BOTT-CHERN-AEPPLI AND FROLICHER ON COMPLEX 3-FOLDS

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ABSTRACT. We derive some Bott-Chern-Aeppli cohomology for (non-Kahler) compact complex manifolds. We also show that the hodge numbers of the Frolicher spectral sequence terms can be given in terms of information from decompositions of the Aeppli cohomology.

We then show that if one of the decomposition terms, \( L^{n-1,n-1}_\partial \), is zero, then every hermitian metric is conformal to a strongly Gauduchon metric. As a corollary, we duplicate a result already shown by Popovici and Ugarte [18]: if \( h^{0,1}_\partial = h^{0,1}_{BC} \), then every hermitian metric is conformal to a strongly Gauduchon metric.

We show (what is essentially given by Tosatti [20]) that

\[
h^{1,1}_{BC} = \hat{h}^1(\mathcal{H}) = 2h^{0,1}_\partial - b^1 + \dim_R(H^{1,1}_{dR}(\mathbb{R}))
\]

where \( \mathcal{H} \) is the sheaf of pluri-harmonic functions on \( X \) and \( H^{1,1}_{dR}(\mathbb{R}) \) is the subgroup of \( H^2_{dR}(X, \mathbb{R}) \) consisting of classes with a real \( d \)-closed 1,1-form as a representative. We finally give the complete Bott-Chern-Aeppli cohomology for compact complex 3-folds in terms of Dolbeault, Frolicher, a bi-degree DeRham-like type of cohomology, \( K^{p,q} \), defined as

\[
K^{p,q} = \frac{\ker(\partial) \cap \ker(\bar{\partial})}{\im(\partial) \cap \ker(\bar{\partial}) + \im(\bar{\partial}) \cap \ker(\partial)}
\]

and \( \hat{H}^1(\mathcal{H}) \). We then give the complete Bott-Chern-Aeppli cohomology for a hypothetical complex structure on \( S^6 \) in terms of Dolbeault and Frolicher. Finally, we show agreement of our results with the calculation by Angella [5] of the Bott-Chern-Aeppli cohomology for small Kuranishi deformations of the Iwasawa manifold.

1. Introduction

The Aeppli cohomology of a complex manifold is defined by the vector spaces (see Aeppli [11] and also Angella [5] and Popovici [17]):

\[
H^{p,q}_A = \frac{\ker(\bar{\partial} : C^\infty p,q \to C^\infty p+1,q+1)}{(\im(\bar{\partial} : C^\infty p-1,q \to C^\infty p,q) + \im(\partial : C^\infty p,q-1 \to C^\infty p,q))}
\]

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The Bott-Chern cohomology of a complex manifold is defined by the vector spaces (see Bott and Chern\cite{4} and also Angella\cite{5} and Popovici\cite{17}) :

\[ H^{p,q}_{BC} = \ker(\bar{\partial} : C^{\infty}_{p,q} \to C^{\infty}_{p,q+1}) \cap \ker(\partial : C^{\infty}_{p+1,q} \to C^{\infty}_{p,q}) \]  

\[ \text{im}(\partial \bar{\partial} : C^{\infty}_{p-1,q-1} \to C^{\infty}_{p,q}) \]

On compact complex manifolds, there is a harmonic theory due to Schweitzer\cite{19} for each of these cohomologies which ensures that they are finite dimensional complex vector spaces. Schweitzer’s harmonic theory shows that the two cohomologies are dual to each other.

Let \( h^{p,q}_A = \dim(H^{p,q}_A) \) and \( h^{p,q}_{BC} = \dim(H^{p,q}_{BC}) \). We have then (see Schweitzer\cite{19} and also Angella\cite{5} and Popovici\cite{17}) that \( h^{p,q}_A = h^{q,p}_A \), \( h^{p,q}_{BC} = h^{q,p}_{BC} \) and \( h^{n-p,n-q}_A = h^{n-p,n-q}_{BC} \). We mention as a historical note that results on compact complex manifolds about finiteness and duality between Aeppli and Bott-Chern cohomology also appear in Bigolin\cite{3}.

We define a bigraded DeRham-like cohomology,  

\[ K^{p,q} = \frac{\ker(\partial : C^{\infty}_{p,q} \to C^{\infty}_{p+1,q}) \cap \ker(\bar{\partial} : C^{\infty}_{p,q} \to C^{\infty}_{p,q+1})}{\text{im}(\partial) \cap \ker(\partial) + \text{im}(\bar{\partial}) \cap \ker(\bar{\partial})} \]

We set \( k^{p,q} = \text{dim}_C(K^{p,q}) \). Since  

\[ \ker(\partial) \cap \ker(\bar{\partial}) \subseteq \ker(d) , \]

\[ \ker(\partial) \cap \ker(\bar{\partial}) \subseteq \ker(\bar{\partial}) , \]

\[ \ker(\partial) \cap \ker(\bar{\partial}) \subseteq \ker(\partial) \]

and  

\[ \text{im}(d) \subseteq \text{im}(\partial) + \text{im}(\bar{\partial}) , \]

\[ \text{im}(\bar{\partial}) \subseteq \text{im}(\partial) + \text{im}(\bar{\partial}) , \]

\[ \text{im}(\partial) \subseteq \text{im}(\partial) + \text{im}(\bar{\partial}) . \]

It can easily be shown that \( k^{p,q} \leq h^{p,q}_A, k^{p,q} \leq h^{p,q}_{BC}, k^{p,q} \leq h^{p,q} \leq h^{q,p} \).

We also see that \( K^{p,q} \) embeds into \( H^{p,q}_A \) by the obvious map, \([\kappa]_K \mapsto [\kappa]_A\). This is easily seen to be one-to-one. Thus \( k^{p,q} \leq h^{p,q}_A \).

Since  

\[ \text{Im}(\partial \bar{\partial}) \subseteq (\text{Im}(\partial) + \text{Im}(\bar{\partial})) , \]

we have \( k^{p,q} \leq h^{p,q}_{BC} \) from the definitions of \( H^{p,q}_{BC} \) and \( K^{p,q} \). We also easily have from the definition of \( K^{p,q} \) that \( k^{p,q} = h^{p,q}_{BC} \).

Bott-Chern/Aeppli cohomology has been studied extensively by a number of mathematicians. Popovici\cite{17} utilizes Aeppli cohomology, in particular, \( H^{n-1,n-1}_A \) to study Gauduchon metrics on complex manifolds. Tseng and Yau\cite{21} point out the importance of understanding of Bott-Chern/Aeppli cohomology, in particular, \( H^{2,2}_{BC} \), for the study of
Strominger’s system of supersymmetric equations in type IIB theory on complex 3-folds.

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2. SEQUENCES OF MAPS OF COHOMOLOGY

Consider the following sequences of maps of cohomology on a compact \( N \)-dimensional complex manifold \( X \) with, \( p = 0, \cdots, n \),

\[
0 \rightarrow H^{p,0}_{BC} \xrightarrow{\text{im}(\bar{\partial})} H^{p,0}_{\bar{\partial}} \xrightarrow{\text{im}(\bar{\partial})+\text{im}(\partial)} H^{p,0}_{A} \xrightarrow{\bar{\partial}} H^{p,1}_{BC} \xrightarrow{\text{im}(\bar{\partial})} \cdots
\]

\[
\cdots \rightarrow H^{p,q}_{BC} \xrightarrow{\text{im}(\bar{\partial})} H^{p,q}_{\bar{\partial}} \xrightarrow{\text{im}(\bar{\partial})+\text{im}(\partial)} H^{p,q}_{A} \xrightarrow{\bar{\partial}} H^{p,q+1}_{BC} \xrightarrow{\text{im}(\bar{\partial})} \cdots
\]

\[
/(\text{im}(\bar{\partial})+\text{im}(\partial)) \quad H^{p,n-1}_{A} \xrightarrow{\bar{\partial}} H^{p,n}_{BC} \xrightarrow{\text{im}(\bar{\partial})} H^{p,n}_{\bar{\partial}} \xrightarrow{\text{im}(\bar{\partial})+\text{im}(\partial)} H^{p,n}_{A} \xrightarrow{\bar{\partial}} H^{p,n+1}_{BC} \rightarrow 0 .
\]

The above sequence of maps for the case of \( X \), a hypothetical complex 3-fold diffeomorphic to \( S^6 \), was also given in McHugh\[14\]. We give some claims and lemmas below about this sequence of maps.

**Lemma 2.1.** The sequence of maps above is exact at \( H^{p,q}_{BC} \). Namely,

\[
\ker(\text{im}(\bar{\partial}) : H^{p,q}_{BC} \rightarrow H^{p,q}_{\bar{\partial}}) = \text{im}(\bar{\partial} : H^{p,q-1}_{A} \rightarrow H^{p,q}_{BC}).
\]

**Lemma 2.2.** The sequence of maps above is exact at \( H^{p,q}_{A} \). Namely,

\[
\text{im}(/(\text{im}(\bar{\partial})+\text{im}(\partial)) : H^{p,q}_{\bar{\partial}} \rightarrow H^{p,q}_{A}) = \ker(\bar{\partial} : H^{p,q}_{A} \rightarrow H^{p,q+1}_{BC}).
\]

We have the following

**Proposition 2.3.**

\[
\dim(\text{Im}(/(\text{im}(\bar{\partial}) : H^{p,q}_{BC} \rightarrow H^{p,q}_{\bar{\partial}}))
\]

\[
= k^{p,q} + \dim(\ker(/(\text{im}(\bar{\partial}) + \text{im}(\partial)) : H^{p,q}_{\bar{\partial}} \rightarrow H^{p,q}_{A})).
\]
Proof We first show that
\[ \ker((/im(\bar{\partial}) + im(\partial)) : H_{\bar{\partial}}^{p,q} \to H_{\bar{\partial}}^{p,q}) \subseteq \text{Im}((/im(\bar{\partial}) : H_{BC}^{p,q} \to H_{\bar{\partial}}^{p,q}) \to H_A^{p,q}) \].

Indeed, if we have a \( \bar{\partial} \)-closed \( p,q \)-form, \( \phi \) such that \( \phi = \partial \lambda + \bar{\partial} \chi \), then \( \partial \lambda \) is \( \bar{\partial} \)-closed and represents the same element as \( \phi \) in \( H_{\bar{\partial}}^{p,q} \). Clearly, \( \partial \lambda \in \ker(\partial) \cap \ker(\bar{\partial}) \) and thus
\[ [\phi] = [\partial \lambda] \in \text{Im}((/im(\bar{\partial}) : H_{BC}^{p,q} \to H_{\bar{\partial}}^{p,q}) \to H_A^{p,q}). \]

Denote \( V^{p,q} = \text{Im}((/im(\bar{\partial}) : H_{BC}^{p,q} \to H_{\bar{\partial}}^{p,q}) \). We also have by what we have just shown that
\[ \ker((/im(\bar{\partial}) + im(\partial)) : H_{\bar{\partial}}^{p,q} \to H_{A}^{p,q}) \]

\[ = \ker((/im(\bar{\partial}) + im(\partial)) : V_{p,q} \to H_{A}^{p,q})) \].

If we apply the Rank Theorem from basic Linear Algebra, to

\[ (/im(\bar{\partial}) + im(\partial)) : V_{p,q} \to H_{A}^{p,q} \]

we get the result of our proposition:

\[ \dim(V_{p,q}) = \dim(\ker((/im(\bar{\partial}) + im(\partial)) : V_{p,q} \to H_{A}^{p,q})) + \dim(\text{im}((/im(\bar{\partial}) + im(\partial)) : V_{p,q} \to H_{A}^{p,q})) \]

or

\[ \dim(\text{Im}(/im(\bar{\partial}) : H_{BC}^{p,q} \to H_{\bar{\partial}}^{p,q})) = \dim(\ker((/im(\bar{\partial}) + im(\partial)) : H_{\bar{\partial}}^{p,q} \to H_{A}^{p,q})) + \dim(\frac{\ker(\bar{\partial}) \cap \ker(\partial)}{\text{im}(\partial) + \text{im}(\partial)}) \]

End of Proof

Again using the Rank theorem from basic Linear Algebra we have:
Notice that we have trivially added $k^{p,q}$ to both sides of the equation from the Rank theorem for $h^{p,q}_\partial$. Creating an alternating sum of both sides of the equations above and using the lemmas and proposition on exactness, we have:

**Proposition 2.4.**

\[
\sum_{q=0}^{n} (-1)^q(h^{p,q}_{BC} - (h^{p,q}_\partial + k^{p,q}) + h^{p,q}_A) = 0
\]

or using the duality, $h^{p,q}_A = h^{n-p,n-q}_{BC}$.

**Proposition 2.5.** On an $n$-dimensional compact complex manifold, $X$,

\[
\sum_{q=0}^{n} (-1)^q(h^{p,q}_{BC} + h^{n-p,n-q}_{BC}) = \sum_{q=0}^{n} (-1)^q(h^{p,q}_\partial + k^{p,q})
\]
Using the Riemann-Roch-Hirzebruch Theorem, we can also write this equation as
\[ \sum_{q=0}^{n} (-1)^q (h_{BC}^{p,q} + h_{BC}^{n-p,n-q} - k^{p,q}) = \sum_{q=0}^{n} (-1)^q h_{\bar{\partial}}^{p,q} \]
\[ = \chi(\Omega^p) \]
\[ = \int_X ch(\Omega^p) td(X) \]
where \( \Omega^p \) is the bundle of holomorphic \( p,0 \)-forms. Thus,

**Proposition 2.6.** On an \( n \)-dimensional compact complex manifold, \( X \),
\[ \sum_{q=0}^{n} (-1)^q (h_{BC}^{p,q} + h_{BC}^{n-p,n-q} - k^{p,q}) = \int_X ch(\Omega^p) td(X) \]
is a “topological” invariant of our compact complex manifold (in that it depends only on the topological structure of \( \Omega^p \) and \( TX \)).

We shall use the (almost exact) sequences above in calculating the Bott-Chern cohomology. In more specific situations the sequences split into two or more useful sequences.

3. SOME GENERAL RESULTS FOR AEPPLI/BOTT-CHERN COHOMOLOGY

We have the following straight forward result (see McHugh[14]) when \( b^1 = 0 \)

**Proposition 3.1.** If \( b^1 = 0 \) then \( H^{1,0}_{BC} = H^{0,1}_{BC} = H^{n-1,n}_A = H^{n,n-1}_A = 0 \).

**Proof** Let \( [\phi] \in H^{1,0}_{BC} \) where \( \phi \) is a 1,0-form such that \( \partial \phi = 0 \) and \( \bar{\partial} = 0 \). Specifically, \( d\phi = 0 \) and so since \( b^1 = 0 \) we have \( \phi = df \) for some global function \( f \). Now \( \phi = \partial f + \bar{\partial} f \). Since \( \phi \) is a 1,0-form, we have \( \bar{\partial} f = 0 \). Thus, \( f \) is a global holomorphic function on a compact complex manifold and thus must be a constant function. Hence, \( \bar{\partial} f = 0 \), and we have \( \phi = 0 \).

We can generalize this to the case \( b^1 \neq 0 \). Consider the following portion of our sequence of maps:
\[ 0 \rightarrow H^{0,0}_{BC} \rightarrow H^{0,0}_{\bar{\partial}} \rightarrow H^{0,0}_A \rightarrow H^{0,1}_{BC} \rightarrow H^{0,1}_{\bar{\partial}} \rightarrow \ldots \]
For a connected compact complex manifold, \( b^0 = h^{0,0}_{\bar{\partial}} = k^{0,0} = 1 \), and it is easy to see that \( h^{0,0}_{BC} = 1 \). Thus the image of the map, \( H^{0,0}_{\bar{\partial}} \rightarrow H^{0,0}_A \) is one-dimensional. Also, \( H^{0,0}_A \) consists of all pluri-harmonic scalar
functions on our compact manifold. These must be constant by the maximum modulus theorem so $h_{0,0}^{0,0} = 1$. (I thank Michael Albanese for pointing this out to me.) By exactness, the kernel of the map, $H_{BC}^{0,1} \to H_{0,1}^{0,1}$ is zero. Since,

$$K^{0,1} = (\ker(d) \cap E^{0,1}) / (\text{Im}(\partial) \cap E^{0,1} + \text{Im}(\bar{\partial}) \cap E^{0,1})$$

we have,

**Proposition 3.2.** On a connected, compact complex manifold with $b^1 \neq 0$,

$$h_{BC}^{0,1} = k^{0,1}.$$

We will also show the following:

**Proposition 3.3.** For any $n$-dimensional complex manifold, $X$,

$$h_{BC}^{n-1,0} = h_{\bar{\partial}}^{n-1,0}.$$

Before we prove this we need the following lemma:

**Lemma 3.4.** For any $n$-dimensional complex manifold, $X$, a $\bar{\partial}$-closed, $\partial$-exact $n,0$-form is zero. Explicitly, let $\phi$ be an $n-1,0$-form. If $\partial \phi$ is $\bar{\partial}$-closed, then $\partial \phi = 0$.

**Proof of Lemma** Following in the same manner almost verbatim as Lemma 2.2 in Brown[7],

$$\int_X \partial \phi \wedge \bar{\partial} \phi = \int_X \bar{\partial}(\partial \phi \wedge \bar{\phi}) - \int_X (\bar{\partial}(\partial \phi)) \wedge \bar{\phi}$$

$$= \int_X d(\partial \phi \wedge \bar{\phi}) - \int_X (\bar{\partial}(\partial \phi)) \wedge \bar{\phi} = 0 + 0 = 0$$

by Stokes theorem.

Locally,

$$\partial \phi = f dz^1 \wedge \cdots \wedge dz^n$$

and so

$$\int_X \partial \phi \wedge \bar{\partial} \phi = \int_X |f|^2 dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n$$

$$= \int_X (-1)^{n(n-1)/2} |f|^2 dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n = 0$$

Thus $f = 0$ and $\partial \phi = 0$.

**Proof of Proposition** We have the sequence,

$$0 \to H_{BC}^{n-1,0} \to H_{\bar{\partial}}^{n-1,0} \to H_{\partial}^{n-1,0} \to \cdots$$
which is exact at $H_{BC}^{n-1,0}$. Thus $H_{BC}^{n-1,0} \hookrightarrow H_{\bar{\partial}}^{n-1,0}$ injectively. We shall show that the map is surjective.

Let $[\phi] \in H_{\bar{\partial}}^{n-1,0}$, with of course, $\bar{\partial}\phi = 0$. We have that $\partial\phi$ is a $\bar{\partial}$-closed, $\partial$-exact $n,0$-form and thus $\partial\phi = 0$. We have

$$\phi \in \ker(\partial) \cap \ker(\bar{\partial})$$

and $[\phi]_{\bar{\partial}} = [\phi]_{BC}/(im(\bar{\partial}))$. Hence $H_{BC}^{n-1,0} = H_{\bar{\partial}}^{n-1,0}$ and $h_{BC}^{n-1,0} = h_{\bar{\partial}}^{n-1,0}$.

We shall also show in almost exactly the same way that $h_{BC}^{n,0} = h_{\bar{\partial}}^{n,0} = k^{n,0}$.

We repeat the argument for completeness. See also Angella[5], Section 1.4.4.

**Proposition 3.5.** For any $n$-dimensional complex manifold, $X$,

$$h_{BC}^{n,0} = h_{\bar{\partial}}^{n,0} = k^{n,0}.$$  

**Proof of Proposition** We have the sequence,

$$0 \rightarrow H_{BC}^{n,0} \rightarrow H_{\bar{\partial}}^{n,0} \rightarrow H_{A}^{n,0} \rightarrow \cdots$$

which is exact at $H_{BC}^{n,0}$. Thus $H_{BC}^{n,0} \hookrightarrow H_{\bar{\partial}}^{n,0}$ injectively. We shall show that the map is surjective.

Let $[\phi] \in H_{\bar{\partial}}^{n,0}$, with of course, $\bar{\partial}\phi = 0$. We have that $\partial\phi = 0$ since $\phi$ is an $n,0$-form. We have

$$\phi \in \ker(\partial) \cap \ker(\bar{\partial})$$

and $[\phi]_{\bar{\partial}} = [\phi]_{BC}/(im(\bar{\partial}))$. Hence $H_{BC}^{n,0} = H_{\bar{\partial}}^{n,0}$ and $h_{BC}^{n,0} = h_{\bar{\partial}}^{n,0}$.

Now consider a nonzero, $[\phi]_{BC} \in H_{BC}^{n,0}$. By our lemma above, $\phi$ is not $\partial$-exact. Thus the map,

$$/(im(\partial) + im(\bar{\partial})): H_{BC}^{n,0} \rightarrow K^{n,0}$$

is injective and $h_{BC}^{n,0} \leq k^{n,0}$. Combining this together we get

$$h_{BC}^{n,0} \leq k^{n,0} \leq h_{\bar{\partial}}^{n,0} = h_{BC}^{n,0}$$

and thus $h_{BC}^{n,0} = h_{\bar{\partial}}^{n,0} = k^{n,0}$. 

4. Some more general results for Aeppli/Bott-Chern Cohomology: Calculating $h_{BC}^{p,0}$ and $h_{A}^{p,0}$

Recall the following results just shown above.

On a connected, compact complex manifold with $b^1 = 0$,

$$H_{BC}^{1,0} = H_{BC}^{0,1} = H_A^{n-1,n} = H_A^{n,n-1} = 0.$$ 

On a connected, compact complex manifold with $b^1 \neq 0$,

$$h_{BC}^{0,1} = k_{BC}^{0,1}.$$ 

For any $n$-dimensional complex manifold, $X$,

$$h_{BC}^{n-1,0} = h_{BC}^{n,n-1}.$$ 

$$h_{BC}^{n,0} = k_{BC}^{n,0}.$$ 

We will be deriving the “extreme” rows/columns of our Bott-Chern and Aeppli cohomology tables, i.e. $h_{BC}^{p,0}$ and $h_{A}^{p,0}$ for all $p$.

We first derive a formula for $h_{BC}^{p,0}$. We have the beginning of the sequence,

$$0 \to H_{BC}^{0,0} \to H_{\bar{\partial}}^{p,0} \to H_{A}^{p,0} \to \cdots$$

We can consider the sequence,

$$0 \to H_{BC}^{p,0} \to H_{\bar{\partial}}^{p,0} \to \text{Im}(/(\text{im}(\partial) + \text{im}(\bar{\partial})): H_{\bar{\partial}}^{p,0} \to H_{A}^{p,0}) \to 0$$

which is exact except in the middle with the non-exactness measured by $K_{p,0}$. Thus

$$h_{BC}^{p,0} - (h_{\bar{\partial}}^{p,0} + k_{p,0}) + \text{dim}(\text{Im}(/(\text{im}(\partial) + \text{im}(\bar{\partial})): H_{\bar{\partial}}^{p,0} \to H_{A}^{p,0})) = 0$$

and

$$h_{BC}^{p,0} = h_{\bar{\partial}}^{p,0} - (\text{dim}(\text{Im}(/(\text{im}(\partial) + \text{im}(\bar{\partial})): H_{\bar{\partial}}^{p,0} \to H_{A}^{p,0})) - k_{p,0}).$$

We claim that

$$\text{dim}(\text{Im}(/(\text{im}(\partial) + \text{im}(\bar{\partial})): H_{\bar{\partial}}^{p,0} \to H_{A}^{p,0})) - k_{p,0} = \text{dim}(\text{Im}(\partial : H_{\bar{\partial}}^{p,0} \to H_{\bar{\partial}}^{p+1,0}))$$

where $\partial : H_{\bar{\partial}}^{p,0} \to H_{\bar{\partial}}^{p+1,0}$ is the map from the Frolicher spectral sequence. Note that we may consider

$$K_{p,0} \hookrightarrow \text{Im}(/(\text{im}(\partial) + \text{im}(\bar{\partial})): H_{\bar{\partial}}^{p,0} \to H_{A}^{p,0}))$$

and thus

$$\text{dim}(\text{Im}(/(\text{im}(\partial) + \text{im}(\bar{\partial})): H_{\bar{\partial}}^{p,0} \to H_{A}^{p,0})) - k_{p,0} = \text{dim}(\text{Im}(/(\text{im}(\partial) + \text{im}(\bar{\partial})): H_{\bar{\partial}}^{p,0} \to H_{A}^{p,0})/K_{p,0}).$$
Consider an element, 
\[ [\theta]_A / K^{p,0} \in \text{Im}(/(\text{im}(\partial) + \text{im}(\bar{\partial})): H^p_\partial \to H^p_A) / K^{p,0} \]
for \( \theta \), a nonzero \( \bar{\partial} \)-closed, \( p, 0 \)-form. We will show that \( \partial[\theta]_\partial \neq 0 \). We can not have \( \partial \theta = \bar{\partial} \nu \) as this is \( p + 1, 0 \) form and if we have \( \partial \theta = 0 \) then \( [\theta]_A \in K^{p,0} \) contradicting \( [\theta]_A / K^{p,0} \neq 0 \). Thus 
\[ \partial[\theta]_\partial \in \text{Im}(\partial : H^p_\partial \to H^p_{\partial + 1}) \]
and is nonzero. This is thus an injective linear map from 
\[ \text{Im}(/(\text{im}(\partial) + \text{im}(\bar{\partial})): H^p_\partial \to H^p_A) / K^{p,0} \]
to 
\[ \text{Im}(\partial : H^p_\partial \to H^p_{\partial + 1}) \].
We still need to show it is onto. Clearly, the map is well defined.

Now consider a nonzero element, 
\[ \partial[\phi]_\partial \in \text{Im}(\partial : H^p_\partial \to H^p_{\partial + 1}) \]
for \( \phi \), a nonzero \( \bar{\partial} \)-closed, \( p, 0 \)-form. If \( [\phi]_A / K^{p,0} = 0 \), then \( \phi = \kappa + \partial \mu \) where \( \kappa \) is a local representative of \( K^{p,0} \). But then \( \partial \phi = 0 \) and this is a contradiction. Hence, \( [\phi]_A / K^{p,0} \) is a non-zero and thus \( \partial[\phi]_\partial \) is in the image of our map just above. Our map is onto. Thus 
\[ \text{Im}(/(\text{im}(\partial) + \text{im}(\bar{\partial})): H^p_\partial \to H^p_A) / K^{p,0} \]
\[ = \text{Im}(\partial : H^p_\partial \to H^p_{\partial + 1}) \]
and 
\[ \text{dim}(\text{Im}(/(\text{im}(\partial) + \text{im}(\bar{\partial})): H^p_\partial \to H^p_A)) - k^{p,0} = \text{dim}(\text{Im}(\partial : H^p_\partial \to H^p_{\partial + 1})) \].

We now calculate \( \text{dim}(\text{Im}(\partial : H^p_\partial \to H^p_{\partial + 1})) \). Recall the Frolicher Sequence,
\[ 0 \to H^{0,0}_\partial \xrightarrow{\partial} \cdots \xrightarrow{\partial} H^{p,0}_\partial \xrightarrow{\partial} H^{p+1,0}_\partial \xrightarrow{\partial} \cdots \xrightarrow{\partial} H^{q,0}_\partial \xrightarrow{\partial} 0 \]
where the non-exactness is measured by the spectral sequence terms,
\[ H^q_2 = \text{Ker}(H^{q,0}_\partial \xrightarrow{\partial} H^{q+1,0}_\partial) / \text{Im}(H^{q-1,0}_\partial \xrightarrow{\partial} H^q_\partial) \].

Let 
\[ \text{ker}_q = \text{dim}(\text{Ker}(H^{q,0}_\partial \xrightarrow{\partial} H^{q+1,0}_\partial)) \]
and 
\[ \text{im}_q = \text{dim}(\text{Im}(H^{q-1,0}_\partial \xrightarrow{\partial} H^q_\partial)) \]
and thus 
\[ h^{q,0}_2 = \text{ker}_q - \text{im}_q \].

Using, 
\[ h^{q,0}_\partial = \text{ker}_q + \text{im}_{q+1} = \text{im}_q + h^{q,0}_2 + \text{im}_{q+1} \]
we have,

\[ h_0^{0,0} = \text{im}_0 + h_2^{0,0} + \text{im}_1 , \]
\[ h_1^{1,0} = \text{im}_1 + h_2^{1,0} + \text{im}_2 , \]
\[ h_2^{2,0} = \text{im}_2 + h_2^{2,0} + \text{im}_3 , \]
\[ \vdots \]
\[ h_{p-1,0}^{p-1,0} = \text{im}_{p-1} + h_2^{p-1,0} + \text{im}_p , \]
\[ h_p^{0,0} = \text{im}_p + h_2^{p,0} + \text{im}_{p+1} . \]

and

\[ h_0^{0,0} - h_2^{0,0} = \text{im}_0 + \text{im}_1 , \]
\[ h_1^{1,0} - h_2^{1,0} = \text{im}_1 + \text{im}_2 , \]
\[ h_2^{2,0} - h_2^{2,0} = \text{im}_2 + \text{im}_3 , \]
\[ \vdots \]
\[ h_{p-1,0}^{p-1,0} - h_2^{p-1,0} = \text{im}_{p-1} + \text{im}_p , \]
\[ h_p^{0,0} - h_2^{p,0} = \text{im}_p + \text{im}_{p+1} . \]

Thus

\[ \text{im}_{p+1} = (h_{0,0}^{p,0} - h_2^{p,0}) - (h_{0,0}^{p-1,0} - h_2^{p-1,0}) + \cdots + (-1)^{p-1}(h_{0,0}^{1,0} - h_2^{1,0}) + (-1)^p(h_{0,0}^{0,0} - h_2^{0,0}) . \]

For compact complex manifolds, \( h_0^{0,0} = h_2^{0,0} = 1 \), so we can write

\[ \text{im}_{p+1} = \sum_{j=1}^{p} (-1)^{p+j}(h_{0,0}^{j,0} - h_2^{j,0}) . \]

Finally, we have

\[ h_{BC}^{0,0} = h_{0,0}^{p,0} - \sum_{j=1}^{p} (-1)^{p+j}(h_{0,0}^{j,0} - h_2^{j,0}) . \]

We now derive formulas for \( h_{A,0}^{n,0} = h_{BC}^{n,n-p} \). We will proceed by induction on \( q \) for calculating \( h_{BC}^{n,q} \). We start the induction on deriving \( h_{BC}^{n,0} \) and \( h_{BC}^{n,1} \). We know from Proposition 3.5 that \( h_{A,0}^{n,0} = h_{BC}^{n,0} = k_1^{n,0} \). Thus the map, \( \bar{\partial} : H_{A,0}^{n,0} \to H_{BC}^{n,1} \) is zero and we have the clipped sequence,

\[ 0 \to H_{BC}^{n,1} \to H_{\bar{\partial}}^{n,1} \to H_{A,0}^{n,1} \to H_{BC}^{n,2} \to \cdots \]

We have that \( H_{BC}^{n,1} \) embeds into \( H_{\bar{\partial}}^{n,1} \). We have

\[ H_{\bar{\partial}}^{n,1} = \text{Ker}(\text{im}(\bar{\partial})+\text{im}(\partial)) : H_{\bar{\partial}}^{n,1} \to H_{A,0}^{n,1} \]

\[ + \text{Im}(\text{im}(\bar{\partial})+\text{im}(\partial)) : H_{\bar{\partial}}^{n,1} \to H_{A,0}^{n,1} . \]
so that
\[ h_{\bar{\partial}}^{n,1} = \dim(\text{Im}(\text{im}(\bar{\partial}))) - k^{n,1} + \dim(\text{Im}(\text{im}(\bar{\partial})) : H_{\bar{\partial}}^{n,1} \to H_A^{n,1}) \]
\[ = h_{BC}^{n,1} + \dim(\text{Im}(\text{im}(\bar{\partial})) : H_{\bar{\partial}}^{n,1} \to H_A^{n,1}) - k^{n,1} \]

We claim that
\[ \dim(\text{Im}(\text{im}(\bar{\partial})) : H_{\bar{\partial}}^{n,1} \to H_A^{n,1}) - k^{n,1} = 0 \]

Note that we may consider
\[ K^{n,1} \hookrightarrow \text{Im}(\text{im}(\bar{\partial})) : H_{\bar{\partial}}^{n,1} \to H_A^{n,1}) \]
and thus
\[ \dim(\text{Im}(\text{im}(\bar{\partial})) : H_{\bar{\partial}}^{n,1} \to H_A^{n,1}) - k^{n,1} \]
\[ = \dim(\text{Im}(\text{im}(\bar{\partial})) : H_{\bar{\partial}}^{n,1} \to H_A^{n,1})/K^{n,1} \]

Indeed, consider an element,
\[ [\theta]_A/K^{n,1} \in \text{Im}(\text{im}(\bar{\partial})) : H_{\bar{\partial}}^{n,1} \to H_A^{n,1})/K^{n,1} \]
for \( \theta \), a nonzero \( \bar{\partial} \)-closed, but not \( \bar{\partial} \)-exact, \( n, 1 \)-form. We automatically have that \( \partial \theta = 0 \) as it is an \( n, 1 \)-form. Thus we can write
\[ \theta = \kappa + \partial \mu + \bar{\partial} \nu \]
and \([\theta]_A/K^{n,1} = 0\). Hence
\[ \text{Im}(\text{im}(\bar{\partial})) : H_{\bar{\partial}}^{n,1} \to H_A^{n,1})/K^{n,1} = 0 \]
and
\[ \dim(\text{Im}(\text{im}(\bar{\partial})) : H_{\bar{\partial}}^{n,1} \to H_A^{n,1}) - k^{n,1} = 0 \]

Thus

**Proposition 4.1.**
\[ h_{A}^{0,n-1} = h_{BC}^{n,1} = h_{\bar{\partial}}^{n,1} \]

We shall now proceed with our induction on \( q \) in \( h_{BC}^{n,q} \). Assume that we can write formulas for \( h_{BC}^{n,j} \) in terms of \( h_1^{*,*}, h_2^{*,*}, \) and \( k^{*,*} \), for \( 0 \leq j \leq q \). We note that we already can write formulas for \( h_{A}^{n,j} \), \( 0 \leq j \leq n \), in such terms since \( h_{A}^{n,j} = h_{BC}^{n,j} = h_{BC}^{n,j,0} \). Consider again the sequence,
\[ 0 \to H_{BC}^{n,1} \to \cdots \to H_{BC}^{n,q} \to H_{\bar{\partial}}^{n,q} \to H_A^{n,q} \]
\[ \to H_{BC}^{n,q+1} \to H_{\bar{\partial}}^{n,q+1} \to H_A^{n,q+1} \to \cdots \]
and shorten it to
\[ 0 \to H_{BC}^{n,1} \to \cdots \to H_{BC}^{n,q} \to H_{\bar{\partial}}^{n,q} \to H_A^{n,q} \to H_{BC}^{n,q+1} \to H_{\bar{\partial}}^{n,q+1} \]
We shall show that $\text{Im}((\text{im}(\partial) + \text{im}(\bar{\partial})): H^{n,q+1}_{\bar{\partial}} \to H^{n,q+1}_{A}) = K^{n,q+1}$. It is clear that $K^{n,q+1} \subset \text{Im}((\text{im}(\partial) + \text{im}(\bar{\partial})): H^{n,q+1}_{\bar{\partial}} \to H^{n,q+1}_{A})$.

If $[\phi]_A \in \text{Im}((\text{im}(\partial) + \text{im}(\bar{\partial})): H^{n,q+1}_{\bar{\partial}} \to H^{n,q+1}_{A})$ then $\bar{\partial}\phi = 0$ and $\partial\phi = 0$ since $\phi$ is an $n,q+1$-form. Thus $[\phi]_A \in K^{n,q+1}$ and

$$\text{Im}((\text{im}(\partial) + \text{im}(\bar{\partial})): H^{n,q+1}_{\bar{\partial}} \to H^{n,q+1}_{A}) \subset K^{n,q+1}.$$ 

Thus

$$\text{Im}((\text{im}(\partial) + \text{im}(\bar{\partial})): H^{n,q+1}_{\bar{\partial}} \to H^{n,q+1}_{A}) = K^{n,q+1}.$$ 

We have then that

$$\sum_{l=1}^{q}(-1)^{l-1}(h^{n,l}_{BC} - (h^{n,l}_{\bar{\partial}} + k^{n,l}) + h^{n,l}_{A}) + (-1)^{q}(h^{n,q+1}_{BC} - (h^{n,q+1}_{\bar{\partial}} + k^{n,q+1} + k^{n,q+1})) = 0$$

and hence,

**Proposition 4.2.**

$$h^{n,q+1}_{BC} = h^{n,q+1}_{\bar{\partial}} + \sum_{l=1}^{q}(-1)^{q+l}(h^{n,l}_{BC} - (h^{n,l}_{\bar{\partial}} + k^{n,l}) + h^{n,l}_{A}).$$

Thus

**Proposition 4.3.**

$$h^{n-p,0}_{A} = h_{\bar{\partial}}^{0,p} + \sum_{l=1}^{p-1}(-1)^{p-1+l}(h^{n-l,0}_{A} - (h^{0,n-l}_{\bar{\partial}} + k^{n,l}) + h^{n-l,0}_{BC}).$$

We now know $h^{p,0}_{BC}$ and $h^{p,0}_{A}$ in terms of $h^{*,*}_{\bar{\partial}}, h^{*,*}_{2}$, and $k^{*,*}$ for all $p$ such that $0 \leq p \leq n$.

### 5. Decomposition of $H^{p,q}_{A}$

In this section we define some groups/vector spaces associated with $H^{p,q}_{A}$ and prove some results with regard to these spaces. These will be helpful in some later sections in calculating further Bott-Chern/Aeppli cohomology of compact complex 3-folds.

Consider the map:

$$\partial: H^{p,q}_{A} \to H^{p+1,q}_{\bar{\partial}}.$$
We define
\[ G^{p,q}_\partial = \ker(\partial : H_A^{p,q} \rightarrow H_{\partial \partial}^{p,q+1} ) , \]
\[ G^p_q = G^{p,q}_\partial / \text{Im}(\partial (\partial + \partial) : H_{\partial \partial}^{p,q} \rightarrow H_A^{p,q} ) , \]
and
\[ L^{p,q}_\partial = H_A^{p,q} / G^{p,q}_\partial . \]
Following terminology of Popovici\cite{17}, we suggest calling \( G^{p,q}_\partial \), strongly Gauduchon cohomology and calling \( L^{p,q}_\partial \), weakly Gauduchon cohomology. (Popovici\cite{17} calls an hermitian metric on a complex n-fold, strongly Gauduchon, if the \( n-1 \) power of its associated \( 1,1 \)-form, \( \omega \), is such that \( \partial(\omega^{n-1}) = \partial \eta \) for some \( n,n-2 \)-form, \( \eta \).) We also define the hodge numbers, \( g^{p,q}_\partial = \dim_C(G^{a,b}_\partial) \) and \( l^{p,q}_\partial = \dim_C(L^{a,b}_\partial) \).

We define \( G^{p,q}_\partial, G^{p,q}_0, L^{p,q}_\partial, G^p_q, \) and \( L^p_q \) completely analogously:
\[ G^{p,q}_\partial = \ker(\bar{\partial} : H_A^{p,q} \rightarrow H_{\bar{\partial} \partial}^{p,q+1} ) , \]
\[ G^p_q = G^{p,q}_\partial / \text{Im}(\bar{\partial} (\bar{\partial} + \bar{\partial}) : H_{\bar{\partial} \partial}^{p,q} \rightarrow H_A^{p,q} ) , \]
and
\[ L^{p,q}_\partial = H_A^{p,q} / G^{p,q}_\partial \]
with also \( g^{p,q}_\partial = \dim_C(G^{a,b}_\partial) \) and \( l^{p,q}_\partial = \dim_C(L^{a,b}_\partial) \). We note that \( g^{p,q}_\partial = \dim_C(G^{n-1,q}_\partial) \) and \( l^{p,q}_\partial = l^{n-2,q-2}_\partial \). The first two equations are obvious from the definitions. We shall give a proof of the last equation later on. We give a proof of the third equation now.

**Proposition 5.1.** \( g^{p,q}_\partial = g^{p+1,q-1}_\partial \)

We shall show an isomorphism, \( \phi \), between \( G^{p,q}_\partial \) and \( G^{p+1,q-1}_\partial \). Indeed, let \([\mu]_{G^{p,q}_\partial} \in G^{p,q}_\partial \), where \( \mu \) is a \( p,q \)-form such that \( \partial \mu = \bar{\partial} \nu \) for some \( p+1, q-1 \)-form \( \nu \) such that \([\nu]_{G^{p+1,q-1}_\partial} \in G^{p+1,q-1}_\partial \). Thus we will show
\[ \phi : G^{p,q}_\partial \rightarrow G^{p+1,q-1}_\partial \]
\[ \phi( [\mu]_{G^{p,q}_\partial} ) = [\nu]_{G^{p+1,q-1}_\partial} \]
is a well defined and bijective linear map. Let us proceed to show that it is well defined. If \( \tilde{\mu} \) is another \( p,q \)-form such that \([\tilde{\mu}]_{G^{p,q}_\partial} = [\mu]_{G^{p,q}_\partial} \), then
\[ \tilde{\mu} = \mu + \chi + \bar{\partial} \sigma + \partial \tau \]
where \( \partial \chi = 0 \). Thus,
\[ \partial \tilde{\mu} = \bar{\partial} (\nu - \partial \tau) \]
and
\[ \phi( [\tilde{\mu}]_{G^{p,q}_\partial} ) = [\nu - \partial \tau]_{G^{p+1,q-1}_\partial} = [\nu]_{G^{p+1,q-1}_\partial} \]
Our map, \( \phi \), is clearly linear. Now to show \( \phi \) is one-to-one. If \([\hat{\mu}]_{G^p_0} \) is such that
\[
\phi([\hat{\mu}]_{G^p_0}) = [\nu]_{G^{p+1}_0}\]
then
\[
\partial \mu - \partial \hat{\mu} = \bar{\partial} \nu - \bar{\partial}(\nu + \partial \tau) .
\]
Thus
\[
\hat{\mu} = \mu + \chi - \bar{\partial} \tau
\]
for some \( p, q \)-form, \( \chi \) such that \( \partial \chi = 0 \) and
\[
[\hat{\mu}]_{G^p_0} = [\mu]_{G^p_0}.
\]
Thus our map is one-to-one. To show our map is onto, let
\[
[\nu]_{G^{p+1}_0} \in G^{p+1,q-1}_0.
\]
Then
\[
\bar{\partial} \nu = \partial \mu
\]
for some \( p, q \)-form \( \mu \). We see that
\[
[\mu]_{G^p_0} \in G^{p,q}
\]
and that
\[
\phi([\mu]_{G^p_0}) = [\nu]_{G^{p+1,q-1}_0}.
\]
Thus our map is onto. Thus we have that \( G^{p,q} \) is isomorphic to \( G^{p+1,q-1}_0 \).

Underlying much of our analysis will be the following decompositions of \( H^{p,q}_{BC} \) and \( H^{p,q}_A \) (dependent on some, always existing, choice of hermitian metric):

**Lemma 5.2.**

\[
H^{p,q}_{BC} = \partial G^{p-1,q} \oplus \partial L^{p-1,q} \oplus \bar{\partial} L^{p,q-1} \oplus K^{p,q} ,
\]
\[
H^{p,q}_{BC} = \bar{\partial} G^{p,q-1} \oplus \bar{\partial} L^{p,q-1} \oplus \partial L^{p-1,q} \oplus K^{p,q} ,
\]
\[
H^{p,q}_A = H^{p,q}_\delta / (\bar{\partial} L^{p,q-1}_\delta \oplus G^{p,q}_\delta \oplus L^{p,q}_\delta) ,
\]
\[
H^{p,q}_A = H^{p,q}_\delta / (\partial L^{p,q-1}_\delta \oplus G^{p,q}_\delta \oplus L^{p,q}_\delta) ,
\]

The proof of the second two statements is straightforward from definitions. We give a proof of the first two statements at least with respect to the hodge numbers:

We notice that
\[
\dim(\text{im}(\bar{\partial} : H^{p,q}_A \to H^{p,q+1}_{BC})) = g^{p,q}_\delta + \bar{p}^{p,q}_\delta
\]
since
\[ \dim_C(\ker(\bar{\partial} : H^p_A \to H^{p+1}_A)) = \dim_C(\text{im}(H^p_A \to H^p_A)) \]

We also know that
\[ l^{p-1,q}_\partial + k^{p,q} = \dim_C(\text{im}(\bar{\partial}/\text{im}(\partial) : H^p_A \to H^p_A)) \]
and thus that
\[ \dim_C(\text{im}(\bar{\partial}/\text{im}(\partial) : H^p_{BC} \to H^p_{BC})) = \dim_C(\text{im}(\bar{\partial}/\text{im}(\partial) : H^{p+1}_A \to H^{p+1}_A)) \].

By the exactness of our sequence at \( H^{p+1}_{BC} \) we have
\[ \dim_C(\ker(\bar{\partial}/\text{im}(\partial) : H^p_{BC} \to H^p_{BC})) = g^{p,q}_\partial + p^{p,q} \]
and by the rank theorem,
\[ h^{p,q}_{BC} = g^{p,q}_\partial + l^{p,q-1}_\partial + k^{p,q+1}_\partial = g^{p,q}_\partial + l^{p,q-1}_\partial + k^{p,q+1}_\partial \]
or
\[ h^{p,q}_{BC} = g^{p,q}_\partial + l^{p,q-1}_\partial + k^{p,q} \]
and
\[ h^{p,q}_{BC} = g^{p,q}_\partial + l^{p,q-1}_\partial + k^{p,q} \].

We also give here the formulas from the decompositions for \( H^p_A \):
\[ h^{p,q}_A = h^{p,q}_\partial - l^{p,q-1}_\partial + g^{p,q}_\partial + k^{p,q} \]
and
\[ h^{p,q}_A = h^{p,q}_\partial - l^{p,q-1}_\partial + g^{p,q}_\partial + k^{p,q} \].

We have the following result which gives the formula:

**Lemma 5.3.** For \( 0 \leq p \leq n - 2 \) and \( 0 \leq q \leq n \), we have:
\[ l^{p-1,q}_\partial + k^{p,q} = h^{p+1,q-1}_\partial - h^{p,q}_\partial - k^{p+1,q}_\partial \]

**Proof** Consider the portion of our sequence
\[ \ldots \to H^{p+1,q-1}_A \to H^{p+1,q}_A \to H^{p+1,q}_\partial \to H^{p+1,q}_A \to \ldots \]
We have by our exactness results
\[ \text{Im}(H^{p+1,q}_B \to H^{p+1,q}_\partial) = \partial L^{p,q}_\partial + K^{p+1,q}_\partial \].

Thus
\[ 0 \to H^{p+1,0}_B \to \ldots \to H^{p+1,q-1}_A \to H^{p+1,q}_B \to \partial L^{p,q}_\partial + K^{p+1,q}_\partial \to 0 \]
and
\[ l^{p,q}_\partial + k^{p+1,q} = h^{p+1,q}_B - h^{p+1,q-1}_A + \ldots + (-1)^{q+1}(q)h^{p+1,0}_B \].
In a similar manner we consider the portion of our sequence:
\[ \ldots \to H_{BC}^{n-p-1,n-q} \to H_{\partial}^{n-p-1,n-q} \to H_{A}^{n-p-1,n-q} \to \ldots \]
and with \( \text{Ker}(H_{\partial}^{n-p-1,n-q} \to H_{A}^{n-p-1,n-q}) = \partial L_{\partial}^{n-p-2,n-q} \) we have
\[ 0 \to \partial L_{\partial}^{n-p-2,n-q} \to H_{\partial}^{n-p-1,n-q} \to H_{A}^{n-p-1,n-q} \to \ldots \to H_{A}^{n-p-1,n} \to 0 \]
Thus,
\[ l_{\partial}^{n-p-2,n-q} = h_{A}^{n-p-1,n-q} - h_{A}^{n-p-1,n} + \ldots + \epsilon(h^{n-p-1,n}) \]
Thus
\[ l_{\partial}^{n-p-2,n-q} = h_{\partial}^{p+1,q} - l_{\partial}^{p,q} - k_{\partial}^{p+1,q} \]

We also will use the following:

**Lemma 5.4.**

\( \text{Ker}(\partial : H_{\partial}^{p,q} \to H_{\partial}^{p+1,q}) = G_{\partial}^{p,q} \cap (H_{\partial}^{p,q}/\text{im}(\partial)) \oplus K_{\partial}^{p,q} \oplus \partial L_{\partial}^{p-1,q} \).

**Proof** Recall from Lemma 5.2 the decomposition of \( H_{BC}^{p,q} \):
\[ H_{BC}^{p,q} = \partial G_{p,q-1} \oplus \partial L_{p-1,q} \oplus \partial L_{p-1,q} \oplus K_{p,q} \]
and thus that
\[ \text{Im}(/(\text{im}(\partial)) : H_{BC}^{p,q} \to H_{\partial}^{p,q}) \cong K_{\partial}^{p,q} \oplus \partial L_{\partial}^{p-1,q} \]
We see that
\[ \text{Im}(/(\text{im}(\partial)) : H_{BC}^{p,q} \subset \text{Ker}(\partial : H_{\partial}^{p,q} \to H_{\partial}^{p+1,q}) \]
We also have that
\[ H_{\partial}^{p,q} \cong H_{\partial}^{p,q}/\text{im}(\partial) \oplus \partial L_{\partial}^{p-1,q} \]
\[ \cong H_{A}^{p,q} \cap H_{\partial}^{p,q}/\text{im}(\partial) \oplus \partial L_{\partial}^{p-1,q} \]
\[ \cong (H_{\partial}^{p,q}/(\text{im}(\partial)) \oplus G_{\partial}^{p,q} \oplus L_{\partial}^{p,q}) \cap H_{\partial}^{p,q}/\text{im}(\partial) \oplus \partial L_{\partial}^{p-1,q} \]
\[ \cong H_{\partial}^{p,q}/(\text{im}(\partial)) \cap H_{\partial}^{p,q}/\text{im}(\partial) \oplus G_{\partial}^{p,q} \cap H_{\partial}^{p,q}/\text{im}(\partial) \oplus L_{\partial}^{p,q} \cap H_{\partial}^{p,q}/\text{im}(\partial) \oplus \partial L_{\partial}^{p-1,q} \]
\[ \cong K_{\partial}^{p,q} \oplus G_{\partial}^{p,q} \oplus L_{\partial}^{p,q} \oplus H_{\partial}^{p,q}/\text{im}(\partial) \oplus \partial L_{\partial}^{p-1,q} \]
where we have used the fact that
\[ K_{\partial}^{p,q} = H_{\partial}^{p,q}/(\text{im}(\partial)) \cap H_{\partial}^{p,q}/\text{im}(\partial) \]
By the definitions of \( G_{\partial}^{p,q} \) and \( L_{\partial}^{p,q} \) we see that
\[ \text{Ker}(\partial : H_{\partial}^{p,q} \to H_{\partial}^{p+1,q}) = G_{\partial}^{p,q} \cap (H_{\partial}^{p,q}/\text{im}(\partial)) \oplus K_{\partial}^{p,q} \oplus \partial L_{\partial}^{p-1,q} \]
Now let $|V|$ denote the dimension over $\mathbb{C}$ of a complex vector space, $V$. Using the Frolicher spectral sequence we claim we can write,

$$|\text{Ker}(\partial : H^{p,q}_\bar{\partial} \to H^{p+1,q}_\bar{\partial})| = h^{p,q}_2 + \sum_{j=1}^{p-1} (-1)^{p+1+j}(h^{j,q}_\bar{\partial} - h^{j,q}_2).$$

To see this consider a $q$-version of equation (1),

$$\text{im}_{p,q} = \sum_{j=1}^{p-1} (-1)^{p+1+j}(h^{j,q}_\bar{\partial} - h^{j,q}_2).$$

Thus,

$$\text{ker}_{p,q} - h^{p,q}_2 = \sum_{j=1}^{p-1} (-1)^{p+1+j}(h^{j,q}_\bar{\partial} - h^{j,q}_2).$$

Combining this with Lemma 5.4 we have

**Proposition 5.5.**

$$|G^{p,q}_\bar{\partial} \cap (H^{p,q}_\bar{\partial} / \text{im}(\partial))| + p^{p,q} - k^{p,q} = h^{p,q}_2 + \sum_{j=1}^{p-1} (-1)^{p+1+j}(h^{j,q}_\bar{\partial} - h^{j,q}_2).$$

### 6. Frolicher in terms of Dolbeault, $G$, and $L$

In this section we show how to express the Frolicher spectral sequence Hodge numbers in terms of $G^{p,q}_\bar{\partial}$, $G^{p,q}_\partial$, $L^{p,q}_\bar{\partial}$, $L^{p,q}_\partial$, $H^{p,q}_\bar{\partial}/L^{p,q}_\bar{\partial}$, $H^{p,q}_\partial/L^{p,q}_\partial$ and their intersections. (This section can be skipped over on a first reading of this article as sections after it are not dependent on it.) We will be using in this section results of Codero, Fernandez, Gray, and Ugarte\[9\] characterizing terms in the Frolicher spectral sequence. For a complex manifold, $M$, they define the spaces $X^{p,q}_r$, and $Y^{p,q}_r$ as (see \[9\], pp. 76-77):

$$X^{p,q}_1(M) = \{ \theta \in \bigwedge^{p,q}(M) | \bar{\partial}\theta = 0 \}, \quad Y^{p,q}_1(M) = \{ \bar{\partial}(\bigwedge^{p,q-1}(M)) \},$$

for $r = 1$, and for $r \geq 2$,

$$X^{p,q}_r(M) = \{ \theta^{p,q} \in \bigwedge^{p,q}(M) | \bar{\partial}\theta^{p,q} = 0, \text{ and such that} \exists \theta^{p+i,q-i} \in \bigwedge^{p+i,q-i}(M) \text{ such that} \},

\partial \theta^{p+i,q-i} + \bar{\partial}\theta^{p+i,q-i} = 0, \text{ for } 1 \leq i \leq r - 1,$$
\[ Y_r^{p,q}(M) = \{ \partial \theta^{p-1,q} + \bar{\partial} \theta^{p,q-1} \in \Lambda^{p,q}(M) \mid \exists \theta^{p-i,q+i-1} \in \Lambda^{p-i,q+i-1}(M), \ 2 \leq i \leq r-1, \text{ such that,} \]
\[ \bar{\partial} \theta^{p-i,q+i-2} + \partial \theta^{p-i,q+i-1} = 0, \]
\[ \bar{\partial} \theta^{p-r+1,q+r-2} = 0 \}, \]

Just for clarity, we give explicitly for \( r = 2 \):

\[ Y_2^{p,q}(M) = \{ \partial \theta^{p-1,q} + \bar{\partial} \theta^{p,q-1} \in \Lambda^{p,q}(M) \mid \bar{\partial} \theta^{p-1,q} = 0 \} \]

Codero, Fernandez, Gray, and Ugarte\[9\] have the results that

\[ \frac{X_r^{p,q}}{Y_r^{p,q}} \cong E_r^{p,q}, \]

\[ \frac{X_{r+1}^{p,q}}{Y_{r+1}^{p,q}} \cong \text{Ker}(d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}), \]

\[ \frac{Y_{r+1}^{p,q}}{Y_r^{p,q}} \cong d_r(E_r^{p-r,q+r-1}). \]

where \( E_r^{p,q} \) are the terms and \( d_r \) are the maps,

\[ E_r^{p,q} \xrightarrow{d_r} E_r^{p+r,q-r+1}, \]

in the Frolicher spectral sequence.

We will now show how we can compute the hodge numbers of the entire Frolicher spectral sequence from the Dolbeault cohomology and the decomposition of the Aeppli cohomology. Recall our formula,

\[ \text{Ker}(\partial : H^{p,q}_\bar{\partial} \to H^{p+1,q}_\bar{\partial}) = G^{p,q}_\bar{\partial} \cap (H^{p,q}_\bar{\partial}/\text{Im}(\partial)) \oplus K^{p,q} \oplus \partial L^{p-1,q}_\bar{\partial}. \]

We can also see that

\[ \text{Im}(\partial : H^{p-1,q}_\bar{\partial} \to H^{p,q}_\bar{\partial}) = \partial(L^{p-1,q}_\bar{\partial} \cap H^{p-1,q}_\bar{\partial}/\text{Im}(\partial)). \]

Thus, since

\[ E_2^{p,q} = \text{Ker}(\partial : H^{p,q}_\bar{\partial} \to H^{p+1,q}_\bar{\partial})/\text{Im}(\partial : H^{p-1,q}_\bar{\partial} \to H^{p,q}_\bar{\partial}) \]

we have

\[ E_2^{p,q} = G^{p,q}_\bar{\partial} \cap (H^{p,q}_\bar{\partial}/\text{Im}(\partial)) \oplus K^{p,q} \oplus \partial(L^{p-1,q}_\bar{\partial}/(L^{p-1,q}_\bar{\partial} \cap H^{p-1,q}_\bar{\partial}/\text{Im}(\partial))). \]

Hence

\[ h_2^{p,q} = |G^{p,q}_\bar{\partial} \cap (H^{p,q}_\bar{\partial}/\text{Im}(\partial))| + K^{p,q} + L^{p-1,q}_\bar{\partial} - |L^{p-1,q}_\bar{\partial} \cap H^{p-1,q}_\bar{\partial}/\text{Im}(\partial)|. \]

We see that if we have the data

\[ k^{p,q}_\bar{\partial}, l^{p,q}_\bar{\partial}, |G^{p,q}_\bar{\partial} \cap (H^{p,q}_\bar{\partial}/\text{Im}(\partial))| \]

and

\[ |L^{p-1,q}_\bar{\partial} \cap H^{p-1,q}_\bar{\partial}/\text{Im}(\partial)| \]
then we can calculate, \( h_{2,p} \), for all \( p \) and \( q \).

Let us next consider \( E_{3}^{p,q} \). From the work of Codero, Fernandez, Gray, and Ugarte\(^9\) we have

\[
\text{ker}(d_2 : E_2^{p,q} \rightarrow E_2^{p+2,q-1}) = X_3^{p,q} / Y_2^{p,q}.
\]

We note that

\[
\text{Im}(\bar{\partial} : \bigwedge^{p,q-1} \rightarrow \bigwedge^{p,q}) \subseteq X_r^{p,q} \subseteq \text{Ker}(\bar{\partial} : \bigwedge^{p,q} \rightarrow \bigwedge^{p,q+1}),
\]

\[
\text{Im}(\bar{\partial} : \bigwedge^{p,q-1} \rightarrow \bigwedge^{p,q}) \subseteq Y_{r-1}^{p,q}
\]

and

\[
Y_r^{p,q} \subseteq \partial L_{\bar{\partial}}^{p,q} + \text{im}(\bar{\partial} : \bigwedge^{p,q-1} \rightarrow \bigwedge^{p,q}) \subseteq \text{Ker}(\bar{\partial} : \bigwedge^{p,q} \rightarrow \bigwedge^{p,q+1}).
\]

For \( r = 3 \) we have from before and from the definition of \( Y_{2}^{p,q} \),

\[
Y_{2}^{p,q} / \text{im}(\bar{\partial}) = \partial(L_{\bar{\partial}}^{p-1,q} \cap H_{\bar{\partial}}^{p-1,q} / \text{im}(\bar{\partial})).
\]

We note that

\[
\text{Im}(\bar{\partial} : \bigwedge^{p,q-1} \rightarrow \bigwedge^{p,q}) \subseteq Y_{r}^{p,q} \subseteq \partial L_{\bar{\partial}}^{p,q} + \text{im}(\bar{\partial} : \bigwedge^{p,q-1} \rightarrow \bigwedge^{p,q}) \subseteq \text{Ker}(\bar{\partial} : \bigwedge^{p,q} \rightarrow \bigwedge^{p,q+1}).
\]

In fact,

\[
\text{Im}(\bar{\partial} : \bigwedge^{p,q-1} \rightarrow \bigwedge^{p,q}) \subseteq Y_{r}^{p,q} \subseteq \partial L_{\bar{\partial}}^{p,q} + \text{im}(\bar{\partial} : \bigwedge^{p,q-1} \rightarrow \bigwedge^{p,q}) \subseteq \text{Ker}(\bar{\partial} : \bigwedge^{p,q} \rightarrow \bigwedge^{p,q+1}).
\]

(We are using notation where \( V / \text{im}(\bar{\partial}) \equiv V / (V \cap \text{im}(\bar{\partial})). \))

We see altogether from the definition of \( X_{3}^{p,q} \), that

\[
X_{3}^{p,q}(M) = \{ \theta^{p,q} \in \bigwedge^{p,q}(M) \mid \bar{\partial}\theta^{p,q} = 0, \text{and such that} \}
\]

\[
\exists \theta^{p+1,q-1}, \theta^{p+2,q-2} \text{ such that,}
\]

\[
\partial\theta^{p,q} + \bar{\partial}\theta^{p+1,q-1} = 0, \quad \partial\theta^{p+2,q-2} = 0
\]

If \( \partial\theta^{p,q} \neq 0 \) then the first equation in the \( \theta \)'s in the definition of \( X_{3}^{p,q} \) tells us that \( \theta^{p,q} \in G_{\bar{\partial}}^{p,q} \cap (H_{\bar{\partial}}^{p,q} / \text{im}(\bar{\partial})) \) with \( \phi(\theta^{p,q}) = -\theta^{p+1,q-1} \) where \( \phi \) is the isomorphism,

\[
G_{\bar{\partial}}^{p,q} \rightarrow G_{\bar{\partial}}^{p+1,q-1}.
\]

The second equation in the \( \theta \)'s in the definition of \( X_{3}^{p,q} \) tells us that \( \theta^{p+1,q-1} \in G_{\bar{\partial}}^{p+1,q-1} \). Thus,

\[
\theta^{p,q} \in (G_{\bar{\partial}}^{p,q} \cap (H_{\bar{\partial}}^{p,q} / \text{im}(\bar{\partial}))) \cap \phi^{-1}(G_{\bar{\partial}}^{p+1,q-1} \cap G_{\bar{\partial}}^{p+1,q-1})
\]
and
\[ X_3^{p,q}/\text{im}(\bar{\partial}) = K^{p,q} \oplus \partial L^{p,q}_\partial \oplus (G^{p,q}_\partial \cap (H^{p,q}_\partial / \text{im}(\partial))) \cap \phi^{-1}(G^{p+1,q-1}_\partial \cap G^{p+1,q-1}_\partial) \]
\[ = K^{p,q} \oplus (G^{p,q}_\partial \cap (H^{p,q}_\partial / \text{im}(\partial))) \cap \phi^{-1}(G^{p+1,q-1}_\partial \cap G^{p+1,q-1}_\partial) \oplus \partial L^{p,q}_\partial \]

Thus,
\[ \text{ker}(d_2 : E_2^{p,q} \to E_2^{p+2,q-1}) = X_3^{p,q}/Y_2^{p,q} = (X_3^{p,q}/\text{im}(\bar{\partial}))/Y_2^{p,q}/\text{im}(\bar{\partial}) \]
\[ = K^{p,q} \oplus (G^{p,q}_\partial \cap (H^{p,q}_\partial / \text{im}(\partial))) \cap \phi^{-1}(G^{p+1,q-1}_\partial \cap G^{p+1,q-1}_\partial) \]
\[ \oplus \partial(L^{p+1,q}_\partial/(L^{p+1,q}_\partial \cap H^{p+1,q}_\partial / \text{im}(\partial))) , \]
\[ = K^{p,q} \oplus (H^{p,q}_\partial / \text{im}(\partial)) \cap \phi^{-1}(G^{p+1,q-1}_\partial \cap H^{p+1,q-1}_\partial / \text{im}(\partial) \oplus G^{p+1,q-1}_\partial) \]
\[ \oplus \partial(L^{p+1,q}_\partial \cap (G^{p+1,q-1}_\partial \oplus L^{p+1,q}_\partial)) . \]

We consider \( d_2(E_2^{p,q+1}) = Y_3^{p+1,q}(M)/Y_2^{p+1,q}(M) \) where
\[ Y_3^{p+1,q}(M) = \{ \partial \theta^{p,q} + \bar{\partial} \theta^{p+1,q-1} \in \Lambda^{p+1,q}(M) \mid \exists \theta^{p,q-1} \in \Lambda^{p,q-1}(M) \]
\[ \text{such that } \bar{\partial} \theta^{p,q} + \partial \theta^{p+1,q-1} = 0, \bar{\partial} \theta^{p+1,q-1} = 0 \} , \]
and
\[ Y_2^{p+1,q}(M) = \{ \partial \theta^{p,q} + \bar{\partial} \theta^{p+1,q-1} \in \Lambda^{p+1,q}(M) \mid \bar{\partial} \theta^{p,q} = 0 \} . \]

We see from the equations in the theta’s in the definition of \( Y_3^{p+1,q}(M) \)
that
\[ Y_3^{p+1,q}(M) = \partial(G^{p,q}_\partial \oplus L^{p,q}_\partial \cap (H^{p,q}_\partial / \text{im}(\partial)) \cap \phi(G^{p+1,q-1}_\partial \cap H^{p+1,q-1}_\partial / \text{im}(\partial))) \]
\[ + \partial(L^{p+1,q}_\partial/(L^{p+1,q}_\partial \cap H^{p+1,q}_\partial / \text{im}(\partial))) \]
\[ + \bar{\partial}(\Lambda^{p+1,q-1}) \]
( where we have used \( \partial G^{p,q}_\partial \subseteq \bar{\partial}(\Lambda^{p+1,q-1}) \)) and
\[ Y_2^{p+1,q}(M) = \partial(G^{p,q}_\partial \oplus L^{p,q}_\partial \cap H^{p,q}_\partial / \text{im}(\partial)) + \bar{\partial}(\Lambda^{p+1,q-1}) \]
\[ = \partial(L^{p,q}_\partial \cap H^{p,q}_\partial / \text{im}(\partial)) + \bar{\partial}(\Lambda^{p+1,q-1}) \]
Thus
\[ Y_3^{p+1,q}/Y_2^{p+1,q} = \partial(L^{p,q}_\partial \cap \phi(G^{p+1,q-1}_\partial \cap H^{p+1,q-1}_\partial / \text{im}(\partial))) \]
where \( \phi \) is the isomorphism, \( \phi : G^{p+1,q-1}_\partial \to G^{p,q}_\partial \).

Thus,
\[ d_2(E_2^{p-1,q+1}) = \partial(L^{p,q}_\partial \cap \phi(G^{p+1,q-1}_\partial \cap H^{p+1,q-1}_\partial / \text{im}(\partial))) \]
and

\[ \text{Im}(d_2 : E^{p-2,q+1}_2 \rightarrow E^{p,q}_2) = d_2(E^{p-2,q+1}_2) = \partial(L^{p-1,q}_{\partial} \cap \phi(G^{p-2,q+1}_{\partial} \cap H^{p-2,q+1}_{\partial} / \text{im}(\partial))) \]

Altogether we have

\[ E^{p,q}_3 = \ker(d_2 : E^{p,q}_2 \rightarrow E^{p+2,q-1}_2) / \text{im}(d_2 : E^{p-2,q+1}_2 \rightarrow E^{p,q}_2) \]

\[ = (K^{p,q} \oplus (H^{p,q}_{\partial} / \text{im}(\partial))) \cap \phi^{-1}(G^{p+1,q-1}_{\partial} \cap (H^{p+1,q-1}_{\partial} / \text{im}(\partial) \oplus G^{p+1,q-1}_{\partial})) \]

\[ \oplus \partial(L^{p-1,q}_{\partial} \cap (G^{p-1,q}_{\partial} \oplus L^{p-1,q}_{\partial}))) \]

\[ / (\partial(L^{p-1,q}_{\partial} \cap \phi(G^{p-2,q+1}_{\partial} \cap H^{p-2,q+1}_{\partial} / \text{im}(\partial)))) \]

\[ = K^{p,q} \oplus (H^{p,q}_{\partial} / \text{im}(\partial)) \cap \phi^{-1}(G^{p+1,q-1}_{\partial} \cap (H^{p+1,q-1}_{\partial} / \text{im}(\partial) \oplus G^{p+1,q-1}_{\partial})) \]

\[ \oplus \partial(L^{p-1,q}_{\partial} \cap (G^{p-1,q}_{\partial} \oplus L^{p-1,q}_{\partial}))) \cap \phi^{-1}(G^{p+2,q+2}_{\partial} \cap H^{p+2,q+2}_{\partial} / \text{im}(\partial)) \]

\[ \oplus \partial(L^{p-1,q}_{\partial} \cap L^{p-1,q}_{\partial}) \]

Considering in general,

\[ E^{p,q}_r = \frac{X^{p,q}_r}{Y^{p,q}_r} , \]

we have

\[ X^{p,q}_r = H^{p,q}_{\partial} / \text{im}(\partial) \cap (H^{p,q}_{\partial} / \text{im}(\partial)) \oplus \phi^{-1}(G^{p+1,q-1}_{\partial} \cap (H^{p+1,q-1}_{\partial} / \text{im}(\partial)) \]

\[ \oplus \phi^{-1}(G^{p+2,q+2}_{\partial} \cap (H^{p+2,q+2}_{\partial} / \text{im}(\partial)) \oplus \phi^{-1}(G^{p+3,q+3}_{\partial} \cap (H^{p+3,q+3}_{\partial} / \text{im}(\partial)) \oplus \)

\[ \ldots \oplus \phi^{-1}(G^{p+r+2,q+r+2}_{\partial} \cap (H^{p+r+2,q+r+2}_{\partial} / \text{im}(\partial)) \oplus G^{p+r+2,q+r+2}_{\partial}) \]

\[ \oplus \partial L^{p-1,q}_{\partial} \oplus \bar{\partial}(\Lambda^{p,q-1}) \]

and also

\[ Y^{p,q}_r = \partial(L^{p-1,q}_{\partial} \cap (H^{p,q}_{\partial} / \text{im}(\partial)) \oplus \phi(G^{p-2,q+1}_{\partial} \cap (H^{p-2,q+1}_{\partial} / \text{im}(\partial)) \oplus \phi(G^{p-3,q+2}_{\partial} \cap (H^{p-3,q+2}_{\partial} / \text{im}(\partial)) \oplus \)

\[ \ldots \oplus \phi^{-1}(G^{p+r+2,q+r+2}_{\partial} \cap (H^{p+r+2,q+r+2}_{\partial} / \text{im}(\partial)) \oplus G^{p+r+1,q+r-2}_{\partial}) \]

\[ \oplus \partial L^{p-1,q}_{\partial} \oplus \bar{\partial}(\Lambda^{p,q-1}) \]

Thus, (remembering that \( H^{p,q}_{\partial} / \text{im}(\partial) \cap H^{p,q}_{\partial} / \text{im}(\partial) = K^{p,q} \),

\[ E^{p,q}_r = K^{p,q} \oplus H^{p,q}_{\partial} / \text{im}(\partial) \cap \phi^{-1}(G^{p+1,q-1}_{\partial} \cap (H^{p+1,q-1}_{\partial} / \text{im}(\partial)) \]

\[ \oplus \phi^{-1}(G^{p+2,q+2}_{\partial} \cap (H^{p+2,q+2}_{\partial} / \text{im}(\partial)) \oplus \phi^{-1}(G^{p+3,q+3}_{\partial} \cap (H^{p+3,q+3}_{\partial} / \text{im}(\partial)) \oplus \)

\[ \ldots \oplus \phi^{-1}(G^{p+r+2,q+r+2}_{\partial} \cap (H^{p+r+2,q+r+2}_{\partial} / \text{im}(\partial)) \oplus G^{p+r+1,q+r-2}_{\partial}) \]
For the term
\[ (\partial L^{-1,q}_{\overline{\partial}})/Y_r^{p,q} \]
above, we can write
\[ (\partial L^{-1,q}_{\overline{\partial}})/Y_r^{p,q} = \partial(L^{-1,q}_{\overline{\partial}} \cap H^{-1,q}_{\overline{\partial}}/im(\partial) \oplus L^{-1,q}_{\overline{\partial}} \cap G^{-1,q}_{\overline{\partial}} \oplus L^{-1,q}_{\overline{\partial}} \cap L^{-1,q}_{\overline{\partial}}) \]
\[ = \partial(L^{-1,q}_{\overline{\partial}} \cap G^{-1,q}_{\overline{\partial}}/Z^{p-1,q}_{\overline{\partial}}) \oplus \partial(L^{-1,q}_{\overline{\partial}} \cap L^{-1,q}_{\overline{\partial}}), \]
where
\[ Z^{p-1,q}_{\overline{\partial}} = \phi(G^{p-2,q+1}_{\overline{\partial}} \cap (H^{-2,q+1}_{\overline{\partial}}/im(\partial)) \oplus \phi(G^{p-3,q+2}_{\overline{\partial}} \cap \ldots \oplus \phi(G^{p+r-2,q+r-2}_{\overline{\partial}} \cap (H^{p+r-2,q+r-2}_{\overline{\partial}}/im(\partial)) \ldots)) \]
If we define
\[ \tilde{Z}^{p,q}_{\overline{\partial}} = \phi^{-1}(G^{p+1,q-1}_{\overline{\partial}} \cap (H^{p+1,q-1}_{\overline{\partial}}/im(\partial))) \oplus \phi^{-1}(G^{p+2,q-2}_{\overline{\partial}} \cap (H^{p+2,q-2}_{\overline{\partial}}/im(\partial)) \oplus \ldots \oplus \phi^{-1}(G^{p+r-2,q+r-2}_{\overline{\partial}} \cap (H^{p+r-2,q+r-2}_{\overline{\partial}}/im(\partial)) \oplus G^{p+r-2,q+r-2}_{\overline{\partial}}) \ldots) \]
we can write
\[ E^{p,q}_{\overline{\partial}} = K^{p,q} \oplus (H^{p,q}_{\overline{\partial}}/im(\partial) \cap \tilde{Z}^{p,q}_{\overline{\partial}}) \oplus \partial(L^{-1,q}_{\overline{\partial}} \cap G^{-1,q}_{\overline{\partial}}/Z^{p-1,q}_{\overline{\partial}}) \oplus \partial(L^{-1,q}_{\overline{\partial}} \cap L^{-1,q}_{\overline{\partial}}). \]
Note we could have defined recursively, \( Z^{p-1,q}_{\overline{\partial}} = \{0\} \) and
\[ Z^{p-1,q}_{\overline{\partial}} = \phi(G^{p-2,q+1}_{\overline{\partial}} \cap (H^{p-2,q+1}_{\overline{\partial}}/im(\partial)) \oplus \phi(G^{p-3,q+2}_{\overline{\partial}} \cap \ldots \oplus \phi(G^{p+r-2,q+r-2}_{\overline{\partial}} \cap Z^{p-1,q}_{\overline{\partial}}) \ldots)) \]
We could also have defined recursively, \( \tilde{Z}^{p,q}_{\overline{\partial}} = G^{p,q}_{\overline{\partial}} \) and
\[ \tilde{Z}^{p,q}_{\overline{\partial}} = \phi^{-1}(G^{p+1,q-1}_{\overline{\partial}} \cap (H^{p+1,q-1}_{\overline{\partial}}/im(\partial)) \oplus \phi^{-1}(G^{p+2,q-2}_{\overline{\partial}} \cap \ldots \oplus \phi^{-1}(G^{p+r-2,q+r-2}_{\overline{\partial}} \cap \tilde{Z}^{p+q-1}_{\overline{\partial}}) \ldots) \]
Noting that \( Z^{p-2,q+1}_{\overline{\partial}} \subseteq G^{p-2,q+1}_{\overline{\partial}} \) and that we can decompose \( G^{p-1,q}_{\overline{\partial}} \) as
\[ G^{p-1,q}_{\overline{\partial}} = \phi(G^{p-2,q+1}_{\overline{\partial}} \cap H^{p-2,q+1}_{\overline{\partial}}) \oplus \phi(G^{p-3,q+2}_{\overline{\partial}} \cap G^{p-2,q+1}_{\overline{\partial}}) \]
\[ \oplus \phi(G^{p-2,q+1} \cap L^{p-2,q+1}_{\overline{\partial}}) \],
we can write,
\[ E^{p,q}_{\overline{\partial}} = K^{p,q} \oplus H^{p,q}_{\overline{\partial}}/im(\partial) \cap \phi^{-1}(G^{p+1,q-1}_{\overline{\partial}} \cap H^{p+1,q-1}_{\overline{\partial}}/im(\partial)) \]
\[ \oplus H^{p,q}_{\overline{\partial}}/im(\partial) \cap \phi^{-1}(G^{p+1,q-1}_{\overline{\partial}} \cap \tilde{Z}^{p+q-1}_{\overline{\partial}}) \]
\[ \begin{align*}
\oplus & \partial(L_{p-1,q}^{p-1,q} \cap \phi(G_{p-2,q+1}^{p-2,q+1} \cap \frac{G_{p-2,q+1}}{Z_{p-1,q}^{p-2,q+1}})) \\
\oplus & \partial(L_{p-1,q}^{p-1,q} \cap \phi(G_{p-2,q+1}^{p-2,q+1} \cap L_{p-1,q}^{p-2,q+1})) + \partial(L_{p-1,q}^{p-1,q} \cap L_{p-1,q}^{p-1,q}) \\
= & K^{p,q} \oplus H_{p,q}^{p,q} / \text{im}(\partial) \cap \phi^{-1}(G_{p+1,q-1}^{p+1,q-1} \cap H_{p+1,q-1}^{p+1,q-1} / \text{im}(\partial)) \\
+ & H_{p,q}^{p,q} / \text{im}(\partial) \cap \phi^{-1}(G_{p+1,q-2}^{p+1,q-2} \cap H_{p+1,q-2}^{p+1,q-2} / \text{im}(\partial)) \\
+ & H_{p,q}^{p,q} / \text{im}(\partial) \cap \phi^{-1}(G_{p+2,q-3}^{p+2,q-3} \cap H_{p+2,q-3}^{p+2,q-3} / \text{im}(\partial)) \\
\vdots \\
+ & H_{p,q}^{p,q} / \text{im}(\partial) \cap (\prod_{j=1}^{r-2}(\phi^{-1}(G_{p+q-j}^{p+q-j} \cap \ast))H_{p+q-r}^{p+q-r}) \\
+ & \partial(L_{p-1,q}^{p-1,q} \cap L_{p-1,q}^{p-1,q}) \\
+ & \partial(L_{p-1,q}^{p-1,q} \cap \phi(G_{p-2,q+1}^{p-2,q+1} \cap L_{p-1,q}^{p-2,q+1})) \\
\vdots \\
+ & \partial(L_{p-1,q}^{p-1,q} \cap (\prod_{j=1}^{r-2}(\phi(G_{p-j-1,q+j}^{p-j-1,q+j} \cap \ast))L_{p}^{p+1-r,q+r-2})) \\
+ & \partial(L_{p-1,q}^{p-1,q} \cap (\prod_{j=1}^{r-2}(\phi(G_{p-j-1,q+j}^{p-j-1,q+j} \cap \ast))G_{p+1-r,q+r-2})) \\
\end{align*} \]

**Theorem 6.1.**

\[ E_r^{p,q} = K^{p,q} \oplus \bigoplus_{i=1}^{r-2} H_{p,q}^{p,q} / \text{im}(\partial) \cap (\prod_{j=1}^{i}(\phi^{-1}(G_{p+q-j}^{p+q-j} \cap \ast))H_{p+q-i}^{p+q-i} / \text{im}(\partial)) \]

\[ \oplus H_{p,q}^{p,q} / \text{im}(\partial) \cap (\prod_{j=1}^{r-2}(\phi^{-1}(G_{p+q-j}^{p+q-j} \cap \ast))G_{p+q-r}^{p+q-r+2}) \]

\[ \oplus \partial(L_{p-1,q}^{p-1,q} \cap L_{p-1,q}^{p-1,q}) \]

\[ \bigoplus_{i=1}^{r-2} \partial(L_{p-1,q}^{p-1,q} \cap (\prod_{j=1}^{i}(\phi(G_{p-j-1,q+j}^{p-j-1,q+j} \cap \ast))L_{p-1,q}^{p-i-1,q+i})) \]

\[ \oplus \partial(L_{p-1,q}^{p-1,q} \cap (\prod_{j=1}^{r-2}(\phi(G_{p-j-1,q+j}^{p-j-1,q+j} \cap \ast))G_{p+1-r,q+r-2})) \]
We shall now find a formula for $h_{r}^{p,q} - h_{r+1}^{p,q}$. We see that

$$E_{r+1}^{p,q} \oplus H_{\partial}^{p,q}/im(\partial) \cap (\prod_{j=1}^{r-2}(\phi^{-1}(G_{\partial}^{p+j,q-j} \cap *))G_{\partial}^{p+r-2,q-r+2})$$

$$\oplus \partial(L_{\partial}^{p-1,q} \cap (\prod_{j=1}^{r-2}(\phi(G_{\partial}^{p-j,1,q+j} \cap *))G_{\partial}^{p+1-r,q+r-2}))$$

$$= E_{r}^{p,q} \oplus H_{\partial}^{p,q}/im(\partial) \cap (\prod_{j=1}^{r-1}(\phi^{-1}(G_{\partial}^{p+j,q-j} \cap *))H_{\partial}^{p+r-1,q-r+1}/im(\partial)$$

$$\oplus H_{\partial}^{p,q}/im(\partial) \cap (\prod_{j=1}^{r-1}(\phi(G_{\partial}^{p+j,q-j} \cap *))G_{\partial}^{p+r-1,q-r+1})$$

$$\oplus \partial(L_{\partial}^{p-1,q} \cap (\prod_{j=1}^{r-1}(\phi(G_{\partial}^{p-j,1,q+j} \cap *))L_{\partial}^{p-r,q+r-1}))$$

$$\oplus \partial(L_{\partial}^{p-1,q} \cap (\prod_{j=1}^{r-1}(\phi(G_{\partial}^{p-j,1,q+j} \cap *))G_{\partial}^{p-r,q+r-1}))$$

Noticing that

$$G_{\partial}^{p+r-2,q-r+2} = \phi^{-1}(G_{\partial}^{p+r-1,q-r+1} \cap H_{\partial}^{p+r-1,q-r+1}/im(\partial))$$

$$\oplus \phi^{-1}(G_{\partial}^{p+r-1,q-r+1} \cap G_{\partial}^{p+r-1,q-r+1})$$

$$\oplus \phi^{-1}(G_{\partial}^{p+r-1,q-r+1} \cap L_{\partial}^{p+r-1,q-r+1})$$

and

$$G_{\partial}^{p+1-r,q+r-2} = \phi(G_{\partial}^{p-r,q+r-1} \cap H_{\partial}^{p-r,q+r-1}/im(\partial))$$

$$\oplus \phi(G_{\partial}^{p-r,q+r-1} \cap G_{\partial}^{p-r,q+r-1})$$

$$\oplus \phi(G_{\partial}^{p-r,q+r-1} \cap L_{\partial}^{p-r,q+r-1}),$$

we have,

$$E_{r}^{p,q} = E_{r+1}^{p,q} \oplus H_{\partial}^{p,q}/im(\partial) \cap (\prod_{j=1}^{r-1}(\phi^{-1}(G_{\partial}^{p+j,q-j} \cap *))L_{\partial}^{p+r-1,q-r+1})$$

$$\oplus \partial(L_{\partial}^{p-1,q} \cap (\prod_{j=1}^{r-1}(\phi(G_{\partial}^{p-j,1,q+j} \cap *))H_{\partial}^{p-r,q+r-1}))$$
We can conclude explicitly that for $r \geq 2$,

$$h_r^{p,q} = h_r^{p,q} + |H_0^{p,q}/im(\partial) \cap (\prod_{j=1}^{r-1}(\phi^{-1}(G_0^{p+j,q-j} \cap *))L_0^{p+r-1,q-r+1})|$$

$$+ |\partial(L_0^{p-1,q} \cap (\prod_{j=1}^{r-1}(\phi(G_0^{p-j,q+j} \cap *))H_0^{p-r,q+r-1})| \ .$$

or

$$h_{r+1}^{p,q} - h_r^{p,q} = -|H_0^{p,q}/im(\partial) \cap (\prod_{j=1}^{r-1}(\phi^{-1}(G_0^{p+j,q-j} \cap *))L_0^{p+r-1,q-r+1})|$$

$$- |\partial(L_0^{p-1,q} \cap (\prod_{j=1}^{r-1}(\phi(G_0^{p-j,q+j} \cap *))H_0^{p-r,q+r-1})| \ .$$

7. Hermitian metrics

We shift gears a little, to considering consequences of the vanishing of the weakly gauduchon cohomology, $L_0^{n-1,n-1}$. Recall that an hermitian metric with associated $1,1$-form $\omega$, is called gauduchon if and only if $\partial \bar{\partial} \omega^{n-1} = 0$. Gauduchon proved that every hermitian metric is conformal to such a metric. An hermitian metric is called strongly gauduchon if and only if $\partial \omega^{n-1} = \bar{\partial} \mu$ for some $n,n-2$-form, $\mu$. Strongly gauduchon metrics were introduced by Popovici. Recall that we can write (with respect to some background hermitian metric),

$$H_A^{n-1,n-1} = H_0^{n-1,n-1}/im(\partial) \oplus G_0^{n-1,n-1} \oplus L_0^{n-1,n-1}$$

$$= G_0^{n-1,n-1} \oplus L_0^{n-1,n-1}$$

Every hermitian metric can be associated through a conformal gauduchon hermitian metric to a class in $H_A^{n-1,n-1}$ and if $L_0^{n-1,n-1} = 0$, that class is in $G_0^{n-1,n-1}$. In other words, if $\omega$ is the associated $1,1$-form to the gauduchon metric, then $\partial \omega^{n-1} = \bar{\partial} \mu$ for some $n,n-2$-form $\mu$. Thus the gauduchon metric is strongly gauduchon.

**Theorem 7.1.** If $l_0^{n-1,n-1} = 0$ for a compact complex manifold, $X$, then every hermitian metric is conformal to a strongly gauduchon hermitian metric.

In the language of Popovici and Ugarte(2014), $X$ is an SGG compact complex manifold, i.e. one in which every gauduchon hermitian metric is a strongly gauduchon hermitian metric.
Recall that an hermitian metric is called balanced if and only if 
\[ d(\omega^{n-1}) = 0. \] Balanced metrics were introduced by Michelsohn\cite{15} in 1982. A theorem of Popovici\cite{17}, states that on a complex manifold, \( X \), with vanishing second betti number, \( b^2 = 0 \), strongly gauduchon metrics and balanced metrics are the same. Thus,

**Corollary 7.2.** If \( b^2 = h_0^{1,n-1} = 0 \) for a compact complex manifold, \( X \), then every hermitian metric is conformal to a balanced hermitian metric.

Recall our formula,

\[ \text{Ker}(\partial : H^{p,q}_\partial \to H^{p+1,q}_\partial) = G^{p,q}_\partial \cap (H^{p,q}_\partial/im(\partial)) \oplus K^{p,q} \oplus \partial L^{p-1,q}_\partial. \]

If \( p = n \) and \( q = n - 1 \), then

\[ \text{Ker}(\partial : H^{n,n-1}_\partial \to H^{n+1,n-1}_\partial) = H^{n,n-1}_\partial \]

and

\[ H^{n,n-1}_\partial = G^{n,n-1}_\partial \cap (H^{n,n-1}_\partial/im(\partial)) \oplus K^{n,n-1} \oplus \partial L^{n-1,n-1}_\partial. \]

or

\[ h^{n,n-1}_\partial = |G^{n,n-1}_\partial \cap (H^{n,n-1}_\partial/im(\partial))| + k^{n,n-1} + l^{n-1,n-1}_\partial. \]

Thus, if \( h^{0,1}_\partial = h^{n,n-1}_\partial = 0 \) then we must have \( l^{n-1,n-1}_\partial = 0 \). Hence,

**Corollary 7.3.** If \( h^{0,1}_\partial = 0 \) for a compact complex manifold, \( X \), then every hermitian metric is conformal to a strongly gauduchon hermitian metric.

and

**Corollary 7.4.** If \( b^2 = h^{0,1}_\partial = 0 \) for a compact complex manifold, \( X \), then every hermitian metric is conformal to a balanced hermitian metric.

Examples of a compact complex three-folds with \( b^2 = h^{0,1}_\partial = 0 \) are the non-Kahler Calabi-Yau complex structures of Friedman\cite{10} and Lu-Tian\cite{13} on connected sums of \( S^3 \times S^3 \). Recall the exact sequence of complex vector spaces:

\[ 0 \to \partial L^{n-1,n-1}_\partial \to H^{n,n-1}_\partial \to H^{n,n-1}_A \to 0 \]

From this we conclude,

\[ l^{n-1,n-1}_\partial = h^{n,n-1}_\partial - h^{n,n-1}_A \]

or

\[ l^{n-1,n-1}_\partial = h^{n,n-1}_\partial - k^{0,1} \]

Thus
Corollary 7.5. If \( h^{0,1}_\partial = k^{0,1} \) for a compact complex manifold, \( X \), then every hermitian metric is conformal to a strongly gauduchon hermitian metric.

and

Corollary 7.6. If \( b^2 = 0 \) and \( h^{0,1}_\partial = k^{0,1} \) for a compact complex manifold, \( X \), then every hermitian metric is conformal to a balanced hermitian metric.

Noting that \( k^{0,1} = h^{0,1}_{BC} \), Popovici and Ugarte\[18\] actually have the stronger result on strongly gauduchon metrics:

Theorem 7.7. (Popovici and Ugarte) On a compact complex manifold, \( X \), every hermitian metric is conformal to a strongly gauduchon hermitian metric if and only if \( h^{0,1}_\partial = h^{0,1}_{BC} \).

Thus one can easily also conclude that

Corollary 7.8. On a compact complex manifold, \( X \), there exists a gauduchon metric which is not strongly gauduchon if and only if \( h^{n-1,n-1}_\partial \neq 0 \) or equivalently \( h^{0,1}_\partial \neq h^{0,1}_{BC} \).

8. BC-A COHOMOLOGY ON GENERIC COMPACT COMPLEX 3-FOLDS

We shall now complete a table for a generic compact complex 3-fold. Using the formulas above for \( h^{p,0}_{BC} \) and \( h^{p,n}_{BC} \), one obtains

\[
\begin{align*}
    h^{0,0}_{BC} &= 1 \\
    h^{1,0}_{BC} &= k^{1,0} \\
    h^{2,0}_{BC} &= h^{2,0}_\partial \\
    h^{3,0}_{BC} &= h^{3,0}_\partial = k^{3,0} \\
    h^{3,1}_{BC} &= h^{3,1}_\partial = h^{0,2}_\partial \\
    h^{3,2}_{BC} &= h^{0,1}_\partial + h^{2,0}_\partial - k^{3,1} \\
    h^{3,3}_{BC} &= 1
\end{align*}
\]

Using the Bigolin resolution\[23\] of the sheaf of pluri-harmonic functions, \( \mathcal{P} \mathcal{H} \), we can also derive a formula for \( h^{1,1}_{BC} \) for any compact complex manifold:

Theorem 8.1. For any compact complex manifold,

\[
H^{1,1}_{BC} = \tilde{H}^1(\mathcal{P} \mathcal{H}) .
\]

Thus, \( h^{1,1}_{BC} = h^{n-1,n-1}_{A} = |\tilde{H}^1(\mathcal{P} \mathcal{H})| .\)
Proof: Let $\mathcal{E}^{p,q}$ denote the sheaf of $C^\infty p,q$-forms. The Bigolin resolution for $\mathcal{P}\mathcal{H}$,

$$0 \to \mathcal{P}\mathcal{H} \hookrightarrow \mathcal{E}^{0,0} \xrightarrow{\partial} \mathcal{E}^{1,1} \xrightarrow{d} \mathcal{E}^{2,1} \oplus \mathcal{E}^{1,2} \to \text{Coker}(d) \to 0$$

is acyclic (except at the last term, Coker$(d)$). Thus

$$\tilde{H}^1(\mathcal{P}\mathcal{H}) = \frac{\text{ker}(d : \Gamma(\mathcal{E}^{1,1}) \to \Gamma(\mathcal{E}^{2,1} \oplus \mathcal{E}^{1,2}))}{\text{im}(\partial \bar{\partial} : \Gamma(\mathcal{E}^{0,0}) \to \Gamma(\mathcal{E}^{1,1}))} = \frac{\text{ker}(d : C^{1,1} \to C^{2,1} \oplus C^{1,2})}{\text{im}(\partial \bar{\partial} : C^{0,0} \to C^{1,1})} = H^{1,1}_{BC}$$

Following Bigolin\cite{3}, we note that

$$\mathcal{P}\mathcal{H} = \mathcal{O} + \bar{\mathcal{O}}$$

and that we have the short exact sequence

$$0 \to C \to \mathcal{O} \oplus \bar{\mathcal{O}} \to \mathcal{O} + \bar{\mathcal{O}} \to 0$$

or

$$0 \to C \to \mathcal{O} \oplus \bar{\mathcal{O}} \to \mathcal{P}\mathcal{H} \to 0$$

Hence we have the following portion of the subsequent long exact sequence,

$$0 \to \tilde{H}^0(C) \to \tilde{H}^0(\mathcal{O} \oplus \bar{\mathcal{O}}) \to \tilde{H}^0(\mathcal{P}\mathcal{H}) \to \tilde{H}^1(C) \to \tilde{H}^1(\mathcal{O} \oplus \bar{\mathcal{O}}) \to \tilde{H}^1(\mathcal{P}\mathcal{H}) \to \tilde{H}^2(C) \to \cdots$$

which on a compact complex manifold becomes (using the fact that global holomorphic, global anti-holomorphic and global pluri-harmonic functions are constant)

$$0 \to C \to C \oplus C \to C \to \tilde{H}^1(C) \to \tilde{H}^1(\mathcal{O} \oplus \bar{\mathcal{O}}) \to \tilde{H}^1(\mathcal{P}\mathcal{H}) \to \tilde{H}^2(C) \to \cdots.$$

Exactness at the first three terms of the sequence gives that the map $C \to \tilde{H}^1(C)$ must be the zero map. Hence we have the clipped long exact sequence,

$$0 \to \tilde{H}^1(C) \to \tilde{H}^1(\mathcal{O} \oplus \bar{\mathcal{O}}) \to \tilde{H}^1(\mathcal{P}\mathcal{H}) \to \tilde{H}^2(C) \to \cdots.$$

We rewrite this as,

$$0 \to H^1(C) \to H^0 \oplus H^{1,0} \to H^{1,1}_{BC} \to H^2(C) \to \cdots.$$

If $b^2 = 0$ then $H^2(C) = 0$ and we have the short exact sequence,

$$0 \to H^1(C) \to H^0 \oplus H^{1,0} \to H^{1,1}_{BC} \to 0.$$

Thus,
Theorem 8.2. On a compact complex manifold with $b^2 = 0$, we have
\[ h_{BC}^{1,1} = h^n_A - 1 - 1 = 2h_0^{0,1} - b^1. \]

Our formula for $h_{BC}^{1,1}$ generalizes even further by results of Tosatti [20] which we follow here virtually verbatim. Tosatti gives the short exact sequence of sheaves,
\[ 0 \to R \to \mathcal{O} \xrightarrow{Im} \mathcal{P} \to 0 \]
where $\mathcal{P}$ is the sheaf of real valued pluriharmonic functions. He thus also gives the resulting, long exact sequence in sheaf cohomology,
\[ 0 \to H^0(R) \to H^0(\mathcal{O}) \to H^0(\mathcal{P}) \]
\[ \to H^1(R) \to H^1(\mathcal{O}) \to H^1(\mathcal{P}) \to H^2(R) \to \cdots \]
The first three terms form a short exact sequence,
\[ 0 \to R \to \mathbb{R}^2 \to \mathbb{R} \to 0 \]
so that one has,
\[ 0 \to H^1(R) \to H^1(\mathcal{O}) \to H^1(\mathcal{P}) \to H^2(R) \xrightarrow{\pi^{0,2}} H^2(\mathcal{O}) \to \cdots. \]
The map, $\pi^{0,2}$, is the projection to the 0, 2 part of $H^2(R) \cong H^2_{dR}(\mathbb{R})$ followed by the isomorphism, $H^2(\mathcal{O}) \cong H^0_{\bar{\partial}}$. We have
\[ ker(\pi^{0,2}) = H^{1,1}(R) \]
where $H^{1,1}(R)$ is the subgroup of $H^2(R)$ of deRham classes which have a representative which is a real $d$-closed 1, 1-form. Thus one has the exact sequence of real vector spaces,
\[ 0 \to H^1(R) \to H^1(\mathcal{O}) \to H^1(\mathcal{P}) \to H^{1,1}(R) \to 0 \]
Note that
\[ dim_{\mathbb{R}}(H^1(R)) = dim_{\mathbb{C}}(H^1(\mathbb{C})) = b^1 \]
\[ dim_{\mathbb{R}}(H^1(\mathcal{O})) = 2dim_{\mathbb{C}}(H^1(\mathcal{O})) = 2h_0^{0,1} \]
and
\[ dim_{\mathbb{R}}(H^1(\mathcal{P})) = dim_{\mathbb{C}}(H^1(\mathcal{PH})) = h_{BC}^{1,1}. \]
Thus
\[ b^1 - 2h_0^{0,1} + h_{BC}^{1,1} - dim_{\mathbb{R}}(H^{1,1}(R)) = 0 \]
and
Theorem 8.3. On a compact complex manifold, we have
\[ h_{BC}^{1,1} = 2h_0^{0,1} - b^1 + dim_{\mathbb{R}}(H^{1,1}(R)). \]
If we can calculate $h_{BC}^{1,2}$ or $h_{BC}^{2,2}$, then we can know the BC-A cohomology completely in terms of Dolbeault and Frolicher terms.

Consider the following expression of $h_{A}^{2,1}$,
\[
h_{A}^{2,1} = h_{\bar{\partial}}^{2,1} - l_{\bar{\partial}}^{2,0} + g_{\bar{\partial}}^{2,1} + l_{\bar{\partial}}^{2,1}
\]
\[
= h_{\bar{\partial}}^{2,1} - l_{\bar{\partial}}^{0,2} + g_{\bar{\partial}}^{2,1} + l_{\bar{\partial}}^{2,1}
\]
We know from the almost exact sequence
\[
0 \to L_{\bar{\partial}}^{2,1} \to H^{3,1}_{\bar{\partial}} \to H^{3,1}_{A} \to H^{3,2}_{BC} \to H^{3,2}_{\bar{\partial}} \to H^{3,2}_{A} \to 0
\]
(which we note is exact at each term except at $H^{3,2}_{\bar{\partial}}$) that
\[
l_{\bar{\partial}}^{2,1} = h_{\bar{\partial}}^{3,1} - h_{A}^{3,1} + h_{BC}^{3,2} - (h_{\bar{\partial}}^{3,2} + k^{3,2}) + h_{A}^{3,2}.
\]
We have from Proposition 4.2 and Proposition 4.1 that
\[
h_{BC}^{3,2} = h_{\bar{\partial}}^{3,2} + (-1)^{2}(h_{BC}^{3,1} - (h_{\bar{\partial}}^{3,1} + k^{3,1}) + h_{A}^{3,1})
\]
\[
= h_{\bar{\partial}}^{3,1} - h_{\bar{\partial}}^{3,1} - h_{\bar{\partial}}^{0,1} - k^{2,0} + h_{\bar{\partial}}^{2,0}
\]
\[
= h_{\bar{\partial}}^{0,1} - k^{2,0}.
\]
Plugging this into the expression for $l_{\bar{\partial}}^{2,1}$ just above, we have
\[
l_{\bar{\partial}}^{2,1} = h_{\bar{\partial}}^{0,2} - h_{\bar{\partial}}^{2,0} + h_{\bar{\partial}}^{0,1} - k^{2,0} + h_{\bar{\partial}}^{2,0} - (h_{\bar{\partial}}^{0,1} + k^{3,2}) + k^{1,0}
\]
\[
= h_{\bar{\partial}}^{0,2} - k^{2,0}.
\]
Notice that this calculation follows through without using the assumptions, $b^{1} = 0$ or $b^{2} = 0$. We have instead used the fact that $k^{3,2} = k^{1,0}$, from a Serre duality that can be proved using harmonic representations of $H_{A}$ and $H_{BC}$:
\[
K^{p,q} = H_{\bar{\partial}}^{p,q} / \text{im}(\bar{\partial}) \cap H_{\bar{\partial}}^{p,q} / \text{im}(\bar{\partial}) \subseteq H_{A}^{p,q},
\]
and thus
\[
\ast K^{p,q} = K^{n-p,n-q} \subseteq H_{BC}^{n-p,n-q}.
\]
We will now show $g_{\bar{\partial}}^{2,1} = 0$. We know that
\[
g_{\bar{\partial}}^{2,1} = g_{\bar{\partial}}^{3,0} = g_{\bar{\partial}}^{0,3}.
\]
Consider the following expression of $h_{A}^{0,3}$,
\[
h_{A}^{0,3} = h_{\bar{\partial}}^{0,3} - l_{\bar{\partial}}^{0,2} + g_{\bar{\partial}}^{0,3} + l_{\bar{\partial}}^{0,3}
\]
\[
= h_{\bar{\partial}}^{0,3} - l_{\bar{\partial}}^{2,0} + g_{\bar{\partial}}^{0,3} + l_{\bar{\partial}}^{0,3}
\]
Previously, we proved For any $n$-dimensional complex manifold, $X$, a $\bar{\partial}$-closed, $\bar{\partial}$-exact $n,0$-form is zero. Explicitly, let $\phi$ be an $n-1,0$-form. If $\partial\phi$ is $\bar{\partial}$-closed, then $\partial\phi = 0$. In other words, for any compact
complex manifold, \( l^{n-1,0}_\partial = 0 \). Thus, we have above, \( l^{2,0}_\partial = 0 \). We also showed previously that \( \bar{h}_{A}^{0,3} = h_{A}^{0,3} = k^{0,3} \). Thus \( g_{\partial}^{2,1} = g_{\bar{\partial}}^{0,3} = 0 \) and \( l^{0,3}_\partial = 0 \).

We proceed to calculate \( l^{0,2}_\partial \). Recall that

\[
l^{n-p-2,m-q}_\partial = h^{p+1,q}_\partial - t^{p,q}_\partial - k^{p+1,q}.
\]

Thus

\[
l^{0,2}_\partial = h^{2,1}_\partial - l^{1,1}_\partial - k^{2,1}.
\]

To calculate \( l^{1,1}_\partial \) we consider the equation,

\[
\operatorname{Ker}(\partial : H^{p,q}_\partial \to H^{p+1,q}_\partial) = G^{p,q}_\partial \cap (H^{p,q}_\partial/\text{im}(\partial)) \oplus K^{p,q} \oplus \partial L^{p-1,q}_\partial
\]

with \( p, q = 2, 1 \). Thus

\[
|\operatorname{Ker}(\partial : H^{2,1}_\partial \to H^{3,1}_\partial)| = k^{2,1} + |G^{2,1}_\partial \cap (H^{2,1}_\partial/\text{im}(\partial))| + l^{1,1}_\partial.
\]

We have \( g_{\partial}^{2,1} = 0 \) so

\[
|G^{2,1}_\partial \cap (H^{2,1}_\partial/\text{im}(\partial))| = 0.
\]

Thus

\[
l^{1,1}_\partial = |\operatorname{Ker}(\partial : H^{2,1}_\partial \to H^{3,1}_\partial)| - k^{2,1}.
\]

Define as temporary shorthand,

\[
\ker_{p,q} = |\operatorname{Ker}(\partial : H^{p,q}_\partial \to H^{p+1,q}_\partial)|
\]

so that we have,

\[
l^{1,1}_\partial = \ker_{2,1} - k^{2,1}.
\]

Also define as temporary shorthand,

\[
im_{p,q} = |\text{Im}(\partial : H^{p-1,q}_\partial \to H^{p,q}_\partial)|.
\]

We shall calculate \( \ker_{2,1} \) from the Frolicher spectral sequence as we did above for \( \im_{p,0} \). We have

\[
h^{p,q}_\partial = \ker_{p,q} - \im_{p,q}
\]

and

\[
h^{p,q}_\partial = \ker_{p,q} + \im_{p+1,q} = \ker_{p,q} + \ker_{p+1,q} - h^{2+1,q}_2.
\]

Thus,

\[
h^{p,q}_\partial = \ker_{p,q} + \ker_{p+1,q} - h^{p+1,q}_2
\]

and

\[
h^{p+1,q}_\partial = \ker_{p+1,q} + \ker_{p+2,q} - h^{p+2,q}_2
\]

\[
\vdots
\]

\[
h^{n-1,q}_\partial = \ker_{n-1,q} + \ker_{n,q} - h^{n,q}_2
\]

and

\[
h^{n,q}_\partial = \ker_{n,q}
\]
and
\[ \sum_{j=p}^{n} (-1)^{j-p} h_{j}^{2,q} = \ker_{p,q} - \sum_{j=p}^{n-1} (-1)^{j-p} h_{j+1}^{2,q}. \]

Thus,
\[ \ker_{p,q} = \sum_{j=p}^{n} (-1)^{j-p} h_{j}^{2,q} + \sum_{j=p}^{n-1} (-1)^{j-p} h_{j+1}^{2,q}. \]

For \( n = 3 \) and \( p, q = 2, 1 \) this is simply,
\[ \ker_{2,1} = h_{2}^{2,1} - h_{2}^{3,1} + h_{2}^{3,1}. \]

Hence,
\[ l_{1}^{1,1} = h_{2}^{2,1} - h_{2}^{3,1} + h_{2}^{3,1} - k_{2,1} \]
and
\[ l_{0}^{0,2} = h_{2}^{2,1} - l_{1}^{1,1} - k_{2,1} \]
\[ = h_{2}^{2,1} - (h_{2}^{2,1} - h_{2}^{3,1} + h_{2}^{3,1} - k_{2,1}) - k_{2,1} \]
\[ = h_{2}^{3,1} - h_{2}^{3,1} \]

Finally,
\[ h_{A}^{2,1} = h_{2}^{1,2} - l_{0}^{0,2} + g_{2}^{2,1} + l_{2}^{2,1} \]
\[ = h_{2}^{1,2} - (h_{2}^{3,1} - h_{2}^{3,1}) + 0 + (h_{2}^{0,2} - k_{2,0}) \]
\[ = h_{2}^{1,2} + h_{2}^{3,1} - k_{2,0} \]

Thus

**Proposition 8.4.** On a general compact complex three-fold,

\[ h_{BC}^{1,2} = h_{2}^{1,2} + h_{2}^{3,1} - k_{2,0} \]

In Proposition 2.4 we had used the long almost exact sequence to arrive at the equation,
\[ \sum_{q=0}^{n} (-1)^{q}(h_{BC}^{p,q} + h_{BC}^{n-p,n-q} - k^{p,q}) = \sum_{q=0}^{n} (-1)^{q}h_{\partial}^{p,q} \]
For \( n = 3 \) and \( p = 2 \), this gives
\[
\hat{h}^{2,2}_{BC} = -\hat{h}^{2,0}_A + (h^{2,0}_A + k^{2,0}_A) - h^{2,0}_B + h^{2,1}_B - (h^{2,1}_B + k^{2,1}_B) + h^{2,1}_A \\
+ (h^{2,2}_A + k^{2,2}_A) - h^{2,2}_B + h^{2,3}_B - (h^{2,3}_B + k^{2,3}_B) + h^{2,3}_A \\
= -\hat{h}^{2,0}_\bar{\partial} + h^{2,0}_\bar{\partial} + k^{2,0}_\bar{\partial} - h^{0,2}_\bar{\partial} + h^{1,2}_\bar{\partial} + h^{3,1}_\bar{\partial} - k^{2,0}_\bar{\partial} - h^{2,1}_\bar{\partial} - k^{2,1}_\bar{\partial} \\
+ h^{1,2}_\bar{\partial} + h^{3,1}_\bar{\partial} - k^{2,0}_\bar{\partial} + h^{1,1}_\bar{\partial} + k^{1,1}_\bar{\partial} - 2h^{0,1}_\bar{\partial} + b^1 - \dim_\mathbb{R}(H^{1,1}(\mathbb{R})) \\
+ h^{0,1}_\bar{\partial} + h^{2,0}_\bar{\partial} - k^{2,0}_\bar{\partial} - h^{1,0}_\bar{\partial} - k^{1,0}_\bar{\partial} + k^{1,0}_\bar{\partial} \\
= h^{2,0}_\bar{\partial} + h^{1,2}_\bar{\partial} - h^{0,2}_\bar{\partial} + h^{1,1}_\bar{\partial} - h^{1,0}_\bar{\partial} - h^{0,1}_\bar{\partial} + b^1 - \dim_\mathbb{R}(H^{1,1}(\mathbb{R})) \\
+ 2h^{3,1}_\bar{\partial} - 2k^{2,0} + k^{1,1} - k^{1,2} .
\]

So,
\[
\hat{h}^{2,2}_{BC} = -h^{0,1}_\bar{\partial} - h^{0,2}_\bar{\partial} - h^{1,0}_\bar{\partial} + h^{1,1}_\bar{\partial} + h^{1,2}_\bar{\partial} + h^{2,0}_\bar{\partial} \\
+ 2h^{3,1}_\bar{\partial} + k^{1,1} - k^{1,2} - 2k^{2,0} + b^1 - \dim_\mathbb{R}(H^{1,1}(\mathbb{R})) .
\]

We also note for future use that we can write the formula for \( h^{2,2}_{BC} \) in terms of \( h^{1,1}_{BC} = \hat{h}^1(\mathcal{D}\mathcal{H}) \), Dolbeault, Frohlicher and \( k^{i,j} \) terms:
\[
\hat{h}^{2,2}_{BC} = -h^{2,0}_\bar{\partial} + (h^{2,0}_\bar{\partial} + k^{2,0}_\bar{\partial}) - h^{2,0}_A + h^{2,1}_B - (h^{2,1}_B + k^{2,1}_B) + h^{2,1}_A \\
+ (h^{2,2}_A + k^{2,2}_A) - h^{2,2}_B + h^{2,3}_B - (h^{2,3}_B + k^{2,3}_B) + h^{2,3}_A \\
= -h^{2,0}_\bar{\partial} + h^{2,0}_\bar{\partial} + k^{2,0}_\bar{\partial} - h^{0,2}_\bar{\partial} + h^{1,2}_\bar{\partial} + h^{3,1}_\bar{\partial} - k^{2,0}_\bar{\partial} - h^{2,1}_\bar{\partial} - k^{2,1}_\bar{\partial} + h^{1,2}_\bar{\partial} + h^{3,1}_\bar{\partial} - k^{2,0}_\bar{\partial} \\
+ h^{1,1}_\bar{\partial} + k^{1,1}_\bar{\partial} - h^{1,1}_\bar{\partial} + h^{0,1}_\bar{\partial} + h^{1,0}_\bar{\partial} + h^{0,1}_\bar{\partial} - h^{1,0}_\bar{\partial} + h^{1,1}_\bar{\partial} + h^{3,1}_\bar{\partial} - k^{2,0}_\bar{\partial} - h^{1,0}_\bar{\partial} - k^{1,0}_\bar{\partial} + k^{1,0}_\bar{\partial} \\
= h^{2,0}_\bar{\partial} + h^{1,2}_\bar{\partial} - h^{0,2}_\bar{\partial} + h^{1,1}_\bar{\partial} - h^{1,0}_\bar{\partial} + h^{0,1}_\bar{\partial} - h^{1,1}_\bar{\partial} \\
+ 2h^{3,1}_\bar{\partial} - 2k^{2,0} + k^{1,1} - k^{1,2} .
\]

So,
\[
\hat{h}^{2,2}_{BC} = -h^{1,1}_\bar{\partial} + h^{0,1}_\bar{\partial} - h^{0,2}_\bar{\partial} - h^{1,0}_\bar{\partial} + h^{1,1}_\bar{\partial} + h^{1,2}_\bar{\partial} + h^{2,0}_\bar{\partial} \\
+ 2h^{3,1}_\bar{\partial} + k^{1,1} - k^{1,2} - 2k^{2,0} .
\]

We summarize with a table of our results.

| Bott-Chern cohomology on a Compact Complex 3-fold |
|-----------------------------------------------|
| \( \hat{h}^{3,0}_\bar{\partial} \) | \( \hat{h}^{0,2}_\bar{\partial} \) | \( \hat{h}^{0,1}_\bar{\partial} + h^{2,0}_\bar{\partial} - k^{2,0}_\bar{\partial} \) | 1 |
| \( h^{2,0}_\bar{\partial} \) | \( h^{1,2}_\bar{\partial} + h^{3,1}_\bar{\partial} - k^{2,0}_\bar{\partial} \) | \( h^{2,2}_{BC} \) | \( h^{0,1}_\bar{\partial} + h^{2,0}_\bar{\partial} - k^{2,0}_\bar{\partial} \) |
| \( h^{1,1}_\bar{\partial} \) | \( h^{1,2}_\bar{\partial} + h^{3,1}_\bar{\partial} - k^{2,0}_\bar{\partial} \) | \( h^{2,0}_\bar{\partial} \) | \( h^{0,2}_\bar{\partial} \) |
| \( h^{1,0}_\bar{\partial} \) | \( h^{1,1}_\bar{\partial} \) | \( h^{2,0}_\bar{\partial} \) | \( k^{3,0} \) |
where

\[ h^{1,1}_{BC} = 2h^{0,1}_\partial - b^1 + \dim_H(H^{1,1}(\mathbb{R})) \]

and \( h^{2,2}_{BC} \) is given in terms of \( b^1, \dim_H(H^{1,1}(\mathbb{R})) \), Dolbeault, Frolicher and \( k^{p,q} \) terms in the equations just preceding the table.

9. Examples of Bott-Chern/Aeppli Cohomology on Compact Complex 3-folds

9.1. BC-A cohomology on complex \( S^6 \). For a hypothetical complex structure on \( S^6 \), it has been shown by Brown \[7\] that

\[ h^{3,1}_2 = h^{0,2}_2 \]

by Gray\[12\] that

\[ h^{3,0}_\partial = 0 \]

and by Ugarte\[22\] that

\[ h^{0,1}_\partial = 1 + h^{0,2}_\partial \]

and

\[ h^{1,1}_\partial = 1 + h^{1,2}_\partial + h^{1,0}_\partial - h^{2,0}_\partial \]

(We also note of course that since \( b^j = 0 \) for \( 0 < j < 6 \), we have \( k^{p,q} = 0 \) for \( 0 < p + q < 6 \).) So for complex \( S^6 \) we can write from our results just above,

\[ h^{1,2}_{BC} = h^{1,2}_\partial + h^{0,2}_2. \]

We can also write from our results just above,

\[ h^{1,1}_{BC} = 2h^{0,1}_\partial = 2 + 2h^{0,2}_\partial \]

and

\[ h^{2,2}_{BC} = -h^{1,1}_{BC} + h^{0,1}_\partial - h^{0,2}_\partial - h^{1,0}_\partial + h^{1,1}_\partial + h^{1,2}_\partial + h^{2,0}_\partial + 2h^{3,1}_2 \]

\[ = -2 - 2h^{0,2}_\partial + 1 + h^{0,2}_\partial - h^{0,2}_\partial - h^{1,0}_\partial + h^{1,1}_\partial + h^{1,2}_\partial + h^{2,0}_\partial + 2h^{3,1}_2 \]

\[ = -1 - 2h^{0,2}_\partial - h^{1,0}_\partial + (1 + h^{1,2}_\partial + h^{1,0}_\partial - h^{2,0}_\partial) + h^{1,2}_\partial + h^{2,0}_\partial + 2h^{0,2}_2 \]

\[ = 2h^{1,2}_\partial + 2h^{0,2}_\partial - 2h^{0,2}_\partial \]

We note that these agree with the formula

\[ h^{1,1}_{BC} + h^{2,2}_{BC} = 2h^{1,2}_{BC} + 2 \]

derived by the author in \[14\]. We summarize this with a table:
### Bott-Chern cohomology on complex $S^6$

$\begin{array}{|c|c|c|c|}
\hline
0 & h^{0,2}_\partial & 1 + h^{0,2}_\partial + h^{2,0}_\partial & 1 \\
\hline
h^{2,0}_\partial & h^{1,2}_\partial + h^{0,2}_2 & 2h^{1,2}_\partial + 2h^{0,2}_2 - 2h^{0,2}_\partial & 1 + h^{0,2}_\partial + h^{2,0}_\partial \\
\hline
0 & 2 + 2h^{0,2}_\partial & h^{1,2}_\partial + h^{0,2}_2 & h^{0,2}_\partial \\
\hline
1 & 0 & h^{2,0}_\partial & 0 \\
\hline
\end{array}$

where the table is oriented so that $h^{0,0}_\partial$ is in the lower lefthand corner and $h^{3,0}_{BC}$ in in the lower righthand corner.

9.2. **Bott-Chern Aeppli cohomology of a Calabi-Eckmann 3-fold.** We shall compute the Bott-Chern/Aeppli cohomology for a more concrete case.

Consider the Calabi-Eckmann 3-fold, diffeomorphic to $S^3 \times S^3$. We will calculate the Bott-Chern/Aeppli cohomology in this case, replicating the calculation of Angella and Tomassini[6].

The Dolbeault cohomology of Calabi-Eckmann manifolds was originally calculated by Borel. For our particular Calabi-Eckmann complex 3-fold ( see for example [8]) the Hodge diamond for the Dolbeault cohomology is

**Dolbeault cohomology of the Calabi-Eckmann Complex 3-fold**

$\begin{array}{c}
1 \\
1 0 \\
0 1 0 \\
0 1 1 0 \\
0 1 0 \\
0 1 \\
1 \\
\end{array}$

Note, that we have $b^1 = b^2 = 0$, $h^{2,1}_\partial = 1$ and $h^{0,2} = h^{3,1} = 0$. Thus $h^{3,1}_2 = 0$. We also must have $k^{1,2}_\partial = 0$ or $k^{1,2}_\partial = 1$. We show that $k^{1,2}_\partial = 0$. Indeed, the Dolbeault cohomology in this case (see [6]) has nonzero, $\theta \in H^{0,1}_\partial$ and nonzero $\partial\theta \in H^{1,1}_\partial$. One can check that $H^{2,1}_\partial$ is generated by $\bar{\theta}\partial\theta = \partial(\bar{\theta}\theta)$. Thus the map $H^{2,1}_\partial \rightarrow H^{2,1}_A$ in our sequence
above is the 0-map and we conclude that $k^{2,1} = k^{1,2} = 0$. Thus our Bott-Chern cohomology is given by the hodge diamond

Bott-Chern cohomology of the Calabi-Eckmann 3-fold

\[
\begin{array}{cccc}
1 & & & \\
0 & 0 & & \\
0 & 2 & 0 & \\
0 & 1 & 1 & 0 \\
0 & 1 & 0 & \\
1 & 1 & & \\
1 & & & \\
\end{array}
\]

This agrees with the calculation of the Bott-Chern cohomology for the Calabi-Eckmann 3-fold done by Angella and Tomassini [6].

9.3. Comparison with Angella’s calculation on the Iwasawa manifold and its deformations. Angella [5] gives the De Rham and Dolbeault cohomologies and calculates the Bott-Chern/Aeppli cohomologies on the Iwasawa manifold, $\mathbb{I}_3$, and its small deformations in the Kuranishi family of deformations due to Nakamura [16]. The Iwasawa manifold is a holomorphically parallelizable compact complex three-fold with a global holomorphic coframe given by (see Angella [5], p. 39)

\[
\begin{align*}
\phi^1 &= dz^1 \\
\phi^2 &= dz^2 \\
\phi^3 &= dz^1 - z^1 dz^2
\end{align*}
\]

where $z^1, z^2, z^3$ are local coordinates given from $\mathbb{I}_3$ being a quotient space of $\mathbb{C}^3$ by a discrete group action. The structure equations are easily seen to be

\[
\begin{align*}
d\phi^1 &= 0 \\
d\phi^2 &= 0 \\
d\phi^3 &= -\phi^1 \wedge \phi^2
\end{align*}
\]

For small deformations in the Kuranishi family of deformations of $\mathbb{I}_3$, there is a global coframe of $\bigwedge^{1,0} \mathbb{I}_3$, \{\phi^1_t, \phi^2_t, \phi^3_t\}, where $t \in \Delta(\epsilon, 0) \subset \mathbb{C}^6$. Define

\[
D(t) = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}.
\]
The different classes of the small Kuranishi deformations of $\mathbb{I}_3$ are defined according to the parameter, $t$:

(i) $t_{11} = t_{12} = t_{21} = t_{22} = 0$ (Note, generically, $t_{3k} \neq 0$, $t_{k3} \neq 0$.)

(ii) $(t_{11}, t_{12}, t_{21}, t_{22}) \neq (0,0,0,0)$, $D(t) = 0$

(iii) $D(t) \neq 0$

Define

$$S = \begin{pmatrix} \bar{\sigma}_{11} & \bar{\sigma}_{22} & \bar{\sigma}_{21} & \bar{\sigma}_{12} \\ \sigma_{11} & \sigma_{22} & \sigma_{21} & \sigma_{12} \end{pmatrix}$$

The subclasses (ii.a) and (ii.b) of class (ii) are defined by

(ii.a) $D(t) = 0$ and $S$ has rank 1.

(ii.b) $D(t) = 0$ and $S$ has rank 2.

Similarly, the subclasses (iii.a) and (iii.b) of class (iii) are defined by

(iii.a) $D(t) \neq 0$ and $S$ has rank 1.

(iii.b) $D(t) \neq 0$ and $S$ has rank 2.

The structure equations for class (i) are:

$$d\phi^1 = 0$$

$$d\phi^2 = 0$$

$$d\phi^3 = -\phi^1 \wedge \phi^2$$

For classes (ii) and (iii), the structure equations are given by

$$d\phi_i^1 = 0$$

$$d\phi_i^2 = 0$$

$$d\phi_i^3 = \sigma_{12}\phi_i^1 \wedge \phi_i^2$$

$$+\sigma_{11}\phi_i^1 \wedge \bar{\phi}_i^1 + \sigma_{12}\phi_i^1 \wedge \bar{\phi}_i^2$$

$$+\sigma_{21}\phi_i^2 \wedge \bar{\phi}_i^1 + \sigma_{22}\phi_i^2 \wedge \bar{\phi}_i^2$$

We shall be using notation as in Angella[5]: $\phi^{IJ} = \phi^{i_1} \wedge \cdots \wedge \phi^{i_p} \wedge \bar{\phi}^{j_1} \wedge \cdots \wedge \bar{\phi}^{j_q}$ for multi-indices, $I$ and $J$.

One can check that $h_2^{3,1} = 2$ for all the classes of deformations of $\mathbb{I}_3$.

One can check that, a priori, $H_3^{3,1} = \ker(\partial)$ and that,

$$H_3^{3,1} = \mathbb{C}_< \phi_t^{1231}, \phi_t^{1232} > .$$
Since,
\[ H^{2,1}_\partial \subseteq \mathbb{C} < \phi^{121}_t, \phi^{132}_t, \phi^{231}_t, \phi^{232}_t > \]
for class (i) and
\[ H^{2,1}_\partial \subseteq \mathbb{C} < \phi^{121}_t, \phi^{132}_t, \phi^{131}_t \]
for classes (ii) and (iii), we see that \( \partial \) acts on \( H^{2,1}_\partial \) as the zero map. Thus \( Im(\partial : H^{2,1}_\partial \to H^{2,1}_\partial) = 0 \) and \( h^{2,1}_\partial = 2 \).

We see from the 2, 1 -forms above, that \( \phi^{121}_t = \partial \phi^{3,1}_t \) and \( \phi^{122}_t = \partial \phi^{3,2}_t \) and thus \( k^{2,1} \leq 4 \). We shall show that \( k^{1,2} = 4 \) on the Iwasawa manifold and its classes of deformations (i), (ii) and (iii). In fact, Angella\[[5]\] shows that the 2, 1 -forms,
\[ \{ \phi^{131}_t, \phi^{132}_t, \phi^{231}_t, \phi^{232}_t \} \]
for class (i) and
\[ \{ \phi^{131}_t - \frac{\sigma^{22}}{\sigma^{12}} \phi^{123}_t, \phi^{132}_t - \frac{\sigma^{22}}{\sigma^{12}} \phi^{231}_t, \phi^{231}_t - \frac{\sigma^{12}}{\sigma^{12}} \phi^{132}_t, \phi^{232}_t - \frac{\sigma^{11}}{\sigma^{12}} \phi^{123}_t \} \]
for classes (ii) and (iii), are \( \partial \omega_t \)-harmonic with respect to the hermitian metric \( g_t = \phi^1_t \circ \phi^3_t + \phi^2_t \circ \phi^2_t + \phi^3_t \circ \phi^3_t \).

We can show that for the class of deformations (i), the 2, 1-forms
\[ \{ \phi^{131}_t, \phi^{132}_t, \phi^{231}_t, \phi^{232}_t \} \]
are also \( \partial \omega_t \)-harmonic: It is easy to see from the structure equations that these are \( \partial \)-closed. Now observe that (using the notation of Wells\[[23]\])
\[
\begin{align*}
\ast \phi^{131}_t &= \phi^{233}_t \\
\ast \phi^{132}_t &= - \phi^{213}_t \\
\ast \phi^{231}_t &= - \phi^{123}_t \\
\ast \phi^{232}_t &= \phi^{113}_t 
\end{align*}
\]
and that these are also \( \partial \)-closed. Thus these 2, 1-forms for class (i) are \( \partial^* \)-closed and hence \( \partial \omega_t \)-harmonic. Since they are also \( \partial \omega_t \)-harmonic, we know that
\[ \mathbb{C} < \phi^{131}_t, \phi^{132}_t, \phi^{231}_t, \phi^{232}_t > \]
is orthogonal to \( Im(\partial) + Im(\partial^*) \). We conclude that \( k^{2,1} \geq 4 \) for the class (i). Thus, \( k^{2,1} = 4 \) for class (i).
For deformation classes, \((ii)\) and \((iii)\), consider the five dimensional complex vector space,
\[
W = \mathbb{C} \prec \phi_{12}^{33}, \phi_t^{131}, \phi_t^{132}, \phi_t^{231}, \phi_t^{232} \succ .
\]
One can check that this is also
\[
W = \mathbb{C} \prec \bar{\partial}^* \phi_{12}^{131}, \phi_t^{131} - \sigma_{12}^{11} \phi_t^{123}, \phi_t^{132} - \sigma_{12}^{21} \phi_t^{123}, \phi_t^{231} - \sigma_{12}^{12} \phi_t^{123}, \phi_t^{232} - \sigma_{12}^{11} \phi_t^{123} \succ .
\]
Recalling the orthogonal decomposition,
\[
\mathcal{E}^{2,1} = \mathcal{H}^{2,1}_\partial \oplus \text{Im}(\bar{\partial}) \oplus \text{Im}(\partial^*),
\]
we see that \(W\) is orthogonal to \(\text{Im}(\bar{\partial})\).

We also have
\[
W = \mathbb{C} \prec \partial \phi^{33}, \phi_t^{131} + \sigma_{12}^{11} \phi_t^{123}, \phi_t^{132} + \sigma_{12}^{21} \phi_t^{123}, \phi_t^{231} + \sigma_{12}^{12} \phi_t^{123}, \phi_t^{232} + \sigma_{12}^{22} \phi_t^{123} \succ
\]
noting that for the first basis element we have
\[
\partial \phi^{33} = \sigma_{12} \phi_t^{123} + \sigma_{11} \phi_t^{131} + \sigma_{21} \phi_t^{132} + \sigma_{12} \phi_t^{231} + \sigma_{22} \phi_t^{232}
\]
and that the other four basis elements are \(\partial\)-harmonic. Thus we see that \(W \cap \text{Im}(\partial) = \mathbb{C} \prec \partial \phi^{33} \succ\). We also have that
\[
W = \mathbb{C} \prec \frac{1}{\sigma_{12}} \partial \phi^{33}, \phi_t^{131} - \frac{\sigma_{22}}{\sigma_{12}} \phi_t^{123}, \phi_t^{132} - \frac{\sigma_{21}}{\sigma_{12}} \phi_t^{123}, \phi_t^{231} - \frac{\sigma_{12}}{\sigma_{12}} \phi_t^{123}, \phi_t^{232} - \frac{\sigma_{11}}{\sigma_{12}} \phi_t^{123} \succ
\]
since for small \(t\), the matrix,
\[
\begin{pmatrix}
\frac{1}{\sigma_{12}} & -\sigma_{12} & -\sigma_{12} & -\sigma_{12} & -\sigma_{12} \\
\frac{\sigma_{11}}{\sigma_{12}} & 1 & 0 & 0 & 0 \\
\frac{\sigma_{21}}{\sigma_{12}} & 0 & 1 & 0 & 0 \\
\frac{\sigma_{12}}{\sigma_{12}} & 0 & 0 & 1 & 0 \\
\frac{\sigma_{22}}{\sigma_{12}} & 0 & 0 & 0 & 1
\end{pmatrix}
\]
has non-zero determinant. Note that the basis elements
\[
\{\phi_t^{131} - \frac{\sigma_{22}}{\sigma_{12}} \phi_t^{123}, \phi_t^{132} - \frac{\sigma_{21}}{\sigma_{12}} \phi_t^{123}, \phi_t^{231} - \frac{\sigma_{12}}{\sigma_{12}} \phi_t^{123}, \phi_t^{232} - \frac{\sigma_{11}}{\sigma_{12}} \phi_t^{123}\}
\]
are \(\bar{\partial}\)-closed and \(\partial\)-closed. Denote,
\[
V = \mathbb{C} \prec \phi_t^{131} - \frac{\sigma_{22}}{\sigma_{12}} \phi_t^{123}, \phi_t^{132} - \frac{\sigma_{21}}{\sigma_{12}} \phi_t^{123}, \phi_t^{231} - \frac{\sigma_{12}}{\sigma_{12}} \phi_t^{123}, \phi_t^{232} - \frac{\sigma_{11}}{\sigma_{12}} \phi_t^{123} \succ
\]
Thus,
\[
V/(V \cap (\text{Im}(\bar{\partial}) + \text{Im}(\partial))) \subseteq K^{2,1}.
\]
Since
\[ V/(V \cap (\text{Im}(\partial) + \text{Im}(\bar{\partial}))) = V/(V \cap \mathbb{C} < \partial \phi^{33} >) \]
we have that \( \text{dim}_{\mathbb{C}}(V/(V \cap (\text{Im}(\partial) + \text{Im}(\bar{\partial})))) = 4 \) and \( k^{2,1} \geq 4 \). We conclude that \( k^{2,1} = 4 \) for deformation classes, \((ii)\) and \((iii)\).

It is clear from
\[ H^{0,1}_K = \mathbb{C} < \phi^1, \phi^2 > \]
that \( k^{1,0} = k^{0,1} = 2 \).

We now calculate \( k^{2,0} \) for each of the three classes. For class \((i)\), we have that
\[ H^{2,0}_K = \mathbb{C} < \phi_{12}^{1}, \phi_{13}^{1}, \phi_{23}^{1} > \]
where we note that these basis elements are also \( \partial \)-closed and that \( \phi_{12}^{1} = \partial \phi_{i}^{3} \). Thus for class \((i)\), \( k^{2,0} = 2 \).

For class \((ii)\), we have that
\[ H^{2,0}_K = \mathbb{C} < \phi_{12}^{1}, \alpha \phi_{13}^{1} + \beta \phi_{23}^{1} > \]
with \( \alpha \) and \( \beta \) not both zero. As before, the generators are also \( \partial \)-closed and \( \phi_{12}^{1} = \partial \phi_{i}^{3} \). Thus for class \((ii)\), \( k^{2,0} = 1 \).

For class \((iii)\), we have that
\[ H^{2,0}_K = \mathbb{C} < \phi_{12}^{1} > \]
and as before, \( \phi_{12}^{1} = \partial \phi_{i}^{3} \). Thus for class \((iii)\), \( k^{2,0} = 0 \).

We now calculate \( k^{1,1} \) for each of the three classes. For class \((i)\), \( K^{1,1} \) has a basis consisting of \( \{ \phi^{1T}, \phi^{1T}, \phi^{2T}, \phi^{3T} \} \). Thus, \( k^{1,1} = 4 \) for class \((i)\).

For classes \((ii.a)\) and \((iii.a)\),
\[ \partial \phi^{3} = \sigma_{11} \phi_{1}^{11} + \sigma_{12} \phi_{1}^{12} + \sigma_{21} \phi_{1}^{21} + \sigma_{22} \phi_{1}^{22} \]
and
\[ \overline{\partial \phi^{3}} = -(\overline{\sigma_{11} \phi_{1}^{11}} + \overline{\sigma_{12} \phi_{1}^{12}} + \overline{\sigma_{21} \phi_{1}^{21}} + \overline{\sigma_{22} \phi_{1}^{22}}) \]
are non-zero but linearly dependent over \( \mathbb{C} \) and thus a non-zero linear combination of \( \{ \phi^{1T}, \phi^{1T}, \phi^{2T}, \phi^{3T} \} \) is in \( \text{Im}(\partial) + \text{Im}(\bar{\partial}) \). Hence, \( k^{1,1} = 3 \) in classes \((ii.a)\) and \((iii.a)\).

For classes \((ii.b)\) and \((iii.b)\),
\[ \partial \phi^{3} = \sigma_{11} \phi_{1}^{11} + \sigma_{12} \phi_{1}^{12} + \sigma_{21} \phi_{1}^{21} + \sigma_{22} \phi_{1}^{22} \]
and
\[ \overline{\partial \phi^{3}} = -(\overline{\sigma_{11} \phi_{1}^{11}} + \overline{\sigma_{12} \phi_{1}^{12}} + \overline{\sigma_{21} \phi_{1}^{21}} + \overline{\sigma_{22} \phi_{1}^{22}}) \]
are non-zero and linearly independent over \( \mathbb{C} \). Thus two linearly independent non-zero linear combinations of \( \{ \phi^{1T}, \phi^{1T}, \phi^{2T}, \phi^{3T} \} \) are in \( \text{Im}(\partial) + \text{Im}(\bar{\partial}) \). Hence, \( k^{1,1} = 4 - 2 = 2 \) in classes \((ii.b)\) and \((iii.b)\).

Clearly, \( k^{3,0} = h^{0,3}_K = 1 \) for all the classes of deformations of the Iwasawa manifold being considered.
We summarize the $k^{i,j}$ with the table,

| $k^{i,j}$ | $k^{1,0}$ | $k^{1,1}$ | $k^{2,0}$ | $k^{3,0}$ |
|-----------|-----------|-----------|-----------|-----------|
| (i)       | 2         | 4         | 4         | 2         |
| (ii.a)    | 2         | 3         | 4         | 1         |
| (ii.b)    | 2         | 2         | 4         | 1         |
| (iii.a)   | 2         | 3         | 4         | 0         |
| (iii.b)   | 2         | 2         | 4         | 0         |

Let us calculate $h_{BC}^{1,1}$ for all the classes of small Kuranishi deformations of $\mathcal{G}_3$. We have

$$h_{BC}^{1,1} = 2h_{\delta}^{0,1} - b_1 + \dim_{\mathbb{R}}(H_{dR}^{1,1}(\mathbb{R})) .$$

One can check that for all five classes of deformations we have that

$$\mathbb{R} < i\phi_{t_1}, i\phi_{t_2}^2, \phi_{t_1}^2 - \phi_{t_1}^2, i(\phi_{t_1}^{12} + \phi_{t_2}^{21}) > \subseteq H_{dR}^{1,1}(\mathbb{R}) .$$

Thus, $\dim_{\mathbb{R}}(H_{dR}^{1,1}(\mathbb{R})) \geq 4$. We wish to show $\dim_{\mathbb{R}}(H_{dR}^{1,1}(\mathbb{R})) = 4$ for all the classes of deformations. Any $d$-closed $1,1$-form must also be $\bar{\partial}$-closed and $\partial$-closed. Such a $1,1$-form independent of

$$\mathbb{R} < i\phi_{t_1}, i\phi_{t_2}^2, \phi_{t_1}^2 - \phi_{t_1}^2, i(\phi_{t_1}^{12} + \phi_{t_2}^{21}) >$$

will be written as

$$\psi = \alpha \phi_{13}^{13} + \beta \phi_{23}^{23} + \gamma \phi_{33}^{33} + \delta \phi_{32}^{32} + \epsilon \phi_{33}^{33}$$

where the $\alpha, \beta, \gamma, \delta$ and $\epsilon$ are global smooth complex valued functions.

Applying $\bar{\partial}$ and $\partial$-closedness and projecting onto $\mathbb{C} < \phi_{112}, \phi_{212}, \phi_{121}, \phi_{122} >$ gives the matrix equation

$$
\begin{pmatrix}
-\sigma_{12} & 0 & -\sigma_{12} & -\sigma_{11} \\
0 & -\sigma_{12} & -\sigma_{22} & -\sigma_{21} \\
\sigma_{12} & -\sigma_{11} & \sigma_{12} & 0 \\
\sigma_{22} & -\sigma_{21} & 0 & \sigma_{12}
\end{pmatrix}
\begin{pmatrix}
\alpha \\
\beta \\
\gamma \\
\delta
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix} .
$$

We note that this is similar to the equation in Angella[p.47]. The matrix above has rank 4 for small deformations in classes (i), (ii), and (iii). Thus we must have $\alpha = \beta = \gamma = \delta = 0$. We then can easily show that $\epsilon = 0$:

$$\partial(\epsilon \phi_{33}^{33}) = (\partial \epsilon) \wedge \phi_{33}^{33} - \epsilon \phi_{123}^{123} + \epsilon \phi_{3}^{3} \wedge (\bar{\partial} \phi_{3}^{3}) = 0 .$$
Each of these three terms are linearly independent, i.e. the only term, when all expanded, in $\phi^{123}$ is the middle term. Thus we must have $\epsilon = 0$. Hence we conclude that for the Iwasawa manifold and its small deformations in the Kuranishi family, $\dim_{\mathbb{R}}(H_{dR}^{1,1}(\mathbb{R})) = 4$. One can see that we obtain for all the classes of deformation $(i)$, $(ii)$, and $(iii)$,

$$h^{1,1}_{BC} = 2h^{0,1}_{\partial} - b^1 + \dim_{\mathbb{R}}(H_{dR}^{1,1}(\mathbb{R}))$$

$$= 2 \times 2 - 4 + 4$$

$$= 4$$

This agrees with Angella\[5\] p.49.

Using this value of $h^{1,1}_{BC}$, the values for $k^{i,j}$ in the table above, Angella’s calculations of $h^{i,j}_{\partial}$ in the above table for $h^{i,j}_{BC}$, we get agreement with Angella’s calculation of the rest of the Bott-Chern cohomology. For example,

$$h^{2,2}_{BC} = -h^{1,1}_{BC} + h^{0,1}_{\partial} - h^{0,2}_{\partial} - h^{1,0}_{\partial} + h^{1,1}_{\partial} + h^{1,2}_{\partial} + h^{2,0}_{\partial}$$

$$+ 2h^{3,1}_{2} + k^{1,1} - k^{1,2} - 2k^{2,0}$$

For class $(i)$ this is

$$h^{2,2}_{BC} = -4 + 2 - 2 - 3 + 6 + 6 + 3$$

$$+ 2(2) + 4 - 4 - 2(2)$$

$$= 8.$$  

For class $(ii.a)$ this is

$$h^{2,2}_{BC} = -4 + 2 - 2 - 2 + 5 + 5 + 2$$

$$+ 2(2) + 3 - 4 - 2(1)$$

$$= 7.$$  

For class $(ii.b)$ this is

$$h^{2,2}_{BC} = -4 + 2 - 2 - 2 + 5 + 5 + 2$$

$$+ 2(2) + 2 - 4 - 2(1)$$

$$= 6.$$  

For class $(iii.a)$ this is

$$h^{2,2}_{BC} = -4 + 2 - 2 - 2 + 5 + 4 + 1$$

$$+ 2(2) + 3 - 4 - 2(0)$$

$$= 7.$$
For class \( (iii.b) \) this is
\[
\begin{align*}
 h^{2,2}_{BC} &= -4 + 2 - 2 + 5 + 4 + 1 + 2(2) + 2 - 4 - 2(0) \\
&= 6 .
\end{align*}
\]
Let us also also calculate \( h^{1,2}_{BC} \), for each of the three classes \((i), \( (ii), \) and \((iii)\).
\[
\begin{align*}
 h^{1,2}_{BC} &= h^{1,2}_\partial + h^{3,1}_2 - k^{2,0} \\
\end{align*}
\]
For class \((i)\), this is
\[
\begin{align*}
 h^{1,2}_{BC} &= 6 + 2 - 2 \\
&= 6 .
\end{align*}
\]
For class \((ii)\), this is
\[
\begin{align*}
 h^{1,2}_{BC} &= 5 + 2 - 1 \\
&= 6 .
\end{align*}
\]
And, for class \((iii)\), this is
\[
\begin{align*}
 h^{1,2}_{BC} &= 4 + 2 - 0 \\
&= 6 .
\end{align*}
\]
References

[1] A. Aeppli, *On the cohomology structure of Stein manifolds*, Proc. Conf. Complex Analysis (Minneapolis, Minn., 1964), Springer, Berlin, 1965, pp. 5870.

[2] Michael Albanese [http://mathoverflow.net/users/21564/michael-albanese], Are there compact complex manifolds with non-constant pluriclosed functions?, URL (version: 2017-02-24): http://mathoverflow.net/q/263003

[3] B. Bigolin, Gruppi di Aeppli, Ann. Scuola Norm. Sup. Pisa (3) 23 (1969), no. 2, 259287.

[4] R. Bott and S. S. Chern, *Hermitian vector bundles and the equidistribution of the zeroes of their holomorphic sections*, Acta Math. 114 (1965), no. 1, 71112.

[5] D. Angella, *Cohomological Aspects of Non-Kahler Manifolds*, Ph.D. Thesis, Universita di Pisa, [arXiv:1302.0524] [math.DG] .

[6] D. Angella and A. Tomassini, On Bott-Chern cohomology and formality. Journal of Geometry and Physics 93 (2015) 52-61.

[7] J.R. Brown, *Properties of Exotic Complex Structures on \( \mathbb{C}P^3 \)*, MATHEMATICA BOHEMICA, 132, (2007), pp. 59-74

[8] J. Cirici, A Short Course on the Interactions of Rational Homotopy Theory and Complex (Algebraic) Geometry. [PDF] from fu-berlin.de

[9] L. Codero, M. Fernandez, L. Ugarte, and A. Gray, A general description of the terms in the Frolicher spectral sequence, Differential Geometry and Its Applications 7 (1997) 75-84

[10] R. Friedman, *On Threefolds with Trivial Canonical Bundle*, Proceeding of Symposia on Pure Mathematics, Volume 53, (1991)
[11] P. Gauduchon, Le theoreme de l’excencierte nulle, C.R. Acad.Sc. Paris, Serie A, t. 285 (1977), 387-390.
[12] A. Gray, A property of a hypothetical complex structure on the six sphere. Bol. Un. Mat. Ital. Suppl. fasc., 2:251-255, 1997.
[13] Lu, P. and Tian, G., The complex structures on connected sums of $S^3 \times S^3$. Manifolds and geometry (Pisa, 1993), 284293, Sympos. Math., XXXVI, CUP.
[14] A. McHugh, Narrowing Cohomology on Complex $S^6$. European Journal of Pure and Applied Mathematics, Vol 10, No. 3 (2017).
[15] M. L. Michelsohn, On the existence of special metrics in complex geometry. Acta Math.,149, (1982), no. 1, 261295.
[16] I. Nakamura, Complex parallelisable manifolds and their small deformations, J. Differ. Geom. 10 (1975), 85112.
[17] D. Popovici, Aeppli Cohomology Classes Associated with Gauduchon Metrics on Compact Complex Manifolds. arXiv e-print DG 1310.3685v1
[18] D. Popovici and L. Ugarte, The SGG Class of Compact Complex Manifolds arXiv:1407.5070 [math.DG]
[19] M. Schweitzer, Autour de la cohomologie de Bott-Chern, Prepublication de l’Institut Fourier no. 703 (2007), arXiv:0709.3528v1 [math.AG].
[20] V. Tosatti, " 483-1 Algebraic Geometry, Northwestern University, Solution of Homework 6", www.math.northwestern.edu/~tosatti/hw6_ag_sol.pdf.
[21] L.S. Tseng, S.T. Yau, Non-Kahler Calabi-Yau Manifolds, Proceedings of Symposia in Pure Mathematics, Volume 85, 2012
[22] L. Ugarte, Hodge numbers of a hypothetical complex structure on the six sphere. Geom. Dedicata, 81:173-179, 2000.
[23] R.O. Wells, Differential analysis on complex manifolds, second ed., Graduate Texts in Mathematics, vol. 65, Springer, New York, 1979.