Understanding Discharge Voltage Inconsistency in Lithium-Ion Cells via Statistical Characteristics and Numerical Analysis

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\textbf{ABSTRACT} In electric vehicles (EVs), owing to the necessity of large amounts of energy and power, lithium-ion batteries need to be used in series and parallel configurations. However, the performance of the battery pack is lower than that of any single cell within the pack because of the inconsistency among the packed cells. Herein, the inconsistent voltages of unpacked cells due to varying capacities during discharge are analyzed to provide mechanical reason for inconsistency of battery pack. In terms of dispersion and symmetry, the statistical characteristics of voltage distribution are described using Weibull parameters and is investigated using a numerical analysis of the characteristic voltage curve. The numerical analysis results agree well with the experimental and statistical ones, which confirms that voltage inconsistency originating from manufacturing processes is primarily related with capacity inconsistency and the features of the voltage curves. Furthermore, this numerical approach can provide not only significant theoretical insight into the formation and evaluation of voltage inconsistency; but also practical guidance for controlling the quality of cell production and state estimation for the battery pack due to its low computational cost.

\textbf{INDEX TERMS} Voltage inconsistency, inconsistency model, weibull probability model, 4-d probability nephogram, dispersion and symmetry of voltage distribution.

I. INTRODUCTION

To fulfill the requirements of high-energy and high-power EVs, lithium-ion batteries need to be packed in series and parallel configurations \cite{1}, \cite{2}. The inconsistency among individual packed cells results in lower battery pack performance compared with individual cells and decreases the estimation accuracy of the battery-management system (BMS) \cite{3}. However, the inconsistency in packed cells is inevitable and increases with cycling, wherein cells with inconsistent performances experience different current or voltage loads; this phenomenon directly leads to a spread of cell aging rates \cite{4} as well as with safety issues \cite{5}. In addition, the performance of a battery pack is directly influenced by the inconsistency of the cells within \cite{6}; thus, the performance relationship of the inconsistent in-pack cells and the entire battery pack is closely associated and should not be ignored \cite{7}, \cite{8}. Moreover, the relationship between the inconsistent performance parameters of cells is the key to predicting inconsistent characteristics of unpacked cells \cite{9} or battery packs \cite{10}, \cite{11}. Because of the enormous amount of inconsistency information and the limited computing capacity, the weakest cells in packs are regarded as the determining factors for battery pack performance \cite{12}–\cite{14}, which barely ensures battery reliability control and prediction. Besides, the inconsistent cells need larger space for data storage during operation in EVs, bringing challenges to the data transmitting and storage \cite{15}–\cite{17}. Therefore, it is significantly important to comprehensively understand the formation and evolution of cell inconsistency in a simplified and accurate way.

Inconsistency information of in-pack cells is random, which primarily originates from the source of manufacturing...
uncertainty. Experimental measurement and numerical simulation results have been proposed to understand the formation of cell inconsistency caused by the manufacturing process from the perspective of electrochemical theory [18]–[21]. Constant-current and constant-voltage discharge are common testing approaches adopted to extract non-uniform performance parameters from a batch of cells. Dubarry [22] separates the source of cell inconsistency into thermodynamic and kinetic factors based on different discharge rates; the results of that study revealed that discharge behavior deviates because of inconsistent capacity at a low discharge rate and because of resistance at a high discharge rate. An [19] traces the thermodynamic and kinetic factors to their detailed origins in the cell manufacturing process. Rumpf [20] qualifies the discharge behavior variation based on capacity variation, which is determined using resistance. Hence, the discharge-voltage inconsistency is found to be mainly caused by capacity and resistance in unpacked cells. To trace the origin of cell inconsistency to the manufacturing process, numerical simulation methods have been applied to model the formation of cell inconsistency. The detailed inconsistency evolution of discharge behavior is displayed by Edourad [23], wherein voltage variation in 10 cells is investigated through the Newman electrochemistry model with equipspaced input parameters. Santhanagopalan and White [21] simulates various AC impedance spectroscopies with varying porosities, tortuosities, and particle sizes of electrodes as inconsistency inputs. Kenney [24] simulates the initial and aging capacity inconsistency with uncertain electrode parameters of thickness, the weight fraction of the active material, and area density. The issue of cell inconsistency can be quantificationally understood via numerical simulation, and the inconsistency characteristic can be explained in terms of electrode structure and electrochemical process. However, research conducted via testing and simulation are isolated, and they have failed to unify the experimental and numerical analysis of cell inconsistency.

Statistical method are also used to characterize the inconsistency in cell parameters for both unpacked and packed cells. The Normal Model, as the most widely used probabilistic model, can analyze a symmetrically distributed set of data [22], in which the differences between the average value and the maximum and minimum values are equal. The probability density function (PDF) of Normal Model is used to analyze inconsistent capacity [19], [24], resistance [19], and voltage [23], [25] by considering the parameters to be a symmetric distribution. However, because lithium-ion cells comprise a nonlinear system, their parameter distribution is not always symmetrical. When the differences between the average value and the maximum and minimum values are unequal, the eigenvalues of Normal Model cannot capture this asymmetrical characteristic. Regardless of whether the cells are packed, the incomplete statistical data from Normal Model result in inaccurate probability distributions of inconsistent parameters, particularly in terms of resistance and capacity [26], [27]. Fortunately, this issue can be solved using Weibull Model. Owing to the shape parameter in Weibull Model, the probability distribution of life data with a symmetric or an asymmetric shape is easy to identify, which is an advantage over Normal Model. The PDF of Weibull Model is introduced to assess life reliability based on the inconsistencies in parameters including capacity, resistance, energy, and power [28]–[31]. Chiodo [30] statistically analyzes life distribution via accelerated life testing, the results show that the distribution of peak power of the cells is obvious right-skew distributed and those of energy and life are left-skew distributed, divergent from normal distribution. Harris [27] applies two- and three-parameter Weibull Models on cell inconsistency. The cell-to-cell variation is observed to be assuredly asymmetric and more similar using Weibull distribution. Schuster [26] indicated that two-parameter Weibull distribution is a good function for degraded capacity and resistance variation, and the impact of resistance on capacity is revealed through the shape of Weibull Model. However, with respect to the statistical characteristic of performance parameters, the three-parameter Weibull Model can provide more detailed and more accurate information on cell inconsistency.

In summary, experimental testing and numerical simulation enable the analysis of cell inconsistency characteristics from the perspective electrochemistry theory; however, the isolation between the two types of methods limits their validity and implementation of the conclusion for battery packs. Regarding the statistical model used to describe inconsistency information, the Weibull Model performs better than the Normal Model, and three-parameter Weibull Model is a promising approach for providing detailed information on cell inconsistency. Additionally, the understanding of the relationship is still experimental and cannot be quantified; hence, its further application, such as that to the state of charge (SOC) algorithm of battery packs considering the inconsistency of in-packed cells, is limited. Therefore, constructing an accurate and effective numerical analysis approach is crucial to understand the inconsistency characteristics and to further optimizing SOC estimation models the general.

Herein, an inconsistency model is proposed based on the relationship between voltage and capacity inconsistency, considering statistical and electrochemical theories, results of which agrees well with the experimental results and quantitatively explain the formation and evolution of voltage distribution during discharge. Besides, the statistical characteristics of discharge voltage are represented by three-parameter Weibull Model than Normal Model. Furthermore, a visual 4-dimension (4-D) probability nephogram is presented for an intuitive observation of voltage distribution during discharge, and the parameters of characteristic voltage and characteristic capacity are firstly defined to replace the average of symmetry distribution, which can provide a more accurate cognition on statistical information of asymmetry voltage and capacity distribution. The research results can not only contribute to theoretical understanding of the issue of inconsistency at the cell level but also expect to be applied to control the quality...
of produced cells and optimize the SOC estimation model of abattery pack at system level.

II. EXPERIMENT AND METHODS
A. EXPERIMENT
The practical tested samples are 80 format LiNi$_{0.5}$Co$_{0.2}$Mn$_{0.3}$ O$_2$/Graphite pouch cells with a nominal capacity of 25 Ah, produced by the China Automotive Battery Research Institute LLC. They were randomly selected from 33000 cells produced in the same batch. The regime of the discharge voltage curves has a galvanometric C/3 rate from 4.2 V to 2.8 V at an ambient temperature of 27.0 ± 0.3°C.

Depending on the testing sample cells selected from a batch of produced cells, the inconsistent discharge voltage curves are simulated using the Newman Electrochemical model [32]–[34]. The simulated result is regarded as the object for the statistical and numerical analysis, wherein hardly any statistical error is introduced because of the environment and manual operation. The simulated results are verified by the average of experimental voltage curves, as shown in Fig. 4.

B. STATISTICAL CHARACTERISTICS
The three-parameter Weibull Model [33], [35] is used to analyze the inconsistent capacity and voltage of the cells. The Weibull PDF is presented in Eq. 1, and Weibull cumulative probability function (CPF) in Eq. 2.

$$f(x) = \begin{cases} \frac{B}{A} \left( \frac{x-C}{A} \right)^{B-1} e^{-\left( \frac{x-C}{A} \right)^{B}} & x \geq C \\ 0 & x < C \end{cases}$$  \hspace{1cm} (1)

$$F(x) = \begin{cases} 1 - e^{-\left( \frac{x-C}{A} \right)^{B}} & x \geq C \\ 0 & x < C \end{cases}$$  \hspace{1cm} (2)

where, $A$ is the scale parameter; $B$ is the shape parameter; and $C$ is the location parameter.

Fig. 1 clearly explains the meaning of the three parameters in the Weibull Model. The scale parameter $A$ in Eq. 1 represents the difference between the starting point and the peak of the Weibull PDF, which is used to characterize the dispersion of the data distribution. As shown in Fig. 1(a), the CPF is constantly 63.7% at the point of $x = A$, and the probability density function widens with $A$ increasing. The shape parameter $B$ varies with the location of the PDF peak, which is used to characterize the symmetry of the data distribution. As shown in Fig. 1(b), when $1 < B < 2.6$, the CPF increases faster before $x = A$ than after $x = A$ and PDF peak position appeared on the left of the mid-value and closer to the minimum; the distribution is thus left-skewed. When $B > 3.7$, the CPF increases more slowly before $x = A$ than after $x = A$ and PDF peak position appeared on the right of the mid-value and closer to the maximum; the distribution is thus right-skewed. When $2.6 < B < 3.7$, the peak of the PDF appeared approximatively in the middle, and the CPF increases with the same rate, thus the distribution is symmetric and similar to normal distribution. When $B \leq 1$, the distribution is exponential without a peak, which is not discussed in this study. The location parameter $C$ in Fig. 1(c) represents the starting point of the Weibull PDF and CPF. When $x = C$, the PDF and the CPF are equal to 0. [36] Because of the introduction of C, the Weibull Model can be adapted for capacity and voltage distributions.

In the analysis of the discharge curves, the Weibull Model was utilized for analyzing the voltage and capacity distributions, and these parameters are estimated via the maximum likelihood estimation [37], [38]. The parameters of voltage distribution are marked as $A_{Whd}$, $B_{Whd}$, and $C_{Whd}$, and those of capacity distribution marked as $A_{Whd}(C_i)$, $B_{Whd}(C_i)$, $C_{Whd}(C_i)$.

C. NUMERICAL ANALYSIS
In this section, we explore the formation of the voltage dynamic distribution by focusing on the relationship between voltage and capacity. Firstly, the probability theorem, according to which the derivatives of random variables are also random [39], is adopted to interpret the characteristics of...
random variables in a numerical manner in terms of dispersion and symmetry. Secondly, a mathematical relationship is established to recover the voltage curve of any cell in a same batch by introducing the characteristic voltage curve \(V_{0}\) and capacity \(C_{0}\) for the batch of cells. Thirdly, the inconsistency model combining dispersion and symmetry parameters is constructed via the derivatives of voltage function with various capacity, which contributes to understanding the statistical characters via Weibull Model, by combining of the probability and electrochemistry theory.

1) THEOREM OF FUNCTION PROBABILITY

Let us consider the issue in terms of a simple function \(h\), which is defined as a monotone function for the sake of convenience. Variable \(y\) can be obtained using the function \(h\) of variable \(x\), as shown in Eq. 3.

\[
y = h(x)
\]

Transforming the variables into sets, \(X\) and \(Y\) are defined as the sets of random variables \(x\) and \(y\). When the distribution of \(X\) is known, \(Y\) can be calculated via the mapping presented in Eq. 4.

\[
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n \\
\end{pmatrix} \rightarrow 
\begin{pmatrix}
h(x_1) \\
h(x_2) \\
\vdots \\
h(x_n) \\
\end{pmatrix}
\]

As shown in Fig. 2, to understand the effect of function \(h\) on the distribution of \(Y\), three examples are considered and the distribution feature of these three \(Y\) sets can be compared, which are mapped from the same \(X\) set, as a normal distribution between 0 and 1 with a mean (marked as \(\mu\)) of 0.5 and standard deviation (marked as \(\sigma\)) of 0.166. In Fig. 2, seven specific variables in \(X\), marked as red scatters, are placed with the center \(\mu\) and equidifference of \(\sigma\). Transformed into \(Y_1, Y_2, \) and \(Y_3\), the probability density of the red scatters is calculated based on Eq. 5 [40], the theory of random variable function probability.

\[
f_Y = f(h(x)) = f_X \left( h^{-1}(y) \right) \cdot \left| \frac{1}{h^{-1}(y)} \right| \tag{5}\]

In Fig. 2 (c.1–3), the scatters and green histograms clearly explain the relationship between the function \(h(x)\) and the distribution of \(Y\). With respect to the linear \(h_1(x)\), the histogram and scatters are identical to those of \(X\). As regards \(h_2(x)\), the concave function leads the peak of the probability distribution closer to 0. Then, the difference in the scatter between the maximum PD and 1 becomes farther than that between the maximum PD and 0 because of the decelerating slope of \(h_2(x)\). The feature for convex \(h_3(x)\) is opposite. As can be observed in Fig. 2, dispersion of the variables from a function is controlled by the slope of the function, and symmetry is determined by the concavity or convexity of the function [41]. As these characteristics of a function can be expressed by derivatives of the function, the proposed numerical analysis is based on statistical rules of random variables and functions from the perspective of derivatives.

The dispersion of \(y\) is defined by \(\Delta y\), the difference from \(y_0 = h(x_0)\) to \(y = h(x)\), and it can be deduced using the Taylor series, shown as Eq. 6.

\[
\Delta y = h(x) - h(x_0) \\
= \Delta x \cdot h'(x_0) + o(x_0) \tag{6}
\]

The first term of Eq. 6 indicates that the steep function is easier to discretize \(y\) than the gentle one because of \(h'(x)\) being the coefficient of \(\Delta x\). The second term of Eq. 7 is negligible when \(\Delta x\) is small enough. Additionally, \(\Delta x\) is the essential reason for dispersion of \(\Delta y\). Thus, the dispersion of \(y\) can be expressed in a differential form.

Chosen of \(x_0\) and \(y_0\) is generally based on some statistical characteristics, such as average, mid-value or the value with maximum probability of a set of random variables. For the application of Weibull Model, the definitions of \(x_0\) and \(y_0\) are the values with maximum probability, which can be marked as \(x_{\text{peak}}\) and \(y_{\text{peak}}\) as well.

As known above, concavity or convexity of the function controls the symmetry of \(y\), which can be expressed by the sign of the second-order derivation of the function. To verify the relationship if symmetry of \(y\) distribution and the second-order derivation, a deduction is made in Appendix, and the resulting Eq. 15 or Eq. A4 present the effect of \(h''(x)\) on symmetry of \(y\).

\[
h''(x) \bigg|_{h(x) = y_{\text{peak}}} = f_{X}(x) \cdot h'(x) \tag{7}
\]

As for a monotone function, sign of \(h'(x)\) is invariable, and the \(f(x)\) is positive all the time. Then the sign of \(h'(x)\) is directly related with \(f_X(x)\), where the sign of \(f_X(x)\) is positive.
when \( x \) is less than \( x_{\text{peak}} \) (the \( x \) with maximum probability), and negative when \( x \) is more than \( x_{\text{peak}} \). The position of \( y_{\text{peak}} \) is consistent with that of \( x_{\text{peak}} \) when \( h''(x) \) is 0 and \( h(x) \) is linear; otherwise \( y_{\text{peak}} \) moves with \( h''(x) \). The peak of \( f_y(y) \), \( y_{\text{peak}} \), moves to the right of \( x_{\text{peak}} \) when \( h''(x) \) is negative and \( h(x) \) is convex. The peak of \( f_y(y) \) moves to the left of \( x_{\text{peak}} \) when \( h''(x) \) is positive and \( h(x) \) is concave. This indicates that the shape of the function is the reason for the symmetry of the distribution, which agrees with the conclusion drawn from Fig. 2.

In order to know the absolute symmetry of \( y \), symmetry of \( x \) distribution should be considered. The symmetry of \( y \) distribution is changed from that of \( x \) distribution by the variety of \( h''(x) \) or the sign of \( h''(x) \), the comparison of the dispersion on the both sides of the \( y_{\text{peak}} \) of function distribution in a differential way. Then, symmetry of \( y \) distribution can be expressed as \( \Delta^2y \) in Eq. 8.

\[
\Delta^2y = \frac{\Delta y_1 - \Delta y_2}{\Delta x_1 + \Delta x_2} \tag{8}
\]

where, \( \Delta x_1 \) and \( \Delta x_2 \) are the difference from \( x_{\text{peak}} \) to the maximum \( (y_{\text{max}}) \) and minimum \( (y_{\text{min}}) \) of \( x \) distribution, \( \Delta y_1 \) and \( \Delta y_2 \) are the difference from \( y_{\text{peak}} \) to the maximum \( y_{\text{max}} \) and minimum \( y_{\text{min}} \) of \( y \) distribution. The expressions are shown as follow.

\[
\Delta x_1 = x_{\text{max}} - x_{\text{peak}} \tag{9}
\]
\[
\Delta x_2 = x_{\text{peak}} - x_{\text{min}} \tag{10}
\]
\[
\Delta y_1 = y_{\text{max}} - y_{\text{peak}} \tag{11}
\]
\[
\Delta y_2 = y_{\text{peak}} - y_{\text{min}} \tag{12}
\]

When the distribution of \( x \) is symmetry, \( \Delta x_1 \) and \( \Delta x_2 \) are equal. Then Eq. 8 can be simplified by Eq. 13, the differential form of \( h''(x) \).

\[
\Delta^2y = \frac{y_{\text{max}} - 2y_{\text{peak}} + y_{\text{min}}}{(\Delta x)^2} \tag{13}
\]

Hence, there is a slight difference between \( \Delta^2y \) and \( h''(x) \) when distribution of \( x \) is asymmetry. Conversely, \( \Delta^2y \) can be replaced by \( h''(x) \) when the distribution of \( x \) is symmetry. Then the indicator of \( h''(x) \) for symmetry \( y \) distribution is still available.

Meanwhile, if the symmetry of \( y \) distribution is discussed at the point of \( y_0 = h(x_0) \), matched with Eq. 17, \( \Delta x_1 \) can be simplified as \( \Delta x \), and \( \Delta^2y \) can be rewritten as Eq. 14.

\[
\Delta^2y(x_0) = \frac{y_{\text{max}} - 2y_0 + y_{\text{min}}}{\Delta x^2} \tag{14}
\]

In the subsequent analysis, the link between the function \( h(x) \) and distribution of \( Y \) is explained. The discharge voltage inconsistency evolution of lithium-ion cells caused by capacity inconsistency can thus be calculated and predicted.

2) FOUNDATION OF VOLTAGE FUNCTION

In this section, three preconditions should be declared beforehand: (1) the inconsistency of the capacity of cells is dominant in the voltage variation, (2) the relationship between the voltage and SOC functions is unique; (3) all the cells are discharged from the same SOC. The reason for these ideal conditions is that the disturbing sources in the experimental testing, such as the initial SOC, resistance, and ambient temperature, are only marginally eliminated. However, the disturbing sources are not more influential in the voltage inconsistency than the capacity. As a preliminary study, it is necessary to reduce the variables that contribute to examining the basic laws and to investigate the more general phenomenon in subsequent studies.

In Eq. 15, voltage can be expressed as the function of the SOC based on the second prediction; in Eq. 16, the SOC can be expressed as the function of the discharge capacity and capacity of a cell based on the third prediction. Therefore, the relationship between the cell capacity and voltage is established with the SOC as the bridge.

\[
V = V(SOC) \tag{15}
\]
\[
SOC = SOC_{\text{int}} - \frac{C}{C_I} \tag{16}
\]

where \( V \) is discharge voltage; \( SOC \) is the state of charge of the cell; \( SOC_{\text{int}} \) is the initial SOC of the cell; \( C \) is the discharge capacity; \( C_I \) is the cell capacity.

Eq. 15 is the identical functional relationship between the voltages and SOC of the cells produced in the same batch and tested by \( 1/3 \) \( C \) discharge current, where voltage is determined by the open-circuit voltage of the cells and all the electro-chemical reactions and mass transport processes occur while varying the SOC. In Eq. 16, the SOC is determined by the discharge capacity \( C \) and capacity \( C_I \) during the discharge from the initial SOC (\( SOC_{\text{int}} \)).

Eq. 15 and Eq. 16 can then be integrated into Eq. 17, where the voltage function can be abbreviated as \( V(C, C_I) \), as follows:

\[
V(C, C_I) = V(SOC_{\text{int}} - \frac{C}{C_I}) \tag{17}
\]

From section II-C1, the distribution of the function variables \( y \) is dependent on the distribution of \( x \) and the function of \( y = h(x) \), so that the statistical characteristics of the voltage should be determined by the distribution of the capacity and the function of the voltage and capacity. Referring to the Weibull PDF, the dispersion and symmetry parameters are deduced in order to understand the statistical characteristics of the voltage distribution during discharge.

Depending on the first prediction, the distribution of voltage curves can be determined by Eq. 17 with the unique uncertain source of \( C_I \). In other words, the voltage curves are identical with respect to the SOC according to Eq. 15, but vary with respect to the discharge capacity based on Eq. 17. Unfortunately, the voltage curves from the actual experiment were recorded with the discharge capacity and failed to account for the distribution of the voltage curves based on the SOC. In order to approximate the actual situation and application of the testing regime, a typical voltage curve
that varies with the discharge capacity and a typical capacity can be selected from the various voltage curves, represented by \( V_0 - C \) and \( C_{t_0} \), as shown in Eq. 18 and Eq. 19.

\[
\begin{align*}
V_0 &= A_{Wbl} + B_{Wbl} \\
C_{t_0} &= A_{Wbl}(C_t) + B_{Wbl}(C_t)
\end{align*}
\] (18) (19)

\( V_0 \) and \( C_{t_0} \) are set by the values with maximum PD based on the peak of the Weibull PDF and are defined as the characteristic voltage and capacity as shown in Eq. 18 and 19 (This designation originates from the application of the Weibull Model in the reliability analysis on the life distribution of the components, which defines the sum of the scale and location parameters as the characteristic life [42]). Meanwhile, \( V_0 \) and \( C_{t_0} \) in Eq. 18 and Eq. 19 are calculated based on the Weibull parameters of simulated voltage curves, thereby avoiding the external disturbance during the practical experiment. On the one hand, the value with the maximum probability is occupied by most samples, and it is the most common observation for the characteristic of the voltage distribution. On the other hand, the expression for the characteristic voltage and capacity based on the Weibull parameters makes it possible to describe the statistical characteristics of voltage distribution from statistical and the numerical methods subsequently.

Nevertheless, the curve of \( V_0 - C \) obtained from Eq. 18 is insufficient for recovering the whole voltage cluster because the Weibull parameters are only recorded from the beginning of the discharge just until the first cell reaches the cut-off voltage. (The statistical parameter is invalid when the sample size changes). To obtain the complete voltage curves of \( V_0 - C \), a simulated cell whose voltage curve coincides best with \( V_0 \) in Eq. 18 before some cell reaches the cut-off voltage and capacity is closest to \( C_{t_0} \) in Eq. 19. According to Eq. 18 and Eq. 19, a reference cell is assumed with a characteristic capacity and voltage curve. The voltage function of the reference cell can be expressed as Eq. 20.

\[
V(C, C_{t_0}) = V_0(C)
\] (20)

where the voltage curve of \( V_0 - C \) is expressed as \( V_0(C) \), only varying with the discharge capacity, \( C \) with a capacity of \( C_{t_0} \).

Accordingly, the SOC function of the reference cell can be expressed as Eq. 21.

\[
SOC_0 = SOC_{int} - \frac{C}{C_{t_0}}
\] (21)

Similar to Eq. 21, \( SOC_0 \) is the only function of discharge capacity, \( C \), because of the constant capacity of \( C_{t_0} \). In addition, when the capacity of a cell is not \( C_{t_0} \) but \( C_{t_i} \), \( SOC_0 \) can be calculated by Eq. 22.

\[
SOC_i(C_i, C_{t_i}) = SOC_{int} - \frac{C_i}{C_{t_i}}
\] (22)

In order to determine the voltage curves and the derivation of the dispersion and symmetry parameters of the voltage distribution from \( V_0 \) and \( C_{t_0} \), the inconsistent voltage curves should be transformed from \( V_0(C) \). The SOC of the cells with various \( C_{t_i} \)s are different when discharged by the same current from the same initial SOC, so the voltage curve of \( V(C, C_{t_i}) \) should be obtained from the transformation of \( V_0(C) \). As for the same SOC for the cell with capacity of \( C_{t_i} \) and \( C_{t_0} \), the relationship of discharge capacity, marked as \( C_i \) and \( C_0 \), can be determined by Eq. 23.

\[
SOC_{int} - \frac{C_i}{C_{t_i}} = SOC_{int} - \frac{C_0}{C_{t_0}}
\] (23)

where \( C_i \) is the discharge capacity for the cell with a capacity of \( C_{t_i} \). Based on the second precondition as mentioned above, the relation of \( C_i \) and \( C_0 \) can be solved from Eq. 23, as shown in Eq. 24.

\[
C_0 = C_i \cdot \frac{C_{t_0}}{C_{t_i}}
\] (24)

In Eq. 24, an equivalent of \( C_0 \) formed by \( C_i \) is obtained. Substituting Eq. 24 into \( V_0(C) \), the voltage curve for the cell with capacity of \( C_{t_i} \) can be given, as shown in Eq. 25, which is different from Eq. 17, a general function of voltage with respect to the capacity \( C \) and discharge capacity \( C_{t_i} \). The derivation of Eq. 25 is drawn as a diagram in Fig. 3 for a clear description.

\[
V(C, C_{t_i}) = V_0 \left( C \cdot \frac{C_{t_0}}{C_{t_i}} \right)
\] (25)

Thus, the voltage curves with various capacity can be described based on Eq. 25, which intersect at the beginning of discharge and disperse from the curve of \( V_0-C \) gradually with increasing discharge capacity, \( C \).

3) INCONSISTENCY MODEL OF VOLTAGE DISTRIBUTION

According to section II-C1, the dispersion and symmetry of the voltage distribution are related to the first- and second-order derivatives of \( V(C, C_{t_i}) \), respectively. Due to the chain relationship of voltage, \( C_i \) and \( SOC \), the derivative of \( V(C, C_{t_i}) \) with respect to \( SOC \) can be solved as a nested function from a chain derivative, where the derivative of \( SOC(C, C_{t_i}) \) should be previously solved.
According to Eq. 16, the total differential of $SOC(C, C_t)$ is shown as Eq. 26.

$$dSOC = -\frac{1}{C_t} dC + \frac{C}{C_{t_0}} dC_t$$  \hspace{1cm} (26)

As a result of the constant discharge current for the cells tested in this study, the values of the discharge capacity ($C$) are identical to each other during discharge, so that there is no difference caused by $dC$, as shown in Eq. 27.

$$dSOC = \frac{C}{C_{t_0}} dC_t$$  \hspace{1cm} (27)

Based on Eq. 6, the dispersion of the discharge voltage from $C_{t_0}$, expressed by Taylor series, is shown in Eq. 28.

$$\Delta V = \left| \frac{\partial V(C_t, C)}{\partial C_t} \right|_{C_t=C_{t_0}} \cdot \Delta C_t$$  \hspace{1cm} (28)

In Eq. 28, the Taylor series can be used for the dispersion voltage ($\Delta V$) caused by the dispersion capacity ($\Delta C_t$). The absolute value sign of $\frac{\partial V(C_t, C)}{\partial C_t}$ when $C_t$ is $C_{t_0}$, which confirms that the SOC is the media of dispersion transmission from $C_t$ to $V$. The transfer of dispersion from $C_t$ to $V$ is achieved via the media variable, $SOC$.

According to Eq. 21 and Eq. 20, $\frac{dV(SOC)}{dSOC} \left|_{C_t=C_{t_0}} \right.$ can be derived as Eq. 30.

$$\frac{dV(SOC)}{dSOC} \left|_{C_t=C_{t_0}} \right. = \frac{dV_0(C)}{dSOC_0}$$  \hspace{1cm} (30)

As both of $V_0(C)$ and $SOC_0$ are functions of the discharge capacity ($C$), Eq. 30 can be solved as Eq. 31.

$$\frac{dV(SOC)}{dSOC} \left|_{C_t=C_{t_0}} \right. = -\frac{dV_0(C)}{dSOC_0} \cdot \frac{dSOC_0}{dSOC}$$  \hspace{1cm} (31)

Based on Eq. 16, $\frac{dSOC_0}{dSOC}$ in Eq. 29 is known. Therefore, the terms related to the $SOC$ in Eq. 29 can be eliminated, and the expression of Eq. 29 is transformed as Eq. 32.

$$\frac{\partial V(C_t, C)}{\partial C_t} = -\frac{C}{C_{t_0}} \cdot \frac{dV_0(C)}{dC}$$  \hspace{1cm} (32)

Substituting Eq. 32 into Eq. 28, the dispersion voltage caused by the capacity is numerically derived, as shown Eq. 33. The absolute value sign of $\frac{\partial V_0(C)}{dC}$ is added in Eq. 33 because the value of $\Delta V$ is simply the absolute difference, without considering positive or negative directions.

$$\Delta V(C) = -\frac{C}{C_{t_0}} \cdot \frac{|dV_0(C)|}{dC} \cdot \Delta C_t$$  \hspace{1cm} (33)

In Eq. 21, the dispersion voltage ($\Delta V$), caused by the dispersion capacity ($\Delta C_t$) is expressed as a function of the discharge capacity. Similar to the scale parameter of the Weibull PDF, $\Delta V$ is the difference of $V$ and $V_0$, and $\Delta C_t$ is the difference of $C_t$ and $C_{t_0}$. $V_0$ and $C_{t_0}$ are regarded as the Weibull peaks of voltage and capacity distributions in Eq. 18 and Eq. 19; hence, $\Delta V$ can be solved as the scale parameter of the voltage distribution ($A_{Wbl}$) as long as $\Delta C_t$ is set as the scale parameter of the capacity distribution($A_{Wbl}(C_t)$). Thus, a numerical method is provided on the statistical characteristics of the voltage distribution, which differs from the statistical method. Eq. 33 can be written from Eq. 34 as the dispersion parameter of voltage during discharge.

$$A_{Num} = \frac{C}{C_{t_0}} \cdot \frac{|dV_0(C)|}{dC} \cdot A_{Wbl}(C_t)$$  \hspace{1cm} (34)

In comparison with zero, the sign of $\frac{\partial V_0}{\partial C}$ indicates the movement direction of the voltage distribution relative to the peak of the capacity distribution. Thus, the absolute location of the peak of the voltage distribution depends on the peak of capacity distribution and the second-order of $V(C, C_t)$. $\frac{\partial V_0}{\partial C}$ provides a similar expression for the symmetry of voltage distribution compared with the shape parameter of the Weibull Model, so that it can be regarded as the symmetry parameter of voltage distribution and represented as $B_{Num}$ in the following discussion, as shown in Eq. 36.

$$B_{Num} = \frac{C}{C_{t_0}^2} \cdot \left( \frac{\partial V_0}{\partial C} + C \cdot \frac{\partial^2 V_0}{\partial C^2} \right)$$  \hspace{1cm} (36)

Through the above derivations, an inconsistency model is established based on the Weibull Probability Model to understand the statistical characteristics of the voltage distribution. The inconsistency model is established based on the characteristic voltage and capacity ($V_0$ and $C_{t_0}$). This includes the dispersion parameter ($A_{Num}$) and symmetry parameter ($B_{Num}$), corresponding to the probability peak, scale parameter, and shape parameter of the Weibull Model, which transforms the Weibull parameters into a numerical description.
Two theories are introduced to the derivation of the dispersion and symmetry parameter: the functional relationship between the voltage and capacity is determined by the electrochemical theory. The derivatives are solved based on the probability theorem of random variables. From this, the formation and evolution of the voltage distribution caused by the capacity inconsistency can be understood via the inconsistency model.

c: VALIDATION

According to Eq. 9–13, the essence of dispersion and symmetry of voltage distribution is related with the distances from $V_0$ to the minimum of voltage distribution ($\Delta V_{0_{\text{min}}}$) and the maximum ($\Delta V_{0_{\text{max}}}$). Likewise, capacity distribution is the same description on $C_{0_{\text{min}}}$ and $C_{0_{\text{max}}}$. Meanwhile, when the Weibull parameters of voltage and capacity distributions are given, these differential parameters can be calculated by Weibull PDF in Eq. 1 and the CPF in Eq. 2, as shown in Eq. 37–40 (The minimum of the distribution is defined as the value of local parameter of Weibull Model, and the maximum is the value with CPF of 99.9%).

$$\Delta V_{0_{\text{min}}} = A_\text{Wbl}$$
$$\Delta V_{0_{\text{max}}} = A_\text{Wbl} \cdot (\ln 1000)_{\text{Wbl}}^{-1} - A_\text{Wbl}$$
$$\Delta C_{0_{\text{min}}} = A_\text{Wbl}(C_1)$$
$$\Delta C_{0_{\text{max}}} = A_\text{Wbl}(C_1) \cdot (\ln 1000)_{\text{Wbl}}^{-1} - A_\text{Wbl}(C_1)$$

In Eq. 37, $\Delta V_{0_{\text{min}}}$ is consistent with the dispersion described by $A_\text{Wbl}$ and $A_\text{Num}$, which can be used for validation of $A_\text{Num}$.

Based on Eq. 22, symmetry of voltage distribution can be expressed by the differential parameters in Eq. 8–40, as shown in Eq. 41.

$$\Delta^2 V = \frac{\Delta V_{0_{\text{max}}} - \Delta V_{0_{\text{min}}}}{\Delta C_{0_{\text{max}}} + \Delta C_{0_{\text{min}}}}$$

When the Weibull parameters plugged into Eq. 41, symmetry of voltage distribution is expressed by the statistical parameters in a differential form, as shown in Eq. 42.

$$\Delta^2 V = \frac{2A_\text{Wbl}}{A_\text{Wbl}(C_1)^2} \cdot (\ln 1000)_{\text{Wbl}}^{-1} - 1$$

Compared with $B_{\text{Num}}$, symmetry expressed by $\Delta^2 V$ is comparison of the dispersion on both sides of $V_0$, which is the indicator for change of voltage symmetry from capacity symmetry. Considering capacity distribution, $\Delta^2 V$ is the differential expression of voltage symmetry, corresponding with $B_{\text{Num}}$. Thus, $B_{\text{Num}}$ can be validated by $\Delta^2 V$ subsequently.

III. RESULTS AND DISCUSSION

In Fig. 4, the R-Square of the Newman Model compared with the experimental average voltage curve is 99.87%, indicating the reliability of the model. To validate the statistical characteristics of the simulated voltage, as the characteristic voltage, $V_0$ is compared with the corresponding experimental $V_0$ in Fig. 5, which are calculated by Eq. 28. The R-square of simulated and experimental $V_0$s is 95.07%, indicating the reliability of voltage curves recovery method. Simultaneously, the Weibull parameters of capacity distribution are estimated and listed in Tab. 1, and the error of the simulated $C_{0_{\text{min}}}$ from the experimental one is only 0.29%, so that the distribution of simulated capacity is proved valid. The probability histograms of the simulated and experimental capacity are displayed in Fig. 7. The difference in the experimental and simulated capacity distribution is due to inevitable uncertainty during the testing of cells, such as ambient temperature and contact resistance of the equipment. Considering the validation of voltage curves and capacity distribution, the recovered voltage curves are reliable, and the voltage curves of experiment, simulation and recovery are shown in Fig. 6. Therefore, $V_0$ and $C_{0_{\text{min}}}$ can be used to calculate $A_{\text{Num}}$ and $B_{\text{Num}}$, respectively.

A. STATISTICAL CHARACTERISTICS

Before analyzing the voltage distribution, a 4-D probability nephogram is used, as a visual tool, to present the distribution probability of discharge voltage curves from the experiment.
As shown in Fig. 8, the axis is composed of three mathematical axes and a color axis. The X, Y, and Z axes are related to discharge capacity, nominal voltage, and voltage in sequence, and the color axis is related to the probability of voltage at each recorded moment where the color bar represents the scale.

From the original view in Fig. 8(-.1), the varying colored surfaces represent the voltage distributions of the voltage values, where bright yellow indicates the peak of the voltage distribution. Comparing the three nephograms in original view, the yellow patterns of the experimental voltage distribution in Fig. 8(a.1) is indistinct, while the patterns of simulated and recovered voltage distribution in Fig. 8(b.1) and Fig. 8(c.1) are distinct and clear, which is because inevitable factors presented in the experiment contribute to extra disturbance in voltage inconsistency. Fig. 8(-.2) appears similar to the discharge curves; however, it is different. The figures in the X–Z view provide the scopes of the voltage surfaces in Fig. 8(.1), representing the scopes of voltage distribution. The widening trend of voltage surf in the X–Z view agrees with the divergence of voltage curves in Fig. 6. The detailed information of voltage distribution is recorded by the color patterns and filled in narrow scopes, which are difficult to read in Fig. 8(-.2), while it can be expanded for a clear view in Fig. 8(-.3). The voltage surfs in the X-Y view present the voltage distribution in a normalized scope without varying voltage values with discharge. The nominal nephograms can clarify the evolution of voltage distribution during discharge, despite the narrowest distributing scopes at the beginning of discharge. In the X–Y view, locations of voltage with PDF peaks of the Weibull Model are displayed by red lines. Depending on the indication of the red lines for the distribution, it confirms that voltage is not always symmetrically distributed and cannot be described by the normal probability model.

The nominal nephograms can clarify the evolution of voltage distribution during discharge, despite the narrowest distributing scopes at the beginning of discharge. In the X–Y view of the nephogram in Fig. 8(-.3), position of voltage with PDF peaks of the Weibull Model and Normal Model are displayed by red and green lines respectively. The peak of the Normal Model is the average of voltage distribution and the peak of Weibull Model is the “characteristic” average of voltage distribution. The difference proves that voltage distribution is not symmetry and varied with discharge capacity. The trends of red line and green line is similar in experimental, simulated and recovered nephogram, which indicates the validity of parameter estimation of Weibull Model. The red lines are above the green ones, which means the voltage distributions are asymmetrical or right-skewed.

The method of $\chi^2$ test can measure the goodness of fitting between a given probability model and the sample distribution [43], [44]. Using the method of $\chi^2$ test, validity of Weibull and Normal Model used for voltage distribution are as shown in Fig. 9. The estimated distributions with the $\chi^2$ value less than the critical values of 38.89 (for Weibull distribution, freedom degree is 26) or 40.11 (Normal distribution, freedom degree is 27) [45] are considered as acceptable results at the confidence level of 95%. In Fig. 9(a), except the range of 23.51–25.65 Ah, the $\chi^2$ values of Weibull and Normal distributions fail to satisfy the verification condition, that is, the values of experimental voltage are not distributed typically. The external disturbances lead to disorder of voltage distribution, which is hard to be discerned by the Weibull or Normal Models. In Fig. 9(b), the $\chi^2$ of Weibull and Normal...
distributions are less than the critical values before 25.67 Ah. The $\chi^2$ of Normal distribution is more than 40.11 after 25.67 Ah, which means the Normal Model is not suitable of the right-skewed voltage distribution at the end of discharge. In Fig. 9(c), the $\chi^2$ of Weibull and Normal distributions is much less than that in Fig. 9(b), and the $\chi^2$ of Normal distribution is more than 40.11 after 25.51 Ah, which appears the similar phenomenon in Fig. 9(b). Therefore, Weibull Model is more adaptive to voltage distribution, especially at the end period of discharge, and the description of Weibull parameters is closer to the characteristics of voltage distribution.

The three parameters of the Weibull Model quantify the evolution of voltage inconsistency. With the MLE method, the three Weibull parameters are plotted depending on the discharge capacity in Fig. 10.

As stated in Section II-A, the scale parameter of the Weibull Model, distance from $C_{Wbl}$ to the peak of the Weibull distribution, reflects the dispersion of random variables. In Fig. 10(a), $A_{Wbl}$ quantitatively describes the evolution of voltage dispersion, related to the X–Z axis view of the nephogram presented in Fig. 8(b). In detail, Fig. 10(a) shows three smooth peaks at the previous stage $A_{Wbl}$ and a leap at the end for experimental and simulated voltage distribution. The first peak is obtained in the experiment and simulation during discharge capacity from 0 to 4.5 Ah resulting from resistant inconsistency. In the case of the simulated voltage, the first two peaks of $A_{Wbl}$ fluctuate within a small range from $0.6 \times 10^{-2}$ to $1 \times 10^{-2}$. In this period, the voltage gently declines with slight dispersion. For the third peak from 17.3 Ah to 24.2 Ah, the voltage change rate gradually rises such that the peak begins at $7.0 \times 10^{-3}$ and ends at $1.1 \times 10^{-2}$. The voltage distribution begins to disperse in a small region. The final surge leaps over two orders of magnitude (0.8) when the cell with the least capacity reaches cutoff voltage. Therefore, the rapid voltage declining corresponds to the surge of voltage dispersion. Similarly, Edouard [23] analyzed the voltage dispersion of LiFePO$_4$/Graphite cells using standard deviation and the Normal Model statistics. The parameter $A_{Wbl}$ of the experimental voltage exhibited a similar trend compared to $A_{Wbl}$ of the simulated voltage; however, the
peak value of the experimental $A_{\text{Wbl}}$ is much higher than that of the simulated voltage. The first and third peaks of the experimental $A_{\text{Wbl}}$ reach $3 \times 10^{-2}$, and the leakage at the end begins after 25 Ah. The experimental values of $A_{\text{Wbl}}$ are blur, which makes it difficult to find out the rules for voltage inconsistency, while $A_{\text{Wbl}}$ of the recovered voltage displays a clear and accessible evolution of voltage dispersion. Without the first peak value of the experimental and simulated voltage, $A_{\text{Wbl}}$ of the recovered voltage jumps from a minimal value close to zero to the second peak value. The values of $A_{\text{Wbl}}$ for the recovered voltage after 17.3 Ah enter to a sloping peak, where the trend is more well-defined compared with the trend of the simulated voltage. The difference between the recovered and simulated voltage is dominant at the beginning of the discharge and it weakens as the discharge proceeds, implying that the effect of capacity on voltage dispersion is slight at the initial stage and dominant during the middle and later stages. Thus, capacity inconsistency is the basis of voltage inconsistency.

$B_{\text{Wbl}}$ in Fig. 10(b) allow identifying the voltage distribution symmetry. Considering the simulated voltage distribution, the first peak value ranging from 0 to 6.5 Ah, up to 6.4, is much greater than the second peak value, which is different from $A_{\text{Wbl}}$. Subsequently, three weak peaks are in the $B_{\text{Wbl}}$ range of 4–4.5, during the middle of the discharge process from 6.5 to 23.8 Ah. The third peak value occurs from 16.3 to 24.4 Ah. It is composed of two small peaks very close to each other, which is relevant to the third upswept peak when the voltage begins declining at an accelerated rate and dispersion gradually increases. The experimental behavior of $B_{\text{Wbl}}$ in Fig. 10(b), containing substantial perturbation, is more sensitive to the change of voltage with some cell reaching cutoff voltage extremely early. Thus, the curve of experimental $B_{\text{Wbl}}$ is fuzzy and incomplete, comparing with simulated and recovered $B_{\text{Wbl}}$ curves; hence, the experimental $B_{\text{Wbl}}$ cannot be considered as the reference for the evolution of voltage distribution. Conversely, $B_{\text{Wbl}}$ of the recovered voltage completely retains capacity variation without any other noise and exhibits the fundamental trend of voltage symmetry during discharge. Before the capacity of 15 Ah, $B_{\text{Wbl}}$ of the recovered voltage is maintained in the range of 3.74 and 3.81, which means that the voltage distribution is symmetric and agrees with the yellow patterns in Fig. 8(c.3). The adjacent double peaks from 15 Ah to 23.68 Ah and the leap after 23.68 Ah are consistent in trend with the simulated $B_{\text{Wbl}}$, indicating the decisive effect of capacity inconsistency on voltage inconsistency.

In Fig. 10(c), $C_{\text{Wbl}}$ represents the minimum voltage. Despite some perturbation in the experimental $C_{\text{Wbl}}$ at the beginning and bend of the curve, the excellent fitting of the three $C_{\text{Wbl}}$ curves verifies the reliability and rationality of the simulated and recovered voltage curves.

The statistical characteristics can be clearly compared, considering nephograms of voltage probability in Fig. 8, and plots of Weibull parameters in Fig. 10. In terms of the experimental voltage, the unavoidable and complex external disturbance distorts the trend of voltage dispersion which is challenging to identify. Hence, the statistical results of the experimental voltage curves are only used for the basic verification of simulated and recovered voltage curves. The voltage curves simulated using the Newman electrochemical model restore the instinct effects on voltage inconsistency from the uncertainty of the cell production process. As the experimental data verify the reliability of this model, the statistical results of the simulated voltage curves can be used instead of the statistical results of the experiment. Based on capacity inconsistency, the recovered voltage curves extract the relationship between voltage and capacity inconsistency. According to Fig. 8 and Fig. 10, capacity inconsistency is the dominant reason for voltage inconsistency. Moreover, the statistical characteristics and trend of the evolution of recovered voltage inconsistency agree with that of the simulated voltage. Regardless of the difference at the beginning of discharge, the statistical results of the recovered voltage are the basis of the subsequent discussion on the formation of voltage inconsistency evolution.

**FIGURE 10.** Weibull parameters of voltage distribution varying with discharge capacity: (a) Scale parameter, $A_{\text{Wbl}}$; (b) shape parameter of voltage, $B_{\text{Wbl}}$ (the mathematical treatment is performed for a detailed plot in a semilog axis); and (c) voltage location parameter, $C_{\text{Wbl}}$. 

![Graphs showing Weibull parameters of voltage distribution](image-url)
B. NUMERICAL ANALYSIS

The results in Section III-B indicate that the evolution of voltage inconsistency during discharge is not random; instead, a pattern exists. As concluded above, capacity inconsistency is the primary origin of voltage inconsistency; the inconsistency model in Section II-C2 can be used to capture the statistical rule and understand it from a numerical perspective. Therefore, the proposed numerical analysis method is used to recover the statistical characteristics of voltage distribution and make further analysis on the formation and evolution of voltage inconsistency based on the theory of electrochemistry.

1) DISPERSION

In Fig. 11, the dispersion parameters of $A_{Num}$ and $\Delta V_{0-min}$ are highly fitted with each other, which not only indicates the effectiveness of numerical analysis method but also further verifies the reliability of voltage recovery method. Considering experimental $A_{Wbl}(C_t)$ less than the simulated $A_{Wbl}(C_t)$, it is reasonable that the curve of $A_{Num}(EXP)$ is lower than the others before 23.34 Ah. In fact, $A_{Num}(EXP)$ is deviated from the similar trend with $A_{Num}(RCV)$ and increases to coincide with $A_{Num}(RCV)$ at 25.36 Ah, which is caused by the other factors than capacity. The dispersion of experimental voltage caused by capacity inconsistency presents the similar rules, so that the formation of voltage dispersion can be explained by the component of the dispersion parameter in Eq. 34. The product of three terms in Eq. 34 indicates that the three factors affect the evolution of voltage dispersion. The first term, $\frac{C}{\tau_0}$, records the rate of the discharge process and the relevant SOC. Thus, the dispersion of voltage distribution begins with minimal value and generally increases with increasing SOC. The second term, $\left|\frac{\partial V_0}{\partial C}\right|$, controls the trend of dispersion evolution. Compared with the voltage curves in Fig. 6(b), the peaks of $A_{Num}$ correspond to the rapid descending period of discharge, such as 10.78 Ah and 22.03 Ah. Conversely, when the voltage curve gradually declines, $A_{Num}$ is in the trough. The third term, $A_{Wbl}(C_t)$, determines the range of dispersion according to the assumption that capacity inconsistency is the origin of voltage inconsistency. The parameter $A_{Wbl}(C_t)$ is assumed to be the input of the dispersion parameter, and the product of $\frac{C}{\tau_0}$ and $\left|\frac{\partial V_0}{\partial C}\right|$ is considered as the varying coefficient, forming the process of voltage dispersion during discharge.

2) SYMMETRY

The sign of the symmetry parameter indicates the movement direction of voltage distribution peak relative to capacity distribution peak. In Fig. 12, the symmetry parameters of $B_{Num}$ and $\Delta^2V$ present a good fitting with each other, so that the numerical $B_{Num}$ can reflect the statistical characteristics of symmetry identified with that from statistical method. The slight difference between $B_{Num}$ and $\Delta^2V$ is caused by the not absolutely symmetric capacity distribution, but the error less than $10^{-3}$ in not effective on the assessment and prediction of symmetry of voltage distribution. With external disturbance, $B_{Num}(EXP)$ surging up and down appears an unstable statistical rules, which is corresponding to the divergency of the $A_{Num}(EXP)$ in Fig. 11. Except the external disturbance, the numerical symmetry parameter is available for experimental voltage as well. Thus, the symmetry of voltage distribution can be analyzed by the structure of $B_{Num}$ expression. According to Eq. 36, the function comprises three components. The first component, $\frac{C}{\tau_0}$, is identical with that of Eq. 36 such that the fluctuation of voltage symmetry increases with discharge capacity. Accordingly, the value of $B_{Num}$ becomes close to 0 before 7.28 Ah, proving Eq. 7. Furthermore, the value of $B_{Num}$ being close to 0 implies that the relationship between voltage and capacity is approximately linear; hence, the distributions of voltage and capacity are considered to be the same, which can be verified by Fig. 10(b) that $B_{Wbl}$ and $B_{Wbl}(C_t)$ are almost equal before 7.28 Ah. The subsequent trend reflects the second and third components in the bracket of Eq. 36, quantitatively explaining the formation of the symmetry evolution. The third component in Eq. 36 is the product of discharge and second-order derivative of voltage, i.e., the effect of $\frac{\partial^2 V_0}{\partial C^2}$ gradually displays with the discharge process, whereas the effect of $\frac{\partial V_0}{\partial C}$ is dominant in the initial; however, it weakens with discharge. Thus, the downward peak of $B_{Num}$ at 13.87 Ah corresponds to the reduction in $\frac{\partial V_0}{\partial C}$, and the neighboring peaks from 17.92 Ah to 23.64 Ah are the mixing result of $\frac{\partial V_0}{\partial C}$ and $\frac{\partial^2 V_0}{\partial C^2}$ as the effect of $\frac{\partial^2 V_0}{\partial C^2}$ is additionally amplified by $\frac{\partial V_0}{\partial C}$. After 23.64 Ah, $B_{Num}$
The manifestation of electrochemical thermodynamics in the Lithium-ion cell causes the voltage inconsistency evolution due to capacity inconsistency. According to the principle of the cell [46], the equilibrium potential is determined by SOC, with the occurrence of specific electrochemical reactions. In this study, a cell voltage of 1/3C is close to the equilibrium potential; thus, Eq. 26 provides the root cause for the formation of SOC inconsistency. Due to inconsistent capacity, SOC is scattered, and it increasingly spreads out through the discharge progress. Hence, the trends of $A_{Num}$ and $B_{Num}$ are not only the single derivatives of $V$-SOC, but also the chain derivatives of $V(C, C_t)$ with respect of $C_t$, transferring the effort of capacity inconsistency from SOC to $V$. Transformed into the formulation containing derivatives of $V_0 - C$, $A_{Num}$ and $B_{Num}$ can be connected with the characteristic of voltage curves; they are easy to apply for assessing and predicting the dispersion and symmetry of voltage distribution from practical tests. Owing to the floating capacity for a batch of cells, the function curve of $V_0 - C$ is easier to obtain compared with the function curve of $V_0$ - SOC. Because of the low computational complexity and easy accessibility, $A_{Num}$ and $B_{Num}$ can be applied to the algorithm of battery pack SOC estimation when $V_0$ - C and Weibull parameters of the capacity distribution are known.

It is worth emphasizing that the numerical parameters agree well with the differential parameters ($\Delta V$ and $\Delta^2V$), which means that on one hand, the numerical method can reflect the statistical characteristics voltage inconsistency and can act as the basis of analysis on formation and evolution of voltage inconsistency, so that the inconsistency model can replace the statistical method with a lower computational cost; on the other hand, reliability of numerical method provides the analytical basis on formation and evolution of voltage inconsistency, so that the voltage behaviors of dispersion and symmetry can be understood in an electrochemical way, because of the derivative process considering the theory of electrochemistry.

**IV. CONCLUSION**

In this study, the statistical characteristics of voltage inconsistency of Lithium-ion pouch cells during 1/3C galvanostatic discharge are investigated in way of statistical characteristics and numerical analysis. The conclusion is as follows.

(i) The inconsistency model is proposed to understand formation and evolution of capacity-dependent recovered voltage distribution from the view of dispersion and symmetry in a numerical way. Based on the characteristic voltage and capacity ($V_0$ and $C_0$) defined based on Weibull Model, the dispersion and symmetry parameters are derived. With the precondition of capacity is as the only reason of voltage inconsistency, the formation of voltage inconsistency can be separated into three components: rate of discharge proceeding, distribution of capacity, and characteristic of voltage curve, including slope and concavity, thus the evolution is determined by the discharge behavior and capacity inconsistency, varying with discharge proceed.

(ii) The statistical characteristics of voltage distribution indicated in the 4-D axis visually proves that voltage distribution gradually diverges from the character voltage and it is not constantly symmetry, especially after 24.4 Ah. Weibull Probability Model is confirmed as an adaptive method for the asymmetry voltage distribution, and quantify the evolution of voltage distribution from views of dispersion and symmetry.

(iii) It is concluded that the variations of discharge voltages from manufacturing uncertainty can be manifested as the capacity inconsistency and affect the voltage behavior, which provides basis for the inconsistency model.

This research provides theoretical and methodological support for the assessment and prediction of voltage inconsistency. The combination of statistical and electrochemical theories realizes the transformation of the uncertain issue of voltage inconsistency into a certain one, establishing the model of voltage inconsistency depending on capacity distribution. Owing to the advantage of simple computational complexity, the formulation of $A_{Wbl}$ and $B_{Wbl}$ is promising in terms of providing reference for the screening standard of cell capacity. The $A_{Num}$ and $B_{Num}$ of the inconsistency model can be introduced into state estimation algorithm for battery packs, eliminating the error from voltage inconsistency caused by capacity variation. In future work, the inconsistency model considering asymmetric distributed capacity will be developed for a higher accuracy, and the formation of voltage inconsistency will be investigated considering ohmic resistance, polarization resistance, and initial SOC together for a derivative-based state estimation algorithm of a battery packs with multiple inconsistent parameters.

**GLOSSARY**

**Abbreviation**

| Abbreviation | Description          |
|--------------|----------------------|
| 4-D          | 4-Dimension;         |
| PDF          | Probability density function; |
| CPF          | Cumulative probability function; |
| PD           | Probability density; |
| MLE          | Max likelihood estimation; |
| SOC          | State of charge; |
| V-SOC        | Function relationship of voltage and SOC; |
| V-C          | Function relationship of voltage and discharge capacity; |
| $V_0$-C      | Function relationship of characteristic voltage and discharge capacity; |
| RCV          | Recovery; |
| EXP          | Experiment; |

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**Symbol**

- **A**: Scale parameter of Weibull PDF;
- **B**: Shape parameter of Weibull PDF;
- **C**: Location parameter of Weibull PDF, discharge capacity;
- **$A_{Wbl}$**: Scale parameter of voltage distribution;
- **$B_{Wbl}$**: Shape parameter of voltage distribution;
- **$C_{Wbl}$**: Location parameter of voltage distribution;
- **$A_{Wbl}(C_t)$**: Scale parameter of capacity distribution;
- **$B_{Wbl}(C_t)$**: Shape parameter of capacity distribution;
- **$C_{Wbl}(C_t)$**: Location parameter of capacity distribution.

**Greek Letters**

- **$\Delta C_{t_{0-max}}$**: Difference between the maximum capacity and $C_{t_0}$;
- **$\Delta^2 V$**: Symmetry parameter of voltage distribution in a differential form.

**APPENDIX**

Deduction of Eq. 7 is provided in this section. Retaining the first term on the right side of Eq. 6, as transformed into Eq. A1, the dispersion of $y$ depends on $|h'(x_0)|$ and $\Delta x$. Because dispersion in this study is the absolute value of the variables, $|h'(x_0)|$ is used in Eq. A1 instead of $h'(x_0)$. $\Delta y$ is the absolute difference caused by $\Delta x$, the absolute difference of the random variable $x$ and the specific variable $x_0$.

$$\Delta y = \Delta x \cdot |h'(x_0)| \tag{A1}$$

As observed in Fig. 2, the symmetry of the $Y$ distribution is related to the trend of $h(x)$. To eliminate the absolute value sign of $|h'(x_0)|$, $h(x)$ is assumed to be a monotone increasing function. Then, the derivative of Eq. 6 is considered, as shown in Eq. A2.

$$\frac{df_y(y)}{dy} = \frac{df(x)}{dx} \tag{A2}$$

As for the left side of Eq. A2, $\frac{df_y(y)}{dy}$ equals 0 when $f_y(y)$ is the peak, and $y$ is marked as $y_{peak}$. Thus, Eq. A2 can be further expanded as Eq. A3.

$$\frac{df_x(x)}{h(x)} \bigg|_{h(x)=y_{peak}} = \frac{f'_x(x) \cdot h(x) - f_x(x) \cdot h''(x)}{h'(x)^2} = 0 \tag{A3}$$

Based on Eq. A3, the solution of $h''(x)$ is obtained, as shown in Eq. A3.

$$h''(x)\bigg|_{h(x)=y_{peak}} = \frac{f'_x(x) \cdot h'(x)}{f_x(x)} \tag{A4}$$

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L. Wang et al.: Understanding Discharge Voltage Inconsistency in Lithium-Ion Cells via Statistical Characteristics

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