THE SHARP ESTIMATES OF EIGENVALUES OF POLYHARMONIC OPERATOR AND HIGHER ORDER STOKES OPERATOR

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Abstract. In this paper, we establish some lower bounds for the sums of eigenvalues of the polyharmonic operator and higher order Stokes operator, which are sharper than the recent results in [6, 11]. At the same time, we obtain some certain bounds for the sums of positive and negative powers of eigenvalues of the polyharmonic operator.

1. Introduction

Let $\Omega$ be a bounded domain in an $n$-dimensional Euclidean space $\mathbb{R}^n$ ($n \geq 2$). The Dirichlet eigenvalue problem of the polyharmonic operator is described by

\begin{equation}
\begin{cases}
(-\Delta)^l u = \lambda u, & \text{on } \Omega, \\
u|_{\partial\Omega} = \frac{\partial u}{\partial \nu}|_{\partial\Omega} = \cdots = \frac{\partial^{l-1} u}{\partial \nu^{l-1}}|_{\partial\Omega} = 0,
\end{cases}
\end{equation}

where $\Delta$ is the Laplacian and $\nu$ denotes the outward unit normal vector field of $\partial\Omega$. As we known, this problem has a real and discrete spectrum:

$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \rightarrow \infty,$

where each eigenvalue repeats with its multiplicity.

When $l = 1$, problem (1.1) is called the Dirichlet Laplacian problem or the fixed membrane problem. The asymptotic behavior of its $k$-th eigenvalue $\lambda_k$ relates to geometric properties of $\Omega$ when $k \rightarrow \infty$. In fact, the following Weyl’s asymptotic formula asserts that

\begin{equation}
\lambda_k \sim C_n \left( \frac{k}{|\Omega|} \right)^{\frac{2}{n}}, \quad \text{as } k \rightarrow \infty,
\end{equation}

where $C_n = 4\pi \Gamma\left(1 + \frac{n}{2}\right)^{2}$ and $|\Omega|$ denote the volume of $\Omega$. Here $\Gamma(m)$ denotes the Gamma function $\Gamma(m) = \int_0^\infty t^{m-1} e^{-t} dt$ for $m > 0$. In 1961, Pólya proved in [22] that

\begin{equation}
\lambda_k \geq C_n \left( \frac{k}{|\Omega|} \right)^{\frac{2}{n}},
\end{equation}

for tiling domain in $\mathbb{R}^n$. Moreover, he conjectured that (1.3) holds for any bounded domain in $\mathbb{R}^n$. There have been some results in this direction. In 1980, Lieb [18] proved

$$
\lambda_k \geq \tilde{C}_n \left( \frac{k}{|\Omega|} \right)^{\frac{2}{n}},
$$

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where $\tilde{C}_n$ differs from $C_n$ in (1.3) by a factor. In 1983, Li and Yau [17] proved

$$(1.4) \quad \sum_{j=1}^{k} \lambda_j \geq 4\pi \frac{n}{n+2} \left( \frac{\Gamma(1 + \frac{n}{2})}{|\Omega|} \right)^{\frac{2}{n}} k^{1 + \frac{2}{n}}.$$ 

It has been pointed out in [15] that by using the Legendre transform, (1.4) is equivalent to the inequality derived by Berezin [4]. Hence, (1.4) is also called the Berezin-Li-Yau inequality. Using the similar approach, Kröger [12] has obtained the sharp upper bound for the Neumann eigenvalues. Improvements to the Berezin–Li–Yau inequality in (1.4) for the case of Dirichlet Laplacian have appeared recently (for example, see [13, 19, 27]). In particular, Melas [19] improved (1.4) to

$$(1.5) \quad \sum_{j=1}^{k} \lambda_j \geq 4\pi \frac{n}{n+2} \left( \frac{\Gamma(1 + \frac{n}{2})}{|\Omega|} \right)^{\frac{2}{n}} k^{1 + \frac{2}{n}} + \frac{1}{24(n+2)I(\Omega)} k,$$

where $I(\Omega) = \min_{a \in \mathbb{R}^n} \int_{\Omega} |x-a|^2 \, dx$ is the moment of inertia of $\Omega$ and $a$ is a constant vector in $\mathbb{R}^n$. In 2010, Ilyin [10] obtained the following asymptotic lower bound for eigenvalues of this problem:

$$(1.6) \quad \sum_{j=1}^{k} \lambda_j \geq 4\pi \frac{n}{n+2} \left( \frac{\Gamma(1 + \frac{n}{2})}{|\Omega|} \right)^{\frac{2}{n}} k^{1 + \frac{2}{n}} + \frac{n}{48 I(\Omega)} \left( 1 - \varepsilon_n(k) \right),$$

where $0 \leq \varepsilon_n(k) = O(k^{-\frac{1}{2}})$ is a infinitesimal of $k^{-\frac{1}{2}}$. In 2013, Y. Yolcu and T. Yolcu [28] proved that

$$(1.7) \quad \sum_{j=1}^{k} \lambda_j \geq 4\pi \frac{n}{n+2} \left( \frac{\Gamma(1 + \frac{n}{2})}{|\Omega|} \right)^{\frac{2}{n}} k^{1 + \frac{2}{n}} + \frac{2\sqrt{n}}{n+2} \left( \frac{|\Omega|}{I(\Omega)} \right)^{\frac{1}{2}} \left( \frac{\Gamma(1 + \frac{n}{2})}{|\Omega|} \right)^{\frac{1}{2}} k^{1 + \frac{2}{n}} - \frac{5}{8(n+2)I(\Omega)} k + \frac{1}{16 \sqrt{n+2}} \left( \frac{|\Omega|}{I(\Omega)} \right)^{\frac{1}{2}} \left( \frac{\Gamma(1 + \frac{n}{2})}{|\Omega|} \right)^{\frac{1}{2}} k^{1 - \frac{1}{n}}.$$ 

When $l = 2$, problem (1.1) is called the clamped plate problem. For the developments of eigenvalues of the clamped plate problem, we refer the readers to [1, 5, 7, 10, 11, 16, 21, 29].

For any order $l$, Levine and Protter [16] proved

$$(1.8) \quad \sum_{j=1}^{k} \lambda_j \geq (4\pi)^l \frac{n}{n+2l} \left( \frac{\Gamma(1 + \frac{n}{2})}{|\Omega|} \right)^{\frac{2l}{n}} k^{1 + \frac{2l}{n}}.$$ 

Recently, Cheng, Sun, Wei and Zeng [6] proved (see also Theorem 3 in [11])

$$(1.9) \quad \sum_{j=1}^{k} \lambda_j \geq (4\pi)^l \frac{n}{n+2l} \left( \frac{\Gamma(1 + \frac{n}{2})}{|\Omega|} \right)^{\frac{2l}{n}} k^{1 + \frac{2l}{n}} + (4\pi)^{l-1} \frac{nl}{48} \left( \frac{\Gamma(1 + \frac{n}{2})}{|\Omega|} \right)^{\frac{2l-2}{n}} \frac{|\Omega|}{I(\Omega)} \left( \frac{|\Omega|}{I(\Omega)} \right)^{\frac{2l-2}{n}} \left( 1 - \varepsilon_n(k) \right),$$

where $0 \leq \varepsilon_n(k) = O(k^{-\frac{2l+1}{2n}})$ is a infinitesimal of $k^{-\frac{2l+1}{2n}}$.

In this article, we obtain the following estimates for the sum of eigenvalues of problem 1.1.
**Theorem 1.1.** For any bounded domain $\Omega \subset \mathbb{R}^n (n \geq 2), k \geq 1$ and $1 \leq l < \frac{n+1}{2}$, the eigenvalues of (1.1) satisfy

\begin{equation}
\sum_{j=1}^{k} \lambda_j \geq \frac{n(4\pi)^{1/l}}{n + 2l} \left( \frac{(4\pi)^{1/l}}{\left| \Omega \right|} \right)^{\frac{2n}{n+2l}} k^{1+\frac{2n}{n+2l}} + \frac{(4\pi)^{2/l - 1}}{8(2n + 2l)I(\Omega)^{2/l}} \left| \Omega \right|^{\frac{2l - 2}{l}} \left( \frac{1}{2} \right)^{\frac{2l - 2}{l}} k^{1+\frac{2l - 2}{l}}
\end{equation}
(1.10)

Remark 1.2. Theorem (1.1) is a generalization of (1.7) for problem (1.1). The inequality (1.10) gives an improvement for (1.9).

Motivated by the work of [11, 26, 29], we obtain the following estimates of the negative and positive power of eigenvalues of problem 1.1:

**Theorem 1.3.** For $0 < q \leq 1$, $l \in \mathbb{N}$ and $2 \leq n$, the sums of positive powers of eigenvalues of the polyharmonic Laplacian problem (1.1) on $\Omega$ satisfy

\begin{equation}
\sum_{j=1}^{k} \lambda_j^q \geq (4\pi)^{q/l} \frac{n}{n + 2lq} \left( \frac{\Gamma(1 + n/2)}{\left| \Omega \right|} \right)^{\frac{2n}{n + 2lq}} k^{1+\frac{2n}{n + 2lq}}
\end{equation}
(1.11)

\begin{equation}
+ (4\pi)^{q - 1} \frac{n l q}{48 l q} \frac{\left| \Omega \right|}{I(\Omega)} \left( \frac{\Gamma(1 + n/2)}{\left| \Omega \right|} \right)^{\frac{2n - 2}{n + 2lq}} k^{1+\frac{2n - 2}{n + 2lq}} + O(k^{1+\frac{2n - 4}{n}}).
\end{equation}

**Corollary 1.4.** For $0 < q \leq 1$, $l \in \mathbb{N}$ and $2 \leq n$, the sums of positive powers of eigenvalues of the polyharmonic Laplacian problem (1.1) on $\Omega$ satisfy

\begin{equation}
\sum_{j=1}^{k} \lambda_j^q \geq \frac{n}{n + 2lq} (4\pi)^{q/l} \left( \frac{\Gamma(1 + n/2)}{\left| \Omega \right|} \right)^{\frac{2n}{n + 2lq}} k^{1+\frac{2n}{n + 2lq}}.
\end{equation}
(1.12)

**Theorem 1.5.** For $l \in \mathbb{N}$, $2 \leq n$ and $0 < p < \frac{q}{2n}$, the sums of negative powers of eigenvalues of the polyharmonic Laplacian problem (1.1) on $\Omega$ satisfy

\begin{equation}
\sum_{j=1}^{k} \lambda_j^{-p} \leq (4\pi)^{p/l} \frac{n}{n - 2lp} \left( \frac{\left| \Omega \right|}{\Gamma(1 + n/2)} \right)^{\frac{2n}{n - 2lp}} k^{1-\frac{2n}{n - 2lp}}.
\end{equation}
(1.13)

Remark 1.6. The coefficients of $k^{1+\frac{2n}{n + 2lq}}$ (rep. $k^{1-\frac{2n - 2}{n - 2lp}}$) appeared in the inequalities (1.11) and (1.12)(rep. (1.13)) are the best possible from the Weyl’s asymptotic formula. Moreover, Corollary 1.4 and Theorem 1.5 generalize the eigenvalue estimates of the Dirichlet Laplacian problem ($l = 1$ in (1.1)) in [26] and the clamped plate problem ($l = 2$ in (1.1)) [29] respectively.
Another work of this paper is to consider the following eigenvalue problem defined by

\[
\begin{cases}
(-\Delta)^l u_k + \nabla p_k = \mu_k u_k, & \text{in } \Omega, \\
\text{div } u_k = 0, & \text{in } \Omega, \\
u_k|_{\partial \Omega} = \cdots = \frac{\partial^{l-1} u_k}{\partial n^{l-1}}|_{\partial \Omega} = 0,
\end{cases}
\]

(1.14)

where \( l \in \mathbb{N} \) and \( \Omega \) is a bounded domain with smooth boundary in \( \mathbb{R}^n \).

For \( l = 1 \), (1.14) is the eigenvalue problem of the classical Stokes operator. Li-Yau type lower bounds for the eigenvalues of the classical Stokes operator were obtained in [9]:

\[
\sum_{k=1}^{\infty} \mu_k \geq 4\pi \frac{n}{n+2} \left( \frac{\Gamma(1 + \frac{n}{2})}{(n-1)|\Omega|} \right)^{2/n} k^{1+2/n}.
\]

(1.15)

The coefficient of \( k^{1+2/n} \) in (1.15) is sharp in view of the asymptotic formula (cf. [2] when \( n = 3 \) and [20] when \( n \geq 2 \) )

\[
\mu_k \sim \left( \frac{(2\pi)^n}{\omega_n (n-1)|\Omega|} \right)^{2/n} k^{2/n}, \quad \text{as } k \to \infty.
\]

(1.16)

Based on Melas’s approach in [19], Ilyin [10] proved

\[
\sum_{k=1}^{\infty} \mu_k \geq 4\pi \frac{n}{n+2} \left( \frac{\Gamma(1 + \frac{n}{2})}{(n-1)|\Omega|} \right)^{2/n} k^{1+2/n} + \frac{(n-1)|\Omega|}{48 I(\Omega)} k \left( 1 - \epsilon_n(k) \right),
\]

(1.17)

where \( 0 \leq \epsilon_n(k) = O(k^{-2/n}) \). In 2012, Y. Yolcu and T. Yolcu [28] obtained

\[
\sum_{k=1}^{\infty} \mu_k \geq 4\pi \frac{n}{n+2} \left( \frac{\Gamma(1 + \frac{n}{2})}{(n-1)|\Omega|} \right)^{2/n} k^{1+2/n} + \frac{2\sqrt{\pi}}{n+2} \left( \frac{(n-1)|\Omega|}{n I(\Omega)} \right)^{1/2} \left( \frac{(1 + \frac{n}{2})}{(n+1)|\Omega|} \right)^{1/2} k^{1+\frac{2}{n}}
\]

(1.18)

- \frac{5(n-1)}{8n(n+2) I(\Omega)} k + \frac{1}{16 \sqrt{\pi}} \left( \frac{n-1}{n} \frac{|\Omega|}{I(\Omega)} \right)^{1/2} \left( \frac{(1 + \frac{n}{2})}{(n+1)|\Omega|} \right)^{1/2} k^{1-\frac{2}{n}}.

For the eigenvalue problem (1.14), Ilyin [11] proved

\[
\sum_{k=1}^{\infty} \mu_k \geq (4\pi)^l \frac{n}{n+2} \left( \frac{\Gamma(1 + \frac{n}{2})}{(n-1)|\Omega|} \right)^{2/n} k^{1+\frac{2}{n}}
\]

(1.19)

\[
+ (4\pi)^{l-1} \frac{l}{48 \frac{(n-1)|\Omega|}{I(\Omega)}} \left( \frac{(1 + \frac{n}{2})}{(n+1)|\Omega|} \right)^{\frac{2l-2}{n}} k^{1+\frac{2}{n}} \left( 1 + \epsilon_n(k) \right),
\]

where \( 0 \leq \epsilon_n(k) = O(k^{-\frac{2}{n}}) \) is a infinitesimal of \( k^{-\frac{2}{n}} \).
Lemma 2.1. The function $F$ defined by (2.1) satisfies

$$F(x) = \sum_{j=1}^{k} |\hat{u}_j(\xi)|^2.$$
(2.5) $\int_{\mathbb{R}^n} F(\xi) d\xi = k,$

(2.6) $\sum_{j=1}^{k} |\nabla \hat{u}_j(\xi)|^2 \leq \frac{I(\Omega)}{(2\pi)^n},$

(2.7) $|\nabla F(\xi)| \leq L,$

(2.8) $\frac{n}{n+2} \left( \frac{|\Omega|}{\omega_n} \right)^{\frac{2}{n}} \leq \frac{I(\Omega)}{\Omega}$

and

(2.9) $\frac{|\Omega|^{1+\frac{1}{n}}}{(2\pi)^n \omega_n^{\frac{1}{n}}} \leq L.$

Let $F^*(\xi)$ denote the decreasing radial rearrangement of $F(\xi).$ Therefore, by approximating $F(\xi),$ we may assume that there exists a real valued absolutely continuous function $\phi : [0, \infty) \rightarrow [0, M]$ such that $F^*(\xi) = \phi(|\xi|).$ From [6, 11], we have

**Lemma 2.2.** The function $\phi(s)$ satisfies

(2.10) $n\omega_n \int_0^{\infty} t^{n-1} \phi(t) dt = k,$

(2.11) $\sum_{j=1}^{k} \lambda_j \geq n\omega_n \int_0^{\infty} t^{n+2l-1} \phi(t) dt$

and

(2.12) $0 \leq -\phi(t) \leq L.$

### 2.2. Stokes operator with higher order

We firstly recall the functional definition of the Stokes operator [8, 14, 25] and its generalization [11]. Let $\mathcal{V}$ denote the set of smooth divergence-free vector functions with compact supports

$$\mathcal{V} = \{u : \Omega \rightarrow \mathbb{R}^n, \ u \in C^\infty(\Omega), \ \text{div} \ u = 0\}.$$

Let $L$ and $V$ denote the the closures of $\mathcal{V}$ in $L_2(\Omega)$ and $H^l_0(\Omega)$ ($1 \leq l \in \mathbb{N}$) respectively. Moreover, we note that $L_2(\Omega)$ can be written as $L_2(\Omega) = L \oplus L^\perp$ (see for instance, [25]), where

$$L = \{u \in L_2(\Omega) | \text{div} u = 0, u \cdot \nu|_{\partial \Omega} = 0\},$$

(2.13) $L^\perp = \{u \in L_2(\Omega) | u = \nabla p, p \in L^2_0(\Omega)\}.$

Define the operator $A : V \rightarrow V'$ by

$$(Au, v)_{V' \times V} := ((-\Delta)\frac{i}{2} u, (-\Delta)\frac{i}{2} v)$$

for $u, v \in V.$ The operator $A$ is an isomorphism between $V$ and $V'.$ For a sufficient smooth $u,$ we have

(2.14) $Au = P(-\Delta)^{\frac{i}{2}} u,$
where \( P \) is orthogonal projection mapping \( L_2(\Omega) \) to \( L \), i.e. \( P : L_2(\Omega) \rightarrow L \). For \( l = 1 \), (2.14) corresponds to the classical Stokes operator. The operator \( A \) is a self-adjoint positive definite operator with the following discrete spectrum

\[
A u_j = \mu_j u_j, \quad 0 < \mu_1 \leq \mu_2 \cdots ,
\]

where \( \{ u_j \}_{j=1}^{\infty} \in V \) are the orthonormal vector eigenfunctions and

\[
\mu_j = \| (\Delta)^{\frac{1}{2}} u_j \|.
\]

For the orthonormal family \( \{ u_k \}_{k=1}^{\infty} \in L \), we set

\[
(2.15) \quad F_S(\xi) = \sum_{k=1}^{m} |\hat{u}_k(\xi)|^2,
\]

where \( \hat{u}_k(\xi) \) is the Fourier transform of \( u_j(x) \) defined by

\[
(2.16) \quad \hat{u}_j(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\Omega} u_j(x) e^{ix \cdot \xi} dx.
\]

**Lemma 2.3.** The function \( F_S(\xi) \) defined by (2.15) satisfies

\[
0 \leq F_S(\xi) \leq M_S, \quad |\nabla F_S(\xi)| \leq L_S, \quad \int F_S(\xi) d\xi = k, \quad \int |\xi|^{2l} F_S(\xi) d\xi = \sum_{k=1}^{k} \| (\Delta)^{\frac{l}{2}} u_j \|^2 = \sum_{k=1}^{k} \mu_j.
\]

Supposing that \( F_S^*(\xi) \) denotes the decreasing radial rearrangement of \( F_S(\xi) \), by approximating \( F_S(\xi) \), we may infer that there exists a real valued absolutely continuous function \( \phi_S : [0, \infty) \rightarrow [0, M_S] \) such that \( F_S(\xi) = \phi_S(|\xi|) \). Form [11], we have

**Lemma 2.4.** The function \( \phi_S(s) \) satisfies

\[
(2.18) \quad n\omega_n \int_0^{\infty} t^{n-1} \phi_S(t) dt = k, \quad n\omega_n \int_0^{\infty} t^{n+2l-1} \phi_S(t) dt \leq \sum_{j=1}^{k} \mu_j, \quad 0 \leq -\phi_S(t) \leq L_S.
\]

3. SOME LEMMAS

In order to prove our main results, we first establish the following lemmas which are motivated by Melas’ work in [19].

**Lemma 3.1.** For \( l \in \mathbb{N}, 2 \leq n \), two positive real numbers \( s \) and \( t \), we have the following inequalities

\[
(3.1) \quad nt^{n+2l} - (n + 2l)t^{n}s^{2l} + 2ls^{n+2l} \geq \left( 2ls^{n+2l-2} + 4lt^{3n+2l-3} \right) (s - t)^2.
\]
Proof. By some direct calculation, for \( y > 0 \), we obtain
\[
ny^{n+2l} - (n + 2l)y^n + 2l = (y - 1)^2 \left( n \sum_{j=2}^{2l} (j-1)y^{n+2l-j} + 2l \sum_{j=1}^{n} (n-j+1)y^{n-j} \right)
\]
\[
= (y - 1)^2 \left( n \sum_{j=2}^{2l} (j-1)y^{n+2l-j} + 2l \sum_{j=1}^{n-2} (n-j+1)y^{n-j} \right)
\]
\[+ (2l + 4ly)(y - 1)^2.\]
Therefore, we have
\[
ny^{n+2l} - (n + 2l)y^n + 2l - (2l + 4ly)(y - 1)^2
= (y - 1)^2 \left( n \sum_{j=2}^{2l} (j-1)y^{n+2l-j} + 2l \sum_{j=1}^{n} (n-j+1)y^{n-j} \right) \geq 0.
\]
Setting \( y = \frac{4}{l} \) in above formula, we get (3.1). \( \square \)

Lemma 3.2. Suppose that \( \beta : [0, \infty) \rightarrow [0, 1] \) such that
\[
(3.2) \quad \int_{0}^{\infty} \beta(t)dt = 1, \int_{0}^{\infty} t^n \beta(t)dt < \infty, \int_{0}^{\infty} t^{n+2l} \beta(t)dt < \infty.
\]
Then, there exists \( \delta \geq 0 \) such that
\[
(3.3) \quad \int_{\delta}^{\delta+1} t^n dt = \int_{0}^{\infty} t^n \beta(t)dt
\]
and
\[
(3.4) \quad \int_{\delta}^{\delta+1} t^{n+2l} dt \leq \int_{0}^{\infty} t^{n+2l} \beta(t)dt.
\]
Proof. Note that
\[
(3.5) \quad (t^n - 1)(\beta(t) - \chi_{[0,1]}(t)) \geq 0, \quad \text{on} \quad [0, \infty).
\]
Integrating (3.5) over \([0, \infty)\), it gives
\[
(3.6) \quad \int_{0}^{\infty} t^n \beta(t)dt \geq \int_{0}^{1} t^n = \frac{1}{n+1}.
\]
Therefore, there exists \( \delta \geq 0 \) such that (3.3) holds.

According to Cramer’s rule, we can find two positive number \( a_1 \) and \( a_2 \) such that the function
\[
(3.7) \quad q(t) = t^{n+2l} - a_1 t^n + a_2
\]
satisfies \( q(\delta) = q(\delta + 1) = 0 \). Since \( q'(s) \) has at most one zero in \([0, \infty)\), we conclude that
\[
q(s) \begin{cases} < 0, & \text{in} \ (\delta, \delta + 1), \\ > 0, & \text{in} \ [0, \infty)/[\delta, \delta + 1]. \end{cases}
\]
The assumptions on \( \beta(t) \) imply that
\[
(3.8) \quad q(t)(\chi_{[\delta,\delta+1]}(t) - \beta(t)) \leq 0, \quad \text{on} \quad [0, \infty).
\]
Integrating the inequality (3.8), taking into account the choice of \( \delta \) and using (3.5), we have (3.4).
Lemma 3.3. Let \( n \geq 2, D, A > 0 \) be constants and \( \psi : [0, \infty) \rightarrow [0, \infty) \) be a decreasing and absolutely continuous function such that

\[
-D \leq \psi'(t) \leq 0
\]

and

\[
\int_{0}^{\infty} t^{n-1} \psi(t) dt = A.
\]

Then

\[
\int_{0}^{\infty} t^{n+2l-1} \psi(t) dt \geq \frac{1}{n + 2l} (nA)^{1+\frac{2l}{n}} \psi(0)^{-\frac{2l}{n}} + \frac{2l \varepsilon}{n(n+2l)} (nA)^{1+\frac{2l}{n}} \psi(0)^{1-\frac{2l}{n}} D^{-1}
\]

\[
- \frac{5l}{2n(n+2l)} (nA)^{1+\frac{2l}{n}} \psi(0)^{2-\frac{2l}{n}} D^{-2} + \frac{\varepsilon l}{n(n+2l)} (nA)^{1+\frac{2l}{n}} \psi(0)^{3-\frac{2l}{n}} D^{-3},
\]

where \( \tau \in (0, 1] \) and \( \varepsilon \in (0, 1] \).

Proof. We assume that \( B = \int_{0}^{\infty} t^{n+2l-1} \psi(t) dt < \infty \). Otherwise there is nothing to prove. Define

\[
S(t) = \frac{1}{\psi(0)} \psi \left( \frac{\psi(0)}{L} t \right) \quad \text{and} \quad h(t) = -S'(t), \quad t \geq 0.
\]

Note that \( S(0) = 1, 0 \leq h(t) \leq 1 \) and \( \int_{0}^{\infty} h(t) dt = S(0) = 1 \). Now we define

\[
\tilde{A} = \int_{0}^{\infty} t^{n-1} S(t) dt
\]

and

\[
\tilde{B} = \int_{0}^{\infty} t^{n+2l-1} S(t) dt.
\]

Using the similar arguments as in [28], we can infer

\[
\lim \inf_{t \to \infty} t^{n+2l} S(t) = 0.
\]

Moreover, it is not difficult to observe that \( \lim \inf_{t \to \infty} t^n S(t) = 0 \) as well. Using the integration by parts, we have

\[
\int_{0}^{\infty} t^n h(t) dt = n \tilde{A}
\]

and

\[
\int_{0}^{\infty} t^{n+2l} h(t) dt = (n + 2l) \tilde{B}.
\]

Since \( h(t) \) satisfy the conditions in Lemma 3.2, there exists \( \delta \geq 0 \) such that

\[
\int_{\delta}^{\delta + 1} t^n dt = n \tilde{A}
\]

and

\[
\int_{\delta}^{\delta + 1} t^{n+2l} dt \leq \int_{0}^{\infty} t^{n+2l} h(t) dt = (n + 2l) \tilde{B}.
\]
Making use of the Jensen’s inequality, we derive
\[
\frac{1}{2^n} \leq n\bar{\Lambda}.
\]
By Lemma 3.1, integrating (3.1) in \(t\) from \(\delta\) to \(\delta + 1\), we obtain
\[
n \int_{\delta}^{\delta + 1} t^{n+2l} dt \geq (n + 2l)s^2 \int_{\delta}^{\delta + 1} t^n dt - 2ls^{n+2l} + 2ls^{n+2l} - 2 \int_{\delta}^{\delta + 1} (t - s)^2 dt
\]
(3.17)\[+ 4l\delta^{n+2l-3} \int_{\delta}^{\delta + 1} t(t - s)^2 dt.
\]
Since
\[
\int_{\delta}^{\delta + 1} (t - s)^2 dt \geq \frac{1}{12}
\]
and
\[
\int_{\delta}^{\delta + 1} t(t - s)^2 dt \geq \frac{1}{2}s^2 - \frac{2}{3}s + \frac{1}{4},
\]
setting \(s = (n\bar{\Lambda})^{\frac{1}{2}}\), we obtain
\[
n(n + 2l)\bar{B} \geq n(n\bar{\Lambda})^{\frac{1}{2}} + 2l(n\bar{\Lambda})^{\frac{1}{2}} + 2(n\bar{\Lambda})^{\frac{1}{2}} - \frac{5}{2}l(n\bar{\Lambda})^{\frac{1}{2}} + l(n\bar{\Lambda})^{\frac{1}{2}}
\]
by using (3.15) and (3.16). Finally, by the definitions of (3.12) and (3.13), we get (3.11). \(\Box\)

4. Proofs of Theorem 1.1 and Theorem 1.7

Proof of Theorem 1.1. From Lemma 2.2 and Lemma 3.3, we have
\[
\sum_{j=1}^{k} \lambda_j \geq \frac{n}{n + 2l} \omega_n^{\frac{2l}{n}} \psi(0)^{\frac{2l}{n}} k^{1+\frac{2l}{n}} + \frac{2le}{(n + 2l)L} \omega_n^{\frac{2l}{n}} \psi(0)^{1-\frac{2l}{n}} k^{1+\frac{2l}{n}}
\]
(4.1)
\[- \frac{5l}{2(n + 2l)L^2} \omega_n^{\frac{2l}{n}} \psi(0)^{2-\frac{2l}{n}} k^{1+\frac{2l}{n}} + \frac{e}{(n + 2l)L^3} \omega_n^{\frac{2l}{n}} \psi(0)^{3-\frac{2l}{n}} k^{1+\frac{2l}{n}}.
\]
Now define two functions
\[
F_1(x) = c_1 x^{-\frac{2l}{n}} + c_2 x^{1-\frac{2l}{n}} \quad \text{and} \quad F_2(x) = c_3 x^{3-\frac{2l}{n}} - c_4 x^{2-\frac{2l}{n}},
\]
(4.2)
for \(x \in (0, \frac{\Omega}{(2\pi)^n})\), where the constants
\[
c_1 = \frac{n}{n + 2l} \omega_n^{\frac{2l}{n}} k^{1+\frac{2l}{n}}, \quad c_2 = \frac{2le}{(n + 2l)L} \omega_n^{\frac{2l}{n}} k^{1+\frac{2l}{n}},
\]
\[
c_3 = \frac{e}{(n + 2l)L^3} \omega_n^{\frac{2l}{n}} k^{1+\frac{2l}{n}}, \quad c_4 = \frac{5l}{2(n + 2l)L^2} \omega_n^{\frac{2l}{n}} k^{1+\frac{2l}{n}}.
\]
For \(1 \leq l < \frac{n+1}{2}\), \(F_1(x)\) is decreasing if \(0 < x \leq \left(\frac{lc_1}{(n-2l+1)c_2}\right)^{\frac{n}{n+1}}\) and \(F_2(x)\) is decreasing if \(x \leq \left(\frac{(2n+2-2lc_4)}{(3n+3-2lc_3)}\right)^{\frac{n}{n+1}}\). The function \(F_1(x) + F_2(x)\) is decreasing for \(x \in (0, \frac{\Omega}{(2\pi)^n})\) when we have
\[
\frac{|\Omega|}{(2\pi)^n} \leq \min \left\{ \left(\frac{lc_1}{(n-2l+1)c_2}\right)^{\frac{n}{n+1}}, \left(\frac{(2n+2-2lc_4)}{(3n+3-2lc_3)}\right)^{\frac{n}{n+1}} \right\}
\]
(4.3)
Since $k \geq 1$, by (2.9), it is sufficient to find the upper bound of $\varepsilon$. In fact, we have
\begin{equation}
\varepsilon \leq \min_{1 \leq l < \frac{n}{n-1}, 2 \leq n} \left\{ \frac{n}{n-2l+1} \Gamma(1 + \frac{n}{2})^\frac{2}{3}, \frac{10(n+1-l)}{3n+3-2l} \Gamma(1 + \frac{n}{2})^\frac{2}{3} \right\}.
\end{equation}

Obviously, it holds that $\frac{n}{n-1} \leq \frac{n}{n-2l+1}$ and $\frac{1}{4} < \frac{n+1-l}{3n+3-2l}$ for $1 \leq l < \frac{n+1}{2}$ and then $\frac{n}{n-1} < \frac{5}{2}$ for $n \geq 2$. From the definition of $\varepsilon$ in Lemma 3.3, we can choose
\[\varepsilon = \min \left\{ 1, \frac{n}{n-1} \Gamma(1 + \frac{n}{2})^\frac{2}{3} \right\} = 1\]
to guarantee the function $F_1(x) + F_2(x)$ satisfying the conditions
\[\int_{\Omega} F(x) \, dx \leq k \varepsilon \leq \frac{n+1}{2} \Gamma(1 + \frac{n}{2})^\frac{2}{3}\]
\[\text{for } x \in \left(0, \frac{\sqrt{n}}{2\pi^{\frac{n}{2}}l_0}\right),\]
and then $\varepsilon = 1$ and $\psi(0) = \frac{\Omega}{(2\pi)^n}$ in (4.1), we can infer (1.10).

Proof of Theorem 1.7. From Lemma 3.3, we have the inequality
\[\sum_{j=1}^{k} \mu_j \geq \frac{n}{n+2l} \omega_n^2 \psi(0)^{-\frac{2}{n}} k^{1+\frac{2}{n}} \frac{2l\varepsilon}{(n+2l)L_S^2} \omega_n^2 \psi(0)^{1-\frac{2l}{n}} k^{1+\frac{2l}{n}} \]
\[-\frac{5l}{2(n+2l)L_S^2} \omega_n^2 \psi(0)^{-\frac{2l}{n}} k^{1+\frac{2l}{n}} \frac{\varepsilon \omega_n^2 \psi(0)^{-\frac{2l}{n}} k^{1+\frac{2l}{n}}}{(n+2l)L_S^2} \omega_n^2 \psi(0)^{1-\frac{2l}{n}} k^{1+\frac{2l}{n}} \]
Similar to the proof of Theorem 1.1, we can give the proof of Theorem 1.7 by only replacing $\lambda_j, L, M$ by $\mu_j, L_S, M_S$ in the proof of Theorem 1.1 respectively. So we omit its proof.

5. Proofs of Theorem 1.3 and Theorem 1.5

In order to prove Theorem 1.3, we need the following lemma derived by Ilyin [10, 11].

Lemma 5.1. (Lemma 3.1 in [10]) Let
\[\Psi_j(r) = \begin{cases} M, & \text{for } 0 \leq r \leq s, \\ M - L(r-s), & \text{for } s \leq r \leq s + \frac{M}{L}, \\ 0, & \text{for } r \geq s + \frac{M}{L}, \end{cases}\]
where $M, L$ is given by (2.1). Suppose that $\int_0^{+\infty} r^d \Psi_j(r) dr = m^*$ and $d \geq b$. Then for any decreasing and absolutely continuous function $F$ satisfying the conditions
\[0 \leq F \leq M, \quad \int_0^{+\infty} r^d F(r) dr = m^*, \quad 0 \leq -F' \leq L,\]
the following inequality holds:
\[\int_0^{+\infty} r^d F(r) dr \geq \int_0^{+\infty} r^d \Psi_j(r) dr.\]

Now we give the proof Theorem 1.3.

Proof of Theorem 1.3. Assume that $F(\xi)$ is defined as in (2.3). Define
\[R = \left( \frac{n+2l}{n} \frac{2\pi^n}{\Omega \omega_n} \sum_{j=1}^{k} \lambda_j^0 \right)^{\frac{1}{2lq}}\]
(5.4)
and

\( Y(\xi) = \frac{|\Omega|}{(2\pi)^n} \chi_{B_2(0)}(\xi), \)

such that

\[
\int_{\mathbb{R}^n} |\xi|^{2q} Y(\xi) d\xi = \sum_{j=1}^{k} \lambda_j^q.
\]

Since \( \int_{\mathbb{R}^n} |\hat{u}_j(\xi)|^2 d\xi = 1 \) and \( x \mapsto x^q \) is concave for \( x \geq 0 \) and \( q \in (0, 1) \), by using the Jesen’s inequality, we can derive

\[
\sum_{j=1}^{k} \lambda_j^q \geq \int_{\mathbb{R}^n} |\xi|^{2q} F(\xi) d\xi.
\]

Let \( F^*(\xi) \) denote the decreasing radial rearrangement of \( F(\xi) \). According to the Hardy-Littlewood inequality, we have

\[
\int_{\mathbb{R}^n} |\xi|^{2q} F^{*}(\xi) d\xi \geq \int_{\mathbb{R}^n} |\xi|^{2q} F^{*}(r) d\xi.
\]

By using (5.7) and (5.8), we can get

\[
\sum_{j=1}^{k} \lambda_j^q \geq \int_{\mathbb{R}^n} |\xi|^{2q} F^{*}(\xi) d\xi = n\omega_n \int_{0}^{\infty} r^{n-1+2q} F^{*}(r) dr,
\]

where \( F^{*}(r) = F^{*}(|r|) \). It is easy to find \( F(\xi) \) and \( F^{*}(|\xi|) \) satisfy the conditions of Lemma 5.1 according to Lemma 2.1. Therefore, we have

\[
\int_{0}^{\infty} r^{n-1+2q} F^{*}(r) dr \geq \int_{0}^{\infty} r^{n-1+2q} \Psi_s(r) dr.
\]

Using Corollary 1 in [11], we find that

\[
n\omega_n \int_{0}^{\infty} r^{n-1+2ql} \Psi_s(r) dr \geq \frac{n}{n+2lq} \left( \frac{1}{\omega_n M} \right)^{\frac{2q}{n}} n\omega_n k^{1+\frac{2q}{n}} + \frac{2lqnM^2}{24} \left( \frac{1}{\omega_n M} \right)^{\frac{2q}{n}} n\omega_n k^{1+\frac{2q}{n}} + O(k^{1+\frac{2q-4}{n}})
\]

\[
= (4\pi)^q \frac{n}{n+2lq} \left( \frac{\Gamma(1+n/2)}{|\Omega|} \right)^{\frac{2q}{n}} k^{1+\frac{2q}{n}} + O(k^{1+\frac{2q-4}{n}}),
\]

where we use the definitions of \( M \) and \( L \) given by (2.1). Finally, from (5.9)–(5.11), we obtain (1.11).

\[\square\]

Now we give the proof of Theorem 1.5.
Proof of Theorem 1.5. Define

\[
\tilde{R} = \left( \frac{(n-2lp)(2\pi)^n}{n} \sum_{j=1}^{k} \frac{\lambda_j^{-p}}{|\Omega|\omega_n} \right)^{\frac{1}{2lp}}
\]

and

\[
\tilde{Y}(\xi) = \frac{|\Omega|}{(2\pi)^n} \chi_{B_R(0)}(\xi)
\]

such that

\[
\int_{\mathbb{R}^n} \frac{\tilde{Y}(\xi)}{|\xi|^{2lp}} d\xi = \sum_{j=1}^{k} \lambda_j^{-p}.
\]

Since \(|\hat{u}_j(\xi)|^2 d\xi\) defines a probability measure on \(\mathbb{R}^n\), \(x \mapsto x^{-p}\) is convex for \(x > 0\) and \(p > 0\), we derive

\[
\int_{\mathbb{R}^n} \frac{F(\xi)}{|\xi|^{2lp}} d\xi \geq \sum_{j=1}^{k} \lambda_j^{-p}
\]

according to the Jensen’s inequality. Similar to the arguments in Lemma 1 in [17], one can infer

\[
\int_{\mathbb{R}^n} F(\xi) d\xi \geq \int_{\mathbb{R}^n} \tilde{Y}(\xi) d\xi.
\]

By using (2.5) and (5.12), we obtain (1.13). \(\Box\)

Remark 5.2. The proof of Theorem 1.4 can also be given directly by using a similar approach as the proof of Theorem 1.5.

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