Increasing Domain Infill Asymptotics for Stochastic Differential Equations Driven by Fractional Brownian Motion

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Abstract

Although statistical inference in stochastic differential equations (SDEs) driven by Wiener process has received significant attention in the literature, inference in those driven by fractional Brownian motion seem to have seen much less development in comparison, despite their importance in modeling long range dependence. In this article, we consider both classical and Bayesian inference in such fractional Brownian motion based SDEs, observed on the time domain $[0, T]$. In particular, we consider asymptotic inference for two parameters in this regard; a multiplicative parameter $\beta$ associated with the drift function, and the so-called “Hurst parameter” $H$ of the fractional Brownian motion, when $T \to \infty$. For unknown $H$, the likelihood does not lend itself amenable to the popular Girsanov form, rendering usual asymptotic development difficult. As such, we develop increasing domain infill asymptotic theory, by discretizing the SDE into $n$ discrete time points in $[0, T]$, and letting $T \to \infty$, $n \to \infty$, such that either $n/T^2$ or $n/T$ tends to infinity. In this setup, we establish consistency and asymptotic normality of the maximum likelihood estimators, as well as consistency and asymptotic normality of the Bayesian posterior distributions. However, classical or Bayesian asymptotic normality with respect to the Hurst parameter could not be established. We supplement our theoretical investigations with simulation studies in a non-asymptotic setup, prescribing suitable methodologies for classical and Bayesian analyses of SDEs driven by fractional Brownian motion. Applications to a real, close price data, along with comparison with standard SDE driven by Wiener process, is also considered. As expected, it turned out that our Bayesian fractional SDE triumphed over the other model and methods, in both simulated and real data applications.

Keywords: Asymptotic normality; Fractional Brownian motion; Increasing domain infill asymptotics; Posterior asymptotics; Stochastic differential equations; Transformation based Markov chain Monte Carlo.

1 Introduction

Stochastic differential equations (SDEs) are well-placed in both probability and statistics literature for their probabilistic importance as well as in statistical modeling of random phenomena varying continuously with time. The most important development in this regard have been for SDEs driven by Brownian motion (Wiener process). Accessible books on the theory of SDEs driven by Brownian motion can be found in [Arnold (1974), Ikeda and Watanabe (1981), Liptser and Shiryaev (2001), Øksendal (2003), Mao (2011), Karatzas and Shreve (2019)] while statistical inference for such SDEs have been considered in [Basawa and Rao (1980), Iacus (2008), Bishwal (2008) and Fuchs (2013), the last containing interesting applications from life sciences. Perhaps the most important applications of SDEs have been developed in the fields...
of biology and financial modeling; see, for example, Allen (2010), Donnet and Samson (2013), Crépey (2013) and Braumann (2019).

However, Wiener processes, that form the core of the SDE literature, are inadequate for modeling phenomena that exhibit long-range dependence. Such long range dependence is very much evident in asset returns and financial time series such as stock prices, foreign exchange rates, market indices and macroeconomics (see, for example, Henry and Zafforoni (2003)). Geophysical time series (Witt and Malamud (2013)), internet traffic (Park et al. (2011)), etc. are also associated with long-range dependence. Theory and applications of long-range dependence can be found in Doukhan et al. (2003). Note that long-range dependence is associated with scale-invariant self-similar processes with stationary increments, as under suitable conditions, the latter can be shown to exhibit long-range dependence.

Continuous-time phenomena with long-range dependence can be more appropriately handled by SDEs driven by fractional Brownian motion, which is a self-similar Gaussian process with stationary increments, consisting of a fractal index $H \in (0,1)$, usually termed as the “Hurst parameter”. Long-range dependence results when $H > 1/2$. Detailed of fractional Brownian motion and that of SDEs driven by fractional Brownian motion can be found in Norros et al. (1999), Biagini et al. (2008), Mishura (2008), Jien and Ma (2009), Xu and Luo (2018). Statistical inference, including asymptotic theory in such SDEs, assuming that $H > 1/2$ and known, is detailed in Rao (2010); see also Mishura and Ralchenko (2014), Neuenkirch and Tindel (2014), Hu et al. (2017), where again $H$ is assumed to be known. As we shall subsequently discuss, attempts have been made to estimate $H$, but such attempts are at best heuristic, and there does not seem to exist any theory for validation of such heuristic procedures. For instance, we are unaware of any asymptotic theory of such heuristic inference regarding $H$. Moreover, to our knowledge, unless $H$ is known, there does not exist any asymptotic theory of inference for the other parameters. Our goal in this article is to establish asymptotic theory of both classical and Bayesian inference for fractional Brownian motion based SDEs, considering $H$ and other relevant parameters to be unknown. Among the other relevant parameters, here we focus on a single multiplicative real-valued parameter, $\beta$, associated with the “drift function” of the underlying SDE.

As we shall point out subsequently, for unknown $H$, the likelihood function does not admit expression in terms of the convenient Girsanov formula that is usually employed for inference when $H$ is assumed to be known (see Norros et al. (1999), Kleptsyna et al. (2000) and Rao (2010) for example). In any case, in reality, continuous trajectories of the underlying stochastic process is not available, and the data are observed only at discrete time points. Hence, we construct a likelihood by discretizing the time domain $[0, T]$ ($T > 0$) of the fractional SDE into $n$ time points, and quantifying the joint distribution associated with the $n$ time points. Our asymptotic setup is an “increasing domain infill asymptotics” framework where both $T \to \infty$ and $n \to \infty$ such that $n$ grows faster than $T$. In particular, we consider two cases, namely, $n/T \to \infty$ and $n/T^2 \to \infty$. Thus, the time domain is increased, but the number of discrete points in the time domain is increased faster than $T$, attempting to fill up the domain. This is the essence of increasing domain infill asymptotics that we employ in this article.

In our asymptotics framework, we establish strong consistency of the maximum likelihood estimators (MLEs) of $\beta$ and $H$ and asymptotic normality of the MLE of $\beta$, under mild assumptions. We also prove strong consistency of the posterior distribution of $(\beta, H)$ and asymptotic normality of the posterior of $\beta$, in the same increasing domain infill asymptotics framework.

For convenience, in the rest of our paper, we shall often denote “SDE driven by fractional Brownian motion” simply by “fractional SDE”. The roadmap of our paper is structured as
follows. In Section 2, we provide an overview of the \textit{SDE} driven by fractional Brownian motion that we undertake in this work, and in Section 3 we include a brief discussion of the main issues involved in the existing attempts of statistical inference in fractional \textit{SDE}s. The modeled and the true likelihoods under discretization are presented in Section 4. We establish strong consistency of the \textit{MLE} of \((\beta, H)\) in Section 5, assuming compact parameter space. Asymptotic normality of the \textit{MLE} of \(\beta\) is established in Section 6 when \(H\) is estimated by its corresponding \textit{MLE}. In the same section it is shown that compact parameter space for \(\beta\) is unnecessary for either strong consistency or asymptotic normality of the \textit{MLE} of \(\beta\). Inconsistency of the \textit{MLE} of \(\beta\) when \(H\) is estimated by any inconsistent estimator, is established in Section 7. In Section 8, we prove strong consistency of the posterior distribution of \((\beta, H)\), which is shown to hold even when the parameter space of \(\beta\) is non-compact. In the same section we also prove that when \(H\) is estimated by its \textit{MLE}, the posterior distribution of \(\beta\) asymptotically concentrates on Kullback-Leibler neighborhoods. Asymptotic normality of the posterior of \(\beta\) when \(H\) is estimated by its \textit{MLE}, is established in Section 9. Finally, in Section 10, we summarize our contributions, make concluding remarks and provide directions for future research. Some technical lemmas and their proofs, as well as some existing technical overview are relegated to a supplement, whose sections, tables and figures have the prefix “S-” when referred to in this paper. Additionally, in the supplement, we provide details on a simulation experiment and a real data analysis involving close price.

\section{Overview of our \textit{SDE} driven by fractional Brownian motion}

We consider the following \textit{SDE} driven by fractional Brownian motion:

\begin{equation}
\begin{aligned}
    dX_t = \beta B(t, X_t) dt + C(t, X_t) dW_t^H
\end{aligned}
\end{equation}

where the Hurst parameter \(H \in (0, 1)\), and \(\beta \in \mathbb{R}\), the real line, and \(X_0 = x\) is the initial value of the stochastic process \(X_t\), which is assumed to be continuously observed on the time interval \([0, T]\). The function \(B(t, \cdot)\) is a known, real-valued function on \(\mathbb{R}\); this function is known as the drift function. The function \(C(t, \cdot) : \mathbb{R} \mapsto \mathbb{R}\) is the known diffusion coefficient.

According to Theorem D.1.3 of Biagini et al. (2008) we consider the following assumptions under which (2.1) will have a unique solution.

\begin{itemize}
    \item[(H1)] \(B(t, u), C(t, u)\) be Lipschitz continuous in \(u\), uniformly with respect to \(t\), that is, for some \(L > 0\)
    \begin{align*}
        |B(t, u_1) - B(t, u_2)| + |C(t, u_1) - C(t, u_2)| \leq L|u_1 - u_2|.
    \end{align*}
    \item[(H2)] The initial values \(x_0, y_0 \in \mathbb{R}\) and \(\alpha_b > \alpha_1, \alpha_\sigma > 2\alpha_2\) such that
    \begin{align*}
        B(\cdot, x_0) \in L^{\alpha_b}([0, T]), \quad C(\cdot, y_0) \in L^{\alpha_\sigma}([0, T])
    \end{align*}
    where \(\alpha_i = (1 - i|H - 1/2|)^{-1}\) for \(i = 1, 2\).
\end{itemize}

Let \((\Omega, \mathcal{F}, (\mathcal{F}_t), P)\) be a stochastic basis satisfying the usual conditions. The natural filtration of a process is understood as the \(P\)-completion of the filtration generated by this process. In our considered \textit{SDE} (2.2), \(B = \{B(t, X_t), t \geq 0\}\) is an \((\mathcal{F}_t)\)-adapted process and \(C(t, X_t)\) is a non-vanishing, random function.
2.1 True, data generating model and our inferential model

We assume that data \( x = \{x_t : t \in [0, T]\} \) is generated from (2.1) with unknown true parameters \( \theta_0 = (\beta_0, H_0) \). We denote this true model, as well as the probability distribution associated with it by \( P_{\theta_0} \).

However, given data \( x \), our inference regarding \( \theta = (\beta, H) \) will proceed by first creating a new fractional SDE which is the same as (2.1) but \( B(t, X_t) \) and \( C(t, X_t) \) replaced with the deterministic quantities \( B(t, x_t) \) and \( C(t, x_t) \), respectively. In other words, given data \( x \), for inference we consider the following fractional SDE

\[
dX_t = \beta B(t, x_t)dt + C(t, x_t)dW^H_t.
\]

Observe that given data \( x \), inference about \( \theta \) remains exactly the same with respect to both the fractional SDEs (2.1) and (2.2). However, the former has no semimartingale associated with it, making inferential procedures difficult, while the latter admits a Gaussian martingale (see Section 2.2), thereby significantly simplifying inference, and allowing great generality.

Note that although in principle the entire set \( x \) is observed, in reality, \( x_t \) are observed only at discrete time points \( t = t_i \), for \( i = 1, \ldots, n \), for some positive integer \( n \). The issue of discretization will be made precise in Section 4 as we formally introduce our premise. We establish classical and Bayesian consistency and asymptotic normality of \( (\beta, H) \) when the data are generated from \( P_{\theta_0} \) but for inferential purposes (2.2) is considered.

2.2 Gaussian martingale for (2.2)

Even though the process \( X \) is not a semimartingale, one can associate a semimartingale \( Z = \{Z_t, t \geq 0\} \) which is called a fundamental semimartingale such that the natural filtration (\( Z_t \)) of the process \( Z \) coincides with the natural filtration (\( X_t \)) of the process \( X \) (Kleptsyna et al. (2000)). Define, for \( 0 < s < t \),

\[
k_H = 2H \Gamma \left( \frac{3}{2} - H \right) \Gamma \left( H + \frac{1}{2} \right), \quad (2.3)
\]

\[
k_H(t, s) = k_H^{-1} s^{\frac{1}{2} - H} (t - s)^{\frac{1}{2} - H}, \quad (2.4)
\]

\[
\lambda_H = \frac{2H \Gamma(3 - 2H) \Gamma \left( H + \frac{1}{2} \right)}{\Gamma \left( \frac{3}{2} - H \right)}, \quad (2.5)
\]

\[
w_t^H = \lambda_H^{-1} t^{2 - 2H}, \quad (2.6)
\]

and

\[
M_t^H = \int_0^t k_H(t, s)dW^H_s, \quad t \geq 0. \quad (2.7)
\]

The process \( M^H \) is a Gaussian martingale (Norros et al. (1999)). Furthermore, the natural filtration of the martingale \( M^H \) coincides with the natural filtration of the fractional Brownian motion \( W^H \). In fact the stochastic integral

\[
\int_0^t C(s, x_s)dW^H_s
\]

can be represented in terms of the stochastic integral with respect to the martingale \( M^H \). For a
measurable function \( f \) on \([0, T]\), let
\[
K_H^f(t, s) = -2H \frac{d}{ds} \int_s^t f(r) r^{H - \frac{1}{2}} (r - s)^{H - \frac{1}{2}} dr, \quad 0 \leq s \leq t
\] (2.8)
when the derivative exists in the sense of absolute continuity with respect to the Lebesgue measure (Samko and Marichev (1993)).

Consider the sample \( \{ B(t, x_t)/C(t, x_t), t \geq 0 \} \) to be smooth enough (Samko and Marichev (1993)) such that
\[
Q_\theta(t) = \beta d w_H^{H_t} \int_0^t k_H(t, s) \frac{B(s, x_s)}{C(s, x_s)} ds, \quad t \in [0, T]
\] (2.9)
is well defined, where \( w_H \) and \( k_H(t, s) \) are as defined in (2.4) and (2.6) respectively and the derivative is understood in the sense of absolute continuity. Next we state the following theorem by Kleptsyna et al. (2000) which associates a fundamental semimartingale \( Z \) associated with the process \( X \) such that the natural filtration \( (Z_t) \) coincides with the natural filtration \( (X_t) \) of \( X \).

**Theorem 1** (Kleptsyna et al. (2000)) Suppose the sample paths of the process \( Q_\theta \) defined by (2.9) belong \( P \)-a.s. to \( L^2([0, T], dw_H^{H}) \) where \( w_H \) is as defined by (2.6). Let the process \( Z = \{ Z_t, t \in [0, T] \} \) be defined by
\[
Z_t = \int_0^t k_H(t, s) [C(s, x_s)]^{-1} dX_s
\] (2.10)
where the function \( k_H(t, s) \) is as defined in (2.4). Then the following results hold:

(i) The process \( Z \) is an \((\mathcal{F}_t)\)-semimartingale with the decomposition
\[
Z_t = \int_0^t Q_\theta(s) d w_H^{H_s} + M_t^H
\] (2.11)
where \( M^H \) is the fundamental martingale defined by (2.7).

(ii) The process \( X \) admits the representation
\[
X_t = \int_0^t K^C_H(t, s) dZ_s
\] (2.12)
where the function \( K^C_H(\cdot, \cdot) \) is as defined in (2.8).

(iii) The natural filtrations \((Z_t)\) and \((X_t)\) coincide.

By Theorem 3.1 of Norros et al. (1999), the Gaussian martingale \( M^H \) has independent increments with variance function
\[
E \left[ (M_t^H)^2 \right] = c_2 t^{2-2H},
\]
where
\[
c_2 = \frac{c_H}{2H(2-2H)^{1/2}} \quad \text{and} \quad c_H = \left( \frac{2H \Gamma \left( \frac{3}{2} - H \right)}{\Gamma(H + \frac{1}{2}) \Gamma(2 - 2H)} \right)^{1/2}.
\]
3 Discussion on existing attempts regarding inference in fractional SDEs

As already mentioned, consistency of the drift parameter has been studied by various authors assuming that the Hurst parameter $H$ is known. In relatively simple specific situations, such as in a linear model and Ornstein-Uhlenbeck process driven by fractional Brownian motion, Le Breton (1998) and Kleptsyna and Le Breton (2002) investigated statistical inference, and the latter proved consistency of the MLE of the parameter, assuming that $H$ is known. Chiba (2020) studied asymptotic properties of an $M$-estimator for the drift parameter when the diffusion coefficient is a known constant and $H \in (1/4, 1/2)$ is known. Rao (2010) considered a more general setup, namely, the class of linear fractional SDEs, and investigated the asymptotic theory of MLE and Bayes estimators, again assuming known $H$. In particular, he established strong consistency and asymptotic normality of the MLE and the Bayes estimator, under appropriate assumptions. His asymptotic theory hinges upon construction of an appropriate likelihood when $H$ is known. This likelihood, which can be perceived as the Girsanov formula for the fractional Brownian motion case, has been derived by Kleptsyna et al. (2000). However, this relies upon the assumption that entire $x$ is observed, which is perhaps not quite realistic.

Indeed, when $H$ is of known value, say $H_0$, then the distribution of the driving fractional Brownian motion $W_{t}^{H_0}$, is known. This then facilitates obtaining a Radon-Nikodym derivative of the measure of the original process associated with the SDE with other model parameters $\psi$, with respect to that of an SDE with known parameters, also driven by $W_{t}^{H_0}$, which is treated as the likelihood function. The form of this likelihood is given by

$$L_T(\psi) = \exp \left\{ -\int_0^T Q_\psi(s)dZ_s + \frac{1}{2} \int_0^T Q_\psi^2(s)dw^H_0 \right\}, \quad (3.1)$$

which has been exploited by the authors for inference regarding $\psi$, assuming $H$ to be known. However, a severe drawback of this Girsanov-based approach is that the entire theory leading to (3.1) breaks down when $H$ is unknown. Since in reality $H$ is always unknown, alternatives must be sought.

An alternative is to obtain some good estimator of $H$ and plug-in its value in (3.1), pretending it to be $H_0$. If the estimator is good enough in the sense of at least being consistent, then for large $T$, such a plug-in method need not be too unrealistic. Unfortunately, regarding the Hurst parameter $H$, the literature is concerned with only various types of heuristic estimators, such as the $R/S$ estimator (Mandelbrot and Wallis (1969)), Whittle estimator (Whittle (1953)), log-periodogram estimator (Geweke and Porter-Hudak (1983)), to name a few. Theoretical validation of such estimators, asymptotic or otherwise, have not been hitherto undertaken in the SDE context. It is thus not clear if a plug-in estimator of $H$ will work even in empirical studies. In our experience, the aforementioned estimators work only when estimated from the data on fractional Brownian motion, but unfortunately, in the SDE context, such data are not available. Simulation studies reported in Filatova et al. (2007) first estimate $H$ from simulated fractional Brownian motion, and then estimate the other parameters given such estimate of $H$. In realistic situations this is of course not possible.

Kubilius and Mishura (2012) study asymptotic properties of two estimators for fractional SDE where $H \in (1/2, 1)$ and where there are no other parameters, in a fixed-domain asymptotics setup, where the time interval $[0, T]$ is partitioned uniformly, but $T$ remains fixed. Brouste and Iacus (2013) consider a special case of fractional SDE leading to fractional Ornstein-Uhlenbeck motion.
Uhlenbeck process and investigate asymptotic properties of estimators in the discretized case; however, even such special scenario, their results for the drift coefficient are obtained only when $\frac{1}{3} < H < \frac{3}{4}$. In special cases of fractional SDEs when the diffusion function is equal to one, and when the time interval is $[0, 1]$ or $[0, \infty)$ with $H \leq 1/2$, and when there are no unknown parameters other than $H$, Gairing et al. (2020) provide two estimators for $H$, proving their strong consistency and weak convergence to normality as the discretization becomes more and more fine. Proofs of their results also require separate restrictions for different ranges of $H$.

Thus, the approaches of Kubilius and Mishura (2012) and Gairing et al. (2020) does not seem to be widely applicable in practice. Besides, the Bayesian counterpart has not been considered at all by these authors.

Using a method based on conic multivariate adaptive regression spline, numerical estimation of $H$ in fractional SDEs when there are no other parameters and when the diffusion coefficient is a constant, is considered by Yerlikaya-Özkurt et al. (2014), but no theoretical investigation has been undertaken.

Our goal in this paper is to develop both classical and Bayesian asymptotic theories for general fractional SDEs. We also develop methods that are applicable to practical, non-asymptotic scenarios, and demonstrate their importance with simulation experiments. Because of the futility of (3.1) in realistic situations, we do away entirely with the Girsanov formula, and consider discretization of the SDE in question on the domain $[0, T]$, so that it consists of $n$ observations, and establish both classical and Bayesian increasing domain infill asymptotic theory for the model parameters as well as for $H$. Details follow.

4 Modeled and true likelihoods under the increasing domain infill asymptotics framework

Realizations of $X$ are assumed to arise from the form (2.2) with $\theta_0 = (\beta_0, H_0)$ being the unknown true parameters; these realizations are then assumed to be converted to realizations of $Z$ using (2.10) given parameter $\theta$ of (2.2). In theory, this requires the continuous trajectory $\{X_s : s \in [0, t]\}$, for $t \in [0, T]$, but for practical computation we approximate the integral (2.10) numerically using the available discrete trajectories of the realizations of $X$. Below we provide details on the likelihood construction, assuming that data are available at discrete time points.

We consider the discretized version of (2.2) on $[0, T]$ with step length $\frac{T}{n}$. Thus, denoting $Z_t$ by $Z_t^\theta$ to indicate its dependence on $\theta$, using (2.11) and Euler’s discretization scheme, we obtain

$$Z_{t_{i+1}}^\theta - Z_{t_i}^\theta = \int_{t_i}^{t_{i+1}} Q_\theta(s)dw^H_s + M^H_{t_{i+1}} - M^H_{t_i}$$

$$\approx Q_\theta(t_i)(w^H_{t_{i+1}} - w^H_{t_i}) + M^H_{t_{i+1}} - M^H_{t_i}$$

(4.1)

where $t_i = T\frac{i}{n}$ for $i = 1, \cdots, n$ and either $\frac{n_i}{T} \to \infty$ or $\frac{n}{T} \to \infty$ as both $T \to \infty$ and $n \to \infty$. We re-write (4.1) as

$$\Delta Z_{t_i}^\theta = Q_\theta(t_i)\Delta w^H_{t_i} + \Delta M^H_{t_i},$$

(4.2)

replacing “$\approx$” in (4.1) with “$=$”. Note that $\Delta M^H_{t_i}$ are independently distributed as $N(0, \nu^H_{M_{t_i}})$,
the normal distribution with mean zero and variance \( \nu_{M_i}^2 \), where
\[
\nu_{M_i}^2 = \frac{\gamma (i^{2-2H} - i^{2-2H})}{2}.
\]

The notation \( \Delta M_i^H \) aptly indicates that the distribution is independent of \( \beta \).

The true value of \( \theta = (\beta, H) \) is denoted by \( \theta_0 = (\beta_0, H_0) \). We write the likelihood function in terms of the distribution of \( \Delta M_i^H = \Delta Z_{t_i}^\theta - Q_\theta(t_i) \Delta w_{t_i}^H \). Thus, the true likelihood corresponds to \( \Delta M_i^{H_0} = \Delta Z_{t_i}^{\theta_0} - Q_{\theta_0}(t_i) \Delta w_{t_i}^{H_0} \) and the modeled likelihood is associated with \( \Delta M_i^H = \Delta Z_{t_i}^\theta - Q_\theta(t_i) \Delta w_{t_i}^H \).

Thus, the likelihood given the true parameter \( \theta_0 \) is
\[
L_{T,n}(\theta_0) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi \nu_{M_i}^{H_0}}} e^{-\frac{(\Delta M_i^{H_0})^2}{2 \nu_{M_i}^{H_0}}} = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi \nu_{M_i}^{H_0}}} e^{-\frac{(-\Delta z_{t_i}^{H_0} - Q_{\theta_0}(t_i) \Delta w_{t_i}^{H_0})^2}{2 \nu_{M_i}^{H_0}}},
\]
and the modeled likelihood is given by
\[
L_{T,n}(\theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi \nu_{M_i}^H}} e^{-\frac{(\Delta M_i^H)^2}{2 \nu_{M_i}^H}} = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi \nu_{M_i}^H}} e^{-\frac{(-\Delta z_{t_i}^{H_0} - Q_{\theta}(t_i) \Delta w_{t_i}^{H})^2}{2 \nu_{M_i}^H}}.
\]

In what follows, for any two sequences \( \{a_k : k \geq 1\} \) and \( \{b_k : k \geq 1\} \), \( a_k \sim b_k \) as \( k \to \infty \) denotes \( \frac{a_k}{b_k} \to 1 \), as \( k \to \infty \). In other words, \( a_k \sim b_k \) as \( k \to \infty \) denotes that the sequences \( a_k \) and \( b_k \) are “asymptotically equivalent”. Also, “\( a_k \) would denote “almost sure” convergence.

5 Strong consistency of the MLE of \( \theta \)

**Theorem 2** Let data \( x \) be generated from \( P_{\theta_0} \) and given \( x \) consider the SDE given by (2.2) for inference. In \( P_{\theta_0} \) and (2.2), let \( B(t, x_t) = B(x_t) \) and \( C(t, x_t) = C(x_t) \) and assume (H1) and (H2). Also, let \( \Theta = \mathcal{B} \times [\eta, 1-\eta] \) be the parameter space for \( \theta = (\beta, H) \), where \( \eta > 0 \) and \( \mathcal{B} \) is a compact subset of the real line \( \mathbb{R} \). Further assume that there exist real valued continuous functions \( C_1 \) and \( C_2 \) such that for any \( H \in [\eta, 1-\eta] \),
\[
\int_0^1 u^{\frac{1}{2}-H} (1-u)^{\frac{1}{2}-H} \frac{B(x_{t,u})}{C(x_{t,u})} du \to C_1(H, x),
\]
almost surely, as \( i \to n; \ n \to \infty; \ T \to \infty; \) \quad \quad(5.1)
\[
\frac{d}{dt} \int_0^1 u^{\frac{1}{2}-H} (1-u)^{\frac{1}{2}-H} \frac{B(x_{t,u})}{C(x_{t,u})} du \bigg|_{t=t_i} \to C_2(H, x),
\]
almost surely, as \( i \to n; \ n \to \infty; \ T \to \infty, \) \quad \quad(5.2)

and that the convergences (5.1) and (5.2) hold uniformly for \( H \in [\eta, 1-\eta] \). Then the MLE of \( \theta = (\beta, H) \) is strongly consistent in the sense that \( \hat{\theta}_{T,n} \to \theta_0 \) almost surely with respect to \( P_{\theta_0} \), as \( T \to \infty, \ n \to \infty \), such that \( n/T^2 \to \infty \).
Proof. Let $\ell_{T,n}(\theta) = -2 \log L_{T,n}(\theta)$. We shall show that 
$$\inf_{\theta \in \Theta \mid \| \theta - \theta_0 \| \geq \delta} \{ \ell_{T,n}(\theta) - \ell_{T,n}(\theta_0) \} \xrightarrow{a.s.} \infty$$
as $T \to \infty$, $n \to \infty$ such that $T^2/n \to 0$. Now,
$$\ell_{T,n}(\theta) = n \log 2\pi + \sum_{i=1}^{n} \log v_{M_i}^2 + \sum_{i=1}^{n} \left( \frac{\Delta M_i^H}{v_{M_i}^2} \right)^2$$
Hence,
$$\ell_{T,n}(\theta) - \ell_{T,n}(\theta_0) = \sum_{i=1}^{n} \log \frac{v_{M_i}^2}{v_{M_i}^2} + \sum_{i=1}^{n} \left( \frac{\Delta M_i^H}{v_{M_i}^2} \right)^2 - \sum_{i=1}^{n} \left( \frac{\Delta M_i^{H_0}}{v_{M_i}^2} \right)^2$$
$$= - \sum_{i=1}^{n} \log \frac{v_{M_i}^2}{v_{M_i}^2} + \sum_{i=1}^{n} \left( \frac{\Delta M_i^H - \Delta M_i^{H_0}}{v_{M_i}^2} \right)^2 - \sum_{i=1}^{n} \left( \frac{\Delta M_i^{H_0}}{v_{M_i}^2} \right)^2$$
$$= - \sum_{i=1}^{n} \log \frac{v_{M_i}^2}{v_{M_i}^2} - \sum_{i=1}^{n} \left( \frac{\Delta M_i^{H_0}}{v_{M_i}^2} \right)^2 + \sum_{i=1}^{n} \left( \frac{\Delta M_i^H - \Delta M_i^{H_0}}{v_{M_i}^2} \right)^2 \cdot \frac{v_{M_i}^2}{v_{M_i}^2}$$
$$+ 2 \sum_{i=1}^{n} \left( \frac{\Delta M_i^H - \Delta M_i^{H_0}}{v_{M_i}^2} \right) \cdot \frac{\Delta M_i^{H_0}}{v_{M_i}^2} + \sum_{i=1}^{n} \left( \frac{\Delta M_i^{H_0}}{v_{M_i}^2} \right)^2 (5.3)$$
Recall that $H \in [\eta, 1 - \eta]$ and $\beta \in \mathcal{B}$, where $\mathcal{B}$ is compact. Hence for any given data sequence \{x_i; i = 1, \ldots, n\}, the infimum of $\ell_{T,n}(\theta) - \ell_{T,n}(\theta_0)$ with respect to $\theta \in \Theta$ such that $\| \theta - \theta_0 \| \geq \delta$, will yield a sequence \{\theta_{T,n} : T > 0, n \geq 1\} such that $T^2/n \to 0$. Since $\Theta$ is compact, there must exist a convergent subsequence \{\theta_{T_k,n_k} = (\beta_{T_k,n_k}, H_{T_k,n_k}) : k = 1, 2, \ldots\} of the above sequence which converges to a limit, say $\theta = (\beta, H)$, as $T \to \infty$, $n \to \infty$ such that $T^2/n \to 0$. Thus, $\beta_{T_k,n_k} \sim \beta$ and $H_{T_k,n_k} \sim H$, almost surely, as $k \to \infty$. For simplicity of notation, we shall denote $\beta_{T_k,n_k}, H_{T_k,n_k}, \theta_{T_k,n_k}, \ell_{T_k,n_k}(\theta_{T_k,n_k})$ and $\ell_{T_k,n_k}(\theta_0)$ by $\beta_k$, $H_k$, $\theta_k$, $\ell_k(\theta_k)$ and $\ell_k(\theta_0)$, respectively. If we can show that
$$\ell_k(\theta_k) - \ell_k(\theta_0) \to \infty,$$\quad \text{almost surely, as } k \to \infty, \quad (5.4)$$then it would follow that 
$$\inf_{\theta \in \Theta \mid \| \theta - \theta_0 \| \geq \delta} \{ \ell_{T,n}(\theta) - \ell_{T,n}(\theta_0) \} \xrightarrow{a.s.} \infty$$\quad \text{as } T \to \infty, \quad n \to \infty\quad \text{such that } T^2/n \to 0.$$In what follows, we show that (5.4) indeed holds.
For $t_{ik} = T_k \frac{i}{n_k}$ where $i = 1, \ldots, n_k$, let $Y_{t_{ik}} = \frac{\Delta M_i^{H_0}}{v_{M_i}^2}$. Note that $Y_{t_{ik}}$ for $i = 1, 2, \ldots, n_k$, are $N(0,1)$ random variables.
Note that, $t_{ik} = T_k \frac{i}{n_k}$ for $i = 1, \ldots, n_k$. By the mean value theorem it follows that,
$$t_{i+1,k}^2 - t_{ik}^2 = (T_k \frac{i+1}{n_k})^2 - (T_k \frac{i}{n_k})^2 = T_k (2 - 2H) \xi_{t_{ik}}^2$$\quad \text{(where } \xi_{t_{ik}} \in (t_{ik}, t_{i+1,k}).) \quad (5.5)$$Now note that if $H \leq 1/2$, then $T_k \left( \frac{i}{n_k} \right)^{1-2H} < \xi_{t_{ik}}^{1-2H} < T_k \left( \frac{i+1}{n_k} \right)^{1-2H}$. If on the
other hand $H > 1/2$, then $T_k^{1-2H} (i+1/n_k)^{1-2H} < \xi_{i k}^{1-2H} < T_k^{1-2H} (i/n_k)^{1-2H}$. In both the cases, 
\[
\lim_{k \to \infty; i \to n_k} (i/n_k)^{1-2H} = \lim_{k \to \infty; i \to n_k} (i+1/n_k)^{1-2H} = 1. \text{ Hence,} 
\]
\[
\xi_{i k}^{1-2H} \sim T_k^{1-2H}, \text{ as } i \to n_k \text{ and } k \to \infty. 
\] (5.6)

Hence, from (5.5) and (5.6) we obtain 
\[
n_k \left( t_{i+1,k}^2 - t_{ik}^2 \right) \sim T_k^{(2-2H)(2-2H)}, \text{ as } i \to n_k; \; k \to \infty. 
\] (5.7)

Note that the same holds even if $H$ is replaced with $H_k$, since $H_k \in [\eta, 1 - \eta]$. Using (5.7) we obtain 
\[
\frac{u_{M_{ik}}^2}{u_{M_{ik}}^2} = \frac{c^2_{H_0}}{4H_0^2(2-2H_0)} n_k (t_{i+1,k}^2 - t_{ik}^2 H_0) 
\]
\[
\sim \frac{c^2_{H_0}}{4H_0^2(2-2H_0)} (2 - 2H) T_k^{2-2H} 
\]
\[
= \frac{c^2_{H_0}}{H_0^2} T_k^{2(H_k-H_0)}, \text{ as } i \to n_k; \; k \to \infty. 
\] (5.8)

Hence, 
\[
\lim_{i \to n_k, k \to \infty} \frac{1}{T_k^{2(H_k-H_0)}} \frac{u_{M_{ik}}^2}{u_{M_{ik}}^2} = \frac{c^2_{H_0}}{c^2_H H_0^2} = K \text{ (say).} 
\] (5.9)

Now, equation (5.3) can be re-written as 
\[
\ell_k (\theta_k) - \ell_k (\theta_0) = - \sum_{i=1}^{n_k} \log \frac{\nu_{M_{ik}}^2}{\nu_{M_{ik}}^2} - \sum_{i=1}^{n_k} V_{t_{ik}}^2 + \sum_{i=1}^{n_k} \frac{V_{t_{ik}}^2}{\nu_{M_{ik}}^2} + 
\]
\[
\sum_{i=1}^{n_k} \frac{\Delta M_{t_{ik}} H_{ik} - \Delta M_{t_{ik}} H_{0}}{\nu_{M_{ik}}^2} + 2 \sum_{i=1}^{n_k} \frac{\Delta M_{t_{ik}} H_{ik} - \Delta M_{t_{ik}} H_{0}}{\nu_{M_{ik}}^2} 
\]
\[
= \frac{\nu_{M_{ik}}^2}{\nu_{M_{ik}}^2} \frac{\nu_{M_{ik}}^2}{\nu_{M_{ik}}^2} \frac{\nu_{M_{ik}}^2}{\nu_{M_{ik}}^2}. 
\] (5.10)
Consider first three terms of the right hand side of above equation, namely,

\[- \sum_{i=1}^{n_k} \log \frac{u^2_{M_{ti}^0}}{M_{ti}^H} - \sum_{i=1}^{n_k} Y_{tik}^2 + \sum_{i=1}^{n_k} \frac{u^2_{M_{ti}^0}}{M_{ti}^H} \]

\[= n_k \left( \frac{T_k^{2(H_k - H_0)}}{n_k} \sum_{i=1}^{n_k} Y_{tik}^2 \left( \frac{1}{T_k^{2(H_k - H_0)}} \frac{u^2_{M_{ti}^0}}{M_{ti}^H} \right) - \frac{1}{n_k} \sum_{i=1}^{n_k} \log \left( \frac{1}{T_k^{2(H_k - H_0)}} \frac{u^2_{M_{ti}^0}}{M_{ti}^H} \right) - \log T_k^{2(H_k - H_0)} - \frac{1}{n_k} \sum_{i=1}^{n_k} Y_{tik}^2 \right) . \quad (5.11)\]

Using Lemma 16 (see Appendix) in conjunction with (5.9) and the strong law of large numbers applied to independent standard normal random variables, it follows that

\[\frac{T_k^{2(H_k - H_0)}}{n_k} \sum_{i=1}^{n_k} Y_{tik}^2 \left( \frac{1}{T_k^{2(H_k - H_0)}} \frac{u^2_{M_{ti}^0}}{M_{ti}^H} \right) \sim KT_k^{2(H_k - H_0)} , \text{ almost surely, as } k \to \infty. \quad (5.12)\]

As for the second term of (5.11), note that due to (5.9) and Lemma 16,

\[- \frac{1}{n_k} \sum_{i=1}^{n_k} \log \left( \frac{1}{T_k^{2(H_k - H_0)}} \frac{u^2_{M_{ti}^0}}{M_{ti}^H} \right) \sim - \log K , \text{ as } k \to \infty. \quad (5.13)\]

Since the fourth term of (5.11) tends to 1 as \( k \to \infty \) by the strong law of large numbers, then along with (5.12) and (5.13), it follows that (5.11) has the following asymptotic equivalence:

\[- \frac{1}{n_k} \sum_{i=1}^{n_k} \log \frac{u^2_{M_{ti}^0}}{M_{ti}^H} - \sum_{i=1}^{n_k} Y_{tik}^2 + \sum_{i=1}^{n_k} \frac{u^2_{M_{ti}^0}}{M_{ti}^H} \sim n_k K T_k^{2(H_k - H_0)} - n_k \log K - n_k \log T_k^{2(H_k - H_0)} - n_k , \text{ as } k \to \infty. \quad (5.14)\]

From equation (2.9), in conjunction with (2.6), we obtain

\[Q_\theta(t) \Delta u_t^H = \beta \frac{k^{-1}((t + 1)^{2-2H} - t^{2-2H})}{t^{1-2H}(2 - 2H)} \frac{d}{dt} \int_0^t s^{\frac{1}{2} - H} (t - s)^{\frac{1}{2} - H} B(x_s) C(x_s) ds \]

\[= \beta \left[ \frac{k^{-1}((t + 1)^{2-2H} - t^{2-2H})}{(2 - 2H)} \right] \left[ (2 - 2H) \int_0^1 u^{\frac{1}{2} - H} (1 - u)^{\frac{1}{2} - H} B(x_{tu}) C(x_{tu}) du \right] + t \frac{d}{dt} \int_0^1 u^{\frac{1}{2} - H} (1 - u)^{\frac{1}{2} - H} B(x_{tu}) C(x_{tu}) du . \quad (5.15)\]

In the last equality we have used the substitution \( u = \frac{s}{t} \).
Now consider the last term of equation (5.10). Using (5.15) we obtain

\[
2 \sum_{i=1}^{n_k} \frac{\left(\Delta M_{t_{ik}} - \Delta M_{t_{0}}\right) \Delta M_{t_{0}}}{M_{t_{ik}}^2} \frac{v_{t_{0}}^2}{M_{t_{ik}}^2} = 2 \sum_{i=1}^{n_k} \left(\Delta M_{t_{ik}}^2 - \Delta M_{t_{0}}^2\right) \frac{v_{t_{ik}}}{M_{t_{ik}}^2}
\]

It follows from (5.1) and (5.2) that

\[
\text{Now, since (5.1) and (5.2) hold uniformly in } \eta^k \in [\eta, 1 - \eta] \text{ and } C_1, C_2 \text{ are continuous in } H, \text{ it}
\]

\[
\int_0^1 u^{1/2 - H_0}(1 - u)^{1/2 - H_0} B(x_{t_{ik}}) \frac{C(x_{t_{ik}})}{C(x_{t_{0}})} du \to C_1(H_0, x), \text{ almost surely, as } i \to n_k; k \to \infty; (5.18)
\]

\[
\frac{d}{dt} \int_0^1 u^{1/2 - H_0}(1 - u)^{1/2 - H_0} B(x_{t_{ik}}) \frac{C(x_{t_{ik}})}{C(x_{t_{0}})} du \bigg|_{t=t_{ik}} \to C_2(H_0, x), \text{ almost surely, as } i \to n_k; k \to \infty; (5.19)
\]

Now, since (5.1) and (5.2) hold uniformly in } H \in [\eta, 1 - \eta] \text{ and } C_1, C_2 \text{ are continuous in } H, \text{ it
follows from Lemma 17 (see Appendix) that

\[ \int_0^1 u^{\frac{1}{2}-H_k}(1-u)^{\frac{1}{2}-H_k} \frac{B(x_{t_i}, u)}{C(x_{t_i})} du \rightarrow C_1(\hat{H}, x), \quad \text{almost surely, as } i \rightarrow n_k; \quad k \rightarrow \infty; \quad (5.20) \]

\[ \frac{d}{dt} \int_0^1 u^{\frac{1}{2}-H_k}(1-u)^{\frac{1}{2}-H_k} \frac{B(x_{tu})}{C(x_{tu})} du \bigg|_{t=t_{ik}} \rightarrow C_2(\hat{H}, x), \quad \text{almost surely, as } i \rightarrow n_k; \quad k \rightarrow \infty. \quad (5.21) \]

Using (5.18) – (5.21) and the result \( t_{i+1,k}^{2-2H} - t_{ik}^{2-2H} \sim (2 - 2H) \frac{2^{2-2H}}{n_k} \) in (5.17), we obtain

\[ w_{ik} \sim c_{H_0} \frac{4\hat{H}^2}{c_H^2} H_0^2 (2 - 2\hat{H}) T_k^{1-H_0} \times \left[ \frac{\dot{\beta}_0 k^{-1}}{2 - 2H_0} \left( \frac{1 - H_0}{1 - H} \right) T_k^{2(H_k-H_0)} \{ (2 - 2H_0)C(H_0, x) + T_k C_2(H_0, x) \} \right] - \frac{\dot{\beta}_k}{2 - 2H} \left\{ (2 - 2\hat{H})C(\hat{H}, x) + T_k C_2(\hat{H}, x) \right\} + \Delta Z_{t_{ik}}^k - \Delta Z_{t_{ik}}^0, \]

almost surely, as \( i \rightarrow n_k; \quad k \rightarrow \infty. \quad (5.22) \]

Now observe that due to (4.2) and normality of \( \Delta M_{t_{ik}}^{H_k} \), \( \Delta Z_{t_{ik}}^k \sim N(0, 1) \), almost surely, as \( k \rightarrow \infty \). Thus, \( \frac{\Delta Z_{t_{ik}}^k - \dot{Q}_k(t_{ik}) \Delta w_{t_{ik}}^{H_k}}{c_2 \sqrt{2 - 2H_k} k^{1-H_k^+\nu_k} / \sqrt{n_k}} \rightarrow 0 \), almost surely, as \( k \rightarrow \infty \), for any \( u_k > 0 \). Using (5.15), (5.18) – (5.21) and the result \( t_{i+1,k}^{2-2H} - t_{ik}^{2-2H} \sim (2 - 2H) \frac{2^{2-2H}}{n_k} \), it is also easy to see that \( \frac{\dot{Q}_k(t_{ik}) \Delta w_{t_{ik}}^{H_k}}{c_2 \sqrt{2 - 2H_k} k^{1-H_k^+\nu_k} / \sqrt{n_k}} \rightarrow 0 \), almost surely, as \( k \rightarrow \infty \), so that \( \frac{\Delta Z_{t_{ik}}^0}{T_k^{2-H_0+u_k} / \sqrt{n_k}} \rightarrow 0 \), almost surely, as \( k \rightarrow \infty \). In the same way, \( \frac{\Delta Z_{t_{ik}}^0}{T_k^{2-H_0+u_k} / \sqrt{n_k}} \rightarrow 0 \), almost surely, as \( k \rightarrow \infty \). Using these results, it follows from (5.22) that

\[ \tilde{w}_{ik} \sim c_2 \frac{T_k^{2-H_0+u_k}}{\sqrt{n_k}}, \quad \text{almost surely, as } i \rightarrow n_k; \quad k \rightarrow \infty. \quad (5.23) \]

where, if \( \hat{H} > H_0 \), then \( u_k = 2(H_k - H_0) \) and

\[ c = \beta_0 \frac{c_{H_0}}{c_H^2} \frac{4\hat{H}^2}{H_0} k^{-1} C_2(H_0, x); \quad (5.24) \]

if \( \hat{H} < H_0 \), then \( u_k = 0 \) and

\[ c = -\dot{\beta} \frac{c_{H_0}}{c_H^2} \frac{4\hat{H}^2}{H_0} k^{-1} C_2(\hat{H}, x); \quad (5.25) \]

and if \( \hat{H} = H_0 \), then \( u_k = 0 \) and

\[ c = (\beta_0 - \dot{\beta}) \frac{4H_0}{c_{H_0}} k^{-1} C_2(H_0, x). \]
Hence, Lemma 16 together with (5.16), (5.17), (5.22) and (5.23), yields
\[
\sum_{i=1}^{n_k} Y_{ik} \tilde{w}_{ik} \sim c \frac{T_k^{2-H_0 + u_k}}{\sqrt{n_k}} \sum_{i=1}^{n_k} Y_{ik}, \text{ almost surely, as } k \to \infty. \tag{5.26}
\]
Hence, combining (5.26) with (5.14) yields, almost surely,
\[
- \sum_{i=1}^{n_k} \log \frac{\nu^2_{M_t k}}{\nu^2_{M_{tik}}} - \sum_{i=1}^{n_k} Y_{ik}^2 + \sum_{i=1}^{n_k} \frac{\nu^2_{M_t k}}{\nu^2_{M_{tik}}} + 2 \sum_{i=1}^{n_k} (\Delta M_{tik}^H - \Delta M_{tik}^H) \Delta M_{tik}^H \frac{\nu^2_{M_t k}}{\nu^2_{M_{tik}}} \sim n_k K T_k^{2(H_k - H_0)} - n_k \log K - n_k \log T_k^{2(H_k - H_0)} - n_k + c \frac{T_k^{2-H_0 + u_k}}{\sqrt{n_k}} \sum_{i=1}^{n_k} Y_{ik}, \text{ as } k \to \infty. \tag{5.27}
\]
To calculate the limit of the right hand side of (5.27), let us first consider the case \( \tilde{H} > H_0 \).
Since \( H_k \to H \) as \( k \to \infty \), it follows that for any \( \epsilon > 0 \), there exists \( k_0(\epsilon) \geq 1 \), such that for \( k \geq k_0(\epsilon) \), \( 2(\tilde{H} - H_0) - \epsilon < 2(H_k - H_0) < 2(\tilde{H} - H_0) + \epsilon \). Hence, for \( k \geq k_0(\epsilon) \),
\[
T_k^{2(H_k - H_0)} < T_k^{2(H_k - H_0)} < T_k^{2(H_k - H_0)} + \epsilon \quad \text{Hence, for } \tilde{H} > H_0, T_k^{2(H_k - H_0)} \to \infty, \text{ as } k \to \infty.
\]
Now, since \( \frac{T_k^{2-H_0 + u_k}}{n_k} \to 0 \) as \( k \to \infty \), it follows that
\[
\frac{T_k^{2-H_0 + u_k}}{n_k} \to 0, \text{ as } k \to \infty. \tag{5.28}
\]
Now note that \( \frac{1}{\sqrt{n_k}} \sum_{i=1}^{n_k} Y_{ik} \sim N(0, 1) \), for \( k \geq 1 \). That is, \( \frac{1}{\sqrt{n_k}} \sum_{i=1}^{n_k} Y_{ik} \) remains almost surely finite, as \( k \to \infty \). Hence, it follows using \( T_k^{2(H_k - H_0)} \to \infty \), as \( k \to \infty \), and (5.28), that almost surely,
\[
n_k K T_k^{2(H_k - H_0)} - n_k \log K - n_k \log T_k^{2(H_k - H_0)} - n_k + c \frac{T_k^{2-H_0 + u_k}}{\sqrt{n_k}} \sum_{i=1}^{n_k} Y_{ik} \sim n_k K T_k^{2(H_k - H_0)} \to \infty, \text{ as } k \to \infty.
\]
In other words, the right hand side of (5.27) tends to infinity as \( k \to \infty \), when \( \tilde{H} > H_0 \).
Now consider the case where \( \tilde{H} < H_0 \). Then the right hand side of (5.27) is equal to \( \frac{n_k K}{T_k^{2(H_0 - H_k)}} - n_k \log K + n_k \log T_k^{2(H_k - H_0)} - n_k + c \frac{T_k^{2-H_0 + u_k}}{\sqrt{n_k}} \sum_{i=1}^{n_k} Y_{ik} \). Now observe that
\[
\frac{n_k \log T_k^{2(H_0 - H_k)}}{T_k^{2(H_k - H_0)}} = \frac{1}{K} T_k^{2(H_0 - H_k)} \log T_k^{2(H_0 - H_k)} \to \infty, \text{ as } k \to \infty; \tag{5.29}
\]
\[
\frac{n_k \log T_k^{2(H_0 - H_k)}}{T_k^{2-H_0}} = 2(H_0 - H_k) \times \frac{n_k}{T_k^{2}} \times T_k^{H_0} \log T_k \to \infty, \text{ as } k \to \infty. \tag{5.30}
\]
Using (5.29) and (5.30) we see that, almost surely
\[
\frac{n_k K}{T_k^{2(H_0 - H_k)}} - n_k \log K + n_k \log T_k^{2(H_k - H_0)} - n_k + c \frac{T_k^{2-H_0 + u_k}}{\sqrt{n_k}} \sum_{i=1}^{n_k} Y_{ik} \sim n_k \log T_k^{2(H_k - H_0)} \to \infty, \text{ as } k \to \infty. \tag{5.31}
\]
In other words, for \( \tilde{H} \neq H_0 \), even if \( \tilde{\beta} = \beta_0 \),

\[
- \sum_{i=1}^{n_k} \log \frac{u_{M_{iH}}^2}{u_{M_{iH}}^2} - \sum_{i=1}^{n_k} Y_{i H}^2 + \sum_{i=1}^{n_k} Y_{i H}^2 \frac{u_{M_{iH}}^2}{u_{M_{iH}}^2} \Delta M_{iH}^H - \Delta M_{iH}^{H_0} \frac{u_{M_{iH}}^2}{u_{M_{iH}}^2} \Delta M_{iH}^H \rightarrow \infty, \text{ almost surely, as } k \rightarrow \infty. \tag{5.32}
\]

Since the term \( \sum_{i=1}^{n_k} \frac{(\Delta M_{iH}^H - \Delta M_{iH}^{H_0})^2}{u_{M_{iH}}^2} \Delta M_{iH}^H \rightarrow \infty \), almost surely, as \( k \rightarrow \infty \),

Now consider the case where \( H_k = H_0 \), for \( k = 1, 2, \ldots \), but \( \tilde{\beta} \neq \beta_0 \). In this case,

\[
- \sum_{i=1}^{n_k} \log \frac{u_{M_{iH}}^2}{u_{M_{iH}}^2} - \sum_{i=1}^{n_k} Y_{i H}^2 + \sum_{i=1}^{n_k} Y_{i H}^2 \frac{u_{M_{iH}}^2}{u_{M_{iH}}^2} = 0, \tag{5.33}
\]

and it follows from (5.28) and (5.26) that

\[
2 \sum_{i=1}^{n_k} \frac{(\Delta M_{iH}^H - \Delta M_{iH}^{H_0})^2}{u_{M_{iH}}^2} \Delta M_{iH}^H \frac{u_{M_{iH}}^2}{u_{M_{iH}}^2} \Delta M_{iH}^H \rightarrow (\beta_0 - \tilde{\beta}) \times \frac{4H_0}{cH_0} \times k^{-\delta} C_2(H_0, x) \times \frac{T^2 - H_0}{\sqrt{n_k}} \sum_{i=1}^{n_k} Y_{i H}, \tag{5.34}
\]

almost surely, as \( k \rightarrow \infty \).

Also, in a similar way of obtaining (5.26), we see that

\[
\frac{(\Delta M_{iH}^H - \Delta M_{iH}^{H_0})^2}{u_{M_{iH}}^2} \frac{u_{M_{iH}}^2}{u_{M_{iH}}^2} \Delta M_{iH}^H \rightarrow \frac{4H_0}{cH_0} \times \frac{1}{n_k} \left\{ (2 - 2H_0)C_1(H_0, x) + T_k C_2(H_0, x) \right\}^2, \text{ almost surely, as } i \rightarrow n_k; \ k \rightarrow \infty.
\]

\[
= \frac{4H_0^2}{n_k c^2 H^2 T_k^{-2}} \times \frac{1}{n_k} \left\{ (2 - 2H_0)C_1(H_0, x) + T_k C_2(H_0, x) \right\}^2. \tag{5.35}
\]
Hence, if, for \( k \geq 1, H_k = \hat{H} = H_0 \) and \( \beta \neq \beta_0 \), then it holds due to (5.35) that
\[
\sum_{i=1}^{n_k} \left( \frac{\Delta M_{i,k}^{H}}{v_{n,k}^2} - \frac{\Delta M_{i,k}^{H_0}}{v_{n,k}^2} \right)^2 \sim (\hat{\beta} - \beta_0)^2 \times \frac{4H_0^2}{c^2_H} \times \left\{ k_{H_0}^{-1}C_2(H_0, x) \right\}^2 \times T_k^{1-2H_0},
\]
almost surely, as \( k \to \infty \).

Combining (5.33), (5.34) and (5.36) yields in the case \( H_k = H_0 \), for \( k = 1, 2, \ldots \), and \( \hat{\beta} \neq \beta_0 \), yields
\[
\ell_k(\theta_k) - \ell_k(\theta_0) \sim (\beta_0 - \tilde{\beta}) \times \frac{AH_0^2}{c^2_H} \times k_{H_0}^{-1}C_2(H_0, x) \times \frac{T_k^{2-H_0}}{\sqrt{n_k}} \sum_{i=1}^{n_k} Y_{i,k}
\]
\[
+ (\hat{\beta} - \beta_0)^2 \times \frac{4H_0^2}{c^2_H} \times \left\{ k_{H_0}^{-1}C_2(H_0, x) \right\}^2 \times T_k^{1-2H_0}
\]
\[
\sim (\hat{\beta} - \beta_0)^2 \times \frac{4H_0^2}{c^2_H} \times \left\{ k_{H_0}^{-1}C_2(H_0, x) \right\}^2 \times T_k^{1-2H_0}
\]
\[
\to \infty, \text{ almost surely, as } k \to \infty.
\]

Now note that \( \ell_{T,n}(\theta) \) is continuous in \( \theta, T \) and \( n \). Using continuity, in conjunction with (5.20) and (5.21), it follows that for any other subsequence \( H_k \) converging to \( H_0 \), with \( \beta \neq \beta_0 \), it must hold that \( \ell_k(\theta_k) - \ell_k(\theta_0) \to \infty \), almost surely, as \( k \to \infty \).

We thus conclude that for any \( \delta > 0 \),
\[
\inf_{\theta \in \Theta : \| \theta - \theta_0 \| \geq \delta} \{ \ell_{T,n}(\theta) - \ell_{T,n}(\theta_0) \} \to \infty, \text{ almost surely, as } n \to \infty, \text{ } T \to \infty, \text{ such that } n/T^2 \to \infty.
\]
In other words, \( \hat{\theta}_{T,n} = (\hat{\beta}_{T,n}, \hat{H}_{T,n}) \xrightarrow{a.s.} \theta_0 = (\beta_0, H_0) \), as \( n \to \infty, T \to \infty \), such that \( n/T^2 \to \infty \). With stronger assumption, we can replace the assumption \( n/T^2 \to \infty \) with \( n/T \to \infty \), as the following theorem shows.

**Theorem 3** Let data \( x \) be generated from \( P_\theta \) and given \( x \) consider the SDE given by (2.2) for inference. In \( P_\theta \) and (2.2), let \( B(t, x_t) = B(x_t) \) and \( C(t, x_t) = C(x_t) \) and assume (H1) and (H2). Also assume that \( B/C \) is a constant. If \( \Theta = \mathfrak{B} \times [\eta, 1-\eta] \) is the parameter space for \( \theta = (\beta, H) \), where \( \eta > 0 \) and \( \mathfrak{B} \) is a compact subset of the real line \( \mathbb{R} \), then \( \theta_{T,n} \to \theta_0 \) almost surely with respect to \( P_\theta \), as \( T \to \infty, n \to \infty \), such that \( n/T \to \infty \).

**Proof.** Since \( B/C \) is a constant, we have
\[
\frac{d}{dt} \int_0^1 u^{\frac{1}{2}-H}(1-u)^{\frac{1}{2}-H}B(x_{tu}) \frac{C(x_{tu})}{C(x_t)} du \bigg|_{t=t_i} = 0
\]
and \( C_1(H, x) \equiv C_1(H) \). In this case it follows from (5.22) that
\[
\bar{w}_{i,k} \sim c \frac{T_k^{1-H_0+u_k}}{\sqrt{n_k}}, \text{ almost surely, as } i \to n_k; \text{ } k \to \infty.
\]
where, if \( \hat{H} > H_0 \), then \( u_k = 2(H_k - H_0) \) and
\[
c = \beta_0 \frac{c_{H_0}}{c_H} \frac{4H^2}{H_0} k_{H_0}^{-1}(2-2H_0)C_1(H_0);
\]
if \( \tilde{H} < H_0 \), then \( u_k = 0 \) and

\[
c = -\beta \frac{c^{C_H} 4\tilde{H}^2}{c_H} \frac{H_0}{k \tilde{H}^2} (2 - 2\tilde{H})C_1(\tilde{H});
\]

and if \( \tilde{H} = H_0 \), then \( u_k = 0 \) and

\[
c = (\beta_0 - \tilde{\beta}) \frac{4H_0}{c_H} k^{-1}(2 - 2H_0)C_1(H_0);
\]

These imply that

\[
2 \sum_{i=1}^{n_k} \frac{(\Delta M_{t_{ik}}^H - \Delta M_{t_{ik}}^H) \Delta M_{t_{ik}}^H \frac{v^2_{M_{t_{ik}}}}{v^2_{M_{t_{ik}}^H}}}{v^2_{M_{t_{ik}}^H}} = \sum_{i=1}^{n_k} Y_{t_{ik}} \tilde{u}_{ik} \sim c \frac{T_{k^{-1} - H_0 + u_k}}{\sqrt{n_k}} \sum_{i=1}^{n_k} Y_{t_{ik}}, \text{ almost surely, as } k \to \infty.
\]

and

\[
\sum_{i=1}^{n_k} \frac{(\Delta M_{t_{ik}}^H - \Delta M_{t_{ik}}^H) \Delta M_{t_{ik}}^H \frac{v^2_{M_{t_{ik}}}}{v^2_{M_{t_{ik}}^H}}}{v^2_{M_{t_{ik}}^H}} \sim (\tilde{\beta} - \beta_0)^2 \times \frac{4H_0^2}{c_H^2} \times \left\{ k^{-1}(2 - 2H_0)C_1(H_0) \right\}^2 \times T^{2 - 2H_0},
\]

almost surely, as \( k \to \infty \). \hfill (5.39)

The rest of the proof follows in the same way as that of Theorem 2 by setting \( T_k/n_k \to 0 \) instead of \( T_k^2/n_k \to 0 \), as \( k \to \infty \). When \( H_k = H_0 \) and \( \tilde{\beta} \neq \beta_0 \), we must replace \( T_k^{4 - 2H_0} \) with \( T_k^{2 - 2H_0} \) in (5.36).

6 Asymptotic normality of \( \beta \)

**Theorem 4** Let data \( x \) be generated from \( P_{\theta_0} \) and given \( x \) consider the SDE given by (2.2) for inference. In \( P_{\theta_0} \) and (2.2), let \( B(t, x_t) = B(x_t) \) and \( C(t, x_t) = C(x_t) \) and assume (H1) and (H2). Also, let \( \Theta = \mathbb{R} \times [\eta, 1 - \eta] \) be the parameter space for \( \theta = (\beta, H) \). Further assume that there exist real valued functions \( C_1 \) and \( C_2 \) such that for any \( H \in [\eta, 1 - \eta] \), where \( \eta > 0 \), (5.1) and (5.2) hold. Then almost surely with respect to \( P_{\theta_0} \),

\[
\frac{T^{2 - H_{T,n}}}{a_{H_{T,n}}} (\tilde{\beta}_{T,n} - \beta_0) \to \tilde{Z} \sim N(0, 1), \text{ as } T \to \infty; \quad n \to \infty; \quad \frac{n}{T^2} \to \infty.
\]

where, for any \( H \in [\eta, 1 - \eta] \), \( a_H = \frac{k_{H} c_{H}}{2 c_{H}^2 (H, x)} \).

**Proof.** To obtain the asymptotic distribution of \( \beta \), let us minimize \( \ell_{T,n}(\theta) \) by differentiation with respect to \( \beta \), fixing \( H = H_0 \). In this case, since \( \ell_{T,n}(\beta, H_0) \) is a quadratic form in \( \beta \), it is easy to verify that differentiating \( \ell_{T,n}(\beta, H_0) \) with respect to \( \beta \) and solving for \( \beta \) by setting the differential equal to zero, and finally applying asymptotic equivalence to the solution, is the same as first obtaining the asymptotic equivalence of \( \ell_{T,n}(\beta, H_0) - \ell_{T,n}(\beta_0, H_0) \), and then differentiating the asymptotic equivalent with respect to \( \beta \) and finding the solution by setting the differential to zero.
Indeed, in the same way as (5.37),

\[
\ell_{T,n}(\beta, H_0) - \ell_{T,n}(\beta_0, H_0) \sim (\beta_0 - \beta) \times \frac{4H_0}{c_{H_0}} \times k_{H_0}^{-1}C_2(H_0, x) \times \frac{T^{2-H_0}}{\sqrt{n}} \sum_{i=1}^{n} Y_i, \\
+ (\beta - \beta_0)^2 \times \frac{4H_0^2}{c_{H_0}^2} \times \left\{ k_{H_0}^{-1}C_2(H_0, x) \right\}^2 \times T^{4-2H_0},
\]

almost surely, as \( T \to \infty; \ n \to \infty; \ \frac{n}{T^2} \to \infty. \) \( (6.1) \)

Differentiating the asymptotic equivalent on the right hand side of (6.1) with respect to \( \beta \) and setting equal to zero we obtain the solution \( \hat{\beta}_{T,n} \) as

\[
T^{2-H_0}(\hat{\beta}_{T,n} - \beta_0) = a_{H_0}Z_{T,n},
\]

where

\[
a_{H_0} = \frac{k_{H_0}c_{H_0}}{2H_0C_2(H_0, x)}; \quad (6.3)
\]

\[
Z_{T,n} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_i. \quad (6.4)
\]

Note that for any \( T > 0 \) and \( n \geq 1 \), \( Z_{T,n} \) is equivalent in distribution to \( Z \), where \( Z \sim N(0, 1) \). From (6.2), (6.3) and (6.4), it follows that

\[
T^{2-H_0}(\hat{\beta}_{T,n} - \beta_0) \to a_{H_0}Z \sim N(0, a_{H_0}^2), \text{ almost surely, as } T \to \infty; \ n \to \infty; \ \frac{n}{T^2} \to \infty. \quad (6.5)
\]

Hence, continuity of logarithm ensures that

\[
(2-H_0) \log T - \log |a_{H_0}| + \log |\hat{\beta}_{T,n} - \beta_0| \to \log |Z|, \text{ almost surely, as } T \to \infty; \ n \to \infty; \ \frac{n}{T^2} \to \infty. \quad (6.6)
\]

Now note that for any fixed \( \beta \in \mathbb{R} \), the same method of the proof of Theorem 2 shows that \( \hat{H}_{T,n} \to H_0 \). Thus, due to continuity of \( a_H \) with respect to \( H \),

\[
(2-H_0) \log T - \log |a_{H_0}| + \log |\hat{\beta}_{T,n} - \beta_0| \sim (2-\hat{H}_{T,n}) \log T - \log |a_{\hat{H}_{T,n}}| + \log |\hat{\beta}_{T,n} - \beta_0|,
\]

almost surely, as \( T \to \infty; \ n \to \infty; \ \frac{n}{T^2} \to \infty. \) Hence,

\[
(2-\hat{H}_{T,n}) \log T - \log |a_{\hat{H}_{T,n}}| + \log |\hat{\beta}_{T,n} - \beta_0| \to \log |Z|, \text{ almost surely, as } T \to \infty; \ n \to \infty; \ \frac{n}{T^2} \to \infty. \quad (6.7)
\]

Using continuity of exponential we obtain from (6.7) that

\[
\frac{T^{2-\hat{H}_{T,n}}}{a_{\hat{H}_{T,n}}} |\hat{\beta}_{T,n} - \beta_0| \to |Z|, \text{ almost surely, as } T \to \infty; \ n \to \infty; \ \frac{n}{T^2} \to \infty. \quad (6.8)
\]

Now let \( \tilde{Z} \) be a random variable that takes \( |Z| \) and \( -|Z| \) with probabilities 1/2 and 1/2. Then \( \tilde{Z} \sim N(0, 1) \). Since due to (6.8), \( |Z| \) and \( -|Z| \) are the asymptotic forms of \( \frac{T^{2-\hat{H}_{T,n}}}{a_{\hat{H}_{T,n}}} |\hat{\beta}_{T,n} - \beta_0| \).
and \(-\frac{q^{2-H_{T,n}}}{\alpha_{H_{T,n}}} |\hat{\beta}_{T,n} - \beta_0|\), respectively, it follows that

\[
\frac{T^{2-H_{T,n}}}{\alpha_{H_{T,n}}} \left( \hat{\beta}_{T,n} - \beta_0 \right) \to \tilde{Z} \sim N(0, 1), \text{ almost surely, as } T \to \infty; \ n \to \infty; \ \frac{n}{T^2} \to \infty.
\]

This completes the proof. \( \blacksquare \) As in the case for consistency, the stronger assumption that \( B/C \) is constant, allows replacement of \( T^{2-H_{T,n}} \) in the asymptotic normality of Theorem 4 with \( T^{1-H_{T,n}} \), which is associated with \( n/T \to \infty \) instead of \( n/T^2 \to \infty \) associated with the former rate.

**Theorem 5** Let data \( x \) be generated from \( P_{\theta_0} \) and given \( x \) consider the SDE given by (2.2) for inference. In \( P_{\theta_0} \) and (2.2), let \( B(t, x_t) = B(x_t) \) and \( C(t, x_t) = C(x_t) \) and assume \((H1)\) and \((H2)\). Also, let \( \Theta = \mathbb{R} \times [\eta, 1-\eta] \) be the parameter space for \( \theta = (\beta, H) \), where \( \eta > 0 \). Further assume that \( B/C \) is constant. Then almost surely with respect to \( P_{\theta_0} \),

\[
\frac{T^{1-H_{T,n}}}{\alpha_{H_{T,n}}} \left( \hat{\beta}_{T,n} - \beta_0 \right) \to \tilde{Z} \sim N(0, 1), \text{ as } T \to \infty; \ n \to \infty; \ \frac{n}{T} \to \infty.
\]

where, for any \( H \in [\eta, 1-\eta] \), \( \alpha_H = \frac{k_H c_H}{4H(1-H)c_1(H)} \).

**Proof.** In this case we have

\[
\ell_{T,n}(\beta, H_0) - \ell_{T,n}(\beta_0, H_0) \sim (\beta_0 - \beta) \times \frac{4H_0}{c_{H_0}} \times k_{H_0}^{-1}(2 - 2H_0)C_1(H_0) \times \frac{T^{1-H_0}}{\sqrt{n}} \sum_{i=1}^{n} Y_i \times \left( 1 + \beta - \beta_0 \right)^2 \times \frac{4H_0^2}{c_{H_0}^2} \times \left( k_{H_0}^{-1}(2 - 2H_0)C_1(H_0) \right)^2 \times T^{2-2H_0},
\]

almost surely, as \( T \to \infty; \ n \to \infty; \ \frac{n}{T} \to \infty. \)

The rest of the proof follows in the same way as that of Theorem 4. \( \blacksquare \)

**Remark 6** It is worth noting that Theorems 4 and 5 show that for asymptotic normality of \( \beta \), the parameter space is no longer needed to be compact, unlike Theorems 2 and 3. Here the entire real line is allowed for \( \beta \). These theorems also ensure that \( \hat{\beta}_{T,n} \overset{a.s.}{\longrightarrow} \beta_0 \).

### 7 Inconsistency of the MLE of \( \beta \) when the estimator of \( H \) is inconsistent

Remark 6 shows that consistency of \( \hat{\beta}_{T,n} \) continues to hold even without the assumption that \( \mathcal{B} \) is compact. However, if \( H \) is estimated with an inconsistent estimator, say, \( H_{T,n} \) which does not converge to \( H_0 \), then even consistency does not hold for the MLE of \( \beta \) for \( \mathcal{B} \) either compact or non-compact. Theorems 7 and 8 formalize this in the situations where \( B/C \) is non-constant and constant, respectively. Notably, the results hold simply as \( T \to \infty \) and \( n \to \infty \), irrespective of whether or not \( n \) grows faster than \( T \).

**Theorem 7** Let data \( x \) be generated from \( P_{\theta_0} \) and given \( x \) consider the SDE given by (2.2) for inference. In \( P_{\theta_0} \) and (2.2), let \( B(t, x_t) = B(x_t) \) and \( C(t, x_t) = C(x_t) \) and assume \((H1)\)
and (H2). Also, let \( \Theta = \mathbb{R} \times [\eta, 1 - \eta] \) be the parameter space for \( \theta = (\beta, H) \). Further assume that there exist real valued functions \( C_1 \) and \( C_2 \) such that for any \( H \in [\eta, 1 - \eta] \), where \( \eta > 0 \), (5.1) and (5.2) hold. For each \( T > 0 \) and \( n \geq 1 \), let \( H \) be estimated by \( H_{T,n}^* \), where \( H_{T,n}^* \overset{a.s.}{\rightarrow} H \) with respect to \( P_{\theta_0} \), with \( \tilde{H} \neq H_0 \). Then with respect to \( P_{\theta_0} \), the MLE of \( \beta \) is almost surely an inconsistent estimator of \( \beta_0 \), for \( \mathcal{B} \) either compact or non-compact.

**Proof.** Differentiating \( \ell_{T,n}(\beta, H_{T,n}^*) - \ell_{T,n}(\beta_0, H_0) \) with respect to \( \beta \) and solving for \( \beta \) after setting the differential equal to zero yields the solution \( \beta = \tilde{\beta}_{T,n} \), where

\[
\tilde{\beta}_{T,n} \sim \beta_0 \left( \frac{d_{H_0} T^{2-2H_0} + b_{H_0} T^{3-2H_0}}{d_{H} T^{2-2H_{T,n}^*} + b_{H} T^{3-2H_{T,n}^*}} \right) + \frac{C_3 Z_{T,n} T^{2-H_0} \left( [2 - 2\tilde{H}] T^{1-2H_{T,n}^*} C_1(\tilde{H}, x) + T^{2-2H_{T,n}^*} C_2(\tilde{H}, x) \right)}{(d_{H} T^{2-2H_{T,n}^*} + b_{H} T^{3-2H_{T,n}^*})^2},
\]

almost surely, as \( T \to \infty \) and \( n \to \infty \),

where \( Z_{T,n} = n^{-1/2} \sum_{i=1}^{n} Y_i \sim N(0, 1) \) for any \( T > 0 \) and \( n \geq 1 \), \( C_3 = \frac{c_{H_0}}{H_0} \times k_{\tilde{H}}^{-1} \) and for any \( H \in [\eta, 1 - \eta] \), \( d_H = 2k_{\tilde{H}}^{-1} (1 - H) C_1(H, x) \); \( b_H = k_{\tilde{H}}^{-1} C_2(H, x) \). The second term (7.2) is asymptotically equivalent to \( T^{-H_0-2(1-H_{T,n}^*)} \), which tends to zero almost surely, as \( T \to \infty \), irrespective of \( \tilde{H} > H_0 \) or \( \tilde{H} < H_0 \). However, the first term (7.1) goes to infinity if \( \tilde{H} > H_0 \), and to zero if \( \tilde{H} < H_0 \). In other words, \( \tilde{\beta}_{T,n} \) tends to infinity if \( \tilde{H} > H_0 \) and to zero if \( \tilde{H} < H_0 \), almost surely, as \( T \to \infty \). Hence, \( \tilde{\beta}_{T,n} \) given above is an inconsistent estimator of \( \beta_0 \). Note that the conclusion does not depend upon compactness or non-compactness of the parameter space \( \mathcal{B} \), since in neither case \( \tilde{\beta}_{T,n} \) can converge to \( \beta_0 \), for all \( \beta_0 \in \mathcal{B} \).

**Theorem 8** Let data \( x \) be generated from \( P_{\theta_0} \) and given \( x \) consider the SDE given by (2.2) for inference. In \( P_{\theta_0} \) and (2.2), let \( B(t, x_t) = B(x_t) \) and \( C(t, x_t) = C(x_t) \) and assume (H1) and (H2). Also, let \( \Theta = \mathbb{R} \times [\eta, 1 - \eta] \) be the parameter space for \( \theta = (\beta, H) \). Further assume that \( B/C \) is a constant. For each \( T > 0 \) and \( n \geq 1 \), let \( H \) be estimated by \( H_{T,n}^* \), where \( H_{T,n}^* \overset{a.s.}{\rightarrow} \tilde{H} \) with respect to \( P_{\theta_0} \), with \( \tilde{H} \neq H_0 \). Then the MLE of \( \beta \) is almost surely an inconsistent estimator of \( \beta_0 \), for \( \mathcal{B} \) either compact or non-compact.

**Proof.** When \( B/C \) is a constant, differentiating \( \ell_{T,n}(\beta, H_{T,n}^*) - \ell_{T,n}(\beta_0, H_0) \) with respect to \( \beta \) and solving for \( \beta \) after setting the differential equal to zero yields the solution \( \beta = \tilde{\beta}_{T,n} \), where

\[
\tilde{\beta}_{T,n} \sim \beta_0 \left( \frac{d_{H_0} H_{T,n}^* - H_0}{d_{H} (H_{T,n}^* - H_0)} \right) + C_3 Z_{T,n} \left( \frac{2 - 2\tilde{H}}{d_{H}^2} \right) T^{1-H_0-2(1-H_{T,n}^*)},
\]

almost surely, as \( T \to \infty \); \( n \to \infty \),

where the definitions of \( Z_{T,n} \) and \( C_3 \) remain the same as in the proof of Theorem 7, and \( d_{H} = 2k_{\tilde{H}}^{-1} (1 - H) C_1(H) \). Note that (7.3) follows from the previous step since

\[
\frac{T^{2(H_{T,n}^* - H_0)}}{T^{1-H_0-2(1-H_{T,n}^*)}} = T^{1-H_0} \to \infty, \text{ as } T \to \infty; \ n \to \infty,
\]

since \( H_0 < 1 \).
Hence if \( \hat{H} > H_0 \) and \( \hat{H} < H_0 \), then using (7.3) it follows that \( \hat{\beta}_{T,n} \to \infty \) and \( \hat{\beta}_{T,n} \to 0 \), respectively, almost surely, as \( T \to \infty \) and \( n \to \infty \).

Again, note that the conclusion does not depend upon compactness or non-compactness of the parameter space \( \mathcal{B} \), since in neither case \( \hat{\beta}_{T,n} \) can converge to \( \beta_0 \), for all \( \beta_0 \in \mathcal{B} \).

**Remark 9** Theorems [7] and [8] show that if the estimator of \( H \) is inconsistent, then this induces inconsistency in the MLE of \( \beta \). Thus, these results caution against simple-minded usage of estimators of \( H \) that lack theoretical validation of consistency. Hence, the heuristic estimators of \( H \), discussed in Section 6, need not be automatic choices, as generally treated by practitioners. Since we showed strong consistency of the MLE of \( H \) in Theorems [7] and [8], we recommend MLE as the appropriate estimator of \( H \).

### 8 Bayesian posterior consistency

For any set \( A \) in the Borel sigma-algebra of the parameter space \( \Theta \), let \( I_A(\theta) = 1 \) if \( \theta \in A \) and 0, otherwise.

**Theorem 10** Let data \( x \) be generated from \( P_{\theta_0} \) and given \( x \), consider the SDE given by (2.2) for inference. In \( P_{\theta_0} \) and (2.2), let \( B(t, x_t) = B(x_t) \) and \( C(t, x_t) = C(x_t) \) and assume (H1) and (H2). Also, let \( \Theta = \mathbb{R} \times [\eta, 1 - \eta] \) be the parameter space for \( \theta = (\beta, H) \), where \( \eta > 0 \). Further assume (5.1) and (5.2). Then the posterior distribution of \( \theta = (\beta, H) \) is strongly consistent in the sense that for any prior on \( \Theta \) that includes the true value \( \theta_0 \) in its support, \( \pi(\theta \in A|x_t; i = 1, \ldots, n) \xrightarrow{\text{a.s.}} I_A(\theta_0) \) with respect to \( P_{\theta_0} \), as \( T \to \infty \), \( n \to \infty \), such that \( n/T^2 \to \infty \), for any set \( A \) in the Borel sigma-algebra of \( \Theta \).

**Proof.** As argued in Remark 6, the MLE \( \hat{\theta}_{T,n} \) is strongly consistent for \( \theta_0 \), even when the parameter space of \( \beta \) is \( \mathbb{R} \). Hence, the proof follows by a simple application of Doob’s theorem (see, for example, Theorem 7.78 of Schervish (1995)), which is easily seen to hold valid in our case as \( T \to \infty \), \( n \to \infty \), such that \( n/T^2 \to \infty \). \( \square \) For constant \( B/C \), the following holds for \( n/T \to \infty \), as to be expected.

**Theorem 11** Let data \( x \) be generated from \( P_{\theta_0} \) and given \( x \), consider the SDE given by (2.2) for inference. In \( P_{\theta_0} \) and (2.2), let \( B(t, x_t) = B(x_t) \) and \( C(t, x_t) = C(x_t) \) and assume (H1) and (H2). Also assume that \( B/C \) is a constant. If \( \Theta = \mathbb{R} \times [\eta, 1 - \eta] \) is the parameter space for \( \theta = (\beta, H) \), where \( \eta > 0 \), then the posterior distribution of \( \theta = (\beta, H) \) is strongly consistent in the sense that for any prior on \( \Theta \) that includes the true value \( \theta_0 \) in its support, \( \pi(\theta \in A|x_t; i = 1, \ldots, n) \xrightarrow{\text{a.s.}} I_A(\theta_0) \) with respect to \( P_{\theta_0} \), as \( T \to \infty \), \( n \to \infty \), such that \( n/T \to \infty \), for any set \( A \) in the Borel sigma-algebra of \( \Theta \).

**Proof.** Again, the proof is a simple application of Doob’s theorem. \( \square \)

Note that although Doob’s theorem ensures consistency of the posterior distributions, it does not specify which neighborhoods the posteriors concentrate on. The theorems below show the type of neighborhoods where the posterior of \( \beta \), given \( H = \hat{H}_{T,n} \), concentrates on. It is easy to verify that such neighborhoods are also the Kullback-Leibler neighborhoods.

**Theorem 12** Let data \( x \) be generated from \( P_{\theta_0} \) and given \( x \), consider the SDE given by (2.2) for inference. In \( P_{\theta_0} \) and (2.2), let \( B(t, x_t) = B(x_t) \) and \( C(t, x_t) = C(x_t) \) and assume (H1) and (H2). Further assume (5.1) and (5.2). Let \( N_0 \) be any neighborhood of \( \beta_0 \) containing \( C_\epsilon = \{ \beta : (\beta - \beta_0)^2 g(H_0) < \epsilon \} \), for any \( \epsilon > 0 \), where, for any \( \epsilon \in [\eta, 1 - \eta] \), \( g(H) = \frac{4H^2}{c_H} \times \)
\[ \{ k_H^{-1}C_2(H, x) \}^2 \]. Let \( \pi(\beta) \) be any Lebesgue-dominated prior for \( \beta \) that satisfies \( \pi(C_\epsilon) > 0 \), for any \( \epsilon > 0 \). Then, the posterior distribution of \( \beta \in \mathbb{R} \) given \( H = H_{T,n} \) satisfies
\[
\pi \left( \beta \in N_0 | H = H_{T,n}, x_i; \; i = 1, \ldots, n \right) \overset{a.s.}{\longrightarrow} 1,
\]
with respect to \( P_{\theta_0} \), as \( T \to \infty, n \to \infty \), such that \( n/T^2 \to \infty \).

**Proof.** Note that
\[
\pi(\beta \in N_0 | H = H_0, x_i; \; i = 1, \ldots, n) = \frac{\int_{N_0} \exp \left\{ -\frac{1}{2} (\ell_{T,n}(\beta, H_0) - \ell_{T,n}(\beta_0, H_0)) \right\} d\pi(\beta)}{\int_{N_0} \exp \left\{ -\frac{1}{2} (\ell_{T,n}(\beta, H_0) - \ell_{T,n}(\beta_0, H_0)) \right\} d\pi(\beta)}
\]
(8.1)
If we can show that
\[
\frac{\int_{N_0} \exp \left\{ -\frac{1}{2} (\ell_{T,n}(\beta, H_0) - \ell_{T,n}(\beta_0, H_0)) \right\} d\pi(\beta)}{\int_{N_0} \exp \left\{ -\frac{1}{2} (\ell_{T,n}(\beta, H_0) - \ell_{T,n}(\beta_0, H_0)) \right\} d\pi(\beta)} \to \infty, \text{ almost surely, as } T \to \infty; \; n \to \infty; \; \frac{n}{T^2} \to \infty,
\]
(8.2)
then it would prove that
\[
\pi(\beta \in N_0 | H = H_0, x_i; \; i = 1, \ldots, n) \overset{a.s.}{\longrightarrow} 1.
\]
(8.3)
Now,
\[
\pi(\beta \in N_0 | H = H_0, x_i; \; i = 1, \ldots, n) = E(I_{N_0}(\beta) | H = H_0, x_i; \; i = 1, \ldots, n),
\]
(8.4)
and the expression on the right hand side is a martingale. By the martingale convergence theorem (see, for example, Theorem B.118 of Schervish [1995]),
\[
E(I_{N_0}(\beta) | H = H_0, x_i; \; i = 1, \ldots, n) \overset{a.s.}{\longrightarrow} E(I_{N_0}(\beta) | H = H_0, x_i; \; t \geq 0), \text{ as } T \to \infty, \; n \to \infty \text{ and } n/T^2 \to \infty.
\]
This, combined with (8.3) and (8.4), shows that
\[
E(I_{N_0}(\beta) | H = H_0, x_i; \; t \geq 0) = 1.
\]
(8.5)
Since
\[
\pi \left( \beta \in N_0 | H = H_{T,n}, x_i; \; i = 1, \ldots, n \right) = E \left( I_{N_0}(\beta) | H = H_{T,n}, x_i; \; i = 1, \ldots, n \right)
\]
is also a martingale, by the martingale convergence theorem, almost sure consistency of \( H_{T,n} \) and (8.5),
\[
\pi \left( \beta \in N_0 | H = H_{T,n}, x_i; \; i = 1, \ldots, n \right) \overset{a.s.}{\longrightarrow} E \left( I_{N_0}(\beta) | H = H_0, x_i; \; t \geq 0 \right) = 1,
\]
as \( T \to \infty, \; n \to \infty, \; \frac{n}{T^2} \to \infty.\)
(8.6)
Thus, we only require to prove that (8.2) holds. In this regard, first let \( \tilde{C}_\epsilon = \{ \beta : (\beta - \beta_0)^2 g(H_0) \leq \epsilon \} \),
the closure of $C$, and note that
\[
\int_{N_0} \exp \left\{ -\frac{1}{2} (\ell_{T,n}(\beta, H_0) - \ell_{T,n}(\beta_0, H_0)) \right\} \, d\pi(\beta)
\geq \int_{C} \exp \left\{ -\frac{T^{4-2H_0}}{2} \sup_{\beta \in C} (\ell_{T,n}(\beta, H_0) - \ell_{T,n}(\beta_0, H_0)) \right\} \, d\pi(\beta). \tag{8.7}
\]

Now, from (6.11), we see that, point-wise with respect to $\beta \in \bar{C}$,
\[
\frac{(\ell_{T,n}(\beta, H_0) - \ell_{T,n}(\beta_0, H_0))}{T^{4-2H_0}} \xrightarrow{a.s.} (\beta - \beta_0)^2 g(H_0), \text{ as } T \to \infty, \, n \to \infty, \, \frac{n}{T^2} \to \infty. \tag{8.8}
\]

Hence, for $\beta_1, \beta_2 \in \bar{C}$, it follows from (8.8) and continuity of the absolute function, that
\[
\frac{|\ell_{T,n}(\beta_1, H_0) - \ell_{T,n}(\beta_2, H_0)|}{T^{4-2H_0}} \xrightarrow{a.s.} g(H_0) |\beta_1 - \beta_2| |\beta_1 + \beta_2 - 2\beta_0|, \text{ as } T \to \infty, \, n \to \infty, \, \frac{n}{T^2} \to \infty. \tag{8.9}
\]

Since $\beta_1, \beta_2 \in \bar{C}$, they are bounded, and so, $|\beta_1 + \beta_2 - 2\beta_0| \leq M$, for some $M > 0$. This, and (8.9) implies that
\[
\frac{(\ell_{T,n}(\beta, H_0) - \ell_{T,n}(\beta_0, H_0))}{T^{4-2H_0}} \text{ is stochastically equicontinuous, and hence (see, for example, Newey (1991)),}
\]
\[
\sup_{\beta \in C} \left| \frac{(\ell_{T,n}(\beta, H_0) - \ell_{T,n}(\beta_0, H_0))}{T^{4-2H_0}} - (\beta - \beta_0)^2 g(H_0) \right| \xrightarrow{a.s.} 0. \tag{8.10}
\]

Now, due to (8.10), for $\epsilon > 0$, there exists $n_0(\epsilon) \geq 1$ and $T_0(\epsilon) > 0$ such that $n \geq n_0(\epsilon)$ and $T \geq T_0(\epsilon)$,
\[
\sup_{\beta \in C} \frac{(\ell_{T,n}(\beta, H_0) - \ell_{T,n}(\beta_0, H_0))}{T^{4-2H_0}} \leq \sup_{\beta \in C} \left| \frac{(\ell_{T,n}(\beta, H_0) - \ell_{T,n}(\beta_0, H_0))}{T^{4-2H_0}} - (\beta - \beta_0)^2 g(H_0) \right| + \sup_{\beta \in C} (\beta - \beta_0)^2 g(H_0)
\leq \epsilon + (\beta^* - \beta_0)^2 g(H_0)
\leq 2\epsilon, \tag{8.11}
\]

where $\beta^* \in \bar{C}$ is such that $\sup_{\beta \in \bar{C}} (\beta - \beta_0)^2 g(H_0) = (\beta^* - \beta_0)^2 g(H_0) \leq \epsilon$, thanks to compactness of $\bar{C}$. It follows from (8.11) and (8.7), that for $n \geq n_0(\epsilon)$ and $T \geq T_0(\epsilon)$,
\[
\int_{N_0} \exp \left\{ -\frac{1}{2} (\ell_{T,n}(\beta, H_0) - \ell_{T,n}(\beta_0, H_0)) \right\} \, d\pi(\beta)
\geq \int_{C} \exp \left\{ -\epsilon T^{4-2H_0} \right\} \, d\pi(\beta) = \pi(\bar{C}) \exp \left\{ -\epsilon T^{4-2H_0} \right\}. \tag{8.12}
\]
Now observe that
\[
\int_{N_0} \exp \left\{ -\frac{1}{2} \left( \ell_{T,n}(\beta, H_0) - \ell_{T,n}(\beta_0, H_0) \right) \right\} d\pi(\beta) \\
\leq \int_{C_\epsilon} \exp \left\{ -\frac{T^{4-2H_0}}{2} \inf_{\beta \in C_\epsilon} \left( \ell_{T,n}(\beta, H_0) - \ell_{T,n}(\beta_0, H_0) \right) \right\} d\pi(\beta). \tag{8.13}
\]

Also, there exists \( \tilde{\beta} \neq \beta_0 \) such that
\[
\frac{\inf_{\beta \in C_\epsilon} (\ell_{T,n}(\beta, H_0) - \ell_{T,n}(\beta_0, H_0))}{T^{4-2H_0}} \geq \frac{\left( \ell_{T,n}(\tilde{\beta}, H_0) - \ell_{T,n}(\beta_0, H_0) \right)}{T^{4-2H_0}} \rightarrow (\tilde{\beta} - \beta_0)^2 g(H_0),
\]
almost surely, as \( T \rightarrow \infty; n \rightarrow \infty; \frac{n}{T^2} \rightarrow \infty. \tag{8.14}

It follows due to (8.14) that for any \( \epsilon > 0 \), there exists \( n_1(\epsilon) \geq 1 \) and \( T_1(\epsilon) > 0 \) such that \( n \geq n_1(\epsilon) \) and \( T \geq T_1(\epsilon) \),
\[
\frac{\inf_{\beta \in C_\epsilon} (\ell_{T,n}(\beta, H_0) - \ell_{T,n}(\beta_0, H_0))}{T^{4-2H_0}} \geq (\tilde{\beta} - \beta_0)^2 g(H_0) - \epsilon. \tag{8.15}
\]

Using (8.15) in (8.12) we obtain for \( n \geq n_1(\epsilon) \) and \( T \geq T_1(\epsilon) \),
\[
\int_{N_0} \exp \left\{ -\frac{1}{2} \left( \ell_{T,n}(\beta, H_0) - \ell_{T,n}(\beta_0, H_0) \right) \right\} d\pi(\beta) \\
\leq \int_{C_\epsilon} \exp \left\{ -\frac{T^{4-2H_0}}{2} \left( (\tilde{\beta} - \beta_0)^2 g(H_0) - \epsilon \right) \right\} d\pi(\beta) \\
= \pi(C_\epsilon) \exp \left\{ -\frac{T^{4-2H_0}}{2} \left( (\tilde{\beta} - \beta_0)^2 g(H_0) - \epsilon \right) \right\}. \tag{8.16}
\]

From (8.12) and (8.16) we obtain for \( n \geq \max\{n_0(\epsilon), n_1(\epsilon)\} \) and for \( T \geq \max\{T_0(\epsilon), T_1(\epsilon)\} \), that
\[
\int_{N_0} \exp \left\{ -\frac{1}{2} \left( \ell_{T,n}(\beta, H_0) - \ell_{T,n}(\beta_0, H_0) \right) \right\} d\pi(\beta) \\
\int_{N_0} \exp \left\{ -\frac{1}{2} \left( \ell_{T,n}(\beta, H_0) - \ell_{T,n}(\beta_0, H_0) \right) \right\} d\pi(\beta) \geq \frac{\pi(C_\epsilon)}{\pi(C_\epsilon)} \exp \left\{ -\frac{T^{4-2H_0}}{2} \left( (\tilde{\beta} - \beta_0)^2 g(H_0) - 2\epsilon \right) \right\} \\
\rightarrow \infty, \text{ almost surely, as } T \rightarrow \infty; n \rightarrow \infty; \frac{n}{T^2} \rightarrow \infty.
\]

This proves the theorem. \( \blacksquare \)

We now consider the case \( n/T \rightarrow \infty \); the result is summarized below as Theorem 13.

**Theorem 13** Let data \( x \) be generated from \( P_{\theta_0} \) and given \( x \) consider the SDE given by (2.2) for inference. In \( P_{\theta_0} \) and (2.2), let \( B(t, x_t) = B(x_t) \) and \( C(t, x_t) = C(x_t) \) and assume (H1) and (H2). Further assume that \( B/C \) is a constant. Let \( N_0 \) be any neighborhood of \( \beta_0 \) containing \( C_\epsilon = \{ \beta : (\beta - \beta_0)^2 g(H_0) < \epsilon \} \), for any \( \epsilon > 0 \), where, for any \( H \in [\eta, 1 - \eta] \),
\[
g(H) = \frac{4H^2}{c_\eta} \times \left\{ k_\eta^{-1} (2 - 2H) C_1(H) \right\}^2.
\]
Let \( \pi(\beta) \) be any Lebesgue-dominated prior for \( \beta \)
that satisfies $\pi(C_{\epsilon}) > 0$, for any $\epsilon > 0$. Then, the posterior distribution of $\theta \in \mathbb{R}$ given $H = \hat{H}_{T,n}$ satisfies $\pi \left( \beta \in N_0 | H = \hat{H}_{T,n}, x_i; \ i = 1, \ldots, n \right) \overset{a.s.}{\rightarrow} 1$ with respect to $P_{\theta_0}$, as $T \rightarrow \infty$, $n \rightarrow \infty$, such that $n/T \rightarrow \infty$.

**Proof.** The proof is similar to that of Theorem 12. \hfill \blacksquare

9 Asymptotic posterior normality

For asymptotic posterior normality of $\beta$ given $H = \hat{H}_{T,n}$, we make use of Theorem 7.89 of Schervish (1995). In Section 9.2, we state the general setup, regularity conditions referred to as (1) – (6), and the main result presented in Schervish (1995). It is worth noting that the setup, conditions and the result are not established for the purpose of increasing domain infill asymptotics as in our setup, namely, $T \rightarrow \infty$, $n \rightarrow \infty$ such that $n/T^2 \rightarrow \infty$ or $n/T \rightarrow \infty$. Nevertheless, our setup can be adapted to that of Schervish (1995), which continues to hold even in the increasing domain infill asymptotics situation.

9.1 Asymptotic posterior normality for fractional SDE

In our case of increasing domain infill asymptotics when $n/T^2 \rightarrow \infty$, we obtain the following result using the result in Schervish (1995):

**Theorem 14** Let data $x$ be generated from $P_{\theta_0}$ and given $x$ consider the SDE given by (2.2) for inference. In $P_{\theta_0}$ and (2.2), let $B(t, x_t) = B(x_t)$ and $C(t, x_t) = C(x_t)$ and assume (H1) and (H2). Further assume (5.1) and (5.2). Let $\Psi_{T,n} = \frac{T^{2-\tilde{H}_{T,n}}}{a_{\tilde{H}_{T,n}}} (\beta - \beta_0)$, where, for any $H \in [\eta, 1 - \eta]$, and

$$a_H = \frac{c_H k_H}{2HC_2(H, x)}.$$  \hspace{1cm} (9.1)

Let $\pi_{T,n}$ denote the posterior distribution of $\Psi_{T,n}$ given $x_i; i = 1, \ldots, n$, and $H = \hat{H}_{T,n}$, with respect to any Lebesgue-dominated prior. Then for each compact subset $B$ of $\mathbb{R}$ and each $\epsilon > 0$, the following holds:

$$\lim_{T \rightarrow \infty, n \rightarrow \infty, \frac{n}{T^2} \rightarrow \infty} P_{\theta_0} \left( \sup_{\psi \in B} |\pi_{T,n}(\psi) - \phi(\psi)| > \epsilon \right) = 0.$$  \hspace{1cm} \hspace{1cm} \hspace{1cm}

where $\phi(\cdot)$ denotes the density of the one-dimensional standard normal distribution.

**Proof.** Note that $\tilde{\ell}_{T,n}(\theta) = -\ell_{T,n}(\theta)/2$, where we have replaced $\hat{\ell}_{T,n}(\theta)$ with $\ell_{T,n}(\theta)$ for obvious reasons. For our purpose, we shall deal with $\tilde{\ell}_{T,n}(\beta, H_0)$. For the relevant Taylor’s series expansion considered in the proof of Theorem 7.89 provided in Schervish (1995), we consider the expansion of $\ell_{T,n}(\beta, H_0)$ around the MLE $\theta_{T,n} = (\beta_{T,n}, H_{T,n})$.

To obtain $\tilde{\ell}_{T,n}(\beta, H_0)$, we consider the above Taylor’s series expansion of $\tilde{\ell}_{T,n}(\beta, H_0)$ around $\hat{\theta}_{T,n}$ consisting of at least the first four terms, that is, at least till the term consisting of three derivatives. It is then easy to verify by direct calculation that for all $\beta \in \mathbb{R}$ and also for $\beta = \hat{\beta}_{T,n}$,

$$\tilde{\ell}_{T,n}(\beta, H_0) \sim -a_{\tilde{H}_{T,n}}^2 T^{4-2\tilde{H}_{T,n}}, \text{ almost surely, as } T \rightarrow \infty; \ n \rightarrow \infty; \ \frac{n}{T^2} \rightarrow \infty,$$  \hspace{1cm} (9.2)
where, for any $H$, $a_H$ is given by (9.1). Thus,

$$
\Sigma_{T,n} = -\left(\tilde{\ell}''_{T,n}(\hat{\beta}_T,n, H_0)\right)^{-1} a_H^2 T^{-4+2\hat{H}_{T,n}}, \text{ almost surely, as } T \to \infty; n \to \infty; \frac{n}{T^2} \to \infty. \tag{9.3}
$$

Now observe that the first three regularity conditions of Section S-2.0.1 are trivially satisfied, and because of (9.3), condition (4) is also satisfied.

Now note that condition (5), associated with $\rho_{T,n}$, is required to control equation (7.97) of the proof of Schervish (1995). In this regard, first observe that the aforementioned equation remains valid even if we consider the Taylor’s series expansion of $\tilde{\ell}_{T,n}(\beta, H_0)$ in equation (7.97) around $(\hat{\beta}_T,n, H_0)$ instead of around $(\hat{\beta}_n, \hat{H}_{T,n})$. To obtain $\rho_{T,n}$ corresponding to this case, we expand $\tilde{\ell}_{T,n}(\beta, H_0)$ $(\hat{\beta}_T,n, H_0)$ till at least the first four terms, as before. This yields $\rho_{T,n} \sim a_H^2 T^{-4+2H_0}$, almost surely, as $T \to \infty; n \to \infty; \frac{n}{T^2} \to \infty$.

Now, since $\tilde{\ell}_{T,n}(\beta, H_0)$ is quadratic in $\beta$, the quantity $\sup_{|\beta-\beta_0|>\delta} \rho_{T,n} \left[\tilde{\ell}_{T,n}(\beta, H_0) - \tilde{\ell}_{T,n}(\beta_0, H_0)\right]$ is $\rho_{T,n} \left[\tilde{\ell}_{T,n}(\beta^*, H_0) - \tilde{\ell}_{T,n}(\beta_0, H_0)\right]$, where $\beta^* = \beta_0 + \delta$ or $\beta_0 - \delta$. Hence,

$$
\sup_{|\beta-\beta_0|>\delta} \rho_{T,n} \left[\tilde{\ell}_{T,n}(\beta, H_0) - \tilde{\ell}_{T,n}(\beta_0, H_0)\right] < -K(\delta)
\iff \rho_{T,n} \left[\tilde{\ell}_{T,n}(\beta^*, H_0) - \tilde{\ell}_{T,n}(\beta_0, H_0)\right] < -K(\delta). \tag{9.4}
$$

The left hand side of (9.4) converges almost surely to $-(\beta^* - \beta_0)^2/2 = -\delta^2/2$, as $T \to \infty; n \to \infty; \frac{n}{T^2} \to \infty$. Hence, for $K(\delta) < \delta^2/2$, condition (5) holds.

To verify condition (6), note that by (9.2), it follows that for any $\epsilon > 0$, $\left|1 + \gamma^T \Sigma_2 \tilde{\ell}_n'(\theta) \Sigma_2^{-1} \gamma\right| < \epsilon$ as $T \to \infty; n \to \infty; \frac{n}{T^2} \to \infty$, for $\|\gamma\| = 1$. Here $\gamma$ is a scalar and so $\gamma^T = \gamma$ and $\|\gamma\| = |\gamma|$. Hence, condition (6) holds trivially.

Thus, the proof of Theorem 14 is complete.

The following theorem is regarding asymptotic posterior normality when $B/C$ is a constant, which allows $n/T \to \infty$, instead of $n/T^2 \to \infty$, as before.

**Theorem 15** Let data $x$ be generated from $P_{\theta_0}$ and given $x$ consider the SDE given by (2.2) for inference. In $P_{\theta_0}$ and (2.2), let $B(t, x_t) = B(x_t)$ and $C(t, x_t) = C(x_t)$ and assume (H1) and (H2). Further assume that $B/C$ is a constant. Let $\Psi_{T,n} = \mathbb{E}_{\pi_{T,n}}[\alpha_{H_{T,n}}(\beta - \beta_0)]$, where, for any $H \in [\eta, 1 - \eta]$, and

$$
\alpha_H = \frac{c_H k_H}{4H(1-H)C(1)}. 
$$

Let $\pi_{T,n}$ denote the posterior distribution of $\Psi_{T,n}$ given $x_t; i = 1, \ldots, n,$ and $H = \hat{H}_{T,n}$, with respect to any Lebesgue-dominated prior. Then for each compact subset $B$ of $\mathbb{R}$ and each $\epsilon > 0$, the following holds:

$$
\lim_{T \to \infty, n \to \infty, \frac{n}{T^2} \to \infty} P_{\theta_0} \left(\sup_{\psi \in B} |\pi_{T,n}(\psi) - \phi(\psi)| > \epsilon\right) = 0.
$$

where $\phi(\cdot)$ denotes the density of the one-dimensional standard normal distribution.

**Proof.** The proof is similar to that of Theorem 14. ■
10 Summary, conclusion and future direction

Although \textit{SDE}s driven by fractional Brownian motion have undeniable importance in modeling continuous time with long-range dependence, hitherto the statistical literature does not seem to have recognized its full potential. The theory of inference, specifically, asymptotic theory of inference, seem to be far less developed compared to \textit{SDE}s based on Wiener process. One reason for such negligence is probably the difficulty of statisticians in imbibing the somewhat involved theory of fractional \textit{SDE}s compared to traditional \textit{SDE}s. Indeed, note that for \textit{SDE}s driven by Wiener process, the Girsanov formula is readily available, provided that the diffusion coefficient is known. However, this is not the case for fractional \textit{SDE}s if $H$ is unknown. Thus, most of the methods developed for inference in traditional \textit{SDE}s fail to be carried over to fractional \textit{SDE}s. Discretization is probably the only way out in this situation, and also realistic, since in practice, the observed data will invariably be in discretized forms. Our increasing domain infill asymptotics theory established in this article shows that statisticians can expect accurate inference for the model parameters, as well as the Hurst parameter, both in the classical and Bayesian paradigms, provided that the time domain is sufficiently large and enough observations are collected in the time domain. Our simulation studies testify to this even when the time domain is not large. The simulation experiments also demonstrate that the Bayesian framework is preferable. Moreover, our classical and Bayesian applications of both fractional and standard \textit{SDE} to a real, close price data also brought out the superiority of our Bayesian fractional \textit{SDE} over the other model and methods. However, Theorems 7, 8 and Remark 9 provide the necessary caveat against non-judgmental usage of the existing estimators of $H$.

We are currently in the process of extending our work to systems of \textit{SDE}s driven by fractional Brownian motion. In the realm of \textit{SDE}s driven by Wiener processes, asymptotic theory based on the Girsanov formula have been investigated by these authors in a wide variety of contexts; see Maitra and Bhattacharya (2016), Maitra and Bhattacharya (2015), Maitra and Bhattacharya (2018), Maitra and Bhattacharya (2019), Maitra and Bhattacharya (2020c), Maitra and Bhattacharya (2020b), Maitra and Bhattacharya (2020a). These works deserve to be extended to \textit{SDE}s driven by fractional Brownian motion. Since the Girsanov formula is ineffective for unknown $H$, our increasing domain infill asymptotics framework seems to be a very viable alternative for such extensions.

A somewhat disconcerting issue is that we are unable to obtain the asymptotic distribution of the MLE of $H$. The problem lies in the presence of $H$ even as the power of $T$, in the likelihood. Although this does not hinder consistency, asymptotic normality or otherwise could not be addressed because of the complications involved that prevent even a convenient Taylor’s series expansion needed for the asymptotic distribution of the MLE of $H$. We look forward to resolving the issue in our future work, to be communicated elsewhere.
Supplementary Material

This document is an addendum to the theory developed in the main manuscript. This supplementary material is organized as follows.

In Section [S-1] we present some results and their proofs; these results are used to prove the key results in MB and in Section [S-2] we present the posterior asymptotic normal setup of Schervish (1995). In Section [S-3] we illustrate classical and Bayesian inference in a simulation experiment where the time domain is relatively small, a situation important for practical considerations, while applications of classical and Bayesian inference for both fractional SDE and standard SDE driven by Brownian motion to a real, close price data is detailed in Section [S-4] along with comparisons.

S-1 Some technical results

Lemma 16 If \( a_k \sim b_k \) as \( k \to \infty \), then it holds that \( \frac{1}{r_k} \sum_{i=1}^{m_k} a_k \sim \frac{1}{r_k} \sum_{i=1}^{m_k} b_k \), as \( k \to \infty \), provided that \( m_k \to \infty \), \( r_k \to \infty \) and \( \frac{1}{r_k} \sum_{i=1}^{m_k} b_k \) remains non-zero, as \( k \to \infty \).

Proof. Note that \( a_k \sim b_k \) implies that for any \( \epsilon > 0 \), there exists \( k_0 \geq 1 \) depending upon \( \epsilon \) such that for \( k > k_0 \), \( (1 - \epsilon)b_k < a_k < (1 + \epsilon)b_k \). Hence, for \( k > k_0 \), \((1 - \epsilon)\frac{1}{r_k} \sum_{i=1}^{m_k} b_k < \frac{1}{r_k} \sum_{i=m_{k_0}+1}^{m_k} b_k \) as \( k \to \infty \). That is,

\[
\frac{1}{r_k} \sum_{i=m_{k_0}+1}^{m_k} a_k \sim \frac{1}{r_k} \sum_{i=m_{k_0}+1}^{m_k} b_k, \text{ as } k \to \infty. \tag{S-1}
\]

Since by hypothesis, \( \frac{1}{r_k} \sum_{i=1}^{m_k} b_k \) remains non-zero, as \( k \to \infty \), it follows that \( \frac{\sum_{i=1}^{m_{k_0}} a_k}{\sum_{i=1}^{m_k} b_k} \to 0 \) and \( \frac{\sum_{i=1}^{m_{k_0}} b_k}{\sum_{i=1}^{m_k} b_k} \to 0 \), as \( k \to \infty \). Using these and (S-1), it follows that

\[
\frac{1}{r_k} \sum_{i=1}^{m_k} a_k = \sum_{i=1}^{m_{k_0}} a_k + \sum_{i=m_{k_0}+1}^{m_k} a_k \\
\frac{1}{r_k} \sum_{i=1}^{m_k} b_k = \sum_{i=1}^{m_{k_0}} b_k + \sum_{i=m_{k_0}+1}^{m_k} b_k \to 1, \text{ as } k \to \infty. \tag{S-2}
\]

This completes the proof. ■

Lemma 17 Let \( \{ f_k : k \geq 1 \} \) be a sequence of functions such that \( f_k(x) \to f(x) \) uniformly in \( x \in \mathcal{X} \), as \( k \to \infty \), for some \( \mathcal{X} \subseteq \mathbb{R} \), where \( f \) is continuous. Suppose that \( x_k \to \bar{x} \), as \( k \to \infty \).

Then \( f_k(x_k) \to f(\bar{x}) \), as \( k \to \infty \).

Proof. Note that

\[
|f_k(x_k) - f(\bar{x})| \leq |f_k(x_k) - f(x_k)| + |f(x_k) - f(\bar{x})|. \tag{S-3}
\]

Now, due to uniform convergence, there exists \( k_0(\epsilon) \geq 1 \) such that for \( k \geq k_0(\epsilon) \),

\[
|f_k(x_k) - f(x_k)| \leq \sup_{x \in \mathcal{X}} |f_k(x) - f(x)| < \frac{\epsilon}{2}. \tag{S-4}
\]

Also, due to continuity of \( f \), there exists \( k_1(\epsilon) \geq 1 \) such that for \( k \geq k_1(\epsilon) \),

\[
|f(x_k) - f(\bar{x})| < \frac{\epsilon}{2}. \tag{S-5}
\]
Combining (S-3), (S-4) and (S-5) we see that for \( k \geq \max\{k_0(\epsilon), k_1(\epsilon)\} \),

\[ |f_k(x_k) - f(\tilde{x})| < \epsilon. \]

This completes the proof. ■

**S-2 Posterior asymptotic normal setup and the main result in [Schervish (1995)]**

Let \( \tilde{\ell}_n(\theta) = \log L_n(\theta) \) denote the log-likelihood associated with \( n \) observations \( x_1, \ldots, x_n \) having density \( f(x_1, \ldots, x_n|\theta) \) with parameter \( \theta \in \Theta \), where \( \Theta \subseteq \mathbb{R}^p \) is the parameter space, for \( p \geq 1 \). Now denoting by \( \hat{\theta}_n \) the MLE associated with \( n \) observations, let

\[
\Sigma_n^{-1} = \begin{cases} -\tilde{\ell}''_n(\hat{\theta}_n) & \text{if the inverse and } \hat{\theta}_n \text{ exist} \\ \mathbb{I}_p & \text{if not,} \end{cases} \tag{S-1}
\]

where for any \( t \in \mathbb{R}^p \),

\[
\tilde{\ell}''_n(t) = \left( \frac{\partial^2}{\partial \theta_i \partial \theta_j} \tilde{\ell}_n(\theta) \bigg|_{\theta = t} \right), \tag{S-2}
\]

and \( \mathbb{I}_p \) is the identity matrix of order \( p \). Thus, \( \Sigma_n^{-1} \) is the observed Fisher’s information matrix.

### S-2.0.1 Regularity conditions

1. The true value \( \theta_0 \) is a point interior to \( \Theta \).
2. The prior distribution of \( \theta \) has a density with respect to Lebesgue measure that is positive and continuous at \( \theta_0 \).
3. There exists a neighborhood \( \mathcal{N}_0(\delta) \subseteq \Theta \) of \( \theta_0 \) on which \( \tilde{\ell}_n(\theta) = \log f(x_1, \ldots, x_n|\theta) \) is twice continuously differentiable with respect to all co-ordinates of \( \theta \), almost surely with respect to the true, data-generating distribution \( P_{\theta_0} \).
4. The largest eigenvalue of \( \Sigma_n \) goes to zero in probability.
5. For \( \delta > 0 \), define \( \mathcal{N}_0(\delta) \) to be the open ball of radius \( \delta \) around \( \theta_0 \). Let \( \rho_n \) be the smallest eigenvalue of \( \Sigma_n \). If \( \mathcal{N}_0(\delta) \subseteq \Theta \), there exists \( K(\delta) > 0 \) such that

\[
\lim_{n \to \infty} P_{\theta_0} \left( \sup_{\theta \in \Theta \setminus \mathcal{N}_0(\delta)} \rho_n \left[ \tilde{\ell}_n(\theta) - \tilde{\ell}_n(\theta_0) \right] < -K(\delta) \right) = 1. \tag{S-3}
\]

6. For each \( \epsilon > 0 \), there exists \( \delta(\epsilon) > 0 \) such that

\[
\lim_{n \to \infty} P_{\theta_0} \left( \sup_{\theta \in \mathcal{N}_0(\delta(\epsilon))} \left| 1 + \gamma^T \Sigma_n^{-1/2} \tilde{\ell}_n''(\theta) \Sigma_n^{-1/2} \gamma \right| < \epsilon \right) = 1. \tag{S-4}
\]

**Theorem 18 (Theorem 7.89 of [Schervish (1995)])** Assume the above six regularity conditions. Letting \( \Psi_n = \Sigma_n^{-1/2} (\theta - \hat{\theta}_n) \) and denoting by \( \pi_n \) the posterior distribution of \( \Psi_n \) given
for each compact subset $B$ of $\mathbb{R}^p$ and each $\epsilon > 0$, the following holds:

$$\lim_{n \to \infty} P_{\theta_0} \left( \sup_{\psi \in B} |\pi_n(\psi) - \phi(\psi)| > \epsilon \right) = 0,$$

(S-5)

where $\phi(\cdot)$ denotes the density of the $p$-dimensional standard normal distribution.

S-3 Simulation study

We perform simulation experiments by generating data from a specific fractional SDE as follows:

$$dX_t = \beta(1 - x_t)dt + (1 - x_t)dW_t^H.$$

(S-1)

For simplicity we choose the ratio $B(t, x_t)/C(t, x_t)$ to be constant in the considered SDE. We fix $T = 5$, $n = 100$ for our experiments, so that the non-asymptotic flavour can be retained. We conduct simulation experiments in both classical and Bayesian setups, detailed in Sections S-3.1 and S-3.2 respectively.

S-3.1 Simulation study in the classical setup

We perform simulated annealing aided by transformation based Markov chain Monte Carlo (TMCMC) (Dutta and Bhattacharya (2014)) to obtain the MLE of $\theta = (\beta, H)$. We provide our algorithm as Algorithm S-3.1, where we use the additive transformation.

Algorithm S-3.1 Simulated annealing aided by additive TMCMC

1. Begin with an initial value $\theta^0 = (\beta^{(0)}, H^{(0)})$.
2. For $t = 1, \ldots, N$, do the following:
   
   (i) Generate $\varepsilon \sim N(0, 1)$, $b_1 \sim U\{-1, 1\}$ for $j = 1, 2$, and set $\tilde{\beta} = \beta^{(t-1)} + b_1 \varepsilon |\varepsilon|$ and $\tilde{H} = H^{(t-1)} + b_2 \varepsilon |\varepsilon|$. Let $\tilde{\theta} = (\tilde{\beta}, \tilde{H})$.
   
   (ii) Evaluate
   
   $$\alpha = \min \left\{ 1, \exp \left\{ \frac{\ell_{T,n}(\tilde{\theta}) - \ell_{T,n}(\theta^{(t-1)})}{\tau_t} \right\} I_{(0,1)}(\tilde{H}) \right\},$$

   where $\tau_t = \frac{1}{\log(\log(t)))}$ when $t \geq 3$ and $t^{-1}$ otherwise. In the above, $I_{(0,1)}(\tilde{H}) = 1$ if $\tilde{H} \in (0, 1)$ and zero otherwise.
   
   (iii) Set $\theta^{(t)} = \tilde{\theta}$ with probability $\alpha$, else set $\theta^{(t)} = \theta^{(t-1)}$.
3. From the sample $\left\{ \theta^{(j)}; j = 0, 1, \ldots, N \right\}$, set $\hat{\theta}_{T,n} = \theta^{(\ast)}$ where $\theta^{(\ast)} = \arg\max_{j=0,1,\ldots,N} \ell_{T,n}(\theta^{(j)})$. 

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For our experiments, setting $N = 20000$ and $a_1 = a_2 = 0.0001$ produced MLEs that are reasonably close to the true values. For the initial value $\theta^{(0)}$, we conduct a grid search on a grid consisting of equispaced values of $\beta$ and $H$; the maximizer of the log-likelihood on the grid is taken as $\theta^{(0)}$.

It is also important to obtain the distribution of the MLE so that desired confidence intervals can be obtained. For $\hat{H}_{T,n}$, at least asymptotic normality is provided by Theorem 5 of MB, but for $\hat{H}_{T,n}$ even asymptotic normality is not available. In any case, it is important to provide the distribution of the MLE for any sample size. We numerically approximate the distribution by generating 1000 data samples from the SDE (S-1) and then computing the MLE of $\theta$ in each case using Algorithm (S-3.1). The results of our experiments are tabulated in Table S-1. The 95% confidence intervals (CIs) are obtained by setting the lower and upper ends to be the 2.5% and 97.5% quantiles obtained from the 1000 realizations of the MLEs. Note that most of the MLEs $\hat{H}_{T,n}$ and $\hat{\beta}_{T,n}$ are reasonably close to the true values $H_0$ and $\beta_0$, respectively. Also, most of the 95% CIs contain the true values. In the third case, where $H_0 = 0.3$ and $\beta_0 = -1$, the 95% CI of $H$ marginally misses $H_0$, and that of $\beta$ misses $\beta_0$ by a larger margin. In cases 8 and 9, where $(H_0, \beta_0) = (0.8, -1.5)$ and $(0.9, 1.4)$, respectively, the variabilities of the distribution of $\hat{H}_{T,n}$ seem to be too small to include the true value. These apparent fail-
ers demonstrate the inadequacy of the small size (1000) of the number of realizations from the distribution of the MLEs. However, obtaining even 1000 realizations implemented in the $R$ software, takes about 5 hours and 40 minutes on our ordinary laptop, precluding accurate evaluation of the distributions of the MLEs. Moreover, the choices of $a_1 = a_2 = 0.0001$ need not be adequate for all the 1000 samples and for all the nine cases shown in Table S-1. This problem of missing the true value from the desired confidence intervals can be avoided in the Bayesian setup, as we shall demonstrate in Section S-3.2.

Table S-1: Results of our simulation study in the classical setup.

| $H_0$ | $\hat{H}_{T,n}$ | 95% CI | $\beta_0$ | $\hat{\beta}_{T,n}$ | 95% CI |
|-------|-----------------|--------|----------|----------------------|--------|
| 0.1   | 0.095           | [0.086,0.116] | 1.0      | 0.899                | [0.773,1.035] |
| 0.2   | 0.191           | [0.101,0.262] | -1.0     | -1.0                 | [-1.093,-0.725] |
| 0.3   | 0.289           | [0.232,0.296] | -0.5     | -0.490               | [-0.562,-0.374] |
| 0.4   | 0.391           | [0.315,0.452] | 0.7      | 0.689                | [0.620,0.792] |
| 0.5   | 0.489           | [0.454,0.547] | -1.2     | -1.191               | [-1.225,-1.190] |
| 0.6   | 0.590           | [0.583,0.600] | -1.5     | -1.522               | [-1.538,-1.494] |
| 0.7   | 0.675           | [0.674,0.686] | 1.4      | 1.415                | [1.396,1.439] |
| 0.8   | 0.773           | [0.763,0.777] | 1.0      | 0.899                | [0.773,1.035] |
| 0.9   | 0.856           | [0.851,0.865] | 1.0      | 0.899                | [0.773,1.035] |

S-3.2 Simulation study in the Bayesian setup

For the Bayesian investigation we focus on the same cases as shown in Table S-1 and analyze the same datasets for valid comparison between the classical and Bayesian paradigms.

In the Bayesian setup, we first reparameterize $H$ as $H = \exp(\nu)/(1 + \exp(\nu))$, where we assume a priori that $\nu \sim N(\nu_{T,n}, \sigma_\nu^2)$. We set $\nu_{T,n} = \log \left( \frac{\beta_{T,n}}{1-H_{T,n}} \right)$ and $\sigma_\nu = 0.2$ for the first seven cases and 0.02 for the last two cases. We also assume a priori that $\beta \sim N\left(\beta_{T,n}, \sigma_\beta^2\right)$, where $\sigma_\beta = 0.1$ in the first seven cases and 0.01 in the last two cases. These choices quantify the prior belief that the true value $\theta_0$ is located around the MLE $\hat{\theta}_{T,n}$. We considered a larger
variance for $H$ so that it has lesser risk of missing the true value compared to that for $\beta$. Indeed, Table S-1 for the classical paradigm demonstrates that without any prior, $H$ is more at risk of missing the true values compared to $\beta$. In the Bayesian setup we exploited this wisdom gained in the form of the priors to counter the problem of omitting the true values. For cases 8 and 9, we set much smaller prior variances, so that even in these extreme cases the posteriors have less propensity to concentrate around less extreme values. Note that even though this is a simulation study where the true values are known which makes it easy to set the prior variances as explained, in reality the MLE is expected to be close to the truth, and hence even in practice, such prior variances are easy to envisage. It is also important to remark that a Beta prior on $H$ seems to be natural, but it does not have the flexibilities of setting the mean and variance as desired, compared to our normal prior distribution.

For Bayesian inference, we propose an additive TMCMC algorithm, enriched with the provision for further enhancement of mixing properties. The details are provided as Algorithm S-3.2.

**Algorithm S-3.2** TMCMC for SDE driven by fractional Brownian motion

1. Begin with an initial value $\theta^0 = (\beta^{(0)}, \nu^{(0)})$; set $H^{(0)} = \exp(\nu^{(0)})/(1 + \exp(\nu^{(0)}))$.
2. For $t = 1, \ldots, N$, do the following:
   
   (i) Generate $\varepsilon_1 \sim N(0, 1)$, $b_j \overset{\text{ind}}{\sim} U\{\{−1, 1\}\}$ for $j = 1, 2$, and set $\tilde{\beta} = \beta^{(t-1)} + b_1a_1|\varepsilon_1|$ and $\tilde{\nu} = \nu^{(t-1)} + b_2a_2|\varepsilon_1|$. Let $\theta = (\tilde{\beta}, \tilde{H})$, where $\tilde{H} = \exp(\tilde{\nu})/(1 + \exp(\tilde{\nu}))$.
   
   (ii) Evaluate
   $$\alpha_1 = \min\left\{1, \exp\left\{\ell_{T,n}(\tilde{\theta}) - \ell_{T,n}\left(\theta^{(t-1)}\right)\right\}\right\}.$$  

   (iii) Set $\theta^{(t)} = \tilde{\theta}$ with probability $\alpha_1$, else set $\theta^{(t)} = \theta^{(t-1)}$.

   (iv) Generate $\varepsilon_2 \sim N(0, 1)$ and $U \sim U(0, 1)$. If $U < 1/2$, create $\beta^* = \beta^{(t)} + a_1|\varepsilon_2|$ and $H^* = \exp(\nu^*)/(1 + \exp(\nu^*))$, where $\nu^* = \nu^{(t)} + b_2a_2|\varepsilon_2|$. Else, set $\beta^* = \beta^{(t)} - a_1|\varepsilon_2|$ and $H^* = \exp(\nu^*)/(1 + \exp(\nu^*))$, where $\nu^* = \nu^{(t)} - a_2|\varepsilon_2|$. Let $\theta^* = (\beta^*, H^*)$.

   (v) Evaluate
   $$\alpha_2 = \min\left\{1, \exp\left\{\ell_{T,n}(\theta^*) - \ell_{T,n}\left(\theta^{(t)}\right)\right\}\right\}.$$  

   (vi) With probability $\alpha_2$ accept $\theta^*$, else remain at $\theta^{(t)}$. Abusing notation, let us continue to denote either $\theta^*$ or $\theta^{(t)}$ as $\theta^{(t)}$, to be understood as the final value of each iteration.

3. Store the realizations $\{\theta^{(t)}; t = 0, 1, \ldots, N\}$ for Bayesian inference.

Note that steps (2) (iv) – (2) (vi) is actually not required for convergence of TMCMC but often has the important advantage of improving mixing properties of the underlying Markov chain. See Liu and Sabatti (2000) and the supplement of Dutta and Bhattacharya (2014) for details.

In our implementation of Algorithm S-3.2, we choose $N = 5 \times 10^3$ for all the nine cases. Also, we set $a_1 = a_2 = 0.05$ for all the cases except cases 2 and 7, where we set $a_1 = a_2 = 0.01$ and $a_1 = a_2 = 1$, respectively. The initial values are set to the MLEs. We discard the first 1000 TMCMC iterations as burn-in in all the cases and make our Bayesian inference based on the remaining iterations. Our implementation of Algorithm S-3.2 yielded excellent mixing.
properties in all the cases, which is evident from the trace plots shown in Figures S-1, S-2 and S-3 after discarding the first 1000 iterations and subsequently plotting thinned samples (one in every 50 TMCMC realizations) to reduce file size. Our implementation of Algorithm S-3.2 in the R software takes about 13 minutes on our ordinary laptop to generate $5 \times 10^5$ TMCMC realizations.

Table S-2 summarizes the results of our Bayesian analyses for cases 1–9. Note that in contrast to Table S-1, the true values of $H$ and $\beta$ are contained in the 95% Bayesian credible intervals (BCIs) in all the cases. Moreover, the lengths of such intervals are generally substantially shorter than their classical counterpart, as shown in Table S-1. Indeed, the 95% BCI for $H$ are shorter than the corresponding classical 95% CI in all the cases, and for $\beta$, the 95% BCI are shorter than the corresponding 95% CI in all the cases except only cases 6 and 7. The average lengths of the 95% BCI and 95% CI for $H$ are 0.049 and 0.060, respectively, and for $\beta$, the lengths are 0.127 and 0.134, respectively.

In other words, our simulation experiments vouch for the Bayesian framework for analysing SDEs driven by fractional Brownian motion using the priors and the TMCMC algorithm that we propose.

### S-4 Application to real data

To deal with real data we collect the close price of a particular company from stock market data (460 observations during the time range August 5, 2013, to June 30, 2015) which is available on www.nseindia.com.

A simple autocorrelation plot of the close price data revealed autocorrelations close to one, even for lag greater than 25, suggesting long-range dependence. Thus, an SDE driven by a fractional Brownian motion seems to be appropriate here. Moreover, for simplicity we fix the choice of the ratio $B(t, x_t)/C(t, x_t)$ as a constant; in fact, we set $B(t, x_t) = C(t, x_t) = x_t^{\nu}$, where $\nu > 0$ is to be determined appropriately, following the discussion below.

In other words, we propose the following form of such SDE:

$$dX_t = \beta x_t^\nu dt + \gamma x_t^\nu dW_t^H,$$

where $\nu > 0$ is to be chosen such that both classical and Bayesian analyses of the data using the above SDE supports values of $H$ close to one. As it turned out, $\nu = 3/2$ is an appropriate
Figure S-1: TMCMC trace plots of $H$ and $\beta$. 
Figure S-2: TMCMC trace plots of $H$ and $\beta$. 
Figure S-3: TMCMC trace plots of $H$ and $\beta$. 
MLE obtain the with a TMCMC algorithm that has provision for further enhancement of mixing properties to \( \theta \).

### S-4.1 Real data analysis in the classical setup

Starting from arbitrary initial values of \( \theta = (\beta, H) \) we performed simulated annealing aided with a TMCMC algorithm that has provision for further enhancement of mixing properties to obtain the MLE of \( \theta \). Our algorithm is provided as Algorithm S-4.1.

**Algorithm S-4.1** Simulated annealing aided by TMCMC with enhanced mixing properties

1. Begin with an initial value \( \theta^0 = (\beta^{(0)}, H^{(0)}) \).
2. For \( t = 1, \ldots, N \), do the following:
   
   (i) Generate \( \varepsilon_1 \sim N(0,1) \), \( \varepsilon_2 \sim U(-1,1) \), \( b_j \sim_{id} U\{\{-1,1\}\} \) for \( j = 1, 2 \), \( U_1 \sim U(0,1) \). If \( U_1 < 1/2 \), create \( \tilde{\beta} = \beta^{(t-1)} + b_1 a_1 |\varepsilon_1| \) and \( \tilde{H} = H^{(t-1)} + b_2 a_2 |\varepsilon_1| \). Else, set \( \tilde{\beta} = \beta^{(t-1)} \varepsilon_2 b_1 \) and \( \tilde{H} = H^{(t-1)} \varepsilon_2 b_2 \). Let \( \tilde{\theta} = (\tilde{\beta}, \tilde{H}) \).
   
   (ii) Evaluate
   
   \[
   \alpha_1 = \min \left\{ 1, \exp \left\{ \frac{\ell_{T,n}(\tilde{\theta}) - \ell_{T,n}(\theta^{(t-1)}) + \log(J)}{\tau_t} \right\} I_{(0,1)}(\tilde{H}) \right\},
   \]

   where \( \tau_t = \frac{1}{\log(t)} \) when \( t \geq 100 \) and \( t^{-1} \) otherwise and \( J \) is the Jacobian of corresponding transformation. (\( J = 1 \) in additive case and \( J = |\varepsilon_2| \sum b_j \) in multiplicative case)

   (iii) Set \( \theta^{(t)} = \tilde{\theta} \) with probability \( \alpha_1 \), else set \( \theta^{(t)} = \theta^{(t-1)} \). 

   To provide further mixing,

   (iv) Generate \( \varepsilon_3 \sim N(0,1) \), \( U_2 \sim U(0,1) \). If \( U_2 < 1/2 \),

   Further generate \( u_1 \sim U(0,1) \). If \( u_1 < 1/2 \), create \( \beta^* = \beta^{(t)} + a_1 |\varepsilon_3| \) and \( H^* = H^{(t)} + a_2 |\varepsilon_3| \). Else, set \( \beta^* = \beta^{(t)} - a_1 |\varepsilon_3| \) and \( H^* = H^{(t)} - a_2 |\varepsilon_3| \).

   If \( u_2 \geq 1/2 \),

   Generate \( \varepsilon_4 \sim U(-1,1) \), \( u_2 \sim U(0,1) \). If \( u_2 < 1/2 \), create \( \beta^* = \beta^{(t)} \varepsilon_4 \) and \( H^* = H^{(t)} \varepsilon_4 \). Else, set \( \beta^* = \beta^{(t)} / \varepsilon_4 \) and \( H^* = H^{(t)} / \varepsilon_4 \). Let \( \theta^* = (\beta^*, H^*) \).
(v) Evaluate
\[
\alpha_2 = \min \left\{ 1, \exp \left( \frac{\ell_{T,n}(\hat{\theta}) - \ell_{T,n}(\theta^{(t-1)}) + \log(J)}{\tau_t} \right) \right\}
\]
where \( \tau_t = \frac{1}{\log(t)} \) when \( t \geq 100 \) and \( t^{-1} \) otherwise and \( J \) is the Jacobian of corresponding transformation. (In the case of \( U_2 < 1/2 \), \( J = |\varepsilon_2|^2 \) or \( J = |\varepsilon_2|^{-2} \) correspondingly)

(vi) Accept \( \theta^* \) with probability \( \alpha_2 \) or remain at \( \theta^{(t)} \). To avoid notational complexity whether it is \( \theta^* \) or \( \theta^{(t)} \) we write final value of each iteration as \( \theta^{(t)} \).

(3) From the sample \( \left\{ \theta^{(j)}; j = 0, 1, \ldots, N \right\} \), set \( \hat{\theta}_{T,n} = \theta^{(j^*)} \) where \( \theta^{(j^*)} = \arg \max_{j=0,1,\ldots,N} \ell_{T,n}(\theta^{(j)}) \).

In our application, initial values are arbitrarily chosen and we set \( N = 10^4 \) and \( a_1 = a_2 = 0.0005 \). With the obtained MLE given a specific value of \( \gamma \) we predicted the close price of next 60 days for that company in two ways. In the first way, at every step each data point is predicted with the consideration that data values at the previous time point is given beforehand, that is, one-step-ahead forecast is obtained. In the other way, starting from the 400th data point, predictions for the next 60 data points are obtained, which is the 60-step-ahead forecast. To obtain desired confidence intervals for the forecasts, we generated 10,000 realizations of size 60 from the SDE (S-1) given the first 400 data points. We repeated the calculations for different values of \( \gamma \), from which we obtained the final value of \( \gamma \) as the one which yielded minimum average length of the 95% confidence interval. Associated with the final value of \( \gamma \), we also report the number of held-out data points that fail to lie within the 95% and the 50% confidence intervals.

We repeated the entire procedure of classical analysis for the case where the SDE is assumed to be driven by Brownian motion instead of fractional Brownian motion. In the latter case, as \( H = 0.5 \) corresponds to Brownian motion, the only unknown parameter is \( \beta \) and \( \gamma \) is fixed in the same way as in the fractional set up. This analysis facilitates comparison of fractional SDE model with the more usual SDE models driven by Brownian motion.

For the classical analysis \( \gamma \) is found to be 165 and 0.4 for our SDEs driven by fractional and standard Brownian motion, respectively. The results are tabulated in Tables S-1 and S-2 showing the MLEs of \( H \) and \( \beta \), the values of \( \gamma \) for one step ahead and 60 steps ahead forecasts, the number of held-out data points outside the corresponding 50% and 95% confidence intervals (CI) and the average length of the 95% CIs of the held-out data points. Observe that compared to fractional SDE, the average lengths of the 95% CI are small for standard SDE, but a significantly larger number of held-out observations are excluded from the 50% CIs of the standard SDE. This indicates that compared to standard SDE, most of the held-out observations are concentrated in the high-density regions of the fractional SDE. Hence, the fractional SDE clearly outperforms the standard SDE.

The densities associated with fractional and standard SDEs are displayed in Figures S-1 and S-2, respectively, where higher densities are represented by progressively intense colours; the thick, black line stands for the held-out observed data. These figures are the detailed versions of Tables S-1 and S-2.

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Figure S-1: One step ahead and 60 steps ahead forecasts in the classical setup corresponding to fractional $SDE$.

Figure S-2: One step ahead and 60 steps ahead forecasts in the classical setup corresponding to standard $SDE$. 
Table S-1: Results of classical analysis of real data modeled by fractional SDE.

| $H_{T,n}$ | $\beta_{T,n}$ | $\gamma$ | step | # outside 50% CI | # outside 95% CI | Average length of 95% CI |
|-----------|----------------|-----------|------|------------------|------------------|--------------------------|
| 0.995     | -0.002578      | 165       | 1    | 24               | 1                | 31.11014                 |
| 0.995     | -0.002578      | 165       | 60   | 17               | 0                | 155.5142                 |

Table S-2: Results of classical analysis of real data modeled by standard SDE.

| $H$   | $\beta_{T,n}$ | $\gamma$ | step | # outside 50% CI | # outside 95% CI | Average length of 95% CI |
|-------|----------------|-----------|------|------------------|------------------|--------------------------|
| 0.5   | 0.00021        | 0.4       | 1    | 38               | 2                | 25.20115                 |
| 0.5   | 0.00021        | 0.4       | 60   | 24               | 0                | 122.7201                 |

S-4.2 Real data analysis in the Bayesian setup

In the Bayesian setup, we first reparameterize $H$ as $H = \exp(\nu)/(1 + \exp(\nu))$ as in Section S-3.2 and assume a priori that $\nu \sim N(\nu_{T,n}, \sigma^2_\nu)$ and $\beta \sim N(\mu_\beta, \sigma^2_\beta)$, where $\mu_\beta = 0.001, \sigma_\nu = \sigma_\beta = 0.1$. As in Section S-3.2, $\nu_{T,n}$ is set as $\log(H_{T,n}^{*} - H_{T,n}^{*})$.

For Bayesian inference, a TMCMC algorithm with enhanced mixing properties is considered similar to Algorithm S-4.1 where $\nu$ is updated instead of $H$ and the acceptance probability becomes

$$\alpha = \min\left\{1, \exp\left\{\ell_{T,n}(\tilde{\theta}) - \ell_{T,n}(\theta^{(t-1)}) + \log(J)\right\}\right\}.$$  

In our application, we set $N = 10^4$ and $a_1 = a_2 = 0.05$.

Posterior predictive distributions of the 60 held-out observations are obtained in two ways similar to that of the classical set up. In each case, whether it is one step ahead or 60 steps ahead predictions, 95% and 50% Bayesian credible intervals are obtained and the number of data points lying outside corresponding regions are also noted. Again, $\gamma$ is treated as a fixed parameter whose value is obtained by minimizing the average length of the 95% credible intervals subject to their inclusion of all the respective held-out observations. Once again we remind the reader that this fixed value of $\gamma$ would be used even for actual forecasting future observations.

The whole analysis is carried out for the standard SDE driven by Wiener process, that is, with $H = 0.5$ and in this case we are interested in the posterior of $\beta$ only as $\gamma$ is fixed in the way as explained above.

In this Bayesian set up, we obtain $\gamma = 158$ and $\gamma = 1$ for fractional and standard SDE, respectively, corresponding to which Tables S-3 and S-4 provide the results of our Bayesian analyses. In the tables $H^{*}$ and $\beta^{*}$ stand for the respective posterior medians and Bayesian credible interval is denoted by BCI. Note that the average length of 95% BCI for standard SDE is much larger compared to that for fractional SDE and hence it is no surprise that even those for 50% BCIs would be much larger for the former and hence excludes less number of observations. Indeed, the average lengths for standard SDE are much larger even compared to both the classical analyses reported in Tables S-1 and S-2, and thus it is clear that the standard SDE fails to outperform the fractional SDE. It is also clear, in terms of adequate average lengths of the 95% BCI and the number of held-out observations falling in the high-density
Table S-4: Results of Bayesian analysis of real data modeled by standard SDE.

| $H$ | $\beta^*$ | $\gamma$ | step | # outside 50% BCI | # outside 95% BCI | Average length of 95% BCI |
|-----|-----------|----------|------|-------------------|-------------------|--------------------------|
| 0.5 | 0.00035   | 1        | 1    | 6                 | 0                 | 62.98049                 |
| 0.5 | 0.00035   | 1        | 60   | 0                 | 0                 | 334.7808                 |

Figure S-3: One and 60 steps ahead forecasts in the Bayesian setup corresponding to fractional SDE.

regions, that our Bayesian fractional SDE is the best performer.

The posterior predictive densities associated with fractional and standard SDEs are displayed in Figures S-3 and S-4, respectively, where higher posterior predictive densities are represented by progressively intense colours; the thick, black line stands for the held-out observed data, as before.
1 step prediction, $H = 0.5, \gamma = 1$

Bayesian Forecasts

400 410 420 430 440 450 460

250 300 350 400

60 step prediction, $H = 0.5, \gamma = 1$

Days

Bayesian Forecasts

400 410 420 430 440 450 460

150 250 350 450 550 650

(a)

(b)

Figure S-4: One and 60 steps ahead forecasts in the Bayesian setup corresponding to standard $SDE$.

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