Chiral asymmetry and axial anomaly in magnetized relativistic matter

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The induced axial current and the chiral anomaly are studied in the normal phase of magnetized relativistic matter. A special attention is paid to the role of the chiral shift parameter $\Delta$, leading to a relative shift of the longitudinal momenta in the dispersion relations of opposite chirality fermions. In the Nambu-Jona-Lasinio model, it is shown directly from the form of the gap equation that $\Delta$ necessarily exists in the normal phase in a magnetic field. By making use of the gauge invariant point-splitting regularization, we then show that the presence of $\Delta$ essentially modifies the form of the axial current, but does not affect the conventional axial anomaly relation. By recalculating the axial current with the proper-time regularization, we conclude that the result is robust with respect to a specific regularization scheme used.

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I. INTRODUCTION

Several types of dense relativistic matter exist in compact stars. For example, a relativistic electron plasma forms and plays an essential role in white dwarfs. Also, electrons form a relativistic fluid inside nuclear matter in the interior of neutron stars. If quark stars exist in nature, the corresponding dense quark matter in the core will be a strongly coupled version of relativistic matter. Often, such matter is subject to strong magnetic fields. In white dwarfs, e.g., the magnetic fields reach up to $10^9$ G, while, in neutron stars, they may be up to $10^{15}$ G \cite{1,2}. Relativistic matter in a strong magnetic field is also created in heavy ion collisions \cite{3} that can lead to the chiral magnetic effect \cite{4}.

Many physical properties of the stellar matter under extreme conditions realized inside compact stars are understood theoretically and could be tested to some extent through observational data. However, as was pointed out in Refs. \cite{5–12}, the dense relativistic matter in a strong magnetic field may hold some new theoretical surprises. In particular, a topological contribution in the axial current at the lowest Landau level (LLL) was revealed in Ref. \cite{6}. More recently, it was shown in Ref. \cite{7}, that in the normal phase of dense relativistic matter in a magnetic field, there exists a contribution to the axial current associated with a relative shift of the longitudinal momenta in the dispersion relations of opposite chirality fermions, $k^3 \rightarrow k^3 \pm \Delta$, where the momentum $k^3$ is directed along magnetic field and $\Delta$ is the chiral shift parameter intimately connected with the induced axial current $j_5^3$. Unlike the topological contribution in $j_5^3$ at the lowest Landau level (LLL) \cite{6}, the dynamical one appears only in interacting matter and affects the fermions in all Landau levels, including those around the Fermi surface. The induced axial current and the shift of the Fermi surfaces of the left-handed and right-handed fermions are expected to play an important role in transport and emission properties of matter in various types of compact stars as well as in heavy ion collisions.

The main goal of this Letter is to study some general and subtle features of the dynamics with the chiral shift parameter $\Delta$. One of such issues is whether the form of the induced axial $j_5^3$ current coincides with the result in the theory of noninteracting fermions in a magnetic field \cite{6} or whether it is affected by interactions (for related discussions, see Refs. \cite{6,6a,13}). This question is intimately related to that of the connection of the structure of the induced $j_5^3$ with the axial anomaly \cite{14}. By using the Nambu-Jona-Lasinio (NJL) model in a magnetic field, it will be shown that while the dynamics responsible for the generation of the chiral shift $\Delta$ essentially modifies the form of this current, it does not affect the form of the axial anomaly; the latter is connected only with the topological part in the LLL \cite{13}. Moreover, while the topological contribution in the axial current is generated in the infrared kinematic...
region (at the LLL), the contribution of $\Delta$ in this current is mostly generated in ultraviolet, which implies that higher Landau levels are primarily important in that case.

II. MODEL: GENERAL PROPERTIES

As in Ref. [7], in order to illustrate this phenomenon in the clearest way, we will utilize the simplest NJL model with one fermion flavor, whose Lagrangian density is

$$\mathcal{L} = \bar{\psi} \left( iD_{\mu} + \mu_0 \delta_{\mu}^0 \right) \gamma^\mu \psi - m_0 \bar{\psi} \psi + \frac{G_{\text{int}}}{2} \left[ (\bar{\psi} \psi)^2 + (\bar{\psi} i \gamma^5 \psi)^2 \right],$$

(1)

where $m_0$ is the bare fermion mass and $\mu_0$ is the chemical potential. By definition, $\gamma^5 \equiv i \gamma^0 \gamma^1 \gamma^2 \gamma^3$. The covariant derivative $D_{\mu} = \partial_{\mu} + i e A_{\mu}$ includes the external gauge field $A_{\mu}$, which is assumed to be in the Landau gauge, $A^\nu = x B \delta_{\nu}^z$ [16]. Here $B$ is the strength of the external magnetic field pointing in the $z$-direction. The $(3 + 1)$-dimensional Lorentz symmetry in the model is explicitly broken down to the $SO(2)$ symmetry of rotations around the $z$-axis in the presence of this magnetic field. Also, except parity $P$, all the discrete symmetries $C$, $T$, $CP$, $CT$, $PT$, and $CPT$ are broken (here $C$ and $T$ are charge conjugation and time reversal, respectively).

In the chiral limit, $m_0 = 0$, this model possesses the chiral $U(1)_c \times U(1)_R$ symmetry. In the vacuum state ($\mu_0 = 0$), however, this chiral symmetry is known to be spontaneously broken because of the magnetic catalysis phenomenon [17]. In essence, such spontaneous breaking results from the enhanced pairing dynamics of fermions and antifermions in the infrared. The enhancement results from the non-vanishing density of states in the lowest Landau level that is subject to an effective dimensional reduction $D \rightarrow D - 2$. At a sufficiently large value of the chemical potential, the chiral symmetry is expected to be restored. As we shall see below, this is indeed the case, but the corresponding normal ground state is characterized by a nonzero chiral shift parameter $\Delta$.

We will analyze model (1) in the mean field approximation, which is reliable in the weakly coupled regime when the dimensionless coupling constant $g \equiv G_{\text{int}} \Lambda^2 / (4 \pi^2) \ll 1$, where $\Lambda$ is an ultraviolet cutoff. Note that here the coupling $g$ is defined in such a way that $g_{cr} = 1$, where $g_{cr}$ is the critical value for generating a fermion dynamical mass in the NJL model without magnetic field.

In this approximation, the full fermion propagator does not allow any wave function renormalization different from 1. Thus, the general ansatz for the (inverse) full propagator is given by [11]

$$i G^{-1}(u, u') = \left[ (i \partial_\mu + \mu) \gamma^\mu -(\pi \cdot \gamma) + i \mu' \gamma^1 \gamma^2 + \Delta \gamma^3 \gamma^5 - m \right] \delta^4(u-u'),$$

(2)

where $u = (t, \mathbf{r})$, $\pi^k = i(\partial^k + i e A^k)$ is the canonical momentum, $m$ is the constituent (medium-modified) fermion mass, $\mu$ is an effective chemical potential in the quasiparticle dispersion relation, $\mu'$ is an anomalous magnetic moment, and $\Delta$ is the chiral shift parameter. Note that $\mu$ in the full propagator may differ from the thermodynamical chemical potential $\mu_0$ in Lagrangian density (see below). As is shown in Appendix A in the mean field approximation one has $\mu' = 0$ in a self-consistent solution to the gap equation in this model.

Let us now demonstrate that one can get an important insight into the properties of the solutions in this model already from the form of the gap equation for the parameters $\mu$, $\Delta$, and $m$. As described in more detail in Appendix A utilizing the approach based on the effective action for composite operators [18], one can show that the gap equation in the mean field approximation reduces to the following set of equations:

$$\mu - \mu_0 = -\frac{1}{2} G_{\text{int}} (j^0),$$

(3)

$$\Delta = -\frac{1}{2} G_{\text{int}} (j^3_0),$$

(4)

$$m - m_0 = -G_{\text{int}} \langle \bar{\psi} \psi \rangle,$$

(5)

where the chiral condensate $\langle \bar{\psi} \psi \rangle$, the vacuum expectation values of the fermion density $j^0$ and the axial current density $j^3_0$ are

$$\langle j^0 \rangle = -\text{tr} \left[ \gamma^0 G(u,u) \right],$$

(6)

$$\langle j^3_0 \rangle = -\text{tr} \left[ \gamma^3 \gamma^5 G(u,u) \right],$$

(7)

$$\langle \bar{\psi} \psi \rangle = -\text{tr} \left[ G(u,u) \right].$$

(8)

Let us now consider the case of the normal phase in the chiral limit, when $m = m_0 = 0$ and $\langle \bar{\psi} \psi \rangle = 0$. It is realized when the chemical potential $\mu_0 > m_{\text{dyn}} / \sqrt{2}$ [11], where $m_{\text{dyn}}$ is a dynamical fermion mass in a magnetic field at zero
chemical potential and zero temperature. Let us analyze Eqs. (3) and (11) in perturbation theory in the coupling constant $g$. In the zero approximation, we have a theory of free fermions in a magnetic field. To this order, $\mu = \mu_0$ and $\Delta = 0$. However, even in this case the fermion density $\langle j^0 \rangle$ and the axial current density $\langle j^3 \rangle$ are nonzero. The former can be presented as a sum over the Landau levels:

$$\langle j^0 \rangle_0 = \frac{\mu_0 |eB|}{2\pi^2} + \frac{\text{sign}(|\mu_0|) |eB|}{\pi^2} \sum_{n=1}^{\infty} \sqrt{\mu_0^2 - 2n|eB|} \theta \left( |\mu_0| - \sqrt{2n|eB|} \right),$$

and the latter is

$$\langle j^3 \rangle_0 = \frac{eB}{2\pi^2} \mu_0. \quad (10)$$

Then, to the next order in the coupling constant, one finds from Eq. (11) that $\Delta \propto G_{\text{int}}(\langle j^3 \rangle_0 \neq 0$. Thus, in the normal phase of this theory, there necessarily exists a nonzero shift parameter $\Delta$. In fact, this is one of the main results of Ref. [7]. Let us also emphasize that $\Delta$ is generated by perturbative dynamics, which is directly connected with the fact that the vanishing $\Delta$ is not protected by any symmetry [recall that $C = +1$, $P = +1$, and $T = -1$ for the axial current density $j^3$, and beside parity $P$, all the discrete symmetries are broken in model (1)].

Similarly, one finds from Eq. (3) that $\mu = \mu_0 \propto G_{\text{int}}(\langle j^0 \rangle_0 \neq 0$, i.e., $\mu$ and $\mu_0$ are different. One can trace the origin of this difference to the Hartree terms in the gap equation [see the last two terms in Eq. (A2)]. This seems to be robust in the NJL model with a local four-fermion interaction and a chemical potential, associated with a global charge, such as a baryon (or lepton) charge for example. Note that when the conserved charge is related to a gauge symmetry, as in the case of the electric charge, the situation may be different. In that case, a neutrality condition imposed by the Gauss law takes place [19]. The latter is necessary for providing the thermodynamic equilibrium in a system. This is likely to result in $\mu^{(e)} = \mu_0^{(e)}$ when $\mu^{(e)}$ is the chemical potential for electric charge. Note that usually there are chemical potentials of both types in dense relativistic matter. While being of importance for potential applications in principle, we expect that this fact will not change our main conclusion regarding the chiral shift parameter.

By noting from Eq. (4) that the chiral shift $\Delta$ is induced by the axial current, it is naturally to ask whether $\Delta$ itself affects this current. The answer to this question is affirmative [7]. Another natural question is whether the divergence of this modified current satisfies the conventional anomaly equation [14]. As will be shown in the next section, the answer to this question is also affirmative.

### III. Induced Axial Current and Axial Anomaly

In this section, using the gauge invariant point-splitting regularization, we study the influence of the shift parameter $\Delta$ on the form of the axial current and the axial anomaly (for reviews of this regularization, see Refs. [20, 21]). The analysis is model independent and is based only on the form of the fermion propagator with $\Delta$ in an external electromagnetic field. Our main conclusion is that while including $\Delta$ essentially changes the form of the axial current, it does not modify the axial anomaly. Moreover, while the contribution of the chemical potential in the axial current is generated in the infrared kinematic region (at the LLL) [8], the contribution of $\Delta$ in the current is mostly generated in ultraviolet (at all Landau levels).

#### A. Axial current

We consider the case of a constant electromagnetic field. Since it is known that the axial anomaly is insensitive to chemical potential [22, 23], the latter will be omitted. Then, the general form of the fermion propagator is

$$G(u, u') = e^{i\Phi(u, u')} \bar{G}(u - u') \quad (11)$$

with the Schwinger phase

$$\Phi(u, u') = e \int_{u'}^{u} dx^\nu A^\nu, \quad (12)$$

where the integration is performed along the straight line. The translation invariant part $\bar{G}(u - u')$ depends only on the field strength $F_{\mu\nu}$.
Therefore, in the normal phase with \( m = 0 \), the inverse propagator (2) (with \( \mu = 0 \)) can be rewritten as
\[
iG^{-1} = iD_νγ^ν + Δγ^5 = (iD_νγ^ν - Δs_+γ^3) P^-_5 + (iD_νγ^ν + Δs_+γ^3) P^+_5,
\]
where \( s_+ \equiv \text{sign}(eB) \), \( D_ν = ∂_ν + ieA_ν \), and \( P^±_5 = (1 ± s_+γ^5)/2 \). This equation implies that the effective electromagnetic vector potential equals \( \tilde{A}_ν = A_ν + (s_+Δ/e)δ^ν_5 \) and \( \tilde{A}_ν = A_ν - (s_+Δ/e)δ^ν_5 \) for the and + chiral fermions, respectively. Since the field strength \( F_{μν} \) for \( \tilde{A}_ν \) is the same as for \( A_ν \), \( Δ \) affects only the Schwinger phase (12):
\[
\Phi^\pm(u, u') = \Phi(u, u') + s_+Δ(u^3 - u'^3),
\]
(14)
\[
\tilde{\Phi}^\pm(u, u') = \Phi(u, u') - s_+Δ(u^3 - u'^3).
\]
(15)
Thus, we find
\[
G(u, u') = \exp[is_+Δ(u^3 - u'^3)] P^-_5 \tilde{G}_0(u, u') + \exp[-is_+Δ(u^3 - u'^3)] P^+_5 \tilde{G}_0(u, u'),
\]
where \( \tilde{G}_0 \) is the propagator with \( Δ = 0 \). Note that \( Δ \) appears now only in the phase factors.

According to Eq. (7), the axial current density is equal to
\[
\langle j^\mu_5(u) \rangle = -\text{tr} \left[ γ^\muγ^5 G(u, u + e) \right]_{e→0},
\]
(17)
On the other hand, the fermion propagator in an electromagnetic field has the following singular behavior for \( \epsilon \rightarrow 0 \) [20, 21]:
\[
\langle j^\mu_5(u) \rangle \text{singular} = \frac{iεs_+}{2\pi^2ε^4} \left( e^{-is_+Δε^3} - e^{is_+Δε^3} \right) + \frac{iεF_{μν}ε_{βδ}e^{2μεσ}}{8\pi^2ε^2} \left( e^{-is_+Δε^3} + e^{is_+Δε^3} \right)_{ε→0}.\]
(19)
Taking into account that the limit \( ε \rightarrow 0 \) should be taken in this equation symmetrically [20, 21], i.e., \( ε^με^ν/ε^2 \rightarrow 1/2g^{μν} \), and the fact that its second term contains only odd powers of \( ε \), we arrive at
\[
\langle j^\mu_5 \rangle_{\text{singular}} = -\frac{Δ}{2π^2ε^2} g^{μ}_5 \sim \frac{A^2Δ}{2π^2} g^{μ}_5.
\]
(20)
It is clear that \(-1/ε^2 \) plays the role of a Euclidean ultraviolet cutoff \( A^2 \). This feature of expression (20) agrees with the results obtained in Ref. [6]. As was shown there, while the contribution of each Landau level into the axial current density \( \langle j^μ_5 \rangle \) is finite at a fixed \( Δ \), their total contribution is quadratically divergent. However, the important point is that since the solution of gap equation (11) for the dynamical shift \( Δ \) yields \( Δ \sim gμeB/A^2 \), the axial current density is actually finite. The explicit expressions for \( Δ \) and \( \langle j^μ_5 \rangle \) are [11, 12]:
\[
Δ \sim -gμ\frac{eB}{A^2(1 + 2ag)},\]
(21)
\[
\langle j^μ_5 \rangle \sim \frac{eB}{2π^2μ} + \frac{A^2}{π^2} Δ \sim \frac{eB}{2π^2} \frac{μ}{(1 + 2ag)},\]
(22)
where \( a \) is a dimensionless constant of order one [14], which is determined by the regularization scheme used, and \( g \) is the coupling constant defined in Sec. [11] Note that both the topological and dynamical contributions are included in \( \langle j^μ_5 \rangle \). (Terms of higher order in powers of \( |eB|/A^2 \) are neglected in both expressions.)

In Ref. [7], a gauge noninvariant regularization (with a cutoff in a sum over Landau levels) was used. One can show that the main features of the structure of the axial current in model [11] remain the same also in the gauge invariant proper-time regularization [22]. In this regularization,
\[
Δ \sim -gμ\frac{eB}{A^2(1 + g/2)},\]
(23)
\[
\langle j^μ_5 \rangle \sim \frac{eB}{2π^2μ} + \frac{\sqrt{π}}{2(2πl)^2} \frac{e^{-sΔ^2/2}}{\sqrt{s}} \text{erfi}(\sqrt{s}Δ) \text{coth}(eB) \bigg|_{s=1/A^2} \sim \frac{eB}{2π^2} \frac{μ}{(1 + g/2)},\]
(24)
where \( \text{erfi}(x) \equiv -i \text{erf}(ix) \) is the imaginary error function. Note that in this case the parameter \( a \) equals 1/4 [compare with Eq. (22)].

We conclude that interactions leading to the shift parameter \( Δ \) essentially change the form of the induced axial current in a magnetic field. It is important to mention that unlike the topological contribution in \( \langle j^μ_5 \rangle \) [6], the dynamical one is generated by all Landau levels.
B. Axial anomaly

In this subsection we will show that the shift parameter $\Delta$ does not affect the axial anomaly.

In the gauge invariant point-splitting regularization, the divergence of the axial current in the massless theory equals \[20, 21\]

$$
\partial_{\mu} j^\mu_{\alpha}(u) = i e \epsilon^\alpha \bar{\psi}(u + \epsilon) \gamma^\mu \gamma^5 \psi(u) F_{\alpha\mu} \big|_{\epsilon \to 0}.
$$

Then, calculating the vacuum expectation value of the divergence of the axial current, we find

$$
\langle \partial_{\mu} j^\mu_{\alpha}(u) \rangle = -i e \epsilon^\alpha F_{\alpha\mu} \text{tr} \left[ \gamma^\mu \gamma^5 G(u, u + \epsilon) \right] \big|_{\epsilon \to 0} = i e \epsilon^\alpha F_{\alpha\mu} \langle j^\mu_{\alpha}(u) \rangle,
$$

where $G(u, u')$ is the fermion propagator in Eq. (16). Let us check that the presence of $\Delta$ in $G(u, u')$, which modifies the axial current, does not affect the standard expression for the axial anomaly.

We start by considering the first term in the axial current density in Eq. (19):

$$
\frac{i e \mu s}{\pi^2 e^4} \left( e^{-i s \Delta e^3} - e^{i s \Delta e^3} \right) \simeq \frac{2 \Delta e^3}{\pi^2 e^4} \left( 1 - \frac{\Delta^2 e^3}{6} + ... \right).
$$

Its contribution to the right-hand side of Eq. (25) is

$$
\frac{2 i e \epsilon^\alpha e^3}{\pi^2 e^4} \left( 1 - \frac{\Delta^2 e^3}{6} + ... \right) e F_{\alpha\mu}.
$$

Since this expression contains only odd powers of $\epsilon$, it gives zero contribution after symmetric averaging over space-time directions of $\epsilon$.

Thus, only the second term in Eq. (19) is relevant for the divergence of axial current in Eq. (20):

$$
\langle \partial_{\mu} j^\mu_{\alpha}(u) \rangle = -\frac{e^2 \beta \mu \lambda \sigma}{8 \pi^2 e^2} F_{\alpha\mu} F_{\lambda\sigma} \epsilon^\beta \epsilon^\lambda \epsilon^\sigma \left( e^{-i s \Delta e^3} + e^{i s \Delta e^3} \right) \rightarrow -\frac{e^2}{16 \pi^2} e^\beta \mu \lambda \sigma F_{\beta\mu} F_{\lambda\sigma}
$$

for $\epsilon \to 0$ and symmetric averaging over space-time directions of $\epsilon$. Therefore, the presence of the shift parameter $\Delta$ does not affect the axial anomaly indeed.

IV. DISCUSSION

The emphasis in this Letter was on studying the structure of the induced axial current and the chiral anomaly in the normal phase in magnetized relativistic matter. Our conclusion is that there are two components in this current, the topological component, induced only in the LLL, and the dynamical one provided by the chiral shift $\Delta$ (and generated in all Landau levels). While the former is intimately connected with the axial anomaly, the latter does not affect the form of the anomaly at all. Thus one can say that while the topological component of the current is anomalous, the dynamical one is normal.

The present analysis was realized in the NJL model. It would be important to extend it to renormalizable field theories, especially, QED and QCD. In connection with that, we would like to note the following. The expression for the chiral shift parameter, $\Delta \sim g \mu e B / \Lambda^2$, obtained in the NJL model implies that both fermion density and magnetic field are necessary for the generation of $\Delta$. This feature should also be valid in renormalizable theories. As for the cutoff $\Lambda$, it enters the results only because of the nonrenormalizability of the NJL model. Similar studies of chiral symmetry breaking in the vacuum ($\mu_0 = 0$) QED and QCD in a magnetic field show that the cutoff scale $\Lambda$ is replaced by $\sqrt{|eB|}$ there \[20\]. Therefore, one might expect that in QED and QCD with both $\mu$ and $B$ being nonzero, $\Lambda$ will be replaced by a physical parameter, such as $\sqrt{|eB|}$. This in turn suggests that a constant chiral shift parameter $\Delta$ will become a running quantity that depends on the longitudinal momentum $k^3$ and the Landau level index $n$.

It has been recently suggested in Refs. 10, 11, that a chiral magnetic spiral solution is realized in the chirally broken phase in the presence of a strong magnetic field. Like the present solution with the chiral shift parameter $\Delta$, the chiral magnetic spiral one is anisotropic, but beside that it is also inhomogeneous. It is essential, however, that the solution with the chiral shift is realized in the normal phase of matter, in which the fermion density and the axial current density are non-vanishing. It would be interesting to clarify whether there is a connection between these two solutions describing the dynamics in the two different phases of magnetized relativistic matter.

In this Letter, we concentrated on the basic and delicate questions regarding the chiral shift parameter $\Delta$, the induced axial current, and the axial anomaly, but did not address many specific details regarding the dynamics, e.g., those related to the chiral asymmetry of the Fermi surface \[3\] and a dependence of $\Delta$ on the temperature and the current fermion mass. These issues, which are of great interest because of their potential applications in neutron stars and in heavy ion collisions, will be considered elsewhere \[25\].
Appendix A: Derivation of gap equation

In order to derive the gap equation, it is convenient to utilize the formalism of the effective action for composite operators [18]. In the mean field approximation that we use, the corresponding effective action $\Gamma$ has the following form:

$$
\Gamma(G) = -i \text{Tr} \left[ \log G^{-1} + S^{-1}G - 1 \right] + \frac{G_{\text{int}}}{2} \int dt \int d^3x \left\{ \left( \text{tr} \left[ G(x, x) \right] \right)^2 - \left( \text{tr} \left[ \gamma^5 G(x, x) \right] \right)^2 \right. \\
- \text{tr} \left[ G(x, x) G(x, x) \right] + \text{tr} \left[ \gamma^5 G(x, x) \gamma^5 G(x, x) \right] \right\},
$$

(A1)

where the trace, the logarithm, and the product $S^{-1}G$ are taken in the functional sense. Here $S$ and $G$ are the tree level fermion propagator and the full one, respectively. The free energy density $\Omega$ is expressed through $\Gamma$ as $\Omega = -\Gamma/TV$, where $TV$ is a space-time volume.

The stationarity condition $\delta \Gamma(G)/\delta G = 0$ leads to the gap equation

$$
G^{-1}(u, u') = S^{-1}(u, u') - iG_{\text{int}} \left\{ G(u, u) - \gamma^5 G(u, u) \gamma^5 - \text{tr} [G(u, u)] + \gamma^5 \text{tr} [\gamma^5 G(u, u)] \right\} \delta^4(u - u').
$$

(A2)

Here while the first two terms in the curly brackets describe the exchange (Fock) interactions, the last two terms describe the direct (Hartree) interactions.

The structure of $G^{-1}(u, u')$ is shown in Eq. (2), and the structure of the inverse tree level fermion propagator $S^{-1}(u, u')$ is determined from the Lagrangian density in Eq. (1):

$$
i S^{-1}(u, u') = \left[ (i\partial_t + m_0)\gamma^0 - (\pi \cdot \gamma) - \pi^3 \gamma^3 - m_0 \right] \delta^4(u - u').
$$

(A3)

Comparing this expression with that in Eq. (2), we see that the inverse full fermion propagator $G^{-1}$ contains two new types of dynamical parameters that are absent at tree level in $S^{-1}$: $\tilde{\mu}$ and $\Delta$. It is clear that $\tilde{\mu}$ plays the role of an anomalous magnetic moment.

It should be emphasized that the Dirac mass and the chemical potential terms in the full propagator are determined by $m$ and $\mu$ that may differ from their tree level counterparts, $m_0$ and $\mu_0$. While $m_0$ is a bare fermion mass, $m$ is a constituent one, which in general depends on the density and temperature of the matter, as well as on the strength of interactions. Concerning the chemical potentials, it is $\mu_0$ that is the chemical potential in the thermodynamic sense. The value of $\mu$, on the other hand, is an “effective” chemical potential that determines the quasiparticle dispersion relations for fermion quasiparticles in interacting theory. As was already pointed out in Sec. [11, Eq. 3] implies that at any nonzero fermion density $\langle \rho \rangle$, $\mu_0$ and $\mu$ are different if $G_{\text{int}} \neq 0$.

In order to determine the values of the parameters $m$, $\mu$, $\Delta$ and $\tilde{\mu}$ in the model at hand, we use gap equation (A2). As one can see, the right-hand side of this equation depends only on the full fermion propagator $G(u, u')$ at $u' = u$. This fact greatly simplifies the analysis. Of course, it is related to the fact that we use the local four-fermion interaction. This feature will be lost in more realistic models with long-range interactions. The main disadvantage of the local four-fermion interaction is the nonrenormalizability of the model. Therefore, model (a) should be viewed only as a low-energy effective model reliable at the energy scales below a certain cutoff energy $\Lambda$.

The propagator $G(u, u)$ has the following Dirac structure:

$$
G(u, u) = -\frac{1}{4} \left[ \gamma^0 A + B + i\gamma^1 \gamma^2 C + \gamma^3 \gamma^5 D \right].
$$

(A4)
The four coefficient functions can be defined through the following traces:

\[
A = - \text{tr} \left[ \gamma^0 G(u,u) \right] \equiv \langle \bar{\psi} \gamma^0 \psi \rangle = \langle j^0 \rangle, \quad (A5)
\]
\[
B = - \text{tr} \left[ G(u,u) \right] \equiv \langle \bar{\psi} \psi \rangle, \quad (A6)
\]
\[
C = - \text{tr} \left[ \gamma^1 \gamma^2 G(u,u) \right] \equiv \langle \bar{\psi} i \gamma^1 \gamma^2 \psi \rangle, \quad (A7)
\]
\[
D = - \text{tr} \left[ \gamma^3 \gamma^5 G(u,u) \right] \equiv \langle \bar{\psi} \gamma^3 \gamma^5 \psi \rangle = \langle j^5 \rangle. \quad (A8)
\]

Then the gap equation \((A2)\) can be rewritten in the following form:

\[
(\mu - m_0) \gamma^0 + i \bar{\mu} \gamma^1 \gamma^2 + \Delta \gamma^3 \gamma^5 - m + m_0 = -\frac{1}{2} G_{\text{int}} \left[ \gamma^0 A + \gamma^3 \gamma^5 D \right] + G_{\text{int}} B. \quad (A9)
\]

Note that function \(C\), related to the anomalous magnetic moment, see Eq. \((A7)\), does not enter the right-hand side of the gap equation. Therefore, in the mean field approximation used here, no nontrivial solution for \(\bar{\mu}\) is allowed \cite{22}. The matrix equation \((A9)\) with \(\bar{\mu} = 0\) is equivalent to the set of three algebraic Eqs. \((3) - (5)\) in Sec. \(\S\).

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