Painlevé IV, Chazy II, and asymptotics for recurrence coefficients of semi-classical Laguerre polynomials and their Hankel determinants

Chao Min¹ | Yang Chen²

¹School of Mathematical Sciences, Huaqiao University, Quanzhou, China
²Department of Mathematics, Faculty of Science and Technology, University of Macau, Macau, China

Correspondence
Chao Min, School of Mathematical Sciences, Huaqiao University, Quanzhou 362021, China.
Email: chaomin@hqu.edu.cn

This paper studies the monic semi-classical Laguerre polynomials based on previous work by Boelen and Van Assche, Filipuk et al., and Clarkson and Jordaan. Filipuk et al. proved that the diagonal recurrence coefficient $a_n(t)$ satisfies the fourth Painlevé equation. In this paper, we show that the off-diagonal recurrence coefficient $\beta_n(t)$ fulfills the first member of Chazy II system. We also prove that the sub-leading coefficient of the monic semi-classical Laguerre polynomials satisfies both the continuous and discrete Jimbo–Miwa–Okamoto $\sigma$-form of Painlevé IV. By using Dyson’s Coulomb fluid approach together with the discrete system for $a_n(t)$ and $\beta_n(t)$, we obtain the large $n$ asymptotic expansions of the recurrence coefficients and the sub-leading coefficient. The large $n$ asymptotics of the associated Hankel determinant (including the constant term) is derived from its integral representation in terms of the sub-leading coefficient.

KEYWORDS
asymptotic expansions, Chazy II system, Hankel determinants, Painlevé IV, recurrence coefficients, semi-classical Laguerre polynomials

MSC CLASSIFICATION
42C05, 33E17, 41A60

1 INTRODUCTION

Orthogonal polynomials play an important role in mathematical physics (e.g., random matrix theory and integrable systems), approximation theory, mechanical quadrature, and so forth. The relationship between recurrence coefficients of semi-classical orthogonal polynomials and Painlevé equations has been studied extensively over the past decade; see [1–8] for reference. Semi-classical orthogonal polynomials are defined as orthogonal polynomials whose weight functions $w(x)$ satisfy the Pearson equation

$$\frac{d}{dx}(\rho(x)w(x)) = \tau(x)w(x), \quad (1.1)$$

where $\rho(x)$ and $\tau(x)$ are polynomials with $\deg \rho > 2$ or $\deg \tau \neq 1$ [8, Section 1.1.1].

Boelen and Van Assche [1] considered the orthonormal polynomials with respect to the so-called semi-classical Laguerre weight

$$w(x) = w(x; t) := x^t e^{-x^2+tx}, \quad x \in \mathbb{R}^+, \quad (1.2)$$

This paper considers the monic semi-classical Laguerre polynomials based on previous work by Boelen and Van Assche, Filipuk et al., and Clarkson and Jordaan. Filipuk et al. proved that the diagonal recurrence coefficient $a_n(t)$ satisfies the fourth Painlevé equation. In this paper, we show that the off-diagonal recurrence coefficient $\beta_n(t)$ fulfills the first member of Chazy II system. We also prove that the sub-leading coefficient of the monic semi-classical Laguerre polynomials satisfies both the continuous and discrete Jimbo–Miwa–Okamoto $\sigma$-form of Painlevé IV. By using Dyson’s Coulomb fluid approach together with the discrete system for $a_n(t)$ and $\beta_n(t)$, we obtain the large $n$ asymptotic expansions of the recurrence coefficients and the sub-leading coefficient. The large $n$ asymptotics of the associated Hankel determinant (including the constant term) is derived from its integral representation in terms of the sub-leading coefficient.
with \( \lambda > -1, \ t \in \mathbb{R} \). We mention that we just follow the terminology of [1]. In fact, the semi-classical Laguerre polynomials are different from the classical Laguerre polynomials, since the former are related to the parabolic cylinder functions [3, Appendix 1]. It is easy to check that (1.2) is indeed a semi-classical weight since it satisfies the Pearson equation (1.1) with

\[
\phi(x) = x, \quad r(x) = -2x^2 + tx + 1 + \lambda.
\]

It was shown in [1, Theorem 1.1] that the recurrence coefficients of the semi-classical Laguerre polynomials satisfy a discrete system, which is related to an asymmetric discrete Painlevé IV equation. Later, Filipuk et al. [6, Theorem 1.1] proved that the recurrence coefficient \( b_n(t) \), which is equal to \( a_n(t) \) below, satisfies the (continuous) Painlevé IV equation.

More recently, Clarkson and Jordaan [3] studied the monic orthogonal polynomials with respect to the weight (1.2), that is,

\[
\int_0^\infty p_m(x; t) p_n(x; t) w(x; t) \, dx = h_n(t) \delta_{mn}, \quad m, n = 0, 1, 2, \ldots .
\]  

(1.3)

Here, \( p_n(x; t) \) has the monomial expansion

\[
p_n(x; t) = x^n + p(n, t)x^{n-1} + \cdots + p_n(0; t), \quad n = 0, 1, 2, \ldots ,
\]  

(1.4)

and \( p(n, t) \) denotes the coefficient of \( x^{n-1} \) and we set \( p(0, t) = 0 \).

It is well known that the orthogonal polynomials \( p_n(x; t) \) obey the three-term recurrence relation of the form [9]

\[
x P_n(x; t) = P_{n+1}(x; t) + a_n(t) P_n(x; t) + \beta_n(t) P_{n-1}(x; t),
\]  

(1.5)

with the initial conditions

\[
P_0(x; t) = 1, \quad \beta_0(t) P_{-1}(x; t) = 0.
\]

The combination of (1.3), (1.4), and (1.5) gives

\[
a_n(t) = p(n, t) - p(n+1, t), \quad \beta_n(t) = \frac{h_n(t)}{h_{n-1}(t)}.
\]  

(1.6, 1.7)

Taking a telescopic sum of (1.6), we have

\[
\sum_{j=0}^{n-1} a_j(t) = -p(n, t).
\]  

(1.8)

Based on the results in [1] and [6], Clarkson and Jordaan [3] showed the following two lemmas.

**Lemma 1.1.** The recurrence coefficients \( a_n(t) \) and \( \beta_n(t) \) satisfy the discrete system:

\[
a_n(2a_n - t) + 2\beta_n + 2\beta_{n+1} = 2n + 1 + \lambda, \quad (1.9a)
\]

\[
(2a_n - t)(2a_{n-1} - t)\beta_n = (2\beta_n - n)(2\beta_n - n - \lambda). \quad (1.9b)
\]

**Lemma 1.2.** The recurrence coefficients \( a_n(t) \) and \( \beta_n(t) \) are given by

\[
a_n(t) = \frac{1}{2} q_n(s) + \frac{1}{2} t, \quad \beta_n(t) = -\frac{1}{8} q_n'(s) - \frac{1}{8} q_n^2(s) - \frac{1}{4} s q_n(s) + \frac{1}{2} n + \frac{1}{4} \lambda.
\]
Statement of main results

In this subsection, we present the main results obtained in this paper, which are not considered in previous works [1, 3, 6]. For convenience, we will take $\lambda$ in the weight (1.2) to be strictly positive in the following discussions. This is due to two reasons. First, it makes the weight vanish at the endpoints of the orthogonality interval and then the ladder operator approach can be applied. Second, in this case, the potential for the weight is convex such that the equilibrium density discussed in Section 4 is supported in a single interval (the so-called one-cut case).

For brevity, we will not show the $t$-dependence of all the quantities, such as the recurrence coefficients $\alpha_n$ and $\beta_n$, considered in this paper from now on. By applying the ladder operators to the monic semi-classical Laguerre polynomials, we have the following theorem.

The Hankel determinant generated by the semi-classical Laguerre weight, satisfies the fourth Painlevé equation [10]

$$q_n''(s) = \frac{(q_n'(s))^2}{2q_n(s)} + \frac{3}{2} q_n^3(s) + 4 s q_n^2(s) + 2(s^2 - 2n - 1 - \lambda)q_n(s) - \frac{2\lambda^2}{q_n(s)}. \tag{1.10}$$

The Hankel determinant generated by the weight (1.2) is defined by

$$D_n(t) := \det(M_{t+j})(n-1)_{j=0} = \begin{vmatrix} \mu_0(t) & \mu_1(t) & \cdots & \mu_{n-1}(t) \\ \mu_1(t) & \mu_2(t) & \cdots & \mu_n(t) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1}(t) & \mu_n(t) & \cdots & \mu_{2n-2}(t) \end{vmatrix},$$

where $\mu_j(t)$ is the $j$th moment given by

$$\mu_j(t) := \int_0^\infty x^j w(x; t) dx \quad = \frac{1}{2} \Gamma\left(\frac{j + 1 + \lambda}{2}\right)_{1F1}\left(\frac{j + 1 + \lambda}{2}; \frac{1}{2}; \frac{t^2}{4}\right) + t \Gamma\left(\frac{j + 2 + \lambda}{2}\right)_{1F1}\left(\frac{j + 2 + \lambda}{2}; \frac{3}{2}; \frac{t^2}{4}\right),$$

where $_1F_1(\cdot; \cdot; \cdot)$ is Kummer’s confluent hypergeometric function (see, e.g., [11, p. 1023]). The moments can also be expressed in terms of the parabolic cylinder functions [3].

It is a known fact that the Hankel determinant can be expressed as a product of the square of $L^2$ norms for the monic orthogonal polynomials [12, (2.1.6)],

$$D_n(t) = \prod_{j=0}^{n-1} h_j(t). \tag{1.11}$$

In addition, it was shown in [3] that the Hankel determinant $D_n(t)$, generated by the semi-classical Laguerre weight, satisfies the Toda molecule equation [13]

$$\frac{d^2}{dt^2} \ln D_n(t) = \frac{D_{n+1}(t)D_{n-1}(t)}{D_n^2(t)}.$$

The rest of the paper is organized as follows. In the next subsection, we give a summary of the main results obtained in this paper. In Section 2, we recall some important results of the paper by Filipuk et al. [6], from which we derive the second-order differential equation for the semi-classical Laguerre polynomials. In Section 3, we show that the auxiliary quantities $R_n(t)$ and $r_n(t)$ satisfy the coupled Riccati equations. This enables us to obtain the second-order differential equation for $\beta_n(t)$, which is equivalent to a Chazy type equation under the suitable transformation. Furthermore, we find that $p(n, t)$, the sub-leading coefficient of the monic semi-classical Laguerre polynomials, satisfies both the continuous and discrete Jimbo–Miwa–Okamoto $\sigma$-form of Painlevé IV. In Section 4, we derive the large $n$ asymptotics of the recurrence coefficients, the sub-leading coefficient, and the Hankel determinant by using the Coulomb fluid approach.

1.1 Statement of main results

In this subsection, we present the main results obtained in this paper, which are not considered in previous works [1, 3, 6]. For convenience, we will take $\lambda$ in the weight (1.2) to be strictly positive in the following discussions. This is due to two reasons. First, it makes the weight vanish at the endpoints of the orthogonality interval and then the ladder operator approach can be applied. Second, in this case, the potential for the weight is convex such that the equilibrium density discussed in Section 4 is supported in a single interval (the so-called one-cut case).

For brevity, we will not show the $t$-dependence of all the quantities, such as the recurrence coefficients $\alpha_n$ and $\beta_n$, considered in this paper from now on. By applying the ladder operators to the monic semi-classical Laguerre polynomials, we have the following theorem.
\textbf{Theorem 1.3.} The semi-classical Laguerre polynomials $P_n(x)$ satisfy the linear second-order ordinary differential equation

$$P_n''(x) + \Phi_n(x)P_n'(x) + \Psi_n(x)P_n(x) = 0,$$

(1.12)

where $\Phi_n(x)$ and $\Psi_n(x)$ are expressed in terms of $\alpha_n$ and $\beta_n$ as follows:

$$\Phi_n(x) = t - 2x + \frac{\lambda}{x} - \frac{t - 2\alpha_n}{x(t - 2\alpha_n)},$$

(1.13)

$$\Psi_n(x) = 2n - \frac{nt - 4\alpha_n \beta_n}{x} - \frac{2(n - 2\beta_n)(n + \lambda - 2\beta_n)}{x(t - 2\alpha_n)} + \frac{n - 2\beta_n}{x^2} + \frac{(t - 2\alpha_n)(n - 2\beta_n)}{x^2(t - 2\alpha_n)}.$$  

(1.14)

\textbf{Theorem 1.4.} The recurrence coefficient $\beta_n(t)$ satisfies the second-order differential equation

$$[2\beta_n'' + 12\beta_n^2 - 4(2n + \lambda)\beta_n + n(n + \lambda)]^2 = t^2 \left[(\beta_n')^2 + 4\beta_n^3 - 2(2n + \lambda)\beta_n^2 + n(n + \lambda)\beta_n \right].$$

(1.15)

Let $t = \sqrt{2} z$ and

$$\beta_n(t) = \frac{2n + \lambda}{6} - \frac{v(z)}{2}.$$

Then $v(z)$ admits the first member of Chazy II system [14, (1.17)]

$$\left(v'' - 6v^2 - \tilde{a}_1 \right) = z^2 \left((v')^2 - 4v^3 - 2\tilde{a}_1 v - \tilde{\beta}_1 \right),$$

(1.16)

with the parameters

$$\tilde{a}_1 = -\frac{2}{3}(n^2 + n\lambda + \lambda^2), \quad \tilde{\beta}_1 = -\frac{4}{27}(2n^3 + 3n^2\lambda - 3n\lambda^2 - 2\lambda^3).$$

(1.17)

\textbf{Remark 1.} We use here $v(\cdot)$ to match with [14, (1.17)]. Do not confuse it with the notation $v(\cdot)$ for the potential in (2.3).

The Chazy equations were first found by Chazy [15, 16] and subsequently derived by a number of authors. These equations also arose in the problems on the Gaussian, Laguerre, and Jacobi weights with jump discontinuities [17–19]. The following theorem reveals the relation between the sub-leading coefficient $p(n, t)$ and the Painlevé IV equation.

\textbf{Theorem 1.5.} Let

$$\sigma_n(s) := -2p(n, t) - (n + \lambda)t, \quad s = \frac{1}{2} t.$$  

(1.18)

Then $\sigma_n(s)$ satisfies the Jimbo–Miwa–Okamoto $\sigma$-form of Painlevé IV [20, (C.37)]:

$$(\sigma_n''(s))^2 = 4(\sigma_n'(s) - \sigma_n(s))^2 - 4(\sigma_n'(s) + v_0)(\sigma_n'(s) + v_1)(\sigma_n'(s) + v_2),$$

(1.19)

with the parameters

$$v_0 = 0, \quad v_1 = 2\lambda, \quad v_2 = 2n + 2\lambda.$$  

Moreover, $\sigma_n(s)$ also admits the discrete $\sigma$-form of Painlevé IV:

$$2[\sigma_n + n(\sigma_{n-1} - \sigma_{n+1}) + 2\lambda s][\sigma_n + (n + \lambda)(\sigma_{n-1} - \sigma_{n+1})] = [\sigma_n + 2(n + \lambda)s][\sigma_{n+1} - \sigma_{n-1} + 2s][\sigma_{n-1} - \sigma_n][\sigma_n - \sigma_{n+1}].$$

(1.20)

The following results are concerned with the large $n$ asymptotics of the recurrence coefficients $\alpha_n$ and $\beta_n$, the sub-leading coefficient $p(n, t)$, and the Hankel determinant $D_n(t)$. The derivation is based on the Coulomb fluid approach together with the discrete system satisfied by the recurrence coefficients.
Theorem 1.6. The recurrence coefficients $a_n$ and $b_n$ have the asymptotic expansions as $n \to \infty$:

\[
\begin{align*}
\alpha_n &= \sqrt{\frac{2n}{3}} \left[ t + t^2 + 12(1 + \lambda) \frac{24\sqrt{6n}}{2304\sqrt{6n^{3/2}}} \right] + \frac{t(9\lambda^2 - 2)}{144n^2} + \frac{t^2 + 24t^2(1 + \lambda) - 48(6\lambda^2 - 6\lambda - 5)}{110592\sqrt{6n^{5/2}}} + \frac{t \left[t^2(27\lambda^2 - 7) - 12(9\lambda^3 + 9\lambda^2 - 2\lambda - 2)\right]}{1728n^3} + O(n^{-7/2}), \\
\beta_n &= \frac{n}{6} + \frac{t\sqrt{n}}{6\sqrt{6}} + \frac{t^2 + 6\lambda}{72} + \frac{t(t^2 + 12\lambda)}{288\sqrt{6n}} - \frac{t(t^2 + 18\lambda)}{216} + \frac{-t^4 + 24\lambda t^2 + 3168\lambda^2 - 816}{1152(n^2)} + \frac{t \left[t^2(7 - 27\lambda^2) + 4\lambda(9\lambda^2 - 2)\right]}{165888\sqrt{6n^{3/2}}} + \frac{t \left[t^2(27\lambda^2 - 7) + 12\lambda(2 - 9\lambda^2)\right]}{3456n^2} + O(n^{-5/2}).
\end{align*}
\]

Theorem 1.7. The sub-leading coefficient $p(n, t)$ has the large $n$ asymptotic expansion

\[
\begin{align*}
p(n, t) &= -\frac{2}{3} \sqrt{\frac{2}{3}} \frac{n^{3/2}}{n} - \frac{nt}{6} - \frac{(t^2 + 12\lambda)\sqrt{n}}{12\sqrt{6}} - \frac{t(t^2 + 18\lambda)}{216} - \frac{-t^4 + 24\lambda t^2 - 288\lambda^2 + 48}{1152\sqrt{6n}} + \frac{t(9\lambda^2 - 2)}{144n^2} + \frac{t^6 + 36\lambda t^4 + 144t^2(66\lambda^2 - 17) - 1728\lambda(8\lambda^2 - 1)}{165888\sqrt{6n^{3/2}}} + \frac{t \left[t^2(27\lambda^2 - 7) + 12\lambda(2 - 9\lambda^2)\right]}{3456n^2} + O(n^{-5/2}).
\end{align*}
\]

Theorem 1.8. The Hankel determinant $D_n(t)$ has the large $n$ expansion

\[
\begin{align*}
\ln D_n(t) &= \frac{1}{2} n^2 \ln n - \frac{3 + 2 \ln 6}{4} n^2 + \frac{2}{5} \sqrt{\frac{2}{3}} n^{3/2} t + \frac{\lambda}{2} n \ln n + C_1 n + \frac{t(t^2 + 36\lambda)\sqrt{n}}{36\sqrt{6}} + \frac{3\lambda^2 - 1}{6} \ln n + C_2 + \frac{t \left[t^4 + 40\lambda t^2 + 240(1 - 6\lambda^2)\right]}{5760\sqrt{6n}} - \frac{(9\lambda^2 - 2)t^2 - 12\lambda(5\lambda^2 - 2)}{288n} + O(n^{-3/2}),
\end{align*}
\]

where

\[
C_1 = \frac{t^2}{12} + \ln(2\pi) - \frac{\lambda(1 + \ln 6)}{2},
\]

\[
C_2 = \frac{t^2(t^2 + 36\lambda)}{864} + \frac{1}{24} \left[48\zeta^2(-1) - 24\ln G(\lambda + 1) - 12\lambda^2 \ln \frac{3}{2} + 12\lambda \ln(2\pi) - 4\ln 2 + 3\ln 3 \right],
\]

and $\zeta(\cdot)$ is the Riemann zeta function and $G(\cdot)$ is the Barnes G-function, which satisfies the relation [21, 22]

\[
G(z + 1) = \Gamma(z)G(z), \quad G(1) := 1.
\]
Finally, we make a remark about the differential equation (1.12) as $n \to \infty$. Substituting the large $n$ expansion of $\alpha_n$ and $\beta_n$ in Theorem 1.6 into (1.13) and (1.14), we obtain

$$\Phi_n(x) = t - 2x + \frac{1+\lambda}{x} + O(n^{-1/2}),$$
$$\Psi_n(x) = \frac{4\sqrt{6} n^{3/2}}{9x} + O(n).$$

Considering the equation

$$\ddot{P}_n(x) + \left( t - 2x + \frac{1+\lambda}{x} \right) \dot{P}_n(x) + \frac{4\sqrt{6} n^{3/2}}{9x} P_n(x) = 0,$$ 

one would find that this is the biconfluent Heun equation (BHE) [23, p. 194 (1.2.5)]. The relations between orthogonal polynomials and Heun’s differential equations have been discussed in recent years; see [24–26] for reference.

### 2 | LADDER OPERATORS AND SECOND-ORDER DIFFERENTIAL EQUATION

The ladder operator approach has been applied to solve problems on orthogonal polynomials for many years. This approach is especially useful to establish the relations between Painlevé equations and recurrence coefficients of semi-classical orthogonal polynomials. See [2, 5–8] for reference.

Following the general set-up (see, e.g., [2]) and noting that $w(0) = w(\infty) = 0$ since we require $\lambda > 0$ in (1.2), Filipuk et al. [6] showed that the monic semi-classical Laguerre polynomials $P_n(x)$ satisfy the following ladder operator equations:

$$\left( \frac{d}{dx} + B_n(x) \right) P_n(x) = \beta_n A_n(x) P_{n-1}(x), \quad (2.1)$$
$$\left( \frac{d}{dx} - B_n(x) - v'(x) \right) P_{n-1}(x) = -A_{n-1}(x) P_n(x), \quad (2.2)$$

where $v(x)$ is the potential

$$v(x) = -\ln w(x) = x^2 - tx - \lambda \ln x \quad (2.3)$$

and

$$A_n(x) = 2 + \frac{R_n(t)}{x}, \quad (2.4)$$
$$B_n(x) = \frac{r_n(t)}{x}, \quad (2.5)$$

with

$$R_n(t) := \frac{\lambda}{h_n} \int_0^\infty P_n^2(y)y^{n-1}e^{-y^2+ty}dy,$$
$$r_n(t) := \frac{\lambda}{h_{n-1}} \int_0^\infty P_n(y)P_{n-1}(y)y^{n-1}e^{-y^2+ty}dy.$$

Substituting (2.4) and (2.5) into the compatibility conditions for the ladder operators, Filipuk et al. [6] obtained the following results.

**Lemma 2.1.** The auxiliary quantities $R_n(t)$, $r_n(t)$ and the recurrence coefficients $\alpha_n$, $\beta_n$ satisfy the relations:

$$R_n(t) = 2\alpha_n - t, \quad (2.6)$$
$$r_n(t) + r_{n+1}(t) = \lambda - \alpha_n R_n(t), \quad (2.7)$$
\begin{align*}
    r_n(t) & = 2\beta_n - n, \tag{2.8} \\
    r_n^2(t) - \lambda r_n(t) & = \beta_n R_n(t) R_{n-1}(t), \tag{2.9} \\
    \sum_{j=0}^{n-1} R_j(t) - tr_n(t) & = 2\beta_n (R_n(t) + R_{n-1}(t)). \tag{2.10}
\end{align*}

We would like to point out that one will obtain the results in Lemma 1.1 by substituting (2.6) and (2.8) into (2.7) and (2.9), respectively. We now prove Theorem 1.3.

\textbf{Proof of Theorem 1.3.} It was shown in [2] that the orthogonal polynomials \( P_n(x) \) satisfy the second-order differential equation

\[ P_n''(x) - \left( \frac{A'_n(x)}{A_n(x)} \right) P'_n(x) + \left( B'_n(x) - B_n(x) \frac{A'_n(x)}{A_n(x)} + \sum_{j=0}^{n-1} A_j(x) \right) P_n(x) = 0, \tag{2.11} \]

which is obtained by eliminating \( P_{n-1}(x) \) from the ladder operators (2.1) and (2.2).

Next, we will express the coefficients in the above equation in terms of \( \alpha_n \) and \( \beta_n \). Inserting (2.6) into (2.4) and (2.8) into (2.5) gives

\[ A_n(x) = 2 + \frac{2\alpha_n - t}{x}, \quad B_n(x) = \frac{2\beta_n - n}{x}. \tag{2.12} \]

From (2.4) and with the aid of (2.10), we have

\[ \sum_{j=0}^{n-1} A_j(x) = 2n + \frac{\sum_{j=0}^{n-1} R_j(t)}{x} = 2n + \frac{tr_n(t) + 2\beta_n R_n(t) + 2\beta_n R_{n-1}(t)}{x}. \]

It follows from (2.9) that

\[ \beta_n R_{n-1}(t) = \frac{r_n(t)(r_n(t) - \lambda)}{R_n(t)}. \]

By making use of (2.6) and (2.8), we then obtain

\[ \sum_{j=0}^{n-1} A_j(x) = 2n - \frac{nt - 4\alpha_n \beta_n}{x} - \frac{2(n - 2\beta_n)(n + \lambda - 2\beta_n)}{x(t - 2\alpha_n)}. \tag{2.13} \]

Substituting (2.12) and (2.13) into (2.11), we establish the theorem. \qed

\section{Chazy II and \( \sigma \)-Form of Painlevé IV}

In this section, we will prove that the recurrence coefficient \( \beta_n \) is related to a Chazy type equation, and the sub-leading coefficient \( p(n, t) \) satisfies the Jimbo–Miwa–Okamoto \( \sigma \)-form of Painlevé IV. To prove the results, we introduce the following lemma at first.

\textbf{Lemma 3.1.} The auxiliary quantities \( R_n(t) \) and \( r_n(t) \) admit the coupled Riccati equations:

\[ r_n'(t) = \frac{n + r_n(t)}{2} R_n(t) - \frac{r_n^2(t) - \lambda r_n(t)}{R_n(t)}, \tag{3.1} \]

\[ R_n'(t) = \lambda - 2r_n(t) - \frac{R_n(t)(t + R_n(t))}{2}. \tag{3.2} \]
Proof. From (1.3) we have
\[ \int_0^\infty P_n^2(x; t)x^4 e^{-x^2 + \mu x} \, dx = h_n(t) \]
and
\[ \int_0^\infty P_n(x; t)P_{n-1}(x; t)x^2 e^{-x^2 + \mu x} \, dx = 0. \]
By taking derivatives with respect to \( t \), we obtain
\[ \frac{d}{dt} \ln h_n(t) = \alpha_n = \frac{t + R_n(t)}{2} \] \( (3.3) \)
and
\[ \frac{d}{dt} p(n, t) = -\beta_n = \frac{n + r_n(t)}{2}, \]
respectively. Taking account of (1.7), it follows from (3.3) that
\[ 2\beta_n' = \beta_n R_n(t) - \beta_n R_{n-1}(t). \] \( (3.5) \)
Using (2.8) and (2.9), we arrive at (3.1).
On the other hand, taking a derivative in (1.6) and in view of (3.4), we find
\[ 2\alpha_n' = 1 + r_{n+1}(t) - r_n(t) = 1 + \lambda - \alpha_n R_n(t) - 2r_n(t), \]
where use has been made of (2.7) in the second equality. Finally we obtain (3.2) with the aid of (2.6). \( \Box \)

Proof of Theorem 1.4. Solving \( R_n(t) \) from (3.1), we have two solutions:
\[ R_n(t) = \frac{r_n'(t) \pm \sqrt{(r_n'(t))^2 - 2r_n(t)(n + r_n(t))(\lambda - r_n(t))}}{n + r_n(t)}. \]
Substituting either solution into (3.2), we obtain the second-order differential equation satisfied by \( r_n(t) \) after removing the square roots:
\[ 4[r_n''(t) + 3r_n'(t) + 2(n - \lambda)r_n(t) - n\lambda] = t^2 [(r_n'(t))^2 + 2r_n^3(t) + 2(n - \lambda)r_n^2(t) - 2n\lambda r_n(t)]. \]
Then Equation (1.15) follows from the relation (2.8). Under the given transformation, Equation (1.15) turns into the Chazy equation (1.16). \( \Box \)

Remark 2. In the appendix of the paper [14], Cosgrove gave the relationship between Chazy II system and Painlevé equations. It is easy to check that our results in Theorem 1.4 are coincident with (A.1) and (A.2) in [14] by using Lemma 1.2. To be specific, we have \( z = \sqrt{2}x \) and \( v(z) = \frac{1}{4}q_n'(s) + \frac{1}{4}q_n^3(s) + \frac{1}{2}q_n(s) - \frac{n}{3} - \frac{\lambda}{6} \), where \( q_n(s) \) satisfies the Painlevé IV equation (1.10). This corresponds to \( A = \frac{1}{2}, \epsilon_1 = 1 \) and \( q = \frac{1}{3}(2n + \lambda) \) (one solution of the cubic equation \( 4q^3 + 2\alpha_1q + \beta_1 = 0 \) with the values of \( \alpha_1 \) and \( \beta_1 \) given by (1.17)) in [14, (A.2)].

Remark 3. From (3.2) we have
\[ r_n(t) = \frac{1}{4} \left( 2\lambda - tR_n(t) - R_n^2(t) - 2R_n'(t) \right). \]
Substituting it into (3.1), we obtain the second-order differential equation for \( R_n(t) \):
\[ 8R_n(t)R_n''(t) - 4(R_n'(t))^2 - 3R_n^3(t) - 4tR_n^3(t) + (8n + 4\lambda + 4 - t^2)R_n^2(t) + 4\lambda^2 = 0. \]
Letting \( t = 2s \) and \( R_n(t) = q_n(s) \), we find that \( q_n(s) \) satisfies the Painlevé IV equation

\[
q_n''(s) = \frac{(q_n'(s))^2}{2q_n(s)} + \frac{3}{2} q_n'(s) + 4s q_n^2(s) + 2(s^2 - 2n - \lambda - 1)q_n(s) - \frac{2\lambda^2}{q_n(s)}.
\]

These results are equivalent to those in Lemma 1.2.

**Proof of Theorem 1.5.** Recall that from (3.4) we have

\[
\beta_n = -\frac{d}{dt} p(n, t). \tag{3.6}
\]

Then Equation (3.5) becomes

\[
\beta_n R_n(t) - \beta_{n-1} R_n(t) = -2 \frac{d^2}{dt^2} p(n, t). \tag{3.7}
\]

On the other hand, from (1.8) and (2.6) we find

\[
\sum_{j=0}^{n-1} R_j(t) = -2p(n, t) - nt. \tag{3.8}
\]

The combination of (2.8) and (3.6) gives

\[
r_n(t) = -2 \frac{d}{dt} p(n, t) - n. \tag{3.9}
\]

In view of (3.8) and (3.9), Equation (2.10) turns into

\[
\beta_n R_n(t) + \beta_{n-1} R_n(t) = t \frac{d}{dt} p(n, t) - p(n, t). \tag{3.10}
\]

The sum and difference of (3.7) and (3.10) produce

\[
2\beta_n R_n(t) = t \frac{d}{dt} p(n, t) - p(n, t) - 2 \frac{d^2}{dt^2} p(n, t) \tag{3.11}
\]

and

\[
2\beta_{n-1} R_{n-1}(t) = t \frac{d}{dt} p(n, t) - p(n, t) + 2 \frac{d^2}{dt^2} p(n, t), \tag{3.12}
\]

respectively. The product of (3.11) and (3.12) gives

\[
4\beta_n \cdot \beta_{n-1}(t) R_{n-1}(t) = \left( t \frac{d}{dt} p(n, t) - p(n, t) \right)^2 - 4 \left( \frac{d^2}{dt^2} p(n, t) \right)^2. \tag{3.13}
\]

From (2.9) and (3.9) we have

\[
\beta_n R_n(t) R_{n-1}(t) = \left( n + 2 \frac{d}{dt} p(n, t) \right) \left( n + \lambda + 2 \frac{d}{dt} p(n, t) \right). \tag{3.14}
\]

Substituting (3.6) and (3.14) into (3.13), we obtain

\[
4 \left( \frac{d^2}{dt^2} p(n, t) \right)^2 = \left( t \frac{d}{dt} p(n, t) - p(n, t) \right)^2 + 4 \frac{d}{dt} p(n, t) \left( n + 2 \frac{d}{dt} p(n, t) \right) \left( n + \lambda + 2 \frac{d}{dt} p(n, t) \right). \tag{3.15}
\]

This equation is converted into (1.19) under the transformation (1.18).
Next, we derive the second-order difference equation satisfied by \( p(n, t) \). Substituting (2.6) and (2.8) into (2.10) and using (3.8), we have
\[
p(n, t) = \beta_n (t - 2a_n - 2a_{n-1}).
\] (3.16)

Taking account of (1.6), we can express \( \beta_n \) in terms of \( p(n, t) \) and \( p(n \pm 1, t) \):
\[
\beta_n = \frac{p(n, t)}{t + 2p(n + 1, t) - 2p(n - 1, t)}.
\] (3.17)

Substituting (2.6) and (2.8) into (2.9) gives
\[
(2\beta_n - n)(2\beta_n - n - \lambda) = \beta_n (2a_n - t)(2a_{n-1} - t)
\] (3.18)
\[
= \beta_n (2p(n, t) - 2p(n + 1, t) - 2p(n - 1, t) - 2p(n, t) - t).
\]

Inserting (3.17) into (3.18), we obtain the second-order difference equation satisfied by \( p(n) := p(n, t) \):
\[
[2p(n) - n (t + 2p(n + 1) - 2p(n - 1))] [2p(n) - (n + \lambda) (t + 2p(n + 1) - 2p(n - 1))]
\] (3.19)
\[
= p(n) (t + 2p(n + 1) - 2p(n - 1)) (t + 2p(n + 1) - 2p(n) (t + 2p(n) - 2p(n - 1))).
\]

Equation (1.20) follows from the transformation (1.18). The proof is complete. \( \square \)

**Remark 4.** Let \( H_n(t) \) be the logarithmic derivative of the Hankel determinant, that is,
\[
H_n(t) := \frac{d}{dt} \ln D_n(t).
\] (3.19)

From (1.11) we have
\[
H_n(t) = \sum_{j=0}^{n-1} \frac{d}{dt} \ln h_j(t).
\]

Using the first equality in (3.3) and (1.8), we obtain
\[
H_n(t) = -p(n, t).
\] (3.20)

In this case, Equation (1.19) is equivalent to the result obtained in [3, Theorem 4.11].

### 4 Large \( n \) Asymptotics of the Recurrence Coefficients and the Hankel Determinant

In random matrix theory (RMT), it is known that our Hankel determinant \( D_n(t) \) can be viewed as the partition function for the semi-classical Laguerre unitary ensemble [12, Corollary 2.1.3]. That is,
\[
D_n(t) = \frac{1}{n!} \int_{(0, \infty)^n} \prod_{1 \leq i < j \leq n} (x_i - x_j)^2 \prod_{k=1}^{n} x_k^2 e^{-x_k^2 + tx_k} dx_k,
\]

where \( x_1, x_2, \ldots, x_n \) are the eigenvalues of \( n \times n \) Hermitian matrices from the ensemble with the joint probability density function
\[
p(x_1, x_2, \ldots, x_n) = \frac{1}{n! D_n(t)} \prod_{1 \leq i < j \leq n} (x_i - x_j)^2 \prod_{k=1}^{n} x_k^2 e^{-x_k^2 + tx_k}.
\]

See [27–29] for more discussions of this topic.
Dyson’s Coulomb fluid approach [30] shows that the collection of eigenvalues (particles) can be approximated as a continuous fluid with a density \( \sigma(x) \) supported in \( J \), a subset of \( \mathbb{R} \), when \( n \) is sufficiently large. It is easy to see that the potential \( v(x) \) in (2.3) is convex for \( x \in \mathbb{R}^+ \) when \( \lambda > 0 \). In this case, \( J \) is a single interval denoted by \((a, b)\); see Chen and Ismail [31] and also [32, p. 198].

Following [31], the equilibrium density \( \sigma(x) \) is determined by the constrained minimization problem:

\[
\min_{\sigma} F[\sigma] \quad \text{subject to} \quad \int_a^b \sigma(x) dx = n,
\]

where

\[
F[\sigma] := \int_a^b \sigma(x) v(x) dx - \int_a^b \int_a^b \sigma(x) \ln |x - y| \sigma(y) dx dy
\]

and \( v(x) \) is the potential given by (2.3).

It follows that the density \( \sigma(x) \) satisfies the integral equation

\[
v(x) - 2 \int_a^b \ln |x - y| \sigma(y) dy = A, \quad x \in (a, b),
\]

where \( A \) is the Lagrange multiplier that fixes the constraint. Differentiating the above equation with respect to \( x \) gives the singular integral equation

\[
v'(x) - 2P \int_a^b \frac{\sigma(y)}{x - y} dy = 0, \quad x \in (a, b),
\]

where \( P \) represents the Cauchy principal value. The solution of (4.2) subject to the boundary condition \( \sigma(a) = \sigma(b) = 0 \) is

\[
\sigma(x) = \frac{\sqrt{(b - x)(x - a)}}{2\pi^2} P \int_a^b \frac{v'(y)}{(y - x)\sqrt{(b - y)(y - a)}} dy,
\]

with a supplementary condition

\[
\int_a^b \frac{v'(x)}{\sqrt{(b - x)(x - a)}} dx = 0.
\]

In addition, using (4.3), the normalization condition \( \int_a^b \sigma(x) dx = n \) becomes

\[
\frac{1}{2\pi} \int_a^b \frac{x v'(x)}{\sqrt{(b - x)(x - a)}} dx = n.
\]

The endpoints \( a \) and \( b \) are determined by (4.4) and (4.5). Furthermore, it is shown in [31] that

\[
\alpha_n = \frac{a + b}{2} + O \left( \frac{\partial^3 A}{\partial n^3} \right),
\]

\[
\beta_n = \left( \frac{b - a}{4} \right)^2 \left( 1 + O \left( \frac{\partial^4 A}{\partial n^3} \right) \right).
\]

Substituting (2.3) for \( v(x) \) into (4.4) and (4.5) respectively, we get two equations for \( a \) and \( b \):

\[
(X - t)^2 Y = \lambda^2,
\]

\[
3X^2 - 2tX - 4Y = 8n + 4\lambda.
\]
where
\[ X = a + b, \quad Y = ab. \]

Eliminating \( Y \) from (4.7) and (4.8), we have the following result.

**Lemma 4.1.** The quantity \( X = a + b \) satisfies a quartic equation
\[ (X - t)^2(3X^2 - 2tX - 8n - 4\lambda) = 4\lambda^2. \] (4.9)

Using MATHEMATICA, we find that Equation (4.9) has only one positive solution when \( n \to \infty \) and the series expansion reads
\[
X = 2\sqrt{\frac{2n}{3}} + \frac{t^2 + 12\lambda}{12\sqrt{6n}} - \frac{t^4 + 24\lambda t^2 - 288\lambda^2}{1152\sqrt{6}n^{3/2}} + \frac{a^2 t}{8n^2} \\
+ \frac{t^6 + 36\lambda t^4 + 9504\lambda^2 t^2 - 13824\lambda^3}{55296\sqrt{6}n^{5/2}} + \frac{\lambda^2 t(t^2 - 4\lambda)}{32n^3} + O(n^{-7/2}).
\]

It follows that
\[
a + b \over 2 = \frac{\sqrt{2n}}{3} + \frac{t^2 + 12\lambda}{24\sqrt{6n}} - \frac{t^4 + 24\lambda t^2 - 288\lambda^2}{2304\sqrt{6}n^{3/2}} + \frac{\lambda^2 t}{16n} \\
+ \frac{t^6 + 36\lambda t^4 + 9504\lambda^2 t^2 - 13824\lambda^3}{110592\sqrt{6}n^{5/2}} + \frac{\lambda^2 t(t^2 - 4\lambda)}{64n^3} + O(n^{-7/2}),
\] (4.10a)

and
\[
\left( b - a \over 4 \right)^2 = \frac{X^2 - 4Y}{16} = \frac{4n + 2\lambda + tX - X^2}{8} \\
= \frac{n}{6} + \frac{t\sqrt{n}}{6\sqrt{6}} + \frac{t^2 + 6\lambda}{72} + \frac{t(t^2 + 12\lambda)}{288\sqrt{6n}} - \frac{\lambda^2 t}{16n} - \frac{t(t^4 + 24\lambda t^2 + 3168\lambda^2)}{27648\sqrt{6}n^{3/2}} \\
- \frac{\lambda^2 (3t^2 - 4\lambda)}{128n^2} + \frac{t(t^6 + 36\lambda t^4 - 35424\lambda^2 t^2 + 110592\lambda^3)}{1327104\sqrt{6}n^{5/2}} + O(n^{-3}).
\] (4.10b)

where use has been made of (4.8) in the second equality.

By using the similar method in [33], we evaluate the Lagrange multiplier \( A \) in the following lemma. The proof will be omitted. The key is that we multiply by \( 1/\sqrt{(b - x)(x - a)} \) on both sides of Equation (4.1) and then integrate with respect to \( x \) from \( a \) to \( b \).

**Lemma 4.2.** We have
\[
A = \frac{3a^2 + 2ab + 3b^2}{8} - \frac{(a + b)t}{2} - \lambda \ln \frac{a + b + 2\sqrt{ab}}{4} - 2n \ln \frac{b - a}{4} \\
= \frac{4n + 2\lambda + tX}{4} - \lambda \ln \frac{X^2 - tX + 2\lambda}{4(X - t)} - n \ln \frac{4n + 2\lambda + tX - X^2}{8}.
\]

Then as \( n \to \infty \),
\[
A = -n \ln n + n(1 + \ln 6) - t\sqrt{\frac{2n}{3}} - \frac{\lambda}{2} \ln n + \frac{6\lambda \ln 6 - t^2}{12} - \frac{t(t^2 + 36\lambda)}{72\sqrt{6n}} - \frac{\lambda^2}{2n} \\
+ \frac{t(t^4 + 40\lambda t^2 - 1440\lambda^2)}{11520\sqrt{6}n^{3/2}} - \frac{\lambda^2 (3t^2 - 20\lambda)}{96n^2} + O(n^{-5/2}).
\] (4.11)
**Proof of Theorem 1.6.** From (4.6), (4.10) and (4.11), we see that \( \alpha_n \) and \( \beta_n \) have the large \( n \) expansion form

\[
\alpha_n = \sqrt{\frac{2n}{3}} + \sum_{j=0}^{\infty} \frac{a_j}{n^j} \tag{4.12a}
\]

and

\[
\beta_n = \frac{n}{6} + \sum_{j=-1}^{\infty} \frac{b_j}{n^j}, \tag{4.12b}
\]

respectively. Substituting (4.12) into the discrete system (1.9) and taking a large \( n \) limit, we obtain the expansion coefficients \( a_j \) and \( b_j \) recursively by equating the powers of \( n \):

\[
a_0 = \frac{t}{6}, \quad b_{-1} = \frac{t}{6\sqrt{6}}, \quad a_1 = \frac{t^2 + 12(1 + \lambda)}{24\sqrt{6}}, \quad b_0 = \frac{t^2 + 6\lambda}{72},
\]

\[
a_2 = 0, \quad b_1 = \frac{t(t^2 + 12\lambda)}{288\sqrt{6}}, \quad a_3 = \frac{t^4 + 24t^3(1 + \lambda) - 48(6\lambda^2 - 6\lambda - 5)}{2304\sqrt{6}}, \quad b_2 = \frac{2 - 9\lambda^2}{144},
\]

and so on. The theorem is then established.

**Remark 5.** Recently, Clarkson and Jordaan studied the generalized Airy polynomials and derived the large \( n \) formal asymptotic expansions for the recurrence coefficients; see [34, Lemma 3.15]. We expect Dyson’s Coulomb fluid approach can be applied to justify the assumption of the expansion forms for the recurrence coefficients in the proof [34, (46)]. That is, there should also be an algebraic equation similar to (4.9) to determine the endpoints of the support interval of the equilibrium density.

**Proof of Theorem 1.7.** Recall that \( p(n, t) \) can be expressed in terms of \( \alpha_n \) and \( \beta_n \) (see (3.16)):

\[
p(n, t) = \beta_n(t - 2\alpha_n - 2\alpha_{n-1}).
\]

Substituting (1.21) and (1.22) into the above, we obtain (1.23) after taking a large \( n \) limit.

**Proof of Theorem 1.8.** Following the similar development in [7, 33] and using the fact

\[
\beta_n = \frac{D_{n+1}(t)D_{n-1}(t)}{D_n^2(t)},
\]

we obtain the large \( n \) asymptotic expansion of \( D_n(t) \):

\[
\ln D_n(t) = \frac{1}{2} n^2 \ln n - \frac{3 + 2 \ln 6}{4} n^2 + \frac{\sqrt{2}}{3} n^{3/2} t + \frac{\lambda}{2} n \ln n + C_1 n + \frac{t(t^2 + 36\lambda) \sqrt{n}}{36\sqrt{6}} + \frac{3\lambda^2 - 1}{6} \ln n + C_2 + \frac{t [t^4 + 40\lambda t^2 + 240(1 - 6\lambda^2)]}{5760\sqrt{6} n} - \frac{(9\lambda^2 - 2)t^2 - 12\lambda(5\lambda^2 - 2)}{288n} + O(n^{-3/2}),
\]

where \( C_1 \) and \( C_2 \) are two undetermined constants independent of \( n \). We proceed to determine them in the following analysis.

It is easy to see from (3.19) and (3.20) that

\[
\ln \frac{D_n(t)}{D_n(0)} = \int_0^t H_n(u) du = -\int_0^t p(n, u) du.
\]
Taking account of (1.23), we find
\[
\ln \frac{D_n(t)}{D_n(0)} = \frac{2}{3} \sqrt{\frac{2}{3}} n^{3/2} t + \frac{n t^2}{12} + \frac{t(t^2 + 36 \lambda) \sqrt{n}}{36 \sqrt{6}} + \frac{t^2(t^2 + 36 \lambda)}{864} + \frac{t^3(2 - 9 \lambda^2) t^2}{288 n} + \frac{t^4(5 t^6 + 252 \lambda t^4 + 1680(66 \lambda^2 - 17)t^2 - 60480 \lambda(8 \lambda^2 - 1))}{5760 \sqrt{6n}} \\
+ \frac{t^5(7 - 27 \lambda^2) t^2 + 24 \lambda(9 \lambda^2 - 2)}{13824 n^2} + O(n^{-3/2}).
\]

Next, we will use the results of Deaño and Simm [35] (see also [36]) to evaluate \(D_n(0)\). By making a simple change of variables, we have
\[
D_n(0) = \frac{n^{n(1+\lambda)/2}}{n!} Z_n(1),
\]

where
\[
Z_n(s) := \int_{(0, \infty)^n} \prod_{1 \leq i < j \leq n} (x_i - x_j)^2 \prod_{k=1}^n x_k^s e^{-n(x_k^2 + n(x_k^2 - x_i))} dx_k, \quad 0 \leq s \leq 1.
\]

It follows that
\[
\ln D_n(0) = \frac{1}{2} n^2 \ln n + \frac{\lambda}{2} n \ln n - n \ln n - n \ln \Gamma(n) + \ln \frac{Z_n(1)}{Z_n(n)} + \ln Z_n(0).
\]

Taking account of (2.31) and (A.3) in [35] and with the aid of Stirling's formula (see, e.g., [11, p. 895]), we obtain
\[
\ln D_n(0) = \frac{1}{2} n^2 \ln n - \frac{3 + 2 \ln 6}{4} n^2 + \frac{\lambda}{2} n \ln n + \left[ \ln(2\pi) - \frac{\lambda (1 + \ln 6)}{2} \right] n + \frac{3 \lambda^2 - 1}{6} \ln n \\
+ \frac{1}{24} \left[ 48 \zeta(-1) - 24 \ln G(\lambda + 1) - 12 \lambda^2 \ln \frac{3}{2} + 12 \lambda \ln(2\pi) - 4 \ln 2 + 3 \ln 3 \right] \\
+ O(n^{-1}),
\]

where \(\zeta(.)\) is the Riemann zeta function and \(G(.)\) is the Barnes G-function [21, 22].

The combination of (4.13) and (4.14) shows that
\[
C_1 = \frac{t^2}{12} + \ln(2\pi) - \frac{\lambda (1 + \ln 6)}{2},
\]
\[
C_2 = \frac{t^2(t^2 + 36 \lambda)}{864} + \frac{1}{24} \left[ 48 \zeta(-1) - 24 \ln G(\lambda + 1) - 12 \lambda^2 \ln \frac{3}{2} + 12 \lambda \ln(2\pi) - 4 \ln 2 + 3 \ln 3 \right].
\]

This completes the proof. \(\square\)

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**CONFLICT OF INTEREST STATEMENT**

The authors have no conflicts of interest to declare that are relevant to the content of this article.

**ORCID**

Chao Min https://orcid.org/0000-0002-6682-3830
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