Compact Finite Difference Scheme with Hermite Interpolation for Pricing American Put Options Based on Regime Switching Model

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Abstract
American put options with the regime-switching model is a system of coupled free boundary problems. In this study, we present an accurate finite difference method coupled with the Hermite interpolation for solving this system. To this end, we first employ the logarithmic transformation to map the free boundary for each regime to a fixed interval and then eliminate the first-order derivatives in the transformed model by taking derivatives to obtain a system of partial differential equations which we call the asset-delta-gamma-speed equations. We then discretize the system using the fourth-order compact scheme coupled with the Crank–Nicholson method. At the same time, the influence of other asset options and option sensitivities are estimated based on the third-order Hermite interpolation. As such, the overall scheme consists of four tridiagonal linear systems, which can be easily solved using the Thomas algorithm and the Gauss–Seidel iteration. The obtained scheme is then applied for the model with two, four, and sixteen regimes, respectively. Our results show that the scheme provides an accurate solution that is fast in computation as compared with other existing numerical methods.

Keywords American put options with regime switching · Logarithmic transformation · Optimal exercise boundary · Compact finite difference method · Hermite interpolation
1 Introduction

The well-known Black–Scholes model has been used over decades in options valuation. This model constructs a delta hedging portfolio with an assumption of the frictionless market, no-arbitrage, and constant risk-free interest and volatility (Ugur, 2009). To remove this ideal assumption and reproduce the actual market price, risk, behavior, and dynamics, researchers have proposed several improvements which include stochastic volatility (Chockalingam & Muthuraman, 2011; Company et al., 2020; Cuomo et al., 2020; Düring & Fournié, 2012; Garnier & Søløna, 2017; Huang et al., 2011; Hull & White, 1987; Ikonen & Toivanen, 2007; Zhylyevskyy, 2009), jump-diffusion (Kou, 2002; Cont & Tankov, 2004; Bingham, 2006; Kim et al., 2017; Patel & Mehra, 2017; Chen et al., 2019, Akbari et al., 2019; Kumar & Kumar, 2019), and regime-switching model (Company et al., 2016a, 2016b, 2016c, 2016d; Egorova et al., 2016; Huang et al., 2011; Khaliq & Liu, 2009; Mamon & Rodrigo, 2005).

The regime-switching model for American option valuation, first introduced by Hamilton (1989), has gained broader interest after the seminal work of Buffington and Elliott (2002). It defines a finite number of market states known as regimes. Each regime has its own set of market variables, and the market randomly switches among different regimes (Chiarella et al., 2016). The model for option valuation with regime-switching involves a system of partial differential equations with free boundaries for which the analytical solution is very difficult to obtain in general. Thus, numerical techniques for solving the option pricing equation with regime-switching have been proposed. These numerical methods include the penalty method (Khaliq & Liu, 2009; Nielsen et al., 2002; Zhang et al., 2013), the method of line (MOL) (Chiarella et al., 2016; Meyer & van der Hoek, 1997), the lattice method (Han & Kim, 2016; Shang & Bryne, 2019), the fast Fourier transform (Boyarchenko & Levendorskii, 2008; Liu et al., 2006), radial basis function (Li et al., 2019) and the front-fixing techniques (Egorova et al., 2016). The main challenge for numerical methods lies in tracking the optimal exercise boundary (Shang & Bryne, 2019). The penalty method removes the free boundary by introducing a penalty term (Khaliq & Liu, 2009). The MOL method calculates the asset and delta options and the optimal exercise boundary simultaneously during computation. However, there are still some complications with the MOL method due to the singularity of the solution and infinite interval.

To the best of our knowledge, the above numerical methods provide up to second-order accurate solutions. Moreover, beyond the two-regime example, the gamma solution in each regime obtained in some existing methods exhibit spurious oscillation near the optimal exercise boundary. As Chiarella et al. (2016) pointed out, the regime-switching model has not been fully explored and exploited.

The motivation of this research is to employ the high order compact finite difference method coupled with the Hermite interpolation to develop a numerical scheme for solving the American options with regime-switching, and hence to present a simple numerical method that provides more accurate approximations of the optimal exercise boundary, asset option, and option Greeks in each regime.

To this end, we first use a front fixing transformation (Blackwell & Hogan, 1994; Company et al., 2016b, 2016c; Landau, 1950; Mitchell & Vynnycky, 2009, 2012) to map the free boundary for each regime to multi-fixed intervals and the non-smooth
initial condition is transferred to the left end of the fixed interval. We then eliminate the first derivatives by further differentiating the transformed model and obtain a system of partial differential equations which we call the asset-delta-gamma-speed equations. As such, the fourth-order compact finite difference for the second-order derivatives and the Crank–Nicholson method can be used for solving this system. On the other hand, the influence of other assets, delta, gamma, speed options in the present regime is estimated based on Hermite interpolations. The numerical scheme for solving each PDE becomes a tridiagonal linear system which can be easily solved using the Thomas algorithm and Gauss-Seidel iteration, which predicts the optimal exercise boundary, option value, and option Greeks in each regime.

The rest of the paper is organized as follows. In Sect. 2, we consider a regime-switching model and its transformations. We transform the model to obtain coupled partial differential equations for option values, delta, gamma, and speed options in each regime. In Sect. 3, we develop an accurate numerical method and its algorithm for solving these equations and obtaining the option values, optimal exercise boundary, and the Greeks in each regime. In Sect. 4, we test our algorithm using examples with the two-, four-, and sixteen-regime and compare our results with the existing methods. We conclude the paper in Sect. 5.

2 Regime Switching Model and Its Transformations

2.1 Regime Switching Model

Consider a continuous-time Markov chain whose states are labeled as \( m = 1, 2, \ldots, I \). Let \( Q = (q_{ml})_{I \times I} \) represent the generator matrix with the entry elements \( q_{ml} \) satisfying the condition below (Norris, 1998):

\[
q_{mm} = - \sum_{l \neq m} q_{ml}, \quad q_{ml} \geq 0, \quad \text{for } l \neq m, \quad l = 1, 2, \ldots, I.
\]  

(1)

Assuming a risk-neutral measure (Elliott et al., 2007), the underlying asset follows a stochastic process

\[
dS_t = S_t\left(r_{\alpha_t}dt + \sigma_{\alpha_t}dB_t\right), \quad 0 \leq t < \infty,
\]  

(2)

where \( r_{\alpha_t} \) and \( \sigma_{\alpha_t} \) are the interest rate and volatility of the asset, respectively, and are dependent on the Markov chain state with

\[
r_{\alpha_t|\alpha_t} = r_m, \quad \sigma_{\alpha_t|\alpha_t} = \sigma_m, \quad m = 1, 2, \ldots, I.
\]  

(3)

We consider an American put option written on the asset \( S_t \) with strike price \( K \) and expiration time \( T \). Let \( V_m(S, t) \) denote the option price and \( \tau = T - t \). Then, \( V_m(S, \tau) \) satisfies the following parabolic PDEs with free boundaries (Khaliq & Liu, 2009):
\[- \frac{\partial V_m(S, \tau)}{\partial \tau} + \frac{1}{2} \sigma^2 m S^2 \frac{\partial^2 V_m(S, \tau)}{\partial S^2} + r_m S \frac{\partial V_m(S, \tau)}{\partial S} - r_m V_m(S, \tau) + \sum_{l \neq m} q_{ml} [V_l(S, \tau) - V_m(S, \tau)] = 0, \quad \text{for } S > s_f(m)(\tau), \quad (4)\]

\[V_m(S, \tau) = K - S, \quad \text{for } S < s_f(m)(\tau). \quad (5)\]

Here the initial and boundary conditions are given as:

\[V_m(S, 0) = \max(K - S, 0), \quad s_f(m)(0) = K; \quad (6)\]

\[V_m\left(s_f(m), \tau \right) = K - s_f(m)(\tau), \quad V_m(0, \tau) = K, \quad V_m(\infty, \tau) = 0, \quad \frac{\partial}{\partial S} V_m\left(s_f(m), \tau \right) = -1, \quad (7)\]

where \(s_f(m)(\tau)\) is the optimal exercise boundary for the \(m\)th regime. Since \(s_f(m)(\tau)\) is moving with time, the above model involves a free boundary which is often troublesome for numerical methods.

### 2.2 Logarithmic Transformation

To overcome the free boundary problem, we employ a transformation (Wu & Kwok, 1997; Sevcovic, 2007; Egorova et al., 2016) on multi-variable domains as

\[x_m = \ln \frac{S}{s_f(m)(\tau)} = \ln S - \ln s_f(m)(\tau), \quad m = 1, 2, \ldots, I,\]

(8)

where the variable \(x_m\) exists in a positive domain \(x_m \in (0, \infty)\) if \(S > s_f(m)\). The transformed \(m\) option value functions \(U_m(x_m, \tau)\) are related to the original \(m\) option value functions \(V_m(S, \tau)\) by the dimensionless transformation

\[U_m(x_m, \tau) = V_m(S, \tau), \quad m = 1, 2, \ldots, I. \quad (9a)\]

Applying this transformation, we obtain the following relations:

\[\frac{\partial x_m}{\partial S} = \frac{1}{S}, \quad \frac{\partial x_m}{\partial \tau} = -\frac{s'_f(m)(\tau)}{s_f(m)(\tau)}, \quad \frac{\partial V_m}{\partial S} = \frac{1}{S} \frac{\partial U_m}{\partial x_m}; \quad (9b)\]

\[\frac{\partial^2 V_m}{\partial S^2} = \frac{1}{S^2} \left( -\frac{\partial U_m}{\partial x_m} + \frac{\partial^2 U_m}{\partial x_m^2} \right), \quad \frac{\partial V_m}{\partial \tau} = \frac{\partial U_m}{\partial \tau} - \frac{s'_f(m)(\tau)}{s_f(m)(\tau)} \frac{\partial U_m}{\partial x_m}. \quad (9c)\]
Because our interest is to also calculate speed, delta decay, and color options and apply the Hermite interpolation later, we differentiate further to obtain higher derivatives of the option value function in each regime as follows:

\[
\frac{\partial^3 V_m}{\partial S^3} = \frac{1}{S^3} \left( 2 \frac{\partial U_m}{\partial x_m} - 3 \frac{\partial^2 U_m}{\partial x_m^2} + \frac{\partial^3 U_m}{\partial x_m^3} \right), \quad \frac{\partial^2 V_m}{\partial S \partial \tau} = \frac{1}{S} \left( \frac{\partial^2 U_m}{\partial x_m \partial \tau} - \frac{s_f'(m)(\tau)}{s_f(m)(\tau)} \frac{\partial^2 U_m}{\partial x_m^2} \right), \quad (9b)
\]

\[
\frac{\partial^3 V_m}{\partial S^2 \partial \tau} = \frac{1}{S^2} \left( \frac{\partial^3 U_m}{\partial x_m^3 \partial \tau} - \frac{\partial^2 U_m}{\partial x_m \partial \tau} + \frac{s_f'(m)(\tau)}{s_f(m)(\tau)} \frac{\partial^2 U_m}{\partial x_m^2} - \frac{s_f'(m)(\tau)}{s_f(m)(\tau)} \frac{\partial^3 U_m}{\partial x_m^3} \right), \quad (9e)
\]

Let \( l \) represent the coupled regime(s) in the \( m \) free boundary PDEs. It also has a variable

\[
x_l = \ln \frac{S}{s_f(l)(\tau)} = \ln S - \ln s_f(l)(\tau), \quad l \neq m, \quad l = 1, 2, \ldots, I. \quad (10)
\]

Eliminating \( S \) in the \( l \)th and \( m \)th equations, we obtain a relationship between these two different regimes as

\[
x_l = x_m - \ln \frac{s_f(l)(\tau)}{s_f(m)(\tau)}. \quad (11)
\]

Substituting (8)–(9) into (4) and (5) (i.e., \( S = s_f(m)(\tau)e^{x_m} \)), the model can be changed to

\[
\frac{\partial U_m(x_m, \tau)}{\partial \tau} - \frac{1}{2} \sigma_m^2 \frac{\partial^2 U_m(x_m, \tau)}{\partial x_m^2} - \left( \frac{s_f'(m)(\tau)}{s_f(m)(\tau)} + r_m - \frac{\sigma_m^2}{2} \right) \frac{\partial U_m(x_m, \tau)}{\partial x_m} + r_m U_m(x_m, \tau)
- \sum_{l \neq m} q_m l [U_l(x_m, \tau) - U_m(x_m, \tau)] = 0, \quad \text{for } x_m > 0; \quad (12)
\]

\[
U_m(x_m, \tau) = K - S = K - s_f(m)(\tau)e^{x_m}, \quad \text{for } x_m < 0; \quad (13)
\]

where the initial condition (5) is changed to

\[
U_m(x_m, 0) = \max(K - Ke^{x_m}, 0) = 0, \quad x_m \geq 0, \quad s_f(m)(0) = K. \quad (14)
\]

By letting \( x_m \rightarrow 0^- \), we obtain from (13) together with (7), the boundary condition for (12) as

\[
U_m(0, \tau) = K - s_f(m)(\tau), \quad U_m(\infty, \tau) = 0. \quad (15)
\]

It can be seen in (15) that the free boundary has been transformed into a fixed boundary at \( x_m = 0 \). To apply the high order compact finite difference method, we
further transform the system in (12)–(15) by eliminating the first-order derivative. To this end, we let \( W_m(x_m, \tau) \) represent the derivative of the option value in each regime known as the delta option and given as

\[
W_m(x_m, \tau) = \frac{\partial U_m(x_m, \tau)}{\partial x_m}, \quad \text{for } x_m > 0 \text{ and } x_m < 0. \tag{16}
\]

Differentiating (12) and (13) with respect to \( x_m \), respectively, we generate a system of partial differential equations in terms of delta option as

\[
\frac{\partial W_m(x_m, \tau)}{\partial \tau} - \frac{1}{2} \sigma_m^2 \frac{\partial^2 W_m(x_m, \tau)}{\partial x_m^2} - \left( \frac{s_{f(m)}}{s_{f(m)}} + r_m - \frac{\sigma_m^2}{2} \right) \frac{\partial^2 U_m(x_m, \tau)}{\partial x_m^2} + r_m W_m(x_m, \tau)
-
\sum_{l \neq m} q_{ml} (W_l(x_m, \tau) - W_m(x_m, \tau)) = 0, \quad \text{for } x_m > 0; \tag{17}
\]

\[
W_m(x_m, \tau) = -s_{f(m)} e^{x_m}, \quad \text{for } x_m < 0; \tag{18}
\]

where the initial condition for (17) is obtained based on (14) as:

\[
W_m(x_m, 0) = 0, \quad x_m \geq 0. \tag{19}
\]

By letting \( x_m \to 0^- \) in (18) together with (7), we obtain the boundary condition for (17) as

\[
W_m(0, \tau) = -s_{f(m)}, \quad W_m(\infty, \tau) = 0. \tag{20}
\]

It should be pointed out that for \( W_m(x_m, \tau) \) at \( x_m = 0 \) when \( \tau = 0 \), its value obtained based on the initial condition and the boundary condition are different. This happens in many PDE problems. Since we are mostly concerned with what happens for \( W_m(x_m, \tau) \) and other functions in \( x_m \geq 0 \) when \( \tau > 0 \), we set \( W_m(0, 0) = 0 \) in this study. We have used the average of the limit from the left and right as the value of \( W_m(0, 0) \) and the smoothstep method to approximate \( W_m(0, 0) \). However, when comparing them with our choice of \( W_m(0, 0) = 0 \), we found no significant difference.

Furthermore, we let \( Y_m(x_m, \tau) \) represent the derivative of the delta option in the \( m \)th regime known as the gamma option and obtain

\[
Y_m(x_m, \tau) = \frac{\partial W_m(x_m, \tau)}{\partial x_m} = \frac{\partial^2 U_m(x_m, \tau)}{\partial x_m^2}, \quad \text{for } x_m > 0 \text{ and } x_m < 0. \tag{21}
\]

Differentiating (17) and (18) with respect to \( x_m \), respectively, we generate a system of gamma option PDEs for each regime of the form
\[
\frac{\partial Y_m(x_m, \tau)}{\partial \tau} - \frac{1}{2} \sigma_m^2 \frac{\partial^2 Y_m(x_m, \tau)}{\partial x_m^2} - \left( \frac{s'_f(m)}{s_f(m)} + r_m - \frac{\sigma_m^2}{2} \right) \frac{\partial^2 W_m(x_m, \tau)}{\partial x_m^2} + r_m Y_m(x_m, \tau)
\]
\[
= -\sum_{l \neq m} q_{ml}(Y_l(x_m, \tau) - Y_m(x_m, \tau)) = 0, \quad \text{for } x_m > 0; \quad (22)
\]
\[
Y_m(x_m, \tau) = -s_f(m)e^{x_m}, \quad \text{for } x_m < 0; \quad (23)
\]
where the initial condition for (22) is obtained based on (19) as
\[
Y_m(x_m, 0) = 0, \quad x_m \geq 0. \quad (24)
\]

By letting \(x_m \to 0^-\) in (23) together with (7), we obtain the boundary condition for (22)
\[
Y_m(0, \tau) = -s_f(m), \quad Y_m(\infty, \tau) = 0. \quad (25)
\]
Finally, we let \(Z_m(x_m, \tau)\) represent the derivative of the gamma option known as the speed option and obtain
\[
Z_m(x_m, \tau) = \frac{\partial Y_m(x_m, \tau)}{\partial x_m} = \frac{\partial^2 W_m(x_m, \tau)}{\partial x_m^2}
\]
\[
= \frac{\partial^2 U_m(x_m, \tau)}{\partial x_m^2}, \quad \text{for } x_m > 0 \text{ and } x_m < 0, \quad (26)
\]
Differentiating (22), (23) with respect to \(x_m\), we generate a system of speed option PDEs as
\[
\frac{\partial Z_m(x_m, \tau)}{\partial \tau} - \frac{1}{2} \sigma_m^2 \frac{\partial^2 Z(x_m, \tau)}{\partial x_m^2} - \left( \frac{s'_f(m)}{s_f(m)} + r_m - \frac{\sigma_m^2}{2} \right) \frac{\partial^2 Y_m(x_m, \tau)}{\partial x_m^2} + r_m Z_m(x_m, \tau)
\]
\[
= -\sum_{l \neq m} q_{ml}(Z_l(x_m, \tau) - Z_m(x_m, \tau)) = 0, \quad \text{for } x_m > 0; \quad (27)
\]
\[
Z_m(x_m, \tau) = -s_f(m)e^{x_m}, \quad \text{for } x_m < 0; \quad (28)
\]
where the initial and boundary conditions for (27) are given as:
\[
Z_m(x_m, 0) = 0, \quad x_m \geq 0, \quad Z_m(0, \tau) = -s_f(m), \quad Z_m(\infty, \tau) = 0. \quad (29)
\]
Thus, a set of asset-delta-gamma-speed option PDEs for the \(m\) regimes can be written as follows:
\[
\frac{\partial U_m}{\partial \tau} - \frac{1}{2} \sigma_m^2 \frac{\partial^2 U_m}{\partial x_m^2} - \left( \frac{s'_f(m)}{s_f(m)} + r_m - \frac{\sigma_m^2}{2} \right) W_m + (r_m - q_{mm})U_m - \sum_{l \neq m} q_{ml} U_l = 0, \quad (30a)
\]
\[
\frac{\partial W_m}{\partial \tau} - \frac{1}{2} \sigma_m^2 \frac{\partial^2 W_m}{\partial x_m^2} - \left( \frac{s_f(m)}{s_f(m)} + r_m - \frac{\sigma_m^2}{2} \right) \frac{\partial^2 U_m}{\partial x_m^2} + (r_m - q_{mm}) W_m - \sum_{l \neq m} q_{ml} W_l = 0, \tag{30b}
\]

\[
\frac{\partial Y_m}{\partial \tau} - \frac{1}{2} \sigma_m^2 \frac{\partial^2 Y_m}{\partial x_m^2} - \left( \frac{s_f(m)}{s_f(m)} + r_m - \frac{\sigma_m^2}{2} \right) \frac{\partial^2 W_m}{\partial x_m^2} + (r_m - q_{mm}) Y_m - \sum_{l \neq m} q_{ml} Y_l = 0, \tag{30c}
\]

\[
\frac{\partial Z_m}{\partial \tau} - \frac{1}{2} \sigma_m^2 \frac{\partial^2 Z_m}{\partial x_m^2} - \left( \frac{s_f(m)}{s_f(m)} + r_m - \frac{\sigma_m^2}{2} \right) \frac{\partial^2 Y_m}{\partial x_m^2} + (r_m - q_{mm}) Z_m - \sum_{l \neq m} q_{ml} Z_l = 0, \tag{30d}
\]

where \( m = 1, 2, \ldots, I \), \( x_m > 0 \), and the initial and boundary conditions for \( U_m(x_m, \tau) \), \( W_m(x_m, \tau) \), \( Y_m(x_m, \tau) \), and \( Z_m(x_m, \tau) \) are given as:

\[
U_m(x_m, 0) = 0, \quad W_m(x_m, 0) = 0, \quad Y_m(x_m, 0) = 0; \quad Z_m(x_m, 0) = 0, \quad s_f(m)(0) = K, \quad x_m \geq 0; \tag{31a}
\]

\[
U_m(0, \tau) = K - s_f(m)(\tau), \quad W_m(0, \tau) = Y_m(0, \tau) = Z_m(0, \tau) = -s_f(m)(\tau); \quad (31c)
\]

\[
U_m(\infty, \tau) = 0; \quad W_m(\infty, \tau) = 0, \quad Y_m(\infty, \tau) = 0, \quad Z_m(\infty, \tau) = 0. \tag{31d}
\]

On the other hand, for \( x_m < 0 \),

\[
U_m(x_m, \tau) = K - s_f(m)(\tau)e^{x_m}, \quad W_m(x_m, \tau) = -s_f(m)(\tau)e^{x_m}; \quad (32a)
\]

\[
Y_m(x_m, \tau) = -s_f(m)(\tau)e^{x_m}, \quad Z_m(x_m, \tau) = -s_f(m)(\tau)e^{x_m}. \quad (32b)
\]

Note that \( U_l \), \( W_l \), \( Y_l \), and \( Z_l \) are unknown functions in the \( l \)th regime. Therefore, the whole system must be solved numerically.

**Remark 1** The advantage of the formulation in (30) for solving the regime-switching model using the front-fixing approach are as follows:

- To implement a fourth-order compact scheme for approximating the asset option, options Greeks and optimal exercise boundary in each regime simultaneously as seen in the next section.
To avoid approximating the Greeks from the numerical solution of the asset option. Because of the coupled regime in the regime-switching model and the sensitivity associated with the Greek parameters, using the numerical solution of the asset option to approximate the Greeks could further introduce some errors which could be substantial when the number of regimes increases.

- To utilize the numerical values of the Greeks for computing the optimal exercise boundary.

### 3 Numerical Formulation

To solve the above asset-delta-gamma-speed option PDEs numerically, we first design a uniform grid \([0, \infty) \times [0, T]\) for each regime taking into consideration how the \(m\)th regime’s interval relates to the \(l\)th regime’s interval when using the Hermite interpolation technique. The infinite boundary is replaced with the far estimate boundary, which we denote as \((x_m)_M\). Denote \(i\) as the node point in the \(m\)th regime’s interval, \(j\) as the node point in the \(l\)th regime’s interval and \(n\) as the time level. Given positive integers \(M\) and \(N\) representing the numbers of grid points and time steps, respectively, we have

\[
\begin{align*}
(x_m)_i &= i h, \quad (x_l)_j = j h, \quad \tau_n = n k, \quad h = \frac{(x_m)_M}{M}, \quad k = \frac{T}{N}, \quad i, j \in [0, M], \quad k \in [0, N].
\end{align*}
\]

Here, we choose the same length of intervals so that we have the same \(h\) for all intervals. We denote the numerical solutions of \(U_m, U_l, W_m, W_l, Y_m, Y_l, Z_m, Z_l, s_f(m),\) and \(s_f(l)\) as \((u_m)_i^n, (u_l)_j^n, (w_m)_i^n, (w_l)_j^n, (y_m)_i^n, (y_l)_j^n, (z_m)_i^n, (z_l)_j^n, s_{f(m)}^n,\) and \(s_{f(l)}^n\), respectively.

#### 3.1 Compact Finite Difference Scheme

In the numerical discretization for the asset, delta, gamma, and speed options in each regime, the higher-order compact finite difference method for the second-order derivatives is used in space, while the second-order Crank–Nicolson method is used in time.

To simultaneously compute the optimal exercise boundary, asset options, and option Greeks, we present an approach based on the relationship between the asset and delta options at \((x_m)_0 = 0\) and optimal exercise boundary which is given in (15) and (20). To achieve this, we first develop a fourth-order compact scheme for the asset option based on the compact finite difference described in the following lemma.

**Lemma** Assume \(f(x) \in C^6[x_0, x_1]\), then it holds

\[
\begin{align*}
\frac{7}{4} f''(x_0) + \frac{3}{4} f''(x_1) &= \frac{5}{h^2} \left[ f(x_1) - f(x_0) \right] - \frac{5}{h} f'(x_0) \\
& \quad - \frac{h}{4} f^{(3)}(x_0) + \frac{h}{6} f^{(3)}(x_1) + O(h^4).
\end{align*}
\] (34)
**Proof** Applying the Taylor expansion at \(x_0\), we obtain

\[
\frac{h}{12} f''(x_1) - \frac{h}{12} f''(x_0) = \frac{h^2}{12} f^{(3)}(x_0) + \frac{h^3}{24} f^{(4)}(x_0) + \frac{h^4}{72} f^{(5)}(x_0) + \ldots, \tag{35a}
\]

\[
\frac{5}{3} \left[ \frac{f(x_1) - f(x_0)}{h} \right] - \frac{5}{3} f'(x_0) - \frac{5h}{6} f''(x_0) = \frac{5h^2}{18} f^{(3)}(x_0) + \frac{5h^3}{72} f^{(4)}(x_0) + \frac{h^4}{72} f^{(5)}(x_0) + \ldots. \tag{35b}
\]

Eliminating the fifth-order derivative by subtracting (35b) from (35a), we obtain

\[
- \frac{5}{3} \left[ \frac{f(x_1) - f(x_0)}{h} \right] + \frac{5}{3} f'(x_0) + \frac{h}{12} f''(x_1) + \frac{9h}{12} f''(x_0) = -\frac{7h^2}{36} f^{(3)}(x_0) - \frac{h^3}{36} f^{(4)}(x_0) + O(h^5). \tag{36}
\]

On the other hand, we have (Hirsh, 1975)

\[
\frac{2}{3} f'(x_0) + \frac{1}{3} f'(x_1) = \left[ \frac{f(x_1) - f(x_0)}{h} \right] - \frac{h}{6} f'''(x_0) + O(h^3), \tag{37a}
\]

for which we multiply it by \(h^2/6\), differentiate twice and then rearrange. This gives

\[
\frac{h}{6} f''(x_0) - \frac{h}{6} f''(x_1) = -\frac{2h^2}{18} f^{(3)}(x_0) - \frac{h^2}{18} f^{(3)}(x_1) - \frac{h^3}{36} f^{(4)}(x_0) + O(h^5). \tag{37b}
\]

We then subtract (37b) from (36) to eliminate the fourth-order derivative and obtain

\[
- \frac{5}{3} \left[ \frac{f(x_1) - f(x_0)}{h} \right] + \frac{5}{3} f'(x_0) + \frac{7h}{12} f''(x_0) + \frac{3h}{12} f''(x_1) = -\frac{3h^2}{36} f^{(3)}(x_0) + \frac{h^2}{18} f^{(3)}(x_1) + O(h^5). \tag{38}
\]

Dividing (38) by \(h/3\) and rearranging the terms give Eq. (34) and hence the proof has been completed.

To avoid evaluating the third derivative of the asset option at the boundary, we delicately shift the third derivative of the asset option in (34) away from the optimal exercise boundary by using the fourth-order compact approximation as follows:

\[
h f^{(3)}(x_0) = \frac{12}{h} \left[ f'(x_0) - 2 f'(x_1) + f'(x_2) \right] - 10h f^{(3)}(x_1) - h f^{(3)}(x_2) + O(h^5). \tag{39a}
\]
Substituting (39a) to (34), we then obtain

\[
\frac{7}{4} f''(x_0) + \frac{3}{4} f''(x_1) = \frac{5}{h^2} [f(x_1) - f(x_0)] - \frac{1}{h} [8 f'(x_0) - 6 f'(x_1) + 3 f'(x_2)] \\
+ \frac{32h}{12} f^{(3)}(x_1) + \frac{3h}{12} f^{(3)}(x_2) + O(h^4). \tag{39b}
\]

Thus, using (39b) for the second-order derivative term in (30a), we obtain

\[
\frac{1}{2} \sigma_m^2 \left[ \frac{7}{4} \frac{\partial^2 U_m((x_m)_0, \tau_{n+1/2})}{\partial x_m^2} + \frac{3}{4} \frac{\partial^2 U_m((x_m)_1, \tau_{n+1/2})}{\partial x_m^2} \right] \\
= \frac{5\sigma_m^2}{2} \left[ \frac{U_m((x_m)_1, \tau_{n+1/2}) - U_m((x_m)_0, \tau_{n+1/2})}{h^2} - \frac{1}{h} \frac{\partial U_m((x_m)_0, \tau_{n+1/2})}{\partial x_m} \right] \\
- \frac{3\sigma_m^2}{2h} \left[ \frac{\partial U_m((x_m)_0, \tau_{n+1/2})}{\partial x_m} - 2 \frac{\partial U_m((x_m)_1, \tau_{n+1/2})}{\partial x_m} + \frac{\partial U_m((x_m)_2, \tau_{n+1/2})}{\partial x_m} \right] \\
- \frac{\sigma_m^2 h}{2} \left[ \frac{32}{12} \frac{\partial^3 U_m((x_m)_1, \tau_{n+1/2})}{\partial x_m^3} + \frac{3}{12} \frac{\partial^3 U_m((x_m)_2, \tau_{n+1/2})}{\partial x_m^3} \right] + O(h^4). \tag{40}
\]

Note that in (40), we have first-order and third-order derivatives that we need to approximate further. To evaluate the first-order derivative in (40) at \((x_m)_0\), we use (15) and (20) to obtain a Robin boundary condition as follows:

\[
\frac{\partial U_m((x_m)_0, \tau_{n+1/2})}{\partial x_m} - U_m((x_m)_0, \tau_{n+1/2}) = W_m((x_m)_0, \tau_{n+1/2}) - U_m((x_m)_0, \tau_{n+1/2}) = -K. \tag{41}
\]

To evaluate the third-order derivative term in (40) at \((x_m)_1\) and \((x_m)_2\), we use (30b) and discretize \(\partial W_m/\partial t\). This gives

\[
\frac{32\sigma_m^2 h}{24} \frac{\partial^3 U_m((x_m)_0, \tau_{n+1/2})}{\partial x_m^3} + \frac{3\sigma_m^2 h}{24} \frac{\partial^3 U_m((x_m)_2, \tau_{n+1/2})}{\partial x_m^3} \\
= \frac{32h}{12} \left[ \frac{W_m((x_m)_1, \tau_{n+1}) - W_m((x_m)_1, \tau_{n})}{k} \right] + \frac{3h}{12} \left[ \frac{W_m((x_m)_2, \tau_{n+1}) - W_m((x_m)_2, \tau_{n})}{k} \right] \\
- \frac{32h}{12} \frac{(\omega_m)_n+1/2 Y_m((x_m)_1, \tau_{n+1/2})}{(\omega_m)_n+1/2 Y_m((x_m)_1, \tau_{n+1/2})} - \frac{3h}{12} \frac{(\omega_m)_n+1/2 Y_m((x_m)_2, \tau_{n+1/2})}{(\omega_m)_n+1/2 Y_m((x_m)_2, \tau_{n+1/2})} \\
+ \frac{32h}{12} \frac{(r_m - q_{mm}) W_m((x_m)_1, \tau_{n+1/2})}{(r_m - q_{mm}) W_m((x_m)_1, \tau_{n+1/2})} + \frac{3h}{12} \frac{(r_m - q_{mm}) W_m((x_m)_2, \tau_{n+1/2})}{(r_m - q_{mm}) W_m((x_m)_2, \tau_{n+1/2})} \\
- \frac{32h}{12} \sum_{l \neq m} q_{ml} W_l((x_m)_j, \tau_{n+1/2}) - \frac{3h}{12} \sum_{l \neq m} q_{ml} W_l((x_m)_j, \tau_{n+1/2}) + O(k^2). \tag{42a}
\]
where \((x_m)_{j^*|i=1}\) and \((x_m)_{j^*|i=2}\) are the locations in the space for the \(l\)th equation corresponding to \((x_m)_1\) and \((x_m)_2\) in the \(m\)th equation, respectively, and we denote

\[
(\omega_m)_{n+1/2} = \frac{2}{k} \left( \frac{s_{f(m)}^{n+1} - s_{f(m)}^{n}}{s_{f(m)}^{n+1} + s_{f(m)}^{n}} \right) + r_m - \frac{\sigma_m^2}{2}.
\] (42b)

Here \((\omega_m)_{n+1/2}\) is the discretized nonlinear coefficient in the set of PDEs in (30). Substituting (41) and (42a) into (40), we obtain

\[
\frac{1}{2} \sigma_m^2 \left[ \frac{7}{4} \frac{\partial^2 U_m((x_m)_0, \tau_{n+1/2})}{\partial x_m^2} + \frac{3}{4} \frac{\partial^2 U_m((x_m)_{1}, \tau_{n+1/2})}{\partial x_m^2} \right]
\]

\[
= \frac{5 \sigma_m^2}{2} \left[ \frac{U_m((x_m)_1, \tau_{n+1/2}) - U_m((x_m)_0, \tau_{n+1/2})}{h^2} \right] - \frac{3 \sigma_m^2}{2} \left[ \frac{U_m((x_m)_0, \tau_{n+1/2}) - 2U_m((x_m)_1, \tau_{n+1/2}) + U_m((x_m)_2, \tau_{n+1/2})}{h^2} \right]
\]

\[
+ \frac{h}{2} \left[ \frac{32 W_m((x_m)_1, \tau_{n+1}) - W_m((x_m)_1, \tau_n)}{k} + \frac{3}{k} \frac{W_m((x_m)_2, \tau_{n+1}) - W_m((x_m)_2, \tau_n)}{k} \right]
\]

\[
- \frac{h}{12} (\omega_m)_{n+1/2} (32 Y_m((x_m)_1, \tau_{n+1/2}) + 3 Y_m((x_m)_2, \tau_{n+1/2}))
\]

\[
+ \frac{h(r_m - q_{mm})}{12} \left[ 32 W_m((x_m)_1, \tau_{n+1/2}) + 3 W_m((x_m)_2, \tau_{n+1/2}) \right]
\]

\[
- \frac{h}{12} \sum_{l \neq m} q_{ml} (32 W_l((x_m)_{j^*|i=1}, \tau_{n+1/2}) + 3 W_l((x_m)_{j^*|i=2}, \tau_{n+1/2})) + O(h^4).
\] (43)

Thus, applying the Crank–Nicholson method in time for (43), we obtain the compact finite difference scheme at \((x_m)_0 = 0\) as

\[
\frac{7}{4} \left[ \frac{(u_m)_0^{n+1} - (u_m)^n_0}{k} \right] + \frac{3}{4} \left[ \frac{(u_m)_1^{n+1} - (u_m)^n_1}{k} \right] - \frac{5 \sigma_m^2}{4h} \left[ \frac{(u_m)^{n+1}_1 - (u_m)^n_0}{h} - (u_m)^n_0 \right]
\]

\[
- \frac{5 \sigma_m^2}{4h} \left[ \frac{(u_m)^n_1 - (u_m)^n_0}{h} - (u_m)^n_0 \right] - \frac{5 \sigma_m^2}{2h} K
\]

\[
+ \frac{3 \sigma_m^2}{4} \left[ \frac{(w_m)^{n+1}_0 + (w_m)^n_0}{h} - 2 \frac{(w_m)^{n+1}_1 + (w_m)^n_1}{h} + (w_m)^{n+1}_2 + (w_m)^n_2 \right]
\]

\[
+ \frac{(r_m - q_{mm})}{8} \left[ 7 \frac{(u_m)^{n+1}_0 + (u_m)^n_0}{k} + 3 \frac{(u_m)^{n+1}_1 + (u_m)^n_1}{k} \right]
\]

\[
- \frac{h}{12} \left[ 32 \frac{(w_m)^{n+1}_1 - (w_m)^n_1}{k} + 3 \frac{(w_m)^{n+1}_2 - (w_m)^n_2}{k} \right]
\]

\[
- \frac{h(r_m - q_{mm})}{24} \left[ 32 \frac{(w_m)^{n+1}_1 + (w_m)^n_1}{k} + 3 \frac{(w_m)^{n+1}_2 + (w_m)^n_2}{k} \right]
\]

\[
- \frac{(\omega_m)_{n+1/2}}{8} \left[ 7 \frac{(w_m)^{n+1}_0 + (w_m)^n_0}{k} + 3 \frac{(w_m)^{n+1}_1 + (w_m)^n_1}{k} \right]
\]
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The third derivative of the asset option in (40) using (30b) which is the highest order computing the third derivative term as shown in (39). Moreover, we have evaluated from the optimal exercise boundary with fourth-order compact approximation when To ensure satisfying the condition of Lemma 1, we slightly shift away Remark 2 for (30a)–(30d), we obtain

\[
\frac{1}{12} f''(x_{i-1}) + \frac{10}{12} f''(x_i) + \frac{1}{12} f''(x_{i+1}) = \frac{1}{h^2} \left[ f(x_{i-1}) - 2f(x_i) + f(x_{i+1}) \right] + O\left(h^4\right).
\] (44)

with the truncation error of \(O(k^2 + h^4)\). Here, \(j^*|i = 0,1\) indicate the values of \(j^*\) given at \((x_m)_0\) and \((x_m)_1\), respectively, where \((u_l)^{n+1}_{j^*|i}\) and \((w_l)^{n+1}_{j^*|i}\) need to be evaluated from \((u_l)^{n+1}\) and \((w_l)^{n+1}\), respectively, in the grid for the \(l\)th regime.

**Remark 2** To ensure satisfying the condition of Lemma 1, we slightly shift away from the optimal exercise boundary with fourth-order compact approximation when computing the third derivative term as shown in (39). Moreover, we have evaluated the third derivative of the asset option in (40) using (30b) which is the highest order derivative involved in our discretize boundary equation.

At each interior grid point, \((x_m)_i = 1, 2, \ldots, M - 1\), using the compact finite difference scheme (Adam, 1975; 1976; Zhao et al., 2007; Liao & Khaliq, 2009; Dremlakov & Ehrhardt, 2011; Cao et al., 2011; Gao & Sun, 2013; Yan et al., 2019) as

\[
\frac{1}{12} \left[ \frac{(u_m)^{n+1}_{i-1} - (u_m)^n_{i-1}}{k} \right] + \frac{10}{12} \left[ \frac{(u_m)^{n+1}_i - (u_m)^n_i}{k} \right] + \frac{1}{12} \left[ \frac{(u_m)^{n+1}_{i+1} - (u_m)^n_{i+1}}{k} \right] - \frac{\sigma^2}{4h^2} \left[ \frac{(w_m)^{n+1}_{i-1} - 2(u_m)^{n+1}_i + (w_m)^n_{i-1}}{h^2} \right] - \frac{\sigma^2}{4h^2} \left[ \frac{(w_m)^{n+1}_i - 2(u_m)^n_i + (w_m)^n_i}{h^2} \right] - \frac{\sigma^2}{4h^2} \left[ \frac{(w_m)^{n+1}_{i+1} - 2(u_m)^n_{i+1} + (w_m)^n_{i+1}}{h^2} \right] + \frac{(\rho_m)^{n+1/2}}{24} \left[ \frac{(w_m)^{n+1}_{i-1} - 2(u_m)^{n+1}_i + (w_m)^n_{i-1}}{h^2} \right] + \frac{(\rho_m)^{n+1/2}}{24} \left[ \frac{(w_m)^{n+1}_i - 2(u_m)^n_i + (w_m)^n_i}{h^2} \right] + \frac{(\rho_m)^{n+1/2}}{24} \left[ \frac{(w_m)^{n+1}_{i+1} - 2(u_m)^n_{i+1} + (w_m)^n_{i+1}}{h^2} \right] \right.

\[
\left. - \frac{(\rho_m)^{n+1/2}}{24} \left[ \frac{(w_m)^{n+1}_{i-1} - 2(u_m)^{n+1}_i + (w_m)^n_{i-1}}{h^2} \right] + \frac{(\rho_m)^{n+1/2}}{24} \left[ \frac{(w_m)^{n+1}_i - 2(u_m)^n_i + (w_m)^n_i}{h^2} \right] + \frac{(\rho_m)^{n+1/2}}{24} \left[ \frac{(w_m)^{n+1}_{i+1} - 2(u_m)^n_{i+1} + (w_m)^n_{i+1}}{h^2} \right] \right]
\]

\[
= 0,
\] (46)

where \(j^*\) represents the location for the \(l\)th regime corresponding to \((x_m)_l\), and the truncation error is \(O(k^2 + h^4)\). The optimal exercise boundary is then calculated from the asset option as follows:

\[
s_{f(m)}^{n+1} = K - (u_m)^{n+1}_0.
\] (47a)
The boundary conditions for the option Greeks at \((x_m)_0\) are calculated as follows:

\[
(w_m)^{n+1}_0 = -s_f^{n+1}, \quad (y_m)^{n+1}_0 = -s_f^{n+1}, \quad (z_m)^{n+1}_0 = -s_f^{n+1};
\]  

(47b)

The truncated boundary conditions at the endpoint are calculated as follows:

\[
(u_m)^{n+1}_M = 0, \quad (w_m)^{n+1}_M = 0, \quad (y_m)^{n+1}_M = 0, \quad (z_m)^{n+1}_M = 0.
\]  

(47c)

Subsequently, we employ a compact finite difference scheme to the interior point for the delta, gamma, and speed options as follows:

\[
\frac{1}{12} \left[ \frac{(w_m)^{n+1}_i - (w_m)^n_i}{k} \right] + \frac{10}{12} \left[ \frac{(w_m)^{n+1}_i - (w_m)^n_i}{k} \right] + \frac{1}{12} \left[ \frac{(w_m)^{n+1}_i - (w_m)^n_i}{k} \right] 
- \frac{\sigma_m^2}{4h^2} \left[ \frac{\sigma_m^2}{4h^2} \right] + \frac{\sigma_m^2}{4h^2} \left[ \frac{\sigma_m^2}{4h^2} \right] 
- \frac{(\omega_m)^{n+1}_i}{2h^2} \left[ \frac{(u_m)^{n+1}_i + (u_m)^n_i}{2} \right] - \frac{(\omega_m)^{n+1}_i}{2h^2} \left[ \frac{(u_m)^{n+1}_i + (u_m)^n_i}{2} \right] 
+ \frac{(r_m - q_{mm})}{24} \left[ \frac{(w_m)^{n+1}_i + (w_m)^n_i}{2} \right] + \frac{(r_m - q_{mm})}{24} \left[ \frac{(w_m)^{n+1}_i + (w_m)^n_i}{2} \right] 
- \frac{\sum_{l \neq m} q_{ml}}{24} \left[ \frac{(w_m)^{n+1}_i + (w_m)^n_i}{2} \right] + \frac{\sum_{l \neq m} q_{ml}}{24} \left[ \frac{(w_m)^{n+1}_i + (w_m)^n_i}{2} \right] 
= 0,
\]  

(48a)

\[
\frac{1}{12} \left[ \frac{(y_m)^{n+1}_i - (y_m)^n_i}{k} \right] + \frac{10}{12} \left[ \frac{(y_m)^{n+1}_i - (y_m)^n_i}{k} \right] + \frac{1}{12} \left[ \frac{(y_m)^{n+1}_i - (y_m)^n_i}{k} \right] 
- \frac{\sigma_m^2}{4h^2} \left[ \frac{\sigma_m^2}{4h^2} \right] + \frac{\sigma_m^2}{4h^2} \left[ \frac{\sigma_m^2}{4h^2} \right] 
- \frac{(\omega_m)^{n+1}_i}{2h^2} \left[ \frac{(y_m)^{n+1}_i + (y_m)^n_i}{2} \right] - \frac{(\omega_m)^{n+1}_i}{2h^2} \left[ \frac{(y_m)^{n+1}_i + (y_m)^n_i}{2} \right] 
+ \frac{(r_m - q_{mm})}{24} \left[ \frac{(y_m)^{n+1}_i + (y_m)^n_i}{2} \right] + \frac{(r_m - q_{mm})}{24} \left[ \frac{(y_m)^{n+1}_i + (y_m)^n_i}{2} \right] 
- \frac{\sum_{l \neq m} q_{ml}}{24} \left[ \frac{(y_m)^{n+1}_i + (y_m)^n_i}{2} \right] + \frac{\sum_{l \neq m} q_{ml}}{24} \left[ \frac{(y_m)^{n+1}_i + (y_m)^n_i}{2} \right] 
= 0,
\]  

(48b)

\[
\frac{1}{12} \left[ \frac{(z_m)^{n+1}_i - (z_m)^n_i}{k} \right] + \frac{10}{12} \left[ \frac{(z_m)^{n+1}_i - (z_m)^n_i}{k} \right] + \frac{1}{12} \left[ \frac{(z_m)^{n+1}_i - (z_m)^n_i}{k} \right] 
- \frac{\sigma_m^2}{4h^2} \left[ \frac{\sigma_m^2}{4h^2} \right] + \frac{\sigma_m^2}{4h^2} \left[ \frac{\sigma_m^2}{4h^2} \right] 
- \frac{(\omega_m)^{n+1}_i}{2h^2} \left[ \frac{(z_m)^{n+1}_i + (z_m)^n_i}{2} \right] - \frac{(\omega_m)^{n+1}_i}{2h^2} \left[ \frac{(z_m)^{n+1}_i + (z_m)^n_i}{2} \right] 
+ \frac{(r_m - q_{mm})}{24} \left[ \frac{(z_m)^{n+1}_i + (z_m)^n_i}{2} \right] + \frac{(r_m - q_{mm})}{24} \left[ \frac{(z_m)^{n+1}_i + (z_m)^n_i}{2} \right] 
+ \frac{\sum_{l \neq m} q_{ml}}{24} \left[ \frac{(z_m)^{n+1}_i + (z_m)^n_i}{2} \right] + \frac{\sum_{l \neq m} q_{ml}}{24} \left[ \frac{(z_m)^{n+1}_i + (z_m)^n_i}{2} \right] 
= 0,
\]  

(48c)
\[-\sum_{l \neq m} \frac{q_{ml}}{24} \left[ (z_l)^{j_{l-1}+1} + (z_l)^{j_{l-1}} + 10(z_l)^{j_{l+1}+1} + (z_l)^{j_{l+1}} \right] \]
\[= 0. \quad (48c)\]

The initial conditions for the asset option and option sensitivities in each regime are calculated as

\[(u_m)^0_i = (w_m)^0_i = (y_m)^0_i = (z_m)^0_i = 0, \quad i = 1, 2, \ldots, M. \quad (49)\]

Once the asset, delta, gamma, and speed options for the \(m\) regimes are obtained from the above numerical method, one may further compute the theta, delta decay, and color options for the \(m\) regime given as

\[
\frac{\partial U_m ((x_m)_i, \tau_n)}{\partial \tau} = (\Theta_m)_i^n, \quad \frac{\partial W_m ((x_m)_i, \tau_n)}{\partial \tau} = (K_m)_i^n, \\
\frac{\partial Y_m ((x_m)_i, \tau_n)}{\partial \tau} = (\Gamma_m)_i^n, \quad (50)\]

respectively. Here, for \(n = 1\), we approximate the Greeks using first-order backward finite differences

\[(\Theta_m)_i^1 \approx \frac{(u_m)_i^1 - (u_m)_i^0}{k}, \quad (K_m)_i^1 \approx \frac{(w_m)_i^1 - (w_m)_i^0}{k}, \quad (\Gamma_m)_i^1 \approx \frac{(y_m)_i^1 - (y)_i^0}{k}. \quad (51a)\]

Subsequently, we use the second-order backward finite difference approximations as

\[(\Theta_m)_i^{n+1} \approx \frac{3(u_m)_i^{n+1} - 4(u_m)_i^n + (u_m)_i^{n-1}}{2k}, \quad (K_m)_i^{n+1} \approx \frac{3(w_m)_i^{n+1} - 4(w_m)_i^n + (w_m)_i^{n-1}}{2k}; \quad (51b)\]

\[(\Gamma_m)_i^{n+1} \approx \frac{3(y_m)_i^{n+1} - 4(y_m)_i^n + (y_m)_i^{n-1}}{2k}. \quad (51c)\]

The initial conditions of the theta, delta decay, and color options for each regime are calculated as

\[(\Theta_m)_i^0 = 0, \quad (K_m)_i^0 = 0, \quad (\Gamma_m)_i^0 = 0, \quad i = 0, 1, \ldots, M. \quad (52)\]

The above scheme in (44), (46), and (48) consist of tridiagonal linear systems for solving \(u_m, w_m, y_m, z_m\) in time level \(n + 1\), once \(u_l, w_l, y_l, z_l\) are given. Note that \(u_l, w_l, y_l, z_l\) are the option prices and their derivatives. These relationships allow us to employ the third-order Hermite interpolation to approximate these \(u_l, w_l, y_l, z_l\), for which we give a detailed description in the following subsection. This is one of the key ideas in the present method.
3.2 Hermite Interpolation

To evaluate \((u_l)^n_j\), \((w_l)^n_j\), \((y_l)^n_j\), and \((z_l)^n_j\) in (40), (46), and (48), we need to consider the relationship between the fixed interval (and the mesh) for the \(l\)th regime and the fixed interval (and the mesh) for the \(m\)th regime after the logarithmic transformation. If \(s_{f(l)}(\tau_n) = s_{f(m)}(\tau_n)\), then the fixed interval for the \(l\)th regime overlaps completely with the fixed interval for the \(m\)th regime. Hence, \((x_m)_i = (x_l)_j\). If \(s_{f(l)}(\tau_n) \neq s_{f(m)}(\tau_n)\), there are three possible cases as shown in Fig. 1.

Figure 1a shows that there exists a possibility for \((x_m)_i\) corresponding to \((x_l)_j < 0\) in the \(l\)th interval. For this case, \((u_l)^n_i = K - s_{f(l)}(\tau_n)e^{(x_l)_j}\) and \((w_l)^n_j = (y_l)^n_j = (z_l)^n_j = s_{f(l)}(\tau_n)e^{(x_l)_j}\) based on (13)–(16). Figure 1b shows that there exists a possibility for \((x_m)_i\) corresponding to a point in the interval \((0, (x_l)_M)\). For this case, \((u_l)^n_j\), \((w_l)^n_j\), \((y_l)^n_j\), and \((z_l)^n_j\) have to be evaluated using interpolation based on \((u_l)^n_j\), \((w_l)^n_j\), \((y_l)^n_j\), and \((z_l)^n_j\). Note that \((u_l)^n_j\), \((w_l)^n_j\), \((y_l)^n_j\), and \((z_l)^n_j\) are for the \(l\) option prices and their derivatives. These relationships allow us to employ the third-order Hermite interpolation (Burden et al., 2016) to obtain higher-order accuracy. Figure 1c shows that there exists a possibility for \((x_m)_i\) corresponding to \((x_l)_j > (x_l)_M\). For

\[\text{Fig. 1 Relationship between the } l\text{th and } m\text{th intervals and the location of the } (x_m)_i \text{ in the } l\text{th interval} \]

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this case, we set \((u_t)^n_{j*} = (w_t)^n_{j*} = (y_t)^n_{j*} = (z_t)^n_{j*} = 0\). Overall, we can evaluate \((u_t)^n_{j*}, (w_t)^n_{j*}, (y_t)^n_{j*}\) and \((z_t)^n_{j*}\) based on the following formulas:

\[
(u_t)^n_{j*} = \begin{cases} 
K - s_f(l) (\tau_n) e^{(x_t)}{j*}, \\
a (u_t)^n_j + b (u_t)^n_{j+1} + c (w_t)^n_j + d (w_t)^n_{j+1}, (x_t)_j \leq (x_m)_j - \ln \frac{s_f(l)(\tau_n)}{s_f(m)(\tau_n)} \leq (x_t)_{j+1}; \\
0, \\
\end{cases} 
\]

\[
(w_t)^n_{j*} = \begin{cases} 
-s_f(l) (\tau_n) e^{(x_t)}{j*}, \\
e (u_t)^n_j + f (u_t)^n_{j+1} + g (w_t)^n_j + h (w_t)^n_{j+1}, (x_t)_j \leq (x_m)_j - \ln \frac{s_f(l)(\tau_n)}{s_f(m)(\tau_n)} \leq (x_t)_{j+1}; \\
0, \\
\end{cases} 
\]

\[
(y_t)^n_{j*} = \begin{cases} 
-s_f(l) (\tau_n) e^{(x_t)}{j*}, \\
a (y_t)^n_j + b (y_t)^n_{j+1} + c (z_t)^n_j + d (z_t)^n_{j+1}, (x_t)_j \leq (x_m)_j - \ln \frac{s_f(l)(\tau_n)}{s_f(m)(\tau_n)} \leq (x_t)_{j+1}; \\
0, \\
\end{cases} 
\]

\[
(z_t)^n_{j*} = \begin{cases} 
-s_f(l) (\tau_n) e^{(x_t)}{j*}, \\
a (z_t)^n_j + b (z_t)^n_{j+1} + c (z_t)^n_j + d (z_t)^n_{j+1}, (x_t)_j \leq (x_m)_j - \ln \frac{s_f(l)(\tau_n)}{s_f(m)(\tau_n)} \leq (x_t)_{j+1}; \\
0, \\
\end{cases} 
\]

where the coefficients are given based on the cubic Hermite interpolation as follows:

\[
a = \frac{1}{h^2} \left[ 1 + 2 \left( \frac{(x_t)_{j*} - (x_t)_j}{h} \right) \right] (x_t)_{j*} - (x_t)_{j+1}, \quad (54a)
\]

\[
b = \frac{1}{h^2} \left[ 1 - 2 \left( \frac{(x_t)_{j*} - (x_t)_{j+1}}{h} \right) \right] (x_t)_{j*} - (x_t)_j, \quad (54b)
\]

\[
c = \frac{1}{h^2} \left[ (x_t)_{j*} - (x_t)_j \right] (x_t)_{j*} - (x_t)_{j+1}, \quad d \]

\[
e = \frac{2}{h^2} \left[ 1 + \frac{2 (x_t)_{j*} - (x_t)_j}{h} \right] (x_t)_j - (x_t)_{j+1} + \frac{2}{h^3} (x_t)_{j*} - (x_t)_{j+1}, \quad (54d)
\]
$$f = \frac{2}{h^2} \left[ 1 - \frac{2[(x_i)_{j^*} - (x_i)_{j+1}]}{h} \right] [(x_i)_{j^*} - (x_i)_j] + \frac{2}{h^3} [(x_i)_{j^*} - (x_i)_j]^2, \quad (54e)$$

$$g = \frac{2}{h^2} [(x_i)_{j^*} - (x_i)_j][(x_i)_{j^*} - (x_i)_{j+1}] + \frac{1}{h^2} [(x_i)_{j^*} - (x_i)_{j+1}]^2, \quad (54f)$$

$$o = \frac{2}{h^2} [(x_i)_{j^*} - (x_i)_{j+1}][(x_i)_{j^*} - (x_i)_j] + \frac{1}{h^2} [(x_i)_{j^*} - (x_i)_j]^2. \quad (54g)$$

It should be pointed out that we have compared with other third-order interpolation methods such as the cubic spline when estimating these $(u_1)_{j^*}$, $(w_1)_{j^*}$, $(y_1)_{j^*}$ and $(z_1)_{j^*}$. Hermite interpolation proves to be accurate, more efficient, and very fast in computation. Moreover, it is worth noting that $(\dot{z})_{j^*}$ (the derivative of $(z_1)_{j^*}$) is employed in the cubic Hermite interpolation. To evaluate $(\dot{z})_{j^*}$ with fourth-order accuracy, we obtain it from the speed option PDE by further taking derivative.

### 3.3 Stability Analysis

The stability analysis of our numerical schemes is carried out using the matrix form of the von Neumann method (Hirsch, 2001; Liao & Khaliq, 2009). Due to the complex system of the present method, we ignore the influence of $(u_1)_n$, $(w_1)_n$, $(y_1)_n$ and $(z_1)_n$. Furthermore, we ignore the effect due to the Neumann boundary condition we implemented in (39b). Let

$$(u_m)_n = \lambda^m e^{i\beta i h}, \quad (w_m)_n = \Phi^m e^{i\beta i h}, \quad (y_m)_n = \Psi^m e^{i\beta i h}, \quad \kappa = \sqrt{-1}. \quad (55)$$

Denote

$$\mu = \frac{\sigma^2 m k}{4h^2}, \quad \kappa = (r_m - q_{mm}) k, \quad \omega_{n+1/2} = \left( \frac{2 (s_{f(m)}^{n+1} - s_{f(m)}^n)}{(s_{f(m)}^{n+1} + s_{f(m)}^n) k} + r_m - \frac{\sigma^2 m}{2} \right) k. \quad (56)$$

Here we freeze the nonlinear coefficient in (56) and assume that $\omega \equiv \omega_{n+1/2}$. Substituting (55), (56) into (46), and (48), we obtain

$$\lambda^m_{n+1} \left[ 1 - \frac{1}{3} \sin^2 \left( \frac{\beta h}{2} \right) + 4\mu \sin^2 \left( \frac{\beta h}{2} \right) + \frac{\kappa}{2} - \frac{\kappa}{2} \sin^2 \left( \frac{\beta h}{2} \right) \right]$$

$$- \lambda^m_{n} \left[ 1 - \frac{1}{3} \sin^2 \left( \frac{\beta h}{2} \right) - 4\mu \sin^2 \left( \frac{\beta h}{2} \right) + \frac{\kappa}{2} - \frac{\kappa}{2} \sin^2 \left( \frac{\beta h}{2} \right) \right]$$

$$- \omega \left[ \Phi^{m}_{n+1} \left( \frac{1}{2} - \frac{1}{6} \sin^2 \left( \frac{\beta h}{2} \right) \right) + \Phi^{m}_{n} \left( \frac{1}{2} - \frac{1}{6} \sin^2 \left( \frac{\beta h}{2} \right) \right) \right] = 0. \quad (57a)$$
\[ \Phi^{n+1}_m \left[ 1 - \frac{1}{3} \sin^2 \left( \frac{\beta h}{2} \right) + 4\mu \sin^2 \left( \frac{\beta h}{2} \right) + \frac{\kappa}{2} - \frac{\kappa}{2} \sin^2 \left( \frac{\beta h}{2} \right) \right] \\
- \Phi^n_m \left[ 1 - \frac{1}{3} \sin^2 \left( \frac{\beta h}{2} \right) - 4\mu \sin^2 \left( \frac{\beta h}{2} \right) + \frac{\kappa}{2} - \frac{\kappa}{2} \sin^2 \left( \frac{\beta h}{2} \right) \right] \\
- \omega \left[ -4\lambda^{n+1}_m \sin^2 \left( \frac{\beta h}{2} \right) - 4\lambda^n_m \sin^2 \left( \frac{\beta h}{2} \right) \right] = 0, \quad (57b) \]

\[ \Upsilon^{n+1}_m \left[ 1 - \frac{1}{3} \sin^2 \left( \frac{\beta h}{2} \right) + 4\mu \sin^2 \left( \frac{\beta h}{2} \right) + \frac{\kappa}{2} - \frac{\kappa}{2} \sin^2 \left( \frac{\beta h}{2} \right) \right] \\
- \Upsilon^n_m \left[ 1 - \frac{1}{3} \sin^2 \left( \frac{\beta h}{2} \right) - 4\mu \sin^2 \left( \frac{\beta h}{2} \right) + \frac{\kappa}{2} - \frac{\kappa}{2} \sin^2 \left( \frac{\beta h}{2} \right) \right] \\
- \omega \left[ -4\Upsilon^{n+1}_m \sin^2 \left( \frac{\beta h}{2} \right) - 4\Upsilon^n_m \sin^2 \left( \frac{\beta h}{2} \right) \right] = 0, \quad (57c) \]

\[ \Psi^{n+1}_m \left[ 1 - \frac{1}{3} \sin^2 \left( \frac{\beta h}{2} \right) + 4\mu \sin^2 \left( \frac{\beta h}{2} \right) + \frac{\kappa}{2} - \frac{\kappa}{2} \sin^2 \left( \frac{\beta h}{2} \right) \right] \\
- \Psi^n_m \left[ 1 - \frac{1}{3} \sin^2 \left( \frac{\beta h}{2} \right) - 4\mu \sin^2 \left( \frac{\beta h}{2} \right) + \frac{\kappa}{2} - \frac{\kappa}{2} \sin^2 \left( \frac{\beta h}{2} \right) \right] \\
- \omega \left[ -4\Psi^{n+1}_m \sin^2 \left( \frac{\beta h}{2} \right) - 4\Psi^n_m \sin^2 \left( \frac{\beta h}{2} \right) \right] = 0, \quad (57d) \]

which can be simplified to

\[ p\lambda^{n+1}_m - q\lambda^n_m = r\Phi^{n+1}_m + r\Phi^n_m, \quad p\Phi^{n+1}_m - q\Phi^n_m = s\lambda^{n+1}_m + s\lambda^n_m, \quad (58a) \]

\[ p\Upsilon^{n+1}_m - q\Upsilon^n_m = s\Phi^{n+1}_m + s\Phi^n_m, \quad p\Psi^{n+1}_m - q\Psi^n_m = s\Upsilon^{n+1}_m + s\Upsilon^n_m, \quad (58b) \]

where

\[ p = 1 - \frac{1}{3} \sin^2 \left( \frac{\beta h}{2} \right) + 4\mu \sin^2 \left( \frac{\beta h}{2} \right) + \frac{\kappa}{2} - \frac{\kappa}{2} \sin^2 \left( \frac{\beta h}{2} \right), \quad r = -\frac{2\omega}{h^2} \sin^2 \left( \frac{\beta h}{2} \right), \quad (59a) \]

\[ s = \omega \left[ \frac{1}{2} - \frac{1}{6} \sin^2 \left( \frac{\beta h}{2} \right) \right], \quad q \]

\[ = 1 - \frac{1}{3} \sin^2 \left( \frac{\beta h}{2} \right) - 4\mu \sin^2 \left( \frac{\beta h}{2} \right) + \frac{\kappa}{2} - \frac{\kappa}{2} \sin^2 \left( \frac{\beta h}{2} \right). \quad (59b) \]

Thus, we obtain a system of equations from (58) and present it in matrix–vector form as

\[
\begin{bmatrix}
\lambda^{n+1}_m \\
\Phi^{n+1}_m \\
\Upsilon^{n+1}_m \\
\Psi^{n+1}_m
\end{bmatrix} =
\begin{bmatrix}
p & -r & 0 & 0 \\
-s & p & 0 & 0 \\
0 & -s & p & 0 \\
0 & 0 & -s & p
\end{bmatrix}^{-1}
\begin{bmatrix}
q & r & 0 & 0 \\
s & q & 0 & 0 \\
0 & s & q & 0 \\
0 & 0 & s & q
\end{bmatrix}
\begin{bmatrix}
\lambda^n_m \\
\Phi^n_m \\
\Upsilon^n_m \\
\Psi^n_m
\end{bmatrix}
\]
\[
\begin{bmatrix}
\frac{p}{p^2-sr} & \frac{r}{p^2-sr} & 0 & 0 \\
\frac{s}{p^2-sr} & \frac{p}{p^2-sr} & 0 & 0 \\
\frac{s^2}{p(p^2-sr)} & \frac{s}{p^2-sr} & \frac{1}{p} & 0 \\
\frac{s^3}{p(p^2-sr)^2} & \frac{1}{p^2} & \frac{s}{p} & 1
\end{bmatrix}
\begin{bmatrix}
q \\
r \\
s \\
0
\end{bmatrix}
= \begin{bmatrix}
\lambda_m \\
\Phi_m \\
\gamma_m \\
\Psi_m
\end{bmatrix}\]

\[
\begin{bmatrix}
\frac{pq+rs}{p^2-sr} & \frac{pq+rs}{p^2-sr} & 0 & 0 \\
\frac{ps+qs}{p^2-sr} & \frac{pq+rs}{p^2-sr} & 0 & 0 \\
\frac{ps^2+qs^2}{p(p^2-sr)^2} & \frac{pq+rs}{p^2-sr} & \frac{q}{p} & 0 \\
\frac{ps^3+qs^3}{p^2(p^2-sr)^3} & \frac{pq+rs}{p^2-2} & \frac{q}{p} & \frac{q}{p}
\end{bmatrix}
= A
\begin{bmatrix}
\lambda_m \\
\Phi_m \\
\gamma_m \\
\Psi_m
\end{bmatrix}.
\] (60)

Here, \(A\) represents the amplification matrix. To show our numerical method to be unconditionally stable, we need to confirm that the modulus of the eigenvalue of the matrix \(A\) is less than or equal to 1. Denoting the eigenvalue of the matrix \(A\) as \(\varphi\), we obtain the equation below

\[
\varphi^2 - 2\varphi \frac{pq+rs}{p^2-sr} + \frac{(pq+rs)^2}{(p^2-sr)^2} = 0.
\] (61)

Note that

\[
rs = -\frac{2\omega^2}{h^2} \sin^2\left(\frac{\beta h}{2}\right) \left[1 - \frac{1}{6} \sin^2\left(\frac{\beta h}{2}\right)\right] \leq 0, \quad p \geq q.
\] (62)

Since \(\mu > 0\), we obtain \(p > q\) and hence

\[
\left(\frac{q}{p} - \varphi\right) \left(\frac{q}{p} - \varphi\right) = 0, \quad |\varphi_{1,2}| = \left|\frac{q}{p}\right| < 1.
\] (63)

Furthermore, we need to obtain \(\varphi_{3,4}\) by solving

\[
\varphi^2 - 2\varphi \frac{pq+rs}{p^2-sr} + \frac{(pq+rs)^2}{(p^2-sr)^2} - \frac{(pr+qr)(ps+qs)}{(p^2-sr)^2} = 0,
\] (64)

which gives

\[
\varphi_{3,4} = \frac{(pq+rs) \pm (p+q)\sqrt{rs}}{p^2-sr}.
\] (65)

Noticing \(\varpi = -rs \geq 0\), we obtain the complex conjugate values of the eigenvalues as

\[
\varphi_{3,4} = \frac{(pq - \varpi) \pm (p+q)\sqrt{\varpi}}{p^2+\varpi}.
\] (66)

Thus, we have
\[ |\varphi_{3,4}|^2 = \frac{(pq - \sigma)^2 + (p + q)^2 \sigma}{(p^2 + \sigma)^2} = \frac{(p^2 + \sigma)(q^2 + \sigma)}{(p^2 + \sigma)^2} = \frac{(q^2 + \sigma)}{(p^2 + \sigma)} \leq 1. \quad (67) \]

Based on the matrix form of the von Neumann analysis, we have proved that our numerical schemes are unconditionally stable.

### 3.4 Computational Procedure

The system in (40) and (46)–(54) must be solved iteratively. Here, we present an iterative procedure based on the Gauss–Seidel (GS) method. In detail, we initialize \( s^n_{f(m)} \), \( (u_m)^n_i \), \( (w_m)^n_i \), \( (y_m)^n_i \), \( (z_m)^n_i \), \( (\Theta_m)^n_i \), and \( (K_m)^n_i \) where \((u_l)^n_j, (w_l)^n_j, (y_l)^n_j, \) and \((z_l)^n_j\) are calculated based on (53) and (54). We assume that \( (u_m)^{n+1}_{i(l=0)} = (u_m)^n_i \), \( (w_m)^{n+1}_{i(l=0)} = (w_m)^n_i \), \( (y_m)^{n+1}_{i(l=0)} = (y_m)^n_i \), and \( (z_m)^{n+1}_{i(l=0)} = (z_m)^n_i \), where “\( l \)” is the iteration counter. Next, \( (u_m)^{n+1}_{i(l=1)} \) is computed and \( s^{n+1}_{f(m)} \) is obtained from \( (u_m)^{n+1}_{i(l=1)} \). Subsequently, we compute \( (u_m)^{n+1}_{i(l=1)} \), \( (w_m)^{n+1}_{i(l=1)} \), \( (y_m)^{n+1}_{i(l=1)} \), \( (z_m)^{n+1}_{i(l=1)} \), \( (\Theta_m)^{n+1}_{i(l=1)} \), \( (K_m)^{n+1}_{i(l=1)} \), and \( (\Gamma_m)^{n+1}_{i(l=1)} \). The iterative process continues until the convergence criterion of both \( \max_m \left| s^{n+1}_{f(m)} - s^{n+1}_{f(m)} \right| < \varepsilon \) and \( \max_m, i \left| (u_m)^{n+1}_{i(l=1)} - (u_m)^{n+1}_{i(l=1)} \right| < \varepsilon \) is satisfied.

An algorithm for obtaining the numerical solutions of the optimal exercise boundary, asset option, and the option Greeks in each regime using the GS method is described below.

**Algorithm**

1. Initialize \( s^n_{f(m)} \), \( (u_m)^n_i \), \( (w_m)^n_i \), \( (y_m)^n_i \), \( (z_m)^n_i \), \( (\Theta_m)^n_i \), \( (K_m)^n_i \), and \( (\Gamma_m)^n_i \) for \( i = 0, 1, ..., M \) and \( m = 1, 2, ..., I \).
2. **for** \( n = 1 \) to \( N \)
3. Compute \( (u_l)^n_j \) and \( (w_l)^n_j \), \( (y_l)^n_j \) and \( (z_l)^n_j \) for \( l = 1, 2, ..., I \) and \( l \neq m \) based on (53) and (54).
4. Set \( s^{n+1}_{f(m)} = s^n_{f(m)} \), \( (u_m)^{n+1}_{i(l=0)} = (u_m)^n_i \), \( (w_m)^{n+1}_{i(l=0)} = (w_m)^n_i \), \( (y_m)^{n+1}_{i(l=0)} = (y_m)^n_i \), \( (z_m)^{n+1}_{i(l=0)} = (z_m)^n_i \).
5. **while** true
6. Compute \( (u_l)^{n+1}_{i(l=1)} \), \( (w_l)^{n+1}_{i(l=1)} \), \( (y_l)^{n+1}_{i(l=1)} \) and \( (z_l)^{n+1}_{i(l=1)} \) for \( l = 1, 2, ..., I \) and \( l \neq m \) based on (53) and (54).
7. **for** \( m = 1 \) to \( I \)
8. Compute \( (u_m)^{n+1}_{i(l=1)} \) and evaluate \( s^{n+1}_{f(m)} \) based on (44), (46), and (47) using the GS iteration.
9. Evaluate \( (w_m)^{n+1}_{i(l=1)} \), \( (y_m)^{n+1}_{i(l=1)} \), \( (z_m)^{n+1}_{i(l=1)} \) based on (48a)-(48c).
10. **end**
11. if \( \max_m \left| s^{n+1}_{f(m)} - s^n_{f(m)} \right| < \varepsilon \) and \( \max_m, i \left| (u_m)^{n+1}_{i(l=1)} - (u_m)^n_i \right| < \varepsilon \)
12. Calculate \( (\Theta_m)^{n+1}_{i(l=1)} \), \( (K_m)^{n+1}_{i(l=1)} \), and \( (\Gamma_m)^{n+1}_{i(l=1)} \) based on (50) - (52).
13. Set \( s^{n+1}_{f(m)} = s^{n+1}_{f(m)} \), \( (u_m)^{n+1}_{i(l=1)} = (u_m)^{n+1}_{i(l=1)} \), \( (w_m)^{n+1}_{i(l=1)} = (w_m)^{n+1}_{i(l=1)} \), \( (y_m)^{n+1}_{i(l=1)} = (y_m)^{n+1}_{i(l=1)} \), and \( (z_m)^{n+1}_{i(l=1)} = (z_m)^{n+1}_{i(l=1)} \).
14. **else**
15. Set \( (\Theta_m)^{n+1}_{i(l=1)} = (\Theta_m)^{n+1}_{i(l=1)} \), \( (K_m)^{n+1}_{i(l=1)} = (K_m)^{n+1}_{i(l=1)} \), and \( (\Gamma_m)^{n+1}_{i(l=1)} = (\Gamma_m)^{n+1}_{i(l=1)} \).
16. **end**
17. **end**
18. **end**
4 Numerical Experiments

To test the accuracy and applicability of the present scheme, we consider the American put options pricing problems with the two-, four-, and sixteen-regime, respectively. The numerical code was written with MATLAB 2019a on Intel Core i5-3317U CPU 1.70 GHz 64-bit ASUS Laptop. The numerical procedures were carried out on the mesh with a uniform grid size. Furthermore, to ensure a stable non-oscillatory solution due to the implementation of the Crank–Nicholson scheme, we maintain a mesh ratio of $k/h^2 = 1$ in the numerical examples.

4.1 Examples 1: Two Regimes

We consider the American put options with the two-regime. We label our present method as “FF-CS” which we denote as the front fixing-compact scheme with cubic Hermite interpolation. We further compare them with MTree (Liu, 2010), IMS1, IMS2 (Khaliq and Liu, 2009), MOL (Chiarella et al., 2016), RBF-FD (Li et al., 2019), FF-expl (Egorova et al., 2016), ETD-CN (Khaliq et al., 2013), Iterated Optimal Stopping and Local Optimal Iteration (Babbin et al., 2014).

Case 1 Consider a switching regime problem with the strike price chosen to be $K = 9$ at the expiration time $T = 1$. In our computation, we chose the interval $0 \leq x_m \leq 3$ with the grid size $h = 0.05$, and 0.01. The parameters were given as

$$Q = \left[ -66 \atop 9 - 9 \right], \quad r = \left[ 0.10 \atop 0.05 \right], \quad \sigma = \left[ 0.80 \atop 0.30 \right], \quad \varepsilon = 10^{-8}. \quad (68)$$

Figures 2 and 3 and Tables 1 and 2 show the profiles of the option prices, Greek parameters, and optimal exercise boundaries. In Tables 1, 2 and 3, we observed that from $h = 0.05$, the data obtained based on FF-CS is very close to the one from the MOL, MTree, and RBF-FD methods. This is an indication that we can obtain more accurate numerical solutions with FF-CS using coarse grids. Furthermore, Chiarella

![Fig. 2 Asset options and optimal exercise boundaries for the two-regime example when $\tau = T$ (case 1)](image)
et al. (2016) mentioned that MTree results were used as a benchmark in the work of Khaliq and Liu (2009).

Furthermore, from Tables 1 and 5, we observe that when the coupled regime is treated explicitly by following the work of Egorova et al. (2016) [as seen in Eq. (36) in Ref. 25], the numerical solutions converge very fast. However, this approach reduces the accuracy of our numerical approximation which is more substantial beyond the two-regime example (see further Fig. 7 below in the four-regime example).
| S | MTree | IMS1  | IMS2  | MOL  | FF-CS |
|---|-------|-------|-------|------|-------|
|    |       |       |       |      |       | Coupled regime treated explicitly |
|    |       |       |       |      |       |
|    |       |       |       |      |       |
|    |       |       |       |      |       |
| Regime 1 |
| 3.5 | 5.5000 | 5.5001 | 5.5001 | 5.5000 | 5.5000 |
| 4.0 | 5.0066 | 5.0067 | 5.0066 | 5.0033 | 5.0036 |
| 4.5 | 4.5432 | 4.5486 | 4.5482 | 4.5433 | 4.5442 |
| 6.0 | 3.4144 | 3.4198 | 3.4184 | 3.4143 | 3.4142 |
| 7.5 | 2.5844 | 2.5877 | 2.5867 | 2.5842 | 2.5854 |
| 8.5 | 2.1560 | 2.1598 | 2.1574 | 2.1559 | 2.1553 |
| 9.0 | 1.9722 | 1.9756 | 1.9731 | 1.9720 | 1.9722 |
| 9.5 | 1.8058 | 1.8090 | 1.8064 | 1.8056 | 1.8062 |
| 10.5 | 1.5186 | 1.5214 | 1.5187 | 1.5185 | 1.5190 |
| 12.0 | 1.1803 | 1.1827 | 1.1799 | 1.1803 | 1.1803 |
| Regime 2 |
| 3.5 | 5.5000 | 5.5012 | 5.5012 | 5.5000 | 5.5000 |
| 4.0 | 5.0000 | 5.0016 | 5.0016 | 5.0000 | 5.0000 |
| 4.5 | 4.5117 | 4.5194 | 4.5190 | 4.5119 | 4.5128 |
Table 1 (continued)

| S   | MTree | IMS1  | IMS2  | MOL  | FF-CS |
|-----|-------|-------|-------|------|-------|
|     | Coupled regime treated explicitly |       |       |      |       |
|     | h = 0.05 | 0.01 | h = 0.05 | 0.01 |
| 6.0 | 3.3503 | 3.3565 | 3.3550 | 3.3507 | 3.3509 | 3.3507 |
| 7.5 | 2.5028 | 2.5078 | 2.5056 | 2.5033 | 2.5044 | 2.5033 |
| 8.5 | 2.0678 | 2.0722 | 2.0695 | 2.0683 | 2.0685 | 2.0683 |
| 9.0 | 1.8819 | 1.8860 | 1.8832 | 1.8825 | 1.8819 | 1.8825 |
| 9.5 | 1.7143 | 1.7181 | 1.7153 | 1.7149 | 1.7148 | 1.7149 |
| 10.5| 1.4267 | 1.4301 | 1.4272 | 1.4273 | 1.4272 | 1.4273 |
| 12.0| 1.0916 | 1.0945 | 1.0916 | 1.0923 | 1.0928 | 1.0923 |
Table 2  Further Comparison of American put option price for case 1

| S     | RBF-FD Reg. 1 | ETD-CN Reg. 1 | FF-expl Reg. 1 | FF-CS (h = 0.01) Reg. 1 | RBF-FD Reg. 2 | ETD-CN Reg. 2 | FF-expl Reg. 2 | FF-CS (h = 0.01) Reg. 2 |
|-------|---------------|---------------|----------------|--------------------------|---------------|---------------|----------------|--------------------------|
| 9.0   | 1.9718        | 1.9756        | 1.9713         | 1.9720                   | 1.8825        | 1.8859        | 1.8817         | 1.8825                   |
| 10.5  | 1.5185        | 1.5213        | 1.5177         | 1.5185                   | 1.4274        | 1.4301        | 1.4265         | 1.4273                   |
| 12.0  | 1.1803        | 1.1825        | 1.1796         | 1.1804                   | 1.0924        | 1.0945        | 1.0915         | 1.0923                   |

Table 3  Comparison of American put option price with no jump between regimes for case 1

| S     | MOL Regime 1 | FF-CS Regime 1 | MOL Regime 2 | FF-CS Regime 2 |
|-------|--------------|----------------|--------------|----------------|
| 6.00  | 3.66676242   | 3.66675302     | 3.0000        | 3.0000         |
| 9.00  | 2.37538560   | 2.37541371     | 0.888311178  | 0.888422077    |
| 12.00 | 1.60485395   | 1.60491666     | 0.203543056  | 0.203650387    |

Remark 3  We have tested treating the coupled regime explicitly with the Newton iteration. We found that our numerical solution using the Newton iteration converges as fast as using the GS iteration and reduces the accuracy. Furthermore, without the explicit treatment, the Newton method is actually slow in convergence as compared with the GS iteration for the system in (30).

We further investigate the performance of our method when there is no jump between regimes (Chiarella et al., 2016) as shown below.

\[
Q = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},
\]

(69)

and compared our result with the MOL. With no transition between states, we are simply solving two separate American options. The obtained result was listed in Table 3. From Table 3, the result that we obtained closely resembles the one from the MOL method up to five digits.

To check the accuracy of our present method, we calculated the convergence rate from the asset option in regime 1. To obtain the maximum errors and convergence rates of our numerical scheme, we used a fixed time step \( k = 2.5 \times 10^{-6} \), \( T = 0.1 \), varied grid sizes \( h = 0.2, 0.1, 0.05, \) and \( 0.025 \), and displayed the result in Table 4.

The convergence rate for space that we obtained from Table 4 is around 2.8 which is higher than 2.0 for the existing methods. We expect the convergence rate to be higher if we find out a more suitable local mesh refinement at \( x_m = 0 \) and use a higher-order interpolation to deal with the derivative of the asset option.

From Tables 5 and 6, one may see that the computational speed of our method is very fast even though we simultaneously compute the optimal exercise boundary,
Table 4 The maximum errors and convergence rates in space in regime 1 

| \( h \) | Maximum error | Convergence rate |
|---------|---------------|-----------------|
| \( 2 \times 10^{-1} \) | 5.945 \times 10^{-1} | 2.60 |
| 1 \times 10^{-1} | 9.677 \times 10^{-2} | 2.60 |
| \( 5 \times 10^{-2} \) | 1.406 \times 10^{-2} | 2.80 |

Table 5 Average CPU time(s) per each time step for the two-regime example 

| \( h \) | CPU time(s) |
|---------|-------------|
| Coupled regime treated implicitly | Coupled regime treated explicitly |
| \( h \) | \( h \) | Newton iterative method | Gauss–Seidel Iteration |
|-----|-----|-----------------|-----------------|
| 0.1 | 0.19 | 0.06 | 0.04 |
| 0.05 | 0.25 | 0.09 | 0.07 |
| 0.01 | 0.89 | 0.45 | 0.30 |

Remark 4 We would like to point out that we used only two neighboring points to approximate the coupled regimes with the piecewise cubic Hermite interpolation based on (53–54). It presents an advantage in approximating the coupled regime faster. Moreover, for cubic spline interpolation, we envisaged that because it uses more nodal points for function approximation coupled with some matrix inversion, it is likely that it will be much slower than piecewise cubic Hermite interpolation when the number of grid points increases.

Case 2 Consider the example presented in the work of Babbin et al. (2014). The strike price was chosen to be \( K = 10 \) at the expiration time \( T = 1 \). The grid size was
chosen to be $h = 0.01$. The parameters were given as

$$Q = \begin{bmatrix} -3 & 3 \\ 2 & -2 \end{bmatrix}, \quad r = \begin{bmatrix} 0.05 \\ 0.05 \end{bmatrix}, \quad \sigma = \begin{bmatrix} 0.3 \\ 0.4 \end{bmatrix}. \quad (70)$$

The convergence criterion $\varepsilon = 10^{-8}$ was chosen. Here, we focus on comparing the option sensitivities result from FF-CS with the one obtained from the RBF-FD method. The reason is that the asset option’s values we obtained are very close to the ones obtained from RBF-FD, MOL, and MTree methods. The plots of the gamma and speed options obtained from our method were displayed in Fig. 4a. In Fig. 4b, we present the plot of the gamma option obtained from the RBF-FD method.

![Option sensitivities for the two-regime example when $\tau = T$ (case 2) with FF-CS. Gamma options for the two-regime example with RBF-FD (Li et al., 2019)](image-url)
Computing the sensitivities accurately in regime-switching problems is very challenging. Moreover, in some of the existing literature, the sensitivities’ results are either unreported, unavailable, or inaccurate in some regime examples. From Fig. 4a, we see that our method provides smooth curves for gamma and speed options in each regime, which improves the oscillation near the optimal exercise boundary. Hence, we can validate that our method performs better in approximating the option sensitivities when compared with the RBF-FD method.

4.2 Examples 2: Four Regimes

We consider the American put options pricing problems with four regimes. The strike price and expiration time were chosen to be $K = 9$ and $T = 1$, respectively. In our computation, we chose the interval $0 \leq x_m \leq 3$ with the grid size $h = 10^{-2}$ and time step $k = 10^{-4}$. The four-regime example was computed with the convergent criterion of $\epsilon = 10^{-8}$. The parameters were given as
Figures 5 and 6 plot the profiles of the option prices, option Greeks, and optimal exercise boundaries for the four regimes. Table 7 lists the option prices of the four regimes.
regimes using the asset values in the interval of $3.5 \leq S \leq 12$.

From Table 7, the numerical solutions of the asset options are very accurate when close to those obtained from the MTree and RBF-FD methods. Here, we like to point out that the plots for the gamma option and its derivative for each regime in Fig. 5 are very smooth. This is different from those obtained from the RBF-FD method, which is non-smooth and exhibits spurious oscillation near the optimal exercise boundary. It indicates that the present method produces more accurate numerical solutions, especially for the Greek parameters. Moreover, the gamma profile for the four-regime example was not provided in the work of Egorova et al. (2016).

Furthermore, the plot of the optimal exercise when the coupled regime is treated explicitly based on the work of Egorova et al. (2016) is displayed in Fig. 7. From Fig. 7, the solution profile of the optimal exercise exhibits some oscillation which indicates that treating the coupled regime explicitly (when an implicit approach is implemented) might not be an ideal approach for the regime-switching model if a more accurate solution is desired.

By considering a free boundary problem, we implemented a front-fixing approach to fix the free boundary and compute the optimal exercise boundary simultaneously with the option. It is worth mentioning that a recent article (Egorova et al., 2016) also implemented front-fixing a similar approach for solving the regime-switching problem. However, our model formulation and the numerical scheme are entirely different from the one presented in the work of Egorova et al. (2016) and require some novel ideas. Egorova et al. (2016) implemented a second-order numerical scheme with a linear interpolation method for the numerical approximation. In our work, we presented a coupled system of PDEs consisting of the asset option and option Greeks. It enabled us to formulate a special boundary treatment for computing the optimal exercise boundary and implement cubic Hermite interpolation for approximating the coupled regime. We then solved the system of coupled PDEs using a high-order compact scheme. We showed in the numerical examples above that our formulation and numerical scheme present more accurate numerical solution for the option sensitivity when compared with the existing methods.
Table 7 Comparison of American put options price for the four-regimes example

| S    | MTree | RBF-FD | ff-expl |
|------|-------|--------|---------|
|      | Reg 1 | Reg 2  | Reg 3   | Reg 4   | Reg 1 | Reg 2  | Reg 3   | Reg 4   | Reg 1 | Reg 2  | Reg 3   | Reg 4   |
| 7.5  | 3.1433 | 2.2319 | 2.6746  | 1.6574  | 3.1424 | 2.2320 | 2.6744  | 1.6576  | 3.1421 | 2.2313 | 2.6739  | 1.6573  |
| 9.0  | 2.5576 | 1.5834 | 2.0568  | 0.9855  | 2.5564 | 1.5835 | 2.0566  | 0.9857  | 2.5563 | 1.5827 | 2.0559  | 0.9850  |
| 10.5 | 2.1064 | 1.1417 | 1.6014  | 0.6533  | 2.1052 | 1.1415 | 1.6013  | 0.6554  | 2.1047 | 1.1406 | 1.6004  | 0.6540  |
| 12.0 | 1.7545 | 0.8377 | 1.2625  | 0.4708  | 1.7527 | 0.8377 | 1.2625  | 0.4708  | 1.7524 | 0.8368 | 1.2614  | 0.4700  |

| S    | ETD-CN | FF-CS (0.01) |
|------|--------|--------------|
| 7.5  | 3.1513 | 3.1418       |
| 9.0  | 2.5641 | 2.5549       |
| 10.5 | 2.1113 | 2.1015       |
| 12.0 | 1.7578 | 1.7527       |
4.3 Examples 3: Sixteen Regimes

To further investigate the efficiency of our method, especially in handling the sensitivity of the option Greeks beyond four regimes, we consider a sixteen regimes example. We chose an interval of $0 \leq x_m \leq 3$ with the following parameter:

$$
\varepsilon = 10^{-7}, \; K = 9, \; T = 1, \; h = 10^{-2}, \; k = 10^{-4}
$$

$$
r = [0.04 \; 0.15 \; 0.30 \; 0.13 \; 0.12 \; 0.10 \; 0.18 \; 0.08 \; 0.25 \; 0.06 \; 0.20 \; 0.21 \; 0.07 \; 0.12 \; 0.19],
$$

$$
\sigma = [0.07 \; 0.30 \; 0.90 \; 0.80 \; 0.25 \; 0.15 \; 0.12 \; 0.28 \; 0.85 \; 0.35 \; 0.39 \; 0.72 \; 0.45 \; 0.18 \; 0.20 \; 0.25],
$$

$$
Q = \begin{bmatrix}
-3 & 0.2 & 0.2 & 0.2 & 0.2 & \cdots & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\
0.2 & -3 & 0.2 & 0.2 & 0.2 & \cdots & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\
0.2 & 0.2 & -3 & 0.2 & 0.2 & \cdots & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\
0.2 & 0.2 & 0.2 & -3 & 0.2 & \cdots & 0.2 & 0.2 & 0.2 & 0.2 & 0.2 \\
& & & & & & \cdots & \cdots & \cdots & \cdots & \cdots \\
0.2 & 0.2 & 0.2 & 0.2 & \cdots & 0.2 & -3 & 0.2 & 0.2 & 0.2 \\
0.2 & 0.2 & 0.2 & 0.2 & \cdots & 0.2 & 0.2 & -3 & 0.2 & 0.2 \\
0.2 & 0.2 & 0.2 & 0.2 & \cdots & 0.2 & 0.2 & 0.2 & -3 & 0.2 \\
0.2 & 0.2 & 0.2 & 0.2 & \cdots & 0.2 & 0.2 & 0.2 & 0.2 & -3
\end{bmatrix}
$$

Figure 8 plots the profiles of the option prices, option Greeks, and optimal exercise boundaries for the sixteen regimes. Table 8 lists the option prices of the sixteen regimes using the asset values in the interval of $3.5 \leq S \leq 12$. 

![Fig. 7 Optimal exercise boundaries for the four-regimes example when the coupled regime is treated explicitly ($\tau = T, \; h = 0.01$)](image-url)
Fig. 8 Optimal exercise boundaries, asset options, and option Greeks for the sixteen-regime example when $\tau = T$. 
Table 8 American put options price for the sixteen-regime example using FF-CS

| S   | Reg. 1 | Reg. 2 | Reg. 4 | Reg. 6 | Reg. 8 | Reg. 12 | Reg. 16 |
|-----|--------|--------|--------|--------|--------|---------|---------|
| 3.5 | 5.5000 | 5.5000 | 5.5000 | 5.5000 | 5.5000 | 5.5000  | 5.5000  |
| 4.0 | 5.0074 | 5.0000 | 5.0000 | 5.0000 | 5.0000 | 5.0000  | 5.0000  |
| 4.5 | 4.5385 | 4.5000 | 4.5000 | 4.5000 | 4.5000 | 4.5000  | 4.5000  |
| 6.0 | 3.2833 | 3.0000 | 3.0872 | 3.0000 | 3.0000 | 3.1064  | 3.0000  |
| 7.5 | 2.3075 | 1.7145 | 2.1058 | 1.6227 | 1.6624 | 2.1162  | 1.6248  |
| 8.5 | 1.8260 | 1.2060 | 1.6495 | 1.1088 | 1.1533 | 1.6496  | 1.1144  |
| 9.0 | 1.6287 | 1.0207 | 1.4661 | 0.9311 | 0.9730 | 1.4618  | 0.9378  |
| 9.5 | 1.4560 | 0.8696 | 1.3071 | 0.7898 | 0.8272 | 1.2990  | 0.7963  |
| 10.5| 1.1718 | 0.6434 | 1.0483 | 0.5842 | 0.6112 | 1.0346  | 0.5883  |
| 12.0| 0.8614 | 0.4285 | 0.7690 | 0.3928 | 0.4079 | 0.7508  | 0.3936  |

From Fig. 8, one can easily confirm the consistency in the smoothness of the gamma and speed options in the sixteen-regime example as we observed from two and four-regime examples. Again, this is an indication that our method provides more accurate solution profiles for the Greek parameters.

5 Conclusion

We have developed an accurate numerical method for solving American put options with regime-switching. Through the front-fixing transformation, we were able to map the optimal exercise boundary for each regime to a fixed interval. The derivative transformation enables us to employ the higher-order compact finite difference method coupled with the Hermite interpolation for solving the system of the asset, delta, gamma, and speed options while capturing the optimal exercise boundary and theta, delta decay, and color options for each regime. Our numerical discretization also presents a system where the coefficient matrix is tridiagonal and positive definite with constant entries, which enables the implementation of Thomas algorithm and Gauss—Seidel iteration with simple computation. The present scheme has been tested in two-, four-, and sixteen-regime cases. The numerical results show that the method provides an accurate solution and is fast in computation as compared with the existing methods. In particular, the present method produces accurate and non-oscillating solutions of the option Greeks for the cases beyond two regimes that are often difficult to obtain correctly using the existing methods.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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