Local regularity conditions on initial data for local energy solutions of the Navier–Stokes equations

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Published online: 21 December 2021
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Abstract
We study the regular sets of local energy solutions to the Navier–Stokes equations in terms of conditions on the initial data. It is shown that if a weighted $L^2$ norm of the initial data is finite, then all local energy solutions are regular in a region confined by space-time hypersurfaces determined by the weight. This result refines and generalizes Theorems C and D of Caffarelli et al. (Comm. Pure Appl. Math. 35(6):771–831, 1982) and our recent paper (Kang et al., Pure Appl. Anal. arXiv:2006.13145) as well.

Keywords Navier–Stokes equations · Regular sets · Local energy solutions

Mathematics Subject Classification 35Q30 · 76D05 · 76D03

Dedicated to Hideo Kozono on the occasion of his 60th birthday.

This article is part of the topical collection dedicated to Prof. Hideo Kozono on the occasion of his 60th birthday, edited by Kazuhiro Ishige, Tohru Ozawa, Senjo Shimizu, and Yasushi Taniuchi.

We thank Professor Reinhard Farwig for valuable suggestions on Theorem 1.3. The research of Kang was partially supported by NRF-2019R1A2C1084685 and NRF-2015R1A5A1009350. The research of Miura was partially supported by JSPS Grant 16H06339 and 17K05312. The research of Tsai was partially supported by NSERC Grant RGPIN-2018-04137.

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1 Introduction

1.1 Regular sets for local energy solutions

We are concerned with the regularity of weak solutions of the incompressible Navier–Stokes equations

\[ \partial_t v - \Delta v + v \cdot \nabla v + \nabla p = 0, \quad \text{div } v = 0 \]  

(\text{NS})

in \( \mathbb{R}^3 \), associated with the initial value \( v|_{t=0} = v_0 \) with \( \text{div } v_0 = 0 \). The global existence of weak solutions for finite energy initial data was established by Leray [21], and extended to domains by Hopf [10]. Despite a lot of efforts since their foundational work, the global regularity of the weak solutions remains a longstanding open problem. It is known from works [7,9,16] that for \( q \) in the critical case \( q = 3 \) in \([1,14]\); see also [25] for a condition on the initial enstrophy and [1] for further extension to the \( L^{3,\infty} \) space and the critical Besov spaces. In our previous work [15], we revisited this problem from a slightly different view point. Let us define the scaled local energy solution as follows:

\[ \left\{ (x,t) \in B_1 \times (0, \infty) : c N_{(1)}^2 |x|^2 \leq t < t_1 \right\} \]

(1.2)

for \( \alpha > 0 \). We showed that if \( N_{(1)} \) (with \( \alpha = 1 \)) is sufficiently small, any local energy solution is regular and satisfies the estimate (1.1) in the region

\[ (x,t) \in B_1 \times (0, \infty) : c N_{(1)}^2 |x|^2 \leq t < t_1 \]

for some \( t_1 > 0 \); see Sect. 2 for the definition of the local energy solutions. Concerning the question (R), the above result shows that if the local Morrey norm defined by

\[ \| f \|_{m^{2,1}(B_2)} := \sup_{x_0 \in B_2, r \in (0,1)} \left( \frac{1}{r} \int_{B_r(x_0) \cap B_2} |f(x)|^2 \, dx \right)^{\frac{1}{2}} \]

is sufficiently small, the local energy solution is regular in \( B_1 \times (0, t_1) \) for some \( t_1 > 0 \) and satisfies the \( L^\infty \) bound (1.1). By the well-known relation \( m^{2,1} \leftrightarrow L^3 \), the latter recovers the local regularity results [1,14] for \( L^3 \) data. The goal of the present work is to study the local in space regularity of the solutions with data satisfying the condition \( N_{(\alpha)} < \infty \) for general \( \alpha > 1 \). Intuitively, the bigger \( \alpha \) is, the more regular the data is, and so is the solution at least locally. We justify it in the following sense: The solution is regular in a larger region and it also satisfies an improved \( L^\infty \) estimate in a slightly different region. More precisely, it can be stated as follows.
Theorem 1.1 Assume that a local energy solution \( v \) in \( \mathbb{R}^3 \times (0, T) \) with the initial data \( v_0 \in L^2_{uloc} \) satisfies \( N(\alpha) < \infty \) for some \( \alpha > 1 \). Then there exists \( T_1 = T_1(\|v_0\|_{L^2_{uloc}}, N(\alpha), \alpha, T) > 0 \) such that \( v \) is regular in the region

\[
\Pi_1 = \left\{ (x, t) \in B_{\frac{1}{2}} \times (0, T) : c_1 N_{(\alpha)}^2 |x|^{2\alpha} \leq t < T_1 \right\}
\]

and satisfies

\[
|v(x, t)| \leq Ct^{-\frac{1}{2}} \quad \text{for} \quad (x, t) \in \Pi_1.
\]

Moreover, if \( \alpha \in (1, 4) \), there exist \( T_2 = T_2(\|v_0\|_{L^2_{uloc}}, N(\alpha), \alpha, T) > 0 \) and \( M = M(\|v_0\|_{L^2_{uloc}}, N(\alpha), \alpha) > 0 \) such that

\[
|v(x, t)| \leq Mt^{\frac{\alpha-3}{4}} \quad \text{for} \quad (x, t) \in \Pi_2
\]

holds with

\[
\Pi_2 = \left\{ (x, t) \in B_{\frac{1}{2}} \times (0, T) : \frac{c_2 N_{(\alpha)}^2}{(1 + N(\alpha))^{2 + \frac{2}{\alpha}}} |x|^2 \leq t < T_2 \right\}.
\]

Here \( c_1, c_2, \) and \( C \) are positive constants depending only on \( \alpha \).

Comments for Theorem 1.1

(a) Theorem 1.1 is also valid for \( \alpha = 1 \) if we further assume smallness of \( N(1) \). In that case we recover Theorem 3.4 of [15].

(b) In Theorem 3.1 we will give a more general statement for \( \alpha \in (1, 4) \) in terms of

\[
N(\alpha, R) := \sup_{r \in (R, 1]} \frac{1}{r^\alpha} \int_{B_r(0)} |v_0|^2 \quad \left( 0 \leq R \leq \frac{1}{2} \right).
\]

(c) \( T_1 \) will be given by (3.30). \( T_2 \) will be given by (3.2) as \( S_2 \) with \( N_R = N(\alpha, 0) \). None is necessarily larger than the other.

As one of applications of our result to some specific cases, we show a regularizing estimate of the form (1.1) for locally \( L^q \) data, \( 3 < q \leq \infty \).

Theorem 1.2 Let \( (v, p) \) be a local energy solution in \( \mathbb{R}^3 \times (0, T) \) with the initial data \( v_0 \in L^2_{uloc} \). If \( v_0 \in L^q(B_2) \) for some \( q \in (3, \infty] \), then there exists \( T_3 = T_3(q, \|v_0\|_{L^q(B_2)}, \|v_0\|_{L^2_{uloc}}) > 0 \) such that \( v \) is regular in \( B_1 \times (0, T_3) \) and satisfies

\[
\|v(t)\|_{L^\infty(B_1)} \leq C(1 + \|v_0\|_{L^q(B_2)})^{3 - \frac{6}{q}} \|v_0\|_{L^q(B_2)} t^{-\frac{1}{2q}} \quad \text{for} \quad t \in (0, T_3)
\]
Estimate (1.5) does not explicitly depend on $\|v_0\|_{L^2_{uloc}}$. Its dependence on $\|v_0\|_{L^2_{uloc}}$ is through the time upper bound $T_3$. In [2, Theorem 3], Barker and Prange obtained an $L^\infty$ bound in some time interval for locally $L^6$ initial data with the aim of the behavior of the $L^3$ norm near the singular point.

In [5], Caffarelli, Kohn, and Nirenberg established estimates of the regular sets for the suitable weak solutions for data in $L^2$ weighted spaces. Define weighted $L^2$ norm

$$\|v_0\|_{L^2, \beta} := \||x|^{\beta} v_0\|_{L^2(\mathbb{R}^3)} (1.6)$$

for $\beta \in \mathbb{R}$. It is shown in [5] that if the finite energy data satisfies $\|v_0\|_{L^2, 1} < \infty$, then there exists a suitable weak solution which is regular in the set of points satisfying $t > \min(C, K|x|^{-2})$. Moreover it is also shown that if the finite energy data satisfies $\|v_0\|_{L^2, -1} < \epsilon_0$ with some absolute small constant $\epsilon_0 > 0$, then there exists a suitable weak solution which is regular in the set $t > C|x|^2$; see [6] for its refinement. In [15, Corollary 1.4], we extended estimates of the regular sets for general data in $L^{2, \beta}$, $\beta \geq -1$. By using Theorem 1.1, we are able to show similar results for $\beta$ below $-1$ at least locally in time.

**Theorem 1.3** Let $(v, p)$ be a local energy solution in $\mathbb{R}^3 \times (0, T)$ for the initial data $v_0 \in L^2_{uloc}$. Assume that $v_0 \in L^{2, \beta}(\mathbb{R}^3)$ for some $\beta < -1$. Then there exist positive constants $T_4 = T_4(v_0)$ and $c(v_0)$ such that $v$ is regular in the set

$$\{(x, t) \in \mathbb{R}^3 \times (0, T) : c(v_0)|x|^{-2\beta} \leq t < T_4(v_0)\}.$$

The proofs of Theorems 1.1, 1.2 and 1.3 are based on a local-in-space a priori estimate for the scaled local energy defined by

$$E_r(t) := \text{ess sup}_{0 < s < t} \frac{1}{r^\alpha} \int_{B_r} |v(s)|^2 + \frac{1}{r^\alpha} \int_0^t \int_{B_r} |\nabla v|^2,$$

which is useful to focus on the local regularity at the origin and is used in [15] for the case $\alpha = 1$. In order to deal with the non-local effect of the pressure term, we apply the decomposition formula of [17,19] to the local energy estimate. Our key observation is that the (global) $L^2_{uloc}$ norm of the initial data is sufficient to control the non-local effect at least locally in time; see Lemma 3.2. It should be noted that our strategy is different from previous works [1,12,14] and that ours provides a rather direct and transparent approach.

1.2 Outline of the paper and notation

In Sect. 2, we introduce the notion of local energy solutions and recall the local regularity criterion due to [5] as well as some technical lemmas. Section 3 is devoted to stating and proving our main results including Theorem 1.1.

Throughout this paper, $C \in (0, \infty)$ denotes an absolute constant which may change line by line.

2 Preliminaries

In this section, we recall some notions about the weak solution to (NS) and some results such as the $\epsilon$-regularity theorems and a priori estimates for the solutions.
For any domain $\Omega \subset \mathbb{R}^3$ and open interval $I \subset (0, \infty)$, we say $(v, p)$ is a suitable weak solution in $\Omega \times I$ if it satisfies (NS) in the sense of distributions in $\Omega \times I$,

$$v \in L^\infty(I; L^2(\Omega)) \cap L^2(I; \dot{H}^1(\Omega)), \quad p \in L^{3/2}(\Omega \times I),$$

and the local energy inequality:

$$\int_\Omega |v(t)|^2 \phi(t) \, dx + 2 \int_0^t \int_\Omega |\nabla v|^2 \phi \, dx \, dt \leq \int_0^t \int_\Omega |v|^2 (\partial_t \phi + \Delta \phi) \, dx \, dt + \int_0^t \int_\Omega (|v|^2 + 2p)(v \cdot \nabla \phi) \, dx \, dt$$

for all non-negative $\phi \in C^\infty_c(\Omega \times I)$. Note that no boundary condition is required.

We next define the notion of local energy solutions. The following definition is formulated in [4], which is slightly revised from the notions of the local Leray solution defined in [19], the local energy solution in [17] and the Leray solution in [12]. We refer to [14, Sect. 2] for discussion of their relation.

**Definition 2.1 (Local energy solutions [4])** A vector field $v \in L^2_{uloc}(\mathbb{R}^3 \times [0, T))$ is a local energy solution to (NS) with divergence free initial data $v_0 \in L^2_{uloc}$ if

1. for some $p \in L^{3/2}_{uloc}(\mathbb{R}^3 \times [0, T))$, the pair $(v, p)$ is a distributional solution to (NS),
2. for any $R > 0$,

$$\sup_{0 \leq t < \min(R^2, T)} \sup_{x_0 \in \mathbb{R}^3} \int_{B_R(x_0)} |v|^2 \, dx + \sup_{x_0 \in \mathbb{R}^3} \int_0^{\min(R^2, T)} \int_{B_R(x_0)} |\nabla v|^2 \, dx \, dt < \infty,$$

3. for all compact subsets $K$ of $\mathbb{R}^3$ we have $v(t) \to v_0$ in $L^2(K)$ as $t \to 0^+$,
4. $(v, p)$ satisfies the local energy inequality (2.1) for all non-negative functions $\phi \in C^\infty_c(Q)$ with all cylinder $Q$ compactly supported in $\mathbb{R}^3 \times (0, T)$,
5. for every $x_0 \in \mathbb{R}^3$ and $r > 0$, there exists $c_{x_0, r} \in L^{3/2}(0, T)$ such that

$$p(x, t) - c_{x_0, r}(t) = \frac{1}{3} |v(x, t)|^2 + p.v. \int_{B_r(x_0)} K(x - y) : v(y, t) \otimes v(y, t) \, dy$$

$$+ \int_{\mathbb{R}^3 \setminus B_r(x_0)} (K(x - y) - K(x_0 - y)) : v(y, t) \otimes v(y, t) \, dy$$

in $L^{3/2}(B_{2r}(x_0) \times (0, T))$, where $K(x) = \nabla^2(\frac{1}{4\pi |x|})$, and
6. for any compact supported functions $w \in L^2(\mathbb{R}^3)^3$,

$$\text{The function } t \mapsto \int_{\mathbb{R}^3} v(x, t) \cdot w(x) \, dx \text{ is continuous on } [0, T).$$

Let us also recall the uniformly local $L^q$ spaces for $1 \leq q < \infty$. We say $f \in L^q_{uloc}$ if $f \in L^q_{loc}(\mathbb{R}^3)$ and

$$\|f\|_{L^q_{uloc}} = \sup_{x \in \mathbb{R}^3} \|f\|_{L^q(B_1(x))} < \infty.$$  

(2.5)

Local in time existence of local energy solutions for initial data in $L^2_{uloc}$ and also global existence for initial data in $E^2 := \overline{\cap_{0}^{\infty} L^2_{uloc}}$ and the weighted $L^2$ spaces are established in
There are absolute constants Lemma 2.3 [5, Proposition 1]. It is formulated in the present form in [22,24].

Assume that $f$ Lemma 2.5 where $C$ is a universal constant.

for some $\alpha > 0$.

Finally we also show the following elementary bound for the scaled energy.

Lemma 2.5 Assume that $f \in L_{\text{loc}}^2(\mathbb{R}^3)$ and let $N_R = N_{(\alpha),R}(f) := \sup_{r \in (R,1)} \frac{1}{r^\alpha} \int_{B_r} |f|^2 < \infty$

for some $\alpha > 0$ and $R \in [0, \frac{1}{2}]$. 

\[ E_{\text{uloc}}(t) := \text{ess sup}_{0 \leq \tau \leq t} \|v(\tau)\|_{L_2}^2 + \sup_{x_0 \in \mathbb{R}^3} \int_0^t \int_{B_1(x_0)} \|\nabla v\|^2 \leq 2 \|v_0\|_{L_2}^2, \] (2.6)

\[ \sup_{x_0 \in \mathbb{R}^3} \|p - c_{x_0} \|_{L^q(0,T;L^q(\mathbb{R}^3))} \leq C(s,q) \|v_0\|_{\text{uloc}}^2 \] (2.7)

for $t \leq T_{\text{uloc}} := \min\left( \frac{T}{c_0}, \frac{\|v_0\|_{L_2}^2}{1 + \|v_0\|_{L_2}^2} \right)$ with a universal constant $c_0 > 0$. Similar estimates with $B_1$ replaced by $B_r$ are valid.

We now recall the scaled version of the $\epsilon$-regularity theorem of Caffarelli-Kohn-Nirenberg [5, Proposition 1]. It is formulated in the present form in [22,24].

Lemma 2.3 There are absolute constants $\epsilon_{\text{CKN}}$ and $C_{\text{CKN}} > 0$ with the following property. Suppose that $(v, p)$ is a suitable weak solution of (NS) in $B_r(x_0) \times (t_0 - r^2, t_0)$, $r > 0$, with

\[ \frac{1}{r^2} \int_{t_0 - r^2}^{t_0} \int_{B_r(x_0)} |v|^3 + |p|^{3/2} dx dt \leq \epsilon \] for some $\epsilon \leq \epsilon_{\text{CKN}}$, then $v \in L^\infty(B_{r}(x_0) \times (t_0 - \frac{r^2}{T}, t_0))$ and

\[ \|v\|_{L^\infty(B_r \times (t_0 - \frac{r^2}{T}, t_0))} \leq \frac{C_{\text{CKN}} \epsilon^{\frac{1}{3}}}{r}. \] (2.8)

We recall a useful Gronwall-type inequality from [4, Lemma 2.2].

Lemma 2.4 Suppose $f \in L_{\text{loc}}^\infty([0,T_0); [0, \infty))$ (which may be discontinuous) satisfies, for some $a, b > 0$, and $m \geq 1$,

\[ f(t) \leq a + b \int_0^t (f(s) + f(s)^m)ds \quad \text{for } t \in (0, T_0), \]

then we have $f(t) \leq 2a$ for $t \in (0, T)$ with

\[ T = \min\left( T_0, \frac{C}{b(1 + a^{m-1})} \right), \]

where $C$ is a universal constant.

Finally we also show the following elementary bound for the scaled energy.
(i) If $\delta \geq 2^\alpha N_R$, then for any $x_0 \in B_{\frac{1}{2}}$, we have
\[
\sup_{R(x_0) < r \leq 1 - |x_0|} \frac{1}{r^\alpha} \int_{B_r(x_0)} |f|^2 \leq \delta \tag{2.9}
\]
with $R(x_0) = \max\left(\frac{R}{2}, 2\left(\frac{N_R}{\delta}\right)^{\frac{1}{\alpha}}|x_0|\right)$.

(ii) If $\alpha \geq 1$ and $\delta \geq 2N_R$, then for any $x_0 \in B_{\frac{1}{2}}$, we have
\[
\sup_{R'(x_0) < r \leq 1 - |x_0|} \frac{1}{r} \int_{B_r(x_0)} |f|^2 \leq \delta \tag{2.10}
\]
with $R'(x_0) = \max\left(\frac{R}{2}, \frac{2^\alpha N_R}{\delta}|x_0|^{\alpha}\right)$.

**Proof** (i) Assume that $r \in (R(x_0), 1 - |x_0|]$. If $1/2 < r \leq 1 - |x_0|$, we have
\[
\frac{1}{r^\alpha} \int_{B_r(x_0)} |f|^2 \leq \frac{1}{r^\alpha} \int_{B_{1/2}(0)} |f|^2 \leq \frac{1}{r^\alpha} N_R \leq \delta.
\]
If $|x_0| \leq r \leq 1/2$, (2.9) clearly holds by the following estimate:
\[
\frac{1}{r^\alpha} \int_{B_{r}(x_0)} |f|^2 \leq \frac{1}{r^\alpha} \int_{B_2(0)} |f|^2 \leq 2^\alpha N_R \leq \delta,
\]
where we have used $|x_0| + r \leq 2r$. Finally, if $R(x_0) < r < |x_0|$ (this case is empty if $|x_0| \leq R/2 \leq R(x_0)$), we have $|x_0| + r < 2|x_0|$, and hence
\[
\frac{1}{r^\alpha} \int_{B_{r}(x_0)} |f|^2 \leq \frac{1}{r^\alpha} \int_{B_{2|x_0|}(0)} |f|^2 \leq \left(\frac{2|x_0|}{r}\right)^\alpha N_R.
\]
The right hand side is bounded by $\delta$ since $r > R(x_0) \geq 2\left(\frac{N_R}{\delta}\right)^{\frac{1}{\alpha}}|x_0|$. Therefore we have verified (2.9).

(ii) We note that if $R/2 < r < |x_0|$, we have
\[
\frac{1}{r} \int_{B_{r}(x_0)} |f|^2 \leq \frac{1}{r} \int_{B_{2|x_0|}(0)} |f|^2 \leq \frac{(2|x_0|)^\alpha}{r} N_R.
\]
Hence the right hand side is bounded by $\delta$ when $r \geq \frac{2^\alpha N_R}{\delta}|x_0|^{\alpha}$. The other case is similar to the proof of (i), and we omit the details. \qed

**Remark 2.6** In our previous paper [14], we have used the scale critical Herz norm in $\mathbb{R}^3$:
\[
\|f\|_{K_{p, \infty}}^{\frac{1}{q}} := \sup_{x_0 \neq 0} \|x_0|^{1 - \frac{3}{p}} \|f\|_{L^p(B_{|x_0|}(x_0))}
\]
for $p \geq 3$. We easily observe that
\[
\|f\|_{K_{\frac{1}{2}, \infty}}^{\frac{1}{q}} \leq C \sup_{r > 0} \frac{1}{r} \int_{B_r} |f|^2 \ dx \leq C' \|f\|_{K_{\frac{1}{2}, \infty}}^{\frac{1}{q}}
\]
with some absolute constants $C$ and $C' > 0$. 

\[\text{Springer}\]
3 Main results

In this section we prove our main results, Theorems 1.1–1.3, regarding the regularity of the local energy solutions. They are obtained as consequences of the following theorem.

**Theorem 3.1** Let \( v \) be a local energy solution of (NS) in \( \mathbb{R}^3 \times (0, T) \) associated with initial data \( v_0 \in L^2_{uloc} \), and

\[
N_R = N(\alpha, R(v_0)) := \sup_{r \in (R, 1)} \frac{1}{r^\alpha} \int_{B_r(0)} |v_0|^2 < \infty
\]

for some \( \alpha \in (1, 4) \) and \( R \in [0, \frac{1}{2}] \). There are absolute constants \( c \) and \( C > 0 \) such that the following holds.

(i) If \( N_R \leq 1 \) and \( R \leq c \sqrt{S_1} \) with

\[
S_1 := \min \left( T, \frac{c}{1 + \|v_0\|_{L^2_{uloc}}^{12}} \right),
\]

then we have

\[
|v(x, t)| \leq C \left[ 1 + \|v_0\|_{L^2_{uloc}} \right] \frac{1}{2} (\alpha - 3) \frac{t^{1/4} (\alpha - 3)}{2^{(4 - \alpha)}}, \quad \text{for } |x| \leq \frac{1}{2},
\]

\[
c \max \left( R^2, N_R^{2/\alpha} \right) \leq t \leq S_1. \tag{3.1}
\]

(ii) If \( N_R \geq 1 \), assume \( R \leq cN_R \sqrt{S_2} \) with

\[
S_2 = \min \left( T, \frac{c}{1 + \|v_0\|_{L^2_{uloc}}^{12}}, \frac{c}{\left( 1 + \|v_0\|_{L^2_{uloc}}^{4/3} \right) N_R^2} \right), \tag{3.2}
\]

then we have

\[
|v(x, t)| \leq C \left[ N_R^{(\alpha + 1)/2} + N_R^{3/2} \|v_0\|_{L^2_{uloc}}^{1/4} \right] \frac{1}{2} (\alpha - 3) \frac{t^{1/4} (\alpha - 3)}{2^{(4 - \alpha)}},
\]

\[
\text{for } |x| \leq \frac{1}{2}, \quad \frac{c}{N_R^2} \max \left( R^2, |x|^2 \right) \leq t \leq S_2. \tag{3.3}
\]

The following local energy estimate is our key ingredient for the proof of Theorem 3.1, which shows the smallness of the scaled local energy \( E_r \) defined in (1.7) is propagated at least locally in time and some scales.

**Lemma 3.2** Let \( v \) be a local energy solution of (NS) in \( \mathbb{R}^3 \times (0, T) \) associated with initial data \( v_0 \in L^2_{uloc} \), and \( N_R = \sup_{R < r \leq 1} \frac{1}{r^\alpha} \int_{B_r(0)} |v_0|^2 < \infty \) for some constants \( \alpha \in [1, 4] \) and \( R \in [0, 1] \). Let \( \delta \geq 70N_R \). Then for any \( r \in (R, 1] \) we have

\[
E_r(t) \leq \delta \quad \text{for } 0 < t \leq \min \left( \lambda r^2, S_1 \right), \tag{3.4}
\]

where

\[
\lambda = \frac{c}{1 + \delta^2} \leq 1 \quad \text{and} \quad S_1 = \min \left( T, \frac{c \min \{1, \delta^4\}}{1 + \|v_0\|_{L^2_{uloc}}^{12}} \right).
\]
Moreover, if $R < R_1 := \min \left( \sqrt{\frac{3\lambda}{2}}, 1 \right)$, there exists $c_2(t) \in \mathbb{R}$ such that
\[
\frac{1}{r^2} \int_0^{r^2} \int_{B_r} |v|^3 + |p - c_2(t)|^2 \, dx \, dt \leq C(\lambda \delta^6_r r^{2(\alpha-1)} + \lambda \|v_0\|_{\mathcal{L}^2_{\text{loc}}}^3 r^{\frac{9}{2}}) \quad \text{for all } r \in (R, R_1).
\] (3.5)

Above $c$ and $C > 0$ are absolute constants.

**Proof of Lemma 3.2** By the definition of the local energy solution, for $r \in (R, 1]$, there exists $c_r = c_r(t)$ such that the pressure admits the following decomposition (2.3):
\[
p - c_r = -\frac{|v|^2}{3} + p_{\text{loc}} + p_{\text{nonloc}}
\]
\[
:= -\frac{|v|^2}{3} + \text{p.v.} \int_{B_{3r}} K(x - y)(v \otimes v)(y) \, dy
\]
\[
+ \int_{\mathbb{R}^3 \setminus B_{3r}} (K(x - y) - K(-y))(v \otimes v)(y) \, dy
\] (3.6)
in $L^{3/2}(B_{2r}(0) \times (0, T))$. By considering (2.3) in a larger region $B_2 \times (0, T)$, we may remove the $r$-dependence of $c_r(t)$ by choosing $c_r(t) = c_1(t)$. Since $(v, p - c_r)$ is a suitable weak solution to (NS) in $B_{2r}$, the local energy inequality (2.1) with the test function $\phi \in C^\infty_0(B_{2r})$ such that $0 \leq \phi \leq 1$ in $B_{2r}$ with $\phi = 1$ in $B_r$ and $\|\nabla^2 \phi\|_{L^\infty} \leq C_k r^{-k}$ readily yields
\[
E_r(t) \leq \frac{2^\alpha}{(2r)^{\alpha}} \int_{B_{2r}} |v_0|^2 + C \int_0^t \int_{B_{2r}} |v|^2 + C \int_0^t \int_{B_{2r}} |v|^3 + |v||p - c_r|.
\] (3.7)

By (3.6) together with Hölder and Young inequalities, the last term involving the pressure can be estimated as follows:
\[
\frac{1}{r^{\alpha+1}} \int_0^t \int_{B_{2r}} |v||p - c_r| \leq \left( \frac{1}{r^{\alpha+1}} \int_0^t \int_{B_{2r}} |v|^3 \right)^{\frac{1}{3}} \left( \frac{1}{r^{\alpha+1}} \int_0^t \int_{B_{2r}} |p - c_r|^2 \right)^{\frac{2}{3}}
\]
\[
\leq \frac{1}{r^{\alpha+1}} \int_0^t \int_{B_{2r}} |v|^3 + \frac{C}{r^{\alpha+1}} \int_0^t \int_{B_{2r}} |p - c_r|^2
\]
\[
\leq \frac{1}{r^{\alpha+1}} \int_0^t \int_{B_{2r}} |v|^3 + C \int_0^t \int_{B_{2r}} |p_{\text{loc}}|^\frac{3}{2} + |p_{\text{nonloc}}|^\frac{3}{2}.
\]
Noting that $\frac{3\alpha}{4} + \frac{5}{4} \leq \alpha + 1 \leq \frac{3\alpha}{2} + \frac{1}{2}$ for $\alpha \geq 1$, we have
\[
E_r(t) \leq 2^\alpha N_R + C \int_0^t \int_{B_{2r}} |v|^2 + C \int_0^t \int_{B_{2r}} |v|^3 + |p_{\text{loc}}|^\frac{3}{2} + |p_{\text{nonloc}}|^\frac{3}{2}
\]
\[
=: 2^\alpha N_R + I_{\text{lin}} + I_{\text{nonlin}} + I_{\text{loc}} + I_{\text{nonloc}}.
\] (3.8)

We divide the estimate into two cases.

**Case I**: $R < r \leq \frac{1}{6}$. This case is empty if $R > \frac{1}{6}$. For the simplicity of notation, let $E_r(t) := \sup_{r \in (r, 1]} E_r(t)$. We easily see
\[
I_{\text{lin}} \leq C \int_0^t E_R(s) \, ds.
\] (3.9)

By the interpolation and Young’s inequalities, we also have
\[
I_{\text{nonlin}} \leq C \int_0^t \left( \int_{B_r} |\nabla v|^2 \right)^{\frac{3}{4}} \left( \int_{B_r} |v|^2 \right)^{\frac{3}{4}} + \left( \frac{1}{r} \int_{B_r} |v|^2 \right)^{\frac{3}{2}} \, ds
\]
\[ I_{ploc} \leq \frac{C}{r^{\frac{3\alpha}{4} + \frac{\alpha}{2}}} \int_0^t \int_{B_{3r}} |v|^3 ds \]

This estimate and the Calderón–Zygmund estimate give

\[ I_{pnonloc} \leq \frac{C}{r^{\frac{3\alpha}{4} + \frac{\alpha}{2}}} \int_0^t \int_{B_{3r}} |v|^3 ds \]

On the other hand, since \(|x - y| \geq |y|/3\) for \(x \in B_{2r}\) and \(y \in \mathbb{R}^3 \setminus B_{3r}\), we see

\[ |p_{\text{nonloc}}(x)| \leq \int_{\mathbb{R}^3 \setminus B_{3r}} |K(x - y) - K(-y)||v(y)|^2 \, dy \]

\[ \leq Cr \int_{\mathbb{R}^3 \setminus B_{3r}} \frac{1}{|x - y|^\alpha} |v(y)|^2 \, dy \]

\[ \leq Cr \int_{\mathbb{R}^3 \setminus B_{3r}} \frac{|v(y)|^2}{|y|^4} \, dy \]

\[ \leq Cr^{\alpha - 3} \sum_{k=1}^{[-\log_2 r - 1]} \int_{B_{2^{k+1}r} \setminus B_{2^{k}r}} \frac{|v(y)|^2}{|y|^4} \, dy + Cr \int_{\mathbb{R}^3 \setminus B_{1/4}} \frac{|v(y)|^2}{|y|^4} \, dy \]

\[ \leq Cr^{\alpha - 3} E_R + Cr \|v_0\|^2_{L^2_{\text{uloc}}} \]

provided \(t \leq T_{\text{uloc}} = \min \left( T, \frac{c_0}{1 + \|v_0\|_{L^2_{\text{uloc}}}^2} \right)\) and \(\alpha < 4\), where we have used Lemma 2.2 in the last line. We then obtain

\[ I_{pnonloc} \leq \frac{C}{r^{\frac{3\alpha}{4} + \frac{\alpha}{2}}} \int_0^t r^3 \left( r^{\frac{3\alpha}{4} - 3} E_R(s) + r^{\frac{3\alpha}{4}} \|v_0\|_{L^2_{\text{uloc}}}^3 \right) ds \]

\[ \leq C \int_0^t E_R(s)^3 ds + C tr^{\frac{3\alpha}{4} - 3} \|v_0\|_{L^2_{\text{uloc}}}^3 \]

\[ \leq C \int_0^t E_R(s)^3 ds + \delta \frac{\delta}{10}, \]

provided \(t \leq \left( c_\delta \frac{\|v_0\|_{L^2_{\text{uloc}}}^2}{\|v_0\|_{L^2_{\text{uloc}}}^2} \right)\) with a small absolute constant \(c > 0\). We have used \(\alpha \in [1, 4]\) in the second inequality. Hence plugging (3.9), (3.10), (3.11), and (3.13) into (3.8) with \(\varepsilon = 1/6\), we obtain

\[ \sup_{R < r < 1/6} E_r(t) \leq \delta \frac{\delta}{3} + \frac{1}{3} E_R(t) + \frac{C}{R^2} \int_0^t E_R(s) + E_R(s)^3 ds \]

for \(\delta \geq 70N_R\).
Case II: $\frac{1}{6} \leq r \leq 1$. In order to bound the right hand side of (3.8), we observe from Lemma 2.2 that

$$\sup_{1/3 \leq r \leq 2} E_r(t) \leq CE_{uloc}(t) \leq C\|v_0\|_{L^2_{uloc}}^2$$
holds for $t \leq T_{uloc}$.

(3.14)

This shows

$$I_{lin} \leq C \int_0^t \sup_{1/3 \leq r \leq 2} E_r(s)ds \leq Ct\|v_0\|_{L^2_{uloc}}^2,$$

and hence if $t \leq \left( T_{uloc}, \frac{c\delta}{\|v_0\|_{L^2_{uloc}}^2} \right)$ with a suitable small constant $c > 0$, we have

$$I_{lin} \leq \frac{\delta}{40}.$$

(3.15)

For the nonlinear term in (3.8), we have

$$I_{nonlin} \leq C \int_0^t \int_{B_2} |v|^3 \leq C \left( \int_0^t \int_{B_2} |\nabla v|^2 \right)^{\frac{3}{4}} \left( \int_0^t \left( \int_{B_2} |v|^2 \right)^{\frac{3}{2}} \right)^{\frac{1}{2}} + C \int_0^t \int_{B_2} |v|^2 \leq C(t^{\frac{1}{4}} + t)\|v_0\|_{L^2_{uloc}}^3.$$

(3.16)

By the Calderón–Zygmund estimate it also implies

$$I_{ploc} \leq C \int_0^t \int_{B_3} |v|^3 \leq C(t^{\frac{1}{4}} + t)\|v_0\|_{L^2_{uloc}}^3.$$

(3.17)

Thus the right hand sides in (3.16) and (3.17) are bounded by $\frac{\delta}{40}$, provided

$$t \leq \min \left( T_{uloc}, \frac{c\delta^4}{\|v_0\|_{L^2_{uloc}}^{\frac{1}{2}}}, \frac{c\delta}{\|v_0\|_{L^2_{uloc}}^3} \right)$$

with some small absolute constant $c > 0$. On the other hand, in the same way as in (3.12), we have

$$|p_{nonloc}(x)| \leq C \int_{\mathbb{R}^3 \setminus B_1/2} \frac{|v(y)|^2}{|y|^4}dy \leq C\|v_0\|_{L^2_{uloc}}^2,$$

which implies

$$I_{pnonloc} \leq \frac{\delta}{40} \quad \text{for} \quad t \leq \frac{c\delta}{\|v_0\|_{L^2_{uloc}}^3}.$$

Making use of these estimates in (3.8), we obtain

$$\sup_{1/6 \leq r \leq 1} E_r(t) \leq \frac{\delta}{3}.$$
By choosing $c_1 > 0$ sufficiently small, we may take
\[
S_1 := \min \left( T, \frac{c_1 \min \{1, \delta^4 \}}{1 + \|v_0\|_{L^2_{u_{\text{loc}}}}^2} \right) \leq \min \left( T_{u_{\text{loc}}}, \frac{c\delta}{\|v_0\|_{L^2_{u_{\text{loc}}}}^2}, \frac{c\delta^4}{\|v_0\|_{L^2_{u_{\text{loc}}}}^2}, \frac{c\delta}{\|v_0\|_{L^2_{u_{\text{loc}}}}^2} \right).
\]
Summarizing the conclusions of the cases I and II, we have
\[
\mathcal{E}_R(t) \leq \frac{\delta}{2} + \frac{C}{R^2} \int_0^t \mathcal{E}_R(s) + \mathcal{E}_R(s)^3 \, ds
\]
for $t \leq S_1$. Applying Lemma 2.4 we obtain
\[
\mathcal{E}_R(t) \leq \delta \quad \text{for} \quad t \leq \min \left( \lambda R^2, S_1 \right), \quad \lambda = \frac{c}{1 + \delta^2}. \tag{3.18}
\]
Since $\delta \geq 70N_R \geq 70N_r$ for any $r \in (R, 1]$, we may replace $R$ by $r$ in (3.18), and thus we have verified (3.4) for $R < r \leq 1$.

To prove (3.5), we see that if $\lambda r^2 \in (0, S_1]$,
\[
\frac{1}{r^{\frac{3\alpha}{2} + \frac{1}{2}}} \int_0^{\lambda r^2} \int_{B_r} |v|^3 \leq \frac{C}{r^{\frac{3\alpha}{2} + 1}} \int_0^{\lambda r^2} \left( \int_{B_r} |\nabla v|^2 \right)^{\frac{3}{2}} \left( \int_{B_r} |v|^2 \right)^{\frac{3}{2}} \, ds + \frac{C}{r^{\frac{3\alpha}{2} + 1}} \int_0^{\lambda r^2} \left( \frac{1}{r} \int_{B_r} |v|^2 \right)^{\frac{3}{2}} \, ds
\]
\[
\leq \frac{C}{r^{\frac{3\alpha}{2} + 1}} \int_0^{\lambda r^2} \left( \int_{B_r} |\nabla v|^2 \right)^{\frac{3}{2}} \, ds \frac{3\alpha}{2} E_r(\lambda r^2)^{\frac{3}{2}} + \frac{C}{r^{\frac{3\alpha}{2} + 1}} \int_0^{\lambda r^2} \frac{3\alpha}{2} E_r(\lambda r^2)^{\frac{3}{2}} + C \lambda E_r(\lambda r^2)^{\frac{3}{2}}
\]
\[
\leq C(\lambda^{\frac{1}{2}} + \lambda) E_r(\lambda r^2)^{\frac{3}{2}}.
\]
Hence taking $R_1 := \min \left( \sqrt{\frac{S_1}{\lambda}}, 1 \right)$, we have
\[
\frac{1}{r^{\frac{3\alpha}{2} + 1}} \int_0^{\lambda r^2} \int_{B_r} |v|^3 \leq C^{\frac{3}{2}} \delta^{\frac{3}{2}} \quad \text{for} \quad r \in (R, R_1]. \tag{3.19}
\]
By the Calderón–Zygmund estimate we also have
\[
\frac{1}{r^{\frac{3\alpha}{2} + 1}} \int_0^{\lambda r^2} \int_{B_r} |p_{\text{loc}}|^3 \leq C^{\frac{1}{3}} \delta^{\frac{3}{2}} \quad \text{for} \quad r \in (R, R_1]. \tag{3.20}
\]
From (3.12) we have
\[
\frac{C}{r^{\frac{3\alpha}{2} + 1}} \int_0^{\lambda r^2} \int_{B_{2r}} |p_{\text{nonloc}}|^3 \leq \frac{C}{r^{\frac{3\alpha}{2} + 1}} \int_0^{\lambda r^2} r^3 \left( \frac{3\alpha}{2} - 3 \right) \mathcal{E}_R(s)^{\frac{3}{2}} + r^3 \int_0^1 \|v_0\|_{L^2_{u_{\text{loc}}}}^3 \, ds
\]
\[
\leq C\lambda (\delta^{\frac{3}{2}} + r^{3\alpha - 3} \|v_0\|_{L^2_{u_{\text{loc}}}}^3).
\]
Combining this with (3.19) and (3.20) we obtain (3.5). \[\square\]
**Proof of Theorem 3.1** By Lemma 2.5 using $R \leq 1/2$, for each $\delta \geq 2^a N_R$ and $x_0 \in B_{1/2}$

$$\sup_{R(x_0) < r \leq \rho} \frac{1}{r^a} \int_{B_r(x_0)} |v_0|^2 dx \leq \delta, \quad \rho = 1 - |x_0|, \quad R(x_0) = \max \left( \frac{R}{2}, \left( \frac{N_R}{\delta} \right)^{\frac{1}{a^a}} |x_0| \right).$$

Let $v_{x_0}(x, t) = \rho v(x_0 + \rho x, \rho^2 t), p_{x_0}(x, t) = \rho^2 p(x_0 + \rho x, \rho^2 t)$. Since $(v_{x_0}, p_{x_0})$ solves (NS) in $\mathbb{R}^3 \times (0, \rho^{-2} T)$, corresponding to $(v, p)$ in $\mathbb{R}^3 \times (0, T)$, and $1/2 \leq \rho \leq 1$, we have

$$\|v_{x_0}(t = 0)\|_{L^2_{\text{loc}}} \leq C\|v_0\|_{L^2_{\text{loc}}},$$

$$N_{\rho^{-1} R(x_0)}(v_{x_0}) = \sup_{\rho^{-1} R(x_0) < r \leq \rho} \frac{1}{r^a} \int_{B_r(x_0)} |v_{x_0}(t = 0)|^2 \leq \sup_{R(x_0) < r \leq \rho} \frac{\rho^{a-1}}{r^a} \int_{B_r(x_0)} |v_0|^2 \leq \delta.$$

This ensures the assumption of Lemma 3.2 for $v_{x_0}$. If we take $\delta x_0 = 70\delta$, then from (3.5), there exists $c'(x_0)(t)$ such that

$$\sup_{\rho^{-1} R(x_0) < r \leq R_1'} \frac{1}{r^a} \int_{0}^{\lambda} \int_{B_r(x_0)} |v_{x_0}|^3 + |p_{x_0} - c'(x_0)(t)|^\frac{3}{2} \leq C(\lambda^\frac{3}{2} \delta^\frac{3}{2} r^\frac{3}{2}(a-1) + \|v_0\|_{L^2_{\text{loc}}}^3 r^\frac{9}{2}).$$

Here

$$R'_1 = \min \left( \sqrt{\frac{S'_1}{\lambda}}, 1 \right) \quad \text{with} \quad S'_1 = \min \left( \rho^{-2} T, \frac{c_1 \min\{1, \delta^4\}}{1 + \|v_0\|_{L^2_{\text{loc}}}^2} \right).$$

This implies

$$\sup_{R(x_0) \leq r \leq \rho R'_1} \frac{1}{\rho^a} \int_{0}^{\lambda} \int_{B_{\sqrt{S'_1}}(x_0)} |v|^3 + |p - c(x_0)(t)|^\frac{3}{2} \leq C(\lambda^{-\frac{3}{2}} \delta^\frac{3}{2} r^\frac{3}{2}(a-1) + \|v_0\|_{L^2_{\text{loc}}}^3 r^\frac{9}{2}).$$

Denote

$$\epsilon(x_0, r) := \frac{1}{r^2} \int_{0}^{\lambda} \int_{B_r(x_0)} |v|^3 + |p - c(x_0)(t)|^\frac{3}{2}. $$

For $\sqrt{\lambda} R(x_0) \leq r \leq \sqrt{\lambda} R'_1$ we have

$$\epsilon(x_0, r) \leq C(\lambda^{-\frac{3}{2}} \delta^\frac{3}{2} (\lambda^{-1/2} r)\frac{3}{2}(a-1) + \|v_0\|_{L^2_{\text{loc}}}^3 (\lambda^{-1/2} r)\frac{9}{2})$$

$$\leq C(1 + \delta^\frac{5}{2})\delta^\frac{3}{2} r^\frac{3}{2}(a-1) + C(1 + \delta)\frac{9}{2} r^\frac{9}{2} \|v_0\|_{L^2_{\text{loc}}}^3. $$

(3.22)

We now choose $r > 0$ as

$$r \leq R_2 := \min \left( \left( \frac{c}{(1 + \delta^2)\delta} \right)^{\frac{1}{3}} \frac{c}{\|v_0\|_{L^2_{\text{loc}}}^{1/3}} (1 + \delta) \right)$$

with a small constant $c > 0$ so that $\epsilon(x_0, r)$ is smaller than $\epsilon_{\text{CKN}}$ in Lemma 2.3. This enables us to apply Lemma 2.3 for $x_0 \in B_{1/2}$ and

$$t = r^2 \in \left( \lambda R(x_0)^2, \min \left( \lambda (\rho R'_1)^2, R_2^2 \right) \right]$$
to see that $v$ is regular at $(x_0, t)$ and, by (3.22),

$$
|v(x_0, t)| \leq C \kappa (x_0, t^{\frac{1}{2}})^{\frac{1}{2}} t^{-\frac{1}{2}}
$$

$$
\leq C (1 + \delta^{\frac{a}{2}}) \delta^{\frac{1}{2}} t^{\frac{1}{4}} (\alpha - 3) + C (1 + \delta)^{\frac{3}{2}} \|v_0\|_{L^2_{uloc}} t^{\frac{1}{4}}
$$

$$
= C \left[ (1 + \delta^{\frac{a}{2}}) \delta^{\frac{1}{2}} + (1 + \delta)^{\frac{3}{2}} \|v_0\|_{L^2_{uloc}} t^{\frac{3}{4}} (4 - \alpha) \right] t^{\frac{1}{4}} (\alpha - 3).
$$

(3.23)

Since $1/2 \leq \rho \leq 1$, we may take $c > 0$ so small that

$$
\lambda (\rho R_1')^2 = \rho^2 \min(S', \lambda) = \rho^2 \min \left( \frac{\rho^{-2} T, c \min\{1, \delta^4\}}{1 + \|v_0\|_{L^2_{uloc}}^{12}}, \lambda \right)
$$

$$
\geq \min \left( T, \frac{c \min\{1, \delta^4\}}{1 + \|v_0\|_{L^2_{uloc}}^{12}}, \frac{c}{(1 + \delta)^2} \right).
$$

Thus (3.23) is valid for $x_0 \in B_{1/2}$ and $S_a < t < S_b$, where

$$
S_a := \frac{c}{1 + \delta^2} \max \left( R^2, C (N_R/\delta)^{\frac{2}{a}} |x_0|^2 \right),
$$

$$
S_b := \min \left( T, \frac{c \min\{1, \delta^4\}}{1 + \|v_0\|_{L^2_{uloc}}^{12}}, \frac{c}{(1 + \delta^a)^{\frac{2}{a-1}}}, \frac{c}{(1 + \|v_0\|_{L^2_{uloc}}^{12}) (1 + \delta)^2} \right).
$$

(3.24)

If $N_R \geq 1$, we can take $\delta = 70 N_R$. Then

$$
S_a \leq \frac{c}{N_R^2} \max \left( R^2, |x_0|^2 \right),
$$

$$
S_b \geq S_2 = \min \left( T, \frac{c}{1 + \|v_0\|_{L^2_{uloc}}^{12}}, \frac{c}{N_R^{\frac{2a+2}{a-1}} \delta^{\frac{1}{2}}}, \frac{c}{(1 + \|v_0\|_{L^2_{uloc}}^{12}) N_R^2} \right),
$$

and

$$
|v(x_0, t)| \leq C \left[ N_R^{(a+1)/2} + N_R^{3/2} \|v_0\|_{L^2_{uloc}} S_2^{\frac{1}{3}} (4 - \alpha) \right] t^{\frac{1}{4}} (\alpha - 3).
$$

(3.25)

If $N_R \leq 1$, we can take $\delta = 70$. Then

$$
S_a \leq c \max \left( R^2, N_R^{2/\alpha} |x_0|^2 \right),
$$

$$
S_b \geq S_1 = \min \left( T, \frac{c}{1 + \|v_0\|_{L^2_{uloc}}^{12}} \right).
$$

(This $S_1$ may have a smaller $c$ than $S_1$ in Part (i)), and
\[ |v(x_0, t)| \leq C \left[ 1 + \|v_0\|_{L^2_{\text{loc}}} \delta^{\frac{1}{2}(4-\alpha)} \right] t^\frac{1}{2}(\alpha-3) \]

for \( c \max \left( R^2, N^{2/\alpha}_R |x_0|^2 \right) \leq t \leq S_1. \) (3.26)

**Proof of Theorem 1.1** For the proof of (1.3), we first claim that there exist \( \epsilon_* \) and \( c > 0 \) such that if \( N_0 := N(\alpha, 0) := \sup_{r \in (0, 1]} \frac{1}{r^2} \int_{B_r} |v_0(x)|^2 \, dx < \epsilon_* \), then for any \( x_0 \in B_{\frac{1}{2}} \) and \( r \in (cN_0|x_0|^\alpha, c(1 - |x_0|)R_1] \),

\[
\frac{1}{r^2} \int_0^r \int_{B_r(x_0)} \left| v \right|^3 + \left| p - c(x_0)(t) \right|^3 \, dx \, dt \leq \epsilon_{\text{CKN}} \quad (3.27)
\]

holds for some \( c(x_0)(t) \in \mathbb{R} \) and \( R'_1 > 0 \) defined below. Here \( \epsilon_{\text{CKN}} \) is the small constant in Lemma 2.3. We observe from Lemma 2.5 (ii) that for each \( \eta \geq 2N_0 \) and \( x_0 \in B_{\frac{1}{2}} \)

\[
\|v_x(t = 0)\|_{L^2_{\text{loc}}} \leq C \|v_0\|_{L^2_{\text{loc}}},
\]

\[
\sup_{r \leq \rho \leq 1} \frac{1}{r} \int_{B_r(0)} |v_0|^2 \, dx \leq \eta, \quad \rho = 1 - |x_0|, \quad R(x_0) = \frac{2^\alpha N_0}{\eta} |x_0|^\alpha.
\]

Let \( v_{x_0}(x, t) = rv(x_0 + rx, \rho^2 t) \), \( p_{x_0}(x, t) = \rho^2 p(x_0 + rx, \rho^2 t) \) and \( \delta = 70\eta \). As in the proof of Theorem 3.1 we have

\[
\|v_{x_0}(t = 0)\|_{L^2_{\text{loc}}} \leq C \|v_0\|_{L^2_{\text{loc}}},
\]

\[
\sup_{r \leq \rho \leq 1} \frac{1}{r} \int_{B_r(0)} |v_{x_0}(t = 0)|^2 \, dx = \sup_{r \leq \rho \leq 1} \frac{1}{r} \int_{B_r(x_0)} |v_0|^2 \, dx \leq \eta = \frac{\delta}{70}.
\]

This guarantees the assumption of Lemma 3.2 for \( v_{x_0} \) and \( \alpha = 1 \), and hence there exists \( c'(x_0)(t) \) such that

\[
\sup_{r \leq \rho \leq 1} \frac{1}{r} \int_{B_r(0)} \left| v_{x_0}(t = 0) \right|^3 + \left| p_{x_0} - c'(x_0)(t) \right|^3 \leq C \frac{\delta^2}{\lambda^2} \delta^2. \]

Here

\[
R'_1 = \min \left( \sqrt{\frac{S_1'}{\lambda}}, 1 \right) \quad \text{with} \quad S'_1 = \min \left( \rho^{-2}T, \frac{c_1 \min\{1, \delta^4\}}{1 + \|v_0\|_{L^2_{\text{loc}}}^{12}} \right).
\]

This implies

\[
\sup_{R(x_0) \leq r \leq \rho R'_1} \frac{1}{r^2} \int_0^{\rho r^2} \int_{B_{\sqrt{\lambda r}}(x_0)} \left| v \right|^3 + \left| p - c(x_0)(t) \right|^3 \, dx \, dt \leq \frac{C \delta^2}{\lambda^2} \leq C (1 + \delta^2 \delta^2). \quad (3.28)
\]

Take a constant \( \delta_0 > 0 \) so small that \( C(1 + \delta_0^2) \delta_0^3 \leq \epsilon_{\text{CKN}} \) and assume that \( v_0 \) satisfies \( N_0 \leq \delta_0/70 \). Then we may choose \( \delta = \delta_0 \) since \( \delta_0 \geq 70N_0 \). With this choice and with \( \lambda_0 = \lambda(\delta_0) \), we have verified (3.27) for \( \lambda_0 \frac{1}{70} R(x_0) < r \leq \lambda_0 \frac{1}{70} R'_1 \).

This enables us to apply Lemma 2.3 for \( x_0 \in B_{1/2} \) and \( t_0 = r^2 \in \left( C\lambda_0 \left( \frac{N_0}{\delta_0} \right)^2 |x_0|^{2\alpha}, \lambda_0(\rho R'_1)^2 \right) \) to see

\[
|v(x_0, t_0)| \leq C_{\text{CKN}} r^{-1} = C_{\text{CKN}} r_0^{-\frac{1}{2}},
\]

\( \epsilon \) Springer
and hence $v$ is regular at $(x_0, t_0)$. Since $\lambda_0$ and $\delta_0$ are absolute constants and $1/2 \leq \rho \leq 1$, we may choose $c > 0$ so small that

\[
\lambda_0(\rho R_1')^2 = \rho^2 \min(S_1', \lambda_0) = \rho^2 \min \left( \rho^{-2} T, \frac{c \min \{1, \delta_0^4\}}{1 + \|v_0\|_{L^2_{\text{uloc}}}^2}, \lambda_0 \right)
\]

\[
\geq \min \left( T, \frac{c}{1 + \|v_0\|_{L^2_{\text{uloc}}}^2} \right).
\]

Thus $|v(x_0, t_0)| \leq C_{\text{CKN}} t_0^{-\frac{1}{2}}$ for $x_0 \in B_{1/2}$ and $c N_0^2 |x_0|^{2\alpha} \leq t_0 \leq \min \left( T, \frac{c}{1 + \|v_0\|_{L^2_{\text{uloc}}}^2} \right)$. This proves (1.3) for the case $N_0 \leq \epsilon_*$.

In order to consider the case $N_0 > \epsilon_*$, let $u(x, t) = \rho v(\rho x, \rho^2 t), q(x, t) = \rho^2 p(\rho x, \rho^2 t)$, $u_0(x) = \rho v_0(\rho x)$ with $0 < \rho < 1$ to be given below. We easily see

\[
\|u_0\|_{L^2_{\text{uloc}}} = \rho^{-\frac{1}{2}} \sup_{x_0 \in \mathbb{R}^3} \|v_0\|_{L^2(B(\rho x_0))} \leq C \rho^{-\frac{1}{2}} \|v_0\|_{L^2_{\text{uloc}}},
\]

\[
N_0(u_0) = \sup_{r \in (0, 1)} \frac{1}{r^{\alpha}} \int_{B_r} |u_0|^2 = \sup_{r \in (0, 1)} \frac{1}{\rho r^{\alpha}} \int_{B_{\rho r}} |v_0|^2 = \sup_{r \in (0, \rho)} \frac{\rho^{\alpha-1}}{r^{\alpha}} \int_{B_r} |v_0|^2 \leq \rho^{\alpha-1} N_0.
\]

(3.29)

Let $\rho = \left( \frac{N_0}{\epsilon_*} \right)^{1/\alpha}$ so that $N_0(u_0) \leq \epsilon_*$. Then it follows from Step 1 that $u$ is regular and satisfies $|u(x, t)| \leq C_{\text{CKN}} t^{-\frac{1}{2}}$ in $\Pi$ in the region

\[
\left\{ (x, t) \in B_{\frac{1}{2}} \times (0, \rho^{-2} T); \ c N_0(u_0)^2 |x|^{2\alpha} \leq t \leq \frac{c}{1 + \|u_0\|_{L^2_{\text{uloc}}}^2} \right\}.
\]

Rescaling back to $(v, p)$ and using (3.29), we see that $v$ is regular and satisfies $|v(x, t)| \leq C_{\text{CKN}} t^{-\frac{1}{2}}$ in the region

\[
\left\{ (x, t) \in B_{\frac{1}{2}} \times (0, T); \ c N_0^2 |x|^{2\alpha} \leq t \leq \frac{c \rho^2}{1 + \rho^{-6} \|v_0\|_{L^2_{\text{uloc}}}^2} \right\}.
\]

By taking $c > 0$ sufficiently small, we see that this region contains

\[
\Pi_1 = \left\{ (x, t) \in B_{\frac{1}{2}} \times (0, T); \ c N_0^2 |x|^{2\alpha} \leq t \leq T_1 \right\}, \quad T_1 := \frac{c}{N_0 \frac{6}{\alpha+1} \left( 1 + N_0 \frac{6}{\alpha+1} \|v_0\|_{L^2_{\text{uloc}}}^2 \right)}.
\]

(3.30)

This completes the proof of (1.3).

Est. (1.4) is the special case of Theorem 3.1 with $R = 0$. Indeed for all $N_0 < \infty$ we have

\[
|v(x, t)| \leq M t^{\frac{\alpha-3}{\alpha}} \left( C_1 |x|^2 < t < S_2 \right),
\]

(3.31)
where \( M = C \left[ 1 + \| v_0 \|_{L^2_{\text{loc}}} S_1^{\frac{1}{2}(4-\alpha)} \right] \) if \( N_0 \leq 1 \), and \( M = C \left[ N_0^{(\alpha+1)/2} + N_0^{3/2} \| v_0 \|_{L^2_{\text{loc}}} S_2^{\frac{1}{2}(4-\alpha)} \right] \) if \( N_0 \geq 1 \). Hence we can take
\[
C_0 = C \left[ 1 + N_0^{(\alpha+1)/2} + (1 + N_0^{3/2}) \| v_0 \|_{L^2_{\text{loc}}} S_1^{\frac{1}{2}(4-\alpha)} \right].
\]
We also have \( C_1 = c N_0^{2/\alpha} \) if \( N_0 \leq 1 \), and \( C_1 = c N_0^{-2} \) if \( N_0 \geq 1 \). Hence we can take
\[
C_1 = \frac{c N_0^{2/\alpha}}{(1 + N_0)^{2+2/\alpha}}.
\]
Therefore (3.31) implies (1.4) of Theorem 1.1. This completes the proof. \( \square \)

**Proof of Theorem 1.2** The proof is similar to that of (1.4), but we give the details for completeness. We first observe that
\[
N(x_0) := \sup_{r \in (0,1]} \frac{1}{r^\alpha \int_{B_r(x_0)} |v_0|^2} \leq C \| v_0 \|_{L^q}^2 < \infty \quad (3.32)
\]
for \( x_0 \in B_1 \) with \( \alpha = 3 - \frac{6}{q} \in (1, 3) \) for \( 3 < q \leq \infty \). Here we write \( \| \cdot \|_{L^q} = \| \cdot \|_{L^q(B_1)} \).

For each \( x_0 \in B_1 \) we may adapt the proof of (1.4) for \( v(x + x_0, t) \) and \( R = 0 \). Indeed for any \( \delta \geq 70N(x_0) \) restricting the time variable in (3.23) and (3.24) as \( \tilde{S}_a \leq t \leq \tilde{S}_b \) with
\[
\tilde{S}_a := \frac{c}{1 + \delta^2} \left( \frac{N(x_0)}{\delta} \right)^{\frac{2}{\alpha}} |x - x_0|^2,
\]
\[
\tilde{S}_b := \min \left( S_b, \frac{1}{1 + \delta^2} \left( \frac{c \delta}{\| v_0 \|_{L^2_{\text{loc}}}^2} \right)^{\frac{4}{4-\alpha}} \right),
\]
we have
\[
|v(x, t)| \leq C \left[ (1 + \delta^2)\delta^\frac{\alpha}{2} + (1 + \delta)^{\frac{\alpha}{2}} \| v_0 \|_{L^2_{\text{loc}}} t^{\frac{1}{2}(4-\alpha)} \right] t^{\frac{1}{2}(4-\alpha)} \leq C (1 + \delta^2)\delta^\frac{1}{2} \tilde{S}_a^{\frac{1}{2}} (\tilde{S}_a^{-1})^{\frac{4}{4-\alpha}}.
\]

The second inequality is by the second argument of \( \tilde{S}_b \). By (3.32), taking \( C > 0 \) sufficiently big, we may choose \( \delta = C \| v_0 \|_{L^q}^2 \) uniformly on \( x_0 \in B_1 \). Taking \( x = x_0 \), this implies
\[
\| v(t) \|_{L^\infty(B_1)} \leq C (1 + \| v_0 \|_{L^q}^8) \| v_0 \|_{L^q} t^{\frac{1}{2}(4-\alpha)}
\]
for \( 0 < t < T_3 \) where
\[
T_3 = \min \left( T, \frac{c}{1 + \| v_0 \|_{L^2_{\text{loc}}}^2}, \frac{1}{(1 + \| v_0 \|_{L^2_{\text{loc}}}^2) \| v_0 \|_{L^q}^2}, \left( \frac{c}{(1 + \| v_0 \|_{L^2_{\text{loc}}}^2) \| v_0 \|_{L^q}^2} \right)^{\frac{4}{4-\alpha}}, \frac{1}{1 + \| v_0 \|_{L^2_{\text{loc}}}^2} \left( \frac{c \| v_0 \|_{L^q}^2}{(1 + \| v_0 \|_{L^2_{\text{loc}}}^2) \| v_0 \|_{L^q}^2} \right)^{\frac{4}{4-\alpha}} \right)
\]
\[
(3.34)
\]
This shows the desired claim. \( \square \)
Proof of Theorem 1.3  For \( \alpha := -\beta > 1 \) we observe that
\[
\frac{1}{r^\beta} \int_{B_r} |v_0|^2 \leq \int_{B_r} |x|^\beta |v_0|^2 \leq L := \|v_0\|_{L^{2,\beta}(\mathbb{R}^3)}^2 < \infty.
\]
Hence \( N(\alpha)(v_0) = \sup_{r \in (0,1)} \frac{1}{r^\beta} \int_{B_r} |v_0|^2 \leq L \). By (1.3) of Theorem 1.1, \( v \) is regular and satisfies \( |v(x, t)| \leq C t^{-1/2} \) in the region
\[
\left\{(x, t) \in B_{\frac{1}{2}} \times (0, \infty) : c_1 L |x|^{2\alpha} \leq t < T_1 \right\}
\]
with \( T_1 = T_1(\|v_0\|_{L^{2,\alpha}_{\text{loc}}}, L, \alpha) \), while \( c_1 \) and \( C \) depend only on \( \alpha \). Let \( T_4 = \min(T_1, c_1 L 2^{-2\alpha}) \). The region
\[
\left\{(x, t) \in \mathbb{R}^3 \times (0, \infty) : c_1 L |x|^{2\alpha} \leq t < T_4 \right\}
\]
is a subset of (3.35). This finishes the proof of Theorem 1.3. \( \square \)

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