Quiver Gauge Theory of Nonabelian Vortices and Noncommutative Instantons in Higher Dimensions

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Abstract

We construct explicit BPS and non-BPS solutions of the Yang-Mills equations on the noncommutative space $\mathbb{R}_\theta^{2n} \times S^2$ which have manifest spherical symmetry. Using $SU(2)$-equivariant dimensional reduction techniques, we show that the solutions imply an equivalence between instantons on $\mathbb{R}_\theta^{2n} \times S^2$ and nonabelian vortices on $\mathbb{R}_\theta^{2n}$, which can be interpreted as a blowing-up of a chain of D0-branes on $\mathbb{R}_\theta^{2n}$ into a chain of spherical D2-branes on $\mathbb{R}_\theta^{2n} \times S^2$. The low-energy dynamics of these configurations is described by a quiver gauge theory which can be formulated in terms of new geometrical objects generalizing superconnections. This formalism enables the explicit assignment of D0-brane charges in equivariant K-theory to the instanton solutions.
1 Introduction and summary

One of the most basic questions that arises in trying to understand the nonperturbative structure of string theory concerns the classification of vector bundles over real and complex manifolds. In the presence of D-branes one encounters gauge theories in spacetime dimensionalities up to ten. Already more than 20 years ago, BPS-type equations in higher dimensions were proposed [1, 2] as a generalization of the self-duality equations in four dimensions. For nonabelian gauge theory on a Kähler manifold the most natural BPS condition lies in the Donaldson-Uhlenbeck-Yau equations [3], which arise, for instance, in compactifications down to four-dimensional Minkowski spacetime as the condition for at least one unbroken supersymmetry.

While the criteria for solvability of these BPS equations are by now very well understood, in practice it is usually quite difficult to write down explicit solutions of them. One recent line of attack has been to consider noncommutative deformations of these field theories [4]–[6]. In certain instances, D-branes can be realized as noncommutative solitons [7], which is a consequence [8, 9] of the relationship between D-branes and K-theory [10]–[13]. All celebrated BPS configurations in field theories, such as instantons [14], monopoles [15] and vortices [16], have been generalized to the noncommutative case, originally in [17], in [18] and in [19], respectively (see [20] for reviews and further references). Solution generating techniques such as the ADHM construction [21], splitting [22] and dressing [23] methods have also been generalized to the noncommutative setting in [17, 24] and in [25]. Solutions of the generalized self-duality equations [1, 2] were investigated in [2, 26], for example. Noncommutative instantons in higher dimensions and their interpretations as D-branes in string theory have been considered in [27]–[30]. In all of these constructions the usual worldvolume description of D-branes emerges from the equivalence between analytic and topological formulations of K-homology.

In this paper we will complete the construction initiated in [29, 30] of multi-instanton solutions of the Yang-Mills equations on the manifold which is the product of noncommutative euclidean space \( \mathbb{R}^{2n} \) with an ordinary two-sphere \( S^2 \). We consider both BPS and non-BPS solutions, and extend previous solutions to those which are explicitly SU(2)-equivariant for any value of the Dirac monopole charge characterizing the gauge field components along the \( S^2 \) directions. Dimensional reduction techniques are used to establish an equivalence between multi-instantons on \( \mathbb{R}^{2n} \times S^2 \) and nonabelian vortices on \( \mathbb{R}^{2n} \). The configurations can be interpreted in Type IIA superstring theory as chains of branes and antibranes with Higgs-like open string excitations between neighbouring sets of D-branes. The equivalence between instantons and vortices may then be attributed to the decay of an unstable configuration of D(2n)-branes into a state of D0-branes (There are no higher brane charges induced because \( \mathbb{R}^{2n} \) is equivariantly contractible). The D0-brane charges are classified by SU(2)-equivariant K-theory and the low-energy dynamics may be succinctly encoded into a simple quiver gauge theory. Unlike the standard brane-antibrane systems, the effective action cannot be recast using the formalism of superconnections [31] but requires a more general formulation in terms of new geometrical entities that we call “graded connections”. This formalism makes manifest the interplay between the assignment of K-theory classes to the explicit instanton solutions and their realization in terms of a quiver gauge theory.

The organisation of this paper is as follows. The material is naturally divided into two parts. Sections 2–5 deal with ordinary gauge theory on a generic Kähler manifold of the form \( M_{2n} \times \mathbb{C}P^1 \) in order to highlight the geometric structures that arise due to dimensional reduction and which play a prominent role throughout the paper. Sections 6–10 are then concerned with the noncommutative deformation \( \mathbb{R}^{2n} \times \mathbb{C}P^1 \to \mathbb{R}^{2n}_\theta \times \mathbb{C}P^1 \) and they construct explicit solutions of the dimensionally reduced Yang-Mills equations, emphasizing their interpretations in the context of equivariant K-theory, quiver gauge theory, and ultimately as states of D-branes. In Section 2 we introduce basic definitions and set some of our notation, and present the field equations that are to be solved. In
Section 3 we write down an explicit ansatz for the gauge field which is used in the SU(2)-equivariant dimensional reduction. In Section 4 we describe three different interpretations of the ansatz as configurations of D-branes, as charges in equivariant K-theory, and as field configurations in a quiver gauge theory (Later on these three descriptions are shown to be equivalent). In Section 5 the dimensional reduction mechanism is explained in detail in the new language of graded connections and the resulting nonabelian vortex equations, arising from reduction of the Donaldson-Uhlenbeck-Yau equations, are written down. In Section 6 we introduce the noncommutative deformations of all these structures. In Section 7 we find explicit BPS and non-BPS solutions of the noncommutative Yang-Mills equations and show how they naturally realize representations of the pertinent quiver. In Section 8 we develop an SU(2)-equivariant generalization of the (noncommutative) Atiyah-Bott-Shapiro construction, which provides an explicit and convenient representation of our solution in terms of K-homology classes. In Section 9 we compute the topological charge of our instanton solutions directly in the noncommutative gauge theory, and show that it coincides with the corresponding K-theory charge, which then allows us to assign D0-brane charges to the solutions from a worldvolume perspective. Finally, in Section 10 we construct some novel BPS solutions in the vacuum sectors of the noncommutative field theory and describe their relation to stable states of brane-antibranes.

2 Yang-Mills equations

In this section we will introduce the basic definitions and notation that will be used throughout this paper, as well as the pertinent field equations that we will solve.

The manifold \( \mathcal{M}_q \times S^2 \). Let \( \mathcal{M}_q \) be a real \( q \)-dimensional lorentzian manifold with nondegenerate metric of signature \((- + \cdots +)\), and \( S^2 \cong \mathbb{CP}^1 \) the standard two-sphere of constant radius \( R \). We shall consider the manifold \( \mathcal{M}_q \times S^2 \) with local real coordinates \( x' = (x'^\mu) \in \mathbb{R}^q \) on \( \mathcal{M}_q \) and coordinates \( \vartheta \in [0, \pi], \varphi \in [0, 2\pi] \) on \( S^2 \). In these coordinates the metric on \( \mathcal{M}_q \times S^2 \) reads

\[
\mathrm{d}s^2 = g_{\hat{\mu}\hat{\nu}} \, \mathrm{d}x^{\hat{\mu}} \, \mathrm{d}x^{\hat{\nu}} + g_{\mu'\nu'} \, \mathrm{d}x^{\mu'} \, \mathrm{d}x^{\nu'} + R^2 \left( \mathrm{d}\vartheta^2 + \sin^2 \vartheta \, \mathrm{d}\varphi^2 \right),
\]

where hatted indices \( \hat{\mu}, \hat{\nu}, \ldots \) run over \( 0, 1, \ldots, q + 1 \) while primed indices \( \mu', \nu', \ldots \) run through \( 0, 1, \ldots, q - 1 \). We use the Einstein summation convention for repeated spacetime indices.

The Kähler manifold \( M_{2n} \times \mathbb{CP}^1 \). As a special instance of the manifold \( \mathcal{M}_q \) we shall consider the product \( \mathcal{M}_q = \mathbb{R}^1 \times M_{2n} \) of dimension \( q = 2n + 1 \) with metric

\[
g_{\mu'\nu'} \, \mathrm{d}x^{\mu'} \, \mathrm{d}x^{\nu'} = - (\mathrm{d}x^0)^2 + g_{\mu\nu} \, \mathrm{d}x^\mu \, \mathrm{d}x^\nu.
\]

Here \( M_{2n} \) is a Kähler manifold of real dimension \( 2n \) with local real coordinates \( x = (x^\mu) \in \mathbb{R}^{2n} \), where the indices \( \mu, \nu, \ldots \) run through \( 1, \ldots, 2n \). The cartesian product \( M_{2n} \times \mathbb{CP}^1 \) is also a Kähler manifold with local complex coordinates \( (z^1, \ldots, z^n, y) \in \mathbb{C}^{n+1} \) and their complex conjugates, where

\[
z^a = x^{2a-1} - i \, x^{2a} \quad \text{and} \quad z^{\bar{a}} = x^{2a-1} + i \, x^{2a} \quad \text{with} \quad a = 1, \ldots, n
\]

while

\[
y = \frac{R \sin \vartheta}{1 + \cos \vartheta} \exp(-i\varphi) \quad \text{and} \quad \bar{y} = \frac{R \sin \vartheta}{1 + \cos \vartheta} \exp(i\varphi)
\]

are stereographic coordinates on the northern hemisphere of \( S^2 \). In these coordinates the metric on \( M_{2n} \times \mathbb{CP}^1 \) takes the form

\[
\mathrm{d}s^2 = g_{ab} \, \mathrm{d}z^a \, \mathrm{d}\bar{z}^b + \frac{4R^4}{(R^2 + y\bar{y})^2} \, \mathrm{d}y \, \mathrm{d}\bar{y},
\]
while the Kähler two-form \( \omega \) is given by
\[
\omega = \frac{1}{2} \omega_{\mu\nu} \, dx^\mu \wedge dx^\nu + R^2 \sin \theta \, d\theta \wedge d\varphi = -2i g_{ab} \, dz^a \wedge d\bar{z}^b - \frac{4i R^4}{(R^2 + y\bar{y})^2} \, dy \wedge d\bar{y} .
\] (2.6)

**Yang-Mills equations.** Consider a rank \( k \) hermitean vector bundle \( \mathcal{E} \to M_q \times S^2 \) with gauge connection \( A \) of curvature \( F = dA + A \wedge A \). In local coordinates, wherein \( A = A_\mu \, dx^\mu \), the two-form \( F \) has components \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \), where \( \partial_\mu := \partial / \partial x^\mu \). Both \( A_\mu \) and \( F_{\rho\sigma} \) take values in the Lie algebra \( u(k) \). For the usual Yang-Mills lagrangian

\[
L_{YM} = -\frac{1}{4} \sqrt{g} \, \text{tr}_{k \times k} F_{\rho\sigma} F^{\rho\sigma}
\] (2.7)

the equations of motion are

\[
\frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} \, F^{\mu\rho}) + [A_\mu, F^{\rho\sigma}] = 0 ,
\] (2.8)

where \( g = |\det(g_{\mu\nu})| \). The curvature two-form can be written in local coordinates on \( M_q \times \mathbb{C}P^1 \) as

\[
F = \frac{1}{2} F_{\mu'\nu'} \, dx^{\mu'} \wedge dx^{\nu'} + F_{\mu'y} \, dx^{\mu'} \wedge dy + F_{\mu'\bar{y}} \, dx^{\mu'} \wedge d\bar{y} + F_{y\bar{y}} \, dy \wedge d\bar{y}
\] (2.9)

and the Yang-Mills lagrangian becomes

\[
L_{YM} = -\frac{1}{4} \sqrt{g} \, \text{tr}_{k \times k} \left[ F_{\mu'\nu'} F^{\mu'\nu'} + \left( \frac{R^2 + \bar{y}y}{R^4} \right)^2 \right] \text{tr}_{k \times k} F_{\rho\sigma} F^{\rho\sigma} - \frac{1}{2} \left( \frac{(R^2 + \bar{y}y)^2}{R^4} \right) \text{tr}_{k \times k} F_{\rho\sigma} F^{\rho\sigma}
\] (2.10)

**Donaldson-Uhlenbeck-Yau equations.** For static field configurations in the temporal gauge \( A_0 = 0 \), the Yang-Mills equations (2.8) on \( \mathbb{R}^1 \times M_{2n} \times \mathbb{C}P^1 \) reduce to equations on \( M_{2n} \times \mathbb{C}P^1 \). Their stable solutions are provided by solutions of the Donaldson-Uhlenbeck-Yau (DUY) equations which can be formulated on any Kähler manifold [3]. The importance of these equations derives from the fact that they yield the BPS solutions of the full Yang-Mills equations.

The DUY equations on \( M_{2n} \times \mathbb{C}P^1 \) are

\[
* \omega \wedge F = 0 \quad \text{and} \quad F^{0,2} = 0 ,
\] (2.11)

where * is the Hodge duality operator and \( F = F^{2,0} + F^{1,1} + F^{0,2} \) is the Kähler decomposition of the gauge field strength. In the local complex coordinates \((z^a, y)\) these equations take the form

\[
g^{ab} F_{z^a \bar{z}^b} + \frac{(R^2 + \bar{y}y)^2}{2 R^4} F_{y\bar{y}} = 0 ,
\] (2.12)

\[
F_{z^a \bar{z}^b} = 0 = F_{z^a \bar{b}} ,
\] (2.13)

\[
F_{z^a \bar{y}} = 0 = F_{z^a y} ,
\] (2.14)

where the indices \( a, b, \ldots \) run through \( 1, \ldots, n \). Eq. (2.12) is a hermitean condition on the gauge field strength tensor, while eqs. (2.13) and (2.14) are integrability conditions implying that the bundle \( \mathcal{E} \) endowed with a connection \( A \) is holomorphic. It is easy to show that any solution of these \( n(n+1)+1 \) equations also satisfies the full Yang-Mills equations.

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1The Yang-Mills coupling constant \( g_{YM} \) can be introduced via the redefinition \( A \to g_{YM} A \).
3 Invariant gauge fields

In this section we shall write down the fundamental form of the gauge potential $A$ on $M_q \times \mathbb{C}P^1$ that will be used later on to dimensionally reduce the Yang-Mills equations for $A$ to equations on $M_q$. This will be achieved by prescribing a specific $\mathbb{C}P^1$ dependence for $A$, which we proceed to describe first.

**Monopole bundles.** Consider the hermitean line bundle $L^m \rightarrow \mathbb{C}P^1$ over the sphere with $L^m := (L)^{\otimes m}$ and unique SU(2)-invariant unitary connection $a_m$ having, in the local complex coordinate $y$ on $\mathbb{C}P^1$, the form

$$a_m = \frac{m}{2(R^2 + y\bar{y})} (\bar{y} \, dy - y \, d\bar{y}) \, ,$$

(3.1)

where $m$ is an integer. The curvature of this connection is

$$f_m = da_m = -\frac{mR^2}{(R^2 + y\bar{y})^2} \, dy \wedge d\bar{y} \, .$$

(3.2)

The topological charge of this gauge field configuration is given by the first Chern number (equivalently the degree) of the associated complex line bundle as

$$\deg L^m = \frac{i}{2\pi} \int_{\mathbb{C}P^1} f_m = m \, .$$

(3.3)

In terms of the spherical coordinates $(\vartheta, \varphi)$ the configuration (3.1,3.2) has the form

$$a_m = -\frac{i}{2} m(1 - \cos \vartheta) \, d\varphi \quad \text{and} \quad f_m = da_m = -\frac{i}{2} m \sin \vartheta \, d\vartheta \wedge d\varphi \, .$$

(3.4)

It describes $|m|$ Dirac monopoles or antimonopoles sitting on top of each other.

The $m$-monopole bundle is classified by the Hopf fibration $S^1 \hookrightarrow S^3 \rightarrow S^2$. For each $m \in \mathbb{Z}$ there is a one-dimensional representation $\nu_m = (\nu_1)^{\otimes m}$ of the circle group $U(1) \cong S^1$ defined by

$$\nu_m : v \mapsto \zeta \cdot v = \zeta^m v \quad \text{with} \quad \zeta \in S^1 \quad \text{and} \quad v \in \mathbb{C} \, .$$

(3.5)

We denote this irreducible U(1)-module by $S_m \cong \mathbb{C}$. Regarding the sphere as the homogeneous space $\mathbb{C}P^1 \cong SU(2)/U(1)$, the SU(2)-equivariant line bundle $L^m \rightarrow \mathbb{C}P^1$ corresponds to the representation $\nu_m$ in the sense that it can be expressed as

$$L^m = SU(2) \times_U(1) S_m \, .$$

(3.6)

where the quotient on $SU(2) \times S_m$ is by the U(1) action $\zeta \cdot (g, v) = (g \zeta^{-1}, \zeta^m v)$ for $g \in SU(2)$, $v \in S_m$ and $\zeta \in U(1)$. The action of SU(2) on $SU(2) \times S_m$ given by $g' \cdot (g, v) = (g'g, v)$ descends to an action on (3.6). Any SU(2)-equivariant hermitean vector bundle over the sphere is a Whitney sum of bundles (3.6).

There is an alternative description in terms of the holomorphic line bundle $O(m) \rightarrow \mathbb{C}P^1$ defined as the $m$-th power of the tautological bundle over the complex projective line. The universal complexification of the Lie group SU(2) is SL(2,$\mathbb{C}$), and we may regard the sphere as a projective variety through the natural diffeomorphism $\mathbb{C}P^1 \cong SU(2)/U(1) \cong SL(2,\mathbb{C})/P$, where P is the parabolic subgroup of lower triangular matrices in SL(2,$\mathbb{C}$). The SU(2) action on (3.6) lifts to a smooth SL(2,$\mathbb{C}$) action, and the complexification of (3.6) is realized as the SL(2,$\mathbb{C}$)-equivariant line bundle

$$O(m) = SL(2,\mathbb{C}) \times_P S_m$$

(3.7)
over $\mathbb{C}P^1$. Only the Cartan subgroup $\mathbb{C}^\times \subset P$ of non-zero complex numbers acts non-trivially on the modules $\mathbb{C}_m$, with the $\mathbb{C}^\times$ action defined analogously to (3.5). The two descriptions are equivalent after the introduction of a hermitean metric on the fibres of $\mathcal{O}(m)$. This holomorphic line bundle has transition function $y^m$ transforming sections from the northern hemisphere to the southern hemisphere of $S^2$. However, the monopole connection (3.1) is transformed on the intersection of the two patches covering $\mathbb{C}P^1$ via the transition function $(y/\bar{y})^{m/2}$, which is the unitary reduction of the holomorphic transition function $y^m$. Thus the bundle $\mathcal{O}(m)$ regarded as a hermitean line bundle has transition function $(y/\bar{y})^{m/2}$ and can be substituted for the monopole bundle $\mathcal{L}^m$.

**SU(2)-invariant gauge potential.** The form of our ansatz for the gauge connection on $\mathcal{M}_q \times \mathbb{C}P^1$ is fixed by imposing invariance under the SU(2) isometry group of $\mathbb{C}P^1$ acting through rigid rotations of the sphere. Let $\mathcal{E} \to \mathcal{M}_q \times \mathbb{C}P^1$ be an SU(2)-equivariant U($k$)-bundle, with the group SU(2) acting trivially on $\mathcal{M}_q$ and in the standard way on $\mathbb{C}P^1 = SU(2)/U(1)$. Let $\mathcal{A}$ be a connection on $\mathcal{E}$. Imposing the condition of SU(2)-equivariance means that we should look for representations of the group SU(2) inside the U($k$) structure group, i.e. for homomorphisms $\rho : SU(2) \to U(k)$. The ansatz for $\mathcal{A}$ is thus given by $k$-dimensional representations of SU(2). Up to isomorphism, for each positive integer $d$ there is a unique irreducible SU(2)-module $\mathcal{V}_d \cong \mathbb{C}^d$ of dimension $d$. Therefore, for each positive integer $m$, the module

$$
\mathcal{V} = \bigoplus_{i=0}^{m} \mathcal{V}_{k_i} \quad \text{with} \quad \sum_{i=0}^{m} k_i = k
$$

(3.8)

gives a representation $\rho$ of SU(2) inside U($k$). The total number of such homomorphisms is the number of partitions of the positive integer $\text{rank}(\mathcal{E}) = k$ into $\leq (m + 1)$ components. The original U($k$) gauge symmetry is then broken down to the centralizer subgroup of $\rho(SU(2))$ in U($k$) as

$$
U(k) \longrightarrow \prod_{i=0}^{m} U(k_i) .
$$

(3.9)

It is natural to allow for gauge transformations to accompany the SU(2) action [32], and so some “twisting” can occur in the reduction of the connection $\mathcal{A}$ on $\mathcal{M}_q \times \mathbb{C}P^1$. The $\mathbb{C}P^1$ dependence in this case is uniquely determined by the above SU(2)-invariant Dirac monopole configurations [33, 34]. The $u(k)$-valued gauge potential $\mathcal{A}$ thus splits into $k_i \times k_j$ blocks $\mathcal{A}^{ij}$,

$$
\mathcal{A} = (\mathcal{A}^{ij}) \quad \text{with} \quad \mathcal{A}^{ij} \in \text{Hom}(\mathcal{V}_{k_j}, \mathcal{V}_{k_i}) ,
$$

(3.10)

where the indices $i, j, \ldots$ run over $0, 1, \ldots, m$, $k_0 + k_1 + \ldots + k_m = k$ and

$$
\mathcal{A}^{i\bar{t}} = \mathcal{A}^i(x') \otimes 1 + 1_{k_i} \otimes a_{m-2i}(y) ,
$$

(3.11)

$$
\mathcal{A}^{i+j} = - (\mathcal{A}^{i+1})^\dagger = - (\Phi_{i+1})^\dagger = -\phi_{i+1}^\dagger (x') \otimes \beta(y) ,
$$

(3.12)

$$
\mathcal{A}^{i+l} = 0 = \mathcal{A}^{i+1} \quad \text{for} \quad l \geq 2 .
$$

(3.13)

Here

$$
\beta = \frac{R}{R^2 + y\bar{y}} \quad \text{and} \quad \bar{\beta} = \frac{R}{R^2 + y\bar{y}}
$$

(3.15)

are the unique covariantly constant, SU(2)-invariant forms of type $(1,0)$ and $(0,1)$ such that the Kähler $(1,1)$-form on $\mathbb{C}P^1$ is $4R^2 \beta \wedge \bar{\beta}$. They respectively take values in the bundles $\mathcal{L}^2$ and $\mathcal{L}^{-2}$. 

5
Here we have defined the gauge potential \( A \) and the field strength tensor. Covariant derivatives that the complexified cotangent bundle of \( C \) depends only on the coordinates \( x' \) on \( M_q \). Every SU(2)-invariant unitary connection \( A \) on \( M_q \times CP^1 \) is of the form given in (3.10)–(3.14) [34], which follow from the fact that the complexified cotangent bundle of \( CP^1 \) is \( \mathcal{L}^2 \oplus \mathcal{L}^{-2} \). This ansatz amounts to an equivariant decomposition of the original rank \( k \) SU(2)-equivariant bundle \( E \rightarrow M_q \times CP^1 \) in the form

\[
E = \bigoplus_{i=0}^{m} E_i \quad \text{with} \quad E_i = E_{k_i} \otimes \mathcal{L}^{m-2i}, \quad (3.16)
\]

where \( E_{k_i} \rightarrow M_q \) is a hermitean vector bundle of rank \( k_i \) with typical fibre the module \( V_{k_i} \), and \( E_i \rightarrow M_q \times CP^1 \) is the bundle with fibres \( (E_i)_{(x',y)} = (E_{k_i})_{x'} \otimes (\mathcal{L}^{m-2i})_{(y,y)} \). By regarding \( \Phi_i \in \text{Hom}(E_i,E_{i-1}) \cong H^0(M_q \times CP^1; E_{i-1} \otimes E_i) \) for \( i = 1, \ldots, m \) and defining \( \Phi_0 := 0 =: \Phi_{m+1} \), we can summarize our ansatz through the following chain of bundles:

\[
\begin{array}{cccccccc}
\bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet & \rightarrow & \bullet \\
E_m & \Phi_m & \mid & \Phi_{m-1} & \mid & \Phi_{m-2} & \mid & \Phi_1 & \rightarrow 0 \\
\end{array}
\quad (3.17)
\]

**Field strength tensor.** The calculation of the curvature (2.9) for \( A \) of the form (3.10)–(3.14) yields

\[
\mathcal{F} = (\mathcal{F}^{ij}) \quad \text{with} \quad \mathcal{F}^{ij} = dA^{ij} + \sum_{l=0}^{m} A^{il} \wedge A^{lj}, \quad (3.18)
\]

where

\[
\begin{align*}
\mathcal{F}^{ii} &= F^{i} + f_{m-2i} + (\phi_{i+1} \phi_i^\dagger - \phi_i^\dagger \phi_{i+1}) \beta \wedge \bar{\beta}, \quad (3.19) \\
\mathcal{F}^{i+1i} &= D\phi_{i+1} \wedge \bar{\beta}, \quad (3.20) \\
\mathcal{F}^{i+1i} &= - (\mathcal{F}^{i+1i})^\dagger = - (D\phi_{i+1})^\dagger \wedge \beta, \quad (3.21) \\
\mathcal{F}^{i+i+l} &= 0 = \mathcal{F}^{i+i+l} \quad \text{for} \quad l \geq 2. \quad (3.22)
\end{align*}
\]

Here we have defined \( F^i := dA^i + A^i \wedge A^i = \frac{1}{2} F_{\mu\nu}(x') \, dx^\mu \wedge dx^\nu \) and introduced the bi-fundamental covariant derivatives \( D\phi_{i+1} := d\phi_{i+1} + A^i \phi_{i+1} - \phi_{i+1} A^{i+1} \).

From (3.19)–(3.22) we find the non-vanishing field strength components

\[
\begin{align*}
\mathcal{F}_{\mu'\nu'}^{ii} &= F_{\mu'\nu'}^{i}, \quad (3.24) \\
\mathcal{F}_{\mu'\bar{y}}^{i+1i} &= \frac{R}{R^2 + y\bar{y}} D_{\mu'} \phi_{i+1} = - (\mathcal{F}_{\mu'\bar{y}}^{i+1i})^\dagger, \quad (3.25) \\
\mathcal{F}_{\bar{y}\bar{y}}^{ii} &= - \frac{R^2}{(R^2 + y\bar{y})^2} \left( m - 2i + \phi_i^\dagger \phi_i - \phi_{i+1}^\dagger \phi_{i+1} \right). \quad (3.26)
\end{align*}
\]
4 Description of the ansatz

In this section we shall clarify some features of the ansatz constructed in the previous section from three different points of view. To set the stage for the string theory interpretations of the solutions that we will construct later on, we begin by indicating how the ansatz can be interpreted in terms of configurations of D-branes in Type II superstring theory. This leads into a discussion of how the ansatz is realized in topological K-theory, which classifies the Ramond-Ramond charges of these brane systems, and we will derive the decomposition (3.16) directly within the framework of SU(2)-equivariant K-theory. We will then explain how seeking explicit realizations of our ansatz is equivalent to finding representations of the $A_{m+1}$ quiver. One of the goals of the subsequent sections will be to establish the precise link between these three descriptions, showing that they are all equivalent.

Physical interpretation. Before entering into the formal mathematical characterizations of the ansatz of the previous section, let us first explain the physical situation which they will describe. Our ansatz implies an equivalence between brane-antibrane systems on $\mathcal{M}_q \times \mathbb{C}P^1$. In the standard D-brane interpretation, our initial rank $k$ hermitean vector bundle $\mathcal{E} \to \mathcal{M}_q \times \mathbb{C}P^1$ corresponds to $k$ coincident D($q+1$)-branes wrapping the worldvolume manifold $\mathcal{M}_q \times \mathbb{C}P^1$. The condition of SU(2)-equivariance imposed on this bundle fixes the dependence on the coordinates of $\mathbb{C}P^1$ and breaks the gauge group $U(k)$ as in (3.9). The rank $k_1$ sub-bundle $E_{k_1} \to \mathcal{M}_q$ of this bundle is twisted by the Dirac multi-monopole bundle $\mathcal{L}^{m-2i} \to \mathbb{C}P^1$. The system of $k$ coincident D($q+1$)-branes thereby splits into blocks of $k_0+k_1+\ldots+k_m = k$ coincident D($q+1$)-branes, associated to irreducible representations of SU(2) and wrapping a common sphere $\mathbb{C}P^1$ with the monopole fields. This system is equivalent to a system of $k_0+k_1+\ldots+k_m = k$ D($q-1$)-branes carrying different magnetic fluxes on their common worldvolume $\mathcal{M}_q$. The D($q-1$)-branes which carry negative magnetic flux have opposite orientation with respect to the D($q-1$)-branes with positive magnetic flux, i.e. they are antibranes. This will become evident from the K-theory formalism, which will eventually lead to an explicit worldvolume construction, and also from the explicit calculation of the topological charges of the instanton solutions. In addition to the usual Chan-Paton gauge field degrees of freedom $A^i \in \text{End}(E_{k_i})$ living on each block of branes, the field content on the brane configuration contains bi-fundamental scalar fields $\phi_{i+1} \in \text{Hom}(E_{k_{i+1}}, E_{k_i})$ corresponding to massless open string excitations between neighbouring blocks of $k_i$ and $k_{i+1}$ D($q-1$)-branes. Other excitations are suppressed by the condition of SU(2)-equivariance.

However, as we shall see explicitly in the following, the fields $\phi_{i+1}$ should not be regarded as tachyon fields, but rather only as (holomorphic) Higgs fields responsible for the symmetry breaking (3.9). Only the brane-antibrane pairs whose constituents carry equal and opposite monopole charges are neutral and can thus annihilate to the vacuum, which carries no monopole charge (although it can carry a K-theory charge from the virtual Chan-Paton bundles over $\mathcal{M}_q$). Other brane pairs are stable because their overall non-vanishing Chern number over $\mathbb{C}P^1$ is an obstruction to decay, and the monopole bundles thereby act as a source of flux stabilization for such brane pairs by giving them a conserved topological charge. In particular, neighbouring blocks of D($q-1$)-branes are marginally bound by the massless open strings stretching between them. In this sense, the SU(2)-invariant reduction of D-branes on $\mathcal{M}_q \times \mathbb{C}P^1$ induces brane-antibrane systems on $\mathcal{M}_q$. Note that while the system on $\mathcal{M}_q$ is generically unstable, the original brane configuration on $\mathcal{M}_q \times \mathbb{C}P^1$ can be nonetheless stable.

K-theory charges. Given that the charges of configurations of D-branes in string theory are classified topologically by K-theory [10, 11, 13], let us now seek the K-theory representation of the above physical situation. The one-monopole bundle $\mathcal{L}$ is a crucial object in establishing the Bott
periodicity isomorphism

\[ K(\mathcal{M}_q \times \mathbb{C}P^1) = K(\mathcal{M}_q) \]  

(4.1)
in topological K-theory. The isomorphism is generated by taking the K-theory product of the tachyon field \( \phi_1 : E_{k_1} \to E_{k_0} \) of a virtual bundle \( [E_{k_0}, E_{k_1}; \phi_1] \in K(\mathcal{M}_q) \) with that of the class of the line bundle \( L \) which represents the Bott generator of \( K(\mathbb{C}P^1) = \mathbb{Z} \) [11]. The topological equivalence (4.1) then implies the equivalence of brane-antibrane systems on \( \mathcal{M}_q \times \mathbb{C}P^1 \) and \( \mathcal{M}_q \), with the brane and antibrane systems each carrying a single unit of monopole charge. When they carry \( m > 1 \) units of charge, the isomorphism breaks down, and it is necessary to introduce the notion of “D-operations” to establish the relationship [30]. While these operations are natural, they are not isomorphisms and they reflect the fact that the explicit solutions in this setting are not SU(2)-invariant, so that the equivalence breaks down due to spurious moduli dependences of the system of branes on the \( \mathbb{C}P^1 \) factor. In what follows we will derive a modification of the relation (4.1) in equivariant K-theory which will naturally give the desired isomorphism, reflecting the equivalence of the brane-antibrane systems for arbitrary monopole charge, and bypass the need for introducing D-operations. This is only possible by augmenting the basic brane-antibrane system to a chain of \((m + 1)\) branes and antibranes with varying units of monopole charge as described above, and we will thereby arrive at an independent purely K-theoretic derivation of our ansatz.

The representation ring \( R_G \) of a group \( G \) [35] is the Grothendieck ring of the category of finite dimensional representations of \( G \), with addition induced by direct sum of vector spaces, \([\mathcal{V}'] + [\mathcal{V}''] := [\mathcal{V}' \oplus \mathcal{V}''],\) and multiplication induced by tensor product of modules, \([\mathcal{V}] \cdot [\mathcal{V}'] := [\mathcal{V} \otimes \mathcal{V}']\). As an abelian group it is generated by the irreducible representations of \( G \). Alternatively, since the isomorphism class of a \( G \)-module \( \mathcal{V} \) is completely determined by its character \( \chi_\mathcal{V} : G \to \mathbb{C} \), the map \( \mathcal{V} \mapsto \chi_\mathcal{V} \) identifies \( R_G \) as a subring of the ring of \( G \)-invariant functions on \( G \). If \( \mathcal{M}_q \) is a \( G \)-space, then the Grothendieck group of \( G \)-equivariant bundles over \( \mathcal{M}_q \) is called the \( G \)-equivariant K-theory group \( K_G(\mathcal{M}_q) \). This group unifies ordinary K-theory with group representation theory, in the sense that for the trivial space \( K_G(\text{pt}) = R_G \) is the representation ring of \( G \), while for the trivial group \( K_{\text{id}}(\mathcal{M}_q) = K(\mathcal{M}_q) \) is the ordinary K-theory of \( \mathcal{M}_q \). The former property implies that \( K_G(\mathcal{M}_q) \) is an \( R_G \)-module and the coefficient ring in equivariant K-theory is \( R_G \), rather than just \( \mathbb{Z} \) as in the ordinary case. If the \( G \)-action on \( \mathcal{M}_q \) is trivial, then any \( G \)-equivariant bundle \( E \to \mathcal{M}_q \) may be decomposed as a finite Whitney sum

\[ E = \bigoplus_{\mathcal{V} \in \text{Rep}(G)} \text{Hom}_G(\mathbb{I}_\mathcal{V}^G, E) \otimes \mathbb{I}_\mathcal{V}^G \]  

(4.2)
where \( \mathbb{I}_\mathcal{V} = \mathcal{M}_q \times \mathcal{V} \) is the trivial bundle over \( \mathcal{M}_q \) with fibre the irreducible \( G \)-module \( \mathcal{V} \). It follows that for trivial \( G \)-actions the equivariant K-theory takes the simple form

\[ K_G(\mathcal{M}_q) = K(\mathcal{M}_q) \otimes R_G \ . \]  

(4.3)
The \( K_G \)-functor behaves analogously to the ordinary K-functor, and in addition \( K_G \) is functorial with respect to group homomorphisms. A useful computational tool is the equivariant excision theorem. If \( F \) is a closed subgroup of \( G \) and \( \mathcal{M}_q \) is an \( F \)-space, then the inclusion \( i : F \hookrightarrow G \) induces an isomorphism [35]

\[ i^* : K_G(G \times F \mathcal{M}_q) \cong K_F(\mathcal{M}_q) \ , \]  

(4.4)
where the quotient on \( G \times \mathcal{M}_q \) is by the \( F \)-action \( f \cdot (g, x') = (gf^{-1}, f \cdot x') \) for \( g \in G, x' \in \mathcal{M}_q \) and \( f \in F \). The \( G \)-action on \( G \times F \mathcal{M}_q \) descends from that on \( G \times \mathcal{M}_q \) given by \( g' \cdot (g, x') = (g'g, x') \).

Let us specialize to our case of interest by taking \( G = \text{SU}(2), F = \text{U}(1) \) and the trivial action of \( \text{SU}(2) \) on the space \( \mathcal{M}_q \). Using (4.3) and (4.4) we may then compute

\[ K_{\text{SU}(2)}(\mathcal{M}_q \times \mathbb{C}P^1) = K_{\text{SU}(2)}(\text{SU}(2) \times_\text{U}(1) \mathcal{M}_q) \]
\[ = K_{\text{U}(1)}(\mathcal{M}_q) = K(\mathcal{M}_q) \otimes R_{\text{U}(1)} \ . \]  

(4.5)
This K-theoretic equality asserts a one-to-one correspondence between classes of SU(2)-equivariant bundles over $\mathcal{M}_q \times \mathbb{C}P^1$ and classes of U(1)-equivariant bundles over $\mathcal{M}_q$ with U(1) acting trivially on $\mathcal{M}_q$. The isomorphism (4.5) of equivariant K-theory groups is constructed explicitly as follows [35]. Given an SU(2)-equivariant bundle $E \to M$ of a SU(2)-equivariant bundle, where the quotient on SU(2) bundles over $M$, this K-theoretic equality asserts a one-to-one correspondence between classes of SU(2)-equivariant bundles and classes of U(1)-equivariant bundles. The crucial difference now is that virtual bundles over $M$ are multiplied by arbitrary powers of the $\nu_m$ given by (3.5), the representation ring of U(1) is the ring of formal Laurent polynomials in the variable $\nu_1$, $R_{U(1)} = \mathbb{Z}[\nu_1, \nu_1^{-1}]$. Using (3.6) we can associate the monopole bundle $L$ to the generator $\nu_1$, and thereby identify (4.6) as the Laurent polynomial ring

$$K_{SU(2)}(\mathbb{C}P^1) = R_{U(1)} \tag{4.6}$$

which establishes a one-to-one correspondence between classes of homogeneous vector bundles over the sphere $\mathbb{C}P^1$ and classes of finite-dimensional representations of U(1). Since the corresponding irreducible representations are the $\nu_m$ given by (3.5), the representation ring of U(1) is the ring of formal Laurent polynomials in the variable $\nu_1$, $R_{U(1)} = \mathbb{Z}[\nu_1, \nu_1^{-1}]$. Using (3.6) we can associate the monopole bundle $L$ to the generator $\nu_1$, and thereby identify (4.6) as the Laurent polynomial ring

$$K_{SU(2)}(\mathbb{C}P^1) = \mathbb{Z}[L, L^\vee] \tag{4.7}$$

In particular, the relationship (4.5) can be expressed as

$$K_{SU(2)}(\mathcal{M}_q \times \mathbb{C}P^1) = K(\mathcal{M}_q) \otimes \mathbb{Z}[L, L^\vee] \tag{4.8}$$

This is the appropriate modification of the Bott periodicity isomorphism (4.1) to the present setting. The crucial difference now is that virtual bundles over $\mathcal{M}_q$ are multiplied by arbitrary powers of the one-monopole bundle, allowing us to extend the equivalence to arbitrary monopole charges $m \in \mathbb{Z}$. In the equivariant setting, there is no need to use external twists of the monopole bundle, nor the ensuing K-theory product as done in [30]. The monopole fluxes are now naturally incorporated by the coefficient ring $R_{U(1)}$ of the U(1)-equivariant K-theory, superseding the need for introducing D-operations.

It is instructive to see precisely how the correspondence (4.8) works. For this, it is convenient to work instead in the category of holomorphic SL(2, $\mathbb{C}$)-equivariant bundles [34]. If $E$ is an SU(2)-equivariant vector bundle over $\mathcal{M}_q \times \mathbb{C}P^1$, then the action of SU(2) can be extended to an SL(2, $\mathbb{C}$) action. Everything we have said above carries through by replacing the group SU(2) with its complexification SL(2, $\mathbb{C}$) and the Cartan torus $U(1) \subset SU(2)$ with the subgroup $P \subset SL(2, \mathbb{C})$ of lower triangular matrices. We are then interested in $P$-equivariant bundles over $\mathcal{M}_q$ with $P$ acting trivially on $\mathcal{M}_q$. The Lie algebra $sl(2, \mathbb{C})$ is generated by the three Pauli matrices

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{4.9}$$

with the commutation relations

$$[\sigma_3, \sigma_\pm] = \pm 2 \sigma_\pm \quad \text{and} \quad [\sigma_+, \sigma_-] = \sigma_3. \tag{4.10}$$

The Lie algebra of the subgroup $P$ is generated by the elements $\sigma_3$ and $\sigma_-$, while the Cartan subgroup $\mathbb{C}^\times \subset P$ is generated by the element $\sigma_3$ with the corresponding irreducible representations being the $\nu_m$ given by (3.5).
Since the manifold $\mathcal{M}_q$ carries a trivial action of the subgroup $\mathbb{C}^\times$, any $\mathbb{C}^\times$-equivariant bundle $E' \to \mathcal{M}_q$ can be written using (4.2) as a finite Whitney sum

$$E' = \bigoplus_{l \in \Delta(E')} E'_l \otimes S^l,$$

where $\Delta(E') \subset \mathbb{Z}$ is the set of eigenvalues for the $\mathbb{C}^\times$-action on $E'$ and $E'_l \to \mathcal{M}_q$ are bundles carrying the trivial $\mathbb{C}^\times$-action. The rest of the $\mathcal{P}$-equivariant structure is determined by the generator $\sigma_-$. Since $[\sigma_3, \sigma_-] = -2 \sigma_-$, the action of $\sigma_-$ on $E'_l \otimes S^l$ corresponds to holomorphic bundle morphisms $E'_l \to E'_{l-2}$ and the trivial $\sigma_-$-action on the irreducible $\mathbb{C}^\times$-modules $S^l$. Thus every indecomposable $\mathcal{P}$-equivariant bundle $E' \to \mathcal{M}_q$ has weight set of the form $\Delta(E') = \{m_0, m_0 + 2, \ldots, m_1 - 2, m_1\}$ for some $m_0, m_1 \in \mathbb{Z}$ with $m_0 \leq m_1$. After an appropriate twist by a $\mathbb{C}^\times$-module and a relabelling, the $\sigma_3$-action is given by the $\mathbb{C}^\times$-equivariant decomposition

$$E = \bigoplus_{i=0}^m E_{k_i} \otimes S_{m-2i},$$

while the $\sigma_-$-action is determined by a chain

$$0 \to E_{k_m} \xrightarrow{\phi_m} E_{k_{m-1}} \xrightarrow{\phi_{m-1}} \cdots \xrightarrow{\phi_2} E_{k_1} \xrightarrow{\phi_1} E_{k_0} \to 0$$

of holomorphic bundle maps between consecutive $E_{k_i}$'s. We can now consider the underlying $\mathrm{U}(1)$-equivariant hermitean vector bundle defined by the unitary $\mathrm{U}(k)$ reduction of the $\mathrm{GL}(k, \mathbb{C})$ structure group of the holomorphic bundle (4.12), after introducing a hermitean metric on its fibres. Then the corresponding bundle $\mathcal{E} \to \mathcal{M}_q \times \mathbb{CP}^1$ is given by

$$\mathcal{E} = \mathrm{SU}(2) \times \mathrm{U}(1) \cdot E.$$  

Using (3.6) one finds that (4.14) coincides with the original equivariant decomposition (3.16). Conversely, given an $\mathrm{SU}(2)$-equivariant bundle $\mathcal{E} \to \mathcal{M}_q \times \mathbb{CP}^1$, its restriction $E = \iota^* \mathcal{E}$ defines a $\mathrm{U}(1)$-equivariant bundle over $\mathcal{M}_q$ which thereby admits an isotopical decomposition of the form (4.12) and $\mathcal{E}$ may be recovered from (4.14).

**Quiver gauge theory.** The ansatz for the gauge potential on $\mathcal{M}_q \times \mathbb{CP}^1$, represented symbolically by the bundle chain (3.17), corresponds to the disjoint union of two copies of the quiver

$$\mathcal{A}_{m+1} : \quad \begin{array}{cccccc}
\bullet & \phi_m & \bullet & \phi_{m-1} & \cdots & \bullet & \phi_1 & \bullet
\end{array}$$

with the second copy obtained from (4.15) by reversing the directions of the arrows and replacing $\phi_i$ with $\phi_i^\dagger$ for each $i = 1, \ldots, m$. The vertices of the quiver are labelled by the degrees of the monopole bundles $L^{m-2i}$, while the arrows correspond to module morphisms $\phi_i : V_{k_i} \to V_{k_{i+1}}$ (locally at each point $x' \in \mathcal{M}_q$). Equivalently, the vertices may be labelled by irreducible chiral representations of the group $\mathcal{P}$. Thus our ansatz determines a representation of the quiver $\mathcal{A}_{m+1}$ in the category of complex vector bundles over the manifold $\mathcal{M}_q$ [36]. Such a representation is called an $\mathcal{A}_{m+1}$-bundle. Many properties of the explicit solutions that we construct later on find their most natural explanation in the context of such a quiver gauge theory, which provides a more refined description of the brane configurations than just their K-theory charges. This framework encompasses the algebraic and representation theoretic aspects of the problem [37].

The quiver graph (4.15) is identical to the Dynkin diagram of the Lie algebra $\mathcal{A}_{m+1}$. The adjacency matrix of the quiver has matrix elements specifying the number of links between each
pair of vertices \( m - 2i, m - 2j \), and in the case (4.15) it is given by \( \text{Adj}(A_{m+1}) = (\delta_{i,j-1})_{i,j=0,1,\ldots,m} \). The matrix elements \( C_{ij} = 2\delta_{ij} - \text{Adj}(A_{m+1})_{ij} \) are then identical to those of the Cartan matrix \( C_{ij} = \varepsilon_i \cdot \varepsilon_j \), where \( \varepsilon_i, i = 0, 1, \ldots, m \) are the simple roots of \( A_{m+1} \). Corresponding to the gauge symmetry breaking (3.9), the dimension vector \( \vec{k} := (k_0, k_1, \ldots, k_m) \) can be regarded as a positive root of \( A_{m+1} \) associated with the Cartan matrix \( C = (C_{ij}) \) by writing it as

\[
\vec{k} := \sum_{i=0}^{m} k_i \vec{e}_i \quad \text{with} \quad \vec{k} := \sum_{i=0}^{m} k_i = k . \tag{4.16}
\]

By Kac’s theorem [37], there is a one-to-one correspondence between the isomorphism classes of indecomposable representations of the quiver \( A_{m+1} \) and the set of positive roots of the Lie algebra \( A_{m+1} \). This property is a consequence of the SU(2)-invariance of our ansatz.

Let us focus for a while on the case \( M_q = \text{pt} \). In this case eq. (3.16), with the \( m \)-monopole bundles \( \mathcal{L}^m \) substituted everywhere by the holomorphic line bundles (3.7), gives a relation between the categories of homogeneous holomorphic vector bundles over \( \mathbb{C}P^1 = \text{SL}(2, \mathbb{C})/\mathbb{P} \) and of finite-dimensional chiral representations of \( \mathbb{P} \), while the quiver representation further gives a relation with the abelian category of finite-dimensional representations of \( A_{m+1} \) [36]. To describe this latter category, it is convenient to introduce the notion of a path \( \mathcal{P} \) in \( A_{m+1} \), which is generally defined as a sequence of arrows of the quiver which compose. In the present case any path is of the form

\[
\mathcal{P} : \quad m_0 \xrightarrow{\phi_{m-m_0}} m_0 + 2 \xrightarrow{\phi_{m-m_{0+2}}} \cdots \xrightarrow{\phi_{m-m_1}} m_1 + 1
\]

(4.17)

with \(-m \leq m_0 \leq m_1 \leq m\). We will denote it by the formal vector \( |m_0, \ldots, m_1\rangle \). The non-negative integer \( |\mathcal{P}| := \frac{1}{2} (m_1 - m_0) \) is the length of the path (4.17). The trivial path of length 0 based at a single vertex \( m_0 \) is denoted \( |m_0\rangle \). The path algebra \( \mathbb{C} \mathcal{A}_{m+1} \) of the quiver (4.15) is then defined as the algebra generated by all paths \( \mathcal{P} \) of \( A_{m+1} \). i.e. as the vector space

\[
\mathbb{C} \mathcal{A}_{m+1} = \bigoplus_{m_0, m_1 = -m}^{m} \mathbb{C}|m_0, \ldots, m_1\rangle
\]

(4.18)

together with the \( \mathbb{C} \)-linear multiplication induced by (left) concatenation of paths where possible,

\[
|m_0, \ldots, m_1\rangle \cdot |n_0, \ldots, n_1\rangle = \delta_{m_1n_0} \langle m_0, \ldots, n_1\rangle . \tag{4.19}
\]

This makes \( \mathbb{C} \mathcal{A}_{m+1} \) into a finite-dimensional quasi-free algebra. The path algebra has a natural \( \mathbb{Z}_{m+1} \)-grading by path length,

\[
\mathbb{C} \mathcal{A}_{m+1} = \bigoplus_{i=0}^{m} (\mathbb{C} \mathcal{A}_{m+1})_i \quad \text{with} \quad (\mathbb{C} \mathcal{A}_{m+1})_i = \bigoplus_{m_0 = -m}^{m-2i} \mathbb{C}|m_0, \ldots, m_0 + 2i\rangle , \tag{4.20}
\]

and can thereby be alternatively described as the tensor algebra over the ring

\[
C_0 = \bigoplus_{i=0}^{m} \mathbb{C}|m - 2i\rangle \cong \mathbb{C}^{m+1} \tag{4.21}
\]

of the \( C_0 \)-bimodule

\[
C_1 = \bigoplus_{i=0}^{m} \mathbb{C}|m - 2i, m - 2i + 2\rangle . \tag{4.22}
\]
The importance of the path algebra stems from the fact that the category of representations of the quiver $A_{m+1}$ is equivalent to the category of (left) $\mathbb{C}A_{m+1}$-modules [37]. Given a representation $W_{m-2i} \xrightarrow{\eta_i} W_{m-2i+2}$, $i = 1, \ldots, m$, of $A_{m+1}$, the associated $\mathbb{C}A_{m+1}$-module $W$ is

$$W = \bigoplus_{i=0}^{m} W_{m-2i}$$

(4.23)

with multiplication extended $\mathbb{C}$-linearly from the definitions

$$|m-2i| \cdot w_j = \delta_{ij} w_j \quad \text{and} \quad |m-2i, m-2i+2| \cdot w_j = \delta_{i,j+1} \eta_j(w_j)$$

(4.24)

for $w_j \in W_{m-2j}$. Conversely, given a left $\mathbb{C}A_{m+1}$-module $W$, we can set $W_{m-2i} := |m-2i| \cdot W$ for $i = 0, 1, \ldots, m$ and define $\eta_i : W_{m-2i} \rightarrow W_{m-2i+2}$ for $i = 1, \ldots, m$ by

$$\eta_i(w_i) = |m-2i, m-2i+2| \cdot w_i$$

(4.25)

One can further show that morphisms of representations of $A_{m+1}$ correspond to $\mathbb{C}A_{m+1}$-module homomorphisms [37]. Thus, the problem of determining finite-dimensional representations of the quiver $A_{m+1}$, or equivalently homogeneous vector bundles over $\mathbb{C}P^1$, is equivalent to finding representations of its path algebra.

As an example, consider the $A_2$ quiver

$$A_2 : \begin{array}{c} \bullet \\ -1 \end{array} \xrightarrow{\phi_1} \begin{array}{c} \bullet \\ +1 \end{array}.$$  

(4.26)

It represents the standard brane-antibrane system, and as expected SU(2)-equivariance implies that it can only carry $m = 1$ unit of monopole charge [30]. The corresponding path algebra is

$$\mathbb{C}A_2 = \mathbb{C}[-1] \oplus \mathbb{C}[1] \oplus \mathbb{C}[-1, +1] = \begin{pmatrix} \mathbb{C} & \mathbb{C} \\ 0 & \mathbb{C} \end{pmatrix}.$$  

(4.27)

Representations of this algebra yield the standard superconnections characterizing the low-energy field content on the worldvolume of a brane-antibrane system [31]. In the next section we will show how to generalize the superconnection formalism to account for representations of generic path algebras (4.18). Later on we shall write down explicit solutions with generic monopole charge $m \in \mathbb{Z}$ that also correspond to the basic brane-antibrane system.

Our technique for generating D-branes from a quiver gauge theory on $\mathcal{M}_q$ arises via a quotient with respect to a generalized SU(2)-action on Chan-Paton bundles over $\mathcal{M}_q \times \mathbb{C}P^1$. This new construction is rather different from the well-known quiver gauge theories that arise from orbifolds with respect to the action of a discrete group $G$ [38]. In the latter case the nodes of a quiver represent the irreducible representation fractional branes into which a regular representation D-brane decays into when it is taken to an orbifold point of $\mathcal{M}_q / G$, and they can be thought of in terms of a projection of branes sitting on the leaves of the covering space $\mathcal{M}_q$. While our quiver gauge theory is fundamentally different, it shares many of the physical features of orbifold theories of D-branes. For instance, the blowing up of vortices on $\mathcal{M}_q$ into instantons on $\mathcal{M}_q \times \mathbb{C}P^1$ is reminescent of the blowing up of fractional D($q-1$)-branes into D($q+1$)-branes wrapping a non-contractible $\mathbb{C}P^1$ that is used to resolve the orbifold singularity in $\mathcal{M}_q / G$. Our solutions provide explicit realizations of this blowing up phenomenon, but in a completely smooth setting.
5 Dimensional reduction

The condition of SU(2)-equivariance uniquely prescribes a specific $\mathbb{C}P^1$ dependence for the gauge potential $A$ and reduces the Yang-Mills equations (2.8) on $\mathcal{M}_q \times \mathbb{C}P^1$ to equations on $\mathcal{M}_q$. In this section we will formulate this reduction in detail and relate it to representations of the path algebra (4.18). This will be done by developing a new formalism of $\mathbb{Z}_{m+1}$-graded connections which describes the field content corresponding to the bundle chains (3.17) and (4.13), and which generalizes the standard superconnection field theories on the worldvolumes of brane-antibrane systems [31]. This formalism will be the crux to merging together the three interpretations of the previous section.

Reduction of the Yang-Mills functional. The dimensional reduction of the Yang-Mills equations can be seen at the level of the Yang-Mills lagrangian (2.7). Substituting (3.24)–(3.26) into (2.10) and performing the integral over $\mathbb{C}P^1$ we arrive at the action\

\[ S_{\text{YM}} := \int_{\mathcal{M}_q \times \mathbb{C}P^1} d^{q+2}x \ L_{\text{YM}} = \pi R^2 \int_{\mathcal{M}_q} d^{q}x' \sqrt{g'} \sum_{i=0}^{m} \text{tr}_{k_i \times k_i} \left[ (F^{i \mu \nu'})^\dagger (F^{i \mu \nu'}) + \frac{1}{R^2} (D_{\mu'} \phi_{i+1}) (D_{\mu'} \phi_{i+1})^\dagger \right. \\
+ \frac{1}{R^2} (D_{\mu} \phi_i) (D_{\mu} \phi_i) + \left. \frac{1}{2 R^4} \left( m - 2i + \phi_i \phi_i - \phi_{i+1} \phi_{i+1} \right) \right]^2 , \quad (5.1) \]

where $g' = |\text{det}(g_{\mu \nu'})|$. In the remainder of this paper we shall only consider static field configurations on $\mathcal{M}_q = \mathbb{R}^1 \times M_{2n}$ in the temporal gauge $A_0 = 0$. In this case one can introduce the corresponding energy functional\

\[ E_{\text{YM}} = \pi R^2 \int_{M_{2n}} d^{2n}x \sqrt{g_n} \sum_{i=0}^{m} \text{tr}_{k_i \times k_i} \left[ (F^{i \mu \nu}) (F^{i \mu \nu}) + \frac{1}{R^2} (D_{\mu} \phi_{i+1}) (D_{\mu} \phi_{i+1})^\dagger \right. \\
+ \frac{1}{R^2} (D_{\mu} \phi_i) (D_{\mu} \phi_i) + \left. \frac{1}{2 R^4} \left( m - 2i + \phi_i \phi_i - \phi_{i+1} \phi_{i+1} \right) \right]^2 , \quad (5.2) \]

where $g_n = \text{det}(g_{\mu \nu})$. The functional (5.2) is non-negative.

Graded connections. The energy functional (5.2) is analysed most efficiently by introducing a framework specific to connections on the rank $k$ $\mathbb{Z}_{m+1}$-graded vector bundle\

\[ E := \bigoplus_{i=0}^{m} E_k \]

over $M_{2n}$ whose typical fibre is the module (3.8). The endomorphism algebra bundle corresponding to (5.3) is given by the direct sum decomposition\

\[ \text{End}(E) = \bigoplus_{i=0}^{m} \text{End}(E_k) \oplus \bigoplus_{i,j=0 \atop i \neq j}^{m} \text{Hom}(E_k, E_j) . \quad (5.4) \]

We may naturally associate to (5.4) a distinguished representation of the $A_{m+1}$ quiver. For this, we note that the path algebra $\mathbb{C}A_{m+1}$ is itself a $\mathbb{C}A_{m+1}$-module, and that the elements $|m - 2i| \in$\^[2]A set of Yang-Mills coupling constants $g_{Y_M}^i$, $i = 0, 1, \ldots, m$ can be introduced via the redefinitions $A' \mapsto g_{Y_M}^i A'$.
$\mathbb{C} A_{m+1}$ define a complete set of orthogonal projectors of the path algebra, i.e. $|m-2i| \cdot |m-2j| = \delta_{ij} |m-2i|$ for $i, j = 0, 1, \ldots, m$ with $\sum_{i=0}^{m}|m-2i| = 1$. Analogously to the construction of (4.23)–(4.25), we may thereby define a projective $\mathbb{C} A_{m+1}$-module $P_i := |m-2i| \cdot \mathbb{C} A_{m+1}$ for each $i = 0, 1, \ldots, m$ [37], which is the subspace of $\mathbb{C} A_{m+1}$ generated by all paths which start at the $i$-th vertex of the quiver $A_{m+1}$. Then $(P_i)_{m-2j} \cong \mathbb{C}$ is the vector space generated by the path from the $i$-th vertex to the $j$-th vertex, and the corresponding dimension vector is

$$\tilde{k}_{P_i} = \sum_{j=i}^{m} \tilde{e}_j \quad . \quad (5.5)$$

The modules $P_i, i = 0, 1, \ldots, m$ are exactly the set of all indecomposable projective representations of the $A_{m+1}$ quiver [37], with

$$\mathbb{C} A_{m+1} = \bigoplus_{i=0}^{m} P_i \quad . \quad (5.6)$$

The importance of this path algebra representation stems from the fact that, for any quiver representation (3.8), there is a natural isomorphism [37]

$$\text{Hom}(P_i, \mathbb{C}) \cong V_{k_i} \quad . \quad (5.7)$$

We may thereby identify $\text{Hom}(V_{k_j}, V_{k_i})$ in terms of appropriate combinations of the spaces

$$\text{Hom}(P_j, P_i) \cong |m-2j| \cdot \mathbb{C} A_{m+1} \cdot |m-2i| \cong \mathbb{C} \quad . \quad (5.8)$$

This is the vector space generated by the path from the $i$-th vertex to the $j$-th vertex of $A_{m+1}$. A natural representation of this path is by a matrix of dimension $(m+1) \times (m+1)$ with 1 in its $(ij)$-th entry and 0’s everywhere else. The path algebra (5.6) is thereby identified with the algebra of upper triangular $(m+1) \times (m+1)$ complex matrices [37]. For a given quiver representation (3.8), this algebra may be represented by assembling the chiral Higgs fields $\phi_1, \ldots, \phi_m$ into the $k \times k$ matrix

$$\phi_{(m)} := \begin{pmatrix} 0 & \phi_1 & 0 & \cdots & 0 \\ 0 & 0 & \phi_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \phi_m \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \quad . \quad (5.9)$$

with respect to the decomposition (5.3). This object generates a representation of the path algebra in the category of complex vector bundles over $M_{2n}$, corresponding to the off-diagonal $i < j$ components of the decomposition (5.4). The finite dimensionality of $\mathbb{C} A_{m+1}$ is reflected in the property that generically

$$\phi_{(m)}, (\phi_{(m)})^2, \ldots, (\phi_{(m)})^m \neq 0 \quad \text{but} \quad (\phi_{(m)})^{m+1} = 0 \quad . \quad (5.10)$$

The field configuration (5.9) generates the basic zero-form component of a geometric object that we shall refer to as a "$\mathbb{Z}_{m+1}$-graded connection" on $M_{2n}$. For $m = 1$ it corresponds to a standard superconnection [39], while for $m > 1$ it is the appropriate entity that constructs representations corresponding to the enlargement of the path algebra $\mathbb{C} A_{m+1}$. Its matrix form is similar to (3.10)–(3.14), but without the one-forms on $\mathbb{C} P^1$.

To formulate the definition precisely, we note that the algebra $\Omega(M_{2n}, E)$ of differential forms on $M_{2n}$ with values in the bundle (5.3) has a natural $\mathbb{Z} \times \mathbb{Z}_{m+1}$ grading, where the $\mathbb{Z}$-grading is by form degree. We can thereby induce a total $\mathbb{Z}_{m+1}$-grading by the decomposition

$$\Omega^*(M_{2n}, E) = \bigoplus_{p=0}^{m} \Omega(p)(M_{2n}, E) \quad \text{with} \quad \Omega(p)(M_{2n}, E) = \bigoplus_{i+j \equiv m+1 \atop i \geq 0} \Omega^i(M_{2n}, E_{k_j}) \quad , \quad (5.11)$$

14
where \(\equiv_{m+1}\) denotes congruence modulo \((m + 1)\). By using (5.4) and the usual tensor product grading, this induces a \(\mathbb{Z}_{m+1}\)-grading on the corresponding endomorphism algebra as

\[
\Omega_{\bullet}(M_{2n}, \text{End } E) = \bigoplus_{p=0}^{m} \Omega_{(p)}(M_{2n}, \text{End } E) \tag{5.12}
\]

with

\[
\Omega_{(p)}(M_{2n}, \text{End } E) = \bigoplus_{i=0}^{m} \bigoplus_{a=0}^{p} \bigoplus_{i_a = m+1} (p-a) \Omega^{i_a}(M_{2n}) \otimes \text{Hom}(E_{k_i}, E_{k_{i+a}}) \tag{5.13}
\]

A graded connection on (5.11) is defined to be a linear operator \(\Omega_{(1)}(M_{2n}, E) \rightarrow \Omega_{(1)+1}(M_{2n}, E)\) which shifts the total \(\mathbb{Z}_{m+1}\)-grading by 1 modulo \((m + 1)\), i.e. an element of

\[
\Omega_{(1)}(M_{2n}, \text{End } E) = \bigoplus_{i=0}^{m} \bigoplus_{i_{a=m+1}^{1}} \Omega^{i_{a}}(M_{2n}) \otimes \text{End}(E_{k_i}) \tag{5.14}
\]

and which satisfies the usual Leibniz rule on \(\Omega(M_{2n})\). As in the standard cases, the \(\mathbb{Z}_{m+1}\)-graded connections form an affine space modelled on a set of local operators.

In our case we retain only the \(i_0 = 0\) and \(i_1 = 1\) components of (5.14) corresponding to the lowest lying massless degrees of freedom on the given configuration of D-branes. From the Leibniz rule it follows that the pertinent graded connections are then of the form \((d + A^{(m)} + (\phi_{(m)}) + (\phi_{(m)})^\dagger)\), where

\[
A^{(m)} := \sum_{i=0}^{m} A^i \otimes \Pi_i \tag{5.15}
\]

and \(\Pi_i : E \rightarrow E_{k_i}\) are the canonical orthogonal projections of rank 1,

\[
\Pi_i \Pi_j = \delta_{ij} \Pi_i , \tag{5.16}
\]

which may be represented, with respect to the decomposition (5.3), by diagonal matrices \(\Pi_i = (\delta_{ji} \delta_{li})_{j,l=0,1,\ldots,m}\) of unit trace. In this geometric framework all \(\phi_i\) are assumed to anticommute with a given local basis \(dx^\mu\) of the cotangent bundle of the Kähler manifold \(M_{2n}\), as if they were \(m\) basic odd complex elements of a superalgebra. This requisite property may be explicitly realized by extending the graded connection formalism to \(M_{2n} \times \mathbb{C}P^1\). For this, we rewrite the ansatz (3.10)–(3.15) in terms of the above field configurations as

\[
A_{\mu} = (A^{(m)})_\mu \otimes 1 , \tag{5.17}
\]

\[
A_y = 1_k \otimes (a^{(m)})_y - (\phi_{(m)})^\dagger \otimes \beta_y , \tag{5.18}
\]

\[
A_{\bar{y}} = 1_k \otimes (a^{(m)})_{\bar{y}} + (\phi_{(m)}) \otimes \bar{\beta}_{\bar{y}} , \tag{5.19}
\]

where

\[
a^{(m)} := \sum_{i=0}^{m} a_{m-2i} \otimes \Pi_i \tag{5.20}
\]

and \(\Pi_i : \mathcal{E} \rightarrow \mathcal{E}_i\) are the canonical projections on (3.16). The coupling of \(\phi_{(m)}\) to d\(\bar{y}\) in (5.19) yields the desired anticommutativity with \(dx^\mu\).
Alternatively, we may use the canonical isomorphism $\Omega(M_{2n} \times \mathbb{C}P^1) \cong \mathcal{C}(M_{2n} \times \mathbb{C}P^1)$ to map the cotangent basis $dz^\mu \mapsto \Gamma^\mu$ onto the generators of the Clifford algebra

$$
\Gamma^\mu \Gamma^\nu + \Gamma^\nu \Gamma^\mu = -2g^{\mu \nu} \mathbf{1}_{2n+1} \quad \text{with} \quad \mu, \nu = 1, \ldots, 2n + 2 .
$$

(5.21)

The gamma-matrices in (5.21) may be decomposed as

$$
\{ \Gamma^\mu \} = \{ \Gamma^\mu, \Gamma^y, \Gamma^{\bar{y}} \} \quad \text{with} \quad \Gamma^\mu = \gamma^\mu \otimes \mathbf{1}_2 , \quad \Gamma^y = \gamma \otimes \gamma^y \quad \text{and} \quad \Gamma^{\bar{y}} = \gamma \otimes \gamma^{\bar{y}} ,
$$

(5.22)

where the $2^n \times 2^n$ matrices $\gamma^\mu = -(\gamma^\mu)^\dagger$ act on the spinor module $\Delta(M_{2n})$ over the Clifford algebra $\mathcal{C}(M_{2n})$,

$$
\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = -2g^{\mu \nu} \mathbf{1}_{2n} \quad \text{with} \quad \mu, \nu = 1, \ldots, 2n ,
$$

(5.23)

while

$$
\gamma = \frac{i^n}{(2n)! \sqrt{g_n}} \epsilon_{\mu_1 \ldots \mu_{2n}} \gamma^\mu_1 \ldots \gamma^\mu_{2n} \quad \text{with} \quad (\gamma)^2 = \mathbf{1}_{2n} \quad \text{and} \quad \gamma \gamma^\mu = -\gamma^\mu \gamma
$$

(5.24)

is the corresponding chirality operator. Here $\epsilon_{\mu_1 \ldots \mu_{2n}}$ is the Levi-Civita symbol with $\epsilon_{12 \ldots 2n} = +1$. The action of the Clifford algebra $\mathcal{C}(\mathbb{C}P^1)$ on the spinor module $\Delta(\mathbb{C}P^1)$ is generated by

$$
\gamma^y = \frac{1}{R^2} (R^2 + y\bar{y}) \sigma^y \quad \text{and} \quad \gamma^{\bar{y}} = \frac{1}{R^2} (R^2 + y\bar{y}) \sigma^{\bar{y}}
$$

(5.25)

with constant $2 \times 2$ Pauli matrices $\sigma^y, \sigma^{\bar{y}}$ obeying $[\sigma^y, \sigma^{\bar{y}}] = -\sigma_3$. The gauge potential (3.10)–(3.14) may then be written in an algebraic form as

$$
\hat{A} := \Gamma^\mu A^\mu
$$

$$
= \gamma^\mu (A^{(m)})^\mu_\mu \otimes \mathbf{1}_2 + (\phi^{(m)})^\gamma \otimes \gamma^y \bar{\beta}_y - (\phi^{(m)})^\gamma \otimes \gamma^\gamma \beta_y
$$

$$
+ \gamma \otimes \left( \gamma^y (a^{(m)})^y_\gamma + \gamma^{\bar{y}} (a^{(m)})^{\bar{y}}_\gamma \right) ,
$$

(5.26)

and the coupling of (5.9) with the chirality operator (5.24) realizes the desired anticommutativity with the one-form representatives $\gamma^\mu$. Note that the products

$$
(\phi^{(m)})^\gamma \otimes \gamma^y \bar{\beta}_y = \frac{1}{R} (\phi^{(m)})^\gamma \otimes \sigma^y \quad \text{and} \quad (\phi^{(m)})^\gamma \gamma^y \beta_y = \frac{1}{R} (\phi^{(m)})^\gamma \otimes \sigma^y
$$

(5.27)

are independent of the coordinates $(y, \bar{y}) \in \mathbb{C}P^1$.

The curvature $(d + A^{(m)} + (\phi^{(m)}) + (\phi^{(m)})^\dagger)^2 \in \Omega(2)(M_{2n}, \text{End } E)$ of the graded connection is also most elegantly expressed through dimensional reduction from $M_{2n} \times \mathbb{C}P^1$. From (3.18)–(3.26) it is given by

$$
\hat{F} := \frac{i}{4} [\Gamma^\mu, \Gamma^\nu] F_{\mu \nu}
$$

$$
= \frac{i}{4} \left[ \gamma^\mu, \gamma^\nu \right] (F^{(m)})_{\mu \nu} \otimes \mathbf{1}_2 - \frac{1}{R} \gamma (\gamma^\mu D_\mu \phi^{(m)})^\dagger \otimes \sigma^y - \frac{1}{R} \gamma (\gamma^\mu D_\mu \phi^{(m)}) \otimes \sigma^{\bar{y}}
$$

$$
+ \frac{1}{2R^2} \left( Y^{(m)} + (\phi^{(m)})^\dagger (\phi^{(m)}) - (\phi^{(m)}) (\phi^{(m)})^\dagger \right) \mathbf{1}_{2n} \otimes \sigma_3
$$

(5.28)

where $F^{(m)} := dA^{(m)} + A^{(m)} \wedge A^{(m)}$ and

$$
Y^{(m)} := \sum_{i=0}^m (m - 2i) \Pi_i .
$$

(5.29)
The contribution (5.29) is generated by the monopole connection on $\mathbb{CP}^1$ in (5.26), while the Higgs potentials in (5.28) are produced by (5.27). The graded curvature is independent of $(y, \bar{y}) \in \mathbb{CP}^1$, and the standard gamma-matrix trace formulas

$$
\text{Tr}_{C^{2n+1}} \left( \gamma^\mu \gamma^\nu \otimes 1_2 \right) = -2^{n+1} g^{\mu\nu} ,
$$

$$
\text{Tr}_{C^{2n+1}} \left( \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\rho \otimes 1_2 \right) = 2^{n+1} \left( g^{\mu\nu} g^{\lambda\rho} + g^{\mu\rho} g^{\nu\lambda} - g^{\mu\lambda} g^{\nu\rho} \right) ,
$$

$$
\text{Tr}_{C^{2n+1}} \left( \left[ \gamma^\mu , \gamma^\nu \right] \left[ \gamma^\lambda , \gamma^\rho \right] \otimes 1_2 \right) = 2^{n+3} \left( g^{\mu\rho} g^{\nu\lambda} - g^{\mu\lambda} g^{\nu\rho} \right) ,
$$

$$
\text{Tr}_{C^{2n+1}} \left( \gamma^\mu \gamma^\nu \gamma \otimes \sigma^y \sigma^y \right) = -2^n g^{\mu\nu} = \text{Tr}_{C^{2n+1}} \left( \gamma^\mu \gamma^\nu \gamma \otimes \sigma^y \sigma^y \right) \tag{5.33}
$$

imply that the energy functional (5.2) can be compactly written in terms of (5.28) as

$$
E_{YM} = \frac{\pi R^2}{2^n} \int_{M_{2n}} d^{2n}x \sqrt{g_n} \text{Tr}_{k \times k} \text{Tr}_{C^{2n+1}} \bar{F}^2 .
$$

**Nonabelian coupled vortex equations.** Let us now examine the reduction of the DUY equations on $M_{2n} \times \mathbb{CP}^1$ for a gauge potential of the form proposed in Section 3 (with static configurations in the gauge $A_0 = 0$). Substituting (3.19)–(3.22) into (2.12)–(2.14), we obtain

$$
g^{ab} F^{i}_{ab} = \frac{1}{2 R^2} \left( m - 2i + \phi_i^\dagger \phi_i - \phi_{i+1} \phi_{i+1}^\dagger \right) ,
$$

$$
F^{i}_{ab} = 0 = F^{i}_{ab} \tag{5.35}
$$

and

$$
\partial_a \phi_{i+1} + A^i_a \phi_{i+1} - \phi_{i+1} A^{i+1}_a = 0 \tag{5.36}
$$

for each $i = 0, 1, \ldots, m$, where $\phi_0 := 0 =: \phi_{m+1}$. Recall that there is no summation over $i$ in these equations. We have abbreviated $F^{i}_{ab} := F^{i}_{a} \gamma^{ab} \gamma_{ab}$ etc., and defined the derivatives $\partial_a := \partial_{a} = \frac{1}{2} (\partial_{a-1} + i \partial_{a})$ and $\partial_{a} := \partial_{a} = \frac{1}{2} (\partial_{a-1} - i \partial_{a})$ with $a, b = 1, \ldots, n$. We shall call (5.35)–(5.37) the nonabelian coupled vortex equations.

Eq. (5.36) implies that the vector bundles $E_{k_i} \rightarrow M_{2n}$ are holomorphic, while eq. (5.37) implies that the Higgs fields $\phi_{i+1} : E_{k_{i+1}} \rightarrow E_{k_i}$ are holomorphic maps. By using a Bogomolny-type transformation [33] one can show that solutions to these equations realize absolute minima of the energy functional (5.2). These field configurations describe supersymmetric BPS states of D-branes.

**Seiberg-Witten monopole equations.** For $n = 2, m = 1$ and $k_0 = k_1 = 1$ (so that $k = k_0 + k_1 = 2$), the equations (5.35)–(5.37) coincide with the perturbed abelian Seiberg-Witten monopole equations on a Kähler four-manifold $M_4$ [40]. In this case we have

$$
A^0 = -A^1 =: A \in \mathfrak{u}(1) , \quad F^0 = -F^1 =: F \quad \text{and} \quad \phi_1 =: \phi \in \mathbb{C} \tag{5.38}
$$

and the equations (5.35)–(5.37) reduce to

$$
g^{\bar{a}b} F_{\bar{a}b} = \frac{1}{2 R^2} \left( 1 - \phi \bar{\phi} \right) ,
$$

$$
F_{\bar{a}b} = 0 = F_{ab} \tag{5.39}
$$

and

$$
\partial_{\bar{a}} \phi + 2 A_{\bar{a}} \phi = 0 .
$$

The perturbation, i.e. the term $\frac{1}{2 R^2}$ in (5.39), is needed whenever $M_4$ has non-negative scalar curvature in order to produce a non-trivial and non-singular moduli space of finite energy $L^2$-solutions. It is usually introduced into the Seiberg-Witten equations by hand. In the present context, it arises automatically from the extra space $\mathbb{CP}^1$ and the reduction from $M_4 \times \mathbb{CP}^1$ to $M_4$. 

17
6 Noncommutative gauge theory

To build further on the interpretation of our ansatz in terms of configurations of D-branes as described in Section 4, we should now proceed to construct explicit solutions of the reduced Yang-Mills equations on $M_{2n}$. Unfortunately, even solutions of the vortex equations (5.35)–(5.37) are difficult to come by and there is no known general method for explicitly constructing the appropriate field configurations. As we will demonstrate in the following, explicit realizations of these D-brane states are possible in the context of noncommutative gauge theory, which can be mapped afterwards onto commutative worldvolume configurations. For this, we will now specialize the Kähler manifold $M_{2n} \times \mathbb{C}P^1$ to be $\mathbb{R}^{2n} \times \mathbb{C}P^1$ with metric tensor $g_{\mu \nu} = \delta_{\mu \nu}$ on $\mathbb{R}^{2n}$ and pass to a noncommutative deformation of the flat part of the space, i.e. $\mathbb{R}^{2n} \times \mathbb{C}P^1 \rightarrow \mathbb{R}_{\theta}^{2n} \times \mathbb{C}P^1$. Note that the $\mathbb{C}P^1$ factor remains a commutative space throughout this paper. Then we will deform the Yang-Mills, DUY and nonabelian coupled vortex equations, and in the subsequent sections construct various solutions of them.

Noncommutative deformation. Field theory on $\mathbb{R}_\theta^{2n}$ may be realized in an operator formalism which turns Schwartz functions $f$ on $\mathbb{R}^{2n}$ into compact operators $\hat{f}$ acting on the $n$-harmonic oscillator Fock space $H$ [6]. The noncommutative space $\mathbb{R}_\theta^{2n}$ is then defined by declaring its coordinate functions $\hat{x}_1, \ldots, \hat{x}_{2n}$ to obey the Heisenberg algebra relations

$$[\hat{x}^\mu, \hat{x}^\nu] = i \theta^{\mu \nu}$$

with a constant real antisymmetric tensor $\theta^{\mu \nu}$. Via an orthogonal transformation of the coordinates, the matrix $\theta = (\theta^{\mu \nu})$ can be rotated into its canonical block-diagonal form with non-vanishing components

$$\theta^{2a-1 \ 2a} = -\theta^{2a \ 2a-1} =: \theta^a$$

for $a = 1, \ldots, n$. We will assume for definiteness that all $\theta^a > 0$. The noncommutative version of the complex coordinates (2.3) has the non-vanishing commutators

$$[\hat{z}^a, \hat{\bar{z}}^b] = -2 \delta^{ab} \theta^a =: \theta^{ab} = -\theta^{b a} \leq 0 .$$

(6.3)

Taking the product of $\mathbb{R}_\theta^{2n}$ with the commutative sphere $\mathbb{C}P^1$ means extending the noncommutativity matrix $\theta$ by vanishing entries along the two new directions.

The Fock space $H$ may be realized as the linear span

$$H = \bigoplus_{r_1, \ldots, r_n = 0}^{\infty} \mathbb{C}|r_1, \ldots, r_n\rangle ,$$

(6.4)

where the orthonormal basis states

$$|r_1, \ldots, r_n\rangle = \prod_{a=1}^{n} (2 \theta^a r_a!)^{-1/2} (\hat{z}^a)^{r_a}|0, \ldots, 0\rangle$$

(6.5)

are connected by the action of creation and annihilation operators subject to the commutation relations

$$\left[ \frac{\hat{\bar{z}}^b}{\sqrt{2 \theta^b}} , \frac{\hat{z}^a}{\sqrt{2 \theta^a}} \right] = \delta^{ab} .$$

(6.6)

In the Weyl operator realization $f \mapsto \hat{f}$, coordinate derivatives are given by inner derivations of the noncommutative algebra according to

$$\hat{\partial}_{\bar{z}^a} f = \theta_{ab} [\hat{\bar{z}}^b, \hat{f}] =: \partial_{\bar{z}^a} \hat{f} \quad \text{and} \quad \hat{\partial}_{\bar{z}^a} f = \theta_{ab} [\hat{z}^b, \hat{f}] =: \partial_{\bar{z}^a} \hat{f} ,$$

(6.7)
where $\theta_{\bar{a}b}$ is defined via $\theta_{bc} \theta^{cn} = \delta^n_0$ so that $\theta_{\bar{a}b} = -\theta_{b\bar{a}} = \frac{\delta_{a\bar{b}}}{2\theta^n}$. On the other hand, integrals are given by traces over the Fock space $\mathcal{H}$ as

$$\int_{\mathbb{R}^{2n}} d^{2n} x \ f(x) = \left( \prod_{a=1}^n 2\pi \theta^a \right) \text{Tr}_\mathcal{H} \hat{f}.$$  

(6.8)

The transition to the noncommutative Yang-Mills and DUY equations is trivially achieved by going over to operator-valued objects everywhere. In particular, vector bundles $E \to \mathbb{R}^{2n}$ whose typical fibres are complex vector spaces $\mathbb{V}$ are replaced by the corresponding (trivial) projective modules $\mathbb{V} \otimes \mathcal{H}$ over $\mathbb{R}^{2n}$. The field strength components along $\mathbb{R}^{2n}$ in (2.8) and (2.12)–(2.14) read

$$\hat{F}_{\mu\nu} = \partial_{x^\mu} \hat{A}_\nu - \partial_{x^\nu} \hat{A}_\mu + [\hat{A}_\mu, \hat{A}_\nu],$$

where $\hat{A}_\mu$ are simultaneously $u(k)$ and operator valued. To avoid a cluttered notation, we drop the hats over operators from now on. Thus all our equations have the same form as previously but are considered now as operator equations.

**Noncommutative coupled vortex equations.** By reducing the noncommutative version of the DUY equations on $\mathbb{R}^2 \times \mathbb{C}P^1$ to $\mathbb{R}^{2n}$ we obtain the noncommutative nonabelian coupled vortex equations. Instead of working with the gauge potentials $A^i_\mu$ we shall use the operators $X^i_\mu$ defined by

$$X^i_a := A^i_a + \theta_{ab} \bar{z}^b \quad \text{and} \quad X^i_{\bar{a}} := \bar{A}^i_{\bar{a}} + \theta_{ab} \bar{z}^b. \quad (6.9)$$

In terms of these operators the field strength tensor reads

$$F^i_{ab} = [X^i_a, X^i_b] + \theta_{ab}, \quad F^i_{\bar{a}b} = [X^i_{\bar{a}}, X^i_b] \quad \text{and} \quad F^i_{\bar{a}\bar{b}} = [X^i_{\bar{a}}, X^i_{\bar{b}}], \quad (6.10)$$

while the bi-fundamental covariant derivatives become

$$D_a \phi_{i+1} = X^i_a \phi_{i+1} - \phi_{i+1} X^i_a \quad \text{and} \quad D_{\bar{a}} \phi_{i+1} = X^i_{\bar{a}} \phi_{i+1} - \phi_{i+1} X^i_{\bar{a}}. \quad (6.11)$$

The nonabelian vortex equations (5.35)–(5.37) can then be rewritten as

$$\delta^a \delta^b \left( [X^i_a, X^i_b] + \theta_{ab} \right) = \frac{1}{4R^2} \left( m - 2i \phi_{i+1} \right) \quad \text{and} \quad F^0_{ab} = 0 = F^0_{\bar{a}b}, \quad (6.12)$$

$$[X^i_a, X^i_b] = 0 = [X^i_{\bar{a}}, X^i_b], \quad (6.13)$$

$$X^i_a \phi_{i+1} - \phi_{i+1} X^i_a = 0 \quad \text{for} \ i = 0, 1, \ldots, m. \quad (6.14)$$

for $i = 0, 1, \ldots, m$. Note that for $m = 1$ we obtain the equations

$$\delta^a \delta^b F^0_{ab} = \frac{1}{4R^2} \left( 1 - \phi_1 \phi_1^\dagger \right) \quad \text{and} \quad F^0_{\bar{a}b} = 0 = F^0_{\bar{a}\bar{b}}, \quad (6.15)$$

$$\delta^a \delta^b F^1_{ab} = -\frac{1}{4R^2} \left( 1 - \phi_1^\dagger \phi_1 \right) \quad \text{and} \quad F^1_{\bar{a}b} = 0 = F^1_{\bar{a}\bar{b}}, \quad (6.16)$$

$$\delta_a \phi_1 + A^0_a \phi_1 - \phi_1 A^0_{\bar{a}} = 0 \quad (6.17)$$

which are considered in [29, 30]. In particular, for $n = 2$ and $k_0 = k_1 = 1$ the equations (6.15)–(6.17) coincide with the perturbed Seiberg-Witten $U_+ (1) \times U_- (1)$ monopole equations on $\mathbb{R}^4$ as considered in [41].
7 Explicit solutions of the noncommutative Yang-Mills equations

We are now ready to construct solutions to the Yang-Mills equations on $\mathbb{R}^{2n} \times \mathbb{C}P^1$. We shall first present the generic non-BPS solutions of the full Yang-Mills equations, and then proceed to solve the nonabelian coupled vortex equations (6.12)–(6.14), and thus the DUY equations on $\mathbb{R}^{2n} \times \mathbb{C}P^1$, which describe the stable BPS states. Our technique will make use of appropriate partial isometry operators $T_{N_i}$ in the noncommutative space.

**Ansatz for explicit solutions.** Let us fix a monopole charge $m > 0$ and an arbitrary integer $0 < r \leq k$. Consider the ansatz

$$X_a^i = \theta_{ab} T_{N_i} z^b T_{N_i}^\dagger \quad \text{and} \quad X_\alpha^i = \theta_{ab} T_{N_i} z^b T_{N_i}^\dagger, \quad (7.1)$$

$$\phi_{i+1} = \alpha_{i+1} T_{N_i} T_{N_i+1}^\dagger \quad \text{and} \quad \phi_\dagger_{i+1} = \bar{\alpha}_{i+1} T_{N_i+1} T_{N_i}^\dagger \quad (7.2)$$

for $i = 0, 1, \ldots, m$, where $\alpha_i \in \mathbb{C}$ are some constants with $\alpha_0 = \alpha_{m+1} = 0$. Denoting by $\mathcal{H}$ the $n$-oscillator Fock space, the Toeplitz operators $T_{N_i} : \mathbb{C}^r \otimes \mathcal{H} \to \mathbb{V}_{k_i} \otimes \mathcal{H}$ are partial isometries described by rectangular $k_i \times r$ matrices (with operator entries acting on $\mathcal{H}$) possessing the properties

$$T_{N_i}^\dagger T_{N_i} = 1_r \quad \text{while} \quad T_{N_i} T_{N_i}^\dagger = 1_{k_i} - P_{N_i}, \quad (7.3)$$

where $P_{N_i}$ is a hermitean projector of finite rank $N_i$ on the Fock space $\mathbb{V}_{k_i} \otimes \mathcal{H}$ so that

$$P_{N_i}^2 = P_{N_i} = P_{N_i}^\dagger \quad \text{and} \quad \text{Tr}_{\mathbb{V}_{k_i} \otimes \mathcal{H}} P_{N_i} = N_i. \quad (7.4)$$

From (7.3) it follows that the operator $T_{N_i}$ has a trivial kernel, while the kernel of $T_{N_i}^\dagger$ is the $N_i$-dimensional subspace of $\mathbb{V}_{k_i} \otimes \mathcal{H}$ corresponding to the range of $P_{N_i}$. Thus

$$\text{dim ker} T_{N_i} = 0 \quad \text{but} \quad \text{dim ker} T_{N_i}^\dagger = N_i. \quad (7.5)$$

Substituted into (6.10) this ansatz yields the gauge field strength

$$F_{ab}^i = \theta_{ab} P_{N_i} = \frac{1}{2 \theta^2} \delta_{ab} P_{N_i} \quad \text{and} \quad F_{ab}^i = 0 = F_{ab}^i, \quad (7.6)$$

while from (6.11) one finds the covariant derivatives

$$D_a \phi_{i+1} = 0 = D_a \phi_{i+1}. \quad (7.7)$$

Thus our ansatz describes holomorphic fields, and the projector $P_{N_i}$ defines a noncommutative gauge field configuration of rank $N_i$ and constant curvature in the subspace $\text{ker} T_{N_i}^\dagger \subset \mathbb{V}_{k_i} \otimes \mathcal{H}$. In particular, the Higgs fields $\phi_{i+1}$ are covariantly constant with

$$\phi_{\dagger_i} \phi_i = |\alpha_i|^2 (1_{k_i} - P_{N_i}) \quad \text{and} \quad \phi_{i+1} \phi_{\dagger_i+1} = |\alpha_{i+1}|^2 (1_{k_i} - P_{N_i}) \quad (7.8)$$

The ranks $N_i$ are generically non-negative integers. If some $N_i = 0$, then we should formally set $P_{N_i} = 0$, $T_{N_i} = 1$ and $\phi_{i+1} = \alpha_{i+1}$ in the $i$-th component of the ansatz. Then

$$X_a^i = \theta_{ab} z^b \quad \text{and} \quad X_\alpha^i = \theta_{ab} z^b \quad (7.9)$$

which leads to the vacuum gauge field configuration

$$A^i = 0 \quad \text{and} \quad F^i = 0. \quad (7.10)$$
These matter fields correspond to open strings with one end on a D-brane and the other end on the closed string vacuum.

Our ansatz has a natural interpretation in quiver gauge theory. Consider the module

$$\mathcal{I} := \bigoplus_{i=0}^{m} \ker T_{N_i}^\dagger \text{ with } \ker \mathcal{I} = \sum_{i=0}^{m} N_i \bar{e}_i \quad (7.11)$$

over the quiver $A_{m+1}$, which is a finite-dimensional submodule of the infinite-dimensional representation $\mathcal{V} \otimes \mathcal{H}$ of $A_{m+1}$ given by the noncommutative quiver bundle. Let us fix an integer $0 \leq s \leq m$, and take $N_i \neq 0$ for all $i \leq s$ and $N_i = 0$ for all $i > s$. The quiver representation (7.11) is a combination of the indecomposable projective representations $P_i$ of $A_{m+1}$ that we encountered in Section 5. The $P_i$’s form a complete set of projective representations in the sense that any quiver representation has a projective resolution in terms of sums of them [37]. In particular, the canonical Ringel resolution of (7.11) is given by the exact sequence

$$0 \rightarrow \bigoplus_{i=1}^{s} P_{i-1} \otimes \ker T_{N_i}^\dagger \rightarrow \bigoplus_{i=0}^{s} P_i \otimes \ker T_{N_i}^\dagger \rightarrow \mathcal{I} \rightarrow 0 \quad (7.12)$$

Solving the Yang-Mills equations. We shall now demonstrate that the field configurations (7.1)–(7.3) yield solutions of the full Yang-Mills equations on $\mathbb{R}^{2n} \times \mathbb{C}P^1$ for any values of $m$, $N_0, N_1, \ldots, N_m$ and $\alpha_1, \ldots, \alpha_m$. For this, we write the ansatz in the form

$$A_a - \theta_{ab} \bar{z}^b = \sum_{i=0}^{m} X_i \otimes \Pi_i = \theta_{ab} \sum_{i=0}^{m} T_{N_i} \bar{z}^b T_{N_i}^\dagger \otimes \Pi_i , \quad (7.13)$$

$$A_a - \theta_{ab} \bar{z}^b = \sum_{i=0}^{m} \bar{X}_i \otimes \Pi_i = \theta_{ab} \sum_{i=0}^{m} T_{N_i} \bar{z}^b T_{N_i}^\dagger \otimes \Pi_i . \quad (7.14)$$

We also have

$$A_i^j = \frac{(m-2i) \bar{y}^i}{2} \frac{1}{R^2 + y \bar{y}} 1_{k_i} , \quad (7.15)$$

$$A_i^j = -\frac{(m-2i) y^i}{2} \frac{1}{R^2 + y \bar{y}} 1_{k_i} , \quad (7.16)$$

$$A_i^{j+1} = \frac{R}{R^2 + y \bar{y}} \phi_{i+1} T_{N_i} T_{N_{i+1}}^\dagger , \quad (7.17)$$

$$A_i^{j+1} = -\frac{R}{R^2 + y \bar{y}} \phi_{i+1} \bar{T}_{N_{i+1}} T_{N_i}^\dagger , \quad (7.18)$$

with

$$A_i^{j} = 0 = A_i^{i+1} \quad \text{for } j \neq i, i+1 . \quad (7.19)$$

Thus for the ansatz (7.1)–(7.3) the field strength tensor is given by

$$F_{ab} = \theta_{ab} \sum_{i=0}^{m} P_{N_i} \otimes \Pi_i \quad , \quad (7.20)$$

$$F_{yy} = -\frac{R^2}{(R^2 + y \bar{y})^2} \sum_{i=0}^{m} \left( m - 2i + (|\alpha_i|^2 - |\alpha_{i+1}|^2) (1_{k_i} - P_{N_i}) \right) \otimes \Pi_i , \quad (7.21)$$
with all other components of $F_{\hat{\mu}\hat{\nu}}$ vanishing.

Let us now insert these expressions into the Yang-Mills equations (2.8) (for static configurations with $A_0 = 0$). It is enough to consider the cases $\hat{\nu} = c$ and $\hat{\nu} = \bar{y}$, since the cases $\hat{\nu} = \bar{c}$ and $\hat{\nu} = y$ can be obtained by hermitean conjugation of (2.8) due to the anti-hermiticity of $A_{\hat{\mu}}$ and $F_{\hat{\mu}\hat{\nu}}$. For $\hat{\nu} = c$, eq. (2.8) becomes

$$\delta^a \delta^b (\partial_c F_{ab} + [A_c, F_{ab}]) = 0$$

which is equivalent to

$$\delta^a \delta^b [A_c - \theta_{cb} z^b, F_{ab}] = 0.$$  \hspace{1cm} (7.23)

Substituting (7.14) and (7.20), we see that (7.23) is satisfied due to the identities (5.16) and (7.24).

In the case $\hat{\nu} = \bar{y}$, eq. (2.8) simplifies to

$$\partial_y (\sqrt{g} F_{y\bar{y}}) + \sqrt{g} [A_y, F_{y\bar{y}}] = 0$$

with $\sqrt{g} = 2 R^4/(R^2 + y\bar{y})^2$. Substituting (7.15), (7.18), (7.19) and (7.21), we find that (7.25) is also satisfied due to the identities (5.16) and (7.24). Hence, the Yang-Mills equations on $\mathbb{R}^{2n} \times \mathbb{C}P^1$ are solved by our choice of ansatz.

**Finite-energy solutions.** The arbitrary coefficients $\alpha_i \in \mathbb{C}$ can be fixed (up to a phase) by demanding that the solution (7.1)–(7.3) yield finite-energy field configurations. For this, we evaluate the energy functional (5.2) using (6.8). From (7.6) we may compute

$$\left( F_{\mu\nu}^i \right)^\dagger \left( F^{i\mu\nu} \right) = 8 \delta^{ac} \delta^{db} F_{a\bar{b}}^i F_{d\bar{c}}^i = 2 \left( \sum_{a=1}^n \frac{1}{(\theta^a)^2} \right) P_{N_i},$$  \hspace{1cm} (7.26)

and combining this with (7.7) and (7.8) we find the noncommutative Yang-Mills energy

$$E_{YM} = 2\pi R^2 \left( \prod_{a=1}^n 2\pi \theta^a \right) \sum_{i=0}^m \text{Tr}_{\mathcal{K}_{k_i} \otimes \mathcal{H}} \left[ \left( \sum_{b=1}^n \frac{1}{(\theta^b)^2} \right) P_{N_i} \right]$$

$$+ \frac{1}{4 R^4} \left( m - 2i + (|\alpha_i|^2 - |\alpha_{i+1}|^2) \left( 1_{k_i} - P_{N_i} \right) \right)^2.$$  \hspace{1cm} (7.27)

Because of the trace over the infinite-dimensional Fock space $\mathcal{H}$, the constant terms in (7.27) which are not proportional to the projectors $P_{N_i}$ must all vanish in order for the energy to be finite. This leads to the finite-energy conditions

$$m - 2i + |\alpha_i|^2 - |\alpha_{i+1}|^2 = 0$$  \hspace{1cm} (7.28)

for each $i = 0, 1, \ldots, m$.

With $\alpha_0 = \alpha_{m+1} = 0$, the constraints (7.28) are solved by

$$|\alpha_{i+1}|^2 = (i + 1) m - 2 \sum_{j=0}^i j = (i + 1) (m - i)$$  \hspace{1cm} (7.29)

and the energy (7.27) can thereby be written as

$$E_{YM} = 2\pi R^2 \left( \prod_{a=1}^n 2\pi \theta^a \right) \sum_{i=0}^m (N_i + N_{m-i}) \left[ \left( \sum_{b=1}^n \frac{1}{(\theta^b)^2} \right) + \frac{(m - 2i)^2}{4 R^4} \right].$$  \hspace{1cm} (7.30)
The original noncommutative DUY equations are fixed by the positive integers as a stable bound state (i.e. a vortex-like solution on ansatz (3.10)–(3.14) and (7.1)–(7.3) is labelled by the collection of positive integers (s

For s > 0 the conditions (7.31) are incompatible with one another, implying that the ansatz (7.1)–(7.3) with s > 0 does not allow for BPS configurations. For s = 0, the equation (7.31) relates the radius R of the sphere to the noncommutativity parameters θ

The configurations with i > 0 correspond to the vacuum gauge fields (7.10) with trivial bundle maps φ

The BPS conditions (6.12)–(6.14) force us to take k1 = \cdots = km corresponding to the gauge symmetry breaking U(k) → U(k0) × U(k1)m, so that r = k1, k0 + mk1 = k with k0 > 0 and k1 > 0. The configurations with i > 0 correspond to the vacuum gauge fields (7.10) with trivial bundle maps φ

These solutions have a natural physical interpretation along the lines described in Section 4. The original noncommutative DUY equations are fixed by the positive integers n and k. Our ansatz (3.10)–(3.14) and (7.1)–(7.3) is labelled by the collection of positive integers (m,k_i,N_i) with i = 0,1,...,s. According to the standard identification of D-branes as noncommutative solitons [42], the configuration (7.32,7.33) with s = 0 describes a collection of m N_0 BPS D0-branes as a stable bound state (i.e. a vortex-like solution on \mathbb{R}_0^{2n} \times \mathbb{C}P^1) in a system of D0-branes (vortices) in a D(2n)-brane-antibrane system, because deg \mathcal{L}^{m-2i}m = m - 2i for each i = 0,1,...,s. Again they form a system of spherical D2-branes (i.e. an SU(2)-symmetric multi-instanton) in the initial brane-antibrane system on \mathbb{R}_0^{2n} \times \mathbb{C}P^1. Their orientation depends on the sign of the magnetic charge m - 2i
for each $i = 0,1,\ldots,s$, which determines whether we have D2-branes or D2-antibranes. If more than one $N_i \neq 0$ then the ansatz either describes pairs of D0-branes with overall non-vanishing monopole charges, or both D0-branes and anti-D0-branes. Such systems cannot be stable, i.e. the corresponding configuration (7.1)–(7.3) cannot satisfy the noncommutative vortex and DUY equations.

The distinction between BPS versus non-BPS solutions is very natural in quiver gauge theory. The BPS configurations are described by the simple Schur representations $\mathbf{l}_i$, $i = 0,1,\ldots,m$ of the $A_{m+1}$ quiver given by a one-dimensional vector space at vertex $i$ with all maps equal to 0, i.e. the $A_{m+1}$-module with $(\mathbf{l}_i)_{m-2j} = \delta_{ij}$ $C$ and dimension vector $\vec{k}_{\mathbf{l}_i} = \vec{e}_i$. The BPS states constructed above then correspond to the quiver representations $(\mathbf{l}_0)^{\oplus \delta_{0}}>$. Together with the projective modules $\mathcal{P}_i$, the Schur modules $\mathbf{l}_i$ admit the projective resolutions

$$0 \rightarrow \mathcal{P}_0 \rightarrow \mathbf{l}_0 \rightarrow 0 \ ,$$

$$0 \rightarrow \mathcal{P}_{i-1} \rightarrow \mathcal{P}_i \rightarrow \mathbf{l}_i \rightarrow 0 \quad \text{for} \quad i = 1,\ldots,s$$

and satisfy the relations [37]

$$\text{Hom}(\mathbf{l}_i, \mathbf{l}_j) = \delta_{ij} \ C \ = \ \text{Hom}(\mathcal{P}_i, \mathbf{l}_j) \ .$$

(7.37)

The resolutions (7.12) and (7.35,7.36) exhibit a sharp homological distinction between BPS and non-BPS solutions. The constituent D-branes at the vertices of the quiver $A_{m+1}$ are associated with the basic representations $\mathbf{l}_i$; Sums $(\mathbf{l}_i)^{\oplus N_i}$ for fixed $i$ correspond to BPS states, associated generally with the symmetry breaking $U(k) \rightarrow U(k_1) \times U(k_{i+1})$ $m$, which are constructed analogously to (7.32,7.33) but with the vacuum Higgs configurations $\phi_j = \alpha_j 1_{k_i}$ for $j < i$ and $\phi_j = \alpha_j 1_{k_{i+1}}$ for $j > i$. A generic non-BPS state, associated to the quiver representation (7.11), corresponds to the decay of the original SU(2)-symmetric branes wrapped on $\mathbb{R}^2_{\theta} \times \mathbb{C}P^1$ into the collection of constituent branes $(\mathbf{l}_0)^{\oplus \delta_{0}} \oplus (\mathbf{l}_1)^{\oplus \delta_{1}} \oplus \cdots \oplus (\mathbf{l}_s)^{\oplus \delta_{s}}$ in $\mathbb{R}^2_{\theta}$. For $s > 0$ this collection is unstable. In the quiver gauge theory, we have thereby arrived at a natural construction of the unstable D-brane configurations in terms of stable BPS states of D-branes, which may be succinctly summarized through the sequence of distinguished triangles of quiver representations

$$(\mathbf{l}_0)^{\oplus \delta_{0}} = \mathcal{T}_0 \quad \rightarrow \quad \mathcal{T}_1 \quad \rightarrow \cdots \quad \mathcal{T}_{m-1} \quad \rightarrow \quad \mathcal{T}_m = \mathcal{T}$$

$$(\mathbf{l}_1)^{\oplus \delta_{1}} \quad \quad \cdots \quad \quad (\mathbf{l}_s)^{\oplus \delta_{s}}$$

(7.38)

where $\mathcal{T}_s := \bigoplus_{i=0}^s \ker T_{N_i} \rightarrow \ker T_{N_0}^{\dagger} \oplus T_{s-1}$ and the horizontal maps are the canonical inclusions of submodules. This exact sequence expresses the fact that, for each $s = 1,\ldots,m$, the non-BPS module $\mathcal{T}_s$ is an extension of the BPS module $(\mathbf{l}_s)^{\oplus \delta_{s}}$ by the non-BPS module $\mathcal{T}_{s-1}$.

8 Generalized Atiyah-Bott-Shapiro construction

In this section we shall construct an explicit realization of the basic partial isometry operators $T_{N_i}$ which will be particularly useful for putting the D-brane interpretation of our noncommutative multi-instanton solutions on firmer ground. It is based on an SU(2)-equivariant generalization of the (noncommutative) Atiyah-Bott-Shapiro (ABS) construction of tachyon field configurations [9]–[11].

**Equivariant ABS construction.** If $G$ is a group and $\mathcal{C}l_{2n} := \mathcal{C}l(\mathbb{R}^{2n})$, we denote by $\mathcal{R}_{\text{Spin}}(\mathbb{C}l_{2n})$ the Grothendieck group of isomorphism classes of finite-dimensional $\mathbb{Z}_2$-graded $G \times \mathcal{C}l_{2n}$ modules,
i.e. Clifford modules possessing an even ($\mathbb{Z}_2$-degree preserving) $G$-action which commutes with the $\text{Cl}_{2n}$-action. More precisely, we consider representatives of $\mathbb{C}[G] \otimes \text{Cl}_{2n}$ with $\mathbb{C}[G]$ the group ring of $G$. The inclusion $i(2n) : \text{Cl}_{2n} \hookrightarrow \text{Cl}_{2n+1}$ of Clifford algebras induces a restriction map

$$i_G(2n)^* : R_{\text{Spin}}^G(2n+1) \to R_{\text{Spin}}^G(2n)$$

(8.1)
on equivariant Clifford modules. Following the standard ABS construction [43], we may then obtain the $G$-equivariant K-theory $K_G(\mathbb{R}^{2n})$ (with compact support) through the descendent isomorphism

$$K_G(\mathbb{R}^{2n}) = \text{coker } i_G(2n)^* = R_{\text{Spin}}^G(2n)/i_G(2n)^* R_{\text{Spin}}^G(2n+1) \quad \text{.}$$

(8.2)
The image of $i_G(2n)^*$ in $R_{\text{Spin}}^G(2n)$ contains classes of Clifford modules $[V]$ which admit a $G \times \text{Cl}_{2n}$-equivariant involution $\bar{V} \cong V^\vee$, where $V^\vee$ is the Clifford module $V$ with its $\mathbb{Z}_2$-parity reversed.

In our case, we take $G = U(1) \subset \text{SU}(2)$ acting trivially on $\mathbb{R}^{2n}$, and thereby consider $U(1) \times \text{Cl}_{2n}$-modules with the $U(1)$-action commuting with the Clifford action. Any such module is a direct sum of tensor products of a $U(1)$-module and a spinor module, and hence

$$R_{\text{Spin}}^{U(1)}(2n) = R_{\text{Spin}}(2n) \otimes R_{U(1)} \quad \text{and } \quad i_{U(1)}(2n)^* = i(2n)^* \otimes 1 \quad \text{.}$$

(8.3)

Since from the standard ABS construction one has the isomorphism [43]

$$K(\mathbb{R}^{2n}) = \text{coker } i(2n)^* = R_{\text{Spin}}(2n)/i(2n)^* R_{\text{Spin}}(2n+1)$$

(8.4)
of abelian groups, we can reduce (8.2) for $G = U(1)$ to the isomorphism

$$K_{U(1)}(\mathbb{R}^{2n}) = K(\mathbb{R}^{2n}) \otimes R_{U(1)}$$

(8.5)
of $R_{U(1)}$-modules, where $K(\mathbb{R}^{2n}) \cong \mathbb{Z}$ (Note that the isomorphism $K_{U(1)}(\mathbb{R}^{2n}) \cong R_{U(1)}$ also follows from the fact that $\mathbb{R}^{2n}$ is equivariantly contractible to a point). We may describe the isomorphism (8.5) along the lines explained in Section 4. In particular, the spinor module $\Delta_{2n} := \Delta(\mathbb{R}^{2n})$ admits the isotopical decomposition

$$\Delta_{2n} = \bigoplus_{i=0}^m \Delta_i \otimes S_{m-2i} \quad \text{with } \quad \Delta_i = \text{Hom}_{U(1)}(S_{m-2i}; \Delta_{2n})$$

(8.6)

obtained by restricting $\Delta_{2n}$ to representations of $U(1) \subset \text{Spin}(2n) \subset \text{Cl}_{2n}$. The $\Delta_i$’s in (8.6) are the corresponding multiplicity spaces.

The most instructive and useful way to explicitly realize the decomposition (8.6) is to use the equivariant excision theorem (4.5) directly and consider the SU(2)-invariant dimensional reduction of spinors from $\mathbb{R}^{2n} \times \mathbb{C}P^1$ to $\mathbb{R}^{2n}$. For this, we introduce the twisted Dirac operator on $\mathbb{R}^{2n} \times \mathbb{C}P^1$ using the graded connection formalism of Section 5 to write the $\mathbb{Z}_{m+1}$-graded Clifford connection

$$\tilde{D} := \Gamma^\mu D_\mu = \gamma^\mu D_\mu \otimes 1 + (\phi_{(m)}) \gamma \otimes \gamma^\gamma \beta_\gamma - (\phi_{(m)})^\dagger \gamma \otimes \gamma^\gamma \beta_\gamma + \gamma \otimes \mathcal{D}_{\mathbb{C}P^1} \quad ,$$

(8.7)

where

$$\mathcal{D}_{\mathbb{C}P^1} := \gamma^\mu D_\mu + \gamma^\gamma D_\gamma = \gamma^\mu \left( \partial_\mu + \omega_\mu + (a^{(m)})_\mu \right) + \gamma^\gamma \left( \partial_\gamma + \omega_\gamma + (a^{(m)})_\gamma \right)$$

(8.8)

and $\omega_\gamma, \omega_\gamma$ are the components of the Levi-Civita spin connection on the tangent bundle of $\mathbb{C}P^1$. From (8.7) we see that the monopole charges $m-2i$ in the Yang-Mills energy functional (5.2) can be understood as originating from the Dirac operator (8.8) on $\mathbb{C}P^1$. The operator (8.7) acts on spinors $\Psi$ which are sections of the bundle

$$\Psi = \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} \in \bigoplus_{i=0}^m (E_{k_i} \otimes \Delta_{2n}) \otimes (L_{m-2i+1}^{m-2i-1})$$

(8.9)
over $\mathbb{R}^{2n} \times \mathbb{C}P^1$, where $\mathcal{L}^{m-2i+1} \oplus \mathcal{L}^{m-2i-1}$ are the twisted spinor bundles of rank 2 over the sphere $\mathbb{C}P^1$. We are therefore interested in the twisted spinor module $\Delta \mathbb{R}^{2n} \times \mathbb{C}P^1 \otimes \mathbb{C}l$ which is the product of the spinor module $\Delta \mathbb{R}^{2n} \otimes \Delta (\mathbb{C}P^1)$ with the fundamental representation $\mathcal{C}(\mathbb{R}^{2n} \times \mathbb{C}P^1)$ of the gauge group $U(k)$ broken as in (3.9).

The symmetric fermions on $\mathbb{R}^{2n}$ that we are interested in correspond to $SU(2)$-invariant spinors on $\mathbb{R}^{2n} \times \mathbb{C}P^1$. They belong to the kernel of the Dirac operator (8.8) on $\mathbb{C}P^1$ and will be massless on $\mathbb{R}^{2n}$. One can write

$$\mathcal{D}_{\mathbb{C}P^1} = \bigoplus_{i=0}^{m} \mathcal{D}_{m-2i} = \bigoplus_{i=0}^{m} \left( \begin{array}{cc} 0 & \mathcal{D}_{m-2i}^- \\ \mathcal{D}_{m-2i}^+ & 0 \end{array} \right),$$

where

$$\mathcal{D}_{m-2i}^+ = \frac{1}{R^2} \left[ \left( R^2 + y \bar{y} \right) \partial_y - \frac{1}{2} (m - 2i + 1) y \right],$$

$$\mathcal{D}_{m-2i}^- = -\frac{1}{R^2} \left[ \left( R^2 + y \bar{y} \right) \partial_y + \frac{1}{2} (m - 2i - 1) y \right].$$

The operator (8.10) acts on sections of the bundle (8.9) which we write with respect to this decomposition as

$$\Psi = \bigoplus_{i=0}^{m} \left( \psi_{m-2i}^+ \psi_{m-2i}^- \right),$$

where $\psi_{m-2i}^\pm$ are sections of $\mathcal{L}^{m-2i \pm 1}$ taking values in $\mathcal{D}_{m-2i} \otimes \mathcal{V}_{k_i}$ with coefficients depending on $x \in \mathbb{R}^{2n}$.

To describe the kernel of the Dirac operator (8.10), we need to solve the differential equations

$$\mathcal{D}_{m-2i}^+ \psi_{(m-2i)}^+ = 0 \quad \text{and} \quad \mathcal{D}_{m-2i}^- \psi_{(m-2i)}^- = 0$$

for the positive and negative chirality spinors $\psi_{(m-2i)}^+$ and $\psi_{(m-2i)}^-$ in $\ker \mathcal{D}_{m-2i}^+$ and $\ker \mathcal{D}_{m-2i}^-$. By recalling the form of the transition functions for the monopole bundles from Section 3, one easily sees that the only solutions of these equations which are regular on both the northern and southern hemispheres of $S^2$ are of the form

$$\psi_{(m-2i)}^+ = \frac{1}{(R^2 + y \bar{y})^{t_i/2}} \sum_{\ell=0}^{t_i} \psi_{(m-2i)}^\ell(x) y^\ell \quad \text{and} \quad \psi_{(m-2i)}^- = 0 \quad \text{for} \quad m - 2i < 0$$

and

$$\psi_{(m-2i)}^- = \frac{1}{(R^2 + y \bar{y})^{t_i/2}} \sum_{\ell=0}^{t_i} \psi_{(m-2i)}^\ell(x) \bar{y}^\ell \quad \text{and} \quad \psi_{(m-2i)}^+ = 0 \quad \text{for} \quad m - 2i > 0 .$$

Here $t_i = |m - 2i| - 1$ and the component functions $\psi_{(m-2i)}^\ell(x)$ on $\mathbb{R}^{2n}$ with $\ell = 0, 1, \ldots, t_i$ form the irreducible representation $\mathcal{V}_{t_i+1}$ of the group $SU(2)$. Thus the chirality grading is by the sign of the magnetic charges.

This analysis is valid when the monopole charge $m$ is an even or odd integer. However, when $m$ is even there is precisely one term in (8.9) with $m = 2i$ for which the sub-bundle $\mathcal{E}_k \mathcal{W} \to \mathbb{R}^{2n}$ is twisted by the ordinary spinor bundle $\mathcal{L} \oplus \mathcal{L}^\vee \to \mathbb{C}P^1$ of vanishing magnetic charge. This bundle admits an infinite-dimensional vector space of symmetric $L^2$-sections comprised of spinor harmonics $\Psi_{lq} \in \mathbb{C}^2$ with $l \in \mathbb{N}_0 + \frac{1}{2}$, $q \in \{-l, -l+1, \ldots, l-1, l\}$ and $\mathcal{D}_0 \Psi_{lq} \neq 0$ [44]. The spectrum
of the (untwisted) Dirac operator $\mathcal{D}_D$ consists of the eigenvalues $\pm (l + \frac{1}{2})$, each of even multiplicity $p + 1 = 2l + 1$. After dimensional reduction, this produces an infinite tower of massive spinors on $\mathbb{R}^{2n}$, and such fermions of zero magnetic charge have no immediate interpretation in the present context. However, one has dimker $\mathcal{D}_D = 0$, and this will be enough for our purposes. We will therefore fix one of these vector spaces, such that after integration over $\mathbb{C}P^1$ it corresponds to the space

$$H_p \cong \mathbb{C}^2 \otimes \mathbb{C}^{p+1} \quad \text{with} \quad p = 1, 3, 5, \ldots \quad (8.17)$$

All of our subsequent results will be independent of the particular choice of eigenspace $(8.17)$.

We have thereby shown that the SU(2)-equivariant reduction of the twisted spinor representation of $Cl(\mathbb{R}^{2n} \times \mathbb{C}P^1)$ decomposes as a $\mathbb{Z}_2$-graded bundle giving

$$\Delta_{\mathcal{V}}(\mathbb{R}^{2n} \times \mathbb{C}P^1)^{SU(2)} = \Delta_{2n} \otimes \left( \Delta_{\mathcal{V}}^+ \oplus \Delta_{\mathcal{V}}^- \right) \quad \text{for } m \text{ odd ,}$$

where

$$\Delta_{\mathcal{V}}^+ = \bigoplus_{i=m_+}^m V_{k_i} \otimes V_{|m-2i|} \quad \text{and} \quad \Delta_{\mathcal{V}}^- = \bigoplus_{i=0}^{m_-} V_{k_i} \otimes V_{m-2i} \quad (8.19)$$

with $m_+ = \lfloor \frac{m+1}{2} \rfloor$ and $m_- = \lfloor \frac{m-1}{2} \rfloor$. When $m$ is an even integer, one should also couple the eigenspace $(8.17)$ giving

$$\Delta_{\mathcal{V}}(\mathbb{R}^{2n} \times \mathbb{C}P^1)^{SU(2)} = \Delta_{2n} \otimes \left( \Delta_{\mathcal{V}}^+ \oplus \left( V_{2n} \otimes H_p \right) \oplus \Delta_{\mathcal{V}}^- \right) \quad \text{for } m \text{ even } (8.20)$$

with $m_+ = \lfloor \frac{m+3}{2} \rfloor$ and $m_- = \lfloor \frac{m-1}{2} \rfloor$. It remains to work out the corresponding action of Clifford multiplication

$$\mu_{\mathcal{V}} : \Delta_{\mathcal{V}}^+ \rightarrow \Delta_{\mathcal{V}}^+ \quad (8.21).$$

For this, we recall from Section 4 that the action of the generators of the parabolic subgroup $P \subset SL(2, \mathbb{C})$ on the equivariant decomposition $(8.18,8.19)$ is given by $\sigma_3(V_{k_i} \otimes V_{|m-2i|}) = (m - 2i)(V_{k_i} \otimes V_{|m-2i|})$ and $\sigma_+ : V_{k_i} \otimes V_{|m-2i|} \rightarrow V_{k_{i+1}} \otimes V_{|m-2i|}$. Since the Clifford action is required to commute with this action, the map $(8.21)$ is thereby uniquely fixed on the isotopical components in the form

$$\mu_{\mathcal{V}} \circ \Pi_i : V_{k_i} \otimes V_{|m-2i|} \rightarrow V_{k_{i+1}} \otimes V_{|m-2i|} \quad \text{for} \quad i = 0, 1, \ldots, m_-. \quad (8.22)$$

Furthermore, since $\sigma_3(\mathcal{H}_p) = 0$ for all $p$, the space of spinor harmonics must lie in the kernel of the Clifford map and one has

$$\mu_{\mathcal{V}} \circ \Pi_{m_+} = 0 \quad \text{for } m \text{ even .} \quad (8.23)$$

It is also illuminating to formulate this equivariant dimensional reduction from a dynamical point of view, as we did for the gauge fields in Section 5. Using the gauged Dirac operator $(8.7)$ we may define a fermionic energy functional on the space of sections of the bundle $(8.9)$ by

$$E_D := \int_{\mathbb{R}^{2n} \times \mathbb{C}P^1} \sqrt{g} \Psi^\dagger \mathcal{D} \Psi \quad (8.24)$$

One has

$$\Psi^\dagger \left( (\phi_0) \otimes \sigma^y - (\phi_0)^\dagger \otimes \sigma^y \right) \Psi = \left( (\Psi^+) \otimes (\Psi^-) \right) \left( (\phi_0)^\dagger (\Psi^-) \right) \left( (\phi_0) (\Psi^+) \right) \quad (8.25)$$
Substituting (8.10)–(8.16), we see that (8.25) vanishes on symmetric spinors and after integration over $\mathbb{C}P^1$ the energy functional (8.24) for $m$ odd becomes

$$
E_D = 4\pi R^2 \int_{\mathbb{R}^{2n}} d^{2n} x \left[ \sum_{i=m_+}^m \sum_{\ell=0}^{[m-2i]-1} (\psi^+_{(m-2i) \ell})^\dagger \gamma^\mu D^\mu (\psi^+_{(m-2i) \ell}) + \sum_{i=0}^{m-1} \sum_{\ell=0}^{m-2i-1} (\psi^-_{(m-2i) \ell})^\dagger \gamma^\mu D^\mu (\psi^-_{(m-2i) \ell}) \right].
$$

(8.26)

The symmetric fermion energy functional for $m$ even also contains mass terms for fermions of vanishing magnetic charge which are proportional to the multiplicity $(p+1)$ of the spinor harmonics.

**Explicit form of the operators $T_{N_i}$.** The operators $T_{N_i}$ parametrizing the solutions of the previous section may be realized explicitly by appealing to a noncommutative version of the above construction. For this, we first note that the (trivial) action of $U(1) \subset SU(2)$ on $\mathbb{R}^{2n}$ induces an action on functions $f$ on $\mathbb{R}^{2n}$ by $(\zeta \cdot f)(x) := f(\zeta^{-1} \cdot x)$ for $\zeta \in U(1)$. This in turn defines an action of $U(1)$ on the noncommutative space $\mathbb{R}_b^{2n}$ through automorphisms $\hat{f} \mapsto \hat{\zeta} \hat{f}$ of the Weyl operator algebra, i.e. a representation of $U(1)$ in the automorphism group of the algebra. We will assume that the Fock space (6.4) carries a unitary representation of $U(1)$. We can then decompose it into its isotopical components in the usual way as

$$
\mathcal{H} = \bigoplus_{i=0}^m \mathcal{H}_i \otimes S_{m-2i}.
$$

(8.27)

For $\zeta \in U(1)$ we denote the corresponding unitary operator on $\mathcal{H}$ by $\hat{\zeta}$. If we demand that the representations of $U(1)$ above are covariant with respect to each other [45],

$$
\hat{\zeta} \hat{f} \hat{\zeta}^{-1} = \hat{\zeta} \cdot \hat{f} \ ,
$$

(8.28)

then they define a representation of the crossed-product of the algebra of Weyl operators with the group $U(1)$. This defines the (trivial) noncommutative $U(1)$-space $\mathbb{R}_b^{2n} \rtimes U(1)$, and equivariant field configurations are operators belonging to the commutant of $U(1)$ in the crossed-product algebra. In quiver gauge theory, the pertinent representation of $A_{m+1}$ thus labels isotopical components of the Hilbert space of the noncommutative gauge theory. Since the $U(1)$-action is trivial here, the isotopical components of the Fock space (8.27) are given by $\mathcal{H}_i \cong \mathcal{H}$ for each $i = 0, 1, \ldots, m$. Note that one has an isomorphism $(\mathcal{H})^{\oplus (m+1)} \cong \mathcal{H}$ by the usual Hilbert hotel argument.

We will now construct a representation on (8.27) of the partial isometry operators $T_{N_i}$ in $\mathbb{R}_b^{2n} \rtimes U(1)$. For this, let us put $r := 2^{n-1}$ and consider the operators [9]

$$
\Sigma = (\sigma \cdot x)^\dagger = \frac{1}{\sqrt{(\sigma \cdot x)(\sigma \cdot x)^\dagger}} \quad \text{and} \quad \Sigma^\dagger = \frac{1}{\sqrt{(\sigma \cdot x)(\sigma \cdot x)^\dagger}} (\sigma \cdot x) ,
$$

(8.29)

where $\sigma \cdot x := \sigma^\mu x^\mu$, $\mu, \nu, \lambda = 1, \ldots, 2n$ and the $r \times r$ matrices $\sigma_{\mu}$ are subject to the anticommutation relations

$$
\sigma^\mu^\dagger \sigma^\nu + \sigma^\nu^\dagger \sigma^\mu = 2 \delta^\mu_\nu 1_r = \sigma^\mu \sigma^\nu^\dagger + \sigma^\nu \sigma^\mu^\dagger .
$$

(8.30)

Eq. (8.30) implies that the matrices

$$
\gamma_\mu = \begin{pmatrix} 0 & \sigma^\mu \\
-\sigma^\mu & 0 \end{pmatrix} \quad \text{with} \quad \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = -2 \delta^\mu_\nu 1_{2r}
$$

(8.31)
generate the Clifford algebra $C\ell_{2n}$. Note that for $n = 1$ we have $r = 1$, $\sigma_1 = 1$ and $\sigma_2 = i$, which yields

$$\Sigma^\dagger = \frac{1}{\sqrt{z^2 - 2}} z^1 = \sum_{\ell=1}^\infty |(\ell - 1)\ell\rangle\langle \ell|$$

(8.32)

and we obtain the standard shift operator $(\Sigma)^N$ on the Fock space $\mathcal{H}$ in this case. Generally, the operators (8.29) obey

$$(\Sigma^\dagger)^{Ni} (\Sigma)^{Ni} = 1_r \quad \text{and} \quad (\Sigma)^{Ni} (\Sigma^\dagger)^{Ni} = 1_r - P_{Ni}^0,$$

(8.33)

where $P_{Ni}$ is a projector of rank $N_i$ on the vector space $\Delta^+_n \otimes \mathcal{H}$, and $\Delta^\pm_n \cong \mathbb{C}^r$ are the irreducible chiral spinor modules of dimension $r = 2^{n-1}$ (with $\Delta_{2n} = \Delta^+_n \oplus \Delta^-_n$) on which the matrices $\sigma_\mu$ act.

The partial isometry operators $(\Sigma)^{Ni}$ in $\mathbb{R}^{2n}_+$ do not act on the isotopical decomposition (8.27) and thus do not properly incorporate the SU(2)-equivariant reduction of the original system of D-branes. The desired operators $T_{Ni}$ in $\mathbb{R}^{2n}_+ \times \text{U}(1)$ are obtained by first projecting these partial isometries onto constituent brane subspaces. With $\Pi_i$ the rank 1 projector onto the $i$-th isotopical component in (8.27), we thereby define the $r \times r$ matrices

$$T_{Ni}^{(0)} = 1_r \otimes (1 - \Pi_i) + (\Sigma)^{Ni} \otimes \Pi_i.$$  

(8.34)

The operator $T_{Ni}^{(0)}$ acts as the shift operator $(\Sigma)^{Ni}$ on $\mathcal{H}_i$ and as the identity operator $1_r$ on $\mathcal{H}_j$ for all $j \neq i$. It is easy to see that these matrices satisfy the equations

$$(T_{Ni}^{(0)})^\dagger (T_{Ni}^{(0)}) = 1_r \quad \text{and} \quad (T_{Ni}^{(0)}) (T_{Ni}^{(0)})^\dagger = 1_r - P_{Ni}^{(0)}$$

with

$$P_{Ni}^{(0)} = P_{Ni} \otimes \Pi_i$$

(8.35)

(8.36)

a projector of rank $N_i$ on the Fock space $\Delta^+_n \otimes \mathcal{H}$. They also satisfy the algebra

$$(T_{Ni}^{(0)})^N = T_{Ni}^{(0)} N_i \quad \text{and} \quad T_{Ni}^{(0)} T_{N_j}^{(0)} = T_{N_i}^{(0)} + T_{N_j}^{(0)} - 1_r = T_{N_j}^{(0)} T_{Ni}^{(0)} \quad \text{for} \quad i \neq j.$$  

(8.37)

The operator (8.34) may be regarded as a linear map

$$T_{Ni}^{(0)} : \Delta^+_n \otimes \mathcal{H} \longrightarrow \Delta^+_n \otimes \mathcal{H}.$$  

(8.38)

In particular, the map $(T_1^{(0)})$ has a trivial kernel, while $(T_1^{(0)})^\dagger$ has a one-dimensional kernel which is spanned by the vector $|\psi\rangle \otimes |0, \ldots, 0\rangle$ where $|\psi\rangle$ denotes the lowest-weight spinor of SO(2n).

Finally, the desired rectangular $k_i \times r$ Toeplitz operators $T_{Ni}$ may be realized in terms of the partial isometries (8.34) by appealing to the Hilbert hotel argument. For this, we introduce a lexicographic ordering $N_0^a \sim N_0$ on the Fock space $\mathcal{H}$ so that $|r_1, \ldots, r_n\rangle = |q\rangle$ with $q = 0, 1, 2, \ldots$, and fix an orthonormal basis $\bar{\rho}_0, \bar{\rho}_1, \ldots, \bar{\rho}_{r-1}$ of the chiral spinor representation space $\Delta^+_n \cong \mathbb{C}^r$. Then $\bar{\rho}_a \otimes |q\rangle$, $a = 0, 1, \ldots, r - 1$ is an orthonormal basis for $\Delta^+_n \otimes \mathcal{H}$ and there is a one-to-one correspondence $\bar{\rho}_a \otimes |q\rangle \leftrightarrow |r q + a\rangle$ of basis states. Similarly, by fixing an orthonormal basis $\tilde{\lambda}^+_a, \tilde{\lambda}^-_1, \ldots, \tilde{\lambda}^{-}_{k_i-1}$ of the SU(2) representation space $V_{k_i} \cong \mathbb{C}^{k_i}$, there is a one-to-one correspondence $\tilde{\lambda}^+_a \otimes |q_i\rangle \leftrightarrow |k_i q_i + a_i\rangle$, $a_i = 0, 1, \ldots, k_i - 1$ for the corresponding orthonormal basis of $V_{k_i} \otimes \mathcal{H}$. Let us now introduce unitary isomorphisms $U_i : \Delta^+_n \otimes \mathcal{H} \rightarrow V_{k_i} \otimes \mathcal{H}$ and $U_i^\dagger : V_{k_i} \otimes \mathcal{H} \rightarrow \Delta^+_n \otimes \mathcal{H}$. 

29
by the formulas

\[ U_i = \sum_{a=0}^{r-1} \sum_{a_i=0}^{k_i-1} \sum_{q,q_i=0}^{\infty} a_i \langle r \ q \ a \ q_i \rangle \]

(8.39)

\[ U_i^\dagger = \sum_{a=0}^{r-1} \sum_{a_i=0}^{k_i-1} \sum_{q,q_i=0}^{\infty} a_i \langle r \ q \ a_i \ q_i \rangle \]

(8.40)

By using the shift operators (8.34), we then define the operators

\[ T_{N_i} = U_i (T_{N_i}^{(0)}) \quad \text{and} \quad T_{N_i}^\dagger = (T_{N_i}^{(0)})^\dagger U_i^\dagger \]

(8.41)

on \( \Delta_{2n}^+ \otimes \mathcal{H} \rightarrow \mathcal{V}_{k_i} \otimes \mathcal{H} \) and \( \mathcal{V}_{k_i} \otimes \mathcal{H} \rightarrow \Delta_{2n}^- \otimes \mathcal{H} \). They satisfy the requisite equations (7.3), with the \( k_i \times k_i \) matrix

\[ P_{N_i} = U_i \left( \mathcal{P}_{N_i} \otimes \Pi_i \right) U_i^\dagger \]

(8.42)

a projector of rank \( N_i \) on the Fock space \( \mathcal{V}_{k_i} \otimes \mathcal{H} \).

Notice that the rank \( r = 2^{n-1} \) used in this construction is an even integer for \( n \geq 2 \). To work with odd ranks \( r \) one may introduce the \( (2^{n-1} + 1) \times (2^{n-1} + 1) \) matrices

\[ T_{N_i}^{(0)} = \left( T_{N_i}^{(0)} - 1 \right) \]

(8.43)

where \( T_{N_i}^{(0)} \) is defined as above and

\[ \Sigma_i' = \sum_{\ell=1}^{\infty} |0, \ldots, 0, \ell, \ell-1, 0, \ldots, 0| \]

(8.44)

is a shift operator on the Fock space \( \mathcal{H} \). Then the operators (8.43) satisfy the equations (8.35) with

\[ P_{N_i}^{(0)} = \left( \mathcal{P}_{N_i} \otimes \Pi_i \right) \]

(8.45)

a projector of rank \( N_i \) on the Fock space \( \mathcal{V}_{k_i} \otimes \mathcal{H} \cong \mathbb{C}^r \otimes \mathcal{H} \), where \( r = 2^{n-1} + 1 \). In this case the Toeplitz operators \( T_{N_i} \) are obtained by substituting (8.43) into (8.41) with the replacement \( \Delta_{2n}^+ \rightarrow \Delta_{2n}^\dagger \). Note also that the partial isometry operator

\[ T^{(0)} := \prod_{i=0}^{m} T_{N_i}^{(0)} \]

\[ = \ 1_r + \sum_{i=0}^{m} (T_{N_i}^{(0)} - 1_r) = 1_r + \sum_{i=0}^{m} (\Sigma^N_i - 1_r) \]

(8.46)
together with the above representations of the U(1) group on the Weyl operator algebra of \( \mathbb{R}^{2n} \) and on the Fock space \( \mathcal{H} \), defines a cycle in the U(1)-equivariant analytic K-homology \( K^a(\mathbb{R}^{2n} \times U(1)) \cong K^a(U(1)(\mathbb{R}^{2n})) \). After a twisting appropriate to the inclusion of the pertinent magnetic monopole bundles, it describes the SU(2)-invariant configurations of D-branes as branes on the (trivial) quotient space \( \mathbb{R}^{2n} / U(1) \). The charge of this class is the same as that of the cocycle \( \{ \Delta_{2n}^+, \Delta_{2n}^{-1} ; \mu_{2n}^+ \} \) built earlier in the topological K-theory (8.5) from the standard ABS brane-antibrane class \( \Delta_{2n} \) which is the generator of (8.4) [9]–[11],[43]. The computation of the topological charge, as well as the equivalence between the commutative (topological) and noncommutative (analytic) K-homology descriptions of the D-brane configurations, will be presented in the next section.

Moduli space of solutions. The realization (8.34) can be generalized in order to introduce \( 2n \sum_{i=0}^{m} N_i \) real moduli into the solution which specify the locations of the various noncommutative solitons in \( \mathbb{R}^{2n} \) [45]. For this, one first has to introduce “shifted ground states” centered at \( (b_{i}^{\mu}) \), \( \ell_i = 1, \ldots, N_i \) for each \( i = 0, 1, \ldots, m \). The operators (8.34) are rewritten as
\[
T_{N_i}^{(0)} = 1_r \otimes (1 - \Pi_i) + (\Sigma_1^i \Sigma_2^i \cdots \Sigma_{N_i}^i) \otimes \Pi_i ,
\]
where each \( \Sigma_{\ell_i}^i, \ell_i = 1, \ldots, N_i, i = 0, 1, \ldots, m \) is of the form of the shift operator \( \Sigma \) in (8.29) but with the coordinates \( x \) shifted to \( x_{\ell_i} := x - b_{\ell_i}^{\mu} \). They behave just like \( \Sigma \) except that now the kernel of \( (\Sigma_{\ell_i}^i)^\dagger \) is spanned by the vector \( |\psi\rangle \otimes |\tilde{b}_{\ell_i}^{\mu}\rangle \), where \( |\psi\rangle \) is the fermionic ground state and the shifted ground state \( |\tilde{b}_{\ell_i}^{\mu}\rangle \) is a coherent state in the \( n \)-oscillator Fock space \( \mathcal{H} \), i.e. \( \tilde{x}_{\ell_i}^{\dagger} \tilde{b}_{\ell_i}^{\mu} = 0 \). The states \( |\psi\rangle \otimes |\tilde{b}_{\ell_i}^{\mu}\rangle \) and \( \Sigma_1^i \cdots \Sigma_{\ell_i-1}^i (|\psi\rangle \otimes |\tilde{b}_{\ell_i}^{\mu}\rangle) \) for \( \ell_i = 2, \ldots, N_i \) span the kernel of the operator \( (T_{N_i}^{(0)})^\dagger \) given by (8.47), and we find that the equations (8.35) are obeyed with \( P_{N_i}^{(0)} \) the orthogonal projection onto \( \text{ker}(T_{N_i}^{(0)})^\dagger \).

The space of partial isometries (8.46) may thereby be described as the complex manifold \( \prod_{i=0}^{m} (\mathbb{C}^n)^{N_i} \). After a quotient by the appropriate discrete symmetry group, the moduli space for the full solution consisting of the rectangular Toeplitz operators (8.41) is given by
\[
\mathcal{M}(n; \vec{k}_{\nu}, \vec{k}_{\tau}) = \mathcal{Q}(\vec{k}_{\nu}) \times \prod_{i=0}^{m} \text{Hilb}^{N_i}(\mathbb{C}^n) ,
\]
where \( \text{Hilb}^{N_i}(\mathbb{C}^n) \) is the moduli space of \( N_i \) noncommutative solitons on \( \mathbb{R}^{2n} \) [46] which is given as the (singular) Hilbert scheme of \( N_i \) points in \( \mathbb{C}^n \), i.e. the set of ideals \( \mathcal{I} \) of codimension \( N_i \) in the polynomial ring \( \mathbb{C}[b_1^1, \ldots, b_{N_i}^n] \). The factor \( \mathcal{Q}(\vec{k}_{\nu}) \) is the moduli space of isomorphism classes of quiver representations (3.8) of dimension [37]
\[
\dim \mathcal{Q}(\vec{k}_{\nu}) = 1 - \frac{1}{2}\vec{k}_{\nu} \cdot C\vec{k}_{\nu} = 1 + \sum_{i=0}^{m} k_i (k_{i+1} - k_i) .
\]
Note that real roots (having \( \vec{k}_{\nu} \cdot C\vec{k}_{\nu} = 2 \)) correspond to rigid representations of the quiver \( A_{m+1} \) with no moduli, while imaginary roots (having \( \vec{k}_{\nu} \cdot C\vec{k}_{\nu} \leq 0 \)) carry moduli associated to the gauge symmetry breaking (3.9). The points of the moduli space (8.48) label the positions of well-separated D-branes, and it coincides in the low-energy limit with the moduli space of the commutative brane description [45].
9 D-brane charges

In this section we will compute the topological charge of our multi-instanton solutions in essentially two distinct ways. The first one is a direct field theoretic calculation of the \((n+1)\)-th Chern number of our gauge field configurations on \( \mathbb{R}^2_n \times S^2 \), which can also be computed using the \(\mathbb{Z}_{m+1}\)-graded connection formalism of Section 5. The second one is a homological calculation of the index class of our solutions in K-theory, which is also equivalent to the Euler-Ringel character of the pertinent representations of the quiver \(A_{m+1} \). The equivalence of these two calculations will then lead us directly into a worldvolume description whereby we can interpret the topological charge in terms of cycles in topological equivariant K-homology, yielding the claimed D-brane interpretation of our solutions. The results of this section bridge together the descriptions presented in Section 4 and justify the brane interpretations that have been given throughout this paper thus far.

Field theory calculation. We will first compute the topological charge of the configurations \( (7.1) \)–\( (7.3) \). For this, it is convenient to parametrize the two-sphere by the angular coordinates \( 0 \leq \varphi \leq 2\pi \) and \( 0 \leq \theta < \pi \) defined in \((2.4)\). In these coordinates

\[
F_{y\bar{y}} = \frac{i}{2} \sin \varphi \frac{\partial \varphi}{\partial y} \ ,
\]

and we have

\[
F_{2a-1 \bar{a}} \ = \ 2i \ F_{a\bar{a}} \ = \ -i \frac{\theta}{\theta^2} \sum_{i=0}^{m} P_{N_i} \otimes \Pi_i ,
\]

\[
F_{\varphi \varphi} \ = \ -i \frac{\sin \theta}{2} \sum_{i=0}^{m} (m-2i) \ P_{N_i} \otimes \Pi_i ,
\]

giving

\[
F_{12} \ F_{34} \cdots F_{2n-1 \bar{a}} \ F_{\varphi \varphi} \ = \ (-i)^{n+1} \frac{\sin \theta}{n} \left( \sum_{i=0}^{m} P_{N_i} \otimes \Pi_i \right) \left( \sum_{j=0}^{m} (m-2j) \ P_{N_j} \otimes \Pi_j \right)
\]

\[
\[ (-i)^{n+1} \frac{\sin \theta}{n} \sum_{i=0}^{m} (m-2i) \ P_{N_i} \otimes \Pi_i ,
\]

where we have used the definitions \((7.4)\) and \((5.16)\) of the projectors \(P_{N_i}\) and \(\Pi_i\).

The instanton charge is then given by the \((n+1)\)-th Chern number

\[
Q := \frac{1}{(n+1)!} \left( \frac{i}{2\pi} \right)^{n+1} \left( \prod_{a=1}^{n} 2\pi \theta^a \right) \int_{S^2} \text{Tr}_{\mathcal{H}} \ F_{12} \cdots F_{n+1} \cdot \left( \sum_{i=0}^{m} \ (m-2i) \ P_{N_i} \otimes \Pi_i \right)
\]

\[
= \left( \frac{i}{2\pi} \right)^{n+1} \left( \frac{-1}{2} \right)^{n+1} \left( \prod_{a=1}^{n} 2\pi \theta^a \right) \sum_{i=0}^{m} (m-2i) N_{i} \int_{S^2} \sin \theta \, d\theta \wedge d\varphi .
\]

After splitting the sum over \(i\) into contributions from monopoles and antimonopoles analogously to \( (7.30)\), this becomes

\[
Q = \sum_{i=0}^{m} (m-2i) (N_i - N_{m-i}) ,
\]
where we recall that $N_i \geq 0$ for $i = 0, 1, \ldots, m$. The formula (9.6) clarifies the D-brane interpretation of the configuration (7.1)–(7.3). It describes a collection of $(m-2i) N_i$ D0-branes for $2i < m$ and $(2i - m) N_i$ anti-D0-branes for $2i > m$ as a bound state (i.e. a vortex-like configuration on $\mathbb{R}^{2n}_\theta$) in a system of $k_0 + k_1 + \ldots + k_m = k$ D$(2n)$ branes and antibranes. However, from the point of view of the initial brane-antibrane system on $\mathbb{R}^{2n}_\theta \times S^2$, they are spherical $|m-2i| N_i$ D2-branes or D2-antibranes depending on the sign of the monopole charge $m-2i$. Note that the vortices with $2i = m$, which always exist for even $m$, have vanishing instanton charge since they couple with the trivial line bundle $L^0 = S^2 \times \mathbb{C}$. Thus they are not extended to instantons on $\mathbb{R}^{2n}_\theta \times S^2$, but are rather unstable and simply decay into the vacuum.

The topological charge can be alternatively computed within the graded connection formalism of Section 5. Recalling the equivariant ABS construction (8.18)–(8.20), we note that the $\mathbb{Z}_{m+1}$-graded vector space (3.8) (the fibre of the $\mathbb{Z}_{m+1}$-graded bundle (5.3)) also has a natural $\mathbb{Z}_2$-grading by the sign of the magnetic charge, i.e. by the involution $\varepsilon : \mathcal{V} \to \mathcal{V}$ defined by $\varepsilon(v_i) := \text{sgn}(m-2i) v_i$ for $v_i \in \mathcal{V}_{k_i}$, where throughout we use the convention $\text{sgn}(0) := 0$. The corresponding supertrace is given by

$$
\text{str}_{k \times k} X := \text{tr}_{k \times k}(\varepsilon \circ X) = \sum_{i=0}^{m} \text{sgn}(m-2i) \text{tr}_{k_i \times k_i} X_i
$$

for any linear operator $X \in \text{End}(\mathcal{V})$ with block-diagonal components $X_i \in \text{End}(\mathcal{V}_{k_i})$. This extends to a supertrace $\text{STr}_{\mathcal{V} \otimes \mathcal{H}} := \text{Tr}_{\mathcal{H}} \text{str}_{k \times k}$ which we may use to express the Chern number in terms of the graded curvature (5.28) as

$$
Q = \frac{R^2}{2n(n+1)!} \left( \frac{i}{2\pi} \right)^{n+1} \left( \prod_{a=1}^{n} 2\pi \theta^a \right) \text{STr}_{\mathcal{V} \otimes \mathcal{H}} \text{Tr}_{\mathcal{C}^{2n+1}} \left( \Gamma \hat{F}^{n+1} \right)_{\text{asym}},
$$

where $\Gamma := \frac{2}{\sqrt{g}} \Gamma^1 \cdots \Gamma^{2n+2} = \gamma \otimes \sigma_3$ and the antisymmetrized product of gamma-matrices

$$
(\Gamma^{a_1} \cdots \Gamma^{a_q})_{\text{asym}} := \frac{1}{q!} \sum_{\pi \in S_q} \text{sgn}(\pi) \Gamma^{a_{\pi(1)}} \cdots \Gamma^{a_{\pi(q)}}
$$

mimics the algebraic structure of the exterior product of differential forms. The formula (9.6) follows from (9.8) upon repeated application of the Clifford algebra and the trace identities (5.30)–(5.33), with the supertrace (9.7) giving the appropriate sign alternations.

**K-theory calculation.** The origin of the topological charge lies in the graded Chern character $\text{ch}(\mathcal{V} \otimes \mathcal{H}) := \text{str}_{k \times k} \exp \hat{F}/2\pi i$. Standard transgression arguments can be used to show that the cohomology class defined by this closed differential form is independent of the choice of graded connection [39]. In particular, we may either compute it by setting the off-diagonal Higgs fields $\phi_i = 0$ or by setting the diagonal gauge fields $A^i = 0$. It is instructive to recall how this works in the case $m = 1$ corresponding to the basic brane-antibrane system represented by the chain (4.26) [8, 31]. In the former case we would obtain the difference $\text{ch}(\mathcal{V}_{k_1} \otimes \mathcal{H}) - \text{ch}(\mathcal{V}_{k_0} \otimes \mathcal{H})$ of topological charges on the branes and antibranes. In the latter case we would compute the index of the tachyon field $\phi_1$, or equivalently the Euler characteristic of the two-term complex $0 \to \mathcal{V}_{k_1} \otimes \mathcal{H} \xrightarrow{\phi_1} \mathcal{V}_{k_0} \otimes \mathcal{H} \to 0$. The virtual Euler class generated by the cohomology of this complex is the analytic K-homology class $[\phi_1] \in K^a(\mathbb{R}^{2n})$ of the brane configuration. The equivalence of the two computations is asserted by the index theorem.

The situation for $m > 1$ is more subtle. The action of the graded connection zero-form (5.9) on the bundle (5.3) produces the holomorphic chain (4.13). In general this is not a complex because, according to (5.10), $(\phi_{(m)})^2 \neq 0$ for $m > 1$, i.e. $\phi_i \phi_{i+1} \neq 0$. The only physical instance in which
such a chain generates a complex is when it corresponds to an alternating sequence of branes and antibranes [47]. But if one has a tachyon field which is a holomorphic map from an antibranes to a brane, then the adjoint map is antiholomorphic. Recalling (5.37), we see that in our chain (4.13) all maps \( \phi_i \) are *holomorphic* and thus do not represent tachyon fields between pairs of branes and antibranes. Furthermore, the maps \( \phi_i \) obtained as solutions of the vortex equations, which can be associated with the \( A_{m+1} \) quiver and are obtained by SU(2)-invariant reduction, can never satisfy the constraints \( \phi_i \phi_{i+1} = 0 \) [34, 36].

The solution to this problem is to fold the given holomorphic chain into maps between branes and antibranes. Let us first carry out the calculation in the case that the monopole Chern number \( m \) is an odd integer. By using the \( \mathbb{Z}_2 \)-grading \( \varepsilon : \mathcal{V} \to \mathcal{V} \) introduced above, we explicitly decompose (3.8) as a \( \mathbb{Z}_2 \)-graded module into the \( \pm 1 \) eigenspaces of the involution \( \varepsilon \) giving

\[
\mathcal{V} = \mathcal{V}_+ \oplus \mathcal{V}_- \quad \text{with} \quad \mathcal{V}_+ = \bigoplus_{i=0}^{m_-} \mathcal{V}_{k_i} \quad \text{and} \quad \mathcal{V}_- = \bigoplus_{i=m_+}^{m} \mathcal{V}_{k_i} . \tag{9.10}
\]

Using (5.9) and (5.10) we now introduce the operator

\[
T_{(m)} := (\phi_{(m)})^{\lfloor \frac{m}{2} \rfloor +1} . \tag{9.11}
\]

With respect to the \( \mathbb{Z}_2 \)-grading (9.10), it is an odd map

\[
T_{(m)} : \mathcal{V}_- \otimes \mathcal{H} \to \mathcal{V}_+ \otimes \mathcal{H} \quad \text{with} \quad (T_{(m)})^2 = 0 . \tag{9.12}
\]

Thus the triple \( \{\mathcal{V}_+ \otimes \mathcal{H}, \mathcal{V}_- \otimes \mathcal{H}; T_{(m)}\} \) defines a two-term complex and represents a brane-antibranes system with tachyon field given in terms of the graded connection by (9.11). The corresponding index class \( [T_{(m)}] \in K^\varepsilon(\mathbb{R}^{2m}) \) is thus the analytic K-homology class of our configuration of D-branes. In particular, on isotopical components one has

\[
T_{(m)} \circ \Pi_{i+1+\lfloor \frac{m}{2} \rfloor} = \phi_{i+1} \cdots \phi_{i+1+\lfloor \frac{m}{2} \rfloor} = (\alpha_{i+1} \cdots \alpha_{i+1+\lfloor \frac{m}{2} \rfloor}) \ T_N \ T_{N_i}^\dagger \ T_{N_{i+1+\lfloor \frac{m}{2} \rfloor}} \tag{9.13}
\]

while \( T_{(m)} \circ \Pi_i = 0 \), where \( i = 0, 1, \ldots, m_- \). The tachyon field is thus a holomorphic map between branes of equal and opposite magnetic charge,

\[
T_{(m)} \circ \Pi_{i+\lfloor \frac{m}{2} \rfloor +1} : \mathcal{V}_{k_i+\lfloor \frac{m}{2} \rfloor +1} \otimes \mathcal{H} \to \mathcal{V}_{k_i} \otimes \mathcal{H} , \tag{9.14}
\]

and from (7.5) it follows that it has a finite dimensional kernel and cokernel with

\[
\dim \ker (T_{(m)} \circ \Pi_{i+\lfloor \frac{m}{2} \rfloor +1}) = N_{i+\lfloor \frac{m}{2} \rfloor +1} \quad \text{and} \quad \dim \ker (T_{(m)} \circ \Pi_{i+\lfloor \frac{m}{2} \rfloor +1})^\dagger = N_i . \tag{9.15}
\]

To incorporate the twistings by the magnetic monopole bundles, we use the ABS construction (8.18)–(8.23) to define the tachyon field

\[
\mathcal{T}_{(m)} := T_{(m)} \otimes 1 : \mathcal{A}_{\mathcal{V}}^+ \otimes \mathcal{H} \to \mathcal{A}_{\mathcal{V}}^- \otimes \mathcal{H} . \tag{9.16}
\]

It behaves like a noncommutative version of Clifford multiplication \( \mu_{\mathcal{V}}^\dagger \), in (8.21,8.22). Since \( \dim \mathcal{V}_{m-2i} = |m-2i|, \) from (9.15) it follows that the index of the tachyon field (9.16) is given by

\[
\text{index } \mathcal{T}_{(m)} = \dim \ker(\mathcal{T}_{(m)}) - \dim \ker(\mathcal{T}_{(m)})^\dagger = \sum_{i=m_+}^{m} |m-2i| N_i - \sum_{i=0}^{m_-} |m-2i| N_i = -Q . \tag{9.17}
\]
Thus the K-theory charge of the noncommutative soliton configuration (7.1)–(7.3) coincides with the Yang-Mills instanton charge (9.5, 9.6) on $\mathbb{R}^2_n \times S^2$.

When the monopole charge $m$ is even, we introduce the tachyon field $T_{(m)}$ by the same formula (9.11). The only difference now is that the subspace $V_{k_{2m}} \otimes \mathcal{H}$ is annihilated by both operators $(T_{(m)})$ and $(T_{(m)})^\dagger$ so that

$$V_{k_{2m}} \otimes \mathcal{H} \subset \ker(T_{(m)}) \cap \ker(T_{(m)})^\dagger.$$  \hspace{1cm} (9.18)

According to (8.20), this subspace should be coupled to the eigenspace (8.17) of spinor harmonics on $\mathbb{C}P^1$ when defining the extended tachyon field (9.16). Analogously to (8.23), one then has

$$\ker(T_{(m)} \circ \Pi_{m}) = \ker(T_{(m)} \circ \Pi_{2m})^\dagger = V_{k_{2m}} \otimes \mathcal{H} \otimes \mathcal{H}.$$  \hspace{1cm} (9.19)

With a suitable regularization of the infinite dimensions of the kernel and cokernel of the operator $T_{(m)} \circ \Pi_{m}$, these subspaces will make no contribution to the index (9.17). This statement will be justified below by the fact that index $P_0 = 0$ and that the index class of the noncommutative tachyon field coincides with that of the twisted SU(2)-invariant Dirac operator on $\mathbb{R}^2 \times \mathbb{C}P^1$.

We can give a more detailed picture of how the topological charge of the system of D-branes arises by relating the index to a homological computation in the corresponding quiver gauge theory, which shows precisely how the original brane configuration folds itself into branes and antibranes. Consider the $A_{m+1}$ quiver (7.11) defined by a generic (non-BPS) solution of the Yang-Mills equations on $\mathbb{R}^2_n \times \mathbb{C}P^1$, and let

$$\mathcal{W} = \bigoplus_{i=0}^m W_i$$

with

$$\mathcal{k}_{\mathcal{W}} = \sum_{i=0}^m w_i e_i.$$  \hspace{1cm} (9.20)

be any quiver representation. Applying the functor $\text{Hom}(\_, \mathcal{W})$ to the projective resolution (7.12) gives a complex whose cohomology in the $p$-th position defines the extension groups $\text{Ext}^p(T, \mathcal{W}) \cong H^p(\mathbb{R}^2_n; \mathcal{W} \otimes \mathcal{L}^p)$, with $\text{Ext}^0 = \text{Hom}$ and $\text{Ext}^1 = \text{Ext}$. We may then define the relative Euler character between these two representations through the corresponding Euler form

$$\chi(T, \mathcal{W}) := \sum_{p \geq 0} (-1)^p \dim \text{Ext}^p(T, \mathcal{W}).$$  \hspace{1cm} (9.21)

Since the $A_{m+1}$ quiver has no relations, one has $\text{Ext}^p(T, \mathcal{W}) = 0$ for all $p \geq 2$ in the present case [37].

By using (5.7), the resolution (7.12) induces an exact sequence of extension groups given by

$$0 \to \text{Hom}(T, \mathcal{W}) \to \bigoplus_{i=0}^s \text{Hom}(\ker T_{N_i}^\dagger, W_i) \to$$

$$\to \bigoplus_{i=0}^{s-1} \text{Hom}(\ker T_{N_{i+1}}^\dagger, W_i) \to \text{Ext}(T, \mathcal{W}) \to 0.$$  \hspace{1cm} (9.22)

from which we may compute the Euler form (9.21) explicitly to get

$$\chi(T, \mathcal{W}) = \dim \text{Hom}(T, \mathcal{W}) - \dim \text{Ext}(T, \mathcal{W})$$

$$= \sum_{i=0}^s \dim \text{Hom}(\ker T_{N_i}^\dagger, W_i) - \sum_{i=0}^{s-1} \dim \text{Hom}(\ker T_{N_{i+1}}^\dagger, W_i)$$

$$= \sum_{i=0}^m N_i w_i - \sum_{i=0}^{m-1} N_{i+1} w_i.$$  \hspace{1cm} (9.23)

35
Thus the relative Euler character depends only on the dimension vectors of the corresponding representations and coincides with the Ringel form \( \langle \mathbf{r}, \mathbf{w} \rangle \) on the representation ring \( R_{A_{m+1}} \) of the \( A_{m+1} \) quiver [37]. The map \( [\mathbf{W}] \mapsto \mathbf{w} \) gives a linear map \( R_{A_{m+1}} \rightarrow \mathbb{Z}^{m+1} \) which is an isomorphism of abelian groups since \( R_{A_{m+1}} \) is generated by the Schur modules \( L_i, i = 0, 1, \ldots, m \). By using (7.2) and (7.5) we can write this bilinear pairing in the suggestive form

\[
\chi(\mathbf{I}, \mathbf{W}) = -\sum_{i=0}^{m} w_i \text{index}(\phi_{i+1}). \tag{9.24}
\]

The appropriate representation \( \mathbf{w} \) to couple with in the present case is dictated by the correct incorporation of magnetic charges. As before, the fact that the Higgs fields \( \phi_{i+1} \) in (9.24) themselves are not tachyonic, i.e. do not generate a complex, means that we have to fold the \( SU(2) \) representations \( V_{\mid m-2\mid} \) appearing in the ABS construction (8.19) appropriately. The correct folding is expressed by the collection of distinguished triangles (7.38) which shows that we should couple an increasing sequence \( W_0 \subset W_1 \subset \cdots \subset W_m \) of representations as we move along the chain of constituent D-branes of the quiver, so that the \( SU(2) \)-module \( W_i \) gives an extension of the monopole field carried by the elementary brane state at vertex \( i \) by the \( SU(2) \)-module \( W_{i-1} \). Thus we take \( W_i = \bigoplus_{j=0}^{i} V_{|m-2j|} = V_{|m-2i|} \oplus W_{i-1} \) and embed its class into the representation ring \( R_{A_{m+1}} \) using the \( \mathbb{Z}_2 \)-grading above as the element

\[
[W_i] = \sum_{j=0}^{i} \text{sgn}(m-2j) \left[ V_{|m-2j|} \right] = \text{sgn}(m-2i) \left[ V_{|m-2i|} \right] + [W_{i-1}] \tag{9.25}
\]

of virtual dimension

\[
w_i = \sum_{j=0}^{i} (m-2j) = (i+1)(m-i) \tag{9.26}
\]

for each \( i = 0, 1, \ldots, m \). In this case the Euler-Ringel form (9.23) becomes

\[
\chi(\mathbf{I}, \mathbf{W}) = \sum_{i=0}^{m} (i+1)(m-i) (N_i - N_{i+1}) = \sum_{i=0}^{m} (m-2i) N_i = Q \tag{9.27}
\]

and it also coincides with the instanton charge of the gauge field configurations on \( \mathbb{R}^2 \times S^2 \). The equivalence of the relative Euler character with the index of the tachyon field above is a consequence of the Grothendieck-Riemann-Roch theorem.

**Worldvolume construction.** We can now present a very explicit geometric description of the equivalence between the brane configurations on \( \mathbb{R}^2 \times \mathbb{C}P^1 \) and on \( \mathbb{R}^2 \). The crux of the formulation is the well-known map in K-theory between analytic (noncommutative) and topological (commutative) descriptions [9, 12, 30, 48]. If \( \mathcal{D} := -i \sigma \cdot \partial : L^2(\mathbb{R}^2, \Delta_{2n}^-) \rightarrow L^2(\mathbb{R}^2, \Delta_{2n}^+ ) \) is the usual Dirac operator on \( \mathbb{R}^2 \), then its index coincides with that of the noncommutative ABS configuration (8.29) giving

\[
\text{index } \Sigma = \text{index } \mathcal{D}. \tag{9.28}
\]

This coincides with the K-theory charge of the Bott class \( [\Delta_{2n}^+ , \Delta_{2n}^- ; \mu] \in K(\mathbb{R}^{2n}) \) given by the ordinary ABS construction [43], where \( \mu_x = \frac{\sigma \cdot x}{|x|} : \Delta_{2n}^- \rightarrow \Delta_{2n}^+ \) is Clifford multiplication by \( x \in \mathbb{R}^2 \). In particular, the Dirac operator itself can be used to represent the analytic K-homology class \( \Sigma = [\mathcal{D}] \) described by the noncommutative ABS field.

Let us represent a system of \( k \) Type IIA D-branes wrapped on \( \mathbb{R}^2 \times \mathbb{C}P^1 \) with virtual Chan-Paton bundle \( \Xi \in K(\mathbb{R}^2 \times \mathbb{C}P^1) \) by the K-cycle \( [\mathbb{R}^2 \times \mathbb{C}P^1, \Xi, \text{id}] \) in the topological K-homology.
K^t(\mathbb{R}^{2n} \times CP^1). Its equivalence class is invariant under the usual relations of bordism, direct sum and vector bundle modification [12, 30, 48]. There is an isomorphism K^t(\mathbb{R}^{2n} \times CP^1) \cong K^a(\mathbb{R}^{2n} \times CP^1) of abelian groups which sends this K-cycle to the analytic K-homology class [\hat{\mathcal{D}}_\Xi] defined by the corresponding twisted Dirac operator on \mathbb{R}^{2n} \times CP^1. Similarly, if \xi \in K(\mathbb{R}^{2n}) and \iota : \mathbb{R}^{2n} \hookrightarrow \mathbb{R}^{2n} \times CP^1 is the slice induced by the inclusion U(1) \hookrightarrow SU(2) of groups, then the topological K-cycle \([\mathbb{R}^{2n}, \xi, \iota] \in K^a(\mathbb{R}^{2n} \times CP^1)\) corresponds to the analytic K-homology class \(\iota_* [\hat{\mathcal{D}}_\xi] \in K^a(\mathbb{R}^{2n} \times CP^1)\), where \(\hat{\mathcal{D}}_\xi\) is the twisted Dirac operator on \(\mathbb{R}^{2n}\).

Now consider the SU(2)-equivariant reduction of these cycles. From the construction of the previous section with \(\phi_{(m)} = 0\) and the equivariant excision theorem of Section 4 we have the equality

\[ [\hat{\mathcal{D}}_\Xi]^{SU(2)} = \iota_* [\hat{\mathcal{D}}_{\iota \Xi}]^{U(1)} \]  

in \(K^a_{SU(2)}(\mathbb{R}^{2n} \times CP^1)\) which leads to

\[ [\mathbb{R}^{2n}, \xi, \iota] = [\mathbb{R}^{2n} \times CP^1, \Xi, \text{id}] \quad \text{with} \quad \Xi = SU(2) \times_{U(1)} \xi \]  

in \(K^t_{SU(2)}(\mathbb{R}^{2n} \times CP^1)\). The left-hand side of (9.30) corresponds to the class of D(2n) brane-antibrane pairs wrapping \(\mathbb{R}^{2n}\), while the right-hand side corresponds to D(2n + 2) brane-antibrane pairs wrapping \(\mathbb{R}^{2n} \times CP^1\). This is just the equivalence between instantons on \(\mathbb{R}^{2n} \times CP^1\) and vortices on \(\mathbb{R}^{2n}\). We note that in the case \(m = 1\), the monopole field is automatically spherically symmetric on \(CP^1\) and one can formulate the equivalence (9.30) using only the requirement of vector bundle modification in ordinary topological K-homology [30], which is equivalent to Bott periodicity (4.1). In contrast, for \(m > 1\) one must appeal to an SU(2)-equivariant framework and the identification (9.30) of K-cycles is far more intricate. In this case it is a result of the equivariant excision theorem, and not of Bott periodicity in equivariant K-theory. It is this intricacy that leads to a more complicated brane-antibrane system when \(m > 1\).

Using the equivariant ABS construction of the previous section, the K-homology class of the multi-instanton solution (7.1)–(7.3) is given by the left-hand side of (9.30) with

\[ \xi = [\Delta^+_E, \Delta^-_E; \mu_{N_0, N_1, \ldots, N_m}] , \]  

where

\[ \Delta^+_E = \bigoplus_{i=m+}^m E_{k_i} \otimes V_{m-2i} \quad \text{and} \quad \Delta^-_E = \bigoplus_{i=0}^m E_{k_i} \otimes V_{m-2i} \]  

(9.32)

while

\[ \mu_{N_0, N_1, \ldots, N_m} = \prod_{i=0}^m (\mu_E \circ \Pi_i)^{N_i} \prod_{j=m+}^m (\mu^*_E \circ \Pi_j)^{N_j} \]  

(9.33)

with \(\mu_E : \Delta^-_E \rightarrow \Delta^+_E\) acting fibrewise as Clifford multiplication (8.21,8.22). The class (9.31) is the K-theory class of the noncommutative soliton field (8.46). The relation (9.30) equates the resulting K-homology class with that defined by

\[ \Xi = [SU(2) \times_{U(1)} \Delta^+_E, SU(2) \times_{U(1)} \Delta^-_E; \pi^* \circ \mu_{N_0, N_1, \ldots, N_m} \circ \iota^*] , \]  

(9.34)

where the projection \(\pi : \mathbb{R}^{2n} \times CP^1 \rightarrow \mathbb{R}^{2n}\) is a left inverse to the inclusion \(\iota\), i.e. \(\pi \circ \iota = \text{id}\). Through the standard process of tachyon condensation on the system of D(2n + 2) branes and antibranes wrapping \(\mathbb{R}^{2n}\), the right-hand side of (9.30) then describes \(\sum_{2i \leq m} (m-2i) N_i\) D2-branes and \(\sum_{2i > m} (m-2i) N_i\) D2-antibranes. On the left-hand side of (9.30), these are instead D0-branes corresponding to vortices left over from condensation in the transverse space \(\mathbb{R}^{2n}\).
One can also compute the topological charge in this worldvolume picture and explicitly demonstrate that the K-theory charges on both sides of (9.30) are the same. The natural charge of branes defined by elements of equivariant K-theory is given by the equivariant index $\text{index}_{\text{SU}(2)}(\hat{D}_\Xi) \in R_{\text{SU}(2)}$, which may be computed by using the SU(2)-index theorem [49]

$$\text{index}_{\text{SU}(2)}(\hat{D}_\Xi) = - \int_{\mathbb{R}^{2n} \times \mathbb{C}P^1} \text{ch}_{\text{SU}(2)}(\Xi) \wedge \tilde{A}(\mathbb{R}^{2n} \times \mathbb{C}P^1),$$

(9.35)

where $\text{ch}_{\text{SU}(2)} : K_{\text{SU}(2)}(\mathbb{R}^{2n} \times \mathbb{C}P^1) \to H^*_\text{SU}(2)(\mathbb{R}^{2n} \times \mathbb{C}P^1; \mathbb{Q})$ is the equivariant Chern character taking values in SU(2)-equivariant rational cohomology. Since this index depends only on the equivariant K-homology class of the Dirac operator on $\mathbb{R}^{2n} \times \mathbb{C}P^1$, we may explicitly use (9.29) and perform the dimensional reduction to write the index (9.35) as

$$\text{index}_{\text{SU}(2)}(\hat{D}_\Xi) = - \int_{\mathbb{R}^{2n}} \text{ch}_{\text{SU}(2)}(\xi).$$

(9.36)

Since the Chern character in (9.36) is a ring homomorphism between $K_{\text{SU}(2)}(\mathbb{R}^{2n}) \cong R_{\text{SU}(2)}$ and $H^*(\mathbb{R}^{2n}; \mathbb{Q}) \otimes R_{\text{SU}(2)}$, upon substitution of (9.31,9.32) we can use its additivity and multiplicativity to compute

$$\text{ch}_{\text{SU}(2)}(\xi) = \text{ch}_{\text{SU}(2)}(\Delta_E^+ \oplus \Delta_E) = \sum_{i=m_+}^m \text{ch}(E_{k_i}) \otimes \chi_{V_{m-2i}} - \sum_{i=0}^{m_-} \text{ch}(E_{k_i}) \otimes \chi_{V_{m-2i}},$$

(9.37)

where $\chi_{V_{m-2i}} : \text{SU}(2) \to \mathbb{C}$ are the characters of the SU(2) representations $V_{m-2i} \cong \mathbb{C}^{m-2i}$. This enables us to write the equivariant index on $\mathbb{R}^{2n} \times \mathbb{C}P^1$ in terms of ordinary indices on $\mathbb{R}^{2n}$ to get

$$\text{index}_{\text{SU}(2)}(\hat{D}_\Xi) = \sum_{i=0}^{m_-} \text{index}(\hat{D}_{E_{k_i}}) \otimes \chi_{V_{m-2i}} - \sum_{i=m_+}^m \text{index}(\hat{D}_{E_{k_i}}) \otimes \chi_{V_{m-2i}}.$$

(9.38)

We can turn (9.38) into a linear map $K_{\text{SU}(2)}(\mathbb{R}^{2n} \times \mathbb{C}P^1) \to \mathbb{Z}$ by composing it with the projection $\pi_0 : R_{\text{SU}(2)} \to \mathbb{Z}$ onto the trivial representation. Acting on the character ring this gives

$$\pi_0(\chi_{V_{m-2i}}) = \chi_{V_{m-2i}}(\text{id}) = \text{dim} V_{m-2i} = |m-2i|$$

(9.39)

and one finally arrives at

$$\pi_0(\text{index}_{\text{SU}(2)}(\hat{D}_\Xi)) = \sum_{i=0}^{m_-} |m-2i| \text{index}(\hat{D}_{E_{k_i}}) - \sum_{i=m_+}^m |m-2i| \text{index}(\hat{D}_{E_{k_i}}).$$

(9.40)

Alternatively, one may arrive at the same formula by directly computing the ordinary index of the Dirac operator (8.7) with $\phi_{(m)} = 0$ using (8.10) and (8.14)-(8.16). Since

$$\text{index} \hat{D}_{m-2i} = \text{dim ker} \hat{D}^+_{m-2i} - \text{dim ker} \hat{D}^-_{m-2i} = -(m-2i),$$

(9.41)

the index of (8.7) acting on sections of the bundle (8.9) coincides with (9.40). For a gauge field configuration appropriate to the K-theory class defined by the tachyon field (9.33), these topological charges coincide with (9.6).
10 Vacuum solutions

The extremal cases for which the Higgs fields have the configurations \( \{ \phi_{i+1} = 0, \ i = 0, 1, \ldots, m-1 \} \) and \( \{ \partial_\mu \phi_{i+1} = 0, \ i = 0, 1, \ldots, m-1 \} \), fall outside of the general scope of the previous analysis and are worth special consideration. They correspond to vacuum sectors of the noncommutative gauge theory and are associated with indecomposable representations of the quiver \( A_{m+1} \) that have no arrows. Nevertheless, these vacuum sectors admit non-trivial BPS solutions which signal the presence of stable D-branes attached to the closed string vacuum after condensation on the brane-antibrane system. We shall now study them in some detail.

**Monopole vacuum.** Let us first look at the case \( \partial_\mu \phi_{i+1} = 0, \ i = 0, 1, \ldots, m-1 \). The nonabelian coupled vortex equations (6.12)–(6.14) then imply

\[
A^0 = A^1 = \ldots = A^m =: A \quad \text{and} \quad F^0 = F^1 = \ldots = F^m =: F , \tag{10.1}
\]

which is only possible in the equal rank case \( r = k_0 = k_1 = \ldots = k_m \) corresponding to the gauge symmetry breaking pattern \( U(k) \to U(r)^{m+1} \) with \( k = (m+1)r \). Thus we take \( \phi_{i+1} = \alpha_{i+1} 1_r \) and \( \phi^\dagger_{i+1} = \bar{\alpha}_{i+1} 1_r \), with \( i = 0, 1, \ldots, m-1 \), where \( \alpha_{i+1} \) are given in (7.29). In quiver gauge theory, the BPS conditions in this sector thus correspond to the representation of \( A_{m+1} \) which is \( r \) copies of the indecomposable quiver representation \( \mathbb{L}_0 \oplus \mathbb{L}_1 \oplus \cdots \oplus \mathbb{L}_m \). They also require

\[
\delta^{\bar{a} \bar{b}} F_{ab} = 0 \quad \text{and} \quad F_{\bar{a} \bar{b}} = 0 = F_{ab} , \tag{10.2}
\]

which are simply the DUY equations on \( \mathbb{R}^{2n}_D \). Note that (3.26) implies \( \mathcal{F}_{yy} = 0 \) in this case, giving the trivial dimensional reduction to \( \mathbb{R}^{2n}_D \). After switching to matrix form via (6.9), we obtain

\[
\delta^{\bar{a} \bar{b}} [X_a , X_b] + \delta^{\bar{a} \bar{b}} \theta_{ab} = 0 \quad \text{and} \quad [X_{\bar{a}} , X_{\bar{b}}] = 0 = [X_a , X_b] . \tag{10.3}
\]

The obvious solution to (10.3) is the trivial one with \( X_a = \theta_{ab} \bar{z}_b \), giving \( F_{\bar{a} \bar{b}} = 0 \). This sector can be understood physically as the endpoint of tachyon condensation, wherein the Higgs fields \( \phi_{i+1} \) have rolled to their minima at \( \phi_{i+1} = \alpha_{i+1} 1_r \) and the fluxes have been radiated away to infinity. Here the D0-branes have been completely dissolved into the D(2n)-branes.

However, non-trivial solutions of the equations (10.3) also exist. For this, let us restrict ourselves to the abelian case \( r = 1 \) and simplify matters by taking \( \theta^a = \theta \) for all \( a = 1, \ldots, n \). We fix an integer \( l \geq 1 \) and consider the ansatz [28]

\[
X_a = \theta_{ac} \Sigma^\dagger_l f(\mathcal{N}) \bar{z}_c \Sigma_l \quad \text{and} \quad X_{\bar{a}} = \theta^{\bar{a}c} \Sigma^\dagger_l z^c f(\mathcal{N}) \Sigma_l , \tag{10.4}
\]

where \( f \) is a real function of the “total number operator”

\[
\mathcal{N} := \frac{1}{2\theta} \sum_{a=1}^n z^a \bar{z}_{\bar{a}} \tag{10.5}
\]

with the property that \( f(r) = 0 \) for \( r \leq l-1 \). The shift operator \( \Sigma_l \) in (10.4) is defined to obey

\[
\Sigma_l \Sigma_l^\dagger = 1 \quad \text{while} \quad \Sigma_l^\dagger \Sigma_l = 1 - P_l \tag{10.6}
\]

with

\[
P_l := \sum_{|\vec{r}| \leq l-1} |r_1, \ldots, r_n\rangle \langle r_1, \ldots, r_n| , \tag{10.7}
\]
where \( \vec{r} = (r_1, \ldots, r_n) \) with \( |\vec{r}| := r_1 + \ldots + r_n \). Note that

\[
\Sigma_l^\dagger P_l = P_l \Sigma_l = 0 \quad \text{and} \quad f(\mathcal{N}) P_l = P_l f(\mathcal{N}) = 0 , \tag{10.8}
\]

and \( \Sigma_l^\dagger \) projects all states with \( |\vec{r}| < l \) out of the Fock space \( \mathcal{H} \).

One easily sees that (10.4) fulfills the homogeneous equations in (10.3). Remembering that \( \theta_{ab} = -\theta_{ba} = \frac{1}{2\theta} \delta_{ab} \), we also obtain

\[
[X_a, X_b] = \theta_{ac} \theta_{bd} \Sigma_l^\dagger \left\{ f(\mathcal{N}) \bar{z}^c (1 - P_l) z^d f(\mathcal{N}) - z^d f(\mathcal{N}) (1 - P_l) f(\mathcal{N}) \bar{z}^c \right\} \Sigma_l
\]

\[
= \frac{1}{4\theta^2} \delta_{ac} \delta_{bd} \Sigma_l^\dagger \left\{ f^2(\mathcal{N}) \bar{z}^c z^d - f^2(\mathcal{N} - 1) \bar{z}^c z^d \right\} \Sigma_l \tag{10.9}
\]

with the help of the identities \( \bar{z}^c P_l = P_{l-1} \bar{z}^c \) where \( P_0 := 0 \). We have also used

\[
\bar{z}^c f(\mathcal{N}) = f(\mathcal{N} + 1) \bar{z}^c \quad \text{and} \quad z^d f(\mathcal{N}) = f(\mathcal{N} - 1) z^d . \tag{10.10}
\]

Substituting (10.9) into (10.3), we employ

\[
\delta_{cd} z^d \bar{z}^c = 2 \theta \mathcal{N} \quad \text{and} \quad \delta_{cd} \bar{z}^c z^d = 2 \theta (\mathcal{N} + n) \tag{10.11}
\]

to find the conditions

\[
0 = \delta^{ab} [X_a, X_b] + \delta^{ab} \theta_{ab}
\]

\[
= -\frac{1}{2\theta} \Sigma_l^\dagger \left\{ f^2(\mathcal{N}) (\mathcal{N} + n) - f^2(\mathcal{N} - 1) \mathcal{N} \right\} \Sigma_l + \frac{n}{2\theta} \tag{10.12}
\]

on the operator \( f \). With the initial conditions \( f(0) = f(1) = \cdots = f(l-1) = 0 \) and the finite-energy condition \( f(r) \to 1 \) as \( r \to \infty \), these recursion relations are solved by

\[
f^2(\mathcal{N}) = \left( 1 - \frac{Q \ n!}{(\mathcal{N} + 1) \cdots (\mathcal{N} + n)} \right) (1 - P_l) , \tag{10.13}
\]

where

\[
Q := \frac{l(l+1) \cdots (l+n-1)}{n!} \tag{10.14}
\]

is the number of states in \( \mathcal{H} \) with \( \mathcal{N} \leq l - 1 \), i.e. the number of states removed by the operator \( \Sigma_l^\dagger \).

We arrive finally at the non-trivial gauge field configuration given by

\[
X_a = \frac{1}{2\theta} \Sigma_l^\dagger \sqrt{1 - \frac{Q \ n!}{(\mathcal{N} + 1) \cdots (\mathcal{N} + n)} (1 - P_l)} \delta_{ac} \bar{z}^c \Sigma_l . \tag{10.15}
\]

The field strength \( F \) on \( \mathbb{R}^{2n}_g \) obtained from (10.15) has finite \( n \)-th Chern number \( Q \) [28]. The topological charge \( Q \) given by (10.14) is calculated here via an integral over \( \mathbb{R}^{2n}_g \). However, the \((n+1)\)-th Chern number for this configuration considered as a gauge field on \( \mathbb{R}^{2n}_g \times \mathbb{C}P^1 \) with \( \mathcal{F}_{\bar{y}y} = 0 = \mathcal{F}_{\partial \bar{x}} \) vanishes. Moreover, this configuration has finite energy (5.2) proportional to the topological charge [28],

\[
E_{\text{BPS}} = (2\pi)^{n+1} R^2 n (n - 1) Q , \tag{10.16}
\]

as usual for a BPS instanton solution.
Higgs vacuum. The choice \( \phi_{i+1} = 0 \) for all \( i = 0, 1, \ldots, m - 1 \) is somewhat more interesting since from (3.26) and (5.29) we then have \( \mathcal{F}_{y\bar{y}} \neq 0 \) with

\[
\mathcal{F}_{y\bar{y}} = -\frac{R^2}{(R^2 + y\bar{y})^2} \phi_{i} \cdot (m - 2i) \frac{2n}{4R^2} \phi_{i}^* \, (m - 2i) \theta \phi_{i}^* = 0 \quad \text{and} \quad F_{ab}^i = 0 = F_{ab}^i.
\]

(10.17)

This configuration gives the local maximum of the Higgs potential corresponding to the open string vacuum containing D-branes. In this case the vortex equations (6.12)–(6.14) reduce to

\[
\begin{align*}
\delta^{ab} F_{ab}^i &= \frac{m - 2i}{4R^2} \quad \text{and} \quad F_{ab}^i = 0 = F_{ab}^i.
\end{align*}
\]

(10.18)

After switching to matrix form via (6.9) we obtain

\[
\begin{align*}
\delta^{ab} [X_a^i, X_b^j] + \delta^{ab} \left( 1 - \frac{(m - 2i) \theta}{2n R^2} \right) \theta_{ab}^i &= 0 \quad \text{and} \quad [X_a^i, X_b^j] = 0 = [X_a^i, X_b^j],
\end{align*}
\]

(10.19)

where we have used the formula \( \theta_{ab}^i = \frac{1}{2} \delta_{ab} \). Recall that there is no summation over the index \( i = 0, 1, \ldots, m \) in the equations (10.19).

By comparing (10.19) and (10.3), we conclude that (10.19) can be solved for each \( i \) by the same ansatz as for (10.3). For this, let us restrict ourselves again to the abelian case for all nodes \( i = 0, 1, \ldots, m \) (so that \( k = m + 1 \)), and fix \( m + 1 \) positive integers \( l_0, l_1, \ldots, l_m \). We take

\[
\begin{align*}
X_a^i &= \theta_{ac} \Sigma_{l_0} f_i (N) \tilde{\xi} \xi \Sigma_{l_i} \quad \text{and} \quad X_a^i = \theta_{ac} \Sigma_{l_0} \tilde{\xi} \xi f_i (N) \Sigma_{l_i},
\end{align*}
\]

(10.20)

analogously to (10.4)–(10.7). Producing then the same calculations as before, we obtain the gauge field configuration

\[
X_a^i = \frac{1}{2} \theta^i \Sigma_{l_i} \left( 1 - \frac{Q_i \, n!}{(N+1) \cdots (N+n)} \right) (1 - P_{l_i}) \theta_{ac} \tilde{\xi} \xi \Sigma_{l_i},
\]

(10.21)

where

\[
\theta^i := \frac{\theta}{\sqrt{1 - \frac{(m - 2i) \theta}{2n R^2}}}
\]

(10.22)

and

\[
Q_i = \frac{l_i (l_i + 1) \cdots (l_i + n - 1)}{n!}.
\]

(10.23)

We have chosen the radius \( R \) of the sphere so that \( R^2 > \frac{m \theta}{2n} \).

The solutions (10.21) coincide with those given by (10.15) if one assigns different noncommutativity parameters \( \theta^i \) to the worldvolumes of D(2\( n \))-branes carrying different magnetic fluxes proportional to \( m - 2i \). Then the field strength \( F^i(\theta^i) \) on \( \mathbb{R}^{2n}_y \) obtained from (10.21) will have finite topological charge \( Q_i \) given by (10.23) and corresponding finite BPS energy analogous to (10.16), and the configuration thus described extends to instantons on \( \mathbb{R}^{2n}_y \times \mathbb{C}P^1 \). The interesting idea of introducing distinct noncommutativity parameters on multiple coincident D-branes, generated by different magnetic fluxes on their worldvolumes [50], was discussed in [51] as a means (among other things) of stabilizing brane-antibrane systems. This proposal gains support from our Higgs vacuum BPS solutions (10.21) which carry different magnetic fluxes on different branes.

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