Estimation of general parameters under stratified adaptive cluster sampling based on dual use of auxiliary information

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Abstract. Auxiliary information is used mostly in conjunction with study variables to enhance the efficiency of estimators for population mean, total, and variance. Thompson introduced adaptive cluster sampling as an appropriate sampling scheme for rare and clustered populations. This paper presents difference-type and difference-cum-exponential-ratio-type estimators utilizing two auxiliary variables for estimating general parameters under stratified adaptive cluster sampling. The proposed estimators utilize auxiliary information in terms of ranks, variances, and means of auxiliary variables in hth stratum. Expressions for bias and mean square error of the proposed estimators are derived using first-order approximation. This numerical study aims to evaluate the performance of the proposed estimators.

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1. Introduction

Thompson [1] introduced Adaptive Cluster Sampling (ACS) as an efficient sampling scheme under rare, hidden, and clustered population such as drug addicts, endangered species of animals, fisheries, contagious diseases, rare and precious plants, minerals, and natural resources. ACS begins by taking an initial sample using traditional sampling designs, e.g., simple random sampling with or without replacement, systematic and strip ACS, stratified sampling, inverse sampling, ranked set sampling, two-stage sampling, partial systematic sampling, double sampling, sampling via probability proportional to size, and simple Latin square sampling. Then, the sample using information of neighboring units, which satisfies the pre-specified condition, adaptively increases. Recent advances in the area of ACS include the works of Chutman et al. [2], Gattone et al. [3], Yasmeen and Thompson [4], Qureshi et al. [5], Bakh [6], Younis and Shabbir [7-10].

In case of a sufficient correlation between the study and auxiliary variables, auxiliary information is used to enhance the precision of estimators. Haq et al. [11] and Shabbir [12] proposed that ranks of auxiliary variables could also be used to increase the efficiency of estimators. In this article, difference-type and difference-cum-exponential-ratio-type estimators were presented utilizing two auxiliary variables to estimate general parameters under Stratified Adaptive Cluster Sampling (SACS). Estimators are proposed assuming that population parameters are known for one auxiliary variable (say $y$) and unknown for another auxiliary variable (say $x$). We adopt the two-phase sampling scheme using ACS as follows:

1. In Phase 1, a large sample of size $n'$ is drawn and information on the auxiliary variables ($x$ and $y$) is recorded;
2. In Phase 2, a sub sample of size $n$ is drawn...
from phase one $n'$ and information on the study variable ($y$) and the auxiliary variables ($x$ and $z$) is accessible.

1.1. Symbols and Notations

Consider a finite population of $N$ units partitioned into $L$ strata such that:

$$\sum_{h=1}^{L} N_h = N.$$  

Let $y_{hi}$ and $(x_{hi}, z_{hi})$, be the observed values of the study variable $y$ and the auxiliary variables $(x, z)$, respectively, in the $h$th stratum. Let $r(x_{hi}), r(z_{hi})$ be the ranks of two auxiliary variables $(x, z)$ in the $h$th stratum.

Let $\bar{y}_{wh}$, $\bar{x}_{wh}$, $\bar{z}_{wh}$, $\bar{R}(w_u)_h$, and $\bar{R}^*(w_u)_h$, be the sample means corresponding to population means $\bar{y}_{wh}$, $\bar{x}_{wh}$, $\bar{z}_{wh}$, $\bar{R}(w_u)_h$, and $\bar{R}^*(w_u)_h$, respectively, in the $h$th stratum. Let $s^2_{w_{uh}}, s^2_{w_{zu}},$ and $s^2_{w_{yz}}$ be the sample variances corresponding to the population variances $S^2_{w_{uh}}, S^2_{w_{zu}}$, and $S^2_{w_{yz}}$, respectively, in the $h$th stratum. Also, let $\hat{C}_{w_{uh}}, \hat{C}_{w_{zu}},$ and $\hat{C}_{w_{yz}}$ be the sample coefficients of variation corresponding to population coefficients of variation $C_{w_{uh}}, C_{w_{zu}},$ and $C_{w_{yz}}$, respectively, in the $h$th stratum.

The following notations are used:

$$\bar{R}(w_u)_h = \sum_{h=1}^{L} \frac{N_h}{N} \bar{R}(w_u)_h, \quad \bar{R}^*(w_u)_h = \frac{1}{N_h} \sum_{i=1}^{N_h} r(w_{ui})_h \bar{R}(w_u)_h = \frac{1}{n_h} \sum_{i=1}^{n_h} r(w_{ui})_h,$$

$$S^2_{r(w_u)_h} = \frac{1}{N_h} \sum_{i=1}^{N_h} \left\{ r(w_{ui})_h - \bar{R}(w_u)_h \right\}^2, \quad C_r(w_{uh}).$$

Error terms are defined as:

$$\zeta_{wh} = \frac{y_{wh} - \bar{y}_{wh}}{\bar{y}_{wh}}, \quad \zeta_{z_{wh}} = \frac{z_{wh} - \bar{z}_{wh}}{\bar{z}_{wh}},$$

$$\zeta_{x_{wh}} = \frac{x_{wh} - \bar{x}_{wh}}{\bar{x}_{wh}},$$

$$\zeta_{x_{wh}}^* = \frac{x_{wh} - \bar{x}_{wh}}{\bar{x}_{wh}}, \quad \zeta_{z_{wh}}^* = \frac{z_{wh} - \bar{z}_{wh}}{\bar{z}_{wh}},$$

$$\zeta_{x_{wh}} = \frac{x_{wh} - \bar{x}_{wh}}{\bar{x}_{wh}}, \quad \zeta_{z_{wh}} = \frac{z_{wh} - \bar{z}_{wh}}{\bar{z}_{wh}}.$$

such that:

$$E(\zeta_{iwh}) = 0 \quad \forall i = 1, 2, 3, 4, 5, 6, 7.$$

$$E(\zeta_{iwh}^*) = \theta_h \lambda^2_{iwh},$$

$$E(\zeta_{iwh}^*) = \theta_h \lambda^2_{iwh}.$$
where:

\[ E(\hat{\theta}_h) = E(\hat{\theta}_h \mid \hat{\theta}_h) = E(\hat{\theta}_h \mid \hat{\theta}_h) = \theta_h, E(\hat{\theta}_h). \]

\[ \theta_h = \left( \frac{1}{n_h} - \frac{1}{N_h} \right), \]

\[ C_{w,h}(w_h) = C_{w,h} C_{w,h} \rho_{w_r, w_h}, \]

\[ C_w, r(w_h) = C_w, r(w_h) \rho_{w_r, w_h}, \]

\[ \theta_h = \theta_h - \frac{1}{N_h}, \]

\[ C_{w,h} w_h = C_{w,h} C_w, w_h \rho_{w,w_h}, \]

\[ C_w, w_h = C_w, w_h \rho_{w,w_h}, \]

\[ \lambda_{v_{vh}} = \frac{\mu_{v_{vh}}}{\mu_{v_{000000}} \mu_{v_{000200}} \mu_{v_{000200}} \mu_{v_{000000}}}, \]

\[ C_r(w_h, r(w_h)) = C_r(w_h) C_r(w_h) \rho_{r_r, r_h}, \]

\[ \lambda_{v_{vh}} = \lambda_{v_{vh}} - 1, \]

\[ C_{w,r}(w_h) = C_{w,r} C_r(w_h) \rho_{w_r, r_h}, \]

\[ C_{w,r}(w_h) = C_{w,r} C_r(w_h) \rho_{w_r, r_h}, \]

\[ \mu_{v_{vh}} = \frac{1}{N_h} - \sum_{i=1}^{N_h} \left\{ \left( w_h \bar{X}_{w_h} \right)^r \left( w_h \bar{X}_{w_h} \right)^r \left( w_h \bar{R}_{w_h} \right)^r \left( w_h \bar{R}_{w_h} \right)^r \right\}. \]

2. Existing estimators

2.1. Estimators for population mean

Some of the existing estimators for population mean under Simple Random Sampling (SRS) using two auxiliary variables are discussed in this section under SACS.

1. Usual sample mean in SACS is given by:

\[ t_{S-1m} = \frac{1}{N} \sum_{h=1}^{L} N_h \bar{Y}_{y_h}. \] (1)

The Mean Square Error (MSE) of \( t_{S-1m} \) to the first-order approximation is given by:

\[ MSE(t_{S-1m}) \simeq \frac{1}{N^2} \sum_{h=1}^{L} N_h^2 \theta_h \bar{Y}_{w_h} \bar{C}_{w_h}. \] (2)

2. Usual ratio estimator for population mean in SACS is given by:

\[ t_{S-2m} = \frac{1}{N} \sum_{h=1}^{L} N_h \bar{Y}_{y_h} \left( \frac{\bar{Y}_{y_h}}{\bar{Y}_{w_h}} \right) \left( \frac{\bar{Y}_{w_h}}{\bar{Y}_{w_h}} \right). \] (3)

The bias and MSE of \( t_{S-2m} \) to first-order approximation are given by:

\[ \text{Bias}(t_{S-2m}) \simeq \frac{1}{N} \sum_{h=1}^{L} N_h \bar{Y}_{w_h} \left[ \theta_h \left( C_{w_h}^2 - C_{w_h} w_h \right) + \theta_h' \left( C_{w_h}^2 - C_{w_h} w_h \right) \right]. \] (4)

and:

\[ MSE(t_{S-2m}) \simeq \frac{1}{N^2} \sum_{h=1}^{L} N_h^2 \bar{Y}_{w_h} \left[ \theta_h C_{w_h}^2 + \theta_h' \left( C_{w_h}^2 - 2C_{w_h} w_h \right) \right]. \] (5)

3. Traditional exponential ratio-type estimator for population mean in SACS is given by:

\[ t_{S-3m} = \frac{1}{N} \sum_{h=1}^{L} N_h \bar{Y}_{y_h} \exp \left( \frac{\bar{Y}_{w_h}}{\bar{Y}_{w_h} + \bar{Y}_{w_h}} \right) \exp \left( \frac{\bar{Y}_{w_h}}{\bar{Y}_{w_h} + \bar{Y}_{w_h}} \right). \] (6)

The bias and MSE of \( t_{S-3m} \) to first-order approximation are given by:

\[ \text{Bias}(t_{S-3m}) \simeq \frac{1}{N} \sum_{h=1}^{L} N_h \bar{Y}_{w_h} \left[ \theta_h \left( \frac{3C_{w_h}^2}{8} - \frac{C_{w_h} w_h}{2} \right) \right. \left. + \theta_h' \left( \frac{3C_{w_h}^2}{8} - \frac{C_{w_h} w_h}{2} \right) \right]. \] (7)

and:

\[ MSE(t_{S-3m}) \simeq \frac{1}{N^2} \sum_{h=1}^{L} N_h^2 \bar{Y}_{w_h} \left[ \theta_h C_{w_h}^2 + \theta_h' \left( C_{w_h}^2 + C_{w_h} w_h \right) \right]. \] (8)

4. Traditional difference-type estimator for population mean in SACS is given by:
\[ t_{S_{-4m}} = \frac{1}{N} \sum_{h=1}^{L} N_h \left[ \bar{w}_{yh} + k_{1h} \left( \bar{w}_{xh} - \bar{w}_{zh} \right) \right. \]
\[ + k_{2h} \left( \bar{Z}_{wh} - \bar{w}_{zh} \right) \left. \right] , \tag{9} \]

where \( k_{1h} \) and \( k_{2h} \) are constants. The MSE of \( t_{S_{-4m}} \) to first-order approximation is given by:

\[ MSE(t_{S_{-4m}})_{\min} \approx \frac{1}{N^2} \sum_{h=1}^{L} N_h^2 \bar{w}_{wh} C_{wh}^2 \left( \theta_h - \theta_h^\prime \rho_{w,w,h} - \theta_h^\prime \rho_{z,w,h} \right)^2. \tag{10} \]

where:

\[ k_{1h,\text{opt}} = \frac{\bar{Y}_{wh} C_{wh} \rho_{w,w,h}}{F_{wh} C_{wh}} , \]

\[ k_{2h,\text{opt}} = \frac{\bar{Y}_{wh} C_{wh} \rho_{w,w,h}}{Z_{wh} C_{wh}} . \]

5. Based on Shabbir and Gupta [13], the exponential ratio-type estimator for population mean in SACS is given by:

\[ t_{S_{-5m}} = \frac{1}{N} \sum_{h=1}^{L} N_h \bar{w}_{yh} \exp \left( \frac{\xi_{th} \left( \bar{w}_{xh} - \bar{w}_{zh} \right)}{\bar{w}_{zh} + (k_{th} - 1) \bar{w}_{zh}} \right) \]
\[ \exp \left( \frac{\xi_{th} \left( \bar{w}_{xh} - \bar{v}_{zh} \right)}{\bar{w}_{zh} + (k_{th} - 1) \bar{w}_{zh}} \right) \exp \left( \frac{\xi_{th} \left( \bar{w}_{xh} - \bar{Z}_{wh} \right)}{\bar{w}_{zh} + (k_{th} - 1) \bar{Z}_{wh}} \right) , \tag{11} \]

where \( \xi_{th}, \xi_{th}, \xi_{th} \) and \( (k_{th}, k_{th}, k_{th}) \) are constants. The bias and minimum MSE of \( t_{S_{-5m}} \) at optimum values of constants to first-order approximation are given by:

\[ Bias(t_{S_{-5m}}) \approx \frac{1}{N^2} \sum_{h=1}^{L} N_h F_{wh} \left[ \theta_h^\prime \left( \frac{C_{wh}}{k_{th}} - \frac{C_{w,w,h}^2}{2 k_{th}^2} \right) \right. \]
\[ + \theta_h^\prime \left( \frac{C_{w,w,h}}{k_{th}} - \frac{C_{w,w,h}^2}{2 k_{th}^2} \right) \]
\[ + \left. \frac{C_{w,w,h}^2}{k_{th}} - \frac{C_{w,w,h}^2}{2 k_{th}^2} + \frac{C_{w,w,h}^2}{k_{th}^2} \cdot \left] \right) \tag{12} \]

and:

\[ MSE(t_{S_{-5m}})_{\min} \approx \frac{1}{N^2} \sum_{h=1}^{L} N_h^2 \bar{w}_{wh} C_{wh}^2 \left[ \theta_h - \theta_h^\prime \rho_{w,w,h} - \theta_h^\prime \rho_{z,w,h} \right] \left( \rho_{w,w,h} \right)^2 \tag{13} \]

where:

\[ (\xi_{k})_{3h,\text{opt}} = \frac{C_{w,w,h}}{C_{w,w,h} (1 - \rho_{w,w,h})} , \]
\[ (\xi_{k})_{4h,\text{opt}} = \frac{C_{w,w,h} - \rho_{w,w,h} \rho_{w,w,h}}{C_{w,w,h} (1 - \rho_{w,w,h})} , \]
\[ (\xi_{k})_{5h,\text{opt}} = \frac{-C_{w,w,h}}{C_{w,w,h}} . \]

6. Based on Gupta and Shabbir [14], the ratio-type estimator for population mean in SACS is given by:

\[ t_{S_{-6m}} = \frac{1}{N} \sum_{h=1}^{L} N_h \bar{w}_{yh} \exp \left( \frac{\xi_{th} \left( \bar{w}_{xh} - \bar{w}_{zh} \right)}{\bar{w}_{zh} + (k_{th} - 1) \bar{w}_{zh}} \right) \]
\[ \exp \left( \frac{\xi_{th} \left( \bar{w}_{xh} - \bar{v}_{zh} \right)}{\bar{w}_{zh} + (k_{th} - 1) \bar{w}_{zh}} \right) \exp \left( \frac{\xi_{th} \left( \bar{w}_{xh} - \bar{Z}_{wh} \right)}{\bar{w}_{zh} + (k_{th} - 1) \bar{Z}_{wh}} \right) , \tag{14} \]

where \( (k_{th}, k_{th}) \) are constants. The bias and minimum MSE of \( t_{S_{-6m}} \) at optimum values of constants to first-order approximation are given by:

\[ Bias(t_{S_{-6m}}) \approx \frac{1}{N^2} \sum_{h=1}^{L} N_h F_{wh} \left[ \theta_h^\prime \left( \frac{k_{th} (k_{th} + 1)}{2} \right) \right. \]
\[ \left. \left( C_{w,w,h} - \frac{C_{w,w,h}^2}{2 k_{th}^2} \right) \right] \]
\[ + \theta_h^\prime \left( \frac{k_{th} (k_{th} + 1)}{2} \right) \left( C_{w,w,h} - \frac{C_{w,w,h}^2}{2 k_{th}^2} \right) \right\} \tag{15} \]

and:

\[ MSE(t_{S_{-6m}})_{\min} \approx \frac{1}{N^2} \sum_{h=1}^{L} N_h^2 \bar{w}_{wh} C_{wh}^2 \left[ \theta_h - \theta_h^\prime \rho_{w,w,h} - \theta_h^\prime \rho_{z,w,h} \right] \left( \rho_{w,w,h} \right)^2 \tag{16} \]

where:

\[ k_{th,\text{opt}} = \frac{C_{w,w,h}}{C_{w,w,h}} \quad \text{and} \quad k_{th,\text{opt}} = \frac{C_{w,w,h}}{C_{w,w,h}} . \]

7. According to Singh et al. [15], the exponential ratio-type estimator for population mean in SACS is given by:

\[ t_{S_{-7m}} = \frac{1}{N} \sum_{h=1}^{L} N_h \bar{w}_{yh} \exp \left( \frac{\xi_{th} \left( \bar{w}_{xh} - \bar{w}_{zh} \right)}{\bar{w}_{zh} + (k_{th} - 1) \bar{w}_{zh}} \right) \]
\[ \exp \left( \frac{\xi_{th} \left( \bar{w}_{xh} - \bar{v}_{zh} \right)}{\bar{w}_{zh} + (k_{th} - 1) \bar{w}_{zh}} \right) \exp \left( \frac{\xi_{th} \left( \bar{w}_{xh} - \bar{Z}_{wh} \right)}{\bar{w}_{zh} + (k_{th} - 1) \bar{Z}_{wh}} \right) , \tag{17} \]

where \( (k_{th}, k_{th}) \) are constants. The bias and mini-
Minimum MSE of $t_{S^{-\gamma}m}$ at optimum values of constants to first-order approximation are given by:

$$
Bias(t_{S^{-\gamma}m}) \cong \frac{1}{N} \sum_{h=1}^{L} N_h V_{wh} \left[ \theta_h \left( \frac{k_{wh}}{2} C_{w,wh} \right) + \frac{k_{wh}}{4} C_{w,wh} - \frac{k_{wh}}{2} C_{w,wh,h} \right] + \theta_h' \left( \frac{k_{wh}}{8} C_{w,wh} - \frac{k_{wh}}{4} C_{w,wh} + \frac{k_{wh}}{2} C_{w,wh,h} \right),
$$

and:

$$
MSE(t_{S^{-\gamma}m})_{\min} \cong \frac{1}{N^2} \sum_{h=1}^{L} N_h^2 V_{wh}^2 C_{w,wh,h} \left[ \theta_h - \theta_h' \rho_{w,wh,h} - \theta_h' \rho_{w,wh,h} \right],
$$

where:

$$
k_{wh,\text{opt}} = \frac{2C_{w,wh,h}}{C_{w,wh,h}} \quad \text{and} \quad k_{wh,\text{opt}} = -\frac{2C_{w,wh,h}}{C_{w,wh,h}}.
$$

9. Hamad et al. [17] presented the difference ratio-type estimator for population mean in SACS below:

$$
t_{S^{-\delta}m} = \frac{1}{N} \sum_{h=1}^{L} N_h \left[ \bar{y}_{wh} + k_{11h} \left( \bar{w}_{wh} - \bar{w}_{wh} \right) \right] + \left[ k_{12h} \frac{\bar{w}_{wh}}{\bar{w}_{wh}} + \left( 1 - k_{12h} \right) \frac{\bar{w}_{wh}}{\bar{w}_{wh}} \right],
$$

where $(k_{11h}, k_{12h})$ are constants. The bias and minimum MSE of $t_{S^{-\delta}m}$ at optimum values of constants to first-order approximation are given by:

$$
Bias(t_{S^{-\delta}m}) \cong \frac{1}{N} \sum_{h=1}^{L} N_h V_{wh} \left[ \theta_h \left( \frac{C_{w,wh,h}}{C_{w,wh,h}} - 2C_{w,wh,h} \right) \right] + \theta_h' \left( C_{w,wh,h} + k_{11h} \left( C_{w,wh,h}^2 - 2C_{w,wh,h} \right) \right) + \theta_h' \left( \frac{C_{w,wh,h} + k_{11h} \left( C_{w,wh,h}^2 - 2C_{w,wh,h} \right) }{C_{w,wh,h}} \right),
$$

and:

$$
MSE(t_{S^{-\delta}m})_{\min} \cong \frac{1}{N^2} \sum_{h=1}^{L} N_h^2 V_{wh}^2 \left( \frac{A_h + F_h G_h}{B_h D_h} - F_h H_h^2 \right),
$$

where:

$$
A_h = \theta_h C_{w,wh}^2 + \theta_h' \left( 2C_{w,wh,h} + C_{w,wh,h} \right), \quad B_h = 4\theta_h C_{w,wh}^2, \quad F_h = 4\theta_h C_{w,wh,h}, \quad D_h = \theta_h' C_{w,wh,h} \quad G_h = 4\theta_h' \left( C_{w,wh,h} - C_{w,wh,h} \right), \quad k_{11h,\text{opt}} = \frac{\bar{w}_{wh} (2H_h B_h - F_h G_h)}{\bar{w}_{wh} (4B_h D_h - F_h^2)}, \quad H_h = 2\theta_h' \left( C_{w,wh,h} - C_{w,wh,h} \right),
$$

10. Chutiman [18], Yadav et al. [19], and Qureshi et al. [20] proposed the ratio-type estimator for population mean in SACS as follows:

$$
t_{S^{-10m}j} = \frac{1}{N} \sum_{h=1}^{L} N_h \bar{y}_{wh} \left( k_{13h,j} \bar{w}_{wh} + k_{14h,j} \bar{w}_{wh} \right) \left( k_{13h,j} \bar{w}_{wh} + k_{14h,j} \bar{w}_{wh} \right),
$$

where:

$$
A_h = \theta_h C_{w,wh,h}^2 + F_h, \quad B_h = \theta_h' C_{w,wh,h}^2 + \theta_h' C_{w,wh,h}, \quad k_{10h,\text{opt}} = F_h / B_h.
$$
Table 1. Combinations of constants for $t_{S-10m_j}$.

| $j$ | $k_{13h,j}$ | $k_{14h,j}$ | $k_{13h,j}$ | $k_{14h,j}$ |
|-----|-------------|-------------|-------------|-------------|
| 1   | $C_{w,h}$   | $C_{w,h}$   | $C_{w,h}$   | $C_{w,h}$   |
| 2   | $\beta_2(w,h)$ | $\beta_2(w,h)$ | $C_{w,h}$   | $C_{w,h}$   |
| 3   | $\beta_2(w,h)$ | $C_{w,h}$   | $\beta_2(w,h)$ | $C_{w,h}$   |
| 4   | $C_{w,h}$   | $\beta_2(w,h)$ | $C_{w,h}$   | $\beta_2(w,h)$ |

$\rightarrow$ [18]

| $j$ | $\rho_{w,w,h}$ | $\rho_{w,w,h}$ |
|-----|----------------|----------------|
| 5   | $\beta_2(w,h)$ | $\beta_2(w,h)$ |
| 6   | $\beta_2(w,h)$ | $\beta_1(w,h)$ | $\beta_2(w,h)$ |
| 7   | $\beta_2(w,h)$ | $\beta_2(w,h)$ | $\beta_1(w,h)$ | $\beta_2(w,h)$ |

$\rightarrow$ [20]

| $j$ | $MR(w,h)$ | $MR(w,h)$ | $MR(w,h)$ |
|-----|-----------|-----------|-----------|
| 8   | $\beta_1(w,h)$ | $MR(w,h)$ | $MR(w,h)$ |
| 9   | $TM(w,h)$ | $TM(w,h)$ | $TM(w,h)$ |
| 10  | $HL(w,h)$ | $HL(w,h)$ | $HL(w,h)$ |
| 11  | $TM(w,h)$ | $TM(w,h)$ | $TM(w,h)$ |

where $(k_{13h,j}, k_{14h,j}, k_{13h,j}, k_{14h,j})$ are the constants that assume different values for $j = 1, 2, ..., 11$ as given in Table 1.

The bias and MSE of $t_{S-10m_j}$ to first-order approximation are given by:

$$Bias(t_{S-10m_j}) = \frac{1}{N} \sum_{h=1}^{L} N_h \bar{Y}_{w,h}$$

$$MSE(t_{S-10m_j}) = \sum_{h=1}^{L} N_h \bar{Y}_{w,h}^2$$

$$\left[ \frac{3}{8} Q_{z,h} C_{w,h}^2 - \frac{1}{2} \bar{Q}_h C_{w,h}, h \right]$$

and:

$$MSE(t_{S-10m_j}) = \sum_{h=1}^{L} N_h \bar{Y}_{w,h}^2$$

$$\left[ \frac{3}{8} Q_{z,h} C_{w,h}^2 - \frac{1}{2} \bar{Q}_h C_{w,h}, h \right]$$

where:

$$Q_{z,h} = \frac{k_{13h,j} Z_{w,h}}{k_{13h,j} X_{w,h} + k_{14h,j}}$$

12. Singh and Khalid [22] gave the exponential ratio-type estimator for population mean in SACS below:

$$t_{S-12m} = \frac{1}{N} \sum_{h=1}^{L} N_h \bar{Y}_{w,h} \exp$$

$$\left( \frac{1}{2 \bar{Q}_h} \bar{Z}_{w,h} - \frac{1}{2 \bar{Q}_h} \bar{X}_{w,h} \right)$$

where:

$$Z_{w,h} = \frac{N_h Z_{w,h} - N_h \bar{Y}_{w,h}}{N_h \bar{Y}_{w,h}}$$

11. Vishwakarma and Gangele [21] presented the exponential ratio-type estimator for population mean in SACS:
\[MSE(t_{S-12m}) \approx \frac{1}{N^2} \sum_{h=1}^{L} N_h^2 \sigma_{w_{h}}^2 \]

\[\theta_C^2 \sigma_{w_{h}} + \theta_C^2 \left( \frac{C_{w_{h}}^2}{4} - C_{w_{h},w_{h}} \right) + \theta_C^2 \left( \frac{C_{w_{h}}^2}{4} - C_{w_{h},w_{h}} \right) \times (34)\]

where \( a_h = \frac{\gamma_n}{\gamma_{n+1}}. \)

13. Khan and AI-Hossain [23] proposed a difference-type estimator for population mean in SACS:

\[t_{S-13m} = \frac{1}{N} \sum_{h=1}^{L} N_h \left[ \bar{Z}_{w_{h}} + k_{17h} \left( \frac{\bar{Z}_{w_{h}}}{\bar{w}_{h}} - \bar{w}_{h} \right) \right. \]

\[+ k_{18h} \left( \frac{\bar{Z}_{w_{h}}}{\bar{w}_{h}} - \bar{w}_{h} \right) \times (35)\]

where \((k_{17h}, k_{18h})\) are constants. The bias and minimum MSE of \(t_{S-13m}\) at optimum values of constants to first-order approximation are given by:

\[Bias(t_{S-13m}) \approx \frac{1}{N} \sum_{h=1}^{L} N_h \bar{Z}_{w_{h}} \theta_C^2 \times (36)\]

and:

\[MSE(t_{S-13m}) \approx \frac{1}{N^2} \sum_{h=1}^{L} N_h^2 \sigma_{w_{h}}^2 \]

\[\theta_C^2 - \frac{1}{A_h B_h + E_h^2} \times (37)\]

where:

\[A_h = \theta_C^2 C_{w_{h}} + \theta_C^2 C_{w_{h},w_{h}}. \]

\[B_h = A_h + 2\theta_C^2 C_{w_{h},w_{h}}. \]

\[E_h = A_h + \theta_C^2 C_{w_{h},w_{h}}. \]

\[C_h = \theta_C^2 C_{w_{h},w_{h}} + \theta_C^2 C_{w_{h},w_{h}}. \]

\[k_{17h} = \frac{\bar{Z}_{w_{h}}(B_h C_h - D_h E_h)}{X_{w_{h}}(A_h B_h - E_h^2)} \times (38)\]

14. Based on Khan [24], the exponential-type estimator for population mean in SACS is given by:

\[t_{S-14m} = \frac{1}{N} \sum_{h=1}^{L} N_h \left[ \bar{w}_{y_{h}} \exp \left( \frac{\bar{Z}_{w_{h}}}{\bar{w}_{h}} - \bar{w}_{h} \right) \right. \]

\[+ k_{19h} \left( \frac{\bar{Z}_{w_{h}}}{\bar{w}_{h}} - \bar{w}_{h} \right) \times (39)\]

where \((k_{19h}, k_{20h})\) are constants. The bias and minimum MSE of \(t_{S-14m}\) at optimum values of constants to first-order approximation are given by:

\[Bias(t_{S-14m}) \approx \frac{1}{N} \sum_{h=1}^{L} N_h \bar{Z}_{w_{h}} \theta_C^2 \times (40)\]

and:

\[MSE(t_{S-14m}) \approx \frac{1}{N^2} \sum_{h=1}^{L} N_h^2 \sigma_{w_{h}}^2 \]

\[\theta_C^2 - \frac{1}{A_h B_h + E_h^2} \times (41)\]

where:

\[k_{19h} = \frac{4C_{w_{h},w_{h}}}{C_{w_{h},w_{h}} + \frac{1}{2} C_{w_{h},w_{h}}} \times \]

\[+ \frac{1}{8} \theta_C^2 C_{w_{h},w_{h}} \times (42)\]

15. According to Singh et al. [25], the ratio-type estimator for population mean in SACS is given by:

\[t_{S-15m} = \frac{1}{N} \sum_{h=1}^{L} N_h \bar{w}_{y_{h}} \left( k_{21h+j} Z_{w_{h}} + k_{22h+j} \bar{w}_{w_{h}} \right) \times \]

\[X_{w_{h}}(A_h B_h - E_h^2) \times (43)\]

where \((k_{21h+j}, k_{22h+j}, k_{23h+j}, k_{24h+j}, y_i = x, z)\) are constants that assume different values for \(j = 1, 2, 3\), as given in Table 2.

The bias and MSE of \(t_{S-15m}\) to the first-order approximation are given by:
Table 2. Combinations of constants for \( t_{S-15m_j} \).

| \( j \) | \( k_{21h,\delta_{ij}} \) | \( k_{23h,\delta_{ij}} \) | \( k_{23h,\delta_{ij}} \) | \( k_{31h,\delta_{ij}} \) | \( k_{22h,\delta_{ij}} \) | \( k_{32h,\delta_{ij}} \) | \( k_{24h,\delta_{ij}} \) |
|-----|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 1   | \( C_{w,h}^{2} \) | \( -\rho_{w,w,h} \) | \( C_{w,h}^{2} \) | \( -\rho_{w,w,h} \) | \( C_{w,h}^{2} \) | \( -\rho_{w,w,h} \) | \( C_{w,h}^{2} \) |
| 2   | \( \beta_{2}(w,h) \) | \( -C_{w,h} \) | \( -C_{w,h} \) | \( -C_{w,h} \) | \( -C_{w,h} \) | \( -C_{w,h} \) |
| 3   | \( C_{w,h}^{2} \) | \( C_{w,h}^{2} \) | \( C_{w,h}^{2} \) | \( C_{w,h}^{2} \) | \( C_{w,h}^{2} \) | \( C_{w,h}^{2} \) | \( C_{w,h}^{2} \) |

\[ \text{Bias}(t_{S-15m_j}) \approx \frac{1}{N} \sum_{h=1}^{L} N_h Y_{wh} \]

\[ \left[ -\theta_h C_{w,h} \delta_{1h} - \theta_h C_{w,h} \delta_{2h} \right] \]

\[ \left\{ q_{x1h} \delta_{1h} + q_{x2h} \delta_{2h} - q_{x1h} \delta_{2h} \right\} \]

\[ \left\{ q_{x2h} \delta_{1h} - q_{x2h} \delta_{2h} \right\} \]

and:

\[ \text{MSE}(t_{S-15m_j}) \approx \frac{1}{N^2} \sum_{h=1}^{L} N_h^2 \bar{Y}_{wh}^2 \]

\[ \theta_h \left\{ C_{w,h}^{2} + \delta_{1h} \delta_{2h} \right\} \]

\[ + \theta_h \left\{ C_{w,h}^{2} \left( \delta_{1h}^2 + 2 \delta_{1h} \delta_{2h} \right) + C_{w,h}^{2} \delta_{1h} \delta_{2h} \right\} \]

\[ - 2 \delta_{1h} C_{w,h} \delta_{2h} - 2 \delta_{2h} \delta_{3h} + C_{w,h} \delta_{3h} \]

\[ +\left( 2 \delta_{1h} \delta_{2h} + 2 \delta_{2h} \delta_{3h} \right) \}

(42)

where:

\[ q_{x1h} = \frac{k_{21h,\delta_{ij}}}{k_{21h,\delta_{ij}} + k_{22h,\delta_{ij}}} \]

\[ q_{x2h} = \frac{k_{22h,\delta_{ij}}}{k_{21h,\delta_{ij}} + k_{22h,\delta_{ij}}} \]

16. Shabbir and Gupta [26] presented the difference-cum-exponential ratio-type estimator for population mean in SACS as follows:

\[ t_{S-15m} = \frac{1}{N} \sum_{h=1}^{L} N_h \left\{ k_{25h} \bar{w}_{y} + k_{26h} \right\} \]

\[ \left( \bar{w}_{y} - \pi_{y,h} \right) + k_{27h} \left( Z_{wh} - \pi_{y,h} \right) \}

\[ \exp \left( \frac{\pi_{y,h} - \pi_{x,h}}{\bar{w}_{y} + \pi_{x,h}} \right) \]

(44)

where \( (k_{25h}, k_{26h}, k_{27h}) \) are constants. The bias and minimum MSE of \( t_{S-15m} \) at optimum values of constants to first-order approximation are given by:

\[ \text{Bias}(t_{S-15m}) \approx \frac{1}{N} \sum_{h=1}^{L} N_h \]

\[ \left[ k_{25h} \bar{Y}_{wh} \theta_h^2 \left\{ \frac{3}{8} C_{w,h}^{2} - \frac{1}{2} C_{w,h} \right\} \right. \]

\[ + k_{26h} \bar{X}_{wh} \theta_h^2 C_{w,h} + \bar{Y}_{wh} \left( k_{25h} - 1 \right) \]

(45)

and:

\[ \text{MSE}(t_{S-15m})_{\text{min}} \approx \frac{1}{N^2} \sum_{h=1}^{L} N_h \left( \frac{L_h - H_h^2}{B_h} \right) \]

(46)

where:

\[ A_h = 1 + \theta_h C_{w,h}^{2} + \theta_h \left( C_{w,h}^{2} - 2 C_{w,h} \right) \]

\[ F_h = \theta_h \left( C_{w,h}^{2} - C_{w,h} \right) \]

\[ E_h = \theta_h \left( C_{w,h}^{2} - C_{w,h} \right) \]

\[ k_{26h,w} = \frac{\bar{Y}_{wh} \left( E_{h,w} - F_h H_h \right)}{X_{wh} B_h L_h} \]

\[ B_h = \theta_h C_{w,h}^{2} \]

\[ D_h = 1 + \theta_h \left( \frac{3}{8} C_{w,h}^{2} - \frac{1}{2} C_{w,h} \right) \]
\[ H_h = D_h - \frac{E_h F_h}{B_h}, \quad C_h = \theta_h^2 C_{w,w_h}, \]
\[ G_h = \theta_h^2 C_{w,w_h}, \quad L_h = A_h - \frac{F_h^2}{B_h} - \frac{G_h^2}{C_h}, \]
\[ k_{27,\text{opt}} = \frac{Y_{wh} G_h H_h}{\sum_{h=1}^{L} C_h L_h}. \]

17. According to Muneer et al. [27], difference-cum-exponential estimators for population mean in SACS are given by:
\[
t_s^{17m} = \frac{1}{N} \sum_{h=1}^{L} N_h \left[ \left\{ k_{28h} \bar{y}_{gh} + k_{29h} \right\} \left( \bar{w}_{xh} - \bar{w}_{xh} \right) + \left( \frac{\bar{w}_{xh} - Z_{wh}}{\bar{w}_{xh} + Z_{wh}} \right) \right], \tag{47}
\]
and:
\[
t_s^{18m} = \frac{1}{N} \sum_{h=1}^{L} N_h \left[ \left\{ k_{30h} \bar{y}_{gh} + k_{31h} \right\} \left( \bar{w}_{xh} - \bar{w}_{xh} \right) \exp \left( \frac{Z_{wh} - \bar{w}_{xh}}{\bar{w}_{xh} + \bar{w}_{xh}} \right) \right], \tag{48}
\]
where \( k_{28h}, k_{29h}, k_{30h}, k_{31h} \) are constants. The bias and minimum MSE of \( t_s^{17m} \) and \( t_s^{18m} \) at optimum values of constants to first-order of approximation are given by:
\[
\text{Bias}(t_s^{17m}) \approx \frac{1}{N} \sum_{h=1}^{L} N_h \left[ Y_{wh} (k_{28h} - 1) + k_{28h} Y_{wh} \theta_h^2 \left( \frac{1}{8} C_{w,w_h}^2 - \frac{1}{2} C_{w,w_h} \right) \right], \tag{49}
\]
\[
\text{Bias}(t_s^{18m}) \approx \frac{1}{N} \sum_{h=1}^{L} N_h \left[ Y_{wh} (k_{30h} - 1) + k_{30h} Y_{wh} \theta_h^2 \left( \frac{3}{8} C_{w,w_h}^2 - \frac{1}{2} C_{w,w_h} \right) \right], \tag{50}
\]
\[
\text{MSE}(t_s^{17m})_{\text{min}} \approx \frac{1}{N} \sum_{h=1}^{L} N^2_h \left[ Y_{wh}^2 \frac{M_{1h} - B^2_{1h}}{M_{1h}} \right], \tag{51}
\]
and:
\[
\text{MSE}(t_s^{18m})_{\text{min}} \approx \frac{1}{N} \sum_{h=1}^{L} N^2_h \left[ Y_{wh}^2 \frac{M_{2h} - B^2_{2h}}{M_{2h}} \right], \tag{52}
\]
where:
\[ A_{1h} = 1 + \theta_h C_{w,w_h}^2 + \theta_h \left( \frac{1}{2} C_{w,w_h}^2 - 2 C_{w,w_h} \right), \]
\[ B_{1h} = 1 + \theta_h^2 \left( \frac{1}{8} C_{w,w_h}^2 - \frac{1}{2} C_{w,w_h} \right), \]
\[ k_{26h,\text{opt}} = \frac{B_{1h}}{M_{1h}}, \]
\[ A_{2h} = 1 + \theta_h C_{w,w_h}^2 + \theta_h \left( C_{w,w_h}^2 - 2 C_{w,w_h} \right), \]
\[ B_{2h} = 1 + \theta_h^2 \left( \frac{3}{8} C_{w,w_h}^2 - \frac{1}{2} C_{w,w_h} \right), \]
\[ D_h = C_{w,w_h}, \]
\[ F_h = C_{w,w_h}, \quad k_{30h,\text{opt}} = \frac{B_{2h}}{M_{2h}}, \]
\[ M_{1h} = A_{1h} - \theta_h^2 \frac{F_{2h}}{D_h}, \]
\[ k_{30h,\text{opt}} = \frac{Y_{wh} F_h B_{1h}}{Y_{wh} D_h M_{1h}}, \]
\[ M_{2h} = A_{2h} - \theta_h^2 \frac{F_{2h}}{D_h}, \]
\[ k_{31h,\text{opt}} = \frac{Y_{wh} F_h B_{2h}}{Y_{wh} D_h M_{2h}}. \]

18. Shabbir [12] found the difference-type estimator for population mean in SACS below:
\[
t_s^{19m} = \frac{1}{N} \sum_{h=1}^{L} N_h \left[ \bar{y}_{gh} + k_{32h} \{ \bar{w}_{ygh} - \bar{w}_{xh} \} \right. \]
\[ \left. + k_{32h} \{ \bar{w}_{ygh} - \bar{w}_{xh} \} \right] \tag{53}
\]
where \( k_{32h}, k_{33h}, k_{34h}, k_{35h} \) are constants. The bias and minimum MSE of \( t_s^{19m} \) at optimum values of constants to first-order of approximation are given by:
\[
\text{Bias}(t_s^{19m}) = 0, \tag{54}
\]
and:
\[
\text{MSE}(t_s^{19m})_{\text{min}} \approx \frac{1}{N} \sum_{h=1}^{L} N^2_h \left[ \theta_h^2 \frac{M_{1h} - B^2_{1h}}{M_{1h}} \right], \tag{55}
\]
where:
\[ R_{z_h} = \frac{\rho^2_{w, w, h} + \rho^2_{w, r(w), h} - 2 \rho_{w, w, h} \rho_{w, r(w), h} \rho_{w, r(w), h}}{1 - \rho^2_{w, r(w), h}} \]

\[ R_{\bar{z}_h} = \frac{\rho^2_{w, w, h} + \rho^2_{w, r(w), h} - 2 \rho_{w, w, h} \rho_{w, r(w), h} \rho_{w, r(w), h}}{1 - \rho^2_{w, r(w), h}} \]

\[ k_{\bar{z}_h, w_h} = \frac{S_{w_h} \{ \rho_{w, w, h} - \rho_{w, r(w), h} \rho_{w, r(w), h} \}}{S_{w_h} \left( 1 - \rho^2_{w, r(w), h} \right)} \]

\[ k_{\bar{z}_h, \bar{w}_h} = \frac{S_{w_h} \{ \rho_{w, r(w), h} - \rho_{w, r(w), h} \rho_{w, w, h} \}}{S_{w_h} \left( 1 - \rho^2_{w, r(w), h} \right)} \]

\[ k_{\bar{z}_h, \bar{w}_h} = \frac{S_{w_h} \{ \rho_{w, w, h} - \rho_{w, r(w), h} \rho_{w, r(w), h} \}}{S_{w_h} \left( 1 - \rho^2_{w, r(w), h} \right)} \]

\section*{2.2. Estimators for population variance}

Some of the existing estimators for population variance using two auxiliary variables are discussed in this section under SACS and SRS. Usual sample variance in SACS is given by:

1. Usual sample variance in SACS is given by:

\[ t_{S_{-1v}} = \frac{1}{\sqrt{N}} \sum_{h=1}^{L} N_h \hat{s}^2_{w_h}. \]  

The MSE of \( t_{S_{-1v}} \) to first-order approximation is given by:

\[ MSE(t_{S_{-1v}}) \approx \frac{1}{N^2} \sum_{h=1}^{L} N_h^2 \theta_h S^4_{w_h} \lambda^4_{100h}. \]  

2. Usual ratio estimator (using variance of auxiliary variables) for population variance in SACS is given as follows:

\[ t_{S_{-2v}} = \frac{1}{\sqrt{N}} \sum_{h=1}^{L} N_h \hat{s}^2_{w_h} \frac{s^2_{w_h}}{s^2_{w_h}} \left( \frac{s^2_{w_h}}{s^2_{w_h}} + s^2_{w_h} \right). \]  

The bias and MSE of \( t_{S_{-2v}} \) to first-order approximation are given by:

\[ Bias(t_{S_{-2v}}) \approx \frac{1}{N} \sum_{h=1}^{L} N_h S^2_{w_h} \]

\[ \left[ \theta'_h \left( \lambda^*_{040h} - \lambda^*_{220h} \right) + \theta'_h \left( \lambda^*_{040h} + \lambda^*_{220h} \right) \right]. \]  

and:

\[ MSE(t_{S_{-2v}}) \approx \frac{1}{N^2} \sum_{h=1}^{L} N_h^2 S^4_{w_h} \]

\[ \left[ \theta_h \lambda^*_{400h} + \theta'_h \left( \lambda^*_{040h} - 2 \lambda^*_{220h} \right) \right]. \]  

3. Usual ratio estimator (using mean of auxiliary variables) for population variance in SACS is given by:

\[ t_{S_{-3v}} = \frac{1}{\sqrt{N}} \sum_{h=1}^{L} N_h \hat{s}^2_{w_h} \left( \frac{\bar{w}_{wh}}{\bar{w}_{z_h}} \right) \left( \frac{\bar{Z}_{wh}}{\bar{Z}_{z_h}} \right). \]  

The bias and MSE of \( t_{S_{-3v}} \) to first-order approximation are given by:

\[ Bias(t_{S_{-3v}}) \approx \frac{1}{N} \sum_{h=1}^{L} N_h S^2_{w_h} \]

\[ \left[ \theta'_h \left( C^2_{w_h} - C_{w_h} \lambda_{210h} \right) + \theta'_h \left( C^2_{w_h} - C_{w_h} \lambda_{210h} \right) \right]. \]  

and:

\[ MSE(t_{S_{-3v}}) \approx \frac{1}{N^2} \sum_{h=1}^{L} N_h^2 S^4_{w_h} \]

\[ \left[ \theta_h \lambda^*_{400h} + \theta'_h \left( C^2_{w_h} - 2 C_{w_h} \lambda_{210h} \right) \right]. \]  

4. Traditional exponential ratio-type estimator for population variance in SACS is given by:

\[ t_{S_{-4v}} = \frac{1}{\sqrt{N}} \sum_{h=1}^{L} N_h \hat{s}^2_{w_h} \exp \left( \frac{s^2_{w_h} - s^2_{w_h}}{s^2_{w_h} + s^2_{w_h}} \right) \exp \left( \frac{s^2_{w_h} - s^2_{w_h}}{s^2_{w_h} + s^2_{w_h}} \right). \]  

The bias and MSE of \( t_{S_{-4v}} \) to first-order approximation are given by:

\[ Bias(t_{S_{-4v}}) \approx \frac{1}{N} \sum_{h=1}^{L} N_h S^2_{w_h} \]

\[ \left[ \theta'_h \left( \frac{3 \lambda^*_{040h}}{8} - \lambda^*_{220h} \right) + \theta'_h \left( \frac{3 \lambda^*_{0404h}}{8} - \lambda^*_{220h} \right) \right]. \]  

and:
\[ MSE(t_{S-\delta}) \approx \frac{1}{N^2} \sum_{h=1}^{L} N_h^2 S_{\upsilon_{wh}}^2 \]
\[ \left[ \theta_h \lambda_{0400h}^* + \theta''_h \left( \frac{\lambda_{0400h}^*}{4} - \frac{\lambda_{0200h}^*}{4} \right) + \theta''_h \left( \frac{\lambda_{0400h}^*}{4} - \frac{\lambda_{0200h}^*}{4} \right) \right]. \] (66)

5. Traditional difference-type estimator for population variance in SACS is given by:
\[ t_{S-\delta} = \frac{1}{N} \sum_{h=1}^{L} N_h \left[ \frac{S_{\upsilon_{wh}}^2}{\lambda_{0400h}} + p_{1h} \left( s_{w_r,wh}^2 - s_{w_r,rh}^2 \right) \right] + p_{2h} \left( s_{w_r,wh}^2 - s_{w_r,rh}^2 \right), \] (67)
where \( p_{1h} \) and \( p_{2h} \) are constants. The minimum MSE of \( t_{S-\delta} \) at optimum values of constants to first-order approximation is given by:
\[ MSE(t_{S-\delta})_{\text{min}} \approx \frac{1}{N^2} \sum_{h=1}^{L} N_h^2 S_{\upsilon_{wh}}^2 \left[ \theta_h \lambda_{0400h}^* - \theta''_h \left( \frac{\lambda_{0400h}^*}{4} - \frac{\lambda_{0200h}^*}{4} \right) - \theta''_h \left( \frac{\lambda_{0200h}^*}{4} - \frac{\lambda_{0004h}^*}{4} \right) \right], \] (68)
where:
\[ p_{1h,\text{opt}} = \frac{S_{w_{r,h},\upsilon_{wh}}^2}{\lambda_{0400h}^*}, \quad p_{2h,\text{opt}} = \frac{S_{w_{r,h},\upsilon_{wh}}^2}{\lambda_{0200h}^*}. \]

6. Singh et al. [28] presented the exponential ratio-type estimator for population variance in SACS below:
\[ t_{S-\delta} = \frac{1}{N} \sum_{h=1}^{L} N_h \frac{S_{\upsilon_{wh}}^2}{\lambda_{0400h}} \left[ p_{3h} \exp \left( \frac{S_{w_{r,h}}^2 - S_{w_{r,rh}}^2}{S_{w_{r,h}}^2 + S_{w_{r,rh}}^2} \right) \right] + (1 - p_{3h}) \exp \left( \frac{S_{w_{r,h}}^2 - S_{w_{r,rh}}^2}{S_{w_{r,h}}^2 + S_{w_{r,rh}}^2} \right). \] (69)
where \( p_{3h} \) is a constant. The bias and minimum MSE of \( t_{S-\delta} \) at optimum values of constants to first-order approximation are given by:
\[ \text{Bias}(t_{S-\delta}) \approx \frac{1}{N} \sum_{h=1}^{L} N_h S_{\upsilon_{wh}}^2 \left[ p_{3h} \left\{ \frac{\theta''_h}{8} \left( \frac{\lambda_{0400h}^*}{4} - \frac{\lambda_{0200h}^*}{4} \right) \right\} \right.
\[ - \left. \theta''_h \left( \frac{3}{8} \lambda_{0004h}^* - \frac{1}{2} \lambda_{0200h}^* \right) \right\} + \theta''_h \left( \frac{3}{8} \lambda_{0004h}^* - \frac{1}{2} \lambda_{0200h}^* \right) \right], \] (70)
and:
\[ MSE(t_{S-\delta})_{\text{min}} \approx \frac{1}{N^2} \sum_{h=1}^{L} N_h^2 S_{\upsilon_{wh}}^2 \left( D_h - B_h^2 \right)^2 / A_h^2, \] (71)
where:
\[ A_h = \frac{\theta''_h}{4} \lambda_{0400h}^* + \frac{\theta''_h}{4} \lambda_{0004h}^*, \quad p_{1h,\text{opt}} = -B_h / A_h \]
\[ B_h = \theta''_h \left( \frac{1}{2} \lambda_{0200h}^* + \frac{1}{4} \lambda_{0004h}^* \right) - \frac{\theta''_h}{2} \lambda_{0004h}, \]
\[ D_h = \theta''_h \lambda_{0400h}^* + \theta''_h \left( \frac{1}{4} \lambda_{0004h}^* - \lambda_{0200h}^* \right). \]

7. As proposed by Singh and Solanki [29], the ratio-type estimator for population variance in SACS is given by:
\[ t_{S-\tau_{ij}} = \frac{1}{N} \sum_{h=1}^{L} N_h \frac{S_{w_{r,h}}^2}{\lambda_{0400h}} \left( \frac{p_{4h,i,j} S_{w_{r,h}}^2 + p_{5h,j}}{p_{4h,i,j} S_{w_{r,h}}^2 + p_{5h,j}} \right) \]
\[ \left( \frac{p_{4h,i,j} S_{w_{r,h}}^2 + p_{5h,j}}{p_{4h,i,j} S_{w_{r,h}}^2 + p_{5h,j}} \right), \] (72)
where \( p_{4h,i,j}, p_{5h,j} \) are constants that assume different values for \( j = 1, 2, 3, 4 \) as given in Table 3.

The bias and MSE of \( t_{S-\tau_{ij}} \) to first-order approximation are given by:
\[ \text{Bias}(t_{S-\tau_{ij}}) \approx \frac{1}{N} \sum_{h=1}^{L} N_h S_{\upsilon_{wh}}^2 \left[ \theta''_h \left\{ Q_{2h}^2 \lambda_{0400h}^* - Q_{2h} \lambda_{20000h}^* \right\} \right.
\[ + \theta''_h \left( Q_{2h}^2 \lambda_{0400h}^* - Q_{2h} \lambda_{20000h}^* \right) \right], \] (73)
and:
\[ MSE(t_{S-\tau_{ij}}) \approx \frac{1}{N^2} \sum_{h=1}^{L} N_h^2 S_{\upsilon_{wh}}^2 \left[ \theta''_h \left\{ Q_{2h}^2 \lambda_{0400h}^* - 2 Q_{2h} \lambda_{20000h}^* \right\} \right]. \] (73)

Table 3. Combinations of constants for \( t_{S-\tau_{ij}} \).

| \( j \) | 1 | 2 | 3 | 4 |
|-------|---|---|---|---|
| \( p_{4h,i,j} \) | 1 | 1 | 1 | 1 |
| \( p_{5h,j} \) | \( \beta_{2(h_{wh})} \) | \( \beta_{2(h_{wh})} \) | \( \beta_{2(h_{wh})} \) | \( \beta_{2(h_{wh})} \) |
| \( p_{4h,i,j} \) | \( \beta_{2(h_{wh})} \) | \( \beta_{2(h_{wh})} \) | \( \beta_{2(h_{wh})} \) | \( \beta_{2(h_{wh})} \) |
| \( p_{5h,j} \) | \( \beta_{2(h_{wh})} \) | \( \beta_{2(h_{wh})} \) | \( \beta_{2(h_{wh})} \) | \( \beta_{2(h_{wh})} \) |
| \( p_{5h,j} \) | \( \beta_{2(h_{wh})} \) | \( \beta_{2(h_{wh})} \) | \( \beta_{2(h_{wh})} \) | \( \beta_{2(h_{wh})} \) |
+ \theta_h \lambda^*_k \chi_{000000h} + \theta'_h \left\{ Q^2_{zh} \lambda^*_{000000h} + 2Q^2_{zh} \lambda^*_{200000h} \right\}, \tag{74}

\text{where:}

Q^2_{zh} = \frac{p_{zh} \beta_{zh} S_{\omega_{zh}}^2}{p_{zh} \beta_{zh} S_{\omega_{zh}}^2 + p_{zh} \beta_{zh}}.

Q^2_{zh} = \frac{p_{zh} \beta_{zh} S_{\omega_{zh}}^2}{p_{zh} \beta_{zh} S_{\omega_{zh}}^2 + p_{zh} \beta_{zh}}.

8. Olfat and Kadilar \cite{30} presented the ratio-type estimator for population variance in SACS as follows:

\[ t_{S-\omega_0} = \frac{1}{N} \sum_{h=1}^{L} N_h \lambda^*_k \left( s^2_{\omega_{zh}} + s^2_{\omega_{zh}} \right) \left( p_{zh} \beta_{zh} S_{\omega_{zh}}^2 \right)^p_{zh} \left( p_{zh} \beta_{zh} S_{\omega_{zh}}^2 \right)^p_{zh}, \tag{75}

\text{where } p_{zh} \text{ and } p_{zh} \text{ are constants. The bias and minimum MSE of } t_{S-\omega_0} \text{ at optimum values of constants to first-order approximation are given by:}

\text{Bias}(t_{S-\omega_0}) \approx \frac{1}{N} \sum_{h=1}^{L} N_h \lambda^*_k \left( p_{zh} \beta_{zh} S_{\omega_{zh}}^2 \right)^p_{zh} \left( p_{zh} \beta_{zh} S_{\omega_{zh}}^2 \right)^p_{zh}.

\text{And:}

\text{MSE}(t_{S-\omega_0})_{\text{min}} \approx \frac{1}{N^2} \sum_{h=1}^{L} N_h^2 \lambda^*_k \left( p_{zh} \beta_{zh} S_{\omega_{zh}}^2 \right)^p_{zh} \left( p_{zh} \beta_{zh} S_{\omega_{zh}}^2 \right)^p_{zh}.

\text{where:}

p_{zh} = \frac{\lambda^*_k \left( \frac{\lambda^*_k}{\lambda^*_600000h} - \frac{\lambda^*_k}{\lambda^*_600000h} \right)}{2},

p_{zh} = \frac{\lambda^*_k \left( \frac{\lambda^*_k}{\lambda^*_600000h} - \frac{\lambda^*_k}{\lambda^*_600000h} \right)}{2}.

9. Amin et al. \cite{31} presented the ratio-type estimators for population variance in SACS below:

\[ t_{S-\omega_0} = \frac{1}{N} \sum_{h=1}^{L} N_h \lambda^*_k \left( s^2_{\omega_{zh}} + s^2_{\omega_{zh}} \right) \left( p_{zh} \beta_{zh} S_{\omega_{zh}}^2 \right)^p_{zh} \left( p_{zh} \beta_{zh} S_{\omega_{zh}}^2 \right)^p_{zh}, \tag{78}

\text{and:}

\[ t_{S-\omega_0} = \frac{1}{N} \sum_{h=1}^{L} N_h \lambda^*_k \left( s^2_{\omega_{zh}} + s^2_{\omega_{zh}} \right) \left( p_{zh} \beta_{zh} S_{\omega_{zh}}^2 \right)^p_{zh} \left( p_{zh} \beta_{zh} S_{\omega_{zh}}^2 \right)^p_{zh}, \tag{79}

\text{The minimum MSE values for } t_{S-\omega_0} \text{ and } t_{S-\omega_0} \text{ at optimum values of constants to first-order approximation are given by:}

\text{MSE}(t_{S-\omega_0})_{\text{min}} \approx \frac{1}{N^2} \sum_{h=1}^{L} N_h^2 \lambda^*_k \left( p_{zh} \beta_{zh} S_{\omega_{zh}}^2 \right)^p_{zh} \left( p_{zh} \beta_{zh} S_{\omega_{zh}}^2 \right)^p_{zh}, \tag{80}

\text{and:}

\[ t_{S-\omega_0} = \frac{1}{N} \sum_{h=1}^{L} N_h \lambda^*_k \left( s^2_{\omega_{zh}} + s^2_{\omega_{zh}} \right) \left( p_{zh} \beta_{zh} S_{\omega_{zh}}^2 \right)^p_{zh} \left( p_{zh} \beta_{zh} S_{\omega_{zh}}^2 \right)^p_{zh}.

\text{The minimum MSE values for } t_{S-\omega_0} \text{ and } t_{S-\omega_0} \text{ at optimum values of constants to first-order approximation are given by:}

\text{MSE}(t_{S-\omega_0})_{\text{min}} \approx \frac{1}{N^2} \sum_{h=1}^{L} N_h^2 \lambda^*_k \left( p_{zh} \beta_{zh} S_{\omega_{zh}}^2 \right)^p_{zh} \left( p_{zh} \beta_{zh} S_{\omega_{zh}}^2 \right)^p_{zh}.

\text{And:}

\text{MSE}(t_{S-\omega_0})_{\text{min}} \approx \frac{1}{N^2} \sum_{h=1}^{L} N_h^2 \lambda^*_k \left( p_{zh} \beta_{zh} S_{\omega_{zh}}^2 \right)^p_{zh} \left( p_{zh} \beta_{zh} S_{\omega_{zh}}^2 \right)^p_{zh}.

\text{And:}

\[ p_{zh} = \frac{\lambda^*_k \left( \frac{\lambda^*_k}{\lambda^*_600000h} - \frac{\lambda^*_k}{\lambda^*_600000h} \right)}{2},

\text{and:}

\[ p_{zh} = \frac{\lambda^*_k \left( \frac{\lambda^*_k}{\lambda^*_600000h} - \frac{\lambda^*_k}{\lambda^*_600000h} \right)}{2}.

3. Proposed estimators

3.1. Difference-type estimator

The following difference-type estimator is proposed for general parameters under SACS:

\[ t_{S-(\alpha, \beta)_y} = \frac{1}{N} \sum_{h=1}^{L} N_h \left( \tilde{t}_{(\alpha, \beta)_y} + f_{zh} \{ \bar{w}_{zh} - \bar{w}_{zh} \} \right) + \frac{f_{zh} \{ s^2_{\omega_{zh}} + s^2_{\omega_{zh}} \} + \{ \tilde{\tau}(w_{zh}) \} - \{ \tau(w_{zh}) \} }{\{ \tilde{R}(w_{zh}) \} - \{ \tau(w_{zh}) \}}. \tag{82}

\text{where } f_{zh} = \{ i = 1, 2, 3, 4 \} \text{ are constants whose values are to be determined. Estimators for population mean } \left( t_{S-(\alpha, \beta)_y} \right) \text{ and variance } \left( t_{S-(\alpha, \beta)_y} \right) \text{ can be obtained by substituting } (\alpha = 1, \beta = 0) \text{ and } (\alpha = 0, \beta = 2) \text{ in Eq. (82), respectively. Rewriting Eq. (82) in terms of errors, we get:}
\[
\hat{\tau}_{S-\alpha(y)\beta} - \tau_{\alpha(\beta)y} = \frac{1}{N} \sum_{h=1}^{L} N_h \tau_{\alpha(\beta)y}^h \left[ \left\{ \alpha \hat{Q}_h + \frac{\beta}{2} \hat{G}_h + \frac{\alpha (\alpha - 1)}{2} \hat{Q}_h^2 \right\} + \frac{\beta (\beta - 2)}{8} \hat{G}_h^2 + \frac{\alpha \beta}{2} \hat{Q}_h \hat{G}_h \right]
+ f_{1h} \hat{X}_{\omega(h)} (\hat{Q}_h - \hat{G}_h) + f_{2h} \hat{S}^2_{\omega(h)} (\hat{Q}_h - \hat{G}_h) + \hat{R}(w_h)(\hat{Q}_h - \hat{G}_h)
- f_{1h} \hat{Z}_{\omega(h)} \hat{Q}_h - f_{1h} \hat{S}^2_{\omega(h)} \hat{Q}_h^2
- \hat{R}(w_h) \hat{Q}_h \right].
\]

Taking expectations of both sides, we get:

\[
\text{Bias}(\hat{\tau}_{S-\alpha(y)\beta}) = \frac{1}{N} \sum_{h=1}^{L} N_h \tau_{\alpha(\beta)y}^h \left\{ \frac{\alpha (\alpha - 1)}{2} C_{\omega(h)^2} + \frac{\beta (\beta - 2)}{8} \lambda^4_{40000h} + \frac{\alpha \beta}{2} C_{\omega(h)^3} \lambda_{30000h} \right\}.
\]

(83)

Bias of the proposed estimator for population mean ($t_{S-\alpha(y)\beta}$) and variance ($t_{S-\alpha(y)\beta}$) can be obtained by substituting ($\alpha = 1, \beta = 0$) and ($\alpha = 0, \beta = 2$) in Eq. (84), respectively. Squaring Eq. (83) and considering first-order approximation, we get:

\[
[\hat{\tau}_{S-\alpha(y)\beta}]^2 = \frac{1}{N^2} \sum_{h=1}^{L} N_h^2 \left[ \tau_{\alpha(\beta)y}^h \left( \alpha \hat{Q}_h + \frac{\beta}{2} \hat{G}_h \right) + f_{1h} \hat{X}_{\omega(h)} (\hat{Q}_h - \hat{G}_h) + f_{2h} \hat{S}^2_{\omega(h)} (\hat{Q}_h - \hat{G}_h) + \hat{R}(w_h)(\hat{Q}_h - \hat{G}_h) - f_{1h} \hat{Z}_{\omega(h)} \hat{Q}_h - f_{1h} \hat{S}^2_{\omega(h)} \hat{Q}_h^2 - \hat{R}(w_h) \hat{Q}_h \right]^2.
\]

(85)

Taking expectations of both sides, we get:

\[
MSE(\hat{\tau}_{S-\alpha(y)\beta}) = \frac{1}{N^2} \sum_{h=1}^{L} N_h^2 \left\{ \tau_{\alpha(\beta)y}^h \left( \alpha \hat{Q}_h + \frac{\beta}{2} \hat{G}_h \right) + f_{1h} \hat{X}_{\omega(h)} (\hat{Q}_h - \hat{G}_h) + f_{2h} \hat{S}^2_{\omega(h)} (\hat{Q}_h - \hat{G}_h) + \hat{R}(w_h)(\hat{Q}_h - \hat{G}_h) - f_{1h} \hat{Z}_{\omega(h)} \hat{Q}_h - f_{1h} \hat{S}^2_{\omega(h)} \hat{Q}_h^2 - \hat{R}(w_h) \hat{Q}_h \right]^2.
\]

(86)

where:

\[
A_{y^2} = \alpha^2 C_{\omega(h)^2} + \frac{\beta^2}{4} \lambda^4_{40000h} + \alpha \beta C_{\omega(h)^3} \lambda_{30000h},
\]

\[
A_{z_h} = \hat{X}_{\omega(h)} C_{\omega(h)^2},
\]

\[
B_{z_h} = S^2_{\omega(h)} \lambda^*_{40000h},
\]

\[
F_{z_h} = \hat{X}_{\omega(h)} S^2_{\omega(h)} C_{\omega(h)} \lambda_{30000h},
\]

\[
D_{z_h} = \tau_{\alpha(\beta)y} \hat{X}_{\omega(h)} \left( \alpha C_{\omega(h)^2} + \frac{\beta}{2} C_{\omega(h)^3} \lambda_{20000h} \right)
- \hat{X}_{\omega(h)} \hat{R}(w_h) C_{\omega(h)} \lambda_{30000h},
\]

\[
E_{z_h} = \tau_{\alpha(\beta)y} S^2_{\omega(h)} \left( \alpha C_{\omega(h)^2} \lambda_{20000h} + \frac{\beta}{2} \lambda^*_{20000h} \right)
- \hat{S}^2_{\omega(h)} \hat{R}(w_h) C_{\omega(h)} \lambda_{30000h},
\]

\[
G_{z_h} = \hat{R}(w_h) C^2_{\omega(h)} - 2 \tau_{\alpha(\beta)y} \hat{R}(w_h) \left( \alpha C_{\omega(h)^2} + \frac{\beta}{2} C_{\omega(h)^3} \lambda_{20000h} \right).
\]

From Eq. (86), the optimum values of $f_{iz}$ ($i = 1, 2, 3, 4$) are:

\[
f_{iz, opt} = \frac{B_{z_h} D_{z_h} - E_{z_h} F_{z_h}}{A_{z_h} B_{z_h} - F^2_{z_h}},
\]

\[
f_{iz, opt} = \frac{A_{z_h} E_{z_h} - F_{z_h} D_{z_h}}{A_{z_h} B_{z_h} - F^2_{z_h}}.
\]
\[ f_{th,\varphi} = \frac{B_{ih} D_{zh} - E_{ih} F_{zh}}{A_{ih} B_{zh} - F_{zh}^2}, \]

\[ f_{th,\varphi} = \frac{A_{ih} E_{zh} - F_{ih} D_{zh}}{A_{ih} B_{zh} - F_{zh}^2}. \]

By substituting optimum values of \( f_{th}(i = 1, 2, 3, 4) \) in Eq. (86), the minimum MSE of the proposed difference-type estimator for general parameters is obtained as follows:

\[
MSE(\hat{S}_{-(a,b)}_{yP_1})_{min} \approx \frac{1}{N^2} \sum_{h=1}^{L} N_h^2 \left[ \tau_{(a,b)}^2 y_{h} \theta_{h} A_{y_{h}} + \theta'_{h} Q_{x_{h}} + \theta'_{h} Q_{z_{h}} \right],
\]

where:

\[ Q_{h} = G_{ih} + \frac{2D_{ih} E_{ih} F_{ih} - B_{ih} D_{zh}^2 - A_{ih} E_{zh}^2}{A_{ih} B_{zh} - F_{zh}^2}, \]

\[ \forall i = x, z. \]

Minimum MSE of the proposed estimator for population mean \((t_{S-P_1})\) and variance \((t_{S-P_1})\) can be obtained by substituting \((\alpha = 1, \beta = 0)\) and \((\alpha = 0, \beta = 2)\) in Eq. (87), respectively.

### 3.2. Difference-cum-exponential-ratio-type estimator

The following difference-cum-exponential-ratio-type estimator is proposed for general parameters under SACS:

\[
\hat{\tau}_{S-(a,b)}_{yP_2} = \frac{1}{N} \sum_{h=1}^{L} N_h \left[ g_{ih} \tau_{(a,b)} y_{h} \right.
\]

\[ + g_{ih} \left( \overline{m}_{x_{h}} - \overline{m}_{x_{h}} \right) + g_{ih} \left( \overline{m}_{y_{h}} - \overline{m}_{y_{h}} \right) \]

\[ \exp \left( \frac{s_{x_{h}}' - s_{y_{h}}'}{s_{x_{h}}' + s_{y_{h}}'} \right), \]

\[ \exp \left( \frac{\tau_{(x_{h})}}{\tau_{(y_{h})}} + \frac{\tau_{(x_{h})}}{\tau_{(y_{h})}} \right), \] \( (88) \)

where \( g_{ih}(i = 1, 2, 3) \) are constants whose values are to be determined. Estimators for population mean \((t_{S-P_2})\) and population variance \((t_{S-P_2})\) can be obtained by substituting \((\alpha = 1, \beta = 0)\) and \((\alpha = 0, \beta = 2)\) in Eq. (88), respectively. Rewriting Eq. (88) in terms of errors and considering first order approximation, we get:

\[
\hat{\tau}_{S-(a,b)}_{yP_2} \approx \frac{1}{N} \sum_{h=1}^{L} N_h \left[ g_{ih} \tau_{(a,b)} y_{h} \right.
\]

\[ + \alpha \xi_{ih} + \frac{\beta}{2} \xi_{ih} + \frac{\alpha (\alpha - 1)}{2} Q_{h}^2 \]

\[ + \frac{\alpha \beta}{2} \xi_{ih} Q_{h} + \frac{\beta (\beta - 2)}{8} Q_{h}^2 \]

\[ + g_{ih} \xi_{x_{h}} y_{h} \left( \xi_{y_{h}} - \xi_{h} \right) - g_{ih} \xi_{x_{h}} y_{h} \xi_{h} \]

\[ + \frac{\alpha \beta}{2} \xi_{ih} Q_{h} + \frac{\beta (\beta - 2)}{8} Q_{h}^2 \] \( \right) \]

\[ + g_{ih} \xi_{x_{h}} y_{h} \xi_{h} \xi_{h} \]

\[ \left\{ 1 + \frac{\xi_{ih} - \xi_{h}}{2} + \frac{3 (\xi_{ih} - \xi_{h})^2}{8} \right\} \]

\[ \left\{ 1 + \frac{\xi_{ih} - \xi_{h}}{2} + \frac{3 (\xi_{ih} - \xi_{h})^2}{8} \right\}. \] \( (89) \)

Taking expectations of both sides, we get:

\[
Bias(\hat{\tau}_{S-(a,b)}_{yP_2}) \approx \frac{1}{N} \sum_{h=1}^{L} N_h \left[ g_{ih} \tau_{(a,b)} y_{h} \left( \theta_{h} Q_{h} + \theta_{h} T_{h} \right) + g_{ih} \xi_{x_{h}} y_{h} \frac{\theta_{h} Q_{h}}{2} U_{ih} \right.
\]

\[ + \tau_{(a,b)} y_{h} \left( g_{ih} - 1 \right). \] \( (90) \)

where:

\[ Q_{1h} = 3 \frac{\lambda_{w_{ih}}}{8} + 3 \frac{\lambda_{w_{ih}}^2}{8} C_{r_{(w_{ih})}} - \frac{\alpha}{2} \]

\[ (C_{w_{ih}} \lambda_{2b00h} + C_{w_{ih}} r_{(w_{ih})}) \]

\[ + \frac{1}{4} (C_{r_{(w_{ih})}} \lambda_{2b00h} - 2 \lambda_{2b00h} + C_{r_{(w_{ih})}} \lambda_{2b00h}) \]

\[ T_{1h} = \frac{\alpha (\alpha - 1)}{2} C_{w_{ih}} + \frac{\beta (\beta - 2)}{8} \lambda_{4b00h} \]

\[ + \frac{\alpha \beta}{2} C_{w_{ih}} \lambda_{3b00h}, \]

\[ U_{1h} = C_{w_{ih}} \lambda_{3b00h} + C_{w_{ih}} r_{(w_{ih})}. \]

Bias of the estimator for population mean \((t_{S-P_2})\) and population variance \((t_{S-P_2})\) can be obtained by substituting \((\alpha = 1, \beta = 0)\) and \((\alpha = 0, \beta = 2)\) in Eq. (90), respectively. Squaring Eq. (89) and considering first-order approximation, we have:

\[
[\hat{\tau}_{S-(a,b)}_{yP_2} - \tau_{(a,b)} y_{h}]^2 \approx \frac{1}{N^2} \sum_{h=1}^{L} N_h^2 \left[ g_{ih} \tau_{(a,b)} y_{h} \right.
\]

\[ + \frac{2 \alpha \beta}{2} g_{ih} \xi_{ih} y_{h} \left( \xi_{h} - \xi_{ih} \right) + \frac{2 \alpha \beta}{2} g_{ih} \xi_{ih} y_{h} \xi_{h} \]

\[ + \frac{2 \alpha \beta}{2} g_{ih} \xi_{ih} y_{h} \xi_{h} \xi_{h} \]

\[ \right)^2 - 2 g_{ih} \xi_{ih} y_{h} \frac{\theta_{h} Q_{h}}{2} U_{ih} \xi_{x_{h}} y_{h} \xi_{h} \]

\[ \left\{ 1 + \frac{\xi_{ih} - \xi_{h}}{2} + \frac{3 (\xi_{ih} - \xi_{h})^2}{8} \right\}^2 \]

\[ \left\{ 1 + \frac{\xi_{ih} - \xi_{h}}{2} + \frac{3 (\xi_{ih} - \xi_{h})^2}{8} \right\}. \]
\[
\left\{ \frac{C_1^h C_2^h - C_1^h C_1^h}{2} + g_{2h}^2 X_{wh}^2 \left( \frac{C_1^h - C_1^h}{2} \right)^2 \\
+ g_{2h}^2 X_{wh}^2 C_2^h + \tau_{(\alpha, \beta)yh}^2 (g_{1h} - 1)^2 \\
- 2g_{2h}g_{2h} \tau_{(\alpha, \beta)yh} Z_{wh} \left( \frac{\alpha C_{wh}^h C_1^h}{2} \right) \\
+ \frac{\beta}{2} (C_1^h C_2^h + C_1^h C_2^h) + 2g_{2h}g_{2h} \tau_{(\alpha, \beta)yh} Z_{wh} \frac{\alpha C_{wh}^h C_2^h}{2} \\
+ \frac{\alpha (C_2^h C_1^h - \frac{\alpha}{2} C_1^h) + \beta}{2} \\
+ C_1^h C_1^h - C_1^h C_1^h + C_1^h C_1^h - C_1^h C_1^h \\
+ \frac{\beta}{4} \left( C_1^h C_2^h - C_1^h C_2^h + C_1^h C_2^h - C_1^h C_2^h \right) \\
+ \frac{\alpha (\alpha - 1)}{2} C_2^h + \frac{3}{8} \left( C_1^h C_2^h - C_1^h C_2^h \right) \\
+ \frac{\alpha \beta}{2} \frac{\beta}{4} C_1^h C_2^h \\
+ \frac{\alpha C_2^h C_1^h - C_1^h C_2^h}{4} + 2g_{2h} (g_{1h} - 1) \\
\right \} \\
X_{wh} \tau_{(\alpha, \beta)yh} \left[ \frac{C_1^h C_2^h - C_1^h C_1^h}{2} \\
+ \frac{\alpha C_2^h C_1^h - C_1^h C_2^h}{2} \right].
\]

Taking expectations of both sides, we have:

\[
M E(\tau_{(\alpha, \beta)yh} | P_2) \approx \frac{1}{N^2} \sum_{h=1}^{L} \sum_{\lambda}^L \lambda^2
\]

\[
\left[ \tau_{(\alpha, \beta)yh}^2 + g_{2h}^2 A_{1h} + g_{2h}^2 B_{1h} + g_{2h}^2 D_{1h} \\
- 2g_{1h} E_{1h} - 2g_{2h} F_{1h} + 2g_{1h} g_{1h} H_{1h} \\
- 2g_{1h} g_{1h} J_{1h} \right].
\]

where:

\[
A_{1h} = \tau_{(\alpha, \beta)yh}^2 (1 + \theta_h a_{xh} + \theta_h^A a_{z1h}).
\]

\[
E_{1h} = \tau_{(\alpha, \beta)yh}^2 (1 + \theta_h a_{xh} + \theta_h^E a_{z1h}).
\]

\[
a_{xh} = (2\alpha^2 - \alpha) C_{wh}^2 + \frac{(\beta^2 - \beta)}{2} \lambda_{04000h}^* \\
+ 2\alpha C_{wh}^2 \lambda_{30000h}.
\]

\[
B_{1h} = \theta_h^B b_{kh}. 
\]

\[
a_{z1h} = C_{r(w, b)} \lambda_{02100h} - 2\alpha (C_{w, w} \lambda_{12000h} + C_{w, r(w, b)}) \\
\lambda_{04000h}^* - \beta (C_{22000h} + C_{r(w, b)}) + C_{r(w, b)}^2.
\]

\[
J_{1h} = \theta_h^J j_{zh}. 
\]

\[
e_{xh} = \frac{\alpha (\alpha - 1)}{2} C_{wh}^2 + \frac{\beta (\beta - 2)}{8} \lambda_{04000h}^* \\
+ \frac{\alpha \beta}{2} \lambda_{30000h}.
\]

\[
b_{zh} = \frac{\alpha (\alpha - 1)}{2} C_{wh}^2 + \frac{\beta (\beta - 2)}{8} \lambda_{04000h}^* \\
+ \frac{\alpha \beta}{2} \lambda_{30000h}.
\]

\[
j_{zh} = \tau_{(\alpha, \beta)yh} Z_{wh} \left( \alpha C_{w, w, w} + \frac{\beta}{2} C_{w, w, w} \lambda_{20100h} \right),
\]

\[
D_{1h} = \theta_h^D d_{zh}. 
\]

\[
d_{zh} = \frac{\alpha (\alpha - 1)}{2} C_{wh}^2 + \frac{\beta (\beta - 2)}{8} \lambda_{04000h}^* \\
+ \frac{\alpha \beta}{2} \lambda_{30000h}.
\]

\[
e_{zh} = \frac{\alpha (\alpha - 1)}{2} C_{wh}^2 + \frac{\beta (\beta - 2)}{8} \lambda_{04000h}^* \\
+ \frac{\alpha \beta}{2} \lambda_{30000h}.
\]

\[
h_{zh} = \tau_{(\alpha, \beta)yh} X_{wh} \left( C_{w, w, w} \lambda_{03000h} + C_{w, r(w, b)} \right) \\
- \alpha C_{w, w, w} - \frac{\beta}{2} C_{w, w, w} \lambda_{21000h}^*.
\]

\[
f_{zh} = \frac{\tau_{(\alpha, \beta)yh} X_{wh}}{2} (C_{w, w, w} \lambda_{03000h} + C_{w, r(w, b)}).
\]

\[
F_{1h} = \theta_h^F f_{zh}. 
\]

\[
F_{1h} = \theta_h^F f_{zh}. 
\]

\[
H_{1h} = \theta_h^H h_{zh}. 
\]

From Eq. (92), the optimum values of \(g_{1h}, (i = 1, 2, 3)\) are as follows:

\[
g_{1h} = \frac{L_{1h}}{M_{1h}}, 
\]

\[
g_{2h} = \frac{F_{1h} M_{1h} - H_{1h} L_{1h}}{B_{1h} M_{1h}},
\]

\[
g_{3h} = \frac{J_{1h} L_{1h}}{D_{1h} M_{1h}}.
\]

where:
\[ M_{1h} = A_{1h} - \frac{H^2_{1h}}{B_{1h}} - \frac{F_{1h}}{D_{1h}} \]

\[ L_{1h} = E_{1h} - \frac{F_{1h} H_{1h}}{B_{1h}}. \]

By substituting optimum values of \( g_k(i = 1, 2, 3) \) in Eq. (92), the minimum MSE of the proposed exponential ratio-type estimator for general parameters is as follows:

\[
MSE(\hat{t}_{S_{-(\alpha, \beta)}yp_k})_{\min} = \frac{1}{N^2} \sum_{h=1}^{L} N^2_h \left[ \frac{F_{1h}^2 M_{1h}}{B_{1h}} \right].
\]  

The minimum MSE of the proposed exponential ratio-type estimator for population mean \( (t_{S_{-p_2m}}) \) and variance \( (t_{S_{-p_2v}}) \) can be obtained by substituting \( (\alpha = 1, \beta = 0) \) and \( (\alpha = 0, \beta = 2) \) in Eq. (93), respectively.

4. Numerical study

Data of teals from Smith et al. [32] are considered to make a numerical comparison between the existing and the proposed estimators. The data of Blue-winged teal is used as a study variable for stratum 1 and data of Green-winged teal is used as a study variable for stratum 2. Auxiliary variables \( (x_h \; \text{and} \; z_h) \) are generated using the concept given by Dryver and Chao [33] and Chao et al. [34] as follows:

\[ x_i = \begin{cases} y_i \times \text{Poi}(600) + \epsilon_i \quad \text{if} \quad y_i < 100 \\ y_i \quad \text{otherwise} \end{cases} \]

where \( \epsilon_i \sim N(0, \epsilon_i) \) and Pois represents random generation from Poisson distribution. Data statistics at different levels of correlation are given below:

1. \( N = 400, N_1 = 200, N_2 = 200, n_1 = 20, n_2 = 20, n'_1 = 50, n'_2 = 50, E(v_1) = 38, \rho_{w,w_1} = 0.47, \rho_{w,w_2} = 0.998, \rho_{w,v_1} = 0.987, \bar{Y}_{w_1} = 70.60485, \bar{Y}_{w_2} = 12.01, S^2_{w_1} = 130872.4, S^2_{w_2} = 12816.53, E(v_2) = 22, \bar{X}_{w_1} = 367.81, \bar{X}_{w_2} = 47.64, S^2_{w_1} = 1473602, S^2_{w_2} = 116807.1, \bar{Z}_{w_1} = 391.035, \bar{Z}_{w_2} = 59.24, S^2_{w_1} = 1713164, S^2_{w_2} = 172156.

2. \( \rho_{w,v_1} = 0.66, \rho_{w,v_2} = 0.61, \rho_{w,w_1} = 0.59, \rho_{w,w_2} = 0.58, \rho_{w_1,v_1} = 0.993, \rho_{w_2,v_1} = 0.989, \bar{Y}_{w_1} = 70.60485, \bar{Y}_{w_2} = 12.01, S^2_{w_1} = 130872.4, S^2_{w_2} = 12816.53, \bar{X}_{w_1} = 208.79, \bar{X}_{w_2} = 35.33, S^2_{w_1} = 430393.9, S^2_{w_2} = 55797.76, Z_{w_1} = 232.095, Z_{w_2} = 36.255, S^2_{w_1} = 563897.9, S^2_{w_2} = 60042.

3. \( \rho_{w,u_1} = 0.88, \rho_{w,u_2} = 0.83, \rho_{w,v_1} = 0.84, \rho_{w,v_2} = 0.78, \rho_{w_1,v_1} = 0.995, \rho_{w_2,v_2} = 0.996, \bar{Y}_{w_1} = 70.60485, \bar{Y}_{w_2} = 12.01, S^2_{w_1} = 130872.4, S^2_{w_2} = 12816.53, \bar{X}_{w_1} = 125.165, \bar{X}_{w_2} = 22.22, S^2_{w_1} = 191574.7, S^2_{w_2} = 22789.27, \bar{Z}_{w_1} = 138.9, \bar{Z}_{w_2} = 24.545, S^2_{w_1} = 216293.5, S^2_{w_2} = 27145.

4. \( \rho_{w_1,v_1} = 0.92, \rho_{w_2,v_2} = 0.94, \rho_{w,v_1} = 0.89, \rho_{w,v_2} = 0.83, \rho_{w_1,v_1} = 0.997, \rho_{w_2,v_2} = 0.964, \bar{Y}_{w_1} = 70.60485, \bar{Y}_{w_2} = 12.01, S^2_{w_1} = 130872.4, S^2_{w_2} = 12816.53, \bar{X}_{w_1} = 115.655, \bar{X}_{w_2} = 18.485, S^2_{w_1} = 174073.3, S^2_{w_2} = 17558.42, \bar{Z}_{w_1} = 121.36, \bar{Z}_{w_2} = 20.845, S^2_{w_1} = 186673.5, S^2_{w_2} = 21254.

For the data sets discussed above, Absolute Relative Bias (ARB) and Percent Relative Efficiency (PRE) are calculated for the existing and proposed estimators. Results of ARB and PRE of the existing and proposed estimators for population mean are presented in Tables 4 and 5. Similarly, results of ARB and Relative Efficiency (RE) for the existing and proposed estimators for population variance are given in Tables 6 and 7. Results presented in Tables 4–7 reveal that for the proposed difference cum exponential-ratio-type estimator for population mean and variance, ARB decreases upon an increase in the correlation between the study and auxiliary variables. The proposed difference-type estimator population mean \( (t_{S_{-p_1 m}}) \) and for population variance \( (t_{S_{-p_1 v}}) \) is unbiased. Thus, ARB remains zero at all correlation levels.

The proposed difference-cum-exponential-ratio-type estimator for population mean \( (t_{S_{-p_2 m}}) \) is more efficient when the correlation between the study and auxiliary variables is low or moderate. The proposed difference-type estimator for population mean \( (t_{S_{-p_1 m}}) \) outperforms all other estimators when the correlation is high. When the correlation between the study and auxiliary variables is low or moderate, \((t_{S_{-p_1 m}}) \) is most efficient among all existing estimators. Thus, the comparison between \( t_{S_{-p_2 m}} \) and \( t_{S_{-p_1 m}} \) is given in Figure 1(a). At high correlation levels, \( t_{S_{-p_1 m}} \) outperforms all existing estimators. Thus, the comparison between \( t_{S_{-p_1 m}} \) and \( t_{S_{-p_2 m}} \) is given in Figure 1(b). Finally, the comparison between \( t_{S_{-p_1 m}} \) and \( t_{S_{-p_2 m}} \) is given in Figure 1(c).

The proposed difference-cum-exponential-ratio-type estimator for population variance \( t_{S_{-p_2 v}} \) is more efficient when the correlation between the study and auxiliary variables is low or moderate. The proposed difference-type estimator for population variance \( t_{S_{-p_1 v}} \) outperforms all other estimators when the correlation is high. When the correlation between the study and auxiliary variables is low or moderate, \( t_{S_{-p_1 v}} \) is most efficient among all existing estimators. Thus, the comparison between \( t_{S_{-p_2 v}} \) and \( t_{S_{-p_1 v}} \) is given in Figure 2(a). At high levels of correlation, \( t_{S_{-p_1 v}} \) performs better than all existing estimators. Thus, the comparison between \( t_{S_{-p_1 v}} \) and \( t_{S_{-p_2 v}} \) is given in Figure 2(b).
Table 4. Absolute Relative Bias (ARB) of different estimators for population mean.

\[
\begin{array}{cccccc}
\text{ARB} & (\rho_{wegr}, \rho_{wegk1} : & (\rho_{wegr}, \rho_{wegk1} : & (\rho_{wegr}, \rho_{wegk2} : & (\rho_{wegr}, \rho_{wegk2} : \\
\text{Estimators} & (0.42, 0.40 : & (0.66, 0.39 : & (0.88, 0.84 : & (0.92, 0.89 : \\
& 0.47, 0.41) & 0.51 & 0.78) & 0.94 & 0.83) \\
\hline
\hat{t}_{s_{1m}} & 0.000 & 0.000 & 0.000 & 0.000 \\
\hat{t}_{s_{2m}} & 0.5515 & 0.1493 & 0.2145 & 0.3068 \\
\hat{t}_{s_{3m}} & 0.1023 & 0.0807 & 0.2696 & 0.3321 \\
\hat{t}_{s_{4m}} & 0.000 & 0.000 & 0.000 & 0.000 \\
\hat{t}_{s_{5m}} & 1.3852 & 1.1827 & 0.6127 & 1.0908 \\
\hat{t}_{s_{6}} & 0.1654 & 0.0622 & 0.1285 & 0.1892 \\
\hat{t}_{s_{7}} & 0.1654 & 0.0622 & 0.1285 & 0.1892 \\
\hat{t}_{s_{8}} & 0.6076 & 0.2005 & 0.3638 & 0.5285 \\
\hat{t}_{s_{9m}} & 27.8494 & 16.4177 & 82.6546 & 88.8825 \\
\hat{t}_{s_{10m_{1}}}, & 0.3695 & 0.0134 & 0.3697 & 0.4766 \\
\hat{t}_{s_{10m_{2}}}, & 0.0173 & 0.1951 & 0.3524 & 0.3536 \\
\hat{t}_{s_{10m_{3}}}, & 0.5488 & 0.1454 & 0.2200 & 0.3122 \\
\hat{t}_{s_{10m_{4}}}, & 0.2729 & 0.0480 & 0.3832 & 0.4903 \\
\hat{t}_{s_{10m_{5}}}, & 0.5371 & 0.1295 & 0.2454 & 0.3470 \\
\hat{t}_{s_{10m_{6}}}, & 0.5482 & 0.1451 & 0.2203 & 0.3127 \\
\hat{t}_{s_{10m_{7}}}, & 0.3110 & 0.0370 & 0.3789 & 0.4682 \\
\hat{t}_{s_{10m_{8}}}, & 0.5514 & 0.1492 & 0.2151 & 0.3072 \\
\hat{t}_{s_{10m_{9}}}, & 0.5515 & 0.1493 & 0.2148 & 0.3068 \\
\hat{t}_{s_{10m_{10}}}, & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
\hat{t}_{s_{11m}} & 0.1004 & 0.1621 & 0.5210 & 0.6334 \\
\hat{t}_{s_{12m}} & 0.6897 & 0.7432 & 0.9310 & 1.0603 \\
\hat{t}_{s_{13m}} & 3.7853 & 1.0862 & 10.7265 & 14.3532 \\
\hat{t}_{s_{14m}} & 353.436 & 5727.64 & 10317.7 & 14189.8 \\
\hat{t}_{s_{15m_{1}}}, & 0.5534 & 0.2024 & 0.1532 & 0.2112 \\
\hat{t}_{s_{15m_{2}}}, & 0.3454 & 0.4002 & 0.7428 & 0.8782 \\
\hat{t}_{s_{15m_{3}}}, & 0.4599 & 0.4245 & 0.2804 & 0.1944 \\
\hat{t}_{s_{15m_{4}}}, & 0.6559 & 0.6233 & 0.5304 & 0.4702 \\
\hat{t}_{s_{16m}} & 0.6000 & 0.5664 & 0.4558 & 0.3608 \\
\hat{t}_{s_{17m}} & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
\hat{t}_{s_{p1m}} & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
\hat{t}_{s_{p2m}} & 0.2978 & 0.3798 & 0.2527 & 0.1442 \\
\end{array}
\]

Figure 2(b). Thus, the comparison between \( t_{s_{-P1}} \) and \( t_{s_{-P2}} \) is given in Figure 2(c).

5. Conclusion

In this article, difference-type and difference-sum-exponential-ratio-type estimators were recommended for general parameters under stratified adaptive cluster sampling. Estimators were proposed using two auxiliary variables. The proposed estimators utilized auxiliary information in terms of mean, variance, and ranks of auxiliary variates in the \( h \)th stratum. Based on the numerical study, it became clear that the proposed estimators for population mean were more efficient than the usual mean, ratio, exponential-ratio, difference estimator, and estimators of Gupta and Shabbir [14], Singh et al. [15], Chudhury and Singh [16], Hamad et al. [17], Chutiman [18], Yadav et al. [19], Vishwakarma and Gangele [21], Singh and Khalid [22], Khan and Al-Hossain [23], Khan [24],
Table 5. Percent Relative Efficiency (PRE) of different estimators for population mean.

| Estimators | \( \rho_{wyre.1} \) : | \( \rho_{wyre.1} : \rho_{wyre.2} \) | \( \rho_{wyre.2} \) : \( \rho_{wyre.2} \) |
|------------|----------------|----------------|----------------|
| \( t = 1 \) | (0.42, 0.40) | (0.66, 0.59) | (0.88, 0.84) |
| \( t = 2 \) | 0.47, 0.41 | 0.61, 0.58 | 0.83, 0.78 |
| \( t = 3 \) | 0.47, 0.41 | 0.61, 0.58 | 0.83, 0.78 |
| \( t = 4 \) | 0.47, 0.41 | 0.61, 0.58 | 0.83, 0.78 |
| \( t = 5 \) | 0.47, 0.41 | 0.61, 0.58 | 0.83, 0.78 |
| \( t = 6 \) | 0.47, 0.41 | 0.61, 0.58 | 0.83, 0.78 |
| \( t = 7 \) | 0.47, 0.41 | 0.61, 0.58 | 0.83, 0.78 |
| \( t = 8 \) | 0.47, 0.41 | 0.61, 0.58 | 0.83, 0.78 |
| \( t = 9 \) | 0.47, 0.41 | 0.61, 0.58 | 0.83, 0.78 |

Likewise, the proposed estimators for population variance were found more efficient than usual variance, ratio, exponential-ratio, difference estimator, and estimators of Singh et al. [28], Singh and Solanki [29], Ohadi and Kadilar [30], and Noor-ul-Amin et al. [31] under SACS.

At a low or moderate correlation, the proposed difference-sum-exponential-ratio-type estimator was the most efficient one, in addition, at a high correlation between the study and auxiliary vari-
Table 6. Absolute Relative Bias (ARB) of different estimators for population variance.

| Estimators | \((0.42, 0.40 : 0.47, 0.41)\) | \((0.66, 0.59 : 0.61, 0.58)\) | \((0.88, 0.84 : 0.83, 0.78)\) | \((0.92, 0.89 : 0.94, 0.83)\) |
|------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|
| \(t_{s=1v}\) | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| \(t_{s=2v}\) | 1.9450 | 0.6879 | 0.2080 | 0.278 |
| \(t_{s=3v}\) | 0.6095 | 0.1882 | 0.2036 | 0.3146 |
| \(t_{s=4v}\) | 0.6990 | 0.1487 | 0.2912 | 0.3701 |
| \(t_{s=5v}\) | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| \(t_{s=6v}\) | 0.3290 | 0.0590 | 0.2392 | 0.3182 |
| \(t_{s=7v_{j=1}}\) | 1.9446 | 0.6876 | 0.2084 | 0.278 |
| \(t_{s=7v_{j=2}}\) | 1.9422 | 0.6858 | 0.2106 | 0.2826 |
| \(t_{s=7v_{j=3}}\) | 1.9446 | 0.6876 | 0.2084 | 0.278 |
| \(t_{s=7v_{j=4}}\) | 1.9450 | 0.6880 | 0.2081 | 0.278 |
| \(t_{s=8v}\) | 0.1664 | 0.1730 | 0.1234 | 0.1610 |
| \(t_{s=P1v}\) | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| \(t_{s=P2v}\) | 0.3023 | 0.3897 | 0.2751 | 0.1722 |

Table 7. Relative Efficiency (RE) of different estimators for population variance.

| Estimators | \((0.42, 0.40 : 0.47, 0.41)\) | \((0.66, 0.59 : 0.61, 0.58)\) | \((0.88, 0.84 : 0.83, 0.78)\) | \((0.92, 0.89 : 0.94, 0.83)\) |
|------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|
| \(t_{s=1v}\) | 21.37727 | 21.37727 | 21.37727 | 21.37727 |
| \(t_{s=2v}\) | 15.71694 | 30.27605 | 176.42810 | 347.89820 |
| \(t_{s=3v}\) | 23.25597 | 34.30783 | 72.82164 | 96.14209 |
| \(t_{s=4v}\) | 20.77428 | 28.58016 | 51.40028 | 59.05843 |
| \(t_{s=5v}\) | 21.92377 | 31.33513 | 263.27930 | 603.49040 |
| \(t_{s=6v}\) | 21.88700 | 26.52829 | 36.26390 | 38.08119 |
| \(t_{s=7v_{j=1}}\) | 15.71700 | 30.27619 | 176.41870 | 347.85300 |
| \(t_{s=7v_{j=2}}\) | 15.71754 | 30.27718 | 176.37800 | 347.60100 |
| \(t_{s=7v_{j=3}}\) | 15.71704 | 30.27624 | 176.41580 | 347.83580 |
| \(t_{s=7v_{j=4}}\) | 15.71694 | 30.27605 | 176.42760 | 347.89630 |
| \(t_{s=8v}\) | 21.92377 | 31.33513 | 263.27930 | 603.49040 |
| \(t_{s=9v}\) | 21.92377 | 31.33513 | 263.27930 | 603.49040 |
| \(t_{s=10v}\) | 33.84675 | 61.00437 | 189.48520 | 402.68180 |
| \(t_{s=P1v}\) | 45.67044 | 37.93838 | 764.39190 | 2077.81700 |
| \(t_{s=P2v}\) | 58.36229 | 65.44724 | 141.49770 | 203.16210 |

ables, the proposed difference-type estimator was of highest Percent Relative Efficiency (PRE) among all other estimators. Thus, the proposed difference-type and difference-exponential-ratio type estimators are suggested to make an efficient estimation of population mean and variance under rare and clustered pop-
ulations like pollution concentration, drug addiction, and epidemiological studies of AIDS and HIV.

The present study considered ranks of the auxiliary variables for efficient estimation of general parameters under SACS design. Zamanzade and Vock [35] found that when actual quantification of the concomi-
Figure 1. Percent Relative Efficiency (PRE) of estimators for mean in Stratified Adaptive Cluster Sampling (SACS).

Figure 2. Relative Efficiency (RE) of estimators for variance in Stratified Adaptive Cluster Sampling (SACS).
tant variable was available, the ranked set sampling would be more efficient than usual double sampling. A rewarding area for further study is to incorporate ranked set sampling under Adaptive Cluster Sampling (ACS) and SACS designs.

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