Complete structural restoring of transferred multi-qubit quantum state

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Keywords: spin chain; quantum state transfer; XX-Hamiltonian; unitary transformations; quantum control

PACS: 75.10.Pq Spin chain models; 05.60.Gg Quantum transport 03.67.-a Quantum information; 03.67.Hk Quantum communication;

Abstract

In this work, we study optimal transfer of quantum states via spin chains. We develop the protocol for structural restoring of multi-quantum coherence matrices of multi-qubit quantum states transferred from sender to receiver along a spin-1/2 chain. We also develop a protocol for constructing such 0-order coherence matrix that can be perfectly transferred in this process. The restoring protocol is based on the specially constructed unitary transformation applied to the state of the extended receiver. This transformation for a given length of the chain and Hamiltonian is universally optimal in the sense that ones constructed it can be applied to optimally restore any higher-order coherence matrices.

PACS numbers:
I. INTRODUCTION

The problem of quantum state transfer along the spin-1/2 chain was first formulated in the famous paper by Bose [1], where transfer of an arbitrary pure state along a homogeneous spin chain with Heisenberg Hamiltonian was considered. In this case the state transfer was not perfect and fidelity [2] averaged over pure initial states was used to estimate the deviation of the transferred state from the expected state. Similar to the teleportation problem [3–5], this subject attracted great attention of researchers. A number of works were performed considering the ideal transfer [6, 7] and high-probability state transfer along the inhomogeneous chains [8]. Various aspects of state transfer were studied [9–14]. Robustness to perturbations was widely investigated [15–18] and it was shown that chain which provides perfect state transfer in the unperturbed case is equivalent to chain providing high-probability state transfer for perturbed Hamiltonian [17]. Later the remote state transfer method was introduced as an alternative to the state transfer. Initially this method was proposed for photon systems [19–22], and then for spin systems [23–25]. In Refs. [24, 25], the extended receiver (i.e., a subsystem including the receiver and its few nearest sites) subjected to specially constructed unitary transformation was introduced for controlling the receiver’s state in the state-creation process. In Refs. [26–28] a method of optimal transfer of quantum states via spin-1/2 chains was considered, where optimal transfer is understood as quantum state transfer with minimal and well characterized deformation of the transferred density matrix.

Our paper is further development of the method underlined in Refs. [26–28]. Before exploring the novelty of our paper, we briefly summarize the results of Refs. [26–28]. First of all, recall that the density matrix $\rho$ of an $N$-qubit state of a spin system can always be written in the form:

$$\rho = \sum_{-N}^{N} \rho^{(n)}.$$  \hspace{1cm} (1)

Here $n$-order coherence matrix $\rho^{(n)}$ includes elements of the density matrix $\rho$ which correspond to state transitions changing the total spin $z$-projection by $n$. It is shown in Ref. [26] that evolution of a spin chain under the Hamiltonian conserving the excitation number in the system (for instance, for nuclear spins in the strong magnetic field, the number of excited spins is the number of spins directed opposite the magnetic field; the Hamiltonian
of the dipole-dipole interaction in the strong magnetic field conserves this number \( \rho^{(n)} \) does not mix the coherence matrices of different orders \( \rho^{(n)} \). This fact stimulated studying the state transfer from the sender to the receiver without such mixing in \[27, 28\].

In Refs. \[27, 28\] the sender (\( S \)) interacts with the receiver (\( R \)) through a spin chain called the transmission line (\( TL \)). However the initial state of the subsystem \( TL \cup R \) is different in those works: it is the thermodynamic equilibrium state in \[27\] and ground state (i.e., the state without excited spins) in \[28\].

In Ref. \[27\], the block-scalable and block-scaled quantum states were found. A special property of these states is that the receiver’s state registered at some time instant \( t_0 \) differs from the sender’s initial state by multiplication of the \( n \)-order \((|n| > 0)\) coherence matrices by certain scale factors \( \lambda^{(n)} \). But the 0-order coherence matrix must satisfy the trace-normalization condition and therefore can not be completely scaled. However, in contrast to non-zero-order coherence matrices, the 0-order coherence matrix can be perfectly transferred, i.e., transfer with \( \lambda^{(0)} = 1 \) is possible. We emphasize that in Refs. \[26, 27\] the extended receiver with optimizing unitary transformation was not involved into the control process.

As was shown in \[27\], for block-scaled state transfer each pair of \( \pm n \)-order coherence matrices can transfer not more than one arbitrary complex parameter. Therefore the information capacity of the block-scaled state transfer is small. This makes important finding its modifications to increase the information capacity and motivates development of protocols for restoring the structure of the non-diagonal part of the density matrix of the transferred state using special unitary transformation of the extended receiver so that the non-diagonal elements of the receiver state would differ only by fixed scale factors from the corresponding elements of the initial sender’s state. Then each non-diagonal element would transfer its own parameter. An example of such protocol for restoring a 2-qubit state was proposed in Ref. \[28\]. But the method of unitary transformations does not work for restoring diagonal elements because they have different structures. Therefore, the problem of restoring the diagonal elements was completely disregarded in \[28\].

The goal of our paper is two-fold. First, we extend the 2-qubit structural restoring protocol of Ref. \[28\] and, using the tool of multi-index technique, derive general formulas for restoring non-diagonal elements of multi-qubit quantum states using the optimizing unitary transformation of the extended receiver. Second, we explore the method of manipulating with the elements of 0-order coherence matrix developing the idea of \[27\]. Similar to Ref. \[28\],
the subsystem $TL \cup R$ is in the ground initial state without excitations. This general theoretical analysis is then applied to state-restoring protocol for a 2-qubit state evolved under $XX$-Hamiltonian which, together with other aspects of a 2-qubit state restoring given in [27, 28], provides a comprehensive description of the state-restoring process.

We emphasize that the scale factors found in Ref. [27], the scale factors together with the optimizing unitary transformation of the extended receiver found in Ref. [28] and in the present work are universal objects in the sense that they are defined only by the evolution Hamiltonian (in particular, by the chain length), by the initial state of the subsystem $TL \cup R$ and by the time instance fixed for the receiver state registration and do not depend on the particular initial state of the sender to be transferred. Thus, the registered receiver’s state depends on the initial sender’s state, but the scale factors obtained in the restoring process and the appropriate unitary transformation do not depend on the initial state of sender. Hence once constructed, this unitary transformation can be applied to restore non-diagonal elements of any transferred sender’s initial state. This universality notion is related to work [30], where the most general class of universally optimal Kraus maps was described and investigated for quantum control (Kraus maps are most general transformations of quantum states [31]). The state-restoring method developed below is applicable to any ground and thermodynamically equilibrium initial states of the subsystem $TL \cup R$. However, using the ground initial state allows us to reduce the dimension of the Hilbert space of the whole $N$-qubit chain (whose dynamic is to be described) from $2^N$ to $\sum_{k=1}^{N(S)} C_k^{N(S)}$ (where $N(S)$ is the number of qubits in the sender, $N(S) < N$) and therefore simplifies calculations. This remark motivates our choice of the ground state of the subsystem $TL \cup R$. It is important that the restoring unitary transformation of the extended receiver is not unique [28], and it can be used for further optimization (for instance, maximization by the absolute values) of the scale factors in the restorer state. We perform some rough optimization in this work. Global optimization techniques which were applied for controlling spins [32–36] could be used as well.

We remark that the perfect (and almost perfect up to one diagonal element) state transfer was achieved in Ref. [27] by the method which is not applicable to the case of ground initial state of the subsystem $TL \cup R$. Therefore in this work for ground initial state we develop a different method and show that there exists a special 0-order coherence matrix which can be almost perfectly transferred, up to two diagonal elements, which must satisfy the trace-
normalization condition. Furthermore, there exists such 0-order coherence matrix that can be perfectly transferred from the sender to the receiver up to the trivial exchange of two its diagonal elements. We call such transfer as perfect transfer of 0-order coherence matrix.

The paper is organized as follows. Evolution of the sender’s initial state and structure of the receiver’s state under the unitary transformation of the extended receiver is analyzed in Sec. II. General protocol for state restoring based on the optimizing unitary transformation of the extended receiver is developed in Sec. III. In the same section, we also find a 0-order coherence matrix which can be perfectly transferred to the receiver. An example of a two-qubit state restoring is constructed in Sec. IV. Conclusions are provided in Sec. V. Some additional important formulas are given in the Appendix Sec. VI.

II. EVOLUTION OPERATOR AND UNITARY TRANSFORMATION OF THE EXTENDED RECEIVER

We consider the communication line shown in Fig. 1 and consisting of the sender \( S \), where the state to be transferred is initiated, the transmission line \( TL \), along which the state is transferred from the sender to the receiver, and the receiver \( R \), where the transferred state is registered. We also will use an extended receiver \( ER \) which is necessary to handle in a desired manner the state of the receiver via controlled unitary transformation. The receiver is a part of the extended receiver. By \( TL \) denote the transmission line without nodes of the extended receiver. The sender and the receiver are considered to have the same dimension:

\[
N^{(S)} = N^{(R)} = N. \tag{2}
\]
Evolution of the density matrix of the spin chain is given by unitary transformation with Hamiltonian $H$,

$$\rho(t) = V(t)\rho(0)V^\dagger(t), \quad V(t) = e^{-iHt},$$  \hspace{1cm} (3)

To prevent mixing of the multi-quantum coherence matrices of different orders we require that the Hamiltonian $H$ satisfies the commutation condition \[26\]

$$[H, I_z] = 0,$$  \hspace{1cm} (4)

and that the initial state $\rho(0)$ has tensor-product form,

$$\rho(0) = \rho^{(S)}(0) \otimes \rho^{(TL,R)}(0).$$  \hspace{1cm} (5)

Here $\rho^{(S)}(0)$ is the sender’s initial density matrix whose structure will be discussed below, and $\rho^{(TL,R)}(0)$ is the initial density matrix of the subsystem $TL \cup R$, which is the ground state:

$$\rho^{(TL,R)}(0) = \text{diag}(1, 0, \ldots, 0).$$  \hspace{1cm} (6)

It is important that $\rho^{(TL,R)}(0)$ contains only the 0-order coherence matrix. This condition together with the commutation condition \[14\] provides transfer of a quantum state from the sender to the receiver without interaction among multi-quantum coherence matrices \[26\].

At some time instant $t_0$ we apply to the extended receiver a unitary transformation $U$ which depends on the set of free parameters $\varphi$. This transformation is required to conserve the excitation number, similarly to $H$, i.e.,

$$[I_{z}^{ER}, U(\varphi)] = 0,$$  \hspace{1cm} (7)

where $I_{z}^{ER}$ is the z-projection of the total spin moment in the state-space of the extended receiver $ER$. Then the state of the system becomes

$$\rho(t_0, \varphi) = W(t_0, \varphi)\rho(0)W^\dagger(t_0, \varphi),$$  \hspace{1cm} (8)

where

$$W(t_0, \varphi) = (I_{S,TL} \otimes U(\varphi))V(t_0).$$  \hspace{1cm} (9)

Here $I_{S,TL}$ is the identity operator in the subspace of the subsystem $S \cup TL$. Notice that the dynamics of the initial state \[5\] with $N$-qubit sender can be described in the subspace
of up to $N$ excitations due to the commutation relation $[W, I_z] = 0$ (which follows from (4) and (7)) and the initial state (5), (6). Therefore both operators $U$ and $W$ should have the following block-diagonal form:

$$U(\varphi) = \text{diag}(1, U^{(1)}(\varphi^{(1)}), \ldots, U^{(N)}(\varphi^{(N)})), \quad (10)$$

$$W = \text{diag}(1, W^{(1)}(\varphi^{(1)}), \ldots, W^{(N)}(\varphi^{(N)})), \quad (11)$$

$$\varphi = (\varphi^{(1)}, \ldots, \varphi^{(N)}).$$

where $W^{(k)}$ is the block in the subspace of $k$ excited spins and $\varphi^{(k)}$ is the list of free parameters in this block, $k = 0, \ldots, N$.

The dimension of the $k$th block,

$$C_N^k \times C_N^k, \quad C_N^k = \frac{N!}{k!(N-k)!}, \quad (12)$$

defines the number of free real parameters in the block $U^{(k)}$:

$$\varphi^{(k)} = (\varphi_1^{(k)}, \ldots, \varphi_{C_N^k-1}^{(k)}). \quad (13)$$

State of the receiver at the time instant $t_0$ can be found from $\rho(t_0, \varphi)$ by tracing over $S$ and $TL$:

$$\rho^{(R)}(t_0, \varphi) = \text{Tr}_{S,TL}(\rho(t_0, \varphi)). \quad (14)$$

Hereafter for shortness of notations we will sometimes omit the parameters $t_0$ and $\varphi$ from the arguments.

A. Structure of the transformation (8)

Now we proceed to study the structure of the transformation (8). Before writing this equation in components, we introduce the multi-indexes for the subsystems $S$, $TL$, and $R$, which we denote by the capital latin letters with the corresponding subscripts. For instance, $I_S$ is the multi-index linked to the sender $S$. The multi-index is the set of zeros and ones, the cardinality of this set equals to the number of qubits in the subsystem. The number 1 at a certain position indicates that the corresponding spin is excited, while the number 0 indicates that the spin is in the ground state. We also introduce the norm of the multi-index $| \cdot |$, which is the number of excited spins in the subsystem (and equals to the sum of all
elements in this multi-index). The notation of indexed 1 and 0 (for instance, $1_s$, $0_r$) are used to indicate the indexes with all ones or zeros. Since both sender and receiver have the same dimension\(^2\), the multi-indexes with subscripts $R$ and $S$ are of equal length, so that in particular, $1_r = 1_s$ and $0_r = 0_s$. In these terms, the scalar blocks $W^{(0)}$ and $W^{(N)}$ of the unitary transformation $W$ given by Eq. (11) have the form:

$$W^{(0)} = W^{(0)}_{0s0TL0r:0s0TL0r} = 1,$$

$$W^{(N)} = W^{(N)}_{1s1TL1r:1s1TL1r}.$$

In general, elements of the $n$th block $W^{(n)}_{isITLi_r:j_sj_TLj_r}$ satisfy

$$|I_s| + |I_TL| + |I_r| = |J_s| + |J_TL| + |J_r| = n.$$  

Now the density matrix $\rho(t_0, \varphi)$ (8) can be written in the following component form:

$$\rho_{NsNTLN_R:M_SM_TLM_R} = \sum_{I_s,J_s} W^{(|I_s|)}_{NsNTLN_R:I_s0TL0r} \rho^{(S)}_{I_s,J_s} (W^{(|J_s|)})^\dagger_{J_s0TL0r:M_SM_TLM_R}. $$

### B. Structure of the receiver’s state (14)

Tracing over $S$ and $TL$ gives the receiver’s density matrix $\rho^{(R)}$:

$$\rho^{(R)}_{N_R:M_R} = \sum_{N_s,N_TL} \sum_{|I_s|,|J_s|} W^{(|I_s|)}_{NsNTLN_R:I_s0TL0r} \rho^{(S)}_{I_s,J_s} (W^{(|J_s|)})^\dagger_{J_s0TL0r:N_sNTLM_R}. $$

Expression (19) can be written in the matrix form as follows:

$$\rho^{(R)} = \sum_{N_s,N_TL} \tilde{W}_{NsNTL} \rho^{(S)} \tilde{W}_{NsNTL}^\dagger, $$

where $\tilde{W}_{NsNTL}$ are Kraus operators\(^3\) which satisfy

$$\sum_{N_s,N_TL} \tilde{W}_{NsNTL}^\dagger \tilde{W}_{NsNTL} = I_s;$$

and have elements

$$(\tilde{W}_{NsNTL})_{N_R:I_S} = \begin{cases} W_{NsNTLN_R:I_s0TL0r}^{(|I_s|)}, & |N_s| + |N_TL| + |N_R| = |I_s|, \\ 0, & |N_s| + |N_TL| + |N_R| \neq |I_s|. \end{cases}$$

Here $I_s$ is the identity operator in the subspace of $S$. In (20) and (21), the sum is over all $N_s$ and $N_{TL}$.
Now we prove that there is no mixing of elements of different multi-quantum coherence matrices of the matrix $\rho^{(S)}(0)$ in (19). In fact, by virtue of (17) one has that (19)

$$|N_S| + |N_{TL}| + |N_R| = |I_S|, \quad |N_S| + |N_{TL}| + |M_R| = |J_S|.$$  \tag{23}

Since by definition the elements $\rho^{(S)}_{I_S, J_S}$ and $\rho^{(R)}_{N_{R}, M_R}$ are included in, respectively, the coherence matrices of the order $n = |J_S| - |I_S|$ and $m = |M_R| - |N_R|$, taking into account (23) gives:

$$m = |M_R| - |N_R| = (|J_S| - |N_S| - |N_{TL}|) - (|I_S| - |N_S| - |N_{TL}|) = |J_S| - |I_S| = n. \tag{24}$$

This means that elements of the $n$-order coherence matrix of the sender become equal to elements of the $n$-order coherence matrix of the receiver. Then, using the expansion of $\rho^{(S)}$ and $\rho^{(R)}$ in terms of the multi-quantum coherence matrices $\rho^{(S;n)}$ and $\rho^{(R;n)}$,

$$\rho^{(S)} = \sum_{-N^{(S)}}^{N^{(S)}} \rho^{(S;n)},$$  \tag{25}

$$\rho^{(R)} = \sum_{-N^{(S)}}^{N^{(S)}} \rho^{(R;n)},$$  \tag{26}

Eq. (19) can be written as

$$\rho^{(R;n)}_{N_{R}, M_R} = \sum_{|I_S, J_S| \leq |N_S| + |N_{TL}| = |I_S| - |N_R|}^{N_{S}, N_{TL}} \sum_{W^{(I_S)}_{N_{S}N_{TL}N_{R}; I_S0TL0R}} \rho^{(S;n)}_{I_S, J_S} \left(W^{(J_S)}_{I_S0TL0R; N_{S}N_{TL}M_R}\right)^\dagger.$$

\[ -N \leq n \leq N. \tag{27} \]

C. Multi-quantum coherence matrices

Now we consider the structure of the multi-quantum coherence matrices of positive order. The coherence matrices of negative order are Hermitian conjugate of these matrices. 0-order and higher order coherence matrices play different roles in state transfer. Only higher-order coherence matrices of the sender’s initial state can be effectively restored at the receiver’s state and thus serve to transfer arbitrary parameters from the sender to the receiver. On the contrary, sender’s 0-order coherence matrix can not be completely restored at the receiver side. But this matrix can be fixed in such a way that most of its elements (or even all of them) are perfectly transferred to the receiver. In this way we provide the minimal deformation of the sender’s initial state during its transfer to the receiver.
1. Higher-order coherence matrices

a. N-order coherence matrix. We select the N-order coherence matrix because the condition \(|M_R| - |N_R| = N\) can be satisfied only if \(N_R = 0, M_R = 1\), i.e., when there is only one element in this matrix. Eq. (27) yields in view of (15)

\[
\rho^{(R:N)}_{R,1_R} = W^{(0)}_{0_S0_{TL}0_R;0_S0_{TL}0_R}\rho^{(S:N)}_{0_S1_S}(W^{(N)})^\dagger_{1_S0_{TL}0_R;0_S0_{TL}1_R}
\]

(28)

Following Ref. 28, we use the intensity of this coherence to fix the optimal time instant \(t_0\) for receiver’s state registration. Namely, we take the time instant \(t_0\) corresponding to the maximal absolute value of the scale factor in \(\rho^{(R:N)}_{0_R,1_R}\) with zero values of all parameters \(\varphi\):

\[
\max_t |W^{(N)}_{0_S0_{TL}1_R;1_S0_{TL}0_R}|_{\varphi=0} = |W^{(N)}_{0_S0_{TL}1_R;1_S0_{TL}0_R}|_{\varphi=0}.
\]

(29)

b. k-order coherence matrix, \(1 \leq k < N\). For matrix elements \(\rho^{(R:k)}_{N_R,M_R}\) of the k-order coherence matrix one has \(|M_R| - |N_R| = k\). First, select the following family of elements: \(|M_R| = N\) (i.e., \(M_R = 1\)) and \(|N_R| = N - k\). For these elements, one can write

\[
\rho^{(R:k)}_{N_R,1_R} = \sum_{|I_S|=N-k} W^{(N-k)}_{0_S0_{TL}N_R;I_S0_{TL}0_R}\rho^{(S:k)}_{I_S,1_S}(W^{(N)})^\dagger_{1_S0_{TL}0_R;0_S0_{TL}1_R}.
\]

(30)

For other elements, one has \(0 < |M_R| < N\), and one can write the general formula (which reduces to (30) for \(|M_R| = N\)):

\[
\rho^{(R:k)}_{N_R,M_R} = \sum_{N_SN_{TL}} \sum_{|I_S|=N-k; |J_S| \geq |N_R|; |J_S|-|I_S|=k} W^{(|I_S|)}_{N_SN_{TL}N_R;I_S0_{TL}0_R}\rho^{(S:k)}_{I_S,J_S}(W^{(|J_S|)})^\dagger_{J_S0_{TL}0_R;N_SN_{TL}M_R}.
\]

2. 0-order coherence matrix

For 0-order coherence matrix one has in (19):

\[
|N_R| = |M_R|, \quad |I_S| = |J_S|.
\]

(32)

Then the matrix elements have the following representation:

\[
\rho^{(R:0)}_{N_R,M_R} = \sum_{N_SN_{TL}} \sum_{I_S,J_S} W^{(|I_S|)}_{N_SN_{TL}N_R;I_S0_{TL}0_R}\rho^{(S:0)}_{I_S,J_S}(W^{(|J_S|)})^\dagger_{J_S0_{TL}0_R;N_SN_{TL}M_R}.
\]

(33)
To check the normalization condition, we set $M_R = N_R$ and calculate the sum of eqs. (33) over $N_R$ to obtain:

$$\sum_{N_R} \rho^{(R,0)}_{N_R,N_R} = \sum_{I_S} \rho^{(S,0)}_{I_S,I_S} = 1.$$  \hspace{1cm} (34)

We analyze the structure of Eq. (33) for the two particular cases.

First, let $|N_R| = |M_R| = N$. This is possible only if $N_R = M_R = 1_R$. Then (33) yields

$$\rho^{(R,0)}_{1_R,1_R} = W^{(N)}_{0S0TL1_R;1S0TL0R} \rho^{(S,0)}_{1S,1S} (W'(N))_{1S0TL0R;0S0TL1_R}$$

$$= |W^{(N)}_{0S0TL1_R;1S0TL0R}|^2 \rho^{(S,0)}_{1S,1S},$$

i.e., the element $\rho^{(R,0)}_{1_R,1_R}$ is proportional to $\rho^{(S,0)}_{1S,1S}$.

Second, let $|N_R| = |M_R| = 0$. Then $N_R = M_R = 0_R$ and Eq. (33) yields

$$\rho^{(R,0)}_{0_R,0_R} = \sum_{I_S} \sum_{I_T|S\rangle} W^{(|I_S|)}_{N_S,N_T;0S0TL0R} \rho^{(S,0)}_{I_S,J_S} (W'(|J_S|))_{J_S0TL0R;N_S,N_T,0_R}.$$  \hspace{1cm} (35)

**Remark.** If all elements of $\rho^{(S,0)}$ equal to zero except the single one $\rho^{(S,0)}_{0S,0S} = 1$, i.e.,

$$\rho^{(S,0)}_{N_S,M_S} = \delta_{N_S,0_S} \delta_{M_S,0_S},$$  \hspace{1cm} (37)

where $\delta$ is the Kronecker symbol, then Eq. (33) reduces to the following one

$$\rho^{(R,0)}_{N_R,M_R} = \sum_{I_S} W^{(0)}_{0S0TL1_R;0S0TL0R} \rho^{(S,0)}_{0S,0S} (W'(0))_{0S0TL0R;0S0TL1_R}$$

$$= |W^{(0)}_{0S0TL1_R;0S0TL0R}| \delta_{N_R,0_R} \delta_{M_R,0_R} \rho^{(S,0)}_{0_R,0_R},$$  \hspace{1cm} (38)

where we take into account that $W^{0}_{0S0TL0R;0S0TL0R} \equiv 1$. Thus, if the 0-order coherence matrix of the sender’s initial state is

$$\rho^{(S,0)} = \text{diag}(1,0,\ldots,0),$$  \hspace{1cm} (39)

then this matrix is transferred to the state of the receiver without any change, so that $\rho^{(R,0)} \equiv \rho^{(S,0)} \equiv \text{diag}(1,0,\ldots,0)$. However, any correction to (39) destroys the positivity of the density matrix and therefore it can not be used as a 0-order coherence matrix of the sender’s initial state for the purpose of transferring the higher order coherence matrices to the receiver.
III. GENERAL STATE RESTORING PROTOCOL

In summary, the restoring of the sender’s initial state at the receiver includes three steps.

1. Fixing the time instant for the receiver’s state registration.

2. Constructing the optimizing unitary transformation of the extended receiver.

3. Constructing the 0-order coherence matrix of special form in the sender’s initial state.

The time instant $t_0$ was fixed in Sec. [II C 1a] by Eq. (29). Two other points are explored below.

A. Unitary transformation of the extended receiver

As stated above, the unitary transformation of the extended receiver is needed to obtain density matrix $\rho^{(R)}$ at some time instant such that elements of its higher order coherence matrices satisfy the condition

$$\rho^{(R;n)}_{N_R,M_R} = \lambda^{(n)}_{N_R,M_R} \rho^{(S;n)}_{N_R,M_R}, \quad |n| > 0.$$  \hspace{1cm} (40)

We remind that the subscripts $S$ and $R$ are equivalent in multi-index notation. Below we study this condition in detail.

a. $N$-order coherence matrix. According to the definition (40) and formulae of Sec.[II C 1a] the $N$-order coherence matrix has one element and therefore does not require restoring:

$$\lambda^{(N)}_{0_R,1_R} = (W^{(N)})^\dagger_{1S0_{TL}0_R0_S0_{TL}1_R} = W^{(N)}_{0S0_{TL}1R;1S0_{TL}0_R}.$$ \hspace{1cm} (41)

b. $k$-order coherence matrices, $1 \leq k < N$. Eq. (30) yields

$$W^{(N-k)}_{0_S0_{TL}N_R;I_S0_{TL}0_R} (W^{(N)})^\dagger_{1S0_{TL}0_R0_S0_{TL}1_R} = \lambda^{(k)}_{N_R4_R} \delta_{N_R,I_S}, \quad |N_R| = |I_S| = N - k.$$ \hspace{1cm} (42)

Eq. (31) yields

$$\sum_{N_S,N_{TL}} W^{(|I_S|)}_{N_SN_{TL}N_R;I_S0_{TL}0_R} (W^{(|I_S|)})^\dagger_{I_S0_{TL}0_R;N_SN_{TL}M_R} = \lambda^{(k)}_{N_RM_R} \delta_{N_R,I_S} \delta_{M_R,I_S}. \hspace{1cm} (43)$$

$|J_S| - |I_S| = |M_R| - |N_R| = k, \quad 0 < |M_R| < N.$

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Eq. (42) with zero rhs yields the constraint
\[ W^{(N-k)}_{0_00_00_00_0} = 0, \quad N_R \neq I_S, \quad |N_R| = |I_S| = N - k. \] (44)

Eq. (42) with non-zero rhs results in the following expressions for the scale factors:
\[ \lambda_{N_R}^{(k)} = W^{(N-k)}_{0_00_00_00_0} (W^{(N)}_{1_00_00_00_0})^\dagger_{1_00_00_00_0}, \quad |N_R| = N - k. \] (45)

Eq. (43) with zero rhs yields the additional constraints
\[ \sum_{N_S, N_T L \in |N_S|+|N_T L|=|I_S|-|N_R|} W^{(|I_S|)}_{N_S N_T L N_R I_S 0_00_00_00_0} (W^{(|J_S|)}_{J_S 0_00_00_00_0})^\dagger_{J_S 0_00_00_00_0}, N_R \neq I_S \quad \text{or} \quad M_R \neq J_S, \quad 0 < |M_R| < N, \quad |J_S| - |I_S| = |M_R| - |N_R| = k. \] (46)

Eq. (43) with non-zero rhs results in the following expressions for the scale factors:
\[ \lambda_{N_R M_R}^{(k)} = W^{(|N_R|)}_{0_00_00_00_00_0} (W^{(|M_R|)}_{M_R 0_00_00_00_0})^\dagger_{M_R 0_00_00_00_0}, \] (47)
\[ |M_R| - |N_R| = k, \quad 0 < |M_R| < N. \]

\[c. \quad \text{Optimization of the scale factors.} \] The parameters \( \varphi \) of the unitary transformation must satisfy the constraints (44) and (46). We call the unitary transformation \( U \) with parameters satisfying these constraints as the optimizing unitary transformation \( U_{opt} \) of the extended receiver. In addition, we also can use the parameters \( \varphi \) to optimize (for instance, to maximize the absolute values) the scale factors given in expressions (41), (45) and (47).

The number of the parameters \( \varphi \) must be large enough to satisfy all the above constraints and requirements. This number depends on the dimension of the unitary transformation which in turn is defined by the dimension of the extended receiver.

\[B. \quad \text{Construction of the 0-order coherence matrix} \]

\[1. \quad \text{Elements which can be perfectly transferred and the normalization condition} \]

In this section we find the structure of the 0-order coherence matrix, which can be either nearly perfectly transferred (up to two elements included into the blocks of 0 and \( N \) excitations) or perfectly transferred.
First of all one has to take into account constraints (44) and (46) obtained in restoring the higher order coherence matrices. Eq. (35) for \(|N_R| = |M_R| = N\) remains the same. Eq. (33) for \(0 < |N_R| = |M_R| < N\) yields

\[
\rho_{N_R,M_R}^{(R;0)} = W_{0S0TLN_R;N_R0TL0R}^{(S;0)} \rho_{N_R,M_R}^{(S;0)} (W_{0TL0R}^{(|M_R|)})^\dagger M_R0TL0R0S0TL0R + \sum_{N_S,N_{TL}} \sum_{|S_j| \neq |S|} W_{NSN_{TL}N_R;S_00TL0R}^{(|I_S|)} \rho_{IS,J_S}^{(S;0)} (W_{0TL0R}^{(|J_S|)})^\dagger J_S0TL0R0S0TL0R + \sum_{N_S,N_{TL}} W_{NSN_{TL}0R;1R0TL0R}^{(S;0)} \rho_{1R,1R}^{(S;0)} (W_{0TL0R}^{(N)})^\dagger 1R0TL0R0S0TL0R \sum_{N_S,N_{TL}} N \neq 1 \ \\
\sum_{N_S,N_{TL}} W_{NSN_{TL}0R;1R0TL0R}^{(S;0)} \rho_{1R,1R}^{(S;0)} (W_{0TL0R}^{(N)})^\dagger 1R0TL0R0S0TL0R \sum_{N_S,N_{TL}} N \neq 1.
\]

Finally, Eq. (36) for \(|N_R| = |M_R| = 0\) reads

\[
\rho_{0R,0R}^{(R;0)} = \rho_{0R,0R}^{(S;0)} + \sum_{N_S,N_{TL}} \sum_{I_S,J_S \neq |S|} W_{NSN_{TL}0R;I_S0TL0R}^{(|I_S|)} \rho_{IS,J_S}^{(S;0)} (W_{0TL0R}^{(|J_S|)})^\dagger J_S0TL0R0S0TL0R + \sum_{N_S,N_{TL}} W_{NSN_{TL}0R;1R0TL0R}^{(S;0)} \rho_{1R,1R}^{(S;0)}.
\]

Now we look for such elements which can be perfectly transferred to the receiver. For the element \(\rho_{1R,1R}^{(R;0)}\) given in (35) this requirement \(\rho_{1R,1R}^{(R;0)} = \rho_{1R,1R}^{(S;0)}\) may not be satisfied for nonzero \(\rho_{1R,1R}^{(S;0)}\), because \(W_{0S0TL1R;S_00TL0R}^{(N)} < 1\) for \(t > 0\). Therefore this element can not be perfectly transferred. There must be at least one more non-perfectly transferred diagonal element to provide the normalization

\[
\sum_{N_R} \rho_{N_R,N_R}^{(R;0)} = 1.
\]

Let \(\rho_{0R,0R}^{(S;0)}\) be such element.

The perfect transfer of the elements \(\rho_{N_R,M_R}^{(S;0)}\) with \(0 < |N_R| = |M_R| < N\) can be arranged using Eq. (48). Setting \(\rho_{N_R,M_R}^{(R;0)} = \rho_{N_R,M_R}^{(S;0)}\) in this system, we obtain in this case the following
equation for \( \rho^{(S;0)}_{N_R,M_R} \):

\[
\rho^{(R;0)}_{N_R,M_R} = \rho^{(S;0)}_{N_R,M_R} \\
= W^{(|N_R|)}_{0S0TLN_R;NR0TL0R} \rho^{(S;0)}_{N_R,M_R} (W^{(|M_R|)})_{MR0TL0R;0S0TLM_R}^\dagger \\
+ \sum_{N_S,N_{TL}} \sum_{|S|+|N_{TL}|=|S|=|N_R|} W^{(|S|)}_{N_SN_TLN_R;IS0TL0R} \rho^{(S;0)}_{IS,J_S} (W^{(|J_S|)})_{J_S0TL0R;NSN_TLM_R}^\dagger \\
+ \sum_{|N_S,N_{TL}|=N-|N_R|} W^{(N)}_{NSN_TLN_R;1R0TL0R} \rho^{(S;0)}_{1S,1R} (W^{(N)})_{1R0TL0R;NSN_TLM_R}^\dagger.
\]

(51)

Thus, among the elements of the 0-order coherence matrix, only two elements can not be perfectly transferred, namely \( \rho^{(S;0)}_{0R,0R} \) and \( \rho^{(S;0)}_{1R,1R} \), at that the normalization (50) becomes equivalent to

\[
\rho^{(S;0)}_{0R,0R} + \rho^{(S;0)}_{1R,1R} = \rho^{(R;0)}_{0R,0R} + \rho^{(R;0)}_{1R,1R},
\]

(52)

i.e., the sum of the elements from 0- and \( N \)-excitation blocks is conserved.

However, there is a way to overcome this obstacle and make the perfect transfer of all elements of the 0-order coherence matrix. First, we require

\[
\rho^{(R;0)}_{1R,1R} = \rho^{(S;0)}_{0R,0R}.
\]

(53)

Then Eq. (52) yields

\[
\rho^{(R;0)}_{0R,0R} = \rho^{(S;0)}_{1R,1R}.
\]

(54)

Now apply the unitary transformation to the receiver (not to the extended receiver) which exchanges the first and last rows (as well as the first and last columns) of the density matrix. Of course, this transformation does not commute with the \( z \)-projection of the total spin moment, and therefore it exchanges some elements between \( n \)- and \(-n\)-order coherence matrices \( (n = 1, \ldots, N - 1) \). This problem does not appears if these rows and columns are zero, i.e., the sender’s initial density matrix has the following block-diagonal form:

\[
\rho^{(S)}(0) = \begin{pmatrix}
\rho^{(S;0)}_{0S,0S} & 0_{2N-2} \\
0_{2N-2}^T & \tilde{\rho}^{(S)}_{2N-2} \\
0_{2N-2} & \rho^{(S;0)}_{1S,1S}^T
\end{pmatrix},
\]

(55)
where $0_{2N-2}^T$ and $0_{2N-2}$ are, respectively, the row and column of $2N-2$ zeros, and $\rho^{(S)}$ collects elements of the density matrix in the subspace of states with $1$, $2$, $\ldots$, $N-1$ excitations. Otherwise, if $\rho^{(S)}(0)$ is a full matrix, one still can restore the structure of the higher-order coherence matrices by applying to the extended receiver a unitary transformation which combines the elements of $\pm n$-order coherence matrices with fixed $n$. Such transformation does not satisfy the commutation condition (7); we leave this problem beyond the scope of the present work.

2. Restoring of nondiagonal elements of the 0-order coherence matrix

Along with the elements of the higher order coherence matrix, the nondiagonal elements of 0-order coherence matrix can also be restored, as was shown in [28] for the two-qubit state. The nondiagonal elements are described by Eq. (48) with $N_R \neq M_R$. Restoring these elements reduces to the definition of $\lambda^{(0)}_{N_R,M_R}$:

$$\lambda^{(0)}_{N_R,M_R} = W_{0S0TLN_R;NR0TL0R}(W^{(0)}_{1R0TL0R;0S})^\dagger_{M_R0TL0R;0S0TLM_R}, \quad N_R \neq M_R$$

and set of constraints for the parameters $\varphi$:

$$0 = \sum_{|N_R|,|M_R|,|I_S|,|I_J|=N-|N_R|} W_{N_SN_TLN_R;I_S0TL0R}(W^{(0)}_{1R0TL0R;N_SN_TLM_R})^\dagger_{M_R0TL0R} \quad (56)$$

$$0 = \sum_{|N_R|,|M_R|,|I_S|,|I_J|=N-|N_R|} W_{N_SN_TLN_R;I_S0TL0R}(W^{(1)}_{1R0TL0R;N_SN_TLM_R})^\dagger_{M_R0TL0R} \quad (57)$$

where at least one of the conditions $M_R \neq N_R$ or $I_S \neq J_S$ holds. These constraints eliminate nondiagonal elements $\rho^{(S)}_{I_S,J_S}$ ($I_S \neq J_S$) from the right hand side (r.h.s) of Eqs. (48) with $N_R = M_R$ and remove extra terms from eqs. (48) with $N_R \neq M_R$ to provide Eq. (56). The diagonal part of system (48) with $M_R = N_R$ takes the form

$$\rho^{(R;0)}_{N_R,N_R} = \sum_{|N_R|,|I_S|,|I_J|=N-|N_R|} W_{N_SN_TLN_R;I_S0TL0R}(W^{(0)}_{1R0TL0R;N_SN_TLM_R})^\dagger_{M_R0TL0R} \quad (58)$$

$$\rho^{(R;0)}_{N_R,N_R} = \sum_{|N_R|,|I_S|,|I_J|=N-|N_R|} W_{N_SN_TLN_R;I_S0TL0R}(W^{(1)}_{1R0TL0R;N_SN_TLM_R})^\dagger_{M_R0TL0R} \quad (59)$$

$$\rho^{(R;0)}_{N_R,N_R} = \sum_{|N_R|,|I_S|,|I_J|=N-|N_R|} W_{N_SN_TLN_R;I_S0TL0R}(W^{(2)}_{1R0TL0R;N_SN_TLM_R})^\dagger_{M_R0TL0R} \quad (60)$$
We also have to eliminate nondiagonal elements from (49), imposing on $\varphi$ more constraint for $\varphi$:

$$
\sum_{N_{S},N_{TL}} |N_{S}| + |N_{TL}| = |I_{S}| \sum_{N_{S},N_{TL}} |N_{S}| = |I_{S}| = |J_{S}| < N.
$$

Then Eq. (49) reduces to

$$
\rho^{(R,0)}_{0,0} = \rho^{(S,0)}_{0,0} + \sum_{N_{S},N_{TL}} \sum_{|N_{S}| + |N_{TL}| = |I_{S}| < N} W_{N_{S},N_{TL}}^{(|I_{S}|)} |I_{S}| R_{0,0,0,0}^{(S,0)} = 0
$$

Now only diagonal elements of $\rho^{(R,0)}$, described by (59), can satisfy the condition of perfect transfer given by Eq. (51) with $M_{R} = N_{R}$ and $0 < |N_{R}| < N$.

Thus, equations (57)–(60) have to be added to Eqs. (44) and (46) in constructing the optimizing unitary transformation of the extended receiver $U_{opt}$. Constraint (53) is also applicable in this case.

C. State-restoring protocol

Finally, the general protocol for state restoring is the following:

1. Create the initial state $\rho^{(0)}$ of the form (25), (5), where $\rho^{(S,n)}(0) \ (|n| > 0)$ are arbitrary matrices, the elements of $\rho^{(S,0)}$ satisfy (51) and (52) (and perhaps (53) with the initial state in form (55)), and $\rho^{(T_{L},R_{R})}(0)$ is

$$
\rho^{(T_{L},R_{R})} = \text{diag}(1, 0, 0, \ldots, 0).
$$

2. Run evolution of the density matrix up to the time instant $t_{0}$.

3. At the time instant $t = t_{0}$, apply to the extended receiver the universally optimal transformation $U_{opt}$, constructed in Secs. III B 1 and III B 2

4. Construct the matrix $\rho^{(R)}$ by tracing the obtained density matrix over the subsystems $T_{L}$ and $S$. 
5. If necessary, apply the unitary transformation exchanging the matrix elements $\rho^{(R,0)}_{0R,0R}$ and $\rho^{(R,0)}_{1R,1R}$ to completely restore the 0-order coherence matrix. This step is applicable if (53) was used to construct $\rho^{(S,0)}(0)$ and the block-diagonal structure (55) of the sender’s initial state $\rho^{(S)}(0)$.

It is important to emphasize that both $U_{opt}$ and $\rho^{(S,0)}$ are universal objects associated with a particular quantum system and time instant $t_0$ for the receiver’s state registration. Once constructed, they can be used for transferring and restoring any allowed higher order coherence matrices. Thus, 0-order coherence matrix serves as a core for transferring the parameters encoded into the higher order coherence matrices. It is also important that equations (51) describing the perfect transfer of some elements of $\rho^{(S,0)}$, include the parameters $\varphi$, which are fixed in constructing the optimized transformation $U_{opt}$. Therefore the matrix $\rho^{(S,0)}$ depends on the optimization transformation $U_{opt}$ and must be constructed after fixing parameters $\varphi$.

In context of quantum control, a general notion of a Kraus map which is simultaneously optimal for all initial states for a given quantum control problem was introduced in [30] and called as universally optimal Kraus maps [37]. Similarly, we call the unitary transformation $U_{opt}$ and induced by it Kraus map as universally optimal for state recovering since they are optimal for recovering simultaneously all non-zero order coherence matrices.

IV. EXAMPLE: 2-QUBIT STATE RESTORING

In this section we consider an example of two-qubit state restoring. Now $N = 2$ and three multi-quantum coherence matrices have to be considered. Restoring of the non-diagonal part of the 2-qubit transferred matrix was proposed in Ref. [28]. Considering 2-qubit sender and receiver states we explore other perspectives of 2-qubit state transfer and restoring which were missed in Ref. [28]. For instance, the structure of the 0-order coherence matrix was not optimized therein. Here we represent restoring of the 2- and 1-order coherence matrices, requiring a special structure for the 0-order coherence matrix which can be almost perfectly transferred up to 2 diagonal elements which provide the trace-normalization (50). Then we give remarks on restoring the nondiagonal part of the 0-order coherence matrix and, finally, construct the 0-order coherence matrix which can be perfectly transferred along the spin chain up to the trivial exchange of two its diagonal elements, see Eq. (73).
We deal with formulas of Sec III written for the 2-qubit sender and receiver. These formulas are similar to those derived in [28], therefore we move them to Appendix, Sec VI. Results below in this section are obtained using those formulas.

Let us consider the dynamics of the spin-1/2 chain governed by the \( XX \)-Hamiltonian taking into account dipole-dipole interactions among all nodes:

\[
H = \sum_{j>i} D_{ij} (I_i; I_j + I_i; I_j),
\]

where \( D_{ij} = \gamma^2 \hbar / r_{ij}^3 \) is the coupling constant, \( \gamma \) is the gyromagnetic ratio, \( \hbar \) is the Planck constant. For the homogeneous chain \( r_{i,i+1} = r \) and therefore the coupling constants between the nearest neighbors are the same.

Following Ref. [28], we consider the chain of 42 nodes with two pairs of adjusted coupling constants [38] and chose the time instant \( t_0 \) for state registration according to (29):

\[
\delta_1 = \delta_{N-1} = 0.3005\delta, \quad \delta_2 = \delta_{N-2} = 0.5311\delta, \quad \delta t_0 = 58.9826,
\]

where \( \delta \) is the nearest-neighbor coupling constant between inner nodes of the chain. We take the extended receiver consisting of four nodes and unitary transformation having the block-diagonal form, see (10)–(13),

\[
U = \text{diag}(1, U^{(1)}(\varphi^{(1)}), U^{(2)}(\varphi^{(2)})),
\]

\[
\varphi = (\varphi^{(1)}, \varphi^{(2)}), \quad \varphi^{(1)} = (\varphi^{(1)}_1, \ldots, \varphi^{(1)}_{12}), \quad \varphi^{(2)} = (\varphi^{(2)}_1, \ldots, \varphi^{(2)}_{30}),
\]

where \( U^{(1)} \) and \( U^{(2)} \) are, respectively, \( 4 \times 4 \) and \( 6 \times 6 \) unitary transformations in the subspaces of one and two excited spins.

At this stage we shall give a remark. In the above discussion, the multi-index notations were quite formal. Now we point on the desirable symmetry between nodes of the sender and receiver. According to this symmetry, the nodes of the receiver must be counted in the reverse order, i.e., the first and the second nodes of the receiver are the \( N \)th and the \( (N - 1) \)th nodes of the chain. Hereafter we follow this remark.

System (75), (76) consists of 6 independent complex equations for the parameters \( \varphi \) of the unitary transformation. Solution of this system is not unique. Our purpose is to find such unitary transformation which maximizes the scale factors in formulas (74), (77) and (78). As an objective function \( J \) for maximization, we take the sum of the absolute values
of these parameters,

\[ J = |\lambda_{00,01}^{(1)}| + |\lambda_{00,10}^{(1)}| + |\lambda_{01,11}^{(1)}| + |\lambda_{10,11}^{(1)}| + |\lambda_{00,11}^{(2)}| \to \max, \]

(66)

and perform the rough maximization \[28\] which gives

\[ J = 2.391. \]

(67)

Then we fix \[ J \], find 1000 solutions of the system \((75), (76)\) and

\[ |\lambda_{00,01}^{(1)}| + |\lambda_{00,10}^{(1)}| + |\lambda_{01,11}^{(1)}| + |\lambda_{10,11}^{(1)}| + |\lambda_{00,11}^{(2)}| = 2.391, \]

(68)

and select the solution which corresponds to the maximal value of the minimal absolute values of the scale factors, which we denote by \(|\lambda_{\min}|\). In our case, the role of \(\lambda_{\min}\) in the above 1000 solutions is played by either \(\lambda_{01,11}\) or \(\lambda_{10,11}\) with the maximal by absolute value is \(|\lambda_{01,11}| = 0.315\). The histogram of calculated values of \(|\lambda_{\min}|\) is given in Fig. 2.

Each solution is found by the Newton method starting with the random set of initial values for parameters \(\varphi\) in the range \((0, 2\pi)\). Then we solve system \((83)\) with

\[ (n_1, n_2, m_1, m_2) = (0, 1, 1, 0), \ (0, 1, 0, 1), \ (1, 0, 1, 0) \]

(69)

for \(\rho_{01,10}^{(S,0)}, \rho_{01,01}^{(S,0)}, \rho_{10,10}^{(S,0)}\). As the result, we obtain \(\rho^{(R)}\) (the scale factors and the elements \(\rho_{01,10}^{(S,0)}, \rho_{01,01}^{(S,0)}, \rho_{10,10}^{(S,0)}\) are at the appropriate positions in the formula for \(\rho^{(R)}\))

\[
\rho^{(R)} =
\begin{pmatrix}
1 - 0.517(1 - \rho_{00,00}^{(S,0)}) & 0.575e^{-1.812i} \rho_{00,01}^{(S,0)} & 0.616e^{3.06i} \rho_{00,10}^{(S,0)} & 0.548e^{0.257i} \rho_{00,11}^{(S,0)} \\
0.575e^{1.812i} \rho_{00,01}^{(S,0)} & 0.214(1 - \rho_{00,00}^{(S,0)}) & 0.031e^{-2.432i} (1 - \rho_{00,00}^{(S,0)}) & 0.315e^{2.070i} \rho_{01,11}^{(S,0)} \\
0.616e^{-3.06i} \rho_{00,10}^{(S,0)} & 0.031e^{2.432i} (1 - \rho_{00,00}^{(S,0)}) & 0.095(1 - \rho_{00,00}^{(S,0)}) & 0.337e^{-2.804i} \rho_{10,11}^{(S,0)} \\
0.548e^{-0.257i} \rho_{00,11}^{(S,0)} & 0.315e^{-2.070i} \rho_{01,11}^{(S,0)} & 0.337e^{2.804i} \rho_{10,11}^{(S,0)} & 0.208(1 - \rho_{00,00}^{(S,0)})
\end{pmatrix}
\]
and the sender’s initial state $\rho^{(s)}(0)$ in the form

$$
\rho^{(s)} = \begin{pmatrix}
\rho_{00,00}^{(s)} & \rho_{00,01}^{(s)} & \rho_{00,10}^{(s)} & \rho_{00,11}^{(s)} \\
\rho_{00,01}^{(s)*} & 0.214(1 - \rho_{00,00}^{(s)}) & 0.031e^{-2.432i}(1 - \rho_{00,00}^{(s)}) & \rho_{01,11}^{(s)} \\
\rho_{00,10}^{(s)*} & 0.031e^{2.432i}(1 - \rho_{00,00}^{(s)}) & 0.095(1 - \rho_{00,00}^{(s)}) & \rho_{10,11}^{(s)} \\
\rho_{00,11}^{(s)*} & (\rho_{01,11}^{(s)})^* & (\rho_{10,11}^{(s)})^* & 0.691(1 - \rho_{00,00}^{(s)})
\end{pmatrix} .
$$

(71)

In formulas (70) and (71), as well as in other density matrices of this section, we use the basis of two-qubit states (0, 0), (0, 1), (1, 0), (1, 1), where (0, 1) means the excited edge spin. The non-zero-order coherence matrices include five arbitrary complex parameters: $\rho_{00,01}^{(s)}$, $\rho_{00,10}^{(s)}$, $\rho_{00,11}^{(s)}$, $\rho_{01,11}^{(s)}$. We do not provide the explicit form of $U_{opt}$.

Further, we consider the constraint (63) together with the block structure (65). In our case of two-qubit sender and receiver, the above block structure includes only 0-order coherence matrix, so that there is no higher order coherence matrices. Therefore we disregard $U_{opt}$ (or set $\varphi = 0$). We solve the system (63) with $(n_1, n_2, m_1, m_2)$ from (69) together with Eq. (53) (which now reads $\rho_{11,11}^{(R,0)} = \rho_{00,00}^{(s)}$) for the matrix elements $\rho_{01,10}^{(S,0)}$, $\rho_{01,01}^{(S,0)}$, $\rho_{10,10}^{(S,0)}$, $\rho_{00,00}^{(S,0)}$ taking into account the normalization $\sum_{i,j=0}^{1} \rho_{ij,ij}^{(S,0)} = 1$. After exchanging the first and last columns and rows in the derived matrix one gets

$$
\rho^{(R)}(t_0) = \rho^{(s)}(0) = \begin{pmatrix}
0.130 & 0 & 0 & 0 \\
0 & 0.487 & 0.110e^{-1.290i} & 0 \\
0 & 0.110e^{1.290i} & 0.085 & 0 \\
0 & 0 & 0 & 0.298
\end{pmatrix} .
$$

(72)

Notice that there are no free parameters in the density matrix (72). However, according to Sec. III B 2, the non-diagonal elements of 0-order coherence matrix can carry an arbitrary parameter. For that, we have to satisfy condition (84). We perform the rough maximization similar to maximization in deriving state (70) with objective $J = |\lambda_{01,10}^{(0)}|$ to result in the state with single arbitrary parameter $\rho_{01,10}^{(S,0)}$:

$$
\rho^{(R)}(t_0) = \begin{pmatrix}
0.006 & 0 & 0 & 0 \\
0 & 0.604 & 0.560e^{2.393i} & \rho_{01,10}^{(S,0)} \\
0 & 0.560e^{-2.393i} & (\rho_{01,10}^{(S,0)})^* & 0.032 \\
0 & 0 & 0 & 0.358
\end{pmatrix} .
$$

(73)
V. CONCLUSIONS AND DISCUSSION

We consider the problem of constructing such mixed initial state of the multi-qubit sender which can be transferred to the receiver through the spin chain with minimal and well-characterizable deformation. This deformation is the scaling of the elements of the higher-order coherence matrices (and perhaps non-diagonal elements of the 0-order coherence matrix) of the transferred state and perfect (or almost perfect) transfer of the 0-order coherence matrix.

For this purpose, we consider a spin-1/2 chain consisting of sender $S$, transmission line $TL$ and receiver $R$, where the sender and the receiver have the same dimension. The system’s dynamics is governed by Hamiltonian preserving the number of excitation in the system. As was shown in [26], such dynamics prevents multi-quantum coherence matrices from mixing. However, mixing elements inside each coherence matrix had remained an open problem. Partially this problem was resolved in [28] for a 2-qubit sender and receiver with the ground initial state of $TL \cup R$, where a protocol for restoring non-diagonal elements of the transferred matrix via special unitary transformation of the extended receiver was proposed. But diagonal elements have special properties and can not be optimized in this way.

Here we generalize the protocol of Ref. [28] to the multi-qubit sender and receiver of equal dimensions and also study the problem of optimization of the 0-order coherence matrix. The state of the receiver must be registered at a certain time instant $t_0$. We fix $t_0$ as a time instant maximizing the $N$-order coherence intensity [28] ($N = 2$ in example of Sec.IV) without applying the unitary transformation to the extended receiver.

For the restoring purpose, the mentioned above extended receiver is introduced into the chain. It includes the receiver as a subsystem and serves to handle the receiver’s state through a unitary transformation. The dimension of the extended receiver must be large enough so that the appropriate unitary transformation possesses enough number of free parameters $\varphi$ to restore the structure of the elements of the higher order coherence matrices and, if needed, to optimize the scale factors and to restore the nondiagonal elements of the 0-order coherence matrix. We describe in detail the protocol for constructing this optimizing unitary transformation for the sender and the receiver of arbitrary dimension. This constructed transformation is universally optimal, i.e., it fixes factors ahead of the re-
stored elements and, ones constructed, can be used to structurally restore any higher-order coherence matrix of the sender’s initial state.

The above unitary transformation is not unique. The variety of such transformations allows to perform further optimization of state restoring. We perform rough optimization calculating 1000 independent unitary transformations and selecting one of them which results in the maximal sum of the absolute values of all scale factors with the maximal value of the smallest term in this sum.

We also find such 0-order coherence matrix that can be almost perfectly, up to two diagonal elements whose sum is conserved, transferred to the receiver and use this matrix as a block of the sender’s initial state in the proposed protocol for restoring the higher-order coherence matrices. Further, if the initial state of the N-qubit sender is embedded into the up to N-excitation space then there is the perfectly transferable 0-order coherence matrix up to the exchange of two elements corresponding to the blocks of states with 0 and N excitations. Furthermore, if the sender’s initial state $\rho^{(S)}(0)$ has the block structure (55), then the complete structural restoring of the transferred matrix is possible. Notice that each element of the higher-order coherence matrix transfers a free parameter. On the contrary, the 0-order coherence matrix with perfectly transferred nondiagonal elements does not carry any free parameter because all its elements are fixed. However, we can extend the restoring protocol to the non-diagonal elements of the 0-order coherence matrix so that free parameters of the initial sender’s state appear in the non-diagonal part of 0-order coherence matrix. An example of a two-qubit state restoring is explored in details.

This work was funded by Russian Federation represented by the Ministry of Science and Higher Education (grant number 075-15-2020-788).

VI. APPENDIX: GENERAL FORMULAS FOR 2-QUBIT STATE RESTORING

Let $N = 2$ and $k = 1$ in formulas (41)-(47). All multi-indexes associated with the sender and receiver consist of two entries $\{n_1, n_2\}$, $n_i = 0, 1$, $i = 1, 2$. In particular $1_R = 1_S = \{1, 1\}$, $0_R = 0_S = \{0, 0\}$. Eq. (41) yields

$$\lambda^{(2)}_{11,11} = (W^{(2)})_{11,0_{RL,00},0_{RL,11}}.$$  (74)
Eqs. (44) and (46) yield two equations:

\[ W_{00,0TL,n1n2;i1i2,0TL,00}^{(1)} = 0 \quad \text{for} \quad (n_1, n_2) \neq (i_1, i_2), \quad n_1 + n_2 = i_1 + i_2 = 1 \tag{75} \]

and four equations

\[
\sum_{i_1,i_2,N_{TL}\mid N_{TL}=1} W_{i_1i_2,0,00}^{(1)}(W_{11,0TL,00;i1i2,0TL,00}^{(2)})^\dagger = 0, \quad |I_S| = |M_R| = 1. \tag{76}
\]

In addition, eqs. (45) and (47) yield the following definitions of the scale factors:

\[
\lambda_{n_1n_2,11}^{(1)} = W_{00,0TL,n_1n_2;i1i2,0TL,00}^{(1)} W_{11,0TL,00;00,0TL,11}^{(2)}, \quad n_1 + n_2 = 1 \tag{77}
\]

\[
\lambda_{00,m_1m_2}^{(2)} = (W_{11,0TL,00;00,0TL,11}^{(1)})^\dagger_{n_1m_2,00,00,00,00,00} W_{11,0TL,00;00,0TL,11}^{(2)}, \quad m_1 + m_2 = 1. \tag{78}
\]

For the elements \(\rho_{11,11}^{(R:0)}\) and \(\rho_{00,00}^{(R:0)}\) of the 0-order coherence matrix (35), (49), we have

\[
\rho_{11,11}^{(R:0)} = |W_{00,0TL,11;11,0TL,00}^{(2)}|^2 \rho_{11,11}^{(S:0)} \tag{79}
\]

\[
\rho_{00,00}^{(R:0)} = \rho_{00,00}^{(S:0)} + \sum_{k_1,k_2,N_{TL}\mid N_{TL}=2} |W_{k_1k_2,0TL,00;11,0TL,00}^{(2)}|^2 \rho_{11,11}^{(S:0)} + \sum_{k_1,k_2,N_{TL}\mid N_{TL}=1} W_{k_1k_2,0TL,00;i1i2,0TL,00}^{(1)} (W_{i1i2,0TL,00;j1j2,0TL,00}^{(1)})^\dagger \rho_{i1i2,j1j2}^{(S:0)}. \tag{80}
\]

The last term in Eq. (80) can be simplified using unitarity of \(W^{(1)}\) which assumes

\[
\sum_{k_1,k_2,N_{TL}\mid N_{TL}=1} W_{k_1k_2,0TL,00;i1i2,0TL,00}^{(1)} (W_{i1i2,0TL,00;j1j2,0TL,00}^{(1)})^\dagger = 0, \quad i_1 + i_2 = j_1 + j_2 = 1. \tag{81}
\]

and condition (75), so that

\[
\rho_{00,00}^{(R:0)} = \rho_{00,00}^{(S:0)} \tag{82}
\]

\[
+ \sum_{k_1,k_2,N_{TL}\mid N_{TL}=2} |W_{k_1k_2,0TL,00;11,0TL,00}^{(2)}|^2 \rho_{11,11}^{(S:0)} + \sum_{k_1,k_2,N_{TL}\mid N_{TL}=1} |W_{k_1k_2,0TL,00;i1i2,0TL,00}^{(1)}|^2 \rho_{i1i2,i1i2}^{(S:0)}.
\]

Other elements of \(\rho^{(R:0)}\) satisfy Eq. (51) which now takes the form

\[
\rho_{n_1n_2,m_1m_2}^{(R:0)} = \rho_{n_1n_2,m_1m_2}^{(S:0)} \tag{83}
\]

\[
= W_{00,0TL,n_1n_2;n_1n_2,0TL,00}^{(1)} W_{m_1m_2,0TL,00;00,0TL,m_1m_2}^{(1)} + \sum_{k_1,k_2,N_{TL}\mid N_{TL}=1} W_{k_1k_2,0TL,n_1n_2;11,0TL,00}^{(2)} (W_{11,0TL,00;00,0TL,11}^{(1)})^\dagger \rho_{11,11}^{(S:0)} \rho_{n_1n_2,m_1m_2}^{(S:0)}.
\]

\[n_1 + n_2 = m_1 + m_2 = 1.\]
The condition (52) is satisfied in this case, the constrain (53) can also be imposed.

From the structure of Eqs. (82) and (83) it follows that, for restoring the non-diagonal part of $\rho^{(S,0)}$, we have the equation:

$$
\sum_{k_1,k_2,N_{TL}, k_1+k_2+N_{TL}=1} W^{(2)}_{k_1k_2,N_{TL},01;11,0_{TL},00}(W^{(2)}_{11,0_{TL},00;01,k_1k_2,N_{TL},0}) = 0
$$

with the additional scale factor

$$
\lambda^{(0)}_{01,10} = W^{(1)}_{00,0_{TL},01;01,0_{TL},00}(W^{(1)}_{01,0_{TL},00;00,0_{TL},10})
$$
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