The Prime Index Graph of a Group

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Abstract

Let $G$ be a group. The prime index graph of $G$, denoted by $\Pi(G)$, is the graph whose vertex set is the set of all subgroups of $G$ and two distinct comparable vertices $H$ and $K$ are adjacent if and only if the index of $H$ in $K$ or the index of $K$ in $H$ is prime. In this paper, it is shown that for every group $G$, $\Pi(G)$ is bipartite and the girth of $\Pi(G)$ is contained in the set $\{4, \infty\}$. Also we prove that if $G$ is a finite solvable group, then $\Pi(G)$ is connected.

1 Introduction

Let $\Gamma$ be a graph. We say that $\Gamma$ is connected if there is a path between any two distinct vertices of $\Gamma$. We denote by $d(v)$, the degree of a vertex $v$ in $\Gamma$. A graph in which every vertex has the same degree is called a regular graph. If all vertices have degree $k$, then the graph is said to be $k$-regular. The girth of $\Gamma$, denoted by $gr(\Gamma)$, is the length of a shortest cycle in $\Gamma$ (We say that $gr(\Gamma) = \infty$ if $\Gamma$ contains no cycle). A null graph is a graph with no edges. A forest is a graph with no cycle. We denote the complete graph, the path and the cycle of order $n$ by $K_n$, $P_n$ and $C_n$, respectively. We use $n$-cycle to denote the cycle of order $n$, where $n \geq 3$. The Cartesian product of two graphs $\Gamma$ and $\Omega$ is denoted by $\Gamma \boxtimes \Omega$. The hypercube graph $Q_s$ is the Cartesian product of $s$ copies of $P_2$.

Let $G$ be a group. We denote the identity element of $G$ by $e$. The derived subgroup of $G$ is denoted by $G'$ and $G^{(n+1)} = (G^{(n)})'$, where $n$ is a positive integer. For any subgroup $H$ of $G$, the intersection of all the conjugates of $H$ in $G$ is denoted by $Core_G(H)$. Let $x \in G$. Then the subgroup generated by $x$ is denoted by $\langle x \rangle$. As usual, $\mathbb{Z}_n$, $A_n$ and $S_n$ denote the group of integers modulo $n$, the alternating group and the symmetric group of degree $n$, respectively. For a fixed prime $p$, the quasicyclic $p$-group is denoted by $\mathbb{Z}(p^\infty)$. Also the projective special linear group of degree $n$ over the field $\mathbb{Z}_p$ is denoted by $\text{PSL}(n, p)$.

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There are several graphs associated with groups, for instance non-commuting graph of a group, intersection graph of subgroups of a group, and subgroup graph of a group. (See [1, 2, 3].) The subgroup graph of a group $G$ is defined as the graph of its lattice of subgroups, that is, the graph whose vertices are the subgroups of $G$ such that two subgroups $H$ and $K$ are adjacent if one of $H$ or $K$ is maximal in the other. In this article, we introduce and investigate the prime index graph of $G$, denoted by $\Pi(G)$. It is an undirected graph whose vertices are all subgroups of $G$ and two distinct comparable vertices $H$ and $K$ are adjacent if and only if $[H : K]$ or $[K : H]$ is prime. Clearly, the prime index graph of $G$ is a subgraph of the subgroup graph of $G$ and whenever $G$ is a nilpotent group, see [12, p.143], then these two graphs are coincide. In follows the prime index graphs of $S_3$ and $A_4$ are given. Note that $H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, for $i = 1, 2, 3$ and $L_j \cong \mathbb{Z}_3$, for $j = 1, \ldots, 4$.

Here we show that for every group $G$, $\Pi(G)$ is a bipartite graph and $gr(\Pi(G)) \in \{4, \infty\}$. We prove that for any finite abelian group $G$, $\Pi(G)$ is a regular graph if and only if $\Pi(G)$ is a hypercube graph. Finally, we study the connectivity of $\Pi(G)$ and we show that for every finite solvable group $G$, $\Pi(G)$ is a connected graph. Among other results, we prove that if $\Pi(G)$ is a connected graph and $N$ is a normal subgroup of $G$, then both graphs $\Pi(N)$ and $\Pi(G/N)$ are connected.

2 The Prime Index Graphs are Bipartite

In this section, we show that the prime index graph of a group $G$ is bipartite. To see this, we prove a stronger result. First we define a directed graph $\Gamma(G)$. It is a directed graph whose vertex set is the set of all subgroups of $G$ and for every two distinct vertices $H$ and $K$, there is an arc from $H$ to $K$, whenever $H \subseteq K$ and $[K : H] = r$, for some positive integer $r$. Suppose that $r$ is the weight of the arc from $H$ to $K$.

**Theorem 1.** Let $C$ be a cycle of $\Gamma(G)$. Then the product of weights of all clockwise arcs of $C$ is equal to the product of weights of all counter-clockwise arcs of $C$.

**Proof.** Let $C$ be a cycle of $\Gamma(G)$ of length $n$. We prove the theorem by induction on $n$. Clearly, for $n = 3$ the assertion holds. Now, suppose that $n > 3$ and the assertion is true for every integer $m$,
3 ≤ m < n. If C contains a directed path P of length 2, such as \( H \xrightarrow{r} K \xrightarrow{p} L \), then we replace P with the path \( H \xrightarrow{r'} L \). Hence by the induction hypothesis the result holds. Otherwise, C contains a path of the form \( H \xrightarrow{r} K \xrightarrow{p} L \xrightarrow{r} M \). We consider two cases:

**Case 1.** If \( H \cap L \) is not a vertex of C, then we replace \( H \xrightarrow{r} K \xrightarrow{p} L \) with the path \( H \xrightarrow{r'} H \cap L \xrightarrow{r} L \), where \([L : H \cap L] = r' \) and \([H : H \cap L] = s' \). Note that \( s'r = r's \) and so \( r/s = r'/s' \). Next, we replace \( H \cap L \xrightarrow{r} L \xrightarrow{s} M \) with \( H \cap L \xrightarrow{r} M \). Thus we find a cycle \( C_1 \) of length \( n - 1 \) and by the induction hypothesis \( r'ta = s'b \), where a is the product of weights of all clockwise arcs of \( C_1 \) except the weight of \( H \cap L \xrightarrow{r} M \) and b is the product of weights of all counter-clockwise arcs of \( C_1 \) except the weight of \( H \xrightarrow{r'} \). Hence \( r/s = s'/s \) and so \( r'ta = sb \). It is clear that \( r'ta \) is the product of weights of all clockwise arcs of \( C \) and \( s'b \) is the product of weights of all counter-clockwise arcs of \( C \). The result holds.

**Case 2.** Assume that \( H \cap L \) is a vertex of C. Clearly, \( H \cap L \neq H \) or \( H \cap L \neq L \). With no loss of generality, suppose that \( H \cap L \neq H \). By adding the arc \( H \cap L \xrightarrow{s} H \), we find two cycles \( C_1 \) and \( C_2 \) of length less than \( n \). Let \([H : H \cap L] = s' \). Assume that the arc \( H \cap L \xrightarrow{s} H \) is a clockwise arc of \( C_1 \). So \( H \cap L \xrightarrow{s'} H \) is a counter-clockwise arc of \( C_2 \). Now, by the induction hypothesis, \( s' = b_1/a_1 = a_2/b_2 \), where \( a_1 \) is the product of weights of all clockwise arcs of \( C_1 \) except the weight of \( H \cap L \xrightarrow{s} H \), \( a_2 \) is the product of weights of all counter-clockwise arcs of \( C_1 \), and \( b_2 \) is the product of weights of all counter-clockwise arcs of \( C_2 \) except the weight of \( H \cap L \xrightarrow{s} H \). Thus \( a_1a_2 = b_1b_2 \). The proof is complete.

\[ \square \]

Now, we are in a position to prove the following corollary.

**Corollary 1.** Let \( G \) be a group. Then \( \Pi(G)\ ) is bipartite.

**Proof.** We show that every cycle of \( \Pi(G)\ ) is an even cycle. If \( \Pi(G)\ ) has a cycle \( C \), we may assume that \( C \) is a cycle in \( \overline{\Pi}(G)\ ). Now, by Theorem \([\square] \), since all weights are primes, the number of clockwise arcs of \( C \) is equal to the number of counter-clockwise arcs of \( C \). Hence \( C \) is an even cycle. This implies that \( \Pi(G)\ ) is a bipartite graph.

\[ \square \]

If \( G \) is a non-trivial group and \( e \neq x \in G \), then \( \langle x \rangle \) contains a subgroup of prime index and hence \( d(\langle x \rangle) \geq 1 \). So \( \Pi(G)\ ) is not a null graph.

**Lemma 1.** Let \( G \) be a group. Then \( \Pi(G)\ ) is a complete bipartite graph if and only if \( G \) is a cyclic group of prime order or \( |G| = pq \), for some primes \( p \) and \( q \).

**Proof.** Clearly, if \( G \cong \mathbb{Z}_p \), then \( \Pi(G)\ ) \cong K_2 \). Also if \( |G| = pq \), then \( \Pi(G)\ ) is a complete bipartite graph whose one part contains all subgroups of \( G \) of orders \( p \) or \( q \) and the other part contains \( \{e\} \) and \( G \). Conversely, assume that \( \Pi(G)\ ) is complete bipartite. If \( \{e\} \) and \( G \) are contained in two different parts of \( \Pi(G)\ ), then \( G \cong \mathbb{Z}_p \), where \( p \) is a prime number. Otherwise, there exists a subgroup \( H \) of \( G \) adjacent to both \( \{e\} \) and \( G \). Thus \( |G| = pq \), for some primes \( p \) and \( q \).
The following theorem shows that if \( \Pi(G) \) contains a cycle \( C \), then \( gr(\Pi(G)) = 4 \).

**Theorem 2.** Let \( G \) be a group. Then \( gr(\Pi(G)) \in \{4, \infty\} \).

**Proof.** First assume that \( G \) is finite and \( |G| = p_1^{n_1} \cdots p_s^{n_s} \), where \( p_1, \ldots, p_s \) are distinct primes and \( n_1, \ldots, n_s \) are positive integers. Suppose that \( L_i \) is a Sylow \( p_i \)-subgroup of \( G \), for \( i = 1, \ldots, s \). First assume that \( L_i \) contains two distinct maximal subgroups \( H \) and \( K \), for some \( i \). Since \( H \) and \( K \) are normal subgroups of \( L_i \), so \( HK = L_i \). This implies that \( |H \cap K| = p_i^{n_i-2} \) and hence \( L_i - H - H \cap K - K - L_i \) is a 4-cycle in \( \Pi(G) \). So by Corollary 1 \( gr(\Pi(G)) = 4 \). Next, assume that \( L_i \) contains a unique maximal subgroup, for \( i = 1, \ldots, s \). Hence all Sylow subgroups of \( G \) are cyclic. Now, by [10, Theorem 10.26], \( G \) is a supersolvable group. If \( s \geq 2 \), then \( G \) has a subgroup \( K \) of order \( p_1p_2 \) ([11, p.292]). Let \( H_i \) be a subgroup of \( K \) of order \( p_i \), for \( i = 1,2 \). Hence \( \{e\} - H_1 - K - H_2 - \{e\} \) is a 4-cycle in \( \Pi(G) \) and so by Corollary 1 \( gr(\Pi(G)) = 4 \). If \( s = 1 \), then \( G \cong Z_{p_1^{n_1}} \). Thus \( \Pi(G) \cong P_{n_{i+1}} \) and \( gr(\Pi(G)) = \infty \).

Now, suppose that \( G \) is infinite and \( \Pi(G) \) contains a cycle \( C \). It is easy to see that \( C \) should contain a path of the form \( M - H - N \), where \( H, M \) and \( N \) are subgroups of \( G \) and furthermore \( M \) and \( N \) are maximal subgroups of \( H \). If both \( M \) and \( N \) are normal subgroups of \( H \), then \( [M : M \cap N] = [MN : N] = [H : N] \) and similarly \( [N : M \cap N] = [H : M] \). Thus \( H - M - M \cap N - N - H \) is a 4-cycle in \( \Pi(G) \).

Now, assume that \( M \) is not a normal subgroup of \( H \). Then \( M - H - xMx^{-1} \) is a path in \( \Pi(G) \), for some \( x \in G \). Therefore, \( M/\text{Core}_H(M) - H/\text{Core}_H(M) - xMx^{-1}/\text{Core}_H(M) \) is a path in \( \Pi(H/\text{Core}_H(M)) \).

Clearly, \( H/\text{Core}_H(M) \) is a finite group which is not a cyclic \( p \)-group. So by the previous paragraph, \( gr(\Pi(H/\text{Core}_H(M))) = 4 \) and hence \( gr(\Pi(G)) = 4 \). \( \square \)

By the proof of the previous theorem, we have the following corollary.

**Corollary 2.** If \( G \) is a finite group or an infinite abelian group, then \( \Pi(G) \) is a forest if and only if \( G \) is isomorphic to either \( Z_{p^n} \) or \( Z(p^\infty) \), where \( p \) is a prime and \( n \) is a positive integer.

**Proof.** Suppose that \( \Pi(G) \) is a forest. If \( G \) is finite, then by the proof of Theorem 2 \( G \cong Z_{p^n} \), for some prime number \( p \) and positive integer \( n \). If \( G \) is an infinite abelian group, then \( G \) is a torsion \( p \)-group. (Note that \( gr(\Pi(Z)) = 4 \) and if \( G \) has two elements of orders \( p \) and \( q \), then \( Z_{pq} \) is a subgroup of \( G \), where \( p, q \) are distinct primes.) Also by the proof of Theorem 2 every finite subgroup of \( G \) is cyclic. Thus \( G \) has no non-trivial direct summand. Now, by [11, p.110], \( G \cong Z(p^\infty) \), for some prime \( p \). Clearly, \( \Pi(Z(p^\infty)) \) is a disjoint union of an isolated vertex and an infinite path. The proof is complete. \( \square \)

In the following theorem, we consider the prime index graph of cyclic groups.

**Theorem 3.** Let \( n = p_1^{n_1} \cdots p_s^{n_s} \), where \( p_1, \ldots, p_s \) are distinct primes and \( n_1, \ldots, n_s \) are positive integers. Then \( \Pi(Z_n) \cong P_{n_{i+1}} \square \cdots \square P_{n_{i+1}} \).

**Proof.** We know that \( Z_n \cong Z_{p_1^{\alpha_1}} \times \cdots \times Z_{p_s^{\alpha_s}} \). If \( H \) and \( K \) are two distinct subgroups of \( Z_n \), then \( H \cong Z_{p_1^{\beta_1}} \times \cdots \times Z_{p_s^{\beta_s}} \) and \( K \cong Z_{p_1^{\beta_1}} \times \cdots \times Z_{p_s^{\beta_s}} \), where \( 0 \leq \alpha_i, \beta_i \leq n_i \) for \( i = 1, \ldots, s \). So \( H \) and \( K \) are
adjacent if and only if there exists an integer \( j, 1 \leq j \leq s \), such that \( \alpha_i = \beta_j \) for \( i \neq j \) and \( \alpha_j = \beta_j \pm 1 \). Thus \( \Pi(Z_n) \cong \Pi(Z_{p_1^{n_1}}) \square \cdots \square \Pi(Z_{p_s^{n_s}}) \) and \( \Pi(Z_n) \cong P_{n_1+1} \square \cdots \square P_{n_s+1} \). \( \square \)

**Theorem 4.** Let \( G \) be a finite abelian group. If \( \Pi(G) \) is regular, then \( G \cong \mathbb{Z}_{p_1 \cdots p_s} \) and \( \Pi(G) \cong Q_s \), where \( p_1, \ldots, p_s \) are distinct prime numbers.

**Proof.** Let \( |G| = p_1^{\alpha_1} \cdots p_s^{\alpha_s} \) where \( p_1, \ldots, p_s \) are distinct primes and \( \alpha_1, \ldots, \alpha_s \) are positive integers. Assume that \( G \cong \mathbb{Z}_{p_1^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p_1^{\alpha_1}} \times \cdots \times \mathbb{Z}_{p_s^{\alpha_s}} \times \cdots \times \mathbb{Z}_{p_s^{\alpha_s}}, \) where \( k_i \) is a positive integer and \( \alpha_i + \cdots + \alpha_i = \alpha_i \), for \( i = 1, \ldots, s \). We claim that \( k_i = 1 \) for each \( i, 1 \leq i \leq s \). By contradiction assume that \( k_i \neq 1 \), for some \( i, 1 \leq i \leq s \). Let \( n(k_i, p_i) \) be the number of subgroups of order \( p_i \) in \( \mathbb{Z}_{p_i^{\alpha_i}} \times \cdots \times \mathbb{Z}_{p_i^{\alpha_i}} \). Obviously, the number of subgroups of order \( p_i \) in two groups \( \mathbb{Z}_{p_i^{\alpha_i}} \times \cdots \times \mathbb{Z}_{p_i^{\alpha_i}} \) and \( \mathbb{Z}_{p_i^{\alpha_i}} \) is the same. Hence by [3] p.59, we have \( n(k_i, p_i) = (p_i^{k_i} - 1)/(p_i - 1) \). Clearly, \( d(Z_{p_i^{\alpha_i}}) = 1 + n(k_i, p_i) + \sum_{j \neq i} n(k_j, p_j) \) and \( d(\{e\}) = \sum_{j=1}^{s} n(k_j, p_j) \). Since \( \Pi(G) \) is a regular graph, so \( n(k_i, p_i) = 1 + n(k_i, p_i) \). This implies that \( p_i^{k_i-1} = 1 \) and hence \( k_i = 1 \), a contradiction. The claim is proved. Thus \( G \) is a cyclic group of order \( p_1^{\alpha_1} \cdots p_s^{\alpha_s} \) and by Theorem [3] \( \Pi(G) \cong P_{\alpha_1+1} \square \cdots \square P_{\alpha_s+1} \). Now, since \( \Pi(G) \) is a regular graph, \( \alpha_1 = \cdots = \alpha_s = 1 \) and \( \Pi(G) \cong Q_s \). \( \square \)

**Theorem 5.** Let \( G \) be a finite group. If \( \Pi(G) \) is a 2-regular graph, then \( \Pi(G) \cong C_4 \) and \( G \cong \mathbb{Z}_{pq} \), where \( p \) and \( q \) are distinct primes.

**Proof.** Since \( d(\{e\}) = 2 \), the order of \( G \) has at most two distinct prime divisors. Clearly, a \( p \)-group cannot have exactly two subgroups of order \( p \). So assume that \( p \) and \( q \) are two distinct prime divisors of \( |G| \). Suppose that \( H \) and \( K \) are subgroups of \( G \) such that \( |H| = p \) and \( |K| = q \). Since \( d(\{e\}) = 2 \), \( H \) and \( K \) are normal subgroups of \( G \). Hence \( HK \) is a subgroup of \( G \) and \( \{e\} = H - HK - K = \{e\} \) is a cycle in \( \Pi(G) \). Now since \( \Pi(G) \) is a 2-regular graph, \( H \) is a Sylow \( p \)-subgroup and \( K \) is a Sylow \( q \)-subgroup of \( G \). Thus \( G = HK \cong \mathbb{Z}_{pq} \) and \( \Pi(G) \cong C_4 \). \( \square \)

### 3 Connectivity

In this section, we study those groups whose prime index graphs are connected. First we have the following lemma.

**Lemma 2.** Let \( G \) be an infinite group. Then \( \Pi(G) \) is not connected. Moreover, if \( G \) is a simple group, then \( G \) is an isolated vertex in \( \Pi(G) \).

**Proof.** It is clear that if \( G \) is an infinite group, then there is no path between \( \{e\} \) and \( G \) in \( \Pi(G) \). So \( \Pi(G) \) is not connected. If \( G \) is an infinite simple group, by [10] Corollary 4.15, \( G \) cannot have a proper subgroup of finite index. Hence \( G \) is an isolated vertex of \( \Pi(G) \). \( \square \)
By [10, p.292], a finite group $G$ is supersolvable if and only if each subgroup of $G$ satisfies the converse of Lagrange’s Theorem. So for finite supersolvable groups $G$ such as finite abelian groups and finite $p$-groups, $\Pi(G)$ is connected. (Note that every subgroup of $G$ is connected to $\{e\}$.)

**Theorem 6.** Let $G$ be a finite group and $N$ be a normal subgroup of $G$. If $\Pi(N)$ is a connected graph and also for every subgroup $H/N$ of $G/N$, $\Pi(H/N)$ is a connected graph, then $\Pi(G)$ is connected.

**Proof.** Assume that $H$ is a subgroup of $G$. Hence $\Pi(HN/N)$ is a connected graph. Since $HN/N \cong H/(H \cap N)$, so the graph $\Pi(H/H \cap N)$ is connected. This implies that there is a path between $H$ and $H \cap N$ in $\Pi(G)$. Now, since $\Pi(N)$ is connected, there is a path between $H \cap N$ and $\{e\}$ in $\Pi(N)$. Thus every subgroup of $G$ is connected to $\{e\}$. Therefore $\Pi(G)$ is connected. $\square$

Now, we prove that the prime index graph of every finite solvable group is connected.

**Theorem 7.** Let $G$ be a finite solvable group. Then $\Pi(G)$ is connected.

**Proof.** Since $G$ is a solvable group, $G^{(n)} = \{e\}$, for some positive integer $n$. We prove the theorem by applying the induction on $n$. If $G' = \{e\}$, then $G$ is an abelian group and so $\Pi(G)$ is a connected graph. Assume that $n > 1$ and $G^{(n)} = \{e\}$. By the induction hypothesis, $\Pi(G')$ is connected. Now, by Theorem 6, $\Pi(G)$ is a connected graph. $\square$

If $G$ is a group of odd order, then $G$ is solvable (Feit-Thompson Theorem [5]) and by Theorem 4, $\Pi(G)$ is connected. Moreover, suppose that $|G| = 2^m n$, where $m$ and $n$ are positive integers with $m$ odd. If $G$ has a cyclic Sylow 2-subgroup, then by [4, p.148], $G$ has a normal subgroup of order $m$ and hence $G$ is a solvable group. Thus $\Pi(G)$ is a connected graph. Since every subgroup of a solvable group is solvable, by Theorems 6 and 7, we have the next result.

**Corollary 3.** Let $G$ be a finite group and $N$ be a normal subgroup of $G$. If $\Pi(N)$ is a connected graph and $G/N$ is a solvable group, then $\Pi(G)$ is connected.

**Theorem 8.** Let $G$ be a group and $N$ be a normal subgroup of $G$. If $\Pi(G)$ is a connected graph, then $\Pi(N)$ and $\Pi(G/N)$ are connected graphs.

**Proof.** First we prove that $\Pi(N)$ is a connected graph. Let $H$ and $K$ be two distinct subgroups of $N$. Since $\Pi(G)$ is a connected graph, so there is a path $H - L_1 - \cdots - L_t - K$ from $H$ to $K$ in $\Pi(G)$. We claim that by removing the same consecutive vertices in $H - L_1 \cap N - \cdots - L_t \cap N - K$ and keeping one of them we obtain a walk from $H$ to $K$ in $\Pi(N)$, With no loss of generality, assume that $L_t \subseteq L_{t+1}$ and $[L_{t+1} : L_t] = p$, for some prime number $p$. Thus $L_t \cap N \subseteq L_{t+1} \cap N$ and we have

$$[L_{t+1} \cap N : L_t \cap N] = \frac{|L_{t+1} \cap N|}{|L_t \cap N|} = \frac{|L_t N|}{|L_{t+1} N|} \frac{|L_{t+1}|}{|L_t|}.$$
Hence \[[L_{i+1}N : L_iN][L_{i+1} \cap N : L_i \cap N] = p\]. Therefore \(L_{i+1} \cap N = L_i \cap N\) or \([L_{i+1}N : L_iN] = p\). So the claim is proved. Hence there is a path from \(H\) to \(K\) in \(\Pi(N)\) which implies that \(\Pi(N)\) is connected.

Next, assume that \(H\) and \(K\) are two distinct subgroups of \(G\) containing \(N\). Suppose that \(H - L_1 - \cdots - L_t - K\) is a path from \(H\) to \(K\) in \(\Pi(G)\). Similar to the previous case, one can prove that \(H/N - L_1N/N - \cdots - L_tN/N - K/N\) is a walk from \(H/N\) to \(K/N\) in \(\Pi(G/N)\). Thus \(\Pi(G/N)\) is also a connected graph. □

Now, we propose the following problem.

**Problem.** Let \(G\) be a group and \(N\) be a normal subgroup of \(G\). If \(\Pi(N)\) and \(\Pi(G/N)\) are both connected, then is it true that \(\Pi(G)\) is connected?

By Theorem 8, we have the next corollary.

**Corollary 4.** Let \(G \cong H \times K\), for some groups \(H\) and \(K\). If \(\Pi(G)\) is connected, then both \(\Pi(H)\) and \(\Pi(K)\) are connected.

We close this article by the study of the connectivity of \(\Pi(A_n)\) and \(\Pi(S_n)\). Moreover, we show that the prime index graph of all groups up to 500 elements is connected except for \(A_6\).

**Remark.** Let \(n\) be a positive integer. Then \(\Pi(A_n)\) is connected if and only if \(n \leq 5\). Also, \(\Pi(S_n)\) is a connected graph if and only if \(n \leq 5\). To prove the remark first assume that \(n \leq 4\). Hence \(A_n\) is a solvable group and by Theorem 7, \(\Pi(A_n)\) is a connected graph. If \(n = 5\), we know that every proper subgroup of \(A_5\) is solvable and \(A_5\) contains a maximal subgroup of prime index, then \(\Pi(A_5)\) is connected.

Also if \(n \leq 5\), since \(A_n\) is a normal subgroup of \(S_n\) and \(\Pi(A_n)\) is connected, by Corollary 3 \(\Pi(S_n)\) is connected. Now, assume that \(n > 5\). If \(n\) is not a prime number, then by [6, p.305], \(A_n\) has no subgroup of prime index and hence \(A_n\) is an isolated vertex of \(\Pi(A_n)\). Otherwise, if \(H\) is a maximal subgroup of \(A_n\) of prime index, then \(H \cong A_{n-1}\) (see [6, p.305]). Since \(n - 1\) is not a prime number, so \(\Pi(A_n)\) is not connected. Thus by Theorem 8 \(\Pi(S_n)\) is not connected.

**Theorem 9.** Let \(G\) be a group and \(|G| \leq 500\). If \(\Pi(G)\) is not connected, then \(G \cong A_6\).

**Proof.** Suppose that \(G\) is the smallest group such that \(\Pi(G)\) is not a connected graph. By Theorem 6 one can see that \(G\) is a simple group. Note that by the remark, \(\Pi(A_6)\) is not a connected graph.

On the other hand by [11, p.295], if \(G\) is a non-abelian simple group of order at most 500, then \(G\) is isomorphic to one of the groups \(A_5\), \(PSL(2,7)\), or \(A_6\). By remark, \(\Pi(A_5)\) is connected. Also by [13, Theorem 6.26], \(PSL(2,7)\) contains a maximal subgroup of index 7 and by [11, p.191], all subgroups of \(PSL(2,7)\) are solvable. Hence \(\Pi(PSL(2,7))\) is connected. Thus \(G \cong A_6\). Finally, for every non-abelian group \(G\) with \(360 < |G| \leq 500\), since \(G\) is not a simple group, so \(G\) has a non-trivial proper normal subgroup \(N\). Clearly, \(|N|\) and \(|G/N|\) are both less than 360. Thus by Theorem 6 \(\Pi(G)\) is a connected graph. □
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