On finite-time and fixed-time consensus algorithms for dynamic networks switching among disconnected digraphs

David Gómez-Gutiérrez\textsuperscript{a,b}, Carlos Renato Vázquez\textsuperscript{b}, Sergej Čelikovský\textsuperscript{c}, Juan Diego Sánchez-Torres\textsuperscript{d} and Javier Ruiz-León\textsuperscript{e}

\textsuperscript{a}Multi-agent Autonomous Systems Lab, Intel Labs, Intel Tecnología de México, Av. del Bosque 1001, 45019, Zapopan, Jalisco, Mexico; \textsuperscript{b}Tecnologico de Monterrey, Escuela de Ingeniería y Ciencias, Av. General Ramón Corona 2514, 45201, Zapopan, Jalisco, Mexico; \textsuperscript{c}The Czech Academy of Sciences, Institute of Information Theory and Automation, Pod vodárenskou věží 4, 182 08 Prague, Czech Republic, \texttt{celikovs@utia.cas.cz}; \textsuperscript{d}Research Laboratory on Optimal Design, Devices and Advanced Materials -OPTIMA-, Department of Mathematics and Physics, ITESO, Periférico Sur Manuel Gómez Morín 8585 C.P. 45604, Tlaquepaque, Jalisco, México; \textsuperscript{e}CINVESTAV Unidad Guadalajara, Av. del Bosque 1145, Zapopan, 45019, Jalisco, Mexico

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ABSTRACT
This paper aims to analyze the stability of a class of consensus algorithms with finite-time or fixed-time convergence for dynamic networks composed of agents with first-order dynamics. In particular, in the analyzed class a single evaluation of a nonlinear function of the consensus error is performed per each node. The classical assumption of switching among connected graphs is dropped here, allowing to represent failures and intermittency in the communications between agents. Thus, conditions to guarantee finite and fixed-time convergence, even while switching among disconnected graphs, are provided. Moreover, the algorithms of the considered class are computationally simpler than previously proposed finite-time consensus algorithms for dynamic networks, which is an essential feature in scenarios with computationally limited nodes and energy efficiency requirements such as in sensor networks. Simulations illustrate the performance of the proposed consensus algorithms. In the presented scenarios, results show that the settling time of the considered algorithms grows slower than other consensus algorithms for dynamic networks as the number of nodes increases.

KEYWORDS
Finite-time consensus, Fixed-time consensus, dynamical networks, multi-agent systems, multiple interacting autonomous agents, self-organizing systems

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CONTACT C. R. Vázquez: cr.vazquez@itesm.mx
1. Introduction

Inspired by the ability of certain social insects to self-organize and mutually cooperate by relying only on neighbor-to-neighbor communication, there has been an increasing interest during the last decade in distributed problems to control the behavior of an agent’s network by local interactions. Of particular interest is the consensus problem, which deals with allowing a network of agents to agree on a common value for its internal state by using only communication among neighbors (see e.g. Cai (2012); Y. Chen, Lu, Yu, and Hill (2013); Cortés (2006); Jiang and Wang (2009); Lewis, Zhang, Hengster-Movric, and Das (2014); Olfati-Saber and Murray (2004)). Consensus algorithms have application, for instance, in distributed formation control (S. Li & Wang, 2013; Ren, Beard, & Atkins, 2005), distributed resource allocation (Xu, Han, et al., 2017; Xu, Yan, Cai, & Lim, 2017) and multi-agent rendezvous at the globally optimal point (Adibzadeh, Suratgar, Menhaj, & Zamani, 2018).

There are several published works proposing consensus algorithms in which the agents are first-order integrator systems (L. Wang & Xiao, 2010), second-order integrator systems (Guan, Sun, Wang, & Tao-Li, 2012; Tian, Lu, Zuo, & Yang, 2018) or high-order integrator dynamics (Tian, Zuo, & Wang, 2017; Zuo, Tian, Defoort, & Ding, 2018). Some works consider static communication topologies while others consider dynamic topologies, modeling intermittency in the communications, movement of the agents and the switching between different transmission/reception power levels.

Regarding first-order agents, it is known that if the graph topology is strongly connected, then consensus can be achieved by the standard protocol (Olfati-Saber, Fax, & Murray, 2007). Convergence to the average of the agent’s initial states is achieved if the network topology is balanced (identical number of in-neighbors and out-neighbors). For unbalanced graphs, the standard algorithm can be modified by adding a surplus dynamic, to still achieve consensus on the average value (Cai, 2012). These algorithms are linear, and thus the convergence is asymptotic.

Nonlinear protocols have been used to achieve finite-time convergence, in particular, binary protocols have been broadly investigated, achieving consensus to the average value (Franceschelli, Giua, Pisano, & Usai, 2011), the average-min-max value (Cortés, 2006; C. Li & Qu, 2014), the median value (Franceschelli, Giua, & Pisano, 2017), and the maximum or minimum value (B. Liu, Lu, & Chen, 2015) of the agent’s initial conditions. Continuous finite-time protocols have been introduced in Hui, Haddad, and Bhat (2008); Shang (2012); L. Wang and Xiao (2010); Zhu, Guan, and Luo (2013). Moreover, in Ning, Jin, Zheng, and Man (2018); Parsegov, Polyakov, and Shcherbakov (2013); Zuo and Tiq (2014); Zuo, Yang, Tie, and Meng (2014) there have been proposed protocols with fixed-time convergence, i.e., there exists a bound for the convergence time that is independent of the initial conditions. Consensus algorithms where the convergence time is set a priori have been introduced in Y. Liu, Zhao, Ren, and Chen (2018); Yong, Guangming, and Huiyang (2012), using linear consensus with a time-varying gain; unfortunately, these methods require that all nodes have a common reference-time which is often restrictive. Multiple variations of the consensus problem have been derived recently. For instance, in Meng and Jia (2016) the multi-scale consensus problem has been addressed, where the nodes agree on a common quantity, but each one with its predetermined scale. In that paper, protocols with asymptotic, finite-time and fixed-time convergence were analyzed for the case of static networks. In Meng and Zuo (2016), the signed-average consensus problem is considered, where the nodes converge to values that are equal in magnitude but may be different in sign. For this problem, fixed-time convergent
algorithms were proposed for the static network.

It can be demonstrated that some of the previously mentioned algorithms achieve consensus under dynamic networks switching among connected topologies (Cai & Ishii, 2014; Olfati-Saber et al., 2007), while maintaining finite-time convergence (Franceschelli et al., 2013; L. Wang & Xiao, 2010) or fixed-time convergence (Zuo et al., 2014). Moreover, some protocols have been shown to achieve consensus in dynamic networks composed by disconnected topologies provided that they form a connected graph in a “joint sense”, for instance, in X. Chen, Hao, and Shao (2015) an algorithm with asymptotic convergence is proposed for the event-triggered consensus problem (where the control action is triggered only when an event is satisfied). In Lin, Qin, Zhao, and Sun (2012), the average consensus problem with time-delay is addressed using asymptotic convergent algorithms. In B. Liu et al. (2015), a discontinuous protocol is analyzed using nonsmooth stability theory.

There exist several works regarding consensus for double integrator agents. Frequently, matching disturbances are considered (for instance, Khoo, Xie, and Man (2009); S. Li, Du, and Lin (2011); L.-W. Zhao and Hua (2014)). Most of the papers propose finite-time protocols (Cao, Ren, & Meng, 2010; Guan et al., 2012; L.-W. Zhao & Hua, 2014; Y. Zhao, Duan, Wen, & Zhang, 2013). Most of the works consider a fixed topology (for instance, Cao et al. (2010); Khoo et al. (2009); S. Li et al. (2011); L.-W. Zhao and Hua (2014); Y. Zhao et al. (2013)). From the current literature, only few works consider dynamic topologies (Dai & Guo, 2017; Guan et al., 2012), where additional conditions about the graph connection must hold. On the other hand, the consensus in high-order agents has been recently considered. In general, agents are described as linear systems (e.g., Z. Li, Duan, Chen, and Huag (2010); Seo, Shim, and Back (2009)), but there are few works in which agents are non-linear (Mondal & Su, 2016; Mu, Xiao, Liu, & Zhang, 2014). Frequently, the information shared by each agent is its output, then, observers are used to estimate the relative errors and thus evaluate the consensus protocol. Generally, the protocol has the form of linear feedback (for instance, Z. Li et al. (2010); Seo et al. (2009); You, Li, and Xie (2013)), in some cases with an adaptive gain, leading to asymptotic convergence, but non-linear protocols have also been applied (Mondal & Su, 2016; Mu et al., 2014). In most of the works the topology is fixed (e.g., Z. Li et al. (2010); Mondal and Su (2016); Seo et al. (2009)), however, some works consider dynamic topologies (Mu et al., 2014; Qin & Yu, 2014; You et al., 2013) but requiring certain restrictions, for instance, that the topology graphs be jointly connected (Cai & Ishii, 2014; Mu et al., 2014; Qin & Yu, 2014).

### 1.1. Paper contribution

In this work, we revisit the consensus problem for first-order agents considering dynamic topologies. The goal is to analyze a class of consensus algorithms from a common framework, rather than to propose a particular protocol. In the analyzed consensus class, each agent applies a protocol defined as a nonlinear function evaluated on the sum of the errors of all neighboring nodes. By using nonsmooth stability analysis and results from finite-time stability and homogeneity theory, conditions for asymptotic, finite-time and fixed-time consensus are derived for this class of protocols. Three cases are considered: when the communication topology is static; when the communication topology switches among connected graphs; and when the communication topology switches among disconnected graphs. We emphasize that,
in the literature, continuous algorithms of this class have only been demonstrated to achieve consensus on static networks. In fact, a detailed comparison to the related protocols existing in the literature is presented in Subsection 2.3. The efficacy of the analyzed consensus class on dynamic networks is proven in this paper. Additionally, it is shown through simulations that the settling time of the analyzed class grows slower when the number of nodes increases than with other related algorithms (Shang, 2012; L. Wang & Xiao, 2010; Zuo & Tie, 2014). Another advantage of the analyzed class is that they are computationally simpler than existing finite-time and fixed-time algorithms for dynamic networks (Franceschelli et al., 2013; Hui, Haddad, & Bhat, 2010; L. Wang & Xiao, 2010; Zuo & Tie, 2014), which is relevant in applications with energy efficiency requirements and limited computing resources.

The rest of the paper is organized as follows: In Section 2, mathematical preliminaries on graph theory and finite-time stability are presented. In Section 3, the main result is derived, and illustrative examples are introduced. In Section 4, a comparison between consensus algorithms for dynamic networks is presented, analyzing how their convergence time varies as the size of the network increases. Finally, the conclusions and future work are presented in Section 5.

2. Preliminaries

2.1. Graph Theory

The following notation and preliminaries on graph theory are taken mainly from Godsil and Royle (2001).

Definition 1. A directed graph (also called digraph) \( X \) consists of a vertex set \( \mathcal{V}(X) \) and an edge set \( \mathcal{E}(X) \) where an edge (also called node) is an ordered pair of distinct vertices of \( X \). Writing \((i,j)\) denotes an edge with direction from vertex \( i \) to vertex \( j \). The set of in-neighbors of a vertex \( i \) in the graph \( X \) is denoted by \( N^{-i}(X) = \{ j \in \mathcal{V}(X) : (j,i) \in \mathcal{E}(X) \} \). A graph is said to be undirected if \((i,j) \in \mathcal{E}(X)\) implies that \((j,i) \in \mathcal{E}(X)\).

A path from \( i \) to \( j \) in a digraph is a sequence of edges \((i,k_1) (k_1,k_2) \cdots (k_{n-1},k_n) (k_n,j)\) starting in node \( i \) and ending in node \( j \). A digraph is said to be connected if for every pair of vertices \( i, j \in \mathcal{V}(X) \) either there is a path from \( i \) to \( j \) or a path from \( j \) to \( i \), otherwise it is said to be disconnected. A subgraph of \( X \) is a graph \( Y \) such that \( \mathcal{V}(Y) \subseteq \mathcal{V}(X) \) and \( \mathcal{E}(Y) \subseteq \mathcal{E}(X) \). A subgraph \( Y \) of \( X \) is called a proper subgraph of \( X \). A subgraph \( Y \) of \( X \) is called an induced subgraph if for any two vertices \( i, j \in \mathcal{V}(Y) \), there is an edge \((i,j) \in \mathcal{E}(Y)\) if and only if \((i,j) \in \mathcal{E}(X)\). An induced subgraph \( Y \) of \( X \) that is connected is called maximal if it is not a proper subgraph of another connected subgraph of \( X \). A connected induced subgraph of \( X \) that is maximal is called a connected component of \( X \). The edge connectivity \( k_1(X) \) of \( X \) is the minimum number of edges that are needed to be removed to decrease the number of connected components.

Definition 2. A weighted digraph is a digraph together with a weight function \( W : \mathcal{E}(X) \rightarrow \mathbb{R}_+ \). The adjacency matrix \( A = [a_{ij}] \in \mathbb{R}^{n \times n} \) of a graph with \( n \) vertices is a square matrix where \( a_{ij} \) corresponds to the weight of the edge \((j,i)\) if \( j \in N^{-i}(X) \) and \( a_{ij} = 0 \) otherwise.

For an undirected graph \( a_{ij} = a_{ji} \). The Laplacian of \( X \) is the matrix \( Q(X) = \Delta - A \). 

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where \( \Delta = \text{diag}(d_1 \cdots, d_n) \) with \( d_i = \sum_{j=1}^{n} a_{ij} \). If the graph \( \mathcal{X} \) is connected, then the eigenvalue \( \lambda_1(\mathcal{Q}) = 0 \) has algebraic multiplicity one with eigenvector \( 1 = [1 \cdots 1]^T \), i.e. the right annihilator of \( \mathcal{Q}(\mathcal{X}) \) is \( \ker \mathcal{Q}(\mathcal{X}) = \{x : x_1 = \ldots = x_n\} \). The algebraic connectivity of \( \mathcal{X} \) is the second smallest eigenvalue of \( \mathcal{Q} \), \( \lambda_2(\mathcal{Q}) \), which is lower or equal to the edge connectivity of \( \mathcal{X} \), i.e. \( \lambda_2 \leq \kappa_1(\mathcal{X}) \) (Godsil & Royle, 2001).

### 2.2. Finite-time and Fixed-time Stability

Some results on homogeneity theory (Center & Kawski, 1995, Hermes, 1991, Rosier, 1992), finite-time (Bhat & Bernstein, 2000, 2005) and fixed-time stability (Andriuen, Praly, & Astolfi, 2008; Polyakov, Efimov, & Perruquetti, 2016), which will be useful in the exposition later on, are recalled in this subsection. Let us first present some definitions.

**Definition 3.** (Perruqueti, Floquet, & Moulay, 2008) Let \( f : \mathbb{R}^n \to \mathbb{R}^n \) be a piecewise continuous function with \( f(0) = 0 \), \( \psi(t,x_0) \) is said to be a right-maximally defined solution of

\[
\dot{x} = -f(x) \tag{1}
\]

if \( \psi(t,x_0) \) is such that

\[
\frac{d\psi(t,x_0)}{dt} = -f(\psi(t,x_0)), \quad \psi(0,x_0) = x_0, \quad \forall t \in [t_0,T_m(x_0)], \forall x_0 \in \mathcal{D} \setminus \{0\}
\]

where \( T_m(x_0) \in (0, \infty) \) is the maximal possible real number with the above property, or plus infinity. Moreover, \( \mathbb{I} \) is said to have a unique solution in forward time if for any \( x_0 \in \mathbb{R}^n \) and two right-maximally defined solutions of \( \mathbb{I} \), \( \psi(t,x_0) \) and \( \phi(t,x_0) \) defined on \( [t_0,T_m^\psi] \) and \( [t_0,T_m^\phi] \), respectively, there exists \( t_0 < T_m(x_0) < \min(T_m^\psi(x_0),T_m^\phi(x_0)) \) such that \( \psi(t,x_0) = \phi(t,x_0) \) for all \( t \in [t_0,T_m(x_0)) \).

Solutions to \( \mathbb{I} \) are understood in the sense of Filippov and are assumed to be unique in forward-time.

**Definition 4.** A switched nonlinear system

\[
\dot{x} = -f_{\sigma(t)}(x) \tag{2}
\]

is defined by the tuple \( \langle \mathcal{P}, \sigma \rangle \) where \( \mathcal{P} = \{f_1, \ldots, f_m\} \) is a family of nonlinear vector fields such that \( \dot{x} = -f_k(x) \), \( k \in \{1, \ldots, m\} \), has a unique solution in forward time (understood in the sense of Filippov) and \( \sigma : \mathbb{R}^+ \to \{1, \ldots, m\} \) is the switching signal defining the active vector field, such that \( f_{\sigma(t)}(x) = f_k(x) \) whenever \( \sigma(t) = k \), with the property that only a finite number of switchings occur in any finite interval, i.e. Zeno behavior is excluded.

The solution \( \psi(t,x_0) \) of \( \mathbb{I} \) is absolutely continuous and it is such that, if \( \sigma(t) = k \), \( \forall t \in [t_i,t_j] \) then \( \psi(t,x_0) = \psi_k(t-t_i,x(t_i)) \), \( \forall t \in [t_i,t_j] \) where \( \psi_k(t-t_i,x(t_i)) \) is the unique solution in forward-time of \( \dot{y}(\hat{t}) = -f_k(y(\hat{t})) \) with \( \hat{t} = t-t_i \) and initial condition \( y(t_0) = x(t_i) \).
Definition 5. (Bhat & Bernstein, 2005) The origin of (2) is called finite-time convergent if there exists an open neighborhood \( M \subseteq \mathbb{R}^n \) around the origin and a function \( T : M \setminus \{0\} \to (0, \infty) \), called the settling-time function, such that for every \( x_0 \in M \setminus \{0\} \), the solution \( \psi(t, x_0) \) is defined on \([0, T(x_0))\), \( \psi(t, x_0) \in M \setminus \{0\} \) for all \( t \in [0, T(x_0)) \) and \( \lim_{t \to T(x_0)^-} \psi(t, x_0) = 0 \). Because of the uniqueness of \( \psi(t, x_0) \), it follows that \( T(x_0) = \min \{ t \in \mathbb{R}_+ \mid \psi(t, x_0) = 0 \} \). Furthermore, the origin is said to be finite-time stable if it is stable and finite-time convergent. Similarly, the origin is said to be globally finite-time stable if it is finite-time stable with \( M = \mathbb{R}^n \).

The origin is called fixed-time convergent for (1) if it is finite-time convergent and \( \forall x_0 \in \mathbb{R}^n \) the settling time \( T(x_0) \) is bounded by some \( T_{\max} > 0 \). Furthermore, the origin is said to be fixed-time stable if it is stable and fixed-time convergent.

The following definitions introduce the concept of homogeneity in functions and vector fields, which will be used for finite-time and fixed-time stability analysis.

Definition 6. (Bhat & Bernstein, 2005) A function \( g : \mathbb{R}^n \to \mathbb{R} \) is called homogeneous of degree \( l \) with respect to the “standard dilation” \( \Delta_\lambda(x) = \lambda x \) if and only if

\[
g(\lambda x) = \lambda^l g(x)
\]

for all \( \lambda > 0 \).

A vector field \( f(x) \), where \( x \in \mathbb{R}^n \), is homogeneous of degree \( d \) with respect to the standard dilation if

\[
f(x) = \lambda^{-(d+1)} f(\lambda x).
\]

Definition 7. (Andrieu et al., 2008; Polyakov et al., 2016) A function \( g : \mathbb{R}^n \to \mathbb{R} \), such that \( g(0) = 0 \), is said to be homogeneous in the \( \lambda_0 \)-limit with degree \( d_{\lambda_0} \) if the function \( g_{\lambda_0} : \mathbb{R}^n \to \mathbb{R} \), defined as

\[
g_{\lambda_0}(x) = \lim_{\lambda \to \lambda_0} \lambda^{-d_{\lambda_0}} g(\lambda x),
\]

is homogeneous of degree \( d_{\lambda_0} \) with respect to the standard dilation.

A vector field \( f : \mathbb{R}^n \to \mathbb{R}^n \) is said to be homogeneous in the \( \lambda_0 \)-limit with degree \( d_{\lambda_0} \) if the vector field \( f_{\lambda_0} : \mathbb{R}^n \to \mathbb{R}^n \), defined as

\[
f_{\lambda_0}(x) = \lim_{\lambda \to \lambda_0} \lambda^{-(d_{\lambda_0}+1)} f(\lambda x),
\]

is homogeneous of degree \( d_{\lambda_0} \) with respect to the standard dilation.

The following results provide sufficient conditions for finite-time stability and fixed-time stability, respectively.

Theorem 8. (Bhat & Bernstein, 2005, Theorem 7.1) Let \( f(x) \), with \( x \in \mathbb{R}^n \), be an homogeneous vector field of degree \( q \) with respect to the standard dilation. Then the origin of \( \dot{x} = -f(x) \) is globally finite-time stable if and only if it is globally asymptotically stable and \( d < 0 \), where \( d \) is the homogeneity degree of \( f(x) \).

Theorem 9. (Bhat & Bernstein, 2005, Theorem 7.4) Suppose \( f(x) = f_1(x) + \ldots + f_k(x) \), where \( f(0) = 0 \) and for each \( i = 1, \ldots, k \), the vector field \( f_i(x) \) is continuous, homogeneous of degree \( d_i \) with respect to the standard dilation and \( d_1 < \cdots < d_k \). If
the origin is a finite-time-stable equilibrium under \( f_1(x) \), then the origin is a finite-time-stable equilibrium under \( f(x) \).

The results in [Bhat and Bernstein (2005)] required continuous vector fields. This restriction was eliminated in [Levant (2005); Orlov (2004)] and extended for the finite-time stability analysis of switched systems and differential inclusions. Thus, Theorems 8 and 9 hold even if \( f(x) \) is not continuous.

**Theorem 10.** ([Andrieu et al., 2008; Polyakov et al., 2016]) Let the vector field \( f : \mathbb{R}^n \to \mathbb{R}^n \) be homogeneous in the 0-limit with degree \( d_0 < 0 \) and homogeneous in the \( +\infty \)-limit with degree \( d_\infty > 0 \). If for the dynamic systems \( \dot{x} = -f(x) \), \( \dot{x} = -f_0(x) \) and \( \dot{x} = -f_\infty(x) \) the origin is globally asymptotically stable (where \( f_0 \) and \( f_\infty \) are obtained from (3) with \( \lambda_0 = 0 \) and \( \lambda_0 = +\infty \), respectively), then the origin of \( \dot{x} = -f(x) \) is a globally fixed-time stable equilibrium.

The following lemma introduces vector fields that guarantee finite-time and fixed-time stability.

**Lemma 11.** The origin of the nonlinear system (1) is globally

- **finite-time stable if**
  \[
  f(x) = k \text{sign}(x), \quad \text{with } k > 0, \tag{4}
  \]

- **finite-time stable if**
  \[
  f(x) = k |x|^{\alpha}, \quad \text{with } k > 0 \text{ and } \alpha \in (0, 1), \tag{5}
  \]

- **fixed-time stable if**
  \[
  f(x) = k_1 |x|^p + k_2 |x|^q, \quad \text{with } q > 1 > p \geq 0, \quad k_1, k_2 > 0, \tag{6}
  \]

where \( |x|^\alpha = |x|^\alpha \text{sign}(x) \) and

\[
\text{sign}(x) = \begin{cases} 
1 & \text{if } x > 0 \\
0 & \text{if } x = 0 \\
-1 & \text{if } x < 0.
\end{cases}
\]

**2.3. On finite-time and fixed-time consensus for first-order agents**

**Definition 12.** A switched dynamic network (or simply dynamic network) \( \mathcal{X}_{\sigma(t)} \) is described by the tuple \( \mathcal{X}_{\sigma(t)} = (\mathcal{F}, \sigma) \) where \( \mathcal{F} = \{\mathcal{X}_1, \ldots, \mathcal{X}_m\} \) is a collection of undirected graphs having the same vertex set \( \mathcal{V}(\mathcal{X}_{\sigma(t)}) \) and \( \sigma : [t_0, \infty) \to \{1, \ldots, m\} \) is a switching signal that determines the topology of the dynamic network at each instant of time, i.e. \( \mathcal{X}_{\sigma(t)} = \mathcal{X}_i \) when \( \sigma(t) = i \).

Furthermore, each vertex \( i \in \mathcal{V}(\mathcal{X}_{\sigma(t)}) \) is associated to an agent \( i \), with a first-order integrator dynamics:

\[
\dot{x}_i(t) = u_i(t) \tag{7}
\]
where \( x_i(t) \in \mathbb{R} \) is the state of the \( i \)-th agent and \( u_i(t) \) is its consensus protocol. The network is said to achieve consensus if the evolution of the network converges to \( x_1 = \cdots = x_n \).

At a given time \( t \geq 0 \), an agent \( i \) has access to the state of its in-neighbors agents \( \mathcal{N}_i^-(x_{\sigma(t)}) \).

In this paper, we assume that \( \sigma(t) \) is exogenously generated and that there is a minimum dwell time \( \tau_{\min} \) between consecutive switchings in such a way that Zeno behavior in the network’s dynamic is excluded, i.e., there is a finite number of switchings in any finite time interval.

Consider a continuous nonlinear function \( f : \mathbb{R} \rightarrow \mathbb{R} \), with \( f(0) = 0 \), such that the origin of \( \dot{x} = -f(x(t)) \) is globally asymptotically stable. Then, two directions (approaches) can be taken to derive nonlinear consensus algorithms, where the protocol for each node \( i \) is based on the states of the nodes in its in-neighbor set \( \mathcal{N}_i^-(x_{\sigma(t)}) \).

On the one hand, an approach is defined by applying the function \( f(x) \) on the error described by each neighboring node, i.e.,

\[
 u_i(t) = \sum_{j \in \mathcal{N}_i^-(x_{\sigma(t)})} a_{ij} f(e_{ij}), \quad e_{ij} = x_j(t) - x_i(t). \tag{8}
\]

On the other hand, another approach is defined by applying the \( f(x) \) on the sum of the errors of all neighboring nodes, i.e.,

\[
 u_i(t) = f(e_i), \quad e_i = \sum_{j \in \mathcal{N}_i^-(x_{\sigma(t)})} a_{ij} e_{ij} = \sum_{j \in \mathcal{N}_i^-(x_{\sigma(t)})} a_{ij}(x_j(t) - x_i(t)). \tag{9}
\]

**Definition 13.** Approaches \([8]\) and \([9]\) are said to be direction \([8]\) and direction \([9]\), respectively.

Based on standard feedback controllers presented in Lemma \([11]\) and considering the directions \([8]\) and \([9]\) in the sense of Definition \([13]\), Table \([1]\) provides particular consensus algorithms of the form \( f(x) = g_r(x), \; r = 1 \ldots 4 \).

| \( f(x) \)               | Direction \([8]\) | Direction \([9]\) |
|-------------------------|------------------|------------------|
| \( g_1(x) = kx \)       | \( u_i = k \sum_{j \in \mathcal{N}_i^-(x_{\sigma(t)})} a_{ij}e_{ij} \) \tag{10} | \( u_i = ke_i \) |
| \( g_2(x) = k \text{sign}(x) \) | \( u_i = k \sum_{j \in \mathcal{N}_i^-(x_{\sigma(t)})} a_{ij} \text{sign}(e_{ij}) \) \tag{11} | \( u_i = k \text{sign}(e_i) \) \tag{12} |
| \( g_3(x, a) = k|x|^a \) | \( u_i = k \sum_{j \in \mathcal{N}_i^-(x_{\sigma(t)})} a_{ij} |e_{ij}|^a \) \tag{13} | \( u_i = k|e_i|^a \) \tag{14} |
| \( g_4(x, p, q) = k_1|x|^p + k_2|x|^q \) | \( u_i = \sum_{j \in \mathcal{N}_i^-(x_{\sigma(t)})} a_{ij} (k_1|e_{ij}|^p + k_2|e_{ij}|^q) \) \tag{15} | \( u_i = k_1|e_i|^p + k_2|e_i|^q \) \tag{16} |

**Table 1.** Some examples of consensus algorithms derived following direction \([8]\) and direction \([9]\).

**Remark 14.** The basic stabilizing functions given in Table \([7]\) can be combined to generate consensus algorithms following either direction \([5]\) or direction \([9]\), by taking \( f(x) = l_1(x)g_1(x) + l_2(x)g_2(x) + l_3(x)g_3(x, a) + l_4(x)g_4(x, p, q) \) where \( l_r(x), \; r = 1 \ldots 4 \) are nonnegative piecewise constant functions (when \( l_r(x) \) is constant we simply write \( l_r \) not all zero at the same time. In this paper, our focus is on protocols obtained from \( f(x) \) following direction \([9]\); and we will derive conditions on \( f(x) \) such that a finite-time or a fixed-time consensus algorithm for dynamic networks is obtained.

In the following, some common consensus protocols proposed in the literature will
be presented as particular cases of the combination given in Remark 14.

The standard consensus algorithm (10) proposed in Olfati-Saber et al. (2007) is derived from \( f(x) = g_1(x) = kx \), since, for this case, \( f(\cdot) \) is a linear function then direction (8) and direction (9) are equivalent and its convergence is asymptotic.

Regarding finite consensus following direction (8), in G. Chen, Lewis, and Xie (2011); Hui et al. (2010); Sayyaadi and Doostmohammadian (2011) the discontinuous consensus algorithm in (11) was shown to achieve finite-time convergence, while Franceschelli et al. (2013) showed that consensus is also achieved in the presence of disturbances and dynamic networks switching among connected topologies. The protocol (13) was shown in Hui et al. (2008); Xiao, Wang, and Chen (2009) to be a finite-time consensus algorithm for static networks, while L. Wang and Xiao (2010) showed that it provides finite-time convergence for dynamic networks switching among connected topologies. In X. Liu, Lam, Yu, and Chen (2016) a protocol switching between (11) and (13) was proposed, exhibiting finite-time convergence, whereas in Cao and Ren (2014) a finite-time consensus algorithm was obtained by switching between (10) and (13). In X. Wang et al. (2018) finite-time consensus for switching dynamic networks was obtained from \( f(x) = l_1g_3(e_{ij}, \alpha) + l_2g_1(x) \).

Regarding fixed-time consensus following direction (8). In Parsegov et al. (2013); Zuo and Tie (2014), the protocol (15) was shown to be a fixed-time consensus algorithm for static networks, later, fixed-time convergence in networks switching among connected topologies was demonstrated in Zuo et al. (2014). In Hong, Yu, Wen, and Yu (2017) different consensus protocols for static networks were proposed derived from \( f(x) = l_1g_2(x) + l_2g_3(x, \alpha) + l_3g_4(x, p, q) \) with \( l_1, l_2, l_3 \geq 0 \) and \( l_2, l_3 \) not both zero. In Sharghi, Baradaraninia, and Hashemzadeh (2016) a fixed-time consensus based on \( f(x) = l_1g_1(x) + l_2g_4(x, p, q) \) with \( l_1, l_2 > 0 \), was proposed for the leader-follower consensus problem in static networks.

Deriving finite and fixed-time consensus following direction (9) has been less explored and, as shown below, its analysis has been mainly focused on static networks. In Cortés (2006) it was shown that (12) is a finite-time consensus for static networks, while Franceschelli, Pisano, Gina, and Usai (2015) showed that finite-convergence is maintained in the presence of disturbances and under dynamic networks switching among connected topologies. Furthermore, in C. Li and Qu (2014); B. Liu et al. (2015) it was shown that (12) is still a consensus algorithm even when switching among disconnected topologies. Although (14) and (16) have been shown to achieve finite-time and fixed-time convergence in Gómez-Gutiérrez, Ruiz-León, Celikovsky, and Sánchez-Torres (2018); Shang (2012); L. Wang and Xiao (2010); Xiao et al. (2009) and Zuo et al. (2014), respectively, the results of these papers are restricted to static connected networks. In Defoort, Polyakov, Demesure, Djemai, and Veluvolu (2015) a fixed-time consensus for the leader-follower consensus problem was presented for static networks.

Recently, in Ning, Jin, and Zheng (2017); Ning et al. (2018) a discontinuous consensus algorithm for static networks was proposed showing that if the protocol is the sum of the linear protocol in Olfati-Saber et al. (2007) and (12) finite-time consensus is obtained; whereas if the protocol is the sum of (16) and (12) fixed-time consensus is obtained.

A comparison among the different papers addressing the finite-time and the fixed-time consensus problem following direction (9) is summarized in Table 2. As it can be noted, for papers based on \( g_3(x, \alpha) \) and \( g_4(x, p, q) \) no formal proofs have been presented in the literature to show that these methods can be applied for networks switching among connected graphs nor for networks forming a jointly connected graph, the main reason is that their analysis is based on Lyapunov functions candidates that are graph
dependent. Thus, the argument of a common Lyapunov function cannot be made in their case to show convergence in switched dynamic networks.

| Reference                      | $f(x)$, $l_1, l_2, l_3 > 0$ | Network Type | Convergence |
|-------------------------------|-----------------------------|--------------|-------------|
| Cortes (2006)                 | $g_2(x)$                    | Static       | finite-time |
| Xiao et al. (2009)            | $g_3(x,\alpha)$             | Static       | finite-time |
| L. Wang and Xiao (2010)       | $g_3(x,\alpha)$             | Static       | finite-time |
| Shang (2012)                  | $g_3(x,\alpha)$             | Static       | finite-time |
| C. Li and Qu (2014)           | $g_2(x)$                    | JC           | finite-time |
| Zuo et al. (2014)             | $g_4(x, p, q)$              | Static       | fixed-time |
| Franceschelli et al. (2015)   | $g_2(x)$                    | SC           | finite-time |
| B. Liu et al. (2015)          | $g_2(x)$                    | JC           | finite-time |
| Defoort et al. (2015)         | $l_1 g_1(x) + l_2 g_4(x, 2, 0)$ | Static   | fixed-time |
| Tu, Yu, and Xia (2017)        | $g_3(x,\alpha)$             | Static       | finite-time |
| Shang and Ye (2017)           | $l_1 g_2(x) + l_2 g_4(x, p, q)$ | Static   | fixed-time |
| Ning et al. (2017)            | $l_2 g_2(x) + l_4 g_4(x)$   | Static       | finite-time |
| Ning et al. (2018)            | $l_1 g_1(x) + l_2 g_2(x)$   | Static       | finite-time |
| Ning et al. (2018)            | $l_1 g_1(x) + l_2 g_2(x) + l_3 g_4(x, p, q)$ | Static   | fixed-time |

Table 2. Comparison of papers presenting finite-time and fixed-time consensus algorithms following direction (9). Here SC stands for networks switching among connected topologies and JC stands for networks forming a jointly connected graph and the functions $g_i(\bullet)$, $i = 1, \ldots, 4$ are defined in Table 1.

Remark 15. The aim of this paper is to analyze finite-time and fixed-time consensus algorithms derived following direction (9) to show, by using nonsmooth stability analysis, that finite-time and fixed-time consensus is achieved also in dynamic networks. We analyze dynamic networks switching among connected topologies as well as dynamic networks composed of disconnected topologies but forming a connected graph in a “joint sense”.

The considered class includes asymptotic, finite-time and fixed-time convergent protocols. Notice that, if the initial conditions are known to belong to a bounded set, fixed-time algorithms may not represent an advantage over finite-time or asymptotic algorithms, because the gain of the latter ones may be selected to provide a desired settling time for any initial condition in the set. On the other hand, under the same topology and initial conditions, fixed-time protocols may require more energy than finite-time protocols to achieve consensus at the same convergence time (which can be seen in the benchmark herein presented), similarly, finite-time protocols may require more energy than asymptotic protocols. Of course, fixed-time consensus algorithms have a great advantage when the initial conditions are unknown and unbounded, because they guarantee the existence of a bound for the convergence time.

3. Main Result

If a consensus algorithm based on direction (9) is applied to a dynamic network, then the closed-loop behavior can be compactly represented using a vectorial notation. For this, let $x = [x_1, \ldots, x_n]^T$ be the state vector of the agents of the dynamic network. Let $e_i = \sum_{j \in \mathcal{N}_i} a_{ij} (x_j - x_i)$ be the consensus error at node $i$ and let $e = [e_1, \ldots, e_n]^T$ be the consensus error vector. Then, it can be shown that $e = -Q(x_{\sigma(t)}^*) x$, thus the
network’s behavior is
\[
\dot{x} = -F(Q(\mathcal{X}_{\sigma(t)})x) = F(e) \quad \text{where} \quad F(e) = \begin{bmatrix} f(e_1) \\ \vdots \\ f(e_n) \end{bmatrix}.
\] (17)

Note that \(\mathcal{X}_{\sigma(t)}\) is a switched dynamic network and (17) is a switched nonlinear system.

Given a dynamic network, in this section it is proved that consensus is reached by using the direction (9). Namely, taking
\[
\dot{x}_i = u_i, \quad u_i = f(e_i), \quad e_i = \sum_{j \in \mathcal{N}_i^{-}(\mathcal{X}_{\sigma(t)})} a_{ij}(x_j(t) - x_i(t))
\] (18)

with \(f : \mathbb{R} \to \mathbb{R}\) such that \(f(0) = 0\) and the origin is a globally asymptotically stable equilibrium of the system (1). Moreover, convergence is guaranteed not only for static or dynamic networks switching among connected graphs, but also for dynamic networks switching among disconnected graphs, provided that \(\exists \tau < \infty\) such that for any time interval \([\bar{t}, \bar{t} + \tau]\) the graph \(\bar{\mathcal{X}}\) with vertex set \(\mathcal{V}(\bar{\mathcal{X}}) = \mathcal{V}(\mathcal{X}_{\sigma(t)})\) and edge set \(\mathcal{E}(\bar{\mathcal{X}}) = \mathcal{E}(\mathcal{X}_{\sigma(t_1)}) \cup \cdots \cup \mathcal{E}(\mathcal{X}_{\sigma(t_k)})\) is connected, where \(t_1, \ldots, t_k\) are the successive switching times in the time interval \([\bar{t}, \bar{t} + \tau]\). Furthermore, it is shown that if the function \(f(\bullet)\) is such that the origin is a globally finite-time (resp. fixed-time) stable equilibrium of the system (1), and \(f(\bullet)\) satisfies the conditions of Theorem 8 (resp. Theorem 10), then the network’s closed-loop system reaches consensus in finite-time (resp. fixed-time).

**Remark 16.** Consensus algorithms obtained by using direction (9) are computationally less expensive than previously proposed finite-time consensus algorithms for dynamic networks that use direction (8), particularly L. Wang and Xiao (2010) and Zuo et al. (2014). In detail, direction (9) only requires a single evaluation of the nonlinear function \(f(\bullet)\) for each node, whereas direction (8) requires a number of evaluations of \(f(\bullet)\) for each node equal to the number of its in-neighbors, a number that grows in highly connected topologies.

### 3.1. Consensus over static networks

Assuming that the communication topology is static and connected, the asymptotic convergence to the consensus state of the standard consensus algorithm is shown in this subsection by using the Lyapunov theory. Afterwards, it is shown, by using homogeneity results (Andrieu et al., 2008; Center & Kawski, 1995; Hermes, 1991; Polyakov et al., 2016; Rosier, 1992), that if \(f(\bullet)\) satisfies the conditions of Theorem 8 (resp. Theorem 10) then the consensus algorithm is finite-time (resp. fixed-time) convergent.

The convergence to the consensus state will be demonstrated by showing that, the non-smooth function

\[
V(x) = \max\{x_1, \ldots, x_n\} - \min\{x_1, \ldots, x_n\},
\] (19)

also introduced in Sayyaadi and Doostmohammadian (2011) and B. Liu et al. (2015)
Moreover, by using a simple Taylor expansion argument it can be proved that there exists $\varepsilon_1^1 > 0$ such that

$$x_j(t) < x_{\max}(t) \quad \forall j \notin \{i_1, \ldots, i_k\}, \quad \forall t \in [t^* - \varepsilon_1^1, t^* + \varepsilon_1^1].$$

Moreover, by using a simple Taylor expansion argument it can be proved that there exists $\varepsilon_1^2 > 0$ such that

$$x_i(t) = \ldots = x_{i_k}(t^*) = x_{\max}(t^*), \quad x_j(t^*) < x_{\max}(t^*) \quad \forall j \notin \{i_1, \ldots, i_k\},$$

$$\frac{d}{dt} x_{i_1}(t^*) \leq \ldots \leq \frac{d}{dt} x_{i_{k-1}}(t^*) < \frac{d}{dt} x_{i_k}(t^*).$$
exists $\varepsilon^2_t > 0$ such that

$$x_{i_k}(t) > x_{i_{k-1}}(t) \geq \ldots \geq x_{i_1}(t), \quad \forall t \in [t^*, t^* + \varepsilon^2_t].$$

As a consequence,

$$x_{\text{max}}(t) = \max\{x_{i_k}(t), x_{i_{k-1}}(t), \ldots, x_{i_1}(t)\}, \quad \forall t \in (t^*, t^* + \varepsilon_{t^*}], \quad \varepsilon_{t^*} := \min\{\varepsilon_{t^*}, \varepsilon_{t^*}^2\},$$

and therefore

$$x_{\text{max}}(t) = x_{i_k}(t) > x_j(t) \forall j \in \{1, 2, \ldots, n\} \setminus \{i_k\}, \quad \forall t \in (t^*, t^* + \varepsilon_{t^*}], \quad \varepsilon_{t^*} := \min\{\varepsilon_{t^*}, \varepsilon_{t^*}^2\},$$

which in turn guarantees that $x_{\text{max}}(t)$ is differentiable $\forall t \in (t^*, t^* + \varepsilon_{t^*}].$

On the other hand, by a Taylor expansion argument, there exists $\varepsilon^3_t > 0$ such that

$$x_{i_1}(t) \geq x_{i_2}(t) \geq \ldots > x_{i_k}(t), \quad \forall t \in [t^* - \varepsilon^3_t, t^*),$$

and thus

$$x_{\text{max}}(t) = x_{i_k}(t) \geq x_{i_j}(t) \forall j \in \{1, 2, \ldots, n\} \setminus \{i_1\}, \quad \forall t \in [t^* - \varepsilon_{t^*}, t^*), \quad \varepsilon_{t^*} := \min\{\varepsilon_{t^*}, \varepsilon_{t^*}^3\},$$

which guarantees that $x_{\text{max}}(t)$ is differentiable $\forall t \in [t^* - \varepsilon_{t^*}, t^*].$ Thus, the claim of the lemma is proved.

**Theorem 20.** Consider a dynamic network $X_{\sigma(t)} = (F, \sigma)$ such that $\sigma(t) = r, \forall t \geq t_0,$ and $X_r$ is a connected graph. Consider a consensus algorithm defined by direction [9] and a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that the origin is a globally asymptotically stable equilibrium of the system $\dot{x} = -f(x).$ Then, the equilibrium of the network’s closed-loop system is globally asymptotically stable.

**Proof.** Notice that [19] is radially unbounded and $V(x) > 0$ if $x \notin \ker Q(X_r),$ where $\ker Q(X_r) = \{x : x_1 = \ldots = x_n\}$ is the set of equilibrium points of the network’s closed-loop system [17], i.e. the consensus states. Notice that $V(x) = 0$ implies $x \in \ker Q(X_r).$

Moreover, [19] is Lipschitz continuous by Lemma [18]. Thus, according to [Bacciotti & Rosier, 2006, Lemma 6.1], $V(x)$ is nonincreasing along the network’s closed-loop behavior [17] if $\dot{V}(x) \leq 0$ for almost every $x \notin \ker Q(X_k),$ which will be demonstrated in the sequel.

Now, according to Lemma [19], $V(x(t))$ is continuously differentiable except on a set of isolated points $\{t^*_1, t^*_2, \ldots\}.$ Let $\{t^*_1, t^*_2, \ldots, k\}$ be the set of points where $V(t)$ is not differentiable, then $\forall t \in (t^*_k, t^*_k + )$, $V(x)$ is differentiable with time derivative, $\dot{V}(x) = (f(x_j) - f(e_k)),$ where $x_j = x_{\text{max}}$ and $x_k = x_{\text{min}}.$ Since $x_j = x_{\text{max}}$ and $e_j = \sum_{i \in X_r} a_{ji}(x_i - x_j),$ then $\text{sign}(e_j) = -1$ and thus $f(e_j) = -|f(e_j)|.$ By using a similar argument, it can be shown that $\text{sign}(e_k) = 1$ with $f(e_k) = |f(e_k)|$ and therefore

$$\dot{V} = -(|f(e_j)| + |f(e_k)|) \leq 0, \quad \forall t \in (t^*_k, t^*_k + ).$$

Next, asymptotic convergence can be proved by using LaSalle’s invariance principle [Khalil & Grizzle, 2002]. To this aim, let $E = \{x \in \mathbb{R}^n \setminus \ker Q(X_r) | \dot{V}(x) = 0\}.$ Let $x(t) \in E$ for a nonzero subinterval $(t^*_k, t^*_k + ) \subseteq (t^*_k, t^*_k + ).$ Thus, according
to (20) $\dot{V}(x) = 0$ implies $f(e_j) = f(e_k) = 0$, which implies $e_j = e_k = 0$
because of the theorem’s conditions on $f(\bullet)$. Now, since $x_j$ is the maximum, $e_j = \sum_{i \in N_j^{-}(X_t)} a_{ji}(x_i - x_j) = 0$ implies $x_j = x_i \forall i \in N_j^{-}(X_t)$ and $\forall t \in (t_{\delta}, t^*_{\delta+1})$.
Furthermore, $e_i = 0$ for all $i \in N_j^{-}(X_t)$ (otherwise, if $e_i > 0$ then $f(e_i) > 0$ and thus $x_i(t) < 0$, which implies $x_i(t - \delta) > x_j(t - \delta)$ for a small enough $\delta > 0$ with $t - \delta \in (t_{\delta}, t^*_{\delta+1})$, i.e. a contradiction; by an analogous reason $e_i < 0$ cannot occur), which implies $x_j = x_i = x_t, \forall i \in N_j^{-}(X_t), \forall t \in N_i^{-}(X_t)$. By iterating this reasoning it can be concluded that $x_j = x_p$ for any node $p$ such that there exists a path from $p$ to $j$. In particular, since $X_t$ is connected, there exists a path from $k$ to $j$, hence $\max(x_1, \ldots, x_n) = x_j = x_k = \min(x_1, \ldots, x_n)$, which clearly implies that $x_1 = \ldots = x_n$. Thus, the equality holding in (20) for a nonzero interval implies that consensus is achieved. Since $V(x(t))$ is absolutely continuous along the closed-loop trajectory (17) and according to Lemma 19, it is differentiable almost everywhere, then by (20) and LaSalle’s invariance principle, $V < 0$ for almost every $t$ such that $V(t) \neq 0$, therefore $V(t)$ is decreasing excepting at the consensus states, i.e. the closed-loop system asymptotically converges to the consensus state.

In the following theorem, additional conditions are given for finite-time and fixed-time convergence of the consensus protocols.

**Theorem 21.** Consider a dynamic network $X_{\sigma(t)} = (F, \sigma)$ such that $\sigma(t) = r, \forall t \geq t_0,$ and $X_r$ is a connected graph. Consider a consensus algorithm defined by direction (9) and a continuous function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that the origin is a globally asymptotically stable equilibrium of the system $\dot{x} = -f(x)$. Then, the consensus algorithm

1. is a finite-time consensus algorithm if the vector field $f(x)$ can be written as $f(x) = f_1(x) + \cdots + f_k(x)$ and for each $i = 1, \ldots, k$, the vector field $f_i(x)$ is homogeneous of degree $d_i$ with respect to the standard dilation and $d_1 < \cdots < d_k$ with $d_k < 0$.
2. is a finite-time consensus algorithm if $f(x)$ is a piecewise function such that there exist a nonzero constant $b$ where $\forall x \in \{x : |x| < b\}, f(x)$ satisfies condition (7).
3. is a fixed-time consensus algorithm if the vector field $f(x)$ is homogeneous in the 0-limit with degree $d_0 < 0$, homogeneous in the $+\infty$-limit with degree $d_\infty > 0$ and the origin is a globally asymptotically stable equilibrium of the dynamic systems $\dot{x} = -f_0(x)$ and $\dot{x} = -f_\infty(x)$ (where $f_0$ and $f_\infty$ are obtained from (5) with $\lambda_0 = 0$ and $\lambda_0 = +\infty$, respectively).

**Proof.** Theorem 20 states that the closed-loop behavior (17) converges asymptotically to its equilibrium. Thus, based on Theorem 8 and Theorem 9, statement 1 holds since the condition in statement 1 implies that the vector field $F(e) e = -Q(X_t)x$, can be written as $F(e) = F_1(e) + \cdots + F_k(e)$ such that for each $i = 1, \ldots, k$, $F_i(e)$ is homogeneous of degree $d_i$ with respect to the standard dilation, where $d_1 < 0$ is the smallest degree. This property holds given that if $f_i(x)$ is homogeneous of degree $d_i$ then $F(-Q(X_t)x) = \lambda^{d_{i+1}}F_i(-Q(X_t)x)$ for all $\lambda > 0$. To show that statement 1 holds, notice that asymptotic convergence to the consensus state implies that after a finite-time the trajectory $x(t)$ will belong to the nonempty set $\{x : |Q(X_t)x|_{\infty} < b\}$ from which the conditions of item 2 are satisfied, thus achieving finite-time convergence to a consensus state where $Q(X_t)x = 0$.

Now, let us demonstrate the statement 3. Consider the closed-loop behavior (17) and a parameter $d_\lambda \in \mathbb{R}$. Next, by considering $F_{\lambda}(\bullet)$ and $f_{\lambda}(\bullet)$ as defined in (3), it
follows

\[
F_{\lambda_0}(e) = \lim_{\lambda \to \lambda_0} \lambda^{-(d_{\lambda_0}+1)} F(e) = \begin{bmatrix} -\lim_{\lambda \to \lambda_0} \lambda^{-(d_{\lambda_0}+1)} f(e_1) \\ \vdots \\ -\lim_{\lambda \to \lambda_0} \lambda^{-(d_{\lambda_0}+1)} f(e_n) \end{bmatrix} = \begin{bmatrix} -f_{\lambda_0}(e_1) \\ \vdots \\ -f_{\lambda_0}(e_n) \end{bmatrix}.
\] (21)

On the other hand, by Definition 7, the condition of statement 3 implies that \( f_{\lambda_0}(e_i) \) is homogeneous with respect to the standard dilation for \( \lambda_0 = 0 \) with degree \( d_0 < 0 \) and for \( \lambda_0 = \infty \) with degree \( d_\infty > 0 \). Thus, by (21), \( F_{\lambda_0}(e) \) is homogeneous with respect to the standard dilation for \( \lambda_0 = 0 \) with degree \( d_0 < 0 \) and for \( \lambda_0 = \infty \) with degree \( d_\infty > 0 \), which implies that \( F(e) \) is homogeneous in the \( 0^- \)-limit with degree \( d_0 < 0 \) and in the \( +\infty^- \)-limit with degree \( d_\infty > 0 \), in accordance to Definition 7. Moreover, according to Theorem 20, if the origin is a globally asymptotically stable equilibrium of \( \dot{x} = -f_{\lambda_0}(x) \) then the network’s closed-loop system \( \dot{x} = -F_{\lambda_0}(Q(X_k)x) \) converges to a globally asymptotically stable equilibrium. Then, it follows from Theorem 10 that the equilibrium of the closed-loop system is globally fixed-time stable.

The following corollary states, based on Theorem 21, that protocol (14) is finite-time convergent and protocol (16) is fixed-time convergent. These are particular protocols of the analyzed class, but more finite-time and fixed-time protocols can be derived.

Corollary 22. Let \( \sigma(t) = r, \forall t \geq t_0 \), and let \( \mathcal{X}_r \) be a connected graph and let \( g_1(x) = kx, g_2(x) = k \text{sign}(x), g_3(x, \alpha) = k|x|^\alpha \) and \( g_4(x, p, q) = k_1|x|^p + k_2|x|^q \).

1. If a consensus protocol \( u_i \) is obtained from \( f(x) = l_1g_1(x) + l_2g_2(x, \alpha) \) where \( \alpha \in (0, 1), l_1 \geq 0 \) and \( l_2 > 0 \), following direction [9], i.e. \( u_i = f(e_i) \), then \( u_i \) is a continuous consensus algorithm with finite-time convergence.

2. If a consensus protocol \( u_i \) is obtained from \( f(x) = l_1g_1(x) + l_2g_2(x) + l_3g_3(x, \alpha) \) where \( \alpha \in (0, 1), l_1, l_3 \geq 0 \) and \( k_2 > 0 \), following direction [9], i.e. \( u_i = f(e_i) \), then \( u_i \) is a discontinuous consensus algorithm with finite-time convergence.

3. If a consensus protocol \( u_i \) is obtained from \( f(x) = l_1g_1(x) + l_2g_2(x, \alpha) + l_3g_3(x, p, q), l_1, l_3 \geq 0, l_2, q > \alpha > p, q > 1 > p > 0 \), following direction [9], i.e. \( u_i = f(e_i) \), then \( u_i \) is a continuous consensus algorithm with fixed-time convergence.

4. If a consensus protocol \( u_i \) is obtained from \( f(x) = l_1g_1(x) + l_2g_2(x) + l_3g_3(x, \alpha) + l_4g_4(x, p, q), l_1, l_3 \geq 0, l_2, l_4 > 0, q > \alpha > p \) and \( q > 1 \), following direction [9], i.e. \( u_i = f(e_i) \), then \( u_i \) is a discontinuous consensus algorithm with fixed-time convergence.

Proof. Notice that, with respect to the standard dilation, \( g_1(x) \) is homogeneous of degree \( d_1 = 0 \), \( g_2(x) \) is homogeneous of degree \( d_2 = -1 \), \( g_3(x, \alpha) \) is homogeneous of degree \( \alpha - 1 \). Thus, for statement 1 \( f(x) \) can be written as the sum of two homogeneous functions where the smallest degree is \( \alpha - 1 < 0 \). Thus, the proof for statement 1 follows from Theorem 21. The same argument applies for statement 2 but since \( k_2 > 0 \) the smallest degree is \( d_2 = -1 \) from \( g_2(x) \).

To prove the statement 3 it is easy to verify that \( f_0(x) \) defined as

\[
f_0(x) = \lim_{\lambda \to 0} \lambda^{-(d_0+1)} f(\lambda x) = l_4k_1 [e_i]^p
\] (22)

is homogeneous of degree \( p - 1 < 0 \) with respect to the standard dilation. Thus, \( f(\bullet) \)
is homogeneous in the $0−$limit with degree $d_0 = p − 1 < 0$. In a similar way

$$f_∞(x) = \lim_{\lambda \to +\infty} \lambda^{-(d_∞+1)} f(\lambda x) = l_4 k_2 |x|^q$$

(23)

is homogeneous of degree $q − 1 > 1$ with respect to the standard dilation. Thus, the vector field \[ \text{(17)} \]
is homogeneous in the $+\infty−$limit with degree $d_∞ = q−1$. Furthermore, the origin of $\dot{x} = −l_4 k_1 |x|^p$ and $\dot{x} = −l_4 k_2 |x|^q$ is a globally asymptotically stable equilibrium.

Thus, according to Theorem \[ \text{21} \] statement \[ \text{3} \] holds. Statement \[ \text{4} \] follows from a similar argument by noticing that if $k_2 > 0$ then $f(x)$ is homogeneous in the $+\infty−$limit with degree $d_∞ = q−1 > 1$ and homogeneous in the $0−$limit with degree $d_∞ = −1$. \Box

Remark 23. The use of homogeneity theory for finite-time and fixed-time convergence analysis does not provide a bound for the convergence-time. However, this approach will allow to demonstrate that protocols derived by following direction \[ \text{(9)} \] (for instance derived from $f(x)$ in Table \[ \text{3} \]) achieve finite/fixed-time convergence even under dynamic networks.

3.2. Consensus over dynamic networks switching among connected topologies

In the proof of Theorem \[ \text{20} \] it was shown that the function \[ \text{(19)} \] is a Lyapunov function, valid for any given connected topology. On the other hand, the stability theory for switching systems (Liberzon 2003) states that a switching system, composed of a collection of nonlinear systems and an arbitrary switching signal determining the currently evolving nonlinear system, is asymptotically stable if there exists a Lyapunov function valid for all the nonlinear systems in the collection. In this way, \[ \text{(19)} \] is a common Lyapunov function for a dynamic network under arbitrary switching, provided the communication topology is always connected, and thus it can be proved that the consensus state is a globally asymptotic equilibrium of the dynamic network. This is formally stated in the following theorem.

Theorem 24. Consider a dynamic network $\mathcal{X}_{\sigma(t)} = (\mathcal{F}, \sigma)$ such that $\sigma(t) \in \{1, \ldots, m\}$ and $\forall r \in \{1, \ldots, m\}$ the graph $\mathcal{X}_r$ is connected. Consider a consensus algorithm defined by direction \[ \text{(9)} \] and a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that the origin is a globally asymptotically stable equilibrium of the system $\dot{x} = −f(x)$. Then, the consensus state is a globally asymptotically stable equilibrium of the network’s closed-loop system under an arbitrary switching signal $\sigma(t)$. Moreover, the consensus algorithm

(1) is a finite-time consensus algorithm if the vector field $f(x)$ can be written as $f(x) = f_1(x) + \ldots + f_k(x)$ and for each $i = 1, \ldots, k$, the vector field $f_i(x)$ is homogeneous of degree $d_i$ with respect to the standard dilation and $d_1 < \cdots < d_k$ with $d_1 < 0$.

(2) is a finite-time consensus algorithm if $f(x)$ is a piecewise function such that there exist a nonzero constant $b$ where $\forall x \in \{x : |x| < b\}$, $f(x)$ satisfies condition \[ \text{(7)} \].

(3) is a fixed-time consensus algorithm if the vector field $f(x)$ is homogeneous in the $0−$limit with degree $d_0 < 0$, homogeneous in the $+\infty−$limit with degree $d_∞ > 0$ and the origin is a globally asymptotically stable equilibrium of the dynamic systems $\dot{x} = −f_0(x)$ and $\dot{x} = −f_∞(x)$ (where $f_0$ and $f_∞$ are obtained
Figure 1. Convergence of the consensus algorithm for Example 26 with \( k = 1 \). from (3) with \( \lambda_0 = 0 \) and \( \lambda_0 = +\infty \), respectively).

Proof. According to Theorem 20, the Lyapunov function (19) asymptotically converges to zero regardless of the current connected topology \( X_r \), i.e. \( V(x) \) defined as in (19) is a common Lyapunov function. Thus, by (Liberzon, 2003, Theorem 2.1), the equilibrium of the network’s closed-loop system is globally asymptotically stable under arbitrary switching of the communication topology. Moreover, since the graph \( X_\sigma \) is connected, \( Q(X_\sigma(t))x = 0 \) implies that \( x \in \ker Q(X_\sigma(t)) \), i.e. \( x_1 = \ldots = x_n \) and consensus is achieved.

The proof for finite/fixed-time stability follows the same argument as in Theorem 21, i.e., by using homogeneity and Theorem 8 and Theorem 10 for finite-time convergence and fixed-time convergence, respectively.

Remark 25. Notice that, for the case of switching among connected graphs, the convergence of the consensus algorithm (12), obtained from (4) following direction (9), is independent of the network topology, because if \( x_j = x_{\text{max}} \) and \( x_k = x_{\text{min}} \) then \( \dot{V} = -2k \) with \( V(x) \) as in (19), regardless of the network topology or the number of nodes. However, this steady convergence rate is not obtained for the consensus algorithm (11) because a neighbor \( x_i \) of \( x_j = x_{\text{max}} \) (resp. \( x_k = x_{\text{min}} \)) may satisfy \( \text{sign}(e_i) = \text{sign}(e_j) \). Thus, \( \dot{V} \) will have different values that depend on the topology and the state of the neighbors.

Example 26. Consider a network composed of 10 vertices and two different graphs, \( X_0 \) and \( X_1 \). Let \( X_1 \) be such that the i-th vertex is adjacent to the \( j = (i + 1)(\text{mod } 10) \) vertex, where \( x(\text{mod } 10) \) stands for the common residue of \( x \) modulo 10, and let \( X_0 \) be such that the i-th vertex is adjacent to the \( j = (i + 3)(\text{mod } 10) \) vertex. Let \( \sigma(t) \) be the switching signal and let the initial condition be \( x(t_0) = \begin{bmatrix} 0 & -5 & 10 & 3 & -8 & -2 & 5 & 3 & -1 & 4 \end{bmatrix} \). Figure 1 shows the convergence of the finite-time consensus algorithm (14) in Table 1, obtained from (5) following direction (9), under the graph topology \( X_{\sigma(t)} \) and switching signal \( \sigma(t) \).
3.3. Consensus over dynamic networks switching among disconnected topologies

Theorem 24 guarantees consensus along the network under arbitrary switching. A particular case occurs when \( \sigma(t) = i, \forall t \in [0, \infty) \), i.e. the system remains in the same topology without switching. Thus, a necessary condition for consensus under arbitrary switching signal is that each possible topology is connected. Otherwise, each connected component could reach a different consensus since there will not be communication among components. This connectivity condition for each network topology can be relaxed by requiring a connected graph in a “joint sense”. This is formalized in the following.

Definition 27. Let \( X_{\sigma(t)} = (F, \sigma) \) be a dynamic network with \( \sigma(t) \in \{1, \ldots, m\} \). The switching signal \( \sigma(t) \) is said to generate a \( \tau \)-jointly connected graph if there exists \( \tau < \infty \) such that for all \( i \geq 0 \), the graph \( \mathcal{X} \) with vertex set \( V(\mathcal{X}) = V(X_{\sigma(t)}) \) and edge set \( \mathcal{E}(\mathcal{X}) = \mathcal{E}(X_{\sigma(t)}) \cup \cdots \cup \mathcal{E}(X_{\sigma(t_{k})}) \) is connected, where \( t_{i}, \ldots, t_{k} \) are the successive switching times in the time interval \([t, t + \tau]\).

Theorem 28. Let \( X_{\sigma(t)} = (F, \sigma) \) be a dynamic network such that the switching signal \( \sigma(t) \) generates a \( \tau \)-jointly connected graph.

Consider a consensus algorithm defined by direction (9) and a continuous function \( f : \mathbb{R} \rightarrow \mathbb{R} \) such that the origin is a globally asymptotically stable equilibrium of the system \( \dot{x} = -f(x) \). Then, the consensus state is a globally asymptotically stable equilibrium of the consensus evolution (17).

Proof. Similarly as in the proof of Theorem 20, we will show the convergence of \( x \) to a consensus state, under the switched dynamic topology \( X_{\sigma(t)} \), by using the candidate Lyapunov function (19) and showing that \( g(t) = \dot{V}(x(t)) \) along the trajectory of the system converges to zero provided that the switching signal generates a \( \tau \)-jointly connected graph. To this end, notice that if \( X_{k} \) is the current graph topology, not necessarily connected, then according to Lemma 19, \( V(x) \) in (19) is continuously differentiable except on a set of points \( \{t_{1}^{*}, t_{2}^{*}, \ldots\} \).

Thus, the time derivative of \( V(x) \) along the trajectory of (17) in the time interval \((t_{i}^{*}, t_{i+1}^{*})\) is given by

\[
\dot{V} = -(|f(e_{j})| + |f(e_{k})|) \leq 0 \quad \forall t \in (t_{i}^{*}, t_{i+1}^{*}).
\]  

(24)

It was shown in the proof of Theorem 20 that, if the current graph topology is connected, the equality in (24) holds for a nonzero interval only if consensus is achieved, i.e. \( x_{1} = \ldots = x_{n} \). However, if the current graph topology \( X_{k} \) is not connected then the equality can hold, for a nonzero interval, whenever \( \exists x_{j}, x_{k} \) and connected components \( K \) and \( L \) of \( X_{k} \) such that \( x_{j} = \max(x_{1}, \ldots, x_{n}), x_{k} = \min(x_{1}, \ldots, x_{n}) \), \( j \in K, k \in L \) and consensus is achieved along \( K \) and \( L \).

Nonetheless, since \( \sigma(t) \) generates a \( \tau \)-jointly connected graph within any time interval of length \( \tau \), a graph \( X_{k} \) will become active when there exists a node \( \hat{j} \) adjacent to a node \( \hat{i} \) such that \( x_{j} = \max(x_{1}, \ldots, x_{n}) \) and \( x_{j} > x_{i} \) (a similar argument applies for a node \( x_{k} = \min(x_{1}, \ldots, x_{n}) \)). Thus, for each \( x \notin \ker \mathcal{Q}(\mathcal{X}) \) such that \( \dot{V} = 0 \) and every time interval \([t, t + \tau]\) of length \( \tau \) there exists a graph \( X_{k} \), that will become active in \([t, t + \tau]\) such that \( \dot{V}(x) < 0 \). Thus, by LaSalle’s invariance principle Khalil and Grizzle (2002) it follows that \( g(t) = V(x(t)) \) will asymptotically converge to zero.
Example 29. Consider a dynamic network composed of 10 vertices and 10 graphs. Let $X_i$, $i \in \{1, \ldots, 10\}$, be a graph with vertex set $V(X_i) = \{1, \ldots, 10\}$ and edge set $E(X_i) = \{ij, ji\}$ such that $j = i + 1(\mod 10)$. The initial condition is $x(t_0) = [0 \ 5 \ 3 \ 2 \ 4 \ -9 \ 10 \ 5 \ -5 \ -3]$. The evolution of the consensus algorithm on the switched dynamic network $X_{\sigma(t)}$ for two different switching signals $\sigma_1(t) = \lfloor t \rfloor (\mod 10) + 1$ and $\sigma_2(t) = \lfloor 100t \rfloor (\mod 10) + 1$ (where $\lfloor \cdot \rfloor$ denotes the floor function) is shown in Figure 2-a and Figure 2-b, respectively. Notice that the switching signals as defined above generate $\tau$-jointly connected graphs with $\tau = 10$ and thus consensus is achieved. Moreover, notice that $\sigma_2$ has a faster switching frequency than $\sigma_1$, thus the behavior of the network with $\sigma_2$ seems to be smoother.

Corollary 30. Let $X_{\sigma(t)} = \langle F, \sigma \rangle$ be a dynamic network, and $\tau$ a finite number such that within each time interval of length $\tau$, a strongly connected graph is active during a nonzero interval. Consider a consensus algorithm defined by direction (9) and a continuous function $f : \mathbb{R} \to \mathbb{R}$ such that the origin is a globally finite-time (respectively, fixed-time) stable equilibrium of the system $\dot{x} = -f(x)$. Then, the consensus state is a globally finite-time (respectively, fixed-time) stable equilibrium of the consensus evolution (17).

Remark 31. In this paper, the analysis has been focused on the class of protocols that follow direction (9), however, a similar analysis can be performed for the class of protocols that follow direction (8), by using the same candidate Lyapunov function (19).

4. Benchmark: Convergence time vs Graph connectivity

In this section, experiments are performed to evaluate the convergence time of a network's closed-loop system under different consensus algorithms. In particular, it is investigated how the convergence time increases when the graph's algebraic connectivity decreases.
4.1. Description and Motivation

The motivation of this work is to analyze a class of algorithms that may work in a wide range of (possible unanticipated) situations. Imagine for instance a company developing low-power nodes of a sensor network, which must achieve consensus to provide an output sensing value, and whose interest is in enabling users to apply its solution in either small or large networks with minimum additional configurations. For a given topology, and assuming that a bound is known for the initial consensus error (a realistic assumption in sensor consensus), the gains of any consensus protocol can be adjusted to obtain a proper convergence (or settling) time. However, an important desired property of the implemented consensus algorithm is that the convergence time is maintained within an acceptable range, without the need of additional configuration, when the network’s connectivity changes by either the connection or disconnection of sensors. This property is investigated in this benchmark, by comparing the convergence time of different protocols when the algebraic connectivity changes.

In detail, we compare algorithms based on the direction (9) against existing finite-time and fixed-time consensus algorithms for dynamics networks that were designed following direction (8). Generally, the convergence time of a consensus algorithm grows when the algebraic connectivity of the graph decreases, which occurs when the network size increases. However, it will be shown that such increment in the convergence time is slower in nonlinear algorithms based on the direction (9) than in algorithms based on the direction (8). Thus, the analyzed direction (9) can be applied, with the same parameters selection, to graphs with either high or low algebraic connectivity, still achieving consensus in a satisfactory amount of time.

4.2. Methodology

The next methodology was used to benchmark the direction (9) in two experiments that illustrate how the convergence time of each algorithm increases as the algebraic connectivity decreases. To this aim, switched networks are generated in such a way that the algebraic connectivity decreases as the number of nodes increases. For this, circular undirected graphs of \( n \) nodes are defined, which are denoted by \( C_n \), satisfying \( \lambda_2(C_n) = 2 - 2 \cos \left( \frac{2\pi}{n} \right) \) (where \( \lambda_2(\cdot) \) denotes the second eigenvalue of the Laplacian of the argument network).

- In the first experiment, the finite-time consensus algorithms of Table 1 are compared. Namely, the algorithm (13), proposed in L. Wang and Xiao (2010), versus the algorithm (14), which applies the same nonlinear function of (13) but following direction (9).
- In the second experiment, the fixed-time consensus algorithms of Table 1 are compared. Namely, the algorithm (15), proposed in Zuo and Tie (2014), versus the algorithm (16), which applies the same nonlinear function of (15) but following direction (9).
- A dynamic network is considered, described by two undirected graphs of \( n \) nodes, \( \mathcal{X}_0 \) and \( \mathcal{X}_1 \), where \( \mathcal{X}_0 \) is such that \( (i,j) \in \mathcal{E}({\mathcal{X}_0}) \) if and only if \( j - i \equiv \pm 1 \pmod{n} \) and \( \mathcal{X}_1 \) is such that \( (i,j) \in \mathcal{E}({\mathcal{X}_0}) \) if and only if \( j - i \equiv \pm h \pmod{n} \), where \( h = \max \{ h \in \{1, \ldots, [n/2]\} | n \pmod{n/2} \equiv 1 \} \). The switching signal is given by \( \sigma(t) = \lfloor 5t \mod 2 \rfloor \). An example of these graphs for the case of a graph with \( n = 25 \) nodes is illustrated in Figure 3.
- The initial conditions are set equally for the different algorithms using the linear
congruential generator (Brunner & Uhl, 1999),

\[ z_{i+1} = rz_i + s \mod M \]

such that \( z_0 = M \) and \( r = 45, s = 1, M = 1024, l = 20 \) and \( m = 10 \) and \( n \) is the number of nodes in the graph. This iterative procedure produces a pseudo-random sequence of initial conditions \( x_i(t_0) \) in the interval \([-10, 10]\).

- The exponents of the consensus protocols are set equal, \( \alpha = 0.5 \) for the first experiment and \( q = \frac{3}{2}, p = \frac{1}{2} \) for the second experiment. Additionally, the gains are experimentally set (for the second experiment \( k = k_1 = k_2 \)) such that in a network of 25 nodes, both algorithms achieve \( V(x) = 0.05 \) at 1.00 s.
- To measure the control effort of each approach, the Integrated Squared Control Effort (ISCE) of the network is computed as

\[ E_{tot}(t) = \sum_{i=1}^{n} E_i(t), \text{ where } E_i(t) = \left( \int_{t_0}^{t} u_i^2 \right)^\frac{1}{2}. \]

- Experiments are performed varying from 25 to 1000 nodes. The convergence time and the ISCE of each test are compared.
- The simulations are performed in OpenModelica® using Euler’s integration method with interval 0.0001 s.

**4.3. Results**

The results for the first experiment, comparing the finite-time consensus algorithms in Table 1, is presented in Figure 4 a). It is important to highlight that, even if both algorithms achieve \( V = 0.05 \) at time \( t_f = 1 \) for \( n = 25 \), the ISCE of (13) is \( E_{tot}(t_f) = 361.31 \) whereas the ISCE of the proposed method (14) is \( E_{tot}(t_f) = 273.57 \).

The results for the second experiment, comparing the fixed-time consensus algorithms in Table 1 are presented in Figure 4 b). The ISCE of (15) in a network of 25 nodes is \( E_{tot}(t_f) = 616.15 \), while the ISCE of (16) for the same network is \( E_{tot}(t_f) = 588.15 \).

The results of these experiments suggest, first that the ISCE required to achieve
consensus at a given time is lower by following the direction \( (9) \); second, that the convergence time growing with the decreasing of the algebraic connectivity is significantly slower with the algorithms based on the direction \( (9) \) than with the finite/fixed-time algorithms of L. Wang and Xiao (2010); Zuo and Tie (2014) based on the direction \( (8) \). As notice in Table 2, previous results on finite-time and fixed-time consensus algorithms obtained following direction \( (9) \) does not justify the convergence to the consensus state in this example, since those results are restricted to static networks.

5. Conclusions and Future Work

In this work, a class of consensus algorithms for dynamic networks with finite/fixed-time convergence were analyzed by using homogeneity theory and switching stability theory. In particular, it was shown that the analyzed class, identified as direction \( (9) \), in which a nonlinear function of the consensus error is evaluated per each node, achieves finite/fixed-time consensus even if the communication topologies are disconnected. This feature is an essential advantage concerning other finite-time consensus algorithms that require that the sum of the time intervals for which the topology is connected be sufficiently large. Thus, the analyzed class allows the application of finite/fixed-time consensus algorithms with intermittent connections.

Among the advantages of the analyzed consensus algorithms over other previously proposed finite/fixed-time consensus algorithms for dynamic networks, the analyzed algorithms are computationally simpler, use lower control effort to achieve consensus at a given time and have slower growth in the convergence time as the algebraic connectivity decreases.

Future work concerns the analysis of the considered consensus class under noisy measurements as well as the implementation of its discrete version over robotic swarms. Moreover, the extension to high-order agents will be studied.

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