A WEAK LAW OF LARGE NUMBERS FOR REALISED COVARIATION IN A HILBERT SPACE SETTING

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Abstract. This article generalises the concept of realised covariation to Hilbert-space-valued stochastic processes. More precisely, based on high-frequency functional data, we construct an estimator of the trace-class operator-valued integrated volatility process arising in general mild solutions of Hilbert space-valued stochastic evolution equations in the sense of Da Prato & Zabczyk (2014). We prove a weak law of large numbers for this estimator, where the convergence is uniform on compacts in probability with respect to the Hilbert-Schmidt norm. In addition, we show that the conditions on the volatility process are valid for most common stochastic volatility models in Hilbert spaces.

1. Introduction

Stochastic volatility and covariance estimation are of key importance in many fields. Motivated in particular by financial applications, a lot of research has been devoted to constructing suitable (co-) volatility estimators and to deriving their asymptotic limit theory in the setting when discrete, high-frequency observations are available. Initially, the main interest was in (continuous-time) stochastic models based on (Itô) semimartingales, where the so-called realised variance and covariance estimators (and their extensions) proved to be powerful tools. Relevant articles include the works by Barndorff-Nielsen & Shephard (2002, 2003, 2004), Andersen et al. (2003) and Jacod (2008), amongst many others, and the textbooks by Jacod & Protter (2012) and Aït-Sahalia & Jacod (2014).

Subsequently, the theory was extended to cover non-semimartingale models, see, for instance, Corcuera et al. (2006), Barndorff-Nielsen et al. (2011), Barndorff-Nielsen et al. (2013), Corcuera et al. (2013), Corcuera et al. (2014) and the survey by Podolskij (2015), where the proofs of the asymptotic theory rely on Malliavin calculus and the famous fourth-moment theorem, see Nualart & Peccati (2005). The multivariate theory has been studied in Granelli & Veraart (2019), Passeggeri & Veraart (2019).

Common to these earlier lines of investigation is the fact that the stochastic processes considered have finite dimensions. In this article, we extend the concept of realised covariation to an infinite-dimensional framework.
The estimation of covariance operators is elementary in the field of functional data analysis and was elaborated mainly for discrete-time series of functional data (see e.g. Ramsay & Silverman (2005), Ferraty & Vieu (2006), Yao et al. (2005), Bosq (2012), Horváth & Kokoszka (2012), Panaretos & Tavakoli (2013)). However, spatio-temporal data that can be considered as functional might also be sampled densely in time, like forward curves for interest rates or commodities and data from geophysical and environmental applications.

In this paper, we consider a separable Hilbert space $H$ and study $H$-valued stochastic processes $Y$ of the form

$$
Y_t = S(t)h + \int_0^t S(t-s)\alpha_s ds + \int_0^t S(t-s)\sigma_s dW_s, \quad t \in [0, T],
$$

for some $T > 0$. Here $(S(t))_{t \geq 0}$ is a strongly continuous semigroup, $\alpha := (\alpha_t)_{t \in [0, T]}$ a predictable and almost surely integrable $H$-valued stochastic process, $\sigma := (\sigma_t)_{t \in [0, T]}$ is a predictable operator-valued process, $h \in H$ some initial condition and $W$ a so called $Q$-Wiener process on $H$ (see Section 2 below for details).

Our aim is to construct an estimator for the integrated covariance process

$$
\left( \int_0^t \sigma_s Q\sigma_s^* ds \right)_{t \in [0, T]}.
$$

More precisely, we denote by

$$
\left( \frac{[t/\Delta_n]}{\Delta_n} \sum_{i=1}^{[t/\Delta_n]} (Y_{t_i} - S(\Delta_n) Y_{t_{i-1}})^\otimes 2, \right.
$$

the \textit{semigroup-adjusted realised covariance (SARCV)} for an equally spaced grid $t_i := i\Delta_n$ for $\Delta_n = 1/n$, $i = 1, \ldots, [t/\Delta_n]$. We prove uniform convergence in probability (ucp) with respect to the Hilbert-Schmidt norm of the SARCV to the integrated covariance process under mild conditions on the volatility.

This framework differs from common high-frequency settings mainly due to peculiarities that arise from infinite dimensions. First, observe that the main motivation to consider processes in this form, is that a vast amount of parabolic stochastic partial differential equations posses only mild (in opposition to analytically strong) solutions, which are of the form 1. That is, $Y$ is (under weak conditions) the mild solution of a stochastic partial differential equation

$$
(\text{SPDE}) \quad dX_t = (AX_t + \alpha_t)dt + \sigma_t dW_t, \quad X_0 = h, \quad t \in [0, T].
$$

(cf. Da Prato & Zabczyk (2014), Peszat & Zabczyk (2007) or Mandrekar & Gawarecki (2011)).

In contrast to finite-dimensional stochastic diffusions, this is a priori not an $H$-valued semimartingale, but rather an $H$-valued Volterra process.

Various recent developments related to statistical inference for (parabolic) SPDEs based on discrete observations in time and space have emerged, see e.g. Cialenco & Huang (2020), Bibinger & Trabs (2020), Chong (2020), Chong & Dalang (2020).

To the best of our knowledge, our paper is the first one considering high-frequency estimation of (co-) volatility of infinite-dimensional stochastic evolution equations in an operator setting. This is of interest for various reasons. For instance, a simple and important application might be the parameter estimation for $H$-valued Ornstein-Uhlenbeck process (that is, $\sigma = \sigma$ is a constant operator). Elementary techniques such as functional principal component analysis might then be
considered on the level of volatility. In a multivariate setting, dynamical dimension reduction was conducted for instance in A¨ıt-Sahalia & Xiu (2019). Furthermore, it can be used as a tool for inference of infinite-dimensional stochastic volatility models as in Benth et al. (2018) or Benth & Simonsen (2018). In the special case of a semigroup that is continuous with respect to the operator norm, the framework also covers the estimation of volatility for $H$-valued semimartingales.

We organize the paper as follows: First, we recall the main technical preliminaries of our framework in Section 2. In Section 3 we establish the weak law of large numbers. For that, we discuss the conditions imposed on the volatility process in Section 3.1 and state our main result, given by Theorem 3.3, in Section 3.2. Afterwards, we show how to weaken the assumptions on the volatility by a localization argument in Section 3.3. In Section 4, we study the behaviour of the estimator in special cases of semigroups and volatility. We discuss conditions for particular examples of semigroups to determine the speed of convergence of the estimator in Section 4.1. In Section 4.2, we validate our assumptions for some stochastic volatility models in Hilbert spaces. Section 5 is devoted to the proofs of our main results, while in Section 6 we discuss our results and methods in relation to some existing literature and provide some outlook into further developments. Some technical proofs are relegated to the Appendix.

2. Notation and some preliminary results

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ denote a filtered probability space satisfying the usual conditions. Consider two separable Hilbert spaces $U, H$ with scalar products denoted by $\langle \cdot, \cdot \rangle_U, \langle \cdot, \cdot \rangle_H$ and norms $\| \cdot \|_U, \| \cdot \|_H$, respectively. We denote $L(U, H)$ the space of all linear bounded operators $K : U \to H$, and use the shorthand notation $L(U) = L(U, U)$. Equipped with the operator norm, $L(U, H)$ becomes a Banach space. The adjoint operator of a $K \in L(U, H)$ is denoted by $K^*$, and is an element on $L(H, U)$.

Following Peszat & Zabczyk (2007, Appendix A) we use the following notations: An operator $K \in L(U, H)$ is called nuclear or trace class if the following representation holds

$$Ku = \sum_k b_k \langle u, a_k \rangle_U, \text{ for } u \in U,$$

where $\{a_k\} \subset U$ and $\{b_k\} \subset H$ such that $\sum_k \|a_k\|_U \|b_k\|_H < \infty$. The space of all nuclear operators is denoted by $L_1(U, H)$: it is a separable Banach space and its norm is denoted by

$$\|K\|_1 := \inf \{ \sum_k \|a_k\|_U \|b_k\|_H : Ku = \sum_k b_k \langle u, a_k \rangle_U \}.$$

We denote by $L^+_1(U, H)$ the class of all symmetric, non-negative-definite nuclear operators from $U$ to $H$. We write $L_1(U)$ and $L^+_1(U)$ for $L_1(U, U)$ and $L^+_1(U, U)$, resp. Frequently, nuclear operators are also called trace class operators.

For $x \in U$ and $y \in H$, we define the tensor product $x \otimes y$ as the linear operator in $L(U, H)$ defined as $x \otimes y(z) := \langle x, z \rangle_U y$ for $z \in U$. We note that $x \otimes y \in L_1(U, H)$ and $\|x \otimes y\|_1 = \|x\|_U \|y\|_H$, see Peszat & Zabczyk (2007, p. 107).
The operator $K \in L(U, H)$ is said to be a *Hilbert-Schmidt operator* if
\[
\sum_k \|Ke_k\|_H^2 < \infty,
\]
for any orthonormal basis (ONB) $(e_k)_{k \in \mathbb{N}}$ of $U$. The space of all Hilbert-Schmidt operators is denoted by $L_{\text{HS}}(U, H)$. We can introduce an inner product by
\[
\langle K, L \rangle_{\text{HS}} := \sum_k \langle Ke_k, Le_k \rangle_H, \quad \text{for } K, L \in L_{\text{HS}}(U, H).
\]
The induced norm is denoted $\| \cdot \|_{\text{HS}}$. As usual, we write $L_{\text{HS}}(U)$ in the case $L_{\text{HS}}(U, U)$.

We have the following convenient result for the space of Hilbert-Schmidt operators. Although it is well-known, we include the proof of this result in the Appendix A for the convenience of the reader:

**Lemma 2.1.** Let $U, V, H$ be separable Hilbert spaces. Then $L_{\text{HS}}(U, H)$ is a separable Hilbert space. Moreover, if $K \in L_{\text{HS}}(U, V), L \in L_{\text{HS}}(V, H)$, then $LK \in L_{\text{HS}}(U, H)$ and
\[
\|LK\|_{\text{HS}} \leq \|L\|_{\text{op}} \|K\|_{\text{HS}} \leq \|L\|_{\text{HS}} \|K\|_{\text{HS}},
\]
where the HS-norms are for the spaces in question.

### 2.1. Hilbert-space-valued stochastic integrals.

Fix $T > 0$ and assume that $0 \leq t \leq T$ throughout. Let $W$ denote a Wiener process taking values in $U$ with covariance operator $Q \in L_1^+(U)$.

**Definition 2.2.** A stochastic process $(W_t)_{t \geq 0}$ with values in $U$ is called Wiener process with covariance operator $Q \in L_1^+(U)$, if $W_0 = 0$ almost surely, $W$ has independent and stationary increments, and for $0 \leq s \leq t$, we have $W_t - W_s \sim N(0, (t-s)Q)$.

**Remark 2.3.** Recall that a $U$-valued random variable $X$ is normal with mean $a \in U$ and covariance operator $Q \in L_1^+(U)$ if $\langle X, f \rangle_U$ is a real-valued normally distributed random variable for each $f \in U$, with mean $\langle a, f \rangle_U$ and
\[
E[\langle X, f \rangle_U \langle X, g \rangle_U] = \langle Qf, g \rangle_U, \forall f, g \in U.
\]

We introduce the space $L_{2,T}(U, H)$ of predictable $L(U, H)$-valued stochastic processes $Z = (Z_t)_{t \geq 0}$ such that
\[
E \left[ \int_0^T \|Z_sQ^{1/2}\|_{\text{HS}}^2 ds \right] < \infty,
\]
for $T < \infty$. Then $L_{2,T}(U, H)$ will be the space of integrable processes with respect to the $Q$-Wiener process $W$ on $[0, T]$.

Let $\sigma = (\sigma_t)_{t \geq 0}$ denote a stochastic volatility process where $\sigma_t \in L_{2,T}(U, H)$ for some fixed $T < \infty$. The stochastic integral
\[
Y_t := \int_0^t \sigma_s dW_s
\]
can then be defined as in [Peszat & Zabczyk 2007, Chapter 8] and takes values in the Hilbert space $H$. 
We denote the tensor product of the stochastic integral $Y$ by $(Y_t)^\otimes 2 = Y_t \otimes Y_t$, and define the corresponding stochastic variance term as the operator angle bracket (not to be confused with the inner products introduced above!) given by

$$\langle\langle Y \rangle\rangle_t = \int_0^t \sigma_s Q^1/2 \sigma_s^* ds = \int_0^t (\sigma_s Q^{1/2})(\sigma_s Q^{1/2})^* ds,$$

see Peszat & Zabczyk (2007, Theorem 8.7, p. 114).

**Remark 2.4.** As in Da Prato & Zabczyk (2014, p. 104), we note that $(\sigma_s Q^{1/2}) \in L_{HS}(U, H)$ and $(\sigma_s Q^{1/2})^* \in L_{HS}(H, U)$. Hence the process $(\sigma_s Q^{1/2})(\sigma_s Q^{1/2})^*$ for $s \in [0, T]$ takes values in $L_1(H, H)$.

**Remark 2.5.** The integral $\int_0^t \sigma_s Q \sigma_s^* ds$ is interpreted as a Bochner integral in the space of Hilbert-Schmidt operators $L_{HS}(H)$. Indeed, $\sigma_s Q \sigma_s^*$ is a linear operator on $H$, and we have

$$\int_0^t \mathbb{E}[\| \sigma_s Q \sigma_s^* \|_{HS}] ds = \int_0^t \mathbb{E}[\| \sigma_s Q^{1/2}(\sigma_s Q^{1/2})^* \|_{HS}] ds \leq \int_0^t \mathbb{E}[\| \sigma_s Q^{1/2} \|^2_{HS}] ds < \infty,$$

by appealing to Lemma 2.1 and the assumption on $\sigma$ being an integrable process with respect to $W$. This means that the Bochner integral is a.s. defined. If we relax integrability to go beyond $L^2$, this argument fails, but we still have a well-defined Bochner integral as we can argue pathwise.

**Remark 2.6.** From Peszat & Zabczyk (2007, Theorem 8.2, p. 109) we deduce that the process $(M_t)_{t \geq 0}$ with

$$M_t = (Y_t)^\otimes 2 - \langle\langle Y \rangle\rangle_t$$

is an $L_1(H)$-valued martingale w.r.t. $(\mathcal{F}_t)_{t \geq 0}$. Thus, the operator angle bracket process can be called the quadratic covariation process of $Y_t$, which we shall do from now on.

We end this section with a general expression for the even moments of an increment of the Wiener process. Later we will need the fourth moment in our analysis.

First, we introduce the $p$-trace of an operator $K \in L(U)$: We denote by $\text{Tr}_p(K)$ the $p$-trace of $K$, $p \in \mathbb{N}$, defined as

$$\text{Tr}_p(K) = \sum_{i=1}^{\infty} \langle Ke_i, e_i \rangle_U^p,$$

whenever this converges. Here, $(e_i)_{i \in \mathbb{N}}$ is an ONB in $U$. We denote by $\text{Tr}$ the classical trace, given by $\text{Tr} = \text{Tr}_1$. Consider now the positive definite symmetric trace class operator $Q$. If we organize the eigenvalues $(\lambda_i)_{i=1}^{\infty} \subset \mathbb{R}_+$ of $Q$ in decreasing order, letting $(e_i)_{i \in \mathbb{N}}$ be the ONB of eigenvectors, we have

$$\text{Tr}_p(Q) \leq \lambda_p^{p-1} \sum_{i=1}^{\infty} \lambda_i = \text{Tr}(Q),$$

and hence the $p$-trace is bounded by the trace for any $p > 1$, and therefore also finite. The proof of the following result is relegated to Appendix A.
Lemma 2.7. Let $W$ be a $Q$-Wiener process on $U$ and $q \in \mathbb{N}$ and define a generic increment as $\Delta W_t := W_{t+\Delta} - W_t$ for $\Delta > 0$. Furthermore, let $(e_k)_{k \in \mathbb{N}}$ be the ONB in $U$ of eigenvectors of $Q$ with associated eigenvalues $(\lambda_k)_{k \in \mathbb{N}}$. Then, for any $t \geq 0$ and $m \in \mathbb{N}$ it holds that

$$E \left[ \left\| \Delta W_t \right\|_U^{2q} \right] = (-i)^q \lim_{m \to \infty} \Phi_m^q(0),$$

where

$$\Phi_m(x) = \exp \left( -\frac{1}{2} \sum_{k=1}^{m} \ln(1 - 2ix\lambda_k) \right),$$

for $x \in \mathbb{R}$. In particular,

$$E[\left\| \Delta W_t \right\|_U^{4}] = \Delta^2 (\text{Tr}(Q)^2 + 2\text{Tr}_2(Q)).$$

This finishes our section with preliminary results.

3. The weak law of large numbers

In this section, we show our main result on the law of large numbers for Volterra-type stochastic integrals in Hilbert space with operator-valued volatility processes. Consider

$$Y_t := \int_0^t S(t-s)\sigma_s dW_s,$$

where $W$ is a $Q$-Wiener process on the separable Hilbert space $U$, $\sigma$ is an element of $L_{2,T}(U,H)$ and $S$ is a $C_0$-semigroup on $H$. We assume that we observe $Y$ at times $t_i := i\Delta_n$ for $\Delta_n = 1/n$, $i = 1, \ldots, \lfloor t/\Delta_n \rfloor$ and define the semigroup-adjusted increment

$$\tilde{\Delta}_n Y := Y_{t_i} - S(\Delta_n)Y_{t_{i-1}} = \int_{t_{i-1}}^{t_i} S(t_i-s)\sigma_s dW_s.$$

We define the process of the semigroup-adjusted realised covariance (SARCV) as

$$t \mapsto \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} (\tilde{\Delta}_n^i Y)^{\otimes 2}.$$  

The aim is to prove the following weak law of large numbers for the SARCV

$$\sum_{i=1}^{\lfloor t/\Delta_n \rfloor} (\tilde{\Delta}_n^i Y)^{\otimes 2} \overset{ucp}{\longrightarrow} \int_0^t \sigma_s Q\sigma^*_s ds,$$

as $n \to \infty$, in the ucp-topology, that is, for all $\epsilon > 0$ and $T > 0$

$$\lim_{n \to \infty} P \left( \sup_{0 \leq t \leq T} \left\| \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} (\tilde{\Delta}_n^i Y)^{\otimes 2} - \int_0^t \sigma_s Q\sigma^*_s ds \right\|_{HS} \right) = 0.$$
3.1. Technical assumptions. We need some technical assumptions on the stochastic volatility process \( \sigma \).

**Assumption 1.** Assume that the volatility process satisfies the following Hölder continuity property: For all \( T > 0 \) and \( s, t \in [0, T] \) we have
\[
\mathbb{E} \left[ \| \sigma_t - \sigma_s \|^2_{HS} \right]^{\frac{1}{2}} \leq C_1(T) |t - s|^\alpha,
\]
for some \( \alpha > 0 \) and a constant \( C_1(T) > 0 \) (depending on \( T \)).

Notice that we assume only local mean-square Hölder continuity for the paths of the volatility process. This allows for including volatility processes with càdlàg paths in our considerations, as we will see later.

We shall also need a moment condition to hold for the volatility process:

**Assumption 2.** Assume that the volatility process satisfies for all \( T > 0 \) the following moment conditions:
\[
\mathbb{E} \left[ \| \sigma_s Q_{\frac{1}{2}}^2 \|^4_{HS} \right] \leq C_2(T) \quad \forall s \in [0, T],
\]
for some constant \( C_2(T) > 0 \) (depending on \( T \)).

**Remark 3.1.** Using the Cauchy-Schwarz inequality, we can deduce under Assumption 2 for each \( T > 0 \)
\[
\sup_{s \in [0, T]} \mathbb{E} \left[ \| \sigma_s Q_{\frac{1}{2}}^2 \|^2_{HS} \right] \leq \sup_{s \in [0, T]} \sqrt{\mathbb{E} \left[ \| \sigma_s Q_{\frac{1}{2}}^2 \|^4_{HS} \right]} \leq \sqrt{C_2(T)}.
\]
Moreover, we find that for all \( t \in [0, T] \), also
\[
\mathbb{E} \left[ \int_0^t \| \sigma_s Q_{\frac{1}{2}}^2 \|^2_{HS} ds \right] \leq t \sqrt{C_2(T)} < \infty.
\]
Thus, the integrability condition on \( (\sigma_t)_{t \in [0, T]} \) holds for adapted processes satisfying Assumption 2.

The semigroup is in general not continuous with respect to time in the operator norm, but only strongly continuous. This makes it more involved to verify convergence in Hilbert-Schmidt norms, like (7), since then the semigroup component \( S(\Delta_n) \) in the adjusted increment \( \sigma_t Q_{\frac{1}{2}}^2 \) converges just strongly to the identity. However, we can make use of compactness of the closure of the image of the operators \( \sigma_s Q_{\frac{1}{2}}^2 \) for each \( s \in [0, T] \), and show the convergence of the semigroup to the identity operator on compacts by the subsequent argument in Theorem 5.2. This line of argument necessitates one of the following two alternative assumptions:

**Assumption 3.** (a) Assume we can find a mean-square continuous process \( (\mathcal{K}_s)_{s \in \mathbb{R}_+} \in L^2(\Omega \times \mathbb{R}_+; L(U, H)) \) of compact operators and a Hilbert-Schmidt operator \( T \in L_{HS}(U) \) such that almost surely \( \sigma_s Q_{\frac{1}{2}}^2 = \mathcal{K}_s T \) for each \( s \in [0, t] \).

(b) The semigroup \( (S(t))_{t \geq 0} \) is uniformly continuous, that is \( S(t) = e^{At} \) for some bounded operator \( A \in L(H) \).

Observe, that Assumption 3(a) is fulfilled in the following cases:

(i) \( \sigma \) satisfies Assumption 1 and \( \sigma_t \) is almost surely compact (for instance itself a Hilbert-Schmidt operator) for each \( t \in [0, T] \). In this case we can choose \( \mathcal{K}_s := \sigma_s \) and \( T := Q_{\frac{1}{2}} \).
Theorem 3.3. Assume that Assumptions 1, 2 and either 3(a) or 3(b) hold. For each $T > 0$ there is a constant $L(T) > 0$ such that

\[
E \left[ \sup_{0 \leq t \leq T} \left\| \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} (\Delta_n^i Y)^{\otimes 2} - \int_0^t \sigma_s Q\sigma_s^* ds \right\|_{HS} \right] \leq L(T)(\Delta_n^\alpha + b_n^2(T)),
\]

where

\[
b_n(T) := \sup_{r \in [0,T]} E[\sup_{x \in [0,\Delta_n]} \| (I - S(x))\sigma_s Q\sigma_s^* \|_{op}^2].
\]

In particular, for all $T > 0$

\[
\lim_{n \to \infty} E \left[ \sup_{0 \leq t \leq T} \left\| \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} (\Delta_n^i Y)^{\otimes 2} - \int_0^t \sigma_s Q\sigma_s^* ds \right\|_{HS} \right] = 0.
\]

Before we prove this result in Section 5 we will make a couple of remarks and discuss uniform continuity of semigroups on compact sets.

Remark 3.4. The factor $L(T)$ in the Theorem above is actually not just depending on $T$, but also shrinks when $n$ gets larger. Effectively, the constant can be precisely computed by careful inspection of the estimates (14), (15), (16) and (17) in the proof of Theorem 3.3. However, the expression becomes rather extensive and we refrain from stating it here.

That $(b_n)_{n \in \mathbb{N}}$ converges to 0 is an implication of the following Proposition 5.2. The magnitude of this sequence essentially determines the rate of convergence of the realised covariation by virtue of inequality (19). We will come back to the magnitude of the $b_n$'s in specific cases in Section 4.1.

Denote for $t \geq 0$

\[
M(t) := \sup_{x \in [0,t]} \| S(x) \|_{op},
\]

which is finite by the Hille-Yosida bound on the semigroup. Often in stochastic modelling one also has a drift present. The following remark shows that our results are not altered by this:

Remark 3.5. Observe that we could easily extend $Y$ to posses a drift and an "initial condition", that is

\[
Y_t = S(t)h + \int_0^t S(t-s)\alpha_s ds + \int_0^t S(t-s)\sigma_s dW_s,
\]
for a predictable and almost surely Bochner-integrable stochastic process \((\alpha_t)_{t \in [0,T]}\), such that

\[
\sup_{t \in [0,T]} \mathbb{E}[\|\alpha_t\|^2] < \infty,
\]

and for an initial value \(h \in H\). In this case

\[
\Delta_n^i Y := Y_{t_i} - S(\Delta_n)Y_{t_{i-1}} = \int_{t_{i-1}}^{t_i} S(t_i - s)\alpha_s ds + \int_{t_{i-1}}^{t_i} S(t_i - s)\sigma_s dW_s.
\]

We can then argue that

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \left\| \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} (\Delta_n^i Y)^{\odot 2} - \int_0^t \sigma_s^2 \alpha_s^2 ds \right\|_{H^2} \right] \\
\leq \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left\| \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \left( \int_{t_{i-1}}^{t_i} S(t_i - s)\alpha_s ds \right)^{\odot 2} \right\|_{H^2} \right] \\
+ \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left\| \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \left( \int_{t_{i-1}}^{t_i} S(t_i - s)\sigma_s dW_s \right)^{\odot 2} - \int_0^t \sigma_s \alpha_s dW_s \right\|_{H^2} \right] \\
= (1) + (2)
\]

Summand (2) can be estimated with Theorem 3.3. For Summand (1) we find

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \left\| \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \left( \int_{t_{i-1}}^{t_i} S(t_i - s)\alpha_s ds \right)^{\odot 2} \right\|_{H^2} \right] \\
\leq \mathbb{E} \left[ \left\| \sum_{i=1}^{\lfloor T/\Delta_n \rfloor} \left( \int_{t_{i-1}}^{t_i} S(t_i - s)\alpha_s ds \right)^{\odot 2} \right\|_{H^2} \right] \\
\leq \sum_{i=1}^{\lfloor T/\Delta_n \rfloor} \Delta_n^2 M^2(T) \sup_{r \in [0,T]} \mathbb{E} \left[ \|\alpha_r\|^2_H \right] \\
\leq M^2(T)T \sup_{r \in [0,T]} \mathbb{E} \left[ \|\alpha_r\|^2_H \right] \Delta_n,
\]

where we appealed to the bound (12) on the semigroup. Hence, Summand (1) is \(O(\Delta_n)\) and will not impact the estimation of the covariation (in the limit).

### 3.3. Extension by localisation

In general, we have the following result:

**Theorem 3.6.** Let \((\Omega_m)_{m \in \mathbb{N}}\) be a sequence of measurable subsets such that \(\Omega_m \uparrow \Omega\). Suppose Assumptions 1, 2 and 3 hold for \(\sigma^{(m)} := \sigma \mathbf{1}_{\Omega_m}\) for all \(m \in \mathbb{N}\). Then

\[
\lim_{n \to \infty} \mathbb{P} \left( \sup_{0 \leq s \leq t} \left\| \sum_{i=1}^{\lfloor s/\Delta_n \rfloor} (\Delta_n^i Y)^{\odot 2} - \int_0^s \sigma_u \sigma_u^* du \right\|_{H^2} > \epsilon \right) = 0,
\]

for any \(\epsilon > 0\), that is, convergence holds in ucp of the realized covariation.

We can apply the localization on volatility processes \(\sigma\) with almost sure Hölder-continuous paths:
Corollary 3.7. Assume \( \sigma \) is almost surely \( \alpha \)-Hölder-continuous on \([0, T]\) with respect to the operator norm, satisfies Assumption 3 and that the initial value has a finite fourth moment, i.e.

\[
E[\|\sigma_0\|_{op}^4] < \infty. \tag{14}
\]

Then the ucp convergence in Eq. \( \text{(13)} \) holds.

Proof. We know that

\[
C(T) := \sup_{s \neq t \in [0, T]} \frac{\| (\sigma_t - \sigma_s) Q^\frac{1}{2} \|_{HS}}{|t - s|^\alpha} < \infty, \quad \text{a. s.} \tag{15}
\]

Then \( C(T) \) is a random variable and the set \( \Omega_m := \{ \omega \in \Omega : C(T) \leq m \} \) is measurable and \( \Omega_m \uparrow \Omega \) since \( (15) \) is satisfied automatically. The \( \alpha \)-Hölder continuity is obtained since

\[
E[\| (\sigma^{(m)}_t - \sigma^{(m)}_s) Q^\frac{1}{2} \|_{HS}^2] \leq m^2 |t - s|^{2\alpha} \text{Tr}(Q),
\]

and the fourth moment is finite since

\[
E[\|\sigma^{(m)}_t\|_{op}^4] \leq E[\|\sigma^{(m)}_t - \sigma^{(m)}_0\|_{op}^4] + E[\|\sigma^{(m)}_0\|_{op}^4] \leq m^4 t^{4\alpha} + E[\|\sigma_0\|_{op}^4] < \infty.
\]

The proof is complete. \( \square \)

4. Applications

In this section, we give an overview of potential settings and scenarios for which we can use the techniques described above to infer volatility.

Stochastic integrals of the form \( (5) \) arise naturally in correspondence to mild or strong solutions to stochastic partial differential equations. Take as a simple example a process given by

\[
\text{(SPDE)} \begin{cases}
    dY_t = AY_t dt + \sigma_t dW_t, & t \geq 0 \\
    Y_0 = h_0 \in H,
\end{cases}
\]

where \( A \) is the generator of a \( C_0 \)-semigroup \((S(t))_{t \geq 0}\) on the separable Hilbert space \( H \), \( W \) is a \( Q \)-Wiener process on a separable Hilbert space \( U \) for some positive semidefinite and symmetric trace class operator \( Q : U \to U \) and \( \sigma \in L^T_{2,\infty}(U, H) \).

There are three components in this model, which need to be estimated in practice: the covariance operator \( Q \) of the Wiener process, the generator \( A \) (or the semigroup \((S(t))_{t \geq 0}\) respectively) and the stochastic volatility process \( \sigma \).

4.1. Semigroups. The essence of the convergence result in Theorem 3.3 is that we can infer on \( Q \) and \( \sigma \) based on observing the path of \( Y \), given that we know the semigroup \((S(t))_{t \geq 0}\). Certainly, this is not always the case, since we may just have knowledge about the infinitesimal generator \( A \). However, if we know the precise form of the semigroup it is sometimes possible to estimate the speed of convergence, that is, a bound on the \( b_n(T) \)'s given in \( (10) \).

\(^1\text{At least a convergence to a set with full measure} \)
4.1.1. Martingale case. For $A = 0$ and $S(t) = I$ and for all $t \geq 0$, we have the solution

$$Y_t = \int_0^t \sigma_s dW_s,$$

for the stochastic partial differential equation (16). Clearly in this case we have

$$b_n(T) = 0.$$

4.1.2. Uniformly continuous semigroups. Assume that $(S(t))_{t \geq 0}$ is continuous with respect to the operator norm. This is equivalent to $A \in L(H)$ and $S(t) = e^{tA}$.

Lemma 4.1. If the semigroup $(S(t))_{t \geq 0}$ is uniformly continuous, we have, for $b_n$ given in (10), that

$$b_n(T) \leq \Delta_n \|A\|_{op} \cdot \Delta_n \sup_{r \in [0,T]} E[\|\sigma_r Q^{\frac{1}{2}}\|_{HS}^2].$$

In particular, if Assumptions [2] and [3] are valid, we have

$$b_n(T) \leq \Delta_n \|A\|_{op} \cdot \Delta_n \sqrt{C_2(T) Tr(Q)}.$$

Proof. Recall the following fundamental equality from semigroup theory (cf. Engel & Nagel (1999, Lemma 1.3)):

$$\begin{align*}
(S(x) - I)h &= \int_0^x A S(s) h ds, \quad \forall h \in H \\
&= \int_0^x S(s) A h ds, \quad \forall h \in D(A).
\end{align*}$$

(17) (18)

Using (17), we get

$$\sup_{x \in [0,\Delta_n]} \|I - S(x)\|_{op} = \sup_{x \in [0,\Delta_n]} \sup_{\|h\| = 1} \left\| \int_0^x A S(s) h ds \right\|_H \leq \sup_{x \in [0,\Delta_n]} x \|A\|_{op} e^{\|A\|_{op} \Delta_n} = \Delta_n \|A\|_{op} e^{\|A\|_{op} \Delta_n}.$$

It follows that

$$b_n^2(T) = \sup_{r \in [0,T]} E\left[ \sup_{x \in [0,\Delta_n]} \|I - S(x)\|_{op}^2 \|\sigma_r Q^{\frac{1}{2}}\|_{op}^2 \right] \leq \sup_{x \in [0,\Delta_n]} \|I - S(x)\|_{op}^2 \sup_{r \in [0,T]} E[\|\sigma_r Q^{\frac{1}{2}}\|_{HS}^2] \leq \Delta_n^2 \|A\|_{op}^2 e^{\|A\|_{op} \Delta_n} \sup_{r \in [0,T]} E[\|\sigma_r Q^{\frac{1}{2}}\|_{HS}^2].$$

For uniformly continuous semigroups we obtain a convergence speed of the order $\min(\Delta_n^2, \Delta_n^2)$ for the convergence of the realized covariation to the quadratic covariation in Theorem 3.3.

Remark 4.2. Note that, if the semigroup is uniformly continuous and under Assumptions [2] and [3] we can get back to the martingale case of Section 4.1.1 if we operate on the values of $Y_t$ in any of the following two ways:
(i) We continue as in the martingale case for the realised covariation of \( \tilde{Y}_t := S(-t)Y_t \). This can be done since \( S(t) = \exp(At) \) and we have

\[
Y_t = \int_0^t S(t-s)\sigma_s ds = S(t) \int_0^t S(-s)\sigma_s dW_s.
\]

Thus \( \tilde{Y}_t \) is a martingale.

(ii) We continue as in the martingale case for the realised covariation of \( \tilde{Y}_t := Y_t - AY_t \). This can be done since the process \((Y_t)_{t \in [0,T]}\) is the strong solution to (16) with the continuous linear generator \( A \) and \( h_0 \equiv 0 \in H \), since \( D(A) = H \) (see for instance Theorem 3.2 in Mandrekar & Gawarecki (2011)). That is, in particular,

\[
Y_t = AY_t + \int_0^t \sigma_s dW(s), \quad \forall t \in [0,T].
\]

Let us turn our attention to a case of practical interest coming from financial mathematics applied to commodity markets.

4.1.3. Forward prices in commodity markets: the Heath-Jarrow-Morton approach.

A case of relevance for our analysis is inference on the volatility for forward prices in commodity markets as well as for forward rates in fixed-income markets. The Heath-Jarrow-Morton-Musiela equation (HJMM-equation) describes the term structure dynamics in both of these settings (see Filipović (2001) for a detailed motivation for the use in interest rate modelling and Benth & Krühner (2014) for its use in commodity markets) and is given by

\[
\begin{cases}
    dX_t = \left( \frac{d}{dx}X_t + \alpha_t \right) dt + \sigma_t dW_t, & t \geq 0 \\
    X_0 = h_0 \in H,
\end{cases}
\]

(19) (HJMM)

where \( H \) is a Hilbert space of functions \( f : \mathbb{R}_+ \to \mathbb{R} \) (the forward curve space), \((\alpha_t)_{t \geq 0}\) is a predictable and almost surely locally Bochner-integrable stochastic process and \( \sigma \) and \( W \) are as before. Conveniently, the states of this forward curve dynamics are realized on the separable Hilbert space

\[
H = H_\beta = \{ h : \mathbb{R}_+ \to \mathbb{R} : h \text{ is absolutely continuous and } \| h \|_\beta < \infty \},
\]

(20)

for fixed \( \beta > 0 \), where the inner product is given by

\[
\langle h, g \rangle_\beta = h(0)g(0) + \int_0^\infty h'(x)g'(x)e^{\beta x} dx,
\]

and norm \( \| h \|_\beta^2 = \langle h, h \rangle_\beta \). This space was introduced and analysed in Filipović (2001). As in Filipović (2001), one may consider more general scaling functions in the inner product than the exponential \( \exp(\beta x) \). However, for our purposes here this choice suffices. The suitability of this space is partially due to the following result:

**Lemma 4.3.** The differential operator \( A = \frac{d}{dx} \) is the generator of the strongly continuous semigroup \((S(t))_{t \geq 0}\) of shifts on \( H_\beta \), given by \( S(t)h(x) = h(x + t) \), for \( h \in H_\beta \).

*Proof.* See for example Filipović (2001). \( \square \)
The HJMM-equation (19) possesses a mild solution (see e.g. Peszat & Zabczyk (2007))

\[ f_t = S(t) f_0 + \int_0^t S(t-s) \alpha_s ds + \int_0^t S(t-s) \sigma_s dW_s. \] (21)

Since forward prices and rates are often modelled under a risk neutral probability measure, the drift has in both cases (commodities and interest rates) a special form. In the case of forward prices in commodity markets, it is zero under the risk neutral probability, whereas in interest rate theory it is completely determined by the volatility via the no-arbitrage drift condition

\[ \alpha_t = \sum_{j \in \mathbb{N}} \sigma_j^t \Sigma_j^t , \quad \forall t \in [0, T], \] (22)

where \( \sigma_j^t = \sqrt{\lambda_j} \sigma_j(t) \) and \( \Sigma_j^t = \int_0^t \sigma_s^j ds \) for some eigenvalues \( (\lambda_j)_{j \in \mathbb{N}} \) and a corresponding basis of eigenvectors \( (e_j)_{j \in \mathbb{N}} \) of the covariance operator \( Q \) of \( W \) (cf. Lemma 4.3.3 in Filipovic (2001)).

**Lemma 4.4.** Assume that the volatility process \( (\sigma_t)_{t \in [0,T]} \) satisfies Assumption 2 and that for each \( t \in [0,1] \) the operator \( \sigma_t \) maps into \( H_0^0 = \{ h \in H_\beta : \lim_{x \to \infty} h(x) = 0 \} \).

Then the drift given by (22) has values in \( H_\beta \), is predictable, satisfies (12) and is almost surely Bochner integrable. Thus, the conditions of Remark 3.5 are satisfied.

**Proof.** That the drift is well defined follows from Lemma 5.2.1 in Filipovic (2001). Predictability follows immediately from the predictability of the volatility. We have by Theorem 5.1.1 from Filipovic (2001) that there is a constant \( K \) depending only on \( \beta \) such that

\[ \| \sigma_j^t \Sigma_j^t \|_{H_\beta} \leq K \| \sigma_j^t \|_{H_\beta}^2. \]

Therefore, we get by the triangle inequality that

\[ \| \alpha_t \|_{H_\beta} \leq K \sum_{j \in \mathbb{N}} \| \sigma_j^t \|_{H_\beta}^2 = K \| \sigma_t Q^{\frac{1}{2}} \|_{HS}^2. \]

Using Cauchy-Schwarz inequality we obtain

\[ \sup_{t \in [0,T]} \mathbb{E}[\| \alpha_t \|_{H_\beta}^2] \leq \sup_{t \in [0,T]} \mathbb{E}[\| \sigma_t Q^{\frac{1}{2}} \|_{HS}^4], \]

which is finite by Assumption 2. This shows (12). Moreover, the Bochner integrability follows, since we have the stronger

\[ \mathbb{E}\left[ \int_0^T \| \alpha_t \|_{H_\beta} dt \right] \leq \int_0^T \mathbb{E}[\| \alpha_t \|_{H_\beta}^2] dt \leq \sup_{t \in [0,T]} \mathbb{E}[\| \sigma_t Q^{\frac{1}{2}} \|_{HS}^4] < \infty. \]

The result follows. \( \square \)

**Remark 4.5.** Since we know the exact form of the semigroup \( (S(t))_{t \geq 0} \), we can recover the adjusted increments \( \Delta_t f \) efficiently from forward curve data by a simple shifting in the spatial (e.g., time-to-maturity) variable of these curves. Theorem 3.3 (and Remark 3.3 in case of a nonzero drift in interest rate theory) can therefore be applied in practice to make inference on \( \sigma \) under Assumptions 1, 2 and 3, in which case the ucp-convergence (7) holds.
Lemma 4.7. Assume for some constants $\alpha, C_1$ and $C_2$ that for all $s, t \in [0, T]$ we have

$$\mathbb{E} \left[ \left\| (\Sigma_t - \Sigma_s) \right\|^2_{op} \right]^{\frac{1}{2}} \leq C_1(T)^2 \frac{t - s}{T(t)}^{2\alpha}$$

and

$$\sup_{s \in [0, T]} \mathbb{E} \left[ \| \Sigma_s \|^2_{op} \right] \leq C_2(T).$$

Then $\sigma$ satisfies Assumptions 4 and 3 with corresponding constants $\alpha, C_1$ and $C_2$. 

The shift semigroup is strongly, but not uniformly, continuous, leaving us with the question to determine the convergence speed of the estimator established in Corollary 4. We close this subsection by deriving a convergence bound under regularity condition of the volatility in the space variable (that is time to maturity).

Observe that by Theorem 4.11 in [Benth & Krühner 2014] we know that for all $r \in [0, T]$ there exist random variables $c_r$ with values in $\mathbb{R}$, $f_r, g_r$ with values in $H$ such that $g_r(0) = 0 = f_r(0)$ and $p_r$ with values in $L^2(\mathbb{R}_+^2)$ such that we have

$$\sigma_r Q^r(x) = c_r h(0) + (g_r, h)_{\beta} + h(0) f_r(x) + \int_{0}^{\infty} q_r(x, z) h^r(z) dz,$$

where $q_r(x, z) = \int_{0}^{x} p_r(y, z) e^{\beta z - y} dy$. We denote by $C^{1, \gamma}_{loc} := C^{1, \gamma}_{loc}(\mathbb{R}_+)$ the space of continuously differentiable functions with locally $\gamma$-Hölder continuous derivative for $\gamma \in (0, 1)$.

Theorem 4.6. Assume that $f_r, q_r(\cdot, z) \in C^{1, \gamma}_{loc}$ for all $z \geq 0$, $r \in [0, T]$ and that the corresponding local Hölder constants $L^{1, \gamma}_{loc}(x)$ of $e^{\frac{\beta}{2}} f_r^r(\cdot)$ and $L^{2, \gamma}_{loc}(z, x)$ of $p_r$ are square integrable in $x$ and in $(x, z)$ respectively such that

$$\hat{L} := \sup_{r \in [0, T]} \mathbb{E} \left[ \left( \left| f_r^r(\cdot) \right| + \left\| L^{1, \gamma}_{loc}(x) \right\|_{L^2(\mathbb{R}_+)} + \left\| L^{2, \gamma}_{loc}(x, z) \right\|_{L^2(\mathbb{R}_+^2)} + \frac{\beta}{2} \left\| p_r \right\|_{L^2(\mathbb{R}_+^2)} \right)^2 \right] < \infty.$$

Then for $b_n(T)$ as given in (10), we can estimate

$$b_n(T) \leq \hat{L} \Delta^2 T^\gamma.$$

In the next section, we investigate the validity of assumptions for volatility models.

4.2. Stochastic volatility models. In this section different models for stochastic volatility in Hilbert spaces are discussed. So far, infinite-dimensional stochastic volatility models are specified by stochastic partial differential equations on the positive cone of Hilbert-Schmidt operators (see [Benth et al. 2018], [Benth & Simonsen 2018]). As such, Assumption 3 is trivially fulfilled. We will check therefore, which models satisfy Assumptions 4 and 3.

Throughout this section, we take $H = U$ for simplicity. The volatility is often-times given as the unique positive square-root of a process $\Sigma_t$, e.g.,

$$(\Sigma_t := \Sigma_0)^{\frac{1}{2}},$$

where $\Sigma$ takes values in the set of positive Hilbert-Schmidt operators on $H$.

Before we proceed with the particular models, we state the following result:

Lemma 4.7. Assume for some constants $\alpha, C_1$ and $C_2$ that for all $s, t \in [0, T]$ we have

$$\mathbb{E} \left[ \left\| (\Sigma_t - \Sigma_s) \right\|^2_{op} \right]^{\frac{1}{2}} \leq C_1(T)^2 \frac{t - s}{T(t)}^{2\alpha}$$

and

$$\sup_{s \in [0, T]} \mathbb{E} \left[ \left\| \Sigma_s \right\|^2_{op} \right] \leq C_2(T).$$

Then $\sigma$ satisfies Assumptions 4 and 3 with corresponding constants $\alpha, C_1$ and $C_2$. 

Proof. By the inequality in Lemma 2.5.1 of [Bogachev (2018)], the Hölder inequality and (24)
\[E[\| (\sigma_t - \sigma_s)^{\frac{\alpha}{2}} \|_{HS}^2] \leq E[\| (\Sigma_t^{\alpha} - \Sigma_s^{\alpha}) \|_{op}] \text{Tr}(Q) \]
\[\leq E[\| (\Sigma_t - \Sigma_s) \|_{op}] \text{Tr}(Q) \]
\[\leq E[\| (\Sigma_t - \Sigma_s)^{\frac{2}{\alpha}} \|_{op}] \text{Tr}(Q) \]
\[\leq C_1(T)(t - s)^{\alpha}.\]

Moreover, Assumption 2 is satisfied, since
\[\sup_{s \in [0, T]} E[\| \sigma_s \|^4_{op}] = \sup_{s \in [0, T]} E[\| \Sigma_s^{\frac{2}{\alpha}} \|^4_{op}] = \sup_{s \in [0, T]} E[\| \Sigma_s \|^2_{op}] \leq C_2(T).\]
The proof is complete. \(\Box\)

4.2.1. Barndorff-Nielsen & Shephard (BNS) model. We assume \(\Sigma\) is given by the Ornstein-Uhlenbeck dynamics
\[(BNS) \begin{cases} d\Sigma_t = B\Sigma_t dt + d\mathcal{L}_t, \\ \Sigma_0 = \Sigma \in L_{HS}(H), \end{cases}\]
where \(B\) is a positive bounded linear operator on the space of Hilbert-Schmidt operators \(L_{HS}(H)\) and \(\mathcal{L}\) is a square integrable Lévy subordinator on the same space. \(B\) is then the generator of the uniformly continuous semigroup given by \(S(t) = \exp(Bt)\) and the equation has a mild solution given by
\[\Sigma_t = S(t)\Sigma_0 + \int_0^t S(t-s)\mathcal{L}_s ds,\]
which defines a process in \(L_{T,2}(H, H)\) (see [Benth et al. (2018)]). Stochastic volatility models with OU-dynamics were suggested in [Benth et al. (2018)], extending the BNS-model introduced in [Barndorff-Nielsen & Shephard (2001)] to infinite dimensions.

Lemma 4.8. For all \(s, t \in [0, T]\) such that \(t - s \leq 1\) we have
\[E[\| (\Sigma_t - \Sigma_s)^{\frac{2}{\alpha}} \|_{HS}^\alpha] \leq \tilde{L}(T)(t - s)^{\frac{\alpha}{2}},\]
where we denote
\[\tilde{L}(T) := \sqrt{3(Ce^\|C\|^\|\sigma\|^T\|_{HS} e^\|C\|^\|\sigma\|^T \text{Tr}(Q)e^T(1 + Ce^\|C\|^\|\sigma\|^T) \text{Tr}(Q))}.\]
In particular, \(\sigma\) satisfies Assumptions 1 and 2 with corresponding constants \(C_1(T) = \sqrt{\tilde{L}(T) \text{Tr}(Q)}\) and \(C_2(T) = Ce^\|C\|^\|\sigma\|^T (\|\Sigma_0\|_{HS} + \text{Tr}(Q))^{\frac{\alpha}{2}}.\)

It is also possible to derive ucp convergence for rough volatility models, which we present in the following section.

4.2.2. Rough volatility models. In [Benth & Harang (2020)] pathwise constructions of Volterra processes are established and suggested for the use in stochastic volatility models. In this setting, a process is mostly known to be Hölder continuous almost surely of some particular order.

Therefore we fix an almost surely Hölder continuous process \((\gamma_t)_{t \in [0, T]}\) of order \(\alpha\) with values in \(H\). Without any further knowledge of the process, we do not know
whether the corresponding Hölder constant, that is the random variable $C(T)$ such that

\[
C(T) := \sup_{s,t \in [0,T]} \frac{\|Y_t - Y_s\|_H}{|t - s|^{\alpha}},
\]

is square-integrable, and therefore we cannot verify Assumptions 1 or 2 without additional assumptions. However, for various models we can use Corollary 3.7. If $H$ is a Banach algebra (like the forward curve space defined by (20)), we can define the volatility process by

\[
\sigma_t h := \exp(Y_t) h.
\]

This is a direct extension of the volatility models proposed in Gatheral et al. (2018).

**Lemma 4.9.** Assume that $H$ be a commutative Banach algebra and $\sigma$ is defined by (27). Moreover assume that $\mathbb{E}[\exp(4\|Y_0\|_H)] < \infty$. Then the ucp-convergence in (7) holds.

**Proof.** Since in commutative Banach algebras $\exp(f + g) = \exp(f) \exp(g)$ holds for all $f, g \in H$, we have

\[
\| \exp(f) - \exp(g) \|_\text{op} \leq \exp(\|f - g\|_H) \|\exp(g) - \exp(-f + 2g)\| \leq 2\exp(2\|f\|_H + 2\|g\|_H)\|f - g\|_H.
\]

This implies the local $\alpha$-Hölder continuity of $\sigma$. Due to Corollary 3.7 the assertion holds. $\square$

5. Proofs

In this section, we will present the proofs of our previously stated results.

5.1. Proofs of results in Section 3

5.1.1. Uniform continuity of semigroups on compact sets. In order to verify that $b_n(T)$ defined in (10) converges to 0 and to prove Theorem 3.3 we need to establish some convergence properties of semigroups on compacts.

Let $X$ be a compact Hausdorff space. Recall that a subset $F \subset C(X; \mathbb{R})$ is equicontinuous, if for each $x \in X$ and $\epsilon > 0$ there is a neighbourhood $U_x$ of $x$ in $X$ such that for all $y \in U_x$ and for all $f \in F$ we have

$$ |f(x) - f(y)| \leq \epsilon. $$

$F$ is called pointwise bounded, if for each $x \in X$ the set $\{|f(x)| : f \in F\}$ is bounded in $\mathbb{R}$. $F$ is called relatively compact (or conditionally compact), if its closure is compact. For convenience, we recall the Arzelá-Ascoli Theorem (see for example Theorem IV.6.7 in Dunford & Schwartz (1958)):

**Theorem 5.1.** Let $X$ be a compact Hausdorff space. A subset $F \subset C(X; \mathbb{R})$ is relatively compact in the topology induced by uniform convergence, if and only if it is equicontinuous and pointwise bounded.

The next proposition follows from the Arzelá-Ascoli Theorem and will be important for our analysis:

**Proposition 5.2.** The following holds:
(i) Let $\mathcal{C} \subset H$ be a compact set. Then
\[
\sup_{h \in \mathcal{C}} \sup_{x \in [0, \Delta_n]} \|(I - S(x))h\|_H \to 0, \quad \text{as } n \to \infty.
\]

(ii) If $\sigma \in L^p(\Omega; L(U, H))$ for some $p \in [1, \infty)$ is an almost surely compact random operator, we get that
\[
\sup_{x \in [0, \Delta_n]} \|(I - S(x))\sigma\|_{op} \to 0, \quad \text{as } n \to \infty,
\]
where the convergence holds almost surely and in $L^p(\Omega; \mathbb{R})$.

(iii) Let $(\sigma_s)_{s \in [0, T]}$ in $L^p(\Omega \times [0, T]; L(U, H))$ for some $p \in [1, \infty)$ be a stochastic process, such that $\sigma_s$ is almost surely compact for all $s \in [0, t]$. If in addition the volatility process is continuous in the $p$'th mean, we obtain
\[
\sup_{r \in (0, t]} \mathbb{E}\left( \sup_{x \in [0, \Delta_n]} \|(I - S(x))\sigma_r\|_{p} \right) \to 0 \quad \text{as } n \to \infty.
\]

Proof. We want to apply the Arzelà-Ascoli Theorem for the subset $F := \{ h \mapsto \sup_{x \in [0, \Delta_n]} \|(I - S(x))h\|_H : n \in \mathbb{N} \} \subset C(\mathcal{C}; \mathbb{R})$.

It is clear that $F$ is pointwise bounded and the equicontinuity holds, since there is a common Lipschitz-constant (independent of $n$):
\[
\left| \sup_{x \in [0, \Delta_n]} \|(I - S(x))h\|_H - \sup_{x \in [0, \Delta_n]} \|(I - S(x))g\|_H \right| \\
\leq \sup_{x \in [0, \Delta_n]} \|(I - S(x))(h - g)\|_H \\
\leq \sup_{x \in [0, \Delta_n]} \|(I - S(x))\|_H \|h - g\|_H,
\]
for all $g, h \in \mathcal{C}$. This implies the relative compactness of $F$ with respect to the sup-norm on $C(\mathcal{C}; \mathbb{R})$. Therefore, there exists a subsequence such that, for $n \to \infty$, we have
\[
\sup_{h \in \mathcal{C}} \sup_{x \in [0, \Delta_n]} \|(I - S(x))h\| \to 0.
\]

Since the sequence $\sup_{x \in [0, \Delta_n]} \|(I - S(x))\cdot\|$ is monotone in $n$, we obtain convergence for the whole sequence. This shows (28).

Let $B_0(1) := \{ h \in H : \|h\|_H = 1 \}$ be the unit sphere in $H$ and fix $\omega \in \Omega$, such that $\sigma(\omega)$ is compact. Since $\sigma(\omega)$ is compact, $\mathcal{C} := \sigma(\omega)(B_0(1))$ is compact in $H$. The set $F(\omega)$ of functionals of the form
\[
f_n := \sup_{x \in [0, \Delta_n]} \|(I - S(x))\cdot\|_H : \mathcal{C} \to \mathbb{R}
\]
forms an equicontinuous and pointwise bounded subset of $C(\mathcal{C}; \mathbb{R})$. Thus, by (28)
\[
\sup_{x \in [0, \Delta_n]} \|(I - S(x))\sigma(\omega)\|_{op} = \sup_{x \in [0, \Delta_n]} \sup_{\|h\| = 1} \|(I - S(x))\sigma(\omega)h\|_H \\
\leq \sup_{g \in \mathcal{C}} f_n(g) \\
\to 0, \quad \text{as } n \to \infty.
\]

This gives almost sure convergence. Since the sequence is uniformly bounded by $(1 + M(T))\|\sigma\|_{op}$, which has finite $p$th moment, we obtain $L^p(\Omega; \mathbb{R})$-convergence by the dominated convergence theorem, and therefore (29) holds.
To verify the convergence (30) we argue as follows: Defining

$$g_n(s) := \left( \mathbb{E} \left[ \sup_{x \in [0, \Delta_n]} \|(I - S(x))\sigma_s\|_{op}^p \right] \right)^{\frac{1}{p}},$$

we obtain pointwise boundedness with the bound \((1 + M(T))\mathbb{E}[\|\sigma_s\|_{op}^p]^{\frac{1}{p}}\) and equicontinuity of \(\{g_n : n \in \mathbb{N}\} \subset C([0, T]; \mathbb{R})\) by the continuity in the \(p\)th mean of the process \((\sigma_s)_{s \in [0, T]}\), since by the Minkowski inequality

$$|g_n(t) - g_n(s)| \leq \left( \mathbb{E} \left[ \sup_{x \in [0, \Delta_n]} \|(I - S(x))(\sigma_t - \sigma_s)\|_{op}^p \right] \right)^{\frac{1}{p}} \leq (I + M(T)) \left( \mathbb{E} \left[ \|\sigma_t - \sigma_s\|_{op}^p \right] \right)^{\frac{1}{p}}.$$

By the Arzelá-Ascoli Theorem this induces the convergence of a subsequence of \((b_n)_{n \in \mathbb{N}}\) in the sup-norm and thus since \(b_n\) decreases pointwise with \(n\), the convergence of the whole sequence. For all \(s \in [0, T]\) we have by (29) that \(((I - S(x))\sigma_s)_{s \in [0, T]}\) goes to zero as \(n \to \infty\) almost surely. By uniqueness of the limit (in probability), this implies that \(b_n(s)\) converges to zero and thus, \(\sup_{s \in [0, T]} b_n(s)\) goes to zero. \(\Box\)

Recall also the following fact:

**Lemma 5.3.** The family \((S(t)^*)_{t \geq 0}\) of adjoint operators of the \(C_0\)-semigroup \((S(t))_{t \geq 0}\) forms again a \(C_0\)-semigroup on \(H\).

**Proof.** See Section 5.14 in *Engel & Nagel (1999).* \(\Box\)

Now we can proceed with the proof of our main theorem in the next subsection.

5.1.2. **Proof of Theorem 3.3** The operator bracket process for the semigroup-adjusted increment takes the form

$$(\langle \tilde{\Delta}_n^i Y \rangle) = \int_{t_{i-1}}^{t_i} S(t_i - s)\sigma_s Q\sigma_s^* S(t_i - s)^* ds.$$

For \(i \in \{1, \ldots, \lfloor t / \Delta_n \rfloor \}\) we denote by \(\Delta_n^i W := W_{t_i} - W_{t_{i-1}}\) and:

$$\tilde{\beta}_n^i := S(t_i - t_{i-1})\sigma_{t_{i-1}} \Delta_n^i W;$$

$$\tilde{\chi}_i^n := \int_{t_{i-1}}^{t_i} [S(t_i - s)\sigma_s - S(t_i - t_{i-1})\sigma_{t_{i-1}}] dW_s.$$

Then

$$\tilde{\Delta}_n^i Y = \tilde{\beta}_n^i + \tilde{\chi}_i^n.$$

To this end, fix some \(T > 0\). Using the triangle inequality, we can estimate

$$\sup_{t \in [0, T]} \left\| \sum_{i=1}^{\lfloor t / \Delta_n \rfloor} (\tilde{\Delta}_n^i Y)^{\odot 2} - \int_0^t \sigma_s Q\sigma_s^* ds \right\|_{HS} \leq \sup_{t \in [0, T]} \left\| \sum_{i=1}^{\lfloor t / \Delta_n \rfloor} (\tilde{\Delta}_n^i Y)^{\odot 2} - (\tilde{\beta}_n^i)^{\odot 2} \right\|_{HS}$$

$$(32) \leq \sup_{t \in [0, T]} \left\| \sum_{i=1}^{\lfloor t / \Delta_n \rfloor} (\tilde{\Delta}_n^i Y)^{\odot 2} - (\tilde{\beta}_n^i)^{\odot 2} \right\|_{HS}$$
Using the Itô isometry, see Peszat & Zabczyk (2007, Corollary 8.7, p. 123), we have

\begin{align}
&\quad \text{Proof.} \\
&\quad \text{Under Assumptions 1, 2 and either 3(a) or 3(b), we have} \\
&\quad \text{For the fourth moment, we argue as follows: By the independent increment property of } \mathcal{W}, \text{ we have that } \Delta_t^n W \text{ is independent of the } \mathcal{F}_{(i-1)\Delta_n} \text{-measurable random variable } \sigma_{(i-1)\Delta_n}. \text{ Thus, again by using the bound } (11) \text{ on the semigroup gives} \\
&\quad \text{Lemma 5.4. Under Assumption 2 we have} \\
&\quad E \left[ \|\tilde{\beta}^n_i\|_H^2 \right] \leq M(\Delta_n) \sqrt{T - Q(\sqrt{C_2(T)\Delta_n^2}}/2, \\
&\quad E \left[ \|\tilde{\beta}^n_i\|_H^4 \right] \leq M(\Delta_n)^2 Tr(Q) \sqrt{C_2(T)\Delta_n^2}, \\
&\quad E \left[ \|\tilde{\beta}^n_i\|_H^2 \right] \leq M(\Delta_n)^4 (Tr(Q) + 2 Tr_2(Q)) C_2(T) \Delta_n^2. \\
&\quad \text{Under Assumptions 2 and either 3(a) or 3(b), we have} \\
&\quad E \left[ \|\hat{\beta}^n_i\|_H^2 \right] \leq M(\Delta_n) \sqrt{T - Q(\sqrt{C_2(T)\Delta_n^2}}/2, \\
&\quad E \left[ \|\hat{\beta}^n_i\|_H^4 \right] \leq M(\Delta_n)^2 Tr(Q) \sqrt{C_2(T)\Delta_n^2}, \\
&\quad \text{for some constant } K(T) > 0 \text{ and a sequence } (a_n(T))_{n \in \mathbb{N}} \text{ of real numbers converging to zero.} \\
&\quad \text{Proof. First notice that the trace class property of } Q \text{ yields } \|Q^{1/2}\|_{HS} = Tr(Q) < \infty. \text{ Using the Itô isometry, see Peszat & Zabczyk (2007, Corollary 8.7, p. 123), we deduce from Assumption 2 that} \\
&\quad E \left[ \|\tilde{\beta}^n_i\|_H^2 \right] = \Delta_n E \left[ \|S(t_i - t_{i-1})\sigma_{t_{i-1}}Q^{1/2}\|_{HS}^2 \right] \\
&\quad \leq M(\Delta_n)^2 \Delta_n E \left[ ||\sigma_{t_{i-1}}||_{op}^2 \right] \|Q^{1/2}\|_{HS}^2 \\
&\quad \leq M(\Delta_n)^2 Tr(Q) \sqrt{C_2(T)\Delta_n^2}, \\
&\quad \text{where } M(\Delta_n) \text{ is given by } (11). \text{ An application of the Cauchy-Schwarz inequality gives} \\
&\quad E \left[ \|\tilde{\beta}^n_i\|_H^2 \right] \leq \sqrt{E \left[ \|\beta^n_i\|_H^4 \right]}, \\
&\quad \text{which leads to the result for } p = 1. \\
&\quad \text{For the fourth moment, we argue as follows: By the independent increment property of } \mathcal{W}, \text{ we have that } \Delta_t^n W \text{ is independent of the } \mathcal{F}_{(i-1)\Delta_n} \text{-measurable random variable } \sigma_{(i-1)\Delta_n}. \text{ Thus, again by using the bound } (11) \text{ on the semigroup gives} \\
&\quad E \left[ \|\hat{\beta}^n_i\|_H^4 \right] \leq M(\Delta_n)^4 E \left[ ||\sigma_{t_{i-1}}\Delta_n||_{op}^4 \|\Delta_t^n W||_H^4 \right] \\
&\quad = M(\Delta_n)^4 E \left[ ||\sigma_{t_{i-1}}||_{op}^4 \right] E \left[ ||\Delta_t^n W||_H^4 \right] \\
&\quad \leq M(\Delta_n)^4 C_2(T) (Tr(Q)^2 + 2 Tr_2(Q)) \Delta_n^2, \\
&\quad \text{after appealing to Lemma 2.7 and Assumption 2.}
We have, by Assumption 3(b) that
\[
\sup_{s \in (t_{i-1}, t_i]} \mathbb{E} \left[ \|(\sigma_s - \sigma_{t_{i-1}})Q^{1/2}\|_{HS}^2 \right] \leq C^2_1(T) \Delta_n^{2a}.
\]
Hence, for all \( i \in \{1, \ldots, \lfloor t/\Delta_n \rfloor \} \)
\[
\int_{t_{i-1}}^{t_i} \mathbb{E} \left[ \|(\sigma_s - \sigma_{(i-1)\Delta_n})Q^{1/2}\|_{HS}^2 \right] ds \leq C^2_1(T) \Delta_n^{1+2a}.
\]
By the Itô isometry
\[
\mathbb{E} \left[ \|\tilde{\chi}^2 \|_{H}^2 \right] = \int_{t_{i-1}}^{t_i} \mathbb{E} \left[ \|S(t_i - s)\sigma_s - S(t_i - t_{i-1})\sigma_{t_{i-1}}\|_{HS}^2 \right] ds
\]
\[
\leq \int_{t_{i-1}}^{t_i} \mathbb{E} \left[ M(\Delta_n)^2 \|S(t_i - s)\sigma_s - S(t_i - t_{i-1})\sigma_{t_{i-1}}\|_{HS}^2 \right] ds
\]
\[
\leq 2M(\Delta_n)^2 \int_{t_{i-1}}^{t_i} \mathbb{E} \left[ \|\sigma_s - \sigma_{t_{i-1}}\|_{HS}^2 + \|S(t_i - s)\sigma_s - S(t_i - t_{i-1})\sigma_{t_{i-1}}\|_{HS}^2 \right] ds,
\]
where we used the fact that \( S(t_i - t_{i-1}) = S(t_i - s)S(s - t_{i-1}) \) in the first inequality.

Assume now Assumption 3(a) holds and denote by \( \sigma_sQ^{\frac{1}{2}} = K_sT \) the corresponding decomposition. We obtain
\[
\mathbb{E} \left[ \|\tilde{\chi}^2 \|_{H}^2 \right] \leq 2M(\Delta_n)^2 \int_{t_{i-1}}^{t_i} \mathbb{E} \left[ \|S(t_i - s) - I\|_{op} \|K_{t_{i-1}}\|_{op}^2 \right] \|T\|_{HS}^2 ds
\]
\[
\quad + \mathbb{E} \left[ \|\sigma_s - \sigma_{t_{i-1}}\|_{HS}^2 \right] ds
\]
\[
\leq 2M(\Delta_n)^2 \left( \Delta_n \mathbb{E} \left[ \sup_{x \in [0, \Delta_n]} \|I - S(x)K_{t_{i-1}}\|_{op}^2 \right] \|T\|_{HS}^2 + C^2_1(T) \Delta_n^{1+2a} \right).
\]
The assertion follows with
\[
a_n(T) = 2M(\Delta_n)^2 \left( \sup_{s \in [0, T]} \mathbb{E} \left[ \sup_{x \in [0, \Delta_n]} \|I - S(x)K_s\|_{op}^2 \right] \|T\|_{HS}^2 + C^2_1(T) \Delta_n^{2a} \right),
\]
by (30) in Corollary 5.2, since \( (K_s)_{s \in [0, T]} \) is mean square continuous and \( K_s \) is almost surely a compact operator for all \( s \in [0, T] \).

Assume now Assumption 3(b) holds. By (41) and (40) and Assumption 2 we obtain
\[
\mathbb{E} \left[ \|\tilde{\chi}^2 \|_{H}^2 \right]
\]
\[
\leq 2M(\Delta_n)^2 \int_{t_{i-1}}^{t_i} \mathbb{E} \left[ \|\sigma_s - \sigma_{t_{i-1}}\|_{HS}^2 + \|S(t_i - s)\sigma_s - S(t_i - t_{i-1})\sigma_{t_{i-1}}\|_{HS}^2 \right] ds
\]
\[
\leq 2M(\Delta_n)^2 \left( \int_{t_{i-1}}^{t_i} \mathbb{E} \left[ \|\sigma_s - \sigma_{t_{i-1}}\|_{HS}^2 + \sup_{r \in [0, \Delta_n]} \|S(r) - I\|_{op}^2 \|\sigma_{t_{i-1}}Q^{\frac{1}{2}}\|_{HS}^2 \right] ds \right)
\]
\[
\leq 2M(\Delta_n)^2 \left( C^2_1(T) \Delta_n^{1+2a} + \Delta_n \sup_{r \in [0, \Delta_n]} \|S(r) - I\|_{op}^2 \sqrt{C^2_2(T) \text{Tr}(Q)} \right).
\]
This shows the assertion with
\[ a_n(T) = 2M(\Delta_n)^2 \left( \sup_{r \in [0, \Delta_n]} \| (S(r) - I) \|_{op}^2 \sqrt{C_2(T) \text{Tr}(Q) + C_1^2(T) \Delta_n^{2\alpha}} \right), \]
since, by the uniform continuity of the semigroup, \( \sup_{r \in [0, \Delta_n]} \| (S(r) - I) \|_{op} \) converges to zero as \( n \to \infty \).

**Remark 5.5.** In the following, we need Assumption 3 only if we want to apply Lemma 5.4, where we needed it to verify that the sequence \( a_n \) converges to zero. The convergence rate of \( a_n \) is determined by both, the path-regularity of the volatility process as well as the convergence rate of the semigroup (on compacts) as \( t \to 0 \). The convergence speed of this sequence will essentially determine the rate of convergence of the sequence \( b_n \) from Theorem 3.3.

**Remark 5.6.** We notice that for the first and second moment estimates of \( \| \tilde{\beta}^n_I \|_{H} \), we could relax the assumption on \( \sigma \) slightly by assuming \( \| \sigma Q^{1/2} \|_{HS} \) having finite second moment. However, the fourth moment of \( \| \tilde{\beta}^n_I \|_{H} \) is most conveniently estimated based on a fourth moment condition on the operator norm of \( \sigma \).

With the results in Lemma 5.4 at hand, we prove convergence of the four components \( \tilde{\beta}^n_i \). First, we show the convergence of \( \tilde{\beta}^n_i \) from Theorem 3.3.

**Proposition 5.7.** Under Assumptions 1, 2 and 3 we have
\[
\lim_{n \to \infty} E \left[ \sup_{t \in [0, T]} \left\| \sum_{i=1}^{[t/\Delta_n]} \tilde{\xi}^n_i \right\|_{HS}^2 \right] = 0.
\]

**Proof.** Define
\[
\tilde{\xi}^n_i := (\tilde{\Delta}^n_i Y)^{\otimes 2} - (\tilde{\beta}^n_i)^{\otimes 2} = (\tilde{\beta}^n_i + \tilde{\chi}^n_i)^{\otimes 2} - (\tilde{\beta}^n_i)^{\otimes 2} = (\tilde{\chi}^n_i)^{\otimes 2} + \tilde{\beta}^n_i \otimes \tilde{\chi}^n_i + \tilde{\chi}^n_i \otimes \tilde{\beta}^n_i.
\]

By the triangle inequality, we note that
\[
\| \tilde{\xi}^n_i \|_{HS} \leq \| (\tilde{\chi}^n_i)^{\otimes 2} \|_{HS} + \| \tilde{\beta}^n_i \otimes \tilde{\chi}^n_i \|_{HS} + \| \tilde{\chi}^n_i \otimes \tilde{\beta}^n_i \|_{HS}
\]
\[
= \| \tilde{\chi}^n_i \|_H^2 + 2 \| \tilde{\beta}^n_i \|_H \| \tilde{\chi}^n_i \|_H.
\]

Again appealing to the triangle inequality, it follows
\[
\sup_{t \in [0, T]} \left\| \sum_{i=1}^{[t/\Delta_n]} \tilde{\xi}^n_i \right\|_{HS} \leq \sup_{t \in [0, T]} \sum_{i=1}^{[t/\Delta_n]} \| \tilde{\xi}^n_i \|_{HS} \leq \sum_{i=1}^{[T/\Delta_n]} \| \tilde{\xi}^n_i \|_{HS}.
\]

Applying (39) in Lemma 5.4 leads to
\[ E \left[ \| \tilde{\chi}^n_i \|_H^2 \right] \leq \Delta_n a_n(T). \]

We next apply the Cauchy-Schwarz inequality to obtain, using the notation \( K_n(T) = M(\Delta_n)^2 \text{Tr}(Q) \sqrt{C_2(T)} \),
\[
E \left[ \| \tilde{\beta}^n_i \|_H \| \tilde{\chi}^n_i \|_H \right]^2 \leq E \left[ \| \tilde{\beta}^n_i \|_H^2 \right] E \left[ \| \tilde{\chi}^n_i \|_H^2 \right] \leq K_n(T) \Delta_n^2 a_n(T),
\]
by (30) and (39) in Lemma 5.4. Altogether we have, since \( a_n \to 0 \) as \( n \to \infty \), that

\[
(43) \quad \mathbb{E} \left[ \sup_{t \in [0,T]} \left\| \left( t/\Delta_n \right) \sum_{i=1}^{\lceil t/\Delta_n \rceil} \tilde{\zeta}_i^n \right\|_{HS} \right] \leq \frac{\Delta_n}{T/\Delta_n} (\Delta_n a_n(T) + 2 \sqrt{K_n(T)a_n(T)\Delta_n}),
\]

converges to zero as \( n \to \infty \) by Lemma 5.4. □

Now we prove the convergence of (33).

**Proposition 5.8.** Under Assumptions 1, 2 and 3 we have,

\[
\lim_{n \to \infty} \mathbb{E} \left[ \sup_{t \in [0,T]} \left\| \sum_{i=1}^{\lceil t/\Delta_n \rceil} (\tilde{\beta}_i^n)^{\otimes 2} - S(t_i - t_{i-1})\sigma_{t_{i-1}}Q\sigma_{t_i}^* S(t_i - t_{i-1})^* \Delta_n \right\|_{HS}^2 \right] = 0.
\]

**Proof.** We define

\[
\tilde{\zeta}_i^n := (\tilde{\beta}_i^n)^{\otimes 2} - S(t_i - t_{i-1})\sigma_{t_{i-1}}Q\sigma_{t_i}^* S(t_i - t_{i-1})^* \Delta_n.
\]

First we show that \( \sup_{t \in [0,T]} \left\| \sum_{i=1}^{\lceil t/\Delta_n \rceil} \tilde{\zeta}_i^n \right\|_{HS} \) has finite second moment. By the triangle inequality and Lemma 2.1,

\[
\sup_{t \in [0,T]} \left\| \sum_{i=1}^{\lceil t/\Delta_n \rceil} \tilde{\zeta}_i^n \right\|_{HS} \leq \sum_{i=1}^{\lceil t/\Delta_n \rceil} \left\| \tilde{\zeta}_i^n \right\|_{HS}
\]

\[
\leq \sum_{i=1}^{\lceil t/\Delta_n \rceil} \left\| (\tilde{\beta}_i^n)^{\otimes 2} \right\|_{HS}
\]

\[
+ \Delta_n \sum_{i=1}^{\lceil t/\Delta_n \rceil} \left\| S(t_i - t_{i-1})\sigma_{t_{i-1}}Q\sigma_{t_i}^* S(t_i - t_{i-1})^* \right\|_{HS}
\]

\[
\leq \sum_{i=1}^{\lceil t/\Delta_n \rceil} \left\| \tilde{\beta}_i^n \right\|_H^2 + \Delta_n \sum_{i=1}^{\lceil t/\Delta_n \rceil} \left\| S(t_i - t_{i-1})\sigma_{t_{i-1}}Q^{1/2} \right\|_{HS}^2
\]

\[
\leq \sum_{i=1}^{\lceil t/\Delta_n \rceil} \left\| \tilde{\beta}_i^n \right\|_H^2 + \Delta_n \text{Tr}(Q)M(\Delta_n)^2 \sum_{i=1}^{\lceil t/\Delta_n \rceil} \left\| \sigma_{t_{i-1}} \right\|_{op}^2.
\]

Considering \( \mathbb{E} \left[ \sup_{t \in [0,T]} \left\| \sum_{i=1}^{\lceil t/\Delta_n \rceil} \tilde{\zeta}_i^n \right\|_{HS}^2 \right] \), we get a finite sum of terms of the type \( \mathbb{E} \left[ \left\| \tilde{\beta}_i^n \right\|_H^4 \right] \), \( \mathbb{E} \left[ \left\| \sigma_{t_{i-1}} \right\|_{op}^4 \right] \) and \( \mathbb{E} \left[ \left\| \tilde{\beta}_i^n \right\|_H^2 \left\| \sigma_{t_{i-1}} \right\|_{op}^2 \right] \). The first is finite due to Lemma 5.4, while the second is finite by the imposed Assumption 2. For the third, we apply the Cauchy-Schwarz inequality and argue as for the first two. In conclusion, we obtain a finite second moment as desired.

Note that \( R_t = \int_0^t h_s dW(s) \) where \( h_s = \sum_{i=1}^n S(t_i - t_{i-1})\sigma_{t_{i-1}}^1 \mathbf{1}_{(t_{i-1}, t_i]}(s) \) defines a martingale, such that \( R_{t_m} = \sum_{j=1}^m \tilde{\beta}_j^n \). Then the squared process is

\[
\int_0^{t_m} h_s dW(s)^{\otimes 2} = \sum_{i,j=1}^m \langle \tilde{\beta}_i^n, \tilde{\beta}_j^n \rangle
\]

and

\[
\langle \int_0^{t_m} h_s dW(s) \rangle_{t_m}
\]
Proposition 5.9. Hence, the proposition follows. We obtain that

\[ (44) \]

forms a sequence of martingale differences with respect to \((F_{t_{j-1}})_{j \in \mathbb{N}}\), by Remark 2.4. This implies in particular, after double conditioning, that for \(1 \leq i \neq j \leq \lfloor t/\Delta_n \rfloor\),

\[ \mathbb{E}\left[ \langle \hat{\zeta}^n_i, \hat{\zeta}^n_j \rangle_{HS} \right] = 0. \]

By Doob’s martingale inequality we obtain

\[
\mathbb{E} \left[ \sup_{t \in [0,T]} \left\| \frac{t}{\Delta_n} \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \hat{\zeta}^n_i \right\|_{HS}^2 \right] \leq 4 \mathbb{E} \left[ \left\| \sum_{i=1}^{\lfloor T/\Delta_n \rfloor} \hat{\zeta}^n_i \right\|_{HS}^2 \right] = 4 \sum_{i=1}^{\lfloor T/\Delta_n \rfloor} \mathbb{E} \left[ \left\| \hat{\zeta}^n_i \right\|_{HS}^2 \right].
\]

Applying the triangle inequality and the basic inequality \((a+b)^2 \leq 2(a^2 + b^2)\), we find

\[
\|\hat{\zeta}^n_i\|_{HS}^2 \leq 2 \left( \|\hat{\beta}^n_i\|_{HS}^2 + \|S(t_i - t_{i-1})\sigma_{t_{i-1}} Q\sigma_{t_{i-1}}^* S(t_i - t_{i-1})^* \|_{HS}^2 \Delta_n^2 \right)
\]

\[
\leq 2 \left( \|\hat{\beta}^n_i\|_{HS}^2 + \|\sigma_{t_{i-1}} Q\sigma_{t_{i-1}}^* \|_{HS}^2 M(\Delta_n)^4 \Delta_n^2 \right).
\]

Denoting again \(K_n(T) = M(\Delta_n)^2 \text{Tr}(Q) \sqrt{C_2(T)}\), we can now apply Lemma 5.4 to conclude that

\[
\sum_{i=1}^{\lfloor T/\Delta_n \rfloor} \mathbb{E} \left[ \|\hat{\zeta}^n_i\|_{HS}^2 \right] \leq 2 \left( K_n(T) \frac{T}{\Delta_n} \Delta_n^2 + M(\Delta_n)^4 \Delta_n \left[ \sum_{i=1}^{\lfloor T/\Delta_n \rfloor} \|\sigma_{t_{i-1}} Q\sigma_{t_{i-1}}^* \|_{HS}^2 \Delta_n \right] \right) 
\]

\[ \rightarrow 0, \text{ as } n \rightarrow \infty, \]

since the expectation operator on the right-hand side of the inequality above converges to

\[
\mathbb{E} \left[ \int_0^T \|\sigma_s Q\sigma_s^* \|_{HS}^2 ds \right] < \infty.
\]

Hence, the proposition follows. \(\square\)

Next, we prove the convergence of \((44)\).

**Proposition 5.9.** Assume that Assumptions [1] and [2] hold. Then

\[
\lim_{n \to \infty} \mathbb{E} \left[ \sup_{t \in [0,T]} \left\| \sum_{i=1}^{\lfloor t/\Delta_n \rfloor} \int_{(t_{i-1})}^{t_i} (S(t_i - t_{i-1})\sigma_{t_{i-1}} Q\sigma_{t_{i-1}}^* S(t_i - t_{i-1})^* - S(t_i - s)\sigma_s Q\sigma_s^* S(t_i - s)^* ds) \right\|_{HS} \right] = 0.
\]
Proof. From the triangle and Bochner inequalities, we get
\[
\| \sum_{i=1}^{[T/\Delta_n]} \int_{t_{i-1}}^{t_i} (S(t_i - t_{i-1})\sigma_{t_{i-1}} Q\sigma_{t_{i-1}}^* S(t_i - t_{i-1})^* - S(t_i - s)\sigma_s Q\sigma_s^* S(t_i - s)^*) \|_{HS} \leq \sum_{i=1}^{[T/\Delta_n]} \int_{t_{i-1}}^{t_i} \| (S(t_i - t_{i-1})\sigma_{t_{i-1}} Q\sigma_{t_{i-1}}^* S(t_i - t_{i-1})^* - S(t_i - s)\sigma_s Q\sigma_s^* S(t_i - s)^*) \|_{HS} ds.
\]

Note that for \( s \in (t_{i-1}, t_i) \), we have
\[
S(t_i - t_{i-1})\sigma_{t_{i-1}} Q\sigma_{t_{i-1}}^* S(t_i - t_{i-1})^* - S(t_i - s)\sigma_s Q\sigma_s^* S(t_i - s)^* = (S(t_i - t_{i-1})\sigma_{t_{i-1}} - S(t_i - s)\sigma_s) Q\sigma_{t_{i-1}}^* S(t_i - t_{i-1})^* + S(t_i - s)\sigma_s Q(\sigma_{t_{i-1}}^* S(t_i - t_{i-1})^* - \sigma_s^* S(t_i - s)^*).
\]

Hence, using the triangle inequality and then the Cauchy-Schwarz inequality, we have
\[
\begin{align*}
E \left[ \| (S(t_i - t_{i-1})\sigma_{t_{i-1}} Q\sigma_{t_{i-1}}^* S(t_i - t_{i-1})^* - S(t_i - s)\sigma_s Q\sigma_s^* S(t_i - s)^*) \|_{HS} \right]^2 &= E \left[ \| (S(t_i - t_{i-1})\sigma_{t_{i-1}} - S(t_i - s)\sigma_s) Q\sigma_{t_{i-1}}^* S(t_i - t_{i-1})^* \right]^2 \\
&+ 2E \left[ \| S(t_i - s)\sigma_s Q\sigma_{t_{i-1}}^* S(t_i - t_{i-1})^* \|_{op} \right]^2 = 2E \left[ \| S(t_i - s)\sigma_s Q\sigma_{t_{i-1}}^* \|_{op} \right]^2 \| S(t_i - t_{i-1})^* \|_{op}^2 \\
&\leq 2E \left[ \| (S(t_i - t_{i-1})\sigma_{t_{i-1}} - S(t_i - s)\sigma_s) Q\sigma_{t_{i-1}}^* \|_{op} \right]^2 \| S(t_i - t_{i-1})^* \|_{op}^2 \| S(t_i - s)\sigma_s Q\sigma_{t_{i-1}}^* \|_{op}^2 \\
&\leq 2E \left[ \| (S(t_i - t_{i-1})\sigma_{t_{i-1}} - S(t_i - s)\sigma_s) Q\sigma_{t_{i-1}}^* \|_{op} \right]^2 \| S(t_i - t_{i-1})^* \|_{op}^2 \| S(t_i - s)\sigma_s Q\sigma_{t_{i-1}}^* \|_{op}^2 \\
&\leq 2E \left[ \sup_{r \in [0, T]} \| S((s - t_{i-1})\sigma_{t_{i-1}} - s) Q\sigma_{t_{i-1}}^* \|_{op} \right]^2 \| S(s - t_{i-1})^* \|_{op}^2 \| S(s)\sigma_s Q\sigma_{t_{i-1}}^* \|_{op}^2 \\
&\leq 4M(\Delta_n)^4 \sup_{r \in [0, T]} \| S((s - t_{i-1})\sigma_{t_{i-1}} - s) Q\sigma_{t_{i-1}}^* \|_{op} \| S(s - t_{i-1})^* \|_{op}^2 \| S(s)\sigma_s Q\sigma_{t_{i-1}}^* \|_{op}^2 .
\end{align*}
\]

By Assumption 2, we know that
\[
A_n := 4M(\Delta_n)^4 \sqrt{C_2(T)} \geq 2M(\Delta_n)^4 \sup_{r \in [0, T]} \| S((s - t_{i-1})\sigma_{t_{i-1}} - s) Q\sigma_{t_{i-1}}^* \|_{op} \| S(s - t_{i-1})^* \|_{op}^2 .
\]

Using Assumption 1, this gives the following estimate:
\[
E \left[ \| (S(t_i - t_{i-1})\sigma_{t_{i-1}} Q\sigma_{t_{i-1}}^* S(t_i - t_{i-1})^* - S(t_i - s)\sigma_s Q\sigma_s^* S(t_i - s)^*) \|_{HS} \right]^2 \leq 4M(\Delta_n)^4 \sup_{r \in [0, T]} \| S((s - t_{i-1})\sigma_{t_{i-1}} - s) Q\sigma_{t_{i-1}}^* \|_{op} \| S(s - t_{i-1})^* \|_{op}^2 \| S(s)\sigma_s Q\sigma_{t_{i-1}}^* \|_{op}^2 .
\]
By Lemma 2.1 and the Cauchy-Schwarz inequality we obtain

\[ \leq A_n(T) \mathbb{E} \left[ \sup_{x \in [0, \Delta_n]} \| (S(x)\sigma_{t_{i-1}} - \sigma_{t_{i-1}} + \sigma_{t_{i-1}} - \sigma_s)Q_{\Delta_n}^\perp \|_{\text{op}}^2 \right] \]

\[ \leq A_n(T) 2 \mathbb{E} \left[ \sup_{x \in [0, \Delta_n]} \| (S(x) - I)\sigma_{t_{i-1}}Q_{\Delta_n}^\perp \|_{\text{op}}^2 \right] + \mathbb{E} \left[ \| (\sigma_{t_{i-1}} - \sigma_s)Q_{\Delta_n}^\perp \|_{\text{op}}^2 \right] \]

\[ \leq A_n(T) 2 (b_n(T) + C_1^2(T)\Delta_n^{2\alpha}) \]

where \( b_n(T) := \sup_{s \in [0, T]} \mathbb{E} [\sup_{x \in [0, \Delta_n]} \| (I - S(x))\sigma_s Q_{\Delta_n}^\perp \|_{\text{op}}^2 ] \) as before. We have that \( (b_n(T))_{n \in \mathbb{N}} \) is a real sequence converging to 0 by (30) in Corollary 5.2 since for each \( s \in [0, T] \) the operator \( \sigma_s Q_{\Delta_n}^\perp \) is almost surely compact as a Hilbert-Schmidt operator and the process \((\sigma_s Q_{\Delta_n}^\perp)_{s \in [0, T]} \) is mean square continuous by Assumption II.

Summing up, we obtain

\[ (45) \]

\[ \mathbb{E} \left[ \sum_{i=1}^{[T/\Delta_n]} \int_{t_{i-1}}^{t_i} (S(t_i - t_{i-1})\sigma_{t_{i-1}}Q_{\Delta_n}^\perp S(t_i - t_{i-1})^* - S(t_i - s)\sigma_s Q_{\Delta_n}^\perp S(t_i - s)^* ds \right]_{\text{HS}} \]

\[ \leq \sum_{i=1}^{[T/\Delta_n]} \int_{t_{i-1}}^{t_i} (A_n(T) 2 (C_1^2(T)\Delta_n^{2\alpha} + b_n(T))) \frac{1}{2} ds \]

\[ = [T/\Delta_n] \Delta_n (A_n(T) 2 (C_1^2(T)\Delta_n^{2\alpha} + b_n(T))) \frac{1}{2} \to 0, \text{ as } n \to \infty, \]

and the proof is complete. \( \square \)

Finally, we prove the convergence of (45).

**Proposition 5.10.** Suppose that Assumption I and II hold. Then

\[ \lim_{n \to \infty} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left\| \sum_{i=1}^{[t/\Delta_n]} \langle \Delta_{i} Y \rangle - \int_0^t \sigma_s Q_{\Delta_n}^\perp ds \right\|_{\text{HS}} \right] = 0. \]

**Proof.** Recall the expression for \( \langle \Delta_{i} Y \rangle \) in (31). By the triangle and Bochner inequalities, we find,

\[ \sup_{t \in [0, T]} \left\| \int_0^{[t/\Delta_n]} \sigma_s Q_{\Delta_n}^\perp ds - \sum_{i=1}^{[t/\Delta_n]} \int_{t_{i-1}}^{t_i} S(t_i - s)\sigma_s Q_{\Delta_n}^\perp S(t_i - s) ds \right\|_{\text{HS}} \]

\[ \leq \sup_{t \in [0, T]} \sum_{i=1}^{[t/\Delta_n]} \int_{t_{i-1}}^{t_i} \| \sigma_s Q_{\Delta_n}^\perp - S(t_i - s)\sigma_s Q_{\Delta_n}^\perp S(t_i - s)^* \|_{\text{HS}} ds \]

\[ \leq \sum_{i=1}^{[T/\Delta_n]} \int_{t_{i-1}}^{t_i} \| \sigma_s Q_{\Delta_n}^\perp - S(t_i - s)\sigma_s Q_{\Delta_n}^\perp S(t_i - s)^* \|_{\text{HS}} ds. \]

By Lemma 2.1 and the Cauchy-Schwarz inequality we obtain

\[ \mathbb{E} \left[ \sup_{0 \leq t \leq T} \left\| \sum_{i=1}^{[t/\Delta_n]} \langle \Delta_{i} Y \rangle - \int_0^t \sigma_s Q_{\Delta_n}^\perp ds \right\|_{\text{HS}} \right] \]
Since $\sigma$ and $s$ sequence converging to 0 by (30) in Corollary 5.2, since for each
Define
Using Assumption 2, we can estimate
\[ \leq \sum_{i=1}^{\lceil T/\Delta_n \rceil} \int_{t_{i-1}}^{t_i} \mathbb{E}[\| (I - S(t_i - s))\sigma_sQ\sigma_s^* \|_{HS}] \]
\[ + \mathbb{E}[\| S(t_i - s)\sigma_sQ\sigma_s^*(I - S(t_i - s)^* )\|_{HS}]ds \]
\[ + \int_{t_n}^{T} \mathbb{E}[\| \sigma_sQ\sigma_s^* \|_{HS}]ds \]
\[ \leq \sum_{i=1}^{\lceil T/\Delta_n \rceil} \int_{t_{i-1}}^{t_i} \mathbb{E}[\| (I - S(t_i - s))\sigma_sQ\frac{1}{2}\|_{op}\| Q\frac{1}{2}\sigma_s^* \|_{HS}] \]
\[ + M(\Delta_n)\mathbb{E}[\| \sigma_sQ\frac{1}{2}\|_{HS}\| Q\frac{1}{2}\sigma_s^*(I - S(t_i - s)^* )\|_{op}ds \]
\[ + \int_{t_n}^{T} \mathbb{E}[\| \sigma_sQ\frac{1}{2}\|_{HS}]ds \]
\[ \leq \sup_{r \in [0,T]} \mathbb{E} \left[ \sup_{x \in [0,\Delta_n]} \| (I - S(x))\sigma_rQ\frac{1}{2}\|_{op}^2 (1 + M(\Delta_n)) \int_{0}^{T} \mathbb{E}[\| Q\frac{1}{2}\sigma_s^* \|_{HS}^2]ds \right] \]
\[ + \int_{t_n}^{T} \mathbb{E}[\| \sigma_sQ\frac{1}{2}\|_{HS}]ds. \]

Using Assumption 2 we can estimate
\[ \mathbb{E} \left[ \sup_{0 \leq s \leq T} \left\| \int_{0}^{t} \sigma_sQ\sigma_s^* ds \right\|_{HS} \right] \]
\[ \leq (b_n(T))^\frac{1}{2} (1 + M(\Delta_n))T(\sqrt{C_2(T)\text{Tr}(Q)})^\frac{1}{2} + (T - t_n)\sqrt{C_2(T)\text{Tr}(Q)} \]
\[ \to 0 \quad \text{as} \quad n \to \infty. \]

Here again $b_n(T) := \sup_{s \in [0,T]} \mathbb{E}[\sup_{x \in [0,\Delta_n]} \| (I - S(x))\sigma_sQ\frac{1}{2}\|_{op}^2]$, which is a real sequence converging to 0 by (30) in Corollary 5.2 since for each $s \in [0,T]$ the operator $\sigma_sQ\frac{1}{2}$ is almost surely compact as a Hilbert-Schmidt operator and the process $(\sigma_sQ\frac{1}{2})_{s \in [0,T]}$ is mean square continuous by Assumption 1.

5.1.3. Proof of Theorem 5.6

Proof of Theorem 5.6 Define
\[ Y^{(m)}_t := \int_{0}^{t} S(t - s)\sigma_s^{(m)} dW_s, \]
and
\[ Z_n^m := \sup_{0 \leq s \leq t} \left\| \sum_{i=1}^{\lfloor s/\Delta_n \rfloor} (\Delta_n^i Y^{(m)})^\otimes 2 - \int_{0}^{s} \sigma_u^{(m)}Q\sigma_u^{(m)*} du \right\|_{HS}, \]
\[ Z_n^m := \sup_{0 \leq s \leq t} \left\| \sum_{i=1}^{\lfloor s/\Delta_n \rfloor} (\Delta_n^i Y)^\otimes 2 - \int_{0}^{s} \sigma_u Q\sigma_u^* du \right\|_{HS}. \]

Since $\sigma^{(m)}$ satisfies the conditions of Theorem 3.3 we obtain that for all $m \in \mathbb{N}$ and $\epsilon > 0$
\[ \lim_{n \to \infty} \mathbb{P}[Z_n^m > \epsilon] = 0. \]
5.2. Proofs of Section 4

We will now present the longer proofs of the results presented in Section 4.

5.2.1. Proof of Theorem 4.6

Proof of Theorem 4.6. Since for all \( h \in H_\beta \) it is \( |h(0)| \leq \|h\|_\beta \) we have for \( \|h\|_\beta = 1 \) that
\[
\left\| (I - S(x) )\sigma_r Q^\frac{\theta}{2} h \right\|_\beta \leq \left\| (I - S(x)) f_r \right\|_\beta + \left\| (I - S(x)) \int_0^\infty q_r(\cdot, z) h'(z) dz \right\|_\beta
\]
\[
= (1) + (2).
\]

The first summand can be estimated as follows, for some \( \zeta \in (0, t) \) and \( x < 1 \):
\[
(1) = \left( |f_r(x)|^2 + \int_0^\infty (f'_r(y + x) - f'_r(y))^2 e^{\beta y} dy \right)^{\frac{1}{2}}
\]
\[
\leq \left( |f'_r(\zeta)|^2 x^2 + x^{2\gamma} |\|L_1\|_{L^2(\mathbb{R}^+)}|^2 \right)^{\frac{1}{2}} \leq x^{\gamma} (|f'_r(\zeta)| + |\|L_1\|_{L^2(\mathbb{R}^+)}|).
\]

We can show, using Hölder inequality, for all \( h \in H_\beta \) such that \( \|h\|_\beta = 1 \), that
\[
(2) = \left( \int_0^\infty \left[ \partial_y \int_0^\infty (q_r(y + x, z) - q_r(y, z)) h'(z) dz \right]^2 e^{\beta y} dy \right)^{\frac{1}{2}}
\]
\[
= \left( \int_0^\infty \left[ \int_0^\infty \left( e^{-\frac{\beta}{2} x} p_r(y + x, z) - p_r(y, z) \right) e^{\frac{\beta}{2} z - y} h'(z) dz \right]^2 e^{\beta y} dy \right)^{\frac{1}{2}}
\]
\[
= \left( \int_0^\infty \left[ \int_0^\infty \left( e^{-\frac{\beta}{2} x} p_r(y + x, z) - p_r(y, z) \right) e^{\frac{\beta}{2} z} h'(z) dz \right]^2 dy \right)^{\frac{1}{2}}
\]
\[
\leq \left( \int_0^\infty \int_0^\infty (e^{-\frac{\beta}{2} x} p_r(y + x, z) - p_r(y, z))^2 d\|h\|_\beta dy \right)^{\frac{1}{2}}.
\]

Now we can estimate, for \( x < 1 \),
\[
(2) \leq \left( \int_0^\infty \int_0^\infty (e^{-\frac{\beta}{2} x} (p_r(y + x, z) - p_r(y, z)))^2 dz dy \right)^{\frac{1}{2}}
\]
Combining (49) and (50), we obtain, for
\[ \text{L} \]
\[ \|S(t)\|_{L^2(\mathbb{R}_+^2)} + \|r\|_{L^2(\mathbb{R}_+^2)} \leq x^2 (\|S(t)\|_{L^2(\mathbb{R}_+^2)} + \|r\|_{L^2(\mathbb{R}_+^2)}). \]

Combining (49) and (50), we obtain, for \( \|h\|_\beta = 1, \)
\[ \|S(t)\|_{L^2(\mathbb{R}_+^2)} + \|r\|_{L^2(\mathbb{R}_+^2)} \leq x^2 (\|S(t)\|_{L^2(\mathbb{R}_+^2)} + \|r\|_{L^2(\mathbb{R}_+^2)}). \]

Now we can conclude that
\[ \text{L} \]
\[ \|S(t)\|_{L^2(\mathbb{R}_+^2)} + \|r\|_{L^2(\mathbb{R}_+^2)} \leq x^2 (\|S(t)\|_{L^2(\mathbb{R}_+^2)} + \|r\|_{L^2(\mathbb{R}_+^2)}). \]

5.2.2. Proof of Lemma 4.8
Proof of Lemma 4.8 We have
\[ \Sigma_t - \Sigma_s = (S(t) - S(s))\Sigma_0 + \int_s^t (S(t - u) - S(s - u)) d\mathcal{L}_u \]
\[ \Sigma_t - \Sigma_s = (S(t) - S(s))\Sigma_0 + \int_s^t (S(t - u) - S(s - u)) d\mathcal{L}_u \]
\[ := (1) + (2) + (3). \]

As the semigroup \( (S(t))_{t \geq 0} \) is uniformly continuous, we can again use the fundamental equality \[ \|S(t)\|_{\mathcal{B}} \]
and the triangle inequality for Bochner integrals to deduce, for \( s, t \in [0, T] \) and \( t \geq s, \)
\[ \|S(t) - S(s)\|_{\mathcal{B}} = \left\| e^{B_s} \int_s^t e^{B_x} \mathcal{B} dx \right\|_{\mathcal{B}} \]
\[ \leq \left\| e^{B_s} \int_s^t e^{B_x} \mathcal{B} dx \right\|_{\mathcal{B}} \leq e^{\|B\|_{\mathcal{B}} T} \left\| \mathcal{B} \right\|_{\mathcal{B}} (t - s). \]

Denoting \( U := e^{\|B\|_{\mathcal{B}} T} \left\| \mathcal{B} \right\|_{\mathcal{B}} \), this gives
\[ \|S(t) - S(s)\|_{\mathcal{B}} \leq U \|S_0\|_{\mathcal{B}} (t - s). \]

This induces \( E\|S(t) - S(s)\|_{\mathcal{B}}^2 \leq U \|S_0\|_{\mathcal{B}} (t - s). \) Moreover, by the Itô isometry
\[ E\|S(t) - S(s)\|_{\mathcal{B}}^2 \leq U \|S_0\|_{\mathcal{B}} (t - s). \]

where \( Q_x \) denotes the covariance operator of \( L \). Finally, we can show again, by the Itô isometry and the mean value inequality, that
\[ \|S(t) - S(s)\|_{\mathcal{B}}^2 \leq \left( \int_s^t \|S(t) - S(s)\|_{\mathcal{B}}^2 \right) \frac{1}{2} \]
\[ \leq U^2 (t - s)^2 \int_s^t \|S(t) - S(s)\|_{\mathcal{B}}^2 \frac{1}{2} \]
\[ \leq U (t - s)^2 e^{\|B\|_{\mathcal{B}} T} \text{Tr}(Q(x)). \]
Summing up, we obtain, for $t - s \leq 1$,

$$E[\|\Sigma_t - \Sigma_s\|_{\text{HS}}^2]^{\frac{1}{2}} \leq (E[\|\Sigma(1)\|_{\text{HS}}^2]^{\frac{1}{2}} + E[\|\Sigma(2)\|_{\text{HS}}^2]^{\frac{1}{2}} + E[\|\Sigma(3)\|_{\text{HS}}^2]^{\frac{1}{2}})$

$$\leq (U\|\Sigma_0\|_{\text{HS}} + e\|B\|_{\text{op}}\|T\text{Tr}(Q_L)\|_{\text{op}}^{-\frac{1}{2}} (1 + U))(t - s)^{-\frac{1}{2}}.

Since also by the Itô isometry, we obtain

$$\sup_{t \in [0,T]} E[\|\Sigma_t\|_{\text{HS}}^2]^{\frac{1}{2}} \leq \sup_{t \in [0,T]} \left(\|S(t)\Sigma_0\|_{\text{HS}} + E\left[\left\|\int_0^t S(t-u)dL_u\right\|_{\text{HS}}^2\right]^{\frac{1}{2}}\right)

\leq \sup_{t \in [0,T]} \left(\|S(t)\Sigma_0\|_{\text{HS}} + \left(\int_0^T \|S(t-u)Q_L\|_{\text{HS}}^2 du\right)^{\frac{1}{2}}\right)

\leq e\|B\|_{\text{op}}\|T\text{Tr}(Q_L)\|_{\text{op}}^{-\frac{1}{2}} \|\Sigma_0\|_{\text{HS}} + e\|C\|_{\text{op}}\|T\text{Tr}(Q_L)\|_{\text{op}}^{-\frac{1}{2}} T^{\frac{1}{2}},

the additional assertion follows by Lemma 4.7. \qed

6. DISCUSSION AND OUTLOOK

Our paper develops a new asymptotic theory for high-frequency estimation of the volatility of infinite-dimensional stochastic evolution equations in an operator setting. We have defined the so-called semigroup-adjusted realised covariation (SARCV) and derived a weak law of large numbers based on uniform convergence in probability with respect to the Hilbert-Schmidt norm. Moreover, we have presented various examples where our new method is applicable.

Many articles on (high-frequency) estimation for stochastic partial differential equations rely on the so-called spectral approach and assume therefore the applicability of spectral theorems to the generator $A$ (cf. the survey article Cialenco (2018)). This makes it difficult to apply these results on differential operators that do not fall into the symmetric and positive definite scheme, as for instance $A = \frac{d}{dx}$ in the space of forward curves presented in Section 4.1.3, a case of relevance in financial applications that is included in our framework. Moreover, a lot of the related work assumes the volatility as a parameter of estimation to be real-valued (cf. the setting in Cialenco (2018)). An exception is the spatio-temporal volatility estimation in the recent paper by Chong (2020) (see also Chong & Dalang (2020) for limit laws for the power variation of fractional stochastic parabolic equations). Here, the stochastic integrals are considered in the sense of Walsh (1986) and the generator is the Laplacian. In our analysis, we operate in the general Hilbert space framework in the sense of Peszat and Zabczyck for stochastic integration and semigroups.

In our framework, we work with high-frequency observations of Hilbert-space valued random elements, hence we have observations, which are discrete in time but not necessarily in space. Recent research on inference for parabolic stochastic partial differential considered observation schemes which allow for discreteness in time and space, cf. Cialenco & Huang (2020), Bibinger & Trabs (2020), Chong (2020), Chong & Dalang (2020). However, as our approach falls conveniently into the realm of functional data analysis, we might reconstruct data in several cases corresponding to well-known techniques for interpolation or smoothing. Indeed, in practice, a typical situation is that the Hilbert space consists of real-valued functions (curves) on $\mathbb{R}^d$ (or some subspace thereof), but we only have access to discrete observations of the curves. We may have data for $Y_t(x_j)$ at locations $x_j, j = 1, \ldots, m,$
or possibly some aggregation of these (or, in more generality, a finite set of linear functionals of $Y_t^i$). For example, in commodity forward markets, we have only a finite number of forward contracts traded at all times, or, like in power forward markets, we have contracts with a delivery period (see e.g. Benth et al. (2008)) and hence observations of the average of $Y_t^i$ over intervals on $\mathbb{R}_+$. In other applications, like observations of temperature and wind fields in space and time, we may have accessible measurements at geographical locations where meteorological stations are situated, or, from atmospheric reanalysis where we have observations in grid cells regularly distributed in space. From such discrete observations, one must recover the Hilbert-space elements $Y_t^i$. This is a fundamental issue in functional data analysis, and several smoothing techniques have been suggested and studied. We refer to Ramsay & Silverman (2005) for an extensive discussion of this. However, smoothing introduces another layer of approximation, as we do not recover $Y_t^i$ but some approximate version $Y_t^m$, where the superscript $m$ indicates that we have smoothed based on the $m$ available observations. The construction of a curve from discrete observations is not a unique operation as this is an inverse problem. In future research, it will be interesting to extend our theory to the case when (spatial) smoothing has been applied to the discrete observations.

Interestingly, when we compare our work to recent developments on high-frequency estimation for volatility modulated Gaussian processes in finite dimensions, see e.g. Podolskij (2015) for a survey, it appears that a scaling factor is needed in the realised (co)variation so that an asymptotic theory for Volterra processes can be derived. This scaling factor is given by the variogram of the associated so-called Gaussian core process, and depends on the corresponding kernel function. However, in our case, due to the semigroup property, we are in a better situation than for general Volterra equations, since we actually have (or can reconstruct) the data in order to compute the semigroup-adjusted increments. We can then develop our analysis based on extending the techniques and ideas that are used in the semimartingale case. In this way, the estimator becomes independent of further assumptions on the remaining parameters of the equation. However, the price to pay for this universality is that the convergence speed cannot generally be determined. The semigroup-adjustment of the increments effectively forces the estimator to converge at most at the same rate as the semigroup converges to the identity on the range of the volatility as $t$ goes to 0. At first glance, it seems that the strong continuity of the semigroup suggests that we can obtain convergence just with respect to the strong topology. This would make it significantly harder to apply methods from functional data analysis, even for constant volatility processes. Fortunately, the compactness of the operators $\sigma_t Q^\frac{1}{2}$ for $t \in [0,T]$ comes to the rescue and enables us to prove that convergence holds with respect to the Hilbert-Schmidt norm. In this case, we obtain reasonable convergence rates for the estimator.

**Appendix A. Proofs of some technical results**

*Proof of Lemma 2.1.* It is well-known that $L_{HS}(U, H)$ is a separable Hilbert space (see e.g. Peszat & Zabczyk (2007, Appendix A.2, p. 356)). Indeed, an ONB is $(e_i \otimes f_j)_{i,j \in \mathbb{N}}$ where $(e_i)_{i \in \mathbb{N}}$ is ONB for $U$ and $(f_j)_{j \in \mathbb{N}}$ for $H$. 


Notice that for any \( x \in U \), we have for \( L \in L_{HS}(U, H) \)
\[
\|Lx\|_{H}^2 = \sum_{i=1}^{\infty} \langle Lx, e_i \rangle_{H}^2 = \sum_{i=1}^{\infty} \langle x, L^* e_i \rangle_{H}^2 \leq \|x\|_{H}^2 \sum_{i=1}^{\infty} \|L^* e_i\|_{H}^2 = \|x\|_{D}^2 \|L^*\|_{HS}^2,
\]
where \( (e_i)_{i=1}^{\infty} \) is an ONB in \( U \) and we applied the Cauchy-Schwarz inequality. Hence, \( \|L\|_{op} \leq \|L^*\|_{HS} = \|L\|_{HS} \). It can be seen directly from definition of the Hilbert-Schmidt norm that for \( L \in L_{HS}(V, H), K \in L_{HS}(U, V) \), it holds
\[
\|LK\|_{HS} \leq \|L\|_{op} \|K\|_{HS} \leq \|L\|_{HS} \|K\|_{HS},
\]
and the claimed algebraic structure of Hilbert-Schmidt operators follows. \( \square \)

**Proof of Lemma 2.7.** As \( Q \) is a symmetric positive definite trace class operator, there exists an ONB \( (e_k)_{k=1}^{\infty} \) in \( U \) being the eigenvectors of \( Q \). Further, recall by Fernique’s Theorem (see e.g. [Peszat & Zabczyk 2007, Thm. 3.31]) that all moments of \( \|\Delta W_t\|_U \) exists (in fact, exponential moments are finite up to a certain degree).

By Parseval’s identity,
\[
\|\Delta W_t\|_U^{2q} = \left( \sum_{k=1}^{\infty} \langle \Delta W_t, e_k \rangle_U^2 \right)^q.
\]
Obviously, \( \sum_{k=1}^{m} \langle \Delta W_t, e_k \rangle_U^2 \) is increasing in \( m \), and it follows from Tonelli’s Theorem that
\[
\mathbb{E}[\|\Delta W_t\|_U^{2q}] = \lim_{m \to \infty} \mathbb{E}\left[\left( \sum_{k=1}^{\infty} \langle \Delta W_t, e_k \rangle_U^2 \right)^q\right]
\]
We find the characteristic function of \( \sum_{k=1}^{m} \langle \Delta W_t, e_k \rangle_U^2 \): By independence of the sequence of random variables \( (\langle \Delta W_t, e_k \rangle_U)_{k \in \mathbb{N}} \) and the fact that \( \langle \Delta W_t, e_k \rangle_U \sim \mathcal{N}(0, \Delta \lambda_k) \), it follows for \( x \in \mathbb{R} \)
\[
\mathbb{E}\left[ e^{ix \Sigma_{k=1}^{m} \langle \Delta W_t, e_k \rangle_U^2} \right] = \prod_{k=1}^{m} \mathbb{E}\left[ e^{ix \langle \Delta W_t, e_k \rangle_U^2} \right] = \prod_{k=1}^{m} \mathbb{E}\left[ e^{ix \Delta \lambda_k Z^2} \right],
\]
with \( Z \sim \mathcal{N}(0, 1) \). But \( Z^2 \) is \( \chi^2(1) \)-distributed, and thus,
\[
\mathbb{E}\left[ e^{ix \Sigma_{k=1}^{m} \langle \Delta W_t, e_k \rangle_U^2} \right] = \prod_{k=1}^{m} (1 - 2ix\Delta \lambda_k)^{-1/2}
= \prod_{k=1}^{m} \exp\left(-\frac{1}{2} \ln(1 - 2ix\Delta \lambda_k)\right)
= \exp\left(-\frac{1}{2} \sum_{k=1}^{m} \ln(1 - 2ix\Delta \lambda_k)\right).
\]
As the \( q \)th moment of a random variable multiplied by \( i^q \) is given by the \( q \)th derivative of its characteristic function evaluated at zero, the first part of the Lemma follows.

For the second part, we first find that
\[
\Phi''_m(x) = -\Delta^2 \Phi(x) \left( \sum_{k=1}^{m} \frac{\lambda_k}{1 - 2ix\Delta \lambda_k} \right)^2 + 2 \sum_{k=1}^{m} \frac{\lambda_k^2}{(1 - 2ix\Delta \lambda_k)^2}
\]
and therefore,

$$\mathbb{E} \left[ \| \Delta W_t \|_{L^4}^4 \right] = (-i)^2 \lim_{m \to \infty} \Phi_m''(0) = \Delta^2 \left( \sum_{k=1}^{\infty} \lambda_k \right)^2 + 2\Delta^2 \sum_{k=1}^{\infty} \lambda_k^2.$$ 

The result follows. □

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