Weak Convergence of Approximate reflection coupling and its Application to Non-convex Optimization

Keisuke Suzuki
Biometrics Research Laboratories, NEC Corporation, 1753, Shimonumabe, Nakahara-Ku, Kawasaki, Kanagawa 211-8666, Japan

Abstract
In this paper, we propose a weak approximation of the reflection coupling (RC) for stochastic differential equations (SDEs), and prove it converges weakly to the desired coupling. In contrast to the RC, the proposed approximate reflection coupling (ARC) need not take the hitting time of processes to the diagonal set into consideration and can be defined as the solution of some SDEs on the whole time interval. Therefore, ARC can work effectively against SDEs with different drift terms. As an application of ARC, an evaluation on the effectiveness of the stochastic gradient descent in a non-convex setting is also described. For the sample size \( n \), the step size \( \eta \), and the batch size \( B \), we derive uniform evaluations on the time with orders \( n^{-1}, \eta^{-1/2} \), and \( \sqrt{(n-B)/B(n-1)} \), respectively.

Keywords: Reflection Coupling; Stochastic Differential Equation; Gradient Descent; Non-convex Optimization

1. Introduction
Finding a good coupling \( \gamma \) between two probability measures \( \mu \) and \( \nu \) is important for evaluating the difference between them. Here, \( \gamma \) is said to be a coupling between \( \mu \) and \( \nu \) if each marginal distributions of \( \gamma \) coincide with \( \mu \) and \( \nu \), respectively. In fact, the Wasserstein distance \( \text{W}_2 \), which measures the difference between two probability measures through good couplings of them, is bounded by the Kullback–Leibler divergence \( \text{KL} \) and is one direction to connect the probability theory with the information theory. In particular, it is worth finding a good coupling between laws of solutions of stochastic differential equations (SDEs), which appear frequently in applications.

For a good coupling between laws of solutions of SDEs, \( \text{RC} \) introduced the reflection coupling (RC). For a continuously differentiable function \( H : \mathbb{R}^d \rightarrow \mathbb{R} \) and a \( d \)-dimensional Brownian motion \( W \), for example, we consider Langevin dynamics along with the gradient \( \nabla H \) of \( H \).

\[
dx_t = -\nabla H(X_t)dt + dW_t. \tag{1.1}\]

Then, the RC for \( \text{RC} \) is defined by

\[
dY_t = -\nabla H(Y_t)dt + (I_d - 2e_ie_i^\top)\,dW_t, \quad t < T, \quad Y_t = X_t, \quad t \geq T. \tag{1.2}\]

Here, \( I_d \) denotes the \( d \times d \) identity matrix, \( T = \inf\{t \geq 0 \mid X_t = Y_t\} \) is the hitting time of \((X,Y)\) to the diagonal set, and \( e_i = (X_t - Y_t)/\|X_t - Y_t\| \) is the orthogonal matrix \( I_d - 2e_ie_i^\top \) defines a plane symmetric transformation with respect to a plane orthogonal to \( e_i \). Therefore, denoting the indicator function of a set \( A \) by \( \chi_A \), \( W'_t = \int_0^t (I_d - 2\chi_{\{s<T\}}e_i e_i^\top)\,dW_s \) defines the Brownian motion whose instantaneous increments are plane symmetric with those of \( W \). Thus, for each \( t < T \), the orthogonal matrix \( I_d - 2e_ie_i^\top \) defines a process that approaches to the diagonal set by the symmetry of \( W \). In particular, \( Y \) is also a weak solution of \( \text{RC} \) by the Markov property of \( X \), and \((X,Y)\) is a process that approaches to the diagonal set by the symmetry of \( W \) and \( W' \). In fact, \( \text{RC} \) proved the inequality \( E[\rho_2(X_t,Y_t)] \leq e^{-ct}E[\rho_2(X_0,Y_0)] \) for some \( c > 0 \) and a function \( \rho_2 : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R} \) that satisfies \( \|x - y\|_2 \leq C\rho_2(x,y) \) for some \( C > 0 \). Thus, the 1-Wasserstein distance between Langevin dynamics \( \text{RC} \) with different initial values converges to 0 as \( t \) tends to infinity.

However, the RC does not work for Langevin dynamics with different or functional drift terms since, in this case, \( Y \) is not the weak solution of the SDE it should solve because of the definition of it for \( t \geq T \). For Langevin dynamics with different drift terms, \( \text{ARC} \) proposed the sticky coupling as a substitute for the RC. However, the sticky coupling can evaluate only the probability that this coupling is out of the diagonal set for each fixed time. That
is, we cannot evaluate important quantities like 1-Wasserstein distance by the sticky coupling. Therefore, we can conclude that good couplings for Langevin dynamics that are more general than (1.1) have not been found.

In this paper, we propose the following approximate reflection coupling (ARC) that also works for Langevin dynamics that are more general than (1.1).

\[ dY_t^{(e)} = -\nabla H(Y_t^{(e)})dt + (I_d - 2h_\varepsilon(\|X_t - Y_t^{(e)}\|_{\mathbb{R}^d})e_t^{(e)}e_t^{(e)\top})dW_t, \quad t \geq 0. \quad (1.3) \]

Here, \( e_t^{(e)} = (X_t - Y_t^{(e)})/\|X_t - Y_t^{(e)}\|_{\mathbb{R}^d} \) and \( h_\varepsilon : \mathbb{R} \rightarrow [0, 1] \) is an arbitrary \( C^1 \)-function that values 0 in a neighborhood of the origin and 1 outside of another neighborhood. (1.3) has an advantage in that it can define \( Y_t^{(e)} \) as the solution of the SDE defined on the whole time interval. In other words, in contrast to the RC, (1.3) does not need the particular definition of \( Y_t^{(e)} \) for \( t \) after the hitting time to the diagonal set and can handle the case of different or functional drift terms. The first main result, Theorem 2.2, states that the ARC defined between a semi-martingale and Langevin dynamics, which is a more general case than (1.1) and (1.3), converges weakly to the desired coupling.

The second main result is an application of Theorem 2.2 to the theoretical analysis of stochastic gradient Langevin dynamics (SGLD) in a non-convex setting. As we will see later, the problem to evaluate the effectiveness of SGLD is equivalent to the problem to evaluate the difference between Langevin dynamics with different drift and functional drift terms. Thus, we can derive a sharp evaluation of the effectiveness of SGLD by using ARC.

This paper is organized as follows. In Section 2, we give accurate statements of our main results of weak convergence of ARC and evaluations for SGLD. Section 3 describes the proof of the first main result, Theorem 2.2 and Sections 4, 5, and 6 are devoted to the proofs of the three inequalities given in the second main result, Theorem 2.4. Finally, auxiliary results are given in Appendix.

2. Main Result

To formulate our first main result, we introduce the following notations. \( \{F_t\} \) is a filtration that satisfies usual condition (Definition 1.2.25 in [13]) and \( W \) is a \( d \)-dimensional \( \{F_t\} \)-Brownian motion. \( X_0 \) and \( Y_0 \) are \( F_0 \)-measurable \( \mathbb{R}^d \)-valued random variables and \( V \) is a \( d \)-dimensional \( \{F_t\} \)-adapted continuous process with bounded variation and initial value \( V_0 = 0 \). \( C([0, \infty); \mathbb{R}^d) \) denotes the set of all continuous functions from \([0, \infty)\) to \( \mathbb{R}^d \) and a functional \( G : [0, \infty) \times C([0, \infty); \mathbb{R}^d) \rightarrow \mathbb{R}^d \) is progressively measurable (Definition 3.5.15, [13]). Then, we impose the following assumption. Here, \( L^p(\Omega; \mathbb{R}^d) \) is the set of all \( p \)-th integrable random variables from \( \Omega \) to \( \mathbb{R}^d \) and \( \dot{V} \) is the total variation of \( V \).

**Assumption 2.1.** For some \( p > 2 \), \( X_0, Y_0 \in L^p(\Omega; \mathbb{R}^d) \) and \( E[\|\dot{V}_t\|_{\mathbb{R}^d}^p] < \infty \) holds for all \( t \geq 0 \). In addition, there exists a constant \( K(t) \) for all \( t \geq 0 \) such that the following inequalities hold for all \( \varphi, \psi \in C([0, \infty); \mathbb{R}^d) \).

\[
\|G(s, \varphi) - G(s, \psi)\|_{\mathbb{R}^d} \leq K(t) \sup_{0 \leq u \leq s} \|\varphi(u) - \psi(u)\|_{\mathbb{R}^d}, \quad 0 \leq s \leq t, \quad (2.1)
\]

\[
\|G(s, \varphi)\|_{\mathbb{R}^d} \leq K(t) \left( 1 + \sup_{0 \leq u \leq s} \|\varphi(u)\|_{\mathbb{R}^d} \right), \quad 0 \leq s \leq t. \quad (2.2)
\]

Under Assumption 2.1, for a constant \( \sigma \neq 0 \), we define the semi-martingale \( X \) as

\[ X_t = X_0 + V_t + \sigma W_t, \quad t \geq 0 \quad (2.3) \]

and denote the solution of the functional SDE

\[ Y_t = Y_0 + \int_0^t G(s, Y_s)ds + \sigma W_t, \quad t \geq 0 \quad (2.4) \]

by \( Y \). Thus, Gronwall’s lemma yields \( E[\sup_{0 \leq s \leq t} \|Y_t\|_{\mathbb{R}^d}^p] < \infty \) for all \( t \geq 0 \).

Under the aforementioned notations, for all \( \varepsilon > 0 \), we define the ARC between \( X \) and \( Y \) by

\[
\begin{cases}
    dY_t^{(e)} = G(t, Y_t^{(e)})dt + \sigma(I_d - 2h_\varepsilon(\|X_t - Y_t^{(e)}\|_{\mathbb{R}^d})e_t^{(e)}e_t^{(e)\top})dW_t, \\
    Y_0^{(e)} = Y_0.
\end{cases}
\quad (2.5)
\]

Here, \( e_t^{(e)} = (X_t - Y_t^{(e)})/\|X_t - Y_t^{(e)}\|_{\mathbb{R}^d} \) and \( h_\varepsilon : \mathbb{R} \rightarrow [0, 1] \) is an arbitrary \( C^1 \)-function that satisfies

\[
\begin{cases}
    h_\varepsilon(a) = 0, \quad |a| \leq \varepsilon, \\
    h_\varepsilon(a) = 1, \quad |a| \geq 2\varepsilon.
\end{cases}
\quad (2.6)
\]

Our first main result is stated as follows, where \( \mathcal{L}(Z) \) denotes the law of a random variable \( Z \).
Theorem 2.2. Under Assumption $\mathcal{A}$, we can take a subsequence $\varepsilon_t$ and a coupling $\gamma$ between $L(X)$ and $L(Y)$ so that $L(X,Y(\varepsilon_t))$ converges weakly to $\gamma$.

Next, to formulate our second main result, we introduce the following notations. $Z$ is the set of all data points and $\ell(w;z)$ denotes the loss on $z \in Z$ for a parameter $w \in \mathbb{R}^d$. $z_1, \ldots, z_n$ are independent and identically distributed (IID) samples generated from the distribution $D$ on $Z$. For the batch size $B \leq n$, $\{I_k\}_{k=1}^{\infty}$ denotes the sequence of random extraction from $\{1, \ldots, n\}$ with size $B$. Finally, for each parameter $w \in \mathbb{R}^d$, we define the expected loss, the empirical loss, and its mini-batch by $L(w) = E_{z \sim D}[\ell(w; z)]$, $L_n(w) = \frac{1}{n} \sum_{i=1}^{n} \ell(w; z_i)$ and $L_n,k(w) = \frac{1}{k} \sum_{i \in I_k} \ell(w; z_i)$, respectively. For the step size $\eta > 0$ and the inverse temperature $\beta > 0$, we define the SGLD $X^{(n,\eta,B)}$ as

$$X_t^{(n,\eta,B)} = X_{kn}^{(n,\eta,B)} - (t-k\eta)\nabla L_n,k(X_{kn}^{(n,\eta,B)}) + \sqrt{2/\beta}(W_t-W_{kn}), \quad k\eta \leq t < (k+1)\eta. \quad (2.7)$$

There are many existing works [1, 3, 4, 8, 12, 14, 16, 18, 20, 21, 22, 23, 24] aimed at evaluating the effectiveness of SGLD. In almost all of them, the goal is to derive a sharp bound for fixed large $t$ to the quantity

$$E[L(X_t^{(n,\eta,B)})] - \min_{w \in \mathbb{R}^d} L(w) \quad (2.8)$$

in terms of $n, \eta, B$ and $\beta$.

Let two processes $X^{(n)}$ and $X^{(n,\eta)}$ defined by

$$dX_t^{(n)} = -\nabla L_n(X_t^{(n)})dt + \sqrt{2/\beta}dW_t, \quad t \geq 0, \quad (2.9)$$

$$X_t^{(n,\eta)} = -(t-k\eta)\nabla L_n(X_{kn}^{(n,\eta)}) + \sqrt{2/\beta}(W_t-W_{kn}), \quad k\eta \leq t < (k+1)\eta \quad (2.10)$$

have the same initial values as $X^{(n,\eta,B)}$. Then, the quantity (2.8) can be decomposed as

$$E[L(X_t^{(n,\eta,B)})] - \min_{w \in \mathbb{R}^d} L(w) = \{E[L(X_t^{(n,\eta,B)})] - E[L_n(X_t^{(n)})]\} + \{E[L_n(X_t^{(n)})] - \min_{w \in \mathbb{R}^d} L(w)\}. \quad (2.11)$$

According to (3.26) in [20] and the result in [6], the second term in the R.H.S of (2.11) is bounded by the form of constant times $e^{-ct} + \beta^{-1}\log(\beta/d+1)$. Thus, the problem to derive a bound to (2.8) can be reduced to the problem to evaluate the first term in the R.H.S of (2.11), which is the difference between Langevin dynamics with different drift terms, and ARC can be applied to evaluate it.

To evaluate the first term in the R.H.S of (2.11), we impose the following assumption, which is commonly used in previous works. Here, $C^k(\mathbb{R}^d; \mathbb{R})$ is the set of all $C^k$-functions from $\mathbb{R}^d$ to $\mathbb{R}$.

Assumption 2.3. The same initial value of (2.7), (2.9) and (2.10) belongs to $L^4(\Omega; \mathbb{R}^d)$. The loss $\ell(w;z)$ is nonnegative and satisfies $\sup_{z \in Z} \ell(0;z) < \infty$ and $\sup_{z \in Z} \|\ell(0;z)\|_{\mathbb{R}^d} \leq A$ for some $A > 0$. Thus, the expected loss $L(w) = E_{z \sim D}[\ell(w;z)]$ is well-defined. In addition, $\ell(\cdot;z) \in C^1(\mathbb{R}^d; \mathbb{R})$ satisfies the following two conditions for all $z \in Z$.

1. $(m,b)$-dissipative for some $m,b > 0$. Here, $H \in C^1(\mathbb{R}^d; \mathbb{R})$ is said to be $(m,b)$-dissipative when the following inequality holds.

$$\langle \nabla H(x), x \rangle_{\mathbb{R}^d} \geq m\|x\|_{\mathbb{R}^d}^2 - b, \quad x \in \mathbb{R}^d. \quad (2.12)$$

2. $M$-smooth for some $M > 0$. Here, $H \in C^1(\mathbb{R}^d; \mathbb{R})$ is said to be $M$-smooth when the following inequality holds.

$$\|\nabla H(x) - \nabla H(y)\|_{\mathbb{R}^d} \leq M\|x-y\|_{\mathbb{R}^d}, \quad x,y \in \mathbb{R}^d. \quad (2.13)$$

As in previous works [18, 20, 24], we decompose the first term in the R.H.S of (2.11) to the sum of $E[L(X_t^{(n,\eta,B)})] - E[L(X_t^{(n)})]$, $E[L(X_t^{(n)})] - E[L(X_t^{(n,\eta,B)})]$ and $E[L(X_t^{(n,\eta,B)})] - E[L_n(X_t^{(n)})]$, and prove the following bounds. Here, $f = O_\alpha(g)$ means that there exists a constant $C_\alpha > 0$ depends only on $\alpha$ such that $f \leq C_\alpha g$.

Theorem 2.4. Under Assumption 2.3 for some $\eta_0 = O_{m,M}(1)$, the following inequalities hold uniformly on $0 < \eta \leq \eta_0$. Here, $\alpha_0 = (m,b,M,\beta,A,E[\|X_0\|_{\mathbb{R}^d}^4], d)$.

1. $|E[L(X_t^{(n)})] - E[L_n(X_t^{(n)})]| \leq O_{\alpha_0}(n^{-1})$, 3
(2) \(|E[L(X_{t}^{(n,B)})] - E[L(X_{t}^{(n)})]| \leq O_{\alpha}(n^{1/2}),
(3) \)|E[L(X_{t}^{(n,B)})] - E[L(X_{t}^{(n)})]| \leq O_{\alpha}(n^{1/2} + \sqrt{(n - B)/B(n - 1)}).

Since \(t\) is large, Theorem 2.4 (2) and (3) are refinements of Corollary 2.9 in [24] and Theorem 3.6 in [23], which are the sharpest evaluation on \(\eta\) and \(B\) among previous works, respectively. The latter claim can be proved by Gronwall’s lemma.

3. Proof of First Main Result

In this section, we prove our first main result, Theorem 2.4. The scheme of our proof is as follows. First, we prove the tightness of \((X, Y^{(c)})\) and the existence of its weak limit \((\tilde{X}, \tilde{Y})\). Next, we confirm that \(\tilde{Y}\) solves the martingale problem corresponding to (2.4) and the law of \(\tilde{Y}\) coincides with that of \(Y\). Therefore, the weak limit \((\tilde{X}, \tilde{Y})\) defines a coupling between \(L(X)\) and \(L(Y)\), and Theorem 2.4 is proved.

3.1. Auxiliary lemmas

To prove Theorem 2.4, we prepare the following three lemmas assuming Assumption 2.1.

Lemma 3.1. There exists the strong solution \((X, Y^{(c)})\) of (2.4) uniquely and \(\sup_{\epsilon > 0} E[\sup_{0 \leq s \leq \epsilon} \|X_s, Y^{(c)}_s\|_{\bar{R}^d}] < \infty\) holds for each fixed \(t > 0\).

Proof. The existence and uniqueness are standard. The latter claim can be proved by Gronwall’s lemma.

Lemma 3.2. The family \(\{(X, Y^{(c)})\}_{\epsilon > 0}\) is tight.

Proof. According to Theorem 2.4.10 and Problem 2.4.11 in [24], we only have to prove the following two conditions since \(X\) is independent of \(\epsilon > 0\).

\(\text{T1} \sup_{\epsilon > 0} E[\|Y^{(c)}_0\|_{\bar{R}^d}] < \infty,\)

\(\text{T2} \sup_{\epsilon > 0} E[\|Y^{(c)}_t - Y^{(c)}_s\|_{\bar{R}^d}] \leq C_T(t-s)^{1+r}, \quad T > 0, \quad 0 \leq s \leq t \leq T.\)

Here, \(\nu > 0\) and \(q, r > 0\) are constants independent of \(T\), while \(C_T > 0\) may depend on \(T\).

For \(\text{T1}\), we can take \(\nu = p\) by Lemma 3.1. For \(\text{T2}\), we have

\[E[\|Y^{(c)}_t - Y^{(c)}_s\|_{\bar{R}^d}] \leq 2^p \left\{(t-s)^{1/2} \int_s^t E[\|G(u, Y^{(c)}_u)\|_{\bar{R}^d}] du + O_{\nu,p} \left(t-s)^{p/2}\right)\right\}\]

by Burkholder–Davis–Gundy inequality. Thus, \(\text{T2}\) holds for \(q = p, r = \min\{p/2 - 1, p/(p-1)\}\) by Lemma 3.1.

Lemma 3.3. For any subsequence \(\epsilon_{t_k}\), we can extract a further subsequence \(\epsilon_{t_{l_k}}\) so that

\[\lim_{k \to \infty} \{1 - h_{\epsilon_{t_{l_k}}}(\|Z^{(c)}_{t_{l_k}}\|_{\bar{R}^d})\} h_{\epsilon_{t_{l_k}}}(\|Z^{(c)}_{t_{l_k}}\|_{\bar{R}^d}) = 0\]

holds almost everywhere on \([0, \infty) \times \Omega\). Here, \(Z^{(c)}_{t} = X_t - Y^{(c)}_t\).

Proof. Since \(e^{(c)}_{i,s} = Z^{(c)}_{i,s}/\|Z^{(c)}_{i,s}\|_{\bar{R}^d}\), for all \(\delta > 0\), we have

\[\sum_{i,j=1}^d \frac{\delta_{ij} \delta_{i,s}^d}{\|Z^{(c)}_{i,s}\|_{\bar{R}^d}^2 + \delta} - \frac{Z^{(c)}_{i,s} Z^{(c)}_{j,s}}{\|Z^{(c)}_{i,s}\|_{\bar{R}^d}^2 + \delta} e^{(c)}_{i,s} e^{(c)}_{j,s} = \frac{\delta}{\|Z^{(c)}_{i,s}\|_{\bar{R}^d}^2 + \delta}.\]

Here, \(\delta_{ij}\) denotes the Kronecker’s delta. Thus, Ito’s formula yields

\[\langle Z^{(c)}_{i,s}\rangle_{\bar{R}^d} = \langle Z^{(c)}_{0} \rangle_{\bar{R}^d} + \delta + \int_0^t \frac{1}{\|Z^{(c)}_{i,s}\|_{\bar{R}^d}^2 + \delta} (Z^{(c)}_{i,s}, dV_s - G(s, Y^{(c)}) ds)_{\bar{R}^d} + 2\sigma \int_0^t \frac{\delta_{ij} \langle Z^{(c)}_{i,s}\rangle_{\bar{R}^d}^2}{\|Z^{(c)}_{i,s}\|_{\bar{R}^d}^2 + \delta} ds.\]
Since $h_\varepsilon$ is not 0 only on $\{|a| \geq \varepsilon\}$,
\[
\frac{Z_s^{(\varepsilon)}}{\|Z_s^{(\varepsilon)}\|_{\mathbb{R}^d}^2 + \delta} \to X_{\{Z_s^{(\varepsilon)} \neq 0\}}^{(\varepsilon)}, \quad \frac{\|Z_s^{(\varepsilon)}\|_{\mathbb{R}^d} h_\varepsilon(Z_s^{(\varepsilon)})}{\|Z_s^{(\varepsilon)}\|_{\mathbb{R}^d}^2 + \delta} \to h_\varepsilon(Z_s^{(\varepsilon)}), \quad \frac{\delta h_\varepsilon(\|Z_s^{(\varepsilon)}\|_{\mathbb{R}^d})^2}{\|Z_s^{(\varepsilon)}\|_{\mathbb{R}^d}^2 + \delta} \to 0
\]
hold as $\delta \to 0$. Therefore, taking the limit $\delta \to 0$, we find that $\|Z_t^{(\varepsilon)}\|_{\mathbb{R}^d}$ is a one-dimensional semi-martingale satisfying
\[
\|Z_t^{(\varepsilon)}\|_{\mathbb{R}^d} = \|Z_0^{(\varepsilon)}\|_{\mathbb{R}^d} + \int_0^t X_{\{Z_s^{(\varepsilon)} \neq 0\}}^{(\varepsilon)} ds + \int_0^t h_\varepsilon(Z_s^{(\varepsilon)}) dW_s,
\]
According to (3.7.10) in [13], the local time $\Lambda^{(\varepsilon)}(t, a)$ of $\|Z_t^{(\varepsilon)}\|_{\mathbb{R}^d}$ is given by
\[
\Lambda^{(\varepsilon)}(t, x) = \|Z_t^{(\varepsilon)}\|_{\mathbb{R}^d} - a - \|Z_0^{(\varepsilon)}\|_{\mathbb{R}^d} - a - \int_0^t \text{sgn}(\|Z_s^{(\varepsilon)}\|_{\mathbb{R}^d} - a) dM_s^{(\varepsilon)} - \int_0^t \text{sgn}(\|Z_s^{(\varepsilon)}\|_{\mathbb{R}^d} - a) dV_s^{(\varepsilon)}.
\]
In particular, for each fixed $t$, $\sup_{t > 0, |a| \leq 1} E[\Lambda^{(\varepsilon)}(t, a)] < \infty$ holds by Lemma 3.1. In addition, by the definition of the local time, we have
\[
\int_0^t v(\|Z_s^{(\varepsilon)}\|_{\mathbb{R}^d}) dM_s^{(\varepsilon)} = \int_{\mathbb{R}} v(a) \Lambda^{(\varepsilon)}(t, a) da.
\]
for all measurable function $v : \mathbb{R} \to [0, \infty)$. Taking $v = 1 - h_\varepsilon$ in (3.1), since $1 - h_\varepsilon$ is not 0 only on $\{|a| \leq 2\varepsilon\}$, we obtain
\[
\int_0^t (1 - h_\varepsilon(\|Z_s^{(\varepsilon)}\|_{\mathbb{R}^d})) h_\varepsilon(\|Z_s^{(\varepsilon)}\|_{\mathbb{R}^d}) ds = \frac{1}{4\delta^2} \int_0^t (1 - h_\varepsilon(a)) \Lambda^{(\varepsilon)}(t, a) da = \frac{1}{4\delta^2} \int_{-2\varepsilon}^{2\varepsilon} (1 - h_\varepsilon(a)) \Lambda^{(\varepsilon)}(t, a) da.
\]
Thus,
\[
\lim_{\varepsilon \to 0} \frac{\varepsilon}{2} \sup_{t > 0, |a| \leq 2\varepsilon} E[\Lambda^{(\varepsilon)}(t, a)] = \lim_{\varepsilon \to 0} \frac{\varepsilon}{2} \sup_{t > 0, |a| \leq 2\varepsilon} E[\Lambda^{(\varepsilon)}(t, a)] = 0
\]
holds and we can extract a subsequence $\varepsilon_{\ell_k}$ so that $\{1 - h_{\varepsilon_{\ell_k}}(\|Z^{(\varepsilon_{\ell_k})}_{s\varepsilon_{\ell_k}}\|_{\mathbb{R}^d})\} h_{\varepsilon_{\ell_k}}(\|Z^{(\varepsilon_{\ell_k})}_{s\varepsilon_{\ell_k}}\|_{\mathbb{R}^d}) \to 0$ holds almost everywhere on $[0, t] \times \Omega$. Applying the diagonal argument, we obtain the desired result.

3.2. Proof of Theorem 2.2

By Lemma 3.2 and Prohorov’s theorem, we can extract a subsequence $\varepsilon_{\ell}$ so that $(X, Y^{(\varepsilon)})$ has its weak limit $(\tilde{X}, \tilde{Y})$. To confirm that $\mathcal{L}^{(\tilde{X}, \tilde{Y})}$ is a coupling between $\mathcal{L}(X)$ and $\mathcal{L}(Y)$, we only have to show $\mathcal{L}(\tilde{Y}) = \mathcal{L}(Y)$ since $\mathcal{L}(\tilde{X}) = \mathcal{L}(X)$ is obvious. Furthermore, the pathwise uniqueness of (2.3) yields the weak uniqueness of itself by corollary to Lemma 5.1.2 in [11]. Therefore, according to Propositions 5.4.6 and 5.4.11 in [11], we only have to prove that $M^f$ defined by (3.2) below is a martingale for all compact supported $f \in C^2(\mathbb{R}^d; \mathbb{R})$.

\[
M_t^{(\varepsilon, f)} := f(Y_t^{(\varepsilon)}) - f(Y_0^{(\varepsilon)}) - \int_0^t \langle G(s, Y^{(\varepsilon)}), \nabla f(Y_s^{(\varepsilon)}) \rangle ds - \frac{\sigma^2}{2} \int_0^t \Delta f(Y_s^{(\varepsilon)}) ds.
\]
For each $s \geq 0$, let $B_s$ be the smallest $\sigma$-algebra on $C([0, \infty); \mathbb{R}^d)$ such that the map $C([0, \infty); \mathbb{R}^d) \ni \varphi \mapsto \varphi(\min\{s, t\})$ is $B_s$-measurable. By Ito’s formula,
\[
M_t^{(\varepsilon, f)} := f(Y_t^{(\varepsilon)}) - f(Y_0^{(\varepsilon)}) - \int_0^t \langle G(s, Y^{(\varepsilon)}), \nabla f(Y_s^{(\varepsilon)}) \rangle ds - \frac{\sigma^2}{2} \int_0^t \Delta f(Y_s^{(\varepsilon)}) ds + 2\sigma^2 \sum_{i,j=1}^d \int_0^t \partial_{x_i}^2 f(Y_s^{(\varepsilon)}) d\langle Z^{(\varepsilon)}\rangle_{i,j} - h_{\varepsilon}(\|Z_s^{(\varepsilon)}\|_{\mathbb{R}^d})\varepsilon_{i,j,s}^{(\varepsilon, f)(\varepsilon)}.
\]
is a martingale for all $\varepsilon > 0$. Thus, for all $s \leq t$ and $B_s$-measurable bounded continuous functional $F : C([0, \infty); \mathbb{R}^d) \to \mathbb{R}$, we have
\[
E[M_t^{(\varepsilon, f)} F(Y^{(\varepsilon)})] = E[M_s^{(\varepsilon, f)} F(Y^{(\varepsilon)})].
\]
By Lemma [3,3] extracting a further subsequence of \( \varepsilon_t \) if needed, we may assume that \( \{1-h_{4t}(\|Z^{(\varepsilon_t)}_s\|_{\mathbb{R}^d})\}h_{4t}(\|Z^{(\varepsilon_t)}_s\|_{\mathbb{R}^d}) \to 0 \) holds almost everywhere. As a result, taking the limit \( \varepsilon \to 0 \) along with \( \varepsilon_t \) in (3.4), Lemma A.2 yields

\[
E[M_F^t F(\tilde{Y})] = E[M_F^t F(\tilde{Y})].
\] (3.5)

\[\square\]

4. Bounds on the Difference between Langevin Dynamics with Different Drift Terms

In this section, we describe the first application of Theorem 2.2. We apply Theorem 2.2 to the evaluation of the difference between Langevin dynamics along with gradients of \( F,G \in C^4(\mathbb{R}^d; \mathbb{R}) \), and prove Theorem 2.4 (1) as its corollary. Here, \( F \) and \( G \) are assumed to be \((m,b)\)-dissipative and \( M \)-smooth.

Let \( X \) and \( Y \) be the solutions of SDEs with initial values \( X_0 \) and \( Y_0 \)

\[
dX_t = -\nabla F(X_t)dt + \sqrt{2/\beta}dW_t,
\]

\[
dY_t = -\nabla G(Y_t)dt + \sqrt{2/\beta}dW_t,
\]

respectively. First, adopting the technique developed in [6], we derive a bound on the difference between \( \mathcal{L}(X_t) \) and \( \mathcal{L}(Y_t) \) on the bases of the ARC \((X,Y^{(\varepsilon)})\) of \((X,Y)\) defined by

\[
\begin{align*}
\begin{cases}
dY^{(\varepsilon)}_t &= -\nabla G(Y^{(\varepsilon)}_t)dt + \sqrt{2/\beta}(I_d - 2h_{\varepsilon}(\|X_t - Y^{(\varepsilon)}_t\|_{\mathbb{R}^d}))\epsilon_t^{(\varepsilon)}(\epsilon_t^{(\varepsilon)})^TdW_t, \\
Y^{(\varepsilon)}_0 &= Y_0.
\end{cases}
\end{align*}
\] (4.3)

Here, \( \epsilon_t^{(\varepsilon)} = (X_t - Y^{(\varepsilon)}_t)/\|X_t - Y^{(\varepsilon)}_t\|_{\mathbb{R}^d} \).

4.1. Notations form [6]

Before going into the details, we prepare the special case of notations used in [6]. For \( p > 0 \), define \( V_p : \mathbb{R}^d \to \mathbb{R} \) by \( V_p(x) = \|x\|_{\mathbb{R}^d}^p \) and let \( \tilde{V}_p(x) = 1 + V_p(x) \). For constants \( C(p) \) and \( \lambda(p) \) defined in Lemma A.3 let

\[
C = C(2) + \lambda(2), \quad \lambda = \lambda(2)
\] (4.4)

and let

\[
\begin{align*}
S_1 &:= \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d \mid \tilde{V}_2(x) + \tilde{V}_2(y) \leq 2\lambda^{-1}C\}, \\
S_2 &:= \{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d \mid \tilde{V}_2(x) + \tilde{V}_2(y) \leq 4C(1 + \lambda^{-1})\}.
\end{align*}
\] (4.5) (4.6)

The diameters of \( S_1 \) and \( S_2 \) are denoted by \( R_1 \) and \( R_2 \), respectively, where the diameter of a set \( \Gamma \subset \mathbb{R}^d \) is defined by \( \text{sup}_{x,y \in \Gamma} \|x - y\|_{\mathbb{R}^d} \).

For a constant \( \kappa \) defined by [4.10], we define \( Q(\kappa) \) by

\[
Q(\kappa) := \sup_{x \in \mathbb{R}^d} \frac{\|\nabla \tilde{V}_2(x)\|_{\mathbb{R}^d}}{\max\{V_2(x),\kappa^{-1}\}} = \sup_{x \in \mathbb{R}^d} \frac{2\|x\|_{\mathbb{R}^d}}{\max\{1 + \|x\|_{\mathbb{R}^d}^2,\kappa^{-1}\}} = 2\sqrt{\kappa - \kappa^2} \in (0,1]
\] (4.7)

and functions \( \varphi, \Phi : [0, \infty) \to [0, \infty) \) by

\[
\varphi(r) := \exp\left(-\frac{M\beta}{8}r^2 - 2Q(\kappa)r\right), \quad \Phi(r) = \int_0^r \varphi(s)ds,
\] (4.8)

respectively. For constants \( \zeta \) and \( \xi \) defined by

\[
\frac{1}{\zeta} := \int_0^{R_2} \Phi(s)\varphi(s)^{-1}ds, \quad \frac{1}{\xi} := \int_0^{R_1} \Phi(s)\varphi(s)^{-1}ds,
\] (4.9)

let

\[
g(r) := 1 - \frac{\zeta}{4} \int_0^{\min\{r,R_2\}} \Phi(s)\varphi(s)^{-1}ds - \frac{\xi}{4} \int_0^{\min\{r,R_1\}} \Phi(s)\varphi(s)^{-1}ds.
\] (4.10)
Furthermore, for
\[
 f(r) := \begin{cases} 
 \int_0^{\min\{r, R_2\}} \varphi(s)g(s)ds, & r \geq 0 \\
 r, & r < 0 
\end{cases} \tag{4.11}
\]
and \( U(x, y) := 1 + \kappa \bar{V}_2(x) + \kappa \bar{V}_2(y) \), let
\[
 \rho_2(x, y) = f(||x - y||_{\mathbb{R}^d})U(x, y), \quad x, y \in \mathbb{R}^d. \tag{4.12}
\]
Finally, for probability measures \( \mu \) and \( \nu \) on \( \mathbb{R}^d \), denoting the set of all couplings between them by \( \Pi(\mu, \nu) \), let
\[
 W_{\rho_2}(\mu, \nu) := \inf_{\gamma \in \Pi(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \rho_2(x, y)\gamma(dx\,dy). \tag{4.13}
\]
Here, for random variables \( Z_1 \) and \( Z_2 \), \( W_{\rho_2}(\mathcal{L}(Z_1), \mathcal{L}(Z_2)) \) may be abbreviated as \( W_{\rho_2}(Z_1, Z_2) \).

4.2. Uniform bound on the time for the difference between \( (4.1) \) and \( (4.2) \)

The following Proposition 4.1 is an extension of Theorem 2.2 in [10] to the case of different drift terms. Although Proposition 4.1 is proved in the same manner as Theorem 2.2 in [10], we give the complete proof of it to explain that the error terms caused by adopting ARC do not disturb the proof.

**Proposition 4.1.** Let \( X_0, Y_0 \in L^4(\Omega; \mathbb{R}^d) \) and let \( c := \min\left\{ \frac{\lambda}{\beta}, \frac{1}{2}\lambda C \kappa \right\} \). Then, for the solutions \( X \) and \( Y \) of \((4.2)\) and \((4.2)\), we have
\[
 W_{\rho_2}(X_t, Y_t) \leq e^{-ct} E\rho_2(X_0, Y_0) + O_\alpha \left( e^{-ct} \int_0^t e^{c\theta} E[\|\nabla F(Y_s) - \nabla G(Y_s)\|_{\mathbb{R}^d}^2]^{1/2} ds \right). \tag{4.14}
\]
Here, \( \alpha = (m, b, M, \beta, \|\nabla F(0)\|_{\mathbb{R}^d}, \|\nabla G(0)\|_{\mathbb{R}^d}, E[\|X_0\|_{\mathbb{R}^d}^4], E[\|Y_0\|_{\mathbb{R}^d}^4], d) \).

**Proof.** Fix \( \varepsilon > 0 \) and let \( Z^{(\varepsilon)}_t = X_t - Y^{(\varepsilon)}_t \) for \( X_t \) and \( Y^{(\varepsilon)}_t \) defined by \( (4.3) \).

**Step 1** Evaluation of \( f(\|Z^{(\varepsilon)}_t\|_{\mathbb{R}^d}) \).

As in the proof of Lemma 3.3, we can show that \( \|Z^{(\varepsilon)}_t\|_{\mathbb{R}^d} \) is an one-dimensional semi-martingale satisfying
\[
 \|Z^{(\varepsilon)}_t\|_{\mathbb{R}^d} = \|Z^{(\varepsilon)}_0\|_{\mathbb{R}^d} - \int_0^t \int_{\{Z^{(\varepsilon)}_s \neq 0\}} \langle e^{(\varepsilon)}_s, \nabla F(X_s) - \nabla G(Y^{(\varepsilon)}_s) \rangle_{\mathbb{R}^d} ds + 2 \sqrt{\frac{2}{\beta}} \int_0^t h_\varepsilon(\|Z^{(\varepsilon)}_s\|_{\mathbb{R}^d}, (e^{(\varepsilon)}_s, dW_s)_{\mathbb{R}^d}).
\]
Let \( \Lambda^{(\varepsilon)}(t, a) \) be the local time of \( \|Z^{(\varepsilon)}\|_{\mathbb{R}^d} \). Since \( f \) is a concave function by Lemma \( \Lambda.10 \) Tanaka’s formula (Theorem 3.7.1 in [13]) yields
\[
 f(\|Z^{(\varepsilon)}_t\|_{\mathbb{R}^d}) - f(\|Z^{(\varepsilon)}_0\|_{\mathbb{R}^d}) = -\int_0^t f'(\|Z^{(\varepsilon)}_s\|_{\mathbb{R}^d}) \int_{\{Z^{(\varepsilon)}_s \neq 0\}} \langle e^{(\varepsilon)}_s, \nabla F(X_s) - \nabla G(Y^{(\varepsilon)}_s) \rangle_{\mathbb{R}^d} ds
 \]
\[
 + 2 \sqrt{\frac{2}{\beta}} \int_0^t f'(\|Z^{(\varepsilon)}_s\|_{\mathbb{R}^d}) h_\varepsilon(\|Z^{(\varepsilon)}_s\|_{\mathbb{R}^d}, (e^{(\varepsilon)}_s, dW_s)_{\mathbb{R}^d}) + \frac{1}{2} \int_{-\infty}^t \Lambda^{(\varepsilon)}(t, a)ds. \tag{4.15}
\]
Here, \( f' \) and \( \mu_f \) denote the left derivative and the second derivative measure of \( f \), respectively. Whereas, by the definition of the local time, for a measurable function \( \nu = \chi_{\{R_1, R_2\}} \), we have
\[
 \frac{8}{\beta} \int_0^t \chi_{\{R_1, R_2\}}(\|Z^{(\varepsilon)}_s\|_{\mathbb{R}^d}) h_\varepsilon(\|Z^{(\varepsilon)}_s\|_{\mathbb{R}^d})^2 ds = \int_{\mathbb{R}} \chi_{\{R_1, R_2\}}(a) \Lambda^{(\varepsilon)}(t, a)da = 0. \tag{4.15}
\]
Since \( h_\varepsilon(a) \) is equal to 1 if \( |a| \geq 2\varepsilon \), the occupation time of \( \|Z^{(\varepsilon)}\|_{\mathbb{R}^d} \) on \( \{R_1, R_2\} \) must be 0 by \( (4.15) \) if \( \varepsilon \) is sufficiently small for \( R_1 \) and \( R_2 \). Thus, we may assume \( f \in C^2(\mathbb{R}; \mathbb{R}) \) when we consider its value at \( \|Z^{(\varepsilon)}_s\|_{\mathbb{R}^d} \).
Similarly, since \( \mu_f(\{R_1, R_2\}) \leq 0 \) by Lemma A.14, the definition of the local time yields
\[
\int_{-\infty}^{\infty} \Lambda^{(e)}(t, a) \mu_f(da) \leq \int_{-\infty}^{\infty} \chi_{R \setminus \{R_1, R_2\}}(a) \Lambda^{(e)}(t, a) \mu_f(da)
= \int_{-\infty}^{\infty} \chi_{R \setminus \{R_1, R_2\}}(a) f''(a) \Lambda^{(e)}(t, a) da
= \frac{8}{\beta} \int_{0}^{t} f''(\|Z_{e}^{(e)}\|_{R^d}) h_{\varepsilon}(\|Z_{e}^{(e)}\|_{R^d})^{2} ds.
\]

Therefore, we obtain
\[
df(\|Z_{e}^{(e)}\|_{R^d}) \leq -f'(\|Z_{e}^{(e)}\|_{R^d}) \chi_{\{Z_{e}^{(e)} \neq 0\}} \langle \varepsilon \cdot \nabla F(X_t) - \nabla G(Y_{t}^{(e)}) \rangle_{R^d} dt
+ 2 \sqrt{\frac{\beta}{3}} f'\langle Z_{e}^{(e)} \rangle_{R^d} h_{\varepsilon}(\|Z_{e}^{(e)}\|_{R^d}) \langle \varepsilon_{t}^{(e)} \rangle_{R^d} dt
+ \frac{4}{\beta} f''(\|Z_{e}^{(e)}\|_{R^d}) h_{\varepsilon}(\|Z_{e}^{(e)}\|_{R^d})^{2} dt.
\]

Finally, by M-smoothness of \( F \), we have
\[
\langle \varepsilon_{t}^{(e)} \rangle_{R^d} \leq M \|Z_{e}^{(e)}\|_{R^d}.
\tag{4.16}
\]

Furthermore, by Lemma A.12, we have also
\[
f''(\|Z_{e}^{(e)}\|_{R^d}) \leq -\left( \frac{M \beta}{4} \|Z_{e}^{(e)}\|_{R^d} + 2Q(\kappa) \right) f'(\|Z_{e}^{(e)}\|_{R^d}) \chi_{(0, R_2)}(\|Z_{e}^{(e)}\|_{R^d}) - \frac{\kappa}{4} f(\|Z_{e}^{(e)}\|_{R^d}) \chi_{(0, R_1)}(\|Z_{e}^{(e)}\|_{R^d}).
\]

As a result, noting \( 0 \leq f' \leq 1 \), we obtain
\[
df(\|Z_{e}^{(e)}\|_{R^d}) \leq \nabla F(Y_{t}^{(e)}) - \nabla G(Y_{t}^{(e)}) dt + M \|Z_{e}^{(e)}\|_{R^d} \{ 1 - h_{\varepsilon}(\|Z_{e}^{(e)}\|_{R^d})^{2} \} dt
+ 2 \sqrt{\frac{\beta}{3}} f'(\|Z_{e}^{(e)}\|_{R^d}) h_{\varepsilon}(\|Z_{e}^{(e)}\|_{R^d}) \langle \varepsilon_{t}^{(e)} \rangle_{R^d} dt
- \frac{\kappa}{4} f(\|Z_{e}^{(e)}\|_{R^d}) \chi_{(0, R_1)}(\|Z_{e}^{(e)}\|_{R^d}) dt. \tag{4.17}
\]

**Step 2** Evaluation of \( U(X_t, Y_{t}^{(e)}) \).

Ito’s formula yields
\[
dU(X_t, Y_{t}^{(e)}) = \kappa(\mathcal{L}_F \tilde{V}_2(X_t) + \mathcal{L}_G \tilde{V}_2(Y_{t}^{(e)}))dt - \frac{4 \kappa h_{\varepsilon}(\|Z_{e}^{(e)}\|_{R^d})(1 - h_{\varepsilon}(\|Z_{e}^{(e)}\|_{R^d}))}{\beta} \sum_{i,j=1}^{d} \varepsilon_{i,t}^{(e)} e_{i,j}^{(e)} \sigma_{i,j}^{2} \tilde{V}_2(Y_{t}^{(e)}) dt
+ \sqrt{\frac{\beta}{3}} \nabla \tilde{V}_2(X_t) + \nabla \tilde{V}_2(Y_{t}^{(e)}), dW_t \}_{R^d} - 2 \sqrt{\frac{\beta}{3}} h_{\varepsilon}(\|Z_{e}^{(e)}\|_{R^d}) \langle \varepsilon_{t}^{(e)} \rangle_{R^d} \nabla \tilde{V}_2(Y_{t}^{(e)}) dt. \tag{4.18}
\]

Here, \( \mathcal{L}_H = -\langle \nabla H, \nabla \rangle_{R^d} - \beta^{-1} \Delta \), for \( H \in C^1(\mathbb{R}^d; \mathbb{R}) \). According to Lemma A.8 by the \( (m, b) \)-dissipativity of \( F \) and \( G \), we have
\[
\mathcal{L}_F \tilde{V}_2(X_t) + \mathcal{L}_G \tilde{V}_2(Y_{t}^{(e)}) \leq 2C - \lambda \tilde{V}_2(Y_{t}^{(n)}(X_t)) - \tilde{V}_2(Y_{t}^{(e)})).
\]

Furthermore, by Lemma A.9 and [4.9], we have also
\[
2C \kappa \leq \frac{\lambda}{\beta} \left( \int_{0}^{R_l} \varphi(s)^{-1} \Phi(s) ds \right)^{-1} = \frac{\xi}{\beta}.
\]

Thus, A.8 and A.9 yield
\[
\kappa(\mathcal{L}_F \tilde{V}_2(X_t) + \mathcal{L}_G \tilde{V}_2(Y_{t}^{(e)})) \leq \kappa(\mathcal{L}_F \tilde{V}_2(X_t) + \mathcal{L}_G \tilde{V}_2(Y_{t}^{(e)})) \chi_{(0, R_1)}(\|Z_{e}^{(e)}\|_{R^d}) - \frac{\lambda}{2} \min\{1, 4C \kappa\} U(X_t, Y_{t}^{(e)}) \chi_{(R_2, \infty)}(\|Z_{e}^{(e)}\|_{R^d})
\leq \frac{\xi}{\beta} \chi_{(0, R_1)}(\|Z_{e}^{(e)}\|_{R^d}) - \frac{\lambda}{2} \min\{1, 4C \kappa\} U(X_t, Y_{t}^{(e)}) \chi_{(R_2, \infty)}(\|Z_{e}^{(e)}\|_{R^d})
\leq \frac{\xi}{\beta} U(X_t, Y_{t}^{(e)}) \chi_{(0, R_1)}(\|Z_{e}^{(e)}\|_{R^d}) - \frac{\lambda}{2} \min\{1, 4C \kappa\} U(X_t, Y_{t}^{(e)}) \chi_{(R_2, \infty)}(\|Z_{e}^{(e)}\|_{R^d}).
\]
As a result, since the Hessian matrix of $\hat{V}_2$ is nonnegative-definite, we obtain

$$dU(X_t, Y_t^{(c)}) \leq \left( \frac{\xi}{\beta}U(X_t, Y_t^{(c)})\chi_{(0, R_1)}(\|Z_t^{(c)}\|_{\mathbb{R}^d}) - \frac{\lambda}{2} \min\{1, 4C\kappa\}U(X_t, Y_t^{(c)})\chi_{(R_2, \infty)}(\|Z_t^{(c)}\|_{\mathbb{R}^d}) \right) dt$$

$$+ \kappa \sqrt{\frac{2}{\beta}}(\nabla \hat{V}_2(X_t) + \nabla \hat{V}_2(Y_t^{(c)}))dW_t_{\mathbb{R}^d} - 2\kappa \sqrt{\frac{2}{\beta}}h\varepsilon(\|Z_t^{(c)}\|_{\mathbb{R}^d}) (\langle e_t^{(c)}, \nabla \hat{V}_2(Y_t^{(c)}) \rangle_{\mathbb{R}^d} \langle e_t^{(c)}, dW_t \rangle_{\mathbb{R}^d}).$$

(4.19)

**Step 3** Evaluation of $\rho_2(X_t, Y_t^{(c)})$.

According to the representations of $f(\|Z_t^{(c)}\|_{\mathbb{R}^d})$ and $U(X_t, Y_t^{(c)})$ as semi-martingales,

$$d(f(\|Z_t^{(c)}\|_{\mathbb{R}^d}), U(X, Y^{(c)}))_t = \frac{4\kappa}{\beta}f'(\|Z_t^{(c)}\|_{\mathbb{R}^d})h\varepsilon(\|Z_t^{(c)}\|_{\mathbb{R}^d})^2 (\langle e_t^{(c)}, \nabla \hat{V}_2(X_t) - \nabla \hat{V}_2(Y_t^{(c)}) \rangle_{\mathbb{R}^d} dt$$

$$+ \frac{4\kappa}{\beta}f'(\|Z_t^{(c)}\|_{\mathbb{R}^d})\{1 - h\varepsilon(\|Z_t^{(c)}\|_{\mathbb{R}^d})\} h\varepsilon(\|Z_t^{(c)}\|_{\mathbb{R}^d}) (\langle e_t^{(c)}, \nabla \hat{V}_2(X_t) + \nabla \hat{V}_2(Y_t^{(c)}) \rangle_{\mathbb{R}^d} dt$$

holds. By the definition of $Q(\kappa)$, for all $x \neq y$ we have

$$\kappa \left\langle \nabla \hat{V}_2(x) - \nabla \hat{V}_2(y), \frac{x - y}{\|x - y\|_{\mathbb{R}^d}} \right\rangle_{\mathbb{R}^d} \leq \kappa \|\nabla \hat{V}_2(x) - \nabla \hat{V}_2(y)\|_{\mathbb{R}^d}$$

$$\leq U(x, y) \left( \frac{\|\nabla \hat{V}_2(x)\|_{\mathbb{R}^d}}{\kappa - 1 + \hat{V}_2(x)} + \frac{\|\nabla \hat{V}_2(y)\|_{\mathbb{R}^d}}{\kappa - 1 + \hat{V}_2(y)} \right)$$

$$\leq 2Q(\kappa)U(x, y).$$

Thus,

$$d(f(\|Z_t^{(c)}\|_{\mathbb{R}^d}), U(X, Y^{(c)}))_t \leq \frac{8Q(\kappa)}{\beta}f'(\|Z_t^{(c)}\|_{\mathbb{R}^d})U(X_t, Y_t^{(c)})h\varepsilon(\|Z_t^{(c)}\|_{\mathbb{R}^d})^2 dt$$

$$+ \frac{4\kappa}{\beta}(1 - h\varepsilon(\|Z_t^{(c)}\|_{\mathbb{R}^d})) h\varepsilon(\|Z_t^{(c)}\|_{\mathbb{R}^d}) (\langle e_t^{(c)}, \nabla \hat{V}_2(X_t) + \nabla \hat{V}_2(Y_t^{(c)}) \rangle_{\mathbb{R}^d} dt$$

(4.20)

holds.

For the first term in the R.H.S of

$$d(f(\|Z_t^{(c)}\|_{\mathbb{R}^d}), U(X_t, Y_t^{(c)})) = U(X_t, Y_t^{(c)})df(\|Z_t^{(c)}\|_{\mathbb{R}^d}) + f(\|Z_t^{(c)}\|_{\mathbb{R}^d})dU(X_t, Y_t^{(c)})$$

(4.21)

the following holds by Step 1.

$$U(X_t, Y_t^{(c)})df(\|Z_t^{(c)}\|_{\mathbb{R}^d})$$

$$\leq -\frac{\xi}{\beta}\rho_2(X_t, Y_t^{(c)})\chi_{(0, R_2)}(\|Z_t^{(c)}\|_{\mathbb{R}^d}) dt - \frac{\xi}{\beta}\rho_2(X_t, Y_t^{(c)})\chi_{(0, R_1)}(\|Z_t^{(c)}\|_{\mathbb{R}^d}) dt$$

$$+ U(X_t, Y_t^{(c)})||\nabla F(Y_t^{(c)}) - \nabla G(Y_t^{(c)})||_{\mathbb{R}^d} dt - \frac{8Q(\kappa)}{\beta}f'(\|Z_t^{(c)}\|_{\mathbb{R}^d})U(X_t, Y_t^{(c)})h\varepsilon(\|Z_t^{(c)}\|_{\mathbb{R}^d})^2 dt$$

$$+ 2\sqrt{\frac{2}{\beta}}f'(\|Z_t^{(c)}\|_{\mathbb{R}^d})U(X_t, Y_t^{(c)})h\varepsilon(\|Z_t^{(c)}\|_{\mathbb{R}^d}) (\langle e_t^{(c)}, dW_t \rangle_{\mathbb{R}^d} + MU(X_t, Y_t^{(c)})\|Z_t^{(c)}\|_{\mathbb{R}^d} (1 - h\varepsilon(\|Z_t^{(c)}\|_{\mathbb{R}^d})^2) dt.$$

Similarly, for the second term in the R.H.S of (4.21), the following holds by Step 2.

$$f(\|Z_t^{(c)}\|_{\mathbb{R}^d})dU(X_t, Y_t^{(c)})$$

$$\leq \left( \frac{\xi}{\beta}\rho_2(X_t, Y_t^{(c)})\chi_{(0, R_1)}(\|Z_t^{(c)}\|_{\mathbb{R}^d}) - \frac{\lambda}{2} \min\{1, 4C\kappa\} \rho_2(X_t, Y_t^{(c)})\chi_{(R_2, \infty)}(\|Z_t^{(c)}\|_{\mathbb{R}^d}) \right) dt$$

$$+ \kappa \sqrt{\frac{2}{\beta}}f(\|Z_t^{(c)}\|_{\mathbb{R}^d}) (\langle \nabla \hat{V}_2(X_t) + \nabla \hat{V}_2(Y_t^{(c)}), dW_t \rangle_{\mathbb{R}^d} - 2\kappa \sqrt{\frac{2}{\beta}}f(\|Z_t^{(c)}\|_{\mathbb{R}^d}) h\varepsilon(\|Z_t^{(c)}\|_{\mathbb{R}^d}) (\langle e_t^{(c)}, \nabla \hat{V}_2(Y_t^{(c)}) \rangle_{\mathbb{R}^d} \langle e_t^{(c)}, dW_t \rangle_{\mathbb{R}^d}).$$
As a result, there exists a martingale $M^{(\varepsilon)}$ such that

\[
\begin{align*}
\frac{d\rho_2(X_t,Y_t^{(\varepsilon)})}{dt} & \leq -\min \left\{ \frac{\varepsilon}{2}, 2C\lambda \right\} \rho_2(X_t,Y_t^{(\varepsilon)}) + U(X_t,Y_t^{(\varepsilon)})\|\nabla F(Y_t^{(\varepsilon)}) - \nabla G(Y_t^{(\varepsilon)})\|_{\mathbb{R}^d} dt \\
& \quad + MU(X_t,Y_t^{(\varepsilon)})\|Z_t^{(\varepsilon)}\|_{\mathbb{R}^d} \{1 - h_\varepsilon(\|Z_t^{(\varepsilon)}\|_{\mathbb{R}^d})\}^2 dt + \frac{4K}{\beta} \left\{ 1 - h_\varepsilon(\|Z_t^{(\varepsilon)}\|_{\mathbb{R}^d}) \right\} h_\varepsilon(\|Z_t^{(\varepsilon)}\|_{\mathbb{R}^d}) (c_t^{(\varepsilon)}, \nabla V_2(X_t) + \nabla \tilde{V}_2(Y_t^{(\varepsilon)}))_{\mathbb{R}^d} dt \\
& \quad + dM^{(\varepsilon)}.
\end{align*}
\]  

(4.22)

Since $\|Z_t^{(\varepsilon)}\|_{\mathbb{R}^d} \{1 - h_\varepsilon(\|Z_t^{(\varepsilon)}\|_{\mathbb{R}^d})\}^2 \leq 2\varepsilon$, taking expectation in both sides of (4.22) and the limit $\varepsilon \to 0$, Lemma A.3 and Theorem 2.4 yield

\[
W_{\rho_2}(X_t,Y_t) \leq E[\rho_2(X_0,Y_0)] - c \int_0^t W_{\rho_2}(X_s,Y_s) ds + \int_0^t E[U(X_s,Y_s)]\|\nabla F(Y_s) - \nabla G(Y_s)\|_{\mathbb{R}^d} ds.
\]

Thus, $d(e^{ct}W_{\rho_2}(X_t,Y_t)) \leq e^{ct}E[U(X_t,Y_t)]\|\nabla F(Y_t) - \nabla G(Y_t)\|_{\mathbb{R}^d} dt$, and therefore we obtain

\[
W_{\rho_2}(X_t,Y_t) \leq e^{-ct}E[\rho_2(X_0,Y_0)] + e^{-ct}\int_0^t e^{cs}E[U(X_s,Y_s)]\|\nabla F(Y_s) - \nabla G(Y_s)\|_{\mathbb{R}^d} ds
\]

\[
\leq e^{-ct}E[\rho_2(X_0,Y_0)] + \sup_{u \geq 0} E[U(X_u,Y_u)]^{1/2} e^{-ct}\int_0^t e^{cs}E[\|\nabla F(Y_s) - \nabla G(Y_s)\|_{\mathbb{R}^d}^2]^{1/2} ds.
\]

Since $\sup_{u \geq 0} E[U(X_u,Y_u)]^{1/2} \leq O_\alpha(1)$ by Lemma A.4, the proof is completed.

As a corollary to Proposition 4.1, we can bound the difference between Gibbs measures $\pi_{\beta,F}(dx) \propto e^{-\beta F(x)} dx$ and $\pi_{\beta,G}(dx) \propto e^{-\beta G(x)} dx$ in term of the difference between gradients $\nabla F$ and $\nabla G$. The integral of a function $h$ with respect to measure $\mu$ is denoted by $\langle h \rangle_{\mu}$.

**Corollary 4.2.** Let $F$ and $G$ be nonnegative and let $H \in C^1(\mathbb{R}^d;\mathbb{R})$ be $M'$-smooth. Then, we have

\[
|\pi_{\beta,F}(H) - \pi_{\beta,G}(H)| \leq O_{m,b,M',\beta,\|\nabla H(0)\|_{\mathbb{R}^d},\|\nabla F(0)\|_{\mathbb{R}^d},\|\nabla G(0)\|_{\mathbb{R}^d},d} \left( \int_{\mathbb{R}^d} \|\nabla F(x) - \nabla G(x)\|_{\mathbb{R}^d}^2 \pi_{\beta,G}(dx) \right)^{1/2} .
\]

(4.23)

**Proof.** According to (5.4.58) in [11], Gibbs measures $\pi_{\beta,F}$ and $\pi_{\beta,G}$ are invariant measures for $X$ and $Y$ defined by (4.11) and (4.12), respectively. Thus, applying Proposition 4.1 for $\mathcal{L}(X_0) = \pi_{\beta,F}$ and $\mathcal{L}(Y_0) = \pi_{\beta,G}$ and taking the limit $t \to \infty$, we obtain by Lemma A.1 that

\[
W_{\rho_2}(\pi_{\beta,F},\pi_{\beta,G}) \leq O_{m,b,M',\beta,\|\nabla H(0)\|_{\mathbb{R}^d},\|\nabla F(0)\|_{\mathbb{R}^d},\|\nabla G(0)\|_{\mathbb{R}^d},d} \left( \int_{\mathbb{R}^d} \|\nabla F(x) - \nabla G(x)\|_{\mathbb{R}^d}^2 \pi_{\beta,G}(dx) \right)^{1/2} .
\]

Since we have by Lemma A.15

\[
|\pi_{\beta,F}(H) - \pi_{\beta,G}(H)| \leq O_{m,b,M',\beta,\|\nabla H(0)\|_{\mathbb{R}^d},d}(W_{\rho_2}(\pi_{\beta,F},\pi_{\beta,G})) ,
\]

the result follows from Proposition 4.1.

4.3. Proof of Theorem 2.4 (1)

For $z_1,\ldots,z_n$ and $z'_1,\ldots,z'_n$, suppose that there exists at most one $i_0$ such that $z_{i_0} \neq z'_{i_0}$. Then, setting $L_n(w) = \frac{1}{n} \sum_{i=1}^n \ell(w; z_i)$ and denoting $X^{(n)}$ by $Y^{(n)}$ when the $L_n$ is replaced by $L'_n$ in (2.9), Proposition 4.1 and Lemma A.14 yield the following for all $z \in \mathcal{Z}$,

\[
|E[\ell(X_t^{(n)}; z)] - E[\ell(Y_t^{(n)}; z)]| \leq O_{\alpha_0} \left( e^{-ct} \int_0^t e^{cs}E[\|\nabla L_n(Y_s^{(n)}) - \nabla L'_n(Y_s^{(n)})\|_{\mathbb{R}^d}^2]^{1/2} ds \right) \leq O_{\alpha_0} \left( n^{-1} \right) .
\]

Here, we used Lemma A.4 in the second inequality. In particular, by Theorem 2.2 in [10], if $z_1,\ldots,z_n$ are IID generated from $\mathcal{D}$, then

\[
|E[L(X_t^{(n)})] - E[L_n(X_t^{(n)})]| \leq O_{\alpha_0} \left( n^{-1} \right) holds.
\]
5. Bounds on the Discretization Error for Langevin Dynamics

In this section, we prove Theorem 2.3 (2) as the second application of Theorem 2.2. In our proof, it is important that Theorem 2.2 admits the case of $G$ is a functional.

In the following, we fix $\eta > 0$ and assume $F \in C^1(\mathbb{R}^d; \mathbb{R})$ is $(m, b)$-dissipative and $M$-smooth. For initial values $X_0$ and $Y_0$, we consider the solutions $X$ and $Y$ of (4.1) and

\[ dY_t = -\nabla F(Y_{(t/\eta)\eta}) dt + \sqrt{2/\beta} dW_t, \]

respectively. Here, $\lfloor \cdot \rfloor$ denotes the floor function. Then we define the ARC between $X$ and $Y$ by

\[
\begin{cases}
    dY_t^{(c)} = -\nabla F(Y_{(t/\eta)\eta}) dt + \sqrt{2/\beta}(I_d - 2h_\varepsilon(\|X_t - Y_t^{(c)}\|_2)\varepsilon_t(e_t^c)^T) dW_t, \\
Y_0^{(c)} = Y_0.
\end{cases}
\]

Note that the functional $G$ defined by $G(t, \varphi) = \nabla F(\varphi([t/\eta]\eta))$ for $\varphi \in C([0, \infty); \mathbb{R}^d)$ is progressively measurable and satisfies (4.1) and (4.2).

5.1. Uniform bound on the time for the difference between (4.1) and (5.1)

Proposition 5.1 below gives a uniform evaluation on the time to the discretization error of (4.1) with order $\eta^{1/2}$. Since the proof of Proposition 5.1 is quite similar to that of Proposition 4.1, we only explain the important differences of those in the following proof.

**Proposition 5.1.** Let $X$ and $Y$ be the solutions of (4.1) and (5.1) with initial values $X_0, Y_0 \in L^4(\Omega; \mathbb{R}^d)$, respectively. Then there exists some $\eta_0 = O_{m, M}(1)$ and the following inequality holds uniformly on $0 < \eta \leq \eta_0$.

\[ \mathcal{W}_{p_2}(X_t, Y_t) \leq e^{-ct} E[p_2(X_0, Y_0)] + O_{m, b, \beta, \|F(0)\|_{l_4}, \|\nabla X_0\|_{l_4}, \|\nabla Y_0\|_{l_4}, \eta^{1/2}). \]

Here, $c > 0$ is the constant defined in Proposition 4.1.

**Proof.** Fix $\varepsilon > 0$ and let $Z_t^{(c)} = X_t - Y_t^{(c)}$ for $X_t$ and $Y_t^{(c)}$ defined by (5.2).

**Step 1** Evaluation of $f(\|Z_t^{(c)}\|_R^4)$.

In this case, (4.10) is replaced by

\[ \langle e_t^{(c)}, \nabla F(X_t) - \nabla F(Y_{(t/\eta)\eta}^{(c)}) \rangle_{R^d} \leq M \|Y_t^{(c)} - Y_{(t/\eta)\eta}^{(c)}\|_{R^d} + M \|Z_t^{(c)}\|_{R^d}. \]

Thus, $f(\|Z_t^{(c)}\|_R^4)$ is a semi-martingale that satisfies

\[
df(\|Z_t^{(c)}\|_{R^d}) \leq M \|Y_t^{(c)} - Y_{(t/\eta)\eta}^{(c)}\|_{R^d} dt + M \|Z_t^{(c)}\|_{R^d} \{1 - h_\varepsilon(\|Z_t^{(c)}\|_{R^d})^2\} dt \\
+ 2\sqrt{\frac{2}{\beta}} f'(\|Z_t^{(c)}\|_{R^d}) h_\varepsilon(\|Z_t^{(c)}\|_{R^d}) \langle e_t^{(c)}, dW_t \rangle_{R^d} - \frac{8Q(\kappa)}{\beta} f'(\|Z_t^{(c)}\|_{R^d}) h_\varepsilon(\|Z_t^{(c)}\|_{R^d})^2 dt \\
- \frac{\zeta}{\beta} f(\|Z_t^{(c)}\|_{R^d}) \chi_{(0, R_1)}(\|Z_t^{(c)}\|_{R^d}) dt - \frac{\xi}{\beta} f(\|Z_t^{(c)}\|_{R^d}) \chi_{(0, R_1)}(\|Z_t^{(c)}\|_{R^d}) dt.
\]

**Step 2** Evaluation of $U(X_t, Y_t^{(c)})$.

In this case, the term $2\kappa \langle Y_t^{(c)}, \nabla F(Y_{(t/\eta)\eta}^{(c)}) - \nabla F(Y_t^{(c)}) \rangle_{R^d} dt$ is added to (4.18). Thus, we have

\[
dU(X_t, Y_t^{(c)}) \leq \left( \frac{\xi}{\beta} U(X_t, Y_t^{(c)}) \chi_{(0, R_1)}(\|Z_t^{(c)}\|_{R^d}) - \frac{\lambda}{2} \min \{1, 4C\kappa\} U(X_t, Y_t^{(c)}) \chi_{(R_2, \infty)}(\|Z_t^{(c)}\|_{R^d}) \right) dt \\
+ 2M\kappa \|Y_t^{(c)}\|_{R^d} M \|Y_t^{(c)} - Y_{(t/\eta)\eta}^{(c)}\|_{R^d} dt \\
+ \kappa \sqrt{\frac{2}{\beta}} (\nabla \tilde{V}_t(X_t) + \nabla \tilde{V}_t(Y_t^{(c)}), dW_t)_{R^d} - 2\kappa \sqrt{\frac{2}{\beta}} h_\varepsilon(\|Z_t^{(c)}\|_{R^d}) \langle e_t^{(c)}, \nabla \tilde{V}_t(Y_t^{(c)}) \rangle_{R^d} dt + \frac{2}{\beta} h_\varepsilon(\|Z_t^{(c)}\|_{R^d}) \langle e_t^{(c)}, dW_t \rangle_{R^d}. \]
Step 3 Evaluation of $\rho_2(X_t, Y_t^{(\varepsilon)})$.

The martingale parts of semi-martingales $f(\|Z_t^{(\varepsilon)}\|_{\mathbb{R}^d})$ and $U(X_t, Y_t^{(\varepsilon)})$ are the same as those in the proof of Proposition 4.1. Therefore, (2.11) holds. For (2.12), in this case, $M\|Y_t^{(\varepsilon)} - Y_{[t/\eta]}^{(\varepsilon)}\|_{\mathbb{R}^d}dt$ and $2M\kappa\|Y_t^{(\varepsilon)}\|_{\mathbb{R}^d}Y_t^{(\varepsilon)} - Y_{[t/\eta]}^{(\varepsilon)}\|_{\mathbb{R}^d}dt$ are added to the upper bounds of $df(\|Z_t^{(\varepsilon)}\|_{\mathbb{R}^d})$ and $dU(X_t, Y_t^{(\varepsilon)})$, respectively. As a result, there exists a martingale $M^{(\varepsilon)}$ such that

$$d\rho_2(X_t, Y_t^{(\varepsilon)})$$

$$\leq -\min\left\{\frac{\lambda}{2C\kappa}, \frac{2C\kappa}{\lambda}\right\} \rho_2(X_t, Y_t^{(\varepsilon)})dt + M\{U(X_t, Y_t^{(\varepsilon)}) + 2\kappa f(\|Z_t^{(\varepsilon)}\|_{\mathbb{R}^d})\|Y_t^{(\varepsilon)}\|_{\mathbb{R}^d}\} \|Y_t^{(\varepsilon)} - Y_{[t/\eta]}^{(\varepsilon)}\|_{\mathbb{R}^d}dt$$

$$+ \frac{4\kappa}{\beta}\{1 - h_{\varepsilon}(\|Z_t^{(\varepsilon)}\|_{\mathbb{R}^d})\}h_{\varepsilon}(\|Z_t^{(\varepsilon)}\|_{\mathbb{R}^d})\epsilon_t^{(\varepsilon)} - \nabla V_2(X_t) + \nabla V_2(Y_t^{(\varepsilon)})\}dt + dM_t^{(\varepsilon)}.$$ 

Hence, as in the proof of Proposition 4.1 we obtain

$$W_{\rho_2}(X_t, Y_t) \leq E[\rho_2(X_0, Y_0)] - c\int_0^t W_{\rho_2}(X_s, Y_s)ds + M\int_0^t E\{U(X_s, Y_s) + 2\kappa f(R_2)\|Y_s\|_{\mathbb{R}^d}\} \|Y_s - Y_{[s/\eta]}\|_{\mathbb{R}^d}ds.$$ 

Thus, since

$$d(e^{ct}W_{\rho_2}(X_t, Y_t)) \leq Me^{ct}E\{U(X_t, Y_t) + 2\kappa f(R_2)\|Y_t\|_{\mathbb{R}^d}\} \|Y_t - Y_{[t/\eta]}\|_{\mathbb{R}^d}dt,$$ 

we obtain

$$W_{\rho_2}(X_t, Y_t) \leq e^{-ct}E[\rho_2(X_0, Y_0)] + Me^{-ct}\int_0^t e^{cs}E\{U(X_s, Y_s) + 2\kappa f(R_2)\|Y_s\|_{\mathbb{R}^d}\} \|Y_s - Y_{[s/\eta]}\|_{\mathbb{R}^d}ds$$

$$\leq e^{-ct}E[\rho_2(X_0, Y_0)] + M\sup_{a\geq 0} E\{U(X_s, Y_s) + 2\kappa f(R_2)\|Y_s\|_{\mathbb{R}^d}\}^{1/2}e^{-ct}\int_0^t e^{cs}E[\|Y_s - Y_{[s/\eta]}\|_{\mathbb{R}^d}]^{1/2}ds.$$ 

Lemmas A.5 and A.6 complete the proof.

5.2 Proof of Theorem 2.4 (2)

By Lemma A.13 we have

$$E[L(X_t^{(\eta, \eta)})] - E[L(X_t^{(\eta)})] \leq O_{m,b,M,\beta,A,d}(W_{\rho_2}(X_t^{(\eta, \eta)}, X_t^{(\eta)})).$$

Thus, applying Proposition 5.1 to $F = L_n$, we obtain the desired result.

6. Bounds on the Mini-batch Error for Stochastic Gradient Langevin Dynamics

In this section, we prove Theorem 2.4 (3) as the third application of Theorem 2.2. As in the previous section, it is important that Theorem 2.2 admits the case of $G$ is a functional.

In the following, we fix $\eta > 0$ and for simplification, abbreviate $X^{(n, \eta, B)}$ and $X^{(n, \eta)}$ defined by (2.7) and (2.11) as $X$ and $Y$, respectively. Then, we define the ARC between $X$ and $Y$ by

$$\begin{cases} dY_t^{(\varepsilon)} = -\nabla L_n(Y_t^{(\varepsilon)}_{[t/\eta]})dt + \sqrt{2/\beta}(I_d - 2h_{\varepsilon}(\|X_t - Y_t^{(\varepsilon)}\|_{\mathbb{R}^d})\epsilon_t^{(\varepsilon)}\epsilon_t^{(\varepsilon)\top})dW_t, \\ Y_t^{(\varepsilon)} = X_0. \end{cases}$$

6.1 Uniform bound on the time for the difference between (2.7) and (2.11)

Proposition 6.1 below gives a uniform evaluation on the time to the mini batch error of (2.11) with order $\sqrt{(n - B)/B(n - 1)}$. As in the previous section, we only explain the important difference between the proof of the following Proposition 6.1 and that of Proposition 4.1 in the following proof.

**Proposition 6.1.** Under Assumption 2.3 there exists some $\eta_0 = O_{m,M}(1)$ and the following inequality holds uniformly on $0 < \eta \leq \eta_0$.

$$W_{\rho_2}(X_t, Y_t) \leq O_{a_0}(\eta^{1/2} + \sqrt{(n - B)/B(n - 1)}).$$

$$\text{(6.2)}$$
Proof. Fix $\varepsilon > 0$ and let $Z_t^{(c)} = X_t - Y_t^{(c)}$ for $X_t$ and $Y_t^{(c)}$ defined by \([1.1]\).

**Step 1** Evaluation of $f(\|Z_t^{(c)}\|_{\mathbb{R}^d})$.

In this case, \([4.20]\) is replaced by
\[
\langle e_t^{(c)}, \nabla L_{n,[t/\eta]}(X_{[t/\eta]}) - \nabla L_n(Y_{[t/\eta]}) \rangle_{\mathbb{R}^d} \\
\leq M \|X_t - X_{[t/\eta]}\|_{\mathbb{R}^d} + M \|Z_t^{(c)}\|_{\mathbb{R}^d} + M \|\nabla L_n(Y_{[t/\eta]})(Y_{[t/\eta]})(Y_{t})\|_{\mathbb{R}^d} + M \|Y_t^{(c)} - Y_{[t/\eta]}\|_{\mathbb{R}^d}.
\]

Thus, $f(\|Z_t^{(c)}\|_{\mathbb{R}^d})$ is a semi-martingale that satisfies
\[
df(\|Z_t^{(c)}\|_{\mathbb{R}^d}) \leq \|\nabla L_n(Y_{t}^{(c)}) - \nabla L_{n,[t/\eta]}(Y_{t}^{(c)})\|_{\mathbb{R}^d} dt + M \|Z_t^{(c)}\|_{\mathbb{R}^d} \left\{ 1 - h_\varepsilon(\|Z_t^{(c)}\|_{\mathbb{R}^d})^2 \right\} dt \\
+ 2 \sqrt{\frac{\gamma}{\beta}} \|Z_t^{(c)}\|_{\mathbb{R}^d} h_\varepsilon(\|Z_t^{(c)}\|_{\mathbb{R}^d}) (\langle e_t^{(c)}, dW_t \rangle)_{\mathbb{R}^d} - \frac{8Q(\kappa)}{\beta} f'(\|Z_t^{(c)}\|_{\mathbb{R}^d}) h_\varepsilon(\|Z_t^{(c)}\|_{\mathbb{R}^d})^2 dt \\
- \frac{c}{\beta} f(\|Z_t^{(c)}\|_{\mathbb{R}^d}) \chi(0,R_0)(\|Z_t^{(c)}\|_{\mathbb{R}^d}) dt.
\]

**Step 2** Evaluation of $U(X_t, Y_t^{(c)})$.

In this case, the terms $2\kappa(Y_t^{(c)}, \nabla L_n(Y_{[t/\eta]}) - \nabla L_n(Y_{t}^{(c)}))_{\mathbb{R}^d}$ are added to the upper bounds of \([1.1]\). Thus, we have
\[
dU(X_t, Y_t^{(c)}) \leq \left( \frac{\lambda}{\beta} U(X_t, Y_t^{(c)}) \chi(0,R_0)(\|Z_t^{(c)}\|_{\mathbb{R}^d}) - \frac{\lambda}{2} \min\{1, 4C\kappa\} U(X_t, Y_t^{(c)}) \chi(0,R_\infty)(\|Z_t^{(c)}\|_{\mathbb{R}^d}) \right) dt \\
+ 2M\kappa \|Y_t^{(c)}\|_{\mathbb{R}^d} - Y_{[t/\eta]}\|_{\mathbb{R}^d} dt + 2M\kappa \|X_t\|_{\mathbb{R}^d} X_t - X_{[t/\eta]}\|_{\mathbb{R}^d} dt \\
+ \kappa \sqrt{\frac{\gamma}{\beta}} (\nabla V_2(X_t) + \nabla V_2(Y_t^{(c)}), dW_t)_{\mathbb{R}^d} - 2\kappa \sqrt{\frac{\gamma}{\beta}} h_\varepsilon(\|Z_t^{(c)}\|_{\mathbb{R}^d}) (\langle e_t^{(c)}, \nabla V_2(Y_t^{(c)}) \rangle_{\mathbb{R}^d} \langle e_t^{(c)}, dW_t \rangle_{\mathbb{R}^d}.$

**Step 3** Evaluation of $\rho_2(X_t, Y_t^{(c)})$.

The martingale parts of semi-martingales $f(\|Z_t^{(c)}\|_{\mathbb{R}^d})$ and $U(X_t, Y_t^{(c)})$ are the same as those in the proof of Proposition \([4.1]\). Therefore, \([4.21]\) holds. For \([4.21]\), in this case,
\[
\|\nabla L_n(Y_t^{(c)}) - \nabla L_{n,[t/\eta]}(Y_t^{(c)})\|_{\mathbb{R}^d} dt + M \|Y_t^{(c)} - Y_{[t/\eta]}\|_{\mathbb{R}^d} dt + M \|X_t - X_{[t/\eta]}\|_{\mathbb{R}^d} dt
\]
and $2M\kappa \|Y_t^{(c)}\|_{\mathbb{R}^d} - Y_{[t/\eta]}\|_{\mathbb{R}^d} dt + 2M\kappa \|X_t\|_{\mathbb{R}^d} X_t - X_{[t/\eta]}\|_{\mathbb{R}^d} dt$ are added to the upper bounds of $df(\|Z_t^{(c)}\|_{\mathbb{R}^d})$ and $dU(X_t, Y_t^{(c)})$, respectively. As a result, there exists a martingale $M_t^{(c)}$ such that
\[
d\rho_2(X_t, Y_t^{(c)}) \\
\leq -\min \left\{ \frac{\lambda}{\beta} \frac{\lambda}{2} 2C\kappa \right\} \rho_2(X_t, Y_t^{(c)}) dt + U(X_t, Y_t^{(c)}) \|\nabla L_n(Y_t^{(c)}) - \nabla L_{n,[t/\eta]}(Y_t^{(c)})\|_{\mathbb{R}^d} dt \\
+ M \{ U(X_t, Y_t^{(c)}) + 2\kappa f(\|Z_t^{(c)}\|_{\mathbb{R}^d})\|Y_t^{(c)}\|_{\mathbb{R}^d} \} \|Y_t^{(c)} - Y_{[t/\eta]}\|_{\mathbb{R}^d} dt \\
+ M \{ U(X_t, Y_t^{(c)}) + 2\kappa f(\|Z_t^{(c)}\|_{\mathbb{R}^d})\|X_t\|_{\mathbb{R}^d} \} \|X_t - X_{[t/\eta]}\|_{\mathbb{R}^d} dt \\
+ MU(X_t, Y_t^{(c)}) \|Z_t^{(c)}\|_{\mathbb{R}^d} \left\{ 1 - h_\varepsilon(\|Z_t^{(c)}\|_{\mathbb{R}^d})^2 \right\} dt + \frac{4\kappa}{\beta} \left\{ 1 - h_\varepsilon(\|Z_t^{(c)}\|_{\mathbb{R}^d}) \right\} h_\varepsilon(\|Z_t^{(c)}\|_{\mathbb{R}^d}) (\langle e_t^{(c)}, \nabla V_2(X_t) + \nabla V_2(Y_t^{(c)}) \rangle_{\mathbb{R}^d} dt \\
+ dM_t^{(c)}.
\]

Hence, as in the proof of Proposition \([4.1]\) we obtain
\[
W_{\rho_2}(X_t, Y_t) \leq -c \int_0^t W_{\rho_2}(X_s, Y_s) ds + \int_0^t E[U(X_s, Y_s)\|\nabla L_n(Y_{[s/\eta]}n) - \nabla L_{n,[s/\eta]}(Y_{[s/\eta]}n)\|_{\mathbb{R}^d}] ds \\
+ M \int_0^t E\{ \{ U(X_s, Y_s) + 2\kappa f(2R_2)(\|Y_s\|_{\mathbb{R}^d} + \|X_s\|_{\mathbb{R}^d}) \} (\|Y_s - Y_{[s/\eta]}\|_{\mathbb{R}^d} + \|X_s - X_{[s/\eta]}\|_{\mathbb{R}^d}) ds,
\]

13
Thus, since
\[
d(e^{ct}W_{\rho_2}(X_t, Y_t)) \leq e^{ct}E[U(X_t, Y_t)]\|\nabla L_n(Y_{t/\eta}) - \nabla L_n_{t/\eta}(Y_{t/\eta})\|_{\mathbb{R}^d}dt \\
+ Me^{ct}E[\{U(X_t, Y_t) + 2\kappa f(R_2)(|Y_t|_{\mathbb{R}^d} + |X_t|_{\mathbb{R}^d})\}(|Y_t - Y_{t/\eta}|_{\mathbb{R}^d} + |X_t - X_{t/\eta}|_{\mathbb{R}^d})]dt
\]
and \(M\sup_{u \geq 0} E[\{U(X_u, Y_u) + 2\kappa f(R_2)(|Y_u|_{\mathbb{R}^d} + |X_u|_{\mathbb{R}^d})\}^2]^{1/2} \leq O_n(1)\) holds by Lemma A.3, we obtain
\[
W_{\rho_2}(X_t, Y_t) \leq O_n \left( e^{-ct} \int_0^t e^{cs}E[\|\nabla L_n(Y_{s/\eta}) - \nabla L_n_{s/\eta}(Y_{s/\eta})\|_{\mathbb{R}^d}^{1/2}ds + e^{-ct} \int_0^t e^{cs}\{E[|Y_s - Y_{s/\eta}|^{2}_{\mathbb{R}^d}]^{1/2} + E[|X_s - X_{s/\eta}|^{2}_{\mathbb{R}^d}]^{1/2}\}ds \right).
\]

Lemmas A.6 and A.7 complete the proof. \(\square\)

6.2. Proof of Theorem 2.4 (3)
By Lemma A.14 we have
\[
|E[L(X_{t}^{(n,\eta,B)})] - E[L(X_{t}^{(n,\eta)})]| \leq O_{m,b,M,\beta,d}(W_{\rho_2}(X_t^{(n,\eta)}, X_t^{(n,\eta,B)})).
\]
Thus, Proposition 6.1 proves Theorem 2.4 (3). \(\square\)

A. Appendix

A.1. Lemmas on weak convergence

Lemma A.1. Let a continuous function \(H: \mathbb{R}^d \to \mathbb{R}\) satisfy \(|H(x)| \leq K(1 + \|x\|_{\mathbb{R}^d})^p\) for \(p > 0\) and \(K > 0\). If a \(\mathbb{R}^d\)-valued process \(Y_t\) satisfies \(\sup_{t \geq 0} E[\|Y_t\|_{\mathbb{R}^d}] < \infty\) for some \(q > p\) and converges weakly to \(Y\) as \(t \to \infty\), then
\[
\lim_{t \to \infty} ce^{-ct} \int_0^t e^{cs}E[H(Y_s)]ds = E[H(Y)]
\]
holds for all \(c > 0\).

Proof. Fix an arbitrary \(\varepsilon > 0\). By the assumption, we can take \(R > 0\) so that
\[
E[H(Y_t)]; \|Y_t\|_{\mathbb{R}^d} \geq R \leq \frac{KE[(1 + \|Y_t\|_{\mathbb{R}^d})^q]}{R^{q-p}} \leq \varepsilon
\]
holds. For this \(R\), if we define
\[
\phi_R(a) = \begin{cases} 1, & a \leq R, \\ 1 - (a - R), & R \leq a \leq R + 1, \\ 0, & a \geq R + 1 \end{cases}
\]
and \(H_R(x) = H(x)\phi_R(|x|_{\mathbb{R}^d})\), then we have \(|H(x) - H_R(x)| = |H(x)(1 - \phi_R(x))| \leq |H(x)|_x|\chi_{|x|_{\mathbb{R}^d} \geq R}|\). In particular, \(|E[H(Y_t)] - E[H_R(Y_t)]| \leq E[|H(Y_t)|]; \|Y_t\|_{\mathbb{R}^d} \geq R| \leq \varepsilon\) holds for all \(t \geq 0\). Therefore, we only have to handle the case of \(H\) is bounded.

Fix arbitrarily \(\delta > 0\). Since \(Y_t\) converges weakly to \(Y\) and \(H\) is bounded and continuous, we can take \(T > 0\) so that \(E[H(Y_t)]; \|Y_t\|_{\mathbb{R}^d} \geq R \leq \delta\) holds for all \(t \geq T\). Thus, if \(t \geq T\), then
\[
\left|ce^{-ct} \int_0^t e^{cs}E[H(Y_s)]ds - E[H(Y)]\right| \leq ce^{-ct} \int_0^T e^{cs}|E[H(Y_s)] - E[H(Y)]|ds + ce^{-ct}\sup_{x \in \mathbb{R}^d}|H(x)|
\]
\[
\leq ce^{-ct} \int_0^T e^{cs}|E[H(Y_s)] - E[H(Y)]|ds + ce^{-ct}\int_T^t e^{cs}ds + ce^{-ct}\sup_{x \in \mathbb{R}^d}|H(x)|
\]
\[
\leq 3e^{-c(t-T)}\sup_{x \in \mathbb{R}^d}|H(x)| + \delta
\]
holds. \(\square\)

As in the proof of Lemma A.1, we can prove the following by reducing the proof to the case of \(H(t, \cdot)\) is bounded.

Lemma A.2. For a functional \(H: [0, \infty) \times C([0, \infty); \mathbb{R}^d) \to \mathbb{R}\), assume that \(H(t, \cdot)\) is continuous and satisfies (2.4) for all \(t \geq 0\). Furthermore, for a sequence of \(C([0, \infty); \mathbb{R}^d)\)-valued random variables \(Z^{(n)}\), assume that for each \(t \geq 0\), there exists some \(\rho(t) > 1\) such that \(\sup_{n \in \mathbb{N}} E[\sup_{0 \leq s \leq t} |Z_s^{(n)}|]^{\rho(t)} < \infty\). If \(Z^{(n)}\) converges weakly to \(Z\) as \(n \to \infty\), then we have \(\lim_{n \to \infty} E[H(t, Z^{(n)})] = E[H(t, Z)]\) for all \(t \geq 0\).
A.2. Lemmas on Langevin Dynamics

**Lemma A.3.** Let $H \in C^1(\mathbb{R}^d; \mathbb{R})$ be $(m, b)$-dissipative and let

$$\mathcal{L}_H f(x) := -\nabla H(x) \cdot \nabla f(x) + \frac{1}{\beta} \Delta f(x), \quad f \in C^2(\mathbb{R}^d; \mathbb{R}). \quad \text{(A.1)}$$

Then, for any $p \geq 2$ and $x \in \mathbb{R}^d$,

$$\mathcal{L}_H V_p(x) \leq C(p) - \lambda(p)V_p(x)$$

holds, where

$$L(p) = \left\{ \frac{2}{m} \left( \frac{d + p - 2}{\beta} + b \right) \right\}^{1/2}\quad \text{and}\quad \lambda(p) = \frac{mp}{2}, \quad C(p) = \lambda(p) L(p)^p. \quad \text{(A.2)}$$

**Proof.** Since $\nabla V_p(x) = p\|x\|_{\mathbb{R}^d}^{-2}x$ and $\Delta V_p(x) = p(d + p - 2)\|x\|_{\mathbb{R}^d}^{-2}$, by the $(m, b)$-dissipativity of $H$, we have

$$\langle \nabla H(x), \nabla V_p(x) \rangle_{\mathbb{R}^d} = p\|x\|_{\mathbb{R}^d}^{-2}(\nabla H(x), x)_{\mathbb{R}^d} \geq p\|x\|_{\mathbb{R}^d}^{-2}(m\|x\|_{\mathbb{R}^d}^2 - b).$$

Thus, we obtain

$$-\langle \nabla H(x), \nabla V_p(x) \rangle_{\mathbb{R}^d} + \frac{\Delta V_p(x)}{\beta} \leq -mp\|x\|_{\mathbb{R}^d}^2 + bp\|x\|_{\mathbb{R}^d}^2 + \frac{p(d + p - 2)}{\beta} \|x\|_{\mathbb{R}^d}^2 = -\frac{mp}{2}\|x\|_{\mathbb{R}^d}^2 - \left( \frac{p(d + p - 2)}{\beta} + bp - \frac{mp}{2}\|x\|_{\mathbb{R}^d}^2 \right) \|x\|_{\mathbb{R}^d}^2.$$ 

In particular, if $\|x\|_{\mathbb{R}^d} \geq L(p)$, then $\mathcal{L}_H V_p(x) \leq -\lambda(p)V_p(x)$ holds. On the other hand, if $\|x\|_{\mathbb{R}^d} \leq L(p)$, then

$$\mathcal{L}_H V_p(x) \leq -\lambda(p)V_p(x) + \left( \frac{p(d + p - 2)}{\beta} + bp \right) L(p)^p = C(p) - \lambda(p)V_p(x)$$

holds. \hfill \Box

**Lemma A.4.** Let $H \in C^1(\mathbb{R}^d; \mathbb{R})$ be $M$-smooth and let $p \geq 2$. For a $\{\mathcal{F}_t\}$-adapted process $\gamma_t$ and $Y_0 \in L^p(\Omega; \mathbb{R}^d)$, there exists uniquely the strong solution $Y^{(\varepsilon)}$ of

$$\begin{cases}
    dY^{(\varepsilon)}_t = -\nabla H(Y^{(\varepsilon)}_t)dt + \sqrt{2/\beta}(I_d - 2h_\varepsilon(\|\gamma_t - Y^{(\varepsilon)}_t\|_{\mathbb{R}^d})e^{(\varepsilon)}_t e^{(\varepsilon)\top} )dW_t, & t \geq 0, \\
    Y^{(\varepsilon)}_0 = Y_0.
\end{cases} \quad \text{(A.3)}$$

Here, $e^{(\varepsilon)}_t = (\gamma_t - Y^{(\varepsilon)}_t)/\|\gamma_t - Y^{(\varepsilon)}_t\|_{\mathbb{R}^d}$. Furthermore, if $H$ is $(m, b)$-dissipative, then for all $t \geq 0$, $Y^{(\varepsilon)}$ satisfies

$$E[\|Y^{(\varepsilon)}_t\|_{\mathbb{R}^d}^p] \leq e^{-\lambda(p)t}E[\|Y_0\|_{\mathbb{R}^d}^p] + \frac{C(p)}{\lambda(p)} (1 - e^{-\lambda(p)t}). \quad \text{(A.4)}$$

The same bound as (A.4) holds also for the solution of

$$dY_t = -\nabla H(Y_t)dt + \sqrt{2/\beta}W_t, \quad t \geq 0. \quad \text{(A.5)}$$

**Proof.** For each $\varepsilon > 0$, the map $\mathbb{R}^d \ni x \mapsto h_\varepsilon(\|x\|_{\mathbb{R}^d}) (x/\|x\|_{\mathbb{R}^d})(x/\|x\|_{\mathbb{R}^d})^\top \in \mathbb{R}^d \otimes \mathbb{R}^d$ is Lipschitz continuous. Thus, (A.3) has the unique strong solution $Y^{(\varepsilon)}_t$. Denoting $Z^{(\varepsilon)}_t = \gamma_t - Y^{(\varepsilon)}_t$, Ito’s formula yields

$$dV_p(Y^{(\varepsilon)}_t) = \mathcal{L}_H V_p(Y^{(\varepsilon)}_t)dt - \frac{4h_\varepsilon(\|Z^{(\varepsilon)}_t\|_{\mathbb{R}^d})}{\beta}(1 - h_\varepsilon(\|Z^{(\varepsilon)}_t\|_{\mathbb{R}^d})) \sum_{i,j=1}^d e^{(\varepsilon)}_i e^{(\varepsilon)}_j \partial_{ij} V_p(Y^{(\varepsilon)}_t)dt \quad + \sqrt{2/\beta} \langle \nabla V_p(Y^{(\varepsilon)}_t) , dW_t \rangle_{\mathbb{R}^d} - 2 \sqrt{2/\beta} h_\varepsilon(\|Z^{(\varepsilon)}_t\|_{\mathbb{R}^d}) \langle e^{(\varepsilon)}_t , \nabla V_p(Y^{(\varepsilon)}_t) \rangle_{\mathbb{R}^d} \langle e_t , dW_t \rangle_{\mathbb{R}^d}. $$
Since $V_p$ is convex for $p \geq 2$, its Hessian matrix is nonnegative-definite. Thus, by Lemma A.3, we have
\[ dV_p(Y_t^{(e)}) \leq \{C(p) - \lambda(p)\}V_p(Y_t^{(e)})dt + \sqrt{\frac{2}{\beta}}(\nabla V_p(Y_t^{(e)}), dW_t)_{\mathbb{R}^d} - 2\sqrt{\frac{2}{\beta}}b_e(\|Z_t^{(e)}\|_{\mathbb{R}^d})(e_t^{(e)}, \nabla V_p(Y_t^{(e)}))_{\mathbb{R}^d}(e_t^{(e)}, dW_t)_{\mathbb{R}^d}. \]

Hence, since
\[ E[V_p(Y_t^{(e)})] \leq E[V_p(Y_0)] + C(p)t - \lambda(p)\int_0^t E[V_p(Y_s^{(e)})]ds \]
holds, by \( d(e^{\lambda(p)t}E[V_p(Y_t^{(e)})]) \leq C(p)(e^{\lambda(p)t}t, dt) \), we obtain
\[ e^{\lambda(p)t}E[V_p(Y_t^{(e)})] \leq E[V_p(Y_0)] + \frac{C(p)}{\lambda(p)}(e^{\lambda(p)t} - 1). \]

The bound for \( A.5 \) can be proved in the same way. \( \square \)

The following lemma is an extension of Lemma 3.2 in [24] based on Young’s inequality.

**Lemma A.5.** Assume that \( F_k \in C^1(\mathbb{R}^d; \mathbb{R}) \) is \((m, b)\)-dissipative and \( M\)-smooth for each \( k \) and satisfies \( \sup_{k \in \mathbb{N}} \|\nabla F_k(0)\|_{\mathbb{R}^d} \leq A \). For fixed \( \eta > 0 \), we define the process \( Y \) as
\[ Y_t = Y_{k\eta} - (t - k\eta)\nabla F_k(Y_{k\eta}) + \sqrt{2/\beta}(W_t - W_{k\eta}), \quad k\eta \leq t < (k + 1)\eta. \]

Then, for all \( \ell \in \mathbb{N} \), there exists some \( \eta_0 = \alpha_{m, M, \beta, \ell}(1) \) and the following inequality holds uniformly on \( 0 < \eta \leq \eta_0 \).
\[ \sup_{\ell \geq 0} E[\|Y_t^{(\ell)}\|_{\mathbb{R}^d}^2] \leq O_{m,b,M,\beta,A,d,\ell}(1 + E[\|Y_0\|_{\mathbb{R}^d}^2]). \]

**Proof.** Let \( k\eta \leq t < (k + 1)\eta \) and define \( \Delta_{k,\ell} = Y_{k\eta} - (t - k\eta)\nabla F_k(Y_{k\eta}) \) and \( U_{k,\ell} = \sqrt{2/\beta}(W_t - W_{k\eta}) \). If \( s \in \mathbb{N} \) is odd, then we have \( E[(\Delta_{k,\ell}, U_{k,\ell})_{\mathbb{R}^d} | Y_{k\eta}] = 0 \). Thus, there exist constants \( a_j \) that depend only on \( \ell \in \mathbb{N} \) such that
\[ E[\|Y_t^{(\ell)}\|_{\mathbb{R}^d}^2 | Y_{k\eta}] = \|\Delta_{k,\ell}\|_{\mathbb{R}^d}^2 + 2(\Delta_{k,\ell}, U_{k,\ell})_{\mathbb{R}^d} + \|U_{k,\ell}\|_{\mathbb{R}^d}^2 \leq \|\Delta_{k,\ell}\|_{\mathbb{R}^d}^2 + \sum_{j=1}^{\ell} a_j E[\|\Delta_{k,\ell}\|_{\mathbb{R}^d}^{2\ell - 2j} \|U_{k,\ell}\|_{\mathbb{R}^d}^{2j} | Y_{k\eta}]. \]

By Young’s inequality, for any \( \varepsilon > 0 \) and \( 1 \leq j \leq \ell - 1 \),
\[ a_j \|\Delta_{k,\ell}\|_{\mathbb{R}^d}^{2\ell - 2j} \|U_{k,\ell}\|_{\mathbb{R}^d}^{2j} = (\varepsilon \|\Delta_{k,\ell}\|_{\mathbb{R}^d}^{2\ell - 2j})(a_j \|U_{k,\ell}\|_{\mathbb{R}^d}^{2j}/\varepsilon) \leq \frac{\ell - j}{\varepsilon} \|\Delta_{k,\ell}\|_{\mathbb{R}^d}^{2\ell} + \frac{\varepsilon \ell}{\ell - j} \|U_{k,\ell}\|_{\mathbb{R}^d}^{2j} \]
holds. In particular, setting \( \varepsilon = \{\ell^{-1}m(t - k\eta)\}^{\ell-1} \), we obtain
\[ a_j \|\Delta_{k,\ell}\|_{\mathbb{R}^d}^{2\ell - 2j} \|U_{k,\ell}\|_{\mathbb{R}^d}^{2j} \leq \frac{m(t - k\eta)}{\ell} \|\Delta_{k,\ell}\|_{\mathbb{R}^d}^{2\ell} + \frac{\varepsilon \ell}{\ell - j} \|U_{k,\ell}\|_{\mathbb{R}^d}^{2j}. \]
Hence, since \( E[\|U_{k,\ell}\|_{\mathbb{R}^d}^{2\ell}] = O_{\beta,\ell}(\ell - k\eta)^{\ell} \) and \( \ell - \frac{\lambda\eta}{\beta} \geq 1 \), for \( \eta \leq 1 \), there exists a constant \( C_1 \) independent of \( k \) such that
\[ E[\|Y_t^{(\ell)}\|_{\mathbb{R}^d}^2 | Y_{k\eta}] \leq \{1 + m(t - k\eta)\} \|\Delta_{k,\ell}\|_{\mathbb{R}^d}^{2\ell} + (t - k\eta)C_1. \]

For the first term, if \( \eta \leq 1 \), then there exist constants \( b_j \) that depend only on \( \ell \in \mathbb{N} \) such that
\[ \|\Delta_{k,\ell}\|_{\mathbb{R}^d}^{2\ell} = \|Y_{k\eta}\|_{\mathbb{R}^d}^{2\ell} - 2(t - k\eta)(Y_{k\eta}, \nabla F_k(Y_{k\eta}))_{\mathbb{R}^d} + (t - k\eta)^2 \|\nabla F_k(Y_{k\eta})\|_{\mathbb{R}^d}^{2\ell} \]
\[ \leq \|Y_{k\eta}\|_{\mathbb{R}^d}^{2\ell} - 2\ell(t - k\eta)\|Y_{k\eta}\|_{\mathbb{R}^d}^{2\ell - 2} (Y_{k\eta}, \nabla F_k(Y_{k\eta}))_{\mathbb{R}^d} + (t - k\eta)^2 \sum_{j=1}^{\ell} b_j \|Y_{k\eta}\|_{\mathbb{R}^d}^{2\ell - 2j} \|\nabla F_k(Y_{k\eta})\|_{\mathbb{R}^d}. \]

Since the map \( r \mapsto r^q \) is convex for \( q \geq 2 \), by \( M\)-smoothness of \( F_k \) and the boundedness \( \sup_{k \in \mathbb{N}} \|\nabla F_k(0)\|_{\mathbb{R}^d} \leq A \), we have
\[ \|\nabla F_k(Y_{k\eta})\|_{\mathbb{R}^d}^{q} \leq (M\|Y_{k\eta}\|_{\mathbb{R}^d} + A)^q \leq 2^{q-2}(M^2\|Y_{k\eta}\|_{\mathbb{R}^d}^2 + A^q). \]
Thus, by the \((m, b)\)-dissipativity of \(F_k\), we can find constants \(C_2\) and \(C_3\) independent of \(k\) so that
\[
\| \Delta_{k,t} \|^2_{\mathbb{R}^d} \leq \left\{ 1 - \frac{3}{2} m(t - k\eta) \right\} \| Y_{k\eta} \|^2_{\mathbb{R}^d} + (t - k\eta) \{ b \| Y_{k\eta} \|^2_{\mathbb{R}^d} + C_3 \}
\]
holds. Thus, taking \(K \geq 0\) sufficiently large so that \(\frac{1}{4} \ell m K^2 \geq b K^{2\ell} + C_3 \geq 0\), if \(\| Y_{k\eta} \|_{\mathbb{R}^d} \geq K\), then we have
\[
\| \Delta_{k,t} \|^2_{\mathbb{R}^d} \leq \left\{ 1 - \frac{3}{2} m(t - k\eta) \right\} \| Y_{k\eta} \|^2_{\mathbb{R}^d} \leq \left\{ 1 - \frac{3}{2} m(t - k\eta) \right\} \| Y_{k\eta} \|^2_{\mathbb{R}^d} + (t - k\eta) \{ b K^{2\ell} + C_3 \}.
\] (A.6)
Since (A.6) holds when \(\| Y_{k\eta} \|_{\mathbb{R}^d} \leq K\), (A.6) is always true. Therefore, by
\[
\left( 1 - \frac{3}{2} m(t - k\eta) \right) (1 + m(t - k\eta)) = 1 - \frac{3}{2} m(t - k\eta) + m(t - k\eta) - \frac{3}{2} m^2(t - k\eta)^2 \leq 1 - \frac{3}{2} m(t - k\eta),
\]
we can find a constant \(C_4\) independent of \(k\) such that
\[
E[\| Y_t \|^2_{\mathbb{R}^d} | Y_{k\eta}] \leq \left( 1 - \frac{1}{2} m(t - k\eta) \right) \| Y_{k\eta} \|^2_{\mathbb{R}^d} + (t - k\eta) C_4.
\]
From the aforementioned, we obtain
\[
E[\| Y_t \|^2_{\mathbb{R}^d}] \leq \left( 1 - \frac{1}{2} m(t - k\eta) \right) E[\| Y_{k\eta} \|^2_{\mathbb{R}^d}] + (t - k\eta) C_4
\]
\[
\leq (t - k\eta) C_4 + \left( 1 - \frac{1}{2} m(t - k\eta) \right) \left\{ 1 - \frac{1}{2} m(t - k\eta) \right\} \| Y_{(k-1)\eta} \|^2_{\mathbb{R}^d} + \eta C_4
\]
\[
\leq \eta C_4 \left\{ 1 + (1 - \frac{1}{2} m(t - k\eta)) + (1 - \frac{1}{2} m(t - k\eta)(1 - \frac{1}{2} m(t - k\eta)) \| Y_{(k-1)\eta} \|^2_{\mathbb{R}^d} \right\}
\]
\[
\leq \eta C_4 \left\{ 1 + (1 - \frac{1}{2} m(t - k\eta)) \sum_{j=0}^{k-1} (1 - \frac{1}{2} m(t - k\eta))^j \right\} + (1 - \frac{1}{2} m(t - k\eta)) \| Y_{(k-3)\eta} \|^2_{\mathbb{R}^d}
\]
\[
\leq \eta C_4 \left\{ 1 + (1 - \frac{1}{2} m(t - k\eta)) \sum_{j=0}^{k-1} (1 - \frac{1}{2} m(t - k\eta))^j \right\} + \frac{1}{1 - \frac{1}{2} m(t - k\eta)) (1 - \frac{1}{2} m(t - k\eta))^k \| Y_0 \|^2_{\mathbb{R}^d}
\]
\[
\leq \frac{2 C_4}{m} + (1 - \frac{1}{2} m(t - k\eta))(1 - \frac{1}{2} m(t - k\eta))^k \| Y_0 \|^2_{\mathbb{R}^d}.
\]
Thus, \(E[\| Y_t \|^2_{\mathbb{R}^d}] \leq 2C_4/m + E[\| Y_0 \|^2_{\mathbb{R}^d}]\) holds for all \(t > 0\). \(\square\)

**Lemma A.6.** Let \(H : [0, \infty) \times \mathbb{R}^d \to \mathbb{R}\) and let \(H(t, \cdot) \in C^1(\mathbb{R}^d; \mathbb{R})\) be \(M\)-smooth for each \(t \geq 0\). For fixed \(\eta > 0\), if we define \(Y\) by
\[
dY_t = -\nabla_x H([t/\eta] \eta, Y_{[t/\eta] \eta}) dt + \sqrt{2/\beta} dW_t, \quad t \geq 0,
\] (A.7)
then, for each \(t \geq 0\), we have
\[
E[\| Y_t - Y_{[t/\eta] \eta} \|^2_{\mathbb{R}^d}] \leq \eta^2 E[\| M(Y_{[t/\eta] \eta}) \|^2_{\mathbb{R}^d} + \| \nabla_x H([t/\eta] \eta, 0) \|^2_{\mathbb{R}^d}] + (2\eta \beta)/\beta.
\]
Proof. According to (A.7), we have
\[
Y_t - Y_{[t/\eta] \eta} = -([t/\eta] \eta) \nabla_x H([t/\eta] \eta, Y_{[t/\eta] \eta}) dt + \sqrt{2/\beta} (W_t - W_{[t/\eta] \eta}).
\]
Thus,
\[
E[\|Y_t - Y_{[t/\eta]}\|_{\mathbb{R}^d}^2] = \eta^2 E[\|\nabla x H([t/\eta]^\tau, Y_{[t/\eta]}^\tau)\|_{\mathbb{R}^d}^2] + (2d\eta)/\beta
\leq \eta^2 E[\|M\|Y_{[t/\eta]}\|_{\mathbb{R}^d}^2 + \|\nabla x H([t/\eta], 0)||_{\mathbb{R}^d}^2] + (2d\eta)/\beta
\]
holds.

The following lemma can be proved similarly to Lemma C.5 in [23].

**Lemma A.7.** For each \( k \in \mathbb{N} \), we have
\[
E[\|\nabla L_n(w) - \nabla L_n,k(w)\|_{\mathbb{R}^d}^2] \leq 4(n - B)/(B(n - 1)(M\|w\|_{\mathbb{R}^d} + A))^2, \quad w \in \mathbb{R}^d.
\]

**A.3. Lemmas from [6]**

In this subsection, we use the notations introduced in subsection 4.1. The following lemma is in Section 2 in [6].

**Lemma A.8.** Let \( F, G \in C^1(\mathbb{R}^d; \mathbb{R}) \) be \((m, b)\)-dissipative and \( M\)-smooth. Under the notations of (4.5) and (4.6), if \((x, y) \notin S_1\), then
\[
\mathcal{L}_F \tilde{V}_2(x) + \mathcal{L}_G \tilde{V}_2(y) < 0 \tag{A.8}
\]
holds, where for \( H \in C^1(\mathbb{R}^d; \mathbb{R})\), \( \mathcal{L}_H = -(\nabla H, \nabla)_{\mathbb{R}^d} - \beta^{-1}\Delta \). Furthermore, if \((x, y) \notin S_2\), then for any \( \kappa > 0 \),
\[
\kappa \mathcal{L}_F \tilde{V}_2(x) + \kappa \mathcal{L}_G \tilde{V}_2(y) \leq -\frac{\lambda}{2} \min\{1, 4C\kappa\} \{1 + \kappa \tilde{V}_2(x) + \kappa \tilde{V}_2(y)\} \tag{A.9}
\]
holds.

The following condition (A.11) is the counterpart of (2.25) in [6].

**Lemma A.9.** Let
\[
\kappa := \min\left\{ \frac{1}{2}, \frac{2}{C\beta(e^{2R_1} - 1 - 2R_1)} \exp\left\{ -\frac{M\beta}{8} R_1^2 \right\} \right\} \in (0, 1). \tag{A.10}
\]

Then we have
\[
\frac{1}{2C\beta\kappa} \geq \int_0^{R_1} \Phi(s)\varphi(s)^{-1} ds. \tag{A.11}
\]

**Proof.** Since \( Q(\kappa) \in (0, 1], \)
\[
\int_0^{R_1} \Phi(s)\varphi(s)^{-1} ds = \int_0^{R_1} \int_0^s \exp\left\{ \frac{M\beta}{8} (s^2 - r^2) + 2Q(\kappa)(s - r) \right\} dr ds
\]
\[
\leq \int_0^{R_1} \int_0^s \exp\left\{ \frac{M\beta}{8} (s^2 - r^2) + 2(s - r) \right\} dr ds
\]
\[
= \exp\left\{ \frac{M\beta}{8} R_1^2 \right\} \int_0^{R_1} \int_0^s e^{2(s-r)} dr ds
\]
\[
= \exp\left\{ \frac{M\beta}{8} R_1^2 \right\} \int_0^{R_1} \int_0^s e^{2(s-r)} dr ds
\]
\[
= \frac{1}{2} \left\{ e^{2R_1} - 1 - 2R_1 \right\} \exp\left\{ \frac{M\beta}{8} R_1^2 \right\}
\]
holds.

The following three lemmas are in Section 5 in [6].
Lemma A.10. The function $f$ defined by (4.11) is nonnegative and bounded on $[0, \infty)$ and satisfies $f(0) = 0$. Furthermore, $f$ is continuous, increasing and concave on $\mathbb{R}$, and
\[
r \varphi(R_2) \leq \Phi(r) \leq 2f(r) \leq 2\Phi(r) \leq 2r, \quad 0 \leq r \leq R_2
\]
holds.

Lemma A.11. $\mu_f((-\infty,0] \cup (R_2, \infty)) = 0$ and $\mu_f(\{R_1\}) \leq 0$ hold.

Lemma A.12. For any $r \in (0, R_1) \cup (R_1, R_2)$, we have
\[
f''(r) \leq -\left(\frac{M\beta}{4} r + 2Q(\kappa)\right) f'(r) - \frac{\zeta}{4} f(r)\chi_{(0,R_2)}(r) - \frac{\zeta}{4} f(r)\chi_{(0,R_1)}(r).
\]

A.4. Inequalities based on couplings

Lemma A.13. (Lemma 6 in [20]) Let $H \in C^1(\mathbb{R}^d; \mathbb{R})$ satisfy $\|\nabla H(x)\|_{\mathbb{R}^d} \leq c_1\|x\|_{\mathbb{R}^d} + c_2$. Then for any probability measures $\mu$ and $\nu$ on $\mathbb{R}^d$ and any $\gamma \in \Pi(\mu, \nu)$, we have
\[
\left|\int_{\mathbb{R}^d} H(x)\mu(dx) - \int_{\mathbb{R}^d} H(y)\nu(dy)\right| \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} \left(\frac{c_1}{2}\|x\|_{\mathbb{R}^d} + \frac{c_1}{2}\|y\|_{\mathbb{R}^d} + c_2\right) \|x - y\|_{\mathbb{R}^d}\gamma(dx,dy).
\]

Lemma A.14. Let $A > 0$. For $\rho_2$ defined by (4.12),
\[
\left(\frac{M}{2}\|x\|_{\mathbb{R}^d} + \frac{M}{2}\|y\|_{\mathbb{R}^d} + A\right) \|x - y\|_{\mathbb{R}^d} \leq 2 \exp\left(\frac{M\beta R_2^2}{8} + 2R_2\right) \max\left\{1, \frac{1}{R_2}\right\} \max\left\{A + M, \frac{1}{\kappa}\left(A + M\right)\right\} \rho_2(x,y)
\]
holds.

Proof. For $r \in [0, R_2)$, we have
\[
f'(r) = \varphi(r)g(r) \geq \frac{1}{2} \exp\left(-\frac{M\beta R_2^2}{8} - 2R_2\right).
\]
Thus, since $f(0) = 0$, if $\|x - y\|_{\mathbb{R}^d} \leq R_2$, then Taylor’s theorem yields
\[
f(\|x - y\|_{\mathbb{R}^d}) \geq \frac{1}{2} \exp\left(-\frac{M\beta R_2^2}{8} - 2R_2\right) \|x - y\|_{\mathbb{R}^d}.
\]
Therefore, if $\|x - y\|_{\mathbb{R}^d} \leq R_2$, then by $r \leq 2^{-1}(1 + r^2)$, we obtain
\[
\left(\frac{M}{2}\|x\|_{\mathbb{R}^d} + \frac{M}{2}\|y\|_{\mathbb{R}^d} + A\right) \|x - y\|_{\mathbb{R}^d} \leq 2 \exp\left(\frac{M\beta R_2^2}{8} + 2R_2\right) \left(A + M, \frac{M}{4}\|x\|_{\mathbb{R}^d} + \frac{M}{4}\|y\|_{\mathbb{R}^d}\right) f(\|x - y\|_{\mathbb{R}^d})
\]
\[
\leq 2 \exp\left(\frac{M\beta R_2^2}{8} + 2R_2\right) \max\left\{A + M, \frac{M}{4}\kappa\right\} \rho_2(x,y).
\]
On the other hand, if $\|x - y\|_{\mathbb{R}^d} > R_2$, then we have
\[
f(\|x - y\|_{\mathbb{R}^d}) = f(R_2) \geq \frac{R_2}{2} \exp\left(-\frac{M\beta R_2^2}{8} - 2R_2\right)
\]
and therefore
\[
\left(\frac{M}{2}\|x\|_{\mathbb{R}^d} + \frac{M}{2}\|y\|_{\mathbb{R}^d} + A\right) \|x - y\|_{\mathbb{R}^d} \leq A + \left(\frac{A}{2} + M\right) \left(\|x\|_{\mathbb{R}^d}^2 + \|y\|_{\mathbb{R}^d}^2\right)
\]
\[
\leq \frac{2}{R_2} \exp\left(\frac{M\beta R_2^2}{8} + 2R_2\right) \max\left\{A, \frac{1}{\kappa}\left(A + M\right)\right\} \rho_2(x,y)
\]
holds.

The following lemma is a simple corollary to Lemmas A.13 and A.14

Lemma A.15. Let $H \in C^1(\mathbb{R}^d; \mathbb{R})$ be $M$-smooth and let $\mu$ and $\nu$ be probability measures on $\mathbb{R}^d$. Then we have
\[
|\mu(H) - \nu(H)| \leq O_{m,b,M,\beta,\|\nabla H(0)\|_{\mathbb{R}^d}}(W_{\rho_2}(\mu, \nu)). \quad \text{(A.12)}
\]
References

[1] Dimitri P Bertsekas and John N Tsitsiklis. Gradient convergence in gradient methods with errors. *SIAM Journal on Optimization*, 10(3):627–642, 2000.

[2] François Bolley and Cédric Villani. Weighted csiszár-kullback-pinsker inequalities and applications to transportation inequalities. In *Annales de la Faculté des sciences de Toulouse: Mathématiques*, volume 14, pages 331–352, 2005.

[3] Huy N Chau, Chaman Kumar, Miklós Rásonyi, and Sotirios Sabanis. On fixed gain recursive estimators with discontinuity in the parameters. *ESAIM: Probability and Statistics*, 23:217–244, 2019.

[4] Ngoc Huy Chau, Éric Moulines, Miklos Rásonyi, Sotirios Sabanis, and Ying Zhang. On stochastic gradient langevin dynamics with dependent data streams: The fully nonconvex case. *SIAM Journal of Mathematics of Data Science*, 3(3):959–986, 2021.

[5] Xiang Cheng, Niladri S Chatterji, Yasin Abbasi-Yadkori, Peter L Bartlett, and Michael I Jordan. Sharp convergence rates for langevin dynamics in the nonconvex setting. *arXiv preprint arXiv:1805.01648*, 2018.

[6] Andreas Eberle, Arnaud Guillin, and Raphael Zimmer. Quantitative harris-type theorems for diffusions and mckean–vlasov processes. *Transactions of the American Mathematical Society*, 371(10):7135–7173, 2019.

[7] Andreas Eberle and Raphael Zimmer. Sticky couplings of multidimensional diffusions with different drifts. In *Annales de l’Institut Henri Poincaré, Probabilités et Statistiques*, volume 55, pages 2370–2394. Institut Henri Poincaré, 2019.

[8] Murat A Erdogdu, Lester Mackey, and Ohad Shamir. Global non-convex optimization with discretized diffusions. *Advances in Neural Information Processing Systems*, 31, 2018.

[9] Rong Ge, Furong Huang, Chi Jin, and Yang Yuan. Escaping from saddle points-online stochastic gradient for tensor decomposition. In *Conference on learning theory*, pages 797–842. PMLR, 2015.

[10] Moritz Hardt, Ben Recht, and Yoram Singer. Train faster, generalize better: Stability of stochastic gradient descent. In *International conference on machine learning*, pages 1225–1234. PMLR, 2016.

[11] Nobuyuki Ikeda and Shinzo Watanabe. *Stochastic differential equations and diffusion processes*. Elsevier, 2014.

[12] Chi Jin, Rong Ge, Praneeth Netrapalli, Sham M Kakade, and Michael I Jordan. How to escape saddle points efficiently. In *International Conference on Machine Learning*, pages 1724–1732. PMLR, 2017.

[13] Ioannis Karatzas and Steven Shreve. *Brownian motion and stochastic calculus*, volume 113. Springer Science & Business Media, 2012.

[14] Chris Junchi Li, Lei Li, Junyang Qian, and Jian-Guo Liu. Batch size matters: A diffusion approximation framework on nonconvex stochastic gradient descent. *stat*, 1050:22, 2017.

[15] Torgny Lindvall and L Cris G Rogers. Coupling of multidimensional diffusions by reflection. *The Annals of Probability*, pages 860–872, 1986.

[16] Mateusz B Majka, Aleksandar Mijatović, and Lukasz Szpruch. Nonasymptotic bounds for sampling algorithms without log-concavity. *The Annals of Applied Probability*, 30(4):1534–1581, 2020.

[17] Panayotis Mertikopoulos, Nadav Hallak, Ali Kavis, and Volkan Cevher. On the almost sure convergence of stochastic gradient descent in non-convex problems. *Advances in Neural Information Processing Systems*, 33:1117–1128, 2020.

[18] Wenlong Mou, Liwei Wang, Xiyu Zhai, and Kai Zheng. Generalization bounds of sgld for non-convex learning: Two theoretical viewpoints. In *Conference on Learning Theory*, pages 605–638. PMLR, 2018.

[19] Boris Muzellec, Kanji Sato, Mathurin Massias, and Taiji Suzuki. Dimension-free convergence rates for gradient langevin dynamics in rkhs. *arXiv preprint arXiv:2003.00306*, 2020.
[20] Maxim Raginsky, Alexander Rakhlin, and Matus Telgarsky. Non-convex learning via stochastic gradient langevin dynamics: a nonasymptotic analysis. In Conference on Learning Theory, pages 1674–1703. PMLR, 2017.

[21] Taiji Suzuki. Generalization bound of globally optimal non-convex neural network training: Transportation map estimation by infinite dimensional langevin dynamics. Advances in Neural Information Processing Systems, 33:19224–19237, 2020.

[22] Cédric Villani. Optimal transport: old and new, volume 338. Springer, 2009.

[23] Pan Xu, Jinghui Chen, Difan Zou, and Quanquan Gu. Global convergence of langevin dynamics based algorithms for nonconvex optimization. Advances in Neural Information Processing Systems, 31, 2018.

[24] Ying Zhang, Ömer Deniz Akyildiz, Theodoros Damoulas, and Sotirios Sabanis. Nonasymptotic estimates for stochastic gradient langevin dynamics under local conditions in nonconvex optimization. arXiv preprint arXiv:1910.02008, 2019.

[25] Yuchen Zhang, Percy Liang, and Moses Charikar. A hitting time analysis of stochastic gradient langevin dynamics. In Conference on Learning Theory, pages 1980–2022. PMLR, 2017.