On variations in teleparallelism theories

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Abstract

The variation procedure on a teleparallel manifold is studied. The main problem is the non-commutativity of the variation with the Hodge dual map. We establish certain useful formulas for variations and restate the master formula due to Hehl and his collaborators. Our approach is different and sometimes easier for applications. By introducing the technique of the variational matrix we find necessary and sufficient conditions for commutativity (anti-commutativity) of the variation derivative with the Hodge dual operator. A general formula for the variation of the quadratic-type expression is obtained. The described variational technique are used in two viable field theories: the electro-magnetic Lagrangian on a curved manifold and the Rumpf Lagrangian of the translation invariant gravity.

1 Introduction

One of the biggest challenges of the theoretical physics is to unify the standard model of particles interactions with Einstein’s theory of gravity. Both these theories are in a good according with all known observable phenomena. But their

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fundamental concept are rather differ. Standard model is a quantum field theory on a flat Minkowskian space-time. In contrast to that the Einstein’s description of gravity is a classical field theory essential connected with the geometrical properties of the pseudo-Riemannian manifold. One can expect the combination of these different theories only after principal modification of every one of them. Great attempts are making recently in modernization the standard model in accordance with the idea of superstrings. The resent description of the situation in this area can be found in [1]. The second partner of the future couple should also make some necessary preparations. The Einsteinian theory of general relativity (GR) is almost a unique theory of gravity that can be formulated in the framework of the pseudo-Riemannian manifold. Thus an essential modification can be made only on a basis of some generalized geometrical construction. The study of various geometries relevant to gravity began at once after Einstein had proposed his general relativity. The most general geometrical theory of gravity is the metric-affine theory [3]. In the present paper we study a rather simpler generalization of the pseudo-Riemann geometry - the theories of teleparallelism. The teleparallel space was introduced for the first time by Cartan [4] and used by Einstein [5], [6] in a certain variant of his unified theory of gravity and electromagnetism. The work of Weitzenböck [5] was the first denoted to the investigation of the geometric structure of teleparallel spaces. The theories based on this geometrical structure appear in physics time to time in order to give an alternative model of gravity or to describe the spin properties of matter. For the resent investigations in this area see Ref. [6], [7], [10], [11], [12], [13].

It is convenient and useful to have a Lagrangian formulation of a field theory. A designated functional, denoted by action, is the basis for the Lagrangian formalism. The field equations of the classical field theory are the representation of the critical point condition for the prescribed action functional.

Let us take a brief look on the main properties of the action functionals in the classical relativistic field theory.

- The action functional is an integral taken on the whole 4-dimensional differential manifold.
- The integrand is a certain differential 4-form - \textit{Lagrangian density}.
- The Lagrangian density is a scalar invariant under the group of transformations of the considered system.
- The Lagrangian density incorporates only the squares of the first-order derivatives of the field variables in the same point - local densities.
We consider the last condition as a necessary one, because almost all the physically meaningful field equations are second-order partial differential equations of hyperbolic type. The applying of the variational principle on the teleparallel framework is connected with various problems, as it emphasized in [14]. The main source of this problems connects with the fact that the operators of a certain type, namely the Hodge dual operator, the interior product and the coderivative operator, non-commute with the variational derivative. In the present paper we study the the variation procedure (free and constrained) on the teleparallel manifold. The overview of the work is as following. We begin with a brief overview of the variational procedure in the general relativity. In the second section we describe the preferences that bring the consideration of a teleparallel space instead of a pseudo-Riemann manifold. The most advantage of such generalization is a possibility to describe the gravity by the quadratic Lagrangians, similar to the other field theories. Variational procedure on the teleparallel spaces is described in the third section. We prove the commutative relation which coincides in the case of pseudo-orthonormal coframe with the master formula of Muench, Gronwald and Hehl [14]. The analogous relation described also in [15]. Using the commutative relation we derive the formula for variation of the general quadratic Lagrangian. The fourth section is devoted for describing the various types of the covariant constrains on a teleparallel structure. We study the relation of such constrains with the commutativity and anti-commutativity of the variational derivative with the Hodge dual map. The last two sections are devoted to the application of our variational formula for viable field theories. We consider the Maxwell Lagrangian on a teleparallel space and the Rumpf Lagrangian for a translation invariant gravity.

2 Actions on a teleparallel manifold.

The teleparallel space was considered originally as a 4D-space endowed with a smooth field of frame (an ordered set of 4 independent vectors). Thus one can give a sense to parallelism of two vectors taken in different points of the space. In order to furnish the action with a differential 4-form and to apply the technique of the differential forms we consider a coframe field instead of the frame field. Thus

Definition 2.1: A teleparallel manifold is a pair \((M, \Gamma)\), where \(M\) is a differential
$4D$-manifold and $\Gamma = \{ \vartheta^a(x^\mu) \}, \ a = 0, 1, 2, 3$ is a fixed smooth cross-section of the coframe bundle $F M$.

The field variable is a 4-tuple of 1-forms $\vartheta^a$ and they constitute a basis of the covector space $T^*_x M$ at every point $x$ of $M$. We endow this vector space with the Lorentzian metric $\eta^{ab} = \eta_{ab} = \text{diag}(-1, 1, 1, 1)$ and require the 1-forms $\vartheta^a$ to be pseudo-orthonormal relative to this metric. Let $C^\infty$ be a set of smooth real valued functions on $M$. Denote by $\Omega^p$ the $C^\infty$-modulo of differential $p$-forms on $M$.

The following algebraic operations are defined on a $n$-dimensional teleparallel manifold:

1. The exterior (wedge) product $\wedge : \Omega^p \times \Omega^q \to \Omega^{p+q}$ of two differential forms $\alpha \in \Omega^p$ and $\beta \in \Omega^q$, which is associative, $C^\infty$-bilinear and in general not commutative

$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha. \quad (2.1)$$

2. The Hodge dual map $\ast : \Omega^p \to \Omega^{n-p}$. This is a $C^\infty$-linear map $\ast : \Omega^p \to \Omega^{n-p}$, which acts on the wedge product monomials of the basis 1-forms as

$$\ast(\vartheta^{a_1 \cdots a_p}) = \epsilon^{a_1 \cdots a_n} \vartheta_{a_{p+1} \cdots a_n}, \quad (2.2)$$

where $\vartheta^a$ is the down indexed 1-forms $\vartheta^a = \eta_{ab} \vartheta^b$ and $\epsilon^{a_1 \cdots a_n}$ is the total antisymmetric pseudo-tensor.

3. The interior product $v \lrcorner \alpha$ of an arbitrary differential form $\alpha \in \Omega^p$ with an arbitrary vector $v \in T_x M$. This is a $C^\infty$-bilinear map $\lrcorner : \Omega^p \to \Omega^{p-1}$, which satisfies the following properties:

$$v \lrcorner (\alpha \wedge \beta) = (v \lrcorner \alpha) \wedge \beta + (-1)^{\text{deg} \alpha} \alpha \wedge (v \lrcorner \beta), \quad (2.3)$$

$$e_a \lrcorner \vartheta^b = \delta_a^b, \quad (2.4)$$

where $e_a$ is a basis vector of $T_x M$ and $\vartheta^a$ is a basis 1-form of $T^*_x M$.

\[\text{We use the abbreviated notations for the wedge product monomials: } \vartheta^{a_1 \cdots a_p} := \vartheta^a \wedge \vartheta^b \wedge \cdots.\]
Define a product for an arbitrary 1-form \( w \) and an arbitrary \( p \)-form as
\[
 w \star \alpha := \ast (w \wedge \ast \alpha). \tag{2.5}
\]
It is easy to see that this operation is a \( C^\infty \)-bilinear map \( \Omega^p \to \Omega^{p-1} \), which satisfies the properties (2.3, 2.4). In particular, we have
\[
 \vartheta_a \vartheta^b := \ast (\vartheta_a \wedge \ast \vartheta^b) = \delta_a^b. \tag{2.6}
\]
Note that the product defined by (2.5) is really a multiplication of two exterior forms thus this type of definition justifies the term “interior product”. We use in this paper the definition of the interior product in the form (2.5).

The following first-order differential operators are defined on a teleparallel manifold:

1. The exterior derivative operator \( d : \Omega^p \to \Omega^{p+1} \)
\[
d\alpha = d\left(\alpha_{a_1\ldots a_p} dx^{a_1} \wedge \cdots \wedge dx^{a_p}\right) = d\alpha_{a_1\ldots a_p} \wedge dx^{a_1} \wedge \cdots \wedge dx^{a_p}. \tag{2.7}
\]
This is an anti-derivative relative to the wedge product of forms
\[
d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg\alpha} \alpha \wedge d\beta. \tag{2.8}
\]
2. The coderivative operator \( d : \Omega^p \to \Omega^{p-1} \) defined by
\[
d^\perp \alpha = \ast d \ast \alpha. \tag{2.9}
\]
Using the operators described above the action functional can be accepted on the teleparallel manifold in the quadratic (Dirichlet) form. The simplest choice of the Lagrangian density can be made in the form of the Yang-Mills Lagrangian:
\[
 S[\vartheta^a] = \int_{M} d\vartheta^a \wedge \ast d\vartheta_a. \tag{2.10}
\]
Observe that the action (2.10) satisfies the following conditions:

1. It is independent on a particular choice of a local coordinate system.
2. It includes only the first order derivatives of the field variables (coframe 1-forms) so the corresponding field equation is at most of the second order.
3 It is a functional of a Dirichlet-type so it provides an equation of a harmonic type.

4 It is invariant under the global group of $SO(1, 3)$ transformations of the coframe field $\vartheta^a$.

Therefore the generalization of the pseudo-Riemannian structure to a teleparallel structure allows to consider general Lagrangians for gravity similar to the quadratic Lagrangians of the other (already quantized) field theories. In this way one can hope to define the energy of gravity field in a local, covariant form.

3 Variational procedure

The variational procedure on a teleparallel manifold for the action functional $S = S[\vartheta^a]$ can be described as follows.

Let $\vartheta^a(\lambda)$ be a smooth 1-parametric family of cross-sections of $FM$, with the following initial conditions

$$\vartheta^a(\lambda = 0) = \vartheta^a \quad \text{and} \quad \frac{\partial \vartheta^a}{\partial \lambda} \bigg|_{\lambda = 0} = \delta \vartheta^a.$$  \hspace{1cm} (3.1)

The critical points of the functional $S[\vartheta^a]$ are defined by the condition:

$$\delta S = \frac{dS}{d\lambda} \bigg|_{\lambda = 0} \quad d\lambda = 0.$$ \hspace{1cm} (3.2)

The Hamilton principle of the least action postulates that the field equation of the physical system coincides with the critical point condition for an appropriative action functional.

The variation $\delta$ ($\lambda$-differential) is independent on the space-time coordinates so it satisfies the following rules.

1. The ordinary Leibniz rule for the wedge product

$$\delta(\alpha \wedge \beta) = \delta \alpha \wedge \beta + \alpha \wedge \delta \beta.$$ \hspace{1cm} (3.3)

2. The commutativity with the exterior derivative

$$\delta \circ d = d \circ \delta.$$ \hspace{1cm} (3.4)
The non-commutativity (in general) with the Hodge star-operator
\[ \delta \circ * \neq * \circ \delta \] (3.5)

The non-commutativity (in general) with the coderivative operator
\[ \delta \circ d^+ \neq d^+ \circ \delta \] (3.6)

The following lemma is useful for the actual calculations of the variations.

Lemma 3.1: A variation of the wedge product monomials satisfies
\[ \delta \vartheta^{a_1 \ldots a_p} = \delta \vartheta^m \wedge (\vartheta_m \downarrow \vartheta^{a_1 \ldots a_p}), \] (3.7)

and for the Hodge dual monomials
\[ \delta * \vartheta^{a_1 \ldots a_p} = \delta \vartheta^m \wedge (\vartheta_m \downarrow * \vartheta^{a_1 \ldots a_p}). \] (3.8)

Proof: Using the Leibniz rule (3.3) we obtain
\[ \delta \vartheta^{a_1 \ldots a_p} = \delta \vartheta^1 \wedge \vartheta^{a_2 \ldots a_p} + \delta \vartheta^2 \wedge \vartheta^{a_1 \ldots a_p} + \ldots + \delta \vartheta^{a_1 \ldots a_{p-1}} \wedge \vartheta^p. \] (3.9)

The property of the interior product yields
\[ \vartheta_m \downarrow \vartheta^{a_1 \ldots a_p} = \delta \vartheta^a_1 \wedge \vartheta^{a_2 \ldots a_p} - \delta \vartheta^a_2 \wedge \vartheta^{a_1 a_3 \ldots a_p} + \ldots. \] (3.10)

Thus
\[ \delta \vartheta^m \wedge (\vartheta_m \downarrow \vartheta^{a_1 \ldots a_p}) = \delta \vartheta^a_1 \wedge \vartheta^{a_2 \ldots a_p} - \delta \vartheta^a_2 \wedge \vartheta^{a_1 a_3 \ldots a_p} + \ldots. \] (3.11)

Comparing (3.9) and (3.11) we obtain the required formula (3.7). The second formula (3.8) is the consequence of the previous one, because the Hodge dual of a wedge product monomial is also a monomial. □

Therefore, we have the following formula for variation of the dual form
\[ \delta * \vartheta^a = \delta \vartheta_m \wedge * (\vartheta^m \wedge \vartheta^a) \] (3.12)

Thus the variation of the dual form is essentially differ from the variation \( \delta \vartheta^a \) of the basis form \( \vartheta^a \).

For the variation of the volume element the formula (3.3) yields
\[ \delta * 1 = \delta \vartheta^m \wedge (e_m \downarrow * 1) = \delta \vartheta^m \wedge * (\vartheta_m \wedge *^2 1) = -\delta \vartheta^m \wedge * \vartheta_m. \] (3.13)
Proposition 3.2: For an arbitrary differential form $\alpha \in \Omega^p$ on a $n$-dimensional manifold the following commutative relation holds

$$\delta * \alpha - *\delta \alpha = *(\delta \vartheta_m J(\vartheta^m \wedge \alpha) - \delta \vartheta_m \wedge (\vartheta^m J\alpha)).$$  \hfill (3.14) 

Proof: Evaluating the $p$-form $\alpha$ in the basis component forms $\alpha = \alpha_{a_1...a_p} \vartheta^{a_1...a_p}$ and using the relation (3.8) we derive

$$\delta * \alpha = \delta(\alpha_{a_1...a_p}) \vartheta^{a_1...a_p} + \alpha_{a_1...a_p} \delta \vartheta^{a_1...a_p}$$

$$= \delta(\alpha_{a_1...a_p}) \vartheta^{a_1...a_p} + \alpha_{a_1...a_p} \delta \vartheta^m \wedge (e_m J * \vartheta^{a_1...a_p})$$

$$= \delta(\alpha_{a_1...a_p}) \vartheta^{a_1...a_p} + \delta \vartheta^m \wedge (\vartheta^m \wedge \alpha).$$

On the other hand

$$\delta \alpha = \delta(\alpha_{a_1...a_p}) \vartheta^{a_1...a_p} + \alpha_{a_1...a_p} \delta \vartheta^{a_1...a_p}$$

$$= \delta(\alpha_{a_1...a_p}) \vartheta^{a_1...a_p} + \delta \vartheta^m \wedge (\vartheta^m \wedge \alpha).$$

Therefore

$$\delta * \alpha - *\delta \alpha = \delta \vartheta_m \wedge (\vartheta^m J \alpha) - *(\delta \vartheta_m \wedge (\vartheta^m J \alpha))$$

$$= \left[(-1)^{p(n-p)+1} \delta \vartheta_m \wedge *(\vartheta^m \wedge \alpha) - *\left(\delta \vartheta_m \wedge (\vartheta^m J \alpha)\right)\right]$$

This proves the relation (3.14).

It is easy to see from the proof of the proposition above that in a particular case of basis forms the following (anti)commutativity relations hold

$$\delta \vartheta^{a_1...a_p} \pm *\delta \vartheta_{a_1...a_p} = *(\delta \vartheta_m J \vartheta^{ma_1...a_p} \pm \delta \vartheta_m \wedge (\vartheta^m J \vartheta^{a_1...a_p})).$$ \hfill (3.15)
4 Variations and constrains.

The ordinary variational problem is used to study the critical points of a real-valued functional. The variations of the field variables are considered to be independent - free variations. In order to study a constraint physical system, for instance a motion of a particle on a fixed surface, one consider a critical point of a functional restricted by appropriate constrains. This restriction is usually incorporated by the method of Lagrangian multipliers. Another view on the constrained variational problem can be proposed by consideration of constraint variations. These variations satisfy some additional conditions. In the framework of a global $SO(1,3)$ invariant and diffeomorphic covariant theory we are interesting in covariant constraint conditions. Denote the variation of the basis 1-form $\vartheta^a$ by

$$\delta \vartheta^a = \delta \vartheta^a = \epsilon^a_b \vartheta^b. \quad (4.1)$$

We will refer to the matrix $\epsilon_{ab} = \eta_{ac} \epsilon^c_b$ as the variational matrix. The variational matrix admits a natural covariant decomposition

$$\epsilon_{ab} = \epsilon^{(1)}_{ab} + \epsilon^{(2)}_{ab} + \epsilon^{(3)}_{ab}, \quad (4.2)$$

where

$$\epsilon^{(1)}_{ab} = \frac{1}{2}(\epsilon_{ab} - \epsilon_{ba}) \quad -\text{antisymmetric variation}, \quad (4.3)$$

$$\epsilon^{(2)}_{ab} = \frac{1}{2}(\epsilon_{ab} + \epsilon_{ba}) - \epsilon_{ab} \quad -\text{traceless symmetric variation}, \quad (4.4)$$

$$\epsilon^{(3)}_{ab} = \epsilon_{ab} \quad -\text{trace variation}. \quad (4.5)$$

We use here and later the notation $\epsilon = \epsilon^a_a = \epsilon_{ab} \eta^{ab}$ for a trace of the variation matrix. We can study now a free variation with an arbitrary variational matrix as well as various constrains variations, determined by a special choice of the variational matrix $\epsilon_{ab}$.

Let us examine the conditions given by the different constrains.

The question is: What condition should be imposed on the variations of $\{\vartheta^a\}$ in order to obtain some commutativity conditions with the Hodge star operator?

Consider two following cases

$$\delta * \vartheta^a = * \delta \vartheta^a \quad -\text{commutativity} \quad (4.6)$$
\[ \delta \ast \vartheta^a = - \ast \delta \vartheta^a \] - anticommutativity (4.7)

Using (3.12) in the right hand sides of (4.6,4.7) and considering they together we obtain

\[ \ast \delta \vartheta^a = \pm \delta \vartheta_m \wedge \ast (\vartheta^m \wedge \vartheta^a). \] (4.8)

Hence the (anti)commutativity conditions take the form

\[ \ast \delta \vartheta^a = \pm \delta \vartheta_m \vartheta^{ma}. \] (4.9)

Inserting the variational matrix notation we derive

\[ \epsilon^a_b \vartheta^b = \pm \epsilon^k_m \vartheta_k \vartheta^{ma} = \pm \epsilon^k_m (\vartheta^a_k - \vartheta^a_m) = \pm \epsilon^a_b = \pm \left( \epsilon \delta^a_b - \epsilon b^a \right). \]

Therefore in the notations of the variational matrix the (anti)commutativity conditions (4.9) take the form

\[ \epsilon^a_b = \pm \left( \epsilon \delta^a_b - \epsilon b^a \right). \] (4.10)

Taking the trace in two sides of this matrix equation we obtain

\[ \epsilon = \pm 3 \epsilon. \]

Thus the first necessary condition for (anti)commutativity of the variation with the Hodge dual is

\[ \epsilon = 0. \] (4.11)

The equation (4.10) gives the second condition for (anti)commutativity

\[ \epsilon_{ab} \pm \epsilon_{ba} = 0. \] (4.12)

The consideration above can be summarized in the following

**Proposition 4.1:**

*The variation commutes with the Hodge dual of a basis 1-form

\[ \delta \ast \vartheta^a = \ast \delta \vartheta^a \] (4.13)*

*if and only if the variational matrix \( \epsilon^a_b \) is antisymmetric.*

*The variation anticommutates with the Hodge dual of a basis 1-form

\[ \delta \ast \vartheta^a = - \ast \delta \vartheta^a \] (4.14)*

*if and only if the variational matrix \( \epsilon^a_b \) is traceless and symmetric.*
Consider now the variation of the metric tensor

\[ \delta g_{\mu\nu} = \delta(\eta_{ab}\vartheta^a_{\mu}\vartheta^b_{\nu}) = \eta_{ab}(\delta \vartheta^a_{\mu}\vartheta^b_{\nu} + \vartheta^a_{\mu}\delta \vartheta^b_{\nu}). \]

But

\[ \delta \vartheta^a_b = \epsilon^a_b \vartheta^b = \delta(\vartheta^a_{\mu}dx^\mu) = \delta \vartheta^a_{\mu}dx^\mu = \vartheta^b_{\mu}\delta \vartheta^a_{\nu}. \]

Therefore

\[ \vartheta^a_{\mu} \delta \vartheta^b_{\nu} = \epsilon^a_b, \]

or

\[ \delta \vartheta^a_{\mu} = \epsilon^a_b \vartheta^b_{\mu}. \]

Hence the variation of the metric tensor is

\[ \delta g_{\mu\nu} = \eta_{ab}(\epsilon^c_a \vartheta^c_{\mu}\vartheta^b_{\nu} + \epsilon^c_b \vartheta^a_{\mu}\vartheta^c_{\nu}) = \epsilon_{bc}(\vartheta^c_{\mu}\vartheta^b_{\nu} + \vartheta^b_{\mu}\vartheta^c_{\nu}). \] (4.15)

Note that the expression in the brackets is symmetric under the permutation of the indices \(c\) and \(b\). Thus the variation of the metric tensor \(\delta g_{\mu\nu}\) is identically equal to zero if and only if the matrix \(\epsilon_{bc}\) is antisymmetric. Hence the commutativity of the variational operator with the Hodge dual map results in a zero variation of the metric tensor.

In the second (anti-commutativity) case the variation of the metric tensor is generally differ from zero. This variation is traceless and symmetric so it represents a 9-parametric family of transformations.

The traceless variations preserve the determinant of the metric tensor \(g^{\mu\nu}\). Indeed the relation (4.15) yields:

\[ g^{\mu\nu}\delta g_{\mu\nu} = 2\epsilon_{bc}\eta^{bc} = 2\epsilon. \] (4.16)

The variation of the determinant of the metric tensor can be written as

\[ \delta g = gg^{\mu\nu}\delta g_{\mu\nu}. \] (4.17)

Therefore in the case of the traceless variations

\[ \delta g = 2\epsilon g = 0. \] (4.18)

Note a connected result. Since \(\delta \ast 1 = \epsilon \ast 1\) it follows that the volume element is preserved if and only if \(\epsilon = 0\). Summarize the conclusions in the following table:
Constrained variations and commutativity rules.

| Type of variation | Variational matrix | Metric tensor | Variation of volume | Commutativity rules |
|-------------------|---------------------|---------------|---------------------|---------------------|
| free              | $\epsilon_{ab}$    | $\delta g^{\mu\nu} \neq 0$ | $\delta * 1 \neq 0$ |                     |
| antisymmetric     | $\epsilon_{ab}^{(1)}$ | $\delta g^{\mu\nu} = 0$ | $\delta * 1 = 0$ | $\delta * \vartheta^a = *\delta \vartheta^a$ |
| symmetric         | $\epsilon_{ab}^{(2)} + \epsilon_{ab}^{(3)}$ | $\delta g^{\mu\nu} \neq 0$ | $\delta * 1 \neq 0$ |                     |
| traceless symmetric | $\epsilon_{ab}^{(2)}$ | $\delta g^{\mu\nu} \neq 0$ | $\delta * 1 = 0$ | $\delta * \vartheta^a = - * \delta \vartheta^a$ |

We have established the commutativity conditions only for the variations of the basis 1-forms, but they can be extended straightforward for basis forms of an arbitrary degree. For instance, for a form $\alpha$ of an arbitrary degree the formula \((3.14)\) yields

$$
\delta * \vartheta^a - * \delta \vartheta^a = * \epsilon_{mk}^k \left( \vartheta_k \mathcal{J} (\vartheta^m \wedge \alpha) - \vartheta_k \wedge (\vartheta^m \mathcal{J} \alpha) \right)
$$

The first term in the last expression vanish in the case of a traceless variation. The expression in the brackets is symmetric under the underchange of the indices $k$ and $m$. Thus in the case of an antisymmetric variational matrix $\epsilon_{mk}$, the variation commutes with the Hodge dual. In the case of a basic form of an arbitrary degree the relation \((3.15)\) yields to the anti-commutative condition for traceless symmetric variation

$$
\delta * \vartheta^{a_1 \ldots a_p} = - * \delta \vartheta^{a_1 \ldots a_p}.
$$

We see that the commutativity (anti-commutativity) of the variation with the Hodge dual operator is connected with a certain special case of constrained variation of the coframe field. The special case of commutativity was proved for the first time in [14].
5 General quadratic Lagrangian

A typical Lagrangian used in the field theory is quadratic. It is useful to have a formula for a variational derivative of a such type expression.

**Proposition 5.1:** For \( \alpha, \beta \in \Omega^n \) on a \( n \)-dimensional manifold the variation of the square-norm expression \( L = \alpha \wedge * \beta \) takes a following form

\[
\delta (\alpha \wedge * \beta) = \delta \alpha \wedge * \beta + \alpha \wedge * \delta \beta - \delta \vartheta^m \wedge J^m, \tag{5.1}
\]

where

\[
J^m = (\vartheta^m \downarrow \beta) \wedge * \alpha - (-1)^p \alpha \wedge (\vartheta^m \downarrow * \beta). \tag{5.2}
\]

**Proof:** Using the Leibniz rule for the variation (3.3) we write

\[
\delta (\alpha \wedge * \beta) = \delta \alpha \wedge * \beta + \alpha \wedge \delta * \beta. \tag{5.3}
\]

The formula for commutator (3.14) yields

\[
\delta (\alpha \wedge * \beta) = \delta \alpha \wedge * \beta + \alpha \wedge \delta * \beta + \alpha \wedge (\delta \vartheta^m \downarrow (\vartheta^m \wedge \beta) - \delta \vartheta^m \wedge (\vartheta^m \downarrow * \beta)). \tag{5.4}
\]

Compute the third term on the right hand side of (5.4)

\[
\alpha \wedge * \left( \delta \vartheta^m \downarrow (\vartheta^m \wedge \beta) \right) = \alpha \wedge * \left( \delta \vartheta^m \wedge * (\vartheta^m \wedge \beta) \right)
\]

\[
= (-1)^p (n-p+1) \alpha \wedge \delta \vartheta^m \wedge * (\vartheta^m \wedge \beta)
\]

\[
= (-1)^p \delta \vartheta^m \wedge \alpha \wedge (\vartheta^m \downarrow * \beta).
\]

The fourth term on the right hand side of (5.4) is

\[
\alpha \wedge * \left( \delta \vartheta^m \wedge (\vartheta^m \downarrow \beta) \right) = \delta \vartheta^m \wedge (\vartheta^m \downarrow \beta) \wedge * \alpha.
\]

These prove the statement. □

The \( n - 1 \) form \( J^m \) represents a certain type of a field current. Because of a symmetry of the expression (5.1) under the permutation of \( \alpha \) and \( \beta \) we can rewrite the value \( J^m \) in an explicitly symmetric form:

\[
J^m = -\frac{1}{2} \left[ \left[ (\vartheta^m \downarrow \beta) \wedge * \alpha + (\vartheta^m \downarrow \alpha) \wedge * \beta \right.ight.
\]

\[
\left. -(-1)^p [\alpha \wedge (\vartheta^m \downarrow * \beta) + \beta \wedge (\vartheta^m \downarrow * \alpha)] \right]. \tag{5.5}
\]
In the special case $\alpha = \beta$ we obtain

$$J^m = (\vartheta^m J^\alpha) \wedge *\alpha - (-1)^p \alpha \wedge (\vartheta^m J \ast \alpha). \quad (5.6)$$

The $n - 1$ form of current $J^m$ can be written component-wisely as

$$J^m = J^{mn} \ast \vartheta_n. \quad (5.7)$$

Using

$$*(\vartheta^k \wedge J^m) = J^{mn} * (\vartheta^k \wedge \vartheta_n) \quad (5.8)$$

we obtain the explicit expression for the matrix $J^{mn}$

$$J^{mn} = * (\vartheta^m \wedge J^m). \quad (5.9)$$

The two indexed object $J^{mn}$ can be considered as an analog of the energy-momentum tensor.

**Proposition 5.2**: The matrix $J^{mn}$ is symmetric

$$J^{mn} = J^{nm}. \quad (5.10)$$

**Proof**: For the proof we use a simpler expression (5.2) for $J^{mn}$, but take in account that this object is actually symmetric for a permutation of the forms $\alpha$ and $\beta$. Using (5.2, 5.9) we derive

$$J^{mn} = \ast (\vartheta^n \wedge J^m) = \ast \left[ \vartheta^n \wedge (\vartheta^m \beta) \wedge *\alpha - (-1)^p \alpha \wedge (\vartheta^m J \ast \beta) \right]$$

$$= (-1)^{p-1} * \left( (\vartheta^m \beta) \wedge (\vartheta^n \wedge *\alpha) + (\vartheta^n \wedge \alpha) \wedge (\vartheta^m \ast \beta) \right).$$

Write

$$\vartheta^n \wedge *\alpha = (-1)^{\sigma_1} \ast^2 \vartheta^n \wedge *\alpha = (-1)^{\sigma_1} * (\vartheta^n J^\alpha), \quad (5.11)$$

$$\vartheta^m J \ast \beta = * (\vartheta^m \wedge *^2 \beta) = (-1)^{\sigma_2} * (\vartheta^m \wedge \beta), \quad (5.12)$$

where $\sigma_1, \sigma_2$ are two integers. Thus we obtain

$$J^{mn} = (-1)^{p-1} * \left( (-1)^{\sigma_1} (\vartheta^m \beta) \wedge * (\vartheta^n J^\alpha) + \right.$$

$$(-1)^{\sigma_2} (\vartheta^n \wedge \alpha) \wedge * (\vartheta^m \wedge \beta) \right). \quad (5.13)$$
Now the symmetry of $J_{mn}$ in the form (5.13) under the permutation of the forms $\alpha$ and $\beta$ yields the symmetry under the permutation of the indices $m$ and $n$. Calculate the trace of the matrix $J_{mn}$.

$$J_{m m} = *\left(\partial_m \wedge J^m \right) = *\left(\partial_m \wedge \left(\partial^m \wedge \beta \right) \wedge *\alpha - (-1)^p \alpha \wedge \left(\partial^m \wedge *\beta \right)\right)$$

$$= *\left(\partial_m \wedge \left(\partial^m \wedge \beta \right) \wedge *\alpha - \alpha \wedge \partial_m \wedge \left(\partial^m \wedge *\beta \right)\right). \quad (5.14)$$

Use the formula

$$\partial_m \wedge (\partial^m w) = pw, \quad (5.15)$$

where $w$ is an arbitrary $p$-form. The trace takes the form

$$J_{m m} = p\beta \wedge *\alpha - (n - p)\alpha \wedge *\beta = -(n - 2p) * (\alpha \wedge *\beta). \quad (5.16)$$

Thus we obtain

**Proposition 5.3:** For a quadratic Lagrangian

$$\mathcal{L} = \alpha \wedge *\beta \quad \text{deg} \alpha = \text{deg} \beta = p \quad (5.17)$$

on a manifold $M$ the matrix $J_{mn}$ is traceless $J_{m m} = 0$ if and only if the dimensional of the manifold $M$ is even.

The second condition, namely $\text{deg} \alpha = \text{deg} \beta = \frac{n}{2}$, does not so crucial as it can be supposed on a first look. Indeed, for a 4-dimensional manifold three different cases are possible

$$\text{deg} \alpha = \text{deg} \beta = 2, \quad (5.18)$$

$$\text{deg} \alpha = \text{deg} \beta = 1, \quad (5.19)$$

$$\text{deg} \alpha = \text{deg} \beta = 2. \quad (5.20)$$

In the second case we can rewrite the Lagrangian as

$$\mathcal{L} = \alpha \wedge *\beta = \partial^a \wedge \alpha \wedge \left(\partial_a \wedge *\beta \right) = \left(\partial^a \wedge \alpha \right) \wedge *\left(\partial_a \wedge \beta \right). \quad (5.21)$$

Now this is a product of two forms of a second degree. In the third case (5.20) we can apply the same transformation twice.

The variational procedure described above can be straightforward extended for a
general situation with an external field on the manifold \( \{M, \vartheta^a\} \). Consider, for instance, the Lagrangian

\[
\mathcal{L} = \alpha(\phi, d\phi, \vartheta^a) \wedge *\beta(\phi, d\phi, \vartheta^a),
\]

where \( \phi \) is a certain external field (possible multicomponent) differential form of degree \( q \). Write the variation of the forms \( \alpha, \beta \) as

\[
\delta \alpha = \delta \phi \wedge \frac{\delta \alpha}{\delta \phi} + \delta d\phi \wedge \frac{\delta \alpha}{\delta d\phi} + \delta \vartheta^m \wedge \frac{\delta \alpha}{\delta \vartheta^m},
\]

\[
\delta \beta = \delta \phi \wedge \frac{\delta \beta}{\delta \phi} + \delta d\phi \wedge \frac{\delta \beta}{\delta d\phi} + \delta \vartheta^m \wedge \frac{\delta \beta}{\delta \vartheta^m}.
\]

The relations (5.1) yields

\[
\delta \mathcal{L} = (\delta \phi \wedge \frac{\delta \alpha}{\delta \phi} + \delta d\phi \wedge \frac{\delta \alpha}{\delta d\phi} + \delta \vartheta^m \wedge \frac{\delta \alpha}{\delta \vartheta^m}) \wedge *\beta +
(\delta \phi \wedge \frac{\delta \beta}{\delta \phi} + \delta d\phi \wedge \frac{\delta \beta}{\delta d\phi} + \delta \vartheta^m \wedge \frac{\delta \beta}{\delta \vartheta^m}) \wedge *\alpha - \delta \vartheta^m \wedge J^m. (5.25)
\]

Thus we obtain two Euler-Lagrange field equations

\[
\frac{\delta \alpha}{\delta \phi} \wedge *\beta + \frac{\delta \beta}{\delta \phi} \wedge *\alpha - (\delta \frac{\delta \alpha}{\delta d\phi} \wedge *\beta + \delta \frac{\delta \beta}{\delta d\phi} \wedge *\alpha) = 0,
\]

\[
\frac{\delta \alpha}{\delta \vartheta^m} \wedge *\beta + \frac{\delta \beta}{\delta \vartheta^m} \wedge *\alpha = J^m.
\]

In two following sections we apply the variational procedure described above for two important cases: the Maxwell Lagrangian for electromagnetism and the Rumpf Lagrangian for translation invariant gravity.

## 6 Maxwell Lagrangian on a teleparallel space

The Lagrangian of the Maxwell theory of electromagnetism can be written as

\[
\mathcal{L} = dA \wedge *dA,
\]

(6.1)
where $A$ is a 1-form of electro-magnetic potential. This form of Lagrangian represents the classical theory of electromagnetism on a flat Minkowskian space. If one consider the field $A$ to be defined on a curved pseudo-Riemannian manifold the 4-form $\mathcal{L}$ is well defined (invariant under the transformation of coordinates). The Hodge star operator in this situation depends on the metric $g$ on the manifold. Thus it is better to write the electro-magnetic Lagrangian on a pseudo-Riemannian manifold as

$$\mathcal{L} = dA \wedge *_g dA.$$  \hfill (6.2)

The variation of the Lagrangian have to be applied by using a free variation of the electro-magnetic potential $A$ as well as a free variation of the metric $g$. On a teleparallel manifold the Lagrangian can be taken in the same form, but now the Hodge star operator depends on the coframe field $\vartheta^a$

$$\mathcal{L} = dA \wedge *_{\vartheta^a} dA.$$  \hfill (6.3)

The variation of the Lagrangian (6.3) can be produced using the formulas (5.1, 5.2).

$$\delta \mathcal{L} = \delta (dA) \wedge *dA + dA \wedge *\delta (dA) - \delta \vartheta^m \wedge J^m,$$  \hfill (6.4)

where

$$J^m = (\vartheta^m dA) \wedge *dA - dA \wedge (\vartheta^m \downarrow dA).$$  \hfill (6.5)

Extracting the total derivative in (6.4)

$$\begin{align*}
\delta \mathcal{L} &= 2d(\delta A) \wedge *dA - \delta \vartheta^m \wedge J^m \\
&= 2d(\delta A \wedge *dA) + 2\delta A \wedge d * dA - \delta \vartheta^m \wedge J^m
\end{align*}$$

we obtain two field equations

$$d \star dA = 0$$  \hfill (6.6)

and

$$J^m = 0,$$  \hfill (6.7)

where

$$J^m = (\vartheta^m \downarrow dA) \wedge *dA - dA \wedge (\vartheta^m \downarrow dA).$$  \hfill (6.8)
The equation (6.6) has exactly the same form as the ordinary electro-magnetic field equation, but it is an equation on a curved manifold (operator $*$ depends on the coframe $\varpi^a$). For interpretation the equations (6.6,6.7) introduce the strength notation for electro-magnetic field

$$dA = F.$$  \hfill (6.9)

The first field equation (6.6) takes the form of ordinary Maxwell equations

$$dF = 0,$$  \hfill (6.10)
$$d * F = 0.$$  \hfill (6.11)

The equation (6.7) gives an additional relation

$$J^m = (\varpi^m \lrcorner F \rightharpoonup *) F - F \lrcorner (\varpi^m \lrcorner F).$$  \hfill (6.12)

The 2-form $F$ can be explained component-wisely as

$$F = F_{mn} \varpi^{mn}. \hfill (6.13)$$

Note that the coefficients of $F$ are antisymmetric by definition $F_{mn} = -F_{nm}$.

Consider the first term of $J^m$ (6.12)

$$(\varpi^m \lrcorner F \rightharpoonup *) F = (\varpi^m \lrcorner F^{\delta^k_n} \varpi_n) \lrcorner *F_{pq} \varpi^{pq} = 2F^{kn}F_{pq} \delta^m_k \varpi_n \lrcorner *\varpi^{pq} = 4F^{mn}F_{pq} \varpi^m \varpi^q,$$

The second term of (6.12) takes the form

$$F \lrcorner (\varpi^m \lrcorner F) = F^{kn} \varpi_k \lrcorner (\varpi^m \lrcorner F_{pq} \varpi^{pq}) = -F^{kn}F_{pq} \varpi_k \lrcorner *\varpi^{mpq} = F^{kn}F_{pq} \varpi_k \lrcorner (\delta^m_n \varpi^{pq} - 2\delta^m_p \varpi^{mq}) = F^{kn}F_{pq} \varpi^m = 2F^{mn}F_{pq} \varpi^m \varpi^q + 2F_{pq} \varpi^m,$$

Thus the object $J^m$ takes the form

$$J^m = 8(F^{mn}F_{nk} - \frac{1}{4}F_{pq}F_{pq}\delta^m_k) \varpi^k. \hfill (6.14)$$

or in a component-wise form

$$J^m_q = 8(F^{mn}F_{nk} - \frac{1}{4}F_{pq}F_{pq}\delta^m_k). \hfill (6.15)$$
We see now what the field equation (5.8) mean. It is a vanishing condition for the trace of the matrix $J^m_q$. Note that due to the propositions (5.3-5.4) the traceless condition as well as the symmetric condition are the consequences of the definition of $J_{ab}$. Thus in the case of the Maxwell Lagrangian the object $J^m_n$ coincides (up to a numerical coefficient equal to $-\frac{1}{32\pi}$) with the classical expression of the energy-momentum tensor. Note that by the variational procedure described above we obtain it in a symmetric and a traceless form.

7 Translational Lagrangian in gravity

In the teleparallel approach to gravity the coframe field $\vartheta^a$ is the basic gravitational field variable. A general Lagrangian density for the coframe field $\vartheta^a$ (quadratic in the first order derivatives) described by the gauge invariant translation Lagrangian of Rumpf [17]. Up to the $\Lambda$-term it can be written as

$$L = \frac{1}{2\ell^2} \sum_{I=1}^{3} \rho_I (1) V, \quad (7.1)$$

where

$$(1) \ L = d\vartheta^a \wedge *d\vartheta_a, \quad (7.2)$$
$$(2) \ L = (d\vartheta_a \wedge \vartheta^a) \wedge *(d\vartheta_b \wedge \vartheta^b), \quad (7.3)$$
$$(3) \ L = (d\vartheta_a \wedge \vartheta^b) \wedge *(d\vartheta_b \wedge \vartheta^a), \quad (7.4)$$

and $\ell$ is the Plank length constant.

Note that the Lagrangian (7.1) is a linear combination of three independent terms every one of which is of the prescribed form $\alpha \wedge *\beta$ and we can apply the procedure described above. Using the proposition (5.1) we obtain for the first Lagrangian (7.3)

$$\delta (1) L = 2\delta (d\vartheta_a) \wedge *d\vartheta_a - \delta (\vartheta_m) \wedge (1) J^m \quad (7.5)$$

where

$$(1) \ J^m = (\vartheta^m \bigwedge d\vartheta_a) \wedge *d\vartheta^a - d\vartheta^a \wedge (\vartheta^m \bigwedge *d\vartheta_a)$$
$$= 2(\vartheta^m \bigwedge d\vartheta_a) \wedge *d\vartheta^a - \vartheta^m \bigwedge (d\vartheta_a \wedge *d\vartheta_a) \quad (7.6)$$
Thus the contribution of the Lagrangian \((1) \mathcal{L}\) in the field equation is:

\[
2\rho_1 \delta(\vartheta_a) \wedge d^* d\vartheta^a - 2\rho_1 (\vartheta^a J(d\vartheta_m) \wedge *d\vartheta^m) - \rho_1 \vartheta^a J(d\vartheta_m \wedge *d\vartheta^m) \tag{7.7}
\]

Variation of the second Lagrangian \((7.4)\) takes the form:

\[
\delta \left( (2) \mathcal{L} \right) = 2\delta(\vartheta_a \wedge \vartheta^a) \wedge *(d\vartheta_b \wedge \vartheta^b) - \delta(\vartheta_m) \wedge (2) J^m
\]

\[
= 2d(\vartheta_a \wedge \vartheta^a) \wedge *(d\vartheta_b \wedge \vartheta^b) + 2d\vartheta_a \wedge \delta \vartheta^a \wedge *(d\vartheta_b \wedge \vartheta^b) - \delta(\vartheta_m) \wedge (2) J^m
\]

\[
= 2d(\vartheta_a \wedge \vartheta^a) \wedge *(d\vartheta_b \wedge \vartheta^b) + 2\delta \vartheta_a \wedge d\vartheta^a \wedge *(d\vartheta_b \wedge \vartheta^b) - \delta(\vartheta_m) \wedge (2) J^m
\]

\[
= 2d(\vartheta_a \wedge \vartheta^a) \wedge *(d\vartheta_b \wedge \vartheta^b) - 2\delta \vartheta_a \wedge \vartheta^a \wedge d*(d\vartheta_b \wedge \vartheta^b) + 4\delta \vartheta^a \wedge d\vartheta_a \wedge *(d\vartheta_b \wedge \vartheta^b) - \delta(\vartheta_m) \wedge (2) J^m, \tag{7.8}
\]

where the current term:

\[
(2)^{} J^m = \left( \vartheta^m \wedge (d\vartheta_b \wedge \vartheta^b) \right) \wedge *(d\vartheta_a \wedge \vartheta^a) + \left( d\vartheta_a \wedge \vartheta^a \right) \wedge \left( \vartheta^m \wedge * (d\vartheta_b \wedge \vartheta^b) \right)
\]

\[
= -\vartheta^m \left( (d\vartheta^a \wedge \vartheta_a) \wedge *(d\vartheta_b \wedge \vartheta_b) \right) + 2\vartheta^m \left( d\vartheta^a \wedge \vartheta_a \wedge *(d\vartheta_b \wedge \vartheta_b) \right)
\]

\[
= -\vartheta^m \left( (d\vartheta^a \wedge \vartheta_a) \wedge *(d\vartheta_b \wedge \vartheta_b) \right) + 2\vartheta^m \left( d\vartheta^a \wedge \vartheta_a \wedge *(d\vartheta_b \wedge \vartheta_b) \right)
\]

\[
+ 2d\vartheta^m \wedge * (d\vartheta_b \wedge \vartheta_b) \tag{7.9}
\]

Thus the contribution of the Lagrangian \((2) \mathcal{L}\) in the field equation is:

\[
-2\rho_2 \vartheta^a \wedge d*(d\vartheta_b \wedge \vartheta^b) + \rho_2 \vartheta^a \wedge (d\vartheta_m \wedge \vartheta^m) (d\vartheta_m \wedge \vartheta^m) + \rho_2 \vartheta^a \wedge * (d\vartheta_m \wedge \vartheta^m) \tag{7.10}
\]

As for the third laplacian \((7.3)\) by the proposition \((5.1)\) we obtain:

\[
\delta \left( (3) \mathcal{L} \right) = 2\delta(\vartheta_a \wedge \vartheta^b) \wedge *(d\vartheta_b \wedge \vartheta^a) - \delta(\vartheta_m) \wedge (3) J^m = -\delta(\vartheta_m) \wedge (3) J^m + 2d(\delta \vartheta_a \wedge \vartheta^b \wedge *(d\vartheta_b \wedge \vartheta^a) + 2\delta \vartheta^b \wedge d\vartheta_a \wedge *(d\vartheta_b \wedge \vartheta^a)
\]

\[
= 2d(\delta \vartheta_a \wedge \vartheta^b \wedge *(d\vartheta_b \wedge \vartheta^a) - 2\delta \vartheta_a \wedge \vartheta^b \wedge d*(d\vartheta_b \wedge \vartheta^a) + 4\delta \vartheta^b \wedge d\vartheta_a \wedge *(d\vartheta_b \wedge \vartheta^a) - \delta(\vartheta_m) \wedge (3) J^m. \tag{7.11}
\]
As for the current term
\[ (3) J^m = \left( \vartheta^m \mathbf{J} (d\vartheta_a \wedge \vartheta^b) \right) \wedge * (d\vartheta_b \wedge \vartheta^a) + (d\vartheta_a \wedge \vartheta^b) \wedge \left( \vartheta^m \mathbf{J} (d\vartheta_a \wedge \vartheta^a) \right) \]
\[ = -\vartheta^m \mathbf{J} \left( (d\vartheta_a \wedge \vartheta^b) \wedge * (d\vartheta_b \wedge \vartheta^a) + 2 \vartheta^m \mathbf{J} (d\vartheta_a \wedge \vartheta^a) \wedge * (d\vartheta_b \wedge \vartheta^a) \right) \]
\[ = -\vartheta^m \mathbf{J} \left( (d\vartheta_a \wedge \vartheta^b) \wedge * (d\vartheta_b \wedge \vartheta^a) + 2 (\vartheta^m \mathbf{J} d\vartheta_a) \wedge \vartheta^b \wedge * (d\vartheta_b \wedge \vartheta^a) \right) + \\
= 2d\vartheta_a \wedge * (d\vartheta^m \wedge \vartheta^a) \]
(7.12)

Hence the contribution of the Lagrangian \((3) \mathcal{L}\) in the field equation is

\[ -2\rho_3 \vartheta^b \wedge d \ast (d\vartheta_b \wedge \vartheta^a) + \rho_3 \vartheta^a \mathbf{J} \left( (d\vartheta_m \wedge \vartheta^n) \wedge * (d\vartheta_n \wedge \vartheta^m) \right) \]
\[ -2\rho_3 (\vartheta^a \mathbf{J} d\vartheta_m) \wedge \vartheta^m \wedge * (d\vartheta_n \wedge \vartheta^m) + 2 \rho_3 d\vartheta_b \wedge * (d\vartheta^a \wedge \vartheta^b) \]
(7.13)

The relations \((7.7, 7.10, 7.13)\) yield the field equation generated by free variations of the Rumpf Lagrangian cf. Kopczyński [8] in the form [14].

\[-2 \ell^2 \Sigma_a = 2 \rho_1 d \ast d\vartheta_a - 2 \rho_2 \vartheta_a \wedge d \ast (d\vartheta_b \wedge \vartheta) - 2 \rho_3 \vartheta_b \wedge d \ast (\vartheta_a \wedge d\vartheta_b) \]
\[ + \rho_1 \left[ e_a \mathbf{J} (d\vartheta^b \wedge * d\vartheta_b) - 2 (e_a \mathbf{J} d\vartheta^b) \wedge * d\vartheta_b \right] \]
\[ + \rho_2 \left[ 2d\vartheta_a \wedge * (d\vartheta^b \wedge \vartheta_b) + e_a \mathbf{J} (d\vartheta^c \wedge \vartheta_c \wedge * (d\vartheta^b \wedge \vartheta_b)) \right. \]
\[ - 2 (e_a \mathbf{J} d\vartheta^b) \wedge \vartheta_b \wedge * (d\vartheta^c \wedge \vartheta_c) \]
\[ + \rho_3 \left[ 2d\vartheta_b \wedge * (\vartheta_a \wedge d\vartheta^b) + e_a \mathbf{J} (\vartheta_c \wedge d\vartheta^b \wedge * (d\vartheta^c \wedge \vartheta_b)) \right. \]
\[ - 2 (e_a \mathbf{J} d\vartheta^b) \wedge \vartheta_c \wedge * (d\vartheta^c \wedge \vartheta_b) \],
(7.14)

where \(\Sigma_a\) depends on matter fields.

8 Conclusions

We discuss the variational procedure on a teleparallel manifold. The situation is rather differ from the variation on a pseudo-Riemannian manifold. We have re-proved a part of the propositions exhibited in the paper [14] and have generalized some of them. The commutativity and anti-commutativity of the variation with the Hodge dual map are related with the special covariant types of the constraint
variation. We derive a general relation for variation of the quadratic type Lagrangians and apply it to the viable cases of the electro-magnetic Lagrangian and to the translation invariant of Rumpf. The established formulas can be applied for study the various material fields on teleparallel background. We hope that these techniques will also be useful for Lagrangian formulation of the dynamic of $p$-branes.

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