Raising type twisted Pieri formulas for Jack polynomials and their applications to interpolation Jack polynomials

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Abstract

We propose new Pieri type formulas for Jack polynomials, which is another kind of Pieri type formulas than the ones in the previous paper (G. Shibukawa, arXiv:2004.12875). From these new Pieri type formulas, we give yet another proof of difference and Pieri formulas for interpolation Jack polynomials. Further, we also generalize a falling type twisted Pieri formula to the binomial type polynomials including Jack and multivariate Bernoulli polynomials.

1 Introduction

Let $r$ be a positive integer and $d \neq 0$ be a complex parameter. We denote the symmetric group of degree $r$ by $S_r$ and partition set of the length $\leq r$ by

$$\mathcal{P} := \{ \mathbf{m} = (m_1, \ldots, m_r) \in \mathbb{Z}^r | m_1 \geq \cdots \geq m_r \geq 0 \}.$$ 

For any partition $\mathbf{m} = (m_1, \ldots, m_r) \in \mathcal{P}$ and the variables $z = (z_1, \ldots, z_r)$, the Jack polynomials $P_m(z; \frac{d}{2})$ are a family of homogeneous symmetric polynomials defined by the following two conditions [M], [St], [VK]:

\begin{align*}
(1) \quad & D(z)P_m\left(z; \frac{d}{2}\right) = P_m\left(z; \frac{d}{2}\right) \sum_{j=1}^{r} m_j (m_j - 1 + d(r - j)), \\
(2) \quad & P_m\left(z; \frac{d}{2}\right) = \sum_{k \leq m} c_{m_k} m_k(z), \quad c_{m_k} \in \mathbb{Q}(d), \quad c_{mm} = 1.
\end{align*}
Here $D(z)$ is the second-order differential operator
\[ D(z) := \sum_{j=1}^{r} z_j^2 \partial^2 z_j + d \sum_{1 \leq j \neq l \leq r} \frac{z_j^2}{z_j - z_l} \partial z_j, \]
m$_{k}(z)$ is the monomial symmetric polynomial
\[ m_{k}(z) := \sum_{n \in \mathcal{S}_r k} z^n \]
and $k \leq m$ is the dominance order
\[ k \leq m \iff \begin{cases} \sum_{i=1}^{k_l} k_i \leq \sum_{i=1}^{m_l} m_i & (i = 1, \ldots, r - 1) \\ k_1 + \cdots + k_r = m_1 + \cdots + m_r \end{cases} \].

Similarly, the interpolation Jack polynomials (or shifted Jack polynomials) $P_{m}^{ip}(z; \frac{d}{2})$ are a family of homogeneous symmetric polynomials defined by the following two conditions [Sa1], [KS], [OO]:
\begin{align*}
(1)^{ip} P_{k}^{ip} \left( m + \frac{d}{2}; \frac{d}{2} \right) &= 0, & \text{unless } k \subseteq m \in \mathcal{P} \\
(2)^{ip} P_{m}^{ip}(z; \frac{d}{2}) &= P_{m}(z; \frac{d}{2}) + \text{(lower degree terms)}. 
\end{align*}

Here $\delta$ denotes the staircase partition $(r - 1, r - 2, \ldots, 2, 1, 0)$ and $k \subseteq m$ is the inclusion partial order defined by
\[ k \subseteq m \iff k_i \leq m_i \quad i = 1, \ldots, r. \]

For convenience, we introduce two kinds of normalization for Jack polynomials,
\begin{align*}
\Phi_{m}^{(d)}(z) &:= \frac{P_{m}(z; \frac{d}{2})}{P_{m}(1; \frac{d}{2})}, \\
\Psi_{m}^{(d)}(z) &:= \frac{P_{m}(z; \frac{d}{2})}{P_{m}^{ip}(m + \frac{d}{2}; \frac{d}{2})} = \frac{P_{m}(1; \frac{d}{2})}{P_{m}^{ip}(m + \frac{d}{2}; \frac{d}{2})} \Phi_{m}^{(d)}(z),
\end{align*}
where $1 := (1, \ldots, 1)$.

In the previous paper [Sh2], we gave some new Pieri type formulas for Jack polynomials, which we call twisted Pieri formulas. These formulas explicitly express the action of twisted Sekiguchi operators
\[ \frac{(ad \partial z)^l}{l!} S_{l}^{(d)}(u; z) \quad (l = 0, 1, \ldots, r) \quad (1.1) \]
on the Jack polynomials $\Phi^{(d)}_m(z)$ or $\Psi^{(d)}_m(z)$. Here

$$|\partial_z| := \sum_{j=1}^r \partial z_j, \quad \partial z_j := \frac{\partial}{\partial z_j}$$

and

$$(\text{ad} A)(B) := AB - BA,$$

and $S^{(d)}_r(u;z)$ is the Sekiguchi operator defined by

$$S^{(d)}_r(u;z) := \sum_{p=0}^r H^{(d)}_{r,p}(z) u^{-p},$$

$$H^{(d)}_{r,p}(z) := \sum_{l=0}^p \left( \frac{2}{d} \right)^{p-l} \sum_{J \subseteq [r], |J| = l} \left( \prod_{i \in J} \Delta z_i \right) \sum_{J' \subseteq [r] \setminus J, |J'| = p-l} \left( \prod_{j \in J'} z_j \partial z_j \right)$$

where $[r] = \{1, 2, \ldots, r\}$ and

$$\Delta(z) := \prod_{1 \leq i < j \leq r} (z_i - z_j).$$

Twisted Pieri formulas for Jack polynomials \text{(Sh2)} For $l = 0, 1, \ldots, r$, we have

$$\left[ \frac{(\text{ad} |\partial_z|)^l}{l!} S^{(d)}_r(u;z) \right] \Phi^{(d)}_x(z) = \sum_{J \subseteq [r], |J| = l} \Phi^{(d)}_{x^{-\epsilon_J}}(z) I^{(d)}_{J^c}(u;x) A^{(d)}_{-\epsilon_J}(x) \prod_{j \in J} \left( x_j + \frac{d}{2} (r - j) \right), \quad (1.2)$$

$$\left[ \frac{(\text{ad} |\partial_z|)^l}{l!} S^{(d)}_r(u;z) \right] \Psi^{(d)}_x(z) = \sum_{J \subseteq [r], \epsilon_J \setminus J = l, x^{-\epsilon_J} \in \mathcal{P}} \Psi^{(d)}_{x^{-\epsilon_J}}(z) I^{(d)}_{J^c}(u;x) A^{(d)}_{+\epsilon_J}(x - \epsilon_J), \quad (1.3)$$

where $J^c := [r] \setminus J$, $\epsilon_J := \sum_{j \in J} \epsilon_j$, $\epsilon_j := (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{Z}^r$ and

$$A^{(d)}_{\pm \epsilon_J}(x) := \prod_{j \in J, l \in J^c} \frac{x_j - x_l - \frac{d}{2} (j - l) \pm \frac{d}{2}}{x_j - x_l - \frac{d}{2} (j - l)},$$
Further, by comparing the coefficients for \( u^r l \) of the twisted Pieri (1.2) and (1.3), we also derived
\[
\left( \frac{d}{2} \right) ^l \left[ \frac{(ad|\partial_k)|^l}{l!} H^{(d)}_{r,l}(z) \right] \Phi^{(d)}_x(z) = \sum_{J \subseteq [r], |J| = l, x-\epsilon_J \in \mathcal{P}} \Phi^{(d)}_{x-\epsilon_J}(z) A^{(d)}_{-\epsilon_J}(x) \cdot \prod_{j \in J} \left( x_j + \frac{d}{2} (r - j) \right), \tag{1.4}
\]
\[
\left( \frac{d}{2} \right) ^l \left[ \frac{(ad|\partial_k)|^l}{l!} H^{(d)}_{r,l}(z) \right] \Psi^{(d)}_x(z) = \sum_{J \subseteq [r], |J| = l, x-\epsilon_J \in \mathcal{P}} \Psi^{(d)}_{x-\epsilon_J}(z) A^{(d)}_{+\epsilon_J}(x - \epsilon) \tag{1.5}
\]
for any \( z \in \mathbb{C}^r \) and \( l = 0, 1, \ldots, r \).

These formulas (1.2), (1.3) and (1.4), (1.5) are natural generalizations of the relations for Sekiguchi operators \[ \text{Se}, \text{D}, \text{M} \]
\[
H^{(d)}_{r,p}(z) P_m \left( z; \frac{d}{2} \right) = P_m \left( z; \frac{d}{2} \right) e_{r,p} \left( m + \frac{d}{2} \delta \right), \tag{1.6}
\]
\[
S^{(d)}_r(u; z) P_m \left( z; \frac{d}{2} \right) = P_m \left( z; \frac{d}{2} \right) I^{(d)}_r(u; m), \tag{1.7}
\]
where \( e_{r,k}(z) \) is the elementary symmetric polynomial
\[
e_{r,k}(z) := \sum_{1 \leq i_1 < \cdots < i_k \leq r} z_{i_1} \cdots z_{i_k} \quad (k = 1, \ldots, r), \quad e_{r,0}(z) := 1
\]
and
\[
I^{(d)}_r(u; m) := \prod_{k=1}^r \left( u + r - k + \frac{2}{d} m_k \right) = \left( \frac{2}{d} \right) ^r \prod_{k=1}^r \left( m_k + \frac{d}{2} (u + r - k) \right).
\]
Since the first-order Sekiguchi operator \( H^{(d)}_{r,1}(z) \) equals to the Euler operator \( \sum_{i=1}^r z_i \partial_{z_i} \) essentially
\[
H^{(d)}_{r,1}(z) = \frac{2}{d} \sum_{i=1}^r z_i \partial_{z_i} + \frac{r(r - 1)}{2}
\]
and
\[(\text{ad } |\partial_z|) H^{(d)}_{r,1}(z) = (\text{ad } |\partial_z|) \left( \frac{2}{d} \sum_{i=1}^{r} z_i \partial_{z_i} \right) = \frac{2}{d} |\partial_z|,\]
the formulas (1.2), (1.3) are also generalizations of Pieri type formulas [L]
\[|\partial_z| \Phi^{(d)}_{x}(z) = \sum_{i=1}^{r} \Phi^{(d)}_{x-\epsilon_i}(z) A^{(d)}_{-i}(x),\]
\[|\partial_z| \Psi^{(d)}_{x}(z) = \sum_{i=1}^{r} \Psi^{(d)}_{x-\epsilon_i}(z) A^{(d)}_{+i}(x-\epsilon_i).\]

We remark that if \(x-\epsilon_i \notin P\) (resp. \(x+\epsilon_i \notin P\)) then \(A^{(d)}_{-i}(x) = 0\) (resp. \(A^{(d)}_{+i}(x) = 0\)).

The previous twisted Pieri formulas (1.2) and (1.3) are the falling type, i.e., lowering the degree of the Jack polynomials. In the previous paper [Sh2], as applications of these falling type twisted Pieri formulas, we gave another proofs of the following difference equations for the interpolation Jack polynomials proved by Knop-Sahi [KS].

**Theorem 1.1** (Knop-Sahi). For any \(x \in \mathbb{C}^r\) and \(k \in \mathcal{P}\), we have

\[D_{\frac{d}{2}}^{ip}(u; x) P_{k}^{ip}\left(x + \frac{d}{2} \delta; \frac{d}{2}\right) = P_{k}^{ip}\left(x + \frac{d}{2} \delta; \frac{d}{2}\right) I_{\frac{d}{2}}^{ip}(u; k),\]  
(1.8)

where

\[D_{\frac{d}{2}}^{ip}(u; x) := \sum_{J \subseteq [r]} (-1)^{|J|} I_{J}^{ip}(u; x) A_{-J}^{(d)}(x) \prod_{j \in J} \left( x_j + \frac{d}{2} (r - j) \right) T_x^J,\]

\[T_x f(x) := f(x - \epsilon_j), \quad T_x^J := \prod_{j \in J} T_{x_j}.\]

Further, we derived the following Pieri formula for interpolation Jack polynomials.

**Theorem 1.2** ([Sh2]). For any \(x \in \mathbb{C}^r\) and \(k \in \mathcal{P}\), we have

\[I_{\frac{d}{2}}^{ip}(u; x) \frac{P_{k}^{ip}\left(x + \frac{d}{2} \delta; \frac{d}{2}\right)}{P_{k}^{ip}\left(1; \frac{d}{2}\right)} = \sum_{J \subseteq [r], \ k + \epsilon_j \in \mathcal{P}} \frac{P_{k+\epsilon_j}^{ip}\left(x + \frac{d}{2} \delta; \frac{d}{2}\right)}{P_{k+\epsilon_j}^{ip}\left(1; \frac{d}{2}\right)} I_{\frac{d}{2}}^{ip}(u; k) A_{+J}^{(d)}(k).\]  
(1.9)
The purpose of this paper is to provide the raising type twisted Pieri formulas for Jack polynomials, which are generalizations of (1.6), (1.7) and Pieri formulas [St], [M]:

\[ z \Phi_x^{(d)}(z) = \sum_{i=1}^{r} \Phi_{x + \epsilon_i}^{(d)}(z) A_{z,i}^{(d)}(x), \] (1.10)

\[ z \Psi_x^{(d)}(z) = \sum_{1 \leq i \leq r, x + \epsilon_i \in P} \Psi_{x + \epsilon_i}^{(d)}(z) \left( x_i + 1 + \frac{d}{2}(r - i) \right) A_{z,i}^{(d)}(x + \epsilon_i), \] (1.11)

where \( z := z_1 + \ldots + z_r = (\text{ad } |z|) \left( \sum_{i=1}^{r} z_i \partial z_i \right) = \frac{d}{2} (\text{ad } |z|) H_{r,1}^{(d)}(z). \)

**Theorem 1.3** (Raising type twisted Pieri formulas for Jack polynomials). For \( l = 0, 1, \ldots, r, \) we have

\[ \left[ \frac{(-\text{ad } |z|)^l}{l!} S_r^{(d)}(u, z) \right] \Phi_k^{(d)}(z) = \sum_{J \subseteq [r], |J| = l} \Phi_{k + \epsilon_J}^{(d)}(z) I_{j_c}^{(d)}(u, k) A_{j_c}^{(d)}(k), \] (1.12)

\[ \left[ \frac{(-\text{ad } |z|)^l}{l!} S_r^{(d)}(u, z) \right] \Psi_k^{(d)}(z) = \sum_{J \subseteq [r], |J| = l} \Psi_{k + \epsilon_J}^{(d)}(z) I_{j_c}^{(d)}(u, k) A_{j_c}^{(d)}(k + \epsilon_J) \prod_{j \in J} \left( k_j + 1 + \frac{d}{2}(r - j) \right). \] (1.13)

Further, from these raising type twisted Pieri formulas (2.3), (2.4) and the binomial formulas for Jack polynomial [VK], [L]

\[ e^{|x|} \Phi_k^{(d)}(z) = \sum_{k \subseteq x} \frac{P_k^{(d)}(x + \frac{d}{2} \delta; \frac{d}{2})}{P_k^{(d)}(1; \frac{d}{2})} \Phi_x^{(d)}(z), \] (1.14)

\[ e^{|x|} \Psi_k^{(d)}(z) = \sum_{k \subseteq x} \frac{P_k^{(d)}(x + \frac{d}{2} \delta; \frac{d}{2})}{P_k^{(d)}(k + \frac{d}{2} \delta; \frac{d}{2})} \Psi_x^{(d)}(z), \] (1.15)

we give yet another proof of Theorem 1.1 and Theorem 1.2.

The content of this article is as follows. In Section 2, we prove the raising type twisted Pieri formulas for Jack polynomials. From these twisted Pieri formulas for Jack polynomials, we give yet another proof of Theorem 1.1 in
Section 3 and another proof of Theorem 1.2 in Section 4. We mention some future works for raising and falling type twisted Pieri formulas and their applications in Section 5. Finally, in Appendix, we derive a differential-difference relation for the binomial type polynomials including Jack and multivariate Bernoulli polynomials, which is a generalization of the falling type twisted Pieri formula (1.4) from the Jack polynomial to the binomial type polynomials.

2 Raising twisted Pieri formulas for Jack polynomials

To prove Theorem 1.3, we refer the following summation.

Lemma 2.1 ([Sh2] Lemma 2.1). For any \( I \subseteq [r] \) and \( x = (x_1, \ldots, x_r) \in \mathbb{C}^r \), we have

\[
\sum_{i \in I} \left( x_i + 1 + \frac{d}{2}(r - i) \right) A^{(d)}_{-i, I \setminus i}(x + \epsilon_i) A^{(d)}_{+i, I \setminus i}(x) - \sum_{i \in I} \left( x_i + \frac{d}{2}(r - i) \right) A^{(d)}_{+i, I \setminus i}(x - \epsilon_i) A^{(d)}_{-i, I \setminus i}(x) = |I|, \tag{2.1}
\]

where

\[
A^{(d)}_{\pm i, I \setminus i}(x) := \prod_{j \in I \setminus i} \frac{x_i - x_j - \frac{d}{2}(i - j) \pm \frac{d}{2}}{x_i - x_j - \frac{d}{2}(i - j)}.
\]

Proof of Theorem 1.3 We prove Theorem 1.3 by induction on \( l \). Since the proofs of (2.3) and (2.3) are similar, we prove only (2.3). From the property of the Sekiguchi operator (1.7), the case of \( l = 0 \) holds;

\[
S_r^{(d)}(u; z)\Phi_k^{(d)}(z) = \Phi_k^{(d)}(z)I_r^{(d)}(u; k). \tag{2.2}
\]

If \( l = 1 \), then

\[
\left[ (-\text{ad } |z|)S_r^{(d)}(u; z) \right] \Phi_k^{(d)}(z)
= -|z|\Phi_k^{(d)}(z)I_r^{(d)}(u; k) + S_r^{(d)}(u; z) \sum_{i=1}^r \Phi_{k+\epsilon_i}^{(d)}(z)A^{(d)}_{+i}(k)
= \sum_{i=1}^r \Phi_{k+\epsilon_i}^{(d)}(z)A^{(d)}_{+i}(k)(I_r^{(d)}(u; k + \epsilon_i) - I_r^{(d)}(u; k)).
\]
\[
\sum_{J \subseteq [r], \lvert J \rvert = 1} \Phi_{k + \epsilon, j}^{(d)} (z) J_{j_c}^{(d)} (u; k) A_{+, J}^{(d)} (k).
\]

Assume the \( n = l \) case holds. Hence,
\[
\left[ \frac{(-ad |z|)^{l+1}}{(l+1)!} S_r^{(d)}(u; z) \right] \Phi_k^{(d)} (z)
= -\frac{1}{l+1} |z| \left[ \frac{(-ad |z|)^l}{l!} S_r^{(d)}(u, z) \right] \Phi_k^{(d)} (z)
+ \left[ \frac{(-ad |z|)^l}{l!} S_r^{(d)}(u, z) \right] \frac{1}{l+1} |z| \Phi_k^{(d)} (z)
= \frac{1}{l+1} \sum_{J \subseteq [r], \lvert J \rvert = l} \sum_{\nu = 1}^r
\cdot \left\{ \Phi_{k + \epsilon, j, \cup \{\nu\}}^{(d)} (z) A_{+, \nu}^{(d)} (k + \epsilon, j) J_{j_c}^{(d)} (u, k) A_{+, J}^{(d)} (k)
- \Phi_{k + \epsilon, j, \cup \{\nu\}}^{(d)} (z) J_{j_c}^{(d)} (u, k + \epsilon, \nu) A_{+, j}^{(d)} (k + \epsilon, \nu) A_{+, \nu}^{(d)} (k) \right\}.
\]

From a simple calculation, we have
\[
\left[ \frac{(-ad |z|)^{l+1}}{(l+1)!} S_r^{(d)}(u, z) \right] \Phi_k^{(d)} (z)
= \sum_{I \subseteq [r], \lvert I \rvert = l+1} \frac{1}{l+1} \Phi_{k + \epsilon, \cup \{\nu\}}^{(d)} (z) J_{I_c}^{(d)} (u, k) A_{+, I}^{(d)} (k)
\cdot \sum_{i \in I} \left\{ \left( s_i + 1 + \frac{d}{2} u \right) A_{-, \{i\}, \cap \{\nu\}}^{(d)} (k + \epsilon, \nu) A_{+, \nu, \cap \{i\}}^{(d)} (k)
- \left( s_i + \frac{d}{2} u \right) A_{-, \{i\}, \cup \{\nu\}}^{(d)} (k) A_{+, \{i\}, \cup \{\nu\}}^{(d)} (k - \epsilon, \nu) \right\}
= \sum_{I \subseteq [r], \lvert I \rvert = l+1} \Phi_{k + \epsilon, j}^{(d)} (z) J_{I_c}^{(d)} (u, k) A_{+, I}^{(d)} (k),
\]
where
\[
\begin{align*}
\frac{d}{2} (r - j).
\sum_{i \in I} \left\{ \left( s_i + 1 + \frac{d}{2} u \right) A_{-, \{i\}, \cap \{\nu\}}^{(d)} (k + \epsilon, \nu) A_{+, \nu, \cap \{i\}}^{(d)} (k)
- \left( s_i + \frac{d}{2} u \right) A_{-, \{i\}, \cup \{\nu\}}^{(d)} (k) A_{+, \{i\}, \cup \{\nu\}}^{(d)} (k - \epsilon, \nu) \right\}
\end{align*}
\]
\[- \left( s_i + \frac{d}{2}u \right) A_{-i,J \setminus \{i\}}^{(d)}(k) A_{+,i,J \setminus \{i\}}^{(d)}(k - \epsilon_i) \right) = l + 1 \]

is the summation (2.1) exactly. Then we obtain the conclusion. \(\square\)

By comparing the coefficients for \(u^r - l\) of the formulas (2.3) and (2.3), we obtain the following twisted Pieri type formulas.

**Corollary 2.2.** For any \(z \in \mathbb{C}^r\) and \(l = 0, 1, \ldots, r\),

\[
\left( \frac{d}{2} \right)^l \left[ \frac{(-\text{ad} |z|)^l}{l!} H_{r,l}^{(d)}(z) \right] \Phi_k^{(d)}(z) = \sum_{J \subseteq [r] \mid |J| = l} \Phi_{k+\epsilon_J}^{(d)}(z) A_{+,J}^{(d)}(k), \tag{2.3}
\]

\[
\left( \frac{d}{2} \right)^l \left[ \frac{(-\text{ad} |z|)^l}{l!} H_{r,l}^{(d)}(z) \right] \Psi_k^{(d)}(z) = \sum_{J \subseteq [r] \mid |J| = l} \Psi_{k+\epsilon_J}^{(d)}(z) A_{-,J}^{(d)}(k + \epsilon_J) \prod_{j \in J} \left( k_j + 1 + \frac{d}{2}(r - j) \right). \tag{2.4}
\]

### 3 Yet another proof of difference equations for interpolation Jack polynomials

In the previous paper [Sh2], we calculated

\[
S_r^{(d)}(u; z) \Phi_x^{(d)}(1 + z)
\]

in two different ways and provided another proof of Theorem 1.1 proved by Knop-Sahi [KS]. In this section, we compute

\[
[e^{\text{ad} |z|} S_r^{(d)}(u, z)] e^{\text{ad} |z|} \Psi_k^{(d)}(z)
\]

in two different ways and give yet another proof of Theorem 1.1.

**Proof of Theorem 1.1** From the binomial formula (1.15) and (1.17),

\[
[e^{\text{ad} |z|} S_r^{(d)}(u, z)] e^{\text{ad} |z|} \Psi_k^{(d)}(z) = e^{\text{ad} |z|} S_r^{(d)}(u, z) e^{-\text{ad} |z|} e^{\text{ad} |z|} \Psi_k^{(d)}(z)
\]

\[
= e^{\text{ad} |z|} S_r^{(d)}(u, z) \Psi_k^{(d)}(z)
\]

\[
= e^{\text{ad} |z|} \Psi_k^{(d)}(z) J_r^{(d)}(u, k)
\]

\[
= \sum_{k \in \chi} ^{\Phi_k^{(d)}(z)} \frac{P_k^{(d)}}{P_k^{(d)}} \left( x + \frac{d}{2} \delta; \frac{d}{2} \right) J_r^{(d)}(u, k).
\]
On the other hand, since the highest derivative in $H_{r,j}^{(d)}(z)$ has degree $p$, the sum

$$e^{ad[z]}S_{r}^{(d)}(u, z) = \sum_{l \geq 0} \frac{(ad[z])^l}{l!}S_{r}^{(d)}(u; z)$$

terminates after $(ad[z])^r$. Then, we have

$$[e^{ad[z]}S_{r}^{(d)}(u, z)]\psi_{k}^{(d)}(z)$$

$$= \sum_{k \leq x} \frac{f_{k}^{(d)}(x + \frac{d}{2}; \frac{d}{2})}{f_{k}^{(d)}(k + \frac{d}{2}; \frac{d}{2})} [e^{ad[z]}S_{r}^{(d)}(u; z)]\psi_{x}^{(d)}(z)$$

$$= \sum_{k \leq x} \frac{f_{k}^{(d)}(x + \frac{d}{2}; \frac{d}{2})}{f_{k}^{(d)}(k + \frac{d}{2}; \frac{d}{2})} \sum_{l=0}^{r} \frac{(ad[z])^l}{l!}S_{r}^{(d)}(u; z) \psi_{x}^{(d)}(z)$$

$$= \sum_{k \leq x} \frac{f_{k}^{(d)}(x + \frac{d}{2}; \frac{d}{2})}{f_{k}^{(d)}(k + \frac{d}{2}; \frac{d}{2})} \cdot \sum_{J \subseteq [r]} (-1)^{|J|} \sum_{J \subseteq \{ \frac{ad[z]}{d} \} \psi_{x}^{(d)}(z) I_{j,j}^{(d)}(u; x) A_{\epsilon, j}^{(d)}(x + \epsilon, j) \prod_{j \in J} \left( x_j + 1 + \frac{d}{2}(r - j) \right)$$

$$= \sum_{k \leq x} \psi_{x}^{(d)}(z)$$

$$\sum_{J \subseteq [r]} (-1)^{|J|} I_{j,j}^{(d)}(u; x - \epsilon, j) A_{\epsilon, j}^{(d)}(x) \prod_{j \in J} \left( x_j + \frac{d}{2}(r - j) \right) \frac{f_{k}^{(d)}(x - \epsilon, j + \frac{d}{2}; \frac{d}{2})}{f_{k}^{(d)}(k + \frac{d}{2}; \frac{d}{2})}.$$
in two different ways [Sh2]. Here we calculate

\[ S_r^{(d)}(u, z)e^{[z]} \Phi_k^{(d)}(z) \]

in two ways and derive Theorem 1.2.

**Proof of Theorem 1.2** As with the proof of Theorem 1.1, it is enough to prove (1.9) for \( x \in P \). From the binomial formula (1.14) and (1.7), we have

\[ S_r^{(d)}(u, z)e^{[z]} = \sum_{k \subseteq x} \frac{P_k^p(x + \frac{d}{2}; \frac{d}{2})}{P_k(1; \frac{d}{2})} S_r^{(d)}(u, z) \Psi_x^{(d)}(z) \]

On the other hand, a simple calculation shows that

\[ S_r^{(d)}(u, z)e^{[z]} = e^{[z]}e^{-[z]}S_r^{(d)}(u, z)e^{[z]} = e^{[z]}[e^{-[z]} S_r^{(d)}(u, z)] \]

Then, from the twisted Pieri formula (2.4) and binomial formula (1.13), we have

\[ S_r^{(d)}(u, z)e^{[z]} \Phi_k^{(d)}(z) = e^{[z]}[e^{-[z]} S_r^{(d)}(u, z)] \Phi_k^{(d)}(z) \]

\[ = e^{[z]} \left[ \sum_{l=0}^{r} \frac{(-adz)^l}{l!} S_r^{(d)}(u, z) \right] \Phi_k^{(d)}(z) \]

\[ = \sum_{l=0}^{r} \sum_{J \subseteq [r], |J| = l} e^{[z]} \Phi_k^{(d)}(z) I_{J}^{(d)}(u, k) A_{+J}^{(d)}(k) \]

\[ = \sum_{k \subseteq x} \Psi_x^{(d)}(z) \cdot \sum_{J \subseteq [r], k+\epsilon_J \in P} \frac{P_k^{p+\epsilon_J}(x + \frac{d}{2}; \frac{d}{2})}{P_k^{p+\epsilon_J}(1; \frac{d}{2})} I_{J}^{(d)}(u, k) A_{-J}^{(d)}(k). \]

By comparing of the coefficients for \( \Psi_x^{(d)}(z) \), we obtain the conclusion. \( \square \)

5 Concluding remarks

In this article and [Sh2], we propose the falling and raising type twisted Pieri formulas and (1.2), (1.3) and (2.3), (2.4), and apply these formulas to the
proofs of Theorem 1.1 and Theorem 1.2. As the next stage, it is desirable to write down explicitly the next mixed-type twisted Pieri formulas

\[
\left[ \frac{(ad | z)_m}{m!} \frac{(ad | \partial_z)_n}{n!} S_r^{(d)}(u; z) \right] \Phi_k^{(d)}(z) =?,
\]

\[
\left[ \frac{(ad | z)_m}{m!} \frac{(ad | \partial_z)_n}{n!} S_r^{(d)}(u; z) \right] \Psi_k^{(d)}(z) =?.
\]

We expect these more difficult twisted Pieri formulas to contribute to the description of some formulas for type $BC$.

A A twisted Pieri formula for binomial type polynomials

We proved the following intertwining relations for a kernel function of Jack polynomials [VK], [L]

\[
0\mathcal{F}_0^{(d)}(z, u) := \sum_{m \in \mathcal{P}} \Psi_m^{(d)}(z)\Phi_m^{(d)}(u) = \sum_{m \in \mathcal{P}} \Phi_m^{(d)}(z)\Psi_m^{(d)}(u).
\]

Lemma A.1 ([Sh2]). For any $l = 0, 1, \ldots, r$, we have

\[
\left( \frac{d}{2} \right)^l \left[ \frac{(ad | \partial_z)^l}{l!} H_r^{(d)}(z) \right] 0\mathcal{F}_0^{(d)}(z, u) = 0\mathcal{F}_0^{(d)}(z, u) e_{r,l}(u). \quad (A.1)
\]

In this section, we mention some applications of these intertwining relations (A.1) and show that twisted Pieri formula (1.2) can be generalized to the binomial type polynomials including Jack and multivariate Bernoulli polynomials. First we give a variation of usual Pieri formulas for the ordinary Jack polynomials [St], [M]:

\[
e_{r,l}(z) \Phi_m^{(d)}(z) = \sum_{J \subseteq [r], |J| = l, \begin{array}{c} \mathcal{P} \end{array}} \Phi_{m+\epsilon_J}^{(d)}(z)A_{+,J}^{(d)}(m) \quad (l = 0, 1, \ldots, r).
\]

Lemma A.2. For $l = 0, 1, \ldots, r$,

\[
e_{r,l}(z) \Psi_m^{(d)}(z) = \sum_{J \subseteq [r], |J| = l, \begin{array}{c} \mathcal{P} \end{array}} \Psi_{m+\epsilon_J}^{(d)}(z)A_{-,J}^{(d)}(m + \epsilon_J) \prod_{i \in J} \left( m_j + 1 + \frac{d}{2}(r - j) \right).
\]

(A.2)

In particular, the case of $l = 1$ is (2.4) exactly.
proof. A simple calculation shows that

\[
\sum_{m \in P} \Phi_m^{(d)}(u) e_{r,l}(z) \Psi_m^{(d)}(z) = \mathcal{F}_0^{(d)}(z, u) e_{r,l}(z)
\]

\[
= \left( \frac{d}{2} \right)^l \left[ \frac{(ad\lvert \partial u \rvert)^l}{l!} H_{r,l}^{(d)}(u) \right] \mathcal{F}_0^{(d)}(z, u)
\]

\[
= \sum_{m \in P} \Psi_m^{(d)}(z) \left( \frac{d}{2} \right)^l \left[ \frac{(ad\lvert \partial u \rvert)^l}{l!} H_{r,l}^{(d)}(u) \right] \Phi_m^{(d)}(u)
\]

\[
= \sum_{m \in P} \sum_{\substack{J \subseteq \{r\}, \lvert J \rvert = l, \ m_{-\epsilon_j}, \epsilon_j \in P}} \Phi_m^{(d)}(u) \frac{d}{2} (r - j)
\]

\[
\cdot \prod_{j \in J} \left( m_j + \frac{d}{2} (r - j) \right)
\]

\[
= \sum_{m \in P} \sum_{\substack{J \subseteq \{r\}, \lvert J \rvert = l, \ m_{+\epsilon_j}, \epsilon_j \in P}} \Psi_m^{(d)}(z) A_{-\epsilon_j}(m + \epsilon_j)
\]

\[
\cdot \prod_{j \in J} \left( m_j + 1 + \frac{d}{2} (r - j) \right).
\]

Here the second and fourth equalities follow from (A.1) and (1.4) respectively.

We assume that \( F(u) \) is a symmetric function and \( \mathcal{F}_0^{(d)}(z, u) F(u) \) has the series expansion

\[
\mathcal{F}_0^{(d)}(z, u) F(u) = \sum_{m \in P} f_m^{(d)}(z) \Psi_m^{(d)}(u)
\]

of absolute convergence at \( |u_1| + \cdots + |u_r| \ll 1 \). Under the assumption, we define binomial type polynomials \( f_m^{(d)}(z) \) associated with \( F(u) \) by the generating function (A.3).

Example A.3. (1) If \( F(u) = 1 \), then \( f_m^{(d)}(z) \) is the Jack polynomial \( \Phi_m^{(d)}(z) \) exactly.

(2) If

\[
F(u) = \frac{|u|}{e^{\lvert u \rvert} - 1},
\]

then \( f_m^{(d)}(z) \) is the multivariate Bernoulli polynomial \( B_m^{(d)}(z) \) (see \[Sh1\]).
The binomial type polynomial satisfies the binomial formula.

**Proposition A.4.** For any partition \( m \), we have

\[
\begin{align*}
\tilde{f}_m(d)(1+z) &= \sum_{k \subseteq m} \frac{P^p_k(m + \frac{d\delta}{2}; \frac{d}{2})}{P^p_k(k + \frac{d\delta}{2}; \frac{d}{2})} f_k(d)(z).
\end{align*}
\]

**(A.4)**

**proof.** It follows from the index law

\[
\begin{align*}
0\mathcal{F}_0^{(d)}(1+z, u) &= e^{|u|} 0\mathcal{F}_0^{(d)}(z, u) F(u) \\
&= \sum_{k \in \mathcal{P}} \tilde{f}_k(d)(z) e^{|u|} \Psi_k(u) \\
&= \sum_{m \in \mathcal{P}} \Psi_m(d)(u) f_m(d)(1+z) = 0\mathcal{F}_0^{(d)}(1+z, u) F(u) \\
&= \sum_{m \in \mathcal{P}} \Psi_m(d)(u) f_m(d)(z).
\end{align*}
\]

Finally, we derive a twisted Pieri formula for binomial type polynomials \( f_m(d)(z) \) that is a generalization of the twisted Pieri formula (A.4).

**Theorem A.5.** For any \( z \in \mathbb{C}^r \) and \( l = 0, 1, \ldots, r \), we have

\[
\begin{align*}
\left( \frac{d}{2} \right)^l \left[ \frac{\partial |z|}{l!} H_{r,l}^{(d)}(z) \right] f_m(d)(z) &= \sum_{J \subseteq [r], |J| = l, \ m - \epsilon_j \in \mathcal{P}} f_{m - \epsilon_j}(z) A_{m,J}^{(d)}(m) \\
& \quad \cdot \prod_{j \in J} \left( m_j + \frac{d}{2}(r-j) \right).
\end{align*}
\]

**(A.5)**

**proof.** From (A.1) and (A.2), we have

\[
\begin{align*}
\sum_{m \in \mathcal{P}} \left( \frac{d}{2} \right)^l \left[ \frac{\partial |z|}{l!} H_{r,l}^{(d)}(z) \right] f_m(d)(z) \Psi_m(d)(u) \\
&= \left( \frac{d}{2} \right)^l \left[ \frac{\partial |z|}{l!} H_{r,l}^{(d)}(z) \right] 0\mathcal{F}_0^{(d)}(z, u) F(u)
\end{align*}
\]
\[ \sum_{m \in \mathcal{P}} \Psi_{m}(z) e_{r,l}(u) \Psi_{m}(u) \]

\[ \sum_{m \in \mathcal{P}} f_{m}^{(d)}(z) \Psi_{m}(u) \]

\[ \sum_{m \in \mathcal{P}} f_{m}^{(d)}(z) \sum_{J \subseteq [r], |J| = l, m+\epsilon_{J} \in \mathcal{P}} \Psi_{m+\epsilon_{J}}^{(d)}(z) A_{m+\epsilon_{J}}^{(d)}(m) \prod_{j \in J} \left( m_{j} + \frac{d}{2}(r - j) \right) \]

\[ \sum_{m \in \mathcal{P}} \Psi_{m}(z) \sum_{J \subseteq [r], |J| = l, m-\epsilon_{J} \in \mathcal{P}} f_{m-\epsilon_{J}}^{(d)}(z) A_{m-\epsilon_{J}}^{(d)}(m) \prod_{j \in J} \left( m_{j} + \frac{d}{2}(r - j) \right) \]

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