Integrable partial differential equations and Lie–Rinehart algebras

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Abstract
We develop the method for constructing Lax representations of PDEs via the twisted extensions of their algebras of contact symmetries by generalizing the construction to the Lie–Rinehart algebras. We present examples of application of the proposed technique.

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1. Introduction

Theory of integrable partial differential equations is an important part of modern mathematics, and numerous applications thereof are of big significance in physics. Lax representations are widely recognized as the key feature of integrable PDEs, being the starting point for such techniques as the inverse scattering transformations, the bi-Hamiltonian structures, the Bäcklund transformations, the recursion operators, the nonlocal symmetries, the Darboux transformations, etc., see [45, 46, 42, 33, 15, 1, 23, 34, 11, 3] and references therein. Therefore the problem of finding intrinsic properties that ensure existence of a Lax representation for a given PDE is of great interest. In the series of papers [27] — [32] we proposed the method to attack this problem via the technique of the twisted extensions of the Lie algebras of...
symmetries of the PDEs under the study. This approach is of a limited scope and can not be used in some examples. Analysis of such examples reveals that the invariants of the symmetry algebras of both the PDE and the Lax representation have to be included into the construction. This can be achieved by considering the Lie–Rinehart algebras associated to the symmetry algebras of PDEs.

In the present paper we generalize the approach of [27]—[32] for the Lie–Rinehart algebras. We discuss the twisted extensions of the Lie–Rinehart algebras as well as the extensions by appending an integral of a non-trivial 1-cocycle. Then we expose examples of constructing Lax representations via these extensions of the Lie–Rinehart algebras.

2. Preliminaries and notation

The presentation in this section closely follows [17]—[21] and [44]. Let \( \pi: \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n \), \( \pi: (x^1, \ldots, x^n, u^1, \ldots, u^m) \mapsto (x^1, \ldots, x^n) \), be a trivial bundle, and \( J^\infty(\pi) \) be the bundle of its jets of the infinite order. The local coordinates on \( J^\infty(\pi) \) are \( (x^i, u^\alpha, u^\alpha_I) \), where \( I = (i_1, \ldots, i_n) \) are multi-indices with \( i_k \geq 0 \), and for every local section \( f: \mathbb{R}^n \rightarrow \mathbb{R}^n \times \mathbb{R}^m \) of \( \pi \) the corresponding infinite jet \( j^\infty(f) \) is a section \( j^\infty(f): \mathbb{R}^n \rightarrow J^\infty(\pi) \) such that

\[
u^\alpha_I(j^\infty(f)) = \frac{\partial^{\#I} f^\alpha}{\partial x^I} = \frac{\partial^{i_1 + \ldots + i_n} f^\alpha}{(\partial x^1)^{i_1} \ldots (\partial x^n)^{i_n}}.
\]

We put \( u^\alpha = u^\alpha_{(0, \ldots, 0)} \). Also, we will simplify notation in the following way: e.g., in the case of \( n = 3 \), \( m = 1 \) we denote \( x^1 = t \), \( x^2 = x \), \( x^3 = y \), and \( u^1_{(i,j,k)} = u_{t \ldots t \ldots x \ldots x \ldots y \ldots y} \) with \( i \) times \( t \), \( j \) times \( x \), and \( k \) times \( y \).

The vector fields

\[
D_{x^k} = \frac{\partial}{\partial x^k} + \sum_{#I \geq 0} \sum_{\alpha=1}^m u^\alpha_{I+1k} \frac{\partial}{\partial u^\alpha_I}, \quad k \in \{1, \ldots, n\},
\]

\((i_1, \ldots, i_k, \ldots, i_n) + 1_k = (i_1, \ldots, i_k + 1, \ldots, i_n)\), are called total derivatives. They commute everywhere on \( J^\infty(\pi): [D_{x^i}, D_{x^j}] = 0 \).

The evolutionary vector field associated to an arbitrary vector-valued smooth function \( \varphi: J^\infty(\pi) \rightarrow \mathbb{R}^m \) is the vector field

\[
E_\varphi = \sum_{#I \geq 0} \sum_{\alpha=1}^m D_I(\varphi^\alpha) \frac{\partial}{\partial u^\alpha_I}
\]

with \( D_I = D_{(i_1, \ldots, i_n)} = D_{x^{i_1}} \circ \cdots \circ D_{x^{i_n}} \).
A system of PDEs \( F_r(x^i, u_{\alpha I}^i) = 0 \) of the order \( s \geq 1 \) with \( \#I \leq s \), \( r \in \{1, \ldots, R\} \) for some \( R \geq 1 \), defines the submanifold \( \mathcal{E} = \{(x^i, u_{\alpha I}^i) \in J^\infty(\pi) \mid D_K(F_r(x^i, u_{\alpha I}^i)) = 0, \#K \geq 0\} \) in \( J^\infty(\pi) \).

A function \( \varphi: J^\infty(\pi) \to \mathbb{R}^m \) is called a (generator of an infinitesimal) symmetry of equation \( \mathcal{E} \) when \( E_\varphi(F) = 0 \) on \( \mathcal{E} \). The symmetry \( \varphi \) is a solution to the defining system

\[
\ell_\mathcal{E}(\varphi) = 0,
\]

where \( \ell_\mathcal{E} = \ell_F|_\mathcal{E} \) with the matrix differential operator

\[
\ell_F = \left( \sum_{\#I \geq 0} \frac{\partial F_r}{\partial u_{\alpha I}^i} D_I \right).
\]

The symmetry algebra \( \text{Sym}(\mathcal{E}) \) of equation \( \mathcal{E} \) is the linear space of solutions to (1) endowed with the structure of a Lie algebra over \( \mathbb{R} \) by the Jacobi bracket

\[
\{\varphi, \psi\} = E_\varphi(\psi) - E_\psi(\varphi).
\]

The algebra of contact symmetries \( \text{Sym}_0(\mathcal{E}) \) is the Lie subalgebra of \( \text{Sym}(\mathcal{E}) \) defined as \( \text{Sym}(\mathcal{E}) \cap C^\infty(J^1(\pi)) \).

Let the linear space \( \mathcal{W} \) be either \( \mathbb{R}^N \) for some \( N \geq 1 \) or \( \mathbb{R}^\infty \) endowed with local coordinates \( w^a, a \in \{1, \ldots, N\} \) or \( a \in \mathbb{N} \), respectively. Variables \( w^a \) are called pseudopotentials \([45]\). Locally, a differential covering of \( \mathcal{E} \) is a trivial bundle \( \pi: J^\infty(\pi) \times \mathcal{W} \to J^\infty(\pi) \) equipped with extended total derivatives

\[
\tilde{D}_{x^k} = D_{x^k} + \sum_a T^a_k(x^i, u_{\alpha I}^i, w^b) \frac{\partial}{\partial w^a}
\]

such that \( [\tilde{D}_{x^i}, \tilde{D}_{x^j}] = 0 \) for all \( i \neq j \) if and only if \( (x^i, u_{\alpha I}^i) \in \mathcal{E} \). Define the partial derivatives of \( w^a \) by \( w^s_{x^k} = \tilde{D}_{x^k}(w^s) \). This yields the over-determined system of PDEs

\[
w^a_{x^k} = T^a_k(x^i, u_{\alpha I}^i, w^b) \tag{2}
\]

which is compatible if and only if \( (x^i, u_{\alpha I}^i) \in \mathcal{E} \). System (2) is referred to as the covering equations or the Lax representation of equation \( \mathcal{E} \).

Dually, the differential covering is defined by the Wahlquist-Estabrook forms

\[
\tau^a = dw^a - \sum_{k=1}^m T^a_k(x^i, u_{\alpha I}^i, w^b) \, dx^k \tag{3}
\]

as follows: when \( w^a \) and \( u^a \) are considered to be functions of \( x^1, \ldots, x^n \), forms (3) are equal to zero if and only if system (2) holds.
3. Lie–Rinehart algebras and their extensions

While Élie Cartan was well aware of the constructions underlying Lie–Rinehart algebras, see [6], at first time these algebras were introduced explicitly by J.-C. Herz [12] under the name of ‘Lie pseudo-algebras’. Then they were examined by R. Palais [37] under the name ‘d-Lie rings’ and studied by G. Rinehart [41]. The geometric counterpart of the Lie–Rinehart algebras are the Lie algebroids, see survey [22].

The notion of the twisted Lie algebroid cohomology was defined in [9]. The first principle study of the LR algebra extensions were done (albeit, in a different language) in [14]. The extensive and proper study of the Lie algebroid/Lie–Rinehart algebra extensions were done in [5] and (in full generality) in [2]. The very natural LR algebra construction was proposed in the framework of the geometric approach to PDEs. These Lie algebroid/LRA structures (under the name ”Der-modules”) were introduced by A.M. Vinogradov, I.S. Krasil’shchik and V.V. Lychagin in their various works in 1970–1986, see [16] and references therein. This algebras naturally appear in geometry of jet spaces. The cohomology of Der-complexes (including the extensions) were studied in 1980 thesis of V.N. Rubtsov and summarized in [43].

In this section we follow [41, 13, 22] in exposition of the basic definitions of the theory of Lie–Rinehart algebras. Then we discuss the twisted extensions of these algebras as well as the extensions by appending an integral of a non-trivial 1-cocycle.

DEFINITION 1. Let \( R \) be a commutative ring, \( \mathcal{A} \) be a commutative \( R \)-algebra, and let \( \mathfrak{g}_\mathcal{A} \) be a Lie algebra over \( R \) equipped with two structures:

1. a structure of a left \( \mathcal{A} \)-module on \( \mathfrak{g}_\mathcal{A} \), that is, a map \( \mathfrak{g}_\mathcal{A} \otimes \mathcal{A} \rightarrow \mathfrak{g}_\mathcal{A} \), \( a \otimes x \mapsto a \cdot x \), such that
   \[(a \cdot b) \cdot x = a \cdot (b \cdot x)\];

2. a map \( \Psi: \mathfrak{g}_\mathcal{A} \rightarrow \text{Der}(\mathcal{A}) \) called the anchor which is a homomorphism of Lie algebras over \( R \) and a homomorphism of \( \mathcal{A} \)-modules, that is
   \[\Psi([x, y]) = [\Psi(x), \Psi(y)]\]
   and
   \[\Psi(a \cdot x)(b) = a \cdot (\Psi(x)(b))\]

\( \otimes \) the unadorned tensor product symbol will refer to the tensor product over \( R \).
for \( x, y \in \mathfrak{g}_A \) and \( a, b \in \mathcal{A} \).

Then \( \mathfrak{g}_A \) is referred to as a \textit{Lie–Rinehart algebra over} \( \mathcal{A} \) provided there holds

\[
[x, a \cdot x] = a \cdot [x, y] + \Psi(x)(a) \cdot y.
\]

**DEFINITION 2.** A \textit{Lie-Rinehart module over a Lie–Rinehart algebra} \( \mathfrak{g}_A \) is a vector space \( V \) equipped with two operations

\[
\mathfrak{g}_A \otimes V \to V, \quad x \otimes v \mapsto x(v)
\]

and

\[
\mathcal{A} \otimes V \to V, \quad a \otimes v \mapsto a \cdot v
\]

such that the first map makes \( V \) into a Lie algebra module over the Lie \( \mathcal{R} \)-algebra \( \mathfrak{g}_A \), while the second map makes \( V \) into an \( \mathcal{A} \)-module and additionally there hold

\[
(a \cdot x)(v) = a \cdot (x(v)),
\]

\[
x(a \cdot v) = a \cdot x(v) + \Psi(x)(a) \cdot v.
\]

**DEFINITION 3.** Let \( V \) be a Lie–Rinehart module over the Lie–Rinehart algebra \( \mathfrak{g}_A \). Put \( C^0(\mathfrak{g}_A, V) = V \) and \( C^k(\mathfrak{g}_A, V) = \text{Hom}_\mathcal{A}(\Lambda^k(\mathfrak{g}_A), V) \) for \( k \geq 1 \). For \( k \geq 0 \) define differential

\[
d: C^k(\mathfrak{g}_A, V) \to C^{k+1}(\mathfrak{g}_A, V)
\]

by

\[
d\theta(x_1, ..., x_{k+1}) = \sum_{j=1}^{k+1} (-1)^{j+1} \Psi(x_j) (\theta(x_1, ..., \hat{x}_j, ..., x_{k+1}))
\]

\[+ \sum_{1 \leq i < j \leq k+1} (-1)^{i+j+1} \theta([x_i, x_j], x_1, ..., \hat{x}_p, ..., \hat{x}_q, ..., x_{k+1}). \tag{5}\]

The cohomology groups of the complex

\[
C^0(\mathfrak{g}_A, V) \xrightarrow{d} C^1(\mathfrak{g}_A, V) \xrightarrow{d} \ldots \xrightarrow{d} C^k(\mathfrak{g}_A, V) \xrightarrow{d} C^{k+1}(\mathfrak{g}_A, V) \xrightarrow{d} \ldots
\]

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are
\[ H^k(g_A, V) = \frac{Z^k(g_A, V)}{B^k(g_A, V)} = \ker d: C^k(g_A, V) \to C^{k+1}(g_A, V) \]
\[ \text{im } d: C^{k-1}(g_A, V) \to C^k(g_A, V). \]

**REMARK 1.** It is natural to consider \( H^k(g_A, \mathcal{A}) \) as the cohomology groups of \( g_A \) with trivial coefficients. These groups will be denoted as \( H^k(g_A) \). Likewise, we denote \( C^k(g_A, \mathcal{A}) = C^k(g_A) \), \( Z^k(g_A, \mathcal{A}) = Z^k(g_A) \), and \( B^k(g_A, \mathcal{A}) = B^k(g_A) \).

Below we consider Lie–Rinehart algebras within the following specific setting:

1. \( \mathbb{R} = \mathbb{R} \),
2. \( \mathcal{A} \) is the algebra of smooth or real-analytic functions \( f(w) = f(w^1, \ldots, w^n) \) defined on an open set \( W \subseteq \mathbb{R}^n \),
3. the Lie algebra \( g_A \) is a free \( \mathcal{A} \)-module with finite or countable set of generators \( v_m \), where \( m \in \{1, \ldots, M\} \) for some \( M \geq 1 \) or \( m \in \mathbb{N} \). In the last case elements of \( g_A \) are linear combinations \( \sum_m f_m(w) v_m \) with finite number of non-zero functions \( f_m \).

Commutators of the basis elements
\[ [v_i, v_j] = \sum_k c^k_{ij}(w) v_k \]  \hspace{1cm} (6)

define the *structure functions* \( c^k_{ij}(w) \), and the anchor has the form
\[ \Psi(v_i) = \sum_{q=1}^{n} h^q_i(w) \partial_{w^q} \]  \hspace{1cm} (7)

for some functions \( h^q_i(w) \). The skew-symmetry of commutator entails \( c^k_{ij}(w) = -c^k_{ji}(w) \). The Jacobi identity \( \sum_{\text{cycl}(i,j,k)} [v_i, [v_j, v_k]] = 0 \) gives
\[ \sum_{\text{cycl}(i,j,k)} \left( \sum_{q} h^q_i \partial_{w^q} c^m_{jk} + \sum_{l} c^l_{jk} c^m_{li} \right) = 0, \]
while from (7) it follows that
\[ \sum_{s} (h^q_i \partial_{w^s} h^q_j - h^q_j \partial_{w^s} h^q_i) = \sum_k c^k_{ij} h^q_k. \]
Consider $A$-linear functions $\theta^i: \mathfrak{g}_A \to A$ defined by $\theta^i(v_j) = \delta_{ij}$. Then (5), (6), and (7) yield the structure equations
\[
\begin{align*}
d\theta^i &= -\sum_{j<k} c^i_{jk}(w) \theta^j \wedge \theta^k, \\
dw^q &= \sum_i h^q_i(w) \theta^i.
\end{align*}
\]
of the Lie–Rinehart algebra $\mathfrak{g}_A$.

In all the examples below the image of the anchor is finite-dimensional, in other words, the sums in the RHS of equations for $dw^q$ are finite. For such Lie–Rinehart algebras we can assume without loss of generality that $\text{rank}(h^q_i) = n = \dim W$, since otherwise we can reduce the number of functionally independent variables $w^q$. We rename $\sum_i h^q_i \theta^i =: \eta^q$, then we have $dw^q = \eta^q$ and $d\eta^q = 0$, so $B^1(\mathfrak{g}_A) = \langle \eta^1, \ldots, \eta^n \rangle$. Furthermore, for a Lie–Rinehart algebra with the finite-dimensional image of the anchor we can write the structure equations in the form
\[
\begin{align*}
d\theta^i &= \sum_{j<k} P^i_{jk}(w) \vartheta^j \wedge \vartheta^k + \sum_{j,q} Q^i_{jq}(w) \vartheta^j \wedge \eta^q + \sum_{q<s} R^i_{qs}(w) \eta^q \wedge \eta^s, \\
d\eta^q &= 0, \\
dw^q &= \eta^q
\end{align*}
\]
with some functions $P^i_{jk}, Q^i_{jq}, R^i_{qs}$ and 1-forms $\vartheta^j$ such that collection $\{\eta^q, \vartheta^j\}$ provides a basis for $C^1(\mathfrak{g}_A)$.

**DEFINITION 4.** Consider a Lie–Rinehart algebra $\mathfrak{g}_A$ with $H^1(\mathfrak{g}_A) \neq \{[0]\}$. Let $\alpha$ be a non-trivial 1-cocycle, that is, $d\alpha = 0$ and $\alpha \notin B^1(\mathfrak{g}_A)$. For a constant $c \in \mathbb{R}$ define the twisted differential $d^t_{c\alpha}: C^k(\mathfrak{g}_A) \to C^{k+1}(\mathfrak{g}_A)$ by the formula
\[
d^t_{c\alpha}\theta = d\theta - c \alpha \wedge \theta.
\]
Then $d^2_{c\alpha} = 0$. The cohomology groups $H^*_\alpha(\mathfrak{g}_A)$ of the associated complex are referred to as the twisted cohomology groups of $\mathfrak{g}_A$.

**DEFINITION 5.** Suppose $H^2_{\alpha}(\mathfrak{g}_A) \neq \{[0]\}$ for some $c \in \mathbb{R}$ and $\Omega$ is a non-trivial twisted 2-cocycle. Then equation
\[
d\sigma = c \alpha \wedge \sigma + \Omega
\]
with unspecified 1-form $\sigma$ is compatible with the structure equations of $\mathfrak{g}_A$. The Lie–Rinehart algebra $\tilde{\mathfrak{g}}_A$ with the structure equations obtained by appending (8) to the structure equations of $\mathfrak{g}_A$ is referred to as the twisted extension of $\mathfrak{g}_A$.

**EXAMPLE 1.** Consider the Lie–Rinehart algebra

$$\mathfrak{g}_A = \left\{ \sum_{k=1}^{4} f_k(w) v_k \mid f_k \in C^\infty(\mathbb{R}) \right\}$$

over $\mathcal{A} = C^\infty(\mathbb{R})$ with non-zero commutators

$$[v_1, v_2] = -v_2, \quad [v_1, v_3] = -v_3, \quad [v_2, v_4] = -v_3$$

of the basis elements $v_1, \ldots, v_4$ and the anchor

$$\Psi(v_k) = \begin{cases} 0, & 1 \leq k \leq 3, \\ \partial_w, & k = 4. \end{cases}$$

The structure equations of $\mathfrak{g}_A$ read

$$\begin{cases} d\theta^1 &= 0, \\ d\theta^2 &= \theta^1 \wedge \theta^2, \\ d\theta^3 &= \theta^1 \wedge \theta^3 + \theta^2 \wedge \theta^4, \\ d\theta^4 &= 0, \\ dw &= \theta^4. \end{cases} \quad (9)$$

We have $H^1(\mathfrak{g}_A) = \{[\theta^1]\}$, and the straightforward computations give

$$H^2_{\omega\theta}(\mathfrak{g}_A) = \begin{cases} \langle [\theta^1 \wedge \theta^2], [\theta^1 \wedge (w \theta^2 + \theta^3)] \rangle, & c = 1, \\ \langle [\theta^2 \wedge \theta^3], & c = 2, \\ 0, & c \notin \{1, 2\}. \end{cases}$$

Therefore we have the three-dimensional twisted extension of $\mathfrak{g}_A$ defined by appending equations

$$\begin{cases} d\sigma^1 &= \theta^1 \wedge \sigma^1 + \theta^1 \wedge \theta^2, \\ d\sigma^2 &= \theta^1 \wedge \sigma^2 + \theta^1 \wedge (w \theta^2 + \theta^3), \\ d\sigma^3 &= 2 \theta^1 \wedge \sigma^3 + \theta^2 \wedge \theta^3 \end{cases}$$

to system (9). Then in the basis $\langle v_1, \ldots, v_7 \rangle$ dual to forms $\theta^k, \sigma^j$ the non-zero commutators for the extended Lie–Rinehart algebra are

$$[v_1, v_2] = -v_2 - v_5 - w v_6, \quad [v_1, v_3] = -v_3 - v_6, \quad [v_1, v_5] = -v_5.$$
\[ [v_1, v_6] = -v_6, \quad [v_1, v_7] = -2v_7, \quad [v_2, v_3] = -v_7, \quad [v_2, v_4] = -v_3, \]

and for the anchor we have \( \Psi(v_k) = 0 \) when \( k \in \{5, 6, 7\} \).

**DEFINITION 6.** Suppose we have \( H^1(\mathfrak{g}_A) \neq \{0\} \) for a Lie–Rinehart algebra \( \mathfrak{g}_A \), and \( \alpha \) is a non-trivial 1-cocycle on \( \mathfrak{g}_A \). Then we extend \( \mathcal{A} \) and thus \( \mathfrak{g}_A \) by considering algebra \( \tilde{A} = C^\infty(W \times \mathbb{R}) \) of functions \( f(w^1, \ldots, w^{n+1}) \) and extending the anchor by \( dw^{n+1} = \alpha \). We refer this extension as appending an integral of \( \alpha \). Notice that \( \alpha \in B^1(\tilde{\mathfrak{g}}_\tilde{A}) \).

**REMARK 2.** The procedure of extension by appending an integral of a 1-cocycle is applicable to a Lie algebra over \( \mathbb{R} \) with non-trivial first cohomology group. If \( H^1(\mathfrak{a}) \neq 0 \) for a Lie algebra \( \mathfrak{a} \) and \( \alpha \) is a non-trivial 1-cocycle, then the extended algebra is the Lie–Rinehart algebra \( \mathfrak{a}_{C^\infty(\mathbb{R})} \), where \( C^\infty(\mathbb{R}) \) consists of smooth functions \( f(w) \) of \( w \in \mathbb{R} \) and the structure equations of \( \mathfrak{a}_{C^\infty(\mathbb{R})} \) are obtained by appending equation \( dw = \alpha \) to the structure equations of \( \mathfrak{a} \).

**DEFINITION 7.** For a Lie–Rinehart algebra \( \mathfrak{g}_A \) with non-trivial second twisted cohomology group we can combine the procedures of twisted extension and appending an integral. Namely, if \( \alpha \) is a non-trivial 1-cocycle and \( \Omega \) is non-trivial twisted 2-cocycle with \( d\Omega = c \alpha \wedge \Omega \) for \( c \in \mathbb{R} \), we define the combined extension of \( \mathfrak{g}_A \) in two steps: first, constructing the twisted extension \( \tilde{\mathfrak{g}}_A \) of \( \mathfrak{g}_A \), and then extending \( \mathcal{A} \) to \( \tilde{\mathcal{A}} \) by appending an integral \( w \) of 1-cocycle \( \alpha \). The resulting Lie–Rinehart algebra \( \tilde{\mathfrak{g}}_\tilde{A} \) is not a twisted extension of \( \mathfrak{g}_A \) anymore, since \( \alpha \in B^1(\tilde{\mathfrak{g}}_\tilde{A}) \). The structure equations of \( \tilde{\mathfrak{g}}_\tilde{A} \) are obtained from the structure equations of \( \tilde{\mathfrak{g}}_A \) by adding equations \( d\sigma = c \alpha \wedge \sigma + \Omega \) and \( dw = \alpha \).

4. Lax representations via extensions of Lie–Rinehart algebras

In this section we expose three examples of constructing Lax representations via the procedures of the combined extension of a Lie–Rinehart algebra and extension of a Lie algebra by appending an integral of a non-trivial 1-cocycle. To the best of our knowledge the results of Examples 2 and 3 can not be recovered by the method of [27]. Example 4 exposes new Lax representation for the hyper-CR equation of Einstein–Weyl structures [19].
EXAMPLE 2. Consider equation $\mathcal{E}_1$

$$u_{yy} = \frac{u_{tx}}{u_{xy}} + F(u_{xx})u_{xy}^2, \quad (10)$$

where function $F$ is a solution to Chazy’s equation

$$F''' + 12F F'' - 18 (F')^2 = 0. \quad (11)$$

Equation (10) was introduced in \cite{38}, the Lax representation thereof was presented in \cite{40} in implicit form and in \cite{8} in explicit form.

The algebra $\text{Sym}_0(\mathcal{E}_1)$ of contact symmetries for equation (10) has generators:

$$\begin{align*}
\varphi_0(A_0) &= -A_0 u_t - \frac{1}{3} A_0' y u_y - \frac{1}{18} A_0'' y^3, \\
\varphi_1(A_1) &= -A_1 u_y - \frac{1}{2} A_1' y^2, \\
\varphi_2(A_2) &= A_2 y, \\
\varphi_3(A_3) &= A_3, \\
\psi_0 &= 3 u - \frac{3}{2} x u_x - y u_y, \\
\psi_1 &= -u_x, \\
\psi_2 &= x.
\end{align*}$$

where $A_i = A_i(t)$ are arbitrary smooth functions of $t$. The action of $\text{Sym}_0(\mathcal{E}_1)$ on $J^2(\pi)$ with $\pi: (t, x, y, u) \mapsto (t, x, y)$ has two invariants $u_{xx}$ and $(u_{xy} u_{yy} - u_{tx}) u_{xy}^{-3}$. These invariants are functionally dependent when restricted to $\mathcal{E}_1$: $(u_{xy} u_{yy} - u_{tx}) u_{xy}^{-3} = F(u_{xx})$. Using the technique of moving frames \cite{35, 7, 36} the structure equations of $\text{Sym}_0(\mathcal{E}_1)$ can be written in the form

$$\begin{cases}
d\alpha_0 &= 0, \\
d\alpha_1 &= \alpha_0 \wedge \alpha_1, \\
d\alpha_2 &= \alpha_0 \wedge \alpha_2 - \eta \wedge \alpha_1, \\
d\eta &= 0, \\
d\Theta &= h_0 \alpha_0 \wedge \partial_{h_0} \Theta + \partial_{h_1} \Theta \wedge (\Theta - \frac{2}{3} h_0 \partial_{h_0} \Theta), \\
d\theta_{3,-1} &= 2 \alpha_0 \wedge \theta_{3,-1} + \theta_{3,0} \wedge \theta_{0,0} + \frac{1}{3} \theta_{2,0} \wedge \theta_{1,0} + \alpha_1 \wedge \alpha_2, \\
dw &= \eta,
\end{cases} \quad (12)$$

\footnote{We carried out computations of generators of contact symmetries in the Jets software \cite{4}.}
where
\[ \Theta = \sum_{k=0}^{3} \sum_{m=0}^{\infty} \frac{1}{m!} h^k_0 h^m_1 \theta_{k,m}, \]

\( h^k_0 = 0 \) when \( k > 3 \), \( dh_i = 0 \), and \( w = u_{xx} \). Equations for \( d\alpha_0 \), \( d\alpha_2 \), \( d\alpha_2 \), \( d\eta \), and \( dw \) differ only in notation from system (9), therefore, according to Example 1 and Definition 7, the Lie–Rinehart algebra with the structure equations (12) admits the combined extension whose structure equations are obtained by appending equations

\[ d\sigma = \alpha_0 \wedge \sigma + \alpha_0 \wedge \alpha_1 \quad (13) \]

and

\[ dq = \alpha_0 \]

to system (12). In these equations \( \sigma \) is an unspecified 1-form and \( q \) is new invariant. In what follows we need explicit expressions for the Maurer–Cartan forms

\[ \alpha_1 = e^q dx, \]

\[ \alpha_2 = e^q (du_x - u_{xx} dt), \]

\[ \eta = du_{xx}, \]

\[ \theta_{0,0} = e^q u_{xy}^3 dt, \]

\[ \theta_{1,0} = e^q (u_{xy} dy + (u_{tx} - 2F u_{xy}^3) dt), \]

\[ \theta_{3,-1} = e^{2q} (du - ut dt - u_x dx - u_y dy). \]

Integration of equation (13) yields

\[ \sigma = e^q (dv + q dx). \]

To find the Wahlquist–Estabrook form of a Lax representation for equation (10) we consider the linear combination

\[ \sigma - P_1 \theta_{0,0} - P_2 \theta_{1,0} = e^q \left( dv + q dx - P_2 u_{xy} dy - (P_1 u_{xy}^3 + P_2 (u_{tx} - 2F u_{xy}^3)) dt \right), \]
where coefficients $P_i$ are functions of invariants $u_{xx}$ and $q$. This 1-form defines the Lax representation

$$
\begin{align*}
    v_t &= P_1 u_{xy}^3 + P_2 (u_{tx} - 2 F u_{xy}^3), \\
    v_y &= P_2 u_{xy},
\end{align*}
$$

(14)

provided $q = -v_x$ and thus $P_i = P_i(u_{xx}, v_x)$. System (14) differs only in notation from the Lax representation found in [8]. Analysis of compatibility of (14) yields

$$
P_1 = \frac{1}{2} (P_{2,u_{xx}} + P_{2,v_x}) + 2 P_2 F
$$

and the over-determined system

$$
\begin{align*}
P_{2,v_x,v_x} &= \frac{2 P_3^2 - F_{u_{xx}u_{xx}} - 6 F_{u_{xx}} P_{2,v_x} - 6 F P_{2,v_x}^2}{P_{2,u_{xx}} + P_{2,v_x}}, \\
P_{2,v_x,u_{xx}} &= \frac{P_2 F_{u_{xx}u_{xx}} + 3 F_{u_{xx}} (P_{2,v_x} P_{2,v_x} - P_{2,u_{xx}}) - 6 F P_{2,v_x} P_{2,u_{xx}} + 2 P_{2,v_x}^2 P_{2,u_{xx}}}{P_{2,u_{xx}} + P_{2,v_x}}, \\
P_{2,u_{xx}u_{xx}} &= \frac{2 P_{2,v_x} P_{2,u_{xx}}^2 - 6 F P_{2,u_{xx}}^2 - P_{2,u_{xx}}^2 F_{u_{xx}u_{xx}} + 6 P_2 F_{u_{xx}} P_{2,u_{xx}}}{P_{2,u_{xx}} + P_{2,v_x}},
\end{align*}
$$

for function $P_2$. In its turn this system is compatible if and only if equation (11) holds.

REMARK 3. While each equation $(u_{xy} u_{yy} - u_{tx}) u_{xy}^{-3} = G(u_{xx})$ with an arbitrary function $G$ admits $\text{Sym}_0(\mathcal{E}_1)$ as the symmetry algebra, this equation possesses the Lax representation if and only if $G$ is a solution to Chazy’s ODE (11), cf. [8].

EXAMPLE 3. Equation $\mathcal{E}_2$

$$
u_{yy} = u_g (u_{ty} + u_x u_{xy} - u_y u_{xx})
$$

(15)

was introduced in [26]. Algebra $\text{Sym}_0(\mathcal{E}_2)$ of contact symmetries of this equation is generated by functions

$$
\varphi_0(A_0) = -A_0 u_t - A_0' u_x + A_0' u + \frac{1}{2} A_0'' x^2,
$$
\[ \varphi_1(A_1) = -A_1 \ u_x + A_1' \ x, \]
\[ \varphi_2(A_2) = A_2, \]
\[ \psi_0 = -y \ u_y, \]
\[ \psi_1 = -u_y, \]
where \( A_i = A_i(t) \) are arbitrary functions of \( t \). The structure equations of \( \text{Sym}_0(\mathcal{E}_2) \) can be written in the form

\[
\begin{cases}
  d\alpha_0 &= 0, \\
  d\alpha_1 &= \alpha_0 \wedge \alpha_1, \\
  d\Theta &= \partial h_1 \Theta \wedge \Theta,
\end{cases}
\]

where

\[ \Theta = \sum_{k=0}^{2} \sum_{m=0}^{\infty} \frac{1}{m!} h_k^m \theta_{k,m}, \]

\( h_k = 0 \) for \( k > 2 \), and \( dh_i = 0 \). From these equations it follows that

\[ H^1(\text{Sym}(\mathcal{E}_2)) = \langle [\alpha_0] \rangle \]

and

\[ H^2_{\alpha_0}(\text{Sym}(\mathcal{E}_2)) = \begin{cases} \langle [\alpha_0 \wedge \alpha_1] \rangle, & c = 1, \\ \{[0]\}, & c \neq 1. \end{cases} \]

The non-trivial twisted 2-cocycle defines the twisted extension of the Lie algebra \( \text{Sym}(\mathcal{E}_2) \) with the additional structure equation

\[ d\sigma = \alpha_0 \wedge \sigma + \alpha_0 \wedge \alpha_1. \]

In accordance with Remark 2 the obtained Lie algebra admits extension by appending integral of \( \alpha_0 \). The resulting Lie–Rinehart algebra has the following Maurer–Cartan forms

\[ \alpha_0 = dq, \]
\[ \alpha_1 = e^q \ dy, \]
\[ \theta_{0,0} = u_y^{-1} e^q \ dt, \]
\[ \theta_{1,0} = u_y^{-1} e^q (dx - u_x dt), \]
\[ \theta_{2,0} = u_y^{-1} e^q (du - u_t dt - u_x dx), \]
\[ \sigma = e^q (dv + q dy). \]

Consider the linear combination
\[ \tau = \sigma - Q_1 \theta_{1,0} - Q_2 \theta_{0,0} = e^q \left( dv - Q_1 u_y^{-1} dx - \frac{Q_2 - Q_1 u_x}{u_y} dt + q dy \right), \]
where \( Q_i \) are functions of \( q \). Upon setting \( \tau = 0 \) we obtain the overdetermined system for function \( v = v(t, x, y) \). This system yields \( q = -v_y \) and hence \( Q_i = Q_i(v_y) \). Analysis of compatibility of two other equations
\[
\begin{aligned}
    v_t &= \frac{Q_2 - Q_1 u_x}{u_y}, \\
    v_x &= \frac{Q_1 u_y}{u_y},
\end{aligned}
\]
(16)
of the system gives
\[ Q_1 = \frac{1}{\Phi'}, \quad Q_2 = \frac{\Phi}{\Phi'}, \]
(17)
where \( \Phi = \Phi(v_y) \) is a solution to ODE
\[ \Phi'' = \Phi (\Phi')^2. \]
(18)
Up to a change of notation this equation defines function \( \Phi(v_y) \) implicitly by formula
\[ v_y = \text{erf}(\Phi) = \frac{2}{\sqrt{\pi}} \int_0^\Phi e^{-z^2} \, dz. \]

In another notation system (16), (17), (18) was found in [26] by the method of contact integrable extensions proposed in [25].

\textbf{EXAMPLE 4}. Consider the hyper-CR equation of Einstein–Weyl structures \( \mathcal{E}_3 \)
\[ u_{yy} = u_{tx} + u_y u_{xx} - u_x u_{xy}. \]
(19)
introduced independently in [24, 39, 10], where an ‘isospectral’ Lax representation for this equation was found. As we show in [29, 30], this Lax representation as well as its ‘nonisospectral’ generalization can be derived from the twisted extension of the symmetry algebra Sym(E) of (19). In this example we apply the technique described in Remark 2 to find the further generalization of the Lax representation from [30].

As we show in [29], the structure equations of the Lie algebra Sym(E) read

\[
\begin{align*}
\text{d} \alpha_0 &= 0, \\
\text{d} \alpha_1 &= \alpha_0 \wedge \alpha_1, \\
\text{d} \Theta &= \nabla_1(\Theta) \wedge \Theta + (h_0 \alpha_0 + h_0^2 \alpha_1) \wedge \nabla_0(\Theta),
\end{align*}
\]

where

\[
\Theta = \sum_{k=0}^{3} \sum_{m=0}^{\infty} h_0^k h_1^m \frac{m!}{m!} \theta_{k,m},
\]

with the formal parameters \( h_0 \) and \( h_1 \) such that \( h_0^k = 0 \) when \( k > 3 \). The additional structure equation for the twisted extension of Sym(E) has the form

\[
\text{d} \sigma = \alpha_0 \wedge \sigma + \alpha_0 \wedge \alpha_1.
\]

Just as in papers [29, 30], we need the following Maurer–Cartan forms for constructing the Lax representations of equation (19). \( \alpha_0 = dq, \alpha_1 = -e^q ds, \theta_{0,0} = r \, dt, \theta_{1,0} = r \, e^q(dy - (u_x - 2s) \, dt), \theta_{2,0} = r \, e^{2q}(dx - (u_x - s) \, dy - (uy + s \, s - s^2) \, dt), \theta_{3,0} = r \, e^{3q}(du - ut \, dt - ux \, dx - uy \, dy), \) and \( \sigma = e^q(dv - q \, ds), \) where \( q, s, v, \) and \( r \) are free parameters. We choose the linear combination

\[
\tau = \sigma - \sum_{k=0}^{2} S_k \theta_{k,0} = e^q(dv - q \, ds - S_2 r \, e^q \, dx - r(S_1 + S_2 e^q(s - u_x)) \, dy
\]

\[
- r(S_0 e^{-q} + S_1 (2s - u_x) + S_2 e^q(s^2 - s \, u_x - u_y)) \, dt)
\]

of the form \( \sigma \) and the basic horizontal forms \( \theta_{0,0}, \theta_{1,0}, \theta_{2,0} \) as the Wahlquist–Estabrook form of a Lax representation. Unlike the computations in [24,25], we now treat coefficients \( S_k \) as functions of the integral \( q \) of form \( \alpha_0 \in H^1(\text{Sym}(E)) \) rather than constants. Since the restriction of form \( \tau \) to the sections of the bundle \((t, x, y, u, v) \mapsto (t, x, y)\) has to be equal to zero, we
put $q = v_s$. By renaming $r$ we obtain without loss of generality $S_2 = 1$ and $r = v_x \exp (-v_s)$. Then the form

$$\tau = e^q \left( dv - v_t \, ds - v_x (dx + (s - u_x + S_1 e^{-v_s}) \, dy \\
+ (s^2 - s \, u_x - u_y + S_1 e^{-v_s} (2 \, s - u_x) + S_0 e^{-2v_s}) \, dt) \right)$$

is equal to zero whenever there hold

$$\begin{cases}
v_t &= (s^2 - s \, u_x - u_y + S_1 e^{-v_s} (2 \, s - u_x) + S_0 e^{-2v_s}) \, v_x, \\
v_y &= (s - u_x + S_1 e^{-v_s}) \, v_x.
\end{cases} \quad (20)$$

Just as in paper [30], the analysis of compatibility condition $(v_t)_y = (v_y)_t$ for system (20) leads to $S_0 = S_1^2$. Denoting $R = S_1 e^{-v_s}$ we obtain the Lax representation

$$\begin{cases}
v_t &= (s^2 - s \, u_x - u_y + R (2 \, s - u_x) + R^2) \, v_x, \\
v_y &= (s - u_x + R) \, v_x.
\end{cases} \quad (21)$$

of equation (19) with an arbitrary function $R = R(v_s)$. When $R = 0$, this system coincides with the Lax representation from [24, 30, 10], while when $R = e^{-v_s}$ we get the Lax representation from [30].

\section{5. Concluding remarks}

We have proposed the generalization of the method for constructing Lax representations based on twisted extensions of Lie algebras to the Lie-Rinehart algebras and showed that new technique allows one to recover in a simple manner known results as well as to find new Lax representations. We hope that further examples will clarify this technique and the limits of its applicability. The very important issue to address in the future research is to establish relationship between extensions of Lie-Rinehart algebras and the method of contact integrable extensions of Lie symmetry pseudo-groups proposed in [25].

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