Evolution of a Network of Vortex Loops in HeII. Exact Solution of the ”Rate Equation”.

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Evolution of a network of vortex loops in HeII due to the fusion and breakdown of vortex loops is studied. We perform investigation on the base of the ”rate equation” for the distribution function $n(l)$ of number of loops of length $l$ proposed by Copeland and coauthors. By using the special ansatz in the ”collision” integral we have found the exact power-like solution of ”kinetic equation” in stationary case. That solution is the famous equilibrium distribution $n(l) \propto l^{-5/2}$ obtained earlier in numerical calculations. Our result, however, is not equilibrium, but on the contrary, it describes the state with two mutual fluxes of the length (or energy) in space of the vortex loop sizes. Analyzing this solution we drew several results on the structure and dynamics of the vortex tangle in the superfluid turbulent helium. In particular, we obtained that the mean radius of the curvature is of order of interline space. We also obtain that the decay of the vortex tangle obeys the Vinen equation, obtained earlier phenomenologically. We evaluate also the full rate of reconnection events.

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A network of one-dimensional singularities appears in various physical systems affecting important properties of them. As examples we would point out quantized vortices in quantum fluids both in turbulent regime (see e.g. book [1] and papers [2, 3, 4]) and thermodynamically equilibrium state ([5, 6]). Other examples are the flux tubes in superconductors [7], dislocations in solids [8], global cosmic strings [10], [11] polymer chains [9]. To evaluate contribution of mentioned structure in various effects, such as thermodynamic and kinetic properties, phase transition etc. we have to know structure and dynamics of network of singularities.

The whole evolution of network of chaotic set of line consists of two main ingredients. The first is the motion of the individual elements of each of lines, due to specific equations of motion, different for each of cases listed above. For instance elements of vortex filaments in turbulent superfluid $^4$He move obeying the Biot-Savart law supplemented by the friction force and the external flow/counterflow if any. Besides of motion of elements there is another very important element of the whole dynamics, common for all systems - fusion and breakdown of the loops or recombination of open strings due to reconnection processes. Further for definiteness we will talk about vortex loops in superfluid helium.

It is widely appreciated that the reconnection processes influence both the structure and evolution of vortex tangle. For instance, Feynman in his pioneering paper devoted to superfluid turbulence proposed scenario how the vortex tangle decays in absence of the external counterflow. According to this scenario a fusion of small vortex rings into larger ones as well as a breakdown into smaller ones is possible at the moments of the reconnection events. In assumption that on the average the last property dominates, the cascade like process of formation of smaller and smaller loops forms. When the scale of the small rings becomes of the order of the interatomic distances, which is the final stage of the cascade, the vortex motion is degenerated into thermal excitations. The idea that degeneration of the vortex tangle occurs due to cascade-like transferring of the length in space of scale of sizes of vortex loops was indirectly confirmed only in numerical calculations, where the procedure of artificial elimination of small loops had been used.

In spite of the recognized importance of the reconnecting loops kinetics, the numerical results remain main source of information about this process. The obvious lack of theoretical investigations interferes with the deep insight in the nature of this phenomena (this situation had been recently discussed in [17]). For instance it is not clear how the cascade of length in space of vortex loops sizes is formed, what mechanisms are responsible for this, what quantities determine intensity of cascade, and why at all the breakdown of the loop prevail. The scarcity of analytic investigations related to incredible complexity of the problem. Indeed we have to deal with a set of objects with not fixed number of elements, they can born and die. Thus, some analog of the secondary quantization method is required with the difference that objects (vortex loops) themselves possess an infinite number of degree of freedom with very involved dynamics. Clearly this problem can hardly be resolved in nearest future. Recently in [10] much more modest approach, based on the ”rate equation” for distribution function $n(l, t)$ was elaborated in context of cosmic strings. Following this work we introduce distribution function $n(l, t)$ of density of loop in ”space” of their lengths. It is defined as the number of loops (per unit volume) with lengths lying between $l$ and $l + dl$. Due to reconnection processes $n(l, t)$ can vary. We discriminate two types of processes, namely the fusion of two loops into the larger single loop and the breakdown of single loop into two daughter loops. The kinetic of vortex tangle is affected by the intensity of the introduced processes. The intensity of the first process is characterized by the rate of collision.
\( A(l_1, l_2, l) \) of two loops with lengths \( l_1 \) and \( l_2 \) and forming the loop of length \( l = l_1 + l_2 \). The intensity of the second process is characterized by the rate of self-intersection \( B(l, l_1, l_2) \) of the loop with length \( l \) into two daughter loops with lengths \( l_1 \) and \( l_2 \). In view of exposed above we can directly write out the master "kinetic" equation for rate of change the distribution function \( n(l, t) \).

\[
\frac{\partial n(l, t)}{\partial t} = \int \int A(l_1, l_2, l)n(l_1)n(l_2)\delta(l - l_1 - l_2)dl_1dl_2 \quad l_1 + l_2 \to l \quad (1)
\]

\[
- \int \int A(l_1, l_2, l_2)\delta(l_2 - l_1 - l)n(l)n(l_1)dl_1dl_2 \quad l_1 + l \to l_2 \quad (2)
\]

\[
- \int \int A(l_2, l_1, l_1)\delta(l_1 - l_2 - l)n(l)n(l_1)dl_1dl_2 \quad l_2 + l \to l_1 \quad (3)
\]

\[
- \int \int B(l_1, l_2, l_2)n(l)\delta(l - l_1 - l_2)dl_1dl_2 \quad l \to l_1 + l_2 \quad (4)
\]

\[
+ \int \int B(l_1, l_2, l_1)\delta(l_1 - l_1 - l_2)\delta(l_1)dl_1dl_2 \quad l_1 \to l + l_2 \quad (5)
\]

\[
+ \int \int B(l_1, l_1, l_2)\delta(l_2 - l_1 - l_1)n(l_1)dl_1dl_2 \quad l_2 \to l + l_1 \quad (6)
\]

All processes are depicted at the left of each line. Coefficient \( A \) and \( B \) were evaluated on the base of qualitative considerations in [10]. A bit more rigorous way, having universal character was elaborated in work [18]. Both models give similar expressions of type

\[
A(l_1, l_2, l) = b_mVll_1l_2, \quad B(l_1, l_1, l) = b_s\frac{Vl}{(\xi_0l)^{3/2}}. \tag{2}
\]

Here \( V \) is some characteristic velocity or elements of line, \( b_m \) and \( b_s \) are some constants depending on the model. For instance in work [10] on evolution of network of cosmic strings autors offered values about \( 0.1 \pm 0.3 \) for both coefficients (not necessary equal). In paper [18] on vortex loops in superfluid turbulent HeII there was offered \( b_m \sim 1/3, \quad b_s \approx 0.0164772 \). The quantity \( \xi_0 \) appears in as the persistency length of theory of randomly walking chains modelling chaotic cosmic strings. For the vortex tangle in superfluid helium this quantity \textit{is} associated with the mean radius of curvature (see [19]). Both these approaches are failed for scales near \( \xi_0 \), therefore usually this value appears as a low cut-off. Equation [10] had been studied (mainly numerically) in papers [18], [11], where some conclusions about evolution of network of cosmic string were made.

In the present work we demonstrate that the master "rate equation" [10] has exact stationary power-like solution of form \( n(l) = C*l^s \). We discuss the physical meaning of this solution and apply it to describe some properties of vortex tangle in the superfluid turbulent helium. To find power-like solution of form \( n(l) = C*l^s \) we use the Zakharov ansats, which is the special treatment of the "collision" integral in equation [10]. This trick was elaborated by Zakharov for the wave turbulence (see e.g. [27]), now we will show how it works in our case. Let us take for instance first and second integrals in the "collision term" of [10]. Let us further perform in the second integral the following change of variables.

\[
l = \tilde{l}_2 \left( \frac{l}{l_2} \right), \quad l_1 = \tilde{l}_1 \left( \frac{l}{l_2} \right), \quad l_2 = \tilde{l} \left( \frac{l}{l_2} \right). \tag{3}
\]

Under this change of variables various factors in the integrand of the second integral transforms as follows

\[
\delta(l_2 - l_1 - l) \to \left( \frac{l}{l_2} \right)^{-1} \delta(l - \tilde{l}_1 - \tilde{l}_2).
\]

\[
n(l) \to n(\tilde{l}_2) \left( \frac{l}{l_2} \right)^s, \quad n(l_1) \to n(\tilde{l}_1) \left( \frac{l}{l_2} \right)^s,
\]

\[
A(l_1, l_2) \to \frac{1}{2}V\tilde{l}_1\tilde{l}_2 \left( \frac{l}{l_2} \right)^2 = A(\tilde{l}_1, \tilde{l}_2, l) \left( \frac{l}{l_2} \right)^2.
\]
As result the second integral in the "collision" term takes a form (additional term 3 in the power counting appears from the Jacobian of transformation)

\[
\int \int \left( \frac{l_1}{l_2} \right)^{2+2s-1+3} A(l_1, l_2) \delta(l - l_1 - l_2) dl_1 dl_2.
\] (4)

It is easy to see that the transformed second term in the "collision" integral in rhs of master kinetic equation turns into first integral with additional factor \( \left( \frac{l}{l_2} \right)^{4+2s} \) in the integrand. Performing the same procedure for all lines we conclude that the "collision integral" of the "rate equation" can be written as

\[
\int \int A(l_1, l_2, l) n(l_1) n(l_2) \left( 1 - \left( \frac{l_1}{l_2} \right)^{4+2s} - \left( \frac{l}{l_1} \right)^{4+2s} \right) \delta(l - l_1 - l_2) dl_1 dl_2
\]

\[
- \int \int B(l_1, l_2, l) n(l) \left( 1 - \left( \frac{l_1}{l} \right)^{s+3/2} - \left( \frac{l}{l_1} \right)^{s+3/2} \right) \delta(l - l_1 - l_2) dl_1 dl_2
\] (5)

For \( s = -5/2 \) both expressions \( 1 - \left( \frac{l_1}{l_2} \right)^{4+2s} - \left( \frac{l}{l_1} \right)^{4+2s} \) and \( 1 - \left( \frac{l_1}{l} \right)^{s+3/2} - \left( \frac{l}{l_1} \right)^{s+3/2} \) are equal to \((l - l_1 - l_2)/l\). Thus the integrands of both integrals in (5) include expressions of type \( x \delta(x) \) and these integrals vanish. This implies in stationary case the power-like solution \( n = C l^{-5/2} \) for distribution function \( n(l, t) \) of density of loop in "space" of their lengths takes place.

Let us discuss the physical meaning of the solution obtained. First of all we stress that it does not relate to detailed balance, it rather corresponds to nonequilibrium state. To clarify what is happening we introduce density \( L(t) \) (in space of sizes \( l \)) of full length per unit of volume

\[
L(l, t) = l * n(l, t) = \frac{\text{total length}}{\text{unit of volume} \cdot \text{interval of length}}.
\] (6)

The total length (per unit volume), or the vortex line density

\[
\mathcal{L}(t) = \int L(l, t) dl = \int l * n(l, t) dl
\]

is obviously conserved during the reconnections events \( d\mathcal{L}(t)/dt = 0 \). Conservation of the vortex line density can be expressed in form of continuity equation:

\[
\frac{\partial L(t)}{\partial t} + \frac{\partial P(t)}{\partial l} = 0.
\] (7)

This form of equation states that the rate of change of length is associated with "flux" of length in space of sizes of the loops. Term "flux" here means just redistribution of length among the loops due to reconnections. Expression for \( P(l) \) is obtained by multiplication kinetic equation by \( l \) and by rewriting the "collision" term in the shape of derivative with respect to \( l \). Result is (substitutions \( l_1/l = x \) and \( l_2/l = y \) had been used below)

\[
P = \frac{(l^{5+2s})}{(5+2s)} \int \int \frac{1}{2} b_m V_l x y C^2 x^s y^s \left( 1 - \left( \frac{l_1}{l} \right)^{4+2s} - \left( \frac{l}{l_1} \right)^{4+2s} \right) \delta(1 - x - y) dx dy
\]

\[
- \left( \frac{l^{s+5/2}}{s+5/2} \right) \int \int \frac{1}{2} b_s V_l \frac{1}{(\xi_0 x)^{3/2}} C \left( 1 - \left( \frac{l_1}{l} \right)^{s+3/2} - \left( \frac{l}{l_1} \right)^{s+3/2} \right) \delta(1 - x - y) dx dy.
\] (8)

Both integrals in relation \( \text{[5]} \) coincide with integrals in \( \text{[3]} \), therefore they vanish for \( s = -5/2 \). However they have pre-integral factor with the denominators, which also vanish for \( s = -5/2 \). Calculating indeterminacy 0/0 we obtain the final expressions for "flux" of length in space of sizes of the loops

\[
P = \frac{12.555}{2} C^2 b_m V_l - \frac{5.545}{2\xi_0^{3/2}} C b_s V_l
\] (9)
The positive sign of the first corresponds to flux of length in direction of large scales. This is justified, since the fusion processes lead to formation of large loops. The negative sign of the second term corresponds to flux of length in direction of small scales. This is justified, since the breaking down processes lead to formation of small loops.

The approach elaborated above allows to draw several conclusions concerning the structure and dynamics of real vortex tangle in turbulent HeII. To do it we have to specify quantity $V_l$, which enters into the rates coefficients of the both merging and breaking down processes. Since the averaged radius of curvature is $\xi_0$, we estimate the velocity factor $V_l$ to be of order of $\kappa/\xi_0$ ($\kappa$ is the quantum of circulation). Thus the only parameters of the whole theory are the quantum of circulation $\kappa$ and the mean radius of curvature $\xi_0$.

**Vortex Line Density.** In steady case the positive flux of length in $V_L$ exactly compensates the negative flux. Equating these two terms we have

$$C = \frac{5.455}{12.555} \frac{b_\kappa}{b_m} \frac{1}{\xi_0^{3/2}} = C_{VLD} \frac{1}{\xi_0^{3/2}}. \quad (10)$$

New numerical parameter $C_{VLD} \approx 0.2 \div 1$. Thus the power-like solution of the master Kinetic equation $n(l)$ is

$$n(l) = \frac{C_{VLD}}{\xi_0^{3/2}} l^{-5/2} \quad (11)$$

Accordingly the total length $L$ per unit of volume (we recall that quantity $\xi_0$ serves as the low cut-off).

$$L = \int_{\xi_0}^{\infty} \frac{l}{l} \cdot n(l)dl = \frac{2}{3} \frac{C_{VLD}}{\xi_0^{3/2}} \quad (12)$$

Result (12) is remarkable. The idea that interline space $\delta = \mathcal{L}^{-1/2}$ is of order of mean radius of curvature $\xi_0$ was launched by Schwarz [13]. Earlier it was confirmed only in numerical simulations and the nature of this phenomenon was not clear. We proved that this relation appears due to kinetics of colliding vortex loops.

**Decay of Vortex Tangle (Vinen Equation).** Let us suppose that there is some sink for loops of small sizes. Then the steady situation is reached when there is some source which generates length. This is the mutual friction for superfluid turbulence. If we switch off the source of length, the total length will decrease. The attenuation of vortex line density is related to negative flux $P_{neg}$ (in direction of small scales) of the full flux $V_L$ and can be evaluated from the continuity equation for density of length $L(l, t)$ in space of the loop sizes, which we integrate over $l$

$$\frac{\partial L(l, t)}{\partial t} + \frac{\partial P(l)}{\partial l} = 0 \Rightarrow \frac{d \mathcal{L}(t)}{dt} + P_{neg} = 0.$$ 

Evaluating $P_{neg}$ from relation for full flux $V_L$, where constant $C$ is used from (10) we conclude that

$$\frac{d \mathcal{L}(t)}{dt} = - C_{Vinen} \kappa \mathcal{L}^2. \quad (13)$$

Here $C_{Vinen}$ is new numerical factor. For parameters, which we adopt here $C_{Vinen} \approx 0.05 \div 0.2$. Expression (13) is the famous Vinen equation [5] obtained him from experiment. It is remarkable that this relation appears due to the reconnection processes. The own dynamics of filament specific for various systems is absorbed by the persistency length $\xi_0$, which has dropped out the Vinen equation at all. Thus the (13) has universal character and can be applied for other systems, e.g. for cosmic strings. We also would like to note that our calculations confirmed the brilliant Feynman's conjecture on the formation of cascade-like breakdown of vortex loops.

**The Full Rate of Reconnection.** The full rate of reconnection $\dot{N}_{rec}$ can be evaluated directly from master "rate equation" [11]. Indeed, this equation describes change of $n(l)$ due to reconnection events. It takes into account sign of events, depending on whether the loop of size $l$ appears or dies in result of reconnection. Therefore, if we take all terms in collision integral with the plus sign we obtain the total number of reconnections. The according calculations lead to result

$$\dot{N}_{rec} = \frac{1}{3} \kappa (b_\kappa C_{VLD} + b_m^2 C_{VLD}) \frac{1}{\xi_0^{5/2}} = C_{rec} \kappa \mathcal{L}^{5/2},$$

where $C_{rec}$ one more constant of order $0.1 - 0.5$. That results agrees with the recent numerical investigation by Barenghi and Samuels [13].
The problem of nonequilibrium dynamics of vortex loops, which merge and break down due to reconnections has been considered. The description was performed on the base of the "rate equation" for distribution of number of loops in space of their lengths. By use of special substitution of variables in the collision integral (Zakharov ansatz) we found the power-like solution of the kinetic equation. That is non-equilibrium solution characterized by two mutual fluxes of length (energy) in space of loop sizes.

The result obtained were used to draw some conclusion about the structure of the vortex tangle appeared in the superfluid turbulent HeII. (i) In particular we obtained that vortex line density is of order of inverse squared mean curvature of lines. (ii) We also found that the rate of decay of the vortex line density is proportional to squared vortex line density itself multiplied by quantum of circulation and numerical factor of order of unity. (iii) we also estimate the total number of reconnections, which turned out to be of order of quantum of circulation multiplied by the vortex line density in power 5/2.

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