Transitionless quantum drivings for the harmonic oscillator

J G Muga1, X Chen1,2, S Ibáñez1, I Lizuain1 and A Ruschhaupt3

1 Departamento de Química-Física, UPV-EHU, Apdo 644, 48080 Bilbao, Spain
2 Department of Physics, Shanghai University, 200444 Shanghai, People’s Republic of China
3 Institut für Theoretische Physik, Leibniz Universität Hannover, Appelstraße 2, 30167 Hannover, Germany

Received 21 December 2009, in final form 9 February 2010
Published 6 April 2010
Online at stacks.iop.org/JPhysB/43/085509

Abstract

Two methods to change a quantum harmonic oscillator frequency without transitions in a finite time are described and compared. The first method, a transitionless-tracking algorithm, makes use of a generalized harmonic oscillator and a non-local potential. The second method, based on engineering an invariant of motion, only modifies the harmonic frequency in time, keeping the potential local at all times.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Changing the external parameters of the Hamiltonian is a fundamental and standard operation to probe, control or prepare a quantum system. In many cases, it is desirable to go from an initial parameter configuration to a final one without inducing transitions, as in the expansions performed in fountain clocks [1], the release of atoms from optical lattices [2] or compressions to increase the spatial density [2]. In fact, most of the current experiments with cold atoms are based on a cooling stage and then an adiabatic drive of the system to some desired final trap or regime [3]. These ‘transitionless’ [4] or ‘frictionless’ [5] adiabatic processes may require exceedingly large times and become impractical [2], even impossible [3], or quite simply a faster process is desirable, e.g. to increase the repetition rate of a cycle or signal-to-noise ratio. This motivates the generic objective of achieving the same final state as the slow adiabatic processes, possibly up to phase factors, but in a much shorter time. One may try to fulfill that goal in two different ways: (a) designing appropriate ‘parameter trajectories’ of the Hamiltonian from the initial to the final times or (b) applying entirely new interactions that modify the Hamiltonian beyond a simple parameter evolution of the original form, for example, by adding different terms to it. In this paper, we shall analyse and discuss, for the harmonic oscillator, two recently proposed methods, whose relation had not been investigated, which actually implement these two different routes. While most of the treatment is applicable to an ‘abstract’ harmonic oscillator, we shall discuss physical implementations specific for ultracold atoms or ions. Indeed, harmonic traps and their manipulation are basic workhorses of this field.

For the harmonic oscillator, the parameter we consider is the trap frequency, which should go from $\omega_0$ to $\omega_f$ in a time $t_f$, preserving the populations of the levels $P_n(t_f) = P_n(0)$. ‘$n$’ labels the instantaneous $n$th eigenstate of the initial and final harmonic oscillator Hamiltonians,

$$\begin{align*}
H_0(0)|n(0)\rangle &= \hbar \omega_0 (n + 1/2)|n(0)\rangle, \\
H_0(t_f)|n(t_f)\rangle &= \hbar \omega_f (n + 1/2)|n(t_f)\rangle.
\end{align*}$$

(1)

One of the methods we shall discuss here relies on a general framework set by Kato in a proof of the adiabatic theorem [6], and has been formulated recently by Berry [4]. We shall term it ‘transitionless-tracking’ approach, or TT for short; the other one [7, 8] engineers the Lewis–Riesenfeld invariant [9] by an inverse method [10] to satisfy the desired boundary conditions; we shall call this method ‘inverse-invariant’, or II for short. In the basic version of TT, the dynamics is set to follow at all intermediate times the adiabatic path defined by an auxiliary Hamiltonian $\tilde{H}_0(t)$ (in our case a regular harmonic oscillator with frequency $\omega(t)$ and boundary conditions $\omega(t) = \omega_0$ and $\omega(t = t_f) = \omega_f$), and its instantaneous eigenvectors $|n(t)\rangle$, up to phase factors. Instead, in the II approach, the auxiliary object is an engineered Lewis–Riesenfeld invariant $I(t)$ set to commute with $\tilde{H}_0(0)$ at $t = 0$ and with $\tilde{H}_0(t_f)$ at $t_f$. In both cases, intermediate states may be highly non-adiabatic with respect to the instantaneous eigenstates of the Hamiltonians actually applied, $\tilde{H}_1(t)$ and $\tilde{H}_II(t)$. 


We shall provide first the equations characterizing the two approaches and then comment on possible physical implementations.

2. Transitionless tracking algorithm

2.1. General formalism

For the general formalism, we follow [4] closely. Assume a time-dependent Hamiltonian \( \hat{H}_0(t) \) with initial and final values (1), instantaneous eigenvectors \( |n(t)\rangle \) and eigenvalues \( E_n(t) \).

\[
\hat{H}_0(t)|n(t)\rangle = E_n(t)|n(t)\rangle. \tag{2}
\]

A slow change would preserve the eigenvalue and eigenvector along the dynamical evolution times a phase factor, and the adiabatic approximation of the wavefunction is

\[
|\psi_n(t)\rangle = \exp \left\{ \frac{i}{\hbar} \int_0^t dt' E_n(t') \right\} |n(t)\rangle. \tag{3}
\]

We now seek a Hamiltonian \( \hat{H}(t) \) such that the adiabatic approximation \( |\psi_n(t)\rangle \) represents the exact dynamics,

\[
\hat{i}\hbar \partial_t |\psi_n(t)\rangle = \hat{H}(t)|\psi_n(t)\rangle, \tag{4}
\]

\( \hat{H}(t) \) (which is \( \hat{H}_{TT} \) if distinction with the other method is needed) is related to the corresponding unitary operator by

\[
\hat{U}(t) = \sum_n E_n(t)|n\rangle \langle n| + \hat{i} \hbar \sum_n (|\hat{n}| \langle n| - |n\rangle \langle \hat{n}|) (\langle n| \langle n|) - |n\rangle \langle \hat{n}|)\langle n|), \tag{5}
\]

we find from (6),

\[
\hat{H}(t) = \sum_n E_n(t)|n\rangle \langle n| + \hat{i} \hbar \sum_n (|\hat{n}| \langle n| - |n\rangle \langle \hat{n}|) (\langle n| \langle n|) - |n\rangle \langle \hat{n}|)\langle n|), \tag{7}
\]

where we have simplified the notation \( |n\rangle = |n(t)\rangle \). It is also possible to choose other phases in (3) [4]. The simplest case is \( \hat{U}(t) = \sum_n |n(t)\rangle \langle n(0)| \), without phase factors, which leads to \( \hat{H}(t) = \hat{i} \hbar \sum_n |\hat{n}| \langle n| \). Note that with this choice, \( \hat{H}_0(t) \) has been formally suppressed in \( \hat{H}(t) \) but still plays a role through its eigenfunctions \( |n(t)\rangle \).

2.2. Application to the harmonic oscillator

We now apply the above to the harmonic oscillator

\[
\hat{H}_0(t) = \hat{p}_i^2/(2m) + m\omega(t)^2\hat{x}_i^2/2 = \hbar\omega(t) (\hat{a}_i^\dagger \hat{a}_i + 1/2), \tag{9}
\]

where \( \hat{a}_i \) and \( \hat{a}_i^\dagger \) are the (Schrödinger picture!) annihilation and creation operators at time \( t \),

\[
\hat{x} = \sqrt{\frac{\hbar}{2m\omega(t)}} (\hat{a}_i + \hat{a}_i^\dagger), \tag{10}
\]

\[
\hat{p} = i\sqrt{\frac{\hbar m\omega(t)}{2}} (\hat{a}_i^\dagger - \hat{a}_i), \tag{11}
\]

\[
\hat{a}_i = \sqrt{\frac{\hbar m\omega(t)}{2}} \left( \hat{x} + \frac{i}{\hbar\omega(t)} \hat{p} \right), \tag{12}
\]

\[
\hat{a}_i^\dagger = \sqrt{\frac{\hbar m\omega(t)}{2}} \left( \hat{x} - \frac{i}{\hbar\omega(t)} \hat{p} \right). \tag{13}
\]

This time dependence may be misleading and rather unusual at first so we insist that since the frequency depends on time, the ‘instantaneous’ ladder operators \( \hat{a}_i, \hat{a}_i^\dagger \) create and annihilate different ‘instantaneous’ states, adapted to the corresponding frequency. Thus, the ladder operators with different time labels do not commute in general, although some combinations, e.g. those equivalent to powers of \( \hat{x} \) and/or \( \hat{p} \), do commute, as we shall see later.

The instantaneous eigenstates \( |n(t)\rangle \) can be written in a coordinate representation as

\[
\langle x|n(t)\rangle = \frac{1}{\sqrt{2\pi n!}} \left( \frac{m\omega(t)}{\pi\hbar} \right)^{1/4} \times \exp \left\{ -\frac{1}{2\hbar} \left( \frac{m\omega(t)}{\hbar} \right) x^2 \right\} H_n \left( \sqrt{\frac{m\omega(t)}{\hbar}} x \right), \tag{14}
\]

and their derivative with respect to time is

\[
\langle x|\partial_t n(t)\rangle = \frac{1}{4} \sqrt{n(n-1)} \frac{\omega}{\omega(t)} |n\rangle + \frac{1}{2} \sqrt{(n+1)(n+2)} \frac{\omega}{\omega(t)} |n\rangle + \frac{1}{2} \sqrt{n(n+1)} \frac{\omega}{\omega(t)} |n\rangle, \tag{15}
\]

so that \( \hat{H}_1(t) \) can be written as

\[
\hat{H}_1(t) = \hat{i} \hbar \sum_n |\partial_t n(t)| \langle n| \hat{n} \rangle \langle n| \sum_n |n(t)\rangle \langle n|, \tag{16}
\]

\[
\times \left[ \left( \frac{1}{4} - \frac{m\omega(t)}{2\hbar} \right) x^2 \right] |n\rangle \langle n| + \sqrt{\frac{m\omega(t)}{2\hbar}} \hat{x} \sqrt{n(n+1)} \langle n| |n\rangle. \tag{17}
\]

Using \( \hat{a}_i = \sum_n \sqrt{n} |n-1\rangle \langle n| \), and the relations between \( \hat{x}, \hat{p}, \hat{a}_i, \hat{a}_i^\dagger \) written above,

\[
\hat{H}_1(t) = \hat{i} \hbar \frac{\omega}{\omega(t)} \sum_n \left[ \frac{1}{4} - \frac{m\omega(t)}{2\hbar} \right] x^2 + \sqrt{\frac{m\omega(t)}{2\hbar}} \hat{x} \hat{a}_i, \tag{18}
\]

With \( [\hat{x}, \hat{p}] = i\hbar \), we finally write the Hamiltonian \( \hat{H}_1(t) \) in the following simple forms:

\[
\hat{H}_1(t) = -\frac{\omega}{4\omega} (\hat{x} \hat{p} + \hat{p} \hat{x}) = \frac{\hbar}{4\omega} (\hat{a}_i - (\hat{a}_i^\dagger)^2). \tag{19}
\]

In the last expression, the subscript \( t \) in \( \hat{a} \) and \( \hat{a}_i^\dagger \) has been dropped because the squeezing combination \( \hat{a}^2 - (\hat{a}_i^\dagger)^2 \) is
assumption in this Hamiltonian is that the same vibrational frequency, i.e. the same external potential for atomic motion, is applicable to the ground and excited states $|g\rangle$ and $|e\rangle$. This is the standard case for ions in Paul traps, whereas for neutral atoms, it may be realized in optical traps too, at least for suitably chosen species and transitions (for example, alkali-earth atoms and spin forbidden transitions), by means of ‘magic wavelength’ compensating techniques by which the light shifts and thus the potentials for atomic motion in the $|g\rangle$ and $|e\rangle$ states can be controlled and made identical [15, 16].

2.3.2. Interaction picture. Let us now write the above Hamiltonian in an interaction picture defined by the Hamiltonian $\hat{H}_0 = \hat{H}_T + \hbar \omega_L |e\rangle \langle e|$, where $\omega_L = (\omega_1 + \omega_2)/2$ has been introduced. The interaction Hamiltonian $\hat{H}_I = e^{i\omega_1/\hbar}(\hat{H}_R - \hbar \bar{\eta}) e^{-i\omega_1/\hbar}$ reads

$$\hat{H}_I(t) = -\hbar \hat{\Delta} |e\rangle \langle e| + \sum_{j=1}^2 \hbar \Omega_j \left( e^{i\eta_j(t)+\phi_j(t)} e^{-i\omega_L} e^{-i\phi_j} |e\rangle \langle g| + \text{H.c.} \right),$$

where $\hat{\Delta} = \omega_L - \omega$, and now $\hat{\Delta}(t) = \hat{\Delta} e^{-i\omega_1 t}$ and $\hat{\Delta}^\dagger(\hat{\Delta})^\dagger$ are the time-dependent Heisenberg annihilation and creation operators respectively. Note also that the fast oscillating off-resonant $e^{i\omega_1(t)\eta_j(t)^\dagger}$ terms have been neglected in the rotating wave approximation (RWA). The parameter $\eta_j = k_j x_0$ is known as the Lamb–Dicke (LD) parameter, where $x_0 = \sqrt{\hbar/(2m\omega)}$ is the extension (square root of the variance) of the ion’s ground state, i.e. $\hat{x} = x_0 (\hat{\Delta} + \hat{\Delta}^\dagger)$.

2.3.3. Adiabatic elimination and effective Hamiltonian. For a general wavefunction (in the corresponding interaction picture) such as

$$|\psi_f(t)\rangle = \sum_{n=0}^\infty |g_n(t)\rangle |g, n\rangle + e_n(t) |e, n\rangle,$$

the differential equations of motion for the probability amplitudes $g_n(t)$ and $e_n(t)$ are obtained from the Schrödinger equation $i\hbar \partial_t |\psi_f(t)\rangle = \hat{H}_I |\psi_f(t)\rangle$,

$$i \dot{g}_n(t) = \frac{1}{2} \sum_{j=1}^2 \sum_{n'=0}^\infty \Omega_j e^{i(\theta_j + \phi_j)} |n\rangle \langle n',|e_n(t),$$

$$i \dot{e}_n(t) = -\hat{\Delta} e_n(t) + \frac{1}{2} \sum_{j=1}^2 \sum_{n'=0}^\infty \Omega_j e^{-i(\theta_j + \phi_j)} |n\rangle \langle n',|g_n(t),$$

where $\theta_j = \omega_j - \omega_L$. For large detunings, i.e. for $|\Delta| \gg \Omega_j, \omega$, see figure 1, and for an ion initially in the ground state, one may assume that the excited state $|e\rangle$ is scarcely populated and it may be adiabatically eliminated. Then, setting $\dot{e}(t) = 0$, $e_n(t)$ may be written as a function of $g_n(t)$ from equation (27), and substituting this result into (26) results in a differential equation for the ground state probability amplitude,

$$i \dot{g}_n(t) = s g_n(t) + \frac{\Omega}{2} \sum_{n'=0}^\infty F_{n,n'}(t) g_{n'}(t),$$

where $s$ is the transition rate between the excited state $|e\rangle$ and the ground state $|g\rangle$. This is the so-called adiabatic elimination of the excited state. The differential equation (28) is then solved for $g_n(t)$, and the probabilities $e_n(t)$ are then calculated from equation (26).

Figure 1. Schematic electronic and vibrational level structure for a two-photon transition in an ion trapped with frequency $\omega$. $\omega_1$ and $\omega_2$ are the laser frequencies, and $\omega_j$ is the transition frequency between the ground and excited states. See the text for further details.
where
\[ s = \frac{\Omega_1^2 + \Omega_2^2}{4\Delta}, \]

\[ \mathcal{F}_{\alpha,\nu}(t) = \langle n | e^{-i[\hat{d}_\nu^+(t) + \hat{d}_\nu(t)]} | n' \rangle e^{i\hat{d}_\nu \phi}, \]
and where the effective two-photon Raman parameters, denoted by tildes, are given by
\[ \tilde{\delta} = \delta_1 - \delta_2, \quad \tilde{\eta} = \eta_1 - \eta_2, \quad \tilde{\phi} = \phi_1 - \phi_2, \quad \frac{\tilde{\Omega}}{2} = \frac{\Omega_1 \Omega_2}{4\Delta}. \]

The equation for the ground state probability amplitude corresponds to an effective Hamiltonian
\[ \hat{H}_{\text{eff}} = \hbar \delta \langle g | \hat{g} + \frac{\hbar \tilde{\Omega}}{2} (e^{i[\hat{d}_\nu^+(t) + \hat{d}_\nu(t)]} + \hat{d}_\nu \hat{d}_\nu^+ e^{-i\hat{d}_\nu \phi} + \hat{d}_\nu^+ \hat{d}_\nu e^{i\hat{d}_\nu \phi} + \text{H.c.}) | g \rangle \]

Note that the Stark shift produced by off-resonant driving is included in \( s \), which is a constant of motion and produces no effect on the Raman coupling between sidebands. Thus, we have adiabatically eliminated the excited state |e⟩ ending with a Hamiltonian of the same form as (24) where the transitions between electronic levels are not present.

2.3.4. Two-photon Jaynes–Cummings Hamiltonian in the Raman scheme: vibrational RWA. Using the Baker–Campbell–Hausdorff (BCH) identity, the exponential in the effective Hamiltonian (32) may be expanded in the power series of \( \tilde{\eta} \) [17, 18].

\[ \hat{H}_{\text{eff}} = \frac{\hbar \tilde{\Omega}}{2} \sum_{nn'} \frac{(i\tilde{\eta})^{n-n'}}{n!n'!} \hat{a}^\dagger_{n'} \hat{a}^\dagger_{n} e^{-i\hat{d}_\nu \phi} e^{-i\hat{d}_\nu \phi} + \text{H.c.} \]

If the effective detuning is \( \tilde{\delta} = \delta_1 - \delta_2 = 2\omega \), the second blue sideband becomes resonant, and we may neglect rapidly oscillating terms in a second or vibrational RWA [18]. The above Hamiltonian is then simplified to a two-photon Jaynes–Cummings–like Hamiltonian without electronic transitions. To leading order in \( \tilde{\eta} \), it takes the form
\[ \hat{H}_{\text{eff}} \approx \tilde{\eta} \frac{\hbar \Omega}{4} (\hat{a}^\dagger \hat{a} \delta^2 + \hat{a}^\dagger \hat{a} \delta^2 + \hat{a}^\dagger \hat{a} \delta^2 - \hat{a}^\dagger \hat{a} \delta^2), \]
where, in the last step, a relative phase between the applied fields \( \phi = \phi_1 - \phi_2 = -\pi/2 \) has been assumed.

2.3.5. Validity for time-dependent \( \omega \). Unfortunately, the above formal manipulations and approximations cannot be carried out in general for a time-dependent \( \omega \). The interaction picture elaborated in section 2.3.2, in particular, assumes a constant \( \hat{\omega} \). A time-dependent one would require a more complex approach with time-ordering operators [19]. Similarly, the vibrational rotating wave approximation requires the stability of the frequency for times larger than a period to avoid off-resonant couplings. One may still obtain (34) for a sufficiently slowly varying \( \omega \), the criterion being that the change of the time-dependent trapping frequency in one time period \( T \) has to be much smaller than the frequency itself. We can write this condition as \( \dot{\omega}(t) T \ll \omega(t) \) or
\[ \frac{\dot{\omega}(t)}{\omega(t)^2} \ll 1, \]
which turns out to be the adiabaticity condition for the harmonic oscillator. Of course, if this condition were satisfied, the whole idea of applying the TT method would be useless. These arguments are far from constituting a proof that the TT method cannot be implemented for the harmonic oscillator. They simply leave this as an unresolved issue.

3. Engineering the Lewis–Riesenfeld invariant

In this section, we describe a different method for the transitionless dynamics of the harmonic oscillator [7]. A harmonic oscillator such as \( \hat{H}_0(t) \) in equation (9) has the following time-dependent invariant [9]:
\[ I(t) = \frac{1}{2} \left( \frac{\dot{\hat{x}}^2}{\hbar^2 m \omega_0^2} + \frac{\hat{p}^2}{m} \right), \]
where \( \dot{\hat{x}} = b(t) \hat{p} - \frac{m \omega_0^2}{\hbar} \hat{x} \) plays the role of a momentum conjugate to \( \hat{x}/\hbar \), the dots are derivatives with respect to time and \( \omega_0 \) is in principle an arbitrary constant. The scaling, dimensionless function \( b = b(t) \) satisfies the subsidiary condition
\[ b + \omega(t)^2 b = \omega_0^2/b^3, \]
for the Ermakov equation where real solutions must be chosen to make \( I \) Hermitian. \( \omega_0 \) is frequently rescaled to unity by a scale transformation of \( b \) [9]. Another common and convenient choice, which we shall adopt here, is \( \omega_0 = \omega(0) \). The eigenstates of \( I(t) \) become, with appropriate phase factors, the solutions of the time-dependent Schrödinger equation,
\[ \Psi_n(x, t) = \left( \frac{m \omega_0}{\pi \hbar} \right)^{1/4} \frac{1}{(2^n n! 1/2)^{1/2}} \exp \left[ -i (n + 1/2) \int_0^t \frac{\omega_0}{b(t)^2} \right] \times \exp \left[ \frac{m}{\hbar} \left( \frac{b(t)}{b(t)} + \frac{i \omega_0}{b(t)^2} \right) \right] \frac{m \omega_0}{b(t)} \]
and form a complete basis to expand any time-dependent state, \( \psi(x, t) = \sum_n c_n \psi_n(x, t) \), with the amplitudes \( c_n \) constant. A method to achieve frictionless, population preserving processes is to leave \( \omega(t) \) undetermined first, and set \( b \) so that \( \dot{I}(0) = \dot{\omega}(0) \) and \( [I(t_j), \hat{H}_0(t_j)] = 0 \). This guarantees that the eigenstates of \( I \) and \( \hat{H}_0 \) are common at initial and finite times and can be done by setting
\[ b(0) = 1, \quad b(0) = 0, \quad b = 0, \]
and interpolating \( b(t) \) with some real function that satisfies these boundary conditions. The simplest choice is a polynomial
\[ b(t) = \sum_{j=0}^s a_j t^j. \]
may be formed by magnetic and/or optical means and their frequencies are routinely varied in time as part of many cold atom experiments. In magnetic traps, for example, the frequency is modulated harmonically to look for collective excitation modes of a condensate [21], and ramped down adiabatically to change its conditions (critical temperature, particle number, spatial extension) [21, 22] or as a preliminary step to superimpose an optical lattice [23]. Some experiments involve both time-dependent magnetic and optical traps or antitrap [24]. Purely optical traps are also manipulated in time, e.g. for adiabatic cooling of single neutral atoms [25]. In particular, laser beams detuned with respect to the atomic transition form effective potentials for the ground state depending on Rabi frequency $\Omega$ and detuning $\Delta$ as $\Omega^2/4\Delta$ by adiabatic elimination of the excited state, thus forming attractive or repulsive potentials. This effective interaction can be made time dependent by varying the laser intensity, the frequency or both [1] since the optical frequencies are many orders of magnitude larger than Rabi frequencies or detunings, and the changes will be slowly varying in the scale of optical periods. To vary the sign of $\omega^2(t)$, two dipole beams locked respectively on the blue and red side of the line may be superimposed (see figures 3 and 4). The intensity of a dipole trap can be changed by three or four orders of magnitude in 100 ns using the acousto-optics or electro-optics modulators. This affects the trap radius, which can also be controlled [2].

4. Discussion

We have compared and distinguished two different methods: a ‘transitionless-tracking’ (TT) algorithm, and an ‘inverse-invariant’ (II) method, to achieve transitionless dynamics for a fast frequency change of a quantum harmonic oscillator. They imply different driving Hamiltonians $H_{\text{TT}}$ and $H_{\text{II}}$. The one in the II method can be implemented for ultracold atoms or ions by varying the trap frequency in time so as to enforce at initial and final times the commutativity between the Hamiltonian and an invariant of motion $I$ with the form of a generalized harmonic oscillator. In this method, the states are at all times eigenstates of the invariant if initially so. Extending this method to other (anharmonic) potentials or systems may be difficult and remains an open question; in principle one should find in each case the corresponding state-carrying invariant and be able to engineer it. A generalization to Bose–Einstein condensates has been worked out [8] without invoking the invariant concept, but it relies on scaling laws applicable to the harmonic oscillator. The TT method does not explicitly make use of invariants and provides a Hamiltonian for which the state dynamics follows exactly the adiabatic approximation of a time-dependent process defined by a reference Hamiltonian $H_0$. In this respect, it has the advantage of being, at least formally, more generally applicable than the II approach. The feasibility of the actual realization is quite another matter and has to be studied in each case. An application example for two-level systems is discussed in [4]. For the harmonic oscillator studied here, $H_{\text{TT}}$ becomes, as $I$, a generalized harmonic oscillator too, but it is different from $I$ and plays a different role. In particular,
the ‘carrying states’ are not instantaneous eigenstates of $\hat{H}_T$, but of the reference, harmonic oscillator Hamiltonian $\hat{H}_0$. $\hat{H}_T$ includes, in addition to a harmonic term in $\hat{H}_0$ with a time-dependent frequency going from the initial to the final values, a time-dependent (squeezing) non-local potential with the symmetrized product of position and momentum operators. We have found a realization for cold atoms that, unfortunately, requires a time scale larger than the times needed for an adiabatic process. Therefore the applicability of the method remains an open question. Trap expansions and compressions play such an important role in cold atom experiments that investigating further the fundamental properties of both approaches and their technical realization is definitely worthwhile.

Acknowledgments

We thank M V Berry and E Torrontegui for discussions and acknowledge funding by projects no G1U07/40, FIS2009-12773-C02-01, 60806041, 08QA14030, 2007CG52, S30105, and Juan de la Cierva Program.

Appendix. Relation to the squeezing operator

The evolution operator takes a particularly simple form when using the simplified case $E_\omega(t) = 0$, so that $\hat{H}(t) = \hat{H}_1(t)$. Taking into account that $[\hat{H}_1(t), \hat{H}_1(t')] = 0$, we can write

$$\hat{U}(t) = e^{-i\int_0^t \hat{H}_1(t') \, dt'}/\hbar.$$  (A.1)

This may be evaluated explicitly with (19) fixing the time of the creation and annihilation operators to 0,

$$\hat{U}(t) = e^{\frac{i}{\hbar} \ln (\sqrt{\frac{\omega(t)}{\omega(0)}} |\hat{a}|^2)} = 1 + \frac{\hat{S}(r(t))}{\sqrt{n}}.$$  (A.2)

which is a squeezing operator with real argument $r(t) = \ln (\sqrt{\omega(t)/\omega(0)})$. It is unitary with inverse $[\hat{S}(r)]^{-1} = \hat{S}(-r)$. Using the relations

$$\hat{a}_t + \hat{a}_r = \frac{\omega(T)}{\omega(0)} (\hat{a}_0 + \hat{a}_0^\dagger),$$

$$\hat{a}_t^\dagger - \hat{a}_r = \frac{\omega(T)}{\omega(0)} (\hat{a}_0^\dagger - \hat{a}_0),$$  (A.3)

and the formal properties of $\hat{S}$, see e.g. [26], it is easy to prove that

$$\hat{S}(r)\hat{a}_0\hat{S}(-r) = \hat{a}_r, \quad \hat{S}(r)\hat{a}_r^\dagger\hat{S}(-r) = \hat{a}_t.$$  (A.4)

In fact, any combination of powers of $\hat{a}_0$ and $\hat{a}_0^\dagger$ is mapped to the same combination of powers of $\hat{a}_r$ and $\hat{a}_r^\dagger$ by this unitary transformation. $|0_t\rangle \equiv \hat{S}(0_t)|0_t\rangle$ is indeed the vacuum at time $t$ as

$$\hat{a}_t|0_t\rangle = \hat{S}(r)|\hat{S}(-r)a\hat{S}(r)|0_t\rangle = \hat{S}(r)|\hat{a}_0|0_t\rangle = 0.$$  (A.5)

In a similar way, we note that, consistently,

$$\hat{S}(r)|n(0)\rangle = \hat{S}(r)\frac{1}{\sqrt{n!}}(\hat{a}_0^\dagger)^n|0_n\rangle$$

$$= \frac{1}{\sqrt{n!}}\hat{S}(r)(\hat{a}_0^\dagger)^n \hat{S}(-r)\hat{S}(r)|0_n\rangle$$

$$= \frac{1}{\sqrt{n!}}(\hat{a}_r)^n|0_t\rangle = |n(t)\rangle.$$  (A.6)

References

[1] Bize S et al 2005 J. Phys. B: At. Mol. Opt. Phys. 38 S449
[2] Han D J, DePue M T and Weiss D S 2001 Phys. Rev. A 63 023405
[3] Polkovnikov A and Gritsev V 2008 Nature Phys. 4 477
[4] Berry M V 2009 J. Phys. A: Math. Theor. 42 365303
[5] Rezek Y and Kosloff R 2006 New. J. Phys. 8 83
[6] Kato T 1950 J. Phys. Soc. Japan 5 345
[7] Chen X, Ruschhaupt A, Schmidt S, del Campo A, Guéry-Odelin D and Muga J G 2010 Phys. Rev. Lett. 104 063002
[8] Muga J G, Chen Xi, Ruschhaupt A and Guéry-Odelin D 2009 J. Phys. B: At. Mol. Opt. Phys. 42 241001
[9] Lewis H R and Riesenfeld W B 1969 J. Math. Phys. 10 1458
[10] Palao J P, Muga J G and Sala R 1998 Phys. Rev. Lett. 80 5469
[11] Berry M V 1985 J. Phys. A: Math..Gen. 18 15
[12] Meekhof D M, Monroe C, King B E, Itano W M and Wineland D J 1996 Phys. Rev. Lett. 76 1796
[13] Zeng H and Lin F 1998 Phys. Lett. A 201 139
[14] Zeng H 1998 Phys. Lett. A 247 273
[15] Katori H, Ido T and Kuwata-Gonokami M 1999 J. Phys. Soc. Japan 68 2479
[16] Ido T and Katori H 2003 Phys. Rev. Lett. 91 053001
[17] Orszag M 2000 Quantum Optics (Berlin: Springer)
[18] Lizzaun I, Muga J G and Eschner J 2008 Phys. Rev. A 77 053817
[19] Schrade G, Bardroff P J, Glauber R J, Leichtle C, Yakovlev V and Schleich W P 1999 Appl. Phys. B 64 181
[20] Khaykovich L et al 2002 Science 296 1290
[21] Jin D S, Ensher J R, Matthews M R, Wieman C E and Cornell E A 1996 Phys. Rev. Lett. 77 470
[22] Miesner H J, Stamper-Kurn D M, Andrews M R, Durfee D S, Inouye S and Ketterle W 1998 Science 279 1005
[23] Tuchman A K, Li W, Chien H, Dettmer S and Kasevich M A 2006 New J. Phys. 8 311
[24] Ho T-L and Zhou Q 2007 Phys. Rev. Lett. 99 120404
[25] Tuchendler C, Lance A M, Browaeys A, Sortais Y R P and Grangier P 2008 Phys. Rev. A 78 033425
[26] Barnett A M and Radmore P M 1997 Methods in Theoretical Quantum Optics (Oxford: Oxford University Press)