Quark-Gluon Coupling in the Global Colour Model of QCD

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November 1997

Abstract

The Global Colour Model of QCD is used in conjunction with a pure-gluon lattice correlator (by Marenzoni et al.) to extract from meson data a momentum-dependent quark-gluon coupling down to $s \approx 0.3 GeV^2$. This is compared with a lattice calculation (by Skullerud) of the quark-gluon coupling.

Keywords: Quantum Chromodynamics, Global Colour Model, Quark-Gluon Coupling

PACS numbers: 12.38.Lg, 13.75.Cs, 11.10.St, 12.38.Aw

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A key feature of Quantum Chromodynamics (QCD) is that the quark-gluon coupling varies strongly with gluon momentum $q$ over the range $0 < q < 2\text{GeV}$ relevant to low energy hadronic physics. Here we extract from meson data this quark-gluon coupling $g(q^2)$ down to $q = 0.5\text{GeV}$, using the Global Colour Model (GCM) of QCD in conjunction with a pure-gluon lattice correlator by Marenzoni et al. [1]. The extracted quark-gluon coupling (see Fig.2.) is compared with a recent lattice calculation by Skullerud [2].

The GCM modelling of QCD is based on the idea that because the hadronic correlators are related by specific functional identities (Non-scripted $G$’s will denote constituent correlators, as defined later). However the GCM does not derive from these equations/identities, its nature follows instead from an analytical continuum estimation procedure for the functional integrations. Direct numerical estimation procedures are used in lattice modelings of the functional integrals.

The correlators in (1) may be extracted from the generating functional of QCD

$$Z_{QCD}[ar{\eta}, \eta, J] = \int D\pi Dq DA D\bar{C} D\bar{C} \exp(-S_{QCD}[A, \bar{\eta}, q, \bar{C}, C] + \bar{\eta} q + \bar{C} + JA).$$

(2)

The functional transformations which lead to the GCM are discussed in Tandy [3]; briefly and not showing source terms for convenience, the gluon and ghost integrations are formally performed

$$\int D\pi Dq DA D\bar{C} D\bar{C} \exp(-S_{QCD}[A, \bar{\eta}, q, \bar{C}, C])$$

$$= \int D\pi Dq \exp(- \int \bar{\pi}(-\gamma \partial + M)q +$$

$$+ \frac{g_0^2}{2} \int j^a_\mu(x) j^a_\nu(y) G_{\mu\nu}(x - y) + \frac{g_0^3}{3!} \int j^a_\mu j^b_\rho j^c_\sigma G^{abc}_{\mu\rho\sigma} + ......)$$

(3)

where $j^a_\mu(x) = \bar{q}(x) i \gamma_\mu q(x)$, $g_0$ is the bare coupling constant, and $G_{\mu\nu}(x)$ is the gluon correlator with no quark loops but including ghosts

$$G_{\mu\nu}(x - y) = \frac{\int DAD\bar{C} D\bar{C} A^a_\mu(x) A^a_\nu(y) \exp(-S_{QCD}[A, \bar{C}, C])}{\int DAD\bar{C} D\bar{C} \exp(-S_{QCD}[A, \bar{C}, C])}.$$  

(4)

A variety of techniques for computing $G_{\mu\nu}(x)$ exist: the gluon-ghost DSE [12], and the gluon only DSE [13] and lattice simulations [14]. The terms of higher order than the term quartic in the quark fields are difficult to explicitly retain in any analysis. However we can model, in part, the effect of these higher order terms by replacing the coupling constant $g_0$ by a momentum dependent quark-gluon coupling $g(s)$, and neglecting terms like $G^{abc}_{\mu\rho\sigma}$ and higher order. This $g(s)$ is a restricted form of vertex function. This modification $g_0^2 G_{\mu\nu}(p) \rightarrow D_{\mu\nu}(p) = g(p^2) G_{\mu\nu}(p)$
and truncation in (3) then defines the GCM. However we make one further modification: we shall use lattice results with ghosts neglected for $G_{\mu\nu}(x)$ [1]. Then $g(s)$ models as well the effect of the ghosts in both the gluon correlator and the quark-gluon vertex. See [12] for an analysis of these ghost effects. We call $D_{\mu\nu}(p)$ the effective gluon correlator.

The GCM is equivalent to using a quark-gluon field theory with the action

$$S_{GCM}[A, \bar{q}, q] = \int \left( \bar{q}(-\gamma \cdot \partial + \mathcal{M} + i A_\mu^a \frac{\lambda^a}{2} \gamma_\mu)q + \frac{1}{2} A_\mu^a D_{\mu\nu}^{-1}(i\partial) A_\nu^a \right). \tag{5}$$

Here $D_{\mu\nu}^{-1}(p)$ is the matrix inverse of $D_{\mu\nu}(p)$, which in turn is the Fourier transform of $D_{\mu\nu}(x)$. This action is invariant under $q \to Uq, \bar{q} \to \bar{q}U^\dagger$, and $A_\mu^a \lambda^a \to UA_\mu^a \lambda^a U^\dagger$ (where $U$ is a global $3 \times 3$ unitary colour matrix) - the global colour symmetry of the GCM. The gluon self-interactions that arise as a consequence of the local colour symmetry in (4) and the ghost and vertex effects lead to $D_{\mu\nu}^{-1}(p)$ being non-quadratic. Hence, in effect, the GCM models the QCD local gluonic action $\int F_{\mu\nu,a}[A] F_{\mu\nu,a}[A]$, having local colour symmetry, in $S_{QCD}$ of (1), by a highly nonlocal action, having global colour symmetry, in the last term of (5). The success of this modelling has been amply demonstrated [3]. The form for $g(p^2)$ is here determined by comparing the meson data determined $D_{\mu\nu}(p)$ to the pure-gluon lattice-determined correlator $G_{\mu\nu}(x)$.

Hadronisation of the functional integrations in (1) involves a sequence of changes of variables involving, in part, the transformation to bilocal boson fields, and then to the usual local hadron fields (sources not shown):

$$Z \approx \int Dq D\bar{q} DA \exp(-S_{GCM}[A, \bar{q}, q] + \bar{q}q + q\bar{q}) \quad \text{(GCM)} \tag{6}$$

$$= \int DB \ldots \exp(-S[B, \ldots]) \quad \text{(bilocal fields)}$$

$$= \int D\pi D\rho D\omega \ldots \exp(-S_{had}[\pi, \rho, \omega, \ldots]) \quad \text{(local fields)} \tag{7}$$

The bilocal fields in (6) naturally arise and correspond to the fact that, for instance, mesons are extended states. This bosonisation/hadronisation arises by functional integral calculus changes of variables that are induced by generalized Fierz transformations that emerge from the colour, spin and flavour structure of QCD [13]. The final functional integrations in (7) over the hadrons give the hadronic observables, and amounts to dressing each hadron by, mainly, lighter mesons. The basic insight is that the quark-gluon dynamics, in (5), is fluctuation dominated, whereas the hadronic functional integrations in (7) are not.

The second key idea in the GCM is that in proceeding from (1) to (7) one expands $S[B, \ldots]$ about the configuration $B_{CQ}$ that minimises it; giving the GCM Constituent Quark (CQ) equations.

$$\frac{\delta S}{\delta B(x, y)}|_{B_{CQ}} = 0. \tag{8}$$

Thus for all hadrons one assumes a universal dominant configuration. This amounts to assuming that all hadrons share a common dynamical feature. Of the set $B(x, y)_{CQ}$ only $A(x - y)$ and $B(x - y)$ are non-zero translation-invariant bilocal fields characterising the dominant configuration. Then writing out the translation invariant CQ equations we find that the dominant
configuration is indeed simply the constituent quark effect as they may be written in the form 3.

\[ G^{-1}(p) = i\gamma + m + \frac{4}{3} \int \frac{d^4q}{(2\pi)^4} D_{\mu\nu}(p-q)\gamma_\mu G(q)\gamma_\nu, \]  

(9)

and we see that this is the gluon dressing of a constituent quark; and is exact in the GCM. Here

\[ G(q) = (iA(q)q\gamma + B(q) + m)^{-1} = -iq_\gamma\sigma_\nu(q) + \sigma_s(q). \]  

(10)

In the chiral limit there are more \( B_{CQ} \) fields that are non-zero, and a resultant degeneracy of the dominant configuration is responsible for the masslessness of the pion 3.

The constituent quark \( G \) correlator should not be confused with the complete quark correlator \( G \) from 3. This \( G \) would be needed to analyse the existence or otherwise of free quarks. The \( G \) on the other hand relates exclusively to the internal structure of hadrons, and to the fact that this appears to be dominated by the constituent quark effect. The evaluation of \( G \) is a very difficult task, even in the GCM. \( G \) is however reasonably easy to study using 3.

The hadronic effective action in 3 arises when \( S[B,..] \) is expanded about the dominant CQ configuration: the 1st derivative is zero by 3, and the 2nd derivatives, or curvatures, give the constituent or core meson correlators \( G(q, p; P) \)

\[ G^{-1}(q, p; P) = F.T. \left( \frac{\delta^2 S}{\delta B(x, y)\delta B(u, v)} \right) |_{B_{CQ}} \]  

(11)

after exploiting the translation invariance and Fourier transforming. Higher order derivatives lead to couplings between the meson cores. The \( G(q, p; P) \) are given by ladder-type correlator equations, see 3. The non-ladder effects can be inserted by the final functional integrals in 3, giving the complete GCM meson correlators \( G(q, p; P) \). In the present analysis the \( \omega \) and \( a_1 \) mesons are described by these constituent meson correlators; that is, we ignore meson dressings of these mesons. The mass \( M \) of these states is determined by finding the pole position of \( G(q, p; P) \) in the meson momentum \( P^2 = -M^2 \), this leads to the homogeneous vertex equation

\[ \Gamma(p; P) = -\frac{4}{3} \int \frac{d^4q}{(2\pi)^4} D_{\mu\nu}(q-p)\gamma_\mu G(q + \frac{P}{2})\Gamma(q; P)G(q - \frac{P}{2})\gamma_\nu. \]  

(12)

To solve 3 for various \( D_{\mu\nu}(p) \) and then to proceed to use \( A(s) \) and \( B(s) \) in meson correlator equations for fitting observables to meson data is particularly difficult. A robust numerical technique is to use a separable expansion for \( D_{\mu\nu}(p) \) 16-17. In Landau gauge

\[ D_{\mu\nu}(p) = (\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2})D(p^2), \quad \text{and} \quad G_{\mu\nu}(p) = (\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2})D(p^2). \]  

(13)

We expand \( D(p - q) \) in 3 into \( O(4) \) hyperspherical harmonics

\[ D(p - q) = D_0(p^2, q^2) + q_\mu p_\nu D_1(p^2, q^2) + ... \]  

(14)

where

\[ D_0(p^2, q^2) = \frac{2}{\pi} \int_0^\pi d\beta \sin^2\beta D(p^2 + q^2 - 2pq\cos\beta), ... \]  

(15)

We then introduce multi-rank separable expansions for each term

\[ D_0(p^2, q^2) = \sum_{i=1, n} \Gamma_i(p^2)\Gamma_i(q^2), ... \]  

(16)
Introduction of the separable expansion clearly breaks translational invariance and must be regarded purely as a numerical procedure, much like a lattice breaks translation invariance. Translation invariance is restored as the rank of the separability is increased. Here we use a rank $n = 3$ form for $D_0$, and rank 1 form for $D_1$. The constituent quark equations then have solutions of the form

$$B(s) = \sum B_i(s); \quad B_i(s) = b_i \Gamma_i(s),...$$

(17)

where the $b_i,...$ are easily determined, in the chiral limit, to be

$$b_i^2 = \frac{16}{3} \pi^2 \int_0^{\infty} s ds B_i(s) \sigma_s(s).$$

(18)

where

$$B_i(s) = \frac{\sigma_s(s)_i}{s \sigma_v(s)^2 + \sigma_s(s)^2}$$

(19)

and $\sigma_s$ and $\sigma_v$ are seen to have the form of sums

$$\sigma_s(s) = \sum_{i=1,n} \sigma_s(s)_i, \quad \sigma_v(s) = \sum_{i=1,k} \sigma_v(s)_i,$$

(20)

However rather than specifying $\Gamma_i$ in (14) we proceed by parametrising forms for the $\sigma_s$ and $\sigma_v$; then the $\Gamma_i$ follow from (17) and (19):

$$\sigma_s(s)_i = c_i \exp(-d_i s), \quad i = 1, 2; \quad \sigma_s(s)_3 = c_3 \left( \frac{2s - d_3(1 - \exp(-2s/d_3))}{2s^2} \right)^2;$$

$$\sigma_v(s) = \frac{2s - \beta^2(1 - \exp(-2s/\beta^2))}{2s^2}.$$  \hspace{1cm} (21)

As these forms are entire functions we avoid spurious singularities developing in $G$. The asymptotic form of $\sigma_s(s) \sim 1/s^2$ for $s \to \infty$ is described by the $\sigma_s(s)_3$ term. With these parametrised forms we can numerically relate the mass of the $a_1$ and $\omega$ mesons, from (18) and $f_\pi$ (for $N_f = 2$) to the chiral-limit parameter set $\{c_1, c_2, c_3, d_1, d_2, d_3, \beta\}$ in a robust and stable manner. The parameter values are shown in Table 1. The chiral limit expression for $f_\pi$ is, see 3,

$$f_\pi = 6 \int \frac{d^4q}{(2\pi)^4} \left( \sigma_v^2 - 2(\sigma_s \sigma_v' + s \sigma_v \sigma_v') - s(\sigma_s \sigma_v'' - (\sigma_v')^2) - s^2(\sigma_v \sigma_v'' - (\sigma_v')^2) \right) B(q)^2.$$ \hspace{1cm} (22)

The translation invariant form for the effective gluon correlator is easily reconstructed by using $D(p^2) = D_0(p^2, 0)$ from (15)

$$D(p^2) = \sum_i \frac{1}{b_i^2 \sigma_s(0)_i^2} \frac{\sigma_s(p^2)_i}{p^2 \sigma_v(p^2)_i^2 + \sigma_s(p^2)_i^2}.$$ \hspace{1cm} (23)

With the parameter set in Table 1, (18) gives $b_1 = 0.0210 \text{ GeV}^2$, $b_2 = 0.0251 \text{ GeV}^2$ and $b_3 = 0.0351 \text{ GeV}^2$ and the resulting $D(p^2)$ is shown in Fig.1; it has estimated uncertainties of 5%. Shown in Fig.1 for the pure gluon correlator is $D(p^2)$ from the lattice calculations, corresponding to the value $\beta = 6.0$, of Marenzoni et al 1 where the errors arise from a 5% uncertainty in the lattice spacing; $a = 0.50 \pm 0.025 \text{GeV}^{-1}$. In Fig.2 we show the form of $g(s)$, where $g^2(s) = D(s)/D(0)$, see (13), that then follows from our analysis. Here the error bars now indicate combined uncertainties. This extracted quark-gluon coupling extends down to $0.3 \text{GeV}^2$, and shows infrared (IR) enhancement. Below this limit the separable expansion becomes unreliable unless more terms and more fitting data are used. We have not corrected
for either lattice spacing dependence or for quark loops; corrections for these would require further development. It is possible to identify where the IR effect arises. If we artificially lessen this effect at small $s$ then we find that the main consequence is an increase in the value of $f_\pi$. Indirectly, then, we can show that the IR signature is the (inverse) pion size in comparison with the $a_1$ and $\omega$ masses. The pion size enters through $f_\pi$ because in (22) in the chiral limit the pion form factor $\Gamma_\pi(q;0) = B(q)$, see Tandy [3]. However the GCM extraction of this effect does not explain what aspect of QCD drives it. In [12] it is argued that the QCD origin of this IR enhancement is due to the ghost correlator presence in the quark-gluon vertices. We also report various condensate values that arise from the present work: $<\bar{q}q> = (211.4\,\text{MeV})^3 |_{1\text{GeV}}$, $<g\overline{q}F_{\mu\nu}\sigma^{\mu\nu}q> = (491.5\,\text{MeV})^5 |_{1\text{GeV}}$, and $<\frac{\alpha_s}{\pi}F_{\mu\nu}F_{\mu\nu}> = 0.026\,\text{GeV}^4 |_{1\text{GeV}}$, ignoring quark-loop contributions.

Our most significant result follows from comparing, in Fig.2, the GCM-meson-data/lattice-gluon determined quark-gluon coupling with that determined recently by Skullerud [2] using a lattice calculation with $\beta = 6.0(\alpha = 0.5\text{GeV}^{-1})$ and a lattice size of $16^3 \times 48$. For comparison we also show the perturbative coupling derived from the two-loop beta function

$$g^2(s) = \left( b_0 \ln\left(\frac{s}{\Lambda^2}\right) + \frac{b_1}{b_0} \ln \ln\left(\frac{s}{\Lambda^2}\right) \right)^{-1},$$

with $b_0 = 11/16\pi^2, b_1 = 102/(16\pi^2)^2$ for $\Lambda = 0.420\text{GeV}$. Fig.2 indicates a general agreement of all three methods down to $s \approx 0.7\text{GeV}^2$. The most significant difference being the decreasing lattice $g(s)$ in the deep IR; however this could be due to the finite lattice size which induces an IR cutoff, or to the absence of the ghost effects [12]. The Skullerud data is similar to the running coupling extracted from the 3-gluon vertex [18]. In Fig.3 we show $\alpha_s = g^2/(4\pi)$ against $q(GeV)$. These results indicate that QCD may now be sufficiently well modelled by the GCM in the low energy regime that detailed hadronic calculations may be performed, particularly for the nucleon properties; the GCM having the advantage of easily dealing with the near chiral limit needed for the nucleon, in contrast to lattice studies. Fig.2 shows that the lattice results for the gluon correlator and the quark-gluon coupling may be combined to form $D(s)_{\text{lat}} = g^2(s)_{\text{lat}} D(s)_{\text{lat}}$; a lattice derived effective gluon correlator for $[5]$, except for the deep IR where we should be guided by the meson data fitting. We thus have a meeting of the continuum and lattice approaches. The deep IR behaviour remains undetermined, but the region of uncertainty mainly affects questions of absolute confinement and will have little effect upon low energy hadronic phenomena.

We thank N. Stella for assistance with the lattice results in [1]. Research supported by an ARC Grant from Flinders University. This work is part of the activities of the Special Research Centre for the Subatomic Structure of Matter, University of Adelaide.
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Table 1: $\sigma_s(s)$ and $\sigma_v(s)$ Parameters

| $c_1$ | 0.1732GeV$^{-1}$               | $d_1$ | 1.389GeV$^{-2}$               |
|------|-------------------------------|------|-------------------------------|
| $c_2$ | 1.527GeV$^{-1}$               | $d_2$ | 4.982GeV$^{-2}$               |
| $c_3$ | 0.3435GeV$^3$                 | $d_3$ | 1.971GeV$^2$                  |
| $\beta$ | 0.4807GeV                  |      |                               |
Figure Captions

**Figure 1** The effective gluon correlator $D(s)$ (solid line) extracted by fitting the GCM to meson data. Also shown are the lattice results for the pure gluon correlator $\mathcal{D}(s)$ from Marenzoni et al. (1995). The error bars indicate uncertainties arising from the value of the lattice spacing $a = 0.50 \pm 0.025 \text{GeV}^{-1}$.

**Figure 2** The GCM quark-gluon coupling $g(s)$ (boxes). Here $g^2(s)$ was obtained by dividing the GCM effective gluon correlator $D(s)$ (solid line in Fig.1) by the lattice gluon correlator $\mathcal{D}(s)$. The error bars arise from the lattice spacing uncertainty and from systematic errors in the fitting of the GCM to the meson data. Also shown is $g(s)$ from the lattice calculation of Skullerud (circles) (1997). The curve shows the two-loop form for $\Lambda = 0.420 \text{GeV}$.

**Figure 3** Here we replot, for convenience, the GCM quark-gluon coupling in the form $\alpha_s = g^2/(4\pi)$ against $q(\text{GeV})$. 
