THE HESSIAN SOBOLEV INEQUALITY AND ITS EXTENSIONS

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Abstract. The Hessian Sobolev inequality of X.-J. Wang, and the Hessian Poincaré inequalities of Trudinger and Wang are fundamental to differential and conformal geometry, and geometric PDE. These remarkable inequalities were originally established via gradient flow methods. In this paper, direct elliptic proofs are given, and extensions to trace inequalities with general measures in place of Lebesgue measure are obtained. The new techniques rely on global estimates of solutions to Hessian equations in terms of Wolff’s potentials, and duality arguments making use of a non-commutative inner product on the cone of \( k \)-convex functions.

1. Introduction

Let \( F_k (k = 1, 2, \ldots, n) \) be the \( k \)-Hessian operator defined by

\[
F_k[u] = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k},
\]

where \( \lambda_1, \ldots, \lambda_n \) are the eigenvalues of the Hessian matrix \( D^2 u \) on \( \mathbb{R}^n \). In other words, \( F_k[u] \) is the sum of the \( k \times k \) principal minors of \( D^2 u \), which coincides with the Laplacian \( F_1[u] = \Delta u \) if \( k = 1 \); the operator \( F_2[u] = \frac{1}{2} \left[ (\Delta u)^2 - |D^2 u|^2 \right] \) if \( k = 2 \); and the Monge–Ampère operator \( F_n[u] = \det (D^2 u) \) if \( k = n \).

The basic existence and regularity theory for fully nonlinear equations of Monge–Ampère type which involve the \( k \)-Hessian \( F_k[u] \) is presented in [2]. In [17–18], Trudinger and Wang introduced the concept of the Hessian measure \( \mu[u] \) associated with \( F_k[u] \), and established the fundamental weak continuity theorem. Using these tools, Labutin [8] obtained local pointwise estimates for solutions of Hessian equations in terms of nonlinear Wolff’s potentials, and used them to deduce an analogue of the Wiener criterion. The corresponding global Wolff’s potential estimates were recently derived by N. C. Phuc and the author, and applied to Hessian equations of Lane–Emden type [10], [11]. These estimates are used extensively below, along with the underlying existence theory for Hessian and quotient Hessian equations [2], [13].

The Hessian Sobolev inequality states that, for \( k \)-convex \( C^2 \)-functions \( u \) with zero boundary values,

\[
||u||_{L^q(\Omega)} \leq C \left( \int_{\Omega} |u| F_k[u] \, dx \right)^{\frac{1}{q^*+1}},
\]

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where \( q = \frac{n(k+1)}{n-2k}, \) \( 1 \leq k < \frac{n}{2}, \) and \( \Omega \) is a \((k-1)\)-convex domain with \( C^2 \) boundary in \( \mathbb{R}^n. \) (The corresponding best constant \( C \) and radial minimizers \( u \) are known for \( \Omega = \mathbb{R}^n. \)) When \( k = 1 \) and \( q = \frac{2n}{n-2}, \) this is the classical Sobolev inequality.

The case \( k = \frac{n}{2}, \) for even \( n, \) requires the usual modifications involving exponential integrability.

This fully nonlinear inequality was originally established by X.-J. Wang [24] using the gradient flow method and estimates of solutions of parabolic Hessian equations, with a good control of the constants as \( t \to \infty. \) It effectively reduces the problem to radially symmetric functions in a ball. Earlier work on Monge–Ampère type integrals and their extensions is due to K. Tso [20], [21] for convex functions.

The Hessian Sobolev inequality is central to the theory of Hessian equations and \( k \)-convexity, differential and conformal geometry, in particular, the Yamabe problem for higher order curvatures (\[12], [14], [18], and the literature cited there).

We develop a direct elliptic approach to (1.2) which leads to significant extensions where the lack of symmetry makes a reduction to the radial setup infeasible. It is also applicable to the Hessian Poincaré inequalities

\[
|u|_{L^1(\Omega)} \leq C \left( \int_{\Omega} |F_k[u]| \, dx \right)^{\frac{1}{k+1}},
\]

\[
||Du||_{L^2(\Omega)} \leq C \left( \int_{\Omega} |F_k[u]| \, dx \right)^{\frac{1}{k+1}}.
\]

The preceding inequalities, along with their higher order analogues, were obtained by Trudinger and Wang [16] using the Hessian flow method, which gives the best constants for individual \((k-1)\)-convex domains \( \Omega. \)

In Sec. 3, we will give transparent proofs of (1.3) with the best constant, and (1.4), as well as its higher order versions, with explicit constants, using a nonlinear duality argument discussed below.

The Hessian Sobolev inequality (1.2) is more complex than Poincaré inequalities, which is manifested by the presence of the critical exponent \( q = \frac{n(k+1)}{n-2k}, \) \( k = 1, 2, \ldots, \lceil \frac{n}{2} \rceil. \) Our proofs of (1.2) and its extensions are based in part on the idea of the integral representation, which is reminiscent of Sobolev’s original proof of the classical Sobolev inequality in the linear case. For Hessian functionals and \( k \)-convex functions, integral representations are provided by means of Th. Wolff’s potentials,

\[
W_{\alpha, p, \mu}(x) = \int_0^R \left[ \frac{\mu(B_t(x))}{t^{n-\alpha p}} \right]^\frac{1}{p-1} \, dt, \quad x \in \Omega,
\]

where \( \mu \) is a positive Borel measure on \( \Omega, \) \( \alpha = \frac{2k}{k+1}, \) \( p = k + 1, \) and \( B_t(x) \) is a ball of radius \( t \) centered at \( x. \) For a bounded domain \( \Omega, \) we will set \( R = 2 \text{diam } \Omega. \) If \( \Omega = \mathbb{R}^n, \) then \( R = +\infty; \) in this case we drop the superscript \( R \) and use the notation

\[
W_{\alpha, p, \mu}(x) = \int_0^{+\infty} \left[ \frac{\mu(B_t(x))}{t^{n-\alpha p}} \right]^\frac{1}{p-1} \, dt, \quad x \in \mathbb{R}^n.
\]

Potentials \( W_{\alpha, p, \mu} \) appeared in [9] in relation to Th. Wolff’s inequality, and L. I. Hedberg’s solution of the spectral synthesis problem for Sobolev spaces (see [1], Sections 4.5 and 9.1). Local estimates involving Wolff’s potentials have been used extensively in quasilinear and Hessian equations, especially in relation to analogues of the Wiener criterion [7], [8], [19].
We will use global estimates of solutions to Hessian equations [10] which effectively invert the Hessian operator \( F_k \) on \( \mathbb{R}^n \) with zero boundary conditions at \( \infty \):

\[
C_1 W_{\alpha,p} \mu(x) \leq -u(x) \leq C_2 W_{\alpha,p} \mu(x), \quad x \in \mathbb{R}^n,
\]

where \( \alpha = \frac{2k}{k+1} \), \( p = k + 1 \), and \( C_1 \) and \( C_2 \) are positive constants depending only on \( n \) and \( k \). Here \( u \) is an entire solution to \( F_k[u] = \mu \) which is \( k \)-convex, and vanishes at \( \infty \).

Analogous global estimates for solutions to the Dirichlet problem associated with \( F_k[u] = \mu \) in a bounded domain \( \Omega \) are given by ([10], Theorem 2.2):

\[
C_1 W_{r,\alpha,p} \mu(x) \leq -u(x) \leq C_2 W_{r,\alpha,p} \mu(x), \quad x \in \Omega,
\]

where \( r = \frac{1}{2} \text{dist} (x, \partial \Omega) \), \( R = 2 \text{diam} \Omega \); as above, \( \alpha = \frac{2k}{k+1} \), \( p = k + 1 \), and \( C_1 \), \( C_2 \) are positive constants depending only on \( n \) and \( k \).

Another important tool established in Sec. 3 is a fully nonlinear analogue of Schwarz’s inequality which enables us to employ powerful duality arguments:

\[
\left| \int_{\Omega} u F_k[v] \right| \leq \left( \int_{\Omega} |u| F_k[u] \right)^{\frac{1}{2}} \left( \int_{\Omega} |v| F_k[v] \right)^{\frac{1}{2}},
\]

where \( u, v \) are \( k \)-convex functions with zero boundary values in \( \Omega \).

We observe that the expression \( \langle u, v \rangle = \int_{\Omega} u F_k[v] \) on the left-hand side of the preceding inequality defines a non-commutative inner product (for \( k \geq 2 \)) on the cone of \( k \)-convex functions which plays a role of mutual energy for the corresponding Hessian measures.

This paper is also concerned with extensions of the Hessian Sobolev inequality to a class of the so-called trace inequalities, with general Borel measures \( d\omega \) in place of Lebesgue measure \( dx \) on the left-hand side in ([12], 2). Such inequalities have numerous applications in linear and nonlinear PDE (see [1], [9], [11], [22]).

We will give in Sec. 4 a characterization of the Hessian trace inequality

\[
||u||_{L^q(\Omega, d\omega)} \leq C \left( \int_{\Omega} |u| F_k[u] \, dx \right)^{\frac{1}{q}},
\]

We show that (1.10) holds for \( q > k + 1 \) and \( 1 \leq k < \frac{n}{2} \) if

\[
\omega(B) \leq c |B|^{(1 - \frac{k}{q})^{\frac{1}{q}}},
\]

for all balls \( B \). This is an extension of the well-known theorem of D. R. Adams in the linear case \( k = 1 \) (see [11], Sec. 7.2). In particular, if \( \omega \) is Lebesgue measure on a hyperplane, (1.10) characterizes integrability properties of traces of \( k \)-convex functions on lower dimensional cross-sections of \((k-1)\)-convex bodies \( \Omega \).

In the more difficult case \( q = k + 1 \), we will show that (1.10) holds if

\[
\omega(E) \leq c \text{cap}_k(E),
\]

for every compact set \( E \subset \Omega \). Here \( \text{cap}_k(\cdot) \) is the Hessian capacity introduced by Trudinger and Wang [18], and studied subsequently in [8]. In [10], a complete description of the Hessian capacity is given in terms of the well-understood fractional capacity \( \text{cap}_{2k}\alpha_{2k}(\cdot) \) associated with the Sobolev space \( W^{2k,2k}(\Omega) \).

In the linear case, conditions of the type (1.12) that characterize admissible measures for trace inequalities were introduced by V. G. Maz’ya, and used extensively in the spectral theory of the Schrödinger operator (see [9]).
We observe that (1.11) and (1.12) are sharp, and in fact necessary for (1.10) to hold if \( \Omega = \mathbb{R}^n \), or if \( \omega \) is compactly supported in \( \Omega \).

Other conditions equivalent to (1.12) which do not use capacities are readily available. In particular, we obtain the following fully nonlinear version of the C. Fefferman and D. H. Phong inequality [3]. Suppose that \( d\omega = w \, dx \), where \( w \geq 0 \) is a weight such that
\[
\int_{B_R \cap \Omega} w^{1+\epsilon} \, dx \leq c R^{n-2k(1+\epsilon)},
\]
for some \( \epsilon > 0 \), and every ball \( B_R \). Then (1.10) holds with \( q = k + 1 \).

The above results, including complete proofs of the Hessian Schwarz and Poincaré inequalities, were presented in the special session “Nonlinear Elliptic Equations and Geometric Inequalities” of the AMS meeting at Courant Institute in March 2008. They were also announced at the Oberwolfach Workshop “Real Analysis, Harmonic Analysis and Applications” in July, 2011 (see [23]). Further developments involving the relationship between the Hessian energy and the fractional Laplacian energy are established in [4], [5].

2. HESSIAN OPERATORS AND \( k \)-CONVEX FUNCTIONS

Let \( \Omega \) be an open set in \( \mathbb{R}^n \), \( n \geq 2 \). Let \( k = 1, \ldots, n \). For \( u \in C^2(\Omega) \), let
\[
F_k[u] = S_k(\lambda(D^2u)),
\]
where \( \lambda(D^2u) = (\lambda_1, \ldots, \lambda_n) \) is the spectrum of the Hessian matrix \( D^2u \), and \( S_k \) is the \( k \)-th elementary symmetric function on \( \mathbb{R}^n \), that is,
\[
S_k(\lambda) = \sum_{1 \leq i_1 < \cdots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}.
\]
Equivalently,
\[
F_k[u] = [D^2u]_k,
\]
where \([A]_k\) stands for the sum of the \( k \times k \) principal minors of a symmetric matrix \( A \).

A function \( u \in C^2_{\text{loc}}(\Omega) \) is called \( k \)-convex if
\[
F_j[u] \geq 0 \quad \text{in} \quad \Omega \quad \text{for all} \quad j = 1, \ldots, k.
\]

The notion of \( k \)-convexity can be defined for more general classes of functions in terms of viscosity solutions [17], [18]. A function \( u : \Omega \to [-\infty, \infty) \) is said to be \( k \)-convex in \( \Omega \) if it is upper semi-continuous, and \( F_k[q] \geq 0 \) for any quadratic polynomial \( q \) such that \( u - q \) has a local finite maximum in \( \Omega \).

The class of all \( k \)-convex functions in \( \Omega \) which are not identically equal to \(-\infty\) in each component of \( \Omega \) will be denoted by \( \Phi^k(\Omega) \). We observe that \( \Phi^n(\Omega) \subset \Phi^{n-1}(\Omega) \cdots \subset \Phi^1(\Omega) \) (see [17], [18]). Here \( \Phi^1(\Omega) \) coincides with the set of all subharmonic functions (with respect to the Laplacian) in \( \Omega \), and \( \Phi^0(\Omega) \) is the set of functions convex on each component of \( \Omega \). By \( \Phi^k_0(\Omega) \) we denote the class of all functions from \( \Phi^k(\Omega) \) with zero boundary values.

The theory of \( k \)-convex functions is based on the weak continuity result [18]. It states that for each \( u \in \Phi^k(\Omega) \), there exists a nonnegative Borel measure \( \mu_k[u] \) in \( \Omega \) such that

(i) \( \mu_k[u] = F_k[u] \) for \( u \in C^2(\Omega) \), and
(ii) if \( \{u_m\} \) is a sequence in \( \Phi^k(\Omega) \) converging in \( L^1_{\text{loc}}(\Omega) \) to a function \( u \in \Phi^k(\Omega) \),
then the sequence of the corresponding measures \( \{\mu_k[u_m]\} \) converges weakly to \( \mu_k[u] \).

The measure \( \mu = \mu_k[u] \) in the above theorem is called the \( k \)-Hessian measure associated with \( u \). Property (i) justifies writing \( F_k[u] \) in place of \( \mu_k[u] \) even when \( u \in \Phi^k(\Omega) \) does not belong to \( C^2(\Omega) \).

In what follows we will assume that the domain \( \Omega \) is uniformly \((k - 1)\)-convex, i.e., \( H_j[\partial \Omega] > 0 \) on \( \partial \Omega \) for \( j = 1, 2, \ldots, k - 1 \). Here \( H_j[\partial \Omega] \) is the \( j \)-th mean curvature defined by \( H_j[\partial \Omega] = S_j(\kappa_1, \kappa_2, \ldots, \kappa_{n-1}) \) where \( S_j \) is the \( j \)-th elementary symmetric function of the principal curvatures \( \kappa_1, \kappa_2, \ldots, \kappa_{n-1} \) of \( \partial \Omega \).

3. Duality for Hessian integrals and Poincaré inequalities

In this section we prove a duality theorem for Hessian integrals which will serve as a powerful tool in our approach to Hessian Sobolev and Poincaré type inequalities. It can be regarded as a nonlinear version of the usual Schwarz inequality.

Let \( u, v \in \Phi^k_0(\Omega) \), where \( k = 1, 2, \ldots, n \) and let \( \mu, \nu \in M^+(\Omega) \) be the corresponding Hessian measures. We introduce the following notation and terminology analogous to the classical case \( k = 1 \). By \( \mathcal{E}_k[\mu, \nu] \) we denote the mutual Hessian energy of \( \mu \) and \( \nu \) defined by

\[
\mathcal{E}_k[\mu, \nu] = \langle u, v \rangle = \int_{\Omega} (-v) F_k[u],
\]

where \( \langle u, v \rangle \) is the non-commutative (for \( k \geq 2 \)) inner product on the cone of \( k \)-convex functions mentioned in the Introduction. Since \( v \leq 0 \) by the maximum principle, and \( F_k[u] \geq 0 \), it follows that \( \mathcal{E}_k[\mu, \nu] \geq 0 \). By \( \mathcal{E}_k[\mu] = \mathcal{E}_k[\mu, \mu] \) we denote the Hessian integral

\[
\mathcal{E}_k[\mu] = \int_{\Omega} (-u) F_k[u].
\]

When \( k = 1 \),

\[
\mathcal{E}_1[\mu, \nu] = \int_{\Omega} G[\nu] d\mu = \int_{\Omega} G[\mu] d\nu,
\]

where \( G[\mu](x) = \int_{\Omega} G(x, y) d\mu(y) \) is the Green potential of \( \mu \), and \( G(x, y) \) is the Green function of the Dirichlet Laplacian on \( \Omega \). As is well known,

\[
\mathcal{E}_1[\mu, \nu] = \int_{\Omega} DG[\mu] \cdot DG[\nu] \, dx \leq \mathcal{E}_1[\mu]^{1/2} \mathcal{E}_1[\nu]^{1/2}, \tag{3.4}
\]

where

\[
\mathcal{E}_1[\mu] = \int_{\Omega} G[\mu] d\mu = \int_{\Omega} |DG[\mu]|^2 \, dx
\]
denotes the electrostatic energy of \( \mu \).

We now prove a nonlinear analogue of Schwarz’s inequality (3.4) for Hessian integrals mentioned in the Introduction:

\[
\langle u, v \rangle = \mathcal{E}_k[\mu, \nu] \leq \mathcal{E}_k[\mu]^{1/2k} \mathcal{E}_k[\nu]^{1/2k}. \tag{3.5}
\]

The proof employs a convexity argument that was used earlier by X.-J. Wang in his proof of the Hessian Minkowski inequality (22, Theorem 5.1).

\[
\mathcal{E}_k[\mu + \nu]^{1/1k} \leq \mathcal{E}_k[\mu]^{1/k} + \mathcal{E}_k[\nu]^{1/k}. \tag{3.6}
\]

In the linear case \( k = 1 \), (3.5) and (3.6) are known to be equivalent, but for \( k \geq 2 \) the relationship between them is less obvious.
Theorem 3.1. Let $k = 1, 2, \ldots, n$. Let $\Omega$ be a bounded uniformly $(k - 1)$-convex domain with $C^2$ boundary. Let $u, v \in \Phi^k_0(\Omega)$, and let $\mu, \nu$ be the corresponding Hessian measures. Then (3.5) holds.

Proof. We will use some standard notation and basic facts of the Hessian theory. Let $r = (r_{ij})_{i,j=1}^n$ be a real symmetric matrix, and $\lambda[r]$ its eigenvalues. By $S_k(\lambda[r]) = [r]_k$ ($k = 1, 2, \ldots, n$) we denote the sum of all the $k \times k$ principal minors of $r$. Let $S_k^{ij}[r] = \frac{\partial}{\partial r_{ij}} S_k(\lambda[r])$ ($i, j = 1, 2, \ldots, n$). Then (see [24], p. 29)

\begin{equation}
S_k(\lambda[r]) = \frac{1}{k} \sum_{i,j} r_{ij} S_k^{ij}[r].
\end{equation}

In particular, if $r = D^2 u$, then

\begin{equation}
S_k(\lambda[D^2 u]) = \frac{1}{k} \sum_{i,j} u_{ij} S_k^{ij}[D^2 u] = \frac{1}{k} \sum_{i,j} D_j(D_i u S_k^{ij}[D^2 u]),
\end{equation}

since

\begin{equation}
\sum_{j} D_j(S_k^{ij}[D^2 u]) = 0,
\end{equation}

for every $i = 1, 2, \ldots, n$. Hence,

\begin{equation}
S_k(\lambda[D^2(u + tv)]) = \frac{1}{k} \sum_{i,j} (u_{ij} + tv_{ij}) S_k^{ij}[D^2(u + tv)].
\end{equation}

We will need the following formula for the derivative of $S_k(\lambda[D^2(u + tv)])$:

\begin{equation}
\frac{d}{dt} S_k(\lambda[D^2(u + tv)]) = \sum_{i,j} v_{ij} S_k^{ij}[D^2(u + tv)].
\end{equation}

The preceding equation is deduced using R. C. Reilly’s identities, as in [24] in the case of equation (3.12):

\begin{equation}
S_k(\lambda[D^2(u + tv)]) = \frac{1}{k!} \sum_{i_1, \ldots, i_k} \delta_{j_1, \ldots, j_k} \prod_{s=1}^{k} (u_{i_s j_s} + tv_{i_s j_s}),
\end{equation}

where $\delta_{j_1, \ldots, j_k}$ is the generalized Kronecker delta. This yields (see equation (2.4) in [24])

\begin{equation}
S_k^{ij}[D^2(u + tv)] = \frac{1}{(k - 1)!} \sum_{i_1, \ldots, i_{k-1}} \delta_{j_1, \ldots, j_{k-1}} \prod_{s=1}^{k-1} (u_{i_s j_s} + tv_{i_s j_s}).
\end{equation}

Differentiating both sides of (3.12) with respect to $t$, and invoking the preceding equation, together with the symmetry properties of the Kronecker delta, we see
that \( \frac{d}{dt} S_k(\lambda[D^2(u + tv)]) \) equals

\[
\frac{1}{k!} \sum_{i_1, \ldots, i_k} \delta_{i_1, \ldots, i_k} \sum_{j_1, \ldots, j_k} v_{i_m, j_m} \prod_{s=1}^{k} (u_{i_s, j_s} + tv_{i_s, j_s})
\]

\[
= \frac{1}{k!} \sum_{m=1}^{k} \sum_{i_m, j_m} v_{i_m, j_m} \sum_{i_1, \ldots, i_{m-1}, i_m+1, \ldots, i_k} \delta_{i_1, \ldots, i_k} \prod_{s=1}^{k} (u_{i_s, j_s} + tv_{i_s, j_s})
\]

\[
= \frac{(k-1)!}{k!} \sum_{m=1}^{k} \sum_{i_m, j_m} v_{i_m, j_m} S_k^{i_m, j_m} (D^2[u + tv]) = \sum_{i, j} v_{ij} S_k^{ij} (D^2[u + tv]).
\]

This completes the proof of (3.11).

For \( t \in [0, 1] \), let

\[
h(t) = \int_{\Omega} -(u + tv) S_k(\lambda[D^2(u + tv)]) dx.
\]

Using (3.8) and integrating by parts, we get

\[
h(t) = \frac{1}{k} \int_{\Omega} \sum_{i, j} D_i (u + tv) D_j (u + tv) S_k^{ij} [D^2(u + tv)] dx.
\]

Notice that the matrix \( \left( S_k^{ij} [D^2(u + tv)] \right) \) is nonnegative since \( u + tv \) is \( k \)-convex.

Differentiating both sides of (3.13) with respect to \( t \), and integrating by parts using (3.9)–(3.11), we deduce (see [24], p. 37):

\[
h'(t) = (k + 1) \int_{\Omega} (-v) S_k(\lambda[D^2(u + tv)]) dx
\]

\[
= \frac{k+1}{k} \int_{\Omega} \sum_{i, j} D_i v D_j (u + tv) S_k^{ij} [D^2(u + tv)] dx.
\]

Differentiating further, we obtain

\[
h''(t) = (k + 1) \int_{\Omega} (-v) \frac{d}{dt} S_k(\lambda[D^2(u + tv)]) dx
\]

\[
= (k + 1) \int_{\Omega} \sum_{i, j} D_i v D_j v S_k^{ij} [D^2(u + tv)] dx.
\]

From these calculations, we deduce using Schwarz’s inequality,

\[
h(t)h''(t) \geq \frac{k}{k+1} h'(t)^2, \quad 0 \leq t \leq 1.
\]

In other words, \( h^{\frac{k+1}{k}} \) is convex on \([0, 1]\) ([24], p. 37). This fact yields the Hessian Minkowski inequality (3.6). We now observe that by the convexity of \( h^{\frac{k+1}{k}} \) on \([0, 1]\),

\[
h(0)^{\frac{k+1}{k}} + \frac{d}{dt} h^{\frac{k+1}{k}}(0) \leq h(1)^{\frac{k+1}{k}}.
\]

Notice that

\[
\mathcal{E}_k[\mu, \nu] = \int_{\Omega} (-v) F_k[u] = \frac{1}{k} \int_{\Omega} \sum_{i, j} u_i v_j S_k^{ij} (D^2u) dx.
\]
Thus by (3.13) and (3.15),
\[ h(0) = \mathcal{E}_k[\mu], \quad h'(0) = (k+1)\mathcal{E}_k[\mu, \nu], \quad h(1) = \mathcal{E}_k[\mu + \nu], \]
and hence
\[ \frac{d}{dt} h^{\frac{1}{k+1}}(0) = \frac{1}{k+1} h(0) h'(0) = \mathcal{E}_k[\mu] \mathcal{E}_k[\mu, \nu]. \]
Consequently by (3.16) and (3.6),
\[ \mathcal{E}_k[\mu] \mathcal{E}_k[\mu + \nu] \mathcal{E}_k[\mu, \nu] \leq \mathcal{E}_k[\mu + \nu] \leq \mathcal{E}_k[\mu] \mathcal{E}_k[\nu]. \]
Thus,
\[ \mathcal{E}_k[\mu] \mathcal{E}_k[\mu + \nu] \mathcal{E}_k[\mu, \nu] \leq \mathcal{E}_k[\nu]. \]
This completes the proof of (3.5). □

An immediate consequence of Theorem 3.1 is a global integral inequality of Poincaré type due to Trudinger and Wang [18]. It is used in the proof of both local and global Wolff potential estimates obtained in [8], [10], which play a crucial role in our proof of the Hessian Sobolev inequality and its extensions given below.

Corollary 3.2. Let \( \Omega \) be as in Theorem 3.1, and \( k = 1, 2, \ldots, n \). Let \( u \in \Phi^0_0(\Omega) \cap C^2(\Omega) \). Then

\[ (3.17) \quad \int_{\Omega} (-u) \, dx \leq C \left( \int_{\Omega} (-u) \mathcal{F}_k[u] \, dx \right)^{\frac{1}{k+1}}, \]

where the best constant is

\[ (3.18) \quad C = \left( \int_{\Omega} (-w) \, dx \right)^{\frac{1}{k+1}}. \]

Here \( w \in \Phi^0_0(\Omega) \cap C^2(\Omega) \) is the unique solution to the Dirichlet problem \( \mathcal{F}_k[w] = 1 \); the corresponding minimizer is \( u = w \).

Proof. By the existence theorem for Hessian equations [2], one can find \( w \in \Phi^0_0(\Omega) \cap C^2(\Omega) \) such that \( \mathcal{F}_k[w] = 1 \) in \( \Omega \). By Theorem 3.1

\[ \int_{\Omega} (-u) \, dx \leq \left( \int_{\Omega} (-u) \mathcal{F}_k[u] \, dx \right)^{\frac{1}{k+1}} \left( \int_{\Omega} (-w) \, dx \right)^{\frac{1}{k+1}}. \]

Clearly, \( u = w \) gives equality in (3.17). □

If \( B_R \) is a ball of radius \( R \), then \( w = c(|x|^2 - R^2) \) where \( c = \frac{1}{2}(C^k_n)^{-\frac{k}{n}} \) in the above proof, and we arrive at the following corollary.

Corollary 3.3. If \( B_R \) is a ball of radius \( R > 0 \), and \( c \) is as above, then

\[ \frac{1}{R^n} \int_{B_R} (-u) \, dx \leq c \left( \int_{B_R(0)} (1 - |x|^2) \, dx \right)^{\frac{1}{k+1}} \times \left( \frac{1}{R^{n-2k}} \int_{B_R} (-u) \mathcal{F}_k[u] \, dx \right)^{\frac{1}{k+1}}, \]

for all \( u \in \Phi^0_0(B_R) \cap C^2(B_R) \).

More generally, we can obtain higher order Hessian Poincaré inequalities with sharp constants on balls.
Corollary 3.4. If $B_R$ is a ball of radius $R > 0$ and $k = 1, 2, \ldots, n$, then

\begin{equation}
(3.19) \quad \left( \int_{B_R} (-u) F_{k-1}[u] \, dx \right)^{\frac{1}{k}} \leq C \left( \int_{B_R} (-u) F_k[u] \, dx \right)^{\frac{1}{k+1}},
\end{equation}

for all $u \in \Phi_k^{(0)}(B_R) \cap C^2(B_R)$, where the best constant

\[ C = \left( \int_{B_R} (-w) F_k[w] \, dx \right)^{\frac{1}{k+1}} \]

is attained when $u = w$, where $w = c(|x|^2 - R^2)$, $c = \frac{k}{2(n-k+1)}$.

Proof. We observe that $w = c(|x|^2 - R^2)$ where $c = \frac{k}{2(n-k+1)}$ is a convex solution to the equation $F_k[w] = F_{k-1}[w]$ such that $w = 0$ on $\partial B_R$. Then using the divergence form of $F_k[w]$ (3.8), and integrating by parts, we obtain,

\begin{align*}
(3.20) \quad \int_{B_R} (-w) F_k[u] \, dx &= \frac{1}{k} \sum_{i,j} S_{ij}^k D^2u (-w) w_{ij} dx \\
(3.21) &= \frac{2}{k} \int_{B_R} \sum_i S_{ii}^k D^2u (-w) \, dx \\
(3.22) &= \frac{2(n-k+1)}{k} \int_{B_R} (-u) F_{k-1}[u] \, dx.
\end{align*}

In the last line we used the identity (see [24], Proposition 2.2)

\[ \sum_{i=1}^{n} S_{ii}^k [D^2u] = (n - k + 1) S_{k-1}[D^2u]. \]

From (3.20) – (3.22), applying Theorem 3.1 to estimate the left-hand side, we deduce

\[ \left( \int_{B_R} (-u) F_k[u] \, dx \right)^{\frac{1}{k}} \leq \left( \frac{2c(n-k+1)}{k} \right)^{-\frac{1}{k}} \times \left( \int_{B_R} (-u) F_k[u] \, dx \right)^{\frac{1}{k+1}}. \]

Since $F_k[w] = F_{k-1}[w]$, it follows

\[ \left( \frac{2c(n-k+1)}{k} \right)^{-\frac{1}{k}} \left( \int_{B_R} (-w) F_k[w] \, dx \right)^{\frac{1}{k+1}} = \frac{\left( \int_{B_R} (-w) F_{k-1}[w] \, dx \right)^{\frac{1}{k}}}{\left( \int_{B_R} (-u) F_k[u] \, dx \right)^{\frac{1}{k+1}}}. \]

Thus, setting $u = w$ gives equality in (3.19). \qed

Remark 3.5. Hessian Poincaré inequalities (3.19) for all convex domains $\Omega$ in place of balls, and $k = 1, 2, \ldots, n$, with sharp constant, can be deduced in a similar way.

We next give a simple proof of the higher order Hessian Poincaré inequalities of Trudinger and Wang [10] for general $(k-1)$-convex domains, with explicit constants.
Proof. According to the existence theory for quotient Hessian equations [13], there is a unique solution $w \in \Phi_k^0(\Omega) \cap C^2(\overline{\Omega})$ to the Dirichlet problem $F_k[w] = F_l[w]$. Then by the concavity in $D^2[u]$ of the operators $(F_l[u])^\dagger$ and $(F_k[u]/F_l[u])^\dagger$ on the cone of $k$-convex functions (see [2], [13]), it follows

\[
\int_\Omega (-u) F_k[u + w] \, dx = \int_\Omega (-u) F_l[u + w] \frac{F_k[u + w]}{F_l[u + w]} \, dx \\
\geq \int_\Omega (-u) F_l[u] \frac{F_k[w]}{F_l[w]} \, dx = \int_\Omega (-u) F_l[u] \, dx.
\]

Applying Theorem 3.1 to estimate the left-hand side, we deduce

\[
\int_\Omega (-u) F_l[u] \, dx \leq \int_\Omega (-u) F_k[u + w] \, dx
\]

\[
\leq \left( \int_\Omega (-u) F_k[u] \, dx \right)^{\frac{1}{1+\varepsilon}} \left( \int_\Omega -(u + w) F_k[u + w] \, dx \right)^{\frac{\varepsilon}{1+\varepsilon}}.
\]

Using the Hessian Minkowski inequality (3.4) to estimate the second factor on the right-hand side, we obtain

\[
\int_\Omega (-u) F_l[u] \, dx \leq \left( \int_\Omega (-u) F_k[u] \, dx \right)^{\frac{1}{1+\varepsilon}} \\
\times \left[ \left( \int_\Omega (-u) F_k[u] \, dx \right)^{\frac{1}{1+\varepsilon}} + \left( \int_\Omega -(w - u) F_k[w] \, dx \right)^{\frac{\varepsilon}{1+\varepsilon}} \right]^k.
\]

Replacing $u$ by $cu$, where $c > 0$, in the preceding inequality and minimizing over $c$ yields $\Box$.

\section{Hessian Sobolev and Trace Inequalities on Bounded Domains}

In this section we give a new proof of the Hessian Sobolev inequality [24], and extend it to trace inequalities on a bounded $(k-1)$-convex domain $\Omega \subset \mathbb{R}^n$. We will use duality of Hessian integrals developed in Sec. 3 and the classical Sobolev inequality for fractional integrals.

\begin{theorem}
Let $1 \leq k < \frac{n}{2}$ and $0 < q < \frac{n(k+1)}{n-2k}$. Let $\Omega$ be a bounded uniformly $(k-1)$-convex domain with $C^2$ boundary. Then, for $u \in \Phi_k^0(\Omega) \cap C^2(\overline{\Omega})$,

\[
\left( \int_\Omega |u|^q \, dx \right)^{\frac{1}{q}} \leq C \left( \int_\Omega |u| F_k[u] \, dx \right)^{\frac{1}{1+\varepsilon}},
\]

where $C$ is a positive constant that depends only on $n, k$, and $q$.
\end{theorem}
The Wolff potential \( W \) for \( \Omega = \mathbb{R}^n \) with the best constant which depends on \( \Omega \). The best constant in (4.1) is known for \( \Omega = \mathbb{R}^n \) and \( q = \frac{n(k+1)}{n-2k} \) (see [24]).

**Proof.** Since \( \Omega \) is bounded, we can assume that \( q = \frac{n(k+1)}{n-2k} \) so that \( q > 1 \). By the maximum principle, \( u \leq 0 \) in \( \Omega \). Clearly, \( C_0^\infty(\Omega) \) is dense in \( L^q(\Omega) \) where \( \frac{1}{p} + \frac{1}{q} = 1 \). Hence by duality, (4.1) is equivalent to

\[
\int_\Omega (-u) w \, dx \leq C \left( \int_\Omega (-u) F_k[u] \, dx \right)^{\frac{1}{k+1}} ||w||_{L^{q'}(\Omega)},
\]

where \( C \) does not depend on \( w \in L^q(\Omega) \cap C^\infty(\Omega), w \geq 0 \). Denote by \( v \in \Phi^k_0(\Omega) \cap C^2(\Omega) \) the unique solution to the equation \( F_k[v] = w \). Then by Theorem 3.1

\[
\int_\Omega (-u) w \, dx \leq C \left( \int_\Omega (-u) F_k[u] \, dx \right)^{\frac{1}{k+1}} \left( \int_\Omega (-v) F_k[v] \, dx \right)^{\frac{1}{k+1}}.
\]

We now invoke a global estimate of the solution \( v \) to the Hessian equation \( F_k[v] = w \) by means of Wolff’s potential defined by (1.5) (see [10], Theorem 2.2):

\[
|v(x)| \leq C W_{\alpha,p}^{2\text{diam}(\Omega)} w(x), \quad x \in \Omega,
\]

where \( \alpha = \frac{2k}{k+1} \) and \( p = k+1 \), and \( C \) depends only on \( k \) and \( n \). Hence,

\[
\int_\Omega (-v) F_k[v] \, dx \leq C \int_\Omega W_{\alpha,p}^{2\text{diam}(\Omega)} w \, w \, dx.
\]

The Wolff potential \( W_{\alpha,p}^{2\text{diam}(\Omega)} w \) on the right-hand side is obviously bounded by \( W_{\alpha,p} w \) which corresponds to \( \Omega = \mathbb{R}^n \), see (1.4). Here we extend \( w \) to \( \mathbb{R}^n \) so that \( w = 0 \) outside \( \Omega \).

We next replace \( W_{\alpha,p} w \) with the Havin–Maz’ya potential \( I_\alpha(I_\alpha w)^{\frac{1}{p-1}} \) using the pointwise estimate ([9], Sec. 10.4.2)

\[
W_{\alpha,p} w(x) \leq C I_\alpha(I_\alpha w)^{\frac{1}{p-1}}(x), \quad x \in \mathbb{R}^n,
\]

where \( I_\alpha = (-\Delta)^{-\frac{\alpha}{2}} \) is a Riesz potential of order \( \alpha \), and \( C \) is a positive constant depending only on \( \alpha, p, n \). We deduce

\[
\int_\Omega (-v) F_k[v] \, dx \leq C \int_\Omega W_{\alpha,p} w \, w \, dx
\]

\[
\leq C \int_\Omega I_\alpha(I_\alpha w)^{\frac{1}{p-1}} w \, dx = C \int_{\mathbb{R}^n} (I_\alpha w)^{\frac{1}{p-1}} \, dx.
\]

Now applying the Sobolev inequality for fractional integrals of order \( \alpha = \frac{2k}{k+1} \) (see, e.g., [9], Sec. 8.3), we obtain

\[
\int_\Omega (-v) F_k[v] \, dx \leq C \int_{\mathbb{R}^n} (I_\alpha w)^{\frac{1}{p-1}} \, dx \leq C ||w||_{L^{q'}(\Omega)},
\]

where \( \frac{p}{p-1} = \frac{k+1}{k} \). Combining the preceding inequality with (4.3), we arrive at (4.2).

**Remark 4.2.** In the special case \( q = 1 \), inequality (4.1) was proved in Corollary 3.2 with the best constant which depends on \( \Omega \). The best constant in (4.1) is known for \( \Omega = \mathbb{R}^n \) and \( q = \frac{n(k+1)}{n-2k} \) (see [24]).

By inspecting the proof of Theorem 4.1 we deduce the following inequality for Hessian integrals which can be regarded as a dual form of (4.1).
Corollary 4.3. Let $1 \leq k < \frac{n}{2}$. Let $v \in \Phi^k_0(\Omega) \cap C^2(\Omega)$. Then

$$\left( \int_\Omega (-v) F_k[v] \, dx \right)^\frac{1}{k+1} \leq C \left( \int_\Omega (F_k[v])^{q'} \, dx \right)^\frac{1}{q'}$$

(4.5)

where $q' = \frac{n(k+1)}{(n+2-k)}$ is dual to the Hessian Sobolev exponent $q = \frac{n(k+1)}{n-2k}$.

Theorem 4.4. Let $1 \leq k < \frac{n}{2}$, and $q > k + 1$. Let $\omega$ be a positive Borel measure on a bounded uniformly $(k - 1)$-convex domain $\Omega$ with $C^2$ boundary. Suppose that $\omega$ obeys

$$\mathcal{K}(\omega) = \sup_{B} \frac{\omega(B \cap \Omega)}{|B|^{\frac{1}{n-k}} + \frac{n}{k+1}} < \infty,$$

where the supremum is taken over all balls $B$ in $\mathbb{R}^n$. Then, for every $u \in \Phi^k_0(\Omega) \cap C^2(\Omega)$,

$$||u||_{L^q(\Omega, \omega)} \leq C \mathcal{K}(\omega)^{\frac{1}{p}} \left( \int_\Omega |u| F_k[u] \, dx \right)^\frac{1}{k+1},$$

(4.7)

where the constant $C$ depends on $n, k, q$, and $\Omega$.

Remark 4.5. If $k = \frac{n}{2}$ for even $n$, then (4.7) holds if

$$\mathcal{K}(\omega) = \sup_{B} \frac{\omega(B \cap \Omega)}{\left( \log \frac{2}{|B|} \right)^\frac{1}{n-k}} < \infty,$$

(4.8)

for every ball $B$ such that $|B| < 1$.

Proof. Let $\alpha = \frac{2k}{k+1}$, $p = k + 1$, where $k = 1, 2, \ldots, \left( \frac{n}{2} \right)$. Let $\mu = \mu_k[u]$ be the Hessian measure associated with $u$ supported in $\Omega$. As in the proof of Theorem 4.1 by duality (4.7) is equivalent to

$$\int_\Omega (-u) \omega \, dw \leq C \mathcal{K}(\omega)^{\frac{1}{p}} \left( \int_\Omega (-u) F_k[u] \, dx \right)^\frac{1}{k+1} ||u||_{L^{q'}(\Omega, \omega)},$$

(4.9)

where $C$ does not depend on $w \in L^{q'}(\Omega, \omega) \cap C_0^\infty(\Omega), w \geq 0$. Without loss of generality we may assume that $\omega$ is compactly supported in $\Omega$. Denote by $v \in \Phi^k_0(\Omega)$ a solution to the equation $F_k[v] = \mu$, where $d\mu = w \, dw$. Then by Theorem 3.4 and the global Wolff potential estimate (4.3), it suffices to prove the inequality

$$\int_\Omega W_{\alpha, p} \mu \, dw \leq C \mathcal{K}(\omega)^{\frac{1}{p'}} ||u||_{L^{q'}(\Omega, \omega)},$$

(4.10)

As in the proof of the Hessian Sobolev inequality above,

$$\int_\Omega W_{\alpha, p} \mu \, dw \leq C ||I_{\alpha} \mu||_{L^{q'}(\mathbb{R}^n)},$$

where $C$ depends only on $n, p, \alpha$. It remains to notice that

$$||I_{\alpha} \mu||_{L^{q'}(\mathbb{R}^n)} = ||I_{\alpha} (w \, dw)||_{L^{q'}(\mathbb{R}^n)} \leq C \mathcal{K}(\omega)^{\frac{1}{p'}} ||w||_{L^{q'}(\Omega, \omega)},$$

since by duality it is equivalent to D. Adams’s inequality for Riesz potentials $I_{\alpha}$ (II, Sec. 7.2):

$$||I_{\alpha} f||_{L^p(\mathbb{R}^n, \omega)} \leq C \mathcal{K}(\omega)^{\frac{1}{p}} ||f||_{L^p(\mathbb{R}^n)},$$

for all $f \in L^p(\mathbb{R}^n)$.
By using the same argument as in the preceding theorem, but with a capacitary characterization of the trace inequality for fractional integrals ([1], Sec. 7.2; [9], Sec. 11.3) in place of D. Adams’s theorem, we arrive at a characterization of (4.7) in the case $q = k + 1$ in terms of the classical Riesz capacity $\text{cap}_{\alpha, p}(\cdot)$.

**Theorem 4.6.** Let $q = k + 1$ and $1 \leq k \leq \frac{n}{2}$. Let $\omega$ and $\Omega$ be as in Theorem 4.4.

Suppose that $\omega$ obeys

\begin{equation}
\omega(E) \leq c \text{cap}_{k, k+1}(E),
\end{equation}

for every compact set $E \subset \Omega$. Then, for every $u \in \Phi_0(\Omega) \cap C^2(\Omega)$,

\begin{equation}
||u||_{L^q(\Omega, \omega)} \leq C \left( \int_{\Omega} |u| F_k[u] \, dx \right)^{\frac{1}{k}},
\end{equation}

where the constant $C$ depends only on $n, k$, and $c$.

The Hessian version of the Fefferman–Phong inequality stated in the Introduction is a consequence of Theorem 4.6.

**Corollary 4.7.** Suppose $q = k + 1$ and $1 \leq k < \frac{n}{2}$. Under the assumptions of Theorem 4.6, suppose that $d\omega = w \, dx$, where $w \geq 0$ is a weight on $\Omega$ such that

\begin{equation}
\int_{B_R \cap \Omega} w^{1+\epsilon} \, dx \leq C R^{n-2k(1+\epsilon)},
\end{equation}

for some $\epsilon > 0$, and every ball $B_R$. Then (4.12) holds.

**Proof.** Suppose (4.13) holds. By the classical version of the Fefferman–Phong inequality for fractional integrals the operator $I_\alpha : L^q(\mathbb{R}^n) \to L^q(\Omega, \omega)$ is bounded for $q = k + 1$, which yields (4.11) (see, e.g., [22]). Then by Theorem 4.6 inequality (4.12) holds. \hfill \Box

**Remark 4.8.** Other Hessian inequalities, in particular (4.12) in the case $q < k + 1$, are easily characterized using the same proof as in Theorem 5.1 along with the corresponding trace inequalities for fractional integrals ([22, Theorem 1.13]).

## 5. Hessian Trace Inequalities on $\mathbb{R}^n$

In this section we prove general trace inequalities on the entire Euclidean space $\mathbb{R}^n$. In this case there is no need to use Hessian duality. Our main tools are global estimates of solutions to Hessian equations (1.7) and Wolff’s inequality [6]. A simplified proof of Wolff’s inequality, along with a thorough discussion of related facts of nonlinear potential theory, can be found in [1], Sections 4.5 and 9.3.

Suppose $q = \frac{n(k+1)}{n-2k}$, and $k$ is an integer such that $1 \leq k < \frac{n}{2}$. If $u$ is a $k$-convex $C^2$-function on $\mathbb{R}^n$ which vanishes at infinity, then the Hessian Sobolev inequality

\begin{equation}
||u||_{L^q(\mathbb{R}^n, dx)} \leq C \left( \int_{\mathbb{R}^n} |u| F_k[u] \, dx \right)^{\frac{1}{q}}
\end{equation}

holds. The preceding statement is a special case of the following theorem.

**Theorem 5.1.** Let $\omega$ be a positive Borel measure on $\mathbb{R}^n$, $1 \leq k < \frac{n}{2}$, and $q > k + 1$. Suppose that $\omega$ obeys

\begin{equation}
\mathcal{H}(\omega) = \sup_B \frac{\omega(B)}{|B|^\left(1 - \frac{2k}{n} \right) \frac{q}{q-k}} < \infty,
\end{equation}

where $B$ is a ball in $\mathbb{R}^n$. If $u \in C^2(\mathbb{R}^n)$ satisfies $\Delta^k u = 0$, then

\begin{equation}
||u||_{L^q(\mathbb{R}^n, \omega)} \leq C \left( \int_{\mathbb{R}^n} |u| F_k[u] \, dx \right)^{\frac{1}{q}}
\end{equation}

for some constant $C$ depending only on $n, k, q, \mathcal{H}(\omega), \text{cap}_{\alpha, p}(\cdot)$.
where the supremum is taken over all balls $B$ in $\mathbb{R}^n$. Then, for every $k$-convex $C^2$-function $u$ on $\mathbb{R}^n$ such that $\sup_{x \in \mathbb{R}^n} u(x) = 0$,

\begin{equation}
||u||_{L^q(\mathbb{R}^n, d\omega)} \leq C \varphi(\omega)^{\frac{1}{k}} \left( \int_{\mathbb{R}^n} |u| F_k[u] \, dx \right)^{\frac{1}{k-1}},
\end{equation}

where the constant $C$ depends only on $n, k, q$.

Conversely, if (5.3) is valid for $q > k + 1$ and $1 \leq k < \frac{n}{2}$, then (5.2) holds. If $k = \frac{n}{2}$ for even $n$, then (5.2) holds only if $\omega = 0$.

Proof. Let $\alpha = \frac{2k}{k+1}$, $p = k + 1$, where $k = 1, 2, \ldots, [\frac{n}{2}]$. Let $\mu = \mu_k[u]$ be the Hessian measure associated with $u$. By the global Wolff potential estimate (1.7), it suffices to prove the inequality

\begin{equation}
\int_{\mathbb{R}^n} (W_{\alpha, p})^q \, d\omega \leq C \varphi(\omega) \mu_k[u]^q,
\end{equation}

where $\omega, \mu$ are positive Borel measures on $\mathbb{R}^n$, and the Wolff energy $\mu_k[u]$ is defined by

\begin{equation}
\mu_k[u] = \int_{\mathbb{R}^n} W_{\alpha, p} \, d\mu.
\end{equation}

It was shown in [10] that $\mu_k[u]$ is equivalent to the Hessian energy

\begin{equation}
E_k[u] = \int_{\mathbb{R}^n} (-u) F_k[u].
\end{equation}

Let $g = (I_\alpha \mu)^{\frac{1}{k+1}}$, where $I_\alpha = (-\Delta)^{-\frac{n}{2}}$ is a Riesz potential of order $\alpha$ on $\mathbb{R}^n$. By Wolff’s inequality [6] (see also [1], Sec. 4.5),

\begin{equation}
\mathcal{E}_k[\mu] = \int_{\mathbb{R}^n} W_{\alpha, p} \, d\mu \geq C(p, \alpha, n) \int g^p \, dx.
\end{equation}

On the other hand, consider the Havin–Maz’ya nonlinear potential

\begin{equation}
U_{\alpha, p} \mu = I_\alpha g = I_\alpha (I_\alpha \mu)^{\frac{1}{k+1}}.
\end{equation}

It is well-known that $U_{\alpha, p} \mu$ dominates $W_{\alpha, p} \mu$ (see, e.g., [2], Sec. 10.4.2), i.e.,

\begin{equation}
W_{\alpha, p} \mu(x) \leq C U_{\alpha, p} \mu(x) = C I_\alpha g(x), \quad x \in \mathbb{R}^n,
\end{equation}

where $C$ depend only on $n, \alpha, p$. Thus, (5.4) reduces to

\begin{equation}
||I_\alpha g||_{L^p(dx)} \leq C \varphi(\omega) ||g||_{L^p(dx)},
\end{equation}

which follows from D. Adams’s inequality for Riesz potentials ([2], Sec. 7.2), or the classical Sobolev inequality for fractional integrals in case $d\omega$ is Lebesgue measure.

We now prove the necessity of condition (5.2). We fix a compact set $E \subset \mathbb{R}^n$, and denote by $\mu$ the equilibrium measure on $E$ in the sense of the $k$-Hessian capacity (see [3]). Denote by $u$ a $k$-convex solution to the equation $F_k[u] = \mu$ that vanishes at $\infty$. Then $u \geq 1$ on $E$, and the Hessian energy of $\mu$ is equivalent to the Hessian capacity $\text{cap}_k(E)$. Thus, $\omega(E) \leq C \text{cap}_k(E)$. If $E$ is a ball $B$ of radius $r$, then $\text{cap}_k(B) = C r^{n-2k}$, if $k < \frac{n}{2}$, and (5.2) holds; if $k = \frac{n}{2}$ then $\text{cap}_k(B) = 0$ and consequently $\omega(B) = 0$ for every ball $B$ (see [8], [10]), \hfill \square
Remark 5.2. As in the case of bounded domains treated in Sec. 4 for $q = k + 1$ there is a similar characterization of (5.3) in terms of Riesz capacities. Noncapacitary characterizations of (5.3) for $q = k + 1$, including the Fefferman–Phong condition (1.13), along with similar results for $q < k + 1$, are easily deduced as well using the argument employed above combined with the corresponding results for fractional integrals (see [22]).

6. Remarks on the proof of the Hessian Sobolev inequality

It is worth observing that our approach to the Hessian Sobolev inequality requires certain modifications of some proofs in the theory of Hessian equations ([18], [8], [10]). They have to be redone in order to avoid an apparent circle argument since the Hessian Sobolev inequality was referred to in the original proofs.

Notice that the proofs of the global existence theorems [2], [13] along with the interior gradient estimates [15], and local integral estimates of the type [18]

\[
\left( \frac{1}{R^{n-2k}} \int_{B_{r/2}} F_k[u] \, dx \right)^{\frac{1}{k}} \leq C \frac{1}{R^n} \int_{B_R} |u| \, dx,
\]

for $u \in \Phi^k(B_R)$, $u \leq 0$, do not use the global inequality (1.2). The proofs of the comparison principle [17], and the weak continuity theorem, which is essentially local ([18], [19]), do not require the use of (1.2) either.

Some changes are needed in the proofs of the pointwise Wolff potential estimates [8], [10] where the full strength of the Hessian Sobolev inequality is not necessary. What is actually used (see, e.g., the proof of estimate (2.19) in [8], p. 13) is a version of the Poincaré inequality for $u \in \Phi^0(B_R)$ on balls $B_R$:

\[
\frac{1}{R^n} \int_{B_R} |u| \, dx \leq C \left( \frac{1}{R^{n-2k}} \int_{B_R} |u| F_k[u] \, dx \right)^{\frac{1}{k+1}}.
\]

The preceding estimate was deduced above (Corollary 3.3) using a straightforward duality argument.

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