THE ZHANG TRANSFORMATION AND $\mathcal{U}_q(\text{osp}(1, 2l))$-VERMA MODULES ANNIHILATORS

EMMANUEL LANZMANN

Abstract. In [Zh], R. B. Zhang found a way to link certain formal deformations of the Lie algebra $\mathfrak{o}(2l + 1)$ and the Lie superalgebra $\text{osp}(1, 2l)$. The aim of this article is to reformulate the Zhang transformation in the context of the quantum enveloping algebras à la Drinfeld-Jimbo and to show how this construction can explain the main theorem of [GL2]: the annihilator of a Verma module over the Lie superalgebra $\text{osp}(1, 2l)$ is generated by its intersection with the centralizer of the even part of the enveloping algebra.

1. Introduction

A well known theorem of Duflo claims that the annihilator of a Verma module over a complex semi-simple Lie algebra is generated by its intersection with the centre of the enveloping algebra. In [GL2] we show that in order for this theorem to hold in the case of the Lie superalgebra $\text{osp}(1, 2l)$ one has to replace the centre by the centralizer of the even part of the enveloping algebra. The purpose of this article is to show how quantum groups can elucidate this phenomenon.

Let $\mathfrak{k}$, $\mathfrak{g}$ be respectively the complex simple Lie algebra $\mathfrak{o}(2l + 1)$ and the complex superalgebra $\text{osp}(1, 2l)$. From many points of view, the algebras $\mathfrak{g}$ and $\mathfrak{k}$ are very similar objects. For instance, identifying properly Cartan subalgebras of $\mathfrak{k}$ and $\mathfrak{g}$, the root systems $\Delta_\mathfrak{k}$, $\Delta_\mathfrak{g}$ are contained one into the other, and the set of irreducible roots of $\Delta_\mathfrak{g}$ is $\Delta_\mathfrak{k}$. Moreover, given a simple finite dimensional $\mathfrak{g}$-module, the corresponding simple $\mathfrak{k}$-module of the same highest weight is also finite dimensional and has the same formal character (and even the same crystal graph). Nevertheless, there is no obvious direct way to link the algebras $\mathfrak{g}$, $\mathfrak{k}$. To bridge the gap, one has to go through the quantum level: in his article published in 1992, R. B. Zhang (see [Zh], 3) found a recipe to pass from a certain formal deformation of $\mathcal{U}(\mathfrak{k})$ to a formal deformation of $\mathcal{U}(\mathfrak{g})$.

In this article we present a reformulation of the Zhang transformation in the more algebraic context of the quantizations à la Drinfeld-Jimbo. The idea is to start with the Drinfeld-Jimbo quantum enveloping algebra $\mathcal{U} := \mathcal{U}_q(\mathfrak{o}(2l + 1))$ and to extend the torus by the finite group $\Gamma := \Delta_\mathfrak{g}/2\Delta_\mathfrak{g}$. In other words, we introduce the semi-direct product

The author was partially supported by the EC TMR network Algebraic Lie Representations Grant No. ERB FMRX-CT97-0100 and Minerva grant 8337.
\( \hat{U} := U \times k \Gamma \) where \( \Gamma \) acts on \( U \) in an obvious manner. Twisting the generators of \( U \) by elements of \( \Gamma \), we build a subalgebra \( \overline{U} \) of \( \hat{U} \) isomorphic to the quantum enveloping algebra \( U_{-q}(g) \), and such that \( \hat{U} \simeq \overline{U} \times k \Gamma \). This construction provides an involution of the vector space \( \hat{U} \) mapping \( U \) onto \( \overline{U} \). We call it the Zhang transformation.

A first obvious consequence of this construction is that the algebras \( U \) and \( \overline{U} \) have the “same” finite dimensional modules. To be more precise, the quantum simple spinorial \( U \)-modules, viewed as \( \overline{U} \)-modules (via \( \hat{U} \)), are not deformation of \( g \)-modules (even up to tensorization by one dimensional modules). Moreover, roughly speaking, given a simple finite dimensional \( U \)-modules which is not of spinorial type, the specialization \( q \rightarrow 1 \) provides a simple \( \mathfrak{f} \)-module and the specialization \( q \rightarrow -1 \) a simple \( g \)-module. These classical \( \mathfrak{f} \) and \( g \)-modules have therefore the same formal character. We believe that the characters of their Demazure modules are also equal, and this construction might be an interesting approach to this problem.

Another consequence, which is the crucial observation for our purpose, is that the Zhang transformation maps the centre of \( Z(U) \) to \( A(U) \), the commutant of the even part of the \( \overline{U} \). As in the classical case, \( A(U) \) turns out to be the direct sum of the centre \( Z(U) \) of \( U \) and of the anticentre \( Z(\overline{U}) \), the subspace of elements which commute with even elements and anticommute with odd elements. The subspace \( Z(\overline{U}) \) is a cyclic module over the centre \( Z(U) \). We construct a generator of this module which is a quantization of the element \( T \) introduced in 4.4.1 [GL2]. The set \( Z(\overline{U}) \) has another interpretation: this is the set of invariant elements under a certain twisted superadjoint action. This twisted action is the quantum analogue of the “non-standard” action considered in the classical case by Arnaudon, Bauer, Frappat (see [ABF], 2). More generally, we show that the locally finite part \( F(U) \) of \( U \) for the adjoint action is mapped to the direct sum \( \overline{F(U)} \oplus \overline{F}(\overline{U}) \) of the locally finite parts of \( \overline{U} \) for the superadjoint action and for its twisted version. This allows us to deduce from the work of Joseph and Letzter (see [JL]) that the annihilator of a \( \overline{U} \)-Verma module in \( \overline{F(U)} \oplus \overline{F}(\overline{U}) \) is generated by its intersection with \( A(U) \) (theorem 6.5). We also prove a quantum analogue of theorem 7.1 [GL].

The article is organized as follows. In section 3 we present the algebra \( \hat{U} \), the main object of this article, and we define the Zhang transformation. In section 4 we study the locally finite parts of \( \hat{U} \) for different actions and analyse how these actions are transformed by the Zhang transformation. We define in 5 the subalgebra \( U \). We show that \( \overline{U} \) is isomorphic to \( U_{-q}(g) \) and we deduce from section 4 algebraic structure theorems for \( \overline{U} \). Section 6 is devoted to a proof of the annihilation theorem for \( \overline{F(U)} \oplus \overline{F}(\overline{U}) \). In section 7 we show that \( A(U) \) coincides with the centralizer of the even part of \( \overline{U} \).

Acknowledgement. I am grateful to M. Gorelik for reading earlier versions of this paper and making numerous important remarks. I would like to thank T. Joseph for his support and his comments. I wish to express my gratitude to P. Cartier, M. Duflo, T. Levasseur and G. Perets for helpful discussions.
2. Background

2.1. Notations. Let \( \mathfrak{g} \) be the complex simple Lie algebra \( o(2l + 1), \ l \geq 1 \). Fix a Cartan subalgebra \( \mathfrak{h} \) of \( \mathfrak{g} \) and denote by \( \Delta_\mathfrak{g} \) the root system of \( \mathfrak{g} \). We fix a basis of simple roots \( \pi \) of \( \Delta_\mathfrak{g} \). Denote by \( W \) the Weyl group of \( \Delta_\mathfrak{g} \), and set \( \rho := \sum_{\alpha \in \Delta_\mathfrak{g}^+} \alpha \). Denote by \( (-,-) \) the non-degenerate bilinear form on \( \mathfrak{h}^* \) coming from the restriction of the Killing form of \( \mathfrak{g}_0 \) to \( \mathfrak{h} \). For any \( \lambda, \mu \in \mathfrak{h}^* \), \( (\mu, \mu) \neq 0 \) one defines \( \langle \lambda, \mu \rangle := 2(\lambda, \mu)/(\mu, \mu) \). One has the following useful realization of \( \Delta_\mathfrak{g} \). Identify \( \mathfrak{h}^* \) with \( \mathbb{C}^l \) and consider \( (-,-) \) as an inner product on \( \mathbb{C}^l \). Then there exists an orthonormal basis \( \{ \beta_1, \ldots, \beta_l \} \) such that

\[
\pi = \{ \beta_1 - \beta_2, \ldots, \beta_{l-1} - \beta_l, \beta_l \}, \quad \Delta_\mathfrak{g} = \{ \pm \beta_i \pm \beta_j, 1 \leq i < j \leq l, \pm \beta_i, 1 \leq i \leq l \}
\]

and the Weyl group \( W \) is just the group of the signed permutations of the \( \beta_i \). We set \( \alpha_i := \beta_i - \beta_{i+1} \) with the convention \( \beta_{l+1} := 0 \). Let also \( w_i, 1 \leq i \leq l \), be the elements \( w_i := \beta_1 + \ldots + \beta_i \). With these notations, the set of weights \( P_\mathfrak{g}(\pi) \) of \( \Delta_\mathfrak{g} \) is

\[
P_\mathfrak{g}(\pi) := \mathbb{Z}w_1 \oplus \ldots \oplus \mathbb{Z}w_{l-1} \oplus \mathbb{Z}w_l = \mathbb{Z} \alpha_1 \oplus \ldots \oplus \mathbb{Z} \alpha_{l-1} \oplus \mathbb{Z} \alpha_l.
\]

We consider also the set of dominant weights \( P_\mathfrak{g}^+(\pi) \).

2.1.1. Let \( \mathfrak{g} \) be the complex Lie superalgebra \( osp(1,2l) \). Denote by \( \mathfrak{g}_0 \) the even part of \( \mathfrak{g} \) and by \( \mathfrak{g}_1 \) its odd part. We identify \( \mathfrak{h} \) with a Cartan subalgebra of \( \mathfrak{g} \) in such a way that \( \pi \) is also a basis of simple roots of the root system \( \Delta_\mathfrak{g} \) of \( \mathfrak{g} \) with respect to \( \mathfrak{h} \). The sets of even and odd roots of \( \mathfrak{g} \) equal respectively \( \{ \pm \beta_i \pm \beta_j, 1 \leq i < j \leq l, \pm 2 \beta_i, 1 \leq i \leq l \} \) and \( \{ \pm \beta_i, 1 \leq i \leq l \} \). The set of weights \( P_\mathfrak{g}(\pi) \) of \( \Delta_\mathfrak{g} \) is

\[
P_\mathfrak{g}(\pi) := \mathbb{Z}w_1 \oplus \ldots \oplus \mathbb{Z}w_{l-1} \oplus \mathbb{Z}w_l = \mathbb{Z} \alpha_1 \oplus \ldots \oplus \mathbb{Z} \alpha_{l-1} \oplus \mathbb{Z} \alpha_l.
\]

In a standard manner we define the set of dominant weights \( P_\mathfrak{g}^+(\pi) \).

2.1.2. Let \( q \) be an indeterminate and \( k = \mathbb{C}(\sqrt{q}) \). Let \( \nu \in k \), be such that \( \nu \neq 0, \pm 1 \). For all \( n \in \mathbb{N} \), we set \( [n]_\nu := \frac{\nu^n - \nu^{-n}}{\nu - \nu^{-1}} \) and \( [n]_\nu! := [n]_\nu \times [n - 1]_\nu \times \ldots \times [1]_\nu \) with the convention \( [0]_\nu = 1 \). If \( 1 \leq m \leq n \in \mathbb{N} \) we set \( \left[ \frac{n}{m} \right]_\nu := \frac{[n]_\nu!}{[m]_\nu![n - m]_\nu!} \).

2.2. The algebra \( \mathcal{U} \). Let \( \mathcal{U} \) be the algebra over the field \( k \) generated by the elements \( E_i, F_i, 1 \leq i \leq l, K_\mu, \mu \in P_\mathfrak{g}(\pi) \) under the relations

\[
K_0 = 1, \quad K_\lambda K_\mu = K_{\lambda + \mu} \tag{1}
\]

\[
K_\lambda E_j K_\lambda^{-1} = q^{(\lambda, \alpha_j)} E_j, \quad K_\lambda F_j K_\lambda^{-1} = q^{-(\lambda, \alpha_j)} F_j \tag{2}
\]

\[
E_i F_j - F_j E_i = \delta_{ij} \frac{K_{\alpha_i} - K_{-\alpha_i}}{q - q^{-1}} \tag{3}
\]
\[
\sum_{k=0}^{1-\langle \alpha_j, \alpha_i \rangle} (-1)^k \left[ 1 - \frac{\langle \alpha_j, \alpha_i \rangle}{k} \right] E_i^{1-\langle \alpha_j, \alpha_i \rangle} - k E_i^{1-\langle \alpha_j, \alpha_i \rangle} E_j E_i^k = 0 \quad (4)
\]
\[
\sum_{k=0}^{1-\langle \alpha_j, \alpha_i \rangle} (-1)^k \left[ 1 - \frac{\langle \alpha_j, \alpha_i \rangle}{k} \right] F_i^{1-\langle \alpha_j, \alpha_i \rangle} - k F_i^{1-\langle \alpha_j, \alpha_i \rangle} F_j F_i^k = 0 \quad (5)
\]

where \( q_i := q^{\frac{\langle \alpha_j, \alpha_i \rangle}{2}} \) and \( \langle \alpha_j, \alpha_i \rangle := 2(\alpha_j, \alpha_i)/(\alpha_i, \alpha_i) \). Relations (4), (5) are called the quantum Serre relations.

2.2.1. Replacing \( E_i \) by \( \left( \frac{q - q^{-1}}{\sqrt{q} - \sqrt{q}^{-1}} \right) E_i \) in the above equations, we see that \( \mathcal{U} \) is just the Drinfeld-Jimbo quantum enveloping algebra \( \mathcal{U}_q \).

2.2.2. Let \( \mathcal{U}^+ \) (resp. \( \mathcal{U}^- \)) be the subalgebra of \( \mathcal{U} \) generated by the \( E_i \) (resp. \( F_i \)). We also denote by \( \mathcal{U}^0 \) the subalgebra generated by the \( K_\lambda \). If \( T \) stands for the multiplicative group \( \{ K_\mu, \mu \in P_\mathfrak{g}(\pi) \} \), then the group algebra of \( T \) identifies with \( \mathcal{U}^0 \). One has the triangular decomposition \( \hat{\mathcal{U}} \simeq \mathcal{U}^- \otimes \mathcal{U}^0 \otimes \mathcal{U}^+ \), this isomorphism of vector spaces being given by multiplication.

2.2.3. The algebra \( \mathcal{U} \) is a \( k \)-Hopf algebra with counit \( \varepsilon \), coproduct \( \Delta \) and antipode \( S \) defined by

\[
\varepsilon(K_\lambda) = 1, \quad \varepsilon(E_i) = \varepsilon(F_i) = 0
\]
\[
\Delta(K_\lambda) = K_\lambda \otimes K_\lambda, \quad \Delta(E_i) = E_i \otimes 1 + K_{\alpha_i} \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_{\alpha_i}^{-1} + 1 \otimes F_i
\]
\[
S(K_\lambda) = K_\lambda^{-1}, \quad S(E_i) = -K_{\alpha_i}^{-1} E_i, \quad S(F_i) = -F_i K_{\alpha_i}
\]

2.2.4. Representations. Let \( \hat{T} \) be the group of characters of \( T \) with values in \( k \). Given a character \( \Lambda \in \hat{T} \), we say that \( \Lambda \) is linear if there exists \( \lambda \in P_\mathfrak{t}(\pi) \) such that \( \Lambda(K_\mu) = q^{(\lambda, \mu)} \forall \mu \in P_\mathfrak{g}(\pi) \). In that case we write \( \Lambda := q^\lambda \).

Let \( M \) be a \( T \)-module. An element \( m \in M \) is said to be an element of weight \( \Lambda \in \hat{T} \) if \( K_\mu m = \Lambda(K_\mu) m \forall \mu \in P_\mathfrak{g}(\pi) \). We also call \( m \) a \( T \)-weight element. We denote by \( M_\Lambda \) the subspace of elements of weight \( \Lambda \).

For any \( \Lambda \in \hat{T} \), we denote by \( M(\Lambda) \) the \( \mathcal{U} \)-Verma module of highest weight \( \Lambda \), and by \( V(\Lambda) \) its unique simple quotient. We recall now basic properties of the representation theory of \( \mathcal{U} \) (see for instance [12], 4.4.9). If \( \Lambda = q^\lambda \) is linear, and if there exists \( \alpha \in \Delta_\mathfrak{t}^+ \) such that \( \langle \lambda + \rho, \alpha \rangle \in \mathbb{N}^+ \) then \( M(q_{\alpha, \lambda}) \) is a submodule of \( M(q^\lambda) \). The simple module \( V(\Lambda) \) is finite dimensional if and only if \( \Lambda = \phi q^\lambda \) where \( \lambda \in P_\mathfrak{t}^+(\pi) \) and \( \phi \in \hat{T} \) is such that \( \phi(K_\mu) = \pm 1 \) for all \( \mu \in P_\mathfrak{g}(\pi) \). Any finite dimensional \( \mathcal{U} \)-module is completely reducible.
2.2.5. The group $T$ acts on $\mathcal{U}$ by inner automorphisms. Thus we can speak of weight elements in $\mathcal{U}$. By a slight abuse of notation we shall say that $u \in \mathcal{U}$ is of weight $\lambda \in P_\pi$ if it is actually of weight $q^\lambda$.

3. The algebra $\hat{\mathcal{U}}$

In this section we introduce the main object of this article.

3.1. Definition and basic properties.

3.1.1. Definition. Let $\Gamma$ be the multiplicative group $\{\xi_{\mu}, \mu \in P_{g}(\pi)\} \cong P_{g}(\pi)/2P_{g}(\pi)$ and $k\Gamma$ its group algebra. The group $\Gamma$ acts on $\mathcal{U}$ in the following natural way:

$$\xi_{\mu}.u = (-1)^{(\mu, \lambda)}u, \quad \forall u \in \mathcal{U} \text{ of weight } \lambda, \forall \lambda, \mu \in P_{g}(\pi).$$

Using this action of $\Gamma$ we introduce the algebra $\hat{\mathcal{U}} := \mathcal{U} \rtimes k\Gamma$.

Throughout this article, we shall use the shortened notation $u\xi_{\mu}$ for an element $u \otimes \xi_{\mu} \in \hat{\mathcal{U}}$.

3.1.2. The Hopf structure. Recall that the algebra $k\Gamma$ is a Hopf algebra for the counit $\varepsilon$, coproduct $\Delta$ and antipode $S$ defined by

$$\varepsilon(\xi_{\mu}) = 1, \quad \Delta(\xi_{\mu}) = \xi_{\mu} \otimes \xi_{\mu}, \quad S(\xi_{\mu}) = \xi_{\mu}.$$ Since $\Gamma$ acts on $\mathcal{U}$ by Hopf algebra automorphisms, $\hat{\mathcal{U}} = \mathcal{U} \rtimes k\Gamma$ is a Hopf algebra for the obvious coalgebra structure (resp. antipode) on tensor product.

3.1.3. A $\mathbb{Z}_2$-gradation. The algebra $\hat{\mathcal{U}}$ is also endowed with the following $\mathbb{Z}_2$-gradation. Introduce $\xi := \xi_{w_1}$, and define

$$\forall i \in \mathbb{Z}_2, \quad \hat{\mathcal{U}}_{i} = \{x \in \hat{\mathcal{U}}, \xi x \xi = (-1)^{i} x\}.$$ An element $x \in \hat{\mathcal{U}}_{0} \cup \hat{\mathcal{U}}_{1}$ is called $\mathbb{Z}_2$-homogenous, and we write $|x| = i$ if $x \in \hat{\mathcal{U}}_{i}$, $x \neq 0$.

3.1.4. The Harish-Chandra projection. Recall the triangular decomposition given in 2.2.2.

Introduce the subalgebra $\hat{\mathcal{U}}^{o} := \mathcal{U}^{o} \otimes k\Gamma$. One has the triangular decomposition

$$\hat{\mathcal{U}} \simeq \mathcal{U}^- \otimes \hat{\mathcal{U}}^{o} \otimes \mathcal{U}^+.$$ Let $\mathcal{U}^+$ (resp. $\mathcal{U}^-$) be the augmentation ideal of $\mathcal{U}^+$ (resp. $\mathcal{U}^-$). The triangular decomposition of $\hat{\mathcal{U}}$ implies $\hat{\mathcal{U}} = (\mathcal{U}^- \hat{\mathcal{U}} + \hat{\mathcal{U}}^{++}) \oplus \hat{\mathcal{U}}^{o}$ which allows to define a Harish-Chandra projection $\Upsilon : \hat{\mathcal{U}} \longrightarrow \hat{\mathcal{U}}^{o}$ with respect to this decomposition.

3.2. Representations of $\hat{\mathcal{U}}$. 

3.2.1. **Generalities.** Let $\hat{\Gamma}$ be the group of characters of $\Gamma$, identified with the set of group morphisms $P_\vartheta(\pi)/2P_\vartheta(\pi) \rightarrow \{1, \pm 1\}$. Any $\lambda \in P_\vartheta(\pi)$ defines a character $(-1)^\lambda \in \hat{\Gamma}$ by the formula: $(-1)^\lambda(\mu) := (-1)^{\langle \lambda, \mu \rangle}$. Observe that $\Gamma$ embeds in $\hat{\Gamma}$ as the set $\{\Lambda \in \hat{\Gamma} \mid \forall \mu \in P_\vartheta(\pi), \Lambda(K_\mu) = \pm 1\}$.

Recall 2.2.4. Let $M$ be a $T \times \Gamma$-module. We say that an element $m \in M$ is a $T \times \Gamma$-weight element of weight $(\Lambda, \theta) \in \hat{T} \times \hat{\Gamma}$ if $\xi_\mu K_\mu m = \Lambda(K_\mu)\theta(\mu')m \forall \mu, \mu' \in P_\vartheta(\pi)$.

Take $\Lambda \in \hat{T}$ and $\theta \in \hat{\Gamma}$. There is an obvious way to endow $M(\Lambda)$ with a structure of a $\hat{\mathcal{U}}$-module. Define for any $x \in M(\Lambda)$ of weight $q^{-\nu}\Lambda, \nu \in P_\vartheta(\pi)$,

$$\xi_\mu x := (-1)^{\langle \mu, \nu \rangle}\theta(\mu)x \forall \mu \in P_\vartheta(\pi).$$

We call this $\hat{\mathcal{U}}$-module a $\hat{\mathcal{U}}$-Verma module and we denote it by $M(\Lambda, \theta)$. By definition, $M(\Lambda, \theta)$ and $M(\Lambda)$ have the same submodules. Let $\Lambda = q^\lambda$ be linear. Assume that there exists $\alpha \in \Delta_k^+$ such that $\langle \lambda + \rho, \alpha \rangle \in \mathbb{N}^+$ and define $\theta' := (-1)^{s_\alpha\lambda - \lambda}\theta$. Then $M(q^{s_\alpha\lambda}, \theta')$ is a $\hat{\mathcal{U}}$-submodule of $M(q^\lambda, \theta)$. The $\hat{\mathcal{U}}$-module $M(\Lambda, \theta)$ has a unique simple quotient, $V(\Lambda, \theta)$. As a $\mathcal{U}$-module, $V(\Lambda, \theta) \simeq V(\Lambda)$.

3.2.2. **Lemma.** All finite dimensional $\hat{\mathcal{U}}$-modules are completely reducible. Moreover, any simple finite dimensional $\hat{\mathcal{U}}$-module $M$ is isomorphic to a $V(q^\lambda\phi, \theta), \lambda \in P_\vartheta^+(\pi), \phi, \theta \in \hat{\Gamma}$.

**Proof.** Let $M^{\mathcal{U}^+}$ be the subspace of $M$ of invariant vectors by $\mathcal{U}^+$. This subspace is stable by the action of the commutative algebra $\hat{\mathcal{U}}^\circ$. Since $\Gamma$ is finite and $M$ is a $\mathcal{U}$-module of finite dimension, $\hat{\mathcal{U}}^\circ$ acts diagonally on $M$ and hence on $M^{\mathcal{U}^+}$. Let $\{v_1, \ldots, v_r\}$ be basis of $M^{\mathcal{U}^+}$ composed of $T \times \Gamma$-weight vectors. The representation theory of $\mathcal{U}$ (see [Jan] chap. 5) asserts that on the one hand $M_i := \mathcal{U}v_i = \hat{\mathcal{U}}v_i$ is a simple $\mathcal{U}$-module, and so a $\hat{\mathcal{U}}$-module of the form $V(q^{s_\alpha\lambda}\phi_i, \theta_i)$ with $\lambda_i \in P_\vartheta^+(\pi), \theta, \phi \in \hat{\Gamma}$, and on the other hand that $M = \oplus M_i$.

3.2.3. **The $\mathbb{Z}_2$-gradations.** Both $M(\Lambda, \theta)$ and $V(\Lambda, \theta)$ inherits a $\mathbb{Z}_2$-gradation. Let $v_\Lambda$ be the highest weight vector of $M(\Lambda, \theta)$. They are two natural $\mathbb{Z}_2$-gradations on $M(\Lambda, \theta)$.

Fix $j \in \mathbb{Z}_2$ and define $M(\Lambda, \theta)_i = \hat{\mathcal{U}}^{i+j}v_\Lambda \forall i \in \mathbb{Z}_2$. The gradations on $V(\Lambda, \theta)$ are defined similarly using the highest weight vector of $V(\Lambda, \theta)$.

3.3. **Three gradations and the Zhang transformation.**

3.3.1. **The gradation by the weights.** The considerations of 2.2.3 extend to $\hat{\mathcal{U}}$. We shall denote by $\nu(x)$ the weight of a weight element $x \in \hat{\mathcal{U}}$ and by $\hat{\mathcal{U}}_{\nu}$ the subspace of elements
of weight \( \nu \). The algebra \( \hat{U} \) is graded by its weight subspace:
\[
\hat{U} := \bigoplus_{\nu \in \mathcal{P}_g(\pi)} \hat{U}_\nu.
\]

### 3.3.2. The \( \mu \)-gradation and the Zhang transformation.

Define the \( \mathcal{P}_g(\pi)/2\mathcal{P}_g(\pi) \)-gradation
\[
\hat{U} := \bigoplus_{\mu \in \mathcal{P}_g(\pi)/2\mathcal{P}_g(\pi)} \mu \hat{U}
\]
for which \( \xi_\lambda \in \hat{U}_0, K_\lambda \in \hat{U}_1, E_i \in \hat{U}_{i+1}, \) and \( F_i \in \hat{U}_{i} \) for all \( \lambda \in \mathcal{P}_g(\pi), 1 \leq i \leq l \). If \( x \in \mu \hat{U} \), we set \( \mu(x) := \mu \). We call Zhang transformation the involution of vector space \( \Psi : \hat{U} \rightarrow \hat{U} \) such that
\[
\Psi(x) := \xi_\mu x \quad \forall x \in \mu \hat{U}, \forall \mu \in \mathcal{P}_g(\pi)/2\mathcal{P}_g(\pi).
\]
We shall show in 5.2 that \( \Psi(\hat{U}) \) is a subalgebra isomorphic to \( U_q(\mathfrak{g}) \) (see [MZ] for the definition of this algebra). For any homogenous elements \( a, b \) for the respective gradations \( (\hat{U}_\nu) \) and \( (\mu \hat{U}) \),
\[
\Psi(ab) = (-1)^{(\nu(a),\mu(b))}\Psi(a)\Psi(b).
\]

### 3.3.3. The \( \delta \)-gradation.

We introduce another \( \mathcal{P}_g(\pi)/2\mathcal{P}_g(\pi) \)-gradation on \( \hat{U} \) (compare this gradation with the filtration defined in 5.3.1).
\[
\hat{U} = \bigoplus_{\delta \in \mathcal{P}_g(\pi)/2\mathcal{P}_g(\pi)} \hat{U}_\delta
\]
for which \( \xi_\mu, E_i \in \hat{U}_0, K_\mu \in \hat{U}_1, F_i \in \hat{U}_{\alpha_i} \) for all \( \mu \in \mathcal{P}_g(\pi), 1 \leq i \leq l \). A glance at the defining relations of \( \hat{U} \) ensures that this does define a gradation on \( \hat{U} \). If \( x \in \hat{U}_\delta \), we shall use the notation \( \delta(x) := \delta \).

### 3.3.4. Compatibilities between the gradations.

The gradation \( (\hat{U}_\delta) \) is invariant under the action of \( T \) (see 3.3.1). Thus the \( \hat{U}_\delta \) are direct sums of their weight subspaces. If \( \delta, \nu \in \mathcal{P}_g(\pi) \), we set \( \hat{U}_\nu \delta := \hat{U}_\delta \cap \hat{U}_\nu \). One has the bigradation on \( \hat{U} \)
\[
\hat{U} = \bigoplus_{\nu \in \mathcal{P}_g(\pi)} \bigoplus_{\delta \in \mathcal{P}_g(\pi)/2\mathcal{P}_g(\pi)} \hat{U}_\nu \delta.
\]

The relation between the gradations \( (\hat{U}_\nu), (\mu \hat{U}), (\hat{U}_\delta) \) reads as follows. The gradation \( (\mu \hat{U}) \) is also \( T \)-invariant, and hence induces a bigradation \( \hat{U} = \bigoplus (\mu \hat{U} \cap \hat{U}_\nu) \). Then, this bigradation coincides with the bigradation [7]. To be more precise, one has
\[
\hat{U}_\nu \delta = \delta + \eta(\nu) \hat{U}_\nu \cap \hat{U}_\nu,
\]
where $\eta : P_{\mathfrak{g}}(\pi) \to P_{\mathfrak{g}}(\pi)/2P_{\mathfrak{g}}(\pi)$ is the map defined by

$$\eta(\sum n_i \alpha_i) := \sum n_i \beta_{i+1}.$$  

Indeed, it is enough to check (8) on the generators. One has $\delta(E_i) + \eta(\nu(E_i)) = \beta_{i+1} = \mu(E_i)$, $\delta(F_i) + \eta(\nu(F_i)) = \alpha_i + \beta_{i+1} = \beta_i = \mu(F_i)$, $\delta(K_\mu) + \eta(\nu(K_\mu)) = \mu = \mu(K_\mu)$, $\delta(\xi_\mu) = \nu(\xi_\mu) = \mu(\xi_\mu) = 0$.

3.3.5. Recall the $\mathbb{Z}_2$-gradation defined in 3.1.3. If $\nu \in P_{\mathfrak{g}}(\pi)$, we set $|\nu| := (\nu, w_1) \pmod{2}$. Observe that for all $\nu, \nu' \in P_{\mathfrak{g}}(\pi)$, the following identity holds in $\mathbb{Z}_2$:

$$(\nu, \eta(\nu')) + (\eta(\nu), \nu') + (\nu, \nu') = |\nu||\nu'| \quad (9)$$

Since both sides of (9) are bilinear in $\nu, \nu'$, the identity (9) reduces to the case where $\nu = \alpha_i, \nu' = \alpha_j$. In that case, the left hand side of (9) is equal (in $\mathbb{Z}_2$) to

$$(\alpha_i, \beta_{j+1}) + (\beta_{i+1}, \alpha_j) + (\alpha_i, \alpha_j) = (\beta_i, \beta_j) + (\beta_{i+1}, \beta_{j+1}) = \begin{cases} 0 & \text{if } (i, j) \neq (l, l) \\ 1 & \text{if } i = j = l \end{cases} = |\alpha_i||\alpha_j|$$

3.3.6. Recall the definition of $\hat{\mathcal{U}}^\delta_\nu$ (see 3.3.3). One has

$$\Delta(\hat{\mathcal{U}}^\delta_\nu) \subset \sum_{\nu_1 + \nu_2 = \nu} \hat{\mathcal{U}}^{\delta + \nu_2}_\nu \otimes \hat{\mathcal{U}}^{\delta}_\nu \quad (10)$$

and $S(\hat{\mathcal{U}}^\delta_\nu) \subset \hat{\mathcal{U}}^{\delta + \nu}_\nu$.

Indeed, according to 2.2.3 3.1.2 these inclusions are satisfied for the generators of $\hat{\mathcal{U}}$.

3.4. A Hopf superalgebra structure on $\hat{\mathcal{U}}$. The algebra $\Psi(\mathcal{U})$ is not a Hopf subalgebra of $\hat{\mathcal{U}}$ for the coproduct and antipode defined in 2.2.3 3.1.2. In this subsection, we endow $\hat{\mathcal{U}}$ with a structure of Hopf superalgebra for which $\Psi(\mathcal{U})$ is a Hopf subalgebra.

Recall definitions of $\Psi$ (see 3.3.2) and $\hat{\mathcal{U}}^\delta_\nu$ (see 3.3.3). Define for all homogenous elements $a \in \hat{\mathcal{U}}$ for the bigradation $\hat{\mathcal{U}}^\delta_{\nu}$:

$$\Delta(\Psi(a)) := (1)(\nu(a_1), \nu(a_2)) \Psi(a_1) \otimes \Psi(a_2), \quad (11)$$

$$\Delta(\hat{\mathcal{U}}^\delta_\nu) \subset \sum_{\nu_1 + \nu_2 = \nu} \hat{\mathcal{U}}^{\delta + \nu_2}_\nu \otimes \hat{\mathcal{U}}^{\delta}_\nu$$

and $S(\hat{\mathcal{U}}^\delta_\nu) \subset \hat{\mathcal{U}}^{\delta + \nu}_\nu$.

Indeed, according to 2.2.3 3.1.2 these inclusions are satisfied for the generators of $\hat{\mathcal{U}}$.

3.4.1. Introduce $e_i := \Psi(E_i), f_i := \Psi(F_i), k_\mu := \Psi(K_\mu)$. On the generators the definitions (11) give:

$$\Delta(\xi_\mu) = \xi_\mu \otimes \xi_\mu, \quad \Delta(k_\lambda) = k_\lambda \otimes k_\lambda,$$

$$\Delta(e_i) = e_i \otimes 1 + k_{\alpha_1} \otimes e_i, \quad \Delta(f_i) = f_i \otimes k^{-1}_{\alpha_1} + 1 \otimes f_i,$$

$$S(\xi_\mu) = \xi_\mu, \quad S(k_\lambda) = k^{-1}_\lambda, \quad S(e_i) = -k^{-1}_{\alpha_1} e_i, \quad S(f_i) = -f_i k_{\alpha_1}.$$
3.4.2. Lemma. \( \hat{U} \) endowed with \((\bar{\Delta}, \bar{S}, \varepsilon)\) is a Hopf superalgebra.

Proof. Retain the definition of the \( \mathbb{Z}_2 \)-gradation on \( \hat{U} \) (see [3.1.3]). Let us prove that for any \( a, b \in \hat{U} \) homogenous for the bigradation \((\hat{U}_\delta)\)

\[
\bar{\Delta}(\Psi(a)\Psi(b)) = (-1)^{|a_2||b_1|} \bar{\Delta}(a) \bar{\Delta}(b)
\]

(12)

with Sweedler notation \( \Delta(a) = a_1 \otimes a_2, \Delta(b) = b_1 \otimes b_2 \). Recall [8], [10] which assert in particular that \( a, b, a_1, a_2, b_1, b_2 \) are graded for the three gradations \((\hat{U}_\nu), (\hat{\mu} \hat{U}), (\hat{U}_\delta)\). By definition of \( \bar{\Delta} \) and by (11) one has

\[
\bar{\Delta}(\Psi(a)\Psi(b)) = (-1)^{\nu(a)\mu(b)} \bar{\Delta}(ab)
\]

\[
= (-1)^{\nu(a)\mu(b) + \nu(a_1b_1, a_2b_2) + \eta\nu(a_2b_2)} \Psi(a_1b_1) \otimes \Psi(a_2b_2)
\]

\[
= (-1)^s \Psi(a_1) \Psi(b_1) \otimes \Psi(a_2) \Psi(b_2)
\]

where \( s \in \mathbb{Z}_2 \) and

\[
S := (\nu(a), \mu(b)) + (\nu(a_1b_1), \nu(a_2b_2) + \eta\nu(a_2b_2) + (\nu(a_1), \mu(b_1)) + (\nu(a_2), \mu(b_2)).
\]

According to (8) and (10), one has \( \mu(b) = \delta(b) + \eta\nu(b), \mu(b_1) = \delta(b) + \nu(b_2) + \eta\nu(b_1), \mu(b_2) = \delta(b) + \eta\nu(b_2) \), and hence

\[
s = (\nu(a_1b_2), \eta\nu(b_1b_2)) + (\nu(a_1b_1), \nu(a_2b_2) + \eta\nu(a_2b_2)) + (\nu(a_1), \nu(b_2))
\]

\[
+ (\nu(a_1), \eta\nu(b_1)) + (\nu(a_2), \eta\nu(b_2)).
\]

Expanding all scalar products in the above expression of \( s \), we find

\[
s = (\nu(a_1), \nu(a_2) + \eta\nu(a_2)) + (\nu(b_1), \nu(b_2) + \eta\nu(b_2))
\]

\[
+ (\nu(a_2), \nu(b_1)) + (\nu(b_1), \eta\nu(a_2)) + (\nu(a_2), \eta\nu(b_1)).
\]

Using (11), \( s \) can be rewritten as

\[
s = (\nu(a_1), \nu(a_2) + \eta\nu(a_2)) + (\nu(b_1), \nu(b_2) + \eta\nu(b_2)) + |a_2||b_1|
\]

and therefore

\[
\bar{\Delta}(\Psi(a)\Psi(b)) = (-1)^{|a_2||b_1|} \left((-1)^{\nu(a_1, a_2) + \eta\nu(a_2)} \Psi(a_1) \otimes \Psi(a_2)\right)
\]

\[
\times \left((-1)^{\nu(b_1), b_2 + \eta\nu(b_2)} \Psi(b_1) \otimes \Psi(b_2)\right)
\]

\[
= (-1)^{|a_2||b_1|} \bar{\Delta}(a) \bar{\Delta}(b)
\]

which proves (12).

Let \( m : \hat{U} \otimes \hat{U} \longrightarrow \hat{U} \) be the multiplication map. We show next that for any element \( a \in \hat{U} \) homogenous for the bigradation \( \hat{U}_\delta \),

\[
m(1 \otimes \bar{S}) \bar{\Delta}(a) = m(\bar{S} \otimes 1) \bar{\Delta}(a) = \varepsilon \Psi(a)
\]

(13)

By definition of \( \bar{\Delta} \), and \( \bar{S} \)

\[
m(1 \otimes \bar{S}) \bar{\Delta}(a) = (-1)^{\nu(a_1, a_2) + \eta\nu(a_2)} \Psi(a_1) \otimes \bar{S} \Psi(a_2)
\]

\[
= (-1)^{\nu(a_1, a_2) + \eta\nu(a_2)} + (\nu(a_2), \delta(a_2)) \Psi(a_1) \otimes \Psi(Sa_2).
\]
According to (10), \( S(a_2) \in \widehat{U}_{\rho(a_2)} \) and \( \delta(a_2) = \delta(a) \). Then formula (3) gives 
\[
\Psi(a_1) \Psi(Sa_2) = (-1)^{[\rho(a_1),\delta(a)]+\nu(a)+\eta(a_2)} \Psi(a_1 Sa_2),
\]
and we obtain finally 
\[
m(1 \otimes \overline{S}) \Delta \Psi(a) = (-1)^{[\rho(a),\delta(a)]} \Psi(\varepsilon(a)) = (-1)^{[\rho(a),\delta(a)]} \varepsilon \Psi(a) = \varepsilon \Psi(a)
\]
since \( \varepsilon \Psi(a) = 0 \) if \( \nu(a) \neq 0 \). The other equality of (13) can be established in the same way.

It remains to check that 
\[
(\varepsilon \otimes 1) \Delta \Psi(a) = (1 \otimes \varepsilon) \Delta \Psi(a) = \Psi(a)
\]
\[
(\Delta \otimes 1) \Delta \Psi(a) = (1 \otimes \Delta) \Delta \Psi(a).
\]
These identities are straightforward from the definition of \( \Delta \). \( \square \)

4. Twisted adjoint actions and locally finite parts

The object of this section is to compute the locally finite parts of \( \widehat{U} \) for certain twisted adjoint actions.

4.1. A general construction. Let \( X = X_0 \oplus X_1 \) be a Hopf superalgebra. We recall that the Hopf structure of \( X \) provides an adjoint action defined by the formula \( \text{ad} a(x) = (-1)^{|a||x|} a_1 x S(a_2) \), using the Sweedler notation: \( \Delta(a) = a_1 \otimes a_2 \). There is an elementary way to construct new actions by twisting the adjoint action by an algebra morphism (similar twisted actions have been considered by Joseph in [3]). One proceeds as follows. Let \( \psi : X \to X \) be an algebra morphism. For any \( \mathbb{Z}_2 \)-homogenous element \( a, x \in X \), set:
\[
\text{ad}_\psi a(x) = (-1)^{|a||x|} a_1 x S(\psi(a_2)).
\]
This formula defines an action since for any homogenous elements \( a, b, x \)
\[
(\text{ad}_\psi a)(\text{ad}_\psi b)(x) = (-1)^{|b_2||x|+|a_2|(|b_1|+|x|+|b_2|)} a_1 b_1 x S(\psi(b_2)) S(\psi(a_2)) = (-1)^{|x|(|a_2|+|b_2|)+|b_1||a_2|} a_1 b_1 x S(\psi(a_2) \psi(b_2)) = (-1)^{|x|(|a_2|+|b_2|)} a_1 b_1 x S(\psi(a_2 b_2)) = (-1)^{|x||ab|} (ab)_1 x S(\psi((ab)_2)) = \text{ad}_\psi(ab)(x)
\]

4.2. Let \( \mu \in \widehat{P}_g(\pi)/2 \widehat{P}_g(\pi) \), and \( \psi_\mu \) be the inner automorphism of \( \widehat{U} \) defined by \( \psi_\mu(a) = \xi_\mu a \xi_\mu^{-1} \). In what follows we shall consider the twisted adjoint actions obtained by applying the construction 4.1 to the cases of

— the genuine Hopf algebra \( \widehat{U} \) (for the Hopf structure given in 2.2.3) and the morphisms \( \psi_\mu \).

— the Hopf superalgebra \( \widehat{U} \) (for the Hopf superstructure given in 3.4) and the morphisms \( \psi_\mu \).
In order to avoid any confusion, we shall write ad the adjoint action of \( \hat{U} \), and \( \overrightarrow{ad} \) the super adjoint action. The twisted actions are denoted respectively by \( \text{ad}_\mu := \text{ad}_{\psi_\mu}, \overrightarrow{\text{ad}}_\mu := \overrightarrow{\text{ad}}_{\psi_\mu} \). Of course, \( \text{ad}_0 = \text{ad} \) and \( \overrightarrow{\text{ad}}_0 = \overrightarrow{\text{ad}} \). In the case \( \mu = w_l \), we shall often prefer to write \( \text{ad} \) instead of \( \overrightarrow{\text{ad}}_w \). The twisted adjoint action \( \text{ad} \) is the quantum version of the “non-standard” adjoint action considered by Arnaudon, Bauer, Frappat in [ABF], 2. Recall 3.4.1. By definition, \( \overrightarrow{\text{ad}} a = \text{ad} a \) for any generator \( a \in \{k_\lambda, \xi_\lambda \lambda \in P_0(\pi); \ e_i, f_i, 1 \leq i < l \} \) and

\[
\begin{align*}
\overrightarrow{\text{ad}}(e_i)x &= e_i x - (-1)^{|x|}k_i x k_i^{-1} e_i, \\
\overrightarrow{\text{ad}}(f_i)x &= f_i x k_i - (-1)^{|x|} x f_i k_i \\
\text{ad} \lambda x &= e_i x + (-1)^{|x|} k_i x k_i^{-1} e_i, \\
\text{ad} \gamma x &= f_i x k_i + (-1)^{|x|} x f_i k_i \\
\end{align*}
\]

(14)

4.2.1. If \( N \) is any \( \text{ad}\lambda \)-stable (resp. \( \overrightarrow{\text{ad}}\lambda \)-stable) subspace of \( \hat{U} \), we denote by \( F_\lambda(N) \) (resp. \( \overrightarrow{F}_\lambda(N) \)) its locally finite part for the action \( \text{ad}_\lambda \) (resp. \( \overrightarrow{\text{ad}}_\lambda \)). If \( \mu = 0 \) we shall write respectively \( F(\gamma), \overrightarrow{F}(\gamma) \) instead of \( F_0(N), \overrightarrow{F}_0(N) \). Also, we shall often prefer to use the notation \( \overrightarrow{F}_\lambda(N) \) instead of \( \overrightarrow{F}_w(N) \).

4.2.2. Let \( \lambda \in P_0(\pi) \). By definition of \( \text{ad}_\lambda \), for all \( x \in \hat{U} \), and for all weight element \( a \in \hat{U} \), \( \text{ad}_\lambda(a)(\xi_\lambda x) = a_1 \xi_\lambda x S(\xi_\lambda a_1 \xi_\lambda) = \xi_\lambda(\xi_\lambda a_1 \xi_\lambda) x S(\xi_\lambda a_1 \xi_\lambda) = (-1)^{(\lambda, \lambda)}(\xi_\lambda \text{ad} a) x \). The same holds replacing \( \text{ad} \) by \( \overrightarrow{\text{ad}} \). It follows that

\[
F_\lambda(\hat{U}) = \xi_\lambda F(\hat{U}) \quad \text{and} \quad \overrightarrow{F}_\lambda(\hat{U}) = \xi_\lambda \overrightarrow{F}(\hat{U}).
\]

4.2.3. Lemma. Let \( \lambda \) be in \( P_0(\pi)/2P_0(\pi) \). Then

(i) \( F_\lambda(U) = 0 \) if \( \lambda \neq 0 \)

(ii) \( F_\lambda(\hat{U}) = \xi_\lambda F(U) \)

Proof. Assume that \( \lambda \neq 0 \), and let \( V \subset U \) be a simple \( \text{ad}_\lambda U \)-module. Take \( a \) an element of lowest weight of \( V \). Since \( \lambda \neq 0 \), there exists \( \alpha_i \in \pi \) such that \( (\lambda, \alpha_i) = 1 + 2\mathbb{Z} \). Hence \( 0 = \text{ad}_\mu F_i a = (F_i a + a F_i) K_{\alpha_i} \). Proposition 1.7, [DK], forces \( a = 0 \). This establishes the assertion (i).

Let \( a \in F(\hat{U}) \). Write \( x = \sum_{\mu \in P_0(\pi)/2P_0(\pi)} \xi_\mu a_\mu, \ x_\mu \in U \). According to 4.2.2 \( \text{ad}(\hat{U}) x = \text{ad}(U) x = \bigoplus_{\mu \in P_0(\pi)/2P_0(\pi)} \xi_\mu \text{ad}_\mu(U)(x_\mu) \). Thus \( \text{ad}_\mu(U)(x_\mu) \in F_{\mu}(U) \) and hence \( x_\mu = 0 \) if \( \mu \neq 0 \) by (i). This proves \( F(\hat{U}) = F(U) \) and (ii) follows from 4.2.2. \( \square \)

4.3. Retain the definitions of 3.3.3. If follows from (11) and (11) that the gradation \( (\hat{U}^\delta) \) possesses a very striking property: it is invariant by the actions \( \text{ad}_\lambda, \overrightarrow{\text{ad}}_\lambda \). Recall the definition of \( \Psi \) (see 3.3.2).

Lemma. Fix \( \lambda, \delta \in P_0(\pi)/2P_0(\pi) \). Let \( a, x \in \hat{U} \) be homogenous elements for the bigradation \( (\hat{U}^\delta) \). Then

\[
\Psi(\text{ad}_\lambda a(x)) = \pm \overrightarrow{\text{ad}}_{\lambda + \delta} \Psi(a)(\Psi(x)).
\]

(15)
Proof. Let \( a, x \in \hat{\mathcal{U}} \) be as in the Lemma. By definition, \( \text{ad}_\lambda ax = (-1)^{(\nu(a_2), \lambda)} a_1 x S(a_2) \), where \( \Delta(a) = a_1 \otimes a_2 \) in Sweedler notation. Recall (8), (10) which assert in particular that \( a, x, a_1, a_2 \) are graded for the gradations \( (\hat{\mathcal{U}}_\nu), (\hat{\mathcal{U}}^0), (\hat{\mathcal{U}}^\phi). \) Using (3) one obtains

\[
\Psi(\text{ad}_\lambda ax) = (-1)^{(\nu(a_2), \lambda) + (\nu(a_1), \mu(x)) + (\nu(x), \mu(Sa_2)) + (\nu(a_1), \mu(Sa_2))} \Psi(a_1) \Psi(x) \Psi(Sa_2) = (-1)^s \Psi(a_1) \Psi(x) \Psi(Sa_2)
\]

where \( s \in \mathbb{Z}_2, \) and

\[
s = \frac{(\nu(a_2), \lambda) + (\nu(a_1), \mu(x)) + (\nu(x), \mu(Sa_2)) + (\nu(a_1), \mu(Sa_2)) + (\nu(a_2), \delta(a_2))}{s_2}.
\]

According to (10) \( \mu(x) = \delta(x) + \eta \nu(x), \delta(a_2) = \delta(a), \mu(Sa_2) = \delta(a) + \nu(a_2) + \eta \nu(a_2) \), so

\[
s_1 = (\nu(a), \delta(a)) + (\nu(a_1), \nu(a_2) + \eta \nu(a_2))
\]

and

\[
s_2 = (\nu(a_2), \lambda) + (\nu(a_2) + \nu(a), \delta(x) + \eta \nu(x)) + (\nu(x), \nu(a_2) + \eta \nu(a_2) + \delta(a)) = (\nu(a), \mu(x)) + (\nu(x), \delta(x)) + \delta \lambda + \delta(x)
\]

\[
+ (\nu(a_2), \eta \nu(x)) + (\nu(x), \eta \nu(a_2)) + (\nu(x), \nu(a_2))
\]

\[
= (\nu(a), \mu(x)) + (\nu(x), \delta(a)) + (\nu(a_2), \lambda + \delta(x)) + |a_2| |x| \text{ by (3)}.
\]

Consequently

\[
s = s_1 + s_2 = t + |a_2| |x| + (\nu(a_2), \lambda + \delta(x)) + (\nu(a_1), \nu(a_2) + \eta \nu(a_2))
\]

where \( t = (\nu(a), \delta(a)) + (\nu(a), \mu(x)) + (\nu(x), \delta(a)) \) depends only on the “degrees” (for the different gradations) of \( a \) and \( x \). Substituting the above expression of \( s \) in (16), and using definitions of \( \underline{\Delta}, \underline{S} \) we derive that

\[
\Psi(\text{ad}_\lambda ax) = (-1)^t \underline{\text{ad}}_{\lambda + \delta(x)} \Psi(a) \Psi(x)
\]

as required. \( \square \)

4.4. We recall the results of Joseph and Letzter (see [12]):

\[
F(\mathcal{U}) = \bigoplus_{\lambda \in \mathbb{P}^+_{\pi}} (\text{ad} \mathcal{U}) K_{-2\lambda}
\]

and each \( (\text{ad} \mathcal{U}) K_{-2\lambda} \) contains a unique (up to a non-zero scalar) central element denoted by \( z_{2\lambda} \). The centre \( \mathcal{Z}(\mathcal{U}) \) of \( \mathcal{U} \) is the polynomial algebra

\[
\mathcal{Z}(\mathcal{U}) = \mathbb{C}[z_{2w_1}, \ldots, z_{2w_{l-1}}, z_{w_l}].
\]

We shall need the following submodules of \( F(\mathcal{U}) \):

\[
N_0 := \bigoplus_{\lambda \in \mathbb{P}^+_{\pi}} (\text{ad} \mathcal{U}) K_{-2\lambda}, \quad N_1 := \bigoplus_{\lambda \in \mathbb{P}^+_{\pi} \setminus \mathbb{P}^+_{\pi}} (\text{ad} \mathcal{U}) K_{-2\lambda}.
\]
If \( \lambda \in P_\mathfrak{g}^+(\pi) \), then \( \delta(K_{-2\lambda}) = 2\lambda = 0 \) and so \( N_0 \subset U^0 \). If \( \lambda \in P_\mathfrak{g}^+(\pi) \), \( \lambda' \in P_\mathfrak{g}^+(\pi) \), then \( \delta(K_{-2\lambda}) = w_l \) and so \( N_1 \subset U^{wl} \). Consequently

\[
F(U) \cap U^0 = N_0 \quad F(U) \cap U^{wl} = N_1.
\]

(19)

It follows from Lemma 4.3 that \( F(\hat{U}) \cap \hat{U}^\mu = \Psi(F_\mu(\hat{U}) \cap \hat{U}^\mu) \). By Lemma 4.2.3 we know that \( F_\mu(\hat{U}) = \xi_\mu F(U) \). Hence, using (19) and recalling 4.2.2, we obtain

\[
F_\lambda(\hat{U}) = \bigoplus_{\mu} \xi_{\lambda+\mu} \Psi(F(U) \cap U^\mu) = \xi_\lambda(\Psi(N_0) \oplus \xi \Psi(N_1)).
\]

(20)

\[\text{5. Algebraic structures of } \mathcal{U}_q(\mathfrak{g})\]

Recall the definition of \( \Psi \) (see 3.3.2). We define \( \overline{U} := \Psi(U) \). By definition of \( \Psi \),

\[\hat{U} \simeq U \rtimes k\Gamma.\]

With the notations of 3.4.1, \( \overline{U} \) is the subalgebra of \( \hat{U} \) generated by the \( e_i, f_i, k_\lambda \). The algebra \( U \) is graded for all the different gradations we defined on \( \hat{U} \). By definition of \( \overline{\Delta}, \overline{\mathcal{S}} \), the subalgebra \( \overline{U} \) is a Hopf subalgebra of \( (\hat{U}, \overline{\Delta}, \overline{\mathcal{S}}, \varepsilon) \).

We shall show in 5.2 that \( \overline{U} \simeq \mathcal{U}_{q^\lambda}(\mathfrak{g}) \).

\[\text{5.1. Representations of } \overline{U}.\]

We shall now give the classification of the finite dimensional \( \overline{U} \)-modules.

\[\text{5.1.1. Generalities.}\]

Let us prove that every simple \( \overline{U} \)-module is the restriction of a simple \( \hat{U} \)-module and that every finite dimensional \( \overline{U} \)-modules is completely reducible.

Let \( V \) be a simple finite dimensional \( \overline{U} \)-module. Assume for the moment that we work over the algebraic closure \( \overline{k} \) of \( k \) (we extend the scalars of all our objects). Take any non-zero weight vector of \( V \) (that is a common eigenvector for the \( k_\lambda \)). The simplicity forces \( V = \overline{U} v \). Choose any character \( \theta \in \Gamma \). Then the following formula defines an action of \( \hat{U} \) on \( V : \xi_\mu a_\nu v = \theta(\mu)(-1)^{(\mu,\nu)} a_\nu v \), for any \( \nu \in P_\mathfrak{g}(\pi) \) and any \( a_\nu \in \overline{U} \) of weight \( \nu \).

Indeed, the vector \( v \) being a weight vector, the annihilator \( \text{Ann}_{\overline{U}} v \) is the sum of its weight subspaces, and hence the previous formula makes sense. As a \( \overline{U} \)-module, \( V \) is necessarily simple, and thus is a \( V(\phi q^\lambda, \theta) \) by lemma 3.2.2 (which obviously also holds over \( \overline{k} \)). This shows in particular that all the eigenvalues of the \( k_\lambda \) actually lie in our ground field \( k \). Therefore, we could have chosen \( v \) to be in the \( k \)-vector space \( V \), and so all that precedes actually holds over \( k \). We have proved that \( V \) is the restriction of a simple \( \hat{U} \)-module.

Remark that we have just showed that the group \( \{k_\mu, \mu \in P_\mathfrak{g}(\pi)\} \) acts diagonally on a simple \( \overline{U} \)-module. Hence the restriction of a simple finite dimensional \( \hat{U} \)-module to \( \overline{U} \) is also simple.
Consider now $M$, a finite dimensional $\mathcal{U}$-module. The induced module $\text{Ind}_U^{\hat{U}}M$ is a finite dimensional $\hat{U}$-module, and therefore completely reducible by lemma 3.2.2. Thus, $\text{Ind}_U^{\hat{U}}M$ is also completely reducible as a $\hat{U}$-module (see the previous remark). But as a $\mathcal{U}$-module, $M$ lies in $\text{Ind}_U^{\hat{U}}M$. Hence $M$ is completely reducible.

5.1.2. It follows from what precedes, that for any fixed $\theta \in \hat{\Gamma}$, the set $\{V(q^\lambda \phi, \theta), (\lambda, \phi) \in P_+(\pi) \times \hat{\Gamma}\}$ is a complete set of non-isomorphic finite dimensional simple modules for both $\mathcal{U}$ and $\mathcal{U}$.

5.1.3. Remark. Recall that we shall prove in 5.2 that $\mathcal{U} \simeq U_{-q}(\mathfrak{g})$. The classification of the finite dimensional modules over the “quantum” enveloping algebra of $\mathfrak{g}$ has been obtained by R. B. Zhang (see [Zh]) in the context of formal deformations and by Zou (see [Zou]) for the Drinfeld-Jimbo quantization $U_q(\mathfrak{g})$, through the standard approach.

5.1.4. Crystals. We admit for a moment that $\mathcal{U} \simeq U_{-q}(\mathfrak{g})$. Fix $\theta \in \hat{\Gamma}$ and let $V(\lambda)$, $\lambda \in P_+(\pi)$ be the simple finite dimensional $\hat{U}$-module $V(q^\lambda, \theta)$. A priori, one can associate to $V(\lambda)$ two crystals. One is given by the work of Kashiwara (see [Kash]), considering $V(\lambda)$ as a $\mathcal{U}$-module. We denoted it by $B(\lambda)$. The other one, $B(\lambda)$, follows from the work of Musson and Zou (see [MZ]), viewing this time $V(\lambda)$ as a $\hat{U}$-module. Both sets $\{B(\lambda), \lambda \in P_+(\pi)\}$, $\{B(\lambda), \lambda \in P_+(\pi)\}$ are closed family of highest weight normal crystals, in the sense of [J2] 6.4.21. Hence there are equal (up to isomorphisms) by proposition 6.4.21, [J2]. Another way to see that $B(\lambda) \simeq B(\lambda)$ is to remark (keeping the notations of [Kash] and [MZ]) that $L(\lambda) = L(\lambda)$ and that the crystalline operators of Musson and Zou act on a given weight subspace of $L(\lambda)$ as the crystalline operators of Kashiwara up to signs (depending on the weight of the subspace and on the “color” of the operators).

5.2. Recall the definition of $U_q(\mathfrak{g})$ given in [MZ]. Let $Z(\mathcal{U})$ (resp. $\overline{Z}_*(\mathcal{U})$) be the supercentre (resp. the anticentre) of $\mathcal{U}$, that is the subspace of invariants elements of $\mathcal{U}$ with respect to $\overline{ad}$ (resp $\overline{ad}_*$). One has $\overline{Z}_*(\mathcal{U}) := \{a \in \mathcal{U}, ax = (-1)^{|x|}xa \forall \mathbb{Z}_2\text{-homogenous } x \in \mathcal{U}\}$. We also introduce the algebra $A(\mathcal{U}) := \overline{Z}(\mathcal{U}) + \overline{Z}_*(\mathcal{U})$.

We deduce from section 4 the (compare (i) with 3.3 [Zh])

Theorem.

(i) The subalgebra $\mathcal{U}$ is isomorphic to $U_{-q}(\mathfrak{g})$.
(ii) One has $\Psi(F(\mathcal{U})) = F(\hat{U}) \oplus F_*(\hat{U})$ with

$$F(\hat{U}) = \Psi(N_0) = \bigoplus_{\lambda \in P_+(\pi)} \overline{ad} \mathcal{U} k_{-2\lambda}$$
\[
\hat{F}_\ast(U) = \Psi(N_1) = \bigoplus_{\lambda \in \mathfrak{p}^+} \text{ad}k_{-2\lambda}
\]

(iii) Recall that \( \xi := \xi_{\beta_i} \) and (13). One has \( A(U) = \overline{Z}(U) \oplus \overline{Z}_\ast(U) = \Psi(Z(U)) \) and
\[
\overline{Z}(U) = \mathbb{C}[z_{w_1}, \ldots, z_{2w_{l-1}}, z_{w_l}^2]
\]
\[
\overline{Z}_\ast(U) = (\xi_{z_{w_l}})\overline{Z}(U)
\]

Proof. We start by proving (i). The relations (2), (3), (4), (5) can be respectively rewritten as
\[
\text{ad}K_\mu E_i = q^{(\mu,\alpha_i)}E_i, \quad \text{ad}K_\mu F_i = q^{-(\mu,\alpha_i)}F_i, \quad \text{ad}F_i E_j = \delta_{ij}(1 - K_{\alpha_i}^2)/(q - q^{-1}), \quad \text{ad}E_i \text{ad}F_j = 0, \quad \text{ad}F_i \text{ad}F_j = 0.
\]
Take the image of these relations by \( \Psi \). According to Lemma 4.3 (and more precisely to formula (17)) we obtain
\[
\text{ad}k_\mu f_i = (-q)^{-(\mu,\alpha_i)}f_i, \quad \text{ad}f_i e_j = -\delta_{ij}(-1)^{\delta_{ij}}(1 - k_{\alpha_i}^2)/(q - q^{-1}), \quad \text{ad}e_i \text{ad}f_j = 0, \quad \text{ad}f_i \text{ad}f_j = 0.
\]
It is easy to see that these relations are exactly the relations defining \( U_{-\lambda}(\mathfrak{g}) \).

The assertion (ii) results from (20). And (ii) implies that \( \Psi(Z(U) \cap N_0) = \overline{Z}(U) \) and \( \Psi(Z(U) \cap N_1) = \overline{Z}_\ast(U) \). On the other hand, elements of the centre \( Z(U) \) are of weight zero. Therefore combining (14) with (8), one has \( \Psi(Z(U) \cap N_0) = Z(U) \cap N_0 \) and \( \Psi(Z(U) \cap N_1) = \xi(Z(U) \cap N_1) \), which ends the proof.

5.2.1. Remark. The element \( \xi_{z_{w_l}} \) is a quantization of the element \( T \) introduced in 4.4.1 [GL2]. See also formula (12) and remark 4.4.1. Notice also that this element coincides with the \( s \)-Casimir element constructed in [AB] for the algebra \( U_{(osp(1,2))} \).

5.2.2. Remark. Let \( \overline{F}_\ast(U(\mathfrak{g})) \) be the locally finite part of \( U(\mathfrak{g}) \) for the “non-standard” action of Arnaudon, Bauer, Frappat (see [ABF], 2). Then \( \overline{F}_\ast(U(\mathfrak{g})) = \overline{F}(U(\mathfrak{g})) \) since \( \mathfrak{g}_1 \) is finite dimensional. The situation in the quantum case is therefore radically different on this point.

5.3. The separation theorem for \( \overline{F}(U) \oplus \overline{F}_\ast(U) \). In [LJ3], Joseph and Letzter established a separation theorem for the algebra \( F(U) \). They proved the existence of ad-submodules \( \mathcal{H}(U)(\lambda) \) of \( (\text{ad}U)K_{-2\lambda} \) such that if \( \mathcal{H}(U) := \oplus_{\lambda \in \mathfrak{p}^+} \mathcal{H}(U)(\lambda) \), then the multiplication \( \mathcal{H}(U) \otimes Z(U) \rightarrow U \) is an isomorphism of \( \text{ad}U \)-modules. Introduce
\[
\overline{\mathcal{H}}(U) := \Psi(N_0 \cap \mathcal{H}(U)) \quad \text{and} \quad \overline{\mathcal{H}}_\ast(U) := \Psi(N_1 \cap \mathcal{H}(U))
\]
Let \( h_i \in \mathcal{H}(U) \) be weight elements, and \( z_i \in Z(U) \cap N_{j_i} \), \( j_i \in \{0, 1\} \). It follows from (19) and (8) that \( \sum \Psi(h_i)\Psi(z_i) = \Psi(\sum \pm h_iz_i) \). This is enough to deduce the separation theorem for \( \overline{F}(U) \oplus \overline{F}_\ast(U) \):

5.3.1. Proposition. The multiplication
\[
(\overline{\mathcal{H}}(U) \oplus \overline{\mathcal{H}}_\ast(U)) \otimes A(U) \rightarrow \overline{F}(U) \oplus \overline{F}_\ast(U)
\]
is an isomorphism.
6. The annihilation theorem

The goal of this section (theorem 6.5) is to establish that the annihilator of any \( \mathcal{U} \)-Verma module in \( \overline{F(U)} \oplus \overline{F^*(U)} \) is generated by its intersection with \( A(U) \).

6.1. By definition (see \([3.2.1]\)), a \( \hat{U} \)-Verma module is a \( U \)-Verma module and a character of \( \Gamma \) which describes the action of \( \Gamma \) on the highest weight vector. Of course, the same holds replacing \( U \) by \( \hat{U} \). Hence, throughout this subsection, we shall not make any distinctions between the \( U \), \( \hat{U} \) and \( \hat{U} \)-Verma modules.

We fix now once for all a \( \hat{U} \)-Verma module \( M := M(\Lambda, \theta) \) and we define for all \( \mu \in P_\theta(\pi) \),

\[ \Lambda(k_\mu) := \Lambda(K_\mu) \theta(\xi_\mu). \]

6.2. The Verma module \( M \) being \( \mathbb{Z}_2 \)-graded (see \([3.2.3]\)), \( \text{End}(M) \) inherits the natural gradation:

\[ \text{End}(M)_{ij} = \{ f \in \text{End}(M), \forall i \in \mathbb{Z}_2 f(M|_i) \subset M|_{i+j} \} \]

Consider the adjoint action of \( \hat{U} \) on \( \text{End}(M) \) defined by \((\text{ad}\ a)f(x) = a_1(f(S(a_2)x))\) for all \( a \in \hat{U} \), \( f \in \text{End}(M) \) and all \( x \in M \). Let \( F(M, M) \) be the locally finite part of \( \text{End}(M) \) for the adjoint action \( \text{ad} \). The subspace \( F(M, M) \) is \( \mathbb{Z}_2 \)-graded for the above gradation. The restriction of \( \hat{U} \twoheadrightarrow \text{End}(M) \) induces a morphism of \( \text{ad} \hat{U} \)-modules: \( F(\hat{U}) \twoheadrightarrow F(M, M) \). Its image coincides with the image of \( F(U) \twoheadrightarrow F(M, M) \).

We recall (see Lemma 8.3. in \([J2]\)) that \( F(M, M) \) is a domain.

6.3. \textbf{Lemma.} Let \( f \in F(M, M) \) and \( i \in \mathbb{Z}_2 \) be such that \( f(M|_i) = 0 \). Then \( f = 0 \).

\textbf{Proof.} Let \( f \) be as in the lemma. By definition of the \( \mathbb{Z}_2 \)-gradation on \( F(M, M) \) we may assume that \( f \) is \( \mathbb{Z}_2 \)-homogenous, and hence that \( f^2 \) is even. Take any non-zero \( p \in F(M, M)_{|_i} \) (obviously such \( p \) exists; for instance \( \text{ad} E_{l}K_{-w_l} = (1 - q^{-1})E_{l}K_{-w_l} \in F(U) \) has a non-trivial image in \( F(M, M)_{|_i} \)). Then \( f^2pf^2 = 0 \) which implies \( f = 0 \) since \( F(M, M) \) is a domain. \( \square \)

6.4. Recall that \( \Lambda(k_\mu) := \Lambda(K_\mu) \theta(\xi_\mu) \).

\textbf{Lemma.} For any \( \mathcal{U} \)-Verma module \( M \), the following equivalence holds

\[ \text{Ann}_{A(\mathcal{U})} M = A(\overline{\mathcal{U}}) \text{Ann}_{\mathcal{Z}(\mathcal{U})} M \iff \forall 1 \leq i \leq l, \ \Lambda(k_{\beta_i}) \neq \pm iq^{-(\rho, \beta_i)}. \]

\textbf{Proof.} By proposition \([5.2]\), one has \( A(\overline{\mathcal{U}}) = (k \oplus k(\xi_{w_l})) \otimes \mathcal{Z}(\overline{\mathcal{U}}) \). The centre \( \mathcal{Z}(\overline{\mathcal{U}}) \) acts by scalars on \( M \). Hence the equality \( \text{Ann}_{A(\mathcal{U})} M = A(\overline{\mathcal{U}}) \text{Ann}_{\mathcal{Z}(\mathcal{U})} M \) is equivalent to \( \text{Ann}_{k \oplus k(\xi_{w_l})} M = 0 \). The element \( \xi_{w_l} \) acts on the \( \mathbb{Z}_2 \)-graded components of \( M \) by the two
opposite scalars \( \pm \Lambda(\mathcal{Y}(z_{w_l})) \), \( \Lambda \) being linearly extended to \( \mathcal{U}^o \). Thus, \( \text{Ann}_{\mathbb{C}[\mathfrak{k}]}(\xi_{w_l}) M = 0 \) is equivalent to \( \Lambda(\mathcal{Y}(z_{w_l})) \neq 0 \). Retain notation of 3.1.4. By [JL] 7.1.19,

\[
\mathcal{Y}(z_{w_l}) = \sum_{\mu \in \text{Pr}(\pi)} \dim V(q^{\mu}) q^{-2(\rho, \mu)} K_{-2\mu}
\]

where \( V(q^{\mu}) \) is simple \( \mathcal{U} \)-module of highest weight \( \frac{\mu}{2} \). Since \( \frac{\mu}{2} \) is a minuscule weight, \( \dim V(q^{\mu}) = 1 \) if \( \mu \in W(\frac{\mu}{2}) = \{ \frac{1}{2} \sum_{i=1}^{l} \varepsilon_i \beta_i, \varepsilon_i = \pm 1 \} \) and 0 otherwise. So (21) can be rewritten

\[
\mathcal{Y}(z_{w_l}) = \prod_{i=1}^{l} (q^{-\langle \rho, \beta_i \rangle} K_{-\beta_i} + q^{\langle \rho, \beta_i \rangle} K_{\beta_i}).
\]

The assertion follows. \( \square \)

6.4.1. Remark. If \( \widehat{T}_d := \{ \Lambda \in \widehat{T}, \exists 1 \leq i \leq t \text{ such that } \Lambda(K_{\beta_i}) = \pm iq^{-\langle \rho, \beta_i \rangle} \} \) then the formula (22) implies that

\[
\xi z_{w_l} \in \bigcap_{\Lambda \in \widehat{T}_d, \theta \in \Gamma} \text{Ann}_{\mathcal{M}} M(\Lambda, \theta).
\]

This is the quantum version of a property satisfied by the element \( T \) constructed in [GL2] (see [GL2] 4.4.1 and 6.1.3).

6.5. In [JL], Joseph and Letzter prove the annihilation theorem of Duflo for \( \mathcal{F}(\mathcal{U}) \). We deduce from this result the (compare with Theorem 7.1 [GL], and Theorem 6.2 [GL2])

**Theorem.** Let \( M \) be a \( \mathcal{U} \)-Verma module. Then

(i) For any \( i \in \mathbb{Z}_2 \), \( \text{Ann}_{\mathcal{F}(\mathcal{U}) \oplus \mathcal{F}(\mathcal{U})} M_i = (\mathcal{F}(\mathcal{U}) \oplus \mathcal{F}(\mathcal{U})) \text{Ann}_{\mathcal{A}(\mathcal{U})} M_i \).

(ii) \( \text{Ann}_{\mathcal{F}(\mathcal{U}) \oplus \mathcal{F}(\mathcal{U})} M = (\mathcal{F}(\mathcal{U}) \oplus \mathcal{F}(\mathcal{U})) \text{Ann}_{\mathcal{A}(\mathcal{U})} M \).

(iii) \( \text{Ann}_{\mathcal{F}(\mathcal{U}) \oplus \mathcal{F}(\mathcal{U})} M = (\mathcal{F}(\mathcal{U}) \oplus \mathcal{F}(\mathcal{U})) \text{Ann}_{Z(\mathcal{U})} M \iff \forall 1 \leq i \leq l, \Lambda(K_{\beta_i}) \neq \pm iq^{-\langle \rho, \beta_i \rangle} \)

**Proof.** We start by (i). Recall Theorem 5.2 (iii). As the centre \( Z(\mathcal{U}) \) acts by scalars on \( M \), the algebra \( \mathcal{A}(\mathcal{U}) \) acts by scalars on the homogenous components \( M_i \). It follows from Proposition 5.3.1 that (i) is equivalent to the statement \( \forall i \in \mathbb{Z}_2, \text{Ann}_{\mathcal{F}(\mathcal{U}) \oplus \mathcal{F}(\mathcal{U})} M_i = 0 \). Let \( \Psi(h), \Psi(h') \in \mathcal{H}(\mathcal{U}), \mathcal{H}(\mathcal{U}) \) and \( i \in \mathbb{Z}_2 \) be such that \( \Psi(h) + \Psi(h') \in \text{Ann}_{\mathcal{H}(\mathcal{U}) \oplus \mathcal{H}(\mathcal{U})} M_i \).

Since \( M_i \) is \( T \)-invariant, we can assume that \( \Psi(h), \Psi(h') \) (and hence \( h, h' \)) are elements of the same weight \( \nu \). Combining (19) and (8) one has \( \Psi(h) = \xi_{\eta(\nu)} h \) and \( \Psi(h') = \xi_{\eta(\nu) + w_l} h' \). Hence \( h + \xi h' \in \text{Ann}_{\mathcal{H}(\mathcal{U})} M_i \). The element \( \xi \) acts by \( \pm id \) on \( M_i \), so we can assume that \( h + \xi h' \in \text{Ann}_{\mathcal{H}(\mathcal{U})} M_i \). From Lemma 5.3 we derive that \( h + \xi h' \in \text{Ann} M \). Therefore \( h' = -h \) using 4.2, [JL]. But \( h \in N_0, h' \in N_1 \) and \( N_0 \cap N_1 = 0 \), which forces finally \( h = h' = 0 \). This finishes the proof of (i).
For (ii), one has the equalities
\[
\text{Ann}_{\Gamma(U) \oplus_{\Upsilon(U)} M} = \bigcap_i \text{Ann}_{\Gamma(U) \oplus_{\Upsilon(U)} M_i}
\]
\[
= \bigcap_i (\Gamma(U) \oplus_{\Upsilon(U)} \text{Ann}_{\Gamma(U)} M_i)
\]
\[
= \bigcap_i (\Gamma(U) \oplus_{\Upsilon(U)} \text{Ann}_{\Gamma(U)} M_i)
\]
by (i)
\[
= \bigcap_i (\text{Ann}_{\Gamma(U)} \oplus_{\Upsilon(U)} \text{Ann}_{\Gamma(U)} M_i)
\]
by Proposition 5.3.1
\[
= (\Gamma(U) \oplus_{\Upsilon(U)} \text{Ann}_{\Gamma(U)} M)
\]
by Proposition 5.3.1 and of Lemma 6.4.

And (iii) is a consequence of (ii), Proposition 5.3.1 and of Lemma 6.4.

6.5.1. Remark. We believe that (i) should also hold in the classical case.

7. \(\mathcal{A}(\mathcal{U})\) is the commutant of \(\mathcal{U}_{\downarrow 0}\)

In this section we shall prove that \(\mathcal{A}(\mathcal{U})\) is the commutant of the even part of \(\mathcal{U}\), that is \(\mathcal{A}(\mathcal{U}) = \mathcal{C}(\mathcal{U}_{\downarrow 0})\). Let \(\mathcal{A}(\mathcal{U})\) be the subalgebra \(\mathcal{A}(\mathcal{U}) := \mathcal{Z}(\mathcal{U}) \oplus \xi \mathcal{Z}(\mathcal{U})\). Since \(\mathcal{C}(\mathcal{U}_{\downarrow 0}) \cap \mathcal{U} = \mathcal{C}(\mathcal{U}_{\downarrow 0})\) and \(\mathcal{A}(\mathcal{U}) \cap \mathcal{U} = \mathcal{A}(\mathcal{U})\) (recall Theorem 5.2 (iii)) it is enough to prove the equality \(\mathcal{A}(\mathcal{U}) = \mathcal{C}(\mathcal{U}_{\downarrow 0})\). For this, we shall proceed by quantizing the mechanics of 4 [GL2].

7.1. Lemma. For every subset \(\Omega\) of \(P_k(\pi)\) dense for the Zariski topology, one has
\[
\bigcap_{\lambda \in \Omega} \text{Ann}_{\mathcal{U}} V(q^\lambda, \theta) = 0.
\]

Proof. Fix \(\lambda \in \Omega\). One has
\[
\bigcap_{\theta \in \Gamma} \text{Ann}_{\mathcal{U}} V(q^\lambda, \theta) = (k\Gamma) \text{Ann}_{\mathcal{U}} V(q^\lambda).
\] (23)

where \(V(q^\lambda)\) stands for the \(\mathcal{U}\)-simple module of highest weight \(q^\lambda\).

Indeed, let \(e_\chi \in k\Gamma, \chi \in \Gamma\), be the projector corresponding to \(\chi\), that is the projector such that \(ge_\chi = \chi(g)e_\chi\), \(\forall g \in \Gamma\). Let \(x \in \mathcal{U}\) and write \(x = \sum_{\chi \in \Gamma} x_\chi e_\chi, x_\chi \in \mathcal{U}\). As a \(\mathcal{U}\)-module, \(V(q^\lambda, \theta)\) canonically identifies with the \(\mathcal{U}\)-module \(V(q^\lambda)\) (see 3.2.1). Under this identification, \(x\) acts on the subspace of \(T\)-weight \(q^{\lambda - \nu}\) of \(V(q^\lambda, \theta)\) as \(x_{(-1)^\nu \theta}\) on the subspace of the same weight of \(V(q^\lambda)\). Hence, \(x \in \bigcap_{\theta \in \Gamma} \text{Ann}_{\mathcal{U}} V(q^\lambda, \theta)\) implies that \(x_{(-1)^\nu \theta}\) vanishes on \(V(q^\lambda)\) for all \(\nu\) and \(\theta\). This gives (23).

7.2. As in 4.1 [GL2], the previous lemma implies

Lemma. The algebra \(\mathcal{C}(\mathcal{U}_{\downarrow 0})\) coincides with the subalgebra of elements of \(\mathcal{U}\) acting by scalars on the homogenous components of simple highest weight modules.
7.3. Retain the notation of [3.1.4]. Take \( x \in \hat{U}^o \) and write \( x := \sum a_{\mu,\mu'} \xi_{\mu} K_{\mu'} \), \( a_{\mu,\mu'} \in k \).

For any \( (\lambda, \theta) \in P_\pi(\pi) \times \hat{\Gamma} \), we set \( x(\lambda, \theta) := \sum a_{\mu,\mu'} \theta(q^{(\lambda, \mu')} \)."

With these conventions, \( a \in C(\hat{U}_0) \) acts by the scalar \( \Upsilon(a)(\lambda, \theta) \) on the homogenous component of \( V(q^\lambda, \theta) \) containing the highest weight vector.

7.4. **Lemma.** The restriction of \( \Upsilon \) to \( C(\hat{U}_0) \) is injective.

*Proof.* For all \( (\lambda, \theta) \in P_\pi^+(\pi) \times \hat{\Gamma} \) denote by \( v_\lambda \) the highest weight vector of \( V(q^\lambda, \theta) \). Let \( a \) be in \( A(\hat{U}) \). We recall that \( a \) acts on \( \hat{U}_0 v_\lambda \) by the scalar \( \Upsilon(a)(\lambda, \theta) \). On the other hand, if \( \lambda \in \Omega := \{ \lambda \in P_\pi^+(\pi), \langle s_{\beta_i}(\lambda) + \rho, \beta_i \rangle \in 2N + 1 \} \) and \( \theta' := (-1)^{\langle s_{\beta_i}, \lambda \rangle} \theta \) we claim that \( a \) acts on \( \hat{U}_0 v_\lambda \) by the scalar \( \Upsilon(a)(s_{\beta_i} \lambda, \theta') \). Indeed, assume that \( \lambda \in \Omega \) and \( \theta' = (-1)^{\langle s_{\beta_i}, \lambda \rangle} \theta \). Then (see [3.2.1]) \( M(q^\lambda, \theta) \subset M(q^{s_{\beta_i} \lambda}, \theta') \). Moreover, if \( u_\lambda, u_{s_{\beta_i} \lambda} \) are the respective highest weight vectors of these Verma modules, one has \( \hat{U}_0 u_\lambda \subset \hat{U}_0 u_{s_{\beta_i} \lambda} \) and the claim follows. Hence \( \Upsilon(a) = 0 \) implies \( a \in Ann_{\hat{U}_0} V(q^\lambda, \theta) \) for all \( (\lambda, \theta) \in \Omega \times \hat{\Gamma} \). The density of \( \Omega \) allows us to use [3.1] and then to conclude.

7.5. Set

\[
U^o_{ev} := \sum_{\mu \in P_\pi(\pi)} kK_{2\mu} \subset U^o.
\]

The Weyl group \( W \) acts on \( U^o \) and on \( U^o_{ev} \) in the following way:

\[
w.K_\mu := q^{(\mu, w^{-1} \rho - \rho)} K_{w\mu}.
\]

**Lemma.** \( \Upsilon(C(\hat{U}_0)) \subset (U^o_{ev})^w \oplus \xi(U^o_{ev})^w \)

*Proof.* Firstly, we shall check that

\[
\Upsilon(C(\hat{U}_0)) \subset (U^o)^w \oplus \xi(U^o)^w. \tag{24}
\]

We start by fixing some notations. Recall [2.2.3]. For any \( 1 \leq i \leq l \), we define \( \Gamma_i := \{ \xi_\mu, \mu \in \bigoplus_{j \neq i} (Z/2Z) w_j \} \), and the subalgebra \( U^o_\iota := (k \Gamma_i) U^o \). By definition, \( U^o = \hat{U}^o \oplus \xi_{w_i} \hat{U}^o \).

Fix \( a \in C(\hat{U}_0) \). For each \( i = 1, \ldots, l \), write

\[
\Upsilon(a) = P_i + \xi_{w_i} Q_i \tag{25}
\]

with \( P_i, Q_i \in \hat{U}^o_i \). We fix \( i < l \) and show that \( Q_i = 0 \).

Let \( (\lambda, \theta) \) be in \( P_\pi(\pi) \times \hat{\Gamma} \) such that \( \langle \lambda, \alpha_i \rangle \in N \). Consider \( \theta' \) defined by \( \theta' := (-1)^{\langle \lambda + \rho, \alpha_i \rangle} \theta \). In other words, \( \theta'(w_j) = \theta(w_j) \forall j \neq i \) and \( \theta'(w_i) = (-1)^{\langle \lambda + \rho, \alpha_i \rangle} \theta(w_i) \). According to [3.2.1], \( M(q^{s_{\alpha_i} \lambda}, \theta') \) is a submodule of \( M(q^\lambda, \theta) \). Moreover, as \( i < l \), \( \hat{U}_0 v_{s_{\alpha_i} \lambda} \subset
\[ \mathcal{U}_\alpha v_\lambda \] where \( v_\lambda, v_{s_\alpha, \lambda} \) stand for the vectors of highest weight of \( M(q^\lambda, \theta) \) and \( M(q^{s_\alpha, \lambda}, \theta') \).

It follows that

\[ \Upsilon(a)(s_\alpha, \lambda, (\lambda + p, \alpha_i)\alpha_i \lambda) = \Upsilon(a)(\lambda, \theta) \]

(26)

if \( \langle \lambda, \alpha_i \rangle \in \mathbb{N} \). We shall now check that formula (26) extends to all \( \lambda \in P_\theta(\pi) \). Indeed if \( \langle \lambda, \alpha_i \rangle = -1 \) then \( \lambda + p, \alpha_i \lambda = \lambda \) and (26) is obvious. If \( \langle \lambda, \alpha_i \rangle = -p - 2, \ p \geq 0 \), then \( s_\alpha \lambda = \lambda + (p + 1)\alpha_i \) and \( \langle s_\alpha, \lambda, \alpha_i \rangle = p \). We can then apply (26) to \( s_\alpha \lambda \), which establishes (26) for \( \lambda \).

Using notation (25), (26) can be rewritten as follows

\[ P_l(s_\alpha, \lambda, \theta_i) + \theta(w_i)(-1)^{(\lambda + p, \alpha_i)\alpha_i \lambda}Q_i(s_\alpha, \lambda, \theta_i) = P_l(\lambda, \theta_i) + \theta(w_i)Q_i(\lambda, \theta_i) \]

where \( \theta_i \) is the restriction of \( \theta \) to \( \Gamma_i \). Taking successively \( \theta(w_i) = \pm \) in the last equation, we obtain

\[ P_l(s_\alpha, \lambda, \theta_i) = P_l(\lambda, \theta_i) \]

(27)

\[ Q_i(s_\alpha, \lambda, \theta_i) = (-1)^{(\lambda + p, \alpha_i)\alpha_i \lambda}Q_i(\lambda, \theta_i) \]

(28)

Write \( Q_i = \sum_{(\mu, \gamma) \in P_\theta(\pi) \times \Gamma_i} a_{\mu, \gamma} K_\mu \xi_\gamma \). Then (28) implies that for all \( (\lambda, \theta) \in P_\theta(\pi) \times \widehat{\Gamma}_i \),

\[ \sum_{(\mu, \gamma) \in P_\theta(\pi) \times \Gamma_i} a_{\mu, \gamma} (-1)^{(\lambda, \alpha_i)\alpha_i \lambda}q(\mu, s_\alpha, \lambda)\theta(\gamma) + \sum_{(\mu, \gamma) \in P_\theta(\pi) \times \Gamma_i} a_{\mu, \gamma} q(\mu, \lambda)\theta(\gamma) = 0. \]

Since the characters \( P_\theta(\pi) \times \widehat{\Gamma}_i \rightarrow k, (\lambda, \theta) \mapsto (-1)^{(\lambda, \alpha_i)\alpha_i \lambda}q(\mu, s_\alpha, \lambda)\theta(\gamma) \) and \( (\lambda, \theta) \mapsto q(\mu, \lambda)\theta(\gamma) \) are pairwise distinct, the lemma of linear independence of the characters of Dedekind forces \( Q_i = 0 \).

Finally, \( i \) running from 1 to \( l \), we have proved that

\[ \Upsilon(a) = P + \xi Q \]

where \( P, Q \in \mathcal{U}^c \) are invariant under the action of the subgroup of \( W \) generated by the \( s_i, i < l \).

If \( \lambda \) is such that \( \langle \lambda + \rho, \beta_i \rangle \in 2\mathbb{N} + 2 \), then \( M(q^{s_\beta_i, \lambda}, \theta) \subset M(q^{\lambda}, \theta) \) with \( \mathcal{U}_\alpha v_{s_\beta_i, \lambda} \subset \mathcal{U}_\alpha v_\lambda \), and one shows, proceeding as above, that

\[ P(s_\beta_i, \lambda) = P(\lambda), \quad Q(s_\beta_i, \lambda) = Q(\lambda) \]

(29)

for all \( \lambda \in P_\theta(\pi) \) such that \( \langle \lambda + \rho, \beta_i \rangle \in 2\mathbb{Z} \). We shall check that (29) actually holds for all \( \lambda \in P_\theta(\pi) \). Let us treat the case of \( P \) for instance. The identity (29) can be rewritten as \( (s_\beta_i, P - P)(\frac{w_i}{2} + \lambda') = 0 \) for all \( \lambda' \in P_\theta(\pi) \). Write \( P = \sum_{\mu \in P_\theta(\pi)} a_\mu K_\mu \). Then

\[ \sum_{\mu \in P_\theta(\pi)} q^{(\mu, w_i)}(a_{s_\beta_i, \mu} q^{(\mu, p - s_\beta_i, \rho)} - a_\mu)q^{(\mu, \lambda')} = 0 \]

for all \( \lambda' \in P_\theta(\pi) \). The linear independence of the characters \( P_\theta(\pi) \rightarrow k, \lambda \mapsto q^{(\mu, \lambda)} \) forces the equalities \( a_{s_\beta_i, \mu} q^{(\mu, p - s_\beta_i, \rho)} = a_\mu \) and therefore \( s_\beta_i, P = P \).

Finally, \( P, Q \) are \( W \)-invariant and we have proved (24).
It remains to show that $P, Q$ are actually elements of $\mathcal{U}_{ev}^\phi$. For this, we should reproduce the reasoning above, analyzing now the action of $a$ on the $M(q^\phi, \theta)$ where $\phi \in \Gamma$.

Another way to do it is to imitate [Jan] 6.6, that is to consider for each $\phi \in \hat{\Gamma}$, the automorphism $\sigma_\phi$ which keeps $\mathcal{C}(\hat{\mathcal{U}}_{0})$ invariant, and sends $K_{\alpha_i}, E_i, F_i, \xi_i$ respectively to $\phi(\alpha_i)K_{\alpha_i}, \phi(\alpha_i)E_i, F_i, \xi_i$.

7.6. Proposition. $\mathcal{C}(\hat{\mathcal{U}}_{0}) = \mathcal{A}(\hat{\mathcal{U}})$.

Proof. By [J2] 7.17, $\Upsilon(\mathcal{Z}(\mathcal{U})) = (\mathcal{U}_{ev}^\phi)^W$. Since $\mathcal{A}(\hat{\mathcal{U}}) \subset \mathcal{C}(\hat{\mathcal{U}}_{0})$, we deduce from lemma [J2] that $\Upsilon(\mathcal{A}(\hat{\mathcal{U}})) = \Upsilon(\mathcal{C}(\hat{\mathcal{U}}_{0}))$. And lemma [J2] ends the proof.

REFERENCES

[AB] D. Arnaudon, M. Bauer, Scasimir operator, scentre and representations of $\mathcal{U}_q(\text{osp}(1,2))$, Lett. Math. Phys., 40 (1997), No.4, p. 307–320.

[ABF] D. Arnaudon, M. Bauer and L. Frappat, On Casimir’s ghost, Comm. Math. Phys, 187, No.2, (1997), p. 429–439.

[DK] C. De Concini and V. G. Kac, Representations of quantum groups at roots 1, Progress in Math., 92, Birkhäuser, Boston 1990, p. 471–506.

[GL] M. Gorelik and E. Lanzmann, The annihilation theorem for the completely reducible Lie superalgebras, Inv. Math., 137 (3) 1999, p. 651–680.

[GL2] M. Gorelik and E. Lanzmann, The minimal primitive spectrum of the enveloping algebra of the Lie algebra $\text{osp}(1, 2l)$, preprint.

[Jan] J. C. Jantzen, Lectures on Quantum Groups, Graduate studies in Math, Vol 6, Am. Math. Soc., 1995.

[J2] A. Joseph, Quantum groups and their primitive ideals, Springer Verlag (1995).

[J3] A. Joseph, On the prime and primitive spectra of the algebra of functions on a quantum group, J. Algebra, 169 (1994), no 2, p. 441–551.

[GL] A. Joseph and G. Letzter, Verma modules annihilators and quantized enveloping algebras, Ann. Ec. Norm. Sup. série 4, t. 28 (1995), p.493—526.

[JL2] A. Joseph and G. Letzter, Local finiteness for the adjoint action for quantized enveloping algebras, J. of Algebra, 153 (1992), p.289–318.

[JL3] A. Joseph and G. Letzter, Separation of variables for quantized enveloping algebras, Amer. J. Math., 116 (1994), p.127–177.

[Kash] M. Kashiwara, On crystal bases of the Q-analogue of universal enveloping algebras, Duke Math. J., 63 (1991), 2, p. 465–516.

[MZ] Ian. M. Musson and Yi. Ming Zou, Crystal Bases for $U_q(\text{osp}(1,2r))$, J. of Algebra, 210 (1998), p. 514–534

[Zh] R. B. Zhang, Finite-Dimensional Representations of $U_q(\text{osp}(1,2n))$ and its connection with Quantum $\text{so}(2n + 1)$, Letters in Math. Phys., 25 (1992), p. 317–325.

[Zou] Y. M. Zou, Integrable representations of $\mathcal{U}_q(\text{osp}(1,2n))$, J. Pure Appl. Algebra, 130 (1998), p. 99–112.

DEPT. OF THEORETICAL MATHEMATICS, THE WEIZMANN INSTITUTE OF SCIENCE, REHOVOT 76100, ISRAEL, EMAIL: LANZMANN@WISDOM.WEIZMANN.AC.IL