The $k$th smallest Dirac operator eigenvalue and the pion decay constant

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Abstract

We derive an analytical expression for the distribution of the $k$th smallest Dirac eigenvalue in QCD with an imaginary isospin chemical potential in the Dirac operator for arbitrary gauge field topology $\nu$. Because of its dependence on the pion decay constant $F_\pi$ through the chemical potential in the epsilon regime of chiral perturbation theory, this can be used for lattice determinations of that low-energy constant. On the technical side, we use a chiral random-two matrix theory, where we express the $k$th eigenvalue distribution through the joint probability of the ordered $k$ smallest eigenvalues. The latter can be computed exactly for finite and infinite $N$, for which we derive generalizations of Dyson’s integration theorem and Sonine’s identity.

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(Some figures may appear in colour only in the online journal)

1. Introduction

It is by now well known how the spontaneous breaking of chiral symmetry in QCD leads to remarkably strong predictions for the spectral properties of the Dirac operator in that theory. Based first exclusively on the relation to the effective field theory for the associated Nambu–Goldstone bosons at fixed gauge field topology [1], an intriguing relation to universal random matrix theory (RMT) was also pointed out [2]. It has subsequently become clear how these two alternative formulations are related, and all $n$-point spectral correlation functions have been shown to be identical in these two formulations to leading order (LO) in a $1/L$-expansion (where $L \equiv V^{1/4}$ gives the extent of the spacetime volume $V$) [3, 4]. This also then holds for individual distributions of Dirac operator eigenvalues [5].

If one seeks sensitivity to the pion decay constant $F_\pi$, it turns out to be useful to consider the Dirac operator of quark doublets with isospin chemical potential $\mu$. Based on the chiral Lagrangian formulation [6], it has been suggested to use a spectral two-point function of the two associated Dirac operators with imaginary isospin chemical potential. The advantage of
an imaginary chemical potential lies in the fact that the corresponding Dirac operator retains its anti-hermiticity. To LO, all results can be expressed in terms of the simple finite-volume scaling variable $\hat{\mu} = \mu F_\pi \sqrt{V}$. In this way, $F_\pi$ can be extracted from fits that vary $\mu$ and/or $V$. There is also sensitivity to $F_\pi$ in other observables that couple to chemical potential [7–11].

The LO chiral Lagrangian computations of [6] have been given a reformulation in terms of a chiral random two-matrix theory in [12]. In this way, all spectral correlation functions associated with the two Dirac operators $D_1$ and $D_2$, with respective chemical potentials $\mu_1$ and $\mu_2$, have been computed analytically in [12] for both the quenched and the full theory with $N_f$ light flavours. It also includes all spectral correlation functions, where the imaginary isospin chemical potential only enters in the Dirac operator whose eigenvalues are being computed, while the gauge field configurations are obtained in the usual way at vanishing chemical potential. In analogy with what is being done when varying quark masses away from the value used for generating the gauge field configurations, we call this ‘partial quenching’.

In an earlier paper [13], it was shown how all probability distributions of individual Dirac operator eigenvalues can be computed by means of a series expansion in higher $n$-point spectral correlation functions. In [13], an explicit analytical formula was also given for the lowest non-zero eigenvalue distribution for arbitrary combinations of flavours $N_1$ and $N_2$ of the Dirac operators $D_1$ and $D_2$, respectively, at gauge field topology $\nu = 0$, in an approach quite close to that of [14]. Two obvious questions remained open that we will answer in this paper: how to extend this to non-zero topology $\nu > 0$, and how to compute the distribution of the second, third or general $k$th eigenvalue as these are known for zero chemical potential [15]. However, the path of [13] is not very suitable in particular for the derivation of the distributions of higher eigenvalues in a compact analytical manner. Our setup will closely follow [15] at vanishing chemical potential. We shall present here a new formalism that immediately allows for the analytical determination of the distributions of these higher eigenvalues. The extension of the first approach [13] to $\nu > 0$ for the first eigenvalue is presented in appendix C, as an alternative formulation and analytical check to part of our new approach. The benefit of our new results should be twofold: while expressions for higher topology $\nu > 0$ allow for an independent determination of $\Sigma_1$ and $F_\pi$ from different lattice configurations, the expressions for higher eigenvalues should allow for a better determination using the same configurations as for the first eigenvalue.

How much of the program proposed in [6] to determine $F_\pi$ on the lattice has been realized in the meantime? Based on a preliminary account [16] of [13], the expansion of the first eigenvalue was first used in simulations in [17]. However, the question remained how large the finite-volume corrections to the LO epsilon expansion are, in which the chiral Lagrangian–RMT correspondence holds. In a series of papers this question has been addressed and answered: in [18] the next to LO corrections (NLO) and in [19] next to next to LO (NNLO) corrections in the epsilon expansion were computed. As a result of these computations, at NLO all RMT expressions for arbitrary $n$-point density correlations functions (and thus for all individual eigenvalues too) remain valid. The infinite-volume expressions simply get renormalized by finite-volume corrections, one only has to replace $\Sigma_1$ and $F_\pi$ by $\Sigma_{\text{eff}}$ and $F_{\pi \text{ eff}}$ in the corresponding RMT expressions. Here, the subscript ‘eff’ for effective encodes the corrections that match those computed earlier in [20] and [21, 11], respectively. Only at NNLO non-universal, non-RMT corrections appear. It was further noticed in [18, 19] that the size of the corrections at each order depends considerably on the lattice geometry, in particular when using asymmetric geometries. In order to keep the NNLO corrections small, in [22] the authors used a specific optimized geometry where they could apply RMT predictions and effective couplings at NLO only, and obtained realistic values for $\Sigma_1$ and $F_\pi$ from partially quenched lattice data for small chemical potential. Technically speaking, $\Sigma_1$ was determined
there from the first eigenvalue distribution at vanishing chemical potential, and $F_\pi$ from the shift of the eigenvalues compared to zero chemical potential. We refer to [22] for a more detailed discussion of these fits.

Motivated by these findings, we have completed the computation for the $k$th Dirac eigenvalues for all $\nu \geq 0$ in the RMT setting, in order to have a more complete mathematical toolbox at hand.

Our paper is organized as follows. In the next section, we briefly define the notation and remind the reader of the definition of chiral random two-matrix theory. We introduce a certain joint probability density and describe how it can be used to derive individual eigenvalue distributions. We give the explicit finite-$N$ solution here in terms of new polynomials and a new sequence of matrix model kernels. We take the scaling limit relevant for QCD in section 4, and write out explicitly and discuss the physically most important examples such as partially quenched $N_f = 2$ results. Section 5 contains our conclusions and a suggestion for a quite non-trivial but important extension of these results. Because some of the relevant technical details have been described in [13], we have relegated many of the mathematical details in this paper to appendices. In addition, in appendix A we describe an explicit construction of the polynomials needed to compute the first eigenvalue distribution in sectors of non-trivial gauge field topology if one alternatively uses the method of [13].

2. Chiral random two-matrix theory

Before turning to the relevant random two-matrix theory, we first briefly outline the set-up in the language of the gauge field theory. We are considering QCD at finite four-volume $V$, and we assume that chiral symmetry is spontaneously broken at infinite volume. We consider two Dirac operators $D_{1,2}$ with different imaginary baryon (quark) chemical potential $\mu_{1,2}$,

$$D_1 \psi_1^{(n)} \equiv \left( D(A) + i \mu_1 \gamma_0 \right) \psi_1^{(n)} = i \lambda_1^{(n)} \psi_1^{(n)},$$

$$D_2 \psi_2^{(n)} \equiv \left( D(A) + i \mu_2 \gamma_0 \right) \psi_2^{(n)} = i \lambda_2^{(n)} \psi_2^{(n)}.$$  \hspace{2cm} (2.1)

When $\mu \equiv \mu_1 = -\mu_2$ this is simply imaginary isospin chemical potential, but we can stay with the more general case. We thus consider $N_1$ light quarks coupled to quark chemical potential $\mu_1$, and $N_2$ light quarks coupled to quark chemical potential $\mu_2$. Let us first consider the conceptually simplest case where $N_1 + N_2 = N_f$, and later comment on the changes needed to deal with partial quenching.

In the chiral Lagrangian framework, the terms that depend on $\mu_{1,2}$ are easily written down on the basis of the usual correspondence with external vector sources. Going to the $\epsilon$-regime of chiral perturbation theory in sectors of fixed gauge field topology [1], the leading term in the effective partition function including imaginary $\mu_{1,2}$ reads [6, 23] (see also [24] for QCD-like theories)

$$Z^{(N_f)}_{\nu} = \int_{U(N_f)} dU (\det U)^\nu e^{\frac{i}{2} F^2_{\epsilon} \text{Tr}(U, B)[U^\dagger, B] + \frac{i}{2} \Sigma \text{Tr}(\mathcal{M} U + \mathcal{M} U^\dagger)}.$$ \hspace{2cm} (2.2)

In (2.2), the $N_f \times N_f$ matrix,

$$B = \text{diag}(\mu_1 \mathbf{1}_{N_1}, \mu_2 \mathbf{1}_{N_2}),$$  \hspace{2cm} (2.3)

is made out of the chemical potentials, and the quark mass matrix is

$$\mathcal{M} = \text{diag}(m_1, \ldots, m_{N_f}).$$  \hspace{2cm} (2.4)

The partition function (2.2) is a simple zero-dimensional group integral. The leading contribution to the effective low-energy field theory at finite volume $V$ in the $\epsilon$-regime is thus well known.
We consider now the limit in which \( V \to \infty \), while \( \hat{m} = m_\text{eff} \Sigma V \) and \( \hat{\mu} = \mu F_{\text{eff}} \sqrt{V} \) are kept fixed. In this limit, to LO in the \( \epsilon \)-expansion, the effective partition function of this theory and all the spectral correlation functions of its Dirac operator eigenvalues are completely equivalent to the chiral random two-matrix theory with imaginary chemical potential that was introduced in [12]. The equivalence for the two-point function follows from [6], for all higher density correlations it was proven in [4]. Therefore, since we have proven [13] that the probability distribution of the \( k \)th smallest eigenvalue can be computed in terms of this infinite sequence of spectral correlation functions, we are free to use the chiral random two-matrix theory when performing the actual analytical computation.

As already mentioned, it has been shown in [18] that also to NLO in the \( \epsilon \)-expansion the random matrix expressions [12] for density correlation functions remain valid, when replacing \( \Sigma \) and \( F_{\text{eff}} \) by the renormalized constants \( \Sigma_{\text{eff}} \) and \( F_{\text{eff}} \) that encode the finite-volume corrections. Only to NNLO non-universal corrections to the random matrix setting appear.

The partition function of chiral random two-matrix theory is, up to an irrelevant normalization factor, defined as

\[
\mathcal{Z}^{(N)}_v = \int d\Phi d\Psi \ e^{-N\text{Tr}(\Phi^\dagger \Phi + \Psi^\dagger \Psi)} \prod_{j_i=1}^{N_1} \det[D_1 + m_{j_i}] \prod_{j_i=1}^{N_2} \det[D_2 + m_{j_i}],
\]

where \( D_{1,2} \) are given by

\[
D_{1,2} = \begin{pmatrix}
0 & 0 \\
\Phi^\dagger & \Phi^\dagger + i\mu_{1,2} \Psi
\end{pmatrix}.
\]

The operator remains anti-Hermitian because the chemical potentials are imaginary, as shown explicitly. Both \( \Phi \) and \( \Psi \) are complex rectangular matrices of size \( N \times (N + \nu) \), where both \( N \) and \( \nu \) are integers. The index \( \nu \) corresponds to gauge field topology in the usual way. The aforementioned correspondence to chiral perturbation theory holds in the following microscopic large-\( N \) limit:

\[
\lim_{N \to \infty} \mathcal{Z}^{(N)}_v = \mathcal{Z}_v^{(N_1)}, \quad \text{with} \quad \hat{m} = 2Nm, \quad \hat{\mu} = \sqrt{2N} \mu.
\]

In the framework of chiral random two-matrix theory, it is particularly simple to consider the situation corresponding to what we call partial quenching. Here, one simply considers eigenvalues of one of the matrices, say \( D_1 \), that then does not enter into the actual integration measure of (2.5) by setting \( N_1 = 0 \). In the language of the chiral Lagrangian, this needs to be done in terms of graded groups or by means of the replica method.

Reverting to [12] for details, we immediately write down the corresponding representation in terms of eigenvalues \( x_i^2 \) and \( y_i^2 \) of \( D_1 \) and \( D_2 \), respectively,

\[
\mathcal{Z}_v^{(N_1)} = \int_0^\infty dx_1 \cdots dx_N \int_0^\infty dy_1 \cdots dy_N \mathcal{P}_v^{(N_1)}([x], [y]),
\]

up to an irrelevant (mass-dependent) normalization factor. The integrand is the joint probability distribution function, which is central for what follows:

\[
\mathcal{P}_v^{(N_1)}([x], [y]) \equiv \prod_{i=1}^N \left( x_i y_i \right)^{\nu+1} e^{-N(x_i^2 + y_i^2)} \prod_{j_i=1}^{N_1} \left( x_i^2 + m_{j_i}^2 \right) \prod_{j_i=1}^{N_2} \left( y_i^2 + m_{j_i}^2 \right) \times \Delta_N(|x_i^2|) \Delta_N(|y_i^2|) \det_{1 \leq i, j \leq N} [I_c(2dN x_i y_j)].
\]

Because the integration in equation (2.5) was over \( \Phi \) and \( \Psi \) separately, the matrices now become coupled in the exponent. The corresponding unitary group integral leads to the
determinant of modified J-Bessel functions, and removes one of the initially two Vandermonde determinants, which is defined as $\Delta_N((x^2)) = \prod_{j=1}^{N} (x_j^2 - x_i^2)$. The precise connection between the constants and $\mu_{1, 2}$ is given by
\begin{align}
c_1 &= (1 + \mu_2^2) / \delta^2, \quad c_2 = (1 + \mu_1^2) / \delta^2, \\
d &= (1 + \mu_1\mu_2) / \delta^2, \quad \delta = \mu_2 - \mu_1,
\end{align}
(2.10)
where the latter is defined for later convenience. We need the joint probability distribution to be normalized to unity, which is done trivially by dividing by $Z^2_N$ (cf equation 2.8).

3. The kth eigenvalue at finite-$N$ for arbitrary $\nu \geq 0$

We now follow the derivation of [15] rather closely. We are able to do that because we focus here on the distributions of individual $x$-eigenvalues only—which are those we may partially quench. For that purpose, it is convenient to first consider the joint probability distribution of the $k$ smallest $x$-eigenvalues, ordered such that $0 \leq x_1 \leq x_2 \leq \cdots \leq x_k$:
\begin{align}
\Omega^{(N)}_\nu(x_1, \ldots, x_k) &= \frac{N!}{Z^{(N)}_\nu(N-k)!} \int_{x_1}^\infty \mathrm{d}x_{k+1} \cdots \int_{x_k}^\infty \mathrm{d}x_N \int_0^\infty \prod_{i=1}^N \mathrm{d}y_i \mathcal{P}^{(N)}_\nu(x_i, \{y_i\}).
\end{align}
(3.1)
This quantity is then used to generate the $k$th $x$-eigenvalue distribution through the following integration:\footnote{Compared to [15] we are already working with squared variables here. Translating to that picture the integration bounds in equations (3.1) and (3.2) remain the same as in [15].}
\begin{align}
p^{(N, \nu)}_k(x_k) &= \int_0^\infty \mathrm{d}x_1 \int_{x_1}^\infty \mathrm{d}x_2 \cdots \int_{x_{k-1}}^\infty \mathrm{d}x_{k-1} \Omega^{(N)}_\nu(x_1, \ldots, x_k).
\end{align}
(3.2)
Note that for $k = 1$, no integration is needed, and $p^{(N, \nu)}_1(x_1) = \Omega^{(N)}_\nu(x_1)$.

The computation of mixed or conditional individual eigenvalue distributions, e.g. to find the joint distribution of the first $x$- and $y$-eigenvalues, remains an open problem.

We next proceed as in [13], and integrate out all $y$-eigenvalues exactly. Because of this we note that in equation (3.1) we can replace the determinant over the Bessel functions by $N!$ times its diagonal part, after having made use of the antisymmetry property of $\Delta_N((y^2))$.

After inserting a representation of the Bessel function in terms of a factorized infinite sum over Laguerre polynomials (see equation (7.7) in [12]), we obtain
\begin{align}
\int_0^\infty \prod_{i=1}^N \mathrm{d}y_i \mathcal{P}^{(N)}_\nu(x_i, \{y_i\}) &= N! \int_0^\infty \prod_{i=1}^N \left( \prod_{j=1}^{N_i} \frac{(x_j^2 + m_j^2)}{\delta_j^2} \right) \Delta_N((x^2)) \\
&\times \Delta_N((y^2)) \prod_{i=1}^N (N\delta)^{2v+1}(\delta_1 \cdots \delta_N)^{2v+1} e^{-N(x_1^2 + \cdots + x_N^2)} \\
&\times \sum_{n_i=0}^{\infty} \frac{n_i!(1-\tau)^{m_i}}{(n_i + v)!} L^{(n_i)}_m(N\tau(x_1 \cdots x_N^2)) L^{(n_i)}_m(N\tau(y_1 \cdots y_N^2))
\end{align}
(3.3)
where the Laguerre polynomials $L^{(n_i)}_m(N\tau y_1 \cdots y_N^2)$ now appear with their corresponding weight function $y^{2v+1} e^{-N\tau y^2}$ due to the identity used. Next we include the set of $N_2$ masses, $\{m_2\}$,
into $\Delta_N(|y|^2)$ to form a larger Vandermonde determinant of size $N + N_2$, and then replace it by a determinant of in general arbitrary Laguerre polynomials normalized to be monic:

$$
\Delta_N(|y|^2) \prod_{i=1}^N \prod_{j_1=1}^{N_1} \left( y_i^2 + m^2_{j_1} \right) = \begin{vmatrix}
\hat{L}_0^\nu(N\tau c_2(im_{j_1}=1)^2) & \cdots & \frac{1}{(N\tau c_2)^{N+N_2-1}} \hat{L}_N^\nu(N\tau c_2(im_{j_1}=1)^2) \\
\vdots & \ddots & \vdots \\
\hat{L}_0^\nu(N\tau c_2(im_{N_1})^2) & \cdots & \frac{1}{(N\tau c_2)^{N+N_2-1}} \hat{L}_N^\nu(N\tau c_2(im_{N_1})^2) \\
\vdots & \ddots & \vdots \\
\hat{L}_0^\nu(N\tau c_2(y_N^2)) & \cdots & \frac{1}{(N\tau c_2)^{N+N_2-1}} \hat{L}_N^\nu(N\tau c_2(y_N^2))
\end{vmatrix}
\Delta_N(|(im|^2)) \quad (3.4)
$$

Here, the index of the Laguerre polynomials $\nu$ is arbitrary. The monic Laguerre polynomials relate to ordinary Laguerre polynomials in a $\nu$-independent manner:

$$
\hat{L}_j^\nu(x) \equiv (-1)^n n! L_j^\nu(x) = \sum_{j=0}^n (-1)^{n+j} \frac{n!(n+\nu)!}{(n-j)!(\nu+j)!} x^j = x^n + O(x^{n-1}). \quad (3.5)
$$

In equation (3.4), the inverse powers $(N\tau c_2)^{-j}$ can be taken out of the determinant. Inserting this back into equation (3.3) for $\nu = \tilde{\nu}$, we can use the orthogonality of the Laguerre polynomials in the integrated variables $y_1,...,N$, killing the infinite sums from the expanded Bessel functions. The Laguerre polynomials in $x$ thus replace those in $y$ inside the determinant, times the norm from the integration. We obtain

$$
\int_0^\infty \prod_{j=1}^N \frac{d\gamma_j}{P^{(N_j)}_\nu}(\{x\}, \{y\}) = \frac{N!((N\tau c_2)^{N+N_2-1} \prod_{j=0}^{N} (1 - \tau)^j (N\tau c_2)^{-j})}{\Delta_N(|(im|^2)) 2^N (N\tau c_2)^{N(N+1)}} \times \prod_{j=1}^N \left( x_j^{2\nu+1} e^{-N\tau c_2 x_j^2} \prod_{j_1=1}^{N_1} (x_j^2 + m^2_{j_1}) \right) \Delta_N(|x|^2) \times 
\begin{vmatrix}
\hat{L}_0^\nu(N\tau c_2(im_{j_1}=1)^2) & \cdots & \frac{1}{(1-\tau)^{N+N_2-1}} \hat{L}_N^\nu(N\tau c_2(im_{j_1}=1)^2) \\
\vdots & \ddots & \vdots \\
\hat{L}_0^\nu(N\tau c_2(im_{N_1})^2) & \cdots & \frac{1}{(1-\tau)^{N+N_2-1}} \hat{L}_N^\nu(N\tau c_2(im_{N_1})^2) \\
\vdots & \ddots & \vdots \\
\hat{L}_0^\nu(N\tau c_2(y_N^2)) & \cdots & \frac{1}{(1-\tau)^{N+N_2-1}} \hat{L}_N^\nu(N\tau c_2(y_N^2))
\end{vmatrix}
\quad (3.6)
$$

after taking out common factors of the determinant. The determinant in equation (3.6), which we call $D_{N+N_2}$, can almost be mapped to a Vandermonde determinant, using an identity proved
We can now change variables in appendix A in [13]

\[
D_{N+N_1}(\{m_2^2\}; \{x^2\}) = \left| \begin{array}{cccc}
\hat{L}_0 \left( \frac{1}{\tau} M^2_{\hat{f}_1} \right) & \cdots & \frac{\tau^{N+N_1-1}}{(1-\tau)^{N+N_1-1}} \hat{L}_0 \left( \frac{1}{\tau} M^2_{\hat{f}_1} \right) \\
\vdots & \ddots & \vdots & \vdots \\
\frac{\tau^{N+N_1-1}}{(1-\tau)^{N+N_1-1}} \hat{L}_0 \left( \frac{1}{\tau} M^2_{\hat{f}_1} \right) & \cdots & \frac{\tau^{N+N_1-1}}{(1-\tau)^{N+N_1-1}} \hat{L}_0 \left( \frac{1}{\tau} M^2_{\hat{f}_1} \right) & 1 \\
\end{array} \right|
\]

where we have defined

\[
M^2_{\hat{f}_1} \equiv N\tau c_2 (i m_{f_1})^2 \quad \text{and} \quad X_j^2 \equiv N\tau c_1 x_j^2.
\]

This fact can be used below to perform the \(N - k\) remaining integrations in the generating quantity \(\Omega_k\), after inserting equation (3.7) into equations (3.6) and (3.1). This leads to

\[
\Omega^{(N_1)}(x_1, \ldots, x_k) = C \prod_{j=j+2}^{k} (x_j^2 - x_1^2) \int_{x_2}^{\infty} dx_2 \cdots \int_{x_k}^{\infty} dx_k
\times \prod_{j=i+2}^{N} (x_j^2 - x_i^2) \prod_{i=1}^{k} (x_i^2 - x_i^2)
\times \prod_{i=1}^{N} \left( x_i^{2i+1} e^{-N\tau c_1 x_i^2} \prod_{j=1}^{N_1} (x_i^2 + m_j^2)^{N_1} \right) D_{N+N_1}(\{m_2^2\}; \{x^2\})
\]

where we have split the Vandermonde determinant \(\Delta_N(\{x^2\})\) into integrated and unintegrated variables, and defined the following constant:

\[
C \equiv \frac{(N!)^2 (Nd)^{N^2} e^{-\tau N^2}}{2^{N^2} \tau^{N^{(N+1)}} 2N \nu \tau^{(N+1)} \Delta_N(1(i m_{2}^2))}.
\]

We can now change variables \(x_j \rightarrow u_j = x_j^2\) for \(j = k + 1, \ldots, N\), and then perform the shift \(u_j \rightarrow z_j = u_j - x_k^2\) to obtain integrations \(\int_0^\infty dz_j\) in equation (3.9):

\[
\Omega^{(N_1)}(x_1, \ldots, x_k) = C \prod_{j=j+2}^{k} (x_j^2 - x_1^2) \prod_{i=1}^{k} \left( x_i^{2i+1} e^{-N\tau c_1 x_i^2} \prod_{j=1}^{N_1} (x_i^2 + m_j^2) \right) \int_0^\infty \frac{1}{2^{N-k}} e^{-N(N-k)\tau c_1 x_k^2}
\times \prod_{j=i+2}^{N} \left( dz_j e^{-N\tau c_1 z_j} \prod_{j=1}^{k} (z_j + x_k^2 - x_i^2) \prod_{j=1}^{N_1} (z_j + x_k^2 + m_j^2) \right)
\times \prod_{j=i+2}^{N} \left( z_j - z_i \right) D_{N+N_1}(\{m_2^2\}; x_1^2, \ldots, x_k^2, z_{k+1}^2 + x_k^2, \ldots, z_N + x_k^2).
\]
We thus obtain an integral with \(\nu + k - 1\) extra mass terms of flavour-type ‘1’, in addition to the \(N_1\) shifted masses. The weight

\[ w(z) = z^3 e^{-N\tau c_1 z} \]  

(3.12)
is now of Laguerre-type corresponding to a fixed topological charge of \(\bar{\nu} = 1\), irrespective of the actual topological charge \(\nu\) of the given gauge field sector we started with. We will therefore call \(\bar{\nu}\) spurious topology. Compared with the corresponding derivation in the case of vanishing chemical potential [15], this can be seen to differ by one unit, compared to spurious topology \(\bar{\nu} = 2\) at vanishing \(\mu_{1,2}\) in [15]. The reason for this difference is easily traced to the different integration measure for the \(x\)-eigenvalues, which has one power less in the Vandermonde determinant compared to the case of vanishing chemical potential\(^4\). It is an interesting and quite non-trivial check on our present calculation that we recover the results of [15] in the limit of vanishing chemical potential. In particular, the shift from spurious topology \(\bar{\nu} = 1\) to spurious topological charge \(\bar{\nu} = 2\) in the integration measure will now arise due to recurrence relations of Laguerre polynomials. Some details of this will be given below.

When replacing the Vandermonde determinant in the variables \(z_j\) as well as \(D_{N+k_2}\) by a determinant containing Laguerre polynomials, we thus choose polynomials \(L_j^1(N\tau c_1 z)\) in order to be able to exploit the orthogonality properties with respect to the measure \(w(z)\) equation (3.12).

For the new masses times \(\Delta_{N+k}(\{z\})\) this is an easy task. We can include them into a bigger determinant of size \(N - k + N_1 + \nu + k - 1\), following the identity equation (3.4). Here we replace the \(N\) variables \(y_i^2\) by \(N - k\) variables \(z_i\), and the set of \(N_2\) masses by the following set of \(N_1 + \nu + k - 1\) masses:

\[
\begin{align*}
    m_{f_1}^2 &\equiv m_{f_1}^2 + x_k^2 & \text{for } f_1 = 1, \ldots, N_1, \\
    m_{N_1+j}^2 &\equiv x_k^2 + \epsilon_j^2 & \text{for } j = 1, \ldots, \nu, \\
    m_{N_1+\nu+j}^2 &\equiv x_k^2 - x_i^2 & \text{for } i = 1, \ldots, k - 1,
\end{align*}
\]  

(3.13)

and likewise we define

\[
M_j^2 \equiv N\tau c_1 (im_j')^2 \quad \text{for } j = 1, \ldots, N_1 + \nu + k - 1.
\]  

(3.14)

For computational simplicity, we first set the \(\nu\) degenerate masses to be different by adding small pairwise different constants, \(\epsilon_j^2\), and then set \(\epsilon_j = 0\) at the end of the computation. Also, we may choose spurious topology \(\bar{\nu} = 1\) in equation (3.4). The prefactors in front of the Laguerre polynomials inside the determinant can be taken out.

To express the determinant \(D_{N+k_2}\) of the shifted arguments in equation (3.11) in terms of Laguerre polynomials requires a bit more algebra:

\(^4\) Of course, the additional pieces due to the \(y\)-integrations are what ensures equivalence to those corresponding one-matrix model results in the limit \(\mu_j \to 0\).


where for convenience we have defined $M_{N+1}^2 = 0$, as well as

$$Z_k \equiv N \tau c_1 z_k.$$  \hfill (3.16)

In the first step in equation (3.15), we have used the invariance of the determinant to undo the shift in $x_i^2$ of the $z_i$ variables. This leads to a shift in variables $X_i$ and to linear combinations of the Laguerre polynomials in the $N_2$ masses. In the second step, we have added columns from the left to the right to replace monic powers in $Z_l$ and $M_{N+1}^2$ by polynomials $\tilde{L}^1_l$. Because the determinant $D_{N+1}$ is not an invariant Vandermonde this leads to a further sum in the first $N_2$ rows, invoking the following new polynomials:

$$q''_l(M_{j_l}) = (-)^n n! \sum_{l=0}^{n} \frac{1}{(1 - \tau)^l} L^l_j(M_{j_l}) L_{N-1}^{N}(-X_k^2).$$  \hfill (3.17)

The form given here is derived in appendix A using identities for Laguerre polynomials. For later reference, we note already that in the limit of zero chemical potential, i.e. in the limit $\tau \rightarrow 0$, we obtain Laguerre polynomials of shifted mass from the $q''_l$:

$$\lim_{\tau \rightarrow 0} q''_l(M_{j_l}) \frac{1}{(-)^n n!} = \sum_{j=0}^{n} L^l_j(-Nm^2) L_{N-1}^{N}(-Nz_k^2) = L^1_l(-N(m^2 + z_k^2)).$$  \hfill (3.18)
In this way we recover, after the use of a few identities for Laguerre polynomials, the results of [15] in the limit of vanishing chemical potential.

We now proceed with the integration over the variables $z_{k+1}, \ldots, z_N$ in equation (3.11). Using the rewriting discussed above, we have

$$\Omega_\nu^{(N)}(x_1, \ldots, x_k) = C \prod_{j+1}^k (x_j^2 - x_i^2) \prod_{i=1}^k \left( x_i^{2v+1} e^{-\nu \tau_1 x_i^2} \prod_{f_i=1}^{N_f} (x_i^2 + m_{f_i}^2) \right)$$

$$\times \frac{\Delta_{k+1-k-1}((im)^2)}{\Delta_{k+1-k}((im)^2)} \int_0^\infty \prod_{j=k+1}^N (dz_j e^{-\nu \tau_1 z_j^2}) \prod_{j=0}^{N+N_f+k-2} (\nu \tau_1)^{-j}$$

$$\times \left| \begin{array}{cccc}
\hat{L}_0^k(M_1^2) & \cdots & \hat{L}_0^{N+N_f+k-2}(M_1^2) \\
\vdots & \ddots & \vdots \\
\hat{L}_0^k(M_1^2+N_f+k-1) & \cdots & \hat{L}_0^{N+N_f+k-2}(M_1^2+N_f+k-1) \\
\hat{L}_0^k(Z_k+1) & \cdots & \hat{L}_0^{N+N_f+k-2}(Z_k+1) \\
\vdots & \ddots & \vdots \\
\hat{L}_0^k(Z_N) & \cdots & \hat{L}_0^{N+N_f+k-2}(Z_N) \\
q_0^k(M_1^2) & \cdots & q_0^{N+N_f+k-1}(M_1^2) \\
q_0^k(M_1^2+N_f+k) & \cdots & q_0^{N+N_f+k-1}(M_1^2+N_f+k) \\
\vdots & \ddots & \vdots \\
q_0^k(M_1^2+N_f+k-1) & \cdots & q_0^{N+N_f+k-1}(M_1^2+N_f+k-1) \\
\hat{L}_0^k(Z_k+1) & \cdots & \hat{L}_0^{N+N_f+k-2}(Z_k+1) \\
\vdots & \ddots & \vdots \\
\hat{L}_0^k(Z_N) & \cdots & \hat{L}_0^{N+N_f+k-2}(Z_N) \\
\end{array} \right|_{(3.19)}.$$ 

This expression can be simplified somewhat by noting the identity

$$\prod_{f_i=1}^{N_f} \left( x_i^{2v+1} \prod_{f_i=1}^{N_f} (x_i^2 + m_{f_i}^2) \right)$$

$$\frac{\Delta_{k+1-k-1}((im)^2)}{\Delta_{k+1-k}((im)^2)} \frac{x_i^{2v+1}}{\Delta_{k+1-k}((im)^2)} \prod_{f_i=1}^{N_f} (x_i^2 + m_{f_i}^2) (1 + O(\epsilon)).$$

The Vandermonde determinant $\Delta_{N_f}((im)^2)$ also occurs as a factor in the partition function, and thus cancels.

The integrals over $z_j$, $j = k + 1, \ldots, N$ can be performed, again exploiting orthogonality properties of the Laguerre polynomials with respect to our weight function (3.12). This is done by means of a generalization of the original Dyson theorem (see e.g. [25]), now for two determinants of different size, of which different sets of entries are not even of Laguerre type, and not integrated over either. The needed generalization of this theorem was essentially provided in [26], and we only need a slight extension of this more general theorem here. We have relegated the proof of the new theorem to appendix B, and will only quote the result here.
Similarly, for J. Phys. A: Math. Theor. 45 (2012) 115205

To express the result, we define two kernels (both corresponding to the kernel $K'$ of appendix B for different values of the indices):

$$K(M_1^2, M_2^2) = \sum_{j=0}^{\Lambda} \frac{1}{j+1} L_j^1(M_1^2) L_j^1(M_2^2),$$

$$\tilde{K}(M^2, M^2) = \sum_{j=0}^{\Lambda} (-)^j (j+1)! L_j^1(M^2) q_j^1(M^2),$$

with $\Lambda = \min(N + N_1 + \nu - 2, N + N_2 - 1)$, and the associated matrix of size $(N_2 + k) \times (N_1 + \nu + k - 1)$:

$$K = \begin{pmatrix}
\tilde{K}(M_1^2, M_2^2) & \ldots & \tilde{K}(M_{N_1+v+k-1}^2, M_{f_2=1}^2) \\
\vdots & \ddots & \vdots \\
\tilde{K}(M_{N_1+v+k}^2, M_{N_2}^2) & \ldots & \tilde{K}(M_{N_1+v+k}^2, M_{f_2=1}^2) \\
K(M_1^2, M_{N_1+v+1}^2) & \ldots & K(M_{N_1+v+k-1}^2, M_{N_1+v+1}^2) \\
\vdots & \ddots & \vdots \\
K(M_{N_1+v+k}^2, M_{N_1+v+k}^2) & \ldots & K(M_{N_1+v+k}^2, M_{N_1+v+k}^2)
\end{pmatrix}.$$  \hspace{1cm} (3.21)

We can now give the answer for $\Omega_{\nu}^{(N_1)}$, where we have to distinguish three different cases, depending on the respective sizes of the two determinants in (3.19).

(i) The simplest case is $N_2 = N_1 + \nu - 1$ (this implies that $K$ is square). Applying appendix B to the integral of (3.19) and using equation (3.20), we obtain to leading order in mass difference $\epsilon$:

$$\Omega_{\nu}^{(N_1)}(x_1, \ldots, x_k) = C_k \sum_{j=0}^{N_1} \prod_{l=1}^{N_1} (x_l e^{-N \tau c_1 x_l^2} x_l^2 - x_l^2) \prod_{f=1}^{N_2} (x_f^2 + m_f^2)^j (N \tau c_1)^{-2(N-k)}$$

$$\times e^{-\nu(N-k+1)\tau c_1 x_k^2} \prod_{j=0}^{N_1+N_2} (N \tau c_1)^{-j}$$

$$\times \prod_{j=0}^{\Lambda} ((j+1)! j!) (N-k)! (-)^{(N_1+N_2+v+1)-(N-k)} \det K.$$  \hspace{1cm} (3.23)

A typical example for this case would be $N_1 = N_2 = \nu = 1$.

(ii) For $N_2 > N_1 + \nu - 1$, the expression is the same but with additional columns in the determinant in the second line of equation (3.23):

$$K = \begin{pmatrix}
q_{N_1+N_1+v+1}^1(M_{f_2}^2) & \ldots & q_{N_1+N_1+v+1}^1(M_{f_2}^2) \\
\vdots & \ddots & \vdots \\
q_{N_1+N_2}^1(M_{f_2}^2) & \ldots & q_{N_1+N_2}^1(M_{f_2}^2) \\
\hat{L}_{N_1+N_1+v+1}(M_{f_2}^2) & \ldots & \hat{L}_{N_1+N_2-1}(M_{f_2}^2) \\
\hat{L}_{N_1+N_1+v+1}(M_{f_2}^2) & \ldots & \hat{L}_{N_1+N_2-1}(M_{f_2}^2)
\end{pmatrix},$$

$$f_2 = 1, \ldots, N_2, \hspace{1cm} j = 1, \ldots, k.$$  \hspace{1cm} (3.24)

This case typically occurs for partial quenching $N_1 = 0$, $N_2 = 2$ and small topology $\nu = 0, 1, 2$.

(iii) Similarly, for $N_2 < N_1 + \nu - 1$, which typically occurs for higher topology, the determinant in equation (3.23) is replaced by

$$K^T = \begin{pmatrix}
\hat{L}_{N_1+N_1+v+2}(M_f^2) & \ldots & \hat{L}_{N_1+N_1+v+2}(M_f^2) \\
\vdots & \ddots & \vdots \\
\hat{L}_{N_1+N_2}(M_f^2) & \ldots & \hat{L}_{N_1+N_2}(M_f^2) \\
\hat{L}_{N_1+N_1+v+2}(M_f^2) & \ldots & \hat{L}_{N_1+N_2}(M_f^2)
\end{pmatrix},$$

$$f_1 = 1, \ldots, N_2, \hspace{1cm} j = 1, \ldots, N_1 + \nu + k - 1.$$  \hspace{1cm} (3.25)
We note that all three cases contain degeneracies for \( \nu > 1 \), and that a Taylor expansion must be performed when sending \( \epsilon_j \to 0 \).

Finally, we are ready to compute the \( k \)th individual eigenvalue probability distribution \( p_k^{(\nu)}(x) \) by means of inserting the above results into equation (3.2):

\[
p_k^{(Nf,\nu)}(x_k) = \int_0^{x_k} dx_1 \int_{x_1}^{x_k} dx_2 \cdots \int_{x_{k-2}}^{x_k} dx_{k-1} \Omega_1^{(\nu)}(x_1, \ldots, x_k).
\]

An alternative, equivalent formulation for the first eigenvalue \( p_1^{(\nu)}(x) \) at arbitrary \( \nu \) follows from the derivative of its cumulative distribution given in appendix C, where we generalize the approach of [13] to \( \nu > 0 \).

We refrain from giving further explicit examples for finite \( N \) and instead turn to the large-\( N \) limit.

4. The microscopic scaling limit

Having obtained the explicit solution for any finite \( N \), we are now ready to take the appropriate microscopic \( N \to \infty \) scaling limit. In the language of QCD, this corresponds to a finite-volume scaling in \( V \), where \( V \) is the spacetime volume. It is only in this limit that we expect to obtain universal results which do not depend on having chosen Gaussian measures for the two original matrices \( \Phi \) and \( \Psi \). We remind the reader that the appearance of Laguerre polynomials in the finite-\( N \) solution is directly linked to these two original measures having been chosen Gaussian. As we have seen, there are numerous instances where our derivation makes explicit reference to specific identities for Laguerre polynomials, and also the measure factor was specific to these Gaussian integrals. It is therefore a quite non-trivial task, and an interesting challenge, to generalize the universality proof of the last reference in [2] to this more general setting. The universality has been implicitly checked by the equivalence proof of [4], but an explicit proof of universality directly in the framework of the chiral random two-matrix theory remains to be constructed.

In QCD terminology, we keep \( \hat{m}_i \equiv m_i \Sigma V \) and \( \mu \equiv \mu F \pi / \sqrt{V} \) fixed, while we take \( V \to \infty \). In the language of our chiral random two-matrix theory, we take the \( N \to \infty \) limit while keeping

\[
\hat{\chi}_j \equiv 2N\chi_j, \quad \hat{m}_j \equiv 2Nm_j, \quad \hat{\mu}_{1,2} \equiv 2N\mu_{1,2}
\]

(4.1)

fixed (we also scale the masses \( \epsilon_j \) in the same way, before taking them to zero at the end of the calculation). In addition, we introduce the following relevant quantity, the difference in rescaled chemical potential, \( \hat{\delta} \equiv \hat{\mu}_2 - \hat{\mu}_1 \). We will follow [13] closely, without giving a detailed derivation.

Defining the following continuum indices

\[
t \equiv j/N, \quad \text{and} \quad r \equiv l/j,
\]

(4.2)

we replace the sum by an integral, \( \sum_{j=0}^{N-1} \to \int_0^1 \text{d}t \), and correspondingly for index \( l \). In addition, we use the identities

\[
\lim_{N,j,l \to \infty} \frac{1}{(1 - \tau)^\nu} = \exp \left[ \frac{1}{2} r \delta^2 \right],
\]

\[
\lim_{N,j,l \to \infty} j^{-\nu} L_j^\nu(M^2 = -N\tau c_1 m^2) = \left( \frac{4r}{\hat{m}^2} \right)^{\nu/2} I_\nu(\sqrt{rt} \hat{m}).
\]

(4.3)
The scaling limit of the kernel $K$ is obtained using the Christoffel–Darboux identity:

$$
K_S(\hat{\nu}_1, \hat{\nu}_2) = \lim_{N \to \infty} \frac{1}{N^2} K_N(M^2_1, M^2_2) = \lim_{N \to \infty} \frac{1}{N^2} L_{N+1}^1(M^2_1) L_{N}^1(M^2_2) - L_{N+1}^1(M^2_2) L_{N}^1(M^2_1)
$$

$$
= 8 \frac{\hat{\nu}_1 I_0(\hat{\nu}_1) I_1(\hat{\nu}_2) - \hat{\nu}_1 I_0(\hat{\nu}_2) I_1(\hat{\nu}_1)}{\hat{\nu}_1^2 - \hat{\nu}_2^2}.
$$

(4.4)

The scaling limit of the new polynomials $q^\nu_N$ requires a little more care. For $\nu > 0$ one needs to treat the terms corresponding to indices $l = N - \nu + 1, \ldots, N$ in (3.17) separately. For those ‘anomalous’ terms, the scaling limit is

$$
q^\nu_{S\lambda}(\hat{\nu}; t) = \lim_{N, j \to \infty} j^{-1} \sum_{l=\nu-1}^j \frac{1}{(1 - \tau)^l} L_l^1(M^2) L_{j-l}^1(-X^2_\nu)
$$

$$
= \lim_{N, j \to \infty} \sum_{p=0}^{j-\nu-1} \frac{1}{p!} \left( -\frac{t^2_2}{4} + O(j^{-1}) \right)^{p}
$$

$$
\times \sum_{j=\nu}^{j-\nu-1} (-1)^j \frac{1}{j - l - p} \frac{1}{(1 - \tau)^l} j^{-l} X^2_\nu
$$

$$
= \sum_{p=0}^{j-\nu-1} \frac{1}{p!} \left( \frac{t^2_2}{4} \right)^{p} \frac{\partial^{p-1}}{\partial r^{p-1}} \left[ e^{\frac{4r}{\sqrt{t^2_2}m}} I_{\nu/2} \right].
$$

(4.5)

Here we write out $L_{j-l}$ and use the fact that the binomial weight kills the terms of order lower than $\nu - p - 1$ of the expansion in $r$. Note that naively $q^\nu_{S\lambda}$ is proportional to $j^{-1}$, but that the pattern of cancellation is exactly such that the limit is finite. The notation in equation (4.5) is chosen such that for $\nu = 0$ the sum is void, and thus the anomalous term is absent for $\nu = 0$, $q^\nu_{S\lambda} = 0$.

For the remainder of the terms, the sum turns into an integral:

$$
q^\nu_{S\beta}(\hat{\nu}; t) = \lim_{N, j \to \infty} j^{-1} \sum_{l=0}^{j-\nu} \frac{1}{(1 - \tau)^l} L_l^1(M^2) L_{j-l}^1(-X^2_\nu)
$$

$$
= \left( \frac{\hat{\nu}}{m} \right)^{\nu} \int_0^1 dr e^{\frac{4r}{\sqrt{t^2_2}m}} I_{(\nu/2)} (\sqrt{(1 - r)t_0^\nu}).
$$

(4.6)

The final scaling limit of the new polynomials is then the sum

$$
q^\nu_5(\hat{\nu}; t) = \lim_{N, j \to \infty} \frac{(-1)^j}{j!} q^\nu_j(M^2) = q^\nu_{S\beta}(\hat{\nu}; t) + q^\nu_{S\lambda}(\hat{\nu}; t).
$$

(4.7)

Let us give the limiting polynomials for the first topology $\nu = 0, 1, 2$ as examples:

$$
q^0_{S\beta}(\hat{\nu}; t) = q^0_{S\beta}(\hat{\nu}; t) = \int_0^1 dr e^{\frac{4r}{\sqrt{t_0^2m}}} I_0(\sqrt{t_0^\nu}) I_0(\sqrt{(1 - r)t_0^\nu}),
$$

(4.8)

$$
q^1_{S\beta}(\hat{\nu}; t) = \frac{\hat{\nu}}{m} \int_0^1 dr e^{\frac{4r}{\sqrt{t_0^2m}}} \left( \frac{r}{1 - r} \right)^{1/2} I_1(\sqrt{t_0^\nu}) I_1(\sqrt{(1 - r)t_0^\nu}) + e^{\frac{4r}{\sqrt{t_0^2m}} - \frac{2}{\sqrt{t_0^2m}}} I_1(\sqrt{t_0^\nu}),
$$

(4.9)

5 For integer $\nu > 0$ and fixed $j, z$, the limit $\lim_{n \to \infty} n L_j^{-\nu}(z/n)$ is only finite for $j \gg \nu$. 

13
\[ q^{(2)}_S (\hat{m}; t) = \left( \frac{\lambda}{\hat{m}} \right)^2 \int_0^1 \frac{d \rho_1}{1 - \rho_1} \frac{r}{1 - r} l_2 (\sqrt{\tau} \hat{m}) l_2 (\sqrt{(1 - r)} \hat{d}), \]
\[ + e^{i \hat{d}^2} \left( \frac{2 \hat{d}^2}{\hat{m}^2} l_2 (\sqrt{\tau} \hat{m}) + \frac{2}{\sqrt{\tau} \hat{m}} l_1 (\sqrt{\tau} \hat{m}) \right). \]

(4.10)

For comparison see equation (C.12) for the limiting polynomials of the previous approach [13] for the first eigenvalue extended to \( \nu \geq 0 \). Because of technical reasons the anomalous terms already appear there at \( \nu = 0 \).

The last building block we need is the new kernel \( \hat{K}^\nu (M^2, M^2) \), equation (3.21). It contains both the regular and anomalous terms \( q^{(2)}_S = q^{(2)}_{SR} + q^{(2)}_{SA} \) inside the integral (originating from the sum), but no further complications occur. Hence, we have the scaling limit

\[ \hat{K}^\nu (\hat{m}', \hat{m}) \equiv \lim_{N \to \infty} \frac{1}{N^2} \hat{K}^\nu_N (M^2, M^2) = \frac{2}{\hat{m}^2} \int_0^1 dt \sqrt{\tau} l_1 (\sqrt{\tau} \hat{m}) q^{(2)}_S (\hat{m}; t). \]

(4.11)

Using these building blocks, we can calculate the main quantity of interest, the scaling limit of the eigenvalue distributions defined as

\[ P^{(N, \nu)}_S (\hat{x}_k) \equiv \int_0^{\hat{x}_k} d \hat{x}_1 \int_{\hat{x}_1}^{\hat{x}_2} d \hat{x}_2 \ldots \int_{\hat{x}_{k-2}}^{\hat{x}_{k-1}} d \hat{x}_{k-1} \Omega^{(N)}_{S^\nu} (\hat{x}_1, \ldots, \hat{x}_k), \]

with

\[ \Omega^{(N)}_{S^\nu} (\hat{x}_1, \ldots, \hat{x}_k) \equiv \lim_{N \to \infty} (2N)^{-k} \Omega^{(N)}_{S^\nu} (x_1 = \frac{\hat{x}_1}{2N}, \ldots, x_k = \frac{\hat{x}_k}{2N}). \]

(4.12)

4.1. Example partial quenching

As pointed out in [5], there is no dependence in the fully quenched \( (N_1 = N_2 = 0) \) theory. The simplest case to consider is then also what is probably the physically most relevant situation, namely the partially quenched case, where \( N_1 = 0 \). On the lattice gauge theory side, this corresponds to generating the gauge configurations using \( N_2 \) dynamical quarks with zero chemical potential, and then looking at the spectrum of the Dirac operator \( D_1 \) with \( \hat{\mu}_1 = -\hat{\delta} \).

In the simplest case \( N_1 = 1, \nu = 0 \) and \( k = 1 \) (case ii), the matrix \( K \) is empty, and (3.24) is just a \( 2 \times 2 \) determinant of polynomials, with \( M'_1 = 0 \). The two columns are degenerate, so we perform a Taylor expansion in \( t \). With the abbreviation \( \hat{m} = \hat{m}_{1=1} \), the limiting distribution is then found to be

\[ P^{(0+1, 0)}_S (\hat{x}_1) = - \frac{e^{i \frac{1}{2} \hat{d}^2}}{2l_0 (\hat{m})} \hat{x}_1 e^{i \frac{1}{2} \hat{d}^2} \left| q^{(0)}_S (\hat{m}; t = 1) - \frac{\partial}{\partial t} q^{(0)}_S (\hat{m}; t) \right|_{t=1}, \]

(4.14)

in agreement (after partial integration) with the expression in [5]. By introducing primed masses, and changing the index of the (scaling limit of the) new polynomials \( q^{(0)}_S \) and the partition function, we obtain the higher topology distributions which are new.

For \( \nu = 1 \) (case ii), we can set \( \epsilon_1 = 0 \) in equation (3.15) from the beginning, and find

\[ P^{(0+1, 1)}_S (\hat{x}_1) = - \frac{\hat{m}_1^2 e^{i \frac{1}{2} \hat{d}^2}}{16l_1 (\hat{m})} \left| K^{(1)}_S (\hat{x}_1, \hat{m}) - \frac{\partial}{\partial t} q^{(1)}_S (\hat{m}; t = 1) \right|_{t=1}, \]

(4.16)

where we note that the singularities of \( K (\hat{m}_1', \hat{m}_2') \) at \( \hat{m}_2' \to \hat{m}_1' \) and \( \hat{m}_2' \to 0 \) are removable. Increasing the topology further, the primed masses become degenerate, so we take the scaling
limit with the (scaled) $\epsilon_j s$ to be finite, and then let $\epsilon_j \to 0$. The Vandermonde $\Delta_s((i\epsilon)^2)$ of (3.23) ensures that this limit is non-trivial. We then find for $\nu = 2$ (case i),

$$p_{S1}^{(0+1,2)}(\xi_1) = -\frac{m^3 x_1^4 e^{-\frac{x_1^2}{2\delta^2}}}{64I_3(\delta)} \begin{bmatrix} \hat{K}_3^2(\xi_1, \delta) & \partial_{\hat{m}} \hat{K}_3^2(\xi_1, \delta) \\ \delta_{\hat{m}}^2 \hat{K}_3^2(\xi_1, \delta) & \partial_{\delta_{\hat{m}}} \hat{K}_3^2(\xi_1, \delta) \end{bmatrix}, \quad (4.17)$$

and for $\nu = 3$ (case iii),

$$p_{S1}^{(0+1,3)}(\xi_1) = -\frac{m^3 x_1^4 e^{-\frac{x_1^2}{2\delta^2}}}{64I_3(\delta)} \begin{bmatrix} \hat{K}_3^3(\xi_1, \delta) & \partial_{\hat{m}} \hat{K}_3^3(\xi_1, \delta) & \delta_{\hat{m}}^2 \hat{K}_3^3(\xi_1, \delta) \\ 1/\xi_1 I_1(\xi_1) & 1/\xi_1 I_2(\xi_1) & \partial_{\xi_1} \left[ 1/\xi_1 I_2(\xi_1) \right] \end{bmatrix}, \quad (4.18)$$

where by $\partial_{\delta}$ we mean the derivative with respect to the first argument (which, incidentally, is not the same as $\partial_{\xi_1}$). Our new results for $\nu = 1, 2$ and 3 are illustrated in figure 1. For alternative expressions in an equivalent formulation see appendix C. The first eigenvalue distribution alone is in general not sufficient to fit both low-energy constants (LEC) $\Sigma$ and $F_\pi$, see the discussion in [22]. The benefit of computing the first eigenvalue distribution at higher topology here is to be able to fit both LEC independently from different lattice configurations, where typically $\nu = 1$ offers better statistics than $\nu = 0$.

Continuing with the new expressions for the $k$th lowest eigenvalues for $N_f = 0$ with $k > 1$, we will, for simplicity, focus on $\nu = 0$. Results at higher topology are easily obtained following the same approach as above. For $k = 2$ and $k = 3$ the joint probability distribution is found to be, respectively,

$$\Omega_{S0}^{(0+1)}(\xi_2, \xi_3) = \frac{e^{-\frac{x_2^2+x_3^2}{2\delta^2}}}{16\delta_0(\delta)} \hat{K}_3^0(\xi_2, \delta) q_0^2(\delta; t = 1) \partial_{\delta} q_0^2(\delta; t)|_{t=1} \times \begin{bmatrix} \hat{K}_3^3(\xi_2, \delta) \\ \partial_{\xi_2} \hat{K}_3^3(\xi_2, \delta) \\ \partial_{\delta} \hat{K}_3^3(\xi_2, \delta) \end{bmatrix} \begin{bmatrix} \xi_2 \xi_3 \\ I_1(\xi_2) \\ I_2(\xi_2) \end{bmatrix}, \quad (4.19)$$
and

\[ \Omega_{S0}^{(0+1)}(\hat{x}_1, \hat{x}_2, \hat{x}_3) = -\frac{e^{-\frac{1}{2}x^2}}{128I_0(m)} \hat{x}_1\hat{x}_2\hat{x}_3(x_1^2 - \hat{x}_1^2)(x_2^2 - \hat{x}_2^2) e^{-\frac{1}{2}x^2} \]

\[ \begin{vmatrix}
K_S^0(\hat{x}_3, \hat{x}_1) & \tilde{K}_S^0(\hat{x}_3, \hat{x}_2) & q_S^0(\hat{m}_1; t = 1) & \partial_t^2 q_S^0(\hat{m}_1; t) \bigg|_{t=1} \\
K_S(\hat{x}_3, \hat{x}_1) & K_S(\hat{x}_3, \hat{x}_2) & \frac{2}{\hat{x}_3} I_1(\hat{x}_3) & I_2(\hat{x}_3) \\
K_S(\hat{x}_3, \hat{x}_1) & K_S(\hat{x}_3, \hat{x}_2) & \frac{2}{\hat{x}_3} I_1(\hat{x}_3) & I_2(\hat{x}_3) \\
K_S(\hat{x}_3, 0) & K_S(\hat{x}_3, 0) & 1 & 0
\end{vmatrix} \]

(4.20)

Here we have used the shorthand \( \hat{x}_i \equiv \sqrt{x_i^2 - \hat{x}_i^2} \). The eigenvalue distributions then follow by applying equation (4.12).

The extension of these results to higher values of \( N_2 \) is straightforward. We will restrict ourselves here to \( N_2 = 2 \) and \( v = 0 \). Defining the normalization constant

\[ \Lambda_r^{(0+2)} = e^{\frac{1}{2}x^2} \begin{vmatrix}
I_0(\hat{m}_1) & \hat{m}_1 I_1(\hat{m}_1) \\
I_0(\hat{m}_2) & \hat{m}_2 I_1(\hat{m}_2)
\end{vmatrix}, \]

(4.21)

we have

\[ p_{S1}^{(0+2),0}(\hat{x}_1) = \frac{\hat{x}_1 e^{-\frac{1}{2}x^2}}{\Lambda_r^{(0+2)}} \begin{vmatrix}
\partial_t q_S^0(\hat{m}_1; t) \bigg|_{t=1} & \partial_t^2 q_S^0(\hat{m}_1; t) \bigg|_{t=1} \\
\partial_t q_S^0(\hat{m}_2; t) \bigg|_{t=1} & \partial_t^2 q_S^0(\hat{m}_2; t) \bigg|_{t=1}
\end{vmatrix}. \]

(4.22)

Note that an alternative representation for the integral of this quantity, its cumulative distribution, was given in equation (4.44) in [13].

For \( k = 2 \) and \( k = 3 \), we obtain the following new expressions:

\[ \Omega_{S0}^{(0+2)}(\hat{x}_1, \hat{x}_2) = -\frac{\hat{x}_1 \hat{x}_2 (\hat{x}_1^2 - \hat{x}_2^2) e^{-\frac{1}{2}x^2}}{8\Lambda_r^{(0+2)}} \begin{vmatrix}
K_S^0(\hat{x}_2, \hat{m}_1) & q_S^0(\hat{m}_1; t = 1) & \partial_t^2 q_S^0(\hat{m}_1; t) \bigg|_{t=1} \\
K_S^0(\hat{x}_2, \hat{m}_2) & q_S^0(\hat{m}_2; t = 1) & \partial_t^2 q_S^0(\hat{m}_2; t) \bigg|_{t=1} \\
K_S(\hat{x}_2, 0) & 0 & 0
\end{vmatrix} \]

(4.23)

and

\[ \Omega_{S0}^{(0+2)}(\hat{x}_1, \hat{x}_2, \hat{x}_3) = \frac{\hat{x}_1 \hat{x}_2 \hat{x}_3 (\hat{x}_1^2 - \hat{x}_2^2)(\hat{x}_2^2 - \hat{x}_3^2) e^{-\frac{1}{2}x^2}}{2^{2}\Lambda_r^{(0+2)}} \begin{vmatrix}
K_S^0(\hat{x}_3, \hat{m}_1) & \tilde{K}_S^0(\hat{x}_3, \hat{m}_2) & q_S^0(\hat{m}_1; t = 1) & \partial_t^2 q_S^0(\hat{m}_1; t) \bigg|_{t=1} \\
K_S^0(\hat{x}_3, \hat{m}_2) & \tilde{K}_S^0(\hat{x}_3, \hat{m}_1) & q_S^0(\hat{m}_2; t = 1) & \partial_t^2 q_S^0(\hat{m}_2; t) \bigg|_{t=1} \\
K_S(\hat{x}_3, \hat{m}_1) & K_S(\hat{x}_3, \hat{m}_2) & \frac{2}{\hat{x}_3} I_1(\hat{x}_3) & I_2(\hat{x}_3) \\
K_S(\hat{x}_3, 0) & K_S(\hat{x}_3, 0) & 1 & 0
\end{vmatrix} \]

(4.24)

after inserting them into equation (4.12). We illustrate the distributions of individual eigenvalues following from these equations in figure 2 for non-degenerate masses. In the case of equal masses, two rows become degenerate in both the numerator and denominator, necessitating a Taylor expansion. Figure 2 offers a further graphical consistency check for our results, when comparing the sum of the individual eigenvalues to the actual density. The two curves nicely agree almost up to the third local maximum for all parameter values.
4.2. Two light flavours

For \( N_1 = N_2 = 1, \nu = 0 \), the distribution of the first eigenvalue is given by

\[
p^\|(1+0.1)(\hat{\chi}_1) = - \frac{\hat{\chi}_1(\hat{m}_1^2 + \hat{x}_1^2)}{8N^{(1+1)}} e^{i \hat{x}_1^2} |K_0^1(\hat{m}_1', \hat{m}_2)\rangle q_0^n(\hat{m}_2; t = 1),
\]

(4.25)

\[
N^{(1+1)} = \int_0^1 dt e^{i \hat{x}_1^2} I_0(\sqrt{\hat{m}_2}),
\]

(4.26)

with \( \hat{m}_1 = \hat{m}_{s_1}, \hat{m}_2 = \hat{m}_{s_2} \) and \( \hat{m}' = \sqrt{\hat{m}_1^2 + \hat{x}_1^2} \). For its cumulative distribution see equation (4.33) in [13].

The expressions for distribution of the second and third eigenvalues which are new results follow in a similar fashion from

\[
\Omega^{(1+1)}_{00}(\hat{\chi}_1, \hat{\chi}_2) = \frac{\hat{\chi}_1 \hat{\chi}_2 (\hat{x}_1^2 - \hat{x}_2^2) (\hat{m}_1^2 + \hat{x}_1^2) e^{i \hat{x}_2^2}}{64N^{(1+1)}} |K_0^0(\hat{m}'_1, \hat{m}_2)\rangle |K_0^0(\hat{m}_1', \hat{x}_{s_1})\rangle q_0^n(\hat{m}_{s_2}; t = 1) |K_0^0(\hat{m}_1, \hat{x}_{s_2})\rangle |K_0^0(\hat{m}_1', \hat{x}_{s_2})\rangle |K_0^0(\hat{m}_1, \hat{x}_{s_2})\rangle |K_0^0(\hat{m}_1', \hat{x}_{s_2})\rangle |K_0^0(\hat{m}_1, \hat{x}_{s_2})\rangle |1\rangle.
\]

(4.27)

\[
\times |K_0^0(\hat{m}'_1, \hat{m}_2)\rangle |K_0^0(\hat{m}_1', \hat{x}_{s_1})\rangle |K_0^0(\hat{m}_1, \hat{x}_{s_2})\rangle |K_0^0(\hat{m}_1', \hat{x}_{s_2})\rangle |K_0^0(\hat{m}_1, \hat{x}_{s_2})\rangle |K_0^0(\hat{m}_1', \hat{x}_{s_2})\rangle |K_0^0(\hat{m}_1, \hat{x}_{s_2})\rangle |1\rangle.
\]

(4.28)

The corresponding figures are shown in figure 3.
Both figures 2 and 3 illustrate the influence of chemical potential and thus the possibility to determine $F_\pi$ from individual eigenvalue distributions. In particular, not only the shape of the first eigenvalue, but also the positions of the maxima of the second and third eigenvalues change considerably when comparing the left and right plots, especially in the partially quenched setting in figure 2. Let us briefly compare to how $\Sigma_1$ and $F_\pi$ were determined from lattice data in [22]. In order to avoid two-parameter fits, there $\Sigma$ was first determined at $\mu_1, \mu_2 = 0$ from the first eigenvalue alone. Then, the two-point density correlation function $\rho(\hat{x}, \hat{y})$ of finding one $\hat{x}$- and one $\hat{y}$-eigenvalues was expanded for small $\hat{\delta}$, and $F_\pi$ was fitted to the resulting Gaussian repulsion between $\hat{x}$ and $\hat{y}$. We would expect that the detailed knowledge of the shape and spacing of several individual eigenvalues combined with independent measurements at different topologies could provide an alternative way to determine both low-energy constants.

5. Conclusions

We have derived new analytical expressions for the probability distribution of the $k$th lowest Dirac operator eigenvalue in QCD with three (or more) colours with spontaneous breaking of chiral symmetry. Extending earlier results, we have shown how these distributions become modified when the Dirac operator couples to imaginary chemical potential, for arbitrary topology and including the partially quenched case. Because the deformation parameter of the spectrum is $\tilde{\mu} = \mu F_\pi \sqrt{V}$ (where $\mu$ is the externally supplied isospin chemical potential), one can use this to extract the magnitude of $F_\pi$ through lattice gauge theory simulations. This has already been done successfully for the first eigenvalue at $\nu = 0$, after including first-order finite-volume corrections via $F_\pi^{\text{eff}}$. Our results thus offer further independent checks for higher topology or higher eigenvalues, given they remain in the $\epsilon$-regime.

There is a clear sensitivity to $F_\pi$ in these individual eigenvalue distributions, but it requires quark masses to be quite small. Because present-day lattice gauge configurations may tend to be available for relatively large masses, far from the scale of eigenvalues considered here, the eigenvalue distributions will look close to quenched and there is then little variation in the distributions as $\mu$ is introduced. For this reason, it would be extremely helpful if one
could derive analogous analytical expressions for mixed or conditional individual eigenvalue distributions, say to find simultaneously the first eigenvalue of $\mathcal{D}_1$ at $x$ and the first eigenvalue of $\mathcal{D}_2$ at $y$. In the same way as the mixed spectral two-point density correlation function $\rho(x, y)$ considered previously in the literature, which develops a delta function in the $\mu \to 0$ limit, this quantity should lead to a more dramatic numerical signal as chemical potential is turned on and thus to an easier way of measuring $F_\pi$ by means of such a technique. It is an open challenge to find new mathematical tricks that will be needed to perform this extension of our results.

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Appendix A. Identity for Laguerre polynomials

In this appendix, we derive the form of the new polynomials given in equation (3.17). They first appear in the manipulations leading to equation (3.15) in the following form:

$$q_n^i(M_2^i) \equiv \sum_{j=0}^{n} \frac{(-)^{n+j}n!(n+1)!}{(n-j)!j!(j+1)!} \sum_{l=0}^{j} \frac{\left( M_2^i \right)^{(2)}}{1 - \tau} L_j^l \left( -X_2^i \right)^{j-l} \binom{M_2^i}{l} \left( \left( \frac{X_2^i}{\tau} \right)^{j-l} \right)$$

$$= (-)^n n! \sum_{l=0}^{n} \frac{\tau^l}{(1 - \tau)^l} L_j^l \left( M_2^i \right) \left( \left( \frac{X_2^i}{\tau} \right)^{j-l} \right)$$

$$= (-)^n n! \sum_{j=0}^{n} L_{n-j}^{j+1}(-X_2^i) \sum_{l=0}^{j} \frac{(-)^{j+l}(l+v)!}{(l-j)!(j+v)!} L_j^l \left( M_2^i \right)$$

$$= (-)^n n! \sum_{j=0}^{n} \frac{1}{(1 - \tau)^l} L_j^l \left( M_2^i \right) \sum_{i=0}^{n-j} \frac{(-)^{j+l}(i+j+v)!}{(j+v)!} L_{n-j}^{j+1}(-X_2^i). \quad (A.1)$$

The first line originates from generating polynomials $L_j^l_N$ in the lower rows in equation (3.15). Next we have swapped the sums, $\sum_{j=0}^{n} \sum_{l=0}^{j} \to \sum_{j=0}^{n} \sum_{l=0}^{n-j}$, so that the sum in $-X_2^i$ gives $L_{n-j}^{j+1}(-X_2^i)$. Then we use the following identity:

$$L_j^l(zw) = \sum_{j=0}^{n} \frac{(n+v)!}{(n-j)!(v+j)!} w^j(1-w)^{n-j} L_j^l(z) \quad (A.2)$$

to eliminate the argument $1/\tau$ from the first Laguerre polynomial, choosing $z = M_2^i$ and $w = 1/\tau$. The newly appearing sum is swapped again to $\sum_{j=0}^{n} \sum_{l=0}^{n-j}$. The remaining sum can be simplified with the help of the identity

$$L_{m}^{-(n+1)}(x) = \sum_{i=0}^{m} \frac{(-)^{n+v+m+i}!}{(n+v+m)!} \tau_{m-i}^{m-n+1+i}(x), \quad (A.3)$$

with $m = n-j$ and $x = -X_2^i$.  

19
The polynomials $Q_n^\nu(x)$ and their derivation differ slightly from the polynomials $Q_n(x)$ in [13]. First, we allow for non-zero topology here, in contrast to there. Second, in the related formula to (3.15) the Laguerre polynomials $L_n^\nu$ have to be generated there, instead of $L_n^0$ here.

For completeness, we also define the generalization of the polynomials $Q_n(x)$ that are needed to compute the averages of the product of two determinants of different size that both contain different variables and

$$Q_n^\nu(M_2^\nu) \equiv \sum_{j=0}^n \frac{(-)^{n+j}(n!)^2}{(n-j)!j!} \frac{\tau^j}{(1-\tau)^j} L_j^\nu\left(\frac{M_2^2}{\tau}\right) (-S^2)^{j-1} \left(\begin{array}{c} j \\ j \end{array}\right)$$

$$= (-)^n n! \sum_{j=0}^n \frac{1}{(1-\tau)^j} L_j^\nu(M_2^\nu) L_{n-j}^{-\nu-1} (-S^2), \quad (A.4)$$

with $S^2 = +N\tau c_1 s^2$. The derivation goes along exactly the same lines as above and we skip the details here. The single difference compared to equation (3.17) is the shift in the negative index from $-\nu \rightarrow -\nu - 1$ in the final result. In appendix C below, we give the corresponding expressions for the first eigenvalue from [13] for non-zero $\nu$.

Appendix B. Integration theorem over determinants

In this appendix, we compute averages of the product of two determinants of different size that both contain different variables and different polynomials. The theorem we give is a slight generalization of theorem 1 in [26]. It was already used in [13]; but since the form we need here is more general, we provide a short derivation. In particular, it generalizes Dyson’s theorem (see theorem 5.1.4 in [25]), valid only for determinants of the same size and of the same polynomials.

Suppose we have two sets of (bi)orthonormal functions (this includes the case of polynomials orthogonal w.r.t. a given weight, since the weight can be absorbed into the functions) $\varphi_{j=0,1,...}(z)$ and $\tilde{\varphi}_{j=0,1,...}(z)$:

$$\int dz \varphi_j(z) \tilde{\varphi}_l(z) = \delta_{jl} \quad \text{for } j, l = 0, 1, \ldots. \quad (B.1)$$

Furthermore, we assume that we have two matrices (independent of $z_1, \ldots, z_N$), $A_{ij}$ and $B_{ij}$. We can then compute the following average over $N$ variables $z_j$ with $k \geq 1$:

$$\mathcal{T} \equiv \int_0^\infty \prod_{j=1}^N dz_j \begin{vmatrix} A_{1,0} & \cdots & A_{1,N+k-1} \\ \vdots & \ddots & \vdots \\ A_{k,0} & \cdots & A_{k,N+k-1} \end{vmatrix} \begin{vmatrix} B_{1,0} & \cdots & B_{1,N+k-1} \\ \vdots & \ddots & \vdots \\ B_{k,0} & \cdots & B_{k,N+k-1} \end{vmatrix}, \quad (B.2)$$

where without loss of generality we assume $k \geq k'$. We need to define four different objects that we refer to as kernels:

$$K_{N+k}^I(i, j) = \sum_{l=0}^{N+k-1} A_{i,l} B_{j,l}, \quad K_{N+k}^II(i, z) = \sum_{l=0}^{N+k-1} A_{i,l} \tilde{\varphi}_l(z), \quad (B.3)$$

$$K_{N+k}^III(z, j) = \sum_{l=0}^{N+k-1} \varphi_l(z) B_{j,l}, \quad K_{N+k}^IV(z, z') = \sum_{l=0}^{N+k-1} \varphi_l(z) \tilde{\varphi}_l(z').$$

J. Phys. A: Math. Theor. 45 (2012) 115205 G Akemann and A C Ipsen
Only the last kernel has the self-reproducing properties required for Dyson’s theorem to apply:

\[ \int d\zeta K_{N+k}^{IV}(\zeta, \zeta')K_{N+k}^{IV}(\zeta', \zeta'') = K_{N+k}^{IV}(\zeta, \zeta''), \]

(B.4)

\[ \int d\zeta' w(\zeta')K_{N+k}^{IV}(\zeta', \zeta') = N + k'. \]

Obviously, the contraction of mixed kernels leads to mixed kernels,

\[ \int d\zeta K_{N+k}^{II}(i, \zeta')K_{N+k}^{III}(\zeta', j) = K_{N+k}^{I}(i, j), \]

\[ \int d\zeta K_{N+k}^{II}(i, \zeta')K_{N}^{IV}(\zeta', z) = K_{N+k}^{II}(i, z), \]

(B.5)

\[ \int d\zeta K_{N+k}^{IV}(\zeta, \zeta')K_{N+k}^{III}(\zeta', i) = K_{N+k}^{IV}(\zeta, i). \]

The result is now given in terms of the matrix

\[ B_{ij} = \left\{ \begin{array}{ll} K_{N+k}^{I}(i, j) & i = 1, \ldots, k, j = 1, \ldots, k' \\ A_{i, j+N-1} & i = 1, \ldots, k, j = k' + 1, k' + 2, \ldots, k. \end{array} \right. \]

(B.6)

By choosing \( A_{ij} \) and \( B_{ij} \) to be appropriate polynomials of the unintegrated variables, letting \( \psi_i(z) \) and \( \tilde{\psi}_i(z) \) be Laguerre polynomials times the weight factor, and choosing \( N - k \) variables \( z_1, \ldots, z_N \mapsto Z_{k+1}, \ldots, Z_k \), we arrive at the integral of (3.19). Taking the integration norm into account, we see that (3.23), (3.24) and (3.25) follow from (B.6).

The proof of the formula is obtained by induction on \( k \). We first consider the base case \( k = k' \), where (B.6) is just \( N! \det[K^I(i, j)] \). To derive this, we first transpose one of the matrices in (B.2), and then multiply them inside a common determinant. The resulting matrix will then consist of blocks with each of the four types of kernels. Using the self-reproducing property of \( K^{IV} \) and Dyson’s theorem [25], we can carry out the integration, yielding the stated result.

For the step \( k \rightarrow k + 1 \), we expand the first determinant of (B.2) in the last column:

\[ \sum_{i=1}^{k+N+1} (-)^{k+N+1+i} \int_0^\infty \left( \prod_{j=1}^N dz_j \right) (M_A)_{i,k+N+1} \det[m_A(i)] \det[M_B], \]

with \( M_A \) (\( M_B \)) being the first (second) matrix of (B.2) and \( m_A(i) \) being \( M_A \) with the row \( i \) and column \( k+N+1 \) removed. The integrands with \( i > k + 1 \) are proportional to \( \tilde{\psi}_{k+N}(z_{i-k-1}) \), but since there is no corresponding \( \tilde{\psi}_{k+N}(z_{i-k-1}) \) factor in the expansion of the \( M_B \) determinant, these terms are killed by orthogonality. Using the induction hypothesis on the remaining terms (proportional to \( A_{i,k+N} \)) and comparing to the expansion of \( \det B \) in the last column, we verify (B.6).

Appendix C. Non-zero topology for \( p_{k=1}^{(N, \nu)}(x) \): an alternative formulation

In this appendix we derive an alternative formulation of the distribution of the first eigenvalue for \( \nu > 0 \). It is based on an earlier paper [13] to where we refer for more details, explicit results were given there only for \( \nu = 0 \). Because the calculations in the main part of this paper are quite involved it is useful to have an independent, equivalent result as a cross check.
In [13] one computes first the gap probability, that the interval $[0, s]$ is empty of all eigenvalues $\xi_i$:
\[
E^{(N_f)}_v(s) \equiv \frac{1}{Z_v^{(N_f)}} \int_0^\infty dx_1 \cdots dx_N \int_0^\infty dy_1 \cdots dy_N \mathcal{P}_v^{(N_f)}(\{x\}, \{y\}; [m_1], [m_2]). \tag{C.1}
\]

From this the distribution of the first eigenvalue follows by differentiation, $p_1^{(N_f,v)}(s) = -\partial_s E^{(N_f)}_v(s)$. In steps very similar to the main body of this paper we obtain
\[
E^{(N_f)}_v(s) \sim \frac{1}{Z_v^{(N_f+N_v)}} \sum_{\nu} \frac{1}{\Delta_{N_2+N_v}^{(M_2^2)}} \int_0^\infty dz_1 \cdots dz_N e^{-N^2 \sum_{i=1}^n \tau_{c_i} z_i} \times \begin{bmatrix}
\hat{L}_0(M_1^2) & \cdots & \hat{L}_{N+N_2-1}(M_1^2) \\
\cdots & \cdots & \cdots \\
\hat{L}_0(Z_1) & \cdots & \hat{L}_{N+N_2-1}(Z_1) \\
\cdots & \cdots & \cdots \\
\hat{L}_0(Z_N) & \cdots & \hat{L}_{N+N_2-1}(Z_N)
\end{bmatrix}, \tag{C.2}
\]

where we have suppressed all mass independent normalization factors. The Laguerre kernel:
\[
\nu (x, M) \equiv -\nu(x) = (-)^{k} k! L_k(x) \text{ in monic normalization contain the following masses:}
\]
\[
M_j^2 \equiv -N \tau c_j m_j^2, \quad j = 1, \ldots, N_2 \tag{C.3}
\]

for the flavours of type $N_2$, and for flavour $N_1$ we have $v$ additional masses
\[
M_j^2 \equiv -N \tau c_j m_j^2 \equiv -N \tau c_1 (m_j^2 + s^2), \quad j = 1, \ldots, N_1, \tag{C.4}
\]

The degeneracy of the latter can be reinstated by setting $\epsilon_j = 0$ at the end of the calculation. Alternatively, we could write out the resulting derivatives acting on the first determinant in equation (C.2) for $v > 1$. Last but not least we have introduced a set of new polynomials containing now generalized Laguerre polynomials, as defined in (A.4). This completes in principle the computations of [13] for non-zero topology. The derivation from the first to the last line in equation (A.4) goes along the same lines as the previous appendix A for the polynomials $d_n^v(x)$, apart from the slightly different identity to be used:
\[
L_{n-j}^{v-1}(-x) = \sum_{i=0}^{n-j} (-1)^{i+j+v+i} \binom{n-j+i}{j+v} L_{n-j-i}^{v-1}(x). \tag{C.5}
\]

Applying the theorem derived in the next appendix D the $N$ integrals over variables $z_k$ can now be easily performed. Instead of giving the most general result, distinguishing between different cases depending on the numbers $N_1, N_2$ and $v$, we give a few simple examples for illustration. These can be stated in terms of the above polynomials as well as the following kernel:
\[
K_N^v(M_1^2, M_2^2) \equiv \sum_{j=0}^{N-1} (-1)^j j! L_j^0(M_1^2) Q_j^v(M_2^2). \tag{C.6}
\]

Note only that the second polynomial $Q_j^v$ gets modified when $v \neq 0$, compared to [13]. The kernel is obviously not symmetric in its arguments.

\footnote{It is denoted there by $E_{0,0}(s, t = 0)$.}
In the first example we choose $N_1 = N_2 = 1$ to obtain

$$E^{(1+1)}_{s=0}(s) \sim \frac{e^{-N \tau_1 s^2}}{Z_0^{(1+1)}(m_1; m_2)} K^{\nu=0}_{N+1}(M^2_1, M^2_2),$$

$$E^{(1+1)}_{s=1}(s) \sim \frac{e^{-N \tau_1 s^2}}{Z_1^{(1+1)}(m_1; m_2)(m_1^2 - s^2)} \begin{vmatrix} K^{\nu=0}_{N+1}(M^2_1, M^2_2) & K^{\nu=0}_{N+1}(S^2, M^2_2) \\ L^0_{N+1}(M^2_2) & L^0_{N+1}(S^2) \end{vmatrix},$$

etc., with more rows with Laguerre polynomials for higher $\nu$.

The second example which is probably most relevant for applications is partially quenched, with $N_1 = 0$ and $N_2 = 2$. Here the size of the determinant does not grow immediately as the masses $m_{1,2}$ get paired with those generated by topology, $\nu = 1, 2$ in our examples:

$$E^{(0+2)}_{s=0}(s) \sim \frac{e^{-N \tau_1 s^2}}{Z_0^{(0+2)}(m_{1,2}) (m_1^2 - m_2^2)} \begin{vmatrix} Q^{\nu=0}_N(M^2_1) & Q^{\nu=0}_N(M^2_2) \\ Q^{\nu=0}_{N+1}(M^2_1) & Q^{\nu=0}_{N+1}(M^2_2) \end{vmatrix},$$

$$E^{(0+2)}_{s=1}(s) \sim \frac{e^{-N \tau_1 s^2}}{Z_1^{(0+2)}(m_{1,2}) (m_1^2 - m_2^2)} \begin{vmatrix} K^{\nu=0}_{N+1}(S^2, M^2_1) & K^{\nu=0}_{N+1}(S^2, M^2_2) \\ Q^{\nu=0}_{N+1}(M^2_1) & Q^{\nu=0}_{N+1}(M^2_2) \end{vmatrix},$$

$$E^{(0+2)}_{s=2}(s) \sim \frac{e^{-N \tau_1 s^2}}{Z_2^{(0+2)}(m_{1,2}) (m_1^2 - m_2^2) 2x} \begin{vmatrix} K^{\nu=0}_{N+1}(S^2, M^2_1) & K^{\nu=0}_{N+1}(S^2, M^2_2) \\ \partial_\nu K^{\nu=0}_{N+1}(S^2, M^2_1) & \partial_\nu K^{\nu=0}_{N+1}(S^2, M^2_2) \end{vmatrix}.$$ (C.8)

In all cases one further differentiation with respect to $s$ yields the distribution of the first eigenvalue.

### C.1. The large-$N$ limit

We will sketch here first how to take the microscopic large-$N$ limit of the main building blocks: the generalized Laguerre polynomials $L_{j\nu}$, the new polynomials $Q_j$, and the kernel $K^\nu_N$. Then we will give two explicit examples for the distribution of the first eigenvalue at $\nu > 0$.

The scaling limit is exactly as described in the main body of this paper (and in [13]), so we can be brief. We begin with the new polynomials $Q_j\nu$. The sum can be replaced by an integral except for a few terms, and we need the following ingredients in the limit $N \to \infty$ together with $\mu_{1,2} \to 0$.

$$\lim_{N,j,k \to \infty} (1 - \tau)^{-k} = \exp \left[ \frac{1}{2} \tau \delta^2 \right], \quad \text{where} \quad t = j/N, \quad r = k/j.$$ (C.9)

$$\lim_{N \to \infty} \tau_{c_{1,2}} = 1.$$

For the Laguerre polynomials the following scaling holds:

$$\lim_{N,j \to \infty} L^\nu_j(M^2) = N^\nu (r)^{\nu/2} (2/\sqrt{m})(\sqrt{\tau},) L_0(\sqrt{\tau}),$$

$$\lim_{N,j,k \to \infty} L_{j-k+\nu}^{\nu-1}(-S^2) = N^{-\nu-1} (t(1 - r))^{-(\nu+1)/2} (\delta/2)^{\nu+1} L_{\nu+1}(\delta \sqrt{1 - r}).$$ (C.10) (C.11)

In comparison to $\nu = 0$, where we had to split off the $s$-independent part $L_{\nu+1}^0 = 1$ from the sum in equation (A.4), we now have to separate a total number of $\nu + 1$ terms $L_{\nu+1}^0$ from the sum. In the large-$N$ limit these terms will seem to be of higher order, but after some cancellations taking place they will give a contribution of the same order as the sum itself.

To illustrate this we give the first two examples (including the known $\nu = 0$ case) for the limiting new polynomials $\lim_{j \to \infty} \left( \frac{e^{\nu \tau}}{\tau^j} \right) Q_j(M^2) \equiv Q_{j\nu}(\hat{m}; t)$:

$$Q_{2\nu}^0(\hat{m}; t) = \hat{t} \int_0^1 \frac{e^{\nu \tau \hat{t}^2}}{\sqrt{1 - r}} h_0(\hat{m} \sqrt{\tau \hat{r}}) h_1(\hat{r} \sqrt{(1 - r) \hat{r}}),$$
\[Q^{(\nu)}_s(\hat{\nu}; t) = \frac{\hat{s}^2}{2\hat{m}} \int_0^1 dr \frac{\hat{t} e^{\frac{\hat{s}^2}{\hat{m}}}}{1 - r} I_1(\hat{m} \sqrt{1 - r}) I_3(\hat{\nu} \hat{s} \sqrt{1 - r}) + I_0(\hat{m}) = I_0(\sqrt{\hat{m}^2 + \hat{s}^2}), \] (C.12)

The fact that we obtain the following relation \(\lim_{\mu_1, \mu_2 \to 0} (-\mu^2 Q^2(M^2)) / n! = L^0(\mu^2 (m^2 + s^2))\) from equation (A.4) in the limit of vanishing chemical potentials, implies a series of generalized Sonine identities, e.g. for \(\nu = 1\):

\[\frac{s^2}{2m} \left( \int_0^1 dr \frac{\hat{t} e^{\frac{\hat{s}^2}{\hat{m}}}}{1 - r} I_1(\hat{m} \sqrt{1 - r}) I_3(\hat{s} \sqrt{1 - r}) + I_0(\hat{m}) \right) + I_0(\hat{m}) = I_0(\sqrt{\hat{m}^2 + \hat{s}^2}), \] (C.13)

etc. The general identity obtained in this way is stated in the following appendix and proven by induction.

The asymptotic limit of \(Q^2_s\) then easily translates into the limiting form of the microscopic kernel:

\[\lim_{N \to \infty} \frac{1}{N} K^s_{\nu}(M^2_1, M^2_2) = \frac{1}{2} \int_0^1 \frac{dr I_0(\hat{m} \sqrt{1 - r}) Q^{(\nu)}_s(\hat{\nu}; t) \bigg|_{\hat{m} = \hat{m}_1, \hat{m}_2}}{I_1(\hat{m})}. \] (C.14)

We are now ready to give some explicit examples for gap probabilities at non-zero topology. The first example is for one flavour, the simplest partially quenched case, with \(N_1 = 0\) and \(N_2 = 1\) with mass \(\hat{m}\). While for \(\nu = 0\) we had [13]

\[E^{(0+1)}_{S, v = 0}(\hat{s}) = \exp \left[ -\frac{1}{4} \hat{s}^2 - \frac{1}{2} \hat{\delta}^2 \right] Q^{(0)}(\hat{\nu}; t = 1) \] (C.15)

our new results for topology \(\nu = 1, 2\) are

\[E^{(0+1)}_{S, v = 1}(\hat{s}) = \exp \left[ -\frac{1}{4} \hat{s}^2 - \frac{1}{2} \hat{\delta}^2 \right] \frac{\hat{m}^2}{I_1(\hat{m})} K^{(0)}_{\nu}(\hat{s}, \hat{m}), \] (C.16)

\[E^{(0+1)}_{S, v = 2}(\hat{s}) = \exp \left[ -\frac{1}{4} \hat{s}^2 - \frac{1}{2} \hat{\delta}^2 \right] \frac{\hat{m}^2}{I_1(\hat{m})} \left| K^{(0)}_{\nu}(\hat{s}, \hat{m}) \right| I_0(\hat{s}) \left| I_1(\hat{s}) \right|. \] (C.17)

Note that the normalizing partition function is chosen such that it does not vanish at zero mass, e.g. \(Z^{(0+1)}(\hat{m}) = I_0(\hat{m}) / \hat{m}\). It also gives rise to an extra factor \(e^{\hat{\delta}^2}\) per unpaired flavour. The corresponding eigenvalues are shown in figure C1 together with the corresponding densities from [12]:

\[\rho^{(0+1)}_v(\hat{s}) = \rho^{\text{quen}}_v(\hat{s}) - \exp \left[ -\frac{1}{2} \hat{s}^2 \right] \frac{\hat{m}}{I_1(\hat{m})} \int_0^1 dT e^{\frac{1}{2} T^2} I_1(T \hat{m}) J_v(T \hat{s}), \] (C.18)

where the quenched one-matrix model spectral density reads [2]

\[\rho^{\text{quen}}_v(\hat{s}) = \frac{1}{2} \left( J_v(\hat{s})^2 - J_{v-1}(\hat{s}) J_{v+1}(\hat{s}) \right). \] (C.19)

It can be seen that our new results for individual eigenvalue distributions follow nicely the corresponding known spectral densities almost up to the first local maximum, for the two different values of topology and various values of \(\hat{m}\) and \(\hat{\delta}\) chosen.
The second set of examples is the gap for two flavours. First we give the case $N_1 = 1 = N_2$ at topology $\nu = 1$ (and 0):

$$E_{S=0}^{(1+1)}(\hat{s}) = \frac{e^{-\hat{s}^2/4}}{Z_{S=0}^{(1+1)}(\hat{m}_1; \hat{m}_2)} K_{S}^{v=0}(\hat{m}_1', \hat{m}_2),$$

$$E_{S=1}^{(1+1)}(\hat{s}) = \frac{e^{-\hat{s}^2/4}}{Z_{S=1}^{(1+1)}(\hat{m}_1; \hat{m}_2)(\hat{m}_1^2 - \hat{s}^2)} \begin{vmatrix} K_{S}^{v=1}(\hat{m}_1', \hat{m}_2) & K_{S}^{v=1}(\hat{m}_1, \hat{m}_2) \\ I_0(\hat{m}_1') & I_0(\hat{m}_1) \end{vmatrix},$$

with

$$Z_{v}^{(1+1)}(\hat{m}_1; \hat{m}_2) = \frac{1}{2(\hat{m}_1 \hat{m}_2)^{\nu}} \int_{0}^{1} dt \ e^{\hat{s}^2/4} I_v(\hat{m}_1 \sqrt{t}) I_v(\hat{m}_2 \sqrt{t}).$$

The corresponding first eigenvalue distributions obtained by differentiating these quantities with respect to $s$ are plotted in figure C2 against the corresponding spectral density [12],

$$\rho_{v}^{(1+1)}(\hat{s}) = \rho_{v}^{\text{quen}}(\hat{s}) - \hat{s} \int_{0}^{1} dt J_v(\hat{s} \hat{\delta}) I_v(\hat{m}_1') \int_{0}^{1} dt e^{\hat{s}^2/4} J_v(\hat{s} \hat{\delta}) I_v(\hat{m}_2') \int_{0}^{1} dt e^{\hat{s}^2/4} I_v(\hat{m}_1') I_v(\hat{m}_2').$$

Here, the difference between 1 + 1 flavours (left) and partial quenching (right) becomes visible. Once more the individual eigenvalue distributions agree nicely with the corresponding spectral densities.

The partially quenched two-flavour case, with $N_1 = 0$ and $N_2 = 2$ reads as follows in the large-$N$ limit:

$$E_{\nu=1}^{(0+2)}(s) = \frac{e^{-s^2/4 - \hat{s}^2} \hat{m}_1 \hat{m}_2}{\hat{m}_1 I_2(\hat{m}_1) I_1(\hat{m}_2) - \hat{m}_2 I_2(\hat{m}_2) I_1(\hat{m}_1)} \begin{vmatrix} K_{S}^{v=1}(s, \hat{m}_1') & K_{S}^{v=1}(s, \hat{m}_2') \\ Q_{S}^{v=1}(\hat{m}_1, t = 1) & Q_{S}^{v=1}(\hat{m}_2, t = 1) \end{vmatrix},$$

Note again the proper normalization of the partition function in masses and $\hat{s}$. The corresponding density to compare with was derived in [12] where it is given as a determinant.
The case \( t \neq 1 \) is easily reestablished when rescaling \( \hat{m}, \hat{t} \to \sqrt{\hat{m}}, \sqrt{\hat{t}} \).

\footnote{The case \( t \neq 1 \) is easily reestablished when rescaling \( \hat{m}, \hat{t} \to \sqrt{\hat{m}}, \sqrt{\hat{t}} \).}
\[ T_1 \equiv \frac{s^2}{m} \int_0^1 \frac{dx}{1-x^2} I_0(mx)I_0(s\sqrt{1-x^2}) + \frac{s^2}{2m} I_1(m) + I_0(m) - I_0(\sqrt{m^2+s^2}) = 0, \quad (D.2) \]

\[ T_2 \equiv \frac{s^3}{m^2} \int_0^1 \frac{dx}{(1-x^2)^2} I_0(mx)I_0(s\sqrt{1-x^2}) + \frac{s^4}{8m^2} I_2(m) + \frac{s^2}{2m} I_1(m) + I_0(m) - I_0(\sqrt{m^2+s^2}) = 0. \quad (D.3) \]

The first equation is known as the Sonine identity, see e.g. [27], [28] as well as [13] for an independent derivation. It is easier to state and prove the difference between two consecutive integral identities of this kind:

\[ 0 = T_v - T_{v-1} = \frac{s^{v+1}}{m^v} \int_0^1 \frac{dx}{(1-x^2)^{v+1/2}} I_0(mx)I_{v+1}(s\sqrt{1-x^2}) + \frac{s^{2v}}{2^v v! m^{v+1}} I_v(m) - \frac{s^v}{m^{v+1}} \int_0^1 \frac{dx}{(1-x^2)^{v+1/2}} I_{v-1}(mx)I_v(s\sqrt{1-x^2}). \quad (D.4) \]

Because the way to prove the induction start \( T_1 - T_0 = 0 \) and the induction step is the same, using integration by parts, we will be brief. If we use

\[ \partial_y \left( \frac{I_v(y)}{y^v} \right) = \frac{I_{v+1}(y)}{y^v}, \quad (D.5) \]

upon choosing \( y = s\sqrt{1-x^2} \) we can write

\[ \int_0^1 \frac{dx}{(1-x^2)^{v+1/2}} I_0(mx)I_{v+1}(s\sqrt{1-x^2}) = -\partial_y \left( \frac{I_v(s\sqrt{1-x^2})}{s(1-x^2)^{v/2}} \right). \quad (D.6) \]

We can therefore rewrite the first integral in equation (D.4) as follows:

\[ \int_0^1 \frac{dx}{(1-x^2)^{v+1/2}} I_0(mx)I_{v+1}(s\sqrt{1-x^2}) = -\frac{x^v I_v(mx)I_v(s\sqrt{1-x^2})}{s(1-x^2)^{v/2}} \bigg|_0^1 + \int_0^1 \frac{dx}{(1-x^2)^{v/2}} x^m I_{v-1}(mx) \frac{s^v}{v!} \int_0^1 \frac{dx}{(1-x^2)^{v/2}} I_{v-1}(mx)I_v(s\sqrt{1-x^2}), \quad (D.7) \]

where we have used a Bessel identity \( pI_v(p) + vI_v(p) = pI_{v-1}(p) \) as well as the series representation of the Bessel function to determine the limit at the upper bound \( x = 1 \). Inserting this for \( v = 1 \) and then general \( v \) yields the induction start and induction step.

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