Space of Invariant bilinear forms under representation of a group of order 8

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Abstract

Let $G$ be a group of order 8 and $F$ an algebraically closed field with $\text{char}(F) \neq 2$. In this paper we compute the number of $n$ degree representations of $G$ and subsequent dimensions of the corresponding spaces of invariant bilinear forms over the field $F$. We explicitly discuss about the existence of non-degenerate invariant bilinear forms.

Keywords: Invariant bilinear forms, Representation of groups, Vector spaces, Direct sums

2010 MSC: 15A63, 11E04, 06B15, 15A03

1. Introduction

Definition 1.1. A homomorphism $\rho : G \rightarrow GL(V(F))$ is called a representation of the group $G$, where $V(F)$ is a finite dimensional vector space over $F$. $V(F)$ is also called a representing space of $G$. The dimension of $V(F)$ over $F$ is called degree of the representation $\rho$.

Definition 1.2. A bilinear form on a finite dimensional vector space $V(F)$ is said to be invariant under the representation $\rho$ of a finite group $G$ if

$$B(\rho(g)x, \rho(g)y) = B(x, y), \quad \forall \ g \in G \text{ and } x, y \in V(F).$$

Let $\Xi$ denotes the space of bilinear forms on the vector space $V(F)$ over $F$.

Definition 1.3. The space of invariant bilinear forms under the representation $\rho$ is given by

$$\Xi_G = \{B \in \Xi | B(\rho(g)x, \rho(g)y) = B(x, y), \quad \forall \ g \in G \text{ and } x, y \in V(F)\}.$$ 

Let $G$ be a group of order 8, $n \in \mathbb{Z}^+$, $F$ an algebraically closed field with $\text{char} \neq 2$ & $(\rho, V(F))$ an $n$ degree representation of $G$ over $F$. Then the corresponding set $\Xi_G$ of invariant bilinear forms on $V(F)$ under $\rho$, forms a subspace of $\Xi$. In this paper our investigation pertains to following questions.

Question. How many $n$ degree representations (upto isomorphism) of $G$ can be there? What is the dimension of $\Xi_G$ for every $n$ degree representation? What are the necessary and sufficient conditions for the existence of a
The representation $(\rho, \mathbb{V}(\mathbb{F}))$ is irreducible if there doesn’t exist any proper invariant subspace of $\mathbb{V}(\mathbb{F})$ under the representation $\rho$. Frobenius (see pp 319, Theorem (5.9) [1]) showed that there are only finitely many irreducible representations of $G$. Therefore the number of irreducible characters is finite. Also by Maschke’s theorem (see pp 316, corollary (4.9) [1]) every $n$ degree representation of $G$ can be written as a direct sum of copies of irreducible representations $\rho_i$, $i = 1, 2, 3, \ldots, r$, where $r$ is the number of irreducible representations which is same as the number of conjugacy classes $|\mathcal{C}|$ of $G$. For $\rho = \oplus_{i=1}^r k_i \rho_i$, an $n$ degree representation of $G$, the coefficient of $\rho_i$ is $k_i$, $1 \leq i \leq r$, so that $\sum_{i=1}^r d_i k_i = n$, and $\sum_{i=1}^r d_i^2 = |G|$, where $d_i$ is the degree of $\rho_i$ and $d_j||G|$ with $d_j \geq d_i$ when $j > i$. It is already well understood in the literature that the invariant space $\Xi_G$ under $\rho$ can be expressed by the set $\Xi_G^\rho = \{ X \in M_n(\mathbb{F}) \mid C_{\rho(g)}^t X C_{\rho(g)} = X, \forall g \in G \}$ with respect to an ordered basis $\mathcal{B}$ of $\mathbb{V}(\mathbb{F})$, where $M_n(\mathbb{F})$ is the set of square matrices of order $n$ as the entries from $\mathbb{F}$ and $C_{\rho(g)}$ is the matrix representation of the linear transformation $\rho(g)$.

In the discrete perspective this question has been studied in the literature. Gongopadhyay and Kulkarni [3] investigated the existence of T-invariant non-degenerate symmetric (resp. skew-symmetric) bilinear forms. Kulkarni and Tanti [8] investigated the dimension of space of T-invariant bilinear forms. Gongopadhyay, Mazumder and Sardar [6] investigated for an invertible linear map $T : V \rightarrow V$, when does the vector space $V$ over $\mathbb{F}$ admit a T-invariant non-degenerate c-hermitian form. Chen [2] discussed the all matrix representation of the real numbers. The application of representation of group, the character for a reducible representation is important for the physical problem [3]. For a cubic crystal has many symmetry operations and therefore many classes and many irreducible representations. The connection between group theory and quantum mechanics [4], the group of symmetry operators which leave the Hamiltonian invariant. These operators are symmetry operations of the system and the symmetry operators commute with the Hamiltonian. The symmetry operators are said to form the group of Schrödinger’s equation.

In this paper we investigate about the counting of $n$ degree representations of a group of order 8, dimensions of their corresponding spaces of invariant bilinear forms and establish a characterization criteria for existence of a non-degenerate invariant bilinear form. Our investigations are stated in the following three main theorems.

**Theorem 1.1.** The number of $n$ degree representations (up to isomorphism) of a group $G$ of order 8 is $\binom{n+7}{7}$ when $G$ is abelian and $\sum_{s=0}^{\binom{n}{2}} \binom{n+2s+3}{3}$ otherwise.

**Theorem 1.2.** The space $\Xi_G$ of invariant bilinear forms of a group $G$ of order 8 under an $n$ degree representation $(\rho, \mathbb{V}(\mathbb{F}))$ is isomorphic to the direct sum of the subspaces $\mathbb{W}_{(i,j)\in A_G}$ of $M_n(\mathbb{F})$, i.e., $\Xi_G = \bigoplus_{(i,j)\in A_G} \mathbb{W}_{(i,j)\in A_G}$, where $A_G = \{(i,j) \mid \rho_i$ and $\rho_j$ dual to each other $\}$ and $\mathbb{W}_{(i,j)\in A_G} = \{ X \in M_n(\mathbb{F}) \mid (i,j)^{th}$ block $X_{i,j}^{ij} = C_{k_i,\rho_i(g)}^t X_{d_i,k_i \times d_j,k_j} C_{k_j,\rho_j(g)}$, $\forall g \in G$ and rest block is zero $\}$. Also the dimension of $\mathbb{W}_{(i,j)\in A_G} = k_i k_j$.

**Theorem 1.3.** (Characterization theorem for an $n$ degree representation of a group of order 8 having a non-degenerate invariant bilinear form.)
degenerate invariant bilinear form).

A. For \( G = D_4 \) and \( \rho = \bigoplus_{i=1}^{5} k_i \rho_i \) an \( n \) degree representation, \( \rho \) always has a non-degenerate invariant bilinear form.

B. For \( G = Q_8 \) and \( \rho = \bigoplus_{i=1}^{5} k_i \rho_i \) an \( n \) degree representation, \( \rho \) always has a non-degenerate invariant bilinear form.

C. For \( G = \mathbb{Z}_8 \) and \( \rho = \bigoplus_{i=1}^{8} k_i \rho_i \) an \( n \) degree representation, \( \rho \) has a non-degenerate invariant bilinear form iff \( k_3 = k_4 \), \( k_5 = k_6 \) & \( k_7 = k_8 \).

D. For \( G = \mathbb{Z}_4 \times \mathbb{Z}_2 \) and \( \rho = \bigoplus_{i=1}^{8} k_i \rho_i \) an \( n \) degree representation, \( \rho \) has a non-degenerate invariant bilinear form iff \( k_5 = k_6 \) & \( k_7 = k_8 \).

E. For \( G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \) and \( \rho = \bigoplus_{i=1}^{8} k_i \rho_i \) an \( n \) degree representation, \( \rho \) always has a non-degenerate invariant bilinear form.

Remark 1.1. Thus we get the necessary and sufficient condition for existence of a non-degenerate invariant bilinear form under an \( n \) degree representation from above characterization theorem. It is to remark that these results also hold equally good for a field (not necessarily algebraically closed) of characteristic \( \equiv 1 \pmod{8} \).

This is to note that concerning the existence of a non-degenerate invariant bilinear form on a given representation, it amounts to checking whether this representation is isomorphic to its dual. This can be done by looking at the character tables of the irreducible representations, inverting every character, and checking which dual (irreducible) representation is obtained. Then, in an isomorphism class of representations, specified by multiplicities of irreducible ones, one just needs to check if multiplicities are equal on every irreducible representation and its dual. One can see that at one hand the Theorem 1.3 takes care of all these conversations and on the other hand it has been proved in an elementary way as purely an application of Matrix theory.

2. Preliminaries

In this section \((\rho, V(\mathbb{F}))\) stands for an \( n \) degree representation over an algebraically closed field \( \mathbb{F} \) with \( \text{char} \neq 2 \) of a group \( G \) of order 8. Also \( \omega \in \mathbb{F} \) is primitive eighth root of unity.

2.1. Irreducible representations of a group \( G \) of order 8

Here we present the table of irreducible representations for each group of order 8. We denote the table of \( G \) by \( T_G \).

2.1.1. \( D_4 = \langle a, b \mid a^4 = b^2 = 1, ba = a^3b \rangle \)

\[
T_{D_4} =
\begin{array}{cccccc}
\rho_1 & \rho_2 & \rho_3 & \rho_4 & \rho_5 \\
\hline
a & \omega^8 & \omega^8 & \omega^4 & 0 & \omega^4 \\
& \omega^4 & \omega^4 & \omega^8 & \omega^8 & 0 \\
b & \omega^8 & 0 & \omega^4 & 0 & \omega^4 \\
\end{array}
\]
2.1.2. \( Q_8 = \langle a, b \mid a^4 = 1, a^2 = b^2, ba = a^3b \rangle \)

\[
\begin{array}{c|ccccc}
& \rho_1 & \rho_2 & \rho_3 & \rho_4 & \rho_5 \\
\hline
a & \omega_8 & \omega_8 & \omega^4 & \omega^4 & \omega^2 \\
b & \omega_8 & \omega^4 & \omega^4 & \omega^8 & 0 \\
\end{array}
\]

2.1.3. \( Z_8 = \langle a \mid a^8 = 1 \rangle \)

\[
\begin{array}{c|cccccc}
& \rho_1 & \rho_2 & \rho_3 & \rho_4 & \rho_5 \\
\hline
a & \omega^8 & \omega^4 & \omega & \omega^7 & \omega^2 \\
\end{array}
\]

2.1.4. \( Z_4 \times Z_2 = \langle a, b \mid a^4 = b^2 = 1, ab = ba \rangle \)

\[
\begin{array}{c|cccccc}
& \rho_1 & \rho_2 & \rho_3 & \rho_4 \\
\hline
a & \omega^8 & \omega^4 & \omega^8 & \omega^4 \\
b & \omega^8 & \omega^4 & \omega^8 & \omega^4 \\
\end{array}
\]

2.1.5. \( Z_2 \times Z_2 \times Z_2 = \langle a, b, c \mid a^2 = b^2 = c^2 = 1, ab = ba, ac = ca, bc = cb \rangle \)

\[
\begin{array}{c|cccccc}
& \rho_1 & \rho_2 & \rho_3 & \rho_4 \\
\hline
a & \omega^8 & \omega^4 & \omega^8 & \omega^4 \\
b & \omega^8 & \omega^4 & \omega^8 & \omega^4 \\
c & \omega^8 & \omega^4 & \omega^8 & \omega^4 \\
\end{array}
\]

**Definition 2.1.** The character of \( \rho \) is a function \( \chi : G \to \mathbb{F} \), \( \chi(g) = tr(\rho(g)) \) and is also called character of the group \( G \).

**Theorem 2.1.** (Maschke’s Theorem): If \( \text{char}(\mathbb{F}) \) does not divide \( |G| \), then every representation of \( G \) is a direct sum of irreducible representations.

**Proof.** See pp 316, corollary (4.9) [1]. \( \square \)

Now as

\[
\rho = k_1\rho_1 \oplus k_2\rho_2 \oplus \ldots \ldots \oplus k_r\rho_r,
\]

where for every \( 1 \leq i \leq r \), \( k_i\rho_i \) stands for the direct sum of \( k_i \) copies of the irreducible representation \( \rho_i \).

Let \( \chi \) be the corresponding character of the representation \( \rho \), then

\[
\chi = k_1\chi_1 + k_2\chi_2 + \ldots \ldots + k_r\chi_r.
\]

Where \( \chi_i \) is the character of \( \rho_i \), \( \forall \ 1 \leq i \leq r \). Dimension of the character \( \chi \) is being calculated at identity element of a group. i.e,

\[
dim(\rho) = \chi(1) = tr(\rho(1))
\]

\[
\Rightarrow d_1k_1 + d_2k_2 + \ldots \ldots + d_rk_r = n.
\]
Note 2.1. equation (3) holds in more general case which helps us in finding all possible distinct r-tuples \((k_1, k_2, \ldots, k_r)\), which correspond to the distinct n degree representations (up to isomorphism) of a finite group.

Theorem 2.2. Two representations \((\rho, \mathbb{V}(\mathbb{F}))\) and \((\rho', \mathbb{V}(\mathbb{F}))\) are isomorphic iff their character tables are same i.e, \(\chi(g) = \chi'(g)\) for all \(g \in G\).

Proof. See pp 319, corollary (5.13) \[4\].

3. Existence of non-degenerate invariant bilinear form.

The space of invariant bilinear forms under a finite group may have non-degenerate and degenerate forms. Sometimes all the elements of the space are degenerate with respect to a particular representation, such a space is called a degenerate invariant space in this paper. How many such representations exist out of total representations, is a matter of investigation. Some of the spaces contains both non-degenerate and degenerate invariant bilinear forms under a particular representation. In this section we compute the number of such representations of the group \(G\) of order 8.

Remark 3.1. The space \(\Xi_G\) of invariant bilinear forms under an n degree representation \(\rho\) contains only those \(X \in \mathbb{M}_n(\mathbb{F})\) whose \((i, j)\)th block is a \(O\) sub-matrix of order \(d_i k_i \times d_j k_j\) when \((i, j) \notin A_G = \{(i, j) | \rho_i \text{ and } \rho_j \text{ is dual to each other}\}\) whereas the \((i, j)\)th block of order \(d_i k_i \times d_j k_j\), is given by

For \(G \neq \mathbb{Z}_8\) and \((i, j) = (1, 1), (2, 2), (3, 3), (4, 4)\), it is

\[
X_{d_i k_i \times d_j k_j}^{ij} = \begin{bmatrix}
    x_{11}^{ij} & x_{12}^{ij} & \cdots & \cdots & x_{1k_j}^{ij} \\
    x_{21}^{ij} & x_{22}^{ij} & \cdots & \cdots & x_{2k_j}^{ij} \\
    \vdots & \vdots & \ddots & \cdots & \vdots \\
    x_{k_i,1}^{ij} & x_{k_i,2}^{ij} & \cdots & \cdots & x_{k_i k_j}^{ij}
\end{bmatrix},
\]

and for \(G = \mathbb{Z}_8\) and \((i, j) = (1, 1), (2, 2), (3, 3), (4, 3), (5, 6), (6, 5), (7, 8), (8, 7)\), it is

\[
X_{d_i k_i \times d_j k_j}^{ij} = \begin{bmatrix}
    x_{11}^{ij} & x_{12}^{ij} & \cdots & \cdots & x_{1k_j}^{ij} \\
    x_{21}^{ij} & x_{22}^{ij} & \cdots & \cdots & x_{2k_j}^{ij} \\
    \vdots & \vdots & \ddots & \cdots & \vdots \\
    x_{k_i,1}^{ij} & x_{k_i,2}^{ij} & \cdots & \cdots & x_{k_i k_j}^{ij}
\end{bmatrix},
\]

for \(G = \mathbb{Z}_4 \times \mathbb{Z}_2\) and \((i, j) = (5, 6), (6, 5), (7, 8), (8, 7)\), it is

\[
X_{d_i k_i \times d_j k_j}^{ij} = \begin{bmatrix}
    x_{11}^{ij} & x_{12}^{ij} & \cdots & \cdots & x_{1k_j}^{ij} \\
    x_{21}^{ij} & x_{22}^{ij} & \cdots & \cdots & x_{2k_j}^{ij} \\
    \vdots & \vdots & \ddots & \cdots & \vdots \\
    x_{k_i,1}^{ij} & x_{k_i,2}^{ij} & \cdots & \cdots & x_{k_i k_j}^{ij}
\end{bmatrix},
\]
for \((i, j) = (5, 5)\), with \(G = D_4\) it is
\[
X_{d, k_i}^{ij} = \begin{bmatrix}
x_{i1}^{ij}I_2 & x_{i2}^{ij}I_2 & \cdots & x_{i(2k_i - 1)}^{ij}I_2 \\
x_{i3}^{ij}I_2 & x_{i4}^{ij}I_2 & \cdots & x_{i(2k_i - 1)}^{ij}I_2 \\
\vdots & \vdots & \ddots & \vdots \\
x_{(2k_i - 1)1}^{ij}I_2 & x_{(2k_i - 1)2}^{ij}I_2 & \cdots & x_{(2k_i - 1)(2k_i - 1)}^{ij}I_2
\end{bmatrix},
\]

& for \((i, j) = (5, 5)\), with \(G = Q_8\) it is
\[
X_{d, k_i}^{ij} = \begin{bmatrix}
x_{i1}^{ij}I_2 & x_{i2}^{ij}I_2 & \cdots & x_{i(2k_i - 1)}^{ij}I_2 \\
x_{i3}^{ij}I_2 & x_{i4}^{ij}I_2 & \cdots & x_{i(2k_i - 1)}^{ij}I_2 \\
\vdots & \vdots & \ddots & \vdots \\
x_{(2k_i - 1)1}^{ij}I_2 & x_{(2k_i - 1)2}^{ij}I_2 & \cdots & x_{(2k_i - 1)(2k_i - 1)}^{ij}I_2
\end{bmatrix},
\]

where \(I_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \).

**Note 3.1.** Note that \(X \in \Xi'_s\) is invariant under \(\rho\) if and only if for every \((i, j) \in A_G\), we have \(X_{d, k_i \times d, k_j}^{ij} = C_{k_i, \rho_i(g)}X_{d, k_i \times d, k_j}^{ij}C_{k_j, \rho_j(g)}\), \(\forall g \in G\) and \(X = [X_{d, k_i \times d, k_j}^{ij}]_{(i, j) \in A_G}\).

**Lemma 3.1.** If \(X \in \Xi'_s\), then \(X_{d, k_i \times d, k_j}^{ij}\) is non-singular, with \((i, j) \in A_G\) iff \(X\) is non-singular.

**Proof.** With reference to the above remark and note, for every \(X \in \Xi'_s\), we have \(X_{d, k_i \times d, k_j}^{ij} = C_{k_i, \rho_i(g)}X_{d, k_i \times d, k_j}^{ij}C_{k_j, \rho_j(g)}\), \(\forall g \in G\). Suppose \(X_{d, k_i \times d, k_j}^{ij}\) is non-singular, then \(X_{d, k_i \times d, k_j}^{ij}\) is square sub-matrix for \((i, j) \in A_G\) and rows (columns) of \(X\) is linearly linearly independent. Thus the result follows. Converse part is easy to see.

To prove following lemmas, from remark 3.1 we will choose only those \(X \in M_n(F)\) whose \((i, j)\)th block is zero for \((i, j) \notin A_G\) and for \((i, j) \in A_G\), the \((i, j)\)th block \(X_{d, k_i \times d, k_j}^{ij}\) is non-singular.

**Lemma 3.2.** For \(n \in \mathbb{Z}^+\), the number of \(n\) degree representations of group \(D_4\) or \(Q_8\), whose corresponding spaces of invariant bilinear forms contain non-degenerate bilinear forms is \(\sum_{s=0}^{\lfloor n/2 \rfloor} \binom{n-2s+3}{3}\).

**Proof.** From (2) we have \(k_1 + k_2 + k_3 + k_4 + 2k_5 = n\) and we have to choose \(X \in M_n(F)\) such that \(X = \text{Diag}[X_{k_1}^{11}, X_{k_2}^{22}, X_{k_3}^{33}, X_{k_4}^{44}, X_{k_5}^{55}]\). For \(1 \leq i \leq 5\), the chosen sub-matrices \(X_{d, k_i}^{ij}\) is non-singular. Thus \(X = \text{Diag}[X_{k_1}^{11}, X_{k_2}^{22}, X_{k_3}^{33}, X_{k_4}^{44}, X_{k_5}^{55}]\) is non-singular and \(X = C_{\rho(g)}X_{\rho(g)}\), \(\forall g \in D_4\) or \(Q_8\) with \(k_1 + k_2 + k_3 + k_4 + 2k_5 = n\).

We have a collection of distinct 5-tuples \((k_1, k_2, k_3, k_4, k_5)\) of size \(\sum_{s=0}^{\lfloor n/2 \rfloor} \binom{n-2s+3}{3}\), which is same as the number of representations of degree \(n\), whose corresponding spaces contain non-degenerate bilinear forms.

**Lemma 3.3.** For every \(n \in \mathbb{N}\), the number of \(n\) degree representations of group \(Z_8\), whose corresponding spaces of invariant bilinear forms contain non-degenerate bilinear forms is \(\sum_{s=0}^{\lfloor n/2 \rfloor} \binom{n}{s+2} \binom{n-2s+1}{1}\).
Proof. Let $G = \mathbb{Z}_8$. Then $k_1 + k_2 + k_3 + k_4 + k_5 + k_6 + k_7 + k_8 = n$ and we have to choose $X \in \mathbb{M}_n(\mathbb{F})$ such that

$$X = \text{Diag} \left[ X_{k_1 \times k_1}^{11}, X_{k_2 \times k_2}^{22}, \begin{bmatrix} O & X_{k_3 \times k_4}^{34} \\ X_{k_3 \times k_3}^{33} & O \end{bmatrix}, \begin{bmatrix} O & X_{k_5 \times k_6}^{56} \\ X_{k_5 \times k_5}^{55} & O \end{bmatrix}, \begin{bmatrix} O & X_{k_7 \times k_8}^{78} \\ X_{k_8 \times k_7}^{78} & O \end{bmatrix} \right]$$

with $k_3 = k_4, k_5 = k_6 \& k_7 = k_8$. Since the chosen sub-matrices $X_{d_1 k_1 \times d_j k_j}^{ij}$ is non-singular for $(i, j) \in \{(1, 1), (2, 2), (3, 4), (4, 3), (5, 6), (6, 5), (7, 8), (8, 7)\}$. The rows or columns of $X$ are linearly independent thus $X$ is non-singular and $X = C_{\rho(g)}^t X C_{\rho(g)}$, $\forall g \in \mathbb{Z}_8$ with

$$k_1 + k_2 + 2k_3 + 2k_5 + 2k_7 = n.$$

As the number of such 5-tuples $(k_1, k_2, k_3, k_5, k_7)$ of size $\sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} \binom{s+1}{2} \binom{n-2s+1}{1}$, the number of representations of degree $n$, whose corresponding spaces contain non-degenerate bilinear forms is $\sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} \binom{s+1}{2} \binom{n-2s+1}{1}$. \hfill $\square$

Lemma 3.4. For every $n \in \mathbb{N}$, the number of $n$ degree representations of the group $\mathbb{Z}_4 \times \mathbb{Z}_2$, whose corresponding spaces of invariant bilinear forms contain non-degenerate bilinear forms is $\sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} \binom{s+1}{2} \binom{n-2s+3}{3}$. 

Proof. Let $G = \mathbb{Z}_4 \times \mathbb{Z}_2$. Then $k_1 + k_2 + k_3 + k_4 + k_5 + k_6 + k_7 + k_8 = n$ and we have to choose $X \in \mathbb{M}_n(\mathbb{F})$ such that

$$X = \text{Diag} \left[ X_{k_1 \times k_1}^{11}, X_{k_2 \times k_2}^{22}, X_{k_3 \times k_3}^{33}, X_{k_4 \times k_4}^{44}, \begin{bmatrix} O & X_{k_5 \times k_6}^{56} \\ X_{k_5 \times k_5}^{55} & O \end{bmatrix}, \begin{bmatrix} O & X_{k_7 \times k_8}^{78} \\ X_{k_8 \times k_7}^{78} & O \end{bmatrix} \right]$$

with $k_5 = k_6 \& k_7 = k_8$. Since the chosen sub-matrices $X_{d_1 k_1 \times d_j k_j}^{ij}$ is non-singular for $(i, j) \in \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 6), (6, 5), (7, 8), (8, 7)\}$. The rows or columns of $X$ are linearly independent thus $X$ is non-singular and $X = C_{\rho(g)}^t X C_{\rho(g)}$, $\forall g \in \mathbb{Z}_4 \times \mathbb{Z}_2$ with

$$k_1 + k_2 + k_3 + k_4 + 2k_5 + 2k_7 = n.$$

As the number of such 6-tuples $(k_1, k_2, k_3, k_4, k_5, k_7)$ of size $\sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} \binom{s+1}{2} \binom{n-2s+3}{3}$, the number of representations of degree $n$, whose corresponding spaces contain non-degenerate bilinear forms is $\sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} \binom{s+1}{2} \binom{n-2s+3}{3}$. \hfill $\square$

Lemma 3.5. For every $n \in \mathbb{N}$, the number of $n$ degree representations of a group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, whose corresponding spaces of invariant bilinear forms contain non-degenerate bilinear forms is $(\binom{n+7}{7})$.

Proof. From the equation (2), we have $k_1 + k_2 + k_3 + k_4 + k_5 + k_6 + k_7 + k_8 = n$ and we have to choose $X \in \mathbb{M}_n(\mathbb{F})$ such that

$$X = \text{Diag} \left[ X_{k_1 \times k_1}^{11}, X_{k_2 \times k_2}^{22}, X_{k_3 \times k_3}^{33}, X_{k_4 \times k_4}^{44}, X_{k_5 \times k_5}^{55}, X_{k_6 \times k_6}^{66}, X_{k_7 \times k_7}^{77}, X_{k_8 \times k_8}^{88} \right].$$

For $1 \leq i \leq 8$, the chosen sub-matrices $X_{d_i k_1 \times d_i k_i}^{ii}$ is non-singular. The rows or columns of $X$ are linearly independent thus $X$ is non-singular and $X = C_{\rho(g)}^t X C_{\rho(g)}$, $\forall g \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ with

$$k_1 + k_2 + k_3 + k_4 + k_5 + k_6 + k_7 + k_8 = n.$$

As the number of such 8-tuples $(k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8)$ is $(\binom{n+7}{7})$, the number of representations of degree $n$, whose corresponding spaces contain non-degenerate bilinear forms is $(\binom{n+7}{7})$. \hfill $\square$

Remark 3.2. Since $\mathbb{F}$ is algebraically closed, it has infinitely many non zero elements, hence if there is one non-degenerate invariant bilinear form in the space $\Xi_G$, it has infinitely many.
Thus from Lemmas 3.2 to 3.5, we find that the number of $n$ degree representations of a group $G$ of order 8, whose corresponding spaces of invariant bilinear forms contain non-degenerate forms are

$$\sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} \left( \binom{n-2s+3}{3} \right), \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} \left( \binom{n-2s+3}{3} \right), \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} \left( \binom{s^2}{1} \right), \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} \left( \binom{s^2}{1} \right), \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} \left( \binom{s^2}{1} \right)$$

and

$$\sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} \left( \binom{s+1}{1} \right), \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} \left( \binom{s+1}{1} \right), \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} \left( \binom{s+1}{1} \right)$$

and

$$\sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} \left( \binom{s+2}{1} \right), \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} \left( \binom{s+2}{1} \right), \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} \left( \binom{s+2}{1} \right).$$

3.1. Characterization of invariant bilinear forms under an $n$ degree representation of a group of order 8

**Lemma 3.6.** For $G = D_4$ and $\rho = \bigoplus_{i=1}^{5} k_i \rho_i$ an $n$ degree representation of $G$, $\rho$ always has a non-degenerate bilinear form.

*Proof.* Follows from the proof of Lemma 3.2

**Lemma 3.7.** For $G = Q_8$ and $\rho = \bigoplus_{i=1}^{5} k_i \rho_i$ an $n$ degree representation of $G$, $\rho$ always has a non-degenerate bilinear form.

*Proof.* Follows from the proof of Lemma 3.2

**Lemma 3.8.** For $G = Z_8$ and $\rho = \bigoplus_{i=1}^{8} k_i \rho_i$ an $n$ degree representation of $G$, $\rho$ has a non-degenerate bilinear form iff $k_3 = k_4, k_5 = k_6$ and $k_7 = k_8$.

*Proof.* Follows from the proof of Lemma 3.3

**Lemma 3.9.** For $G = Z_4 \times Z_2$ and $\rho = \bigoplus_{i=1}^{8} k_i \rho_i$ an $n$ degree representation of $G$, $\rho$ has a non-degenerate bilinear form iff $k_5 = k_6$ and $k_7 = k_8$.

*Proof.* Follows from the proof of Lemma 3.4

**Lemma 3.10.** For $G = Z_2 \times Z_2 \times Z_2$ and $\rho = \bigoplus_{i=1}^{8} k_i \rho_i$ an $n$ degree representation of $G$, $\rho$ always has a non-degenerate bilinear form.

*Proof.* Follows from the proof of Lemma 3.5

**Definition 3.1.** The space $\Xi_G$ of invariant bilinear forms is called degenerate if all its elements are degenerate.

We will discuss about the degenerate invariant space in the later section.

4. Dimensions of spaces of invariant bilinear forms under representations of groups of order 8

The space of invariant bilinear forms under an $n$ degree representation is finite dimensional and so are the symmetric subspace and the skew-symmetric subspace. In this section we calculate the dimension of the space of invariant bilinear forms under a representation of a group of order 8.

**Theorem 4.1.** If $\Xi_G$ is the space of invariant bilinear forms under an $n$ degree representation $\rho = \bigoplus_{i=1}^{r} k_i \rho_i$ of a group $G$ of order 8, then $\text{dim}(\Xi_G) = \sum_{(i,j) \in A_G} k_i k_j$. 

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Proof. Since for every $X \in \Xi'_G$, the non zero sub-matrices $X_{i,j}^{k_l}$, for each $(i, j) \in A_G$ we have, $X_{i,j}^{k_l} = C_{i,j}^{k_l}(g)X_{i,j}^{k_l}$. Now with reference of remark 3.1 to span $(i, j)^{th}$ sub-matrix of $X \in \Xi'_G$, it needs maximum $k_i, k_j$ linearly independent vectors from $M_n(\mathbb{F})$. This completes the proof.

Corollary 4.1. The space of invariant symmetric bilinear forms under an $n$ degree representation $\rho = \oplus_{i=1}^{r} k_i \rho_i$ of a group $G \neq Q_8$ of order 8 has dimension $= \sum_{i=1}^{r} k_i(k_i+1) + \sum_{i \neq j}^{r} (k_i k_j)$.

Proof. Follows from the proof of theorem 4.1 and remark 3.1

Corollary 4.2. The space of invariant skew-symmetric bilinear forms under an $n$ degree representation $\rho = \oplus_{i=1}^{r} k_i \rho_i$ of a group $G \neq Q_8$ of order 8 has dimension $= \sum_{i=1}^{r} k_i(k_i+1) - \sum_{i \neq j}^{r} (k_i k_j)$.

Proof. Follows from the proof of theorem 4.1 and remark 3.1

Corollary 4.3. The space of invariant symmetric (skew-symmetric) bilinear forms under an $n$ degree representation $\rho = \oplus_{i=1}^{r} k_i \rho_i$ of a group $Q_8$ has dimension $= \sum_{i=1}^{4} k_i(k_i+1) + \frac{k_5(k_5+1)}{2}$.

Proof. Follows from the proof of theorem 4.1 and remark 3.1

5. Main results

Here we present the proofs of main theorems stated in the Introduction section.

Proof of theorem 1.1 Since $G$ is a group of order 8 and dimension of the vector space $V(\mathbb{F})$ is n, if $G$ is $D_4$ or $Q_8$ then $r=5, d_i = 1$ for $i=1,2,3,4$ and $d_5 = 2$. Now from equation (2) we have

$$k_1 + k_2 + k_3 + k_4 + 2k_5 = n.$$

From the proof of the lemma 3.2 the number of distinct 5-tuples $(k_1, k_2, k_3, k_4, k_5)$ is $\sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} (n-2s+3)$.

If $G$ any of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_4 \times \mathbb{Z}_2, \mathbb{Z}_8$ we have $r = 8$ and $d_i = 1$ for $i = 1, 2, ..., 8$. Now from equation (2), we have

$$k_1 + k_2 + k_3 + k_4 + k_5 + k_6 + k_7 + k_8 = n,$$

so the number of such 8-tuples $(k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8)$ is $\binom{n+7}{7}$.

Thus from (2) and Theorem 2.2 the number of $n$ degree representations (upto isomorphism) of a group $G$ of order 8 is $\sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} (n-2s+3)$ for non abelian and $\binom{n+7}{7}$ for abelian.

5.1. Degenerate invariant spaces

From Theorem 1.1 and Lemmas 3.2 to 3.5 we have the number of $n$ degree representations whose corresponding invariant spaces of bilinear forms contain only degenerate invariant bilinear forms are $\binom{n+7}{7} - \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} (s+1) \binom{n-2s+3}{3}$, $\binom{n+7}{7} - \sum_{s=0}^{\lfloor \frac{n}{2} \rfloor} \binom{s+2}{2} (n-2s+1), 0, 0, 0$ of the groups $\mathbb{Z}_2 \times \mathbb{Z}_4, \mathbb{Z}_8, Q_8, D_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, respectively. The groups $\mathbb{Z}_8$ & $\mathbb{Z}_2 \times \mathbb{Z}_4$ have representations whose corresponding spaces of invariant forms are degenerate and the groups $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2, D_4 & Q_8$ have no degenerate spaces.
Proof of theorem \[\text{Let } \mathbb{W}_{(i,j)\in A_G} \text{ be the subspaces of } M_n(F) \text{ and } \Xi'_G \text{ the space of invariant bilinear forms of a group } G. \text{ Let } X \text{ be an element of } \Xi'_G \text{ then}
\]
\[
C^t_{\rho(g)}XC_{\rho(g)} = X \text{ and } X = [X^ij_{d_i,k_i\times d_j,k_j}] \in A_G
\]
Existence:
Let \(X \in \Xi'_G\) then for every \((i,j) \in A_G\), there exists at least one \(X_{(i,j)} \in \mathbb{W}_{(i,j)\in A_G}\), such that \(\sum_{(i,j)\in A_G} X_{(i,j)} = X\).

Uniqueness:
For every \((i,j) \in A_G\), suppose there exists \(Y_{(i,j)} \in \mathbb{W}_{(i,j)\in A_G}\), such that \(\sum_{(i,j)\in A_G} Y_{(i,j)} = X\), then \(\sum_{(i,j)\in A_G} X_{(i,j)} = \sum_{(i,j)\in A_G} Y_{(i,j)}\) i.e., \(Y_{(i',j')} - X_{(i',j')} = \sum_{(i,j)\neq (i',j')} (X_{(i,j)} - Y_{(i,j)})\). Therefore \(Y_{(i',j')} - X_{(i',j')} \in \sum_{(i,j)\neq (i',j')} \mathbb{W}_{(i,j)\in A_G}\) hence \(Y_{(i',j')} - X_{(i',j')} = 0\) or \(Y_{(i',j')} = X_{(i',j')}\) for all \((i',j') \in A_G\).

Thus we have
\[
\Xi'_G = \oplus_{(i,j)\in A_G} \mathbb{W}_{(i,j)\in A_G} \text{ and } \dim(\Xi'_G) = \sum_{(i,j)\in A_G} \dim(\mathbb{W}_{(i,j)\in A_G}). \quad (3)
\]

Now as \(\mathbb{W}_{(i,j)\in A_G} = \{ X \in M_n(F) \mid (i,j)^{t}h \text{ block } X^{ij}_{d_i,k_i\times d_j,k_j} \text{ a sub - matrix of order } d_i k_i \times d_j k_j \text{ satisfying } X^{ij}_{d_i,k_i\times d_j,k_j} = C^{t}_{k_i,\rho_1(g)} X^{ij}_{d_i,k_i\times d_j,k_j} C^{t}_{k_j,\rho_1(g)}, \forall g \in G \text{ and rest block is zero } \}\), from the remark 3.1 we see that for \((i,j) \in A_G\), the sub-matrices \(X^{ij}_{d_i,k_i\times d_j,k_j} \in \mathbb{W}_{(i,j)\in A_G}\) have \(k_i k_j\) free variables & \(\mathbb{W}_{(i,j)\in A_G} \cong M_{k_i \times k_j}(F)\). Thus \(\Xi'_G \cong \oplus_{(i,j)\in A_G} M_{k_i \times k_j}(F)\) and \(\dim(\mathbb{W}_{(i,j)\in A_G}) = k_i k_j\).

Thus substituting this in equation (3) we get the dimension of \(\Xi'_G\).

Proof of theorem [1,3] Follows immediately from Lemmas 3.9 to 3.10. \(\square\)

6. Representations over a field of characteristic 2.

Remark 6.1. If characteristic of the field \(F\) is 2 then a group \(G\) of order 8 has only trivial irreducible representation. Therefore \(n\) copies of irreducible representation is written as
\[
\rho(g) = n \rho_1(g),
\]
where \(\rho_1\) is the trivial representation of a group \(G\) of degree 1. So the representation \(\rho\) is a trivial representation of degree \(n\), i.e,
\[
\rho(g) = I_n, \text{ for all } g \in G.
\]

Proposition 6.1. The space of invariant bilinear forms under an \(n\) degree trivial representation of a group \(G\) of order 8 with \(\text{char}(F) = 2\) is isomorphic to \(M_n(F)\).

Proposition 6.2. The space of symmetric invariant bilinear forms is the direct sum of space of skew-symmetric invariant forms and space of diagonal invariant forms.
Note 6.1. If \( \text{char}(\mathbb{F}) = 2 \), the Maschke’s theorem does not hold and there are composite representations that are not direct sums.

Thus here we have completely characterised the representations of a group of order 8 for having a non-degenerate invariant bilinear form over an algebraically closed field. Note that these results hold equally good when considered over a field of characteristic \( \equiv 1 \pmod{8} \).

Acknowledgement The first author would like to thank UGC, India for providing the research fellowship and authors are thankful to the Central University of Jharkhand, India for support to carry out this research work. The second author is thankful to the Babasaheb Bhimrao Ambedkar University, Lucknow, India where he revised the paper.

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