ON THE CANONICAL DEGREES OF GORENSTEIN
THREEFOLDS OF GENERAL TYPE

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Abstract. Let $X$ be a Gorenstein minimal projective 3-fold with
at worst locally factorial terminal singularities. Suppose that the
canonical map is generically finite onto its image. C. Hacon showed
that the canonical degree is universally bounded by 576. We im-
proved Hacon’s universal bound to 360. Moreover, we gave all the
possible canonical degrees of $X$ if $X$ is an abelian cover over $\mathbb{P}^3$
and constructed all the examples with these canonical degrees.

1. Introduction

The study of the canonical maps of projective varieties of general
type is one of the central problems in algebraic geometry. For the
case of surfaces, Persson ([Per]) constructed a surface of general type
with canonical degree 16 in 1978. About the same time, Beauville
([Bea]) proved that the degree of the canonical map is less than or
equal to 36 and with the equality holds if and only if $X$ is a ball
quotient surface with $K_X^2 = 36$, $p_g = 3$, $q = 0$, and $|K_X|$ is base
point free. Later, Xiao also found some restrictions on surfaces with
high canonical degrees ([Xiao]). Since the canonical degree is bounded
above, next interesting question is to determine which positive integers
$d$’s occur as the degrees of the canonical map. There are plenty of
examples (see [Bea], [Cat1], [V-Z] ) with canonical degrees being 2.
For $d = 3$ and $d = 5$, Tan ([Tan]) and Pardini ([Par1]) constructed
several surfaces independently. When $p_g(\Sigma) = 0$, Beauville ([Bea])
constructed surfaces with $\chi(O_X)$ arbitrarily large and the canonical
degrees 2, 4, 6 and 8 . For $d = 9$, Tan also constructed a surface in

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Later, Casnati ([Cas]) constructed surfaces of canonical degree from 3 to 9 as subvarieties in some projective bundle given by Pfaffians of alternating matrices. The authors ([D-G]) classified the surfaces whose canonical maps are abelian covers over $\mathbb{P}^2$ and constructed these surfaces by explicit defining equations. Recently, Rito in his series papers ([Rit], [Rit2], [Rit3]) constructed some new surfaces of general type with canonical degrees 12, 16 and 24 respectively.

In dimension at least three, the situation seems much less clear. M. Chen studied the canonical map of fiber type ([Ch], [C-H]) and posted an open problem in [Ch] as follows: Let $X$ be a Gorenstein minimal projective 3-fold with at worst locally factorial terminal singularities. Suppose that the canonical map is generically finite onto its image. Is the generic degree of the canonical map universally upper bounded? Hacon gave a positive answer to Chen’s problem. More precisely, he showed that the canonical degree is at most 576.

In this paper, we improve Hacon’s upper bound by showing the following main theorem.

**Theorem 1.1.** Let $X$ be a Gorenstein minimal complex projective 3-fold of general type with locally factorial terminal singularities. Suppose that $|K_X|$ defines a generically finite map $\phi_X : X \dasharrow \mathbb{P}^{p_g-1}$, then $\deg \phi_X \leq 360$ and with the equality holds if and only if $p_g(X) = 4$, $q(X) = 2$, $\chi(\omega_X) = 5$, $K^3 = 360$ and $|K_X|$ is base point free.

Since the canonical degree is bounded above, it is quite interesting to consider a parallel problem as surfaces that which positive integers $d$’s occur as the degrees of the canonical map of Gorenstein minimal projective 3-fold. As far as we know, there are quite few examples about 3-fold of general type with higher canonical degree. Cai ([Cai]) constructed some examples of 3-fold with canonical degrees 32 and 64 based on the existence of the surface with canonical degree 16 which was constructed by Persson. We show that if the canonical map is an abelian cover over $\mathbb{P}^3$ then the only possible canonical degrees of a Gorenstein minimal projective 3-fold are $2^m$ ($1 \leq m \leq 5$), by explicit constructions.

**2. PROOF OF THE MAIN THEOREM**

Let $X$ be a Gorenstein minimal complex projective 3-fold of general type with locally factorial terminal singularities. Suppose that $|K_X|$ defines a generically finite map $\phi_X : X \dasharrow \mathbb{P}^{p_g-1}$. We will base on Hacon’s beautiful arguments to improve the universal upper bound of the canonical degree.
Proof. Since \( \phi_X \) is generically finite, one has that \( p_g(X) \geq 4 \). Let \( d = \text{deg } \phi_X \). By the Miyaoka-Yau inequality (\[M1\]), we have

\[
d(p_g(X) - 3) \leq K^3_X \leq 72 \chi(\omega_X).
\]

If we can show \( \chi(\omega_X) \leq p_g(X) + 1 \), then

\[
d \leq 72 \frac{\chi(\omega_X)}{p_g(X)} - 3 \leq 72 \left( p_g(X) + 1 \right) = 72 \left( 1 + \frac{4}{p_g(X) - 3} \right) \leq 360. \tag{2.1}
\]

If \( q(X) \leq 2 \), then \( \chi(\omega_X) \leq p_g(X) + q(X) - 1 \leq p_g(X) + 1 \).

Now we can assume hereafter that \( q(X) \geq 3 \). Consider the Albanese map \( alb_X \) of \( X \) and the Stein factorization \( f \) of \( alb_X \) as follows:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
& \downarrow{alb_X} & \\
& Alb(X) & \\
\end{array}
\]

By Hacon’s argument (see the proof of \[Ha\], Theorem 1.1), one has

(1) \( \chi(\omega_X) \leq p_g(X) \), if \( \dim Y \geq 2 \);
(2) \( \chi(\omega_X) \leq p_g(X) + \chi(\omega_Y) \) and \( \chi(\omega_Y)p_g(F) \leq p_g(X) \), where \( F \) is the general fiber of \( f \), if \( \dim Y = 1 \).

Hence if \( \dim Y \geq 2 \), by (2.1), the statement holds. More precisely,

\[
d \leq 72 \frac{p_g(X)}{p_g(X) - 3} \leq 288.
\]

We only need to consider \( \dim Y = 1 \).

If \( p_g(F) \leq \dim X - 1 \), then

\[
h^0(\mathcal{O}_X(K_X) \otimes \mathcal{O}_F) \leq h^0(\mathcal{O}_X(K_X + F) \otimes \mathcal{O}_F) = h^0(\mathcal{O}_F(K_F)) \leq \dim X - 1,
\]

which means that \( \dim \text{Im}(\phi_X|_F) \leq \dim X - 2 \), and hence \( \dim \text{Im}\phi_X \leq \dim X - 1 \), which contradicts the assumption that \( \phi_X \) is generically finite. So we have that \( p_g(F) \geq \dim X = 3 \) and then \( p_g(X) \geq \chi(\omega_Y)p_g(F) = (q(X) - 1)p_g(F) \geq 6 \).

Therefore

\[
d \leq 72 \frac{\chi(\omega_X)}{p_g(X) - 3} \leq 72 \frac{p_g(X) + \chi(\omega_Y)}{p_g(X) - 3} \leq 72 \left( 1 + \frac{1}{p_g(F)} \right) \frac{p_g(X)}{p_g(X) - 3} \leq 192.
\]

From the argument above, we know that the equality of (2.1) holds if and only if \( p_g(X) = 4, q(X) = 2, \chi(\omega_X) = 5, K^3_X = 360 \) and \( |K_X| \) is base point free. \( \square \)
Remark 2.1. If $X$ is nonsingular and the canonical divisor $K_X$ is ample then the equality in the main theorem holds if and only if $X$ is a ball quotient. We guess that such a ball quotient with those invariants exists. For the parallel case of surfaces with the maximal canonical degree, the surface of general type with canonical degree 36 does exist which was constructed as some fake projective plane by S. Yeung recently ([Yeung]).

3. Canonical maps defined by abelian covers

The theory of cyclic covers of algebraic surfaces was studied first by Comessatti in [Com]. Then F. Catanese ([Cat2]) studied smooth abelian covers in the case $(\mathbb{Z}_2)^{\oplus 2}$ and R. Pardini ([Par2]) analyzed the general case. In this section, we shall recall some basic definitions and results for abelian covers and construct minimal 3-folds of general type whose canonical maps are abelian covers over $\mathbb{P}^3$. Since our point of view is to find the defining equations, we use the second author’s expressions and notations appearing in [Gao].

Let $\varphi : X \to Y$ be an abelian cover associated to abelian group $G \cong \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k}$, i.e., function field $\mathbb{C}(X)$ of $X$ is an abelian extension of the rational function field $\mathbb{C}(Y)$ with Galois group $G$. Without lose of generality, we can assume $n_1|n_2|\cdots|n_k$.

**Definition 3.1.** The dates of abelian cover over $Y$ with group $G$ are $k$ effective divisors $D_1, \ldots, D_k$ and $k$ linear equivalent relations $D_1 \sim n_1L_1, \ldots, D_k \sim n_kL_k$.

Let $\mathcal{L}_i = \mathcal{O}_Y(L_i)$ and $f_i$ be the defining equation of $D_i$, i.e., $D_i = \text{div}(f_i)$, where $f_i \in H^0(Y, \mathcal{L}_i^{n_i})$. Denote $V(\mathcal{L}_i) = \text{Spec} S(\mathcal{L}_i)$ to be the line bundle corresponding to $\mathcal{L}_i$, where $S(\mathcal{L}_i)$ is the sheaf of symmetric $\mathcal{O}_Y$ algebra. Let $z_i$ be the fiber coordinate of $V(\mathcal{L}_i)$. Then the abelian cover can be realized by the normalizing of surface $V$ defined by the system of equations

$$z_1^{n_1} = f_1, \ldots, z_k^{n_k} = f_k.$$

So we have the following diagram:
Sometimes we call $X$ is defined by these equations if there is no confusions in the context.

We summerize our main results as follows.

**Theorem 3.2.** (See [Gao]) Denote by $[Z]$ the integral part of a $\mathbb{Q}$-divisor $Z$, $-L_g = - \sum_{i=1}^{k} g_i L_i + \left\lfloor \frac{m_i}{n_i} D_i \right\rfloor$. Then
\[
\varphi_* \mathcal{O}_X = \bigoplus_{g \in G} \mathcal{O}_Y(-L_g).
\]
(3.1)

where $g = (g_1, \ldots, g_k) \in G$.

So the decomposition of $\varphi_* \mathcal{O}_X$ is totally determined by the abelian cover.

**Corollary 3.3.** If $X$ is non-singular, $D$ is the divisor on $Y$, then
\[
h^i(X, \varphi^* \mathcal{O}_Y(D)) = \sum_{g \in G} h^i(Y, \mathcal{O}_Y(D - L_g)).
\]

If the canonical map of $X$ is an abelian cover over $\mathbb{P}^3$ then we have the explicit decomposition of $\varphi_* \mathcal{O}_X$.

**Lemma 3.4.** If $\varphi = \phi_X$ is a finite abelian cover of degree $d$ over $\mathbb{P}^3$, then $\varphi_* \mathcal{O}_X = \mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(-2)^{\oplus d/2 - 1} \oplus \mathcal{O}_{\mathbb{P}^3}(-3)^{\oplus d/2 - 1} \oplus \mathcal{O}_{\mathbb{P}^3}(-5)$.

**Proof.** Because $\varphi$ is a finite abelian cover, $\varphi_* \mathcal{O}_X$ is a direct sum of the line bundles by Theorem 3.2.

\[
\varphi_* \mathcal{O}_X = \mathcal{O}_{\mathbb{P}^3} \oplus \bigoplus_{i=1}^{d-1} \mathcal{O}_{\mathbb{P}^3}(-l_i).
\]

Assume $0 < l_{d-1} \leq l_{d-2} \leq \cdots \leq l_1$.

Since $K_X = \varphi^*(\mathcal{O}_{\mathbb{P}^3}(1))$, for any $m \geq 1$,
\[
P_m(X) = h^0(mK_X) = h^0(\varphi^*(\mathcal{O}_{\mathbb{P}^3}(m))) = h^0(\mathcal{O}_{\mathbb{P}^3}(m)) + \sum_{i=1}^{d-1} h^0(\mathcal{O}_{\mathbb{P}^3}(m-l_i)).
\]

Because $p_g(X) = h^0(\varphi^*(\mathcal{O}_{\mathbb{P}^3}(H))) = h^0(\mathcal{O}_{\mathbb{P}^3}(1)) = 4$, we see that
\[
h^0(\mathcal{O}_{\mathbb{P}^3}(1-l_i)) = 0, \quad 1 \leq i \leq d - 1.
\]

So $l_i \geq 2$.

And $p_g = h^3(\varphi_* \mathcal{O}_X) = h^3(\mathcal{O}_{\mathbb{P}^3}) + \sum_{i=1}^{d-1} h^3(\mathcal{O}_{\mathbb{P}^3}(-l_i))$,

So $4 = \sum_{i=1}^{d-1} h^0(\mathcal{O}_{\mathbb{P}^3}(l_i - 4))$, then $l_i \leq 5$.

Therefore, we have two cases as follows:

1. $l_1 = 5, \quad l_2, \ldots, l_{d-1} < 4$, and
2. $l_1 = l_2 = l_3 = l_4 = 4, \quad l_5, \ldots, l_{d-1} < 4$. 

Let \( m = 2 \), we have the second plurigenus of \( X \) \( P_2(X) = \chi(2K_X) = \frac{1}{2}K_X^3 + \frac{1}{6}K_Xc_2 + \chi(\mathcal{O}_X) = \frac{d}{2} + 3\chi(K_X) = \frac{d}{2} + 9 \). On the other hand, \( P_2(X) = h^0(\mathcal{O}_{\mathbb{P}^3}(2)) + \sum_{i=1}^{d-1} h^0(\mathcal{O}_{\mathbb{P}^3}(2 - l_i)) \). So
\[
\sum_{i=1}^{d-1} h^0(\mathcal{O}_{\mathbb{P}^3}(2 - l_i)) = \frac{d}{2} - 1,
\]
then there are exact \( (\frac{d}{2} - 1) \) 2’s among \( l_i \)’s.

Let \( m = 3 \), we have the third plurigenus of \( X \) \( P_3(X) = \chi(3K_X) = \frac{5}{2}K_X^3 + \frac{1}{4}K_Xc_2 + \chi(\mathcal{O}_X) = \frac{5d}{2} + 5\chi(K_X) = \frac{5d}{2} + 15 \). On the other hand, \( P_3(X) = h^0(\mathcal{O}_{\mathbb{P}^3}(3)) + \sum_{i=1}^{d-1} h^0(\mathcal{O}_{\mathbb{P}^3}(3 - l_i)) \). So
\[
\sum_{i=1}^{d-1} h^0(\mathcal{O}_{\mathbb{P}^3}(3 - l_i)) = \frac{5d}{2} - 5,
\]
The second case does not satisfy the equation. So the lemma is proved.

Now let \( \varphi : X \to \mathbb{P}^3 \) be an abelian cover associated to an abelian group \( G \cong \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k} \). Then \( X \) is the normalization of the 3-fold defined by
\[
z_1^{n_1} = f_1 = \prod_\alpha p_\alpha^{\alpha_1}, \ldots, z_k^{n_k} = f_k = \prod_\alpha p_\alpha^{\alpha_k},
\]
where \( p_\alpha \)'s are coprime and \( \alpha = (\alpha_1, \ldots, \alpha_k) \in G \). Denote \( x_\alpha \) to be the degree of \( p_\alpha \), \( e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in G, 1 \leq i \leq k \), and \( l_g \) be the degree of \( L_g, g \in G \). So \( x_\alpha \) and \( l_g \) are all integers. Then
\[
n_i e_i = \sum_\alpha \alpha_i x_\alpha \quad i = 1, \ldots, k,
\]
\[
l_g = \sum_{i=1}^{k} g_i l_{e_i} - \sum_\alpha \left[ \sum_{i=1}^{k} \frac{g_i \alpha_i}{n_i} \right] x_\alpha.
\]

**Lemma 3.5.** Using the notation as above, if \( \varphi = \phi_X \), then there exists \( g' = (g'_1, \ldots, g'_k) \in G \cong \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k} \) and a partition of \( G \) set-theoretically, \( G = \{0\} \cup \{g'\} \cup S_1 \cup S_2 \), where the cardinalities of \( S_1 \)
and $S_2$ are equal, such that $x_\alpha$ satisfies the following equation

\[
\begin{align*}
\left\{ \begin{array}{l}
n_i l_{e_i} &= \sum_\alpha \alpha_i x_\alpha \\
l_g' &= \sum_{i=1}^k g_i l_{e_i} - \sum_\alpha \left[ \sum_{i=1}^k \frac{g_i \alpha_i}{n_i} \right] x_\alpha = 5 \\
l_g &= \sum_{i=1}^k g_i l_{e_i} - \sum_\alpha \left[ \sum_{i=1}^k \frac{g_i \alpha_i}{n_i} \right] x_\alpha = 3, \quad g \in S_1 \\
l_g &= \sum_{i=1}^k g_i l_{e_i} - \sum_\alpha \left[ \sum_{i=1}^k \frac{g_i \alpha_i}{n_i} \right] x_\alpha = 2, \quad g \in S_2
\end{array} \right.
\]

Proof. It comes from Lemma 3.4 directly.

By the above lemma, finding 3-folds whose canonical map are abelian covers over $\mathbb{P}^3$ is equivalent to finding the integral roots $\{x_\alpha\}$ of the above equations.

**Theorem 3.6.** Let $X$ be a Gorenstein minimal complex projective 3-fold of general type with locally factorial terminal singularities and $\varphi : X \to \mathbb{P}^3$ is an abelian cover. If $\varphi = \phi_X$ then the canonical degree can only be $2^m$, $1 \leq m \leq 5$.

Proof. By Lemma 3.5, we only need to find the integral solutions of the equations (\(*\)). So the only possible degrees are 2, 4, 8, 16 and 32 by using computer calculations. Moreover, we have the defining equations of the examples of all the degrees as follows.

- **Degree 2:**
  \[
  z^2 = f;
  \]

- **Degree 4:**
  \[
  \begin{cases}
  z_1^2 = s \\
  z_2^2 = q;
  \end{cases}
  \]

- **Degree 8:**
  \[
  \begin{cases}
  z_1^2 = t_1 q \\
  z_2^2 = t_2 q \\
  z_3^2 = t_3 q;
  \end{cases}
  \]

- **Degree 16:**
  \[
  \begin{cases}
  z_1^2 = h_1 h_4 t_1 t_2 \\
  z_2^2 = h_2 h_4 t_2 t_3 \\
  z_3^2 = h_3 h_4 t_1 t_3 \\
  z_4^2 = h_2 h_3 t_3;
  \end{cases}
  \]
Degree 32:

\[
\begin{align*}
z_1^2 &= h_1 h_2 h_3 h_{10} \\
z_2^2 &= h_4 h_5 h_6 h_{10} \\
z_3^2 &= h_2 h_3 h_6 h_7 \\
z_4^2 &= h_1 h_3 h_5 h_8 \\
z_5^2 &= h_7 h_8 h_9 h_{10}
\end{align*}
\]

where the degree of \( h \)'s is 1, \( t \)'s is 2, \( q \)'s is 4, \( s \)'s is 6, \( f \)'s is 10 and they all define nonsingular surfaces in \( \mathbb{P}^3 \) and intersect normal crossingly.

We want to show these 3-folds are smooth after normalization. Since the arguments are similar, we only prove the most complicated case with canonical degree 32.

Actually, we only need to consider the intersections of the branch locus locally. Let \( \ell_{ij} \)'s be the intersection lines of \( h_i \) and \( h_j \). Around the general point of \( \ell_{ij} \) (except the intersection of three planes), the cover is locally defined by

\[
\begin{align*}
z_1^2 &= x^{a_{i1}} y^{a_{i2}}, \\
z_2^2 &= x^{a_{21}} y^{a_{22}}, \\
z_3^2 &= x^{a_{31}} y^{a_{32}}, \\
z_4^2 &= x^{a_{41}} y^{a_{42}},
\end{align*}
\]

where \( a_{ij} = 0 \) or 1 for all \( i, j \).

It is easy to check that \( \{(a_{i1}, a_{i2})\} \subsetneq \{(1, 1), (0, 0)\} \) i.e., at least one pair \( \{(a_{i1}, a_{i2})\} \subseteq \{(1, 0)\} \text{ or } \{(0, 1)\} \). Without lose of generality, we can assume \( (a_{11}, a_{21}) = (1, 0) \), i.e. \( z_1^2 = x \). After normalization, the cover is branched along the smooth surfaces. So the 3-fold is smooth at the preimages of the general points of \( \ell_{ij} \)'s under the normalization map.

Let \( p_{ijk} \) be the intersection point of \( h_i \), \( h_j \) and \( h_k \). The arguments are similar. Let us take \( p_{123} \) for example. The cover is locally defined by

\[
\begin{align*}
z_1^2 &= xyw, \\
z_3^2 &= yw, \\
z_4^2 &= w.
\end{align*}
\]

After normalization, the cover is locally defined by

\[
\begin{align*}
z_1^2 &= x, \\
z_3^2 &= y, \\
z_4^2 &= w.
\end{align*}
\]

So the 3-folds are smooth.

It is easy to see that these 3-folds are all nonsingular with \( p_g(X) = 4, q(X) = h^{2,0} = 0, \chi(\mathcal{O}_X) = -3, K_X = \varphi^*(\mathcal{O}_{\mathbb{P}^3}(1)) \) and \( K^3_X \) equals each degree of the covers. □

**Remark 3.7.** By Lemma 3.5, we have the integral solution of the equations (*) for degree 6 and 18. But the isolated singularities of corresponding 3-fold are not Gorenstein terminal since they are not cDV ([Rei2]).
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