QUATERNIONIC PROJECTIVE INVARIANCE OF THE $k$-CAUCHY-FUETER COMPLEX AND APPLICATIONS I.

WEI WANG

Abstract. The $k$-Cauchy-Fueter complex in quaternionic analysis is the counterpart of the Dolbeault complex in complex analysis. In this paper, we find the explicit transformation formula of these complexes under $\text{SL}(n+1,\mathbb{H})$, which acts on $\mathbb{H}^n$ as quaternionic fractional linear transformations. These transformation formulae have several interesting applications to $k$-regular functions, the quaternionic counterpart of holomorphic functions, and geometry of domains. They allow us to construct the $k$-Cauchy-Fueter complex over locally projective flat manifolds explicitly and introduce various notions of pluripotential theory on this kind of manifolds. We also introduce a quaternionic projectively invariant operator from the quaternionic Monge-Ampère operator, which can be used to find projectively invariant defining density of a domain, generalizing Fefferman’s construction in complex analysis.

1. Introduction

Since 1980s, people have been interested in developing analysis of several quaternionic variables [35]. The quaternionic counterpart of the Cauchy-Riemann operator is a family of operators acting on $\otimes^k \mathbb{C}^2$-valued functions, called the $k$-Cauchy-Fueter operator, $k = 0, 1, \ldots$, because the group $\text{SU}(2)$ of unit quaternionic numbers has a family of irreducible representations $\otimes^k \mathbb{C}^2$, while the group of unit complex numbers has only one irreducible representation space $\mathbb{C}$. As the quaternionic counterpart of the Dolbeault complex, the $k$-Cauchy-Fueter complexes on the flat space $\mathbb{H}^n$ are known explicitly now (cf. [9] [10] [18] [19] [44] and references therein):

$$0 \to \Gamma(D, \mathcal{V}_0) \xrightarrow{D_0} \Gamma(D, \mathcal{V}_1) \xrightarrow{D_1} \cdots \xrightarrow{D_{2n-2}} \Gamma(D, \mathcal{V}_{2n-1}) \to 0,$$

for a domain $D$ in $\mathbb{H}^n$, where $\Gamma(D, \mathcal{V}_j)$ is the space of $\mathcal{V}_j$-valued smooth functions, and

$$\mathcal{V}_j := \left\{ \begin{array}{ll}
\otimes^j \mathbb{C}^2 \otimes \wedge^j \mathbb{C}^{2n*}, & j = 0, \ldots, k, \\
\otimes^{j-k-1} \mathbb{C}^2 \otimes \wedge^{j+1} \mathbb{C}^{2n*}, & j = k + 1, \ldots, 2n-1.
\end{array} \right.$$ 

They have several interesting applications to quaternionic analysis, e.g. to the quaternionic Monge-Ampère operator and quaternionic plurisubharmonic functions (cf. [43] [47] and references therein). A function $f \in \Gamma(D, \otimes^k \mathbb{C}^2)$ is called $k$-regular if $D_0 f = 0$ on $D$. The space of all $k$-regular functions on $D$ is denoted by $\mathcal{O}_k(D)$. Because of Hartogs’ phenomenon for $k$-regular functions (see e.g. [18] [35] [44]), it is a fundamental problem to characterize domains of $k$-regularity, the quaternionic counterpart of domains of holomorphy, and domains with vanishing cohomology of the $k$-Cauchy-Fueter complex.

Key words and phrases. $\text{SL}(n+1,\mathbb{H})$-invariance; the $k$-Cauchy-Fueter complex; quaternionic fractional linear transformations; $k$-regular functions; locally quaternionic projective flat manifolds; quaternionic linearly convex domains; the quaternionic Monge-Ampère operator; Fefferman defining density.

Supported by National Nature Science Foundation in China (No. 11971425).

Department of Mathematics, Zhejiang University, Zhejiang 310027, P. R. China, Email: wwang@zju.edu.cn.
It is a useful and important fact in complex analysis that the product of two holomorphic functions is also holomorphic, and so is the composition of two holomorphic transformations. Moreover, the Cauchy-Riemann operator and Dolbeault complex are invariant under biholomorphic transformations, and so they exist on complex manifolds. The counterpart of a holomorphic transformation is the notion of a regular transformation
\[ f : \mathbb{H}^n \to \mathbb{H}^n, \]
i.e. each component of \( f \) is a 1-regular \( \mathbb{H} \)-valued function. But the composition of a \( k \)-regular function with a regular transformation is usually not \( k \)-regular because of non-commutativity. It is necessary to known under which transformations of \( \mathbb{H}^n \) the \( k \)-regularity is preserved, i.e. to clarify the invariant group of the \( k \)-Cauchy-Fueter operator and complex. Liu-Zhang \cite{31} constructed and investigated invariant operators on the quaternionic hyperbolic space under the action of \( \text{Sp}(n,1) \) by using representation theory, which coincide with the \( k \)-Cauchy-Fueter operator for \( k \geq 1 \).

The Cauchy-Riemann operator is unique in the sense that it has an invariant group of infinite dimensions, while for all known generalizations, such as the Dirac operator in Clifford analysis and the tangential Cauchy-Riemann operator etc., their invariant groups are only of finite dimensions. But they are still large enough to have various important applications (cf. e.g. \cite{34, 38}). More generally, it is an active direction to investigate the function theory of conformally invariant operators of higher spins (cf. \cite{7, 8, 20, 21, 22, 30} and references therein). On 4-dimensional Minkowski space, they are massless field operators for higher spins in physics, which are systematically investigated by Penrose et. al. \cite{36, 37} with the help of conformal invariance. Operators of higher spins on the Euclidean space and on the Minkowski space have the same complexification. They are also explored from the point of view of representation theory by Frenkel-Libine \cite{25, 26}.

The 1-Cauchy-Fueter complex has been studied by using commutative algebra and computer algebra method since 90s (cf. \cite{18} and references therein). Meanwhile, Baston \cite{5} constructed a family of quaternionic complexes over complexified quaternionic-Kähler manifolds by using the twistor method, generalizing Eastwood-Penrose-Wells result for \( n = 1 \) \cite{23}. The twistor construction implies the invariance of complexes under the action of \( \text{SL}(2n+2, \mathbb{C}) \), which was used to find explicit form of operators in the complexified version of complexes \cite{9, 10, 19}. See also \cite{13, 14, 17, 39} and references therein for the construction of invariant differential operators and complexes. Several interesting differential complexes over curved manifolds have been constructed from BGG sequences \cite{15, 16} associated to a semisimple Lie algebra \( g \) and a parabolic subalgebra. This construction can be applied to quaternionic manifolds. But the kernel of the first operator of a BGG sequence is a finite dimensional irreducible representation of \( g \), while for the \( k \)-Cauchy-Fueter complex, the kernel of the first operator (i.e. the \( k \)-Cauchy-Fueter operator) is of infinite dimensional. So it is not a BGG sequence. In this paper, we find the transformation formulae of each operator \( D_j \) in \( (1.1) \) under the action of \( \text{SL}(n+1, \mathbb{H}) \), which acts on \( \mathbb{H}^n \) as quaternionic fractional linear transformations. This transformation formulae have several important applications to \( k \)-regular functions and geometry of domains.

Let \( C^2 \) be the standard \( GL(1, \mathbb{H}) \)-module and let \( C^{2n} \) be the standard \( GL(n, \mathbb{H}) \)-module. Let \( C^{2*} \) and \( C^{2n*} \) be modules dual to \( C^2 \) and \( C^{2n} \), respectively. They are trivially extended to be
\[
G_0 = S(GL(1, \mathbb{H}) \times GL(n, \mathbb{H})) = (GL(1, \mathbb{H}) \times GL(n, \mathbb{H})) \cap \text{SL}(n+1, \mathbb{H})
\]
modules. It is convenient to identify a \( \otimes^\sigma C^2 \otimes \wedge^\tau C^{2n*} \)-valued function \( f \) with a function in variables \( q \in \mathbb{H}^n, s_{A'} \in C^2 \) and Grassmannian variables \( \omega^A \), which is homogeneous of degree \( \sigma \) in \( s_{A'} \) and of degree \( \tau \) in \( \omega^A \), i.e.
\[
f = f^A_{A'}(q)s_{A'}\omega^A
\]
where \( s_{A'} := s_{A'_1} \cdots s_{A'_r} \) for the multi-index \( A' = A'_1 \cdots A'_r \), and \( \omega^A := \omega^{A_1} \cdots \omega^{A_r} \) for the multi-index \( A = A_1 \cdots A_r \) \((A_j = 0, \ldots, 2n - 1, A'_j = 0', 1')\). Here and in the sequel, We use the Einstein convention of taking summation for repeated indices.

Write an element \( g \in \text{SL}(n+1, \mathbb{H}) \) as

\[
(1.3) \quad g^{-1} = \left( \begin{array}{c|c} a_{1 \times 1} & b_{1 \times n} \\ \hline n_{1 \times 1} & d_{n \times n} \end{array} \right)
\]

where \( a, b, c \) and \( d \) are quaternionic matrices. It defines a fractional linear transformation of \( \mathbb{H}^n \):

\[
(1.4) \quad T_{g^{-1}} : z \mapsto g^{-1}.q := (c + dq)(a + bq)^{-1},
\]

Denote

\[
J_1(g^{-1}, q) := a + bq \in \mathbb{H},
\]

\[
J_2(g^{-1}, q) := d - (c + dq)(a + bq)^{-1}b \in \text{GL}(n, \mathbb{H})
\]

If \( j \leq k \), an element \( g \) in (1.3) acts on \( f \in \Gamma(\mathbb{H}^n, \mathcal{V}_j) \) given by (1.2) as

\[
(1.5) \quad [\pi_j(g)f](q) := \frac{f^A'(g^{-1}.q)}{|a + bq|^{2(j+1)}} J_1(g^{-1}, q)^{-1}.s^{A'}J_2(g^{-1}, q).\omega^A.
\]

If \( j \geq k \), we write \( f \in \Gamma(\mathbb{H}^n, \mathcal{V}_j) \) as

\[
(1.6) \quad f = f_{A'A}^{\omega^A}.s^{A'}\omega^A,
\]

where we use \( s^{A'} \) as coordinate functions on \( \mathbb{C}^2^s \). An element \( g \) in (1.3) acts as

\[
(1.7) \quad [\pi_j(g)f](q) := \frac{f^A'(g^{-1}.q)}{|a + bq|^{2(j+1)}} J_1(g^{-1}, q)^{-1}.s^{A'}J_2(g^{-1}, q).\omega^A.
\]

\( \pi_j \) is not a real representation on \( \Gamma(\mathbb{H}^n, \mathcal{V}_j) \), because for \( f \in \Gamma(\mathbb{H}^n, \mathcal{V}_j) \), \( \pi_j(g)f \) is singular on the quaternionic hyperplane

\[
\mathcal{L}_g := \{ q \in \mathbb{H}^n; a + bq = 0 \}.
\]

But outside of singularities, it still satisfies the identity of a representation:

\[
(1.8) \quad \pi_j(g_1)\pi_j(g_2)f = \pi_j(g_1g_2)f.
\]

**Theorem 1.1.** \( D_j \) is \( \text{SL}(n+1, \mathbb{H}) \)-invariant, i.e.

\[
D_j(\pi_j(g)f) = \pi_{j+1}(g)D_jf.
\]

for any \( g \in \text{SL}(n+1, \mathbb{H}) \) and \( f \in \Gamma(\mathbb{H}^n, \mathcal{V}_j) \).

The invariance implies that if \( f \) is \( k \)-regular on a domain \( D \subset \mathbb{H}^n \), then

\[
(1.9) \quad \frac{1}{|a + bq|^2}(a + bq)^{-1}.f((c + dq)(a + bq)^{-1})
\]

is also \( k \)-regular on \( g.D \setminus \mathcal{L}_g \) for any \( g \in \text{SL}(n+1, \mathbb{H}) \). In particular, if we take a \( \mathcal{V}_0 \)-valued constant function \( f = s_{A'} \), then the rational function

\[
(1.10) \quad \frac{1}{|a + bq|^2}(a + bq)^{-1}.s_{A'}
\]

is \( k \)-regular. This allows us to introduce the quaternionic version of the Fantappiè transformation (1.9), and leads to an interesting question when any \( k \)-regular function on a subset of \( \mathbb{H}^n \) is the superposition of the simple rational functions of the form (1.10).
A domain $D \subset \mathbb{H}^n$ is called *(quaternionic) linearly convex* if for any $p \in \partial D$, there is an hyperplane of quaternionic dimension $n - 1$ passing through $p$ and not intersecting $D$. This notion is the generalization of the complex one. As a consequence, a linearly convex domain is a domain of $k$-regularity.

A manifold is called *locally (quaternionic) projective flat* if it has coordinates charts $\{(U_\alpha, \phi_\alpha)\}$ with $\phi_\alpha : U_\alpha \to \mathbb{H}^n$ and transition maps
\begin{equation}
\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)
\end{equation}
given by the induced action (1.2) for some $g \in \text{SL}(n + 1, \mathbb{H})$. If $\Gamma$ is a discrete subgroup of $\text{SL}(n + 1, \mathbb{H})$, $\mathbb{H}P^n/\Gamma$ is a locally projective flat manifold. The quaternionic hyperbolic space can be realized as the unit ball $B^{4n}$, whose group of isometric automorphisms is $\text{Sp}(n, 1) \subset \text{SL}(n + 1, \mathbb{H})$. If $\Gamma$ is a discrete subgroup of $\text{Sp}(n, 1)$, then $B^{4n}/\Gamma$ is a locally projective flat manifold. In particular, if $\Gamma$ is a cocompact or convex cocompact subgroup of $\text{Sp}(n, 1)$, then $B^{4n}/\Gamma$ is a compact locally projective flat manifold without or with boundary (a spherical quaternionic contact manifold) (32).

$J_\mu$ is a cocycle, i.e.
\begin{equation}
J_\mu(g_2^{-1}g_1^{-1}, q) = J_\mu(g_2^{-1}, g_1^{-1}.q)J_\mu(g_1^{-1}, q), \quad \mu = 1, 2.
\end{equation}

$J_1^{-1}$ can be used to glue trivial $\mathbb{C}^2$-bundles to obtain the bundle $H$. We use $J_1$ to glue trivial $\mathbb{C}^{2n}$-bundles to obtain the bundle $H^*$. Here the action $J_1$ on the representation $\mathbb{C}^{2n}$ is dual to the action of $J_1^{-1}$. While $J_2$ can be used to glue trivial $\mathbb{C}^{2n+1}$-bundles to obtain the bundle $E^*$. Let $\wedge^T E^*$ be the $\tau$-th exterior product of $E^*$, and let $\wedge^\sigma H$ and $\wedge^\sigma H^*$ be the $\sigma$-th symmetric products of $H$ and $H^*$, respectively. There also exists a distinguished line bundle $\mathbb{R}[-1]$ so that
\begin{equation}
\wedge^{4n} T^* M \cong \mathbb{R}[-2n - 2],
\end{equation}
where $\mathbb{R}[-l] = \otimes^l \mathbb{R}[-1]$, and $\mathbb{C}[-1] \cong \wedge^2 H^*$. Denote $V[-l] := V \otimes \mathbb{R}[-l]$ for a vector bundle $V$. On a locally projective flat manifold $M$, we have the the $k$-Cauchy-Fueter complex:
\begin{equation}
0 \to \Gamma(M, \mathcal{V}_0) \xrightarrow{D_0} \Gamma(M, \mathcal{V}_1) \xrightarrow{D_1} \cdots \xrightarrow{D_{2n-2}} \Gamma(M, \mathcal{V}_{2n-1}) \to 0,
\end{equation}
where
\begin{equation*}
\mathcal{V}_j := \left\{ \begin{array}{l l}
\wedge^k H \otimes \wedge^j E^*[-j - 1], & j = 0, \cdots, k; \\
\wedge^{j-k-1} H^* \otimes \wedge^{j+1} E^*[-j - 1], & j = k + 1, \cdots, 2n - 1.
\end{array} \right.
\end{equation*}

For $k = 0$, $D_0$ is the Baston operator $\triangle : \Gamma(M, \mathbb{R}[{-1}]) \to \Gamma(M, \wedge^2 E^*[{-2}])$. A upper semicontinuous section of $\mathbb{R}[{-1}]$ is said to be plurisubharmonic if $\Delta u$ is a closed positive 2-current. The quaternionic Monge-Ampère operator on a locally projective flat manifold is defined as $(\Delta u)^n : \Gamma(M, \mathbb{R}[{-1}]) \to \Gamma(M, \wedge^{2n} E^*[{-2n}])$.

Recall that a quaternionic-Kähler manifold $M$ is a Riemannian manifold whose Levi-Civita connection preserves the quaternionic structure, i.e. the frame bundle of $M$ reduces to a principal $\text{Sp}(n)\text{Sp}(1)$-bundle with a torsion-free connection. The quaternionic hyperbolic space $B^{4n}$ is quaternionic-Kähler, and so is $B^{4n}/\Gamma$ for a discrete subgroup $\Gamma$ of the isometric group $\text{Sp}(n, 1)$ of the quaternionic hyperbolic metric. But for a discrete subgroup $\Gamma$ of $\text{SL}(n + 1, \mathbb{H})$, the locally projective flat manifold $\mathbb{H}P^n/\Gamma$ is not quaternionic-Kähler in general, since the manifold may have nonvanishing torsion. The construction of locally projective flat manifolds is easy, because we don’t need to construct special connections on them.

Alesker [3] constructed and investigated the quaternionic Monge-Ampère operator on quaternionic-Kähler manifolds by the twistor method and method of complexification of such manifolds by Baston [23]. The quaternionic Monge-Ampère operator in [3] is defined in terms of the quaternionic-Kähler connection, while on locally projective flat manifolds, the quaternionic Monge-Ampère operator is easily
defined. Moreover, it allows us to introduce various notions of pluripotential theory on this kind of manifolds, in particular, closed positive currents and their “integrals”, etc.

The paper is organized as follows. In Section 2, we describe the complexified version of the $k$-Cauchy-Fueter complex over the complex space $\mathbb{C}^{2n \times 2}$, on which $\text{SL}(2n + 2, \mathbb{C})$ acts as complex fractional linear transformations. In Section 3, the $\text{SL}(2n + 2, \mathbb{C})$-invariance of the complexified $k$-Cauchy-Fueter complex is proved. It is reduced to its $\mathfrak{sl}(2n + 2, \mathbb{C})$-invariance, which can be checked more easily and directly. In Section 4, the $\text{SL}(n + 1, \mathbb{H})$-invariance in Theorem 1.1 is deduced from the $\text{SL}(2n + 2, \mathbb{C})$-invariance by using the embedding of $\mathbb{H}^n$ to $\mathbb{C}^{2n \times 2}$. Cocycles $J_{\mu}$’s are used to construct the bundles $H$, $H^*$ and $E^*$ over locally projective flat manifolds and the $k$-Cauchy-Fueter complex exists over such manifolds. In Section 5, we introduce various notions of pluripotential theory on locally projective flat manifolds. In Section 6, we construct a quaternionic projectively invariant operator from the quaternionic Monge-Ampère operator, which can be used to find projectively invariant defining density of a domain, as Fefferman [24] did in the complex case and Sasaki [11] and Marugame [32, 33] did for locally real projective flat manifolds. This defining density will be used to constructed various projectively invariant operators on boundaries in the subsequent part. It is also interesting to consider the quaternionic version of the generalization of Fefferman-type constructions to curved projective manifolds [12].

2. The complexified version of the $k$-Cauchy-Fueter complex

2.1. $\text{SL}(n + 1, \mathbb{H})$ and its complexification. The Lie algebra of $G = \text{SL}(n + 1, \mathbb{H})$ is $\mathfrak{g} = \mathfrak{sl}(n + 1, \mathbb{H}) = \{ A \in \mathfrak{gl}(n + 1, \mathbb{H}); \text{Re} \text{Tr} A = 0 \}$. Let $\mathfrak{g}_0 = \mathfrak{sl}(1, \mathbb{H}) \oplus \mathfrak{gl}(n, \mathbb{H}) = \mathfrak{sl}(1, \mathbb{H}) \oplus \mathfrak{sl}(n, \mathbb{H}) \oplus \mathbb{R}$. $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ with the grading easily visible in a block form with blocks of sizes $1, 2n$:

$$
\mathfrak{g}_{-1} = \left\{ \left( \begin{array}{cc} 0 & 0 \\ * & 0 \end{array} \right) \right\}, \quad \mathfrak{g}_0 = \left\{ \left( \begin{array}{cc} * & 0 \\ 0 & * \end{array} \right) \right\}, \quad \mathfrak{g}_1 = \left\{ \left( \begin{array}{cc} 0 & * \\ 0 & 0 \end{array} \right) \right\}.
$$

Thus, $\mathfrak{g}_{-1} \cong \mathbb{H}^n \cong \mathfrak{g}_1$. Its Lie brackets are given by $[X, Y] = XY - YX$ for $X, Y \in \mathfrak{g}$.

Write an element $g$ of $G^C = \text{SL}(2n + 2, \mathbb{C})$ as $(g_{\beta}^\alpha)$ with $\alpha, \beta = 0', 1', 0, 1, \cdots, 2n - 1$. We adopt the following index notations: $A, B, C, \cdots \in \{ 0, 1, \cdots, 2n - 1 \}$, $A', B', C', \cdots \in \{ 0', 1' \}$. Then we can write

$$
g = (g_{\beta}^\alpha) = \left( \begin{array}{cc} g_B^{A'} & g_B^A \\ g_B^{A'} & g_B^A \end{array} \right),
$$

where lower indices are column ones, while upper indices are row ones. It acts on vector $\left( \begin{array}{c} u_{A'} \\ u_A \end{array} \right) \in \mathbb{C}^{2(n+1)}$. As a dual module, element $(u_{A'}, u_A)$ in $\mathbb{C}^{2(n+1)*}$ is acted by matrix (2.1) from right, i.e. $g. v_{\alpha} = (u_{A'}, u_A) g^{-1}$.

Denote by $\mathfrak{g}_{-1}^C$, $\mathfrak{g}_0^C$ and $\mathfrak{g}_1^C$ subalgebras of the following forms

$$
\left( \begin{array}{cc} 0 & 0 \\ g_A^{A'} & 0 \end{array} \right), \quad \left( \begin{array}{cc} g_B^{A'} & 0 \\ 0 & g_B^A \end{array} \right), \quad \left( \begin{array}{cc} 0 & g_A^{A'} \\ 0 & 0 \end{array} \right),
$$

and by $\mathfrak{e}_\beta^\alpha$ the matrix with all entries zero except for the entry in $\alpha$-th column and $\beta$-th row to be one. Then

$$
[e_\beta^\alpha, e_\gamma^\delta] = \delta_\gamma^\alpha e_\beta^\delta - \delta_\beta^\gamma e_\delta^\alpha,
$$

where $\delta_{\gamma\alpha}$ is the Kronecker delta.
in particular,

\[
\begin{align*}
[\epsilon^A, \epsilon^B'] &= \delta^A_{B'} \epsilon^B - \delta^B_{A'} \epsilon^A, \\
\epsilon^A', \epsilon^B' &= -\delta^A_{B'} \epsilon^B, \\
\epsilon^A', \epsilon^C' &= \delta^A_{C'} \epsilon^C, \\
\epsilon^B', \epsilon^C' &= \delta^B_{C'} \epsilon^C.
\end{align*}
\] (2.3)

**Remark 2.1.** The matrix $g$ in (2.1) can be written as $g^\alpha_\beta \epsilon^\alpha_\beta$. The column and row indices of the tuple $(g^\alpha_\beta)$ and that of the basis $\epsilon^\alpha_\beta$ are exchanged. We use the upper indices of the tuple $(g^\alpha_\beta)$ as row indices as in differential geometry [27].

The parabolic subalgebra is

\[ p^C := g^C_0 \oplus g^C_1, \]

and let $P^C$ be corresponding subgroup. Then

\[ G_0^C = S (GL(2, \mathbb{C}) \times GL(2n, \mathbb{C})) = (GL(2, \mathbb{C}) \times GL(2n, \mathbb{C})) \cap \text{SL}(2n + 2, \mathbb{C}). \]

$\mathbb{C}^{2n+2} = \mathbb{C}^2 \oplus \mathbb{C}^{2n}$ as the defining representation of $\text{SL}(2n + 2, \mathbb{C})$ is a $P^C$-module. It is obvious that $\mathbb{C}^2$ in this decomposition is a $P^C$-module, and so is $\mathbb{C}^{2n*}$ in the decomposition $\mathbb{C}^{2(n + 1)*} = \mathbb{C}^{2*} \oplus \mathbb{C}^{2n*}$. They are also $G_0^C$-modules, and $\mathbb{C}^{2*}$ and $\mathbb{C}^{2n*}$ are $G_0^C$-modules dual to $\mathbb{C}^2$ and $\mathbb{C}^{2n}$, respectively. Then as $G_0^C$-modules,

\[ g^C_{-1} \cong \mathbb{C}^{2n} \otimes \mathbb{C}^{2*}, \quad g^C \cong \mathbb{C}^2 \otimes \mathbb{C}^{2n*}. \]

### 2.2. $G/P$

Let $G$ be a real or complex semisimple Lie group and $P$ is a parabolic subgroup. A point of the homogeneous space $G/P$ is a coset $hP$ for some $h \in G$. $g \in G$ acts on $G/P$ as

\[ g \cdot (hP) = ghP. \] (2.4)

Since for a function on $G/P$, the action defined by

\[ g \cdot f(hP) = f \left( g^{-1}hP \right) \]

is a group action, i.e. $g_2 \cdot (g_1 \cdot f) = (g_2 g_1) \cdot f$, we have to know the action of $g^{-1}$ on the homogeneous space.

In our case, $G$ is $\text{SL}(n + 1, \mathbb{H})$ or its complexification $\text{SL}(2n + 2, \mathbb{C})$. For an element $g \in \text{SL}(2n + 2, \mathbb{C})$, write

\[ g^{-1} = \left( \begin{array}{cc} a_{2 \times 2} & b_{2 \times 2n} \\ c_{2n \times 2} & d_{2n \times 2n} \end{array} \right) \]

where $a, b, c$ and $d$ are complex matrices. The parabolic subgroup $P$ consisting matrices of the form

\[ \left( \begin{array}{cc} a & b \\ 0 & d \end{array} \right), \]

and

\[ \left( \begin{array}{cc} 1_2 & 0 \\ z & 1_{2n} \end{array} \right) P, \quad z \in \mathbb{C}^{2n \times 2}, \]

constitute an open subset of $G/P$, which is holomorphically diffeomorphic to $\mathbb{C}^{2n \times 2}$, where $1_l$ is the $l \times l$ identity matrix.
Proposition 2.1. The action \([g \cdot z] = (c + dz)(a + bz)^{-1}z\) for \(SL(2n + 2, \mathbb{C})\) on \(\mathbb{C}^{2n} \times 2\) is given by
\[T^{-1}_g : z \rightarrow g^{-1}z = (c + dz)(a + bz)^{-1},\]
for \(g^{-1}\) in \((2.3)\).

Proof. This is because
\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\begin{pmatrix}
I_2 & 0 \\
z & I_{2n}
\end{pmatrix}
= \begin{pmatrix}
(a + bz) & b \\
c + dz & d
\end{pmatrix}
\begin{pmatrix}
I_2 & -(a + bz)^{-1}b \\
0 & I_{2n}
\end{pmatrix}
= \begin{pmatrix}
I_2 & 0 \\
0 & I_{2n}
\end{pmatrix}
\]
if we denote
\[J_1(g^{-1}, z) := a + bz,\]
\[J_2(g^{-1}, z) := d - (c + dz)(a + bz)^{-1}b.\]
Then \((2.7)\) mod \(P\) gives us the result.

Since the action \((2.6)\) is induced from \((2.4)\), it is a group action, i.e.
\[(g_2^{-1}.g_1^{-1}z) = (g_2^{-1}g_1^{-1}z).\]

Proposition 2.2. \(J_\mu(g, z)\) is a cocycle, i.e.
\[(2.8)\]
\[J_\mu(g_2^{-1}g_1^{-1}, z) = J_\mu(g_2^{-1}g_1^{-1}, z)J_\mu(g_1^{-1}, z), \quad \mu = 1, 2.\]

Proof. Denote \(g_\alpha^{-1} = \begin{pmatrix}
a_\alpha & b_\alpha \\
c_\alpha & d_\alpha
\end{pmatrix}\) \((\alpha = 1, 2)\) in \(SL(2n + 2, \mathbb{C})\). Then, by using \((2.7)\) twice, we get
\[
g_2^{-1}g_1^{-1} \begin{pmatrix}
I_2 & 0 \\
z & I_{2n}
\end{pmatrix}
= \begin{pmatrix}
a_2 & b_2 \\
c_2 & d_2
\end{pmatrix}
\begin{pmatrix}
I_2 & 0 \\
zg_1^{-1} & I_{2n}
\end{pmatrix}
\begin{pmatrix}
J_1(g_2^{-1}, z) & 0 \\
0 & J_2(g_2^{-1}, z)
\end{pmatrix}
\begin{pmatrix}
I_2 & J_1(g_2^{-1}, z)^{-1}b \\
0 & I_{2n}
\end{pmatrix}
= \begin{pmatrix}
I_2 & 0 \\
g_2^{-1}.g_1^{-1} & I_{2n}
\end{pmatrix}
\begin{pmatrix}
J_1(g_2^{-1}, g_1^{-1}, z) & 0 \\
0 & J_2(g_2^{-1}, g_1^{-1}, z)
\end{pmatrix}
\begin{pmatrix}
I_2 & * \\
0 & I_{2n}
\end{pmatrix}
\]
This together with the decomposition \((2.7)\) for \((g_1g_2)^{-1}\) and \((2.8)\) implies the cocycle condition \((2.9)\). □
for \( \mathbf{a} = (a_{ij}') \in \text{GL}(2, \mathbb{C}) \), since
\[
\mathbf{a}(\tilde{\mathbf{a}}. s_{A'}) = \mathbf{a}(\tilde{\mathbf{a}}_{A'} s_{B'}) = a'_{C'} \tilde{\mathbf{a}}_{A'} s_{C'} = (\mathbf{a} \tilde{\mathbf{a}}). s_{A'}
\]
for the other \( \tilde{\mathbf{a}} \in \text{GL}(2, \mathbb{C}) \). Let \( \{\omega^A; A = 0, \ldots, 2n - 1\} \) be a basis of \( \mathbb{C}^{2n*} \). They are Grassmannian variables, i.e. \( \omega^A \omega^B = -\omega^B \omega^A \). \( \mathbf{d} \in \text{GL}(2n, \mathbb{C}) \) acts on \( \mathbb{C}^{2n*} \) as
\[
d. \omega^A = d_{B}^A \omega^B.
\]
This action is not a representation, but \( \varrho(\mathbf{d}) \omega^A = d^{-1}_A \omega^A \) defines the dual representation of defining representation \( \mathbb{C}^{2n} \) of \( \text{GL}(2n, \mathbb{C}) \), since
\[
\varrho(\mathbf{d}) \varrho(\mathbf{d}) \omega^A = d^{-1}_A (\varrho(\mathbf{d}^{-1}_A \omega^B)) = (\varrho(\mathbf{d}^{-1}_B) \omega^B)_C (\varrho(\mathbf{d})^{-1})_1 \omega^A = \varrho(\mathbf{d} \mathbf{d}) \omega^A.
\]
for \( \mathbf{d}, \tilde{\mathbf{d}} \in \text{GL}(2n, \mathbb{C}) \). For \( j \geq k \), we will denote by \( s^A \) coordinate functions of \( \mathbb{C}^{2\tau} \) with the action
\[
\mathbf{a}. s^A = a''_{B} s^B.
\]
Similarly, \( \varrho(\mathbf{a}) \omega^A = a^{-1}_A s^A \) defines the dual representation of \( \mathbb{C}^2 \).

For a vector space \( V \), let \( \Gamma(\mathbb{C}^{2n \times 2}, V) \) be the space of \( V \)-valued holomorphic functions. Let
\[
(2.10)\quad \partial^A_{\mathbf{A}} := \frac{\partial}{\partial z^A}.
\]
Denote \( \omega^A = \omega^A_1 \cdots \omega^A_\tau \) for a \( \tau \)-tuple \( \mathbf{A} = A_1 \cdots A_\tau \) for some \( \tau \). An element of \( \Gamma(\mathbb{C}^{2n \times 2}, \wedge^\tau \mathbb{C}^{2n}) \) can be written as \( f = f_{\mathbf{A}} \omega^A \) with \( f_{\mathbf{A}} \) antisymmetric under permutation of indices. Define \( \mathcal{d}^A_{\mathbf{A}} : \Gamma(\mathbb{C}^{2n \times 2}, \wedge^\tau \mathbb{C}^{2n}) \rightarrow \Gamma(\mathbb{C}^{2n \times 2}, \wedge^\tau+1 \mathbb{C}^{2n}) \) as
\[
(2.11)\quad \mathcal{d}^A_{\mathbf{A}} f := \partial^A_{\mathbf{A}} f_{\mathbf{A}} \omega^A \omega^A A.
\]
Corresponding to a notation on \( \mathbb{H}^n \), its counterpart on \( \mathbb{C}^{2n \times 2} \) is usually denoted by the same symbol with underline.

**Proposition 2.3.** [13] Proposition 2.2] (f) \( \mathcal{d}^A_{\mathbf{A}} d^A_{\mathbf{A}'} = -d^A_{\mathbf{A}'} d^A_{\mathbf{A}} \).

(2) \( (\mathcal{d}^A_{\mathbf{A}})^2 = (\mathcal{d}^A_{\mathbf{A}})^2 = 0 \).

(3) For \( F \in \Gamma(\mathbb{C}^{2n \times 2}, \wedge \mathbb{C}^{2n}) \), \( G \in \Gamma(\mathbb{C}^{2n \times 2}, \wedge \mathbb{C}^{2n}) \), we have
\[
\mathcal{d}^{A'}(F \cdot G) = \mathcal{d}^{A'} F \cdot G + (-1)^T F \cdot \mathcal{d}^{A'} G, \quad A' = 0', 1'.
\]

**Proof.** We give its simple proof here for convenience of readers. For \( F = F_{\mathbf{A}} \omega^A \) with \( |\mathbf{A}| = \tau \),
\[
\mathcal{d}^{A'} \mathcal{d}^{B'} F = \partial^{A'} \partial^{B'} F_{\mathbf{A}} \omega^A \omega^B \omega^A = -\partial^{B'} \partial^{A'} F_{\mathbf{A}} \omega^B \omega^A \omega^A = -\mathcal{d}^{B'} \mathcal{d}^{A'} F,
\]
and for \( G = G_{\mathbf{A}} \omega^A \) with \( |\mathbf{G}| = \chi \), we have
\[
\mathcal{d}^{A'}(F \wedge G) = \partial^{A'} (F_{\mathbf{A}} G_{\mathbf{B}}) \omega^A \omega^A \omega^B = \partial^{A'} F_{\mathbf{A}} \omega^A \omega^A (G_{\mathbf{B}} \omega^B) + (-1)^T F_{\mathbf{A}} \omega^A (\partial^{A'} G_{\mathbf{B}} \omega^A \omega^B).
\]
The proposition is proved. \( \square \)

We will also use the following notations:
\[
(2.12)\quad \mathbf{a}. \mathcal{d}^{A'} = a_{B}^{A'} \mathcal{d}^{B'}, \quad \mathbf{d}. \mathcal{d}^{A'} = \mathbf{d} . \omega^A \partial^{A'}.
\]
for \( \mathbf{a} \in \text{gl}(2, \mathbb{C}) \), \( \mathbf{d} \in \text{gl}(2n, \mathbb{C}) \).

The \( \sigma \)-th symmetric power \( \otimes^{\sigma} \mathbb{C}^2 \) is an irreducible \( \text{GL}(2, \mathbb{C}) \)-module. It is convenient to realize \( \otimes^{\sigma} \mathbb{C}^2 \) as the space \( \mathcal{P}_\sigma(\mathbb{C}^2) \) of homogeneous polynomials of degree \( \sigma \) on \( \mathbb{C}^2 \) [30]. Denote \( s_{A'} = s_{A'_1} \cdots s_{A'_\sigma} \).
for \( A' = A'_1 \cdots A'_s \). Set \( |A'_1 \cdots A'_s| = \sigma \). \( a.s_{A'} = a.s_{A'_1} \cdots a.s_{A'_s} \) for \( a \in \text{GL}(2, \mathbb{C}) \). The action of the Lie algebra \( \mathfrak{gl}(2, \mathbb{C}) \) is given by

\[
(2.13) \quad a.s_{A'} = \sum_{j=1}^{\sigma} s_{A'_1} \cdots (a.s_{A'_j}) \cdots s_{A'_s} = a.s_{A'} \cdot \partial^{A'} s_{A'}, \quad \partial^{A'} = \frac{\partial}{\partial s_{A'}}.
\]

for \( a \in \text{GL}(2, \mathbb{C}) \). The \( \tau \)-th exterior power \( \wedge^\tau \mathbb{C}^{2n} \) is a representation of \( \mathfrak{gl}(2n, \mathbb{C}) \) with induced action

\[
(2.14) \quad d.\omega^A = \sum_{j=1}^{\tau} \omega^{A_{j-1}}(d.\omega^{A_j})\omega^{A_{j+1}} \cdots = d.\omega^A \cdot \partial_A \omega^A, \quad \partial_A = \frac{\partial}{\partial \omega^A}.
\]

**Remark 2.2.** It is convenient to use derivatives and multiplications with respect to variables \( s_{A'} \in \mathbb{C}^2 \) or Grassmannian variables \( \omega^A \) to represent linear transformations on the space \( \otimes^\sigma \mathbb{C}^2 \) or \( \wedge^\tau \mathbb{C}^{2n} \).

The complexified version of the \( \text{k-Cauchy-Fueter complex} \) is

\[
0 \rightarrow \Gamma(\mathbb{C}^{2n+2}, \mathcal{V}_0) \xrightarrow{\mathcal{D}_0} \Gamma(\mathbb{C}^{2n+2}, \mathcal{V}_1) \xrightarrow{\mathcal{D}_1} \cdots \xrightarrow{\mathcal{D}_{2n-2}} \Gamma(\mathbb{C}^{2n+2}, \mathcal{V}_{2n-1}) \rightarrow 0.
\]

A section of \( f \in \Gamma \left( \mathbb{C}^{2n+2}, \otimes^\sigma \mathbb{C}^2 \otimes \wedge^\tau \mathbb{C}^{2n} \right) \) is a function in complex variables \( z_A^a, \ s_{A'} \in \mathbb{C}^2 \) and Grassmannian variables \( \omega^A \), which is homogeneous of degree \( \sigma \) in \( s_{A'} \) and homogeneous of degree \( \tau \) in \( \omega^A \). It is a function in supervariables as

\[
f(z) = f^A'(z)s_{A'}\omega^A,
\]

where \( f^A' \) is invariant under permutations of indices \( A' = A'_1 \cdots A'_s \) and is antisymmetric under permutation of indices \( A = A_1 \cdots A_s \). Define the action for \( g \in \text{SL}(2n+2, \mathbb{C}) \) given by (2.5) as

\[
(2.15) \quad [\mathcal{T}_j(g)f](z) := \frac{f^A'(g^{-1}z)}{\det(a + bz)^{j+1}}(a + bz)^{-1} s_{A'} \cdot [d - (c + dz)(a + bz)^{-1}b] \cdot \omega^A,
\]

**Remark 2.3.** If \( g \) given by (2.5) belongs to \( G^C_0 \), i.e. \( g = \begin{pmatrix} a^{-1} & 0 \\ 0 & d^{-1} \end{pmatrix} \), then (2.15) becomes

\[
[\mathcal{T}_j(g)f](z) := \frac{f^A'(dza^{-1})}{\det(a)^{j+1}}a^{-1} s_{A'} \cdot d.\omega^A.
\]

Namely, \( g \) acts on \( s_{A'} \) by the defining representation of \( \text{GL}(2, \mathbb{C}) \) and trivially by \( \text{GL}(2n, \mathbb{C}) \), meanwhile, it acts on \( \omega^A \) by the representation dual to the defining representation of \( \text{GL}(2n, \mathbb{C}) \) and trivially by \( \text{GL}(2, \mathbb{C}) \). Thus \( \mathcal{V}_j \) as \( G^C_0 \)-module is \( \otimes^{k-j} \mathbb{C}^2 \otimes \wedge^{j+1} \mathbb{C}^{2n} \). While for the case \( j \geq k \) in (2.10), \( g \) acts on \( s_{A'} \) by the representation dual to the defining representation of \( \text{GL}(2, \mathbb{C}) \) and trivially by \( \text{GL}(2n, \mathbb{C}) \).

If \( j < k \), let \( \mathcal{D}_j := \mathcal{D}_k \), where

\[
\mathcal{D}_k := \partial^{A'} \circ d^{B'} = \frac{1}{2} \partial^{A'} \circ d^{B'} - \frac{1}{2} \partial^{B'} \circ d^{A'},
\]

is a derivative of second order on functions in variables \( z_A^a, \ s_{A'} \) and \( \omega^A \). The operator is zero if \( A' = B' \), and the nontrivial one is unique up to a sign for \([A'B'] = [0'1'] \) or \([1'0'] \). Here and in the sequel, the antisymmetrization is explained by the formula above, while \( \circ \) is the composition of operators acting on such functions, and is usually omitted.

If \( j = k \), let

\[
\mathcal{D}_k := d^{A'} \circ d^{B'}.
\]

If \( j \geq k \), \( f \in \Gamma(\mathbb{C}^{2n+2}, \otimes^\sigma \mathbb{C}^2 \otimes \wedge^\tau \mathbb{C}^{2n}) \) \( (\sigma = j-k-1, \tau = j+1) \) can be written as

\[
f = f^{A'A'} s_{A'} \omega^A.
\]
with $|A'| = \sigma$, $|A| = \tau$. Define

$$[\pi_j(g)f](z) := \frac{f_A|A(g^{-1},z)}{\det(a + bz)^{j+1}} (a + bz) \cdot \omega_A.'$$

(2.16)

Let $\tilde{D} := \hat{D}$ with

$$\hat{D} := s[A' \circ d'B].$$

(2.17)

For $a \in GL(2, \mathbb{C})$, denote

$$a, \partial^{A'} := a^B_{|B'}, \quad a, \partial_{A'} := a^B_{|B'},$$

by the following lemma, $\partial^{[A', d'B]}$ and $s[A', d'B]$ are both invariant under the action of $GL(2, \mathbb{C})$, but not invariant under the action of $GL(2, \mathbb{C})$.

**Lemma 2.1.** For $a \in \mathfrak{gl}(2, \mathbb{C})$, we have

$$\partial^{[A' \circ d'B]} + \partial^{[A' \circ a \cdot d'B]} = \text{tr} a \cdot \partial^{[A' \circ \partial] B},$$

$$s^{[A' \circ d'B]} + s^{[A' \circ a \cdot d'B]} = \text{tr} a \cdot \partial^{[A' \circ \partial] B}. \quad (2.18)$$

**Proof.** Let $\left( \varepsilon^{A' B'} \right) = \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)$. Then, we have

$$\partial^{[A' \circ d'B]} + \partial^{[A' \circ a \cdot d'B]} = a_C^{A'} \cdot \partial^{[C' \circ d'B]} + a_C^{B'} \cdot \partial^{[A' \circ d'] B}$$

$$= \left( a_C^{A'} \cdot C' B' - a_C^{B'} \cdot C' A' \right) \cdot \partial^{[C' \circ \partial] B}$$

$$= \text{tr} a \cdot \partial^{[A' \circ d'B]},$$

by $a \varepsilon - (a \varepsilon)' = a \varepsilon + \varepsilon a' = \text{tr} a \varepsilon$. This is because if we write $a = \left( \begin{array}{cc} \alpha & \gamma \\ \beta & \delta \end{array} \right)$, then we have

$$\left( \begin{array}{cc} \alpha & \gamma \\ \beta & \delta \end{array} \right) \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) + \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right) = \left( \begin{array}{cc} 0 & \alpha + \delta \\ -\alpha - \delta & 0 \end{array} \right).$$

The lemma is proved. \square

**Proposition 2.4.** For $f \in \Gamma(\mathbb{C}^{n \times 2}, V_j)$, $\pi_j(g_1)\pi_j(g_2)f = \pi_j(g_1 g_2)f$ outside of singularities.

**Proof.** Note that if we identify an element $f \in \Gamma(\mathbb{C}^{2n \times 2}, V_j)$ with the tuple $(f_{A'A}(z))$, the representation $\pi_j(g)$ in (2.15) for $j \leq k$ maps the tuple $(f_{A'A}(z))$ to the tuple $\left( [\pi_j(g)f]^{A'}_A(z) \right)$ with

$$[\pi_j(g)f]^{A'}_A(z) := \frac{1}{\det(J_1(g^{-1},z))^{j+1}} [J_1(g^{-1},z)^{-1}]^{A'}_{B'} J_2(g^{-1},z) \cdot f^{B'} (g^{-1},z).$$

(2.19)

where $B = B_1 \ldots B_j$, $B' = B'_1 \ldots B'_{k-j}$, $A = A_1 \ldots A_j$, $A' = A'_1 \ldots A'_{k-j}$, and

$$[J_1(g^{-1},z)^{-1}]^{A'}_{B'} := \prod_{\alpha=1}^{k-j} [J_1(g^{-1},z)^{-1}]^{A'_{\alpha}}_{B'_{\alpha}},$$

$$J_2(g^{-1},z)_{B'} := \prod_{\beta=1}^{j} J_2(g^{-1},z)_{A'_{\beta}}^{A_{\beta}}.$$
Then
\[ \left( \pi_j(g_1) \pi_j(g_2) \right) f}{A'}^{A'}_A(z) = \frac{1}{\det(J_1(g_1^{-1}, z)^{j+1})} \left[ J_1(g_1^{-1}, z)^{-1} \right]^{A'}_B J_2(g_1^{-1}, z)^B_A + \frac{1}{\det(J_1(g_2^{-1}, g_1^{-1}, z)^{j+1})} \left[ J_1(g_2^{-1}, g_1^{-1}, z)^{-1} \right]^{B'}_C J_2(g_2^{-1}, g_1^{-1}, z)^C_B f^B_C \]
\[ = \frac{1}{\det(J_1(g_2^{-1}, g_1^{-1}, z)^{j+1})} \left[ J_1(g_2^{-1}, g_1^{-1}, z)^{-1} \right]^{A'}_C J_2(g_2^{-1}, g_1^{-1}, z)^C_J f^C_J \]
by using the cocycle condition (2.9). Namely, \( \pi_j(g_1) \pi_j(g_2) f = \pi_j(g_1 g_2) f \) outside of singularities. It is similar for \( j > k \).

**Theorem 2.1.** \( \bar{D}_j : \Gamma(C^{2n}\times 2, \mathcal{V}_j) \to \Gamma(C^{2n}\times 2, \mathcal{V}_{j+1}) \) is \( \text{SL}(2n+2, \mathbb{C}) \)-invariant, i.e.
\[ \bar{D}_j \circ \pi_j(g) = \pi_j(g_{j+1}) \circ \bar{D}_j. \]
for any \( g \in \text{SL}(2n+2, \mathbb{C}) \).

3. The \( \text{SL}(2n+2, \mathbb{C}) \)-invariance of the complexified version

To prove the \( \text{SL}(2n+2, \mathbb{C}) \)-invariance, it is sufficient to check the invariance under the action of its Lie algebra \( sl(2n+2, \mathbb{C}) \) by the following Proposition 3.1 which is more easy. See [28] for construction of invariant operators for \( \text{Mp}(n, R) \) and \( \text{SU}(n, n) \) by this method. Since we do not find appropriate reference for this proposition, we give the proof here for convenience of readers.

For \( X \in sl(2n+2, \mathbb{C}) \), consider the subgroup of one parameter \( g_t = e^{tX} = I + tX + O(t^2) \). The action of Lie algebra is defined as
\[ d\pi_j(X)f = \left. \frac{d}{dt} \pi_j(g_t) f \right|_{t=0}. \]

**Proposition 3.1.** If
\[ \bar{D}_j \circ d\pi_j(X)f = d\pi_{j+1}(X) \circ \bar{D}_j f \]
for any \( X \in \text{sl}(2n+2, \mathbb{C}) \) and \( f \in \Gamma(C^{2n}\times 2, \mathcal{V}_j) \), then \( \bar{D}_j \) is \( \text{SL}(2n+2, \mathbb{C}) \)-invariant.

**Proof.** It is sufficient to prove the result for \( f \) to be a holomorphic polynomial. Let \( g_t \) be a subgroup of one parameter generated by \( X \in \text{sl}(2n+2, \mathbb{C}) \), i.e. \( g_t = \exp(tX) \). Differentiate \( \pi_j(g_{t+s}) f = \pi_j(g_{t}) \pi_j(g_s) \) with respect to \( s \) at \( s = 0 \) to get
\[ \frac{d}{dt} \pi_j(g_t) f = \pi_j(g_t) \frac{d}{ds} \pi_j(g_s) f \left|_{s=0} = \pi_j(g_t) d\pi_j(X) f. \right. \]
Differentiating it with respect to \( t \) repeatedly, we get
\[ \frac{d^m}{dt^m} \pi_j(g_t) f = \pi_j(g_t) (d\pi_j(X))^m f, \]
in particular,
\[ \frac{d^m}{dt^m} \pi_j(g_t) f \left|_{t=0} = (d\pi_j(X))^m f. \right. \]
For fixed $z \in \mathbb{C}^{2n \times 2}$, let

$$F_1(t) = \mathcal{D}_j \pi_j(g_t)f(z), \quad F_2(t) = \pi_{j+1}(g_t)\mathcal{D}_j f(z).$$

Without loss of generality, we only need to check $F_1(t) \equiv F_2(t)$ for $z$ near the origin, since both sides are rational function of $z$. Moreover, we only need to check it for $g_t = \exp(tX)$ with $\|X\|$ small, because the identity for $\exp(X)$ can be obtained by using the identity for $\exp(X/m)$ for $m$ times. For such $z$ and $g$, $f(g_t^{-1}z)$ is not singular for $t \in [0,1]$, and so it is a real analytic curve in $t$. Then $F_1(t)$ and $F_2(t)$ are both real analytic functions in $t \in [0,1]$, and

$$F^{(m)}(0) = \mathcal{D}_j((\pi_j(X))^{m}f(z)), \quad F^{(m)}_2(0) = (d\pi_{j+1}(X))^{m}\mathcal{D}_j f(z)$$

by (3.2). But

$$\mathcal{D}_j((\pi_j(X))^{m}f(z)) = (d\pi_{j+1}(X))^{m}\mathcal{D}_j f(z)$$

by using the assumed invariance (3.1) of $\mathcal{D}_j$ under action of Lie algebra repeatedly. Therefore, $F_1^{(m)}(0) = F_2^{(m)}(0)$ for $m = 0, 1, \ldots$, and so $F_1 \equiv F_2$ as real analytic functions of $t$. At $t = 1$, we get $\pi_{j+1}(g)\mathcal{D}_j f(z) = \mathcal{D}_j((\pi_j(g)f)(z))$. \hfill \Box

3.1. **Proof of the invariance for the case $j < k$.** We need to check (3.1) for

$$X = \begin{pmatrix} 0 & b \\ 0 & 0 \\ a & 0 \\ 0 & d \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

**Case i.**

Since $g_t^{-1} = e^{-tX} = \begin{pmatrix} 1 & -tb \\ 0 & 1 \\ 2n \end{pmatrix} + O(t^2)$, for $X = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$,

$$g_t^{-1}z = z(1_2 - t(bz))^{-1} + O(t^2) = z + tz^2b + O(t^2),$$

we get

$$\det(1_2 - t(bz)) = 1 - t \text{tr}(bz) + O(t^2).$$

The **infinitesimal vector field** $Y$ of transformations $g_t^{-1}$ on $\mathbb{C}^{2n \times 2}$ is defined by

$$Yf(z) = \left. \frac{d}{dt} f(g_t^{-1}z) \right|_{t=0}$$

for any holomorphic functions $f$. It follows from (3.4) that

$$Y = (zbz)^{\frac{\partial}{\partial A}} \cdot \partial_A f.$$

By differentiation the action (2.15) of $\pi_j(g)$ and using the Leibniz law, we see that

$$d\pi_j(X)f = \left. \frac{d}{dt} \pi_j(g_t)f \right|_{t=0}$$

(3.6)

$$= \frac{d}{dt} \left[ \frac{\partial A}{\partial A'}(g_t^{-1}z) \right](1 - t(bz))^{-1} + O(t^2)$$

$$= \left. \left[ Y + (j + 1) \text{tr}(bz) \right] f + zb, (A') \cdot \partial_A f + zb, \omega_A \cdot \partial f \right|_{t=0}$$

by (2.13) - (2.14). It follows that

$$\mathcal{D} \circ d\pi_j(X) - d\pi_{j+1}(X) \circ \mathcal{D} = \left[ \partial [A']^B, [A']^B \right], Y + (j + 1) \text{tr}(bz) - \text{tr}(bz) \partial [A']^B$$

(3.7)

$$+ \left[ \partial [A']^B, (zb), (A') \cdot \partial_A f \right] + \left[ \partial [A']^B, z, \omega_A \cdot \partial_A f \right].$$

$$$\square$$
This is an identity of operators acting on functions in variables $z^A, s_A$, and $\omega^A$. To show it vanishing, we need to calculate commutators or anticommutators (i.e., \{A, B\}; by Lemma 3.1.

\begin{align*}
(1) \quad & [\mathbf{A}^A', z b, \omega^A] = \omega^A \Omega^A', \quad \text{where } \Omega^A' := b_A^A \omega^A; \\
(2) \quad & [\mathbf{A}^A', \text{tr}(b z)] = \Omega^A' \\
(3) \quad & [\mathbf{A}^A', Y] = b z \mathbf{A}^A' + z b \mathbf{A}^A'.
\end{align*}

**Proof.** Noting that

$$\partial_A^A z_{B'}^B = \delta^B_A \delta_{B'}^A,$$

by definition, we get

$$\{\mathbf{A}^A, z b, \omega^A\} = \mathbf{A}^A (z b, \omega^A) = \omega^B \partial_B^A \left(z_b^A b^C_D\right) \omega^D = \omega^A b^D_D \omega^D = \omega^A \Omega^A',$$

and

$$[\mathbf{A}^A, \text{tr}(b z)] = \mathbf{A}^A \text{tr}(b z) = \omega^A \partial_A^A \left(b^B_D z_B^D\right) = b^A_A \omega^A = \Omega^A',$$

by definition, we get

$$[\mathbf{A}^A, Y] = \mathbf{A}^A (z b z)_{B'}^B = \omega^A \partial_A^A \left(z_b^A b^C_D z_B^D\right) \partial_B^B = \omega^A (z b)_{B'}^B \partial_B^B.$$

The result follows by using notations (2.12). \hfill \Box

For the third commutator in (3.7), note that

$$[\mathbf{A}^A, s_c, \partial^C] = \mathbf{A}^A (s_c, \partial^C) = \omega^B \partial_B^A (b^D_D z_C^D) s_D, \partial^C = s_D, \partial^C = s_D, \Omega^D \partial^B',$$

by \(\mathbf{A}^A\) commuting \(\partial^B\), and the operator identity for functions in supervariables:

$$[U V, W] = U V W - U W V + U W V - W U V = U [V, W] + [U, W] V.$$

Then we get

\begin{align}
[\partial^A' \mathbf{A}^A, [s_c, \partial^C]] &= \frac{1}{2} \partial^A' [\mathbf{A}^B, [s_c, \partial^C]] + \frac{1}{2} [\partial^A', [s_c, \partial^C]] \mathbf{A}^B - A' \leftrightarrow B' \\
&= \frac{1}{2} \partial^A' s_c, \partial^C, \partial^B' + \frac{1}{2} (b z)_{B'}^A \partial^C, \partial^B' - A' \leftrightarrow B' \\
&= \Omega^A' \partial^B' + b z \partial^A' \mathbf{A}^B',
\end{align}

since

$$[\partial^A', s_c^C] = \delta^C_C, \quad [\partial^A', \partial^B'] = 0.$$

Similarly, we have

\begin{align}
\mathbf{A}^B', [z b, \omega^A \partial_A] &= \omega^A \Omega^B' \partial_A f - z b, \omega^A \partial_A f - z b, \omega^A \partial_A \mathbf{A}^B' f \\
&= - j \Omega^B' f - z b, \mathbf{A}^B' f
\end{align}

by using the Leibnitz law, Lemma 3.1 (1) and \(\omega^A \partial_A f = j f\) if \(f\) is homogeneous of degree \(j\) in \(\omega\), and

$$\{\mathbf{A}^B', \partial_A\} = \mathbf{A}^B' \partial_A + \partial_A \mathbf{A}^B' = \partial_A^B'$$

as an anti-commutator. Consequently,

$$[\partial^A' \mathbf{A}^B'], [z b, \omega^A \partial_A] = - j [\partial^A' \Omega^B'] - [\partial^A' z b, \mathbf{A}^B'],$$

where \(\Omega^B' = b_A^A \omega^A\).
since $\partial^A$ commute with other operators.

On the other hand, we have

\begin{equation}
\left[\partial^A, \partial_z B\right], Y] = \partial^A \left[\partial_z B\right], Y] = \partial^A \left[\partial_z B\right], Y] + \partial^A \left[\partial_z B\right], Y],
\end{equation}

by Lemma 3.1 (3) and $\partial^A$ commuting $\partial_z B$ and $Y$, and

\begin{equation}
[\partial^A, \partial_z B], \text{tr}(b \text{z})] = \partial^A [\partial_z B], \text{tr}(b \text{z})] = \partial^A \Omega_B,
\end{equation}

by Lemma 3.1 (2).

Now substituting (3.10) (3.13)-(3.15) into (3.7), we see that terms $\partial^A \Omega_B$ and $\partial^A \partial_z B$ are cancelled each other. So we get

\[ P \circ \partial_z \pi_j(X) - \partial_z \pi_{j+1}(X) \circ P = \partial^A [\partial_z B], Y] + \text{tr}(b \text{z})] = 0
\]

by using Lemma 2.1.

**Case ii.** Since

\begin{equation}
g_t^{-1} = e^{-tX} = \left(\begin{array}{cc}
1 - ta & 0 \\
0 & 12n - td
\end{array}\right) + O(t^2), \quad \text{for } X = \left(\begin{array}{cc}
a & 0 \\
0 & d
\end{array}\right),
\end{equation}

we have

\[ d\pi_j(X)f = \left. \frac{d}{dt} \pi_j(g_t) \right|_{t=0} f = \bar{Y} f + (j + 1) \text{tr} a f + a \cdot s d \cdot \partial B f - d \cdot \omega A \cdot \partial A f,
\]

by differentiate the action 2.15 of $\pi_j(g_t)$, where

\begin{equation}
\bar{Y} := \left[za \right]C - \left(\text{dz}_C\right]C \partial C
\end{equation}

is the infinitesimal vector field of transformations

\[ g_t^{-1} \cdot z = (12n - td)z(12 - ta)^{-1} = z + t(za - dz) + O(t^2).
\]

Thus

\[ P \circ \partial_z \pi_j(X) - \partial_z \pi_{j+1}(X) \circ P = \partial^A \left[\partial_z B\right], \bar{Y}] - \text{tr} a \cdot f + \left[\partial^A \partial_z B, a \cdot s d\right]\partial C - \partial^A \left[\partial_z B, d \cdot \omega A \partial A\right],
\]

by $\partial^A$ commuting $\partial B$, $\partial A$ and $d \cdot \omega A \cdot \partial A$. Here and in the sequel, we use antisymmetrisation:

\[ \partial^A \left[\partial_z B, \bar{Y}] = \frac{1}{2} \partial^A \left[\partial_z B, \bar{Y}] - \frac{1}{2} \partial B \left[\partial^A, \bar{Y}] - \frac{1}{2} \partial B \left[\partial^A, \bar{Y}].
\]

Note that

\begin{equation}
[\partial_z B, \bar{Y}] = \omega A \partial A \left[za \right]C - \left(\text{dz}_C\right]C \partial C = \omega A \left[a \cdot \partial z, \partial z\right]C - \partial A \left[\partial z, \partial z\right]C = a, \partial B - d, \partial B,
\end{equation}

and

\begin{equation}
[\partial z, d \cdot \omega A \partial A] = -d \cdot \omega A \partial A \partial A \partial B - d \cdot \omega A \partial A \partial B - d \cdot \omega A \partial A \partial B = -d \cdot \omega A \partial A \partial B - d \cdot \partial B,
\end{equation}

by using the Leibnitz law and $\left\{\omega A, \partial A\right\} = \delta A$. Now we see that

\[ P \circ \partial_z \pi_j(X) - \partial_z \pi_{j+1}(X) \circ P = \partial^A \left[a, \partial B\right] - \text{tr} a \cdot f + a \cdot \partial z, \partial z\right]C - \partial A \left[\partial z, \partial z\right]C = \partial^A \left[a, \partial B\right] - \text{tr} a \cdot f + a \cdot \partial A \partial B = 0
\]

by using (3.11) and Lemma 2.1 again.

**Case iii.** The last case is trivial, since it is direct to see that $\pi_j(X)f(z) = f(z + c)$.
3.2. Proof of the invariance for the case $j > k$. Case i. Differentiate (2.16) for $g_t^{-1}$ given by (3.3) to get

$$d_{g_t^{-1}}(X)f = \frac{d}{dt} d_{g_t^{-1}} (e^{-tX}) \bigg|_{t=0} = [Y + (j + 1) \text{tr}(bz)]f - bz.A^f + bz. \omega^A \cdot \partial_A f,$$

by (3.4), where $Y$ is given by (3.5). Then we have

$$\hat{D} \circ d_{g_j}(X) - d_{g_{j+1}}(X) \circ \hat{D} = \hat{D} Y + (j + 1) \left[ \hat{D} \text{tr}(bz) - \text{tr}(bz) \hat{D} \right] - \left[ \hat{D} bz.A^f + \hat{D} bz. \omega^A \cdot \partial_A f \right].$$

(3.20)

Note that

$$\left[ \hat{D} Y \right] = s[A^f, \hat{D} Y] = s[A^f, zb.dB] + s[A^f, zb.dA^f]$$

by Lemma 3.1 (3), $s^A$ commuting $Y$ and $d^B$, and

$$\left[ \hat{D} \text{tr}(bz) \right] = s[A^f, \text{tr}(bz)] = s[A^f, \Omega B].$$

By 3.12, we have

$$\left[ \hat{D} zb. \omega^A \cdot \partial_A f \right] = s[A^f, zb.dA^f] - (j + 1)s[A^f, \Omega B] - s[A^f, zb.dB].$$

(3.23)

Here $\omega^A \partial_A f = (j + 1) f$ for $f \in \Gamma(\mathbb{C}^{2n \times 2}, Y_j)$ homogeneous of degree $j + 1$ in $\omega$. Since

$$[d^B, zb.s^C \partial_C] = d^B (zb.s^C \partial_C) = \omega^B \partial_B (zb.s^C \partial_C) s^B.\partial_C = s^B.\Omega^C \partial_C,$$

we get

$$\left[ \hat{D} zb. s^C \partial_C \right] = s[A^f, zb.s^C \partial_C] + s[A^f, zb.s^C \partial_C] d^B)

(3.24)

$$= s[A^f, \Omega B] - bz. s[A^f, d^B].$$

by 3.9.

Now substitute (3.21)-(3.24) into (3.20) to get

$$\hat{D} \circ d_{g_j}(X) - d_{g_{j+1}}(X) \circ \hat{D} = s[A^f, zb.dB] + bz. s[A^f, d^B] - \text{tr}(bz) \hat{D} = 0$$

by using Lemma 2.1 again.

Case ii. Differentiate (2.16) for $g_t^{-1} = e^{-tX}$ with $X = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ to get

$$[d_{g_t^{-1}}(X)f](z) = \frac{d}{dt} d_{g_t^{-1}} (e^{-tX}) \bigg|_{t=0} = \tilde{Y} f + (j + 1) \text{tr}(a) f - a.A^f + d. \omega^A \cdot \partial_A f,$$

where $\tilde{Y}$ is given by (3.17). Thus

$$\hat{D} \circ d_{g_j}(X) - d_{g_{j+1}}(X) \circ \hat{D} = \hat{D} \tilde{Y} - \text{tr}(a) \hat{D} - a.A^f + d. \omega^A \cdot \partial_A f \right]$$

$$= s[A^f, d^B] - s[A^f, d^B] - \text{tr}(a) \hat{D} + a.A^f + s[A^f, d^B] + s[A^f, d^B]$$

$$= s[A^f, d^B] - \text{tr}(a) \hat{D} + a.A^f + s[A^f, d^B] = 0.$$
by using (3.18),
\[
\{\hat{D}, \partial_A\} = \hat{D}\partial_A + \partial_A\hat{D} = s^{A} \partial_A^{B'},
\]
and Lemma (2.7) again.

Case iii. This is trivial.

3.3. Proof of the invariance for the case \( k = j \). Case i. For \( X = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \), differentiate representations (2.15) for \( j = k \) and (2.16) for \( j = k + 1 \) to get
\[
\left[ d_{\hat{\pi}_{k}}(X)f \right](z) = \frac{d}{dt} d_{\hat{\pi}_{k}}(e^{-tX}) \bigg|_{t=0} = f = \left[ Y + (k + 1) \text{tr}(bz) \right] f + zb.\omega^{A}\partial_{A}f,
\]
\[
\left[ d_{\hat{\pi}_{k+1}}(X)F \right](z) = \frac{d}{dt} d_{\hat{\pi}_{k+1}}(e^{-tX}) \bigg|_{t=0} = F = \left[ Y + (k + 2) \text{tr}(bz) \right] F + zb.\omega^{A}\partial_{A}F,
\]
for \( f \in \Gamma(C^{2n\times 2}, \nu_{k}), F \in \Gamma(C^{2n\times 2}, \nu_{k+1}) \). So we have
\[
(3.25) \quad \hat{D}_{k} \circ d_{\hat{\pi}_{k}}(X) - d_{\hat{\pi}_{k+1}}(X) \circ \hat{D}_{k} = \left[ \hat{D}_{k}, Y + (k + 1) \text{tr}(bz) \right] - \text{tr}(bz) \hat{D}_{k} + \left[ \hat{D}_{k}, zb.\omega^{A}\partial_{A} \right].
\]
Note that \( \hat{D}_{k} = d^{0'} d^{1'} \) and
\[
\left[ \hat{D}_{k}, zb.\omega^{A}\partial_{A} \right] = d^{0'} \left[ d^{1'}, zb.\omega^{A}\partial_{A} \right] + \left[ d^{0'}, zb.\omega^{A}\partial_{A} \right] d^{1'}
\]
\[
= -k d^{0'} \circ \Omega^{1'} = d^{0'} \circ zb.\omega^{A}\partial_{A}' \circ \Omega^{1'} - (k + 1) \Omega^{0'} d^{1'} - zb. d^{0'} \circ d^{1'}
\]
\[
= k \Omega^{1'} d^{0'} - \omega^{A} \Omega^{0'} \partial_{A}' + zb. d^{1'} d^{0'} - (k + 1) \Omega^{0'} d^{1'} - zb. d^{0'} \circ d^{1'}
\]
\[
= -2 k \Omega^{0'} d^{1'} - 2 zb. d^{0'} \circ d^{1'}
\]
by using (3.12) repeatedly, Lemma (3.1) (1) and \( d^{0'}(\Omega^{1'} f) = -\Omega^{1'} d^{0'} f \). We also have
\[
[\hat{D}_{k}, Y] = d^{0'} \left[ d^{1'}, Y \right] + \left[ d^{0'}, Y \right] d^{1'}
\]
\[
= \left[ d^{0'}, (bz) Y \right] d^{1'} + (bz) Y d^{0'} + \left[ d^{0'}, zb.\omega^{B} \right] \partial_{B}' - zb.\omega^{B} \partial_{B}' d^{0'}
\]
\[
+ (bz) Y d^{0'} \circ d^{1'}
\]
\[
= -\Omega^{1'} d^{0'} + \text{tr}(bz) d^{0'} d^{1'} - \Omega^{0'} d^{1'} - zb. d^{1'} d^{0'} + zb. d^{0'} \circ d^{1'}
\]
\[
= -2 \Omega^{0'} d^{1'} + \text{tr}(bz) d^{1'} + 2 zb. d^{0'} \circ d^{1'}
\]
by using Lemma (3.1) (1) (3) and anti-commutativity of \( d^{A'} \)'s in Proposition (2.3) and
\[
[\hat{D}_{k}, \text{tr}(bz)] = d^{0'} \left[ d^{1'}, \text{tr}(bz) \right] + \left[ d^{0'}, \text{tr}(bz) \right] d^{1'}
\]
\[
= d^{0'} \circ \Omega^{1'} + \Omega^{0'} d^{1'} = 2 \Omega^{0'} d^{1'},
\]
by Lemma (3.1) (2). Substitute the above three identities into (3.25) to see its vanishing.

Case ii. For \( X = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \),
\[
\left[ d_{\hat{\pi}_{k}}(X)f \right](z) = \left[ Y + (k + 1) \text{tr}(a) \right] f - d.\omega^{A}\partial_{A}f,
\]
\[
\left[ d_{\hat{\pi}_{k+1}}(X)F \right](z) = \left[ Y + (k + 2) \text{tr}(a) \right] F - d.\omega^{A}\partial_{A}F,
\]
where $\tilde{Y}$ is given by (3.17). Then
\[
D_k \circ d_{\bar{\alpha}_j}(X) - d_{\bar{\alpha}_{j+1}}(X) \circ D_k = [D_k, \tilde{Y}] - \text{tr}(a)D_k - [D_k, d\omega^A \partial_A] = 0,
\]
since
\[
[D_k, \tilde{Y}] = \tilde{d}' \left[ d', \tilde{Y} \right] + \left[ \tilde{d}', \tilde{Y} \right] \tilde{d}'
\]
\[
= \tilde{d}' \circ a d' - d' \circ d a' + a \tilde{d}' \tilde{d}' - d \tilde{d}' \tilde{d}'
\]
\[
= \text{tr} a \tilde{d}' \tilde{d}' - d \tilde{d}' \tilde{d}'
\]
by (3.18) and anti-commutativity of $d^{A'}$'s, and
\[
[D_k, d\omega^A \partial_A] = \tilde{d}' \left[ d', d\omega^A \partial_A \right] + \left[ \tilde{d}', d\omega^A \partial_A \right] d'
\]
\[
= -d' \circ d d' - d \tilde{d}' d'
\]
by (3.19).

Case iii. This is trivial.

4. The invariance on $\mathbb{H}^n$ and complexes on locally projective flat manifolds

4.1. The invariance on $\mathbb{H}^n$. Let $a = (a_{jk})_{p \times m}$ be a quaternionic $(l \times m)$-matrix and write $a_{jk} = a_{jk}^1 + ia_{jk}^2 + ja_{jk}^3 + ka_{jk}^4 \in \mathbb{H}$. We define $\tau(a)$ to be the complex $(2p \times 2m)$-matrix

\[
\tau(a) = \begin{pmatrix}
\tau(a_{00}) & \tau(a_{01}) & \cdots & \tau(a_{0(m-1)}) \\
\tau(a_{10}) & \tau(a_{11}) & \cdots & \tau(a_{1(m-1)}) \\
\vdots & \vdots & \ddots & \vdots \\
\tau(a_{(p-1)0}) & \tau(a_{(p-1)1}) & \cdots & \tau(a_{(p-1)(m-1)})
\end{pmatrix},
\]

where $\tau(a_{jk})$ is the complex $(2 \times 2)$-matrix

\[
\begin{pmatrix}
a_{jk}^1 + ia_{jk}^2 & -a_{jk}^3 - ia_{jk}^4 \\
-a_{jk}^3 - ia_{jk}^4 & a_{jk}^2 - ia_{jk}^1
\end{pmatrix}.
\]

This is motivated by the embedding of quaternionic numbers into $2 \times 2$-matrices $\mathbb{H}$.

**Proposition 4.1.** [8] Proposition 2.1] (1) $\tau(ab) = \tau(a)\tau(b)$ for a quaternionic $(p \times m)$-matrix $a$ and a quaternionic $(m \times l)$-matrix $b$. In particular, for $q' = aq$ with $q, q' \in \mathbb{H}^n$ and a quaternionic $(n \times n)$-matrix $a$, we have

\[
\tau(q') = \tau(a)\tau(q)
\]

as complex $(2n \times 2)$-matrices.

By Proposition 4.1, $\tau$ is an isomorphism from $\mathfrak{sl}(n + 1, \mathbb{H})$ to a subalgebra of $\mathfrak{sl}(2n + 2, \mathbb{C})$, and so is an isomorphism from $\text{SL}(n + 1, \mathbb{H})$ to a subgroup of $\text{SL}(2n + 2, \mathbb{C})$. By Proposition [1], $\mathbb{C}^2$ and $\mathbb{C}^{2n*}$ have the actions of $\text{GL}(1, \mathbb{H})$ given by $q.s_{A'} := \tau(q).s_{A'}$ and $q.s_{A''} := \tau(q).s_{A''}$, respectively, and $\mathbb{C}^{2n*}$ have
the action of \( \text{GL}(2n, \mathbb{H}) \) given by \( d_\omega^A := \tau(d) \omega^A \). By embedding \( \tau \), we have

\[
\tau(q) = (z^A(q)^{1}) := \begin{pmatrix}
 x_0 + i x_1 & -x_2 - i x_3 \\
 x_2 - i x_3 & x_0 - i x_1 \\
 \vdots & \vdots \\
 x_{4l} + i x_{4l+1} & -x_{4l+2} - i x_{4l+3} \\
 x_{4l+2} - i x_{4l+3} & x_{4l} - i x_{4l+1} \\
 \vdots & \vdots 
\end{pmatrix}, \quad \text{for} \quad q = \begin{pmatrix}
 q_1 \\
 q_2 \\
 \vdots \\
 q_n 
\end{pmatrix},
\]

where \( q_l = x_{4l} + i x_{4l+1} + j x_{4l+2} + k x_{4l+3}, \ l = 0, \ldots, n-1 \). \( \tau(\mathbb{H}^n) \) is a \( 4n \)-dimensional totally real subspace of \( \mathbb{C}^{2n \times 2} \). Note that we have the inverse \( \tau^{-1} : \tau(\mathbb{H}^n) \to \mathbb{H}^n \). By applying \( \tau^{-1} \) to (2.6), we get the fractional linear action \( \tau \) on \( \mathbb{H}^n \) for \( g^{-1} \in \text{SL}(n+1, \mathbb{H}) \) in (1.3). The action (1.4) a group action:

\[
g_2^{-1}(g_1^{-1} \cdot q) = (g_2^{-1} g_1^{-1}) \cdot q
\]

for \( g_1, g_2 \in \text{SL}(n+1, \mathbb{H}) \) by applying \( \tau^{-1} \) to (2.8).

Restricted to \( \tau(\mathbb{H}^n) \), derivatives \( \frac{\partial}{\partial x^A_{4l}} \) can be realized as the first-order differential operators

\[
(\nabla^A_{4l}) := \frac{1}{2} \begin{pmatrix}
 \partial x_0 + i \partial x_1 & -\partial x_2 - i \partial x_3 \\
 -\partial x_2 + i \partial x_3 & \partial x_0 + i \partial x_1 \\
 \vdots & \vdots \\
 -\partial x_{4l} + i \partial x_{4l+1} & -\partial x_{4l+2} + i \partial x_{4l+3} \\
 \partial x_{4l+2} - i \partial x_{4l+3} & \partial x_{4l} - i \partial x_{4l+1} \\
 \vdots & \vdots 
\end{pmatrix}
\]

\[
= \frac{1}{2} \begin{pmatrix}
 \partial x_0 - i \partial x_1 & \partial x_2 + i \partial x_3 & \cdots & \partial x_{4l} - i \partial x_{4l+1} & -\partial x_{4l+2} + i \partial x_{4l+3} & \cdots \\
 -\partial x_2 + i \partial x_3 & \partial x_0 - i \partial x_1 & \cdots & \partial x_{4l} + i \partial x_{4l+1} & \partial x_{4l+2} - i \partial x_{4l+3} & \cdots 
\end{pmatrix}.
\]

Now define \( d^A' : \Gamma(\mathbb{H}^n, \wedge^r \mathbb{C}^{2n}) \to \Gamma(\mathbb{H}^n, \wedge^{r+1} \mathbb{C}^{2n}) \) as

\[
d^A' F := \nabla^A_{4l} f_A \omega^A \omega^A,
\]

for \( F = f_A \omega^A \in \Gamma(\mathbb{H}^n, \wedge^r \mathbb{C}^{2n}) \). Denote \( \triangle u = d'' d' u \). Operators in the \( k \)-Cauchy-Fueter complex \( (1.1) \) are given by

\[
D_j = \begin{cases}
 \frac{\partial}{\partial A'B'}, & \text{if } j = 0, \ldots, k-1, \\
 d''^{A'B'}, & \text{if } j = k, \\
 s(A'B'), & \text{if } j = k + 1, \ldots, 2n+1.
\end{cases}
\]

We take the nontrivial one with \([A'B'] = [0'1']\).

**Proof of Theorem 1.1.** It is direct to check that

\[
\nabla^A_{4l} \tau(q)^{B'}_j = \delta^{B'}_A \delta^A_{B'},
\]

(cf. [43] Lemma 3.1) for \( \nabla^A_{4l} \) given by (4.1) and \( \tau(q)^{B'}_j \) given by (4.3). Therefore,

\[
\nabla^A_{4l} [F(\tau(q))] = \frac{\partial F}{\partial x^A_{4l}}(\tau(q))
\]
for any holomorphic function $F$ on $\mathbb{C}^{2n\times 2}$. By embedding $U$, it is easy to see that a complex valued polynomial $P(x_0, \ldots, x_{4n-1})$ on $\mathbb{R}^{4n}$ can be extended naturally to a holomorphic polynomial

$$P(z) := P\left(\frac{z_0^1 + z_1^1}{2}, \frac{z_0^1 - z_1^1}{2i}, \frac{z_3^1 - z_4^1}{2}, \frac{z_0^0 + z_1^1}{2i}, \ldots\right)$$

on $\mathbb{C}^{4n}$ satisfying $P(q) = \tilde{P}(\tau(q))$. Then by (1.8), we have

$$d^{A'}P(q) = \left(d^{A'}\tilde{P}\right)(\tau(q)).$$

If $j \leq k$, for a $\mathcal{V}_j$-polynomial $f = f_A^A(q)s_A^\omega_A$ on $\mathbb{H}^n$, we can construct a holomorphic $\mathcal{V}_j$-polynomial $E = E_A^A(z)s_A^\omega_A$ on $\mathbb{C}^{2n\times 2}$ so that $f(q) = E(\tau(q))$. Then

$$[\pi_j(g)f](q) = [\mathcal{Z}_j(\tau(g))E](\tau(q))$$

for $g \in \text{SL}(n+1, \mathbb{H})$, by comparing definition $\pi_j(g)$ in (1.5) with $\pi_j(\tau(g))$ in (2.15). Moreover, we have

$$\partial^{[A',d^{B']}}f(q) = \left(\partial^{[A',d^{B']}}\tilde{E}\right)(\tau(q)).$$

So the invariance in Theorem 1.1 follows from the identity (3.1) in Theorem 3.1. It is similar for the case $j \geq k$. \hfill \square

**Corollary 4.1.** $\pi_j(g)$ in (1.5) and (1.7) satisfy the identity (1.8) of the representations outside singularities.

A domain $D$ is called a domain of $k$-regularity if ones cannot find two nonempty open sets $D_1$ and $D_2$ such that (1) $D_1$ is connected, $D_1 \not\subset D$ and $D_2 \subset D_1 \cap D$; (2) for each $f \in \mathcal{O}_k(D)$, there is a $\tilde{f} \in \mathcal{O}_k(D_1)$ satisfying $f = \tilde{f}$ on $D_2$.

**Corollary 4.2.** A linearly convex domain is a domain of $k$-regularity.

**Proof.** Let $D$ be a linearly convex domain. Then for any $p \in \partial D$, there is a hyperplane of quaternionic dimension $n-1$ passing through $p$ and not intersecting $D$. We can write the hyperplane as

$$a + bq = 0$$

for some $a \in \mathbb{H}, b \in \mathbb{H}^n$. Then the $k$-regular function (1.10) tends to infinity as $q \to p$, i.e. any boundary point is not $k$-regularly extendible. So $D$ is a domain of $k$-regularity. \hfill \square

By definition, it is easy to see that each convex domain in $\mathbb{H}^n$ or the product $D = D_1 \times \cdots \times D_n$ of domains in $\mathbb{H}$ is a linearly convex domain. In particular, any domain in $\mathbb{H}$ is linearly convex. Thus any domain in $\mathbb{H}$ is a domain of $k$-regularity. But ones expect that the cohomologies of the $k$-Cauchy-Fueter complex on a domain $D \subset \mathbb{H}$ vanish if and only if the domain $D$ is $k$-pseudoconvex [10]. This is different from the complex case, because the $\overline{\partial}$-complex on $\mathbb{C}$ is trivial, while the $k$-Cauchy-Fueter complex on $\mathbb{H}$ is not when $k > 1$.

The *quaternionic projective space* $\mathbb{H}P^n$ of dimension $n$ is the set of right quaternionic lines in $\mathbb{H}^{n+1}$. More precisely, $\mathbb{H}P^n := (\mathbb{H}^{n+1}\setminus\{0\})/\sim$, where $\sim$ is the equivalent relation: $p \sim q$ in $\mathbb{H}^{n+1}$ if there is a non-zero quaternion number $\lambda$ such that $p = q\lambda$. For a subset $E$ of $\mathbb{H}P^n$, the dual complement $E^*$ is defined to be the set of hyperplane not intersecting $E$. For simplicity, assuming $E, E^* \subset \mathbb{H}^n$ and $0 \in E$, then the quaternionic version of the Fantappiè transformation is defined as

$$\int_{E^*} \frac{(1 + bq)^{-1}s_{A'}^\omega_A}{|1 + bq|^2} d\mu_{A'}(b), \quad (4.9)$$
which is \( k \)-regular, where \( \mu^A \)'s are measures on \( E^* \). It is an interesting question when any \( k \)-regular function on a set \( E \) is the superposition of the simple fractions of the form (1.10). In the complex case, it is known that the result holds if and only if \( E \) is \( \mathbb{C} \)-convex \([3] \S 3.6\).

4.2. Complexes over locally projective flat manifolds. Let the parabolic subgroup \( P \) of \( \text{SL}(n+1, \mathbb{H}) \) consist matrices of the form

\[
\begin{pmatrix}
a & b \\
0 & d
\end{pmatrix}.
\]

Then the homogeneous space \( \text{SL}(n+1, \mathbb{H})/P \) is the quaternionic projective space \( \mathbb{H}P^n \) of dimension \( n \).

\[
\begin{pmatrix} 1 & 0 \\ q & 1_n \end{pmatrix} P, \quad q \in \mathbb{H}^n,
\]

constitute an open subset of \( \mathbb{H}P^n \), which is diffeomorphic to quaternionic space \( \mathbb{H}^n \).

Recall that \( \text{Sp}(n,1) \) is the group of all \( (n+1) \times (n+1) \) quaternionic matrices which preserve the following hyperhermitian form:

\[ Q(q, p) = -\overline{q}p_1 - \cdots - \overline{q}p_{n+1} + \overline{q}p_{n+1}, \]

where \( q = (q_1, \ldots, q_{n+1}) \), \( p = (p_1, \ldots, p_{n+1}) \in \mathbb{H}^{n+1} \). It is a subgroup of \( \text{SL}(n+1, \mathbb{H}) \). Under the induced action of \( \text{Sp}(n,1) \) on \( \mathbb{H}P^n \), \( D_+ := \{ q \in \mathbb{H}P^n; Q(q,q) > 0 \} \) is an invariant subset which is equivalent to the quaternionic hyperbolic space \([42] \). In this case we must have \( q_{n+1} \neq 0 \), and a point in \( D_+ \) is equivalent to \((q_1,q_{n+1}^{-1}, \cdots, q_n,q_{n+1}^{-1})\). So we have the ball model for quaternionic hyperbolic space:

\[ B^{4n} = \{ q \in \mathbb{H}^n; |q| < 1 \}. \]

Thus the space \( \mathcal{O}_k(B^{4n}) \) of all \( k \)-regular functions on the ball \( B^{4n} \) is invariant under the action of the rank-1 Lie group \( \text{Sp}(n,1) \).

A group \( \Gamma \) is called discrete if the topology on \( \Gamma \) is the discrete topology. We say that \( \Gamma \) acts discontinuously on a space \( X \) at point \( q \) if there is a neighborhood \( U \) of \( q \), such that \( g(U) \cap U = \emptyset \) for all but finitely many \( g \) of \( \Gamma \). Let \( \Gamma \) be a discrete subgroup of \( \text{SL}(n+1, \mathbb{H}) \). Then \( \mathbb{H}P^n/\Gamma \) is a locally projective flat manifold. In particular, if \( \Gamma \) is a discrete subgroup of \( \text{Sp}(n,1) \subset \text{SL}(n+1, \mathbb{H}) \), then \( B^{4n}/\Gamma \) is a locally projective flat manifold. If \( \Gamma \) is a cocompact or convex cocompact subgroup of \( \text{Sp}(n,1) \), then \( B^{4n}/\Gamma \) is a compact manifold without or with boundary, respectively. In the latter case, the boundary is a spherical quaternionic contact manifold (cf. \([42] \)).

Let \( M \) be a locally projective flat manifold with coordinates charts \( \{(U_\alpha, \phi_\alpha)\} \) with \( \phi_\alpha : U_\alpha \rightarrow \mathbb{H}^n \), whose transition maps \( \phi_\beta \circ \phi_\alpha^{-1} \) in (1.11) are given by \( g_{\beta \alpha} \in \text{SL}(n+1, \mathbb{H}) \) with the induced action (1.4):

\[
\begin{array}{ccc}
U_\alpha & \rightarrow & U_\beta \\
\phi_\alpha & \mapsto & \phi_\beta \\
\phi_\alpha(U_\alpha \cap U_\beta) & \xmapsto{g_{\beta \alpha}^{-1}} & \phi_\beta(U_\alpha \cap U_\beta)
\end{array}
\]

It is obvious that

\[ g_{\alpha \beta}^{-1} = g_{\beta \alpha}. \]

\( J_2 \) can be used to glue trivial bundles \( \phi_\alpha(U_\alpha) \times \mathbb{C}^{2n^*} \) to obtain the bundle \( E^* \) by the transition functions of bundles given by

\[
(4.10) \quad \tilde{g}_{\beta \alpha} : \phi_\alpha(U_\alpha \cap U_\beta) \times \mathbb{C}^{2n^*} \rightarrow \phi_\beta(U_\alpha \cap U_\beta) \times \mathbb{C}^{2n^*}, \quad (q, \omega^A) \mapsto \left(g_{\alpha \beta}^{-1}q, J_2 \left(g_{\alpha \beta}^{-1}q, \omega^A \right) \right).
\]
Because the transition functions satisfy the compatibility condition:

\[ \hat{g}_{\gamma \beta} \circ \hat{g}_{\beta \alpha}(q, \omega^A) = \hat{g}_{\gamma \beta} \left( g_{\alpha \beta}^{-1} q, J_2 \left( g_{\gamma \beta}^{-1} q \right) \right) = \left( g_{\alpha \beta}^{-1} \hat{g}_{\gamma \beta}^{-1} q, J_2 \left( g_{\gamma \beta}^{-1} \hat{g}_{\gamma \beta}^{-1} q \right) \right) \cdot \omega^A \]

by applying the cocycle identity \( (1.12) \) in Proposition \( 2.2 \) to \( g_2^{-1} = g_{\beta \gamma}^{-1}, g_1^{-1} = g_{\alpha \beta}^{-1} \).

Similarly, \( J_1^{-1} \) can be used to glue trivial bundles \( \phi_\alpha(U_\alpha) \times \mathbb{C}^2 \) to obtain the bundle \( H \) by the transition functions of bundles given by

\[ \hat{g}_{\alpha \beta} : \phi_\alpha(U_\alpha \cap U_\beta) \times \mathbb{C}^2 \to \phi_\beta(U_\alpha \cap U_\beta) \times \mathbb{C}^2, \quad (q, s_A) \mapsto \left( g_{\alpha \beta}^{-1} q, J_1^{-1} \left( g_{\alpha \beta}^{-1} q \right), s_A \right). \]

\( J_1 \) can be used to construct the bundle \( H^* \). \( \wedge^\nu E \) is the \( \nu \)-th exterior product of the bundle \( E \), while \( \circ^\sigma H^* \) is \( \sigma \)-th symmetric product of the bundle \( H^* \). \( \wedge^2 H^* \) is a complex line bundle defined similarly.

Moreover, we can define real line bundle \( \mathbb{R}[-1] \) by the transition functions of bundles given by

\[ \hat{g}_{\alpha \beta} : \phi_\alpha(U_\alpha \cap U_\beta) \times \mathbb{R} \to \phi_\beta(U_\alpha \cap U_\beta) \times \mathbb{R}, \quad (q, t) \mapsto \left( g_{\alpha \beta}^{-1} q, J_1^{-1} \left( g_{\alpha \beta}^{-1} q \right)^2 t \right), \]

where

\[ \left| J_1^{-1} \left( g_{\alpha \beta}^{-1} q \right) \right|^2 = \frac{1}{a + bq^2}, \quad \text{for} \quad g_{\alpha \beta} = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in \text{SL}(n+1, \mathbb{H}). \]

We can also define the bundles \( \mathbb{R}_+[-1] \) and \( \mathbb{R}_-[-1] \) by \( \mathbb{R} \) replaced by \( \mathbb{R}_+ = [0, \infty) \), \( \mathbb{R}_- = (-\infty, 0] \) and \( \mathbb{R}_- = [-\infty, 0] \), respectively. By definition, we have the isomorphism of complex line bundles:

\[ \wedge^2 H^* \cong \mathbb{C}[-1]. \]

When \( j \leq k \), a global section of \( \circ^{k-j} H \otimes \wedge^j E^*[-j - 1] \) is given by a family of local sections \( f_\beta \in \Gamma(\phi_\alpha(U_\beta), \mathcal{V}_j) \) such that

\[ \hat{g}_{\alpha \beta}^* \left( f_\beta \right) = f_\alpha, \]

where \( \mathcal{V}_j = \circ^{k-j} C^2 \otimes \wedge^j \mathbb{C}^{2n}[-j - 1] \). If writing

\[ f_\beta(q) = (f_\beta)^{A_\beta}(q) s_A \omega^A \]

as in \( (1.2) \), substituting \( (4.10)-(4.12) \) into \( (4.14) \) and comparing it with the definition of representation \( \pi_j(g) \) in \( (1.5) \), we see that \( (4.14) \) can be rewritten as

\[ \left[ \pi_j(g_{\alpha \beta}) f_\beta \right](q) = f_\alpha(q). \]

Thus a family of local sections \( f_\beta \in \Gamma(\phi_\beta(U_\beta), \mathcal{V}_j) \) give us a global section of \( \circ^{k-j} H^* \otimes \wedge^j E[-j - 1] \) if and only if \( (1.14) \) is satisfied.

It follows from the \( \text{SL}(n+1, \mathbb{H}) \)-invariance of \( D_j \) in Theorem \( 1.4 \) that

\[ D_j f_\alpha(q) = D_j \left[ \pi_j(g_{\alpha \beta}) f_\beta \right](q) = \left[ \pi_{j+1}(g_{\alpha \beta}) D_j f_\beta \right](q). \]

Namely, \( \{ D_j f_\alpha \} \) gives us a section of \( \circ^{k-j-1} H^* \otimes \wedge^{j+1} E[-j - 2] \). Therefore, \( D_j \) is a well defined operator between bundles:

\[ D_j : \Gamma \left( M, \circ^{k-j} H \otimes \wedge^j E^*[-j - 1] \right) \to \Gamma \left( M, \circ^{k-j-1} H \otimes \wedge^{j+1} E^*[-j - 2] \right). \]
It is similar for \( j \geq k \). Thus we get the \( k \)-Cauchy-Fueter complex \((1.13)\) on locally projective flat manifolds. In particular, the \( k \)-th operator in the \( k \)-Cauchy-Fueter complex give us

\[
(4.17) \quad D_k = d^\omega d^\omega : \Gamma (M, \wedge^k E^*[-k+1]) \to \Gamma (M, \wedge^{k+2} E^*[-k+2]).
\]

If \( k = 0 \), it is the Baston operator \( \triangle : \Gamma (M, \wedge^2 E^*[-1]) \to \Gamma (M, \wedge^3 E^*[-2]). \)

**Remark 4.1.** In \( [4.16] \), we apply the \( \text{SL}(n+1, \mathbb{H}) \)-invariance Theorem \( [7.7] \) to holomorphic functions \( f_\alpha \) locally defined on \( \phi_\alpha(U_\alpha) \). But functions in the theorem are globally defined. This can be done by approximating holomorphic functions \( f_\alpha \) on given convex domain by polynomials.

5. The quaternionic Monge-Ampère operator on locally projective flat manifolds

5.1. The cone bundle \( \text{Sp}^{2p} E^* \) of strongly positive \( 2p \)-elements. Recall that a \( 2p \)-form \( \omega \in \wedge^2 \mathbb{C}^{2n*} \) is said to be elementary strongly positive (cf. e.g. [43 §3.1] [47]) if there exist linearly independent right \( \mathbb{H} \)-linear mappings \( \eta_j : \mathbb{H}^n \to \mathbb{H} \), \( j = 1, \ldots, p \), such that

\[
(5.1) \quad \omega = \eta_{1*} \omega' \wedge \eta_{1*} \omega' \wedge \cdots \wedge \eta_{p*} \omega' \wedge \eta_{p*} \omega',
\]

where \( \{\omega', \omega'\} \) is a basis of \( \mathbb{C}^2 \). The right \( \mathbb{H} \)-linear mapping \( \eta_j \) is identified with a row vector in \( \mathbb{H}^n \), and so \( \tau(\eta_j) \) is a \( 2 \times 2n \)-complex matrix. Thus,

\[
(5.2) \quad \eta_{j*} \omega^A = \tau(\eta_j)^A \omega^A.
\]

In this section, we use the wedge product to denote the product of Grassmannian variables, which are consistent with notations in pluripotential theory.

A \( 2p \)-element \( \omega \) is called strongly positive if it belongs to the convex cone \( \text{Sp}^{2p} \mathbb{C}^{2n*} \) generated by elementary strongly positive \( 2p \)-elements. It is said to be positive if for any strongly positive element \( \eta \in \text{Sp}^{2n-2p} \mathbb{C}^{2n*}, \omega \wedge \eta \) is positive. The trivial cone bundles \( U_\alpha \times \text{Sp}^{2p} \mathbb{C}^{2n*} \) can be glued by \( J_2 \) to a cone bundle \( \text{Sp}^{2p} E^* \), a subbundle of \( \wedge^{2p} E^* \), by the following proposition.

**Proposition 5.1.** If \( \omega \) is (elementary strongly or strongly) positive \( 2p \)-form, then \( J_2(g^{-1}, q) \omega \) is (elementary strongly or strongly) positive \( 2p \)-form for \( g \in \text{SL}(n+1, \mathbb{H}) \).

**Proof.** If \( \omega \) is a elementary strongly positive \( 2p \)-form with \( \omega \) given by \((5.1)\), then we have

\[
J_2(g^{-1}, q) \omega = J_2(g^{-1}, q) \eta_{j*} \omega' \wedge J_2(g^{-1}, q) \eta_{j*} \omega' \wedge \cdots \wedge J_2(g^{-1}, q) \eta_{p*} \omega' \wedge J_2(g^{-1}, q) \eta_{p*} \omega',
\]

for \( g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(n+1, \mathbb{H}) \), i.e. \( J_2(g^{-1}, q) \omega \) is also a elementary strongly positive \( 2p \)-form, where

\[
\hat{\eta}_j = \eta_j \cdot (d - (c + dq)(a + bq)^{-1} b) : \mathbb{H}^n \to \mathbb{H}, \quad j = 1, \ldots, p
\]

are linearly independent right \( \mathbb{H} \)-linear mappings (row vectors), since

\[
J_2(g^{-1}, q) \eta_{j*} \omega^A = \tau(\hat{\eta}_j)^A \omega^A = \eta_{j*} \omega^A.
\]

Consequently, a \( 2p \)-form in the convex cone \( \text{Sp}^{2p} \mathbb{C}^{2n*} \) is mapped by \( g^{-1} \in \text{SL}(n+1, \mathbb{H}) \) to a form also in this cone. So (strongly) positivity is preserved. \( \square \)
5.2. Closed positive currents and “integrals”.

Proposition 5.2. For $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(n+1, \mathbb{H})$, 
\begin{equation}
T_{g^{-1}}^* \text{Vol} = \frac{1}{|a + bq|^{4n+3}} \text{Vol}
\end{equation}
where Vol is the standard volume form of $\mathbb{R}^{4n}$.

Proof. Recall that fractional linear transformation $T_{g^{-1}}$ in (2.6) is a holomorphic mapping from $\mathbb{C}^{2n \times 2}$ minus a subspace to $\mathbb{C}^{2n \times 2}$, and 
\[ \text{Vol}_{\mathbb{C}} := \sum_{A=0}^{2n-1} \left( \prod_{A=0}^{2n-1} dz^A \right) \]
is a $(4n, 0)$-form on $\mathbb{C}^{2n \times 2}$. Its pull-back by $T_{g^{-1}}$ is 
\begin{equation}
T_{g^{-1}}^* \text{Vol}_{\mathbb{C}} = \frac{1}{\det(a + bq)^{2n+2}} \text{Vol}_{\mathbb{C}}.
\end{equation}

If we denote $X = \frac{d}{dt} T_{g^{-1}}^* \big|_{t=0}$, it is sufficient to prove 
\begin{equation}
X \text{Vol}_{\mathbb{C}} = -(2n+2) \text{tr}(\hat{a} + \hat{b}z) \text{Vol}_{\mathbb{C}}, \quad \text{for } X = \begin{pmatrix} \hat{a} & \hat{b} \\ \hat{c} & \hat{d} \end{pmatrix} \in \mathfrak{sl}(2n+2, \mathbb{C}).
\end{equation}

To show (5.5), note that for $g^{-1} \in \text{SL}(2n+2, \mathbb{C})$ given by (2.3), we have 
\begin{equation}
T_{g^{-1}}^* dz^A_{A'} = \partial (c + dz) (a + bz)^{-1} A' = \partial \left( (c + dz)(a + bz)^{-1} \right)^A_{A'}
\end{equation}
\begin{equation}
= d^B d^B \left( (a + bz)^{-1} \right)^{A'}_{A'} - (c + dz)(a + bz)^{-1} B'\left( (a + bz)^{-1} \right)^{A'}_{A'}
\end{equation}
\begin{equation}
= [d - (c + dz)(a + bz)^{-1} B']^A dz^B_{B'} \left( (a + bz)^{-1} \right)^{A'}_{A'}
\end{equation}
\begin{equation}
= [J_2(g^{-1}, z)]^A dz^B_{B'} [J_1(g^{-1}, z) (a + bz)^{-1}]^{A'}_{A'}.
\end{equation}

Consider the subgroup of one parameter: $g_t^{-1} = \exp(tX) = I + tX + O(t^2)$ for 
\begin{equation}
(0 \ 0 \ B)
\end{equation}
$\in \mathfrak{sl}(2n+2, \mathbb{C})$. Differentiate (5.6) for $g_t^{-1}$ to get 
\begin{equation}
X = \partial [\hat{b}]^A_B \partial [\hat{b}]^B_{A'} - \partial [\hat{b}]^B_{B'} \partial [\hat{b}]^A_A.
\end{equation}

Then it is direct to see that 
\begin{equation}
X \text{Vol}_{\mathbb{C}} = - \sum_{A=0}^{2n-1} \left( \prod_{A=0}^{2n-1} dz^A \right) \left( \prod_{A=0}^{2n-1} dz^A_{A'} \right) \left( \prod_{A=0}^{2n-1} dz^A_{A'} \right) \left( \prod_{A=0}^{2n-1} dz^A_{A'} \right) \left( \prod_{A=0}^{2n-1} dz^A_{A'} \right)
\end{equation}
\begin{equation}
= -2 \partial (\hat{b})^A_B \partial [\hat{b}]^B_{A'} \partial [\hat{b}]^A_A \text{Vol}_{\mathbb{C}}
\end{equation}
\begin{equation}
= -(2n + 2) \partial (\hat{b})^A_B \partial [\hat{b}]^B_{A'} \text{Vol}_{\mathbb{C}},
\end{equation}
by tr($\hat{b}$) = $\hat{b}^A_A$, $\hat{b}^A_A = \text{tr}(\hat{b})$. 

\[ \text{Vol}_{\mathbb{C}} \]
For $X = \begin{pmatrix} \hat{a} & 0 \\ 0 & \hat{d} \end{pmatrix}$ with $\text{tr}(\hat{a}) + \text{tr}(\hat{d}) = 0$, differentiate \([5.0]\) for $g_t^{-1} = \exp(tX)$ to get $X dz_A = d\hat{A}^1_i dz_{B_1}^i - d\hat{A}_{B_1} dz_{A_1}^i$. Similarly, we get

$$X. \text{Vol}_C = (2 \text{tr}(\hat{d}) - 2n \text{tr}(\hat{a})) \text{Vol}_C = -(2n + 2) \text{tr}(\hat{a}) \text{Vol}_C.$$ 

While for $X = \begin{pmatrix} 0 & 0 \\ \hat{c} & 0 \end{pmatrix}$, $X dz_A = 0$ and so $X. \text{Vol}_C = 0$. The transformation formula \([5.3]\) is proved.

When pulled back to $\mathbb{R}^{4n}$ by $\tau$,

$$\tau^* (dz_A^i) := \begin{pmatrix} dx_{4l_1} + idx_{4l_1+1} & -dx_{4l_2} - idx_{4l_2+3} \\ dx_{4l_2} - idx_{4l_2+3} & dx_{4l_1} - idx_{4l_1+1} \end{pmatrix}.$$ 

Thus

$$\tau^* (dz_{21}^0 \land dz_{02}^1 \land dz_{21}^1 \land dz_{12}^{21+1}) = 4 dx_{4l_1} \land dx_{4l_1+1} \land dx_{4l_2} \land dx_{4l_2+3}$$

and so $\tau^* \text{Vol}_C = 4^n \text{Vol}$, where $\text{Vol} = dx_0 \land \cdots \land dx_{4n-1}$. Since $T_{g^{-1}} = \tau^{-1} \circ T_{\tau(g)}^{-1} \circ \tau$, we find that

$$T_{g^{-1}} \text{Vol} = 4^{-n} T_{g^{-1}}^* \tau^* \text{Vol}_C = (-4)^{-n} \tau^* T_{\tau(g)}^{-1} \text{Vol}_C$$

$$= 4^{-n} \tau^* \left[ \frac{1}{\det(\tau(a) + \tau(b)\tau(q))^{2n+2}} \text{Vol}_C \right] = \frac{1}{|a + bq|^{4n+4}} \text{Vol}$$

by \([5.3]\). The Proposition is proved. \(\square\)

**Remark 5.1.** \([5.0]\) implies the well known decomposition of the complexified cotangent bundle into a tensor product $CT^* M \cong H \otimes E^*$ as $G_0$-modules. It is Salamon’s EH formalism \([10]\).

**Corollary 5.1.** On a locally projective flat manifold $M$, $\wedge^{2n} E^* \cong \mathbb{R}[-1]$ and $\wedge^{4n} T^* M \cong \mathbb{R}[-2n - 2]$.

**Proof.** It is direct to check that

$$\tau^* (dz_A^i) \left( \nabla_B^p \right) = \delta_A^B \delta_B^p.$$ 

For $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(n + 1, \mathbb{H})$, $\tau(g^{-1}) \in \text{SL}(2n + 2, \mathbb{C})$. So the identity \([2.7]\) implies that

$$\det \left[ \tau(d) - \tau(c + dq) \tau(a + bq)^{-1} \tau(b) \right] = \det \left[ \tau(a + bq) \right]^{-1} = \frac{1}{|a + bq|^2}.$$ 

The first isomorphism holds by definition \([4.12]\) of $\mathbb{R}[-1]$. The second one follows from Proposition \(5.2\). \(\square\)

On a locally projective flat manifold $M$, denote by $\mathcal{D}(M, \wedge^p E^*[-p])$ the space $C^\infty_0(M, \wedge^p E^*[-p])$, elements of which are often called $p$-forms. An element $\eta \in \mathcal{D}(M, \text{SP}^{2p}E^* \otimes \mathbb{R}[-l])$ is called a strongly positive $2p$-form, while $\psi \in \mathcal{D}(M, \wedge^p E^*[-p])$ is called closed if $\tilde{\nabla} \psi = 0$ where

$$\tilde{\nabla} = s^{[A^i, d^l]} : \Gamma (\wedge^p E^*[-p]) \to \Gamma (H^* \otimes \wedge^{p+1} E^*[-p - 1])$$

given by \([2.17]\) is an invariant operator, which is the $(p - 1)$-th operator in the $(p - 2)$-Cauchy-Fueter complex (here we assume $p \geq 2$ for simplicity). It is equivalent to

$$d^p \psi = d^{l'}\psi = 0$$
locally. Note that \( d^{A^t} \) is not an invariant operator, but \( s[A^t d^{B^t}] \) is.

As in the flat case, we can define “integral” for a \( 2n \)-form. Assuming \( M \) is orientable, there exists a global nonvanishing section of \( \wedge^{4n} T^* M \), the volume form, say \( dV \). By Corollary \( 5.1 \) we have

\[
\wedge^{2n} E^*[-2n - 1] \cong \mathbb{R}[-2n - 2] \cong \wedge^{4n} T^* M.
\]

Thus for a section \( \omega \in \mathcal{D}(M, \wedge^{2n} E^*[-2n - 1]) \), there exists a function \( f \) on \( M \) such that \( \omega \cong f dV \), and so the functional on \( \mathcal{D}(M, \wedge^{2n} E^*[-2n - 1]) \) defined by

\[
(5.10) \quad \int_M \omega := \int_M f dV,
\]

is well defined.

**Corollary 5.2.** For \( u_0, \cdots, u_n \in \Gamma(M, \mathbb{R}[-1]) \), \( u_0 \triangle u_1 \wedge \cdots \wedge \triangle u_n \in \Gamma(M, \wedge^{2n} E^*[-2n - 1]) \) and

\[
\int_K u_0 \triangle u_1 \wedge \cdots \wedge \triangle u_n
\]

for a compact subset \( K \) is well defined.

**Proof.** Since \( \triangle u_j \in \Gamma(M, \wedge^2 E^*[-2]) \), we have \( \triangle u_1 \wedge \cdots \wedge \triangle u_n \in \Gamma(M, \wedge^{2n} E^*[-2n]) \) by definition. \( \square \)

An element of the space \( [\mathcal{D}(M, \wedge^{2n-p} E^*[-(2n - p) - 1])]' \) dual to \( \mathcal{D}(M, \wedge^{2n-p} E^*[-(2n - p) - 1]) \) is called a \( p \)-current. \( \psi \in \mathcal{D}(M, \wedge^p E^*[-p]) \) defines a \( p \)-current by

\[
(5.11) \quad T_\psi(\eta) := \int_M \psi \wedge \eta
\]

for any \( \eta \in \mathcal{D}(M, \wedge^{2n-p} E^*[-(2n - p) - 1]) \), since \( \psi \wedge \eta \in \mathcal{D}(M, \wedge^{2n} E^*[-2n - 1]) \). A \( p \)-current \( T \) is called closed if

\[
(\overline{\partial} T)(\eta) := T(\partial \eta) = 0
\]

for any \( \eta \in \mathcal{D}(M, H \otimes \wedge^{2n-p-1} E^*[-(2n - p)]) \), where \( \partial \eta \in \mathcal{D}(M, \wedge^{2n-p} E^*[-(2n - p) - 1]) \) by definition. It is direct to check that \( \overline{\partial} T_\psi = (-1)^{p+1} T_{\overline{\partial} \psi} \) with the natural extension of \( \overline{\partial} \) to the dual pair between \( H \otimes \wedge^{2n-p} E^*[-(2n - p)] \) and \( H^* \otimes \wedge^{p+1} E^*[-p - 1] \). We omit details.

A \( 2p \)-current \( T \) is said to be positive if we have \( \overline{\partial} T(\eta) \geq 0 \) for any strongly positive form \( \eta \in \mathcal{D}(M, SP^{2n-2p} E^* \otimes \mathbb{R}_+[-(2n - 2p) - 1]) \). A upper semicontinuous section of \( \mathbb{R}[-1] \) is said to be plurisubharmonic if \( \triangle u \) is a closed positive \( 2 \)-current. The space of plurisubharmonic section on \( M \) is denoted by \( \text{PSH}(M) \). For \( u \in \text{PSH}(M) \cap C^2(M, \mathbb{R}[-1]) \), \( \triangle u \) is a closed strongly positive \( 2 \)-form.

Now we fix a metric \( h \) for the line bundle \( \mathbb{R}[-1] \).

**Theorem 5.1.** On a locally projective flat manifold \( M \), for any \( u_0, \cdots, u_n \in \text{PSH}(M) \cap C^2(M, \mathbb{R}[-1]) \), we have

\[
(5.12) \quad 0 \leq \int_M -u_0 \triangle u_1 \wedge \cdots \wedge \triangle u_n \leq C \prod_{i=0}^n \| u_i \|_{L^\infty},
\]

where \( L^\infty \)-norms are defined in terms of the metric \( h \).

Its proof is reduced to known Chern-Levine-Nirenberg estimate \( 243 \) on flat quaternionic space \( \mathbb{H}^n \) by the unit partition. As a corollary, for \( u_j \in \text{PSH}(M) \cap C(M, \mathbb{R}[-1]) \),

\[
-u_0 \triangle u_1 \wedge \cdots \wedge \triangle u_n
\]
defines a measure on $M$. The capacity of a compact subset $K$ of $M$ can be defined as
\[
\text{cap}(K, M) := \sup \left\{ -\int_K u_0 \Delta u_1 \wedge \cdots \wedge \Delta u_n; u_j \in \text{PSH}(M) \cap C(M, \mathbb{R}_{-[-1]}), |u_j|_k \leq 1 \right\}.
\]
The Monge-Ampère equation and pluripotential theory on a locally projective flat manifold will be discussed in the subsequent part.

6. Projective Invariance

Write $(\Delta u)^n = M(u)\Omega_{2n}$ locally, where $\Omega_{2n} := \omega^0 \wedge \omega^1 \wedge \cdots \wedge \omega^{2n-2} \wedge \omega^{2n-1}$.

**Proposition 6.1.** $\frac{M(u)}{\Omega_{2n}}$ is quaternionic projectively invariant for $u \in \Gamma(M, \mathbb{R}_{[-1]})$.

**Proof.** Note that for $u \in \Gamma(M, \mathbb{R}_{[-1]})$, we have locally
\[
\Delta \left( \frac{1}{|a + bq|^2} u(g^{-1}, q) \right) = \frac{1}{|a + bq|^4} J_2(g^{-1}, q), \Delta u(g^{-1}, q),
\]
by the $\text{SL}(n+1, \mathbb{H})$-invariance of $D_0$ in the $0$-Cauchy-Fueter complex in Theorem 1. Thus,
\[
(\Delta \left( \frac{1}{|a + bq|^2} u(g^{-1}, q) \right))^n = \frac{1}{|a + bq|^{4n+2}} M(u)(g^{-1}, q), (\Delta u(g^{-1}, q))^n.
\]
Consequently, we get
\[
M \left( \frac{1}{|a + bq|^2} u(g^{-1}, q) \right) \Omega_{2n} = \frac{1}{|a + bq|^{4n}} M(u)(g^{-1}, q) J_2(g^{-1}, q), \Omega_{2n}
\]
\[
= \frac{1}{|a + bq|^{4n+2}} M(u)(g^{-1}, q) \Omega_{2n},
\]
by using (5.9), and so
\[
M \left( \frac{1}{|a + bq|^2} u(g^{-1}, q) \right) = \frac{1}{|a + bq|^{4n+2}} M(u)(g^{-1}, q).
\]
It implies $\frac{M(u)}{\Omega_{2n}}$ is a quaternionic projectively invariant for $u \in \Gamma(M, \mathbb{R}_{[-1]})$. \qed

Now consider the problem:
\[
(6.2) \quad \begin{cases} 
M(u) = 1, \\
u|_{bD} = \infty.
\end{cases}
\]
for $u \in C(D, \mathbb{R}_{-[-1]})$, where $D$ is a domain in $M$. Letting $u = -\frac{1}{\varrho}$, then $\varrho$ is a section of $\mathbb{R}_{+}[1]$. Note that locally $\Delta = d^{\varrho'} d'\varrho$ and
\[
\Delta \left( -\frac{1}{\varrho} \right) = \frac{\Delta \varrho}{\varrho^2} - \frac{2 d^{\varrho'} \varrho \wedge d' \varrho}{\varrho^3}.
\]
Then,
\[
(6.3) \quad \left( \Delta \left( -\frac{1}{\varrho} \right) \right)^n = \frac{1}{\varrho^{2n+1}} \left[ \varrho(\Delta \varrho)^n - 2 n d^{\varrho'} \varrho \wedge d' \varrho \wedge (\Delta \varrho)^{n-1} \right].
\]
If we define $J(\varrho)$ locally by
\[
(6.4) \quad -J(\varrho) \Omega_{2n} := \varrho(\Delta \varrho)^n - 2 n d^{\varrho'} \varrho \wedge d' \varrho \wedge (\Delta \varrho)^{n-1},
\]
then (6.3) implies

\[ J(\varphi)\Omega_{2n} = \frac{M(u)}{u^{2n+1}}\Omega_{2n}. \]

By Proposition 6.1, \( J(\varphi) \) is quaternionic projectively invariant, i.e. it is a well defined scalar function on \( M \). Therefore, the problem (6.2) is equivalent to the Dirichlet problem

\[
\begin{cases}
J(\varphi) = 1, \\
\varphi|_{\partial D} = 0.
\end{cases}
\] (6.5)

Fefferman [24] used the complex Monge-Ampère operator to construct a holomorphically invariant defining density of a strictly pseudoconvex domain, and CR invariant differential operators on the boundary. We construct a projectively invariant defining density.

**Theorem 6.1.** For a domain \( D \) in a locally projective flat manifold \( M \), there exists a defining density \( \varphi \), a section of \( \mathbb{R}_+[1] \), such that

\[ J(\varphi) = 1 + O(\varphi^{2n+2}). \]

Any smooth local approximate solution \( \varphi \in C^\infty(\overline{D}, \mathbb{R}_+[1]) \) to this equation is uniquely determined up to order \( 2n + 2 \).

**Proof.** It is sufficient to show the result locally. For a defining function \( \varphi \) of the domain \( D = \{ \varphi > 0 \} \) and \( \text{grad} \varphi \neq 0 \) on \( \partial D \). We can assume \( J(\varphi) = 1 \) on \( \partial D \). This is because for any smooth function \( \eta \),

\[ J(\eta\varphi)|_{\partial D} = \eta^{n+1}J(\varphi)|_{\partial D}, \]

we can choose \( \eta = J(\varphi) \).

Now suppose that for \( s \geq 2 \), we have \( J(\varphi) = 1 + O(\varphi^{s-1}) \). We want to solve this equation for \( s \) replaced by \( s + 1 \), i.e.

\[ J(\varphi) = 1 + O(\varphi^s). \] (6.6)

Take \( \varphi = \varphi + \eta\varphi^s \). Then

\[ -J(\varphi)\Omega_{2n} = \varphi(\Delta \varphi)^n - 2n(1 + \eta\varphi^{s-1})^2d^0 \varphi \wedge d^1 \varphi \wedge (\Delta \varphi)^{n-1} + O(\varphi^s), \] (6.7)

where

\[ \Delta \varphi = (1 + \eta\varphi^{s-1})\Delta \varphi + s(s - 1)\eta\varphi^{s-2}d^0 \varphi \wedge d^1 \varphi + s\varphi^{s-1}(d^0 \eta \wedge d^1 \varphi + d^1 \varphi \wedge d^0 \eta), \] (6.8)

and so

\[ d^0 \varphi \wedge d^1 \varphi \wedge (\Delta \varphi)^{n-1} = (1 + \eta\varphi^{s-1})^{n-1}d^0 \varphi \wedge d^1 \varphi \wedge (\Delta \varphi)^{n-1}, \] (6.9)

by \( d^A \varphi \wedge d^A \varphi = 0 \). By substituting (6.8) into (6.7) and absorbing terms having factor \( \varphi^s \) into \( O(\varphi^s) \), we find that

\[ -J(\varphi)\Omega_{2n} = (1 + \eta\varphi^{s-1})^{n+1}\varphi \left[ \Delta \varphi + s(s - 1)(1 + \eta\varphi^{s-1})^{-1}\varphi^{s-2}d^0 \varphi \wedge d^1 \varphi \right]^n \\
- 2n(1 + \eta\varphi^{s-1})^{n+1}d^0 \varphi \wedge d^1 \varphi \wedge (\Delta \varphi)^{n-1} + O(\varphi^s) \\
= (1 + \eta\varphi^{s-1})^{n+1} \left[ J(\varphi)\Omega_{2n} + ns(s - 1)(1 + \eta\varphi^{s-1})^{-1}\varphi^{s-1}d^0 \varphi \wedge d^1 \varphi \wedge (\Delta \varphi)^{n-1} \right] + O(\varphi^s) \\
= (1 + \eta\varphi^{s-1})^{n+1} \left[ -J(\varphi)\Omega_{2n} + \frac{1}{2}s(s - 1)(1 + \eta\varphi^{s-1})^{-1}\varphi^{s-1}J(\varphi)\Omega_{2n} \right] + O(\varphi^s) \\
= \frac{-J(\varphi)\Omega_{2n}}{1 - [s(n + 1) - s(s - 1)/2]\eta\varphi^{s-1}} + O(\varphi^s), \]
where we have used
\[(1 + s^n \varphi^{s-1}) \varphi = \varphi + O(\varphi^s), \quad (\varphi^{s-1})^2 = O(\varphi^s).\]

Thus \(J(\varphi) = 1 + O(\varphi^s)\) is equivalent to
\[
\frac{s}{2} (2n + 3 - s) \eta \varphi^{s-1} = 1 - J(\varphi) + O(\varphi^s)
\]
Namely, if we take
\[
\varrho = \varphi \left(1 + 2 \frac{1 - J(\varphi)}{s(2n + 3 - s)}\right),
\]
(6.6) is solvable for \(s = 2, \ldots, 2n + 2\).

We call the defining density given by Theorem [6.1] a Fefferman defining density. If \(u = -\frac{1}{\varrho}\) is a PSH section of \(\mathbb{R}_{-1}\), it defines a projectively invariant positive 2-form
\[
\Delta u = \frac{\Delta \varrho}{\varrho^2} - \frac{2 d^\varrho \wedge d^{\varrho}}{\varrho^3},
\]
on \(D\). This can be viewed as the quaternionic version of the Blaschke metric on strictly convex domains [33, 11]. Discussion of this form and applications to quaternionic strictly pseudoconvex boundary will appear in the subsequent part.

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