ON UNIQUENESS OF DISTRIBUTION OF A RANDOM
VARIABLE WHOSE INDEPENDENT COPIES SPAN A
SUBSPACE IN $L_p$

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Abstract. Let $1 \leq p < 2$ and let $L_p = L_p[0,1]$ be the classical $L_p$-space of all (classes of) $p$-integrable functions on $[0,1]$. It is known that a sequence of independent copies of a mean zero random variable $f \in L_p$ spans in $L_p$ a subspace isomorphic to some Orlicz sequence space $l_M$. We present precise connections between $M$ and $f$ and establish conditions under which the distribution of a random variable $f \in L_p$ whose independent copies span $l_M$ in $L_p$ is essentially unique.

1. Introduction

It is well known that the class of all subspaces of $L_1 = L_1(0,1)$ is very rich and still does not have any reasonable description. If we consider only symmetric subspaces of $L_1$, that is, subspaces with a symmetric basis or isomorphs of some symmetric function spaces, then these subspaces are known to be isomorphic to averages of Orlicz spaces [6, 13]. Far more information is available on subspaces of $L_1$ isomorphic to Orlicz spaces. First of all, an isomorph of an Orlicz sequence space $l_M \neq l_1$ in $L_1$ can always be given by the span of a sequence of independent identically distributed (i.i.d) random variables. The latter fact was discovered by M.I. Kadec in 1958 [8], who proved that for arbitrary $1 \leq p < q < 2$ there exists a symmetrically distributed function $f \in L_p$ (a $q$-stable random variable) such that the sequence $\{f_k\}_{k=1}^{\infty}$ of independent copies of $f$ spans in $L_p$ a subspace isomorphic to $l_q$.

This direction of study was taken further by J. Bretagnolle and D. Dacunha-Castelle (see [4, 5, 6]). In particular, D. Dacunha-Castelle showed that for every given mean zero $f \in L_p = L_p(0,1)$, the sequence $\{f_k\}_{k=1}^{\infty}$ of its independent copies is equivalent in $L_p$ to the unit vector basis of some Orlicz sequence space $l_M$ [6, Theorem 1, p.X.8]. Moreover, J. Bretagnolle and D. Dacunha-Castelle proved that an Orlicz function space $L_M = L_M[0,1]$ can be isomorphically embedded into the space $L_p$ if and only if $M$ is equivalent to a $p$-convex and $2$-concave Orlicz function on $[0,\infty)$ [5, Theorem IV.3]. Later on some of these results were independently rediscovered by M. Braverman [2, 3].

Note that the methods used in [1, 4, 5, 2, 3] depend heavily on the techniques related to the theory of random processes. In a recent paper [11], two first named co-authors suggested a different approach to study of this problem, which is based
on methods and ideas from the interpolation theory of operators. In addition, it should be pointed out that papers [11, 5, 6, 2, 3] concern only with the verification of existence of a function \( f \) such that the sequence of its independent copies is equivalent in \( L_p \) to the unit vector basis in some Orlicz sequence space and do not address the question concerning the determination of \( f \), whereas [11] is mainly focused on revealing precise connections between the Orlicz function and the distribution of corresponding random variable \( f \). Among other results, in [11], it is shown the following. Let \( 1 \leq p < 2 \) and let \( M \) be a \( p \)-convex and \( 2 \)-concave Orlicz function on \([0, \infty)\) such that \( M(t) \neq t^p \) for small \( t > 0 \) and the function

\[
S(u) := -2pM(u) + (p + 1)uM'(u) - u^2M''(u)
\]

is positive on \((0, \infty)\), increasing and bounded on \((0, 1)\). Then, under some technical conditions on \( M \) (see [11] Proposition 12 and Theorem 15) the unit vector basis in \( l_M \) is equivalent in \( L_p \) to the sequence \( \{f_k\}_{k=1}^{\infty} \) of independent copies of an arbitrary mean zero function \( f \in L_p \) such that its distribution function

\[
n_f(\tau) := \lambda\{u : |f(u)| > \tau\}, \quad \tau > 0
\]

(\( \lambda \) is the Lebesgue measure) is equivalent to the function \( S(1/\tau) \) for \( \tau \geq 1 \).

The present paper continues this direction of research. Our main result (Theorem 1) is a somewhat surprising fact that in the case, when an Orlicz function \( M \) is ‘far’ from the extreme functions \( t^p \) and \( t^2 \), \( 1 \leq p < 2 \), the distribution of a random variable \( f \in L_p \) whose independent copies span \( l_M \) essentially is equivalent to that of the function

\[
m(t) = \frac{1}{M^{-1}(t)}, \quad t > 0.
\]

**Theorem 1.** Let \( 1 \leq p < 2 \) and let \( M \) be a \( p \)-convex and \( 2 \)-concave Orlicz function. The following conditions are equivalent:

(i) The function \( M \) is \((p + \varepsilon)\)-convex and \((2 - \varepsilon)\)-concave for some \( \varepsilon > 0 \);

(ii) If a sequence \( \{f_k\}_{k=1}^{\infty} \) of independent copies of a mean zero random variable \( f \in L_p \) is equivalent in \( L_p \) to the unit vector basis \( \{e_k\}_{k=1}^{\infty} \) in \( L_M \), then the distribution function \( n_f(\tau) \) is equivalent to that of \( m \) for large \( \tau \).

(iii) The function \( m \in L_p \) and any sequence of independent copies of a mean zero random variable \( m \) is equivalent in \( L_p \) to the unit vector basis in \( l_M \).

Observe that even in the simplest case, when \( 1 \leq p < q < 2 \) and \( M(t) = t^q, t \geq 0 \), the theorem above complements the above-mentioned classical Kadec result [3], by establishing the uniqueness of the distribution of a mean zero random variable \( f \) whose independent copies span \( l_q \) in \( L_p \).

It is worth noting that the assertion of Theorem 1 is in a sense sharp. Namely, in Proposition 13 we show that there exist two random variables \( x \) and \( y \) with non-equivalent distribution for large \( \tau \) whose independent copies span in \( L_1 \) the same Orlicz space \( l_M \), where \( M \) is equivalent to the function \( t/\log(e/t) \) for small \( t > 0 \).

Note that in the special case \( p = 1 \), another attempt to describe the connection between the distribution of a random variable \( f \in L_p \) and the corresponding Orlicz function \( M \) can be found in [12]. However, the methods used in [12] have a strong combinatorial flavor and formulas obtained there seem to be less accessible. Moreover, in [12] the question of uniqueness of distribution of \( f \) is not raised at all.
The proof of Theorem 1 is presented in Section 4. Two important components of the proof are Proposition 6 and Theorem 9, which are given in Sections 2 and 3, respectively.

We propose the following conjecture.

**Conjecture 2.** Let $1 \leq p < 2$ and let $M$ be a $p$–convex and $2$–concave Orlicz function. If there is a unique (up to equivalence near 0) mean zero function $f$ whose independent copies are equivalent in $L_p$ to the unit vector basis in $l_M$, then $M$ is $(p+\varepsilon)$–convex and $(2-\varepsilon)$–concave for some $\varepsilon > 0$.

### 2. Preliminaries and auxiliary results

#### 2.1. Orlicz functions and spaces.

For the theory of Orlicz spaces we refer to [9, 11].

Let $M$ be an Orlicz function, that is, an increasing convex function on $[0, \infty)$ such that $M(0) = 0$. To any Orlicz function $M$ we associate the Orlicz sequence space $l_M$ of all sequences of scalars $a = (a_n)_{n=1}^{\infty}$ such that

$$\sum_{n=1}^{\infty} M\left(\frac{|a_n|}{\rho}\right) < \infty$$

for some $\rho > 0$. When equipped with the norm

$$\|a\|_{l_M} := \inf \left\{ \rho > 0 : \sum_{n=1}^{\infty} M\left(\frac{|a_n|}{\rho}\right) \leq 1 \right\},$$

$l_M$ is a Banach space. Clearly, if $M(t) = t^p$, $p \geq 1$, then the Orlicz space $l_M$ is the familiar space $l_p$. Moreover, the sequence $\{e_n\}_{n=1}^{\infty}$ given by

$$e_n = (0, \ldots, 0, 1, 0, \ldots)$$

$n - 1$ times

is a Schauder basis in every Orlicz space $l_M$ provided that $M$ satisfies the $\Delta_2$–condition at zero, i.e., there are $u_0 > 0$ and $C > 0$ such that $M(2u) \leq CM(u)$ for all $0 < u < u_0$.

Similarly, if $M$ is an Orlicz function, then the Orlicz function space $L_M = L_M[0,1]$ consists of all measurable functions $x$ on $[0,1]$ such that the norm

$$\|x\|_{L_M} = \inf \left\{ u > 0 : \int_0^1 M(|x(t)|/u) \, dt \leq 1 \right\}$$

is finite.

Let $1 \leq p < q < \infty$. Given an Orlicz function $M$, we say that $M$ is $p$-convex if the map $t \mapsto M(t^{1/p})$ is convex, and is $q$-concave if the map $t \mapsto M(t^{1/q})$ is concave. Throughout this paper, we assume that $M(1) = 1$ and that $M : [0, \infty) \to [0, \infty)$ is a bijection.

Careful inspection of the proof of [1] Lemma 5] establishes the following two lemmas.

**Lemma 3.** Let $1 \leq p < \infty$. An Orlicz function $M : [0, \infty) \to [0, \infty)$ satisfying $\Delta_2$-condition at 0 is equivalent to a $p$-convex Orlicz function on the segment $[0,1]$. 
if and only if there exists a constant $C > 0$ such that for all $0 < s < 1$ and all $0 < t \leq 1$ we have

$$M(st) \leq C s^p M(t).$$

**Lemma 4.** Let $1 < q < \infty$. An Orlicz function $M : [0, \infty) \to [0, \infty)$ is equivalent to a $q$-concave Orlicz function on the segment $[0, 1]$ if and only if there exists a constant $C > 0$ such that for all $0 < s < 1$ and all $0 < t \leq 1$ we have

$$C^{-1} s^q M(t) \leq M(st).$$

In what follows, by $f^*$ we will denote the non-increasing right-continuous rearrangement of a random variable $f$, that is,

$$f^*(s) := \inf \{ t : n_f(t) \leq s \},$$

where $n_f$ is the distribution function of the random variable $f$. One says that random variables $f$ and $g$ are equimeasurable if $f^*(t) = g^*(t), \ 0 < t \leq 1$ (equivalently, $n_f(\tau) = n_g(\tau), \ \tau > 0$). Finally, given two positive functions (quasinorms) $f$ and $g$ are said to be equivalent (we write $f \sim g$) if there exists a positive finite constant $C$ such that $C^{-1} f \leq g \leq C f$. Sometimes, we say that these functions are equivalent for large (or small) values of the argument, meaning that the preceding inequalities hold only for its specified values.

2.2. **A condition for independent copies of a mean zero $f$ to be equivalent in $L_p$ to the unit vector basis of $l_M$.** For a fixed $f \in L_1(0,1)$, every $k \in \mathbb{N}$, and $t > 0$ we set

$$\overline{f}_k(t) := \begin{cases} f(t - k + 1), t \in [k-1, k), \\ 0, \text{ otherwise.} \end{cases}$$

The following assertion is an immediate consequence of the famous Rosenthal inequality [14] (or, its more general version due to Johnson and Schechtman [7]). It establishes a connection between the behaviour in $L_p$ of an arbitrary sequence $\{f_k\}_{k=1}^{\infty}$ of independent copies of a mean zero random variable $f \in L_p$ and that of corresponding sequence $(\overline{f}_k)_{k=1}^{\infty}$ in the Banach sum $(L_p + L_2)(0, \infty)$ of the Lebesgue spaces $L_p(0, \infty)$ and $L_2(0, \infty).

**Lemma 5.** Let $1 \leq p \leq 2$. For every finitely supported $a = (a_k)_{k=1}^{\infty}$ and for a mean zero random variable $f \in L_p(0, 1)$ we have

$$\left\| \sum_{k=1}^{\infty} a_k f_k \right\|_p \sim \left\| \sum_{k=1}^{\infty} a_k \overline{f}_k \right\|_{L_p + L_2}.$$ 

**Lemma** allows us to investigate sequences of independent identically distributed mean zero random variables in $L_p = L_p(0, 1)$.

**Proposition 6.** Let $1 \leq p \leq 2$ and let $f \in L_p$ be a mean zero random variable. Then, a sequence $\{f_k\}_{k=1}^{\infty}$ of independent copies of the random variable $f$ is equivalent (in $L_p$) to the unit vector basis in $l_M$ if and only if

$$\frac{1}{M^{-1}(t)} \sim \left( \frac{1}{t} \int_0^t f^*(s)^p \, ds \right)^{1/p} + \left( \frac{1}{t} \int_t^1 f^*(s)^2 \, ds \right)^{1/2}, \quad 0 < t \leq 1.$$

Proposition 7. Let \( m \) function \( L_2, \Delta_1 \) Since

Proof. At first, we assume that a sequence \( \{ f_k \}_{k=1}^\infty \) of independent copies of \( f \) is equivalent in \( L_p \) to the unit vector basis in \( l_M \). Then, we have

\[
\left\| \sum_{k=1}^n e_k \right\|_{l_M} \sim \left\| \sum_{k=1}^n f_k \right\|_p^M \sim \left\| \sum_{k=1}^n f_k \right\|_{L_p+L_2}.
\]

Since \( 1 \leq p \leq 2 \), it follows that

\[
\|x\|_{L_p+L_2} \sim \left( \int_0^1 x^*(s)^p \, ds \right)^{1/p} + \left( \int_1^\infty x^*(s)^2 \, ds \right)^{1/2}.
\]

Therefore, from the equalities

\[
\left( \sum_{k=1}^n f_k \right)^*(s) = f^*\left( \frac{s}{n} \right), \quad s > 0,
\]

and

\[
\left\| \sum_{k=1}^n e_k \right\|_{l_M} = \inf \left\{ \rho > 0 : nM\left( \frac{1}{\rho} \right) \leq 1 \right\} = \frac{1}{M^{-1}(1/n)}, \quad n \geq 1,
\]

it follows that

\[
\frac{1}{M^{-1}(1/n)} \sim \left( \int_0^{1/n} (f^*(s))^p \, ds \right)^{1/p} + \left( \int_{1/n}^{1} (f^*(s))^2 \, ds \right)^{1/2} = \left( n \int_0^{1/n} (f^*(s))^p \, ds \right)^{1/p} + \left( n \int_{1/n}^{1} (f^*(s))^2 \, ds \right)^{1/2}, \quad n \geq 1.
\]

Let \( t \in (1/(n+1), 1/n) \) for some \( n \geq 1 \). We clearly have \( M^{-1}(1/n) \sim M^{-1}(t) \) and

\[
\left( n \int_0^{1/n} (f^*(s))^p \, ds \right)^{1/p} + \left( n \int_{1/n}^{1} (f^*(s))^2 \, ds \right)^{1/2} \sim \left( \frac{1}{t} \int_0^1 (f^*(s))^p \, ds \right)^{1/p} + \left( \frac{1}{t} \int_1^1 (f^*(s))^2 \, ds \right)^{1/2}.
\]

The assertion \( 1 \) follows immediately from the equivalences above.

Conversely, by [6] Theorem 1, p.X.8 (see also [1] Theorem 9), for every given mean zero \( f \in L_p(0,1) \) the sequence \( \{ f_k \}_{k=1}^\infty \) of independent copies of \( f \) is equivalent in \( L_p \) to the unit vector basis in some Orlicz sequence space \( l_N \). Arguing in the same way as in the first part of the proof, we conclude that

\[
\frac{1}{N^{-1}(t)} \sim \left( \frac{1}{t} \int_0^t f^*(s)^p \, ds \right)^{1/p} + \left( \frac{1}{t} \int_1^1 f^*(s)^2 \, ds \right)^{1/2}, \quad t \in (0,1).
\]

Taken together with \( 1 \) the equivalence above yields that the Orlicz functions \( M \) and \( N \) are equivalent on the segment \([0,1]\) and thus, \( l_N = l_M \). This completes the proof. \( \square \)

3. When does the equivalence \( 1 \) hold for the function \( f = m \)?

The following proposition provides necessary and sufficient conditions for the function \( m^p \) to be equivalent to its Cesaro transform.

Proposition 7. Let \( 1 \leq p < \infty \) and let \( M \) be a \( p \)-convex Orlicz function satisfying \( \Delta_2 \)-condition at 0. The following conditions are equivalent:

(i) The function \( M \) is equivalent on the segment \([0,1]\) to a \((p+\varepsilon)\)-convex Orlicz function for some \( \varepsilon > 0 \);
(ii) \[
\frac{1}{t} \int_0^t m^p(s) \, ds \leq \text{const} \cdot m^p(t), \quad t \in (0, 1).
\]

Proof. Let the function \( \varphi \) be defined by setting
\[
\varphi(t) = tm^p(t), \quad t \in (0, 1).
\]

(i) \( \rightarrow \) (ii). It suffices to show that
\[
\int_0^t \frac{\varphi(s) \, ds}{s} \leq \text{const} \cdot \varphi(t), \quad t \in (0, 1).
\]

It follows directly from the definitions that, for all \( s \in (0, 1) \),
\[
\sup_{0 < t \leq 1} \frac{\varphi(st)}{\varphi(t)} = s \cdot \sup_{0 < t \leq 1} \left( \frac{M^{-1}(t)^{p+\varepsilon}}{(M^{-1}(st))^{p+\varepsilon}} \right)^{\frac{1}{p+\varepsilon}}.
\]
Since \( M \) is \((p + \varepsilon)-\)convex, the mapping
\[
t \mapsto (M^{-1}(t))^{p+\varepsilon}, \quad t \in (0, 1],
\]
is concave. In particular, we have
\[
\frac{(M^{-1}(t))^{p+\varepsilon}}{(M^{-1}(st))^{p+\varepsilon}} \leq s^{-1}, \quad 0 < s, t \leq 1.
\]
Therefore,
\[
\sup_{t \in (0, 1)} \frac{\varphi(st)}{\varphi(t)} \leq s \cdot \frac{1}{s^{\frac{1}{p+\varepsilon}}}, \quad 0 < s \leq 1.
\]
Applying now Lemma II.1.4 from [10], we infer (2) and this completes the proof of implication (i) \( \rightarrow \) (ii).

(ii) \( \rightarrow \) (i). Since \( M \) is \( p \)-convex, it follows that
\[
\frac{M(s)}{s^p} \leq \frac{M(t)}{t^p}, \quad 0 \leq s \leq t \leq 1.
\]
Replacing \( s \) with \( M^{-1}(s) \) and \( t \) with \( M^{-1}(t) \), we infer that \( \varphi \) is increasing.

By the assumption, we have
\[
\int_0^t \frac{\varphi(s) \, ds}{s} \leq C \varphi(t), \quad t \in (0, 1),
\]
for some \( C > 0 \). Take \( s_0 < e^{-2C} \). We claim that
\[
\sup_{t \in (0, 1)} \frac{\varphi(st)}{\varphi(t)} < 1.
\]
Indeed, suppose that supremum in (3) equals 1. In particular, there exists \( t \in (0, 1) \) such that \( \varphi(st) > \varphi(t)/2 \). Since \( \varphi \) is increasing and since \( \log(s_0^{-1}) > 2C \), it follows that
\[
\int_0^t \frac{\varphi(s) \, ds}{s} \geq \int_{s_0t}^t \frac{\varphi(s) \, ds}{s} \geq \varphi(s_0t) \log\left( \frac{t}{s_0t} \right) > C \varphi(t).
\]
This contradiction proves the claim.

According to (3), we can fix \( a \in (0, 1) \) such that
\[
\varphi(s_0t) \leq a \varphi(t), \quad t \in (0, 1).
\]
Without loss of generality, we can assume \( a > s_0^{\frac{1}{p+\varepsilon}} \). Hence, there exists \( \varepsilon \in (0, 1) \) such that \( a = s_0^{\frac{1}{p+\varepsilon}} \).

For an arbitrary \( s \in (0, 1] \) there exists \( n \in \mathbb{N} \) such that \( s \in (s_0^{n+1}, s_0^n) \). Since \( \varphi \) is increasing, it follows that
\[
\varphi(st) \leq \varphi(s_0^n t) \leq s_0^{\frac{n}{p+\varepsilon}} \varphi(t), \quad t \in (0, 1).
\]
Hence, we have
\[
\varphi(st) \leq \text{const} \cdot s^{\frac{n}{p+\varepsilon}} \varphi(t), \quad s, t \in (0, 1)
\]
or, equivalently,
\[
(st)^{\frac{n}{p+\varepsilon}} \varphi(st) \leq \text{const} \cdot t^{\frac{n}{p+\varepsilon}} \varphi(t), \quad s, t \in (0, 1).
\]
Therefore, it follows from the definition of \( \varphi \) that
\[
M(st) \leq \text{const} \cdot s^{p+\varepsilon} \cdot M(t), \quad s, t \in (0, 1).
\]
The argument is completed, by referring to Lemma 3. 

Now, we prove a dual result.

**Proposition 8.** Let \( M \) be a \( q \)-concave Orlicz function for some \( 1 < q < \infty \). The following conditions are equivalent:

(i) The function \( M \) is equivalent to a \((q - \varepsilon)\)-concave Orlicz function for some \( \varepsilon > 0 \) on the segment \([0, 1]\);

(ii) \[
\frac{1}{t} \int_0^t m^q(s) \, ds \leq \text{const} \cdot m^q(t), \quad t \in (0, 1).
\]

**Proof.** Define the function \( \psi \) by setting
\[
\psi(t) := t m^q(t), \quad t \in (0, 1).
\]

(i) \(\Rightarrow\) (ii). It suffices to verify that
\[
\int_t^1 \frac{\psi(s) \, ds}{s} \leq \text{const} \cdot \psi(t), \quad t \in (0, 1).
\]
We have
\[
\sup_{t} \frac{\psi(st)}{\psi(t)} = s \cdot \sup \left( \frac{(M^{-1}(t))^{q-\varepsilon}}{(M^{-1}(st))^{q-\varepsilon}} \right)^{\frac{1}{q-\varepsilon}},
\]
where the supremums are taken over all \( t \in (0, 1) \) and \( s > 1 \) such that \( 0 < st \leq 1 \). Since \( M \) is \((q - \varepsilon)\)-concave, it follows that the mapping 
\[
t \to (M^{-1}(t))^{q-\varepsilon}, \quad t \in (0, 1),
\]
is convex. In particular, we have
\[
\frac{(M^{-1}(t))^{q-\varepsilon}}{(M^{-1}(st))^{q-\varepsilon}} \leq s^{-1}, \quad s > 1, \quad 0 < st \leq 1.
\]
Therefore,
\[
\sup_{t} \frac{\psi(st)}{\psi(t)} \leq s^{-\frac{1}{q-\varepsilon}} < 1,
\]
where again the supremum is taken over all \( t \in (0, 1) \) and \( s > 1 \) such that \( 0 < st \leq 1 \).

Applying now Lemma II.1.5 in [10], we infer (5).
Indeed, suppose that supremum in (6) equals 1 for some $C > 0$, appealing to the fact that $\psi$ is decreasing. Without loss of generality, we have $\psi(st) \leq \psi(t)/2$. Since $\psi$ is decreasing, it follows that

$$\int_t^1 \frac{\psi(s) ds}{s} \leq C\psi(t), \quad t \in (0,1),$$

for some $C > 0$. Take $s_0 > e^{2C}$. We claim that

$$(6) \quad \sup_{t \in (0, s_0^{-1})} \frac{\psi(s_0 t)}{\psi(t)} < 1.$$ 

Indeed, suppose that supremum in (6) equals 1. In particular, there exists $t \in (0, s_0^{-1})$ such that $\psi(s_0 t) = \psi(t)/2$. Since $\psi$ is decreasing, it follows that

$$\int_t^1 \frac{\psi(s) ds}{s} = \int_t^{s_0 t} \frac{\psi(s) ds}{s} = \int_{s_0 t}^t \frac{\psi(s) ds}{s} \geq \psi(s_0 t) \log\left(\frac{s_0 t}{t}\right) > C\psi(t).$$

This contradiction proves the claim.

According to (6), we can fix $b \in (0,1)$ such that

$$(7) \quad \psi(s_0 t) \leq b\psi(t), \quad t \in (0, s_0^{-1}).$$

Without loss of generality, $b > s_0^{-1}$. Hence, there exists $\varepsilon > 0$ such that $b = s_0^{-\frac{\varepsilon}{p'}}$. Let $s > 1$ and $0 < t < s^{-1}$. We can find $n \in \mathbb{N}$ such that $s \in (s_0^n, s_0^{n+1})$. Again appealing to the fact that $\psi$ is decreasing, we have

$$\psi(st) \leq \psi(s_0^n t) \leq \psi(s_0^{-\frac{\varepsilon}{p}} \psi(t)) \leq \psi(s_0^{-\frac{\varepsilon}{p}} s^{-\frac{\varepsilon}{p}} \psi(t)).$$

It follows that

$$\psi(st) \leq \text{const} \cdot s^{-\frac{\varepsilon}{p}} \psi(t), \quad s > 1, t \in (0, s^{-1})$$

or, equivalently,

$$s^{\frac{\varepsilon}{p}} \psi(s) \leq \text{const} \cdot t^{\frac{\varepsilon}{p}} \psi(t), \quad 0 \leq t \leq s \leq 1.$$ 

Therefore, from the definition of $\psi$, we have

$$\frac{s}{M^{-1}(s)^{q-\varepsilon}} \leq \text{const} \cdot \frac{t}{M^{-1}(t)^{q-\varepsilon}}, \quad 0 \leq t \leq s \leq 1.$$ 

or

$$\text{const} \cdot s^{q-\varepsilon} \cdot M(t) \leq M(st), \quad \forall t, s \in (0,1].$$

Applying Lemma 4, we complete the proof.

The following theorem answers the question stated in the title of the present section.

**Theorem 9.** Let $1 \leq p < 2$ and let $M$ be a $p$--convex and $2$--concave Orlicz function. The following conditions are equivalent:

(i) Equivalence (11) holds for $f = m$.

(ii) $M$ is $(p + \varepsilon)$--convex and $(2 - \varepsilon)$--concave for some $\varepsilon > 0$. 
Proof. (ii) ⇒ (i). If $M$ is $(p + \varepsilon)$–convex for some $\varepsilon > 0$, then it follows from Proposition [7] that
\begin{equation}
\left( \frac{1}{t} \int_{0}^{t} m^p(s) \, ds \right)^{1/p} \leq \text{const} \cdot m(t), \quad t \in (0, 1).
\end{equation}
If $M$ is $(2 - \varepsilon)$–concave for some $\varepsilon > 0$, then Proposition [8] implies
\begin{equation}
\left( \frac{1}{t} \int_{t}^{1} m^2(s) \, ds \right)^{1/2} \leq \text{const} \cdot m(t), \quad t \in (0, 1).
\end{equation}
Observe now that the inequality
\begin{equation}
m(t) \leq \left( \frac{1}{t} \int_{0}^{t} m^p(s) \, ds \right)^{1/p}, \quad t \in (0, 1)
\end{equation}
holds trivially, due to the fact that $m$ is decreasing. The equivalence (1) for $f = m$ follows immediately from [8], [9] and [10].

(i) ⇒ (ii). Suppose that (1) holds for $f = m$. Then, we have [8] and [9]. Applying Propositions [7] and [8] we obtain that $M$ is $(p + \varepsilon)$–convex and $(2 - \varepsilon)$–concave for some $\varepsilon > 0$, and the proof is completed. \hfill \blacksquare

4. When does equivalence (1) hold for a unique $f$ (up to equivalence near 0)?

This section contains the proof of Theorem [1].

Proof of Theorem [1]. The implication (ii) → (iii) is obvious and the implication (iii) → (i) follows by combining results of Proposition [8] and Theorem [9].

(i) → (ii). We begin with the following technical lemma.

Lemma 10. Let $1 \leq p < \infty$, $1 < q < \infty$ and let $M$ be an Orlicz function.

(i) If $M$ is $(q - \varepsilon)$–concave for some $\varepsilon > 0$, then
\[ N \sup_{t > 0} \frac{m^q(Nt)}{m^q(t)} \to 0, \quad N \to \infty. \]

(ii) If $M$ is $(p + \varepsilon)$–convex for some $\varepsilon > 0$, then
\[ \frac{1}{N} \sup_{t > 0} \frac{m^p\left(\frac{t}{N}\right)}{m^p(t)} \to 0, \quad N \to \infty. \]

Proof. Proofs of (i) and (ii) are very similar. So, we prove (i) only. Since $M$ is $(q - \varepsilon)$–concave, it follows that the mapping $t \to \frac{M(t)}{t^{q-\varepsilon}}$, $t > 0$, is decreasing. Hence, the mapping $t \to t m^{q-\varepsilon}(t) = \frac{t}{(M^{-1}(t))^{q-\varepsilon}}$, $t > 0$, is also decreasing. Therefore,
\[ N^{\frac{q}{q-\varepsilon}} \sup_{t > 0} \frac{m^q(Nt)}{m^q(t)} = \left( \sup_{t > 0} \frac{N t m^{q-\varepsilon}(Nt)}{tm^{q-\varepsilon}(t)} \right)^{\frac{q}{q-\varepsilon}} \leq 1, \]
whence
\[ N \sup_{t > 0} \frac{m^q(Nt)}{m^q(t)} \leq N^{-\frac{1}{q}} \to 0 \quad \text{if} \quad N \to \infty. \]

Now, let \( M \) be a \((p + \varepsilon)\)-convex and \((2 - \varepsilon)\)-concave Orlicz function and let \( f \) be a mean zero function from \( L^p \). Suppose that the sequence \( \{f_k\}_{k=1}^\infty \) of independent copies of \( f \) is equivalent to the unit vector basis \( \{e_k\}_{k=1}^\infty \) in \( l_M \). It suffices to show that the functions \( f^* \) and \( m \) are equivalent for small values of argument. For simplicity we abuse the notation assuming that \( f = f^* \).

By Proposition 6 we know that the equivalence (1) holds for \( f \), that is,
\[ m(t) \sim \frac{1}{t} \int_0^t f(s) \, ds \] for some \( C > 0 \) such that
\[ f(t) \leq C_1 \cdot m(t), \quad t \in (0, 1), \]
for some \( C_1 > 0 \) follows immediately from (11) and the (already used) inequality
\[ f(t) \leq \left( \frac{1}{t} \int_0^t f(s)^p \, ds \right)^{1/p}, \quad t \in (0, 1). \]

Thus, we need to show that the estimate
\[ m(t) \leq \text{const} \cdot f(t), \quad t \in (0, 1), \]
holds for all sufficiently small \( t \in (0, 1) \). By Propositions 7 and 8 there exists a constant \( C_0 > 0 \) such that
\[ \frac{1}{t} \int_0^t m^p(s) \, ds \leq C_0^p m^p(t), \quad t \in (0, 1), \]
\[ \frac{1}{t} \int_t^1 m^2(s) \, ds \leq C_0^2 m^2(t), \quad t \in (0, 1). \]

Moreover, there is a constant \( C > 0 \) such that for a given \( t \in (0, 1) \), from (11) it follows that either
\[ \left( \frac{1}{t} \int_t^1 f^2(s) \, ds \right)^{1/2} \geq \frac{1}{2C} m(t), \]
or
\[ \left( \frac{1}{t} \int_0^t f^p(s) \, ds \right)^{1/p} \geq \frac{1}{2C} m(t). \]

By Lemma 10 we can fix \( N \) so large that
\[ \sup_{t > 0} \frac{m^2(Nt)}{m^2(t)} \leq \frac{1}{8NC^2C_1^2}, \quad \sup_{t > 0} \frac{m^p(t)}{m^p(t)} \leq \frac{N}{2p+1C_1^pC^p}. \]
Let $t \in (0, 1/N)$. Firstly, we consider the situation when (17) holds. Taking squares in this inequality and then applying (13), we obtain
\[
\frac{1}{4C^2} m^2(t) \leq \frac{1}{t} \int_t^{1} f^2(s) \, ds = \frac{1}{t} \int_{Nt}^{1} f^2(s) \, ds + \frac{1}{t} \int_t^{1} f^2(s) \, ds
\]
\[
\leq (N - 1)f^2(t) + \frac{NC_1^2}{Nt} \int_{Nt}^{1} m^2(s) \, ds.
\]
Hence, by (16), we have
\[
\frac{1}{4C^2} m^2(t) \leq (N - 1)f^2(t) + NC_1^2 C_0^2 m^2(Nt).
\]
Combining the latter estimate with the first inequality in (19), we obtain
\[
(N - 1)f^2(t) \geq (N - 1)f^2(t) \geq \frac{1}{4C^2} m^2(t) - NC_1^2 C_0^2 m^2(Nt) \geq \frac{1}{8C^2} m^2(t).
\]
If (18) holds, then
\[
\frac{1}{2pC^p} m^p(t) \leq \frac{1}{t} \int_t^{1/2} f^p(s) \, ds = \frac{1}{t} \int_0^{t/N} f^p(s) \, ds + \frac{1}{t} \int_{t/N}^{1} f^p(s) \, ds.
\]
Taking (13) and (15) into account, we obtain
\[
\frac{1}{2pC^p} m^p(t) \leq \frac{C_1^p}{t/N} \int_0^{t/N} m^p(s) \, ds + (1 - \frac{1}{N})f^p(t/N)
\]
\[
\leq \frac{1}{N} C_1^p C_0^p m^p(t/N) + (1 - \frac{1}{N})f^p(t/N).
\]
We infer from this estimate and the second inequality in (19) that
\[
(1 - \frac{1}{N})f^p(t/N) \geq \frac{1}{2pC^p} m^p(t) - \frac{1}{N} C_1^p C_0^p m^p(t/N) \geq \frac{1}{2p+1} C^p m^p(t).
\]
In either case, we have
\[
f(t/N) \geq \text{const} \cdot m(t), \quad t \in (0, \frac{1}{N}),
\]
for a universal constant. Since $m(t) \sim m(t/N)$, it follows that
\[
f(t) \geq \text{const} \cdot m(t), \quad t \in (0, \frac{1}{N^2}).
\]
The latter inequality together with (13) suffices to conclude the proof of implication (i) $\rightarrow$ (ii).

5. Sharpness of Theorem 1

Let $\{h_k\}_{k=1}^\infty$ (respectively, $\{g_k\}_{k=1}^\infty$) be a sequence of pairwise disjoint measurable subsets of $(0, 1)$ such that $\lambda(h_k) = 2^{-k-2^k}$ (respectively, $\lambda(g_k) = 4^{-k-4^k}$), $k \geq 1$. We define functions $x, y \in L_1(0, 1)$ by setting
\[
x = \sum_{k=1}^\infty 2^k \chi_{h_k}, \quad y = \sum_{k=1}^\infty 4^k \chi_{g_k},
\]
($\chi_c$ is the indicator function of a set $c$).
Lemma 11. We have
\[ \int_0^1 \min\{x(s), tx^2(s)\} \, ds \sim \int_0^1 \min\{y(s), ty^2(s)\} \, ds \sim \frac{1}{\log(e/t)}, \quad 0 < t \leq 1. \]

Proof. It is clear that
\[ \int_0^1 \min\{x(s), tx^2(s)\} \, ds = \sum_{2^k \geq 1/t} 2^{-k} \cdot 2^{k-2k} + t \cdot \sum_{2^k < 1/t} 2^{k+1} \cdot 2^{-k-2k}. \]

Let \( t < 1/4 \). If \( m \) is the maximal positive integer such that \( 2^m < 1/t \), then
\[ \int_0^1 \min\{x(s), tx^2(s)\} \, ds = \sum_{k=m+1}^\infty 2^{-k} + t \cdot \sum_{k=1}^m 2^{k-2k} = 2^{-m} + t \cdot \sum_{k=1}^m 2^{k-k}. \]
Also, we have
\[ \sum_{k=1}^m 2^{k-k} \leq 2^{m-m} + (m-1) \cdot 2^{m-1-m+1} \leq 2^{m-m} + 2^{m-1} \leq 2 \cdot 2^{m-m}. \]
Therefore, we obtain
\[ 2^{-m} \leq \int_0^1 \min\{x(s), tx^2(s)\} \, ds \leq 2^{-m} + 2t \cdot 2^{m-m} \leq 3 \cdot 2^{-m}. \]
it follows now from the definition of the number \( m \) that
\[ \frac{1}{\log_2(1/t)} \leq \int_0^1 \min\{x(s), tx^2(s)\} \, ds \leq \frac{6}{\log_2(1/t)}. \]
The similar equivalence for \( y \) follows mutatis mutandi. \( \square \)

Lemma 12. Distributions of the functions \( x \) and \( y \) are not equivalent.

Proof. Suppose that \( n_x(Ct) \leq Cn_y(t), t > 0 \). Fix \( k \) such that
\[ 2^{2k+1} > \log_2 C + 1 \]
and select \( t \) such that both \( t \) and \( Ct \) belong to the interval \((2^{2k+1}, 2^{2k+2})\). Then, we have
\[ n_x(Ct) = n_x(2^{2k+1}) \geq 2^{-(2k+2)-2^{2k+2}} \]
and
\[ n_y(t) = n_y(4^k) \leq 2 \cdot 4^{-(k+1)-k} = 2^{-2k-1-2k+3}. \]
It follows from the preceding inequalities that
\[ 2^{2k+2+2^{2k+2}} \geq \frac{1}{C} \cdot 2^{2k+1+2^{2k+3}} \]
or, equivalently,
\[ 2k + 2 + 2^{2k+2} \geq -\log_2(C) + 2k + 1 + 2^{2k+3}. \]
Clearly, the latter inequality contradicts the choice of \( k \). \( \square \)

Let \( \{x_k\}_{k=1}^\infty \) (respectively, \( \{y_k\}_{k=1}^\infty \)) be a sequence of independent copies of a mean zero random variable equimeasurable with \( x \) (respectively, \( y \)), where \( x \) and \( y \) are defined in (20). Let us show that the sequences \( \{x_k\}_{k=1}^\infty \) and \( \{y_k\}_{k=1}^\infty \) span in \( L_1 \) the same Orlicz space \( l_M \), where \( M \) is equivalent to the function \( t/\log(e/t) \) for small \( t > 0 \). Note that \( M \) does not satisfy condition (i) of Theorem 1 more
precisely, \( M \) is not \((1 + \varepsilon)\)-convex for any \( \varepsilon > 0 \). Taking into account Lemma 5, it suffices to prove the following proposition.

**Proposition 13.** For every finitely supported \( a = (a_k)_{k=1}^{\infty} \), we have

\[
\left\| \sum_{k=1}^{n} a_k x_k \right\|_{L_1 + L_2} \sim \left\| \sum_{k=1}^{n} a_k y_k \right\|_{L_1 + L_2} \sim \| (a_k)_{k=1}^{\infty} \|_{l_M}.
\]

**Proof.** Define the Orlicz function \( N \) by setting

\[
N(t) = \begin{cases} 
  t^2, & t \in (0, 1) \\
  2t - 1, & t \geq 1.
\end{cases}
\]

It is easy to check that \( \| z \|_{L_1 + L_2} \sim \| z \|_{L_N} \) for every \( z \in L_1 + L_2 \), where \( L_N \) is the function Orlicz space on \([0, 1] \).

Setting

\[
M(t) = \int_0^1 N(tx(s)) \, ds, \quad t > 0,
\]

we obtain

\[
\left\| \sum_{k=1}^{\infty} a_k x_k \right\|_{L_N} \leq 1 \iff \int_0^\infty N\left( \sum_{k=1}^{\infty} |a_k| x_k(s) \right) \, ds \leq 1 \iff \sum_{k=1}^{\infty} \int_0^1 N(|a_k| x_k(s)) \, ds \leq 1 \iff \sum_{k=1}^{\infty} M(a_k) \leq 1 \iff \| a \|_{l_M} \leq 1.
\]

Therefore,

\[
\left\| \sum_{k=1}^{\infty} a_k x_k \right\|_{L_1 + L_2} \sim \| a \|_{l_M}.
\]

Since \( N(t) \sim \min\{t, t^2\} \ (t > 0) \), it follows that

\[ M(t) \sim \int_0^1 \min\{tx(s), (tx(s))^2\} \, ds, \]

and from Lemma 11 it follows that

\[ M(t) \sim \frac{t}{\log(e/t)}, \quad 0 < t \leq 1. \]

This proves the assertion for the sequence \( \{x_k\} \). The proof of the similar assertion for \( \{y_k\} \) is the same. \( \square \)

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