Comparing quantumness criteria

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Abstract – Measuring the quantumness of a system can be done with a variety of methods. In this article we compare different criteria, namely quantum discord, Bell inequality violation and non-separability, for systems placed in a Gaussian state. When the state is pure, these criteria are equivalent, while we find that they do not necessarily coincide when decoherence takes place. Finally, we prove that these criteria are essentially controlled by the semi-minor axis of the ellipse representing the state’s Wigner function in phase space.

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Introduction. – The characterisation of “classicality” and “quantumness” in quantum systems has become a topic of major importance in several branches of modern physics. Indeed, maybe surprisingly, it is not always trivial to establish whether a system behaves “classically” or “quantum-mechanically”. This question is especially important when one tries to understand the nature of a physical phenomenon.

For instance, in cosmology, it is well-known that primordial perturbations are very well reproduced \cite{1} by vacuum quantum fluctuations, amplified by gravitational instability \cite{2–7} during an early epoch of accelerated expansion named inflation \cite{8–13}. However, the quantum origin of those primordial perturbations has never been tested directly and, in practice, they are mostly treated by astronomers as classical, stochastic fluctuations. The reason why this is possible is that, under peculiar circumstances, and for certain observables, a quantum system can be mimicked by a classical one \cite{14–16}. However, if a genuine quantum signature could be detected in cosmological observables, that would shed light on fundamental issues such as the need to quantise gravitational degrees of freedom or the emergence of classicality at cosmological scales \cite{17–22}.

The same need to distinguish classical from quantum processes appears in analogue gravity, where phenomena involving gravitational physics are mapped to condensed-matter systems. In these setups, particles can either be created by quantum channels or by the classical amplification of a thermal bath \cite{23}. The latter mechanism is always present when conducting experiments at finite temperature. A quantum test is a way to tell the two populations apart and to demonstrate the existence of a quantum channel in these experiments \cite{23–26}.

In quantum technologies, the distinction between quantum and classical behaviours is also central, since “quantumness” is a crucial resource, \textit{e.g.}, in quantum computing \cite{27} and quantum cryptography \cite{28,29}.

This has led various notions of “quantumness” to be put forward. One possible approach is to consider correlations between sub-parts of a given system, and to determine whether or not they can be reproduced by classical random variables. This route gave rise to the celebrated Bell inequalities \cite{30–32}, quantum steering \cite{33}, different measures of entanglement (non-separability \cite{34}, multipartite entanglement \cite{35}, entanglement witnesses \cite{36}, etc.), quantum discord \cite{37–39}, etc.

Another possible approach, leading to a second class of criteria, is to make use of phase-space formulations of quantum mechanics. For instance, the non-positivity of the Wigner function \cite{40} or the absence of the P-representation \cite{41,42} have been viewed as criteria signalling the quantumness of a system \cite{43,44}.

How these different criteria are related is a non-trivial question. In pure states, it is known that quantum discord reduces to entanglement entropy \cite{37}, which only

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vanishes in separable states, and that all non-separable states violate a Bell inequality [34]. For mixed states however, these relations become more elusive (for instance non-separability is only a necessary condition for Bell-inequality violation [34]).

In this article, our goal is to investigate the relations between different criteria in a subclass of quantum states where explicit calculations can be performed. We want to determine in which cases they lead to the same conclusion regarding the quantumness of a system, and in which cases they differ. In practice, we consider two continuous degrees of freedom placed in two-mode squeezed thermal states and analyze the link between three quantum criteria: non-separability, quantum discord and a Bell inequality violation [34].

Two-mode squeezed vacuua (TMSV) are Gaussian states whose covariance matrix depends on two parameters, \( r \) and \( \varphi \), respectively called squeezing amplitude and squeezing angle, and reads [47–49]

\[
\gamma_{\text{TMSV}} = \begin{pmatrix} \gamma_{11}^{\text{TMSV}} & \gamma_{12}^{\text{TMSV}} \\ \gamma_{21}^{\text{TMSV}} & \gamma_{22}^{\text{TMSV}} \end{pmatrix},
\]

with

\[
\gamma_{11} = \gamma_{22} \equiv \cosh(2r) I_2,
\]

and

\[
\gamma_{12} = \gamma_{21} \equiv - \sinh(2r) \begin{pmatrix} \cos 2\varphi & \sin 2\varphi \\ \sin 2\varphi & - \cos 2\varphi \end{pmatrix}.
\]

TMSV are ubiquitous in modern physics: they appear in quantum optics [47–49], cold atoms [50,51] as well as in the study of inflation [52–55] and Hawking radiation [56,57]. Using eq. (5) one can check that they are pure. In general, TMSV may become mixed as an effect of decoherence [58–60]. We will consider the class of two-mode squeezed thermal states which are defined as Gaussian states with covariance matrices of the form

\[
\gamma = \gamma_{\text{TMSV}}^{\text{SM}},
\]

where one can check from eq. (5) that \( p \) is indeed the purity of the state. These states arise for instance for cosmological perturbations linearly coupled to an environment while preserving statistical homogeneity [61,62], or when an initial TMSV interacts with two identical independent thermal baths [63,64], or when the modes are sent through a pure-loss or an additive Gaussian noise channel [65].

The two latter channels are described by simple transformations of the covariance matrix, respectively given by

\[
\gamma = \eta \gamma_{\text{TMSV}} + (1 - \eta) I_4
\]

where the efficiency parameter \( 0 \leq \eta \leq 1 \) encodes the level of loss/damping experienced across the channel, and \( \gamma = \gamma_{\text{TMSV}} + \Delta I_4 \) where \( \Delta \geq 0 \) encodes the level of noise. Both matrices can then be put in the form (9), with effective squeezing and purity parameters given in eqs. (A.55) and (A.59) of the Supplementary Material. Its covariance matrix reads

\[
W(R_{1/2}) = \frac{1}{\pi^2 \sqrt{\text{det} \gamma}} \exp(-R_{1/2}^T \gamma^{-1} R_{1/2}).
\]

Let us also introduce the purity \( p \equiv \text{Tr}(\hat{\rho}^2) \), which determines whether the state is pure \( (p = 1) \) or mixed \( (p < 1) \). For a Gaussian state, the purity is directly related to the determinant of the covariance matrix [46]

\[
p = \frac{1}{\sqrt{\text{det} \gamma}}.
\]

In the following we work in terms of these effective squeezing and purity parameters, such that all setups mentioned above are encompassed in the analysis. Decoherence is expected to play a key role in the emergence of classicality, and this simply parameterised class of states will allow us to study how different criteria respond to it.

Under a canonical transformation, \( \hat{R} \rightarrow \hat{T} \hat{R} \hat{T}^T \), where \( \hat{T} \) is a symplectic matrix (i.e., it preserves commutation relations), the covariance matrix changes according to

\[
\gamma \rightarrow \hat{T} \gamma \hat{T}^T.
\]

This implies that the covariance matrix depends on the set of canonical variables used to describe a system.
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For instance, there exists a partition \( \hat{R}_D \) where the covariance matrix is block diagonal,

\[
\gamma^D = \frac{1}{\sqrt{p}} \begin{pmatrix}
\gamma_{\text{OMSV}} & 0 \\
0 & \gamma_{\text{OMSV}}
\end{pmatrix},
\]

with

\[
\gamma_{\text{OMSV}} \equiv \begin{pmatrix}
\gamma_{qq} & \gamma_{qp} \\
\gamma_{pq} & \gamma_{pp}
\end{pmatrix}
\]

and

\[
\gamma_{qq} = \cosh(2r) - \cos(2\varphi) \sinh(2r),
\]

\[
\gamma_{pq} = \gamma_{qp} = -\sin(2\varphi) \sinh(2r),
\]

\[
\gamma_{pp} = \cosh(2r) + \cos(2\varphi) \sinh(2r),
\]

such that the Wigner function factorises according to \( W^D(\mathbf{R}^D) = W(q^D_1, p^D_1)W(q^D_2, p^D_2) \). In this basis, the quantum state is nothing but the product of two identical and uncorrelated one-mode squeezed (thermal) states. If \( p = 1 \) they are one-mode squeezed vacua (OMSV).

This also implies that quantumness criteria, which characterise the correlations between two subsystems, obviously depend on the way the system is partitioned (for instance, the way quantum discord depends on the choice of partition has been studied in refs. [15,62]).

In practice, there often exists a “preferred” basis of operators corresponding to separately measurable physical degrees of freedom [66,67]. The factorised partition (10) is nonetheless useful as it provides a simple geometric representation of the quantum state: the contours of \( \hat{W} \) are ellipses in the phase space \( (q^D_i, p^D_i) \), as displayed in fig. 1. Their eccentricity is controlled by \( r, \varphi \) and the area contained in the ellipses is proportional to \( 1/p \).

Quantumness criteria. – Since the quantum states we consider are fully characterised by the three parameters \( r, \varphi \) and \( p \), let us express the three quantumness criteria in terms of these parameters, in order to compare them.

Quantum discord. A first way to characterise the presence of quantum correlations between two sub-parts of a system is by quantum discord [38,39]. The idea is to introduce two measures of correlation that coincide for classically correlated setups thanks to Bayes’ theorem, but that may differ for quantum systems. The first measure is the so-called mutual information \( I \), which is the sum between the von-Neumann entropy of both reduced sub-systems, minus the entropy of the entire system. The second measure \( J \) evaluates the difference between the entropy contained in the first subsystem, and the entropy contained in that same subsystem when the second subsystem has been measured, where an extremisation is performed over all possible ways to “measure” the second subsystem. \( J \) can be shown to be always less than \( I \). Quantum discord \( \mathcal{D} \) is defined as the difference between these two measures and is thus a positive quantity that only vanishes for classical systems.

For Gaussian states, \( I, J \) and \( \mathcal{D} \) can be expressed in terms of the local symplectic invariants of the covariance matrix [46]\(^1\). It is shown in [68–70] that, for covariance matrices of the form (9), quantum discord depends on \( r \) and \( p \) only and is given by

\[
\mathcal{D}(p, r) = f(\sigma(p, r)) - 2f(p^{-1/2}) + f \left[ \frac{\sigma(p, r) + p^{-1}}{\sigma(p, r) + 1} \right],
\]

where the function \( f(x) \) is defined for \( x \geq 1 \) by

\[
f(x) \equiv \left( x \frac{-1}{2} \right) \log_2 \left( x \frac{+1}{2} \right) - \left( x \frac{+1}{2} \right) \log_2 \left( x \frac{-1}{2} \right),
\]

and

\[
\sigma(p, r) = \frac{\cosh(2r)}{\sqrt{p}}.
\]

\(^1\)This means that quantum discord is invariant under local symplectic transformations, i.e., those mixing \( q_i \) with \( p_i \) but not with \( q_j \) and \( p_j \). This explains why \( \varphi \) does not appear in the final expression (15), since it can be changed arbitrarily by performing phase-space rotations in each sector.
Note that in the partition (10), where the covariance matrix is block-diagonal, the two sub-systems are uncorrelated hence quantum discord vanishes.

Bell inequality. Another way to characterise the presence of quantum correlations is via Bell inequalities [71]. When violated, they allow one to exclude classical and realistic local theories [72]. Usually designed for discrete observables [31] (such as spins), they can also be applied to continuous variables by means of pseudo-spin operators [30,32] or via projections on coherent states [73]. In this paper we will use the pseudo-spin operators introduced in ref. [32],

\[
\tilde{\sigma}^i_x = \int_{-\infty}^{\infty} \text{sign}(q_i) |q_i\rangle \langle q_i| \, dq_i, \\
\tilde{\sigma}^i_y = -i \int_{-\infty}^{\infty} \text{sign}(q_i) |q_i\rangle \langle -q_i| \, dq_i, \\
\tilde{\sigma}^i_z = \int_{-\infty}^{\infty} |q_i\rangle \langle -q_i| \, dq_i.
\]

One can check that these operators satisfy the SU(2) commutation relations

\[
[\tilde{\sigma}^i_j, \tilde{\sigma}^j_k] = 2i \epsilon_{ijk} \tilde{\sigma}^i_k, \tag{18}
\]

where \( \epsilon_{ijk} \) is the totally anti-symmetric tensor.

From these operators we can build a Bell inequality [16,32],

\[
\langle \tilde{B} \rangle = 2\sqrt{\langle \tilde{\sigma}_1^i \tilde{\sigma}_2^i \rangle^2 + \langle \tilde{\sigma}_1^j \tilde{\sigma}_2^j \rangle^2} \leq 2. \tag{22}
\]

In order to compute the two-point correlation functions of the operators \( \tilde{\sigma}_z \) and \( \tilde{\sigma}_z \), one can derive their Weyl transform and make use of eq. (2). Since \( \tilde{\sigma}_\mu \) and \( \tilde{\sigma}_\nu \) act on different degrees of freedom, the Weyl transform of their product factorises as

\[
\tilde{\sigma}_\mu^i \tilde{\sigma}_\nu^j = \bar{\tilde{\sigma}}_\mu^i \bar{\tilde{\sigma}}_\nu^j, \tag{23}
\]

and in the SM we show that

\[
\tilde{\sigma}_z^i = -\pi \delta(q_i) \delta(p_i), \quad \bar{\tilde{\sigma}}_z = \text{sgn}(q_i), \tag{24}
\]

where \( \delta \) stands for the Dirac distribution. Together with eq. (2), this leads to

\[
\langle \tilde{\sigma}_1^i \tilde{\sigma}_2^i \rangle = p, \tag{25}
\]

\[
\langle \tilde{\sigma}_1^j \tilde{\sigma}_2^j \rangle = -\frac{2}{\pi} \arcsin |\cos(2\varphi)/\tanh(2r)|. \tag{26}
\]

Inserting eqs. (25) and (26) into eq. (22) leads to

\[
\langle \tilde{B} \rangle = 2\sqrt{p^2 + \frac{4}{\pi^2} \arcsin^2 |\cos(2\varphi)/\tanh(2r)|}. \tag{27}
\]

Compared to quantum discord given in eq. (15), one can see that the mean value of the Bell operator \( \langle \tilde{B} \rangle \) depends on the squeezing angle \( \varphi \) in addition to the squeezing amplitude \( r \) and the purity \( p \). This is expected since the operators given in eq. (18) are not invariant under local symplectic transformations.

Non-separability. Finally we consider quantum separability. A state is said to be separable in a certain partition if its density matrix can be written as a statistical mixture of products of density matrices over the two sub-systems, i.e.,

\[
\hat{\rho} = \sum_i \alpha_i \hat{\rho}_1^i \bigotimes \hat{\rho}_2^i, \tag{28}
\]

where \( \alpha_i \) are real coefficients. In general, proving that a state is separable is a non-trivial task, yet, for Gaussian states, the so-called Peres-Horodecki criterion was proven to be necessary and sufficient [74]. In the SM we show how to evaluate this criterion for Gaussian states, in a one-parameter family of partitions that contains both eq. (9) and eq. (10). In the partition corresponding to eq. (10), the state is, as expected, always separable, while for eq. (9) we find that the state is separable if and only if

\[
e^{-2r} \geq \sqrt{p}. \tag{29}
\]
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These results also confirm that decoherence (i.e., smaller value for $p$) is associated to the emergence of classicality. Indeed, for a given squeezing amplitude $r$, there always exists a value of the purity parameter $p$ below which the Bell inequality is not violated, the state is separable and quantum discord is smaller than a given threshold. The required amount of decoherence (i.e., the critical value for the purity parameter $p$), increases (decreases) with the squeezing amplitude. This is because, as $r$ increases, the two subsystems get more entangled, hence it takes more decoherence to erase quantum features. In [64], the authors had considered a similar class of states and studied the robustness of non-classicality measures against decoherence induced by coupling to thermal baths. In this special case it was also found that the state becomes classical in the sense that quantum discord asymptotes zero at large decoherence, and that separability vanishes once decoherence reaches a certain finite threshold.

Our findings also prompt some reservations about the physical relevance of the numerical value of quantum discord. Discord is measured in information bits and, a priori, one may think that it is an extensive quantity, namely the larger the discord the more “quantum” the state. However, one notices in fig. 2 that the value of quantum discord at which the separability or Bell criteria are crossed may be small or large, depending on the squeezing amplitude. For instance, if the state is almost pure $p \approx 1$ and the squeezing weak $r \leq 1$, then one can achieve a non-separable state and/or a Bell inequality violation while keeping a small quantum discord, see point “A”; or for large squeezing and small purity we can both have a large quantum discord and still satisfy the Bell inequality, see point “B”. This suggests that the numerical value of discord itself has no clear interpretation, at least in this setup and in terms of the other quantumness criteria.

The behaviour of these three criteria can be further understood in the phase-space representation. Ignoring the orientation $\varphi$ (which we have set to its optimal value $\varphi = 0$ for Bell inequality violation), the ellipses of fig. 1 have been parameterised so far using their area, via $p$, and their eccentricity, via $r$. Alternatively, one can describe them by means of their semi-major, $a$, and semi-minor, $b$, axes, related to $r$ and $p$ by

$$a = e^r p^{-1/4}, \quad b = e^{-r} p^{-1/4}. \quad (30)$$

In particular, we expect $b$, the size of semi-minor axis, to play a physical role since it encodes the presence or absence of a sub-fluctuant direction in phase space with respect to the vacuum.

Using eq. (30) all criteria can be expressed in terms of $a$ and $b$. The non-separability criterion assumes an extremely simple form as eq. (29) is straightforwardly recast to $b \geq 1$. The fact that the state is non-separable is then equivalent to the existence of a sub-fluctuant direction in phase space (for instance, in fig. 1 the state represented by the green ellipse is non-separable while the one represented by the blue ellipse is separable). The expression of quantum discord and the Bell operator in terms of $a$ and $b$ is not particularly illuminating but in the large-squeezing and small-purity limit, i.e., $a \gg b \gg 1/a$, in the SM we show that the discord also becomes a function of $b$ only (i.e., of the sub-fluctuant mode), namely

$$D(a, b) \to g \left(1 + 2b^2\right) + \log_2 \left(1 + \frac{1}{2b^2}\right), \quad (31)$$

where $g(x)$ is bounded and defined in eq. (A.53) of the SM.
All criteria are displayed as a function of $a$ and $b$ in fig. 3, where one can check that $\langle \hat{B} \rangle$ and $D$ become independent of $a$ in the large-squeezing limit.

Conclusions. – In this letter, we compared three different criteria, quantum discord, Bell inequality violation and non-separability, aimed at assessing whether a system behaves quantum-mechanically or not. We have found that, even in a simple class of Gaussian states, these criteria are inequivalent, i.e., a state can be, at the same time, “quantum” according to one criterion and “classical” according to another one. However, in the large squeezing limit these criteria were found to be mainly controlled by the amplitude of the sub-fluctuant mode. There is no natural threshold for the value of quantum discord at which the other two criteria are crossed, and we found that decoherence always leads to more classical states regardless of the criterion being used.

This analysis could be extended to non-Gaussian states [79], which are known to behave differently under quantum criteria (for instance, according to Hudson theorem [80] their Wigner functions are necessarily non-positive if they are pure).

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