Stochastic resonance and noise delayed extinction in a model of two competing species

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Abstract

We study the role of the noise in the dynamics of two competing species. We consider generalized Lotka-Volterra equations in the presence of a multiplicative noise, which models the interaction between the species and the environment. The interaction parameter between the species is a random process which obeys a stochastic differential equation with a generalized bistable potential in the presence of a periodic driving term, which accounts for the environment temperature variation. We find noise-induced periodic oscillations of the species concentrations and stochastic resonance phenomenon. We find also a nonmonotonic behavior of the mean extinction time of one of the two competing species as a function of the additive noise intensity.

Key words: Statistical Mechanics, Population Dynamics, Noise-induced effects

PACS: 05.40-a, 87.23Cc, 89.75-k

1 Introduction

Recently the influence of noise on population dynamics has been the subject of intense theoretical investigations [1,2,3,4,5]. The problem of the stability of the biological complex systems in the presence of noise for example is one of the more discussed in research activity of these last years [6]. The noise through its interaction with the nonlinearity of the systems can give rise to new counterintuitive phenomena like stochastic resonance [7], noise enhanced

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Preprint submitted to Elsevier Science
stability [8], noise delayed extinction, etc... The principal aim of this work is the investigation of a generalized Lotka-Volterra model with a random interaction parameter between two competing species and in the presence of a multiplicative noise [1]. We find that the noise has a constructive role. In fact it is responsible for the generation of quasi-periodic temporal oscillations and the enhancement of the response of the system to a driving force producing stochastic resonance. Moreover we find noise delayed extinction, i.e. a non-monotonic behavior of the average extinction time of one of the two species as a function of the noise intensity.

2 The model

Time evolution of two competing species is obtained within the formalism of the Lotka-Volterra equations [9] in the presence of a multiplicative noise

\[
\frac{dx}{dt} = \mu_1 x (\alpha_1 - x - \beta_1(t)y) + x \xi_x(t) \\
\frac{dy}{dt} = \mu_2 y (\alpha_2 - y - \beta_2(t)x) + y \xi_y(t),
\]

where \(\xi_x(t)\) and \(\xi_y(t)\) are statistically independent Gaussian white noises with zero mean and correlation function \(\langle \xi_i(t)\xi_j(t') \rangle = \sigma \delta(t-t') \delta_{ij} \) \((i, j = x, y)\). The

![Bistable Potential](image)

**Fig. 1.** The bistable potential \(U(\beta)\) of the interaction parameter \(\beta(t)\). The potential \(U(\beta)\) is centered on \(\beta = 0.99\). The parameters of the potential are \(h = 6.25 \cdot 10^{-3}\), \(\eta = 0.05\), \(\rho = -0.01\).

Time series for the two populations are obtained setting \(\alpha_1 = \alpha_2 = \alpha\), \(\beta_1(t) = \beta_2(t) = \beta(t)\). It is known that for \(\beta < 1\) a coexistence regime takes place, that is both species survives, while for \(\beta > 1\) an exclusion regime is established,
that is one of the two species vanishes after a certain time. Coexistence and exclusion of one of the two species correspond to stable states of the Lotka-Volterra’s deterministic model [10]. Real ecosystems are immersed in a noisy nonstationary environment, so also the interaction parameter is affected by the noise and some other deterministic periodical driving such as the temperature. The change in the competition rate between exclusion and coexistence e. g. occurs randomly because of the coupling between the limiting resources and the noisy environment. A random variation of limiting resources produces a random competition between the species. The noise therefore together with the periodic force determines the crossing from a dynamical regime (\( \beta < 1 \), coexistence) to the other one (\( \beta > 1 \), exclusion). To describe this continuous and noisy behaviour of the interaction parameter \( \beta(t) \) we consider a stochastic differential equation with a bistable potential and a periodical driving force

\[
\frac{d\beta(t)}{dt} = -\frac{dU(\beta)}{d\beta} + \gamma \cos(\omega_0 t) + \xi_\beta(t),
\]

where \( U(\beta) \) is a bistable potential (see Fig.1)

\[
U(\beta) = h(\beta - (1 + \rho))^4/\eta^4 - 2h(\beta - (1 + \rho))^2/\eta^2,
\]

and \( h \) is the height of the potential barrier. The periodic term takes into account for the environment temperature variation. Here \( \gamma = 10^{-1} \) and \( \omega_0/(2\pi) = 10^{-3} \). In Eq.(3) \( \xi_\beta(t) \) is a Gaussian white noise with the usual statistical properties: \( \langle \xi_\beta(t) \rangle = 0 \) and \( \langle \xi_\beta(t)\xi_\beta(t') \rangle = \sigma \delta(t-t') \). Due to the shape of \( U(\beta) \) it is reasonable to expect a coexistence regime for \( \beta(0) < 1 \), when deterministic case \( (\xi_\beta(t) = 0) \) is considered.

2.1 Stochastic resonance

First we investigate the effect of the noise on the time behavior of the species. Since the dynamics of the species strongly depends on the value of the interaction parameter, we initially analyze the time evolution of \( \beta(t) \) for different levels of the additive noise \( \sigma_\beta \). Specifically for \( \sigma_\beta = 0 \) we obtain a periodical behavior of \( \beta(t) \) in the coexistence region (see Fig.2a). For low noise intensity \( (\sigma_\beta \ll h) \) we get the same periodical behavior of deterministic case, slightly perturbed by the noise (Fig.2b). For higher noise intensity \( (\sigma_\beta \sim h) \) the periodical behavior of the interaction parameter jumps between the two values \( \beta = 0.94 < 1 \) and \( \beta = 1.04 > 1 \), which characterize the coexistence and the exclusion regimes. We see in Fig.(2c) the typical picture of stochastic resonance. A further increase of the noise intensity produces a loss of coherence and the dynamical behavior is strongly controlled by the noise (Fig.2d). There
is therefore a cooperation between periodical driving of the temperature, due to some geological cause, and the environmental noise. The noise can synchronize with the periodical driving giving rise to the stochastic resonance effect, which affect considerably the dynamics of the ecosystem. The exact value of $\sigma_\beta = 1.78 \cdot 10^{-3}$ used in Fig.2c is obtained from the statistical synchronization typical of the SR phenomenon

$$\tau_k = T_0/2$$

(5)

where $\tau_k$ is the Kramers time given by

$$\tau_k = \frac{2\pi}{\sqrt{|U''(0.99)|U''(0.94)}} \exp [2h/\sigma_\beta].$$

(6)

and $T_0$ is the period of the driving force. In Eq.(6) $U''(0.99)$ and $U''(0.94)$ are the second derivative respectively calculated in the unstable and stable states of the potential. To analyze the dynamics of the two species we fix the additive noise intensity at the value $\sigma_\beta = 1.78 \cdot 10^{-3}$, corresponding to a competition regime between the two species periodically switched from co-
existence to exclusion. The temporal series of the two species are obtained for different values of the multiplicative noise intensity $\sigma = \sigma_x = \sigma_y$. The initial values of the two species are $x(0) = y(0) = 1$. After a short transient in the coexistence regime, which is not visible in Fig.3 because of the scale used for the x axes, both species reach the value: $x_{st} = y_{st} = \alpha/(1 + \beta) \approx 1/2$. From then on the densities of both species oscillate quasi regularly around this value with amplitudes dependent on the multiplicative noise intensity. In particular for $\sigma = 0$ (see Fig.3a) and for very low levels of multiplicative noise ($\sigma = 10^{-12}$) the two species coexist and the populations show correlated oscil-
lations. For higher level of the multiplicative noise ($\sigma = 10^{-4}$) the amplitude of the oscillations increases, anti-correlated oscillations appear and random periodical inversions of populations occur (see Fig.3c). A further increase of the noise ($\sigma = 10^{-2}$) produces a degradation of the signal and a loss of coherence of the temporal series for the species (see Fig.3d). In the absence of multiplicative noise the Lotka-Volterra equations are symmetric. The choice of the same values for the initial conditions of the two species densities maintains this symmetry, so the species oscillate in phase around the stationary value $\alpha/(1 + \beta)$, even if the additive noise that controls the SR effect of the interaction parameter was fixed at a value that produces many transitions to the exclusion regime (see Fig.3a). In the presence of very weak multiplicative noise ($\sigma = 10^{-12}$) the species oscillate in phase almost regularly with some little anticorrelated spikes. By increasing the amplitude of the multiplicative noise we obtain an enhancement of the amplitude of both species, as we can see from Figs. 3b, 3c and 3d. These figures show this transient behaviour from a regime of quasi correlated oscillations (Fig.3b) to a regime of anticorrelated oscillations (Fig.3d). The presence of the multiplicative noise breaks the symmetric dynamical behaviour of the ecosystem. For $\beta > 1$, i.e. in the exclusion regime, this symmetry breaking determines the "divergent" behaviour of the trajectories of the two species: one tends to survive, while the other one tends to extinguish. From the inspection of Fig.3 we note that in the presence of a bistable potential modulated by a weak periodic force the response of the system may be enhanced by the presence of the noise. The periodicity of the noise-induced oscillations in the time behavior of the population densities shown in Fig.3e is the same of the driving periodic term of Eq. (3). This is the signature of stochastic resonance (SR). In order to underline the presence of SR we analyze the squared difference of population densities $(x - y)^2$. In Fig.4 it is shown the SNR of this quantity as a function of the multiplicative noise intensity $\sigma$, for $\sigma_\beta = 1.78 \cdot 10^{-3}$. We calculate the SNR of $(x - y)^2$ because of the random periodical inversion of population densities. We note that dynamics of $(x - y)$ is mainly affected by the multiplicative noise, as we can see from Eqs.(2). The SNR has been obtained by performing 190 realizations of Eqs. (2). A maximum at $\sigma = 10^{-4}$ is present. From the above analysis it is clear the role of the two noise sources: the additive noise determines the conditions for the different dynamical regimes of the two species, the multiplicative noise produces a coherent response of the system by a mechanism of symmetry breaking of the dynamical evolution of the ecosystem.

2.2 Noise delayed extinction

Now we investigate the average extinction time of one species as a function of the noise intensity $\sigma_\beta$. To do this we fix the multiplicative noise at a low level intensity in such a way that the system is far enough from the SR regime and
the dynamics is not strongly perturbed by the noise. So we choose \( \sigma = 10^{-9} \) because we are not interested in the coherent behavior of the ecological system but we are focused on the effect of the additive noise on the average extinction time of the species. We do simulations of Eqs. (1) and (2) by performing 200 realizations. The results are shown in Fig.(5). We use the bistable potential \( U(\beta) \) of Fig.1 with the initial condition \( \beta(0) = 0.94 \). In this condition the ecosystem is in the coexistence regime, that is the deterministic extinction time of both species is infinite. By introducing noise, exclusion takes place and a finite mean extinction time (MET) appears. By varying the intensity of the additive noise in Eq.3 we obtain, of course, a variation of the average extinction time. The delayed extinction is obtained for noise intensities ranging from the intermediate regime (2 in Fig.5a) to the coexistence regime obtained with higher values of \( \sigma \) (3 in Fig.5a). In real ecosystems we have always a given noise intensity, which corresponds to a finite mean extinction time. Due to some environmental cause the noise intensity can considerably change, as it is observed in experimental data of populations in a very long time interval [12]. Therefore the dynamical behaviour shown in Fig.5 should explain such physical situations, where the variation of the environmental noise produces a delayed extinction of some population. By increasing the noise intensity we obtain noise delayed extinction and the average extinction time grows reaching a saturation value, which corresponds to a situation where the potential barrier is absent. We find nonmonotonic behaviour of the MET as a function of the noise intensity \( \sigma_{\beta} \), with a minimum value \( \tau_{\text{min}} = 40.47 \) at \( \sigma_{\beta} = 2.75 \cdot 10^{-3} \), which is of the same order of magnitude of the barrier height \( h \) (see Fig.5a).
The Kramers time corresponding to this noise intensity is $\tau_k = 41.6$, that is approximately equal to $\tau_{\text{min}}$. This result is due to the noise driven dynamics. In fact for low value of the noise intensities the average time to overcome the potential barrier is very high, i.e. long Kramers times. The ecosystem remains in the coexistence regime for a long time and the extinction time is very large. For noise intensity of the same order of magnitude of the barrier height the system goes towards the exclusion regime of one of two species and the average extinction time is approximately equal to the Kramers time. We get the minimum value of MET. For higher values of noise intensity the Kramers time becomes very small and the representative point of the $\beta$ parameter moves between the two minima in a very short time. In this condition the system "sees" the average value of the interaction parameter ($\beta = 0.99$), which gives a coexistence regime. In Figs.5b,c,d we show the time evolution of the ecosystem corresponding to the points 1, 2 and 3 of Fig.5a. We have coexistence regime in points 1 and 3 and an exclusion regime in point 2. We report finally in the following Fig.6 the behavior of the average extinction time as a function of the Kramers time. This figure confirms our physical description of the extinction dynamics of the system.
Fig. 6. *Semilog plot of the mean extinction time of one species as a function of the Kramers time.*

### 3 Conclusions

We report a study on the role of the noise in the dynamics of two competing species. We consider two noise sources: a multiplicative noise and an additive noise, which produces a random interaction parameter between the species. The noise induces a coherent time behavior of two species giving rise to temporal oscillations and enhancement of the response of the system through stochastic resonance phenomenon. Specifically the additive noise controls the switching between the coexistence and the exclusion dynamical regimes, the multiplicative noise is responsible for coherent oscillations of the two species. The SR in the dynamics of interaction parameter $\beta$ induces SR phenomenon in two competing species. These time behaviors are absent in the deterministic dynamics. We note that our model is useful to describe physical situations in which the amplitude of the periodical driving force, due to the temperature variations, is weak and therefore unable to produce considerable variations of the dynamical regime of the ecosystem. The synergetic cooperation between the nonlinearity of the system and the random and periodical environmental driving forces produces therefore a coherent time behaviour of the ecosystem investigated. The noise is also responsible for a delayed extinction which gives rise to a nonmonotonic behavior of the average extinction time as a function of the additive noise intensity. We note finally that these noise induced effects should be useful to explain the time evolution of ecological species, whose dynamics is strongly affected by the noisy environment [5,6,11,12].
4 Acknowledgments

The authors are grateful to Dr. A. La Barbera for useful discussions concerning numerical simulations. This work was supported by INFM, MIUR and INTAS Grant 01-450.

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