String theories as the adiabatic limit of Yang-Mills theory

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Abstract

We consider Yang-Mills theory with a matrix gauge group $G$ on a direct product manifold $M = \Sigma_2 \times H^2$, where $\Sigma_2$ is a two-dimensional Lorentzian manifold and $H^2$ is a two-dimensional open disc with the boundary $S^1 = \partial H^2$. The Euler-Lagrange equations for the metric on $\Sigma_2$ yield constraint equations for the Yang-Mills energy-momentum tensor. We show that in the adiabatic limit, when the metric on $H^2$ is scaled down, the Yang-Mills equations plus constraints on the energy-momentum tensor become the equations describing strings with a worldsheet $\Sigma_2$ moving in the based loop group $\Omega G = C^\infty(S^1, G)/G$, where $S^1$ is the boundary of $H^2$. By choosing $G = \mathbb{R}^{d-1,1}$ and putting to zero all parameters in $\Omega \mathbb{R}^{d-1,1}$ besides $\mathbb{R}^{d-1,1}$, we get a string moving in $\mathbb{R}^{d-1,1}$. In arXiv:1506.02175 it was described how one can obtain the Green-Schwarz superstring action from Yang-Mills theory on $\Sigma_2 \times H^2$ while $H^2$ shrinks to a point. Here we also consider Yang-Mills theory on a three-dimensional manifold $\Sigma_2 \times S^1$ and show that in the limit when the radius of $S^1$ tends to zero, the Yang-Mills action functional supplemented by a Wess-Zumino-type term becomes the Green-Schwarz superstring action.
1. Introduction. Superstring theory has a long history [1]-[3] and pretends on description of all four known forces in Nature. In the low-energy limit superstring theories describe supergravity in ten dimensions or supergravity interacting with supersymmetric Yang-Mills (SYM) theory. On the other hand, Yang-Mills and SYM theories in four dimensions give description of three main forces in Nature not including gravity [4]-[7]. The aim of this short paper is to show that bosonic strings (both open and closed) as well as type I, IIA and IIB superstrings can be obtained as a subsector of pure Yang-Mills theory with some constraints on the Yang-Mills energy-momentum tensor. Put differently, knowing the action for superstrings with a worldsheet $\Sigma_2$, we introduce a Yang-Mills action functional on $\Sigma_2 \times H^2$ or on $\Sigma_2 \times S^1$ such that the Yang-Mills action becomes the Green-Schwarz superstring action while $H^2$ or $S^1$ shrink to a point. We will work in Lorentzian signature, but all calculations can be repeated for the Euclidean signature of spacetime.

2. Yang-Mills equations. Consider Yang-Mills theory with a matrix gauge group $G$ on a direct product manifold $M = \Sigma_2 \times H^2$, where $\Sigma_2$ is a two-dimensional Lorentz manifold (flat case is included) with local coordinates $x^a, a, b, ... = 1, 2$, and a metric tensor $g_{\Sigma_2} = (g_{ab})$, $H^2$ is the disc with coordinates $x^i, i, j, ... = 3, 4$, satisfying the inequality $(x^3)^2 + (x^4)^2 < 1$, and the metric $g_{H^2} = (g_{ij})$. Then $(x^\mu) = (x^a, x^i)$ are local coordinates on $M$ with $\mu = 1, ..., 4$.

We start with the gauge potential $A = A_\mu dx^\mu$ with values in the Lie algebra $g = \text{Lie} G$ having scalar product $(\cdot, \cdot)$ defined either via trace $\text{Tr}$ or, for abelian groups like $\mathbb{R}^{p,q}$, $T^{p,q}$ etc., via a metric on vector spaces. The gauge field $F = dA + A \wedge A$ is the $g$-valued two-form

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \quad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] . \quad (1)$$

The Yang-Mills equations on $M$ with the metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{ab} dx^a dx^b + g_{ij} dx^i dx^j \quad (2)$$

have the form

$$D_\mu F^{\mu\nu} := \frac{1}{\sqrt{|\det g|}} \partial_\mu (\sqrt{|\det g|} F^{\mu\nu}) + [A_\mu, F^{\mu\nu}] = 0 , \quad (3)$$

where $g = (g_{\mu\nu})$ and $\partial_\mu = \partial/\partial x^\mu$.

The equations (3) follow from the standard Yang-Mills action on $M$

$$S = \frac{1}{4} \int_M d^4x \sqrt{|\det g|} (F_{\mu\nu}, F^{\mu\nu}) , \quad (4)$$

where $(\cdot, \cdot)$ is the scalar product on the Lie algebra $g$. Note that the metric $g_{\Sigma_2}$ on $\Sigma_2$ is not fixed and the Euler-Lagrange equations for $g_{\Sigma_2}$ yield the constraint equations

$$T_{ab} = g^{\lambda\sigma} (F_{a\lambda}, F_{b\sigma}) - \frac{1}{4} g_{ab} (F_{\mu\nu}, F^{\mu\nu}) = 0 \quad (5)$$

for the Yang-Mills energy-momentum tensor $T_{\mu\nu}$, i.e. its components along $\Sigma_2$ are vanishing. Note that these constraints can be satisfied for many gauge configurations, e.g. for self-dual gauge fields not only $T_{ab} = 0$ but even $T_{\mu\nu} = 0$.

3. Adiabatic limit. On $M = \Sigma_2 \times H^2$ we have the obvious splitting

$$A = A_\mu dx^\mu = A_a dx^a + A_i dx^i , \quad (6)$$
\[ F = \frac{1}{2} F_{\mu \nu} dx^\mu \wedge dx^\nu = \frac{1}{2} F_{ab} dx^a \wedge dx^b + F_{ai} dx^a \wedge dx^i + \frac{1}{2} F_{ij} dx^i \wedge dx^j , \] (7)

\[ T = T_{\mu \nu} dx^\mu \wedge dx^\nu = T_{ab} dx^a \wedge dx^b + 2T_{ai} dx^a \wedge dx^i + T_{ij} dx^i \wedge dx^j . \] (8)

By using the adiabatic approach in the form presented in [8, 9], we deform the metric (2) and introduce the metric

\[ ds_\varepsilon^2 = g_{ab} dx^a \wedge dx^b + \varepsilon^2 g_{ij} dx^i \wedge dx^j , \] (9)

where \( \varepsilon \in [0, 1] \) is a real parameter. It is assumed that the fields \( A_\mu \) and \( F_{\mu \nu} \) smoothly depend in \( \varepsilon^2 \), i.e. \( A_\mu = A_\mu^{(0)} + \varepsilon^2 A_\mu^{(1)} + \ldots \) and \( F_{\mu \nu} = F_{\mu \nu}^{(0)} + \varepsilon^2 F_{\mu \nu}^{(1)} + \ldots \). Furthermore, we have \( \det g_\varepsilon = \varepsilon^4 \det(g_{ab}) \det(g_{ij}) \) and

\[ F_{\varepsilon^{ab}} = g_\varepsilon^{ac} g_\varepsilon^{bd} F_{cd} = F_{ab} , \quad F_{\varepsilon}^{ai} = g_\varepsilon^{ac} g_\varepsilon^{ij} F_{cj} = \varepsilon^{-2} F^{ai} \quad \text{and} \quad F_{\varepsilon}^{ij} = g_\varepsilon^{ik} g_\varepsilon^{jl} F_{kl} = \varepsilon^{-4} F^{ij} , \] (10)

where indices in \( F_{\mu \nu} \) are raised by the non-deformed metric tensor \( g^{\mu \nu} \).

For the deformed metric (9) the action functional (4) is changed to

\[ S_\varepsilon = \frac{1}{4} \int_M d^4 x \sqrt{|\det g_{\varepsilon^{ab}}|} \sqrt{|\det g_{H^2}|} \left\{ \varepsilon^2 (F_{ab}, F_{ab}) + 2(F_{ai}, F_{ai}) + \varepsilon^{-2} (F_{ij}, F_{ij}) \right\} . \] (11)

The term \( \varepsilon^{-2} (F_{ij}, F_{ij}) \) in the Yang-Mills Lagrangian (11) diverges when \( \varepsilon \to 0 \). To avoid this we impose the flatness condition

\[ F_{ij}^{(0)} = 0 \quad \Rightarrow \quad \lim_{\varepsilon \to 0} (\varepsilon^{-1} F_{ij}) = 0 \] (12)

on the components of the field tensor along \( H^2 \). Here \( F_{ij}^{(0)} = 0 \) but \( F_{ij}^{(1)} \) etc. in the \( \varepsilon^2 \)-expansion must not be zero. For the deformed metric (9) the Yang-Mills equations have the form

\[ \varepsilon^2 D_a F^{ab} + D_i F^{ib} = 0 , \] (13)

\[ \varepsilon D_a F^{aj} + \varepsilon^{-1} D_i F^{ij} = 0 . \] (14)

In the deformed metric (9) the constraint equations (5) become

\[ T_{\varepsilon}^{ab} = \varepsilon^2 \left\{ g^{cd} (F_{ac}, F_{bd}) - \frac{1}{2} g_{ab} (F_{cd}, F^{cd}) \right\} + g^{ij} (F_{ai}, F_{bj}) - \frac{1}{2} g_{ab} (F_{ci}, F^{ci}) - \frac{1}{4} \varepsilon^{-2} g_{ab} (F_{ij}, F^{ij}) = 0 . \] (15)

In the adiabatic limit \( \varepsilon \to 0 \) the Yang-Mills equations (13) and (14) become

\[ D_i F^{ib} = 0 , \] (16)

\[ D_a F^{aj} = 0 , \] (17)

since the \( \varepsilon^{-1} \)-term vanishes due to (12). We also keep (17) since it follows from the action (11) after taking the limit \( \varepsilon \to 0 \). One can see that the constraint equations (15) are nonsingular in the limit \( \varepsilon \to 0 \) also due to (12):

\[ T_{\varepsilon}^{ab} = g^{ij} (F_{ai}, F_{bj}) - \frac{1}{2} g_{ab} (F_{ci}, F^{ci}) = 0 . \] (18)

Note that for the adiabatic limit of instanton equations [8, 9] the constraints (15) disappear since the energy-momentum tensor for self-dual and anti-self-dual gauge fields vanishes on any four-manifold \( M \).
4. Flat connections. Now we start to consider the flatness equation (12), the equations (16), (17) and the constraint equations (18). From now on we will consider only zero modes in $e^2$-expansions and equations on them. For simplicity of notation we will omit the index “(0)” from all $A^{(0)}$ and $F^{(0)}$ tensor components. In the adiabatic approach it is assumed that all fields depend on coordinates $x^a \in \Sigma_2$ only via moduli parameters $\phi^\alpha(x^a), \alpha, \beta = 1, 2, ..., $ appearing in the solutions of the flatness equation (12).

Flat connection $A_{H^2} := A_i dx^i$ on $H^2$ has the form

$$A_{H^2} = g^{-1} d g \quad \text{with} \quad d = dx^i \partial_i \quad \text{for} \quad \partial_i = \frac{\partial}{\partial x^i},$$

where $g = g(\phi^\alpha(x^a), x^i)$ is a smooth map from $H^2$ into the gauge group $G$ for any fixed $x^a \in \Sigma_2$.

Let us introduce on $H^2$ spherical coordinates: $x^3 = \rho \cos \varphi$ and $x^4 = \rho \sin \varphi$. Using these coordinates, we impose on $g$ the condition $g(\varphi = 0, \rho^2 \rightarrow 1) = \text{Id}$ (framing) and denote by $C^\infty_0(H^2, G)$ the space of framed flat connections on $H^2$ given by (19). On $H^2$, as on a manifold with a boundary, the group of gauge transformations for any fixed $x^a \in \Sigma_2$ is defined as (see e.g. [9, 10, 11])

$$G_{H^2} = \{ g : H^2 \rightarrow G \mid g \rightarrow \text{Id} \quad \text{for} \quad \rho^2 \rightarrow 1 \}.$$  

Hence the solution space of the equation (12) is the infinite-dimensional group $\mathcal{N} = C^\infty_0(H^2, G)$, and the moduli space of solutions is the based loop group $[9, 10, 12]$

$$\mathcal{M} = C^\infty_0(H^2, G)/G_{H^2} = \Omega G.$$  

This space can also be represented as $\Omega G = LG/G$, where $LG = C^\infty(S^1, G)$ is the loop group with the circle $S^1 = \partial H^2$ parameterized by $e^{i \varphi}$.

5. Moduli space. On the group manifold (21) we introduce local coordinates $\phi^\alpha$ with $\alpha = 1, 2, ...$ and recall that $A_i$’s depend on $x^a \in \Sigma_2$ only via moduli parameters $\phi^\alpha = \phi^\alpha(x^a)$. Then moduli of gauge fields define a map

$$\phi : \Sigma_2 \rightarrow \mathcal{M} \quad \text{with} \quad \phi(x^a) = \{ \phi^\alpha(x^a) \}.$$  

These maps are constrained by the equations (16), (17) and (18). Since $A_{H^2}$ is a flat connection for any $x^a \in \Sigma_2$, the derivatives $\partial_\alpha A_i$ have to satisfy the linearized (around $A_{H^2}$) flatness condition, i.e. $\partial_\alpha A_i$ belong to the tangent space $T_\alpha \mathcal{N}$ of the space $\mathcal{N} = C^\infty_0(H^2, G)$ of framed flat connections on $H^2$. Using the projection $\pi : \mathcal{N} \rightarrow \mathcal{M}$ from $\mathcal{N}$ to the moduli space $\mathcal{M}$, one can decompose $\partial_\alpha A_i$ into the two parts

$$T_\alpha \mathcal{N} = \pi^* T_\alpha \mathcal{M} \oplus T_\alpha \mathcal{G} \quad \iff \quad \partial_\alpha A_i = (\partial_\alpha \phi^\beta) \xi_{\beta i} + D_i \epsilon_\alpha,$$

where $\mathcal{G}$ is the gauge group $G_{H^2}$ for any fixed $x^a \in \Sigma_2$, $\{ \xi_\alpha = \xi_{\alpha i} dx^i \}$ is a local basis of tangent vectors at $T_\alpha \mathcal{M}$ (they form the loop Lie algebra $\Omega g$) and $\epsilon_\alpha$ are $g$-valued gauge parameters ($D_i \epsilon_\alpha \in T_\alpha \mathcal{G}$) which are determined by the gauge-fixing conditions

$$g^{ij} D_i \xi_{\alpha j} = 0 \quad \iff \quad g^{ij} D_i D_j \epsilon_\alpha = g^{ij} D_i \partial_\alpha A_j.$$  

Note also that since $A_i(\phi^\alpha, x^j)$ depends on $x^a$ only via $\phi^\alpha$, we have

$$\partial_\alpha A_i = \frac{\partial A_i}{\partial \phi^\beta} \partial_\alpha \phi^\beta \quad \iff \quad \epsilon_\alpha = (\partial_\alpha \phi^\beta) \epsilon_\beta.$$
where the gauge parameters \( \epsilon_\beta \) are found by solving the equations
\[
g^{ij} D_i D_j \epsilon_\beta = g^{ij} D_i \frac{\partial A_i}{\partial \phi^\beta} .
\] (26)

Recall that \( A_i \) are given explicitly by (19) and \( A_a \) are yet free. It is natural to choose \( A_a = \epsilon_a \) [5, 6] and obtain
\[
F_{ai} = \partial_a A_i - D_i A_a = (\partial_a \phi^\beta) \xi_{\beta i} = \pi_s \partial_a A_i \in T_A M .
\] (27)

Thus if we know the dependence of \( \phi^\alpha \) on \( x^a \) then we can construct
\[
(A_\mu) = (A_a, A_i) = \begin{pmatrix} (\partial_a \phi^\beta) \epsilon_\beta, g^{-1}(\phi^\alpha, x^j) \partial_i g(\phi^\beta, x^k) \end{pmatrix},
\] (28)

which are in fact the components \( A_\mu^{(0)} = A_\mu(\epsilon=0) \).

6. Effective action. For finding equations for \( \phi^\alpha(x^a) \) we substitute (27) into (16) and see that (16) are resolved due to (24). Substituting (27) into (17), we obtain the equations
\[
\frac{1}{\sqrt{|\det g_{\Sigma^2}|}} \partial_a \left( \sqrt{|\det g_{\Sigma^2}|} g^{ab} \partial_b \phi^\beta \right) g^{ij} \xi_{\beta j} + g^{ab} g^{ij} (D_a \xi_{\beta j}) \partial_b \phi^\beta = 0 .
\] (29)

We should project (29) on the moduli space \( \mathcal{M} = \Omega G \), metric \( G = (G_{\alpha\beta}) \) on which is defined as
\[
G_{\alpha\beta} = \langle \xi_\alpha, \xi_\beta \rangle = \int_{H^2} d \text{vol} \ g^{ij} (\xi_{\alpha i}, \xi_{\beta j}) .
\] (30)

The projection is provided by multiplying (29) by \( \langle \xi_\alpha, \cdot \rangle \) (cf. e.g. [13, 14]). We obtain
\[
\frac{1}{\sqrt{|\det g_{\Sigma^2}|}} \partial_a \left( \sqrt{|\det g_{\Sigma^2}|} g^{ab} \partial_b \phi^\beta \right) \langle \xi_\alpha, \xi_\beta \rangle + G_{\alpha\beta} \langle \xi_\alpha, D_a \xi_\beta \rangle \partial_b \phi^\beta = 0 .
\] (31)

\[
= G_{\alpha\sigma} \left\{ \frac{1}{\sqrt{|\det g_{\Sigma^2}|}} \partial_a \left( \sqrt{|\det g_{\Sigma^2}|} g^{ab} \partial_b \phi^\sigma \right) + \Gamma_{\beta\gamma}^\sigma g^{ab} \partial_a \phi^\beta \partial_b \phi^\gamma \right\} = 0 ,
\]

where
\[
\Gamma_{\beta\gamma}^\sigma = \frac{1}{2} G^{\sigma\lambda} (\partial_\gamma G_{\beta\lambda} + \partial_\beta G_{\gamma\lambda} - \partial_\lambda G_{\beta\gamma}) \quad \text{with} \quad \partial_\gamma := \frac{\partial}{\partial \phi^\gamma} ,
\] (32)

are the Christoffel symbols and \( \nabla_\gamma \) are the corresponding covariant derivatives on the moduli space \( \mathcal{M} \) of flat connections on \( H^2 \).

The equations
\[
\frac{1}{\sqrt{|\det g_{\Sigma^2}|}} \partial_a \left( \sqrt{|\det g_{\Sigma^2}|} g^{ab} \partial_b \phi^\alpha \right) + \Gamma_{\beta\gamma}^\sigma g^{ab} \partial_a \phi^\beta \partial_b \phi^\gamma = 0
\] (33)
are the Euler-Lagrange equations for the effective action

\[ S_{\text{eff}} = \int_{\Sigma^2} d^4x \sqrt{-\det(g_{ab})} g^{cd} G_{\alpha\beta} \partial_c \phi^\alpha \partial_d \phi^\beta \] (34)

obtained from the action functional (11) in the adiabatic limit \( \varepsilon \to 0 \); it appears from the term \((\mathcal{F}_a, \mathcal{F}^a)\) in (11) (other terms vanish). The equations (33) are the standard sigma-model equations defining maps from \( \Sigma^2 \) into the based loop group \( \Omega G \).

7. Virasoro constraints. The last undiscussed equations are the constraints (18). Substituting (27) into (18), we obtain

\[ g^{ij}(\xi_{\alpha i}, \xi_{\beta j}) \partial_a \phi^\alpha \partial_b \phi^\beta - \frac{1}{2} g_{ab} g^{cd} G_{\alpha\beta} \partial_c \phi^\alpha \partial_d \phi^\beta = 0 . \] (35)

Integrating (35) over \( H^2 \) (projection on \( \mathcal{M} \)), we get

\[ G_{\alpha\beta} \partial_a \phi^\alpha \partial_b \phi^\beta - \frac{1}{2} g_{ab} g^{cd} G_{\alpha\beta} \partial_c \phi^\alpha \partial_d \phi^\beta = 0 . \] (36)

These are equations which one will obtain from (34) by varying with respect to \( g_{ab} \). Thus

\[ T^V_{ab} = G_{\alpha\beta} \partial_a \phi^\alpha \partial_b \phi^\beta - \frac{1}{2} g_{ab} g^{cd} G_{\alpha\beta} \partial_c \phi^\alpha \partial_d \phi^\beta \] (37)

is the traceless stress-energy tensor and equations (36) are the Virasoro constraints accompanying the Polyakov string action (34).

8. B-field. In string theory the action (34) is often extended by adding the B-field term. This term can be obtained from the topological Yang-Mills term

\[ \frac{1}{2} \int_M d^4x \sqrt{\det g^\varepsilon_{H^2}} \varepsilon_{\mu\nu\lambda\sigma}(\mathcal{F}^\mu_{\varepsilon}, \mathcal{F}^\lambda_{\varepsilon}) \] (38)

which in the adiabatic limit \( \varepsilon \to 0 \) becomes

\[ \int_M d^4x \sqrt{\det g_{H^2}} \varepsilon^{ab} \varepsilon^{ij}(\mathcal{F}_{ai}, \mathcal{F}_{bj}) = \int_{\Sigma^2} d^4x \sqrt{-\det g} B_{\alpha\beta} \partial_c \phi^\alpha \partial_d \phi^\beta , \] (39)

where

\[ B_{\alpha\beta} = \int_{H^2} d \text{vol} \varepsilon^{ij}(\xi_{\alpha i}, \xi_{\beta j}) . \] (40)

are components of the two-form \( \mathbb{B} = (B_{\alpha\beta}) \) on the moduli space \( \mathcal{M} = \Omega G \).

9. Remarks on superstrings. The adiabatic limit of supersymmetric Yang-Mills theories with a (partial) topological twisting on Euclidean manifold \( \Sigma \times \bar{\Sigma} \), where \( \Sigma \) and \( \bar{\Sigma} \) are Riemann surfaces, was considered in [15]. Several sigma-models with fermions on \( \Sigma \) (including supersymmetric ones) were obtained. Switching to Lorentzian signature and adding constraints of type (18), which were not considered in [15], one can get stringy sigma-model resembling NSR strings. However, analysis of these sigma-models demands more efforts and goes beyond the scope of our paper.

Another possibility is to consider ordinary Yang-Mills theory (11) but with Lie supergroup \( G \) as the structure group. We restrict ourselves to the \( N=2 \) super translation group with ten-dimensional
Minkowski space $\mathbb{R}^{9,1}$ as bosonic part. This super translation group can be represented as the coset [16, 17]

$$G = \text{SUSY}(N=2)/\text{SO}(9,1),$$

(41)

with coordinates $(X^\alpha, \theta^A)$, where $\theta^p = (\theta^A)$ are two Majorana-Weyl spinors in $d = 10, \alpha = 0, ..., 9, A = 1, ..., 32$ and $p = 1, 2$. The generators of $G$ obey the Lie superalgebra $\mathfrak{g} = \text{Lie} \, G$,

$$\{\xi_{Ap}, \xi_{Bq}\} = (\gamma^\alpha C)_{AB} \delta_{pq} \xi_\alpha, \quad [\xi_\alpha, \xi_{Ap}] = 0, \quad [\xi_\alpha, \xi_\beta] = 0,$$

(42)

where $\gamma^\alpha$ are the $\gamma$-matrices in $\mathbb{R}^{9,1}$ and $C$ is the charge conjugation matrix. On the superalgebra $\mathfrak{g}$ we introduce the standard metric

$$\langle \xi_\alpha \xi_\beta \rangle = \eta_{\alpha\beta}, \quad \langle \xi_\alpha \xi_{Ap} \rangle = 0 \quad \text{and} \quad \langle \xi_{Ap} \xi_{Bq} \rangle = 0,$$

(43)

where $(\eta_{\alpha\beta}) = \text{diag}(-1, 1, ..., 1)$ is the Lorentzian metric on $\mathbb{R}^{9,1}$.

It was shown in [18] that the action functional for Yang-Mills theory on $\Sigma_2 \times H^2$ with the gauge group $G$, defined by (42),

$$S_\varepsilon = \frac{1}{2\pi} \int_{\Sigma_2 \times H^2} d^4x \sqrt{\det g_{\Sigma_2}} \sqrt{\det g_{H^2}} \left\{ \varepsilon^2 \langle F_{ab} F^{ab} \rangle + 2 \langle F_{ai} F^{ai} \rangle + \varepsilon^{-2} \langle F_{ij} F^{ij} \rangle \right\}$$

(44)

plus the Wess-Zumino-type term

$$S_{WZ} = \frac{1}{\pi} \int_{\Sigma_3 \times H^2} dx^a \wedge dx^b \wedge dx^c \wedge dx^2 \wedge dx^4 \int_{\Gamma_{\Delta \Lambda}} \tilde{F}_{ai} \tilde{F}^i_{bj} \tilde{F}^j_{ck} \tilde{F}_{\delta k} \xi^k$$

(45)

yield the Green-Schwarz superstring action [17] in the adiabatic limit $\varepsilon \rightarrow 0$. Here $\Sigma_3$ is a Lorentzian manifold with the boundary $\Sigma_2 = \partial \Sigma_3$ and local coordinates $x^a, \hat{a} = 0, 1, 2$; the structure constants $f_{\Gamma_{\Delta \Lambda}}$ are given in [16] and $(\xi_i) = (\sin \varphi, -\cos \varphi)$ is the unit vector on $H^2$ running the boundary $S^1 = \partial H^2$.

10. **Superstrings from $d = 3$ Yang-Mills.** Here we will show that the Green-Schwarz superstrings with a worldsheet $\Sigma_2$ can also be associated with a Yang-Mills model on $\Sigma_2 \times S^1$. When the radius of $S^1$ tends to zero, the action of this Yang-Mills model becomes the Green-Schwarz superstring action. So, we consider Yang-Mills theory on a direct product manifold $M^3 = \Sigma_2 \times S^1$, where $\Sigma_2$ is a two-dimensional Lorentzian manifold discussed before and $S^1$ is the unit circle parameterized by $x^3 \in [0, 2\pi]$ with the metric tensor $g_{S^1} = (g_{33})$ and $g_{33} = 1$. As the structure group $G$ of Yang-Mills theory we consider the super translation group in $d = 10$ auxiliary dimensions (41) with the generators (42) and the metric (43) on the Lie superalgebra $\mathfrak{g} = \text{Lie} \, G$. As in (20), we impose framing over $S^1$, i.e. consider the group of gauge transformations equal to the identity over $S^1$. Coordinates on $G$ are $X^\alpha$ and $\theta^A\varepsilon$ introduced in the previous section. The one-forms

$$\Pi^\Delta = \{\Pi^\alpha, \Pi^{Ap}\} = \{dX^\alpha - i \delta_{pq} \bar{\theta}^p \gamma^\alpha d\theta^q, \ d\theta^{Ap}\}$$

(46)

form a basis of one-forms on $G$ [16].

By using the adiabatic approach, we deform the metric on $\Sigma_2 \times S^1$ and introduce

$$dx^2 = g_{\mu\nu} \, dx^\mu \, dx^\nu = g_{ab} \, dx^a \, dx^b + \varepsilon^2 (dx^3)^2,$$

(47)
where $\varepsilon \in [0, 1]$ is a real parameter, $a, b = 1, 2, \mu, \nu = 1, 2, 3$. This is equivalent to the consideration of the circle $S^1_\varepsilon$ of radius $\varepsilon$. It is assumed that for the fields $A_\mu$ and $F_{\mu \nu}$ there exist limits $\lim_{\varepsilon \to 0} A_\mu$ and $\lim_{\varepsilon \to 0} F_{\mu \nu}$. Indices are raised by $g^{\mu \nu}_\varepsilon$ and we have

$$F^{\mu \nu}_\varepsilon = g^{ac}_\varepsilon g^{bd}_\varepsilon F_{cd} = F^{ab}_\varepsilon , \quad F^{a3}_\varepsilon = g^{ac}_\varepsilon g^{33}_\varepsilon F_{c3} = \varepsilon^{-2} F^{a3}_3 ,$$

where indices in $F^{\mu \nu}_\varepsilon$ are raised by the non-deformed metric tensor.

We consider the Yang-Mills action of the form

$$S_\varepsilon = \int_{M^3} d^3 x \sqrt{\det g_{\Sigma^2}} \left\{ \varepsilon^2 D_a F^{ab}_\varepsilon + D_3 F^{3b}_\varepsilon = 0 , \quad D_a F^{a3}_\varepsilon = 0 , \quad T^{e}_{ab} = \varepsilon^2 \left( g^{cd} (F_{ac} F^{bd}_\varepsilon) - \frac{1}{4} g_{ab} (F_{cd} F^{cd}_\varepsilon) \right) + (F_{a3} F_{b3}) - \frac{1}{2} g_{ab} (F_{c3} F^{c3}_\varepsilon) \right\} ,$$

which for $\varepsilon = 1$ coincides with the standard Yang-Mills action. Variations with respect to $A_\mu$ and $g_{ab}$ yield the equations

$$D_3 F^{3b}_\varepsilon = 0 , \quad D_a F^{a3}_\varepsilon = 0 , \quad T^0_{ab} = (F_{a3} F_{b3}) - \frac{1}{2} g_{ab} (F_{c3} F^{c3}_\varepsilon) .$$

In the adiabatic limit $\varepsilon \to 0$ equations (50), (51) become

$$D_3 F^{3b}_\varepsilon = 0 , \quad D_a F^{a3}_\varepsilon = 0 , \quad T^0_{ab} = (F_{a3} F_{b3}) - \frac{1}{2} g_{ab} (F_{c3} F^{c3}_\varepsilon) .$$

Notice that as a function of $x^3 \in S^1$, the field $A_3$ belongs to the loop algebra $L_\Omega = g \oplus \Omega g$, where $\Omega g$ is the Lie superalgebra of the based loop group $\Omega G$. Let us denote by $A^0_3$ the zero-mode in the expansion of $A_3$ in $\exp(i x^3) \in S^1$ (Wilson line). The generic $A_3$ can be represented in the form

$$A_3 = h^{-1} A^0_3 h + h^{-1} \partial_3 h ,$$

where $G$-valued function $h$ depends on $x^a$ and $x^3$. For fixed $x^a \in \Sigma_2$ one can choose $h \in \Omega G = \text{Map}(S^1, G) / G$. We denote by $\mathcal{N}$ the space of all $A_3$ given by (54) and define the projection $\pi : \mathcal{N} \to G$ on the space $G$ parametrizing $A^0_3$ since we want to keep only $A^0_3$ in the limit $\varepsilon \to 0$. We denote by $Q$ the fibres of the projection $\pi$.

In the adiabatic approach it is assumed that $A^0_3$ depends on $x^a \in \Sigma_2$ only via the moduli parameters $(X^\alpha, \theta^A p) \in G$. Therefore, the moduli define the maps

$$(X, \theta^P) : \quad \Sigma_3 \to G$$

which are not arbitrary, they are constrained by the equations (52), (53). The derivatives $\partial_a A_3$ of $A_3 \in \mathcal{N}$ belong to the tangent space $T_{A_3} \mathcal{N}$ of the space $\mathcal{N}$. Using the projection $\pi : \mathcal{N} \to G$, one can decompose $\partial_a A_3$ into two parts

$$T_{A_3} \mathcal{N} = \pi^* T_{A^0_3} G \oplus T_{A_3} Q \quad \Leftrightarrow \quad \partial_a A_3 = \Pi^A_\Delta \xi_\Delta + D_3 \varepsilon_a ,$$

where $\Delta = (\alpha, Ap)$ and

$$\Pi^A_\alpha := \partial_a X^\alpha - i \delta_{pq} \bar{\theta}^p \gamma^a \partial_a \theta^q , \quad \Pi^A p := \partial_a \theta^A p .$$
In (56), \( \epsilon_a \) are g-valued parameters \((D_3 \epsilon_a \in T_{A_3}Q)\) and the vector fields \( \xi_{\Delta_3} \) on \( G \) can be identified with the generators \( \xi_{\Delta} = (\xi_{\alpha}, \xi_{\alpha p}) \) of \( G \).

On \( \xi_{\Delta_3} \) we impose the gauge fixing condition

\[
D_3 \xi_{\Delta_3} = 0 \quad \overset{(56)}{\Rightarrow} \quad D_3 D_3 \epsilon_a = D_3 \partial_a A_3 .
\]

Recall that \( A_3 \) is fixed by (54) and \( A_a \) are yet free. In the adiabatic approach one chooses \( A_a = \epsilon_a \) (cf. [5, 6]) and obtains

\[
\mathcal{F}_{a3} = \partial_a A_3 - D_3 A_a = \Pi^\Delta_a \xi_{\Delta_3} \in T_{A_3}G .
\]

Substituting (59) into the first equation in (52), we see that they are resolved due to (58). Substituting (59) into the action \( S_0 = \lim_{\epsilon \to 0} S_\epsilon \) given by (49) and integrating over \( x^3 \), we obtain the effective action

\[
S_0 = 2\pi \int_{\Sigma_2} \sqrt{|\det g_{\Sigma_2}|} g^{ab} \Pi^\alpha_a \Pi^\beta_b \eta_{\alpha\beta} ,
\]

which coincides with the kinetic part of the Green-Schwarz superstring action [17]. One can show (cf. [14]) that the second equations in (52) are equivalent to the Euler-Lagrange equations for \((X^\alpha, \theta^{Ap})\) following from (60). Finally, substituting (59) into (53), we obtain the equations

\[
\Pi^\alpha_a \Pi^\beta_b \eta_{\alpha\beta} - \frac{1}{2} g^{cd} \Pi^\alpha_c \Pi^\beta_d \eta_{\alpha\beta} = 0
\]

which can also be obtained from (60) by variation of \( g^{ab} \).

For getting the full Green-Schwarz superstring action one should add to (60) a Wess-Zumino-type term which is described as follows [16, 17]. One should consider a Lorentzian 3-manifold \( \Sigma_3 \) with the boundary \( \Sigma_2 = \partial \Sigma_3 \) and coordinates \( x^\hat{a}, \hat{a} = 0, 1, 2 \). On \( \Sigma_3 \) one introduces the 3-form [16]

\[
\Omega_3 = i dx^\hat{a} \Pi^\alpha_\hat{a} \wedge (\bar{\partial} \theta^1 \gamma^\beta \wedge \bar{\partial} \theta^1 - \bar{\partial} \theta^2 \gamma^\beta \wedge \bar{\partial} \theta^2) \eta_{\alpha\beta} = \check{d} \Omega_2 ,
\]

where

\[
\Omega_2 = -i \check{d} X^\alpha \wedge (\bar{\partial}^1 \gamma^\beta \check{d} \theta^1 - \bar{\partial}^2 \gamma^\beta \check{d} \theta^2) \quad \text{with} \quad \check{d} = dx^\hat{a} \frac{\partial}{\partial x^\hat{a}} .
\]

Then the term

\[
S_{WZ} = \int_{\Sigma_3} \Omega_3 = \int_{\Sigma_2} \Omega_2
\]

is added to (60) with a proper coefficient \( \kappa \) and \( S_{GS} = S_0 + \kappa S_{WZ} \) is the Green-Schwarz action for the superstrings of type I, IIA and IIB.

To get (63) from Yang-Mills theory we consider the manifold \( \Sigma_3 \times S^1 \) and notice that in addition to (59) we now have the components

\[
\mathcal{F}_{\alpha 3} = \Pi^\alpha_\Delta \xi_{\Delta 3} = (\partial_0 X^\alpha - i \delta_{pq} \bar{\partial}^p \gamma^q \partial_0 \theta^q) \xi_{\alpha 3} + (\partial_0 \theta^{Ap}) \xi_{\alpha p 3} .
\]

Introduce one-forms \( F_3 := \mathcal{F}_{a3}dx^\hat{a} \) on \( \Sigma_3 \), where \( \mathcal{F}_{a3}(\epsilon) \) are general Yang-Mills fields on \( \Sigma_3 \times S^1 \) which take the form (59),(64) only in the limit \( \epsilon \to 0 \), and consider the functional

\[
S_Y M = \int_{\Sigma_3 \times S^1} f_{\Delta \Lambda \Gamma} F^\Delta_3 \wedge F^\Lambda_3 \wedge F^\Gamma_3 \wedge dx^3 ,
\]
where the explicit form of the constant $f_{\Delta \Lambda r}$ can be found in [16]. Therefore, the Yang-Mills action (49) plus (65) in the adiabatic limit $\varepsilon \to 0$ becomes the Green-Schwarz action. This result can be considered as a generalization of the Green result [19] who derived the superstring theory in a fixed gauge from Chern-Simons theory on $\Sigma_2 \times \mathbb{R}$.

11. Concluding remarks. We have shown that bosonic strings and Green-Schwarz superstrings can be obtained via the adiabatic limit of Yang-Mills theory on manifolds $\Sigma_2 \times H^2$ with a Wess-Zumino-type term. Notice that the constraint equations (15) on the Yang-Mills energy momentum tensor with $\varepsilon > 0$ are important for restoring the unitarity of Yang-Mills theory on $\Sigma_2 \times H^2$. More interestingly, the same result is also obtained by considering Yang-Mills theory on three-dimensional manifolds $\Sigma_2 \times S^1_{\varepsilon}$ with the radius of the circle $S^1_{\varepsilon}$ given by $\varepsilon \in [0,1]$. For $\varepsilon \neq 0$ we have well-defined quantum Yang-Mills theory on $\Sigma_2 \times S^1_{\varepsilon}$. For $\varepsilon \to 0$ we get superstring theories. This raises hopes that various results for superstring theories can be obtained from results of the associated Yang-Mills theory on $\Sigma_2 \times S^1_{\varepsilon}$.

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