Central Limit Theorems in Deterministic Systems

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Abstract This is a note on some results of the central limit theorem for deterministic dynamical systems. First, we give the central limit theorem for martingales, which is a main tool. Then we give the main results on the central limit theorem in dynamic system in the cases of martingale and backward martingale.

Keywords: Martingale, Central Limit Theorem, Dynamic System

1 Introduction

I learned the central limit theorem in dynamic systems under the guidance of Teacher Xie Jiansheng. This is a note taking down the main theorems and conclusions in the literature I studied. These results come from [1], [2], [3], [7] in the references. In order to make it easy for readers to understand, I may revise the original literature proof, and I will also use the results in books [4], [5], [6] to support the corresponding proof.

Without specification, the proof of the results in this paper is always based on the following assumption: there is a probability space (Ω, F, P) and a measure preserving mapping of T : Ω → Ω, i.e. for A ∈ F, then P(A) = P(T^(-1)(A)). It is also assumed that if A ∈ F, then T(A) ∈ F, and the measure preserving system (P, T) are ergodic. Suppose X is a measurable function on Ω, denote U : X ↦ U(X) = X ◦ T.

Under these assumptions, for the measurable function f on Ω which satisfies some conditions, we expect the following to be true:

\[ \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} U^i f \rightarrow N(0, \sigma^2) \]

To attain this goal, we will use method of martingale approximation. Set F_0 ⊂ F is a sub σ algebra, and F_k = T^(-k)F_0, k ∈ Z, then there are two general cases at this time

(1) \cdots ⊃ F_1 ⊃ F_0 ⊃ F_1 \supset \cdots
(2) \cdots ⊂ F_1 ⊂ F_0 ⊂ F_1 \subset \cdots

In fact, (1) and (2) correspond respectively to inverted martingales and martingales. Section 2 will describe the central limit theorem of martingales. Results of
inverted martingales and martingale are given respectively in section 3 and section 4.

Remark. 1. Under the above assumptions, if $U$ is considered an operator of $L^2(\Omega) \to L^2(\Omega)$, then $U^*U = I$; further if $T$ is injective, then $U$ is the unitary operator of $L^2(\Omega) \to L^2(\Omega)$.

In fact, since $T$ is measure preserving, $\forall X, Y \in L^2(\Omega)$

$$\int_{\Omega} (X \circ T)(Y \circ T)dP = \int_{\Omega} XYdP$$

Thus $U^*U = I$.

If $T$ is injective, from $P(T(\Omega)) = P(\Omega)$, for almost every $\omega$, we can define $T^{-1}(\omega)$. So for $X \in L^2(\Omega)$, we can define $X \circ T^{-1}(\omega)$, For any $Y \in L^2(\Omega)$,

$$\int_{\Omega} (X \circ T)YdP = \int_{\Omega} X(Y \circ T^{-1})dP$$

From above formula, we see $U^*(Y) = Y \circ T^{-1} = U^{-1}(Y)$, that is, $U^* = U^{-1}$, that is, $U$ is a unitary operator.

Literature [1][2][3] do not emphasize the relationship between the properties of $T$ (such as injective) and $U$ (such as whether it is a unitary transformation). But they add some similar conditions, for example in Theorem 1 of [1] (i.e. Theorem 3.1 of this paper), suppose $E(UU^*\phi|F_1) = E(\phi|F_1)$. In addition, literature [2] suppose $T$ is surjective in the whole text. These conditions are a bit weird at first, but as discussed above, they are reasonable. In contrast, there are no assumptions about $T$ in literature [7], pointing out directly $U$ it is a unitary transformation.

2. Next, what we need to explain is the relationship between the sequence $U^if$ in the introduction and the stationary, ergodic sequence: If $Y_i = U^if$, then the sequence $\{Y_i, i = 1 \ldots\}$ is stationary ergodic sequence. Conversely, given a sequence of stationary random sequences $\{Y_n, i = 1, 2, \ldots\}$, when state space $S$ is good, we can set a probability measure $P$ on $S^N$ so that the random process $\{X_n(\omega) = \omega_N, n = 1, 2, \ldots\}$ has the same distribution as $\{Y_n\}$. For a specific discussion please see [6] example 6.1.4.
2 Central Limit Theorem of Martingale

The core method of all literatures reviewed is to approximate the sequence $U_i f$ with martingale difference (or backward martingale difference). The central limit theorem of martingale is introduced below, which is also the basis of other results.

**Theorem 2.1** ([4]). (Martingale’s Central Limit Theorem) ¹ Let $Y_i$ be a sequence of stationary, ergodic, martingale difference with second-order moments, the central limit theorem holds, that is,

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} Y_i \rightarrow N(0, \sigma^2)$$

Here $\sigma^2 = \text{Var}(Y_i)$.

**Proof.** First, we define

$$\psi(n, j, t) = \exp\left(\frac{\sigma^2 t^2 j}{2n} \right) E\{\exp[it \frac{Y_i + \cdots + Y_j}{\sqrt{n}}]\}$$

$$S_n = Y_1 + \cdots + Y_n$$

$$\theta(n, j, t) = \exp\left(\frac{\sigma^2 t^2 j}{2n} \right) E\{\exp[it \frac{S_{j-1}}{\sqrt{n}}][\frac{(\sigma^2 - Y_j^2)t^2}{2n}]\}$$

$$\theta_k(n, j, t) = \exp\left(\frac{\sigma^2 t^2 kr}{2n} \right) E\{\exp[it \frac{S_{kr}}{\sqrt{n}}][\frac{(\sigma^2 - Y_j^2)t^2}{2n}]\}, \quad kr + 1 \leq j \leq k(r + 1),$$

Here $k$ is an integer.

To prove the theorem, according to the continuity theorem of characteristic function, it suffices to show that for fixed $t$ when $n \rightarrow \infty$, we have

$$\psi(n, n, t) - 1 \rightarrow 0$$

Also notes that

$$|\psi(n, n, t) - 1| = \left| \sum_{j=1}^{n} [\psi(n, j, t) - \psi(n, j - 1, t)] \right|$$

$$\leq \left| \sum_{j=1}^{n} [\psi(n, j, t) - \psi(n, j - 1, t)] - \theta(n, j, t) \right| + \left| \sum_{j=1}^{n} \theta(n, j, t) - \theta_k(n, j, t) \right| + \left| \sum_{j=1}^{n} \theta_k(n, j, t) \right|$$

¹Here Thanks Carlanelo Liverani at University of Rome "Tor Vergata" Life for helping me find this proof.
Next let’s estimate the last three terms in turn, first of all, for \(|t| < T\), we have

\((2.1)\)

\[
||\psi(n, j, t) - \psi(n, j - 1, t)| - \theta(n, j, t)|
\]

\[
= \exp\left[\frac{\sigma^2 t^2 j}{2n}\right]E\{\exp[\frac{S_{j-1}}{\sqrt{n}}] \left[ 1 - it \frac{Y_i}{Y_i} + \frac{Y^2 t^2}{2n} \right] - \exp(-\frac{\sigma^2 t^2}{2n} + \frac{t^2 \sigma^2}{2n}) \}\}
\]

\[
\leq C(T)E\{\left[ 1 - it \frac{Y_i}{Y_i} + \frac{Y^2 t^2}{2n} \right] \} + C(T)\exp(-\frac{\sigma^2 t^2}{2n} + \frac{t^2 \sigma^2}{2n})
\]

The first equation uses the properties of martingales:

\[
E\{\exp[\frac{S_{j-1}}{\sqrt{n}}] \xi_j \} = E(E(\exp[\frac{S_{j-1}}{\sqrt{n}}] \xi_j | S_{j-1})) = E(\exp[\frac{S_{j-1}}{\sqrt{n}}] E(\xi_j | S_{j-1})) = 0
\]

\(C(T)\) is a constant that only relates to \(T\). From the formula (2.1)

\[
\sup_{|t| \leq T} \sum_{j=1}^{n} ||\psi(n, j, t) - \psi(n, j - 1, t)| - \theta(n, j, t)| = \text{no}(\frac{1}{n}) \to 0
\]

Secondly we estimate \(\sum_{j=1}^{n} |\theta(n, j, t) - \theta(n, j, t)|\), select an integer \(k\), large and fixed. Divide \([1, n]\) into blocks of length \(k\) in turn. There may be incomplete blocks at the end.

\((2.2)\)

\[
\sum_{j=1}^{n} |\theta_k(n, j, t) - \theta(n, j, t)| \leq n \sup_{1 \leq j \leq n} |\theta_k(n, j, t) - \theta(n, j, t)|
\]

\[
\leq C(T) \sup_{1 \leq j \leq k} E\{||\exp[\frac{\sigma^2 t^2 j}{2n}]| - 1||\sigma^2 - Y_j^2|\}
\]

According to the control convergence theorem, for each \(k\), when \(n \to \infty\), the above equation tends to 0.

At last we estimate \(\sum_{j=1}^{n} \theta_k(n, j, t)\), by the stationarity of \(Y_i\) and \(r \leq \frac{n}{k}\), we know

\[
\sum_{j=k(r+1)}^{k(r+1)} |\theta_k(n, j, t)| \leq \frac{C(T)}{n} E\{|| \sum_{j=k(r+1)}^{k(r+1)} (\sigma^2 - Y_j^2) ||\} = C(T) \frac{k}{n} \delta(k)
\]

According to Birkhoff’s Ergodic Theorem, when \(k \to \infty\), \(\delta(k) \to 0\). Since the upper estimate holds for all \(r\), and there are at most \(\frac{n}{k}\) blocks, thus we get

\[
\sum_{j=1}^{n} |\theta_k(n, j, t)| \leq C(T) \delta(k)
\]

combining with the formula (2.2), we have

\[
|\sum_{j=1}^{n} \theta(n, j, t)| \leq \sum_{j=1}^{n} |\theta_k(n, j, t)| + \sum_{j=1}^{n} |\theta_k(n, j, t) - \theta(n, j, t)|
\]

\[
\leq C(T) \delta(k) + C(T) \sup_{1 \leq j \leq k} E\{||\exp[\frac{\sigma^2 t^2 j}{2n}]| - 1||\sigma^2 - Y_j^2|\}
\]
Let $n \to \infty$, then let $k \to \infty$, we get $\lim_{n \to \infty} |\sum_{j=1}^{n} \theta(n,j,t)| \to 0$

Combining with formula (2.1), we get

$$\left| \sum_{j=1}^{n} \psi(n,j,t) - \psi(n,j-1,t) \right| \leq |[\psi(n,j,t) - \psi(n,j-1,t)] - \theta(n,j,t)| + |\sum_{j=1}^{n} \theta(n,j,t)| \to 0$$

$\square$

3 Backward martingale difference approximation

Using the martingale central limit theorem in the previous section, the first main result is proved in this section. The result is first proved in [1]

Theorem 3.1 ([1]). Consider a probability space $(\Omega, \mathcal{F}, P)$ and a measure preserving mapping $T: \Omega \to \Omega$. we assume that if $A \in \mathcal{F}$, then $T(A) \in \mathcal{F}$, and the measure preserving system $(P, T)$ is ergodic. we denote $\mathcal{F}_0$ a sub $\sigma$ algebra of $\mathcal{F}$ and define $\mathcal{F}_i = T^{-i}\mathcal{F}_0, i \in \mathbb{Z}$, and $\cdots \supset \mathcal{F}_{-1} \supset \mathcal{F}_0 \supset \mathcal{F}_1 \supset \cdots$ and for $\phi \in L^{\infty}(\Omega)$, we have

(3.1) $E(UU^*\phi|\mathcal{F}_1) = E(\phi|\mathcal{F}_1)$

Then for each $f \in L^{\infty}(\Omega)$ which satisfies:
(1) $E(f) = 0, E(f|\mathcal{F}_0) = f$,
(2) $\sum_{n=0}^{\infty} |E(f U^n f)| < \infty$,
(3) $\sum_{n=0}^{\infty} E(U^n f|\mathcal{F}_0)$ converges almost everywhere,

Then the central limit theorem is valid, that is,

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n} U^i f \to N(0, \sigma^2)$$

Here $\sigma$ satisfies $\sigma^2 \leq -E(f^2) + 2 \sum_{n=0}^{\infty} |E(f U^n f)|$.

Further, $\sigma = 0$ if and only if there exists a $\mathcal{F}_0$ measurable function $g$, such that

$$Uf = Ug - g$$

Finally, if condition (2) converges in the sense of $L^1$, then $\sigma^2 = -E(f^2) + 2 \sum_{n=0}^{\infty} E(f U^n f)$.

Remark. Due to the remark 1 of the section 1, $U^*U = I$ is always valid. Note that this theorem does not assume that $T$ is injective, $U$ is not necessarily a unitary transformation, so the condition (3.1) is to ensure that $UU^* = I$ to some extent.

Before formally proving the theorem, we first prove three lemmas

Lemma 3.2. Set a family of $\sigma$ fields $\mathcal{F}_0 \supset \mathcal{F}_1 \supset \cdots$ and a family of random variables $Y_i, i = 1, 2 \ldots$ satisfies that $Y_i, i \geq 1$ is $F_{i-1}$ measurable, and

$$E(Y_i|\mathcal{F}_i) = 0, i \geq 0$$

Assume also $\sum_{i=0}^{\infty} Y_i$ converges everywhere, then $Y_i$ is backward martingale difference.

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Proof. Denote $X_i = Y_{i+1} + Y_{i+2} + \ldots, i = 0, 1, \ldots$, then $X_i$ is $\mathcal{F}_i$ measurable by the condition with

$$E(X_i|\mathcal{F}_{i+1}) = E(Y_{i+1} + Y_{i+2} + \ldots|\mathcal{F}_{i+1}) = Y_{i+2} + Y_{i+3} \ldots = X_{i+1}$$

So we know that $X_i$ is backward martingale difference. \qed

**Lemma 3.3.** Suppose a sequence of random variable $X_n$ converges to $Z$, $Y_n$ converges to zero in distribution, then $X_n + Y_n$ converges to $Z$ in distribution.

**Proof.** For any continuous point of the distribution function of $Z$, $d$ for example, we have

$$P(X_n + Y_n \leq d) \leq P(X_n \leq d + \epsilon) + P(|Y_n| \geq \epsilon)$$

let $n \to \infty$, and then let $\epsilon \to 0$, we get $\limsup P(X_n + Y_n \leq d) \leq P(Z \leq d)$

For the same reason, from

$$P(X_n + Y_n \leq d) \geq P(X_n \leq d - \epsilon) - P(|Y_n| \geq \epsilon)$$

we get $\liminf P(X_n + Y_n \leq d) \geq P(Z \leq d)$. That is, $\lim P(X_n + Y_n \leq d) = P(Z \leq d)$. Conclusion follows. \qed

**Lemma 3.4.** $E(U^n \phi|\mathcal{F}_n) = U^n E(\phi|\mathcal{F}_0)$.

*This can be obtained from the transformation formula of conditional mathematical expectation integral, see Section 2.4.7 of the Elements of Probability [5] for details*

**proof of theorem 3.1.** We have three steps to prove the first part of the theorem 3.1.

**step1** We want to decompose $U^n f$ as follows:

$$(3.2) \quad U^n f = Y_n + U^n g - U^{n-1} g, n \geq 1$$

Here, $g$ is almost everywhere finite and $\mathcal{F}_0$ measurable. $Y_i$ is the backward martingale difference, in particular, $Y_i$ satisfies $E(Y_i|\mathcal{F}_i) = 0, i \geq 0$.

Taking $n = 1$ in (3.2), we get $U f = Y_1 + U g - g$, taking conditional expectation of $\mathcal{F}_1$ on both side, combining with lemma 3.4, we obtain $UE(f|\mathcal{F}_0) = UE(g|\mathcal{F}_0) - E(g|\mathcal{F}_1)$, notice that $f, g$ is $\mathcal{F}_0$ measurable, and then act on $U^*$, we get

$$(3.3) \quad f = g - U^* E(g|\mathcal{F}_1) = g - U^* E(UU^* g|\mathcal{F}_1) = g - E(U^* g|\mathcal{F}_0)$$

Denote $T_0 : \phi \mapsto E(U^* \phi|\mathcal{F}_0)$, based on the knowledge of functional analysis, the equation

$$f = (I - T_0)g$$

has a unique solution $g = \sum_{n=0}^{\infty} T_0^n f$
claim 3.5. $T_0^n f = E(U^n | F_0)$

In fact, it suffices to notice that

\[ T_0^n E(U^n f | F_0) = U^n E(U^n f | F_0) F_0 \]

\[ = U^n E(U^n f | F_0) = U^n E(U^n f | F_0) F_1 = U^n E(UU^n f | F_0) \]

So we get

\[
(3.4) \quad g = \sum_{n=0}^{\infty} E(U^n f | F_0)
\]

From the assumptions, we can see that $g$ is well defined. Next, we will bring the above formula into (3.2) and prove that the decomposition of (3.2) is reasonable.

First, obviously, $g$ is $F_0$ measurable and after calculation, we get

\[
Y_i = \sum_{i=0}^{\infty} E(U^n f | F_{i-1}) - \sum_{i=0}^{\infty} E(U^n f | F_i), \quad i \geq 1
\]

It is easy to know $Y_i$ is $F_{i-1}$ measurable, and $E(Y_i | F_i) = 0, i \geq 1$, and $\sum_{i=0}^{\infty} Y_i$ converges almost everywhere to $\sum_{i=0}^{\infty} E(U^n f | F_0)$, according to lemma 3.2, $Y_i$ is backward martingale difference.

claim 3.6. $Y_i$ is a stationary sequence

In fact, from the formula (3.2), $Y_i = U^{i-1} Y_1 = Y_1 \circ T^{i-1}$, where $T$ is the measure preserving transformation. For the positive integer $n, m$, and the measurable set $A_i \in F, i = 1 \ldots n$ we have

\[
P(Y_{m+1} \in A_1, Y_{m+2} \in A_2, \ldots, Y_{m+n} \in A_n) = P(Y_1 \in T^{m+1} A_1 \cap \cdots \cap T^{m+n} A_n)
\]

\[
= P(Y_1 \in T^1 A_1 \cap \cdots \cap T^n A_n)
\]

\[
= P(Y_1 \in A_1, Y_2 \in A_2, \ldots, Y_n \in A_n)
\]

thus $Y_i$ is a stationary sequence

step 2 From the above decomposition,

\[
\frac{1}{\sqrt{n}} \sum_{i=0}^{n} U^n f = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} Y_i + \frac{1}{\sqrt{n}} (U^n g - g + f)
\]

Then $\frac{1}{\sqrt{n}} (U^n g - g + f)$ converges to 0 in probability. In fact, for any $\epsilon > 0$, and $T$ is a measure guaranteed mapping, we have

\[
P(|\frac{1}{\sqrt{n}} (U^n g - g + f)| > \epsilon) = P(|(U^n g - g + f)| > \epsilon \sqrt{n})
\]

\[
\leq P(|(U^n g)| > \frac{\epsilon}{2} \sqrt{n}) + P(|(f - g)| > \frac{\epsilon}{2} \sqrt{n})
\]

\[
= P(|g| > \frac{\epsilon}{2} \sqrt{n}) + P(|(f - g)| > \frac{\epsilon}{2} \sqrt{n})
\]
By \( f \in L^\infty(\Omega) \), \( g \) are almost bounded everywhere. It can be seen that the above equation is tend to 0 when \( n \to \infty \).

From lemma 3.3, we know if we want to prove \( \frac{1}{\sqrt{n}} \sum_{i=0}^{n} U_i f \) converges in distribution, it suffices to prove \( Y_i \) is a square integrable, stationary, ergodic backward martingale difference.

That \( Y_i \) is stationary is obtained from the claim 3.6. Note that \( Y_i = U_i^{-1} Y_1 = Y_1 \circ T_i^{-1} \), ergodicity is guaranteed by the measure preserving system, and it only needs to prove \( Y_i \) is square integrable.

**step3** we will prove \( Y_i \) is square integrable in this step.

In fact, the method is similar to the previous one, that is, using martingale difference to approximate \( Y_i \), we want to find \( Y_i(\lambda), \lambda > 1 \), making \( Y_i(\lambda) \) is \( \mathcal{F}_{i-1} \) measurable, and

\[
E(Y_i(\lambda)|\mathcal{F}_i) = 0
\]

as well as

\[
U_i f = Y_i(\lambda) + U_i g(\lambda) - \lambda^{-1} U_i^{-1} g(\lambda)
\]

The same discussion as before shows that \( g(\lambda) = \sum_{n=0}^{\infty} \lambda^{-n} E(U^n f|\mathcal{F}_0) \), note \( E(U^n f|\mathcal{F}_0) \leq ||f||_{L^\infty} \) and \( \lambda > 1 \). So \( g(\lambda) \in L_\infty \subset L_2 \), and \( \lim_{\lambda \to 1} g(\lambda) = g(1) = g \), so \( \lim_{\lambda \to 1} Y_i(\lambda) = Y_i \).

Due to the stationarity and that \( U \) is a unitary transformation, we have

\[
E(U f(U f - U g(\lambda) + \lambda^{-1} g(\lambda))) = E(E(U f(U f - U g(\lambda) + \lambda^{-1} g(\lambda))|\mathcal{F}_1))
\]

\[
= E(U f E((U f - U g(\lambda) + \lambda^{-1} g(\lambda))|\mathcal{F}_1)) = E(U f E((Y_1|\mathcal{F}_1)) = 0
\]

therefore

\[
E(Y_i(\lambda)^2) = E(Y_i(\lambda)^2) = E([U f - U g(\lambda) + \lambda^{-1} g(\lambda)]^2)
\]

\[
= -E((U f)^2) + E([U g(\lambda) - \lambda^{-1} g(\lambda)])
\]

\[
= -E(f^2) + E(U g(\lambda)|U g(\lambda) - \lambda^{-1} g(\lambda)]) - \lambda^{-1} E(g(\lambda U g(\lambda))) + \lambda^{-2} E(U g(\lambda)^2)
\]

\[
= -E(f^2) + 2E(U g(\lambda)|U g(\lambda) - \lambda^{-1} g(\lambda)]) - (1 - \lambda^{-2}) E((g(\lambda)^2)
\]

\[
= -E(f^2) + 2E(g(\lambda)f) - (1 - \lambda^2) E((g(\lambda)^2)
\]

\[
\leq -E(f^2) + 2 \sum_{n=0}^{\infty} \lambda^{-n} E(f U^n f)
\]

\[
= -E(f^2) + 2 \sum_{n=0}^{\infty} \lambda^{-n} E(f U^n f)
\]

\[
\leq -E(f^2) + 2 \sum_{n=0}^{\infty} |E(f U^n f)|
\]

Finally, by the fatous lemma,

\[
E(Y_i^2) = E(\lim_{\lambda \to 1} Y_i(\lambda)^2) \leq \lim_{\lambda \to 1} E(Y_i(\lambda)^2) \leq -E(f^2) + 2 \sum_{n=0}^{\infty} |E(f U^n f)|
\]
Thus $Y_1$ is square integrable. So far, the first part of the theorem has been proved.

For the second part of the theorem, we let $n = 1$ in the formula (3.2) and get

$$\sigma^2 = E(Y_1^2) = E[(Uf - Ug + g)^2]$$

Therefore, $\sigma = 0 \iff \exists F_0$ measurable function $g$, such that $Uf = Ug - g$

For the third part of the theorem, note that

$$|E(Y_1^2) - \{ -E(f^2) + 2 \sum_{n=0}^{\infty} E(fU^n f) \}| \leq \lim_{\lambda \to 1} |E(Y_1(\lambda)^2) - \{ -E(f^2) + 2 \sum_{n=0}^{\infty} E(fU^n f) \}| = 0$$

We prove that the above two formulas tend to 0 when $\lambda \to 1$, so that the conclusion follows. For the convenience of proof, we first consider the second term, combining with the formula 3.5, we have

$$|E(Y_1(\lambda)^2) - \{ -E(f^2) + 2 \sum_{n=0}^{\infty} E(fU^n f) \}| \\
\leq 2 \sum_{n=0}^{\infty} (1 - \lambda^{-n})|E(fU^n f)| + (1 - \lambda^{-2})E(g(\lambda)^2) \\
\leq 2(1 - \lambda^M) \sum_{n=0}^{\infty} |E(fU^n f)| + 2 \sum_{n=M}^{\infty} |E(fU^n f)| + (1 - \lambda^{-2})E(g(\lambda^2))$$

Here, $M$ is a large positive integer and fixed. To estimate $(1 - \lambda^{-2})E(g(\lambda^2))$, for $\lambda > 1, \mu > 1$, we have

$$E(g(\lambda)g(\mu)) = \sum_{n=0, m=0}^{\infty} \lambda^{-n} \mu^{-m} E(U^{*n} f E(U^{*m} f | F_0))$$

$$\leq \sum_{n=0}^{\infty} \lambda^{-n} \sum_{m=0}^{M-1} ||f|| \infty E(|E(U^{*n} f | F_0)|) + \sum_{n=0}^{\infty} \lambda^{-n} \sum_{m=M}^{\infty} ||f|| \infty E(|E(U^{*m} f | F_0)|)$$

$$\leq M ||f|| \infty \sum_{n=0}^{\infty} E(|E(U^{*n} f | F_0)|) + \frac{||f|| \infty}{1 - \lambda^{-1}} \sum_{m=M}^{\infty} E(|E(U^{*m} f | F_0)|)$$

Combining the formula (3.7) and formula (3.8), we can see that the second term of (3.6)

$$\lim_{\lambda \to 1} |E(Y_1(\lambda)^2) - \{ -E(f^2) + 2 \sum_{n=0}^{\infty} E(fU^n f) \}| = 0$$
For the first term (3.6), notice that \( \lambda \geq \mu > 1 \),

\[
E([Y_1(\lambda) - Y_1(\mu)]^2) = E([\lambda^{-1}g(\lambda) - \mu^{-1}g(\mu)][Y_1(\lambda) - Y_1(\mu)])
\]
\[
= E([\lambda^{-1}g(\lambda) - \mu^{-1}g(\mu)]^2) + E([g(\lambda) - g(\mu)]^2)
\]
\[
\leq (1 - \lambda^{-1}\mu^{-1})E(g(\lambda)g(\mu))
\]

This means that in the sense of \( L^2 \), \( \lim_{\lambda \to 1} Y_1(\lambda) = Y_1 \), so

\[
\lim_{\lambda \to 1} E(Y_1^2) - E(Y_1(\lambda)^2) = 0
\]

To sum up, the third part of the theorem is valid \( \Box \)

For the case that \( T \) is a injective, we have the following theorem:

**Theorem 3.7** ([1]). Assume that \( T \) is a invertible measure preserving mapping, \( \mathcal{F}_i \subset \mathcal{F}_{i-1} \), then for \( f \in L^\infty(X) \), \( E(f) = 0 \) satisfies:

1. \( \sum_{n=0}^{\infty} |E(fU^n f)| < \infty \),
2. \( \sum_{n=0}^{\infty} E(U^{-n} f | \mathcal{F}_0) \) convergence in \( L^1 \),
3. \( \exists \alpha > 1 : \sup_{k \in \mathbb{N}} k^\alpha E(|E(f|\mathcal{F}_{-k}) - f|) < \infty \),

Then the central theorem holds, that is

\[
\frac{1}{\sqrt{n}} \sum_{i=0}^{n} U^i f \to N(0, \sigma^2)
\]

Here \( \sigma^2 = -E(f^2) + 2 \sum_{n=0}^{\infty} E(fU^n f) \).

**Remark.** The proof idea is similar to the previous theorem. The following is just the idea of proof. See [1] for the details

**Proof.** Note that we do not assume that \( f \) is \( \mathcal{F}_0 \) measurable, so the main idea is to use \( E(f|\mathcal{F}_{-k}) \) to approximate \( f \), that is, using the method in the previous theorem to prove

\[
S_k^n = \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} U^i E(f|\mathcal{F}_{-k})
\]

converge to \( N(0, \sigma_k^2) \) in distribution, and then let \( k \to \infty \), proving that the left side tends to \( S_n \), right side tend to \( \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} U^i f \). And result follows. \( \Box \)
4 Martingale difference approximation

This section will be discussed under the condition (1.1) (2) in the section 1, and it is assumed that $T$ is a injective (thus, $U$ is a unitary transformation). Let us make some notational conventions first. we denote $H_i = L^2(F_i)$ and $S_k = H_{k+1} ⊕ H_k$. we use $Q$ to represent the span of the elements in the form of the $H_k ⊕ H_j$ in $L^2(\Omega)$, $k$ and $j$ are two integers. $P_{S_k}$ means the projection operator from $L^2(\Omega) → S_k$.

Lemma 4.1 ([3]). Set $f \in L^2(\Omega)$ as well as

\[
\inf_{g \in Q} \limsup_{n \to \infty} n^{-1} E\left[ \sum_{k=0}^{n-1} U^k (f - g) \right]^2 = 0
\]

then

\[
\lim_{n \to \infty} n^{-1} E\left( \sum_{k=0}^{n-1} U^k f \right)^2 = \sigma^2, 0 \leq \sigma^2 < \infty, \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} U^i f \to N(0, \sigma^2).
\]

This lemma is important. The following two theorems will be obtained by this lemma. But since the original literature [3] is written in Russian, which bring many difficulties for me to read and understand the proof. The following is my proof by referring to an English document [7] which annotate [3] and some formulas of [3] itself.

Before formally proving the theorem, we first prove an assertion

claim 4.2. $H_k = U^k H_0$.

In fact, according to 3.4, $\forall f \in H_0$, $E(U^k f | F_k) = U^k E(f | F_0) = U^k f$, so $U^k f$ is $F_K$ is measurable. In addition, $U$ is unitary transformation, $E(U^k f)^2 = E(f^2) < \infty$, which means $U^k f \in H_k$.

On the other hand, $\forall g \in H^k$, we let $f = U^{-n} g$. As above, we have $E(f | F_0) = U^{-n} E(U^k f | F_k) = U^{-n} g = f$, so $f$ is $F_0$ measurable. noting $E(f^2) = E(g^2) < \infty$, it means $f \in H_0$, the conclusion is valid.

proof of lemma 4.1. By condition, $p \in \mathbb{Z}^+, \epsilon_p > 0, \lim_{p \to \infty} \epsilon_p = 0$, there exists $g_p \in Q$ such that

\[
\limsup_{n \to \infty} n^{-1} E[\sum_{k=0}^{n-1} U^k (f - g_p)]^2 < \epsilon_p
\]

step1 we want to prove the existence of random variable $h_p \in S_{-1}$ such that

\[
\limsup_{n \to \infty} n^{-1} E[\sum_{k=0}^{n-1} U^k (f - h_p)]^2 < 2\epsilon_p
\]
in fact,

\[ f = g_p + f - g_p = \sum_{l=\infty}^{\infty} P_{S_l}g_p + f - g_p \]

\[ = \sum_{l=\infty}^{\infty} U^{-(l+1)} P_{S_l}g_p + \sum_{l=\infty}^{\infty} \sum_{m=0}^{-l} U^m P_{S_l}g_p + U \sum_{l=\infty}^{\infty} \sum_{m=0}^{-l} U^m P_{S_l}g_p + f - g_p \]

we denote first term of the above formula is \( h_p \), the second item is \( t_p \), then the third item is \( Ut_p \), by \( g_p \in Q \) we know \( t_p \) is well defined, in addition, we have

\[
\limsup_{n \to \infty} n^{-1} E\left[ \sum_{k=0}^{n-1} U^k(f - h_p) \right]^2 = \lim_{n \to \infty} n^{-1} E\left[ \sum_{k=0}^{n-1} U^k(t_p - Ut_p + f - g_p) \right]^2
\]

\[
\leq \lim_{n \to \infty} 2n^{-1} [E(t_p - U^n t_p)^2 + E\left( \sum_{k=0}^{n-1} U^k(f - g_p)^2 \right)] < 2\epsilon_p
\]

To prove the last inequality, it suffices to prove \( \lim_{n \to \infty} n^{-1} E(t_p - U^n t_p)^2 = 0 \). From the definition of \( Q \) and the claim 4.2, we know that when \( n \) is sufficiently large, \( E(t_p U^n t_p) = 0 \), as a result, we have

\[
\lim_{n \to \infty} n^{-1} [E(t_p - U^n t_p)^2] = \lim_{n \to \infty} n^{-1} [E(t_p^2) + E(U^n t_p)^2 - E(2t_p U t_p)]
\]

\[
= \lim_{n \to \infty} n^{-1} [E(t_p^2) + E(t_p)^2] = 0
\]

**step2** we will prove that \( h_p, p \in \mathbb{Z}^+ \) is a convergence sequence of \( S_{-1} \) in \( L^2 \).

In fact, by \( U^k h_p \in S_{k-1} \), and the orthogonality between \( S_k \) and \( S_l \), \( k \neq l \), we get

\[
E[h_p - h_p']^2 = n^{-1} E[\sum_{k=0}^{n-1} U^k(h_p - h_p')]^2 = n^{-1} E[\sum_{k=0}^{n-1} U^k(f - h_p) - (f - h_p')]^2
\]

\[
\leq \limsup_{n \to \infty} 2n^{-1} E[\sum_{k=0}^{n-1} U^k(f - h_p)]^2 + 2n^{-1} E[\sum_{k=0}^{n-1} U^k(f - h_p')]^2
\]

\[
\leq 2\epsilon_p + 2\epsilon_p'
\]

Thus \( h_p, p \in \mathbb{Z}^+ \) is a convergence sequence of \( S_{-1} \) in \( L^2 \), and \( \lim_{p \to \infty} h_p = h_0 \in S_{-1} \)

**step3** we now prove:

\[
\limsup_{n \to \infty} n^{-1} E[\sum_{k=0}^{n-1} U^k(f - h_0)]^2 = 0
\]
in fact,
\[
\limsup_{n \to \infty} n^{\frac{1}{2}} \left| \sum_{k=0}^{n-1} U^k(f - h_0) \right|^2 \leq \limsup_{n \to \infty} 2n^{\frac{1}{2}} \left| \sum_{k=0}^{n-1} U^k(f - h_p) \right|^2 + n^{-1} \left( \sum_{k=0}^{n-1} U^k(h_p - h_0) \right)^2
\]
\[
\leq \epsilon_p + E[h_p - h'_p]^2
\]
let \( p \to \infty \) conclusion follows.

**step4**
We denote \( r_k = U^k(f - h_0) \), then \( U^k f = U^k h_0 + r_k \). It is easy to see that \( U^k h_0 \) is the martingale difference sequence \(^2\), by step 3, we have
\[
\lim_{n \to \infty} n^{-1} E \left| \sum_{k=0}^{n-1} r_k \right|^2 = 0
\]
Therefore, \( r_k \) converges to 0 in probability, and the conclusion is established by the central limit theorem of theorem 2.1 martingale and lemma 3.3. \( \square \)

Using this lemma, we prove the following theorems:

**Theorem 4.3 ([2])**. Consider a probability space \((\Omega, \mathcal{F}, P)\) And a measure preserving mapping \( T : \Omega \to \Omega \). Here, we also assume that \( T \) is injective, and if \( A \in \mathcal{F} \), then \( T(A) \in \mathcal{F} \), and further assume that \((P,T)\) is ergodic. Let \( \mathcal{F}_0 \) be a sub \( \sigma \) field of \( \mathcal{F} \), defined \( \mathcal{F}_i = T^{-i} \mathcal{F}_0 \), \( i \in \mathbb{Z} \), and \( \cdots \subset \mathcal{F}_{-1} \subset \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \), if \( f \in L^2(\Omega) \), denote
\[
x_r = E(U^r f|\mathcal{F}_0) - E(U^r f|\mathcal{F}_{-1})
\]
If \( \sum_{i \in \mathbb{Z}} x_r = Y_0 \in L^2(\Omega) \), \( E(Y_0^2) = \sigma^2 > 0 \), \( \lim_{n \to \infty} n^{-1} E S_n^2 = \sigma^2 \), then
\[
\frac{1}{\sqrt{n}} \sum_{i=0}^{n} U^i f \to N(0, \sigma^2)
\]
Here \( S_n = \sum_{i=1}^{n} U^i f \).

**Proof.** By condition \( Y_0 \in L^2(\Omega) \), and \( Y_0 \in S_{-1} = H_0 \cap H_{-1} \), We hope to uses lemma 4.1.
we Denote \( Y_j = U^j Y_0 \), \( X_j = U^j f \), and we note that \( Y_j \) is a stationary ergodic martingale difference sequence. \(^3\), If we denote \( T_n = \sum_{i=1}^{n} Y_i \), we only need to prove that when \( n \to \infty \) we have
\[
\frac{1}{n} E(S_n - T_n)^2 \to 0.
\]
Notice
\[
E(S_n - T_n)^2 = ES_n^2 + ET_n^2 - 2ES_nET_n
\]
\(^2\)from \( U^k h_0 \in S_{k-1} \), it can be seen that \( E(U^k h_0|\mathcal{F}_{k-1}) = 0 \) and \( E(U^k h_0) \) is \( \mathcal{F}_k \) measurable, so \( U^k h_0 \) is martingale difference sequence.
\(^3\)it can be calculated that \( U^j Y_0 = \sum_{r \in \mathbb{Z}} E[U^r f|\mathcal{F}_i] - E[U^r f|\mathcal{F}_{i-1}] \), thus \( Y_j \) is naturally martingale difference sequence.
And by condition and the central limit theorem of martingale difference, \( n^{-1}ES_n^2 \rightarrow \sigma^2 \), so it is suffices to prove \( n^{-1}ES_n T_n \rightarrow \sigma^2 \), however,

\[
n^{-1}ES_n T_n = n^{-1} \sum_{i=1}^{n} \sum_{j=1}^{n} E(X_i Y_j) = \sum_{j=-(n-1)}^{n-1} (1 - |j|n^{-1}) E(Y_0 X_j)
\]

so \( n^{-1}ES_n T_n \rightarrow \sum_{j=-\infty}^{\infty} E(Y_0 X_j) \)

On the other hand, \( Y_0 \in H_0 \ominus H_{-1} \), so \( E(Y_0 E(X_j|F_{-1})) = 0 \), so

\[
E(Y_0 X_j) = E(Y_0 E(X_j|F_0)) = E(Y_0 E(X_j|F_0) - Y_0 E(X_j|F_{-1})) = E(Y_0 x_j)
\]

So there are

\[
\sum_{j=-\infty}^{\infty} E(Y_0 X_j) = \sum_{j=-\infty}^{\infty} E(Y_0 x_j) = E(Y_0^2) = \sigma^2.
\]

Lemma 4.1 is a classical result, but its condition (4.1) is usually difficult to verify, so the paper [2] introduce a condition that is stronger than (4.1) but easier to verify, that is, the following theorem:

**Theorem 4.4 ([2]).** Let \( X_j = U f, f \) satisfies \( E(f) = 0, f \in L^2(\Omega) \) and \( F_0 \) measurable, if it satisfies:

1. \( \sum_{k=1}^{\infty} E(X_k E(X_n|F_0)) \) converges for each \( n \geq 0 \)

2. \( \lim_{n \rightarrow \infty} \sum_{k=K}^{\infty} E(X_k E(X_n|F_0)) = 0 \) Uniform convergence for \( K \)

Then \( \lim_{n \rightarrow \infty} n^{-1} ES_n^2 = \sigma^2, \sigma^2 < \infty, \) if \( \sigma^2 > 0 \), then

\[
\frac{1}{\sqrt{n}} \sum_{i=0}^{n} U_i f \rightarrow N(0, \sigma^2).
\]

5 Summary and thanks

In the previous sections, we listed two different approaches to approximate \( U^i f \), one is to use backward martingale difference, the other is martingale difference. Then we use the martingale central limit theorem to obtain the central limit theorem about \( U^i f \).

We can also see that the conditions of these theorems are relatively complex. Whether they have application in other fields is what I still need to learn and explore.

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