THE $A_\ell$ AND $C_\ell$ BAILEY TRANSFORM AND LEMMA

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ABSTRACT. We announce a higher-dimensional generalization of the Bailey Transform, Bailey Lemma, and iterative “Bailey chain” concept in the setting of basic hypergeometric series very well-poised on unitary $A_\ell$ or symplectic $C_\ell$ groups. The classical case, corresponding to $A_1$ or equivalently $U(2)$, contains an immense amount of the theory and application of one-variable basic hypergeometric series, including elegant proofs of the Rogers-Ramanujan-Schur identities. In particular, our program extends much of the classical work of Rogers, Bailey, Slater, Andrews, and Bressoud.

1. Introduction

The purpose of this paper is to announce a higher-dimensional generalization of the Bailey Transform [2] and Bailey Lemma [2] in the setting of basic hypergeometric series very well-poised on unitary [19] or symplectic [14] groups. Both types of series are directly related [14, 18] to the corresponding Macdonald identities. The series in [19] were strongly motivated by certain applications of mathematical physics and the unitary groups $U(n)$ in [10, 11, 15, 16]. The unitary series use the notation $A_\ell$, or equivalently $U(\ell+1)$; the symplectic case, $C_\ell$. The classical Bailey Transform, Lemma, and very well-poised basic hypergeometric series correspond to the case $A_1$, or equivalently $U(2)$.

The classical Bailey Transform and Bailey Lemma contain an immense amount of the theory and application of one-variable basic hypergeometric series [2, 12, 25]. They were ultimately inspired by Rogers’ [24] second proof of the Rogers-Ramanujan-Schur identities [23]. The Bailey Transform was first formulated by Bailey [8], utilized by Slater in [25], and then recast by Andrews [4] as a fundamental matrix inversion result. This last version of the Bailey Transform has immediate applications to connection coefficient theory and “dual” pairs of identities [4], and $q$-Lagrange inversion and quadratic transformations [13].

The most important application of the Bailey Transform is the Bailey Lemma. This result was mentioned by Bailey [8; §4], and he described how the proof would work. However, he never wrote the result down explicitly and thus missed the full power of iterating it. Andrews first established the Bailey Lemma explicitly in [5] and realized its numerous possible applications in terms of the iterative “Bailey chain” concept. This iteration mechanism enabled him to derive many...
q-series identities by “reducing” them to more elementary ones. For example, the Rogers-Ramanujan-Schur identities can be reduced to the q-binomial theorem. Furthermore, general multiple series Rogers-Ramanujan-Schur identities are a direct consequence of iterating suitable special cases of Bailey’s Lemma. In addition, Andrews notes that Watson’s q-analog of Whipple’s transformation is an immediate consequence of the second iteration of one of the simplest cases of Bailey’s Lemma. Continued iteration of this same case yields Andrews’ [3] infinite family of extensions of Watson’s q-Whipple transformation. Even Whipple’s original work [26, 27] fits into the q = 1 case of this analysis. Paule [22] independently discovered important special cases of Bailey’s Lemma and how they could be iterated. Essentially all the depth of the Rogers-Ramanujan-Schur identities and their iterations is embedded in Bailey’s Lemma.

The process of iterating Bailey’s Lemma has led to a wide range of applications in additive number theory, combinatorics, special functions, and mathematical physics. For example, see [2, 5, 6, 7, 9].

The Bailey Transform is a consequence of the terminating \(4\phi_3\) summation theorem. The Bailey Lemma is derived in [1] directly from the \(6\phi_5\) summation and the matrix inversion formulation [4, 13] of the Bailey Transform. We employ a similar method in the \(A_\ell\) and \(C_\ell\) cases by starting with a suitable, higher-dimensional, terminating \(6\phi_5\) summation theorem extracted from [19] and [14], respectively. The \(A_\ell\) proofs appear in [20, 21], and the \(C_\ell\) case is established in [17]. Many other consequences of the \(A_\ell\) and \(C_\ell\) generalizations of Bailey’s Transform and Lemma will appear in future papers. These include \(A_\ell\) and \(C_\ell\) q-Pfaff-Saalschütz summation theorems, q-Whipple transformations, connection coefficient results, and applications of iterating the \(A_\ell\) or \(C_\ell\) Bailey Lemma.

2. Results

Throughout this article, let \(i, j, N,\) and \(y\) be vectors of length \(\ell\) with nonnegative integer components. Let \(q\) be a complex number such that \(|q| < 1\). Define

\[
(2.1a) \quad (\alpha)_\infty \equiv (\alpha; q)_\infty := \prod_{k \geq 0} (1 - \alpha q^k)
\]

and, thus,

\[
(2.1b) \quad (\alpha)_n \equiv (\alpha; q)_n := (\alpha)_\infty / (\alpha^n)_\infty.
\]

Define the Bailey transform matrices, \(M\) and \(M^*\), as follows.

**Definition (\(M\) and \(M^*\) for \(A_\ell\)).** Let \(a, x_1, \ldots, x_\ell\) be indeterminate. Suppose that none of the denominators in (2.2a–b) vanishes. Then let

\[
(2.2a) \quad M(i; j; A_\ell) := \prod_{r,s=1}^{\ell} \left( \frac{x_r^i q^{j_s} - j_s}{x_s^{i_r} q^{j_r} - j_r} \right)^{-1} \prod_{k=1}^{\ell} \left( \frac{a q^{x_k}}{x_k} \right)^{-1} \left( \frac{x_k^{i_r} q^{j_s} - j_s}{x_s^{i_r} q^{j_r} - j_r} \right)^{-1};
\]
and

\[(2.2b)\]
\[M^*(i; j; A_\ell) = \prod_{k=1}^{\ell} \left[ 1 - \frac{x_k}{x_\ell} q^{i_k+\cdots+i_1} \right] \prod_{k=1}^{\ell} \left( a \frac{x_k}{x_\ell} \right)^{j_k+\cdots+i_1 - 1}
\times \prod_{r,s=1}^{\ell} \left( q \frac{x_r}{x_s} q^{j_r-j_s} \right)^{-1} (-1)^{(i_1+\cdots+i_1)-(j_1+\cdots+j_\ell)} q^{(i_1+\cdots+i_\ell-j_1-\cdots-j_\ell)}/2.\]

**Definition** \((M^* and M^* for C_\ell)\). Let \(x_1, \ldots, x_\ell\) be indeterminate. Suppose that none of the denominators in \((2.3a-b)\) vanishes. Then let

\[(2.3a)\]
\[M(i; j; C_\ell) := \prod_{r,s=1}^{\ell} \left[ \left( q \frac{x_r}{x_s} q^{j_r-j_s} \right)^{-1} \left( q x_r x_s q^{j_r+j_s} \right)^{-1} \right] ;\]

and

\[(2.3b)\]
\[M^*(i; j; C_\ell) := \prod_{r,s=1}^{\ell} \left[ \left( q \frac{x_r}{x_s} q^{j_r-j_s} \right)^{-1} \left( q x_r x_s q^{j_r+j_s} \right)^{-1} \right] \prod_{1 \leq r < s \leq \ell} \frac{1 - \frac{x_r x_s q^{j_r+j_s}}{1 - \frac{x_r}{x_s} q^{j_r+j_s}}}{1 - \frac{x_r}{x_s} q^{j_r+j_s}}
\times (-1)^{(i_1+\cdots+i_1)-(j_1+\cdots+j_\ell)} q^{(i_1+\cdots+i_\ell-j_1-\cdots-j_\ell)/2}.\]

As in the classical case \([1]\), we have the following theorem.

**Theorem** \((Bailey Transform for \(A_\ell and C_\ell)\). Let \(G = A_\ell or C_\ell\). Let \(M^* and M^* \) be defined as in \((2.2) and (2.3)\), with rows and columns ordered lexicographically. Then \(M \) and \(M^* \) are inverse, infinite, lower-triangular matrices. That is,

\[(2.4)\]
\[\prod_{k=1}^{\ell} \delta(i_k, j_k) = \sum_{\substack{i_k \leq y_k \leq i_k \in \{1, 2, \ldots, \ell\}}} M(i; y; G) M^*(y; j; G),\]

where \(\delta(r, s) = 1 \) if \(r = s\), and 0 otherwise.

Equations \((2.2) and (2.3)\) motivate the definition of the \(A_\ell and C_\ell\) Bailey pair.

**Definition** \((G-Bailey Pair)\). Let \(G = A_\ell or C_\ell\). Let \(N_k \geq 0 \) be integers for \(k = 1, 2, \ldots, \ell\). Let \(A = \{A_{(y; G)}\} \) and \(B = \{B_{(y; G)}\} \) be sequences. Let \(M \) and \(M^* \) be as above. Then we say that \(A \) and \(B \) form a \(G\)-Bailey Pair if

\[(2.5)\]
\[B_{(N; G)} = \sum_{\substack{0 \leq y_k \leq N_k \leq \ell \in \{1, 2, \ldots, \ell\}}} M(N; y; G) A_{(y; G)}.\]

As a consequence of the Bailey transform, \((2.4)\), and the definition of the \(G\)-Bailey pair, \((2.5)\), we have the following result.
Corollary (Bailey Pair Inversion). A and B satisfy equation (2.5) if and only if

\[
A(\mathcal{N};G) = \sum_{0 \leq y_k \leq N_k} M^*(\mathcal{N}; y; G) B(y;G).
\]

Define the sequences \(A' = \{A'(y;A_i)\}\) and \(B' = \{B'(y;A_i)\}\) by

\[
A'(\mathcal{N};A_i) := \prod_{k=1}^{\ell} \left( \frac{aq x_k}{\rho x_k} \right)^{-1} N_k \prod_{k=1}^{\ell} \left( \frac{\sigma x_k}{x_k} \right)_{N_k} \times \frac{(\rho)_{N_1+\cdots+N_\ell} (aq/\rho \sigma)^N_{1+\cdots+N_\ell} A(\mathcal{N}; A_i)}{(aq/\sigma)^{N_1+\cdots+N_\ell}}
\]

and

\[
B'(\mathcal{N};A_i) := \sum_{0 \leq y_k \leq N_k} \left\{ \prod_{k=1}^{\ell} \left[ \left( \frac{\sigma x_k}{x_k} y_k \right)_{N_k} \left( \frac{aq x_k}{\rho x_k} \right)^{-1} N_k \prod_{r,s=1}^{\ell} \left( q_{x_k} q_{y_r} - y_s \right)_{N_r-y_r} \right] \right\}
\]

Define the sequences \(A' = \{A'(y;C_i)\}\) and \(B' = \{B'(y;C_i)\}\) by

\[
A'(\mathcal{N};C_i) := \prod_{k=1}^{\ell} \left[ \left( \frac{\alpha x_k}{\beta x_k} \right)_{N_k} \left( q x_k \beta^{-1} \right)_{N_k} \left( \frac{\beta}{\alpha} \right)_{N_1+\cdots+N_\ell} A(\mathcal{N}; C_i)
\]

and

\[
B'(\mathcal{N}; C_i) := \sum_{0 \leq y_k \leq N_k} \left\{ \prod_{k=1}^{\ell} \left[ \left( \frac{\alpha x_k}{\beta x_k} \right)_{N_k} \left( q x_k \beta^{-1} \right)_{N_k} \prod_{r,s=1}^{\ell} \left( q_{x_k} q_{y_r} - y_s \right)_{N_r-y_r} \right] \prod_{1 \leq r < s \leq \ell} \left[ \left( q x_r x_s q_{y_r} + y_s \right)_{N_r-y_r}^{-1} \left( q x_r x_s q_{N_r-y_r} \right)_{N_r-y_r}^{-1} \right] \left( \frac{\beta}{\alpha} \right)^{N_1+\cdots+N_\ell} B(y; C_i) \right\}
\]

These definitions lead to our generalization of Bailey’s lemma.

Theorem (The G-generalization of Bailey’s Lemma). Let \(G = A_\ell\) or \(C_\ell\). Suppose \(A = \{A(\mathcal{N};G)\}\) and \(B = \{B(\mathcal{N};G)\}\) form a G-Bailey Pair. If \(A' = \{A'(\mathcal{N};G)\}\) and \(B' = \{B'(\mathcal{N};G)\}\) are as above, then \(A'\) and \(B'\) also form a G-Bailey Pair.
3. Sketches of Proofs

Proof of (2.4). In each case, $A_\ell$ and $C_\ell$, we begin with a terminating $4\phi_3$ summation theorem. In the $C_\ell$ case, it is first necessary to specialize Gustafson’s $C_\ell \psi_6$ summation theorem, see [14], terminate it from below and then from above, and further specialize the resulting terminating $\psi_6$ to yield a terminating $4\phi_3$. In both the $A_\ell$ and $C_\ell$ cases, the $4\phi_3$ is modified by multiplying both the sum and product sides by some additional factors. Finally, that result is transformed term-by-term to yield the sum side of (2.4). □

Proof of (2.6). Equation (2.6) follows directly from the definition, (2.5), and the termwise nature of the calculations in the proof of (2.4). □

Proof of Bailey’s Lemma. The definitions in (2.7) and (2.8) are substituted into (2.5). After an interchange of summation, the inner sum is seen to be a special case of the appropriate $5\phi_5$. The $5\phi_5$ is then summed, and the desired result follows. □

Detailed proofs of the $C_\ell$ case will appear in [17], as will a discussion of the $C_\ell$ Bailey chain and a connection coefficient result associated with the $C_\ell$ Bailey Transform.

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