Systemic Risk with Exchangeable Contagion: Application to the European Banking System *

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Abstract

We propose a model and an estimation technique to distinguish systemic risk and contagion in credit risk. The main idea is to assume, for a set of d obligors, a set of d idiosyncratic shocks and a shock that triggers the default of all them. All shocks are assumed to be linked by a dependence relationship, that in this paper is assumed to be exchangeable and Archimedean. This approach is able to encompass both systemic risk and contagion, with the Marshall-Olkin pure systemic risk model and the Archimedean contagion model as extreme cases. Moreover, we show that assuming an affine structure for the intensities of idiosyncratic and systemic shocks and a Gumbel copula, the approach delivers a complete multivariate distribution with exponential marginal distributions. The model can be estimated by applying a moment matching procedure to the bivariate marginals. We also provide an easy visual check of the good specification of the model. The model is applied to a selected sample of banks for 8 European countries, assuming a common shock for every country. The model is found to be well specified for 4 of the 8 countries. We also provide the theoretical extension of the model to the non-exchangeable case and we suggest possible avenues of research for the estimation.

Keywords: Credit risk, Systemic risk, Contagion, Copula functions, Marshall-Olkin distribution, Financial crisis

1 Introduction

The purpose of this paper is to draw a line between systemic risk and contagion, and to design a method to measure the relative contribution of contagion and systemic risk to the dependence structure

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of a set of credit exposures. The statistical problem of disentangling systemic risk and contagion is of utmost relevance for economic policy. In several problems, such as pollution regulation or banking, recognizing systemic risk, as an event independent of the agents, or contagion, that is a system wide event triggered by one of them, is a major discriminant factor to decide whether the effects should be charged to the community at large or to the individual agents.

The problem is involved for two reasons. The first, already addressed in many studies on the subject, is that the systemic risk factor is not observed and for credit risk applications we are only able to extract the marginal survival probabilities from the market. The second reason, that is the subject of this paper, is whether such dependence is explained by the presence of a systemic risk factor only, or of some infectious elements in the system.

To make the problem clear, assume that we are allowed to observe the systemic risk factor, and we are able to appraise the probability of a systemic crisis. In this situation, the question would naturally arise whether the systemic shock is independent of the other events triggering the default of each component of the set. Answering this question on practical grounds would be obviously easy in this setting, and the dependence between idiosyncratic and systemic triggers of default could be estimated in the usual way, e.g. using copulas.

Notice, however, that even if dependence with the systemic risk factor were observed, a problem of interpretation of these results on theoretical grounds would arise concerning how it would affect the observed dependence structure of the components in the system. In fact, the presence of a systemic factor is sufficient to induce dependence among the components of the system and between each component and the systemic event. In other words, this dependence shows up even if the systemic shock is independent of the idiosyncratic ones. Intuitively, if the idiosyncratic default drivers were linked to the systemic risk trigger by a dependence relationship, the degree of dependence in the system would be even stronger.

In this paper we propose a model to represent these two sources of dependence in a tractable way. The idea of our model is very simple. Given a cluster of $d$ obligors, we assume that the system is subject to a set of $d + 1$ shocks, one of which is common to all the components and leads to simultaneous default of all the oblig-
ors in the cluster, while the others are responsible for the default of each component. Assume further that the times of these shocks are linked by a copula function of dimension $d + 1$. It is immediate to see that this model encompasses the two extreme cases of pure systemic risk and pure contagion. Namely, in the case of a product copula for the $d + 1$ occurrence times of the shocks, we obtain a model with bivariate Marshall-Olkin marginal distributions, representing the pure systemic risk model. The opposite case arises when the systemic shock has zero probability, so that we have a standard survival copula model corresponding to pure contagion. Allowing for positive probability of a systemic crisis and for dependence between this risk and the idiosyncratic credit drivers would then allow to design models that represent both systemic risk and contagion. In these models, the task of disentangling the two is a relevant question.

In its simplest version, our paper assumes the standard restriction of the choice of exchangeable copulas for the credit risk drivers. This means that each idiosyncratic factor is assumed to be linked by the same dependence structure to the systemic risk factor and to the other idiosyncratic ones. We also show that further restrictions change the copula model in a multivariate distribution model with exponential margins. If the model may seem restrictive, on practical grounds we provide a methodology to verify if the assumption is borne out by the data. Moreover, on theoretical grounds, we will also provide the theoretical development for the non-exchangeable version of model.

The plan of the paper is as follows. After reviewing the relevant literature, in Section 2 we motivate and describe in full generality our credit risk model for a basket of issuers, we discuss its main properties and the restrictions that change the copula model in a multivariate distribution with exponential marginals. In Section 3 we discuss the theoretical features of the extension to non-exchangeable dependence of the credit risk drivers. Finally, in Section 4 we illustrate our application to the banking system of a set of countries of the Euro area. In Section 5 we report conclusions and a discussion of the main issues left for future research.
1.1 Related literature

Our paper is related to a large literature on the measurement of systemic risk and contagion, even though to the best of our knowledge it is the first attempt to disentangle the two. Leaving aside any hope of being exhaustive, we may provide a taxonomy of the main contributions according to the structure of models and the data used. As for the methodology involved, a first class of models are based on the application of Granger causality, and related concepts, to the prices of financial assets (Billio et al., 2012). A second set of models is based on the network representation of the relationships among financial institutions (Diebold and Yilmaz, 2011). A third approach is based on the theory of risk measures applied to systemic risk and contagion. Models in this class are based on the measurement of expected losses conditional on an extreme scenario of some systemic risk factor. The technique is the same as expected shortfall, with the difference of conditioning with respect to a systemic variable. These measures are called Marginal Expected Shortfall, MES (Acharya et al., 2010), and CoVaR (Adrian and Brunnermeier, 2011). Cherubini and Mulinacci (2014) give conditions to ensure that coherence requirements be met, and propose examples of measures in this class based on copula functions.

Coming to the kind of data that are used in the empirical analysis, we may distinguish between applications that rely on the analysis of market prices, and those that use flows and balance sheet data. The first choice use equity stock prices (Billio et al., 2012) or volatilities (Diebold and Yilmaz, 2011), credit spreads of bonds and credit derivatives (Baglioni and Cherubini, 2013). With this choice, the focus is on measurement of the effects of systemic risk and contagion, in terms of future cash flows and the default probability that are implied in market quotes. The second choice exploits flows among the financial intermediaries and the focus is more on the means that explain propagation of the shocks through the financial intermediation system. Here the analysis is focussed on flows in the interbank market (Bonaldi, Hortacsy and Kastl, 2013) or on several layers representing other markets (Bargigli et al., 2013), or else on balance sheet indexes such as leverage (Brownlees and Engle, 2010). All these proxies are used as measures of the strength of contagion in the system.

Our paper uses the default probability extracted from CDS and their
dependence structure in order to recognize how much of this dependence is due to relationships among the components of the system, as in network based models, and how much of the co-movement is due to the presence of a systemic risk factor, as in systemic risk models. Moreover, to the best of our knowledge, this is the first attempt to include a dependence structure between each component and the systemic shock, although being the same dependence structure in both cases.

2 The model

Here we introduce the motivation of the model, and its basic setting. The idea is that in a system of $d$ components, the lifetime of each of them can come to an end either for idiosyncratic or systemic shocks, as in a standard Marshall-Olkin setting. Differently from that model, in which all shocks are assumed to be independent, here the idiosyncratic components are infectious. Idiosyncratic defaults can be associated, and they may also represent triggers of the systemic shock, leading to default of the whole system.

Technically, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a $d+1$-vector, $(X_0, X_1, \ldots, X_d)$ whose components have $[0, +\infty)$ as support. $X_0$ denotes the arrival time of the systemic shock and $(X_1, \ldots, X_d)$ are those of the idiosyncratic ones. We assume that the joint survival dependence structure is represented by a strict Archimedean copula, that is

$$
\bar{F}(x_0, x_1, \ldots, x_d) = \psi \left( \psi^{-1}(\bar{F}_0(x_0)) + \cdots + \psi^{-1}(\bar{F}_d) \right)
$$

for $(x_0, \ldots, x_d) \in [0, +\infty)^{d+1}$, where $\bar{F}_i$ (that is assumed to be continuous and strictly decreasing) is the marginal survival function of $X_i$ and $\psi$ is the generator of a strict $d+1$-dimensional Archimedean copula. We recall that $\psi$ is the generator of a $d+1$-Archimedean copula if and only if $\psi : [0, +\infty) \to [0, 1]$ is $d+1$-monotone on $[0, +\infty)$ that is

- it is differentiable on $(0, +\infty)$ up to order $d-1$ and the derivatives satisfy $(-1)^k \psi^{(k)}(x) \geq 0$ for $k = 0, 1, \ldots, d-1$ and $x \in (0, +\infty)$,
- $(-1)^{d-1} \psi^{(d-1)}$ is non-increasing and convex in $(0, +\infty)$. 

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(see McNeil and Nešlehová, 2009, for more details on multidimensional Archimedean copulas).

Since we restrict ourselves to the strict case, we assume $\psi(x) > 0$ for all $x \in [0, +\infty)$. Let us define

$$\tau_k = \min\{X_0, X_k\}, k = 1, \ldots, d.$$ 

This is the standard Marshall-Olkin setting in which the only common shock taken into account is the one affecting all the components in the set. Of course, other specifications are possible, including models with more than one common shock, affecting selected subsets of the components (see, for all, Durante, Hofert and Scherer, 2010). The observed default times $\tau_k$ represent the first arrival time between a common (systemic) shock affecting all the system and the idiosyncratic shocks. We then add an Archimedean type of dependence among the arrival times of the shocks, in order to represent contagion.

The joint survival function of the random vector $\tau = (\tau_1, \ldots, \tau_d)$ can be easily recovered

$$\tilde{F}_\tau(t_1, \ldots, t_d) = \psi\left(\psi^{-1}(\tilde{F}_0(\max_{1 \leq k \leq d} \{t_k\})) + \sum_{k=1}^d \psi^{-1}(\tilde{F}_k(t_k))\right)$$

for $t_1, \ldots, t_d \in [0, +\infty)^d$, while the marginal survival functions are

$$\tilde{F}_{\tau_k}(t) = \psi\left(\psi^{-1}(\tilde{F}_0(t)) + \psi^{-1}(\tilde{F}_k(t))\right) = \psi(H_{0,k}(t)), t \in [0, +\infty)$$

where $H_{0,k}(x) = \psi^{-1}(\tilde{F}_0(x)) + \psi^{-1}(\tilde{F}_k(x))$.

It is also easy to extract the copula function of the observed default times

**Proposition 2.1.** The survival copula $\hat{C}$ of the vector of default times $\tau$ is, for $u \in [0, 1]^d$,

$$\hat{C}(u) = \sum_{j=1}^d \psi\left(\psi^{-1}(u_j) + \sum_{k=1, k \neq j}^d D_k \circ \psi^{-1}(u_k)\right) 1_{A_j}(u)$$

where $D_k(x) = \psi^{-1} \circ F_k \circ H_{0,k}^{-1}(x)$ and

$$A_j = \left\{u \in [0, 1]^d : \max_{1 \leq i \leq d} \{H_{0,i}^{-1} \circ \psi^{-1}(u_i)\} = H_{0,j}^{-1} \circ \psi^{-1}(u_j)\right\}$$
with the convention that if \( u \) satisfies the required condition for more than one index \( j \), it is assumed to belong to the \( A_j \) with the smallest index \( j \).

**Proof.** See Appendix 6. \( \square \)

### 2.1 A multivariate distribution with contagion and exponential marginals

In practical applications it is common to represent and calibrate default times by exponential distributions

\[
\bar{F}_{\tau_k}(x) = \exp(-\mu_k x),
\]

where \( \mu_k \) denotes the intensity parameter. We now discuss which restrictions can be imposed on the model in order to transform the copula model above in a multivariate distribution with exponential marginals.

Starting from the copula model illustrated, constructing such multivariate distribution would imply a data generating process such that: i) the distortion functions \( D_k(x) \) are linear; ii) the dependence is represented by a Gumbel copula.

#### 2.1.1 Linear distortion

A possible assumption about functions \( \psi^{-1}(\bar{F}_i(x)) \), in the spirit of the paper by Muliere and Scarsini (1987), is that they are all proportional to the same function \( K(x) \): that is, \( \psi^{-1}(\bar{F}_i(x)) = \lambda_i K(x) \) for \( \lambda_i > 0 \), for \( i = 0, 1, \ldots, d \). This is equivalent to \( D_i(x) = (1 - \alpha_i)x \) where

\[
\alpha_i = \frac{\lambda_0}{\lambda_i + \lambda_0} \in [0, 1)
\]

and the obtained copula is independent of \( K \).

In the more specific case in which \( \psi \) is completely monotone (that is \( \psi \) is the Laplace transform of some positive random variable), we recover the Scale-Mixture of Marshall-Olkin distributions and copula models (SMMO) studied in Li (2009). The exchangeable case of SMMO model is studied in Mai and Scherer (2013) where it is applied to the pricing of CDOs.
2.1.2 The Gumbel case

A further restriction to yield marginal exponential distributions is to consider the case in which \( \psi \) is the Gumbel generator, that is \( \psi(x) = e^{-x^\theta}, \theta \geq 1 \). Now, equations (1), (2) and (3) take the form

\[
\bar{F}_\tau(t_1, \ldots, t_d) = \exp \left\{ - \left( \lambda_0 K \left( \max_{1 \leq i \leq d} \{t_i\} \right) + \sum_{k=1}^{d} \lambda_k K(t_k) \right)^{\frac{1}{\theta}} \right\}
\]

\[
\bar{F}_{n_k}(t) = \exp \left( - (\lambda_0 + \lambda_k)^{\frac{1}{\theta}} K^{\frac{1}{\theta}}(t) \right)
\]

(5)

\[
\hat{C}(u) = \sum_{j=1}^{d} \exp \left\{ - \left[ (-\ln u_j)^\theta + \sum_{k=1, k\neq j}^{d} (1 - \alpha_k)(-\ln u_k)^\theta \right]^{\frac{1}{\theta}} \right\} 1_{A_j}(u)
\]

Notice that, in this case, \( \psi \) is the Laplace transform of an \( \frac{1}{\theta} \)-stable distributed random variable and so it represents a specification of the SMMO model of Li (2009).

Notice that setting \( K(t) = t^\theta \) in (5) yields exponential marginals as required

\[
\mu_k = (\lambda_0 + \lambda_k)^{\frac{1}{\theta}}
\]

(6)

where \( \mu_k \) is the intensity in equation (4).

2.2 Properties of the model

The main feature of our model, right from the most general setting, is to increase the degree of dependence among the default times, both with respect to the standard Archimedean copula without any systemic risk factor and the Marshall-Olkin copula in which the systemic risk factor is independent of the others.

The dependence structure of the model encompasses both the sensitivity of the default times to the systemic shock, and the dependence among the shocks, represented by Archimedean copulas. Both these elements interact to determine the dependence among default times.
In the general setting, the Kendall’s tau $\tau_{i,k}$ measuring the dependence of the pair of default times $(\tau_i, \tau_k)$ can be written as

$$
\tau_{j,k} = \tau^\psi + 4 \int_{0}^{\infty} (\psi'(x))^2 \cdot T(x)dx
$$

where $\tau^\psi$ denotes the Archimedean Kendall’s tau corresponding to the generator $\psi$ and

$$
T(x) = \psi^{-1} \circ \bar{F}_0 \circ (\psi^{-1} \circ \bar{F}_0 + \psi^{-1} \circ \bar{F}_j + \psi^{-1} \circ \bar{F}_k)^{-1} (x)
$$

where we refer the reader to Mulinacci (2014) for the derivation. Notice that if we are interested in representing the dependence structure between the systemic shock and default times, we have that the Kendall’s tau $\tau_{j,0}$ of the pair $(\tau_j, X_0)$ is

$$
\tau_{j,0} = \tau^\psi + 4 \int_{0}^{\infty} (\psi'(x))^2 \cdot \left(\psi^{-1} \circ \bar{F}_0 \circ (\psi^{-1} \circ \bar{F}_0 + \psi^{-1} \circ \bar{F}_j)^{-1} (x)\right) dx
$$

The first term is simply the Kendall’s tau of the Archimedean copula used in the analysis, while the other term, that is more complex, involves both the generator of the Archimedean copula and the relative relevance of systemic and idiosyncratic shocks.

In the multivariate distribution arising with linear distortions and the Gumbel copula in the model, these relationships simplify substantially. In fact, let $\hat{C}_{j,k}(u,v)$ be the general marginal 2-copula,

$$
\hat{C}_{j,k}(u,v) = \exp \left\{ - \left[ (-\ln u)\theta + (1-\alpha_k)(-\ln v)\theta \right]^\frac{1}{\theta} \right\} \cdot \mathbb{1}_{\{\alpha_j \psi^{-1}(u) \geq \alpha_k \psi^{-1}(v)\}} + \exp \left\{ - \left[ (1-\alpha_j)(-\ln u)\theta + (-\ln v)\theta \right]^\frac{1}{\theta} \right\} \cdot \mathbb{1}_{\{\alpha_j \psi^{-1}(u) < \alpha_k \psi^{-1}(v)\}}
$$

Since this family of copulas represents a particular specification of the Archimax copulas of Capéraà et al. (2000) and of the Archimedean-based Marshall-Olkin copulas of Mulinacci (2014), its Kendall’s tau is known to be

$$
\tau_{j,k} = \frac{\theta - 1}{\theta} + \frac{\tau_{j,k}^{\text{MO}}}{\theta}
$$

where

$$
\tau_{j,k}^{\text{MO}} = \frac{\alpha_j \alpha_k}{\alpha_j + \alpha_k - \alpha_j \alpha_k}
$$

is the Kendall’s tau of the Marshall-Olkin copula.
Now, the dependence between each default time and the time of a systemic shock is linear

\[ \tau_{0,j} = \frac{\theta - 1}{\theta} + \frac{\alpha_j}{\theta} \]  

(8)

This relationship will be used in our estimation strategy in order to verify the specification of the model.

2.3 Estimation strategy

In the estimation of the model we assume to observe a panel set of data \( \mu_k(t_i) \), representing marginal default intensities of \( k = 1, 2, \ldots, d \) components, for \( \{t_1, t_2, \ldots, t_m\} \) dates. Our task is to estimate the set of \( \alpha_k \) parameters, representing the sensitivity of each obligor to the systemic shock, and the parameter \( \theta \), that measures the degree of contagion in the system. We also would like to make a check of the specification of the model.

Since the main feature of our approach is to identify the weight of the sensitivity to the systemic shock and of the degree of contagion in the dependence structure of default times, a natural estimation strategy would be a moment based approach, which resembles the calibration procedure proposed by Genest and Rivest (1993). In particular, our model specification based on linear distortions and Gumbel dependence makes a procedure based on Kendall’s tau calibration very easy.

Since the model is built to be fully characterized by the bivariate marginals, the estimation is naturally performed by calibrating the bivariate Kendall’s tau statistics of the system. For each cluster that we expect to be part of the same exchangeable system, consisting of \( d \) units, we calibrate the set of \( d + 1 \) parameters of our model.

Formally, we first estimate the Kendall’s tau statistics of all the pairs of the sample, and then estimate the set of parameters \( \Theta = \{\alpha_1, \alpha_2, \ldots, \alpha_d, \theta\} \) by solving

\[
\hat{\Theta} = \arg\min_{\{\alpha_1, \alpha_2, \ldots, \alpha_d, \theta\}} \sum_{i=1}^{d-1} \sum_{j=i+1}^{d} \text{dist}(\hat{\tau}_{i,j}, \tau_{i,j}(\alpha_i, \alpha_j, \theta))
\]

where \( \text{dist}(x, y) \) is a suitable distance measure, \( \tau_{i,j}(\alpha_i, \alpha_j, \theta) \) is the theoretical Kendall’s tau based on estimates, and \( \hat{\tau}_{i,j} \) is the corresponding empirical Kendall’s tau statistics. As for the parameters
set, $\alpha_i$ represents the sensitivity of component $i$ to a systemic shock, and $\theta$ represents the contagion parameter, that is assumed to be the same across all pairs.

The structure of the model also provides an easy procedure to check whether the specification of the model provides a good fit to the data. The idea is that if the model is well specified we could use it to estimate the intensity of the systemic shock from market data, and use that information to directly verify the specification of the model. The property of exponentially distributed marginals of the Gumbel specification is particularly useful in this case. Given a panel of $m$ observations of $d$ intensities, we can estimate a time series of $m$ intensities of the systemic shock. Using equation (6) and the definition of $\alpha_k$ it is straightforward to compute

$$\hat{\lambda}_0(t_i) = \frac{\sum_{k=1}^{d} \mu_k^\theta(t_i)}{\sum_{k=1}^{d} \frac{1}{\alpha_k}}$$

(9)

where $\hat{\lambda}_0(t_i)$ denotes the estimate of the systemic shock intensity at time $t_i$.

A straightforward visual check of the specification of the model would then be to estimate the Kendall’s tau value between the arrival time of a systemic shock and marginal default times. If the model is well specified, the Kendall’s tau values should be aligned on the straight line described by equation (8).

In that case, the procedure also provides a new series representing the implied intensity of the systemic shock, that may be usefully applied for further investigation of the cluster and of the system as a whole. As an example, one could verify whether other elements of the system, originally not associated to that cluster, actually have the same dependence with the systemic shock as the other elements of the cluster. As a second example, one could use the estimated systemic shock intensities of different clusters to check the degree of association across clusters.

Our application, that is meant to illustrate this estimation procedure, will be focussed on a set of European banks. We will assume that the banks of the same country constitute a cluster, and we will verify in which case this assumption is borne out by the data.
3 An extension to hierarchical Archimedean risk factors

In this section we will consider a possible extension of the model with exchangeable dependence structure presented above. Clearly, any \( d + 1 \)-dimensional copula can be considered in place of the Archimedean one and the same construction implemented. Among the possible reasonable choices, vine- Archimedean copulas and hierarchical copulas (HAC) could be considered as natural non-exchangeable extensions.

In this paper we will consider \( d+1 \)-dimensional HAC copulas. These are obtained through the composition of simple Archimedean copulas: such composition is recursively applied using different segmentations of the random variables involved. Starting from the initial variables \( u_1, \ldots, u_{d+1} \), these are grouped in \( l_1 \) copulas \( C_{C_1,1}, \ldots, C_{C_1,l_1} \).

Then, these copulas are grouped in \( l_2 \) copulas \( C_{C_2,1}, \ldots, C_{C_2,l_2} \), and up to the last level where we have just one copula. In order to ensure that the so obtained HAC copula is indeed a copula, the generators \( \psi_{i,j} \) of the copulas involved have to be completely monotone and the same must hold for their compositions \( \psi_{i+1,j}^{-1} \circ \psi_{i,k} \) whenever \( C_{C_i,k} \) is an argument of \( C_{C_{i+1},j} \). When the generators \( \psi_{i,j} \) are in the same parametrized family, the described procedure yields a copula if inner copulas have a parameter higher than the outer ones: in this paper we will consider generators belonging to the same family (see Savu and Trede 2008 and McNeil 2008 among the others as references on this topic).

In the fully nested case we have

\[
C(u) = C_d(\ldots C_3(C_2(C_1(u_1, u_2), u_3), u_4), \ldots, u_{d+1}).
\]

If the probability distribution of the systemic shock \( X_0 \) corresponds to \( u_1 \), then, the idiosyncratic risks \( X_i, i \geq 1 \), can be decreasingly ordered with respect to the dependence to \( X_0 \) being

\[
C_{X_0,X_i}(u, v) = C_{i-1}(u, v).
\]

If, instead, the probability of \( X_0 \) corresponds to \( u_{d+1} \), then

\[
C_{X_0,X_i}(u, v) = C_d(u, v).
\]

and the dependence structure between each idiosyncratic risk and the systemic one is the same for all the idiosyncratic triggers.
In the intermediate case in which the probability of \( X_0 \) corresponds to \( u_j \) for some \( j = 2, \ldots, d \), we have that

\[
C_{X_i, X_0}(u, v) = C_{j-1}(u, v)
\]

for those \( X_i \) whose probabilities correspond to those \( u_i \) with \( i < j \), and

\[
C_{X_0, X_i}(u, v) = C_{i-1}(u, v)
\]

for the probabilities of those \( X_i \) correspond to \( u_i \) with \( i > j \).

Of course, under other hierarchical configurations, completely different relationships among the systemic and the idiosyncratic risks can be modelled. For example if

\[
C(u) = C(C_{h,1}(u_1, \ldots, u_{j-1}), C_{h,2}(u_j, u_{j+1}, \ldots, u_{d+1}))
\]

where \( C_{h,1} \) and \( C_{h,2} \) are again HAC copulas, and \( X_0 \) corresponds to \( u_j \), we have that

\[
C_{X_i, X_0}(u, v) = C(u, v)
\]

for all probabilities \( X_i \) that correspond to those \( u_i \) with \( i < j \) and

\[
C_{X_0, X_i}(u, v) = C_{h,2}(u, v)
\]

for \( X_i \) probabilities that correspond to \( u_i \) with \( i > j \). Hence, in the first case, the dependence structure between \( X_i \) and \( X_0 \) is constant and weaker than that in the second case where however it varies according to the structure of \( C_{h,2} \).

Notice that, however, whatever is the case, the dependence structure between \( X_0 \) and \( X_i \) is always Archimedean, exactly as in the exchangeable case investigated in Section 2. As a consequence, the formulas there presented for the Kendall’s tau between the systemic shock and every default time continue to hold. In particular, if: i) all the copulas involved in the hierarchical construction are of Gumbel type and ii) for every idiosyncratic shock arrival time \( X_i \) there exists a function \( K_i \) such that \( \bar{F}_0(t) = \psi_\theta(\lambda_0 K_i(t)) \) and \( \bar{F}_i(t) = \psi_\theta(\lambda_i K_i(t)) \), then (8) applies. Moreover, for all those default times \( \tau_j \) such that the corresponding idiosyncratic shock arrival time \( X_i \) has a dependence relationship with the systemic shock one \( X_0 \) expressed by the same Gumbel copula , the pairs \((\alpha_j, \tau_{0,j})\) must lie on the same straight line (8).
Notice, then, between the fully exchangeable system, and the fully non-exchangeable one, we can identify an intermediate case in which the exchangeability concept is only applied to the bivariate relationships between the systemic shock arrival times and the idiosyncratic shocks, whatever the dependence among the idiosyncratic shocks could be.

3.1 Dependence structure of observed default times

However, since the statistical procedure presented in Section 2.3 is based on the estimation of the pairwise dependence structure of the default times $\tau_j$, we will now compute the Kendall’s function and the Kendall’s tau of any pair of default times.

Clearly, the shocks involved are the systemic one and the two idiosyncratic ones that correspond to the default times we are considering. Formally, let $X_i, X_j, X_k$ be the three shocks arrival times we are considering. Whatever the hierarchical structure is, their joint survival distribution is of type

$$
\bar{F}(x_i, x_j, x_k) = C_{\psi_{\phi}} \left( C_{\psi_{\theta}} \left( \bar{F}_i(x_i), \bar{F}_j(x_j) \right), \bar{F}_k(x_k) \right)
$$

where $C_{\psi_{\phi}}$ and $C_{\psi_{\theta}}$ are bivariate Archimedean copula functions with generators $\psi_{\phi}$ and $\psi_{\theta}$.

Here below we drop the notation according to which the systemic shock arrival time is denoted $X_0$, so that we can move it in different places of the hylrarchical structure. In particular, it is sufficient to study the two cases: the systemic shock is represented by $X_i$ and the case in which it is represented by $X_k$. For the sake of simplicity, we will assume that all marginal survival distributions are differentiable when needed.

3.1.1 $X_i$ is the arrival time of the systemic shock

Assume $X_i$ be the systemic shock’s arrival time and

$$
\tau_j = \min(X_i, X_j), \tau_k = \min(X_i, X_k)
$$

be the considered default times. Then

$$
\bar{F}_{\tau_j, \tau_k}(t_j, t_k) = \psi_{\phi} \left( \psi_{\phi}^{-1} \circ \psi_{\theta} \left( \psi_{\theta}^{-1} \circ \bar{F}_i(\max(t_j, t_k)) + \psi_{\theta}^{-1} \circ \bar{F}_j(t_j) \right) + \psi_{\phi}^{-1} \circ \bar{F}_k(t_k) \right),
$$
\[ F_{\tau_j}(t) = \psi_\theta (\psi_\theta^{-1} \circ F_i(t) + \psi_\theta^{-1} \circ F_j(t)) = \psi_\theta \circ H_{0,j}(t) \]

and

\[ F_{\tau_k}(t) = \psi_\phi (\psi_\phi^{-1} \circ F_i(t) + \psi_\phi^{-1} \circ F_k(t)) = \psi_\phi \circ H_{0,k}(t) \]

where \( H_{0,j}(t) = \psi_\theta^{-1} \circ F_i(t) + \psi_\theta^{-1} \circ F_j(t) \) and \( H_{0,k}(t) = \psi_\phi^{-1} \circ F_i(t) + \psi_\phi^{-1} \circ F_k(t) \). Hence, thanks to Sklar’s Theorem, from

\[ t_j = H_{0,j}^{-1} \circ \psi_\theta^{-1}(u_j) \quad \text{and} \quad t_k = H_{0,k}^{-1} \circ \psi_\phi^{-1}(u_k) \]

we get that the associated survival copula is

\[
\hat{C}_{\tau_j, \tau_k}(u_j, u_k) = \\
= \psi_\phi (\psi_\phi^{-1} \circ \psi_\theta ((\psi_\theta^{-1} \circ F_i)(\text{max}(H_{0,j}^{-1} \circ \psi_\theta^{-1}(u_j), H_{0,k}^{-1} \circ \psi_\phi^{-1}(u_k)))) + \\
\psi_\phi^{-1} \circ \hat{F}_j \circ H_{0,j}^{-1} \circ \psi_\phi^{-1}(u_j) + \psi_\phi^{-1} \circ \hat{F}_k \circ H_{0,k}^{-1} \circ \psi_\phi^{-1}(u_k)).
\]

Set

\[ D_{ij} = \psi_\theta^{-1} \circ \hat{F}_i \circ H_{0,j}^{-1}, \quad D_{ik} = \psi_\phi^{-1} \circ \hat{F}_i \circ H_{0,k}^{-1}, \quad D_{ji} = \psi_\theta^{-1} \circ \hat{F}_j \circ H_{0,j}^{-1}, \quad D_{ki} = \psi_\phi^{-1} \circ \hat{F}_k \circ H_{0,k}^{-1}. \]

Then

\[
\hat{C}_{\tau_j, \tau_k}(u_j, u_k) = \\
= \psi_\phi (\psi_\phi^{-1} \circ \psi_\theta ((\max(D_{ij} \circ \psi_\theta^{-1}(u_j), D_{ik} \circ \psi_\phi^{-1}(u_k)) + D_{ji} \circ \psi_\theta^{-1}(u_j)) + D_{ki} \circ \psi_\phi^{-1}(u_k))) = \\
= \begin{cases} \\
\psi_\phi (\psi_\phi^{-1}(u_j) + D_{ki} \circ \psi_\phi^{-1}(u_k)), u_k \geq h(u_j) \\
\psi_\phi (\psi_\phi^{-1}(u_j) + D_{ik} \circ \psi_\phi^{-1}(u_k) + D_{ji} \circ \psi_\phi^{-1}(u_j)) + D_{ki} \circ \psi_\phi^{-1}(u_k)), u_k < h(u_j) \\
\end{cases}
\]

where

\[ h(x) = \psi_\phi \circ D_{ik}^{-1} \circ D_{ij} \circ \psi_\theta^{-1}(x). \] (10)

**Restriction on the distribution of** \( X_i, X_j, X_k \)

Assume that there exist two functions \( K \) and \( \hat{K} \) such that

\[ \psi_\theta^{-1} \circ F_i(t) = \hat{\lambda}_i \hat{K}(t), \quad \psi_\theta^{-1} \circ F_j(t) = \lambda_j \hat{K}(t) \]

and

\[ \psi_\phi^{-1} \circ F_i(t) = \lambda_i K(t), \quad \psi_\phi^{-1} \circ F_k(t) = \lambda_k K(t) \]

which implies that

\[ \hat{K}(t) = \frac{1}{\lambda_i} \psi_\theta^{-1} \circ \psi_\phi (\lambda_i K(t)). \] (11)
Now, setting $\mu_{ij} = \hat{\lambda}_i + \lambda_j$ and $\mu_{ik} = \lambda_i + \lambda_k$,
\[
H_{0,j}(t) = \hat{\mu}_{ij}\hat{K}(t) \quad \text{and} \quad H_{0,k}(t) = \mu_{ik}K(t)
\]
and
\[
D_{ij}(x) = \frac{\hat{\lambda}_i}{\mu_{ij}}x, \quad D_{ik}(x) = \frac{\hat{\lambda}_i}{\mu_{ik}}x, \quad D_{ji}(x) = \frac{\lambda_j}{\mu_{ij}}x, \quad D_{ki}(x) = \frac{\lambda_k}{\mu_{ik}}x,
\]
from which
\[
\bar{F}_{\tau_j}(t) = \psi_\theta \left( \mu_{ij}\hat{K}(t) \right) \quad \text{and} \quad \bar{F}_{\tau_k}(t) = \psi_\phi \left( \mu_{ik}K(t) \right)
\]
and
\[
\hat{C}_{\tau_j,\tau_k}(u_j, u_k) = \psi_\phi \left( \psi_\phi^{-1} \circ \psi_\theta \left( \max \left( \frac{\hat{\lambda}_i}{\mu_{ij}}\psi_\theta^{-1}(u_j), \frac{\hat{\lambda}_i}{\mu_{ik}}\psi_\phi^{-1}(u_k) \right) + \frac{\lambda_j}{\mu_{ij}}\psi_\theta^{-1}(u_j) \right) + \frac{\lambda_k}{\mu_{ik}}\psi_\phi^{-1}(u_k) \right).
\]

**Remark 3.1. The Gumbel case**

Assume $\psi_\theta(x) = e^{-x^{\theta}}$ and $\psi_\phi(x) = e^{-x^{\phi}}$, with $\theta \geq \phi \geq 1$. Then $\psi_\phi^{-1} \circ \psi_\theta(x) = x^{\#}$ and (12) writes
\[
\hat{C}_{\tau_j,\tau_k}(u_j, u_k) = \exp \left\{ - \left( \max \left( \frac{\hat{\lambda}_i}{\mu_{ij}}(-\log(u_j))^{\theta}, \frac{\hat{\lambda}_i}{\mu_{ik}}(-\log(u_k))^{\phi} \right) + \frac{\lambda_j}{\mu_{ij}}(-\log(u_j))^{\theta} \right)^{\frac{1}{\phi}} + \right.
\]
\[
\left. \frac{\lambda_k}{\mu_{ik}}(-\log(u_k))^{\phi} \right)^{\frac{1}{\phi}}.
\]

Necessarily, by (11), $\hat{\lambda}_i\hat{K} = \lambda_i^{\theta}K^\#$ and an admissible choice is $\hat{\lambda}_i = \lambda_i^{\#}$ and $\hat{K} = K^\#$.

In particular, if $K(t) = t^\theta$ and $\hat{K}(t) = t^\phi$ we recover exponential marginal distributions, that is
\[
\bar{F}_{\tau_j}(t) = e^{-\mu_{ij}^t} \quad \text{and} \quad \bar{F}_{\tau_k}(t) = e^{-\mu_{ik}^t}.
\]
The Kendall’s function and Kendall’s tau

**Theorem 3.1.** If \( \rho = \psi_\phi^{-1} \circ \psi_\theta \), let (see (10))

\[
C(u, v) = \left\{ \begin{array}{ll}
\psi_\phi \left( \psi_\phi^{-1}(u) + D_{ki} \circ \psi_\phi^{-1}(v) \right), & v \geq h(u) \\
\psi_\phi \left( \rho \left( D_{ik} \circ \psi_\phi^{-1}(v) + D_{ji} \circ \psi_\phi^{-1}(u) \right) + D_{ki} \circ \psi_\phi^{-1}(v) \right), & v < h(u)
\end{array} \right.
\]

where

\[
h(x) = \psi_\phi \circ D_{ik}^{-1} \circ D_{ij} \circ \psi_\theta^{-1}(x).
\]

We have that the corresponding Kendall’s function \( K(t) = \mathbb{P}(C(u, v) \leq t) \) and Kendall’s tau are, respectively,

\[
K(t) = t - \psi_\phi' \circ \psi_\phi^{-1}(t) \left[ D_{ki} \circ \psi_\phi^{-1}(t) - \int_{D_{ki} \circ \psi_\phi^{-1}(t)}^{D_{ik} \circ G^{-1} \circ \psi_\phi^{-1}(t)} \rho' \circ \rho^{-1}(\psi_\phi^{-1}(t) - D_{ki} \circ D_{ik}^{-1}(z)) dz \right]
\]

and

\[
\tau = 1 + 4 \int_0^1 \psi_\phi' \circ \psi_\phi^{-1}(t) \cdot D_{ki} \circ \psi_\phi^{-1}(t) dt -
\]

\[
-4 \int_0^1 \psi_\phi' \circ \psi_\phi^{-1}(t) \int_{D_{ki} \circ \psi_\phi^{-1}(t)}^{D_{ik} \circ G^{-1} \circ \psi_\phi^{-1}(t)} \rho' \circ \rho^{-1}(\psi_\phi^{-1}(t) - D_{ki} \circ D_{ik}^{-1}(z)) dz dt
\]

where

\[
G(x) = \psi_\phi^{-1} \circ \psi_\theta \circ D_{ij}^{-1} \circ D_{ik}(x) + D_{ki}(x).
\]

**Proof.** See Appendix 6. \( \square \)

**Remark 3.2. The Gumbel case**

In the setting of Remark 3.1, we get

\[
K(t) = t - \frac{t}{\phi} \left[ \frac{1}{\mu_{ik}} (-\log t)^{\phi} - \frac{\phi}{\theta} \int_{\frac{\theta}{\mu_{ik}} (-\log t)^{\phi}}^{\frac{\lambda_{ki}}{\mu_{ik}} (-\log t)^{\phi}} \left( (-\log t)^{\phi} - \frac{\lambda_{ki}}{\lambda_i} \right)^{1-\phi} \right]^{-\frac{\phi}{\phi - 1}} \frac{dz}{dz}
\]

and

\[
\tau = 1 + \frac{\lambda_{ki}}{\mu_{ik}} \frac{1}{\phi} - \frac{4}{\theta} \int_0^1 t (\log t)^{1-\phi} \left( \int_{\frac{\lambda_{ki}}{\mu_{ik}} (-\log t)^{\phi}}^{\frac{\theta}{\mu_{ik}} (-\log t)^{\phi}} \left( (-\log t)^{\phi} - \frac{\lambda_{ki}}{\lambda_i} \right)^{1-\phi} \right) \frac{dz}{dz} dt
\]

where \( G(x) = \left( \frac{\mu_{ij}}{\mu_{ik}} \right)^{\phi} x^{\phi} + \frac{\lambda_{ki}}{\mu_{ik}} x. \)
3.1.2 $X_k$ is the arrival time of the systemic shock

Here we assume that $X_k$ is the arrival time of the shock and

$$\tau_i = \min(X_i, X_k) \text{ and } \tau_j = \min(X_j, X_k)$$

the considered default times. Then

$$\tilde{F}_{\tau_i, \tau_j}(t_i, t_j) = \psi_{\phi}^{-1} \circ \psi_{\theta} \left( \psi_{\theta}^{-1}(\tilde{F}_i(t_i)) + \psi_{\phi}^{-1}(\tilde{F}_j(t_j)) \right) + \psi_{\phi}^{-1}(\tilde{F}_k(\max(t_i, t_j)))$$

and

$$\tilde{F}_{\tau_i}(t_i) = \psi_{\phi} \circ H_{0,i}(t_i) \text{ and } \tilde{F}_{\tau_j}(t_j) = \psi_{\phi} \circ H_{0,j}(t_j).$$

where

$$H_{0,i} = \psi_{\phi}^{-1} \circ F_i + \psi_{\phi}^{-1} \circ F_k \text{ and } H_{0,j} = \psi_{\phi}^{-1} \circ F_j + \psi_{\phi}^{-1} \circ F_k.$$ 

If $\rho = \psi_{\phi}^{-1} \circ \psi_{\theta}$,

$$\tilde{F}_{\tau_i, \tau_j}(t_i, t_j) = \psi_{\phi} \left[ \rho (\rho^{-1} \circ \psi_{\phi}^{-1} \circ \tilde{F}_i(t_i) + \rho^{-1} \circ \psi_{\phi}^{-1} \circ \tilde{F}_j(t_j)) + \psi_{\phi}^{-1} \circ \tilde{F}_k(\max(t_i, t_j)) \right]$$

and, applying Sklar’s Theorem, we recover the associated survival copula is

$$\tilde{C}_{\tau_i, \tau_j}(u_i, u_j) = \psi_{\phi} \left[ \rho (\rho^{-1} \circ \tilde{F}_i \circ H_{0,i}^{-1} \circ \psi_{\phi}^{-1}(u_i) + \rho^{-1} \circ \tilde{F}_j \circ H_{0,j}^{-1} \circ \psi_{\phi}^{-1}(u_j)) + \psi_{\phi}^{-1} \circ \tilde{F}_k(\max(H_{0,i}^{-1} \circ \psi_{\phi}^{-1}(u_i), H_{0,j}^{-1} \circ \psi_{\phi}^{-1}(u_j))) \right].$$

Set $D_{ik} = \psi_{\phi}^{-1} \circ \tilde{F}_i \circ H_{0,i}^{-1}$, $D_{jk} = \psi_{\phi}^{-1} \circ \tilde{F}_j \circ H_{0,j}^{-1}$, $D_{ki} = \psi_{\phi}^{-1} \circ \tilde{F}_k \circ H_{0,i}^{-1}$ and $D_{kj} = \psi_{\phi}^{-1} \circ \tilde{F}_k \circ H_{0,j}^{-1}$. It follows

$$\tilde{C}_{\tau_i, \tau_j}(u_i, u_j) = \psi_{\phi} \left[ \rho \left( \rho^{-1} \circ D_{ik} \circ \psi_{\phi}^{-1}(u_i) + \rho^{-1} \circ D_{jk} \circ \psi_{\phi}^{-1}(u_j) \right) + \max(D_{ki} \circ \psi_{\phi}^{-1}(u_i), D_{kj} \circ \psi_{\phi}^{-1}(u_j)) \right] =$$

$$= \left\{ \begin{array}{ll} \psi_{\phi} \left[ \rho \left( \rho^{-1} \circ D_{ik} \circ \psi_{\phi}^{-1}(u_i) + \rho^{-1} \circ D_{jk} \circ \psi_{\phi}^{-1}(u_j) \right) + D_{ki} \circ \psi_{\phi}^{-1}(u_i) \right], & u_j \geq h(u_i) \\ \psi_{\phi} \left[ \rho \left( \rho^{-1} \circ D_{ik} \circ \psi_{\phi}^{-1}(u_i) + \rho^{-1} \circ D_{jk} \circ \psi_{\phi}^{-1}(u_j) \right) + D_{kj} \circ \psi_{\phi}^{-1}(u_j) \right], & u_j < h(u_i) \end{array} \right\}$$

where

$$h(x) = \psi_{\phi} \circ D_{kj}^{-1} \circ D_{ki} \circ \psi_{\phi}^{-1}(x).$$

**Restriction on the distribution of $X_i$, $X_j$, $X_k$**

Assume there exists a function $K$ such that $\psi_{\phi}^{-1} \circ \tilde{F}_i(x) = \lambda_i K(x)$ for $v = i, j, k$, and set $\mu_{ik} = \lambda_i + \lambda_k$ and $\mu_{jk} = \lambda_j + \lambda_k$. It follows that
\[ D_{ik}(x) = \frac{\lambda_i}{\mu_{ik}} x, \quad D_{jk}(x) = \frac{\lambda_j}{\mu_{jk}} x, \quad D_{ki}(x) = \frac{\lambda_k}{\mu_{ik}} x \text{ and } D_{kj}(x) = \frac{\lambda_k}{\mu_{jk}} x \]

and the marginal survival distributions can be written as
\[ \overline{F}\tau_s(t) = \psi\phi(\mu_{sk}K(t)), \quad s = i, j \]

while the associated survival copula as
\[ \hat{C}\tau_i,\tau_j(u_i, u_j) = \psi\phi \left[ \rho \left( \frac{1}{\mu_{ik}} \left( \lambda_i \psi^{-1}(u_i) \right) + \frac{1}{\mu_{jk}} \left( \lambda_j \psi^{-1}(u_j) \right) \right) + \max \left( \frac{\lambda_k}{\mu_{ik}} \psi^{-1}(u_i), \frac{\lambda_k}{\mu_{jk}} \psi^{-1}(u_j) \right) \right] . \]

**Remark 3.3. The Gumbel case**

Assume \( \psi\theta(x) = e^{-x^\theta} \) and \( \psi\phi(x) = e^{-x^\phi} \), with \( \theta \geq \phi \geq 1 \). Then \( \rho(x) = x^{\theta/\phi} \) and
\[ \hat{C}\tau_i,\tau_j(u_i, u_j) = \exp \left\{ - \left[ \left( \frac{\lambda_i}{\mu_{ik}} \right)^{\theta/\phi} \left( - \log(u_i) \right)^\theta + \left( \frac{\lambda_j}{\mu_{jk}} \right)^{\theta/\phi} \left( - \log(u_j) \right)^\theta \right]^{\phi/\theta} + \right. \]
\[ + \left. \max \left( \frac{\lambda_k}{\mu_{ik}} (- \log(u_i))^{\phi/\theta}, \frac{\lambda_k}{\mu_{jk}} (- \log(u_j))^{\phi/\theta} \right) \right\} \]

while, if \( K(t) = t^\phi \), we get exponential marginal distributions
\[ \overline{F}\tau_i(t) = e^{-\mu_{ik}t} \text{ and } \overline{F}\tau_j(t) = e^{-\mu_{jk}t}. \]

**The Kendall’s function and Kendall’s tau**

**Theorem 3.2. Let (see [14])**

\[ C(u, v) = \left\{ \begin{array}{ll} \psi\phi \left[ \rho \left( \frac{1}{\mu_{ik}} D_{ik} \circ \psi^{-1}_\phi(u) + \frac{1}{\mu_{jk}} D_{jk} \circ \psi^{-1}_\phi(v) \right) + D_{ki} \circ \psi^{-1}_\phi(u) \right], & v \geq h(u) \\ \psi\phi \left[ \rho \left( \frac{1}{\mu_{ik}} D_{ik} \circ \psi^{-1}_\phi(u) + \frac{1}{\mu_{jk}} D_{jk} \circ \psi^{-1}_\phi(v) \right) + D_{kj} \circ \psi^{-1}_\phi(v) \right], & v < h(u) \end{array} \right. \]

where
\[ h(x) = \psi\phi \circ D_{kj}^{-1} \circ D_{ki} \circ \psi^{-1}_\phi(x). \]
We have that the Kendall’s function $\mathcal{K}(t) = \mathbb{P}(C(u,v) \leq t)$ and the Kerndall’s tau respectively are

\[
\mathcal{K}(t) = t + \psi_\phi' \circ \psi_\phi^{-1}(t).
\]

\[
\cdot \left[ \int_{\rho^{-1} \circ D_{jk} \circ \psi_\phi^{-1}(t)}^{\rho^{-1} \circ F_j \circ G^{-1} \circ \psi_\phi^{-1}(t)} \rho' \circ \rho^{-1} \left\{ \psi_\phi^{-1}(t) - \psi_\phi^{-1} \circ \tilde{F}_k \circ \tilde{F}_j^{-1} \circ \psi_\phi \circ \rho(z) \right\} \, dz + \right.
\]

\[
+ \int_{\rho^{-1} \circ D_{ik} \circ \psi_\phi^{-1}(t)}^{\rho^{-1} \circ F_i \circ G^{-1} \circ \psi_\phi^{-1}(t)} \rho' \circ \rho^{-1} \left\{ \psi_\phi^{-1}(t) - \psi_\phi^{-1} \circ \tilde{F}_k \circ \tilde{F}_i^{-1} \circ \psi_\phi \circ \rho(z) \right\} \, dz +
\]

\[-(D_{kj} + D_{ki}) \circ \psi_\phi^{-1}(t) + 2 \psi_\phi^{-1} \circ \tilde{F}_k \circ G^{-1} \circ \psi_\phi^{-1}(t) \right]
\]

(15)

and

\[
\tau = 1 - 4 \int_0^1 \psi_\phi' \circ \psi_\phi^{-1}(t) \cdot \left[ \int_{\rho^{-1} \circ D_{jk} \circ \psi_\phi^{-1}(t)}^{\rho^{-1} \circ F_j \circ G^{-1} \circ \psi_\phi^{-1}(t)} \rho' \circ \rho^{-1} \left\{ \psi_\phi^{-1}(t) - \psi_\phi^{-1} \circ \tilde{F}_k \circ \tilde{F}_j^{-1} \circ \psi_\phi \circ \rho(z) \right\} \, dz + \right.
\]

\[
+ \int_{\rho^{-1} \circ D_{ik} \circ \psi_\phi^{-1}(t)}^{\rho^{-1} \circ F_i \circ G^{-1} \circ \psi_\phi^{-1}(t)} \rho' \circ \rho^{-1} \left\{ \psi_\phi^{-1}(t) - \psi_\phi^{-1} \circ \tilde{F}_k \circ \tilde{F}_i^{-1} \circ \psi_\phi \circ \rho(z) \right\} \, dz +
\]

\[-4 \int_0^1 \psi_\phi' \circ \psi_\phi^{-1}(t) \cdot (2 \psi_\phi^{-1} \circ \tilde{F}_k \circ G^{-1} - (D_{kj} + D_{ki})) \circ \psi_\phi^{-1}(t) \, dt
\]

where

\[
G(z) = \rho \left\{ \rho^{-1} \circ \psi_\phi^{-1} \circ \tilde{F}_i(z) + \rho^{-1} \circ \psi_\phi^{-1} \circ \tilde{F}_j(z) \right\} + \psi_\phi^{-1} \circ \tilde{F}_k(z).
\]

(16)

\[\text{Proof. See Appendix 6}\]

Remark 3.4. The Gumbel case

In the setting of Remark 3.3, we get

\[
\mathcal{K}(t) = t - \frac{t}{\phi} (-\log t)^{1-\phi} \cdot \left[ \frac{\phi}{\theta} \int_{\rho^{-1} \circ D_{jk} \circ \psi_\phi^{-1}(t)}^{\rho^{-1} \circ F_j \circ G^{-1} \circ \psi_\phi^{-1}(t)} \frac{2}{\phi} \left( -\log t \right)^{\phi} - \frac{\lambda_k}{\lambda_j} \frac{z}{\phi} \right]^{1-\phi} \, dz +
\]

\[
+ \frac{\phi}{\theta} \int_{\rho^{-1} \circ D_{ik} \circ \psi_\phi^{-1}(t)}^{\rho^{-1} \circ F_i \circ G^{-1} \circ \psi_\phi^{-1}(t)} \frac{2}{\phi} \left( -\log t \right)^{\phi} - \frac{\lambda_k}{\lambda_i} \frac{z}{\phi} \right]^{1-\phi} \, dz +
\]

\[- \left( \frac{\lambda_k}{\mu_{ik}} + \frac{\lambda_k}{\mu_{jk}} \right) (-\log t)^{\phi} + 2 \lambda_k G^{-1}((-\log t)^{\phi}) \right]
\]

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and

\[ \tau = 1 - \frac{4}{\theta} \int_0^1 t (-\log t)^{1-\phi} \left[ \int \phi_{ij} (G^{-1}((-\log t)^\phi)) \frac{\phi_i}{\theta} \phi_j (\frac{\theta_i}{\theta_j} (\frac{\phi_j}{\theta_j} (-\log t)^\phi - \frac{\lambda_k}{\lambda_j} z^\phi)) dt \right] + \int \phi_{ij} (G^{-1}((-\log t)^\phi)) \frac{\phi_i}{\theta} \phi_j (\frac{\theta_i}{\theta_j} (\frac{\phi_j}{\theta_j} (-\log t)^\phi - \frac{\lambda_k}{\lambda_j} z^\phi)) dt \]

where

\[ G(z) = (\lambda_i^{\phi_i} + \lambda_j^{\phi_j}) \frac{\phi_i}{\theta} \phi_j (\frac{\theta_i}{\theta_j} (\frac{\phi_j}{\theta_j} (\frac{\lambda_k}{\lambda_j} z^\phi)) + \lambda_k z. \]

4 An application to the European banking sector

In this section we apply the model to the issue of evaluating systemic risk and contagion in a set of European banking systems. Until now, the European banking system has been segmented at the national level, and only after November 4th 2014 it is unified under a common European regulation and supervision setting (the so called SSM, Single Supervisory Mechanism, see, for example, Ferran and Babis, 2013). It is then important to recognize the relevance of systemic risks and contagion at the national level, and address the issue whether they co-move at the cross-country level. The task is to check whether the exchangeable contagion model may provide a good representation of the data. Of course, here our interest is mainly in the illustration of the estimation technique and how it can provide a guide for the specification of the model.

4.1 Data

We apply the model described above to a sample of 35 banks representative of 8 countries of the Euro area. The sample used is the same as in Baglioni and Cherubini (2013). While we refer the reader to that paper for an in-depth description of the data set, here we simply mention that the sample consists of those major European
banks that were subject to the stress test exercise in 2012 and for which a time series of CDS quotes was available on Datastream. The sample consists of daily data of CDS quotes, ranging from January 2007 to end of August 2012, with the exception of Greece for which the sample begins on September 21st 2009, Portugal and Spain, for which the sample starts in January and February 2008, respectively. The survival probabilities were extracted from the 5 year CDS quote using what is called the ”simple rule”, that is assuming a flat default intensity, which is consistent with the model described in Section 2.1. Moreover, since it is well known that data extracted from market prices embed a risk premium, that is are computed under the risk neutral measure, we changed the default probabilities by applying the Sharpe ratio, according to the technique used by the Moody’s rating agency (see Dwyer et al., 2010).

Future research could investigate further the marginal structure including more sophisticated technologies to ”bootstrap” (in the financial literature meaning of the term) the term structure of default intensities (Hull and White, 2000).

4.2 Estimation and results

The estimation procedure applied to the data was the same described in Section 2.3. Namely, we computed pairwise Kendall’s tau values for the survival probabilities of all the banks in the same country. Then, for each country we estimated the $\alpha_k$ parameters and the $\theta$ parameter, minimizing the distance between theoretical and sample Kendall tau’s. In our specific application, we used the quadratic distance. Due to the presence of local minima, that arose in preliminary work, the analysis was finally carried out using a standard global optimization technique, namely simulated annealing.

Our estimation strategy consisted of three steps.

- We first estimated the model on the whole sample for each country.

- Then, for each country we used the model to estimate the intensity of the systemic shock and we computed the Kendall’s tau between the survival probability of each bank and the systemic shock. We verified for which countries the model specification is consistent with the data.
• For the countries where the model specification was considered consistent, we provided an analysis of the stability of parameters, by repeating the estimate in a sequence of rolling windows.

In Table 1 we report the results for the estimates carried out over the whole sample. For each country, we report: i) the $\alpha_k$ parameters for each bank; ii) the average $\bar{\alpha}$ for the country; iii) the contagion parameter $\theta$ for the country. It is worth mentioning that the average parameter $\bar{\alpha}$ is computed as the harmonic mean of the $\alpha_k$ for each country.

Before discussing the parameters estimated, we use them to provide a visual check of the specification of the model. For each country we provided a diagram in which on the horizontal axis we reported the $\alpha_k$ and on the vertical one the Kendall’s tau value. In the diagram we plot the estimated Kendall’s tau statistics between the survival function of a systemic shock, estimated from the intensity in equation (9), and the survival probabilities of the banks in the sample. In the plot we also reported the straight line on which the Kendall’s tau values should lie if the model is well specified, according to equation (8).

Figure 1 shows that the model provides a good specification for Portugal, France, Spain and the Netherlands. In particular, the specification looks very good for Portugal and France. Spain and the Netherlands provide an interesting insight on the model. In both these countries there are banks whose value of $\alpha_k$ is very close to zero. They are Abn Amro in the Netherlands and Santander and BBVA in Spain. These three banks are not affected by the systemic shock, and they are very close to the intercept of the line. Nevertheless, they are linked by a substantial degree of dependence to the systemic risk factor, meaning that an increase of their probability of default may be associated to a higher probability of a country-wide shock.

For the other countries (see Figure 2), that is Italy, Greece, UK and Germany, the model does not seem to fit the data well. In particular, Italy, Greece and UK could have a chance of a better fit if a non-exchangeable model were used. For Germany, instead, the model appears completely wrong, since almost all the Kendall’s tau’s lie below the diagonal in a region, and are not even consistent with the pure systemic risk specification. So, in this case, it seems
| BANK       | α    | BANK       | α    |
|------------|------|------------|------|
| HSH        | 0.288926 | PASTOR    | 0.295905 |
| WEST LB    | 0.783299 | BINTEL    | 0.397199 |
| POSTBANK   | 0.782422 | SABADEL   | 0.699246 |
| DZ BANK    | 0.637563 | POPULA    | 0.842172 |
| BAYERN LB  | 0.885879 | CAJA MADRID | 0.059619 |
| COMMERZ    | 0.869757 | BBVA      | 6.99·10⁻⁷ |
| DB         | 0.752793 | SANTANDER | 8.44·10⁻⁷ |
| θ=1        | α=0.625488 | θ=5.803539 | α=0.197725 |

| BANK       | α    | BANK       | α    |
|------------|------|------------|------|
| UBI        | 0.260332 | SNS        | 0.705556 |
| M-PASCHI   | 0.260332 | ABN AMRO  | 1.279263 |
| INTESA     | 1.0   | RABOBANK   | 0.82459 |
| UNICREDIT  | 0.583622 | ING       | 0.550825 |
| θ=5.589623 | α=0.449591 | θ=1.279262 | α=0.674809 |

| BANK       | α    | BANK       | α    |
|------------|------|------------|------|
| SOC GEN    | 0.690164 | ALPHA     | 0.987782 |
| CA         | 0.469473 | EFG       | 0.199412 |
| BNP        | 0.502889 | NBG       | 0.726229 |
| θ=6.060185 | α=0.5388 | θ=3.311887 | α=0.405181 |

| BANK       | α    | BANK       | α    |
|------------|------|------------|------|
| ESP. SANTO | 0.791219 | LLOYDS    | 0.930215 |
| BCP        | 0.695637 | BARCLAYS  | 2.88·10⁻⁷ |
| CAIXA GERAL | 0.215709 | HSBC      | 4.08·10⁻⁸ |
| θ=5.369255 | α=0.408871 | RBS     | 0.769870 |
|             |       | θ=5.369255 | α=0.842481 |

Table 1: Parameters' values for different banks and countries
that either a model of bivariate relationships without any systemic shock could be preferable, or that there can be some other systemic risk factor missing. Actually, the discussion appeared in newspapers and magazines during the year of the stress testing analysis would suggest that the German banking system is exposed to two key risk factors: the first is the exposure to the so-called ”toxic assets”, coming from the US subprime crisis, the second, less known, is exposure to a specific sector of obligors, namely those linked to the shipping business.

So, our estimation strategy proved able to discriminate cases in which the model provides a good fit to the data from cases in which it does not. For the four cases in which the model seems to work for the entire sample, we now provide an analysis of the stability of parameters, and in particular of the contagion parameter $\theta$ across the sample. We replied the estimation using rolling windows of several lengths, even though here in order to save space we only report the one based on one year of daily data. An alternative more sophisticated approach would be to consider time varying parameters with an estimate performed on GARCH filtered residuals. This on one side could be more accurate, while on the other side it would be inconsistent with the flat intensity assumption, calling for a proper
Figure 2: Kendall’s tau between default times and systemic shock

 specification of a double stochastic model for each marginal intensity curve.

 We also performed the analysis first letting all the parameters change through time, and then assuming the $\alpha_k$ fixed across the sample, allowing only the contagion parameter $\theta$ to change. The reason for the latter choice is twofold. First, since the sensitivity of each bank to shocks mostly depends on its balance sheet, it is reasonable to assume that the parameters $\alpha_k$ remain quite stable across the sample. Second, it was interesting to check whether the estimation of the contagion parameter, that is the main target of our research, was affected by changes in the bank specific parameters. We found that the dynamics of the contagion parameter is almost indistinguishable in the two cases.

 In Figure 3 we report the results of the analysis for the four cases in which the model works. The question we have in mind is whether the contagion parameters increased in the two crucial periods of the crisis. The first was in the first quarter of 2009, when the Lehman crisis of September 2008 propagated to Europe. The second is the sovereign debt crisis triggered by Greece in 2010 and then spread to the other countries of Southern Europe. Figure 3 confirms an interesting co-movement behavior of the contagion parameters for
Spain, Portugal, France and Greece. Intuitively, when financial crisis spread in the international environment, the relevance of contagion from the banks within each country is increasing. Differently from this evidence, however, in the last part of the sample, characterized by the Italian sovereign crisis, only contagion within the French banking system seems to markedly increase, while in the other countries it remains stable or decrease. This could be consistent with the greater involvement of the French banking system with the Italian one. In fact, in a previous version of this work, in which the dynamic analysis had been carried out for Italy as well, the contagion parameter for the Italian market was in that period almost indistinguishable from that of the French one.

![Figure 3: The dynamics of the contagion parameter.](image)

5 Conclusions and future extensions

In this paper we presented a model that includes both systemic risk and contagion. Systemic risk is represented by the presence of a shock that brings about the default of all the elements in a cluster. Contagion is represented by the links between the idiosyncratic shocks specific to each component and the systemic shock.
On theoretical grounds, the analysis can be carried out assuming whatever dependence structure among the non observed components representing the shocks. On empirical grounds, here we provide an estimation procedure for a model in which the dependence structure of the unobserved components is Archimedean and exchangeable. Moreover, we provide a technique to verify whether the specification proposed fits the data. We also show that, including further restrictions may transform the copula model in a new full-fledged multivariate model with exponential marginal distributions.

Given a panel data of observations of marginal intensities, the estimation of the model is carried out on the set of bivariate dependence statistics. Based on estimates, one can extract the time series of the systemic shock, estimate the Kendall’s tau’s of the observed marginals and the systemic shock, and verify whether they are aligned on a straight line, as predicted by the model. We apply this technique to a set of European banks of 8 countries, assuming a systemic shock at the country level, and we found that our model turns out to be well specified for 4 countries: Spain, Portugal, France and The Netherlands. For these countries, we also report an analysis of the dynamics of the contagion parameter, providing empirical evidence of co-movement in periods of international crisis.

Of course, the next step of this line of research would call for estimation of the more general, non-exchangeable setting, that has been also formalized in this paper. More precisely, we see three main promising fields of development

- Estimating the dependence structure of the unobserved components directly. Most likely, this would involve the application of Simulated Maximum Likelihood (SML) or similar techniques, in which one tries to estimate the parameters by simulating data as close as possible to the observed ones. Doing this can be very easy or very complex, depending on the degree of generality that one is willing to accept. As the simplest case, assume one could consistently estimate the parameters \( \alpha_k \) in our model. In this case, it would suffice to estimate the systemic and the idiosyncratic components from the data and study the dependence analysis on those. Exploiting the invariance property of copulas, one could directly obtain a consistent estimate of the contagion parameters. As the most complex case, assume that the dependence structure of the unobserved components must
be handled in full generality. In that case, the concept itself of the $\alpha_k$ parameters would be lost, since there is no guarantee that the same proportionality between the systemic shock intensity and the marginal intensity is maintained through the sample.

- In a similar line of research, one could also decide whether to focus on the full specification of the model, or only in the relationship between the systemic shock and the marginals. In the latter case, the dependence structure among idiosyncratic components would play the role of nuisance parameters. For example, our findings of exchangeable contagion could be consistent with a dependence structure in which some degree of non-exchangeability is present, but it is limited to the idiosyncratic shock dependence. Within this framework, estimation techniques such as those envisaged above could be used to devise formal tests of the weaker concept of exchangeability discussed in the extension of our model, in which only the pairwise dependence between the idiosyncratic shock and the systemic ones are required to have the same copula.

- Finally, on a different line of research, one could use the estimation procedure applied in this paper as an exploratory tool to identify clusters of components that may constitute the same "exchangeable systemic contagion cluster". This could be done evaluating the dependence between new element and the systemic shock representing a cluster. Or it can be obtained by measuring the dependence between the systemic shocks of different clusters to evaluate if some of them can be merged in a single one.

As for our specific application to the banking system, of course, the main challenge would be to extend the analysis to the new unified European banking system, represented by the 130 banks that are since now on under the supervision of the European Central Bank.

6 Appendix

Proof of Proposition 2.1
Proof. By (2), \( t_k = H^{-1}_{0,k} \circ \psi^{-1}(u_k) \). Hence

\[
C(u) = \psi \left( \psi^{-1}(\bar{F}_0 \left( \max_{1 \leq k \leq d} \{ H^{-1}_{0,k} \circ \psi^{-1}(u_k) \} \right)) + \sum_{k=1}^{d} \psi^{-1} \circ \bar{F}_k \circ H^{-1}_{0,k} \circ \psi^{-1}(u_k) \right)
\]

Let

\[
A_j = \left\{ u \in [0,1]^d : \max_{1 \leq i \leq d} \{ H^{-1}_{0,i} \circ \psi^{-1}(u_i) \} = H^{-1}_{0,j} \circ \psi^{-1}(u_j) \right\}
\]

then

\[
\hat{C}(u) 1_{A_j}(u) = \psi \left( \psi^{-1} \left( \bar{F}_0 \left( H^{-1}_{0,j} \circ \psi^{-1}(u_j) \right) \right) + \sum_{k=1, k \neq j}^{d} \psi^{-1} \circ \bar{F}_k \circ H^{-1}_{0,k} \circ \psi^{-1}(u_k) \right) = \psi \left( \psi^{-1}(u_j) + \sum_{k=1, k \neq j}^{d} \psi^{-1} \circ \bar{F}_k \circ H^{-1}_{0,k} \circ \psi^{-1}(u_k) \right)
\]

}\]

Proof of Theorem 3.1

Proof. In the sequel we set \( \partial_1 C(u,v) = \frac{\partial}{\partial u} C(u,v) \) and \( \partial_2 C(u,v) = \frac{\partial}{\partial v} C(u,v) \).

We want to compute the \( C \)-measure of the set

\[
S_t = \{(u,v) \in [0,1]^2 : C(u,v) \leq t \}
\]

Notice that the level curve \( C(u,v) = t \) intersects the graph of the function \( v = h(u) \) in a unique point that we denote with \( (u_t, v_t) \). Hence \( S_t \) can be decomposed as \( S_t = R_t + R_{1,t} + R_{2,t} \) where \( R_t = [0,u_t] \times [0,v_t] \), \( R_{1,t} = \{(u,v) : v \in (v_t,1], C(u,v) \leq t \} \) and \( R_{2,t} = \{(u,v) : u \in (u_t,1], C(u,v) \leq t \} \).

Clearly, the \( C \)-measure of \( R_t \) is \( t \). In order to compute the \( C \)-measure of \( R_{1,t} \) and \( R_{2,t} \), we compute \( u_t \) and \( v_t \). Since \( (u_t, v_t) \) satisfies \( \psi_\phi \left( \psi^{-1}_\phi(u_t) + D_{ki} \circ \psi^{-1}_\phi(v_t) \right) = t \) and \( v_t = h(u_t) \), we get

\[
\psi^{-1}_\phi \circ \psi_\theta \circ D_{ij}^{-1} \circ D_{ik} \circ \psi^{-1}_\phi(v_t) + D_{ki} \circ \psi^{-1}_\phi(v_t) = \psi^{-1}_\phi(t)
\]

from which

\[
v_t = \psi_\phi \circ G^{-1} \circ \psi^{-1}_\phi(t)
\]

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and 
\[ u_t = \psi_\theta \circ D_{ij}^{-1} \circ D_{ik} \circ G^{-1} \circ \psi_\phi^{-1}(t). \]

Let us start with \( R_{1,t} \). Notice that here, \( C(u, v) \leq t \) is equivalent to \( u \leq F_1(t, v) \) where \( F_1(t, v) = \psi_\phi \left( \psi_\phi^{-1}(t) - D_{ki} \circ \psi_\phi^{-1}(v) \right) \). Hence
\[
\mathbb{P}(R_{1,t}) = \int_{v_t}^{1} \mathbb{P}(U \leq F_1(t, v)|V = v) dv = \\
= \int_{v_t}^{1} \partial_2 C(F_1(t, v), v) dv = \\
= \int_{v_t}^{1} \psi'_\phi \circ \psi_\phi^{-1}(t) \cdot \frac{d}{dv} D_{ki} \circ \psi_\phi^{-1}(v) dv = \\
= -\psi'_\phi \circ \psi_\phi^{-1}(t) \cdot D_{ki} \circ \psi_\phi^{-1}(v_t).
\]

Let us now consider \( R_{2,t} \). Notice that here, the inequality \( C(u, v) \leq t \), is equivalent to \( u \leq F_2(t, v) \) where
\[ F_2(t, v) = \psi_\theta \circ D_{ji}^{-1} \left( \rho^{-1} \left( \psi_\phi^{-1}(t) - D_{ki} \circ \psi_\phi^{-1}(v) \right) - D_{ik} \circ \psi_\phi^{-1}(v) \right). \]

But
\[ R_{2,t} = \{(u, v) : u_t < u \leq 1, t < v, C(u, v) \leq t\} \cup \{(u, v) : u_t < u \leq 1, v \leq t\} \]
and
\[ \mathbb{P}(u_t < U \leq 1, V \leq t) = t - C(u_t, t) = \mathbb{P}(U \leq u_t, t < V \leq v_t). \]

Hence
\[
\mathbb{P}(R_{2,t}) = \int_{t}^{v_t} \mathbb{P}(U \leq F_2(t, v)|V = v) dv = \\
= \int_{t}^{v_t} \partial_2 C(F_2(t, v), v) dv = \\
= \int_{t}^{v_t} \psi'_\phi \circ \psi_\phi^{-1}(t) \left\{ \rho' \circ \rho^{-1} \left( \psi_\phi^{-1}(t) - D_{ki} \circ \psi_\phi^{-1}(v) \right) \frac{d}{dv} D_{ik} \circ \psi_\phi^{-1}(v) + \frac{d}{dv} D_{ki} \circ \psi_\phi^{-1}(v) \right\} dv = \\
= \psi'_\phi \circ \psi_\phi^{-1}(t) \left\{ \int_{D_{ik} \circ \psi_\phi^{-1}(v_t)}^{D_{ik} \circ \psi_\phi^{-1}(v)} \rho' \circ \rho^{-1}(\psi_\phi^{-1}(t) - D_{ki} \circ D_{ik}^{-1}(z)) dz + \\
+ D_{ki} \circ \psi_\phi^{-1}(v_t) - D_{ki} \circ \psi_\phi^{-1}(t) \right\}.
\]

From \( \mathbb{P}(S_t) = t + \mathbb{P}(R_{1,t}) + \mathbb{P}(R_{2,t}) \) we get \([13]\).
As a consequence, the Kendall’s tau is
\[
\tau = 3 - 4 \int_0^1 \mathcal{K}(t) dt = \\
= 3 - 4 \int_0^1 \{ t - \psi' \circ \psi^{-1}(t) \} \cdot \left[ D_{ki} \circ \psi^{-1}(t) - \int_{D_{ik} \circ G^{-1} \circ \psi^{-1}(t)} \rho' \circ \rho^{-1}(\psi^{-1}(t) - D_{ki} \circ D_{ik}^{-1}(z)) dz \right] dt = \\
= 1 + 4 \int_0^1 \psi' \circ \psi^{-1}(t) \cdot D_{ki} \circ \psi^{-1}(t) dt - \\
- 4 \int_0^1 \psi' \circ \psi^{-1}(t) \int_{D_{ik} \circ G^{-1} \circ \psi^{-1}(t)} \rho' \circ \rho^{-1}(\psi^{-1}(t) - D_{ki} \circ D_{ik}^{-1}(z)) dz dt.
\]
\[\square\]

Proof of Theorem 3.2

Proof. In the sequel we set \( \partial_1 C(u, v) = \frac{\partial}{\partial u} C(u, v) \) and \( \partial_2 C(u, v) = \frac{\partial}{\partial v} C(u, v) \).

The proof is similar to the one of Theorem 3.1.

Again we decompose the set \( S_t = \{(u, v) \in [0, 1]^2 : C(u, v) \leq t\} \) as \( S_t = R_{t1} + R_{1,t} + R_{2,t} \) where, if \((u_t, v_t)\) is the intersection point of the curves \( C(u, v) = t \) and \( v = h(u) \), \( R_t = [0, u_t] \times [0, v_t] \), \( R_{1,t} = \{(u, v) : u \in (u_t, 1], C(u, v) \leq t\} \) and \( R_{2,t} = \{(u, v) : v \in (v_t, 1], C(u, v) \leq t\} \). Clearly, the \( C \)-measure of \( R_t \) is \( t \). In order to compute the \( C \)-measure of \( R_{1,t} \) and \( R_{2,t} \), we compute \( u_t \) and \( v_t \). Since
\[
\psi \left[ \rho \left( \rho^{-1} \circ D_{ik} \circ \psi^{-1}(u_t) + \rho^{-1} \circ D_{jk} \circ \psi^{-1}(v_t) \right) + D_{kj} \circ \psi^{-1}(v_t) \right] = t
\]
and \( v_t = h(u_t) \), we get
\[
v_t = \psi \circ H_{0,j} \circ G^{-1} \circ \psi^{-1}(t)
\]
and
\[
u_t = \psi \circ H_{0,i} \circ G^{-1} \circ \psi^{-1}(t)
\]
where \( G \) is given by (16).

Let us start with \( R_{1,t} \). Notice that here, \( C(u, v) \leq t \) is equivalent to \( u \leq F_1(t, v) \) where
\[
F_1(t, v) = \psi \circ D_{ik}^{-1} \circ \rho \left( \psi^{-1}(t) - D_{kj} \circ \psi^{-1}(v) \right) - \rho^{-1} \circ D_{jk} \circ \psi^{-1}(v).
\]

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By similar arguments as those used in the proof of Theorem 3.1 we have

\[ P(R_1, t) = \int_0^t P(U \leq F_1(t, v) | V = v) dv = \]

\[ = \int_0^t \partial_2 C(F_1(t, v), v) dv = \]

\[ = \int_0^t \psi' \circ \psi^{-1}(t) \left\{ \rho' \circ \rho^{-1}(\psi^{-1}(t) - D_{kj} \circ \psi^{-1}(v)) \frac{d}{dv} \rho^{-1} \circ D_{kj} \circ \psi^{-1}(v) + \right. \]

\[ + \frac{d}{dv} D_{kj} \circ \psi^{-1}(v) \right\} dv = \]

\[ = \psi' \circ \psi^{-1}(t) \left\{ \int_{\rho^{-1} \circ D_{kj} \circ \psi^{-1}(v)}^{\rho^{-1} \circ D_{kj} \circ \psi^{-1}(t)} \rho' \circ \rho^{-1}(\psi^{-1}(t) - \psi^{-1} \circ \bar{F}_k \circ \bar{F}_j \circ \psi \circ \rho(z)) dz + \right. \]

\[ + D_{kj} \circ \psi^{-1}(v) - D_{kj} \circ \psi^{-1}(t) \right\} . \]

Substituting \( v \) we get

\[ P(R_1, t) = \]

\[ = \psi' \circ \psi^{-1}(t) \left\{ \int_{\rho^{-1} \circ D_{kj} \circ \psi^{-1}(v)}^{\rho^{-1} \circ D_{kj} \circ \psi^{-1}(t)} \rho' \circ \rho^{-1}(\psi^{-1}(t) - \psi^{-1} \circ \bar{F}_k \circ \bar{F}_j \circ \psi \circ \rho(z)) dz + \right. \]

\[ + \psi^{-1} \circ \bar{F}_k \circ G^{-1} \circ \psi^{-1}(t) - D_{kj} \circ \psi^{-1}(t) \right\} . \]

With similar computations we get

\[ P(R_2, t) = \]

\[ = \psi' \circ \psi^{-1}(t) \left\{ \int_{\rho^{-1} \circ D_{ik} \circ \psi^{-1}(v)}^{\rho^{-1} \circ D_{ik} \circ \psi^{-1}(t)} \rho' \circ \rho^{-1}(\psi^{-1}(t) - \psi^{-1} \circ \bar{F}_i \circ \bar{F}_j \circ \psi \circ \rho(z)) dz + \right. \]

\[ + \psi^{-1} \circ \bar{F}_k \circ G^{-1} \circ \psi^{-1}(t) - D_{ki} \circ \psi^{-1}(t) \right\} . \]

From \( P(S_t) = t + P(R_1, t) + P(R_2, t) \) we get 15.
As a consequence, the Kendall’s tau is

\[
\tau = 3 - 4 \int_0^1 K(t) dt = \\
= 1 - 4 \int_0^1 \psi'_\phi \circ \psi^{-1}_\phi(t) \cdot \\
\left[ \int_{\rho^{-1} \circ D_{jk} \circ \psi^{-1}_\phi(t)} \rho' \circ \rho^{-1} \{ \psi^{-1}_\phi(t) - \psi^{-1}_\phi \circ \bar{F}_k \circ \bar{F}_j^{-1} \circ \psi \circ \rho(z) \} \, dz + \\
+ \int_{\rho^{-1} \circ D_{ik} \circ \psi^{-1}_\phi(t)} \rho' \circ \rho^{-1} \{ \psi^{-1}_\phi(t) - \psi^{-1}_\phi \circ \bar{F}_i \circ \bar{F}_k^{-1} \circ \psi \circ \rho(z) \} \, dz \right] dt + \\
- 4 \int_0^1 \psi'_\phi \circ \psi^{-1}_\phi(t) \cdot (2 \psi^{-1}_\phi \circ \bar{F}_k \circ G^{-1} - (D_{kj} + D_{ki})) \circ \psi^{-1}_\phi(t) dt.
\]

\[\Box\]

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