On noncommutative weak Orlicz–Hardy spaces

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Abstract
We introduce noncommutative weak Orlicz spaces associated with a weight and study their properties. We also define noncommutative weak Orlicz–Hardy spaces and characterize their dual spaces.

Keywords Noncommutative Lorentz space · Noncommutative Marcinkiewicz space · Weak noncommutative Orlicz space · Noncommutative weak Orlicz–Hardy space

Mathematics Subject Classification 46L52 · 47L05

1 Introduction
Al-Rashed and Zegarliński [1] introduced the noncommutative Orlicz spaces associated with a normal faithful state on a semifinite von Neumann algebra. In [2], the authors considered a certain class of noncommutative Orlicz spaces, associated with arbitrary faithful normal locally finite weights on a semi-finite von Neumann algebra $\mathcal{M}$. In [19], the authors have investigated weak version of Orlicz spaces and proved the Burkholder–Gundy inequalities of martingales for this weak Orlicz spaces. The weak noncommutative Orlicz spaces were investigated in [3] and it was used for the theory of noncommutative martingales. In this paper, we extend the results of [2] to the weak noncommutative Orlicz space case.

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The dual spaces of commutative weak $L_p$-spaces were characterized in [10, 11], its noncommutative versions were proved in [9, 16]. In [8, 9], Ciach introduced noncommutative Lorentz space and noncommutative Marcinkiewicz space, and discussed their dual spaces. The aim of this paper is to define noncommutative weak Orlicz–Hardy spaces and characterize their dual spaces.

The paper is organized as follows. In Sect. 2, some necessary definitions and notations are collected including the weak Orlicz spaces and the noncommutative weak Orlicz spaces. Using relationship between noncommutative weak Orlicz spaces and noncommutative Marcinkiewicz space, and Ciach’s results to give dual spaces of weak noncommutative Orlicz spaces. The noncommutative weak Orlicz spaces associated with a weight are studied in Sect. 3. In Sect. 4, we characterized the dual spaces of noncommutative weak Orlicz–Hardy spaces.

2 Preliminaries

Let $\Omega = [0, \gamma) (0 < \gamma \leq \infty)$ be equipped with the usual Lebesgue measure $m$. We denote by $L_0(\Omega)$ the space of $m$-measurable real-valued functions $f$ on $(\Omega)$ such that $m(\{\omega \in \Omega : |x(\omega)| > s\}) < \infty$ for some $s$. The decreasing rearrangement function $f^* : [0, \infty) \mapsto [0, \infty]$ for $f \in L_0(\Omega)$ is defined by

$$f^*(t) = \inf\{s > 0 : m(\{\omega \in \Omega : |f(\omega)| > s\}) \leq t\}$$

for $t \geq 0$.

The classical weak $L_p$-space $L_{p,\infty}(\Omega) (0 < p < \infty)$ is defined as the set of all measurable functions $f$ on $\Omega$ such that

$$\|f\|_{L_{p,\infty}} = \sup_{t>0} t^{\frac{1}{p}} f^*(t) < \infty.$$  

However, for $p > 1 L_{p,\infty}(\Omega)$ can be renormed as a Banach space by

$$f \mapsto \sup_{t>0} t^{-1+\frac{1}{p}} \int_0^t f^*(s) ds.$$  

We refer to [15] for more information about weak $L_p$-spaces.

A function $\Phi : (-\infty, \infty) \to [0, \infty)$ is called an N-function if it satisfies the following conditions: (i) $\Phi$ is even and convex, (ii) $\Phi(t) = 0$ iff $t = 0$, (iii) $\lim_{t \to 0} \frac{\Phi(t)}{t} = 0$, $\lim_{t \to \infty} \frac{\Phi(t)}{t} = +\infty$.

Let $\phi(t)$ be the left derivative of $\Phi$. Then $\phi(t)$ is left continuous, nondecreasing on $(0, \infty)$ and satisfies: $0 < \phi(t) < \infty$ for $0 < t < \infty$, $\phi(0) = 0$ and $\lim_{t \to \infty} \phi(t) = \infty$. The left inverse of $\phi$ ($\psi(s) = \inf\{t > 0 : \phi(t) > s\}$ for $s > 0$) will be denoted by $\psi$.

We define a complementary N-function $\Psi$ of $\Phi$ by

$$\Psi(|s|) = \int_0^{|s|} \psi(v) dv.$$
It is clear that $\Phi$ is the complementary N-function of $\Psi$. We call $(\Phi, \Psi)$ is a pair of complementary N-functions.

Let $(\Phi, \Psi)$ be a pair of complementary N-functions, with inverses $\Phi^{-1}, \Psi^{-1}$ (which are uniquely defined on $[0, \infty)$). Then

$$t < \Phi^{-1}(t)\Psi^{-1}(t) < 2t, \quad t > 0. \quad (2.1)$$

An N-function $\Phi$ is said to satisfy the $\triangle_2$-condition for all $t$, written as $\Phi \in \triangle_2$, if there is $K > 2$ such that $\Phi(2t) \leq K\Phi(t)$ for all $t \geq 0$. $\Phi$ is called to satisfy the $\bigtriangleup_2$-condition for all $t$, written as $\Phi \in \bigtriangleup_2$, if there is a constant $c > 1$ such that $\Phi(t) \leq \frac{1}{2c}\Phi(ct)$ for all $t \geq 0$. For a pair of complementary N-functions $(\Phi, \Psi)$, we have that $\Phi \in \triangle_2$ if and only if $\Psi \in \bigtriangleup_2$ (see [25, Theorem 2]).

Let $(\Phi, \Psi)$ is a pair of complementary N-functions. Then the Orlicz space on $\Omega$ associated with $\Phi$ defined by

$$L_\Phi(\Omega) = \left\{ f \in L_0(\Omega) : \int_0^\infty \Phi\left(\frac{|f(t)|}{c}\right) dt < \infty \text{ for some } a > 0 \right\}.$$ 

We define

$$\|f\|_\Phi = \inf \left\{ c > 0 : \int_0^\infty \Phi\left(\frac{|f(t)|}{c}\right) dt \leq 1 \right\}.$$ 

Then for any $f \in L_\Phi(\Omega)$,

$$\|f\|_\Phi \leq \sup \left\{ |\int_0^\infty f(t)g(t)dt| : \int_0^\infty \Psi(|g(t)|)dt \leq 1, \quad g \in L_\Psi(\Omega) \right\} \leq 2\|f\|_\Phi.$$ 

For an N-function $\Phi$, we define

$$a_\Phi = \inf_{r>0} \frac{r\Phi'(t)}{\Phi(t)} \quad \text{and} \quad b_\Phi = \sup_{r>0} \frac{r\Phi'(t)}{\Phi(t)}.$$ 

Then $1 \leq a_\Phi \leq b_\Phi \leq \infty$ and $\Phi \in \triangle_2$ if and only if $b_\Phi < \infty$. It is well-known that

$$L_\Phi(\Omega)^* = L_\Psi(\Omega), \quad (2.2)$$

with equivalent norms. We refer to [25] for the details on Orlicz spaces.

Now we consider the set of all measurable functions

$$L_{\Phi, \infty}(\Omega) = \left\{ f \in L_0(\Omega) : \exists c > 0, \Phi\left(\frac{t}{c}\right)m(|f| > t) \leq 1, \forall t > 0 \right\}$$

and denote

$$\|f\|_{\Phi, \infty} = \inf \left\{ c > 0 : \Phi\left(\frac{t}{c}\right)m(|f| > t) \leq 1, \forall t > 0 \right\}.$$ 

We call $L_{\Phi, \infty}(\Omega)$ is a weak Orlicz space. If $\Phi(t) = t^p$, then $L_{\Phi, \infty}(\Omega) = L_{p, \infty}(\Omega)$ (see [19] for more details).
Recall that
\[ L_{\Phi, \infty}(\Omega) = \left\{ x \in L_0(\Omega) : \exists c > 0 \text{ such that } \sup_{t > 0} t \Phi(f^*(t)/c) < \infty \right\}, \]
and
\[ \|x\|_{\Phi, \infty} = \inf \left\{ c > 0 : t \Phi(f^*(t)/c) \leq 1, \forall t > 0 \right\} = \inf \left\{ c > 0 : \frac{1}{\Phi^{-1}(\frac{1}{c})} \mu_t(x) / c \leq 1, \forall t > 0 \right\} \]
(see [3, Proposition 3.1]).

2.1 Noncommutative weak \( L_p \) spaces

We keep all notations introduced in the above. In rest of this paper, \( \Phi \) will always denote an N-function and \( \Psi \) denote a complementary N-function of \( \Phi \), \( \mathcal{M} \) always denote a semifinite von Neumann algebra acting on a Hilbert space \( \mathbb{H} \) with a normal semifinite faithful trace \( \tau \) (\( \tau(1) = \gamma \)).

For \( 0 < p < \infty \) let \( L_p(\mathcal{M}) \) denote the noncommutative \( L_p \) space with respect to \( (\mathcal{M}, \tau) \). As usual, we set \( L_{\infty}(\mathcal{M}, \tau) = \mathcal{M} \) equipped with the operator norm. Also, let \( L_0(\mathcal{M}) \) denote the topological \( \ast \)-algebra of measurable operators with respect to \( (\mathcal{M}, \tau) \).

For \( x \in L_0(\mathcal{M}) \), we define
\[ \lambda_s(x) = \tau(e_s^+(|x|)) \quad (s > 0) \quad \text{and} \quad \mu_t(x) = \inf \{ s > 0 : \lambda_s(x) \leq t \} \quad (t > 0), \]
where \( e_s^+(|x|) = e_{(s, \infty)}(|x|) \) is the spectral projection of \( |x| \) associated with the interval \( (s, \infty) \). We call the function \( s \mapsto \lambda_s(x) \) the distribution function of \( x \) and \( \mu_t(x) \) is the generalized singular number of \( x \). For simplicity, we denote by \( \lambda(x) \) and \( \mu(x) \) the two functions \( s \mapsto \lambda_s(x) \) and \( t \mapsto \mu_t(x) \), respectively. It is clear that both functions \( \lambda(x) \) and \( \mu(x) \) are decreasing and continuous from the right on \( (0, \infty) \) (for further information, see [14]).

For \( 0 < p < \infty \), the noncommutative weak \( L_p \) space \( L_{p, \infty}(\mathcal{M}) \) is defined as the space of all measurable operators \( x \) such that
\[ \|x\|_{p, \infty} = \sup_{t > 0} t^{\frac{1}{p}} \mu_t(x) < \infty. \]
Equipped with \( \|\cdot\|_{L_{p, \infty}} \), \( L_{p, \infty}(\mathcal{M}) \) is a quasi-Banach space. However, for \( p > 1 \) \( L_{p, \infty}(\mathcal{M}) \) can be renormed as a Banach space by
\[ x \mapsto \sup_{t > 0} t^{-\frac{1}{p} + \frac{1}{p}} \int_0^t \mu_s(x) ds. \]
On the other hand, the quasi-norm admits the following useful description.
\[ \| x \|_{p, \infty} = \inf \left\{ c > 0 : t(\mu_t(x)/c)^p \leq 1, \forall t > 0 \right\}. \quad (2.3) \]

Also, we have a description in terms of distribution function as follows

\[ \| x \|_{p, \infty} = \sup_{s > 0} s \lambda_s(x)^{\frac{1}{p}}. \quad (2.4) \]

Recall that noncommutative weak \( L_p \) spaces can be presented through noncommutative Lorentz spaces, for details, see [12, 27].

### 2.2 Noncommutative weak Orlicz spaces

Let

\[ L_\Phi(M) = \{ x \in L_0(M) : \tau(\Phi(|x|)) = \int_0^{\tau(1)} \Phi(\mu_t(x))dt < \infty \} \]

and

\[ \| x \|_\Phi = \inf \left\{ \lambda > 0 : \tau \left( \Phi \left( \frac{|x|}{\lambda} \right) \right) \leq 1 \right\}, \quad \forall x \in L_\Phi(M). \]

We call \( L_\Phi(M) \) is the noncommutative Orlicz space on \((M, \tau)\). If \( \Phi \in \Delta_2 \), then \( L_\Phi(M) \) is a Banach space.

**Definition 2.1** The noncommutative weak Orlicz space \( L_{\Phi, \infty}(M) \) is defined as following:

\[ L_{\Phi, \infty}(M) = \left\{ x \in L_0(M) : \sup_{t > 0} t \Phi(\mu_t(x)) < \infty \right\}, \]

equipped with

\[ \| x \|_{\Phi, \infty} = \inf \left\{ c > 0 : t \Phi(\mu_t(x)/c) \leq 1, \forall t > 0 \right\}. \]

If \( \Phi(t) = t^p \) with \( 1 \leq p < \infty \), then \( L_{\Phi, \infty}(M) \) is the noncommutative weak \( L_p \)-space.

Recall that if \( \Phi \in \Delta_2 \), then \( L_{\Phi, \infty}(M) \) is a quasi-Banach space, and for any \( x \in L_{\Phi, \infty}(M) \)

\[ \| x \|_{\Phi, \infty} = \inf \left\{ c > 0 : \frac{1}{\Phi^{-1} \left( \frac{1}{t} \right)} \mu_t(x)/c \leq 1, \forall t > 0 \right\} = \sup_{t > 0} \frac{1}{\Phi^{-1} \left( \frac{1}{t} \right)} \mu_t(x). \]

For any \( c > 0 \) we have that
For more information on noncommutative weak Orlicz spaces, see [3].

For any \( x \in L_0(\mathcal{M}) \), set \( \tilde{\mu}_i(x) = \frac{1}{t} \int_0^t \mu_i(x) \, ds \). Then \( \mu_i(x) \leq \tilde{\mu}_i(x) \) for all \( t > 0 \) and the map \( x \mapsto \tilde{\mu}(x) \) is a sublinear operator from \( L_0(\mathcal{M}) \) to \( L_0(\Omega) \).

**Proposition 2.2** If \( 1 < a_\Phi \leq b_\Phi < \infty \), then there exists a constant \( C > 0 \) such that

\[
\sup_{t > 0} t \Phi(\tilde{\mu}_i(x) / c) = \sup_{t > 0} \lambda_i(x) \Phi(s / c), \quad \forall x \in L_0(\mathcal{M}).
\]

(2.5)

In particular, \( \sup_{t > 0} t \Phi(\tilde{\mu}_i(x)) \leq C \sup_{t > 0} t \Phi(\mu_i(x)) \) for all \( x \in L_{\Phi, \infty}(\mathcal{M}) \). Consequently,

\[
\sup_{t > 0} \frac{1}{\Phi^{-1}\left(\frac{1}{t}\right)} \tilde{\mu}_i(x) \leq C \sup_{t > 0} \frac{1}{\Phi^{-1}\left(\frac{1}{t}\right)} \mu_i(x), \quad \forall x \in L_{\Phi, \infty}(\mathcal{M}).
\]

(2.6)

**Proof** Let \( 1 < p_0 < a_\Phi \leq b_\Phi < p_1 < \infty \). By [7, Theorem III.5.5 ], the map \( x \mapsto \tilde{\mu}(x) \) is bounded from \( L_{p_i}(\mathcal{M}) \) to \( L_{p_i}(\Omega) \), \( i = 0, 1 \). Using [3, Corollary 4.4], we obtain the desired result. \( \square \)

We use (2.1) and the above proposition to obtain the following result.

**Corollary 2.3** Let \( 1 < a_\Phi \leq b_\Phi < \infty \). Set

\[
\|x\|_{\Phi', \infty} = \sup_{t > 0} \Psi^{-1}\left(\frac{1}{t}\right) \int_0^t \mu_i(x) \, ds, \quad \forall x \in L^{\Phi, \infty}(\mathcal{M}).
\]

Then \( \|x\|_{\Phi', \infty} \) is an equivalent norm on \( L_{\Phi, \infty}(\mathcal{M}) \).

Set \( \varphi(t) = 1 / \Psi^{-1}(\frac{1}{t}) \). Then \( \varphi \) is an increasing concave function on \( (0, \infty) \) with \( \lim_{t \to 0} \varphi(t) = 0 \) and \( \lim_{t \to \infty} \varphi(t) = \infty \). Let \( \Lambda_{\varphi}(\Omega) \), \( M_{\varphi}(\Omega) \) be the usual Lorentz and Marcinkiewicz spaces with norms defined by

\[
\Lambda_{\varphi}(\Omega) = \left\{ f \in L_0(\Omega) : \|f\|_{\Lambda_{\varphi}} = \int_0^\infty f^+(t) \varphi'(t) \, dt < \infty \right\}
\]

and

\[
M_{\varphi}(\Omega) = \left\{ f \in L_0(\Omega) : \|f\|_{M_{\varphi}} = \sup_{t > 0} \frac{1}{\varphi(t)} \int_0^t f^+(s) \, ds < \infty \right\}.
\]

The Lorentz space \( \Lambda_{\varphi}(\Omega) \) has order continuous norm and \( \Lambda_{\varphi}(\Omega)^* = M_{\varphi}(\Omega) \). If \( M^0_{\varphi}(\Omega) \) denotes the linear subspace of \( M_{\varphi}(\Omega) \) consisting of all \( f \in M_{\varphi}(\Omega) \) for which

\[
\limsup_{t \to 0} \frac{1}{\varphi(t)} \int_0^t \mu_i(x) \, ds = 0 \quad \text{and} \quad \limsup_{t \to \infty} \frac{1}{\varphi(t)} \int_0^t \mu_i(x) \, ds = 0.
\]
Then $M^0_\varphi(\Omega) = \Lambda_\varphi(\Omega)$ (for more details see [18, Chapter II.5]). Since $M^0_\varphi(\Omega)$ is separable, 

$$M^0_\varphi(\mathcal{M}) = \text{closure of } S(\mathcal{M}) \text{ in } M_\varphi(\mathcal{M}).$$

Hence, 

$$\Lambda_\varphi(\mathcal{M})^* = M_\varphi(\mathcal{M}), \quad M^0_\varphi(\mathcal{M})^* = \Lambda_\varphi(\mathcal{M})$$

(see [12, Proposition 5.3], also see [8, Theorem 2.1] and [9, Proposition 2.1]).

Let $1 < a_\varphi \leq b_\varphi < \infty$. By Proposition 2.2,

$$M_\varphi(\mathcal{M}) = L_{\Phi,\infty}(\mathcal{M}).$$

We denote the closure of $S(\mathcal{M})$ in $L_{\Phi,\infty}(\mathcal{M})$ by $L^0_{\Phi,\infty}(\mathcal{M})$ and $\Lambda_\varphi(\mathcal{M})$ by $L_{1,\Psi}(\mathcal{M})$, respectively. Then

$$L_{1,\Psi}(\mathcal{M})^* = L_{\Phi,\infty}(\mathcal{M}) \quad \text{and} \quad L^0_{\Phi,\infty}(\mathcal{M})^* = L_{1,\Psi}(\mathcal{M})$$  \hspace{1cm} (2.8)

It is clear that

$$\lim_{t \to 0} \frac{t}{\varphi(t)} = \lim_{t \to 0} t\Psi^{-1}\left(\frac{1}{t}\right) = 0, \quad \lim_{t \to \infty} \frac{t}{\varphi(t)} = \lim_{t \to \infty} t\Psi^{-1}\left(\frac{1}{t}\right) = \infty.$$

We define continuous seminorms $N_0$ and $N_\infty$ on $L_{\Phi,\infty}(\mathcal{M})$ by

$$N_0(x) = \limsup_{t \to 0} \frac{1}{\varphi(t)} \int_0^t \mu_s(x)ds$$

and

$$N_\infty(x) = \limsup_{t \to \infty} \frac{1}{\varphi(t)} \int_0^t \mu_s(x)ds,$$

for all $x \in L_{\Phi,\infty}(\mathcal{M})$. Using main result in [9], we obtain that

$$L_{\Phi,\infty}(\mathcal{M})^* = L_{1,\Psi}(\mathcal{M}) \oplus S_0 \oplus S_\infty,$$  \hspace{1cm} (2.9)

where

$$S_0 = \{ \ell \in L_{\Phi,\infty}(\mathcal{M})^* : \ell \text{ annihilates all } x \in \mathcal{M} \} = \{ \ell \in L_{\Phi,\infty}(\mathcal{M})^* : \exists C > 0, |\ell(x)| \leq CN_0(x), \quad \forall x \in L_{\Phi,\infty}(\mathcal{M}) \},$$

$$S_\infty = \{ \ell \in L_{\Phi,\infty}(\mathcal{M})^* : \ell \text{ annihilates all } x \in L_{\Phi,\infty}(\mathcal{M}) \text{ with } r(x) \in S(\mathcal{M}) \} = \{ \ell \in L_{\Phi,\infty}(\mathcal{M})^* : \exists C > 0, |\ell(x)| \leq CN_\infty(x), \quad \forall x \in L_{\Phi,\infty}(\mathcal{M}) \}.$$

Let

$$M(t, \varphi) = \sup_{s > 0} \frac{\varphi(ts)}{\varphi(s)}, \quad t > 0.$$
Define
\[ p_\varphi = \lim_{t \searrow 0} \frac{\log M(t, \varphi)}{\log t}, \quad q_\varphi = \lim_{t \nearrow \infty} \frac{\log M(t, \varphi)}{\log t}. \]
Then
\[ [p_\varphi, q_\varphi] \subset \left[ \frac{1}{b_\varphi}, \frac{1}{a_\varphi} \right]. \tag{2.10} \]
(For more details, see [21, Remarks 3 (p.84) and Theorem 11.11] and [20, Theorem 4.2]).

3 Noncommutative weak Orlicz spaces associated with a weight

We denote by \( L_{\text{loc}}(\mathcal{M}) \) the set of all measurable locally measurable operators affiliated with \( \mathcal{M} \). It is well known that \( L_{\text{loc}}(\mathcal{M}) \) is a \( \ast \)-algebra with respect to the strong sum and strong product and \( L_0(\mathcal{M}) \) is a \( \ast \)-subalgebra in \( L_{\text{loc}}(\mathcal{M}) \) (see [22, 23]). Set \( M^+ = \{ x \in \mathcal{M} : x \geq 0 \} \) and \( L_{\text{loc}}(\mathcal{M})^+ = \{ x \in L_{\text{loc}}(\mathcal{M}) : x \geq 0 \} \). Let
\[ \tilde{\tau}(x) = \sup \{ \tau(y) : y \in M^+, y \leq x \}, \quad x \in L_{\text{loc}}(\mathcal{M})^+. \]
Then \( \tilde{\tau} \) is an extension of \( \tau \) to \( L_{\text{loc}}(\mathcal{M})^+ \) (see [23, §4.1]). The extension will be denoted still by \( \tau \).

Definition 3.1
1. A weight on \( \mathcal{M} \) is a map \( \omega : M^+ \to [0, \infty] \) satisfying
\[ \omega(x + \lambda y) = \omega(x) + \lambda \omega(y), \quad \forall x, y \in M^+, \ \forall \lambda \in \mathbb{R} \]
(where \( 0.\infty = 0 \)).
2. A weight \( \omega \) is said to be normal if \( \sup_i \omega(x_i) = \omega(\sup_i x_i) \) for any bounded increasing net \( (x_i) \) in \( M^+ \), faithful if \( \omega(x) = 0 \) implies \( x = 0 \), semifinite if the linear span \( \mathcal{M}_\omega \) of the cone \( M_\omega^+ = \{ x \in M^+ : \omega(x) < \infty \} \) is dense in \( \mathcal{M} \) with respect to the ultra-weak topology, and locally finite if for any non-zero \( x \in M^+ \) there is a non-zero \( y \in M^+ \) such that \( y \leq x \) and \( 0 < \omega(y) < \infty \).

Let \( \omega \) be a faithful normal semifinite weight on \( \mathcal{M} \). Then \( \omega \) has a Radon–Nikodym derivative \( D_\omega \) with respect to \( \tau \) such that \( \omega(\cdot) = \tau(D_\omega \cdot) \) (see [24]). The weight \( \omega \) is locally finite if and only if the operator \( D_\omega \) is locally measurable (see [26]). In the sequel, unless otherwise specified, we always denote by \( \omega \) a faithful normal locally finite weight on \( \mathcal{M} \). Let \( \Phi^{-1} : [0, \infty) \to [0, \infty) \) be the inverse of \( \Phi \) (which is uniquely defined on \( \mathbb{R}^+ \)).

Let \( 0 \leq \alpha \leq 1 \). Set
\[ \mathcal{M}_{\Phi, \infty}^{a, \omega} = \left\{ x \in \mathcal{M} : \sup_{t > 0} t \mu_t(\Phi(|\Phi^{-1}(D_\omega)^{a}x\Phi^{-1}(D_\omega)^{1-a}|)) < \infty \right\} \]

and

\[ \|x\|_{\Phi, \infty, a, \omega} = \inf \left\{ c > 0 : \sup_{t > 0} t \Phi(\mu_t(\Phi^{-1}(D_\omega)^{a}x\Phi^{-1}(D_\omega)^{1-a})/c) \leq 1 \right\}. \]

**Lemma 3.2** Let \( \alpha \in [0, 1] \). If \( \Phi \in \triangle_2 \), then \( \mathcal{M}_{\Phi, \infty}^{a, \omega} \) is a linear subspace in \( \mathcal{M} \).

**Proof** Let \( x \in \mathcal{M}_{\Phi, \infty}^{a, \omega} \) and \( \eta \in \mathbb{C} \). If \( |\eta| \leq 1 \), by Lemma 2.5 in [14] and convexity of \( \Phi \),

\[
\sup_{t > 0} t \mu_t(\Phi(|\Phi^{-1}(D_\omega)^{a}\eta x\Phi^{-1}(D_\omega)^{1-a}|)) = \sup_{t > 0} t \Phi(\mu_t(\Phi^{-1}(D_\omega)^{a}\eta x\Phi^{-1}(D_\omega)^{1-a})) \\
= \sup_{t > 0} t \Phi(\mu_t(\Phi^{-1}(D_\omega)^{a}\eta x\Phi^{-1}(D_\omega)^{1-a})) \\
\leq |\eta| \sup_{t > 0} t \Phi(\mu_t(\Phi^{-1}(D_\omega)^{a}x\Phi^{-1}(D_\omega)^{1-a})) \\
= |\eta| \sup_{t > 0} t \mu_t(\Phi(|\Phi^{-1}(D_\omega)^{a}x\Phi^{-1}(D_\omega)^{1-a}|)) < \infty.
\]

Hence, \( x \in \mathcal{M}_{\Phi, \infty}^{a, \omega} \). If \( |\eta| > 1 \), since \( \Phi \in \triangle_2 \), there exists a constant \( k = k(|\eta|) > 0 \) such that \( \Phi(|\eta|t) \leq k \Phi(t) \) for all \( t > 0 \). Similar to the above, we obtain that \( x \in \mathcal{M}_{\Phi, \infty}^{a, \omega} \).

Now let \( x, y \in \mathcal{M}_{\Phi, \infty}^{a, \omega} \). Using Lemma 2.5 in [14], convexity of \( \Phi \) and \( \Phi \in \triangle_2 \), we get

\[
\sup_{t > 0} t \mu_t(\Phi(|\Phi^{-1}(D_\omega)^{a}x + y\Phi^{-1}(D_\omega)^{1-a}|)) \\
= \sup_{t > 0} t \Phi(\mu_t(\Phi^{-1}(D_\omega)^{a}x + y\Phi^{-1}(D_\omega)^{1-a})) \\
\leq \sup_{t > 0} t \Phi(\mu_{t/2}(\Phi^{-1}(D_\omega)^{a}x\Phi^{-1}(D_\omega)^{1-a}) \\
+ \mu_{t/2}(\Phi^{-1}(D_\omega)^{a}y\Phi^{-1}(D_\omega)^{1-a})) \\
\leq c \sup_{t > 0} t/2 \Phi(\mu_{t/2}(\Phi^{-1}(D_\omega)^{a}x\Phi^{-1}(D_\omega)^{1-a})) \\
+ c \sup_{t > 0} t/2 \Phi(\mu_{t/2}(\Phi^{-1}(D_\omega)^{a}y\Phi^{-1}(D_\omega)^{1-a})) < \infty,
\]

and so \( x + y \in \mathcal{M}_{\Phi, \infty}^{a, \omega} \).

**Proposition 3.3** Let \( \alpha \in [0, 1]. \)

1. If \( \|x\|_{\Phi, \infty, a, \omega} > 0 \) then

\[
\sup_{t > 0} t \Phi(\mu_t(\Phi^{-1}(D_\omega)^{a}x\Phi^{-1}(D_\omega)^{1-a})/\|x\|_{\Phi, \infty, a, \omega}) \leq 1.
\]

2. \( \|x\|_{\Phi, \infty, a, \omega} \) is a quasi-norm on the linear space \( \mathcal{M}_{\Phi, \infty}^{a, \omega} \) and
\[ \|x + y\|_{\Phi, \alpha, \omega} \leq 2\|x\|_{\Phi, \alpha, \omega} + \|y\|_{\Phi, \alpha, \omega}, \quad \forall x, y \in M_{\Phi, \omega}. \] \hfill (3.1)

3. If \( \|x\|_{\Phi, \alpha, \omega} \leq 1 \), then
\[ \sup_{t > 0} t \Phi \left( \mu_t (\Phi^{-1}(D_{\omega})^a_x \Phi^{-1}(D_{\omega})^{-a}) \right) \leq \|x\|_{\Phi, \alpha, \omega}. \]

4. \( \|x\|_{\Phi, \alpha, \omega} \leq \|x\|_{\Phi, \alpha, \omega} \) for any \( x \in M_{\Phi}^a \), where
\[ M_{\Phi}^a = \left\{ x \in M : \Phi^{-1}(D_{\omega})^a_x \Phi^{-1}(D_{\omega})^{-a} \right\}. \]

and \( \|x\|_{\Phi, \alpha, \omega} = \|\Phi^{-1}(D_{\omega})^a_x \Phi^{-1}(D_{\omega})^{-a}\|_{\Phi} \). Consequently, \( M_{\Phi}^{a, \omega} \subset M_{\Phi, \omega} \).

**Definition 3.4** Let \( \omega \) be a faithful normal semifinite weight on \( M \) and \( \alpha \in [0, 1] \). We call the completion of \( (M_{\Phi, \omega}^a, \| \cdot \|_{\Phi, \alpha, \omega}) \) the weak noncommutative Orlicz space associated with \( \Phi, M \) and \( \omega \), denote by \( L_{\Phi, \omega}^a(M, \tau) \).

**Lemma 3.5** Let \( D \) be a positive nonsingular operator in \( L^1(M) \). If \( \alpha \in [0, 1] \) and \( \Phi \in \Delta_2 \), then \( \Phi^{-1}(D)^a \Phi^{-1}(D)^{-a} \) is dense in \( L_{\Phi, \omega}^0(M) \).

**Proof** Set \( e_n = e_{\left(\frac{1}{n}, 1\right)}(D) \), for any \( n \in \mathbb{N} \). Then \( e_n \) increases strongly to 1 and \( \tau(e_n) < \infty \), for any \( n \in \mathbb{N} \). Let \( x \in S(M) \). Then there is a projection \( e \) in \( M \) such that \( \tau(e) < \infty \) and \( ex = xe = x \). Hence, \( x \in L^1(M) \). By [17, Lemma 2.1], we get \( \lim_{n \to \infty} \|xe_n - x\|_1 = 0 \). It follows that \( xe_n - x \to 0 \) in measure as \( n \to \infty \). Using [14, Lemma 3.1], we get for any \( t > 0 \), \( \mu_t(xe_n - x) \to 0 \) as \( n \to \infty \). On the other hand, by Lemma 2.5 in [14], \( \mu_t(xe_n - x) \leq \mu_t(x) \) for all \( t > 0 \). Applying Lebesgue dominated convergence theorem, we get
\[ \lim_{n \to \infty} \tau(\Phi(|xe_n - x|)) = \lim_{n \to \infty} \int_0^\tau(1) \Phi(\mu_t(xe_n - x)dt = 0. \]

Therefore, \( \lim_{n \to \infty} \|xe_n - x\|_{\Phi} = 0 \). Similarly, \( \lim_{n \to \infty} \|e_n^r x - x\|_{\Phi} = 0 \). Using (4) of Proposition 3.3, we obtain that \( \lim_{n \to \infty} \|xe_n - x\|_{\Phi, \omega} = 0 \) and \( \lim_{n \to \infty} \|e_n x - x\|_{\Phi, \omega} = 0 \), and so
\[ \lim_{n \to \infty} \|e_n x - x\|_{\Phi, \omega} \leq 2 \lim_{n \to \infty} \|xe_n - x\|_{\Phi, \omega} + \lim_{n \to \infty} \|e_n x - x\|_{\Phi, \omega} = 0, \]
i.e., the closure of \( \bigcup_{n=1}^{\infty} e_n Me_n \) in \( L_{\Phi, \omega}^0(M) \) contains \( S(M) \). Thus, \( \bigcup_{n=1}^{\infty} e_n Me_n \) is dense in \( L_{\Phi, \omega}^0(M) \).

Next, we prove that \( \Phi^{-1}(D)^a Me^{-n} \Phi^{-1}(D)^{-a} \subset L_{\Phi, \omega}^0(M) \). Set \( \Phi^{(p)}(t) = \Phi(t^p) \), for \( 1 < p < \infty \). Let \( y \in M \). If \( \alpha \in (0, 1) \), then
\[ \lim_{n \to \infty} \tau \left( \Phi^{(\frac{1}{p})}(|\Phi^{-1}(D)^a - e_n \Phi^{-1}(D)^{1-a}|) \right) = \lim_{n \to \infty} \tau(D - De_n) = 0. \]
It follows that \( \lim_{n \to \infty} \| \Phi^{-1}(D)^a - e_n \Phi^{-1}(D)^a \|_{\Phi_{\frac{1}{2}, \infty}} = 0 \). Since \( \Phi^{-1}(\frac{t}{2}) \geq \frac{1}{2} \Phi^{-1}(s) \) for all \( s > 0 \), by Lemma 2.5 in [14], we get
\[
\| \Phi^{-1}(D)^a y \Phi^{-1}(D)^{1-a} - e_n \Phi^{-1}(D)^a y \Phi^{-1}(D)^{1-a} \|_{\Phi, \infty} = \sup_{t>0} \frac{1}{\Phi^{-1}(\frac{t}{2})} \mu_t((1 - e_n) \Phi^{-1}(D)^a y \Phi^{-1}(D)^{1-a}) \\
\leq 2 \| y \| \sup_{t>0} \frac{1}{\Phi^{-1}(\frac{t}{2})} \mu_t((1 - e_n) \Phi^{-1}(D)^a y \Phi^{-1}(D)^{1-a}) \\
\leq 2 \| y \| \| (1 - e_n) \Phi^{-1}(D)^a y \Phi^{-1}(D)^{1-a} \|_{\Phi_{\frac{1}{2}, \infty}} \| \Phi^{-1}(D)^{1-a} \|_{\Phi_{\frac{1}{2}, \infty}}.
\]

Hence, \( \lim_{n \to \infty} \| \Phi^{-1}(D)^a y \Phi^{-1}(D)^{1-a} - e_n \Phi^{-1}(D)^a y \Phi^{-1}(D)^{1-a} \|_{\Phi, \infty} = 0 \). Similar to the above,
\[
\lim_{n \to \infty} \| \Phi^{-1}(D)^a y \Phi^{-1}(D)^{1-a} - e_n \Phi^{-1}(D)^a y \Phi^{-1}(D)^{1-a} e_n \|_{\Phi, \infty} = 0.
\]

On the other hand, \( e_n \Phi^{-1}(D)^a, \Phi^{-1}(D)^{1-a} e_n \in \mathcal{M} \), and so for all \( n \in \mathbb{N} \),
\[
e_n \Phi^{-1}(D)^a y \Phi^{-1}(D)^{1-a} e_n \in e_n \mathcal{M} e_n.
\]

Therefore, \( \Phi^{-1}(D)^a y \Phi^{-1}(D)^{1-a} \in \mathcal{L}_{\Phi, \infty}^0(\mathcal{M}) \). In the case \( a = 0, 1 \), this result also holds. Thus \( \Phi^{-1}(D)^a \mathcal{M} \Phi^{-1}(D)^{1-a} \subset \mathcal{L}_{\Phi, \infty}^0(\mathcal{M}) \).

Finally, we prove that \( \Phi^{-1}(D)^a \mathcal{M} \Phi^{-1}(D)^{1-a} \) is dense in \( \mathcal{L}_{\Phi, \infty}^0(\mathcal{M}) \). For this, it is sufficient to prove that \( \mathcal{U}_{n=1}^\infty e_n \mathcal{M} e_n \subset \Phi^{-1}(D)^a \mathcal{M} \Phi^{-1}(D)^{1-a} \). Since for any \( n \in \mathbb{N} \), \( e_n \Phi^{-1}(D)^{1-a}, \Phi^{-1}(D)^{1-a} e_n \in \mathcal{M} \), we have that
\[
e_n \mathcal{M} e_n = \Phi^{-1}(D)^a e_n \mathcal{M} e_n \Phi^{-1}(D)^{1-a} \\
= \Phi^{-1}(D)^a e_n \mathcal{M} \Phi^{-1}(D)^{1-a} e_n \Phi^{-1}(D)^{1-a} \\
\subset \Phi^{-1}(D)^a \mathcal{M} \Phi^{-1}(D)^{1-a}.
\]

It follows that \( \mathcal{U}_{n=1}^\infty e_n \mathcal{M} e_n \subset \Phi^{-1}(D)^a \mathcal{M} \Phi^{-1}(D)^{1-a} \).

\[\square\]

**Theorem 3.6** Let \( \omega \) be a faithful normal semifinite weight on \( \mathcal{M} \) such that its the Radon–Nikodym derivative \( D_\omega \) with respect to satisfy \( D_\omega \in L^1(\mathcal{M}) \). If \( a \in [0,1] \) and \( \Phi \in \triangle_2 \), then \( L_{\Phi, \infty}^{a, \omega}(\mathcal{M}, \tau) \) and \( L_{\Phi, \infty}^0(\mathcal{M}) \) are isometrically isomorphic.

**Proof** We define \( T : \mathcal{M}_{\Phi, \infty}^{a, \omega} \to L_{\Phi, \infty}^{a, \omega}(\mathcal{M}, \tau) \) by
\[
T(x) = \Phi^{-1}(D)^a x \Phi^{-1}(D)^{1-a}, \quad x \in \mathcal{M}_{\Phi, \infty}^{a, \omega}.
\]

Then \( T \) is a linear isometry from \( \mathcal{M}_{\Phi, \infty}^{a, \omega} \) to \( \Phi^{-1}(D)^a \mathcal{M} \Phi^{-1}(D)^{1-a} \). By the definition of \( L_{\Phi, \infty}^{a, \omega}(\mathcal{M}, \tau) \) and Lemma 3.5, we know that \( \mathcal{M}_{\Phi, \infty}^{a, \omega} \) is dense in \( L_{\Phi, \infty}^{a, \omega}(\mathcal{M}, \tau) \) and \( \Phi^{-1}(D)^a \mathcal{M} \Phi^{-1}(D)^{1-a} \) is dense in \( L_{\Phi, \infty}^0(\mathcal{M}) \). Hence, we can extend to an isometric isomorphism between \( L_{\Phi, \infty}^{a, \omega}(\mathcal{M}, \tau) \) and \( L_{\Phi, \infty}^0(\mathcal{M}) \). \( \square \)
4 Noncommutative weak Orlicz–Hardy spaces

We will assume that $\mathcal{D}$ is a von Neumann subalgebra of $\mathcal{M}$ such that the restriction of $\tau$ to $\mathcal{D}$ is still semifinite. Let $\mathcal{E}$ be the (unique) normal faithful conditional expectation of $\mathcal{M}$ with respect to $\mathcal{D}$ which leaves $\tau$ invariant.

**Definition 4.1** A $w^*$-closed subalgebra $\mathcal{A}$ of $\mathcal{M}$ is called a subdiagonal subalgebra of $\mathcal{M}$ with respect to $\mathcal{E}$ (or $\mathcal{D}$) if

1. $\mathcal{A} + J(\mathcal{A})$ is $w^*$-dense in $\mathcal{M}$, where $J(\mathcal{A}) = \{x^* : x \in \mathcal{A}\}$,
2. $E(xy) = E(x)E(y)$, $\forall x, y \in \mathcal{A}$,
3. $\mathcal{A} \cap J(\mathcal{A}) = \mathcal{D}$.

$\mathcal{D}$ is then called the diagonal of $\mathcal{A}$.

In this section, $\mathcal{A}$ always denotes a subdiagonal subalgebra of $\mathcal{M}$ with respect to $\mathcal{E}$ (or $\mathcal{D}$). We keep all notations introduced in the previous section.

Let $\mathcal{H}_{\Phi, \infty}(\mathcal{A})$ is called noncommutative weak Orlicz–Hardy space associated with $\mathcal{A}$. Similarly, we define $\mathcal{H}_{1, \Psi}(\mathcal{A})$ by

$$\mathcal{H}_{1, \Psi}(\mathcal{A}) = \left\{ x \in L_{1, \Psi}(\mathcal{M}) : \tau(xa) = 0, \forall a \in \mathcal{A}_0 \right\}.$$ 

Let $\mathcal{M}$ be finite. By Propositions 4.3 in [4], we know that $\mathcal{H}_{1, \Psi}(\mathcal{A})$ is the closure of $\mathcal{A}$ in $L_{1, \Psi}(\mathcal{M})$.

**Proposition 4.2** Let $1 < a_\Phi \leq b_\Phi < \infty$. Then

$$L_{1, \Psi}(\mathcal{M}) = \mathcal{H}_{1, \Psi}(\mathcal{A}) \oplus J(\mathcal{H}_{1, \Psi}(\mathcal{A})_0),$$

where $\mathcal{H}_{1, \Psi}(\mathcal{A})_0 = \{ x \in \mathcal{H}_{1, \Psi}(\mathcal{A}) : \mathcal{E}(x) = 0 \}$.

**Proof** Since the lower Boyd index $p_{L_1, \Psi}$ and upper Boyd index $q_{L_1, \Psi}$ of $L_{1, \Psi}(\Omega)$ are $\frac{1}{p_\Psi}$ and $\frac{1}{q_\Psi}$, respectively (see [20, Theorem 4.2]). Using (2.10), we get that $1 < p_{L_1, \Psi} = q_{L_1, \Psi} < \infty$. If $\tau(1) < \infty$, then by [5, Theorem 5], we obtain the desired result. If $\tau(1) = \infty$. Choose that $1 < p < p_{L_1, \Psi} = q_{L_1, \Psi} < q < \infty$. Since there is a bounded projection operator $P$ from $L_p(\mathcal{M})$ onto $H_p(\mathcal{A})$ and from $L_q(\mathcal{M})$ onto $H_q(\mathcal{A})$ (see [6, Theorem 4.2]), by Theorem 3.4 in [13], we know that $P$ is a bounded projection from $L_{1, \Psi}(\mathcal{M})$ onto $\mathcal{H}_{1, \Psi}(\mathcal{A})$. \hfill $\square$

**Theorem 4.3** Let $\Phi 1 < a_\Phi \leq b_\Phi < \infty$. Then
\[ H_{\Phi,\infty}(A) = H_{1,\Psi}(A) \oplus S_0|H_{\Phi,\infty}(A) \oplus S_\infty|H_{\Phi,\infty}(A). \] (4.1)

**Proof** It is clear that
\[ H_{1,\Psi}(A) \oplus S_0|H_{\Phi,\infty}(A) \oplus S_\infty|H_{\Phi,\infty}(A) \subset H_{\Phi,\infty}(A). \]

Let \( \ell' \in H_{\Phi,\infty}(A)^* \). By Hahn–Banach theorem, there is a functional \( \tilde{\ell}' \in L_{\Phi,\infty}(\mathcal{M})^* \) such that \( \ell' = \tilde{\ell}'|H_{\Phi,\infty}(A) \). Using (2.9), we get that
\[ \tilde{\ell}'(x) = \tau(xy^*) + \tilde{\ell}'_1(x) + \tilde{\ell}'_2(x), \quad \forall x \in L_{\Phi,\infty}(\mathcal{M}), \]
where \( y \in L_{1,\Psi}(\mathcal{M}), \tilde{\ell}'_1 \in S_0 \) and \( \tilde{\ell}'_2 \in S_\infty \). Using Proposition 4.2, we obtain that there exist \( h \in H_{1,\Psi}(A) \) and \( z \in H_{1,\Psi}(A)_0 \) such that \( y = h + z^* \). Hence,
\[ \tau(ay^*) = \tau(ah^*) + \tau(az) = \tau(ah^*), \quad a \in H_{\Phi,\infty}(A). \]

Therefore, \( \ell' = h + \tilde{\ell}'_1|H_{\Phi,\infty}(A) + \tilde{\ell}'_2|H_{\Phi,\infty}(A) \). From this, (4.1) follows. \( \square \)

If \( \Phi(t) = \frac{p}{t} (1 < p < \infty) \), then \( \Psi(t) = \frac{p}{q} \) and \( \varphi(t) = \frac{1}{q} t^{1/q} \), where \( \frac{1}{p} + \frac{1}{q} = 1 \).

Hence,
\[ N_0(x) = \limsup_{t \to 0} q^{1/q} \frac{1}{t^{1/q}} \int_0^t \mu_s(x)ds = q^{1/q} \limsup_{t \to 0} t^{1/p-1} \int_0^t \mu_s(x)ds, \]
and
\[ N_\infty(x) = \limsup_{t \to \infty} q^{1/q} \frac{1}{t^{1/q}} \int_0^t \mu_s(x)ds = q^{1/q} \limsup_{t \to \infty} t^{1/p-1} \int_0^t \mu_s(x)ds. \]

**Corollary 4.4** Let \( 1 < p < \infty \). Then
\[ H_{p,\infty}(A)^* = H_{1,q}(A) \oplus S_0|H_{p,\infty}(A) \oplus S_\infty|H_{p,\infty}(A). \]

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