ON A THEOREM OF KISIN

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Let $K$ be a $p$-adic field, i.e., a complete discretely valued field of characteristic 0 with perfect residue field of characteristic $p > 0$, and $\bar{K}$ an algebraic closure of $K$. We fix a uniformiser $\pi \in K$. Let $\Xi = \Xi_{\pi}$ be the corresponding Kummer $\mathbb{Z}_p(1)$-torsor; its elements are sequences $\xi = (\xi_n)_{n \geq 0}$ of elements in $\bar{K}$ such that $\xi_{n+1} = \xi_n, \xi_0 = \pi$. Pick one $\xi$, and set $K_\xi = \bar{K}(\xi)$. Consider the Galois groups $G := \text{Gal}(\bar{K}/K), G_\xi := \text{Gal}(\bar{K}/K_\xi)$; let $\text{Rep}(G), \text{Rep}(G_\xi)$ be the categories of their finite-dimensional $\mathbb{Q}_p$-representations.

The next result was conjectured by Breuil [B] and proved by Kisin [K] 0.2; the proof in loc. cit. is based on theory of Kisin modules. This note provides an alternative argument that uses only basic properties of Fontaine’s rings; its key ingredient (namely, (i) of the lemma below) is the same as in Grothendieck’s proof of the monodromy theorem.

**Theorem.** The restriction functor $\text{Rep}(G) \to \text{Rep}(G_\xi)$ is fully faithful on the subcategory of crystalline representations.

**Proof.** The Galois group $G$ acts on $\Xi$, and $G_\xi$ is the stabilizer of $\xi$. The action is transitive, i.e., $G/G_\xi \sim \Xi$, since polynomials $t^{p^n} - \pi$ are irreducible.

Let $R$ be the ring of continuous $\mathbb{Q}_p$-valued functions on $\Xi$. Let $R_{\text{st}} \subset R_{\phi}$ be the subrings of polynomial, resp. locally polynomial, functions (this makes sense since $\Xi$ is $\mathbb{Z}_p(1)$-torsor). Since $G$ acts on $\Xi$ by affine transformations, its action on $R$ preserves the subrings.

**Lemma.** (i) $R_{\phi}$ is the union of all finite-dimensional $G$-submodules of $R$.
(ii) $R_{\text{st}}$ is the union of all semi-stable $G$-submodules of $R_{\phi}$.
(iii) $\mathbb{Q}_p$ is the only nontrivial crystalline $G$-submodule of $R_{\text{st}}$.

Assuming the lemma, let us prove the theorem. For $V \in \text{Rep}(G_\xi)$ we denote by $I(V)$ the induced $G$-module, that is the space of all continuous maps $f : G \to V$ such that $f(hg) = hf(g)$ for $h \in G$, the action of $G$ is $g(f)(g') = f(g'g)$. It is a $G$-equivariant $R$-module, the $R$-action is $(rf)(g) = r(g^{-1}\xi)f(g)$. For $U \in \text{Rep}(G)$ we have the Frobenius reciprocity $\text{Hom}_{G_\xi}(U,V) \sim \text{Hom}_G(U,I(V))$ that identifies $\alpha : U \to V$ with $\tilde{\alpha} : U \to I(V)$, $\tilde{\alpha}(u)(g) = \alpha(gu), \alpha(u) = \tilde{\alpha}(u)(1)$. For $V \in \text{Rep}(G)$ the image of $\text{id}_V \in \text{Hom}_{G_\xi}(V,V)$ is a $G$-morphism $V \to I(V)$ that yields an identification of $G$-equivariant $R$-modules $V \otimes R \sim I(V)$.

So for $V_1, V_2 \in \text{Rep}(G)$ one has identifications $\text{Hom}_{G_\xi}(V_1, V_2) = \text{Hom}_G(V_1, I(V_2)) = \text{Hom}_G(V_1 \otimes R) = \text{Hom}_G(V_1 \otimes V_2, R) = \text{Hom}_G(V_1 \otimes V_2, R_{\phi}$, the last equality

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comes from (i). If both $V_i$ are crystalline, then this equals \( \text{Hom}_G(V_1 \otimes V_2^*, \mathbb{Q}_p) = \text{Hom}_G(V_1, V_2) \) by (ii), (iii). Thus \( \text{Hom}_G(V_1, V_2) = \text{Hom}_G(V_1, V_2) \), q.e.d. \( \Box \)

**Proof of Lemma.** Let $P$ be the group of all affine automorphisms of $\mathbb{Z}_p(1)$-torsor $\Xi$; it is an extension of $\mathbb{Z}_p$ by $\mathbb{Z}_p(1)$, the choice of $\xi$ gives a splitting. Let $\eta: G \to P$ be the action of $G$ on $\Xi$; its composition with $P \to \mathbb{Z}_p^*$ is cyclotomic character $\chi$.

Consider the filtration $R_{st,n}$ on $R_{st}$ by the degree of the polynomial. Then $G$ acts on $gr_R R_{st}$ by $\chi^{-n}$, i.e., $gr_R R_{st}$ is isomorphic to $\mathbb{Q}_p(-n)$.

There is a canonical morphism $\varepsilon: R_{st} \to B_{st}$ of $\mathbb{Q}_p$-algebras defined as follows. For $\xi \in \Xi$ let $l_\xi : \Xi \to \mathbb{Z}_p(1)$ be the identification of torsors such that $l_\xi(\xi) = 0$. If $\tau$ is a generator of $\mathbb{Z}_p(1)$, then $\tau^{-1}l_\xi \in R_{st}$ is a linear polynomial function, i.e., a free generator of $R_{st}$. We define $\varepsilon$ by formula $\varepsilon(\tau^{-1}l_\xi) = -\tau^{-1}\lambda(\xi)$. Here in the r.h.s. we view $\tau$ as an invertible element of $B_{crys}$ via the embedding $\mathbb{Z}_p(1) \hookrightarrow B_{crys}$ from [F1] 2.3.4, and $\lambda(\xi) \in B_{st}$ is as in [F1] 3.1.4. It follows from the definitions in [F1] 3.1 that $\varepsilon$ does not depend on the auxiliary choice of $\xi$. It evidently commutes with the Galois action. Since $\log(\xi)$ is a free generator of $B_{st}$ over $B_{crys}$, we see that $\varepsilon$ is injective and $R_{st}$ for $n \geq 1$ are non-crystalline semi-stable $G$-modules.

Choose $v$ and $\log$ from [F2] 5.1.2 as $v(\pi) = 1$, $\log(\pi) = 0$. As in [F2] 5.2, this yields the fully faithful tensor functor $D_{st} : \text{Rep}(G)_{st} \to \text{MF}_K(\varphi, N)$.

Consider the polynomial algebra $K_0[t]$. We equip it with Frobenius semi-linear automorphism $\varphi$, $\varphi(t) := pt$, the $K_0$-derivation $N := \partial_t$, and the Hodge filtration $F^i :=$ the $K$-span of $t^{2i}$. The subspaces of polynomials of degree $\leq n$ are filtered \((\varphi, N)\)-modules, so $K_0[t]$ is a ring ind-object of $\text{MF}_K(\varphi, N)$.

There is a canonical isomorphism $K_0[t] \xrightarrow{\sim} D_{st}(R_{st})$ which identifies $t$ with $(\tau^{-1}l_\xi) \otimes \tau + 1 \otimes \lambda(\xi) \in (R_{st} \otimes B_{st})^G = D_{st}(R_{st})$. Thus each $D_{st}(R_{st,n})$ is a single Jordan block for the action of $N$, so every finite-dimensional $G$-submodule of $R_{st}$ equals one of $R_{st,n}$'s, which implies (iii).

Notice that $R_0 = R_0 \otimes R_{st}$, where $R_0$ is the subring of locally constant functions. Since $G$ acts transitively on $\Xi$, one has $R_0^G = \mathbb{Q}_p$ and all finite-dimensional $G$-modules that occur in $R_0$ are generated by $G_\xi$-fixed vectors. These representations are Artinian, hence semisimple, so we have the decomposition $R_0 = \mathbb{Q}_p \oplus R'_0$. Since the map $G_\xi \to \text{Gal}(K^{un}/K)$, where $K^{un} \subset K$ is the maximal unramified extension of $K$, is surjective (for $K^{un} \cap K_0 = K$), every $G$-module in $R_0$ is ramified. Thus every irreducible subquotient of $R'_0 \otimes R_{st}$ is not semi-stable, and we get (ii).

It remains to prove (i). We first show that $\eta(G)$ is an open subgroup of $P$. Since $\chi(G)$ is open in $\mathbb{Z}_p^*$, it suffices to check that $\eta(G) \cap \mathbb{Z}_p(1)$ is open in $\mathbb{Z}_p(1)$. Since every closed nontrivial subgroup of $\mathbb{Z}_p(1)$ is open, we need to check that $\eta(G) \cap \mathbb{Z}_p(1) \neq \{0\}$. If not, then $\eta(G) \sim \chi(G)$ is commutative, so $G$ acts on $\mathcal{R}$ through an abelian quotient. This implies, since $gr_R R_{st} \simeq \mathbb{Q}_p(-n)$ are pairwise non-isomorphic $G$-modules, that filtration $R_{st,n}$ splits, which is not true, q.e.d.

Let $\tau$ be a generator of $\mathbb{Z}_p(1) \subset P$; then $R_0$ is the union of all finite-dimensional $\mathbb{Z}_p(1)$-submodules of $\mathcal{R}$ on which all eigenvalues of $\tau$ are roots of 1. Since $\eta(G)$ has finite index in $\mathcal{R}$, it suffices to show that every finite-dimensional $P$-submodule $V$ of $\mathcal{R}$ has this property. This follows since for $g \in \mathcal{R}$ one has $g \tau^{-1} = \tau^m$, where $m$ is the image of $g$ in $\mathbb{Z}_p^\times$, and there are only finitely many eigenvalues of $\tau$ on $V$. \( \Box \)

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