NILPOTENT CENTRALIZERS AND GOOD FILTRATIONS

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Abstract. Let $G$ be a connected reductive group over an algebraically closed field $k$. Under mild restrictions on the characteristic of $k$, we show that any $G$-module with a good filtration also has a good filtration as a module for the reductive part of the centralizer of a nilpotent element $x$ in its Lie algebra.

1. Introduction

Let $G$ be a connected reductive group over an algebraically closed field $k$ of characteristic $p > 0$, and let $H$ be a connected reductive subgroup. Recall that $(G, H)$ is said to be a Donkin pair or a good filtration pair if every $G$-module with a good filtration still has a good filtration when regarded as an $H$-module.

Now let $x$ be a nilpotent element in the Lie algebra of $G$, and let $G^x \subset G$ be its stabilizer. If $p$ is good for $G$, then the theory of associated cocharacters is available, and this gives rise to a decomposition

$$G^x = G^x_{\text{red}} \ltimes G^x_{\text{unip}}$$

where $G^x_{\text{unip}}$ is a connected unipotent group, and $G^x_{\text{red}}$ is a (possibly disconnected) group whose identity component $(G^x_{\text{red}})^o$ is reductive (cf. [11, 5.10]). The main result of this paper is the following.

Theorem 1.1. Let $G$ be a connected reductive group over an algebraically closed field $k$ of good characteristic. For any nilpotent element $x$ in its Lie algebra, $(G, (G^x_{\text{red}})^o)$ is a Donkin pair.

Now suppose that $H \subset G$ is a possibly disconnected reductive subgroup, i.e., a group whose identity component $H^o$ is reductive. If the characteristic of $k$ does not divide the order of the finite group $H/H^o$, then the category of finite-dimensional $H$-modules is a highest-weight category, as shown in [2]. In particular, it makes sense to speak of good filtrations for $H$-modules, and so the definition of “Donkin pair” makes sense for $(G, H)$.

In order to apply this notion in the case where $H = G^x_{\text{red}}$, we must impose a slightly stronger condition on $p$: we require it to be pretty good in the sense of [9, Definition 2.11]. (In general, this condition is intermediate between “good” and “very good.” It coincides with “very good” for semisimple simply-connected groups, whereas for $GL_n$, all primes are pretty good.) This is equivalent to requiring $G$ to be standard in the sense of [14, §4]. It follows from [8, Theorem 1.8] and [3, Lemma 2.1] that when $p$ is pretty good for $G$, it does not divide the order of $G^x/(G^x)^o \cong G^x_{\text{red}}/(G^x_{\text{red}})^o$ for any nilpotent element $x$. As an immediate consequence of Theorem 1.1 and Lemma 2.2 below, we have the following result.

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Corollary 1.2. Let \( G \) be a connected reductive group over an algebraically closed field \( \mathbb{k} \) of pretty good characteristic. For any nilpotent element \( x \) in its Lie algebra, \((G, G_{\text{red}}^x)\) is a Donkin pair.

This corollary plays a key role in the proof of the Humphreys conjecture [1].

The paper is organized as follows: Section 2 contains some general lemmas on Donkin pairs, along with a lengthy list of examples (some previously known, and some new). Section 3 gives the proof of Theorem 1.1. The proof consists of a reduction to the quasi-simple case, followed by case-by-case arguments.

Remark 1.3. It would, of course, be desirable to have a uniform proof of Theorem 1.1 that avoids case-by-case arguments, perhaps using the method of Frobenius splittings. Thanks to a fundamental result of Mathieu [13], Theorem 1.1 would come down to showing that the flag variety of \( G \) admits a \((G, G_{\text{red}}^x)\)-canonical splitting. According to a result of van der Kallen [16], this geometric condition is equivalent to a certain linear-algebraic condition (called the “pairing condition”) on the Steinberg modules for \( G \) and \((G_{\text{red}}^x)^{\circ}\). Unfortunately, for the moment, the pairing condition for these groups seems to be out of reach.

2. Preliminaries

2.1. General lemmas on Donkin pairs. We begin with three easy statements about good filtrations.

Lemma 2.1. Let \( H \) be a possibly disconnected reductive group over an algebraically closed field \( \mathbb{k} \). Assume that the characteristic of \( \mathbb{k} \) does not divide \(|H/H^0|\). An \( H \)-module \( M \) has a good filtration if and only if it has a good filtration as an \( H^0 \)-module.

Proof. According to [2, Eq. (3.3)], any costandard \( H \)-module regarded as an \( H^0 \)-module is a direct sum of costandard \( H^0 \)-modules. Hence, any \( H \)-module with a good filtration has a good filtration as an \( H^0 \)-module.

For the opposite implication, suppose \( M \) is an \( H \)-module that has a good filtration as an \( H^0 \)-module. To show that it has a good filtration as an \( H \)-module, we must show that \( \text{Ext}^1_H(-, M) \) vanishes on standard \( H \)-modules. As explained in the proof of [2, Lemma 2.18], we have

\[
\text{Ext}^1_H(-, M) \cong (\text{Ext}^1_{H^0}(-, M))^{H/H^0},
\]

and the right-hand side clearly vanishes on standard \( H \)-modules (using [2, Eq. (3.3)] again). \( \square \)

Lemma 2.2. Let \( G \) be a connected, reductive group, and let \( H \subset G \) be a possibly disconnected reductive subgroup. Assume that the characteristic of \( \mathbb{k} \) does not divide \(|H/H^0|\). Then \((G, H)\) is a Donkin pair if and only if \((G^0, (G^0 \cap H)^0)\) is a Donkin pair.

Proof. This is an immediate consequence of Lemma 2.1. \( \square \)

Lemma 2.3. Let \( G \) be a connected, reductive group, and let \( G' \) be its derived subgroup. Let \( H \subset G \) be a connected, reductive subgroup. Then \((G, H)\) is a Donkin pair if and only if \((G', (G' \cap H)^0)\) is a Donkin pair.

Proof. Let \( T \subset B \subset G \) denote a maximal torus and Borel subgroup respectively, and suppose that \( G'' \) is any closed connected subgroup satisfying \( G'' \subset G'' \subset G \). Now let \( T'' = G'' \cap T \), \( B'' = G'' \cap B \), and observe that by [10, I.6.14(1)], we have

\[
\text{Res}^{G'}_{G''} \text{Ind}^G_B M \cong \text{Ind}^{G''}_{B''} \text{Res}^B_{B''} M
\]

(2.1)
for any $B$-module $M$. Thus, for any dominant weight $\lambda \in X(T)^+$ where we set $\lambda'' = \text{Res}^{G''}_{G}(\lambda) \in X(T'')^+$, it follows that $\text{Res}^{G'}_{G'}(\lambda) \cong \text{Ind}^{G'}_{G}(\lambda'')$. Thus $(G, G'')$ is always a Donkin pair. Furthermore, if we let $H' \subseteq H$ be the derived subgroup and $H'' = (G' \cap H)^{\circ}$, then $H' \subseteq H'' \subseteq H$. We can therefore apply [10, I.6.14(1)] again to show that $(H, H'')$ is a Donkin pair.

Now suppose $(G, H)$ is a Donkin pair. In this case it immediately follows from above that $(G, H'')$ is a Donkin pair. Moreover, if we let $T' = G' \cap T$ and $B' = G' \cap B$, then (2.1) actually implies that for any $\lambda' \in X(T')^+$, there exists $\lambda \in X(T)^+$ with $\lambda' = \text{Res}^{T'}_{T}(\lambda)$ such that

$$\text{Res}^{G}_{G'} \text{Ind}^{G'}_{B'}(\lambda) \cong \text{Ind}^{G'}_{B}(\lambda').$$

In particular,

$$\text{Res}^{G'}_{H'} \text{Ind}^{G'}_{B'}(\lambda') \cong \text{Res}^{G'}_{H'} \text{Ind}^{G'}_{B}(\lambda)$$

has a good filtration as an $H''$-module, and hence, $(G', H'')$ is also a Donkin pair.

Conversely, suppose that $(G', H'')$ is a Donkin pair. We can first deduce that $(G, H'')$ is a Donkin pair from the fact that $(G, G')$ is a Donkin pair. Also, by similar arguments as above we can see that for any $\mu'' \in X(H'' \cap T)^+$, there exists some $\mu \in X(H \cap T)^+$ with $\mu'' = \text{Res}^{H'' \cap T}_{H' \cap T}(\mu)$, such that

$$\text{Res}^{H}_{H'} \text{Ind}^{H}_{H' \cap B}(\mu) \cong \text{Ind}^{H''}_{H' \cap B}(\mu'').$$

This implies that an $H$-module $M$ has a good filtration if and only if the $H''$-module $\text{Res}^{H}_{H'} M$ has a good filtration. Therefore, $(G, H)$ is also a Donkin pair. □

2.2. Examples of Donkin pairs. The following proposition collects a number of known examples of Donkin pairs. The last five parts of the proposition deal with various examples where $G$ is quasi-simple and simply connected. For pairs of the form $(\text{Spin}_{n}, H)$, it is usually more convenient to describe the image $H'$ of $H$ under the map $\pi : \text{Spin}_{n} \to \text{SO}_{n}$. Of course, $H$ can be recovered from $H'$, as the identity component of $\pi^{-1}(H)$. We use the notation that

$$(\text{SO}_{n}, H') = (\text{Spin}_{n}, H).$$

It should be noted that the following proposition does not exhaust the known examples in the literature: for instance, according to [5], there is a Donkin pair of type $(B_{3}, G_{2})$, but this example is not needed in the present paper.

**Proposition 2.4.** Let $G$ be a connected, reductive group, and let $H \subset G$ be a closed, connected, reductive subgroup. If the pair $(G, H)$ satisfies one of the following conditions, then it is a Donkin pair.

1. $G = H \times \cdots \times H$, and $H \hookrightarrow G$ is the diagonal embedding.
2. $H$ is a Levi subgroup of $G$.

For the remaining parts, assume that $G$ is quasi-simple and simply connected.

3. $G$ is of simply-laced type, and $H$ is the fixed-point set of a diagram automorphism of $G$:

$$(A_{2n-1}, C_{n}) = (\text{SL}_{2n}, \text{Sp}_{2n}) \quad (D_{4}, G_{2}) = (\text{Spin}_{8}, G_{2})$$

$$(D_{n}, B_{n-1}) = (\text{SO}_{2n}, \text{SO}_{2n-1}) \quad (E_{6}, F_{4})$$

$$\quad (E_{7}, G_{2})$$
(4) Certain embeddings of classical groups:
\[
\begin{align*}
(A_{2n}, B_n) & \quad (\text{SL}_r, \text{SO}_r) \quad (p > 2) \\
(A_{2n-1}, D_n) & \quad (\text{SL}_2n, \text{Sp}_{2n}) \\
(A_{2n-1}, C_n) & \quad (\text{SL}_2n, \text{Sp}_{2n}) \\
(B_{n+m}, B_nD_m) & \quad (\text{SO}_{r+s}, \text{SO}_r \times \text{SO}_s) \sim \quad (p > 2) \\
(D_{n+m}, D_nD_m) & \quad (\text{SO}_{r+s}, \text{SO}_r \times \text{SO}_s) \sim \quad (p > 2) \\
(D_{n+m+1}, B_nB_m) & \quad (\text{SO}_{r+s}, \text{SO}_r \times \text{SO}_s) \sim \\
(C_{n+m}, C_nC_m) & \quad (\text{Sp}_{2n+2m}, \text{Sp}_{2n} \times \text{Sp}_{2m})
\end{align*}
\]

(5) Certain maximal-rank subgroups of exceptional groups:
\[
\begin{align*}
(E_8, A_2E_6) & \quad (p > 5) \\
(E_8, A_1A_5) & \quad (p > 5) \\
(E_8, A_1E_7) & \quad (p > 5) \\
(E_7, A_2D_5) & \quad (p > 5) \\
(E_7, A_1D_6) & \quad (p > 5) \\
(F_4, B_4) & \quad (p > 5) \\
(F_4, A_3A_2) & \quad (p > 5) \\
(G_2, A_1A_1) & \quad (p > 5)
\end{align*}
\]

(6) Certain restricted irreducible representations:
\[
\begin{align*}
(A_n, A_1) & \quad (p > n) \\
(A_7, A_2) & \quad (p > 3) \\
(A_6, G_2) & \quad (p > 3)
\end{align*}
\]

(7) Tensor product embeddings of classical groups (p > 2):
\[
\begin{align*}
(C_{(2n+1)m}, B_n) & \quad (\text{Sp}_{2rm}, \text{SO}_r) \\
(C_{2nm}, D_n) & \quad (\text{SO}_{r+s}, \text{SO}_r) \sim \\
(D_{2nm}, C_n) & \quad (\text{SO}_{2nm}, \text{Sp}_{2n}) \sim \\
(D_{nm}, D_n) & \quad (\text{SO}_{2nm}, \text{Sp}_{2n}) \sim \\
(C_{nm}, C_n) & \quad (\text{Sp}_{2nm}, \text{Sp}_{2n})
\end{align*}
\]

The details of the embeddings in parts (6) and (7) will be described below.

Proofs for parts (2)–(5). Parts (1) and (2) are due to Mathieu [13] (following earlier work of Donkin [6] that covered most cases). Parts (3) and (4), with the exception of the pair \((E_6,F_4)\), are due to Brundan [5]. The pair \((G_2, A_1A_1)\) in part (5) is also due to Brundan [5]. The pair \((E_6,F_4)\) and the pairs in the first column of part (5) are due to van der Kallen [16]. The pairs in the second column of part (5) are due to Hague–McNinch [7].

Proof of part (6). Each pair \((A_n, H) = (\text{SL}_{n+1}, H)\) in this statement arises from some \((n + 1)\)-dimensional representation of \(H\). Call that representation \(V\). The representations \(V\) are as follows:
- \((A_n, A_1)\): the dual Weyl module for \(\text{SL}_2\) of highest weight \(n\)
- \((A_7, A_2)\): the adjoint representation of \(\text{PSL}_3\)
- \((A_6, G_2)\): the 7-dimensional dual Weyl module whose highest weight is the short dominant root

According to [5, Lemma 3.2(iv)] or [7, §3.2.6], to prove the claim, we must show that each exterior algebra \(\wedge^n V\) has a good filtration as an \(H\)-module. For \((A_n, A_1)\), this is shown in [7, §3.4.3]. For \((A_7, A_2)\) and \((A_6, G_2)\), explicit calculations using the
LiE software package [17] show that the character of $\bigwedge^\bullet V$ is the sum of characters of dual Weyl modules whose highest weights are restricted weights when $p > 3$. □

Proof of part (7). To define the group embeddings in this statement, we will assume that $G$ is either $\text{Sp}_{2n}$ or $\text{SO}_n$. However, in the latter case, the proof that $(G, H)$ is a Donkin pair will also imply the corresponding statement for $G = \text{Spin}_n$.

Let $V$ be a vector space equipped with a nondegenerate bilinear form $B_1$ satisfying $B_1(v, w) = \varepsilon_1 B_1(w, v)$, where $\varepsilon_1 = \pm 1$, and let $\text{Aut}(V_1, B_1)^\circ$ be the connected group of linear automorphisms of $V_1$ that preserve $B_1$. This group is either $\text{SO}_{\dim V}$ or $\text{Sp}_{\dim V}$, depending on $\varepsilon_1$. Let $V_2$, $B_2$, $\varepsilon_2$ be another collection of similar data. Then $B_1 \otimes B_2$ is a nondegenerate pairing on $V_1 \otimes V_2$, with sign $\varepsilon_1 \varepsilon_2$. We obtain an embedding

$$\text{Aut}(V_1, B_1)^\circ \times \text{Aut}(V_2, B_2)^\circ \hookrightarrow \text{Aut}(V_1 \otimes V_2, B_1 \otimes B_2)^\circ.$$ 

Now restrict to just one factor:

$$\text{Aut}(V_1, B_1)^\circ \hookrightarrow \text{Aut}(V_1 \otimes V_2, B_1 \otimes B_2)^\circ. \tag{2.2}$$

The four kinds of pairs listed in the statement are all instances of this embedding, depending on the signs $\varepsilon_1$ and $\varepsilon_2$. We will now prove that

$$(G, H) = (\text{Aut}(V_1 \otimes V_2, B_1 \otimes B_2)^\circ, \text{Aut}(V_1, B_1)^\circ)$$

is a Donkin pair. Let $r = \dim V_1$ and $s = \dim V_2$.

Suppose first that $\varepsilon_2 = 1$. Then (2.2) corresponds to either $(\text{SO}_{rs}, \text{SO}_r)$ or $(\text{Sp}_{rs}, \text{Sp}_r)$. In this case, $V_2$ admits an orthonormal basis $x_1, \ldots, x_s$, where

$$B_2(x_i, x_j) = \delta_{ij}.$$ 

Then the group $\text{Aut}(V_1, B_1)^\circ$ preserves each $V_1 \otimes x_i \subset V_1 \otimes V_2$. In this case, the embedding (2.2) can be factored as

$$\text{Aut}(V_1, B_1)^\circ \hookrightarrow \text{Aut}(V_1, B_1)^\circ \times \cdots \times \text{Aut}(V_1, B_1)^\circ \hookrightarrow \text{Aut}(V_1 \otimes V_2, B_1 \otimes B_2)^\circ.$$ 

The first map is a diagonal embedding; it results in a Donkin pair by part (1) of the proposition. The second embedding gives a Donkin pair by part (4).

Next, suppose that $\varepsilon_2 = -1$, and assume for now that $\dim V_2 = 2$. Choose a basis $\{x, y\}$ for $V_2$ such that $B_2(x, y) = 1$. Then $V_1 \otimes x$ and $V_1 \otimes y$ are both maximal isotropic subspaces of $V_1 \otimes V_2$. Define an action of $\text{GL}(V_1)$ on $V_1 \otimes V_2$ as follows:

$$g \cdot (v \otimes x) = (gv) \otimes x,$$

$$g \cdot (v \otimes y) = ((g^t)^{-1}v) \otimes y$$ for $g \in \text{GL}(V_1),$

where $g^t$ denotes the adjoint operator to $g$ with respect to the nondegenerate form on $V_1$. This action defines an embedding of $\text{GL}(V_1)$ in $\text{Aut}(V_1 \otimes V_2, B_1 \otimes B_2)^\circ$. In fact, it identifies $\text{GL}(V_1)$ with a Levi subgroup of $\text{Aut}(V_1 \otimes V_2, B_1 \otimes B_2)^\circ$. (This is the usual embedding of $\text{GL}_r$ as a Levi subgroup in either $\text{SO}_{2r}$ or $\text{Sp}_{2r}$.) The embedding (2.2) then factors as

$$\text{Aut}(V_1, B_1)^\circ \hookrightarrow \text{GL}(V_1) \hookrightarrow \text{Aut}(V_1 \otimes V_2, B_1 \otimes B_2)^\circ.$$ 

The first embedding gives a Donkin pair by part (4) of the proposition, and the second by part (2).
Finally, suppose $\varepsilon_2 = -1$ and $s = \dim V_2 > 2$. This dimension must still be even, say $s = 2m$. Choose a basis $x_1, \ldots, x_m, y_1, \ldots, y_m$ for $V_2$ such that

$$B_2(x_i, x_j) = B_2(y_i, y_j) = 0, \quad B_2(x_i, y_j) = \delta_{ij}.$$ 

Let $V_2^{(i)}$ be the 2-dimensional subspace spanned by $x_i$ and $y_i$. Then $B_2$ restricts to a nondegenerate symplectic form $B_2^{(i)}$ on $V_2^{(i)}$. We factor (2.2) as follows:

$$\Aut(V_1, B_1)^\circ \rightrightarrows \Aut(V_1 \otimes V_2^{(i)}, B_1 \otimes B_2^{(i)}) \times \cdots \times \Aut(V_1 \otimes V_2^{(m)}, B_1 \otimes B_2^{(m)})^\circ \rightrightarrows \Aut(V_1 \otimes V_2, B_1 \otimes B_2)^\circ.$$

Here, the first arrow is a diagonal embedding (part (1) of the proposition); the second arrow is several instances of the embedding from the previous paragraph (since $\dim V_2^{(i)} = 2$); and the last arrow comes from part (4) of the proposition. We thus again obtain a Donkin pair.  

3. Proof of Theorem 1.1

3.1. Reduction to the quasi-simple case. Let $G$ be an arbitrary connected reductive group in good characteristic. For any nilpotent element $x \in \Lie(G)$, there exists a cocharacter $\tau : G_m \to G$ and a Levi subgroup $L_\tau \subset G$ such that $L_\tau$ is the centralizer of the subgroup $\tau(G_m)$, where $G^\text{red}_\tau = L_\tau \cap G^\tau$ (cf. [11, 5.10]).

If we let $G'$ be the derived subgroup of $G$, then any nilpotent element $x$ for $G$ also satisfies $x \in \Lie(G')$, and by [11, 5.9], $\tau(G_m) \subset G' \subset G$. In particular, $(G')^\tau = G' \cap G^\tau$ and $L'_\tau = G' \cap L_\tau$ is the centralizer of $\tau(G_m)$ in $G'$. Thus,

$$(G')^\text{red}_\tau = G' \cap G^\text{red}_\tau.$$

It now follows from Lemma 2.3 that $(G, (G^\text{red}_\tau)\circ)$ is a Donkin pair if and only if $(G', ((G^\tau)_{\text{red}}\circ)^\circ)$ is a Donkin pair. So we can reduce to the case where $G$ is semisimple.

Suppose now that $\pi : G \to \bar{G}$ is an isogeny (i.e. surjective with finite central kernel), where $\bar{G}$ is an arbitrary connected reductive group in good characteristic. Then, by [11, Proposition 2.7(a)], $\pi$ induces a bijection between the nilpotent elements in $\Lie(G)$ and those in $\Lie(\bar{G})$, and for any nilpotent element $x \in \Lie(G)$, we have $\pi(G^\tau) = G^\tau \circ$. Moreover, by similar arguments as above we can also deduce that $\pi(G^\text{red}_\tau) = G^\text{red}_\tau \circ$ (cf. [11, 5.9]). In particular,

$$\pi((G^\text{red}_\tau)\circ) = \pi(G^\text{red}_\tau)\circ = (G^\text{red}_\tau)\circ,$$

since any surjective morphism of algebraic groups takes the identity component to the identity component.

Let $H = G^\text{red}_\tau$ and $\bar{H} = (G^\text{red}_\tau)\circ$, and note that for any $\bar{G}$-module $M$, there is a natural isomorphism

$$\Res^\bar{G}_H \Res^\bar{G}_H M \cong \Res^\bar{H}_H \Res^\bar{H}_H M.$$ 

From this we can see that if $(\bar{G}, \bar{H})$ is a Donkin pair, then $(G, H)$ must also be a Donkin pair, since it is straightforward to check that a $\bar{G}$-module $M$ (resp. an $\bar{H}$-module $N$) has a good filtration if and only if $\Res^\bar{G}_H M$ (resp. $\Res^\bar{H}_H N$) has a
good filtration. This allows us to reduce to the case where $G$ is semisimple and simply connected.

Finally, suppose that that $G = G_1 \times G_2$ where $G_1, G_2$ are connected reductive groups in good characteristic. Let $x = (x_1, x_2) \in \text{Lie}(G_1) \oplus \text{Lie}(G_2)$ be an arbitrary nilpotent element. We can immediately see that

$$(G_{\text{red}}^x)^o = (\text{(}G_1)_{\text{red}}^x)^o \times (\text{(}G_2)_{\text{red}}^x)^o.$$ 

It now follows from the general properties of induction for direct products (see [10, I.3.8]) that $(G, (G_{\text{red}}^x)^o)$ is a Donkin pair if and only if $(G_1, (G_1)_{\text{red}}^x)^o$ and $(G_2, (G_2)_{\text{red}}^x)^o$ are Donkin pairs. Therefore, by the well-known fact that any simply connected semisimple group is a direct product of quasi-simple simply connected groups, we can reduce the proof of Theorem 1.1 to the case where $G$ is quasi-simple.

3.2. Proof for classical groups. We now prove the theorem for the groups $\text{GL}_n$, $\text{Sp}_n$, and $\text{Spin}_n$. For the last case, we will actually describe the group $(G_{\text{red}}^x)^o$ and its embedding in $G$ for $\text{SO}_n$ instead, but the proof of the Donkin pair property will also apply to $\text{Spin}_n$.

Let $x$ be a nilpotent element in the Lie algebra of one of $\text{GL}_n$, $\text{Sp}_n$, or $\text{SO}_n$. Let $s = [s^{r_1}_1, s^{r_2}_2, \ldots, s^r_r]$ be the partition of $n$ that records the sizes of the Jordan blocks of $x$. (This means that $x$ has $r_1$ Jordan blocks of size $s_1$, and $r_2$ Jordan blocks of size $s_2$, etc.) The vector space $V = k^n$ can be decomposed as

$$V = V^{(1)} \oplus V^{(2)} \oplus \cdots \oplus V^{(k)}$$

where each $V^{(i)}$ is preserved by $x$, and $x$ acts on $V^{(i)}$ by Jordan blocks of size $s_i$. (Thus, $\dim V^{(i)} = r_is_i$.) When $G$ is $\text{Sp}_n$ or $\text{SO}_n$, the nondegenerate bilinear form on $V$ restricts to a nondegenerate form of the same type on each $V^{(i)}$.

The description of $(G_{\text{red}}^x)^o$ in [12, Chapter 3] shows that it factors through the appropriate embedding below:

$$\text{GL}(V^{(1)}) \times \cdots \times \text{GL}(V^{(k)}) \hookrightarrow \text{GL}(V)$$

$$\text{Sp}(V^{(1)}) \times \cdots \times \text{Sp}(V^{(k)}) \hookrightarrow \text{Sp}(V)$$

$$\text{SO}(V^{(1)}) \times \cdots \times \text{SO}(V^{(k)}) \hookrightarrow \text{SO}(V)$$

All three of these embeddings give Donkin pairs: in the case of $\text{GL}_n$, it is an inclusion of a Levi subgroup (Proposition 2.4(1)); and in the case of $\text{Sp}_n$ or $\text{SO}_n$, it falls under Proposition 2.4(4).

We can therefore reduce to the case where $x$ has Jordan blocks of a single size. Suppose from now on that $s = [s^r]$. Then there exists a vector space isomorphism

$$V \cong V_1 \oplus V_2$$

where $\dim V_1 = r$ and $\dim V_2 = s$, and such that $x$ corresponds to $\text{id}_{V_1} \otimes N$, where $N : V_2 \to V_2$ is a nilpotent operator with a single Jordan block (of size $s$).

Suppose now that $G = \text{GL}(V)$. Then, according to [12, Proposition 3.8], we have $(G_{\text{red}}^x)^o \cong \text{GL}(V_1)$. Choose a basis $\{v_1, \ldots, v_s\}$ for $V_2$. The embedding of $(G_{\text{red}}^x)^o$ in $G$ factors as

$$\text{GL}(V_1) \hookrightarrow \text{GL}(V_1 \otimes v_1) \times \cdots \times \text{GL}(V_1 \otimes v_s) \hookrightarrow \text{GL}(V).$$

The first map above is a diagonal embedding (Proposition 2.4(1)), and the second is the inclusion of a Levi subgroup (Proposition 2.4(2)), so $(G, (G_{\text{red}}^x)^o)$ is a Donkin pair in this case.
Next, suppose $G = \text{Sp}(V)$ or $\text{SO}(V)$. According to [12, Proposition 3.10], both $V_1$ and $V_2$ can be equipped with nondegenerate bilinear forms $B_1$ and $B_2$ such that $B_1 \otimes B_2$ agrees with the given bilinear form on $V$. Moreover, $(G^r_{\text{red}})^\circ = \text{Aut}(V_1, B_1)^\circ$. We are thus in the setting of Proposition 2.4(7).
3.3. **Proof for** $E_8$. When $x$ is distinguished, $(G_{\text{red}}^x)^{\circ}$ is the trivial group; and when $x = 0$, $(G_{\text{red}}^x)^{\circ} = G$. For all remaining nilpotent orbits, we rely on the very detailed case-by-case descriptions of $(G_{\text{red}}^x)^{\circ}$ given in [12, Chapter 15]. In each case, that description shows that the embedding $(G_{\text{red}}^x)^{\circ} \hookrightarrow G$ factors as a composition of various cases from Proposition 2.4.

These factorizations are shown in Tables 1 and 2. Here is a brief explanation of the notation used in these tables. Nearly all groups mentioned are semisimple, and

| Orbit | $(G_{\text{red}}^x)^{\circ}$ |
|-------|----------------------------|
| $D_4(a_1)A_2$ | $A_2 \hookrightarrow A_7 \hookrightarrow E_8$ |
| $A_5A_1$ | $A_1A_1 \rightarrow A_1G_2 \rightarrow A_1D_4 \rightarrow A_1E_7 \hookrightarrow E_8$ |
| $D_5A_1$ | $A_1A_1 = B_1C_1 \hookrightarrow B_1D_2 \rightarrow B_3B_1D_2 \rightarrow D_8 \hookrightarrow E_8$ |
| $D_6(a_1)A_1$ | $A_1A_1 \rightarrow A_1A_2 \rightarrow A_1F_4 \rightarrow A_1E_6 \rightarrow A_1E_7 \hookrightarrow E_8$ |
| $A_6$ | $A_1^2 \rightarrow A_1E_7 \hookrightarrow E_8$ |
| $D_6$ | $B_2 \rightarrow B_2B_5 \rightarrow D_8 \hookrightarrow E_8$ |
| $D_6(a_1)$ | $A_1A_1 = D_2 \rightarrow D_2D_6 \rightarrow D_8 \hookrightarrow E_8$ |
| $D_6(a_2)$ | $A_1A_1 = D_2 \rightarrow D_2D_6 \rightarrow D_8 \hookrightarrow E_8$ |
| $E_6$ | $G_2 \rightarrow D_4 \rightarrow E_8$ |
| $E_6(a_1)$ | $A_2 \rightarrow A_2E_6 \hookrightarrow E_8$ |
| $E_6(a_3)$ | $G_2 \rightarrow D_4 \rightarrow E_8$ |
| $A_4A_2A_1$ | $A_1 \rightarrow A_1A_1A_1 \rightarrow A_1A_2A_3 \rightarrow E_7 \hookrightarrow E_8$ |
| $A_4A_3$ | $A_1 = B_1 \rightarrow B_7 \rightarrow D_8 \hookrightarrow E_8$ |
| $D_5A_2$ | $T_1$ |
| $D_5(a_1)A_2$ | $A_1 = B_1 \rightarrow B_4 \rightarrow B_4B_4 \rightarrow D_8 \hookrightarrow E_8$ |
| $A_6A_1$ | $A_1^2 \rightarrow E_8$ |
| $E_6A_1$ | $A_1 \rightarrow G_2 \rightarrow D_4 \rightarrow E_8$ |
| $E_6(a_1)A_1$ | $T_1$ |
| $E_6(a_3)A_1$ | $A_1 \rightarrow G_2 \rightarrow D_4 \rightarrow E_8$ |
| $A_7$ | $A_1 = C_1 \rightarrow D_8 \hookrightarrow E_8$ |
| $D_7$ | $A_1 = B_1 \rightarrow B_1B_6 \rightarrow D_8 \hookrightarrow E_8$ |
| $D_7(a_1)$ | $T_1$ |
| $D_7(a_2)$ | $T_1$ |
| $E_7$ | $A_1 \rightarrow A_1E_7 \hookrightarrow E_8$ |
| $E_7(a_1)$ | $A_1 \rightarrow A_1E_7 \hookrightarrow E_8$ |
| $E_7(a_2)$ | $A_1 \rightarrow A_1E_7 \hookrightarrow E_8$ |
| $E_7(a_3)$ | $A_1 \rightarrow A_1E_7 \hookrightarrow E_8$ |
| $E_7(a_4)$ | $A_1 \rightarrow A_1E_7 \hookrightarrow E_8$ |
| $E_7(a_5)$ | $A_1 \rightarrow A_1E_7 \hookrightarrow E_8$ |

Table 2. Nilpotent centralizers in $E_8$, continued
they are recorded in the tables by their root systems. However, the notation “T₁” indicates a 1-dimensional torus; this is used to indicate a reductive group with a 1-dimensional center. In a few cases, nonstandard names for root systems—such as B₁ or C₁, in place of A₁—are used when it is convenient to emphasize the role of a certain classical group. The notation D₁ (meant to evoke SO₂) is occasionally used as a synonym for T₁.

Table 3. Nilpotent centralizers in E₇

| Orbit | (G^x_{\text{red}})° |
|-------|----------------------|
| A₁    | D₆ \rightarrow E₇     |
| A₁⁻²  | A₁B₄ = B₁B₄ \rightarrow D₆ \rightarrow E₇ |
| A₂    | A₅ \rightarrow E₇     |
| A₂A₁  | A₃T₁ \rightarrow E₇   |
| (A₂⁻¹)’ | F₄ \rightarrow E₆ \rightarrow E₇ |
| (A₂⁻¹)’’ | A₁C₃ \rightarrow A₁A₅ \rightarrow E₇ |
| A₃    | A₁B₃ = B₁B₃ \rightarrow D₅ \rightarrow E₇ |
| A₄    | G₂ \rightarrow A₅ \rightarrow E₇ |
| A₂₂A₁ | A₁A₁ = B₁A₁B₁ \rightarrow B₄A₁B₁ \rightarrow A₁D₆ \rightarrow E₇ |
| (A₃⁻¹)’ | (A₁A₁A₁)G₂ \rightarrow A₁D₂D₄ \rightarrow A₁D₆ \rightarrow E₇ |
| (A₃⁻¹)’’ | B₃ \rightarrow D₄ \rightarrow E₇ |
| A₅₁   | A₁A₁ = C₁A₁B₁ \rightarrow D₂A₁B₁ \rightarrow D₂A₁D₂ \rightarrow A₁D₄ \rightarrow E₇ |
| (A₅⁻¹)’’ | A₁A₁ = A₁(A₁A₁) \rightarrow A₁A₃ \rightarrow E₇ |
| A₅₅   | A₁T₁ = B₁D₁ \rightarrow B₁D₃ \rightarrow B₁D₃B₁ \rightarrow D₆ \rightarrow E₇ |
| A₄A₁  | T₂ \rightarrow E₇ |
| D₄A₁  | C₂ \rightarrow A₃ \rightarrow E₇ |
| D₄(A₁)₃ | A₁A₁ \rightarrow E₇ |
| (A₅)’’ | G₂ \rightarrow D₄ \rightarrow E₇ |
| (A₆)’’ | A₁A₁ \rightarrow A₁(A₁A₁A₁) \rightarrow E₇ |
| D₅    | A₁A₁ \rightarrow A₁(A₁A₁) \rightarrow E₇ |
| D₅(A₁) | A₁T₁ \rightarrow E₇ |
| A₃A₂A₁ | A₁ \rightarrow A₁A₁ \rightarrow A₄A₂ \rightarrow E₇ |
| A₄A₂  | A₁ \rightarrow A₁A₁A₂ \rightarrow A₁A₂A₄ \rightarrow E₇ |
| A₅A₁  | A₁ \rightarrow G₂ \rightarrow D₄ \rightarrow E₇ |

Table 3. Nilpotent centralizers in E₇
Table 4. Nilpotent centralizers in $E_7$, continued

| Orbit          | $(G_{\text{red}}^x)^\circ$ |
|---------------|-----------------------------|
| $D_5 A_1$     | $A_1 \rightarrow A_1 A_1 \rightarrow E_7$ |
| $D_5(a_1) A_1$| $A_1 \rightarrow \tilde{A}_2 \rightarrow F_4 \rightarrow E_6 \rightarrow E_7$ |
| $A_6$         | $A_1 \rightarrow E_7$ |
| $D_6$         | $A_1 \rightarrow A_1 A_1 \rightarrow E_7$ |
| $D_6(a_1)$    | $A_1 \rightarrow E_7$ |
| $D_6(a_2)$    | $A_1 \rightarrow E_7$ |
| $E_6$         | $A_1 \rightarrow A_1 A_1 \rightarrow E_7$ |
| $E_6(a_1)$    | $T_1$ |
| $E_6(a_3)$    | $A_1 \rightarrow A_1 A_1 \rightarrow E_7$ |

Table 5. Nilpotent centralizers in $E_6$

| Orbit          | $(G_{\text{red}}^x)^\circ$ |
|---------------|-----------------------------|
| $A_1$         | $A_5 \rightarrow E_6$ |
| $A_2^2$       | $B_3 T_1 \rightarrow D_4 T_1 \rightarrow E_6$ |
| $A_2$         | $A_2 A_2 \rightarrow E_6$ |
| $A_1^3$       | $A_2 A_1 \rightarrow (A_2 A_2) A_1 \rightarrow E_6$ |
| $A_2 A_1$     | $A_2 T_1 \rightarrow E_6$ |
| $A_3$         | $B_2 T_1 \rightarrow D_3 T_1 \rightarrow E_6$ |
| $A_2 A_1^2$   | $A_1 T_1 = C_1 T_1 \rightarrow B_4 T_1 \rightarrow D_5 T_1 \rightarrow E_6$ |
| $A_2^2$       | $G_2 \rightarrow D_4 \rightarrow E_6$ |
| $A_3 A_1$     | $A_1 T_1 = C_1 T_1 \rightarrow D_2 T_1 \rightarrow E_6$ |
| $A_4$         | $A_1 T_1 \rightarrow E_6$ |
| $D_4$         | $A_1 \rightarrow A_2 A_2 \rightarrow E_6$ |
| $D_4(a_1)$    | $T_2$ |
| $A_2^2 A_1$   | $A_1 \rightarrow A_1 A_1 A_1 \rightarrow E_6$ |
| $A_4 A_1$     | $T_1$ |
| $A_5$         | $A_1 \rightarrow E_6$ |
| $D_5$         | $T_1$ |
| $D_5(a_1)$    | $T_1$ |

Finally, we remark that there are two orbits—labeled by $A_6$ and by $A_6 A_1$—where the information given in [12] is insufficient to finish the argument. In each of these cases, $(G_{\text{red}}^x)^\circ$ contains a copy of $A_1$ that is the centralizer of a certain copy of $G_2$ inside $E_7$, and [12] does not give further details on this embedding $A_1 \rightarrow E_7$. However, according to [15, §3.12], this $A_1$ is in fact (the derived subgroup of) a Levi subgroup of $E_7$. □
3.4. **Proof for E7 and E8.** Recall that if \( H \subseteq G \) is a closed subgroup, then there is a natural embedding \( N_H \hookrightarrow N_G \) of nilpotent cones. We also recall that a subgroup \( L \subseteq G \) is a Levi subgroup if and only if it is the centralizer of a torus \( S \subset G \).

**Lemma 3.1.** Let \( G \) be a reductive group, \( S \subseteq G \) a torus with \( L = C_G(S) \) a Levi subgroup, and suppose \( x \in N_L \subseteq N_G \) is such that \( S \subseteq (G^x) \). Then \( (L^x) \) is a Levi subgroup of \( (G^x) \).

**Proof.** We clearly have \( C_{G^x}(S) = G^x \cap L = L^x \). Moreover, the identity component \( (L^x) \) must be contained in \( (G^x) \cap L = C_{(G^x)}(S) \). But since the centralizer of a torus in a connected group is connected, we actually have

\[
(L^x) = C_{(G^x)}(S).
\]

Next, consider the semidirect product decomposition \( (G^x) = (G^x) \rtimes G^x \). Let \( g \in G^x \), and write it as \( g = (g_r, g_u) \), with \( g_r \in (G^x) \), \( g_u \in G^x \). If \( g \) centralizes \( S \), then \( g_r \) and \( g_u \) must individually centralize \( S \) as well. In other words,

\[
C_{(G^x)}(S) = C_{(G^x)}(S) \rtimes C_{G^x}(S).
\]

Here, \( C_{(G^x)}(S) \) is a connected reductive group, and \( C_{G^x}(S) \) is a normal unipotent group (which must be connected, because \( C_{(G^x)}(S) \) is connected). We conclude that (3.1) is a Levi decomposition of \( (L^x) \). In particular, we see that \( (L^x) = C_{(G^x)}(S) \).

\( \square \)
We now let $G$ denote the simple, simply connected group of type $E_8$. If $G_0$ is the simple, simply connected group of type $E_7$ or $E_6$, then as explained in [12, Lemma 11.14], there is a simple subgroup $H$ of type $A_1$ or $A_2$ respectively, such that $G_0 = C_G(H)$. Moreover, by [12, 16.1.2], there exists a torus $S \subset H \subset G$ such that $G_0$ is the derived subgroup of the Levi subgroup $L = C_G(S)$. Explicitly, let $\alpha_1, \ldots, \alpha_8$ be the simple roots for $G$, labelled as in [4], and let $\alpha_0$ be the highest root. The groups $H$ and $G_0$ can be described as follows.

| $G_0$ | simple roots for $H$ | simple roots for $L$ or $G_0$ |
|-------|----------------------|-------------------------------|
| $G_0 = E_7$ | $-\alpha_0, \alpha_8, -\alpha_0$ | $\alpha_1, \ldots, \alpha_7$ |
| $G_0 = E_6$ | $\alpha_1, \ldots, \alpha_6$ |

Now it is explained in [12, 16.1.1], that if $x \in N_{G_0} = N_L \subset N_G$, then the subgroup $(G^x_0)\circ$ must contain a conjugate of $H$. Without loss of generality we can assume that $x$ is chosen so that $H \subset (G^x_0)\circ$. Hence, we can also assume that $S \subset (G^x_0)\circ$. Thus, by Lemma 3.1, $(L^x_0)\circ$ is a Levi subgroup of $(G^x_0)\circ$ and we also have $(L^x_0)\circ' \subset ((G^x_0)\circ)\circ \subset (L^x)\circ$.

By Lemma 2.3, Proposition 2.4(2) and §3.3 we deduce that $(G, (G^x_0)\circ)$ is a Donkin pair.

Finally, to show that $(G_0, ((G^x_0)\circ))$ is a Donkin pair, it will be sufficient to show that every fundamental tilting module for $G_0$ is a summand of the restriction of a tilting module for $G$. In more detail, let $\pi : X_G \to X_{G_0}$ be the map on weight lattices. It is well known that if $\lambda$ is a dominant weight for $G$, then the $G_0$-tilting module $T^G_{G_0}(\pi(\lambda))$ occurs as a direct summand of $\text{Res}^{G_0}_{G}((T_G(\lambda))$. So it is enough to show that every fundamental weight for $G_0$ occurs as $\pi(\lambda)$ for some dominant $G$-weight $\lambda$. Let $\varpi_1, \ldots, \varpi_8$ be the fundamental weights for $G$. A short calculation with (3.2) shows that $\pi(\varpi_1), \ldots, \pi(\varpi_7)$ are precisely the fundamental weights for $E_7$, and that $\pi(\varpi_1), \ldots, \pi(\varpi_6)$ are the fundamental weights for $E_6$.

Remark 3.2. One can also prove the theorem for $E_7$ and $E_6$ directly by writing down the embedding of each centralizer, as we did for $E_8$. Here is a brief summary of how to carry out this approach. Let $G$ be of type $E_8$, and let $G_0$ and $H$ be as in the discussion above. As explained in [12, §16.1], we have

$\left((G^x_0)\circ\right)\circ = C_{(G^x_0)\circ}(H)$.

The computation of $C_{(G^x_0)\circ}(H)$ is explained in [12, §16.1.4], and the results are recorded in Tables 3-5, following the same notational conventions as in the $E_8$ case.

3.5. Proof for $F_4$. Let $G$ be the simple, simply connected group of type $E_8$. Then [12, Lemma 11.7] implies that $G$ contains a simple subgroup $H$ of type $G_2$, and that its centralizer $G_0 = C_G(H)$ is a simple group of type $F_4$. The embeddings of centralizers of nilpotent elements for $G_0 = F_4$ can then be computed using the method explained in Remark 3.2. One caveat is that the name (i.e., the Bala-Carter label) of a nilpotent orbit usually changes when passing from $F_4$ to $E_8$. The correspondence between these names can is given in [12, Proposition 16.10].

We remark that in some cases, the book [12] does not quite give enough details about embeddings of subgroups to establish our result, but in these cases, the relevant details can be found in [15, §3.16]. Here is an example illustrating this. The $F_4$-orbit labelled $\tilde{A}_1$ corresponds (by [12, Proposition 16.10]) to the $E_8$-orbit labelled $A_1^2$. Let $x$ be an element of this orbit. We have see that in $E_8$, $(G^x)_{\circ} = B_0$, with
which embeds in the Levi subgroup $D_7 \subset E_8$. The group $(G^r_{\text{red}})^o$ has a subgroup of type $D_3B_3$, which embeds in $D_3D_4 \subset D_7 \subset D_8$. The explicit construction of $H = G_2$ in [15, §3.16] shows that it is contained in the second factor in each of $D_3B_3 \subset D_3D_4 \subset D_7 \subset D_8$. It follows that $D_3 = A_3$ is contained in $((G^r_0)_{\text{red}})^o = C_{(G^r_{\text{red}})^o}(H)$, and then a dimension calculation shows that in fact $((G^r_0)_{\text{red}})^o = A_3$.

The results of these calculations are recorded in Table 6.

3.6. Proof for $G_2$. In this case, there are only two nilpotent orbits that are neither distinguished nor trivial. From the classification, both of these orbits meet the maximal reductive subgroup $A_1\tilde{A}_1 \subset G_2$, and an argument explained in [12, §16.1.4] shows that if $x$ belongs to either of these orbits, then the reductive part of its centralizer in $A_1\tilde{A}_1$ is equal to the reductive part of its centralizer in $G_2$. See Table 7.

\begin{table}
\centering
\begin{tabular}{|c|c|}
\hline
\textbf{Table 6} & \textbf{Results of Calculations} \\
\hline
\textbf{Nilpotent Orbits} & \textbf{Centralizer} \\
\hline
\textbf{Distinguished} & $(G^r_{\text{red}})^o$ \\
\hline
\textbf{Trivial} & $E_8^r$ \\
\hline
\end{tabular}
\end{table}

\begin{table}
\centering
\begin{tabular}{|c|c|}
\hline
\textbf{Table 7} & \textbf{Distinguished Orbits} \\
\hline
\textbf{Orbit} & \textbf{Centralizer} \\
\hline
$D_3B_3$ & $(G^r_0)_{\text{red}}$ \\
$D_3D_4$ & $(G^r_{\text{red}})^o$ \\
\hline
\end{tabular}
\end{table}

\begin{table}
\centering
\begin{tabular}{|c|c|}
\hline
\textbf{Table 8} & \textbf{Trivial Orbits} \\
\hline
\textbf{Orbit} & \textbf{Centralizer} \\
\hline
$D_3B_3$ & $E_8^r$ \\
$D_3D_4$ & $E_8$ \\
\hline
\end{tabular}
\end{table}

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