Boltzmann Suppression of Interacting Heavy Particles

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I. INTRODUCTION

The computation of number densities of cosmological relics, such as weakly interacting massive particles (WIMPs), neutrinos, baryon and lepton number, is usually accomplished by using a thermally averaged Boltzmann equation [1,2]. The picture emerging in this context is that of a sudden freeze-out: the particle number density follows the equilibrium value \(N(T)\) for temperatures higher than the freeze-out value, \(T_f\), below which the annihilation is frozen. In the WIMP scenario for dark matter, the freeze-out temperature turns out to be typically much lower than the WIMP mass, and one expects a Boltzmann-suppression factor \(\exp(-M/T_f)\) in the final WIMP number density. In a series of papers \([1,2]\), Matsumoto and Yoshimura (MY) have challenged the above conventional conclusion, claiming that two-loop corrections to the number density exhibit only power-law suppression in \(T_f/M\), thus dominating over the Boltzmann-suppressed tree-level contribution if \(T_f\) is low enough. If confirmed, such a finding would imply that the present constraints on dark matter models are actually underestimated and should be carefully reconsidered. To be definite, MY introduced a toy model of two real scalar fields with lagrangian

\[
\mathcal{L} = -\frac{\lambda_{\phi}}{4!}\phi^4 - \frac{\lambda}{4}\phi^2 \chi^2 + \frac{i}{2}\partial_{\mu}\phi \partial^{\mu} \phi - \frac{\lambda_{\chi}}{4!}\chi^4 + \frac{1}{4}\partial_{\mu}\chi \partial^{\mu} \chi - \frac{1}{2}m^2 \chi^2,
\]

with \(M \gg m\). The following relations were assumed among the coupling constants

\[
|\lambda_{\phi}| \ll \lambda^2, \quad |\lambda| \ll |\lambda_{\chi}| < 1,
\]

so that the light \(\chi\) particles act as an efficient heat bath for the heavy \(\phi\)'s. They then considered the quadratic part of the Hamiltonian for \(\phi\),

\[
H_{\phi} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla \phi)^2 + \frac{1}{2}M^2 \phi^2,
\]

and, for temperatures \(T\) such that \(m \ll T < M\) they defined the number density of \(\phi\) particles as

\[
N_{\phi} = \frac{\langle H_{\phi} + \text{counterterms} \rangle_T}{M}
\]

where MY define the thermal average for an operator \(A\) as

\[
\langle A \rangle_T \equiv \frac{\text{tr} Ae^{-H/T}}{\text{tr} e^{-H/T}} = \frac{\langle 0 | A | 0 \rangle}{\langle 0 | 0 \rangle},
\]

\(H\) being the total Hamiltonian. At tree-level, the well known Boltzmann-suppressed contribution is obtained

\[
N_{\phi}^{(0)} = \left(\frac{MT}{2\pi}\right)^{3/2} e^{-M/T}.
\]

The power-law contribution arises at two-loops and is given by

\[
N_{\phi}^{(2)} = c\lambda^2 T^6 \frac{M^3}{M^3}, \quad c = \frac{1}{69120},
\]

which, as anticipated, dominates over the tree-level contribution at low temperatures. The interpretation of \([1,2]\) as the physical number density of heavy \(\phi\) particles has been later questioned by Singh and Srednicki [3,4]. First, they pointed out that the splitting of the full Hamiltonian in two parts – one for the “system” of \(\phi\) particles, and the other for the “environment” of \(\chi\)'s – is problematic, as the interaction energy is not properly taken into account. It was actually shown to be of the same order of the “system” energy in an exactly solvable Caldeira-Leggett model. Second, Srednicki [3] considered a modification of the model in eq. \([1,2]\) by taking complex \(\phi\) fields charged under a \(U(1)\) symmetry. Then, he showed that from the fluctuations of the conserved charge it is possible to define a number density which exhibits the expected Boltzmann-suppression at low temperature. Extrapolating this analysis to the MY model, where no conserved charge exists, he proposed that a physical definition of the number density could be given as \(N_S = (1/M)\rho_{BS}\), where
\(\rho_{BO} \) represents those terms in the total energy density which have a Boltzmann-suppression factor. Moreover, he argued that non-Boltzmann-suppressed terms should be interpreted as corrections to the energy density of the \(\chi\)'s obtained after integrating out the heavy \(\phi\)'s at low temperature. Conclusive insight on the last point was achieved by Braaten and Jia [5], who worked in the effective lagrangian approach. They were able to show that the energy density of the \(\chi\)'s actually receives power-law corrections from the exchange of virtual \(\phi\)'s. However, the leading terms are only \(O(\lambda^2 T^4/M^4)\). A \(O(\lambda^2 T^4/M^2)\) term, which would correspond to the effect claimed by MY, was shown to be eliminated by a field redefinition, thus unphysical. From the above discussion, we see that the problem of the definition of the number density for a heavy particle in a interacting theory is still an open issue. As discussed in [6], the effective lagrangian approach cannot be of help in this respect, as it accounts for the effect of the heavy \(\phi\)'s on the light \(\chi\)'s, whereas what we need is a way to compute the effect of the \(\chi\)'s on the \(\phi\)'s, just the opposite.

In this paper, we will show that it is possible to give a non-ambiguous definition of the number density of an interacting particle at any temperature and at any order in perturbation theory. In the case of the MY model of eq. (6), our definition exhibits the expected Boltzmann-suppression at low temperature. We will find it very convenient to work in the real-time formalism of finite temperature field theory (RTF) [1], which has the advantage of keeping \(T = 0\) and finite temperature effects well separated. Indeed, as we will discuss, the ambiguities in the conventional definitions of the number density are mostly due to the \(T = 0\)-renormalization procedure. Being able to single out \(T = 0\) ultraviolet (UV) divergences turns then out to be a great advantage when one is interested in genuine thermal effects. Moreover, we will eventually be interested in using the defined number density in an out-of-equilibrium context, and in this respect the RTF, due to its close relationship with the Schwinger-Keldysh formalism, is clearly a much better starting point than the imaginary time formalism.

The tree-level propagator for a real scalar particle in the RTF has the structure

\[
D_0 = D_0^{\text{tree}} + D_0^T = iP[D_0] + i(D_0 - \Delta_0)\mathcal{N}(\{q_0\})B, \tag{6}
\]

where \(\Delta_0 = (q^2 - M^2 + (-i\varepsilon)^{-1})^{-1}\) is the tree-level Feynman propagator, \(\mathcal{N}\) the Bose-Einstein (Fermi-Dirac) distribution function. \(\mathcal{N}(x) = (\exp(x/T) - (-1)^{1})^{-1}\) and the two matrices \(P\) and \(B\) are defined as

\[
P[a(q)] = \begin{bmatrix} a(q) & (a - a^*)(q)\theta(-q_0) \\ (a - a^*)(q)\theta(q_0) & -a^*(q) \end{bmatrix},
\]

\[
B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \tag{7}
\]

with \(\theta(x)\) the Heaviside’s step function.

The full propagator has the same structure as the tree-level one, eq. (6), with \(\Delta_0^{(+)}\) replaced by its all-order counterpart \(\Pi(q_0, |q|)\) being the (renormalized) full self-energy at finite temperature. In the following, we will refer to \(iP[D_0] + i\Pi(q_0, |q|)\) as the \(“T = 0”\) part of the tree-level and full propagators, and to \(i(D_0 - \Delta_0)\mathcal{N}(\{q_0\})B\) and \(i(D - \Delta)\mathcal{N}(\{q_0\})B\) as the “thermal” parts. It is important to notice that what we call the \(“T = 0”\) part of the full propagator actually contains thermal corrections in the self-energy, so it is not the full \(T = 0\) propagator.

Our discussion and results may be summarized as follows. We will start recalling how composite operator renormalization induces a mixing between the would-be number (or energy-density) operator for the \(\phi\)-field and operators made up of \(\chi\)-fields (the “counterterms” in eq. (6)), which in general make a definition of a “pure” \(\phi\)-number quite cumbersome. These divergences, and the mixing, arise as the field operators are evaluated at the same point. Diagrammatically, they arise when the ends of the two \(\phi\) legs are joined, provided both legs are \(T = 0\) propagators. It is inherently a \(T = 0\) effect, since if any of the two external legs is thermal, no divergence arises thanks to the UV cut-off provided by the Bose-Einstein function. Then, a thermal averaging procedure can be naturally defined, at least for quadratic operators, which is free from composite operator UV divergences.

If we insist in defining the number density starting from the energy density \(\rho_{tot}\), we are led towards a different splitting of \(\rho_{tot}\) with respect to that considered by MY. The \(\rho_0\), \(\rho_\chi\), and \(\rho_{tot}\) components into which we split \(\rho_{tot}\), are such that: i) each term is closed under operator mixing, so that the splitting is independent on the renormalization scale; ii) no \(O(\lambda^2 T^6/M^2)\) correction to any of the pieces in which \(\rho_{tot}\) is split emerges; iii) the \(\phi\)-energy density is Boltzmann-suppressed. Finally, we will point out that a better definition for the number density, also exhibiting Boltzmann-suppression at \(T < M\) but valid in any temperature range, can be given by considering the thermal part of the full propagator in RTF.

The paper is organized as follows. In sect. II the basic features of composite operator renormalization are illustrated for the case of the \(\phi^2\) operator in the MY model. In sect. III a new definition of thermal average is given, which does not require composite operator renormalization. In sect. IV this definition is applied to the energy-momentum tensor in the MY model, showing that it leads to a splitting of the total energy density in \(\phi\), \(\chi\), and “interaction” pieces which differs to that considered by MY and is not plagued by the usual ambiguities induced by operator mixing. In sect V a definition of the number density valid at any temperatures and extrapolable even to the massless limit is given. Moreover, it is shown how an extension of the MY model in which the heavy fields are complex and carry a conserved charge naturally leads to a Boltzmann-suppressed definition of the heavy particle number density. In sect. VI the case of a heavy particle which is unstable already at \(T = 0\) is discussed, showing that power law corrections do indeed emerge in this case. Finally, in sect. VII we discuss the relation of our work to that of refs. [1] and give our conclusions.
II. REYNORMALIZATION OF $\phi^2$

Before considering the complete energy-momentum tensor, the analysis of the composite operator $\phi^2_0$ (where the 0-label indicates the bare fields, masses and coupling constants) will be useful in illustrating the main points of renormalization and thermal averaging. We start by considering the $T = 0$ case. Due to the local product of fields making up the operator, new UV divergences are induced when it is inserted in a Green function, which are generally not canceled by the lagrangian counterterms. The renormalization of these divergences then requires the introduction of new counterterms. Renormalized operators can be defined, which are generically expressible as linear combinations of all the bare operators of equal or lower canonical dimensionality (for a thorough discussion of composite operator renormalization, see for instance \[10\]). In the model (1) the renormalization of $\phi$ discussion of composite operator renormalization, see for

In minimal-subtraction scheme (MS) the bare parameters can be expressed in terms of the mass counterterms, \[\delta Z^\phi\] any UV divergence. The $\delta Z^\phi$ be expressed in terms of the renormalized ones as

$$\delta Z^\phi = \frac{\partial G^{\lambda \phi \pi}}{\partial m} \delta Z^\phi = \frac{\partial G^{\lambda \phi \pi}}{\partial m} \delta Z^\phi = \frac{\partial G^{\lambda \phi \pi}}{\partial m} \delta Z^\phi = \frac{\partial G^{\lambda \phi \pi}}{\partial m} \delta Z^\phi = \frac{\partial G^{\lambda \phi \pi}}{\partial m} \delta Z^\phi = \frac{\partial G^{\lambda \phi \pi}}{\partial m} \delta Z^\phi = \frac{\partial G^{\lambda \phi \pi}}{\partial m} \delta Z^\phi = \frac{\partial G^{\lambda \phi \pi}}{\partial m} \delta Z^\phi$$

In minimal-subtraction scheme (MS) the bare parameters can be expressed in terms of the renormalized ones as

$$\lambda_i = \mu^{-D} \lambda_i \left( 1 + \sum_{k=1}^{\infty} a^k \left( \frac{\lambda_i^k}{D - 4k} \right) \right),$$

$$M_0^2 = M^2 \left( 1 + \frac{\delta m^2}{M^2} \right) = m^2 \left( 1 + \frac{\delta m^2}{m^2} \right) \sum_{k=1}^{\infty} b^k \left( \frac{\lambda_i^k}{D - 4k} \right),$$

$$\delta Z^\phi$$

where \(\{\lambda_i\} = \{\lambda_0, \lambda, \lambda\}, D\) is the space-time dimensionality, and the important point is that the $a^i$'s, $b^k$'s and $b^\phi$'s are mass, momentum, and temperature independent. Using eqs. \[10\] it is then possible to prove that, for any renormalized green function $G$ the following relation holds

$$M_0^2 \frac{\partial G}{\partial M_0^2} + m_0^2 \frac{\partial G}{\partial m_0^2} = M^2 \frac{\partial G}{\partial M^2} + m^2 \frac{\partial G}{\partial m^2},$$

showing that, being the right hand side manifestly finite, the insertion of the combination $M_0^2 \phi_0^2 + m_0^2 \phi_0^2$ does not induce any UV divergence. The $\delta Z^\phi$, $\delta Z^\phi$ renormalization constants can then be expressed in terms of the mass counterterms as

$$\begin{cases}
\delta Z^\phi = - \frac{\delta m^2}{M^2} \frac{\partial G}{\partial m^2}, \\
\delta Z^\phi = - \frac{\delta m^2}{m^2} \frac{\partial G}{\partial m^2},
\end{cases}$$

whereas the remaining counterterms $\delta Z^\phi$, $\delta Z^\phi$, can be computed as

$$\delta Z^\phi = - \left( \frac{\delta^2 G}{\partial \phi^2} \right)_{\text{DIV}},$$

$$\delta Z^\phi = - \left( \frac{\delta^2 G}{\partial \phi^2} \right)_{\text{DIV}}.$$
both the $\phi$ lines, entering the cross are of $T = 0$ type. It is the cancellation of this divergence which calls into play the mixing, via the renormalization constant $\delta Z_{\phi\chi}$ appearing in the last diagram. At $O(\lambda^2)$ (fig. 3), the same happens, where

is the self energy to $O(\lambda^2)$ renormalized by lagrangian counterterms. The last diagram, containing a tree-level $\chi^2$ operator multiplied by a $O(\lambda^2)$ renormalization constant is required in order to cancel the divergence coming from the integration over the momentum of $\phi$ particles flowing to the cross with $T = 0$ propagators (the first diagram in fig. 1c). Note that, due to the general relation $\sum_{i,j=1,2} \Sigma_{ij} = 0$, $\Sigma_{ij}$ being the self-energy with external thermal indices $i$ and $j$, the contribution with both lines of thermal type also vanishes, once the summation over thermal indices is taken. The remaining loop divergences are cancelled by usual lagrangian counterterms and therefore require no mixing.

The only divergence which is not cancelled by neither lagrangian nor composite operators counterterms is the tree-level one. Then, one usually defines the thermal expectation value of $\langle \phi^2 \rangle$ as the expectation value of $\langle \phi_i^2 \rangle$ subtracted according to one of the two following ways 1,2.

$$
\langle \phi^2 \rangle_T \equiv \frac{\text{tr}[\phi^2 e^{-H/T}]}{\text{tr} e^{-H/T}} - \frac{\langle 0 | \phi^2 | 0 \rangle}{\langle 0 | 0 \rangle} \quad \text{or}
= \frac{\text{tr}[\phi^2 e^{-H/T}]}{\text{tr} e^{-H/T}} - \frac{\langle 0 | \phi^2 | 0 \rangle}{\langle 0 | 0 \rangle} \vert_{\text{free fields}} .
$$

(12)

The definitions above are both finite, and thus a plausible choice. However, both of them include the contributions from the first and the last graphs in figs. 1b and 1c, that is, they require mixing.

The above analysis suggests a new definition of thermal average of $\phi^2$, which represents the main point of this paper, namely

$$
\langle \phi^2 \rangle_{\text{Th}} \equiv \int \frac{d^4 q}{(2\pi)^4} \, D^T(q) = \int \frac{d^4 q}{(2\pi)^4} \, i(\Delta - \Delta^*) (q) N'(q_0) ,
$$

(13)

where $D^T(q)$ is any of the four component of the thermal part of the full propagator.

Expanding $\Delta^*$ as $\Delta^* = \Delta_0^* + \Delta_0^* \Pi \Delta_0^* + \cdots$, and recalling the relations between $\Pi(q_0, [\phi])$ and $\Sigma(q_0, [\phi])_{ij}$, one can check that the perturbative expansion of fig. 1 is reproduced, apart from the ‘problematic’ diagrams, i.e. the tree-level $T = 0$ diagram, the higher order ones with both $T = 0$ lines flowing into the cross, and the diagrams containing composite operators counterterms, which are all omitted.

The definition in (13) then automatically gets rid of all the unpleasant features of the conventional definitions in eq. (12), namely, the need of an arbitrary subtraction and, more importantly, composite operator renormalization and mixing.

We are now ready to define a number density for $\phi$ which will not require any mixing with operators involving the $\chi$-field and, as a welcome byproduct, will exhibit the expected Boltzmann-supression at low temperature.

IV. AVERAGING THE ENERGY-MOMENTUM TENSOR

The renormalization of the energy-momentum tensor is discussed in refs. 3,4. Expressing it in terms of bare parameters, the operator

$$
T^{\mu\nu} = \partial^\mu \phi^0 \partial^\nu \phi^0 + \partial^\mu \chi_0 \partial^\nu \chi_0 - g^{\mu\nu} L + \cdots ,
$$

is finite when inserted in a Green function, that is, it does not require extra counterterms besides those already present in the lagrangian. The dots in the formula above represent pole terms proportional to $(\partial^\mu \partial^\nu - g^{\mu\nu} \partial^2) \phi_0^2$ and to $(\partial^\mu \partial^\nu - g^{\mu\nu} \partial^2) \chi_0^2$. Since we are interested in the thermal average of $T^{\mu\nu}$ alone, translation invariance insures that such pole terms do not contribute. On the other hand, a subtraction of the $T = 0$ contribution as in eqs. (12) is needed. Working with renormalized parameters, $T^{\mu\nu}$ is expressed as

$$
T^{\mu\nu} = T_{\text{can}}^{\mu\nu} + T_{\text{c.t.}}^{\mu\nu} ,
$$

(14)

where $T_{\text{can}}^{\mu\nu}$ has the canonical form and $T_{\text{c.t.}}^{\mu\nu}$ contains only the lagrangian counterterms.

The finiteness of $T^{\mu\nu}$ means that the divergences induced by, e.g., the composite operator $\phi^2 \chi^2$ are canceled by the lagrangian counterterms and the particular combination of operators $\phi^4$, $\chi^4$, $(\partial \phi)^2$, ..., appearing in $T^{\mu\nu}$. In general, if we split $T^{\mu\nu}$ in parts, each part separately will require composite operator renormalization. Indeed, this is what is done by MY in ref. 5. They split the total hamiltonian as

$$
H^{\mu\nu} \equiv H_{\text{can}} + H_X + H_{\text{can}} + H_{\text{c.t.}} ,
$$

and compute thermal average of $H_{\text{can}}$ (given in eq. (2)). In the RTF it gives

$$
\frac{\text{tr} H_{\text{can}} e^{-H/T}}{\text{tr} e^{-H/T}} = \int \frac{d^4 q}{(2\pi)^4} \, [2q^2_0 - (q^2 - M^2)] .
$$

(15)

where, differently from eq. (3), also the $T = 0$ part of the full propagator appears. It is important to recall here what we have noticed after eq. (3), namely that the $T = 0$ full propagator contains thermal effects via the self-energy. Such contributions are not cancelled by purely $T = 0$ subtractions like those defined in eq. (12). By perturbatively expanding $[D^{T=0}]_{12}$ in (14) at $O(\lambda^2)$ we find, apart from divergent contributions which require composite operator counterterms to be added to $H_{\phi}$, the “famous” power-law contribution

$$
\int \frac{d^4 q}{(2\pi)^4} \, \left[2q^2_0 - (q^2 - M^2)\right] [D^{T=0}]_{12}(q) \bigg|_{O(\lambda^2)} =
\int \frac{d^4 q}{(2\pi)^4} \, \left[2q^2_0 - (q^2 - M^2)\right] \theta(-q_0) \text{Re} \Pi - \langle \Delta^2 + \Delta_0^2 \rangle \epsilon(q_0) \text{Im} \Pi
= \text{div.} \frac{1}{69120} \frac{\lambda^2 T^6}{M^2} + \cdots ,
$$

(16)
where $\epsilon(q_0) \equiv \theta(q_0) - \theta(-q_0)$. The same contribution, with opposite sign, is found from the corresponding piece of the thermal average for $H_\chi$.

Paralleling our discussion in the previous section, we will instead define the energy density of the $\phi$ field as

$$\rho'_\phi \equiv \frac{i}{2} \int \frac{d^4q}{(2\pi)^4} \left[ 2 \epsilon(q_0) - (q^2 - M^2) \right] |D_{T12}(q)|^2,$$

and analogously for $\rho'_\chi$. The complete energy density will then be split as

$$\rho_{\text{tot}} \equiv \langle H_{\text{tot}} \rangle_T \equiv \frac{\text{tr} H_{\text{tot}} e^{-H/T}}{\text{tr} e^{-H/T}} - \frac{(0|H_{\text{tot}}|0)}{(0|0)} = \rho'_\phi + \rho'_\chi + \rho_{\text{int}},$$

so that the contribution of eq. (16) and the analogou s

$$\rho_{\text{tot}} \equiv \langle H_{\text{tot}} \rangle_T \equiv \frac{\text{tr} H_{\text{tot}} e^{-H/T}}{\text{tr} e^{-H/T}} - \frac{(0|H_{\text{tot}}|0)}{(0|0)} = \rho'_\phi + \rho'_\chi + \rho_{\text{int}},$$

where $\rho_{\text{int}}$ is finite, the same is true also for $\rho'_\phi$. Notice that neither $\rho'_\phi$ nor $\rho'_\chi$ are affected by the $T = 0$ subtraction.

The definition in eq. (18) exhibits three remarkable properties:

i) The splitting in (13) is closed under operator mixing and composite operator renormalization. No composite operator counterterm is required to make $\rho'_\phi$ or $\rho'_\chi$ separately finite and, since $\rho_{\text{int}}$ is also finite, the same is true also for $\rho_{\text{int}}$. Thus, our definition of number density, differently than that considered by MY, is independent on the renormalization scale $\mu$.

ii) $\rho_{\text{int}}$ is Boltzmann-suppressed for $T < M$. Indeed, writing $|D_{T12}(q)|^2$ explicitly

$$|D_{T12}(q)|^2 = \frac{2 \text{Im} \Pi(q) \epsilon(q_0)}{(q^2 - M^2 - \text{Re} \Pi(q))^2 + \text{Im} \Pi(q)^2} \mathcal{N}(|q_0|),$$

we see that, due to the Bose-Einstein function, non-Boltzmann-suppressed contributions to the integral in eq. (18) might only come from momenta $|q_0| \lesssim T$ ($\ll M$). But, for these values of momenta, it is the imaginary part of the self-energy which is Boltzmann-suppressed, as we show explicitly in the Appendix.

**V. A BETTER DEFINITION FOR $N_\phi$**

Using the $\phi$-energy density as given in eq. (17), we can now define the $\phi$ number density for $T < M$ in a way analogous to MY’s definition, eq. (8), that is

$$N_\phi = \frac{\rho'_\phi}{M} \quad \text{for} \quad T < M.$$  

However a better definition, valid at any value of $T$, and again based on the average of a quadratic operator, can be given as

$$N_\phi \equiv \int \frac{d^4q}{(2\pi)^4} |q_0||D_{T12}(q)|^2,$$

As $\rho'_\phi$, eq. (20) exhibits the nice features of not requiring composite operator renormalization and of being Boltzmann-suppressed at low temperatures, but in addition it may be extended to high temperatures or, equivalently, to the massless limit. The origin of eq. (20) can be understood considering the case of a complex scalar field, $\Phi$, in case an exact $U(1)$ symmetry is imposed to the theory. In this case, a natural definition of number density involves the current operator

$$j_{\phi}(x) = \frac{i}{2} \left[ \Phi^\dagger(x) \partial_\mu \Phi(x) - \Phi(x) \partial_\mu \Phi^\dagger(x) \right].$$

In the RTF the thermal average of this operator corresponds to $i\partial_{\mu} D_C(x - y)|_{x=y}$, where $D_C \equiv |D|_{12} + |D|_{21}$. The thermal part of the propagator for a complex scalar field differs from eq. (4) only in the replacement of $\mathcal{N}(|q_0|)$ with $\mathcal{N}(|q_0|) \equiv \theta(q_0) \mathcal{N}_+(q_0) + \theta(-q_0) \mathcal{N}_-(q_0)$, where $\mathcal{N}_\pm(x) = (\exp(\beta(x \mp \mu_\phi) - 1)^{-1}$ and $\mu_\phi$ is the $\Phi$ chemical potential. Since the current is conserved, the evaluation at $x = y$ does not require composite operator renormalization. Indeed, it can be checked explicitly that, after Fourier transforming, we get

$$\langle j_\phi(x) \rangle_T = \frac{2}{\pi^2} \int \frac{d^4q}{(2\pi)^4} q_0 |D_{T12}(q)|^2,$$

where the $T = 0$ contribution vanishes. Incidentally, this sheds new light on the conclusions of ref. (9) in which it was shown that in case a $U(1)$ symmetry exists, a Boltzmann-suppressed number density can be defined. From what we saw above, it is so because in this case the current operator for $\Phi$ does not mix with any other operator. In absence of mixing, terms like (10) are absent and the number density is automatically Boltzmann-suppressed.

At tree-level we get the well-known result for the charge density

$$Q_\phi = \langle j_\phi(x) \rangle_T = \int \frac{d^4q}{(2\pi)^4} |\mathcal{N}_+(\omega) - \mathcal{N}_-(\omega)|,$$

where $\omega = (|q|^2 + M^2)^{1/2}$.

Now we can see how eq. (21) can be derived in analogy to (20). The number density for the real scalar field $\phi$ should differ from the previous quantity in two respects. First, particle and antiparticle contributions should be summed up, in order to account for their ‘Majorana’ nature. This is obtained by taking the modulus of $|q_0|$ inside (32). Second, a factor $1/2$ is needed, in order to take into account the number of degrees of freedom.

**VI. THE CASE OF DECAYING HEAVY PARTICLES**

The discussion of the previous sections is straightforwardly generalizable to any model in which heavy particles annihilate into lighter ones, both bosons and fermions. The basic conclusions remain unaltered, the main reason being
the Boltzmann-suppression of the imaginary part of the self-energy at small momenta (see the discussion at the end of Sect.IV and in the Appendix).

The situation changes drastically if the heavy particle is unstable at $T = 0$. If, for definitness, it decays into two lighter particles of mass $m < M/2$ the propagator exhibits an imaginary part for $2m < p_0 < M$ already at $T = 0$, so that there is no reason to expect that it is Boltzmann-suppressed for $T < M$. The heavy particle is actually a resonance, whose energy-momentum can take values much lower than the peak at $p^2 = M^2$, of course paying the usual Breit-Wigner suppression far away from the peak. In this case, power-law contributions to the resonance number density do in general emerge, as we will briefly discuss.

We will consider for definitness the annihilation model discussed in ref. [3] and in the first of refs. [3], in which the interaction between the heavy $\Phi$ and the light $\chi$-bosons is due to the term

$$\mathcal{L}_{\text{int}} = \frac{\mu}{2} \Phi \chi^2,$$

where we assume that $\mu \ll M$. In this model, the imaginary part of the self-energy is $\propto \mu^2$, so that, using (19) in (20) we get

$$N_\Phi \sim \frac{\Gamma_0}{M^3} T^5,$$

with $\Gamma_0 \equiv \mu^2/(32\pi M)$ the $T = 0$ decay rate on-shell. In ref. [3] a different behavior, $N_\Phi \sim T^4$ was claimed. Indeed, the following argument clarifies the physical origin of the power-law contribution and confirms the $T^5$ behavior.

In the thermal bath, unstable $\Phi$ particles are continuously produced in $\chi$-$\chi$ annihilations. Being the $\Phi$ a resonance, the production energy, $\omega$, can be much smaller than $M$. The produced $\Phi$'s eventually decay with inverse lifetime $\Gamma = \mu^2/(32\pi \omega) = \Gamma_0 M/\omega$. Thus, the number density of $\Phi$'s of energy around $\omega$, $n^\Phi(\omega)$, obeys the rate equation

$$\frac{dn^\Phi(\omega)}{dt} = \gamma - \Gamma n^\Phi(\omega),$$

where $\gamma$ is the rate per unit volume of the process

$$\chi\chi \rightarrow \Phi \rightarrow \text{everything}.$$  

The above equation leads to the equilibrium value

$$n^\Phi_{\text{eq}}(\omega) = \frac{\gamma}{\Gamma}.$$  

The production rate is given by $\gamma \sim N_\chi^2 \sigma \nu$, where the inclusive cross section $\sigma$ may be computed using the optical theorem as $\sigma \sim \mu^4 \Im \Delta_\Phi/s$, $s$ being the square of the $\Phi$ four-momentum, and $\Delta_\Phi$, the full $\Phi$ propagator.

Taking $m \ll \sqrt{s} \sim T \ll M$, the initial $\chi$ particles are relativistic (i.e. $N_\chi \sim T^3$) and we get $\sigma \sim \mu^4/(M^4 T^2)$. Putting all together, we get a contribution to the total $\Phi$-number density from the region $\omega \sim T$ of order

$$n^\Phi_{\text{eq}}(\omega \sim T) \sim \frac{\mu^2}{M^3} T^5 \sim \frac{\Gamma_0}{M^3} T^5.$$  

The $\Phi$'s with energies closer to the peak of the resonance live longer but their contribution to the number density is Boltzmann-suppressed, as two highly energetic $\chi$'s are required in the initial state to produce them.

As the energy of the relevant $\Phi$'s is typically $O(T)$, the energy density is

$$\rho_\Phi \sim O\left(\frac{\Gamma_0}{M^3} T^6\right) + \text{Boltzmann – suppressed},$$

and $\rho_\Phi/\rho_\chi$ vanishes as $T^2$ at low temperatures.

Before concluding this section, we observe that the definition of a sensible number density along the lines of eq. (15), i.e. starting from the free hamiltonian, does not seem quite appropriate in this case. Indeed, as we have just discussed, the width of the $\Phi$ plays a crucial role, allowing energy values much lower than the tree-level mass-shell. It seems that, if one wants to insist in looking for definitions like eq. (15), a free Hamiltonian for quasi-particles, along the lines discussed for instance in ref. [4], should rather be used.

### VII. CONCLUSIONS

The definition of particle number density in an interacting theory is a delicate matter. The necessity of giving a meaning to divergent composite operators calls into play operator mixing, so that a separation between different particle species turns out, in general, to be a renormalization scale dependent procedure.

Our guideline in this paper was precisely to find a definition of number density free from this problem. We discussed how composite operator renormalization is inherently a zero temperature effect which can be isolated from the physically relevant finite temperature effects. We showed that in the RTF a proper definition of the energy and number densities comes out quite naturally and can be systematically computed at any order in perturbation theory.

In the framework of the MY annihilation model, our definition of thermal average for bilinear operators leads to a different splitting of the total energy density with respect to that considered by MY. Each piece is finite, i.e. it does not require extra counterterms with respect to those already present in the lagrangian. Moreover, as a welcome byproduct, we obtain a Boltzmann-suppressed energy density for the heavy particle $\phi$. Finally, no power-suppressed terms of the type found by MY appear. In this respect, we agree with refs. [4]; the crucial point is how the interaction is treated. In our discussion, we concluded that diverging contributions like eq. (15) should naturally go into the 'interaction' part of the total energy, to make it finite. If this is done consistently, $O(\lambda^3 T^6/M^2)$ terms cancel out in $\rho^\prime_{\text{int}}$. This is consistent with the conclusion by Braaten and Jia [5] that these terms are unphysical.

Our definition of the $\phi$-energy density is close in spirit to that proposed by Srednicki in [5]. However the detail is different. Namely, in Srednicki’s definition, all the Boltzmann-suppressed terms should be included into the $\phi$ energy density, whereas we have such kind of terms also in $\rho^\prime_{\text{int}}$, for instance the contribution depicted in fig. 2.
this appendix that this is not the case as $\Pi$ is Boltzmann-suppressed. The interesting question, now, is how to obtain quantum kinetic equations giving our number densities in the equilibrium limit. From the present investigation it appears that the RTF may provide useful physical insights in this regard.

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APPENDIX

The imaginary part of the $\phi$ self-energy in the MY model emerges at two-loops by the diagram in fig. 3

From eqs. (17), and (19) we know that the contribution to $\rho_{\text{int}}$ with $N_1$'s and one on-shell $\phi$'s is Boltzmann-suppressed. Non Boltzmann-suppressed contributions might only come from $|q_0| \lesssim T$ ($\ll M$), but we will show at the end of this appendix that this is not the case as $\text{Im} \Pi$ is Boltzmann-suppressed in this regime.

According to the standard rules of the RTF [8] we have for the imaginary part:

$$\text{Im} \Pi(q) = -\frac{1}{2} \varepsilon(q_0)(e^{\beta q_0} - 1) \Sigma_{12}(q),$$

where

$$\Sigma_{12}(q) = \frac{\lambda^2}{2} \int \frac{d^4p}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} D_{12}(p)D_{12}(q-k-p)G_{12}(k),$$

and

$$iD_{12}(p) = 2\pi(\theta(-p_0) + N(|p_0|))\delta(p^2)$$

$$iG_{12}(k) = 2\pi(\theta(-k_0) + N(|k_0|))\delta(k^2 - M^2)$$

the $\chi$ and $\phi$ particle propagators respectively. Integrating with respect to $k_0$ and $p_0$ we obtain:

$$\text{Im} \Pi(q) = C \int \frac{d^3p d^3k}{\omega_1 \omega_2 \omega_3} \left[ \delta(q_0 - (\omega_1 + \omega_2 + \omega_3)) - \delta(q_0 + + \omega_1 + \omega_2 + \omega_3) \right] \right.$$ 

$$\left. (1 + N_1)(1 + N_2)(1 + N_3) - N_1 N_2 N_3 \right] + \left[ \delta(q_0 + (\omega_1 + \omega_2 + \omega_3)) - \delta(q_0 + + \omega_1 + \omega_2 + \omega_3) \right] \right.$$ 

$$\left. (1 + N_1)(1 + N_2)(1 + N_3) - N_1 N_2 N_3 \right] + \left[ \delta(q_0 - (\omega_1 + \omega_2 + \omega_3)) - \delta(q_0 + + \omega_1 + \omega_2 + \omega_3) \right] \right.$$ 

$$\left. (1 + N_1)(1 + N_2)(1 + N_3) + \delta(q_0 + (\omega_1 + \omega_2 + \omega_3)) - \delta(q_0 + + \omega_1 + \omega_2 + \omega_3) \right] \right.$$ 

$$\left. (1 + N_1)(1 + N_2)(1 + N_3) \right]$$

(28)

where $C = \frac{\lambda^2}{1024\pi^3}$. $\omega_1 = |\vec{q} - \vec{p} - \vec{k}|$, $\omega_2 = |\vec{p}|$ and $\omega_3 = \sqrt{|k|^2 + M^2}$. Since $\text{Im} \Pi$ is an odd function of $q_0$ we can restrict to $q_0 > 0$. $\text{Im} \Pi$ can then be decomposed as:

$$\text{Im} \Pi = \text{Im} \Pi_D + \text{Im} \Pi_A + \text{Im} \Pi_{LD}$$

where:

$$\text{Im} \Pi_D = C \int \frac{d^3\vec{p} d^3\vec{k}}{\omega_1 \omega_2 \omega_3} \delta(q_0 - (\omega_1 + \omega_2 + \omega_3)) \left[ (1 + N_1)(1 + N_2)(1 + N_3) - N_1 N_2 N_3 \right]$$

$$\text{Im} \Pi_A = C \int \frac{d^3\vec{p} d^3\vec{k}}{\omega_1 \omega_2 \omega_3} \delta(q_0 - (\omega_1 + \omega_2 + \omega_3)) \left[ (1 + N_1)(1 + N_2)(1 + N_3) + \delta(q_0 + (\omega_1 + \omega_2 + \omega_3)) - \delta(q_0 + (\omega_1 + \omega_2 + \omega_3)) \right] \right.$$ 

$$\left. (1 + N_1)(1 + N_2)(1 + N_3) \right]$$

$$\left. (1 + N_1)(1 + N_2)(1 + N_3) \right]$$

(29)

with $N_1 = N(\omega_1)$. $\text{Im} \Pi_D$ corresponds to the decay of an off-shell $\phi$ into two on-shell $\chi$'s and one on-shell $\phi$ (minus the inverse process) (fig. 4a), $\text{Im} \Pi_A$ to the annihilation process, where an off-shell $\phi$ annihilates with an on-shell $\phi$ to produce two $\chi$'s (fig. 4b), and $\text{Im} \Pi_{LD}$ corresponds to Landau damping of the $\phi$ induced by the scattering with a on-shell $\chi$ (fig. 4c).

FIG. 4. a) Decay, b) Annihilation, c) Landau damping .

The computation proceeds as in [10] and, for $\vec{q} = 0$, we get

$$\text{Im} \Pi_D = C' \theta(q_0 - M) \int_0^\infty dv f(v)f(\omega - v)$$

$$\int_0^\infty dv f(v)f(\omega - v)$$

FIG. 2. A diagram giving a Boltzmann-suppressed contribution to $\rho_{\text{int}}$.

FIG. 3. Two-loop contribution to the self energy .

FIG. 4. a) Decay, b) Annihilation, c) Landau damping .
\[ \ln \frac{f(\omega^2)f(\omega^1 + v - \omega)}{f(\omega^1)f(\omega^2 + v - \omega)} = C' \theta(q_0 - M) \int_0^{+\infty} dv + \theta(M - q_0) \int_a^{+\infty} dv \cdot \left[ (f(v) + 1)f(\omega + v - \omega) \ln \frac{f(\omega^4)f(\omega^3 - v - \omega)}{f(\omega^3)f(\omega^4 - v - \omega)} \right] \]

\[ \Im \Pi = C' \left[ \theta(q_0 - M) \int_0^{+\infty} dv + \theta(M - q_0) \int_a^{+\infty} dv \right] \cdot \left[ \frac{(1 + f(\omega^1))f(\omega^2 + v - \omega)}{(1 + f(\omega^2))f(\omega^1 + v - \omega)} + 2C' \theta(q_0 - M) \int_a^{+\infty} dv + \theta(M - q_0) \int_0^{+\infty} dv \right] \cdot \left[ \frac{(1 + f(\omega^3))}{(1 + f(\omega - v + \omega))} \right] \] 

with

\[ C' = \frac{\lambda^2}{128 \pi^2} \epsilon(q_0)(e^{\beta q_0} - 1) \frac{1}{\beta^2} \]

and

\[ a = \frac{M}{T}, \quad \hat{v} = \frac{q_0^2 + M^2}{2q_0 T}, \quad v = \frac{\omega}{T}, \]

\[ q_0 = \frac{q_0}{T}, \quad \omega^{1,2} = \frac{1}{2} \frac{q_0 - \omega^3 \mp k}{T}, \quad \omega^{3,4} = \frac{1}{2} \frac{q_0 + \omega^3 \mp k}{T}, \]

\[ \omega^{5,6} = \frac{1}{2} \frac{\omega^3 - q_0 \mp k}{T}. \]

(31)

From the formulae above it is easy to check that the \( \Im \Pi \) is Boltzmann-suppressed for \( |q_0| \lesssim T (\ll M) \). Physically, it can be understood by looking at fig. 4. For \( |q_0| \lesssim T (\ll M) \) only the annihilation and the Landau damping contribute. In the former case (fig. 4b) the on-shell \( \phi \) particle in the initial state comes from the heat-bath and then carries a \( N \) factor. In the Landau damping case (fig. 4c), the energy to create the on-shell \( \phi \) has to be provided by the \( \chi \) from the heat bath, so that a Boltzmann-suppressed \( N \) has to be payed in this case too.

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