Stationary probability vectors of higher-order two-dimensional transition probability tensors

Zheng-Hai Huang∗ Liqun Qi†

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Abstract

In this paper we investigate stationary probability vectors of higher-order two-dimensional symmetric transition probability tensors. We show that there are two special symmetric transition probability tensors of order $m$ dimension 2, which have and only have two stationary probability vectors; and any other symmetric transition probability tensor of order $m$ dimension 2 has a unique stationary probability vector. As a byproduct, we obtain that any symmetric transition probability tensor of order $m$ dimension 2 has a unique positive stationary probability vector; and that any symmetric irreducible transition probability tensor of order $m$ dimension 2 has a unique stationary probability vector.

Key words: Transition probability tensor; higher-order Markov chain; stationary probability vector; eigenvalue of tensor.

Mathematics Subject Classifications(2000): 15A18; 15A69; 65F15; 60J10; 60J22.

∗School of Mathematics, Tianjin University, Tianjin 300354, P.R. China (huangzhenghai@tju.edu.cn). This author was supported by the National Natural Science Foundation of China (Grant No. 11431002).

†Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong (maqil@polyu.edu.hk). This author’s work was partially supported by the Hong Kong Research Grant Council (Grant No. PolyU 15302114, 15300715, 15301716 and 15300717).
1 Introduction

It is well known that higher-order Markov chains have various applications in many areas [1–6]. An \((m - 1)\)-order \(n\)-dimensional Markov chain is basically characterized by its associated nonnegative tensor \(P\) which is an \(m\)-order \(n\)-dimensional tensor with entries \(p_{i_1i_2\ldots i_m} \geq 0\) for all \(i_j \in \{1, 2, \ldots, n\}\) and \(j \in \{1, 2, \ldots, m\}\) satisfying

\[
0 \leq p_{i_1i_2\ldots i_m} = \text{Prob}(X_{t+1} = i_1|X_t = i_2, \ldots, X_{t-m+2} = i_m) \leq 1 \tag{1.1}
\]

where \(\{X_t : t = 0, 1, \ldots\}\) represents the stochastic process that takes on \(n\) states \(\{1, 2, \ldots, n\}\), and for any \(i_2, \ldots, i_m \in \{1, 2, \ldots, n\}\),

\[
\sum_{i_1=1}^{n} p_{i_1i_2\ldots i_m} = 1. \tag{1.2}
\]

We will use \(\mathbb{T}_{m,n}\) to denote the set of all \(m\)-order \(n\)-dimensional real tensors. For any \(P = (p_{i_1\ldots i_m}) \in \mathbb{T}_{m,n}\), if the entries \(p_{i_1i_2\ldots i_m}\) are invariant under any permutation of their indices, then \(P\) is called a symmetric tensor. A tensor \(P \in \mathbb{T}_{m,n}\) is called a transition probability tensor if it satisfies (1.1) and (1.2). A vector

\[x^* \in \left\{ x \in \mathbb{R}^n : x_i \geq 0 \text{ for all } i \in \{1, 2, \ldots, n\} \text{ and } \sum_{i=1}^{n} x_i = 1 \right\}\]

is called a stationary probability vector of \(P \in \mathbb{T}_{m,n}\) if

\[\sum_{i_2,\ldots,i_m=1}^{n} p_{i_2\ldots i_m} x^*_{i_2} \cdots x^*_{i_m} = x^*_i\]

holds for all \(i \in \{1, 2, \ldots, n\}\), which is just a \(Z_1\)-eigenvector associated with the \(Z_1\)-eigenvalue 1 [7]. It is also closely related to the \(Z\)-eigenvector of a tensor [8,9].

Transition probability tensors and the associated stationary probability vectors are important issues in studies of higher-order Markov chains [7,10–15]. In particular, the uniqueness of the stationary probability vector of transition probability tensors has attracted a lot of interest. Li and Ng [12] proposed some conditions which ensure the uniqueness of the stationary probability vector of transition probability tensors and the linear convergence of the proposed iterative method; Hu and Qi [11] studied the uniqueness of the stationary probability vector of the third order \(n\)-dimensional positive transition probability tensor, and they proved that an irreducible transition probability tensor of order 3 dimension 2 has a unique stationary probability vector; and Chang and Zhang [7] investigated sufficient conditions for transition probability tensors to ensure the uniqueness of the stationary probability vector by using three different methods: contraction mappings, monotone operators, and the Brouwer index of fixed points.
More recently, Culp, Pearson and Zhang [15] investigated symmetric irreducible transition probability tensors of order 4 dimension 2 and of order 3 dimension 3, and showed that a symmetric irreducible transition probability tensor in these orders and dimensions has a unique stationary probability vector.

In this paper, we give a full characterization on the stationary probability vectors of \( m \)-order 2-dimensional symmetric transition probability tensors. We show that there are two special symmetric transition probability tensors of order \( m \) dimension 2, which have and only have two stationary probability vectors, where one is \((\frac{1}{2}, \frac{1}{2})^\top\), and the other one is \((1, 0)^\top\) or \((0, 1)^\top\). In particular, we show that if the concerned symmetric transition probability tensor of order \( m \) dimension 2 is not one of the above two tensors, then it has a unique stationary probability vector, which is \((\frac{1}{2}, \frac{1}{2})^\top\). As a byproduct, we obtain that any symmetric irreducible transition probability tensor of order \( m \) dimension 2 has a unique stationary probability vector. When \( m = 4 \), such a result was obtained by Culp, Pearson and Zhang in [15].

Throughout this paper, we assume that \( m \geq 3 \) is an integer number.

2 Main results

Let \( \mathcal{P} = (p_{i_1i_2...i_m}) \in T_{m, 2} \) be a transition probability tensor. Then \( z = (z_1, z_2)^\top \in \mathbb{R}^2 \) is a stationary probability vector of \( \mathcal{P} \) if and only if \( z_1, z_2 \geq 0 \), \( z_1 + z_2 = 1 \), and

\[
\begin{align*}
f_1(z_1, z_2) := \sum_{i_2, ..., i_m = 1}^2 p_{i_1i_2...i_m} z_{i_2} \cdots z_{i_m} = z_1, \\
f_2(z_1, z_2) := \sum_{i_2, ..., i_m = 1}^2 p_{i_1i_2...i_m} z_{i_2} \cdots z_{i_m} = z_2.
\end{align*}
\] (2.1) (2.2)

In the following, we denote \( x := z_1 \) and \( y := z_2 \), and

\[
\Delta := \{(x, y)^\top \in \mathbb{R}^2 : x \geq 0, y \geq 0, x + y = 1\}.
\] (2.3)

Then, \((x, y)^\top \in \mathbb{R}^2\) is a stationary probability vector of \( \mathcal{P} \) if and only if \((x, y)^\top \in \Delta\) and

\[
f_1(x, y) = x \quad \text{and} \quad f_2(x, y) = y.
\] (2.4)

Throughout this paper, we denote \( a := p_{11...11} \) and \( b := p_{21...11} \).

**Lemma 2.1** Suppose that \( \mathcal{P} = (p_{i_1i_2...i_m}) \in T_{m, 2} \) is a symmetric transition probability tensor. Let \( f_1(\cdot, \cdot) \), \( f_2(\cdot, \cdot) \), and \( \Delta \) be defined by (2.1), (2.2), and (2.3), respectively. Then, for any \((x, y)^\top \in \Delta\), we have the following results.
(i) If $m$ is an even number, then

\[ f_1(x, y) = ax^{m-1} + bC_{m-1}^1 x^{m-2} y + aC_{m-1}^2 x^{m-3} y^2 + bC_{m-1}^3 x^{m-4} y^3 + \cdots + ax^3 y^{m-4} + bC_{m-1}^3 x^2 y^{m-3} + aC_{m-1}^2 x y^{m-2} + by^{m-1}, \]

\[ f_2(x, y) = bx^{m-1} + aC_{m-1}^1 x^{m-2} y + bC_{m-1}^2 x^{m-3} y^2 + aC_{m-1}^3 x^{m-4} y^3 + \cdots + bx^3 y^{m-4} + aC_{m-1}^3 x^2 y^{m-3} + bC_{m-1}^2 x y^{m-2} + ay^{m-1}. \]

(ii) If $m$ is an odd number, then

\[ f_1(x, y) = ax^{m-1} + bC_{m-1}^1 x^{m-2} y + aC_{m-1}^2 x^{m-3} y^2 + bC_{m-1}^3 x^{m-4} y^3 + \cdots + ax^4 y^{m-5} + bx^3 y^{m-4} + aC_{m-1}^3 x^2 y^{m-3} + bC_{m-1}^2 x y^{m-2} + ay^{m-1}, \]

\[ f_2(x, y) = bx^{m-1} + aC_{m-1}^1 x^{m-2} y + bC_{m-1}^2 x^{m-3} y^2 + aC_{m-1}^3 x^{m-4} y^3 + \cdots + bx^4 y^{m-5} + ax^3 y^{m-4} + bC_{m-1}^3 x^2 y^{m-3} + aC_{m-1}^2 x y^{m-2} + ay^{m-1}. \]

**Proof.** Since $\mathcal{P}$ is a symmetric tensor, the above equalities can be rewritten as

\[ f_1(x, y) = p_{1\cdots 11} x^{m-1} + p_{1\cdots 12} C_{m-1}^1 x^{m-2} y + p_{1\cdots 122} C_{m-1}^2 x^{m-3} y^2 + p_{1\cdots 122} C_{m-1}^3 x^{m-4} y^3 + \cdots + p_{11112\cdots 2} x y^{m-2} + p_{112\cdots 2} y^{m-1}, \]

\[ f_2(x, y) = p_{21\cdots 11} x^{m-1} + p_{21\cdots 12} C_{m-1}^1 x^{m-2} y + p_{21\cdots 122} C_{m-1}^2 x^{m-3} y^2 + p_{21\cdots 122} C_{m-1}^3 x^{m-4} y^3 + \cdots + p_{21112\cdots 2} x^2 y^{m-3} + p_{212\cdots 2} y^{m-1} = y. \]

Suppose that $m$ is an even number. Since $\mathcal{P}$ is a transition probability tensor, it follows that

\[ p_{1\cdots 11} + p_{1\cdots 12} + p_{1\cdots 122} = 1, \quad p_{1\cdots 122} + p_{1\cdots 1222} = 1, \quad \ldots, \]

\[ p_{11112\cdots 2} + p_{112\cdots 2} = 1, \quad p_{1112\cdots 2} + p_{112\cdots 2} = 1, \quad p_{11\cdots 2} + p_{12\cdots 2} = 1, \]

\[ p_{21\cdots 11} + p_{21\cdots 12} + p_{21\cdots 122} = 1, \quad p_{21\cdots 122} + p_{21\cdots 1222} = 1, \quad \ldots, \]

\[ p_{21112\cdots 2} + p_{2112\cdots 2} = 1, \quad p_{2112\cdots 2} + p_{212\cdots 2} = 1, \quad p_{212\cdots 2} + p_{22\cdots 2} = 1, \]

and hence,

\[ p_{1\cdots 11} = p_{1\cdots 122} = \cdots = p_{11112\cdots 2} = p_{112\cdots 2} = a, \]

\[ p_{1\cdots 12} = p_{1\cdots 122} = \cdots = p_{1112\cdots 2} = p_{12\cdots 2} = b, \]

\[ p_{21\cdots 11} = p_{21\cdots 122} = \cdots = p_{21112\cdots 2} = p_{212\cdots 2} = b, \]

\[ p_{21\cdots 12} = p_{21\cdots 122} = \cdots = p_{2112\cdots 2} = p_{22\cdots 2} = a. \]

Thus, two equalities given in (i) hold from (2.5).

Suppose that $m$ is an odd number. Then two equalities given in (ii) can be showed similarly. We omit them here. \(\square\)
Denote 
\[ g_1(x) := f_1(x, 1-x) \quad \text{and} \quad g_2(y) := f_2(1-y, y). \] (2.6)

Suppose that \( \mathcal{P} = (p_{i_1 i_2 \cdots i_m}) \in \mathbb{T}_{m,2} \) is a symmetric transition probability tensor. Then, it is obvious that (2.4) has a (unique) solution \( (x^*, y^*) \top \in \Delta \) if and only if \( g_1(x) - x = 0 \) has a (unique) solution \( x^* \in [0,1] \), and \( 1 - x^* \) solves \( g_2(y) - y = 0 \). Thus, (2.4) can be investigated by considering \( g_1(x) - x = 0 \) and \( g_2(y) - y = 0 \). In order to give an appropriate reformulation of the function \( g_1(\cdot) \), we need to use the following combinatorial identity.

**Lemma 2.2** If \( n \) is an even number, then
\[
C_n^0 + C_n^2 + \cdots + C_n^m = C_n^1 + C_n^3 + \cdots + C_n^{m-1} = 2^{n-1};
\]
and if \( n \) is an odd number, then
\[
C_n^0 + C_n^2 + \cdots + C_n^{m-1} = C_n^1 + C_n^3 + \cdots + C_n^m = 2^{n-1}.
\]

Now, we derive a simple expression of the function \( g_1(\cdot) \), which is a key to our discussions later.

**Lemma 2.3** Suppose that \( \mathcal{P} = (p_{i_1 i_2 \cdots i_m}) \in \mathbb{T}_{m,2} \) is a symmetric transition probability tensor, and the function \( g_1(\cdot) \) is defined by (2.6). Then, the following results hold.

(i) If \( m \) is an even number, then
\[
g_1(x) = \frac{a - b}{2} (2x - 1)^{m-1} + b.
\]

(ii) If \( m \) is an odd number, then
\[
g_1(x) = \frac{a - b}{2} (2x - 1)^{m-1} + a.
\]

**Proof.** (i) Suppose that \( m \) is an even number. In this case, we first show that
\[
g_1(x) = (a - b)(m - 1) \sum_{t \in \{1,3,\ldots,m-2\}} \left\{ \frac{2^{m-t-1}}{m-t} C_{m-2}^{t-1} x^{m-t} - \frac{2^{m-t-2}}{m-t-1} C_{m-2}^{t} x^{m-t-1} \right\} + b.
\] (2.7)
Since $m$ is an even number, it follows from Lemma 2.1 that
\[
\begin{align*}
g_1(x) &= ax^{m-1} + bC_{m-1}^1x^{m-2}(1-x) + aC_{m-1}^2x^{m-3}(1-x)^2 \\
&
+ bC_{m-1}^3x^{m-4}(1-x)^3 + \cdots + ax^{3}(1-x)^m-4 \\
&
+ bC_{m-1}^{m-3}x^{m-5}(1-x)^{m-3} + aC_{m-1}^{m-2}x(1-x)^{m-2} + b(1-x)^{m-1}.
\end{align*}
\] (2.8)

Let
\[
g_1(x) = \sum_{t=1}^{m} \alpha_{m-t}x^{m-t},
\]
then it follows from (2.8) that $\alpha_0 = b$, and

- if $t \in \{1, 3, \ldots, m - 1\}$, then
  \[
  \alpha_{m-t} = \sum_{s \in \{0, 2, \ldots, m-t-1\}} \left\{ C_{m-1}^{t-1+s}C_{t-s}^{s}a - C_{m-1}^{t+s}C_{t-s+1}^{s+1}b \right\};
  \]
- if $t \in \{2, 4, \ldots, m - 2\}$, then
  \[
  \alpha_{m-t} = \sum_{s \in \{2, 4, \ldots, m-t-2\}} \left\{ C_{m-1}^{t+s}C_{t-s}^{s}b - C_{m-1}^{t+s+1}C_{t-s+1}^{s+1}a \right\} + C_{m-1}^{m-1}C_{m-1}^{m-t-1}b.
  \]

Since
\[
C_{m-1}^{t-1+s}C_{t-1+s}^{s} = \frac{(m-1)!}{(t-1+s)!(m-t-s)!} \times \frac{(t-1+s)!}{s!(t-1)!}
= \frac{(m-1)!}{(t-1)!} \times \frac{1}{s!(m-t-s)!}
= \frac{m-1}{m-t} \times \frac{(m-2)!}{(t-1)!(m-t-1)!} \times \frac{(m-t)!}{s!(m-t-s)!}
= \frac{m-1}{m-t} C_{m-2}^{t-1}C_{m-t}^{s},
\]
it follows that for any $t \in \{1, 3, \ldots, m - 1\},$
\[
\alpha_{m-t} = \frac{m-1}{m-t} C_{m-2}^{t-1} \sum_{s \in \{0, 2, \ldots, m-t-1\}} \left\{ C_{m-t}^{s}a - C_{m-t}^{s+1}b \right\}
= \frac{m-1}{m-t} C_{m-2}^{t-1} \left[ 2^{m-t-1}a - 2^{m-t-1}b \right]
= 2^{m-t-1}(a-b) \frac{m-1}{m-t} C_{m-2}^{t-1},
\]
where the first equality holds by Lemma 2.2 and for any $t \in \{2, 4, \ldots, m - 2\},$
\[
\alpha_{m-t} = 2^{m-t-1}(b-a) \frac{m-1}{m-t} C_{m-2}^{t-1}.
\]
Thus, (2.7) holds.

Furthermore, it is easy to see that for any $t \in \{1, 2, \ldots, m - 1\}$,

\[
\frac{m - 1}{m - t} C_{m-2}^{t-1} = C_{m-1}^{t-1},
\]

and hence,

\[
g_1(x) = (a - b) \sum_{t \in \{1, 3, \ldots, m - 2\}} \left\{ 2^{-1} C_{m-1}^{t-1} (2x)^m - 2^{-1} C_{m-1}^{t} (2x)^m - 1 \right\} + b
\]

\[
= \frac{a - b}{2} \sum_{t=1}^{m-2} \left\{ C_{m-1}^{t-1} (2x)^m - (1)^{t-1} \right\} + b
\]

\[
= \frac{a - b}{2} (2x - 1)^m + b,
\]

where the last equality follows from the binomial theorem. Thus, we complete the proof of the result in (i).

(ii) Suppose that $m$ is an odd number. In this case, it follows from Lemma 2.1 that

\[
g_1(x) := ax^{m-1} + bC_{m-1}^1 x^{m-2} (1 - x) + aC_{m-1}^2 x^{m-3} (1 - x)^2 + bC_{m-1}^3 x^{m-4} (1 - x)^3 + \cdots + ax^4 (1 - x)^{m-5} + b x^3 (1 - x)^{m-4}
\]

\[
+ aC_{m-1}^{m-3} x^2 (1 - x)^{m-3} + bC_{m-1}^{m-2} x (1 - x)^{m-2} + a(1 - x)^{m-1}.
\]

Let $g_1(x) = \sum_{t=1}^{m} \beta_{m-t} x^{m-t}$. Then, it follows from (2.9) that $\beta_0 = a$, and

- if $t \in \{1, 3, \ldots, m - 2\}$, then
  \[
  \beta_{m-t} = \sum_{s \in \{0, 2, \ldots, m-t-1\}} \left\{ C_{m-1}^{t+s} C_{t+1+s} C_{m-1}^{t+s+1} a - C_{m-1}^{t+s} C_{t+1+s} C_{m-1}^{t+s+1} \right\} + C_{m-1}^{m-t-1} C_{m-1}^{m-t} a;
  \]

- if $t \in \{2, 4, \ldots, m - 1\}$, then
  \[
  \beta_{m-t} = \sum_{s \in \{2, 4, \ldots, m-t-2\}} \left\{ C_{m-1}^{t+s} C_{t+s} C_{m-1}^{t+s+1} b - C_{m-1}^{t+s} C_{t+s} C_{m-1}^{t+s+1} \right\}.
  \]

Thus, similar to (i), we can obtain that

\[
g_1(x) = (a - b)(m - 1) \sum_{t \in \{1, 3, \ldots, m - 2\}} \left\{ \frac{2^{m-t-1}}{m - t} C_{m-2}^{t-1} x^{m-t} - \frac{2^{m-t-2}}{m - t - 1} C_{m-2}^{t} x^{m-t-1} \right\} + a,
\]
and furthermore,
\[
g_1(x) = (a - b) \sum_{t \in \{1, 3, \ldots, m-2\}} \left\{ 2^{-1} C_{m-1}^{t-1}(2x)^{m-t} - 2^{-1} C_{m-2}^{t}(2x)^{m-t} \right\} + a
\]
\[
= \frac{a - b}{2} \sum_{t=1}^{m-2} \left\{ C_{m-1}^{t-1}(2x)^{m-t}(-1)^{t-1} \right\} + a
\]
\[
= \frac{a - b}{2} (2x - 1)^{m-1} + a
\]
which implies that the result in (ii) holds. \qed

**Lemma 2.4** Suppose that \( P = (p_{112 \ldots i_m}) \in \mathbb{T}_{m,2} \) is a symmetric transition probability tensor, and the function \( g_1(\cdot) \) is defined by (2.6). Then,
\[
g'_1(x) = (a - b)(m - 1)(2x - 1)^{m-2}.
\]

**Proof.** The desired result follows from Lemma 2.3 directly. \qed

The following result is a special case of the one in [15, Theorem 3.1].

**Lemma 2.5** Suppose that \( P = (p_{112 \ldots i_m}) \in \mathbb{T}_{m,2} \) is a symmetric transition probability tensor. Then, \( \left( \frac{1}{2}, \frac{1}{2} \right)^\top \) is a stationary probability vector of \( P \).

We now give our main results in this paper.

**Theorem 2.1** Suppose that \( P = (p_{112 \ldots i_m}) \in \mathbb{T}_{m,2} \) is a symmetric transition probability tensor. If \( p_{11 \ldots 1} = 1 \) and \( p_{21 \ldots 1} = 0 \), then the corresponding tensor \( P \) is denoted by \( P_1 \); and if \( p_{1\ldots 1} = 0 \) and \( p_{21 \ldots 1} = 1 \), then the corresponding tensor \( P \) is denoted by \( P_2 \). Then, we have the following results.

(i) The transition probability tensor \( P_1 \) has and only has two stationary probability vectors: \( \left( \frac{1}{2}, \frac{1}{2} \right)^\top \) and \( (0, 1)^\top \).

(ii) The transition probability tensor \( P_2 \) has and only has two stationary probability vectors: \( \left( \frac{1}{2}, \frac{1}{2} \right)^\top \) and \( (1, 0)^\top \).

(iii) If \( P \neq P_1 \) and \( P \neq P_2 \), then \( P \) has a unique stationary probability vector: \( \left( \frac{1}{2}, \frac{1}{2} \right)^\top \).

**Proof.** Let
\[
h(x) := g_1(x) - x. \quad (2.10)
\]
Then, it follows from Lemma 2.4 that
\[ h'(x) := g'_1(x) - 1 = (a - b)(m - 1)(2x - 1)^{m-2} - 1. \] (2.11)

We divide the proof into the following three parts.

**Part 1.** Suppose that \( a = b \). By (2.11) we have that for any \( x \in [0, 1] \),
\[ h'(x) = -1 < 0, \]
which implies that the function \( h(\cdot) \) defined by (2.10) is strictly decreasing on \([0, 1]\). This, together with \( h(\frac{1}{2}) = 0 \) by Lemma 2.1, implies that \( \frac{1}{2} \) is the unique solution of \( h(x) = 0 \) on \([0, 1]\). Furthermore, by Lemma 2.1 it follows that \( (\frac{1}{2}, \frac{1}{2})^\top \) is the unique solution of (2.4) on \([0, 1]\), i.e., \( (\frac{1}{2}, \frac{1}{2})^\top \) is the unique stationary probability vector of the transition probability tensor \( \mathcal{P} \).

**Part 2.** Suppose that \( 1 \geq a > b \geq 0 \). Let \( h'(x^*) = 0 \), then by (2.11), we have
\[ x^* = \frac{1}{2} \left( 1 + \frac{1}{m-2/(a-b)(m-1)} \right). \] (2.12)

We consider the following three cases.

(a) If \((a-b)(m-1) = 1\), then by (2.12), we have \( x^* = 1 \). Since \( m \geq 3 \), it follows that \( 1 > a > b > 0 \). Since \( g_1(1) = a \) by (2.8) and (2.9), it follows by (2.11) that \( h(1) = a - 1 < 0 \), i.e., \( 1 \) is not a solution of \( h(x) = 0 \) on \([0, 1]\). Moreover, by (2.11) it follows that the function \( h(\cdot) \) is strictly decreasing on \([0, 1]\). This, together with \( h(\frac{1}{2}) = 0 \) implies that \( \frac{1}{2} \) is the unique solution of \( h(x) = 0 \) on \([0, 1]\). Thus, \( \frac{1}{2} \) is the unique solution of \( h(x) = 0 \) on \([0, 1]\). This, together with Lemma 2.5, implies that \( (\frac{1}{2}, \frac{1}{2})^\top \) is the unique solution of (2.4) on \([0, 1]\), i.e., \( (\frac{1}{2}, \frac{1}{2})^\top \) is the unique stationary probability vector of the transition probability tensor \( \mathcal{P} \).

(b) If \((a-b)(m-1) < 1\), then by (2.12), we have \( x^* > 1 \). Thus, by (2.11) it follows that the function \( h(\cdot) \) is strictly decreasing on \([0, 1]\). This, together with \( h(\frac{1}{2}) = 0 \), implies that \( \frac{1}{2} \) is the unique solution of \( h(x) = 0 \) on \([0, 1]\). Furthermore, by Lemma 2.5 it follows that \( (\frac{1}{2}, \frac{1}{2})^\top \) is the unique solution of (2.4) on \([0, 1]\), i.e., \( (\frac{1}{2}, \frac{1}{2})^\top \) is the unique stationary probability vector of the transition probability tensor \( \mathcal{P} \).

(c) If \((a-b)(m-1) > 1\), then by (2.12), we have \( \frac{1}{2} < x^* \leq 1 \). Furthermore, by (2.11) it follows that \( h'(x) < 0 \) when \( x \in [0, x^*) \) and \( h'(x) > 0 \) when \( x \in (x^*, 1] \). On one hand, since the function \( h(\cdot) \) is strictly decreasing on \([0, x^*) \) and \( h(\frac{1}{2}) = 0 \) with \( \frac{1}{2} \in [0, x^*) \), it follows that \( \frac{1}{2} \) is the unique solution of \( h(x) = 0 \) on \([0, x^*) \); and in the meantime, we have \( h(x^*) < h(\frac{1}{2}) = 0 \). On the other hand, by (2.8) and (2.9), we have \( g_1(1) = a \), and hence, by (2.10), \( h(1) = a - 1 \leq 0 \). If \( a = 1 \), then since
the function \( h(\cdot) \) is strictly increasing on \((x^*, 1]\) with \( h(1) = 0 \), it follows that 1 is the unique solution of \( h(x) = 0 \) on \((x^*, 1]\); and if \( a < 1 \), then since the function \( h(\cdot) \) is strictly increasing on \((x^*, 1]\) with \( h(1) < 0 \) and \( h(x^*) < 0 \), it follows that \( h(x) = 0 \) has no solution on \((x^*, 1]\). Thus, if \( \mathcal{P} = \mathcal{P}_2 \), then it has and only has two stationary probability vectors: \( (\frac{1}{2}, \frac{1}{2})^\top \) and \( (1, 0)^\top \); otherwise, it has a unique stationary probability vector: \( (\frac{1}{2}, \frac{1}{2})^\top \).

**Part 3.** Suppose that \( 1 \geq b > a \geq 0 \). If \( m \) is an even number, then for any \( x \in [0, 1] \),

\[
h'(x) = (a - b)(m - 1)(2x - 1)^{m-2} - 1 < 0,
\]

and hence, the function \( h(\cdot) \) is strictly decreasing on \([0, 1]\). In this case, \( (\frac{1}{2}, \frac{1}{2})^\top \) is the unique solution of \( (2.4) \) on \([0, 1]\), i.e., \( (\frac{1}{2}, \frac{1}{2})^\top \) is the unique stationary probability vector of the transition probability tensor \( \mathcal{P} \).

In the following, we assume that \( m \) is an odd number. Let \( h'(x^*) = 0 \), then by \( (2.11) \), we have

\[
x^* = \frac{1}{2} \left( 1 - \frac{1}{\sqrt{(b-a)(m-1)}} \right). \tag{2.13}
\]

We consider the following three cases.

(a) If \((b-a)(m-1) = 1\), then by \( (2.13) \), we have \( x^* = 0 \). Since \( m \geq 3 \), it follows that \( 1 > b > a > 0 \). Since \( m \) is an odd number, we have \( g_1(0) = a \) by \( (2.9) \), and hence, it follows by \( (2.10) \) that \( h(0) = a - 0 > 0 \), i.e., \( 0 \) is not a solution of \( h(x) = 0 \). Moreover, by \( (2.11) \) it follows that \( h'(x) < 0 \) on \((0, 1]\), which implies that the function \( h(\cdot) \) is strictly decreasing on \((0, 1]\), and hence, \( \frac{1}{2} \) is a unique solution of \( h(x) = 0 \) on \((0, 1]\). So, \( (\frac{1}{2}, \frac{1}{2})^\top \) is the unique stationary probability vector of the transition probability tensor \( \mathcal{P} \).

(b) If \((b-a)(m-1) < 1\), then by \( (2.13) \), we have \( x^* < 0 \); and by \( (2.11) \), we have \( h'(x) < 0 \) for any \( x \in [0, 1] \). Thus, the function \( h(\cdot) \) is strictly decreasing on \([0, 1]\). This, together with \( h(\frac{1}{2}) = 0 \), implies that \( \frac{1}{2} \) is the unique solution of \( h(x) = 0 \) on \([0, 1]\). Thus, \( (\frac{1}{2}, \frac{1}{2})^\top \) is the unique stationary probability vector of the transition probability tensor \( \mathcal{P} \).

(c) If \((b-a)(m-1) > 1\), then it is easy to see from \( (2.13) \) that \( 0 < x^* < \frac{1}{2} \). Furthermore, by \( (2.11) \), we have \( h'(x) > 0 \) when \( x \in [0, x^*) \) and \( h'(x) < 0 \) when \( x \in (x^*, 1] \). On one hand, since the function \( h(\cdot) \) is strictly decreasing on \((x^*, 1]\) and \( h(\frac{1}{2}) = 0 \) with \( \frac{1}{2} \in (x^*, 1]\), it follows that \( \frac{1}{2} \) is the unique solution of \( h(x) = 0 \) on \((x^*, 1]\). Meantime, we have \( h(x^*) > h(\frac{1}{2}) = 0 \). On the other hand, since \( m \) is an odd number, it follows by \( (2.9) \) that \( g_1(0) = a \), and hence, by \( (2.10) \), \( h(0) = a \geq 0 \). If \( a = 0 \), then \( b = 1 \), and hence, \( \mathcal{P} = \mathcal{P}_2 \). In this case, \( 0 \) is a solution of \( h(x) = 0 \). Otherwise, since
The function \( h(\cdot) \) is strictly decreasing on \((0, x^*)\) with \( h(0) > 0 \) and \( h(x^*) > 0 \), it follows that \( h(x) = 0 \) has no solution on \((0, x^*)\). Thus, if \( \mathcal{P} = \mathcal{P}_2 \), then it has and only has two stationary probability vectors: \((\frac{1}{2}, \frac{1}{2})^T\) and \((1, 0)^T\); otherwise, it has a unique stationary probability vector: \((\frac{1}{2}, \frac{1}{2})^T\).

Therefore, by combining Part 1 with Part 2 and Part 3, we can obtain the desired results.  

By Theorem 2.1, we have the following result immediately.

**Corollary 2.1** Suppose that \( \mathcal{P} = (p_{i_1i_2\ldots i_m}) \in \mathbb{T}_{m,2} \) is a symmetric transition probability tensor. Then, \( \mathcal{P} \) has a unique positive stationary probability vector, which is \((\frac{1}{2}, \frac{1}{2})^T\).

Recall that a tensor \( \mathcal{P} = (p_{i_1i_2\ldots i_m}) \in \mathbb{T}_{m,n} \) is called reducible if there exists a nonempty proper index subset \( I \subset \{1, 2, \ldots, n\} \) such that 

\[ p_{i_1i_2\ldots i_m} = 0, \quad \forall i_1 \in I, \; \forall i_2, \ldots, i_m \notin I. \]

If \( \mathcal{P} \) is not reducible, then it is called irreducible [16]. It is easy to see that both tensors \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) given in Theorem 2.1 are reducible. Thus, by Theorem 2.1 we have the following result immediately.

**Corollary 2.2** Suppose that \( \mathcal{P} = (p_{i_1i_2\ldots i_m}) \in \mathbb{T}_{m,2} \) is a symmetric irreducible transition probability tensor. Then, \( \mathcal{P} \) has a unique stationary probability vector, which is \((\frac{1}{2}, \frac{1}{2})^T\).

When \( m = 4 \), Corollary 2.2 is just Theorem 3.1 given in [15].

### 3 Concluding remarks

In this paper, we gave a full characterization on the stationary probability vectors of \( m \)-order 2-dimensional symmetric transition probability tensors. In particular, for any integer number \( m \geq 3 \), any symmetric irreducible transition probability tensor of order \( m \) dimension 2 has a unique stationary probability vector. In our analysis, the “symmetry” of transition probability tensor plays an important role. It is worthy of studying the stationary probability vectors of \( m \)-order 2-dimensional transition probability tensors in the absence of symmetry. Moreover, it is also worthy of investigating the uniqueness of the stationary probability vectors of \( m \)-order \( n \)-dimensional symmetric transition probability tensors when \( m, n \geq 3 \).
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