UPPER TRIANGULAR FORMS AND SPECTRAL ORDERINGS IN A II\textsubscript{1}-FACTOR

J. NOLES

Abstract. In [3], Dykema, Sukochev and Zanin used a Peano curve covering the support of the Brown measure of an operator \( T \) in a diffuse, finite von Neumann algebra to give an ordering to the support of the Brown measure, and create a decomposition \( T = N + Q \), where \( N \) is normal and \( Q \) is s.o.t.-quasinilpotent. In this paper we prove that a broader class of measurable functions can be used to order the support of the Brown measure giving normal plus s.o.t.-quasinilpotent decompositions.

1. Introduction and description of results

We start with a famous theorem of Schur (see for instance [7]) which will motivate this paper.

Theorem 1. For every matrix \( T \in M_n(\mathbb{C}) \), there exists a unitary matrix \( U \in M_n(\mathbb{C}) \) such that \( U^{-1}TU \) is an upper triangular matrix.

The diagonal entries of \( U^{-1}TU \) are the eigenvalues of \( T \), repeated up to multiplicity, and \( U \) can be chosen so that they appear in any order. Hence each ordering of the spectrum of \( T \) gives a decomposition \( T = N + Q \), where \( N \) is normal and \( Q \) is nilpotent.

In [3], Dykema, Sukochev and Zanin use Haagerup-Schultz projections to prove a related theorem in II\textsubscript{1}-factors.

Theorem 2. Let \( M \) be a diffuse, finite von Neumann algebra with normal, faithful, tracial state \( \tau \) and let \( T \in M \). Then there exist \( N, Q \in M \) such that

\begin{enumerate}
  \item \( T = N + Q \)
  \item the operator \( N \) is normal and the Brown measure of \( N \) equals that of \( T \)
  \item The operator \( Q \) is s.o.t.-quasinilpotent.
\end{enumerate}

The proof of Theorem 2 uses a Peano curve \( \rho : [0, 1] \to \overline{B_{\|T\|}} \). The normal operator \( N \) is created by taking the trace-preserving conditional expectation onto the von Neumann algebra generated by the Haagerup-Schultz projections of the operator \( T \) associated with the sets \( \rho([0, t]) \) for \( t \in [0, 1] \). These projections, along with the normal operator \( N \), are determined by the ordering on the support of the Brown measure of \( T \) given by \( z_1 \leq z_2 \) if and only if \( \min(\rho^{-1}(z_1)) \leq \min(\rho^{-1}(z_2)) \). Theorem 2 generalizes the idea of using an ordering of the spectrum of the operator \( T \) to write it as an uppertriangular form.

2000 Mathematics Subject Classification. 47C15.
In this paper we will further generalize the idea of spectral orderings from the finite dimensional case to II$_1$-factors. We show that normal plus s.o.t.-quasinilpotent decompositions are generated not only by continuous orderings, but by a large class of measurable orderings.

**Theorem 3.** Let $M$ be a II$_1$-factor and $T \in M$. Let $\nu_T$ be the Brown measure of $T$ and for a Borel set $B \subset \overline{B_{\|T\|}}$, let $P_T(B)$ be the Haagerup-Schultz projection for the operator $T$ associated to the set $B$. Let $\psi : [0, 1] \to \overline{B_{\|T\|}}$ be a measurable function such that $\psi([0, t])$ is Borel for all $t \in [0, 1]$ and

$$\nu_T(\{z \in \overline{B_{\|T\|}} : \psi^{-1}(z) \text{ has a minimum}\}) = 1.$$

Then there exists a spectral measure $E$ supported on $\text{supp}(\nu_T)$ such that

1. $E(\psi([0, t])) = P_T(\psi([0, t]))$ for all $t \in [0, 1]$,
2. $\tau(E(B)) = \nu_T(B)$ for all Borel $B \subset \overline{B_{\|T\|}}$, and
3. $T - \int_C zdE$ is s.o.t.-quasinilpotent.

In particular the conclusion holds if $\psi$ is continuous or is a Borel isomorphism.

Note that part 2 of theorem 3 implies that $\int_C zdE$ and $T$ have the same Brown measure.

**2. Background: Brown measure, Haagerup Schultz projections and s.o.t.-quasinilpotent operators**

This section includes some background necessary for the proof of Theorem 3. Throughout this section $M$ is a II$_1$-factor with trace $\tau$, and $T \in M$.

**Definition 4.** In [2], Brown constructed and proved unique a probability measure $\nu_T$ supported on a compact subset of $\text{spec}(T)$ such that for any $\lambda \in \mathbb{C}$,

$$\tau(\log(|T - \lambda|)) = \int_C \log(|z - \lambda|) d\nu_T(z).$$

$\nu_T$ is called the **Brown measure** of $T$.

In the case that $T$ is normal, Brown's construction gives $\nu_T = \tau \circ E$, where $E$ is the projection valued spectral decomposition measure of $T$.

The following theorem of Haagerup and Schultz is the cornerstone of our proof.

**Theorem 5.** Let $M$ be a II$_1$-factor with trace $\tau$ and let $T \in M$. For every Borel set $B \subset \mathbb{C}$, there exists a unique projection $P_T(B) \in M$ such that

1. $\tau(P_T(B)) = \nu_T(B)$, where $\nu_T$ is the Brown measure of $T$,
2. $TP_T(B) = P_T(B)TP_T(B)$,
3. if $P_T(B) \neq 0$, then the Brown measure of $TP_T(B)$ considered as an element of $P_T(B)M_P T(B)$ is supported in $B$ and
4. if $P_T(B) \neq 1$, then the Brown measure of $(1 - P_T(B))T$, considered as an element of $(1 - P_T(B))M(1 - P_T(B))$, is supported in $\mathbb{C} \setminus B$.

Moreover, $P_T(B)$ is $T$-hyperinvariant and if $B_1 \subset B_2 \subset \mathbb{C}$ are Borel sets, then $P_T(B_1) \leq P_T(B_2)$. 
The projection $P_T(B)$ in theorem 5 is called the Haagerup-Schultz projection of $T$

The following two results, from \[4\] and \[5\], respectively, will be crucial to the proof

**Lemma 6.** For any increasing, right-continuous family of $T$-invariant projections $(q_t)_{0 \leq t \leq 1}$ with $q_0 = 0$ and $q_1 = 1$, letting $\mathcal{D}$ be the von Neumann algebra generated by the set of all the $q_t$ and $\mathcal{D}'$ be the relative commutant of $\mathcal{D}$ in $\mathcal{M}$, and letting $\text{Exp}_{\mathcal{D}'}$ be the $\tau$ preserving conditional expectation, the Fuglede–Kadison determinants of $T$ and $\text{Exp}_{\mathcal{D}'}(T)$ agree. Since the same is true for $T - \lambda$ and $\text{Exp}_{\mathcal{D}'}(T) - \lambda$ for all complex numbers $\lambda$, we have that the Brown measures of $T$ and $\text{Exp}_{\mathcal{D}'}(T)$ agree.

**Theorem 7.** If $T \in \mathcal{M}$, and if $p \in \mathcal{M}$ is a projection such that $Tp = pTp$, so that we may write $T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$, where $A = Tp$ and $C = (1 - p)T$, then

$$\Delta_{\mathcal{M}}(T) = \Delta_{p\mathcal{M}p}(A)^{\tau(p)}\Delta_{(1-p)\mathcal{M}(1-p)}(C)^{\tau(1-p)}$$

and

$$\nu_T = \tau(p)\nu_A + \tau(1-p)\nu_C,$$

where $A$ is considered as an element of $p\mathcal{M}p$ and $C$ is considered as an element of $(1 - p)\mathcal{M}(1 - p)$.

**Definition 8.** It was shown in \[4\] that for any $T \in \mathcal{M}$, $((T^*)^nT^n)^{1/2n}$ converges in the strong operator topology as $n$ approaches $\infty$. An operator $T$ is called s.o.t.-quasinilpotent if $((T^*)^nT^n)^{1/2n} \to 0$ in the strong operator topology as $n \to \infty$.

It was also shown in \[6\] that $T$ is s.o.t.-quasinilpotent if and only if the Brown measure of $T$ is concentrated at 0.

We will also need a characterization from \[6\] of the Haagerup-Schultz projection of $T$ associated with the ball $B_r = \{ |z| \leq r \}$.

9. Suppose $\mathcal{M} \subseteq \mathcal{B}({\mathcal{H}})$. Define a subspace $\mathcal{H}_r$ of $\mathcal{H}$ by

$$\mathcal{H}_r = \{ \xi \in \mathcal{H} : \exists \xi_n \to \xi, \text{ with } \limsup_{n \to \infty} \|T^n\xi_n\|^{1/n} \leq r \}.$$ 

Then the projection onto $\mathcal{H}_r$ is equal to $P_T(B_r)$.

3. Construction of the spectral measure

Throughout this section, $\mathcal{M}$, $T$, $\nu_T$, $P_T$ and $\psi$ will be as described in Theorem 3, $Z$ will denote $\{ z \in B_{||T||} : \psi^{-1}(z) \text{ has a minimum} \}$ and $Y$ will denote $B_{||T||} \setminus Z$.

We first define a Borel measure on the unit interval which will be useful in later proofs.

**Lemma 10.** Let $X = \{ \min(\psi^{-1}(z)) : z \in B_{||T||} \}$. If $b \subseteq [0,1]$ is Borel, then $\psi(b \cap X)$ is Borel.

**Proof.** Note first that, for $t \in (0,1]$, we have $\psi([0,t] \cap X) = \psi([0,t]) \setminus Y$ and $\psi([0,t] \cap X) = \psi([0,t]) \setminus Y$, and these sets are Borel. Now, since $\psi$ restricted to $X$ is an injection, we have $\psi((\alpha,\beta) \cap X) = \psi([0,\beta] \cap X) \setminus \psi([0,\alpha] \cap X)$ which is Borel. Since
[0, 1] is second countable, an arbitrary open set \( v = \bigcup_{n \in \mathbb{N}} u_n \) is the countable union of open intervals so that \( \psi(v \cap X) = \psi(\bigcup_{n \in \mathbb{N}} (u_n \cap X)) = \bigcup_{n \in \mathbb{N}} (\psi(u_n \cap X)) \) is Borel.

To complete the proof, we show that the collection of sets

\[
S = \{ b \subset [0, 1] : \psi(b \cap X) \text{ is Borel} \}
\]

forms a \( \sigma \)-algebra. Suppose that \( \psi(b \cap X) \) is Borel. Then \( \psi(b^c \cap X) = \psi(X \setminus (b \cap X)) = Z \setminus \psi(b \cap X) \) is Borel. Now suppose that \( (b_n)_{n \in \mathbb{N}} \subset S \). Then \( \bigcup_{n \in \mathbb{N}} b_n \in S \) by the same argument used for open sets, and we are done. \( \square \)

We now define \( \mu(b) = \nu_T(\psi(b \cap X)) \) for any Borel set \( b \subset [0, 1] \). It is clear that \( \mu \) is countably additive, and hence a Borel probability measure on \( [0, 1] \). That \( \mu \) is a regular measure follows from Theorem 1.1 of [1].

**Observation 11.** For any Borel set \( B \subset B_{\|T\|}, \mu(\psi^{-1}(B)) = \nu_T(B) \).

**Proof.** Since \( \psi \) is a bijection from \( X \) to \( Z \) we have

\[
\mu(\psi^{-1}(B)) = \nu_T(\psi(\psi^{-1}(B) \cap X)) = \nu_T(B \cap Z) = \nu_T(B)
\]

\( \square \)

Prior to constructing the spectral measure, we will need a map from the open subsets of the closed unit interval to the set of projections in \( \mathcal{M} \). For an open interval, define

\[
F(\emptyset) = 0
\]

\[
F((\alpha, \beta)) = P_T(\psi([0, \beta))) - P_T(\psi([0, \alpha]))
\]

\[
F([0, \beta)) = P_T(\psi([0, \beta]))
\]

\[
F((\alpha, 1]) = 1 - P_T(\psi([0, \alpha]))
\]

Since \( P_T(\psi([0, t])) \) and \( P_T(\psi([0, t])) \) are increasing in \( t \), it follows that \( F(u) \) is increasing in \( u \), and \( F(u_1)F(u_2) = 0 \) if \( u_1 \cap u_2 = \emptyset \). For \( u_1 = (\alpha_1, \beta_1) \) and \( u_2 = (\alpha_2, \beta_2) \) with \( \alpha_1 \leq \alpha_2 \leq \beta_1 \leq \beta_2 \),

\[
F(u_1)F(u_2) = (P_T(\psi([0, \beta_1])) - P_T(\psi([0, \alpha_1])))(P_T(\psi([0, \beta_2])) - P_T(\psi([0, \alpha_2])))
\]

\[
= P_T(\psi([0, \beta_1])) - P_T(\psi([0, \alpha_2])) - P_T(\psi([0, \alpha_1])) + P_T(\psi([0, \alpha_1]))
\]

\[
= F(u_1 \cap u_2).
\]

Hence for any open intervals \( u_1 \) and \( u_2 \), \( F(u_1)F(u_2) = F(u_1 \cap u_2) \).

For an arbitrary open set \( v \subset [0, 1] \), we first write \( v = \bigcup_{n \in \mathbb{N}} u_n \), where the \( u_n \) are pairwise disjoint, and all nonempty \( u_n \) are open intervals. Then \( \sum_{n \in \mathbb{N}} F(u_n) \) converges to a projection in the strong operator topology. We define \( F(v) = \sum_{n \in \mathbb{N}} F(u_n) \). Multiplication of the series and application of the corresponding result for intervals gives us \( F(v_1)F(v_2) = F(v_1 \cap v_2) \) for open sets \( v_1, v_2 \subset [0, 1] \).

**Observation 12.** For any open \( v \subset [0, 1] \), \( \tau(F(v)) = \mu(v) \).
Proposition 13. For any Borel set \( E \), we will prove later that \( \mu \) defines a spectral measure. We are now ready to define the spectral measure \( E \). For any Borel set \( B \subset B_{||T||} \), define

\[
E(B) = \wedge \{ F(v) : v \text{ is open and } \psi^{-1}(B) \subset v \}.
\]

Note that \( E \) is increasing and that the range of \( E \) is contained in the von Neumann algebra generated by the projections \( P_T(\psi([0,t])) \) for \( t \in [0,1] \), which is commutative. We will prove later that \( E \) defines a spectral measure.

**Proposition 13.** For any Borel set \( B \subset B_{||T||} \), \( \tau(E(B)) = \nu_T(B) \).

**Proof.** Let \( \epsilon > 0 \) be given. There exist open sets \( v_1, v_2 \subset [0,1] \) such that

1. \( \psi^{-1}(B) \subset v_1 \) and \( \mu(v_1) - \mu(\psi^{-1}(B)) < \epsilon \), and
2. \( \psi^{-1}(B) \subset v_2 \) and \( \tau(F(v_2)) - \tau(E(B)) < \epsilon \).

Applying Observations 11 and 12 to (1), we have

\[
\tau(E(B)) - \nu_T(B) \leq \tau(F(v_1)) - \nu_T(B) = \mu(v_1) - \mu(\psi^{-1}(B)) < \epsilon.
\]

Applying Observations 11 and 12 to (2) gives

\[
\nu_T(B) - \tau(E(B)) = \mu(\psi^{-1}(B)) - \tau(E(B)) \leq \mu(v_2) - \tau(E(B)) = \tau(F(v_2)) - \tau(E(B)) < \epsilon.
\]

Hence we have \( |\tau(E(B)) - \nu_T(B)| < \epsilon \), and we are done. \( \square \)

**Lemma 14.** If \( B_1 \) and \( B_2 \) are Borel subsets of \( B_{||T||} \), then \( E(B_1)E(B_2) = E(B_1 \cap B_2) \).

**Proof.** Noting that whenever \( v_1 \) is an open set containing \( \psi^{-1}(B_1) \) and \( v_2 \) is an open set containing \( \psi^{-1}(B_2) \), \( v_1 \cap v_2 \) is an open superset of \( \psi^{-1}(B_1) \cap \psi^{-1}(B_2) \), we have

\[
E(B_1 \cap B_2) = \wedge \{ F(v) : v \text{ open, } \psi^{-1}(B_1 \cap B_2) \subset v \}
\]

\[
= \wedge \{ F(v) : v \text{ open, } \psi^{-1}(B_1) \cap \psi^{-1}(B_2) \subset v \}
\]

\[
\leq \wedge \{ F(v_1 \cap v_2) : v_1, v_2 \text{ open, } \psi^{-1}(B_1) \subset v_1, \psi^{-1}(B_2) \subset v_2 \}
\]

\[
= \wedge \{ F(v_1)F(v_2) : v_1, v_2 \text{ open, } \psi^{-1}(B_1) \subset v_1, \psi^{-1}(B_2) \subset v_2 \}
\]

\[
= \wedge \{ F(v_1) : v_1 \text{ open, } \psi^{-1}(B_1) \subset v_1 \} \wedge \{ F(v_2) : v_2 \text{ open, } \psi^{-1}(B_2) \subset v_2 \}
\]

\[
= E(B_1)E(B_2).
\]

Now let \( \epsilon > 0 \) be given. There exist open subsets \( v, \tilde{v}_1, \tilde{v}_2 \) of \([0,1]\) such that

1. \( \psi^{-1}(B_1 \cap B_2) \subset v \) and \( \mu(v \setminus \psi^{-1}(B_1 \cap B_2)) < \epsilon \),
2. \( a_1 = \psi^{-1}(B_1) \setminus \psi^{-1}(B_1 \cap B_2) \subset \tilde{v}_1 \) and \( \mu(\tilde{v}_1 \setminus a_1) < \epsilon \), and
3. \( a_2 = \psi^{-1}(B_2) \setminus \psi^{-1}(B_1 \cap B_2) \subset \tilde{v}_2 \) and \( \mu(\tilde{v}_2 \setminus a_2) < \epsilon \).
Let \( v_1 = \tilde{v}_1 \cup v \) for \( i = 1, 2 \). Then \( v_1 \) is an open set containing \( \psi^{-1}(B_1) \) and \( v_2 \) is an open set containing \( \psi^{-1}(B_2) \). We have

\[
\mu(v_1 \cap v_2 \setminus \psi^{-1}(B_1 \cap B_2)) \leq \mu(v \setminus \psi^{-1}(B_1 \cap B_2)) + \mu(\tilde{v}_1 \cap \tilde{v}_2 \setminus \psi^{-1}(B_1 \cap B_2)).
\]

Observing that \( a_1 \cap a_2 = \emptyset \) and

\[
\tilde{v}_1 \cap \tilde{v}_2 = (a_1 \cap a_2) \cup ((\tilde{v}_1 \setminus a_1) \cap a_2) \cup ((\tilde{v}_2 \setminus a_2) \cap a_1) \cup ((\tilde{v}_1 \setminus a_1) \cap (\tilde{v}_2 \setminus a_2))
\]

we have

\[
\mu((v_1 \cap v_2) \setminus \psi^{-1}(B_1 \cap B_2)) < 4\epsilon.
\]

Applying Observations 11 and 12 and Proposition 13, we have

\[
\tau(E(B_1)E(B_2)) - \tau(E(B_1 \cap B_2)) \leq \tau(F(v_1)F(v_2)) - \tau(E(B_1 \cap B_2))
\]

\[
= \tau(F(v_1 \cap v_2)) - \tau(E(B_1 \cap B_2)) < 4\epsilon,
\]

and we conclude \( E(B_1)E(B_2) = E(B_1 \cap B_2) \). \( \square \)

**Lemma 15.** \( E \) is countably additive on disjoint sets, where convergence of the series is in the strong operator topology.

**Proof.** Suppose \( (B_n)_{n \in \mathbb{N}} \) is a countable collection of disjoint Borel subsets of \( B_{||T||} \). By claim 7, \( E(B_i)E(B_j) = 0 \) if \( i \neq j \). Then \( E(\bigcup_{n \in \mathbb{N}} B_n) \) is a superprojection of each \( E(B_n) \), and hence a superprojection of \( \sum_{n \in \mathbb{N}} E(B_n) \). Also, \( \tau(E(\bigcup_{n \in \mathbb{N}} B_n)) = \nu_T(\bigcup_{n \in \mathbb{N}} B_n) = \tau(\sum_{n \in \mathbb{N}} E(B_n)) \). We conclude \( E(\bigcup_{n \in \mathbb{N}} B_n) = \sum_{n \in \mathbb{N}} E(B_n) \). \( \square \)

We are now ready to show that \( E \) is a spectral measure supported on \( \text{supp}(\nu_T) \).

**Proof.** We must show three things:

1. \( E(\emptyset) = 0 \) and \( E(\text{supp}(\nu_T)) = 1 \)
2. \( E(B_1 \cap B_2) = E(B_1)E(B_2) \) for Borel sets \( B_1, B_2 \), and
3. if \( \mathcal{M} \) acts on a Hilbert space \( \mathcal{H} \), and \( x, y \in \mathcal{H} \), then \( \eta(B) = \langle E(B)x, y \rangle \) defines a regular Borel measure on \( C \).

1. Follows from Proposition 13, since \( \tau(E(\emptyset)) = 0 \) and \( \tau(E(\text{supp}(\nu_T))) = 1 \).
2. Was proven as Lemma 14.
3. That \( \eta \) is countably additive on disjoint sets follows from Lemma 15. Regularity of \( \eta \) follows from Theorem 1.1 of [1]. \( \square \)

**4. Proof of Theorem 3**

We first establish several results which will be used to prove Part 3. Throughout this section, \( \mathcal{M}, T, \) and \( \psi \) are as described in Theorem 3, and \( \mu, E \) and \( E_n \) are as defined in Section 3. \( \mathcal{M} \) acts on a Hilbert space \( H \).

We now show that \( \int_C zdE \) is the norm limit of conditional expectations onto an increasing sequence of abelian von Neumann algebras. For each \( n \), divide the \( 3\|T\| \) by \( 3\|T\| \) square centered at 0 into \( 2^n \) by \( 2^n \) squares of equal size indexed \( (A_{n,k})_{k=1}^{2^n} \), \( k \) increasing to the right then down. Include in each \( A_{n,k} \) the top and left edge, excluding the bottom-left and top-right corners, so that for each \( n \), \( A_{n,k} \cap A_{n,j} = \emptyset \)
whenever \( j \neq k \) and \( \overline{B_{\|T\|}} \subset \cup_{k=1}^{2^n} A_{n,k} \). Let \( D_n \) be the von Neumann algebra generated by the (orthogonal) projections \( (E(A_{n,k}))_{k=1}^{2^n} \).

**Proposition 16.** Let \( \mathbb{E}_{D_n}(T) \) denote the conditional expectation of \( T \) onto \( D_n \). Then \( \mathbb{E}_{D_n}(T) \) converges in norm as \( n \to \infty \) to \( \int_C z dE \).

**Proof.** Observe that

\[
\mathbb{E}_{D_n}(T) = \sum_{1 \leq k \leq 2^n} \frac{\tau(E(A_{n,k})TE(A_{n,k}))}{\tau(E(A_{n,k}))} E(A_{n,k}).
\]

Applying Brown’s analog of Lidskii’s theorem (see [2]) gives

\[
\mathbb{E}_{D_n}(T) = \sum_{1 \leq k \leq 2^n} \frac{\int_{A_{n,k}} z d\nu_T(z)}{\nu_T(A_{n,k})} E(A_{n,k}).
\]

For each \( n \), define

\[
f_n(w) = \sum_{1 \leq k \leq 2^n} \frac{\int_{A_{n,k}} z d\nu_T(z)}{\nu_T(A_{n,k})} \chi_{A_{n,k}}(w) + \sum_{1 \leq k \leq 2^n} \frac{\int_{A_{n,k}} z dm(z)}{m(A_{n,k})} \chi_{A_{n,k}}(w),
\]

where \( m \) is the Lebesgue measure on \( C \).

Since \( \nu_T(A_{n,k}) = 0 \) implies \( E(A_{n,k}) = 0 \), \( \int_C f_n dE = \mathbb{E}_{D_n}(T) \). Note that \( f_n \) converges uniformly on \( \text{supp}(E) \) to the inclusion function \( f(z) = z \). Hence \( \int_C f_n dE \) converges in norm to \( \int_C z dE \), and we are done. \[ \Box \]

Let \( D \) be the von Neumann algebra generated by \( (E(\psi([0, t])))_{t \in [0,1]} \) (or equivalently by \( \bigcup_{n=1}^{\infty} D_n \)).

**Proposition 17.** Suppose that \( T \in D' \) and \( B \subset \overline{B_{\|T\|}} \) is Borel with \( \nu_T(B) \neq 0 \). Then the Brown measure of \( E(B)TE(B) \), considered as an element of \( E(B)\mathcal{M}E(B) \), is concentrated in \( B \).

**Proof.** We begin by observing that for any open \( v \subset [0,1] \), with \( \tau(F(v)) \neq 0 \), \( F(v) \in D \) and if \( v = (\alpha, \beta) \) is an open interval, then \( \nu_{TF(v)} \) is concentrated in \( \psi([0, \beta]) \setminus \psi([0, \alpha]) \), and hence is also concentrated in \( \psi((\alpha, \beta)) \cap Z \), where \( Z \) is as described in Section 3. Thus \( \nu_{TF(v)} \) is concentrated in \( \psi((\alpha, \beta) \cap X) \).

Now suppose that \( v = \bigcup_{n=1}^{\infty} u_n \) where all nonempty \( u_n \) are pairwise disjoint open intervals. Let \( \epsilon > 0 \) be given. Let \( N \) be so large that

\[
\tau \left( \sum_{n=1}^{N} F(u_n) \right) > \tau(F(v))(1 - \epsilon).
\]

Then, since each \( F(u_n) \) commutes with \( T \), Theorem 7 gives

\[
\nu_{TF(v)} = \frac{1}{\tau(F(v))} \left( \sum_{n=1}^{N} \tau(F(u_n)) \nu_{TF(u_n)} + \tau \left( \sum_{n=N+1}^{\infty} F(u_n) \right) \nu(\sum_{n=N+1}^{\infty} F(u_n))T \right).
\]
Hence, since each $\nu_{TF(u_n)}$ is concentrated in $\psi(u_n \cap X) \subset \psi(v \cap X)$, we have

$$\nu_{TF(v)}(\psi(v \cap X)) \geq \frac{1}{\tau(F(v))} \left( \sum_{n=1}^{N} \tau(F(u_n)) \right) \nu_{TF(u_n)}(\psi(v \cap X)) > 1 - \epsilon,$$

so that $\nu_{TF(v)}$ is concentrated in $\psi(v \cap X)$.

Now observe that when $v$ is an open set containing $\psi^{-1}(B)$, since

$$\nu_{TF(v)} = \frac{1}{\tau(F(v))} (\tau(E(B))\nu_{TE(B)} + \tau(F(v) - E(B))\nu_{(F(v) - E(B))T}),$$

$\nu_{TE(B)}$ is concentrated in $\psi(v \cap X)$.

Choose an open set $v \subset [0, 1]$ such that $\psi^{-1}(B) \subset v$ and $\mu(v) - \mu(\psi^{-1}(B)) < \epsilon$. Then using Theorem 7 and Observation 11,

$$\epsilon > \nu_T(\psi(v \cap X)) - \nu_T(B)$$

$$= \tau(E(B))\nu_{TE(B)}(\psi(v \cap X) \setminus B) + (1 - \tau(E(B)))\nu_{(1-E(B))T}(\psi(v \cap X) \setminus B)$$

$$\geq \tau(E(B))\nu_{TE(B)}(\psi(v \cap X) \setminus B).$$

Hence

$$\tau(E(B)) - \epsilon < \tau(E(B))(1 - \nu_{TE(B)}(\psi(v \cap X) \setminus B)) = \tau(E(B))(\nu_{TE(B)}(B)).$$

Thus

$$1 - \frac{\epsilon}{\tau(E(B))} < \nu_{TE(B)}(B).$$

Letting $\epsilon$ tend to 0 gives the desired result. 

\[ \square \]

**Lemma 18.** If $T \in D'$, then the Brown measure of $T - \mathbb{E}_{D_n}(T)$ is supported in the ball of radius $\frac{6\sqrt{2}\|T\|}{2n}$. 

**Proof.** The key observation is that for any $\alpha \in \mathcal{C}$, if $\nu_{T-\alpha}$ is the Brown measure of $T-\alpha$, then for any Borel set $B \subset \mathcal{C}$, $\nu_{T-\alpha}(B) = \nu_T(B-\alpha)$. Since whenever $E(A_{n,k}) \neq 0$ the Brown measure of $TE(A_{n,k})$ is supported in $A_{n,k}$, the Brown measure of $(T - \tau(TE(A_{n,k})))E(A_{n,k})$ is supported in the square centered at 0 with edge length $\frac{6\sqrt{2}\|T\|}{2n}$. We complete the proof by observing that $T - \mathbb{E}_{D_n}(T) = \sum_{k=1}^{2n} \left( T - \frac{\tau(TE(A_{n,k}))}{\tau(E(A_{n,k}))} \right) E(A_{n,k})$ and applying Theorem 7 to compute the Brown measure of the sum. \[ \square \]

We now are ready to prove Theorem 3.

**Proof.** (1) Whenever $v$ is an open set containing $\psi^{-1}(\psi([0, t]))$, there exists $\epsilon > 0$ such that $[0, t + \epsilon] \subset v$ so we see that

$$P_T(\psi([0, t])) \leq F([0, t + \epsilon]) \leq F(v).$$

Hence we see that

$$P_T(\psi([0, t])) \leq E(\psi([0, t])).$$

By Proposition 13 and Theorem 5,

$$\tau(P_T(\psi([0, t]))) = \tau(E(\psi([0, t])))$$
so that
\[ P_T(\psi([0, t])) = E(\psi([0, t])). \]

(2) Was proven as Proposition 13.

(3) By Lemma 6, it suffices to assume that \( T \in D' \). We first observe from the proof of Proposition 16 that \( \| E_D(T) - E_{D_n}(T) \| \leq \frac{3\sqrt{2}\|T\|}{2^n} \). The rest of this argument is taken from the proof of Lemma 24 in [3].

We assume without loss of generality that \( \| T \| \leq 1/2 \). Fix \( n \in \mathbb{N} \) and a unit vector \( \xi \in H \). By assumption \( T \in D' \), so we have
\[
(T - E_D(T))^{2m} = \sum_{k=0}^{2m} (-1)^k \binom{2m}{k} (E_D(T) - E_{D_n}(T))^{2m-k} (T - E_{D_n}(T))^k.
\]

Since \( \| T \| \leq 1/2 \), both \( E_D(T) - E_{D_n}(T) \) and \( T - E_{D_n}(T) \) are contractions. For \( k \leq m \) and any \( \eta \in H \), we have
\[
\| (E_D(T) - E_{D_n}(T))^{2m-k} (T - E_{D_n}(T))^k \eta \|_H \leq \| E_D(T) - E_{D_n}(T) \|^m \| \eta \|_H.
\]
For \( k > m \) and any \( \eta \in H \) we have
\[
\| (E_D(T) - E_{D_n}(T))^{2m-k} (T - E_{D_n}(T))^k \eta \|_H \leq \| (T - E_{D_n}(T))^{m+1} \eta \|_H.
\]
Hence for any \( \eta \in H \),
\[
\| (T - E_D(T))^{2m} \|_H \leq 2^{2m} \max \left\{ \left( \frac{3\sqrt{2}\|T\|}{2^n} \right)^m, \| (T - E_{D_n}(T))^{m+1} \|_H \right\}.
\]

By Lemma 18, the Brown measure of \( T - E_{D_n}(T) \) is supported in the ball of radius \( \frac{6\sqrt{2}\|T\|}{2^n} \) centered at 0. By the Haagerup-Schultz characterization (9), there exists a sequence \( \xi_m \to \xi \) such that \( \| \xi_m \|_H = 1 \) and
\[
\limsup_{m \to \infty} \| (T - E_{D_n}(T))^{m} \xi_m \|_H^{1/m} \leq \frac{6\sqrt{2}\|T\|}{2^n}.
\]
Hence there exists \( M \) (depending on \( n \)) such that
\[
\| (T - E_{D_n}(T))^{m} \xi_m \|_H \leq \left( \frac{7\sqrt{2}\|T\|}{2^n} \right)^m, \quad m > M.
\]
Taking \( \eta = \xi_m \) in (1), we have
\[
\| (T - E_D(T))^{2m} \xi_m \|_H^{1/m} \leq \frac{28\sqrt{2}\|T\|}{2^n}, \quad m > M.
\]
Since \( \xi \) was arbitrary, it follows from characterization (9) that the Brown measure of \( (T - E_D(T))^2 \) is supported in the ball of radius \( \frac{28\sqrt{2}\|T\|}{2^n} \) centered at 0. Letting \( n \to \infty \), we obtain that the Brown measure of \( T - E_D(T) \) is \( \delta_0 \). \( \square \).
References

[1] P. Billingsley, *Convergence of Probability Measures*, John Wiley and Sons, New York, 1968.

[2] L. G. Brown, *Lidskii’s theorem in the type II case*, Geometric methods in operator algebras (Kyoto, 1983), Pitman Res. Notes Math. Ser., vol. 123, Longman Sci. Tech., Harlow, 1986, pp. 1–35.

[3] K. Dykema, F. Sukochev, and D. Zanin, *A decomposition theorem in II$_1$–factors*, J. reine angew. Math., to appear, available at http://arxiv.org/abs/1302.1114

[4] [Holomorphic Functional Calculus on Upper Triangular Forms in Finite von Neumann Algebras](http://arxiv.org/abs/1310.2524), preprint, available at http://arxiv.org/abs/1310.2524

[5] U. Haagerup and H. Schultz, *Brown measures of unbounded operators affiliated with a finite von Neumann algebra*, Math. Scand. 100 (2007), 209–263.

[6] [Invariant subspaces for operators in a general II$_1$–factor](http://arxiv.org/abs/1310.2524), Publ. Math. Inst. Hautes Études Sci. 109 (2009), 19-111.

[7] F. Zheng, *Matrix Theory: Basic results and techniques*, Second edition, Universitext, Springer, New York, 2011.

Department of Mathematics, Texas A&M University, College Station, TX, USA. 
E-mail address: jnoles@math.tamu.edu