We construct analytical initial data for a slowly moving and rotating black hole for generic orientations of the linear momentum and the spin. We solve the Hamiltonian constraint approximately and work out the properties of the apparent horizon and show the dependence of its shape on the angle between the spin and the linear momentum. In particular a dimple, whose location depends on the mentioned angle, arises on the 2-sphere geometry of the apparent horizon.

I. INTRODUCTION

Since the first observation of black hole merger [1], there have been many observations of merger of compact objects via gravitational waves. The gravitational waves produced by these mergers are consistent with the numerical solutions of the field equations of General Relativity. Besides the highly accurate numerical results, it always pays to have approximate solutions of relativistic gravitating systems. Here we give an approximate analytical description of a self-gravitating system that has a conserved total energy, total spin and a linear momentum in an asymptotically flat spacetime. The initial configuration is expected to evolve and settle to a single rotating black hole after emitting some gravitational radiation.

The problem was studied in [2] in the case of vanishing linear momentum but with a nonzero spin; and in [3] in the case of vanishing spin with a nonzero linear momentum. See a remarkable exposition in [4]. Here we assume both of these quantities to be nonzero and pointing arbitrarily in three dimensional space. It will turn out that the shape of the apparent horizon depends on the angle between the linear momentum and the spin: even though at the next to leading order, the magnitude of the spin does not appear in the shape of the apparent horizon, its direction does. On the other hand, the shape of the apparent horizon depends on the magnitude of the linear momentum at the first order. The area of the of the apparent horizon does not depend on the angle between the spin and linear momentum. We will also observe that a dimple arises on the 2-sphere geometry of the apparent horizon.
The layout of the paper is as follows. In the next section, we discuss briefly the constraint equations in General Relativity and present the Bowen-York method [5] in finding solutions to the initial value problem. In section III, we give the approximate solution of the Hamiltonian constraint for a slowly rotating and moving black hole. In section IV, we compute the position of the apparent horizon as a function of the angle between the spin and the linear momentum.

II. INITIAL DATA FOR A BLACK HOLE WITH MOMENTUM AND SPIN

Assuming the usual ADM split of the metric [6]

\[ ds^2 = (N_i N^i - N^2) dt^2 + 2 N_i dt dx^i + \gamma_{ij} dx^i dx^j, \quad i, j \in (1, 2, 3), \]

(1)

the Einstein equations in vacuum without a cosmological constant split into constraints and the evolution equations. The constraint equations are given as

\[
- \Sigma R - K^2 + K_{ij} K^{ij} = 0, \\
-2 D_k K^k_i + 2 D_i K = 0,
\]

(2)

where \( \Sigma \) is the Cauchy surface; \( K_{ij} = K_{ij}(t, x^k) \) is its extrinsic curvature defined as

\[ K_{ij} = \frac{1}{2N} \left( \dot{\gamma}_{ij} - D_i N_j - D_j N_i \right), \quad \dot{\gamma}_{ij} = \frac{\partial}{\partial t} \gamma_{ij}, \]

(3)

with the trace \( K := \gamma^{ij} K_{ij} \); and \( D_i \gamma_{kl} = 0 \). For further details of the construction, including the evolution equations which we do not depict here, see the Appendix of [7]. Following Bowen-York [5], let us assume that \( \Sigma \) is conformally flat with the metric

\[ \gamma_{ij} = \psi^4 f_{ij}, \quad \psi > 0, \]

(4)

with \( f_{ij} \) denoting the flat metric in some generic coordinates. One also sets the extrinsic curvature of the hypersurface to be given as \( \hat{K}_{ij} = \psi^{-2} \hat{K}_{ij} \). Furthermore, we assume that \( \Sigma \) is a maximally embedded hypersurface in the spacetime such that the trace of the extrinsic curvature vanishes

\[ K = 0. \]

(5)

Under these conditions the Hamiltonian constraint reduces to a nonlinear elliptic equation

\[
\hat{D}_i \hat{D}^i \psi = -\frac{1}{8} \psi^{-7} \hat{K}_{ij}^2,
\]

(6)

and the momentum constraint reduces to

\[ \hat{D}_i \hat{K}_{ij} = 0, \]

(7)

with \( \hat{D}_i f_{jk} = 0 \). The momentum constraint equations can be solved easily, following [5], let us choose the 6 parameter solution

\[ \hat{K}_{ij} = \frac{3}{2r^2} \left( p_i n_j + p_j n_i + (n_i n_j - f_{ij}) p \cdot n \right) + \frac{3}{r^2} \mathcal{J}^l n_k \left( \epsilon_{kl} n_j + \epsilon_{kj} n_l \right), \]

(8)
where \( n^i \) is the unit normal on a sphere of radius \( r \) and \( \epsilon = \pm 1 \). [For other solutions, see [8].] Assuming the following asymptotic behavior for the conformal factor
\[
\lim_{r \to \infty} \psi(r) = 1 + \frac{E}{2r} + O(1/r^2),
\]
one can easily show that (see [9]) the \( p^i \) in the solution (8) corresponds to the total conserved linear momentum via
\[
P_i = \frac{1}{8\pi} \int_{S_\infty} dS n^i K_{ij} = \frac{1}{8\pi} \int_{S_\infty} dS n^i \hat{K}_{ij}.
\]
Similarly one can show that \( J_i \) corresponds to the total conserved angular momentum expressed in terms of the coordinates and the extrinsic curvature as
\[
J_i = \frac{1}{16\pi} \varepsilon_{ijk} \int_{S_\infty} dS n_l \left( x^j K^{kl} - x^k K^{jl} \right) = \frac{1}{16\pi} \varepsilon_{ijk} \int_{S_\infty} dS n_l \left( x^j \hat{K}^{kl} - x^k \hat{K}^{jl} \right).
\]
Finally the ADM energy
\[
E_{ADM} = \frac{1}{16\pi} \int_{S_\infty} dS n_i \left( \partial_j h^{ij} - \partial_i h^j \right),
\]
becomes
\[
E_{ADM} = -\frac{1}{2\pi} \int_{S_\infty} dS n^i \partial_i \psi,
\]
and so using the asymptotic form (9) one finds \( E_{ADM} = E \). This has been a brief description of the solution of the momentum constraints. Now, the important task is to solve the Hamiltonian constraint, which as we noted, is a nonlinear elliptic equation and thus, generically, it can only be solved numerically. But in the next section, we shall give an approximate solution for small momentum and small rotation.

III. INITIAL DATA WITH SMALL MOMENTUM AND SMALL SPIN

Computation of \( \hat{K}_{ij} \hat{K}^{ij} \) (from equation (8)) yields
\[
\hat{K}_{ij} \hat{K}^{ij} = \frac{9}{2r^4} \left( \vec{p}^2 + 2(\vec{p} \cdot \vec{n})^2 \right) + \frac{18}{r^5} \left( \vec{J} \times \vec{n} \right) \cdot \vec{p} + \frac{18}{r^6} \left( \vec{J} \times \vec{n} \right) \cdot \left( \vec{J} \times \vec{n} \right).
\]
Without loss of generality, let us assume that the direction of the spin is the \( \hat{k} \) direction, namely
\[
\vec{J} = J\hat{k},
\]
and \( \vec{p} \) is lying in the \( xz \) plane and given as
\[
\vec{p} = p \sin \theta_0 \hat{i} + p \cos \theta_0 \hat{k},
\]
with \( \theta_0 \) a fixed angle. To simplify the notation of the following discussion, let us denote
\[
c_1 := \sin \theta_0, \quad c_2 := \cos \theta_0.
\]
The Hamiltonian constraint, after these conventions becomes

$$\hat{D}_i \hat{D}^i \psi = \psi^{-7} \left( \frac{9 J_p}{4r^5} c_1 \sin \theta \sin \phi - \frac{9 J^2}{4r^6} \sin^2 \theta - \frac{9 p^2}{16r^4} (1 + 2(c_1 \sin \theta \cos \phi + c_2 \cos \theta)^2) \right).$$

(18)

As it clear from the right-hand side, the correct perturbative expansion in terms of the momentum and spin reads

$$\psi(r, \theta, \phi) := \psi^{(0)} + J^2 \psi^{(J)} + p^2 \psi^{(p)} + J p \psi^{(Jp)} + \mathcal{O}(p^4, J^4, p^2 J^2),$$

(19)

where the functions on the right-hand side depend on all coordinates \((r, \theta, \phi)\). At the lowest order, one has

$$\hat{D}_i \hat{D}^i \psi^{(0)} = 0.$$  

(20)

To proceed, let us discuss the boundary conditions that we shall employ. Following [10] and [3], we chose the following boundary conditions

$$\lim_{r \to \infty} \psi(r) = 1, \quad \psi(r) > 0,$$

(21)

and

$$\lim_{r \to 0} \psi(r) = \psi^{(0)}.$$  

(22)

At the lowest order, the solution satisfying these boundary conditions reads

$$\psi^{(0)} = 1 + \frac{a}{r}.$$  

(23)

Inserting (19) into (18), one arrives at three linear partial differential equations to be solved:

$$\hat{D}_i \hat{D}^i \psi^{(J)} = -\frac{9}{4} \sin^2 \theta \frac{r}{(r+a)^7}, \quad \hat{D}_i \hat{D}^i \psi^{(Jp)} = \frac{9}{4} c_1 \sin \theta \sin \phi \frac{r^2}{(r+a)^7},$$

(24)

and also

$$\hat{D}_i \hat{D}^i \psi^{(p)} = -\frac{9}{16} \left(1 + 2(c_1 \sin \theta \cos \phi + c_2 \cos \theta)^2\right) \frac{r^3}{(r+a)^7}.$$  

(25)

In finding the solutions to these equations, we will need the following spherical harmonics:

$$Y_0^0(\theta, \phi) = \frac{1}{\sqrt{4\pi}}, \quad Y_1^0(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cos \theta, \quad Y_2^0(\theta, \phi) = \sqrt{\frac{5}{16\pi}} (3 \cos^2 \theta - 1),$$

(26)

$$Y_1^{-1}(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \sin \theta \sin \phi, \quad Y_2^1(\theta, \phi) = \sqrt{\frac{15}{4\pi}} \sin \theta \cos \theta \cos \phi, \quad Y_1^1(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \sin \theta \cos \phi.$$  

(26)

Then ansatz for \(\psi^{(J)}\) can be taken as

$$\psi^{(J)}(r, \theta, \phi) = \psi_0^{(J)}(r) Y_0^0(\theta, \phi) + \psi_1^{(J)}(r) Y_2^0(\theta, \phi),$$

(27)

which upon insertion to the first equation of (24) yields two ordinary differential equations

$$\frac{d}{dr} \left( r^2 \frac{d\psi_0^{(J)}(r)}{dr} \right) = -\sqrt{\pi} \frac{r^3}{(r+a)^7}, \quad \frac{d}{dr} \left( r^2 \frac{d\psi_1^{(J)}(r)}{dr} \right) - 6\psi_1^{(J)}(r) = 3\sqrt{\pi} \frac{r^3}{5 (r+a)^7}. $$

(28)
The solution obeying the boundary conditions \((21)\) reads
\[
\psi^{(J)}(r, \theta, \phi) = \frac{(a^4 + 5a^3r + 10a^2r^2 + 5ar^3 + r^4)}{40a^3(a + r)^5} - \frac{r^2}{40a(a + r)^5}(3\cos^2 \theta - 1). \tag{29}
\]

Similarly setting
\[
\psi^{(Jp)}(r, \theta, \phi) = \psi^{(Jp)}_0(r)(\theta, \phi) + \psi^{(Jp)}_1(r)(\theta, \phi)
\]
and the second equation of \((24)\), one finds that \(\psi^{(Jp)}_0(r) = 0\) satisfies the boundary conditions; and the \(\psi^{(Jp)}_1(r)\) piece satisfies
\[
\frac{d}{dr}\left( r^2 \frac{d\psi^{(Jp)}_1(r)}{dr} \right) - 2\psi^{(Jp)}_1(r) = \frac{3\sqrt{3\pi}}{2} r^4 c_1 (r + a)^7
\]
\[
\tag{31}
\]
of which the solution can be found and one has
\[
\psi^{(Jp)}(r, \theta, \phi) = -\frac{c_1 r (a^2 + 5ar + 10r^2)}{80a(a + r)^5} \sin \theta \sin \phi. \tag{32}
\]

Finally, let us do the \(\psi^{(p)}_1\) part which is slightly more complicated. One sets
\[
\psi^{(p)} = \psi^{(p)}_0(r)(\theta, \phi) + \psi^{(p)}_1(r)(\theta, \phi)^2 + \psi^{(p)}_2(r)(\theta, \phi) + \psi^{(p)}_3(r)(\theta, \phi)^2
\]
\[
\tag{33}
\]
to arrive at four equations two of which are
\[
\frac{d}{dr}\left( r^2 \frac{d\psi^{(p)}_1(r)}{dr} \right) + \frac{3}{\sqrt{\pi}} (\psi^{(p)}_1 + \psi^{(p)}_3) = -\frac{9}{8} \sqrt{\pi} \frac{r^5}{(r + a)^7}, \tag{34}
\]
and
\[
\frac{d}{dr}\left( r^2 \frac{d\psi^{(p)}_1(r)}{dr} \right) - 6\psi^{(p)}_1 = -\frac{3}{2} \pi c_2^2 \frac{r^5}{(r + a)^5}. \tag{35}
\]

\(\psi^{(p)}_2(r)\) equation can be obtained from \((35)\) with the replacement \(c_1^2 \rightarrow \sqrt{\frac{3}{5\pi}} c_1 c_2\) and \(\psi^{(p)}_3(r)\) equation can be obtained from \((35)\) via \(c_1^2 \rightarrow c_2^2\). The solutions read, respectively, as follows
\[
\psi^{(p)}_0(r) = \frac{\sqrt{\pi} (84a^6 + 378a^5r + 653a^4r^2 + 514a^3r^3 + 142a^2r^4 - 35ar^5 - 25r^6)}{80ar^2(a + r)^5} - \frac{21\sqrt{\pi} a}{20r^3} \log \frac{a}{a + r}, \tag{36}
\]
and
\[
\psi^{(p)}_1(r) = \frac{\pi c_1^2 (84a^5 + 378a^4r + 658a^3r^2 + 539a^2r^3 + 192ar^4 + 15r^5)}{40r^2(a + r)^5} + \frac{21\pi ac_2^2}{10r^3} \log \frac{a}{r + a}, \tag{37}
\]
from which one can find \(\psi^{(p)}\), but we do not depict is here since it is a little long.
Recall that for the ADM energy computation, we need the dominant terms up to and including \( O(\frac{1}{r}) \) in \( \psi(r, \theta, \phi) \). Collecting these parts in the above solutions, one gets

\[
\psi(r) = 1 + \frac{a}{r} + \frac{J^2}{40a^3r} + \frac{5p^2}{32ar} + O\left(\frac{1}{r^2}\right). \tag{38}
\]

Therefore from (9), the ADM energy of the solution reads

\[
E_{\text{ADM}} = 2a + \frac{J^2}{20a^3} + \frac{5p^2}{16a}. \tag{39}
\]

Observe that the \( Jp \) term does not contribute to the energy since it is of \( O\left(\frac{1}{r^2}\right) \).

Next, as in [3], let us express the ADM energy in terms of the irreducible mass \( M_{\text{irr}} \) which is defined [11] as

\[
M_{\text{irr}} := \sqrt{\frac{A_{\text{AH}}}{16\pi}} \tag{40}
\]

with \( A_{\text{AH}} \) being the area of the event horizon. But as the event horizon is a 4 dimensional concept, which cannot be derived from the 3 dimensional initial data, we will approximate this with the area of the apparent horizon following [3].

**IV. COMPUTATION OF THE APPARENT HORIZON FOR THE BOOSTED, ROTATING SOLUTIONS**

Let \( S \) be a 2 dimensional subspace of \( \Sigma \) and \( s^i \) be the normalized unit vector of \( S \), i.e. \( s^i s_i = 1 \). Then the metric on \( S \) is the pull-back metric from \( \Sigma \) given as

\[
m_{ij} := \gamma_{ij} - s_is_j. \tag{41}
\]

The expansion of the null geodesic congruence vanishes at the apparent horizon by definition, i.e. it is a marginally trapped surface and the defining equation becomes

\[
\left( \gamma^{ij} - s^i s^j \right) \left( D_is_j - K_{ij} \right) = 0. \tag{42}
\]

Assuming the surface to be defined as a level set of a function

\[
\Phi := r - h(\theta, \phi) = 0, \tag{43}
\]

then the normal one-form reads

\[
s_i := \lambda m_i = \lambda \partial_i \Phi \tag{44}
\]

which explicitly becomes

\[
s_i = \lambda (1, -\partial_\theta h, -\partial_\phi h). \tag{45}
\]

Recall that the metric on \( \Sigma \) is

\[
\gamma_{ij} = \psi^4 \begin{pmatrix}
 1 & 0 & 0 \\
 0 & r^2 & 0 \\
 0 & 0 & r^2 \sin^2 \theta
\end{pmatrix}, \tag{46}
\]

then one has
where we have used Equation (42) reads more explicitly as

\[ \lambda = \left( \gamma^{rr} + \gamma^{\theta\theta} (\partial_\theta h)^2 + \gamma^{\phi\phi} (\partial_\phi h)^2 \right)^{-1/2}. \]  

Equation (42) reads more explicitly as

\[ \gamma^{ij} \partial_i m_j - \gamma^{ij} \Gamma^k_{ij} m_k - \lambda^2 m^i m^j \partial_i m_j + \lambda m^i m^j m_k \Gamma^k_{ij} + \lambda m^i m^j K_{ij} = 0 \]

where we have used \( \gamma^{ij} K_{ij} = K = 0 \). After working out each piece, one arrives at

\[
\begin{align*}
-\gamma^{\theta\theta} \partial^2_\theta h - \gamma^{\phi\phi} \partial^2_\phi h - \frac{1}{2} \left( (\gamma^{rr})^2 \partial_r \gamma_{rr} - \gamma^{\theta\theta} \gamma^{rr} \partial_r \gamma_{\theta\theta} - \gamma^{\phi\phi} \gamma^{rr} \partial_r \gamma_{\phi\phi} + \partial_\theta h \gamma^{\phi\phi} \gamma^{\theta\theta} \partial_\phi \gamma_{\phi\phi} \right) \\
+ \lambda^2 \left( (\gamma^{rr})^2 (\partial_\theta h)^2 \partial^2_\theta h + (\gamma^{\phi\phi})^2 (\partial_\phi h)^2 \partial^2_\phi h + 2 \gamma^{\phi\phi} r^\theta \phi \partial_\theta h \partial_\phi h \partial_\theta \partial_\phi h \right) \\
+ \frac{\lambda^2}{2} \left( (\gamma^{rr})^3 \partial_r \gamma_{rr} + (\gamma^{\theta\theta})^2 (\partial_\theta h)^2 \partial_r \gamma_{\theta\theta} + (\gamma^{\phi\phi})^2 (\partial_\phi h)^2 \partial_r \gamma_{\phi\phi} \\
- (\partial_\theta h)^2 \partial_\phi h (\gamma^{\theta\theta})^2 (\gamma^{\phi\phi})^2 \partial_\theta \gamma_{\phi\phi} \right) \\
+ \lambda \left( (\gamma^{rr})^2 K_{rr} + (\gamma^{\theta\theta})^2 (\partial_\theta h)^2 K_{\theta\theta} + (\gamma^{\phi\phi})^2 (\partial_\phi h)^2 K_{\phi\phi} - 2 \gamma^{rr} \gamma^{\theta\theta} \partial_\theta h K_{r\theta} \\
- 2 \gamma^{rr} \gamma^{\phi\phi} \partial_\phi h K_{r\phi} + 2 \gamma^{\theta\theta} \gamma^{\phi\phi} \partial_\theta h \partial_\phi h K_{\theta\phi} \right) = 0.
\end{align*}
\]

An exact solution to this equation is beyond reach and we do not really need it. All we need is an approximate solution of the form

\[ h(\theta, \phi) = h^0 + ph^p + Jh^J + \mathcal{O}(p^2, J^2, Jp), \]

where

\[ \partial_i h = 0, \quad \partial_r h^0 = 0 = \partial_\theta h^0 = \partial_\phi h^0. \]

Note that to compute the area of the apparent horizon and the irreducible mass up to and including the \( \mathcal{O}(p^2, J^2, Jp) \) terms, one only needs the shape of the horizon up to and including the \( \mathcal{O}(p, J) \) terms which becomes clear when one studies the area integral (See also [3]). Ignoring the higher order terms such as \( (\partial_\theta h)^2 \), \( (\partial_\phi h)^2 \) and \( \partial_\theta h \partial_\phi h \), the apparent horizon equation becomes

\[
\begin{align*}
- \gamma^{\theta\theta} \partial^2_\theta h - \gamma^{\phi\phi} \partial^2_\phi h - \frac{1}{2} \left( (\gamma^{rr})^2 \partial_r \gamma_{rr} - \gamma^{\theta\theta} \gamma^{rr} \partial_r \gamma_{\theta\theta} - \gamma^{\phi\phi} \gamma^{rr} \partial_r \gamma_{\phi\phi} + \partial_\theta h \gamma^{\phi\phi} \gamma^{\theta\theta} \partial_\phi \gamma_{\phi\phi} \right) \\
+ \frac{\lambda^2}{2} \left( (\gamma^{rr})^3 \partial_r \gamma_{rr} + \gamma^{rr} K_{rr} - 2 \gamma^{\theta\theta} \partial_\theta h K_{r\theta} - 2 \gamma^{\phi\phi} \partial_\phi h K_{r\phi} \right) = 0
\end{align*}
\]

To proceed, we need the components of the extrinsic curvature in the \((r, \theta, \phi)\) coordinates. After coordinate transformations, one finds

\[ \tilde{K}_{rr} = \frac{3p}{r^2} \left( c_1 \sin \theta \cos \phi + c_2 \cos \theta \right), \quad \tilde{K}_{r\phi} = \frac{3p}{2r} \left( c_1 \cos \theta \cos \phi - c_2 \sin \theta \right), \]
and
\[ \dot{K}_{r\phi} = -\frac{3p}{2r} c_1 \sin \theta \sin \phi + \frac{3J}{r^2} \sin^2 \theta. \] (55)
Therefore the resulting equation is
\[ \partial^2_\theta h + \frac{1}{\sin^2 \theta} \partial^2_\phi h + \cot \theta \partial_\theta h - 2r - 4r^2 \frac{\partial_\psi}{\psi} + \frac{6J}{\psi^4 r^2} \partial_\phi h - \frac{3p}{\psi^4} (c_1 \sin \theta \cos \phi + c_2 \cos \theta) = 0. \] (56)
At order, \(O(p^0, J^0)\), this equation yields
\[ 1 + 2r \frac{\partial_\psi}{\psi} = 0, \] (57)
where \(\psi = 1 + \frac{a}{r}\). And setting \(r = h\), one finds
\[ h^0 = a. \] (58)
This explains the physical meaning of the parameter \(a\): it is the location of the apparent horizon at the lowest order. The next order contribution, which we shall find below, will be perturbations to this location. At \(O(p)\) and \(O(J)\) we have the following equations, respectively
\[ \partial^2_\theta h^p + \frac{1}{\sin^2 \theta} \partial^2_\phi h^p + \cot \theta \partial_\theta h^p - h^p - \frac{3}{16} \left( c_1 \sin \theta \cos \phi + c_2 \cos \theta \right) = 0, \] (59)
and
\[ \partial^2_\theta h^J + \frac{1}{\sin^2 \theta} \partial^2_\phi h^J + \cot \theta \partial_\theta h^J - h^J = 0. \] (60)
These are linear PDE’s and a close scrutiny shows that \(h^J\) equation is the homogeneous Helmholtz equation on a sphere \((S^2)\), while \(h^P\) equation is the inhomogeneous Helmholtz equation with a non-trivial source. So the next task is to find everywhere finite solutions of the following equation
\[ \left( \vec{\nabla}^2_{S^2} + \lambda \right) f(\theta, \phi) = g(\theta, \phi), \] (61)
where \(\vec{\nabla}^2_{S^2}\) is the Laplacian on \(S^2\):
\[ \vec{\nabla}^2_{S^2} = \partial^2_\theta + \cot \theta \partial_\theta + \frac{1}{\sin^2 \theta} \partial^2_\phi. \] (62)
It is clear that the Green’s function technique is the most suitable approach to this problem. For the Helmholtz operator on the sphere, the Green function \(G(\hat{x}, \hat{x}')\) is defined as
\[ \left( \vec{\nabla}^2_{S^2} + \lambda(n + 1) \right) G(\hat{x}, \hat{x}') = \delta^{(2)}(\hat{x} - \hat{x}'), \] (63)
which can be found to be (for example, see \cite{12})
\[ G(\hat{x}, \hat{x}') = \frac{1}{4 \sin \pi \lambda} \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \frac{\Gamma(n - \lambda) \Gamma(\lambda + 1)}{\Gamma(n + \lambda + 1)} \left( \frac{1 + \hat{x} \cdot \hat{x}'}{2} \right)^n. \] (64)
where \(\hat{x} = \sin \theta \cos \phi i + \sin \theta \sin \phi j + \cos \theta k\) and \(\hat{x}'\) is a similar expression with some other \(\theta\) and \(\phi\). Employing this Green’s function with \(\lambda = \frac{-1 + \sqrt{3}}{2}\) one finds
Figure 1: The shape of the apparent horizon when the angle between $\vec{p}$ and $\vec{J}$ is 45 degrees; and to be able to see the dimple, we have chosen $p/a = 8\sqrt{2}$ which is outside the validity of the approximation we have worked with. But the dimple exists for even small $p$.

$$h^p = -\frac{1}{16} (c_1 \sin \theta \cos \phi + c_2 \cos \theta)$$ (65)

and $h^J = 0$. Therefore the apparent horizon is located at

$$r = h(\theta, \phi) = a - \frac{1}{16} (\vec{p} \cdot \vec{J} \cos \theta + |\vec{p} \wedge \vec{J}| \sin \theta \cos \phi),$$ (66)

where $\vec{J} = \frac{\vec{p}}{J}$. In the limit $\theta_0 = 0$, $h$ reduces to the form given in [3]. Note that the shape of the apparent horizon at this order does not depend on the magnitude of the spin but it does depend on its orientation with respect to the linear momentum. In figure 1, we plot the apparent horizon. To be able to see the dimple clearly in the whole figure, we have chosen a high momentum value.

Let us now evaluate the area of the apparent horizon from the formula

$$A_{AH} = \int_0^{2\pi} d\phi \int_0^\pi d\theta \sqrt{\det m}$$ (67)

which at the order we are working yields

$$A_{AH} = \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta \psi^4 h^2 \left(1 + \frac{1}{h^2} (\partial_\theta h)^2 + \frac{1}{h^2 \sin^2 \theta} (\partial_\phi h)^2 \right)^{1/2},$$ (68)

which is a pretty long computation since the conformal factor is quite complicated. But at the end, one finds

$$A_{AH} = 64\pi a^2 + 4\pi p^2 + \frac{11\pi J^2}{5a^2}.$$ (69)
Note that the angle between the spin and the linear momentum does not appear in the area. Then the irreducible mass \( M_{irr} \) reads

\[
M_{irr} = 2a + \frac{p^2}{16a} + \frac{11J^2}{320a^3}.
\] (70)

Comparing with \( E_{ADM} \) we have

\[
E_{ADM} = M_{irr} + \frac{p^2}{2M_{irr}} + \frac{J^2}{8M_{irr}^3}
\] (71)
in terms of the irreducible mass.

V. CONCLUSIONS

Momentum constraints in General Relativity are easily solved with the method of Bowen-York while the Hamiltonian constraint is a nontrivial elliptic equation. Here, extending earlier works [2], [3] we gave an approximate analytical solution that describes a spinning and moving system with a conserved spin and linear momentum pointing in arbitrary directions. We computed the properties of the apparent horizon, such as its shape and surface area and showed the dependence of the shape on the angle between the spin and the linear momentum. We calculated the relation between the conserved quantities such as the ADM mass, the spin, the linear momentum and the irreducible mass. The area of the apparent horizon does not depend on the angle between the spin and the linear momentum, but a dimple arises in the apparent horizon whose location depends on this angle.

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