ON THE REPRESENTATION OF A DISCRETE GROUP $\tilde{\Gamma}$ WITH SUBGROUP $\Gamma_0$ IN THE CALKIN ALGEBRA OF $l^2(\tilde{\Gamma}/\Gamma_0)$

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ABSTRACT. Given a countable group $\tilde{\Gamma}$ with an infinite, proper subgroup $\Gamma_0$, we find sufficient conditions, such that the unitary representation of $\tilde{\Gamma}$ in the Calkin algebra of $l^2(\tilde{\Gamma}/\Gamma_0)$ be weakly contained in the left regular representation of $\tilde{\Gamma}$. If $\Gamma$ is a discrete, countable group, and $\tilde{\Gamma} = \Gamma \times \Gamma^{op}$ and $\Gamma_0 = \{ (\gamma, \gamma^{-1}) \mid \gamma \in \Gamma \}$, we recover the Akemann-Ostrand property for $\Gamma = \text{PGL}_2(\mathbb{Z}[\frac{1}{p}])$, and for $\Gamma = \text{SL}_3(\mathbb{Z})$. Consequently, these groups have the AO property ([AO]). This implies, using the solidity property of Ozawa ([Oz]), that, for the corresponding group von Neumann algebras, we have: $L(\text{SL}_3(\mathbb{Z})) \not\cong L(\text{SL}_n(\mathbb{Z})), n \geq 4$.

We study the representations of a discrete group $\tilde{\Gamma}$ with a fixed subgroup $\Gamma_0$ into the Calkin algebra of the Hilbert space $l^2(\tilde{\Gamma}/\Gamma_0)$ of left cosets of $\Gamma_0$ in $\tilde{\Gamma}$. We find sufficient conditions for this representation to be weakly contained in the left regular representation of the group $\tilde{\Gamma}$ on $l^2(\tilde{\Gamma})$.

The main example is the case, when given a discrete group $\Gamma$, the group $\tilde{\Gamma}$ is $\Gamma \times \Gamma^{op}$ (where $\Gamma^{op}$ is the same group as $\Gamma$, but with opposite multiplication).

The subgroup $\Gamma_0$ of $\tilde{\Gamma}$ is in this case $\{ (\gamma, \gamma^{-1}) \mid \gamma \in \Gamma \}$. Clearly, the map $\pi$ from $\tilde{\Gamma}/\Gamma_0$ into $\tilde{\Gamma}$, defined by $\pi((g_1, g_2)) = g_1g_2$, $(g_1, g_2) \in \tilde{\Gamma}$, is a bijection. Moreover $\pi$ is $\tilde{\Gamma}$ equivariant, with respect to the action of $\tilde{\Gamma} = \Gamma \times \Gamma^{op}$ on $\Gamma$, defined by $(\gamma_1, \gamma_2)x = \gamma_1x\gamma_2$ for $(\gamma_1, \gamma_2) \in \tilde{\Gamma}, x \in \Gamma$.

For these groups $\tilde{\Gamma}$, we are finding sufficient conditions such that the unitary representation of $\tilde{\Gamma} = \Gamma \times \Gamma^{op}$ (induced by the action described above)

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into the Calkin algebra of $\ell^2(\Gamma)$, be weakly contained in the left regular representation of $\tilde{\Gamma}$.

This property of the group $\Gamma$ (and $\tilde{\Gamma}$) is designated in the literature ([Oz], [AD]) as the property AO. This property was introduced by Akemann and Ostrand in [AO], where they proved that the above property holds for the free groups. As noted explicitly in ([AD]), the property AO is very close to the property S of N. Ozawa ([Oz]).

The property was proven to hold for much larger class of discrete groups, first by Skandalis [Sk], where it was proven to hold true for lattices of Lie groups of rank 1, and then by Ozawa [Oz] for hyperbolic groups (see also [HG]).

Taking the point of view of the action of $\tilde{\Gamma}$ on $\tilde{\Gamma}/\Gamma_0$, we are able to get a few other examples that weren’t covered so far: $\Gamma = \text{PSL}_2\left(\mathbb{Z}\left[\frac{1}{p}\right]\right)$, with $p$ a prime number, or $\Gamma = \text{SL}_3(\mathbb{Z})$. This two groups do not have the property S of N. Ozawa (see [Sa]).

The most easy method (and to the author knowledge the only one so far) to prove that the AO property does not hold, is to construct an infinite subset of the group, having a common, non-amenable normalizer group. The two groups we are considering have certainly no infinite subsets with non amenable normalizers (except for $\text{SL}(3, \mathbb{Z})$ where an order 2 element has a stabilizer isomorphic to $\text{SL}(2, \mathbb{Z})$, however in this case the corresponding orbit order 2 element is isolated from the identity in a convenient profinite completion topology).

We will also use the explicit structure of the countable set consisting in cosets of stabilizer groups, which modulo finite sets, give a “fundamental domain like” paving of the set $\tilde{\Gamma}/\Gamma_0$. We will also assume that in the “profinite” neighborhood of the identity, all the groups have amenable stabilizers The main tool of our approach is a representation for the essential states constructed in [Ra] and a number of conditions on stabilizers of elements in $\tilde{\Gamma}/\Gamma_0$, that are summarized in the following theorem. The idea of the proof is that cosets of amenable subgroups of the group $\tilde{\Gamma}$, become asymptotically disjoint. On such cosets the left and the right action are asymptotically decoupled.

We summarize bellow, the representation method, for the essential states, developped in ([Ra]).

**Remark.** Let $G$ be a discrete countable group acting transitively on a discrete set $D$ (we use the context from book by N. Brown, N. Ozawa ([BO])).
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Let \( \pi_Q : G \to \mathcal{Q}(\ell^2(G)) \) the representation obtained by composing the Koopman representation (\([\text{Ke}])\), \( \pi_K \) of \( G \) into \( B(\ell^2(G)) \) with the projection into the Calkin algebra \( \mathcal{Q}(\ell^2(G)) \).

The positives states on \( G \), obtained through this representation, are (by the work of Calkin \([\text{Ca}]\)), convex combinations of states obtained as follows (we will call in the sequel states coming from the representation into the Calkin algebra, essential states): let \( \xi = (\xi_n)_n \) be a sequence of unit vectors in \( \ell^2(G) \), where for each \( n \), the vector \( \xi_n \) has finite support \( A_n \subseteq G \), and positive coefficients. We assume that the supports avoid eventually any finite subset of \( D \).

The essential states on \( C^*(G) \) obtained through this method also depend on the choice of a free ultrafilter \( \Omega \) on \( \mathbb{N} \). Then the formula of the states is

\[
\omega_{\xi,\Omega}(g) = \lim_{n \to \Omega} \langle \pi_Q(g)\xi_n, \xi_n \rangle, g \in G.
\]

The state \( \omega_{\xi,\Omega} \) on \( C^*(G) \) is doubly positive, in the sense that it is positive definite and takes only positive values on elements \( G \). Let \( \beta(G) \) be the Stone Cech (Gelfand) spectrum of the abelian \( C^* \)-algebra \( \ell^\infty(G) \) and let \( \partial^\beta(G) = \beta(G) \setminus G \) be its boundary. By definition the state \( \omega_{\xi,\Omega} \) has a canonical extension to crossed product \( C^* \)-algebra \( C^*(G \rtimes \partial^\beta(G)) \), which we denote by \( \tilde{\omega}_{\xi,\Omega} \).

We proved in ([Ra]) that the states \( \omega_{\xi,\Omega} \) are weak limits of convex combinations of states \( \phi = \phi(Y,\nu,F,f) \) of the following form: let \( (Y,\nu) \) be an (infinite) measure space, acted freely, by measure preserving transformation by the group \( G \) and \( F \) a measurable subset of \( Y \) of measure 1, whose translates cover \( Y \). Let \( f \) be a positive function in \( L^2(Y,\nu) \) of norm 1. Then \( \phi(g) = \phi(Y,\nu,F,f)(g) \) is defined to be \( (f\nu f)(gF \cap F) = \int_{Y} [\pi_K(g)(f\chi_F)][f\chi_F]d\nu \), for \( g \in G \). Here \( \pi_K \) is the Koopman representation of \( G \) on \( L^2(Y,\nu) \). (We conjecture that in general any doubly positive state on \( G \) has such a an expression, of a weak limit of convex combinations of states as above).

To prove that the representation \( \pi_Q \) of \( G \) is continuous with respect to the norm on \( C^*_{\text{red}}(G) \), it is thus sufficient to prove that the Koopmann representation of \( C^*(G \rtimes L^\infty(Y,\nu)) \) on \( L^2(Y,\nu) \) is continuous with respect to the \( C^*_{\text{red}}(G \rtimes L^\infty(Y,\nu)) \) norm, for every summand, in the convex combination, of states of the form \( \phi(Y,\nu,F,f) \) that yields the state \( \tilde{\omega}_{\xi,\Omega} \).

To prove this we have first to determine what are the states of the form \( \phi(Y,\nu,F,1) \) that appear in the above convex linear combination. We observe that the measures (states) \( \omega_{\xi,\Omega} \) are (weighted) density measures on \( \beta(G) \) ([Ma],
We recall that the vectors $\xi_n$ have finite supports $A_n$. It is natural working with states that are “dominated” by the support sets $(A_n)n$ to find the measures $\omega_{\xi,\Omega}$ as limits of more simple states (which are not extreme points in the set of finitely additive measures, but rather more elementary states dominated by a fixed density measure).

It is proved in ([Ra]), that by transfinite induction and using the Loeb theory of such measures ([Lo]), we find subsets $A_s^n$ forming for every $n$ in $\mathbb{N}$ a partition of $A_n$, after $s$, with the following properties. Let $(C_\Omega((A^n_s)_n), \nu_{\Omega,C_\Omega((A^n_s)_n)})$ be the Loeb probability space associated to the sets $(A^n_s)_n$, $s \in \mathbb{N}$. In addition the corresponding Loeb measures are mutually singular, and there exist positive square integrable functions $f_s$ in the Hilbert space $L^2(C_\Omega((A^n_s)_n), \nu_{\Omega,C_\Omega((A^n_s)_n)})$, such that the state $\tilde{\omega}_{\xi,\Omega}$ is a limit of convex combinations of states of the form

$$f^s \omega = \frac{1}{\sqrt{\text{card}(A^n_s)}} \chi_{A^n_s} \cdot f^s, \ s \in \mathbb{N}.$$ 

The translations by $G$ of the measure spaces $C_\Omega((A^n_s)_n)$ (identified with subsets of $\partial \beta(G)$, have the property that the measures $\nu_{\Omega,C_\Omega((A^n_s)_n)}$ are equal on overlaps, and hence by ”gluing” together this spaces we get a $\Gamma$ invariant (infinite) measure $\nu$ on $\mathcal{Y} = \bigcup_{g \in G} gC_\Omega((A^n_s)_n)$, and a measure 1 subset $F$ equal to $C((A^n_s)_n)$, who’s displacement by $G$ calculates the state $\omega_{\xi,\Omega}$ on $G$.

We note that discarding the essential states corresponding to cosets as above, the essential states on $\tilde{\Gamma}$ are realized in infinite measure spaces, acted freely, by measure preserving transformations, by the group $\tilde{\Gamma}$. The corresponding states are obtained by measuring the displacement, of finite measure subsets, by the action of the group $\tilde{\Gamma}$. We describe this construction in the following theorem:

**Theorem 1.** Let $\tilde{\Gamma}$ be a discrete group which is also exact (see e.g. [Ki]) and let $\Gamma_0$ an infinite subgroup with the following properties:

0) The normalizer $N_{\tilde{\Gamma}}(\Gamma_0) = \{ \gamma \in \tilde{\Gamma} \mid \gamma \Gamma_0 \gamma^{-1} = \Gamma_0 \}$ of $\Gamma_0$ in $\tilde{\Gamma}$ is finite.

1) For every infinite subset $\{x_n \Gamma_0\}_{n \in \mathbb{N}}$ of right coset of $\tilde{\Gamma}$ in $\Gamma_0$ one of the following two properties holds true:

a) The intersection

$$\bigcap_n x_n \Gamma_0 x_n^{-1},$$
of the stabilizers in $\tilde{\Gamma}$ of the cosets $x_n\Gamma_0$, is trivial. (Note that for $y$ in $\tilde{\Gamma}$, the subgroup $y\Gamma_0y^{-1}$ is the stabilizer of the coset $y\Gamma_0$.)

Or either

b) The intersection

$$\bigcap_n x_n\Gamma_0x_n^{-1} = \tilde{\Gamma}_1 \subseteq \tilde{\Gamma}$$

is a non-trivial subgroup of $\tilde{\Gamma}$.

In this second case we require that the following properties (i)-(v) hold true:

(i) There exists a subgroup $\tilde{\mathcal{M}} = \tilde{\mathcal{M}}(\tilde{\Gamma}_1)$ of $\tilde{\Gamma}$, such that $\tilde{\mathcal{M}}\tilde{\Gamma}_1 = \tilde{\Gamma}_1\tilde{\mathcal{M}}$ and such that if $x\Gamma_0$ is any point in $\tilde{\Gamma}/\Gamma_0$ that is normalized by $\tilde{\Gamma}_1$, then the set of cosets in $\tilde{\Gamma}/\Gamma_0$ that are fixed by $\tilde{\Gamma}_1$ (in particular the cosets $x_n\Gamma_0$, $n \in \mathbb{N}$, from the statement of point b above) is contained in $\tilde{\mathcal{M}}x\Gamma_0$. This set is independent on the choice of $x \in \{x_n\Gamma_0 | n \in \mathbb{N}\}$. The choice of the group $\tilde{\mathcal{M}}$ depends on the set $x_n\Gamma_0$, $n \in \mathbb{N}$.

(ii) With the hypothesis in condition (i) there exists a non-trivial subgroup $\tilde{\Gamma}_2$ and a subgroup $\tilde{\mathcal{M}}(\tilde{\Gamma}_2)$, such that $\tilde{\mathcal{M}}(\tilde{\Gamma}_2)x\Gamma_0$ is the set of cosets in $\tilde{\Gamma}/\Gamma_0$ fixed by $\tilde{\Gamma}_2$. Moreover, we require that $\tilde{\mathcal{M}}(\tilde{\Gamma}_2)$ contains $\tilde{\mathcal{M}}(\tilde{\Gamma}_1)$ and that $\tilde{\Gamma}_2$ and $\tilde{\mathcal{M}}(\tilde{\Gamma}_2)$ have the following maximality properties.

If $\tilde{\Gamma}_2 \subseteq \tilde{\Gamma}_1$ is another non-trivial subgroup obtained as the intersection of an infinite set of normalizers as in (1), then also $\tilde{\mathcal{M}}(\tilde{\Gamma}_1) \subseteq \tilde{\mathcal{M}}(\tilde{\Gamma}_2)$. Moreover, $\tilde{\Gamma}_2$ is minimal with these properties (that is if $\tilde{\Gamma}_2^0$ is a smaller non-trivial group with the same properties as $\tilde{\Gamma}_2$ then $\tilde{\mathcal{M}}(\tilde{\Gamma}_2^0) = \tilde{\mathcal{M}}(\tilde{\Gamma}_2)$). The triplet ($\tilde{\Gamma}_2, \tilde{\mathcal{M}}(\tilde{\Gamma}_2), x\Gamma_0$) will be referred to in the rest of the paper, as a maximal block of type $\tilde{\Gamma}_2$ in $\tilde{\Gamma}$. We also assume that every $g \in \tilde{\Gamma}$, belongs the stabilizer of an at most finite subset $F_g$ of maximal blocks $\tilde{\mathcal{M}}(\tilde{\Gamma}_2)$, of the type described above.

(iii) If $(\tilde{\Gamma}_2^\epsilon, \tilde{\mathcal{M}}(\tilde{\Gamma}_2^\epsilon), x\epsilon\Gamma_0)$, $\epsilon = 0, 1$, are two maximal blocks, corresponding to two subgroups $\tilde{\Gamma}_2^\epsilon$, $\epsilon = 0, 1$, as above, then we assume that either the intersection $\bigcap_{\epsilon=0,1} \tilde{\mathcal{M}}(\tilde{\Gamma}_2^\epsilon)x_\epsilon\Gamma_0$ is finite, or either that the two blocks $\tilde{\mathcal{M}}(\tilde{\Gamma}_2^\epsilon)x_\epsilon\Gamma_0$ coincide (and hence the groups $\tilde{\Gamma}_2^\epsilon$ are equal, by the maximality hypothesis in point (ii) ).
(iv) If $\alpha = (\widehat{\Gamma}^2, \widehat{\mathcal{M}}(\widehat{\Gamma}^2), x\Gamma_0)$ is as in property (ii) then we assume that for all $\gamma$ in $\widehat{\Gamma}$ the following intersection

$$(\gamma, \widehat{\mathcal{M}}(\widehat{\Gamma}^2)) x\Gamma_0 \cap \widehat{\mathcal{M}}(\widehat{\Gamma}^2) x\Gamma_0 = (\gamma, \widehat{\mathcal{M}}(\widehat{\Gamma}^2) \gamma^{-1}) \gamma x\Gamma_0 \cap \widehat{\mathcal{M}}(\widehat{\Gamma}^2) x\Gamma_0$$

(corresponding to the maximal blocks $(\gamma, \widehat{\mathcal{M}}(\widehat{\Gamma}^2) \gamma^{-1}, \gamma x\Gamma_0)$ and $\alpha$, is finite if and only if $\gamma$ belongs to $\widehat{\mathcal{M}}(\widehat{\Gamma}^2)$. In particular, the group normalizer of the group $\widehat{\mathcal{M}}(\widehat{\Gamma}^2)$ in $\widehat{\Gamma}$ is $\widehat{\mathcal{M}}(\widehat{\Gamma}^2)$ itself.

(v) For all the subgroups $\widehat{\mathcal{M}}(\widehat{\Gamma}^2)$ as in property (ii), one of the following two properties holds true:

$(\alpha)$ $\widehat{\mathcal{M}}(\widehat{\Gamma}^2)$ is amenable.

$(\beta)$ $If \ x\Gamma_0 \ in \ \widehat{\Gamma} / \Gamma_0 \ is \ such \ that \ (\widehat{\Gamma}^2, \widehat{\mathcal{M}}(\widehat{\Gamma}^2), x\Gamma_0) \ is \ a \ maximal \ block, \ then \ the \ representation \ of \ \widehat{\mathcal{M}}(\widehat{\Gamma}^2) \ on \ the \ Calkin \ algebra \ of \ \ell^2(\widehat{\mathcal{M}}(\widehat{\Gamma}^2) x\Gamma_0)$ is weakly contained in the left regular representation of $\widehat{\mathcal{M}}(\widehat{\Gamma}^2)$.

The conditions (i)-(v) are conditions on the stabilizers and their asymptotic behavior when computing essential states on the pavement realized by cosets of amenable, stabilizer groups. The remaining two conditions are concerning the structure of essential states when the essential states corresponding to the points with non-trivial stabilizer groups are removed.

(vi) There exists a subgroup $H$ of $\widehat{\Gamma}$ and $H_n$ a family of normal subgroups of $H$, with $\bigcap_n H_n = \{e\}$, with the following properties:

a) For every $\gamma \ in \ \Gamma_0$, there exists an inner automorphism $\theta_\gamma$ of $H$ (implemented by a representation of $\Gamma_0$ into the unitary group of the $C^\ast$-algebra associated to $H$), such that $\gamma h = \theta_\gamma(h) \gamma$ for all $\gamma \ in \ \Gamma_0$, $h \ in \ H$. In particular the group $\Gamma_0$ normalizes $H$. We also assume that $\widehat{\Gamma} = H\Gamma_0$.

b) For every $n \ in \ \mathbb{N}$, we have $\Gamma_0(\Gamma_0, \Gamma_0) \subseteq H_n\Gamma_0$, (that is the set of cosets $\{h\Gamma_0 | h \in H_n\}$, is invariant by (left) multiplication by elements in $\Gamma_0$).

(vii) The groups $\Gamma_0, H$ are exact, with infinite conjugacy classes, and the action of $\Gamma_0$ on orbits of $\widehat{\Gamma} / \Gamma_0$ contained in $H_n\Gamma_0 / \Gamma_0$ has amenable stabilizers for large $n \ in \ \mathbb{N}$.

Under these hypothesis, the unitary representation of $\widehat{\Gamma}$ into the Calkin algebra of $\ell^2(\widehat{\Gamma} / \Gamma_0)$ is weakly contained in the left regular representation of $\widehat{\Gamma}$.

Note that in our main example, the group $H$ will be $\Gamma \times 1$ and for $(\gamma, \gamma^{-1})$ in $\Gamma_0$, we have $\theta_{(\gamma, \gamma^{-1})}(h \times 1) = \gamma h \gamma^{-1} \times 1$, for $h \times 1 \ in \ H = \Gamma \times 1$. The groups $H_n, n \ in \ N$, will correspond to a decreasing family of normal subgroups of $\Gamma$, with trivial intersection.
For the purpose of the proof we introduce the following construction, that will be used to represent the states in the Calkin algebra.

**Definition 2.** Let $\omega$ be a free ultrafilter on $\mathbb{N}$ and let $X$ be an infinite, countable set. Let $\beta(X) \subseteq X^{\omega}$ be the subset consisting of all infinite sequences $(x_n)_{n \in \mathbb{N}}$ in $X$, that eventually avoid all finite subsets of $X$.

Fix $A = (A_n)_{n \in \mathbb{N}}$ a sequence of finite subsets of $X$, that are eventually avoiding all finite subsets of $X$.

Define $C_\omega(A) = C_\omega((A_n)_n) = \{(a_n)_{n \in \mathbb{N}} \mid a_n \in A_n \text{ for } n \text{ in a cofinal subset of } \omega\} \subseteq \beta(X)$.

Let $A$ be the boolean algebra of subsets of $C_\omega(A)$, generated by countable unions of disjoint, subsets of $C_\omega(A)$ of the form $C_\omega((B_n)_n)$ where $B_n \subseteq A_n$, for all $n$ in a cofinal subset of $\omega$.

Using the sets $(A_n)$ as a “scale”, we may define a finitely additive probability measure on $(C_\omega(A), A)$. This measure is in fact a particular case of the Loeb measure ([Lo]).

**Proposition 3.** Let $\omega, X, A = (A_n)$ and $A$ as is in the previous definition.

For $B = (B_n)_n$ as in the previous definition, define

$$\mu_{\omega,A}(C_\omega(B)) = \lim_{n \to \omega} \frac{\text{card}(A_n \cap B_n)}{\text{card} A_n}.$$ 

Then $\mu_{\omega,A}$ extends uniquely to a finitely additive probability measure on $(C_\omega(A), A)$.

**Proof.** We remark first that for every family $B = (B_n)_n$ of subsets as above the complement of $C_\omega(B)$ in $C_\omega(A)$ is $C_\omega((C_n)_n)$, where $C_n = A_n \setminus B_n$, $n \in \mathbb{N}$.

Indeed for any $(a_n)$ in $C_\omega(A)$ one of the following subsets of $\mathbb{N}$:

$$\{n \mid a_n \in B_n\} \quad \text{and} \quad \{n \mid a_n \in C_n\}$$

is cofinal in $\omega$.

Because of the formula of the complement, the boolean algebra $A$ from Definition 1 is described as follows.

Let $S, R$ be countable sets of indices. Assume that for every $n \in \mathbb{N}$, $(F_s^n)_{s \in S}$ is a countable disjoint family of subsets of $A_n$. Denote by $F^s$ the family $(F_s^n)_n$. In addition assume that for every $n$, and for every fixed $s \in S$, $(C_s^{s,r})_{r \in R}$ is a countable disjoint family of subsets of $F^s_n$. Denote by $C^{s,r}$, $s \in S, r \in R$, the family $C^{s,r} = (C_s^{s,r})_n$. 
Consider the subset of $C_\omega(A)$ defined by the formula

$$E = \bigcup_{s \in S} \left[ C_\omega(F^s) \setminus \bigcup_{r \in R} C_\omega(C^{s,r}) \right].$$

Then $A$ consists of all subsets as in formula (1).

By finite additivity of limits under the ultrafilter limit, the measure $\mu_{\omega,A}$ on a subset as in formula (1) is given by the formula

$$\mu_{\omega,A}(E) = \sum_{s \in S} \left[ \mu_{\omega,A}(F^s) - \sum_{r \in R} \mu_{\omega,A}(C^{s,r}) \right].$$

This defines a finitely additive measure on $A$ (see [Lo], [Cu]).

We assume in addition that there exists an action of a discrete group $G$ on $X$. Then along with the subsets $A = (A_n)_n$ one may consider the sets $\gamma A = (\gamma A_n)_n$ for $\gamma$ in $G$. In this case $\mu_{\omega,A}$ extends to an infinite finitely additive measure as follows.

The following construction gives a method, given a discrete group acting transitively on an infinite set, to “measure” the dynamics of infinite subsets of a given infinite set.

**Proposition 4.** Let $\omega$, $X$, $A = (A_n)_n$ and $\mu_{\omega,A}$ as above. Assume that a discrete group $G$ acts transitively on $X$.

Let $\gamma_{\omega,A} = \bigcup_{\gamma \in G} \gamma C_\omega(A)$.

Let $B$ the Borel sub-algebra of subsets of $\gamma_{\omega,A}$ generated by translates by $G$ of subsets of the form $C_\omega(B)$, where $B = (B_n) \subseteq A = (A_n)$.

Then, $\mu_{\omega,A}$ extends to an infinite (finitely additive) measure $\nu_{\omega}$ on $\gamma_{\omega,A}$, that is preserved by the action of $G$.

For a countable infinite set $M$ denote by $\beta M$ its Stone-Cech compactification (thus $L^\infty(M) = C(\beta M)$). Denote by $\partial^\beta(M)$ the boundary $\beta M \setminus M$.

We note that the measures $\mu_{\omega,A}$ and $\nu_{\omega}$ are in fact measures on $\partial^\beta(X)$, and hence the measure space $\gamma_{\omega,A}$ is (G equivariantly) identified with a measurable subset of $\partial^\beta(X)$.

**Proof.** The only thing to prove is that $G$ preserves the measure.

Because for all subsets $B_n \subseteq A_n$

$$gC_\omega((B_n)) = C_\omega(gB_n), \quad g \in \Gamma$$

this follows immediately from the fact that $\mu_{\omega,A}$ is an ultrafilter limit of counting measures. □
To deal with more general types of vectors, required to represent the Calkin algebra of $\ell^2(\Gamma/\Gamma_0)$ we introduce the following definition. We are indebted to Taka Ozawa for pointing us that these type of vectors were left out in the first variant of the paper.

**Definition.** With the above notations, assume that $B^s = (B^s_n)_{n \in \mathbb{N}}, s \in S$, is a countable family of disjoint subsets of $A_n$, for each $n$ in a cofinal subset of $\omega$. Assume that $(\lambda^n_s)_{n \in \mathbb{N}}, s \in S$ is a family of positive numbers, such that the sequence

$$(*) \quad \sum \lambda^n_s \frac{\text{card } B^s_n}{\text{card } A_n}$$

is bounded.

Let $C^r = (C^r_n)_{n \in \mathbb{N}}, r \in R$ be another family with the same property as $B$.

Denote by $\xi_{B,\lambda}$ the vector in

$$(**) \quad \xi_{B,\lambda} = \left( \sum \lambda^n_s \chi_{B^s_n} \right)_n.$$

We define

$$(***) \quad \int_{\bigcup_r C_\omega(C^r)} \xi_{B,\lambda} d\mu_{\omega,A} = \sum_r \lim_{n \to \omega} \sum_s \lambda^n_s \frac{\text{card } (B^s_n \cap C^r_n)}{\text{card } A_n}$$

and

$$\int_{C_\omega(A) \setminus \bigcup_r C_\omega(C^r)} \xi_{B,\lambda} d\mu_{\omega,A}$$

as the difference of the “integrals” over $C_\omega(A)$ and $\bigcup_r C_\omega(C^r)$.

We use the integral notation only two suggest a pairing (having similar properties to the integral) between vectors as in $(**)$ and measurable subsets of $C_\omega(A)$. The limit in $(***)$ is convergent because of the boundedness condition $(*)$.

Note that the above integral formula corresponds also to a Loeb measure that might be singular to the previous one in Proposition 3.

We also may define the product of two vectors as in $(**)$ as follows:

For two vectors as in $(**)$

$$\xi_{B,\lambda} = \left( \sum_{s \in S} \lambda^n_s \chi_{B^s_n} \right)_n$$
and
\[ \xi_{C,\mu} = \left( \sum_{r \in R} \mu^n_r \chi_{C_r} \right)_n, \]
where summability in \((\ast)\) is required for the squares of the scalars, we define
\[ \xi_{B,\lambda} \xi_{C,\mu} = \left( \sum_{s \in S, r \in R} \lambda^n_s \mu^n_r \chi_{B_s \cap C_r} \right)_n \]
which this time will verify the summability in \((\ast)\).

**Remark.** In fact as proved in Theorem 71 in [Ra], we may assume that the state induced by \(\xi_{B,\lambda}\) is as follows.

We consider \((A_s)_n\) a family of finite sets in \(\widetilde{\Gamma}/\Gamma_0\), indexed by \(s \in S\), a countable set, which are disjoint for every \(n\).

In addition we assume that for every \(k\), \(\mu_{\omega,(A_k)_n}\) is singular with respect to the measures \(\sum_{l > k} \mu_{\omega,(A_l)_n}\). Also this remains true if we take \(\widetilde{\Gamma}\) translations of the above measures.

Then we may simply work with states on \(G\), measuring the limits of the displacement of the set \((A_s)_n\). We can do this simultaneously for all \(s\), by taking the average (with \(S = \mathbb{N}\)) \(\sum_s \frac{1}{2^s} \mu_{\omega,(A_s)_n}\).

We return to the proof of Theorem 1.

**Proof (of Theorem 1).** Because of the results of Calkin [Ca], the states on the Calkin algebra of \(\ell^2(\Gamma/\Gamma_0)\) are obtained by representing the Calkin algebra into the Hilbert space \(\ell^2(\widetilde{\Gamma}/\Gamma_0)_\omega\) associated to a fixed free ultrafilter \(\omega\) on \(\mathbb{N}\).

The Hilbert space \(\ell^2(\widetilde{\Gamma}/\Gamma_0)_\omega\) consists of all sequences \((\xi_n)_n\) in \(\ell^2(\widetilde{\Gamma}/\Gamma_0)\), weakly convergent to 0, with scalar product given by the formula
\[ \langle (\xi_n), (\mu_n) \rangle_\omega = \lim_{n \to \omega} \langle \xi_n, \mu_n \rangle_{\ell^2(\widetilde{\Gamma}/\Gamma_0)} \]
(where \((\mu_n)_n \in \ell^2(\widetilde{\Gamma}/\Gamma_0)_\omega\).

Let \(A = (A_n)_{n \in \mathbb{N}}\) be a sequence of finite subsets of \(\widetilde{\Gamma}/\Gamma_0\) that is avoiding any given subset of \(\widetilde{\Gamma}/\Gamma_0\). Consider the vector
\[ \xi_A = \left( \frac{1}{(\text{card}(A_n))^{1/2}} \chi_{A_n} \right)_n \in \ell^2(\widetilde{\Gamma}/\Gamma_0)_\omega. \]
Here the characteristic function \(\chi_{A_n}\) of \(A_n\) is viewed as an element of the Hilbert space \(\ell^2(\Gamma/\Gamma_0)\), for all \(n \in \mathbb{N}\).
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By density it is sufficient to consider vectors $\xi = (\xi_n)_n$ of the type $(* *)$. By Theorem 71 in [Ra], we may, in fact, reduce the proof to the case of the vectors of the form $\xi_A$ as above.

Clearly, the subspace generated by such vectors is dense in $l^2(\tilde{\Gamma}/\Gamma_0)_\omega$. To prove the theorem it will be sufficient to show that the positive definite states on $\tilde{\Gamma}$ of the form

$$\varphi_{\omega,A}(\gamma) = \langle \gamma \xi_A, \xi_A \rangle_\omega, \quad \gamma \in \tilde{\Gamma}$$

are contained in the left regular representation of $\tilde{\Gamma}$. We also have to prove the same result for the more general vectors $\xi_B, \lambda$ as in previous definition, with summability as in (*) being required for the square of the coefficients.

Using the terminology from Definition 2 and Proposition 3, we get that

$$(3) \quad \varphi_{\omega,A}(\gamma) = \mu_{\omega,A}(C_\omega(A) \cap \gamma C_\omega(A)), \quad \gamma \in \tilde{\Gamma}. $$

For the general vectors as in the previous definition we get the state

$$\varphi_{\omega,B,\lambda}(\gamma) = \int_{C_\omega(A)} \xi_B, \lambda \gamma(\xi_B, \lambda) \, d\mu_{\omega,A}, \quad \gamma \in \tilde{\Gamma}. $$

Here the integral is as in $(* * *)$.

The only properties of the integral symbol that we will use in the sequel are linearity, and splitting over the domain of integration, which both hold true for this integral (which as we mentioned above is considered as a pairing between vectors as in $(* *)$ and $A$-measurable subsets of $C_\omega(A)$).

Note that the formula (3) obviously defines a positive definite state on $\tilde{\Gamma}$. To see this just consider $C_{\omega,A}$ as a subset of measure 1 of the measure space $(\gamma_{\omega,A}, B, \mu_{\omega,A})$ from Proposition 4. To prove the result we will split $C_{\omega,A}$ into several pieces that have disjoint orbits in $\gamma_{\omega,A}$.

Let $C_{\omega}^0(A)$ be the subset of $C_\omega(A)$ consisting of all sequences in $C_\omega(A)$ of the form:

$$\{ a_n \Gamma_0 | \text{the hypothesis (a), in point (i) holds for } n \text{ in a cofinal subset of } \omega \}. $$

In particular, the points in $C_{\omega}^0(A)$ have trivial stabilizers under the action of $\tilde{\Gamma}$.

If a sequence $(a_n \Gamma_0)_{n \in \mathbb{N}}$ does not belong to $C_{\omega}^0(A)$ then in a cofinal subset $I$ of $\omega$, we will have that

$$\bigcap_{n \in I} a_n \Gamma_0 a_n^{-1} = \tilde{\Gamma}_1,$$
where $\tilde{\Gamma}_1$ is a non-trivial subset of $\tilde{\Gamma}$. By property (ii) there exists a maximal block $\mathcal{M}(\tilde{\Gamma}_2)x\Gamma_0$ of $\tilde{\Gamma}$, associated to the data $(\tilde{\Gamma}_2, \mathcal{M}(\tilde{\Gamma}_2), x\Gamma_0)$ such that

$$(a_n\Gamma_0)_{n\in\mathbb{N}} \in \mathcal{C}_\omega\left((A_n \cap (\mathcal{M}(\tilde{\Gamma}_2)x\Gamma_0))\right).$$

Since the maximal block is in fact determined by $\tilde{\Gamma}_2$ we will denote the set $\mathcal{C}_\omega\left((A_n \cap (\mathcal{M}(\tilde{\Gamma}_2)x\Gamma_0))\right)$ by $\tilde{K}_{\tilde{\Gamma}_2}$.

Clearly, the translates by elements in the group $\tilde{\Gamma}$ of the set $\mathcal{C}_\omega(A)$ are disjoint from the translates under $\tilde{\Gamma}$ of $\tilde{K}_{\tilde{\Gamma}_2}$. By hypothesis (iii) the sets $\tilde{K}_{\tilde{\Gamma}_2}$ are disjoint modulo finite sets (for all groups $\tilde{\Gamma}_2$ as in property (ii)). We take

$$L_{\tilde{\Gamma}_2} = \bigcup_{\gamma \in \tilde{\Gamma} / \tilde{\Gamma}_2} \tilde{K}_{\gamma-1}\tilde{\Gamma}_2\gamma.$$ 

By properties (iv) and (v) the orbits of the points in $L_{\tilde{\Gamma}_2}$ and $L_{\tilde{\Gamma}_2}$ are disjoint, if $\tilde{\Gamma}_2^a$ is not a conjugate of $\tilde{\Gamma}_2^b$ and $\tilde{\Gamma}_2^a$, $\tilde{\Gamma}_2^b$ as in property (ii).

Let

$$\varphi^0_{\omega,A}(\gamma) = \mu_{\omega,A}(\gamma\mathcal{C}_\omega(A) \cap \mathcal{C}_\omega(A)), \quad \gamma \in \tilde{\Gamma};$$

and let

$$(4) \quad \varphi_{\omega,A}(\gamma) = \mu_{\omega,A}(\gamma L_{\tilde{\Gamma}_2} \cap L_{\tilde{\Gamma}_2}), \quad \gamma \in \tilde{\Gamma}.$$ 

Then both $\varphi^0_{\omega,A}$ and $\varphi_{\omega,A}$ are fractions of states (depending on the mass of the corresponding set at $\gamma = e$).

We have

$$(5) \quad \varphi_{\omega,A} = \varphi^0_{\omega,A} + \sum_{\tilde{\Gamma}_2} \varphi_{\omega,A},$$

where $\tilde{\Gamma}_2$ runs over representatives modulo conjugation of groups $\tilde{\Gamma}_2$ as in property (ii) (and a similar decomposition for $\varphi_{\omega,B,\lambda}$).

We analyze first the state $\varphi^0_{\omega,A}$. As we noted above Proposition 4, we may consider the set

$$\mathcal{Y}_{\omega,A}^0 = \bigcup_{\gamma \in \tilde{\Gamma}} \gamma \mathcal{C}_\omega(A) \subseteq \mathcal{Y}_{\omega,A}$$

and $\mu_{\omega,A}$ is a $\tilde{\Gamma}$ invariant measure on $\mathcal{Y}_{\omega,A}^0$ (as in Proposition 4).

Note that there are no fixed points (and no non-amenable stabilizers) for the action of $\tilde{\Gamma}$ on $\mathcal{Y}_{\omega,A}^0$ (which is preserving the measure).

We will prove first the following lemma.
Lemma 5. Let \((\mathcal{X}, \mu)\) be an infinite, \(\sigma\)-finite, measure space. Let \(G\) be a countable discrete group, acting freely, by measure preserving transformations on \(\mathcal{X}\). Assume that there exists two subgroups \(\Gamma_0, H\) in \(G\) with the following properties.

(i) \(\Gamma_0 \cap H = \{e\}\), the trivial element in \(G\).

(ii) There exists a map \(\alpha = (\alpha_{\gamma_0})_{\gamma_0 \in \Gamma_0}\), defined on \(\Gamma_0\) with values into the inner automorphisms of \(H\) (with no obstruction to lifting), such that for every \(\gamma_0 \in \Gamma_0, h \in H\)

\[
h\gamma_0 = \alpha_{\gamma_0}(h)\gamma_0, \quad \text{for all } \gamma_0 \in \Gamma_0, h \in H.
\]

In particular, \(G = \Gamma_0 H = H\Gamma_0\) (thus \(\Gamma_0\) normalizes \(H\)).

(iii) The groups \(G, \Gamma_0, H\) are exact, non-amenable and i.c.c. (with infinite conjugacy classes). We assume that the Koopman representation of \(H\) on \(L^2(\mathcal{X}, \mu)\) is tempered (see also [Ke]), in the following sense:

The crossed product \(C^*\)-algebra \(C^*(H \rtimes L^\infty(\mathcal{X}, \mu))\) is a representation of the crossed product \(C^*\)-algebra \(C^*(H \rtimes \partial^0 H)\) and hence it is nuclear. Consequently, the \(C^*\)-algebra \(A_0 = C^\text{Koop}_*(H \rtimes L^\infty(\mathcal{X}, \mu)) \subseteq B(L^2(\mathcal{X}, \mu))\), generated by the Koopman representation of \(H\) on \(L^2(\mathcal{X}, \mu)\) and by the abelian algebra \(L^\infty(\mathcal{X}, \mu)\) is nuclear, and hence its weak closure \(\mathcal{A} = \widehat{A_0}\) in \(B(L^2(\mathcal{X}, \mu))\) is hyperfinite. Moreover, the hypothesis implies that \(\mathcal{A}_0\) is isomorphic to \(\mathcal{A}_0 = C^\text{red}_*(H \rtimes L^\infty(\mathcal{X}, \mu))\). Note that and \(\mu\) induces a \(H\)-invariant semifinite trace on \(L^\infty(\mathcal{X}, \mu)\) and hence on the reduced crossed product \(C^*\)-algebra \(\mathcal{A}_0\).

(iv) Let \(Z = Z(\mathcal{A}) = Z(\mathcal{A}_0)\) be the center of the von Neumann algebra \(\mathcal{A}\). Then \(Z\) is normalized by \(\Gamma_0\). Moreover (by the i.c.c condition), the algebra \(Z\), which is thus of the form \(L^\infty(\mathcal{Y}) \subseteq L^\infty(\mathcal{X}, \mu)\) for an infinite measure space \(\mathcal{Y}\), carries a canonical \(\Gamma_0\)-invariant, \(\sigma\)-finite measure \(\nu\) induced by \(\mu\).

As proved in (Lemma 74,[Ra]), the algebra \(\mathcal{A}_0\), when disintegrated over \(L^\infty(\mathcal{Y}, \nu)\), has fibers consisting of nuclear \(C^*\) algebras isomorphic to \(C^*(H \rtimes \ell^\infty(\mathcal{H}))\) or to \(C^*(H \rtimes \ell^\infty(H\setminus H_0) \otimes C^\text{red}_*(H_0 \rtimes K_{H_0}))\), where \(H_0\) runs over amenable subgroups of \(H\), and \(K_{H_0}\) are probability measure spaces, depending on the fiber that are acted, by free, measure preserving transformations of \(H_0\). We denote the corresponding spectrum of the associated central pieces, by \(\mathcal{Y}_1\), and respectively \(\mathcal{Y}_{11,\mathcal{H}_0}\). Then \(\mathcal{Y}\) is the disjoint reunion of reunion of \(\mathcal{Y}_1\) and \(\bigcup_{\mathcal{H}_0} \mathcal{Y}_{11,\mathcal{H}_0}\), where \(H_0\) runs over amenable subgroups of \(H\).

We assume that the reunion \(\bigcup_{\mathcal{H}_0} \mathcal{Y}_{11,\mathcal{H}_0}\) is disjoint and that the group \(\Gamma_0\) permutes, through the representation \(\alpha\), the central pieces corresponding
to the amenable subgroups in the above reunion. Hence the representation associated to this part of the spectrum is a tempered representation.

(v) We make the additional assumption that the action constructed above, of the group \( \Gamma_0 \), by measure preserving transformations on the infinite measure space \( (Y, \nu) \) has the property that the \( C^* \)-algebra \( C^*(\Gamma_0 \rtimes L^\infty(Y, \nu)) \) is a representation of \( C^*(\Gamma_0 \rtimes \partial^i \Gamma_0) \), and thus is nuclear.

Then the \( C^* \)-algebra \( B_0 = C^\text{Koop}_0(G \rtimes L^\infty(X, \mu)) \subseteq B(L^2(X, \mu)) \) is nuclear and hence the Koopman representation of \( B_0 \) on \( L^2(X, \mu) \) is tempered, (thus continuous with respect the \( C^\text{red}_1(G \rtimes L^\infty(X, \mu)) \) norm.)

Before giving the proof of the lemma, we explain how this will be used into completing the analysis of the state \( \varphi^{0}_{\omega,A} \). We use the notations from the statement of the theorem. Then \( G \) will be group \( \widetilde{\Gamma} \) and the groups considered in the statement \( \Gamma_0, H \) are the groups \( \Gamma_0 = \{ (\gamma, \gamma^{-1}) \mid \gamma \in \Gamma \} \) and \( H = \{ (\gamma, e) \mid \gamma \in \Gamma \} \). Then, because \( \Gamma/\Gamma_0 \) is isomorphic to \( H \), it follows that \( \mathcal{X} \ast = \mathcal{Y}^{\omega,A}_0 \) with the measure \( \mu = \mu_{\omega,A} \), is an \( H \)-invariant subset of \( \partial^i H \), and hence \( C^*(H \rtimes L^\infty(\mathcal{X}, \mu)) \) is a representation of \( C^*(H \rtimes \partial^i H) \), which is nuclear, since \( H \) is exact ([Oz], [AD]).

The fact that the center \( Z \) of the algebra \( \mathcal{A} \subseteq B(L^2(\mathcal{X}, \mu)) \) (a representation of \( C^*(H \rtimes L^\infty(\mathcal{X}, \mu)) \), is acted by \( \Gamma_0 \) follows from the property (ii). Denote by \( \mathcal{B} \), the weak closure of \( B_0 \) in \( B(L^2(\mathcal{X}, \mu)) \). The group \( \Gamma_0 \) normalizes \( H \), and hence (since \( Z \subseteq L^2(\mathcal{X}, \mu) \), by the i.c.c. condition) it follows that \( \Gamma_0 \) normalizes also \( Z \). The canonical, \( \Gamma_0 \)-invariant state \( \nu \) on \( Z \) is obtained by splitting the center of \( Z \) into summands, as in the statement, where the representation of \( C^*_{\text{red}}(H \rtimes L^\infty(\mathcal{X}, \mu)) \) is either of type I or either of type II.

In either case, if \( \widetilde{F}_0 \) is the central support in \( Z \) of a \( H \) wandering subset \( F_0 \) of \( \mathcal{X} \) (in the type I case) or a \( H/H_0 \) wandering subset \( F_0 \) (in the type II case, for \( H_0 \) an amenable subgroup of \( H \), which generates a type II summand in \( Z \)), then we define \( \nu(\widetilde{F}_0) = \mu(\widetilde{F}_0) \).

By using the construction from Proposition 72, in [Ra], and if \( F = C_\omega((\mathcal{A}_n)_n) \), then by using translations by elements in the group \( H \) on the sets \( C_\omega((\mathcal{A}_n)_n) \) with elements in the groups \( H_n \), then the state on \( \Gamma_0 \) defined by

\[
\varphi_{0}(\gamma_0) = \nu(\gamma_0 \widetilde{F} \cap \widetilde{F}), \quad \gamma_0 \in \Gamma_0,
\]

may be realized in a measure space \( C_\omega((\mathcal{A}'_n)_n), \mu_{\omega,(\mathcal{A}'_n)} \) where \( \mathcal{A}'_n \subseteq H_n \Gamma_0/\Gamma_0 \), \( n \in \mathbb{N} \).
Thus there exists a choice of the sets \((A'_n) \subseteq H_n \Gamma_0 / \Gamma_0, n \in \mathbb{N}\), such that
\[ \varphi_0(\gamma_0) = \mu_{\omega, (A'_n)}(\gamma_0 \mathcal{C}_\omega(\gamma_0 (A'_n)_n) \cap \mathcal{C}_\omega((A'_n)_n)). \]

Then, by the hypothesis that the orbits of \(\Gamma_0\) on \(H_n \Gamma_0 / \Gamma_0\) have amenable stabilizers, by Lemma 73 in [Ra], if follows that the action of \(\Gamma_0\) on \(Y = \bigcup_{\gamma_0 \in \Gamma_0} \mathcal{C}_\omega((A'_n)_n)\) is a representation of \(\Gamma_0 \rtimes \beta \Gamma_0\), and hence the hypothesis on the action of \(\Gamma_0\) on \(Z(A)\) is fulfilled.

Hence, by applying the lemma, one obtains that the action of \(G\) on \(L^\infty(X, \mu)\) is so that \(C_{\text{red}}^*(G \rtimes L^\infty(X, \mu))\) is nuclear, and hence the Koopmann representation of \(G = \Gamma\) on \(X = Y_{\omega, A}\) is tempered and hence continuous with respect the \(C_{\text{red}}^*(G \rtimes L^\infty(X, \mu))\) norm.

**Proof.** (of Lemma 5). Because \(C^*(H \rtimes L^\infty(X, \mu))\) is nuclear it follows that the \(C^*\) algebra \(A_0 = C_{\text{Koop}}^*(H \rtimes L^\infty(X, \mu)) \subseteq B(L^2(X, \mu))\) is isomorphic to the reduced crossed product \(C^*\) algebra \(\widetilde{A}_0 = C_{\text{red}}^*(H \rtimes L^\infty(X, \mu)) \subseteq B(L^2(X, \mu))\), with (semifinite) trace induced by \(\mu\). As in the proof of Lemma 74 in [Ra], we have that the center of the algebra \(\widetilde{A} = \widetilde{A}_0\) disintegrates to hyperfinite factors of type I or of type II. The factors of type I correspond to the situation when there exist a fundamental domain for the action of the group \(H\). The factors of type II correspond to a situation in which we have \(H_0 \subseteq H\) amenable and there exists \(F_0 \subseteq X\), such that \(H_0 F_0 = F_0\) and \(F_0\) is \(H/H_0\) wandering. The corresponding factors are the weak closures corresponding to the \(C^*\) algebra disintegration described in the assumption (iv).

The group \(\Gamma_0\) will map such a type II_1 factor into the type II_1 factor corresponding to an invariant subset of \(X\), for the group \(\gamma_0 H \gamma_0^{-1} \subseteq H\), which is again an amenable subgroup.

We note that, in the setting of Theorem 1, the corresponding pieces of the center \(Y\) are disjoint and acted by conjugation by the group \(\Gamma_0\) since cosets for different amenable subgroups of \(H\), intersect in finite sets of points (because of the first assumptions of the theorem), and because the set \(F_0\), invariant under \(H_0\), is realized by translations of Folner sets for \(H_0\), it follows that the corresponding pieces in the center \(Z\), are disjoint.

We define for \(F_0\) a \(H\) (respectively \(H/H_0\) wandering subset) as above (respectively in the type I case or either in the type II case) the measure \(\nu\) by the formula \(\nu(\chi_{F_0}) = \mu(\chi_{F_0})\), where \(\chi_{F_0}\) is the central support in \(Z\) of \(\chi_{F_0}\).
Then, by the above arguments, the state $\nu$ is a $\Gamma_0$-invariant state on $\mathcal{Z}$. This algebra is identified with the measure space $L^\infty(\mathcal{Y}, \nu)$.

By hypothesis, the action $G$ on $L^\infty(\mathcal{Y}, \nu)$ is so that the $C^*$-algebra $C_0 = C^*(\Gamma_0 \rtimes L^\infty(\mathcal{Y}, \nu))$ is nuclear. Hence $C^*_\text{Koop}(\Gamma_0 \rtimes L^\infty(\mathcal{Y}, \nu)) = C^*_\text{red}(\Gamma_0 \rtimes L^\infty(\mathcal{Y}, \nu))$ and both algebras are identified with the $C^*$ subalgebra $D_0$ of $B(L^2(\mathcal{X}, \nu))$ generated by the image of the Koopman representation of the group $\Gamma_0$ on $L^2(\mathcal{X}, \nu)$ and the subalgebra $L^\infty(\mathcal{Y}, \nu) \subseteq L^\infty(\mathcal{X}, \mu) \subseteq B(L^2(\mathcal{X}, \nu))$.

By using again the same arguments from Lemma 74 in ([Ra]), as above, the center $\mathcal{Z} \subset \mathcal{Z}$ of the von Neumann algebra $C^*_\text{Koop}(\Gamma_0 \rtimes L^\infty(\mathcal{Y}, \nu))'' \subseteq B(L^2(\mathcal{X}, \nu))$ is divided into direct summands $\mathcal{Z}_I$ or $\mathcal{Z}_{II}$, corresponding to summands hyperfinite of type I, or of type II corresponding to amenable subgroups $\Gamma_{00}$ of $\Gamma_0$. In this decomposition we have the corresponding disintegration of the $C^*$-algebra $D_0$ into the corresponding nuclear $C^*$ algebras, that are weakly dense in the summands corresponding to the von Neumann disintegration. In the case of summands of type I, the term in the disintegration of $D_0$, will be a $C^*$ algebra isomorphic to $C^*_\text{red}(\Gamma_0 \rtimes \ell^\infty(\Gamma_0))$ while in the second case corresponding to a type II summand, corresponding to an amenable subgroup $\Gamma_{00}$ of $\Gamma_0$, the summand is described as follows. We have a probability measure space $\mathcal{Y}_{\Gamma_{00}}$ a subset of the disintegration of $\mathcal{Y}$ at the fiber we are considering, acted by measure preserving transformations by $\Gamma_0$. Then the corresponding fiber in the disintegration is the nuclear $C^*$-algebra $C^*_\text{red}(\Gamma_0 \rtimes \ell^\infty(\Gamma_0)) \otimes C^*_\text{red}(\Gamma_{00} \rtimes \mathcal{Y}_{\Gamma_{00}})$.

As observed already in point (iv) of the statement, to prove the lemma we may proceed by reducing to the case when the action of $H$ on $\mathcal{X}$ has no type $II$ components.

Thus we start with $F \subseteq \mathcal{X}$ an $H$ wandering subset, such that $\chi_F$ the central support of $\chi_F$ in $\mathcal{Z}$ sits in $\mathcal{Z}_I$ or $\mathcal{Z}_{II}$. Let $\chi_{\tilde{F}}$ be the central support in $\mathcal{Z}$.

In the first case, when $\chi_{\tilde{F}} \in \mathcal{Z}_I$, we will have that $\chi_{\tilde{F}}B\chi_{\tilde{F}}$ is isomorphic to $\chi_{\tilde{F}}A\chi_{\tilde{F}} \otimes B(\ell^2(\Gamma_0))$ and thus is hyperfinite. In this case if we disintegrate the $C^*$ algebra $\chi_{\tilde{F}}B\chi_{\tilde{F}}$ over its center, it follows by assumption (ii) that the group $\Gamma_0$ normalizes the group $H$, by inner automorphisms, that $\Gamma_0$ normalizes also $\ell^\infty(\tilde{F})$ and hence, for each $\tilde{f} \in \tilde{F}$, an element $\gamma_0 \in \Gamma_0$ will implement an isomorphism from the fiber of $D_0$ at $\tilde{f}$ into the fiber of $D_0$ at $\gamma_0\tilde{f}$, for all $\tilde{f}$ in $\tilde{F}$. Since both the algebras in the fiber of $D_0$ are isomorphic to $C^*_\text{red}(\Gamma_0 \rtimes \ell^\infty(\Gamma_0))$, and these this an inner automorphism, we may correct the action of $\Gamma_0$, fiberwise by this unitary. It follows that the
fibers of \( B_0 \) over elements in \( \tilde{F} \) are isomorphic to the nuclear \( C^* \) algebra \( C^*_{\text{red}}(H \rtimes \ell^\infty(H)) \otimes_{\min} C^*_{\text{red}}(\Gamma_0 \rtimes \ell^\infty(\Gamma_0)) \).

In the second case, we may assume that there exists an amenable subgroup \( \Gamma_{00} \) of \( \Gamma_0 \) such that \( \Gamma_{00} \tilde{F} = \tilde{F} \), where \( \tilde{F} = \Gamma_0/\Gamma_{00} \) wandering, finite measure, subset of \( \mathcal{Y} \). In this case, the subgroup \( \Gamma_{00} \) is leaving the algebra \( \mathcal{A}_{\tilde{F}} = \chi_{\tilde{F}} - \mathcal{A}_F \chi_{\tilde{F}} \) invariant. The central support of this algebra is the projection \( \chi_{\tilde{F}} \). The reduced crossed product von Neumann algebra of \( \mathcal{A}_{\tilde{F}} \) by \( \Gamma_0 \) is hyperfinite and hence so is \( \chi_{\tilde{F}} \mathcal{B} \chi_{\tilde{F}} \), which is then isomorphic to \( W^*_{\text{red}}(\Gamma_{00} \rtimes \chi_{\tilde{F}} \mathcal{A}_F) \otimes B(\ell^2(\Gamma_0/\Gamma_{00})) \). As in the previous case, it is proven that the fibers of the \( C^* \)-algebra \( B_0 \) over this part of its central spectrum are isomorphic to \( C^*_{\text{red}}(H \rtimes \ell^\infty(H)) \otimes_{\min} C^*_{\text{red}}(\Gamma_0 \rtimes \ell^\infty(\Gamma_0/\Gamma_{00})) \otimes_{\min} C^*_{\text{red}}(\Gamma_{00} \rtimes \ell^\infty(\tilde{F}, \nu)) \).

Hence the \( C^* \) algebra \( B_0 = C^*_{\text{Koop}}(G \rtimes \mathcal{X}) \) is nuclear (having nuclear fibers) and thus the norm on the cross product \( C^*_{\text{Koop}}(G \rtimes \mathcal{X}) \) is continuous (equal) with respect to the norm on \( C^*_{\text{red}}(G \rtimes \mathcal{X}) \).

The rest of the proof of the theorem is based on disjointness of cosets (modulo finite sets). Since our “integrals” in the definition after Proposition 4, behave as usual integrals with respect to the operations concerning domain, the proof is identical from here on, for the remaining components of \( \varphi_{\omega, A} \) and \( \varphi_{\omega, B, \lambda} \).

It remains to analyze the states \( \tilde{\varphi}_{\omega,A} \) from the formula (4), where \( \tilde{\Gamma}_2 \) is a group as in property (ii). Here

\[
L_{\tilde{\Gamma}_2} = \bigcup_{\gamma \in \tilde{\Gamma}/\mathcal{M}(\tilde{\Gamma}_2)} \tilde{K}_\gamma \quad \text{and} \quad \tilde{K}_{\tilde{\Gamma}_2} = C_{\omega}((\tilde{\mathcal{M}(\tilde{\Gamma}_2)} x \Gamma_0 \cap A_n)_n),
\]

where \( (\tilde{\Gamma}_2, \mathcal{M}(\tilde{\Gamma}_2), x \Gamma_0) \) is describing a maximal block.

Since \( \gamma C_{\omega}((\mathcal{M}(\tilde{\Gamma}_2) x \Gamma_0 \cap A_n)_n) \) is equal to \( C_{\omega}((\gamma \mathcal{M}(\tilde{\Gamma}_2) \gamma^{-1}) \gamma x \Gamma_0) \cap (\gamma A_n)_n \), it follows by density that we can restrict to the case of \( (A_n)_n \) contained in a single maximal block.

Thus we assume that there exists a group \( \tilde{\Gamma}_2 \) as in property (ii) such that \( (A_n)_n \) are all contained in \( \mathcal{M}(\tilde{\Gamma}_2) x \Gamma_0 \) and thus \( C_{\omega}(A) \) is contained in \( \tilde{K}_{\tilde{\Gamma}_2} \).

But \( \tilde{\Gamma}/\mathcal{M}(\tilde{\Gamma}_2) \) maps \( \tilde{K}_{\tilde{\Gamma}_2} \) into pairwise disjoint subsets. Thus in this case

\[
\varphi_{\omega,A}(g) = \mu_{\omega,A}(gC_{\omega}(A) \cap C_{\omega}(A)), \quad g \in \tilde{\Gamma},
\]
is a state supported on $\widetilde{\mathcal{M}}(\Gamma_2)$. If $\widetilde{\mathcal{M}}(\Gamma_2)$ is amenable we are done since the representation of $\Gamma$ on $\ell^2(\Gamma/\widetilde{\mathcal{M}}(\Gamma_2))$ is weakly contained in the left regular representation. In the other case, since the representation of $\widetilde{\mathcal{M}}(\Gamma_2)$ on the Calkin algebra of $\ell^2(\widetilde{\mathcal{M}}(\Gamma_2)x\Gamma_0)$ is weakly contained in the left regular representation of $\mathcal{M}(\Gamma_2)$, it follows that $\varphi_{\Gamma_2}^{\Gamma_2}|_{\mathcal{M}(\Gamma_2)}$ is continuous on $\mathcal{C}_{\text{red}}(\mathcal{M}(\Gamma_2))$ and hence on $C^*_{\text{red}}(\Gamma)$ (since the state is also supported on $\widetilde{\mathcal{M}}(\Gamma_2)$). \qed

In the rest of the paper, we will present some examples, for which the hypothesis of the theorem holds true.

As explained in the introduction, we consider $\Gamma$ a discrete group, $\widetilde{\Gamma} = \Gamma \times \Gamma^\text{op}$, $\Gamma_0 = \{(\gamma, \gamma^{-1}) \mid \gamma \in \Gamma\} \subset \widetilde{\Gamma}$ and identify $\widetilde{\Gamma}/\Gamma_0$ with $\Gamma$, by the $\widetilde{\Gamma}$-invariant projection map $\pi((g_1, g_2)) = g_1g_2$, for $(g_1, g_2) \in \widetilde{\Gamma}$.

**Proposition 6.** Let $\Gamma$ be a discrete group and let $\widetilde{\Gamma}, \Gamma_0, \pi$ be defined as above. We identify $\widetilde{\Gamma}/\Gamma_0$ with $\Gamma$.

Let $\widetilde{\Gamma}_1 \subset \widetilde{\Gamma}$ be the intersection of the stabilizer groups of an infinite family of distinct elements, $(x_n)_n \subset \Gamma$. Assume that the group $\widetilde{\Gamma}_1$ is non-trivial, and fix an element $x$ in the infinite family $(x_n)_n$.

Then there exists a subgroup $\Gamma_1$ of $\Gamma$ depending only on $\widetilde{\Gamma}_1$ (and not on the particular choice of the element $x$), such that

$$\widetilde{\Gamma}_1 = \{(\gamma_1, x^{-1}\gamma_1^{-1}x) \mid \gamma_1 \in \Gamma_1\}.$$ 

Let $\mathcal{M}(\widetilde{\Gamma}_1) = \Gamma_1'$ be the commutant of the group $\Gamma_1$ in $\Gamma$ (that is the set of the elements $g \in \Gamma$ such that $g\gamma_1g^{-1} = \gamma_1$ for all $\gamma_1 \in \Gamma_1$).

Let $\mathcal{M}(\widetilde{\Gamma}_1) = \{(\gamma_1, \gamma_2) \mid \gamma_1 \in \Gamma_1', \gamma_2 \in x\Gamma_1'x^{-1}\}$. Then $\mathcal{M}(\widetilde{\Gamma}_1)x = x(x^{-1}\mathcal{M}(\widetilde{\Gamma}_1)x)$ is the set of all elements in $\Gamma$ that are stabilized by $\widetilde{\Gamma}_1$.

Obviously, $\mathcal{M}(\widetilde{\Gamma}_1)$ acts on $\mathcal{M}(\widetilde{\Gamma}_1)x$ by left and right translations. Thus if $(\gamma_1, x^{-1}\gamma_2x)$ belongs to $\mathcal{M}(\widetilde{\Gamma}_1)$ (and hence if $\gamma_1, \gamma_2 \in \Gamma_1'$) then for $mx$ in $\mathcal{M}(\widetilde{\Gamma}_1)x$, we have

$$(\gamma_1, x^{-1}\gamma_2x)(mx) = \gamma_1mx(x^{-1}\gamma_2x) = \gamma_1m\gamma_2x.$$ 

If $\mathcal{M}(\widetilde{\Gamma}_1'), \mathcal{M}(\widetilde{\Gamma}_1')$ are two subgroup as above and $x_\alpha, x_\beta$ are the corresponding points stabilized by $\widetilde{\Gamma}_1'$ and $\Gamma_1'$ respectively, then if $\mathcal{M}(\widetilde{\Gamma}_1)x_\alpha \cap \mathcal{M}(\widetilde{\Gamma}_1')x_\beta$ is non void, and thus contain an element of the form $m_\alpha x_\alpha = m_\beta x_\beta$ (where $m_\alpha \in \mathcal{M}(\widetilde{\Gamma}_1')$, $m_\beta \in \mathcal{M}(\widetilde{\Gamma}_1')$) then

$$\mathcal{M}(\widetilde{\Gamma}_1)x_\alpha \cap \mathcal{M}(\widetilde{\Gamma}_1')x_\beta = (\mathcal{M}(\widetilde{\Gamma}_1') \cap \mathcal{M}(\widetilde{\Gamma}_1'))m_\varepsilon x_\varepsilon, \quad \varepsilon = \alpha, \beta.$$

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Proof. Assume that $x, y \in \Gamma$ are two distinct points fixed by an element $(\gamma_1, \gamma_2) \in \widetilde{\Gamma}$. Then $\gamma_1 x \gamma_2 = x$, $\gamma_1 y \gamma_2 = y$ and hence

(5) $\gamma_2 = x^{-1} \gamma_1^{-1} x = y^{-1} \gamma_1^{-1} y$

and hence $(yx^{-1}) \gamma_1^{-1} (yx^{-1})^{-1} = \gamma_1^{-1}$. Denote the elements commuting with an element $g$ of $\Gamma$ by $\{g\}'$. Then $yx^{-1}$ belongs to $\{\gamma_1^{-1}\}' = \{\gamma_1\}'$. Thus there exists $\theta \in \{\gamma_1\}'$ such that

(6) $y = \theta x$.

Note that since $\gamma_2 = x^{-1} \gamma_1^{-1} x$, we have that $\{\gamma_2\}' = x^{-1} \{\gamma_1\}' x$ and hence we obtain that $\theta_1 = x^{-1} \theta x$ belongs to $x^{-1} \{\gamma_1\}' x = \{\gamma_2\}'$.

Clearly, then

(7) $y = \theta x = x(x^{-1} \theta x) = x \theta_1$.

So, we also have the symmetric property of $y$ with respect to $\Gamma_2$ (as $\theta_1 \in \{\gamma_2\}'$).

We now fix an element $x$ fixed by $\widetilde{\Gamma}_1$.

To construct the group $\Gamma_1$ from the statement of the proposition we let

$\Gamma_1 = \{\gamma_1 \in \Gamma \mid \text{there exists } \gamma_2 \in \Gamma \text{ such that } (\gamma_1, \gamma_2) \in \widetilde{\Gamma}_1\}$.

Since because of relation (5) the second component of an element in the group $\widetilde{\Gamma}_1$ must be a conjugate by $x$ of the first component, it follows that

$\widetilde{\Gamma}_1 = \{(\gamma_1, x \gamma_1^{-1} x) \mid \gamma_1 \in \Gamma_1\}$.

Because of (6), (7) this is independent of the choice of $x$ (as long as we choose an $x$ stabilized by $\widetilde{\Gamma}_1$).

From relations (6), (7) we deduce that the elements in the group that are stabilized by $\widetilde{\Gamma}_1$ are the elements of the set

(8) $\Gamma_1 x = \mathcal{M}(\Gamma_1) x = x(x^{-1} \Gamma_1' x)$.

By taking $\widetilde{\mathcal{M}}(\widetilde{\Gamma}_1)$ as in the statement of the proposition (here $\widetilde{\mathcal{M}}(\widetilde{\Gamma}_1)$ is a subgroup of $\widetilde{\Gamma}$), then relation (8) says exactly that the set of elements in the group stabilized by $\widetilde{\Gamma}_1$ is

$\Gamma_1' x = \widetilde{\mathcal{M}}(\widetilde{\Gamma}_1) x$.

The last statement in the proposition is an obvious, general fact about cosets in abstract groups. $\square$

Remark 7. Since we are only considering sequences that avoid eventually any finite set of points, with the notations from the previous proposition,
the groups \( \Gamma_1 \) (and the corresponding groups \( \widetilde{\mathcal{M}}(\Gamma_1) \)) such that \( \Gamma'_1 \) is finite, will not intervene in our computations.

To make the verifications for the conditions in our theorem we will thus have to compute the size of the intersections of the form \( \Gamma'_1 \cap \gamma \Gamma''_1 \gamma^{-1}, \gamma \in \Gamma \), or more generally \( (\Gamma^\alpha)' \cap (\Gamma^\beta)' \) where \( \Gamma^\alpha, \Gamma^\beta \) are groups as in the statement of Proposition 5. Note that the latest intersection is \( (\Gamma^\alpha, \Gamma^\beta)' \), where \( \Gamma^\alpha, \Gamma^\beta \) is the group generated by \( \Gamma^\alpha \) and \( \Gamma^\beta \).

We verify the conditions of the theorem for the group \( \Gamma = \text{PSL}_2(\mathbb{Z}[\frac{1}{p}]) \) where \( p \) is a prime number.

**Proposition 8.** Let \( \Gamma = \text{PGL}_2(\mathbb{Z}[\frac{1}{p}]) \) and let \( \widetilde{\Gamma}, \Gamma_0, \pi \) be as in Proposition 5. Then for a non-trivial group \( \widetilde{\Gamma}_1 \) that stabilizes an infinite set of points we choose \( \widetilde{\mathcal{M}}(\widetilde{\Gamma}_1) \) as in the statement of Proposition 5. The group \( \widetilde{\Gamma}_2 \) will be in this case the commutant of \( \widetilde{\Gamma}_1 \), since in this case the commutant will be maximally abelian. With these choices the hypothesis of Theorem 1 are verified.

**Proof.** Fix \( g \) an element of \( \text{PGL}_2(\mathbb{Z}[\frac{1}{p}]) \). There are two possibilities: either \( g \) viewed as a matrix with real entries has two distinct eigenvalues, or either \( g \) is conjugated to an element in the triangular group

\[
T_p = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in \mathbb{Z}[\frac{1}{p}] \right\} \cap \text{PGL}(2, \mathbb{Z}[\frac{1}{p}]).
\]

Note that here we are taking the quotient modulo the scalars.

In the first case, the commutant of \( g \) will be either finite (e.g., if \( g \) is conjugate to \( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \)) or a maximal abelian subgroup of \( \text{PGL}_2(\mathbb{Z}[\frac{1}{p}]) \) with trivial normalizer (and hence isomorphic to \( \mathbb{Z} \)).

In the second case the commutant will be the group \( T_p \) itself. It is obvious to see that \( T_p \) is a maximal abelian group with trivial normalizer.

Thus the possible groups \( \widetilde{\mathcal{M}}(\widetilde{\Gamma}_2) \) that appear in the statement of the theorem will be of the form

\[
\{(\gamma_2, x^{-1}\gamma_2x) \mid \gamma_2 \in \Gamma_2\}.
\]

Here \( x \) is stabilized by \( \widetilde{\Gamma}_2 \) and the group \( \Gamma_2 \) is either of the form

\[
(a) \quad \Gamma_2 = \{g^n\}, \ g \in \Gamma \text{ has distinct eigenvalues and } \Gamma_2 \cong \mathbb{Z}, \ \Gamma_2 \text{ maximal abelian}
\]

\[
(b) \quad \text{or either } \Gamma_2 \text{ is a conjugate of } T_p.
\]
Clearly, two subgroups as in (α), since they are infinite maximal abelian, if they have infinite intersection, then they coincide. No group of the type in (α) can intersect (except in the trivial element) a group in (β).

A simple computation shows that if \( g \) belongs to \( \Gamma = \text{PSL}_2(\mathbb{Z}[\frac{1}{p}]) \) and \( gT_pg^{-1} \cap T_p \) is non-trivial, then \( g \) must belong to \( T_p \) (this is a stronger property than having trivial normalizer).

Thus the conditions (i)-(v) of Theorem 1 are verified. To finish the proof we simply choose the family of normal subgroups \( H_n \) to be a decreasing family of modular subgroups, with trivial intersection \( \Box \)

Consequently, we get the following corollary.

**Corollary 9.** The group \( \Gamma = \text{PGL}_2(\mathbb{Z}[\frac{1}{p}]) \) has the property AO.

**Remark 10.** As Sergey Neshveyev and Makoto Yamashita kindly pointed out to us, the group \( \Gamma \) does not have the stronger related property \( S \) of Ozawa. Indeed, being a lattice in \( \text{PSL}_2(\mathbb{R}) \times \text{PGL}_2(Q_p) \) (by the work of Ihara), it is stably measurably equivalent to \( F_2 \times F_2 \). But as proven by Sako [S], the property \( S \) is preserved by stably measurable equivalence, and since \( F_2 \times F_2 \) does not have this property, it follows that \( \text{PGL}_2(\mathbb{Z}[\frac{1}{p}]) \) does not have property \( S \), but does have AO.

We will adapt the conditions of Theorem 1 for the group \( \text{SL}_3(\mathbb{Z}) \). For this purpose we introduce the following subgroups of \( \text{SL}_3(\mathbb{Z}) \).

Let \( H \) be the Heisenberg subgroup consisting of all matrices of the form

\[
\begin{pmatrix}
1 & * & * \\
0 & 1 & * \\
0 & 0 & 1
\end{pmatrix}
\]

with integer entries. Let \( \text{SL}_2(\mathbb{Z}) \subseteq \text{SL}_3(\mathbb{Z}) \) be the canonical representation of \( \text{SL}_2(\mathbb{Z}) \) as a subgroup of \( \text{SL}_3(\mathbb{Z}) \), that is the set of all matrices in \( \text{SL}_3(\mathbb{Z}) \) of the form

\[
\begin{pmatrix}
* & * & 0 \\
* & * & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

Let \( E \) be the matrix

\[
\begin{pmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

Let \( H_2 = H \cap \text{SL}_2(\mathbb{Z}) \). This is the abelian subgroup of triangular matrices.

It is an easy computation to see that, as for \( H_2 \), the subgroup \( H \) has the property that for \( \gamma \) in \( \text{SL}_3(\mathbb{Z}) \setminus H \) the intersection \( \gamma H \gamma^{-1} \cap H \) is the trivial subgroup.
In the case of $\text{SL}_3(\mathbb{Z})$, differently from the case of $\text{PSL}_2(\mathbb{Z}[1/\ell])$ the commutant of $E$ is equal to $\text{SL}_2(\mathbb{Z})$, a non-amenable group, and moreover the intersections $g^{-1}\text{SL}_2(\mathbb{Z})g \cap \text{SL}_2(\mathbb{Z})$ might be non-trivial, and infinite, for $g$ not belonging to $\text{SL}_3(\mathbb{Z})$. However, as we prove next, the above intersections of these orbits will be subsets of conjugates of the group $H$. More precisely,

**Remark 11.** Assume that $x, y$ are non-trivial elements of $\text{SL}_2(\mathbb{Z}) \subseteq \text{SL}_3(\mathbb{Z})$ and $g$ is non-trivial element in $\text{SL}_3(\mathbb{Z}) \setminus \text{SL}_2(\mathbb{Z})$ such that $gxg^{-1} = y$.

Then there exists $\gamma$ in $\text{SL}_2(\mathbb{Z})$ such that $\gamma^{-1}x\gamma$ belongs to $H_2$ and there exists $\gamma_0$ in $\text{SL}_2(\mathbb{Z})$ such that $\gamma = \gamma_0(h_\gamma \gamma^{-1})$.

**Proof.** By the results of Olga Tausky ([OT], see also the references in there), the conjugacy classes for elements in $\text{SL}_3(\mathbb{Z})$ are determined by ideal classes in the ring obtained by adjoining to $\mathbb{Z}$ the roots of the characteristic polynomial.

Hence if $x, y$ belong to $\text{SL}_2(\mathbb{Z})$ and are conjugated in $\text{SL}_3(\mathbb{Z})$, they are also conjugate in $\text{SL}_2(\mathbb{Z})$ and hence there exists $\gamma_0$ in $\text{SL}_2(\mathbb{Z})$ such that

$$gxg^{-1} = \gamma_0 x \gamma_0^{-1} = y.$$ But then $(\gamma_0^{-1}g)x(\gamma_0^{-1}g)^{-1} = x$ and hence $\gamma_0^{-1}g \in \text{SL}_3(\mathbb{Z}) \setminus \text{SL}_2(\mathbb{Z})$ commutes with $x$.

Since the only situations in which a non-trivial element $a$ in $\text{SL}_2(\mathbb{Z})$ contains in its commutant an element in $\text{SL}_3(\mathbb{Z}) \setminus \text{SL}_2(\mathbb{Z})$ is the case when $a \in H_2$, and the commutant is equal to the group $H \subseteq \text{SL}_3(\mathbb{Z})$ (or in the case of a conjugate of of $a$, that is in the case of $\gamma a \gamma^{-1}$ (with $a \in H_2$), $\gamma \in \text{SL}_2(\mathbb{Z})$ and then the commutant is $\gamma H \gamma^{-1}$).

Thus there exists $\gamma \in \text{SL}_2(\mathbb{Z})$ such that $x$ belongs to $\gamma H_2 \gamma^{-1}$ and $\gamma_0^{-1}g = h$, where $h$ belongs to $\gamma H\gamma^{-1}$. Thus $g = \gamma_0 h$, where $\gamma_0$ belongs to $\text{PSL}_2(\mathbb{Z})$ and $h$ belongs to $\gamma H \gamma^{-1}$ (and $x$ is in $\gamma H_2 x^{-1}$). \(\square\)

We will adapt the proof of Theorem 1 to the situation of $\text{SL}_3(\mathbb{Z})$. We will use the same setting as in the proof of Theorem 1, with the identifications from Proposition 6.

**Theorem 12.** Let $\Gamma = \text{SL}_3(\mathbb{Z})$. Then the property AO holds true for $\Gamma$.

**Proof.** We consider, exactly as in the settings in the proof of Theorem 1, a free ultrafilter $\omega$, finite sets $(A_n)$ in $\Gamma$ (which in the proof of Theorem 1 is $\bar{\Gamma}/\Gamma_0$) and consider $C_\omega(A) \subseteq \beta(\Gamma)$ and the probability measure $\mu_{\omega,A}$ on $C_\omega(A)$ (or a direct sum of such spaces, with mutually singular Loeb measures) and consider the set $C^0_{\omega,A}$ as before.
The modifications of the proof required to pass from \( \varphi_{\omega,A} \) to the more general states \( \varphi_{\omega,B,\lambda} \) are identically as in the proof of Theorem 1, since the only part that changes in the case of \( \text{SL}_3(\mathbb{Z}) \) is the part concerning cosets, and here because our “integrals” for vector as in the definition after Proposition 4 behave well with respect to domain operations, the proof are identical for \( \varphi^0_{\omega,A} \) and \( \varphi^0_{\omega,B,\lambda} \) (the notations are from the proof of Theorem 1).

As in the proof of Theorem 1, it remains to analyze the state (up to a scalar re-normalizing the total mass to be 1)

\[
\gamma \rightarrow \mu_{\omega,A} \left( (C_{\omega}(A) \setminus C^0_{\omega}(A)) \cap \gamma (C_{\omega}(A) \setminus C^0_{\omega}(A)) \right), \gamma \in \tilde{\Gamma}.
\]

As in the proof of Theorem 1, \( C_{\omega}(A) \setminus C^0_{\omega}(A) \) will consists of disjoint unions of blocks of the form \( C_{\omega}(A) \cap \tilde{M}(\tilde{\Gamma}_2)x \) (\( \tilde{\Gamma}_2 \) as in property (ii)). In this case, however the translates of these blocks have no longer finite intersections, as described in Remark 11.

We start first with the analysis of the possible commutants for elements \( g \) in \( \text{SL}_3(\mathbb{Z}) \) (as in the case of \( \text{PSL}_2(\mathbb{Z}[\frac{1}{p}]) \)).

**Case (\( \alpha \)).** If \( g \) belongs to a conjugate \( \gamma H\gamma^{-1}, \gamma \in \Gamma, \) of the group \( H \), (that is \( g \) has a single eigenvector) then \( \{g\}' \subseteq \gamma H\gamma^{-1}. \) In this situation the group \( M(\tilde{\Gamma}_2) \) will be a subset of \( \gamma H\gamma^{-1}. \)

In this situation, instead of choosing the maximal blocks \( M(\tilde{\Gamma}_2)x \) from the proof of Theorem 1, we will work with cosets \( (\gamma_1 H\gamma_1^{-1})x \) of conjugates of the group \( H. \)

Since the group \( H \) has the property that \( \gamma H\gamma^{-1} \cap H \) is trivial for \( \gamma \) in \( \text{SL}_3(\mathbb{Z}) \setminus H, \) it follows that the cosets \( (\gamma_1 H\gamma_1^{-1})x_1 \) and \( (\gamma_2 H\gamma_2^{-1})x_2 \) have at most finite intersections, unless \( \gamma_1 H = \gamma_2 H \) and \( x_2 \) belongs to \( (\gamma_1 H\gamma_1^{-1})x_1. \)

Hence the sets \( C_{\omega}((A_n \cap (\gamma H\gamma^{-1})x)_n) \) where \( \gamma \) runs over a system of representatives of \( \Gamma/H \) and \( x \) runs over system of cosets representatives, are disjoint. The argument for this part of the state \( \varphi_{\omega,A} \) is then identical to the cases in the Theorem 1.

**Case (\( \beta \)).** If \( g \) has three distinct eigenvalues, no one of them equal to 1, then \( \{g\}' \) is infinite and is isomorphic to \( \mathbb{Z}, \) and the argument is than exactly as for \( \text{PSL}_2(\mathbb{Z}[\frac{1}{p}]) \). The corresponding maximal blocks do not intersect the cosets from \( (\alpha) \) in infinite sets, since the elements \( g \) in this case are not conjugated to elements in the group \( H, \) which are considered in case \( (\alpha). \)

**Case (\( \gamma \)).** The remaining case is the case of elements \( g \) that are conjugated to elements \( g_1 \) in the group \( \text{SL}_2(\mathbb{Z}), \) with distinct eigenvalues.

In this case, the element \( E \) in \( \text{SL}_2(\mathbb{Z}) \) will also commute with \( g_1. \)
The minimal group in property (ii), Theorem 1, will be the corresponding conjugate of the group \( \Gamma_2 = \mathbb{Z}_2 \) and it will be generated by \( E \). The cosets \( \bar{M}(\Gamma_2)x \) do not have the intersection property (iv) from Theorem 1.

However, let \( T \) be the reunion of all the sets, and their translates, considered in \( \alpha \). Excluding this sets, whose contribution to the state \( \varphi_{\omega,A} \) has already been analyzed in point \( \alpha \), we will reduce ourselves to a similar condition as in Theorem 1.

Thus if \( T_0 = \bigcup_{x} C_{\omega}((A_n \cap \gamma H \gamma^{-1} x)_n) \) and \( T = \bigcup gT_0 \), where \( \gamma \) runs over system of representatives for \( \Gamma/H \), \( x \) runs over system of coset representatives, and \( g \) runs over the group \( \bar{\Gamma} \).

In the case \( \alpha \) we proved that the state (up to a positive scalar)

\[
\varphi_{\omega,A}^T(g) = \mu_{\omega,A}(gT_0 \cap T_0), \quad g \in \bar{\Gamma}
\]

is continuous on \( C^*_\text{red}(\bar{\Gamma}) \).

Thus by excluding the state \( \varphi_{\omega,A}^T \) it remains to analyze the state corresponding to the blocks \( C_{\omega}((A_n \cap (\gamma^{-1} \text{SL}_2(\mathbb{Z}) \gamma) x)_n) \), from which we exclude \( T \).

Consequently, for \( \gamma \) running over a system \( X_0 \) of representatives of \( \Gamma/\text{SL}_2(\mathbb{Z}) \) and for \( x \) in a system of representatives \( Y_0 \) for cosets of \( \gamma \text{SL}_2(\mathbb{Z}) \gamma^{-1} \), let

\[
P_{\gamma,x} = C_{\omega}((A_n \cap (\gamma^{-1} \text{SL}_2(\mathbb{Z}) \gamma) x)_n) \setminus T.
\]

By Remark 9, the set \( P_{\gamma,x} \) are disjoint and moreover the orbits \( gP_{\gamma,x}, g \in \bar{\Gamma} \) do not intersect, unless they are equal.

Thus it remains to evaluate the state (state up to a positive factor)

\[
\mu_{\omega,A}(g) = \sum_{(\gamma_1,x_1), (\gamma,x) \in X_0 \times Y_0} \mu_{\omega,A}(P_{\gamma,x} \cap gP_{\gamma_1,x_1}), \quad g \in \bar{\Gamma}.
\]

We may particularize (for the purpose of proving continuity with respect to \( C^*_\text{red}(\bar{\Gamma}) \)), to the case of a finite sum, and then by translation to the case of a single term.

The state given by

\[
\varphi_{\omega,A}^{P_{\gamma,x}}(g) = \mu_{\omega,A}(P_{\gamma,x} \cap gP_{\gamma,x}), \quad g \in \bar{\Gamma}
\]

is a state supported on \( \text{SL}_2(\mathbb{Z}) \subseteq \text{SL}_3(\mathbb{Z}) \). It corresponds to a state on \( \text{SL}_3(\mathbb{Z}) \), where all \( (A_n) \) are concentrated in a coset \( \text{SL}_2(\mathbb{Z})x \subseteq \text{SL}_3(\mathbb{Z}) \).

By the Akemann-Ostrand property of the group \( \text{SL}_2(\mathbb{Z}) \), the restriction of \( \varphi_{\omega,A}^{P_{\gamma,x}} \) to \( \text{SL}_2(\mathbb{Z}) \) is continuous on \( C^*_\text{red}(\text{SL}_2(\mathbb{Z})) \) ([AO], [Oz]). Since \( \varphi_{\omega,A}^{P_{\gamma,x}} \)
is zero on $\text{SL}_3(\mathbb{Z}) \setminus \text{SL}_2(\mathbb{Z})$ it follows that the state is continuous in fact on $C^*_\text{red}(\text{SL}_3(\mathbb{Z}))$. □

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