TOPOLOGICAL VEECH DICHOTOMY AND TESSELLATIONS OF THE HYPERBOLIC PLANE

BY

Duc-Manh Nguyen

Institut de Mathématiques de Bordeaux, CNRS UMR 5251, Université de Bordeaux 351 Cours de la Libération, 33405 Talence, France

e-mail: duc-manh.nguyen@math.u-bordeaux.fr

To the memory of William A. Veech

ABSTRACT

For every half-translation surface with marked points $(M, \Sigma)$, we construct an associated tessellation $\Pi(M, \Sigma)$ of the Poincaré upper half plane whose tiles have finitely many sides and area at most $\pi$. The tessellation $\Pi(M, \Sigma)$ is equivariant with respect to the action of $\text{PSL}(2, \mathbb{R})$, and invariant with respect to (half-)translation covering. In the case $(M, \Sigma)$ is the torus $\mathbb{C}/\mathbb{Z}^2$ with a one marked point, $\Pi(\mathbb{C}/\mathbb{Z}^2, \{0\})$ coincides with the iso-Delaunay tessellation introduced by Veech [25] (see also [1, 2]) as both tessellations give the Farey tessellation. As application, we obtain a bound on the volume of the corresponding Teichmüller curve in the case $(M, \Sigma)$ is a Veech surface (lattice surface). Under the assumption that $(M, \Sigma)$ satisfies the topological Veech dichotomy, there is a natural graph $\mathcal{G}$ underlying $\Pi(M, \Sigma)$ on which the Veech group $\Gamma$ acts by automorphisms. We show that $\mathcal{G}$ has infinite diameter and is Gromov hyperbolic. Furthermore, the quotient $\overline{\mathcal{G}} := \mathcal{G}/\Gamma$ is a finite graph if and only if $(M, \Sigma)$ is actually a Veech surface, in which case we provide an algorithm to determine the graph $\overline{\mathcal{G}}$ explicitly. This algorithm also allows one to get a generating family and a “coarse” fundamental domain of the Veech group $\Gamma$.

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1. Introduction

1.1. Embedded triangles and tessellation of the Poincaré upper half plane. Half-translation surfaces are flat surfaces defined by meromorphic quadratic differentials with at most simple poles on compact Riemann surfaces. If the quadratic differential is the square of an Abelian differential (holomorphic one-form) then we have a translation surface. Otherwise, there is a canonical (ramified) double covering of the Riemann surface such that the pullback of this quadratic differential is the square of a holomorphic 1-form; we will call this the orienting double cover. For a thorough introduction to the subject we refer to [14, 28, 10].

Let $M$ be a half-translation surface defined by a quadratic differential $(X, q)$. Let $\Sigma$ be a finite subset of $M$ that contains all the conical singularities of the flat metric. We will call the pair $(M, \Sigma)$ a half-translation surface with marked points. By a slight abuse of notation, we will call $\Sigma$ the set of singularities of $M$.

**Definition 1.1:** An embedded triangle of $M$ with vertices in $\Sigma$, or an embedded triangle in $(M, \Sigma)$ for short, is the image of a map $\varphi : T \to M$, where $T$ is a triangle in the plane $\mathbb{R}^2$, such that

(i) $\varphi$ maps the vertices $T^{(0)}$ of $T$ to $\Sigma$,
(ii) the restriction of $\varphi$ to $T \setminus T^{(0)}$ is an embedding with image in $M \setminus \Sigma$, and
(iii) $\varphi^* q = dz^2$.

We denote the set of embedded triangles in $M$ with vertices in $\Sigma$ by $\mathbb{T}(M, \Sigma)$.

**Remark 1.2:** Our definition is slightly different from the definition in [24] in that we do not allow a point in the interior of a side of $T$ to get mapped to a point in $\Sigma$.

In what follows, we will sometimes use the same notation for a triangle in $\mathbb{R}^2$ and its image by a map $\varphi$ as above.

Consider now the canonical orienting cover $\pi : \hat{M} \to M$, where $\hat{M}$ is a translation surface defined by a holomorphic 1-form $\hat{\omega}$. By convention, if $M$ is itself a translation surface then we take $\hat{M} = M$, and $\pi = \text{id}$. Let $\hat{\Sigma} = \pi^{-1}(\Sigma)$. If $\pi$ is a double cover, then the pre-image of a saddle connection $a$ in $(M, \Sigma)$ consists of two geodesic segments in $\hat{M}$ with endpoints in $\hat{\Sigma}$. For any directed arc on $\hat{M}$ with endpoints in $\hat{\Sigma}$, the integral of $\hat{\omega}$ along this arc is called its period.
We will call the period of either segment in the pre-image of a its **period**. This is a complex number determined up to sign. If $\pm(a_x + ia_y)$, $a_x, a_y \in \mathbb{R}$, is the period of $a$, we define the slope of $a$ to be

$$k_a := \frac{a_x}{a_y} \in \mathbb{R} \cup \{\infty\}.$$ 

Let $\mathbb{H}$ denote the Poincaré upper half plane. Given an embedded triangle $T$ in $T(M, \Sigma)$, let $k_1, k_2, k_3 \in \mathbb{R} \cup \{\infty\}$ be the slopes of the sides of $T$. We denote by $\Delta_T$ the hyperbolic ideal triangle in $\mathbb{H}$ whose vertices are $\{k_1, k_2, k_3\}$. Denote by $I(M, \Sigma)$ the set of all the ideal triangles arising from elements of $T(M, \Sigma)$. Let $C(M, \Sigma)$ denote the set of points $\mathbb{R} \cup \{\infty\}$ that are vertices of ideals triangles in $I(M, \Sigma)$, and $L(M, \Sigma)$ the sets of hyperbolic geodesics that are sides of elements of $I(M, \Sigma)$.

Recall that a **tessellation** of the upper half plane is a family of convex hyperbolic polygons of finite area (but not necessarily compact) that cover $\mathbb{H}$ such that two polygons in this family intersect in either a common vertex, or a common side. Elements of this family are called **tiles** of the tessellation. Our first result is the following

**Theorem 1.3:** For any half-translation surface with marked points $(M, \Sigma)$, the geodesics in $L(M, \Sigma)$ define a tessellation $\Pi(M, \Sigma)$ of $\mathbb{H}$, each tile of $\Pi(M, \Sigma)$ has finitely many sides and area at most $\pi$. The tessellation $\Pi(M, \Sigma)$ is invariant with respect to half-translation coverings, that is, if $(M', \Sigma')$ is a half-translation covering of $(M, \Sigma)$, then $\Pi(M', \Sigma') = \Pi(M, \Sigma)$.

**Remark 1.4:**

(i) We refer to Section 2 for a detailed discussion on (half-)translation coverings.

(ii) In an earlier version of this paper, we showed that $\Pi(M, \Sigma)$ is a tessellation of $\mathbb{H}$ only for the case $(M, \Sigma)$ satisfies the topological Veech dichotomy. It turns out that the same conclusion holds for any $(M, \Sigma)$.

If $M$ is the standard torus $\mathbb{C}/\mathbb{Z}^2$ and $\Sigma = \{0\}$, then $\Pi(M, \Sigma)$ is the Farey tessellation. Indeed, consider an embedded triangle $T$ in $(\mathbb{C}/\mathbb{Z}^2, \{0\})$ with the slopes of its sides being $k_i = p_i/q_i$, $p_i, q_i \in \mathbb{Z}, \gcd(p_i, q_i) = 1$, $i = 1, 2, 3$. If we cut $M$ along two sides of $T$, we then get a parallelogram with the third side being a diagonal. Since the area of this parallelogram must be equal to the area
of \( M \), we have

\[
\det \begin{pmatrix} p_i & p_{i+1} \\ q_i & q_{i+1} \end{pmatrix} = |p_i q_{i+1} - p_{i+1} q_i| = 1
\]

for \( i = 1, 2, 3 \), with the convention \((p_4, q_4) = (p_1, q_1)\). Thus \( \Pi(\mathbb{C}/\mathbb{Z}^2, \{0\}) \) is the Farey tessellation. It follows from Theorem 1.3 that if \((M, \Sigma)\) is a translation covering of \((\mathbb{C}/\mathbb{Z}^2, \{0\})\), then \( \Pi(M, \Sigma) \) is the Farey tessellation as well.

Recall that a square-tiled surface is a translation surface which is obtained by gluing some copies of the unit square. If \( M \) is a square-tiled surface, and \( \Sigma \) is the set of singularities of the flat metric on \( M \), then \((M, \Sigma)\) is not necessarily a translation covering of \((\mathbb{C}/\mathbb{Z}^2, \{0\})\). This is because the natural map from \( M \) onto \( \mathbb{C}/\mathbb{Z}^2 \) may send some regular points in \( M \) to 0. Thus, in this case \( \Pi(M, \Sigma) \) is not necessarily the Farey tessellation. In Figure 1, we give the tessellation associated to a square-tiled surface \( M \) in \( \mathcal{H}(2) \) which is composed by 4 unit squares, where \( \Sigma \) consists of the unique singularity of \( M \).

Figure 1. Tessellation associated with a square-tiled surface in \( \mathcal{H}(2) \); in the horizontal direction we have a single cylinder composed by 4 squares.

In [25], Veech considered a tessellation of \( \mathbb{H} \) associated to \((M, \Sigma)\) which arises from the Delaunay partition of the surfaces in the orbit \( \text{SL}(2, \mathbb{R}) \cdot (M, \Sigma) \). To define this tessellation, one first identifies \( \mathbb{H} \) with the quotient \( \text{SL}(2, \mathbb{R})/\text{SO}(2, \mathbb{R}) \). Each tile \( P \) of this tessellation corresponds to a subset \( \tilde{P} \) of \( \text{SL}(2, \mathbb{R}) \) such that the Delaunay partition of \( A \cdot (M, \Sigma) \) remains the same as \( A \) varies in \( \tilde{P} \). Each side of \( P \) is a geodesic segment \( \eta \) specified by the following condition: there are two embedded triangles \( T, T' \) in \( \mathcal{T}(M, \Sigma) \) sharing a common side such that for any \( A \) in \( \tilde{\eta} \) (here \( \tilde{\eta} \) is the pre-image of \( \eta \) in \( \text{SL}(2, \mathbb{R}) \)), the quadrilateral \( A \cdot (T \cup T') \) is inscribable in a circle (see [1] for some interesting characterizations of this
tessellation). It turns out that every tile $P$ has at most $6g(M) - 6 + 3|\Sigma|$ sides, where $g(M)$ is the genus of $M$, and $|\Sigma|$ is the cardinality of $\Sigma$. However, there is no known bound on the area of $P$. In [1], it has been conjectured that the area of $P$ is at most $\pi$.

We will refer to the tessellation described above as the iso-Delaunay tessellation associated to $(M, \Sigma)$ (this terminology was introduced in [1, 2]). Iso-Delaunay tessellations can also be seen as the decomposition of the Teichmüller disc generated by $(M, \Sigma)$ which is induced by a cell decomposition of the Teichmüller space. Note that iso-Delaunay tessellations are also invariant by half-translation coverings (see [25, Prop. 4.1]).

Surprisingly, even though the origins of the iso-Delaunay tessellation and of the tessellation $\Pi(M, \Sigma)$ seem to be unrelated, these two tessellations do coincide when $(M, \Sigma)$ is the standard torus with one marked point $(\mathbb{C}/\mathbb{Z}^2, \{0\})$, in which case they both give the Farey tessellation. In [25, Th. 1.3], Veech showed that the $\text{GL}(2, \mathbb{R})$-orbit (of the orienting double cover) of $(M, \Sigma)$ contains a translation cover of the standard torus $(\mathbb{C}/\mathbb{Z}^2, \{0\})$ if and only if the associated iso-Delaunay tessellation is isomorphic to the Farey tessellation. Inspired by this result, we will show

**Theorem 1.5:** The tessellation $\Pi(M, \Sigma)$ is isomorphic to the Farey tessellation if and only if $\text{GL}(2, \mathbb{R}) \cdot (\hat{M}, \hat{\Sigma})$ contains a translation cover of $(\mathbb{C}/\mathbb{Z}^2, \{0\})$, where $(\hat{M}, \hat{\Sigma})$ is the orienting double cover of $(M, \Sigma)$.

**Remark 1.6:** It would be interesting to determine how the iso-Delaunay tessellation and the tessellation in Theorem 1.3 are related in general. We hope to be able to address this question in future work. It is also worth noticing that other tessellations of $\mathbb{H}$ associated to half-translation surfaces have been introduced by Smillie-Weiss [24] (see also [19] for related constructions).

**1.2. Action of the Veech group and a bound on the volume of Teichmüller curves.** An affine automorphism of $(M, \Sigma)$ is an orientation preserving homeomorphism $f : M \to M$ such that $f(\Sigma) = \Sigma$, and there is a matrix $A \in \text{SL}(2, \mathbb{R})$ such that on the local charts given by the flat metric on $M \setminus \Sigma$, $f$ has the form $v \mapsto \pm A \cdot v + c$, where $c \in \mathbb{R}^2$ is constant. The group of affine automorphisms of $(M, \Sigma)$ will be denoted by $\text{Aff}^+(M, \Sigma)$. To each element of $\text{Aff}^+(M, \Sigma)$, we have a corresponding element of $P\text{SL}(2, \mathbb{R})$ by the derivative mapping $D : \text{Aff}^+(M, \Sigma) \to P\text{SL}(2, \mathbb{R})$. The image of $\text{Aff}^+(M, \Sigma)$
under $D$ is called the Veech group of $(M, \Sigma)$ and denoted by $\Gamma(M, \Sigma)$. The pair $(M, \Sigma)$ is called a Veech surface (or equivalently a lattice surface) if $\Gamma(M, \Sigma)$ is a lattice in $\text{PSL}(2, \mathbb{R})$.

Note that by construction, $\Gamma(M, \Sigma)$ permutes the elements of $T(M, \Sigma)$, hence acts naturally on the sets $I(M, \Sigma)$, $L(M, \Sigma)$, and $C(M, \Sigma)$. Denote by $\overline{I}(M, \Sigma)$, $\overline{L}(M, \Sigma)$, $\overline{C}(M, \Sigma)$ the respective quotients. For simplicity, in what follows, once the pair $(M, \Sigma)$ is specified, we will drop it from the notation of the associated objects, i.e., $I, C, L, \Gamma$, etc.

Understanding Veech groups is a central problem in Teichmüller dynamics. Various aspects of this problem have been addressed by several authors; see, for instance, [6, 7, 5, 8, 15, 16, 23, 24, 2, 21, 27, 19]. One of the main goals of this paper is to contribute to the investigation of Veech groups by using the properties of the flat metric. In particular, as a consequence of Theorem 1.3, we get a bound on the volume of the Teichmüller curve.

**Theorem 1.7:** The surface $(M, \Sigma)$ is a Veech surface if and only if $\overline{I}$ is a finite set and we have

$$(1) \quad \text{Area}(\mathbb{H}/\Gamma(M, \Sigma)) \leq \pi \cdot \#\overline{I}.$$ 

**Remark 1.8:** In Section A.2, we will introduce an algorithm to determine the cardinality of $\overline{I}$ in the case $(M, \Sigma)$ is a Veech surface.

### 1.3. The graph of periodic directions.

We say that a half-translation surface satisfies the topological Veech dichotomy if for every $\theta \in \mathbb{RP}^1$, the foliation $\xi_\theta$ of $M$ by straight lines in direction $\theta$ is either periodic, or minimal. This property can also be stated as follows: if there is a saddle connection in direction $\theta$ then the surface is decomposed into cylinders in this direction. Any Veech surface satisfies this dichotomy. However, it is shown in [3, 7, 11] that there exist surfaces that satisfy the topological Veech dichotomy without being Veech surfaces (see also [24]).

Assume from now on that $(M, \Sigma)$ satisfies the topological Veech dichotomy. We will construct a graph $G$ underlying the tessellation $\Pi$ which comes equipped with a natural action of $\Gamma$. One may expect that the quotient graph $G/\Gamma$ captures some geometric properties of the Teichmüller curve $\mathbb{H}/\Gamma$. The graph $G$ is defined as follows:
the vertex set \( G^{(0)} \) of \( G \) is \( C \sqcup I \),
for any pair \((k, \Delta) \in C \times I\), there is an edge connecting \( k \) and \( \Delta \) if and only if \( k \) is a vertex of \( \Delta \),
there is no edge between two elements of \( C \) nor two elements of \( I \).

We set the length of every edge of \( G \) to be \( 1/2 \). We will call \( G \) the graph of periodic directions of \((M, \Sigma)\).

By construction, we have a natural action of \( \Gamma \) on \( G \) by automorphisms. We denote by \( \overline{G} \) the quotient of \( G \) by \( \Gamma \). In the perspective of understanding the Veech group, we study of the geometry of \( G \). In particular, we will show

**Theorem 1.9:** Let \((M, \Sigma)\) be a half-translation surface satisfying the topological Veech dichotomy. Then the graph \( G \) of periodic directions of \((M, \Sigma)\) is connected, has infinite diameter, and is Gromov hyperbolic. The Veech group \( \Gamma \) acts freely on the set of edges of \( G \). Moreover, \((M, \Sigma)\) is a Veech surface if and only if \( \overline{G} \) is a finite graph.

**Example:** In the case \((M, \Sigma) = (\mathbb{C}/\mathbb{Z}^2, \{0\})\), we have \( \Gamma(\mathbb{C}/\mathbb{Z}^2, \{0\}) = \text{PSL}(2, \mathbb{Z}) \), and each of \( \mathcal{I}(\mathbb{C}/\mathbb{Z}^2, \{0\}) \) and \( \mathcal{C}(\mathbb{C}/\mathbb{Z}^2, \{0\}) \) contains a single element. Let \( \Delta_0 \) be the hyperbolic ideal triangle whose vertices are \( \{0, 1, \infty\} \). Since \( \text{PSL}(2, \mathbb{Z}) \) contains an element that fixes \( \Delta_0 \) and permutes its vertices cyclically, namely \( \pm \left( \begin{array}{cc} 1 & -1 \\ 0 & 1 \end{array} \right) \), we deduce that the graph \( \overline{G}(\mathbb{C}/\mathbb{Z}^2, \{0\}) \) consists of a unique segment joining the unique element of \( \mathcal{I}(\mathbb{C}/\mathbb{Z}^2, \{0\}) \) and the unique element of \( \mathcal{C}(\mathbb{C}/\mathbb{Z}^2, \{0\}) \).

Since \( G \) is invariant with respect to (half)-translation coverings and the group \( \Gamma \) acts freely on the set of edges of \( G \), we get

**Corollary 1.10:** If \((M, \Sigma)\) is a translation cover of \((\mathbb{C}/\mathbb{Z}^2, \{0\})\), then the number of edges of \( \overline{G} \) is equals to \([\text{PSL}(2, \mathbb{Z}) : \Gamma]\).

Generally, it would be interesting to determine to what extent the geometry and topology of the hyperbolic surface \( \mathbb{H}/\Gamma \) is encoded in \( \overline{G} \). We hope to return to this problem in the near future.

### 1.4. Fundamental domain and generators of the Veech group.

In the appendix, we propose an algorithm to determine the graph \( \overline{G} \) explicitly in the case \((M, \Sigma)\) is a Veech surface, and hence to get a bound for \( \text{Area}(\mathbb{H}/\Gamma) \) in this case using Theorem 1.7. This algorithm also allows us to calculate a generating set and a “coarse” fundamental domain of the Veech group \( \Gamma \).
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2. Embedded triangles and coverings

2.1. Half-translation covering. Let $(M', \Sigma')$ and $(M, \Sigma)$ be two half-translation surfaces with marked points. Assume that $M'$ and $M$ are defined by two pairs (Riemann surface, quadratic differential) $(X', q')$ and $(X, q)$ respectively. Let $\Sigma$ (resp. $\Sigma'$) be a finite subset of $M$ (resp. of $M'$) that contains all the zeros and (simple) poles of $q$ (resp. of $q'$). A half-translation covering is a ramified covering of Riemann surfaces $f: X' \to X$ which is branched over $\Sigma$ such that $\Sigma' = f^{-1}(\Sigma)$ and $q' = f^*q$. In particular, an orienting double covering map is a half-translation covering. If both $M$ and $M'$ are translation surfaces then such a map is called a translation covering. Note that such coverings are also known as balanced coverings (see e.g. [6]).

Two half-translation surfaces with marked points are said to be affine equivalent if they belong to the same PSL(2, $\mathbb{R}$)-orbit up to scaling. As suggested by Hubert and Schmidt [6], we can define the notion of a tree of half-translation coverings as follows: such a tree is a connected acyclic directed graph whose vertices are equivalence classes of half-translation surfaces with marked points, and two vertices, represented by $(M_1, \Sigma_1)$ and $(M_2, \Sigma_2)$, are connected by a directed edge from the first to the second if there exists a half-translation covering map $f: (M_1, \Sigma_1) \to (M_2', \Sigma'_2)$, where $(M_2', \Sigma'_2)$ is a surface in the equivalence class of $(M_2, \Sigma_2)$. Note that any loop formed by oriented edges of this graph must be trivial (constant). It follows from a result by Möller [17] that if a tree of translation coverings contains a surface which is not a torus cover, then it has a root.

2.2. Invariance of the set of embedded triangles. Throughout this section $(M, \Sigma)$ will be a fixed half-translation surface with marked points, which is defined by a meromorphic quadratic differential $(X, q)$ whose poles are all simple.
Lemma 2.1: Let \( \varphi : T \to M \) be a map from a Euclidean triangle \( T \) to \( M \) such that

- the vertices of \( T \) are mapped to points in \( \Sigma \),
- \( \varphi(T \setminus T^{(0)}) \subset M \setminus \Sigma \), where \( T^{(0)} \) is the set of vertices of \( T \),
- the restriction of \( \varphi \) to \( T \setminus T^{(0)} \) is locally isometric.

Then the restriction of \( \varphi \) to \( T \setminus T^{(0)} \) is an embedding.

Proof. In what follows we identify \( T \) with a subset of \( \mathbb{R}^2 \). Assume that there are two points \( x_1, x_2 \in T \setminus T^{(0)} \) such that \( \varphi(x_1) = \varphi(x_2) \). Since a triangle is a convex subset of the plane, the segment \( x_1x_2 \) is contained in \( T \). Its image by \( \varphi \) is a loop \( \gamma \) in \( M \setminus \Sigma \). Let \( h \in \{ \pm \text{Id} \} \rtimes \mathbb{R}^2 \) be the holonomy of \( \gamma \).

If \( h(v) = -v + c \), then we have \( x_2 = -x_1 + c \). Let \( x_0 \) be the midpoint of \( x_1x_2 \) then \( x_0 = c/2 \). Thus \( h(x_0) = x_0 \), which means that \( x_0 \) is mapped to a singular point with cone angle \( \pi \) of \( M \). By assumption \( \Sigma \) contains all the singularities of \( M \). Since \( x_0 \in T \setminus T^{(0)} \) we have a contradiction to the assumption that \( \varphi(T \setminus T^{(0)}) \subset M \setminus \Sigma \). Hence this case does not occur.

If \( h(v) = v + c \) then \( c = \overrightarrow{x_1x_2} \). Let \( T_c \) denote triangle \( T + c \). Let \( T^*_c \) (resp. \( T^*_c \)) denote the triangles \( T \) (resp. \( T_c \)) with its vertices removed. By assumption, we have \( T^* \cap T^*_c \neq \emptyset \). One readily checks that this condition implies that either \( T^*_c \) contains a vertex of \( T \), or \( T^* \) contains a vertex of \( T_c \). It follows that there is a vertex \( v_0 \) of \( T \) such that either \( v_0 + c \in T^* \) or \( v_0 - c \in T^* \). Since \( v_0 \) and \( v_0 \pm c \) are mapped to the same point in \( M \), we have again a contradiction to the assumption that \( \varphi(T \setminus T^{(0)}) \subset M \setminus \Sigma \). Thus the restriction of \( \varphi \) to \( T \setminus T^{(0)} \) is an embedding.

Lemma 2.2: Let \( f : (M', \Sigma') \to (M, \Sigma) \) be a half-translation covering of half-translation surfaces with marked points. Then we have

\[
T(M', \Sigma') = T(M, \Sigma).
\]

Proof. Consider an embedded triangle \( \varphi : T \to M' \) with \( \varphi(T^{(0)}) \subset \Sigma' \). Composing with \( f \), we get a map

\[
\phi := f \circ \varphi : T \to M
\]

with \( \phi(T^{(0)}) \subset \Sigma \). Since \( f \) is a half-translation covering, the map \( \phi \) is a local isometry on \( T \setminus T^{(0)} \) and satisfies \( \phi^* q = dz^2 \) in the interior of \( T \) (where \( q \) is the quadratic differential defining the flat metric of \( M \)). It follows from Lemma 2.1 that \( f \circ \varphi(T) \) is an embedded triangle in \( (M, \Sigma) \).
On the other hand, given an embedded triangle \( \phi : T \to M \) with vertices in \( \Sigma \), we can lift \( \phi \) to a map \( \hat{\phi} : T \to M' \) which is also a local isometry on \( T \setminus T^{(0)} \). By Lemma 2.1, we have that \( \hat{\phi} : T \to M' \) is also an embedded triangle in \( M' \) with vertices in \( \Sigma' \). Thus the sets \( \mathcal{T}(M', \Sigma') \) and \( \mathcal{T}(M, \Sigma) \) are equal. \( \blacksquare \)

3. Tessellations of the hyperbolic plane

Our goal now is to give the proofs of Theorem 1.3 and Theorem 1.7. By Lemma 2.2, we can replace \((M, \Sigma)\) by its orienting cover. Therefore, throughout this section \((M, \Sigma)\) is a translation surface with marked points. Using the action of \( \text{GL}^+(2, \mathbb{R}) \), we also normalize such that \( \text{Area}(M) = 1 \).

3.1. The ideal triangles in \( \mathcal{I}(M, \Sigma) \) cover \( \mathbb{H} \). We first show

**Lemma 3.1:** Let \( z \) be a point in \( \mathbb{H} \). Then either there is an embedded triangle \( T \in \mathcal{T}(M, \Sigma) \) such that \( z \) is contained in the interior of the ideal triangle \( \Delta_T \) associated with \( T \), or there exist two embedded triangles \( T_1, T_2 \in \mathcal{T}(M, \Sigma) \) such that \( \Delta_{T_1} \cup \Delta_{T_2} \) is an ideal quadrilateral that contains \( z \) in one of its diagonals.

**Proof.** Let \( A \) be a matrix in \( \text{SL}(2, \mathbb{R}) \) such that \( z = A^{-1}(i) \). Consider the surface \((M', \Sigma') := A \cdot (M, \Sigma)\). Let \( \varrho(M', \Sigma') \) denote the length of the shortest saddle connection on \( M' \) (with endpoints in \( \Sigma' \)). Let \( s_0 \) be a saddle connection in \( M' \) such that \( |s_0| = \varrho(M', \Sigma') \). Replacing \( A \) by \( RA \), where \( R \in \text{SO}(2, \mathbb{R}) \), if necessary (note that \((RA)^{-1}(i) = A^{-1}(i) = z\)), we can assume that \( s_0 \) is horizontal.

Consider the vertical separatrices of \((M', \Sigma')\), that is the vertical geodesic rays emanating from the points in \( \Sigma' \). We have two cases:

**Case (a):** \( \text{int}(s_0) \) intersects some vertical separatrices. For each vertical separatrix intersecting \( \text{int}(s_0) \), consider the subsegment from its origin to its first intersection with \( \text{int}(s_0) \). Pick a segment of minimal length \( u_0 \) in this family. Then there is an embedded triangle \( T' \) containing this vertical segment which is bordered by \( s_0 \) and two other saddle connections denoted by \( s_1, s_2 \). This triangle can be constructed as follows: one can identify \( s_0 \) with a horizontal segment and \( u_0 \) with a vertical segment in the plane. Let \( P_1, P_2 \) denote the endpoints of the segment corresponding to \( s_0 \), and \( P_0, Q_0 \) denote the endpoints of the segment corresponding to \( u_0 \), where \( Q_0 \in \overrightarrow{P_1P_2} \). Let \( T' \) be the triangle with vertices \( P_0, P_1, P_2 \). Since the length of \( u_0 \) is minimal among the vertical
segments from a point in \( \Sigma' \) to a point in \( \text{int}(s_0) \), the developing map induces a map \( \varphi : T' \to M' \) which is locally isometric. By Lemma 2.1, the image of \( \varphi \) is an embedded triangle in \( (M', \Sigma') \). By construction \( s_i = \varphi(P_{0i}) \), \( i = 1, 2 \).

Let \( k_1, k_2 \) be the slopes of \( s_1 \) and \( s_2 \) respectively. Note that we always have \( k_1 k_2 < 0 \). Without loss of generality, we can assume that \( k_1 > 0 > k_2 \) (equivalently, \( P_1 \) is the left endpoint of \( s_0 \)). We now claim that

\[
k_1 - k_2 \leq \frac{2}{\sqrt{3}}.
\]

Let \( x_i \) be the length of the segment \( P_i Q_0 \), \( i = 1, 2 \), and \( y \) be the length of the segment \( P_0 Q_0 \). By definition, we have \( k_1 = x_1 / y \) and \( k_2 = -x_2 / y \). Hence

\[
k_1 - k_2 = \frac{x_1 + x_2}{y} = \frac{x}{y}
\]

where \( x = x_1 + x_2 = |s_0| \). By definition, we have \( |s_0| \leq \min\{|s_1|, |s_2|\} \), therefore \( x^2 \leq \min\{x_1^2 + y^2, x_2^2 + y^2\} \). But since \( x = x_1 + x_2 \), we have \( \min\{x_1, x_2\} \leq \frac{2}{3} \).

Thus we have

\[
x^2 \leq \frac{x^2}{4} + y^2 \quad \text{which implies} \quad \frac{x}{y} \leq \frac{2}{\sqrt{3}}
\]

which proves the claim.

Since we have \( k_1 > 0 > k_2 \) and \( k_1 - k_2 \leq 2/\sqrt{3} \), the radius of the half circle perpendicular to the real axis passing through \( k_1 \) and \( k_2 \) is at most \( \frac{1}{\sqrt{3}} < 1 \). Thus it cannot separate \( i \) and \( \infty \). It follows in particular that \( i \) is contained in the ideal triangle \( \Delta_{T'} \) with vertices \( \{\infty, k_1, k_2\} \).

Since \( (M, \Sigma) = A^{-1} \cdot (M', \Sigma') \), \( T = A^{-1}(T') \) is an embedded triangle in \( \mathcal{T}(M, \Sigma) \). Note that the slope of the sides of \( T \) are \( \{A^{-1}(\infty), A^{-1}(k_1), A^{-1}(k_2)\} \) (here we consider the usual action of \( A^{-1} \) on \( \mathbb{R} \cup \{\infty\} = \partial \mathbb{H} \)) which means that \( \{A^{-1}(\infty), A^{-1}(k_1), A^{-1}(k_2)\} \) are the vertices of the ideal triangle \( \Delta_T \), or equivalently \( \Delta_T = A^{-1}(\Delta_{T'}) \). Since \( i \) is contained in the interior of \( \Delta_{T'} \) and \( z = A^{-1}(i) \) by definition, we conclude that \( z \) is contained in the interior of \( \Delta_T \).

Case (b): No vertical saddle connection intersects \( \text{int}(s_0) \). In this case \( s_0 \) is contained in a vertical cylinder \( C \) of \( (M', \Sigma') \). We can realize the cylinder \( C \) as the image of a rectangle \( R \) in the plane under a locally isometric mapping \( \varphi : R \to M \) such that the restriction of \( \varphi \) to \( \text{int}(R) \) is injective, and \( \varphi \) maps both the bottom and top sides of \( R \) onto \( s_0 \).
Let $P_1$ and $P_2$ denote the left and right endpoints of the bottom side of $R$ respectively. There is a subsegment of the left side of $R$, with $P_1$ being an endpoint, that is mapped to a vertical saddle connection $r_1$ in the boundary of $C$. Similarly, there is a subsegment of the right side of $R$, with $P_2$ being an endpoint, that is mapped to a vertical saddle connection $r_2$ in the boundary of $C$. Let $P'_1$ and $P'_2$ denote the upper endpoints of $r_1$ and $r_2$ respectively. Let $s_1$ (resp. $s_2$) denote the saddle connection that is the image of $P'_1P_2$ (resp. of $P_1P'_2$) under $\varphi$. We remark that $s_0, s_i, r_i$ bound an embedded triangle $T'_i$, which is entirely contained in $C$, for $i = 1, 2$.

Let $k_i$ be the slope of $s_i$, then $k_1 < 0 < k_2 < 0$. The vertices of the hyperbolic ideal triangle $\Delta_{T'_i}$ are $\{\infty, 0, k_i\}$. Since $k_1k_2 < 0$, the vertical line from $\infty$ to $0$ is the common side of the ideal triangles $\Delta_{T'_1}$ and $\Delta_{T'_2}$. Hence $i$ is contained in the interior of the ideal quadrilateral formed by $\Delta_{T'_1}$ and $\Delta_{T'_2}$. By the same arguments as in the previous case, we see that there exist two embedded triangles $T_1, T_2$ in $\mathbb{T}(M, \Sigma)$ such that $z$ is contained in a diagonal of the quadrilateral formed by $\Delta_{T_1}$ and $\Delta_{T_2}$. ■

3.2. Locally finite property of $\mathcal{L}$.

**Lemma 3.2:** Let $K$ be a compact subset of $\mathbb{H}$. Then the set

$$\{\gamma \in \mathcal{L}, \gamma \cap K \neq \emptyset\}$$

is finite.

**Proof.** Assume that there is an infinite family of geodesics $\{\gamma_n\}_{n \in \mathbb{N}} \subset \mathcal{L}$ such that $\gamma_n \cap K \neq \emptyset$. For each $n \in \mathbb{N}$, let $p_n, q_n \in \partial \mathbb{H}$ denote the endpoints of $\gamma_n$. Since $\partial \mathbb{H} \simeq \mathbb{RP}^1$ is compact, by extracting a subsequence if necessary, we can suppose that $\lim_{n \to +\infty} p_n = p$ and $\lim_{n \to -\infty} q_n = q$.

We first notice that $p \neq q$; this is because if $p = q$, then for any compact $K$, $\gamma_n \cap K = \emptyset$ when $n$ is large enough. Let $(p', q')$ be a pair of points in $\partial \mathbb{H}$ such that the geodesic joining $p'$ and $q'$ separates $p$ from $q$. Using the action of $\text{PSL}(2, \mathbb{R})$, we can further assume that $p' = \infty$ and $q' = 0$. Without loss of generality, we can assume that $q < 0 < p$, and that all the geodesics in the family $\{\gamma_n\}_{n \in \mathbb{N}}$ cross the vertical half-line $i\mathbb{R}^+$. By definition, each geodesic $\gamma_n$ is associated with an embedded triangle $T_n$ in $\mathbb{T}(M, \Sigma)$. This triangle has two sides $a_n, b_n$ such that the slope of $a_n$ is $p_n$ and the slope of $b_n$ is $q_n$. Since $\gamma_n$ crosses $i\mathbb{R}^+$, we must have $q_n < 0 < p_n$. 
Since the slope of $a_n$ belongs to $(0; +\infty)$, we can suppose that its period is $x_n + iy_n$, where both $x_n$ and $y_n$ are positive real numbers. Similarly, since the slope of $b_n$ belongs to $(-\infty; 0)$, we can suppose that its slope is $-x'_n + iy'_n$, where $x'_n$ and $y'_n$ are both positive. As a consequence

$$\text{Area}(T_n) = \frac{1}{2} \left| \det \begin{pmatrix} x_n & -x'_n \\ y_n & y'_n \end{pmatrix} \right| = \frac{1}{2}(x_n y'_n + y_n x'_n).$$

Since $T_n$ is an embedded triangle, we must have $\text{Area}(T_n) < \text{Area}(M) = 1$. Hence

(3) \hspace{1cm} x_n y'_n + y_n x'_n < 2.

By definition $p_n = x_n/y_n$ and $q_n = -x'_n/y'_n$. Therefore

(4) \hspace{1cm} p_n - q_n = \frac{x_n}{y_n} + \frac{x'_n}{y'_n} = \frac{x_n y'_n + y_n x'_n}{y_n y'_n} < \frac{2}{y_n y'_n}.

Since $p_n$ and $q_n$ converge to $p$ and $q$ respectively, there exists $\alpha > 0$ such that $p_n - q_n > \alpha$ for all $n \in \mathbb{N}$. Thus (4) implies

(5) \hspace{1cm} y_n y'_n < \frac{2}{\alpha}.

We have two possibilities:

- Case 1: both $y_n$ and $y'_n$ are bounded below by a constant $\beta > 0$. Then it follows from (5) that both $y_n$ and $y'_n$ are bounded above by $\frac{2}{\alpha \beta}$. The inequality (3) implies that both $x_n$ and $x'_n$ are also bounded above by $\frac{2}{\beta}$. Thus the lengths of $a_n$ and $b_n$ are bounded by some constant $R$. But it is a well known fact that there are only finitely many saddle connections on $(M, \Sigma)$ that have length at most $R$. This contradiction shows that this case cannot occur.

- Case 2: either $\liminf_{n \to +\infty} y_n = 0$, or $\liminf_{n \to \infty} y'_n = 0$. Let us suppose that $\liminf_{n \to +\infty} y_n = 0$; the case $\liminf_{n \to \infty} y'_n = 0$ follows from the same argument. By extracting a subsequence, we can assume that $\lim_{n \to \infty} y_n = 0$. Since

$$p = \lim_{n \to \infty} p_n = \lim_{n \to \infty} \frac{x_n}{y_n},$$

it follows that $\lim_{n \to \infty} x_n = 0$. In particular, $\{a_n\}_{n \in \mathbb{N}}$ is a sequence of saddle connections that have lengths tending to 0. Since such a sequence cannot exist, we get again a contradiction which proves the lemma. ■
3.3. Proof of Theorem 1.3.

Proof. Let \( \hat{\mathcal{L}} \) denote the union of all the geodesics in \( \mathcal{L} \). It follows from Lemma 3.2 that \( \hat{\mathcal{L}} \) is a closed subset of \( \mathbb{H} \). We need to show that every component of the set \( \mathbb{H} \setminus \hat{\mathcal{L}} \) is a hyperbolic polygon with finitely many sides and area at most \( \pi \).

Let \( z \) be a point in \( \mathbb{H} \setminus \hat{\mathcal{L}} \), and \( P_z \) the component of \( \mathbb{H} \setminus \hat{\mathcal{L}} \) that contains \( z \). By Lemma 3.1, \( z \) is contained in the interior of an ideal triangle \( \Delta \) in \( \mathcal{I} \). Let \( \mathcal{F} \) denote the set of geodesics in \( \mathcal{L} \) that intersect \( \text{int}(\Delta) \). Then \( P_z \) is cut out by the boundary of \( \Delta \) and the geodesics in \( \mathcal{F} \). In particular, \( P_z \) is contained in \( \Delta \). Therefore, we have \( \text{Area}(P_z) \leq \text{Area}(\Delta) = \pi \). It remains to show that \( P_z \) has finitely many sides.

Let \( p_1, p_2, p_3 \) denote the vertices of \( \Delta \). Let \( V_i \) denote the intersection of \( \Delta \) with a horoball tangent to \( \partial \mathbb{H} \) at \( p_i \). We choose the horoballs such that \( V_1, V_2, V_3 \) are pairwise disjoint, and \( z \notin V_1 \cup V_2 \cup V_3 \).

Let \( K := \Delta \setminus (V_1 \cup V_2 \cup V_3) \). We split \( \mathcal{F} \) into two subsets \( \mathcal{F}' \) and \( \mathcal{F}'' \), where \( \mathcal{F}' \) is the set of geodesics in \( \mathcal{L} \) that intersect \( K \). Since \( K \) is compact, it follows from Lemma 3.2 that \( \mathcal{F}' \) is a finite set. Let \( P'_z \) denote the domain containing \( z \) which is cut out by the sides of \( \Delta \) and the geodesics in \( \mathcal{F}' \). Since the set \( \mathcal{F}' \) is finite, \( P'_z \) is a convex polygon with finitely many sides.

Let \( \mathcal{F}'' \) be the set of geodesics in \( \mathcal{L} \) that intersect \( \Delta \) but disjoint from \( K \). Note that if \( \gamma \) is a geodesic in \( \mathcal{F}'' \), then since \( \gamma \) does not cross \( K \), it does not intersect \( V_j \) if \( j \neq i \). Therefore, \( \gamma \) separates \( p_i \) from \( z \), and we have \( \mathcal{F}'' := \mathcal{F}_1'' \sqcup \mathcal{F}_2'' \sqcup \mathcal{F}_3'' \).

For \( i = 1, 2, 3 \), if \( \mathcal{F}_i'' \neq \emptyset \) then we pick a geodesic \( \gamma_i'' \in \mathcal{F}_i'' \). Let \( P''_z \) denote the component of \( P'_z \) containing \( z \) which is cut out by the geodesics \( \gamma_i'' \). By construction, \( P''_z \) is also a polygon with finitely many sides.

- If none of \( \{p_1, p_2, p_3\} \) is a vertex of \( P''_z \), then \( P''_z \) is a compact. Since \( P_z \) is cut out by the geodesics in \( \mathcal{F}'' \) that intersect \( P''_z \), we conclude by Lemma 3.2.
- For \( i = 1, 2, 3 \), if \( p_i \) is a vertex of \( P''_z \) then \( \mathcal{F}_i'' = \emptyset \), because otherwise we have a geodesic that separates \( z \) from \( p_i \). In this case we remove from \( P''_z \) the intersection \( P''_z \cap V_i \). The remaining subset of \( P''_z \) is denoted by \( Q''_z \). Note that \( Q''_z \) is compact.

We now claim that if \( \gamma \) is a geodesic in \( \mathcal{F}'' \) which intersects \( P''_z \), then \( \gamma \) must intersect \( Q''_z \). Indeed, if \( \gamma \) intersects \( P''_z \) but not \( Q''_z \) then it must
intersect one of the set $P_z'' \cap V_i$, where $F_i'' = \emptyset$. But by definition, $F_i''$ contains $\gamma$. Thus we have a contradiction.

The claim implies that $P_z$ is cut out from $P_z''$ by finitely many geodesics in $F''$ (by Lemma 3.2). Therefore, $P_z$ has finitely many sides. The invariance of $\Pi(M, \Sigma)$ with respect to half-translation coverings follows from Lemma 2.2. 

3.4. Proof of Theorem 1.5.

Proof. Since the tessellation associated with $( \mathbb{C}/\mathbb{Z}^2, \{0\})$ is the Farey tessellation, it follows from Lemma 2.2 that if $(M, \Sigma)$ is a translation covering of $( \mathbb{C}/\mathbb{Z}^2, \{0\})$, then $\Pi(M, \Sigma)$ is the Farey tessellation as well. Thus, we only need to show that if $\Pi(M, \Sigma)$ is the Farey tessellation, then up to a rescaling the canonical orienting covering of $(M, \Sigma)$ is a translation covering of $( \mathbb{C}/\mathbb{Z}^2, \{0\})$.

To simplify the arguments, we can assume that $M$ is a translation surface. Recall that every geodesic in the Farey tessellation joins a pair of points $(p/q, p'/q') \in (\mathbb{P}_1^1)^2$ such that $|pq' - qp'| = 1$.

Consider an embedded triangle $T$ in $T$. Since the slopes of the sides of $T$ are in $\mathbb{Q} \cup \{\infty\}$, there is an element $A$ of $\text{SL}(2, \mathbb{Z})$ such that the slopes of the triangle $A \cdot T$ are $(0, 1, \infty)$. Replacing $(M, \Sigma)$ by $A \cdot (M, \Sigma)$, we can then assume that the ideal triangle $\Delta$ associated to $T$ has vertices $\{0, 1, \infty\}$. Recall that the slopes of the horizontal and vertical directions are respectively $\infty$ and $0$. Denote by $s_1, s_2, s_3$ the sides of $T$, where $s_1$ is vertical, the slope of $s_2$ is 1, and $s_3$ is horizontal (see Figure 2). Rescaling $M$ using $\mathbb{R}^*_+$, we can further assume that the lengths of $s_1$ and $s_3$ are both equal to 1. We will show that in this case $(M, \Sigma)$ is a translation cover of the torus $( \mathbb{C}/\mathbb{Z}^2, \{0\})$.

We first show that $T$ is contained in a horizontal cylinder. Indeed, if this is not the case then there would be a horizontal separatrix (that is a horizontal ray emanating from a point in $\Sigma$) that intersects $\text{int}(s_1)$. It follows that there exists an embedded triangle bounded by $s_1$ and two other saddle connections $s'_2, s'_3$, where the slope $k'_2$ of $s'_2$ is negative, and the slope $k'_3$ of $s'_3$ is positive. By definition, $\Pi(M, \Sigma)$ contains the geodesic joining $k'_1$ and $k'_2$. But the Farey tessellation does not contain any such geodesic, hence we have a contradiction.

Let $C$ be the horizontal cylinder containing $T$. We can assume that $s_3$ is contained in the bottom border of $C$. The top border of $C$ contains a horizontal
saddle connection $s_4$ whose right endpoint coincides with the vertex of $T$ opposite to $s_3$. Observe that there is an embedded triangle $T'$ bounded by $s_2, s_4,$ and a third saddle connection denoted by $s_5$, which is contained in $C$.

Let $k_5$ be the slope of $s_5$. Since the slope of $s_2$ is 1, and the slope of $s_4$ is $\infty$, the tessellation $\Pi(M, \Sigma)$ contains a geodesic from $k_5$ to 1 and a geodesic from $k_5$ to $\infty$. But $\Pi(M, \Sigma)$ is the Farey tessellation, thus either $k_5 = 0$, or $k_5 = 2$. By construction, it is clear that $k_5 < 1$. Therefore, we have $k_5 = 0$, that is $s_5$ is vertical. This means that $S := T \cup T'$ is an embedded unit square whose vertices are contained in $\Sigma$.

If the square $S$ does not fill $C$, then the bottom border of $C$ must contain saddle connection $s_6$ whose right endpoint coincides with the left endpoint of $s_3$. Consider the embedded triangle $T''$ which is bounded by $s_6, s_5,$ and a third saddle connection contained in $C$. Repeating the arguments above, we see that the slope of the third side of $T''$ is 1. By induction, we conclude that $C$ is filled by embedded unit squares with vertices in $\Sigma$.

By the same argument, we also conclude that $S$ is also contained in a vertical cylinder $C'$, which is filled out by unit squares with vertices in $\Sigma$. The same argument implies that every square in $C$ and $C'$ is contained in the intersection of a horizontal cylinder and a vertical cylinder, both of which are filled by unit squares with vertices in $\Sigma$. By connectedness, we deduce that $M$ is filled by such squares, which means that $(M, \Sigma)$ is a translation cover of the torus $(\mathbb{C}/\mathbb{Z}^2, \{0\})$. ■
4. The graph of periodic directions

Throughout this section, to simplify the discussion, \((M, \Sigma)\) will be a translation surface satisfying the topological Veech dichotomy. By Lemma 2.2, the results in this section also hold in the case \((M, \Sigma)\) is a half-translation surface. We also normalize \((M, \Sigma)\) using \(\text{GL}^+(2, \mathbb{R})\) such that \(\text{Area}(M) = 1\).

In §1.3, we have defined a graph \(\mathcal{G}\) associated to \((M, \Sigma)\). Recall that the vertices of \(\mathcal{G}\) are elements of \(\mathcal{C} \sqcup \mathcal{I}\), and every edge of \(\mathcal{G}\) must join an element \(\Delta\) of \(\mathcal{I}\) to an element \(k\) of \(\mathcal{C}\) which is a vertex of \(\Delta\). The length of every edge is set to be \(\frac{1}{2}\). The distance on \(\mathcal{G}\) is denoted by \(d\). By construction, the graph \(\mathcal{G}\) has the following properties:

(a) Every vertex representing an element of \(\mathcal{I}\) is the common endpoint of exactly 3 (distinct) edges.

(b) Let \(k, k'\) be two elements of \(\mathcal{C}\). Then \(d(k, k') = 1\) if and only if there is an ideal triangle in \(\mathcal{I}\) that contains \(k\) and \(k'\) as vertices. Equivalently, \(d(k, k') = 1\) if and only if there is a geodesic in \(\mathcal{L}\) that joins \(k\) and \(k'\).

4.1. CONNECTEDNESS. For each \(k \in \mathcal{C}\), let us denote by \(S(k)\) the set of saddle connections in the direction \(k\). The union of the saddle connections in \(S(k)\) will be denoted by \(\hat{S}(k)\). Given \(k, k'\) in \(\mathcal{C}\), we define the ordered intersection number of the pair \((k, k')\) by

\[
i(k, k') = \min\{\#(\text{int}(s) \cap \hat{S}(k')) \mid s \in S(k)\}.
\]

Note that the function \(i\) is not symmetric, that is \(i(k, k')\) and \(i(k', k)\) might not be equal.

**PROPOSITION 4.1**: Let \(k, k'\) be two directions in \(\mathcal{C}\). Then

\[
d(k, k') \leq \log_2(\min\{i(k, k'), i(k', k)\} + 1) + 1.
\]

In particular, the graph \(\mathcal{G}\) is connected.

We first show

**LEMMA 4.2**: If \(\min\{i(k, k'), i(k', k)\} = 0\) then \(d(k, k') = 1\).

**Proof.** Without loss of generality, we can assume \(k = 0\), \(k' = \infty\), and that \(i(k, k') = 0\). This means that \(k\) is the vertical direction, \(k'\) is the horizontal direction, and there is a vertical saddle connection \(s\) which is not crossed by any horizontal saddle connection. By assumption, the horizontal direction is periodic. Since \(s\) does not intersect any horizontal saddle connection in its
interior, $s$ must be contained entirely in a horizontal cylinder $C$. There exists an embedded triangle contained in $C$ whose boundary contains $s$ and a horizontal saddle connection in the boundary of $C$. It follows that $\mathcal{I}$ contains an ideal hyperbolic triangle with vertices $(0, \infty, k'')$, which means that, as vertices of $G$, $0$ and $\infty$ are connected by a path of length one.

Proof of Proposition 4.1. Again, without loss of generality, we can assume that $k = 0$, $k' = \infty$, and $i(k, k') \leq i(k', k)$. Let $n = i(k, k') = \min\{i(k, k'), i(k', k)\}$. If $n = 0$, then by Lemma 4.2 we have $d(0, \infty) = 1$. Let us suppose that $n > 0$.

Consider a vertical saddle connection $s$ such that $\#$\{int$(s)$ $\cap$ $\hat{S}(\infty)$\} = $n$. Let us denote the horizontal saddle connections of $M$ by $a_1, \ldots, a_m$. We choose the orientation of those saddle connections to be from the left to the right. For each $a_i$, let $r_i$ be the distance along $a_i$ from its left endpoint to its first intersection with int$(s)$. If $a_i \cap$ int$(s) = \emptyset$, we set $r_i = +\infty$.

Assume that $r_1 = \min\{r_1, \ldots, r_m\}$. Let $a'_1$ be the subsegment of $a_1$ between its left endpoint and its first intersection with int$(s)$. Using the developing map of the flat metric structure, we see that $a'_1$ is contained in an embedded triangle $T$ bordered by $s$ and two other saddle connections $s_1, s_2$. Let $k_1, k_2$ be the directions of $s_1$ and $s_2$ respectively. By definition there is an ideal hyperbolic triangle with vertices $(0, k_1, k_2)$ in $\mathcal{I}$. Thus we have $d(0, k_1) = d(0, k_2) = 1$ as vertices of $G$. We now observe that

$$\#\{\text{int}(s) \cap \hat{S}(\infty)\} = \#\{\text{int}(s_1) \cap \hat{S}(\infty)\} + \#\{\text{int}(s_2) \cap \hat{S}(\infty)\} - 1.$$ 

Hence $\min\{i(k_1, k'), i(k_2, k')\} < i(k, k')/2$. Replacing $k$ by either $k_1$ or $k_2$, by induction, we get the desired conclusion.

4.2. Action of the Veech group. Since an affine automorphism must send saddle connections to saddle connections and embedded triangles to embedded triangles, we have an action of the group $\Gamma$ on $G$ by automorphisms.

Lemma 4.3: The group $\Gamma$ acts freely on the set of edges of $G$.

Proof. Let $g$ be an element of $\Gamma$. Assume that $g$ fixes an edge of $e$ of $G$. Recall that by construction, one endpoint of $e$ corresponds to an ideal triangle $\Delta$ in $\mathcal{I}$, and the other endpoint corresponds to a vertex $k$ of $\Delta$. Since $g$ fixes $e$, it must fix $\Delta$ and $k$ (this is because $g$ preserves each of the sets $\mathcal{I}$ and $\mathcal{C}$). In particular, $g$ permutes the vertices of $\Delta$. But since $g$ preserves the orientation of $\mathbb{R}P^1 \simeq \partial \mathbb{H}$, if it fixes one vertex of $\Delta$, it must fix all of its vertices. Therefore we must have $g = \pm \text{Id}$.
Recall that $\overline{C}, \overline{L}, \overline{I}, \overline{G}$ are the quotients of $C, L, I, G$ by $\Gamma$ respectively.

**Proposition 4.4:** If $(M, \Sigma)$ is a Veech surface then the quotients $\overline{C}, \overline{L},$ and $\overline{I}$ are all finite. In particular, $\overline{G}$ is a finite graph.

**Proof.** Since every $\Gamma$-orbit in $C$ is a cusp of the corresponding Teichmüller curve, we conclude that the quotient $\overline{C}$ is finite.

Let us show that $\overline{L}$ is finite. Let $k$ be an element of $C$. We can assume that $k = \infty$, that is $k$ is the horizontal direction. Since $M$ is a Veech surface, it is horizontally periodic. Moreover, there is a matrix $A = \left( \begin{smallmatrix} 1 & c \\ 0 & 1 \end{smallmatrix} \right) \in \Gamma$ such that the stabilizer of $\infty$ in $\Gamma$ equals $\{A^n, n \in \mathbb{Z}\}$. Without loss of generality, we can assume that $c > 0$.

Let $\delta$ be the length of the shortest horizontal saddle connections of $(M, \Sigma)$. Consider a geodesic $\gamma \in L$ joining $\infty$ to a point $k' \in \mathbb{R}$. By definition, there is an embedded triangle $T \in T$ whose boundary contains a horizontal saddle connection $s$, and a saddle connection $s'$ in direction $k'$.

We first notice that $|s| \geq \delta$. Let $x' + iy'$, with $y' > 0$, be the period of $s'$. Since $\text{Area}(T) \leq \text{Area}(M) = 1$, we have $y' \leq 2/|s| \leq 2/\delta$. There exists $n \in \mathbb{Z}$ such that $0 \leq x' + ncy' \leq cy' \leq 2c/\delta$. Thus, up to the action of $\{A^n, n \in \mathbb{Z}\}$, we can assume that $0 \leq x' \leq 2c/\delta$. It follows that $|s'|$ is bounded by $\frac{2}{\delta} \sqrt{1 + c^2}$, which implies that $s'$ belongs to a finite set. Hence, up to the action of $\{A^n, n \in \mathbb{Z}\}$, there are only finitely many geodesics in $L$ that contain $\infty$ as an endpoint. Since $\overline{C}$ is finite, we conclude that the set $\overline{L}$ is also finite.

We now claim that any geodesic $\gamma$ in $L$ is contained in finitely many ideal triangles in $I$. Without loss of generality, we can assume that $\gamma$ is the upper half of the imaginary axis. Let $\Delta$ be an ideal triangle in $I$ that contains $\gamma$. By definition, $\Delta$ corresponds to an embedded triangle $\overline{T} \in \mathbb{T}$ whose boundary contains a horizontal saddle connection $s$, and a vertical saddle connection $s'$. Note that the direction of the third side of $T$ is determined up to sign by $|s'|/|s|$. Since there are only finitely many horizontal (resp. vertical) saddle connections, such a triangle belongs to a finite set. Therefore, there are only finitely many elements of $I$ that contain $\gamma$.

Pick a representative for each $\Gamma$-orbit in $L$, and let $L^*$ be the resulting finite family of geodesics in $\mathbb{H}$. By the previous claim, the sets of triangles in $I$ that contain at least one element of $L^*$ is finite. Since every ideal triangle in $I$ is mapped by an element of $\Gamma$ to a triangle that contains a geodesic in the family $L^*$, we conclude that $\overline{I}$ is finite. ■
Proof of Theorem 1.7. Pick a representative element for each $\Gamma$-orbit in $I$. Denote by $I^*$ the resulting family. Let $F \subset \mathbb{H}$ be the union of the ideal triangles in $I^*$. Let $\varphi : \mathbb{H} \to \mathbb{H}/\Gamma$ denote the canonical projection. By Lemma 3.1, we have that $\mathbb{H} = \bigcup_{h \in \Gamma} h(F)$. Therefore, $\varphi(F)$ covers $\mathbb{H}/\Gamma$. For each ideal triangle $\Delta$ in $I^*$, we have

$$\text{Area}(\varphi(\Delta)) \leq \text{Area}(\Delta) = \pi.$$  

Therefore

$$\text{Area}(\mathbb{H}/\Gamma) \leq \sum_{\Delta \in I^*} \text{Area}(\varphi(\Delta)) \leq \# I^* \cdot \pi$$

which proves (1). If $\# I^*$ is finite then $\text{Area}(\mathbb{H}/\Gamma) < \infty$ (by (1)), which means that $\Gamma$ is a lattice in $\text{PSL}(2, \mathbb{R})$, and hence $(M, \Sigma)$ is a Veech surface. Conversely, if $(M, \Sigma)$ is a Veech surface, then it follows from Proposition 4.4 that $\# I^*$ is finite.

5. Geometry of the graph of periodic directions

Our goal now is to give the proof of Theorem 1.9. Throughout this section $(M, \Sigma)$ will be a half-translation surface satisfying the topological Veech dichotomy, which does not need to be a Veech surface.

5.1. Infinite diameter. In this section we will show

PROPOSITION 5.1: The graph $G$ has infinite diameter.

To prove Proposition 5.1, we will make use of the connection between $G$ and the arc and curve graph on a surface with marked points.

5.1.1. Arc and curve graphs. Let $S$ be a topological surface homeomorphic to $M \setminus \Sigma$. We will consider $S$ as a compact surface $\hat{S}$ with a finite set $V$ removed; points in $V$ are called punctures. A simple closed curve in $S$ is non-essential if it is either homotopic to the constant loop, or bounds a disc that contains only one puncture. A simple arc in $S$ is a continuous map $\alpha : I \to \hat{S}$, where $I \subset \mathbb{R}$ is a compact interval, such that the restriction of $\alpha$ to $\text{int}(I)$ is an embedding and $\alpha(I) \cap V = \alpha(\partial I)$. A simple arc is non-essential if it is homotopic relative to its endpoints to the constant map by a homotopy $H : I \times [0, 1] \to S$ such that for all $(t, s) \in \text{int}(I) \times [0, 1)$, $H(t, s) \in S$. A simple closed curve or a simple arc is said to be essential if it is not non-essential.
Define the **curve graph** $\text{Curv}(S)$ to be the graph whose vertices are homotopy classes of essential simple closed curves in $S$, and there is an edge between two vertices if and only if the corresponding simple closed curves can be realized disjointly. Similarly, define the **arc and curve graph** $\text{ACurv}(S)$ to be the graph whose vertices are homotopy classes of essential simple arcs and simple curves on $S$, and there is an edge between two vertices if and only if they can be realized disjointly in $S$. We define the length of every edge of $\text{Curv}(S)$ and of $\text{ACurv}(S)$ to be one. Denote by $d_{AC}$ and $d_C$ the distance in $\text{ACurv}(S)$ and in $\text{Curv}(S)$ respectively. By construction, we have a natural embedding from $\text{Curv}(C)$ into $\text{ACurv}(S)$.

Since the graph of periodic directions $\mathcal{G}$ is unchanged if we replace $(M, \Sigma)$ by a half-translation covering, we can suppose that the genus of $M$ (and hence the genus of $S$) is at least two. We then have the following well known facts (see [13]):

- the graphs $\text{Curv}(S)$ and $\text{ACurv}(S)$ are connected and have infinite diameter,
- the graphs $\text{Curv}(S)$ and $\text{ACurv}(S)$ are quasi-isometric.

A geodesic metric space is said to be **Gromov hyperbolic** if there is a constant $\delta > 0$ such that for any triple of points $(x, y, z)$ in this space, any geodesic from $x$ to $y$ is contained in the $\delta$-neighborhood of the union of a geodesic from $x$ to $z$ and a geodesic from $y$ to $z$. By the celebrated result of Masur–Minsky [12], we know that $\text{Curv}(S)$ (and hence $\text{ACurv}(S)$) is Gromov hyperbolic.

Recall that a measured foliation on $S$ is by definition a measured foliation on $\hat{S}$ which has $k$-pronged singularities with $k \geq 3$ in $S$, and $k$-pronged singularities with $k \geq 1$ at points in $V$ (see [18]). In other words, measured foliations on $S$ are measured foliations on $\hat{S}$ that are modeled by the foliations of meromorphic quadratic differentials with at most simple poles.

A measured foliation is **minimal** if all of its leaves are either dense in $\hat{S}$ or join two singularities, and there is no cycle of leaves. In [9], Klarreich shows that the boundary at infinity $\partial_\infty \text{Curv}(S)$ of $\text{Curv}(S)$ can be identified with the space of topological minimal foliations on $S$. Moreover, we have (see [9, Th. 1.4])

**Theorem 5.2** (Klarreich): *Given a minimal foliation $\mu$ on $S$, a sequence $(c_i)_{i\in\mathbb{N}} \subset \text{Curv}(S)^{(0)}$ converges to the point in $\partial_\infty \text{Curv}(S)$ represented by $\mu$ if and only if for every accumulation point $\nu$ of $(c_i)_{i\in\mathbb{N}}$ in the space of projective measured foliations, $\nu$ is topologically equivalent to $\mu$.***
Let \( \iota \) denote the intersection number function on the space of measured foliations. The following result is proved in [20].

**Theorem 5.3:** If \( \lambda \) is a minimal measured foliation on \( S \), then a measured lamination \( \mu \) is topologically equivalent (Whitehead equivalent) to \( \lambda \) if and only if \( \iota(\lambda, \mu) = 0 \).

For our purpose, we will also need the following result which is due to Smillie [22] (see also [26]).

**Theorem 5.4:** Given any stratum of translation surfaces, there is a constant \( K > 0 \) such that on any surface of area one in this stratum, there exists a cylinder of width bounded below by \( K \).

Since the area of a cylinder is equal to the product of its circumference and its width, if the surface has area one and the width of the cylinder is bounded below by \( K \), then its circumference is at most \( 1/K \). As a consequence of Theorem 5.2, we get the following (see also [4, Prop. 2.4])

**Corollary 5.5:** For \( t \in \mathbb{R} \), let \( M_t \) denote the surface \( a_t \cdot M \), where

\[
a_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.
\]

For \( n \in \{0, 1, \ldots \} \), let \( c_n \) be a regular geodesic on \( M_n \) of length at most \( 1/K \). The existence of such a geodesic is guaranteed by Theorem 5.4. We consider \( (c_n) \) as a sequence of vertices of \( \text{Curv}(S) \) via a homeomorphism \( f : \hat{S} \to M \) sending \( V \) onto \( \Sigma \).

Assume that the vertical foliation \( \mu \) on \( M \) is minimal. Then the sequence \( (c_n) \) defines a point in \( \partial_\infty \text{Curv}(S) \). In particular,

\[
\lim_{n \to \infty} d_{\text{C}}(c_0, c_n) = \infty.
\]

**Proof.** We can consider \( \mu \) as an element of \( \mathcal{MF}(S) \). Since \( \mu \) is minimal by assumption, it represents a point in the boundary at infinity of \( \text{Curv}(S) \). Let \( \mu_t \) denote the (measured) foliation in the vertical direction on \( M_t \). By definition, we have \( \mu_t = e^t \cdot \mu \). We have

\[
e^n \cdot \iota(c_n, \mu) = \iota(c_n, \mu_n) \leq |c_n| \leq 1/K, \quad \text{for all } n \in \mathbb{N}.
\]
Thus

$$\lim_{n \to \infty} \iota(c_n, \mu) = 0.$$  

If \(\nu\) is an element of \(\mathcal{MF}(S)\) representing an accumulation point of \((c_n)\) in the space of projective measured foliations, then \(\iota(\nu, \mu) = 0\). Since \(\mu\) is minimal, by Theorem \(5.3\), \(\nu\) is topologically equivalent to \(\mu\). It follows from Theorem \(5.2\) that \(\mu\) is the limit of \((c_n)\) in \(\partial_\infty \text{Curv}(S)\), and the corollary follows.

5.1.2. Maps to the curve complex and the arc and curve complex. Let us fix a homeomorphism \(f : \hat{S} \to M\) such that \(f^{-1}(\Sigma) = V\). Via the map \(f\), we have two natural “coarse” mappings \(\Psi : C \to \text{ACurv}(S)\) and \(\Psi' : C \to \text{Curv}(S)\) defined as follows: for any \(k \in C\),

- \(\Psi(k)\) is the set of vertices of \(\text{ACurv}(S)\) representing the homotopy classes of the saddle connections and regular geodesics (cylinders) in the direction \(k\), and
- \(\Psi'(k)\) is the set of vertices of \(\text{Curv}(S)\) representing the homotopy classes of the regular geodesics in the direction \(k\).

By construction, \(\text{diam} \Psi(k) = \text{diam} \Psi'(k) = 1\) for any \(k \in C\).

**Lemma 5.6:** Let \(p, q\) be two periodic directions in \(C\) considered as vertices of \(\mathcal{G}\). Then

\[
\text{d}(p, q) \geq \frac{1}{2} \text{d}_{\text{AC}}(\Psi(p), \Psi(q)).
\]

**Proof.** Let \(\beta\) be a path of minimal length from \(p\) to \(q\) in \(\mathcal{G}\). Let \(p = k_0, k_1, \ldots, k_\ell = q\) be the elements of \(C\) that are contained in \(\beta\), where \(\text{d}(p, k_i) = i\). By construction, for each \(i\), there are an element of \(\Psi(k_i)\) and an element of \(\Psi(k_{i+1})\) which are represented by two disjoint arcs in \(S\). Therefore, there is an edge in \(\text{ACurv}(S)\) between a point in \(\Psi(k_i)\) and a point in \(\Psi(k_{i+1})\). Since \(\text{diam} \Psi(k_i) = 1\), it follows that there is a path from a point in \(\Psi(p)\) to a point in \(\Psi(q)\) of length at most \(2\ell\), from which we get inequality \((7)\).

**Proof of Proposition 5.1.** By Lemma 5.6, it is enough to show that \(\text{diam} \Psi(C) = \infty\), which is equivalent to \(\text{diam} \Psi'(C) = \infty\) because the embedding of \(\text{Curv}(S)\) into \(\text{ACurv}(S)\) is a quasi-isometry. Since we can rotate \(M\) such that the vertical foliation is minimal, this follows immediately from Corollary 5.5.
5.2. Hyperbolicity. Our goal now is to show

**Proposition 5.7:** The graph $G$ is Gromov hyperbolic.

For this purpose, we will use the following criterion by Masur–Schleimer [13].

**Theorem 5.8** (Masur–Schleimer): Suppose that $X$ is a graph with all edge lengths equal to one. Then $X$ is Gromov hyperbolic if there is a constant $R \geq 0$, and for all unordered pair of vertices $x, y$ in $X^0$, there is a connected subgraph $g_{x,y}$ containing $x$ and $y$ with the following properties:

- (Local) If $d_X(x, y) \leq 1$ then $g_{x,y}$ has diameter at most $R$,
- (Slim triangle) For any $x, y, z \in X^0$, the subgraph $g_{x,y}$ is contained in the $R$-neighborhood of $g_{x,z} \cup g_{z,y}$.

We will also need the following improvement of Proposition 4.1.

**Lemma 5.9:** There exists a constant $\kappa_0$ depending on the stratum of $(M, \Sigma)$ such that, for any pair of saddle connections $s_1$ and $s_2$ of $(M, \Sigma)$ with directions $k_1$ and $k_2$ respectively, we have

$$d(k_1, k_2) \leq \log_2 (\#(\text{int}(s_1) \cap \text{int}(s_2)) + 1) + \kappa_0.$$  

**Proof.** Assume first that $\#(\text{int}(s_1) \cap \text{int}(s_2)) = 0$, which means that $s_1$ and $s_2$ are disjoint. We can then add other saddle connections to the family $\{s_1, s_2\}$ to obtain a triangulation of $(M, \Sigma)$. Let $\kappa_0$ be the number of triangles in this triangulation. Note that this number only depends on the stratum of $(M, \Sigma)$. Now, since each triangle in this triangulation represents a vertex in $G$ that is connected to the vertices representing the directions of its three sides, we see that there is a path in $G$ from $k_1$ to $k_2$ of length at most $\kappa_0$. Thus we have

$$d(k_1, k_2) \leq \kappa_0.$$  

For the case $\#(\text{int}(s_1) \cap \text{int}(s_2)) > 0$, we use the same induction as in Proposition 4.1 to conclude.  

**Corollary 5.10:** Let $C$ be a cylinder, and $s$ a saddle connection in $(M, \Sigma)$. Let $w(C)$ denote the width of $C$ and $|s|$ the length of $s$. Then the distance in $G$ between the direction of $C$ and the direction of $s$ is at most

$$\log_2 \left( \frac{|s|}{w(C)} + 1 \right) + \kappa_0.$$  

Proof. Let $c$ be a core curve of $C$. Let $m$ be the number of intersections between $c$ and $\text{int}(s)$. Obviously, we only need to consider the case $c$ and $s$ are not parallel. Since $|s| \geq mw(C)$, we have $m \leq |s|/w(C)$. If $s'$ is a saddle connection in the boundary of $C$, then we have

$$\#(\text{int}(s), \text{int}(s')) \leq m \leq \frac{|s|}{w(C)}.$$ 

We then conclude by Lemma 5.9.  

5.2.1. Paths connecting pairs of points in $C$. In view of Theorem 5.8, to simplify the arguments, we will consider another graph, denoted by $G'$, closely related to $G$. The vertices of $G'$ are elements of $C$. Two vertices are connected by an edge if and only if they are two vertices of an ideal triangle in $\mathcal{I}$. The length of every edge is set to be one.

There is a natural map $\Xi: G' \to G$ defined as follows: $\Xi$ is identity on $\mathcal{C} \cong G'_{(0)}$. For each edge $e \in G'_{(1)}$, whose endpoints are $k_1, k_2 \in C$, $\Xi(e)$ is the union of two edges in $G$ that connect $k_1, k_2$ through a vertex representing an ideal triangle $\Delta \in \mathcal{I}$. Recall that by construction, $k_1, k_2$ are two vertices of $\Delta$.

Note that $\Delta$ may not be unique, however the number of admissible $\Delta$ is bounded by a constant depending only on the stratum of $(M, \Sigma)$. Indeed, let $T$ be an embedded triangle associated with $\Delta$ whose sides are denoted by $s_1, s_2, s_3$. We can assume that the directions of $s_1$ and $s_2$ are $k_1$ and $k_2$ respectively. Since each pair of oriented saddle connections are contained in the boundary of at most one embedded triangle, and the number of saddle connections in a given direction is determined by the stratum of $(M, \Sigma)$, the number of ideal triangles in $\mathcal{I}$ that contain $k_1$ and $k_2$ as vertices is bounded by a universal constant depending only on the stratum of $(M, \Sigma)$.

Since every element of $\mathcal{I}$ is of distance $\frac{1}{2}$ from $C$, we get

**Lemma 5.11:** For any pair $(k, k')$ of directions in $C$, the distances between $k$ and $k'$ in $G'$ and in $G$ are the same. The map $\Xi$ is therefore a quasi-isometry.

Our goal now is to show that $G'$ is Gromov hyperbolic. Lemma 5.11 then implies that $G$ is also Gromov hyperbolic. By a slight abuse of notation, we will also denote by $d$ the distance in $G'$.

Our first task is to construct for every pair $(k, k')$ of directions in $C$ a path in $G'$ connecting them. Using $\text{PSL}(2, \mathbb{R})$, we can assume that $k$ is the horizontal direction and $k'$ is the vertical direction. We can further normalize $M$ by a
matrix $a_t := \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$, $t \in \mathbb{R}$, such that the shortest horizontal saddle connection and the shortest vertical saddle connection have the same length.

For any $t \in \mathbb{R}$, let $M_t := a_t \cdot M$. If $c$ is a regular geodesic or a saddle connection on $(M, \Sigma)$, the length of $c$ on $M_t$ will be denoted by $|c|_t$. By Theorem 5.4, there is a cylinder $C_t$ on $M_t$ of width bounded below by $K$. The cylinder $C_t$ may be not unique, but we have

**Lemma 5.12:** If $C'_t$ is another cylinder of width bounded below by $K$ in $M_t$, then the distance in $G'$ between the directions of $C_t$ and $C'_t$ is at most

$$(\log_2(K^{-2} + 1) + \kappa_0).$$

**Proof.** Since $\text{Area}(M_t) = \text{Area}(M) = 1$, the circumference of $C_t$ is at most $K^{-1}$. In particular, a saddle connection $s$ in the boundary of $C_t$ has length at most $K^{-1}$. Let $k$ and $k'$ be the directions of $C_t$ and $C'_t$ respectively. Then Corollary 5.10 implies

$$d(k, k') \leq \log_2 \left( \frac{|s|_t}{w(C'_t)} + 1 \right) + \kappa_0 \leq \log_2(K^{-2} + 1) + \kappa_0.$$

In what follows, for any $t \in \mathbb{R}$, we denote by $C^0_t$ a cylinder of width bounded below by $K$ in $M_t$ and by $k(t)$ in the direction of $a_{-t}(C^0_t)$. Note that we have $k(t) \in C$.

**Lemma 5.13:**

(i) There exists $t_0 > 0$ such that if $t > t_0$, then $k(t) = 0$, and if $t < -t_0$ then $k(t) = \infty$.

(ii) For any $t_1, t_2 \in \mathbb{R}$,

$$d(k(t_1), k(t_2)) \leq \log_2(K^{-2} + 1) + \frac{|t_1 - t_2|}{\ln(2)} + \kappa_0.$$

**Proof.**

(i) If $t > 0$ is large enough, then the width of any vertical cylinder in $M_t$ is at least $1/K$. Thus a non-vertical cylinder in $M_t$ has circumference at least $1/K$, hence its width must be smaller than $K$. Thus we must have $k(t) = 0$. Similar arguments apply for $M_{-t}$.

(ii) Observe that we have, for any saddle connection or regular geodesic $c$ on $M$,

$$\frac{|c|_{t_1}}{|c|_{t_2}} \leq e^{t_1 - t_2}.$$
Since the length of a core curve of $C_{t_1}^0$ on $M_{t_1}$ is at most $K^{-1}$, its length in $M_{t_2}$ is at most $e^{|t_1-t_2|}K^{-1}$. Thus the conclusion follows from Corollary 5.10.  

Define

(9) \[ g^*(k, k') := \{k(i), \ i \in \mathbb{Z}\} \subset \mathcal{C} \simeq G''(0). \]

By Lemma 5.13, the set $g^*(k, k')$ is finite. For any $i \in \mathbb{Z}$, let $\gamma_i$ be a path of minimal length in $G'$ from $k(i)$ and $k(i + 1)$. Let

(10) \[ g(k, k') := \bigcup_{i \in \mathbb{Z}} \gamma_i \subset G'. \]

By construction, $g(k, k')$ is obviously a connected finite subgraph of $G'$. For any subset $A$ of $G'$ and any $r > 0$, let us denote by $\mathcal{N}(A, r)$ the $r$-neighborhood of $A$ in $G'$.

**Lemma 5.14:** There is a constant $R_1 > 0$, depending only on the stratum of $(M, \Sigma)$, such that

(a) $g(k, k') \subset \mathcal{N}(g^*(k, k'), R_1)$, and

(b) for any $t \in \mathbb{R}$, $k(t) \in \mathcal{N}(g^*(k, k'), R_1)$.

**Proof.** Set $R_1 = \log_2(K^{-2} + 1) + \kappa_0 + 1/\ln(2)$. From Lemma 5.13, we have $d(k(i), k(i + 1)) \leq R_1$. Thus every point in $\gamma_i$ is of distance at most $R_1/2$ from either $k(i)$ or $k(i + 1)$, from which we get (a). Again, by Lemma 5.13, any $k(t)$ is of distance at most $R_1$ from a point $k(i)$, with $i \in \mathbb{Z}$, and (b) follows.

5.2.2. Local property.

**Lemma 5.15:** There is a constant $R_2 > 0$ such that if $d(k, k') = 1$, then $\text{diam}(g(k, k')) < R_2$.

**Proof.** We can suppose that $k$ is the horizontal direction, and $k'$ is the vertical direction. By assumption, there are a horizontal saddle connection $s$ and a vertical saddle connection $s'$ that are two sides of an embedded triangle $T$ in $M = M_0$. Recall that $M$ is normalized so that the shortest horizontal saddle connection $s_0$ and the shortest vertical saddle connection $s'_0$ have the same length, say $\delta$. We first have

$\delta^2 \leq |s||s'| = 2\text{Area}(T) < 2.$

Thus $\delta < \sqrt{2}$. 

For any $t \in \mathbb{R}$, the lengths of $s_0$ and $s'_0$ in $M_t$ are respectively $e^t \delta$ and $e^{-t} \delta$. If $i < 0$, then the length of $s_0$ in $M_i$ is smaller than $\sqrt{2}$. It follows from Corollary 5.10 that $d(k, k(i)) \leq \log_2(\sqrt{2}K^{-1} + 1) + \kappa_0$. Similarly, if $i > 0$ then the length of $s'_0$ in $M_i$ is smaller than $\sqrt{2}$, thus 
\[ d(k', k(i)) \leq \log_2(\sqrt{2}K^{-1} + 1) + \kappa_0. \]
Therefore we have 
\[ g^*(k, k') \subset \mathcal{N}(\{k, k'\}, \log_2(\sqrt{2}K^{-1} + 1) + \kappa_0). \]

From Lemma 5.14 (a), we get 
\[ \text{diam}(g(k, k')) \leq R_2 \]
with 
\[ R_2 = 2(R_1 + \log_2(\sqrt{2}K^{-1} + 1) + \kappa_0) + 1. \]

5.2.3. Slim triangle property. Let $(k, k')$ be a pair of directions in $C$. We use $\text{PSL}(2, \mathbb{R})$ to transform $k$ to the horizontal direction, $k'$ to the vertical direction, and such that the shortest horizontal and vertical saddle connections have the same length.

For any $t \in \mathbb{R}$, and $R \in (0, +\infty)$, let $\mathcal{L}_t(k, k', R) \subset C$ denote the set of directions of the cylinders whose circumference in $M_t$ is at most $R$. Since $M_t$ always contains a cylinder of width bounded below by $K$ (hence its circumference is at most $K^{-1}$), for any $R > K^{-1}$, the set $\mathcal{L}_t(k, k', R)$ is non-empty. Define
\[ \hat{g}^*(k, k') = \bigcup_{t \in \mathbb{R}} \mathcal{L}_t(k, k', 2eK^{-1}) \subset \mathcal{G}'(0). \]

By construction, we have $g^*(k, k') \subset \hat{g}^*(k, k')$.

**Lemma 5.16:** The set $\hat{g}^*(k, k')$ is finite, and there exists a constant $R_3$ such that 
\[ \hat{g}^*(k, k') \subset \mathcal{N}(g^*(k, k'), R_3). \]

**Proof.** Observe that for any regular geodesic $c$ in $M$, and any $t_1, t_2 \in \mathbb{R}$, we have $|c|_{t_1}/|c|_{t_2} \leq e^{|t_1 - t_2|}$. It follows that 
\[ \hat{g}^*(k, k') \subset \bigcup_{i \in \mathbb{Z}} \mathcal{L}_i(k, k', 2eK^{-1}). \]

For $i > 0$ large enough, we have 
\[ \mathcal{L}_i(k, k', 2eK^{-1}) = \{k'\} \quad \text{and} \quad \mathcal{L}_{-i}(k, k', 2eK^{-1}) = \{k\}. \]

Since for any fixed $i$, the set of cylinders with circumference at most $2eK^{-1}$ on $M_i$ is finite, we conclude that $\hat{g}^*(k, k')$ is a finite set.
Now, by Corollary 5.10, the direction of a cylinder with circumference at most $2eK^{-1}$ on $M_i$ is of distance at most $\log_2(2eK^{-2} + 1) + \kappa_0$ from $k(i)$. Therefore

$$\hat{g}^*(k, k') \subset \mathcal{N}(g^*(k, k'), R_3)$$

with $R_3 = \log_2(2eK^{-2} + 1) + \kappa_0$. $\blacksquare$

We now show

**Lemma 5.17:** There is a constant $R_4 > 0$ such that for any triple $(k, k', k'')$ of directions in $C(M, \Sigma)$, we have

$$\hat{g}^*(k, k') \subset \mathcal{N}(\hat{g}^*(k, k'') \cup \hat{g}^*(k', k''), R_4).$$

**Proof.** We can renormalize $M$ (using $\text{PSL}(2, \mathbb{R})$) such that $(k, k', k'') = (\infty, 0, 1)$. Note that this normalization is not necessarily the normalization used to define the path $g(k, k')$ in (10). In particular, $g^*(k, k')$ does not necessarily equal the set $\{k(i), i \in \mathbb{Z}\}$. Nevertheless, we obtain the same subset $\hat{g}^*(k, k')$ by (11), that is

$$\hat{g}^*(k, k') = \bigcup_{t \in \mathbb{R}} L_t(k, k', 2K^{-1}).$$

Consider a direction $\hat{k}$ in $\hat{g}^*(k, k')$. By definition, $\hat{k}$ is the direction of a cylinder $C$ whose circumference in $M_t := a_t \cdot M$ is at most $2K^{-1}$ for some $t \in \mathbb{R}$.

**Claim:** If $t \leq 0$, then $\hat{k}$ is contained in the $(\log_2(4K^{-2} + 1) + \kappa_0)$-neighborhood of $\hat{g}^*(k, k'')$.

**Proof of the Claim.** Let $M' := U \cdot M$, where $U = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}$. Note that $U(k) = k = \infty$ and $U(k'') = 0$. By definition, $\hat{g}^*(k, k'')$ is the set of directions $\hat{k}' \in C$ such that, for some $s \in \mathbb{R}$, the circumference of a cylinder in direction $\hat{k}'$ is at most $2K^{-1}$ in $a_s \cdot M'$.

We claim that, for $s = t$, the circumference of $C$ in $a_t \cdot M'$ is at most $4K^{-1}$. To see this, we observe that

$$M'_t := a_t \cdot M' = (a_t \cdot U \cdot a_{-t}) \cdot M_t.$$

Recall that the circumference of $C$ in $M_t$ is at most $2K^{-1}$. Since

$$a_t \cdot U \cdot a_{-t} = \begin{pmatrix} 1 & -e^{2t} \\ 0 & 1 \end{pmatrix},$$

and $t \leq 0$, it follows that the circumference of $C$ in $M'_t$ is at most $4K^{-1}$. 

Let $D^0_t$ be a cylinder of width bounded below by $K$ in $M'_t$ (whose existence is guaranteed by Theorem 5.4). By definition, the direction of $D^0_t$ belongs to $\hat{g}^*(k,k'')$. From Corollary 5.10, it follows that the distance between the directions of $C$ and $D^0_t$ is at most $\log_2(4K^{-2} + 1) + \kappa_0$. The claim is then proved.

It follows immediately from the claim that $L_t(k,k',2K^{-1})$ is contained in the $R_4$-neighborhood of $\hat{g}^*(k,k'')$ if $t \leq 0$, with

$$R_4 = \log_2(4K^{-2} + 1) + \kappa_0.$$  

By similar arguments, one can also show that $L_t(k,k',2K^{-1})$ is contained in the $R_4$-neighborhood of $\hat{g}^*(k',k'')$ if $t \geq 0$. The lemma is then proved.

**Corollary 5.18:** Let $R_5 = R_1 + R_3 + R_4$, where $R_1, R_3, R_4$ are the constants of Lemmas 5.14, 5.16, 5.17 respectively. Then for any triple $(k,k',k'')$ of directions in $C$, we have

$$g(k,k') \subset N(g(k,k'') \cup g(k',k''), R_5).$$

**Proof.** It follows from Lemma 5.17 that we have

$$\hat{g}^*(k,k') \subset N(\hat{g}^*(k,k'') \cup \hat{g}^*(k',k''), R_4).$$

Since $g^*(k,k') \subset \hat{g}^*(k,k')$, Lemma 5.16 implies

$$g^*(k,k') \subset N(g^*(k,k'') \cup g^*(k',k''), R_3 + R_4) \subset N(g(k,k'') \cup g(k',k''), R_3 + R_4).$$

Finally, from Lemma 5.14, we get

$$g(k,k') \subset N(g(k,k'') \cup g(k',k''), R_1 + R_3 + R_4).$$

**Proof of Proposition 5.7.** By Theorem 5.8, Lemma 5.15 and Corollary 5.18 imply that $G'$ is Gromov hyperbolic. Since $G'$ and $G$ are quasi-isometric (cf. Lemma 5.11), this shows that $G$ is Gromov hyperbolic.

**Proof of Theorem 1.9.** The first part of Theorem 1.9 follows from Propositions 4.1, 5.1 and 5.7. By Lemma 4.3, we know that $\Gamma$ acts freely on the set of edges of $G$.

Assume now that $\Gamma$ is a lattice in $\text{PSL}(2,\mathbb{R})$; then $\overline{G}$ is a finite graph by Proposition 4.4. Conversely, if $\overline{G}$ is a finite graph, then in particular $\overline{I}$ is a finite set. Thus, $\Gamma$ has finite covolume by (1), which means that $(M, \Sigma)$ is a Veech surface.
Appendix A. Quotient of the periodic direction graph and generating sets of the Veech group

Throughout this section, we will suppose that \((M, \Sigma)\) is a half-translation Veech surface and that \(\text{Area}(M) = 1\). Our goal is to provide an algorithm to determine the graph \(\overline{G} = G/\Gamma\). In particular, we get a representative families of \(I/\Gamma\), hence an upper bound on the volume of the Teichmüller curve \(\mathbb{H}/\Gamma\) by (3). As a by product, we obtain an algorithm to find a generating set of the Veech group \(\Gamma\). To lighten the notation, we will omit \((M, \Sigma)\) from the notation of the objects constructed from the pair \((M, \Sigma)\).

A.1. Reference domain for a periodic direction. Assume that \((M, \Sigma)\) is horizontally periodic. We then say that \((M, \Sigma)\) is normalized if the shortest horizontal saddle connection of \(M\) has length equal to 1.

Let \(k\) be a periodic direction in \(\mathbb{C}\). There is an element \(A \in \text{PSL}(2, \mathbb{R})\), determined up to the left action of \(\{U_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, t \in \mathbb{R}\}\), such that \(A(k) = \infty\), and \((M', \Sigma') := A \cdot (M, \Sigma)\) is normalized. Let \(\Gamma'\) denote the Veech group of \((M', \Sigma')\). Note that we have \(\Gamma' = A \cdot \Gamma \cdot A^{-1}\).

There exists \(a \in \mathbb{R}_{>0}\) such that the stabilizer \(\text{Stab}_{\Gamma'}(\infty)\) of \(\infty\) in \(\Gamma'\) equals \(\{(1 \ 0 \ a \ 0)\}\). We will call \(a\) the period of the direction \(k\). Note that \(a\) stays unchanged if we replace \(A\) by \(U_t \cdot A\).

Let \(I^*(M', \Sigma', \infty)\) denote the set of hyperbolic ideal triangles \(\Delta \in I(M', \Sigma')\) such that

- \(\infty\) is a vertex of \(\Delta\), and
- \(\Delta\) intersects the vertical strip \((0, a) \times \mathbb{R}_+\).

**Lemma A.1:** Let \(\kappa\) denote the length of the longest horizontal saddle connection in \((M', \Sigma')\). Let \(\Delta\) be an ideal triangle in \(I^*(M', \Sigma', \infty)\), and \(T \in \mathbb{T}(M', \Sigma')\) an embedded triangle which gives rise to \(\Delta\). Denote the sides of \(T\) by \(s_0, s_1, s_2\), where \(s_0\) is a horizontal saddle connection. Then

\[
\min\{|s_1|, |s_2|\} \leq \max\{2\sqrt{1 + a^2}, \sqrt{4 + \kappa^2/4}\}.
\]

**Proof.** For \(i = 1, 2\), let \(x_i + y_i \in \mathbb{C}\) be the period of \(s_i\), and \(k_i = \frac{x_i}{y_i}\). We can always assume that \(y_i > 0\) and \(k_1 < k_2\). Note that we have \(y_1 = y_2\). Since \(T\) is an embedded triangle, \(\text{Area}(T) = \frac{1}{2}y_1|s_0| < 1\). As \((M', \Sigma')\) is normalized, \(|s_0| \geq 1\), hence \(y_1 = y_2 < 2\).
By definition, $[k_1, k_2]$ intersects the interval $(0, a)$. We have two cases:

- If $(0, a) \not\subset [k_1, k_2]$, then at least one of the following holds: $k_1 \in (0, a)$ or $k_2 \in (0, a)$. Assume that $k_1 \in (0, a)$, then $0 < x_1 < ay_1 < 2a$. It follows that $|s_1| < 2\sqrt{1 + a^2}$. By the same argument, if $k_2 \in (0, a)$ then $|s_2| < 2\sqrt{1 + a^2}$.

- If $(0, a) \subset [k_1, k_2]$ then $k_1 \leq 0 < a \leq k_2$. Note that in this case $|s_0| = x_2 - x_1$. Since $x_1 \leq 0 \leq x_2$, it follows that $\min\{-x_1, x_2\} \leq \frac{|s_0|}{2} \leq \frac{\kappa}{2}$. Since $0 < y_1 = y_2 < 2$, we have $\min\{|s_1|, |s_2|\} \leq \sqrt{4 + \kappa^2/4}$.

**Corollary A.2:** The set $\mathcal{I}^*(M', \Sigma', \infty)$ is finite.

**Proof.** We remark that an embedded triangle is uniquely determined by two of its oriented sides (the sides of a triangle are naturally endowed with the induced orientation). Therefore the number of triangles in $\mathcal{I}^*(M', \Sigma', \infty)$ is bounded by the number of pairs $(s, s')$ of oriented saddle connections, where $s$ is horizontal, and $s'$ is non-horizontal with length at most $\max\{2\sqrt{1 + a^2}, \sqrt{4 + \kappa^2/4}\}$. Since the set of saddle connections of length bounded by a constant is finite, it follows that the set $\mathcal{I}^*(M', \Sigma', \infty)$ is finite.

**Remark A.3:** Lemma A.1 provides us with a criterion for the search of ideal triangles in $\mathcal{I}^*(M', \Sigma', \infty)$, namely, we only need to look for embedded triangles bounded by a horizontal saddle connection, and a non-horizontal saddle connection of length at most $\max\{2\sqrt{1 + a^2}, \sqrt{4 + \kappa^2/4}\}$.

Let $\mathcal{D}^*(M', \Sigma', \infty)$ denote the union of the ideal triangles in $\mathcal{I}^*(M', \Sigma', \infty)$.

**Lemma A.4:**

(a) The domain $\mathcal{D}^*(M', \Sigma', \infty)$ is connected.

(b) For any hyperbolic ideal triangle $\Delta$ in $\mathcal{I}(M', \Sigma')$ that has $\infty$ as a vertex, the $\text{Stab}_{\Gamma'}(\infty)$-orbit of $\Delta$ intersects the set $\mathcal{I}^*(M', \Sigma', \infty)$.

**Proof.** (a) To show that $\mathcal{D}^*(M', \Sigma', \infty)$ is connected, it suffices to show that its projection $J$ to the real axis is connected, which means that $J$ is an interval. By definition, the projection of any triangle in $\mathcal{I}^*(M', \Sigma', \infty)$ to the real axis is an interval that intersects $(0, a)$. Therefore, it is enough to show that $(0, a) \subset J$.

Let $k_0$ be any direction in $(0, a)$. Consider the surface $U_{-k_0} \cdot (M', \Sigma')$, where

$$U_{-k_0} = \begin{pmatrix} 1 & -k_0 \\ 0 & 1 \end{pmatrix}.$$
Note that the action of $U_{-k_0}$ on $\mathbb{R} \subset \mathbb{RP}^1$ is the translation by $-k_0$. Let $s$ be (one of) the longest horizontal saddle connection of $U_{-k_0} \cdot (M', \Sigma')$. This saddle connection is contained in the bottom border of a horizontal cylinder, say $C$. There is a singularity in the top border of $C$ such that the downward vertical ray emanating from this singularity hits $s$ before exiting $C$. Thus there is an embedded triangle in $C$ that contains $s$ as a side and the vertical segment above. Let $s_1, s_2$ be the other sides of this triangle, and $k_1, k_2$ be the directions of $s_1$ and $s_2$ respectively. We can assume that $k_1 \leq 0 \leq k_2$. Applying $U_{k_0} = \begin{pmatrix} 1 & k_0 \\ 0 & 1 \end{pmatrix}$, we get an embedded triangle in $(M', \Sigma')$ which corresponds to the hyperbolic ideal triangle $\Delta$ with vertices $\infty, k_1 + k_0, k_2 + k_0$. Since $k_0 \in [k_1 + k_0, k_2 + k_0] \cap (0, a)$, we have $\Delta \in \mathcal{I}^*(M', \Sigma', \infty)$. Thus $k_0 \in [k_1, k_2] \subset J$, and we have $(0, a) \subset J$ as desired.

(b) Let $T \in \mathcal{T}(M', \Sigma')$ be the embedded triangle corresponding to an ideal triangle $\Delta$ which has $\infty$ as a vertex. Let $s_0, s_1, s_2$ denote the sides of $T$, and $k_i \in \mathbb{R} \cup \{\infty\}$ the slope of $s_i$. We can suppose that $k_0 = \infty$ (that is $s_0$ is a horizontal saddle connection), and that $k_1 < k_2$. Since the action of $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ on $\mathbb{R}$ is given by $x \mapsto x + a$, there exists $U \in \text{Stab}_{\Gamma'}(\infty)$ such that $U(k_1) \in [0, a)$, which implies that $U(\Delta) \in \mathcal{I}^*(M', \Sigma', \infty)$. \hfill \qed

We will call $D(k) := A^{-1}(D^*(M', \Sigma', \infty))$ a reference domain for the direction $k$. It follows from Lemma A.4 that $D(k)$ is a polygon in $\mathbb{H}$ with geodesic boundary, which is not necessarily convex. Set $\mathcal{I}^*(k) = A^{-1}(\mathcal{I}^*(M', \Sigma', \infty))$. By definition, $\mathcal{I}^*(k)$ is the set of ideal triangles in $\mathcal{T}(M, \Sigma)$ that compose $D(k)$. Let $N(k)$ denote the set

$$N(k) = \{k' \in C, k' \neq k, k' \text{ is a vertex of some triangle in } \mathcal{I}^*(k)\}.$$ 

In other words, $N(k)$ is the set of vertices of $D(k)$ in $\partial \mathbb{H} \setminus \{k\}$. The following lemma is a reformulation of Lemma A.4.

**Lemma A.5:** Let $\text{Stab}_\Gamma(k)$ denote the stabilizer of $k$ in $\Gamma$. We regard elements of $C$ and $\mathcal{I}$ as vertices of $G$.

(i) For any $k' \in N(k)$, $d(k, k') = 1$.

(ii) If $\Delta \in \mathcal{I}$ such that $d(k, \Delta) = \frac{1}{2}$, then the domain $D(k)$ contains an ideal triangle in the $\text{Stab}_\Gamma(k)$-orbit of $\Delta$.

(iii) If $k' \in C$ such that $d(k, k') = 1$, then $N(k)$ intersects the $\text{Stab}_\Gamma(k)$-orbit of $k'$. 

A.2. Algorithm A: Enumerating Elements of $\overline{C}$ and $\overline{I}$. We first describe an algorithm to enumerate elements of $\overline{C}$ and $\overline{I}$. By Theorem 1.7, this algorithm provides us with a bound for Area$(H/\Gamma)$. In what follows two elements of $C$ (resp. $L, I$) are said to be equivalent if they belong to the same $\Gamma$-orbit.

Remark A.6: To determine if two periodic directions $k$ and $k'$ are equivalent, one can proceed as follows: choose two matrices $A, A'$ such that $A(k) = A'(k') = \infty$, and the surfaces $A \cdot (M, \Sigma)$ and $A' \cdot (M, \Sigma)$ are normalized. Then $k$ and $k'$ are equivalent if and only if, up to the action of $\{U_t, t \in \mathbb{R}\}$ and Dehn twists in the horizontal cylinders, $A \cdot (M, \Sigma)$ and $A' \cdot (M, \Sigma)$ are represented by the same polygon in the plane.

Initialization: Using the action of $\text{PSL}(2, \mathbb{R})$, we can assume that $M$ is horizontally periodic and normalized. Set $C_0^0 = \{\infty\}$. Let $C_0^1$ be a subset of $N(\infty)$ that satisfies

(a) no element of $C_0^1$ is equivalent to $\infty$,
(b) every element of $N(\infty)$ is equivalent to an element of $C_0^1$ or $\infty$,
(c) no pair of elements of $C_0^1$ are equivalent.

The algorithm consists of exploring the graph $G$ from the vertex representing $\infty$ until we get a representative for every element of $C/\Gamma$.

Iteration: Suppose now that we have two finite subsets $C_n^0$ and $C_n^1$ of $C$ satisfying the following:

(1) $C_n^0$ and $C_n^1$ are disjoint,
(2) no pair of directions in $C_n^0 \sqcup C_n^1$ are equivalent,

Assume that $C_n^1 \neq \emptyset$. Set

$$\hat{N}_{n+1} = \bigcup_{k \in C_n^1} N(k) \subset C.$$

Pick a subset $\hat{N}_{n+1}'$ of $\hat{N}_{n+1}$ such that

(a) no element of $\hat{N}_{n+1}'$ is equivalent to an element of $C_n^0 \sqcup C_n^1$,
(b) every element of $\hat{N}_{n+1}$ is either equivalent to an element of $C_n^0 \sqcup C_n^1$, or to a unique element of $\hat{N}_{n+1}'$.

We now set

$$C_{n+1}^0 = C_n^0 \sqcup C_n^1, \quad C_{n+1}^1 := \hat{N}_{n+1}'.$$
Note that we have
\[ C_{n+1}^0 = \{\infty\} \sqcup C_0^1 \sqcup \cdots \sqcup C_n^1. \]
The algorithm stops when \( C_n^1 = \emptyset \).

Consider now the graph \( \mathcal{G} \). By definition, \( \mathcal{G} \) has two types of vertices. Let us denote by \( \mathcal{V} \) the set of vertices of \( \mathcal{G} \) representing the \( \Gamma \)-orbits in \( \mathcal{C} \), and by \( \mathcal{W} \) the set of vertices representing the \( \Gamma \)-orbits in \( \mathcal{I} \). Recall that by construction, every edge of \( \mathcal{G} \) connects a vertex in \( \mathcal{V} \) and a vertex in \( \mathcal{W} \). We denote by \( \mathbf{d} \) the distance in \( \mathcal{G} \). Recall that Proposition 4.1 and Proposition 4.4 imply that \( \mathcal{G} \) is a finite connected graph.

Let \( v_\infty \) denote the vertex of \( \mathcal{G} \) representing the \( \Gamma \)-orbit of \( \infty \).

**Lemma A.7:** Let \( v \) be a vertex in \( \mathcal{V} \). Then \( \mathbf{d}(v_\infty, v) = n \) if and only if \( v \) represents the \( \Gamma \)-orbit of a direction in \( C_{n-1}^1 \).

**Proof.** It is clear from the construction that if \( v \) represents a direction in \( C_0^1 \) then \( \mathbf{d}(v_\infty, v) = 1 \). Conversely, if \( \mathbf{d}(v_\infty, v) = 1 \) then \( v \) represents the \( \Gamma \)-orbit of a direction \( k \in \mathcal{C} \) such that \( \mathbf{d}(\infty, k) = 1 \). By Lemma A.5, \( k \) is equivalent to an element of \( \mathcal{N}(\infty) \). Since \( k \) is not equivalent to \( \infty \), we can choose \( k \) to be an element of \( C_0^1 \).

Assume now that the lemma is true for \( n \leq \ell \), and that \( \mathbf{d}(v_\infty, v) = \ell + 1 \). There exists \( v' \in \mathcal{V} \) such that \( \mathbf{d}(v_\infty, v') = \ell \) and \( \mathbf{d}(v', v) = 1 \). By assumption, \( v' \) represents the \( \Gamma \)-orbit of a direction in \( C_{\ell-1}^1 \). Therefore, \( v \) represents the \( \Gamma \)-orbit of a direction \( k \in \mathcal{N}_\ell \). Note that \( k \) cannot be equivalent to a direction in \( C_{\ell-1}^0 \sqcup C_{\ell-1}^1 \), since otherwise we would have \( \mathbf{d}(v_\infty, v) \leq \ell - 1 \) by the induction hypothesis. Thus \( k \) must be equivalent to a direction in \( \hat{\mathcal{N}}_\ell = C_\ell^1 \). The lemma is then proved.

As a direct consequence of Lemma A.7, we get

**Proposition A.8:** Let \( d_1 \) be the maximal distance in \( \mathcal{G} \) from \( v_\infty \) to another vertex in \( \mathcal{V} \). Then the algorithm stops after \( d_1 \) iterations.

**Proof.** It follows from Lemma A.7 that \( C_{d_1}^1 = \emptyset \), thus the algorithm stops after \( d_1 \) iterations.

Proposition A.8 means that the number of iterations that have been performed when the algorithm stops equals the largest distance from \( v_\infty \) to another vertex in \( \mathcal{V} \subset \mathcal{G} \). By construction, we have a bijection between \( C_{d_1}^0 \) and \( \mathcal{V} \). Thus
this algorithm allows us to get the complete list of elements of \( \mathcal{V} \). Recall that for any \( k \in \mathcal{C} \), \( \mathcal{I}^*(k) \) is the set of ideal triangles that form a reference domain \( \mathcal{D}(k) \) of \( k \). Set

\[
\mathcal{I}^* := \bigcup_{k \in \mathcal{C}_0} \mathcal{I}^*(k).
\]

It follows from Lemma A.4 that every \( \Gamma \)-orbit in \( \mathcal{I} \) has at least a representative in \( \mathcal{I}^* \). Therefore, we can extract from \( \mathcal{I}^* \) a subset \( \mathcal{I}_0^* \) which is in bijection with \( \mathcal{I} \cong \mathcal{W} \). By Theorem 1.7, the cardinality of \( \mathcal{I}_0^* \) provides us with a bound for \( \text{Area}(\mathbb{H}/\Gamma) \). Note also that the domain \( \mathcal{D}_0^* := \bigcup_{k \in \mathcal{I}_0^*} \mathcal{D}(k) \) has finite area and contains a fundamental domain for the action of \( \Gamma \) on \( \mathbb{H} \). Therefore, we can view \( \mathcal{D}_0^* \) as a “coarse” fundamental domain of \( \Gamma \).

A.3. Algorithm B: Finding a Generating Set of \( \Gamma \). We now present an algorithm to obtain a generating set of \( \Gamma \). In the literature, generating sets of a lattice in \( \mathrm{PSL}(2, \mathbb{R}) \) are often obtained from a fundamental domain of the lattice. In this algorithm, we obtain a generating set of \( \Gamma \) without constructing explicitly a fundamental domain. In what follows we will use the same notation as in Section A.2.

Initialization: Let \( g_\infty \) be a generator of the stabilizer of \( \infty \) in \( \Gamma \). Let \( \mathcal{I}^*(\infty) \) be the set of ideal triangles in \( \mathcal{I} \) which compose a reference domain \( \mathcal{D}(\infty) \) of \( \infty \). We set

\[
\mathcal{C}_0 := \{\infty\}, \quad J_0 := \mathcal{I}^*(\infty), \quad F_0 := \{g_\infty\}.
\]

Iteration: Assume now that we have a finite subset \( \mathcal{C}_n \) of \( \mathcal{C} \), and to every element of \( \mathcal{C}_n \) we have an associated finite subset \( \mathcal{I}^*(k) \) of \( \mathcal{I} \), and an element \( g_k \in \Gamma \) satisfying the following: for any \( k \in \mathcal{C}_n \)

(i) the elements of \( \mathcal{I}^*(k) \) represent the ideal triangles which compose a reference domain for \( k \) (in particular, \( k \) is a vertex of every ideal triangle in \( \mathcal{I}^*(k) \)),

(ii) if \( k \) is equivalent to \( \infty \) then \( g_k(\infty) = k \) and \( \mathcal{I}^*(k) = g_k(\mathcal{I}^*(\infty)) \), otherwise \( g_k \) is a generator of \( \text{Stab}_\Gamma(k) \).

We set

\[
J_n = \bigcup_{k \in \mathcal{C}_n} \mathcal{I}^*(k) \subset \mathcal{I} \quad \text{and} \quad F_n = \{g_k, \ k \in \mathcal{C}_n\} \subset \Gamma.
\]

Define \( \mathcal{C}_{n+1} \) to be the subset of \( \mathcal{C} \) consisting of the vertices of the ideal triangles in \( J_n \). Note that \( \mathcal{C}_n \) is a subset of \( \mathcal{C}_{n+1} \). We will associate to each \( k \in \mathcal{C}_{n+1} \) a
finite subset $\mathcal{I}^*(k)$ of $\mathcal{I}$, and an element $g_k$ of $\Gamma$ as follows: if $k \in C_n$, we keep the same $\mathcal{I}^*(k)$ and $g_k$ provided by the previous step. Let $k$ be an element of $C_{n+1} \setminus C_n$. By definition, $k$ is a vertex of a triangle $\Delta \in J_n$. We have two cases:

- Case 1: $k$ is equivalent to $\infty$. In this case, there is an element $g \in \Gamma$ such that $g(\infty) = k$ and $g^{-1}(\Delta) \in \mathcal{I}^*(\infty)$. We define $\mathcal{I}^*(k) = g(\mathcal{I}^*(\infty))$, and $g_k = g$.
- Case 2: $k$ is not equivalent to $\infty$. In this case, we choose a reference domain $D(k)$ such that $\Delta$ is one of the ideal triangles that make up $D(k)$. We then define $\mathcal{I}^*(k)$ to be the family of triangles that compose $D(k)$, and $g_k$ a generator of $\text{Stab}_\Gamma(k)$.

Clearly, $C_{n+1}$ and the mappings $k \mapsto \mathcal{I}^*(k)$ and $k \mapsto g_k$ satisfy the conditions (i) and (ii) above.

**Lemma A.9:** For any $n \in \mathbb{N}$ and any $k \in C_n$, we have:

(a) The subgroup generated by $F_n$ contains the stabilizer of $k$ in $\Gamma$.

(b) As subsets of $\mathcal{G}$, $C_{n+1}$ is contained in the 1-neighborhood of $C_n$.

(c) If $k \in C_{n+1}$, then the distance from $\infty$ to $k$ in $\mathcal{G}$ is at most $n$.

**Proof.** For (a), we only need to consider the case $k$ is equivalent to $\infty$. But in this case, $g_k \cdot g_{\infty} \cdot g_k^{-1}$ is a generator of $\text{Stab}_\Gamma(k)$. For (b), observe that $J_n$ is contained in the $\frac{1}{2}$-neighborhood of $C_n$, and $C_{n+1}$ is contained in the $\frac{1}{2}$-neighborhood of $J_n$. Finally, (c) is an immediate consequence of (b). $\blacksquare$

Let $\Gamma_n$ denote the subgroup of $\Gamma$ that is generated by the elements of $F_n$.

**Lemma A.10:** Let $A$ be an element of $\Gamma$ such that $d(\infty, A(\infty)) \leq n$, where $d$ is the distance on the graph $\mathcal{G}$. Then $A \in \Gamma_n$.

**Proof.** Let $\alpha$ be a path of minimal length from $\infty$ to $A(\infty)$ in $\mathcal{G}$. Let $m = \text{leng}(\alpha) \leq n$. Then $\alpha$ must contain $m + 1$ vertices in $C$. Let us label those vertices by $k_0, k_1, \ldots, k_m$, where $k_0 = \infty, k_m = A(\infty)$ and $d(\infty, k_i) = i$.

For any $k \in C_n$, define

$$N(k) = \{k' \in C, k' \neq k, k' \text{ is a vertex of some triangle in } \mathcal{I}^*(k)\}.$$  

Since $d(k_0, k_1) = 1$, by Lemma A.5, there is an element $B_0 \in \text{Stab}_\Gamma(k_0)$ such that $k_1' := B_0(k_1) \in N(k_0)$. Note that $k'_1 \in C_1$ and $B_0 \in \Gamma_0$. 

Let $k'_2 := B_0(k_2)$. Since $d(k'_1, k'_2) = d(k_1, k_2) = 1$, there is an element $B_1 \in \text{Stab}_\Gamma(k'_1)$ such that $k''_2 := B_1(k'_2) = B_1 \circ B_0(k_2) \in \mathcal{N}(k'_1)$. In particular, we have $k''_2 \in C_2$, and $B_1 \in \Gamma_1$ by Lemma A.9.

By induction, we can find a sequence $(B_0, B_1, \ldots, B_{m-1})$ of elements of $\Gamma$ such that $B_i \in \Gamma_i$, and $B_{m-1} \circ \cdots \circ B_0(k_m) = k_m^{(m)} \in C_m$. Since $k_m$ is equivalent to $\infty$, by construction, there is an element $B_m \in F_m$ such that $B_m(k_m^{(m)}) = \infty$. Hence

$$B_m \circ B_{m-1} \circ \cdots \circ B_0 \circ A(\infty) = \infty,$$

which means that there exists $B \in \text{Stab}_\Gamma(\infty) = \Gamma_0$ such that

$$A = B_0^{-1} \circ \cdots \circ B_m^{-1} \circ B \in \Gamma_m. \quad \Box$$

Let $v_\infty$ be the vertex of $\overline{G}$ that represents the $\Gamma$-orbit of $\infty$ in $C$. Recall that we have defined $d_1$ to be the maximal distance in $\overline{G}$ from $v_\infty$ to another vertex in $\mathcal{V}$ (that is the set of $\Gamma$-orbits in $C$). Note that $d_1$ can be computed by Algorithm A (cf. Proposition A.8).

**Proposition A.11:** We have $\Gamma_{2d_1+1} = \Gamma$.

**Proof.** Let $A$ be an element of $\Gamma$. Let $m := d(\infty, A(\infty))$. We will prove that $A \in \Gamma_{2d_1+1}$ by induction on $m$. For $m \leq 2d_1 + 1$, this follows from Lemma A.10. Thus let us suppose that $m > 2d_1 + 1$, and that the statement is true for any $A$ such that $d(\infty, A(\infty)) < m$.

Let $\alpha$ be any path of minimal length in $G$ from $\infty$ to $A(\infty)$. This path contains $m + 1$ vertices in $C$ that are labeled by $k_0, \ldots, k_m$, where $k_0 = \infty, k_m = A(\infty)$, and $d(k_0, k_i) = i$. Consider the vertex $k_{m-d_1-1}$. Since the vertex of $\overline{G}$ that represents the $\Gamma$-orbit of $k_{m-d_1-1}$ is of distance at most $d_1$ from $v_\infty$, there is a vertex $k \in C$ in the $\Gamma$-orbit of $\infty$ such that $d(k_{m-d_1-1}, k) \leq d_1$. Consequently, $d(\infty, k) \leq (m - d_1 - 1) + d_1 = m - 1$, and $d(k, k_m) \leq d_1 + d_1 + 1 = 2d_1 + 1$.

By assumption, there is an element $A' \in \Gamma$ such that $A'(\infty) = k$. By the induction hypothesis, $A' \in \Gamma_{2d_1+1}$. Consider $k' := A'^{-1}(A(\infty))$. Now, since

$$d(\infty, k') = d(k, A(\infty)) \leq 2d_1 + 1$$

the matrix $A'^{-1} \cdot A$ belongs to $\Gamma_{2d_1+1}$ by Lemma A.10. Thus $A \in \Gamma_{2d_1+1}$, and the proposition is proved. \[ \Box \]

Proposition A.11 implies that we obtain a generating set for the Veech group of $(M, \Sigma)$ after $2d_1 + 1$ iterations of Algorithm B.
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