Volume preserving mean curvature flow in the Hyperbolic space

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Abstract

We prove: “If $M$ is a compact hypersurface of the hyperbolic space, convex by horospheres and evolving by the volume preserving mean curvature flow, then it flows for all time, convexity by horospheres is preserved and the flow converges, exponentially, to a geodesic sphere”. In addition, we show that the same conclusions about long time existence and convergence hold if $M$ is not convex by horospheres but it is close enough to a geodesic sphere.

1 Introduction and Main Results

Given an immersion $X : M \rightarrow \overline{M}$ of a compact $n$-dimensional manifold $M$ into a $(n + 1)$-dimensional Riemannian manifold $\overline{M}$, the mean curvature flow of $X$ is the solution of the partial differential equation

$$\frac{\partial X_t}{\partial t} = -H_t N_t,$$

with the initial condition $X_0 = X$, (1.1)

where $N_t$ is the outward unit normal vector of the immersion $X_t$ and $H_t$ is the trace of the Weingarten map $L_{-N_t} = -L_{N_t}$ of $X_t$ associated to $-N_t$ (then, $H_t$ is $n$ times the usual mean curvature with the sign which makes positive the mean curvature of a round sphere in $\mathbb{R}^{n+1}$). From now on, by $M_t$ we shall denote both the immersion $X_t : M \rightarrow \overline{M}$ and the image $X_t(M)$, as well as the Riemannian manifold $(M_t, g_t)$ with the metric $g_t$ induced by the immersion. The $n$-volume of $M_t$ (from now on called area of $M_t$) decreases along this flow, but no geometric invariant is preserved along it.

A related flow is the volume preserving mean curvature flow, which is defined as a solution of the equation

$$\frac{\partial X_t}{\partial t} = (\overline{H}_t - H_t) N_t,$$

where $\overline{H}_t$ is the averaged mean curvature

$$\overline{H}_t = \frac{\int_{M_t} H_t dv_{g_t}}{\int_{M_t} dv_{g_t}},$$

(1.3)
being $dv_{g_t}$, the volume element on $M_t$. This flow also decreases the area of $M_t$, but preserves the volume of the domain $\Omega_t$ enclosed by $M_t$ (when such $\Omega_t$ exists).

In [16], G. Huisken proved that, when $\overline{M}$ is the Euclidean space $\mathbb{R}^{n+1}$ and $M_0$ is strictly convex, then (1.1) has a maximal solution on a finite time interval $[0,T]$ and $M_t$ converges to a point as $t \to T$. Moreover, after appropriate rescaling of $X_t$ and $t$, $M_t$ converges to a round sphere. In [17], he extended this result to compact hypersurfaces in general Riemannian manifolds (with suitable bounds on curvature).

The flow (1.2) was considered by Huisken in [18], again for $\overline{M} = \mathbb{R}^{n+1}$ and $M_0$ strictly convex; he proved that (1.2) has a solution on $[0,\infty]$, which stays convex all time and converges to a round sphere. However, he noticed the difficulty that the presence of averaged mean curvature in (1.2) causes in order to extend this result to general Riemannian manifolds. In fact, Huisken illustrated this by showing a way to obtain examples of convex hypersurfaces in the sphere $S^{n+1}$ which could lose convexity along the flow. The idea is that, if a piece $M'$ of $M_0$ is a part of a geodesic sphere of mean curvature near to 0 and far from $\overline{M}_0$, $M'$ moves in the outward radial direction of $M'$, soon becoming a totally geodesic hypersurface and, after that, changing the sign of the mean curvature.

The above remark, pointed by Huisken in [18], was really inspiring for us. First we noticed that examples like those in $S^{n+1}$ cannot happen in the Euclidean space, because when a geodesic sphere is moving outward in the the direction of its radius, it becomes of lower and lower normal curvature, but it never becomes a totally geodesic submanifold. Nevertheless, the Euclidean case was already settled by Huisken in [18].

On the other hand, we realized that a similar situation happens in the hyperbolic space: when a geodesic sphere moves outward in the radial direction, its normal curvature decreases, and it becomes nearer and nearer to that of a horosphere (see Remark (ii) below for a definition), but it never gets the curvature of a horosphere. The former intuitive idea was indeed the detonating clue which leads us to hope for a theorem like that of Huisken in [18] for the volume preserving mean curvature flow in the hyperbolic space of a hypersurface convex by horospheres. This paper achieves the realization of such hope by proving the following theorem.

**Theorem 1** Let $M_0^{n+1}_\lambda$ be the complete simply connected $(n + 1)$-dimensional hyperbolic space of sectional curvature $\lambda < 0$. If $M_0$ is a compact hypersurface convex by horospheres, then the equation (1.2) with initial condition $M_0$ has a unique solution $M_t$ such that

(a) it is defined for $t \in [0, \infty[$,

(b) the hypersurfaces $M_t$ stay smooth and convex by horospheres for all time,

(c) and the $M_t$'s converge exponentially (as $t \to \infty$, in the $C^m$ topology for any
fixed \( m \in \mathbb{N} \) to a geodesic sphere of \( \mathcal{M}^{n+1}_\lambda \) enclosing the same volume as \( M_0 \).

Next we include some remarks for a better understanding of the above statement.

**Remarks**

(i) Recall that a horosphere of \( \mathcal{M}^{n+1}_\lambda \) is a hypersurface \( \mathcal{H} \) obtained as the limit of a geodesic sphere of \( \mathcal{M}^{n+1}_\lambda \) when its center goes to the infinity along a fixed geodesic ray, which is equivalent to say that \( \mathcal{H} \) is a complete embedded hypersurface with normal curvature \( \sqrt{\lambda} \). An horoball is the convex domain which boundary is an horosphere.

(ii) A hypersurface \( \mathcal{M} \) of \( \mathcal{M}^{n+1}_\lambda \) is called convex by horospheres (h-convex for short) if it bounds a domain \( \Omega \) satisfying that, for every \( p \in \mathcal{M} = \partial \Omega \), there is a horosphere \( \mathcal{H} \) of \( \mathcal{M}^{n+1}_\lambda \) through \( p \) such that \( \Omega \) is contained in the horoball of \( \mathcal{M}^{n+1}_\lambda \) bounded by \( \mathcal{H} \). This \( \mathcal{H} \) is called a supporting horosphere of \( \Omega \) (and of \( \mathcal{M} \)) through \( p \). One shows that a hypersurface \( \mathcal{M} \) of \( \mathcal{M}^{n+1}_\lambda \) is h-convex if and only if all its normal curvatures are bounded from below by \( \sqrt{\lambda} \).

(iii) Usually two immersions \( X_i : M_i \to \mathcal{M} \), \( i = 1, 2 \) are considered equivalent if there is a diffeomorphism \( \phi : M_1 \to M_2 \) such that \( X_1 = X_2 \circ \phi \). In this case, \( X_1 \) and \( X_2 \) are called parametrizations of the same immersed submanifold. For this reason, dealing with submanifolds, one says that an immersion \( X : M_2 \to \mathcal{M} \) is in a neighborhood \( \mathcal{U} \) of \( Y : M_1 \to \mathcal{M} \) in the \( C^k \)-topology if there is a diffeomorphism \( \phi \) such that \( X \circ \phi \in \mathcal{U} \). In Theorem 1, we use the convergence in the \( C^k \)-topology in this sense.

(iv) Let us notice that \( M_0 \) h-convex implies that it is diffeomorphic to a sphere, and this implies that \( X \) is, in fact, an embedding.

Moreover, a method described in [12] is used to prove that the convergence is exponential. This method relies on maximal regularity theory and is of independent interest. Indeed, its strength allows us to extend statements (a) and (c) in Theorem 1 to certain non-necessarily h-convex initial data. With more precision, as a by-product of the proof of the exponential convergence in Theorem 1, we shall obtain

**Theorem 2** Let \( \mathcal{S} \) be a geodesic sphere of \( \mathcal{M}^{n+1}_\lambda \) and \( 0 < \beta < 1 \). There exists an \( \varepsilon > 0 \) such that, for every embedding \( X : M \to \mathcal{M}^{n+1}_\lambda \) with \( h^{1+\beta} \)-distance
to $S$ lower than $\varepsilon$, the equation (1.2) has a unique solution satisfying $X_0 = X$, defined on $[0, \infty]$ and which converges exponentially to a geodesic sphere in $M^{n+1}_\lambda$ $h^{1+\beta}$-close to $S$ and enclosing the same volume as $X(M)$.

$h^{1+\beta}(M)$ denotes, for a compact manifold $M$, the little Hölder space of order $1 + \beta$, that is, the closure of $C^\infty(M)$ in the usual Hölder norm of $C^{1+\beta}(M)$.

In a recent paper (cf. [1]), Alikakos and Freire proved long time existence for solutions of (1.2) and convergence to constant mean curvature hypersurfaces in general ambient manifolds $\overline{M}$, but with the hypotheses that the initial condition $M_0$ is “close enough” to a geodesic sphere of $\overline{M}$ (although $M_0$ does not need to be convex) and the scalar curvature of $\overline{M}$ has nondegenerate critical points. It may seem that such result includes our Theorem 2, but this is not the case because $M^{n+1}_\lambda$ has constant scalar curvature.

The paper is organized as follows. In section 2, we establish some notation and summarize the basic inequalities for $h$-convex sets which will be used all along the proof of Theorem 1. In section 3, we shall prove that the solution $M_t$ remains $h$-convex along all time it exists. Section 4 contains the main part of the proof: the obtaining of an universal (not depending on $t$) bound for $H_t$ and all its derivatives. As a consequence of this, we get that $M_t$ exists for $t \in [0, \infty]$. Sections 5 and 6 are devoted to prove statement (c) in Theorem 1: first, we find a time sequence $\{t_i\}$ such that $\{M_{t_i}\}$ converges (up to isometries) to a geodesic sphere in $M^{n+1}_\lambda$; later, in section 6, we conclude that the full family $\{M_t\}$ converges $C^m$-uniformly and at exponential rate. Finally, the proof of Theorem 2 is included in section 7.

### 2 Notation and preliminaries on $h$-convex sets

From now on, $\langle \cdot, \cdot \rangle$, $\nabla$, $\Delta$ and $\text{grad}$ will denote the metric, the covariant derivative, the Laplacian and the gradient (respectively) of the ambient manifold $M^{n+1}_\lambda$. For $\overline{\Delta}$ (and the analog rough laplacian on tensor fields) we shall use the following sign convention:

$$\overline{\Delta} f = \text{tr} \nabla^2 f.$$ 

The corresponding operators on $M$ will be denoted by $\nabla$, $\Delta$ and $\text{grad}$.

When $\lambda < 0$, we shall use the notation:

$$s_\lambda(t) = \frac{\sinh(\sqrt{|\lambda|} t)}{\sqrt{|\lambda|}}, \quad c_\lambda(t) = s'_\lambda(t), \quad t\lambda(t) = \frac{s_\lambda(t)}{c_\lambda(t)} \quad \text{and} \quad c\lambda(t) = \frac{c_\lambda(t)}{s_\lambda(t)}.$$

The functions above satisfy the following computational rules:

$$c^2_\lambda + \lambda s^2_\lambda = 1, \quad c_{4\lambda} = c^2_\lambda - \lambda s^2_\lambda, \quad s_{4\lambda} = s_\lambda c_\lambda. \quad (2.1)$$

Given any point $p$ in the ambient space $M^{n+1}_\lambda$, we shall denote by $r_p$ the function “distance to $p$” in $M^{n+1}_\lambda$. Given a function $f : \mathbb{R} \to \mathbb{R}$, $f(r_p)$ will mean
We shall also use the notation $\partial_p = \text{grad} r_p$. In the following lemma, we recall some formulae involving derivatives of $f(r_p)$ that we shall apply later.

**Lemma 3** ([9], [14], [20]) In $M^{n+1}_\lambda$,

$$\langle \nabla X \partial_p, Y \rangle = \nabla^2 r_p(X, Y) = \begin{cases} 0 & \text{if } X = \partial_p \\ \cos\lambda(r_p) \langle X, Y \rangle & \text{if } \langle X, \partial_p \rangle = 0 \end{cases}, \quad (2.2)$$

and

$$\Delta r_p = n \cos\lambda(r_p). \quad (2.3)$$

Moreover, if $f : \mathbb{R} \to \mathbb{R}$ is a $C^2$ function,

$$\Delta(f(r_p)) = f''(r_p) + f'(r_p) \Delta r_p. \quad (2.4)$$

And, for the restriction of $r_p$ to a hypersurface $M$ of $M^{n+1}_\lambda$, one has

$$\Delta r_p = -H \langle N, \partial_p \rangle + \cos\lambda(r_p) \left(n - |\partial_p^\top|^2\right). \quad (2.5)$$

and

$$\Delta(f(r_p)) = f''(r_p) |\partial_p^\top|^2 + f'(r_p) \Delta r_p$$

$$= (f''(r_p) - f'(r_p) \cos\lambda(r_p)) |\partial_p^\top|^2$$

$$+ f'(r_p) \left(n \cos\lambda(r_p) - H \langle N, \partial_p \rangle\right). \quad (2.6)$$

Here $\partial_p^\top$ is the component of $\partial_p$ tangent to $M$, and it satisfies $\partial_p^\top = \text{grad}(r_p | M)$.

Next theorem summarizes some results contained in the quoted references.

**Theorem 4** ([5], [6], [7] and [8]) Let $\Omega$ be a compact $h$-convex domain and let $o$ be the center of an inball of $\Omega$. If $\rho$ is the inradius of $\Omega$ and $\tau = \tan\frac{\lambda}{2}$, then

a) the maximal distance $\max d(o, \partial \Omega)$ between $o$ and the points in $\partial \Omega$ satisfies the inequality

$$\max d(o, \partial \Omega) \leq \rho + \sqrt{|\lambda|} \ln \left(1 + \sqrt{\tau}\right) < \rho + \sqrt{|\lambda|} \ln 2.$$

b) For any interior point $p$ of $\Omega$, $\langle N, \partial_p \rangle \geq \sqrt{|\lambda|} \tan\lambda(\text{dist}(p, \partial \Omega))$, where $\text{dist}$ denotes the distance in the ambient space $M^{n+1}_\lambda$.

Moreover, in section 5 we shall use the elementary result stated below.

**Proposition 5** In the Euclidean space $\mathbb{R}^{n+1}$, let $N \neq \zeta$ be two unit vectors. The maximal value of the (acute) angle between a vector $v$ in the vector hyperplane $N^\perp$ orthogonal to $N$ and its projection onto the hyperplane $\zeta^\perp$ orthogonal to $\zeta$ is attained at the vectors in the intersection line of $N^\perp$ and the plane generated by $N$ and $\zeta$.

We finish this section recalling the following consequence of the inequality between the trace and the norm of an endomorphism that will be used through this paper:

$$|\nabla^m H|^2 \leq n |\nabla^m L|^2 \quad \text{for every } m = 0, 1, 2, 3, \ldots \quad (2.7)$$
3 Preserving \( h \)-convexity

With the notations of Theorem 1, here we shall prove

**Proposition 6** In \( M_{\lambda}^{n+1} \), if \( M_0 \) is \( h \)-convex, under the volume preserving mean curvature flow (1.2), \( M_t \) remains \( h \)-convex for all the time such that the solution exists.

For the proof of this result, we shall use the maximum principle for symmetric tensors as it is stated in [10], page 97. Before, we need some evolution equations.

**Lemma 6.1** For an arbitrary ambient space \( \overline{M} \), the evolution equations of the metric \( g_t \) and the second fundamental form \( \alpha_t \) of a solution \( M_t \) of (1.2) are

\[
\frac{\partial g_t}{\partial t} = 2(\overline{H}_t - H_t)\alpha_t, \\
\frac{\partial \alpha_t}{\partial t} = \Delta_t \alpha_t - 2 H_t \langle L_t^2 \cdot, \cdot \rangle + \overline{H}_t (\langle L_t^2 \cdot, \cdot \rangle - \overline{R} \cdot N_t \cdot N_t) + (|L_t|^2 + \overline{Ric}(N_t, N_t))\alpha_t - \overline{Ric}(\cdot, L_t \cdot) \\
- \overline{Ric}(L_t \cdot, \cdot) + \overline{R}(N_t, L_t \cdot, N_t, \cdot) + \overline{R}(N_t, \cdot, N_t, L_t \cdot) + 2 RiL_t - \nabla \overline{Ric}(\cdot, N_t) - \overline{\delta R} \cdot N_t,
\]

where \( \overline{R} \) and \( \overline{Ric} \) denote, respectively, the curvature and Ricci tensors of \( M \), \( RiL_t(Z,Y) = \sum_{i=1}^{n} \overline{R}_{e_i,Z,L_t e_i,Y} \) and \( \overline{\delta R} = \sum_{i=1}^{n} \nabla e_i(\overline{R})e_i \) for some local orthonormal frame \( \{e_i\} \) of \( M_t \).

**Proof** Formula (3.1) follows from (1.2) by a direct computation as in [16]. Also by this way one obtains

\[
\frac{\partial \alpha_t}{\partial t} = -(\overline{H}_t - H_t)\overline{R} \cdot N_t \cdot N_t + \nabla^2 H_t + (\overline{H}_t - H_t) \langle L_t^2 \cdot, \cdot \rangle,
\]

and, having into account the (generalized) Simons’ formula for the rough Laplacian of the second fundamental form (see, for instance, [4])

\[
\Delta \alpha = \nabla^2 H + H \langle L^2 \cdot, \cdot \rangle - |L|^2 \alpha + H \overline{R} N \cdot N \cdot \alpha - \overline{Ric}(N, N) \alpha - 2 RiL + \overline{Ric}(\cdot, L \cdot) + \overline{Ric}(L \cdot, \cdot) \\
- \overline{R}(N, L \cdot, N, \cdot) - \overline{R}(N, \cdot, N, L \cdot) + \nabla \overline{Ric}(\cdot, N) + \overline{\delta R} \cdot N,
\]

we get (3.2).

**Proof of Proposition 6.** Let us take \( A_t = \alpha_t - \sqrt{|\lambda|} g_t \). Notice that, from the explicit expression of the curvature tensor \( \overline{R} \) of \( M_{\lambda}^{n+1} \), the equation (3.2) becomes

\[
\frac{\partial \alpha_t}{\partial t} = \Delta_t \alpha_t + (\overline{H}_t - 2 H_t) \alpha L_t + (|L_t|^2 - \lambda n) \alpha_t + \lambda(2 H_t - \overline{H}_t) g_t.
\]
where $\alpha L$ is defined by $\alpha L(X,Y) = \alpha(LX,Y)$.

From (3.1) and (3.5), we obtain

$$\frac{\partial A_t}{\partial t} = \Delta_t A_t + B_t,$$

with

$$B_t = (\overline{\mathcal{H}}_t - 2H_t)(\alpha L_t - \lambda g_t) + \left( |L_t|^2 - \lambda n - 2\sqrt{|\lambda|}(\overline{\mathcal{H}}_t - H_t) \right) \alpha_t.$$

Let $V$ be a unitary null vector of $A_t$, that is, $L_t V = \sqrt{|\lambda|} V$. A straightforward computation gives

$$B_t(V,V) = \sqrt{|\lambda|}|L_t|^2 + 2\lambda H_t - n\lambda \sqrt{|\lambda|} \geq \frac{\sqrt{|\lambda|}}{n} H_t^2 + 2\lambda H_t - n\lambda \sqrt{|\lambda|} = \frac{\sqrt{|\lambda|}}{n} (H_t - n\lambda \sqrt{|\lambda|})^2 \geq 0,$$

using (2.7) in the first inequality. Now, the proposition follows by the maximum principle for symmetric tensors quoted above. \qed

4 Long Time Existence

Along this section, we shall denote by $[0,T]$ the maximal interval where the solution of (1.2) is well defined, and want to prove that $T = \infty$.

The main point to establish long time existence is to show that $|L_t|$ has an uniform bound independent of $t$. Since we proved previously that $M_t$ is $h$-convex as long as it exists, then $|L_t|^2 \leq H_t^2$; therefore, it is enough to show that $H_t$ has an upper bound independent of $t$. In order to achieve this, first we shall study the evolution under (1.2) of the function

$$W_t = \frac{H_t}{\sigma_t - c}, \text{ being } \sigma_t = s_\lambda(r_p) \langle N_t, \partial_{r_p} \rangle \text{ and } c \text{ any constant.} \quad (4.1)$$

Before starting the way to obtain the evolution equation for $W_t$, we would like to remark that $\sigma_t$ depends on the choice of the point $p$. This fact will be important later, when we write inequalities.

Let $g_t^\flat$ denote the metric induced on $T^*M$ by $g_t$ through the isomorphism $b_t : TM \rightarrow T^*M$ of lowering indices. The matrix $(g^\flat_{ij})$ of $g_t^\flat$ in some basis is the inverse of the matrix $(g_{ij})$ of $g_t$ in the dual basis. Using this fact and (3.1), one obtains

$$\frac{\partial g_t^\flat}{\partial t} = -2(\overline{\mathcal{H}}_t - H_t) \alpha_t^\flat,$$  \quad (4.2)

where $\alpha_t^\flat$ denotes, again, the tensor induced on $T^*M$ by $\alpha_t$ through the isomorphism $b_t$. From (3.3) and (4.2), we get

$$\frac{\partial H_t}{\partial t} = \Delta_t H_t + (|L_t|^2 + \nabla c(N_t,N_t))(H_t - \overline{\mathcal{H}}_t),$$  \quad (4.3)
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which, by the expression of the curvature tensor \( \overline{R} \) of \( M^{n+1}_\lambda \), becomes

\[
\frac{\partial H_t}{\partial t} = \Delta_t H_t + (|L_t|^2 + n\lambda)(H_t - \overline{H}_t).
\] (4.4)

Another standard computation (similar to that done in [16]) allows to obtain, from (1.2), the evolution equation

\[
\nabla N_t = -\text{grad}(\overline{H}_t - H_t) = \text{grad} H_t.
\] (4.5)

Now, let us note that, for any smooth function \( \varphi : \mathbb{R} \rightarrow \mathbb{R} \), a direct calculation using (2.2) and (1.2) gives

\[
\frac{\nabla (\varphi(r_p)\partial_{r_p})}{\partial t} = (\overline{H}_t - H_t) \left( (\varphi'(r_p) - \varphi(r_p)\cos(\lambda r_p)) \langle \partial_{r_p}, N_t \rangle \partial_{r_p} + \varphi(r_p)\cos(\lambda r_p) N_t \right).
\] (4.6)

Taking \( \varphi = s_\lambda \) and using (4.5), we arrive to

\[
\frac{\partial t}{\partial t} = (\overline{H}_t - H_t) c_\lambda(r_p) + s_\lambda(r_p) \langle \partial_{r_p}, \text{grad} H_t \rangle.
\] (4.7)

From (2.6) (with \( f = s_\lambda \)) and (2.5), we have

\[
\Delta_t(s_\lambda(r_p)) = -\frac{1}{s_\lambda(r_p)}|\partial_{r_p}^T|^2 - c_\lambda(r_p)H_t\langle N_t, \partial_{r_p} \rangle + n\frac{c_\lambda^2(r_p)}{s_\lambda^2(r_p)}. \] (4.8)

Straightforward computations having into account (2.2) give

\[
\langle \text{grad} s_\lambda(r_p), \text{grad} \langle \partial_{r_p}, N_t \rangle \rangle
\]
\[
= -\frac{c_\lambda^2}{s_\lambda}(r_p)\langle \partial_{r_p}, N_t \rangle |\partial_{r_p}^T|^2 + c_\lambda(r_p)\alpha(\partial_{r_p}^T, \partial_{r_p}^T),
\]
\[
\Delta_t \langle \partial_{r_p}, N_t \rangle = \frac{1}{s_\lambda^2(r_p)} \langle \partial_{r_p}, N_t \rangle |\partial_{r_p}^T|^2
\]
\[
+ \cos(r_p)\langle N_t, \partial_{r_p} \rangle^2 H_t - 2 \cos(r_p)\alpha(\partial_{r_p}^T, \partial_{r_p}^T)
\]
\[
+ 2 \cos^2(r_p) \langle N_t, \partial_{r_p} \rangle |\partial_{r_p}^T|^2 + \langle \partial_{r_p}, \text{grad} H_t \rangle + \cos(r_p)H_t - \langle \partial_{r_p}, N_t \rangle |L_t|^2.
\] (4.10)

Joining (4.8), (4.9) and (4.10), we reach

\[
\Delta_t \sigma_t = c_\lambda(r_p)H_t + \langle s_\lambda(r_p) \partial_{r_p}, \text{grad} H_t \rangle - \sigma_t |L_t|^2.
\] (4.11)

By substitution of this expression in (4.7), we obtain the evolution equation

\[
\frac{\partial \sigma_t}{\partial t} = \Delta_t \sigma_t + |L_t|^2 \sigma_t + (\overline{H}_t - 2H_t) c_\lambda(r_p).
\] (4.12)
From (4.1), (4.4) and (4.12), it follows
\[
\frac{\partial W_t}{\partial t} = \frac{1}{\sigma_t - c} \Delta_t H_t + \frac{1}{\sigma_t - c} (H_t - \overline{H}_t) \left( |L_t|^2 + n\lambda \right) - \frac{H_t}{(\sigma_t - c)^2} \left( \Delta_t \sigma_t + |L_t|^2 \sigma_t + (\overline{H}_t - 2 H_t) c_{\lambda(r_p)} \right).
\]

Taking definition (4.1) as starting point, another computation leads to
\[
\Delta_t W_t = \frac{1}{\sigma_t - c} \Delta_t H_t + \frac{2}{(\sigma_t - c)^2} |\text{grad} \, \sigma_t|^2 - \frac{H_t}{(\sigma_t - c)^2} \Delta_t \sigma_t - 2 \left\langle \text{grad} H_t, \frac{1}{(\sigma_t - c)^2} \text{grad} \, \sigma_t \right\rangle.
\]

Replacing (4.14) into (4.13) and doing a few more computations, we can write
\[
\frac{\partial W_t}{\partial t} = \Delta_t W_t + \frac{2}{(\sigma_t - c)} \left\langle \text{grad} W_t, \text{grad} \, \sigma_t \right\rangle - \frac{2}{(\sigma_t - c)} \left( |L_t|^2 + n\lambda \right) - W_t^2 c_{\lambda(r_p)} + 2 W_t^2 c_{\lambda(r_p)} - \frac{c}{(\sigma_t - c)} W_t |L_t|^2 + n\lambda W_t.
\]

To get fine and independent of \( t \) bounds for \( W_t \) from (4.15) by application of the maximum principle, previously we need to bound \( r_p \) and \( \langle N_t, \partial r_p \rangle \). In order to do so, we shall use Theorem 4.

**Lemma 7** Let \( \psi \) be the inverse of the function \( s \mapsto \text{vol}(S^n) \int_0^s s \lambda(\ell) d\ell \) and \( \xi \) the inverse function of \( s \mapsto s + \sqrt{|\lambda|} \ln \left( \frac{1 + \sqrt{\tan(\frac{\pi}{2})}}{1 + \tan(\frac{\pi}{2})} \right) \). If \( V_0 = \text{vol}(\Omega_0) \) and \( \rho_t \) is the inradius of \( \Omega_t \), then
\[
\xi(\psi(V_0)) \leq \rho_t \leq \psi(V_0),
\]
for every \( t \in [0, T] \).

**Proof** Since the flow preserves the enclosed volume, we have \( \text{vol}(\Omega_t) = V_0 \) for all \( t \in [0, T] \). If we take spherical geodesic coordinates in \( M^{n+1}_\lambda \), around a center \( p_t \) of an inball of \( \Omega_t \), we can describe \( M_t \) as the graph of a function \( \ell : S^n \rightarrow \mathbb{R}^+ \), and the volume of \( \Omega_t \) is given by
\[
\text{vol}(\Omega_t) = \int_{S^n} \int_0^{\ell(u)} s \lambda^n(s) \, ds \, du.
\]

But \( \rho_t \leq \ell(u) \leq \max(p_t, M_t) \leq \rho_t + \sqrt{|\lambda|} \ln \left( \frac{1 + \sqrt{\tan(\frac{\pi}{2})}}{1 + \tan(\frac{\pi}{2})} \right) \) (where we have used Theorem 4 a) for the last inequality), thus the lemma follows having into
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account that $\psi^{-1}$ and $\xi^{-1}$ are increasing functions. □

An immediate consequence of the lemma above and Theorem 4 a) is

**Corollary 7.1** For every $t \in [0, T[$, if $p, q \in \Omega_t$, then

$$\text{dist}(p, q) < 2(\psi(V_0) + \sqrt{\lambda} \ln 2).$$

(4.18)

Now, let us continue with the task of bounding $W_t$. First, we fix an arbitrary $t_0 \in [0, T[$. As before, $\rho_t$ will denote the inradius of an inball of $\Omega_t$ and $p_t$ its center. Although $p_0$ does not need to be the center of an inball of $\Omega_t$ when $t \neq t_0$, we can use Corollary 7.1 whereas $p_t \in \Omega_t$ to bound $r_{p_t}(x) \leq 2(\psi(V_0) + \sqrt{\lambda} \ln 2)$ for every $x \in M_t$. Consequently, our next goal is to estimate a time interval $[t_0, t_0 + \tau[$ such that $p_t \in \Omega_t$ for $t \in [t_0, t_0 + \tau[$. To do so, we shall compare the motion of $M_t$ following the equation (1.2) with the motion under (1.1) of a geodesic sphere centered at $p_t$ with radius $\rho_t$ at time $t_0$. As a result, we shall obtain

**Lemma 8** There is $\tau = \tau(\lambda, n, V_0) > 0$ such that $B(p_{t_0}, \rho_{t_0}/2) \subset \Omega_t$ for every $t \in [t_0, t_0 + \min\{\tau, T-t_0]\}$.

**Proof** Let $r_B(t)$ be the radius at time $t$ of a geodesic sphere $\partial B(p_{t_0}, r_B(t))$ centered at $p_{t_0}$, evolving under (1.1) and with the initial condition $r_B(t_0) = \rho_{t_0}$. From (1.1), (2.2) and the fact that the mean curvature of a geodesic sphere centered at $p_{t_0}$ is $\sum r_{p_{t_0}}$, we get

$$\frac{\partial r_B(t)}{\partial t} = -n \cos(r_B(t)),
$$

(4.19)

and the solution of this differential equation satisfying $r_B(t_0) = \rho_{t_0}$ is

$$c_\lambda(r_B(t)) = e^{\lambda n(t-t_0)} c_\lambda(\rho_{t_0}).
$$

(4.20)

Then, for $t \geq t_0$ (and because $c_\lambda$ is an increasing function), $r_B(t) \geq \rho_{t_0}/2$ if and only if $e^{\lambda n(t-t_0)} c_\lambda(\rho_{t_0}) \geq c_\lambda(\rho_{t_0}/2)$, i.e.,

$$r_B(t) \geq \rho_{t_0}/2 \quad \text{if and only if} \quad t - t_0 \leq \frac{1}{\lambda n} \ln \frac{c_\lambda(\rho_{t_0})}{c_\lambda(\rho_{t_0}/2)}
$$

and, as the function $s \mapsto \ln \frac{c_\lambda(s)}{c_\lambda(s/2)}$ is increasing, using (4.16), we have

$$r_B(t) \geq \rho_{t_0}/2 \quad \text{if} \quad t - t_0 \leq \frac{1}{\lambda n} \ln \frac{c_\lambda(\xi(\psi(V_0)))}{c_\lambda(\xi(\psi(V_0))/2)} =: \tau.
$$

(4.21)

For any $x \in M$, let $r(x, t) = r_{p_t}(X_t(x))$. From (1.2), it follows

$$\frac{\partial r}{\partial t} = (\mathbf{H}_t - H_t) \left< N_t, \partial_{p_{t_0}} \right>.
$$

(4.22)
If \( \varphi : \mathbb{R} \to \mathbb{R} \) is a function satisfying \( \varphi'(s) = t\alpha(s) \), and we set \( f(x,t) = \varphi(r(x,t)) - \varphi(r_B(t)) \), from (4.19) and (4.22), we obtain
\[
\frac{\partial f}{\partial t} = t\alpha(r_{p_0}) \left( \frac{\partial f}{\partial t} - H_t \right) + n_r \partial_r + n_t \partial_t + n. \tag{4.23}
\]

On the other hand, from (2.6),
\[
\Delta_t f = \Delta_t (\varphi(r_{p_0})) = \left( \frac{1}{c_A^2(r_{p_0})} - 1 \right) \mid \partial^T_{p_0} \mid^2 + n - H_t t\alpha(r_{p_0}) \left( \frac{\partial f}{\partial t} \right) \tag{4.24}
\]

Now, let \( t_1 = \min\{ t > t_0; p_0 \notin \Omega_t \} \). Because \( \Omega_t \) is \( h \)-convex, \( \left( \frac{\partial f}{\partial t} \right) \geq 0 \) for \( t \in [t_0, t_1] \). By substitution of (4.24) into (4.23), we arrive to
\[
\frac{\partial f}{\partial t} = \Delta_t f + t\alpha(r_{p_0}) \left( \frac{\partial f}{\partial t} - H_t \right) \left( \frac{\partial f}{\partial t} \right) = \Delta_t f, \tag{4.25}
\]

Using the scalar maximum principle for parabolic inequalities (cf. [10], page 94) gives \( f(x,t) \geq 0 \) for \( t_0 \leq t \leq t_1 \) as long as \( f(x,t) \) is well defined. But \( r(x,t) \) is well defined for \( 0 \leq t < T \), and it follows from (4.20) that \( r_B(t) \) is well defined (that is, positive) for \( t \in [t_0, t_0 + \frac{1}{\lambda_0} \ln(c_\lambda(r_{p_0}))] \). Then \( f(x,t) \geq 0 \) on \( [t_0,\min\{ t_0 + \tau, T, t_1 \}] \).

Notice that, by definition of \( t_1 \), \( p_0 \in M_{t_1} = \partial \Omega_{t_1} \). If \( t_1 < \min\{ t_0 + \tau, T \} \), \( f(x,t_1) \geq 0 \) and \( B(p_0, r_B(t_1)) \subset \Omega_{t_1} \), which is a contradiction; therefore, \( t_1 \geq \min\{ t_0 + \tau, T \} \), and the lemma follows.

As a consequence of (4.16), Corollary 7.1 and Lemma 8, on the interval \( [t_0, t_0 + \min\{ \tau, T - t_0 \}] \), and on the hypersurface \( M_t \)
\[
C := \frac{\xi(\psi(V_0))}{2} \leq r_{p_0} \leq 2(\psi(V_0) + \sqrt{\lambda} \ln 2) =: D.
\]

Moreover, having into account Theorem 4 b),
\[
\sigma_t = s(\alpha(r_{p_0})) \left( \frac{\partial f}{\partial t} \right) \geq \sqrt{\lambda} s(C) \alpha(C).
\]

Then, if we take the constant \( c \) in the definition (4.1) as
\[
c = \frac{\sqrt{\lambda} s(C) \alpha(C)}{2},
\]
we get \( \sigma_t - c \geq c > 0 \).
Let us go back to equation (4.15). From the above remark on \( \sigma_t - c \) and the \( h \)-convexity of \( M_t \), we have \( W_t \geq 0 \) and \( H_t + n\lambda \geq 0 \). Moreover, \( \|L_t\|^2 \geq \frac{1}{n}H_t^2 \). Now we can use these inequalities in (4.15) to obtain

\[
\frac{\partial W_t}{\partial t} \leq \Delta_t W_t + \frac{2}{\sigma_t - c} (\text{grad}W_t, \text{grad} \sigma_t) + 2 \frac{\lambda(D)}{\sigma_t - c} W_t^2 - \frac{c}{\sigma_t - c} W_t \frac{H_t^2}{n} \quad (4.27)
\]

By other version of the scalar maximum principle (cf. [10], page 96), in the interval \([t_0, t_0 + \min\{\tau, T - t_0\}]\), \( W_t(x) \) is bounded from above by the solution \( w(t) \) of the ordinary differential equation

\[
w' = \left(2\lambda(D) - \frac{c^2}{n}\right) w^2, \quad \text{with} \quad w(t_0) = \max_{x \in M} W_{t_0}(x).
\]

Observing that \( w'(t) < 0 \) when \( w > \frac{2n\lambda(D)}{c^2} \), it is straightforward to show that \( w(t) \leq \max\{w(t_0), \frac{2n\lambda(D)}{c^2}\} \). Thus we deduce

\[
W_t(x) \leq \max_{x \in M} \{\max_{x \in M} W_{t_0}(x), \frac{2n\lambda(D)}{c^2}\} \quad \text{for every} \quad t \in [t_0, t_0 + \min\{\tau, T - t_0\}].
\]

From the definition of \( W_t \), the election of \( c \) and the upper bound of \( \rho_t \), we have

\[
H_t(x) \leq (\lambda(D) - c) \max_{x \in M} \{\max_{x \in M} W_{t_0}(x), \frac{2n\lambda(D)}{c^2}\}.
\]

Since this occurs for any \( t_0 \) and \( \tau \) does not depend on \( t_0 \), we arrive to

\[
H_t(x) \leq (\lambda(D) - c) \max_{x \in M} \{\max_{x \in M} W_0(x), \frac{2n\lambda(D)}{c^2}\} =: C(n, \lambda, M_0) \quad (4.28)
\]

for every \( t \in [0, T] \).

This implies, by the definition of \( \overline{\mathbf{P}}_t \) and the \( h \)-convexity of \( M_t \),

\[
\overline{\mathbf{P}}_t \leq C(n, \lambda, M_0) \quad \text{and} \quad \|L_t\|^2 \leq H_t^2 \leq C(n, \lambda, M_0)^2 \quad (4.29)
\]

for every \( t \in [0, T] \).

From (3.2), reasoning like in [16] and [15]§13, one can deduce, for every natural number \( m \), the following evolution equation

\[
\frac{\partial}{\partial t} |\nabla^m L_t|^2 = \Delta_t |\nabla^m L_t|^2 - 2|\nabla^{m+1} L_t|^2 + C(m, n, \lambda)|\nabla^m L_t|^2 + \sum_{i+j+k=m} \nabla^i L_t \ast \nabla^j L_t \ast \nabla^k L_t \ast \nabla^m L_t + \overline{\mathbf{P}}_t^2 \sum_{i+j+k=m} \nabla^i L_t \ast \nabla^j L_t \ast \nabla^k L_t \ast \nabla^m L_t
\]

Then, using (4.29) and arguing in the same way as in [18] Theorem 4.1, we conclude
Proposition 9 For every natural number \( m \), there is a constant \( C_m(n, \lambda, M_0) \) such that
\[
|\nabla^m L_t| \leq C_m(n, \lambda, M_0) \tag{4.30}
\]

From (3.1), (4.28), (4.29), and (4.30), it follows (like in [16] pages 257, ff.) that, if \( T < \infty \), then \( X_t \) converges (as \( t \to T \), in the \( C^\infty \)-topology) to a unique smooth limit \( X_T \) which represents a smooth \( h \)-convex hypersurface. Now we can apply the short time existence theorem to continue the solution after \( T \), arriving to a contradiction. In short, the proof that the solution of (1.2) is defined on \([0, \infty]\) (that is, the long time existence statement in Theorem 1) is finished.

5 Convergence to a geodesic sphere

Observe that, to finish the proof of Theorem 1, it remains to deal with the issues related to the convergence of the flow. We begin this task in the present section by proving

Proposition 10 There is a sequence of times \( t_1 < t_2 < \ldots < t_k < \ldots \to \infty \) and isometries \( \varphi_{t_1}, \varphi_{t_2}, \ldots, \varphi_{t_k}, \ldots \) of \( M^{n+1}_\lambda \) such that \( \varphi_{t_k}(M_{t_k}) \) \( C^\infty \)-converges to an embedded geodesic sphere.

Proof The proof is organized in two main steps. Let us begin by showing that if the aforementioned limit exists, it should be a hypersurface in \( M^{n+1}_\lambda \) of constant mean curvature. As \( H_t \) is invariant by the family of isometries \( \{ \varphi_t \} \), it will be enough to prove that \( H_t \) (instead of \( H_t \circ \varphi_t \)) tends to a constant as \( t \to \infty \), in other words,

Step 1. The mean curvature \( H_t \) of the hypersurfaces \( M_t \) which evolve following (1.2) converges to its average, that is,
\[
\lim_{t \to \infty} \sup_{M_t} |H_t - \mathcal{P}_t| = 0. \tag{5.1}
\]

In order to prove the above claim, we shall state a series of auxiliary results.

- ([3], p. 91) Let \( M \) be a Riemannian manifold. If a real function \( f \) on \( M \) satisfies \( f \in L^1(M) \), \( \int_M f dV = 0 \) and \( |\text{grad} f| \in L^r(M) \), then
\[
\sup_M |f| \leq C \|\text{grad} f\|_r \quad \text{for every } r > n. \tag{5.2}
\]

- ([3], p. 93) Let \( M \) be a Riemannian manifold and let \( p, q, r \) be real numbers satisfying \( 1 \leq q, r \leq \infty \) and \( \frac{2}{p} = \frac{1}{q} + \frac{1}{r} \). Every function \( f \in C^\infty_0(M) \) satisfies
\[
\|\text{grad} f\|_p^2 \leq (n^{1/2} + |p - 2|) \|f\|_q \|\nabla^2 f\|_r. \tag{5.3}
\]
• ([3], p. 89) Let $M$ be a Riemannian manifold and let $p, q, r, a$ be real numbers satisfying $1 \leq r < q \leq \infty$, $p \in [r, q]$ and $a = \frac{1/p - 1/q}{1/r - 1/q}$. If $f \in L^r(M) \cap L^q(M)$, then $f \in L^p(M)$ and

$$||f||_p \leq ||f||_r^a ||f||_q^{1-a}. \tag{5.4}$$

Now we are in position to start proving (5.1).

From (3.1) and the expression of $dv_{g_t} = \sqrt{\det(g_{ij})} du^1...du^n$ in local coordinates, a straightforward computation gives

$$\frac{\partial}{\partial t} dv_{g_t} = (\overline{H}_t - H_t) H_t dv_{g_t}. \tag{5.5}$$

This leads to

$$\frac{d}{dt} \text{vol}(M_t) = - \int_M (H_t - \overline{H}_t)^2 dv_{g_t}, \tag{5.6}$$

thus

$$\int_0^\infty \int_M (H_t - \overline{H}_t)^2 dv_{g_t} \, dt = \lim_{t \to \infty} (\text{vol}(M_0) - \text{vol}(M_t)) \leq \text{vol}(M_0). \tag{5.7}$$

On the other hand, from (5.5), (4.4), (4.30) and (2.7) it follows that $\frac{d}{dt} \int_M (H_t - \overline{H}_t)^2 dv_{g_t}$ is uniformly bounded. Therefore, (5.7) implies

$$\lim_{t \to \infty} \int_M (H_t - \overline{H}_t)^2 dv_{g_t} = 0. \tag{5.8}$$

Since $\int_M (H_t - \overline{H}_t)dv_{g_t} = 0$, $H_t - \overline{H}_t$ is smooth and $M$ is compact, then $H_t - \overline{H}_t$ satisfies the hypotheses required to apply (5.2); so

$$\sup_{M_t} |H_t - \overline{H}_t| \leq C ||\text{grad}(H_t - \overline{H}_t)||_p \quad \text{for every } p > n. \tag{5.9}$$

Using (5.4), with $q = \infty$ and $r = 2$, we get

$$\sup_{M_t} |H_t - \overline{H}_t| \leq C ||\text{grad}(H_t - \overline{H}_t)||^{2/p}_2 ||\text{grad}H_t||^{1-2/p}_\infty. \tag{5.10}$$

As a consequence of (2.7) and (4.30), one has the inequality

$$||\text{grad}H_t|| \leq \sqrt{n} C_1. \tag{5.11}$$

Moreover, if we apply (5.3) to $f = (H_t - \overline{H}_t)$, with $p = q = r = 2$, we have

$$||\text{grad}(H_t - \overline{H}_t)||^2 \leq n^{1/2} ||H_t - \overline{H}_t||_2 ||\nabla^2 H_t||_2 \tag{5.12}$$
Replacing (5.11) and (5.12) in (5.10), we obtain that there is a constant $K$ depending only on $n, \lambda$ and $M_0$ such that
\[
\sup_{M_t} |H_t - \overline{H}_t| \leq K \left( \frac{\|H_t - \overline{H}_t\|_2}{\|\nabla^2 H_t\|_2} \right)^{1/p}.
\] (5.13)

But, using again (2.7) and (4.30), and the decrease of $\text{vol}(M_t)$ given by (5.6),
\[
\|\nabla^2 H_t\|_2 = \left( \int_M |\nabla^2 H_t|^2 \, dv_{gt} \right)^{1/2} \leq \sup_{M_t} \|\nabla^2 H_t\| \text{vol}(M_t)^{1/2} \leq K_1(n, \lambda, M_0).
\] (5.14)

By (5.8), (5.13) and (5.14), we reach
\[
\sup_{M_t} |H_t - \overline{H}_t| \leq K_2(n, \lambda, M_0) \left( \int_M (H_t - \overline{H}_t)^2 \, dv_{gt} \right)^{1/(2p)} \overset{t \to \infty}{\longrightarrow} 0,
\]
which finishes the proof of (5.1).

Next step is to show the existence of the convergent sequence claimed in Proposition 10. With more precision,

**Step 2.** There exists a family of isometries \( \{ \varphi_t : M_\lambda^{n+1} \to M_\lambda^{n+1} \} \) such that, if we consider the compositions \( \varphi_t \circ X_t \) with \( X_t \) being a solution of (1.2) on \([0, \infty]\), then \( \{ \varphi_t \circ X_t : M \to M_\lambda^{n+1} \} \) is precompact in the \( C^\infty \)-topology. Moreover, the limit \( M_\infty \) is a compact embedded hypersurface of \( M_\lambda^{n+1} \).

For each \( t \), let us fix a center \( p_t \) of an inball of \( \Omega_t \), and let \( \varphi_t \) be an isometry of \( M_\lambda^{n+1} \) carrying \( p_t \) onto \( p_0 \). Obviously, each \( \varphi_t(X_t(M)) \) is an \( h \)-convex hypersurface with a center of an inball at \( p_0 \) and inradius \( \rho_t \). Then, by Theorem 4 and (4.16), \( \text{dist}(p_0, \varphi_t \circ X_t(x)) \) has an upper bound independent of \( t \) and of \( x \), i.e., the family \( \{ \varphi_t \circ X_t \}_{t \geq 0} \) is uniformly bounded.

Let us denote by \( S^n \) the unit sphere in \( T_{p_0}M_\lambda^{n+1} \). For each \( t \), since \( \varphi_t(X_t(M)) \) is \( h \)-convex, there exists a function \( \tilde{r}_t : S^n \to \mathbb{R}^+ \) such that we can parametrize \( \varphi_t(M_t) \) by a map \( \tilde{X}_t : S^n \to M_\lambda^{n+1} \) satisfying
\[
\tilde{X}_t(u) = \exp_{p_0} \tilde{r}_t(u) u.
\] (5.15)

Notice that \( \tilde{r}_t(u) = r_{p_0}(\tilde{X}_t(u)) \). For any local orthonormal frame \( \{ e_i \}_{i=1}^n \) of \( S^n \), we have
\[
\tilde{X}_{t*}u e_i = \exp_{p_0*}(e_i \tilde{r}_t)(u) u + \exp_{p_0*} \tilde{r}_t(u) e_i = e_i(\tilde{r}_t)(u) \partial_{p_0} + s_{\lambda}(\tilde{r}_t(u)) \tau_s e_i,
\] (5.16)
where \( \tau_s \) denotes the parallel transport along the geodesic starting from \( p_0 \) in the direction of \( u \), and until \( \exp_{p_0} \tilde{r}_t(u) u \).
Let $N_t$ be the outward unit normal vector to $\varphi_t(M_t)$. Observe that, by (5.16), the projection $\pi_\perp \tilde{X}_t e_i$ of $\tilde{X}_t e_i$ onto the space $\partial_{r_{p_0}}^\perp$ orthogonal to $\partial_{r_{p_0}}$ is $s_\lambda(\tilde{r}_t)\tau_s e_i$. Using Proposition 5 (with $\zeta = \partial_{r_{p_0}}$ and $N = N_t$), the angle $\beta$ between $\tilde{X}_t e_i$ and its projection is bounded from above by the angle $\beta_0$ they form in case $\tilde{X}_t e_i$, $\partial_{r_{p_0}}$ and $N_t$ are in the same plane. Then, in general,

$$s_\lambda(\tilde{r}_t) = |\pi_\perp \tilde{X}_t e_i| = |\tilde{X}_t e_i| \cos \beta \geq |\tilde{X}_t e_i| \cos \beta_0 = |\tilde{X}_t e_i| \langle N_t, \partial_{r_{p_0}} \rangle,$$

so

$$|\tilde{X}_t e_i| \leq \frac{s_\lambda(\tilde{r}_t)}{\langle N_t, \partial_{r_{p_0}} \rangle} < \frac{s_\lambda(\rho_t + \sqrt{|\lambda| \ln 2})}{\sqrt{|\lambda|} \tan(\rho_t)} \leq \frac{s_\lambda(\psi(V_0) + \sqrt{|\lambda| \ln 2})}{\sqrt{|\lambda|} \tan(\psi(V_0))},$$

where we have used Theorem 4 for the second inequality and (4.16) for the third one. Moreover, it follows from (5.16) that $|e_i(\tilde{r}_t)| \leq |\tilde{X}_t e_i|$, thus both the first derivatives of $\tilde{X}_t$ and $\tilde{r}_t$ are bounded independently of $t$.

On the other hand, it is clear from the expression (5.15) for $\tilde{X}_t$ that all the higher order derivatives of $\tilde{X}_t$ are bounded if an only if the corresponding derivatives of $\tilde{r}_t$ are bounded. In order to see that such derivatives of $\tilde{r}_t$ are bounded, first we compute the components $\alpha_{ij}$ of the second fundamental form of $\varphi_t(M_t)$ using the parametrization (5.15), that is, $\alpha_{ij} = \alpha(\tilde{X}_t e_i, \tilde{X}_t e_j)$. We shall write $\alpha_{ij}$ in terms of $\tilde{r}_t$ and its derivatives.

If $\xi$ is a vector normal to $\varphi_t(M_t)$ satisfying $\langle \xi, \partial_{r_{p_0}} \rangle = s_\lambda(\rho_{p_0})$, we have

$$0 = \langle \xi, \tilde{X}_t e_i \rangle \circ \tilde{X}_t = e_i(\tilde{r}_t) s_\lambda(\tilde{r}_t) + s_\lambda(\tilde{r}_t) \langle \tau_s e_i, \xi \rangle,$$

and then (without explicit writing of the suitable compositions with the map $\tilde{X}_t$)

$$\langle \tau_s e_i, \xi \rangle = -e_i(\tilde{r}_t), \quad \text{so} \quad \xi = s_\lambda(\tilde{r}_t) \partial_{r_{p_0}} - \sum_{i=1}^{n} e_i(\tilde{r}_t) \tau_s e_i.$$

Consequently, the outward unit normal vector $N_t$ to $\varphi_t(M_t)$ can be written as

$$N_t = \frac{1}{|\xi|} \left( s_\lambda(\tilde{r}_t) \partial_{r_{p_0}} - \sum_{i=1}^{n} e_i(\tilde{r}_t) \tau_s e_i \right), \quad \text{with} \quad (5.18)$$

$$|\xi| = \sqrt{s_\lambda^2(\tilde{r}_t) + |\nabla_{\tilde{r}_t} \tilde{r}_t|^2}.$$

To compute the components $\alpha_{ij}$, we use on $M_{\lambda}^{n+1}$ the spherical coordinates $\gamma: \mathbb{R}^+ \times S^n \rightarrow M_{\lambda}^{n+1}$ defined by $\gamma(s, u) = \exp_{r_{p_0}} s u$. In these coordinates, for a local orthonormal frame $\{E_0 = \partial_{r_{p_0}}, E_1 = \tau_s e_1, \ldots, E_n = \tau_s e_n\}$ of $M_{\lambda}^{n+1}$ and its dual frame $\{d\tau_{r_{p_0}}, \theta^1, \ldots, \theta^n\}$, we have $\gamma^* d\tau_{r_{p_0}} = ds$, $\gamma^* \theta^i = s_\lambda e^i$, being $\{e^1, \ldots, e^n\}$ the dual frame of $\{e_1, \ldots, e_n\}$. Let us denote by $\nabla_S$ and $g_S$ the standard covariant derivative and metric of $S^n$, respectively.
The Cartan connection 1-forms $\omega^j_0$, $\omega^j_i$ of $\nabla$ satisfy $\omega^j_0 = -c_0^j \theta^j$ and $\gamma^* \omega^j_i = s^j_i$, where $s^j_i$ are the connection forms of $\nabla_S$ in the frame $\{e_1, \ldots, e_n\}$. Using these facts and (5.18), after a standard computation, we reach

$$\alpha_{ij} = \left\langle \nabla_{\tilde{X}_t} e_i, \tilde{X}_t e_j, N_t \right\rangle$$

$$= -\frac{1}{|\xi|} (s^\lambda(\tilde{r}_t) \nabla^2_S \tilde{r}_t - s^2(\tilde{r}_t) c^\lambda(\tilde{r}_t) g_S - 2c^\lambda(\tilde{r}_t) d\tilde{r}_t \otimes d\tilde{r}_t) (e_i, e_j).$$

Since each $\varphi_t$ is an isometry of the ambient space, the second fundamental forms of $M_t$ and $\varphi_t(M_t)$ coincide. Then, by (4.30), $\alpha$ and all their derivatives are uniformly bounded and, by (5.19) and the fact that $\tilde{r}_t$ and its first order derivatives are uniformly bounded, we have that all the derivatives of $\tilde{r}_t$ are uniformly bounded. Thus, by the relation (5.15), all the derivatives of $\tilde{X}_t$ are also uniformly bounded.

We are now in conditions to apply Arzel-Ascoli Theorem to conclude the existence of sequences of maps $\tilde{X}_t$ and $\tilde{r}_t$ satisfying (5.15) which $C^\infty$-converge to smooth maps $\tilde{X}_\infty : S^n \to M^{n+1}_\lambda$ and $\tilde{r}_\infty : S^n \to \mathbb{R}^+$ satisfying $\tilde{X}_\infty(u) = \exp_{p_0} \tilde{r}_\infty(u) u$. The last equality implies that $\tilde{X}_\infty$ is an immersion and, since the convergence is smooth and all the hypersurfaces $\tilde{X}_t(S^n)$ are $h$-convex, we can assure that $\mathcal{S} = \tilde{X}_\infty(S^n)$ is $h$-convex. Using Remark (iii) in the Introduction, we say that $\varphi_t(M_t)$ converges to $\mathcal{S}$ as $t_i \to \infty$.

Finally, by Step 1, $\mathcal{S}$ must be a compact embedded hypersurface in $M^{n+1}_\lambda$ of constant mean curvature, that is, a geodesic sphere of $M^{n+1}_\lambda$. This finishes the proof of Proposition 10.

\[\square\]

### 6 Exponential convergence

In order to complete the proof of statement (c) in Theorem 1, our next goal is to show that the $M_t$'s converge to some limit $M_\infty$ exponentially. First, let us fix an instant $t_k \in [0, \infty[$. We can parametrize $M_t$, with $t \geq t_k$, by

$$X_t(x) = \exp_{p_{t_k}} r(t, u(t, x)) u(t, x),$$

where $u(t, x) = \frac{\exp_{p_{t_k}}^{-1} X_t(x)}{r_{p_{t_k}}(X_t(x))}$ and $r(t, u(t, x)) = r_{p_{t_k}}(X_t(x))$. At least for $t$ near to $t_k$, we have $p_{t_k} \in \Omega_t$, and so the map $u_t : M \to S^n \subset T_{p_{t_k}} M^{n+1}_\lambda$ defined by $u_t(x) = u(t, x)$ is a diffeomorphism.

Observe that the map

$$\overline{X}_t(x) = \exp_{p_{t_k}} r(t, u(t_k, x)) u(t_k, x)$$

(6.2)
Volume-preserving M.C.F. in Hyperbolic Space

is another parametrization of \(M_t\). In fact, writing \(\phi_t = u_t^{-1} \circ u_t : M \to M\), we see that \(\overline{X}_t \circ \phi_t = X_t\), i.e., the motions \(X_t\) and \(\overline{X}_t\) differ only by a tangential diffeomorphism \(\phi_t\). Moreover,

\[
\frac{\partial X_t}{\partial t} = \frac{\partial \overline{X}_t}{\partial t} \circ \phi_t + \overline{X}_t \frac{\partial \phi_t}{\partial t}.
\]

(6.3)

Therefore, if \(X_t\) is a solution of (1.2), \(\overline{X}_t\) satisfies the equation

\[
\left< \frac{\partial \overline{X}_t}{\partial t}, N_t \right> = (\overline{H}_t - H_t).
\]

(6.4)

Conversely, it is well known (see, for instance, [11]) that (6.4) is equivalent to (1.2) (by tangential diffeomorphisms).

With the aim of applying some methods from [12], it is convenient to write (6.4) as an equation for the function \(r(t, \cdot)\). Previously, to simplify the notation, we shall compose with the diffeomorphism \(u_t^{-1}k\) in order to consider \(\overline{X}_t\) as a map from \(S^n\) (instead of \(M\)) into \(M^{n+1}_\lambda\), i.e.,

\[
\overline{X}_t(u) = \exp_{p_{uk}} r(t, u) u \quad \text{for every } u \in S^n.
\]

(6.5)

For any local orthonormal frame \(\{e_i\}\) of \(S^n\), a basis of the tangent space to \(M_t\) is given by \(\tilde{e}_i = \overline{X}_{t*}e_i\). In this basis, the outward unit normal vector \(N_t\) to \(M_t\) and the second fundamental form \(\alpha_t\) are given by the expressions (5.18) and (5.19), respectively (with the obvious change of \(\tilde{r}_t\) by \(r(t, \cdot)\)).

From (6.4), (6.5) and (5.18), we obtain

\[
\frac{\partial r}{\partial t} = s_{\lambda}^{-1}(r)(\overline{H}_t - H_t)\sqrt{s_{\lambda}^2(r) + |\text{grad}_{S^n} r|^2}
\]

(6.6)

On the other hand, the components of the metric \(g_t\) in the basis \(\tilde{e}_i\) are

\[
g_{ij} = e_i(r)e_j(r) + s_{\lambda}^2(r)\delta_{ij}
\]

From this, using an elementary algebraic result, we can express the components of the inverse metric as

\[
g^{ij} = \frac{1}{s_{\lambda}^2(r)} \left( \delta^{ij} - \frac{1}{|\xi|^2} e_i(r)e_j(r) \right).
\]

(6.7)

Then, joining (6.7) and (5.19), we get

\[
H_t = - \frac{s_{\lambda}^{-1}(r)}{|\xi|} \left( \Delta_s r - \frac{1}{|\xi|^2} \nabla^2_s r(\text{grad}_{S^n} r, \text{grad}_{S^n} r) \right) + \frac{c_{\lambda}(r)}{|\xi|} \left( n + \frac{|\text{grad}_{S^n} r|^2}{|\xi|^2} \right)
\]

(6.8)
Finally, substituting (6.8) in (6.6), we can write

\[
\frac{\partial r}{\partial t} = s_\lambda^2(r) \left( \Delta_S r - \frac{1}{|\xi|^2} \nabla_S^2 r(\nabla_S r, \nabla_S r) \right) - c_\lambda(r) \left( n + \frac{|\nabla_S r|^2}{|\xi|^2} \right) + s_\lambda^{-1}(r) \bar{H}_t |\xi|.
\] (6.9)

Observe that equation (6.9) coincides with equation (2.1) in [12] when we change \(s_\lambda(r)\) by \(r\) and \(c_\lambda(r)\) by 1. Therefore, (6.9) satisfies all the conditions which allow to apply (vii) in [12], and conclude

**Proposition 11** Given \(m \in \mathbb{N}\) and a constant \(s > 0\), there exists \(\omega > 0\) and a neighborhood \(V\) of \(s\) in \(h^{1+\beta}(\mathbb{S}^n)\) such that for each initial condition \(r_0 \in V\)

(a) The solution \(r(t, \cdot)\) of (6.9) satisfying \(r(0, \cdot) = r_0(\cdot)\) exists on \([0, \infty[^\), and

(b) there exist \(c = c(m, \omega) > 0, T = T(m, \omega) > 0, \) a unique function \(\tilde{\rho}\) (in a space of functions \(M^c\) called center manifold) and \(K = K(r_0, c, \tilde{\rho})\) such that

\[ ||r(t, \cdot) - \tilde{\rho}(\cdot)||_{C^m} \leq K e^{-\omega t} \]

for \(t > T\).

Now we are in position to finish the proof of the exponential convergence of \(M_t\) to a geodesic sphere. Indeed,

**Proof of (c), Theorem 1** Let us apply Proposition 10 to take \(t_k\) big enough so that \(\varphi_{t_k}(M_{t_k})\) is near to the limit geodesic sphere \(S\) of radius \(r\). As \(\varphi_{t_k}\) is an isometry of \(M^{n+1}_\lambda\), we have that \(M_{t_k}\) is close to the geodesic sphere \(\varphi_{t_k}^{-1}(S)\) of radius \(r\). Thus, using spherical geodesic coordinates, we can assure that \(r_{p_{t_k}}\) belongs to a small neighborhood \(V\) of the constant function \(r\) in \(h^{1+\beta}(\mathbb{S}^n)\).

So, applying Proposition 11 with initial condition \(r_{p_{t_k}}\), we can conclude that the solution \(r(t, \cdot)\) of (6.9) starting at \(r_{p_{t_k}}\) is defined on \([0, \infty[^\) and converges exponentially to a unique function \(\tilde{\rho}\). This implies that \(\bar{X}_t(u) = \exp_{p_{t_k}} r(t, u)u\) solves (6.4) and converges exponentially to \(u \mapsto \exp_{p_{t_k}} \tilde{\rho}(u)u\). Therefore, the reparametrization \(X_t\) of \(\bar{X}_t\) given by (6.1) has the same convergence properties; in addition, it is a solution of (1.2) starting at \(p_{t_k}\), and, by uniqueness, \(X_t\) coincides on \([t_k, \infty[^\) with the solution of (1.2) given by part (b) of Theorem 1.

On the other hand, Step 1 in the proof of Proposition 10 says that the mean curvature \(H_t\) of the hypersurfaces \(X_t(M)\) tends to a constant value as \(t \to \infty\). In conclusion, the only possibility is that \(\exp_{p_{t_k}} \tilde{\rho}(u)u\) represents a geodesic sphere in \(M^{n+1}_\lambda\) and, by the volume-preserving properties of the flow, such sphere has to enclose the same volume as the initial condition \(X_0(M)\).
7 A result for certain non-necessarily $h$-convex initial data

A remarkable fact is that in the last section we have not used all the strength of the results on the existence and exponential attractivity of the center manifold $\mathcal{M}^c$. It is precisely this additional power which allows us to extend the claims about long time existence and convergence of Theorem 1 to certain non $h$-convex initial data; in particular, those sufficiently close to a geodesic sphere of $M^{n+1}_\lambda$.

Notice that, if we begin the flow with a non $h$-convex hypersurface $M_0 \subset M^{n+1}_\lambda$, we cannot establish the convergence of a sequence $\{M_t\}$ up to isometries (like in Proposition 10), because the properties of $h$-convexity are strongly used to find $t$-independent bounds for the second fundamental form (together with all its derivatives) of the hypersurfaces $M_t$ evolving under the flow, and recall that these bounds are the key to prove (5.1).

In spite of this, it is not difficult to overcome the absence of $h$-convexity since we are just in the same situation as in [12] (see also [13] for a full understanding of [12]). The only point which need to be checked again in our particular situation is the equality between the center manifold $M^c$ (cf. [12] for its definition) and the equilibria $\mathcal{M}$ of (6.9) in some small neighborhood, as it is obvious that $\mathcal{M}$ is different in equations (6.9) and [12] (2.1). Next we are going to check this identity.

Proposition 12 Let $S$ be a geodesic sphere of $M^{n+1}_\lambda$ of radius $r_S$ and center $p_S$. There is a neighborhood $\mathcal{O}$ of $r_S$ in which $M$ coincides with an open set of the local center manifold $M^c$ for the equation (6.9).

Proof Let us begin by observing that the construction of $M^c$ as a center manifold for (6.9) is identical to that for the equation (2.1) in [12]. Therefore, $M^c$ is a $(n+2)$-dimensional manifold tangent to $\{1\} \oplus H_1$, where $\{1\}$ denotes the space of constant functions on $S^n$ and $H_1$ is the space of eigenfunctions corresponding to the first nonzero eigenvalue of $\Delta_S$.

On the other hand, $\mathcal{M}$ is the space of functions $\rho : S^n \rightarrow \mathbb{R}^+$ such that $\exp_{p_S} \rho(u)u$ parametrizes a constant mean curvature hypersurface of $M^{n+1}_\lambda$, that is, a geodesic sphere of $M^{n+1}_\lambda$. Then, in a small neighborhood of $r_S$

$$U_\varepsilon = \{ r \in C^\infty(S^n) : \| r - r_S \|_{h_{1+\beta}} < \varepsilon \}, \quad (7.1)$$

$\mathcal{M}$ can be parametrized by $(z_0, z) \in \mathbb{R}^{n+2} \equiv \mathbb{R} \oplus T_{p_S} M^{n+1}_\lambda$, being $z_0 + r_S$ the radius of a geodesic sphere $S_z$ and $z = (z_1, ..., z_{n+1})$ the normal coordinates (centered at $p_S$) of its center.

The function $\rho_z : S^n \rightarrow \mathbb{R}^+$ which represents the geodesic sphere $S_z$ has to satisfy $r_S + z_0 = \text{dist}(\exp_{p_S} \rho_z(u)u, \exp_{p_S} z)$. Using hyperbolic trigonometry, we can write this equality under the form

$$c_\lambda(r_S + z_0) = c_\lambda(\rho_z(u))c_\lambda(|z|) + \lambda s_\lambda(\rho_z(u)) \frac{s_\lambda(|z|)}{|z|} \langle u, z \rangle. \quad (7.2)$$
Thus, by implicit differentiation of (7.2), we obtain, at \((z_0, z) = (0, 0)\) \(\in \mathbb{R}^{n+2}\),

\[
\frac{\partial \rho_z}{\partial z_0} \bigg|_{(0,0)} (u) = 1, \quad \frac{\partial \rho_z}{\partial z_i} \bigg|_{(0,0)} (u) = u_i.
\]

(7.3)

Since \(\{1, u_1, \ldots, u_{n+1}\}\) is a basis of \(\{1\} \oplus \mathcal{H}_1\), the differential of the function \(\rho : (z_0, z) \mapsto \rho(z)\) at \((0, 0)\) is an isomorphism between \(\mathbb{R}^{n+2}\) and \(\{1\} \oplus \mathcal{H}_1\). From now on, the equality \(M = M^c\) in a neighborhood \(\mathcal{O}\) of \(r_s\) follows like in (vi) of [12]. \(\square\)

**Remark 1** Arguing as in (vii) of [12], we can conclude that, given \(m \in \mathbb{N}\), there exists \(\omega > 0\) and \(\varepsilon > 0\) such that for each initial condition \(r_0 \in U_\varepsilon\), where \(U_\varepsilon\) is defined by (7.1), it follows the statement (a) in Proposition 11. Moreover, as in part (b) of the same proposition, we can find constants \(c = c(m, \omega) > 0\), \(T = T(m, \omega) > 0\), and a unique \(\tilde{\rho} \in \mathcal{O} \cap M^c\), depending only on \(r_0\), and satisfying

\[
\|r(t, \cdot) - \tilde{\rho}(\cdot)\|_{C^m} \leq K(r_0, c, \tilde{\rho}) e^{-\omega t} \quad \text{for all} \quad t > T.
\]

But applying Proposition 12, we know that \(M^c \cap \mathcal{O} = \mathcal{M}\); so we can find a unique \((z_0, z) \in \tilde{\rho}^{-1}(\mathcal{O})\) such that \(\exp_{p_S} \tilde{\rho}(u)u\) represents a geodesic sphere of \(M^{n+1}_\lambda\) with center \(z\) and radius \(r_s + z_0\).

Thanks to Proposition 12, we are in position to prove Theorem 2.

**Proof of Theorem 2**

Let \(S = \partial B(p_S, r_s)\) be a geodesic sphere in \(M^{n+1}_\lambda\) and \(m \in \mathbb{N}\). By Remark 1, we can find an \(\varepsilon > 0\) and a neighborhood \(U_\varepsilon\), defined as in (7.1), satisfying the property detailed in Remark 1.

Now consider any arbitrary embedding \(X : M \to M^{n+1}_\lambda\) which \(h^{1+\beta}\)-distance to \(S\) is less than \(\varepsilon\); in other words, taking spherical coordinates centered at \(p_S\), the radial distance \(r(\cdot) = \text{dist}(p_S, X(\cdot))\) of \(X(M)\) belongs to the neighborhood \(U_\varepsilon\).

Therefore, Remark 1 assures that the solution \(r_t(\cdot)\) of (6.9) starting at \(r\) exists on \([0, \infty[\) and \(X_t(u) = \exp_{p_S} r_t(u)u\) converges, as \(t \to \infty\), to a geodesic sphere in \(M^{n+1}_\lambda\). Finally, noticing that \(X_t\) is a solution of (6.4), Theorem 2 follows. \(\square\)

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