k)} = z_k, S_n = x) \leq \alpha n^k \sum p(z_1) \cdots p(z_k) P(S_n = x - (z_1 + \cdots + z_k)),
\]
where the summation runs over
\[
z_1 \geq z_2 \geq \cdots \geq z_k \geq 0 \text{ such that, with } z_0 = 0,
\]
\[
z_r \in (\gamma(x - (z_0 + \cdots + z_{r-1}), x - (z_0 + \cdots + z_{r-1}) - C_r a_n)) \text{ for } r = 1, \ldots, k.
\]
Making the change of variable \( x - z_1 = y_1, x - (z_1 + z_2) = y_1 - z_2 = y_2, \ldots \),
Furthermore, since \( k \) (we can use (20) to get \[ k_{-} = y_{k_{-}-1} = y_{k_{-}} \), we deduce the bound

\[
P_{2}^{(k)} \leq cn^{k} \sum_{y_{1}=C_{1}a_{n}}^{(1-\gamma)x} \sum_{y_{k}=C_{k}a_{n}}^{(1-\gamma)y_{k_{-}-1}} p(x-y_{1}) \cdots p(y_{k_{-}-1} - y_{k}) P(S_{n-k} = y_{k})
\]

\[
\leq \frac{cn^{k+1}}{a_{n}} \sum_{k} p(x-y_{1}) \cdots p(y_{k_{-}-1} - y_{k}) F(y_{k}),
\]

where we use (23) and write \( \sum_{k} \) as an abbreviation for the previous sum. (Note we have omitted the requirement \( z_{1} \geq z_{2} \geq \ldots \geq z_{k} \) here.) Next, assume \( a \neq 1/(k + 1) \), so that \( y A(y)^{-r} \) is asymptotically increasing for \( r \leq k + 1 \) and we can use (20) to get

\[
\sum_{y_{k}=C_{k}a_{n}}^{(1-\gamma)y_{k_{-}-1}} p(y_{k_{-}-1} - y_{k}) F(y_{k}) \leq c \sum_{y_{k}=C_{k}a_{n}}^{(1-\gamma)y_{k_{-}-1}} p(y_{k_{-}-1} - y_{k}) A(y_{k})^{2} y_{k}^{-1} \cdot y_{k} A(y_{k})^{-3}
\]

\[
\leq cy_{k_{-}1} A(y_{k_{-}1})^{-3} A(y_{k_{-}1}/y_{k_{-}1}) = cA(y_{k_{-}1})^{-2},
\]

We can then repeat the argument until we get

\[
P_{2}^{(k)} \leq \frac{cn^{k+1}}{a_{n}} \sum_{y_{1}=C_{1}a_{n}}^{(1-\gamma)x} p(x-y_{1}) A(y_{1})^{-k}. \tag{23}
\]

In the case \( \alpha = 1/(k + 1) \) the last step in this procedure requires more care, since \( A(y)^{1-k} = y^{-1} A(y)^{2} y^{-3} A(y)^{-k-1} \) and the last factor here is slowly varying, and not necessarily increasing. But we do have

\[
P_{2}^{(k)} \leq \frac{cn^{k+1}}{a_{n}} \sum_{y_{1}=C_{1}a_{n}}^{(1-\gamma)x} \sum_{y_{2}=C_{2}a_{n}}^{(1-\gamma)y_{1}} p(x-y_{1}) p(y_{1}-y_{2}) A(y_{1})^{1-k}
\]

\[
\leq \frac{cn^{k+1}}{a_{n} \sqrt{A(C_{2}a_{n})}} \sum_{y_{1}=C_{1}a_{n}}^{(1-\gamma)x} \sum_{y_{2}=C_{2}a_{n}}^{(1-\gamma)y_{1}} p(x-y_{1}) p(y_{1}-y_{2}) A(y_{1})^{3/2-k}
\]

\[
\leq \frac{cn^{k+1/2}}{a_{n}} \sum_{y_{1}=C_{1}a_{n}}^{(1-\gamma)x} p(x-y_{1}) A(y_{1})^{1/2-k}. \tag{24}
\]

where we have used the fact that \( y A(y)^{-(k+1)/2} \) is asymptotically increasing. Furthermore, since \( (k + 2) \alpha > 1 \), we deduce that when (23) holds, for any
\[ \delta_0 \in (C\delta_2^*, 1 - \gamma), \]

\[ \sum_{n \in (n_0, \delta A(x))} P_2^{(k)} \leq c \sum_{n \in (n_0, \delta A(x))} \sum_{C_1 a_n} \frac{(1-\gamma)x}{\eta^{k+1} a_n} p(x - y) A(y)^{-k} \]

\[ = c \sum_{y = C_1 a_n} p(x - y) A(y)^{-k} \sum_{n_0} \frac{(1-\gamma)x}{\eta^{k+1} a_n} \]

\[ \leq c \sum_{C_1 a_n} p(x - y) A(y)^{-k} \frac{A(C_1 \delta_2 x \land y)^{k+2}}{C_1 \delta_2 x \land y} \]

\[ \leq c \overline{I}(\delta_0 x) + \frac{c\delta^{k+2} A(x)^{k+2}}{\delta_2 x} \sum_{\delta_0} p(x - y) A(y)^{-k}. \]

Since now \( yA(x)^{-k-2} \) is asymptotically decreasing we see that the second term is bounded asymptotically by \( c\delta^{k+2-n} \), and so is asymptotically negligible. If instead we have (24) a slight variation of this argument gives the same conclusion.

Next, we consider \( P(S_n = x, B_k^\ell) \), which we can bound above by \( \sum_2 P_1^{(j)} + \sum_2 P_3^{(j)} \), where

\[ P_1^{(j)} = P(S_n = x, B_{j-1}, A_1^{(j)}), \text{ with } A_1^{(j)} = (Y_j \leq \gamma(x - Z_n^{(j-1)})) \]

and \( P_3^{(j)} = P(S_n = x, B_{j-1}, A_3^{(j)}), \text{ with } A_3^{(j)} = (Z_n^{(j)} \in [x - C_j a_n, x]) \). Proceeding as above, and using (1), we get the bound

\[ P_1^{(j)} \leq cn^{j-1} \sum_{j-1} p(x - y_1) \cdots p(y_{j-1} - y_{j-2}) P(S_{n-j} = y_{j-1}, Z_{n-j}^{(1)} \leq \gamma y_{j-1}) \]

\[ \leq \frac{cn^{j-1+1/\gamma}}{a_n} \sum_{j-1} p(x - y_1) p(y_1 - y_2) \cdots p(y_{j-2} - y_{j-1}) A(y_{j-1})^{-1/\gamma}. \]

Now

\[ \sum_{y_{j-1} = C_j a_n} p(y_{j-2} - y_{j-1}) A(y_{j-1})^{-1/\gamma} \]

\[ \leq c \sum_{y_{j-1} = C_j a_n} p(y_{j-2} - y_{j-1}) A(y_{j-1})^2 y_{j-1}^{-1} y_{j-1} A(y_{j-1})^{-2-1/\gamma} \]

\[ \leq c A(y_{j-2})^{-1-1/\gamma}, \]

where we have used (20), which we can do since \( 1 + 1/\gamma > \eta \). Then we can repeat the argument, finally getting

\[ P_3^{(1)} \leq cn^{j-1+1/\gamma} \sum_{y_1 = C_1 a_n} (1-\gamma)x p(x - y_1) A(y_1)^{-j+2-1/\gamma} \]

\[ \leq \frac{cn^{j-1+1/\gamma}}{a_n A(x)^{j-1+1/\gamma}}. \]
which gives

\[
\sum_{n \in (n_0, \delta A(x))] P_j^{(1)} \leq \frac{c}{A(x)^{j-1+1/\gamma}} \sum_{n \in (n_0, \delta A(x)]} \frac{n^{j-1+1/\gamma}}{a_n} \\
\leq \frac{c(\delta A(x))^{j+1/\gamma}}{A(x)^{j-1+1/\gamma}} \leq \frac{c(\delta A(x))^{j+1/\gamma}}{A(x)^{j-1+1/\gamma}} \leq \frac{c(\delta A(x))^{j+1/\gamma} A(x)}{A(x)^{j-1+1/\gamma}}.
\]

Thus the term \( P_j^{(1)} \) is asymptotically negligible for \( 2 \leq j \leq k \) and \( \alpha \leq 1/(k+1) \).

Next,

\[
\sum_{n \in (n_0, \delta A(x))] P_j^{(3)} \leq c \sum_{j-1} p(x - y_1) \cdots p(y_{j-2} - y_{j-1}) \Theta(y_{j-1}),
\]

with \( \Theta(y_{j-1}) = \frac{\delta A(x) \wedge A(y_{j-1}/C_j)}{a_n} \sum_{0 \leq z \leq C_j a_n} p(y_{j-1} - z) \leq \frac{\gamma_j y_{j-1} \wedge C_j \delta_x^z}{A(z/C_j)} \sum_{\delta A(x) \wedge A(y_{j-1}/C_j)} \frac{n^j}{a_n} \)

where \( \gamma_j = C_j/C_{j-1} < 1 \). When \( \alpha < 1/(k+1) \) we have \( j+1 \leq k+1 < \eta \), so we can use the bound \( \sum_{A(z/C_j)} \frac{n^j}{a_n} \leq cA(z)^{j+1}/z \) to get

\[
\Theta(y_{j-1}) \leq c \sum_{0} p(y_{j-1} - z) A(z)^{j+1}/z \\
\leq cA(y_{j-1} \wedge \delta_x^z) \sum_{z \leq \gamma_j y_{j-1}} p(y_{j-1} - z) A(z)^j/z \\
\leq cA(y_{j-1} \wedge \delta_x^z) \cdot \frac{A(y_{j-1})^{j-1}}{y_{j-1}}.
\]

Repeating the process, we are finally left with the \( y_1 \) term, which is

\[
\sum_{y_1 = C_0}^{(1-\gamma)x} p(x - y_1) A(y_1) \wedge \delta_x^x \cdot \frac{A(y_1)}{y_1} \\
\leq \sum_{y_1 = C_0}^{(1-\gamma)x} p(x - y_1) A(y_1)^2 / y_1 + A(\delta_x^x) \sum_{y_1 = \delta_0 x}^{(1-\gamma)x} p(x - y_1) A(y_1) / y_1 \\
\leq \tilde{I}(\delta_0 x) + A(\delta_x^x) \sum_{y_1 = \delta_0 x}^{(1-\gamma)x} p(x - y_1) A(y_1)^2 / y_1 \\
\leq \tilde{I}(\delta_0 x) + c \delta \sum_{y_1 = \delta_0 x}^{(1-\gamma)x} p(x - y_1) A(y_1)^2 / y_1
\]

14
Using (20), we see that \(P(j)\) is asymptotically negligible for \(2 \leq j \leq k\). If \(\alpha = 1/(k+1)\) and \(j < k\) the same argument works, so we are left with the case \(\alpha = 1/(k+1)\) and \(j = k\), which is similar to the case \(\alpha = 1/2\). So again we split \(P(S_{n-k} = y_{k-1}, Z_{n-k}^{(1)} \geq y_{k-1} - C_k a_n)\) into two terms, and estimate \(P(S_{n-k} = y_{k-1}, Z_{n-k}^{(1)} \geq y_{k-1} - \lambda a_n)\) as before. We need to deal with the terms

\[
n^k \sum_{k-1} p(x - y_i) \cdot p(y_k - y_{k-1}) \sum_{n \leq z \leq \lambda a_n} z^{-1} e^{-c n/A(z)} p(y_{k-1} - z), \quad (25)
\]

and \(n^k e^{-c n} \sum_{k-1} p(x - y_i) \cdot p(y_k - y_{k-1}) \sum_{1 \leq z \leq n} p(y_{k-1} - z). \quad (26)\)

We have

\[
\sum_{n_0} \delta A(x) \leq c \sum_{y_1 = 0}^{(1-\gamma)x} p(x - y_1) \cdot p(y_k - y_{k-1}) \Omega(y_{k-1}),
\]

where

\[
\Omega(y_{k-1}) = \sum_{n_0} \delta A(x) \sum_{y_1 = 0}^{(1-\gamma)x} \sum_{y_{k-1} = 0} \frac{n^k}{a_n} \sum_{n \leq z \leq \lambda a_n} z^{-1} e^{-c n/A(z)} p(y_{k-1} - z)
\]

\[
\leq c \sum_{n_0} p(y_k - y_{k-1}) \sum_{A(z/\lambda)} n^k e^{-c n/A(z)}
\]

\[
\leq c \sum_{n_0} p(y_k - y_{k-1}) \frac{A(z)^{k+1}}{z}
\]

By repeatedly using the calculation

\[
\sum_{0} p(y_{k-1} - z) \frac{A(z)^{k+1}}{z} \leq A(\gamma y_k - y_{k-1} \wedge C_k \delta^*_x) \sum_{0} p(y_{k-1} - z) \frac{A(z)^k}{z}
\]

\[
\leq c A(\gamma y_{k-1} \wedge \delta^*_x) \frac{A(y_{k-1})^{k-1}}{y_{k-1}}
\]

we deduce that

\[
\sum_{n_0} \delta A(x) \leq c \sum_{y_1 = 0}^{(1-\gamma)x} p(x - y_1) \frac{A(y_1) A(y_1 \wedge \delta^*_x)}{y_1}
\]

This deals with (25), and to bound (26) we use the fact that \(n = o(a_n)\) to see that (7) implies that for sufficiently large \(n\) we have

\[
\sum_{1 \leq z < n} p(y_{k-1} - z) \leq \frac{cn A(y_{k-1})}{y_{k-1}}
\]

\[
= \frac{c A(y_{k-1})^2}{y_{k-1}} \cdot \frac{n}{A(y_{k-1})}
\]

\[
\leq \frac{c A(y_{k-1})^2}{y_{k-1}} \text{ for } y_{k-1} \geq C_k a_n,
\]

15
so that

\[
\sum_{y_{k-1}=C_{k-1}a_n}^{(1-\gamma)y_{k-2}} p(y_{k-2} - y_{k-1}) \sum_{1 \leq z < n} p(y_{k-1} - z) \\
\leq c \sum_{y_{k-1}=C_{k-1}a_n}^{(1-\gamma)y_{k-2}} p(y_{k-2} - y_{k-1}) \frac{A(y_{k-1})^2}{y_{k-1}} \\
\leq \frac{cA(y_{k-2})}{y_{k-2}} \leq \frac{cA(y_{k-2})^2}{ny_{k-2}} \text{ for } y_{k-2} \geq C_{k-2}a_n.
\]

Repeating this we deduce that, for any \( \varepsilon > 0 \),

\[
\sum_{n_0} \delta A(x) \leq c \sum_{n_0} ne^{-cn} \sum_{y_1=C_1a_n}^{(1-\gamma)x} p(x - y_1) \frac{A(y_1)^2}{y_1} \\
\leq \frac{cA(x)}{x} \sum_{n_0} ne^{-cn} \leq \frac{\varepsilon A(x)}{x},
\]

provided \( n_0 \) is chosen sufficiently large. This concludes the proof. \( \blacksquare \)

5 \ Extensions of the renewal process results

1. As previously remarked, in the non-lattice case the obvious analogue of Theorem 4 holds. By this we mean that the NASC for (4) to hold with \( g(x) \) replaced by \( G(x, \Delta) := \sum_{S_n \in (x, x+\Delta]} P(S_n) \) and \( g(\alpha, \rho) \) replaced by \( \Delta g(\alpha, \rho) \) for any fixed \( \Delta > 0 \) is that both

\[
\lim_{x \to \infty} xF(x) P(S_1 \in (x, x + \Delta]) = 0,
\]

and

\[
\lim_{\delta \to 0} \sup_{x \to \infty} xF(x) \int_{1}^{\delta x} \frac{F(x - dw)}{wF(w)^2} = 0.
\]

2. Similarly, whenever \( F \) has a density and a density version of the local limit theorem holds for \( S_n \) a density version of Theorem 3 can be proved in the same manner.

3. In [5], Theorem 3 contains an extension of the SRT to generalized Green’s functions of the form

\[
g_b(x) = \sum_{0}^{\infty} b_n P(S_n = x),
\]

where \( b \) is a non-negative function which is regularly varying at \( \infty \) of index \( \beta \). That result was obtained under the restriction

\[
\sup_{x \geq 1} \omega(x) < \infty, \text{ where } \omega(x) := xp(x)/F(x).
\]

(27)
Here we show that (27) is redundant, by giving a NASC for the same result.

**Theorem 9**  
i) Assume that $F$ is aperiodic, $P(X \geq 0) = 1$, $S \in D(\alpha, 1)$ with $\alpha \in (0, 1)$ and $\beta > -2$. Put $b(A(\cdot)) = B(\cdot)$. Then when $\alpha(2 + \beta) > 1$

$$\lim_{x \to \infty} \frac{x \overline{F}(x) g_b(x)}{B(x)} = g(\alpha, \beta) := \alpha E(Y^{-\alpha(\beta+1)}),$$  

(28)

where $Y$ denotes a random variable having the limiting stable law. When $\alpha(2 + \beta) \leq 1$ (28) holds if and only if, for every fixed $n_0$

$$\lim_{x \to \infty} \frac{x \overline{F}(x) p(x)}{B(x)} = 0,$$

(29)

and

$$\lim \sup_{\delta \to 0} \lim_{x \to \infty} \frac{x \overline{F}(x)}{B(x)} \sum_{n \in (n_0, \delta A(x)]} \frac{p(x-w) B(w)}{w F(w)^2} = 0.$$  

(30)

**Proof.** A careful reading of the proof of Theorem 3 in [5] shows that the only way (27) is used is in establishing

$$\lim \sup_{\delta \to 0} \lim_{x \to \infty} \frac{x}{A(x) B(x)} \sum_{n \in (n_0, \delta A(x)]} b_n P(S_n = x) = 0,$$

which is the analogue of (15). The proof of this, without (27), essentially amounts to repeating the proof of (15) with the difference that whenever we dealt with sums involving $n^j/a_n$ we now need to deal with $n^j b(n)/a_n$; so the difficulties associated with integer values of $\eta$ become associated with integer values of $\eta - \beta$. The details are omitted. —

**Remark 10**  
It should also be mentioned that the restriction on the value of $\beta$ is necessary for $g(\alpha, \beta)$ to be finite.

**Remark 11**  
It should be noted that when (27) holds, both (29) and (30) are automatic.

### 6 Random walks

An obvious question is whether the result for renewal processes in part (ii) of Theorem 4 extends to the random walk case. We make the following

**Conjecture 12**  
The conditions (5) and (6) of Theorem 4 are necessary and sufficient for the SRT (4) to hold for any aperiodic random walk in $D(\alpha, \rho)$ with $\alpha \in (0, 1)$ and $\rho > 0$. 

17
Remark 13  In principle, a variation of our method should establish this result. In fact the proof of the necessity of the conditions is straightforward. Likewise the proof of the sufficiency when \( \alpha > \frac{1}{3} \) is not difficult. Specifically we write \( P(S_n = x) = \sum_1^4 P_1^{(1)} \), where \( P_1^{(1)} \) and \( P_2^{(1)} \) are as before, \( P_3^{(1)} = P(S_n = x, Z_{n_1} \in [x - Ca_n, x + Ca_n]) \), and \( P_4^{(1)} = P(S_n = x, Z_{n_1} > x + Ca_n) \). If we note that (6) is actually equivalent to
\[
\lim_{\delta \to 0} \lim_{x \to \infty} x F(x) \sum_{-\delta x}^{\delta x} \frac{p(x-w)}{wF(w)^2} = 0,
\]
the estimate of \( P_3^{(1)} \) requires only minor changes. Finally a similar argument, but using the result (2) for \(-S\), gives
\[
\sum_{n \in (n_0, \delta A(x))] \ P_4^{(1)} \leq c \sum_{Ca_n} \frac{p(x+z)F(-z)A(z)^3}{z} \ + \frac{\delta^3 A(x)^3}{\delta} \sum_{\delta x} p(x+z)F(-z).
\]
It then follows from a slight variation of the argument following (22) that
\[
\lim_{\delta \to 0} \lim_{x \to \infty} \frac{x}{A(x)} \sum_{n \in (n_0, \delta A(x))] \ P_4^{(1)} = 0,
\]
which completes the proof. However a proof when \( \alpha \in ((k + 2)^{-1}, (k + 1)^{-1}) \) for general \( k \) seems to require consideration of the \( k \) steps which are largest in modulus, and this seems quite complicated.

7 Subordinators

Our proof of Theorem 4 rests on the classical local limit theorems, Theorem 11 and consideration of a finite number of the largest jumps. It is not difficult to see that each of these items can be replicated for an asymptotically stable subordinator, and then essentially the same argument leads to the following result, whose proof is omitted.

Theorem 14 Let \( X \) be any subordinator that is in the domain of attraction of a stable law of index \( \alpha \in (0,1) \) as \( t \to \infty \), and define it’s renewal measure by
\[
G(dx) = \int_0^\infty P(X_t \in dx)dt
\]
Suppose also that its Lévy measure is non-lattice, and write
\[
G_\Delta(x) = G((x, x + \Delta])
\]
Then for any fixed $\Delta > 0$, (i) if $\alpha > 1/2$,
\[
\lim_{x \to \infty} x\Pi(x)G_{\Delta}(x) = \Delta g(\alpha, \rho) = \alpha \Delta E(Y^{-\alpha}),
\]
where $Y$ denotes a random variable having the limiting stable law. (ii) if $\alpha \in (0, 1/2]$ then (31) holds if and only if
\[
\lim_{x \to \infty} x\Pi(x)\Pi((x, x + \Delta]) = 0
\]
and
\[
\lim_{\delta \to 0} \limsup_{x \to \infty} x\Pi(x) \int_1^{\delta x} \frac{\Pi(x - dw)}{w\Pi(w)^2} = 0.
\]

Acknowledgement 15 Almost simultaneously a different proof of the main result of this paper has appeared in Caravenna, [2].

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