COARSE NON-AMENABILITY AND COVERS WITH SMALL EIGENVALUES

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ABSTRACT. Given a closed Riemannian manifold $M$ and a (virtual) epimorphism $\pi_1(M) \twoheadrightarrow \mathbb{F}_2$ of the fundamental group onto a free group of rank 2, we construct a tower of finite sheeted regular covers $\{M_n\}_{n=0}^\infty$ of $M$ such that $\lambda_1(M_n) \to 0$ as $n \to \infty$. This is the first example of such a tower which is not obtainable up to uniform quasi-isometry (or even up to uniform coarse equivalence) by the previously known methods where $\pi_1(M)$ is supposed to surject onto an amenable group.

1. INTRODUCTION

Let $M$ be a closed (that is, compact and without boundary) Riemannian manifold with fundamental group $\pi_1(M)$. A residually finite group $G$, a surjective homomorphism $\pi_1(M) \twoheadrightarrow G$ and a nested sequence of finite index normal subgroups of $G$ with trivial intersection gives rise to a tower of finite sheeted regular covers of $M$; conversely, every tower of finite sheeted regular covers arises in this manner. In summary, writing $G_0 = G$ and $M_0 = M$, we have:

\[
\begin{array}{c}
G_0 \triangleright G_1 \triangleright G_2 \triangleright \cdots, \quad \text{with} \quad \bigcap_{n=0}^\infty G_n = \{1\}, \\
\pi_1(M_n) \to G_n, \text{ and finite groups } \Gamma_n := G/G_n,
\end{array}
\]

\[
\begin{array}{c}
\vdots \\
\Gamma_2 \to M_2 \\
\Gamma_1 \to M_1 \\
\Gamma_0 \to M_0.
\end{array}
\]

In the context of spectral geometry of towers of covers one studies the asymptotic behavior of the first non-zero eigenvalues $\lambda_1(M_n)$ of the Laplacian, that is, of the Laplace-Beltrami operator of the individual Riemannian manifolds $M_n$. In particular, the following questions are classical:

(a) Does there exist a tower with $\lambda_1(M_n) \geq c > 0$ uniformly over $n$?

(b) Does there exist a tower with $\lambda_1(M_n) \to 0$ as $n \to \infty$?

In this note we are concerned with (b). The earliest positive result on this question is due to Randol, who studied the case of cyclic covers using the trace formula [9]. Subsequent results of

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Brooks [3, 2] and Burger [4] were obtained by relating the eigenvalues $\lambda_1(M_n)$ to combinatorial properties of the Cayley graphs of finite groups of deck transformations $\Gamma_n$. Similar results are due to Sunada [12].

In all cases, the method to build a tower of covers satisfying (b) rests on choosing an amenable group $G$ for the construction $(\ast)$. Our main result is that it is possible to obtain such a tower when $G$ is the free group on two generators. In the statement, $H^{(2)}$ denotes the subgroup of the discrete group $H$ generated by the squares of its elements.

**Theorem.** Let $M$ be a closed Riemannian manifold, whose fundamental group admits a virtual\(^1\) surjective homomorphism onto the free group of rank 2. Taking the nested sequence of subgroups in $(\ast)$ to be the sequence of iterated squares in the free group

$$(**): \quad G_0 = F_2, \quad G_1 = F_2^{(2)}, \quad G_2 = (F_2^{(2)})^{(2)}, \quad G_3 = ((F_2^{(2)})^{(2)})^{(2)}, \quad \ldots$$

we obtain a tower of covers of $M$ for which $\lambda_1(M_n) \to 0$ as $n \to \infty$. This tower is not obtainable up to uniform quasi-isometry (or even uniform coarse equivalence) by the construction $(\ast)$ with an amenable $G$.

Observe here that each $G_n$ is normal, even characteristic, in $F_2$.

The hypothesis of the theorem means that the fundamental group is large (the terminology is due to Gromov [5]). It applies to many hyperbolic manifolds [6], in particular, to a closed orientable surface of genus at least two – the fundamental group of such a manifold surjects onto $F_2$.

We conclude the introduction by remarking that in more modern terminology the classical problems above concerning the construction $(\ast)$ can be rephrased in terms of Property $\tau$: (a) asks for $G$ to have Property $\tau$ with respect to the family of subgroups $(G_n)_{n \geq 0}$, whereas (b) asks, after perhaps passing to a subsequence, for $G$ to not have Property $\tau$ with respect to the $(G_n)_{n \geq 0}$. This is explained in the work of Burger and Brooks cited above. Thus, the first assertion in the theorem is essentially equivalent to the assertion that $F_2$ does not have Property $\tau$ with respect to the subgroups appearing in $(**)$. For the definition and relevant facts about Property $\tau$ see [7].

## 2. Eigenvalues

A graph is a collection of vertices and edges. With a small number of exceptions, we permit neither multiple edges nor loops, so that an edge is uniquely determined by its incident vertices. Our graphs are unoriented. The Cheeger constant of a finite graph $\Gamma$ is

$$(2.1) \quad h(\Gamma) = \inf \frac{\#E(A, B)}{\min\{\#A, \#B\}},$$

where the infimum is taken over all decompositions of the vertex set of $\Gamma$ as a disjoint union $A \sqcup B$ and where, for such a decomposition, $E(A, B)$ denotes the set of edges with one incident vertex in $A$ and the other in $B$.

We shall make use of the following result of Brooks which, in the notation of $(\ast)$, relates the eigenvalues of the $M_n$ to the Cheeger constants of the Cayley graphs of the $\Gamma_n$ computed with

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\(^1\)A virtual homomorphism is a homomorphism of a finite-index subgroup.
respect to the canonical images of generators of $G$ and denoted, by an abuse of notation, again by $\Gamma_n$. We shall require only the forward implication, which is the content of [3, Lemma 1].

**Theorem** (Brooks). *In the notation of (*) we have $h(\Gamma_n) \to 0$ precisely when $\lambda_1(M_n) \to 0$. □*

Thus, the first statement in theorem of the introduction is reduced to the following:

1. **Proposition.** Let $G = F_2$ be the free group of rank 2. Consider the tower of iterated squares (***) and the corresponding quotients:
   \[
   \Gamma_0 = \{1\} \leftarrow \Gamma_1 = F_2/F_2^{(2)} \leftarrow \Gamma_2 = F_2/(F_2^{(2)})^{(2)} \leftarrow \ldots
   \]

   Abusing notation, view each $\Gamma_n$ as a Cayley graph with respect to the images of the standard free generators of $F_2$. Then we have $h(\Gamma_n) \to 0$ as $n \to \infty$.

In preparation for the proof we recall the construction of the $\mathbb{Z}/2$-homology cover of a finite graph $\Sigma$. Fix a maximal tree $T$ in $\Sigma$ and let $e_1, \ldots, e_r$ be the edges of $\Sigma$ not in $T$. The vertex and edge sets of the $\mathbb{Z}/2$-homology cover $\tilde{\Sigma}$ are

\[
\tilde{V} = V \times \bigoplus_1^r \mathbb{Z}/2, \quad \tilde{E} = E \times \bigoplus_1^r \mathbb{Z}/2,
\]

where $E$ and $V$ denote the vertex and edge sets of $\Sigma$. Let $e \in E$ and let $v, w \in V$ be the vertices incident with $e$. Consider the edge $(e, \alpha) \in \tilde{E}$. Incidence is defined in two cases:

\[
(e, \alpha) \text{ contains } \begin{cases} 
(v, \alpha) \text{ and } (w, \alpha), & \text{ when } e \text{ belongs to the maximal tree } T \\
(v, \alpha) \text{ and } (w, \alpha + \bar{e}_j), & \text{ when } e = e_j, \text{ for some } 1 \leq j \leq r.
\end{cases}
\]

Here $\bar{e}_j = (\ldots, 1, \ldots)$ is the standard basis vector with a single 1 in the $j$-the position and 0’s elsewhere. Strictly speaking, when defining incidence it is necessary to direct the edges $e_j$. It is quickly verified however that, while the edges are parameterized in a different manner, the underlying undirected graph is independent of the choice. We shall not dwell on this aspect.

**Remark.** The construction given here of the $\mathbb{Z}/2$-homology cover is a special case of the classical construction of a finite sheeted regular cover of $\Sigma$ corresponding to a given normal subgroup of finite index in $\pi_1(\Sigma)$, see, for example, [11, Ch. 2]. Indeed, with $e_1, \ldots, e_r$ as above, and after directing each $e_j$, we identify

\[
\pi_1(\Sigma) \cong F_r = \langle e_1, \ldots, e_r \rangle.
\]

Then the cover corresponding to the kernel of the epimorphism

\[
\pi_1(\Sigma) \cong F_r \twoheadrightarrow F_r/F_r^{(2)} \cong \bigoplus_1^r \mathbb{Z}/2 \quad \text{defined by } \ e_j \mapsto \bar{e}_j,
\]

is the $\mathbb{Z}/2$-homology cover.

2. **Lemma.** Let $\Sigma$ be a finite graph, with vertex set $V$; let $\tilde{\Sigma}$ be its $\mathbb{Z}/2$-homology cover. We have

\[
h(\tilde{\Sigma}) \leq \frac{2}{\#V}.
\]
We employ the notation introduced above for $\tilde{\Sigma}$. We shall exhibit a decomposition of the vertex set $\tilde{V} = A \sqcup B$ for which the quotient in (2.1) is bounded by $2/\#V$. Let
\[ A = \{ (v, \alpha) \in \tilde{V} : \alpha = (\ast, \ldots, \ast, 0) \}, \quad B = \{ (w, \beta) \in \tilde{V} : \beta = (\ast, \ldots, \ast, 1) \}, \]
each of which contains exactly $2^{r-1} \#V$ vertices. The edges in $\tilde{E}$ with one vertex in $A$ and the other in $B$ are exactly those of the form $(e, \gamma)$, for arbitrary $\gamma \in \oplus\mathbb{Z}/2$; thus $E(A, B)$ contains exactly $2^r$ edges.

**Proof of Proposition 1.** The Cayley graph $\Gamma_n$ is the $n$-th iterated $\mathbb{Z}/2$-homology cover of the “figure 8”. Since the number of vertices in $\Gamma_n$ tends to infinity, the result follows from the previous lemma.

**Remark.** A more detailed analysis gives information on the rate of the convergence $h(\Gamma_n) \to 0$. Indeed, let $V_n$ be the set of vertices and $E_n$ the set of edges of (the Cayley graph of) $\Gamma_n$. We have
\[ \frac{\#V_{n+1}}{\#V_n} = \frac{\#E_{n+1}}{\#E_n} = 2^{k \tau_0(\Gamma_n)}. \]
Now, the rank of the fundamental group $\pi_1(\Gamma_n)$ is the number of edges not belonging to a fixed maximal tree in $\Gamma_n$. Since $\#E_n = 2 \cdot \#V_n$, the rank of $\pi_1(\Gamma_n)$ is $\#V_n + 1$. Thus, we get the recursive formula
\[ \#V_{n+1} = \#V_n \cdot 2^{\#V_n + 1}. \]
In particular, $\#V_n$ grows faster than an iterated exponential and, according to the previous lemma, the Cheeger constant $h(\Gamma_{n+1})$ decays as the reciprocal of $\#V_n$.

**3. Non Uniform Coarse Equivalence**

We shall now show that the tower constructed in the previous section cannot be duplicated beginning with an amenable group in $(\ast)$, thus completing the proof of the theorem in the introduction.

Two families $(X_n)_{n \geq 0}$ and $(Y_n)_{n \geq 0}$ of metric spaces are uniformly quasi-isometric if there exist functions $f_n : X_n \to Y_n$ and constants $C \geq 1$ and $D \geq 0$ such that for all $x, y \in X_n$ and $z \in Y_n$, we have
\[ C^{-1}d(x, y) - D \leq d(f_n(x), f_n(y)) \leq Cd(x, y) + D, \]
\[ d(z, f_n(X_n)) \leq D. \]
The families $(X_n)_{n \geq 0}$ and $(Y_n)_{n \geq 0}$ are uniformly coarsely equivalent if there exist functions $f_n : X_n \to Y_n$ with the following two properties:
\[ \forall A \exists B \text{ such that } \forall n \forall x, y \in X_n \text{ we have } d(x, y) \leq A \Rightarrow d(f_n(x), f_n(y)) \leq B, \]
\[ \forall A \exists B \text{ such that } \forall n \forall x, y \in X_n \text{ we have } d(x, y) \geq B \Rightarrow d(f_n(x), f_n(y)) \geq A. \]
If two families are uniformly quasi-isometric then they are uniformly coarsely equivalent. Observe that these notions apply to individual spaces, which we regard as trivial families containing a single space. We say, for example, two spaces are coarsely equivalent.
3. **Proposition** (The Uniform Švarc-Milnor Lemma). Continue with the notation of \((\ast)\). Equip each \(\Gamma_n\) with the word metric associated to a fixed finite generating set for \(G\); equip each \(M_n\) with the path metric associated to its Riemannian structure. The families \((\Gamma_n)_{n \geq 0}\) and \((M_n)_{n \geq 0}\) are uniformly quasi-isometric.

**Remark.** In the situation of \((\ast)\) the group \(G\) is indeed finitely generated. Further, the statement in the proposition is independent of the choice of generators for \(G\).

**Proof of Proposition 3.** The result follows from the Švarc-Milnor Lemma [1, Prop. I.8.19], observing that the inherent quasi-isometry constants (see the proof of the Lemma) depend only on the diameter of a fundamental domain for the action. In detail, \(\Gamma_n\) is the group of deck transformations of the cover \(M_n\) of \(M\); whereas \(G\) is the group of deck transformations of the cover corresponding to the kernel of the surjective homomorphism \(\pi_1(M) \to G\). Further, the image in \(M_n\) of a bounded fundamental domain for the action of \(G\) is a fundamental domain for the action of \(\Gamma_n\), of no greater diameter. \(\square\)

Thus, the second statement in the theorem of the introduction is reduced to the following:

4. **Proposition.** Consider the tower of iterated squares \((\ast\ast)\) of the free group \(\mathbb{F}_2\) and the corresponding quotients

\[
\Gamma_0 = \{1\} \leftarrow \Gamma_1 = \mathbb{F}_2/\mathbb{F}_2(2) \leftarrow \Gamma_2 = \mathbb{F}_2/(\mathbb{F}_2(2))^2 \leftarrow \ldots
\]

Then the family \((\Gamma_n)_{n \geq 0}\) is not uniformly coarsely equivalent to any family of quotients of an amenable group.

Let \(G\) be a finitely generated discrete group, and let \(\ell\) be the word length associated to a fixed finite and symmetric set of generators. Of the many equivalent definitions of amenability we shall work with Reiter’s condition – \(G\) is amenable if for every \(\varepsilon > 0\) and for every \(R > 0\) there exists a finitely supported \(\xi \in \ell^1(G)\) such that \(\xi \geq 0\), \(\|\xi\| = 1\) and

\[
(3.1) \quad \ell(g) \leq R \Rightarrow \|g \cdot \xi - \xi\| < \varepsilon, \quad \xi_x(h) \neq 0 \Rightarrow d(x, y) \leq S.
\]

The analogy with amenability being clear, we say that a metric space having Property A is coarsely amenable whereas one not having Property A is coarsely non-amenable.
Finally, a metric space $X$ is the coarse union of its subspaces $X_n$ if $X = \bigsqcup X_n$ (disjoint union), and if $d(X_n, X_m) \to \infty$ as $n + m \to \infty$. If the $X_n$ are metric spaces each having finite diameter, then there exists a metric space $X$ which is the coarse union of (isometric copies of) the $X_n$. Further, any two such unions are coarsely equivalent. Moreover, if $Y$ is the coarse union of the $Y_n$ then $X$ and $Y$ are coarsely equivalent when the $X_n$ and $Y_n$ are uniformly coarsely equivalent.

We require the following slight generalization of [10, Prop. 11.39]. We include a proof which is both different from other proofs in the literature and convenient for our result.

5. Proposition. Let $G$ be a finitely generated amenable group. Every quotient of $G$ is amenable; the coarse union of any family of finite quotients of $G$ is coarsely amenable.

Proof. Let $H$ be a quotient of $G$ and identify $H$ with a set of cosets $\{gK\}$. Fix a finite and symmetric set of generators for $G$ and equip $G$ with the associated word length; equip $H$ with the word length associated to the induced generators. With these conventions

$$\ell_H(x) \leq R \iff \exists g \in x \text{ such that } \ell_G(g) \leq R$$

and, in particular, the map $G \to H$ is contractive. Given $\varepsilon > 0$ and $R > 0$, obtain $\xi \in \ell^1(G)$ as in (3.1). Define

$$(3.2) \quad \eta(x) = \sum_{g \in x} \xi(g),$$

so that $\eta \geq 0$ and $\|\eta\| = 1$. Further, when $z \in H$ has length at most $R$ we obtain $g \in G$ of length at most $R$ such that $z = gK$. We then calculate

$$\|z \cdot \eta - \eta\| = \sum_{x \in H} |\eta(g^{-1}x) - \eta(x)| \leq \sum_{x \in H} \sum_{h \in x} |\xi(g^{-1}h) - \xi(h)| = \|g \cdot \xi - \xi\| < \varepsilon.$$ 

We conclude that $H$ is amenable.

When dealing with a coarse union the essential observation is that, in the previous argument, if $\xi$ is supported on the elements of length at most $S$ then the same is true of $\eta$. Thus, let $\{H_n\}$ be a family of finite quotients of $G$, each equipped with a length function as above, and let $X$ be a coarse union of the $H_n$. Given $\varepsilon > 0$ and $R > 0$ proceed as above – obtain a Reiter function $\xi$ for $G$ and define $\eta_n$ as in (3.2). For $x \in X$ define

$$\xi_x = \begin{cases} 
\chi_N, & x \in H_n, n \leq N \\
\chi_n \cdot \eta_n, & x \in H_n, n > N,
\end{cases}$$

where $N$ is chosen large enough so that for $n > N$ the distance between $H_n$ and any other $H_m$ is at least $R$ and where $\chi_N$ is the normalized characteristic function of $H_1 \cup \cdots \cup H_N$. Finally, choose $S$ larger than the diameter of $H_1 \cup \cdots \cup H_N$ and large enough so that $\xi$ is supported on elements of length at most $S$ in $G$. The required properties are easily verified. $\Box$

Proof of Proposition 4. The iterated squares are proper characteristic subgroups of the free group, hence, by Levi’s theorem [8, Ch.I, Prop. 3.3], they have trivial intersection, $\cap \{\Pi_2^{(2)} \cdots (2)\} = \{1\}$. Thus, the coarse union of the metric spaces $\Gamma_n$ is an example of a coarsely non-amenable box space. See
We conclude with two remarks. First, we have used a very crude invariant from coarse geometry to distinguish towers constructed from the sequence of iterated squares (**) from those constructed beginning with an amenable group in (*) – the former are coarsely non-amenable while the latter are coarsely amenable. More refined invariants would be needed to establish the existence of coarsely inequivalent towers constructed as in (•) from a given non-amenable group. Second, our construction involving the iterated squares (**) is particular to the free group. It would be interesting to remove the hypothesis of ‘largeness’ from our theorem.

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