Extended Superconformal Symmetry ,
Freudenthal Triple Systems and
Gauged WZW Models

Murat G"unaydin †
Physics Department, 104 Davey Lab.
Penn State University, University Park, PA 16802
March 28, 2022

Abstract: We review the construction of extended ( $N = 2$ and $N = 4$ )
superconformal algebras over triple systems and the gauged WZW models
invariant under them. The $N = 2$ superconformal algebras (SCA) realized
over Freudenthal triple systems (FTS) admit extension to “maximal” $N = 4$
SCA’s with $SU(2) \times SU(2) \times U(1)$ symmetry. A detailed study of the
construction and classification of $N = 2$ and $N = 4$ SCA’s over Freudenthal
triple systems is given. We conclude with a study and classification of
gauged WZW models with $N = 4$ superconformal symmetry.

1 Introduction

It is a singular honor for me to give a talk in the First of the Gürsey Memorial
Conferences which are to be held biannually. I regard Feza Gürsey as my
mentor and a role model I try to emulate. He has had a great influence on
my style of physics.

*Invited talk presented at Gürsey Memorial Conference I, Istanbul, Türkiye (June
6-10, 1994)
†E-mail address: murat@phys.psu.edu
In this talk I will discuss extended superconformal algebras and their connection with triple systems, in particular the Freudenthal Triple Systems. I will also discuss a class of Lagrangian field theories, namely the gauged WZW models, that are invariant under extended superconformal groups. I learned about Freudenthal triple systems more than twenty years ago when I was working with Feza on the questions of physical implications of extending the underlying number system of quantum mechanics from complex numbers to octonions and what possible role exceptional groups may play in physics.

Infinite conformal algebra in two dimensions and its supersymmetric extensions have been studied extensively in recent years. They underlie string and superstring theories as their local gauge symmetries. The classical vacua of string theories are described by conformal field theories. For example, the heterotic string vacua with \( N = 1 \) space-time supersymmetry in four dimensions are described by "internal" \( N = 2 \) superconformal field theories with central charge \( c = 9 \) \([2, 3, 4, 5, 6, 7]\). The \( N = 2 \) space-time supersymmetric vacua of the heterotic string are described by an internal superconformal field theory with four supersymmetries \([8]\). The extended superconformal algebras have important applications to integrable systems \([9]\) and to topological field theories in two dimensions as well \([10]\). The conformal and superconformal algebras have been studied in detail via the coset space method of Goddard, Kent and Olive (GKO) \([11]\) which is a generalization of Sugawara-Sommerfield construction of the Virasoro algebra in terms of bilinears of the generators of a current algebra \([12]\). In the first part of my talk I will review a novel ternary algebraic approach to the construction and study of extended superconformal algebras. The construction of \( N=2 \) SCA’s over Jordan Triple Systems (JTS) was given in \([13]\). This construction was generalized to the realization of \( N=2 \) SCA’s over more general Kantor triple systems (KTS) in \([14, 15]\). The coset spaces associated with the KTS’s are, in general, not symmetric spaces. For a particular subclass of Kantor triple systems, namely the Freudenthal triple systems (FTS) the \( N=2 \) SCA’s admit extensions to “maximal” \( N = 4 \) SCA’s with the gauge group \( SU(2) \times SU(2) \times U(1) \) \([16]\). The construction of \( N = 2 \) and \( N = 4 \) superconformal algebras over FTS’s and their classification will be discussed in detail following \([14, 15, 17, 18]\). The last part of my talk is devoted to the study of gauged WZW models that are invariant under \( N = 4 \) superconformal symmetry \([19]\).
2 Construction of Extended Superconformal Algebras over Triple Systems

The realization of $N = 2$ superconformal algebras over hermitian Jordan triple systems given in [13] is equivalent to their realization over hermitian symmetric spaces a la Kazama and Suzuki [5] which can be compact or non-compact. The connection between hermitian symmetric spaces and hermitian Jordan triple systems arises as follows. If $G/H$ is a hermitian symmetric space then the Lie algebra $g$ of $G$ can be given a 3-graded decomposition with respect to the Lie algebra $g^0$ of $H$:

$$g = g^{-1} \oplus g^0 \oplus g^+$$

(2.1)

where $\oplus$ denotes vector space direct sum and $g^0$ is a subalgebra of maximal rank. We have the formal commutation relations of the elements of various grade subspaces

$$[g^m, g^n] \subseteq g^{m+n} ; m, n = -1, 0, 1$$

(2.2)

where $g^{m+n} = 0$ if $|m+n| > 1$. The Tits-Kantor-Koecher (TKK) construction [20] of the Lie algebra $g$ establishes a mapping between the grade +1 subspace of $g$ and the underlying Hermitian JTS $V$.

The exceptional Lie algebras $G_2, F_4$ and $E_8$ do not admit a TKK type construction. A generalization of the TKK construction to more general triple systems was given by Kantor [21]. All finite dimensional simple Lie algebras admit a construction over these generalized triple systems. The Kantor’s construction of Lie algebras was generalized to a unified construction of Lie algebras and Lie superalgebras in [22] which we shall review briefly restricting our discussion to Lie algebras only.

This more general construction starts from the fact that every simple Lie algebra $g$ with the exception of $SU(2)$ admits a 5-grading (Kantor structure) with respect to some subalgebra $g^0$ of maximal rank [21, 22]:

$$g = g^{-2} \oplus g^{-1} \oplus g^0 \oplus g^+ \oplus g^{+2}$$

(2.3)

One labels the elements of $g^+1$ subspace with the elements of a vector space $V$ [21, 22]:

$$U_a \in g^+1 \iff a \in V$$

(2.4)
g admits a conjugation under which the grade $+m$ subspace gets mapped into the grade $-m$ subspace, which allows one to label the elements of the grade $-1$ subspace by the elements of $V$ as well:

$$U^a \equiv U_a^1 \in g^{-1} \iff U_a \in g^{+1}$$  \hspace{1cm} (2.5)

One defines the commutators of $U_a$ and $U_b$ as

$$[U_a, U_b] = S^b_a \in g^0$$
$$[U_a, U_b] = K_{ab} \in g^{+2}$$
$$[U^a, U^b] = K^{ab} \in g^{-2}$$
$$[S^b_a, U_c] = U_{(abc)} \in g^{+1}$$ \hspace{1cm} (2.6)

where $(abc)$ is the triple product under which the elements of $V$ close. The remaining non-vanishing commutators of $g$ can all be expressed in terms of the triple product $(abc)$:

$$[S^b_a, U^c] = -U^{(bac)}$$
$$[K_{ab}, U^c] = U_{(acb)} - U_{(bca)}$$
$$[K^{ab}, U_c] = -U^{(bca)} + U^{(acb)}$$
$$[S^b_a, S^d_c] = S^{d}_{(abc)} - S^{d}_{(bad)}$$
$$[S^b_a, K_{cd}] = K_{(abc)d} + K_{(abd)c}$$
$$[S^b_a, K^{cd}] = -K^{(bac)d} - K^{(bad)c}$$
$$[K_{ab}, K^{cd}] = S^{d}_{(abc)} - S^{d}_{(bca)} - S^{c}_{(adb)} + S^{c}_{(bda)}$$ \hspace{1cm} (2.7)

The Jacobi identities of $g$ follow from the defining identities of a Kantor triple system (KTS) [21, 22]

$$(ab(cd)) - (cd(ab)) - (a(dcb)x) + ((cda)bx) = 0$$  \hspace{1cm} (2.8)

$$\{(ax(cbd)) - ((cbd)x) + (ab(cx)d) + (c(bax)d)\} - \{c \leftrightarrow d\} = 0$$ \hspace{1cm} (2.9)

In general a given simple Lie algebra can be constructed in several different ways by the above method corresponding to different choices of the subalgebra $g^0$ and different KTS’s. Note that the second defining identity of a KTS is trivially satisfied by a JTS for which the grade $\pm 2$ subspaces vanish.

Consider now the affine Lie algebra $\hat{g}$ defined by the Lie algebra $g$ constructed over a KTS $V$. The commutation relations $\hat{g}$ can be written in the
form of operator products as follows:

\[
U_a(z)U_b(w) = \frac{k_b}{(z-w)^2} + \frac{\Sigma_{bd}(w)}{(z-w)} + \cdots
\]

\[
U^a(z)U_b(w) = \frac{k_b}{(z-w)^2} - \frac{\Sigma_{bd}(w)}{(z-w)} + \cdots
\]

\[
U_a(z)U_b(w) = K_{ab}(w) + \cdots
\]

\[
S_a(z)U_c(w) = \frac{U_{(abc)}(w)}{(z-w)} + \cdots
\]

\[
S_a(z)U^c(w) = -\frac{\Omega_{(bac)}(w)}{(z-w)} + \cdots
\]

\[
S_a(z)S^d(w) = \frac{k_{bd}}{(z-w)^2} - \frac{1}{(z-w)}(S_{(abc)} - S_{(bad)})(w) + \cdots
\]

\[
S_a(z)K_{cd}(w) = \frac{1}{(z-w)}(K_{(abc)d} + K_{(c)}(abd))(w) + \cdots
\]

\[
S_a(z)K^{cd}(w) = -\frac{1}{(z-w)}(K_{(bac)d} + K_{(b)}(cad))(w) + \cdots
\]

\[
K_{ab}(z)K^{cd}(w) = \frac{k_{bd}}{(z-w)^2} + \frac{1}{(z-w)}(S_{(abc)} - S_{(bad)} - S_{(abc)} + S_{(bad)})(w) + \cdots
\]

\[
K_{ab}(z)U^c(w) = \frac{1}{(z-w)}(U_{(abc)} - U_{(bca)})(w) + \cdots
\]

\[
K^{ab}(z)U_c(w) = \frac{1}{(z-w)}(U_{(abc)} - U_{(bca)})(w) + \cdots
\]

(2.10)

where \(\Sigma_{ab}^{cd}\) are the structure constants of the underlying KTS \(V\):

\[
U_{(abc)} = \Sigma_{ac}^{bd}U_d
\]

(2.11)

and the tensor \(\Omega\) is defined as

\[
\Omega_{ab} = \Sigma_{ab}^{cd} - \Sigma_{ab}^{dc}
\]

(2.12)

We choose a basis \(V\) such that

\[
\Omega_{ab} = \frac{2d}{D}(\hat{\gamma} - \hat{s})\delta_b^c
\]

(2.13)

where \(\hat{\gamma}\) and \(\hat{s}\) denote the dual coxeter numbers of the Lie algebra \(g\) and its subalgebra \(s\) generated by the elements of the grade \(\pm 2\) subspaces and their commutant \([g^{-2}, g^2]\), respectively. \(D\) is the dimension of the triple system \(V\) and \(d\) is the dimension of the grade +2 subspace.

Furthermore one introduces fermion fields \(\psi^a(\psi^a)\) and \(\psi_{ab}(\psi_{ab})\) corresponding to the grade +1(−1) and +2(−2) subspaces of \(g\), respectively. They are normalized such that

\[
\psi_a(z)\psi^b(w) = \frac{\delta^a_b}{(z-w)} + \cdots
\]

\[
\psi_{ab}(z)\psi_{cd}(w) = \frac{\Omega_{ab}^{cd}}{(z-w)} + \cdots
\]

(2.14)
The operators of conformal dimension 3/2 defined by the following expressions:

\[ G(z) = \sqrt{\frac{2}{k + \bar{g}}} \left\{ U^a \psi^a + \frac{1}{2(\bar{g} - s)} K_{ab} \psi^a \psi^b - \frac{1}{2} \psi^a \psi^b \right\} (z) \]
\[ \bar{G}(z) = \sqrt{\frac{2}{k + \bar{g}}} \left\{ U^a \psi_a - \frac{1}{2(\bar{g} - s)} K^{ab} \psi_{ab} + \frac{1}{2} \psi^a \psi^b \right\} (z) \] (2.15)

generate an \( N = 2 \) superconformal algebra [14, 15] with the Virasoro generator

\[ T(z) = \frac{1}{k + \bar{g}} \left\{ \frac{1}{2} (U_a U^a + U^a U_a) + \frac{1}{4(\bar{g} - s)} (K_{ab} K^{ba} + K^{ab} K_{ba}) \right\} + \frac{k}{2} \left( \partial \psi^a \psi_a + \partial \overline{\psi}_b \psi^b \right) + \frac{1}{2} \Omega_{ab}^c \partial \psi^a \psi^b \psi^c \] + \frac{1}{4(\bar{g} - s)} S_b^a \psi^a \psi^b + \frac{1}{2(\bar{g} - s)} S_c^b \psi^b \psi^a \psi^c \psi^d \] (2.16)

and the \( U(1) \) current

\[ J(z) = \frac{1}{k + \bar{g}} \left\{ S_a^a - \frac{1}{2(\bar{g} - s)} \Omega_{ca}^b \psi_a^b + k \psi^a \psi_a \right\} + \frac{1}{2} \Omega_{ab}^c \partial \psi^a \psi^b \psi^c \] (2.17)

Its central charge turns out to be:

\[ c = \frac{3}{k + \bar{g}} \left\{ kD + \frac{k}{4(\bar{g} - s)} \Omega_{ab}^{cd} \right\} + \frac{1}{2} \Omega_{ab}^{cd} \] (2.18)

This realization of \( N = 2 \) SCA’s is equivalent to their realization over the coset spaces \( G/H \) where \( G \) and \( H \) are the groups generated by \( g \) and \( g^0 \), respectively. These spaces are, in general, not symmetric spaces. For further details and a complete classification of the \( N = 2 \) SCA’s constructed over KTS’s we refer to [14, 15].

### 3 Construction of \( N = 2 \) Subalgebras of Maximal \( N = 4 \) Superconformal Algebras over Freudenthal Triple Systems

For a very special subclass of Kantor triple systems, namely the Freudenthal triple systems (FTS), the \( N = 2 \) superconformal algebras as constructed in the previous section can be extended to the maximal \( N = 4 \) superconformal
algebras \[14, 15\]. Freudenthal introduced these triple systems in his study of the geometries associated with exceptional groups \[23\]. Kantor and Skopets classified FTS’s and showed that there is a one-to-one correspondence between simple Lie algebras and simple FTS’s with a non-degenerate bilinear form \[24\]. The Freudenthal triple product \((abc)\) can be written in the form:

\[(abc) = \{abc\} - (c, b)a - (c, a)b - (a, b)c\]  

(3.1)

where \(\{abc\}\) is completely symmetric in its arguments and \((, )\) is a skew-symmetric bilinear form defined over the FTS. For a simple Lie algebra \(g\) constructed over a FTS the grade \(\pm 2\) subspaces become one dimensional as a consequence of the identity

\[(abc) - (cba) = 2(a, c)b\]  

(3.2)

The realization of \(N = 2\) superconformal algebra over it corresponds to the coset space \(G/H_0 \times U(1)\) where \(H_0\) is such that \(G/H_0 \times SU(2)\) is the unique quaternionic symmetric space of \(G\).

Let \(g, h_0\) be the Lie algebras of \(G\) and \(H_0\) listed above, respectively. The 5-graded structure of \(g\)

\[g = g^{-2} \oplus g^{-1} \oplus g^0 \oplus g^{+1} \oplus g^{+2}\]  

(3.3)

is such that \(g^0 = h_0 \oplus K_3\) where \(K_3\) is the generator of the \(U(1)\) factor and the elements \(K_{ab}\) and \(K^{ab}\) of grade \(\pm 2\) subspaces can be written in the form

\[K_{ab} = \Omega_{ab}K_+\]

\[K^{ab} = \Omega^{ab}K^+\]

where \(\Omega_{ab}\) is a symplectic form defined over the FTS. The tensor \(\Omega^{ab}\) is the inverse of \(\Omega_{ab}\) and

\[\Omega_{ab}\Omega^{bc} = \delta^c_a\]

\[\Omega^i_{ab} = \Omega^{ba} = -\Omega^{ab}\]  

(3.4)

The elements \(K_+\) and \(K^+\) are hermitian conjugates of each other and can be written as

\[K_+ = K_1 + iK_2 \in g^{+2}\]

\[K^+ \equiv K_- = K_1 - iK_2 \in g^{-2}\]  

(3.5)

\footnote{In a given basis \(e_a\) of the FTS one can set \(\Omega_{ab} = (e_a, e_b)\).}
There is a universal relation between the dual Coxeter number $\hat{g}$ of $G$ and the dimension $D$ of the underlying FTS:

$$D = 2(\hat{g} - 2) \quad (3.6)$$

The generators $K_-, K_+$ and $K_3$ form an $SU(2)$ subalgebra of $g$.

$$[K_+, K_-] = 2K_3$$
$$[K_3, K_{\pm}] = \pm K_{\pm} \quad (3.7)$$

The commutation relations of the $U$’s can now be written in the form

$$[U_a, U_b] = \Omega_{ab} K_+$$
$$[U^a, U^b] = \Omega^{ab} K_-$$
$$[U_a, U^b] = S^b_a \quad (3.8)$$

$\Omega_{ab}$ is an invariant tensor of $H_0$ and $S^b_a$ are the generators of the subgroup $H_0 \times U(1)$. The trace component of $S^b_a$ gives the $U(1)$ generator

$$K_3 = \frac{1}{2(\hat{g} - 2)} S^a_a \quad (3.9)$$

Hence we have the decomposition

$$S^b_a = H^b_a + \delta^b_a K_3 \quad (3.10)$$

where $H^b_a = S^b_a - \frac{1}{D} \delta^b_a S^c_c$ are the generators of the subgroup $H_0$. Note that $H^b_a$ commutes with $K_3$, $K_+$ and $K_-$. The other non-vanishing commutators of $g$ are

$$[K_+, U^a] = \Omega^{ab} U_b$$
$$[K_-, U_a] = \Omega_{ab} U^b$$
$$[K_3, U^a] = -\frac{1}{2} U^a$$
$$[K_3, U_a] = \frac{1}{2} U_a$$
$$[S^b_a, U_c] = \Sigma_{ac}^b U_d$$
$$[S^b_a, U^c] = -\Sigma_{ad}^b U^d$$
$$[S^b_a, S^d_c] = \Sigma_{ae}^b \Sigma_{de}^b \Sigma_{ce}$$

$$\quad (3.11)$$
where $\Sigma_{ab}^{cd}$ are the structure constants of the FTS which in our normalization satisfy

$$
\Sigma_{ab}^{ac} = (\hat{g} - 2)\delta_b^c
$$

$$
\Sigma_{ab}^{bc} = (\hat{g} - 1)\delta_a^c
$$

$$
\Sigma_{ab}^{cd} - \Sigma_{ab}^{dc} = \Omega_{ab}\Omega^{cd}
$$

(3.12)

The complex Fermi fields associated with the grade $\pm 2$ subspaces can be represented as

$$
\psi_{ab}(z) = \Omega_{ab}\psi_+(z)
$$

$$
\psi^{ab}(z) = \Omega^{ab}\psi^+(z)
$$

(3.13)

where $\psi_+(z)$ and $\psi^+(z)$ satisfy [14, 15]:

$$
\psi_+(z)\psi^+(w) = \frac{1}{(z-w)} + \cdots
$$

(3.14)

The supersymmetry generators of the $N=2$ superconformal algebra simplify

$$
G(z) = \sqrt{\frac{2}{k+\hat{g}}} \{ U_a\psi^a + K_+\psi^+ - \frac{1}{2}\Omega_{ab}\psi^a\psi^b\psi_+ \}(z)
$$

(3.15)

$$
\bar{G}(z) = \sqrt{\frac{2}{k+\hat{g}}} \{ U^a\psi_a + K^+\psi_+ - \frac{1}{2}\Omega^{ab}\psi_a\psi_b\psi^+ \}(z)
$$

(3.16)

The Virasoro generator takes the form

$$
T(z) = \frac{1}{k+\hat{g}} \left\{ \frac{1}{2}(U_aU^a + U^aU_a) + \frac{1}{2}(K_+K^+ + K^+K_a) ight. \\
- \frac{k+1}{2}(\psi_a\partial\psi^a + \psi^a\partial\psi_a) - \frac{1}{2}(k+\hat{g}-2)(\psi_+\partial\psi^+ + \psi^+\partial\psi_+) \\
+ H_a^b\psi^a\psi_b + K_3(\psi^a\psi_a + 2\psi^+\psi_+) + \psi_+\psi^+\psi^a\psi_a + \frac{1}{2}\Omega_{ab}\psi^a\psi^b\Omega^{cd}\psi_c\psi_d \right\}(z)
$$

(3.17)

and the $U(1)$ current is given by

$$
J(z) = \frac{1}{k+\hat{g}} \left\{ 2(\hat{g} - 1)K_3 + (k+1)\psi^a\psi_a + (k - \hat{g} + 2)\psi^+\psi_+ \right\}(z)
$$

(3.18)

The central charge of the $N=2$ SCA defined by a FTS is

$$
c = \frac{6(k+1)(\hat{g} - 1)}{(k+\hat{g})} - 3
$$

(3.19)
As stated earlier the above realization of $N = 2$ SCA’s corresponds to the coset $G/H_0 \times U(1)$ where the $U(1)$ generator $K_3$ determines the 5-graded structure of $g$

$$[2K_3, g^m] = mg^m$$

(3.20)

where $g^m$ denotes the subspace of grade $m$ with $m = 0, \pm 1, \pm 2$. The Lie algebra $g$ can also be given a 5-graded structure with respect to $K_1$ as well as $K_2$. Therefore, one can realize the $N = 2$ SCA equivalently over the coset $G/H_0 \times U(1)'$ or the coset $G/H_0 \times U(1)''$, where the generators of $U(1)'$ and $U(1)''$ are $K_1$ and $K_2$, respectively. The generators belonging to grade $\pm 1$ and $\pm 2$ subspaces with respect to $K_1$ are

\[
U_a' = \frac{1}{\sqrt{2}}(U_a + \Omega_{ab} U^b) \\
U^{a'} = \frac{1}{\sqrt{2}}(U^a - \Omega^{ab} U_b) \\
K'_+ = i(K_2 + iK_3) \\
K'_- = -i(K_2 - iK_3)
\]

(3.21)

They satisfy

\[
[U'_a, U'_b] = \Omega_{ab} K'_+ \\
[U'^a, U'^b] = \Omega^{ab} K'_- \\
[K'_+, U'^a] = \Omega^{ab} U'_b \\
[K'_-, U'_a] = \Omega_{ab} U'^b
\]

(3.22)

Whereas the grade $\pm 1$ and $\pm 2$ subspaces with respect to $K_2$ are

\[
U''_a = \frac{1}{\sqrt{2}}(U_a + i\Omega_{ab} U^b) \\
U^{a''} = \frac{1}{\sqrt{2}}(U^a + i\Omega^{ab} U_b) \\
K''_+ = -i(K_3 + iK_1) \\
K''_- = i(K_3 - iK_1)
\]

(3.23)

with analogous commutation relations to (3.22).

For every simple Lie group $G$ (except for $SU(2)$) there exists a subgroup $H_0 \times U(1)$, unique up to automorphisms corresponding to $SU(2)$ rotations, such that its Lie algebra $g$ has a 5-graded structure with respect to the subalgebra $h_0 \oplus K$, where $K$ is the generator of the $U(1)$ subgroup that
determines the 5-grading. This follows from the fact that there is a one-to-one correspondence between simple Lie algebras and simple FTS’s with a non-degenerate bilinear form [24].

4 The Construction of Maximal $N = 4$ Superconformal Algebras

The $N = 2$ superconformal algebras constructed over FTS’s admit extensions to maximal $N = 4$ superconformal algebras of references [16]. To achieve this one needs to introduce a $N = 2$ “matter multiplet” and define two additional supersymmetry generators as well as adding the matter contributions to the first two supersymmetry generators. The required currents of the matter multiplet turn out to be the $U(1)$ current generated by $K_3$ that gives the 5-graded structure of the Lie algebra $g$, and an additional $U(1)$ current whose generator $K_0$ commutes with $g$ together with the associated fermions which we denote as a complex fermion $\chi^+$ and its conjugate $\chi^+$. Then the four supersymmetry generators of the $N = 4$ superconformal algebra can be written as

$$
G^+ \equiv \frac{1}{\sqrt{2}}(G_1 + iG_2) = \sqrt{\frac{2}{k+g}} \left\{ U^a \psi^a + K^+ + K_3 \right\} + i Z \chi^+
$$

$$
G^- \equiv \frac{1}{\sqrt{2}}(G_1 - iG_2) = \sqrt{\frac{2}{k+g}} \left\{ U^a \psi^a + K^+ + K_3 \right\} - i Z \chi^+
$$

$$
G^{+K} \equiv \frac{1}{\sqrt{2}}(G_3 + iG_4) = \sqrt{\frac{2}{k+g}} \left\{ \Omega^{ab} U^a \psi^b + K_+ \right\} + i Z \psi^+
$$

$$
G^{-K} \equiv \frac{1}{\sqrt{2}}(G_3 - iG_4) = \sqrt{\frac{2}{k+g}} \left\{ \Omega^{ab} U^a \psi^b + K_+ \right\} - i Z \psi^+
$$

The Virasoro generator of the maximal $N = 4$ superconformal algebra $\mathcal{A}_g$ is given by

$$
T(z) = \frac{1}{2} \left[ Z^2 - (\chi_+ \partial \chi^+ + \chi^+ \partial \chi_+) - (\psi_+ \partial \psi^+ + \psi^+ \partial \psi_+) \right](z) + \frac{1}{k+g} \left\{ \frac{1}{2} (U^a U^a + U^a U^a) + \frac{1}{2} (K_+ K_+ + K^+ K^+) + K_3^2 
- \frac{k+1}{2} (\psi_a \partial \psi^a + \psi^a \partial \psi_a) + H_b \psi^a \psi_b + \frac{1}{4} \Omega^{ab} \psi^a \psi^b \Omega^{cd} \psi^c \psi_d \right\}(z)
$$
The generators of the two $SU(2)$ currents take the form

$$
V_3^+(z) = K_3(z) + \frac{1}{2}(\psi_+\psi^+ + \chi_+\chi^+)(z)
$$
$$
V_3^-(z) = K_3(z) + \frac{1}{2}(\psi_+\psi^+ + \chi_+\chi^+)(z)
$$
$$
V_4^+(z) = (V_1^+ + iV_2^+)(z) = (K_+ - \psi_+\chi_+)(z)
$$
$$
V_4^-(z) = (V_1^- + iV_2^-)(z) = (K_+ + \psi_+\chi_+)(z)
$$
$$
V_5^+(z) = \frac{1}{2}(\psi^a\psi_a + \psi^+\psi_+ + \chi^+\chi_+)(z)
$$
$$
V_5^-(z) = (V_1^- + iV_2^-)(z) = (\psi^+\chi_+ - \frac{1}{2}\Omega_{ab}\psi^a\psi^b)(z)
$$
$$
V_6^+(z) = (V_1^- - iV_2^-)(z) = (\chi^+\psi_+ - \frac{1}{2}\Omega_{ab}\psi_a\psi_b)(z)
$$

The U(1) current of the $N = 4$ SCA is $Z(z)$ and the four dimension $\frac{1}{2}$ generators are simply the fermion fields $\psi_+(z), \psi^+(z), \chi_+(z)$ and $\chi^+(z)$. One finds that the levels of the two $SU(2)$ currents are $k^+ = k+1$ and $k^- = \tilde{g} - 1$. The central charge of the $N = 4$ SCA turns out to be $c = \frac{6k+k^-}{k+k^+}$. The above realization of the $N = 4$ SCA corresponds to the coset space $G \times U(1)/H$.

By decoupling the four dimension $\frac{1}{2}$ generators and the $U(1)$ current $Z(z)$ one obtains a non-linear $N = 4$ SCA [25, 26] a la Bershadsky and Knizhnik [27, 28]. The realization given above for the $N = 4$ SCA leads to very simple expressions for the supersymmetry generators of the non-linear $N = 4$ algebra

$$
\tilde{G}^+ \equiv \frac{1}{\sqrt{2}}(\tilde{G}_1 + i\tilde{G}_2) = \sqrt{\frac{2}{k+\tilde{g}}} U_a\psi^a
$$
$$
\tilde{G}^- \equiv \frac{1}{\sqrt{2}}(\tilde{G}_1 - i\tilde{G}_2) = \sqrt{\frac{2}{k+\tilde{g}}} U^a\psi_a
$$
$$
\tilde{G}^{+K} \equiv \frac{1}{\sqrt{2}}(\tilde{G}_3 + i\tilde{G}_4) = \sqrt{\frac{2}{k+\tilde{g}}} \Omega_{ab}U_a\psi_b
$$
$$
\tilde{G}^{-K} \equiv \frac{1}{\sqrt{2}}(\tilde{G}_3 - i\tilde{G}_4) = \sqrt{\frac{2}{k+\tilde{g}}} \Omega_{ba}U^a\psi_b
$$

with the central charge $\tilde{c} = c - 3$.

It is clear from the above expressions for the generators that the non-linear $N = 4$ SCA is realized over the symmetric space

$$
G/H \times SU(2)
$$

which is the unique quaternionic symmetric space associated with $G$ [29].
5  \( N = 4 \) Supersymmetric Gauged WZW Models

So far we have been discussing the construction of extended superconformal algebras and study of their chiral rings using algebraic methods. In this section we shall study a certain class of Lagrangian field theories, namely supersymmetric gauged WZW models, that are invariant under extended superconformal groups. Gauged WZW models were studied in \([30, 31]\). The \( N = 1 \) supersymmetric ordinary WZW models were studied in \([32, 33, 34]\) and their gauged versions in \([33]\). \( N = 2 \) supersymmetric gauged WZW models were studied by Witten \([36]\). Witten’s results were extended to \( N = 4 \) gauged WZW models in \([19]\).

The WZW action at level \( k \) is given by

\[
I(g) = \frac{1}{8\pi} \int_{\Sigma} d^2\sigma \sqrt{h} h^{ij} Tr(g^{-1} \partial_i g \cdot g^{-1} \partial_j g) - i\Gamma
\]  

with the WZ functional \([37]\) given by \([38]\)

\[
\Gamma = \frac{1}{12\pi} \int_M d^3\sigma \epsilon^{ijk} Tr(g^{-1} \partial_i g \cdot g^{-1} \partial_j g \cdot g^{-1} \partial_k g)
\]

\( M \) is a three manifold whose boundary is the Riemann surface \( \Sigma \) with metric \( h \). We choose the metric \( h_{zz} = h_{\bar{z}\bar{z}} = 1 \) and work with complex coordinates \( z \) and \( \bar{z} \). \( g \) represents the group element that maps \( \Sigma \) into the group \( G \). The supersymmetric WZW action \( I(g, \Psi) \) is obtained by adding to \( I(g) \) the free action of Weyl fermions \( \Psi_L \) and \( \Psi_R \) in the complexification of the adjoint representation of \( G \) \([36]\)

\[
I(g, \Psi) = I(g) + \frac{i}{4\pi} \int d^2z Tr(\Psi_L \partial_z \Psi_L + \Psi_R \partial_z \Psi_R)
\]

It is invariant under the supersymmetry transformations

\[
\delta g = i\epsilon_- g \Psi_L + i\epsilon_+ \Psi_R g \\
\delta \Psi_L = \epsilon_- (g^{-1} \partial_z g - i\Psi_L^2) \\
\delta \Psi_R = \epsilon_+ (\partial_z gg^{-1} + i\Psi_R^2)
\]  

\(^2\)We shall consider only models that have equal number of supersymmetries in both the left and the right moving sectors.
Gauging any diagonal subgroup $H$ of the $G_L \times G_R$ symmetry of the WZW model leads to an anomaly free theory. The gauge invariant action, which does not involve any kinetic energy term for the gauge fields, can be written as:

$$I(g, A) = I(g) + \frac{1}{2\pi} \int_\Sigma d^2z Tr(A \bar{z} g^{-1} \partial_z g - A_z \partial^2 g g^{-1} + A_z g^{-1} A \bar{z} g - A \bar{z} A_z)$$

where $A_z, A \bar{z}$ are the matrix valued gauge fields belonging to the subgroup $H$. It is invariant under the gauge transformations [36]:

$$\delta g = [u, g]$$
$$\delta A_i = -D_i u = -\partial u - [A_i, u]$$

Denoting as $\mathcal{G}$ and $\mathcal{H}$ the complexifications the Lie algebras of $G$ and $H$ one has the orthogonal decomposition

$$\mathcal{G} = \mathcal{H} \oplus \mathcal{T}$$

where $\mathcal{T}$ is the orthocomplement of $\mathcal{H}$. To supersymmetrize the WZW model with the gauged subgroup $H$ one introduces Weyl fermions with values in $\mathcal{T}$ minimally coupled to the gauge fields and otherwise free

$$I(g, A, \Psi) = I(g, A) + \frac{i}{4\pi} \int d^2z Tr(\Psi_L D_z \Psi_L + \Psi_R D_z \Psi_R)$$

This action is invariant under the supersymmetry transformation laws:

$$\delta g = i\epsilon_- g \Psi_L + i\epsilon_+ \Psi_R g$$
$$\delta \Psi_L = \epsilon_-(1 - \Pi)(g^{-1} D_z g - i\Psi_L^2)$$
$$\delta \Psi_R = \epsilon_+(1 - \Pi)(D_z g g^{-1} + i\Psi_R^2)$$
$$\delta A = 0$$

where $\Pi$ is the orthogonal projection of $\mathcal{G}$ onto $\mathcal{H}$. As shown by Witten if the gauge group $H$ is such that the coset space $G/H$ is Kahler the above action has $N = 2$ supersymmetry [36]. For $G/H$ Kahler the subspace $\mathcal{T}$ can be decomposed as

$$\mathcal{T} = \mathcal{T}_+ \oplus \mathcal{T}_-$$
where $\mathcal{T}_+$ and $\mathcal{T}_-$ are in complex conjugate representations of $H$. The action can then be written in the form \cite{38}

$$I(g, \Psi, A) = I(g, A) + \frac{i}{2\pi} \int d^2z Tr(\beta_L D_z \alpha_L + \beta_R D_z \alpha_R)$$  \hspace{1cm} (5.11)$$

where

$$\begin{align*}
\alpha_L &= \Pi_+ \Psi_L \\
\beta_L &= \Pi_- \Psi_L \\
\alpha_R &= \Pi_+ \Psi_R \\
\beta_R &= \Pi_- \Psi_R
\end{align*}$$  \hspace{1cm} (5.12)$$

with $\Pi_+$ and $\Pi_-$ representing the projectors onto the subspaces $\mathcal{T}_+$ and $\mathcal{T}_-$. Denoting the chiral and anti-chiral supersymmetry generators in left and right moving sectors as $G_L, \bar{G}_L$ and $G_R, \bar{G}_R$, respectively, one finds \cite{38} that they satisfy the $N = 2$ supersymmetry algebra

$$\begin{align*}
\{G_L, \bar{G}_L\} &= -i D_z \\
\{G_R, \bar{G}_R\} &= -i D_z
\end{align*}$$  \hspace{1cm} (5.13)$$

The gauged WZW models are known to be conformally invariant. Hence the $N = 2$ global supersymmetry of these theories implies $N = 2$ superconformal invariance.

Since the existence of a second supersymmetry in a supersymmetric gauged WZW model is guaranteed by the Kählerian property of the coset space $G/H$, i.e when it admits a complex structure, to have $N = 4$ supersymmetry one needs coset spaces with three complex structures which anti-commute with each other and form a closed algebra. Therefore one expects supersymmetric WZW models based on the groups $G \times U(1)$ of the previous section with a gauged subgroup $H$ such that $G/H \times SU(2)$ is a quaternionic symmetric space to actually have $N = 4$ supersymmetry. Let us show that this is indeed the case \cite{14}.

We shall designate the generators of $G \times U(1)$ as we did in previous sections, i.e $K_0, K_1, K_2, K_3, U_a, U^a$ and $H^a_a$, where $K_0$ is the generator of the additional $U(1)$ factor normalized such that

$$Tr K_0^2 = Tr K_1^2 = Tr K_2^2 = Tr K_3^2$$  \hspace{1cm} (5.14)$$
The fermions associated with the grade ±1 subspaces of the Lie algebra of $G$ will be denoted as $\psi_a, \psi^a$ as before and the fermions associated with $K_0$ and $K_i$ will be denoted as $\xi^0, \xi^i (i = 1, 2, 3)$. Thus the fermions in the coset $G \times U(1)/H$ can be represented as

$$\Psi = 2K_0\xi^0 + 2K_i\xi^i + U_a\psi^a + U^a\psi_a$$

(5.15)

for both the left and the right moving sectors. The coset $G \times U(1)/H$ can be given a Kahler decomposition such that

$$\Psi = \alpha + \beta$$

(5.16)

where

$$\alpha = U_a\psi^a + K_+(\xi^1 - i\xi^2) + (K_3 + iK_0)(\xi^3 - i\xi^0)$$

$$\beta = U^a\psi_a + K_-(\xi^1 + i\xi^2) + (K_3 - iK_0)(\xi^3 + i\xi^0)$$

(5.17)

The complex structure $C_3$ corresponding to this Kahler decomposition acts on $\Psi$ as

$$C_3\Psi = -i\alpha + i\beta$$

(5.18)

where the index 3 in $C_3$ signifies the fact in the subspace $G/H \times SU(2)$ its action corresponds to commutation with the generator $-iK_3$. One can similarly give a Kahler decomposition of the coset space $G \times U(1)/H$ which selects out $K_1$ or $K_2$. For the decomposition with respect to $K_1$ we have:

$$\Psi = \alpha' + \beta'$$

$$\alpha' = U_a\psi^a + (K_2 + iK_1)(\xi^2 - i\xi^3) + (K_3 + iK_0)(\xi^1 - i\xi^0)$$

$$\beta' = U^a\psi_a + (K_2 - iK_3)(\xi^2 + i\xi^3) + (K_1 - iK_0)(\xi^1 + i\xi^0)$$

(5.19)

Under the action of the corresponding complex structure $C_1$ we have

$$C_1\Psi = -i\alpha' + i\beta'$$

(5.20)

In the case of $K_2$ one finds

$$\Psi = \alpha'' + \beta''$$

$$\alpha'' = U_a\psi^a + (K_3 + iK_1)(\xi^3 - i\xi^1) + (K_2 + iK_0)(\xi^2 - i\xi^0)$$

$$\beta'' = U^a\psi_a + (K_3 - iK_1)(\xi^3 + i\xi^1) + (K_2 - iK_0)(\xi^2 + i\xi^0)$$

(5.21)
with the complex structure action

\[ C_2 \Psi = -i\alpha'' + i\beta'' \]  

(5.22)

The fermionic part of the action \( I(g, \Psi, A) \) can then be written in three different ways involving the pairs \((\alpha, \beta)\), \((\alpha', \beta')\) and \((\alpha'', \beta'')\).

\[
I(\Psi, A) = \frac{i}{4\pi} \int d^2 z Tr(\Psi_L D_z \Psi_L + \Psi_R D_z \Psi_R)
\]

\[
= \frac{i}{2\pi} \int d^2 z Tr(\beta_L D_z \alpha_L + \beta_R D_z \alpha_R)
\]

\[
= \frac{i}{2\pi} \int d^2 z Tr(\beta'_L D_z \alpha'_L + \beta'_R D_z \alpha'_R)
\]

\[
= \frac{i}{2\pi} \int d^2 z Tr(\beta''_L D_z \alpha''_L + \beta''_R D_z \alpha''_R)
\]

(5.23)

For each form of the action in terms of an \((\alpha, \beta)\) pair one can define a pair of supersymmetry transformations in each sector. Let us denote them as \((G, \bar{G})\), \((G', \bar{G}')\) and \((G'', \bar{G}'')\):

\[
(G, \bar{G}) \leftrightarrow (\alpha, \beta)
\]

\[
(G', \bar{G}') \leftrightarrow (\alpha', \beta')
\]

\[
(G'', \bar{G}'') \leftrightarrow (\alpha'', \beta'')
\]

(5.24)

Each pair of these operators in both sectors satisfy the \( N = 2 \) supersymmetry algebra given by the equations 5.13. However they are not all independent. The sum of each pair gives the manifest \( N = 1 \) supersymmetry generator of the model in both sectors, which we shall denote as \( G^0 \):

\[
G^0 = \frac{1}{\sqrt{2}}(G + \bar{G}) = \frac{1}{\sqrt{2}}(G' + \bar{G}') = \frac{1}{\sqrt{2}}(G'' + \bar{G}'')
\]

(5.25)

They satisfy

\[
\{G^0_L, G^0_L\} = -i D_z
\]

\[
\{G^0_R, G^0_R\} = -i D_{\bar{z}}
\]

(5.26)

One then has three additional supersymmetry generators (in each sector)

\[
G^3 = \frac{1}{i\sqrt{2}}(G - \bar{G})
\]
\[ G^1 = \frac{1}{i\sqrt{2}}(G' - G') \]
\[ G^2 = \frac{1}{i\sqrt{2}}(G'' - G'') \] \hspace{1cm} (5.27)

Each one of these three supersymmetry generators anticommute with \( G^0 \). To prove that \( G^\mu (\mu = 0, 1, 2, 3) \) form an \( N = 4 \) superalgebra we need to further show that the \( G^\nu (i = 1, 2, 3) \) anticommute with each other. To prove this we first note that the complex structures \( C_i \) obey the relation
\[ C_i C_j = C_k \] \hspace{1cm} (5.28)
where \( i, j, k \) are in cyclic permutations of \( (1, 2, 3) \) and the action is invariant under the replacement of \( \Psi \) by \( C_i \Psi \). If we start from an action with \( \Psi \) replaced by \( C_1 \Psi \) then the manifest \( N = 1 \) supersymmetry will be generated by \( G^1 \) and the second supersymmetry generated by the Kahler decomposition with respect to the complex structure \( C_3 \) will be \( G^2 \) since \( C_3 C_1 = C_2 \). Hence by the results of Witten on \( N = 2 \) supersymmetric gauged WZW models we have
\[ \{G^1, G^2\} = 0 \] \hspace{1cm} (5.29)
and by cyclic permutation we find
\[ \{G^2, G^3\} = \{G^3, G^1\} = 0 \] \hspace{1cm} (5.30)

Thus the four supersymmetry generators \( G^\mu \) satisfy the \( N = 4 \) supersymmetry algebra:
\[ \{G^\mu_L, G^\mu_L\} = -i\delta^{\mu\nu} D_z \]
\[ \{G^\mu_R, G^\mu_R\} = -i\delta^{\mu\nu} D_{\bar{z}} \]
\[ \mu, \nu, ... = 0, 1, 2, 3 \] \hspace{1cm} (5.31)

Since the gauged WZW models considered above are known to be conformally invariant we have thus proven that they are invariant under \( N = 4 \) superconformal transformations.

Acknowledgements: This talk was written up during my stay at the Institute for Theoretical Physics of the University of Helsinki. I would like to thank Antti Niemi and other members of the Institute for their kind hospitality.
References

[1] M. Green, J. H. Schwarz and E. Witten, "Superstring Theory", Cambridge Univ. Press, (1987, Cambridge).

[2] T. Banks, L. Dixon, D. Friedan and E. Martinec, *Nucl. Phys.* **B299** (1988) 613.

[3] W. Boucher, D. Friedan and A. Kent, *Phys. Lett.* **B172B** (1986) 316 ; P. DiVecchia, J.L. Petersen and M. Yu, *Phys. Lett.* **B172B** (1986) 211 ; A.B. Zamolodchikov and V.A. Fateev, *Zh. Eksp. Theor. Fiz.* **90** (1986) 1553 and *Sov. Phys. JETP* **6** (1985) 215 ; Z. Qiu, *Phys. Lett.* **188B** (1987) 207; D. Gepner and Z. Qiu, *Nucl. Phys.* **B285** (1987) 423 ;

[4] W. Lerche, C. Vafa and N.P. Warner, *Nucl. Phys.* **B324** (1989) 427.

[5] Y. Kazama and H. Suzuki, *Nucl. Phys.* B321 (1989) 232; *Phys. Lett.* **B216** (1989) 112;

[6] L.J. Dixon, J. Lykken and M.E. Peskin, *Nucl. Phys.* **B325** (1989) 329.

[7] I. Bars, *Nucl. Phys.* **B334** (1990) 125;

[8] N. Seiberg,*Nucl. Phys.* **B303** (1988) 286.

[9] See P. Fendley, W. Lerche, S. D. Mathur and N.P. Warner, "N=2 Supersymmetric Integrable Models from Affine Toda Theories", Preprint CTP1865 (CALT-68-1631; HUTP-90/A036) and the references therein.

[10] E. Witten, *Nucl. Phys.* **B340** (1990) 281 ; E. Witten, *Comm. Math. Phys.* **118** (1988) 411 ; C. Vafa, Harvard Preprint HUTP-90/A064 ; J. Distler and P. Nelson, *Phys. Rev. Lett.* **66** (1991) 1955 ; R. Dijkgraaf, H. Verlinde and E. Verlinde, Preprint PUPT-1217 (IASSNS-HEP-90/80).

[11] P. Goddard, A. Kent and D. Olive, *Phys. Lett.* **152B** (1985) 88 ; *Comm. Math. Phys.* **103** (1986)105 ; P. Goddard, W. Nahm and D. Olive, *Phys. Lett.* **160B** (1985) 111.

[12] H. Sugawara, *Phys. Rev.* **170** (1968) 1659 ; C. Sommerfield, *Phys. Rev.* **176** (1968) 2019.
[13] M. Günyadin, *Phys. Lett.* **B255** (1991) 46.

[14] M. Günyadin and S. Hyun, *Mod. Phys. Lett.* **A6** (1991) 1733.

[15] M. Günyadin and S. Hyun, *Nucl. Phys.* **B373** (1992) 688.

[16] K. Schoutens, *Nucl. Phys.* **B295** (1988) 634 ; A. Sevrin, W. Troost and A. Van Proeyen, *Phys. Lett.* **B208** (1988) 447.

[17] M. Günyadin, “Ternary Algebraic Approach to Extended Superconformal Symmetry”, invited talk to appear in the proceedings of the 5th Regional Conference on Mathematical Physics, Trakya University, Edirne, Türkiye, Dec. 1991.

[18] M. Günyadin, *Int. Jour. Mod. Phys.* **A8** (1993) 301.

[19] M. Günyadin, *Phys. Rev.* **D47** (1993) 3600.

[20] J. Tits, *Nederl. Akad. van Wetens.* **65** (1962) 530 ; I. L. Kantor, *Sov. Math. Dok.** 5** (1964) 1404 ; M. Koecher, *Amer. J. Math.* **89** (1967) 787.

[21] I.L. Kantor, *Trudy Sem. Vektor. Anal.* **16** (1972) 407 ; *Sov. Math. Dokl.* **14** (1973) 254.

[22] I. Bars and M. Günyadin, *J. Math. Phys.* **20** (1979)1977.

[23] H. Freudenthal, *Proc. Konikl.Nederl. Akad. Wet. Ser.* **A57** (1954) 218-230 and 363-408 ; K. Meyberg, ibid **A71** (1968) 162-174 and 175-190.

[24] I.L. Kantor and I.M. Skopets, *Sel. Math. Sov.* **2** (1982)293.

[25] P. Goddard and A. Schwimmer, *Phys. Lett.* **B214** (1988) 209.

[26] M. Günyadin, J.L. Petersen, A. Taormina and A. Van Proeyen, *Nucl. Phys.* **B322** (1989)402.

[27] V.G. Knizhnik, *Theor. Math. Phys.* **66** (1986) 68.

[28] M. Bershadsky, *Phys.Lett.*,174B (1986) 285.

[29] A. Van Proeyen, *Class. Quantum Gravity,* **6** (1989) 1501; A. Sevrin and G. Theodoridis, *Nucl. Phys.*,B332 (1990) 380.
[30] E. Guadagnini, M. Martellini and M. Minchev, *Phys. Lett.*, **B191** (1987) 69; W. Nahm, “Gauging Symmetries of Two Dimensional Conformally Invariant Models”, Davis Preprint, UCD-88-02 (unpublished); M. R. Douglas, “G/H Conformal Theory”, Ph.D. Thesis, Caltech (1988), unpublished; A. Altschuler, K. Bardakci and E. Rabinovici, *Comm. Math. Phys.*, **118** (1988) 157; D. Karabali, Q.H. Park, H. J. Schnitzer and Z. Yang, *Phys. Lett.*, **B216** (1989) 307; D. Karabali and H. J. Schnitzer, *Nucl. Phys.*, **B329** (1990) 649; K. Gawedzki and A. Kupiainen, *Phys. Lett.*, **B215** (1989) 119; *Nucl. Phys.*, **B320** (1989) 649; P. Bowcock, *Nucl. Phys.*, **B316** (1989) 80.

[31] E. Witten, *Comm. Math. Phys.*, **144** (1992) 189.

[32] R. Rohm, *Phys. Rev.*, **D32** (1984) 2849.

[33] P. Di Vecchia, V.G. Knizhnik, J.L. Petersen and P. Rossi, *Nucl. Phys.*, **B253** (1985) 701.

[34] A. Sevrin, W. Troost, A. Van Proeyen and Ph. Spindel, *Nucl. Phys.*, **B308** (1988) 662.

[35] H. J. Schnitzer, *Nucl. Phys.*, **B324** (1989) 412.

[36] E. Witten, *Nucl. Phys.*, **B371** (1992) 191.

[37] J. Wess and B. Zumino, *Phys. Lett.*, **B37** (1971) 95.

[38] E. Witten, *Comm. Math. Phys.*, **92** (1984) 455.