Well-posedness of a non-local abstract Cauchy problem with singular integral

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Abstract

In this work, the abstract Cauchy problem for an initial value system with singular integral is considered. The system is of closed form of two evolution equations for a real-valued function and a function-valued function. By proposing an appropriate Banach space, we prove the existence and uniqueness of classical solutions to the evolution system under assumptions on the boundedness and smoothness of data. Furthermore, we connect by an isomorphism the solution of the evolution system and a class of integral-differential equations with singular convolution kernels and extend our results to the corresponding problem. It is revealed that our findings also improve the understanding of the open problem on the well-posedness of the stationary Wigner equation with inflow boundary conditions.

Keywords: Partial integral-differential equations, Singular integral, Well-posedness, Wigner equation

1 Introduction

We begin by introducing the linear evolution system of the following form

\[
\begin{align*}
\frac{dc(t)}{dt} &= h(t,0)c(t) + \int_{-\infty}^{\infty} K(t,-x')v(t,x') \, dx', \\
\frac{\partial v(t,x)}{\partial t} &= c(t) : h(t,x) - h(t,0) + \int_{-\infty}^{\infty} \frac{K(t,x-x') - K(t,-x')}{x} v(t,x') \, dx',
\end{align*}
\]

(1.1)

where \((t,x) \in (0,T) \times \mathbb{R}, h(t,x), K(t,x)\) are given real-valued functions and \(c:[0,T] \to \mathbb{R}, v:[0,T] \times \mathbb{R} \to \mathbb{R}\) are unknown functions. In this paper, we study the solution \(c,v\) of (1.1) with the initial value

\[
c(0) = c_0, v(0,0) = v_0(x).
\]

(1.2)

Then, we simplify the notation by viewing the linear system (1.1), (1.2) as an abstract Cauchy problem

\[
\begin{align*}
\frac{d}{dt} \begin{pmatrix} c(t) \\ v(t) \end{pmatrix} &= B(t) \begin{pmatrix} c(t) \\ v(t) \end{pmatrix}, \quad t \in (0,T), \\
\begin{pmatrix} c(0) \\ v(0) \end{pmatrix} &= \begin{pmatrix} c_0 \\ v_0 \end{pmatrix},
\end{align*}
\]

(1.3)

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regarding \( v = v(t, \cdot) \) as a function-valued function. For each \( t \in (0, +\infty) \), \( B(t) \) is a linear operator of \( (c, v)^T \)

\[
B(t) \begin{pmatrix} c \\ v \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} ch(x, 0) + \int_{-\infty}^{\infty} K(t, -x') v(x') \, dx' \\ c \cdot h(t, x) - h(t, 0) + \int_{-\infty}^{\infty} \frac{K(t, x - x') - K(t, -x')}{x} v(x') \, dx' \end{pmatrix} v \end{pmatrix}.
\]

(1.4)

In this paper, we put forward an appropriate Banach space \( X \) (it is in fact a direct sum of \( \mathbb{R} \) and \( L^2(\mathbb{R}) \). See section 2), which ensures that \( B(t) \) are bounded linear operators of \( X \), under some bounded and smooth conditions on \( h, K \). Then the well-posedness of the abstract Cauchy problem (1.3) is obtained by using the semigroup theory of linear operators.

On the other side of our work, we assume that \( h(t, x) \) is dependent on \( K(t, x) \) with the formulation

\[
h(t, x) = \int_{-\infty}^{\infty} \frac{K(t, x - x')}{x'} \, dx',
\]

(1.5)

where the right hand side is the integral in the sense of principal value (defined in Section 3), since the integrand function in (1.5) is singular. We then construct a function from the solution \( (c, v)^T \) of this problem by an isomorphism,

\[
u(t, x) = \frac{c(t)}{x} + v(x, t).
\]

(1.6)

It will be verified that the constructed function turns out to be a solution of the initial value problem of a partial integral-differential equation (PIDE)

\[
\begin{cases}
\frac{\partial u(t, x)}{\partial t} = \frac{1}{x} \Psi[K] u(t, x), & t \in (0, T), \\
u(t_0, x) = u_0(x) = \frac{c_0(t)}{x} + v_0(x, t),
\end{cases}
\]

(1.7)

where \( \Psi[K] \) is the convolution operator with the kernel \( K \),

\[
\Psi[K] u(t, x) = \int_{-\infty}^{\infty} K(t, x - x') u(t, x') \, dx'.
\]

(1.8)

Note that if \( c(t) \) in (1.6) is not zero, the solution is not a function in \( L^2(\mathbb{R}) \). Thus, if \( L^2(\mathbb{R}) \) is chosen to be the solution space, there may exist no solution to the initial value problem (1.7) (such circumstance is indeed possible, which will be verified by a forthcoming work).

The problem (1.7) is, however, the one with a singular integral. We solve this problem by applying a “singularity-free” technique (see section 3, Theorem 5), which transform the equation (1.7) into the former case (1.3), with the dependency of \( h \) on \( K \) (1.5). Then the existence and uniqueness of the solution is proved by choosing the appropriate function space, i.e. (1.6) (see section 3 for a strict description), and taking a slight modification on the former problem.

Direct applications of our result take place in the quantum transport simulation (especially in the nano semiconductor simulation), where there is a popular tool called the Wigner transport equation [11, 9]. The stationary Wigner equation with inflow boundary
conditions are used to obtain the current-voltage curves that are an important characteristic of semiconductor devices [3, 4, 6]. The one-dimensional stationary Wigner equation can be written as [7]

\[
v \frac{\partial f(x, v)}{\partial x} + \int_{-\infty}^{\infty} V_w(x, v - v')f(x, v') \, dv' = 0,
\]

where \( f(x, v) \) is the quasi-probability density function in the phase space \((x, v)\), and the Wigner potential \( V_w(x, v) \) is related to the potential \( V(x) \) through

\[
V_w(x, v) = \frac{i}{2\pi} \int_{-\infty}^{\infty} e^{-ivy} [V(x+y/2) - V(x-y/2)] \, dy.
\]

The well-posedness of the stationary Wigner transport equation with inflow boundary conditions have attracted the attention of many mathematicians, but it is still an open problem and is only partially solved in [1, 2, 7, 8]. One big issue is that if \( L^2(\mathbb{R}) \) is a suitable solution space for (1.10). The well-posedness result we obtain in this paper gives a complete answer and provides an optional solution space and theoretical support to solve the open problem of the stationary Wigner equation with inflow boundary conditions.

The rest of this paper is organized as follows. In Section 2, the well-posedness of the Cauchy problem 1.3 for \((c, v)^T\) is proved. Then we establish an isomorphism of the solution space to apply the result to the partial integral-differential equation (1.7) and proved the corresponding well-posedness. Conclusive remarks are given in the last section.

## 2 The well-posedness of the Cauchy problem

In this section, the well-posedness analysis for the Cauchy problem (1.3) is constructed, by the application of some well known results of the semi-group theory of linear operators. Generally, let \( X \) be a Banach space, and \( A(t): D(A(t)) \subset X \to X \) be a linear operator in \( X, \forall t \in T \). Consider the evolution IVP:

\[
\begin{cases}
\frac{du(t)}{dt} = A(t)u(t), & 0 \leq s < t \leq T, \\
u(s) = x \in X.
\end{cases}
\]

The solution of the above equation can be written by a propagator \( U(t, s) \) in the form that \( u(t) = U(t, s)u(s) \). Moreover, \( u \) is called a classical solution if \( u \in C([0, T]; X) \cap C^1((0, T]; X) \). The properties of this IVP then depend on the properties of the operators \( A(t) \) [10]. Two theorems from [10] are listed as follows for future use.

**Theorem 1.** Let \( X \) be a Banach space and for every \( t, 0 \leq t \leq T \), let \( A(t) \) be a bounded linear operator on \( X \). If the function \( t \mapsto A(t) \) is continuous in the uniform operator topology, then for every \( x \in X \) the initial value problem (3.5) has a unique classical solution \( u \).

**Theorem 2.** Suppose that the conditions in Theorem 1 are satisfied. Then for every \( 0 \leq s < t \leq T \), \( U(t, s) \) is a bounded linear operator and

(i) \( \|U(t, s)\| \leq \exp(\int_s^t \|A(r)\| \, dr) \).
(ii) \( U(t, t) = I, U(t, s) = U(t, r)U(r, s) \) for \( 0 \leq s < r < t \leq T \).
(iii) \( (t, s) \mapsto U(t, s) \) is continuous in the uniform operator topology for \( 0 \leq s < t \leq T \).
(iv) \( \frac{\partial U(t, s)}{\partial t} = A(t)U(t, s) \) for \( 0 \leq s < t \leq T \).
(v) \( \frac{\partial U(t, s)}{\partial s} = -U(t, s)A(s) \) for \( 0 \leq s < t \leq T \).
In order to apply the above theorems to study the abstract Cauchy problem (1.3), we need to define an appropriate Banach space $X$, such that $(c(t), v(t))^T \in X$, $\forall t \in \mathbb{R}$, and the operator $B(t)$ on $X$, defined by (1.4), satisfies all the conditions on $A(t)$ in Theorem 1. In other words, $B(t)$ is bounded on $X$ for every $t \in \mathbb{R}$ and is continuous, as an operator function of $t$, in the uniform operator topology.

In this paper, we define the Banach space $X = \mathbb{R} \oplus L^2(\mathbb{R})$, where $(c, v)^T \in X$, if and only if $c \in \mathbb{R}$ and $v \in L^2(\mathbb{R})$. The norm on $X$ is naturally defined by $\|(c, v)^T\|_X = |c| + \|v\|_{L^2}$. Then we rewrite the problem (1.3) into the following form:

$$
\begin{align*}
\begin{cases}
c'(t) = B_1(t)c(t) + B_3(t)v(t, x), & t \in (0, T) \\
\frac{\partial v(t, x)}{\partial t} = B_2(t)c(t) + B_4(t)v(t, x), & t \in (0, T) \\
c(t_0) = c_0, v(t_0, x) = v_0(x),
\end{cases}
\end{align*}
$$

(2.2)

where

$$
\begin{align*}
B_1(t) : \mathbb{R} &\to \mathbb{R}, c \mapsto ch(t, 0), \\
B_2(t) : \mathbb{R} &\to L^2(\mathbb{R}), c \mapsto c\frac{h(t, x) - h(t, 0)}{x}, \\
B_3(t) : L^2(\mathbb{R}) &\to \mathbb{R}, w(x) \mapsto (\tilde{K}(t, \cdot), w(\cdot)), \\
B_4(t) : L^2(\mathbb{R}) &\to L^2(\mathbb{R}),
\end{align*}
$$

where $\tilde{K}(t, x) = K(t, -x)$ and $B_4(t)$, termed the “singularity-free operator” of the convolution kernel $K$, is defined by

$$
B_4(t)w(t, x) = \int_{-\infty}^{\infty} \frac{K(t, x - x') - K(t, -x')}{x} w(t, x') \, dx'.
$$

(2.3)

In order to obtain the boundedness and continuity (in the uniform operator norm) of $B(t)$, it is equivalent to prove the boundedness (for every $t \in T$) and the continuity (in the uniform operator norm) of the four operators $B_i(t)$, $i = 1, 2, 3, 4$.

We discuss the boundedness first. The boundedness for $B_1(t)$ and $B_3(t)$ is obvious, by assuming that $h(t, \cdot)$ is continuous (ensuring the well-definition of $h(t, 0)$), and $K \in L^2(\mathbb{R})$ (since $\|\tilde{K}(t, \cdot), w(\cdot)\| \leq \|K(t, \cdot)\|_{L^2} \cdot \|w(t, \cdot)\|_{L^2} = \|K(t, \cdot)\|_{L^2} \cdot \|w(t, \cdot)\|_{L^2}$). For the boundedness of $B_2(t)$ and $B_4(t)$, we have the following lemmas:

**Lemma 1.** Suppose $w(x)$ is Lipschitz continuous with a Lipschitz constant $M_1$ and is bounded by $\|w(\cdot)\|_{\infty} \leq M_2$. Then $\frac{w(x) - w(0)}{x} \in L^2(\mathbb{R})$, with the norm

$$
\left\| \frac{w(x) - w(0)}{x} \right\|_2 \leq \sqrt{2M_1^2 + 8M_2^2}.
$$

**Proof.** We have

$$
\left\| \frac{w(x) - w(0)}{x} \right\|_2^2 = \int_{\mathbb{R}} \left| \frac{w(x) - w(0)}{x} \right|^2 \, dx = \int_{|x| \leq 1} \left| \frac{w(x) - w(0)}{x} \right|^2 \, dx + \int_{|x| > 1} \left| \frac{w(x) - w(0)}{x} \right|^2 \, dx 
\leq \int_{|x| \leq 1} |M_1|^2 \, dx + \int_{|x| > 1} \left| \frac{2M_2}{x} \right|^2 \, dx
\leq 2M_1^2 + 8M_2^2,
$$

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which concludes the proof.

In the forthcoming, let us use the notation $D^1 w$ to infer the weak derivative of the variable $x$. Here, $w$ can be function of one or two variables, i.e., $w = w(x)$ or $w = w(t, x)$.

**Lemma 2.** Assume $K(t, \cdot) \in H^1(\mathbb{R}) = \{f(v) \in L^2(\mathbb{R}) : D^1 f \in L^2(\mathbb{R})\}$ for every $t \in [0, T]$. Then the singularity-free operators $B_4(t) : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a bounded linear operator with the corresponding operator norm

$$
\|B_4(t)\| \leq 2\sqrt{2}\|K(t, \cdot)\|_{H^1},
$$

where $\|K(t, \cdot)\|_{H^1}$ is the $H^1$ norm of $K(t, \cdot)$.

**Proof.** Note that

$$
B_4(t)u(t, x) = \frac{(K * u)(t, x) - (K * u)(t, 0)}{x}.
$$

Thus, the boundedness of $B_4(t)$ is a direct consequence of Lemma 1, once we can prove the Lipschitz continuity and the boundedness of the function $(K * u)(t, \cdot)$.

We prove the Lipschitz continuity first. Using the Fubini theorem to change the order of integrations, we have

$$
\|(K * u)(t, x_2) - (K * u)(t, x_1)\| = \left|\int_{-\infty}^{+\infty} [K(t, x_2 - x') - K(t, x_1 - x')] u(t, x') \, dx'\right|
$$

$$
= \left|\int_{-\infty}^{+\infty} \left[\int_{x_1}^{x_2} D^1 K(t, \bar{x} - x') \, d\bar{x}\right] u(t, x') \, dx'\right|
$$

$$
\leq \int_{-\infty}^{+\infty} \left[\int_{x_1}^{x_2} D^1 K(t, \bar{x} - x') \cdot |u(t, x')| \, d\bar{x}\right] \, dx'
$$

$$
= \int_{x_1}^{x_2} \left[\int_{-\infty}^{+\infty} D^1 K(t, \bar{x} - x') \cdot |u(t, x')| \, d\bar{x}\right] \, dx'
$$

$$
\leq \int_{x_1}^{x_2} \|D^1 K(t, \bar{x} - \cdot)\|_2 \cdot \|u(t, \cdot)\|_2 \, d\bar{x}
$$

$$
= \|D^1 K(t, \cdot)\|_2 \cdot \|u(t, \cdot)\|_2 \cdot |x_2 - x_1|.
$$

As for the boundedness, with Cauchy-Schwarz inequality, we have

$$
\|(K * u)(t, x)\| = \int_{\mathbb{R}} |K(t, x - x') u(t, x')| \, dx' 
$$

$$
\leq \left(\int_{\mathbb{R}} |K(t, x - x')|^2 \, dx'\right)^{\frac{1}{2}} \cdot \|u(t, \cdot)\|_2
$$

$$
= \|K(t, \cdot)\|_2 \cdot \|u(t, \cdot)\|_2.
$$

Finally, using Lemma 1 on $(K * u)(t, \cdot)$, we have

$$
\|B_4(t)u(t, \cdot)\|_2 = \left\|\frac{(K * u)(t, x) - (K * u)(t, 0)}{x}\right\|_2
$$

$$
\leq \sqrt{2\|D^1 K(t, \cdot)\|_2^2 \|u(t, \cdot)\|_2^2 + 8\|K(t, \cdot)\|_{H^1}^2 \|u(t, \cdot)\|_2^2}
$$

$$
\leq 2\sqrt{2}\|K(t, \cdot)\|_{H^1} \|u(t, \cdot)\|_2,
$$

which concludes the proof. □
We collect the results above and conclude the following

**Theorem 3.** Assume for every \( t \in [0, T] \), \( h(t, \cdot) \) is Lipschitz continuous and \( h(t, \cdot) \in L^\infty(\mathbb{R}) \), \( K(t, \cdot) \in H^1(\mathbb{R}) \). Then \( B(t) \) for the Cauchy problem (1.3) is a bounded linear operator for every \( t \in [0, T] \).

Next, we consider the continuity (in the uniform operator norm) of \( B_i(t) \), which is a little more sophisticated. The idea, however, is similar to the previous discussion. For the sake of brevity, let us write \( w'(t, x) \) (for any function \( w(t, x) \) defined on \( [0, T] \times \mathbb{R} \)) to simplify the notation \( \partial w(t, x) / \partial t \), the \( t \) derivative of \( w \).

In the following part, we assume that \( h(\cdot, x) \) is absolutely continuous, i.e.,

\[
h(t_2, x) - h(t_1, x) = \int_{t_1}^{t_2} h'(\tau, x) \, d\tau,
\]

and that \( h'(t, \cdot) \) is uniformly bounded in the norm of \( L^\infty(\mathbb{R}) \), i.e.,

\[
\|h'(t, \cdot)\|_\infty \leq k, \forall t \in T.
\]

Moreover, we assume \( h'(t, \cdot) \) is Lipschitz continuous with a uniformly bounded Lipschitz constant (also \( k \) without losing generality), i.e.,

\[
|h'(t, x_2) - h'(t, x_1)| \leq k|x_2 - x_1|, \forall t \in T.
\]

Then the continuity of \( B_1(t) \) follows directly. For \( B_2(t) \), we have

\[
\left\| \frac{h(t_2, x) - h(t_1, x)}{x} \right\|_2 \leq \frac{\int_{t_1}^{t_2} \left\| \frac{h'(\tau, x) - h'(\tau, 0)}{x} \right\|_2 \, d\tau}{\int_{t_1}^{t_2} \frac{\|h'(\tau, x) - h'(\tau, 0)\|}{x} \, d\tau} \leq \sqrt{10k|t_2 - t_1|}.
\]

The last inequality is a direct application of Lemma 1.

In order to obtain the continuity of \( B_3(t) \) and \( B_4(t) \), similar assumptions are proposed on \( K(t, x) \) and \( K'(t, x) \). Namely, we assume that \( K(\cdot, x) \) is absolutely continuous, and that \( K'(t, \cdot) \in H^1(\mathbb{R}) \) with a uniform constant \( m \), satisfying

\[
\|K'(t, \cdot)\|_{H^1} \leq m, \quad \forall t \in T.
\]

Then, we have

\[
[B_4(t_2) - B_4(t_1)]u(x) = \int_{-\infty}^{+\infty} \frac{K(t_2, x - x') - K(t_2, -x') - K(t_1, x - x') - K(t_1, -x')}{x} \, u(x') \, dx'
\]

\[
= \int_{-\infty}^{+\infty} \int_{t_1}^{t_2} K'(\tau, x - x') - K'(\tau, -x') \, d\tau \, \frac{u(x')}{x} \, dx'
\]

\[
= \int_{t_1}^{t_2} \int_{-\infty}^{+\infty} K'(\tau, x - x') - K'(\tau, -x') \, u(x') \, dx' \, d\tau.
\]

Define \( B_4'(t) : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \), such that

\[
B_4'(t)u(x) = \int_{-\infty}^{+\infty} \frac{K'(t, x - x') - K'(t, -x')}{x} \, u(x') \, dx',
\]
which is the singularity-free operators of $K'$. From the formulation above, we have

$$[B_4(t_2) - B_4(t_1)]u(x) = \int_{t_1}^{t_2} [B'_4(\tau)u(x)] \, d\tau.$$  

And similar to Theorem 3, the operator norm of the singularity-free operator $B'_4(t, x)$ is bounded by

$$\|B'_4(t)\| \leq 2\sqrt{2}\|K'(t, \cdot)\|_{H^1}.$$  

Now, for $0 \leq t_1 \leq t_2 \leq T$, we have

$$\|B_3(t_2) - B_3(t_1)\| = \sup_{u \in L^2(\mathbb{R})} \|[(\tilde{K}(t_2, \cdot) - \tilde{K}(t_1, \cdot), u(\cdot))] \|_{2} \leq \|\tilde{K}(t_2, \cdot) - \tilde{K}(t_1, \cdot)\|_{2}
= \left\| \int_{t_1}^{t_2} \tilde{K}'(\tau, \cdot) \, d\tau \right\|_2 \leq \int_{t_1}^{t_2} \|\tilde{K}'(\tau, \cdot)\|_2 \, d\tau
\leq \|K'(t, \cdot)\|_{H^1} |t_2 - t_1| \leq 2\sqrt{2}m|t_2 - t_1|,$$

which shows the continuity of $B_3(t)$ and

$$\|B_4(t_2) - B_4(t_1)\|u(\cdot)\|_2 = \left\| \int_{t_1}^{t_2} [B'_4(\tau)u(\cdot)] \, d\tau \right\|_2
\leq \int_{t_1}^{t_2} \|B'_4(\tau)u(\cdot)\|_2 \, d\tau
\leq 2\sqrt{2}\|K'(t, \cdot)\|_{H^1} |t_2 - t_1| \cdot \|u(\cdot)\|_2.$$  

Thus,

$$\|B_4(t_2) - B_4(t_1)\| \leq 2\sqrt{2}\|K'(t, \cdot)\|_{H^1} |t_2 - t_1| \leq 2\sqrt{2}m|t_2 - t_1|$$

and this implies the continuity of $B_4(t)$.

Collecting all the above inferences on boundedness and continuity, the well-posedness result of the Cauchy problem (1.3) is then summarized by

**Theorem 4.** Assume that the real functions $h(t, x)$ and $K(t, x)$ satisfy the following conditions

(i) $\forall x \in \mathbb{R}$, both $h(\cdot, x)$ and $K(\cdot, x)$ is Lipschitz continuous on $[0, T]$.

(ii) $\forall t \in [0, T]$, $h(t, \cdot)$ is Lipschitz continuous and $h(t, \cdot) \in L^\infty(\mathbb{R})$.

(iii) $\forall t \in [0, T]$, $\frac{\partial h}{\partial t}(t, \cdot)$ is Lipschitz continuous and $L^\infty(\mathbb{R})$ bounded. In addition, both the Lipschitz constant and the $L^\infty$ norm is uniformly bounded, i.e., with an upper bound independent of $t$.

(iv) $\forall t \in [0, T]$, $K(t, \cdot) \in H^1(\mathbb{R})$.

(v) $\forall t \in [0, T]$, $\frac{\partial K}{\partial t}(t, \cdot) \in H^1(\mathbb{R})$ and the $H^1$ norm is uniformly bounded.

Then the Cauchy problem (1.3) is well-posed and has a unique classical solution in $C([0, T]; X)$.

3 The construction of the solution in a class of PIDEs

This section is organized in order to build the equivalence between the Cauchy problem (1.3) and the PIDE (1.7), i.e.,

$$\begin{cases}
\frac{\partial u(t, x)}{\partial t} = \frac{1}{x}\Psi[K]u(t, x), & t \in (0, T), \\
u(t_0, x) = u_0(x) = \frac{c_0}{x} + v_0(x),
\end{cases}$$

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functions in the space $L^1$ for the PIDE is put forward in the end of this section. A strict well-posedness result between the Cauchy problem (1.3) and this PIDE is proved (since we have established the well-posedness of the Cauchy problem in the above section). In fact, there are only two operators on $u \in L^2(\mathbb{R})$, which has a natural definition on a subspace of the Schwartz space $[5]$ as $K$ is good enough (for example, Young’s convolution inequality that $\|K\|_x \leq \|K\|_x^0$) is always determined by

$$h(t, x) = \int_{-\infty}^{\infty} \frac{K(t, x - x')}{x'} dx'.$$

We may also write $\Psi[K]u = K \ast u$ and $h = K \ast (1/x)$ for brevity. Our objection is the following theorem:

**Theorem 5.** Assume $\forall t \in [0, T]$, $K(t, \cdot) \in L^2(\mathbb{R})$ and is Lipschitz continuous. Then if the PIDE (1.7) has a solution $u$ such that $u(t, x) = c(t)/x + v(t, x)$ and $v(t, \cdot) \in L^2(\mathbb{R})$, $\forall t \in [0, T]$, $(c(t), v(t, x))^T$ is also a solution of the Cauchy problem (1.3). Conversely, if $(c(t), v(t, x))^T$ is the solution of (1.3) and $v(t, \cdot) \in L^2(\mathbb{R})$, $\forall t \in [0, T]$, $u(t, x) = c(t)/x + v(t, x)$ is also a solution of the PIDE (1.7).

Before we give the proof of this theorem, some explanations must be made. We define a subspace of the Schwartz space $[5]$ as

$$L^2(\mathbb{R}) = \left\{ f(x) = \frac{c}{x} + f_0(x) : c \in \mathbb{R}, f_0(\cdot) \in L^2(\mathbb{R}) \right\}.$$

Thus,

$$u(t, x) = \frac{c(t)}{x} + v(t, x), v(t, \cdot) \in L^2(\mathbb{R}), \forall t \in [0, T],$$

is equivalent to $u(t, \cdot) \in L^2(\mathbb{R})$, $\forall t \in [0, T]$. It can be shown later that the space $L^2(\mathbb{R})$ is proper for the PIDE (1.7) (making the PIDE well posed), if Theorem 5, i.e., the equivalence between the Cauchy problem (1.3) and this PIDE is proved (since we have established the well-posedness of the Cauchy problem in the above section). A strict well-posedness result for the PIDE is put forward in the end of this section.

It can be easily proved that $L^2(\mathbb{R})$ is a Banach space equipped with a natural norm $\|f\|_{L^2(\mathbb{R})} = |c| + \|f_0\|_2$ (equivalent to the norm of $(c, f_0)^T \in X$) by the observation that $1/x \notin L^2(\mathbb{R})$ and that $L^2(\mathbb{R}) = \mathbb{R} \oplus L^2(\mathbb{R})$. Furthermore, we claim that such “odd” functions in the space $L^2(\mathbb{R})$ indeed make sense for all the operators in the PIDE (1.7). In fact, there are only two operators on $u$ in the PIDE (1.7). One is the partial derivative of $t$, which has a natural definition on $u \in L^2(\mathbb{R})$, i.e.,

$$\frac{\partial u(t, x)}{\partial t} = \frac{c(t)}{x} + \frac{\partial v(t, x)}{\partial t}.$$

The other one is the operator $\frac{\partial}{\partial t} \Psi[K]$, which is much more complicated. The crucial problem is the definition of the convolution $K \ast u$ when $u \in L^2(\mathbb{R})$. We know from Young’s convolution inequality that

$$\|f \ast g\|_r \leq \|f\|_p \|g\|_q,$$

if $1/p + 1/q = 1/r + 1$ with $1 \leq p, q, r \leq \infty$. Thus, as long as $K \in L^p(\mathbb{R})$ with $1 \leq p \leq 2$, the convolution $K \ast v$ (in (3.1) is the $L^2$ part of $u$) is well defined. Unfortunately, for the $1/x$ part of $u$, the convolution $K \ast (1/x)$ does not have such good properties, even if $K$ is good enough (for example, $K$ is a mollifier, $K(t, x) = \exp(-1/(1 - |x|^2)), x \in [-1, 1]$}

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and \( K(t, x) = 0, |x| > 1 \). Instead, we consider the convolution on \( 1/x \) in a weaker sense (namely the Cauchy principal integral)

\[
h = K \ast \frac{1}{x} \triangleq \int_0^{+\infty} \frac{K(t, x-x') - K(t, x+x')}{x'} \, dx'.
\]

(3.2)

Alternatively, we can write directly

\[
K \ast u \triangleq \int_0^{+\infty} [K(t, x-x')u(x') - K(t, x+x')u(-x')] \, dx',
\]

(3.3)

for all \( u \in L^2(\mathbb{R}) \).

Similar to Young’s convolution inequality, we have the following

**Lemma 3.** Suppose \( t \in [0, T] \), \( K(t, \cdot) \in L^2(\mathbb{R}) \) is Lipschitz continuous with Lipschitz constant \( k_1(t) \). Then \( h(t, \cdot) = K \ast (1/x)(t, \cdot) \) is well-defined and

\[
\|h(t, \cdot)\|_\infty \leq 2(k_1(t) + \|K(t, \cdot)\|_2).
\]

**Proof.** According to the Lipschitz continuity,

\[
|h(t, x)| = \left| \int_0^{+\infty} \frac{K(t, x-x') - K(t, x+x')}{x'} \, dx' \right|
\leq \left( \int_1^{1} + \int_1^{+\infty} \right) \left| \frac{K(t, x-x') - K(t, x+x')}{x'} \right| \, dx'
\leq \int_0^{1} |2k_1(t)| \, dx' + \int_1^{+\infty} \left| \frac{K(t, x-x')}{x'} \right| \, dx' + \int_1^{+\infty} \left| \frac{K(t, x+x')}{x'} \right| \, dx'.
\]

By Cauchy-Schwarz inequality, we have

\[
\int_1^{+\infty} \left| \frac{K(t, x-x')}{x'} \right| \, dx' \leq \left( \int_1^{+\infty} \left| K(t, x-x') \right|^2 \, dx' \right)^{\frac{1}{2}} \cdot \left( \int_1^{+\infty} \frac{1}{x'}^2 \, dx' \right)^{\frac{1}{2}}
\leq \|K(t, \cdot)\|_2.
\]

Similarly, we have

\[
\int_1^{+\infty} \left| \frac{K(t, x+x')}{x'} \right| \, dx' \leq \|K(t, \cdot)\|_2.
\]

Thus,

\[
|h(t, \cdot)| \leq 2(k_1(t) + \|K(t, \cdot)\|_2),
\]

and the well-definition of \( h(t, \cdot) \), that is, the integrable of (3.4), easily follows form the boundedness proof of \( |h(t, \cdot)| \).

**Lemma 4.** Suppose \( t \in [0, T] \), \( D^1K(t, \cdot) \in L^2(\mathbb{R}) \) is Lipschitz continuous with Lipschitz constant \( k_2(t) \). Then \( h(t, \cdot) = K \ast (1/x)(t, \cdot) \) is Lipschitz continuous with a Lipschitz constant \( 2(k_2(t) + \|D^1K(t, \cdot)\|_2) \).

**Proof.** Let \( V_K(t, x, x') = K(t, x-x') - K(t, x+x') \). Write \( D^1V_K(t, x, x') \) to be the first-order weak derivative concerning variable \( x \). Then \( D^1V_K(t, x, x') = D^1\frac{D^1K(t, x-x') - D^1K(t, x+x')}{x} \). Since \( D^1K(t, \cdot) \in L^2(\mathbb{R}) \subseteq L^1_{\text{loc}}(\mathbb{R}) \), \( D^1V_K(t, \cdot, x') \in L^1_{\text{loc}}(\mathbb{R}) \) for any
fixed $t$ and $x'$. Thus, $V_K(t, \cdot, x')$ is absolutely continuous for any fixed $t$ and $x'$ and $orall x_1 \leq x_2$,

$$V_K(t, x_2, x') - V_K(t, x_1, x') = \int_{x_1}^{x_2} D^1 V_K(t, \bar{x}, x') \, d\bar{x}.$$ 

Therefore, we have

$$|h(t, x_2) - h(t, x_1)| = \left| \int_{0}^{+\infty} V_K(t, x_2, x') - V_K(t, x_1, x') \, dx' \right|$$

$$= \left| \int_{0}^{+\infty} \left[ \int_{x_1}^{x_2} D^1 V_K(t, \bar{x}, x') \, d\bar{x} \right] \frac{1}{x'} \, dx' \right|$$

$$\leq \int_{x_1}^{x_2} \left[ \int_{0}^{+\infty} \left| D^1 V_K(t, \bar{x}, x') \right| \frac{1}{x'} \, dx' \right] \, d\bar{x}$$

$$= \int_{x_1}^{x_2} \left[ \left| D^1 K * \frac{1}{x}(t, \bar{x}) \right| \, d\bar{x} \right].$$

Now, applying Lemma 3 on $D^1 K * (1/x)$, we obtain

$$|h(t, x_2) - h(t, x_1)| \leq \int_{x_1}^{x_2} \left[ 2(k_2(t) + \|D^1 K(t, \cdot)\|_2) \right] \, dx'$$

$$= 2(k_2(t) + \|D^1 K(t, \cdot)\|_2)|x_2 - x_1|,$$

which concludes the proof. \qed

We now give the proof of Theorem 5. Remember that, the convolution $K * (1/x)$ is in the meaning of (3.2).

**Proof of Theorem 5.** Assume

$$u(t, x) = \frac{c(t)}{x} + v(t, x), v(t, \cdot) \in L^2(\mathbb{R}), \forall t \in I$$

is a solution of the PIDE (1.7). Substituting it into (1.7), we obtain

$$c'(t) + x \frac{\partial v}{\partial t}(t, x) = c(t)(K(t, x) * \frac{1}{x}) + K(t, x) * v(t, x). \tag{3.4}$$

According to Lemma 3, $h(t, x) = K(t, x) * (1/x)$ is well-defined. Furthermore, with Young’s inequality,

$$\|(K * v)(t, \cdot)\|_\infty \leq \|K(t, \cdot)\|_2 \|v(t, \cdot)\|_2,$$

$K * v$ is also well-defined. Let $x = 0$ at both sides of (2.2). Note that $h(t, x) = K(t, x) * (1/x)$ and $(K * v)(t, 0) = (\bar{K}(t, \cdot), v(t, \cdot))$. We have

$$c'(t) = c(t)h(t, 0) + (\bar{K}(t, \cdot), v(t, \cdot)), \tag{3.5}$$
which is the first equation of the Cauchy problem (1.3). To derive the evolution of \(v(t, x)\), we substitute the expression of \(c'(t)\), (3.5), into (3.4). We have

\[
\frac{\partial v}{\partial t}(t, x) = \frac{1}{x} \left[ c(t)h(t, x) + K(t, x) * v(t, x) - c'(t) \right]
\]

\[
= \frac{1}{x} \left[ c(t)h(t, x) + (K * v)(t, x) - c(t)h(t, 0) - (K * v)(t, 0) \right]
\]

\[
= c(t)\frac{h(t, x) - h(t, 0)}{x} + \int_{-\infty}^{\infty} \frac{K(t, x - x') - K(t, -x')}{x} v(t, x') \, dx'.
\]

Thus, \((c(t), v(t, x))^T\) is also a solution of the Cauchy problem (1.3). Apparently, the converse proposition also holds by reversing the above deduction.

We have already proved the equivalence between the Cauchy problem (1.3) and the PIDE (1.7). As already mentioned, the well-posedness result of the PIDE then follows from the corresponding result of the Cauchy problem, i.e., Theorem 4. As it should be, certain assumptions must be posed on the convolution kernel \(K\) in order to satisfy the conditions in Theorem 4. We put forward the well-posedness result for the PIDE 1.7 as the end of this section.

**Theorem 6.** Let \(w_1 = K, w_2 = \partial K/\partial t\). Assume \(w_i, i = 1, 2\) satisfy the following conditions:

(a) There exists a constant \(m\) such that \(\|w_i(t, \cdot)\|_{H^1} \leq m, \forall t \in [0, T]\);

(b) \(\forall t \in I, w_i(t, \cdot), D^1w_i(t, \cdot)\) is Lipschitz continuous. Suppose \(k_i(t), k_i'(t)\) are the corresponding Lipschitz constant, then there exists a constant \(k\), such that \(k_i(t), k_i'(t) \leq k, \forall t \in [0, T]\).

Moreover, assume \(\forall x \in \mathbb{R}, K(\cdot, x)\) is absolutely continuous on \([0, T]\). Then the PIDE (1.7) is well-posed and has a unique classical solution in \(C([0, T]; L^2(\mathbb{R}))\).

**Proof.** The only thing is to validate the five conditions in Theorem 4 are all satisfied. Obviously, conditions (iv) and (v) are directly assumed above. Moreover, (ii) follows directly form Lemma 3 and Lemma 4. For (i) and (iii), remember that \(h = K * (1/x)\). For any \(0 \leq t_1 < t_2 \leq T\) and \(x \in \mathbb{R}\), with \(K(\cdot, x)\) absolutely continuous, we have

\[
h(t_2, x) - h(t_1, x) = \int_0^{t_2} \left[ \frac{K(t_2, x + x') - K(t_2, x - x')}{x'} - \frac{K(t_1, x + x') - K(t_1, x - x')}{x'} \right] \, dx'
\]

\[
= \int_0^{t_2} \int_0^{+\infty} K'(\tau, x + x') - K'(\tau, x - x') \, d\tau \, \left[ 1 \frac{1}{x'} \right] \, dx'
\]

\[
= \int_0^{t_2} \int_0^{+\infty} K'(\tau, x + x') - K'(\tau, x - x') \, dx' \, d\tau
\]

\[
= \int_0^{t_2} K'(\tau, x) \frac{1}{x} \, d\tau.
\]

Applying Lemma 3 on \(K'(t, x)\), we have

\[
\left\| K'(\tau, x) \frac{1}{x} \right\|_{\infty} \leq 2(k + m).
\]

(3.6)
which implies that $h(\cdot, x)$ is absolutely continuous and 

$$h'(t, x) = K'(t, x) \ast \frac{1}{x}.$$ 

Thus, condition (i) is directly proved. And applying Lemma 3 and Lemma 4 on $K'$, condition (iii) is immediately validated. \qed

4 Conclusion

By using the semi-group theory of linear operators, we prove the well-posedness of the abstract Cauchy problem of a partial integro-differential system. Furthermore, we apply our result to the abstract Cauchy problem for a class of partial integro-differential equations, which can be applied to the study of the stationary Wigner equation. If both the inflow and outflow conditions are given in the one side of the semiconductor device, our theory shows that the problem is well-posed under some regularity assumptions for the Wigner potential. These results also give us substantial inspiration in the analyse of the stationary Wigner equation with inflow boundary conditions, which will be treated in our future studies.

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