Effective Dissipation and Turbulence in Spectrally Truncated Euler Flows

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ABSTRACT: A new transient regime in the relaxation towards absolute equilibrium of the conservative and time-reversible 3-D Euler equation with high-wavenumber spectral truncation is characterized. Large-scale dissipative effects, caused by the thermalized modes that spontaneously appear between a transition wavenumber and the maximum wavenumber, are calculated using fluctuation dissipation relations. The large-scale dynamics is found to be similar to that of high-Reynolds number Navier-Stokes equations and thus to obey (at least approximately) Kolmogorov scaling.

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Turbulence has been observed in inviscid and conservative systems, in the context of (compressible) low-temperature superfluid turbulence [1, 2, 3]. This behavior has also been reproduced using simple (incompressible) Biot-Savart vortex methods, which amount to Eulerian dynamics with ad hoc vortex reconnection [4]. The purpose of the present letter is to study the dynamics of spectrally truncated 3-D incompressible Euler flows. Our main result is that the inviscid and conservative Euler equation, with a high-wavenumber spectral truncation, has long-lasting transients which behave just as those of the dissipative (with generalized dissipation) Navier-Stokes equation. This is so because the thermalized modes between some transition wavenumber and the maximum wavenumber can act as a fictitious microworld providing an effective viscosity to the modes with wavenumbers below the transition wavenumber.

We thus study general solutions to the finite system of ordinary differential equations for the complex variables \( \hat{v}(k) \) (\( k \) is a 3 D vector of relative integers \( (k_1, k_2, k_3) \)) satisfying \( \sup_\alpha |k_\alpha| \leq k_{max} \)

\[
\partial_t \hat{v}_\alpha(k, t) = -\frac{i}{2} P_{\alpha\beta\gamma}(k) \sum_\mathcal{P} \hat{v}_\beta(p, t) \hat{v}_\gamma(k-p, t) \quad (1)
\]

where \( P_{\alpha\beta\gamma} = k_\beta P_{\alpha\gamma} + k_\gamma P_{\alpha\beta} \) with \( P_{\alpha\beta} = \delta_{\alpha\beta} - k_\alpha k_\beta / k^2 \) and the convolution in (1) is truncated to \( \sup_\alpha |k_\alpha| \leq k_{max} \), \( \sup_\alpha |p_\alpha| \leq k_{max} \) and \( \sup_\alpha |k_\alpha - p_\alpha| \leq k_{max} \).

This system is time-reversible and exactly conserves the kinetic energy \( E = \sum_k E(k, t) \), where the energy spectrum \( E(k, t) \) is defined by averaging \( \hat{v}(k', t) \) on spherical shells of width \( \Delta k = 1 \),

\[
E(k, t) = \frac{1}{2} \sum_{k-\Delta k/2 < |k'| < k + \Delta k/2} |\hat{v}(k', t)|^2. \quad (2)
\]

The discrete equations (1) are classically obtained [5] by performing a Galerkin truncation \( \hat{v}(k) = 0 \) for \( \sup_\alpha |k_\alpha| \leq k_{max} \) on the Fourier transform \( \hat{v}(x, t) = \sum \hat{v}(k, t)e^{ik \cdot x} \) of a spatially periodic velocity field obeying the (unit density) three-dimensional incompressible Euler equations,

\[
\partial_t v + (v \cdot \nabla) v = -\nabla p, \quad \nabla \cdot v = 0. \quad (3)
\]

The short-time, spectrally-converged truncated Eulerian dynamics [1] has been studied [6, 7] to obtain numerical evidence for or against blowup of the original (untruncated) Euler equations [8]. We will study here the behavior of solutions of (1) when spectral convergence to solutions of (3) is lost. Long-time truncated Eulerian dynamics is relevant to the limitations of standard simulations of high Reynolds number (small viscosity) turbulence which are performed using Galerkin truncations of the Navier-Stokes equation [9].

Equations (1) are solved numerically using standard pseudo-spectral methods with resolution \( N \). The solutions are dealiased by spectrally truncating the modes for which at least one wave-vector component exceeds \( N/3 \) (thus a \( 1600^3 \) run is truncated at \( k_{max} = 534 \)). This method allows the exact evaluation of the Galerkin convolution in (1) in only \( N^3 \log N \) operations. Time marching is done with a second-order leapfrog finite-difference scheme, even and odd time-steps are periodically re-coupled using fourth-order Runge-Kutta.

To study the dynamics of (1), we use the so-called Taylor-Green [10] single-mode initial condition of \( u^{TG} = \sin x \cos y \cos z, \quad v^{TG} = -u^{TG}(y, -x, z), \quad w^{TG} = 0 \). Symmetries are employed in a standard way [11] to reduce memory storage and speed up computations. Runs were made with \( N = 256, 512, 1024 \) and 1600.

Figure 1 displays the time evolution (top) and resolution dependence (bottom) of the energy spectra. Each energy spectrum \( E(k, t) \) admits a minimum at \( k = k_{th}(t) < k_{max} \), in sharp contrast with the short-time \( t \leq 4 \) spectrally converged Eulerian dynamics (data not shown, see [6, 11]). For \( k > k_{th}(t) \) the energy spectrum...
obey the scaling law \( E(k,t) = c(t)k^2 \) (see the dashed line at the bottom of the figure). The dynamics thus spontaneously generates a scale separation at wavenumber \( k_{th}(t) \). Figure 1 also shows that \( k_{th} \) slowly decreases with time. For fixed \( k \) inside the \( k^2 \) scaling zone \( E(k,t) \) increases with time but \( E(k,t) \) decreases with time for \( k \) close (but inferior) to \( k_{th}(t) \).

The traditionally expected [5, 12] asymptotic dynamics of the system is to reach an absolute equilibrium, which is a statistically stationary exact solution of the truncated Euler equations, with energy spectrum \( E(k) = ck^2 \). Our new results (see figure 1) show that a time-dependent statistical equilibrium which itself tends towards an absolute equilibrium is given by

\[
E_{th}(t) = \sum_{k_{th}(t)<k} E(k,t) .
\]

(4)

Since the total energy \( E \) is constant, the energy dissipated from large scales into the time dependent statistical equilibrium is given by

\[
\tau_C^2 \partial_{tt} \langle \hat{v}_\alpha(k,t) \hat{v}_\beta(k',0) \rangle_{t=0} = \langle \hat{v}_\alpha(k,0) \hat{v}_\beta(k',0) \rangle ,
\]

(5)

time translation invariance allows to express the second order time derivative as \(-\langle \partial_t \hat{v}_\alpha(k,t) \partial_t \hat{v}_\beta(k',t) \rangle_{t=t'=0} \). Using expression 11 for the time derivatives reduces the evaluation of \( \tau_C \) to that of an equal-time fourth-order moment of a gaussian field with correlation \( \langle \hat{v}_\alpha(k,t) \hat{v}_\beta(-k,t) \rangle = AP_{\alpha\beta}(k) \mathbb{R} \) where \( A = E_{th}/(2k_{max})^3 \). The only non-vanishing contribution is a one loop graph [13]. The correlation time \( \tau_C \) associated to wavenumber \( k \) is found in this way [14] to obey.
the simple scaling law
\[ \tau_C = \frac{C}{k\sqrt{E_{th}}} , \]  
where \( C = 1.43382 \) is a constant of order unity. The time-scale \( \tau_C \) is the eddy turnover time at wavenumber \( k_{th} \). Because of Kolmogorov (K41) behavior (see below) the evolution of \( E_{th} \) is governed by the large-eddy turnover time. The assumption of time-scale separation made above is thus consistent.

This strongly suggests to introduce an effective generalized Navier-Stokes model for the dissipative dynamics of modes \( k \) close to \( k_{th} \). To wit, we make the Ansatz \( \varepsilon(k, t) = \tilde{\nu}|k|E(k, t) \), where \( \tilde{\nu} = \sqrt{E_{th}/C} \) and \( \varepsilon(k, t) = -\partial E(k, t)/\partial t \) is the spectral density of energy dissipation
\[ \varepsilon(t) = \frac{dE_{th}(t)}{dt} . \]  
Assuming that this dissipation takes place in a range of width \( \alpha k_d \) around \( k_d \), we estimate the total dissipation \( \varepsilon \sim \tilde{\nu}k_dE(k_d)/\alpha k_d \). This, together with \( E(k_d) \sim k_d^3 E_{th}/k_{max}^3 \) yields the relation
\[ k_d \sim \left( \frac{\varepsilon}{E_{th}^{3/2}} \right)^{1/4} k_{max}^{3/4} . \]  
The consistency of this estimation of effective dissipation with the results displayed in figure 2 requires that \( k_d \sim k_{th} \). The ratio \( k_d/k_{th} \) is displayed on figure 3. It is seen to be of order unity and is reasonably constant in time and resolution independent (at least for \( N > 256 \)).

Thus the small-scale modes between \( k_{th} \) and \( k_{max} \) act as a fictitious thermostat providing, via the FDT, an effective viscosity to the large-scale modes with wavenumbers below \( k_{th} \). Note that spontaneous equilibration happening in conservative isolated systems, such as the one studied in the present letter, should not be confused with equilibration resulting from interaction with the thermalized degrees of freedom of the molecules constituting a physical fluid. Indeed the reversible dynamics of the isolated system spontaneously generates both the wavenumber at which the fictitious thermostat begins and its temperature.

The previous results indicate scale separation between conservative large-scale and dissipative small-scale dynamics. Furthermore the scale separation increases with resolution. This strongly suggests that large-scale behavior may be identical to that of high-Reynolds number standard Navier-Stokes equations, which is known to obey (at least approximately) K41 scaling.

The energy dissipation rate shown on figure 4 (top, left axis) is in good agreement with the corresponding
data for the Navier-Stokes TG flow (see reference [11], figure 7 and reference [3], figure 5.12). Both the time for maximum energy dissipation $t_{\text{max}} \approx 8$ and the value of the dissipation rate at that time $\varepsilon(t_{\text{max}}) \approx 1.5 \times 10^{-2}$ are in quantitative agreement. Furthermore the long-time quasi-linear behavior of $\varepsilon^{-1/3}$ (shown on right axis) is compatible with K41 self-similar decay $\varepsilon(t) \sim L_2^2 t^{-3}$.

A confirmation for K41 behavior around $t_{\text{max}}$ is displayed on figure 11 (bottom). The value of the inertial-range exponent $n$, obtained by low-$k$ least square fits of the logarithm of the energy spectrum to the function $c - n \log(k)$, is close to 5/3 (horizontal dashed line) when $t \approx t_{\text{max}}$. The $-5/3$ exponent is also shown as the left dashed line on bottom of figure 11 where the dissipative effects can be traced back to the energy spectrum decreasing faster than $k^{-5/3}$ at intermediate wavenumbers.

The mixed K41/absolute equilibrium spectra have already been discussed in the wave turbulence literature (e.g., [15]) and have more recently been studied in connection with the Leith model of hydrodynamic turbulence [17]. In this context, small-scale thermalization may have some bearing on the so-called bottleneck problem if the dissipation wavenumber approaches $k_{\text{max}}$.

Note that the dynamics of spectrally truncated time-reversible nonlinear equations has also been investigated in the special cases of 1-D Burgers-Hopf models [12] and 2-D quasi-geostrophic flows [13]. A central point in these studies was the nature of the statistical equilibrium that is achieved at large times. Several equilibria are a priori possible because both (truncated) 1-D Burgers-Hopf and 2-D quasi-geostrophic flow models admit, besides the energy, a number of additional conserved quantities. The 3-D Euler case is of a different nature because (except for helicity that identically vanishes for the flows considered here) there is no known additional conserved quantity [8] and the equilibrium is thus unique. The central problem in truncated 3-D Eulerian dynamics is therefore the mechanism of relaxation towards equilibrium, as studied in this letter.

In summary, our main result is that the spectrally truncated Euler equation has long-lasting transients behaving just like those of the dissipative Navier-Stokes equation. The small-scale thermalized modes act as a fictitious medium providing an effective viscosity to the large-scale modes. These dissipative effects were estimated using new exact result based on Fluctuation Dissipation relations. Furthermore, the solutions of the truncated Euler equations were shown to obey, at least approximately, K41 scaling. In this context, the spectrally truncated Euler equations appears as a minimal model of turbulence.

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