A Probabilistic proof of the breakdown of Besov regularity in L-shaped domains

Victoria Knopova and René L. Schilling

Abstract We provide a probabilistic approach in order to investigate the smoothness of the solution to the Poisson and Dirichlet problems in L-shaped domains. In particular, we obtain (probabilistic) integral representations (9), (12)–(14) for the solution. We also recover Grisvard’s classic result on the angle-dependent breakdown of the regularity of the solution measured in a Besov scale.

Key Words. Brownian Motion; Dirichlet Problem; Poisson Equation; Conformal Mapping; Stochastic Representation; Besov Regularity.

MSC 2010. 60J65; 35C15; 35J05; 35J25; 46E35.

1 Introduction

Let us consider the (homogeneous) Dirichlet problem

\[ \Delta f = 0 \quad \text{in} \ G, \]
\[ f|_{\partial G} = h \quad \text{on} \ \partial G, \]

where \( G \subset \mathbb{R}^d \) is a domain with Lipschitz boundary \( \partial G \) and \( \Delta \) denotes the Laplace operator, i.e. \( \Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} \). In order to show that there exists a solution to (1) which belongs to some subspace of \( L_p(G) \), say, to the Besov space \( B^{\sigma}_{pp}(G) \), \( \sigma > 0 \), it is necessary that \( h \) is an element of the trace space of \( B^{\sigma}_{pp}(G) \) on \( \partial G \); it is well known that the trace space is given by \( B_{pp}^{\sigma - 1/p}(\partial G) \), see Jerison & Kenig [11, Theorem 3.1],

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a more general version can be found in Jonsson & Wallin [12, Chapter VII], and for domains with $C^\infty$-boundary a good reference is Triebel [18, Sections 3.3.3–4]. The smoothness of the solution $f$, expressed by the parameter $\sigma$ in $B^\sigma_{pp}(G)$, is, however, not only determined by the smoothness of $h$, but also by the geometry of $G$. It seems that Grisvard [10] is the first author to quantify this in the case when $G$ is a non-convex polygon. Subsequently, partly due to its relevance in scientific computing, this problem attracted a lot of attention; for instance, it was studied by Jerison & Kenig [11], by Dahlke & DeVore [7] in connection with wavelet representations of Besov functions, by Mitrea & Mitrea [13] and Mitrea, Mitrea & Yan [14] in Hölder spaces, to mention but a few references.

In this note we use a probabilistic approach to the problem and we obtain a probabilistic interpretation in the special case when $G$ is an $L$-shaped domain of the form $L := \mathbb{R}^2 \setminus \{(x, y) : x, y \geq 0\}$, see Figure 1, and in an $L_2$-setting. This is the

![Fig. 1 The $L$-shaped model domain $L \subset \mathbb{R}^2$.](image-url)

model problem for all non-convex domains with an obtuse interior angle. In this case the Besov space $B^\sigma_{22}(L)$ coincides with the Sobolev–Slobodetskij space $W^\sigma_2(L)$. In particular, we

• give a probabilistic interpretation of the solution to (1) with $G = L$;

• provide a different proof of the fact that the critical order of smoothness of $f$ is $\sigma < \pi/3\pi = \frac{2}{3}$, i.e. even for $h \in C^0(\partial L)$ we may have

$$f \in W^{1,2^\sigma}_{2,\text{loc}}(L), \quad \sigma < \frac{2}{3}, \quad \text{and} \quad f \notin W^{1+\sigma}_{2}(L), \quad \sigma \geq \frac{2}{3}; \quad (2)$$

• apply the “breakdown of regularity” result to the Poisson (or inhomogeneous Dirichlet) problem.

It is clear that this result holds in a more general setting, if we replace the obtuse angle $3\pi/2$ by some $\theta \in (\pi, 2\pi)$.

Results of this type were proved for polygons and in a Hölder space setting by Mitrea & Mitrea [13]. Technically, our proof is close (but different) to that given in [13]—yet our starring idea is different. Dahlke & DeVore [7] proved this regularity result analytically using a wavelet basis for $L_p$-Besov spaces.
Problem (1) is closely related to the Poisson (or nonhomogeneous Dirichlet) problem
\[
\Delta F = g \quad \text{on } G,
\]
\[
F|_{\partial G} = 0 \quad \text{on } \partial G.
\]
(3)

If \( G \) is bounded and has a \( C^\infty \)-boundary, the problems (1) and (3) are equivalent. Indeed, in this case for every right-hand side \( g \in L_2(G) \) of (3) there exists a unique solution \( F \in W^{3/2}_2(G) \), see Triebel [18, Theorem 4.3.3]. Denote by \( N \) the Newtonian potential on \( \mathbb{R}^d \) and define \( w := g * N \); clearly, \( \Delta w = g \) on \( G \) and \( w \in W^{3/2}_2(G) \). Since the boundary is smooth, there is a continuous linear trace operator \( \text{Tr} : W^{3/2}_2(G) \to W^{3/2}_2(\partial G) \) as well as a continuous linear extension operator \( \text{Ex} : W^{3/2}_2(\partial G) \to W^{3/2}_2(G) \), such that \( \text{Tr} \circ \text{Ex} = \text{id} \), cf. Triebel [18]. Hence, the function \( f := w - F \) solves the inhomogeneous Dirichlet problem (1) with \( h = \text{Tr} w \) on \( \partial G \). On the other hand, let \( f \) be the (unique) solution to (1). Since there exists a continuous linear extension operator from \( W^{3/2}_2(\partial G) \) to \( W^{3/2}_2(G) \) given by \( \tilde{h} = \text{Ex} h \), we see that the function \( F := f - \tilde{h} \) satisfies (3) with \( g = \Delta \tilde{h} \).

If the boundary \( \partial G \) is Lipschitz the situation is different. It is known, see for example Jerison & Kenig [11, Theorem B]) that, in general, on a Lipschitz domain \( G \) and for \( g \in L_2(G) \) one can only expect that the solution \( F \) to (3) belongs to \( W^{3/2}_2(G) \); there are counterexamples of domains, for which \( F \) cannot be in \( W^{3/2}_2(G) \) for any \( \alpha > 3/2 \). Thus, the above procedure does not work in a straightforward way. However, by our strategy we can recover the negative result for this concrete domain, cf. Theorem 2: If \( g \in H_1(\mathbb{R}^2) \cap W^{1}_2(\mathbb{L}) \), then the solution \( F \) to (3) is not in \( W^{1+\sigma}_2(\mathbb{L}) \) for any \( \sigma \geq 2/3 \). Here \( H_1(\mathbb{R}^2) \subset L_1(\mathbb{R}^2) \) is the Hardy space, cf. Stein [17].

If \( G \) is unbounded, the solution to (1) might be not unique and, in general, it is only in the local space \( W^{2,\text{loc}}_2(G) \) even if \( \partial G \) is smooth, cf. Gilbarg & Trudinger [9, Chapter 8]. On the other hand, if the complement \( G^c \) is non-empty, if no component of \( G \) reduces to a single point, and if the boundary value \( h \) is bounded and continuous on \( \partial G \), then there exists a unique bounded solution to (1) given by the convolution with the Poisson kernel, see Port & Stone [16, Theorem IV.2.13].

A strong motivation for this type of results comes from numerical analysis and approximation theory, because the exact Besov smoothness of \( u \) is very important for computing \( u \) and the feasibility of adaptive computational schemes, see Dahlke & DeVore [7], Dahlke, Dahmen & DeVore [6], DeVore [8], Cohen, Dahmen & DeVore [5], Cohen [4]; an application to SPDEs is in Cioicka et. al. [2, 3]. More precisely—using the set-up and the notation of [5]—let \( \{ \psi_\lambda, \lambda \in \Lambda \} \) be a basis of wavelets on \( G \) and assume that the index set \( \Lambda \) is of the form \( \Lambda = \bigcup_{i \geq 0} \Lambda_i \) with (usually hierarchical) sets \( \Lambda_i \) of cardinality \( N_i \). By \( u_{\Lambda_i} \) we denote the Galerkin approximation of \( u \) in terms of the wavelets \( \{ \psi_\lambda \}_{\lambda \in \Lambda_i} \) (this amounts to solving a system of linear equations), and by \( e_{\Lambda_i}(u) := \| u - u_{\Lambda_i} \|_p \) the approximation error in this scheme. Then it is known, cf. [5, (4.2) and (2.35)], that
\[
u \in W^{\sigma \ell}_p(G) \implies e_{\Lambda_i}(u) \leq C N_i^{-\sigma/d}, \quad i \geq 1.
\]
(4)
There is also an adaptive algorithm for choosing the index sets \((\Lambda_t)_{t \geq 1}\). Starting with an initial set \(\Lambda_0\), this algorithm adaptively generates a sequence of nested sets \((\Lambda_t)_{t \geq 1}\): roughly speaking, in each iteration step we choose the next set \(\Lambda_{t+1}\) by partitioning the domain of those wavelets \(\psi_\lambda, \lambda \in \Lambda_t\) (i.e. selectively refining the approximation by considering the next generation of wavelets), whose coefficients \(u_\lambda\) make, in an appropriate sense, the largest contribution to the sum \(u = \sum_{\lambda \in \Lambda_t} u_\lambda \psi_\lambda\).

**Notation.** Most of our notation is standard. By \((r, \theta) \in (0, \infty) \times (0, 2\pi]\) we denote polar coordinates in \(\mathbb{R}^2\), and \(\mathbb{H}\) is the lower half-plane in \(\mathbb{R}^2\). We write \(f \approx g\) to say that \(cf(t) \leq g(t) \leq Cf(t)\) for all \(t\) and some fixed constants.

### 2 Setting and the main result

Let \(B = (B_t^x)_{t \geq 0}\) be a Brownian motion started at a point \(x \in G\). Suppose that there exists a conformal mapping \(\varphi : G \to \mathbb{H}\), where \(\mathbb{H} := \{(x_1, x_2) \in \mathbb{R}^2, x_2 \leq 0\}\) is the lower half-plane in \(\mathbb{R}^2\). Using the conformal invariance of Brownian motion, see e.g. Mörters & Peres [15, p. 202], we can describe the distribution of the Brownian motion inside \(G\) in terms of some Brownian motion \(W\) in \(\mathbb{H}\), which is much easier to handle. Conformal invariance of Brownian motion means that there exists a planar Brownian motion \(W = (W_t^y)_{t \geq 0}\) with starting point \(y \in \mathbb{H}\) such that, under the conformal map \(\varphi : G \to \mathbb{H}\) with boundary identification,

\[
(\varphi(B_t^x))_{0 \leq t \leq \tau_G} \quad \text{has the same law as} \quad (W_{\varphi(x)}^\xi(s))_{0 \leq s \leq \tau_{\mathbb{H}}}, \tag{5}
\]

the time-change \(\xi\) is given by \(\xi(t) := \int_0^t |\varphi'(B_s^x)|^2 \, ds\); in particular, \(\xi(\tau_G) = \tau_{\mathbb{H}}\), where \(\tau_G = \inf\{t > 0 : B_t^x \in \partial G\}\) and \(\tau_{\mathbb{H}} := \inf\{t > 0 : W_t^{\varphi(x)} \in \partial \mathbb{H}\}\) are the first exit times from \(G\) and \(\mathbb{H}\), respectively.

Let us recall some properties of a planar Brownian motion in \(\mathbb{H}\) killed upon exiting at the boundary \(\partial \mathbb{H} = \{(w_1, w_2) : w_2 = 0\}\). The distribution of the exit position \(W_{\tau_{\mathbb{H}}}^w\) has the transition probability density

\[
u \mapsto p_{\tau_{\mathbb{H}}}^w(\nu, u) = \frac{1}{\pi} \frac{|w_2|}{|\nu - w_1|^2 + w_2^2}, \quad \nu = (w_1, w_2) \in \mathbb{H}, \tag{6}
\]

cf. Bass [1, p. 91]. Recall that a random variable \(X\) with values in \(\mathbb{R}\) has a Cauchy distribution, \(X \sim C(m, b), m \in \mathbb{R}, b > 0\), if it has a transition probability density of the form

\[
p(u) = \frac{b}{\pi (u - m)^2 + b^2}, \quad u \in \mathbb{R};
\]

if \(X \sim C(m, b)\), then \(Z := (X - m)/b \sim C(0, 1)\). Thus, the probabilistic interpretation of \(W_{\tau_{\mathbb{H}}}^w\) is
Breakdown of Besov regularity in \(L\)-shaped domains

\[
W_{\tau G}^w \sim Z^w \sim C(w_1, |w_2|) \quad \text{or} \quad W_{\tau G}^w \sim \frac{Z - w_1}{|w_2|} \quad \text{where} \quad Z \sim C(0,1). \quad (7)
\]

This observation allows us to simplify the calculation of functionals \(\Theta\) of a Brownian motion \(B\) on \(G\), killed upon exiting from \(G\), in the following sense:

\[
E \Theta(B_{\tau G}^x) = E (\Theta \circ \varphi^{-1}) (\varphi(B_{\tau G}^x)) = E (\Theta \circ \varphi^{-1}) (W_{\tau G}^{\varphi(x)})
\]

\[
= E (\Theta \circ \varphi^{-1}) \left( \frac{Z - \varphi_1(x)}{|\varphi_2(x)|} \right). \quad (8)
\]

In particular, the formula (8) provides us with a probabilistic representation for the solution \(f\) to the Dirichlet problem (1):

\[
f(x) = E h(B_{\tau G}^x) = E (h \circ \varphi^{-1}) (W_{\tau G}^{\varphi(x)}). \quad (9)
\]

**Remark 1.** The formulae in (8) are very helpful for the numerical calculation of the values \(E \Theta(B_{\tau G}^x)\). In fact, in order to simulate \(\Theta(B_{\tau G}^x)\), it is enough to simulate the Cauchy distribution \(Z \sim C(0,1)\) and then evaluate (8) using the Monte Carlo method.

We will now consider the \(L\)-shaped domain \(\mathbb{L}\). It is easy to see that the conformal mapping of \(\mathbb{L}\) to \(\mathbb{H}\) is given by

\[
\varphi(z) = e^{i \frac{2\pi}{3} z^{2/3}} = r^{2/3} \exp \left( \frac{2i}{3} (\theta + \pi) \right) = \varphi_1(r, \theta) + i \varphi_2(r, \theta), \quad (10)
\]

cf. Figure 2, where \(\theta = \arg z \in (0, 2\pi]\).

**Fig. 2** Conformal mapping from \(\mathbb{L}\) to \(\mathbb{H}\) and its behaviour at the boundaries.

The following lemma uses the conformal mapping \(\varphi: \mathbb{L} \to \mathbb{H}\) and the conformal invariance of Brownian motion to obtain the distribution of \(B_{\tau G}^{\varphi(x)}\).

**Lemma 1.** Let \(\mathbb{L}\) be an \(L\)-shaped domain as shown in Fig. 1. The exit position \(B_{\tau G}^x\) of Brownian motion from \(\mathbb{L}\) is a random variable on \(\partial \mathbb{L} = \{0\} \times [0, \infty) \cup [0, \infty) \times \{0\}\) which has the following probability distribution:
Theorem 1. Consider the (homogeneous) Dirichlet problem (1) with a boundary term \( f_0 \), given by (17), and let \( f \) denote the solution to (1).

a. If \( f_0 \in W_2^2(\mathbb{R}) \cap W_2^2(\mathbb{R}) \) satisfies

\[
\liminf_{\varepsilon \to 0} \int_{|x|>\varepsilon} f_0(x) \frac{dx}{x} \neq 0,
\]

then \( f \notin W_2^{1+\sigma}(\mathbb{L}) \), even \( f \notin W_2^{1+\sigma}_{\text{loc}}(\mathbb{L}) \), for any \( \sigma \geq 2/3 \).
b. If \( f_0 \in W^2_p(\mathbb{R}) \cap W^1_p(\mathbb{R}) \), where \( p > \max\{2, \frac{2}{2-3\sigma}\} \), then \( f \in W^{1+\sigma}_{2,\text{loc}}(\mathbb{L}) \) for all \( \sigma \in (0, 2/3) \).

**Remark 2.** By the Sobolev embedding theorem we have \( W^2_p(\mathbb{R}) \cap W^1_p(\mathbb{R}) \subset C_b(\mathbb{R}) \) and \( W^2_p(\mathbb{R}) \cap W^1_p(\mathbb{R}) \subset C_b(\mathbb{R}) \) if \( p > \max\{2, \frac{2}{2-3\sigma}\} \). Hence, the function \( f \) given by (14) is the unique bounded solution to (1).

The idea of the proof of Theorem 1 makes essential use of the results by Jerison & Kenig [11] combined with the observation that it is, in fact, enough to show the claim for \( \mathbb{L} := \mathbb{L} \cap B(0,1) \), where \( B(0,1) := \{ x \in \mathbb{R}^2 : |x| < 1 \} \).

Theorem 1 allows us to prove the negative result for the solution to the Poisson problem, which improves [11, Theorem B]. Recall that \( H_1(\mathbb{R}^2) \subset L_1(\mathbb{R}^2) \) is the usual Hardy space, cf. Stein [17].

**Theorem 2.** Consider the Poisson (inhomogeneous Dirichlet) problem (3) with right-hand side \( g \in H_1(\mathbb{R}^2) \cap W^1_2(\mathbb{L}) \) such that \( f_0(x) := ((\text{Tr}g * N) \circ \varphi^{-1})(x) \) satisfies (17), where \( N(x) := (2\pi)^{-1} \log|x| \) is the Newton kernel. Then the solution \( F \notin W^{1+\sigma}_{2,\text{loc}}(\mathbb{L}) \), even \( F \notin W^{1+\sigma}_{2,\text{loc}}(\mathbb{L}) \), for any \( \sigma \geq 2/3 \).

The proofs of Theorem 1 and 2 are deferred to the next section.

### 3 Proofs

**Proof (Proof of Lemma 1)**. We calculate the characteristic function of \( B^x_\mathbb{L} \). As before, let \( y = (y_1, y_2) \), \( x = (x_1, x_2) \) and \( \varphi(x) = (\varphi_1(x), \varphi_2(x)) \). We have

\[
\mathbb{E}e^{i\xi \cdot B^x_\mathbb{L}} = \mathbb{E}e^{i\xi \cdot \varphi^{-1}(W^\varphi_{\mathbb{R}^2}^{(x)})}
\]

\[
= \int_{\mathbb{R}^2} e^{i\xi \varphi^{-1}(y_1,0)} \|W^\varphi_{\mathbb{R}^2}^{(x)} \| \, dy
\]

\[
= \frac{1}{\pi} \int_{\mathbb{R}} e^{i\xi \varphi^{-1}(y_1,0)} \left| \frac{\varphi_2(x)}{\varphi_1(x) - y_1^2 + |\varphi_2(x)|^2} \right| \, dy_1
\]

\[
= \frac{1}{\pi} \int_{-\infty}^{0} e^{-i\xi u} \left| \frac{\varphi_2(x)}{\varphi_1(x) - u^2 + |\varphi_2(x)|^2} \right| \, du
\]

\[
+ \frac{1}{\pi} \int_{0}^{\infty} e^{i\xi u} \left| \frac{\varphi_2(x)}{\varphi_1(x) - u^2 + |\varphi_2(x)|^2} \right| \, du. \quad \square
\]

For the proof of Theorem 1 we need some preparations. In order to keep the presentation self-contained, we quote the classical result by Jerison & Kenig [11, Theorem 4.1].

**Theorem 3 (Jerison & Kenig).** Let \( \sigma \in (0, 1) \), \( k \in \mathbb{N}_0 \) and \( p \in [1, \infty] \). For any function \( u \) which is harmonic on a bounded domain \( \Omega \), the following assertions are equivalent:

a. \( f \in B^{k+\sigma}_{pp}(\Omega) \):
b. \( \text{dist}(x, \partial \Omega)^{1-\sigma} \mid \nabla^k f \mid + \mid \nabla^k f \mid + |f| \in L_p(\Omega) \).

We will also need the following technical lemma. Recall that \( \hat{\mathbb{L}} = \mathbb{L} \cap B(0, 1) \).

**Lemma 2.** Suppose that \( f_0 \in W^p_2(\mathbb{R}) \) for some \( p > 2 \). Then \( f \in W^1_2(\hat{\mathbb{L}}) \).

**Proof.** Using the representation (16), the H"older inequality and a change of variables, we get

\[
\frac{3}{2\pi^2} \int_{\pi/2}^{\pi/2} \int_0^1 \frac{1}{|f(r, \theta)|^2} r \, dr \, d\theta = \frac{3}{2\pi^2} \int_{\pi/2}^{\pi/2} \int_0^1 \rho^2 \left[ \left( \int_{\mathbb{R}} |f_0(w)|^p \, dw \right)^{1/p} \left( \int_{\mathbb{R}} \left( \frac{1}{\rho^2 + |\rho \sin \Phi_0|^2} \right)^q \, dw \right)^{1/q} \right] \, d\rho \, d\theta
\]

\[
\leq C_1 \int_{\pi/2}^{\pi/2} \int_0^1 \rho^2 \left[ \left( \int_{\mathbb{R}} |f_0(v)|^p \, dv \right)^{1/p} \left( \int_{\mathbb{R}} \frac{1}{v^2 + 1} \, dv \right)^{2/q} \right] \, d\rho \, d\theta
\]

\[
\leq C_2 \int_{\pi/2}^{\pi/2} \int_0^1 \rho^2 |\rho \sin \Phi_0|^{-2+2/q} \left( \int_{\mathbb{R}} \frac{1}{v^2 + 1} \, dv \right)^{2/q} \, d\rho \, d\theta
\]

\[
= C_3 \int_{\pi/2}^{\pi/2} \int_0^1 \rho^2 |\rho \sin \Phi_0|^{-2+2/q} \, d\rho \, d\theta,
\]

where \( p^{-1} + q^{-1} = 1 \). Because of \( p > 2 \) we have \(-2+2/q > -1 \), hence \( q < 2 \). Note that the inequalities \( 2x/\pi \leq \sin x \leq x \) for \( x \in [0, \pi/2] \), imply

\[
\int_{\pi/2}^{\pi/2} |\sin \Phi_0|^{-1+\epsilon} \, d\theta = \int_0^\pi |\sin \varphi|^{-1+\epsilon} \, d\varphi = 2 \int_0^{\pi/2} |\sin \varphi|^{-1+\epsilon} \, d\varphi < \infty.
\]

This shows that \( f \in L_2(\hat{\mathbb{L}}) \).

Recall that the partial derivatives of the polar coordinates are

\[
\frac{\partial}{\partial x_1} r = \cos \theta, \quad \frac{\partial}{\partial x_1} \theta = -\frac{\sin \theta}{r}, \quad \frac{\partial}{\partial x_1} \Phi_0 = \frac{2}{3} \frac{\partial}{\partial x_1} \theta = -\frac{2 \sin \theta}{3r}.
\]

(18)

Therefore, we have for \( \theta \in (\pi/2, 2\pi) \)

\[
\frac{\partial}{\partial x_1} f(r, \theta) = \frac{1}{\pi} \int_{\mathbb{R}} f'(r^{1/3} \cos \Phi_0 - vr^{2/3} \sin \Phi_0) \frac{1}{v^2 + 1} \times
\]

\[
\times \left[ \frac{2\cos \theta}{3r^{1/3}} \left( \cos \Phi_0 - v \sin \Phi_0 \right) \right.
\]

\[
+ \frac{r^{2/3}}{3} \left( -\frac{2 \sin \theta}{3r} \right) \left( -v \cos \Phi_0 - \sin \Phi_0 \right) \, dv
\]

\[
= \frac{2}{3\pi r^{1/3}} \int_{\mathbb{R}} f'(r^{1/3} \cos \Phi_0 - vr^{2/3} \sin \Phi_0) \frac{1}{v^2 + 1} \times
\]

\[
\times \left[ (\cos \Phi_0 - v \sin \Phi_0) \cos \theta + (v \cos \Phi_0 + \sin \Phi_0) \sin \theta \right] \, dv
\]

(19)
where
\[ K(\theta, v) := \cos \omega_\theta - v \sin \omega_\theta, \quad (20) \]
and
\[ \omega_\theta = \frac{1}{3} (2\pi - \theta). \quad (21) \]

Note that \( \Phi_{\pi/2} = \pi \) and \( \omega_{\pi/2} = \pi/2. \)

Let us show that the first partial derivatives of \( f \) belong to \( L_2(\hat{\mathbb{L}}). \) Because of the symmetry of \( \hat{\mathbb{L}}, \) it is enough to check this for \( \frac{\partial}{\partial x_1} f. \)

Using the estimate \( |K(\theta, v)| (1 + v^2)^{-1} \leq C(1 + |v|)^{-1}, \) a change of variables and the Hölder inequality, we get

\[
\int_0^1 \int_{\pi/2}^{2\pi} \left| \frac{\partial}{\partial x_1} f(r, \theta) \right|^2 r \, d\theta \, dr
= \int_0^1 \int_{\pi/2}^{2\pi} \left| \frac{2}{3\pi r^{1/3}} f_0 \left( r^{2/3} \cos \Phi_\theta - vr^{2/3} \sin \Phi_\theta \right) \frac{K(\theta, v)}{1 + v^2} \right|^2 r \, d\theta \, dr
= \frac{2}{3\pi^2} \int_0^1 \int_{\pi/2}^{2\pi} \rho \left| \int_{\mathbb{R}} f_0(r \cos \Phi_\theta - v \rho \sin \Phi_\theta) \frac{K(\theta, v)}{1 + v^2} \, dw \right|^2 d\theta \, d\rho
\leq C_1 \int_0^1 \int_{\pi/2}^{2\pi} \rho \left( \int_{\mathbb{R}} \left| \frac{f_0'(w)}{\rho \sin \Phi_\theta + |w - \rho \cos \Phi_\theta|} \right| \, dw \right)^2 d\theta \, d\rho
\leq C_2 \left( \int_{\mathbb{R}} |f_0'(w)|^p \, dw \right)^{2/p} \left( \int_{\mathbb{R}} \left| \frac{1}{1 + |w|} \right|^q \, dw \right)^{2/q} \times
\int_{\pi/2}^{2\pi} \int_0^1 \rho \left( |\sin \Phi_\theta|^{-1+1/q} \right)^2 d\rho \, d\theta
= C_3 \int_{\pi/2}^{2\pi} \int_0^1 |\sin \Phi_\theta|^{-2+2/q} \rho^{-1+2/q} d\rho \, d\theta < \infty;
\]
in the last line we use again that \(-2 + 2/q > -1.\)

**Proof (Proof of Theorem 1).** It is enough to consider the set \( \hat{\mathbb{L}}. \) We verify that condition b of Theorem 3 holds true. We check whether

\[
\text{dist}(0,.)^{1-\sigma} \left| \frac{\partial^2}{\partial x_1^2} f \right| + \left| \frac{\partial}{\partial x_1} f \right| + |f| \quad \text{is in } L_2(\hat{\mathbb{L}}) \text{ or not.}
\]

From Lemma 2 we already know that \( \left| \frac{\partial}{\partial x_1} f \right| + |f| \in L_2(\hat{\mathbb{L}}). \) Let us check when

\[
\text{dist}(0,.)^{1-\sigma} \left| \frac{\partial^2}{\partial x_1^2} f \right| \in L_2(\hat{\mathbb{L}}).
\]
We will only work out the term $\frac{\partial^2}{\partial x_1^2} f(r, \theta)$ since the calculations for $\frac{\partial^2}{\partial x_2^2} f(r, \theta)$ are similar. We have

$$\frac{\partial}{\partial x_1} K(\theta, v) = \frac{\sin \theta}{3r} (-v \cos \omega_0 - \sin \omega_0) =: \frac{\sin \theta}{3r} K^*(\theta, v),$$

where use that $\frac{\partial}{\partial x_1} \omega_0 = -\frac{1}{3} \frac{\partial}{\partial x_1} \theta = \frac{\sin \theta}{3r}$ and set

$$K^*(\theta, v) := -v \cos \omega_0 - \sin \omega_0. \quad (22)$$

Therefore, differentiating $\frac{\partial}{\partial x_1} f$—we use the representation (19)—with respect to $x_1$ gives

$$\frac{\partial^2}{\partial x_1^2} f(r, \theta) = -\frac{2 \cos \theta}{9 \pi r^{5/3}} \int_B f'_0 \left( r^{2/3} (\cos \Phi \theta - v \sin \Phi \theta) \right) \frac{K(\theta, v)}{1 + v^2} \, dv$$

$$+ \frac{4}{9 \pi r^{5/3}} \int_B f'_0 \left( r^{2/3} (\cos \Phi \theta - v \sin \Phi \theta) \right) \frac{K^2(\theta, v)}{1 + v^2} \, dv$$

$$+ \frac{2 \sin \theta}{9 \pi r^{5/3}} \int_B f'_0 \left( r^{2/3} (\cos \Phi \theta - v \sin \Phi \theta) \right) \frac{K^*(\theta, v)}{1 + v^2} \, dv.$$

Note that

$$\int_{\mathbb{L}} \text{dist}(x, \partial \mathbb{L})^{2-2\alpha} \left| \frac{\partial^2}{\partial x_1^2} f(x) \right|^2 \, dx$$

$$= \int_{0}^{1} \int_{\pi/2}^{2\pi} \text{dist}((r, \theta), \partial \mathbb{L})^{2-2\alpha} \left| \frac{\partial^2}{\partial x_1^2} f(r, \theta) \right|^2 r \, d\theta \, dr \quad (23)$$

Since only the values near the boundary $\Gamma := \partial \mathbb{L} \cap \partial \mathbb{L}$ determine the convergence of the integral, it is enough to check that

$$I = \int_{0}^{1} \int_{\pi/2}^{2\pi} \text{dist}((r, \theta), \Gamma)^{2-2\alpha} \left| \frac{\partial^2}{\partial x_1^2} f(r, \theta) \right|^2 r \, d\theta \, dr \quad (24)$$

is infinite if $\sigma \geq 2/3$ and finite if $\sigma < 2/3$.

We split $\mathbb{L}$ into three parts. For $\delta > 0$ small enough we define, see Figure 3,

$$K_1 := \{(r, \theta) : 0 < r < 1, \frac{\pi}{2} + \delta < \theta < 2\pi - \delta\},$$

$$K_2 := \{(r, \theta) : 0 < r < 1, \frac{\pi}{2} \leq \theta < \frac{\pi}{2} + \delta\},$$

$$K_3 := \{(r, \theta) : 0 < r < 1, 2\pi - \delta < \theta \leq 2\pi\}.$$

Splitting the integral accordingly, we get

$$I = \left( \int_{K_1} + \int_{K_2} + \int_{K_3} \right) \text{dist}((r, \theta), \Gamma)^{2-2\alpha} \left| \frac{\partial^2}{\partial x_1^2} f(r, \theta) \right|^2 r \, d\theta \, dr;$$
in order to show that $I$ is infinite if $\sigma \geq 2/3$, it is enough to see that the integral over $K_1$ is infinite. Noting that in $K_1$ we have $\text{dist}((r, \theta), \Gamma) \approx r$, we get

\[
\begin{align*}
\int_{K_1} \left| r^{1-\sigma} \frac{\partial^2 f(r, \theta)}{\partial r^2} \right|^2 r \, d\theta \, dr &= \int_{K_1} r \left| r^{1-\sigma} \frac{2}{9\pi r^{4/3}} \right|^2 \times \left| \int_{R} f_0' \left( r^{2/3} (\cos \Phi - v \sin \Phi) \right) \frac{K^*(\theta,v) \sin \theta - K(\theta,v) \cos \theta}{1+v^2} \, dv \right|^2 \times \int_{R} f_0'' \left( r^{2/3} (\cos \Phi - v \sin \Phi) \right) \frac{K^*(\theta,v) \sin \theta - K(\theta,v) \cos \theta}{1+v^2} \, dv \\
&= \frac{4}{81\pi^2} \int_{K_1} r^{1/3-2\sigma} \left| \int_{R} f_0' \left( r^{2/3} (\cos \Phi - v \sin \Phi) \right) \frac{K^*(\theta,v) \sin \theta - K(\theta,v) \cos \theta}{1+v^2} \, dv \right|^2 \, dr \, d\theta \\
&= \frac{4}{81\pi^2} \int_{K_1} r^{1/3-2\sigma} \left| J(r^{2/3}, \theta) + I(r^{2/3}, \theta) \right|^2 \, dr \, d\theta \\
&= \frac{2}{27\pi^2} \int_{K_1} \rho^{1-3\sigma} \left| J(\rho, \theta) + I(\rho, \theta) \right|^2 \, d\rho \, d\theta,
\end{align*}
\]

where we use the following shorthand notation
\[
K^{**}(\theta,v) := K^*(\theta,v) \sin \theta - K(\theta,v) \cos \theta = -v \sin(\theta - \omega_0) - \cos(\theta - \omega_0),
\]
\[
J(\rho, \theta) := \int_{R} f_0' \left( \rho (\cos \Phi - v \sin \Phi) \right) \frac{K^{**}(\theta,v)}{1+v^2} \, dv,
\]
\[
I(\rho, \theta) := 2\rho \int_{R} f_0'' \left( \rho (\cos \Phi - v \sin \Phi) \right) \frac{K^2(\theta,v)}{1+v^2} \, dv.
\]

Observe that $\theta - \omega_0 \in (0, 2\pi)$ for $\theta \in (\frac{\pi}{2}, 2\pi)$, and $\theta - \omega_0 \in (\frac{4\pi}{3}, 2\pi - \frac{4\pi}{3})$ whenever $\theta \in (\frac{\pi}{2} + \delta, 2\pi - \delta)$.
Without loss of generality we may assume that \( J(\rho, \theta) + I(\rho, \theta) \neq 0 \) on \( K_1 \). Let us show that \( \lim_{\rho \to 0} |J(\rho, \theta) + I(\rho, \theta)| = C(f_0, \theta) > 0 \). This guarantees that we can choose some \( K_{11} \subset K_1 \) such that

\[
|J(\rho, \theta) + I(\rho, \theta)| \geq C(f_0) > 0 \quad \text{on } K_{11}.
\]

Using the change of variables \( x = \rho \theta \) we get, using dominated convergence,

\[
I(\rho, \theta) = 2 \int_{\mathbb{R}} f_0''(\rho \cos \Phi_\theta - x \sin \Phi_\theta) \frac{(\rho \cos \omega_\theta - x \sin \omega_\theta)^2}{\rho^2 + x^2} \, dx
\]

\[
\xrightarrow{\rho \to 0} 2 \sin^2 \omega_\theta \int_{\mathbb{R}} f_0''(-x \sin \Phi_\theta) \, dx = \frac{2 \sin^2 \omega_\theta}{\sin \Phi_\theta} \int_{\mathbb{R}} f_0''(x) \, dx = 0,
\]

since we assume that \( f_0 \in W^1_2(\mathbb{R}) \).

For \( J(\rho, \theta) \) we have, using the same change of variables,

\[
J(\rho, \theta) = - \int_{\mathbb{R}} f_0'(\rho \cos \Phi_\theta - \rho \sin \Phi_\theta) \frac{\cos(\theta - \omega_\theta) + \rho \sin(\theta - \omega_\theta)}{1 + \rho^2} \, dv
\]

\[
= - \int_{\mathbb{R}} f_0'(\rho \cos \Phi_\theta - x \sin \Phi_\theta) \frac{\rho \cos(\theta - \omega_\theta) + x \sin(\theta - \omega_\theta)}{\rho^2 + x^2} \, dx
\]

\[
= - \left( \int_{|x| > \varepsilon} + \int_{|x| \leq \varepsilon} \right) \cdots \, dx.
\]

The first integral can be treated with the dominated convergence theorem because we have \( f_0' \in L_1(\mathbb{R}) \) and \( \rho(\rho^2 + x^2)^{-1} \leq x^{-2}, x(\rho^2 + x^2)^{-1} \leq x^{-1} \) are bounded for \( |x| > \varepsilon \). Therefore,

\[
\lim_{\rho \to 0} \left[ - \int_{|x| > \varepsilon} \cdots \, dx \right] = - \sin(\theta - \omega_\theta) \int_{|x| > \varepsilon} \frac{f_0'(-x \sin \Phi_\theta)}{x} \, dx.
\]

Now we estimate the two parts of the second integral. For

\[
- \int_{|x| \leq \varepsilon} f_0'(\rho \cos \Phi_\theta - x \sin \Phi_\theta) \frac{\rho \cos(\theta - \omega_\theta)}{\rho^2 + x^2} \, dx
\]

we have \( \rho(\rho^2 + x^2)^{-1} \leq \varepsilon^{-1} \rho x(\rho^2 + x^2)^{-1} \leq \varepsilon^{-1} \), so this term tends to 0 by the dominated convergence theorem. For the second term in this integral we have using a change of variables and the Cauchy–Schwarz inequality,

\[
|\sin(\theta - \omega_\theta)| \cdot \left| \int_{|x| \leq \varepsilon} f_0'(\rho \cos \Phi_\theta - x \sin \Phi_\theta) \frac{x}{x^2 + \rho^2} \, dx \right|
\]

\[
\leq \left| \int_{|w| \leq \varepsilon} \left( f_0'(\rho \cos \Phi_\theta - w \sin \Phi_\theta) - f_0'(\rho \cos \Phi_\theta) \right) \frac{w}{\rho^2 + w^2} \, dw \right|
\]

\[
\leq \int_{|w| \leq \varepsilon} \int_{0}^{1} \left| f_0''(\rho \cos \Phi_\theta - rw \sin \Phi_\theta) \right| \, dr \, dw
\]
Altogether we have upon letting $\rho \to 0$ and then $\varepsilon \to 0$, that
\[
\lim_{\rho \to 0} I(\rho, \theta) = 0,
\]
(27)
\[
\liminf_{\rho \to 0} \liminf_{\varepsilon \to 0} J(\rho, \theta) = \sin(\omega_\theta - \theta) \liminf_{\varepsilon \to 0} \int_{|x|>\varepsilon} \frac{f_0'(x)}{x} \, dx.
\]
(28)
If the “lim inf” diverges, it is clear that (26) holds, if it converges but is still not equal to 0, we can choose $K_{11}$ in such a way that $\sin(\omega_\theta - \theta) \neq 0$. Thus, the integral over $K_1$ blows up as $\int_0^1 \rho^{1-3\sigma} \, d\rho = \infty$ for any $\sigma \geq 2/3$.

To show the convergence result, we have to estimate $I$ and $J$ from above. Write
\[
J(\rho, \theta) = -\int_\mathbb{R} f_0'(\rho (\cos \Phi_\theta - v \sin \Phi_\theta)) \frac{v \sin(\theta - \omega_\theta)}{1 + v^2} \, dv
- \int_\mathbb{R} f_0'(\rho (\cos \Phi_\theta - v \sin \Phi_\theta)) \frac{\cos(\theta - \omega_\theta)}{1 + v^2} \, dv =: J_1(\rho, \theta) + J_2(\rho, \theta).
\]
Since $f_0 \in W^1_p(\mathbb{R})$, using the H"older inequality and a change of variables give
\[
|J_1(\rho, \theta)| \leq \left( \int_\mathbb{R} |f_0'(\rho (\cos \Phi_\theta - v \sin \Phi_\theta))|^p \, dv \right)^{\frac{1}{p}} \left( \int_\mathbb{R} \left( \frac{v}{1 + v^2} \right)^q \, dv \right)^{\frac{1}{q}}
\]
(29)
\[
\leq c |\rho \sin \Phi_\theta|^{-1/p}
\]
for all $\theta \in [\pi/2, 2\pi]$ and $\rho > 0$. An even simpler calculation yields
\[
|J_2(\rho, \theta)| \leq c |\rho \sin \Phi_\theta|^{-1/p}
\]
(30)
for all $\theta \in [\pi/2, 2\pi]$ and $\rho > 0$. Now we estimate $I(\rho, \theta)$. Note that for every $\theta \in [\pi/2, 2\pi]$ we have $K^2(\theta, v)/(1 + v^2) \leq C$. By a change of variables we get
\[
|I(\rho, \theta)| \leq \frac{C_1}{|\sin \Phi_\theta|} \int_\mathbb{R} |f_0''(w + \rho \cos \Phi_\theta)| \, dw \leq \frac{C_2}{|\sin \Phi_\theta|}
\]
(31)
for all $\theta \in [\pi/2, 2\pi]$ and $\rho > 0$. Note that for $\Phi_\theta \in [\pi/2 + \delta, 2\pi - \delta]$ it holds that $|\sin \Phi_\theta| > 0$. Thus, on $K_1$ we have
\[
|I(\rho, \theta) + J(\rho, \theta)| \leq C \rho^{-1/p}, \quad \theta \in [\pi/2 + \delta, 2\pi - \delta], \quad \rho > 0,
\]
(32)
implying
\[
\int_{K_1} \left| \frac{r^{1-\sigma}}{\rho \sin \theta} \frac{\partial^2}{\partial x_1^2} f(r, \theta) \right|^2 \, r \, d\theta \, dr \leq C \int_0^1 \rho^{1-3\sigma-2/p} \, d\rho.
\]
The last integral converges if $\sigma \in (0, 2/3)$ and $p > \frac{2}{3-\sigma}$.

In order to complete the proof of the convergence part, let us show that the integrals over $K_2$ and $K_3$ are convergent for all $\sigma \in (0, 1)$.

In the regions $K_2$ and $K_3$ we have $\text{dist}((r, \theta), \Gamma) \leq r|\cos \theta|$ and $\text{dist}((r, \theta), \Gamma) \leq r|\sin \theta|$, respectively. We will discuss only $K_2$ since $K_3$ can be treated in a similar way. We need to show that

$$
\int_{K_2} \left| r \cos \theta \right|^{1-\sigma} \frac{\partial^2}{\partial x_1^2} f(r, \theta) \right| r \ d\theta < \infty \quad \text{for all } \sigma \in (0, 1). \quad (33)
$$

From (29), (30) and (31) we derive that for all $(\rho, \theta) \in \hat{\Gamma}$

$$
|J(\rho, \theta) + I(\rho, \theta)| \leq C\rho^{-\frac{1}{2}} \left( |\sin \Phi_0|^{-1} + |\sin \Phi_0|^{-\frac{2}{3}} \right) \leq C'\rho^{-\frac{1}{2}} |\sin \Phi_0|^{-1}. \quad (34)
$$

Now we can use a calculation similar to (25) for $K_1$ to show that (33) is finite and, therefore, it is enough to show that

$$
\int_{\frac{\pi}{2}}^{\frac{\pi}{2} + \delta} \left( \frac{|\cos \theta|^{1-\sigma}}{\sin \Phi_0} \right)^2 \ d\theta < \infty. \quad (35)
$$

Observe that $\lim_{\theta \to \frac{\pi}{2}} \cos \frac{1}{3}(\pi + \theta)/\cos \theta = \frac{1}{3}$, implying

$$
\frac{|\cos \theta|^{1-\sigma}}{\sin \Phi_0} = \frac{|\cos \theta|^{1-\sigma}}{2\sin \frac{1}{3}(\pi + \theta) \cos \frac{1}{3}(\pi + \theta)} \sim |\cos \theta|^{-\sigma} \quad \text{as } \theta \to \frac{\pi}{2}.
$$

Therefore, it is sufficient to note that for any $\sigma \in (0, 1)$

$$
\int_{\frac{\pi}{2}}^{\frac{\pi}{2} + \delta} |\cos \theta|^{-2\sigma} \ d\theta \asymp \int_{0}^{1} \frac{dx}{(1-x^2)^\sigma} = \int_{0}^{1} \frac{dx}{(1-x)\sigma(1+x)^\sigma} < \infty.
$$

Summing up, we have shown that

$$
\text{dist}(0, \cdot)^{1-\sigma} \left. \frac{\partial^2}{\partial x_1^2} f \right| \in L_2(\hat{\Gamma}) \quad \text{resp. } \notin L_2(\hat{\Gamma}),
$$

according to $\sigma \in (0, 2/3)$ or $\sigma \in [2/3, 1)$.

**Proof (Proof of Theorem 2).** Let $F$ be the solution to (3) on $\mathbb{L}$ with source function $g$, and define $w = g * N$ for the Newtonian potential $N$ on $\mathbb{R}^2$. As we have already mentioned in the introduction, $f := w - F$ is the solution to (1) on $\mathbb{L}$ with the boundary condition $h := \text{Tr} w$ on $\partial L$. Note that under the condition $g \in H_1(\mathbb{R}^2) \cap W^2_2(\mathbb{L})$ we have $\Delta w = g$ (cf. Stein [17, Theorem III.3.3, p. 114]), which implies $w \in W^2_2(\mathbb{R}^2) \cap W^4_4(\mathbb{L})$. By the trace theorem we have $h \in W^2_2(\partial \mathbb{L}) \cap W^4_4(\partial \mathbb{L})$, which in terms of $f_0$ means $f_0 \in W^2_2(\mathbb{R}) \cap W^4_4(\mathbb{R})$. The explosion result of Theorem 1 requires $f_0 \in W^2_2(\mathbb{R}) \cap W^4_4(\mathbb{R})$ and (17). The latter is guaranteed by the assumption
on the trace in the statement of the theorem. Hence, $f \notin W^{1+\sigma}_{2,\text{loc}}(\mathbb{L})$, $\sigma \geq 2/3$. Since $w \in W^2_{2,\text{loc}}(\mathbb{L})$, this implies that $F \notin W^{1+\sigma}_{2,\text{loc}}(\mathbb{L})$, $\sigma \geq 2/3$. \hfill \Box

Acknowledgement

We thank S. Dahlke (Marburg) who pointed out the reference [11], N. Jacob (Swansea) for his suggestions on the representation of Sobolev–Slobodetskij spaces, and A. Bendikov (Wrocław) who told us about the papers [13], [14]. We are grateful to B. Böttcher for drawing the illustrations and commenting on the first draft of this paper. Financial support from NCN grant 2014/14/M/ST1/00600 (Wrocław) for V. Knopova is gratefully acknowledged.

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