Conformal window in QCD for large numbers of colours and flavours

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We conjecture that the phase transitions in QCD at large number of colours $N \gg 1$ is triggered by the drastic change in the instanton density. As a result of it, all physical observables also experience some sharp modification in the $\theta$ behaviour. This conjecture is motivated by the holographic model of QCD where confinement-deconfinement phase transition indeed happens precisely at temperature $T = T_c$ where $\theta$ dependence of the vacuum energy experiences a sudden change in behaviour: from $N^2 \cos(\theta/N)$ at $T < T_c$ to $\cos \theta \exp(-N)$ at $T > T_c$. This conjecture is also supported by recent lattice studies. We employ this conjecture to study a possible phase transition as a function of $\kappa \equiv N_f/N$ from confinement to conformal phase in the Veneziano limit $N_f \sim N$ when number of flavours and colours are large, but the ratio $\kappa$ is finite. Technically, we consider an operator which gets its expectation value solely from nonperturbative instaton effects. When $\kappa$ exceeds some critical value $\kappa > \kappa_c$ the integral over instanton size is dominated by small-size instatons, making the instanton computations reliable with expected $\exp(-N)$ behaviour. However, when $\kappa < \kappa_c$, the integral over instaton size is dominated by large-size instantons, and the instanton expansion breaks down. This regime with $\kappa < \kappa_c$ corresponds to the confinement phase. We also compute the variation of the critical $\kappa_c(T, \mu)$ when the temperature and chemical potential $T, \mu \ll \Lambda_{QCD}$ slightly vary. We also discuss the scaling $(x_i - x_j)^{-\gamma_{dual}}$ in the conformal phase.

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I. INTRODUCTION

Understanding the phase diagram at nonzero external parameters $T, \mu, \kappa$ and the so-called $\theta$-parameter is one of the most hard problem in QCD. Obviously, this area is a prerogative of numerical lattice computations. However, some insights about the basic features of the phase diagram may be inferred by using some analytical methods. Our study of this hard question with non-zero $\theta$ parameter has been motivated, in fact, by very deep cosmological connections related to the QCD phase transition in early universe when $\theta$ parameter is thought to be nonzero. However, the cosmological connections shall not be elaborated in the present work\(^1\).

The approach we advocate in the present work to attack this hard problem is based on a conjecture originally formulated in refs. \([1, 2]\) that the de-confined phase transition is always accompanied by very sharp changes in $\theta$ behaviour. Therefore, in principle, if our conjecture is correct, one can use any order parameter which nontrivially depends on $\theta$ and study this dependence on two sides of the phase transition line. Very natural question immediately emerges: why and how these two apparently very different things (phase transition vs sharp $\theta$ changes) could be linked? What is the basic motivation for this proposal? First of all, this criteria is motivated by the observation that in holographic model of QCD the confinement-deconfinement phase transition indeed happens precisely at the temperature $T = T_c$ where $\theta$ dependence experiences a sudden change in behaviour, see \([1, 3]\) and many related references therein.

Secondly, the proposal is supported by the numerical lattice results \([4] -[12]\), see also a review article \([13]\), which unambiguously suggest that the topological fluctuations related to $\theta$ (in particular the topological susceptibility) are strongly suppressed in deconfined phase, and this suppression becomes more severe with increasing $N$.

Thirdly, our new criteria is based on a physical picture which can be shortly summarized as follows. On one side of the phase transition line the instanton gas is dilute with density $\sim e^{-N\gamma(\kappa, \mu, T)}$ which implies a strong suppression of the topological fluctuations at large $N$ with $\gamma(\kappa, \mu, T) > 0$, see below some details on generic features of $\gamma(\kappa, \mu, T)$—function.

The calculations in this region are under complete theoretical control and the vacuum energy has a nice analytic behaviour $\sim \cos \theta e^{-N\gamma(\kappa, \mu, T)}$ as function of $\theta$. At the critical value $\gamma(\kappa, \mu, T)$ changes the sign, the instanton expansion suddenly breaks down (at very large $N$), and one should naturally expect that there must be a sharp transition in $\theta$ behavior as simple formula $\sim \cos \theta$ can only be valid when the instanton gas is dilute and semiclassical calculation is justified, which is obviously not the case when the instanton density formally becomes exponentially large $\sim e^N$. This is a region of confinement when the dilute instanton gas approximation can not be trusted anymore. What happens to the well-defined objects (instantons) as the phase transition line is crossed at the critical values? One can argue that the instantons do not completely disappear from the system, but rather dissociate into the instanton quarks\(^2\), the objects with fractional topological charges $\pm 1/N$ which become the dominant pseudo-particles in the confined phase, which is precisely the conjecture advocated in refs. \([1, 2]\).

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\(^1\) Non-vanishing $\theta \neq 0$ implies that the $CP$ symmetry is strongly broken during the cosmological QCD phase transition. At the same time, a non-observation of any $CP$ violating processes in strong interactions at the present epoch implies that the $\theta$ parameter vanishes now. The well known (and generally accepted) resolution of the strong $CP$ problem is provided by the dynamical axion field, which, at the same time, might be a natural dark matter candidate. In different words, a unique and strong source of the $CP$ violation which was available during the QCD phase transition is not available anymore at present time. This unique source of the strong $CP$ violation might be a missing ingredient for understanding the observed the baryon- antibaryon asymmetry existing in nature, see few comments and references on cosmological connection in Conclusion in section IV.

\(^2\) Instanton quarks, also known as “fractional instantons” or “instanton partons”, originally appeared in 2d models. Namely, using an exact accounting and re-summation of the $n$-instanton solutions in 2d CP$^{N−1}$ models, the original problem of a statistical instanton ensemble was mapped unto a 2d Coulomb Gas (CG) system of pseudo-particles with fractional topological charges $\sim 1/N$ \([14]\). This picture leads to the elegant explanation of the confinement phase and other important properties of the 2d CP$^{N−1}$ models \([14]\). We use term the “instanton quarks” to emphasizes that there are precisely $N$ constituents making an instanton, similar to $N$ quarks making a baryon. Other suggested terms such as the “fractional instantons” or the “instanton partons” do not quite reflect this unique $1/N$ feature. These objects do not appear individually in path integral; instead, they appear as configurations consisting $N$ different objects with fractional charge $1/N$ such that the total topological charge of each configuration is always integer. In this case $4Nk$ zero modes for $k$ instanton solution is interpreted as 4 translation zero modes modes accompanied by every single instanton quark. The same counting holds, in fact, for any gauge group $G$, not limited to SU$(N)$ case. While the instanton quarks emerge in the path integral coherently, these objects are highly delocalized: they may emerge on opposite sides of the space time or be close to each other with alike probabilities. Similar objects have been discussed in a number of papers in a different context, see e.g. \([15-20]\). In particular, it has been argued that the well-established $\theta/N$ dependence in confined phase unambiguously implies that the relevant configurations in confined QCD must carry fractional topological charges, see \([1, 18]\) and references on original papers therein. The weakly coupled deformed QCD model \([21–29]\), to be discussed in the next paragraph, is a precise dynamical realization of this idea when the fractionally charged constituents play the dominant role, and when the question on dissociation of the instantons to the instanton quarks can be explicitly tested and studied in the weak coupling regime.
Finally, this entire framework can be in principle tested using some deformations of QCD, which on the one hand preserve all the crucial elements of strongly interacting QCD, including confinement, nontrivial θ dependence, degeneracy of the topological sectors, chiral symmetry breaking, etc. On the other hand the deformations are designed in such a way that they bring the system into the weak coupling regime when all computations are under complete theoretical control. In fact, the corresponding technique is well developed by now, see relevant for present studies references [21–29]. These recent studies essentially support the basic picture that the phase transition occurs as a result of complete reconstruction of the dominant pseudoparticles on two sides of the phase transition line (instantons versus instanton quarks). The system experiences a sharp transition in θ behaviour precisely as a result of this reconstruction of the dominant pseudoparticles. In other words, the driving force of a deconfined phase transition is the dissociation of the instantons into their constituents, the instanton quarks. This reconstruction obviously leads to the drastic changes in θ behaviour on two sides of the phase transition line. We use this sharp alteration in θ behaviour as a signal of the phase transition. It precisely represents the basic conjecture formulated in refs. [1, 2].

Therefore, the phase transition within this framework can be interpreted (with some very important reservations, see below) as Berezinskii-Kosterlitz-Thouless (BKT) -like phase transition [30]: at $T > T_c$ (or $\mu > \mu_c$, or $\kappa > \kappa_c$ depending on a specific slice of the multidimensional phase diagram we are interested in) the constituents prefer to organize a single instanton of a finite size. We coin this phase as a “molecular phase” which corresponds to a de-confined phase in conventional terminology. When one crosses the phase transition line at $T < T_c$ (or $\mu < \mu_c$ or $\kappa < \kappa_c$) the constituents prefer to stay far away from each other. It corresponds to the dissociation of each instanton into N constituents, and we call this state as the “N component plasma phase” in 4d Euclidean space. This regime corresponds to the confined phase in conventional terminology when all instanton quarks are delocalized in 4d Euclidean space. The gap in this confined phase is determined by the Debye correlation length of this 4d plasma. We should comment here that the idea on possibility of the BKT type transition in case when parameter $\kappa$ varies was suggested previously [31], though the arguments and motivation of ref.[31] were very different from those advocated in the present work.

To avoid any confusion in our future discussions, we should emphasize from the very beginning that this analogy should not be taken literally. Indeed, we have pseudo-particles which live in the 4d Euclidean space, instead of real particles/quasi-particles in the original BKT picture. The pseudo-particles in 4d space have an interpretation of an auxiliary objects which describe the tunnelling events, rather than static object which live in real Minkowski space time. Furthermore, our term the “molecular phase” which corresponds to a de-confined phase in conventional terminology should not be confused with molecular phase in the original BKT picture when real particles/quasi-particles make a static bound state. In 4d Euclidean space the “molecular phase” implies that the corresponding Euclidean configurations (instantons) provide the dominant contribution into the path integral. The same comment also applies to the “N component plasma phase” in 4d Euclidean space, which should not be confused with plasma phase in the original BKT picture. This comment obviously implies that conventional entropy arguments can not be applied to 4d “phases” as the merely notion of the entropy does not exist for such Euclidean objects. Nevertheless, this analogy could be quite useful in the description of the universal properties of the phase transitions due to many formal similarities between these two (very different) systems as we discuss in this work. Furthermore, from the 5d (holographic) viewpoint point the 4d Euclidean objects (instantons and the instanton quarks) can be indeed interpreted as some static objects. However, we shall not elaborate on this holographic interpretation in the present work.

To reiterate: we identify the sharp changes in θ behaviour in pure gauge theory with some kind of BKT- like phase transition when in one phase the instanton constituents prefer to be in “molecular phase” while in another phase they prefer to be in the “plasma phase”. In particular, in case with variation of temperature $T$ the corresponding studies have been carried out in [1] while the case when the chemical potential $\mu$ varies, the corresponding results have been presented in [2]. The goal of this work is to study confinement- deconfinement phase transition with variation of the parameter $\kappa$. We argue that the confined phase ceases to exist for sufficiently large $\kappa > \kappa_c$ as a result of reconstruction of the dominant pseudoparticles, similar to our studies of the phase transition lines at $T > T_c$ or $\mu > \mu_c$. It is normally assumed that the asymptotic freedom still holds in this region as plotted on Fig. 1. At zero temperature the confined phase is expected to be replaced by a “non-Abelian Coulomb phase”, conventionally referred to as the “conformal window”, see some original analytical results [32–39], the recent lattice computations[40–45] and review [46] on the subject.

To accomplish our goal we use $\langle \det \bar{\psi}_{L}^{f} \psi_{R}^{f} \rangle$ as the order parameter to analyze the phase transition as function of $\kappa$. This operator breaks $U(1)_A$ symmetry, and gets its expectation value from instantons. The variation of this potential

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3 The BKT picture normally suggests that the correlation length expressed in terms of parameter $(\kappa - \kappa_c)$ should exhibit a very specific behaviour when $(\kappa - \kappa_c)$ crosses the phase transition line. To be more specific, in “molecular phase” the correlation length $\zeta$ should be infinite, while in the “plasma phase” for $(\kappa_c - \kappa) > 0$ it should demonstrate a specific BKT scaling $\zeta \sim \exp(-\frac{1}{\sqrt{\kappa - \kappa_c}})$ in close vicinity of the critical point. Unfortunately, our approach breaks down in the “plasma phase” as we review in next section II. Therefore, while the physical description of the phase transition in terms of the pseudo-particles is very similar to BKT picture, the corresponding analysis of the correlation length $\zeta$ in the vicinity of the phase transition with $(\kappa_c - \kappa) > 0$ can not be carried out within this approach, see also a comment on this subject in concluding section IV.
as a function of $\kappa$ detects the drastic reorganization of the dominant pseudoparticles (instanton vs instanton quarks). Therefore, we study the behaviour of the expectation value $\langle \det \bar{\psi} f_L \psi f_R \rangle$ at $\kappa > \kappa_c$ in the “molecular phase” where the instanton expansion is under complete theoretical control. We slowly approach the regime where the instanton expansion breaks down. We identify this point with the critical point $\kappa_c$. We argue that in large $N$ limit we can approach this point with arbitrary high precision as the instanton density remains to be exponentially suppressed (and therefore instanton expansion remains to be justified) as long as inequality $(\kappa - \kappa_c) \gg 1/N$ holds. We use the same guiding principles formulated in [1, 2] to estimate how this critical value $\kappa_c$ depends on temperature and chemical potential $\kappa_c(T, \mu)$ for sufficiently small $T, \mu$. This analysis essentially makes a solid prediction (of course, within our framework) on how the conformal window extends to non-vanishing temperature and chemical potential, which is the subject of section III. Our presentation starts with section II where we review our basic conjecture and guiding principles originally formulated in refs. [1, 2]. Section IV is our conclusion with few comments on future directions and possible tests of the entire framework.

II. CONFINEMENT- DECONFINEMENT PHASE TRANSITION IN HOT AND DENSE QCD AT LARGE $N$.

We start with a short review of ref.[1] where the conjecture (that the confinement-deconfinement phase transition happens precisely where $\theta$ behavior sharply changes) was implemented for large $N$ QCD at $T \neq 0$. Such a sharp transition is indeed observed in the holographic model of QCD. From quantum field theory viewpoint such a transition can be understood as follows. Instanton calculations are under complete theoretical control in the region $T > T_c$ as the instanton density is parametrically suppressed at large $N$ in deconfined region[1],

$$V_{\text{inst}}(\theta) \sim e^{-\gamma N} \cos \theta, \quad \gamma = \left[ \frac{11}{3} \ln \left( \frac{\pi T}{\Lambda_{\text{QCD}}} \right) - 1.86 \right]. \quad (1)$$

It is assumed that the higher order corrections in the instanton background may change the numerical coefficients in $\gamma(T)$. However, the general structure of eq. (1) is expected to hold. The critical temperature is determined by condition $\gamma = 0$ where exponentially small expansion parameter $\exp(-\gamma N)$ suddenly blows up and becomes exponentially large $\sim \exp N$. Numerically, it happens at

$$\gamma = \left[ \frac{11}{3} \ln \left( \frac{\pi T_c}{\Lambda_{\text{QCD}}} \right) - 1.86 \right] = 0 \quad \Rightarrow \quad T_c(N = \infty) \simeq 0.53 \Lambda_{\text{QCD}}, \quad (2)$$

where $\Lambda_{\text{QCD}}$ is defined in the Pauli-Villars scheme. Our computations are carried out in the regime where the instanton density $\sim \exp(-\gamma N)$ is parametrically suppressed at any small but finite $\gamma(T) = \epsilon > 0$ when $N = \infty$. We identify the regime with $\gamma > 0$ with the “molecular phase” in 4d Euclidean space when the instantons are well defined objects. From eq. (1) one can obtain the following expression for instanton density in vicinity of $T > T_c$,

$$V_{\text{inst}}(\theta) \sim \cos \theta \cdot e^{-\alpha N \left( \frac{T-T_c}{T_c} \right)}, \quad 1 \gg \left( \frac{T-T_c}{T_c} \right) \gg 1/N. \quad (3)$$
where $\alpha = \frac{11}{2}$ and $T_c(N = \infty) \simeq 0.53 \Lambda_{QCD}$ are estimated at one loop level. Such a behavior does imply that the dilute gas approximation is justified even in close vicinity of $T_c$ as long as $\frac{27T_c}{\Lambda_{QCD}} \gg \frac{1}{N}$. Therefore, the $\theta$ dependence, which is sensitive to the topological fluctuations is determined by (3) all the way down to the temperatures very close to the phase transition point from above, $T = T_c + O(1/N)$. The topological susceptibility vanishes $\sim \cos \theta e^{-\gamma N} \to 0$ for $T > T_c$ in deconfined phase.

The instanton expansion breaks down at $T < T_c$. In this regime instantons cease to exist, as they dissociate into the instanton quarks, see some references mentioned in the Introduction. The instanton quarks with fractional topological charges $1/N$ emerge as the dominant pseudoparticles in confined phase. This sharp reconstruction leads to the drastic changes in $\theta$ behaviour. Indeed, the topological susceptibility is order of $\cos(\theta/N)$ in confined phase, while it is of order $\cos \theta e^{-\gamma N}$ in deconfined phase. We identify the confined phase with the “$N$ component plasma phase” in 4d Euclidean space, while deconfined phase with “molecular phase” if one uses the BKT terminology. The strongly coupled regime at $T < T_c$ obviously cannot be not studied using the semiclassical approximation. The key observation of refs. [1, 2] is that we can approach the vicinities of the critical point from above where the semiclassical expansion remains to be justified as instanton density is still parametrically small in large $N$ limit as eq. (3) suggests. This entire picture, in principle, can be theoretically tested for any finite $N$ using the weakly coupled “deformed QCD” as a toy model [21, 22]. Such an analysis in principle allows to study many deep questions as the $\theta$ dependence, topological susceptibility, and other related problems on both sides of the phase transition line, see relevant for present work references [23–29].

Once $T_c$ is determined one can compute a finite segment of the phase transition line $T_c(\mu)$ for relatively small $\mu \ll T_c$. The result can be presented as follows [1]

$$T_c(\mu) = T_c(\mu = 0) \left[ 1 - \frac{3N_f \mu^2}{4N^2 T_c^2(\mu = 0)} \right], \quad \mu \ll \pi T_c, \quad N_f \ll N.$$  

This is a solid prediction of the entire framework in large $N$ limit when many numerical uncertainties are hidden in $T_c(\mu = 0)$. It is amazing that eq. (4) is in excellent numerical agreement with lattice data even for $N = 3$ and $N_f = 2$ though eq. (4) was derived for $N = \infty$, see [1] for the details and references on the original lattice results.

The same guiding principle can be applied for studying the dense matter as well [2]. To be more specific, we identify the point where instanton expansion breaks down with the point $\mu_c$ where the phase transition happens. In the presence of the massless chiral fermions the $\theta$ dependence obviously goes away in QCD in both phases: confined as well as deconfined. This $\theta$-independence of the system in the presence of quarks reflects a simple fact that one can redefine the fermi fields in the chiral limit such that $\theta$ parameter can be rotated away from the partition function. Our conjecture on sharp changes of the system in the presence of massless fermions applies to the instanton induced potential itself which still assumes the same form $\exp(-\gamma N)$ similar to the analysis of the pure gauge system (1). In our estimates below we assume that the colour superconducting phase is realized in deconfined phase for all finite $N$. A precise magnitude of the diquark condensate is not essential for our calculations as it effects only sub-leading terms $\sim 1/N$ which will be consistently ignored in what follows, see item “c” below.

The corresponding estimates for the critical chemical potential $\mu_c$ can be represented as follows [2]:

$$\mu_c(N = \infty) \simeq 2.6 \cdot \sqrt{\frac{N}{N_f}} \cdot T_c(N = \infty, \mu = 0), \quad N_f \ll N,$$

where we express the critical value $\mu_c(N = \infty)$ in terms of the critical temperature (2) to minimize many numerical uncertainties, see footnote 4. One should comment that $\mu_c(N)$ is very large in large $N$ limit and scales as $\mu_c(N) \sim \sqrt{N}$, in contrast with $T_c \sim 1$. The nature of this behaviour can be explained by the observation that a large number of gluons $\sim N^2$ can get excited at $T \sim 1$ while only a relatively small number of quarks in fundamental representation $\sim N$ can get excited at $\mu \sim 1$. Therefore, a very large chemical potential $\mu^2 \sim N$ is required for quarks to play a similar role the gluons play at $T_c \sim 1$. This argument holds as long as $N_f \ll N$, and all quarks belong to the fundamental representation of $SU(N)$.

One can also derive the expression which is analogous to eq. (3) and which is valid in the vicinity of $\mu_c$

$$V_{\text{inst}} \sim e^{-\alpha N \left( \frac{\mu - \mu_c}{\mu_c} \right)}, \quad \frac{1}{N} \ll \left( \frac{\mu - \mu_c}{\mu_c} \right) \ll 1,$$

4 such as the limitations to the leading loop approximation, uncertainty of the numerical value of $T_c$ at $N = \infty$ in terms of $\Lambda_{QCD}$ which itself is expressed in terms of the Pauli -Villars scheme, etc
Numerically, the coefficient $\alpha$ is equal to 11/3 in the leading order, similar to expression (3). Such a behaviour (6) does imply that the dilute gas approximation is justified even in close vicinity of $\mu_c$ as long as $(\mu - \mu_c)/\mu_c \gg N^{-1}$. In this case the diluteness parameter $\sim V_{\text{inst}}$ remains parametrically small. We can not rule out, of course, the possibility that the perturbative corrections may change our numerical estimate for $\mu_c$ as well as for $\alpha$. However, we expect that the qualitative picture of the phase transition advocated by this framework remains unaffected as the corresponding perturbative corrections should be computed, according to our computation scheme, in deconfined phase where the instanton density remains parametrically small.

One can also study the behaviour of the critical chemical potential $\mu_c(T)$ as a function of temperature $T$ when it slightly deviates from zero. One arrives to the following expression for a finite segment of the phase transition line $\mu_c(T)$. The result can be presented as follows [2]:

$$\mu_c(T) = \mu_c(T = 0) \left[1 - \frac{N\pi^2T^2}{3N_f\mu_c^2(T = 0)}\right], \quad \sqrt{NT} \ll \mu_c, \quad N_f \ll N. \quad (7)$$

It is expected that the phase transition line (7) computed at small $T$ and large $\mu_c$ continuously connects to the phase transition line (4) computed at small $\mu$ and large $T_c$.

We conclude this short overview with the following important comments. There are three basic reasons for a generic structure (1) to emerge:

a. The presence of the exponentially large “$T$- independent” and “$\mu$- independent” contributions (e.g. $e^{+1.86N}$ in eq. (1)). This term basically describes the entropy of the configurations and enters the exponent with the positive sign. It is due to a number of contributions such as a number of embedding $SU(2)$ into $SU(N)$ etc;

b. The presence of the “$T$- dependent” and “$\mu$- dependent” contributions to the instanton induced potential $V_{\text{inst}}$ which come from $\int n(\rho)d\rho$ integration. In case of $T \neq 0$ and $\mu = 0$ it is proportional to

$$\left(\frac{\Lambda_{\text{QCD}}}{\pi T}\right)^{\frac{11}{3}N} = \exp \left[-\frac{11}{3}N \ln \left(\frac{\pi T}{\Lambda_{\text{QCD}}}\right)\right]. \quad (8)$$

This term enters the exponent with the negative sign because the corresponding contribution must be strongly suppressed for large $T$ and $\mu$;

c. The fermion- related contributions such as a chiral condensate, diquark condensate or non-vanishing mass term enter the instanton density as follows $\sim \langle \bar{\psi}\psi \rangle N_f \sim e^{N(\kappa \ln \langle \bar{\psi}\psi \rangle)}$. For $\kappa \equiv \frac{N_f}{N} \rightarrow 0$ this term obviously leads to a sub leading effects $1/N$ in comparison with two main terms in the exponent (1). Therefore, such terms can be neglected as they do not change any estimates at $N = \infty$. It is in accordance with the general arguments suggesting that the fundamental fermions can not change the dynamics of the relevant gluon configurations as long as $N_f \ll N$.

The crucial element in this analysis is that both leading contributions (items a and b above) have exponential $\exp(N)$ dependence, and therefore at $N \rightarrow \infty$ for $T > T_c$ the instanton gas is dilute with density $\exp(-\gamma N)$, $\gamma > 0$ which ensures the behaviour (3). For $T < T_c$ the density is exponentially large $\sim \exp(+N)$ which implies that the instanton expansion breaks down. We expect that the instantons will dissociate to the constituents, the instanton quarks, which become the dominant pseudo-particles at $T < T_c$. This happens exactly at the point of the phase transition when the complete reconstruction of relevant pseudo-particles occurs, in close analogy with BKT-like transition. The key observation here is that our predictions are not very sensitive to a precise mechanism of this reconstruction for large $N$. In other words, one can obtain a number of solid relations which follow from this framework, such as (4), (5), (7), without a detail knowledge of the dynamics describing the instanton’s dissociation in the large $N$ limit.

### III. DECONFINEMENT PHASE IN QCD AT LARGE $N \gg 1$ AND $N_f \sim N$.

Our goal here is to generalize the ideas reviewed in section II to include into the system a large number of fermions $N_f \sim N$. The corresponding description can be conveniently expressed in terms of parameter $\kappa \sim 1$. As we mentioned in the Introduction we wish to study the correlation function $\langle \det \bar{\psi}_L^i \psi_R^j(x_i) \rangle$ as a function of parameter $\kappa$ and distances $(x_i - x_j)$. Our main criteria to pinpoint the phase transition line remains the same as formulated above: we study the instanton induced potential as a function of $\kappa$. At small $\kappa < 1$ the integral over $\rho$ diverges in the infrared, which corresponds to the confined phase when the dominant pseudo-particles are the fractional instanton quarks. This corresponds to the strongly coupled regime, the “plasma phase” in BKT terminology. This regime by obvious reasons can not be studied by semiclassical methods used in the present work. As we show below for $\kappa > 1$ the integral over $\rho$ starts to converge at large $\rho$. It is necessary, but obviously not a sufficient, condition for the transition to the “molecular phase” in BKT terminology. The sufficient condition for the “molecular phase” to set in, according to our conjecture, is a sudden $\exp(-N)$ suppression of the instanton density in the deconfined phase similar to eqs (3),
contributions can be effectively incorporated into the parameter $M$. In the present case $\kappa$ justified for sufficiently large $\kappa$ where we keep only zero modes in the chiral limit, assuming that in the dilute gas approximation (which will be justified for sufficiently large $\kappa$ as we shall argue below) all other mode contributions is suppressed by factor $m_\pi \rightarrow 0$. The integration over $d^4x$ corresponds to the integration over the instanton center at point $x$.

In order to study the behaviour of the integral at large $\rho$ we take $x_i = x_j$ and integrate over $d^4x$. One arrives to the following expression

$$\langle \det \bar{\psi}_L \psi_R \rangle = \left( \frac{\pi^2}{3N_f - 1)(3N_f - 2) \right)^{N_f} \int d\rho n(\rho) \rho^4 \cdot \left( \frac{2}{\pi^2 \rho^2} \right)^{N_f},$$  \hspace{1cm} (12)$$

where $n(\rho)$ is defined as before by eq.(9). The combination $\int d\rho n(\rho) \rho^4$ is dimensionless while the dimension of the operator $\langle \det \bar{\psi}_L \psi_R \rangle \sim \rho^{-3N_f} \sim (\text{MeV})^{-3N_f}$ as it should. The next step is to evaluate the integral at large $N$ (and keeping $\kappa = N_f/N$ fixed) by using the standard Stirling formula

$$\Gamma(N + 1) = \sqrt{2\pi N}N^N e^{-N} \left(1 + \frac{1}{12N} + O(\frac{1}{N^2}) \right)$$  \hspace{1cm} (13)$$

One follows the same procedure as before [1, 2] to evaluate integral $\int d\rho$ and take the limit $N \rightarrow \infty$ at the end of computation. In the present case $\kappa = N_f/N \sim 1$ one should keep few additional numerical factors such as $1.34^{N_f}$ from (9) and take into account changes in $b$ due to $N_f$ in eq. (9) which have been previously [1, 2] ignored. However, the most drastic change occurs due to large power of $\rho^{-3N_f}$ in eq. (12). Collecting all leading order contributions we arrive to the following expression for the expectation value $\langle \det \bar{\psi}_L \psi_R \rangle$ in the large $N$ limit at the one-loop level:

$$\langle \det \bar{\psi}_L \psi_R \rangle \sim \Lambda_{QCD}^{N(N_f - \frac{2}{3}\kappa)} \int_{1/M}^{\infty} \frac{d\rho}{\rho} \rho^{N_f N(1 - \kappa)} \cdot \exp \left[ N \left(-1.679 + 2 + 2\ln(\frac{11}{3} - \frac{2}{3}\kappa) + 1.34\kappa - \kappa \ln(\frac{\pi^2}{2}) \right) \right].$$  \hspace{1cm} (14)$$

In our estimate (14) we neglected all $\ln^n(\rho\Lambda_{QCD})$ contributions which enter $\int d\rho$ integral. First, the corresponding logarithmic contributions do not change the convergent properties of the integral, which is the main element of our studies in the present work. Second, the corresponding numerical uncertainty of the integral due to $\ln^n(\rho\Lambda_{QCD})$ contributions can be effectively incorporated into the parameter $M$, which itself is not numerically known as a result

A. Computations and technical details

We start the implementation of this program by analyzing the convergence properties of the $d\rho$ integral at large $\rho$. We use the standard formula for the instanton density at one-loop order [47, 48]

$$n(\rho) = C_N(\beta(\rho))^{2N}\rho^{-5} \exp[-\beta(\rho)] \times \exp[-(N_f\mu^2 + \frac{1}{3}(2N + N_f)\pi^2T^2\rho^2)],$$  \hspace{1cm} (9)$$

where

$$C_N = \frac{0.466e^{-1.679N}1.34^{N_f}}{(N - 1)!(N - 2)!}, \hspace{1cm} \beta(\rho) = -b \ln(\rho\Lambda_{QCD}), \hspace{1cm} b = \frac{11}{3}N - \frac{2}{3}N_f$$  \hspace{1cm} (10)$$

This formula contains, of course, the standard instanton classical action $\exp(-8\pi^2/g^2(\rho)) \sim \exp[-\beta(\rho)]$ which however is hidden as it is expressed in terms of $\Lambda_{QCD}$ rather than in terms of coupling constant $g^2(\rho)$. We inserted the chemical potential $\mu = \mu_B/N$ and temperature $T$ into this expression for future consideration when we shall study the critical value $\kappa_c(\mu, T)$ as a function of $T, \mu$ at small values of these external parameters.

We start with $T = \mu = 0$. In the chiral limit the integral over $\rho$ is reduced to the following expression

$$\langle \det \bar{\psi}_L \psi_R(x_i) \rangle = \int d\rho n(\rho) d^4x \prod_{i}^{N_f} \frac{2\rho^2}{\pi^2[(x - x_i)^2 + \rho^2]},$$  \hspace{1cm} (11)$$

where we keep only zero modes in the chiral limit, assuming that in the dilute gas approximation (which will be justified for sufficiently large $\kappa$ as we shall argue below) all other mode contributions is suppressed by factor $m_\pi \rightarrow 0$.

We identify this point with the critical $\kappa_c$ which is the minimal value of $\kappa$ where the “conformal window” starts, see the left point of the thick black segment plotted on Fig. 1. We shall also study the variation of this critical value $\kappa_c(\mu, T)$ with small variations of the chemical potential $\mu$ and temperature $T$.
of an additional non-perturbative UV divergences\(^5\), see below few comments on physical meaning of parameter \(M\). All other numerical factors in the brackets in the exponent in eq.(14) can be easily traced from the original expressions (9), (10), (11).

Few comments are in order. First of all, the integral (14) is divergent in the ultraviolet (UV) for \(\kappa \geq 1\) at \(\rho \to 0\). This UV divergence has a non-perturbative origin, which was noticed for the first time in [49], see also [50] with related discussions. This UV divergence should be contrasted with conventional perturbative UV divergences in quantum field theory (QFT) which normally emerge when the QFT operators are defined at coinciding points. The UV divergence (14) has no relevance for our studies as we are interested in the convergence properties of this integral in the IR at \(\rho \to \infty\), rather than in UV, as explained in the Introduction and section II. Nevertheless, we have to deal with this non-perturbative UV divergence as its finite portion explicitly enters the expression for the vacuum expectation value \(\langle \det \bar{\psi}_L \psi_R \rangle\). We use conventional for QFT renormalization procedure by subtracting the corresponding UV divergence and introducing a new dimensional parameter, the point of normalization \(M\) where \(\langle \det \bar{\psi}_L \psi_R \rangle_M\) is defined. Essentially, we fix the magnitude of the expectation value of the operator (14) in terms of parameter \(M\). We further elaborate on the physical meaning of this parameter \(M\) later in the text.

Our next comment is as follows: when \(\kappa > 1\) the integral is convergent in the IR for large \(\rho\). The convergence of the integral at large \(\rho\) is necessary but not sufficient condition for the instantons to be in the dilute gas regime. Similarly, the integral over \(\int d\rho\) is convergent for arbitrary small temperature \(T\), but it does not imply, of course, that the instanton density (1) is small. Regime of exponentially small instanton density is achieved when \(T\) in eq. (1) is sufficiently large and \(\gamma(T)\) flips the sign. Equation (14) which is intended to study the conformal window exhibits a similar behaviour for sufficiently large \(\kappa\), as we shall argue below.

Now we return to eq. (14) to elaborate on the physical meaning of the cutoff parameter \(M\). For sufficiently large \(\kappa\) the expression (14) becomes exponentially small. We associate this regime (which is characterized by an exponentially small instanton density) with the deconfined phase similar to our studies of the deconfined phase transition at high temperature \(T > T_c\) [1] and high chemical potential \(\mu > \mu_c\) [2]. The chiral symmetry is also expected to be restored in this phase as the chiral condensate normally forms as a result of strong interactions between the instantons, which obviously can not be the case as the instanton density is exponentially suppressed in this regime. This conclusion is supported by the computations in the instanton liquid model which also suggest that the chiral condensate vanishes when sufficient number of flavours are present in the system [50]. According to Fig. 1 we identify this deconfined phase with the conformal window. In conformal field theory \(\Lambda_{QCD}\) entering the expression (14) is a fictitious scale as there is no dynamically generated scale in the system in this phase. Scale \(M\) was introduced as point of normalization for the operator eq. (14) to remove the non-perturbative UV divergences. It can be also thought as an effective scale (in the conformal window phase) at which the running constant is saturated to its infrared value. Indeed, the \(\rho\) dependence in eq. (14) is essentially originated from \(g(\rho)\) dependence entering the instanton density (9, 10). Therefore, saturation of the integral (14) at \(\rho \sim M^{-1}\) can be interpreted as saturation of the coupling constant at this scale, i.e. \(g(\rho) \simeq g(M) \approx g_{IR}\), where \(g_{IR}\) is the coupling constant at the IR fixed point. The \(\Lambda_{QCD}\) in this system is not an independent parameter, but proportional to \(M\), i.e. \(\Lambda \sim M\), where we dropped subscript \(\sim_{QCD}\) in \(\Lambda\) to emphasize that there is no any dynamically generated scales in this phase. Furthermore, as we discuss below, the correlation function \(\langle \det \bar{\psi}_L \psi_R (x_j)\rangle\) exhibits the power like decay \((x_i - x_j)^{-\gamma_{det}}\) with nontrivial anomalous dimension \(\gamma_{det}\). It implies that the distances \((x_i - x_j)\) are measured in units of the same scale \(M \sim \Lambda\).

To get some feeling about numerical values of relevant parameters when the conformal window may emerge in this framework, it is convenient to represent the integral (14) in the following form

\[
\langle \det \bar{\psi}_L \psi_R \rangle \sim A^{3N_c} \cdot \exp \left[ -N \gamma(\kappa) \right], \quad \gamma(\kappa) = \left[ \frac{11}{3} \ln \left( \frac{M}{A} \right) - 5.65 \right] - \kappa \left[ \frac{11}{3} \ln \left( \frac{M}{A} \right) - 2.63 \right], \quad \kappa > 1,
\]

where we expanded the \(\ln(\frac{11}{3} - \frac{2}{3}\kappa)\) function in vicinity of \(\kappa \simeq 4\) to simplify things and to display the most important features of the \(\gamma(\kappa)\) function where the phase transition is expected to occur according to the lattice studies, see review paper [46]. In this simplified form eq. (15) assumes the structure similar to eq. (1). One can follow exactly the same logic as before to search for a condition when the instanton density suddenly becomes exponentially suppressed \(\sim \exp(-N)\) which is identified with deconfined phase transition. However, there is a crucial difference between present studies represented by eq.(15) and previously analyzed cases \(T > T_c\) and \(\mu > \mu_c\) reviewed in section II. The point is that in our previous studies we had a single unknown parameter such as \(T_c\) which was expressed in terms of \(\Lambda_{QCD}\)

\(^5\) We should remark here that a consistent procedure to collect all \(\ln^\alpha(\Lambda_{QCD})\) factors requires the two-loop computations in the exponent in eq. (14) as \(\exp[\ln(\rho\Lambda_{QCD})] = \ln(\rho\Lambda_{QCD})\), which is beyond the scope of the present work. Furthermore, the corresponding numerical value of \(\Lambda_{QCD}\) should be expressed at the two-loop level for the consistency of this procedure. The corresponding numerical uncertainties can be incorporated in terms of the ratio \(\ln(M/A)\) as we discuss below. We already mentioned in section II that the higher loop corrections in the instanton background obviously modify all numerical estimates within this framework. However, we do not expect that these higher loop corrections can drastically modify the qualitative picture of the phase transition advocated in this work.
defined in the Pauli-Villars scheme (2). In present case we have two unknown parameters: \( \kappa_c \) and \( M \). Therefore, a single condition that \( \gamma(\kappa) \) in eq. (15) flips the sign at the critical value \( \kappa_c \) is not sufficient to estimate \( \kappa_c \). If, instead, we adopt \( \kappa_c \simeq 4 \) as suggested by the lattice simulations at \( N = 3 \), see review paper [46], we infer that \( M/\Lambda \approx 1.55 \) in eq. (15) for \( \gamma(\kappa) \) to flip the sign at \( \kappa_c \simeq 4 \). Of course, it can not be considered as a prediction of our framework. Rather, it should be considered as the consistency check that equation \( \gamma(\kappa_c) = 0 \) has a solution with very reasonable physical parameters. While eq. (15), in contrast with previously considered cases, has not produced an unambiguous prediction for \( \kappa_c \) we shall see in a moment that there are in fact few solid consequences of this framework, which can be in principle tested in the lattice simulations.

One can easily understand the nature of the additional uncertainty (which was not present in previous cases) entering (15) in the form of dimensionless parameter \( M/\Lambda \). Indeed, in our previous studies any uncertainties related to a magnitude of diquark condensate, or specific properties of quarks, etc, could only produce a parametrically small effect \( \sim N_f/N \to 0 \) for finite \( N_f \) and large \( N \gg 1 \), see item c) in section II. In contrast, in present case when we study the vacuum expectation value of the operator \( \langle \det \bar{\psi}_L \psi_R^f \rangle \) with dimensionality of order of \( N \) the uncertainty enters (15) at the leading order \( \sim N \). Precisely this uncertainty prevents us from making a solid prediction for \( \kappa_c \) in large \( N \) limit within this framework. Nevertheless, there are few other firm consequences of this approach to be discussed below.

B. The basic consequences of the framework

While we can not predict the position of the critical value \( \kappa_c \), we can study the behaviour of the system in close vicinity of the critical point in large \( N \) limit. The corresponding expression follows from (15) and reads

\[
\langle \det \bar{\psi}_L \psi_R^f \rangle \sim \Lambda^{3N\kappa} \cdot \exp \left[ -3 \cdot N \cdot \left( \frac{\kappa - \kappa_c}{\kappa_c - 1} \right) \right], \quad \frac{1}{N} \ll (\kappa - \kappa_c) \ll 1.
\]

This formula is very similar in all respects to previously discussed cases (3), (6) when one approaches the critical point from the deconfined side of the phase transition line. The behaviour (16) implies that the dilute gas approximation is justified even in close vicinity of \( \kappa_c \) as long as \( (\kappa - \kappa_c) \gg 1/N \). The instanton density \( \sim \exp(-N) \) is parametrically suppressed in this region for any but finite \( (\kappa - \kappa_c) = \epsilon > 0 \) for large \( N \). We identify the regime with the “molecular phase” in BKT terminology when the instantons are well defined objects with finite sizes. This deconfined regime is identified according to Fig. 1 with the conformal window. When we cross the phase transition line the instanton expansion breaks down at \( \kappa < \kappa_c \). In this regime instantons cease to exist, as they dissociate into their constituents, the instanton quarks, which become the dominant pseudo-particles, see some references mentioned in the Introduction. This regime corresponds to the confined phase where the semiclassical approximation can not be trusted, and we do not even attempt to make any computation in this regime at \( \kappa < \kappa_c \).

The behaviour (16) is very generic feature of this framework which explicitly shows that the transition from one phase to another is very sharp at large \( N \). This behaviour becomes even more pronounced with increasing \( N \), in close analogy with the similar analysis (3) when the temperature \( T \) varies. In the case with temperature’s variations the corresponding lattice simulations [4] - [13] strongly support such drastic changes of the topological fluctuations. We expect exactly the same type of behaviour with variation of \( \kappa \) as the physics behind of eq. (3) and eq. (16) is exactly the same: it is complete reconstruction of the dominant pseudo-particles when \( \kappa \) varies from deconfined regime with \( \kappa = (\kappa_c + \epsilon) \) to confined phase with \( \kappa = (\kappa_c - \epsilon) \).

Now we turn to analysis of the critical behaviour \( \kappa_c(T, \mu) \) when \( T \) and \( \mu \) slightly deviate from zero. We still remain in the conformal region where the instanton based computations are under complete theoretical control. In this case a simple one-loop insertion exp\[ -\left( N_f \mu^2 + \frac{1}{2}(2N + N_f)^2 \left( 2T \right)^2 \rho^2 \right] \] in eq. (9) is justified. We consider this term as a perturbation to the instanton density which is still exponentially suppressed. This term obviously makes the instanton density even smaller. Therefore the transition between confined and deconfined phases happens at a smaller critical value \( \kappa_c(T, \mu) \) in comparison with \( \kappa_c(T = \mu = 0) \). To quantify this argument we notice that the integral (14) is saturated (for small \( T \) and \( \mu \)) by \( \rho \sim M^{-1} \) as before. Therefore, we arrive to the following numerical estimate for this variation of \( \kappa_c(T, \mu) \) in comparison with its \( \kappa_c(T = \mu = 0) \) value:

\[
\left[ \kappa_c(T, \mu) - \kappa_c(T = \mu = 0) \right] \simeq -0.3 \cdot (\kappa_c - 1) \cdot \frac{M^2 + \frac{1}{3}(2 + \kappa_c)^2 T^2}{M^2}, \quad T, \mu \ll M.
\]

Equation (17) is a direct consequence of the entire framework. It is testable, in principle, on the lattice. Essentially, eq (17) predicts how the conformal window (shown on Fig. 1) expands at small, but non-vanishing \( T, \mu \ll M \). We note here that eq (17) is a close analog of the previously discussed eqs. (4) and (7). As we already mentioned the corresponding expression (4) is in very good agreement with available lattice results.
Now we want to demonstrate that the correlation function (11) exhibits a power like decay at large \((x_i - x_j)\) with a specific critical exponent \(\gamma_{\text{det}}\). Such a behaviour is the distinct sign of the conformal phase. Indeed, the integral (11) is convergent in the IR at large \(\rho\), and in fact is numerically exponentially small \(\exp(-N)\) for large \(N\) in deconfined phase at \(\kappa > \kappa_c\) as we argued above. Furthermore, it is perfectly convergent in the UV at \(\rho \to 0\) for non-coinciding points \(x_i \neq x_j\). We are interested in the behaviour of this correlation function\(^6\) when all distances are the same order of magnitude \(\sim R\) and large, i.e. \((x_i - x_j) \sim R \gg M^{-1}\). In this case the integral (11) obviously exhibits the algebraic decay as the integral is saturated by large \(\rho \sim R\). The critical behaviour determined by \(\gamma_{\text{det}}\) of the correlation function \(\langle \bar{\psi}^f_i \psi^f_R(x_i) \rangle\) can be inferred from a simple dimensional argument. It is given by

\[
\langle \bar{\psi}^f_L \psi^f_R(x_i) \rangle \sim \Lambda^{N(4/3 - \frac{2}{3}\kappa)} \cdot P[(x_i - x_j), \gamma_{\text{det}}] \sim \Lambda^{N(4/3 - \frac{2}{3}\kappa)} \cdot \frac{1}{R^{\gamma_{\text{det}}}}, \quad \gamma_{\text{det}} = \frac{11}{3} N \cdot (\kappa - 1), \quad \kappa \geq \kappa_c,
\]

where \(P[(x_i - x_j), \gamma_{\text{det}}]\) in general is a complicated algebraic function of variables \((x_i - x_j)\). Indeed, as one can see from the original expression (11) it must depend only on relative distances \((x_i - x_j)\) as a result of translation invariance. Furthermore, \(P[(x_i - x_j), \gamma_{\text{det}}]\) must be symmetric under exchange of the external points \(x_i \leftrightarrow x_j\) as the original expression (11) exhibits this symmetry.

Few comments are on order. First, the decay law (18) should be compared with behaviour of this correlation function with dimension \(\gamma_{\text{canonical}} = 3N_f\) which corresponds to the decay \(R^{-3N_f}\) representing the canonical behaviour of free massless fermions. The scaling behaviour (18) always exhibits a slower decay power at large distances than the canonical decay as long as the asymptotic freedom holds, i.e. \(\left(\frac{\Lambda}{\rho} - \frac{2}{3}N_f \right) > 0\). We should also comment that the \((\ln R)^n\) corrections which enter the expression for the instanton density (9,10) and which were consistently ignored in our computations will modify a simple scaling property (18) by producing \((\ln R)^n\) corrections in front of eq. (18). Such type of corrections are expected, and in fact, are very generic for conformal theories. Finally, in large \(N\) limit when \(N_f \sim N\) it is instructive to represent the critical exponent \(\gamma_{\text{det}}\) in terms of anomalous dimension per number of flavours, i.e.

\[
\frac{\gamma_{\text{det}}}{N_f} = \frac{11}{3} \frac{(\kappa - 1)}{k}, \quad \kappa \geq \kappa_c.
\]

Formula (19) shows that a properly normalized critical exponent approaches the finite magnitude in large \(N\) limit. The corresponding limiting value in this approach is unambiguously determined by \(\kappa\).

We conclude this section with the following general comment. We advocate a picture that the confinement-deconfinement phase transition is a result of complete reconstruction of the dominant pseudoparticles in the system. The confined phase can be treated as the “plasma phase” in BKT terminology when the instanton’s constituents are de-localized pseudoparticles, while the deconfined phase can be thought as the “molecular phase” when \(N\) coherent instanton’s constituents are well-localized and well-organized in a form of a single instanton with a finite size. As we already emphasized in the Introduction, our system is formulated in terms of pseudo-particles which live in 4d Euclidean space rather in terms of real statical particles/quasiparticles which live in Minkowski space-time. Therefore, one should not confuse the thermodynamical terms such as “phase”, “density” or “entropy” applied to this 4d statistical ensemble with the corresponding entities which are normally applied to a conventional system of static objects in 3+1 Minkowski space. Nevertheless, one can interpret the exponential suppression \(\exp(-N)\) of the “density” of the pseudo-particles in the “molecular phase” as a result of this re-organization of the dominant pseudoparticles in the system.

This picture of complete reconstruction of the dominant pseudo-particles presented above strongly resembles the conventional BKT formulation of the 2d XY model \[30\]. Indeed, 2d XY model can be described at low temperatures as the system when the vortex and anti-vortex pairs make the bound states. This phase corresponds to the conformal field theory with infinite correlation length. At the same time at high temperatures the vortices and anti-vortices are not bounded and can be described as a two component plasma with a finite correlation length \[30\]. This behaviour in conventional BKT system with real physical particles is “opposite” to what we observed in the system of 4d objects when the high temperature phase \((T > T_c\) or \(\mu > \mu_c\) or \(\kappa > \kappa_c\)) corresponds to the “molecular phase” of the bounded objects (instantons), while the low temperature phase \((T < T_c\) or \(\mu < \mu_c\) or \(\kappa < \kappa_c\)) corresponds to the “plasma phase” of unbounded objects (instanton quarks). This “opposite” trend is also quite generic feature of the 4d Euclidean objects. For example, the contribution of the 4d instanton quarks/instantons to the topological susceptibility is always opposite in sign in comparison with the corresponding contributions of any physical propagating degrees of freedom, see e.g. \[25\] where computations in a weakly coupled gauge theory can be explicitly performed.

\[6\] this function obviously has a pure non-perturbative origin, and cannot be generated in the perturbation theory
The conventional BKT picture also suggests that the correlation length $\zeta$ being expressed in terms of parameter $(\kappa - \kappa_c)$ should exhibit a very specific behaviour $\zeta \sim \exp\left(\frac{1}{\kappa - \kappa_c}\right)$ when $\kappa$ approached $\kappa_c$, from the plasma phase side, see comment in footnote 3. Now we can explicitly see from eq. (16) why the correlation length $\zeta$ at $\kappa < \kappa_c$ can not be studied in our approach: the instanton expansion blows up in this region. The break down of the instanton expansion is formally expressed in terms of exponentially large density $\sim \exp(N)$. The corresponding studies in the “plasma phase” require fundamentally different description of the problem formulated from the very beginning in terms of the instanton’s constituents rather than instantons themselves, see also a related comment in the Conclusion.

IV. CONCLUSION AND FUTURE DIRECTIONS

In this paper we advocate an idea that the confinement-deconfinement phase transition in large $N$ limit can be formulated as dissociation of an instanton into its constituents, the so-called instanton quarks, see footnote 2 with description of some generic properties of the constituents, and also with few historical remarks on the subject. In the “molecular phase” the constituents are bound into a finite size instanton while in the “plasma phase” each instanton dissociates into $N$ delocalized objects, the instanton quarks. This picture strongly resembles the well-known BKT formulation of the 2d XY model as mentioned above.

This idea was (successfully) applied previously to the confinement-deconfinement phase transition when temperature $T$ and chemical potential $\mu$ vary. The main objective of the present work is to apply the same ideas and the same guiding principles to study the phase transition at large $N$ when number of flavours $N_f$ (formulated in terms of $\kappa = N_f/N$) varies. We find that for sufficiently large $\kappa$ the phase transition does occur, and we argue that for $\kappa > \kappa_c$, the “molecular phase” in BKT terminology is in fact a conformal field theory, similar to the original BKT studies. The main results of this framework are expressed by eqs. (16, 17, 18, 19). The key observation here is that our predictions in large $N$ limit are not very sensitive to a precise mechanism of dissociation of an instanton to its constituents. In other words, one can obtain a number of solid relations (16, 17, 18, 19) without a detail knowledge of the dynamics describing the instantons dissociation into its constituents, which is indeed a very hard problem of strongly interacting system. We do expect that numerical coefficients in eqs. (14, 15, 16, 17) may change as a result of higher order perturbative corrections in the instanton background. However, we do not expect that entire picture advocated in the present work may experience any drastic changes.

In fact, the basic framework is supported not only by the original arguments motivated by the holographic QCD. It is also supported by the lattice simulations and, though implicitly, by recent computations in the so-called “deformed QCD”, as we already mentioned in the Introduction. As we emphasized previously, the mechanism of the phase transition advocated in the present work can be viewed as à la BKT phase transition when in the “molecular phase” the constituents are bound into finite size instantons while in the “plasma phase” each instanton dissociates into $N$ delocalized objects, the instanton quarks. Apparently, this picture is quite universal and looks very much the same independently on a specific property of the parameter which varies: the temperature $T$, the chemical potential $\mu$ or the number of flavours $N_f = \kappa N$. We conclude this work with some thoughts on possible future directions related to the present studies:

- The key observation of this work is that we can approach the phase transition point $\kappa_c$ with sufficiently high accuracy in large $N$ limit when the instanton induced potential is exponentially small $\exp(-N)$ and our computations are justified (16). In other words, we are not very sensitive to some hard problems related to the dynamical description of the dissociation of the instantons to their constituents. This physics is taking place precisely in the window which is not accessible within our framework: $(\Delta T)/T \sim 1/N$, see eq. (3), or $\Delta \kappa \sim 1/N$, see eq. (16). At large $N$ this unaccessible window shrinks to a point. However, in nature $N = 3$, which is not very large. Therefore, the window $(\Delta T)/T$ numerically could be order of unity, and therefore, this region could be phenomenologically very important. In other words, the window $(\Delta T)/T$ can be experimentally studied in relativistic heavy ion collisions. In fact, some observations and numerical studies apparently suggest that there are additional magnetic degrees of freedom which may play a role in thermodynamics of the quark gluon plasma slightly above $T_c$, see [51–55] and review [56]. The corresponding discussions are beyond the scope of the present work. However, we would like to mention here that the instanton constituents carry the magnetic charges, and they should be considered as the excitations which may contribute into the thermodynamics of the system at $T > T_c$. It would be very exciting if the corresponding magnetic degrees of freedom discussed in [51–56] can be identified with the instanton’s constituents, in which case the corresponding magnetic degrees of freedom must condense exactly at the phase transition point $T_c$. At the temperatures slightly above $T_c$ the constituents can move as they are not sufficiently strong bound to make the instantons. This is because the instantons are still very flimsy and fragile when the temperatures $T$ lies inside the window $(\Delta T)/T \leq 1/N$. When the temperature $T$ lies above this window $(\Delta T)/T \gg N^{-1}$ our picture suggests that $N$ different constituents are strongly correlated to make a well localized instanton with size $\rho \simeq T^{-1}$. We predict that this window must shrink to a point for large $N$. 
The present work is devoted to a study of the conformal window in large $N$ limit when the system consists $N_f$ quarks in fundamental representations with $\kappa = N_f/N$ being fixed. This technique can be easily generalized to any other quark representations. Such a generalization is not a pure academic problem as the corresponding construction can be considered as a phenomenologically viable “walking technicolor” model, see recent review on the subject [57].

In the present work we advocate an idea that the confinement-deconfinement phase transition in large $N$ limit can be viewed as à la BKT phase transition. We also explained (at the end of section III) why the correlation length $\zeta \sim \exp(-\frac{1}{\sqrt{\kappa_\theta - \kappa}})$, which normally accompanies the BKT transition, can not be derived within our approach. We think that the corresponding studies, in principle, can be carried out analytically in the “deformed QCD” model where one can study the correlation length in the “plasma phase” side being in weakly coupled regime [23, 24]. The same hard questions can be also studied using some numerical approaches as advocated recently in ref. [58]. The crucial point is that the partition function in this “plasma phase” (the confinement phase in conventional terminology) should be formulated from the very beginning in terms of the dominant pseudo-particles, the instanton’s constituents. The fact that the the instanton’s constituents rather than the instantons themselves are the dominant pseudo-particles in the confinement phase has been advocating by many authors from different perspective, see footnote 2 with more comments and references. We leave this subject for future studies.

The key ingredient of our approach is the analysis of the instanton induced potential studied in the “molecular side” of the phase diagram where the non-perturbative computations are under theoretical control. In pure gauge theory the corresponding induced potential is linked to the $\theta$ dependence (1). Therefore, a study of the $\theta$ dependence when the phase transition line is crossed is a key element in understanding of the strongly coupled QCD. If a single massless quark is present in the system, the $\theta$ parameter can be rotated away by redefining the quark fields. Such a change of variables does not modify the physics, of course, as the instanton induced potential remains essentially the same in large $N$ limit. However, a non-vanishing quark mass would produce a drastically different $\theta$ behaviour on two sides of the phase diagram border. Such drastic changes in $\theta$ behaviour represents not a pure academical problem. It might also lead to some observational consequences as the $\theta$ parameter is thought to be nonzero during the QCD phase transition in early universe. This unique source of the strong CP violation might be a missing ingredient to understand the dark matter as well as the baryon-antibaryon asymmetry existing in nature, see footnote 1 and ref.[59] with some references and short overview on possible cosmological consequences related to a nonvanishing $\theta$ during the QCD phase transition in early universe.

As we mentioned in the text, one should not confuse the conventional thermodynamical terms such as “phase”, “density” or “entropy” applied to the 4d Euclidean objects with the corresponding entities which are normally applied to a conventional system of static objects in 3+1 Minkowski space. However, from the 5 dimensional (holographic) viewpoint these 4d Euclidean objects can be considered as some static particles. Such a view on the instantons (instanton quarks) as the static objects from 5d perspective might be worthwhile for further investigation.

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7 We would like to make a comment on the terminology which was used in ref. [58], where the corresponding instanton’s constituents were called the “dyons”. This term could be confusing and misleading as the term “dyon” is normally attributed to a static particle in (3+1) dimensional Minkowski space-time, which simultaneously carries the magnetic and electric charges. The instanton constituents have fundamentally different nature: they are the objects defined in 4d Euclidean space, which describe the tunnelling events, see footnote 2 with more comments and references on the subject.
