Propagation of Singularity with Normally Hyperbolic Trapping

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Abstract

We prove microlocal estimates with normally hyperbolic trapping. We use a new type of symbol class which is constructed by blowing up the intersection of the unstable manifold and the fiber infinity. The extra loss of the microlocal estimates compared with the standard propagation of singularities is arbitrarily small.

1 Introduction

1.1 Background

The Kerr de-Sitter spacetime is a manifold $M$ equipped with a Lorentzian metric $g$ such that

$$M = \mathbb{R}_t \times X, \quad X = (r_e, r_c) \times S^2,$$

For detailed definitions of $r_e, r_c, g$, please refer to Section 1.2 and 5.1 $(M, g)$ is describing a rotating black hole. It is determined by two parameters: the angular momentum $a$ and the mass $m$.

Our paper concerns the regularity of solutions to wave equations on Kerr-de Sitter spacetimes. The propagation of singularities is well understood in the case without trapping, see Section 6.1 of [8]. But the loss of regularity becomes delicate when trapping of the Hamilton vector field arises. In [34], Wunsch and Zworski proved that in the stationary case the loss of Sobolev regularity compared with the standard non-trapping propagation estimates
is one logarithmic order. Peter Hintz [14] proved propagation estimates with 1 Sobolev order extra loss (extra compared with non-trapping estimates) and this is an important motivation of our work.

In this paper we prove propagation estimates with arbitrarily small extra loss compared with the classical non-trapping propagation estimates. The novelty of this paper is that we associate different orders to different boundary faces in the blown-up compactified cotengent bundle. This refines and further decomposes the positive commutator argument in [14].

1.2 The main result

In this paper we prove propagation estimates with normally hyperbolic trapping. One of the major applications of estimates of this type is to linearized Einstein equations on the Kerr-de Sitter spacetimes.

The metric $g$ on the Kerr-de Sitter spacetime $M^\circ$ with angular momentum $a$ and mass $m$ is given by

$$g = (r^2 + a^2 \cos^2 \theta) \left( \frac{d^2 r}{\Delta(r)} + \frac{d^2 \theta}{\Delta_\theta} \right) - \frac{\Delta_\theta \sin^2 \theta}{(1 + \frac{\Lambda a^2}{3}) (r^2 + a^2 \cos^2 \theta)} (a dt - (r^2 + a^2)d\varphi)^2$$

$$+ \frac{\Delta(r)}{(1 + \frac{\Lambda a^2}{3}) (r^2 + a^2 \cos^2 \theta)} (dt - a \sin^2 \theta d\varphi)^2,$$

where $\Lambda$ is the cosmological constant and

$$\Delta(r) = (r^2 + a^2)(1 - \frac{\Lambda r^2}{3}) - 2mr, \quad \Delta_\theta = 1 + \frac{\Lambda a^2}{3}, \quad \Lambda \geq 0. \quad (1.2)$$

Then the wave operator $\Box_g$ on $(M^\circ, g)$ is a second order pseudodifferential operator and its principal symbol is the dual metric function $G(z, \zeta) := |\zeta|^{2}_{g^{-1}(z)}$, where $\zeta \in T^*_z M^\circ$. The characteristic set $\Sigma$ is defined to be

$$\Sigma := \{(z, \zeta) \in T^* M^\circ \setminus o : G(z, \zeta) = 0\},$$

where $o$ is the zero section in $T^* M$. Denote momentum variables dual to $t, r, \varphi, \theta$ by $\xi_t, \xi_r, \xi_\varphi, \xi_\theta$ respectively, then $\Sigma$ has two components

$$\Sigma_{\pm} := \Sigma \cap \{ \pm \xi_t > 0 \}.$$
We focus on $\Sigma_+$, the ‘future’ part of $\Sigma$ in our discussion. Since all objects concerned here are homogeneous with respect to the dilation of fiber variables, we can pass our discussion to cosphere bundle which is obtained by identifying orbits of the $\mathbb{R}^+$ action given by dilation on fibers: $S^*M^\circ = (T^*M^\circ \setminus 0)/\mathbb{R}^+$. We still use $\Sigma_+, \Sigma_{\pm}$ to denote their image in $S^*M^\circ$. The rescaled vector field $H_G := (g_0^{-1}(\zeta, dt))^{-1}H_G$ is a homogeneous (with respect to the fiber dilation) vector field of degree 0, and thus can be viewed as a vector field on $S^*M^\circ$. The main feature we use is that the $H_G$ in $\Sigma_+$ is $r$-normally hyperbolic trapping for every $r$ in the sense of [34]. See Section 5 for more discussion.

The trapped set is given by

$$\Gamma_+ := \{(z, \zeta) \in \Sigma_+ : \xi_r = r - r_{\xi_t, \xi_\phi} = G = 0\}.$$  

The definition of $r_{\xi_t, \xi_\phi}$ is given in Proposition 5.1. Null-geodesics starting from $\Gamma_+$ never escape out of the event horizon $\{r = r_e\}$ or to the ‘spatial’ infinity. Instead, when projected to $X$, they stay in the compact set $\{r = r_{\xi_t, \xi_\phi}\}$. In addition, this trapped set is of the form $\Gamma_+ = \Gamma_+^{u/s} \cap \Gamma_{s}^{a}$. $\Gamma_{u/s}^{a}$ are future components of unstable/stable manifolds respectively, consisting of $(z, \zeta) \in \Sigma_+$ such that the backword/forward integral curve starting at $(z, \zeta)$ tends to $\Gamma$. Both $\Gamma_{u/s}^{a}$ are conic codimension 1 submanifolds of $\Sigma_+$. In Kerr-de Sitter spacetimes, they are given by

$$\Gamma_{u/s}^{a} := \{\varphi_{u/s} := \xi_r \mp \text{sgn}(r - r_{\xi_t, \xi_\phi})(1 + \hat{\alpha})\sqrt{\frac{F_{\xi_t, \xi_\phi}(r) - F_{\xi_t, \xi_\phi}(r_{\xi_t, \xi_\phi})}{\Delta(r)}} = 0\} \cap \Sigma_+.$$  

(1.3)

where $r_{\xi_t, \xi_\phi}$ is defined in Proposition 5.1, $F_{\xi_t, \xi_\phi}(r) := \frac{1}{\Delta(0)}((r^2 + a^2)\xi_t + a\xi_\phi)^2$, $\Delta(r)$ is defined in (1.2), and $\hat{\alpha} = \frac{\Lambda a^2}{3}$.

Next we briefly recall basic definitions and facts about the cusp calculus. We recommend [14][21][31] as further references on this topic. Our construction is on Kerr de-Sitter spacetimes, but it is clear that the same construction applies to other manifolds with boundary. First we compactify $M^0$ to be $M = (M^0 \sqcup ([0, \infty)_r \times X))/\sim$, where $\sim$ is identifying $(t, x) \in \mathbb{R}_t \times X$ and $(\tau = t^{-1}, x)$ for $0 < t < \infty$. The cusp vector fields are

$$\mathcal{V}_{cu}(M) := \{V \in \mathcal{V}(M) : V\tau \in \tau^2 C^\infty(M)\}.$$
Away from \( \{ \tau = 0 \} \), \( \mathcal{V}_u(M) \) is the same as \( \mathcal{V}(M) \). Let \( x = (x_1, ..., x_{n-1}) \) be coordinates on \( X \), then near \( \{ \tau = 0 \} \), \( \mathcal{V}_u(M) \) is locally spanned by
\[
\tau^2 \partial_\tau, \partial_{x_1}, \partial_{x_2}, ..., \partial_{x_{n-1}}
\]
as a \( \mathcal{C}^\infty(M) \)-module. And vector fields in \( \mathcal{V}_u(M) \) can be viewed as smooth sections of a vector bundle \( \mathcal{C}^uTM \) called the cusp tangent bundle. \( \text{(1.4)} \) also represents a local frame of \( \mathcal{C}^uTM \). As an element of \( \mathcal{V}_u(M) \), \( \tau^2 \partial_\tau \) is non-vanishing even down to \( \tau = 0 \), which is similar to \( \tau \partial_\tau \) in Melrose’s b-calculus.

The cusp cotangent bundle \( \mathcal{C}^uT^*M \) is the dual bundle of \( \mathcal{C}^uTM \). Locally it is spanned by
\[
\frac{d\tau}{\tau^2}, dx^1, dx^2, ..., dx^{n-1}.
\]
(1.5)

The \( m \)-th order cusp differential operators \( \text{Diff}^m_{cu}(M) \) then consists of products of \( m \) cusp vector fields. Writing covectors in \( \mathcal{C}^uT^*M \) as
\[
-\sigma \frac{d\tau}{\tau^2} + \sum_{i=1}^{n-1} \xi_i dx^i = \sigma dt + \sum_{i=1}^{n-1} \xi_i dx^i,
\]
then to a differential operator
\[
P = \sum_{j+|\alpha|} a_{j\alpha}(\tau, x)(-\tau^2D_\tau)^j D_x^\alpha \in \text{Diff}^m_{cu}(M), \quad a_{j\alpha} \in \mathcal{C}^\infty(M),
\]
we associate a function called its principal symbol:
\[
\sigma^m_{cu}(P) := \sum_{j+|\alpha|=m} a_{j\alpha} \sigma^j \xi^\alpha.
\]

We write \( \sigma(P) \) when there is no confusion about the order and in which algebra we are discussing this operator. More importantly, \( \text{Diff}^m_{cu}(M) \) can be generalized by allowing symbols to be more general functions other than polynomials in fiber variables. Define the symbol class \( S^m_{cu}(M) \) to be functions \( a \in \mathcal{C}^\infty(\mathcal{C}^uT^*M) \) such that
\[
|\partial^j_x \partial^k_\sigma \partial^\beta_\xi a(\tau, x, \sigma, \xi)| \leq C_{j\alpha k\beta}(1 + |\sigma| + |\xi|)^{m-k-|\beta|}
\]
in terms of local coordinates. Away from \( \{ \tau = 0 \} \), since \( \tau^2 \partial_\tau = -\partial_t \), this condition remains the same when we replace \( \partial_\tau \) by \( \partial_t \), which is how we define the standard symbol class.
The quantization $\text{Op}(a)$ of $a \in S^m_{\text{cu}}(M)$ is a pseudodifferential operator acting on smooth functions $u \in \mathcal{C}(\mathcal{M})$ supported near $\{\tau = 0\}$ by

$$\text{Op}(a)u(t, x) = (2\pi)^{-n} \int e^{i((t-t')\sigma + (x-x')\xi)} a(\tau, x, \sigma, \xi) u(t', x') dt' dx' d\sigma d\xi.$$ 

For general $u$, we define this action using a partition of unity. We denote the collection of all such $\text{Op}(a)$ by $\Psi^m_{\text{cu}}(\mathcal{M})$. Then we define the cusp wave front set $\text{WF}^{\text{cu}}(A)$ of an operator $A = \text{Op}(a)$ as the essential support of $a$.

**Definition 1.** For $\zeta \in \text{cu} T^*\mathcal{M}$, we say that $\zeta \notin \text{WF}^{\text{cu}}(A)$ when there exists $\chi \in C^\infty(\text{cu} T^*\mathcal{M})$ with $\chi(\zeta) = 1$ such that $\chi a \in S^{-\infty}_{\text{cu}}(\mathcal{M})$.

This is a conic set and we identify it with its image in the quotient space $\text{cu} S^*\mathcal{M}$. Next we define cusp $L^2$–spaces and Sobolev spaces. The cusp cotangent bundle is locally spanned by $d\tau, dx_1, dx_2, \ldots, dx_{n-1}$, whose wedge product give the cusp density $\nu_{\text{cu}}$. $L^2_{\text{cu}}(\mathcal{M})$ consists of functions that are square integrable with respect to density $\nu_{\text{cu}}$ equipped with norm $\|u\|_{L^2_{\text{cu}}(\mathcal{M})} := (\int_{\mathcal{M}} |u|^2 d\nu_{\text{cu}})^{1/2}$. $L^2_{\text{cu}}(\mathcal{M})$ is unaffected by the blowing up near the fiber infinity. For $r \in \mathbb{R}$, the $r$–weighted cusp $L^2$–space is defined by

$$L^2_{\text{cu}}(M) := \{u \in L^2_{\text{loc}}(M) : \tau^{-r} u \in L^2_{\text{cu}}(M)\},$$

where $L^2_{\text{loc}}(M)$ is the space of locally $L^2$–integrable function class on $M$. We use the notation $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$. For a non-negative integer $s$, we define the weighted cusp Sobolev space as

$$H^{s,r}_{\text{cu}}(M) := \{u \in L^2_{\text{cu}} : Au \in L^2_{\text{cu}} \text{ for all } A \in \text{Diff}^s_{\text{cu}}(M)\}.$$

For general $s > 0$, $H^{s,r}_{\text{cu}}(M)$ is defined by interpolation. For $s < 0$, $H^{s,r}_{\text{cu}}(M)$ is defined to be the dual space of $H^{-s,-r}_{\text{cu}}(M)$. Our main result, the propagation of singularities is as follows.

**Theorem 1.** For $0 < \alpha < \frac{1}{2}$ and $\lambda$ satisfying (3.9), suppose $v \in H^{-N,\mu}(M)$, $Pv = f$ and $\text{WF}_{\text{cu}}^{s+1-\lambda}(v) \cap \Gamma^s = \emptyset$, $\text{WF}_{\text{cu}}^{s+1-\lambda,\mu}(v) \cap (\overline{\Gamma^s} \setminus \Gamma) = \emptyset$; and $\text{WF}_{\text{cu}}^{s-\lambda,\mu}(f) \cap \Gamma = \emptyset$. Then $\text{WF}_{\text{cu}}^{s,\mu}(v) \cap \Gamma = \emptyset$.

**Remark 1.** The condition $0 < \alpha < \frac{1}{2}$ can be relaxed to $0 < \alpha < 1$ when $p$ satisfies the compensable property defined in Definition 4.
1.3 Prior works

The study of Kerr de-Sitter spacetimes has been a vast literature, from which we only list a few as examples.

The close connection between normally hyperbolic trapping and Kerr black holes is observed in [34]. Then this property is extended to the range \(\{|a| < m\}\) by Dyatlov [9]. Kerr-de Sitter case with small angular momentum is discussed by Andás Vasy [30].

Nonnenmacher and Zworski [22] extended results in [34] with weaker conditions. Before Peter Hintz's estimates in [14], Semyon Dyatlov [10] obtained width of the resonance free strip of the modified Laplacian under the same dynamical assumptions. Hintz's work can be viewed as a quantitative version of it. András Vasy [30] gave systematic microlocal treatment of Kerr-de Sitter spacetimes with small angular momentum \(a\) and obtained an expansion of solutions to wave equations on Kerr-de Sitter spacetimes with terms corresponding to quasinormal modes. For the Kerr case (\(\Lambda = 0\)), it is known that results for small \(|a|\) apply to the full subextremal range. See [5][25] and references therein. The mode stability in those cases was investigated in [3][25].

Regarding the stability aspect, the stability of the Schwartzchild black holes is considered as early as in [33]. The linear stability of slowly rotating Kerr black holes is considered in [13]. While the non-linear stability of various families of black holes was considered in [15][35][14].

Local energy and decay estimates in the Schwartzchild case are proved in [24]. For the Kerr case, see [28][27], in which the author proved \(t^{-3}\) local uniform decay rate for linear waves. See also [2][6] for sharp decay estimates. The existence of solutions to semilinear equations with small initial conditions and an extra null condition was considered in [20].

The method of blowing-up the compactified cotangent bundle and the construction of our operator algebra in this paper borrow the idea of the second microlocalization from [19][32][29]. The method of treating Schwarts kernels as paired Lagrangian distributions in [7] also inspired us.

1.4 Structure of this paper

In Section 2, we recall some basic notations of microlocal analysis, and introduce our new symbol classes, operator classes and Sobolev spaces. Then we prove basic facts about them, including mapping properties, composi-
tion laws, elliptic estimates and G\"arding’s inequality. In Section 3 we state assumptions needed for the proof of the main result and a microlocal quantitative version of Theorem 1. Afterwards, Section 4 consists of several steps of positive commutator argument. Finally, in Section 5 we apply the microlocal framework that we constructed to the Kerr-de Sitter spacetimes.

2 Microlocal Analysis

In this section, we are going to develop basic facts about microlocal analysis for the symbol class we construct for our purpose. We assume that all functions are supported in the region \( \{ \tau \leq 1 \} \) (or equivalently, \( \{ t \geq 1 \} \)) throughout our argument, including latter sections. \( S^m_{\text{cu}}(M), \Psi^m_{\text{cu}}(M) \) denote the cusp symbol class and pseudodifferential operator class respectively.

2.1 \( S^m_{\text{cu},\alpha} \) symbols

The spacetime we consider is \( M^\circ = \mathbb{R}_t \times X \), where \( X \) is an \( n-1 \) dimensional closed manifold. \( n = 4 \) in our application to Kerr-de Sitter spacetime. \( M \) is obtained by compactifying \( M^\circ \) on the \( t- \)direction. To be more concrete, we define:

\[
M := (M^\circ \sqcup ([0, \infty), t \times X)) / \sim,
\]  

(2.1)

where \( \sim \) is the identification: \( (t, x) \sim (\tau = t^{-1}, x), x \in X, t \in (0, \infty) \). Use \( \mathcal{V}(M) \) to denote smooth vector fields on \( M \). To facilitate our analysis, we introduce the cusp vector fields:

\[
\mathcal{V}_{\text{cu}}(M) := \{ V \in \mathcal{V}(M) : V\tau \in \tau^2 C^\infty(M) \}.
\]  

(2.2)

Suppose \( x_1, x_2, ..., x_{n-1} \) are local coordinates of \( X \), then \( V_{\text{cu}}(M) \) is locally spanned by

\[
\tau^2 \partial_\tau, \partial_{x_1}, \partial_{x_2}, ..., \partial_{x_{n-1}}.
\]  

(2.3)

cuTM is the vector bundle with local frame consists of vectors in \( (2.3) \). We point out that, as a cusp vector field, \( \tau^2 \partial_\tau \) is nonvanishing down to \( \tau = 0 \). And its dual bundle the cusp cotangent bundle cu\( T^*M \) is locally spanned by

\[
\frac{d\tau}{\tau^2}, dx_1, dx_2, ..., dx_{n-1}.
\]  

(2.4)
We denote the defining function of the fiber infinity by \( \hat{\rho} \), and the radial compactification of \( \text{cu}T^*M \) by \( \text{cu}\bar{T}^*M \). Concretely, let \( \mathbb{R}^n \) denote the radial compactification of \( \mathbb{R}^n \), and in coordinate patches that trivialize \( \text{cu}T^*M \), we replace the \( \mathbb{R}^n \) playing the role of fiber part by \( \mathbb{R}^n \). Concretely, suppose \( U \) is a open set in \( M \) on which \( T^*M \) is trivialized, we define:

\[
\text{cu}\bar{T}^*U_M := U \times \mathbb{R}^n.
\]

We further identify the fiber infinity of \( \text{cu}\bar{T}^*M \) with the sphere bundle \( \text{cu}S^*M \).

Next we consider a submanifold \( Y \) of \( \bar{T}^*M \), locally defined by \( \varphi_Y \). Set \( \bar{Y} := Y \cap \bar{T}^*_{\partial M}M \), and \( Y_0 := (0, \infty)_\tau \times \bar{Y} \) is the stationary extension of \( \bar{Y} \). The defining function of \( \bar{Y} \) in \( \bar{T}^*_{\partial M}M \) is \( \bar{\varphi}_Y := \varphi_Y|_{\bar{T}^*_{\partial M}M} \). And \( \bar{\varphi}_{Y_0} \), the defining function of \( Y_0 \), is the stationary extension of \( \bar{\varphi}_Y \). Next we define the symbol class \( S_{\text{cu},\tilde{m}}^{m,\alpha}(M,Y) \).

In our setting, \( M = (M^\circ \sqcup ([0, \infty)_\tau \times X))/\sim \) is the base manifold before blowing up. \( Z \) is the manifold with corner obtained by blowing up the boundary of \( Y_0 \), where \( Y_0 \) and the fiber infinity intersect each other. For convenience, we define

\[
\rho := |\hat{\rho}|^\alpha, \tilde{\rho} := (\varphi_{Y_0}^2 + \rho^{2\alpha})^{1/2},
\]

where \( \alpha \) is ‘the order of the blow up’ to be determined later. So \( \tilde{\rho} \) is the defining function of the front face. We emphasize that \( Z \) obtained by introducing \( \frac{\tilde{\rho}}{\varphi_{Y_0}} \) and its reciprocal as the coordinate on the front face. This is different from the ‘ordinary’ blow up, which use \( \frac{\rho}{\varphi_{Y_0}} \) and its reciprocal.

**Definition 2.** \( S_{\text{cu},\tilde{m}}^{m,\alpha}(M,Y) \) consists of functions on \( Z \) satisfying:

\[
|W_1...W_k\alpha| \leq C_{k,m,r}\tilde{\rho}^{-m}\rho^{-\tilde{m}}\alpha^{-m}, \tag{2.5}
\]

where \( W_i \) are smooth vector fields on \( \bar{T}^*M \) which are homogeneous of degree 1 in the fiber part, tangent to the front face introduced by the blow up, and has no restriction away from the front face.

In particular, away from the front face, typical examples of \( W_i \) are \( \partial_\tau, \partial_{x_i}, \xi_i \partial_{\xi_j} \).

In the construction of \( Z \) and hence the definition of \( S_{\text{cu},\tilde{m}}^{m,\alpha}(M,Y) \), we used \( Y_0 \) instead of \( Y \). But it is clear that \( \bar{Y} \), and consequently \( Y_0 \) is uniquely determined by \( Y \). Thus \( S_{\text{cu},\tilde{m}}^{m,\alpha}(M,Y) \) is well defined. Next we give a coordinate patch description of \( S_{\text{cu},\tilde{m}}^{m,\alpha}(M,Y) \).
Lemma 1. Suppose the defining functions of $\bar{Y}$ and the fiber infinity are $x_1$ and $\hat{\rho}$, then $S^m_{\text{cu},\bar{\alpha}}(M,Y)$ consists of smooth functions on $Z$, in terms of coordinates on $M$, satisfying:

$$|\partial_\tau^\beta \partial_{\tilde{x}_1}^\gamma \partial_{\xi}^\delta a(\tau, x_1, \tilde{x}_1, \xi_t, \xi)| \leq C \hat{\rho}^{-m-|\gamma|} \hat{\rho}^{-\frac{m-n}{\alpha}-|\beta|},$$

(2.6)

where $\tilde{x}_1$ means all $x_i$ other than $x_1$, $\tilde{\gamma} = (\gamma_\tau, \gamma_1, ..., \gamma_{n-1})$, $\tilde{\xi} = (\xi_t, \xi_1, ..., \xi_{n-1})$.

Proof. Without loss of generality, we assume that $\xi_{n-1}$ is large relative to other $\xi_i$, thus we take $\hat{\rho} = |\xi_{n-1}|^{-1}$. We still write $a \in S^m_{\text{cu},\bar{\alpha}}(M,Y)$ as a function of $\tau, x_i; \xi_t, \xi_i$, but the smoothness near the front face means smoothness on $Z$. Concretely, they are smooth with respect to $\frac{x_1}{|\xi_{n-1}|}$ or its reciprocal instead of $x_1, \xi_{n-1}$ individually. Let $\xi_t$ be the variable dual to $\tau$ and $\xi_i$ be the variable dual to $x_i$. Firstly, we argue that only the case $\mathcal{f} = 0$ need to be considered. In (2.5), write each $W_i$ as

$$W_i = c_{i\tau} \partial_\tau + \bar{W}_i,$$

(2.7)

where $c_{i\tau} \in C^\infty(Z)$ and $\bar{W}_i \tau = 0$. Smoothness on the compactified space implies $c_{i\tau}$ and all of its derivatives are bounded. Consequently, we substitute (2.7) into the left hand side of (2.5) and expand. The final expression is a sum of terms of the form $\partial_\tau^\beta \bar{W}_1 \bar{W}_2 ... a$ with a bounded function coefficient in front of it (possibly with different $\mathcal{f}'$ for different terms). Since the right hand side of both (2.5) and (2.4) are independent of $\tau$, and all $\partial_\tau$ can be commuted to the front, hence in the proof of their equivalence, we can consider $\mathcal{f}$ case first and then add $\partial_\tau$ to (2.6) or each term of the expanded (2.5). In the proof below, we assume $\mathcal{f} = 0$. Our goal is to verify that (2.5) is equivalent to (2.6). We show that (2.6) implies (2.5) first, and then the reverse.

Consider the region $x_1 \leq C \rho = C \hat{\rho}^\alpha$, on which the local coordinates are:

$$\tau, \sigma := \frac{x_1}{|\xi_{n-1}|^{-\alpha}}, x_2, x_3, ..., x_{n-1}, \frac{\xi_t}{\xi_{n-1}}, \frac{\xi_1}{\xi_{n-1}}, ..., \frac{\xi_2}{\xi_{n-1}}, ..., \rho := \hat{\rho}^\alpha.$$

We use the notation $\hat{\xi}_j := \frac{\xi_j}{\xi_{n-1}}, 1 \leq j \leq n-1$ or $j = \tau$. Recall that $\hat{\rho} = |\xi_{n-1}|^{-1}$, we know (assuming $\xi_{n-1} > 0$, the case where $\xi_{n-1} < 0$ is treated in the same manner) $\xi_{n-1} \partial_{\hat{\xi}_{n-1}} = -\hat{\rho} \partial_{\hat{\rho}}$, together with $x_1 = \hat{\rho} \sigma, \hat{\rho} = \rho^{1/\alpha}$, we write b-vectors involving $\sigma$ and $\rho$ as:

$$V_1 := \partial_\sigma = \rho \partial_{x_1} = |\xi_{n-1}|^{-\alpha} \partial_{x_1},$$

(2.8)
\[ V_2 := \rho \partial_\rho = \rho \sigma \partial_{x_1} + \frac{1}{\alpha} \rho^{1/\alpha} \partial_\rho = x_1 \partial_{x_1} - \frac{1}{\alpha} |\xi_{n-1}| \partial_{\xi_{n-1}}. \quad (2.9) \]

We first show that symbol class defined by (2.6) is invariant under application of \( W_i \) defined after (2.5). We only need to verify this for \( V_1 \) and \( V_2 \) since invariance under other \( W_i \) is clear from (2.6) and standard facts in microlocal analysis. For \( V_1 \), we need to show that

\[ |\partial^l_{x_1} \partial^p_{\xi} (|\xi_{n-1}|^{-\alpha} \partial_{x_1} a(\tau, x_1, x_1, \xi, \xi))| \leq C \hat{\rho}^{-m+|\gamma|} \hat{\rho}^{-\frac{m-m}{\alpha} - |l|.} \quad (2.9) \]

When we use Leibniz’s rule to apply \( \partial^2_{\xi_{n-1}} \), it will produce \( \partial^2_{\xi_{n-1}} (|\xi_{n-1}|^{-\alpha} \partial_{x_1} a(\tau, x_1, x_1, \xi, \xi)) \) = \( \sum_{k=0}^{\gamma_{n-1}} \binom{\gamma_{n-1}}{k} \partial^k_{\xi_{n-1}} (|\xi_{n-1}|^{-\alpha}) \partial^2_{\xi_{n-1}} a. \) The \( k \)-th term is \( C |\xi_{n-1}|^{-\alpha-k} \partial^2_{\xi_{n-1}} a. \) Applying triangle inequality we reduce the proof to the case of a single term, and notice that any power of \( |\xi_{n-1}| \) will commute with \( \partial_{x_1} \) in the front, so the inequality is equivalent to:

\[ |\hat{\rho}^{a+k} \partial^l_{x_1} \partial^p_{\xi} |\tilde{\gamma}-(0,\ldots,0,k)| \partial_{x_1} a(\tau, x_1, x_1, \xi, \xi) \| \leq C \hat{\rho}^{-m+|\gamma|} \hat{\rho}^{-\frac{m-m}{\alpha} - |l|}. \]

Move \( \hat{\rho} \) to the right hand side, and since \( \hat{\rho} \) is equivalent to \( \hat{\rho} \) on this patch, we have:

\[ |\partial^l_{x_1} \partial^p_{\xi} \tilde{\gamma}^{- (0, \ldots, 0, k)} a(\tau, x_1, x_1, \xi, \xi) \| \leq C \hat{\rho}^{-m+|\gamma|} \hat{\rho}^{-\frac{m-m}{\alpha} - (|l|+1)}, \quad (2.10) \]

which is (2.6), with \( \tilde{\gamma} \) replaced by \( \tilde{\gamma} - (0, \ldots, 0, k), \) \( I + 1. \)

For \( V_2 \), we consider \( x_1 \partial_{x_1} \) and \( \xi_{n-1} \partial_{\xi_{n-1}} \) respectively. We prove:

\[ |\partial^l_{x_1} \partial^p_{\xi} \tilde{\gamma} (x_1 \partial_{x_1} a(\tau, x_1, x_1, \xi, \xi, \xi)) \| \leq C \hat{\rho}^{-m+|\gamma|} \hat{\rho}^{-\frac{m-m}{\alpha} - |l|} \quad (2.11) \]

and

\[ |\partial^l_{x_1} \partial^p_{\xi} \tilde{\gamma} (\xi_{n-1} \partial_{\xi_{n-1}} a(\tau, x_1, x_1, \xi, \xi, \xi)) \| \leq C \hat{\rho}^{-m+|\gamma|} \hat{\rho}^{-\frac{m-m}{\alpha} - |l|}. \quad (2.12) \]

The proof is the same as \( V_1 \) case (even simpler, we only have 2 terms when we expand the derivative), just notice we need to use \( x_1 \leq C |\xi_{n-1}|^{-\alpha} \) to bound the extra \( x_1 \) factor in (2.11). Thus in order to show (2.5) holds, we only need to consider the case with a single vector. When this vector is one of \( \partial_{x_i}, \xi_i \partial_{\xi_j}, 2 \leq i, j \leq n, \) the inequality is straight forward from (2.6) with one of \( |\tilde{\gamma}| \) or \( |\beta| \) equal to one, and \( I \) and the other being 0. When this vector is \( V_1 \), bound (2.5) follows from the case \( I = 1 \) and \( \beta = \gamma = 0. \) When this vector is
follows from the case 
respectively.

The symbol class defined by (2.5) is invariant under
affect the result. When
are not written here since they commute with everything here and do not
we have:
where \( [-\alpha]_k = (-\alpha)(-\alpha - 1)...(-\alpha - k + 1) \). Since \( S_{\alpha_0,\beta}^m(M, Y) \) is invariant
under \( |\xi_{n-1}|^{-\alpha}\partial_{x_1} \), using the induction hypothesis, each term satisfies (2.6),
which completes the induction step. When \( \beta \) increases, we simply notice that
\( \partial_{x_j}, 2 \leq j \leq n \) are \( b \)-vectors, hence the result is straightforward by (2.5),
we finishes the proof in the patch \( x_1 \leq C\rho \). Next we consider the other patch
\( \rho \leq Cx_1 \). The coordinate system is

\[
\tau, \sigma' := \frac{|\xi_{n-1}|^{-\alpha}}{x_1}, x_2, x_3, ..., x_{n-1}, \frac{\xi_t}{\xi_{n-1}}, \frac{\xi_1}{\xi_{n-1}}, \frac{\xi_2}{\xi_{n-1}}, ..., \rho := \tilde{\rho}^\alpha.
\]
vectors involving $\sigma', \rho$ are

\[
K_1 = \sigma' \partial_{\sigma'} = -x_1 \partial_{x_1},
K_2 = \rho \partial_{\rho} = x_1 \partial_{x_1} + \frac{1}{\alpha} \hat{\rho} \partial_{\rho} = x_1 \partial_{x_1} - \frac{1}{\alpha} \xi_{n-1} \partial_{\xi_{n-1}}.
\]

The argument is the same as in the first patch. We first verify that the symbol defined by (2.6) is invariant under $K_1, K_2$. For $K_1$, the proof is the same as $V_1$ in the first patch, just notice that now $C x_1 \geq \rho$, so $\hat{\rho}$ is equivalent to $x_1$ now. For example: $|x_1 \partial_{x_1} a| \leq \hat{\rho}^{-m} \hat{\rho}^{-\frac{m}{a}}$ is equivalent to $|\partial_{x_1} a| \leq \hat{\rho}^{-m} \hat{\rho}^{-\frac{m}{a}-1}$. For $K_2$, the proof is similar to $V_2$ case in the other patch, both terms are treated similarly. The difference is that, we use $x_1 \leq \hat{\rho}$ to bound $x_1$ in the front.

On the other hand, suppose (2.5) holds on this patch. In order to verify (2.6), we again use induction, the case in which $\beta$ or $\tilde{\gamma}$ increase is the same as before. When $l$ increases, we use $\partial_{x_1} = (x_1)^{-1}(x_1 \partial_{x_1})$, since now $(x_1)^{-1} \approx \hat{\rho}^{-1}$, and then use:

\[
x_1 \partial_{x_1}^{l+1} \partial_{\tilde{\xi}} \partial_{\tilde{\gamma}} a = \partial_{x_1}^{l} \partial_{\tilde{\xi}} \partial_{\tilde{\gamma}} (x_1 \partial_{x_1} a) - \partial_{x_1}^{l} \partial_{\tilde{\xi}} \partial_{\tilde{\gamma}} a.
\]

We have deduced that $x_1 \partial_{x_1} a$ is in the same symbol class, hence we can apply induction hypothesis to the right hand side and use $x_1 \approx \hat{\rho}$, we know:

\[
|\partial_{x_1}^{l+1} \partial_{\tilde{\xi}} \partial_{\tilde{\gamma}} a| \leq C \hat{\rho}^{-m+|\tilde{\gamma}|} \hat{\rho}^{-\frac{m}{a}-|l|-1},
\]

which completes the proof.

Recall that the usual cusp symbol class $S_{\text{cu}, \alpha}^m(M)$ consists of smooth functions on $T^*M$ satisfying:

\[
|\partial_{\tau}^{l} \partial_{x_1}^{\beta} \partial_{\tilde{\xi}} \partial_{\tilde{\gamma}} a(\tau, x_1, \tilde{x}_1, \tilde{\xi}_l, \tilde{\xi}_{l})| \leq C \hat{\rho}^{-m+|\tilde{\gamma}|}, \quad (2.13)
\]

Since smooth functions on $T^*M$ lifts to smooth functions on $Z$, comparing with (2.5), we have the inclusion:

**Corollary 1.** For $m, \tilde{m} \in \mathbb{R}$, $\tilde{m} \geq m$, $M, Y$ as above, we have inclusion relationship between symbol classes:

\[
S_{\text{cu}}^m(M) \subset S_{\text{cu}, \tilde{m}}^{m, \tilde{m}}(M, Y). \quad (2.14)
\]
Next we discuss the quantization procedure. Let $\hat{C}^\infty(M)$ be the class of smooth function on $M$ vanish to infinite order at $\partial M$. For $u \in \hat{C}^\infty(M)$ supported in a coordinate chart near $\{\tau = 0\}$, the action of $\text{Op}(a)$, the left quantization (in short, we use ‘quantization’ below) of $a$ is defined by:

$$\text{Op}(a) u(t,x) := (2\pi)^{-n} \int e^{i((t-t')\xi_t+(x-x')\xi)} a(t^{-1}, x, \xi_t, \xi) u(t', x') dt'dx'd\xi_t d\xi,$$

(2.15)

where $\xi_t$ is the variable dual to $t$. Since we use $\frac{d\tau}{\tau^2}$ in the frame of $\text{cu}T^*M$ and:

$$\xi_t dt + \sum_{i=1}^{n-1} \xi_t dx_i = -\xi_t \frac{d\tau}{\tau^2} + \sum_{i=1}^{n-1} \xi_t dx_i,$$

(2.16)

$\xi_t$ is also dual to $\tau$ up to a sign.

For general functions that are not necessarily supported in a coordinate patch, we use a partition of unity to reduce to the case discussed above. We use $\Psi_{\text{cu},\tilde{\alpha}}^m(M,Y)$ to denote the operator class obtained by quantizing symbols in $S_{\text{cu},\tilde{\alpha}}^m(M,Y)$. It is clear that the symbol class defined by (2.5) is globally defined, hence by the equivalence shown by Lemma 1, (2.6) also defines a symbol class on the manifold $Z$. Next we verify that this class is closed under composition:

**Proposition 2.1.** For $A \in \Psi_{\text{cu},\tilde{\alpha}}^{m_1}(M,Y), B \in \Psi_{\text{cu},\tilde{\alpha}}^{m_2}(M,Y)$ with symbols $a \in S_{\text{cu},\tilde{\alpha}}^{m_1}(M,Y), b \in S_{\text{cu},\tilde{\alpha}}^{m_2}(M,Y)$, $A \circ B \in \Psi_{\text{cu},\tilde{\alpha}}^{m_1+m_2,\tilde{\alpha}}(M,Y)$, and its symbol $a \circ b \in S_{\text{cu},\tilde{\alpha}}^{m_1+m_2,\tilde{\alpha}}(M,Y)$. In addition, the term with $l \in \mathbb{N}^n$ as derivative index belongs to $S_{\text{cu},\tilde{\alpha}}^{m_1+m_2-|l|,\tilde{\alpha}}(M,Y)$

**Proof.** The symbol of $A \circ B$ is:

$$a \circ b(x, \xi) = \sum_{l} \partial^l \xi a \partial^l b,$$

(2.17)

where $l = (l_1, \ldots, l_n)$ runs over $\mathbb{N}^n$. Assuming that both of $a$ and $b$ satisfy (2.6), we verify that (2.17) still satisfy (2.6). We only need to consider the term $\alpha = 0$. Because the only source of potential growth (as $\tilde{\rho} \to 0$) comes from $\partial_{\xi_1} b$, which gives $\tilde{\rho}^{-l_1}$ growth (compared with $\tilde{\rho}^{-m} \tilde{\rho}^{-\frac{n-m}{\alpha}}$). On the other hand, this term is multiplied by $\partial_{\xi_i} a$, which gives $\tilde{\rho}^{l_i}$ decay. Concretely, the $l-$ term
has at least \(|l|, |\ell|(1 - \alpha)\) extra decay order in the first and second indices respectively compared with the typical bound for \(S_{cu,\alpha}^{m_1+m_2,\tilde{m}_1+\tilde{m}_2}(M,Y)\), i.e.
\[
\hat{\rho}^{-\left(m_1+m_2\right)}\hat{\rho}^{-\left(\frac{\tilde{m}_1+\tilde{m}_2-m_1-m_2}{\alpha}\right)}.
\]

In order to verify the symbolic property of \(a \circ b\), we can apply Leibnitz’s rule. Each partial derivative will fall on exactly one of \(\partial_{\xi}^k a\) and \(\partial_{\eta}^\ell b\). The bound on the right hand side of (2.6) with \((m, r)\) being either one of \((m_1 - |l|, \tilde{m}_1)\) and \((m_2, \tilde{m}_2 + \alpha l_1)\) changes in the same manner under each differentiation. Notice that \(l_1 \leq |l|\), hence their product satisfy (2.6), but with \((m_1 + m_2 - |l|, \tilde{m}_1 + \tilde{m}_2 - |l|(1 - \alpha))\) as the symbol class order.

\[
\square
\]

### 2.2 Sobolev spaces and operator classes

Recall (2.4), the cusp cotangent bundle is locally spanned by \(\frac{dr}{\tau}, dx_1, dx_2, \ldots, dx_{n-1}\), whose wedge product give the cusp density \(\nu_{cu}\). \(L_{cu}^2(M)\) consists of functions that are square integrable with respect to density \(\nu_{cu}\) equipped with norm \(\|u\|_{L_{cu}^2(M)} := (\int_M |u|^2 du_{cu})^{\frac{1}{2}}\). \(L_{cu}^2(M)\) is unaffected by the blowing up near the fiber infinity. Recall that for \(r \in \mathbb{R}\), the \(r\)-weighted cusp \(L^2\)-space is defined by

\[
L_{cu}^{2,r}(M) := \{u \in L_{loc}^2(M) : \tau^{-r} u \in L_{cu}^2(M)\},
\]

where \(L_{loc}^2(M)\) is the space of locally \(L^2\)-integrable function class on \(M\). We use the notation \(\langle \cdot \rangle = (1 + |\cdot|^2)^{\frac{1}{2}}\). Define

\[
D := \text{Op}(\rho^{-1}), \langle D \rangle := \text{Op}(\langle \rho^{-1} \rangle),
\]

\[
\tilde{D} := \text{Op}(\hat{\rho}^{-1}), \langle \tilde{D} \rangle := \text{Op}(\langle \hat{\rho}^{-1} \rangle),
\]

\[
D_{m,\tilde{m}} := \text{Op}(\langle \rho^{-1} \rangle^m \langle \hat{\rho}^{-1} \rangle^{\frac{\tilde{m}_1+\tilde{m}_2-m_1-m_2}{\alpha}}).
\]

For \(s, \tilde{s} \geq 0\), the weighted Sobolev space can be characterized as

\[
H_{cu,\alpha}^{s,\tilde{s},r}(M,Y) = \{u \in L_{cu}^{2,r}(M) : D_{s,\tilde{s}} u \in L_{cu}^{2,r}(M)\}. \tag{2.18}
\]

For \(s, \tilde{s} < 0\), \(H_{cu,\alpha}^{s,\tilde{s},r}(M,Y)\) is defined to be the dual space of \(H_{cu,\alpha}^{-s,-\tilde{s},r}(M,Y)\), which has been defined above. For \(s > 0, \tilde{s} < 0\), notice that \(s = \frac{\tilde{s}}{s+|\tilde{s}|}(s + |\tilde{s}|) + \frac{|\tilde{s}|}{s+|\tilde{s}|} \times 0, \tilde{s} = \frac{\tilde{s}}{s+|\tilde{s}|}(s + |\tilde{s}|) + \frac{|\tilde{s}|}{s+|\tilde{s}|}(-s - |\tilde{s}|)\), we know \(H_{cu,\alpha}^{s,\tilde{s},r}(M,Y)\) can be defined by interpolating \(H_{cu,\alpha}^{s+|\tilde{s}|,0,r}(M,Y)\) and \(H_{cu,\alpha}^{0,-s-|\tilde{s}|,r}(M,Y)\). Similarly for the case \(s < 0, \tilde{s} > 0\).
And we can equip it with norm
\[
\|u\|_{s,\tilde{s},r} := \|\tau^{-r}D_{s,\tilde{s}}u\|_{L^2_{\tilde{s}}(M)},
\] (2.19)
The \(\tau\)-weighted version of \(S^{m,\tilde{m},r}_{\text{cu},\alpha}(M,Y)\), \(\Psi^{m,\tilde{m}}_{\text{cu},\alpha}(M,Y)\) are defined by
\[
S^{m,\tilde{m},r}_{\text{cu},\alpha}(M,Y) := \tau^{r}S^{m,\tilde{m}}_{\text{cu},\alpha}(M,Y) = \{\tau^{r}a : a \in S^{m,\tilde{m},r}_{\text{cu},\alpha}(M,Y)\},
\]
\[
\Psi^{m,\tilde{m},r}_{\text{cu},\alpha}(M,Y) := \tau^{r}\Psi^{m,\tilde{m}}_{\text{cu},\alpha}(M,Y) = \{\tau^{r}A : A \in \Psi^{m,\tilde{m}}_{\text{cu},\alpha}(M,Y)\}.
\] (2.20)
The weighted generalization of Proposition [2.1] holds:

**Proposition 2.2.** If \(A \in \Psi^{m_1,\tilde{m}_1,r_1}_{\text{cu},\alpha}(M,Y)\), \(B \in \Psi^{m_2,\tilde{m}_2,r_2}_{\text{cu},\alpha}(M,Y)\), then \(A \circ B \in \Psi^{m_1+m_2,\tilde{m}_1+\tilde{m}_2,r_1+r_2}_{\text{cu},\alpha}(M,Y)\).

2.3 The principal symbol

**Definition 3.** The \((m, \tilde{m}, r)\) principal symbol of \(A = \text{Op}(a) \in \Psi^{m,\tilde{m},r}_{\text{cu},\alpha}(M,Y)\), denoted by \(\sigma_{m,\tilde{m},r}(A)\) is the equivalent class \([a]\) of \(a\) in \(S^{m,\tilde{m},r}_{\text{cu},\alpha}/S^{m-1,\tilde{m}-(1-\alpha),r}_{\text{cu},\alpha}\).

In later arguments, we also call \(a\) the principal symbol of \(A\) if \(a\) is a representative of \(A\)’s principal symbol. An important result we need is the following proposition about the principal symbol of commutators.

**Proposition 2.3.** If \(A \in \Psi^{m,\tilde{m},r}_{\text{cu},\alpha}(M,Y)\), \(B \in \Psi^{m',\tilde{m}',r'}_{\text{cu},\alpha}(M,Y)\), then \([A, B] \in \Psi^{m+m'-1,\tilde{m}+\tilde{m}'-(1-\alpha),r+r'}_{\text{cu},\alpha}(M,Y)\), with principal symbol
\[
\sigma_{m+m'-1,\tilde{m}+\tilde{m}'-(1-\alpha),r+r'}([A, B]) = -iH_ab,
\] (2.21)
where \([a] = \sigma_{m,\tilde{m},r}(a)\), \([b] = \sigma_{m',\tilde{m}',r'}(B)\).

**Proof.** This follows from the symbolic expansion given in the proof of Proposition [2.1]. The principal symbol of \(AB\) and \(BA\) coincide, and going one order further gives [4].

2.4 Wavefront Sets and Ellipticity

To facilitate discussion we define:
\[
S_{\text{cu},\alpha}^{\infty,\infty,r}(M,Y) := \bigcup_{l_1,l_2 \in \mathbb{Z}} S_{\text{cu},\alpha}^{l_1,l_2,r}(M,Y),
\]
\[
S_{\text{cu},\alpha}^{\infty,l_2,r}(M,Y) := \bigcup_{l_1 \in \mathbb{Z}} S_{\text{cu},\alpha}^{l_1,l_2,r}(M,Y),
\]
\[
S_{\text{cu},\alpha}^{l_1,\infty,r}(M,Y) := \bigcup_{l_2 \in \mathbb{Z}} S_{\text{cu},\alpha}^{l_1,l_2,r}(M,Y),
\]
\[
S^{l_1,\infty,r}_{\text{cu,}a}(M,Y) := \cup_{l_2 \in \mathbb{Z}} S^{l_1,l_2,r}_{\text{cu,}a}(M,Y).
\]

And replace union by intersection when we replace \(\infty\) by \(-\infty\):
\[
\begin{align*}
S^{-\infty,-\infty,r}_{\text{cu,}a}(M,Y) := \cap_{l_1,l_2 \in \mathbb{Z}} S^{l_1,l_2,r}_{\text{cu,}a}(M,Y), \\
S^{-\infty,l_2,r}_{\text{cu,}a}(M,Y) := \cap_{l_1 \in \mathbb{Z}} S^{l_1,l_2,r}_{\text{cu,}a}(M,Y), \\
S^{l_1,-\infty,r}_{\text{cu,}a}(M,Y) := \cap_{l_2 \in \mathbb{Z}} S^{l_1,l_2,r}_{\text{cu,}a}(M,Y).
\end{align*}
\]

Similar notation applies to operator classes with \(S^{\infty}_{\text{cu,}a}(M,Y)\) replaced by \(\Psi\) and for Sobolev spaces with \(S\) replaced by \(H\). When we use \(-\infty\) as an order of the Sobolev norm, we mean this estimate hold for \(-N\) with large \(N \in \mathbb{R}^+\).

For \(a \in S^{m,\tilde{m},r}_{\text{cu,}a}(M,Y)\), we define its essential support \(\text{ess supp}(a)\) by defining its complement: for \(\mathfrak{z} \in Z\), we say \(\mathfrak{z} \notin \text{ess supp}(a)\) if there exist \(\chi_\mathfrak{z} \in C^\infty_c(Z)\) being identically 1 near \(\mathfrak{z}\) such that \(\chi_\mathfrak{z} a \in S^{-\infty,-\infty,r}_{\text{cu,}a}(M,Y)\). By this definition, different representatives in a equivalent class in \(S^{m,\tilde{m},r}_{\text{cu,}a}/S^{m-1,\tilde{m}-(1-\alpha),r}_{\text{cu,}a}\) are identified.

For \(A = \text{Op}(a)\), we define its (cusp) wave front set by \(\text{WF}^{\prime}_{\text{cu,}a}(A) := \text{ess supp}(a)\).

Next we state the definition of ellipticity of \(S^{m,\tilde{m},r}_{\text{cu,}a}(M,Y)\) and \(\Psi^{m,\tilde{m},r}_{\text{cu,}a}(M,Y)\) and give the parametrix construction. And then we prove elliptic estimates after we show boundedness between Sobolev spaces.

**Definition 4.** For \(a \in S^{m,\tilde{m},r}_{\text{cu,}a}(M,Y)\), we say that \(a\) is elliptic at \(\mathfrak{z} \in \partial Z\) if there is a neighborhood of \(\mathfrak{z}\) in \(Z\) on which \(a\) satisfies
\[
|a| \geq C\tau^r \rho^{-m - \frac{\tilde{m} - m}{\alpha}}.
\]

(a) is said to be elliptic on \(U\) if it is elliptic on each point of \(U\). \(A = \text{Op}(a)\) is said to be elliptic at a point or on an open set if and only if \(a\) is elliptic at that point or on that open set. The elliptic set of \(A\) (resp. \(a\)) is denoted by \(\text{Ell}(A)\) (resp. \(\text{Ell}(a)\)).

After defining ellipticity, we define the wave front set of a distribution as

**Definition 5.** For \(u \in H^{-\infty,-\infty,r}_{\text{cu,}a}(M,Y)\), we say \(\mathfrak{z} \notin \text{WF}^{\prime}_{\text{cu,}a}(u)\) if and only if there exists \(A \in \Psi^{n,\bar{\alpha},r}_{\text{cu,}a}(M,Y)\) which is elliptic at \(\mathfrak{z}\) such that \(Au \in L^2_{\text{cu}}(M)\)

The parametrix construction using a Neumann series in the classical microlocal analysis generalizes to our situation directly.

**Proposition 2.4.** Suppose \(A \in \Psi^{m,\tilde{m},r}_{\text{cu,}a}(M,Y)\) is elliptic at \(\mathfrak{z} \in \partial Z\), then there exist \(B \in \Psi^{m,\tilde{m},-r}_{\text{cu,}a}(M,Y), E \in \Psi^{0,0}_{\text{cu,}a}(M,Y)\) such that \(\mathfrak{z} \notin \text{WF}^{\prime}_{\text{cu,}a}(E)\), and following identity holds:
\[
B \circ A = \text{Id} + E.
\]
2.5 Mapping properties

Next we state mapping properties of our operator class, which is analogous to the classical case. We first give a square root construction and reduce the general boundedness to the $L^2_{cu}$ boundedness, and then prove $L^2_{cu}$ boundedness using this square root construction.

**Lemma 2.** Suppose $A \in \Psi_{cu,\alpha}^{0,0}(M,Y)$ is an self-adjoint elliptic operator whose principal symbol has a representative $a \in S^{0,0}_{cu}(M,Y)$ which is lower bounded by a positive constant, i.e. $a \geq c > 0$, then there exists $B \in \Psi_{cu,\alpha}^{0,0}(M,Y)$ such that $B$ is self-adjoint and $A = B^2 + E$ with $E \in \Psi^{\infty,\infty,0}(M,Y)$.

**Proof.** The proof is the same as Lemma 5.7 of [31]. The only difference is that the gain of the error term in each inductive is 1 and $1 - \alpha$ on the first and second indices now. But this does not essentially change the proof as long as we have positive gain in first two indices in each step. \qed

**Proposition 2.5.** Suppose $A \in \Psi_{cu,\alpha}^{s,\tilde{s},r'}(M,Y)$, then for $m, \tilde{m}, r' \in \mathbb{R}$, $A \in \mathcal{L}(H^{m,\tilde{m},r}(M,Y), H^{m-s,\tilde{m}-\tilde{s},r+r'}(M,Y))$. That is, $A$ is a bounded linear operator from $H^{m,\tilde{m},r}(M,Y)$ to $H^{m-s,\tilde{m}-\tilde{s},r+r'}(M,Y)$.

**Proof.** According to (2.19), $D_{s,\tilde{s},-r}$ is an isometry mapping $H^{s,\tilde{s},r}(M,Y)$ to $L^2_{cu}(M)$ with inverse $D'_{s,\tilde{s},-r} := \tau r \langle \tilde{D} \rangle^{-\tilde{s}} \langle D \rangle^{-s}$, which is an isometry as well. Consequently, the claimed boundedness $A \in \mathcal{L}(H^{m,\tilde{m},r}(M,Y), H^{m-s,\tilde{m}-\tilde{s},r+r'}(M,Y))$ can be reduced to the claim that $\tilde{A} := D_{m-s,\tilde{m}-\tilde{s},-(r+r')}(D_{m-s,\tilde{m}-\tilde{s},-(r+r')} A D'_{m-s,\tilde{m}-\tilde{s},-(r+r')}) \in \Psi_{cu,\alpha}^{0,0}(M,Y)$ is a bounded map from $L^2_{cu}(M)$ to $L^2_{cu}(M)$. To be more concrete, we write $\tilde{A}$ as

$$A = D'_{-(m-s),-(\tilde{m}-\tilde{s}),-(r+r')}(D_{m-s,\tilde{m}-\tilde{s},-(r+r')} A D'_{m-s,\tilde{m}-\tilde{s},-(r+r')}) D_{m-s,\tilde{m}-\tilde{s},-(r+r')}, \quad (2.24)$$

where two operators outside the bracket are isometries between weighted Sobolev spaces with appropriate indices. The graphic illustration of this conjugation process is given below.

```
```

\[ H^{m,\tilde{m},r}_{cu,\alpha}(M,Y) \xrightarrow{A} H^{m-s,\tilde{m}-\tilde{s},r+r'}_{cu,\alpha}(M,Y) \]

\[ D'_{m-s,\tilde{m}-\tilde{s},r} \uparrow \quad \downarrow D_{m-s,\tilde{m}-\tilde{s},-(r+r')} \]

\[ L^2_{cu}(M) \xrightarrow{\tilde{A}} L^2_{cu}(M) \]
So we only need to show that for any $\tilde{A} = \text{Op}(\tilde{a}) \in \Psi_{cu,\alpha}^{0,0,0}(M,Y)$, we have $\tilde{A} \in \mathcal{L}(L_{cu}^2(M,Y), L_{cu}^2(M,Y))$. Then we apply the proof of Proposition 5.9 of [31] to reduce to the proof of the $L_{cu}^2(M) \rightarrow L_{cu}^2(M)$ boundedness of $E = \text{Op}(e) \in \Psi_{cu,\alpha}^{-\infty,-\infty,0}(M,Y)$. The modification needed is replacing $S_{0,0,\infty,\delta}^{0,0,0}, S_{0,\alpha}^{1,-1,0,0}, \Psi_{cu,\alpha}^{0,0,0}, S_{\infty,\alpha}^{1,-2\delta,0}$ respectively. For a complete statement of the classical Schur's lemma, we refer readers to the Lemma in Appendix A.1 of [11]. Our final task is to verify the condition needed for applying Schur's lemma. Let $K_E(z,z')$ be the Schwartz kernel of $E$, where $z = (t,x), z' = (t',x')$. Then we need to show

$$\sup_z \int |K_E(z,z')| d\nu_{cu}(z') < \infty,$$

$$\sup_z \int |K_E(z,z')| d\nu_{cu}(z) < \infty.$$  \hfill (2.25)

On the other hand, (2.15) gives

$$K_E(z,z') = \mathcal{F}_\xi^{-1}(e(z,\tilde{\xi}))(z,z').$$

Use (2.6) with large (in absolute value) negative $m, \tilde{m}$ large (in each component) positive $\tilde{\gamma}$, we know that we can find $N_E > n$, constant $C_{N_E}$ such that

$$|K_E(z,z')| \leq C_{N_E} |z-z'|^{-N_E},$$

which is sufficient for (2.25). \hfill \square

By (2.23), and the boundedness of $B$ as a map from $H_{cu,\alpha}^{m_1-m,\tilde{m}_1-\tilde{m},-r}(M,Y)$ to $H_{cu,\alpha}^{m_1,m_1,r}(M,Y)$, we obtain the elliptic estimates

**Proposition 2.6.** Suppose $A \in \Psi_{cu,\alpha}^{m,\tilde{m},r}(M,Y)$ is elliptic, then $\forall N \in \mathbb{R}$, we have

$$||u||_{s,\tilde{s},r} \leq ||Au||_{s-m,\tilde{s}-\tilde{m},r} + ||u||_{-N,-N,r}$$ \hfill (2.26)

And in fact when $A$ is not globally elliptic, for $B \in \Psi_{cu,\alpha}^{0,0,r}(M,Y)$ such that $\text{WF}(B) \subset \text{Ell}(A)$, then (2.26) keeps to hold with the left hand side replaced by $||Bu||_{s,\tilde{s},r}$.
2.6 Gårding’s inequality

Next we prove Gårding’s inequality, which exploits bounds on symbols, still hold for our symbol and operator classes.

**Lemma 3.** Let $B, B' \in \Psi^{s,\tilde{s},r}_{\text{cu},\alpha}(M, Y)$ with $\text{WF}_{\text{cu},\alpha}'(B') \subset \text{Ell}(B)$, and suppose that their rescaled symbols $b = \hat{\rho}^s \tilde{\rho}^{-\frac{s}{\alpha}} \sigma(B), b' = \hat{\rho}^s \tilde{\rho}^{-\frac{s}{\alpha}} \sigma(B')$ satisfy: $|b'| \leq b$ on $\text{WF}_{\text{cu},\alpha}'(B')$, then for any $\delta > 0$, there exists a constant $C$ such that:

$$||Bu||_{s,\tilde{s},r} \leq (1 + \delta)||Bu||_{s,\tilde{s},r} + C||u||_{s-\frac{1}{2},\tilde{s}-\frac{1}{2},r}.$$

**Proof.** It is clear that we only need to prove the case $r = 0$, since we can replace $u$ by $\tau^{-r}u$. Consider $(1 + \delta)^2 B^* B - (B')^* B' \in \Psi^{2s,2\tilde{s},2r}_{\text{cu},\alpha}(M, Y)$, whose principal symbol is always strictly positive near $\text{WF}_{\text{cu},\alpha}'(B')$, hence it has a smooth real square root $e \in S^{s,\tilde{s},r}_{\text{cu},\alpha}(M, Y)$. (Away from $\text{WF}_{\text{cu},\alpha}'(B')$, we just set it to be $(1 + \delta)\sigma(B)$.) Then $K := (1 + \delta)^2 B^* B - (B')^* B' - E^* E \in \Psi^{2s-1,2\tilde{s}-(1-\alpha),2r}_{\text{cu},\alpha}(X)$. Apply $K$ to $u$ and pair with $u$ gives the desired inequality. \hfill \Box

3 Statement of the Theorem

In this section, we state assumptions that we need for our microlocal estimates. For more detailed discussion, please refer to [16] and [34]. The definitions of space model $X$ and spacetime $M$ are the same as in Section 2.

3.1 Assumptions near the trapped set

The operator of major concern is $P \in \Psi^m_{\text{cu},\alpha}(M)$ with real principal symbol $p = \sigma^m_{\text{cu}}(P)$ and characteristic set $\Sigma := p^{-1}(0) \subset \text{cu}^* T^* M \backslash o$, where $o$ is the zero section of $\text{cu}^* T^* M$. We use $p_0 := p|_{\tau=0}$ to denote its ‘boundary principal symbol’. We also use $p_0$ to denote the stationary extention of $p_0$ itself: $p_0(t, x, \xi_t, \xi) := p_0(x, \xi_t, \xi)$.

We make following assumptions (c.f. assumption (P.1)-(P.6) of [14], all ‘assumptions’ with indices mentioned in latter sections are assumptions listed here) near the trapped set $\Gamma \subset \Sigma \cap \text{cu}^* S^*_{\partial M} M$:

1. $dp_0 \neq 0$ on $\Sigma$ near $\Gamma$. 

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2. The defining function of fiber infinity \( \hat{\rho} \) is homogeneous of degree \(-1\) and \( H_{p0} \hat{\rho} = 0 \) near \( \Gamma \).

3. The rescaled Hamilton vector field is \( H_{p0} := \hat{\rho}^{-1} H_{p0} \in \mathcal{V}_{cu}(\mathcal{S}^* M) \). It is tangent to \( \Gamma \), and satisfies:

\[
\inf_{\Gamma} H_{p0} t > 0.
\]

4. Stable and unstable manifold at time infinity \( \bar{\Gamma}^{u/s} \) are smooth orientable codimension 1 submanifolds of \( \Sigma \cap \mathcal{S}_{\partial M}^* M \) near \( \Gamma \). They intersect transversally at \( \Gamma \). \( H_{p0} \) is tangent to \( \Sigma \cap \mathcal{S}_{\partial M}^* M \).

5. There exists local defining functions \( \bar{\phi}^{u/s} \in C^\infty(\mathcal{S}_{\partial M}^* M) \) of \( \bar{\Gamma}^{u/s} \) in a neighborhood of \( \Gamma \) in \( \Sigma \) as submanifolds of \( \Sigma \cap \mathcal{S}_{\partial M}^* M \) such that,

\[
H_{p0} \bar{\phi}^u = -\bar{w}^u \bar{\phi}^u, \quad H_{p0} \bar{\phi}^s = -\bar{w}^s \bar{\phi}^s, \quad \hat{\rho}^{-1} H_{\bar{\phi}} \bar{\phi}^s = \hat{\rho}^{-1} \{ \bar{\phi}^u, \bar{\phi}^s \} > 0, \quad (3.1)
\]

where \( \bar{\phi}^{u/s} \) are also considered as functions on \( \mathcal{T}^* M \setminus \) by stationary homogeneous degree 0 extension. And we assume that

\[
\nu_{\min} := \min \{ \inf_{\Gamma} \bar{w}^i, \inf_{\Gamma} \bar{w}^s \} > 0. \quad (3.3)
\]

6. There exists smooth submanifolds \( \Gamma^{u/s} \) of \( \Sigma \) such that \( \Gamma^{u/s} \cap \mathcal{S}_{\partial M}^* M = \Gamma^{u/s} \) and \( H_p := \hat{\rho}^{-1} H_p \) is tangent to \( \Gamma^{u/s} \). There exists defining functions \( \phi^{u/s} \in C^\infty(\mathcal{S}^* M) \) of \( \Gamma^{u/s} \) in \( \Sigma \) such that

\[
\phi^u - \bar{\phi}^u \in \tau C^\infty(\mathcal{S}^* M). \quad (3.4)
\]

In particular, assumption 5 and 6 imply that the defining function in 2. satisfies:

\[
H_p \phi^u = -w^u \phi^u, \quad H_p \phi^s = w^s \phi^s \text{ on } \Sigma, \quad (3.5)
\]

where \( w^u - \bar{w}^u \in C^\infty(\mathcal{S}^* M) \).
We use \( \tilde{w}^{u/s} := \frac{(\phi^{u/s})^2}{(\phi^{u/s})^2 + |\hat{\rho}|^{2\alpha}} w^{u/s} \) to denote the counterpart of \( w^{u/s} \) for \( \tilde{\rho} \), which satisfies

\[
H_p \tilde{\rho} = -\tilde{w}^u \tilde{\rho} + \alpha \frac{|\hat{\rho}|^{2\alpha}}{\tilde{\rho}} \tilde{\rho}^{-1} H_p \tilde{\rho},
\]

and similar equation for \( \tilde{w}^s \) but without the negative sign. The normalized subprincipal symbol of \( P \) is:

\[
p_1 := \hat{\rho}^{m-1} \sigma \left( \frac{1}{2i} (P - P^\ast) \right)
\]

We apply the construction in Section 2 to our current setting. We take \( Y = \Gamma^u \), and all other notations play the same role as in Section 2. In addition, we introduce following regions on \( Z \) to facilitate our discussion:

\[
\tilde{U}_\epsilon := \{ \tau, |\phi^u|, |\phi^s|, |p| < \epsilon \}, \tilde{U}_\epsilon = U_\epsilon \cap \{ \tau = 0 \},
\]

\[
\tilde{U}_\epsilon := \{ \tau, |\phi^u|, \frac{d\phi^u}{d\hat{\rho}}, |\phi^s|, |p| < \epsilon \}, M_{c_1, c_2} := \{ c_1 \leq |\rho \phi^u| \leq c_2 \}.
\]

\[
R_{\eta} := \{ \tilde{\rho} \leq \eta \}, \tilde{R}_\eta := R_\eta \cap \{ \tau = 0 \},
\]

where \( p := \tilde{\rho}^m \hat{\sigma} \) is the normalized symbol of \( p \) and we still use \( \Gamma^{u/s}, \Gamma \) to denote their lifts to \( Z \).

Define \( \nu_{\max} = \sup_{\Gamma} w^u \), \( \nu_{\min} = \inf_{\Gamma} w^u \), \( \nu_{\max, \eta_1, c_1, c_2} = \sup_{M_{c_1, c_2} \cap R_{\eta_1}} w^u \), \( \nu_{\min, \eta_1, c_1, c_2} = \inf_{M_{c_1, c_2} \cap R_{\eta_1}} w^u \). For purposes that will be clear in our positive commutator argument, we fix \( c, C_1 > 0 \) and require following inequality for \( \lambda \) to hold:

\[
-P_1' - c - \lambda \tilde{\rho}^u > 0 \text{ near } M_{0,2C_1} \cap \tilde{U}_{2\eta_1},
\]

where \( P_1' = P_1 - w^u \). Since \( 0 \leq \tilde{w}^u \leq w^u \), we only need

\[
\inf_{M_{0,2C_1} \cap \tilde{U}_{2\eta_1}} (-P_1 + (1 - \lambda)w^u) - c > 0,
\]

which holds if

\[
-s_{1,2\eta_1,0,2C_1} + (1 - \lambda)\nu_{\min,2\eta_1,0,2C_1} - c > 0,
\]
or equivalently:

\[
\lambda < 1 - \frac{s_{1,2\eta_1,0,2C_1} + c}{\nu_{\min,0,2C_1}}.
\]

In our application to the Kerr-de Sitter spacetime, \( p \) has a structural property that is important to improve the loss in our propagation estimates.
Definition 6. We call \( p \in S^{m,\tilde{\alpha}}_{\text{cu},\alpha}(M,\Gamma^u) \) compensable if we can choose coordinate system \((t,x,\xi_t,\xi)\in \mathbb{R}_t \times \mathbb{R}^{n-1} \times \mathbb{R}_{\xi_t} \times \mathbb{R}^{n-1}\) in which \( p \) can be locally decomposed as:

\[
p = x_1 p_1(x,\xi) + p_a,
\]

where \( x_i, \xi_i \) are \( i \)-th component of \( x, \xi \) respectively, \( p_1 \in S^{m,\tilde{\alpha}}_{\text{cu},\alpha}(M,\Gamma^u) \) and \( p_a \) is independent of \( \xi_1 \) in the sense that \( \partial_{\xi_1} p_a = 0 \). In addition, in this coordinate system \( x_1, \xi_1 \) are defining functions of \( \Gamma^u \) and \( \Gamma^s \) respectively.

3.2 The microlocal estimate

Theorem 2. There exists \( B \in \Psi_{\text{cu},\alpha}^{0,0}(M,\Gamma^u) \) which is elliptic on \( \hat{U}_0 \) and the front face, in particular on \( \Gamma. \) \( B_1, G_0 \in \Psi_{\text{cu},\alpha}^{0,0}(M,\Gamma^u) \) and \( \text{WF}_{\text{cu},\alpha}(B_1) \cap \Gamma^u = \emptyset, \text{WF}_{\text{cu},\alpha}(G_0) \cap \Gamma^u = \emptyset, \) and \( \lambda, c \) satisfy \((3.9)\), such that for \( s, N \in \mathbb{R}, \)

\[
||Bv||_{s,r} \lesssim ||B_1v||_{s+1-\lambda \alpha,s,r} + c^{-1}||G_0 P v||_{s-m-2-\lambda \alpha, s-m+2-\alpha,r} + ||v||_{-N,-N,r},
\]

(3.11)

In addition, suppose \( P \) is compensable, then \((3.11)\) holds for \( 0 < \alpha < 1. \)

Remark 2. \( \lambda \in (0,1) \) is the relative order on the front face, introduced when we construct the commutator. In a typical situation \( p_1 = 0, \) hence \( s_1 = 0. \) We take \( c \) small and expect it to be close to 1. Since the orders that capture the main feature of this estimate are: the second order of \( B_0 v \) and the first order of \( B_1 v, \) hence making \( \lambda \) close to 1 will make the loss of propagation \( 1 - \lambda \alpha \) smaller.

Remark 3. Without the compensable assumption, we need \( \alpha < \frac{1}{2} \) to ensure that \( G_1 v' \) appears in \((4.10)\) have Sobolev order strictly less than the principal part of \((4.6)\). The same concern applies to similar terms in later steps.

4 Positive commutator argument

We start our positive commutator argument. In proofs below, we first assume that the prior \(-N\) order regularity of \( v \) is high enough to justify integration by parts and pairing in our proof. Specifically, \(-N = s - \frac{1}{2}. \) And we apply the regularization technique to justify the general \( N \) case.
We sketch the entire proof here. In the first step, we use the energy away from the front face to control the energy on the front face. The second step is divided into three parts. In the first part, we an estimate near the front face. In the second part, we consider the dynamics of \( \phi^u := \phi^u / |\hat{\rho}|^\alpha \). In this part, we control the energy on \( \Gamma \) by the energy on the region away from \( \Gamma^s \). The goal of the third part is to get a ‘reversed’ version of (4.35), using the propagation estimate of \( H_p \) again. In the third step, we combine estimates in previous steps and obtain the estimate in Theorem 2. Lastly, we use the regularization to remove the priori regularity assumption on \( v \).

**Step 1**

In this step, we use the energy on the region away from the front face (\( B_1 \)-term below) to control the energy on the front face (\( B_0 \)-term below). We first consider a simple estimate.

Since \( H_p \hat{\rho} = o(\hat{\rho}) \) by assumption[2] Terms involving it are negligible. By (3.5), we can find \( r^u \in C^\infty(cu^s M) \) such that

\[
H_p \phi^u = -w^u \phi^u + r^u p,
\]

with \( H_p = \hat{\rho}^{m-1} H_p \). For \( [P, \Phi^u] \), we assume

\[
[P, \Phi^u] = iW^u \Phi^u + R_1 P + R_2,
\]

\[
R_1 \in \Psi^{-1,1} \cap (M, \Gamma^u), R_2 \in \Psi^{-2,2} \cap (M, \Gamma^u),
\]

where \( W^u \in \Psi^{-1,1} \cap (M, \Gamma^u) \) is the quantization of \( \hat{\rho}^{-(m-1)} w^u \). Although generally for operators in this operator algebra, the subprincipal part is \( 1 - \alpha \) order lower on the front face. But now all those operators are lifted from the unblown up manifold, the asymptotic expansions of their compositions behave the same as the unblown up case. Hence the subprincipal symbols are 1 order lower on both boundary faces, instead of 1 on fiber infinity and \( 1 - \alpha \) order lower on the front face, it should be 1 order lower on both of them, which explains the orders of \( R_1, R_2 \) above.

Applying both sides of (4.1) to \( v \), and notice that \( P v = f \), we have

\[
P' v' = f',
\]

where

\[
P' := P - iW^u, v' = \Phi^u v, f' = (\Phi^u + R_1) P v + R_2 v.
\]

Using (4.2) and mapping properties of pseudodifferential operators, we have:
Proposition 4.1. With $\Phi^u$, $P, R_1, R_2$ defined above, then for $G, \tilde{G} \in \Psi^{0,0}_{\text{cu},\alpha}(M, \Gamma^u)$ with $\text{WF}'_{\text{cu},\alpha}(G) \subset \text{Ell}(\tilde{G})$, we have:

$$
\|G P v\|_{s-m+2, s-m+2+\lambda \alpha, r} \lesssim \|\tilde{G}(\Phi^u + R_1) P v\|_{s-m+2, s-m+2+\lambda \alpha, r} + \|\tilde{G} R_2 v\|_{s-m+2, s-m+2+\lambda \alpha, r} + \|v\|_{s-1, s+\lambda-(1-\alpha), r}.
$$

(4.3)

There is no restriction on $\lambda$ here, but in latter parts it need to satisfy \((3.9)\). Next we state the main estimate of this step:

Proposition 4.2. There exist operators $B_0, \tilde{G} \in \Psi^{-\infty,0}_{\text{cu},\alpha}(M, \Gamma^u)$, $B_{\text{ff}} \in \Psi^{-\infty,0}_{\text{cu},\alpha}(M, \Gamma^u)$ and constants $C_1, \epsilon_1, \eta_1 > 0$, with $\text{WF}'_{\text{cu},\alpha}(\tilde{G}), \text{WF}'_{\text{cu},\alpha}(B_0), \text{WF}'_{\text{cu},\alpha}(B_1)$ contained in a fixed neighborhood of $\Gamma$, $B_0$ being elliptic on $M_{0,C_1} \cap \mathcal{U}_{\eta_1} = \{0 \leq |\rho| \leq C_1\} \cap \mathcal{U}_{\eta_1}$, $\text{WF}'_{\text{cu},\alpha}(B_1)$ being disjoint from both the lift of $\Gamma^u$ and the front face, such that for $s, N \in \mathbb{R}$, $0 < \alpha < \frac{1}{2}$, $\lambda, c$ satisfying \((3.3)\), we have:

$$
\|B_0 v\|_{-N, s-(1-\lambda)\alpha, r} + \|B_{\text{ff}} v\|_{-N, s+\lambda \alpha-\alpha, r} \lesssim \|B_1 v\|_{s-N} + c^{-1} \|\tilde{G} P v\|_{s-m+1, s-m+1+\lambda \alpha-\alpha, r} + \|\tilde{G} v\|_{s-1, s+\lambda-(1-\alpha), r}.
$$

(4.4)

In particular, we can take $-N = s$.

Remark 4. Recall \((3.4)\), the restriction on $\lambda$. In order to diminish the loss in this estimate by making $\lambda$ close to 1, we should take $c$ small. This comes with the cost that the constant in front of the forcing term becomes large. Concretely, the constant is proportional to $c^{-1}$ because the $\|AP'v\|_{s-m+1, s-\frac{m-1}{2}}$ term in the proof below, which we try to bound in the proof below, has a coefficient $\frac{1}{2c}$.

Proof. The normalized subprincipal part of $P'$ is

$$
\tilde{p}'_1 := \rho^{m-1} \sigma(\frac{1}{2\ell}(P' - (P')^*)) = \tilde{p}_1 - w^u.
$$

(4.5)

Our commutator is

$$
a = \tilde{a}^2, \quad \tilde{a} = \tau^{-\lambda} \rho^{-\lambda-1+\frac{m-1}{2}} \chi_{\text{ff}}(\rho^u) \chi_{\text{inf}}(\hat{\rho}) \chi_{\text{inf}}((\phi^u)^2) \chi_{\text{inf}}((\phi^s)^2) \chi_{\text{T}(\tau)} \chi_{\Sigma}(\tilde{p}),
$$
where $\chi^\inf, \chi^u, \chi^s, \chi_T, \chi_\Sigma$ are chosen to be identically 1 on $[-\eta_1, \eta_1]$, decreasing on $[0, \infty)$, supported on $[-\eta_1, 2\eta_1]$. $\chi^f$ is identically 1 on $[0, C_1]$, decreasing on $[0, \infty)$, supported on $[-C_1, 2C_1]$. Since $W^u \in \Psi_{u, 0}^{m-1, m-1, r}(M, \Gamma^u)$, the principal symbols and corresponding Hamilton vector fields of $P'$ and $P$ are the same., we compute:

$$H_p(\frac{\delta^\alpha}{\phi^u}) = (\phi^u)^{-1} \alpha \delta^\alpha H_p \hat{\rho} - \hat{\rho}^\alpha - w^u \phi^u \hat{\rho}^{-(m-1)} + \hat{\rho}^{-(m-1)} r^u P$$

$$= \frac{\hat{\rho}^\alpha}{\phi^u} (\alpha \hat{\rho}^{-1} H_p \hat{\rho} + w^u \hat{\rho}^{-(m-1)} + \hat{\rho}^{-(m-1)} r^u P)$$

Now take $c > 0$ to be determined later, we have:

$$\hat{\rho} H_p \hat{\rho} + \hat{\rho}^{-m-1} P^u \hat{\rho}^2 = -c \hat{\rho}^{-(m-1)} \hat{\rho}^2 - (\hat{\rho}^{-s-1} \hat{\rho}^{-1} b_0)^2 - (\hat{\rho}^{-s-1} \hat{\rho}^{-s} b_s)^2 + (\hat{\rho}^{-s-1} \hat{\rho}^{-s} b_s)^2 + (\hat{\rho}^{-s-1} \hat{\rho}^{-s} b_s)^2 + h_p + e^{\inf} - b^f \tau^r \hat{\rho}^{-1} \hat{\rho}^{-s-1})^2;$$

(4.6)

where $\chi = \chi^f \chi^\inf \chi^u \chi^s \chi_T \chi_\Sigma$, and $\hat{\chi}^i$ means the product without $\chi^i, i = f, \inf, u, s, T, \Sigma$. $e^{\inf}, b^f$ are terms introduced by taking derivative with respect to $\chi^\inf, \chi^f$:

$$e^{\inf} = \tau^{-2r} \hat{\rho}^{-2s} \hat{\rho}^{-2s-2} \hat{\chi}^\inf (\chi^{\inf})'(\hat{\rho} H_p \hat{\rho} + \hat{\rho} H_p \hat{\rho}),$$

$$b^f = \tau^{-r} \hat{\chi}^f (\chi^f)'(\hat{\rho}^\alpha (\alpha \hat{\rho}^{-1} H_p \hat{\rho} + w^u))^{1/2}.$$

Since $w^u > 0$ near $\Gamma^u$ and $\chi^f f' < 0$, we can choose $\chi^f$ so that square root in $b^f$ is well-defined and smooth. Recall that $\bar{u} = \frac{(\phi^u)^{2}}{\phi^u |\hat{\rho}|} w^u$ and $\hat{\rho}$ satisfies

$$\hat{\rho}^{-1} H_p \hat{\rho} = -\bar{u} + \alpha |\hat{\rho}|^{2\alpha} \rho^{-1} H_p \hat{\rho}. \quad (4.7)$$

$b_0, b^u$ are defined by

$$b_0 = \tau^{-r} \chi(-p^1 + c r \tau^{-2} H_p \tau) + (s + 1 - \frac{m-1}{2}) (\hat{\rho}^{-1} H_p \hat{\rho} + \lambda \hat{\rho}^{-1} H_p \hat{\rho})^{1/2}$$

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$$= \tau^{-r}\chi(-p'_1 - c + r\tau(\tau^{-2}h_p) + (s + 1 - m - 1)\lambda(\hat{\rho}_0^2)(\hat{\rho}^{-1}h_p) - \lambda\tilde{w}^u)^{1/2};$$

$$\chi^{u/s} = \tau^{-r}\hat{\chi}^{u/s}\phi^{u/s}\sqrt{-2w^{u/s}}(\chi^{u/s})'$$

$$b_T = \tau^{-r}\hat{\chi}T\sqrt{\tau\chi T}(\tau^{-2}h_p),$$

$$h = 2\tau^{-2}\hat{\rho}^{-2}\hat{\rho}^{-2s-2+m}\chi(\hat{\chi}^{u/s})'\phi^{u/s} + \chi'(\phi^{u/s})'\phi^{u/s} + + m\hat{\chi}(\phi^{u/s})'(\hat{\rho}_0^2)^{1/2}\phi^{u/s}.$$
left hand side of (4.8) we have:

\[
\text{Im}\langle f', Av' \rangle \leq |\langle f', Av' \rangle | = |\langle P'v', \tilde{A}^* \tilde{A}v' \rangle | = |\langle \tilde{A}P'v', \tilde{A}v' \rangle |
\]

\[
\leq c||\tilde{A}v'||^{1}_{m_1, \frac{1}{2}, \alpha - \frac{1}{2}, r} + \frac{1}{2c}||\tilde{A}P'v'||^{2}_{\frac{1}{2}, \frac{1}{2}, \alpha - \frac{1}{2}, r}.
\]

Combining boundedness of \(\tilde{\rho}^{-1}H_p\tilde{\rho}\) and (4.7), we know that \(\tilde{\rho}^{-1}H_p\tilde{\rho}\) is also bounded. Consequently \(\tilde{\rho}H_p\tilde{\rho} + \tilde{\rho}H_p\tilde{\rho} = \tilde{\rho}\tilde{\rho}(\tilde{\rho}^{-1}H_p\tilde{\rho} + \tilde{\rho}^{-1}H_p\tilde{\rho}) \in S^{-1, -1}_{cu, \alpha}(M, \Gamma^u)\). So we have \(E_{\inf} \in \tau^{-r}_f \Psi^{2s+1, 2s+1 + 2\lambda - \alpha, r}(M, \Gamma^u)\) and it can be absorbed into \(E\)-term.

Combining these inequalities, we have:

\[
c||\tilde{A}v'||^{2}_{\frac{m_1 - 1}{2}, \frac{1}{2}, \alpha, r} + ||B_0v'||^{2}_{s+1, s+1 + \lambda, r} + ||B^s\tilde{v}'||^{2}_{s+1, s+1 + \lambda, r} + ||B^{f\tilde{v}'}||^{2}_{s+1, s+1 + \lambda, r}
\]

\[
\leq ||B^u\tilde{v}'||^{2}_{s+1, s+1 + \lambda, r} + ||B_T\tilde{v}'||^{2}_{s+1, s+1 + \lambda, r} + ||\langle P'v', Hv' \rangle |
\]

\[
+ \langle ||G_1'v'||^{2}_{s+1, s+1 + \lambda, r} + C||\tilde{v}'||^{2}_{s+1, s+1 + \lambda, r} + c||\tilde{A}v'||^{2}_{\frac{m_1 - 1}{2}, \frac{1}{2}, \alpha, r} + \frac{1}{2c}||\tilde{A}P'v'||^{2}_{\frac{1}{2}, \frac{1}{2}, \alpha - \frac{1}{2}, r} + \langle E_{\inf}v', v' \rangle.
\]

The terms in the bracket comes from the fact that (4.6) only concerns principal symbols. They collect terms of order lower than the principal part in (4.6). They are generated from full symbol expansion of composition of \(P'A, AP', (P')^*A\), hence the highest order term among them is 2 order lower in the first index and 2\((1 - \alpha)\) order lower in the second index compared with the product \(P'A\). Here the order refers to the sum of the orders of two operators in a pairing. The is the place where the condition \(\alpha < \frac{1}{2}\) is needed to ensure that this term will not affect the principal part in (4.6). Use Cauchy-Schwartz inequality to control \(|\langle P'v', Hv' \rangle |\). Because of the \(\chi\) factor in \(h\), \(Hv'\) is microlocalized near the support of \(\tilde{a}\) as well, using \(G'_1\) as the microlocalizer again and enlarge is microsupport if necessary, we obtain:

\[
|\langle P'v', Hv' \rangle | \lesssim ||G'_1 P'v'||^{2}_{s-m+2, s-m+2 + \lambda, r} + ||Hv'||^{2}_{s-m+2, -(s-m+2 + \lambda), r} + ||v'||^{2}_{s+\alpha - \frac{1}{2}, s+\alpha + \lambda - \frac{1}{2}, r}
\]

\[
\lesssim ||G'_1 P'v'||^{2}_{s-m+2, s-m+2 + \lambda, r} + ||G'_1v'||^{2}_{s+\alpha, s+\alpha + \lambda, r} + ||v'||^{2}_{s+\alpha - \frac{1}{2}, s+\alpha + \lambda - \frac{1}{2}, r}.
\]

(4.11)

Recall the discussion following (4.8), we obtain \((\text{WF}'_{cu, \alpha}(B^u) \cup \text{WF}'_{cu, \alpha}(B_T)) \cap \bar{U}_1^u = \emptyset\). Combine the first two terms on the right hand side of (4.10) and control them by \(B_1 \in \Psi^{0, 0, r}_{cu, \alpha}(M, \Gamma^u)\) microlocalized in a neighborhood
of \( \Gamma \) but away from \( \Gamma \) itself. They satisfy \( \text{WF}'_{cu,\alpha}(B_u) \cup \text{WF}'_{cu,\alpha}(B_\tau) \subset \text{Ell}(B_1) \) and \( \text{WF}'_{cu,\alpha}(B_1) \cap U_{\eta} = \emptyset \). In particular, \( B_1 \in \tau^{-r}\Psi_{cu,\alpha}^{0,-\alpha,r}(M,\Gamma^u) \).

And \( ||\hat{A}p''||_{-\frac{m-1}{2},-\frac{m-2}{2},r} \) is controlled by mapping properties of \( \hat{A} \). Since \( 0 < \alpha < \frac{1}{2} \), combining \( ||G''_1v'||_{s+\frac{1}{2},s+\lambda\alpha+\frac{1}{2}+\alpha,r} \) and \( ||G''_1v'||_{s,\lambda\alpha+\frac{1}{2}+\alpha,r} \) to be \( ||G''_1v'||_{s+\frac{1}{2},s+\lambda\alpha+\frac{1}{2}+\alpha,r} \), we obtain

\[
||B_0\Phi^u v||_{s+1,s+1+\lambda\alpha,r} + ||Bf\Phi^u v||_{s+1,s+1+\lambda\alpha,r} \leq ||B_1\Phi^u v||_{s+1,s+1+\lambda\alpha,r} + ||G'_1\Phi^u v||_{s+\frac{1}{2},s+\lambda\alpha+\frac{1}{2}+\alpha,r} + ||\Phi^u v||_{s-\frac{1}{2},s+\lambda\alpha-\frac{1}{2}+\alpha,r} + c^{-1}||G'_1f'||_{s-m+2,s-m+2+\lambda\alpha,r}
\]

(4.12)

\( G'_1 \in \tau^{-r}\Psi_{0,\alpha,\alpha}^{0,0,r}(M,\Gamma^u) \) is microlocalized near \( \text{supp} \, \alpha \). Since the support conditions of \( B_0 \) and \( G'_1 \) are the same (possibly with different bounds on \( \frac{\omega}{\rho} \)), we can iterate this estimate to control the \( ||G''_1v'||_{s+\frac{1}{2},s+\lambda\alpha+\frac{1}{2}+\alpha,r} \) term at the cost of enlarging the micro-support of \( G'_1 \). In each iteration we can improve the first index by \( \frac{1}{2} \) and the second index by \( \frac{1}{2} - \alpha \). We need at most \( N_1 := \max\left\{ \left( \frac{s+\lambda}{2} \right), \left[ \frac{s+\lambda+\frac{1}{2}+\alpha-(s+\lambda-\frac{1}{2}+2\alpha)}{\frac{1}{2} - \alpha} \right] \right\} = \max\{2, \left[ \frac{1}{2} - \alpha \right] \} = 2 + \left[ \frac{2\alpha}{1-2\alpha} \right] \) iterations. We denote the new operator playing the role of \( G'_1 \) in the last iteration by \( \hat{G} \). Then we apply Proposition 4.1 to control \( ||\hat{G}f'||_{s-m+2,s-m+2+\lambda\alpha,r} \) and use the mapping property of \( \Phi^u \in \Psi_{cu,\alpha,\alpha}^{0,0,r}(M,\Gamma^u) \) to control norms of \( \Phi^u v \) to get:

\[
||B_0\Phi^u v||_{s+1,s+1+\lambda\alpha,r} + ||Bf\Phi^u v||_{s+1,s+1+\lambda\alpha,r} \leq ||B_1\Phi^u v||_{s+1,s+1+\lambda\alpha,r} + c^{-1}||\hat{G} P v||_{s-m+2,s-m+2+\lambda\alpha-r} + ||\hat{G}v||_{s,s+\lambda\alpha,r} + ||v||_{s-\frac{1}{2},s+\lambda\alpha-\frac{1}{2}+\alpha,r}
\]

(4.13)

where \( \lambda, c \) satisfies (3.3), and \( \text{WF}'_{cu,\alpha}(G'_1) \subset \text{Ell}(\hat{G}) \). In particular, we can require \( \text{WF}_{cu,\alpha}(\hat{G}) \subset \{ \frac{\omega}{\rho} \leq 3C_1 \} \cap U_{3\eta} \). (4.13) will be the same as (4.4) if we can replace \( \Phi^u v \) by \( v \), which we proceed to achieve next. Recall that \( B_1 \in \tau^{-r}\Psi_{cu,\alpha}^{0,-\alpha,r}(M,\Gamma^u) \), hence the second order of the \( B_1 \) term can be taken to be any real number.

Recall that \( \phi^u \geq C_1 \rho \) on the wave front set of \( B_0 \), hence \( \Phi^u \) is elliptic near \( \text{WF}'_{cu,\alpha}(B_0) \) as an operator in \( \tau^{-r}\Psi_{cu,\alpha}^{0,-\alpha,r}(M,\Gamma^u) \), and we can write \( B_0 \Phi^u = \Phi^u B_0 + [B_0, \Phi^u] \) with the commutator term having lower order, hence conclude from above estimate:

\[
||B_0v||_{s+1,s+1+\lambda\alpha-\alpha,r} + ||Bf v||_{s+1,s+1+\lambda\alpha-\alpha,r} \lesssim ||B_1v||_{s+1,s+1+\lambda\alpha-\alpha,r} + ||\hat{G} P v||_{s-m+2,s-m+2+\lambda\alpha-r} + ||\hat{G}v||_{s,s+\lambda\alpha,r} + ||v||_{s-\frac{1}{2},s+\lambda\alpha-\frac{1}{2}+\alpha,r}
\]
This implies, if we replace $s+1$ by $s$, and recall that $B_1 \in \tau^{-r}\Psi_{cu,u}^{0,-\infty,r}(M, \Gamma^u)$:

$$
\|B_0 v\|_{s, s+\lambda_0 - a, r} + \|B^{\text{ff}} v\|_{s, s+\lambda_0 - a, r} \lesssim \|B_1 v\|_{s, -N_1, r} + \|\tilde{G} P v\|_{s-m+1, s-m+1+\lambda_0 - a, r} \\
+ \|\tilde{G} v\|_{s-1, s-1+\lambda_0, r} + \|v\|_{s-\frac{3}{2}, s-\frac{3}{2}+\lambda_0, r},
$$

(4.14)

for any $N_1 \in \mathbb{R}$. The constant implicitly included in ‘$\lesssim$’ depends on $N_1$ as well. The effect of $B^{\text{ff}}$ term here is that we can extend the region where we have control to the area near the boundary of $\text{supp}(\chi^{\text{ff}})$, where $(\chi^{\text{ff}})'$ has a lower bound but $\chi^{\text{ff}}$ is almost vanishing.

Remark 5. The constant in (4.4) depends on how close the wave front set of $B_0$ is to the lift of $\Gamma^u$. We consider $B_0$–term first. What affects the constant in the estimate is, as we approach the lift of $\Gamma^u$, the ellipticity of $\Phi^u$ is becoming weaker and weaker and we do not have a uniform lower bound of its principal symbol. The constant in the elliptic estimate is proportional to the reciprocal of the lower bound of the symbol. Recall that $\text{WF}'_{cu,u}(B^0) \subset \{ \phi^u \geq C_1^{-1} \rho \}$, hence the way that we can ‘push’ the estimate near $\Gamma^u$ is letting $C_1 \to \infty$ and the constant is proportional to $C_1$. The term $\|B^{\text{ff}} v\|_{s+1, s+1+\lambda_0, r}$ is treated in a similar manner. And a more accurate version of (4.14) is:

$$
\|B_0 v\|_{s, s+\lambda_0 - a, r} + \|B^{\text{ff}} v\|_{s, s+\lambda_0 - a, r} \lesssim C_1(\|B_1 v\|_{s, -N_1, r} + \|\tilde{G} P v\|_{s-m+1, s-m+1+\lambda_0 - a, r} \\
+ \|\tilde{G} v\|_{s-1, s-1+\lambda_0, r} + \|v\|_{s-\frac{3}{2}, s-\frac{3}{2}+\lambda_0, r}),
$$

(4.15)

where $C_1$ comes from $\text{WF}'_{cu,u}(B^0) \subset \{ \phi^u \geq C_1^{-1} \rho \}$.

Step 2.1

In this step, our positive commutator argument gives an estimate near the front face.

**Proposition 4.3.** There exist operators $\tilde{B}_0, \tilde{G}_1 \in \Psi_{cu,u}^{0,0,r}(M, \Gamma^u), \tilde{B}_1 \in \Psi_{cu,u}^{0,-\infty,r}(M, \Gamma^u)$ and a small constant $\epsilon_0 > 0$, with $\text{WF}'_{cu,u}(\tilde{B}_0), \text{WF}'_{cu,u}(\tilde{G}_1), \text{WF}'_{cu,u}(B_1)$ contained in a fixed neighborhood of $\Gamma$, $\tilde{B}_0$ being elliptic on $\mathcal{U}_{2\epsilon_0}$ defined in (3.7),
\[ \text{WF}'_{\text{cu},\alpha}(\tilde{B}_1) \cap \{ \frac{|w^u|}{|p^\alpha|} < \epsilon_0, \tau = 0 \} = \emptyset, \text{ we have:} \]

\[
\| \tilde{B}_0 \Phi^u v \|_{s+1,s+1,r} \lesssim \| \tilde{B}_1 \Phi^u v \|_{s+1,s+1,r} + \| \tilde{G}_1 P v \|_{s-m+2,s-m+2,r} + \| \tilde{G}_1 v \|_{s,s,r} + \| v \|_{s-\frac{1}{2},s-\frac{1}{2}+2\alpha,r}, \]

(4.16)

**Proof.** We choose the commutator

\[
a = \hat{a}^2, \hat{a} = \tau^{-r} \hat{\rho}^{-s-1+(m-1)/2} \chi^1_{(u)}(\frac{\phi^u}{|\hat{\rho}|^\alpha}) \chi^s(\tau \chi^T_{m})(\tau \chi^\Sigma_{1}(P)),
\]

where \( p = \hat{\rho}^m p, \chi^i(s) = 1, i = u, s, T, \Sigma \) for \( s \in [-\epsilon_0, \epsilon_0] \), monotonically decrease to 0 when \( s \) grow from \( 2\epsilon_0 \) to \( 3\epsilon_0 \) and extended to \( \mathbb{R} \) in an even manner. Then we consider the pairing

\[
\text{Im}(\langle f', Av' \rangle) = \langle \frac{1}{2} i [P', A] + \frac{P' - (P')^*}{2i} A \rangle v', v' \rangle.
\]

The principal symbol of the operator on the right hand side is

\[
\hat{a}H_p \hat{a} + \hat{\rho}^{-m+1} P'_1 \hat{a}^2 = -c \hat{\rho}^{-m+1} \hat{a}^2 - (\hat{\rho}^{-s-1} \hat{b}_0)^2 - (\hat{\rho}^{-s-1} \hat{b}_s)^2
\]

\[
+ (\hat{\rho}^{-s-1} \hat{b}_r^2) + (\hat{\rho}^{-s-1} \hat{b}_T) + hp.
\]

(4.17)

Defining \( \chi_1 := \chi^u_{1} \chi^s_{1} \chi^T_{1} \chi^\Sigma_{1} \) and defining \( \hat{\chi}_1 \) to be the product without \( \chi^i_1 \),

we have

\[
\hat{b}_0 = \tau^{-r} \chi_1(\mathbf{- p}'_1 - c + r \tau (\tau^{-2} H_p \tau) + (s + 1 - \frac{m-1}{2}) \hat{\rho}^{-1} H_p \hat{\rho})^{1/2}
\]

\[
\hat{b}_s = \tau^{-r} \hat{\chi}_1^s \sqrt{-2w^s \chi^s(\chi^s)'}. \]

We can guarantee that terms in the bracket in \( \hat{b}_0 \) all together is positive when we use \( \chi_T \) to localize to the region on which \( \tau \) is small and use the fact \( \hat{\rho}^{-1} H_p \hat{\rho} = o(1) \) as \( \tau \to 0 \). We compute the derivative of \( \phi^u / |\hat{\rho}|^\alpha \) as

\[
H_p(\phi^u / |\hat{\rho}|^\alpha) = -(w^u \phi^u + r^u p)|\hat{\rho}|^{-\alpha} - \alpha \phi^u |\hat{\rho}|^{-\alpha-1} H_p \hat{\rho}.
\]

We have

\[
\hat{b}^u_s = \tau^{-r} \chi^u_1((-w^u \phi^u |\hat{\rho}|^{-\alpha} - \alpha (\phi^u |\hat{\rho}|^{-\alpha} \hat{\rho}^{-1} H_p \hat{\rho}) \chi^u(\chi^u)'), \]

\[
\hat{b}_T = \tau^{-r} \chi^T_1 (\tau \chi^T_1 (\chi^T_1)(\tau^{-2} H_p \tau))^{1/2}.
\]

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\[ h = 2\tau^{2\rho} \hat{\rho}^{1-2s-2+m} \chi_1(\hat{\chi}_1 u) \partial^u \hat{\rho}^\alpha + \hat{\chi}_1^s(\chi_1) r^s + m \chi_1^s(\chi_1^s)'(\hat{\rho}^{-1} H_\rho \hat{\rho}) \]

We can choose \( \chi_1^u \) so that \( \phi^u \hat{\rho}^\alpha \) is small enough on \( \text{supp} \hat{\rho} \), and notice that \( |\hat{\rho}^{-1} \phi^u| \leq 1, |\hat{\rho}^{-1} \hat{\rho}| \leq 1 \), and \( H_\rho \hat{\rho} = \rho(\hat{\rho}) \) as \( \tau \to 0 \). Evaluating \( \text{Im} \langle f', Av' \rangle \) and use (4.17), we know

\[
\text{Im} \langle f', Av' \rangle = \langle \left( \frac{1}{2i} [P', A] + \frac{P' - (P')^*}{2i} A \right) v', v' \rangle 
\]

\[
= \langle -c ((T_{(m-1)/2})^* (T_{(m-1)/2} H_0) - (T_{s+1} \tilde{B}_0) (T_{s+1} \tilde{B}_0) 
+ (T_{s+1} \tilde{B}_0) (T_s \tilde{B}_T) + (T_{s+1} \tilde{B}_T) (T_{s+1} \tilde{B}_T) + H^* P' + E) v', v' \rangle, \tag{4.18}
\]

with \( \hat{A} \in \Psi^{n+1-(m-1)/2, s+1-(m-1)/2, r} (M, \Gamma^u), \tilde{B}_0, \tilde{B}_u, \tilde{B}_T \in \Psi^{0,0,r} (M, \Gamma^u), H^* \in \Psi^{2s+2-m+\alpha, 2s+2-m+\alpha, r} (M, \Gamma^u), E \in \Psi^{2s+2+s+1+\alpha, r} (M, \Gamma^u) \). \( E \) is the error term introduced because (4.17) concerns only principal symbols. We estimate the left hand side by

\[
\text{Im} \langle f', Av' \rangle \leq \langle f', Av' \rangle = \langle P' v', \hat{A}^* \hat{A} v' \rangle = \langle \hat{A} P' v', \hat{A} v' \rangle 
\]

\[
\leq c ||\hat{A} v'||^{2}_{s\frac{1}{2}, -s\frac{1}{2} + fr} + \frac{1}{2c} ||\hat{A} P' v'||^{2}_{s\frac{1}{2}, -s\frac{1}{2} + fr}. \tag{4.19}
\]

Combining above equations and estimates, the estimate we obtain is

\[
c||\hat{A} v'||^{2}_{(m-1)/2, (m-1)/2, r} + ||\tilde{B}_0 v'||^{2}_{s+1, s+1, r} + ||\hat{B}^s v'||^{2}_{s+1, s+1, r} 
\]

\[
\leq ||\hat{B}^u v'||^{2}_{s+1, s+1, r} + ||\tilde{B}_T v'||^{2}_{s+1, s+1, r} + ||P' v', H v'|| + (||G_1 v'||^{2}_{s\frac{1}{2}, s\frac{1}{2} + fr} + C ||v'||^{2}_{s\frac{1}{2}, s\frac{1}{2} + fr}) 
+ c||\hat{A} v'||^{2}_{(m-1)/2, (m-1)/2, r} + \frac{1}{2c} ||\hat{A} P' v'||^{2}_{(m-1)/2, (m-1)/2, r}. \tag{4.19}
\]

Terms in the bracket are similar terms in the bracket in (4.10). The principal part of this error term is 2 order lower in the first index and 2(1 - \alpha) order lower in the second index compared with the product \( P' A \). Applying Cauchy-Schwartz inequality to \( ||P' v', H v'|| \), since \( \text{supp} h \subset \text{supp} \hat{\rho} \), we can add a microlocalizer at the cost of introducing a lower order error term:

\[
||P' v', H v'|| \lesssim ||G_1 P' v'||^{2}_{s-m+2, s-m+2, r} + ||H v'||^{2}_{(s-m+2), -(s-m+2), r} + ||v'||^{2}_{s+\alpha, s+\alpha, s+\alpha, r}.
\]

Since \( H \in \Psi_{cu, \alpha}^{2s+2-m+\alpha, 2s+2-m+\alpha, r} (M, \Gamma^u) \), we know \( ||H v'||^{2}_{(s-m+2), -(s-m+2), r} \leq ||G_1 v'||^{2}_{s+\alpha, s+\alpha, r} \) for suitable \( G_1 \). The \( c||\hat{A} v'||^{2}_{(m-1)/2, (m-1)/2, r} \) on both sides
cancel out. Dropping the $\tilde{B}^s$‐term on the left hand side of (4.19), we get:

$$
||\tilde{B}_0 \Phi^u v||_{s+1,s+1,r} \leq ||\tilde{B}_1 \Phi^u v||_{s+1,s+1,r} + ||\tilde{G}_1 f'||_{s-m+2,s-m+2,r} + ||\tilde{G}_1 \Phi^u v||_{s+s+1,s+1,r} + C||\Phi^u v||_{s-s+1/2,s+1/2+2\alpha,r}.
$$

where $\tilde{B}_1 \in \Psi^{0,0}_{\text{ca}}(M, \Gamma^u)$ is elliptic on $\text{WF}'_{\text{ca}}(B^u) \cup \text{WF}'_{\text{ca}}(B_T)$. We iterate this estimate to control $||\tilde{G}_1 \Phi^u v||_{s+s+1,s+1+2\alpha,r}$. Then apply Proposition 4.1 to estimate $\tilde{G}_1 f'$‐term:

$$
||\tilde{B}_0 \Phi^u v||_{s+1,s+1,r} \lesssim ||\tilde{B}_1 \Phi^u v||_{s+1,s+1,r} + ||\tilde{G}_1 P v||_{s-m+2,s-m+2,r} + ||\tilde{G}_1 v||_{s,r} + ||v||_{s,s,r} + ||v||_{s-s+1/2,s+1/2+2\alpha,r},
$$

where the microsupport of $\tilde{G}_1$ is enlarged when we iterate the estimate. Our operators satisfy: $\tilde{B}_0$ is elliptic on $\tilde{U}_{\epsilon_0}$, $\text{WF}'_{\text{ca}}(\tilde{B}_1) \cap \{ |\phi^u|/|\hat{\rho}|^\alpha < \epsilon_0, \tau = 0 \} = \emptyset$.

**Step 2.2**

In this step we consider the dynamics of $\tilde{\phi}^u := \phi^u/|\hat{\rho}|^\alpha$, and denote its quantization by $\tilde{\Phi}^u \in \Psi^{0,0}_{\text{ca}}(M, \Gamma^u)$. The estimate we obtain in this step is to control the energy on $\Gamma$ (i.e., the $B_\delta$‐term) by the energy on the region away from $\Gamma^s$ (i.e., the $E_\delta$‐term). The regions of microlocalization depend on a parameter $\delta \in (0, \frac{3}{4})$ to be specified later.

**Proposition 4.4.** There exist $B_\delta, E_\delta, B_2, \tilde{G}_2 \in \Psi^{0,0}_{\text{ca}}(M, \Gamma^u)$ with $B_\delta$ being elliptic on $\tilde{U}_{\epsilon_0} \cap \{ |\phi^u| \leq 3\delta \}$, $E_\delta$ satisfying (4.34) below, $\text{WF}'(B_2) \cap \tilde{U}_{\epsilon_0} = \emptyset$, constants $C, C'$ such that

$$
||B_\delta v||_{s,r} \leq C(\delta^{1/2}||E_\delta v||_{s,r} + ||B_2 v||_{s+1-\alpha,s+1-\alpha,r} + ||\tilde{G}_2 P v||_{s-\alpha-m+2,s-\alpha-m+2,r} + \tilde{C}||v||_{s-s-1/2,s-1/2+2\alpha,r}).
$$

(4.20)

**Proof.** We use a new commutator

$$
a = \tilde{a}^2, \quad \tilde{a} = \hat{\rho}^{s-1/2} \psi_2(\phi^u/|\hat{\rho}|^\alpha) \lambda_2^u T(\tau) \lambda_2 \Sigma(p),
$$

(4.21)

where $\lambda_2^u, \lambda_2^T, \lambda_2 \Sigma$ being identically 1 on $[-\epsilon_0, \epsilon_0]$ and supported in $[-2\epsilon_0, 2\epsilon_0]$. Notice that

$$
H_{\phi^u/|\hat{\rho}|^\alpha} = \phi^u H_{|\hat{\rho}|^{-\alpha}} + |\hat{\rho}|^{-\alpha} H_{\phi^u}
$$

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we compute the principal symbol of \( \frac{i}{2}[\Phi^u, A] \) as two parts corresponding to two terms above:

\[
\hat{a}H_{\hat{a}}\hat{a} = -\alpha \phi^u \hat{\rho}^{-\alpha - 1} H_{\hat{\rho}} + \hat{\rho}^{-\alpha} H_{\phi^u},
\]

Recall our assumptions on \( \phi^u/s \), we can choose \( C_{\phi} > 1 \) such that

\[
C_{\phi}^{-1} \leq H_{\phi^u} \phi^s \leq C_{\phi} \text{ on } \hat{U}_{\delta_0},
\]

where \( H_{\phi^u} := \hat{\rho}^{-1}H_{\phi^u} \) is the normalized Hamilton vector field associated with \( \phi^u \). We recall the construction of the cutoff with respect to \( \phi^s \) in step 2 of the proof of Theorem 3.9 of [14], i.e. we choose smooth \( \psi_2 \) such that: \( \text{supp} \psi_2 \subset [-4\delta, 2\epsilon_0], 0 \leq \psi_2 \leq 1, \psi_2' \geq 0 \) on \(( -\infty, 0 \rbracket, \psi_2(-3\delta) \geq \frac{1}{4}, \psi_2' \geq \frac{1}{12\delta} \) on \([-3\delta, 3\delta], \psi_2' \geq -\frac{1}{\epsilon_0} \) on \([0, \infty), \psi_2 \leq 12\delta \psi_2' \) on \([-4\delta, 3\delta] \). The last inequality automatically hold on \([-3\delta, 3\delta] \) by previous properties, and we can choose appropriate \( \psi_2 \) to extend it to \([-4\delta, 3\delta] \). The construction of \( \psi_2 \) is elementary. Then decompose \( \psi_2 \) as

\[
\psi_2^2 = \psi_{2-}^2 + \psi_{2+}^2,
\]

where \( \psi_{2-} = \psi_2 \) on \((-\infty, 3\delta] \) and \( \text{supp} \psi_{2-} \subset [-4\delta, 4\delta] \), and \( \text{supp} \psi_{2+} \subset (3\delta, 2\epsilon_0] \). Define \( a_{\pm} \) to be symbols obtained from \( a \) by replacing \( \psi_2 \) by \( \psi_{2+} \) and \( \psi_{2-} \) respectively. Hence we have \( \hat{a}^2 = \hat{a}_{+}^2 + \hat{a}_{-}^2 \), and \( e_3 = -\alpha \phi^u \hat{\rho}^{-1} H_{\hat{\rho}} \hat{a}_{+} = \frac{1}{2} \alpha \phi^u \hat{\rho}^{-1} H_{\phi^u} (\hat{a}_{+}^2 + \hat{a}_{-}^2) \). Consequently we have

\[
\psi_2 \psi_2' = \frac{1}{24\delta} \psi_{2-}^2 + \frac{1}{24\delta} \hat{b}_\delta - \hat{e},
\]

where \( \text{supp} \hat{e} \subset (3\delta, 2\epsilon_0], |\hat{e}| \leq \epsilon_0^{-1}, \hat{b}_\delta \geq 0, \text{supp} \hat{b}_\delta \subset [-4\delta, 4\delta] \) and

\[
\hat{b}_\delta \geq \frac{1}{4} \text{ on } [-3\delta, 3\delta].
\]

We consider two terms in \( (4.22) \) separately. For the second term we have:

\[
\hat{a}H_{\phi^u} \hat{a} = \frac{1}{48C_{\phi}} \hat{\rho} \hat{a}_{+}^2 + (\tau^{-r} \hat{\rho}^{-s} \hat{b}_{-})^2 + \frac{1}{24C_{\phi}} (\tau^{-r} \hat{\rho}^{-s} \hat{b}_{+})^2 + \tau^{-2r} \hat{\rho}^{-2s} e_1 - \tau^{-2r} \hat{\rho}^{-2s} e_2,
\]
where

\[ b_- = \psi_2 - \chi_2 \chi_2^T \chi_2^\Sigma \left( \frac{1}{24\delta} (H_{\phi^n} \phi^s - \frac{1}{2} C^{-1}) - r\tau (\tau^{-2} H_{\phi^n} \tau) - (s + 1/2) (\hat{\rho}^{-1} H_{\phi^n} \hat{\rho})) \right)^{1/2}, \]

\[ b_\delta = \chi_2 \chi_2^T \chi_2^\Sigma b_\delta \sqrt{C_{\phi} H_{\phi^n} \phi^s}, \]

\[ e_1 = \psi_2^2 (\tau^2 \chi_2^T \chi_2^\Sigma) (\chi_2^T \chi_2^\Sigma)^{-1} (\tau^{-2} H_{\phi^n} \tau) + (\chi^u)^2 \chi_2 \chi_2^\Sigma (\chi_2^T \chi_2^\Sigma)' (H_{\phi^n} \rho) \]

\[ + (\chi_2^T \chi_2^\Sigma)^2 \chi_2 \chi_2^\Sigma (\chi_2^T \chi_2^\Sigma)' (\phi^u / |\hat{\rho}|^\alpha) \hat{\rho}^{-1} H_{\phi^n} \hat{\rho}, \]

\[ e_2 = (\chi_2^u)^2 (\chi_2^T \chi_2^\Sigma) (H_{\phi^n} \phi^s) \tilde{e} + \psi_2^2 (r\tau (\tau^{-2} H_{\phi^n} \tau) + (s + 1/2) (\hat{\rho}^{-1} H_{\phi^n} \hat{\rho}))). \]

where the term introduced by differentiating \( \chi^u \) is in \( e_1 \). Combining properties of \( \tilde{e}, \psi_{2+} \), we know \( e_1, e_2 \) satisfy

\[ \text{supp} e_1 \cap \hat{U}_{\alpha} = \emptyset, \]

\[ \text{supp} e_2 \subset \{\phi^s \geq 3\delta\}, |e_2| \leq C_{\phi} e_0^{-1} + C, \quad (4.27) \]

where \( C \) is independent of \( \delta \). \( b_\delta \) satisfies

\[ b_\delta \geq \frac{1}{4} \text{ on } \hat{U}_{\alpha} \cap \{ |\phi^s| \leq 3\delta \} \quad (4.28) \]

by (4.23) and (4.25).

Although \( \phi^n \in S^{\alpha,0,r}_{cu,\alpha} (M, \Gamma^u) \), \( \chi^u_2 \) is supported on the region where \( \phi^n_{\rho} \) is bounded, so \( \chi^u_2 \phi^n \in S^{\alpha,0,r}_{cu,\alpha} (M, \Gamma^u) \). Consequently \( e_1 \in S^{\alpha,0,r}_{cu,\alpha} (M, \Gamma^u) \). Then we consider the first term in (4.22). Define \( e_3 := -\alpha \hat{\rho}^u \hat{u} \hat{\rho}^{-1} H_{\phi^n} \hat{a} \). Recall the definition of \( \hat{a} \) in (4.21), factors of terms in \( e_3 \) involving differentiation are:

\( \hat{\rho}^{-1} H_{\phi^n} \phi^s, \hat{\rho}^{-1} H_{\phi^n} \phi^s, \hat{\rho}^{-1} H_{\phi^n} \tau, \hat{\rho}^{-1} H_{\phi^n} \rho \), all of which are all bounded quantities.

In addition, because of the \( \chi^u \) factor in \( \hat{a} \), \( \phi^n \) is bounded on \( WF'_{cu,\alpha} (E_3) \), so \( e_3 \) quantizes to be \( E_3 \in \Psi^{2,2,r}_{cu,\alpha} (M, \Gamma^u) \), which introduces another term \( |\langle E_3 v, v \rangle| \). We define

\[ e_{31} = \frac{1}{2} \hat{\rho}^{-\frac{1+\alpha}{2}} H_{\phi^n} \hat{a}, \quad e_{32} = \hat{a} \hat{\rho}^{-\frac{1+\alpha}{2}}, \]

and we have \( e_3 = e_{31} e_{32} \phi^n \). And define the version truncated by \( \psi_{2+} \) as:

\[ e_{31 \pm} = \frac{1}{2} \hat{\rho}^{-\frac{1+\alpha}{2}} H_{\phi} \hat{a}_{\pm}, \quad e_{32 \pm} = \hat{a}_{\pm} \hat{\rho}^{-\frac{1+\alpha}{2}}. \]

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We define $E_{31}$, $E_{32}$, $E_{31\pm}$, $E_{32\pm}$ to be operators with principal symbols denoted by corresponding lower-case letters. In addition, we require them to be self-adjoint (if not, replace $E_{ij}$ by $\frac{1}{2}(E_{ij}+E_{ij}^\dagger)$). Thus $E_{31\pm} \in \Psi_{\sigma,\alpha}^{s+1/2,s+1/2} (M, \Gamma^u)$, $E_{32\pm} \in \Psi_{\sigma,\alpha}^{s+1/2,s+1/2} (M, \Gamma^u)$. Hence we have $E_3 = (E_{31\pm} E_{32\pm} + E_{31-32-}) \hat{\Phi}^u$ and:

$$
|\langle E_3 v, v \rangle| = |\langle (E_{31\pm} E_{32\pm} + E_{31-32-}) \hat{\Phi}^u, v \rangle| \\
\leq |\langle E_{32\pm} \hat{\Phi}^u v, E_{31\pm} v \rangle| + |\langle E_{32\pm} \hat{\Phi}^u v, E_{31-} v \rangle| \\
\leq ||E_{32\pm} \hat{\Phi}^u||_0^2 + ||E_{31\pm}||_0^2 + ||E_{32\pm} \hat{\Phi}^u||_0^2 + ||E_{31-}||_0^2
$$

We can control $E_{31\pm}$-term up to a constant by $||v||_{s-1+\frac{\alpha}{2}}$, and control $E_{32\pm}$-term by $||\hat{\Phi}^u||_0^2$ by mapping properties and counting the order of operators, where we have used the fact that $e_{32\pm} = \hat{\alpha}_\pm \hat{\rho}^{-\frac{1}{2}}$, hence they are just order-shifted version of each other, i.e. $||E_{32\pm} \hat{\Phi}^u||_0^2$ is equivalent to $||\hat{\Phi}^u||_0^2$. To summarize, we have

$$
|\langle E_3 v, v \rangle| \leq ||E_{32\pm} \hat{\Phi}^u||_0^2 + ||E_{31\pm}||_0^2 + ||E_{32\pm} \hat{\Phi}^u||_0^2 + ||E_{31-}||_0^2 \\
\lesssim ||\hat{\Phi}^u||_0^2 + ||\hat{\Phi}^u||_0^2 + ||\hat{\Phi}^u||_0^2 + ||v||_{s-1+\frac{\alpha}{2}} \\
\text{ (4.29)}
$$

Now we evaluate the pairing $\text{Im}(\hat{\Phi}^u A, v) = \langle \frac{i}{2} [\hat{\Phi}^u, A] v, v \rangle$. First we quantize both sides of (4.22) and then apply it to $v$ and pair with $v$. We have $\hat{A}, \hat{A}_\pm \in \Psi_{\sigma,\alpha}^{s+1/2,s+1/2} (M, \Gamma^u)$, $B_-, B_\delta, E_1, E_2 \in \Psi_{\sigma,\alpha}^{0,0,r} (M, \Gamma^u)$. Setting $\bar{v} := \hat{\Phi}^u v$ to simplify notations, we get:

$$
\frac{1}{48 C_0^2}||\hat{A}_- v||_r^2 + \frac{1}{2\epsilon_0}||\hat{A}_+ v||_r^2 + \frac{1}{2\epsilon_0}||B_- v||_r^2 + \frac{1}{24 C_0^2}||B_\delta v||_r^2 \\
\leq |\langle E_1 v, v \rangle| + |\langle E_2 v, v \rangle| + |\langle \hat{A}_- \bar{v}, v \rangle| + |\langle \hat{A}_- v, \hat{A}_+ v \rangle| + \bar{C} ||v||_{s+\frac{\alpha}{2}}^2 + \frac{1}{48 C_0^2}||\hat{A}_- v||_r^2 + \frac{1}{24 C_0^2}||\hat{A}_- \bar{v}||_{\frac{\alpha}{2}}^2 \\
+ \epsilon_0||\hat{A}_+ \bar{v}||_r^2 + \frac{1}{2\epsilon_0}||\hat{A}_+ v||_r^2 + \frac{1}{2\epsilon_0}||B_- v||_r^2 + \frac{1}{24 C_0^2}||B_\delta v||_r^2 \\
\lesssim |\langle E_1 v, v \rangle| + |\langle E_2 v, v \rangle| + \frac{1}{48 C_0^2}||\hat{A}_- v||_r^2 + \frac{1}{24 C_0^2}||\hat{A}_- \bar{v}||_{\frac{\alpha}{2}}^2 \\
+ \epsilon_0||\hat{A}_+ \bar{v}||_r^2 + \frac{1}{2\epsilon_0}||\hat{A}_+ v||_r^2 + \frac{1}{2\epsilon_0}||B_- v||_r^2 + \frac{1}{24 C_0^2}||B_\delta v||_r^2.
$$

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\[ \tilde{C} \| v \|_{s + \frac{3}{2}, s + \frac{3}{2}, -1 + \alpha, r} = \tilde{C} \| v \|_{s - \frac{1}{2}, s - \frac{1}{2}, -1 + \alpha, r} \text{ arises because } (4.22) \text{ concerns only principal symbols. In the last step, we used } (4.29). \]

Multiply \(24C_\delta \delta\) on both sides and then taking square root, applying Lemma 3, we obtain:

\[
\| B_\delta v \|_{s + \frac{3}{2}, s + \frac{3}{2}, r} \leq C \left( \sqrt{\delta} \right) \| \tilde{E}_\delta v \|_{s + \frac{3}{2}, s + \frac{3}{2}, r} + \| B_2 v \|_{s + \frac{3}{2}, s + \frac{3}{2}, r} + (\delta + \sqrt{\delta \varepsilon_0}) \| \tilde{G}_2 \tilde{\Phi}^u v \|_{s + 1 - \frac{3}{2}, s + 1 - \frac{3}{2}, r} + \tilde{C} \| v \|_{s - \frac{1}{2}, s - \frac{1}{2}, -1 + \alpha, r},
\]

(4.30)

where \( \tilde{G}_2 \) is microlocalized near \( \text{supp} \tilde{a} \), which contains \( \Gamma \). The term \( \delta \| \tilde{G}_2 \tilde{\Phi}^u v \|_{s + 1 - \frac{3}{2}, s + 1 - \frac{3}{2}, r} \) controls \( 24C_\delta \delta \| \tilde{A}_\delta v \|_{s - \frac{3}{2}, s - \frac{3}{2}, -1 + \alpha, r} \) and the term \( \sqrt{\delta \varepsilon_0} \| \tilde{G}_2 \tilde{\Phi}^u v \|_{s + 1 - \frac{3}{2}, s + 1 - \frac{3}{2}, r} \) controls \( \varepsilon_0 \| \tilde{A}_\delta \tilde{\Phi}^u v \|_{s + 1 - \frac{3}{2}, s + 1 - \frac{3}{2}, r} \). \( \delta \| \tilde{G}_2 \tilde{\Phi}^u v \|_{s + 1 - \frac{3}{2}, s + 1 - \frac{3}{2}, r} \) controls \( \tilde{\Phi}^u \). By Lemma 3 and (4.28), we know

\[
WF'(B_2) = \emptyset,
\]

(4.32)

which is possible by (4.27).

Next we estimate \( (\delta + \sqrt{\delta \varepsilon_0}) \| \tilde{G}_2 \tilde{\Phi}^u v \|_{s + 1 - \frac{3}{2}, s + 1 - \frac{3}{2}, r} \), which is equivalent to \( (\delta + \sqrt{\delta \varepsilon_0}) \| \tilde{G}_2 \tilde{\Phi}^u v \|_{s + 1 - \frac{3}{2}, s + 1 - \frac{3}{2}, r} \). We control this term using (4.16). Enlarging the micro-support of \( G_2 \) if necessary, this results in an error term \( (\delta + \sqrt{\delta \varepsilon_0}) \| \tilde{G}_2 v \|_{s + \frac{3}{2}, s + \frac{3}{2}, r} \). By Lemma 3 and (4.28), we know

\[
\| \tilde{G}_2 v \|_{s + \frac{3}{2}, s + \frac{3}{2}, r} \leq 2 \| B_\delta v \|_{s + \frac{3}{2}, s + \frac{3}{2}, r} + 2 \| E_\delta v \|_{s + \frac{3}{2}, s + \frac{3}{2}, r} + \tilde{C} \| B_2 v \|_{s + \frac{3}{2}, s + \frac{3}{2}, r} + \tilde{C} \| v \|_{s - \frac{1}{2}, s - \frac{1}{2}, -1 + \alpha, r}
\]

(4.33)

where \( E_\delta \) satisfies

\[
|\sigma(E_\delta)| \leq 1, \sigma(E_\delta) = 1 \text{ on } \{ 3\delta \leq \phi^u \leq 4\varepsilon_0, |\tau| \leq \varepsilon_0, |\tilde{\phi}^u| \leq \varepsilon_0, |\mathbf{p}| \leq \varepsilon_0 \},
\]

\[
WF'_{\text{cu}, \alpha}(E_\delta) \subset \left\{ \frac{5}{2} \delta \leq \phi^u \leq 5\varepsilon_0, |\tau| \leq 2\varepsilon_0, |\tilde{\phi}^u| \leq 2\varepsilon_0, |\mathbf{p}| \leq 2\varepsilon_0 \right\}.
\]

(4.34)
Substitute \((4.33)\) in \((4.30)\), and choose \(\delta\) small enough so that the \(B_\delta\)-term on the right hand side can be absorbed by the \(B_\delta\)-term on the left hand side and fix \(\epsilon_0\), we get:

\[
\|B_\delta v\|_{s+\frac{\alpha}{2}, s+\frac{\alpha}{2}, r} \leq C(\delta^{1/2}) \|E_\delta v\|_{s+\frac{\alpha}{2}, s+\frac{\alpha}{2}, r} + \|B_2 v\|_{s+1-\frac{\alpha}{2}, s+1-\frac{\alpha}{2}, r} + \|\tilde{G}_2 P v\|_{s-\frac{\alpha}{2}-m+2, s-\frac{\alpha}{2}-m+2, r} + \tilde{C}\|v\|_{s-\frac{1+\alpha}{2}, s-\frac{1+\alpha}{2}, r},
\]

After an overall \(\frac{\alpha}{2}\) shift of differential orders, we can rewrite this as:

\[
\|B_\delta v\|_{s, s, r} \leq C(\delta^{1/2}) \|E_\delta v\|_{s, s, r} + \|B_2 v\|_{s+1-\alpha, s+1-\alpha, r} + \|\tilde{G}_2 P v\|_{s-\alpha-m+2, s-\alpha-m+2, r} + \tilde{C}\|v\|_{s-\frac{1+\alpha}{2}, s-\frac{1+\alpha}{2}, r},
\]  

(4.35)

\[
\square
\]

**Step 2.3**

The goal of this part is to control \(\|E_\delta v\|_{s, s, r}\) by \(\|B_\delta v\|_{s, s, r}\) and get a ‘reversed’ version of \((4.35)\), using the propagation estimate of \(H_p\) again.

**Proposition 4.5.** There exist \(\tilde{B}_1, \tilde{G}_3 \in \Psi_{\alpha,\alpha}^{0,0}(M, \Gamma)\) with \(\text{WF'}(\tilde{B}_1) \subset \tilde{U}_{\tau_0} \setminus \tilde{U}_{\tau}, \tilde{G}_3\) microlocalized near \(\Gamma\), such that

\[
\|E_\delta v\|_{s, s, r} \leq C(\delta^{1/2}) \|B_\delta v\|_{s, s, r} + \|\tilde{B}_1 v\|_{s, s, r} + \|v\|_{s-1/2, s-1/2, r} + \sqrt{C(\|G_3 P v\|_{s-m+1, s-m+1, r} + \|v\|_{s-1/2, s-1/2, r})},
\]

(4.36)

where \(C\) is independent of \(\delta\), and \(\delta \tilde{C}\) is bounded, \(E_\delta, B_\delta\) are the same as in Proposition 4.4.

**Proof.** Consider the commutator:

\[
a = \ddot{a}^2, \quad \ddot{a} = \tau^{-r} \dot{\rho}^{-s+(m-1)/2} |\dot{\phi}^s|^\beta \psi_3 (\log |\dot{\phi}^s/\delta|) \chi_3^u(<\phi^u/|\dot{\rho}|^\alpha) \chi_3^T (\tau) \chi_3^\Sigma (p),
\]

where \(\delta \in (0,1)\) is typically a small parameter. The same as in the previous step, whenever companied with \(\chi_3^u\) or \((\chi_3^u)'\), \(\frac{\partial}{\partial \rho}\) is effectively a symbol in \(S_{\alpha,\alpha}^{0,0}(M, \Gamma)\). For the cut-off \(\psi_3\), we can arrange

\[
\psi_3 \psi_3' = -\tilde{b}_l^2 + \tilde{e},
\]

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where $\tilde{e}$ is supported in $\{\delta \leq |\phi^s| \leq \frac{9}{4} \delta\}$ and it satisfies $|\tilde{e}| \leq 1$. $\tilde{b}_t \geq 0$, and it is supported in $\{\frac{3}{2} \delta \leq |\phi^s| < 6\epsilon_0 \delta^{-1}\}$. And according to the support of $\tilde{b}_t$ and $\tilde{e}$, we use a partition $\psi_3^2 = (\psi_{3-})^2 + (\psi_{3+})^2$; where $\psi_{3+} = \psi_3$ on $\{\frac{3}{4} \delta \leq |\phi^s| < 6\epsilon_0 \delta^{-1}\}$, $\text{supp} \psi_{3-} \subset [0, \log \frac{9}{4})$, $0 \leq \psi_{3+} \leq 1$. Next we consider the pairing

$$\text{Im} \langle P, Av \rangle = \langle \frac{i}{2} [P, A] + \frac{P - P^*}{2i} A \rangle v, v \rangle.$$

For the left hand side, we have

$$\text{Im} \langle P, Av \rangle \leq |\text{Im} \langle P, Av \rangle| \leq |\langle P, Av \rangle|.$$

For the right hand side, its principal symbol is

$$\tilde{a} H_{p}\tilde{a} + \tilde{P}^{-m+1} P_1 \tilde{a}^2 = -c \tilde{P}^{-m+1} \tilde{a}^2 - (\tau^{-r} \tilde{P}^{-s} \tilde{b}_{t-})^2 - (\tau^{-r} \tilde{P}^{-s} \tilde{b}_{t+})^2 + f_1 + f_2 + h p,$$

(4.37)

where

$$b_{t+} = |\phi^s|^{-\beta} \psi_3 \chi^u \Sigma^{\frac{3}{2}} \Sigma \Sigma (\beta w^s - P_1 - c + r \tau (\tau^{-2} H_p \tau) + (s - (m - 1)/2) (\tilde{P}^{-1} H_p \tilde{P} + w^s \tilde{P}^{-1} H_p \tilde{P}))1/2,$$

$$f_1 = \tau^{-2r} \tilde{P}^{-2s} |\phi^s|^{-\beta} \psi_3 \chi^u \Sigma^{\frac{3}{2}} \Sigma (\beta r^s + \frac{\psi_3 \Sigma^{\frac{3}{2}} \Sigma (\beta r^s)}{\psi_3 \Sigma^{\frac{3}{2}} \Sigma (\beta r^s)} + m (\tilde{P}^{-1} H_p \tilde{P}^{\Sigma^{\frac{3}{2}} \Sigma (\beta r^s)})).$$

In the calculation of $f_1$, we used

$$H_p (\phi^u / |\tilde{P}|^{\alpha}) = -(u^u \phi^u + r^u p) |\tilde{P}|^{-\alpha} - \alpha \phi^u |\tilde{P}|^{-\alpha-1} H_p \tilde{P}.$$

$b_{t+}$ is the main term, giving control over the region away from $\Gamma^\perp$. $f_i$ are 'error terms'. On the support of $f_1$, which is away from $\Gamma^\perp$, we have a priori control (because when $\tau$ and $\phi^u / |\tilde{P}|^{\alpha}$ are close enough to 0, $(\chi^u)^{\perp}$ and $\chi^u_T$ will vanish, hence $f_1$ is supported away from these regions). Recall that $\text{supp} \tilde{e}(\cdot) \subset [0, \log (\frac{9}{4})]$, hence $\text{supp} f_2 \subset \{\delta \leq |\phi^s| \leq \frac{9}{4} \delta\}$. $b_{t+}$ and $f_2$ satisfy bounds:

$$b_{t+} \geq c \quad \text{on} \quad \hat{U}_{3\epsilon_0} \cap \{\frac{9}{4} \delta \leq |\phi^s| \leq 5\epsilon_0\},$$

$$|f_2| \lesssim \tau^{-2r} \tilde{P}^{-2s} \delta^{-2\beta} \delta^{-2\beta}.$$

(4.39)
Quantize both sides of (4.37), and apply them to \( v \) and pairing with \( v \), we have:

\[
|c||\tilde{A}v||_{s-m-1/2,(m-1)/2,r}^2 + ||B_{t,+}v||_{s,s,r}^2 + ||B_{t,-}||_{s,s,r}^2 \\
\leq |\langle F_1, v \rangle| + |\langle F_2, v \rangle| + |\langle P, Hv \rangle| + |\langle \tilde{A}Pv, \tilde{A}v \rangle| + \tilde{C}||v||_{s-m+1,m+1,r}^2 + \tilde{C}||\tilde{G}_3v||_{s-1,s-1,r}^2 + c||\tilde{A}v||_{(m-1)/2,(m-1)/2,r}^2 \\
+ \frac{1}{2c}||\tilde{A}Pv||_{(m-1)/2,(m-1)/2,r}^2 + \tilde{C}||v||_{s-1/2,s-1/2,r}^2,
\]

(4.40)

where \( \tilde{G}_3 \in \Psi^{0,0,r}_\text{cu,}\alpha(M,\Gamma^u) \) is the microlocalizer. \( \tilde{G}_3 \) is elliptic near \( \text{supp}\tilde{a} \) with \( \text{WF}^{\prime}_{\text{cu,}\alpha}(\tilde{G}_3) \) contained in a neighborhood of \( \text{supp}\tilde{a} \). \( \tilde{C}||v||_{s-1/2,s-1/2,r}^2 \) arise because (4.37) concerns only principal symbols. Due to the \( \psi_\alpha \) factor, \( |\phi^s|^{-\beta} \lesssim \delta^{-\beta} \) on \( \text{supp}\tilde{a} \) and this estimate gives an upper bound of \( \tilde{a} \) as well. Consequently, for each set of fixed parameters other than \( \delta \), \( \delta^{-2\beta}\tilde{C} \) is bounded. Since \( \beta < \frac{1}{2} \) and we only concern \( \delta \) small, \( \delta\tilde{C} \) is bounded.

The \( \tilde{A}v \)-terms on both sides cancel each other. The \( \tilde{G}_3v \)-term and \( \tilde{G}_3Pv \)-term are introduced to control \( |\langle P, Hv \rangle| \). Use \( \tilde{B}_1 \in \Psi^{0,0,r}_\text{cu,}\alpha(M,\Gamma^u) \) which is elliptic on \( \hat{U}_{\Delta_0} \setminus \hat{U}_{\Delta_2} \) and \( \text{WF}^{\prime}_{\text{cu,}\alpha}(\tilde{B}_1) \subset \hat{U}_{\Delta_0} \setminus \hat{U}_{\Delta_2} \) to control the errors slightly away from \( \Gamma^u \). As we have mentioned, the main term being controlled is \( B_{t,+} \)-term, which is elliptic on a region that is near \( \Gamma^u \) but away from \( \Gamma^s \). By our construction, \( \text{WF}^{\prime}_{\text{cu,}\alpha}(E_\delta) \subset \Ell(B_\delta) \). For \( F_2 \)-term, since \( \hat{U}_{\Delta_0} \cap \{ |\phi^s| \leq 3\delta \} \subset \Ell(B_\delta) \) and \( \hat{U}_{\Delta_0} \setminus \hat{U}_{\Delta_2} \subset \Ell(\tilde{B}_1) \), we know \( \text{WF}^{\prime}_{\text{cu,}\alpha}(F_2) \subset \Ell(B_\delta) \cup \Ell(\tilde{B}_1) \). Consequently, we can control \( |\langle F_2, v \rangle| \) by \( \delta^{-2\beta}||B_\delta v||_{s,s,r}^2 + ||\tilde{B}_1v||_{s,s,r} + ||v||_{s-1/2,s-1/2-\alpha}/2 \) Substitute into (4.40) and take square root on both sides, we have:

\[
||E_\delta v||_{s,s,r} \leq C(\delta^{-\beta}||B_\delta v||_{s,s,r} + ||\tilde{B}_1v||_{s,s,r} + ||v||_{s-1/2,s-1/2-\alpha}/2) \\
+ \sqrt{C}||\tilde{G}_3Pv||_{s-m+1,m+1,r} + ||v||_{s-1/2,s-1/2-\alpha}/2),
\]

where \( C \) is independent of \( \delta \), and \( \delta\tilde{C} \) is bounded.

\[\square\]

**Step 3: Combining Estimates**

Combine (4.30) with (4.35), we obtain an estimate with leading term \( \delta^{1/2-\beta}||B_\delta v||_{s,s} \). By choosing \( \delta \) small enough, we can absorb this term into the left hand side.
and get:
$$\|B_3v\|_{s,s,r} \lesssim \|\tilde{B}_1v\|_{s+1-\lambda, s+1,\alpha, r} + \|\tilde{G}_2Pv\|_{s-m+2-\alpha, s-m+2,\alpha, r} + \|v\|_{s/2, s-(1-\alpha)/2, r},$$
(4.41)

with following properties: $B_3$ is elliptic on (the lift of) $\Gamma$, $WF'_{cu,\alpha}(\tilde{B}_1) \subset U_{\hat{\epsilon}_0} \setminus \hat{U}_3$. In addition, we enlarge $Ell(\tilde{G}_2)$ and $WF'_{cu,\alpha}(\tilde{G}_2)$ to absorb the $\tilde{G}_3$-term. Now combine (4.4) and (4.41), and let $\tilde{B}_1$ in (4.41) play the role of $B_0$ in (4.4). Since the order now is $s+1-\alpha$ instead of $s+\lambda\alpha - \alpha$, so the order need to be shifted by $1-\lambda\alpha$. For $\tilde{B}_1$-term, we localize inside the front face by inserting another cutoff, i.e., we use
$$\|\tilde{B}_1v\|_{s+1-\lambda\alpha, s+1,\alpha, r} \lesssim \|B_1v\|_{s+1-\lambda\alpha, -\infty, r} + \|\tilde{G}Pv\|_{s-m+2-\lambda\alpha, s-m+2,\alpha, r} + \|v\|_{s-\frac{1}{2}-\lambda\alpha, s-\frac{1}{2}+\alpha, r},$$
where these operators satisfy: $U_{\epsilon_0} \setminus U_{\epsilon_0/2} \subset Ell(\tilde{B}_1) \subset WF'_{cu,\alpha}(\tilde{B}_1) \subset U_{\epsilon_0} \setminus \hat{U}_{\epsilon_0/4}$, $WF'_{cu,\alpha}(\hat{G}) \subset \{|\hat{v}| \leq 3C_1\} \cap U_{3\eta_1} = \{|\hat{v}| \geq (3C_1)^{-1}\} \cap U_{3\eta_1}$, and $WF'_{cu,\alpha}(B_1) \cap \hat{U}_{\eta_1} = \emptyset$. In particular, if we choose $C_1, \eta_1, \epsilon_0$ so that $(3C_1)^{-1} > \frac{\eta_0}{2}, 3\eta_1 < 6\epsilon_0$, then $WF'_{cu,\alpha}(\hat{G}) \subset Ell(\tilde{B}_1)$ and we can iterate to improve this error term. Concretely, apply the same estimate to $\tilde{G}v$ with $s$ replaced by $s-1$, and then repeat. The only cost this iteration might cause is the microlocal error introduced when we apply elliptic estimate to $\tilde{B}_1$, the microlocal erro and $\tilde{G}v$-term with one order lower norm can both be absorbed into the last error term $\|v\|_{s-\frac{1}{2}-\lambda\alpha, s-\frac{1}{2}+\alpha}$:
$$\|\tilde{B}_1v\|_{s+1-\lambda\alpha, s+1,\alpha, r} \lesssim \|B_1v\|_{s+1-\lambda\alpha, -\infty, r} + \|\tilde{G}Pv\|_{s-m+2-\lambda\alpha, s-m+2,\alpha, r} + \|v\|_{s-\frac{1}{2}-\lambda\alpha, s-\frac{1}{2}+\alpha, r},$$

Substitute this estimate into (4.41) and we get
$$\|B_3v\|_{s,s,r} \lesssim \|B_1v\|_{s+1-\lambda\alpha, -\infty, r} + \|\tilde{G}Pv\|_{s-m+2-\lambda\alpha, s-m+2,\alpha, r} + \|\tilde{G}_3Pv\|_{s-m+2-\alpha, s-m+2,\alpha, r} + \|v\|_{s-\frac{1}{2}-\lambda\alpha, s-\frac{1}{2}+\alpha, r},$$

$\tilde{G}_3$ is microlocalized in a neighborhood of $\Gamma$ but $WF'_{cu,\alpha}(\tilde{G}_3) \cap \Gamma^s = \emptyset$. $\tilde{G}_3Pv$ and $\tilde{G}Pv$-terms can be combined together using a $G_0$ obtained by enlarging their micro-support. And then we iterate to improve the last error term to obtain
$$\|B_3v\|_{s,s,r} \lesssim \|B_1v\|_{s+1-\lambda\alpha, s, r} + \|G_0Pv\|_{s-m+2-\lambda\alpha, s-m+2,\alpha, r} + \|v\|_{-N, -N, r},$$
where $B_\delta$ is elliptic on $\Gamma$, and $\text{WF}'_{\text{cu},\alpha}(B_1) \cap \bar{\Gamma}^u = \emptyset$, and $\lambda$ satisfies (3.3). In the special case where $P$ is self-adjoint, it becomes

$$\lambda < 1 - \frac{c}{\nu_{\text{max}}}. \quad (4.42)$$

We recall that we can take $c \to 0$, at the cost of making the constant in the estimate $O(c^{-1})$ large. Finally, notice that the estimate keeps to hold when we add intermediate terms in the proof to the left hand side. Concretely, in terms of notations in this section, $||\tilde{\tilde{B}}_1v||_{s+1-\lambda_\alpha,s+1-\alpha,r}$, $||B_1v||_{s+1-\lambda_\alpha,s,r}$ etc. Equivalently we have

$$||Bv||_{s,s,r} \lesssim ||B_1v||_{s+1-\lambda_\alpha,s,r} + ||G_0Pv||_{s-m+2-\lambda_\alpha,s-m+2-\alpha,r} + ||v||_{-N,-N,r},$$

where $B \in \Psi_{\text{cu},\alpha}(M,\Gamma^u)$ is elliptic on both $\hat{\mathcal{U}}_a$ and the front face.

### 4.1 The restriction on $\alpha$

Now we prove the last statement of the theorem relaxing the restriction $\alpha < \frac{1}{2}$ to $\alpha < 1$. Assume that $p$ is compensable as in Definition 6. Locally $p$ can written as:

$$p = x_1p_1 + p_a,$$

where $x_i, \xi_i$ are $i$–th component of $x, \xi$ respectively, $p_a$ is independent of $x_1, \xi_1$. In addition, in this coordinate system $x_1, \xi_1$ are defining functions of $\Gamma^u$ and $\Gamma^s$ respectively. Consider the symbolic expansion of $[P, A]$ obtained from the expansion of compositions $PA$ and $AP$. Since $A = \hat{A}^*\hat{A}$ with $\hat{A} = \text{Op}(\hat{a})$, we may assume $A = \text{Op}(\hat{a}^2) + E$ with $E \in \Psi_{\text{cu},\alpha}^{2s+2-m,2s+2-m+a,r}(M,\Gamma^u)$. From the formulae of symbol of compositions, the symbol of $[P, A]$ is:

$$\sum_{\alpha \in \mathbb{Z}^n} \frac{1}{\alpha!} (\partial_x^\alpha p D_x^\alpha a - D_\xi^\alpha p \partial_\xi^\alpha a). \quad (4.43)$$

Now $\hat{a}^u = x_1$, according to the characterization of $S_{\text{cu},\alpha}^{m,\tilde{m},r}(M,\Gamma^u)$ given in (2.6), the only terms that can possibly cause loss are those $x_1$–derivatives of $a$–term, which will be multiplied by $\xi_1$–derivatives of $p$–term. Recall (4.43), we consider a typical term $\partial_{\xi_1}^k D_1p \partial_{x_1}^k D_2a$, where $D_1$ is an operator differentiating with respect to other momentum variables while $D_2$ is differentiating
with respect to spatial variables corresponding to momentum variables in $D_1$. If we only use the symbolic property of $p$, then $\partial^k_\xi$ will make its asymptotic behaviour at fiber infinity $k(1-\alpha)$ order lower. But now because of this form of $p$, such terms will have an $x_1$ factor, which can be paired with the extra $\hat{\rho}^{-\alpha}$ to offset the loss. So the asymptotic behaviour of this typical term will be $k - (k - 1)\alpha$ order lower. The restriction on $\alpha$ is to require this term to have order lower than the subprincipal part, which means $k - (k - 1)\alpha > 1$, or:

$$\alpha < 1.$$  \hspace{1cm} (4.44)

We first take the argument in Section 4.2 as an example. The term with $k = 2$, which is $2 - \alpha$ order lower in the second index, is the term corresponds to $\hat{G}_1 v'$ in (4.10). Thus the ‘error term’ in (4.10) now becomes

$$\left( ||G'_1 v'||_{s + \frac{1}{2}, s + \lambda \alpha + \frac{3}{2}, \nu}^2 + C||v'||_{s - \frac{1}{2}, s + \lambda \alpha + \frac{3}{2}, \nu}^2 \right).$$

Then when we use iteration to improve the error term, we are improving the first index by $\frac{1}{2}$ and the second index by $\frac{1 - \alpha}{2} > 0$ in each iteration. Thus the argument goes through as before. Similarly, for Section 4.3 the error terms in (4.19) now become

$$\left( ||\hat{G}_1 v'||_{s + \frac{1}{2}, s + \frac{1}{2} + \frac{3}{4}, \nu}^2 + C||v'||_{s - \frac{1}{2}, s + \frac{1}{2} + \frac{3}{4}, \nu}^2 \right).$$

The remaining parts of the argument in this section are the same. The arguments in later sections are unchanged when $\alpha$ exceeds $\frac{1}{2}$. Consequently, under (4.44) and the compensable condition, our propositions and theorems hold for $\alpha \in (0, 1)$.

4.2 Regularization

Only assuming that the right hand side of (3.11) is finite is not sufficient to guarantee that each pairing in our positive commutator is finite and integrations by parts are legal. Potentially some terms in equations (e.g., (4.8) and (4.9)) are not finite with only $(-N, -N)$ order priori control of $v$. In this section we justify pairing and integration by parts in our positive commutator argument by a regularization argument for $-N = s - \frac{1}{2}$ first and then for general $N$ by induction. Starting with Proposition 4.2, we replace $\hat{a}$ by:

$$\tilde{a}_{\eta'} := t_{\eta'}^2 \hat{a},$$  \hspace{1cm} (4.45)
where \( t_{\eta r} = (1 + \eta r \hat{\rho}^{-1})^{-1}, \eta_r > 0 \). In (4.6), this new \( t_r^2 \) factor introduces an extra term given by the \( H_p \)-derivative falling on \( t_{\eta r} \). Direct computation shows

\[
H_p(t_{\eta r}^4) = 4t_{\eta r}^4 H_p t_{\eta r} = 4 \frac{\eta_r}{\eta_r + \hat{\rho}^{-1}} t_{\eta r}^4 (\hat{\rho}^{-1} H_p \hat{\rho}).
\]

Now since \( |\eta_r| \leq 1 \), \( \hat{\rho}^{-1} H_p \hat{\rho} \to 0 \) as \( \tau \to 0 \), hence this term can be made small when we localize near \( \tau = 0 \). Consequently, this term can be absorbed into the \( b_0 \)-term (notice that all terms have an extra \( t_4 \) factor now). For \( \eta_r > 0 \), we know that \( a_{\eta r} = \hat{\alpha}_{\eta r}^2 \) is a symbol of order less in both the indices associated with the fiber infinity and the front face compared with \( a = \hat{\alpha}^2 \). To be concrete, \( a \in S_{2s+2-(m-1),2s+2-(m-1)+2\lambda \alpha, \Gamma^u}(M, \Gamma^u), a_{\eta r} \in S_{2s-m-1,2s-m+2\lambda \alpha, \Gamma^u}(M, \Gamma^u) \). Thus assuming \( -N = s - \frac{1}{2} \) regularity of \( v \) in a priori, all pairing and integration by parts are justified and we obtain an estimate similar to (4.14), but with \( B_0, B_{ff, ...} \) replaced by \( B_{0,\eta r}, B_{ff, ...} \), which are obtained by quantizing the symbol of the same lowercase letter (without \( \eta_r \)) with an extra \( t_{\eta r} \) factor. Finally we let \( \eta_r \to 0 \) and apply the weak-* compactness argument, see for example Section 5.4.4 of [31], we obtain the estimate above (4.14) without \( \eta_r \) and \( B_{0v} \in H^{s+1, s+1+\lambda \alpha - \alpha, \Gamma^u}(M, \Gamma^u) \). The regularization arguments for step 2.1-2.3 are similar. We conduct the argument for step 2.1 individually, but regularize step 2.2, step 2.3 and step 3 together to obtain (4.42) by taking the regularization parameter to 0. And then we prove the general \( N \) case by induction. For \( N \) less than our initial case, the estimate holds automatically by our initial case.

## 5 Application to Kerr-de Sitter spacetime

### 5.1 Kerr-de Sitter spacetime

In this section, we consider a Kerr-de Sitter spacetime with black hole mass \( m \) and angular momentum \( a \in \mathbb{R} \). We assume the black hole is subextremal in the sense that

\[
\Delta(r) = (r^2 + a^2)(1 - \frac{\Lambda r^2}{3}) - 2mr
\]

has four distinct real roots

\[
r_- < r_C < r_e < r_c.
\]
We point out here that some authors use the condition $|a| < m$ to define the subextremal property, which is slightly stronger than the distinct root condition. In the Kerr case ($\Lambda = 0$), the

See [1] for more details.

In the Boyer-Lindquist coordinates, the Kerr-de Sitter spacetime is

$$M^\circ := \mathbb{R}_t \times X,$$

where $X = (r_e, r_c) \times \mathbb{S}^2$. We use $\varphi \in \mathbb{S}^1, \theta \in [0, \pi]$ are spherical coordinates on $\mathbb{S}^2$. $M^\circ$ is equipped with metric

$$g = (r^2 + a^2 \cos^2 \theta)(\frac{dr^2}{\Delta(r)} + \frac{d\theta^2}{\Delta_\theta}) - \frac{\Delta \theta \sin^2 \theta}{(1 + \frac{\Lambda a^2}{3})^2(r^2 + a^2 \cos^2 \theta)}(\text{ad} - (r^2 + a^2)d\varphi)^2$$

$$+ \frac{\Delta(r)}{(1 + \frac{\Lambda a^2}{3})^2(r^2 + a^2 \cos^2 \theta)}(dt - a \sin^2 \theta d\varphi)^2,$$

(5.1)

where

$$\Delta_\theta = 1 + \frac{\Lambda a^2}{3}, \quad \Lambda \geq 0.$$

As discussed in Section 2 concretely (2.1), we use $M = (M^\circ \sqcup ([0, \infty)_t \times X))/\sim$ to denote the spacetime that is compactified at the time infinity. Since (5.1) is independent of $t$, it naturally extends to a metric on $M$. In order to distinguish with our model $\tilde{M}$ in following sections, variables on $T^*M$ are denoted as $(x_M, \xi_M)$. The singularity of (5.1) at the event horizons $\{r = r_e, r_c\}$ can be resolved by change of coordinates, see [1] for more detailed discussion. In our coordinate system, the dual metric is given by

$$G = G_r + G_\theta,$$

$$\begin{align*}
(r^2 + a^2 \cos^2 \theta)G_r &= \Delta(r)\xi_r^2 - \frac{(1 + \frac{\Lambda a^2}{3})^2}{\Delta(r)}((r^2 + a^2)\xi_t + a\xi_\varphi)^2 \\
(r^2 + a^2 \cos^2 \theta)G_\theta &= \Delta_\theta \xi_\theta^2 + \frac{(1 + \frac{\Lambda a^2}{3})^2}{\Delta_\theta \sin^2 \theta}(a \sin^2 \theta \xi_t + \xi_\varphi)^2.
\end{align*}$$

(5.2)

The singularities of $g$ and $G$ at $\{\theta = 0, \pi\}$ can be resolved by coordinate change, we refer to Section 3.1 of [9] for more detailed discussion. Then we denote the rescaled dual metric by

$$p := -(r^2 + a^2 \cos^2 \theta)G,$$

(5.3)
which plays the role of $p$ in our propagation estimates. And other notations in propagation estimates are inherited as well.

5.2 Defining function of the unstable and stable manifolds

In this section, we characterize the trapping phenomena in Kerr-de Sitter spacetime using Theorem 3.2 of [23] restated as Proposition 5.1.

**Proposition 5.1.** For $(\xi_t, \xi_\phi) \in \mathbb{R}^2 \setminus \{(0, 0)\}$, define:

$$F_{\xi_t, \xi_\phi}(r) := \frac{1}{\Delta(r)}(r^2 + a^2)\xi_t + a\xi_\phi)^2.$$  

1. Then either:
   - $F$ vanishes at $r_-$ or $r_+$ and has no critical point in $(r_-, r_+)$.
   - $F$ has exactly one critical point $r_{\xi_t, \xi_\phi} \in (r_-, r_+)$ and $F''(r_{\xi_t, \xi_\phi}) > 0$.

2. $F_{\xi_t, \xi_\phi} > 0$ on $\Sigma$.

3. The trapped set in $M$ is:
   
   $$\Gamma := \bigcup_{(\xi_t, \xi_\phi) \in \mathbb{R}^2 \setminus \{(0, 0)\}} \Gamma_{\xi_t, \xi_\phi},$$
   
   where $\Gamma_{\xi_t, \xi_\phi} := \{\xi_r = r - r_{\xi_t, \xi_\phi} = p = 0\}$.  

4. $\Gamma$ is a smooth connected 5-dimensional submanifold of $T^*M$ with defining function $\xi_r, r - r_{\xi_t, \xi_\phi}, p$.

5. The trapping of the flow of $H := \frac{1}{\mathcal{H}_p}H_p$ in any subextremal Kerr-de Sitter spacetime is eventually absolutely r-normally hyperbolic for every $r$ in the sense of [34]. The unstable(u) and stable(s) manifolds are smooth manifold given by:

$$\Gamma^{u/s} := \{\varphi^{u/s} = 0\} \cap \Sigma,$$

where $\varphi^{u/s} = \xi_r \mp \text{sgn}(r - r_{\xi_t, \xi_\phi})(1 + \hat{\alpha})\sqrt{\frac{F_{\xi_t, \xi_\phi}(r) - F_{\xi_t, \xi_\phi}(r_{\xi_t, \xi_\phi})}{\Delta(r)}}$ and $\hat{\alpha} = \frac{a^2}{3}$.  

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Remark 6. Coupling with $\text{sgn}(r - r_{\xi_t, \xi_\phi})$, the square root function in the definition of $\Gamma_{u/s}$ is smooth because at $r_{\xi_t, \xi_\phi}$, $F'_{\xi_t, \xi_\phi}(r) = 0$ and $F_{\xi_t, \xi_\phi}(r) - F_{\xi_t, \xi_\phi}(r_{\xi_t, \xi_\phi})$ vanishes quadratically at $r_{\xi_t, \xi_\phi}$.

Next we verify that the rescaled version of $\varphi_{u/s}$ above are exactly the defining functions that characterize the normally hyperbolic trapping properties, i.e. they satisfy (3.5). Define $\hat{\varphi}^u := \xi_t^{-1} \varphi^u$, $\hat{\varphi}^s := \varphi^s$.

**Proposition 5.2.** $\hat{\varphi}_{u/s}$ defined in Proposition 5.1 satisfy (3.5) with $\varphi_{u/s}$ replaced by $\hat{\varphi}_{u/s}$.

**Proof.** Since $H_p\xi_t = 0$, thus any factor as a function of $\xi_t$ commute with $H_p$ and we consider $\varphi_{u/s}$ instead, i.e., ignoring the $\xi_t$ power in front of $\cdot$.

$$p(x, \xi) = -(r^2 + a^2 \cos^2 \theta)G(x, \xi)$$
$$= \Delta(r)\xi^2_r + (1 + \hat{\alpha} \cos^2 \theta)\xi^2_\theta + \frac{(1 + \hat{\alpha})^2}{(1 + \hat{\alpha} \cos^2 \theta) \sin^2 \theta} (a\xi_t \sin^2 \theta + \xi_\phi)^2$$
$$- \frac{(1 + \hat{\alpha})^2}{\Delta(r)} ((r^2 + a^2)\xi_t + a\xi_\phi)^2.$$

Thus

$$H_p\xi_t = H_p\xi_\phi = H_p r_{\xi_t, \xi_\phi} = 0,$$
$$H_p\varphi_{u/s} = H_p\varphi_r \pm \text{sgn}(r - r_{\xi_t, \xi_\phi})(1 + \hat{\alpha})H_p \sqrt{\frac{F_{\xi_t, \xi_\phi}(r) - F_{\xi_t, \xi_\phi}(r_{\xi_t, \xi_\phi})}{\Delta(r)}},$$
$$H_p r = 2\Delta(r)\xi_r,$$
$$H_p\xi_r = -\Delta'(r)\xi^2_r + (1 + \hat{\alpha})^2 F'(r),$$
$$H_p\sqrt{\frac{F_{\xi_t, \xi_\phi}(r) - F_{\xi_t, \xi_\phi}(r_{\xi_t, \xi_\phi})}{\Delta(r)}} = \frac{1}{2} (F'_{\xi_t, \xi_\phi}(r)((F_{\xi_t, \xi_\phi}(r) - F_{\xi_t, \xi_\phi}(r_{\xi_t, \xi_\phi}))\Delta(r))^{-1/2}$$
$$- (F_{\xi_t, \xi_\phi}(r) - F_{\xi_t, \xi_\phi}(r_{\xi_t, \xi_\phi}))^{1/2}\Delta(r)^{-3/2}\Delta'(r))H_p r,$$
$$H_p\varphi_{u/s} = H_p\xi_r \pm \text{sgn}(r - r_{\xi_t, \xi_\phi})(1 + \hat{\alpha})H_p \sqrt{\frac{F_{\xi_t, \xi_\phi}(r) - F_{\xi_t, \xi_\phi}(r_{\xi_t, \xi_\phi})}{\Delta(r)}}.$$
\[-\Delta(r)\xi^2 + (1 + \hat{\alpha})^2 F'(r) \mp \text{sgn}(r - r_{\xi_t, \xi}) (1 + \hat{\alpha}) \times \frac{1}{2} (F'_{\xi_t, \xi}(r)(F_{\xi_t, \xi}(r) - F_{\xi_t, \xi}(r_{\xi_t, \xi})))
- F_{\xi_t, \xi}(r_{\xi_t, \xi})) \Delta(r))^{-1/2} - (F_{\xi_t, \xi}(r) - F_{\xi_t, \xi}(r_{\xi_t, \xi}))^{1/2} \Delta(r)^{-3/2} \Delta'(r) H_{\mu r}
\]
\[= \Delta(r)\xi^2 + (1 + \hat{\alpha})^2 F'(r) \mp \text{sgn}(r - r_{\xi_t, \xi}) (1 + \hat{\alpha}) (F'_{\xi_t, \xi}(r)(F_{\xi_t, \xi}(r) - F_{\xi_t, \xi}(r_{\xi_t, \xi}))
\Delta(r))^{-1/2} - (F_{\xi_t, \xi}(r) - F_{\xi_t, \xi}(r_{\xi_t, \xi}))^{1/2} \Delta(r)^{-3/2} \Delta'(r) \Delta(r) \xi_r
\]
\[= (\xi_r \mp \text{sgn}(r - r_{\xi_t, \xi}) (1 + \hat{\alpha}) S_{\xi_t, \xi}(r)) \Delta'(r) \xi_r \mp \text{sgn}(r - r_{\xi_t, \xi})(1 + \hat{\alpha}) F'(r) r_{\xi_t, \xi}(r).
\]
where
\[S_{\xi_t, \xi}(r) = \text{sgn}(r - r_{\xi_t, \xi}) \sqrt{\frac{F_{\xi_t, \xi}(r) - F_{\xi_t, \xi}(r_{\xi_t, \xi})}{\Delta(r)}}.
\]
which is a monotonic increasing smooth function with inverse function $S_{\xi_t, \xi}^{-1}$
when we restrict $r$ close enough to $r_{\xi_t, \xi}$. Next we show that
\[
\frac{(1 + \hat{\alpha}) F'(r)}{S_{\xi_t, \xi}(r)}
\]
is lower bounded. By the characterization of $F_{\xi_t, \xi}$ in Proposition 5.1, we can set
\[F'_{\xi_t, \xi}(r) = (r - r_{\xi_t, \xi}) \tilde{f}_{\xi_t, \xi}(r),
\]
where $c_f(\xi^2 + \xi^2) \leq f_{\xi_t, \xi}(r) \leq C_f(\xi^2 + \xi^2)$ with $c_f, C_f > 0$ when $r$ is close to $r_{\xi_t, \xi}$. The upper bound is easy to show. On the other hand, $c_f$ is obtained by considering the homogeneous degree of $F_{\xi_t, \xi}$ and hence we can restrict $(\xi_t, \xi)$ to a sphere, which is compact. If such $c_f$ does not exist, then by compactness argument we can find $(\xi_t, \xi) \in S^2$ such that $f_{\xi_t, \xi}(r_{\xi_t, \xi}) = 0$, contradicting the simplicity of this critical point. Similarly for the argument about $\tilde{f}_{\xi_t, \xi}$ below. Since $r_{\xi_t, \xi}$ is a critical point of $F_{\xi_t, \xi}$ with $F''(r_{\xi_t, \xi}) > 0$, then we can assume
\[F_{\xi_t, \xi}(r) - F_{\xi_t, \xi}(r_{\xi_t, \xi}) = (r - r_{\xi_t, \xi})^2 \tilde{f}_{\xi_t, \xi}(r),
\]
with $c_f(\xi^2 + \xi^2) \leq \tilde{f}_{\xi_t, \xi}(r) \leq C_f(\xi^2 + \xi^2)$ near $r_{\xi_t, \xi}$. Consequently
\[S_{\xi_t, \xi}(r) = (r - r_{\xi_t, \xi}) \tilde{f}_{\xi_t, \xi}(r) \Delta'(r)^{-1/2}.
\]
(5.6)
Thus

\[
\frac{(1 + \hat{\alpha})F'(r)}{S_{\xi_t, \xi_\varphi}(r)} = (1 + \hat{\alpha})f_{\xi_t, \xi_\varphi}(r)\tilde{f}_{\xi_t, \xi_\varphi}^{-1/2}\Delta(r)^{1/2}
\]

is lower bounded by a positive constant multiple of \((\xi_t^2 + \xi_\varphi^2)^{1/2}\) near \(r = r_{\xi_t, \xi_\varphi}\). In particular, using the characterization of \(\Gamma\) in Proposition 5.1, we can choose a neighborhood of \(\Gamma\) on which \(\xi_r\) is small. Thus the sign of \((-\Delta'_{\xi_t, \xi_\varphi}(r) \xi_r \pm (1 + \hat{\alpha})F'(r))\) is \(\pm\) for \(\varphi_u/\) respectively and \((-\Delta'_{\xi_t, \xi_\varphi}(r) \xi_r \pm (1 + \hat{\alpha})F'(r))\) is a multiple of \((\xi_t^2 + \xi_\varphi^2)^{1/2}\) bounded away from 0, which verifies (3.5) after rescaling by \((\xi_t^2 + \xi_\varphi^2)^{-1/2}\).

5.3 Construction of the symplectomorphism

Define our ‘model manifold’ to be

\[
\tilde{M}^0 := \mathbb{R}_{x_1} \times S^2 \times \mathbb{R}_{x_4}.
\]

(5.7)

Set \(\tilde{X} := \mathbb{R}_{x_1} \times S^2\) and use \(x_2 \in [0, \pi], x_3 \in S^1\) as spherical coordinates. Similar to how we obtain \(M\) from \(M^0\), we define \(\tilde{M}\) to be \(\tilde{M}^0\) compactified by attaching the hypersurface \(\{x_4 = \infty\}\). Formally, using \(x = (x_1, x_2, x_3, x_4)\) as coordinates on \(\tilde{M}^0\), we set

\[
\tilde{M} := (\tilde{M}^0 \sqcup (\tilde{X} \times [0, \infty)_{\tilde{\tau}})) / \sim,
\]

where \(\sim\) is the identification: \((x_1, x_2, x_3, x_4) \sim (x_1, x_2, x_3, \tilde{\tau} = x_4^{-1})\). And \(T^*\tilde{M}\) is equipped with the natural symplectic structure, extending the one on \(T^*\tilde{M}^0\).

In this section, we prove that there exists a homogeneous symplectomorphism \(\text{Sp}\) mapping \(\mathcal{U}\), a conic neighborhood of \(\Gamma\) to a conic set in \(T^*\tilde{M}\). In particular, \(\text{Sp}\) sends \(\Gamma^u\) to \(\{x_1 = 0\}\). We begin by stating our main result.

**Theorem 3.** There exists a homogeneous symplectomorphism \(\text{Sp} : \mathcal{U} \to T^*\tilde{M}\), such that \(\text{Sp}(\Gamma^u \cap \mathcal{U}) = \{x_1 = 0\} \cap \text{Sp}(\mathcal{U}), \text{Sp}(\Gamma^s \cap \mathcal{U}) = \{\xi_1 = 0\} \cap \text{Sp}(\mathcal{U})\). In addition, \(\text{Sp}\) preserves the time infinity in the sense that \(\text{Sp}(\{\tau = 0\}) \subset \{\tilde{\tau} = 0\}\).

**Proof.** We define \(\text{Sp}\) on the coordinate patch \(\mathbb{R}_t \times X\) and then show that \(|t - x_4|\) is bounded, so that \(t\)-infinity and \(x_4\)-infinity coincide and taking \(t \to \infty\) gives the extension of \(\text{Sp}\) down to \(\{\tau = 0\} \subset T^*M\). For
\( z_0 = (\tau = t^{-1}, r, \varphi, \theta) \in \mathcal{U} \), we define \( \text{Sp}(z_0) \) by defining its components \((x_1, x_2, x_3, x_4, \xi_1, \xi_2, \xi_3, \xi_4)\). \( \text{Sp} \) is a homogeneous symplectomorphism if and only if

\[
\begin{align*}
\{ \text{Sp}^*(x_i), \text{Sp}^*(x_i) \} &= \delta_{ij}, \\ 
\{ \text{Sp}^*(\xi_i), \text{Sp}^*(\xi_j) \} &= \delta_{ij}, \\ 
\{ \text{Sp}^*(x_i), \text{Sp}^*(\xi_j) \} &= 0,
\end{align*}
\]

(5.8)

where the Poisson brackets are with respect to the natural symplectic structure on \( T^*M \) and \( G_{t_d} \), respectively. \( \tilde{G}_{t_d} \) are dilating fiber parts by \( t_d \in \mathbb{R} \) on \( T^*M \), respectively. In discussion below, we use \( x_i, \xi_j \) to denote their pull-back by \( \text{Sp} \) when there is no confusion. Define \( x_1 \) by

\[
x_1 := \varphi^u = \xi_t^{-1}(\xi_r - \text{sgn}(r - r_{\xi_t,\xi_\varphi}) \sqrt{F_{\xi_t,\xi_\varphi}(r) - F_{\xi_t,\xi_\varphi}(r_{\xi_t,\xi_\varphi})} / \Delta(r)},
\]

which is the normalized defining function of \( \Gamma^u \).

Since \( H_{\tilde{\varphi}^u} > 0 \) in a neighborhood \( \mathcal{U} \) of \( \Gamma \), we know that starting from each point \( p = (t, r, \theta, \varphi, \xi_t, \xi_\varphi) \) in \( \mathcal{U} \), the \( H_{\tilde{\varphi}^u} \)-flow has exactly one intersection with \( \Gamma^u \) and denote it by \( p_0 = (t_0, r_0, \theta, \varphi_0, \xi_t, \xi_\theta, \xi_\varphi) \), where we used the fact that \( \xi_t, \xi_\theta, \xi_\varphi, \theta \) are constants along \( H_{\tilde{\varphi}^u} \)-flow. Define \( \xi_1 \) to be the time need for traveling from \( p \) to \( p_0 \):

\[
\xi_1 := T^u(p) = \frac{r - r_0}{H_{\tilde{\varphi}^u}r} = \xi_t(r - r_0).
\]

Since \( \tilde{\varphi}^u \) is preserved by \( H_{\tilde{\varphi}^u} \)-flow, we know

\[
\xi_{r_0} - \xi_r = (1 + \alpha)S_{\xi_t,\xi_\varphi}(r_0) - (1 + \alpha)S_{\xi_t,\xi_\varphi}(r), \tag{5.9}
\]

Since \( p_0 \in \Gamma^u \), we know

\[
\xi_{r_0} = -(1 + \alpha)S_{\xi_t,\xi_\varphi}(r_0).
\]

Thus

\[
2\xi_{r_0} = \xi_r - (1 + \alpha)S_{\xi_t,\xi_\varphi}(r) = \tilde{\varphi}^u(p). \tag{5.10}
\]

Combining (5.9) and (5.10), we obtain

\[
S_{\xi_t,\xi_\varphi}(r_0) = \frac{\xi_{r_0} - \xi_r}{1 + \alpha} + S_{\xi_t,\xi_\varphi}(r)
\]

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Thus
\[
\xi_r - (1 + \hat{\alpha})S_{\xi_t,\xi_\varphi}(r) = \frac{\xi_r}{2(1 + \hat{\alpha})} + S_{\xi_t,\xi_\varphi}(r)
\]
\[= -\frac{\xi_r - (1 + \hat{\alpha})S_{\xi_t,\xi_\varphi}(r)}{2(1 + \hat{\alpha})}.
\]

Thus
\[
r_0 = S_{\xi_t,\xi_\varphi}^{-1}(-\frac{\tilde{\varphi}^u(p)}{2(1 + \hat{\alpha})}).
\]

Consequently
\[
T^s(p) = \xi_t(r - S_{\xi_t,\xi_\varphi}^{-1}(-\frac{\tilde{\varphi}^u(p)}{2(1 + \hat{\alpha})})).
\]

By considering dependence of \(x_1 = \hat{\varphi}^u\) and \(\xi_1 = T^s(p)\), we can choose
\[
x_2 = \theta, \xi_2 = \xi_\theta, \xi_3 = \xi_\varphi, \xi_4 = \xi_t.
\]

So \(\theta = 0, \pi\) are sent to \(x_2 = 0, \pi\) respectively. Those \(S^1_\varphi\) degenerate to a single point in \(M\) correspond to those \(S^1_{x_3}\) degenerate to a single point in \(\tilde{M}\), thus \(S_p\) is smooth and well defined on \(M\). Next we try to find \(X_3\) so that
\[
x_3 = \varphi + X_3(x, \xi)
\]
satisfies (5.8), where we are interpreting \(S^1_\varphi\) as \(\mathbb{R}_\varphi/2\pi\mathbb{Z}\) and \(x_3\) is also parametrizing \(S^1_{x_3}\), i.e. \(\mathbb{R}_{x_3}/2\pi\mathbb{Z}\). If two points have the same \(x_3\)-coordinates, but with different \(\varphi, X_3\) respectively, then their \(x_1 = \hat{\varphi}^u, \xi_1 = T^s\) coordinates are different as well because of (5.13) below. On the other hand, when we choose two different representatives of \(\varphi\) for the same point, the \(x_3\) components of their image, which is given by \(\varphi + X_3\), differ by a multiple of \(2\pi\), hence correspond to the same point in \(\mathbb{R}_{x_3}/2\pi\mathbb{Z}\). Thus this map is well-defined and injective. Using (5.8), \(\{\xi_\theta, X_3\} = \{\theta, X_3\} = \{\xi_t, X_3\} = 0\) implies that \(X_3\) is independent of \(\theta, \xi_\theta, t\). \(\{\xi_\varphi, \varphi + X_3\} = 1\) implies \(\{\xi_\varphi, X_3\}\), i.e. \(X_3\) is independent of \(\varphi\). By remaining equations: \(\{\hat{\varphi}^u, \varphi + X_3\} = 0, \{T^s, \varphi + X_3\} = 0\), the ODE system that \(X_3\) satisfies is
\[
\begin{align*}
H_{T^s}X_3 &= -\partial_{\xi_\varphi}T^s(p) \\
H_{\hat{\varphi}^u}X_3 &= -\partial_{\xi_\varphi}(\hat{\varphi}^u) \\
V_DX_3 &= 0,
\end{align*}
\]

(5.13)
where \( V_D = \xi_t \partial_{\xi_t} + \xi_\varphi \partial_{\xi_\varphi} + \xi_r \partial_{\xi_r} \), in which we neglected \( \xi_\theta \partial_{\xi_\theta} \) component since it is not involved in the defining function of \( \Gamma^{u/s} \) and hence not involved in our discussion.

Next we apply Corollary C.1.2 of [18]. Applying equation (21.1.6) of [18], we obtain

\[
\begin{align*}
[H_{\hat{\varphi}^u}, V_D] &= H_{\hat{\varphi}^u}, \\
[H_{T^s}, V_D] &= 0.
\end{align*}
\]

Combining with the fact that \( H_\theta, H_{\xi_\varphi}, H_{\hat{\varphi}^u}, H_{T^s}, H_{\xi_\theta} \) commute with each other, the condition (C.1.2) is satisfied. Next we verify (C.1.4) there:

\[
H_{T^s}(\partial_{\xi_\varphi} \hat{\varphi}^u) - H_{\hat{\varphi}^u}(\partial_{\xi_\varphi} T^s) = 0,
\]

which is equivalent to, using \( H_\varphi = -\partial_{\xi_\varphi} \),

\[
\partial_{\xi_\varphi} H_{T^s} \hat{\varphi}^u + [H_{T^s}, -H_\varphi] \hat{\varphi}^u = \partial_{\xi_\varphi} H_{\hat{\varphi}^u} T^s + [H_{\hat{\varphi}^u}, -H_\varphi] T^s.
\]

Since \( H_{\hat{\varphi}^u} T^s = 1 \) by the definition of \( T^s \), the first terms on both sides vanish, and this is equivalent to

\[
H_{\{T^s, \varphi\}} \hat{\varphi}^u + H_{\{\varphi, \hat{\varphi}^u\}} T^s = 0.
\]

Using Poisson brackets, this is

\[
\{\{T^s, \varphi\}, \hat{\varphi}^u\} + \{\{\varphi, \hat{\varphi}^u\}, T^s\} = 0.
\]

Since \( \{\{\hat{\varphi}^u, T^s\}, \varphi\} = \{1, \varphi\} = 0 \), above equation holds by Jacobi’s identity. Equations in (C.1.4) involving \( V_D \) are

\[
\begin{align*}
H_{\hat{\varphi}^u} 0 - V_D (-\partial_{\xi_\varphi} T^s) &= 0, \\
H_{\hat{\varphi}^u} 0 - V_D (-\partial_{\xi_\varphi} \hat{\varphi}^u) &= -\partial_{\xi_\varphi} \hat{\varphi}^u,
\end{align*}
\]

both of which follow from the fact that \( \partial_{\xi_\varphi} T^s \) and \( \partial_{\xi_\varphi} \varphi \) are homogeneous function of degree 0 and -1 respectively and Euler’s Theorem on homogeneous functions. Thus we can solve the ODE of \( X_3 \) with prescribed initial condition on \( \Theta_1 := \mathcal{U} \cap \{T^s = \hat{\varphi}^u = 0, t = t_1, \theta = \theta_1, \varphi = \varphi_1, \xi_\theta = \xi_{\theta_1}, |\xi| = 1\} \), where \( t_1, \xi_{\theta_1}, \varphi_1, \theta_1 \) are fixed constants and \( |\xi_{\theta_1}| < 1 \). By Proposition 5.1 \( T^s = \hat{\varphi}^u = 0 \) is equivalent to \( r = r_{\xi_\varphi}, \xi_r = 0 \), thus \( \Theta_1 \) is essentially parametrized by \( \xi_\varphi \). We set \( X_3 = 0 \) on this codimension 7 smooth submanifold.
Using the same method, we construct $X_4$ in $x_4 = t + X_4$ such that

$$
\begin{align*}
H_{T^s}X_4 &= -\partial_{\xi t}T^s(p) \\
H_{\hat{\varphi}^u}X_4 &= -\partial_{\xi \hat{\varphi}^u} \\
H_{x_3}X_4 &= -\partial_{\xi_3}x_3, \\
V_DX_4 &= 0,
\end{align*}
$$

where other commutation relations are already satisfied. Condition (C.1.2) and (C.1.4) in [18] are verified in the same manner. We take the first and the third equations as example. We need to verify

$$H_{x_3}\partial_{\xi t}T^s(p) - H_{T^s}\partial_{\xi_3}x_3 = 0.$$  

This is equivalent to

$$\{\{T^s, t\}, x_3\} + \{\{t, x_3\}, T^s\} = 0,$$

which follows from $\{\{x_3, T^s\}, t\} = \{0, T^s\} = 0$ and Jacobi’s identity. The rest conditions involving $V_D$ are verified in the same manner as when we solve $X_3$. And we can assign initial value of $X_4$ at $\Theta_2 := U \cap \{\xi_r = 0, r = r_{\xi, \xi_\varphi}, t = t_1, \theta = \theta_1, \varphi = \varphi_1, \xi_\theta = \xi_{\varphi 1}, \xi_\varphi = \xi_{\varphi 1}, |\xi| = 1\}$ with $|\xi_{\varphi 1}|^2 + |\xi_{\theta 1}|^2 < 1$, which is a single point. We set $X_4 = 0$ at $\Theta_2$. Finally we show that $X_4 = x_4 - t$ is bounded. Since $X_4$ is homogeneous of degree 0, we consider its value on the unit sphere bundle. Then the variables in the ODE from which we obtain $X_4$ are varying over a bounded region and the right hand sides of those ODEs are bounded as well, thus $X_4$ is bounded.

5.4 Quantize Sp

Then we apply Egorov’s theorem of conjugating pseudodifferential operators by multi-valued Fourier integral operators to reduce our claim to the model case. We use a finite open cover $\{U_j\}_{j \in J}$ of $U$ so that $U_i \cap U_j$ is contractible for $i, j \in J$ and $\{(\pi_M(U_j), \varphi_j)\}$ is an atlas of $M$. As a matter of fact, we use $J = \{1, 2\}$, which is possible for our $M$. Here we allow half spaces in the model of charts, so that the compactified $\mathbb{R}_t$ only need one chart to cover and the 2 charts are needed for the $S^2$ component.

We verify the condition needed to quantize $Sp|_{U_i}$ first. Concretely, we apply Theorem 10.1 of [12]. We need to verify that $G(Sp) \subset T^*M \times T^*\bar{M}$,

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the graph of \( \text{Sp} \), has a generating function in the sense of [12]. By Theorem 5.3 of [12], we need to verify that, the projection \( \pi_{\text{Sp}} \) from \( G(\text{Sp}) \) to \( M \times \mathbb{R}^4 \), i.e., the \((\tau, r, \theta, \varphi, \xi_1, \xi_2, \xi_3, \xi_4)\) part is a diffeomorphism. Although the result of Theorem 5.3 there is local, but the proof only relies on the fact that closed differential forms are locally exact, which is true on any contractible region, thus its proof gives a generating function that is ‘global’ on \( U \times T^*\tilde{M} \). Since \( \xi_2 = \xi_\theta, \xi_3 = \xi_\varphi, \xi_4 = \xi_t = -\xi_r \), and the projection from \( G(\text{Sp}) \) to \( T^*\tilde{M} \), i.e., the \((\tau, r, \theta, \varphi, \xi_t, \xi_r, \xi_\theta, \xi_\varphi)\) part is a diffeomorphism, we need to verify

\[
\frac{\partial \xi_1}{\partial \xi_r} \neq 0,
\]

which is sufficient for \( \pi_{\text{Sp}} \) to have full rank. This is straightforward by

\[
\xi_1 = \xi_t(r - S_{\xi_t,\xi_r}^{-1}(-\tilde{\delta}^\alpha(p))) \quad \tilde{\varphi}^\alpha = \xi_r - S_{\xi_t,\xi_r}(r) \quad (S_{\xi_t,\xi_r}^{-1})' \neq 0,
\]

where the last condition holds because by (5.6), when \( r \) is close to \( r_{\xi_t,\xi_r} \), \( S_{\xi_t,\xi_r}'(r) \) is continuous and positive, thus \( (S_{\xi_t,\xi_r}^{-1})' \) is continuous and positive by the inverse function theorem. Consequently, we can apply Theorem 10.1 of [12], which is stated for a small conic region, but its proof goes through on the region where the generating function is valid, hence on \( G(\text{Sp}) \subset U \times T^*\tilde{M} \).

We obtain a Fourier integral operator \( T_j : D'(\pi_M(U_j)) \to D'(\pi_{\tilde{M}}(\text{Sp}(U_j))) \)
quantizing \( \text{Sp}|_{U_j} \), being elliptic on and microlocally unitary on \( U_j \) in the sense that

\[
T_j^*T_j - \text{Id}_M \in \Psi^{-\infty,0}_c(M) \quad \text{on } U_j, \quad T_jT_j^* - \text{Id}_{\tilde{M}} \in \Psi^{-\infty,0}_c(\tilde{M}) \quad \text{on } \text{Sp}(U_j).
\]

The unitary property is not included in [12], but as stated before (2.7) of [17], we can achieve this by adding a smooth factor to the amplitude. In addition, for \( A \in \Psi^{m,r}_{cu,\alpha}(M) \) with principal symbol \( a \) and \( \text{WF}'(A) \subset U_j \), we know \( B := T_jA T_j^* \in \Psi^{m,r}_{cu,\alpha}(\tilde{M}) \). And it has the same symbol modulo \( S^{m-2,r}_{cu,\alpha}(M) \):

\[
b = (\text{Sp}^{-1})^*(a) = a \circ (\text{Sp}^{-1}) \mod S^{m-2,r}_{cu,\alpha}(M). \quad (5.14)
\]

Where the improved Egorov’s property, i.e., their symbol coincide up to the sub-principal level comes from the argument in Section 2 of [17]. Although the argument there is for semiclassical case, but the part starting from equation (2.6) there goes through for classical case as well. \( O(\hbar^\mu) \)—terms are understood as \( \mu \)—order lower symbol or operators in the classical case. And the factors \( \frac{1}{\hbar} \) in phases are not needed. The fact we use is that differences between oscillatory integrals in different patches are purely imaginary. Thus
when we go back to the same point following a closed loop, the product of all these factors, which is the obstruction to the compatibility, is a purely imaginary factor. This factor is cancelled when we apply both the Fourier integral operator and its adjoint.

This property is also discussed in [26], and the global version is mentioned in remark i) in Section 1.2. But the author there did not discuss the phase difference between different coordinate patches in detail.

Let \( \{ \chi_j \}_{j \in J} \) be a partition of unity subordinate to \( \{ \pi_M(U_j) \}_{j \in J} \), shrinking \( U_j \) if necessary, we can choose a family of Fourier integral operators \( T_j \) associated to \( \text{Sp} | U_j \) such that

1. \( T_j^* T_j - \text{Id}_M \in \Psi_{cu, \alpha}^{-\infty,0}(M) \) microlocally on \( U_j \), \( T_j^* T_j - \text{Id}_\tilde{M} \in \Psi_{cu, \alpha}^{-\infty,0}(\tilde{M}) \) microlocally on \( \text{Sp}(U_j) \). \( T_j \) is elliptic on \( U_j \).

2. For \( A \in \Psi_{cu, \alpha}^{m,r}(M) \) with principal symbol \( a \) and \( \text{WF}(a) \subset U_j \), \( B := T_j A T_j^* \in \Psi_{cu, \alpha}^{m,r}(\tilde{M}) \) has principal symbol

\[
\mathbf{b} = (\text{Sp}^{-1})^*(a) = a \circ (\text{Sp}^{-1}).
\] (5.15)

3. \( T_k^* T_j - c_{kj} e^{i\alpha_{kj}} \in \Psi_{cu, \alpha}^{-\infty,0}(M) \) on \( U_j \cap U_k \), where \( c_{kj} > 0, \alpha_{kj} \in \mathbb{R} \).

Define the global Fourier integral operator \( T \) by

\[
Tu = T_1 \chi_1 u + c_{21} e^{i\alpha_{21}} T_2 \chi_2 u, \quad u \in D'(M).
\]

By Property 3 above, two terms here are microlocally equal on \( U_j \cap U_k \), and since \( T_j \) is elliptic on \( U_j \), \( T \) is elliptic on \( U \). Notice that \( T_1^* T_2 T_1^* T_2 T_1 = \text{Id}_M \) microlocally on \( U_1 \cap U_2 \), thus we know

\[
c_{12} = c_{21}^{-1}, \alpha_{12} = -\alpha_{21}.
\]

Direct computation shows

\[
T^* T - \text{Id}_M \in \Psi_{cu, \alpha}^{-\infty,0}(M) \text{ on } U, \quad TT^* - \text{Id}_\tilde{M} \in \Psi_{cu, \alpha}^{-\infty,0}(\tilde{M}) \text{ on } \text{Sp}(U).
\]

Then we consider

\[
\tilde{P} := \sum_{j \in J} T_j (\chi_j P) T_j^*.
\] (5.16)
Microlocally

\[ \tilde{P}Tu = (T_1\chi_1PT_1^* + T_2\chi_2PT_2^*)(T_1\chi_1u + c_{21}e^{i\alpha_{21}T_2\chi_2})u \\
= (T_1\chi_1P\chi_1 + T_2\chi_2Pc_{21}e^{i\alpha_{21}}\chi_1 + T_1\chi_1PT_1^*c_{21}e^{i\alpha_{21}T_2\chi_2} + c_{21}e^{i\alpha_{21}T_2\chi_2}P\chi_2)u + Ru. \]

Since the third term has both \( \chi_1, \chi_2 \) factor, and on \( U_1 \cap U_2 \) we have \( T_1^*T_2 = c_{12}e^{i\alpha_{12}} + R \). So

\[ T_1\chi_1PT_1^*c_{21}e^{i\alpha_{21}T_2\chi_2}u = T_1\chi_1Pc_{21}c_{12}e^{i(\alpha_{21} + \alpha_{12})}\chi_2u = T_1\chi_1P\chi_2u. \]

We have

\[ \tilde{P}Tu = (T_1\chi_1P\chi_1 + T_2\chi_2Pc_{21}e^{i\alpha_{21}}\chi_1 + T_1\chi_1P\chi_2 + c_{21}e^{i\alpha_{21}T_2\chi_2}P\chi_2)u + Ru \\
= (T_1\chi_1 + c_{21}e^{i\alpha_{21}})Pu + Ru \\
= TPu + Ru. \] (5.17)

\( R \) representing the microlocal errors in different steps may represent different operators, but all of them are infinitely smoothing operators and do not affect estimates.

### 5.5 Properties of the conjugated operator

In this part we verify that \( \tilde{P} \) fit into the framework in Section 2 to Section 4. In particular, the microlocal estimate in Section 4 holds with \( P \) replaced by \( \tilde{P} \) and \( v \) replaced by \( Tv \). First we show that \( \tilde{P} \) satisfy the compensable property.

**Proposition 5.3.** \( q := p \circ (Sp^{-1}) \) satisfies the compensable property in the sense of Definition 6.

**Proof.** Direct computation shows

\[ p = \Delta(r)(\xi_t^{-1}\varphi^u)(\xi_t\varphi^s) - F_{\xi_t,\xi_\nu}(r_{\xi_t,\xi_\nu}) + (r^2 + a^2\cos^2\theta)G_\theta, \] (5.18)

where \( G_\theta \) is defined in (5.2). Now define \( p_a := -F_{\xi_t,\xi_\nu}(r_{\xi_t,\xi_\nu}) + (r^2 + a^2\cos^2\theta)G_\theta \), then we show that

\[ \partial_{\xi_\nu}(p_a \circ (Sp)^{-1}) = 0. \]
This is equivalent to \( \{ x_1, p_a \circ (Sp)^{-1} \} = 0 \). Since \( Sp \) is a symplectomorphism, this is equivalent to
\[
\{ \xi_t^{-1}(\xi_r - S_{\xi_t, \xi_r}(r)), p_a \} = 0. \tag{5.19}
\]
Considering which variables are involved in \( \xi_t^{-1}(\xi_r - S_{\xi_t, \xi_r}(r)) \), we know that \( H_{\xi_t^{-1}(\xi_r - S_{\xi_t, \xi_r}(r))} \) is a linear combination of \( \partial_\xi, \partial_r, \partial_{\xi_r}, \partial_\varphi \) with smooth function coefficients, thus \( \text{(5.19)} \) holds.

Next we prove that weighted Sobolev spaces are preserved under \( T \).

**Proposition 5.4.** Suppose \( u \in \tau^\mu H_{cu}^s(M) \) and \( \supp u \subset \pi_M(U) \), then \( Tu \in \tilde{\tau}^\mu H_{cu}^s(\tilde{M}) \). And conversely if \( Tu \in \tilde{\tau}^\mu H_{cu}^s(\tilde{M}) \), then \( u \in \tau^\mu H_{cu}^s(M) \) and
\[
||\tilde{\tau}^{-\mu}Tu||_{H_{cu}^s(\tilde{M})} \lesssim ||\tau^{-\mu}u||_{H_{cu}^s(M)},
\]
\[
||\tau^{-\mu}u||_{H_{cu}^s(M)} \lesssim ||\tilde{\tau}^{-\mu}Tu||_{H_{cu}^s(\tilde{M})} + ||\tau^{-\mu}u||_{H_{cu}^s(M)}. \tag{5.20}
\]

**Proof.** By the definition of \( T \), we only need to prove the same estimates for \( T_j, j = 1, 2 \) respectively. In this proof we denote the coodinantes on \( U_j \) by \((y, \eta) = (t_y, r_y, \varphi_y, \theta_y, \eta_r, \eta_\varphi, \eta_\theta) \) and the coordinates on \( Sp(U_j) \) by \((x, \xi) = (x_1, x_2, x_3, x_4 = \tilde{\tau}^{-1}, \xi_1, \xi_2, \xi_3, \xi_4) \), which is valid down to \( \tau = 0 \) when we consider oscillatory integral expression of \( T_j \). First suppose that \( u \in \tau^\mu H_{cu}^s(M) \) and \( \supp u \subset \pi_M(U_j) \) and we need to show \( T_j u \in \tilde{\tau}^\mu H_{cu}^s(\tilde{M}) \).

By the remark at the end of Section 9 of [12], the generating function is
\[
\varphi(x, \eta) = y \cdot \eta |_{G(Sp)} = \pi_y(\pi_{x, y}^{-1}(x, \eta)) \cdot \eta.
\]
By the argument in [12], this is valid on the entire \( G(Sp) \) since we have shown that the projection from \( G(Sp) \) to \((x, \eta)\) is a diffeomorphism. \( \pi_y(\pi_{x, y}^{-1}(x, \eta)) \) is the \( y \)-component of the point on \( G(Sp) \) parametrized by \((x, \eta)\), which is different, actually independent of, the \( y \) written in the oscillatory integral. \( T_j \) can be written as
\[
T_j u(x) = \int \int e^{i \pi_y(\pi_{x, y}^{-1}(x, \eta)) - y \cdot \eta} a(x, y, \eta) u(y) dy d\eta.
\]
We choose \( a(x, y, \eta) \) to be smooth and uniformly bounded. Denote \( \pi_y(\pi_{x, y}^{-1}(x, \eta)) \) by \( z(x, \eta) = (t_z, r_z, \varphi_z, \theta_z) \), then
\[
T_j u(x) = \int \int e^{i z(x, \eta) - y \cdot \eta} a(x, y, \eta) u(y) dy d\eta. \tag{5.21}
\]

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In the discussion below, subindex $X$ means function spaces or variables coresponds to variables on $X$ and their dual variables. For example, $y_X = (r_X, \varphi_Y, \theta_y)$. Define the Sobolev space $H_{cu}^{s_1, t} H^{s_2}_{cu, X}(M)$ as: $u \in H_{cu}^{s_1, t} H^{s_2}_{cu, X}(M)$ if and only if
\[
\langle \eta \rangle^{s_1} \langle \eta_X \rangle^{s_2} \hat{u}(\eta) \in L^2(\mathbb{R}^4),
\]
where $\eta_X = (\eta_\theta, \eta_\varphi, \eta_r)$. Now suppose $t^\mu u(y) \in H^s_{cu}(M)$, which is equivalent to
\[
|t|^{\mu \alpha_1} D^{\alpha_2}_t u \in L^2_{cu}(M),
\]
for all $\alpha_1 \in \mathbb{R}, \alpha_2 \in \mathbb{R}^{n-1}$ such that $\alpha_1$ and components of $\alpha_2$ have the same sign as $s$ and sum to be $s$. $D^{\alpha_1}_t, D^{\alpha_2}_X$ are understood as the left quantizations of $\eta^{\alpha_1}_t, \eta^{\alpha_2}_X$. Thus we can take partial Fourier transform in $t$ to obtain
\[
|\eta|^{\alpha_1} D^{\alpha_2}_X \mathcal{F}_t u \in H^\mu_{\eta} L^2_{cu, X}(\mathbb{R} \times X).
\]
Integrate against $t_y$ in (5.21) first, we obtain
\[
T_j u(x) = \int \int e^{i(z(x, \eta) - y_X, \eta_X)} \mathcal{F}_t(au)(x, \eta_t, y_X, \eta) d\eta dX d\eta_X
\]
\[
= \int \int e^{it \eta_t} e^{-iX_4(z(x, \eta)) \eta_t} e^{i(z_X - y_X) \cdot \eta_X} \mathcal{F}_t(au)(x, \eta_t, y_X, \eta) d\eta dX d\eta_X.
\]
By the construction process of $X_4$, it is independent of $t_x$, hence we can interchange $e^{-iX_4(z(x, \eta)) \eta_t} e^{i(z_X - y_X) \cdot \eta_X}$ and $\mathcal{F}_t$, and multiplying $e^{-iX_4(z(x, \eta)) \eta_t} e^{i(z_X - y_X) \cdot \eta_X}$ does not affect both smoothness and integrability of $au$, thus
\[
|\eta|^{\alpha_1} D^{\alpha_2}_X \mathcal{F}_t(e^{-iX_4(z(x, \eta)) \eta_t} e^{i(z_X - y_X) \cdot \eta_X} u) \in H^\mu_{\eta} L^2_{X}(\mathbb{R} \times X).
\]
The integration against $\eta_t$ is a inverse Fourier transform evaluated at $t_x$, and by (5.24) we know
\[
\langle t_x \rangle^{\mu \nu} T_j u(x) \in H^s_{cu}(\tilde{M}),
\]
which completes the proof of the first claim. Notice that our procedure above is using equaivalences of norms between different Sobolev spaces, hence we have the first inequality of (5.20):
\[
\|\tau^{-\mu} T u\|_{H^s_{cu}(\tilde{M})} \lesssim \|\tau^{-\mu} u\|_{H^s_{cu}(M)}.
\]
Conversely, the same argument applies to $T^*$, which quantizes $\text{Sp}^{-1}$, and notice that $T^*T = \text{Id}_M + R$, with $R \in \Psi^{-\infty,0}_{\text{cu},\alpha}(M)$, we obtain the second inequality of (5.20).

Since the symplectomorphism preserves the symplectic form, it preserves equations using Hamiltonian vector fields. Hence normally hyperbolic trapping property is preserved under the symplectormorphism. Define $\tilde{\Gamma}^u = \text{Sp}(\Gamma^u), \tilde{\Gamma}^s = \text{Sp}(\Gamma^s), \tilde{H} := \text{Sp}_s H$, where $\Gamma^u/s, H$ are defined in Proposition 5.1 and the Kerr-de Sitter spacetime is assume to be subextremal. We have

**Proposition 5.5.** The trapping of the flow of $\tilde{H}$ is eventually absolutely absolutely $r$-normally hyperbolic for every $r$ in the sense of [34]. The unstable and stable manifolds are $\tilde{\Gamma}^u, \tilde{\Gamma}^s$ respectively, and they have defining function $x_1, (\text{Sp}^{-1})^* \phi^s$ respectively, which satisfy (3.5) with $\phi^u, \phi^s$ replaced by $x_1, (\text{Sp}^{-1})^* \phi^s$ respectively.

Next we prove that wave front sets of functions and operators are preserved under the action of $\text{Sp}$ and its quantization.

**Proposition 5.6.**

$$\text{WF}^s(Tv) = \text{Sp}(\text{WF}^s(v)) \quad (5.25)$$

**Proof.** We consider the complement of $\text{WF}^s(v)$. Suppose $x_0 \notin \text{WF}^s(v)$, then there is an $A \in \Psi^{0,0}_{\text{cu},\alpha}(M)$ such that $Au \in H^s_{cu}(M)$ and $A$ is elliptic at $x_0$. Since $T^*T = \text{Id}_M$ micrilocally, this is equivalent to the existence of $B \in \Psi^{0,0}_{\text{cu},\alpha}(M)$ such that $BTv \in H^s_{cu}(M)$ with $B = TaT^*$, which is elliptic at $\text{Sp}(x_0)$. This shows $\text{Sp}((\text{WF}^s(v))^c) \subset (\text{WF}^s(Tv))^s$. And the converse also holds, which implies $\text{WF}^s(Tv) = \text{Sp}(\text{WF}^s(v))$. \qed

Proposition 5.4 and (5.17) imply that, the estimate (3.11) in which $\tilde{P}$ and $Tv$ play the role of $P$ and $v$ implies the same estiamte for $P$ and $v$:

**Theorem 4.** There exists $B \in \Psi^{0,0,r}_{\text{cu},\alpha}(\tilde{M}, \tilde{\Gamma}^u)$ which is elliptic on $\tilde{\Gamma}$ and the front face in blown-up $T^*\tilde{M}$, $B_1, G_0 \in \Psi^{0,0,r}_{\text{cu},\alpha}(\tilde{M}, \tilde{\Gamma}^u)$ and $\text{WF}^r_{\text{cu},\alpha}(B_1) \cap \tilde{\Gamma}^u = \emptyset, \text{WF}^r_{\text{cu},\alpha}(G_0) \cap \tilde{\Gamma}^u = \emptyset$, and $\lambda, c$ satisfy (3.9), such that for $s, N \in \mathbb{R}, 0 < \alpha < \frac{1}{2}, \lambda, c$ satisfying (3.9), we have:

$$||BTv||_{s,s,r} \lesssim ||B_1Tv||_{s+1-\lambda, s, r} + c^{-1}||G_0TPv||_{s-m+2-\lambda, s-m+2-\alpha, r} + ||Tv||_{-N,-N,r}. \quad (5.26)$$
Proof. Applying (3.11) with \(\tilde{P},Tv\) being \(P,v\) there, we obtain
\[
||BTv||_{s,s,r} \lesssim ||B_1Tv||_{s+1-\lambda, s,r} + c^{-1}||G_0\tilde{P}Tv||_{s-m+2-\lambda, s-m+2-\alpha, r} + ||Tv||_{N,-N,r},
\]
where \(B \in \Psi_{0,\alpha}^{0,0,r}(\tilde{M},\tilde{\Gamma}^u)\) which is elliptic near \(\tilde{\Gamma} = \tilde{\Gamma}^u \cap \tilde{\Gamma}^s \cap \{\tilde{r} = 0\}\) and the front face. \(B_1,G_0 \in \Psi_{0,\alpha}^{0,0,r}(\tilde{M},\tilde{\Gamma}^u)\) and WF\(_{\alpha,u}^r(B_1) \cap \tilde{\Gamma}^u = \emptyset\), WF\(_{\alpha,u}^r(G_0) \cap \tilde{\Gamma}^u = \emptyset\), and \(\lambda, c\) satisfy (3.9), such that for \(s,N \in \mathbb{R}\), \(0 < \alpha < \frac{1}{2}\), \(\lambda, c\) satisfying (3.9) with sup and inf being that of subprincipal part of \(\tilde{P}\) over \(\text{Sp}(M_{0,\Sigma^1} \cap \mathcal{U}_{2\eta})\). Then we apply (5.17) to obtain
\[
||BTv||_{s,s,r} \lesssim ||B_1Tv||_{s+1-\lambda, s,r} + c^{-1}||G_0\tilde{P}Tv||_{s-m+2-\lambda, s-m+2-\alpha, r} + ||Tv||_{N,-N,r},
\]
(5.27)

Away from the trapped set, we apply the real principal type propagation. Combining this, proposition 5.6 and the microlocal estimate in Theorem 4, we obtain Theorem 1.

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