Arithmetic of $\mathcal{N} = 8$ Black Holes

Ashoke Sen

*Harish-Chandra Research Institute*  
*Chhatnag Road, Jhusi, Allahabad 211019, INDIA*

E-mail: sen@mri.ernet.in, ashokesen1999@gmail.com

**Abstract**

The microscopic formula for the degeneracies of 1/8 BPS black holes in type II string theory compactified on a six dimensional torus can be expressed as a sum of several terms. One of the terms is a function of the Cremmer-Julia invariant and gives the leading contribution to the entropy in the large charge limit. The other terms, which give exponentially subleading contribution, depend not only on the Cremmer-Julia invariant, but also on the arithmetic properties of the charges, and in fact exist only when the charges satisfy special arithmetic properties. We identify the origin of these terms in the macroscopic formula for the black hole entropy, based on quantum entropy function, as the contribution from non-trivial saddle point(s) in the path integral of string theory over the near horizon geometry. These saddle points exist only when the charge vectors satisfy the arithmetic properties required for the corresponding term in the microscopic formula to exist. Furthermore the leading contribution from these saddle points in the large charge limit agrees with the leading asymptotic behaviour of the corresponding term in the degeneracy formula.
1 Introduction

We now have a good understanding of the spectrum of BPS dyons in the four dimensional $\mathcal{N} = 8$ supersymmetric string theory obtained by compactifying type II string theory on $T^6$. In particular the exact degeneracies are known for a class of 1/8 BPS dyons in this theory [1, 2, 3, 4, 5, 6]. On the other hand after taking into account the effect of gravitational backreaction these dyons are expected to become black holes with finite area event horizon and hence have finite macroscopic entropy. In the limit of large charges the Bekenstein-Hawking entropy of these black holes match the statistical entropy, – logarithm of the microscopic degeneracy of states carrying the same charges.

In a recent series of papers an algorithm for computing the exact macroscopic degeneracy of extremal black holes was proposed [7, 8, 9]. This algorithm – known as the quantum entropy function – equates the macroscopic degeneracy with the partition function of string theory in the euclidean near horizon geometry of the black hole that contains an $AdS_2$ factor. More precisely the path integral contributing to the partition function includes sum over all configurations which asymptotes to the near horizon geometry of the black hole near the boundary of $AdS_2$. Given this algorithm for computing the exact macroscopic degeneracies it is natural to compare this with the exact microscopic degeneracies which are known.

---

1This gives the contribution from a single centered black hole horizon. The contribution to the degeneracy from a generic multi-centered black hole is obtained by taking the product of the contribution from each horizon and the degeneracies due to the hair modes, – degrees of freedom living outside the horizon [9]. For 1/8 BPS black holes in $\mathcal{N} = 8$ supersymmetric string theories we expect the effect of multi-centered configurations to be absent for the charges of interest to us – both for the wall crossing [5] and also for the total index [10]. Hence we can concentrate on the contribution from single centered black holes represented by a single $AdS_2$ factor.

2Since on the macroscopic side we compute degeneracies but on the microscopic side we compute an appropriate helicity trace index [11, 12] – the 14th helicity trace $B_{14}$ in the present example – one might wonder if it is appropriate to compare the two. It was argued in [9] that as long as the only hair degrees of freedom of the...
A full analysis will be beyond the scope of the present work as it would require explicit evaluation of the path integral of string theory in appropriate backgrounds. Instead in this paper we investigate a particular aspect of this problem. The microscopic result for the degeneracy can be expressed as a sum over finite number of terms. One of these terms, which gives the leading contribution to the entropy in the large charge limit, is a function of the Cremmer-Julia invariant constructed out of the charges \([13, 14]\). The other terms, which give exponentially suppressed contributions in the large charge limit, exist only when the charge vectors satisfy special arithmetic properties. We show that for each of these terms, one can identify saddle points, constructed as appropriate orbifolds (without fixed points) of the near horizon geometry of the black hole, satisfying the following properties:

1. The geometry associated with the saddle point coincides with the near horizon geometry of the black hole near the boundary of \(AdS_2\). This shows that this is a valid configuration to be included in the path integral over the string fields.

2. The saddle point exists only when the charge vectors satisfy special arithmetic properties – the same properties for which the corresponding term in the microscopic degeneracy formula exists.

3. For large charges the contribution to the path integral from this saddle point has the same behaviour as the corresponding term in the degeneracy formula.

Thus these saddle points are the ideal candidates for representing the corresponding terms in the degeneracy formula. This generalizes similar results for \(\mathcal{N} = 4\) supersymmetric string theories \([9]\) (see also \([15, 16]\)).

2 \(\mathcal{N} = 8\) Dyon Spectrum: Microscopic Results

We consider type IIB string theory on \(T^6\), which we shall label as \(T^4 \times S^1 \times \tilde{S}^1\) by regarding two circles inside \(T^6\) as special. In this theory we consider a system of \(D5/D3/D1\) branes wrapped on \(4/2/0\) cycles of \(T^4\) times either \(S^1\) or \(\tilde{S}^1\). We shall denote the charges of the D-branes wrapped on \(S^1\) by \(P\) and the charges of the D-branes wrapped on \(\tilde{S}^1\) by \(Q\). Both \(Q\) and \(P\) are black hole are the fermion zero modes associated with broken supersymmetries – a condition we expect to be satisfied for the configurations involving only D-brane charges as studied here \([9]\) – the macroscopic degeneracy should agree with the microscopic index \(-B_{14}\). For this reason we shall often refer to \(-B_{14}\) as the microscopic degeneracy, and the various microscopic formulae given in the text refer to this index.
8 dimensional vectors reflecting the dimension of the even cohomology of $T^4$. A perturbatively realized symmetry which acts within this class of charges is $SO(4, 4; \mathbb{Z}) \times SL(2, \mathbb{Z})$ where the former is associated with the duality symmetries of $T^4$ and the latter is associated with the global diffeomorphism symmetry of $S^1 \times \tilde{S}^1$. A U-duality transformation maps this system to type IIA string theory on $T^6$ where all charges arise in the NSNS sector, with $Q$ representing the electric charges and $P$ representing the magnetic charges. In this case $SO(4, 4; \mathbb{Z})$ appears as a subgroup of the T-duality group and $SL(2, \mathbb{Z})$ appears as the S-duality group of the theory. In view of this we shall call $SO(4, 4; \mathbb{Z})$ and $SL(2, \mathbb{Z})$ as T- and S-duality symmetries even though in the description we are using both are part of the T-duality group.

The intersection matrix $L$ of the even homology cycles of $T^4$ defines a natural inner product between the charge vectors $Q$ and $P$:

$$Q^2 \equiv Q^T L Q, \quad P^2 \equiv P^T L P, \quad Q \cdot P \equiv Q^T L P. \quad (2.1)$$

We define

$$\ell_1 = \gcd\{Q_i P_j - Q_j P_i\}, \quad \ell_2 = \gcd\left(\frac{Q^2}{2}, \frac{P^2}{2}, Q \cdot P\right), \quad (2.2)$$

where $Q_i$ and $P_i$ are the components of $Q$ and $P$ in some primitive basis of the charge lattice $\Lambda$. $\ell_1$ and $\ell_2$ remain invariant under S- and T-duality transformations of $(Q, P)$. The formula for the dyon degeneracy carrying charges $(Q, P)$ is known in the case $\gcd(\ell_1, \ell_2) = 1$.

$$\gcd(\ell_1, \ell_2) = 1. \quad (2.3)$$

In this case the degeneracy formula for the charge vectors considered here takes the form

$$d(Q, P) = (-1)^{Q^P + 1} \sum_{s|\ell_1 \ell_2} s \hat{c}(\Delta(Q, P)/s^2), \quad (2.4)$$

where $\Delta(Q, P)$ is the Cremmer-Julia invariant $\Delta(Q, P) = Q^2 P^2 - (Q \cdot P)^2, \quad (2.5)$

and $\hat{c}(u)$ is defined through the relations $[1, 2]$

$$- \frac{1}{3} \eta(\tau)^2 \eta(\tau)^{-6} \equiv \sum_{k,l} \hat{c}(4k - l^2) e^{2\pi i(k\tau + lz)}. \quad (2.6)$$

---

3This can be done by first making a T-duality transformation on the circle $\tilde{S}^1$ and then making a $\mathbb{Z}_2$ U-duality transformation that maps $(-1)^{F_L}$ to a geometric symmetry $I_4$ that reverses the signs of all the coordinates of $T^4$ and vice-versa [17]. Thus the $(-1)^{F_L}$ odd gauge fields from the RR sector are mapped to $I_4$ odd gauge fields given by the dimensional reduction of the metric and the NSNS 2-form fields along $T^4$.

4It may be possible to relax this condition by using the results of [1] for non-primitive charge vectors.
\( \vartheta_1(z|\tau) \) and \( \eta(\tau) \) are respectively the odd Jacobi theta function and the Dedekind eta function. The derivation of (2.3), (2.4) has been reviewed in appendix A.

For large charges we have

\[
\hat{c}(\Delta) \sim (-1)^{\Delta+1} \Delta^{-2} \exp(\pi \sqrt{\Delta}).
\]  

(2.7)

Thus the \( s \)-th term in the sum grows as \( \exp(\pi \sqrt{\Delta}/s) \). In this limit the \( s = 1 \) term dominates, and the contribution to the entropy reduces to the Bekenstein-Hawking entropy of the black hole given by \( \pi \sqrt{\Delta} \). However the terms with \( s > 1 \) are significant in that they appear only when the charge vectors satisfy some special arithmetic properties. Thus one should be able to detect the origin of these terms in the macroscopic description by identifying contributions which appear only when the charge vectors satisfy these special arithmetic properties.

### 3 \( N = 8 \) Dyon Spectrum: Macroscopic Viewpoint

According to the proposal of [8, 9], the macroscopic entropy of an extremal black hole is given by the result of path integral over geometries whose asymptotic form coincide with the near horizon geometry of the black hole. In the case under consideration the near horizon metric of the Euclidean black hole carrying charges \((Q,P)\) takes the form:

\[
ds^2 = v \left( \frac{dr^2}{r^2-1} + (r^2-1) d\theta^2 \right) + w(d\psi^2 + \sin^2 \psi d\phi^2) + \frac{R^2}{\tau_2} |dx^4 + \tau dx^5|^2 + \sum_{m,n=6}^{9} \hat{g}_{mn} dx^m dx^n ,
\]

(3.1)

where \( v, w, R \) are real constants, \( \tau = \tau_1 + i\tau_2 \) is a complex constant, and \( \hat{g}_{mn} \) are real constants labelling the metric along \( T^4 \). \((r, \theta)\) label an Euclidean \( AdS_2 \) space, \((\psi, \phi)\) label a 2-sphere, \(x^4\) and \(x^5\) label the coordinates along \( S^1 \) and \( S^1 \) respectively and \( x^6, x^7, x^8, x^9 \) are coordinates along \( T^4 \). Each of the coordinates \( x^4, \ldots x^9, \theta, \phi \) has period \( 2\pi \). The background also contains constant values of various scalar fields and components of \( p \)-form fields along \( T^4 \times S^1 \times S^1 \), and fluxes of various RR fields. In the six dimensional description, in which all the RR field strengths can be regarded as self-dual or anti-self-dual 3-forms after dimensional reduction on \( T^4 \), \( Q \) represents RR fluxes through the 3-cycle spanned by \((x^5, \psi, \phi)\) and \( P \) represents RR fluxes through the 3-cycle spanned by \((x^4, \psi, \phi)\). The (anti-)self-duality constraints on the RR field strengths in six dimensions relate the fluxes through the \((x^4, r, \theta)\) and \((x^5, r, \theta)\) planes to those through the \((x^5, \psi, \phi)\) and \((x^4, \psi, \phi)\) planes. The charges \((Q,P)\) also determine, up to

---

5Different aspects of the relationship between arithmetic and black holes have been studied in [18, 19, 20].
flat directions, the parameters $v, w, R, \tau, \tilde{g}_{mn}$ and the background values of various scalars and $p$-form fields.

In our analysis we shall assume that the $SO(4, 4; \mathbb{Z})_T \times SL(2, \mathbb{Z})_S$ symmetry is a symmetry of string theory in the near horizon geometry, i.e. two different configurations which differ from each other by the action of this symmetry on $(Q, P)$ give identical results for the partition function. This assumption is natural since both of these are perturbative symmetries of the theory in the description in which we are working. We can then make use of these duality symmetries to bring the charge vectors $(Q, P)$ to a specific form and carry out the analysis; the result for a general charge vector can be recovered by making an appropriate $SO(4, 4; \mathbb{Z})_T \times SL(2, \mathbb{Z})_S$ duality transformation.\footnote{More generally we can assume that the full perturbative duality symmetry $SO(6, 6; \mathbb{Z})$ is a symmetry of string theory in the near horizon geometry. In that case our results extend to a more general configuration of D-branes which can be related to the configurations analyzed here by an $SO(6, 6; \mathbb{Z})$ transformation. A general configuration of D-branes wrapped on various cycles of $T^6$ is characterized by a 32 dimensional charge vector transforming in the spinor representation of $SO(6, 6; \mathbb{Z})$, but we do not know under what condition on this charge vector it can be related to a configuration analyzed here by an $SO(6, 6; \mathbb{Z})$ transformation.}

Now it was shown in \cite{21} that given any pair of charge vectors $(Q, P)$ we can use S-duality transformations to bring it to the form

$$(Q, P) = (\ell_1 Q_0, P_0), \quad \gcd\{Q_0 P_0 - Q_0 P_0\} = 1,$$

where $\ell_1$ has been defined in \cite{22} and $Q_0$ and $P_0$ are elements of the charge lattice $\Lambda$. We shall use this representation. Furthermore it follows from the analysis of \cite{22} that if for a charge vector of the form (3.2) we define

$$n = Q_0^2/2, \quad m = P_0^2/2, \quad p = Q_0 \cdot P_0,$$

then with the help of T-duality transformations we can bring $Q_0$ and $P_0$ inside a four dimensional subspace of the full eight dimensional $SO(4, 4)$ lattice, and label them by the vectors

$$Q_0 = \begin{pmatrix} 1 \\ n \\ 0 \\ 0 \end{pmatrix}, \quad P_0 = \begin{pmatrix} 0 \\ p \\ 1 \\ m \end{pmatrix},$$

the metric in this subspace being given by

$$L = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$
Furthermore we can choose the embedding of this four dimensional subspace in the even cohomology of $T^4$ such that the four rows of $Q = \ell_1 Q_0$ represent the RR 5-form fluxes through the $\psi \phi x^5 x^6 x^7$, $\psi \phi x^5 x^8 x^9$, $\psi \phi x^5 x^6 x^8$ and $\psi \phi x^5 x^9 x^7$ cycles respectively, and the four rows of $P = P_0$ represent the RR 5-form fluxes through the $\psi \phi x^4 x^6 x^7$, $\psi \phi x^4 x^8 x^9$, $\psi \phi x^4 x^6 x^8$ and $\psi \phi x^4 x^9 x^7$ cycles respectively. Thus the RR 5-form flux is of the form

$$F = \frac{1}{32\pi^4} \sin \psi d\psi \wedge d\phi \left[ \ell_1 dx^5 \wedge dx^6 \wedge dx^7 + \ell_1 ndx^5 \wedge dx^8 \wedge dx^9 + p dx^4 \wedge dx^8 \wedge dx^9 + \ell_2 dx^4 \wedge dx^6 \wedge dx^8 - m dx^4 \wedge dx^7 \wedge dx^9 \right],$$

(3.6)

where we have normalized the flux so that its integral over any 5-cycle is an integer. From the definition of $\ell_2$ given in (2.2), and (2.3), (3.2), (3.3) it follows that

$$\ell_2 = \gcd(m, n, p), \quad \gcd(m, \ell_1) = 1.$$  

(3.7)

The requirement of self-duality also forces us to have RR 5-form flux through $AdS_2$ times appropriate 3-cycles of $T^6$. They can be determined from (3.6) but we shall not write them down explicitly.

We are now ready to describe our proposal for the macroscopic origin of the different terms appearing in the microscopic formula (2.4). Due to the condition (2.3), any $s$ contributing to the sum in (2.4) must have the form

$$s = s_1 s_2, \quad s_1, s_2 \in \mathbb{Z}, \quad s_1 | \ell_1, \quad s_2 | \ell_2, \quad \gcd(s_1, s_2) = 1.$$  

(3.8)

We propose that the $s$'th term in the sum in (2.4) arises from the orbifold of the geometry (3.1) by a $\mathbb{Z}_{s_1} \times \mathbb{Z}_{s_2}$ transformation. The $\mathbb{Z}_{s_1}$ is generated by

$$(\theta, \phi, x^5) \rightarrow \left( \theta + \frac{2\pi}{s_1}, \phi + \frac{2\pi}{s_1}, x^5 + \frac{2\pi k_1}{s_1}, \right), \quad k_1 \in \mathbb{Z}, \quad \gcd(s_1, k_1) = 1.$$  

(3.9)

On the other hand the $\mathbb{Z}_{s_2}$ action is generated by

$$(\theta, \phi, x^9) \rightarrow \left( \theta + \frac{2\pi}{s_2}, \phi + \frac{2\pi}{s_2}, x^9 + \frac{2\pi k_2}{s_2}, \right), \quad k_2 \in \mathbb{Z}, \quad \gcd(s_2, k_2) = 1.$$  

(3.10)

Since $s_1$ and $s_2$ do not have any common factor, one can also regard this as a $\mathbb{Z}_s$ orbifold generated by

$$(\theta, \phi, x^5, x^9) \rightarrow \left( \theta + \frac{2\pi}{s}, \phi + \frac{2\pi}{s}, x^5 + \frac{2\pi j_1}{s_1}, x^9 + \frac{2\pi j_2}{s_2}, \right), \quad j_1, j_2 \in \mathbb{Z}, \quad \gcd(j_1, s_1) = \gcd(j_2, s_2) = 1.$$  

(3.11)
Since $x^5$ and $x^9$ circles are non-contractible, $\mathbb{Z}_s$ acts freely and hence this orbifold does not have any fixed point.

A simpler description of this orbifold can be given by introducing new coordinates

$$\begin{align*}
\xi = \alpha x^5 + \beta x^9, & \quad \eta = -s_1 \tilde{j}_2 x^5 + s_2 \tilde{j}_1 x^9, \\
\tilde{j}_i \equiv j_i/k, & \quad k \equiv \gcd(j_1, j_2),
\end{align*}$$

(3.12)

where $(\alpha, \beta)$ are chosen such that

$$\alpha s_2 \tilde{j}_1 + \beta s_1 \tilde{j}_2 = 1, \quad \alpha, \beta \in \mathbb{Z}. $$

(3.13)

This is possible since $\gcd(s_2 \tilde{j}_1, s_1 \tilde{j}_2) = 1$ by construction. Since the transformation (3.12) is unimodular, $\xi$ and $\eta$ are both periodic coordinates with period $2\pi$. In terms of these new coordinates $(\xi, \eta)$ we can express the five form flux (3.6) and the orbifold action (3.11) as

$$F = \frac{1}{32\pi^4} \sin \psi \sin d\psi \wedge d\phi \left[ \ell_1 (s_2 \tilde{j}_1 d\xi - \beta d\eta) \wedge dx^6 \wedge dx^7 - \ell_1 n dx^8 \wedge d\xi \wedge d\eta \\
+ p dx^4 \wedge dx^8 \wedge (s_1 \tilde{j}_2 d\xi + \alpha d\eta) + dx^4 \wedge dx^6 \wedge dx^8 \\
- m dx^4 \wedge dx^7 \wedge (s_1 \tilde{j}_2 d\xi + \alpha d\eta) \right],$$

(3.14)

$$\left(\theta, \phi, \xi, \eta\right) \rightarrow \left(\theta + \frac{2\pi}{s}, \phi + \frac{2\pi}{s}, \xi + \frac{2\pi k}{s}, \eta\right).$$

(3.15)

It also follows from the definition of $k$ given in (3.12) that $\gcd(k, s) = 1$.

We are now ready to test the consistency of this orbifold. Since at the origin $r = 1$ of $AdS_2$ the $\theta$ translation has no effect, the effect of taking the $\mathbb{Z}_s$ orbifold (3.15) is to reduce the flux through any 5-cycle sitting at $r = 1$ and containing $(\psi, \phi, \xi)$ to $1/s$ times its original value. Thus in order that the orbifold satisfies the flux quantization laws, the coefficient of every term inside the square bracket in (3.14), containing $d\xi$, must be an integer multiple of $s = s_1 s_2$. Examining (3.14) and using the fact that $\ell_1$ is divisible by $s_1$ and $m, n, p$ are divisible by $s_2$ due to (3.7) we see that this is indeed the case. Thus (3.15) and hence (3.11) describes a consistent orbifold in string theory. Conversely, unless $s_1 | \ell_1$ and $s_2 | \ell_2$, the coefficient of one of the terms containing $d\xi$ inside [ ] will fail to be divisible by $s$ and hence the orbifold (3.11) will not satisfy flux quantization rule.

Following the procedure of [15, 9, 16] one can show that

1. These orbifolds have the same asymptotic behaviour as the near horizon geometry (3.1), and hence must be included in the path integral for computing the quantum entropy function.
2. The classical contribution to the path integral from these orbifolds is given by
\[
\exp(\pi \sqrt{\Delta(Q,P)/s}) \ ,
\]
in agreement with the asymptotic behaviour of the \(s\)-th term in the sum in (2.4).

Since a detailed analysis can be found in [9] (section 6) we shall not repeat it here. One can also show that these orbifolds preserve the necessary amount of supersymmetry so that integration over the fermion zero modes associated with the broken supersymmetry generators does not make the path integral vanish automatically [23]. Thus the contribution to the quantum entropy function from these orbifolds is the ideal candidate for reproducing the \(s\)-th term in the microscopic formula given in (2.4).

Acknowledgement: I wish to thank Atish Dabholkar and Edward Witten for useful discussions. I would like to acknowledge the hospitality of LPTHE, Paris where part of this work was performed. This work was supported by the project 11-R&D-HRI-5.02-0304 and the J.C.Bose fellowship of the Department of Science and Technology, India.

A The Dyon Degeneracy Formula

In this appendix we shall give a derivation of the dyon degeneracy formula given in (2.3), (2.4) from the duality covariant formula described in [6]. Let us consider the configurations considered in the text, labelled by the charges \((Q,P)\). We shall use the symbol \(q\) to denote the pair \((Q,P)\). Working in the duality frame described in footnote 3 one finds that the two discrete duality invariants \(\psi(q)\) and \(\chi(q)\) introduced in [6] take the form:
\[
\psi(q) = \gcd\left(\frac{Q^2}{2}, \frac{P^2}{2}, Q \cdot P, \{Q_i P_j - Q_j P_i\}\right),
\]
(A.1)

and
\[
\chi(q) = \gcd\{q_a \tilde{q}_b - \tilde{q}_a q_b\}, \quad \tilde{q} \equiv (\bar{Q}, \bar{P}) = (Q^2P - (Q \cdot P)Q, -P^2Q + (Q \cdot P)P) \ .
\]
(A.2)

The degeneracy formula given in [6] is valid for charge vectors with \(\psi(q) = 1\) and takes the form
\[
d(q) = (-1)^{Q \cdot P + 1} \sum_{s \in \mathbb{Z}, 2s|\chi(q)} s \tilde{c}(\Delta(Q,P)/s^2) .
\]
(A.3)

\footnote{Derivation of (A.1) and (A.2) can be found in [6]. In a general U-duality frame \(q\) belongs to the 56 representation of the U-duality group \(E_{7(7)}(\mathbb{Z})\), \(\psi(q)\) is the gcd of all the components of the 133 representation constructed from the bilinear \(q_a q_b\), \(\tilde{q}_a\) represents the vector in the 56 representation constructed from the trilinear \(q_a q_b q_c\), and \(\chi(q) = \gcd\{q_a \tilde{q}_b - q_b \tilde{q}_a\}\).}
A detailed derivation of this formula has been given in [5, 6] based on earlier work [2, 3] and a recent discussion can be found in [24].

We shall now show that the restriction $\psi(q) = 1$ and (A.3) lead to (2.3), (2.4). From (A.1) and (2.2) it follows that

$$
\psi(q) = \gcd(\ell_1, \ell_2),
$$

and hence the restriction $\psi(q) = 1$ reduces to $\gcd(\ell_1, \ell_2) = 1$ as given in (2.3). In order to show that (A.3) reduces to (2.4) we need to show that the condition $2s|\chi(q)$ corresponds to the restriction $s|\ell_1\ell_2$ as appears in the sum in (2.4). For this we expand (A.2):

$$
\chi(q) = \gcd \left\{ Q^2(Q_i P_j - Q_j P_i), \ P^2(Q_i P_j - Q_j P_i), \ (-P^2Q_i Q_j - Q^2P_i P_j + 2Q \cdot P Q_i P_j) \right\}.
$$

(A.5)

Now in computing the gcd we can certainly add to the list inside $\{\}$ a term that is obtained by antisymmetrizing the last set of terms in the indices $i$ and $j$. This gives

$$
\chi(q) = \gcd \left\{ Q^2(Q_i P_j - Q_j P_i), \ P^2(Q_i P_j - Q_j P_i), \ 2Q \cdot P(Q_i P_j - Q_j P_i), \ (-P^2Q_i Q_j - Q^2P_i P_j + 2Q \cdot P Q_i P_j) \right\}.
$$

(A.6)

From this we see that a necessary condition for $2s|\chi(q)$ is

$$
2s \mid \gcd \left\{ Q^2(Q_i P_j - Q_j P_i), \ P^2(Q_i P_j - Q_j P_i), \ 2Q \cdot P(Q_i P_j - Q_j P_i) \right\} = \gcd(Q^2, P^2, 2Q \cdot P) \times \gcd(Q_i P_j - Q_j P_i) = 2\ell_1\ell_2,
$$

(A.7)

and hence

$$
s|\ell_1\ell_2
$$

as given in (2.4). However we also need to show that this condition is sufficient i.e. that once (A.8) is satisfied then $2s$ automatically divides the last set of terms inside the list in (A.6).

These terms may be written in two different forms:

$$
\left\{ (-P^2Q_i Q_j - Q^2P_i P_j + 2Q \cdot P Q_i P_j) \right\} = \left\{ -(Q_k P_i - P_k Q_i)(Q_k P_j - Q_j P_k) - Q \cdot P (P_i Q_j - Q_i P_j) \right\}.
$$

(A.9)

The form given in the left hand side shows that (A.9) is divisible by $2\ell_2$ since $2\ell_2 = \gcd(P^2, Q^2, 2Q \cdot P)$. On the other hand the form given on the right hand side shows that it is divisible by $\ell_1$ since $\ell_1 = \gcd(P_i Q_j - Q_j P_i)$. Now if $\ell_1$ is odd, then it follows from (2.3) that $\gcd(2\ell_2, \ell_1) = 1$ and hence (A.9), being divisible by $\ell_1$ and $2\ell_2$, must be divisible by $2\ell_1\ell_2$. On the other hand if $\ell_1$ is even then $Q_k P_i - P_k Q_i$ must be even for every $i, k$ and hence
\( (Q_i P_k - Q_k P_i)(Q_i P_k - Q_k P_i) = 2\{Q^2 P^2 - (Q \cdot P)^2\} \) must be divisible by 4. Since \( Q^2 \) and \( P^2 \) are even, we must have \( (Q \cdot P)^2 \) even and hence \( Q \cdot P \) even. Thus the right hand side of (A.9) is divisible by \( 2\ell_1 \). Furthermore for \( \ell_1 \) even \( \ell_2 \) must be odd since \( \ell_1 \) and \( \ell_2 \) cannot have a common factor. In this case \( \gcd(2\ell_1, \ell_2) = 1 \), and we again conclude that \( 2\ell_1 \ell_2 \) divides (A.9) since \( 2\ell_1 \) and \( \ell_2 \) separately divides (A.9). Thus in either case we see that (A.9) is divisible by \( 2\ell_1 \ell_2 \) and hence by \( 2s \). This shows that (A.8) implies \( 2s|\chi(q) \) and we can express (A.3) as

\[
d(q) = (-1)^{Q \cdot P + 1} \sum_{s \in \mathbb{Z}, s | \ell_1 \ell_2} s \tilde{c}(\Delta(Q, P)/s^2).
\]

(A.10)

This is the relation given in (2.4).

References

[1] J. Maldacena, G. Moore and A. Strominger, “Counting BPS blackholes in toroidal type II string theory,” arXiv:hep-th/9903163.

[2] D. Shih, A. Strominger and X. Yin, “Counting dyons in \( N = 8 \) string theory,” JHEP 0606, 037 (2006) [arXiv:hep-th/0506151].

[3] B. Pioline, “BPS black hole degeneracies and minimal automorphic representations,” JHEP 0508, 071 (2005) [arXiv:hep-th/0506228].

[4] D. Shih and X. Yin, “Exact Black Hole Degeneracies and the Topological String,” JHEP 0604, 034 (2006) [arXiv:hep-th/0508174].

[5] A. Sen, “N=8 Dyon Partition Function and Walls of Marginal Stability,” JHEP 0807, 118 (2008) [arXiv:0803.1014 [hep-th]].

[6] A. Sen, “U-duality Invariant Dyon Spectrum in type II on \( T^6 \),” JHEP 0808, 037 (2008) [arXiv:0804.0651 [hep-th]].

[7] A. Sen, “Entropy Function and AdS(2)/CFT(1) Correspondence,” JHEP 0811, 075 (2008) [arXiv:0805.0095 [hep-th]].

[8] A. Sen, “Quantum Entropy Function from AdS(2)/CFT(1) Correspondence,” arXiv:0809.3304 [hep-th].
[9] A. Sen, “Arithmetic of Quantum Entropy Function,” arXiv:0903.1477 [hep-th].

[10] A. Dabholkar, M. Guica, S. Murthy and S. Nampuri, “No entropy enigmas for N=4 dyons,” arXiv:0903.2481 [hep-th].

[11] A. Gregori, E. Kiritsis, C. Kounnas, N. A. Obers, P. M. Petropoulos and B. Pioline, “R**2 corrections and non-perturbative dualities of N = 4 string ground states,” Nucl. Phys. B 510, 423 (1998) [arXiv:hep-th/9708062].

[12] E. Kiritsis, “Introduction to non-perturbative string theory,” arXiv:hep-th/9708130.

[13] E. Cremmer and B. Julia, “The SO(8) Supergravity,” Nucl. Phys. B 159, 141 (1979).

[14] R. Kallosh and B. Kol, “E(7) Symmetric Area of the Black Hole Horizon,” Phys. Rev. D 53, 5344 (1996) [arXiv:hep-th/9602014].

[15] N. Banerjee, D. P. Jatkar and A. Sen, “Asymptotic Expansion of the N=4 Dyon Degeneracy,” JHEP 0905, 121 (2009) [arXiv:0810.3472 [hep-th]].

[16] S. Murthy and B. Pioline, “A Farey tale for N=4 dyons,” arXiv:0904.4253 [hep-th].

[17] A. Sen and C. Vafa, “Dual pairs of type II string compactification,” Nucl. Phys. B 455, 165 (1995) [arXiv:hep-th/9508064].

[18] G. W. Moore, “Attractors and arithmetic,” arXiv:hep-th/9807056.

[19] G. W. Moore, “Arithmetic and attractors,” arXiv:hep-th/9807087.

[20] G. W. Moore, “Les Houches lectures on strings and arithmetic,” arXiv:hep-th/0401049.

[21] S. Banerjee and A. Sen, “S-duality Action on Discrete T-duality Invariants,” JHEP 0804, 012 (2008) [arXiv:0801.0149 [hep-th]].

[22] S. Banerjee and A. Sen, “Duality Orbits, Dyon Spectrum and Gauge Theory Limit of Heterotic String Theory on T^6”, JHEP 0803, 022 (2008) [arXiv:0712.0043 [hep-th]].

[23] N. Banerjee, S. Banerjee, R. Gupta, I. Mandal and A. Sen, “Supersymmetry, Localization and Quantum Entropy Function,” arXiv:0905.2686 [hep-th].

[24] L. Borsten, D. Dahanayake, M. J. Duff and W. Rubens, “Black holes admitting a Freudenthal dual,” arXiv:0903.5517 [hep-th].

12