Knuth’s coherent presentations of plactic monoids of type A
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Abstract – We construct finite coherent presentations of plactic monoids of type A. Such coherent presentations express a system of generators and relations for the monoid extended in a coherent way to give a family of generators of the relations amongst the relations. Such extended presentations are used for representations of monoids, in particular, it is a way to describe actions of monoids on categories. Moreover, a coherent presentation provides the first step in the computation of a categorical cofibrant replacement of a monoid. Our construction is based on a rewriting method introduced by Squier that computes a coherent presentation from a convergent one. We compute a finite coherent presentation of a plactic monoid from its column presentation and we reduce it to a Tietze equivalent one having Knuth’s generators.

1. Introduction

Plactic monoids. The structure of plactic monoids appeared in the combinatorial study of Young tableaux by Schensted [21] and Knuth [12]. The plactic monoid of rank \( n > 0 \), denoted by \( P_n \), is generated by the set \( \{1, \ldots, n\} \) and subject to the Knuth relations:

\[
xyz = xzy \quad \text{for} \quad 1 \leq x < y < z \leq n, \quad yzx = yxz \quad \text{for} \quad 1 \leq x < y \leq z \leq n.
\]

For instance, the monoid \( P_2 \) is generated by \( \{1, 2\} \) and submitted to the relations 211 = 121 and 221 = 212. The Knuth presentation of the monoid \( P_3 \) has 3 generators and 8 relations. Lascoux and Schützenberger used the plactic monoid in order to prove the Littlewood-Richardson rule for the decomposition of tensor products of irreducible modules over the Lie algebra of \( n \) by \( n \) matrices, [22][16]. The structure of plactic monoids has several applications in algebraic combinatorics and representation theory [15, 16, 14, 5] and several works have generalised the notion of tableaux to classical Lie algebras [1, 25, 10, 19, 23].

Syzygies of Knuth’s relations. The aim of this work is to give an algorithmic method for the syzygy problem of finding all independent irreducible algebraic relations amongst the Knuth relations. A 2-syzygy for a presentation of a monoid is a relation amongst relations. For instance, using the Knuth relations there are two ways to prove the equality 2211 = 2121 in the monoid \( P_2 \), either by applying the first Knuth relation 211 = 121 or the second relation 221 = 212. This two equalities are related by a syzygy. Starting with a monoid presentation, we would like to compute all syzygies for this presentation and in particular to compute a family of generators for the syzygies. For instance, we will prove that in rank 2 the two Knuth relations form a unique generating syzygy for the Knuth relations. For rank greater than 3, the syzygies problem is difficult due to the combinatorial complexity of the relations. In commutative algebra, the theory of Gröbner bases gives algorithms to compute bases for linear syzygies. By a similar method, the syzygy problem for presentation of monoids can be algorithmically solved using convergent rewriting systems.
1. Introduction

Rewriting and plactic monoids. Study presentations from a rewriting approach consists in the orientation of the relations, then called reduction rules. For instance, the relations of the monoid $P_2$ can be oriented with respect to the lexicographic order as follows

$$\eta_{1,1,2} : 211 \Rightarrow 121 \quad \varepsilon_{1,2,2} : 221 \Rightarrow 212.$$ 

In a monoid presented by a rewriting system, two words are equal if they are related by a zig-zag sequence of applications of reductions rules. A rewriting system is convergent if the reduction relation induced by the rules is well-founded and if it satisfies the confluence property. This means that any reductions starting on a same word can be extended to end on a same reduced word. Recently plactic monoids were investigated by rewriting methods [13, 2, 4, 9, 3].

Coherent presentations of plactic monoids. We give a categorical description of 2-syzygies of presentations of the monoid $P_n$ using coherent presentations. Such a presentation extends the notion of a presentation of the monoid by globular homotopy generators taking into account the relations amongst the relations. We compute a coherent presentation of the monoid $P_n$ using the homotopical completion procedure introduced in [8, 6]. Such a procedure extends the Knuth-Bendix completion procedure [11], by keeping track of homotopy generators created when adding rules during the completion. Its correctness is based on the Squier theorem, [24], which states that a convergent presentation of a monoid extended by the homotopy generators defined by the confluence diagrams induced by critical branchings forms a coherent convergent presentation. The notion of critical branching describes the overlapping of two rules on a same word. For instance, the Knuth presentation of the monoid $P_2$ is convergent. It can be extended into a coherent presentation with a unique globular homotopy generator described by the following 3-cell corresponding to the unique critical branching of the presentation between the rules $\eta_{1,1,2}$ and $\varepsilon_{1,2,2}$:

$$\begin{array}{c}
2211 \\
\xrightarrow{\eta_{1,1,2}} \\
1212 \\
\xleftarrow{\varepsilon_{1,2,2}}
\end{array}$$

The Knuth presentation of the monoid $P_3$ is not convergent, but it can be completed by adding 3 relations to get a presentation with 27 3-cells corresponding to the 27 critical branchings. For the monoid $P_4$ we have 4 1-cells and 20 2-cells, for $P_5$ we have 5 1-cells and 40 2-cells and for $P_6$ we have 6 1-cells and 70 2-cells. However, in the last three cases, the completion is infinite and another approach is necessary to compute a finite generating family for syzygies of the Knuth presentation.

The column presentation. Kubat and Okniński showed in [13] that for rank $n > 3$, a finite convergent presentation of the monoid $P_n$ cannot be obtained by completion of the Knuth presentation with the degree lexicographic order. Then Bokut, Chen, Chen and Li in [2] and Cain, Gray and Malheiro in [4] constructed with independent methods a finite convergent presentation by adding column generators to the Knuth presentation. However, on the one hand, the proof given in [4] does not give explicitly the critical branchings of the presentation, which does not permit to use the homotopical completion procedure. On the other hand, the construction in [2] gave an explicit description of the critical branchings of the presentation, but this does not allow to get explicitly the relations amongst the relations.
2. Presentation of plactic monoids by rewriting

The Knuth coherent presentation. We construct a coherent presentation of the monoid $P_n$ that extends the Knuth presentation in two steps. The first step consists in giving an explicit description of the critical branchings of the column presentation. The column presentation of the plactic monoid has one generator $c_u$ for each column $u$, that is, a word $u = x_p \ldots x_1$ such that $x_p > \ldots > x_1$. Given two columns $u$ and $v$, using the Schensted algorithm, we compute the Schensted tableau $P(\Pi uv)$ associated to the word $uv$. One proves that the planar representation of the tableau $P(\Pi uv)$ contains at most two columns. If the planar representation is not the tableau obtained as the concatenation of the two columns $u$ and $v$, one defines a rule $\alpha_{u,v}: c_u c_v \Rightarrow c_w c_w'$ where $w$ and $w'$ are respectively the left and right columns (with one of them possibly empty). We show that the column presentation can be extended into a coherent column presentation whose any 3-cell has at most an hexagonal form. For instance, the column presentation for the monoid $P_2$ has generators $c_1, c_2, c_{21}$, with the rules $\alpha_{2,1}: c_2 c_1 \Rightarrow c_{21}$, $\alpha_{1,21}: c_1 c_{21} \Rightarrow c_{21} c_1$ and $\alpha_{2,21}: c_2 c_{21} \Rightarrow c_{21} c_2$. This presentation has only one critical branching:

\[
\begin{array}{c}
\alpha_{2,1} : c_2 c_1 \Rightarrow c_{21} c_2 \\
\alpha_{1,21} : c_1 c_{21} \Rightarrow c_{21} c_1 \\
\alpha_{2,21} : c_2 c_{21} \Rightarrow c_{21} c_2 \\
\end{array}
\]

and thus the 3-cell of the extended coherent presentation is reduced to this 3-cell defined by this confluence diagram. Note that for column presentations of the monoids $P_3$, $P_4$ and $P_5$ we count respectively 7, 15 and 31 generators, 22, 115 and 531 relations, 42, 621 and 6893 3-cells.

The second step aimed at to reduce the coherent column presentation using Tietze transformations that coherently eliminates redundant column generators and defining relations to the Knuth coherent presentation giving syzygies of the Knuth presentation. For instance, if we apply this Tietze transformation on the column coherent presentation of the monoid $P_2$, we prove that the Knuth coherent presentation of $P_2$ on the generators $c_1, c_2$ and the relations $\eta_{1,1,2}, \varepsilon_{1,2,2}$ has a unique generating 3-cell $2\eta_{1,1,2} \xrightarrow{\varepsilon_{1,2,2}}$ described above.

Organisation of the article. The polygraphical description of string rewriting systems that we will use in this work is briefly recalled in Section 2.1, we refer the reader to [7] for a deeper presentation. In Section 2.2 we define the Knuth 2-polygraph that corresponds to the Knuth relations oriented with respect to the lexicographic order. In Section 2.3 we recall the column presentation introduced in [4]. The proof given in [4] for the convergence of this presentation consists in showing that this presentation has the unique normal form property. We give another proof of the confluence by showing the confluence of all the critical branchings of the column presentation. In Section 3 we recall the notion of coherent presentation of a monoid and we show the first main result of this article, that extends the column presentation into a coherent presentation, Theorem 3.2.2. In Section 4 we reduce the coherent column presentation into a coherent presentation that extends the Knuth presentation and that gives all syzygies of the Knuth’s relations, Theorem 4.4.7. Finally, we explicit a procedure that computes a family of generating syzygies for any plactic monoids of type A.

2. PRESENTATION OF PLACTIC MONOIDS BY REWRITING

In this preliminary section, we recall rewriting notions and some presentations and constructions of plactic monoids used in this article.
2. Presentation of plactic monoids by rewriting

2.1. Presentations of monoids by two-dimensional polygraphs

2.1.1. Two-dimensional polygraphs. In this article, we deal with presentations of monoids by rewriting systems, described by 2-polygraphs with only 0-cell denoted by •. Such a 2-polygraph Σ is given by a pair (Σ₁, Σ₂), where Σ₁ is a set and Σ₂ is a globular extension of the free monoid Σ₁⁺, that is a set of 2-cells β : u ⇒ v relating 1-cells in Σ₁⁺, where u and v denote the source and the target of β, respectively denoted by s₁(β) and t₁(β). If there is no possible confusion, Σ₂ will denote the 2-polygraph itself. Recall that a 2-category (resp. (2, 1)-category) is a category enriched in categories (resp. in groupoids). When two 1-cells, or 2-cells, f and g of a 2-category are 0-composable (resp. 1-composable), we denote by fg (resp. f †g) their 0-composite (resp. 1-composite). We will denote by Σ₂⁺ (resp. Σ₂⁺) the 2-category (resp. (2, 1)-category) freely generated by the 2-polygraph Σ, see [7, Section 2.4.] for expended definitions.

The monoid presented by a 2-polygraph Σ, denoted by Σ, is defined as the quotient of the free monoid Σ⁺ by the congruence generated by the set of 2-cells Σ₂. A presentation of a monoid M is a 2-polygraph whose presented monoid is isomorphic to M. Two 2-polygraphs are Tietze equivalent if they present isomorphic monoids.

2.1.2. Tietze transformations of 2-polygraphs. A 2-cell β of a 2-polygraph Σ is collapsible, if t₁(β) is a 1-cell of Σ₁ and the 1-cell s₁(β) does not contain t₁(β), then t₁(β) is called redundant. Recall from [6 2.1.1.], that an elementary Tietze transformation of a 2-polygraph Σ is a 2-functor with domain Σ₂⁺ that belongs to one of the following four transformations:

i) adjunction t₁⁺ : Σ₂⁺ → Σ₂⁺[x](β) of a redundant 1-cell x with its collapsible 2-cell β.

ii) elimination π₁⁺ : Σ₂⁺ → (Σ₁ \ {x}, Σ₂ \ {β})⁺ of a redundant 1-cell x with its collapsible 2-cell β.

iii) adjunction t₁⁺ : Σ₂⁺ → Σ₂⁺(β) of a redundant 2-cell β.

iv) elimination π₁⁺(γ, β) : Σ₂⁺ → Σ₂⁺/(γ, β) of a redundant 2-cell β.

If Σ and Y are 2-polygraphs, a Tietze transformation from Σ to Y is a 2-functor F : Σ⁺ → Y⁺ that decomposes into sequence of elementary Tietze transformations. Two 2-polygraphs are Tietze equivalent if, and only if, there exists a Tietze transformation between them [6 Theorem 2.1.3.].

Given a 2-polygraph Σ and a 2-cell γ₁ * γ * γ₂ in Σ₂⁺, the Nielsen transformation κ₁γ₁−β is the Tietze transformation that replaces in the (2, 1)-category Σ₂⁺ the 2-cell γ by a 2-cell β : s₁(γ₁) ⇒ t₁(γ₂). When γ₂ is identity, we will denote by κ₁γ₁ the Nielsen transformation which, given a 2-cell γ₁ * γ in Σ₂⁺, replaces the 2-cell γ by a 2-cell β : s₁(γ₁) ⇒ t₁(γ).

2.1.3. Convergence. A rewriting step of a 2-polygraph Σ is a 2-cell of Σ₂⁺ with shape wβw’, where β is a 2-cell of Σ₂ and w and w’ are 1-cells of Σ₁⁺. A rewriting sequence of Σ is a finite or infinite sequence of rewriting steps. A 1-cell u of Σ₁⁺ is a normal form if there is no rewriting step with source u. The 2-polygraph Σ terminates if it has no infinite rewriting sequence.

A branching of the 2-polygraph Σ is a non ordered pair (f, g) of 2-cells of Σ₂⁺ such that s₁(f) = s₁(g). A branching (f, g) is local if f and g are rewriting steps. A branching is aspherical if it is of the form (f, f), for a rewriting step f and Peiffer when it is of the form (fv, ug) for rewriting steps f and g with s₁(f) = u and s₁(g) = v. The overlapping branchings are the remaining local branchings. An overlapping local branching is critical if it is minimal for the order ⊆ generated by the relations (f, g) ⊆ (wfw’, wgw’),
2.2. Plactic monoids

2.2.1. Rows, columns and tableaux. For \( n > 0 \), we denote by \([n]\) the set \( \{1, 2, \ldots, n\} \) totally ordered by \( 1 < 2 < \ldots < n \). A row is a non-decreasing 1-cell \( x_1 \ldots x_k \) in the free monoid \([n]^*\), i.e., with \( x_1 \leq x_2 \leq \ldots \leq x_k \). A column is a decreasing 1-cell \( x_p \ldots x_1 \) in \([n]^*\), i.e., with \( x_p > \ldots > x_2 > x_1 \).

We will denote by \( \text{col}(n) \) the set of non-empty columns in \([n]^*\). We denote by \( \ell(w) \) (resp. \( \text{end}(w) \)) the length of a 1-cell \( w \) (resp. the length of the longest non-decreasing subsequence in \( w \)). A row \( x_1 \ldots x_k \) dominates a row \( y_1 \ldots y_l \), and we denote \( x_1 \ldots x_k \triangleright y_1 \ldots y_l \), if \( k \leq l \) and \( x_k > y_l \), for \( 1 \leq i \leq k \). Any 1-cell \( w \) in \([n]^*\) has a unique decomposition as a product of rows of maximal length \( u_1 \ldots u_k \). Such a 1-cell \( w \) is a tableau if \( u_1 \triangleright u_2 \triangleright \ldots \triangleright u_k \). We will write tableaux in a planar form, with the rows placed in order of domination from bottom to top and left-justified as in 5. The degree lexicographic order is the total order on \( \text{col}(n) \), denoted by \( \triangleright_{\text{deglex}} \), and defined by \( u \triangleright_{\text{deglex}} v \) if \( \ell(u) < \ell(v) \) or \( \ell(u) = \ell(v) \) and \( u \triangleright_{\text{lex}} v \), for all \( u \) and \( v \) in \( \text{col}(n) \), where \( \triangleright_{\text{lex}} \) denotes the lexicographic order on \([n]^*\).

2.2.2. Schensted’s algorithm. The Schensted algorithm computes for each 1-cell \( w \) in \([n]^*\) a tableau denoted by \( \text{P}(w) \), called the Schensted tableau of \( w \) and constructed as follows, 21. Given \( u \) a tableau written as a product of rows of maximal length \( u = u_1 \ldots u_k \) and \( y \) in \([n]\), it computes the tableau \( \text{P}(uy) \) as follows. If \( uy \) is a row, the result is \( u_1 \ldots u_k y \). If \( uy \) is not a row, then suppose \( u_k = x_1 \ldots x_1 \) with \( x_i \) in \([n]\) and let \( j \) minimal such that \( x_j > y \), then the result is \( \text{P}(u_1 \ldots u_{k-1} x_j)v_k \), where \( v_k = x_1 \ldots x_{j-1} y x_{j+1} \ldots x_l \). The tableau \( \text{P}(w) \) is computed from the empty tableau and iteratively applying the Schensted algorithm. In this way, \( \text{P}(w) \) is the row reading of the planar representation of the tableau computed by the Schensted algorithm. The number of columns in \( \text{P}(w) \) is equal to \( \text{end}(w) \), 21. We will denote by \( C(w) \) the column reading of the tableau \( \text{P}(w) \), obtained by reading \( \text{P}(w) \) column-wise from bottom to top and from left to right. We denote by \( C_r(w) \) (resp. \( C_l(w) \)) the reading of the last right (resp. first left) column of the tableau \( \text{P}(w) \).

2.2.3. Knuth’s 2-polygraph and the plactic congruence. The plactic monoid of rank \( n \), denoted by \( \text{P}_n \), is the quotient of the free monoid \([n]^*\) by the congruence \( \sim_{\text{plax}(n)} \), defined by \( u \sim_{\text{plax}(n)} v \) if \( \text{P}(u) = \text{P}(v) \). The Knuth 2-polygraph of rank \( n \) is the 2-polygraph, denoted by \( \text{Knuth}_2(n) \), whose set of 1-cells is \([n]\) and the set of 2-cells is

\[
\{ \text{xy} \xrightarrow{\eta_{xy,1,2}} \text{yx} | 1 \leq x < y < z \leq n \} \cup \{ \text{yx} \xrightarrow{\epsilon_{xy,1,2}} \text{yz} | 1 \leq x < y < z \leq n \}. \tag{1}
\]

The congruence on the free monoid \([n]^*\) generated by the 2-polygraph \( \text{Knuth}_2(n) \) is called the plactic congruence of rank \( n \) and the 2-polygraph \( \text{Knuth}_2(n) \) is a presentation of the monoid \( \text{P}_n \), [12] Theorem 6. Each plactic congruence class contains exactly one tableau, [20] Proposition 5.2.3, and for any 1-cell \( w \), we have that \( w = C(w) \) holds in \( \text{P}_n \), [20] Problem 5.2.4.
2. Presentation of plactic monoids by rewriting

2.3. Column presentation

We recall some presentations of the plactic monoid $P_n$ obtained by adding new generators. In particular, we recall the column presentation of the monoid $P_n$ introduced in [4] which is finite and convergent.

2.3.1. Columns as generators. Let us denote by $Col_1(n) = \{c_u \mid u \in \text{col}(n)\}$ the set of column generators of the monoid $P_n$, and by

$$C_2(n) = \{c_{x_p} \ldots c_{x_1} \xrightarrow{\gamma_{x_{p}}_{x_{1}}} c_u \mid u = x_p \ldots x_1 \in \text{col}(n) \text{ with } \ell(u) \geq 2\}$$

the set of the defining relations for the column generators. We denote by Knuth$^c_2(n)$ the 2-polygraph whose set of 1-cells is $\{c_1, \ldots, c_n\}$ and whose set of 2-cells is given by

$$\{c_c c_x c_y \xrightarrow{\eta_{c,x,y}^{c}} c_c c_x c_y \mid 1 \leq x \leq y < z \leq n\} \cup \{c_y c_c c_x \xrightarrow{\varepsilon_{c,x,y}^{c}} c_y c_x c_z \mid 1 \leq x < y \leq z \leq n\}.$$

By definition, this 2-polygraph is Tietze equivalent to the 2-polygraph Knuth$^c(n)$. In the sequel, we will identify the 2-polygraphs Knuth$^c_2(n)$ and Knuth$^c_2(n)$.

Let us define the 2-polygraph Knuth$^{c*}_2(n)$, whose set of 2-cells is $C_2(n) \cup \text{Knuth}^c_2(n)$. The 2-polygraph Knuth$^{c*}_2(n)$ is a presentation of the monoid $P_n$. Indeed, we add to the 2-polygraph Knuth$^c_2(n)$ all the column generators $c_u$, for all $u = x_p \ldots x_1$ in col$(n)$ such that $\ell(u) \geq 2$, and the corresponding collapsible 2-cell $\gamma_u : c_{x_p} \ldots c_{x_1} \Rightarrow c_u$.

2.3.2. Pre-column presentation. Let us define the 2-polygraph PreCol$_2(n)$ whose set of 1-cells is Col$_1(n)$ and the set of 2-cells is

$$\text{PreCol}_2(n) = \text{PC}_2(n) \cup \{c_c c_u \xrightarrow{\alpha_{c,x,u}^{c'}} c_c x u \mid x u \in \text{col}(n) \text{ and } 1 \leq x \leq n\},$$

where

$$\text{PC}_2(n) = \{c_c c_{y_z} \xrightarrow{\alpha_{c,x,y}^{c'}} c_{x z} c_y \mid 1 \leq x \leq y < z \leq n\} \cup \{c_y c_c c_x \xrightarrow{\varepsilon_{c,x,y}^{c}} c_y c_x c_z \mid 1 \leq x < y \leq z \leq n\}.$$

2.3.3. Proposition. For $n > 0$, the 2-polygraph PreCol$_2(n)$ is a presentation of the monoid $P_n$, called the pre-column presentation of $P_n$.

Proof. We proceed in two steps. The first step consists to prove that the 2-polygraph $\text{CPC}_2(n) := \langle \text{Col}_1(n) \mid C_2(n) \cup \text{PC}_2(n) \rangle$ is Tietze equivalent to the 2-polygraph Knuth$^{c*}_2(n)$. For $1 \leq x \leq y < z \leq n$, consider the following critical branching

$$\xymatrix{ & c_c c_x c_y \ar[r] & c_c c_x c_y \ar[r] & c_c c_x c_y \\
\gamma_{c,x,y}^{c} \ar[r] & \eta_{c,x,y}^{c} & c_c c_x c_y \ar[r] & c_c c_x c_y \ar[r] & c_c c_x c_y}
$$

6
of the 2-polygraph Knuth$^c_2(n)$. Let consider the Tietze transformation

$$\kappa^c_{i,y,z} : \text{Knuth}^c_2(n) \to \text{Knuth}^c_2(n) / (\eta^c_{i,y,z} \leftarrow \alpha^c_{i,y,z}),$$

that substitutes the 2-cell $\alpha^c_{i,y,z} : c_x c_{yz} \Rightarrow c_{xz} c_{y}$ to the 2-cell $\eta^c_{i,y,z}$, for every $1 \leq i \leq y < z \leq n$. We denote by $T_{c_i \leftarrow c_j}$ the successive applications of the Tietze transformation $\kappa^c_{i,y,z} \leftarrow \alpha^c_{i,y,z}$, for every $1 \leq i \leq y < z \leq n$, with respect to the lexicographic order on the triples $(x,y,z)$ induced by the total order on $[n]$.

Similarly, we study in the same way the critical branching $(e^c_{x,y,z}, c_y c_{zx})$ of the 2-polygraph Knuth$^c_2(n)$, for every $1 \leq x < y < z \leq n$, by introducing the Tietze transformation $\kappa^c_{x,y,z} : \text{Knuth}^c_2(n) \to \text{Knuth}^c_2(n) / (e^c_{x,y,z} \leftarrow \alpha^c_{x,y,z})$. We denote by $T_{e_i \leftarrow c_i}$ the successive applications of this Tietze transformation with respect to the lexicographic order on the triples $(x,y,z)$ induced by the total order on $[n]$. In this way, we obtain a Tietze transformation $T_{\eta_i \leftarrow c_i}$ from Knuth$^c_2(n)$ to $CPC_2(n)$ given by the composite $T_{\eta_i \leftarrow c_i} \circ T_{e_i \leftarrow c_i}$.

In a second step, we prove that the 2-polygraph PreCol$^2_2(n)$ is Tietze equivalent to the 2-polygraph $CPC_2(n)$. Let $x_p \ldots x_1$ be a column with $\ell(x_p \ldots x_1) > 2$ and define $\alpha^i_{y,i} := \gamma_{y,i} : c_y c_x \Rightarrow c_{yx}$, for every $x < y$. Consider the following critical branching

$$c_{xp} \gamma_{xp-1 \ldots x_1} c_{xp} c_{xp-1 \ldots x_1},$$

of the 2-polygraph $CPC_2(n)$ and the following Tietze transformation

$$\kappa^i_{y,i} : CPC_2(n) \to CPC_2(n) / (\gamma_{xp-1 \ldots x_1} \leftarrow \alpha^i_{y,i}),$$

that substitutes the 2-cell $\alpha^i_{y,i} : c_{xp} c_{xp-1 \ldots x_1} \Rightarrow c_{xp} c_{xp-1 \ldots x_1}$ to the 2-cell $\gamma_{xp-1 \ldots x_1}$, for each column $x_p \ldots x_1$ such that $p > 2$. Starting from the 2-polygraph $CPC_2(n)$, we apply successively the Tietze transformation $\kappa^i_{y,i} : CPC_2(n) \to CPC_2(n) / (\gamma_{yp-1 \ldots x_1} \leftarrow \alpha^i_{y,i})$, for every column $x_p \ldots x_1$ such that $\ell(x_p \ldots x_1) > 2$, from the bigger to the smaller one with respect to the total order $\leq_{\text{deglex}}$. The composite

$$T_{y \leftarrow \alpha^i} = \kappa^i_{y_{p+1}x_{p+1} \ldots x_{n-1}} \circ \cdots \circ \kappa^i_{y_{n-1}x_{n-1} \ldots x_1} \leftarrow \alpha^i_{y_{p+1}x_{p+1} \ldots x_{n-1}},$$

gives us a Tietze transformation from $CPC_2(n)$ to $\text{PreCol}_2(n)$.

2.3.4. Column presentation. Let $n > 0$. Given columns $u = x_p \ldots x_1$ and $v = y_q \ldots y_1$ in $\text{col}[n]$, the length $\ell^{\text{nd}}_v(uv)$ of the longest non-decreasing subsequence of $uv$ is lower or equal to 2 [4 Lemma 3.1].

We will use graphical notations depending on whether the tableau $P(uv)$ consists in two columns:

- **i)** we will denote $\hat{u} \hat{v}$ if the planar representation of $P(uv)$ is a tableau, that is, $p \geq q$ and $x_i \leq y_i$, for any $i \leq q$.

- **ii)** we will denote $u \bowtie v$ in all the other cases, that is, when $p < q$ or $x_i > y_i$, for some $i \leq q$. 


In the case ii), we will denote \( u^x \) if the tableau \( P(uv) \) has one column and we will denote \( u^{x^2} \) if the tableau \( P(uv) \) has two columns. For every columns \( u \) and \( v \) in \( \text{col}(n) \) such that \( u^x \), we define a 2-cell

\[ \alpha_{u,v} : c_u c_v \Rightarrow c_w c_{w'} \]

where

i) \( w = uv \) and \( c_{w'} = 1 \), if \( u^1 \),

ii) \( w \) and \( w' \) are respectively the left and right columns of the tableau \( P(uv) \), if \( u^{x^2} \).

Let us denote by \( \text{Col}_2(n) \) the 2-polygraph whose set of 1-cells is \( \text{Col}_1(n) \) and the set of 2-cells is

\[ \text{Col}_2(n) = \{ c_u c_v \Rightarrow c_w c_{w'} \mid u, v \in \text{col}(n) \text{ and } u^x \}. \quad (2) \]

The 2-polygraph \( \text{Col}_2(n) \) is a finite convergent presentation of the monoid \( P_n \) [4, Theorem 3.4], called the column presentation of the monoid \( P_n \). Note that Schensted’s algorithm that computes a tableau \( P(w) \) from a 1-cell \( w \), corresponds to the leftmost reduction path in \( \text{Col}_2^*(n) \) from \( w \) to its normal form \( P(w) \), that is, the reduction paths obtained by applying the rules of \( \text{Col}_2(n) \) starting from the left. In particular, we have

2.3.5. Lemma. For any \( u_1, \ldots, u_n \) in \( \text{col}(n) \), the length of the leftmost rewriting path in \( \text{Col}_2(n)^* \) from \( u_1 u_2 \ldots u_n \) to its normal form \( P(u_1 u_2 \ldots u_n) \) is at most \( n \).

3. COHERENT COLUMN PRESENTATION

In this section, we begin by recalling the notion of coherent presentations of monoids from [6]. In a second part, using the homotopical completion procedure, we construct a coherent presentation of the monoid \( P_n \) starting from its column presentation.

3.1. Coherent presentations of monoids

3.1.1. \((3,1)\)-polygraph. A \((3,1)\)-polygraph is a pair \((\Sigma_2, \Sigma_3)\) made of a 2-polygraph \( \Sigma_2 \) and a globular extension \( \Sigma_3 \) of the \((2,1)\)-category \( \Sigma_2^\top \), that is a set of 3-cells \( A : f \Rightarrow g \) relating 2-cells \( f \) and \( g \) in \( \Sigma_2^\top \), respectively denoted by \( s_2(A) \) and \( t_2(A) \) and satisfying the globular relations \( s_1 s_2(A) = s_1 t_2(A) \) and \( t_1 s_2(A) = t_1 t_2(A) \). Such a 3-cell can be represented with the following globular shape:

\[ \begin{array}{c}
\bullet \\
\circ \overset{f}{\underset{\circ}{\phantom{f}}} \bullet \\
\circ \overset{\circ}{\underset{g}{\phantom{g}}} \\
\bullet
\end{array} \quad \text{or} \quad \begin{array}{c}
\bullet \\
\circ \overset{f}{\underset{\circ}{\phantom{f}}} \bullet \\
\circ \overset{\circ}{\underset{A}{\phantom{g}}} \\
\bullet
\end{array} \]

We will denote by \( \Sigma_3^\top \) the free \((3,1)\)-category generated by the \((3,1)\)-polygraph \((\Sigma_2, \Sigma_3)\). A pair \((f, g)\) of 2-cells of \( \Sigma_2^\top \) such that \( s_1(f) = s_1(g) \) and \( t_1(f) = t_1(g) \) is called a 2-sphere of \( \Sigma_2^\top \).
3.1. Coherent presentations of monoids

3.1.2. Coherent presentations of monoids. An extended presentation of a monoid $M$ is a $(3,1)$-polygraph whose underlying 2-polygraph is a presentation of the monoid $M$. A coherent presentation of $M$ is an extended presentation $\Sigma$ of $M$ such that the cellular extension $\Sigma_3$ is a homotopy basis of the $(2,1)$-category $\Sigma^\top_2$, that is, for every 2-sphere $\gamma$ of $\Sigma^\top_2$, there exists a 3-cell in $\Sigma^\top_3$ with boundary $\gamma$.

3.1.3. Tietze transformations of $(3,1)$-polygraphs. We recall the notion of Tietze transformation from [6] Section 2.1]. Let $\Sigma$ be a $(3,1)$-polygraph. A 3-cell $A$ of $\Sigma$ is called collapsible if $t_2(A)$ is in $\Sigma_2$ and $s_2(A)$ is a 2-cell of the free $(2,1)$-category over $(\Sigma_2 \setminus \{t_2(A)\})^\top$, then $t_2(A)$ is called redundant. An elementary Tietze transformation of a $(3,1)$-polygraph $\Sigma$ is a 3-functor with domain $\Sigma^\top_3$ that belongs to one of the following operations:

i) adjunction $t^I_\alpha$ and elimination $\pi_\alpha$ of a 2-cell $\alpha$ as described in 2.1.2

ii) coherent adjunction $t^A_\alpha : \Sigma^\top_3 \to \Sigma^\top_3(\alpha)(A)$ of a redundant 2-cell $\alpha$ with its collapsible 3-cell $A$.

iii) coherent elimination $\pi_\alpha : \Sigma^\top_3 \to \Sigma^\top_3/A$ of a redundant 2-cell $\alpha$ with its collapsible 3-cell $A$.

iv) coherent adjunction $t_\alpha : \Sigma^\top_3 \to \Sigma^\top_3(A)$ of a redundant 3-cell $A$.

v) coherent elimination $\pi_{(B,A)} : \Sigma^\top_3 \to \Sigma^\top_3/(B,A)$ of a redundant 3-cell $A$, that maps $A$ to $B$.

For $(3,1)$-polygraphs $\Sigma$ and $\Upsilon$, a Tietze transformation from $\Sigma$ to $\Upsilon$ is a 3-functor $F : \Sigma^\top_3 \to \Upsilon^\top_3$ that decomposes into a sequence of elementary Tietze transformations. Two $(3,1)$-polygraphs $\Sigma$ and $\Upsilon$ are Tietze-equivalent if there exists an equivalence of 2-categories $F : \Sigma^\top_2/\Sigma_3 \to \Upsilon^\top_2/\Upsilon_3$ and the presented monoids $\Sigma_2$ and $\Upsilon_2$ are isomorphic. Two $(3,1)$-polygraphs are Tietze equivalent if, and only if, there exists a Tietze transformation between them, [6] Theorem 2.1.3.]

3.1.4. Homotopical completion procedure. Following [6] Section 2.2], we recall the homotopical completion procedure that produces a coherent convergent presentation from a terminating presentation. Given a terminating 2-polygraph $\Sigma$, equipped with a total termination order $\preceq$, the homotopical completion of $\Sigma$ is the $(3,1)$-polygraph obtained from $\Sigma$ by successive applications of the Knuth-Bendix completion procedure, [11], and the Squier construction, [24]. Explicitly, for any critical branching $(f, g)$ of $\Sigma$, if $(f, g)$ is confluent one adds a dotted 3-cell $A$:

![Diagram](f)
where the 2-cell $\beta$ is directed from a normal form $\hat{v}$ of $v$ to a normal form $\hat{w}$ of $w$ if $\hat{w} \prec \hat{v}$ and from $\hat{w}$ to $\hat{v}$ otherwise. The adjunction of 2-cells can create new critical branchings, possibly generating the adjunction of additional 2-cells and 3-cells in the same way. This defines an increasing sequence of $(3, 1)$-polygraphs, whose union is called a homotopical completion of $\Sigma$. Following [24, Theorem 5.2], such a homotopical completion of $\Sigma$ is a coherent convergent presentation of the monoid $\Sigma$.

### 3.2. Column coherent presentation

Using the homotopical completion procedure, we extend the 2-polygraph $\text{Col}_2(n)$ into a coherent presentation of the monoid $P_n$.

#### 3.2.1. Column coherent presentation

The presentation $\text{Col}_2(n)$ has exactly one critical branching of the form

$$
\begin{align*}
\alpha_{u,v}c_t & \rightarrow c_e e'_t c_t \\
\alpha_{u,v} & \rightarrow c_e e'_t c_t \\
\alpha_{u,v} & \rightarrow c_e c_b c_{b'} \\
\alpha_{u,v} & \rightarrow c_e c_b c_{b'} \\
\alpha_{u,v} & \rightarrow c_e c_b c_{b'} \\
\alpha_{u,v} & \rightarrow c_e c_b c_{b'} \\
\alpha_{u,v} & \rightarrow c_e c_b c_{b'} \\
\alpha_{u,v} & \rightarrow c_e c_b c_{b'} \\
\end{align*}
$$

(3)

for any $u$, $v$, $t$ in $\text{col}(n)$ such that $u \times v \times t$, where $e$ and $e'$ (resp. $w$ and $w'$) denote the two columns of the tableau $P(uw)$ (resp. $P(ut)$). We prove in this section that all of these critical branchings are confluent and that all the confluence diagrams of these branchings are of the following form:

$$
\begin{align*}
\alpha_{u,v}c_t & \rightarrow c_e e'_t c_t \\
\alpha_{u,v} & \rightarrow c_e e'_t c_t \\
\alpha_{u,v} & \rightarrow c_e c_b c_{b'} \\
\alpha_{u,v} & \rightarrow c_e c_b c_{b'} \\
\alpha_{u,v} & \rightarrow c_e c_b c_{b'} \\
\alpha_{u,v} & \rightarrow c_e c_b c_{b'} \\
\alpha_{u,v} & \rightarrow c_e c_b c_{b'} \\
\alpha_{u,v} & \rightarrow c_e c_b c_{b'} \\
\end{align*}
$$

(4)

where $\alpha$ and $\alpha'$ (resp. $b$ and $b'$) denote the two columns of the tableau $P(uw)$ (resp. $P(e't)$) and $\alpha$, $d$, $b'$ are the three columns of the tableau $P(uvt)$, which is a normal form for the 2-polygraph $\text{Col}_2(n)$. Note that in some cases described below, one or further columns $e'$, $w'$, $\alpha'$ and $b'$ can be empty. In those cases some indicated 2-cells $\alpha$ in the confluence diagram correspond to identities.

Let us denote by $\text{Col}_3(n)$ the extended presentation of the monoid $P_n$ obtained from $\text{Col}_2(n)$ by adjunction of one 3-cell $X_{u,v,t}$ of the form (4), for every columns $u$, $v$ and $t$ such that $u \times v \times t$.

#### 3.2.2. Theorem

**For** $n > 0$, the $(3, 1)$-polygraph $\text{Col}_3(n)$ is a coherent presentation of the monoid $P_n$.

The extended presentation $\text{Col}_3(n)$ is called the column coherent presentation of the monoid $P_n$. The rest of this section consists in a constructive proof of Theorem 3.2.2 that makes explicit all possible forms of 3-cells. Another arguments are given in Remark 3.2.7. Our proof is based on the following arguments. The presentation $\text{Col}_2(n)$ is convergent, thus using the homotopical completion procedure described in [3.1.4], it suffices to prove that the 3-cells $X_{u,v,t}$ with $u \times v \times t$ form a family of generating confluences for the presentation $\text{Col}_2(n)$. There are four possibilities for the critical branching (3) depending on the following four cases:

- $u \times 1 \times 1 t$
- $u \times 2 \times 1 t$
- $u \times 1 \times 2 t$
- $u \times 2 \times 2 t$

Each of these cases is examined in the following four lemmas, where $u = x_p \ldots x_1$, $v = y_q \ldots y_1$ and $t = z_l \ldots z_1$ denote columns of length $p$, $q$ and $l$ respectively.
3.2. Column coherent presentation

3.2.3. Lemma. If \( u^1v^1t \), we have the following confluent critical branching:

\[
\begin{array}{c}
\alpha_{u,v}c_t \rightarrow c_u c_v c_t \\
\alpha_{u,v}c_t \rightarrow c_u c_v c_t
\end{array}
\]

Proof. By hypothesis \( uv \) and \( vt \) are columns, then \( uvt \) is a column. Thus \( u^2v^1t \) and \( u^1v^1t \) and there exist 2-cells \( \alpha_{u,v,t} \) and \( \alpha_{u,v,t} \) in \( \text{Col}_2(\pi) \) making the critical branching (3) confluent, where \( e = uv, w = vt \) and \( e', w' \) are the empty column. \( \square \)

3.2.4. Lemma. If \( u^2v^1t \), we have the following confluent critical branching:

\[
\begin{array}{c}
\alpha_{u,v}c_k \rightarrow c_u c_v c_t \\
\alpha_{u,v}c_k \rightarrow c_u c_v c_t
\end{array}
\]

where \( e \) and \( e' \) (resp. \( s \) and \( s' \)) denote the two columns of the tableau \( P(uv) \) (resp. \( P(uvt) \)).

Proof. By hypothesis, \( vt \) is a column and \( y_1 > z_1 \). The tableau \( P(uv) \) consists of two columns, that we will denote \( e \) and \( e' \), then \( l^{\text{nds}}(uv) = 2 \) and \( x_1 < y_q \). We have \( u^2v^1 \), so that we distinguish the following possible three cases.

Case 1: \( p > q \) and \( x_i > y_i \) for some \( 1 < i < q \). Suppose that \( i_0 = 1 \), that is, \( x_1 > y_1 \). We consider \( y_j \), the biggest element of the column \( v \) such that \( x_j > y_j \), then the smallest element of the column \( e' \) is \( y_j+1 \). By hypothesis, the word \( vt \) is a column, in particular \( y_{j+1} > z_1 \). It follows that \( e' \) is a column. Suppose that \( i_0 > 1 \), then \( x_1 < y_1 \) and the smallest element of \( e' \) is \( y_1 \). Since \( y_1 > z_1 \) by hypothesis, the word \( e' \) is a column. Hence, in all cases, \( e' \) is a column and there is a 2-cell \( \alpha_{e',t} : c_e c_t \Rightarrow c_{e'} t \).

Case 2: \( p < q \) and \( x_i < y_i \) for any \( 1 < i < q \). We have \( e = y_q \ldots y_{p+1} x_p \ldots x_1 \) and \( e' = y_p \ldots y_1 \). By hypothesis, \( y_1 > z_1 \), hence \( e' \) is a column and there is a 2-cell \( \alpha_{e',t} : c_e c_t \Rightarrow c_{e'} t \).

Case 3: \( p < q \) and \( x_i > y_i \) for some \( 1 < i < q \). With the same arguments of Case 1, the smallest element of \( e' \) is \( y_1 \) or \( y_{j+1} \), where \( y_j \) is the biggest element of the column \( v \) such that \( y_j < x_1 \). Hence, \( e' \) is a column and there is a 2-cell \( \alpha_{e',t} : c_e c_t \Rightarrow c_{e'} t \).

In each case, we have \( l^{\text{nds}}(uv) = 2 \), hence \( l^{\text{nds}}(uvt) = 2 \). Thus the tableau \( P(uvt) \) consists of two columns, that we denote \( s \) and \( s' \) and there is a 2-cell \( \alpha_{u,v,t} : c_u c_v \Rightarrow c_{s} c_{s'} \). Moreover, to compute the tableau \( P(uvt) \), one begins by computing \( P(uv) \) and after by introducing the elements of the column \( t \) on the tableau \( P(uvt) \). As \( C(uv) = ee' \), we have \( P(uvt) = P(uvt) \Rightarrow P(ee') \). Hence \( C(ee') = ss' \) and there is a 2-cell \( \alpha_{e,e',t} \) which yields the confluence diagram (5). \( \square \)

3.2.5. Lemma. If \( u^1v^2t \), we have the following confluent critical branching:

\[
\begin{array}{c}
\alpha_{u,v}c_t \rightarrow c_u c_v c_t \\
\alpha_{u,v}c_t \rightarrow c_u c_v c_t
\end{array}
\]
3. Coherent column presentation

where \( w \) and \( w' \) (resp. \( a \) and \( a' \)) denote the two columns of the tableau \( P(vt) \) (resp. \( P(uvw) \)).

Proof. By hypothesis, \( uv \) is a column hence \( x_1 > y_q \). Moreover, the tableau \( P(vt) \) consists of two columns \( w \) and \( w' \), then \( \ell^{\text{ends}}(vt) = 2 \), hence \( y_1 \leq z_1 \). We have \( \nu^2 \), so that we distinguish the three possible following cases.

Case 1: \( q \geq 1 \) and \( y_{i_0} > z_{i_0} \) for some \( 1 \leq i_0 \leq l \). Let us denote \( w = w_1 \ldots w_1 \) and \( w' = w'_1 \ldots w'_1 \). Since \( q \geq 1 \), we have \( w_r = y_q \). By hypothesis, \( x_1 > y_q \). Then the word \( uv \) is a column. As a consequence, there is a \( 2 \)-cell \( \alpha_{u,w} : c_{u,c_w} \Rightarrow c_{u,w} \). In addition, the column \( w \) appears to the left of \( w' \) in the planar representation of the tableau \( P(vt) \), that is, \( \ell(w) \leq \ell(w') \) and \( w_i \leq w'_i \) for any \( i \). Then \( \ell(w) \leq \ell(w') \). We set \( uv = \xi_{i(uvw)} \ldots \xi_1 \) and we have \( \xi_i \leq w'_i \) for any \( i \). Then \( uvw'w' \) and \( c_{uvw}w' \) is a normal form.

On the other hand, the tableau \( P(vt) \) consists of two columns, hence \( \ell^{\text{ends}}(vt) = 2 \). As a consequence, \( \ell^{\text{ends}}(uvt) = 2 \) and the tableau \( P(uvt) \) consists of two columns. Since \( q \geq 1 \), we have \( C(uvt) = uvw'w' \) and \( w' \). Then there is a \( 2 \)-cell \( \alpha_{uvt} : c_{uvt} \Rightarrow c_{uvw}c_{w'} \) which yields the confluence of the critical branching on \( c_{u,c_v,c_t} \), as follows

\[
\begin{diagram}
\node{\alpha_{u,w}c_{u,c_w}} \node{c_{uv}c_t} \node{c_{uvw}c_{w'}} \node{c_{uvt}c_{w'}} \\
\node{c_{u,v,t}c_{c_v,c_t}} \node{c_{u,v,t}c_{u,w}c_{w'}} \node{c_{u,v,t}c_{u,v,t}c_{w'}} \node{c_{u,v,t}c_{u,v,t}c_{w'}} \\
\end{diagram}
\]

Case 2: \( q < 1 \) and \( y_l \leq z_1 \) for any \( i \leq q \). We have \( w = z_1 \ldots z_q+1 \ldots y_1 \ldots y_1 \) and \( w' = z_q \ldots z_1 \). There are two cases along \( uv \) is a column or not.

Case 2 A. If \( x_1 > z_1 \), then \( uv \) is a column. Hence, there is a \( 2 \)-cell \( \alpha_{u,w} : c_{u,c_w} \Rightarrow c_{u,w} \). Moreover, using Schensted’s algorithm we prove that \( C(uvt) = uvw' \) and \( C(uvt) = w' \). Thus there is a \( 2 \)-cell \( \alpha_{u,v,t} : c_{u,w} \Rightarrow c_{uvw}c_{w'} \) which yields the confluence diagram 7.

Case 2 B. If \( x_1 \leq z_1 \), then \( \ell^{\text{ends}}(uv) = 2 \) and \( P(uw) \) consists of two columns, that we denote by \( a \) and \( a' \). Then there is a \( 2 \)-cell \( \alpha_{u,a} : c_{u,c_a} \Rightarrow c_{a,c_a} \). In addition, by Schensted’s algorithm, we deduce that \( a' = z_1 \ldots z_1 \), with \( q + 1 \leq t_1 < \ldots < t_k \leq 1 \). We have \( a'w' = z_1 \ldots z_1 \). Since all the elements of \( a' \) are elements of \( t \) and bigger than \( q \), we have \( z_i > z_q \). It follows that \( a'w' \) is a column and there is a \( 2 \)-cell \( \alpha_{a'} : c_{a,c_a} \Rightarrow c_{a',w'} \) of the

In the other hand, we have two cases whether \( uv^2 \) or \( uv \bar{v} t \). Suppose \( uv^2 \). By Schensted’s algorithm, we have \( C(1) = a \) and \( \alpha(1) = a'w' \). Hence there is a \( 2 \)-cell \( \alpha_{u,a} : c_{u,v} \Rightarrow c_{a,c_{a',w'}} \), which yields the confluence of Diagram 6. Suppose \( uv \bar{v} t \). Then we obtain \( C(uw) = uw \ldots z_q+1 \), and \( C(z_1 \ldots z_q+1,w') = t \). Hence there is a \( 2 \)-cell \( \alpha_{c_1} : c_{c_1} \Rightarrow c_{uvw}c_{w'} \) yielding the confluence diagram

\[
\begin{diagram}
\node{\alpha_{u,v,c_t}} \node{c_{uv}c_t} \node{c_{uvw}c_{z_1 \ldots z_q+1,w'}} \node{c_{u,v,t}c_{w'}} \\
\node{c_{u,v,t}c_{w'}} \node{c_{u,v,t}c_{w'}} \node{c_{u,v,t}c_{w'}} \node{c_{u,v,t}c_{w'}} \\
\end{diagram}
\]

Case 3: \( q < 1 \) and \( y_{i_0} > z_{i_0} \) for some \( 1 \leq i_0 \leq q \). We compute the columns \( w \) and \( w' \) of the tableau \( P(vt) \). If the biggest element of the column \( w \) is \( y_q \), then we obtain the same confluent branching.
as in Case 1. If the first element of \( w \) is \( z_1 \), then one obtains the same confluent critical branchings as in Case 2.

### 3.2.6. Lemma

If \( u^2 \times v^2 \times t \), we have the following confluent critical branching:

\[
\begin{align*}
\alpha_{u,v}c_t & \xrightarrow{\alpha_{u,v}c_t} c_ec_t' & \xrightarrow{\alpha_{c}b,c_{b}'} c_ec_{b}c_{b}' \\
c_uc_vc_t & \xrightarrow{\alpha_{u,v}c_t} c_{u}c_{w}c_{w}' & \xrightarrow{\alpha_{u,v}c_t} c_{u}c_{w}c_{w}' \\
\end{align*}
\]

where \( e, e' \) (resp. \( w, w' \)) denote the two columns of the tableau \( P(uv) \) (resp. \( P(vt) \)) and \( a, a' \) (resp. \( b, b' \)) denote the two columns of the tableau \( P(uw) \) (resp. \( P(e't) \)).

**Proof.** By hypothesis, \( e_{uds}(uv) = 2 \) and \( e_{uds}(vt) = 2 \), hence \( x_1 \leq y_q \) and \( y_1 \leq z_l \). In addition, since \( u^2 \times v \), the tableau \( P(uv) \) consists of two columns, that we denote by \( \alpha \) and \( \alpha' \). Thus there is a 2-cell \( \alpha_{u,v} : c_{u}c_{v} \Rightarrow c_{a}c_{a'} \). Moreover, as \( u^2 \times v \) and \( v^2 \times t \), we have

\[
((p < q) \text{ or } (x_{i_0} > y_{i_0} \text{ for some } i_0 \leq q)) \quad \text{and} \quad ((q < l) \text{ or } (y_{j_0} > z_{j_0} \text{ for some } j_0 \leq l)),
\]

thus we consider the following cases.

**Case 1:** \( p < q < l \) and \( y_1 \leq z_1 \), for all \( i \leq q \), and \( x_1 \leq y_1 \), for all \( i \leq p \). We have

\[
w = z_1 \ldots z_{q+1}y_q \ldots y_1, \quad w' = z_q \ldots z_1, \quad e = y_q \ldots y_{p+1}x_p \ldots x_1 \quad \text{and} \quad e' = y_p \ldots y_1.
\]

Since \( z_1 \geq y_1 \), the tableau \( P(e't) \) consists of two columns, that we denote by \( b \) and \( b' \). Thus there is a 2-cell \( \alpha_{e',t} : c_{e'}c_t \Rightarrow c_{b}c_{b'} \). In addition, we have

\[
b = z_1 \ldots z_{q+1}y_p \ldots y_1, \quad b' = z_p \ldots z_1, \quad a = z_1 \ldots z_{q+1}y_q \ldots y_{p+1}x_p \ldots x_1 \quad \text{and} \quad a' = y_p \ldots y_1.
\]

Since \( z_q \geq y_1 \), the tableau \( P(a'w') \) consists of two columns, that we denote by \( d \) and \( d' \). Thus there is a 2-cell \( \alpha_{a',w'} : c_{a'}c_{w'} \Rightarrow c_{d}c_{d'} \). Since \( z_1 \geq x_1 \), the tableau \( P(eb) \) consists of two columns, that we denote by \( s \) and \( s' \). Then there is a 2-cell \( \alpha_{e,b} : c_{e}c_{b} \Rightarrow c_{s}c_{s'} \). In the other hand, we have

\[
d = z_q \ldots z_{p+1}y_p \ldots y_1, \quad d' = z_p \ldots z_1, \quad s = z_1 \ldots z_{q+1}y_q \ldots y_{p+1}x_p \ldots x_1 \quad \text{and} \quad s' = z_q \ldots z_{p+1}y_p \ldots y_1.
\]

Hence \( a = s, d = s' \) which yields the confluence diagram (8).

**Case 2:** \( \begin{cases} q < l \text{ and } y_1 \leq z_i \text{ for all } i \leq q \\ p \geq q \text{ and } x_{i_0} > y_{i_0} \text{ for some } i_0 \leq q \end{cases} \) or \( \begin{cases} q > l \text{ and } y_1 \leq z_i \text{ for all } i \leq q \\ p < q \text{ and } x_{i_0} > y_{i_0} \text{ for some } i_0 \leq p \end{cases} \)

We have \( w = z_1 \ldots z_{q+1}y_q \ldots y_1 \) and \( w' = z_q \ldots z_1 \). Using Schensted’s algorithm the smallest element of the column \( \alpha' \) is an element of \( v \). Since \( z_q \) is greater or equal than each element of \( v \), the tableau \( P(a'w') \) consists of two columns, that we denote by \( d \) and \( d' \).

On the other hand, all the elements of \( e' \) are elements of \( v \). Since \( z_1 \) is bigger than each element of \( v \), the tableau \( P(e't) \) consists of two columns, that we denote by \( b \) and \( b' \). Thus there is a 2-cell \( \alpha_{e',t} : c_{e'}c_t \Rightarrow c_{b}c_{b'} \). Hence, we consider two cases depending on whether or not \( c_{e}c_{b}c_{b'} \) is a tableau. Suppose \( c_{e}c_{b}c_{b'} \) is a tableau. The column \( e \) does not contain elements from the column \( t \),
then during inserting the column $w$ into the column $u$, we can only insert some elements of $y_q \ldots y_1$ into $u$ and we obtain $a = e$. Since $c_e c_b c_{b'}$ is the unique tableau obtained from $c_{u1} c_e c_t$ and $a = e$, we obtain $C(a'w') = bb'$. As a consequence, there is a 2-cell $\alpha_{a',w'} : c_{a'} c_{w'} \Rightarrow c_b c_{b'}$ yielding the following confluence diagram:

\[
\begin{align*}
\alpha_{u,v,c_t} & \iff c_e c_e' c_t \\
\alpha_{u,v,c_t} & \iff c_e c_{c_b} c_{c_b'}
\end{align*}
\]

Suppose $c_e c_b c_{b'}$ is not a tableau. The first element of the column $b$ is $z_1$. The smallest element of the column $e$ is either $x_1$ or $y_1$, where $y_1$ is the biggest element of the column $v$ such that $y_1 < x_1$. By hypothesis the tableau $P(uw)$ consists of two columns, then $x_1 \leq z_1$. In addition, $z_1$ is greater than each element of $v$ then $y_1 \leq z_1$. Hence, in all cases, the tableau $P(eb)$ consists of two columns. On the other hand, using Schensted’s algorithm, we have $\alpha' = z_{i_k} \ldots z_{i_1} y_{k_1} \ldots y_{j_1}$ with $q + 1 \leq i_1 < \ldots < i_k \leq l$, $1 \leq j_1 < \ldots < j_{k'} \leq q$ and we have $e' = y_{j_1} \ldots y_{j_1}$. In addition, we have $b' = d' = z_{i_{p+1}} \ldots z_{i_1}$ with $l \leq i_1 < \ldots < i_{p'} \leq q$ and $C(eb) = ad$. Hence there is a 2-cell $\alpha_{e,b} : c_e c_b \Rightarrow c_a c_d$ which yields the confluence diagram [3].

**Case 3:** \( \{ q \geq 1 \text{ and } y_{i_0} > z_{i_0} \text{ for some } i_0 \leq l \} \) or \( \{ q < l \text{ and } y_{i_0} > z_{i_0} \text{ for some } i_0 \leq q \} \)

We have $e = y_q \ldots y_{p+1} x_p \ldots x_1$ and $e' = y_p \ldots y_1$. Since $y_1 \leq z_1$, the tableau $P(e't)$ consists of $t$ columns, that we denote by $b$ and $b'$. The first element of the column $b$ is either $z_1$ or $y_p$ which are bigger or equal to $x_1$, then the tableau $P(eb)$ consists of $t$ columns, that we denote by $s$ and $s'$. Suppose $l \leq p$. By Schensted’s insertion algorithm, we have $C(e't) = bw'$ and $w = y_q \ldots y_{p+1} b$. On the other hand, since $x_p < y_{p+1}$, we have $P(uw') = P(u(y_q \ldots y_{p+1} b)) = P(eb)$. Hence, there is a 2-cell $\alpha_{e,b} : c_e c_b \Rightarrow c_a c_{a'}$ which yields the confluence diagram:

\[
\begin{align*}
\alpha_{u,v,c_t} & \iff c_e c_e' c_t \\
\alpha_{u,v,c_t} & \iff c_e c_{c_b} c_{c_b'}
\end{align*}
\]

For $l > p$, we consider two cases depending on whether or not the first element of the column $b$ is $y_p$. If this element is $y_p$, then when computing the tableau $Pvt$ no element of the column $t$ is inserted in $y_q \ldots y_{p+1}$. Hence we have $w = y_q \ldots y_{p+1} b$ and $b' = w'$. On the other hand, by Schensted’s insertion procedure we have $P(uw) = P(eb)$. Hence, there is a 2-cell $\alpha_{e,b} : c_e c_b \Rightarrow c_a c_{a'}$ which yields the confluence diagram [10]. Suppose that the first element of the column $b$ is $z_1$. Then when computing the tableau $Pvt$ some elements of the column $t$ are inserted in $y_q \ldots y_{p+1}$. In this case, we have that the column $w'$ contains more elements than $b'$ and that $c_e c_{c_b} c_{b'}$ is a tableau. Moreover, by Schensted’s insertion procedure, we have $a = s$. Since $c_s c_s' c_{b'}$ is the unique tableau obtained from $c_u c_v c_t$ and $a = s$, we obtain that $C(a'w') = s'b'$. As a consequence, there is a 2-cell $\alpha_{a',w'} : c_{a'} c_{w'} \Rightarrow c_s c_{b'}$ which yields the confluence diagram [8].
4. Reduction of the coherent presentation

Case 4: 
\[
\begin{cases}
q \geq l \text{ and } y_{i_0} > z_{i_0} \text{ for some } i_0 \leq l \\
p \geq q \text{ and } x_{i_0} > y_{i_0} \text{ for some } i_0 \leq q
\end{cases}
\quad \text{or} \quad 
\begin{cases}
q \geq l \text{ and } y_{i_0} > z_{i_0} \text{ for some } i_0 \leq q \\
p < q \text{ and } x_{i_0} > y_{i_0} \text{ for some } i_0 \leq p
\end{cases}
\quad \text{or} \quad 
\begin{cases}
q < l \text{ and } y_{i_0} > z_{i_0} \text{ for some } i_0 \leq q \\
p > q \text{ and } x_{i_0} > y_{i_0} \text{ for some } i_0 \leq q
\end{cases}
\]

By Lemma 3.2.4 the last term of \(e'\) is \(y_j \) or \(y_{j+1}\), where \(y_j \) is the biggest element of \(v\) such that \(y_j < x_1\). Suppose that the last term of \(e'\) is \(y_j\). Since \(z_l \geq y_j\), the tableau \(P(e'^t)\) consists of two columns. Furthermore, if the last term of \(e'\) is \(y_{j+1}\), then we consider two cases: \(z_l \geq y_{j+1}\) or \(z_l < y_{j+1}\). Suppose \(z_l < y_{j+1}\), then the tableau \(P(e'^t)\) consists of one column \(e'^t\). We consider two cases depending on whether or not \(c_e c_{e'^t}\) is a tableau. With the same arguments of Case 2, we obtain a confluence diagram of the following forms:

Suppose the tableau \(P(e'^t)\) consists of two columns. Using the same arguments as in Case 2 and Case 3, we obtain a confluence diagram of the form \(D_{u,v,t}^{(3)}\), \(D_{u,v,t}^{(4)}\), or \(D_{u,v,t}^{(2)}\).

3.2.7. Remark. [17] Thanks to a private communication by Lecouvey, Lemma 2.3.5 and an involution on tableaux can be used to prove the confluence of the critical branching \([3]\) as follows. Let \(u\) be a column in \(\text{col}(n)\) of length \(p\). Schützenberger introduced the \textit{involution} of \(u\), denoted by \(u^*\), as the column of length \(n - p\) obtained by taking the complement of the elements of \(u\). More generally, let \(u_1 \ldots u_r\) be the column reading of a tableau, then \((u_1 \ldots u_r)^* = u_r^* \ldots u_1^*\) and \(u_1^* \ldots u_r^*\) is also the column reading of a tableau. Moreover, if \(w\) is the column reading of a Young tableau, then we have \(P(w^*) = P(w)^*\). In particular, for three columns \(c_u, c_v, c_t\) in \(\text{Col}_1(n)\), we have \(P(c_u c_v c_t) = P(c_u c_v c_t)^*\), see [18].

By Lemma 2.3.5 \((c_u c_v c_t)^*\) is a normal form of \(c_u c_v c_t\), that is, \(P(c_u c_v c_t) = c_u c_v c_t^*\). Then to prove the confluence of the 3-cell \([3]\), it is sufficient to show that \(P(c_u c_v c_t) = c_a C(c_a c_{w'})\). We have

\[c_u c_v c_t \overset{\alpha_{u,v,t}}{\Rightarrow} c_u C(c_v c_t) = c_u c_w c_{w'} \overset{\alpha_{u,w,w'}}{\Rightarrow} C(c_u c_w) c_{w'} = c_a c_{a'} c_{w'} \overset{c_a \alpha_{a'}(w')}{\Rightarrow} c_a C(c_{a'} c_{w'}).\]

By applying the involution on tableaux, we obtain

\[c_t^* c_u^* \overset{c_t^*}{\Rightarrow} C(c_t^* c_u^*) c_u^* = c_{w'} c_{w'}^* c_u^* \overset{c_{w'} c_{w'}^*}{\Rightarrow} c_{w'} C(c_{w'} c_{w'}^*) = c_{w'} c_a c_{w'}^* \overset{c_a}{\Rightarrow} C(c_{a'} c_{w'}^*).\]

By Lemma 2.3.5 we have \(P(c_t^* c_u^*) = C(c_{w'} c_{w'}^*) c_{a^*}\). Since \(P(c_t^* c_u^*) = P(c_u c_v c_t)^*\), we deduce that \(P(c_u c_v c_t)^* = C(c_{w'} c_{w'}^*) c_{a^*}\). Finally, by applying the involution on tableaux, we obtain \(P(c_u c_v c_t) = c_a C(c_{a'} c_{w'}).\) Note that this construction does not give the explicit forms of the 2-sources and the 2-targets of the confluence diagrams of the critical branchings as doing in lemmas above.

4. Reduction of the coherent presentation

In this section, we begin by recalling the homotopical reduction procedure from [6] Section 2.3. We explicit all the reduction steps that we need to reduce the coherent presentation \(\text{Col}_1(n)\) into a smaller finite coherent presentation of the monoid \(P_n\) that extends the Knuth presentation.
4. Reduction of the coherent presentation

4.1. Homotopical reduction procedure

4.1.1. Homotopical reduction procedure. Let \( \Sigma \) be a \((3,1)\)-polygraph. A 3-sphere of the \((3,1)\)-category \( \Sigma_3^+ \) is a pair \((f, g)\) of 3-cells of \( \Sigma_3^+ \) such that \( s_2(f) = s_2(g) \) and \( t_2(f) = t_2(g) \). A collapsible part of \( \Sigma \) is a triple \((\Gamma_2, \Gamma_3, \Gamma_4)\) made of a family \( \Gamma_2 \) of 2-cells of \( \Sigma \), a family \( \Gamma_3 \) of 3-cells of \( \Sigma \) and a family \( \Gamma_4 \) of 3-spheres of \( \Sigma_3^+ \), such that the following conditions are satisfied:

i) every \( \gamma \) of every \( \Gamma_k \) is collapsible, that is, \( t_{k-1}(\gamma) \) is in \( \Sigma_{k-1} \) and \( s_{k-1}(\gamma) \) does not contain \( t_{k-1}(\gamma) \),

ii) no cell of \( \Gamma_2 \) (resp. \( \Gamma_3 \)) is the target of a collapsible 3-cell of \( \Gamma_3 \) (resp. 3-sphere of \( \Gamma_4 \)),

iii) there exists a well-founded order on the cells of \( \Sigma \) such that, for every \( \gamma \) in every \( \Gamma_k \), \( t_{k-1}(\gamma) \) is strictly greater than every generating \((k - 1)\)-cell that occurs in the source of \( \gamma \).

The homotopical reduction of the \((3,1)\)-polygraph \( \Sigma \) with respect to a collapsible part \( \Gamma \) is the Tietze transformation, denoted by \( R_\Gamma \), from the \((3,1)\)-category \( \Sigma_3^+ \) to the \((3,1)\)-category freely generated by the \((3,1)\)-polygraph obtained from \( \Sigma \) by removing the cells of \( \Gamma \) and all the corresponding redundant cells. We refer the reader to \([6, 2.3.1]\) for details on the definition of the Tietze transformation \( R_\Gamma \) defined by well-founded induction as follows. For any \( \gamma \) in \( \Gamma \), we have \( R_\Gamma(t(\gamma)) = R_\Gamma(s(\gamma)) \) and \( R_\Gamma(\gamma) = 1_{R_\Gamma(s(\gamma))} \). In any other cases, the transformation \( R_\Gamma \) acts as an identity.

4.1.2. Generating triple confluences. A local triple branching of a 2-polygraph \( \Sigma \) is a triple \((f, g, h)\) of rewriting steps of \( \Sigma \) with a common source. An aspherical triple branchings have two of their 2-cells equal. A Peiffer triple branchings have at least one of their 2-cells that form a Peiffer branching with the other two. The overlap triple branchings are the remaining local triple branchings. Local triple branchings are ordered by inclusion of their sources and a minimal overlap triple branching is called critical. If \( \Sigma \) is a coherent and convergent \((3,1)\)-polygraph, a triple generating confluence of \( \Sigma \) is a 3-sphere

\[
\begin{array}{c}
\begin{array}{ccc}
\text{f} & \Rightarrow & \text{v} \\
\text{u} & \Rightarrow & \text{w} \\
\text{h} & \Rightarrow & \text{x} \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{ccc}
\text{g}_1 & \Rightarrow & \text{h}'' \\
\text{g}_1' & \Rightarrow & \text{v} \\
\text{h}_2' & \Rightarrow & \text{f}'' \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{ccc}
\text{x'} & \Rightarrow & \text{h}'' \\
\text{u} & \Rightarrow & \text{\hat{u}} \\
\text{h} & \Rightarrow & \text{x} \\
\end{array}
\end{array}
= \omega_{f,g,h}
\end{array}
\]

where \((f, g, h)\) is a triple critical branching of the 2-polygraph \( \Sigma_2 \) and the other cells are obtained by confluence, see \([6, 2.3.2]\) for details.

4.1.3. Homotopical reduction of the polygraph \( \text{Col}_3(n) \). In the rest of this section, we apply three steps of homotopical reduction on the \((3,1)\)-polygraph \( \text{Col}_3(n) \). As a first step, we apply in 4.2 a homotopical reduction on the \((3,1)\)-polygraph \( \text{Col}_3(n) \) with a collapsible part defined by some of the generating triple confluences of the 2-polygraph \( \text{Col}_2(n) \). In this way, we reduce the coherent presentation \( \text{Col}_3(n) \) of the monoid \( P_n \) into the coherent presentation \( \overline{\text{Col}}_3(n) \) of \( P_n \), whose underlying 2-polygraph is \( \text{Col}_2(n) \) and the 3-cells \( \chi_{u,v,t} \) are those of \( \text{Col}_3(n) \), but with \( \ell(u) = 1 \). We reduce in 4.3 the coherent presentation \( \overline{\text{Col}}_3(n) \) into a coherent presentation \( \text{PreCol}_3(n) \) of \( P_n \), whose underlying
2-polygraph is PreCol₂(α). This reduction is given by a collapsible part defined by a set of 3-cells of Col₃(α). In a final step, we reduce in 4.4 the coherent presentation PreCol₃(α) into a coherent presentation Knuth₃(α) of Pₙ whose underlying 2-polygraph is Knuth₂(α). By [6, Theorem 2.3.4], all these homotopical reductions preserve coherence. That is, the (3, 1)-polygraph Col₃(α) being a coherent presentation of Pₙ, the (3, 1)-polygraphs Col₃(α) and Knuth₃(α) are coherent presentations of Pₙ.

4.2. A reduced column presentation

We apply the homotopical reduction procedure in order to reduce the (3, 1)-polygraph Col₃(α) using the generating triple confluenes.

4.2.1. Generating triple confluenes of Col₂(α). Consider the homotopical reduction procedure on the (3, 1)-polygraph Col₃(α) defined using the collapsible part made of generating triple confluenes. By Theorem 3.2.2, the family of 3-cells Xᵤ,ᵥ,τ given in (4) and indexed by columns u, v and t in col(α) such that u, v, t forms a homotopy basis of the (2, 1)-category Col₂(α). Let us consider such a triple (u, v, t) with ℓ(ℓ) ≥ 2. Let xₜ be in [n] such that u = xₜ u₁ with u₁ in col(α). There is a critical triple branching with source cₓₜ,c₁,c₁,c₁. Let us show that the confluence diagram induced by this triple branching is represented by the 3-sphere Ωₓₜ,u₁,v₁,t whose source is the following 3-cell

![Diagram showing a reduced column presentation with source cₓₜ,c₁,c₁,c₁ and target as described.]

and whose target is the following 3-cell

![Diagram showing a reduced column presentation with source cₓₜ,c₁,c₁,c₁ and target as described.]

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4. Reduction of the coherent presentation

In the generating triple confluence, some columns may be empty and thus the indicated 2-cells $\alpha$ may be identities. To facilitate the reading of the diagram, we have omitted the context of the 2-cells $\alpha$.

The 3-sphere $\Omega_{x_p,u_1,v_1,t}$ is constructed as follows. We have $x_p^1 u_1$ and $u_1^x w$, thus $x_p x u_1 w$ is either of the form $A_{x_p,u_1,v_1,w}$ or $C_{x_p,u_1,v_1,w}$. Let us denote by $a_1$ and $a'_1$ the two columns of the tableau $P(u_1 w)$. The 3-cell $x_p u_1 v_1 w$ being confluent, we have $C(x_p a_1) = az$ with $z$ in $[n]$ and $C(z a'_1) = a'$. In addition, from $z^x b_1'$ and $a'_1 ^x w'$, we deduce that $x_p a'_1, w'$ is either of the form $A_{z,a'_1, w'}$ or $C_{z,a'_1, w'}$. From $x_p^1 u_1$ and $u_1^x v$, we deduce that $x_p u_1 v$ is either of the form $A_{x_p,u_1,v}$ or $C_{x_p,u_1,v}$. Let us denote by $s$ and $s'$ the two columns of the tableau $P(u_1 v)$. The 3-cell $x_p u_1 v$ being confluent, we obtain that $C(x_p s) = ey$ with $y$ in $[n]$ and $C(y s') = e'$. From $y^x s'$ and $s'^x t$, we deduce that $x_p s', t$ is either of the form $A_{y,s', t}$ or $C_{y,s', t}$. Denote by $d_1$ and $d'_1$ the two columns of the tableau $P(s't)$. The 3-cell $x_p s', t$ being confluent and $C(e't) = b b'$, we have $C(y d_1) = b s_2$ and $C(2 d'_1) = b'$. On the other hand, the 3-cell $x_p s', t$ is confluent, then we have $C(s_1) = a_1 s_3$ and $C(a'_1 w') = s_3 d'_1$. Finally, since the 3-cell $x_p s', d_1$ is confluent, we obtain $C(z s_3) = d s_2$.

4.2.2. Reduced coherent column presentation. Let us define by $\overline{\text{Col}_3}(n)$ the extended presentation of the monoid $P_n$ obtained from Col$_2(n)$ by adjunction of one family of 3-cells $\text{Col}(u,v,t)$ of the form (4), for every 1-cell $x$ in $[n]$ and columns $v$ and $t$ in $\text{col}(n)$ such that $x^x v^x t$. The following result shows that this reduced presentation is also coherent.

4.2.3. Proposition. For $n > 0$, the $(3,1)$-polygraph $\overline{\text{Col}_3}(n)$ is a coherent presentation of the monoid $P_n$.

Proof. Let $\Gamma_4$ be the collapsible part made of the family of 3-sphere $\Omega_{x_p,u_1,v_1,t}$, indexed by $x_p$ in $[n]$ and $u_1, v, t$ in $\text{col}(n)$ such that $u^x v^x t$ and $u = x_p u_1$. On the 3-cells of Col$_3(n)$, we define a well-founded order $\prec$ by

i) $A_{u,v,t} \prec C_{u,v,t} \prec B_{u,v,t} \prec D_{u,v,t}$.

ii) if $A_{u,v,t} \in \{A_{u,v,t}, B_{u,v,t}, C_{u,v,t}, D_{u,v,t}\}$ and $u' \prec \text{deglex } u$, then $A_{u',v,t} \prec A_{u,v,t}$.

for any $u, v, t$ in $\text{col}(n)$ such that $u^x v^x t$. By construction of the 3-sphere $\Omega_{x_p,u_1,v_1,t}$, its source contains the 3-cell $A_{u,v,t}$ and its target contains the 3-cell $A_{u,v,t}$ with $\ell(u_1) < \ell(u)$. Up to a Nielsen transformation, the homotopical reduction $R_\ell$ applied on the $(3,1)$-polygraph Col$_3(n)$ with respect to $\Gamma_4$ and the order $\prec$ give us the $(3,1)$-polygraph $\overline{\text{Col}_3}(n)$. In this way, the presentation $\overline{\text{Col}_3}(n)$ is a coherent presentation of the monoid $P_n$. ⊓⊔

4.3. Pre-column coherent presentation

We reduce the coherent presentation $\overline{\text{Col}_3}(n)$ into a coherent presentation whose underlying 2-polygraph is PreCol$_2(n)$. This reduction is obtained using the homotopical reduction $R_\ell$ on the $(3,1)$-polygraph Col$_3(n)$ whose collapsible part $\Gamma_3$ is defined by

$$\Gamma_3 = \{ A_{x,v,t} \mid x \in [n], \ v, t \in \text{col}(n) \text{ such that } x^x v^x t \}$$
$$\cup \{ B_{x,v,t} \mid x \in [n], \ v, t \in \text{col}(n) \text{ such that } x^x v^{x+1} t \}$$
$$\cup \{ C_{x,v,t} \mid x \in [n], \ v, t \in \text{col}(n) \text{ such that } x^{x+1} v^x t \}$$,
and the well-founded order defined as follows. Given \( u \) and \( v \) in \( \text{col}(n) \) such that \( u \times v \). We define a well-founded order \( \prec \) on the 2-cells of \( \text{Col}_2(n) \) as follows

\[
\alpha_{u',v'} \prec \alpha_{u,v} \quad \text{if} \quad \left\{ \begin{array}{ll}
\ell(u) > \ell(u') & \text{or} \\
\ell(u) = \ell(u') & \text{and} \quad \ell(u) > \ell(C_r(u'v')) \\
\ell(u) < \ell(C_r(u'v')) \end{array} \right.
\]

for any columns \( u, v, u' \) and \( v' \) in \( \text{col}(n) \) such that \( u \times v \) and \( u \times v' \), where \( \prec \) is the total order on \( \text{col}(n) \) defined by \( u \prec v \) if \( \ell(u) > \ell(v) \) or \( \ell(u) = \ell(v) \) and \( u \prec_{\text{lex}} v \), for all \( u \) and \( v \) in \( \text{col}(n) \).

### 4.3.1. The homotopical reduction \( R_{\text{r}} \).

Consider the well-founded order \( \prec \) on the 2-cells of \( \text{Col}_2(n) \) and the well-founded order \( \prec \) on 3-cells defined in the proof of Proposition 4.2.3. The reduction \( R_{\text{r}} \) induced by these orders can be decomposed as follows. For any \( x \) in \( [n] \) and columns \( v, t \) such that \( x \times^1 v \), \( t \), we have \( \alpha_x \prec \alpha_{xv,t} \), \( \alpha_{vt} \prec \alpha_{xvt} \) and \( \alpha_{xv,t} \prec \alpha_{xvt} \). The reduction \( R_{\text{r}} \) removes the 2-cell \( \alpha_{xvt} \) together with the 3-cell \( A_{xv,t} \) defined in Lemma 3.2.3. By iterating this reduction on the length of the column \( v \), we reduce all the 2-cells of \( \text{Col}_2(n) \) to the following set of 2-cells

\[
\{ \alpha_{u,v} \mid \ell(u) = 1, \ell(v) \geq 2 \text{ and } u \times^1 v \} \cup \{ \alpha_{u,v} \mid \ell(u) = 1, \ell(v) \geq 1 \text{ and } u \times^1 v \}. \tag{11}
\]

For any \( x \) in \( [n] \) and columns \( v, t \) such that \( x \times^1 v \times^1 t \), consider the 3-cell \( C_{xv,t} \) defined in Lemma 3.2.5. The 2-cells \( \alpha_{xv,t} \), \( \alpha_{xvt} \) and \( \alpha_{xv,t}' \) are smaller than \( \alpha_{xvt} \) for the order \( \prec \). The reduction \( R_{\text{r}} \) removes the 2-cell \( \alpha_{xvt} \) together with the 3-cell \( C_{xv,t} \). By iterating this reduction on the length of \( v \), we reduce the set of 2-cells given in (11) to the following set:

\[
\{ \alpha_{u,v} \mid \ell(u) = 1, \ell(v) \geq 2 \text{ and } u \times^2 v \} \cup \{ \alpha_{u,v} \mid \ell(u) = 1, \ell(v) \geq 1 \text{ and } u \times^1 v \}. \tag{12}
\]

For any \( x \) in \( [n] \) and columns \( v, t \) such that \( x \times^2 v \times^1 t \), consider the following 3-cell:

\[
\begin{array}{ccc}
\alpha_x & & \alpha_{xv,t} \\
\downarrow & & \downarrow \\
C_x & & C_{xv,t}
\end{array}
\begin{array}{ccc}
C_x & & C_{xv,t} \\
\downarrow & & \downarrow \\
\alpha_{xv,t} & & \alpha_{xvt}
\end{array}
\begin{array}{ccc}
\alpha_{xv,t} & & \alpha_{xvt} \\
\downarrow & & \downarrow \\
C_x & & C_{xv,t}
\end{array}
\begin{array}{ccc}
C_x & & C_{xv,t} \\
\downarrow & & \downarrow \\
\alpha_{xvt} & & \alpha_{xv,t}'
\end{array}
\]

where \( e, e', s \) and \( s' \) are defined in Lemma 3.2.4. Note that \( \alpha_{e,e'} \) is the 2-cell in (12) obtained from the 2-cell \( \alpha_{e,e'} \) by the previous step of the homotopical reduction by the 3-cell \( C_{xv,t} \). Having \( x \) in \( [n] \), by definition of \( \alpha \) we have \( e' \) in \( [n] \). The 2-cells \( \alpha_{xv,t} \), \( \alpha_{e,e'} \), \( \alpha_{xvt} \) and \( \alpha_{e,e'} \) being smaller than \( \alpha_{xvt} \) for the order \( \prec \), we can remove the 2-cells \( \alpha_{xvt} \) together with the 3-cell \( B_{xv,t} \). By iterating this reduction on the length of the column \( t \), we reduce the set given in (11) to the following set:

\[
\{ \alpha_{u,v} \mid \ell(u) = 1, \ell(v) \geq 2 \text{ and } u \times^2 v \} \cup \{ \alpha_{u,v} \mid \ell(u) = 1, \ell(v) \geq 1 \text{ and } u \times^1 v \}. \tag{13}
\]

### 4.3.2. Lemma.

The set of 2-cells defined in (13) is equal to \( \text{PreCol}_2(n) \).

**Proof.** By definition of \( \text{PreCol}_2(n) \), it is sufficient to prove that

\[
\text{PC}_2(n) = \{ \alpha_{u,v} : c_u c_v \Rightarrow c_v c_u \mid \ell(u) = 1, \ell(v) = 2 \text{ and } u \times^2 v \}.
\]

Consider the 2-cells \( \alpha_{u,v} \) in \( \text{Col}_2(n) \) such that \( \ell(u) = 1, \ell(v) = 2 \) and \( u \times^2 v \). Suppose that \( v = xx' \) with \( x > x' \) in \( [n] \). Since \( u \times^2 v \), we obtain that \( u \leq x \). Hence, we have two cases to consider. If \( u \leq x' \), then \( C(\nu v) = (\nu x)x' \). Hence, the 2-cell \( \alpha_{u,v} \) is equal to the 2-cell \( \alpha'_{u,v} : c_u c_x x' \Rightarrow c_{u x} c_x x' \). In the other case, if \( x' < u \), then \( C(\nu v) = (\nu x')x \). Hence the 2-cell \( \alpha_{u,v} \) is equal to \( \alpha'_{u,v} : c_u c_x x' \Rightarrow c_{u x'} c_x \).
4. Reduction of the coherent presentation

4.3.3. Pre-column coherent presentation. The homotopical reduction \( R_{\Gamma_3} \), defined in 4.3.1, reduces the coherent presentation \( \text{Col}_3(\mathbb{n}) \) into a coherent presentation of the monoid \( \mathbb{P}_n \). The set of 2-cells of this coherent presentation is given by (13), which is PreCol_2(\mathbb{n}) by Lemma 4.3.2. Let us denote by PreCol_3(\mathbb{n}) the extended presentation of the monoid \( \mathbb{P}_n \) obtained from PreCol_2(\mathbb{n}) by adjunction of the 3-cells of type \( R_{\Gamma_3}(C_{x,v,t}) \) where

\[
\begin{align*}
\alpha_{x,v}c_{t} & \rightarrow c_{xv}c'_{t} \\
\alpha_{x,v}c_{t} & \rightarrow c_{xv}c_{w}' \\
\alpha_{x,v}c_{w}' & \rightarrow c_{xw}c_{w}' \\
\end{align*}
\]

with \( x^{1}v^{2}t \), and the 3-cells of type \( R_{\Gamma_3}(D_{x,v,t}) \) where

\[
\begin{align*}
\alpha_{x,v}c_{t} & \rightarrow c_{xv}c'_{t} \\
\alpha_{x,v}c_{t} & \rightarrow c_{xv}c_{w}' \\
\alpha_{x,v}c_{w}' & \rightarrow c_{xw}c_{w}' \\
\end{align*}
\]

with \( x^{2}v^{2}t \). The homotopical reduction \( R_{\Gamma_3} \) eliminates the 3-cells of \( \overline{\text{Col}}_3(\mathbb{n}) \) of the form \( A_{x,v,t}, B_{x,v,t} \) and \( C_{x,v,t} \), which are not of the form \( C_{x,v,t}^{'} \). We have then proved the following result.

4.3.4. Theorem. For \( n > 0 \), the \((3,1)\)-polygraph \( \text{PreCol}_3(\mathbb{n}) \) is a coherent presentation of the monoid \( \mathbb{P}_n \).

4.3.5. Example: coherent presentation of monoid \( \mathbb{P}_2 \). The 2-polygraph Knuth_2(2) has for 2-cells \( \eta_{1,1,2} : 211 \Rightarrow 121 \) and \( \varepsilon_{1,2,2} : 221 \Rightarrow 212 \). It is convergent with only one critical branching with source the 1-cell 2211. This critical branching is confluent:

Following the homotopical completion procedure given in 3.1.4, the 2-polygraph extended by the previous 3-cell is a coherent presentation of the monoid \( \mathbb{P}_2 \). Consider the column presentation \( \text{Col}_2(\mathbb{P}_2) \) of the monoid \( \mathbb{P}_2 \) with 1-cells \( c_1 \), \( c_2 \) and \( c_{21} \) and 2-cells \( \alpha_{2,1}, \alpha_{1,21} \) and \( \alpha_{2,21} \). The coherent presentation \( \text{Col}_3(\mathbb{P}_2) \) has only one 3-cell

\[
\begin{align*}
c_{2}c_{1}c_{21} & \rightarrow c_{21}c_{2}c_{1} \\
c_{2}c_{1}c_{21} & \rightarrow c_{21}c_{2}c_{1} \\
\end{align*}
\]

It follows that the \((3,1)\)-polygraphs \( \overline{\text{Col}}_3(\mathbb{P}_2) \) and \( \overline{\text{Col}}_3(\mathbb{P}_2) \) coincide. Moreover, in this case the set \( \Gamma_3 \) is empty and the homotopical reduction \( R_{\Gamma_3} \) is the identity and thus \( \text{PreCol}_3(\mathbb{P}_2) \) is also equal to \( \text{Col}_3(\mathbb{P}_2) \).
4.3.6. Example: coherent presentation of monoid $P_3$. For the monoid $P_3$, the Knuth presentation has 3 generators and 8 relations. It is not convergent, but it can be completed by adding 3 relations. The obtained presentation has 27 3-cells corresponding to the 27 critical branchings. The column coherent presentation $Col_3(3)$ of $P_3$ has 7 generators, 22 relations and 42 3-cells. The coherent presentation $Col_3(3)$ has 7 generators, 22 relations and 34 3-cells. After applying the homotopical reduction $R_7$, the coherent presentation $PreCol_3(3)$ admits 7 generators, 22 relations and 24 3-cells. We give in 4.4.10 the values of number of cells of the $(3, 1)$-polygraphs $Col_3(n)$ and $PreCol_3(n)$ for plactic monoids of rank $n \leq 10$.

4.4. Knuth’s coherent presentation

We reduce the coherent presentation $PreCol_3(n)$ into a coherent presentation of the monoid $P_n$ whose underlying 2-polygraph is $Knuth_2(n)$. We proceed in three steps developed in the next sections.

**Step 1.** We apply the inverse of the Tietze transformation $T_{y\mapsto\alpha'}$, that coherently replaces the 2-cells $\gamma_{x_p...x_1}$ by the 2-cells $\alpha'_{x_p,x_p...x_1}$, for each column $x_p...x_1$ such that $\ell(x_p...x_1) > 2$.

**Step 2.** We apply the inverse of the Tietze transformation $T_{\eta,\varepsilon\mapsto\alpha'}$, that coherently replaces the 2-cells $\eta_{x,y,z}$ by $\alpha'_{x,y,z}$, for every $1 \leq x < y < z \leq n$, and the 2-cells $\alpha'_{x,y,z}$ by $\varepsilon_{x,y,z}$, for every $1 \leq x < y < z \leq n$.

**Step 3.** Finally for each column $x_p...x_1$, we coherently eliminate the generator $c_{x_p...x_1}$ together with the 2-cell $\gamma_{x_p...x_1}$ with respect to the order $\approx_{\text{deglex}}$.

4.4.1. Step 1. The Tietze transformation $T_{y\mapsto\alpha'} : CPC_2(n)^T \rightarrow PreCol_2(n)^T$ defined in Proposition 2.3.3 substitutes a 2-cell $\alpha'_{x_p,x_p...x_1} : c_{x_p}c_{x_p...x_1} \Longrightarrow c_{x_p...x_1}$ to the 2-cell $\gamma_{x_p...x_1}$ in $C_2(n)$, from the bigger column to the smaller one with respect to the total order $\approx_{\text{deglex}}$.

We consider the inverse of this Tietze transformation $T^{-1}_{y\mapsto\alpha'} : PreCol_2(n)^T \rightarrow CPC_2(n)^T$ that substitutes the 2-cell $\gamma_{x_p...x_1} : c_{x_p}...c_{x_1} \Longrightarrow c_{x_p...x_1}$ to the 2-cell $\alpha'_{x_p,x_p...x_1} : c_{x_p}c_{x_p...x_1} \Longrightarrow c_{x_p...x_1}$, for each column $x_p...x_1$ such that $\ell(x_p...x_1) > 2$ with respect to the order $\approx_{\text{deglex}}$.

Let us denote by $CPC_3(n)$ the $(3, 1)$-polygraph whose underlying 2-polygraph is $CPC_2(n)$, and the set of 3-cells is defined by

$$\{ T^{-1}_{y\mapsto\alpha'}(R_3(C'_{x,v,t})) \text{ for } x'^1v'^2t \} \cup \{ T^{-1}_{y\mapsto\alpha'}(R_3(D_{x,v,t})) \text{ for } x'^2v'^2t \}.$$ 

In this way, we extend the Tietze transformation $T^{-1}_{y\mapsto\alpha'}$ into a Tietze transformation between the $(3, 1)$-polygraphs $PreCol_3(n)$ and $CPC_3(n)$. The $(3, 1)$-polygraph $PreCol_3(n)$ being a coherent presentation of the monoid $P_n$ and the Tietze transformation $T^{-1}_{y\mapsto\alpha'}$ preserves the coherence property, hence we have the following result.

4.4.2. Lemma. For $n > 0$, the monoid $P_n$ admits $CPC_3(n)$ as a coherent presentation.

4.4.3. Step 2. The Tietze transformation $T_{\eta,\varepsilon\mapsto\alpha'}$ from $Knuth_2^c(n)^T$ into $CPC_2(n)^T$ defined in the proof of Proposition 2.3.3 replaces the 2-cells $\eta_{x,y,z}$ and $\epsilon_{x,y,z}$ in $Knuth_2^c(n)$ by composite of 2-cells in $CPC_2(n)$.

Let us consider the inverse of this Tietze transformation $T^{-1}_{\eta,\varepsilon\mapsto\alpha'} : CPC_2(n)^T \rightarrow Knuth_2^c(n)^T$, making the following transformations. For every $1 \leq x < y < z \leq n$, $T^{-1}_{\eta,\varepsilon\mapsto\alpha'}$ substitutes the
4. Reduction of the coherent presentation

2-cell $n_{x,y,z}^c : c_2 c_x c_y \Rightarrow c_x c_z c_y$ to the 2-cell $\alpha'_{x,z,y}$. For every $1 \leq x < y \leq z \leq n$, $T_{n,i,-}^{-1}$ substitutes the 2-cell $\varepsilon_{x,y,z}^c : c_y c_2 c_x \Rightarrow c_x c_y c_x$ to the 2-cell $\alpha_{y,x}$.  

Let us denote by Knuth$_3^c(n)$ the $(3,1)$-polygraph whose underlying 2-polygraph is Knuth$_3^c(n)$ and whose set of $3$-polygraphs is

\[
\{ T_{n,i,-}^{-1}(T_{y,-}^{-1}(R_{l_2}(C_{x,v,t}))) \text{ for } x^1 v^2 t \} \cup \{ T_{n,i,-}^{-1}(T_{y,-}^{-1}(R_{l_3}(D_{x,v,t}))) \text{ for } x^2 v^2 t \}.
\]

We extend the Tietze transformation $T_{n,i,-}^{-1}$, into a Tietze transformation between $(3,1)$-polygraphs

\[
T_{n,i,-}^{-1} : \text{CPC}_3(n)^{-} \rightarrow \text{Knuth}_3^c(n)^{-},
\]

where the $(3,1)$-polygraph CPC$_3(n)$ is a coherent presentation of the monoid $P_n$ and the Tietze transformation $T_{n,i,-}^{-1}$ preserves the coherence property, hence we have the following result.

4.4.4. Lemma. For $n > 0$, the monoid $P_n$ admits Knuth$_3^c(n)$ as a coherent presentation.

4.4.5. Step 3. Finally, in order to obtain the Knuth coherent presentation, we perform an homotopical reduction, obtained using the homotopical reduction $R_{l_2}$ on the $(3,1)$-polygraph Knuth$_3^c(n)$ whose collapsible part $l_2$ is defined by the 2-cells $\gamma_{y}$ of $C_2(n)$ and the well-founded order $\preceq_{\text{deglex}}$. Thus, for every 2-cell $\gamma_{x_p \ldots x_1} : c_{x_p} \ldots c_{x_1} \Rightarrow c_{x_p} \ldots c_{x_1}$ in $C_2(n)$, we eliminate the generator $c_{x_p \ldots x_1}$ together with the 2-cell $\gamma_{x_p \ldots x_1}$, from the bigger column to the smaller one with respect to the order $\preceq_{\text{deglex}}$.

4.4.6. Knuth coherent presentation. Using the Tietze transformations constructed in the previous sections, we consider the following composite of Tietze transformations

\[
\mathcal{R} := R_{l_2} \circ T_{n,i,-}^{-1} \circ T_{y,-}^{-1} \circ R_{l_3}
\]

defined from $\text{Col}_3(n)^{-}$ to Knuth$_3^c(n)^{-}$ as follows. Firstly, the transformation $\mathcal{R}$ eliminates the 3-cells of $\text{Col}_3(n)$ of the form $A_{x,v,t}$, $B_{x,v,t}$ and $C_{x,v,t}$ which are not of the form $C_{x,v,t}'$ and reduces its set of 2-cells to PreCol$_2(n)$. Secondly, this transformation coherently replaces the 2-cells $\gamma_{x_p \ldots x_1}$ by the 2-cells $\alpha'_{x_p \ldots x_1}$, for each column $x_p \ldots x_1$ such that $\ell(x_p \ldots x_1) > 2$, the 2-cells $\alpha'_{y,z}$ by $n_{x,y,z}^c$, for $1 \leq x \leq y < z \leq n$ and the 2-cells $\alpha'_{y,x}$ by $\varepsilon_{x,y,z}^c$, for $1 \leq x < y \leq z \leq n$. Finally, for each column $x_p \ldots x_1$, the transformation $\mathcal{R}$ eliminates the generator $c_{x_p \ldots x_1}$ together with the 2-cell $\gamma_{x_p \ldots x_1}$ with respect to the order $\preceq_{\text{deglex}}$.

Let us denote by Knuth$_3(n)$ the extended presentation of the monoid $P_n$ obtained from Knuth$_2(n)$ by adjunction of the following set of 3-cells

\[
\{ \mathcal{R}(C_{x,v,t}') \text{ for } x^1 v^2 t \} \cup \{ \mathcal{R}(D_{x,v,t}) \text{ for } x^2 v^2 t \}.
\]

The transformation $\mathcal{R}$ being a composite of Tietze transformations, it follows the following result.

4.4.7. Theorem. For $n > 0$, the $(3,1)$-polygraph Knuth$_3(n)$ is a coherent presentation of the monoid $P_n$.  

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4.4. Knuth’s coherent presentation

4.4.8. Example: Knuth’s coherent presentation of the monoid $P_2$. We have seen in Example 4.3.5 that the $(3, 1)$-polygraphs $Col_3(2)$, $Col_1^3(2)$ and $PreCol_3(2)$ are equal. The coherent presentation $PreCol_3(2)$ has three 2-cell $\alpha_{2,1}$, $\alpha_{1,21}$, $\alpha_{2,21}$ and the following 3-cell:

$$
\begin{align*}
\alpha_{2,1}(c_2) & \Rightarrow c_2c_1c_21 \\
\alpha_{1,21}(c_2) & \Rightarrow c_2c_1c_21 \\
\alpha_{2,21}(c_2) & \Rightarrow c_2c_1c_21
\end{align*}
$$

By definition of the 2-cells of $C_2(2)$, we have $\gamma_{21} := \alpha_{2,1}$. Thus we obtain that $T_{\gamma_{21}}^{-1}(C_2(3,1,21)) = C_2(3,1,21)$ up to replace all the 2-cells $\alpha_{2,1}$ in $C_2(3,1,21)$ by $\gamma_{21}$. Hence, the coherent presentation $CPC_3(2)$ is equal to $PreCol_3(2)$. In order to compute the 3-cell $T_{\eta,\epsilon}^{-1}(T_{\gamma_{21}}^{-1}(C_2(3,1,21)))$, the 2-cells $\alpha_{3,21}$ and $\alpha_{2,21}$ in $C_2(3,1,21)$ are respectively replaced by the 2-cells $\eta_1,2$ and $\epsilon_{1,2,2}$ as in the following diagram

$$
\begin{align*}
\gamma_{21} & \Rightarrow c_2c_1c_21 \\
\gamma_{21} & \Rightarrow c_2c_1c_21 \\
\gamma_{21} & \Rightarrow c_2c_1c_21
\end{align*}
$$

where the cancel symbol means that the corresponding 2-cell is removed. Hence the coherent presentation $Knuth_3^{cc}(2)$ of $P_2$ has for 1-cells $c_1$, $c_2$ and $c_{21}$, for 2-cells $\alpha_{2,1}$, $\alpha_{1,21}$ and $\alpha_{2,21}$ and the only 3-cell $[14]$. Let us compute the Knuth coherent presentation $Knuth_3(2)$. The 3-cell $R_{c_2}([14])$ is obtained from $[14]$ by removing the 2-cell $\gamma_{21}$ together with the 1-cell $c_{21}$. Thus we obtain the following 3-cell, where the cancel symbol means that the corresponding element is removed,

$$
\begin{align*}
\gamma_{21} & \Rightarrow c_2c_1c_21 \\
\gamma_{21} & \Rightarrow c_2c_1c_21 \\
\gamma_{21} & \Rightarrow c_2c_1c_21
\end{align*}
$$

Hence, the Knuth coherent presentation $Knuth_3(2)$ of the monoid $P_2$ has generators $c_1$ and $c_2$ subject to the Knuth relations $\eta_{1,2,2} : c_2c_1c_1 \Rightarrow c_1c_2c_2$ and $\epsilon_{1,2,2} : c_2c_1c_2 \Rightarrow c_2c_1c_2$ and the following 3-cell

$$
\begin{align*}
\eta_{1,2,2} & \Rightarrow c_2c_1c_21 \\
\epsilon_{1,2,2} & \Rightarrow c_2c_1c_21
\end{align*}
$$

In this way, we obtain the Knuth coherent presentation of the monoid $P_2$ that we obtain in Example 4.3.5 as a consequence of the fact that the 2-polygraph $Knuth_2(2)$ is convergent.
4. Reduction of the coherent presentation

4.4.9. Procedure to compute the 3-cells of Knuth$_3(n)$. We present a procedure that computes the 2-sources and the 2-targets of the 3-cells of the Knuth coherent presentation Knuth$_3(n)$, using the constructions given in Sections 3 and 4. The first step consists to define a procedure, called ReduceG3($\alpha$), that replaces a 2-cell $\alpha$ of Col2($n$) by a 2-cell of the 2-category PreCol2($n$)$^*$ using a reduction defined in 4.3.1 with respect to the 3-cells $A_{x,v,t}$, $B_{x,v,t}$ and $C_{x,v,t}$, where $x$ is in $[n]$ and $v$ and $t$ are in col($n$).

Given $u$ in col($n$) such that $\ell(u) \geq 2$ and $u = x_0 x_p x_{p-1} \ldots x_2 x_1$, we will denote $x_p$ (resp. $x_1$) by first($u$) (resp. last($u$)) and the column $x_{p-1} \ldots x_1$ (resp. $x_p \ldots x_2$) by rem($u$) (resp. rem$^1(u)$). If $\ell(u) = 1$, we set first($u$) = last($u$) = $u$ and rem$^1(u)$ and rem$^1(u)$ are the empty columns.

ReduceG3($\alpha$):

Input: $\alpha$ in Col2($n$).

\[\alpha = \alpha_{u,v};\]

\[\begin{align*}
\text{case } u^x v & \text { do} \\
\text{if } \ell(u) \geq 2 & \text { then } \\
& x = \text{first}(u); u_2 = \text{rem}(u); \\
& \beta = \text{ReduceG3}(\alpha_{u_2,v}); \\
& \gamma = \alpha_{x,u_2,v} \ast \gamma_1 \ast \gamma_2 \ast \gamma_3; \\
& \text{else return } \alpha;
\end{align*}\]

\[\begin{align*}
\text{case } u^x v & \text { do} \\
\text{if } \ell(u) \geq 2 & \text { and } \ell(v) \geq 2 \text { then } \\
& x = \text{first}(u); u_2 = \text{rem}(u); \\
& w = c_1(u_2 v); w' = c_r(u_2 v); \alpha = c_1(x w); \alpha' = c_r(x w); \\
& \beta = \text{ReduceG3}(\alpha_{u_2,v}); \\
& \gamma = \alpha_{x,u_2,c_0} \ast \gamma_1 \ast \gamma_2 \ast \gamma_3; \\
& \text{if } \ell(u) = 1 & \text { and } \ell(v) \geq 2 \text { then } \\
& v_1 = \text{rem}(v); y = \text{last}(v); \\
& e = c_1(v u_1); e' = c_r(v u_1); \\
& \eta_1 = \text{ReduceG3}(\alpha_{v_1,y}); \eta_2 = \text{ReduceG3}(\alpha_{u,v_1}); \eta_3 = \text{ReduceG3}(\alpha_{e,e'}); \\
& \alpha = c_u \eta_1 \ast \eta_2 \eta' \ast \gamma_1 \ast \gamma_2 \ast \gamma_3; \\
& \text{if } \ell(u) = 1 & \text { and } \ell(v) = 2 \text { then } \\
& \text{return } \alpha;
\end{align*}\]

We define the procedure ElimAlpha($\alpha$) that replaces a 2-cell $\alpha$ of PreCol2($n$) by a 2-cell of the 2-category Knuth$_2^c(n)^*$, using the Tietze transformations given in 4.4.1 and 4.4.3. In the sequel, we will represent every 1-composite $f_1 \ast \ldots \ast f_k$ of 2-cells by a list $[f_1, \ldots, f_k]$ of 2-cells. If $L = [L[0], \ldots, L[k - 1]]$ is a list of length $k$ and $u$ and $v$ are in $[n]^*$, we will denote by $uLv$ the list $[uL[0]v, \ldots, uL[k - 1]v]$.

ElimAlpha($\alpha$):

Input: $\alpha$ in PreCol2($n$).

\[\begin{align*}
\text{case } x^x v & \text { do} \\
\text{if } \ell(v) > 1 & \text { then } \\
& \text{return } [c_y, Y_{zv}, Y_{xv}] \text {; else return } [Y_{xv}]; \\
\text{case } x^x v & \text { do} \\
& z = \text{first}(v); y = \text{last}(v); \\
& \text{if } x \leq y < z & \text { then } \\
& \text{return } [c_x, Y_{zv}, Y_{xv}]; \\
& \text{if } y < x \leq z & \text { then } \\
& \text{return } [c_y, Y_{zv}, Y_{xv}]; \\
\end{align*}\]
We define the procedure \( \text{ElimAG}(f) \) that replaces in a 2-cell \( f \) of the 2-category \( \text{PreCol}_2(n)^* \), every \( \alpha_{x,v} \) in \( \text{PreCol}_2(n) \) by \( \text{ElimAlpha}(\alpha_{x,v}) \). In a second step, it replaces every \( \gamma_u \) in \( C_2(n) \) by \( 1_u \), with respect to the reduction \( R_{\Gamma_2} \) defined in 4.4.5.

\[
\text{ElimAG}(f):
\begin{align*}
\text{Input: } & f = f_1 \star_1 \ldots \star_1 f_k, \text{ where for } i = 1, \ldots, k, f_i = u_i \alpha_i v_i, \\
& \text{with } u_i, v_i \in [n]^* \text{ and } \alpha_i \in \text{PreCol}_2(n). \\
& L = []; \\
& \text{for } i = 0 \text{ to } k - 1 \text{ do} \\
& \quad L[i] = u_{i+1} \text{ElimAlpha}(\alpha_{i+1}) v_{i+1}; \\
& \text{end} \\
& \text{for } i = 0 \text{ to } k - 1 \text{ do} \\
& \quad \text{for } j = 0 \text{ to } \ell(L[i]) - 1 \text{ do} \\
& \quad \quad \text{if } L[i][j] = u_j \beta_j v_j, \text{ with } u_j, v_j \in [n]^* \text{ and } \beta_j \text{ or } \beta_j^{-1} \text{ are in } C_2(n) \text{ then} \\
& \quad \quad \quad L[i][j] = 1_u v_j; \\
& \quad \text{end} \\
& \text{end} \\
& \text{return } L.
\end{align*}
\]

We define a procedure, called \( \text{ComputeD}(n) \), that computes the 2-sources and the 2-targets of the 3-cells \( R(D'_{x,v,t}) \) of the Knuth coherent presentation, where \( R \) is the Tietze transformation defined in 4.4.6.
4. Reduction of the coherent presentation

\textbf{ComputeD}(n):
\begin{itemize}
\item \textbf{Input:} \( n > 0 \).
\item \( K = \emptyset \).
\item \textbf{for} \( x \in [n] \) and \( v \) in \( \text{col}(n) \) such that \( x^2 v^2 t \) \textbf{do}
\item \( e = C_1(xv); e' = C_r(xv); b = C_{l}(P(e't)); b' = C_r(e't); \)
\item \( w = C_1(vt); w' = C_r(vt); a = C_1(xw); a' = C_r(xw); \)
\item \( \alpha_1 = \text{ElimAG}(\text{ReduceG3}(\alpha_{e,v})); \alpha_2 = \text{ElimAG}(\text{ReduceG3}(\alpha_{e',t})); \)
\item \( \alpha_3 = \text{ElimAG}(\text{ReduceG3}(\alpha_{e,b})); \)
\item \( \alpha = [\alpha_1 c_1, c_e \alpha_2, \alpha_3 c_b]; \)
\item \( \alpha'_1 = \text{ElimAG}(\text{ReduceG3}(\alpha_{v,t})); \alpha'_2 = \text{ElimAG}(\text{ReduceG3}(\alpha_{v,w})); \)
\item \( \alpha'_3 = \text{ElimAG}(\text{ReduceG3}(\alpha_{a',w'})); \)
\item \( \alpha'' = [c_x \alpha'_1, \alpha'_2 c_w, c_\alpha \alpha'_3]; \)
\item \( K = K \cup \{ (\alpha, \alpha') \}; \)
\item \textbf{end}
\item \textbf{return} \( K \).
\end{itemize}

Finally, a way to compute the 2-sources and the 2-targets of the 3-cells of the Knuth coherent presentation \( \text{Knuth}_3(n) \) is to apply at the same time the procedures \text{ComputeC}'(n) \text{ ComputeD}(n).

\subsection*{4.4.10. Coherent presentations in small ranks.}
Let us denote by \( \text{Knuth}^\text{KB}_2(n) \) the convergent 2-polygraph obtained from \( \text{Knuth}_2(n) \) by the Knuth-Bendix completion using the lexicographic order. For \( n = 3 \), the polygraph \( \text{Knuth}^\text{KB}_2(3) \) is finite, but \( \text{Knuth}^\text{KB}_2(n) \) is infinite for \( n \geq 4 \). \[13\]. Let us denote by \( \text{Knuth}^\text{KB}_3(n) \) the Squier completion of \( \text{Knuth}^\text{KB}_2(n) \). For \( n \geq 4 \), the polygraph \( \text{Knuth}^\text{KB}_2(n) \) having an infinite set of critical branching, the set of 3-cells of \( \text{Knuth}^\text{KB}_3(n) \) is infinite. However, the \((3,1)\)-polygraph \( \text{Knuth}_3(n) \) is a finite coherent convergent presentation of \( \text{P}_n \). Table 1 presents the number of cells of the coherent presentations \( \text{Knuth}_3(n), \text{Col}_3(n) \) and \( \text{Col}_3(n) \) of the monoid \( \text{P}_n \).

| \( n \) | \( \text{Col}_1(n) \) | \( \text{Knuth}_2(n) \) | \( \text{Knuth}^\text{KB}_2(n) \) | \( \text{Col}_2(n) \) | \( \text{Knuth}^\text{KB}_3(n) \) | \( \text{Knuth}_3(n) \) | \( \text{Col}_3(n) \) | \( \text{Col}_3(n) \) |
|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2 | 3 | 2 | 2 | 3 | 1 | 1 | 1 | 1 |
| 3 | 7 | 8 | 11 | 22 | 27 | 24 | 34 | 42 |
| 4 | 15 | 20 | \( \infty \) | 115 | \( \infty \) | 242 | 330 | 621 |
| 5 | 31 | 40 | \( \infty \) | 531 | \( \infty \) | 1726 | 2225 | 6893 |
| 6 | 63 | 70 | \( \infty \) | 2317 | \( \infty \) | 10273 | 12635 | 67635 |
| 7 | 127 | 112 | \( \infty \) | 9822 | \( \infty \) | 55016 | 65282 | 623010 |
| 8 | 255 | 168 | \( \infty \) | 40971 | \( \infty \) | 275868 | 318708 | 5534197 |
| 9 | 511 | 240 | \( \infty \) | 169255 | \( \infty \) | 1324970 | 1500465 | 48052953 |
| 10 | 1023 | 330 | \( \infty \) | 694837 | \( \infty \) | 6178939 | 6892325 | 410881483 |

Table 1: Number of cells of \((3,1)\)-polygraphs \( \text{Knuth}_3(n), \text{Col}_3(n) \) and \( \text{Col}_3(n) \), for \( 1 \leq n \leq 10 \).

\subsection*{4.4.11. Actions of plactic monoids on categories.}
In \[6\], the authors give a description of the category of actions of a monoid on categories in terms of coherent presentations. Using this description, Theorem \[4.4.7\] allows to present actions of plactic monoids on categories as follows. The category \( \text{Act}(\text{P}_n) \) of actions of the monoid \( \text{P}_n \) on categories is equivalent to the category of 2-functors from the \((2,1)\)-category \( \text{Knuth}_2(n)^\top \) to the category \( \text{Cat} \) of categories, that sends the 3-cells of \( \text{Knuth}_3(n) \) to commutative diagrams in \( \text{Cat} \).
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