We investigate a periodic problem for a linear telegraph equation

\[ u_{tt} - u_{xx} + 2\mu u_t = f(x, t) \]

with Neumann boundary conditions. We prove that the operator of the problem is modeled by a Fredholm operator of index zero in the scale of Sobolev spaces of periodic functions. This result is stable under small perturbations of the equation in which either \( \mu \) becomes variable and discontinuous or an additional zero-order term appears. We also show that the solutions of this problem possess smoothing properties.

1. Introduction

The telegraph equation

\[ u_{tt} - u_{xx} + 2\mu u_t + F(x, t, u) = 0, \quad (x, t) \in (0, 1) \times \mathbb{R}, \]  

(1.1)

combines specific features of the diffusion and wave equations. It describes dissipative wave processes, e.g., in the transmission and propagation of electric signals, dynamical processes in biological populations, economic and ecological systems, etc. (see [1, 6, 9, 10, 21] and references therein). Hillen [8] discusses the appearance of Hopf bifurcations for (1.1) with Neumann boundary conditions in the case where \( \mu \) is a negative constant.

An important step toward a rigorous bifurcation analysis (via the Implicit-Function Theorem and the Lyapunov–Schmidt reduction [4, 12]) is to establish the Fredholm solvability of a linearized problem. Note, in this respect, that the Fredholm property for hyperbolic PDEs is much less studied than for ODEs and parabolic PDEs.

We investigate a linearization of (1.1) known as the damped wave equation

\[ u_{tt} - u_{xx} + 2\mu u_t = f(x, t), \quad (x, t) \in (0, 1) \times \mathbb{R}, \]  

(1.2)

where \( \mu \) is a constant. This equation describes a correlated random walk under the assumption that particles with density \( u \) have a constant speed and a constant turning rate \( \mu \). Furthermore, we impose time-periodicity conditions

\[ u \left( x, t + \frac{2\pi}{\omega} \right) = u(x, t), \quad (x, t) \in [0, 1] \times \mathbb{R}, \]  

(1.3)

\[ u_t \left( x, t + \frac{2\pi}{\omega} \right) = u_t(x, t), \quad (x, t) \in [0, 1] \times \mathbb{R}, \]
where $\omega > 0$ is a speed of particles, and Neumann boundary conditions

$$u_x(0, t) = u_x(1, t) = 0, \quad t \in \mathbb{R}. \quad (1.4)$$

The solvability of periodic problems for equation (1.2) and its nonlinear perturbation (1.1) is investigated, in particular, in [2, 3, 5, 8, 13–15, 19, 22]. Our main result (Theorem 4.2) is the Fredholm alternative for the problem (1.2)–(1.4). Previously, the Fredholm zero-index property was known only in the double-periodic case [13, 15]. Our result is of interest due to the following general question: What function spaces can be used to capture Fredholm solvability for the second-order hyperbolic PDEs (for the first-order hyperbolic PDEs, this problem is investigated in [17, 18]).

Our argument is based on a reduction to a related periodic Neumann problem for a hyperbolic system. As a result of splitting $u = v + w$ into the density $v$ of particles moving to the right and the density $w$ of particles moving to the left, Eqs. (1.2)–(1.4) take the following form:

$$v_t + v_x = g(x, t) + \mu(w - v), \quad (x, t) \in (0, 1) \times \mathbb{R}, \quad (1.5)$$

$$w_t - w_x = g(x, t) + \mu(v - w), \quad (x, t) \in (0, 1) \times \mathbb{R},$$

$$v \left( x, t + \frac{2\pi}{\omega} \right) = v(x, t), \quad (x, t) \in [0, 1] \times \mathbb{R}, \quad (1.6)$$

$$w \left( x, t + \frac{2\pi}{\omega} \right) = w(x, t), \quad (x, t) \in [0, 1] \times \mathbb{R},$$

$$v(0, t) = w(0, t), \quad t \in \mathbb{R}, \quad (1.7)$$

$$v(1, t) = w(1, t), \quad t \in \mathbb{R}.$$

System (1.5) describes a random-walk process introduced by Taylor [23] in which a particle moving right dies with a rate $\mu$ and is reborn as a particle moving left with the same rate. The homogeneous Neumann boundary conditions (1.7) describe the reflection of particles from the boundary.

For (1.5)–(1.7), we proved a Fredholm alternative in [17]. In the present paper, we show the equivalence between the models (1.2)–(1.4) and (1.5)–(1.7) in a certain functional analytic sense, which allows us to derive the Fredholm property for (1.2)–(1.4) from the result obtained for (1.5)–(1.7) in [17].

By the perturbation argument, our results are extended to the equation

$$u_{tt} - u_{xx} + \nu(x, t)u_t + \alpha(x, t)u = f(x, t), \quad (x, t) \in (0, 1) \times \mathbb{R}, \quad (1.8)$$

where $\nu(x, t) - 2\mu$ and $\alpha(x, t)$ are sufficiently small in appropriate function spaces (see Remark 4.3).

In Section 2, we introduce function spaces of solutions and describe their useful properties. The equivalence of the models (1.2)–(1.4) and (1.5)–(1.7) is proved in Section 3. Finally, in Section 4, we prove our main result (Theorem 4.2).
2. Function Spaces and Operators

We also use a different representation of the problem (1.5)–(1.7), namely,

$$u_t + z_x = g(x,t), \quad (x,t) \in (0,1) \times \mathbb{R},$$

$$z_t + u_x = -2\mu z, \quad (x,t) \in (0,1) \times \mathbb{R},$$

$$u \left( x, t + \frac{2\pi}{\omega} \right) = u(x,t), \quad (x,t) \in [0,1] \times \mathbb{R},$$

$$z \left( x, t + \frac{2\pi}{\omega} \right) = z(x,t), \quad (x,t) \in [0,1] \times \mathbb{R},$$

$$z(0,t) = z(1,t) = 0, \quad t \in \mathbb{R},$$

where

$$u = \frac{v + w}{2}, \quad z = \frac{v - w}{2}.$$  \hspace{1cm} (2.4)

Here $2u$ stands for the total density of particles and $2z$ is the flux of particles. The relationship between the models (1.2)–(1.4), (1.5)–(1.7), and (2.1)–(2.3) is discussed in Section 3. We show that they are equivalent in a certain sense.

For the solutions and right-hand sides of the problems (1.2)–(1.4), (1.5)–(1.7), and (2.1)–(2.3) we now introduce pairs of spaces $(U_0^\gamma, H^0,\gamma^{-1})$, $(V_0^\gamma, W_0^\gamma)$, and $(Z_0^\gamma, W_0^\gamma)$, respectively. Here, $\gamma \geq 1$ denotes a real scaling parameter. The subscript $b$ indicates that we construct spaces constrained by boundary conditions, $d$ stands for the diagonal subspace of pairs $(u,u)$, and $0$ for the subspace of pairs $(u,0)$.

Given $l \in \mathbb{N}_0$ and $\gamma \geq 0$, we first introduce the space $H^{l,\gamma}$ of all measurable functions $u: (0,1) \times \mathbb{R} \to \mathbb{R}$ such that

$$u(x,t) = u \left( x, t + \frac{2\pi}{\omega} \right) \quad \text{for a.a.} \quad (x,t) \in [0,1] \times \mathbb{R}$$

and

$$\|u\|_{H^{l,\gamma}}^2 := \sum_{k \in \mathbb{Z}} (1 + k^2)^\gamma \sum_{m=0}^l \int_0^1 \left\| \partial_x^m u(x,t)e^{-ik\omega t} \right\|^2 dx < \infty.$$  \hspace{1cm} (2.5)

It is well-known (see, e.g., [24], Chapter 2.4) that $H^{l,\gamma}$ is a Banach space. In fact, this is the space of all $\frac{2\pi}{\omega}$-periodic maps $u: \mathbb{R} \to H^{l}(0,1)$ that are locally $L^2$-Bochner integrable together with their generalized derivatives up to the (possibly noninteger) order $\gamma$. Furthermore, we define

$$W^\gamma = H^{0,\gamma} \times H^{0,\gamma},$$
\[ W_d^\gamma = \{(g, f) \in W^\gamma : g = \} , \]
\[ W_0^\gamma = \{(g, f) \in W^\gamma : f = 0 \}, \]
\[ V^\gamma = \{(v, w) \in W^\gamma : (v_t + v_x, w_t - w_x) \in W^\gamma \}, \]
\[ Z^\gamma = \{(u, z) \in W^\gamma : (u_t + z_x, z_t + u_x) \in W^\gamma \}, \]
\[ U^\gamma = \{ u \in H^{0,\gamma} : u_x \in H^{0,\gamma-1}, u_{tt} - u_{xx} \in H^{0,\gamma-1} \}, \]

where \( u_t, u_x, u_{tt}, u_{xx}, v_t, v_x, z_t, \) and \( z_x \) are understood in a sense of generalized derivatives. The function spaces \( W^\gamma, V^\gamma, Z^\gamma, \) and \( U^\gamma \) are endowed with the norms

\[ \|(g, f)\|^2_{W^\gamma} = \|g\|^2_{H^{0,\gamma}} + \|f\|^2_{H^{0,\gamma}}, \]
\[ \|(v, w)\|^2_{V^\gamma} = \|(v, w)\|^2_{W^\gamma} + \|(v_t + v_x, w_t - w_x)\|^2_{W^\gamma}, \]
\[ \|(u, z)\|^2_{Z^\gamma} = \|(u, z)\|^2_{W^\gamma} + \|(u_t + z_x, z_t + u_x)\|^2_{W^\gamma}, \]
\[ \|u\|^2_{U^\gamma} = \|u\|^2_{H^{0,\gamma}} + \|u_x\|^2_{H^{0,\gamma-1}} + \|u_{tt} - u_{xx}\|^2_{H^{0,\gamma-1}}. \]

In the following two lemmas, we collect some useful properties of the function spaces \( V^\gamma \) and \( U^\gamma \), respectively.

**Lemma 2.1** ([17], Section 2).

(i) The space \( V^\gamma \) is complete.

(ii) If \( \gamma \geq 1 \), then \( V^\gamma \) is continuously embedded into \( (H^{1,\gamma-1})^2 \).

(iii) For any \( x \in [0,1] \) there exists a continuous trace map

\[ (v, w) \in V^\gamma \mapsto (v(x, \cdot), w(x, \cdot)) \in \left(L^2\left(0, \frac{2\pi}{\omega}\right)\right)^2. \]

Similar properties are encountered in the function spaces \( U^\gamma \).

**Lemma 2.2.**

(i) The space \( U^\gamma \) is complete.

(ii) If \( \gamma \geq 2 \), then \( U^\gamma \) is continuously embedded into \( H^{2,\gamma-2} \).

(iii) If \( \gamma \geq 1 \), then for any \( x \in [0,1] \) there exists a continuous trace map

\[ u \in U^\gamma \mapsto u_x(x, \cdot) \in L^2\left(0, \frac{2\pi}{\omega}\right). \]
Proof. (i) Let \((u_j)\) be a fundamental sequence in \(U^\gamma\). Then \((u_j)\) is fundamental in \(H^{0,\gamma}\) and \((\partial_x u_j)\) and \((\partial^2_t u_j - \partial^2_x u_j)\) are fundamental in \(H^{0,\gamma-1}\). Since \(H^{0,\gamma}\) is complete, there exist \(u \in H^{0,\gamma}\) and \(v, w \in H^{0,\gamma-1}\) such that

\[u_j \to u \quad \text{in} \quad H^{0,\gamma}, \quad \partial_x u_j \to v \quad \text{in} \quad H^{0,\gamma-1}, \quad \text{and} \quad \partial^2_t u_j - \partial^2_x u_j \to w \quad \text{in} \quad H^{0,\gamma-1}\]

as \(j \to \infty\). It remains to show that \(\partial_x u = v\) and \(\partial^2_t u - \partial^2_x u = w\) in the sense of generalized derivatives. For this purpose, we take a smooth function

\[\varphi : (0, 1) \times \left(0, \frac{2\pi}{\omega}\right) \to \mathbb{R}\]

with compact support and note that

\[-\int_0^{2\pi/\omega} \int_0^{2\pi/\omega} u \partial_x \varphi \, dx \, dt = \lim_{j \to \infty} \int_0^{2\pi/\omega} \int_0^{2\pi/\omega} u_j \partial_x \varphi \, dx \, dt = \int_0^{2\pi/\omega} \int_0^{2\pi/\omega} \partial_x u_j \varphi \, dx \, dt = \int_0^{2\pi/\omega} \int_0^{2\pi/\omega} v \varphi \, dx \, dt.\]

Similarly,

\[\int_0^{2\pi/\omega} \int_0^{2\pi/\omega} u (\partial^2_t - \partial^2_x) \varphi \, dx \, dt = \lim_{j \to \infty} \int_0^{2\pi/\omega} \int_0^{2\pi/\omega} u_j (\partial^2_t - \partial^2_x) \varphi \, dx \, dt = \int_0^{2\pi/\omega} \int_0^{2\pi/\omega} (\partial^2_t - \partial^2_x) u_j \varphi \, dx \, dt = \int_0^{2\pi/\omega} \int_0^{2\pi/\omega} w \varphi \, dx \, dt.\]

(ii) We take \(u \in U^\gamma\). Then \(u \in H^{0,\gamma}\) and, hence, \(\partial^2_t u \in H^{0,\gamma-2}\). By the definition of the space \(U^\gamma\), we also have \(\partial^2_x u \in H^{0,\gamma-2}\). Therefore, \(u \in H^{2,\gamma-2}\). Moreover, we get

\[
\|u\|_{H^{2,\gamma-2}}^2 = \|u\|^2_{H^{0,\gamma-2}} + \|\partial_x u\|^2_{H^{0,\gamma-2}} + \|\partial^2_t u\|^2_{H^{0,\gamma-2}} \\
\leq \|u\|^2_{H^{0,\gamma-2}} + c\|\partial_x u\|^2_{H^{0,\gamma-1}} + c\|\partial^2_t u\|^2_{H^{0,\gamma-2}} \\
\leq \|u\|^2_{H^{0,\gamma-2}} + c\|\partial_x u\|^2_{H^{0,\gamma-1}} + c\|\partial^2_t u - \partial^2_x u\|^2_{H^{0,\gamma-2}} + c\|\partial^2_t u\|^2_{H^{0,\gamma-2}} \leq C\|u\|^2_{U^\gamma},
\]

where the constants \(c\) and \(C\) do not depend on \(u\).

Claim (iii) follows from the definition of \(U^\gamma\).

Remark 2.1. By (2.4) and Lemma 2.1 (iii), for any \(x \in [0, 1]\), there exists a continuous trace map

\[\gamma \mapsto \left(u(x, \gamma), z(x, \gamma)\right) \in L^2(0, \frac{2\pi}{\omega})^2.\]
Lemmas 2.1(iii) and 2.2(iii) and Remark 2.1 motivate the consideration of the following closed subspaces in \( V_\gamma, Z_\gamma, \) and \( U_\gamma : \\
\quad V_\gamma^b = \{ (v, w) \in V_\gamma : (1.7) \text{ is satisfied for a.a. } t \in \mathbb{R} \}, \\
\quad Z_\gamma^b = \{ (u, z) \in Z_\gamma : (2.3) \text{ is satisfied for a.a. } t \in \mathbb{R} \}, \\
\quad U_\gamma^b = \{ u \in U_\gamma : (1.4) \text{ is satisfied for a.a. } t \in \mathbb{R} \}.

Finally, we introduce linear operators \( L_{WS}, \tilde{L}_{WS} \in \mathcal{L}(V_\gamma^b; W_\gamma^d) \) by

\[
L_{WS} \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} v_t + v_x - \mu(w - v) \\ w_t - w_x - \mu(v - w) \end{bmatrix},
\]

\[
\tilde{L}_{WS} \begin{bmatrix} v \\ w \end{bmatrix} = \begin{bmatrix} -v_t - v_x - \mu(w - v) \\ -w_t + w_x - \mu(v - w) \end{bmatrix},
\]

a linear operator \( L_{WS}' \in \mathcal{L}(Z_\gamma^b; W_0^\gamma) \) by

\[
L_{WS}' \begin{bmatrix} u \\ z \end{bmatrix} = \begin{bmatrix} u_t + z_x \\ z_t + u_x + 2\mu z \end{bmatrix},
\]

and linear operators \( L_{TE}, \tilde{L}_{TE} \in \mathcal{L}(U_\gamma^b; H^{0,\gamma-1}) \) by

\[
L_{TE}(u) = u_{tt} - u_{xx} + 2\mu u_t,
\]

\[
\tilde{L}_{TE}(u) = u_{tt} - u_{xx} - 2\mu u_t.
\]

3. Equivalence of the Models

We first describe the reduction of problem (1.5)–(1.7) to (1.2)–(1.4) suggested in [11]; see also [7]. Recall that a simple change of variables (2.4) transforms system (1.5)–(1.7) into the form (2.1)–(2.3). We now assume that \( z \) and \( w \) are twice differentiable and differentiate the first equation in (2.1) with respect to \( t \) and the second equation with respect to \( x \). After simple transformations, we come to the problem (1.2)–(1.4) with \( f = g_t + 2\mu g \).

Formally, we will show that problems (1.2)–(1.4) and (1.5)–(1.7) are equivalent in the following sense: there exist isomorphisms

\[
\alpha_1 : V_\gamma^b \rightarrow Z_\gamma^b, \quad \alpha_2 : Z_\gamma^b \rightarrow U_\gamma^b \quad \text{and} \quad \beta_1 : W_\gamma^d \rightarrow W_\gamma^0, \quad \beta_2 : W_0^\gamma \rightarrow H^{0,\gamma-1}
\]
between the respective linear spaces such that the diagram

\[
\begin{array}{ccc}
V_b^\gamma & \xrightarrow{L_{WS}^\gamma} & W_d^\gamma \\
\alpha_1 \downarrow & & \downarrow \beta_1 \\
Z_b^\gamma & \xrightarrow{L'_{WS}^\gamma} & W_0^\gamma \\
\alpha_2 \downarrow & & \downarrow \beta_2 \\
U_b^\gamma & \xrightarrow{L_{TE}^\gamma} & H^{0,\gamma-1}
\end{array}
\]  

(3.1)

is commutative, i.e.,

\[
\beta_1 \circ L_{WS} = L'_{WS} \circ \alpha_1, \quad \beta_2 \circ L'_{WS} = L_{TE} \circ \alpha_2.
\]  

(3.2)

Specifically, we define \(\alpha_1, \beta_1, \alpha_2, \text{ and } \beta_2\) by

\[
\alpha_1(v, w) = \left( \frac{v + w}{2}, \frac{v - w}{2} \right), \quad \beta_1(g, g) = (g, 0),
\]  

(3.3)

\[
\alpha_2(u, z) = u, \quad \beta_2(g, 0) = g_t + 2\mu g.
\]  

(3.4)

Furthermore, let

\[
\alpha = \alpha_2 \circ \alpha_1, \quad \beta = \beta_2 \circ \beta_1.
\]  

(3.5)

The following result reveals the commutativity of the upper part of diagram (3.1) or, the same, the equivalence of problems (1.5)–(1.7) and (2.1)–(2.3). Its proof is straightforward.

**Lemma 3.1.** Suppose \(\gamma \geq 2\). Then

(i) The maps \(\alpha_1 : V_b^\gamma \to Z_0^\gamma\) and \(\beta_1 : W_d^\gamma \to W_0^\gamma\) defined by (3.3) are isomorphisms.

(ii) \(\beta_1 \circ L_{WS} = L'_{WS} \circ \alpha_1\).

To prove the commutativity of the lower part of diagram (3.1), i.e., the equivalence of problems (1.2)–(1.4) and (2.1)–(2.3), we need the following simple lemma:

**Lemma 3.2.** If \(\mu \neq 0\), then the map \(f \in H^{0,\gamma-1} \mapsto g \in H^{0,\gamma}\), where

\[
g_t + 2\mu g = f,
\]  

(3.6)

\[
g \left( x, t + \frac{2\pi}{\omega} \right) = g(x, t),
\]  

is bijective.
**Proof.** Problem (3.6) has a unique solution given by the formula

\[ g(x, t) = \frac{e^{-2\mu(t + \frac{2\pi}{\omega})}}{1 - e^{-2\mu \frac{2\pi}{\omega}}} \frac{2e^{2\mu} f(x, \tau) d\tau}{2\pi} - \int_0^t e^{2\mu(t-\tau)} f(x, \tau) d\tau, \]

which yields the lemma.

**Corollary 3.1.** If \( \mu \neq 0 \), then the map \( u_x \in H^{0, -1} \mapsto z \in H^{0, -1} \), where

\[ z_t + 2\mu z = u_x, \]

\[ z \left( x, t + \frac{2\pi}{\omega} \right) = z(x, t), \]

is bijective. Furthermore,

\[ z(x, t) = \frac{e^{-2\mu(t + \frac{2\pi}{\omega})}}{1 - e^{-2\mu \frac{2\pi}{\omega}}} \frac{2e^{2\mu} u_x(x, \tau) d\tau}{2\pi} - \int_0^t e^{2\mu(t-\tau)} u_x(x, \tau) d\tau. \]

We now state the desired equivalence result:

**Lemma 3.3.** Suppose \( \gamma \geq 2 \). Then

(i) The maps \( \alpha_2 : Z_b^\gamma \to U_b^\gamma \) and \( \beta_2 : W_0^\gamma \to H^{0, -1} \) defined by (3.4) are isomorphisms.

(ii) \( \beta_2 \circ L^1_{WS} = L_{TE} \circ \alpha_2 \).

**Proof.** (i) First, we show that \( \alpha_2 \) maps \( Z_b^\gamma \) to \( U_b^\gamma \) and \( \beta_2 \) maps \( W_0^\gamma \) to \( H^{0, -1} \). The latter is obvious. To show the former, suppose that \( (u, z) \in Z_b^\gamma \) solves (2.1)–(2.3). By the definition of \( Z^\gamma \), \( z_t \in H^{0, -1} \) and \( z_t + u_x \in H^{0, -1} \). Hence, \( u_x \in H^{0, -1} \). Since \( z \in H^{1, -1} \), we have \( z_t \in H^{1, -2} \). Further, as \( z_t + u_x \in H^{0, -1} \), we get \( u_x \in H^{1, -1} \) and, thus, \( u_{xx} \in H^{0, -2} \). Therefore,

\[ z_{tx} + u_{xx} \in H^{0, -2} \quad \text{and} \quad u_{tt} + z_{xt} \in H^{0, -2}. \]

Consequently, \( u_{tt} - u_{xx} \in H^{0, -2} \), as well. This implies that \( u \in U^\gamma \). To check the boundary conditions (1.3) and (1.4), we take into account part (iii) of Lemma 2.2 about the traces of \( u \). Conditions (1.4) now follow from (2.3) and the second equation in (2.1). Conditions (1.3) are a straightforward consequence of (2.2).

Further, we set

\[ \alpha_2^{-1}(u) = (u, z), \quad \text{where} \quad z \quad \text{is given by} \quad (3.9) \]

and

\[ \beta_2^{-1}(f) = (g, 0), \quad \text{where} \quad g \quad \text{is given by} \quad (3.7). \]
We are done if we show that

\[ \alpha_2^{-1} \text{ maps } U_b^\gamma \text{ into } Z_b^\gamma \quad \text{and} \quad \beta_2^{-1} \text{ maps } H^{0,\gamma-1} \text{ into } W_0^\gamma \]  

(3.10)

and that

\[ \beta_2^{-1} \circ \beta_2 = I_{W_0^\gamma}, \quad \beta_2 \circ \beta_2^{-1} = I_{H^{0,\gamma-1}}, \quad \alpha_2 \circ \alpha_2^{-1} = I_{U_b^\gamma}, \quad \alpha_2^{-1} \circ \alpha_2 = I_{Z_b^\gamma}. \]  

(3.11)

We start with proving (3.10). Since the right-hand side of representation (3.9) belongs to \( H^{0,\gamma} \), we get

\[ \partial_t z \in H^{0,\gamma-1}. \]

Differentiating (3.9) with respect to \( t \), we easily arrive at the second equality in (2.1). To meet the first equality, we start from the weak formulation of (1.2)–(1.4): By Lemma 3.2, any function \( f \in H^{0,\gamma-1} \) admits a unique representation \( f = g_t + 2\mu g \), where \( g \in H^{0,\gamma} \). In view of this fact, for any

\[ \varphi \in C^1 \left( [0, 1] \times \left[ 0, \frac{2\pi}{\omega} \right] \right) \quad \text{with} \quad \varphi \left( x, t + \frac{2\pi}{\omega} \right) = \varphi(x, t), \]

we find

\[
0 = \int_0^{2\pi/\omega} \int_0^1 \left[ -u_{tt} \varphi - u_x \varphi_x - 2\mu u_t \varphi + (g_t + 2\mu g) \varphi \right] \, dx \, dt \\
= \int_0^{2\pi/\omega} \int_0^1 \left[ u_t \varphi_t - g \varphi_t + \varphi_x + 2\mu z \varphi_x - 2\mu u_t \varphi + 2\mu g \varphi \right] \, dx \, dt \\
= \int_0^{2\pi/\omega} \int_0^1 \left[ u_t \varphi_t - g \varphi_t + \varphi_x + 2\mu z_x \varphi - 2\mu u_t \varphi + 2\mu g \varphi \right] \, dx \, dt \\
= \int_0^{2\pi/\omega} \int_0^1 \left[ (u_t + z_x - g) \varphi_t - 2\mu (z_x + u_t - g) \varphi \right] \, dx \, dt \\
= \int_0^{2\pi/\omega} \int_0^1 \left[ (u_t + z_x - g) (\varphi_t - 2\mu \varphi) \right] \, dx \, dt.
\]

Taking a constant \( \varphi \), we get

\[
\int_0^{2\pi/\omega} \int_0^1 (u_t + z_x - g) \, dx \, dt = 0. \quad (3.12)
\]
This yields

\[ u_t + z_x - g = 0 \quad \text{a.e. on} \quad (0, 1) \times \left(0, \frac{2\pi}{\omega}\right). \]

The first equality in (2.1) is therefore satisfied. Furthermore, system (2.1) implies that \((u, z) \in Z^\gamma\). The boundary conditions (2.2) and (2.3) follow from (1.3), (1.4), and (2.1).

To complete this part of the proof, it remains to note that the first two equalities in (3.11) follow by Lemma 3.2, the third equality is straightforward, and the last equality follows from (3.9) and the second equality in (2.1).

(ii) Differentiating now the first equality in (2.1) with respect to \(t\) and the second equality with respect to \(x\), subtracting the resulting equations, and then substituting \(z_x\) from the first equation in (2.1), we come to (1.2) with \(f = g_t + 2\mu g\). Hence,

\[ (L_{TE} \circ \alpha_2)(u, z) = g_t + 2\mu g. \]

Moreover, by the definitions of \(L'_{WS}\) and \(\beta_2\), we get \(L'_{WS}(u, z) = (g, 0)\) and \(\beta_2(g, 0) = g_t + 2\mu g\). The desired assertion follows.

Lemma 3.3 is proved.

We are prepared to formulate the main result of this section about the equivalence of the random walk problem (1.5)–(1.7) and the telegraph problem (1.2)–(1.4).

**Theorem 3.1.** Suppose \(\gamma \geq 2\). Then

(i) The maps \(\alpha : V_b^\gamma \to U_b^\gamma\) and \(\beta : W_d^\gamma \to H^{0,\gamma-1}\) defined by (3.5) are isomorphisms.

(ii) \(\beta \circ L_{WS} = L_{TE} \circ \alpha\).

The theorem directly follows from Lemmas 3.1 and 3.3.

4. **Fredholm Alternative**

In the present section, we prove the Fredholm alternative for problem (1.2)–(1.4). From Section 3, we know the relationships between the Fredholm and index properties of the operators of problems (1.2)–(1.4) and (1.5)–(1.7). More precisely, the following lemma is true:

**Lemma 4.1.** Suppose \(\gamma \geq 1\) and \(\mu \neq 0\). Then

(i) \(\dim \ker(L_{WS}) = \dim \ker(L_{TE})\),

(ii) \(\dim \ker(L_{WS}^*) = \dim \ker(L_{TE}^*)\).

The Fredholm solvability of the periodic-Neumann problem for the telegraph equation is now a straightforward consequence of our Fredholm result for the corresponding correlated random walk problem.

**Theorem 4.1** ([17], Theorem 1). Suppose that \(\gamma \geq 1\) and \(\mu \neq 0\). Then

(i) \(L_{WS}\) is a Fredholm operator of index zero from \(V_b^\gamma\) into \(W_d^\gamma\).
(ii) the image of $L_{WS}$ is the set of all $(g, g) \in W^\gamma$ such that

$$
\frac{2\pi}{\omega} \int_0^1 \int_0^1 g(x, t)(\tilde{v}(x, t) + \tilde{w}(x, t)) \, dx \, dt = 0 \quad \text{for all} \quad (\tilde{v}, \tilde{w}) \in \ker(\tilde{L}_{WS}). \tag{4.1}
$$

**Remark 4.1.** As follows from the proof of [17] (Theorem 1), the kernel of problem (1.5)–(1.7) does not depend on $\gamma \geq 1$ and, for given $\gamma > 1$, all $V_0^1$-solutions of problem (1.5)–(1.7) with $f \in W_0^\gamma$ necessarily belong to $V_0^\gamma$ (smoothing effect). This, in particular, implies that if we have (4.1) for $f \in W_0^\gamma$ with $\gamma \geq 2$, then we automatically obtain

$$
\frac{2\pi}{\omega} \int_0^1 \int_0^1 \partial_s g(x, t)(\tilde{v}(x, t) + \tilde{w}(x, t)) \, dx \, dt = 0 \quad \text{for all} \quad (\tilde{v}, \tilde{w}) \in \ker(\tilde{L}_{WS}) \quad \text{and} \quad 0 \leq s \leq \gamma. \tag{4.2}
$$

Note that the smoothing effect does not occur for the corresponding initial-boundary problem with Neumann boundary conditions (see [7, 16, 20]).

We are prepared to formulate our main result.

**Theorem 4.2.** Suppose that $\gamma \geq 2$ and $\mu \neq 0$. Then

(i) $L_{TE}$ is a Fredholm operator of index zero from $U_0^\gamma$ into $H^{0, \gamma - 1}$;

(ii) the image of $L_{TE}$ is the set of all $f \in H^{0, \gamma - 1}$ such that

$$
\frac{2\pi}{\omega} \int_0^1 \int_0^1 f(x, t)u(x, t) \, dx \, dt = 0 \quad \text{for all} \quad u \in \ker(\tilde{L}_{TE}). \tag{4.3}
$$

Theorem 4.2 directly follows from Theorems 3.1 and 4.1 and Lemma 4.1 (see also Remark 4.1).

**Remark 4.2.** As in problem (1.5)–(1.7), one can observe a similar smoothing effect for problem (1.2)–(1.4): The kernel of the problem does not depend on $\gamma$ and, for given $\gamma > 2$, all $U_0^2$-solutions of problem (1.2)–(1.4) with $f \in H^{0, \gamma - 1}$ necessarily belong to $U_0^\gamma$.

**Remark 4.3.** Since the set of Fredholm operators is open, the conclusion of Theorem 4.2 survives under sufficiently small perturbations of the operator $L_{TE} \in L(U^\gamma; H^{0, \gamma - 1})$. The theorem remains true if, instead of $L_{TE}$, we consider the operator of problem (1.8), (1.3), (1.4) with $\nu$ and $\alpha$ such that $\nu(x, t)u_t \in H^{0, \gamma}$, $\nu(x, t)$ is a sufficiently small perturbation of a nonzero constant, $\alpha(x, t)u \in H^{0, \gamma}$, and $\alpha(x, t)$ is sufficiently small. The Fredholm property of the slightly perturbed problem is satisfied independently of whether or not the perturbed problems (1.2)–(1.4) and (1.5)–(1.7) remain equivalent.

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