Generic properties of invariant measures of full-shift systems over perfect separable metric spaces

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Abstract

In this work, we are interested in characterizing typical (generic) dimensional properties of invariant measures associated with the full-shift system, \( T \), in a product space whose alphabet is uncountable. More specifically, we show that the set of invariant measures with upper Hausdorff dimension equal to zero and lower packing dimension equal to infinity is a dense \( G_δ \) subset of \( \mathcal{M}(T) \), the space of \( T \)-invariant measures endowed with the weak topology. We also show that the set of invariant measures with upper rate of recurrence equal to infinity and lower rate of recurrence equal to zero is a \( G_δ \) subset of \( \mathcal{M}(T) \). Furthermore, we also show that the set of invariant measures with upper quantitative waiting time indicator equal to infinity and lower quantitative waiting time indicator equal to zero is residual in \( \mathcal{M}(T) \).

Key words and phrases. Full-shift over an uncountable alphabet, Hausdorff dimension, packing dimension, invariant measures, rates of recurrence.

1 Introduction

Let \((M, \rho)\) be a complete separable (Polish) metric space, and let \( S \) be its \( \sigma \)-algebra of Borel sets. Now, define \((X, \mathcal{B})\) as the bilateral product of a countable number of copies of \((M, S)\). Note that \( \mathcal{B} \) coincides with the \( \sigma \)-algebra of the Borel sets in the product topology. Let \( d \) be any metric in \( X \) which is compatible with the product topology (that is, \( d \) induces an equivalent topology). It is straightforward to show that \((X, d)\) is also a Polish metric space.

One can define in \( X \) the so-called full-shift operator, \( T \), by the action

\[
Tx = y,
\]

where \( x = (\ldots, x_{-n}, \ldots, x_n, \ldots) \), \( y = (\ldots, y_{-n}, \ldots, y_n, \ldots) \), and for each \( i \in \mathbb{Z} \), \( y_i = x_{i-1} \). \( T \) is clearly an homeomorphism of \( X \) into itself. We choose \( d \) in such a way that \( T \) and \( T^{-1} \) are
Lipschitz transformations; set, for instance, for each \( x, y \in X \),
\[
d(x, y) = \sum_{|n| \geq 0} \frac{1}{2^{|n|}} \frac{\rho(x_n, y_n)}{1 + \rho(x_n, y_n)}.
\]

Let \( \mathcal{M}(T) \) be the space of all \( T \)-invariant probability measures, endowed with the weak topology (that is the coarsest topology for which the net \( \{\mu_\alpha\} \) converges to \( \mu \) if, and only if, for each bounded and continuous function \( f \), \( \int fd\mu_\alpha \to \int fd\mu \)). Since \( X \) is Polish, \( \mathcal{M}(T) \) is also a Polish metrizable space (see [10]).

Given \( \mu \in \mathcal{M}(T) \), the triple \( (X, T, \mu) \) is called an \( M \)-valued discrete stationary stochastic process (see [26, 34, 35]; see also [12] for a discussion of the role of such systems in the study of continuous self-maps over general metric spaces).

Some generic properties (in Baire’s sense; see Definition 1.6) of such \( M \)-valued discrete stationary stochastic processes, like ergodicity and zero entropy (this one in case \( M = \mathbb{R} \)), have been studied by Parthasarathy in [26] and Sigmund in [34, 35], respectively. In the last decade, various studies about the full-shift system over an uncountable alphabet have been performed; more specifically, we can mention the works about the Gibbs state in Ergodic Optimization [3], entropy and the variational principle for one-dimensional lattice systems [22], and the variational principle for the specific entropy [1].

In this work, we are interested in the generic dimensional properties of such \( T \)-invariant measures, more specifically, in their Hausdorff and packing dimensions (generally called fractal dimensions). Hence, we recall some basic definitions involving Hausdorff and packing measures, giving a special treatment to the packing dimension of a measure defined in a general metric space (open and closed balls in \( \mathbb{R}^N \) possess nice regularity properties; for example, the diameter of a ball is twice its radius, and open and closed balls of the same radius have the same diameter. In arbitrary metric spaces, the possible absence of such regularity properties means that the usual measure construction based on diameters can lead to packing measures with undesirable features, as was observed by Cutler in [9]).

For a dynamical system, the fractal dimensions of an invariant measure provide more relevant information than the fractal dimensions of its invariant sets, or even the fractal dimensions of its topological support; the point is that invariant sets and topological supports usually contain superfluous sets (in the sense that they have zero measure). Thus, by establishing the fractal dimensions of invariant measures, one has a more precise information about the structure of the relevant sets (of positive measure) of a dynamical system (see [31, 27, 28] for a more detailed discussion).

In what follows, \((X, d)\) is an arbitrary metric space and \( B = B(X) \) is its Borel \( \sigma \)-algebra.

**Definition 1.1** (radius packing \( \phi \)-premeasure, [29]). Let \( \emptyset \neq E \subset X \), and let \( 0 < \delta < 1 \). A \( \delta \)-packing of \( E \) is a countable collection of disjoint closed balls \( \{B(x_k, r_k)\}_k \) with centers \( x_k \in E \) and radii satisfying \( 0 < r_k \leq \delta/2 \), for each \( k \in \mathbb{N} \) (the centers \( x_k \) and radii \( r_k \) are considered part of the definition of the \( \delta \)-packing). Given a measurable function \( \phi \), the radius packing \( (\phi, \delta) \)-premeasure of \( E \) is given by the law
\[
P^\phi_\delta(E) = \sup \left\{ \sum_{k=1}^{\infty} \phi(2r_k) \mid \{B(x_k, r_k)\}_k \text{ is a } \delta \text{-packing of } E \right\}.
\]
Letting $\delta \to 0$, one gets the so-called *radius packing $\phi$-premeasure*

$$P_0^\phi(E) = \lim_{\delta \to 0} P_\delta^\phi(E).$$

One sets $P_\delta^\phi(\emptyset) = P_0^\phi(\emptyset) = 0$.

It is easy to see that $P_0^\phi$ is non-negative and monotone. Moreover, $P_0^\phi$ generally fails to be countably sub-additive. One can, however, build an outer measure from $P_0^\phi$ by applying Munroe’s Method I construction, described both in [24] and [31]. This leads to the following definition.

**Definition 1.2** (radius packing $\phi$-measure, [9]). The radius packing $\phi$-measure of $E \subset X$ is defined to be

$$P^\phi(E) = \inf \left\{ \sum_k P_0^\phi(E_k) \mid E \subset \bigcup_k E_k \right\}. \quad (2)$$

The infimum in (2) is taken over all countable coverings $\{E_k\}_k$ of $E$. It follows that $P$ is an outer measure on the subsets of $X$.

In an analogous fashion, one may define the Hausdorff $\phi$-measure. The theory of Hausdorff measures in general metric spaces is a well-explored topic; see, for example, the treatise by Rogers [31].

**Definition 1.3** (Hausdorff $\phi$-measure, [9]). For $E \subset X$, the outer measure $H^\phi(E)$ is defined by

$$H^\phi(E) = \lim_{\delta \to 0} \inf \left\{ \sum_{k=1}^\infty \phi(\text{diam}(E_k)) \mid \{E_k\}_k \text{ is a } \delta\text{-covering of } E \right\}, \quad (3)$$

where a $\delta$-covering of $E$ is any countable collection $\{E_k\}_k$ of subsets of $X$ such that, for each $k \in \mathbb{N}$, $E \subset \bigcap_k E_k$ and $\text{diam}(E_k) \leq \delta$. If no such $\delta$-covering exists, one sets $H_\phi(E) = \inf \emptyset = \infty$.

Of special interest is the situation where given $\alpha > 0$, one sets $\phi(t) = t^\alpha$. In this case, one uses the notation $P_0^\alpha$, and refers to $P_0^\alpha(E)$ as the $\alpha$-packing premeasure of $E$. Similarly, one uses the notation $P^\alpha(E)$ for the packing $\alpha$-measure of $E$, and $H^\alpha(E)$ for the $\alpha$- Hausdorff measure of $E$.

**Definition 1.4** (Hausdorff and packing dimensions of a set, [9]). Let $E \subset X$. One defines the Hausdorff dimension of $E$ to be the critical point

$$\dim_H(E) = \inf\{\alpha > 0 \mid H^\alpha(E) = 0\};$$

one defines the packing dimension of $E$ in the same fashion.

We note that $\dim_H(X)$ or $\dim_P(X)$ may be infinite for some metric space $X$. One can show that, for each $E \subset X$, $\dim_H(E) \leq \dim_P(E)$ (see Theorem 3.11(h) in [9]), and this inequality is in general strict.
Definition 1.5 (lower and upper packing and Hausdorff dimensions of a measure\cite{23}). Let \( \mu \) be a positive Borel measure on \((X, \mathcal{B})\). The lower and upper packing and Hausdorff dimension of \( \mu \) are defined, respectively, by
\[
\dim^*_K(\mu) = \inf \{ \dim_K(E) \mid \mu(E) > 0, \ E \in \mathcal{B} \}, \\
\dim^+_K(\mu) = \inf \{ \dim_K(E) \mid \mu(X \setminus E) = 0, \ E \in \mathcal{B} \},
\]
where \( K \) stands for \( H \) (Hausdorff) or \( P \) (packing).

Let \( \mu \) be a positive finite Borel measure on \( X \). One defines the upper and lower local dimensions of \( \mu \) at \( x \in X \) by
\[
\overline{d}_\mu(x) = \limsup_{\varepsilon \to 0} \frac{\log \mu(B(x, \varepsilon))}{\log \varepsilon} \quad \text{and} \quad \underline{d}_\mu(x) = \liminf_{\varepsilon \to 0} \frac{\log \mu(B(x, \varepsilon))}{\log \varepsilon},
\]
if, for every \( \varepsilon > 0, \ \mu(B(x; \varepsilon)) > 0 \); if not, \( \overline{d}_\mu(x) := +\infty \).

Proposition 1.1. Let \( \mu \) be a probability measure on \( X \). Then,
\[
\mu\text{-ess}\inf \underline{d}_\mu(x) = \dim^*_H(\mu) \leq \mu\text{-ess}\sup \overline{d}_\mu(x) = \dim^+_H(\mu), \\
\mu\text{-ess}\inf \overline{d}_\mu(x) = \dim^*_P(\mu) \leq \mu\text{-ess}\sup \underline{d}_\mu(x) = \dim^+_P(\mu).
\]

We are also interested in the polynomial returning rates of the \( T \)-orbit of a given point to arbitrarily small neighborhoods of itself (this gives, in some sense, a quantitative description of Poincaré’s recurrence). This question was posed and studied by Barreira and Saussol in \cite{5} (see also \cite{2, 6, 17} for further motivations). Given a separable metric space \( X \) and a Borel measurable transformation \( T \), they have defined the lower and upper recurrence rates of \( x \in X \) in the following way: for each fixed \( r > 0 \), let \( \tau_r(x) = \inf \{ k \in \mathbb{N} \mid T^k x \in \overline{B}(x, r) \} \) be the return time of a point \( x \in X \) into the closed ball \( \overline{B}(x, r) \); then,
\[
\underline{R}(x) = \liminf_{r \to 0} \frac{\log \tau_r(x)}{-\log r} \quad \text{and} \quad \overline{R}(x) = \limsup_{r \to 0} \frac{\log \tau_r(x)}{-\log r}
\]
are, respectively, the lower and upper recurrence rates of \( x \in X \).

Barreira and Saussol have showed (Theorem 2 in \cite{5}) that \( \underline{R}(x) \leq \dim^*_H(\mu) \) for \( \mu\text{-a.e.} \ x \in X \). They also have showed, when \( X \subset \mathbb{R}^n \), that \( \underline{R}(x) \leq \underline{d}_\mu(x) \), and that \( \overline{R}(x) \leq \overline{d}_\mu(x) \) for \( \mu\text{-a.e.} \ x \in X \). Later, Saussol has showed in \cite{32} that, under the hypotheses that \( T \) is a Lipschitz transformation, \( h_\mu(T) > 0 \), and that the decay of the correlations of \((X, T, \mu)\) is super-polynomial, \( \overline{R}(x) = \overline{d}_\mu(x) \), and that \( \underline{R}(x) = \underline{d}_\mu(x) \) for \( \mu\text{-a.e.} \ x \in X \). In this work, we present some results, for \( M \)-valued discrete stationary stochastic processes, relating \( \overline{R} \) and \( \underline{d}_\mu \).

Another dynamical aspect of \( M \)-valued discrete stationary stochastic processes that is explored in this work refers to the quantitative waiting time indicators, defined by Galatolo in \cite{13} as follows: let \( x, y \in X \) and let \( r > 0 \). The first entrance time of \( \mathcal{O}(x) := \{ T^i x \mid i \in \mathbb{Z} \} \), the \( T \)-orbit of \( x \), into the closed ball \( \overline{B}(y, r) \) is given by
\[
\tau_r(x, y) = \min \{ n \in \mathbb{N} \mid T^n(x) \in \overline{B}(y, r) \}.
\]
\(^1\)Actually, the definition presented in \cite{5} uses open balls; it is straightforward to show that they coincide.
Naturally, \( \tau_r(x, x) \) is just the first return time into the closed ball \( B(x, r) \). The so-called quantitative waiting time indicators are defined as

\[
\underline{R}(x, y) = \lim \inf_{r \to 0} \frac{\log \tau_r(x, y)}{\log r} \quad \text{and} \quad \overline{R}(x, y) = \lim \sup_{r \to 0} \frac{\log \tau_r(x, y)}{\log r}.
\]

Let \((X, T)\) be a dynamical system such that \(X\) is a separable metric space and \(T : X \to X\) is a measurable map, and suppose that there exists a \(T\)-invariant measure \(\mu\). Then, Theorem 4 in [13] states that, for each fixed \(y \in X\), one has

\[
\underline{R}(x, y) \geq d_\mu(y) \quad \text{and} \quad \overline{R}(x, y) \geq \overline{d}_\mu(y) \quad \text{for \(\mu\)-a.e.} \ x \in X.
\] (4)

Furthermore, even if \(\mu\) is only a probability measure on \(X\), Theorem 10 in [13] states that for each \(x \in X\), one has \(\underline{R}(x, y) \geq d_\mu(y)\) and \(\overline{R}(x, y) \geq \overline{d}_\mu(y)\) for \(\mu\)-a.e. \(y \in X\).

Before we present our main results, some preparation is required. Recall that a subset \(R\) of a topological space \(X\) is residual if it contains the intersection of a countable family, \(\{U_k\}\), of open and dense subsets of \(X\). A topological space \(X\) is a Baire space if every residual subset of \(X\) is dense in \(X\). By Baire’s Category Theorem, every complete metric space is a Baire space.

**Definition 1.6.** A property \(P\) is said to be generic in the space \(X\) if there exists a residual subset \(R\) of \(X\) such that each \(x \in R\) satisfies property \(P\).

Note that, given a countable family of generic properties \(P_1, P_2, \ldots\), all of them are simultaneously generic in \(X\). This is because the family of residual sets is closed under countable intersections.

We shall prove the following results.

**Theorem 1.1.** Let \((X, T, \mathcal{B})\) be the full-shift dynamical system over \(X = \prod_{-\infty}^{+\infty} M\), where the alphabet \(M\) is a perfect and separable metric space.

I. The set of ergodic measures, \(\mathcal{M}_e\), is residual in \(\mathcal{M}(T)\).

II. The set of invariant measures with full support, \(C_X\), is a dense \(G_\delta\) subset of \(\mathcal{M}(T)\).

III. The set \(HD = \{\mu \in \mathcal{M}_e \mid \dim_H^+ (\mu) = 0\}\) is a dense \(G_\delta\) subset of \(\mathcal{M}_e\).

IV. The set \(PD = \{\mu \in \mathcal{M}_e \mid \dim_P(\mu) = +\infty\}\) is a dense \(G_\delta\) subset of \(\mathcal{M}_e\).

V. The set \(R = \{\mu \in \mathcal{M}_e \mid \underline{R}(x) = 0, \text{ for } \mu\text{-a.e.} \ x\}\) is a dense \(G_\delta\) subset of \(\mathcal{M}_e\).

VI. If \(M\) is also compact, then the set \(\overline{R} = \{\mu \in \mathcal{M}_e \mid \overline{R}(x) = +\infty, \text{ for } \mu\text{-a.e.} \ x\}\) is a dense \(G_\delta\) subset of \(\mathcal{M}_e\).

VII. The set \(\overline{\mathcal{R}} = \{\mu \in \mathcal{M}_e \mid \overline{R}(x, y) = 0, \text{ for } (\mu \times \mu)\text{-a.e.} \ (x, y) \in X \times X\}\) is a dense \(G_\delta\) subset of \(\mathcal{M}_e\).

VIII. If \(M\) is also compact, then the set \(\overline{\mathcal{R}} = \{\mu \in \mathcal{M}_e \mid \overline{R}(x, y) = +\infty, \text{ for } (\mu \times \mu)\text{-a.e.} \ (x, y) \in X \times X\}\) is residual in \(\mathcal{M}_e\).
Item I was proved by Oxtoby in [25] and Parthasarathy in [26], using the fact that the set of $T$-periodic or $T$-closed orbit measures (that is, measures of the form $\frac{1}{k_x} \sum_{i=0}^{k_x-1} \delta_{T^i x}(-)$, where $x$ is a $T$-periodic point of period $k_x$) is dense in $\mathcal{M}(T)$. Sigmund has proved item II in [35] (see also [10]). We have opted to include these results in Theorem 1.1 since they are used in the proofs of items III-VIII, which comprise our main contributions to the problem.

As a direct consequence of Theorem 1.1 we have obtained for typical ergodic measures, $\mu \in \mathcal{RD} = \mathcal{R} \cap \mathcal{R} \cap PD \cap HD$, some relations between $\mathcal{R}$ and $\mathcal{R}_{\mu}$, similar to those obtained by Saussol and Barreira (as discussed above).

**Corollary 1.1.** Let $M$ be a compact and perfect metric space, and let $\mu \in \mathcal{RD} \subset \mathcal{M}_e$. Then, $\mathcal{R}(x) = d_{\mu}(x) = 0$ and $\overline{\mathcal{R}}(x) = \mathcal{R}_{\mu}(x) = \infty$, for $\mu$-a.e. $x \in X$.

It follows from items III and IV in Theorem 1.1 that there exists a dense $G_\delta$ set, $D := PD \cap HD \subset \mathcal{M}_e$, such that each $\mu \in D$ is somewhat similar, in one hand, to a “uniformly distributed” measure, whose lower packing dimension is maximal (for instance, when $X = [0, 1]^2$, the shift Bernoulli measure $\Lambda = \prod_{-\infty}^{\infty} \lambda$, where $\lambda$ is the Lebesgue measure on $[0, 1]$, is an uniformly distributed measure, whose lower packing dimension is infinite) and in the other hand, to a purely point measure, whose upper Hausdorff dimension is zero.

Moreover, by Definition 1.3 each $\mu \in D$ is supported on a Borel set $Z = Z(\mu)$ such that $\dim_{\text{top}}(Z) \leq \dim_H(Z) = 0 < \dim_P(Z) = \infty$, where $\dim_{\text{top}}(Z)$ stands for the topological dimension of $Z$ (see, [18] Sect. 4, page 107, for a proof of the inequality $\dim_{\text{top}}(Z) \leq \dim_H(Z)$). Since $\dim_{\text{top}}(Z) = 0$, if one also assumes that $\text{supp}(\mu) = X$ (just take $\mu \in C_X \cap D$), one concludes that $Z$ is a dense and totally disconnected set in $X$ with zero Hausdorff and infinite packing dimensions. Furthermore, one may take $Z$ as a dense $G_\delta$ subset of $X$ (see Proposition 2.6).

**Remark 1.1.** It is worth noting that although the packing dimension of $X$ is infinite (since, for each $\mu \in D$, $\dim_P(Z) = \infty$), its topological dimension may be finite. This is not unexpected, altogether: there are examples of topological spaces where $\dim_P(X) > \dim_{\text{top}}(X)$ (see example 2.12 in [27], where $0 = \dim_{\text{top}}(X) < \dim_P(X)$).

Items V and VI in Theorem 1.1 say that there exists a dense $G_\delta$ set, $\mathcal{R} := \mathcal{R} \cap \mathcal{R} \subset \mathcal{M}_e$, of ergodic measures such that if $\mu \in \mathcal{R}$, then there exists a Borel set $Z$, with $\mu(Z) = 1$, so that if $x \in Z$, then $\mathcal{R}(x) = 0$ and $\overline{\mathcal{R}}(x) = \infty$. This means that given a very large $\alpha$ and a very small $\beta$, for each $x \in Z$, one has $\mathcal{R}(x) \leq \beta$ and $\overline{\mathcal{R}}(x) \geq \alpha$. So, there exist sequences $(\varepsilon_k)$ and $(\sigma_l)$ converging to zero such that, for each $k, l \in \mathbb{N}$, $\tau_{\varepsilon_k}(x) \leq \varepsilon_k^{-\beta}$ and $\tau_{\sigma_l}(x) \geq \sigma_l^{-\alpha}$, respectively. Setting, for each $k, l \in \mathbb{N}$, $s_k = 1/\varepsilon_k$ and $t_l = 1/\sigma_l$, one has $\tau_{1/s_k}(x) \leq s_k^\beta$ and $\tau_{1/t_l}(x) \geq t_l^\alpha$, respectively.

Therefore, given $x \in Z$, there exists a time sequence (time scale) for which the first incidence of $\mathcal{O}(x)$ to one of its spherical neighborhoods (which depend on time) occurs as fast as possible (that is, it is of order 1; this means that the first return time to those neighborhoods increases sub-polynomially fast); accordingly, there exists a time sequence for which the first incidence of $\mathcal{O}(x)$ to one of its spherical neighborhoods increases as fast as possible (that is, super-polynomially fast).

The following scheme tries to depict how subsequent elements of both sequences are related. Between to consecutive elements of $(\sigma_k)$, there are several elements of $(\varepsilon_l)$:
Here, we also show that the typical measures described in Theorem 1.1 are supported on the dense $G_\delta$ set $\mathcal{R} = \{ x \in X \mid R(x) = 0 \text{ and } \overline{R}(x) = \infty \}$ (Proposition 3.5).

Finally, combining items II, VII and VIII of Theorem 1.1, one concludes that for a typical measure $\mu \in \mathcal{M} \cap \mathcal{R} \cap \mathcal{C}$, almost every $T$-orbit $O(x)$ densely fills the whole space (given that $\mu$ is supported on a dense subset of $X$ and $R(x, y) = 0$ for $(\mu \times \mu)$-a.e $(x, y) \in X \times X$), but not in a homogeneous fashion. Namely, as in the previous analysis, there exists a time scale for which the first entrance time of $O(x)$ to one of the spherical neighborhoods (which depend on time) of $y$ is of order 1; accordingly, there exists a time sequence for which the first entrance time of the $O(x)$ to one of the spherical neighborhoods of $y$ increases as fast as possible. Naturally, these time scales depend on the pair $(x, y) \in X \times X$.

**Remark 1.2.**

i) It is true that the sets defined in items III to VIII of Theorem 1.1 are $G_\delta$ subsets of $\mathcal{M}(T)$ for any topological dynamical system $(X, T)$ such that $X$ is Polish and both $T, T^{-1}$ are Lipschitz transformations (this is particularly true for Axiom A systems on smooth compact Riemannian n-manifolds, $(M, T)$, where $f : M \to M$ is a diffeomorphism: there exists a hyperbolic metric for which both $f$ and $f^{-1}$ are Lipschitz; see Theorem 5.1 in [11]).

ii) It is also true that the sets defined in items III, V and VII of Theorem 1.1 are dense in $\mathcal{M}(T)$ for any topological dynamical system $(X, T)$ such that $X$ is a separable metric space and the set of $T$-periodic measures is dense in $\mathcal{M}(T)$. This is particularly true for any system satisfying the specification property (see [35] for a proof of this proposition and examples of systems that satisfy this property; see also [29]), or even milder conditions (see [14, 16, 19, 20, 21] for a broader discussion involving such conditions).

iii) Since the Axiom A systems described in item i) also have a dense set of $T$-periodic measures (here, $X$ stands for a closed $f$-invariant set and $T := f \upharpoonright X$ is topologically transitive; see [33]), one may combine both properties and obtain the following result.

**Theorem 1.2.** Let $(X, T)$ be an Axiom A system as described in items i) and iii) above. Then, the set $\{ \mu \in \mathcal{M}_e \mid R(x, y) = 0, \text{ for } (\mu \times \mu)$-a.e. $(x, y) \in X \times X \}$ is a dense $G_\delta$ subset of $\mathcal{M}_e$.

iii) The hypothesis that the alphabet $M$ is perfect is crucial for items IV and VI of Theorem 1.1. Namely, the fact that $M$ does not have isolated points is required to guarantee that one can always choose the periodic point $x$ of period $s$ in the statement of Lemma 2.5 so that $x_i \neq x_j$ if $i \neq j$, $1 \leq i, j \leq s$. This result, whose proof relies on the product structure of $X$, is required in the proof of Proposition 2.3, which in turn guarantees that the sets presented in items IV and VI of Theorem 1.1 are dense. Indeed, our strategy depends on the fact that the set of ergodic measures with arbitrarily large entropy is dense in $\mathcal{M}_e$.

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2 By a topological dynamical system we understand a pair $(X, T)$ such that $X$ is a Polish metric space and $T : X \to X$ is a continuous transformation.
here, we explicitly use the fact that the lower packing dimension of an ergodic measure is lower bounded by (up to a constant) its entropy (see Lemma 2.4).

The paper is organized as follows. In Section 2 we present several results (some of those important in themselves) used in the proof of items III and IV of Theorem 1.1. Section 3 is devoted to the proof of items V-VIII of Theorem 1.1 and as in Section 2 we prove some auxiliary results which are the counterparts, for the returning rates and the first entrance rates, of the results stated in Section 2. In Appendix, we present the proof of Proposition 1.1.

2 Sets of ergodic measures with zero Hausdorff and infinity packing dimensions

2.1 $G_\delta$ sets

**Lemma 2.1.** Let $(X, T)$ be a topological dynamical system, where $X$ is endowed with the metric $d$, and let $\mu \in \mathcal{M}(T)$. Then, for each $x \in X$,

$$d_{x}(x) = \lim_{\varepsilon \to 0} \sup(\inf_{\varepsilon} \frac{\log f_{x,\varepsilon}(\mu)}{\log \varepsilon}),$$

where, for each $x \in X$ and each $\varepsilon > 0$,

$$f_{x,\varepsilon}(\cdot) : \mathcal{M}(T) \to [0, 1] \text{ is defined by the law } f_{x,\varepsilon}(\mu) := \int f_{x,\varepsilon}(y) d\mu(y),$$

and $f_{x,\varepsilon} : X \to [0, 1]$ is defined by the law

$$f_{x,\varepsilon}(y) := \begin{cases} 1, & \text{if } d(x, y) \leq \varepsilon, \\ \frac{-d(x, y)}{\varepsilon} + 2, & \text{if } \varepsilon \leq d(x, y) \leq 2\varepsilon, \\ 0, & \text{if } d(x, y) \geq 2\varepsilon. \end{cases}$$

Furthermore, the function $f_\varepsilon(\mu, x) = f_{x,\varepsilon}(\mu) : \mathcal{M}(T) \times X \to [0, 1]$ is jointly continuous.

**Proof.** It follows from the definition of $f_{x,\varepsilon}$ that, for each $x \in X$ and each $\varepsilon > 0$, $f_{x,\varepsilon} \leq \mu(B(x, \varepsilon))$. Then, if $\mu(B(x, \varepsilon)) > 0$, one has $\frac{\log f_{x,\varepsilon}(\mu)}{\log \varepsilon} \geq \frac{\log \mu(B(x, \varepsilon))}{\log \varepsilon} \geq \frac{\log \mu(B(x, \varepsilon))}{\log \varepsilon}$, which proves the first assertion. If $\mu(B(x, \varepsilon)) = 0$, given that $f_{x,\varepsilon} \leq \mu(B(x, \varepsilon))$, just set $\lim_{\varepsilon \to 0} \sup \left(\inf_{\varepsilon} \frac{\log f_{x,\varepsilon}(\mu)}{\log \varepsilon}\right) = +\infty$.

Note that, for each $x \in X$ and each $\varepsilon > 0$, $f_{x,\varepsilon} : X \to \mathbb{R}$ is a continuous function such that, for each $y \in X$, $\chi_{B(x, \varepsilon/2)}(y) \leq f_{x,\varepsilon}(y) \leq \chi_{B(x, \varepsilon)}(y)$. Given that $f_{x,\varepsilon}(y)$ depends only on $d(x, y)$, it is straightforward to show that $f_{x,\varepsilon}$ converges uniformly to $f_{x,\varepsilon}$ on $X$ when $x_n \to x$.

We combine this remark with Theorems 2.13 and 2.15 in [15] in order to prove that $f_\varepsilon(\mu, x)$ is jointly continuous. Let $(\mu_m)$ and $(x_n)$ be sequences in $\mathcal{M}(T)$ and $X$, respectively, such that $\mu_m \to \mu$ and $x_n \to x$. Firstly, we show that

$$\lim_{m \to \infty} \lim_{n \to \infty} f_\varepsilon(\mu_m, x_n) = \lim_{m \to \infty} \lim_{n \to \infty} \int f_{x_n,\varepsilon}(y) d\mu_m(y) = f_\varepsilon(\mu, x).$$
Since, for each \( y \in X \), \(|f_{x_n}^\varepsilon(y)| \leq 1\), it follows from dominated convergence that, for each \( m \in \mathbb{N} \), \( \lim_{n \to \infty} \int f_{x_n}^\varepsilon(y)d\mu_m(y) = \int f_x^\varepsilon(y)d\mu_m(y) \). Now, since \( f_x^\varepsilon \) is continuous, it follows from the definition of weak convergence that
\[
\lim_{m \to \infty} \lim_{n \to \infty} \int f_{x_n}^\varepsilon(y)d\mu_m(y) = \lim_{m \to \infty} \int f_x^\varepsilon(y)d\mu_m(y) = f_x(\mu, x).
\]

The next step consists in showing that, for each \( n \in \mathbb{N} \), the function \( \varphi_n : N \to \mathbb{N} \), defined by the law \( \varphi_n(m) := f_{x_n}(\mu_m, x_n) \), converges uniformly in \( m \in \mathbb{N} \) to \( \varphi(m) := \lim_{n \to \infty} f_{x_n}(\mu_m, x_n) = \int f_x^\varepsilon(y)d\mu_m(y) \). Let \( \delta > 0 \) and fix \( m \in \mathbb{N} \). Since \( f_{x_n}^\varepsilon \) converges uniformly to \( f_x^\varepsilon \), there exists \( N \in \mathbb{N} \) such that, for each \( n \geq N \) and each \( y \in X \), \(|f_{x_n}^\varepsilon(y) - f_x^\varepsilon(y)| < \delta\). Then, one has, for each \( n \geq N \),
\[
|\varphi_n(m) - \varphi(m)| = \left| \int f_{x_n}^\varepsilon(y)d\mu_m(y) - \int f_x^\varepsilon(y)d\mu_m(y) \right| \leq \int |f_{x_n}^\varepsilon(y) - f_x^\varepsilon(y)|d\mu_m(y) < \delta.
\]

It follows from Theorem 2.15 in [15] that \( \lim_{n,m \to \infty} f_x^\varepsilon(\mu_m, x_n) = f_x(\mu, x) \). Given that \( \lim_{n \to \infty} f_x^\varepsilon(\mu_m, x_n) = \int f_x^\varepsilon(y)d\mu_m(y) \) and that \( \lim_{m \to \infty} f_x^\varepsilon(\mu_m, x_n) = \lim_{n \to \infty} f_x^\varepsilon(\mu_m, x_n) = f_x(\mu, x) \) exist for each \( m \in \mathbb{N} \) and each \( n \in \mathbb{N} \), respectively, Theorem 2.13 in [15] implies that
\[
\lim_{m \to \infty} \lim_{n \to \infty} f_x^\varepsilon(\mu_m, x_n) = \lim_{n \to \infty} \lim_{m \to \infty} f_x^\varepsilon(\mu_m, x_n) = \lim_{n,m \to \infty} f_x^\varepsilon(\mu_m, x_n) = f_x(\mu, x).
\]

Hence, if \((\mu_n, x_n)\) is some sequence in \( \mathcal{M}(T) \times X \) (endowed with the product topology) such that \((\mu_n, x_n) \to (\mu, x)\), then \( \lim_{n \to \infty} f_x^\varepsilon(\mu_n, x_n) = f_x(\mu, x) \), showing that \( f_x(\cdot, \cdot) = f_x^\varepsilon(\cdot, \cdot) \) is jointly continuous at \((\mu, x)\).

For each \( t > 0 \), let \( \varepsilon = 1/t \). Since, for each \( x \in X \),
\[
\overline{d}_\mu(x) = \lim_{t \to 0} \frac{\log f_x^\varepsilon(\mu)}{\log \varepsilon} = \lim_{s \to \infty} \sup \left( \inf_{t \geq s} \frac{\log f_{x,1/t}(\mu)}{-\log t} \right),
\]
we set, for each \( s \in \mathbb{N} \),
\[
\overline{\beta}_\mu(x, s) = \sup_{t > s} \frac{\log f_{x,1/t}(\mu)}{-\log t} \quad \text{and} \quad \underline{\beta}_\mu(x, s) = \inf_{t > s} \frac{\log f_{x,1/t}(\mu)}{-\log t};
\]
note that \( \overline{\beta}_\mu(x, s) \) is non-decreasing, whereas \( \underline{\beta}_\mu(x, s) \) is non-increasing in \( s \).

From now on, we assume (save when we explicitly say otherwise) that \((X, d)\) is a Polish metric space and that \( T : X \to X \) is a Lipschitz function, with Lipschitz constant \( \Lambda > 1 \). Assume also that \( T^{-1} : X \to X \) exists as a Lipschitz function, with Lipschitz constant \( \Lambda' > 1 \).

**Proposition 2.1.** Let \( \mu \in \mathcal{M}(T) \). Then, for each \( x \in X \), \( \overline{d}_\mu(x) = \overline{d}_\mu(Tx), \underline{d}_\mu(x) = \underline{d}_\mu(Tx) \).
Furthermore, the sets
\[
Z_1 = \{ x \in X \mid \overline{\beta}_\mu(x, s) \text{ converges uniformly to } \overline{d}_\mu(x) \},
\]
\[
Z_2 = \{ x \in X \mid \underline{\beta}_\mu(x, s) \text{ converges uniformly to } \underline{d}_\mu(x) \}
\]
are \( T \)-invariant.
Proof. It follows from Birkhoff’s Ergodic Theorem that, for each \( z \in X \) and each \( \varepsilon > 0 \), the limit

\[
\hat{\varphi}_{B(z, \varepsilon)}(y) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} f_z^i(T^i(y))
\]

exists for \( \mu \)-a.e. \( y \in X \), and

\[
\int \hat{\varphi}_{B(z, \varepsilon)}(y) d\mu(y) = \int f_z^\varepsilon(y) d\mu(y) = f_{z, \varepsilon}(\mu).
\]

Fix \( x \in \text{supp}(\mu) \). It is straightforward to show that, for each \( y \in X \) and each \( i \in \mathbb{N} \cup \{0\} \), one has \( f_z^{s/L}(T^i(y)) \leq f_z^{s/L}(T^{i+1}(y)) \). Letting \( z = x \) and \( z = Tx \) in (5), respectively, one gets \( \hat{\varphi}_{B(x, \varepsilon/3)}(y) \leq \hat{\varphi}_{B(Tx, \varepsilon)}(y) \) for \( \mu \)-a.e. \( y \in X \), from which follows that \( f_{x, \varepsilon/L}(\mu) \leq f_{T, \varepsilon}(\mu) \).

Case 1: \( x \in \text{supp}(\mu) \). Note that, for each \( \eta > 0 \), \( f_{x, \eta}(\mu) > 0 \). Let \( \varepsilon = 1/t \), \( t = L/\Lambda \) and \( s \geq 1 + 1/\Lambda \); then,

\[
\sup_{t \geq s} \frac{\log f_{T^1/T}(\mu)}{-\log t} \leq \sup_{t \geq \Lambda s} \frac{\log l - \log \Lambda}{-\log l} \leq \frac{\log(\Lambda s)}{-\log l} \sup_{t \geq \Lambda s} \frac{\log f_{T^1/T}(\mu)}{-\log t} = A(s) \sup_{t \geq \Lambda s} \frac{\log f_{T^1/T}(\mu)}{-\log t},
\]

where \( A(s) := \frac{\log s + \log \Lambda}{\log s} \) (since \( s \geq 1 + 1/\Lambda \), one has \( l \geq \Lambda + 1 \)).

Using the same idea, one can prove that \( f_{x, \varepsilon/L'}(\mu) \leq f_{T^{-1}, \varepsilon}(\mu) \); letting \( z = Tx \), one gets \( f_{T^x, \varepsilon/L'}(\mu) \leq f_{x, \varepsilon}(\mu) \). Thus, the previous discussion leads to

\[
\bar{\beta}_\mu(Tx, s) \leq A(s)\bar{\beta}_\mu(x, s) \quad \text{and} \quad \bar{\beta}_\mu(x, s) \leq A(s)\bar{\beta}_\mu(Tx, \Lambda' s);
\]

one can combine these inequalities and obtain, for each \( x \in X \) and each \( s \geq \max\{1 + 1/\Lambda, 1 + 1/\Lambda'\} \),

\[
\bar{\beta}_\mu(Tx, s) \leq A(s)\bar{\beta}_\mu(x, s) \leq A(s)A(\Lambda s)\bar{\beta}_\mu(Tx, \Lambda \cdot \Lambda' s).
\]

Now, taking the limit \( s \to \infty \) in the inequalities above and observing that \( A(s) \) is a decreasing function such that \( \lim_{s \to \infty} A(s) = 1 \), one gets \( \bar{d}_\mu(Tx) = \bar{d}_\mu(x) \).

Case 2: \( x \notin \text{supp}(\mu) \). It follows from the \( T \)-invariance of \( \text{supp}(\mu) \) that \( Tx \notin \text{supp}(\mu) \); thus,

\[
\bar{d}_\mu(Tx) = +\infty = \bar{d}_\mu(x).
\]

The proof that, for each \( x \in X \), \( d_\mu(Tx) = d_\mu(x) \), is analogous; therefore, we omit it.

It remains to prove that \( Z_1 \) and \( Z_2 \) are \( T \)-invariant; since the arguments in both proofs are similar, we just prove the statement for \( Z_1 \).

Write \( Z_1 = (\bigcup_{k \geq 0} B_k) \cup B_\infty \), where for each \( k \in \mathbb{N} \cup \{0\} \), \( B_k = \{ x \in X \mid \bar{\beta}_\mu(x, s) \to \bar{d}_\mu(x) \} \)

uniformly, \( k \leq \bar{d}_\mu(x) < k + 1 \), and \( B_\infty = \{ x \in X \mid \bar{\beta}_\mu(x, s) \to \bar{d}_\mu(x) \} \)

uniformly, \( \bar{d}_\mu(x) = \infty \).

Claim 1. \( T(B_\infty) \subset B_\infty \).
Let $M > 0$. Assume that $x \in \text{supp}(\mu)$. Since $\overline{\beta}_\mu(x, n)$ converges uniformly to $\bar{d}_\mu(x)$ on $B_\infty$, there exists an $n_0 \in \mathbb{N}$ (depending on $M$) such that, for each $x \in B_\infty$ and each, $n \geq n_0$, $\overline{\beta}_\mu(x, n) > 2M$. Let also $n_1 \in \mathbb{N}$ be such that, for each $n \geq n_1$, $A(n) < 2$. It follows from (6) that, for each $x \in B_\infty$ and each $n \geq \max\{n_0, n_1\}$, $\overline{\beta}_\mu(T(x), n) > (A(n))^{-1} \bar{d}_\mu(x, n) > M$. Given that $M$ is arbitrary, one has $Tx \in B_\infty$.

Now, assume that $x \notin \text{supp}(\mu)$. This means that there exists an $\eta > 0$ such that, for each $\varepsilon \in (0, \eta)$, one has $f_{x, \varepsilon}(\mu) = 0$. The claim follows now from the hypothesis that $\overline{\beta}_\mu(x, n)$ converges uniformly to $\bar{d}_\mu(x)$ and from the inequality $f_{T, \varepsilon, \Lambda}(\mu) \leq f_{y, \varepsilon}(\mu)$, valid for each $\varepsilon > 0$ and each $y \in \mathcal{X}$.

Claim 2. For each $k \in \mathbb{N} \cup \{0\}$, $T(B_k) \subset B_k$.

Let $\varepsilon \in (0, 1)$ and let $k \in \mathbb{N}$. Since $\overline{\beta}_\mu(x, n)$ converges uniformly to $\bar{d}_\mu(x)$ on $B_k$, there exists an $n_0 \in \mathbb{N}$ such that for each $x \in B_k$ and each $n \geq n_0$, $|\overline{\beta}_\mu(x, n) - \bar{d}_\mu(x, n)| < \varepsilon/4$. Let also $n_1 \in \mathbb{N}$ be such that, for each $n \geq n_1$, $A(n) < \varepsilon/2(k+1)$. Then, it follows from (6) that, for each $x \in B_k$ and each $n \geq \max\{n_0, n_1\}$,

\[
|\overline{\beta}_\mu(T(x), 2n) - \bar{d}_\mu(T(x))| \leq A(n) \overline{\beta}_\mu(x, n) - \bar{d}_\mu(x) = \bar{d}_\mu(x)(A(n) - 1) + A(n)(\overline{\beta}_\mu(x, n) - \bar{d}_\mu(x))
\]

\[
< (k+1) \frac{\varepsilon}{2(k+1)} + \frac{\varepsilon}{4} \left(1 + \frac{\varepsilon}{2(k+1)}\right) < \varepsilon.
\]

This show that $T(B_k) \subset B_k$.

It follows from Claims 1 and 2 that $T(Z_1) = T\left(\bigcup_{k \geq 0} B_k\right) \cup B_\infty \subset \bigcup_{k \geq 0} T(B_k) \cup T(Z_\infty) \subset Z_1$. Since one can also prove (using the same reasoning as above) that $Z_1 \subset T(Z_1)$, it follows that $Z_1$ is $T$-invariant.

Remark 2.1. If $\mu \in \mathcal{M}_e$, since $T$ and $T^{-1}$ are Lipschitz functions, it follows from (6) that $\bar{d}_\mu(x)$ and $\overline{\beta}_\mu(x, n)$ are constants $\mu$-a.e. (an analogous result can be found in [8], Theorem 4.1.10 chapter 1). Thus, for ergodic measures, the argument above can be simplified.

Lemma 2.2. Let $\alpha > 0$. The following equality of sets is met:

\[
MP := \{\mu \in \mathcal{M}_e \mid \dim \overline{T}_\mu(\mu) < \alpha\} = \bigcup_{s \in \mathbb{N}} \{\mu \in \mathcal{M}_e \mid \overline{\beta}_\mu(x, s) < \alpha, \text{ for } \mu\text{-a.e. } x\} =: \mathcal{M}_\alpha.
\]

Proof. Let $\mu \in MP$, and set

\[
Z_{\text{unif}}^\mu = \{x \in X \mid \overline{\beta}_\mu(x, s_0) \text{ converges uniformly to } \bar{d}_\mu(x), \bar{d}_\mu(x) < \alpha\}.
\]

Since, for each $x \in X$, $\lim_{s \to \infty} \overline{\beta}_\mu(x, s) = \bar{d}_\mu(x)$, it follows from Egoroff’s theorem that given $\gamma > 0$, there exists a measurable $U \subset X$, with $\mu(U) > 1 - \gamma$, such that $\overline{\beta}_\mu(x, s)$ converges uniformly to $\bar{d}_\mu(x)$ on $U$. Given that $U \subset Z_{\text{unif}}^\mu$ and $\mu \in \mathcal{M}_e$, one has $\mu(Z_{\text{unif}}^\mu) = 1$, by Proposition 2.1 (in fact, since there exists an $L \in \mathbb{N} \cup \{0\}$ such that $L \leq \alpha < L + 1$, one has $Z_{\text{unif}}^\mu = \bigcup_{k=0}^L B_k$; the proof that $Z_{\text{unif}}^\mu$ is $T$-invariant follows from the same ideas presented in the proof of Proposition 2.1).

Now, since $\mu \in MP$, it follows from Proposition 2.1 that there exists a measurable subset $Z \subset X$, with $\mu(Z) = 1$, such that for each $x \in X$, $\bar{d}_\mu(x) < \alpha$. 

11
Let $\tilde{Z} := Z \cap Z_{\text{unif}}^\mu$. Given that $\beta_\mu(x, s)$ converges uniformly (and monotonically) on $\tilde{Z}$
to $d_\mu(x) < \alpha$, there exists an $s_0 \in \mathbb{N}$ such that, for each $s \geq s_0$ and each $x \in \tilde{Z}$, one has $\beta_\mu(x, s) \leq d_\mu(x, s_0) < \alpha$. Since $\mu(\tilde{Z}) = 1$, it follows that $\mu \in N_{s_0} := \{\nu \in M_e \mid \beta_\nu(x, s_0) < \alpha, \text{ for } \nu\text{-a.e. } x\}$, and therefore that $\mu \in M_\sigma$.

Reciprocally, let $\mu \in M_\sigma$. Then, there exists an $s_0 \in \mathbb{N}$ such that $\mu \in N_{s_0}$; that is, there exist an $s_0 \in \mathbb{N}$ and a measurable set $Z \subset X$, with $\mu(Z) = 1$, such that for each $x \in Z$, $\beta_\mu(x, s_0) < \alpha$. Given that $\beta_\mu(x, s)$ is a decreasing function of $s$, it follows that for each $s \geq s_0$ and each $x \in Z$, $d_\mu(x) \leq \beta_\mu(x, s) \leq \beta_\mu(x, s_0) < \alpha$. Thus, by Proposition 1.1 $\mu \in MP$. □

**Proposition 2.2.** Let $\alpha > 0$. Then,

$$PD = \{\mu \in M_e \mid \dim_P^-(\mu) \geq \alpha\},$$

$$HD = \{\mu \in M_e \mid \dim_H^-(\mu) = 0\}$$

are $G_\delta$ subsets of $M_e$.

**Proof.** Since the arguments in both proofs are similar, we just prove the statement for $PD$. We show that $M_e \setminus PD = \{\mu \in M_e \mid \dim_P^+(\mu) < \alpha\}$ is an $F_\sigma$ set (it follows from Propositions 1.1 and 2.1) that if $\mu \in M_e$, then $\dim_P^+(\mu) = \dim_P^-(\mu)$).

Let $\mu \in M_e \setminus PD$, let $k, s \in \mathbb{N}$, set $Z_\mu(s, k) = \{x \in X \mid \beta_\mu(x, s) \leq \alpha - 1/k\}$, and set

$$M_{s,k} = \{\mu \in M_e \mid \mu \text{ is supported on } Z_\mu(s, k)\}.$$

**Claim 1.** $Z_\mu(s, k)$ is closed.

Let $(z_n)$ be a sequence in $Z_\mu(s, k)$ such that $z_n \to z$, and let $t > 0$. Since for each fixed $\mu \in M_e$, the mapping $X \ni x \mapsto f_{x,1/t}(\mu) \in [0, 1]$ is continuous (see the proof of Lemma 3.1), the mapping $X \ni x \mapsto \beta_\mu(x, s) \in [0, +\infty]$ is lower semi-continuous, which implies that $z \in Z_\mu(s, k)$.

**Claim 2.** $W_{s,k} = \{(\mu, x) \in M_e \times X \mid \beta_\mu(x, s) > \alpha - 1/k\}$ is open.

This is a consequence of the fact that, by Lemma 2.1 the mapping $M_e \times X \ni (\mu, x) \mapsto \beta_\mu(x, s)$ is lower semi-continuous ($M_e \times X$ is endowed with the induced topology).

Now, we show that $M_{s,k}$ is closed. Let $(\mu_n)$ be a sequence in $M_{s,k}$ such that $\mu_n \to \mu$. Suppose, by absurd, that $\mu \notin M_{s,k}$; then, $\mu(A) > 0$ where, $A = X \setminus Z_\mu(s, k)$.

It follows from Claim 2 that, for each $x \in A$, $(\mu, x)$ is an interior point of $W_{s,k}$; thus, there exists an open set $B_x \subset W_{s,k}$ such that $(\mu, x) \in B_x$. Set $B := \bigcup_{x \in A} B_x \subset W_{s,k}$, which is clearly open in $M_e \times X$. Note that $\{\mu\} \times A \subset B$. Furthermore, $A \subset \pi_2(B)$, with $\pi_2(B)$ an open set in $X$. Now, since $\{\mu\} \times A \subset \{\mu\} \times \pi_2(B) \subset B$ and $B \subset W_{s,k}$ is open, there exists an $\ell \in \mathbb{N}$ such that, for each $n \geq \ell$ and each $x \in A$, $\beta_{\mu_n}(x, s) > \alpha - 1/k$.

On the other hand, one has from the definition of weak convergence that $\liminf_{n \to \infty} \mu_n(A) \geq \mu(A) > 0$ (since $A$ is open, by Claim 1), from which follows that there exists an $\ell_1 \geq \ell$ such that, for each $n \geq \ell_1$, $\mu_n(A) > 0$.

Combining the last results, one concludes that, for each $n \geq \ell_1$, $\mu_n(A) > 0$, and for each $x \in A$, $\beta_{\mu_n}(x, s) > \alpha - 1/k$; this contradicts the fact that $\mu_n \in M_{s,k}$. Hence, $\mu \in M_{s,k}$, and $M_{s,k}$ is closed in $M_e$.  

12
Finally, it follows from Lemma 2.2 that \( \{ \mu \in \mathcal{M}_e \mid \dim^+_p(\mu) < \alpha \} = \bigcup_{s \in \mathbb{N}} \bigcup_{k \in \mathbb{N}} \mathcal{M}_{s,k} \) is an \( F_\sigma \) subset of \( \mathcal{M}_e \), concluding the proof of the proposition. \( \square \)

**Remark 2.2.** It is worth noting that, apart from the equality \( \{ \mu \in \mathcal{M}_e \mid \dim^+_p(\mu) < \alpha \} = \bigcup_{s \in \mathbb{N}} \bigcup_{k \in \mathbb{N}} \mathcal{M}_{s,k} \), all the results presented in the proof of Proposition 2.2 are valid if one replaces \( \mathcal{M}_e \) by \( \mathcal{M}(T) \).

### 2.2 Dense sets

**Lemma 2.3.** Let \( X \) be a compact metric space. If \( \mu \in \mathcal{M}(T) \), then \( \dim^+_p(\mu) \geq \frac{h_\mu(T)}{\log \Lambda} \).

**Proof.** Fix \( x \in X, n \geq 1 \) and \( \varepsilon > 0 \). Given \( y \in B(x, \varepsilon \Lambda^{-n}) \), one has, for each \( 0 \leq i \leq n \), \( \rho(T^i y, T^i x) \leq \Lambda^i \rho(x, y) \leq \Lambda^{i-n} \varepsilon < \varepsilon \), which shows that \( y \in B(x, n, \varepsilon) := \{ z \in X \mid \rho(T^n z, T^n x) < \varepsilon, \forall 0 \leq i \leq n-1 \} \). Hence, for each \( x \in X \) and each \( \varepsilon > 0 \),

\[
\overline{d}_\mu(x) \geq \limsup_{n \to \infty} \frac{\log \mu(B(x, \varepsilon \Lambda^{-n}))}{\log \varepsilon \Lambda^{-n}} \geq \limsup_{n \to \infty} \frac{\log \mu(B(x, n, \varepsilon))}{-n} \frac{1}{-\log \varepsilon + \log \Lambda} \geq \limsup_{n \to \infty} \frac{\log \mu(B(x, n, \varepsilon))}{-n} \frac{1}{\log \Lambda};
\]

it follows that, for \( \mu \)-a.e. \( x \in X \),

\[
\overline{d}_\mu(x) \geq \limsup_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{\log \mu(B(x, \varepsilon \Lambda^{-n}))}{\log \varepsilon \Lambda^{-n}} \geq \limsup_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{\log \mu(B(x, n, \varepsilon))}{-n} \frac{1}{\log \Lambda} = h_\mu(T, x) \frac{1}{\log \Lambda}, \quad (7)
\]

where \( h_\mu(T, x) \) is the so-called local entropy of \( \mu \) at \( x \). One also has, using Brin-Katok Theorem, that \( \int h_\mu(T, x) d\mu(x) = h_\mu(T) \) (the compacity of \( X \) is required in this step; see [7]). Hence, there exists a measurable set \( B \), with \( \mu(B) > 0 \), such that, for each \( x \in B \), \( \overline{d}_\mu(x) \geq \frac{h_\mu(T)}{\log \Lambda} \). The result is now a consequence of Proposition 1.1. \( \square \)

**Lemma 2.4.** Let \( \mu \in \mathcal{M}_e \). Then, \( \dim^-(\mu) \geq \frac{h_\mu(T)}{\log \Lambda} \).

**Proof.** Since \( \mu \) is ergodic, it follows from Proposition 2.1 that \( \overline{d}_\mu(x) \) is constant for \( \mu \)-a.e. \( x \) (this constant may be infinite).

One also has, by Lemma 2.8 in [30], that \( h_\mu(T, x) = \mu \text{-ess inf} h_\mu(T, y) \) is valid for \( \mu \)-a.e. \( x \), and then, by Theorem 2.9 in [30], that \( h_\mu(T, x) \geq h_\mu(T) \) is also valid for \( \mu \)-a.e. \( x \).

Now, if follows from the previous remarks and inequality (7) that there exists a measurable set \( B \), with \( \mu(B) = 1 \), such that for each \( x \in B \), \( \overline{d}_\mu(x) \geq \frac{h_\mu(T)}{\log \Lambda} \). The result is obtained again by an application of Proposition 1.1. \( \square \)

Now, we return to the specific setting where \( X = \prod_{i=-\infty}^{+\infty} M \), with \( M \) a perfect and separable metric space, with \( \rho \) given by (1) and with \( T \) the full-shift over \( X \).

The next result is an extension of Lemma 6 in [31] to \( X = \prod_{i=-\infty}^{+\infty} M \), where \( M \) is perfect and separable (the hypothesis of \( M \) being perfect is required to guarantee that one can always choose the periodic point \( x \) in the statement of Lemma 2.5 in such way that \( x_i \neq x_j \) if \( i \neq j \),
Lemma 2.5 (Lemma 6 in [34]). Let $X = \prod_{\mathbb{R}} M$, where $M$ is perfect and separable, let $\mu \in \mathcal{M}(T)$ be such that $\mu(X) = 1$, and let $s_0 > 0$. Then, $\mu$ can be approximated by a $T$-periodic measure $\mu_x \in \mathcal{M}(T)$ such that $x \in X$ has period $s \geq s_0$ and $x_i \neq x_j$ if $i \neq j$, $i, j = 1, \ldots, s$.

Remark 2.3. Since, for each $x \in X$, $\mu_x(\cdot) = \frac{1}{k_x} \sum_{i=0}^{k_x-1} \delta_{T^i x}(\cdot)$, where $k_x$ is the period of $x$, the measure presented in the statement of Lemma 2.5 is clearly supported on $X$, so it belongs to $\mathcal{M}(T)$.

The next result combines an extension of Lemma 7 in [34] (which is proved using Lemma 2.5) to the space $X = \prod_{\mathbb{R}} M$ (the original result was proved for $M = \mathbb{R}$) with Proposition 6.1 in [25]. We leave the details for the dedicated reader.

Proposition 2.3 (Lemma 7 in [34]). Let $\mu \in \mathcal{M}_\text{e}$ and $L > 0$. Then, every neighborhood $V$ of $\mu$ on $\mathcal{M}_\text{e}$ contains an ergodic measure $\rho$ such that $h_{\rho}(T) \geq L$.

Proposition 2.4. Let $L > 0$. Then, $PD = \{ \mu \in \mathcal{M}_\text{e} \mid \dim_P(\mu) \geq L \}$ is a dense subset of $\mathcal{M}_\text{e}$.

Proof. Let $\delta > 0$, and let $\mu \in \mathcal{M}_\text{e}$. It is straightforward to show that $T$ is a Lipschitz function with constant $\Lambda = 2$. Set $K := L \log 2$. It follows from the proof of Proposition 2.3 (see Lemma 7 in [34]) that given any neighborhood of $\mu$ (in the induced topology) of the form $V_\mu(f_1, \ldots, f_r; \delta) = \{ \nu \in \mathcal{M}_\text{e} \mid \| f_i \nu dV - \int f_i d\mu \| < \delta, i = 1, \ldots, r \}$ (where $\delta > 0$ and each $f_i : X \to \mathbb{C}$ is continuous and bounded; such sets form a sub-basis of the weak topology), there exists a measure $\zeta \in V_\mu(f_1, \ldots, f_r; \delta)$ such that $h_{\zeta}(T) \geq K$. Now, by Lemma 2.4 one has $\dim_P(\mu) \geq \frac{h_{\mu}(T)}{\log 2} = K \frac{1}{\log 2} = L$.

Proposition 2.5. The set $\{ \mu \in \mathcal{M}_\text{e} \mid \dim_H(\mu) = 0 \}$ is dense in $\mathcal{M}_\text{e}$.

Proof. Let $\mu$ be the $T$-periodic measure associated with the $T$-period point $x \in X$, and denote its period by $k$. Naturally, $\mu(\cdot) = \frac{1}{k} \sum_{i=0}^{k-1} \delta_{f_i(x)}(\cdot)$, and for each $i = 0, \ldots, k - 1$, one has

$$\overline{d}_\mu(T^i(x)) = \limsup_{r \to 0} \frac{\log \mu(B(T^i(x), r))}{\log r} = \limsup_{r \to 0} \frac{- \log k}{\log r} = 0.$$ 

The result follows now from the fact that the set of $T$-periodic measures is dense in $\mathcal{M}_\text{e}$ (see Theorem 3.3 in [26]).

Remark 2.4. The result stated in Proposition 2.5 is valid for any topological dynamical system $(X, T)$ such that the set of $T$-periodic measures is dense in $\mathcal{M}_\text{e}$; this is particularly true for systems which satisfy the specification property (see Remark 1.2 for more details).

Proof (Theorem 1.1).

III. Note that, by Propositions 2.2 and 2.3, $PD$ is a countable intersection of dense $G_\delta$ subsets of $\mathcal{M}_\text{e}$.

IV. It follows from Propositions 2.2 and 2.5 that $HD$ is a dense $G_\delta$ subset of $\mathcal{M}_\text{e}$.
The next statement says that each \( \mu \in HD \cap PD \cap CX \) is supported on a dense \( G_\delta \) subset of \( X \).

**Proposition 3.1.** Let \( T^{-s} \) which is an absurd, since \( \mu_x \) for each \( x \):

\[
\text{Proof.} \quad \text{We just present the proof that } \overline{\mathcal{D}}_\mu \text{ is a dense } G_\delta \text{ subset of } X.
\]

\( \overline{\mathcal{D}}_\mu \) is a \( G_\delta \) set in \( X \). Let \( \alpha > 0 \), \( \mathbb{N} \), and set \( Z_{\mu,s}(\alpha) = \{ x \in X \mid \beta_{\mu}(x,s) > \alpha \} \). Following the proof of Claim 1 in Proposition 2.2 it is clear that \( Z_{\mu,s}(\alpha) \) is closed. Thus, taking \( \alpha = n \in \mathbb{N} \), it follows that \( \overline{\mathcal{D}}_\mu = \bigcap_{n \in \mathbb{N}} \bigcap_{s \in \mathbb{N}} (X \setminus Z_{\mu,s}(n)) \) is a \( G_\delta \) subset of \( X \).

\( \overline{\mathcal{D}}_\mu \) is dense in \( X \). Since \( \mu \in PD \), one has \( \mu(\overline{\mathcal{D}}_\mu) = 1 \). Suppose that \( \overline{\mathcal{D}}_\mu \) is not dense; then, there exist \( x \in X \) and \( \varepsilon > 0 \) such that \( B(x,\varepsilon) \cap \overline{\mathcal{D}}_\mu = \emptyset \). This implies that \( 1 = \mu(\overline{\mathcal{D}}_\mu) + \mu(B(x,\varepsilon)) \), which is an absurd, since \( \mu(B(x,\varepsilon)) > 0 \) (recall that \( \text{supp}(\mu) = X \), given that \( \mu \in CX \)).



### 3 Sets of ergodic measures with zero lower and infinity upper rates of recurrence and quantitative waiting time indicators almost everywhere

This section presents the counterparts, for \( \overline{R}(x), \underline{R}(x), \overline{R}(x,y) \) and \( \underline{R}(x,y) \), of the results presented in Section 2. Once again, we assume that \( (X,d) \) is a Polish metric space and that \( T : X \to X \) is a Lipschitz function, with Lipschitz constant \( \Lambda > 1 \). Assume also that \( T^{-1} : X \to X \) exists as a Lipschitz function, with Lipschitz constant \( \Lambda' > 1 \).

**Proposition 3.1.** Let \( \mu \in M_e \). Then, \( \overline{R}(x) \) and \( \underline{R}(x) \) are \( \mu \)-a.e. constants, and each of the sets

\[
R_1 = \{ x \in X \mid \overline{\gamma}(x,s) \text{ converges uniformly to } \overline{R}(x) \},
\]

\[
R_2 = \{ x \in X \mid \underline{\gamma}(x,s) \text{ converges uniformly to } \underline{R}(x) \},
\]

is \( T \)-invariant, where for each \( x \in X \) and each \( s \in \mathbb{N} \), \( \overline{\gamma}(x,s) := \sup_{t>s} \frac{\log \tau_{\mu,t}(x)}{\log t} \) and \( \underline{\gamma}(x,s) := \inf_{t>s} \frac{\log \tau_{\mu,t}(x)}{\log t} \).

**Proof.** It is straightforward to check that, for each \( x \in X \) and each \( \varepsilon > 0 \), \( \tau_{\varepsilon}(Tx) \leq \tau_{\varepsilon}(x) \leq \tau_{\varepsilon}/\Lambda'(Tx) \). Thus, for each \( x \in X \), \( \overline{R}(Tx) = \overline{R}(x) \) and \( \overline{R}(Tx) = \overline{R}(x) \). Given that \( \mu \in M_e \), it follows from the previous result that \( \overline{R}(x) \) and \( \underline{R}(x) \) are \( \mu \)-a.e. constants (that is, they are finite or infinite for \( \mu \)-a.e. \( x \)).

Let \( s \geq 1 + 1/\Lambda \); then,

\[
\sup_{t \geq s} \frac{\log \tau_{\mu,t}(Tx)}{\log t} \leq \sup_{t \geq \Lambda s} \frac{\log t}{\log t} \frac{\log \tau_{\mu,t}(x)}{\log t} \leq \frac{\log \Lambda s}{\log \Lambda t} \sup_{t \geq \Lambda s} \frac{\log \tau_{\mu,t}(x)}{\log t} = A(s) \sup_{t \geq \Lambda s} \frac{\log \tau_{\mu,t}(x)}{\log t},
\]

where \( A(s) = \frac{\log s + \log \Lambda}{\log s} \) and \( t = \Lambda t \) (since \( s \geq 1 + 1/\Lambda \), one has \( t \geq 1 + \Lambda \)). Thus, one gets, for each \( x \in X \) and each \( s \geq 1 + 1/\Lambda \), \( \overline{\gamma}_\mu(Tx,s) \leq A(s)\overline{\gamma}_\mu(x,\Lambda s) \). Now, using the inequality
\( \tau_s(x) \leq \tau_{s/N}(Tx) \), one gets, using the same reasoning as before, for each \( x \in X \) and each \( s \geq 1 + 1/\Lambda' \), \( \tau(x,s) \leq A(s) \tau(x,\Lambda s) \). Combining both inequalities, one has, for each \( x \in X \) and each \( s \geq \max\{1 + 1/\Lambda, 1 + 1/\Lambda'\} \),

\[
\tau(x,s) \leq A(s) \tau(x,\Lambda s) \leq A(s)A(\Lambda s) \tau(x,\Lambda' s).
\]

The rest of the proof follows the same ideas presented in the proof of Proposition 3.1 considering only the cases when there exists a \( K > 0 \) such that \( R(x) \leq K \), or \( R(x) = \infty \) (see Remark 3.1).

**Proposition 3.2.** Let \( \alpha > 0 \). Then, each of the sets

\[
\mathcal{R} = \{ \mu \in \mathcal{M}_e \mid \mu \text{-ess inf } R(x) \geq \alpha \},
\]

\[
\overline{\mathcal{R}} = \{ \mu \in \mathcal{M}_e \mid \mu \text{-ess sup } R(x) = 0 \}
\]

is \( G_\delta \) subset of \( \mathcal{M}_e \).

**Proof.** Since the arguments in both proofs are similar, we just prove the statement for \( \mathcal{R} \). We show that \( \mathcal{M}_e \setminus \mathcal{R} \) is an \( F_\sigma \) set. We begin noting that, by Proposition 3.1, \( \mathcal{R} = \{ \mu \in \mathcal{M}_e \mid \mu \text{-ess sup } R(x) \geq \alpha \} \). Let \( \mu \in \mathcal{M}_e \setminus \mathcal{R} \). It follows from definition of essential supremum that the measurable set \( Z = \{ x \in X \mid \lim_{n \to \infty} \tau(x,s) \leq \alpha \} \) has full \( \mu \)-measure.

Fix \( l, s \in \mathbb{N} \), let \( Z_{s,l} = \{ x \in X \mid \tau(x,s) \leq \alpha - 1/l \} \), and set \( \mathcal{M}_{s,l} = \{ \mu \in \mathcal{M}_e \mid \mu \text{ is supported on } Z_{s,l} \} \).

**Claim.** \( Z_{s,l} \) is closed.

Let \((z_n)\) be a sequence in \( Z_{s,l} \) such that \( z_n \to z \). We show that \( z \in Z_{s,l} \). Given that, for each \( n \in \mathbb{N} \), \( \tau(z_n,s) = \sup_{0 < r \leq s} \frac{\log \tau_s(z_n)}{-\log r} \leq \alpha - 1/l \), it follows that for each \( 0 < r \leq s \), \( \tau_s(z_n) \leq r^{-\alpha+1/l} \). Now, fix \( r \in (0, s] \); then, there exists a sequence \( (k_n) \), \( k_n \in \mathbb{N} \), such that for each \( n \in \mathbb{N} \), \( k_n \leq r^{-\alpha+1/l} \) and \( d(T^{k_n}(z_n), z_n) \leq r \).

Since \((k_n)\) is bounded, there exist a sub-sequence \((k_{n_j})\), a \( k \leq r^{-\alpha+1/l} \), and a \( j_0 \in \mathbb{N} \) such that for each \( j \geq j_0 \), \( k_{n_j} = k \). Since for each \( j \in \mathbb{N} \), \( d(T^{k_{n_j}}(z_{n_j}), z_{n_j}) \leq r \), it follows from the continuity of \( T^k \) and the previous statements that \( d(T^k(z), z) \leq r \), which proves that \( \tau_s(z) \leq r^{-\alpha+1/l} \). Since \( r \leq s \) was take arbitrarily, one gets \( \sup_{0 < r \leq s} \frac{\log \tau_s(z)}{-\log r} \leq \alpha - 1/l \), and therefore, \( z \in Z_{s,l} \).

Now, we show that \( \mathcal{M}_{s,l} \) is closed. Indeed, fix \( s \in \mathbb{N} \) and let \((\mu_n)\) be a sequence in \( \mathcal{M}_{s,l} \) such that \( \mu_n \to \mu \). Suppose that \( \mu \notin \mathcal{M}_{s,l} \); then, \( \mu(A) > 0 \), where \( A = X \setminus Z_{s,l} \). Since, by Claim, \( A \) is open in \( X \) and \( \mu_n \to \mu \), one has \( \lim_{n \to \infty} \mu_n(A) \geq \mu(A) > 0 \), which shows that there exists an \( \ell \in \mathbb{N} \) such that \( \mu_\ell(A) > 0 \). This contradicts the fact that \( \mu_\ell \in \mathcal{M}_{s,l} \). Hence, \( \mu \in \mathcal{M}_{s,l} \).

Given that \( \mathcal{M}_{s,l} \) is closed, it follows that \( \mathcal{N}_s = \bigcup_{l \in \mathbb{N}} \mathcal{M}_{s,l} = \{ \mu \in \mathcal{M}_e \mid \tau(x,s) < \alpha \text{, for } \mu \text{-a.e. } x \} \) is an \( F_\sigma \) subset of \( \mathcal{M}_e \).

The proof of the equality \( \{ \mu \in \mathcal{M}_e \mid \mu \text{-ess sup } R(x) < \alpha \} = \bigcup_{s \in \mathbb{N}} \mathcal{N}_s \) is similar to the argument presented in the proof of Lemma 3.1 and therefore, it is omitted (naturally, Proposition 3.1 is used in this step).

\( \square \)
The next results deal show that such sets are dense.

**Proposition 3.3.** Let \((X,T)\) be the full-shift system, with \(M\) a Polish metric space. Then, 
\[
\mathcal{R} = \{ \mu \in \mathcal{M}_e \mid \mu\text{-ess sup} \mathcal{T}(x) = 0 \}
\]
is a dense subset of \(\mathcal{M}_e\).

**Proof.** Note that if \(\mu_e\) is a \(T\)-periodic measure, then for each \(y \in \mathcal{O}(x)\), \(R(y) = 0\). The result follows, therefore, from the fact that the set of \(T\)-periodic measures is dense in \(\mathcal{M}_e\) (Theorem 3.3 in [26]). \(\Box\)

**Proposition 3.4.** Let \((X,T)\) be the full-shift system, with \(M\) a perfect and compact metric space, and let \(L > 0\). Then, 
\[
\mathcal{R} = \{ \mu \in \mathcal{M}_e \mid \mu\text{-ess inf} \mathcal{T}(x) > L \}
\]
is a dense subset of \(\mathcal{M}_e\).

**Proof.** Fix \(x \in X\), \(n \geq 1\) and \(\varepsilon > 0\). It follows from the argument presented in the proof of Lemma 2.2 that \(B(x,2^{-n}) \subset B(x,n,\varepsilon)\). Note that \(\tau_{2^{-n}}(x) \geq R_n(x,\varepsilon)\), where \(R_n(x,\varepsilon) = \inf\{k \geq 1 \mid T^k(x) \in B(x,n,\varepsilon)\}\) is the \(n\)th return time to the dynamical ball \(B(x,n,\varepsilon)\). Now, as in the proof of Proposition 2.2 for each \(\mu \in \mathcal{M}_e\) and each neighborhood \(V_\mu(f_1,\ldots,f_n;\delta)\) (in the induced topology), there exists a measure \(\zeta \in V_\mu(f_1,\ldots,f_n;\delta)\) such that \(\mu(T(T(\cdot),\zeta)) \geq L \log 2\).

The result is now a consequence of Theorem A and Proposition A in [36], which state that \(\mathcal{R}(x) \geq \frac{h_c(T)}{\log 2} = L\), for \(\zeta\text{-a.e. } x\). \(\Box\)

**Proof (Theorem 1.1).**

V. The result is a consequence of Propositions 3.2 and 3.3.

VI. It follows from Propositions 3.2 and 3.3.

**Remark 3.1.** There is an alternative proof to the fact that \(\mathcal{R} = \{ \mu \in \mathcal{M}_e \mid \mu\text{-ess inf} \mathcal{T}(x) = 0 \}\) is residual in \(\mathcal{M}_e\). In fact, this result can be seen as a direct consequence of Theorem 2 in [5] and Theorem 1.1-III, since it follows that, for each \(\mu \in HD\), \(\mu\text{-ess sup} \mathcal{T}(x) \leq \dim_H^+ (\mu) = 0\).

The following result states that each typical measure obtained in Theorem 1.1 is supported on the dense \(G_\delta\) set \(\mathcal{R} = \{ x \in X \mid \mathcal{R}(x) = 0 \text{ and } \mathcal{R}(x) = \infty \}\).

**Proposition 3.5.** Let \((X,T)\) the full-shift system, where \(M\) is a compact and perfect metric space. Then, each of the sets \(\mathcal{R}^- = \{ x \in X \mid \mathcal{R}(x) < \infty \}\) and \(\mathcal{R}_- = \{ x \in X \mid \mathcal{R}(x) = 0 \}\) is a dense \(G_\delta\) subset of \(X\). Moreover, for each \(\mu \in \mathcal{R} \cap \mathcal{R} \cap C_X\), \(\mu(\mathcal{R}^- \cap \mathcal{R}_-) = 1\).

**Proof.** We just present the proof that \(\mathcal{R}^-\) is a dense \(G_\delta\) subset of \(X\).

\(\mathcal{R}^-\) is a \(G_\delta\) set in \(X\). Let \(\alpha > 0\), \(s \in \mathbb{N}\), and set \(Z_s(\alpha) = \{ x \in X \mid \mathcal{R}(x,s) \leq \alpha \}\). Following the proof of Claim in Proposition 3.2, it is clear that \(Z_s(\alpha)\) is closed. Thus, taking \(\alpha = n \in \mathbb{N}\), it follows that \(\mathcal{R}^- = \bigcap_{n \in \mathbb{N}} \bigcap_{s \in \mathbb{N}} (X \setminus Z_s(n))\) is a \(G_\delta\) set in \(X\).

\(\mathcal{R}^-\) is dense in \(X\). Let \(\mu \in \mathcal{R} \cap C_X\). Then, \(\mu(\mathcal{R}^-) = 1\). Suppose that \(\mathcal{R}^-\) is not dense; so, there exist \(x \in X\) and \(\varepsilon > 0\) such that \(B(x,\varepsilon) \cap \mathcal{R}^- = \emptyset\). This implies that \(1 = \mu(\mathcal{R}^-) + \mu(B(x,\varepsilon))\), which is an absurd, since \(\mu(B(x,\varepsilon)) > 0\). \(\Box\)

Now, we present equivalent results, to those already obtained in this section, for the quantitative waiting time indicators.
Proposition 3.6. Let $\mu \in \mathcal{M}$. Then, $\mathcal{R}(x, y)$ and $\mathcal{R}(x, y)$ are $\mu$-a.e. constants in $X \times X$, and each of the sets

$$W_1 = \{(x, y) \in X \times X \mid \gamma(x, y, s) \text{ converges uniformly to } \mathcal{R}(x, y)\},$$

$$W_2 = \{(x, y) \in X \times X \mid \gamma(x, y, s) \text{ converges uniformly to } \mathcal{R}(x, y)\},$$

is $T \times T$-invariant, where for each $(x, y) \in X \times X$ and each $s \in \mathbb{N}$, $\gamma(x, y, s) := \sup_{t>s} \frac{\log \tau_1(x, y)}{\log t}$ and $\gamma(x, y, s) := \inf_{t>s} \frac{\log \tau_1(x, y)}{\log t}$.

Proof. The proof follows the same ideas presented in the proof of Proposition 3.1.

Proposition 3.7. Let $\alpha > 0$. Then, the set

$$\mathcal{R}_\alpha = \{\mu \in \mathcal{M} \mid (\mu \times \mu) - \text{ess sup } \mathcal{R}(x, y) \leq \alpha\}$$

is a $G_\delta$ subset of $\mathcal{M}$.

Proof. Using the same ideas presented in the proof of Proposition 3.2, we show that $\mathcal{R}_\alpha = \{\mu \in \mathcal{M} \mid (\mu \times \mu) - \text{ess inf } \mathcal{R}(x, y) \leq \alpha\}$ is a $G_\delta$ subset of $\mathcal{M}$ by showing that $\mathcal{M} \setminus \mathcal{R}_\alpha$ is an $F_\sigma$ set (naturally, we use here Proposition 3.3). Let $l, s \in \mathbb{N}$, set $Z_{s,l} = \{(x, y) \in X \times X \mid \gamma(x, y, s) \geq \alpha + 1/l\}$ and $\mathcal{M}_{s,l} = \{\mu \in \mathcal{M} \mid \mu \times \mu \text{ is supported on } Z_{s,l}\}$.

The proofs that $Z_{s,l}$ and $\mathcal{M}_{s,l}$ are closed sets follow the same arguments presented in the proof of Proposition 3.2 (here, one uses Fubini’s theorem for the product measure $\mu \times \mu$). Finally, the proof of the equality $\{\mu \in \mathcal{M} \mid (\mu \times \mu) - \text{ess sup } \mathcal{R}(x, y) \leq \alpha\} = \bigcup_{s \in \mathbb{N}} \bigcup_{l \in \mathbb{N}} \mathcal{M}_{s,l}$ is similar to the argument presented in the proof of Lemma 2.2 and therefore, it is omitted (naturally, Proposition 3.6 is used in this step).

Proof (Theorem 1.1).

VII. Since, by Proposition 3.7, $\mathcal{R} = \{\mu \in \mathcal{M} \mid (\mu \times \mu) - \text{ess sup } \mathcal{R}(x, y) = 0\} = \cap_{k \geq 1} \mathcal{R}_{1/k}$, one just needs to prove that $\mathcal{R}$ is dense. Let $\mu_z$ be a $T$-periodic measure. Then, for each $x, y \in O(z)$, $\mathcal{R}(x, y) = 0$. The result follows from the fact that the set of $T$-periodic measures is dense in $\mathcal{M}$.

VIII. The result is a direct consequence of Theorem 1.1 (IV) and the second inequality in (4) (see Theorem 4 in [13]).

Proposition 3.8. Let $(X, T)$ the full-shift system, where $M$ is a compact and perfect metric space. Then, each of the sets $\mathcal{G}^- = \{(x, y) \in X \times X \mid \mathcal{R}(x, y) = \infty\}$ and $\mathcal{G}_+ = \{(x, y) \in X \times X \mid \mathcal{R}(x, y) = 0\}$ is a dense $G_\delta$ subset of $X \times X$. Moreover, for each $\mu \in \mathcal{R} \cap \mathcal{R} \cap C_X$, $(\mu \times \mu)(\mathcal{G}^- \cap \mathcal{G}_+) = 1$. 


4 Appendix

**Proof (Proposition 1.1).** Since the arguments in both proofs (for Hausdorff and packing dimensions) are similar, we just prove the statement for dim\(^+_\mu\) and dim\(^{-}\mu\).

**a)** \(\dim^+_\mu(x) = \mu\text{-}\text{ess} \sup \overline{d}_\mu(x)\). Let \(\alpha \geq 0\). We show that if \(\mu\text{-}\text{ess} \sup \overline{d}_\mu(x) \leq \alpha\), then \(\dim^+_\mu(x) \leq \alpha\). In fact, since \(\mu\text{-}\text{ess} \sup \overline{d}_\mu(x) = \inf\{a \in \mathbb{R} | \mu\{x | \overline{d}_\mu(x) \leq a\} = 1\} \leq \alpha\), one has \(\mu\{(x \in X | \overline{d}_\mu(x) \leq \alpha\}) = 1\). It follows from the Definition \([15]\) that \(\dim^+_\mu(x) \leq \dim_\mu\{x \in X | \overline{d}_\mu(x) \leq \alpha\}\). Now, by Corollary 3.20(a) in \([9]\), one has \(\dim_\mu\{x \in X | \overline{d}_\mu(x) \leq \alpha\} \leq \alpha\). Thus, \(\dim^+_\mu(x) \leq \alpha\).

Conversely, we show that if \(\dim^+_\mu(x) \leq \alpha\), then \(\mu\text{-}\text{ess} \sup \overline{d}_\mu(x) \leq \alpha\). Suppose that there exists \(\delta > 0\) such that \(\mu\text{-}\text{ess} \sup \overline{d}_\mu(x) \geq \alpha + \delta\); then, by the definition of essential supremum of a measurable function, there exists \(E \in B\), with \(\mu(E) > 0\), such that for each \(x \in E\), \(\overline{d}_\mu(x) \geq \alpha + \delta/2\). Then, by Corollary 3.20(b) in \([9]\), \(\dim_\mu(E) \geq \alpha + \delta/2\), and therefore, \(\dim^+_\mu(x) \geq \alpha + \delta/2\). This contradiction shows that \(\mu\text{-}\text{ess} \sup \overline{d}_\mu(x) \leq \alpha\).

**b)** \(\dim^{-}\mu(x) = \mu\text{-}\text{ess} \inf \overline{d}_\mu(x)\). Let \(\alpha > 0\). We show that if \(\mu\text{-}\text{ess} \inf \overline{d}_\mu(x) \geq \alpha\), then \(\dim^{-}\mu(x) \geq \alpha\). By the definition of essential infimum of a measurable function, \(\mu(A) = 1\), where \(A := \{x \in X | \overline{d}_\mu(x) \geq \alpha\}\). Since, for each \(E \in B\), \(\mu(E) = \mu(A \cap E) (E \setminus A \subset A^c)\), one may only consider, without loss of generality, those sets \(E \in B\) such that \(E \subset A\). Thus, for each \(A \supset E \in B\) so that \(\mu(E) > 0\), it follows from Corollary 3.20(b) in \([9]\) that \(\dim_\mu(E) \geq \alpha\). The result is now a consequence of Definition \([15]\).

Conversely, we show that if \(\dim^{-}\mu(x) \geq \alpha\), then \(\mu\text{-}\text{ess} \inf \overline{d}_\mu(x) \geq \alpha\). Suppose that there exists \(\delta > 0\) such that \(\mu\text{-}\text{ess} \inf \overline{d}_\mu(x) \leq \alpha - \delta/2\); then, by the definition of essential infimum of a measurable function, there exists \(E \in B\), with \(\mu(E) > 0\), such that for each \(x \in E\), \(\overline{d}_\mu(x) \leq \alpha - \delta/2\). Thus, \(E \subset \{x \in X | \overline{d}_\mu(x) \leq \alpha - \delta/2\} = C\) and \(\dim_\mu C \leq \dim_\mu E\). Then, by Corollary 3.20(a) in \([9]\), \(\dim_\mu(E) \leq \alpha - \delta/2\), and therefore, \(\dim^{-}\mu(x) \leq \alpha - \delta/2\). This contradiction shows that \(\mu\text{-}\text{ess} \sup \overline{d}_\mu(x) \geq \alpha\). \(\square\)

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