Stein Estimation for Infinitely Divisible Laws

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Unbiased risk estimation, à la Stein, is studied for infinitely divisible laws with finite second moment

Let us start by briefly recalling the framework and results of Stein (6): Let \(X_i, i = 1, \ldots, n\), be iid \(N(0, \sigma^2)\) random variables and let \(g = (g_1, \ldots, g_n): \mathbb{R}^n \rightarrow \mathbb{R}^n\), be “weakly differentiable.” Then for all \(\theta \in \mathbb{R}^n\),

\[
E\|X + \theta + g(X + \theta) - \theta\|_2^2 = n\sigma^2 + E\|g(X + \theta)\|_2^2 + 2\sigma^2 E \sum_{i=1}^n \frac{\partial}{\partial x_i} g_i(X + \theta),
\]

where \(\| \cdot \|_2\) is the Euclidean norm. Thus the risk of the estimator \(x + g(x)\) can be estimated unbiasedly by \(n\sigma^2 + g(x)^2 + 2\sigma^2 \sum_{i=1}^n \frac{\partial g_i}{\partial x_i}(x)\). This estimate is useful only if the variance of the risk estimate is small compared to the actual risk. This is especially the case if \(g_i\) only depends on \(X_i\), since then the strong law of large numbers kicks in. For normal random variables the existence of the above estimates is based on the identity

\[
\int_{\mathbb{R}} g'(x)e^{-x^2/2}dx = \int_{\mathbb{R}} xg(x)e^{-x^2/2}dx.
\]

We obtain below a corresponding identity for infinitely divisible random variables with finite variance, replacing \(g'\) by \(K(g)\), where \(K\) is an operator commuting with translations.

Let \(f\) be a density on \(\mathbb{R}\), with mean 0 and variance \(\sigma^2\) (for simplicity of notation we concentrate on the univariate case, but see Remark 5). Let \(d(x) = x + g(x)\) be an estimator in the location model induced by \(f\). Let \(F = \{f * \delta_\theta : \theta \in \mathbb{R}\}\), while \(L^2(F)\) and \(L^1(F)\) have their canonical meaning. We want to estimate the risk of \(d\) unbiasedly:

\[
\int_{\mathbb{R}} (d(x + \theta) - \theta)^2 f(x)dx = \int_{\mathbb{R}} g(x + \theta)^2 f(x)dx + \int_{\mathbb{R}} x^2 f(x)dx + 2 \int_{\mathbb{R}} xg(x + \theta)f(x)dx.
\]

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In the above right hand side, the first summand can be estimated unbiasedly, the second is a constant, so we just need to find a function $h \in L^1(F)$ such that

$$
\int_{\mathbb{R}} h(x + \theta)f(x)dx = \int_{\mathbb{R}} xg(x + \theta)f(x)dx.
$$

If $g$ is a polynomial the right-hand side of (2) is itself a polynomial in $\theta$. It is then well-known that there exists an $h$ satisfying (2). But if $g$ is the soft–thresholding operator, i.e., $g(x) = T^\lambda_N(x) = (|x| - \lambda)_{+} \text{sgn}(x)$, then $g$ does not even have a power series expansion. Moreover, $h$ does not have to be unique. Indeed, $h + q$ is also a solution for any function $q$ such that $q \ast f = 0$ (if $\hat{f}$, the Fourier transform of $f$, has zeros such $q$ might exists).

Hence, let us assume that $\hat{f}$ does not have zeros. By computing the generalized Fourier transform of both sides of

$$
\int_{\mathbb{R}} g(-x + \theta)(-x)f(-x)dx = \int_{\mathbb{R}} h(-x + \theta)f(-x)dx,
$$

we get:

$$
\hat{g}(w)\hat{f}'(-w) = i\hat{h}(w)\hat{f}(-w).
$$

This identity shows that if $\hat{g}$ converges to 0 fast enough, e.g., if $\hat{g}$ has compact support, then there exists an $h$ such that (2) holds. Since $\hat{f}$ does not vanish, $h$ is uniquely determined. Hence the set

$$
U_f := \left\{ g \in L^2(F) : \exists h \in L^1(F), \int_{\mathbb{R}} h(x + \theta)f(x)dx = \int_{\mathbb{R}} xg(x + \theta)f(x)dx, \forall \theta \in \mathbb{R} \right\},
$$

is a vector space and clearly there is a unique linear map $K_f : U_f \rightarrow L^1(f)$ with

$$
\int_{\mathbb{R}} K_f(g)(x + \theta)f(x)dx = \int_{\mathbb{R}} g(x + \theta)x f(x)dx.
$$

Let us note some properties of $K_f$:

**Theorem 1.** Let $f, f_1, f_2$ be densities with finite second moment, and let $K_f$, $K_{f_1}$, and $K_{f_2}$ be well defined. Then for all $b \in \mathbb{R}$ and $g \in U_f$, respectively $g \in U_{f_1 \ast f_2}$:

1. $K_f(g\cdot + b) = K_f(g)\cdot + b$
2. $K_{f \ast \delta_b}(g) = K_f(g) + bg(\cdot)$
3. $K_{f_1 \ast f_2}(g) = K_{f_1}(g) + K_{f_2}(g)$
4. $K_{b_f(\cdot)}(g) = K_f(g(\cdot/b))(\cdot/b), \text{ for } b > 0.$
to compute, however from (4) we see that formally
\[ \hat{f} \text{ function or measure, which is the inverse Fourier transform of } h \]
This suggest that \( K \) is quite complicated to compute, however from (4) we see that formally
\[ \hat{K}_f(g)(w) = \hat{g}(w)\hat{f}(-w)/(\hat{f}(-w)i). \]

This suggest that \( h \) can be computed by a convolution of the estimator and of a function or measure, which is the inverse Fourier transform of \( \hat{f}(-w)/(\hat{f}(-w)i) \). Let us try to further formalize this claim. Assume \( K_f(g) := K_f * g \), where \( K_f \in L^1(\mathbb{R}) \)

**Proof.**

1. \[ \int_{\mathbb{R}} g(x + \theta + b)xf(x)dx = \int_{\mathbb{R}} K_f(g)(x + \theta + b)f(x)dx \] and hence \( K_f(g(\cdot + b)) = K_f(g(\cdot + b)) \).
2. \[ \int_{\mathbb{R}} xg(x + \theta)f(x - b)dx = \int_{\mathbb{R}} (K_f(g)(x + \theta + b) + bg(x + \theta + b))f(x)dx \]
   \[ = \int_{\mathbb{R}} (K_f(g)(x + \theta) + bg(x + \theta))f(x - b)dx. \]
3. let \( h_1, h_2 \) be such that \( \int_{\mathbb{R}} g(x + \theta)x_f(x)dx = \int_{\mathbb{R}} h_i(x + \theta)f(x)dx, i = 1, 2, \) then
   \[ \int_{\mathbb{R}} (h_1 + h_2)(z + \theta)(f_1 * f_2)(z)dz \]
   \[ = \int_{\mathbb{R}} \int_{\mathbb{R}} (h_1(x + y + \theta) + h_2(x + y + \theta))f_1(x)f_2(y)dxdy \]
   \[ = \int_{\mathbb{R}} h_1(x + y + \theta)f_1(x)dx f_2(y)dy + \int_{\mathbb{R}} h_2(y + x + \theta)f_2(y)dy f_1(x)dx \]
   \[ = \int_{\mathbb{R}} g(z + \theta)z(f_1 * f_2)(z)dz. \]
4. \[ \int_{\mathbb{R}} g(x + \theta)x(bf(bx))dx = \int_{\mathbb{R}} g(x/b + \theta)xf(x)/bdx \]
   \[ = \int_{\mathbb{R}} g((x + b\theta)/b)xf(x)/bdx \]
   \[ = \int_{\mathbb{R}} K_f(g(\cdot/b))(x + b\theta)/bf(x)dx \]
   \[ = \int_{\mathbb{R}} K_f(g(\cdot/b))((x + \theta)b)/b(bf(xb))dx, \]
and thus \( K_{bf(b)}(g) = K_f(g(\cdot/b))(\cdot/b)/b. \)

Note that the third property presented above is very useful for wavelet analysis, since the law of the noise in a wavelet coefficient is a weighted convolution of the noise in the original data.

For the normal distribution with unit variance the operator \( K \) is defined by \( K(g) = g' \), i.e. \( K \) is the differentiation operator. In general \( K \) is quite complicated to compute, however from (4) we see that formally
\[ \hat{K}_f(g)(w) = \hat{g}(w)\hat{f}(-w)/(\hat{f}(-w)i). \]
and \( \hat{K}_f = \hat{f}'(-\cdot)/(-\cdot)i \). If \( g \in L^\infty(\mathbb{R}) \), then \( K_f \ast g \) does what it is supposed to do:

\[
\int_\mathbb{R} (K_f \ast g)(x + \theta)f(x)dx = \int_\mathbb{R} \int_\mathbb{R} K_f(x - t)g(t + \theta)dtf(x)dx \\
= \int_\mathbb{R} \int_\mathbb{R} K_f(x - t)f(x)dxg(t + \theta)dt \\
= \int_\mathbb{R} \int_\mathbb{R} K_f(-(t - x))f(x)dxg(t + \theta)dt \\
= \int_\mathbb{R} (K_f(-\cdot) \ast f(\cdot))(t)g(t + \theta)dt \\
= \int_\mathbb{R} tf(t)g(t + \theta)dt,
\]

where the last equality follows from the construction of \( K \):

\[
\hat{K}(-\cdot)\hat{f} = \frac{\hat{f}}{i} = (\hat{f}(-\cdot)i)\text{id}.
\]

If \( K_f \) is known, then it can still be a problem to compute \( K_f(g) \), since \( K_f(g) \) is not necessarily as simple as \( g' \). If \( g = \sum_i g_i \) and the \( K_f(g_i) \) are easy to compute, then we can compute \( K_f(g) \) since \( K_f \) is linear. For example, we can take \( g_\lambda^+(x) = (x - \lambda)^+ \) and \( g_\lambda^-(x) := (x - \lambda)^- \) as simple building blocks for functions. Note that \( K_f(g_\lambda^+(x)) = K_f(g_0)(x - \lambda) \) and \( K_f(g_\lambda^-(x)) = \sigma^2 - K_f(g_0) \) if \( \int_\mathbb{R} x^2f(x)dx = \sigma^2 \). For example, the soft thresholding estimator \( T^S_\lambda \) as well as \( T^M_\lambda \) given by \( T^S_\lambda(x) := \lambda 1_{\{x \geq \lambda\}} + 2(|x| - \lambda/2)_+ \text{sgn}(x)1_{\{|x| < \lambda\}} \), have the following decompositions:

\[
T^S_\lambda(x) = x - g^+_\lambda(x) + g^+_\lambda(x) - g^-_\lambda(x) + g^-_\lambda(x),
\]

\[
T^M_\lambda(x) = x - g^+_\lambda(x) + 2g^+_{\lambda/2}(x) - g^+_\lambda(x) - g^-_\lambda(x) + 2g^-_{\lambda/2}(x) - g^-_\lambda(x).
\]

For a further example, assume that \( g : \mathbb{R}^+ \to \mathbb{R} \), is twice continuously differentiable with \( g(0) = 0 \), then \( g(x) = g'(0^+)x^+ + \int_0^\infty (x - y)^+g''(y)dy \).

Another simple example is provided by compound Poisson distributions. Indeed, let \( F \) be a compound Poisson distribution with Fourier transform \( \exp(\lambda(\Psi(w) - 1)) \), where \( \Psi \) is the characteristic function of the density \( f \). Then

\[
\hat{K}_F = \frac{\lambda \Psi'(-w)}{i},
\]

and thus \( K_F(x) = -\lambda f(-x)x \). i.e. \( K_f(g) = K_f \ast g \). Since compound distributions are building blocks for infinitely divisible ones, we have:

**Theorem 2.** Let \( f \) be an infinitely divisible density with finite second moment, i.e., let

\[
\hat{f}(t) = \exp \left(ibt + \int_\mathbb{R} \left( \frac{\exp(ibt) - 1 - ixt}{x^2} \right) M(dx) \right),
\]
where $M$ is a finite positive measure. Let $M(\{0\}) = 0$, let $b = 0$ and let $g$ be Lipschitz. Then

$$K(g)(t) := \int_{\mathbb{R}} \frac{g(t + x) - g(t)}{x} M(dx),$$

is a well defined real valued function, which is moreover bounded and continuous. Furthermore,

$$\int_{\mathbb{R}} K(g)(x + \theta)f(x)dx = \int_{\mathbb{R}} xg(x + \theta)f(x)dx.$$

**Proof.** It is clear that $K(g)$ is well defined, bounded and continuous. If $g$ has compact support then $\int_{\mathbb{R}} |g(x + y) - g(x)|/|y|M(dy)$ and $K(g)$ are in $L^1(\mathbb{R})$ and

$$\hat{K}(g)(t) = \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{g(y + x) - g(y)}{x} M(dx) \exp(ity)dy$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{g(y + x) - g(y)}{x} \exp(ity)dy M(dx)$$

$$= \hat{g}(t) \int_{\mathbb{R}} \exp(-ixt) - \frac{1}{x} M(dx).$$

Since $\int_{\mathbb{R}} \exp(-ixt) - 1/xM(dx) = \hat{f}(-t)/(\hat{f}(-t)i)$, the Fourier transforms of $\int_{\mathbb{R}} K(g)(x + \theta)f(x)dx$ and $\int_{\mathbb{R}} g(x + \theta)xf(x)dx$ are equal and thus these two terms are themselves equal for all $\theta \in \mathbb{R}$.

If $g$ is Lipschitz but does not have compact support, then let $g_n(x) := (1 - |x|/n)_+ g(x)$. Then the Lipschitz constants of the $g_n$ form a bounded set. Clearly, $g_n$ and $K(g_n)$ respectively converge pointwise, respectively to $g$ and $K(g)$, and moreover $\|K(g_n)\|_{\infty}$ is bounded. Hence $\lim_{n \to \infty} \int_{\mathbb{R}} g(x + \theta)xf(x)dx = \int_{\mathbb{R}} g(x + \theta)xf(x)dx$ and $\lim_{n \to \infty} \int_{\mathbb{R}} K(g_n)(x + \theta)f(x)dx = \int_{\mathbb{R}} K(g)(x + \theta)f(x)dx$, for all $\theta$. Thus

$$\int_{\mathbb{R}} K(g)(x + \theta)f(x)dx = \int_{\mathbb{R}} g(x + \theta)xf(x)dx.$$

**Remark 3.** The assumption $b = 0$, is not serious, $b$ is a location parameter of the density and thus we can use Theorem 1. The condition $M(\{0\}) = 0$ is not restrictive either. If $M(\{0\}) = \sigma^2$, then the distribution is the convolution of a centered normal distribution with variance $\sigma^2$ and of an infinitely divisible distribution with Lévy measure without atom at the origin. Again we can use Theorem 1 for this situation. Hence in the general case we obtain:

$$K(g)(t) := bg(t) + M(\{0\})g'(t) + \int_{\mathbb{R}\setminus\{0\}} \frac{g(t + x) - g(t)}{x} M(dx).$$

We also note here that although of little interest to us since we are dealing with mean square errors, the operator $K$ could as well be defined just under a finite first moment assumption on $X$ (in which case, the requirement on $M$ will change too).
Remark 4. Let $f$ be a density with mean zero and variance $\sigma^2$. Without loss of generality let also $K_f(1) = 0$. Let $f_n = \ast_{i=1}^n \sqrt{\pi} f(\sqrt{\pi})$. By the central limit theorem $f_n$ converges in distribution to a normal density. So one would expect that $K_{f_n}$ converges in some sense to $\sigma^2 d/dx$. Assume that $K_f(g)(x) = \int_R (g(x + y) - g(x))/yM(dy)$. Note that if $K_f(g) = Q \ast g$, where $Q$ is a measure, then with the notation $Q^-(A) := Q(-A)$,
\[
(Q \ast g)(x) = \int_R g(x - y) - g(x)Q(dy)
= \int_R g(x + y) - g(x) yQ^-(dy),
\]
where the first equality holds since $K_f(1) = 0$, i.e. $\int_R 1Q(dx) = 0$. Since $\int_R x(x + \theta)f(x)dx = \int_R x^2f(x)dx$, taking $g(x) = x$ gives $M(\mathbb{R}) = K(x)$ and $\int_R K(x)f(x)dx = \int_R x^2f(x)dx = \sigma^2$. As we already know $K_{f_n}(g)(x) = K_f(g(\sqrt{n}))(x\sqrt{n})/\sqrt{n}$. Using the form of $K_f$, we now have
\[
K_{f_n}(g)(x) = \int_R g(x + y/\sqrt{n} - g(x)/\sqrt{n}) M(dy).
\]
Thus, if $g$ is Lipschitz and differentiable, $\lim_{n \to \infty} K_{f_n}(g)(x) = \sigma^2 g'(x)$.

Examples. 1. Let $f(x) = \exp(-\sqrt{2}|x|)/\sqrt{2}$ be the variance normalized Laplace density. It is easy to see that, $\hat{f}(w) = 2/(2 + w^2)$. Thus
\[
\frac{\hat{f}'(w)}{if(w)} = \frac{2iw}{2 + w^2},
\]
and
\[
\hat{K}(w) = \frac{-2iw}{2 + w^2} = -iw\hat{f}(w) = \hat{f}'.
\]
Thus $K(x) = -\exp(-\sqrt{2}|x|) \text{sgn}(x) \in L^1(\mathbb{R})$. Tedious but simple computations yield
\[
K \ast x_+ = \begin{cases} 
\exp(\sqrt{2}x) & : x \leq 0 \\
\frac{2}{1 - \exp(-\sqrt{2}x)} & : x > 0
\end{cases} =: h(x).
\]
Using \(\text{sgn}\) we obtain
\[
K(T^S_\lambda(x) - x) = -h(x) - (1 - h(x)) + h(x - \lambda) + (1 - h(x + \lambda)) = h(x - \lambda) - h(x + \lambda).
\]
Combining these results, we see that for $X$ with a Laplace distribution,
\[
E(T^S_\lambda(X + \theta) - \theta)^2 = 1 + E \min((X + \theta)^2, \lambda^2) + 2(h(X + \theta - \lambda) - h(X + \theta + \lambda)).
\]
2. Let $f_t(x) = \exp(-x)x^{-1}/\Gamma(t)1_{x^+}(x)$ be the density of the Gamma distribution. Since the mean of this distribution is $t$ we want to compute $K_{f_t, x^+}$. Then by Feller
\[ \log(\hat{f}_t(x)) = t \int_0^\infty \frac{\exp(iyx) - 1}{y\exp(-y)} dy \] and thus \( \log(\hat{f}_t)'(x) = it \int_0^\infty \frac{\exp(2iyx)}{y\exp(-y)} dy \).

\[ K_{f_t}(g)(x) = Q \ast g \] where \( Q \in L^1(\mathbb{R}) \) and
\[ \hat{Q}(x) = t \int_0^\infty \frac{\exp(-iyx)\exp(-y)}{y} dy. \]

Hence \( K_{f_t} \ast \delta - t(g)(x) = t \int_0^\infty \frac{\exp(x^2)}{y} dy - t g(x). \) Now, symmetrizing \( f_t \) (to have a zero mean density) gives
\[ \tilde{f}_t(x) = \exp(-|x|) |x| t^{-1/2} \Gamma(t) \] from which the corresponding \( K_{\tilde{f}_t} \) follows.

### 3. Another example is the cosine hyperbolic density,
\[ f(x) = \frac{1}{\cosh(\pi x/2)}, \] again [5, p. 567]
\[ \log(\hat{f}(x)) = \int_{\mathbb{R}} \frac{\exp(2iyx) - 1}{y^2} \frac{y}{\exp(y) - \exp(-y)} dy \]
and thus
\[ K_f(g)(x) = \int_{\mathbb{R}} \frac{g(x + y) - g(x)}{y} \frac{y}{\exp(y) - \exp(-y)} dy. \]

The examples presented above are infinitely divisible distributions and so \( K \) has a nice form. Let us consider a case which is not: The uniform distribution with density \( 2^{-1} 1_{(-1,1)} \). Assume \( g : [-1,1] \to \mathbb{R} \) and \( 2^{-1} \int_{-1}^1 g(x) dx = 0 \). If \( \bar{g} \) is the 2-periodic extension of \( g \) on \( \mathbb{R} \) then \( 2^{-1} g \ast 1_{(-1,1)} = 0 \). Thus unbiased risk estimators are not uniquely determined. Let
\[ r(\theta) = 2^{-1} \int_{-1}^1 x(x + \theta)_+ dx = \begin{cases} 0 : & \theta \leq -1 \\ \frac{1}{3} + \frac{\theta}{4} - \frac{\theta^3}{12} : & \theta \in (-1,1) \\ \frac{1}{3} : & \theta \geq 1 \end{cases} \]
After some tries one finds that with
\[ h(x) = \begin{cases} 0 : & x \leq 0 \\ \frac{x}{2} : & x \geq 0 \end{cases} \]
\( h \) is the 2-periodic extension of \( -x(x - 2)/2 \) defined on \([0,2]\) to \( \mathbb{R}^+ \). \( 2^{-1} \int_{-1}^1 h(x + \theta) dx = r(\theta) \). So with the help of [5] we can now compute an unbiased risk estimator for soft thresholding. Figure 1 shows the unbiased risk estimators for soft thresholding with threshold 2 for the normal distribution, the Laplace distribution, the gamma distribution with \( t = 2 \) and the uniform distribution. The distributions were transformed to have unit variance and mean zero.
Remark 5. As we have seen with (1), for normal random variables, unbiased risk estimation is possible for multivariate means, even if the estimators for the coordinates are not independent. This is also possible for other types of distributions, one has to apply the operator $K$ coordinatewise. Let $X_i, i = 1, \ldots, n$ be random variables, $X_i$ has distribution $F_i$ and $EX_1 = 0, EX_i^2 = \sigma_i^2$. Assume that an operator $K_1$ exists such that $EX_1g(X_1 + \theta_1) = EK_1(g)(X_1 + \theta_1)$ for some $g$. If $g : \mathbb{R}^n \rightarrow \mathbb{R}$ and $\theta \in \mathbb{R}^n$ then $E(X_1 + g(X + \theta) - \theta_1)^2 = \sigma_1^2 + Eg(X + \theta)^2 + 2EK_1(g(\cdot, X_2 + \theta_2, \ldots, X_n + \theta_n))(X_1 + \theta_1)$.

Thus $E(X_1 + g(X + \theta) - \theta_1)^2 = \sigma_1^2 + Eg(X + \theta)^2 + 2EK_1(g(\cdot, X_2 + \theta_2, \ldots, X_n + \theta_n))(X_1 + \theta_1)$.

Remark 6. As the reader might have guessed by now, the motivation for the present paper comes from thresholding methods in wavelet denoising (see [4]). In a function space approach to denoising, the thresholds depend on the sample size $n$, on the Besov space to which the target functions belong to and also on the Besov norm of these targets. In practice it is often not known which threshold is appropriate since the function space to which the signal belongs to as well as the value of its norm are unknown. To bypass this problem, Donoho and Johnstone developed a procedure called SureShrink where thresholds are chosen automatically (see [3]). Their method, based on Stein’s unbiased risk estimate is as follows: for each level (except the highest levels) in the noisy wavelet transform, the largest threshold (smaller than $\sqrt{2 \log n}$) which minimizes the unbiased risk estimate is chosen. For soft thresholding finding this minimum is simple and takes $O(n \log n)$ time.

As noticed in ([1]), the central limit theorem works fast for wavelet coefficients, so it is reasonable to apply the normal adaptive results to the general non-Gaussian framework. However, it is also of interest to understand the scope of SureShrink beyond the normal framework. To do so, we needed to find unbiased risk estimates for other types of distributions. This is what we did here for infinitely divisible and related noise. We could then potentially use the corresponding thresholds found in [1] and [2].

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