Finite-size behavior near the critical point of QCD phase-transition

Chen Lizhu,¹ X. S. Chen,² and Wu Yuanfang¹,3,4

¹Institute of Particle Physics, Hua-Zhong Normal University, Wuhan 430079, China
²Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190, China
³Brookhaven National Laboratory, Physical Department, STAR group, Upton, NY 11973, USA
⁴Key Laboratory of Quark & Lepton Physics (HuaZhong Normal University), Ministry of Education, China

It is pointed out that finite-size effect is not negligible in locating critical point of QCD phase transition at current relativistic heavy ion collisions. The finite-size scaling form of critical related observable is suggested. Its fixed point behavior at critical incident energy can be served as a reliable identification of critical point and nearby boundary of QCD phase transition. How to find experimentally the fixed point behavior is demonstrated by using 3D-Ising model as an example. The validity of the method at finite detector acceptances at RHIC is discussed.

PACS numbers: 25.75.Nq, 12.38.Mh, 21.65.Qr

Quantum colorodynamics (QCD) has predicted quark deconfinement and chiral symmetry restoration at finite temperature and density.⁰ Lattice-QCD has shown that the transition is crossover at vanishing baryon chemical potential $\mu_B$. The QCD based model indicates that the crossover turns to be a first-order phase transition at larger values of $\mu_B$. The endpoint of the first order phase transition to the crossover is referred as critical endpoint, or critical point. All these show that the transition from hadron phase to quark-gluon-plasma (QGP) phase can happen in 3 possible ways.

The data from current relativistic heavy ion experiments show that the QGP has been formed at RHIC energies. But the position of critical point in QCD phase diagram is not clear from the theoretical side. Well defined character of critical point, the divergence of correlation length, warrants the possibility of finding it experimentally. The goal of the beam energy scan at RHIC, in comparison with system size, the system can still be considered as infinite large. The critical behavior under thermal limit is available. This is why the non-monotonic behavior is suggested as an indicator of critical point. In case of first order phase transition, or crossover, some observables also show the non-monotonic behavior. The absence of non-monotonic behavior does not exclude the existence of critical point, such as the maximum cluster size in 3D-Ising model shown in Fig. 1(a).

Moreover, if the correlation length is comparable with system size, the finite-size effect is not negligible. When correlation length is larger than $L$, of system size, it has been shown that the finite-size effect has to be taken into account.

Although, it is still difficult to estimate the size of the formed system and correlation length at critical point in relativistic heavy ion collisions. A rough estimation shows that the system size at freeze-out is less than 12 fm. The correlation length is round 6 fm for typical nuclear collisions. After considering the finite evolution time, or finite-size, it is argued that the maximum of correlation length may not be beyond 2-3 fm at critical point. Base on those estimations, the ratio of correlation length to system size is round $\frac{1}{6}$, which is in the region larger than $\frac{1}{6}$. So in current relativistic heavy ion collisions at RHIC, the finite-size effect most probably has to be taken into account, rather than negligible.

In accounting for the finite-size effects, the critical behavior of all suggested observables, such as the fluctuations and correlations of transverse momentum, multiplicity, conserved charges, and in particular, the higher order moments of baryon numbers, should be re-examined under the frame of finite-size behavior of critical point and nearby boundary.

In the letter, we first discuss the finite-size behavior of the critical point, the first order phase transition, and the crossover in general. Secondly, we suggest the finite-size scaling form of the critical related observable in relativistic heavy ion collisions. Its fixed-point behavior at critical incident energy can be served as a reliable identification of critical point and nearby boundary of QCD phase transition. Thirdly, we demonstrate how to locate the fixed point from experimental observable by using 3D-Ising model as an example. Finally, the validity of the method at finite detector acceptance at RHIC is discussed.

The data from current relativistic heavy ion experiments show that the QGP has been formed at RHIC energies. But the position of critical point in QCD phase diagram is not clear from the theoretical side. Well defined character of critical point, the divergence of correlation length, warrants the possibility of finding it experimentally. The goal of the beam energy scan at RHIC, in comparison with system size, the system can still be considered as infinite large. The critical behavior under thermal limit is available. This is why the non-monotonic behavior is suggested as an indicator of critical point. In case of first order phase transition, or crossover, some observables also show the non-monotonic behavior. The absence of non-monotonic behavior does not exclude the existence of critical point, such as the maximum cluster size in 3D-Ising model shown in Fig. 1(a).

Moreover, if the correlation length is comparable with system size, the finite-size effect is not negligible. When correlation length is larger than $L$, of system size, it has been shown that the finite-size effect has to be taken into account.

Although, it is still difficult to estimate the size of the formed system and correlation length at critical point in relativistic heavy ion collisions. A rough estimation shows that the system size at freeze-out is less than 12 fm. The correlation length is round 6 fm for typical nuclear collisions. After considering the finite evolution time, or finite-size, it is argued that the maximum of correlation length may not be beyond 2-3 fm at critical point. Base on those estimations, the ratio of correlation length to system size is round $\frac{1}{6}$, which is in the region larger than $\frac{1}{6}$. So in current relativistic heavy ion collisions at RHIC, the finite-size effect most probably has to be taken into account, rather than negligible.

In accounting for the finite-size effects, the critical behavior of all suggested observables, such as the fluctuations and correlations of transverse momentum, multiplicity, conserved charges, and in particular, the higher order moments of baryon numbers, should be re-examined under the frame of finite-size behavior of critical point and nearby boundary.

In the letter, we first discuss the finite-size behavior of the critical point, the first order phase transition, and the crossover in general. Secondly, we suggest the finite-size scaling form of the critical related observable in relativistic heavy ion collisions. Its fixed-point behavior at critical incident energy can be served as a reliable identification of critical point and nearby boundary of QCD phase transition. Thirdly, we demonstrate how to locate the fixed point from experimental observable by using 3D-Ising model as an example. Finally, the validity of the method at finite detector acceptance at RHIC is discussed.

For the second order phase transition, the critical behavior is well described by finite-size scaling. It was firstly proposed from phenomenological and renormalization-group theories, and was approved by the Monte Carlo results of finite systems in different universality classes. This scaling form not only describes...
the behavior of the observables at different system sizes, but also indicates the position of the critical point and the critical exponents in infinite system. Therefore, from the finite-size scaling of critical related observables, the position and critical exponents of critical point can be precisely extracted. This has been implemented in locating critical point of multi-fragmentation nuclear liquid-gas phase transition [21].

In contrast to the critical point, the finite-size behavior of first order phase transition has not been well understood in general [22]. But the finite-size scaling behavior of first order phase transition is shown to correspond to so-called discontinuity fixed points of the renormalization group transformations, which are characterized by eigenvalue exponents equal to the spacial dimension [23]. So the finite-size scaling form pertains, and the scaling exponents are the spatial dimension, in contrary to the critical exponents of critical point. The phenomenological theory of finite-size scaling at first-order phase transition is proposed by K. Binder and D.P. Landau and found to be in good agreement with Monte Carlo simulation results [24].

Different from the critical point and the first order phase transition, at the crossover region, there is no singularity in all kinds of observables. The observables are system size independent [2, 25]. But it should be noticed that this holds only when the system size is not too small. When the system size is very small and the finite correlation length is comparable with the system size, the observables will become larger and larger when the system size goes to smaller and smaller.

In heavy ion collisions, the critical related observables are generally considered to be the fluctuations of conserved charges, like baryon number, electric charge, and strangeness [16, 17, 26]. The incident energy and centrality dependence of some related observables are fully investigated in current heavy ion experiments [27].

Incident energy $\sqrt{s}$ is a controlling parameter, like temperature $T$, or external field $h$ in thermodynamic systems. The centrality, i.e., impact parameter, presents the overlapped area of two incident nuclei. It directly related to the size of the formed system, and randomly fluctuates from event to event.

So the finite-size scaling in nuclear collisions can be generalized as following. When the size of the formed matter $L$ is much larger than the microscopic length scale (which is less than 1fm) and incident energy is near the critical one $\sqrt{s}_c$, the critical related observable, e.g., $Q(\sqrt{s}, L)$ in general, can be written in a finite-size scaling form [18, 20],

$$Q(\sqrt{s}, L) = L^{\lambda/\nu} F_Q(\tau L^{1/\nu}).$$  \hspace{1cm} (1)

Where $\tau = (\sqrt{s} - \sqrt{s}_c)/\sqrt{s}_c$ is reduced incident energy. $\nu$ and $\lambda$ are the critical exponents of the correlation length $\xi_0 = \xi_0 \tau^{-\nu}$ and the observable, respectively. They characterize the universal class of the phase transition. Finite-size scaling indicates that the observable at different system sizes can be re-scaled to an identical scaling function $F_Q$ with scaled variable $\tau L^{1/\nu}$.

At critical energy, $\sqrt{s} = \sqrt{s}_c$, the scaled variable $(\tau L^{1/\nu} = 0)$ is independent of system size $L$, and the scaling function becomes a constant,

$$F_Q(0) = Q(\sqrt{s}_c, L) L^{-\lambda/\nu}. \hspace{1cm} \text{(2)}$$

It shows that the fluctuation of critical related observable is self-similar at different size scales. In the case, the energy dependence of the observable at various of system sizes will intersect to this point, i.e., fixed point. The energy of the fixed point indicates the critical incident energy. As an example, we show in Fig. 1(b) the fixed point behavior of the maximum cluster in 3-D Ising model, which is supposed to be the same university of the deconfinement [28]. We can see that the maximum cluster at different lattice sizes intersect exactly at the fixed point, i.e., critical point.

Reversely, if we can find the fixed point from incident energy dependence of properly scaled observable in heavy ion collisions, which are measured at different centralities, i.e., system sizes, it will indicate the existence of critical point.

In order to find the exponent of the scale, and incident energy of the fixed point, we can first present the incident energy dependence of critical related observable at different system sizes, similar to Fig. 1(a). Then multiply a size factor to the observable $Q(\sqrt{s}, L)$, i.e., $Q(\sqrt{s}, L) L^{-a}$, and change the parameter $a$ from $-\infty$ to $\infty$ to see if all size curves interact to a point for a certain value of $a_0$ at a certain incident energy, e.g., Fig. 1(b).

In experiment, the point liked behavior can be quantified by the width of all size points. At a given incident energy, the width is usually defined as the square root of

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig1}
\caption{(Color online) Upper panel: (a) and (b) are the temperature dependence of maximum cluster size and maximum cluster size scaled by $L^{\lambda-\beta/\nu}$, and (c) is the scaling function of maximum cluster size in 3-D Ising model at different lattice sizes $L$. Lower panel: the same measures as in corresponding upper panel, but in a sub-system containing fifty percent of the lattice sites.}
\end{figure}
χ² of all size points, i.e.,

\[
D(\sqrt{s}, a) = \sqrt{\chi^2(\sqrt{s}, L) \over N_L - 1}. \tag{3}
\]

\(N_L\) is the number of points, and \(\chi^2(\sqrt{s}, L) \over N_L - 1\) is error weighted variance of all size points,

\[
\chi^2(\sqrt{s}, L) = \sum_{i=1}^{N_L} \left[ Q(\sqrt{s}, L_i) L_i^{-a} - \langle Q(\sqrt{s}, L) L^{-a} \rangle \right]^2 / w_i^2. \tag{4}
\]

\(w_i = \delta [Q(\sqrt{s}, L_i) L_i^{-a}]\) are the experimental error of \(Q(\sqrt{s}, L_i)\) and system size \(L_i^{-a}\) contribute to. \(\langle Q(\sqrt{s}, L) L^{-a} \rangle\) is also error weighted mean,

\[
\langle Q(\sqrt{s}, L) L^{-a} \rangle = \sum_{i=1}^{N_L} Q(\sqrt{s}, L_i) L_i^{-a} / w_i^2 \over \sum_{i=1}^{N_L} 1 / w_i^2. \tag{5}
\]

For example, in fig. 1(b), this width at a given temperature is the distance between two violet arrows. Therefore, if at a given incident energy, the minimum of \(D(\sqrt{s}, a)\) is round 1 at \(a_0\), i.e., \(D_{\text{min}}(\sqrt{s}, a_0) \sim 1\), it can be recognized as an experimental point. While, if it keeps larger than 1, there is no point like behavior.

For QGP formed system [4], the following 3 cases should be expected. (1) \(D_{\text{min}}(\sqrt{s}, a_0)\) at a certain incident energy is round 1, and at nearby incident energies, it is always larger than 1, and corresponding \(a_0\) is not an integer, as green curve shown in the middle of Fig. 2. This may indicate the existence of the fixed point, i.e., \(\lambda/\nu = a_0\), cf. Fig. 1(b).

In the case, the critical behavior should be further confirmed by the scaling function,

\[
F_Q(\tau L^{1/\nu}) = L^{-a_0} Q(\sqrt{s}, L). \tag{6}
\]

Here the critical exponent of correlation length \(\nu\) is a fitting parameter. If the data at all incident energies and system sizes can be well fitted by the scaling function, the critical point and the critical exponents are finally determined, cf., fig. 1(c).

(2) \(D_{\text{min}}(\sqrt{s}, a_0)\) at a certain incident energy is round 1, and at nearby incident energies, it is always larger than 1, and corresponding \(a_0\) is an integer as green curve shown in the left side of Fig. 2. This indicates also the existence of the fixed point, but scaled power is trivial integer. It implies the region of the first order phase transition. The incident energy of the fixed point is the transition energy of the first order phase transition. The scaling function of the observable should be simply formulated by the spatial dimension, instead of the critical exponents in Eq. (1).

If \(a_0\) is zero, there are two possibilities. It could be the critical point with critical exponent \(\lambda = 0\), like Binder cumulant ratio [29], or the region of the first order phase transition. The final identification is their specified scaling functions, as discussed above.

(3) \(D_{\text{min}}(\sqrt{s}, a_0)\) is round 1 at all incident energies, as green line shown in the right side of Fig. 2, and corresponding \(a_0\) is an integer. This indicates all size curves are overlapped, and there is in fact no fixed point. It corresponds to the transition of crossover.

It should be stressed that the observables we mentioned here are the intensive variable, like susceptibility. If the observables are extensive variable, such as the fluctuation of the particle number, \(\langle (N - \overline{N})^2 \rangle = TV\), the trivial size dependence are included, and can be merged to the power \(a\).

The size of the formed matter in heavy collisions is mainly determined by overlapping area of two incident nuclei. This area is proportional to the number of participant nucleons and is quantified as centrality. The initial size of the formed matter can be approximately estimated by the square root of the number of participants, \(\sqrt{N_{\text{part}}}\). The maximum size is \(\sqrt{2N_A}\), \(N_A\) is the number of nucleons of incident nucleus. The ratio,

\[
L = \sqrt{N_{\text{part}}} \over \sqrt{2N_A}. \tag{7}
\]

presents the relative size of the initial system.

The system size \(L'\) at transition should be larger than initial one \(L\) and monotonically increase function of \(L\), i.e., \(L' = cL^{1+\delta}\) with \(\delta \geq 0\) in general. Whether we take \(L'\) or \(L\) in Eq. (1), the scaling exponents will be different, but the position of critical point will be the same. So the initial size is a good approximation in locating the position of critical point.

It should also be noticed that the detectors at current relativistic heavy ion experiments cover a part of the phase space, and only a part of final state particles is accepted. Even if the critical related information are survived in the final state observables, whether the finite-size behavior of detected subsystem is preserved has to be examined further.

The finite size behavior of a sub-systems is demonstrated in 3-D Ising model. The size of sub-system is chosen to be a certain percent of the whole lattice sites.
Changing the lattice of the whole system, the effective sites of the sub-system, $L_{\text{eff}}$, vary with it. We find that the finite size behavior of sub-system keeps valid as long as the size of sub-system is within the range of finite size scaling.

In the lower panel of Fig. 1, the finite size behavior of the maximum cluster size at various $L_{\text{eff}}$ is presented. Where the size of sub-system is 50% of the whole system. In comparison with the corresponding results of the whole one, but the position of fixed point indicates the same critical temperature, $T_c = 4.51 J$. Moreover, the maximum cluster at different sub-system sizes are well scaled to an identical scaling function. Therefore, the suggested finite size behavior should be visible at a detector with a relative large acceptance, like RHIC/STAR.

In the summary, we point out that the finite-size effects are not negligible in locating the critical point of QCD phase transition at current relativistic heavy ion collisions. At the crossover, critical point and first order QCD phase transition, the finite-size scaling behaviors of the critical related observable are suggested.

The critical point of QCD phase transition can be found by the appearance of the fixed point with a non-integer power in scaled size factor, and the finite-size scaling function of the observable. The region of the first order phase transition is identified by the fixed point with an integer power in scaled size factor and the scaling function which is determined by spatial dimension.

At the region of the crossover, the behavior of the fixed point is absent, and the scaling function reduces to the incident energy dependence of observable, which is system size independent.

At a given incident energy, the width of the observables at various centralities is suggested as a quantification of point like behavior. The energy dependence of the width at different orders of phase transitions are shown. When incident energy scans from high to low, the deviation of minimum width from point like behavior will indicate the appearance of the critical point.

Finally, for a finite acceptance detector, we demonstrate that the finite-size behavior of critical related observables keep valid as long as the detected subsystem is large enough.

The authors would dedicate the work to Prof. Liu Lianshou, who stimulated and appreciated the work when it was infant. The authors are grateful to the valuable suggestions and comments of Dr. V. Koch, Z. B. Xu, and N. Xu. This work is supported in part by the NSFC of China with project No. 10835005 and MOE of China with project No. IRT0624 and No. B08033.

[1] J. C. Collins, M. J. Perry, Phys. Rev. Lett. 34 (1975) 1353; B. A. Freedman, L. D. Lerran, Phys. Rev. D 16 (1977) 1196; E. V. Shuryak, Phys. Lett. B 107 (1981) 103.
[2] Y. Aoki, G. Endrodi, Z. Fodor, S. D. Katz, K.K. Szabo, Nature 443 (2006) 675; Y. Aoki, Z. Fodor, S.D. Katz, K.K. Szabo, Phys. Lett. B 643 (2006) 46.
[3] Z. Fodor and S. D. Katz, JHEP 04 (2004) 050; Z. Fodor, S.D. Katz, and K. K. Szabo, Phys. Lett. B 506 (2003) 73.
[4] M. Gyulassy, L. McLerran, Nucl. Phys. A 750 (2005) 30-63; B. Müller, Annu. Rev. Nucl. and Part. Phys., 1(2006); J. Adams, et al. (STAR Coll.), Nucl.Phys. A 757 (2005) 102-183.
[5] M. A. Stephanov, hep-ph/0402115, Int. J. Mod. Phys. A 20 (2005) 4387.
[6] M. A. Stephanov, Phys. Rev. Lett. 102 (2009) 032301.
[7] M. Asakawa, S. Ejiri, M. Kitazawa, Phys. Rev. Lett. 103 (2009) 262301.
[8] C. Weber, L. Caprriott, G. Misguich, F. Becca, M. Elhajal, and F. Mila, Phys. Rev. Lett. 91 (2003) 177202.
[9] P. Olsson, Phys. Rev. B 55 (1997) 3583.
[10] J. Adams, et. al. (STAR collaboration), Phys. Rev. Lett., 93 (2004) 012301; S. S. Adler, et. al. (PHENIX collaboration), Phys. Rev. Lett. 93 (2004) 152302; R. A. Soltz, J. Phys. G: Nucl. Part. Phys. 31 (2005) S325.
[11] M. A. Stephanov, K. Rajagopal, E. Shuryak, Phys. Rev. D 60 (1999) 114028.
[12] N. G. Antoniou, F. K. Diakonos, and A. S. Kapoyannis, Phys. Rev. Lett. 97 (2006) 032002.
[13] B. Berdnikov and K. Rajagopal, Phys. Rev. D 61 (2000) 105017.
[14] K. Paech, Eur. Phys. J. C 33 (2004) S627.
[15] K. Rajagopal, summary talk at 5th International Workshop on Critical Point and Onset of Deconfinement at BNL, in June 8-12, 2009.
[16] M. A. Stephanov, K. Rajagopal, and E. Shuyak, Phys. Rev. Lett. 81, 4816(1998); H. Heiselberg, Phys. Rept. 351 (2001) 161.
[17] M. A. Stephanov, Phys. Rev. Lett. 102 (2009) 032301; Masayuki Asakawa, Shinji Ejiri, Masakiyo Kitazawa, nucl-th 0904.2089.
[18] M. E. Fisher, in Critical Phenomena, Proceedings of the International School of Physics Enrico Fermi, Course 51, edited by M. S. Green (Academic, New York, 1971).
[19] E. Brézin, J. Phys. (Paris) 43 (1982) 15.
[20] X. S. Chen, V. Dohm, and A. L. Talapov, Physica A 232 (1996) 375; X. S. Chen, V. Dohm, and N. Schultka, Phys. Rev. Lett. 77 (1996) 3641.
[21] M. K. Berkenbusch, et al., Phys. Rev. Lett. 88 (2001) 022701; J. B. Elliott, et al., ibid., (2002) 042702.
[22] K. Binder, Rep. Prog. Phys. 50 (1987) 783.
[23] J. M. van Leeuwen, Phys. Rev. Lett. 34 (1975) 1056; B. Nienhuis and M. Nauenberg, ibid. 35 (1975) 477; M.E. Fisher and A. N. Berker, Phys. Rev. B 26 (1982) 2507.
[24] K. Binder and D. P. Landau, Phys. Rev. B30 (1984) 1477.
[25] A. Ukawa, Lecture on Lattice QCD at finite temperature (1993).
[26] S. Jeon and V. Koch, Phys. Rev. Lett. 85 (2000) 2076; V. Koch, arXiv: 0810.2520.
[27] J. Adams, et. al. (STAR collaboration), Phys. Rev. C 72 (2005) 044902; B. I. Abelev, et. al. (STAR collaboration), hep-ph/0904.2089.
Phys. Rev. C 79 (2009) 024906.

[28] J. García, J. A. Gonzalo, Physica A 326 (2003) 464.

[29] K. Binder, Z. Phys. B 43 (1981) 119.