Topological Test Spaces¹
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Abstract

A test space is the set of outcome-sets associated with a collection of experiments. This notion provides a simple mathematical framework for the study of probabilistic theories – notably, quantum mechanics – in which one is faced with incommensurable random quantities. In the case of quantum mechanics, the relevant test space, the set of orthonormal bases of a Hilbert space, carries significant topological structure. This paper inaugurates a general study of topological test spaces. Among other things, we show that any topological test space with a compact space of outcomes is of finite rank. We also generalize results of Meyer and Clifton-Kent by showing that, under very weak assumptions, any second-countable topological test space contains a dense semi-classical test space.

0. Introduction

A test space in the sense of Foulis and Randall [3, 4, 5], is a pair (X, A) where X is a non-empty set and A is a covering of X by non-empty subsets.² The intended interpretation is that each set E ∈ A represents an exhaustive set of mutually exclusive possible outcomes, as of some experiment, decision, physical process, or test. A state, or probability weight, on (X, A) is a mapping ω : X → [0, 1] summing to 1 over each test.

Obviously, this framework subsumes discrete classical probability theory, which deals with test spaces (E, {E}) having only a single test. It also accommodates quantum probability theory, as follows. Let H be a Hilbert space, let S = S(H) be the unit sphere of H, and let ℱ = ℱ(H) denote the collection of all frames, i.e., maximal pairwise orthogonal subsets of S. The test space (S, ℱ) is a model for the set of maximally informative, discrete quantum-mechanical experiments. As long as dim(H) > 2, Gleason’s theorem [6] tells us that every state ω on (S, ℱ) arises from a density operator W on H via the rule ω(x) = ⟨Wx, x⟩ for all x ∈ S.

In this last example, the test space has a natural topological structure: S is a metric space, and ℱ can be topologized as well (in several ways). The purpose of this paper is to provide a framework for the study of topological test spaces generally. Section 1 develops basic properties of the Vietoris topology, which we use heavily in the sequel. Section 2 considers topological test spaces in general, and locally finite topological test spaces in particular. Section 3 addresses

¹I wish to dedicate this paper to the memory of Frank J. Hague II
²It is also usual to assume that A is irredundant, i.e., that no set in A properly contains another. For convenience, we relax this assumption.
the problem of topologizing the logic of an algebraic topological test space. In section 4, we generalize results of Meyer \[8\] and Clifton and Kent \[2\] by showing that any second-countable topological test space satisfying a rather natural condition contains a dense semi-classical subspace. The balance of this section collects some essential background information concerning test spaces (see \[11\] for a detailed survey). Readers familiar with this material can proceed directly to section 1.

0.1 Events Let \((X, \mathfrak{A})\) be a test space. Two outcomes \(x, y \in X\) are said to be orthogonal, or mutually exclusive, if they are distinct and belong to a common test. In this case, we write \(x \perp y\). More generally, a set \(A \subseteq X\) is called an event for \(X\) if there exists a test \(E \supseteq A\). The set of events is denoted by \(\mathcal{E}(X, \mathfrak{A})\).

There is a natural orthogonality relation on \(\mathcal{E}(X, \mathfrak{A})\) extending that on \(X\), namely, \(A \perp B\) iff \(A \cap B = \emptyset\) and \(A \cup B \in \mathcal{E}(X, \mathfrak{A})\). Every state \(\omega\) on \((X, \mathfrak{A})\) extends to a mapping \(\omega : \mathcal{E}(X, \mathfrak{A}) \to [0, 1]\) given by \(\omega(A) = \sum_{x \in A} \omega(x)\). If \(A \perp B\), then \(\omega(A \cup B) = \omega(A) + \omega(B)\) for every probability weight \(\omega\). Two events \(A\) and \(C\) are complementary — abbreviated \(Aoc\) — if they partition a test, and perspective if they are complementary to a common third event \(C\). In this case, we write \(A \sim B\). Note that if \(A\) and \(B\) are perspective, then for every state \(\omega\) on \((X, \mathfrak{A})\), \(\omega(A) = 1 - \omega(C) = \omega(B)\).

0.2 Algebraic Test Spaces We say that \(X\) is algebraic iff for all events \(A, B, C \in \mathcal{E}(X, \mathfrak{A})\), \(A \sim B\) and \(BocC \Rightarrow AocC\). In this case, \(\sim\) is an equivalence relation on \(\mathcal{E}(X)\). Moreover, if \(A \perp B\) and \(B \sim C\), then \(A \perp C\) as well, and \(A \cup B \sim A \cup C\).

Let \(\Pi(X, \mathfrak{A}) = \mathcal{E}(X, \mathfrak{A})/\sim\), and write \(p(A)\) for the \(\sim\)-equivalence class of an event \(A \in \mathcal{E}(X)\). Then \(\Pi\) carries a well-defined orthogonality relation, namely \(p(A) \perp p(B) \iff A \perp B\), and also a partial binary operation \(p(A) \oplus p(B) = p(A \cup B)\), defined for orthogonal pairs. We may also define \(0 := p(\emptyset)\), \(1 := p(E)\), \(E \in \mathfrak{A}\), and \(p(A') = p(C)\) where \(C\) is any event complementary to \(A\).

The structure \((\Pi, \oplus, ', 0, 1)\), called the logic of \((X, \mathfrak{A})\), satisfies the following conditions:

1. \(p \oplus q = q \oplus p\) and \(p \oplus (q \oplus r) = (p \oplus q) \oplus r\);
2. \(p \oplus p\) is defined only if \(p = 0\);
3. \(p \oplus 0 = 0 \oplus p = p\);
4. For every \(p \in \Pi\), there exists a unique element — namely, \(p'\) — satisfying \(p \oplus p' = 1\).

For the test space \((S, \mathfrak{F})\) of frames of a Hilbert space \(H\), events are simply orthonormal set of vectors in \(H\), and two events are perspective iff they have the same closed span. Hence, we can identify \(\Pi(S, \mathfrak{F})\) with the set of closed subspaces of \(H\), with \(\oplus\) coinciding with the usual orthogonal sum operation.

\(^3\)With one side defined iff the other is.
0.3 Orthoalgebras Abstractly, a structure satisfying (1) through (4) above is called an orthoalgebra. It can be shown that every orthoalgebra arises canonically (though not uniquely) as $\Pi(X, \mathfrak{A})$ for an algebraic test space $(X, \mathfrak{A})$. Indeed, if $L$ is an orthoalgebra, let $X_L = L \setminus \{0\}$ and let $\mathfrak{A}_L$ denote the set of finite subsets $E = \{e_1, ..., e_n\}$ of $L \setminus 0$ for which $e_1 \oplus \cdots \oplus e_n$ exists and equals 1. Then $(X_L, \mathfrak{A}_L)$ is an algebraic test space with logic canonically isomorphic to $L$.

Any orthoalgebra $L$ carries a natural partial order, defined by setting $p \leq q$ iff there exists some $r \in L$ with $p \perp r$ and $p \oplus r = q$. With respect to this ordering, the mapping $p \mapsto p'$ is an orthocomplementation.

0.4 Proposition [3]: If $L$ is an orthoalgebra, the following are equivalent:

(a) $L$ is orthocoherent, i.e., for all pairwise orthogonal elements $p, q, r \in L$, $p \oplus q \oplus r$ exists.

(b) $p \oplus q = p \lor q$ for all $p \perp q$ in $L$.

(c) $(L, \leq,')$ is an orthomodular poset.

Note also that if $(L, \leq,')$ is any orthoposet, the partial binary operation of orthogonal join — that is, $p \oplus q = p \lor q$ for $p \leq q'$ — is associative iff $L$ is orthomodular, in which case, $(L, \oplus)$ is an orthoalgebra, the natural order on which coincides with the given order on $L$ [11]. Thus, orthomodular posets and orthomodular lattices can be regarded as essentially the same things as orthocoherent orthoalgebras and lattice-ordered orthoalgebras, respectively.

1. Background on the Vietoris Topology

General references for this section are [7] and [9]. If $X$ is any topological space, let $2^X$ denote the set of all closed subsets of $X$. If $A \subseteq X$, let

$$[A] := \{F \in 2^X | F \cap A \neq \emptyset\}.$$ 

Clearly, $[A \cap B] \subseteq [A] \cap [B]$ and $\bigcup_i [A_i] = \bigcup_i A_i$. The Vietoris topology on $2^X$ is the coarsest topology in which $[U]$ is open if $U \subseteq X$ is open and $[F]$ is closed if $F \subseteq X$ is closed.\footnote{In particular, $\emptyset$ is an isolated point of $2^X$. Many authors omit $\emptyset$ from $2^X$.} Thus, if $U$ is open, so is $(U) := [U]^c = \{F \in 2^X | F \subseteq U\}$. Let $\mathcal{B}$ be any basis for the topology on $X$: then the collection of sets of the form

$$\langle U_1, ..., U_n \rangle := [U_1] \cap \cdots \cap [U_n] \cap \left(\bigcup_{i=1}^n U_i\right)$$

with $U_1, ..., U_n$ in $\mathcal{B}$, is a basis for the Vietoris topology on $2^X$. Note that $\langle U_1, ..., U_n \rangle$ consists of all closed sets contained in $\bigcup_{i=1}^n U_i$ and meeting each set $U_i$ at least once.
If $X$ is a compact metric space, then the Vietoris topology on $2^X$ is just that induced by the Hausdorff metric. Two classical results concerning the Vietoris topology are Vietoris' Theorem: $2^X$ is compact iff $X$ is compact, and Michael's Theorem: a (Vietoris) compact union of compact sets is compact.\(^5\)

The operation $\cup : 2^X \times 2^X \to 2^X$ is also Vietoris continuous, since
\[
\cup^{-1}(U) = \{(A, B)|A \cup B \in [U]\} = ([U] \times 2^X) \cup (2^X \times [U]),
\]
which is open if $U$ is open and closed if $U$ is closed. In particular, for any fixed closed set $A$, the mapping $f_A : 2^X \to 2^X$ given by $f_A : B \mapsto A \cup B$ is continuous. Notice also that the mapping $\pi : 2^X \times 2^X \to 2^X \times X$ given by $\pi(A, B) = A \times B$ is continuous, as $\pi^{-1}([U \times V]) = [U] \times [V]$ and $\pi^{-1}((U \times V)) = (U) \times (V)$.

Henceforth, we regard any collection $\mathfrak{A}$ of closed subsets of a topological space $X$ as a subspace of $2^X$. In the special case in which $\mathfrak{A}$ is a collection of finite sets of uniformly bounded cardinality, say $|E| < n$ for every $E \in \mathfrak{A}$, there is a more direct approach to topologizing $\mathfrak{A}$ that bears discussion. Let $\mathfrak{A}^o \subseteq X^n$ denote the space of ordered versions $(x_1, ..., x_n)$ of sets $\{x_1, ..., x_n\} \in \mathfrak{A}$, with the relative product topology. We can give $\mathfrak{A}$ the quotient topology induced by the natural surjection $\pi : \mathfrak{A}^o \to \mathfrak{A}$ that “forgets” the order. The following is doubtless well-known, but I include the short proof for completeness.

1.1 Proposition: Let $X$ be Hausdorff and $\mathfrak{A}$, a collection of non-empty finite subsets of $X$ of cardinality $\leq n$ (with the Vietoris topology). Then the canonical surjection $\pi : \mathfrak{A}^o \to \mathfrak{A}$ is an open continuous map. Hence, the Vietoris topology on $\mathfrak{A}$ coincides with the quotient topology induced by $\pi$.

Proof: Let $U_1, ..., U_n$ be open subsets of $X$. Then $\pi((U_1 \times \cdot \cdot \cdot \times U_n) \cap \mathfrak{A}^o) = \langle U_1, ..., U_n \rangle \cap \mathfrak{A}$, so $\pi$ is open. Also
\[
\pi^{-1}(\langle U_1, ..., U_n \rangle \cap \mathfrak{A}) = \bigcup_{\sigma}(U_{\sigma(1)} \times \cdot \cdot \cdot \times U_{\sigma(n)}) \cap \mathfrak{A}^o,
\]
where $\sigma$ runs over all permutations of $\{1, 2, ..., n\}$, so $\pi$ is continuous. It follows immediately that the quotient and Vietoris topologies on $\mathfrak{A}$ coincide. $\square$

2. Topological Test Spaces

We come now to the subject of this paper.

2.1 Definition: A topological test space is a test space $(X, \mathfrak{A})$ where $X$ is a Hausdorff space and the relation $\perp$ is closed in $X \times X$.

\(^5\)More precisely, if $\mathcal{C}$ is a compact subset of $2^X$ with each $C \in \mathcal{C}$ compact, then $\bigcup_{C \in \mathcal{C}} C$ is again compact.
2.2 Examples

(a) Let \( H \) be a Hilbert space. Let \( S \) be the unit sphere of \( H \), in any topology making the inner product continuous. Then the test space \((S, \mathfrak{F})\) defined above is a topological test space, since the orthogonality relation is closed in \( S^2 \).

(b) Suppose that \( X \) is Hausdorff, that every \( E \in \mathfrak{A} \) is finite, and that \((X, \mathfrak{A})\) supports a set \( \Gamma \) of continuous probability weights that are \( \perp \)-separating in the sense that \( p \not\perp q \iff \exists \omega \in \Gamma \text{ with } \omega(p) + \omega(q) > 1 \). Then \( \perp \) is closed in \( X^2 \), so again \((X, \mathfrak{A})\) is a topological test space.

(c) Let \( L \) be any topological orthomodular lattice \([1]\). The mapping \( \phi : L^2 \to L^2 \) given by \( \phi(p, q) = (p, p \land q') \) is continuous, and \( \perp = \phi^{-1}(\Delta) \) where \( \Delta \) is the diagonal of \( L^2 \). Since \( L \) is Hausdorff, \( \Delta \) is closed, whence, so is \( \perp \). Hence, the test space \((L \setminus \{0\}, \mathfrak{A}_L)\) (as described in 0.3 above) is topological.

The following Lemma collects some basic facts about topological test spaces that will be used freely in the sequel.

2.3 Lemma: Let \((X, \mathfrak{A})\) be a topological test space. Then

(a) Each point \( x \in X \) has an open neighborhood containing no two orthogonal outcomes. (We shall call such a neighborhood totally non-orthogonal.)

(b) For every set \( A \subseteq X \), \( A^\perp \) is closed.

(c) Each pairwise orthogonal subset of \( X \) is discrete

(d) Each pairwise orthogonal subset of \( X \) is closed.

Proof: (a) Let \( x \in X \). Since \((x, x) \notin \perp \) and \( \perp \) is closed, we can find open sets \( V \) and \( W \) about \( x \) with \((V \times W) \cap \perp = \emptyset \). Taking \( U = V \cap W \) gives the advertised result.

(b) Let \( y \in X \setminus x^\perp \). Then \((x, y) \notin \perp \). Since the latter is closed, there exist open sets \( U, V \subseteq X \) with \((x, y) \in U \times V \) and \((U \times V) \cap \perp = \emptyset \). Thus, no element of \( V \) lies orthogonal to any element of \( U \); in particular, we have \( y \in V \subseteq X \setminus x^\perp \).

Thus, \( X \setminus x^\perp \) is open, i.e., \( x^\perp \) is closed. It now follows that for any set \( A \subseteq X \), the set \( A^\perp = \bigcap_{x \in A} x^\perp \) is closed.

(c) Let \( D \) be pairwise orthogonal. Let \( x \in D \): by part (b), \( X \setminus x^\perp \) is open, whence, \( \{x\} = D \cap (X \setminus x^\perp) \) is relatively open in \( D \). Thus, \( D \) is discrete.

(d) Now suppose \( D \) is pairwise orthogonal, and let \( z \in \overline{D} \): if \( z \notin D \), then for every open neighborhood \( U \) of \( z \), \( U \cap D \) is infinite; hence, we can find distinct elements \( x, y \in D \cap U \). Since \( D \) is pairwise orthogonal, this tells us that \((U \times U) \cap \perp \neq \emptyset \). But then \((x, x) \) is a limit point of \( \perp \). Since \( \perp \) is closed, \((x, x) \in \perp \), which is a contradiction. Thus, \( z \in D \), i.e., \( D \) is closed. □

It follows in particular that every test \( E \in \mathfrak{A} \) and every event \( A \in \mathcal{E}(X, \mathfrak{A}) \) is a closed, discrete subset of \( X \). Hence, we may construe \( \mathfrak{A} \) and \( \mathcal{E}(X, \mathfrak{A}) \) of as subspaces of \( 2^X \) in the Vietoris topology.
A test space \((X, \mathcal{A})\) is locally finite iff each test \(E \in \mathcal{A}\) is a finite set. We shall say that a test space \((X, \mathcal{A})\) is of rank \(n\) if \(n\) is the maximum cardinality of a test in \(\mathcal{A}\). If all tests have cardinality equal to \(n\), then \((X, \mathcal{A})\) is \(n\)-uniform.

2.4 Theorem: Let \((X, \mathcal{A})\) be a topological test space with \(X\) compact. Then all pairwise orthogonal subsets of \(X\) are finite, and of uniformly bounded size. In particular, \(\mathcal{A}\) is of finite rank.

Proof: By Part (a) of Lemma 2.3, every point \(x \in X\) is contained in some totally non-orthogonal open set. Since \(X\) is compact, a finite number of these, say \(U_1, \ldots, U_n\), cover \(X\). A pairwise orthogonal set \(D \subseteq X\) can meet each \(U_i\) at most once; hence, \(|D| \leq n\). □.

For locally finite topological test spaces, the Vietoris topology on the space of events has a particularly nice description. Suppose \(A\) is a finite event: By Part (a) of Lemma 2.3, we can find for each \(x \in A\) a totally non-orthogonal open neighborhood \(U_x\). Since \(X\) is Hausdorff and \(A\) is finite, we can arrange for these to be disjoint from one another. Consider now the Vietoris-open neighborhood \(V = \langle U_x, x \in A \rangle \cap \mathcal{E}\) of \(A\) in \(\mathcal{E}\): an event \(B\) belonging to \(V\) is contained in \(\bigcup_{x \in A} U_x\) and meets each \(U_x\) in at least one point; however, being pairwise orthogonal, \(B\) can meet each \(U_x\) at most once. Thus, \(B\) selects exactly one point from each of the disjoint sets \(U_x\) (and hence, in particular, \(|B| = |A|\)). Note that, since the totally non-orthogonal sets form a basis for the topology on \(X\), open sets of the form just described form a basis for the Vietoris topology on \(\mathcal{E}\).

As an immediate consequence of these remarks, we have the following:

2.5 Proposition: Let \((X, \mathcal{A})\) be locally finite. Then the set \(\mathcal{E}_n\) of all events of a given cardinality \(n\) is clopen in \(\mathcal{E}(X, \mathcal{A})\).

A test space \((X, \mathcal{A})\) is UDF (unital, dispersion-free) iff for ever \(x \in X\) there exists a \(\{0, 1\}\)-valued state \(\omega\) on \((X, \mathcal{A})\) with \(\omega(x) = 1\). Let \(U_1, \ldots, U_n\) be pairwise disjoint totally non-orthogonal open sets, and and let \(\mathcal{U} = \langle U_1, \ldots, U_n \rangle\): then \(\mathcal{U}\) can be regarded as a UDF test space (each \(U_i\) selecting one outcome from each test in \(\mathcal{V}\)). The foregoing considerations thus have the further interesting consequence that any locally finite topological test space is locally UDF. In particular, for such test spaces, the existence or non-existence of dispersion-free states will depend entirely on the global topological structure of the space.

If \((X, \mathcal{A})\) is a topological test space, let \(\overline{\mathcal{A}}\) denote the (Vietoris) closure of \(\mathcal{A}\) in \(2^X\). We are going to show that \((X, \overline{\mathcal{A}})\) is again a topological test space, having in fact the same orthogonality relation as \((X, \mathcal{A})\). If \((X, \mathcal{A})\) is of finite rank, moreover, \((X, \overline{\mathcal{A}})\) has the same states as \((X, \mathcal{A})\).

2.6 Lemma: Let \((X, \mathcal{A})\) be any topological test space, and let \(E \in \overline{\mathcal{A}}\). Then \(E\) is pairwise orthogonal (with respect to the orthogonality induced by \(\mathcal{A}\)).
Proof: Let $x$ and $y$ be two distinct points of $E$. Let $U$ and $V$ be disjoint neighborhoods of $x$ and $y$ respectively, and let $(E_{\lambda})_{\lambda \in \Lambda}$ be a net of closed sets in $\mathfrak{A}$ converging to $E$ in the Vietoris topology. Since $E \in [U] \cap [V]$, we can find $\lambda_{U,V} \in \Lambda$ such that $E_{\lambda} \in [U] \cap [V]$ for all $\lambda \geq \lambda_{U,V}$. In particular, we can find $x_{\lambda_{U,V}} \in E_{\lambda_{U,V}} \cap U$ and $y_{\lambda_{U,V}} \in E_{\lambda_{U,V}} \cap V$. Since $U$ and $V$ are disjoint, $x_{\lambda_{U,V}}$ and $y_{\lambda_{U,V}}$ are distinct, and hence, - since they belong to a common test $E_{\lambda}$ - orthogonal. This gives us a net $(x_{\lambda_{U,V}}, y_{\lambda_{U,V}})$ in $X \times X$ converging to $(x, y)$ and with $(x_{\lambda_{U,V}}, y_{\lambda_{U,V}}) \in \bot$. Since $\bot$ is closed, $(x, y) \in \bot$, i.e., $x \bot y$. \(\Box\)

It follows that the orthogonality relation on $X$ induced by $\mathfrak{A}$ is the same as that induced by $\mathfrak{a}$. In particular, $(X, \mathfrak{A})$ is again a topological test space.

Let $\mathcal{F}_n$ denote the set of finite subsets of $X$ having $n$ or fewer elements.

2.7 Lemma: Let $X$ be Hausdorff. Then for every $n$,

(a) $\mathcal{F}_n$ is closed in $2^X$.

(b) If $f : X \to \mathbb{R}$ is continuous, then so is the mapping $\hat{f} : \mathfrak{F}_n \to \mathbb{R}$ given by $\hat{f}(A) := \sum_{x \in A} f(x)$.

Proof: (a) Let $F$ be a closed set (finite or infinite) of cardinality greater than $n$. Let $x_1, ..., x_{n+1}$ be distinct elements of $F$, and let $U_1, ..., U_n$ be pairwise disjoint open sets with $x_i \in U_i$ for each $i = 1, ..., n$. Then no closed set in $\mathcal{U} := [U_1] \cap \cdots \cap [U_n]$ has fewer than $n+1$ points – i.e, $\mathcal{U}$ is an open neighborhood of $F$ disjoint from $\mathcal{F}_n$. This shows that $2^X \setminus \mathcal{F}_n$ is open, i.e., $\mathcal{F}_n$ is closed.

(b) By proposition 1.1, $\mathfrak{F}_n$ is the quotient space of $X^n$ induced by the surjection surjection $q : (x_1, ..., x_n) \mapsto \{x_1, ..., x_n\}$. The mapping $\tilde{f} : X^n \to \mathbb{R}$ given by $(x_1, ..., x_n) \mapsto \sum_{i=1}^n f(x_i)$ is plainly continuous; hence, so is $\hat{f}$. \(\Box\)

2.8 Proposition: Let $(X, \mathfrak{A})$ be a rank-$n$ (respectively, $n$-uniform) test space. Then $(X, \mathfrak{A})$ is also a rank-$n$ (respectively, $n$-uniform) test space having the same continuous states as $(X, \mathfrak{A})$.

Proof: If $\mathfrak{A}$ is rank-$n$, then $\mathfrak{A} \subseteq \mathfrak{F}_n$. Since the latter is closed, $\overline{\mathfrak{A}} \subseteq \mathfrak{F}_n$ also. Note that if $\mathfrak{A}$ is $n$-uniform and $E \in \overline{\mathfrak{A}}$, then any net $E_{\lambda} \to E$ is eventually in bijective correspondence with $E$, by Proposition 2.5. Hence, $(X, \overline{\mathfrak{A}})$ is also $n$-uniform. Finally, every continuous state on $(X, \mathfrak{A})$ lifts to a continuous state on $(X, \overline{\mathfrak{A}})$ by Lemma 2.7 (b). \(\Box\)

3. The Logic of a Topological Test Space

In this section, we consider the logic $\Pi = \Pi(X, \mathfrak{A})$ of an algebraic test space $(X, \mathfrak{A})$. We endow this with the quotient topology induced by the canonical surjection $p : \mathcal{E} \to \Pi$ (where $\mathcal{E} = \mathcal{E}(X, \mathfrak{A})$ has, as usual, its Vietoris topology). Our aim is to find conditions on $(X, \mathfrak{A})$ that will guarantee reasonable continuity
properties for the orthogonal sum operation and the orthocomplement. In this connection, we advance the following

3.1 Definition: A topological orthoalgebra is an orthoalgebra $(L, \bot, \oplus, 0, 1)$ in which $L$ is a topological space, the relation $\bot \subseteq L^2$ is closed, and the mappings $\oplus : \bot \to L$ and $' : L \to L$ are continuous.

A detailed study of topological orthoalgebras must wait for another paper. However, it is worth mentioning here that, while every topological orthomodular lattice is a topological orthoalgebra, there exist lattice-ordered topological orthoalgebras in which the meet and join are discontinuous – e.g., the orthoalgebra $L(H)$ of closed subspaces of a Hilbert space, in its operator-norm topology.

3.2 Lemma: Let $(L, \bot, \oplus, 0, 1)$ be a topological orthoalgebra. Then
(a) The order relation $\leq$ is closed in $L^2$
(b) $L$ is a Hausdorff space.

Proof: For (a), note that $a \leq b$ iff $a \bot b'$. Thus, $\leq = f^{-1}(\bot)$ where $f : L \times L \to L \times L$ is the continuous mapping $f(a, b) = (a, b')$. Since $\bot$ is closed, so is $\leq$.

The second statement now follows by standard arguments (cf. Nachbin [10]). □

We now return to the question: when is the logic of a topological test space, in the quotient topology, a topological orthoalgebra?

3.3 Lemma: Suppose $\mathcal{E}$ is closed in $2^X$. Then
(a) The orthogonality relation $\bot_\mathcal{E}$ on $\mathcal{E}$ is closed in $\mathcal{E}^2$.
(b) The mapping $\cup : \bot_\mathcal{E} \to \mathcal{E}$ is continuous.

Proof: The mapping $\mathcal{E}^2 \to 2^X$ given by $(A, B) \mapsto A \cup B$ is continuous; hence, if $\mathcal{E}$ is closed in $2^X$, then so is the set $C := \{(A, B) \in \mathcal{E}^2 | A \cup B \in \mathcal{E}\}$ of compatible pairs of events. It will suffice to show that the set $\mathcal{O} := \{(A, B) \in \mathcal{E} | A \subseteq B^+\}$ is also closed, since $\bot = C \cap \mathcal{O}$. But $(A, B) \in \mathcal{O}$ iff $A \times B \subseteq \bot$, i.e., $\mathcal{O} = \pi^{-1}(\bot) \cap \mathcal{E}$ where $\pi : 2^X \times 2^X \to 2^X$ is the product mapping $(A, B) \mapsto A \times B$. As observed in section 1, this mapping is continuous, and since $\bot$ is closed in $2^X \times 2^X$, so is $(\bot)$ in $2^X \times 2^X$. Statement (b) follows immediately from the Vietoris continuity of $\cup$. □

Remarks: The hypothesis that $\mathcal{E}$ be closed in $2^X$ is not used in showing that the relation $\mathcal{O}$ is closed. If $(X, \mathfrak{A})$ is coherent [10], then $\mathcal{O} = \bot$, so in this case, the hypothesis can be avoided altogether. On the other hand, if $X$ is compact and $\mathfrak{A}$ is closed, then $\mathcal{E}$ will also be compact and hence, closed. (To see this, note that if $X$ is compact then by Vietoris’ Theorem, $2^X$ is compact. Hence, so is the closed set $(E) = \{A \in 2^X | A \subseteq E\}$ for each $E \in \mathfrak{A}$. The mapping $2^X \to 2^X$ given by $E \mapsto (E)$ is easily seen to be continuous. Since $\mathfrak{A}$ is closed, hence compact, in $2^X$, it follows that $\{(E) | E \in \mathfrak{A}\}$ is a compact subset of $2^{2^X}$. By Michael’s theorem, $\mathcal{E} = \bigcup_{E \in \mathfrak{A}}(E)$ is compact, hence closed, in $2^X$.)
In order to apply Lemma 3.3 to show that $\perp \subseteq \Pi^2$ is closed and $\oplus : \perp \rightarrow \Pi$ is continuous, we would like to have the canonical surjection $p : \mathcal{E} \rightarrow \Pi$ open. The following condition is sufficient to secure this, plus the continuity of the orthocomplementation $' : \Pi \rightarrow \Pi$.

3.3 Definition: Call a topological test space $(X, \mathfrak{A})$ is 

stably complemented iff 

for any open set $U$ in $\mathcal{E}$, the set $U^\circ$ of events complementary to events in $U$ is again open.

Remark: If $H$ is a finite-dimensional Hilbert space, it can be shown that the corresponding test space $(S, \mathfrak{F})$ of frames is stably complemented [12].

3.5 Lemma: Let $(X, \mathfrak{A})$ be a topological test space, and let $p : \mathcal{E} \rightarrow \Pi$ be the canonical quotient mapping (with $\Pi$ having the quotient topology). Then the following are equivalent:

(a) $(X, \mathfrak{A})$ is stably complemented

(b) The mapping $p : \mathcal{E} \rightarrow \Pi$ is open and the mapping $' : \Pi \rightarrow \Pi$ is continuous.

Proof: Suppose first that $(X, \mathfrak{A})$ is stably complemented, and let $U$ be an open set in $\mathcal{E}$. Then $p^{-1}(p(U)) = \{A \in \mathcal{E}|\exists B \in U \sim A\}$

$= \{A \in \mathcal{E}|\exists C \in U^\circ \sim A\}$

$= (U^\circ)^{oc}$

which is open. Thus, $p(U)$ is open. Now note that $' : \Pi \rightarrow \Pi$ is continuous iff, for every open set $V \subseteq \Pi$, the set $V' = \{p'|p \in V\}$ is also open. But $p^{-1}(V') = (p^{-1}(V))^{oc}$: since $p$ is continuous and $(X, \mathfrak{A})$ is stably complemented, this last is open. Hence, $V'$ is open.

For the converse, note first that if $'$ is continuous, it is also open (since $a'' = a$ for all $a \in \Pi$). Now for any open set $U \subseteq \mathcal{E}$, $U^{oc} = p^{-1}(p(U))$: Since $p$ and $'$ are continuous open mappings, this last is open as well. □

3.6 Proposition: Let $(X, \mathfrak{A})$ be a stably complemented algebraic test space with $\mathcal{E}$ closed. Then $\Pi$ is a topological orthoalgebra.

Proof: Continuity of $'$ has already been established. We show first that $\perp \subseteq \Pi^2$ is closed. If $(a, b) \not\in \perp$, then for all $A \in p^{-1}(a)$ and $B \in p^{-1}(b)$, $(A, B) \not\in \perp_E$. The latter is closed, by Lemma 3.3 (a); hence, we can find Vietoris-open neighborhoods $\mathcal{U}$ and $\mathcal{V}$ of $A$ and $B$, respectively, with $(\mathcal{U} \times \mathcal{V}) \cap \perp_E = \emptyset$. Since $p$ is open, $U := p(\mathcal{U})$ and $V := p(\mathcal{V})$ are open neighborhoods of $a$ and $b$ with $(U \times V) \cap \perp_E = \emptyset$. To establish the continuity of $\oplus : \perp \rightarrow \Pi$, let $a \oplus b = c$ and let $A \in p^{-1}(a), B \in p^{-1}(b)$ and $C \in p^{-1}(c)$ be representative events. Note that $A \perp B$ and $A \cup B = C$. Let $W$ be an open set containing $c$: then $W := p^{-1}(W)$ is an open set containing $C$. By Lemma 3.3 (b), $\cup : \perp_E \rightarrow \mathcal{E}$ is continuous;
hence, we can find open sets \( U \) about \( A \) and \( V \) about \( B \) with \( A_1 \cup B_1 \in W \) for every \((A_1, B_1) \in (U \times V) \cap \perp\). Now let \( U = p(U) \) and \( V = p(V) \): these are open neighborhoods of \( a \) and \( b \), and for every \( a_1 \in U \) and \( b_1 \in V \) with \( a_1 \perp b_1, a_1 \oplus b_1 \in p(p^{-1}(W)) = W \) (recalling here that \( p \) is surjective). Thus, \((U \times V) \cap \perp \subseteq \oplus^{-1}(W)\), so \( \oplus \) is continuous. \( \square \)

5. **Semi-classical Test Spaces**

From a purely combinatorial point of view, the simplest test spaces are those in which distinct tests do not overlap. Such test spaces are said to be semi-classical. In such a test space, the relation of perspectivity is the identity relation on events; consequently, the logic of a semi-classical test space \((X, \mathfrak{A})\) is simply the horizontal sum of the boolean algebras \(2^E\), \(E\) ranging over \(\mathfrak{A}\). A state on a semi-classical test space \((X, \mathfrak{A})\) is simply an assignment to each \(E \in \mathfrak{A}\) of a probability weight on \(E\). (In particular, there is no obstruction to constructing “hidden variables” models for states on such test spaces.)

Recent work of D. Meyer [8] and of R. Clifton and A. Kent [2] has shown that the test space \((\mathcal{S}(H), \mathfrak{E}(H))\) associated with a finite-dimensional Hilbert space contains (in our language) a dense semi-classical sub-test space. To conclude this paper, I’ll show that the this result in fact holds for a large and rather natural class of topological test spaces.

**4.1 Lemma:** Let \(X\) be any Hausdorff (indeed, \(T_1\)) space, and let \(U \subseteq X\) be a dense open set. Then \((U) = \{F \in 2^X | F \subseteq U\}\) is a dense open set in \(2^X\).

*Proof:* Since sets of the form \((U_1, ..., U_n), U_1, ..., U_n\) open in \(X\), form a basis for the Vietoris topology on \(2^X\), it will suffice to show that \((U) \cap (U_1, ..., U_n) \neq 0\) for all choices of non-empty opens \(U_1, ..., U_n\). Since \(U\) is dense, we can select for each \(i = 1, ..., n\) a point \(x_i \in U \cap U_i\). The finite set \(F := \{x_1, ..., x_n\}\) is closed (since \(X\) is \(T_1\)), and by construction lies in \((U) \cap (U_1, ..., U_n)\). \( \square \)

**4.2 Corollary:** Let \((X, \mathfrak{A})\) be any topological test space with \(X\) having no isolated points, and let \(E\) be any test in \(\mathfrak{A}\). Then open set \((E^c) = [E]^c\) of tests disjoint from \(E\) is dense in \(\mathfrak{A}\).

*Proof:* Since \(E\) is a closed set, its complement \(E^c\) is an open set; since \(E\) is discrete and includes no isolated point, \(E^c\) is dense. The result follows from the preceding lemma. \( \square \)

**4.3 Theorem:** Let \((X, \mathfrak{A})\) be a topological test space with \(X\) (and hence, \(\mathfrak{A}\)) second countable, and without isolated points. Then there exists a countable, pairwise-disjoint sequence \(E_n \in \mathfrak{A}\) such that (i) \(\{E_n\}\) is dense in \(\mathfrak{A}\), and (ii) \(\bigcup_n E_n\) is dense in \(X\).
Proof: Since it is second countable, \( \mathfrak{A} \) has a countable basis of open sets \( \mathcal{W}_k, k \in \mathbb{N} \). Selecting an element \( F_k \in \mathcal{W}_k \) for each \( k \in \mathbb{N} \), we obtain a countable dense subset of \( \mathfrak{A} \). We shall construct a countable dense pairwise-disjoint subsequence \( \{E_j\} \) of \( \{F_k\} \). Let \( E_1 = F_1 \). By Corollary 4.2, \( [E_1]^c \) is a dense open set; hence, it has a non-empty intersection with \( \mathcal{W}_2 \). As \( \{F_k\} \) is dense, there exists an index \( k(2) \) with \( E_2 := F_{k(2)} \in \mathcal{W}_2 \cap [E_1]^c \). We now have \( E_1 \in \mathcal{W}_1, E_2 \in \mathcal{W}_2, \) and \( E_1 \cap E_2 = \emptyset \). Now proceed recursively: Since \( [E_1]^c \cap [E_2]^c \cap \cdots \cap [E_j]^c \) is a dense open and \( \mathcal{W}_{j+1} \) is a non-empty open, they have a non-empty intersection; hence, we can select \( E_{j+1} = F_{k(j+1)} \) belonging to this intersection. This will give us a test belonging to \( \mathcal{W}_{j+1} \) but disjoint from each of the pairwise disjoint sets \( E_1, ..., E_j \). Thus, we obtain a sequence \( E_j := F_{k(j)} \) of pairwise disjoint tests, one of which lies in each non-empty basic open set \( \mathcal{W}_j \) – and which are, therefore, dense.

For the second assertion, it now suffices to notice that for each open set \( U \subseteq X \), \([U]\) is a non-empty open in \( \mathfrak{A} \), and hence contains some \( E_j \). But then \( E_j \cap U \neq \emptyset \), whence, \( \bigcup_j E_j \) is dense in \( X \). \( \square \)

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