A curious relationship between Potts glass models

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Abstract. A Potts glass model, which is proposed by Nishimori and Stephen, is analyzed by means of the replica mean field theory. This model has a gauge symmetry, is a discrete model, and is also called the Potts gauge glass model. By comparing the present results with the results of the conventional Potts glass model, we find coincidences and differences for properties. We find a coincidence that the property for the Potts glass phase in this model is coincident with that in the conventional model at the mean field level. We find a difference that, unlike in the case of the conventional p-state Potts glass model, this system for large p does not become ferromagnetic at low temperature under a concentration of ferromagnetic interaction. The present result supports the act of numerically investigating the present model for study of the Potts glass phase in finite dimensions.

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1 Introduction

The reasons for the slowing down of dynamics of supercooled liquids and the glass transition of the supercooled liquids to amorphous solids are the biggest unsolved problems in the condensed matter physics. There is the Potts glass model as one of abstract models for these problems. There is the Potts glass model as one of abstract models for these problems. A model proposed in Ref. by Nishimori and Stephen is treated in this article, and is also called the Potts gauge glass model. In Ref. two models, in which types of exchange interactions are different, are proposed. One is an extended model of the Sherrington-Kirkpatrick (SK) model that an Ising model has exchange interactions of a Gaussian type. In this article, we call this extended model the GPGG model. There are analysis results of the GPGG model by means of the replica mean field (REPMF) theory in Refs. Another is an extended model of the bimodal model that an Ising model has exchange interactions of a bimodal type. In this article, we call this extended model the present model. The present model is a discrete model. The present model has not been analyzed by means of the REPMF theory. Also, by comparing the present results with the results of the conventional Potts glass model, it is found that there are coincidences and differences for the properties. The conventional Potts glass model is relatively-well studied by means of the REPMF theory. In Ref., it is pointed out that the mean field solution for the GPGG model is coincident with the mean field solution for the conventional Potts glass model when the magnetization vanishes. In this model, also, it is shown that the mean field solution for the present model is coincident with the mean field solution for the conventional Potts glass model when the magnetization vanishes.
when one performs numerical analyses for investigation of the Potts glass phase in finite dimensions \[31,32,33,34,35,36\], since the distribution of the interaction making the system no ferromagnetic in finite dimensions is nontrivial. For this problem, as done in Refs. \[31,32\], there is a method of confirming the conventional Potts glass model in finite dimensions with an unbiased distribution of exchange interaction for \(p > 2\). Also, as done in Refs. \[33,34,35,36\], there are methods of using antiferromagnetically biased distributions roughly estimated from the mean field solution in order to prevent that the system becomes ferromagnetic and to investigate the Potts glass phase in finite dimensions. On the other hand, in order to investigate the Potts glass phase in finite dimensions, there is a method of studying the present model (or the glassy Potts model mentioned below) instead of the conventional model. In this article, it is discussed that the present model does not have the above-mentioned problem.

A Potts glass model, which the magnetization does not appear, is proposed by Marinari, Mossa and Parisi \[9\]. This model is called the glassy Potts model. In the glassy Potts model, the system does not become ferromagnetic due to the symmetry of spins \[9\]. In Ref. \[21\], it is argued that the Potts gauge glass model \[8\], which includes the present and GPGG models, is equivalent to the glassy Potts model. A detailed relationship between the present model and the glassy Potts model is described in this article.

The present result supports the act of numerically investigating the present model or the glassy Potts model for study of the Potts glass phase in finite dimensions.

In Ref. \[22\], it is pointed out for large \(p\) limit that there is a relation between the conventional Potts glass model and a Potts glass model satisfying a gauge symmetry. On the other hand, in this article, it is pointed out by analyzing the mean field solutions that there is a relation between the conventional Potts glass model and the present model for arbitrary \(p\). Also, in Ref. \[22\], the magnetization is not mentioned, while, in this article, the magnetization in the present model is mentioned.

The present model is explained in Section \[2\]. The REPMF theory is applied to this model in Section \[3\]. A discussion utilizing a gauge symmetry is given in Section \[3\]. The comparison between the results of the infinite-range models for the present model and the conventional Potts glass model is made in Section \[4\]. The comparison between the RS approximate solutions for the present and GPGG models is made in Section \[5\]. A detailed relationship between the present model and the glassy Potts model is described in Section \[6\]. The concluding remarks are given in Section \[7\].

### 3 The application of the replica mean field theory

This application is performed based on the method \[37\] by Viana and Bray. In this method, a diluted model with infinite-range interactions is treated. The Hamiltonian of Eq. \[1\] is rewritten as

\[
\mathcal{H} = -\sum_{i<j} J_{ij} \eta^{(p)}(\nu_{ij} + \sigma_i - \sigma_j), \quad J_{ij} = 0, J.
\]

Also, the distribution for the quenched variables \(J_{ij}, \nu_{ij}\) between the spins at sites \(i\) and \(j\) is given by

\[
P(J_{ij}, \nu_{ij}) = \frac{\xi}{N} \delta_{J_{ij}, J} P_\mu(\nu_{ij}) + \frac{1}{p} \left(1 - \frac{\xi}{N}\right) \delta_{\nu_{ij}, \theta}.
\]

The distribution \(P_\mu(\nu_{ij})\) is given in Eq. \[3\]. \(\xi\) is the coordination number, and \(N\) is the total number of spins. In what follows, the average for the distribution \(P(J_{ij}, \nu_{ij})\) is represented as \(\langle \cdot \rangle_J\), and the average for the distribution \(P_\mu(\nu_{ij})\) is represented as \(\langle \cdot \rangle_\eta\).

We use the replica method. In the replica method, a relation \(\langle \log Z \rangle_J = \lim_{n \to 0} \frac{1}{n} \langle (Z^n)_J \rangle_{J=1} \) is used, where \(Z\) is the partition function given by \(Z = \text{Tr}(\sigma_i) \exp(-\beta \mathcal{H})\). \(\beta\) is the inverse temperature, \(\beta = \frac{1}{k_B T}\), \(k_B\) is the Boltzmann constant, and \(T\) is the temperature. We define a function \(\delta^{(p)}(x)\) as

\[
\delta^{(p)}(x) \equiv \frac{1}{p} \left[1 + \eta^{(p)}(x)\right].
\]

### 2 The model

The Hamiltonian for the present model, \(\mathcal{H}\), is given by \[8\]

\[
\mathcal{H} = -J \sum_{<ij>} \eta^{(p)}(\nu_{ij} + \sigma_i - \sigma_j),
\]

where \(<ij>\) denotes nearest-neighbor pairs, and \(\eta^{(p)}(x)\) is defined as

\[
\eta^{(p)}(x) \equiv \sum_{r=1}^{p-1} e^{2\pi i x r/p}.
\]
The function \( \delta^{(p)}(x) \) is similar to the Kronecker delta, but this function is slightly different since \( \delta^{(p)}(0) = \delta^{(p)}(p) = 1 \) for example. By using \( \delta^{(p)} \), \( \langle Z^n \rangle_J \) is written as

\[
\langle Z^n \rangle_J = \text{Tr}[\sigma^z_J \{ \exp \left( \beta \sum_{i<j} \sum_{\alpha} p [\delta^{(p)}(y^\alpha_{ij}) - 1] \right) \}^n].
\]

We expand \( \delta^{(p)}(y^\alpha_{ij}) \delta^{(p)}(y^\beta_{ij}) \) of Eq.\( \ref{eq:expand} \), the terms for \( \eta^{(p)}(\sigma^\alpha_{ij} - \sigma^\beta_{ij}) \eta^{(p)}(\sigma^\beta_{ij} - \sigma^\alpha_{ij}) \) appear, so we take the terms for \( \eta^{(p)}(\sigma^\alpha_{ij} - r) \eta^{(p)}(\sigma^\beta_{ij} - r) \) by expanding \( \delta^{(p)}(y^\alpha_{ij}) \delta^{(p)}(y^\beta_{ij}) \) in addition to the expansions of \( \delta^{(p)}(y^\alpha_{ij}) \), \( \delta^{(p)}(y^\beta_{ij}) \).

In the expansions, a relation \( \delta^{(p)}(\alpha - \beta) = \sum_{\alpha < \beta} (\alpha - \beta) \delta^{(p)}(\alpha - r) - \delta^{(p)}(\alpha - \beta) \delta^{(p)}(\alpha - r) \) is used. Then, we have

\[
\langle Z^n \rangle_J \approx \left[ e^{\beta J(p-1)} + (p-1)e^{-\beta J} \right]^{\frac{(N-1)n}{p}} \times \text{Tr}[\sigma^z_J \exp \left( \sum_{i<j} \sum_{\alpha} n \sum_{\alpha} \eta^{(p)}(\sigma^\alpha_{ij} - r) \right)^n].
\]

As order parameters for investigation of the glass phases in the Potts glass models, two order parameters are used at least. One is that \( q_{rs} = \sum_i \langle \eta^P(\sigma^\alpha_{ij} - \sigma^\beta_{ij} - r) \rangle_T \), and is used in Ref.\( \ref{ref:Yamaguchi} \), where \( \langle \rangle_T \) is the thermal average.

Another is that \( q_{(2)}^\alpha = \frac{1}{N} \sum_i \langle \eta^P(\sigma^\alpha_{ij} - \sigma^\beta_{ij} - r) \rangle_T \), and is used in Ref.\( \ref{ref:Yamaguchi} \). The term for \( \eta^P(\sigma^\alpha_{ij} - \sigma^\beta_{ij} - r) \) of Eq.\( \ref{eq:orderparameter} \) is related to the order parameter \( q_{rs} \). The term for \( \eta^P(\sigma^\alpha_{ij} - \sigma^\beta_{ij} + r) \) of Eq.\( \ref{eq:orderparameter} \) is related to the order parameter \( q^{(2)}_\alpha \). The boundaries of the glass phase by using these two order parameters can be agreed. Then, there can be a relation:

\[
\frac{1}{p^2} \sum_{r=0}^{q-1} \sum_{s=1}^{N} \frac{1}{s} \sum_{\alpha} \eta^P(\sigma^\alpha_{ij} - \sigma^\beta_{ij} - r) \eta^P(\sigma^\beta_{ij} - s) \eta^P(\sigma^\alpha_{ij} - \sigma^\beta_{ij} + r)^2 
\approx \frac{1}{p} \sum_{r=0}^{q-1} \sum_{s=1}^{N} \frac{1}{s} \sum_{\alpha} \eta^P(\sigma^\alpha_{ij} - \sigma^\beta_{ij} + r)^2. \tag{11}
\]

The relation \( \ref{eq:relation} \) is also supported by a discussion based on the gauge symmetry. The discussion is given in Section \ref{sec:gauge}. By using Eqs.\( \ref{eq:relation} \) and \( \ref{eq:orderparameter} \), we obtain

\[
\langle Z^n \rangle_J \approx \left[ e^{\beta J(p-1)} + (p-1)e^{-\beta J} \right]^{\frac{(N-1)n}{p}} \times \text{Tr}[\sigma^z_J \exp \left( \frac{a_1}{2N^2} \sum_{r=0}^{q-1} \sum_{s=1}^{N} \sum_{n} \eta^{(p)}(\sigma^\alpha_{ij} - r) \right)^n]
\times \frac{a_2}{2N^2} \sum_{r=0}^{q-1} \sum_{s=1}^{N} \sum_{n} \eta^{(p)}(\sigma^\alpha_{ij} - r) \eta^{(p)}(\sigma^\beta_{ij} - s)^2 \right],
\]

where \( a_1 \equiv \frac{(p\mu - 1)(e^{\beta J} - 1)}{(p-1)(e^{\beta J} + p - 1)} \xi, \) and

\[
a_2 \equiv \frac{(e^{\beta J} - 1)^2}{p \xi} \tag{14}.
\]

We apply the Stratonovich-Hubbard transformation. By using the Gaussian integral exp\( \sqrt{\frac{N\xi}{2\pi}} \int_\infty d\xi \exp(-\frac{Na^2}{2} + ax \xi) \), we have

\[
\langle Z^n \rangle_J \approx \int_{-\infty}^{\infty} \prod_{\alpha} \prod_{r} dM_{\alpha r} \prod_{\alpha < \beta} \prod_{r} dQ_{\alpha \beta r s} e^{-nNA}, \tag{15}
\]

where \( M_{\alpha r} \) is a parameter for the magnetization, and \( Q_{\alpha \beta r s} \) is a parameter for the order of the Potts glass phase.
obtained as an infinite-range model, and then the Potts glass model are compared in Section 5.

When the site dependence of the spin variable $\sigma_i^a$ is eliminated. By using $\langle Z^a \rangle_j \approx 1 + \max(-nNA)$ and the replica method, the free energy per spin, $f$, is obtained as

$$f = -\frac{\xi}{2\beta} \log \left[ \frac{e^{\beta J(p-1)} + (p-1)e^{-\beta J}}{p} \right] + \frac{a_2}{\beta p} \log \left( \frac{\sum_{\alpha < \beta} \sum_r \sum_s (Q_{\alpha \beta rs})^2}{\sqrt{p}} \right) + \frac{a_1}{2\beta p} \sum_{\alpha < \beta} \sum_r (M_{\alpha r})^2 - \frac{1}{\beta n} \log \text{Tr}_{\{\sigma^n\}} e^{L}.$$

(18)

The disorder for exchange interaction is largest at $\mu = \frac{1}{p}$.

When $\mu = \frac{1}{p}$, we have $a_1 = 0$, and then the term for $M_{\alpha r}$ in Eq. (18) vanishes. This means that, under a concentration of ferromagnetic interaction, the ferromagnetic phase does not appear at low temperature for arbitrary $p$ ! In Section 5 the RS approximation to the model of Eq. (18) is performed.

When $J \to \frac{4}{\sqrt{n}}$ and $\xi \to N$, the model becomes the infinite-range model, and then $-\frac{\beta J}{2} + \frac{\xi}{2} \log \left( \frac{e^{\beta J(p-1)} + (p-1)e^{-\beta J}}{p} \right) \to \frac{(\beta J)^2(p-1)}{4}, a_2 \to (\beta J)^2$. Therefore, when the disorder for exchange interaction is largest $(\mu = \frac{1}{p})$, the free energy per spin in the infinite-range model, $f(\text{Inf})$, is obtained, by using Eq. (18), as

$$f(\text{Inf}) = -\frac{\beta J}{4}(p-1) - \frac{\xi}{2} \log \left[ \frac{\sum_{\alpha < \beta} \sum_r \sum_s (Q_{\alpha \beta rs})^2}{\sqrt{p}} \right],$$

(19)

$$L(\text{Inf}) = \frac{(\beta J)^2}{p} \sum_{\alpha < \beta} \sum_r \sum_s Q_{\alpha \beta rs} \eta^{(P)}(\sigma^a - \eta^{(P)}(\sigma^b - s)).$$

(20)

This free energy and the free energy in the conventional Potts glass model are compared in Section 5.

At high temperature, the system is of $Q_{\alpha \beta rs} = M_{\alpha r} = 0$. Then, by using Eq. (13), the free energy per spin, $f(\text{High})$, is obtained as

$$f(\text{High}) = -\frac{\xi}{2\beta} \left[ \frac{e^{\beta J(p-1)} + (p-1)e^{-\beta J}}{p} \right] - \frac{\log(p)}{\beta}.$$

(21)

This free energy $f(\text{High})$ is coincident with the free energy [22] at high temperature in the random energy model. By using a relation $s = k_B \beta^2 \frac{a_2}{4\beta J}$, the entropy per spin, $s$, is obtained as $s = k_B \log(p) + k_Ba_2 \log \left[ \frac{e^{\beta J(p-1)} + (p-1)e^{-\beta J}}{2e^{\beta J(p-1)} + (p-1)e^{-\beta J}} \right] - \frac{k_Ba_2 \xi(\beta J)^2}{4(p-1)}. By expanding the entropy $s$ close to $\beta = 0$, we have $s \approx k_B \log(p) - \frac{\xi(p-1)(\beta J)^2}{4}$. Thus, the Kauzmann temperature $T_K$ [21,18], where the entropy of the high temperature phase would vanish, is estimated as

$$T_K = \frac{J}{k_B} \sqrt{\frac{\xi(p-1)}{4\beta n}}.$$

(22)

### 4 A discussion based on a gauge symmetry

We describe a discussion utilizing a gauge symmetry to the present model. For this model, the gauge transformation is performed by $8$

$$\nu_{ij} \to \nu_{ij} + \delta_i - \delta_j,$$

(23)

$$\sigma_i \to \sigma_i - \delta_i,$$

where $\delta_i$ is an arbitrary value of 0, 1, . . . , $p-1$ at site $i$.

The condition, that the system is on the Nishimori line, is to satisfy a relation $8$:

$$\beta = \frac{1}{p J} \log \left[ \frac{\mu(p-1)}{1-\mu} \right].$$

(24)

This condition is drawn as a line in the phase diagram for $\mu$ and $\beta$.

The internal energy per spin on the Nishimori line, $u_N$, is exactly $u_N = -\frac{J}{p} \beta_1 (p-1)$ in any dimensions $8$. Also, as in the cases of the Ising model and the GPGG model, the magnetization $m_r$ is written as $m_r = \frac{1}{N} \sum_{i=1}^N \langle \eta(\sigma_i - r) \rangle r$. Then, the Potts glass order parameter $q_r(2)$ is written as $q_r(2) = \frac{1}{N} \sum_{i=1}^N \langle \eta(\sigma_i^a - \sigma_i^b - r) \rangle^{(P)}(\sigma_i^a - \sigma_i^b + r)^2$, and exactly $m_r = q_r(2)$ on the Nishimori line in any dimensions.

We omit the description of the derivation of the relation $m_r = q_r(2)$ since the way of deriving this relation is to use a straightforward generalization of the Ising case $8$. By using Eqs. (9) and (11), we obtain

$$\langle Z^a \rangle_j \approx \frac{[e^{\beta J(p-1)} + (p-1)e^{-\beta J}]^{4(n-1)\mu}}{p} \times \text{Tr}_{\eta^{(P)}} \exp \left[ \frac{a_1}{2Np} \sum_{\alpha=1}^n \sum_{r=0}^{q-1} \sum_{i=1}^N \eta^{(P)}(\sigma_i^a - r)]^2 \right. + \frac{a_2}{2Np} \sum_{\alpha < \beta} \sum_{r=0}^{q-1} \sum_{i=1}^N \eta^{(P)}(\sigma_i^a - \sigma_i^b + r)^2 \right].$$

(25)

This term for $\eta^{(P)}(\sigma_i^a)$ of Eq. (25) is related to the order parameter $m_r$, and the term for $\eta^{(P)}(\sigma_i^a - \sigma_i^b + r)$ of Eq. (26) is related to the order parameter $q_r(2)$. In Eq. (26), when the weight for $m_r$ is equal to the weight for $q_r(2)$, it is expected that $a_1 = a_2$. The relation $a_1 = a_2$ is agreed with the condition for the Nishimori line in Eq. (24). This discussion supports that Eq. (11) holds.

### 5 A comparison with the conventional Potts glass model

The Hamiltonian for the conventional Potts glass model, $\mathcal{H}^{(C)}$, is given by $10$

$$\mathcal{H}^{(C)} = -\sum_{<ij>} J^{(C)} \eta^{(C)}(\sigma_i - \sigma_j).$$

(26)
\[ J_{ij}^{(PG)} \] is a quenched variable, and the distribution
\[ P^{(PG)}(J_{ij}^{(PG)}) \] for \( J_{ij}^{(PG)} \) is given by \[10\]
\[ P^{(PG)}(J_{ij}^{(PG)}) = \sqrt{\frac{N}{2\pi a_{j}}} \exp \left[ -\frac{N}{2a_{j}^{2}} \left( J_{ij}^{(PG)} - J_{ij} \right)^{2} \right]. \] (27)
The infinite-range model with \( p = 2 \) is the SK model.

The free energy per spin in the infinite-range model for
the conventional Potts glass model, \( f^{(PG)} \), is known as
\[ f^{(PG)} = -\frac{\beta^{2}}{2}(p-1) + \lim_{\alpha \to 0} \max_{\alpha < \beta} \left\{ \frac{\beta^{2}(p-2)}{2p^{2}} \sum_{r} \sum_{s} G_{\alpha \beta rs}^{(PG)} \left( e^{\beta(j_{ij}^{(PG)} - j_{ij}) - J_{ij}^{(PG)}} - 1 \right)^{2} \right\} \cdot \frac{1}{\ln \text{Tr}^{L^{(PG)}}}. \]
\[ L^{(PG)} = \left( \frac{\beta^{2} J_{ij}^{2}}{p^{2}} \right) \sum_{r} \sum_{s} G_{\alpha \beta rs}^{(PG)} \eta^{(P)}(\alpha - \beta) \eta^{(P)}(\beta - s) + \frac{\beta}{p} \left[ J_{ij}^{2} - \frac{\beta^{2}(p-2)}{2} \right] \sum_{r} \sum_{s} M_{ar}^{(PG)} \eta^{(P)}(\alpha - \beta) \eta^{(P)}(\alpha - s) \] (10). Surprisingly, \( f^{(PG)} \) without the term for \( M_{ar} \) is coincident with \( f^{(Inf)} \) of Eq.(19) when \( J = J_{1} \).

From the form of the free energy \( f^{(PG)} \), it is suggested that, for numerical investigation of the Potts glass phase in finite dimensions, it is necessary that one chooses a proper negative value of \( J_{0} \) that depends on the values of \( p \) and \( \beta \). The definitive value of \( J_{0} \) is nontrivial for study of the Potts glass phase in finite dimensions. On the other hand, in the present model, it is suggested that one chooses the proper value of \( \mu \) instead of the value of \( J_{0} \). In the present model, the free energy is given in Eq.(19), so it is suggested that, in order to numerically investigate the Potts glass phase in finite dimensions, one investigates the properties at \( a_{1} = 0 (\mu = \frac{1}{p}) \). Then, the value of \( \mu \) does not depend on the value of \( \beta \). Therefore, it is realized from the analytical results by means of the REPMF theory that to set the value of \( J_{0} \) in the conventional Potts glass model to be more difficult than to set the value of \( \mu \) in the present model. In this respect, to investigate the present model is easier than to investigate the conventional Potts glass model. In addition, there is a possibility that, for the present model with \( \mu = \frac{1}{p} \), the system is in the Potts glass phase in the ground state for \( p > 2 \) in finite dimensions. Also, it is described in Section 7 that the present model with \( \mu = \frac{1}{p} \) is equivalent to the glassy Potts model.

From the comparison between the present model and the conventional Potts glass model, it is also understood as follows. In the model of Eq.(19), the replica symmetry ansatz is a poor approximation for \( p \neq 2 \) \[10,28\]. In the model of Eq.(19), for \( p > 4 \), the Potts glass transition occurs with discontinuity of order parameter and without latent heat \[13\]. For the analyses of the model of Eq.(19), see references \[10,13,15,28,30\] for example.

### 6 A comparison with the replica symmetric approximate solution of the Gaussian model

The RS approximate solution of the Potts gauge glass model that has interactions of a Gaussian type (the GPGG model) has already been obtained \[8\]. By using \( J_{ij}^{(r)} \) and \( \lambda_{i} \) defined as \( J_{ij}^{(r)} \equiv J e^{2\pi i \lambda_{ij}} \) and \( \lambda_{i} \equiv e^{2\pi i \nu_{i}} \), Eq.(11) is rewritten as \( H = -\sum_{\nu_{i}} \sum_{\nu_{j}} \nu_{i}^{(r)} \nu_{j}^{(r-\nu_{i})} \). The distribution \( P(J_{ij}^{(r)}) \) for \( J_{ij}^{(r)} \) is given by \( P(J_{ij}^{(r)}) = (2\pi J_{ij}^{2})^{-\frac{1}{2}} \exp \left[ -\frac{1}{2} \sum_{\nu_{r}^{(\nu_{s})}} \sum_{\nu_{r}^{(\nu_{s})}} \lambda_{ij}^{(r)} \lambda_{ij}^{(r-\nu_{i})} \right] \), where \( J_{ij}^{(r)} = J_{ij}^{(r-\nu_{i})} \) are variables for exchange interactions, and are quenched variables. It is assumed that \( J_{ij}^{(r)} \) are complex values. On the other hand, the variables \( \nu_{i} \) for exchange interactions in the present model are discrete values. In numerical estimations, to process discrete values is generally easier than to process complex values. In this respect, to investigate the present model is generally easier than to investigate the GPGG model.

We find the RS approximate solution of the present model. We assume the replica symmetry: \( M_{ar} = M_{r} = \nu \eta^{(P)}(r) (0 \leq m \leq 1), Q_{a \beta rs} = Q_{r s} = \nu \eta^{(P)}(r - s) (0 \leq q \leq 1) \), where \( m \) is the order parameter for ferromagnetic phase, and \( q \) is the order parameter for Potts glass phase. By using Eq.(16), the free energy per spin, \( f^{(RS)} \), is obtained as

\[ f^{(RS)} = -\frac{\xi}{2\beta} \log \left[ \frac{e^{\beta J(p-1)} + (p-1) e^{-\beta J}}{p} \right] + a_{2}(p-1) \frac{4\beta}{4\beta} \]
\[ + \frac{a_{2}(p-1)}{2\beta} \left[ a_{2}(q-1)^{2} + 1 \right] - \frac{1}{\beta} \int_{-\infty}^{\infty} \prod_{r=0}^{p-1} \left( \frac{dz_{r}}{\sqrt{2\pi}} e^{-\frac{z_{r}^{2}}{2}} \right) \log B, \]
\[ B \equiv \exp \left[ \sqrt{a_{2} q z_{0} + a_{1} m(p-1)} \right] + \sum_{r=1}^{p} \exp \left[ \sqrt{a_{2} q z_{r} - a_{1} m} \right]. \]

From a saddle point condition \( \frac{\partial}{\partial m} f^{(RS)} = 0 \), we obtain
\[ m = \frac{1}{p-1} \int_{-\infty}^{\infty} \prod_{r=0}^{p-1} \left( \frac{dz_{r}}{\sqrt{2\pi}} e^{-\frac{z_{r}^{2}}{2}} \right) C \frac{B}{B}, \]
\[ C \equiv (p-1) \exp \left[ \sqrt{a_{2} q z_{0} + a_{1} m(p-1)} \right] - \sum_{r=1}^{p} \exp \left[ \sqrt{a_{2} q z_{r} - a_{1} m} \right]. \]

From a saddle point condition \( \frac{\partial}{\partial q} f^{(RS)} = 0 \), we obtain
\[ q = \frac{p}{p-1} \int_{-\infty}^{\infty} \prod_{r=0}^{p-1} \left( \frac{dz_{r}}{\sqrt{2\pi}} e^{-\frac{z_{r}^{2}}{2}} \right) \frac{D}{B^{2}} + \frac{1}{p-1} \]
\[ D \equiv \exp \left[ 2 \sqrt{a_{2} q z_{0} + 2 a_{1} m(p-1)} \right] + \sum_{r=1}^{p-1} \exp \left[ 2 \sqrt{a_{2} q z_{r} - 2 a_{1} m} \right]. \]
When the ferromagnetic order \( m \) vanishes, the RS solution of the infinite-range model of Eq. (28) is coincident with those solutions of the conventional Potts glass model and the GPGG model. In these RS infinite-range models, the Potts glass phase transition appears to be first order for \( p > 6 \) and to be second order otherwise [26,28].

In the ferromagnetic phase, it is expected that \( q \approx m^2 \). By expanding Eq. (30) with small \( a \), we obtain \( q \approx a_2 q \). This indicates \( a_1 = 1 \) at the ferromagnetic phase transition point. Thus, the ferromagnetic phase transition temperature \( T_F \) is

\[
T_F = \frac{Jp}{k_B \log \left( \frac{(p-1)\xi+1}{(p-1)\xi-p+1} \right)}.
\] (34)

In the Potts glass phase, it is expected that \( m = 0 \). By expanding Eq. (30) with small \( q \), we obtain \( q \approx a_2 q \). This indicates \( a_2 = 1 \) at the Potts glass phase transition point. Thus, the Potts glass phase transition temperature \( T_{PG} \) is

\[
T_{PG} = \frac{Jp}{k_B \log \left( \frac{p+1}{\sqrt{p-1}} \right)}.
\] (35)

At the multiple phase transition point for the ferromagnetic, Potts glass and paramagnetic phases, there can be that \( a_1 = a_2 = 1 \). Thus, the multiple phase transition point \( (T^*, \mu^*) \) is

\[
(T^*, \mu^*) = \left( \frac{Jp}{k_B \log \left( \frac{p+1}{\sqrt{p-1}} \right)}, \frac{1}{p} + \frac{p-1}{\sqrt{p}} \right).
\] (36)

The condition, that the system is on the Nishimori line, is given in Eq. (24), and it is confirmed by using Eq. (24) that the phase transition point \( (T^*, \mu^*) \) for any \( p \) and any \( \xi \) is on the Nishimori line. When \( p = 2 \), the result of Eq. (36) is agreed with the result of Ref. [37] for the ±J Ising model.

Fig. 1 shows a phase diagram of the RS approximate result for the diluted present model with infinite-range interactions. \( \mu \) is the concentration of ferromagnetic interaction. \( T \) is the temperature. The Potts dimension \( p \) is 3, and the coordination number \( \xi \) is 4. The paramagnetic phase (‘Para’), the ferromagnetic phase (‘Ferro’) and the Potts glass phase (‘PG’) are depicted. The Nishimori line (the dashed line) is also depicted. The point ‘M’ is the multiple phase transition point.

\[ q^{(\text{True})} = m^{(\text{True})} \] on the Nishimori line, and the multiple phase transition point of the RS approximate solution for any \( p \) is on the Nishimori line. In the GPGG model, it is shown that \( q^{(\text{True})} = m^{(\text{True})} \) on the Nishimori line [8]. On the other hand, in the case of \( p \to \infty \) in the GPGG model, the multiple phase transition point of the RS approximate solution is not on the Nishimori line [8,39]. Therefore, for the relationship between the Nishimori line and the multiple phase transition point, the present RS result is different from the RS result of the GPGG model. The relations \( q^{(\text{True})} = m^{(\text{True})} \) mentioned in this paragraph are exact relations shown by utilizing the gauge symmetries.

7 A comparison with the glassy Potts model

It is argued in Ref. [21] that the Potts gauge glass model [8], which includes the present and GPGG models, is equivalent to the glassy Potts model [9]. On the other hand, it is not written in Ref. [21] that the M\( q \) model [9] included in the glassy Potts model corresponds to the present model with \( P_m(\xi) = \frac{1}{\xi} \). Here, we show this. The Hamiltonian for the glassy Potts model, \( H^{(\text{GP})} \), is given by [9]

\[
H^{(\text{GP})} = - \sum_{<ij>} \delta_{\sigma_i, \Pi_{ij}(\sigma_j)},
\] (37)

where \( \Pi_{ij} \) is a quenched variable between the spins at sites \( i \) and \( j \). In the M\( q \) model, \( \Pi_{ij} \) represents a random permutation of the Potts spin. By rewriting Eq. (37), we have

\[
H^{(\text{GP})} = - \sum_{<ij>} \delta^{(p)}[\sigma_i - \Pi_{ij}(\sigma_j)]
\]

\[
= - \frac{1}{p} \sum_{<ij>} \{ \eta^{(p)}[\sigma_i - \Pi_{ij}(\sigma_j)] + 1 \}.
\] (38)
By using a random shift variable $\tilde{\nu}_{ij}$ ($\tilde{\nu}_{ij} = 0, \ldots, p - 1$) instead of the random permutation $\Pi_{ij}$, we have

$$H^{(GP)} = -\frac{1}{p} \sum_{i<j} [p^{(p)}(\tilde{\nu}_{ij} + \sigma_i - \sigma_j) + 1]. \quad (39)$$

By comparing Eq. (1) with Eq. (39), it is found that the $M_{gl}$ model included in the glassy Potts model is essentially equivalent to the present model with $P^*_{\mu}(\nu_{ij}) = \frac{\tilde{\nu}_{ij}}{p}$. Also, when $P^*_{\mu}(\nu_{ij}) = \frac{1}{p}$, there is a relation $\mu = \frac{1}{p}$.

### 8 Concluding remarks

In the results of the infinite-range models, the Potts glass phase in the conventional Potts glass model is equivalent to the Potts glass phase in the present model. Also, when one performs numerical analyses, the conventional Potts glass model has the problem of setting the value of the variable for additional antiferromagnetic interaction, while the present model does not have the problem. It is expected that the numerical estimations for the present model in finite dimensions help to understand the Potts glass phase in finite dimensions.

For the relationship between the Nishimori line and the multiple phase transition point, it is shown that the present replica symmetric result is different from the replica symmetric result of the Gaussian model (the GPGG model) when the Potts dimension $p$ is large. It seems that the difference between these results is caused by the difference of exchange interactions. This detailed investigation is a task for the future.

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