Schwarzschild solution in R-spacetime

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Abstract

Here we construct new solution for the Einstein equations – some analog of the Schwarzschild metric in anti-de Sitter-Beltrami space in the \( c \to \infty \) limit (R-space). In this case we derive an adiabatic invariant for finite movement of the massive point particle and separate variables Hamilton-Jacobi equation. Quasi orbital motion is analyzed and its radius time dependence is obtained.
One of the important solutions of Einstein’s Field Equations - this is the Schwarzschild solution, which describes a gravitational field outer the massive spherical symmetric body.

If spacetime at the large distance from the source of this field has negative constant curvature $R$ (Anti de-Sitter spacetime), then the corresponding metric in a spherical coordinate system $x_0, r, \theta, \varphi$ is written out as $\[1\]$:

$$ds^2 = \left(1 - \frac{r^2}{R^2} - \frac{2M}{r}\right) dx_0^2 - \frac{dr^2}{\left(1 - \frac{r^2}{R^2} - \frac{2M}{r}\right)} - r^2 d\Omega^2,$$

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2$. This is the Schwarzschild Anti de-Sitter metric (SAdS). Now we change the metric $\[1\]$ to the Beltrami coordinates. To do that we introduce coordinate transformations $\[2\]$:

$$x_0 \Rightarrow R \arcsin \frac{x_0}{R \sqrt{1 + \frac{x_0^2}{R^2}}}, \quad r \Rightarrow \sqrt{1 + \frac{x_0^2}{R^2} - \frac{r^2}{R^2}}.$$

Putting these transformations $\[2\]$ in metric $\[1\]$, we get

$$ds^2 = \frac{dx^2}{h^2} - \frac{(xdx)^2}{R^2 h^4} - \frac{2Mh}{rh^4} dx_0^2 - \frac{2M}{rh^4} \left(\frac{dr}{R^2 h^2} - \frac{r x_0 dx_0}{R^2 h^2}\right)^2.$$

For abbreviation of formulas we define: $dx^2 \equiv dx_0^2 - dr^2$, $xdx \equiv x_0 dx_0 - \vec{r} d\vec{r}$, $h^2 \equiv 1 + x^2/R^2$, $h^2_0 \equiv 1 + x^2_0/R^2$.

A formula $\[3\]$ is the Schwarzschild-Anti de-Sitter metric in the Beltrami coordinates (SAdSB).

Now this metric is changed into the limit $c \to \infty$ and we define $Mc \equiv g$, so the metric $\[3\]$ will be

$$ds^2 = \frac{R^4}{c^2 t^4} \left(1 - \frac{2gt}{rR}\right) dt^2 - \frac{R^2(tdr - rdt)^2}{c^2 t^4} - \frac{R^2r^2}{c^2 t^2} d\Omega^2.$$

This limiting expression for the metric SAdSB is what we call the Schwarzschild solution in $R$-spacetime.

We can write out an action for classical test particle with mass $m$:

$$S = -mc \int ds = -mR^2 \int \frac{dt}{t^2} \sqrt{1 - \frac{2gt}{rR} - \frac{(t \dot{r} - r)^2}{R^2 \left(1 - \frac{2gt}{rR}\right)} - \frac{r^2 t^2 \dot{\varphi}^2}{R^2}},$$

where $\dot{r} = dr/dt, \dot{\varphi} = d\Omega/dt$. 

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1  Schwarzschild metric in Anti de-Sitter-Beltrami spacetime
Lagrangian of this system has the form

\[ L = -m \frac{R^2}{t^2} \left[ 1 - \frac{2gt}{rR} \right] \sqrt{1 - \frac{2gt}{rR} + \left( \frac{t\dot{r} - r}{R^2} \right)^2 - \frac{r^2 t^2 \dot{\phi}^2}{R^2}}. \] (6)

The investigation of the properties of this Lagrangian will be held in next part.

2 The limit of classical mechanics in R-spacetime

To get the classical mechanics limit we consider a spacetime region \( gt/R \ll r \ll R, \dot{r} \ll R/t \) and expand Lagrangian (6) in the order of the small parameters \( r/R, \dot{r}t/R, gt/(rR) \):

\[ L \approx -m \frac{R^2}{t^2} \left[ 1 - \frac{1}{2} \left( \frac{2gt}{rR} + \frac{(t\dot{r} - r)^2}{R^2 \left( 1 - \frac{2gt}{rR} \right)} + \frac{r^2 t^2 \dot{\phi}^2}{R^2} \right) \right] \]

\[ = -m \frac{R^2}{t^2} + \frac{mgRt}{rt} + \frac{m(t\dot{r} - r)^2}{2t^2} + \frac{mr^2 \dot{\phi}^2}{2}. \] (7)

The equations of motion do not change under the addition of some total derivative to the Lagrangian, and (7) will be written as

\[ L = \frac{m\dot{r}^2}{2} + \frac{mr^2 \dot{\phi}^2}{2} + \frac{mgRt}{rt} + \frac{d}{dt} \left( \frac{R^2}{t} - \frac{r^2}{2t} \frac{d}{dt} \right), \] (8)

and this total derivative will be neglected.

Now we transform to the Hamiltonian formalism and introduce radial momentum

\[ p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \] (9)

and angular momentum

\[ p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2 \dot{\phi} = M_\phi, \] (10)

which is an integral of motion.

The Hamiltonian function will be constructed with the help of the Legendre transformation

\[ H = p_r \dot{r} + p_\phi \dot{\phi} - L = \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} - \frac{mgR}{rt}. \] (11)

The last term in (11) can be considered as the gravitational potential

\[ V = -\frac{mg(t)}{r} \]

with slowly changing gravitational constant

\[ G(t) = \frac{gR}{t}. \]
Hamiltonian (11) contains a parameter which depends on time, therefore we use the method of adiabatic invariant [4].

The stationary Hamiltonian function of our system is represented as

\[ H_0 = \frac{p_r^2}{2m} + \frac{p_\varphi^2}{2mr^2} - \frac{mg_0}{r} = E. \]  

(12)

The Hamilton-Jacobi equation is derived in a simple form

\[ \frac{1}{2m} \left( \frac{\partial S}{\partial r} \right)^2 + \frac{1}{2mr^2} \left( \frac{\partial S}{\partial \varphi} \right)^2 - \frac{mg_0}{r} = E_0, \]  

(13)

where \( S \) is a completed action for considered system. We will find it in the formula

\[ S = S_r(r) + M \varphi - E_0 t. \]  

(14)

We put (14) in the equation (13) and get

\[ \frac{\partial S_r(r)}{\partial r} = \sqrt{2mE_0 + \frac{2m^2g_0}{r} - \frac{M^2}{r^2}} = p_r. \]  

(15)

So the adiabatically invariants are

\[ I_r = \frac{1}{2\pi} \oint p_r \, dr = -M \varphi + g_0 \sqrt{\frac{m^3}{2|E_0|}}, \]  

(16)

\[ I_\varphi = \frac{1}{2\pi} \oint p_\varphi \varphi = M \varphi. \]  

(17)

From the sum of (16) and (17) follows that energy depends on time

\[ E = - \frac{m^3g^2R^2}{2(I_r + I_\varphi)^2} t^2. \]  

(18)

3 The Schwarzschild metric in R-spacetime and the motion on the quasi-circular orbits

In this part we consider the motion of a test point particle with mass \( m \) in the Schwarzschild metric in \( R \)-spacetime [4]. To do that we use the Hamilton-Jacobi method. In General Relativity Theory a Hamilton-Jacobi equation can be represented as [5]

\[ g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} = m^2 c^2. \]

So the Hamilton-Jacobi equation for a metric in \( R \)-spacetime is expressed as

\[ \left[ \frac{t^2}{R^2} \left( \frac{1}{1 - \frac{2gt}{rR}} \right) \frac{\partial S}{\partial t} + \frac{tr}{R^2} \left( \frac{1}{1 - \frac{2gt}{rR}} \right) \frac{\partial S}{\partial r} \right]^2 - \frac{t^2}{R^2} \left( 1 - \frac{2gt}{rR} \right) \left( \frac{\partial S}{\partial r} \right)^2 - \]  

\[ - \frac{t^2}{R^2} \left[ \left( \frac{\partial S}{\partial \theta} \right)^2 + \frac{1}{\sin^2 \theta} \left( \frac{\partial S}{\partial \varphi} \right)^2 \right] = m^2. \]  

(19)
For the motion on the surface $\theta = \frac{\pi}{2}$ the Hamilton-Jacobi equation (19) is simplified

$$\left[ \frac{t^2}{R^2} \frac{1}{\sqrt{1 - \frac{2gt}{rR}}} + \frac{tr}{R^2} \frac{1}{1 - \frac{2gt}{rR}} \frac{\partial S}{\partial r} \right]^2 - \frac{t^2}{R^2} \left( 1 - \frac{2gt}{rR} \right) \left( \frac{\partial S}{\partial r} \right)^2 - \frac{t^2}{R^2r^2 \sin^2 \theta} \left( \frac{\partial S}{\partial \varphi} \right)^2 = m^2. \quad (20)$$

To solve this equation we use the method of separation of variables and represent the full action as a sum of functions of different variables

$$S = S_1(t) + S_2 \left( \frac{r}{t} \right) + M \varphi. \quad (21)$$

When we put (21) in the equation (20) we can find

$$S_1 = -\frac{A}{t}$$

and

$$S_2 = \int \! dx \sqrt{\frac{A^2}{R^2} \left( 1 - \frac{x_g}{x} \right)^2 - \frac{m^2 R^2 + \frac{M^2}{x^2}}{1 - \frac{x_g}{x}}}.$$

where $x \equiv r/t$, $x_g \equiv 2g/R$, $A$ is a constant of variable separation. Finally the action is written out as

$$S = -\frac{A}{t} + \int \! dx \sqrt{\frac{A^2}{R^2} \left( 1 - \frac{x_g}{x} \right)^2 - \frac{m^2 R^2 + \frac{M^2}{x^2}}{1 - \frac{x_g}{x}}} + M \varphi. \quad (22)$$

Two equations of motion are derived from the derivative (22) of two constants $A$ and $M \varphi$:

$$\frac{\partial S}{\partial A} = \text{const} \quad (23)$$

and

$$\frac{\partial S}{\partial M \varphi} = \text{const}. \quad (24)$$

From an equation (23) we can find the dependence $r(t)$

$$\frac{R}{t} = \frac{A}{m} \int \frac{dx}{\left( 1 - \frac{x_g}{x} \right)^\sqrt{\frac{A^2}{m^2} - \left( \frac{x_g}{x} \right) \left( \frac{1 + \frac{M^2}{x^2} + m^2 R^2}{x^2} \right)}}. \quad (25)$$

and an equation of trajectory can be found from the equation (24)

$$\varphi = \int \! dx \sqrt{\frac{x^2}{A^2 - \frac{M^2}{x^2} + m^2 R^2}} \left( 1 - \frac{x_g}{x} \right).$$
We consider now a motion of the test particle on the almost circular orbit. To derive this we rewrite equation (25) in differential form

\[ \frac{1}{1 - \frac{x_g}{x}} \frac{dx}{dt} = -\frac{R}{At^2} \sqrt{A^2 - U^2(x)}, \]

where

\[ U(x) = mR^2 \left( 1 - \frac{x_g}{x} \right) \left( 1 + \frac{M^2}{m^2 R^2 x^2} \right) \]

which play role of the effective potential.

A motion on a circular orbit occurs under following conditions

\[ A = U, \quad \frac{dU}{dx} = 0. \]  

(26)

A solutions of the equation (26) are

\[ r_{\pm} = \frac{1}{2} \frac{M^2}{m^2 g^2 R} \left( 1 \pm \sqrt{1 - \frac{3m^2 R^2 x_g^2}{M^2 \phi^2}} \right). \]  

(27)

Easy to see that \( r_+ \) corresponds to a stable “circular” orbit and \( r_- \) corresponds to a nonstable orbit.

Multiplier \( gt/R \) in the formula (27) has a sense of the Schwarzschild radius. In the limit \( r \gg gt/R \) radicand in (27) is nearly one and radius become

\[ r_+ \simeq \frac{M^2}{m^2 g^2} \left( \frac{gt}{R} \right). \]

If we put \( t = T + \tau \), where \( T \) is the Age of the Universe, then radius of the circular orbit will depend on \( \tau \):

\[ r_+(\tau) = r_+(0) \left( 1 + \frac{\tau}{T} \right). \]  

(28)

For example we consider the annual change of the radius of Moon orbit. Moon orbit radius \( r_m \simeq 4 \cdot 10^8 \text{ m} \), Age of the Univers \( T \simeq 5 \cdot 10^{17} \text{ c} \), duration of the year \( \tau \simeq 2,5 \cdot 10^8 \text{ c} \). From the equation (28) we get the annual increase in the radius of the orbit \( \Delta r \simeq 2 \text{ mm/year} \). This increase is consistent with the observations (34 mm/year), which are explained by the tidal force and decreasing of the Earth angular momentum.

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