Many quantum machine learning (QML) algorithms that claim speed-up over their classical counterparts only generate quantum states as solutions instead of their final classical description. The additional step to decode quantum states into classical vectors normally will destroy the quantum advantage in most scenarios because all existing tomographic methods require runtime that is polynomial with respect to the state dimension. In this Letter, we present an efficient readout protocol that yields the classical vector form of the generated state, so it will achieve the end-to-end advantage for those quantum algorithms. Our protocol suits the case that the output state lies in the row space of the input matrix, of rank $r$, that is stored in the quantum random access memory. The quantum resources for decoding the state in $\ell_2$-norm with $\epsilon$ error require poly($r, 1/\epsilon$) copies of the output state and poly($r, \kappa^2, 1/\epsilon$) queries to the input oracles, where $\kappa$ is the condition number of the input matrix. With our read-out protocol, we completely characterise the end-to-end resources for quantum linear equation solvers and quantum singular value decomposition. One of our technical tools is an efficient quantum algorithm for performing the Gram-Schmidt orthonormal procedure, which we believe, will be of independent interest.

The task of recovering the unknown quantum state from measurements, which is also known as Quantum State Tomography (QST), is one of the fundamental problems in quantum information science. QST has attracted significant interest from both theoretical and experimental perspectives in recent years. The best general tomography method could reconstruct a $d \times d$ density matrix $\rho$ for the unknown state with rank $r$ by using $n = O(rd/\epsilon^2)$ copies to the state, which implies $O(d/\epsilon^2)$ copy complexity for the pure state case $\rho = |v\rangle \langle v|$. Recently, a state tomography protocol has been proposed for the special case that the $d$ dimensional state $|v\rangle$ is prepared by some unitary $U_\epsilon$ with real components, and the query complexity is $O(d \log d/\epsilon^2)$ for the $\ell_2$-norm error bounded in $\epsilon$. We remark that most of QML algorithms that output a $d$-dimensional state as the solution claim the time complexity polylogarithmical to $d$. Thus, directly using state tomography methods for state readout in QML is computationally expensive and would offset the gained quantum speedup. Since the required number $n$ is proven optimal for both cases any further improvement on $n$ could be achieved.

Efficient State Read-out for Quantum Machine Learning Algorithms

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Introduction. Modern machine learning has dramatically advanced research in the artificial intelligence, computer vision and natural language processing fields. However, the large scale deployment of these machine learning algorithms is restricted by their unaffordable computational complexity. In recent years, quantum computing has emerged as one promising solution to this problem, and has evolved into an independent subfield, known as quantum machine learning (QML) [2, 3]. Many quantum machine learning algorithms have been proposed for fundamental machine learning tasks, including solving linear systems [4], support-vector machines [5], singular value decomposition [6] and recommendation systems [7]. These quantum algorithms have shown to achieve exponential or quadratic speed-ups over their classical counterparts.

Despite the claimed quantum speed-up, most QML algorithms suffered from both the input and the readout problems. Specifically, the input problem tackles the issue of efficient state preparation, namely, encoding the classical data, potentially of tantamount size, into quantum states. A few techniques [4, 8–10] have been proposed to address this problem, and among them, the quantum random access memory (QRAM) oracle model [8] has become, arguably, the most popular method in the domain of machine learning applications [2, 4, 11–13]. Generally, for a data matrix $A \in \mathbb{R}^{m \times n}$, the corresponding QRAM oracle could be prepared by using $O(\text{polylog}(mn))$ quantum operations with $O(mn)$ physical resources stored in a binary tree data structure [13]. Although the QRAM oracle is criticized for the requirement of large physical resources, recent work [20] has proven possible the practical implementation of the QRAM oracle.

On the other hand, the readout problem addresses recovery of classical description from the output quantum state that contains the classical solutions. In order to preserve the quantum advantage of the underlining quantum algorithm, the output state needs to be decoded efficiently. For some quantum algorithms, such as the quantum recommendation system [7], the readout issue is relatively mild because the classical solution can be obtained by only a few measurements on the output state. In general, most machine learning problems demand classical solutions in vector form, for example, finding solutions to linear systems. Hence, the readout problem of these quantum algorithms could be critical. However, protocols for efficiently decoding the output quantum states into classical vectors remain little explored [21].
only by assuming special prior knowledge on state $\rho$. For example, QST via local measurements provides efficient estimation for states which can be determined by local reduced density matrices or states with a low-rank tensor decomposition. However, the output states generated by QML algorithms normally do not have these structures.

In contrast with the assumptions in the QST scenarios, the output states generated by most QML algorithms do have inherent relationship between the solution vector and the input data, commonly represented as a matrix. Specifically, the solution vector normally lies in the row space of the input data matrix. Notable examples that satisfy the aforementioned condition include: (1) the quantum SVD algorithm which involves the singular value $\sigma_i$ and corresponding singular vectors $|u_i\rangle$ and $|v_i\rangle$ for matrix $A = \sum_i \sigma_i u_i v_i^T$; and (2) the quantum linear system solver for linear system $Ax = b$ whose solution state $|x\rangle \propto A^{-1} b$ lies in the row space of $A$. Most machine learning problems can be reduced to these two categories. Hence, finding efficient readout protocols for them that go beyond the standard QST limit will be extremely desirable in the field of QML.

In this Letter, we design an efficient state readout protocol that works for QML algorithms which involve a $r$-rank input matrix $A \in \mathbb{R}^{m \times n}$ stored in the quantum random access memory (QRAM), and the output state $|v\rangle$ lies in the row space of $A$. Instead of obtaining coefficients $\{v_i\}$ by measuring the state $|v\rangle = \sum_{i=1}^n v_i |i\rangle$ in the standard orthonormal basis $\{|i\rangle\}$, our key technical contribution is an efficient method to obtain the classical description $x_i$ in the complete basis spanned by the rows $\{A_{g(i)}\}_{i=1}^r$ of $A$, so that $|v\rangle = \sum_{i=1}^r x_i |A_{g(i)}\rangle$. Our state readout protocol requires $O(\text{poly}(r))$ copies of the output states and $O(\text{poly}(r, \kappa^r))$ queries to input oracles, where $r$ is the rank of the input matrix and $\kappa = \sigma_{\text{max}}(A)/\sigma_{\text{min}}(A)$ is the condition number of the input matrix. We remark that the low-rank matrix assumption is common in machine learning models. Compared to previous QST methods which require at least $O(n)$ copies of pure states, our protocol is much more efficient given $r \ll n$ with small condition numbers, and more importantly, the complexity does not depend on the system dimension. Finally, combining our readout protocol with quantum SVD or quantum linear system solver yields an end-to-end complexity that takes $O(\text{poly}(r, \kappa^r, \log(mn)))$ queries to input oracles.

During the whole read-out protocol, we develop a quantum generalization of the Gram-Schmidt Orthonormalization process. Our quantum Gram-Schmidt Process (QGSP) algorithm can construct a complete basis, by sampling a set of rows $\{A_{g(i)}\}_{i=1}^r$ of the input matrix $A$, with $O(\text{poly}(r, \kappa^r))$ queries to QRAM oracles. Since the vector orthonormalization is a crucial procedure in linear algebra as well as machine learning, an efficient quantum algorithm will be of independent interest. Notice that there are some related works for the construction of orthogonal states. However, these results deviate from standard Gram-Schmidt process and their applications are also limited. Ref. is only applicable to the single-qubit system, while Refs. only generate a state that is orthogonal to the input state and their complexity depends on the system dimension. Ref. constructs orthogonal states from original states by lifting the dimension of the original Hilbert space, and cannot select a complete basis as standard Gram-Schmidt process does. Consequently, our proposed QGSP algorithm avoids all these restrictions and can be proven to be efficient.

**Main Result.** The complete statement of our main result is as follows.

**Theorem 1.** For the state $|v\rangle$ lies in the row space of a matrix $A \in \mathbb{R}^{m \times n}$ with rank $r$ and the condition number $\kappa$, the classical form of $|v\rangle$ could be obtained by using $O(r^3 \epsilon^{-2})$ queries to the state $|v\rangle$, $O(r^{11} \kappa^5 \epsilon^{-2} \log(\frac{1}{\delta}))$ queries to QRAM oracles of $A$, and $O(r^2)$ additional inner product operations between rows, such that the $\ell_2$-norm error is bounded in $\epsilon$ with probability at least $1 - \delta$.

Now we explain our protocol in detail. Since $A \in \mathbb{R}^{m \times n}$ is of rank $r$, we can identify a set of $r$ linearly independent vectors $\{|A_{g(i)}\rangle\}_{i=1}^r$ selected from all rows of $A$ so that the output state can be rewritten as $|v\rangle = \sum_{i=1}^r x_i |A_{g(i)}\rangle$. Our goal is accomplished if we can determine $\{x_i\}_{i=1}^r$ efficiently. Following this, our algorithm consists of two major parts, a subroutine to sample a set of $r$ linearly independent rows $\{|A_{g(i)}\rangle\}_{i=1}^r$ from all rows of $A$ and a subroutine to calculate $\{x_i\}$.

We begin with the first subroutine. The Quantum Gram-Schmidt Process (QGSP) in Algorithm 1 is developed to generate a complete row basis, by performing a quantum version of the adaptive sampling. The advantage of our adaptive sampling is that those rows, which have larger orthogonal part to the row space of previous sampled row submatrix, will be sampled with a larger probability. This ensures that the complete basis is non-singular, and will improve the accuracy of the estimation of the coefficients in the second subroutine.

**Algorithm 1 Quantum Gram-Schmidt Process (QGSP)**

**Require:** QRAM oracles $U_A$ and $V_A$ and $A_1$.

**Ensure:** A group of orthonormal states $\{\{t_i\}\}_{i=1}^r$. An index set of the complete basis: $S_t = \{g(i)\}_{i=1}^r$. 1: Initialize the index set $S_t = \emptyset$. 2: for $t = 0$ to $r - 1$ do 3: Run the quantum circuit in Fig. Measure the third register and post-select on result $0$. Measure the first register to obtain an index $g(t + 1)$. Update the index set $S_t = S_t \cup \{g(t + 1)\}$. 4: end for
has been generated in the previous $\ell$ iterations. To proceed to the $(\ell+1)$-th iteration, we perform the quantum circuit illustrated in Fig. 1 where the unitary $R_i = I - 2|t_i\rangle\langle t_i|$. The protocol outputs an index $g(\ell+1)$ from $[m] := \{1, \ldots, m\}$ with probability:

$$P(\ell)(g(\ell+1)) = \frac{\|A_{g(\ell+1)} - \pi S_i (A_{g(\ell+1)})\|^2}{\sum_{i=1}^m \|A_i - \pi S_i (A_i)\|^2},$$

(1)

where $\pi S_i (\cdot)$ denotes the projection on the row space of the row submatrix $A(S_i, \cdot) \in \mathbb{R}^{\ell \times n}$. The new orthonormal state is generated as:

$$|t_{\ell+1}\rangle = \frac{|A_{g(\ell+1)}\rangle - \pi S_i (|A_{g(\ell+1)}\rangle)}{\|A_{g(\ell+1)}\rangle - \pi S_i (|A_{g(\ell+1)}\rangle)}.$$  

The index set is updated as $S_1 = S_1 \cup \{g(\ell+1)\}$. Finally, after $\ell = 0, \ldots, r-1$ iterations, we could obtain the index set $S_1 = \{g(i)\}_{i=1}^r$ such that $\{A_{g(i)}\}_{i=1}^r$ forms a linearly independent basis.

FIG. 1. Quantum circuit for the $(\ell+1)$-th iteration in the QGSP.

The QRAM oracles $U_A$ and $V_A$ in Fig. 1 is defined as

$$|i\rangle|0\rangle \xrightarrow{U_A} |i\rangle|A_i\rangle = \sum_{j=1}^n \frac{A_{ij}}{\|A_i\|_F} |i\rangle|j\rangle, \forall i \in [m],$$

(2)

$$|0\rangle|j\rangle \xrightarrow{V_A} |\tilde{A}\rangle|j\rangle = \sum_{i=1}^m \frac{A_{ij}}{\|A_i\|_F} |i\rangle|j\rangle, \forall j \in [n],$$

(3)

where $A_i$ denotes the $i$-th row of $A$ and $\tilde{A}$ denotes the vector whose $i$-th component is $\|A_i\|_F / \|A_i\|_F$. Then the projections of each row $A_i$ on the sampled row subset $A(S_1, \cdot)$ is subtracted by performing a set of reflection unitaries $R_i = I - 2|t_i\rangle\langle t_i|$, $\forall i \in [\ell]$. We remark that orthonormal states $\{|t_i\rangle\}_{i=1}^\ell$ are generated from $\{|A_{g(i)}\rangle\}_{i=1}^\ell$ by performing Gram-Schmidt orthogonalization. Thus, a group of orthonormal basis is generated after the implementation of Algorithm 1.

The difficulty of constructing the circuit in Fig. 1 comes from efficient implementation of the controlled version of reflection $R_i = I - 2|t_i\rangle\langle t_i|$, since we do not have additional quantum memory to store $\{|t_i\rangle\}$ generated during the algorithm. To overcome this difficulty, we note that the state $|t_i\rangle$ lies in span$\{|A_{g(i)}\rangle\}_{i=1}^\ell$, so that $|t_i\rangle = \sum_{i=1}^\ell z_{it_i} |A_{g(i)}\rangle$ for some coefficients $z_{it_i}$. Instead we could generate $|t_i\rangle$ by the linear combination of unitary (LCU) method [41] with post-selections. Then, given copies of $|t_i\rangle\langle t_i|$, we can implement $R_i = I - 2|t_i\rangle\langle t_i|$ with the help of the Hamiltonian simulation technique developed in Quantum PCA [42]. By considering the error of implementing each $R_i$, we prove that the QGSP algorithm could select a linearly independent basis in time $O(r^{1+5}\kappa^5 \log 1/\delta)$ with probability at least $1 - \delta$. See Appendix A for details of the QGSP algorithm and the proof of Theorem 2.

Theorem 2. By using $O(r^{1+5}\kappa^5 \log 1/\delta)$ queries to input oracles, Algorithm 1 could find a group of row states $\{|A_{g(i)}\rangle\}_{i=1}^\ell$, such that the least singular value of the gram matrix $C$ formed by $\{|A_{g(i)}\rangle\}_{i=1}^\ell$ is greater than $\frac{1}{\sqrt{\kappa^2 \kappa_0^2 \kappa_1 \kappa_2}}$ with probability $1 - \delta$, where $r$ and $\kappa$ is the rank and the condition number of $A$, respectively.

Now we focus on the second subroutine. Once the row basis has been selected, which we denote as $\{|s_i\rangle\}_{i=1}^r$ for simplicity, the read-out problem reduces to obtaining coordinates $\{x_i\}_{i=1}^r$ in the description $|v\rangle = \sum_{i=1}^r x_i |s_i\rangle$. The steps are outlined in Algorithm 2.

The idea of Algorithm 2 is fairly natural. Since the QGSP algorithm generates orthonormal states $\{|t_i\rangle\}_{i=1}^\ell$, we could first calculate the coordinate of state $|v\rangle$ under the basis $\{|t_i\rangle\}_{i=1}^\ell$: $|v\rangle = \sum_{i=1}^r x_i |t_i\rangle$, and then transfer the orthonormal basis to the row basis $\{|s_i\rangle\}_{i=1}^r$:

$$\langle t_1, \ldots, t_r | = \left( \frac{s_1}{\|s_1\|}, \ldots, \frac{s_r}{\|s_r\|} \right) Z, $$

(4)

where $Z = \{z_{ij}\} \in \mathbb{R}^{r \times r}$ is the transformation matrix. The coordinates $\{x_i\}_{i=1}^r$ is given as: $x = Z a$.

Algorithm 2 State Read-out

Require: QRAM oracle $U_A$. Copies of state $|v\rangle$. Orthonormal basis $\{|t_i\rangle\}_{i=1}^\ell$. The precision parameter $\epsilon$.

Ensure: Coordinates $\{x_i\}_{i=1}^r$ in $|v\rangle = \sum_{i=1}^r x_i |s_i\rangle$ that guarantees a $\epsilon$ accuracy under $\ell_2$-norm.

1: Estimate the value $a_i^2 = \langle v | t_i \rangle^2$, for $i \in [r]$ by SWAP Test. Mark $k \equiv \arg \max_{i \in [r]} a_i^2$.

2: Run the circuit in Fig. 2 to estimate $\tilde{a}_i = \langle t_k | v \rangle \langle v | t_i \rangle$ for $i \in [r]$. Normalize the vector $a = \hat{a} / \|a\|$.  

3: Output the solution as $x = Za$.

The crucial part of Algorithm 2 is to calculate the coefficient $a_i = \langle v | t_i \rangle$, $\forall i \in [r]$. However, the overlap estimation techniques based on the Hadamard Test [33] could not be directly employed for estimating the states overlap, since the unitaries for generating the states are required. This drawback limits most quantum algorithms, e.g., the quantum linear system solver, that require post-selection to yield the efficient easily. Another choice is the SWAP test [33] that only requires copies of states. However, directly using the quantum SWAP test could only obtain the estimation to the value $\langle v | t_i \rangle^2$, while $\text{sign}(a_i)$ remains unknown. To overcome this difficulty, we could assume that the state $|v\rangle$ has the positive overlap with one of the basis $|t_k\rangle$, and analyse the value:

$$a_i = \text{sign}(\langle t_k | v \rangle \langle v | t_i \rangle) |\langle v | t_i \rangle|^2 = \frac{\langle t_k | v \rangle \langle v | t_i \rangle}{|\langle t_k | v \rangle|^2}.$$  

(5)
as the state overlap. This assumption is equivalent to adding a global phase 0 or $e^{i\pi} = -1$ on $|v\rangle$, and will not affect the extraction of the classical description.

$$
\begin{align*}
|0\rangle & \quad \text{H} \\
|v\rangle & \quad \text{H} \\
\frac{1}{\sqrt{2}}(|t_k\rangle|0\rangle + |t_i\rangle|1\rangle) & \quad \text{H}
\end{align*}
$$

FIG. 2. Quantum circuit for estimating $\langle t_k|v\rangle\langle v|t_i\rangle$.

We construct a variant of the SWAP Test, illustrated in Fig. 2 for estimating $\tilde{a}_i = \langle t_k|v\rangle\langle v|t_i\rangle$. The probabilistic statistics of measurement results 00 and 11 yields the main result in Theorem 3 with $\tilde{\beta}_i$ statistics of measurement results 00 and 11 yields the classical form of any eigen-

so similar to the SWAP Test, the proposed quantum circuit provides a $\epsilon$-error estimation to the value of $\langle t_k|v\rangle\langle v|t_i\rangle$ with complexity $O(\sqrt{2\epsilon})$ measurements.

The difficulty of implementing the quantum circuit in Fig. 2 is to efficiently prepare the state $|t_k\rangle|0\rangle + |t_i\rangle|1\rangle/\sqrt{2}$ or $|t_i\rangle|0\rangle + |t_k\rangle|1\rangle/\sqrt{2}$ could be prepared with query complexity $O(\sigma_{\min}^{-1/2}(C))$, see Appendix B for detail. By using this circuit along with the SWAP Test, we could approximately calculate the coordinates $\{x_i\}_{i=1}^n$. The error and time complexity of Algorithm 2 is provided in Theorem 3 with proof given in Appendix B.

Theorem 3. Algorithm 2 provides a classical description $v = \sum_{i=1}^n x_i s_i / \|s_i\|$ with $\ell_2$-norm error bounded in $\epsilon$, by using $O(r^3\epsilon^{-2})$ copies of state $|v\rangle$ and $O(r^4\sigma_{\min}^{-1/2}(C)\epsilon^{-2})$ queries to input oracles.

Thus, our state read-out protocol only requires $O(\text{poly}(r)/\epsilon^2)$ copies of the unknown quantum state. The required state copy complexity is independent from the dimension of the state, which makes our algorithm more efficient than previous QST methods in the low-rank case, since the latter needs at least $O(nc^2)$ copies. We remark that the combination of Theorem 2 and Theorem 3 yields the main result in Theorem 1.

Application. As introduced in previous text, our readout protocol suits the case that the output state of the QML algorithm lies in the row space of the input matrix. We remark that this assumption is naturally satisfied for many previous proposed quantum algorithms in the machine learning and linear algebra field. In this section, we discuss the end-to-end versions of two existing QML algorithms: the quantum singular value decomposition algorithm and the quantum linear system solver, when employing our state read-out protocol for generating classical solutions.

We begin with the quantum singular value decomposition protocol. For a given $r$-rank input matrix $A = \sum_{i=1}^r \sigma_i u_i v_i^T \in \mathbb{R}^{m \times n}$, there is:

$$
\sigma_i = \delta_i = \frac{1}{\sigma_i} \sum_{j=1}^m \langle u_i|A_j|v_j\rangle, \forall j \in [m],
$$

so any singular vector $v_i$ lies in the row space $\text{span}\{A_i\}_{i=1}^m$. Given QRAM oracles of the matrix $A$, quantum SVD allows to perform the operation $\sum_{j} \beta_j|v_j\rangle \rightarrow \sum_{j} \beta_j|v_j\rangle|\sigma_j\rangle$ with complexity $O(\text{polylog}(mn)\|A\|_F\epsilon^{-1})$ such that $\sigma_j \in \sigma_j \pm \epsilon$ with high probability. Consider the state $|0\rangle|0\rangle \sum_{i=1}^r \sigma_i|u_i\rangle|v_i\rangle$ as the input to the quantum SVD algorithm to generate the state $\sum_{i=1}^r \sigma_i|u_i\rangle|v_i\rangle|\sigma_i\rangle$. Then the measurement on the eigenvalue register could collapse the state to different eigenstates $|u_i\rangle|v_i\rangle$ with probability $\frac{\sigma_i^2}{\|A\|_F^2}$. Thus, any target state $|v_i\rangle$ could be prepared with complexity $O(\text{polylog}(mn)\|A\|_2^2\Delta_\sigma^{-1}\epsilon^{-2})$, where $\Delta_\sigma$ is the eigen gap of the matrix $A$. Using this result along with Theorem 1, we could derive the end-to-end complexity for SVD as follows.

Corollary 1. The classical form of any eigen-state $|v_i\rangle$ of $A$ could be obtained by using $O(\text{polylog}(mn)\|A\|_F^2\Delta_\sigma^{-1}\epsilon^{-2})$ queries to the input oracle of $A$, such that the $\ell_2$-norm error is bounded in $\epsilon$ with probability at least $1 - \delta$.

Next, consider a quantum solver to the linear system $Ax = b$, where $A \in \mathbb{R}^{n \times n}$, $x \in \mathbb{R}^n$, and $b \in \mathbb{R}^n$. For matrix $A = \sum_{i=1}^r \sigma_i u_i v_i^T$, the solution could be written as:

$$
x = A^+ b,
$$

where $A^+ = \sum_{i=1}^r \frac{1}{\sigma_i} v_i u_i^T$ is the pseudo-inverse matrix of $A$. Equation (7) gives $x = \sum_{i=1}^r \frac{1}{\sigma_i} v_i u_i^T b v_i \in \text{span}\{v_i\}_{i=1}^r$, which means $x$ also lies in the row space $\text{span}\{A_i\}_{i=1}^m$ by using the previous conclusion about eigenvectors.

There has been an increasing interest in quantum machine learning [44, 45] and linear algebra [13, 15] algorithms following the quantum linear system solver proposed by Harrow, et al. [4]. The first quantum linear system solver was proposed especially for the sparse case by Hamiltonian simulation, and several other different linear system solvers [11, 46] have been proposed subsequently for the general case. Here we consider the quantum solver [11] which encodes the input matrix $A \in \mathbb{R}^{n \times n}$ into the QRAM model. For linear system $Ax = b$, the solution state $|x\rangle = |A^+ b\rangle$ could be prepared in time $O(\kappa^2 \text{polylog}(n)\|A\|_F \epsilon^{-1})$ with $\ell_2$-norm error bounded in
The classical form of the solution state $|A^+b\rangle$ for the linear system $Ax = b$ could be obtained by using $O(\kappa^5\text{poly}(r, \log n)\|A\|_{F}\frac{1}{\varepsilon} \log \frac{1}{\delta})$ queries to input oracles of $A$, such that the $\ell_2$-norm error is bounded in $\varepsilon$ with probability at least $1 - \delta$.

Corollary 2. The classical form of the solution state $|A^+b\rangle$ for the linear system $Ax = b$ could be obtained by using $O(\kappa^5\text{poly}(r, \log n)\|A\|_{F}\frac{1}{\varepsilon} \log \frac{1}{\delta})$ queries to input oracles of $A$, such that the $\ell_2$-norm error is bounded in $\varepsilon$ with probability at least $1 - \delta$.

Conclusion. In this letter, we developed an efficient state read-out framework for quantum machine learning algorithms which involve a low-rank input matrix and the output state $|\psi\rangle$ lies in the row space of the input matrix. The proposed framework takes $O(\text{poly}(r)\varepsilon^{-2})$ copies of output state and $O(\text{poly}(r, \kappa)\varepsilon^{-2})$ queries to input oracles for providing $\varepsilon$ error bounded classical description. Thus, our protocol preserves the quantum speed-up at the state read-out step of these quantum ML algorithms for the case that the rank $r$ and the condition number $\kappa$ is not very large. We analyzed the usability of our framework for quantum algorithms including the quantum SVD and the QRAM-based linear system solver in the low-rank case.

We remark that our protocol could be generalized to the case that the output quantum state lies in a known low-rank subspace that is not limited to the input matrix. Moreover, the proposed results about decoding the pure state could be extended into the mixed state case. Namely, we could first employ the quantum PCA [12] to perform the eigen-decompositions, and then to decode the eigenstates using our protocol. Another future direction is to improve our read-out framework such that the complexity is polynomial in both the rank and the condition number.

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In this section, we analyze the Quantum Gram-Schmidt Process (QGSP) algorithm (Algorithm \[1\]) in detail. In the \(\ell\)-th iteration, the quantum circuit first creates the state \(\bigoplus_{j=1}^{m} A_j \sum_{j=1}^{m} j \left| j \right\rangle \left| A_j \right\rangle \left| 0 \right\rangle\) with the help of input oracles \(U_A\) and \(V_A\); then a Hadamard gate is applied to the third register, followed by a list of controlled \(U_j\) gate \(C(R_i) = R_i \otimes |0\rangle\left\langle 0\right| + I \otimes |1\rangle\left\langle 1\right|\), where the unitary \(R_i = I - 2|t_i\rangle\langle t_i|, \forall i \in [\ell]\). The final unitary operation is again a Hadamard gate to the third register. The state before measurement is:

\[
|\phi_1^{(\ell)}\rangle = \frac{1}{\|A\|_F} \sum_{j=1}^{m} \|A_j\| \left| j \right\rangle \otimes \left[ |A_j\rangle - \sum_{i=1}^{\ell} |t_i\rangle \langle t_i|A_j\rangle |1\rangle \right],
\]

Then we measure the third register and post-select on result 0, where the probability of outcome 0 is

\[
P_\ell = \frac{1}{\|A\|_F^2} \sum_{j=1}^{m} \|A_j\|^2 \left\| |A_j\rangle - \sum_{i=1}^{\ell} |t_i\rangle \langle t_i|A_j\rangle \right\|^2, \quad (A1)
\]

and the post-selected state is

\[
|\phi_2^{(\ell)}\rangle = \frac{1}{\sqrt{P_\ell} \|A\|_F} \sum_{j=1}^{m} \|A_j\| \left| j \right\rangle \left[ |A_j\rangle - \sum_{i=1}^{\ell} |t_i\rangle \langle t_i|A_j\rangle \right].
\]

(A2)

Note that we need roughly \(1/P_\ell\) copies of \(|\phi_1^{(\ell)}\rangle\) to generate the state \(|\phi_2^{(\ell)}\rangle\). Finally, we measure the first register for a new basis index \(g(\ell+1)\) and a new orthogonal state \(|t_{\ell+1}\rangle\):

\[
|t_{\ell+1}\rangle = \frac{1}{Z_{\ell+1}} \left[ |A_g(\ell+1)\rangle - \sum_{i=1}^{\ell} |t_i\rangle \langle t_i|A_g(\ell+1)\rangle \right], \quad (A3)
\]

where \(Z_{\ell+1} = \| |A_g(\ell+1)\rangle - \sum_{i=1}^{\ell} |t_i\rangle \langle t_i|A_g(\ell+1)\rangle \|\) is the normalizing constant.

1. Implementation of \(C(R_\ell)\)

The crucial part in Algorithm \[1\] is to implement the controlled operation:

\[
C(R_\ell) = R_\ell \otimes |0\rangle\left\langle 0\right| + I \otimes |1\rangle\left\langle 1\right|, \forall \ell \in [r - 1],
\]

where \(R_\ell = I - 2|t_\ell\rangle\langle t_\ell|\). In the following, we denote \(s_i := A_g(i)\) for the simplicity of notation. By the definition of states \(|t_\ell\rangle\)_{\ell=1}:

\[
|t_\ell\rangle = \frac{1}{Z_\ell} (|s_\ell\rangle - \sum_{i=1}^{\ell-1} |t_i\rangle \langle t_i|s_\ell\rangle), \quad (A4)
\]
each state $|t_{\ell}\rangle$ can be written as the linear combination of states $\{|s_i\rangle\}_{i=1}^\ell$ with some coefficients $\{z_{ij}\}_{j=1}^\ell$, namely,

$$
|t_{\ell}\rangle = \sum_{j=1}^\ell z_{\ell j}|s_j\rangle.
$$

By using the linear combination of unitaries (LCU) and the Hamiltonian simulation methods, we could implement operation $C(R_\ell t_{\ell})$ with given coefficients $\{z_{ij}\}_{j=1}^\ell$. Define the $\ell \times \ell$ matrix $C_\ell, \forall \ell \in [r]$ whose $(i, j)$-th element is $c_{ij} = \langle s_i|s_j\rangle$. The main result in implementing operations $C(R_\ell t_{\ell})$ is provided as follows.

**Theorem 4.** The operation $C(R_\ell t_{\ell})$ can be implemented with error $\epsilon$ by using $O(\ell \epsilon_{\min}^{-1/2}(C_\ell) \epsilon^{-1})$ queries to the oracle $U_A$, $O(\ell^3)$ additional classical operations and $O(\ell^2)$ inner product operations between rows of $A$.

**Proof.** First we provide the following lemma that gives the coefficients $\{z_{ij}\}_{j=1}^\ell$. Notation $|\cdot|$ here denotes the determinant of a matrix.

**Lemma 1.** The coordinates in $|t_{\ell}\rangle = \sum_{i=1}^\ell z_{\ell i}|s_i\rangle$ could be written in the vector form $z_{\ell} = \sqrt{\frac{|C_\ell|}{|C_{\ell-1}|}} C_{\ell-1}^{-1} e_{\ell}$, where $e_{\ell} = (0, 0, \cdots, 0, 1)^T \in \mathbb{R}^\ell$.

**Proof.** Consider the state:

$$
|t_{\ell}\rangle = \sum_{i=1}^\ell z_{\ell i}|s_i\rangle = \frac{1}{Z_{\ell}}(|s_\ell\rangle - \sum_{i=1}^{\ell-1} |t_i\rangle \langle t_i|s_\ell\rangle).
$$

Multiply $\langle t_\ell|$ on both sides could yield:

$$
\langle s_\ell|t_\ell\rangle = \sum_{i=1}^\ell z_{\ell i}\langle s_\ell|s_i\rangle = Z_{\ell}.
$$

The restriction that $|t_\ell\rangle$ is normalized and is orthogonal to states $|s_1\rangle, |s_2\rangle, \cdots, |s_{\ell-1}\rangle$ could yield:

$$
\langle s_j|t_\ell\rangle = \sum_{i=1}^\ell z_{\ell i}\langle s_j|s_i\rangle = 0, \forall j \in [\ell - 1],
$$

$$
\langle t_\ell|t_\ell\rangle = \sum_{j=1}^\ell \sum_{i=1}^\ell z_{\ell i}z_{\ell j}\langle s_j|s_i\rangle = 1.
$$

Rewrite Equation (A5) and (A6) in the vector form:

$$
C_{\ell} z_{\ell} = Z_{\ell} e_{\ell}.
$$

Equation (A7) could be written as:

$$
1 = \sum_{i,j=1}^\ell z_{\ell i}z_{\ell j}\langle s_j|s_i\rangle = z_{\ell}^T C_{\ell} z_{\ell}
= Z_{\ell}^2 e_{\ell}^T C_{\ell-1}^{-1} e_{\ell} = Z_{\ell}^2 \frac{|C_{\ell-1}|}{|C_\ell|},
$$

where the third equation derives from $z_{\ell} = Z_{\ell} C_{\ell}^{-1} e_{\ell}$ by Equation (A8) and the last equation is derived by noticing that the $(\ell, \ell)$-th element of $C_{\ell-1}^{-1}$ is $\frac{|C_{\ell-1}|}{|C_\ell|}$. Thus, we obtain:

$$
Z_{\ell} = \sqrt{\frac{|C_\ell|}{|C_{\ell-1}|}}.
$$

(A9)

Finally, solving (A8) is trivial:

$$
z_{\ell} = Z_{\ell} C_{\ell}^{-1} e_{\ell} = \sqrt{\frac{|C_\ell|}{|C_{\ell-1}|}} C_{\ell-1}^{-1} e_{\ell}.
$$

(A10)

Given coefficients $\{z_{ij}\}_{j=1}^\ell$, now we provide a framework to implement operations $C(R_\ell t_{\ell})$.

We could first prepare the pure state $\rho_{\ell} = |t_{\ell}\rangle \langle t_{\ell}|$ by the linear combination of unitaries method as follows. Firstly, initialize the state $|0\rangle \otimes \log m|0\rangle \otimes \log n|0\rangle$. Then, we apply Hadamard operations on the last log $\ell$ qubits in the first register to create the state:

$$
\frac{1}{\sqrt{\ell}} \sum_{i=1}^\ell |i\rangle |0\rangle |0\rangle.
$$

Next, we employ the operation

$$
U_{\text{index}} = \prod_{i=1}^\ell (I - |i\rangle \langle i| - |g(i)\rangle \langle g(i)| + |i\rangle \langle g(i)| + |g(i)\rangle \langle i|)
$$

(A11)

to swap states $|i\rangle$ and $|g(i)\rangle, \forall i \in [\ell]$, to yield the state:

$$
\frac{1}{\sqrt{\ell}} \sum_{i=1}^\ell |g(i)\rangle |0\rangle |0\rangle.
$$

The unitary $U_{\text{index}}$ could be implemented by $O(\ell)$ operations. Then we employ the oracle $U_A$ on the first and the second register, followed by the unitary $U_{\text{index}}^*$, to yield:

$$
\frac{1}{\sqrt{\ell}} \sum_{i=1}^\ell |i\rangle |A_g(i)\rangle |0\rangle \equiv \frac{1}{\sqrt{\ell}} \sum_{i=1}^\ell |i\rangle |s_i\rangle |0\rangle.
$$

Denote $z_{\ell} \equiv \max_i |z_{\ell i}|$. Then we perform the controlled rotation

$$
\sum_{i=1}^{\ell} |i\rangle \langle i| \otimes e^{-i\pi \sigma_y \arccos(z_{\ell i}/z_{\ell})} + \sum_{i=\ell+1}^{m} |i\rangle \langle i| \otimes I
$$

on the third register, conditioned on the first register $|i\rangle$, to obtain:

$$
\frac{1}{\sqrt{\ell}} \sum_{i=1}^\ell |i\rangle |s_i\rangle \left( z_{\ell i} \langle 0| + \sqrt{1 - \frac{z_{\ell i}^2}{z_{\ell}^2}} \langle 1| \right).
$$
Finally, we employ Hadamard operations on the first register, to obtain the state

\[ \frac{1}{\ell} \sum_{i=1}^{\ell} |0\rangle z_i |s_i\rangle |0\rangle + \text{orthogonal garbage state} \]

or

\[ \frac{1}{\ell} z_\ell |t_\ell\rangle |0\rangle + \text{orthogonal garbage state}. \]

The measurement on the first and the third registers of the final state could yield state \(|t_\ell\rangle\) with success probability \(1/2\ell^2 z_\ell^2\), so we could prepare the state \(|t_\ell\rangle\) with \(O(\ell \cdot z_\ell)\) queries to \(U_A\) by using the amplitude amplification method \((\ref{eq:amplitude})\).

Note that operations \(R_y = I - 2|t_\ell\rangle\langle t_\ell|\) can be viewed as the unitary with Hamiltonian \(\rho_\ell = |t_\ell\rangle\langle t_\ell|\):

\[ e^{-i\pi \rho_\ell} = 1 + (-i\pi \rho_\ell) + \frac{1}{2!}(-i\pi \rho_\ell)^2 + \cdots \]

\[ = 1 - \rho_\ell + \rho_\ell \left[ 1 + (-i\pi) + \frac{1}{2!}(-i\pi)^2 + \cdots \right] \]

\[ = 1 - \rho_\ell + \rho_\ell e^{-i\pi} \]

\[ = I - 2|t_\ell\rangle\langle t_\ell|. \]

Therefore, by using the Hamiltonian simulation method developed in Quantum PCA \((\ref{eq:hamiltonian})\), the controlled version of \(R_y\) could be performed with error \(\epsilon\) consuming \(O(\pi^2/\epsilon) = O(1/\epsilon)\) copies of \(\rho_\ell\). Taking the complexity of generating state \(|t_\ell\rangle\) into account, we could implement operation \(C(R_y)\) with error bounded by \(\epsilon\) by using \(O(\ell \cdot \text{max} |z_\ell|^{-2}\epsilon^{-1})\) queries to \(U_A\). Based on Equation \((\ref{eq:lemma})\), the \(\ell_2\)-norm of vector \(z_\ell\) is bounded as

\[ 1 = z_\ell^T C_\ell z_\ell \geq \|z_\ell\|^2 \sigma_{\min}(C_\ell), \]

which yields:

\[ \max_i |z_{i\ell}| \leq \|z_\ell\| \leq \sigma_{\min}^{-1/2}(C_\ell). \]  

(A12)

So the query complexity for implementing \(C(R_y)\) could be bounded as \(O(\ell \sigma_{\min}^{-1/2}(C_\ell)\epsilon^{-1})\). To obtain the coefficients \(\{z_{ij}\}_{i=1}^\ell\) by Lemma \((\ref{eq:lemma})\) we need to first calculate \(c_{ij} = \langle s_i | s_j \rangle, \forall i, j \in [\ell]\), which takes \(O(\ell^2)\) inner product operations between rows of \(A\). Calculating the determinants \(|C_\ell|\) and \(|C_{\ell-1}|\) takes \(O(\ell^3)\) classical operations by using SVD. Solving the linear system \((\ref{eq:lemma})\) takes \(O(\ell^3)\) classical operations by using SVD. Now we have proved Theorem \((\ref{eq:proof})\). \(\square\)

2. Proof of Theorem \((\ref{eq:proof})\)

In this section, we prove Theorem \((\ref{eq:proof})\). Before we detail the error and time analysis, we first provide some useful theoretical bounds (Lemma \((\ref{eq:lemma})\), Lemma \((\ref{eq:lemma})\), Lemma \((\ref{eq:lemma})\), and Lemma \((\ref{eq:lemma})\)).
Note that
\[ \sum_{k=1}^{r} c_k = \sum_{i=1}^{r} w_i^2 = \sum_{i=1}^{\ell} [WW^T]_{ii} = \ell. \quad (A22) \]
Hence by using Eqs. (A20, A22), we could obtain the lower and upper bounds for \( P_t \) as follows.
\[ P_t \geq 1 - \frac{1}{\|A\|_F^2} \sum_{i=1}^{\ell} \sigma_i^2 = \frac{\sum_{i=\ell+1}^{r} \sigma_i^2}{\|A\|_F^2}, \quad (A23) \]
\[ P_t \leq 1 - \frac{1}{\|A\|_F^2} \sum_{i=r-\ell+1}^{r} \sigma_i^2 = \frac{\sum_{i=1}^{\ell} \sigma_i^2}{\|A\|_F^2}. \quad (A24) \]

**Lemma 3.** Define \( P \) to be the distribution of the adaptive sampling:
\[ P(s_{\ell+1}|s_1, \ldots, s_\ell) = \frac{\|s_{\ell+1} - \pi_{S_{\ell}}(s_{\ell+1})\|^2}{\sum_{s_{\ell+1}=1}^{m} \|s_{\ell+1} - \pi_{S_{\ell}}(s_{\ell+1})\|^2}, \quad (A25) \]
where \( S_{\ell} = (s_1, s_2, \ldots, s_\ell)^T \in \mathbb{R}^{\ell \times n} \) is the row submatrix formed by row vectors of \( A \) sampled in previous \( \ell \) iterations, \( \pi_{S_{\ell}}(x) \) denotes the projection of vector \( x \) on the row space of \( S_{\ell} \), and \( s_{\ell+1} \) denotes the index of the row \( s_{\ell+1} \). Then
\[ \mathbb{E}_P[\sigma_{\min}^{-1}(C_{\ell+1})] \leq \frac{(\ell + 1)^r}{r - \ell} 2^{2\ell}. \]

**Proof.** We can rewrite Eq. (A25) as follows:
\[ P^{(\ell)}(s_{\ell+1}|s_1, \ldots, s_\ell) := P(s_{\ell+1}|s_1, \ldots, s_\ell) \]
\[ = \frac{\|s_{\ell+1}\|^2 \|s_{\ell+1} - \pi_{S_{\ell}}(s_{\ell+1})\|^2}{\sum_{s_{\ell+1}=1}^{m} \|s_{\ell+1} - \pi_{S_{\ell}}(s_{\ell+1})\|^2}, \quad (A26) \]
\[ = \frac{\|s_{\ell+1}\|^2 Z_{\ell+1}}{\sum_{i=1}^{m} \|C_{\ell+1}\|_{C_{\ell+1}}}, \quad (A27) \]
\[ = \frac{\|s_{\ell+1}\|^2 \|C_{\ell+1}\|_{C_{\ell+1}}}{\sum_{i=1}^{m} \|s_{\ell+1} - \pi_{S_{\ell}}(s_{\ell+1})\|^2}, \quad (A28) \]
where, in Eq. (A27), we denote
\[ \Sigma^{(\ell)} = \sum_{s_{\ell+1}=1}^{m} \|s_{\ell+1} - \pi_{S_{\ell}}(s_{\ell+1})\|^2. \quad (A30) \]
Eq. (A28) is derived from Eq. (A4), and Eq. (A29) is due to Eq. (A9). Note that for any \( 1 \leq j \leq \ell, \Sigma^{(j)} \) only depends on matrices \( A \) and \( S_j \), so it can be viewed as the function of \((s_1, s_2, \ldots, s_j)\) when treating \( A \) as the constant matrix, namely,
\[ \Sigma^{(j)} := \Sigma^{(j)}(s_1, s_2, \ldots, s_j) \]
\[ = \|A\|_F^2 P_j, \quad (A31) \]
where Eq. (A32) comes from the definition of \( P_j \) in (A1).

The expectation of the value \( \sigma_{\min}^{-1}(C_{\ell+1}) \) under the distribution \( P \) could be upper bounded as
\[ \mathbb{E}_P[\sigma_{\min}^{-1}(C_{\ell+1})] \]
\[ = \sum_{s_1=1}^{m} \cdots \sum_{s_{\ell+1}=1}^{m} P(s_1, \ldots, s_{\ell+1}) \sigma_{\min}^{-1}(C_{\ell+1}) \]
\[ \leq \sum_{s_1=1}^{m} \cdots \sum_{s_{\ell+1}=1}^{m} \frac{\|s_1\|_2^2 \cdots \|s_{\ell+1}\|_2^2}{\|s_1\|_2^2 \cdots \|s_{\ell+1}\|_2^2} \sum_{s_1=1}^{\ell} C_{\ell+1}^{(j)} \]
\[ \leq \sum_{i=1}^{\ell+1} \sum_{s_{i+1}=1}^{m} \frac{\|s_i\|_2^2}{\|s_{i+1}\|_2^2} \sum_{j=1, j \neq i}^{\ell} \frac{\|s_j\|_2^2}{\|s_{i+1}\|_2^2} C_{\ell+1}^{(j)}, \quad (A35) \]
where Eq. (A34) uses Eq. (A29), Eq. (A35) uses
\[ \sigma_{\min}(C_{\ell+1}) = \sigma_{\max}(C_{\ell+1}) \leq \text{Tr}(C_{\ell+1}^{-1}) = \sum_{i=1}^{\ell+1} C_{\ell+1}^{(i)}, \quad (A36) \]
with \( C_{\ell+1}^{(i)} \in \mathbb{R}^{\ell \times \ell} \) being the principal submatrix of \( C_{\ell+1} \) by removing the \( i \)-th row and column, and Eq. (A36) follows by rearranging the sum order.

Remark that, by employing the lower and upper bounds of \( P_j \) in Eqs. (A23) and (A24), we could bound the function \( \Sigma^{(j)} \) as
\[ \sum_{i=1}^{r} \sigma_i^2(A) \leq \Sigma^{(j)}(s_1, \ldots, s_j) \]
\[ \geq \sum_{i=j+1}^{r} \sigma_i^2(A) \]
\[ \geq \frac{\|s_1\|_2^2 \cdots \|s_{\ell+1}\|_2^2}{\|s_1\|_2^2 \cdots \|s_{\ell+1}\|_2^2} \sum_{i=1}^{\ell+1} C_{\ell+1}^{(j)}, \quad (A37) \]
where Eq. (A37) holds for any choice of linearly independent row vectors for \( \Sigma^{(j)} \). Then Eq. (A32) together with Eq. (A37) yields
\[ \Sigma^{(j)}(s_1, \ldots, s_j) \]
\[ \leq \sum_{i=j+1}^{r} \sigma_i^2(A) \]
\[ \geq \frac{\|s_1\|_2^2 \cdots \|s_{\ell+1}\|_2^2}{\|s_1\|_2^2 \cdots \|s_{\ell+1}\|_2^2} \sum_{i=1}^{\ell+1} C_{\ell+1}^{(j)}, \quad (A38) \]
\[ \geq \frac{\sum_{i=1}^{\ell} \sigma_i^2(A) \|s_i\|_2^2}{\sum_{i=1}^{\ell} \|s_i\|_2^2} \sum_{i=1}^{\ell+1} C_{\ell+1}^{(j)} \]
\[ \geq \frac{\sum_{i=1}^{\ell} \sigma_i^2(A) \|s_i\|_2^2}{\sum_{i=1}^{\ell} \|s_i\|_2^2} \sum_{i=1}^{\ell+1} C_{\ell+1}^{(j)}, \quad (A39) \]
Combining Eq. (A37) and Eq. (A40) yields a further bound on Eq. (A36):
\[ \leq \sum_{i=1}^{\ell+1} \sum_{s_{i+1}=1}^{m} \frac{\|s_i\|_2^2}{\sum_{k=\ell+1}^{\ell} \sigma_k^2(A)} \sum_{j=1, j \neq i}^{\ell+1} \frac{\|s_j\|_2^2}{\sum_{k=0}^{\ell} \sigma_k^2(A)} C_{\ell+1}^{(j)}, \quad (A41) \]
\[ \leq \left( \frac{\sigma_{\max}(A)}{\sigma_{\min}(A)} \right) \cdot \left( \frac{\sum_{i=1}^{\ell} \|s_i\|_2^2}{\sum_{i=1}^{\ell} \|s_i\|_2^2} \right) \sum_{i=1}^{\ell+1} \sum_{s_{i+1}=1}^{m} \sigma_i^2(A) \]
\[ = \sum_{s_{j=1, j \neq i}}^{m} \cdot \sum_{j=1, j \neq i}^{\ell+1} \frac{\|s_j\|_2^2}{\sum_{k=0}^{\ell} \sigma_k^2(A)} C_{\ell+1}^{(j)}, \quad (A42) \]
where Eq. (A37) is applied in Eq. (A41), and in Eq. (A42) we denote

\[ \Sigma(j) = \left\{ \begin{array}{ll}
\Sigma(j)(s_1, s_2, \ldots, s_j), & \forall j < i, \\
\Sigma(j)(s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_{j+1}), & \forall j \geq i.
\end{array} \right. \]  

(A43)

Notice that in Eq. (A42),

\[ \sum_{s_j=1}^{m} \frac{\prod_{j=1}^{\ell+1} s_{j}}{\sum_{j=1, j \neq \ell}^{C(\ell)}} = 1 \]  

(A44)

which can be interpreted as the probability for sampling \((s_1, s_2, \ldots, s_{i-1}, s_{i+1}, \ldots, s_{j+1})\) over all choice of indices. Finally, Eq. (A42) can lead to

\[ E_P[\sigma_{\min}^{-1}(C_{\ell+1})] \leq \left( \frac{2}{r - \ell} \right) \frac{\sum_{i=1}^{m} \sigma_{\max} E P[A(\ell)]}{\sum_{i=1}^{m} \sigma_{\min}^{-1}(A)} \]  

(A45)

\[ \leq \left( \frac{2(r+1)}{r - \ell} \right) \frac{\sigma_{\max}^{2}(A)}{\sigma_{\min}^{2}} \]  

(A46)

\[ \leq \left( \frac{2(r+1)}{r - \ell} \right) \frac{\sigma_{\max}^{2}(A)}{\sigma_{\min}^{2}} \]  

(A47)

\[ \leq \left( \frac{2(r+1)}{r - \ell} \right) \frac{\sigma_{\max}^{2}(A)}{\sigma_{\min}^{2}} \]  

(A48)

\[ = \frac{(r+1)^2}{r - \ell} \frac{1}{\kappa} \]  

(A49)

where \(\sum_{i=1}^{m} \sigma_{\max} E P[A(\ell)] = \|A\|_{\min}^{2} \) and \(\sum_{k=\ell+1}^{r} \sigma_{\min}^{2}(A) \geq (r - \ell) \sigma_{\min}^{2}(A) \) are used in Eq. (A47), and \(\|A\|_{\min}^{2} = \sum_{i=1}^{m} \sigma_{\max}^{2}(A) \) is used in Eq. (A48).

Lemma 4. Denote \( P \) as the distribution of the adaptive sampling:

\[ P(s_{\ell+1}|s_1, \ldots, s_{\ell}) = \frac{\|s_{\ell+1} - \pi S_{s}(s_{\ell+1})\|^2}{\sum_{s_{\ell+1}=1}^{m} \|s_{\ell+1} - \pi S_{s}(s_{\ell+1})\|^2}, \]

as defined in Lemma 3. Then

\[ E_P[\sigma_{\min}^{-1}(C_{\ell+1})] \geq \frac{r - \ell}{(r+1)^2} \frac{1}{\kappa} \]  

Proof. By the Cauchy-Schwarz Inequality, we have:

\[ E_P[\sigma_{\min}(C_{\ell+1}) \cdot E_P[\sigma_{\min}^{-1}(C_{\ell+1})] \geq 1, \]

so:

\[ E_P[\sigma_{\min}(C_{\ell+1})] \geq \frac{1}{E_P[\sigma_{\min}^{-1}(C_{\ell+1})]} \geq \frac{r - \ell}{(r+1)^2} \frac{1}{\kappa} \]  

Instead of the distribution \( P \) defined in Eq. (A25), the adaptive sampling distribution \( \tilde{P} \) employed by noisy gates \( \tilde{R}_i, \forall i \in [r] \), in Algorithm 4 could be written as

\[ \tilde{P}^{(\ell)}(s_{\ell+1}|s_1, \ldots, s_{\ell}) = \frac{1}{\Sigma(s)} \|s_{\ell+1}\|^2 \left\| \frac{\tilde{P}_{\ell} + I}{2} |s_{\ell+1}\|^2 \right\|, \]  

(A50)

with \( \tilde{P}_{\ell} = \prod_{\ell=1}^{r} \tilde{R}_i, \forall \ell \in [r] \), and

\[ \Sigma^{(\ell)} = \sum_{s_{\ell+1}=1}^{m} \|s_{\ell+1}\|^2 \left\| \frac{\tilde{P}_{\ell} + I}{2} |s_{\ell+1}\|^2 \right\|^2, \]

is the normalization factor.

Lemma 5. Given that each quantum operation \( R_i \) in Algorithm 4 is implemented with error:

\[ \|R_i - \tilde{R}_i\| \leq \epsilon = \frac{1}{3r^5} \cdot \kappa^{-2r}, \]  

(A51)

where \( r \) and \( \kappa \) is the rank and the condition number of \( A \), respectively, we have

\[ E_P[\sigma_{\min}(\tilde{C}_r)] \geq \frac{2}{3} E_P[\sigma_{\min}(C_r)], \]

where \( \tilde{P} \) is defined in Eq. (A50), \( P \) is the distribution defined in Lemma 3 and \( \tilde{C}_r \) is the Gram matrix generated by the output of Algorithm 4.

Proof. Denote \( \Pi_{\ell} = \prod_{\ell=1}^{r} R_i \) and \( \tilde{\Pi}_{\ell} = \prod_{\ell=1}^{r} \tilde{R}_i, \forall \ell \in [r] \), then we have \( \|\tilde{\Pi}_{\ell} - \Pi_{\ell}\| \leq \epsilon \) by (48). We could provide a lower bound on the expectation of \( \sigma_{\min}(\tilde{C}_r) \) with the adaptive sampling distribution \( \tilde{P} \) as follows.

\[ E_{\tilde{P}}[\sigma_{\min}(\tilde{C}_r)] = E_{\tilde{P}}[\sigma_{\min}(\tilde{C}_r)] \geq \frac{5r + r - 1}{5r + r} E_{\tilde{P}}[\sigma_{\min}(\tilde{C}_r)] - \frac{1}{6r} E_{\tilde{P}}[\sigma_{\min}(\tilde{C}_r)] \]

(A52)

\[ \geq \frac{5r + r - 2}{6r} E_{\tilde{P}}[\sigma_{\min}(\tilde{C}_r)] - \frac{1}{6r} E_{\tilde{P}}[\sigma_{\min}(\tilde{C}_r)] \]

(A53)

\[ \geq \frac{5r + r - 1}{6r} E_{\tilde{P}}[\sigma_{\min}(\tilde{C}_r)] - \frac{1}{6r} E_{\tilde{P}}[\sigma_{\min}(\tilde{C}_r)] \]

(A54)

\[ = \frac{2}{3} E_{\tilde{P}}[\sigma_{\min}(\tilde{C}_r)], \]

where \( \tilde{P}^{(\ell)} \) is defined in Eq. (A50): the first inequality (A52) requires a bit of work and we will delay the discussion at Eqs. (A55)–(A58): the following inequalities onward repeated the iteration of the first inequality.

The first inequality (A52) follows from, for any \( 1 \leq j \leq r - 1 \), the fact that

\[ E_{\tilde{P}}[\sigma_{\min}(\tilde{C}_r)] \geq \frac{5r + r - 2}{6r} E_{\tilde{P}}[\sigma_{\min}(\tilde{C}_r)] - \frac{1}{6r} E_{\tilde{P}}[\sigma_{\min}(\tilde{C}_r)] \]

(A55)

\[ \geq \frac{5r + r - 1}{6r} E_{\tilde{P}}[\sigma_{\min}(\tilde{C}_r)] - \frac{1}{6r} E_{\tilde{P}}[\sigma_{\min}(\tilde{C}_r)] \]

(A56)

\[ \geq \frac{2}{3} E_{\tilde{P}}[\sigma_{\min}(\tilde{C}_r)], \]

(A57)
\[
\geq \frac{5r + j}{5r + j + 1} \sum_{\rho(i)} P_{tr-1} \sigma_{\min}(\hat{C}_r) - \frac{1}{6r} \sum_{\rho} \sigma_{\min}(C_r).
\]  
(A58)

Eqs. (A57) and (A58) are derived by using \(P_j \geq \frac{\sigma(A)}{|A|} \geq \frac{r-1}{6r^2} \), the definition of \(\epsilon\) in Eq. (A51), and the lower bound in Lemma 4.

\[
\frac{j \epsilon}{2} \leq \frac{j \epsilon}{6r} \leq \frac{r-1}{6r} \leq \frac{1}{6r} \sum_{\rho} \sigma_{\min}(C_r).
\]

Eq. (A60) is derived by noticing that \(\forall j \in \{0, 1, \ldots, r-1\}\), the expectation of any variable \(X = X(s_j) \in [0, 1]\) under the noisy distribution \(\hat{P}^{(j)}\) could be lower bounded as:

\[
E_{\hat{P}^{(j)}}(X) \geq \sum_{s_j+1=1} \hat{P}^{(j)}(s_j+1) X
\]

\[
\geq \sum_{s_j+1=1} \left( \frac{s_j+1}{|A|_F^2} \right) \frac{\|s_j+1\|^2 \|s_j+1\| - \pi S_j(|s_j+1|)\|2 - \frac{j \epsilon}{2}}{P_j + \frac{j \epsilon}{2}} X
\]

\[
\geq P_j \frac{E_{\hat{P}^{(j)}}(X)}{P_j + \frac{j \epsilon}{2}} - \frac{j \epsilon}{2}.
\]  
(A61)

Eq. (A61) is derived by using \(P_j \|A\|_F^2 = \Sigma_{(j)}\) (A32), \(X \in [0, 1]\), and the distribution \(P^{(j)}\) defined in (A27). 

Eq. (A60) is derived by noticing that when \(0 \leq j \leq r-1\), the probability for the \((j+1)\)-th row sampling with gate error could be lower bounded as:

\[
\hat{P}^{(j)}(s_j+1) = \frac{\|s_j+1\|^2 \|s_j+1\| - \pi S_j(|s_j+1|)\|2 - \frac{j \epsilon}{2}}{\Sigma_{(j)}}.
\]

\[
\geq \frac{\|s_j+1\|^2 \|s_j+1\| - \pi S_j(|s_j+1|)\|2 - \frac{j \epsilon}{2}}{P_j + \frac{j \epsilon}{2}}.
\]  
(A62)

Eq. (A66) follows from \(P_j \|A\|_F^2 = \Sigma_{(j)}\) (A32). Eq. (A64) is derived by noticing

\[
\langle s_j+1 | \frac{\Pi_j + I}{2} | s_j+1 \rangle = \frac{\|\Pi_j + I\|^2}{2} - \frac{\|s_j+1\|^2 - \pi S_j(|s_j+1|)\|2 - \frac{j \epsilon}{2}}{P_j + \frac{j \epsilon}{2}}.
\]

Eq. (A66) is derived by noticing:

\[
\Sigma_{(j)} = \sum_{s_j+1=1} \|s_j+1\|^2 \|s_j+1\| - \pi S_j(|s_j+1|)\|2 - \frac{j \epsilon}{2} \geq \frac{2}{3} E P \sigma_{\min}(C_r),
\]

thus,

\[
P_j \geq \frac{2}{3} E P \sigma_{\min}(C_r) \geq \frac{1}{6r^2} \kappa^{-2r+2}.
\]

So we could perform the whole protocol of Algorithm 1

\[
N = O\left( \frac{\log 1/\delta}{P_1} \right) \leq O(\kappa^{-2r+2} \log 1/\delta)
\]

times to guarantee one group of row basis such that \(\sigma_{\min}(C_r) \geq \frac{1}{\kappa^{-2r+2}}\) with probability at least \(1 - \delta\).

Now we move on to analyze the time complexity of Algorithm 1. Denote by \(T_{\text{basis}}\) the required time to implement Algorithm 1 when each \(R_i\) could have an error bounded by \(\epsilon\) defined in Eq. (A51). Let \(T_{\text{R}}\) be the time complexity of oracles \(U_A, V_A\) and let \(T_{\text{R}}\) be the required time to implement each operation \(R_i\). Remark that each operation \(R_i\) in quantum circuit is approximately performed with error. Theorem 4 states that each \(R_i\) with error bounded by \(\epsilon\) can be implemented by using \(O(\kappa^{-1/2} (C_i) \epsilon^{-1})\) queries to input oracles.
We could prepare the state 

\[ \frac{1}{\sqrt{2}} (|0 \rangle + |1 \rangle) / \sqrt{2} \]

Lemma 6. Given coefficients \( \{ z_{jk} \}_{j=1}^k \) such that \( |t_k \rangle = \sum_{j=1}^k z_{jk} |s_j \rangle \) and \( |t_k \rangle = \sum_{j=1}^k z_{jk} |s_j \rangle \), the state \( \frac{1}{\sqrt{2}} (|t_k \rangle + |1 \rangle |t_k \rangle) \) could be prepared with query complexity \( O(r^2 \sigma_{\min}(C) \epsilon^{-1}) \).

Proof. We could prepare the state \( \frac{1}{\sqrt{2}} (|t_k \rangle + |1 \rangle |t_k \rangle) \) by using the circuit in Fig. 2, which depends on the efficiency of preparing the input state \( (|t_k \rangle |0 \rangle + |t_k \rangle |1 \rangle) / \sqrt{2} \). According to Lemma 6, we prove the Theorem 2 below proves that it can be prepared with query complexity \( O(r^2 \sigma_{\min}(C) \epsilon^{-1}) \).

Appendix B: Proof of Theorem 3

We will first demonstrate that the proposed quantum circuit in Fig. 2 is similar to the SWAP test, and provides a \( \epsilon \)-error estimation to the value \( \tilde{a}_i = \langle t_k | v \rangle (v | t_i \rangle) \), \( \forall i \in [r] \), with \( O(1/\epsilon^2) \) measurements.

Firstly, after all unitary operations, the state in Fig. 2 before the measurements is:

\[
\begin{aligned}
&\frac{1}{4} |0 \rangle \left[ |v \rangle |t_k \rangle + |v \rangle |t_i \rangle + |t_k \rangle |v \rangle + |t_i \rangle |v \rangle \right] |0 \rangle \\
&+ \frac{1}{4} |0 \rangle \left[ |v \rangle |t_k \rangle - |v \rangle |t_i \rangle + |t_k \rangle |v \rangle - |t_i \rangle |v \rangle \right] |1 \rangle \\
&+ \frac{1}{4} |1 \rangle \left[ |v \rangle |t_k \rangle + |v \rangle |t_i \rangle - |t_k \rangle |v \rangle - |t_i \rangle |v \rangle \right] |0 \rangle \\
&+ \frac{1}{4} |1 \rangle \left[ |v \rangle |t_k \rangle - |v \rangle |t_i \rangle - |t_k \rangle |v \rangle + |t_i \rangle |v \rangle \right] |1 \rangle.
\end{aligned}
\]

Measuring the first and the last register could result in outcomes 00 and 11 with probability:

\[
\begin{aligned}
P_{00} &= \frac{2 + |\langle v | t_k \rangle|^2 + |\langle v | t_i \rangle|^2 + 2 \langle t_i | v \rangle \langle v | t_k \rangle}{8}, \\
P_{11} &= \frac{2 - |\langle v | t_k \rangle|^2 - |\langle v | t_i \rangle|^2 + 2 \langle t_i | v \rangle \langle v | t_k \rangle}{8}.
\end{aligned}
\]

We remark that the statistics of outcomes 00 and 11 implies the value \( \tilde{a}_i \).

\[
P_{\text{same}} = P_{00} + P_{11} = \frac{1 + \langle t_i | v \rangle \langle v | t_k \rangle}{2} = 1 + \tilde{a}_i.
\]

Then \( \tilde{a} \) defined in Equation 3 could be obtained by normalizing \( \tilde{a} \). The efficiency of the quantum circuit in Fig. 2 depends on the efficiency of preparing the input state \( (|t_k \rangle |0 \rangle + |t_k \rangle |1 \rangle) / \sqrt{2} \). Lemma 6 below proves that it can be prepared with query complexity \( O(r^2 \sigma_{\min}(C) \epsilon^{-1}) \).
Denote by \( \sum \) for reading out the state \( \sum_{j=k+1}^{\ell} |j\rangle|s_j\rangle|1\rangle \) + |1\rangle \sum_{j=1}^{\ell} |j\rangle|s_j\rangle \left( \frac{z_j^{\ell}}{z} |0\rangle + \sqrt{1-\frac{z_j^{\ell}}{z^2}} |1\rangle \right) \)

The unitary \( [47] \) could be performed by using \( O(\ell) \) quantum operations due to the \( O(\ell) \) sparsity. Finally, we employ Hadamard operations on the second register, to obtain the state

\[
\frac{1}{\sqrt{2^\ell z^2}} \sum_{j=1}^{k} |0\rangle |z_j\rangle |s_j\rangle |0\rangle + \frac{1}{\sqrt{2^\ell z}} |1\rangle \sum_{j=1}^{\ell} |z_j\rangle |s_j\rangle |0\rangle
+ \text{orthogonal garbage state}
= \frac{1}{\sqrt{2^\ell z}} \left( |0\rangle |0\rangle |t_k\rangle |0\rangle + |1\rangle |0\rangle |t_k\rangle |0\rangle \right)
+ \text{orthogonal garbage state}.
\]

The measurement on the 2-nd and the 4-th registers of the final state could yield state \( \sqrt{2^\ell z^2} |0\rangle |0\rangle |t_k\rangle |0\rangle \) with probability \( 1/(2^\ell z^2) \), so we could prepare this state with \( O(\ell z) \) queries to \( U_A \) by using the amplitude amplification method \([45]\). By using Equation \([A12]\), the complexity is further bounded as \( O(r \sigma_{\text{min}}^3(C)) \).

Now we begin the proof of Theorem 3 that provides the error analysis of Algorithm 2 for reading out the state \( |v\rangle \).

\textbf{Proof.} Denote by \( \nu' \) the classical description generated by Algorithm 2 and let \( v \) be the ideal classical description of the quantum state \( |v\rangle \). We can express \( \nu' = \sum_{i=1}^{r} a_i' t_i \) in terms of the orthonormal basis \( \{t_i\} \), and similarly \( v = \sum_{i=1}^{r} a_i t_i \).

Then the readout error \( \|v' - v\| \) could be bounded as follows.

\[
\|v' - v\| = \left\| \sum_{i=1}^{r} a_i' t_i - \sum_{i=1}^{r} a_i t_i \right\|
= \sqrt{\sum_{i=1}^{r} (a_i' - a_i)^2}
= \sum_{i=1}^{r} \left( \frac{\tilde{a}_i'}{\|a'\|} - \frac{\tilde{a}_i}{\|a\|} \right)^2,
\]

where \( a' = \tilde{a}'/\|a'\| \) and \( \tilde{a}_i' \) is the \( i \)-th component of \( \tilde{a}' \).

Continuing from Eq. \([B4]\) and, for the time being, assuming \( |a_i' - \tilde{a}_i| \leq \epsilon_i \), for all \( i \), we have

\[
\left| \frac{\tilde{a}_i'}{\|a'\|} - \frac{\tilde{a}_i}{\|a\|} \right| \leq \frac{|\tilde{a}_i' + \epsilon_i|}{\|a\| - \|a'\|} \frac{|\tilde{a}_i|}{\|a\|}
\leq \frac{|\tilde{a}_i| + \epsilon_i}{\|a\| - \|a'\|} \frac{|\tilde{a}_i|}{\|a\|},
\]

where Eq. \([B6]\) follows from the fact that \( \|a' - \tilde{a}\| = \sqrt{\sum_{i=1}^{r} (a_i' - \tilde{a}_i)^2} \leq \sqrt{r} \epsilon_1 \). Eq. \([B7]\) can be further upper bounded by

\[
\left| \frac{\tilde{a}_i'}{\|a'\|} - \frac{\tilde{a}_i}{\|a\|} \right| \leq \frac{\sqrt{r} + 1}{\sqrt{r} - \sqrt{r} \epsilon_1} \frac{|\tilde{a}_i|}{\|a\|},
\]

where Eq. \([B8]\) is derived by using \( \|\tilde{a}\| = |\langle v | t_k \rangle| \) and

\[
|\langle v | t_k \rangle| = \max_{i \in [r]} |\langle v | t_i \rangle| \geq \sqrt{\frac{1}{r} \sum_{i=1}^{r} |\langle v | t_i \rangle|^2} = \frac{1}{\sqrt{r}}.
\]

Eq. \([B9]\) holds because Step 1 in Algorithm 2 finds the index \( k \), such that basis \( |t_k\rangle \) has the largest overlap with the state \( |v\rangle \). Finally,

\[
\left| \frac{\tilde{a}_i'}{\|a'\|} - \frac{\tilde{a}_i}{\|a\|} \right| \leq \frac{|\tilde{a}_i|}{\|a\|} \epsilon,
\]

follows by letting \( \epsilon_1 = \frac{\epsilon}{3\sqrt{r}} \) in Eq. \([B8]\), and we can conclude that the readout error \( \|v' - v\| \leq \epsilon \).

Notice that the error \( |a_i' - \tilde{a}_i| \leq \epsilon_i \) induced by Step 2, namely, the SWAP test, of Algorithm 2 can be achieved using \( O(1/\epsilon_i^2) = O(r^2 \epsilon^{-2}) \) copies of the state \( |v\rangle \) and state \( \frac{1}{\sqrt{2}} (|0\rangle |t_k\rangle + |1\rangle |t_i\rangle) \). Therefore the total required resources are \( O(r^3 \epsilon^{-2}) \) copies of \( |v\rangle \) and \( O(r^4 \sigma_{\text{min}}^{-1/2}(C) \epsilon^{-2}) \) queries to input oracle \( U_A \), followed by Lemma 6. \( \square \)