A posteriori error estimates based on superconvergence of FEM for fractional evolution equations

https://doi.org/10.1515/math-2021-0099
received January 31, 2021; accepted September 5, 2021

Abstract: In this paper, we consider an approximation scheme for fractional evolution equation with variable coefficient. The space derivative is approximated by triangular finite element and the time fractional derivative is evaluated by the $L_1$ approximation. The main aim of this work is to provide convergence and superconvergence analysis and derive a posteriori error estimates. Some numerical examples are presented to demonstrate our theoretical results.

Keywords: a posteriori error estimates, convergence and superconvergence, finite element method, fractional evolution equations

MSC 2020: 65M30, 35R11

1 Introduction

Since the remarkable hereditary and memory properties, fractional partial differential equations (FPDEs) play a very important role in wave propagation, finance, physics, engineering and so on [1]. Since the exact solutions of most FPDEs are very difficult to obtain, numerical methods of FPDEs have been an active research area, such as finite difference methods [2–6], finite element methods (FEMs) [7–11], mixed FEMs [12,13], finite volume methods [14], spectral methods [15–17] and so on.

It is well known that there has been extensive research on the superconvergence of FEMs for partial differential equations (PDEs). A systematic introduction can be found in [18–23]. Generally speaking, there are three types of superconvergence. The first is pointwise superconvergence, namely in certain sampling points the values in derivatives of error have a higher convergent order than elsewhere [24]. The second is global superconvergence, namely the gradient in $L^2$-norm of error between the numerical solution and the projection of exact solution have a greater accuracy than the optimal order of convergence. The third is superconvergence based on post-processing technique, namely reconstruct an improved accuracy gradient. Representative post-processing techniques are developed by interpolation [25], extrapolation [26] and gradient recovery, which include superconvergent patch recovery [27], polynomial preserving recovery [28,29] and supercovergent cluster recovery [30]. In recent years, there has been some research on superconvergence analysis of FEMs for FPDEs. Superconvergence of FEMs and mixed FEMs for FPDEs is inves-
tigated in [31–33] and [34], respectively. In [35,36], superconvergence of nonconforming FEMs for fractional PDEs is established.

Adaptive FEMs are among the most important means to improve the accuracy and efficiency of finite element discretization. The pioneering work was carried out by Babuška and Rheinboldt in [37]. One of the key concepts in adaptive FEMs is a posteriori error estimates, which are computable quantities in terms of the discrete solution and can measure the actual discrete errors without the knowledge of exact solution. A posteriori error estimates of FEMs for elliptic problems are well-developed [38–40]. There are substantial research on a posteriori error estimates of FEMs for integer-order PDEs based on explicit residual [41], local problems [42,43], recovery [27,29,44,45], hierarchical basis [46–49] and equilibrated error [50,51]. However, to the best of our knowledge, a posteriori error estimates of FEMs for evolution equations are less developed, and only a few results can be found in [52–54].

The purpose of this work is to provide a fully discrete finite element approximation for fractional evolution equations and analyze its convergence and superconvergence. Then, we derive a posteriori error estimates based on the superconvergence results and construct an adaptive FEM algorithm for fractional evolution equations with variable coefficients.

We are interested in the following fractional evolution equation:

\[
\begin{align*}
\partial_t^\alpha y(t, x) - \text{div}(A(x)\nabla y(t, x)) &= f(t, x), \quad t \in J, \; x \in \Omega, \\
y(t, x) &= 0, \quad t \in J, \; x \in \partial\Omega, \\
y(0, x) &= y_0(x), \quad x \in \Omega.
\end{align*}
\]

(1)

Here \( J = [0, T] \) \((0 < T < +\infty)\) and \( \Omega \) be a bounded open domain of \( \mathbb{R}^d \) \((1 \leq d \leq 3)\) with smooth boundary \( \partial\Omega \) and \( A(x) = (a_{ij}(x))_{d \times d} \in W^{1,\infty}(\hat{\Omega})^{d \times d} \) is a positive definite matrix, \( f(t, x) \) and \( y_0(x) \) are given smooth functions. \( \partial_t^\alpha \) \((0 < \alpha < 1)\) denotes the \( \alpha \)-order left Caputo derivative defined by

\[
\partial_t^\alpha y(t, x) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-\tau)^\alpha} \frac{\partial y(\tau, x)}{\partial \tau} \, d\tau.
\]

Throughout the paper, \( L^s(J; W^{m,q}(\Omega)) \) denotes all \( L^s \) integrable functions from \( J \) into \( W^{m,q}(\Omega) \) with norm \( \|v\|_{L^s(J; W^{m,q}(\Omega))} = \left( \int_0^T \|v(t)\|_{W^{m,q}(\Omega)}^s \, dt \right)^{1/s} \) for \( s \in [1, \infty) \) and the standard modification for \( s = \infty \), where \( W^{m,q}(\Omega) \) is Sobolev spaces on \( \Omega \). Similarly, one can define \( H^s(J; W^{m,q}(\Omega)) \) and \( C^s(J; W^{m,q}(\Omega)) \) (see e.g. [55]). In addition, \( c \) or \( C \) is a generic positive constant.

The rest of this paper is organized as follows. In Section 2, we give a fully discrete approximation scheme of (1). Convergence analysis results are presented in Section 3. In Section 4, we derive the superconvergence between the numerical solutions and the elliptic projection of exact solutions. In Section 5, we construct a posteriori error estimates based on the superconvergence results. Some numerical experiments are presented to support our theoretical results in Section 6.

## 2 Fully discrete finite element approximation

In this section, we present a fully discrete approximation scheme of (1). To begin with, we introduce triangular FEMs for the spatial discretization. For brevity, we denote \( W^{m\cdot}(\Omega) \) by \( H^m(\Omega) \) and drop \( \Omega \) or \( J \) whenever possible, i.e.,

\[
\begin{align*}
\|\cdot\|_{W^{m,2}\Omega} &= \|\cdot\|_{W^{m,2}}, \\
\|\cdot\|_{L^p(J; W^{m,q}(\Omega))} &= \|\cdot\|_{L^p(W^{m,q})}, \\
\|\cdot\|_{L^p(J; W^{m,q}(\Omega))} &= \|\cdot\|_{L^p(W^{m,q})}.
\end{align*}
\]
Moreover, we set $H_0^1(\Omega) \equiv \{ v \in H^1(\Omega) : v|_{\partial\Omega} = 0 \}$, $W = H_0^1(\Omega)$, $U = L^2(\Omega)$. In addition,

$$a(v, w) = \int_{\Omega} (Av) \cdot \nabla w, \quad \forall v, w \in W,$$

$$(f_1, f_2) = \int_{\Omega} f_1 \cdot f_2, \quad \forall f_1, f_2 \in U.$$  

From the assumption of coefficient matrix $A(x)$, we have

$$a(v, v) \geq c\|v\|_1^2, \quad |a(v, w)| \leq C\|v\|_1\|w\|_1, \quad \forall v, w \in W.$$  

We recast (1) as the following weak formulation:

$$\begin{cases}
\partial \cdot \nabla v + a(v, w) = (f, w), & \forall w \in W, \ t \in J, \\
y(0, x) = y_0(x), & x \in \Omega.
\end{cases} \tag{2}$$

Let $T^h$ be a family of quasi-uniform triangulations of $\Omega$, such that $\bar{\Omega} = \bigcup_{\tau \in T^h} \bar{\tau}$ and $h = \max_{\tau \in T^h} h$, where $h$ is the diameter of the element $\tau$. Furthermore, we set

$$V_h = \{ v_h \in C(\bar{\Omega}) : v_h|_{\partial\Omega} \in \mathbb{P}_1, \ \forall \tau \in T^h \},$$

where $\mathbb{P}_1$ represents the space of all polynomials whose degree at most 1, and $W_h = V_h \cap H_0^1(\Omega)$.

Then a semi-discrete approximation scheme of (2) reads as

$$\begin{cases}
\partial \cdot \nabla y_h + a(y_h, w_h) = (f, w_h), & \forall w_h \in W_h, \ t \in J, \\
y^0_h(x) = y^0_0(x),
\end{cases} \tag{3}$$

where $y^0_h(x) \in W_h$ is a suitable approximation of $y^0_0(x)$.

In the second, we will consider the $L^1$ scheme for the time discretization.

Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a uniform partition of $J$ with time step $\tau = \frac{T}{N}$ and $t_n = n\tau$, $n = 0, 1, \ldots, N$.

We set $\varphi^n = \varphi(t_n, x)$ and $d_t y^n = \frac{y^n - y^{n-1}}{\tau}$. The time fractional derivative can be approximated as follows:

$$\partial_t^\alpha y^n = \frac{1}{\Gamma(1-\alpha)} \int_0^{t_n} \frac{\partial y(s, x)}{\partial s} \, ds$$

$$= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \frac{1}{(t_n - s)^\alpha} \frac{\partial y(s, x)}{\partial s} \, ds$$

$$= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n-1} d_t y^{k+1} \int_{t_k}^{t_{k+1}} \frac{1}{(t_n - s)^\alpha} \, ds + r^n_T$$

$$= \frac{1}{\tau^{\alpha-1}\Gamma(2-\alpha)} \sum_{k=0}^{n-1} b^k d_t y^{n-k} + r^n_T,$$

$$= L^n_T y^n + r^n_T,$$

where $b_k = (k + 1)^{-\alpha} - k^{-\alpha}$, $b_0^n = (n - 1)^{-\alpha} - n^{-\alpha}$, $b_0^n = 1$, $b_k^n = b_{n-k} - b_{n-k-1}$ and $r_T^n$ is the truncation error.

It follows from [25] for $y \in W^{2, \infty}(L^2)$ that

$$|r_T^n| = |\partial_t^\alpha y^n - L^n_T y^n| \leq C\tau^{2-\alpha}. \tag{5}$$
Then a fully discrete approximation scheme of (2) is as follows:
\[
\begin{aligned}
\begin{cases}
(L_t^n u^n_t, w_h) + a(y^n_h, w_h) = (f^n, w_h), & \forall w_h \in W_h, \quad n = 1, 2, \ldots, N, \\
y^n_h = y^n_0(x).
\end{cases}
\end{aligned}
\]
(6)

Usually, we set \(y^n_0(x) = P_h(y_0(x))\), where \(P_h\) will be specified later.

3 Convergence analysis

We will derive the convergence of the numerical scheme (6). Let \(P_h : W \to W_h\) be the elliptic projection operator, for any \(v \in W\) defined by

\[
a(v - P_h v, w_h) = 0, \quad \forall w_h \in W_h.
\]

It has the following approximation properties (see [8]):

\[
|v - P_h v| + h|\nabla(v - P_h v)| \leq Ch^2|v|_2.
\]
(7)

The following conclusions will be used in the following error analysis.

Lemma 3.1. [13] Let \(\{\xi^n\}_n \) be a sequence of functions on \(\Omega\). Then

\[
\left\{ \xi^n, \sum_{k=0}^{n} b^n_k \xi^k \right\} = \frac{1}{2}\left( \|\xi^n\|^2 + \sum_{k=0}^{n-1} b_k^n \|\xi^k\|^2 - \sum_{k=0}^{n-1} b_k^n \|\xi^k - \xi^n\|^2 \right).
\]

Lemma 3.2. [33] Let \(e^k \geq 0, k = 1, 2, \ldots, L\), satisfy \(e^n \leq \sum_{k=1}^{n-1} (b_{k-1} - b_k) e^{n-k} + y\) with \(y > 0\). Then

\[
e^n \leq C r^{-a} y, \quad n = 1, 2, \ldots, L.
\]

Theorem 3.1. Let \(y\) and \(y_h\) be the solutions of (2) and (6), respectively. Suppose that \(y \in W^{2,\infty}(L^2) \cap W^{1,\infty}(H^2)\). Then, for any integer \(1 \leq n \leq N\), we have

\[
|y^n - y^n_h| \leq C(h^2 + \tau^2 - a),
\]
(8)

\[
|y^n - y^n_h|_{H^1} \leq C(h + \tau^2 - a).
\]
(9)

Proof. From (4) and the definition of \(P_h\), we can rewrite (2) at \(t_n\) as

\[
(L_t^n u^n_t, w_h) + a(y^n_h, w_h) = (f^n, w_h), \quad \forall w_h \in W_h.
\]
(10)

Setting \(\theta^n = y^n_h - P_h y^n\) and subtracting (10) from (6), we get

\[
(L_t^n \theta^n, w_h) + a(\theta^n, w_h) = (r^n_t, w_h) + (L_t^n(y^n - P_h y^n), w_h), \quad \forall w_h \in W_h.
\]
(11)

Taking \(w_h = \theta^n\) in (11), we have

\[
(L_t^n \theta^n, \theta^n) + a(\theta^n, \theta^n) = (r^n_t, \theta^n) + (L_t^n(y^n - P_h y^n), \theta^n).
\]
(12)

Note that \(a(\theta^n, \theta^n) \geq 0\), from (5), (7) and (12), we can obtain

\[
\|\theta^n\|^2 \leq -\sum_{k=0}^{n-1} b^n_k \|\theta^k\|^2 + \sum_{k=0}^{n-1} b^n_k \|\theta^k - \theta^n\|^2 + 2r^n t \|\theta^n\|^2 + \|L_t^n(y^n - P_h y^n)\| \|\theta^n\|
\]
\[
\leq -\sum_{k=0}^{n-1} b^n_k \|\theta^k\|^2 + C2r^n t (2 - a) (\tau^2 - a + h^2) \|\theta^n\|.
\]
(13)

Hence, (8) follows from (7), (13) and triangle inequality.
Setting \( w_h = L^a_\theta^n \) in (11), we get
\[
(L^a_\theta^n \theta^n, L^a_\theta^n \theta^n) + a(\theta^n, L^a_\theta^n \theta^n) = (r^n, L^a_\theta^n \theta^n) + (L^a_\theta^n (y^n - P_h y^n), L^a_\theta^n \theta^n). \tag{14}
\]
Similarly, from Hölder’s inequality (5), (7) and (14), we derive
\[
c\|\theta^n\|_1 \|L^a_\theta^n \theta^n\|_1 \leq (\|r_\tau^n\|_1 + \|L^a_\theta^n (y^n - P_h y^n)\|_1) \|L^a_\theta^n \theta^n\|_1 \leq C (\tau^{2-a} + h^2) \|L^a_\theta^n \theta^n\|_1. \tag{15}
\]
Thus, (9) follows from (7), (15) and triangle inequality. \(\square\)

## 4 Superconvergence analysis

In this section, we will derive the global superconvergence results between the finite element solution and the elliptic projection of exact solution.

**Theorem 4.1.** Let \( y \) and \( y_h \) be the solutions of (2) and (6), respectively. Assume all the conditions in Theorem 3.1 are valid. Then, for any integer \( 1 \leq n \leq N \), we have
\[
\|P_h y^n - y_h^n\|_{H^1} \leq C(h^2 + \tau^{2-a}). \tag{16}
\]

**Proof.** Set \( y^n - y_h^n = y^n - P_h y^n + P_h y^n - y_h^n = \eta^n + \zeta^n \). Choosing \( v = w_h \) in (2) and subtracting (2) from (6), we obtain the following error equation:
\[
(L^a_\theta^n (y^n - y_h^n), w_h) + a(y^n - y_h^n, w_h) + (r^n, w_h) = 0. \tag{17}
\]
By using the definition of \( P_h \) and (17), we have
\[
(L^a_\theta^n \zeta^n, w_h) + a(\zeta^n, w_h) = -(L^a_\theta^n \eta^n, w_h) - (r^n, w_h). \tag{18}
\]
Let \( w_h = L^a_\theta \zeta^n \). Using Lemma 3.1, we get
\[
(L^a_\theta \zeta^n, L^a_\theta \zeta^n) + a(\zeta^n, L^a_\theta \zeta^n) \geq (L^a_\theta \zeta^n, L^a_\theta \zeta^n) + c(\nabla \zeta^n, \nabla (L^a_\theta \zeta^n)) = \|L^a_\theta \zeta^n\|^2 + \frac{c}{2\tau^a \Gamma(2-a)} \left( \|\nabla \zeta^n\|^2 + \sum_{k=0}^{n-1} b_k \sum_{i=0}^{n-k} \|\nabla (\zeta^i)\|^2 \right). \tag{19}
\]
According to \( y \in W^{1,\infty}(H^2) \) and (4), there holds
\[
\|L^a_\theta \eta^n\| = \left\| \frac{1}{\Gamma(2-a)} \sum_{k=0}^{n-1} b_k \partial_i \eta^n - k \right\|
\leq \frac{1}{\Gamma(2-a)} \sum_{k=0}^{n-1} b_k \|\partial_i \eta^n - k\|
\leq \frac{1}{\Gamma(2-a)} \sum_{k=0}^{n-1} b_k \int_{t_{n-k-1}}^{t_n} \|\eta\| \, dt
\leq Ch \|y\|_{L^\infty(H^2)}.
\]
Then, by applying Hölder’s inequality, Young’s inequality and (7), we arrive at
\[
(L^a_\theta \eta^n, L^a_\theta \zeta^n) \leq \|L^a_\theta \eta^n\| \|L^a_\theta \zeta^n\| \leq \frac{1}{2} \|L^a_\theta \zeta^n\|^2 + \frac{C}{2} h^4. \tag{21}
\]
Likewise
\[
(r^n, L^a_\theta \zeta^n) \leq \|r^n\| \|L^a_\theta \zeta^n\| \leq \frac{1}{2} \|L^a_\theta \zeta^n\|^2 + \frac{C}{2} \tau^{4-2a}. \tag{22}
\]
Combining (18)–(22) and noting that \( b_k^n < 0 \) \((0 \leq k < n)\), we obtain
\[
\|\nabla \zeta^n\|_2^2 \leq - \sum_{k=0}^{n-1} b_k^n \|\nabla \zeta^n\|_2^2 + C \tau^n (h^2 + \tau^{2-a})^2.
\] (23)

It follows from (23), Lemma 3.2 and Poincaré’s inequality that
\[
\|\zeta^n\| \leq C (h^2 + \tau^{2-a})\] (24)

Hence, we complete the proof of Theorem 4.1.

5 A posteriori error estimates

We introduce recovery operators \( R_h \) and \( G_h \). Similar to the Z–Z patch recovery in [27], \( R_{h^2 v} \) be a continuous piecewise linear function (without zero boundary constraint), the value of \( R_{h^2 v} \) on the nodes is defined by least-squares argument on element patches surrounding the nodes. Then the gradient recovery operator \( G_h = (R_{h_1}, R_{h_2}) \). The details can be found in [56].

**Theorem 5.1.** Let \( y \) and \( y_h \) be the solutions of (2) and (6), respectively. Suppose that \( y \in W^{2,\infty}(L^2) \cap W^{1,\infty}(H^2) \). Then for any integer \( 1 \leq n \leq N \), we have
\[
\|G_{h^2} y_h^n - \nabla y^n\|_{L^2} \leq C (h^2 + \tau^{2-a}).
\] (25)

**Proof.** Let \( y_i^n \) be the piecewise linear Lagrange interpolation of \( y^n \). According to Theorem 2.1.1 in [20], we have
\[
\|P_{h^2} y^n - y_i^n\|_{H^1} \leq Ch^2 \|y^n\|_{H^1}.
\] (26)

From the interpolation error estimate [18], we get
\[
\|G_{h^2} y_h^n - \nabla y^n\| \leq C h \|y^n\|_{H^2}.
\] (27)

From triangle inequality, (16) and (26)–(27), we obtain
\[
\|G_{h^2} y_h^n - \nabla y^n\|_{L^2} \leq \|G_{h^2} y_h^n - G_h P_{h^2} y^n\|_{L^2} + \|G_h P_{h^2} y^n - G_h y_h^n\|_{L^2} + \|G_h y_h^n - \nabla y^n\|_{L^2} \\
\leq C \|y_h^n - P_{h^2} y^n\|_{H^2} + C \|P_{h^2} y^n - y_i^n\|_{H^2} + \|G_h y_h^n - \nabla y^n\|_{L^2} \leq C (h^2 + \tau^{2-a}).
\] (28)

Hence, we complete the proof of (25).

Combining the previous results, we obtain the following a posteriori error estimates of fully discrete finite element approximation for fractional evolution equations.

**Theorem 5.2.** Assume that all the conditions in Theorem 4.1 and Theorem 5.1 are valid. Then
\[
\eta^n = \|G_{h^2} y_h^n - \nabla y_h^n\|_{L^2} = \|\nabla (y_h^n - y^n)\|_{L^2} + O(h^2 + \tau^{2-a}).
\] (29)

**Proof.** According to (25) and triangle inequality, it is easy to get (29).

6 Numerical experiments

In this section, we present some different numerical examples to illustrate the correctness of the convergence and superconvergence results and the reliable and efficient a posteriori error estimates.
For an acceptable iteration error $Tol$, we present an uniformly refined FEM algorithm for the discrete problem (6) of fractional evolution equations.

**Algorithm 6.1.** FEM algorithm

1. Set $i = 1$, initialize mesh $T^h_i$ and $y^0_{h,i}$.
2. Solve the following discrete equations:
   $$(L^a_i y^a_{h,i}, w_h) + a(y^a_{h,i}, w_h) = (f^a, w_h), \quad \forall w_h \in W_h, \ n = 1, 2, \ldots, N.$$ 
3. Uniformly refine the meshes obtain new meshes $T^h_{i+1}$.
4. Calculate the iterative error: $E_i = \|y^a_{h,i} - y^a_{h,i-1}\|_{L^\infty(U^2)}$;
5. If $E_i > Tol$, $i = i + 1$, go to Step 2; else stop.

For an acceptable iteration error $Tol$, by selecting $\eta^n$ in (29) as mesh refinement indicators, we construct the following adaptive FEM algorithm for the discrete problem (6):

**Algorithm 6.2.** Adaptive FEM algorithm

1. Set $i = 1$, initialize mesh $T^h_i$ and $y^0_{h,i}$.
2. Solve the following discretized problems:
   $$(L^a_i y^a_{h,i}, w_h) + a(y^a_{h,i}, w_h) = (f^a, w_h), \quad \forall w_h \in W_h, \ n = 1, 2, \ldots, N.$$ 
3. Obtain numerical solution $y^n_{h,i}(n = 1, 2, \ldots, N)$ on the current meshes $T^h_i$ and calculate the error estimators $\eta^n_i$;
4. Adjust the meshes by using the estimators $\eta^n_i$ obtain new meshes $T^h_{i+1}$;
5. Calculate the iterative error: $E_i = \|y^a_{h,i} - y^a_{h,i-1}\|_{L^\infty(U^2)}$;
6. If $E_i > Tol$, $i = i + 1$, go to Step 2; else stop.

The following examples were dealt numerically with codes developed based on AFEPack, which is freely available and the details can be found in [56]. The discretization was described in Section 2. We denote $\|\cdot\|_{L^\infty(U^t)}$ and $\|\cdot\|_{L^\infty(U^2)}$ by $\|\cdot\|_{H^0}$ and $\|\cdot\|_{L^\infty}$, respectively. The convergence order rate is computed by the following formula:

$$\text{Rate} = \frac{\ln(e_{i+1}) - \ln(e_i)}{\ln(h_{i+1}) - \ln(h_i)},$$

where $e_i (e_{i+1})$ denotes the error when the spatial partition size is $h_i (h_{i+1})$. $E$ is the 2-by-2 identity matrix.

**Example 6.1.** This is a 1D example. The data are as follows:

$$\Omega = (0, 1), \quad T = 1, \quad A(x) = 1, \quad y(t, x) = t \sin(2\pi x), \quad f(t, x) = \left(\frac{1}{t^a(2 - a)} + 4\pi^2\right)y(t, x).$$

This example is solved by Algorithm 6.1. For different $\alpha$ values, the errors on a sequence of uniformly refined meshes’ size $h$ and time step size $\tau$ are shown in Tables 1–3. It is easy to see $\|y - y_h\|_{L^\infty} = O(h^2 + \tau^{2-\alpha})$, $\|y - y_h\|_{H^0} = O(h + \tau^{2-\alpha})$ and $\|P_h y - y_h\|_{L^\infty} = O(h^2 + \tau^{2-\alpha})$. In Figure 1, we show the numerical solution $y_h$ with $\alpha = 0.5$ when $h = \frac{1}{40}$ and $\tau = \frac{1}{90}$.
Table 1: Numerical results with $\alpha = 0.5$, Example 6.1

| $h$ | $\tau$ | $\|y - y_h\|_{L^\infty}$ | Rate | $\|y - y_h\|_{L^1}$ | Rate | $|Py - y_h|_{L^\infty}$ | Rate |
|-----|--------|---------------------------|------|----------------------|------|--------------------------|------|
| $1/10$ | $1/10$ | $1.524550 \times 10^{-3}$ | —    | $7.84280 \times 10^{-1}$ | —    | $5.24997 \times 10^{-2}$ | —    |
| $1/20$ | $1/20$ | $4.069121 \times 10^{-4}$ | 1.9056 | $4.01217 \times 10^{-1}$ | 0.9670 | $1.47561 \times 10^{-2}$ | 1.8310 |
| $1/40$ | $1/40$ | $1.021949 \times 10^{-4}$ | 1.9934 | $2.01314 \times 10^{-1}$ | 0.9949 | $3.90295 \times 10^{-3}$ | 1.9187 |
| $1/80$ | $1/80$ | $2.560060 \times 10^{-5}$ | 1.9971 | $1.00716 \times 10^{-1}$ | 0.9992 | $1.00212 \times 10^{-3}$ | 1.9615 |

Table 2: Numerical results with $\alpha = 0.5$, Example 6.1

| $h$ | $\tau$ | $\|y - y_h\|_{L^\infty}$ | Rate | $\|y - y_h\|_{L^1}$ | Rate | $|Py - y_h|_{L^\infty}$ | Rate |
|-----|--------|---------------------------|------|----------------------|------|--------------------------|------|
| $1/10$ | $1/10$ | $2.14883 \times 10^{-3}$ | —    | $7.84300 \times 10^{-1}$ | —    | $5.02080 \times 10^{-2}$ | —    |
| $1/20$ | $1/20$ | $5.56062 \times 10^{-4}$ | 1.9502 | $4.01220 \times 10^{-1}$ | 0.9670 | $1.41640 \times 10^{-2}$ | 1.8257 |
| $1/40$ | $1/40$ | $1.35887 \times 10^{-4}$ | 2.0328 | $2.01314 \times 10^{-1}$ | 0.9949 | $3.76104 \times 10^{-3}$ | 1.9130 |
| $1/80$ | $1/80$ | $3.32722 \times 10^{-5}$ | 2.0300 | $1.00716 \times 10^{-1}$ | 0.9992 | $9.68909 \times 10^{-4}$ | 1.9567 |

Table 3: Numerical results with $\alpha = 0.95$, Example 6.1

| $h$ | $\tau$ | $\|y - y_h\|_{L^\infty}$ | Rate | $\|y - y_h\|_{L^1}$ | Rate | $|Py - y_h|_{L^\infty}$ | Rate |
|-----|--------|---------------------------|------|----------------------|------|--------------------------|------|
| $1/10$ | $1/10$ | $4.06759 \times 10^{-3}$ | —    | $7.84402 \times 10^{-1}$ | —    | $4.31645 \times 10^{-2}$ | —    |
| $1/20$ | $1/20$ | $1.05515 \times 10^{-3}$ | 1.9467 | $4.01235 \times 10^{-1}$ | 0.9671 | $1.21829 \times 10^{-2}$ | 1.8250 |
| $1/40$ | $1/40$ | $2.56966 \times 10^{-4}$ | 2.0378 | $2.01316 \times 10^{-1}$ | 0.9950 | $3.25106 \times 10^{-3}$ | 1.9059 |
| $1/80$ | $1/80$ | $6.24374 \times 10^{-5}$ | 2.0411 | $1.00717 \times 10^{-1}$ | 0.9992 | $8.42640 \times 10^{-4}$ | 1.9479 |

Figure 1: The numerical solution $y_h$ when $h = 1/40$, $\tau = 1/90$, $\alpha = 0.5$, Example 6.1.
Example 6.2. This is a 2D example. The data are as follows:

\[
\begin{align*}
\Omega &= (0, 1) \times (0, 1), \ T = 1.0, \ A(x) = E, \\
y(t, x) &= t^3 \sin(2\pi x) \sin(2\pi y), \\
f(t, x) &= \left(\frac{6}{t^4} + 8\pi^2\right) y(t, x).
\end{align*}
\]

This example is solved by Algorithm 6.1. In Tables 4–6, the errors \(\|y - y_h\|_{0,\infty}^\alpha, \|y - y_h\|_{1,\infty}^\alpha\) and \(\|P_h y - y_h\|_{1,\infty}^\alpha\) based on different \(a\) values and a sequence of uniformly refined meshes’ size \(h\) and time step size \(\tau\) are shown. It is easy to see that \(\|y - y_h\|_{0,\infty}^\alpha\) and \(\|P_h y - y_h\|_{1,\infty}^\alpha\) are the second-order convergent while \(\|y - y_h\|_{1,\infty}^\alpha\) is the first-order convergent. We plot the profile of the numerical solution \(y_h\) with \(t = 0.5\) and \(\alpha = 0.5\), where \(h = \frac{1}{40}\) and \(\tau = \frac{1}{90}\) in Figure 2.

Table 4: Numerical results with \(\alpha = 0.05\), Example 6.2

| \(h\)  | \(\tau\) | \(\|y - y_h\|_{0,\infty}^\alpha\) | Rate | \(\|y - y_h\|_{1,\infty}^\alpha\) | Rate | \(\|P_h y - y_h\|_{1,\infty}^\alpha\) | Rate |
|-------|-------|--------------------------|-----|--------------------------|-----|--------------------------|-----|
| \(\frac{1}{10}\) | \(\frac{1}{10}\) | 5.45891 \times 10^{-2} | — | 1.35785 | — | 6.88961 \times 10^{-3} | — |
| \(\frac{1}{20}\) | \(\frac{1}{20}\) | 1.42607 \times 10^{-2} | 1.9366 | 6.93043 \times 10^{-1} | 0.9703 | 1.75476 \times 10^{-3} | 1.9731 |
| \(\frac{1}{40}\) | \(\frac{1}{40}\) | 3.60530 \times 10^{-3} | 1.9839 | 3.48335 \times 10^{-1} | 0.9925 | 4.41402 \times 10^{-4} | 1.9911 |
| \(\frac{1}{80}\) | \(\frac{1}{80}\) | 9.03851 \times 10^{-4} | 1.9960 | 1.74395 \times 10^{-2} | 0.9981 | 1.0657 \times 10^{-4} | 1.9960 |

Table 5: Numerical results with \(\alpha = 0.5\), Example 6.2

| \(h\)  | \(\tau\) | \(\|y - y_h\|_{0,\infty}^\alpha\) | Rate | \(\|y - y_h\|_{1,\infty}^\alpha\) | Rate | \(\|P_h y - y_h\|_{1,\infty}^\alpha\) | Rate |
|-------|-------|--------------------------|-----|--------------------------|-----|--------------------------|-----|
| \(\frac{1}{10}\) | \(\frac{1}{10}\) | 5.39968 \times 10^{-2} | — | 1.35788 | — | 1.25360 \times 10^{-2} | — |
| \(\frac{1}{20}\) | \(\frac{1}{20}\) | 1.40961 \times 10^{-2} | 1.9376 | 6.93048 \times 10^{-1} | 0.9703 | 3.24284 \times 10^{-3} | 1.9507 |
| \(\frac{1}{40}\) | \(\frac{1}{40}\) | 3.56467 \times 10^{-3} | 1.9835 | 3.48335 \times 10^{-1} | 0.9925 | 8.03542 \times 10^{-4} | 2.0128 |
| \(\frac{1}{80}\) | \(\frac{1}{80}\) | 8.94056 \times 10^{-4} | 1.9953 | 1.74395 \times 10^{-2} | 0.9981 | 1.97545 \times 10^{-4} | 2.0242 |

Table 6: Numerical results with \(\alpha = 0.95\), Example 6.2

| \(h\)  | \(\tau\) | \(\|y - y_h\|_{0,\infty}^\alpha\) | Rate | \(\|y - y_h\|_{1,\infty}^\alpha\) | Rate | \(\|P_h y - y_h\|_{1,\infty}^\alpha\) | Rate |
|-------|-------|--------------------------|-----|--------------------------|-----|--------------------------|-----|
| \(\frac{1}{10}\) | \(\frac{1}{10}\) | 5.25616 \times 10^{-2} | — | 1.35809 | — | 2.71477 \times 10^{-2} | — |
| \(\frac{1}{20}\) | \(\frac{1}{20}\) | 1.36714 \times 10^{-2} | 1.9428 | 6.93079 \times 10^{-1} | 0.9705 | 7.29615 \times 10^{-3} | 1.8956 |
| \(\frac{1}{40}\) | \(\frac{1}{40}\) | 3.45608 \times 10^{-3} | 1.9840 | 3.48339 \times 10^{-1} | 0.9925 | 1.8208 \times 10^{-3} | 2.0026 |
| \(\frac{1}{80}\) | \(\frac{1}{80}\) | 8.88086 \times 10^{-4} | 1.9604 | 1.74396 \times 10^{-2} | 0.9981 | 4.52293 \times 10^{-4} | 2.0092 |
Example 6.3. This is a 2D example. The data are as follows:

$$\Omega = (0, 1) \times [0, 1], \quad T = 1.0,$$

$$A(x) = \begin{cases} 4E, & x_1 + x_2 \leq 1, \\ E, & x_1 + x_2 > 1, \end{cases}$$

$$y(t, x) = \begin{cases} t^2 \sin(\pi x_1) \sin(\pi x_2), & x_1 + x_2 \leq 1, \\ t^2 \sin(2\pi x_1) \sin(2\pi x_2), & x_1 + x_2 > 1, \end{cases}$$

$$f(t, x) = \left(\frac{2}{t^6} + 8\pi^2\right)y(t, x).$$

We take $\tau = 10^{-2}$ and solve this example by using Algorithms 6.1 and 6.2. Numerical results based on a sequence of uniformly refined meshes and adaptive meshes are listed in Tables 7 and 8, respectively. It is clear that the adaptive meshes generated via the error estimators $\eta^n$ are able to save substantial computational work. We plot the profile of the numerical solution $y_h$ at $t = 0.5$, where adaptive mesh nodes = 1,241 in Figure 3.

Table 7: Numerical results for Example 6.3 on uniform meshes

| Uniform meshes | 1 | 2 | 3 | 4 |
|----------------|---|---|---|---|
| Nodes of $y_h^n$ | 121 | 441 | 1,681 | 6,561 |
| $\|\nabla y - \nabla y_h^0,\|_{L^\infty}$ | $3.85733 \times 10^{-1}$ | $3.72531 \times 10^{-1}$ | $3.65088 \times 10^{-1}$ | $3.60944 \times 10^{-1}$ |
| $\|\nabla y_h - \nabla y_h^0,\|_{L^\infty}$ | $3.91964 \times 10^{-1}$ | $2.22143 \times 10^{-1}$ | $1.62862 \times 10^{-1}$ | $1.17960 \times 10^{-1}$ |

Table 8: Numerical results for Example 6.3 on adaptive meshes

| Adaptive meshes | 1 | 2 | 3 | 4 |
|-----------------|---|---|---|---|
| Nodes of $y_h^n$ | 121 | 383 | 817 | 1,241 |
| $\|\nabla y - \nabla y_h^0,\|_{L^\infty}$ | $3.85733 \times 10^{-1}$ | $3.70120 \times 10^{-1}$ | $3.67693 \times 10^{-1}$ | $3.58027 \times 10^{-1}$ |
| $\|\nabla y_h - \nabla y_h^0,\|_{L^\infty}$ | $3.91964 \times 10^{-1}$ | $2.12143 \times 10^{-1}$ | $1.52864 \times 10^{-1}$ | $1.15966 \times 10^{-1}$ |
7 Conclusion

Although there has been extensive research on FEMs for FPDEs, mostly focused on convergence analysis [7–11]. While there is little work on a posteriori error estimates of FEM for FPDEs. Hence, our results on a posteriori error estimates and adaptive FEM for fractional evolution equations are new.

Funding information: Yuelong Tang is supported by the National Natural Science Foundation of China (11401201), the Natural Science Foundation of Hunan Province (2020JJ4323), the Scientific Research Project of Hunan Provincial Department of Education (20A211), the construct program of applied characteristic discipline in Hunan University of Science and Engineering. Yuchun Hua is supported by the Scientific Research Project of Hunan Provincial Department of Education (20C0854), the scientific research program in Hunan University of Science and Engineering (20XY059).

Conflict of interest: Authors state no conflict of interest.

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