OPTIMAL ORDERING AND PRICING MODELS OF A TWO-ECHelon SUPPLY CHAIN UNDER MULTIPLE TIMES ORDERING

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Abstract. This paper studies ordering and pricing issues under multiple times ordering. A manufacturer and a retailer are involved in our discussion. The definition of a reasonable price is given based on the practical requirement. First, we construct a Stackelberg model in which the manufacturer and the retailer make their decisions respectively. During the process of derivation, both ordering time-points and optimal prices are expressed as functions of number of times of ordering. By solving a quadratic programming model with an undetermined parameter, we demonstrate that the optimal ordering time-points of the retailer are equidistant time points on the given selling period. Second, a cooperative model is developed in which the manufacturer and the retailer jointly make decisions. It is shown that the optimal retail price is lower and the number of times of ordering is more in the cooperative situation than the noncooperative one. Further, an allocation method based on revenue proportions is proposed.

1. Introduction. For a retailer who sells a certain kind of goods, it is imperative to trade off the ordering cost, the stock-holding cost, and the lost sales cost while making his ordering and pricing decisions (Chen et al. 2006). Owing to its practicality and generalizability, a great deal of research has been done with respect to the ordering and pricing model and its various extensions (Yang et al. 2011). Research findings of ordering and pricing policies have been achieved in various aspects. See Chang et al. (2006) for an impressive work of optimal pricing and ordering policies of deteriorating items. You et al. (2007) and You et al. (2010) conducted study for different inventory systems. Yin et al. (2007) and Chen et al. (2012) discussed joint pricing and ordering issues by integrated models. More recently, Zhang et al.
applied pricing and ordering models in global supply chain management to obtain the maximal after-tax profit. For more applications of pricing and ordering models, see, e.g., Serel (2009), Xing et al. (2012), Avinadav et al. (2014).

There is a growing body of literature focuses on inventory optimization when formulating ordering and pricing strategies. Chung et al. (2015) studied a multiperiod discount problem for products that undergo several price cuts in a finite selling period, and proposed optimal pricing and inventory policies. Taleizadeh et al. (2017) considered inventory control strategies in the presence of varying unit purchase price. Li et al. (2017) analyzed pricing and ordering policies for perishable products, and constructed an inventory model by using a discounted cash-flow analysis. Kaya et al. (2017) discussed the problem of jointly determining the optimal pricing and inventory replenishment strategy for a deterministic perishable inventory system. Tiwari et al. (2018) established an inventory model for deteriorating items under a two-level partial trade credit with allowable shortages, and meanwhile obtained the optimal selling price. More recently, Chen et al. (2019) studied a pricing and inventory replenishment problem in the presence of the uncertain demand distribution.

Motivated by the above research, we study the ordering and pricing issues by trading off inventory costs and sales volume. A fixed ordering cost is incurred each time the retailer procures products from the manufacturer. Following the handling in Demirag et al. (2012), the fixed ordering cost of each time mainly refers to the transportation cost. The handling of the consumer's demand is a key point while establishing ordering and pricing models. In general, the demand of the product is often assumed to be linear sensitive with regard to price (see, e.g., Shi et al. 2012; Sajadieh et al. 2009). A good deal of literature applies demand functions to obtain demand amounts. In contrast, we formulate a linear demand rate function with regard to the retail price, just like the handling in Mondal et al. (2003) and Dye (2007). In practice, for nonseasonal products, the linear price-sensitive handling of the demand rate is usual.

The object of this paper is a two-layer supply chain consisting of a manufacturer and a retailer. Both noncooperation and cooperation models are established. Similar to Li et al (2010), one goal of this paper is to compare all the results in the two situations. For the noncooperation case, a Stackelberg game model is constructed to obtain the optimal wholesale price of the manufacturer and the optimal ordering schedule of the retailer. Relations between the optimal sales price, the optimal wholesale price and ordering time-points are thoroughly analyzed. For the centralized decision-making case, similar to Amiri et al. (2020), the inventory management mode is declared before determining the optimal sales level of the supply chain. Although the wholesale price no longer exists under the revenue-sharing contract, the number of times of ordering is still the key issue for the two-echelon supply chain. In addition, an allocation approach based on revenue proportions is proposed for the cooperation case.

The practical background of this research is that the retailer should inform the manufacturer his all ordering time-points at the beginning of the selling period so that the manufacturer has enough time to prepare goods accordingly. In practice, manufacturers who produce multi-process products always need to acquire the ordering information of retailers in advance so as to prepare corresponding amounts of components and parts. For example, cellphone manufacturers often require ordering schedules over a period of time from regional distributors so as to arrange
their components procurement and production schedules. Hence, models based on this background are of practical significance.

In fact, under multiple times ordering, both the manufacture and the retailer should take the number of times of ordering into account when making pricing decisions. Nevertheless, little literature focuses on this aspect of study. This paper yields some significant practical results for this problem. Firstly, the definition of a reasonable sales price is presented according to practical requirements and optimal sales prices of the two models are proved to be reasonable. Secondly, relations between the optimal sales price, the optimal wholesale price and ordering time-points are derived. Thirdly, by constructing a quadratic programming model, optimal ordering time-points are proved to be the equidistant time points on the selling interval. Finally, it is shown that the optimal retail price is lower and the number of times of ordering is more in the cooperation situation than the noncooperative one. Our research reveals the following conclusion: for a two-echelon supply chain with a deterministic demand rate that depends on the sales price, both the wholesale price and the retail price are determined by the number of times of ordering. Further, we show that all decision variables are determined once the number of times of ordering is claimed.

The remainder of this paper is organized as follows. Section 2 describes necessary notations and gives some assumptions. In section 3, we propose a Stackelberg model to deal with the decentralized decision-making issue of the two-layer supply chain. The cooperative case is analyzed in section 4 to discuss the changes on both of prices and the number of times of ordering incurred by cooperation. A numerical illustration is provided in section 5. Section 6 summarizes the paper and reveals the managerial insights of the obtained conclusions.

2. Assumptions and notations. The main assumption of this paper is that the retailer informs the manufacturer his ordering time-points before the manufacturer determines the wholesale price. In practice, this assumption makes sense because the manufacturer needs to produce products before the retailer places an order. Hence, we first regard the ordering time-points as known information both for the manufacturer and the retailer when constructing the Stackelberg model. After the manufacturer declares the wholesale price, the retailer determines his wholesale volume and retail price so as to maximize his revenue. Both the manufacturer and the retailer acquire complete information. In addition, the revenues are transferable in the cooperation situation.

As mentioned previously, shortage is not allowed and the lead time is assumed to be zero. Besides, the demand of the product is assumed to be linear-sensitive with regard to the retail price. Once be determined, the retail price remains constant within the selling period.

The notations used in the following are given by the following table:

Generally, the production cost per item and the stock-holding cost per item cannot be too high, otherwise the retail price should be higher, which will make the product to be unsalable. Hence, we assume that \( \lambda e + \lambda hT \) holds. The fixed ordering cost \( k \) mainly refers to the transportation cost of the retailer.

Besides, the conception of a reasonable retail price is proposed as follows.
Table 1 Model parameters

| Parameters | Definition |
|------------|------------|
| [0, T]    | The given selling period |
| t<sub>i</sub> | The ordering time-point, where t<sub>i</sub> ∈ [0, T] and i ∈ M, M = \{1, ..., m\} |
| p<sub>b</sub> | The wholesale price determined by the manufacturer |
| p<sub>c</sub> | The retail price determined by the retailer |
| q           | The total procurement volume of the retailer |
| e           | The production cost per item of the manufacturer |
| m           | The number of times of ordering of the retailer |
| k           | The fixed ordering cost of each order |
| λ           | The linear price-sensitive coefficient of the demand rate |
| r(p<sub>c</sub>) | The demand rate under price p<sub>c</sub>: r(p<sub>c</sub>) = a - λp<sub>c</sub>, a > 0 |
| h           | The stock-holding cost per item per unit time |
| W           | The revenue of the manufacturer |
| Z           | The revenue of the retailer |
| S           | The total revenue incurred by centralized decision-making |

Definition 1: If a - λp<sub>c</sub> > 0, p<sub>c</sub> is deemed to be a reasonable retail price.

Because neither shortage nor surplus happens under the deterministic situation, thus

\[ q = \int_0^T r(p_c) dt = aT - \lambda T p_c. \]

Let t<sub>1</sub>, ..., t<sub>m</sub> be m time-points of ordering, where t<sub>1</sub> = 0 and t<sub>1</sub> ≤ ... ≤ t<sub>m</sub> ≤ T. Besides, we additionally denote t<sub>m+1</sub> = T and M = \{1, ..., m\}. Then the total stock-holding cost is

\[ h \sum_{i=1}^{m} \int_{t_{i+1}}^{t_{i+1}} \left[ \int_{t_i}^{t_{i+1}} r(p_c) dt - \int_{t_i}^{t} r(p_c) dx \right] dt, \]

in which we use \(\int_{t_i}^{t_{i+1}} r(p_c) dt\) to formulate the ordering volume at time t<sub>i</sub>.

3. The Stackelberg game. This section conducts a Stackelberg model in which complete information is available both for the manufacturer and the retailer. The retailer informs the manufacturer his all ordering time-points at the beginning of the selling period. Hence, all ordering time-points are treated as known information while the manufacturer determines his wholesale price. As the leader, the manufacturer determines the wholesale price first, and then the retailer makes his decision.

By the given parameters, the objective function of the manufacturer is given as follows:

\[ \max W = (p_b - e)q = (p_b - e)(aT - \lambda T p_c). \]  

(1)

Meanwhile, the objective function of the retailer under ordering m times is

\[ \max Z(m) = (p_c - p_b) \int_0^T r(p_c) dt - mk - h \sum_{i=1}^{m} \int_{t_i}^{t_{i+1}} \left[ \int_{t_i}^{t} r(p_c) dx \right] dt - \int_{t_i}^{t} r(p_c) dx dt. \]  

(2)
Although the number of times of ordering influences the revenue of the manufacturer, \( m \) is not a decision variable of him. That’s why \( m \) does not appear in (1).

By substituting \( r(p_c) = a - \lambda p_c \), we obtain

\[
\int_{t_i}^{t_{i+1}} \left[ \int_{t_i}^t r(p_c)dt - \int_{t_i}^t r(p_c)dx \right] dt = \frac{(a - \lambda p_c)(t_{i+1} - t_i)^2}{2}.
\]

Thus, (2) is transformed to

\[
\text{max } Z(m) = -\lambda T p_c^2 + [\lambda T p_b + \frac{\lambda h}{2} \sum_{i=1}^m (t_{i+1} - t_i)^2] p_c - a T p_b - \frac{a h}{2} \sum_{i=1}^m (t_{i+1} - t_i)^2 - mk.
\]

According to the properties of quadratic functions we know that (3) has a unique maximum value for any vector \((t_1, \ldots, t_m)\) that meets the previous setting. Thus, \( p_c^* \) corresponding to max \( Z(m) \) is obtained by the symmetry axes formula of the quadratic function:

\[
p_c^* = \frac{p_b}{2} + \frac{a}{2\lambda} + \frac{h}{4T} \sum_{i=1}^m (t_{i+1} - t_i)^2.
\]

By applying (4), (1) is transformed to

\[
\text{max } W = -\frac{\lambda T}{2} p_b^2 + \left[ \frac{a T}{2} + \frac{\lambda e T}{4} \sum_{i=1}^m (t_{i+1} - t_i)^2 \right] p_b - \frac{a e T}{2} p_c + \frac{\lambda e h}{4} \sum_{i=1}^m (t_{i+1} - t_i)^2.
\]

Let \( p_b^* \) be the solution of (5). Then we have

\[
p_b^* = \frac{a}{2\lambda} + \frac{e}{2} - \frac{h}{4T} \sum_{i=1}^m (t_{i+1} - t_i)^2.
\]

And the expression of \( p_c^* \) is then obtained by (4):

\[
p_c^* = \frac{3a}{4\lambda} + \frac{e}{4} + \frac{h}{8T} \sum_{i=1}^m (t_{i+1} - t_i)^2.
\]

Next, we examine \( p_c^* \) given by (7). For any \( i \in M \) and \( 0 \leq t_i \leq t_{i+1} \leq T \), we have

\[
(t_{i+1} - t_i)^2 + (t_i - t_{i-1})^2
= (t_{i+1} - t_{i-1})^2 + 2t_{i+1}t_{i-1} - 2t_{i+1}t_i + 2t_i^2 - 2t_{i+1}t_{i-1}
= (t_{i+1} - t_{i-1})^2 - 2(t_{i+1} - t_i)(t_i - t_{i-1})
\leq (t_{i+1} - t_{i-1})^2
\]

By analogy, items between \( t_1 \) and \( t_{m+1} \) are cancelled out sequentially as follows:

\[
\sum_{i=1}^m (t_{i+1} - t_i)^2 \leq (t_{m+1} - t_{m-1})^2 + \sum_{i=1}^{m-2} (t_{i+1} - t_i)^2 \leq \cdots \leq (t_{m+1} - t_1)^2 = T^2.
\]
By (8) and $\lambda e + \lambda hT < a$, we have

$$a - \lambda p_e^* = a - \frac{\lambda e}{4} - \frac{\lambda h}{8T} \sum_{i=1}^{m} (t_{i+1} - t_i)^2 \geq a - \frac{\lambda e}{4} - \frac{\lambda hT}{8} > \frac{\lambda hT}{8} > 0.$$  

Hence, $p_e^*$ is a reasonable retail price according to definition 1.

By (6) and (7), (3) is transformed to

$$\max Z(m) = \frac{\lambda h^2}{64T} \sum_{i=1}^{m} (t_{i+1} - t_i)^2 - \frac{a h}{16} - \frac{\lambda e h}{16} \sum_{i=1}^{m} (t_{i+1} - t_i)^2$$

$$+ \frac{a^2 T}{16\lambda} - \frac{ae T}{8} + \frac{\lambda e^2 T}{16} - mk. \tag{9}$$

We denote $f(x) = Z(m)$, where

$$x = \sum_{i=1}^{m} (t_{i+1} - t_i)^2.$$  

Then $Z(m)$ converts into

$$f(x) = \frac{\lambda h^2 x^2}{64T} - \frac{a h}{16} - \frac{\lambda e h}{16} x + \frac{a^2 T}{16\lambda} - \frac{ae T}{8} + \frac{\lambda e^2 T}{16} - mk. \tag{10}$$

First, the minimum value of $f(x)$ is analyzed without considering the value range of $x$. The solution $x^*$ of $\min f(x)$ is obtained by the symmetry axes formula:

$$x^* = \frac{2aT - 2ae T}{\lambda h}.$$

By $\lambda e + \lambda hT < a$ we know that $x^* > 2T^2$. According to the properties of quadratic functions, the value of $x$ must be as far away from $x^*$ as possible so as to maximize $f(x)$. By taking advantage of (8) we further know that the value of $x$ must be as small as possible for maximizing $f(x)$.

For any $m \geq 2$, we consider the following model:

$$\min \sum_{i=1}^{m} (t_{i+1} - t_i)^2$$

s.t. $t_1 = 0,$

$t_{m+1} = T,$

$t_i \leq t_{i+1}, \forall i \in N. \tag{11}$

Firstly, we analyze the objective function of model (11) without considering its constraints. Differentiating the objective function and letting all the first-order derivatives be zero, we have the following equation set:

$$\begin{cases}
\frac{\partial (\sum_{i=1}^{m} (t_{i+1} - t_i)^2)}{\partial t_2} = 2t_2 - 2(t_3 - t_2) = 0 \\
\ldots \\
\frac{\partial (\sum_{i=1}^{m} (t_{i+1} - t_i)^2)}{\partial t_m} = 2(t_m - t_{m-1}) - 2(T - t_m) = 0.
\end{cases} \tag{12}$$

The unique solution of (12) is obtained as follows:

$$t_i = \frac{(i - 1)T}{m}, \forall i \in M/\{1\}. \tag{13}$$

Clearly, the solution determined by (13) meets the constraints of model (11).
Moreover, for any \( t_i \) \((i = 2, \ldots, m)\), the second derivative of the objective function with respect to each variable is

\[
\frac{\partial^2}{\partial t_i^2} \left( \sum_{i=1}^{m} (t_{i+1} - t_i)^2 \right) = 4 > 0, \forall i \in M/\{1\},
\]

which implies that (13) is the solution of model (11).

Given the above, we draw the following conclusion:

**Proposition 1** Optimal ordering time-points are equidistant time points on the selling interval.

The above result shows the regularity of the ordering time-points when maximizing the total profit. Next, we turn to find out the optimal number of times of ordering, by which the optimal sales price and the optimal wholesale price are determined.

By (13), the expressions of \( p^*_b \) and \( p^*_c \) are obtained:

\[
p^*_b = \frac{a}{2\lambda} + \frac{e}{2} - \frac{hT}{4m}, \quad p^*_c = \frac{3a}{4\lambda} + \frac{e}{4} + \frac{hT}{8m}.
\]

(14)

Accordingly, the expression of \( W \) with respect to \( m \) is obtained:

\[
W = \frac{\lambda h^2 T^3}{32m^2} - \frac{ahT^2}{8m} + \frac{\lambda e h T^2}{8m} + \frac{a^2 T}{8\lambda} - \frac{aeT}{4} + \frac{\lambda e^2 T}{8}.
\]

(15)

And the expression of \( Z(m) \) is

\[
Z(m) = \frac{\lambda h^2 T^3}{64m^2} - \frac{ahT^2}{16m} + \frac{\lambda e h T^2}{16m} + \frac{a^2 T}{16\lambda} - \frac{aeT}{8} + \frac{\lambda e^2 T}{16} - mk.
\]

(16)

By differentiating \( Z(m) \), the derivative is as follows:

\[
\frac{\partial Z(m)}{\partial m} = -\frac{\lambda h^2 T^3}{32m^3} + \frac{ahT^2}{16m^2} - \frac{\lambda e h T^2}{16m^2} - k.
\]

Because \( \lambda e + \lambda h T < a \), for any \( m \in Z^+ \) we have

\[
-\frac{\lambda h^2 T^3}{32m^3} + \frac{ahT^2}{16m^2} > \frac{\lambda e h T^2}{16m^2} \quad \Rightarrow \quad \frac{hT^2(2ma - 2m\lambda hT - 2m\lambda e)}{32m^3} > 0
\]

and

\[
\frac{\partial^2 Z(m)}{\partial m^2} = \frac{3\lambda h^2 T^3}{32m^4} - \frac{ahT^2}{8m^3} + \frac{\lambda e h T^2}{8m^3} < \frac{(4m\lambda e + 4m\lambda hT - 4ma)hT^2}{32m^4} < 0,
\]

which suggest that the derivative of \( Z(m) \) is strictly decreasing with the increase of \( m \). In addition, the derivative of \( Z(m) \) tends to \(-k\) when \( m \) tends to \(+\infty\):

\[
\lim_{m \to +\infty} \frac{\partial Z(m)}{\partial m} = \lim_{m \to +\infty} \left( -\frac{\lambda h^2 T^3}{32m^3} + \frac{ahT^2}{16m^2} - \frac{\lambda e h T^2}{16m^2} - k \right) = -k.
\]

Given the above, we draw the following conclusions:

1) If \( Z'(1) \leq 0 \), \( Z'(m) \leq 0 \) holds for any \( m \in Z^+ \), which always means that the fixed ordering cost is relatively high. Then the optimal number of times of ordering \( m^* \) is 1.

2) If \( Z'(1) > 0 \), according to the intermediate value theorem of the continuous function we know that

\[
-\frac{\lambda h^2 T^3}{32m^3} + \frac{ahT^2}{16m^2} - \frac{\lambda e h T^2}{16m^2} - k = 0
\]

has a unique solution. If the solution is an integer, it is exactly \( m^* \) because the second-order derivative of \( Z(m) \) is negative. Otherwise, examining the two integers...
which are closest to the solution, we choose the one which makes \(Z(m)\) bigger to be \(m^*\).

Once \(m^*\) is determined, both \(p_b^*\) and \(p_c^*\) are obtained by (4). Clearly, they are both unique.

4. The cooperative model. This section considers the centralized decision-making between the manufacturer and the retailer. All the costs are shared by the two participants. The production volume, the ordering schedule, and the retail price are determined simultaneously for maximizing the aggregate revenue. All the results of the two models are compared to achieve practical conclusions. Finally, the additional revenue increased by their cooperation is allocated under the assumption that revenues are transferable.

Adding up objective functions (1) and (3), a new objective function is obtained for maximizing the aggregate revenue under \(m\) times ordering:

\[
\max S(m) = -\lambda T p_b^2 + [\lambda e T + a T + \frac{\lambda h}{2} \sum_{i=1}^{m} (t_{i+1} - t_i)^2] p_c - \frac{a h}{2} \sum_{i=1}^{m} (t_{i+1} - t_i)^2 - a e T - m k.
\]

The unique solution of (17) is

\[
p_c^* = \frac{a}{2 \lambda} + \frac{e}{2} + \frac{h}{4 T} \sum_{i=1}^{m} (t_{i+1} - t_i)^2.
\]

Comparing values of the two \(p_c^*\) given by (7) and (18), we have

\[
\frac{a}{2 \lambda} + \frac{e}{2} + \frac{h}{4 T} \sum_{i=1}^{m} (t_{i+1} - t_i)^2 - \left[ \frac{3 a}{4 \lambda} + \frac{e}{4} + \frac{h}{8 T} \sum_{i=1}^{m} (t_{i+1} - t_i)^2 \right]
= - \frac{a}{4 \lambda} + \frac{e}{4} + \frac{h}{8 T} \sum_{i=1}^{m} (t_{i+1} - t_i)^2 \leq - \frac{a}{4 \lambda} + \frac{e}{4} + \frac{h T}{8}
< - \frac{h T}{4} + \frac{h T}{8} < 0
\]

which means that the optimal retail price in the cooperation situation is lower than the one in the noncooperation situation in order to gain more sales volume.

Next, the reasonability of \(p_c^*\) determined by (18) is verified by the following inequality:

\[
a - \lambda p_c^* = \frac{a}{2} - \frac{\lambda e}{2} - \lambda h T \sum_{i=1}^{m} (t_{i+1} - t_i)^2 \geq \frac{a}{2} - \frac{\lambda e}{2} - \frac{\lambda h T}{4} > \frac{a}{2} - \frac{\lambda e}{2} - \frac{\lambda h T}{2} > 0.
\]

Substituting (18) into (17), we have

\[
S(m) = \frac{\lambda h^2}{16 T} \sum_{i=1}^{m} (t_{i+1} - t_i)^2 - \left( \frac{ah e}{4} - \frac{\lambda e h T}{4} \right) \sum_{i=1}^{m} (t_{i+1} - t_i)^2 + \frac{a^2 T}{4 \lambda} - \frac{ae T}{2} + \frac{\lambda e^2 T}{4} - mk.
\]
Similar handling to the previous section, we denote \( f(x) = Z(m) \), where
\[
x = \sum_{i=1}^{m} (t_{i+1} - t_i)^2.
\]

Then function \( S(m) \) is transformed to
\[
g(x) = \frac{\lambda h^2}{16 T}x^2 - \left( \frac{a h}{4} + \frac{\lambda e h}{4} \right)x + \frac{a^2 T^2}{4 \lambda} - \frac{a e T}{2} + \frac{\lambda e^2 T}{4} - mk.
\]

Without considering the value range of \( x \), the solution \( x^* \) of \( \min g(x) \) is obtained:
\[
x^* = \frac{2aT - 2\lambda eT}{\lambda h} > 2T^2.
\]

Hence, the value of \( x \) must be as small as possible for maximizing \( g(x) \).

Substituting all the time-points given by (13) into (19), the aggregate revenue is obtained as follows
\[
S(m) = \frac{\lambda h^2 T^3}{16 m^3} - \frac{ahT^2}{4 m^2} + \frac{\lambda e h T^2}{4 m} + \frac{a^2 T^2}{4 \lambda} - \frac{a e T}{2} + \frac{\lambda e^2 T}{4} - mk.
\]

Comparing (15), (16) and (20), we have
\[
S(m) - W - Z(m) = \frac{\lambda h^2 T^3}{64 m^3} - \frac{ahT^2}{16 m^2} + \frac{\lambda e h T^2}{16 m} + \frac{a^2 T^2}{16 \lambda} - \frac{a e T}{8} + \frac{\lambda e^2 T}{16} = (a - \frac{e}{4} - \frac{hT}{8m})(\frac{aT}{4} - \frac{eT}{4} - \frac{\lambda h T^2}{8m})
\]

By using \( \lambda e + \lambda hT < a \), the following results are obtained:
\[
\frac{a}{4\lambda} - \frac{e}{4} - \frac{hT}{8m} > \frac{a}{4} - \frac{\lambda e}{4} - \frac{\lambda hT}{4} > 0
\]

and
\[
\frac{aT}{4} - \frac{\lambda eT}{4} - \frac{\lambda h T^2}{8m} > \frac{a}{4} - \frac{\lambda e}{4} - \frac{\lambda hT}{4} > 0.
\]

Then
\[
S(m) - W - Z(m) > 0,
\]

i.e., the aggregate revenue of cooperation is higher than the sum revenue of the manufacturer and the retailer of the noncooperation situation as long as the numbers of times of ordering are the same.

Besides, the optimal retail price \( p^*_c \) is obtained by substituting (13) into (18):
\[
p^*_c = \frac{a}{2\lambda} + \frac{e}{2} + \frac{hT}{4m}.
\]

We further seek the optimal number of times of ordering in the cooperation situation. The derivative of \( S(m) \) is obtained as follows:
\[
\frac{\partial S(m)}{\partial m} = -\frac{\lambda h^2 T^3}{8 m^3} + \frac{ahT^2}{4 m^2} - \frac{\lambda e h T^2}{4 m} - k.
\]

For any \( m \in Z^+ \) we have
\[
-\frac{\lambda h^2 T^3}{8 m^3} + \frac{ahT^2}{4 m^2} - \frac{\lambda e h T^2}{4 m} > \frac{hT^2(2ma - 2m\lambda hT - 2m\lambda e)}{8 m^3} > 0
\]

and
\[
\frac{\partial^2 S(m)}{\partial m^2} = \frac{3\lambda h^2 T^3}{8 m^4} - \frac{ahT^2}{2 m^3} + \frac{\lambda e h T^2}{2 m^3} - \frac{(4m\lambda e + 4m\lambda hT - 4ma)hT^2}{8 m^4} < 0,
\]
which implies that the derivative of \( S(m) \) is strictly decreasing with the increase of \( m \). In addition, the derivative of \( S(m) \) tends to \(-k\) when \( m \) tends to \(+\infty\).

Hence, the following conclusions are drawn:

1) If \( S'(1) \leq 0 \), \( S'(m) \leq 0 \) for any \( m \in \mathbb{Z}^+ \), which always suggests that the fixed ordering cost is relatively high. The optimum number of times of ordering \( m^* \) is 1.

2) If \( S'(1) > 0 \), according to the intermediate value theorem of the continuous function we know that

\[-\lambda h^2 T^3 + \frac{ahT^2}{4m^2} - \frac{\lambda ehT^2}{4m^2} - k = 0\]

has a unique solution. If the solution is an integer, it is exactly \( m^* \) because the second-order derivative of \( S(m) \) is negative. Otherwise, examining the two integers which are closest to the solution, we choose the one which makes \( S(m) \) bigger to be \( m^* \).

Comparing the derivatives of \( Z(m) \) and \( S(m) \):

\[\frac{\partial S(m)}{\partial m} - \frac{\partial Z(m)}{\partial m} = -\frac{3\lambda h^2 T^3}{32m^3} + \frac{3ahT^2}{16m^2} - \frac{3\lambda ehT^2}{16m^2} > -\frac{3\lambda h^2 T^3}{32m^3} + \frac{3ahT^2}{16m^2} - \frac{3\lambda ehT^2}{16m^2} > 0,\]

we obtain the following conclusion:

**Proposition 2** The optimal number of times of ordering in the cooperative model is often bigger than the one in the noncooperative case.

We briefly analyze the reason. The demand quantity in the cooperative situation always bigger, which is incurred by a lower retail price. If the number of times of ordering is equal to the one of the noncooperative case, the stock-holding cost will be much higher. Hence, the increase on the number of times of ordering is regarded as a trade-off between the stock-holding cost and the ordering cost.

By comparing (15) and (16), we have

\[\max Z = \frac{1}{2} \max W - mk,\]

which means that the revenue of the manufacturer is far more than the revenue of the retailer. Hence, an allocation approach based on revenue proportions seems to be more reasonable than the equal distribution method.

Let \( W^* \) and \( Z^* \) be the ultimate revenues of the manufacturer and the retailer, respectively. \( W^* \) is given as follows:

\[W^* = \max W + \frac{\max W (\max S - \max W - \max Z)}{\max W + \max Z} = \frac{\max W \max S}{\max W + \max Z}. \quad (22)\]

And \( Z^* \) is

\[Z^* = \max Z + \frac{\max Z (\max S - \max W - \max Z)}{\max W + \max Z} = \frac{\max Z \max S}{\max W + \max Z}. \quad (23)\]

**5. A numerical illustration.** In order to visualize the proposed models, we present a numerical example. Consider the following scenario: \( T = 80, \lambda = 1, a = 180, e = 20, h = 1 \) and \( k = 4000 \).

Firstly, the Stackelberg game is analyzed, in which the manufacturer and the retailer make decisions separately. According to (16), we have

\[Z(m) = \frac{8000}{m^2} - \frac{64000}{m} - 4000m + 128000.\]
Letting the first-order derivative of $Z(m)$ be zero, we have:

$$\frac{\partial Z(m)}{\partial m} = -\frac{16000}{m^3} + \frac{64000}{m^2} - 4000 = 0.$$ 

Clearly, the solution of the above equation locates in the interior of interval (3, 4). Substituting $m = 3$ and $m = 4$ into $Z(m)$, we have $Z(3) < Z(4)$, which means $m^* = 4$. The corresponding time-points of ordering is 0, 20, 40 and 60. According to (14), we obtain $p_c^* = 95$ and $p_c^* = 142.5$. Besides, max $W = 225000$ and max $Z = 96500$ are also obtained by (15) and (16).

Then we turn our attention to the cooperative case. By (20), the aggregate revenue function is obtained under $m$ times ordering:

$$S(m) = \frac{32000}{m^2} - \frac{256000}{m} - 4000m + 512000.$$ 

Letting the first-order derivative of $S(m)$ be zero:

$$\frac{\partial S(m)}{\partial m} = -\frac{64000}{m^3} + \frac{256000}{m^2} - 4000 = 0.$$ 

It is easy to figure out that the solution of the above equation locates in the interior of interval (7, 8). Substituting $m = 7$ and $m = 8$ into $S(m)$ respectively, we have $S(7) < S(8)$, which suggests $m^* = 8$. Then the corresponding time-points of ordering is 0, 10, 20, 30, 40, 50, 60 and 70. Further, $p_c^* = 102.5$ is obtained according to (21). In addition, max $S = 448500$. Given the above, the final revenues of the manufacturer and the retailer are obtained by formulae (22) and (23): $W^* = 313880$, $Z^* = 134620$.

Apparently, both the manufacturer and the retailer significantly enhance their revenue by cooperation.

6. Conclusions. This article presents models for optimal ordering and pricing decisions under multiple times ordering in a two-stage supply chain. During the discussion, all the optimal prices are expressed by undetermined optimal ordering time-points. The optimal ordering time-points are obtained by a mathematical model, in which the objective function is quadratic and the constraints are linear. The optimal number of times of ordering for the retailer is obtained by an equation, which is derived from the differential of the retailer’s objective function. Further, a centralized decision-making model is constructed to deal with the cooperative situation in which the manufacturer and the retailer share the inventory cost. Prices, total revenues, and the number of times of ordering are compared between the two cases. An allocation method of the additional revenue increased by cooperation is proposed according to the proportions of their revenue.

The above results reveal some managerial insights. Firstly, for a market with a linear price-sensitive demand rate, the optimal ordering decision of the retailer is placing orders at periodic intervals. Secondly, in the presence of complete information, the manufacturer (supplier) acquires ordering time-points of the retailer once the retailer informs him about the number of times of ordering. Thirdly, the number of times of ordering increases when the manufacturer and the retailer share the inventory cost and make centralized decisions. —in practice, this is always called as more batches in small quantities.

Nevertheless, some shortcomings exist in this paper. Stochastic factors are not considered when formulating the demand rate function. In reality, the demand rate may be uncertain and vary along with time. Besides, the impact of dual
channels may need to be taken into consideration in pricing decisions with the rapid development of the e-commerce platform.

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