Modelling the Dependence Structure of Financial Assets: A Bivariate Extreme Data Study

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Abstract. Quantifying dependence among variables is the core of all modelling efforts in financial econometrics. In the recent years, copula was introduced to model the dependence structure among financial assets return, and its application developed fast. A large number of studies on copula have been performed, but the study of multivariate extremes related with copulas was quite behind in comparison with the research on copulas. In order to predict extreme losses in finance, extreme value copulas play more important role than copulas. In this paper, we study the modelling of extreme value dependence using extreme value copulas on finance data and introducing a negative dependence between the assets return. This model was applied in the portfolio of the IDX Composite Index (IHSG) and the exchange rate for Indonesia Rupiah to United States Dollar (NT). Each individual asset return is modelled by the AR-GARCH and the joint distribution is modelled using extreme value copulas. As a comparison, the joint distribution is also modelled using the copulas. The result shows that the dependence between IHSG and NT is negative. Therefore, in this paper, we tested the extreme value copulas and the copulas which were rotated 270 degree and 90 degree. The empirical study showed that the extreme value copulas rotated with 270 degree are relatively more appropriate model than the copulas.

Keywords: AR-GARCH, copula, extreme value copula, rotated copula, rotated extreme value copula.
1. Introduction

Quantifying dependence structure among variables is the core of all modelling efforts in financial econometrics. The commonly used dependence measure is Pearson correlation which is not a measure of general, but only of linear dependence. In financial markets, there is often a nonlinear dependence between assets return. Thus alternative methods for capturing codependency should be considered, such as copula-based dependence measures. In the recent years, copula was introduced to model the dependence structure among financial assets return in the portfolio, and its application developed fast. Copula is the function of multivariate distribution whose margins follows the uniform distribution in interval \([0, 1]\). Copula is used as a connector between the joint distribution function \(F\) of random variable vector \(X = (X_1, \ldots, X_d)\) with its marginal distributions \(F_1, \ldots, F_d\). A large number of studies on copula have been performed, such as Aas [1], Joe [15], Embrechts [6], Kolve et al. [16], and McNeil et al. [18], but the study of multivariate extremes related with copulas was quite behind in comparison with the research on copulas.

In finance, modelling extreme events is of high importance. Although they happen rarely, they could cause huge losses for industry and society. Extreme events (big losses) are described as the tail of the loss distribution. In several applications, as in finance, we do not need to model all aspects of the return distribution. Modelling could be focused on the tail of distribution (e.g., Genest et al. [8], Poon et al. [22], Sarma [23], and Haug et al. [13], Embrechts et al. [7]), as for the analysis of financial crisis. Regarding this matter, on one side, we lose information since we only focused on a small part of the distribution. But on the other hand, this could be used as a specific tool to give a comprehensive description of the tail of distribution.

In order to predict extreme losses in finance, extreme value copulas play more important role than copulas. This is due to its special property which can extrapolate data into a far tail region in order to estimate the probability of an extreme event. Poon et al. [22] have done research on modelling extreme value dependence using a nonparametric approach that uses tail dependence. Haug et al. [13] have done the same research using extreme value copulas on insurance data. In this paper, we extended the study that has been done by Haug et al. [13] with two matters, which are: (1) using financial data and (2) introducing a negative dependence between the assets return. Related to the first matter, the distribution of finance data (the distribution of asset return) has the properties of fat-tailed, not independent and identically distributed (i.i.d.), and volatility clustering (McNeil et al. [18]). The specific methodology used for describing volatility clustering is based on the Generalized Autoregressive Conditional Heteroscedasticity (GARCH) methods. It has an additional advantage of providing an i.i.d. innovations series that can be directly modelled by using parametric or nonparametric distribution with tails described by extreme value theory. Related to the second matter, the extreme value copula applies to positive dependence parameter. If the dependence among components is negative, then the extreme value copula must be rotated so that the negative dependency parameter could apply.

The paper is organized as follows. Section 2 introduces the GARCH model and the Extreme Value model. Section 3 presents the concept of copula in general. Section 4 explains estimation and testing procedures for an extreme value copula. Section 5 presents an application to financial data. Section 6 concludes.

2. The GARCH and Extreme Value Model

Before we model the dependencies between the return of financial assets, our first step is to determine the marginal distribution of each assets return. In this section, some concepts required to determine the marginal distribution of each assets return are given. According to McNeil et al. [18], the characteristic of distribution financial assets return is usually fat-tailed, not independent and identically distributed (i.i.d.), and volatility clustering. Related to the characteristic of the assets return distribution, we follow a two-step procedure to determine it.

(i) We fit univariate AR-GARCH (Autoregressive-Generalized Autoregressive Conditional

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Heteroscedasticity) model for each return series (NT and IHSG). In this step, the i.i.d. innovations estimation is obtained.

(ii) We use the peak-over-thresholds method to study the extreme data behaviour. After the thresholds $\upsilon$ is selected, each innovations series (NT and IHSG) for the tail of distribution are estimated using empirical distribution. Then, we transform each innovations series into uniform $(0, 1)$ using the probability-integral transformation.

2.1. Model AR-GARCH

The financial asset return are time series data, thus requires investigation regarding the underlying assumption such as the stationary, autocorrelation, and homogeneity type. AR (Autoregressive) model is used if any correlation between current data and past data is suspected. Whereas McNeil et al. [18] used GARCH model if volatility clustering in the data was suspected. Let $\{R_t\}, t=1,\ldots,T$ is financial asset return series. The AR($p$) model is defined as follows

$$R_t = \sum_{i=1}^{p} \phi_i R_{t-i} + \varepsilon_t$$  \hspace{1cm} (1)

and GARCH($p,q$) model is written as

$$R_t = \mu + \varepsilon_t, \varepsilon_t = \sigma_t Z_t, \sigma_t^2 = \omega_o + \sum_{i=1}^{p} \alpha_i \varepsilon_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2; \forall t \in Z$$  \hspace{1cm} (2)

where $\omega_o > 0$, $\alpha_i, \beta_j > 0; i = 1, \ldots, p; j = 1, \ldots, q$ (McNeil et al. [18]). Equation (2) shows that residual $\varepsilon_t$ is multiplication between volatility, $\sigma_t$, and innovations, $Z_t$. Innovations series $\{Z_t\}, t=1,\ldots,T$ obtained from GARCH($p,q$) model usually has a non-normal distribution. This paper adopts the past research such as that of Genest et al. [10] in which the innovations series in the tail of distribution is estimated using empirical distribution with the probability density function is defined as

$$\hat{F}(z) = \frac{1}{T} \sum_{t=1}^{T} I(Z_t \leq z)$$  \hspace{1cm} (3)

where $I(\cdot)$ is the indicator function.

2.2. Extreme Value (EV)

According to Davidson and Smith [5] and Castillo and Hadi [4], Extreme Value (EV) is quite powerful to study the behavior of data in the tail of distribution. In this paper, the peak-over-threshold method is used to study the extreme data behavior. The important aspect of the peak-over-threshold method is the threshold data behavior and the threshold $\upsilon$ selection. In the practice, it was assumed that those categorized as part of the tail of distribution start from threshold $\upsilon$, which could be selected as $k$ % from the data. The research of McNeil and Frey [17] used $k$ as much as $5 - 13\%$. In this paper, threshold $\upsilon$ is selected so that the extreme data was $5\%$ from the data.

Let the series i.i.d. $\{Z_t\}, t=1,\ldots,T$ has distribution function $F_Z$. We are interested with the realization distribution which exceed a certain threshold $\upsilon$. Let $X = Z - \upsilon$, the distribution function $F_X$ is a conditional distribution function and written as

$$F_X(x) = P(X \leq x | Z > \upsilon)$$
3. Copula

According to Nelsen [19], copula function is a multivariable distribution function with support in \([0, 1]^d\), where its margins have a uniform distribution. In the bivariate case, \(C : [0, 1]^2 \rightarrow [0, 1]\) is a copula function with the following properties:

(i) \(C(u, 1) = u\) and \(C(1, v) = v\),

(ii) \(C\) grounded means \(C(u, 0) = C(0, v) = 0\),

(iii) \(C\) 2-increasing means \(\forall u_1, u_2, v_1, v_2 \in [0, 1], u_1 \leq u_2, v_1 \leq v_2\), then \(C(u_2, v_2) - C(u_1, v_2) - C(u_2, v_1) + C(u_1, v_1) \geq 0\).

Let \((X_1, \ldots, X_d)\) be a random vector with marginal functions \(F_1, \ldots, F_d\) is a continuous function. By applying integral probability transformation to each component of the random vector, then

\[
(U_1, \ldots, U_d) = (F_1(X_1), \ldots, F_d(X_d))
\]

has uniform distribution \([0, 1]\). The copula of a random vector \((X_1, \ldots, X_d)\) is defined as joint cumulative distribution function at \((U_1, \ldots, U_d)\):

\[
C(u_1, \ldots, u_d) = P(U_1 \leq u_1, \ldots, U_d \leq u_d) = P(F_1(X_1) \leq u_1, \ldots, F_d(X_d) \leq u_d) = P(X_1 \leq F_1^{-1}(u_1), \ldots, X_d \leq F_d^{-1}(u_d)) = F(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d))
\]

where \(F\) is joint distribution function and \(F_1^{-1}, \ldots, F_d^{-1}\) is quantile of marginal distribution function \(F_1, \ldots, F_d\). The density \(c(u_1, \ldots, u_d)\) associated to the copula \(C(u_1, \ldots, u_d)\) is defined as:

\[
c(u_1, \ldots, u_d) = \frac{\partial^d C(u_1, \ldots, u_d)}{\partial u_1 \cdots \partial u_d} = \frac{f(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d))}{f_1(F_1^{-1}(u_1)) \cdots f_d(F_d^{-1}(u_d))}
\]  

Sklar Theorem (Nelsen [19]) states that let \(F\) be a joint distribution function with margins \(F_1, \ldots, F_d\), then there exists a copula \(C\) such that for all \(x_1, \ldots, x_d\) in \(R\),

\[
F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d)).
\]  

3.1. The Estimation of Copula Parameter

Firstly, empirical copula is calculated by

\[
C_T(u_1, \ldots, u_d) = \frac{1}{T} \sum_{i=1}^{T} I(U_{i1} \leq u_1, \ldots, U_{id} \leq u_d),
\]

where \(U_{ij} = \frac{T}{t+1} F_T(x_{ij})\) for \(i = 1, \ldots, T\) and \(j = 1, \ldots, d\) is known as pseudo observations, \(F_T(x) = \frac{1}{T} \sum_{i=1}^{T} I(X_{ij} \leq x)\) for \(j = 1, \ldots, d\), is empirical distribution of margins and \(I(\cdot)\) is the indicator function (see Genest and Segers [12]).

The next step is to estimate the parametric copula. For parametric estimation, we focus on fitting a parametric family to the empirical copula. In this paper, we use some popular bivariate parametric copula families, such as the Gaussian, Frank, Gumbel, and Clayton copula (see Nelsen [19], Aas [1], and Schmidt [24]), with the definition as follow. The estimation of copula parameter can be employed by using the Pseudo Maximum Likelihood (PML) estimation, which will be explained in the next section.
• Gaussian copula: for $\theta \in [-1, 1]$,

$$C(u_1, u_2; \theta) = \Phi^{-1}(u_1) \Phi^{-1}(u_2) \exp \left( \frac{-\left( z_1^2 - 2\theta z_1 z_2 + z_2^2 \right)}{2 (1 - \theta^2)} \right) d z_2 d z_1,$$

(7)

and $\Phi$ is standard normal distribution function with inverse $\Phi^{-1}$ (Aas [1]; Schmidt [24]; Haug et al. [13]).

• Frank copula: for $\theta \in \mathbb{R} \setminus \{0\}$,

$$C(u_1, u_2; \theta) = -\frac{1}{n} \log \left( 1 + \frac{(\exp(-\theta u_1) - 1)(\exp(-\theta u_2) - 1)}{\exp(-\theta) - 1} \right),$$

(8)

(Haug et al. [13]).

• Gumbel copula: for $\theta \in [1, \infty)$,

$$C(u_1, u_2; \theta) = \exp \left( - \left( (-\log(u_1))^\theta + (-\log(u_2))^\theta \right)^{1/\theta} \right),$$

(9)

(Nelsen [19]).

• Clayton copula: for $\theta \in (0, \infty)$,

$$C(u_1, u_2; \theta) = \left( u_1^{-\theta} + u_2^{-\theta} - 1 \right)^{-1/\theta},$$

(10)

(Nelsen [19]).

3.2. Goodness-of-fit Tests for Copula

Suppose that $Z = (Z_1, \ldots, Z_d)'$, $d \geq 2$ is a random vector with distribution function $F$ and marginal distribution functions $F_1, \ldots, F_d$. Based on the Sklar Theorem, there is unique copula $C$ so that

$$F(z_1, \ldots, z_d) = C(F_1(z_1), \ldots, F_d(z_d)).$$

Here we want to test, whether the copula $C$ belongs to a particular parametric family $C_{H_0} \in \{ C(\cdot; \theta) : \theta \in \Theta \}$. That is, we test

$$H_0 : C(\cdot) \in \{ C(\cdot; \theta) : \theta \in \Theta \} \quad \text{and} \quad H_1 : C(\cdot) \notin \{ C(\cdot; \theta) : \theta \in \Theta \},$$

(11)

(Hering and Hofert [14]).

Based on a nonparametric estimator $C_T(\cdot)$ and a parametric estimator $C(\cdot; \widehat{\theta}_T)$ of the copula, it is natural to consider rank-based versions of the Cramer-von Mises statistic

$$\int_{[0,1]^d} T \left( C_T(u) - C(u; \widehat{\theta}_T) \right)^2 dC_T(u),$$

to perform the goodness-of-fit test. We can also look the goodness-of-fit test review by Berg [3] and Genest et al. [9].
4. Extreme Value Copula

According to Haug et al. [13], a copula \( C \) is called as an extreme value copula if and only if for all \( m \in \mathbb{N} \)

\[
C(u_1, \ldots, u_d) = \left( C \left( u_1^{1/m}, \ldots, u_d^{1/m} \right) \right)^m
\]

for \( (u_1, \ldots, u_d) \in [0,1]^d \) (see e.g. Gudendorf and Segers [12]). For the bivariate copula case according to Segers [26], a copula \( C \) is called as an extreme value copula if and only if there is a real value function \( A \) in interval \( [0,1] \) so that

\[
C(u_1, u_2) = \exp \left( \log(u_1 u_2) A \left( \frac{\log(u_2)}{\log(u_1 u_2)} \right) \right)
\]

for \( (u_1, u_2) \in [0,1]^2 \). The function \( A \) is the so-called *Pickands dependence function*. Characterizing properties of \( A \) are (i) \( \max(t, 1-t) \leq A(t) \leq 1 \) for all \( t \in [0,1] \), and (ii) \( A \) is convex (Beirlant et al. [2]). Gudendorf and Segers [12] gave a list of several well-known parametric model. Given parametric form of the *Pickands dependence function* \( A(\cdot; \theta) \) for the Gumbel, Galambos, Husler-Reiss, and Tawn copula for \( t \in [0,1] \).

- **Gumbel copula**: for \( \theta \in [1, \infty) \),

\[
A(t; \theta) = \left( t^\theta + (1-t)^\theta \right)^{1/\theta}
\]

\[
C(u_1, u_2) = \exp \left( - \left( -\log u_1 \right)^\theta + \left( -\log u_2 \right)^\theta \right) \left( \frac{t^{\theta}}{t} \right)^{1/\theta}
\]

(14)

- **Galambos copula**: for \( \theta \in (0, \infty) \),

\[
A(t; \theta) = 1 - \left( t^{-\theta} + (1-t)^{-\theta} \right)^{-1/\theta}
\]

\[
C(u_1, u_2) = u_1 u_2 \exp \left( \left( -\log u_1 \right)^{-\theta} + \left( -\log u_2 \right)^{-\theta} \right)^{-1/\theta}
\]

(15)

- **Husler-Reiss copula**: for \( \theta \in (0, \infty) \),

\[
A(t; \theta) = (1-t) \Phi \left( \theta + \frac{1}{2\theta} \log \left( \frac{1-t}{t} \right) \right) + t \Phi \left( \theta + \frac{1}{2\theta} \log \left( \frac{t}{1-t} \right) \right)
\]

\[
C(u_1, u_2) = \exp \left( - \log u_1 \Phi \left( \frac{1}{\theta} + \frac{\theta}{2} \log \left( \frac{\log u_1}{\log u_2} \right) \right) - \log u_2 \Phi \left( \frac{1}{\theta} + \frac{\theta}{2} \log \left( \frac{\log u_2}{\log u_1} \right) \right) \right)
\]

(16)

- **Tawn copula**: for \( \theta \in [0,1] \)

\[
A(t; \theta) = 1 - \theta t + \theta t^2
\]

\[
C(u_1, u_2) = u_1 u_2 \exp \left( -\theta \frac{\log u_1 \log u_2}{\log u_1 u_2} \right)
\]

(17)
4.1. Parameter Estimation for Extreme Value Copula
The estimation of extreme value copula parameter can be employed by using the Pseudo
Maximum Likelihood (PML) estimation. PML estimation is based on the same concept with
IFM estimation. The difference is that in the stage one, the marginal function \( j \)-th is estimated
nonparametrically by empirical distribution function \( \hat{F}_j \), \( j = 1, \ldots, d \), as defined by equation (3).
The PML estimator was introduced by Genest et al. [10], having the characteristic of consistent
and asymptotically normal.

Let realizations \( z_i, i \in \{1, \ldots, T\} \) be independent and identically distributed (i.i.d.) from
random vector \( Z_i, i \in \{1, \ldots, T\} \) with common distribution function \( F \) and \( C \) is copula with an
unknown parameter \( \theta \). Refer to Genest et al. [10], the estimation of dependence parameter \( \theta \)
formally performed with the following two stage,

(i) The marginal distribution function is estimated from the empirical distribution \( \hat{F}_j \) which is
    defined in equation (3).
(ii) The log-likelihood \( L_C \) is maximized over \( \theta \) to get \( \hat{\theta} \),
    \[
    \hat{\theta} = \arg \max_{\theta} L_C(\theta) = \arg \max_{\theta} \sum_{i=1}^{T} \log c \left( \hat{F}_{i1}(z_{i1}), \ldots, \hat{F}_{id}(z_{id}): \theta \right)
    \]

4.2. Goodness-of-fit Test for Extreme Value Copula
For testing whether the extreme value copula comes from particular parametric class, we compare
the distance between nonparametric and parametric estimation of the extreme value copula as
usual. That is, we test

\[
H_0 : C(\cdot) \in \{C(\cdot; \theta) : \theta \in \Theta \} \quad \text{and} \quad H_1 : C(\cdot) \notin \{C(\cdot; \theta) : \theta \in \Theta \}.
\]

Let \((Z_{11}, Z_{12}), \ldots, (Z_{T1}, Z_{T2})\) be independent random vectors with common distribution
function \( F \). Assume that the copula \( C \) of \( F \) is an extreme value copula; i.e., that (12) holds.
Genest and Segers [12] derived the asymptotic distributions of the Pickands dependence function
estimator. It realized that those estimator can be expressed as functionals of the empirical
copula; more precisely, for \( t \in [0, 1] \),

\[
A_T(t) = \left( \int_0^1 C_T \left( u^{-1}, u \right) u^{-1} du \right)^{-1},
\]

where empirical copula \( C_T(\cdot) \) is defined as

\[
C_T(u_1, \ldots, u_d) = \frac{1}{T} \sum_{i=1}^{T} I(u_{i1} \leq u_1, \ldots, u_{id} \leq u_d),
\]

with \( U_{ij} = \frac{T}{T+1} F_{Tj}(X_{ij}) \) for \( i = 1, \ldots, T \) and \( j = 1, \ldots, d \) is called as pseudo observations,
\( F_{Tj}(x) = \frac{1}{T} \sum_{i=1}^{T} I(X_{ij} \leq x) \) for \( j = 1, \ldots, d \) is the empirical distribution of margins and \( I(\cdot) \)
is the indicator function.

Recently, Peng et al. [21] proposed a class of weighted estimator by solving the following equation with respect to \( \alpha \geq 0 \)

\[
\int_0^1 \left\{ C_T \left( u^{-1}, u \right) - u^\alpha \right\} \lambda(u, t) du = 0 \quad ; \alpha \geq 0, t \in [0, 1],
\]

for some weight function \( \lambda(u, t) \geq 0 \). Then the above estimator in (19) correspond to
\( \lambda(u, t) = u^{-1} \).
Let $A^λ(t)$ denote the weighted estimation of Pickands dependence function; i.e., $A^λ(t)$ is solution to (21). According to Haug et al. [13], nonparametric estimator for extreme value copula which is represented by equation (13) is given by

$$
\hat{C}^λ_T(u_1, u_2) = \exp \left( \log(u_1 u_2) A^λ\left( \frac{\log(u_2)}{\log(u_1 u_2)} \right) \right).
$$

Based on nonparametric $\hat{C}^λ_T(\cdot)$ and parametric $C(\cdot; \hat{θ}_T)$ estimator for extreme value copula, Cramér-von Mises statistic test is used

$$
\int_0^1 \int_0^1 T \left( \hat{C}^λ_T(u_1, u_2) - C(u_1, u_2; \hat{θ}_T) \right)^2 dC(u_1, u_2),
$$

to perform the goodness-of-fit test. We can also look the goodness-of-fit test review for copula by Berg [3] and Genest et al. [9]. The goodness-of-fit test review for extreme value copula is given by Genest et al. [8].

5. Empirical Study
5.1. Data
To illustrate these methods empirically, we will analyze the extreme data. As such, observation data at the time interval which involve monetary crisis incident are chosen. More specifically, the 2008 monetary crisis incident is chosen in this research. As such, the IDX Composite Index (IHSG) daily return data and the exchange rate for Indonesia Rupiah to United States Dollar (NT) daily return data from January 2000 to April 2016 is used.

5.2. Result and Discussion
In accordance with the methods, the first step is the investigation of the return data, to see whether there are any autocorrelation and volatility clustering. Figure 1 shows that the return series NT and IHSG are stationary but suspected to be volatility clustering.

Figure 2 shows that there is autocorrelation between lag in the NT return and IHSG return data. The meaning of ACF and PACF graph for NT return in the left part of Figure 2 is that there is autocorrelation between NT return at $t$ with NT return at previous times such as $t - 4, t - 5, t - 9$, which is supported by the Ljung-Box statistic test result. As such, model AR(4), AR(5), and AR(9) are tested on NT. Whereas the meaning of ACF and PACF graph for IHSG return in the right part of Figure 2 is that there is autocorrelation between IHSG return at $t$ with the residual at $t - 1$; also with IHSG return at $t - 1$. 
Figure 1. The plot of exchange rate return NT (left) and the plot of IDX composite index rate return IHSG (right).

Figure 2. The ACF and PACF graph for the exchange rate return NT (left) and the IDX Composite Index IHSG (right).

Thus, MA(1), AR(1), and ARMA(1,1) model are tested on IHSG. To identify the most suitable model, Akaike information criterion (AIC) is used. Table 1 presents the AIC value for
each model tested on NT and IHSG. According to the AIC value in Table 1, the suitable model for the NT return is AR(5) and that for IHSG return is AR(1).

Table 1. Model for the exchange rate return (left) and the IDX index return (right)

| NT model   | AIC       | IHSG model | AIC       |
|------------|-----------|------------|-----------|
| AR(4)      | −27938.73 | MA(1)      | −22512.42 |
| AR(5)      | −27954.14 | AR(1)      | −22512.90 |
| AR(9)      | −27950.22 | ARMA(1,1)  | −22510.92 |

Next, Lagrange Multiplier (LM) Arch test is used to test whether the NT and IHSG return have the volatility clustering characteristic as shown in Figure 1. The statistical value of LM Arch test was 20.66 (p-value ≈ 0) for NT and 461.3 (p-value ≈ 0) for IHSG, which means that the NT and IHSG return are volatility clustering. For NT return, if AR(5) is combined with GARCH(p,q) model, the AIC value increases compared to when GARCH(1,1), GARCH(1,2), and GARCH(1,3) model are tested. Whereas for the IHSG return, AR(1)-GARCH(1,1), AR(1)-GARCH(1,2), and AR(1)-GARCH(1,3) model are tested. The AIC values for the tested models are presented in Table 2. According to the AIC value in Table 2, the best model for NT return is GARCH(1,3) and the best model for IHSG return is AR(1)-GARCH(1,2).

Table 2. The best model for each individual asset

| NT Model       | AIC         | IHSG Model         | AIC         |
|----------------|-------------|-------------------|-------------|
| GARCH(1,1)     | −7.593732   | AR(1)-GARCH(1,1)  | −5.919138   |
| GARCH(1,2)     | −7.607657   | AR(1)-GARCH(1,2)  | −5.919749   |
| GARCH(1,3)     | −7.610265   | AR(1)-GARCH(1,3)  | −5.918709   |

The next step is the determination of estimated marginal distribution for NT and IHSG innovations which are obtained from the best AR-GARCH model. The data used is the extreme data. In this research, extreme data is chosen if NT innovations is greater than υ1 = 0.83 and IHSG innovations is less than υ2 = −0.98. Thus, the sample size become T = 182. Consequently, we do not work with \( P(F_1(Z_1) \leq u_1, F_2(Z_2) \leq u_2), (u_1, u_2) \in [0, 1]^2 \), but with conditional copula

\[
P\left(F_1|\geq u_1, F_2|\leq v_2(Z_1) \leq u_2\right), (u_1, u_2) \in [0, 1]^2
\]

where

\[
F_1|\geq u_1, (z_1) = P(Z_1 \leq z_1 | Z_1 \geq u_1) \text{ and } F_2|\leq v_2, (z_2) = P(Z_2 \leq z_2 | Z_2 \leq v_2).
\]

Next, we use \( F_1 \) and \( F_2 \) notation for conditional distribution function \( F_{1|\geq u_1} \) and \( F_{2|\leq v_2} \). The following are the empirical distribution plot of the extreme value from NT innovations and IHSG innovations. The empirical distribution is tested to be fit with the known extreme value distributions, but none are suitable. Thus in this research, the marginal distribution for NT and IHSG innovations are estimated nonparametrically by empirical distribution function \( F_j \), \( j = 1, 2 \).

Consequently, the estimation of multivariate distribution is performed by the estimation of extreme value copula parameter and the estimation of copula parameter using the Pseudo Maximum Likelihood (PML). In this paper, the extreme value copulas are tested by Gumbel, Galambos, Husler-Reiss, and Tawn copula. As a comparison, the copulas are also tested by Gaussian, Frank, Gumbel and Clyton copula. Based on the parametric model written in equation (9, (10), (14), (15), (16), and (17), the six copulas apply for positive dependence parameter. However, the dependence parameter between IHSG and NT result is negative. As such, 90 (or 270) degree rotation was performed so that the negative parameter could apply. According to
Figure 3. Empirical distribution plot of innovations for the extreme value of exchange rate return (left) and for the extreme value of stock index return (right).

Soto et al. [25], the 270 degree rotation of copula C in equation (10), (14), (15), (16), and (17) become

\[ C_{\text{Rotasi}270^\circ} = u_1 - C(u_1, 1 - u_2; -\theta); \quad -\theta > 0 \]

and the 90 degree rotation of copula C in equation (9) become

\[ C_{\text{Rotasi}90^\circ} = u_2 - C(1 - u_1, u_2; -\theta); \quad -\theta > 0 \]

Goodness-of-fit test of the four extreme value copulas which were rotated 270 degree are presented in Table 3.

Table 3. Goodness-of-fit test of the 270°-rotated extreme value copulas

| Extreme Value Copula | Statistic Cramer-Von-Mises | p-value |
|----------------------|-----------------------------|---------|
| Gumbel               | 0.012716                    | 0.8746  |
| Galambos             | 0.016241                    | 0.8187  |
| Husler-Reiss         | 0.017653                    | 0.7757  |
| Tawn                 | 0.0096834                   | 0.9076  |

Goodness-of-fit test of Gaussian, Frank, the 90° rotated Gumbel, and the 270° rotated Clayton copula are presented in Table 4.

Table 4. Goodness-of-fit test of the copulas

| Copula               | Statistic Cramer-Von-Mises | p-value |
|----------------------|-----------------------------|---------|
| Gaussian             | 0.021576                    | 0.5490  |
| Frank                | 0.018728                    | 0.6299  |
| 90° rotated Gumbel   | 0.013397                    | 0.8776  |
| 270° rotated Clayton | 0.013232                    | 0.8836  |
The smaller value of statistic Cramer-Von-Mises means that a nonparametric estimator \( C_T (\cdot) \) and a parametric estimator \( C (\cdot; \hat{\theta}_T) \) are getting closer. The value of statistic Cramer-Von-Mises for the rotated extreme value copulas in Table 3 are relatively smaller than the value of statistic Cramer-Von-Mises for the copulas in Table 4. Thus, based on Cramer-Von-Mises test criteria, the 270 degree-rotated extreme value copula are relatively more appropriate than the copulas to model the dependence structure between the extreme value of assets return. Based on Sklar Theorem, the 270 degree-rotated extreme value copulas are relatively more suitable than the copulas to model the distribution of extremes multivariate.

Based on the empirical study, Table 3 shows that the highest p-value is obtained for the 270 degree-rotated extreme value Tawn copula. Therefore, the 270 degree-rotated extreme value Tawn copula is the most suitable one to model the extremes multivariate distribution. We would choose this copula if we had to decide between the four of them.

6. Conclusion and Remark

Generally, the extreme value copulas are relatively more appropriate than the copulas to model the dependence structure between the extreme value of assets return (large losses). Especially, the extreme value copulas apply for the positive dependence parameter. Thus, we have to do rotation of extreme value copula so that the negative dependence parameter could apply.

In particular, the results of this study can be used to determine the best portfolio risk model. In general, it can be used to improve risk management in financial institutions.

In this study, we model a portfolio risk consisting only of two individual financial assets (two variables). Further research that can be done is to model a portfolio risk consisting of three or more extreme value of assets return with positive and negative parameters and also to produce the required software.

Acknowledgement

This work funded by Direktorat Riset dan Pengabdian Masyarakat, Direktorat Jenderal Penguatan Riset dan Pengembangan, Kementerian Riset, Teknologi, dan Pendidikan Tinggi, under a number contract 011/SP2H/LT/DRPM/IV/2017.

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