Abstract—We present a coding paradigm that provides a new achievable rate for the primitive relay channel by combining compress-and-forward and decode-and-forward with a chaining construction.

In the primitive relay channel model, the source broadcasts a message to the relay and to the destination; and the relay facilitates this communication by sending an additional message to the destination through a separate channel. Two well-known coding approaches for this setting are decode-and-forward and compress-and-forward: in the former, the relay decodes the message and sends some of the information to the destination; in the latter, the relay does not attempt to decode, but it sends a compressed description of the received sequence to the destination via Wyner-Ziv coding.

In our scheme, we transmit over pairs of blocks and we use compress-and-forward for the first block and decode-and-forward for the second. In particular, in the first block, the relay does not attempt to decode and it sends only a part of the compressed description of the received sequence; in the second block, the relay decodes the message and sends this information plus the remaining part of the compressed sequence relative to the first block. As a result, we strictly outperform both compress-and-forward and decode-and-forward. Furthermore, this paradigm can be implemented with a low-complexity polar coding scheme that has the typical attractive features of polar codes, i.e., quasi-linear encoding/decoding complexity and super-polynomial decay of the error probability. Throughout the paper we consider as a running example the special case of the erasure relay channel and we compare the rates achievable by our proposed scheme with the existing upper and lower bounds.

I. INTRODUCTION

The relay channel was introduced by van der Meulen [1] and it represents the simplest network model: a source wants to communicate with a destination with the help of a relay. As schematized in Figure 1, the source sends $X_s$ to the relay and to the destination, the relay receives $Y_{sr}$ and sends $X_s$ to the destination, and the destination receives $Y_d$ from the source and from the relay. Hence, the relay channel has a broadcast component that goes from the source to the relay and to the destination, and a multiple access component that goes from the source and from the relay to the destination.

In their seminal work [2], Cover and El Gamal introduced two basic achievability lower bounds, namely, decode-and-forward and compress-and-forward, and a general upper bound, namely, the cut-set bound. Since then, many other lower bounds have been discovered, namely, amplify-and-forward, quantize-map-and-forward, compute-and-forward, noisy network coding, hybrid coding, etc [3]–[7]. Furthermore, in most of the cases where the capacity is known, the converse is given by the cut-set bound [2], [8]–[10]. However, the cut-set bound has been shown not be tight for some specific examples of relay channels [11], [12]. New upper bounds tighter than the cut-set bound were recently developed in [13]–[16]. For a literature review on the relay channel, the interested reader is referred to [17] Chapter 16 and [18] Chapter 9.

Recent works have been aimed at providing practical schemes using the paradigm of polar coding, introduced by Arikan in the seminal paper [19]. In particular, polar coding schemes for decode-and-forward are proposed in [20]–[23] for degraded relay channels, where $X_s \rightarrow (X_s, Y_{sr}) \rightarrow Y_d$ forms a Markov chain. Furthermore, a polar coding scheme for compress-and-forward is presented in [22] for relay channels with orthogonal receiver components. Eventually, polar coding schemes for decode-and-forward and compress-and-forward are described in [24] for general relay channels.

Soft decode-and-forward relaying strategies using LDPC codes are developed and analyzed in [25].

In this paper, we focus on relay channels with orthogonal receiver components, also known as primitive relay channels. As schematized in Figure 2, the destination receives separately $Y_{sd}$ from the source and $Y_{rd}$ from the relay. In other words, the
multiple access component that for the general relay channel goes from the source and from the relay to the destination is substituted by two parallel channels. Consequently, without loss of generality, we can assume that the link between relay and destination is a noiseless channel of capacity \(C_{SD}\). Indeed, the relay can always use an optimal point-to-point scheme to communicate with the destination. We also assume that the relay is full-duplex, in the sense that it can listen and transmit simultaneously. Despite this simplification, the capacity of the primitive relay channel is not known in general. For a review on coding techniques for the primitive relay channel, see [26, Proposition 1].

Our main contribution consists in presenting a new coding scheme that, by combining compress-and-forward and decode-and-forward, strictly improves on both these lower bounds. To do so, we code over pairs of blocks and employ a chaining construction: in the first block, we perform a variant of compress-and-forward in which the relay is not required to send the whole compressed sequence to the destination; in the second block, we perform decode-and-forward and the relay sends to the destination the new information bits plus the remaining part of the compressed sequence from the previous block. The idea of chaining was originally introduced in [27] to construct universal codes and in [28] to achieve strong security guarantees on degraded wiretap channels. Since then, it has been exploited in several non-standard scenarios (e.g., broadcast channels [29, 30], asymmetric channels [31, 32], wiretap channels [33]). Our coding paradigm can be implemented with codes that are suitable for compress-and-forward and decode-and-forward. As such, polar codes represent an appealing choice [24]: their encoding/decoding complexity is \(\Theta(n \log n)\) and their error probability scales roughly as \(2^{-\sqrt{n}}\), where \(n\) is the block length.

The reminder of this paper is organized as follows. In Section II we review the existing upper bounds (cut-set and its improvements) and lower bounds (direct transmission, decode-and-forward, and compress-and-forward) for the primitive relay channel and we specialize them to the case of the erasure relay channel. In Section III we state our new lower bound and we show how to achieve it. In Section IV we compare the rates achievable by our new strategy with the existing upper and lower bounds for the special case of the erasure relay channel. In Section V we provide some concluding remarks.

II. EXISTING UPPER AND LOWER BOUNDS

We assume that all channels are binary memoryless and symmetric (BMS). We denote by \(h_2(x) = -x \log_2 x + (1 - x) \log_2 (1 - x)\) the binary entropy function and by \(X_\alpha, Y_\alpha, \text{ and } Y_{\alpha|\alpha}\) the alphabets associated to \(X_\alpha, Y_\alpha, \text{ and } Y_{\alpha|\alpha}\), respectively. We define \(\alpha \circ b = a + b(1 - a)\) for any \(a, b \in \mathbb{R}\).

Throughout the paper, we will use as a running example the special case of the erasure relay channel. As schematized in Figure 3, in the erasure relay channel the links between source and destination and between source and relay are binary erasure channels (BECs) with erasure probabilities \(\varepsilon_{SD}\) and \(\varepsilon_{SR}\), respectively.

A. Cut-Set Upper Bound

For the general relay channel, the cut-set upper bound on the achievable rate \(R\) is given by [17] Theorem 16.1

\[
R \leq \max_{p_{X_S}} \min\{I(X_S, X_S; Y_0) ; I(X_S, Y_{SR}, Y_{SD}|X_S)\}. \tag{1}
\]

For the case of the primitive relay channel, the cut-set bound specializes to [26] Proposition 1

\[
R \leq \max_{p_{X_S}} \min\{I(X_S; Y_{SR}) + C_{SD}; I(X_S; Y_{SR}, Y_{SD})\}. \tag{2}
\]

For the special case of the erasure relay channel, the cut-set bound can be rewritten as

\[
R \leq \min\{1 - \varepsilon_{SD} + C_{SD}; 1 - \varepsilon_{SR}\varepsilon_{SD}\}. \tag{3}
\]

B. Improvements on Cut-Set Upper Bound

For the case of the primitive relay channel, an upper bound demonstrating an explicit gap to the cut-set bound was presented in [13]. Furthermore, two new upper bounds that are generally tighter than cut-set are proposed in [14] for the symmetric primitive relay channel, in which \(Y_{SD}\) and \(Y_{SR}\) are conditionally identically distributed given \(X_S\). The results of [14] are extended to the non-symmetric case and to the Gaussian case in [15] and [16], respectively.

Let us now state the result in [15] Theorem 3.1, which provides an extension of the first bound of [14]. If a rate \(R\) is achievable, then there exists some \(p_{X_S}(x_S)\) and \(a \geq 0\) such that

\[
R \leq I(X_S; Y_{SR}, Y_{SD}),
R \leq I(X_S; Y_{SD}) + C_{SD} - a,
R \leq I(X_S; Y_{SD}, Y_{SR}) + h_2\left(\frac{a \ln 2}{2}\right) + \sqrt{\frac{a \ln 2}{2}} \log_2(|Y_{SR}| - 1) - a, \tag{4}
\]

for any random variable \(Y_{SR}\) with the same conditional distribution as \(Y_{SR}\) given \(X_S\). The evaluation of the term \(I(X_S; Y_{SD}, Y_{SR})\)
that gives the tightest bound is simple in the following special cases:

1) Symmetric \((Y_{\text{in}}\text{ and } Y_{\text{o}})\text{ conditionally identically distributed given } X_{\text{i}}): I(X_{\text{i}}; Y_{\text{in}}, Y_{\text{m}}) = I(X_{\text{i}}; Y_{\text{m}}).

2) Degraded \((Y_{\text{in}})\text{ stochastically degraded version of } Y_{\text{m}}): I(X_{\text{i}}; Y_{\text{in}}, Y_{\text{m}}) = I(X_{\text{i}}; Y_{\text{m}}).

3) Reversely degraded \((Y_{\text{in}})\text{ stochastically degraded version of } Y_{\text{m}}): I(X_{\text{i}}; Y_{\text{in}}, Y_{\text{m}}) = I(X_{\text{i}}; Y_{\text{m}}).

For the special case of the erasure relay channel, the bound can be re-written as

\[
R \leq \max_{a \geq 0} \left\{ 1 - \varepsilon_{\text{in}} + 1 - \varepsilon_{\text{m}} + C_{\text{rd}} - a, \right. \\
\left. 1 - \min\{\varepsilon_{\text{in}}, \varepsilon_{\text{m}}\} + h_2\left(\frac{a \ln 2}{2} + \frac{a \ln 2}{2} - a\right). \right\}
\]

In order to present the second bound of \([14]\), we need some preliminary definitions. Given a channel transition probability \(p(\omega|x)\), for any \(p(x)\) and \(d \geq 0\), we define \(\Delta(p(x), d)\) as

\[
\Delta(p(x), d) = \max_{\mathcal{P}(\omega|x)} \{H(\tilde{p}(\omega|x)p(x)) + D(\tilde{p}(\omega|x)|p(\omega|x)p(x)) - H(p(\omega|x)p(x))\},
\]

subject to the condition

\[
\frac{1}{2} \sum_{(x, \omega)} |p(x)\tilde{p}(\omega|x) - p(x)p(\omega|x)| \leq d,
\]

where \(D(\tilde{p}(\omega|x)|p(\omega|x)p(x))\) is the conditional relative entropy defined as

\[
D(\tilde{p}(\omega|x)|p(\omega|x)p(x)) = \sum_{(x, \omega)} p(x)\tilde{p}(\omega|x) \log_2 \frac{\tilde{p}(\omega|x)}{p(\omega|x)},
\]

and \(H(p(\omega|x)p(x))\) is the conditional entropy similarly defined with respect to \(p(x)p(\omega|x)\). At this point, we can state the result in \([14] \text{ Theorem 4.2.}\) If a rate \(R\) is achievable, then there exists some \(p(x|x)\) and \(a \in [0, \min\{C_{\text{rd}}, H(Y_{\text{m}}|X_{\text{i}})\}]\) such that

\[
\begin{align*}
R &\leq I(X_{\text{i}}; Y_{\text{in}}, Y_{\text{m}}), \\
R &\leq I(X_{\text{i}}; Y_{\text{m}}) + C_{\text{rd}} - a, \\
R &\leq I(X_{\text{i}}; Y_{\text{m}}) + \Delta\left(p(x|x), \frac{a \ln 2}{2}\right).
\end{align*}
\]

As pointed out at the end of Section IV.C of \([14]\), for the special case of the symmetric erasure relay channel, we have that \(\Delta(p(x|x), d) = \infty\) for all \(p(x|x)\) and \(d > 0\). Thus, \((16)\) reduces to the cut-set bound \((3)\).

C. Direct Transmission Lower Bound

In the direct transmission, the source communicates with the destination by using an optimal point-to-point code. The relay transmission is fixed at the most favorable symbol for the channel from the source to the destination.

For the general relay channel, direct transmission allows to achieve the following rate \([17] \text{ Section 16.3]}:\)

\[
R_{\text{DT}} = \max_{p(x; x_{\text{r})}} I(X_{\text{i}}; Y_{\text{m}}|X_{\text{r}} = x_{\text{r}}).
\]

For the case of the primitive relay channel, the direct transmission lower bound specializes to

\[
R_{\text{DT}} = \max_{p(x)} I(X_{\text{i}}; Y_{\text{m}}).
\]

Note that the direct transmission lower bound \((12)\) meets the cut-set upper bound \((3)\) and it equals the capacity of the primitive relay channel when either of the following two conditions holds:

1) the primitive relay channel is reversely degraded, which implies that \(I(X_{\text{i}}; Y_{\text{in}}) = I(X_{\text{i}}; Y_{\text{in}}|Y_{\text{m}});\)

2) \(C_{\text{rd}} = 0\).

For the special case of the erasure relay channel, the direct transmission lower bound can be rewritten as

\[
R_{\text{DT}} = 1 - \varepsilon_{\text{in}}.
\]

The direct transmission lower bound \((13)\) meets the cut-set upper bound \((3)\) and it equals the capacity of the erasure relay channel when either \(1 - \varepsilon_{\text{in}} = 1 - \varepsilon_{\text{in}} + C_{\text{rd}}\) or \(C_{\text{rd}} = 0\).

D. Decode-and-Forward Lower Bound

In decode-and-forward, the relay completely decodes the received sequence and cooperates with the source to communicate the message to the destination.

For the general relay channel, decode-and-forward allows to achieve the following rate \([17] \text{ Theorem 16.2]}:\)

\[
R_{\text{DF}} = \max_{p(x; x_{\text{r})}} \min\{I(X_{\text{i}}; X_{\text{r}}; Y_{\text{in}}), I(X_{\text{i}}; Y_{\text{in}}|X_{\text{r}})\}.
\]

For the case of the primitive relay channel, the decode-and-forward lower bound specializes to \([20] \text{ Proposition 2]}\)

\[
R = \max_{p(x)} \min\{I(X_{\text{i}}; Y_{\text{in}}) + C_{\text{rd}}; I(X_{\text{i}}; Y_{\text{in}})\}.
\]

Note that the decode-and-forward lower bound \((15)\) meets the cut-set upper bound \((2)\) and it equals the capacity of the primitive relay channel when either of the following two conditions holds:

1) the primitive relay channel is degraded, which implies \(I(X_{\text{i}}; Y_{\text{in}}) = I(X_{\text{i}}; Y_{\text{in}}|Y_{\text{m}});\)

2) \(I(X_{\text{i}}; Y_{\text{m}}) \geq I(X_{\text{i}}; Y_{\text{in}}) + C_{\text{rd}}\).

For the special case of the erasure relay channel, the decode-and-forward lower bound can be rewritten as

\[
R = \min\{1 - \varepsilon_{\text{in}} + C_{\text{rd}}, 1 - \varepsilon_{\text{in}}\}.
\]

The decode-and-forward lower bound \((16)\) meets the cut-set upper bound \((3)\) and it equals the capacity of the erasure relay channel when either \(1 - \varepsilon_{\text{in}} = 1 - \varepsilon_{\text{in}} + C_{\text{rd}}\) or \(1 - \varepsilon_{\text{in}} + C_{\text{rd}} \leq 1 - \varepsilon_{\text{in}}\).
E. Compress-and-Forward Lower Bound

In compress-and-forward, the relay does not attempt to decode the received sequence, but it sends a (possibly compressed) description of it, denoted by \( \hat{Y}_{sr} \), to the destination. Since this description is correlated with the sequence received by the destination from the source, Wyner-Ziv coding is used to reduce the rate needed to communicate it to the destination.

For the general relay channel, compress-and-forward allows to achieve the following rate [17, Theorem 16.4]:

\[
R_{CF} = \max_{p_{X,Y}} \min \{ I(X; \hat{Y}_{sr}, Y) - I(Y_{sr}; \hat{Y}_{sr}|X, Y) \},
\]

(17)

where the cardinality of the alphabet associated to \( \hat{Y}_{sr} \) can be bounded as \(|\hat{Y}_{sr}| \leq |X| : |Y_{sr}| + 1\). This expression can be equivalently rewritten as [17, Remark 16.3]

\[
R_{CF} = \max_{p_{X,Y}} \{ I(X; \hat{Y}_{sr}, Y) : I(Y_{sr}; \hat{Y}_{sr}|X, Y) \leq I(X; Y) \}.
\]

The bound is in general not convex, therefore it can be improved via time sharing.

For the case of the primitive relay channel, the compress-and-forward lower bound specializes to [26, Proposition 3]

\[
R_{CF} = \max_{p_{X,Y} \mid Y_{sr}} \{ I(X; \hat{Y}_{sr}, Y) : I(Y_{sr}; \hat{Y}_{sr}|X, Y_{sr}) \leq C_{SD} \},
\]

(19)

with \(|\hat{Y}_{sr}| \leq |Y_{sr}| + 1\).

Note that the compress-and-forward lower bound (19) meets the cut-set upper bound (23) and it equals the capacity of the primitive relay channel when \( H(Y_{sr}|Y_{so}) \leq C_{sd} \). Indeed, in this case, we can pick \( Y_{sr} = \hat{Y}_{sr} \), namely, the relay performs Slepian-Wolf source coding. Therefore, \( R_{CF} = I(X; Y_{sr}, Y_{so}) \), which is one of the two terms in the cut-set bound.

On the contrary, if \( H(Y_{sr}|Y_{so}) > C_{sd} \), then we degrade \( Y_{sr} \) into \( \hat{Y}_{sr} \), namely, the relay performs a step of lossy source coding. The relay transmits this lossy description to the destination that can decode it successfully since \( Y_{sr} \) requires less bits than \( \hat{Y}_{sr} \). However, after that the destination has recovered \( \hat{Y}_{sr} \), there is a penalty loss: we can achieve rates up to \( I(X; \hat{Y}_{sr}, Y_{so}) \), instead of up to \( I(X; Y_{sr}, Y_{so}) \).

For the case of the erasure relay channel, we have that

\[
H(Y_{sr}|Y_{so}) = h_2(\varepsilon_{sr}) + \varepsilon_{so}(1 - \varepsilon_{so}) \text{.}
\]

(20)

Hence, if \( C_{sd} \geq h_2(\varepsilon_{sr}) + \varepsilon_{so}(1 - \varepsilon_{so}) \), then the compress-and-forward lower bound meets the cut-set upper bound and it equals the capacity of the erasure relay channel.

On the contrary, if \( C_{sd} < h_2(\varepsilon_{sr}) + \varepsilon_{so}(1 - \varepsilon_{so}) \), it is not easy to find the best choice of \( Y_{sr} \) even for this simple scenario. Following [25], let us assume that \( Y_{sr} \) is the output of an erasure-erasure channel (EEC) with erasure probability \( \varepsilon_{sr} \) and input \( Y_{sr} \). This means that if \( Y_{sr} = \hat{y} \), then \( Y_{sr} = \hat{y} \) with probability 1; if \( Y_{sr} \in \{0, 1\} \), then \( \hat{Y}_{sr} = \hat{y} \) with probability \( \varepsilon_{sr} \) and \( Y_{sr} = \hat{y} \) with probability \( 1 - \varepsilon_{sr} \). Consequently,

\[
I(X; \hat{Y}_{sr}, Y_{so}) = H(X) - H(X|\hat{Y}_{sr}, Y_{so})
\]

\[
= H(X)(1 - (\varepsilon_{sr} + \varepsilon_{so} \cdot \varepsilon_{sr}) \text{.}
\]

(21)

Clearly, \( I(X; \hat{Y}_{sr}, Y_{so}) \) is maximized by setting \( p_{X,Y} \) to the uniform distribution. Furthermore,

\[
H(Y_{sr}|Y_{so}) = H(\hat{Y}_{sr}|Y_{so}) = (1 - \varepsilon_{sr}) h_2(\varepsilon_{sr}) \text{.}
\]

(22)

As a result, the rate (19) can be rewritten as

\[
R_{CF} = \max_{0 \leq \varepsilon_{sr} \leq 1} \{ 1 - (\varepsilon_{sr} + \varepsilon_{so} \cdot \varepsilon_{sr}) \}.
\]

(23)

III. MAIN RESULT

We are now ready to state our new lower bound for the primitive relay channel.

Theorem 1. Consider the transmission over a primitive relay channel, where the source sends \( X \) to the relay and the destination, the relay receives \( Y_{sr} \) from the source, the destination receives \( Y_{so} \) from the source, and relay and destination are connected via a noiseless link with capacity \( C_{sd} \). Furthermore, denote by \( \hat{Y}_{sr} \) the compressed description of \( Y_{sr} \) transmitted by the relay. Then, the following rate is achievable:

\[
R_{DF-CF} = \frac{C_{sd} - \max \{ 0, I(X; Y_{so}) - I(X; Y_{sr}) \} I(X; \hat{Y}_{sr}, Y_{so})}{I(Y_{sr}; \hat{Y}_{sr}|Y_{so}) - \max \{ 0, I(X; Y_{sr}) - I(X; Y_{so}) \}} I(Y_{sr}; \hat{Y}_{sr}|Y_{so}) + \max \{ 0, I(X; Y_{sr}) - I(X; Y_{so}) \} I(Y_{sr}; Y_{so}) - C_{sd},
\]

(24)

for any joint distribution \( p_{X,Y|Y_{sr}} \) such that

\[
I(X; Y_{sr}) < I(X; Y_{so}) + C_{sd},
\]

(25)

\[
I(Y_{sr}; \hat{Y}_{sr}|Y_{so}) \geq C_{sd},
\]

(26)

and where \(|\hat{Y}_{sr}| \leq |Y_{sr}| + 1\). Furthermore, the rate (24) can be achieved by a polar coding scheme with encoding/decoding complexity \( \Theta(n \log n) \) and error probability \( O(2^{-n}) \) for any \( \beta \in (0, 1/2) \), where \( n \) is the block length.

Remark 2. If (25) does not hold, then decode-and-forward achieves the cut-set bound and it is optimal. Furthermore, if (26) does not hold, then our scheme reduces to compress-and-forward and the achievable rate is given by (19).

The special case of the erasure relay channel is handled by the corollary below.

Corollary 3. Consider the transmission over the erasure relay channel, where \( Y_{sr} \) is obtained from \( X \) via a BEC(\( \varepsilon_{sr} \)), \( Y_{so} \) is obtained from \( X \) via a BEC(\( \varepsilon_{so} \)), \( \hat{Y}_{sr} \) is obtained from \( Y_{sr} \) via an EEC(\( \varepsilon_{sr} \)), and the relay is connected to the destination.
via a noiseless link with capacity $C_{sr}$. Then, the rate (27) is achievable for any $\hat{\epsilon}_s \in [0, 1]$ such that
\[
1 - \epsilon_{sr} < 1 - \epsilon_{dc} + C_{sr},
\]
(28)
\[
h_2(\epsilon_{sr} \circ \hat{\epsilon}_s) + \epsilon_{sd}(1 - \epsilon_{sr} \circ \hat{\epsilon}_s) - (1 - \epsilon_{sr}) h_2(\hat{\epsilon}_s) - \max\{0, \epsilon_{sd} - \epsilon_{sr}\} \geq C_{sd}.
\]
(29)

Furthermore, the rate (27) can be achieved by a polar coding scheme with encoding/decoding complexity $\Theta(n \log n)$ and error probability $O(2^{-n^\beta})$ for any $\beta \in (0, 1/2)$, where $n$ is the block length.

The proof of Corollary 3 easily follows from the application of Theorem 1 and of formulas (21)-(22).

We can now proceed with the proof of our main result.

Proof of Theorem 1: We start by presenting the main idea of our scheme. We split the transmission into two blocks. In the first block, we perform a variant of compress-and-forward: the relay does not decode the received sequence, but it sends a compressed description of it to the destination. However, differently from standard compress-and-forward, we require that (27) holds. Hence, we cannot transmit all the compressed description $\hat{Y}_{sr}$ during the first block. In the second block, we perform decode-and-forward: the relay completely decodes the received sequence. Furthermore, we choose the length of the second block so that the relay can transmit the part of $\hat{Y}_{sr}$ that was not sent in the previous block plus the new information needed to decode the second block.

Let us now describe this scheme more in detail and provide the achievability proof of the rate (24). First, we deal with the case $I(X_s; Y_{sr}) \geq I(X_s; Y_{sr})$.

Consider the transmission of the first block. Denote by $n_1$ and $R_1$ the block length and the rate of the message transmitted by the source, and let $R_1$ approach from below $I(X_s; Y_{sr}, Y_{sd})$. The relay receives $Y_{sr}$ and constructs the compressed description $\hat{Y}_{sr}$. Recall that the destination receives the side information $Y_{sr}$ from the source. Hence, by using Wyner-Ziv coding, the destination needs from the relay a number of bits approaching from above $I(Y_{sr}; Y_{sd} | Y_{sr}, n_1)$, in order to decode the message sent by the source. As $I(Y_{sr}; Y_{sd} | Y_{sr}) \geq C_{sd}$, the relay transmits right away a number of these bits approaching from below $C_{sr}$. The number of remaining bits approaches from above $I(Y_{sr}; Y_{sd} | Y_{sr}, n_1)$ and it is stored by the relay. The destination stores the message received from the relay and the observation $Y_{sr}$ obtained from the source.

Consider the transmission of the second block and define
\[
\alpha = \frac{I(Y_{sr}; Y_{sd} | Y_{sr}) - C_{sd}}{C_{rd} - I(X_s; Y_{sr}) - I(X_s; Y_{sd})}.
\]
(30)

Denote by $n_2$ and $R_2$ the block length and the rate of the message transmitted by the source. Let $n_2 = n_1 \cdot \alpha$ and let $R_2$ approach from below $I(X_s; Y_{sr})$. The relay receives $Y_{sd}$ and successfully decodes the message. Again, the destination receives the side information $Y_{sr}$ from the source. Hence, it needs from the relay a number of bits approaching from above $I(X_s; Y_{sr}) - I(X_s; Y_{sd}) \cdot n_1 \cdot \alpha$, in order to decode the message sent by the source. The relay transmits to the destination the extra $I(Y_{sr}; Y_{sd} | Y_{sr}, n_1)$ · $n_1$ · $\alpha$ information bits plus the $I(Y_{sr}; Y_{sd} | Y_{sr}) - C_{sd}$ · $n_1$ bits remaining from the previous block. This transmission is reliable as (30) implies that
\[
I(X_s; Y_{sr}) - I(X_s; Y_{sd}) \cdot n_1 \cdot \alpha + I(Y_{sr}; Y_{sd} | Y_{sr}) - C_{sd} \cdot n_1 = C_{sr} \cdot n_2.
\]
(31)

At this point, the destination can reconstruct the second block by using the side information received from the source and the extra $I(Y_{sr}; Y_{sd} | Y_{sr}) - C_{sd} \cdot n_1$ bits received from the relay. Furthermore, it can also reconstruct the first block by using the side information previously received from the source and the extra $I(Y_{sr}; Y_{sd} | Y_{sr}) - C_{sr} \cdot n_1$ bits received from the relay (partly in the first and partly in the second block).

The overall block length is $n = n_1 + n_2 = (1 + \alpha)n_1$ and the achievable rate is
\[
R = \frac{R_1 + \alpha R_2}{1 + \alpha},
\]
(32)

which approaches from below
\[
\frac{C_{sr} - I(X_s; Y_{sr}) - I(X_s; Y_{sd})}{I(X_s; Y_{sr}, Y_{sd})} I(X_s; Y_{sr}, Y_{sd})
\]
\[
\cdot \frac{I(Y_{sr}; Y_{sd} | Y_{sr}) - I(X_s; Y_{sd})}{I(X_s; Y_{sr}) - I(X_s; Y_{sd})}
\]
\[
+ \frac{I(Y_{sr}; Y_{sd} | Y_{sr}) - C_{sd}}{I(X_s; Y_{sr}) - I(X_s; Y_{sd})},
\]
(33)

Note that the expression (33) coincides with (24) when $I(X_s; Y_{sr}) \geq I(X_s; Y_{sr})$.

The case $I(X_s; Y_{sr}) < I(X_s; Y_{sr})$ is handled in a similar way. As concerns the transmission of the first block, nothing changes. Denote by $n_1'$ and $R_1'$ the block length and the rate of the message transmitted by the source, and let $R_1'$ approach from below $I(X_s; Y_{sr}, Y_{sd})$. The relay receives $Y_{sr}$ and constructs the compressed description $\hat{Y}_{sr}$. By using Wyner-Ziv coding, the destination needs from the relay a number of bits approaching from above $I(Y_{sr}; \hat{Y}_{sr} | Y_{sr}, n_1')$, in order to decode the message sent by the source. As $I(Y_{sr}; \hat{Y}_{sr} | Y_{sr}) \geq C_{sr}$, the relay transmits right away a number of these bits approaching from below $C_{sr} \cdot n_1'$. The number of remaining bits approaches from above $I(Y_{sr}; \hat{Y}_{sr} | Y_{sr}, n_1') \cdot n_1'$ and it is stored by the relay. The destination stores the message received from the relay and the observation $Y_{sr}$ obtained from the source.
As concerns the transmission of the second block, define
\[
\alpha' = \frac{I(Y_{m}; \hat{Y}_m|Y_m) - C_{RD}}{C_{RD}},
\]
and denote by \(n'_2\) and \(R'_2\) the block length and the rate of the message transmitted by the source. Let \(n'_2 = n'_1 \cdot \alpha'\) and let \(R'_2\) approach from below \(I(X_i; Y_m)\). The relay discards the received message and transmits to the destination the \((I(Y_{m}; \hat{Y}_m|Y_m) - C_{RD}) \cdot n'_1\) bits remaining from the previous block. This transmission is reliable as \((34)\) implies that
\[
(I(Y_{m}; \hat{Y}_m|Y_m) - C_{RD}) \cdot n'_1 = C_{RD} \cdot n'_2.
\]
At this point, the destination can reconstruct the second block and-forward for an interval of values of \(n'_1\). Furthermore, our paradigm can be implemented with a low-complexity polar coding scheme that has the typical attractive features of polar codes, i.e., quasi-linear encoding/decoding complexity and super-polynomial decay of the error probability.

V. CONCLUDING REMARKS

We have proposed a new coding paradigm for the primitive relay channel that combines compress-and-forward and decode-and-forward by means of a chaining construction. The achievable rates obtained by our scheme surpass the state-of-the-art coding approaches (compress-and-forward, decode-and-forward, and the soft decode-and-forward strategy of \([25]\)). Furthermore, our paradigm can be implemented with a low-complexity polar coding scheme that has the typical attractive features of polar codes, i.e., quasi-linear encoding/decoding complexity and super-polynomial decay of the error probability.

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Figure 4: Comparison between the achievable rate provided by our strategy and the existing upper and lower bounds. We use “CF” and “DF” as abbreviations for “compress-and-forward” and “decode-and-forward”, respectively.

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