The edge-vertex inequality in a planar graph and a bipartition for the class of all planar graphs

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Abstract

For a planar graph with a given f-vector \((f_0, f_1, f_2)\), we introduce a cubic polynomial whose coefficients depend on the f-vector. The planar graph is said to be real if all the roots of the corresponding polynomial are real. Thus we have a bipartition of all planar graphs into two disjoint class of graphs, real and complex ones. As a contribution toward a full recognition of planar graphs in this bipartition, we study and recognize completely a subclass of planar graphs that includes all the connected grid subgraphs. Finally, all the 2-connected triangle-free complex planar graphs of 7 vertices are listed.

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The inequality $3(f_0 - 2) \geq f_1$ is an old basic relation that holds for all planar graphs, where its number of vertices $f_0 \geq 3$ and $f_1$ stands for its number of edges, see [1,2]. In other words, for a fixed number of vertices $f_0$ a planar graph can not have more than $3(f_0 - 2)$ edges. Here is a natural question: Is there a tighter upper bound for $f_1$ that holds for all the planar graphs? The answer is; not very likely, due to the fact that maximal planar triangulations have exactly $3(f_0 - 2)$ number of edges. However, this inequality has a stronger version; that is, $2(f_0 - 2) \geq f_1$, and holds for triangle-free planar graphs. The fact that the second inequality is sharper and holds over a subclass of planar graphs is leading us to the idea behind this note. Here, we introduce a different inequality that is also stronger than the first one over a subclass of planar graphs. This new inequality is linked to zeros of certain Euler type polynomials, and leads us to search for a new bipartition of planar graphs.

Let $G$ be a simple planar graph with no loops or parallel edges with the usual f-vector $(f_0, f_1, f_2)$, where $f_0, f_1, f_2$ stand for the number of vertices, edges and faces of $G$. Consider the cubic polynomial

$$p(x) = f_2x^3 + f_1x^2 + f_0x + 2$$

associated with $G$. Here in this note, $p(x)$ will be called Euler polynomial of $G$. Evidently, the Euler polynomial is uniquely defined by $G$, however many non-isomorphic planar graphs may have the same f-vector and hence the same Euler polynomial. It is rather trivial to show that the roots of $p(x)$ are all real if and only if the relation

$$(f_0 + 2)^2 \geq 8(f_1 + 2) \quad (1)$$

holds. Our final aim here is to classify those planar graphs that satisfy this real root property. In other words, we like to construct a bipartition of all planar graphs into two real and complex subclasses. The connection between real root properties of this type of polynomials and the associated class of graphs could open a link between graphs and polynomial root finding. For more on root-finding and its application in polynomiography, another facet of mathematical arts, see [3]. A planar graph that satisfy the inequality above
is defined to be a real graph, and is called a complex graph otherwise. The
inequality can also be stated as:

\[(f_0 - 2)^2 \geq 8f_2 \]  \(2)\),

applying the Euler relation for planar graphs.

It is intuitively evident that the planar graphs with large number of vertices
are real and on the other side the small graphs are complex. The specific
question is what do these large and small mean? and what about those
graphs in the middle levels? The main results of this note, Theorems 3-5,
are toward a complete characterization of those real planar graphs of the
middle levels. It is easy to check that all the planar graphs with 4 vertices or
less are complex. The only planar graphs with 5 vertices that are real must
have \(f_2 = 1\), so they are acyclic that includes all the trees. Similarly, the only
real graphs of 6 vertices are the one with at most one cycle. It is also easy
to show that all the planar graphs with \(f_0 \geq 18\) are real. Hence, the middle
level means those planar graphs with \(7 \leq f_0 \leq 17\). If the working domain
is restricted to the triangle free ones, then the middle level shrinks down to
\(7 \leq f_0 \leq 9\). From now on, we assume all the graphs are connected unless
otherwise is stated. In the following theorem, we begin with some small class
of planar graphs where this real property recognition is straightforward.

Let \(P_n\) denote a path of size \(n\), that is \(P_n\) has \(n\) vertices. A rectangular grid
graph (or simply a grid) \(G_{m,n}\), \(m \leq n\) is defined to be the cartesian product
\(P_m \times P_n\), that has \(mn\) vertices, \(2mn - m - n\) edges and \((m-1)(n-1)+1\) faces.

**Theorem 1.**

(a) A tree is real iff \(f_0 \geq 5\)

(b) A cycle is real iff \(f_0 \geq 6\)

(c) The square grid \(G_n = P_n \times P_n\) is real iff \(n \geq 3\).
Proof. The statements of (a) and (b) can directly be verified. To prove (c), applying the values of \( f_0 \) and \( f_2 \) for the square grid of \( n^2 \) vertices, condition (2) will be stated as

\[
(n^2 - 2)^2 \geq 8(n^2 - 2n + 2) \tag{3}
\]

Let us define \( f(n) = (n^2 - 2)^2 - 8(n^2 - 2n + 2) \). Verifying relations \( f(2) < 0 \), \( f(3) > 0 \) and the fact that \( f \) is increasing for \( n \geq 3 \) shows that the inequality (3) holds, and this completes the proof.

Theorem 2. A rectangular grid \( G_{mn} \), \( m, n \geq 1 \) is always real except the following cases:

(a) \( G_{1,n} \) for \( 1 \leq n \leq 4 \)

(b) \( G_{2,n} \) for \( 2 \leq n \leq 3 \)

Proof. A direct application of the definition shows that \( G_{m,n} \) is complex for all natural numbers \( m, n \), whenever its number of vertices \( mn \leq 4 \). This includes all the paths with 4 vertices or less, and a single square. The grid with \( mn = 5 \), i.e. \( G_{1,5} \), is obviously real. The case of \( mn = 6 \) includes \( G_{1,6} \) that is real and \( G_{2,3} \), the union of two adjacent squares, that is complex. To show that all the cases with \( mn \geq 7 \) are real, consider the inequality that is deduced from (2), that is;

\[
(mn - 2)^2 \geq 8(mn - m - n + 2)
\]

or

\[
(mn - 2)^2 - 8mn \geq 16 - 8(m + n) \tag{4}
\]
To complete the proof, note that the function \( g(x) = (x - 2)^2 - 8x \) is increasing in \( x \) for \( x \geq 7 \). And then the relation (4) holds for all \( mn \geq 7 \), since \( 16 - 8(m + n) < g(7) = -31 \) for all \( mn \geq 7 \).

The real-complex recognition problem for more general planar graphs will be the subject of our work in the following theorems. We first restate the original basic inequalities as the next lemma.

**Lemma 1.** Let \( G \) be any planar graph with \( f_0 \) and \( f_1 \) vertices and edges respectively, where \( f_0 \geq 3 \). Then,

(a) \( 3(f_0 - 2) \geq f_1 \)

(b) Moreover if \( G \) is triangle- free, then \( 2(f_0 - 2) \geq f_1 \)

We proceed by raising the critical question on the initial motivation behind the note. How interesting are these real or complex classes of planar graphs? This depends on the size of the real classes for \( 5 \leq f_0 \leq 17 \). A large class of real graphs means that there is a stronger inequality than the linear one of Lemma 1-a over that special real class.

The inequality (a) of the lemma presents \( 3(f_0 - 2) \) as a linear upper bound for \( f_1 \), however the upper bound for \( f_1 \) in (1), \( \frac{(f_0 + 2)^2}{8} - 2 \), is quadratic and in general will not be superior to the linear one for graphs with very large number of vertices. In fact the quadratic inequality would be interesting exactly when

\[
\frac{(f_0 + 2)^2}{8} - 2 \leq 3(f_0 - 2),
\]

comparing the upper bounds for \( f_1 \) in the two inequalities. The latter implies \( f_0 \leq 17 \). A similar comparison for the triangle free planar graphs shows \( f_0 \leq 9 \). The existence of these bounds form the first half of the next theorem.
Theorem 3.

Any planar graph with $f_0 \geq 18$ is real, in addition, any triangle-free planar graph with $f_0 \geq 10$ is also real. That is:

$$(f_0 - 2)^2 \geq 8f_2$$

holds for planar graphs with $f_0 \geq 18$, and for any triangle-free planar graph with $f_0 \geq 10$. In both cases these are the best possible inequalities, in other words there are appropriate complex planar graphs of 17 and 9 vertices respectively.

Proof. The discussion above concludes that $(f_0 + 2)^2 \geq 8(3f_0 - 4)$ for all $f_0 \geq 18$. Then by applying Lemma 1 (a), $3f_0 - 4 \geq f_1 + 2$, we obtain:

$$(f_0 + 2)^2 \geq 8(f_1 + 2)$$

For the second part, again it is easy to see that $(f_0 + 2)^2 \geq 16(f_0 - 1)$ for all $f_0 \geq 10$. Then by applying Lemma 1 (b), $2(f_0 - 1) \geq f_1 + 2$, we obtain:

$$(f_0 + 2)^2 \geq 8(f_1 + 2)$$

or equivalently

$$(f_0 - 2)^2 \geq 8f_2$$

To complete the proof, maximal complex planar graphs of 17 and maximal complex triangle-free planar graphs of 9 vertices are presented in Figures 1 and 2 respectively.

The complex planar graph of 17 vertices above is a maximal triangulation, that is to say every face including the unbounded face is a 3-cycle. In fact there are more maximal triangulations that are complex.

Corollary. Any maximal planar triangulation graph with $3 \leq f_0 \leq 17$ is complex, and any complex planar graphs of 17 vertices is either a maximal triangulation or can be obtained from such a maximal triangulation by removal of only a single edge.
Proof. Any maximal triangulation of $f_0$ vertices must have $f_1 = 3f_0 - 6$ edges and $f_2 = 2f_0 - 4$ faces. So, to have a complex maximal triangulation we must have $(f_0 - 2)^2 < 8(2f_0 - 4)$. The latter is equivalent to $3 \leq f_0 \leq 17$. Since any maximal triangulation of $f_0 = 17$ vertices has 45 edges (with 30 faces), and any complex planar graph of 17 vertices must have at least 44 edges (and 29 faces), then the last part of the corollary is concluded.

Fig. 1
The complex triangle-free planar graph of 9 vertices above can be used to construct more of such examples with 8 and then again with 7 vertices. This can be done by removal of a degree 2 vertex, applying the following lemma.

**Lemma 2.** Let $G$ be a complex planar graph with $f_0 \geq 7$, and a vertex $x$ of degree 2. Then $G - x$ is also a complex planar graph.

**Proof.** It is a direct consequence of the definition.
The last corollary shows that only for $f_0 = 17$ there is a large class of real graphs and we show more of this in the next theorem. By a grid graph, we mean any graph that can be embedded in the square grid $G_n$ as a subgraph for a large enough $n$. The full characterization of real or complex planar graphs for all $7 \leq f_0 \leq 17$, remains to be an interesting computational problem. However, we proceed to show a complete characterization of real or complex subgraphs of a grid in the next theorem.

**Theorem 4.** Any connected grid graph with the number of vertices $f_0 \geq 7$ is real.

**Proof.** First, consider only the case of 7 vertices, and suppose there is a connected complex grid graph $G$ with $f_0 = 7$. Evidently $G$ is planar, bipartite with at least 4 faces, applying $(f_0 - 2)^2 < 8f_2$. Three of these faces, say $C_1, C_2, C_3$, are cycles. Since $G$ is bipartite and $f_0 = 7$, then, these cycles either must be 4-cycles (unit squares) or 6-cycles (rectangular 6-gons). However none of them can be a 6-cycle, otherwise there would not be any possibility for a second cycle, by the restriction of $f_0 = 7$. Now, let them all to be 4-cycles. Since the union of any two cycles has at least 6 vertices then $C_1 \cup C_2 \cup C_3$ needs at least 8 vertices, a contradiction by $f_0 = 7$. The latter will be apparent if we notice that the connected grid graph $C_1 \cup C_2 \cup C_3$ has only 5 options of non-isomorphic graphs, given that all the cycles are unit squares. On the other hand, if $C_1 \cup C_2 \cup C_3$ is disconnected, then it requires even more than 8 vertices, that is impossible. Finally, for $f_0 = 8$ and $f_0 = 9$ the minimum number of faces are 5 and 7 respectively. Then the minimum number of bounded faces (i.e., 4-cycles) that are needed will be 4 and 6 respectively. We already needed at least 8 vertices for only 3 cycles, so the same contradiction is evident here.

The grid graphs are triangle-free, however the last theorem does not hold for arbitrary triangle-free planar graphs. In fact, we have presented an example of a complex triangle-free planar graph of 9 vertices in Figure 2, and also examples of such complex planar graphs with 8 and 7 vertices, applying Lemma 2. For $f_0 = 7$ all examples of complex triangle-free planar graphs, the only possible number of edges are 9 and 10, applying the inequality (1) and Lemma 1 (b). Hence, the number of edges of such minimal examples is
9, and this is the motivation for the next classification theorem that covers the corresponding class of those examples.

In the following theorem, let $P$ be a polygon and $S$ a finite set of points in the interior of $P$. We will explore all the possibilities for construction of a planar graph $G$ whose vertex set $(\text{vert} P) \cup S$ is fixed, its f-vector is $(f_0, f_1, f_2) = (7, 9, 4)$ and includes the polygon $P$ as a cycle. In this construction, those edges that connect vertices of $S$ to the vertices of $P$ are called connecting edges. In fact, we aim to characterize all the non-isomorphic 2-connected triangle-free planar graphs that satisfy: $f = (f_0, f_1, f_2) = (7, 9, 4)$, and evidently all are complex graphs. Since all the faces of a 2-connected planar graph are cycles, we may consider $G$ to be a 2-connected triangle free planar graph with $(f_0, f_1, f_2) = (7, 9, 4)$ whose unbounded face is denoted by the cycle $P$.

**Theorem 5.** All the minimal complex non-isomorphic 2-connected triangle-free planar graphs with 7 vertices are listed in the figures 3, 4 and 5.

**Proof** We may assume that such graph $G$ has no leaf, or any other cut vertices, since it is 2-connected. Moreover, all of its faces are cycles. Consider a planar drawing of $G$, and let $P$ be the outer face of this drawing, that is a convex polygon that includes all other vertices in its interior. Obviously, $P$ can be a hexagon with only one vertex left inside the hexagon, a pentagon with two vertices inside or a quadrilateral with three vertices inside. Of course, the condition $f_2 = 4, f_1 = 9$ and $f_0 = 7$ holds in all the 3 cases.

Consequently, the characterization is divided into three cases: Case 1: $P$ is a 6-cycle and $|S| = 1$. Case 2: $P$ is a 5-cycle with $|S| = 2$ and in Case 3: $P$ is a 4-cycle with $|S| = 3$.

Figure 3 shows the unique graph $G$, where the three connecting edges of this figure are the only possibility since triangular faces are not allowed. The Case 2 includes 4 graphs where the two internal vertices in $S$ have degree 2 or 3, by the 2- connectivity of $G$. In the first two graphs Figure 4-a, there is one edge incident to the two vertices of $S$, and so there are 3 connecting edges. In the graphs of Figure 4-b there is no internal edge and so there are 4 connecting edges.
In Figure 4-a, again triangles are not allowed, so the two connecting edges that are incident to a vertex of $S$ can only form a 4-cycle or a 5-cycle joint with more edges from the cycle $P$. In Figure 4-b, two connecting edges incident to a vertex in $S$ must also be part of a 4-cycle, or a 5-cycle joint with more edges from the cycle $P$, since triangles are forbidden. The two remaining connecting edges are incident to the other vertex of $S$. Hence, there will remain only two possibilities left in Figure 4-b.

Case 3 includes 7 more graphs, where in all of them $|S| = 3$. However in the first five graphs, Figure 5-a, there are two edges incident to vertices of $S$, and 3 more connecting edges. In the last two graphs, Figure 5-b, there is only one edge incident to the vertices of $S$ and four connecting edges.
Fig. 3

Fig. 4-a

Fig. 4-b
Fig. 5-a

Fig. 5-b
Final remark and more open problems

It is easy to extend the result of Theorem 5 to non-minimal complex planar graphs of 7 vertices, since those planar graphs must either have 9 or 10 edges. Therefore, those planar graphs of 7 vertices with 10 edges can be obtained from the ones of 9 edges by adding one more edges. However, the new extra edge can not create a triangular face, that is, it must be added into a hexagonal face only. Thus, there are only the bipartite graphs of Figures 3 and 5, the ones with the labeled vertices, that should be selected. The complete characterization of real or complex graphs with $7 \leq f_0 \leq 17$ remains to be an interesting problem, and a practical computational project. The notion of real and complex planar graphs that are defined here are directly connected to the quadratic inequality that defines them. So, the first natural question will be if there exist other class of such inequalities. Are there stronger relation that can provide tighter upper bound for $f_0$ in terms of $f_1$?

We have noticed the real class of planar graphs are closely connected to the real root property of Euler polynomials. How about search for higher degree polynomials such as higher degree Euler polynomials for polytopal graphs? The latter type of polynomials can define a new class of graphs and a new link between graphs and polynomial root-finding, see [3].

References

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