Elliptic aspects of statistical mechanics on spheres

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Our earlier results on the temperature inversion properties and the ellipticisation of the finite temperature internal energy on odd spheres are extended to orbifold factors of odd spheres and then to other thermodynamic quantities, in particular to the specific heat. The behaviour under modular transformations is facilitated by the introduction of a modular covariant derivative and it is shown that the specific heat on any odd sphere can be expressed in terms of just three functions. It is also shown that the free energy on the circle can be written elliptically.

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1. Introduction.

The topic of finite temperature free field theories on spheres has recently reappeared specifically in connection with the Verlinde-Cardy relation. In a previous paper, while examining this question, several technical points arose which the present work intends to address. The equivalence of the different ways of approaching the theory results in, or is a result of, various mathematical identities, most of which have been around for some time, many of them classic. It was thought useful to present this material in a particular physical context. The opportunity will also be taken to present some extensions of our previous programme.

In [1] two related aspects were considered. One was the symmetry under temperature inversion and the other the ‘ellipticisation’ of the total internal energy $E$. It was shown, for example, that at certain temperatures, $E$ could be expressed in finite terms involving elliptic integrals.

In the present paper we discuss, firstly, the technical extension to orbifold factors of spheres, $S^d/\Gamma$. This analysis allows us to relate several existing mathematical calculations.

All this detailed work concerns just the internal energy, $E$, and we turn next to our major preoccupation which is a study of the other thermodynamic quantities such as the free energy, entropy and specific heat. Roughly speaking, these are related differentially to $E$.

Our objective, admittedly mathematical, is to determine the behaviour of these quantities under modular transformations (which includes temperature inversion) and to obtain elliptic expressions for them as far as possible.

We find for the positive derivatives of the internal energy, such as the specific heat, that the results generally parallel those of [1] whereas for the negative derivatives (integrals), such as the free energy, there are obstructions to temperature inversion symmetry and ellipticisation. The nature of these obstructions makes contact with some basic modular concepts, such period polynomials, which are not pursued here.

2. Resumé of earlier work.

Our previous paper [1] was restricted to conformally invariant free field theories on the space-times (‘Einstein Universes’) $\mathbb{R} \times S^d$ and, apart from a few comments, we exclusively discussed odd spheres, $S^d$. We will continue to do so here.
The approach was based on individual treatment of the terms in the mode degeneracy, which is just an odd polynomial in the mode label. The complete sphere result was then obtained by simple combination of these individual pieces. From this point of view there is nothing special about the sphere. Any odd polynomial would do.

Other treatments, e.g. [2,3], yield more specific forms. We begin here with an expression given by Chang and Dowker [4] which is somewhat more general than just for the sphere. The situation there discussed was a conformal scalar on an orbifold factoring (‘triangulation’) of the sphere, written $S^d/\Gamma$, where $\Gamma$ is a regular solid symmetry group.

The total free energy at temperature $1/\beta$ on $\mathbb{R} \times S^d/\Gamma$ was given in the form

$$F = E_0 - \frac{1}{\beta} \sum_{m=1}^{\infty} \frac{1}{m} e^{\pm d_0 m \beta / 2a} \prod_{i=1}^{d} \frac{1}{2 \sinh(d_i m \beta / 2a)}$$

where the $d_i$ are the degrees associated with the tiling group $\Gamma$ and $a$ is the radius of the sphere. $E_0$ is the zero temperature, Casimir, value. It can be given in terms of Bernoulli polynomials.

The hemisphere corresponds to $d_0 = 1$; there is only one reflecting plane, and all the degrees $d_i$ are equal to 1. The upper sign in (1) refers to Neumann conditions on the rim of the hemisphere, and the lower sign to Dirichlet conditions. To obtain the full sphere periodic value these two expressions are added, and we will do this now for all tilings so as to replace (1) by

$$F = E_0 - \frac{2}{\beta} \sum_{m=1}^{\infty} \frac{1}{m} \cosh(d_0 m \beta / 2a) \prod_{i=1}^{d} \frac{1}{2 \sinh(d_i m \beta / 2a)}.$$  \hfill (2)

For the time being we will work with just two examples; the hemisphere (giving the periodic full sphere) and the quarter-sphere (giving the periodic half-sphere). The quarter-sphere is a lune of angle $\pi/2$ and, in this case, $d_0 = 2$, corresponding to the two reflecting planes, also $d_1 = 2$, $d_i = 1$ ($i = 2, \ldots, d$). There are conical singularities at the north and south poles.

For these two examples, more specifically,

$$F = E_0 - \frac{1}{2^{d-1} \beta} \sum_{m=1}^{\infty} \frac{1}{m} \frac{\cosh(m \beta / 2a)}{\sinh^d(m \beta / 2a)},$$  \hfill (3)

and

$$F = E_0 - \frac{1}{2^{d-1} \beta} \sum_{m=1}^{\infty} \frac{1}{m} \frac{\coth(m \beta / a)}{\sinh^{d-1}(m \beta / 2a)}.$$  \hfill (4)
Another way of writing (3), which is sometimes useful, is,

\[ F = E_0 + \frac{a}{2^{d-2}(d-1)\beta} \frac{\partial}{\partial \beta} \sum_{m=1}^{\infty} \frac{1}{m^2} \frac{1}{\sinh^{d-1}(m\beta/2a)}. \]

As in our previous article, [1], we will work with the total internal energy

\[ \bar{E} = aE = a\partial(\beta F/\partial \beta), \]

\[ \bar{E} = E_0 + \frac{1}{2d} \sum_{m=1}^{\infty} \left( \frac{d-1}{\sinh^{d-1}(m\beta/2a)} + \frac{d}{\sinh^{d+1}(m\beta/2a)} \right), \tag{5} \]

and

\[ \bar{E} = E_0 + \frac{1}{2^{d+1}} \sum_{m=1}^{\infty} \left( \frac{\text{sech}^2(m\beta/2a)}{\sinh^{d+1}(m\beta/2a)} + \frac{d-1}{\sinh^{d+1}(m\beta/2a)} + \frac{2(d-1)}{\sinh^{d-1}(m\beta/2a)} \right). \tag{6} \]

The full sphere expression (5) involves just the summation

\[ Q_g(x) = \sum_{m=1}^{\infty} \text{cosech}^{2g}mx \tag{7} \]

and we now show how to reduce (6) employing also the sum,

\[ R_f(x) = \sum_{m=1}^{\infty} \text{sech}^{2f}mx. \tag{8} \]

Starting from the more general sum

\[ S_{g,f}(x) = \sum_{m=1}^{\infty} \text{sech}^{2f}mx \text{ cosech}^{2g}mx \tag{9} \]

and writing either

\[ \text{sech}^{2}mx = 1 - \sinh^{2}mx \text{ sech}^{2}mx \tag{10} \]

or

\[ \text{cosech}^{2}mx = \cosh^{2}mx \text{ cosech}^{2}mx - 1 \tag{11} \]

we find, for example from (10), the recursion

\[ S_{g,f} = \sum_{j=0}^{f}(-1)^j \binom{f}{j} S_{g-j,j} \]
which is easily implemented starting from $S_{g,0} = Q_g$ and $S_{0,f} = R_f$. In particular

$$S_{g,1} = \sum_{j=1}^{g} (-1)^{g-j} Q_j + (-1)^g R_1$$

and the energies can be put in the form, if $d = 2r + 1$,

$$E_{2r+1} = E_0 + \frac{1}{2^{2r+1}} \left( 2r Q_r + (2r + 1) Q_{r+1} \right)$$

(12)

and

$$E_{2r+1} = E_0 + \frac{1}{2^{2r+2}} \left( (4r - 1) Q_r + (2r + 1) Q_{r+1} + \sum_{j=1}^{r-1} (-1)^{r+1-j} Q_j - (-1)^r R_1 \right)$$

(13)

where the arguments of all the $Q$’s and $R$’s are $\beta/2a$.

In these two cases at least, we see that the problem reduces to the calculation of the sums, (7) and (8), which are the subjects of the papers by Kiyek and Schmidt [5], Ling [6] and Zucker, [7]. Values of the sum, $Q_g$, for specific $g$ are quite old.

3. Hyperbolic summations.

The methods in the papers just referred to, involve elliptic functions, in one way or another, which is not surprising in view of the partial fraction representation of hyperbolic functions and the double sum form of elliptic functions which originated with Eisenstein, [8].

Going back to the beginning, Eisenstein, following Euler, discussed the single sums (see Hancock [9] p.32 Ex.5),

$$(2g, x) = \sum_{n=-\infty}^{\infty} \frac{1}{(x + n)^{2g}} = \pi^{2g} \sum_{k=1}^{g} (-1)^{k+g} A_{2g,2k} \cosec^{2k} \pi x .$$

(14)

Eisenstein says that the coefficients $A_{2g,2k}$ are simply related to Bernoulli numbers and are given by a recursion derived from continued differentiation.

Transcribing to the hyperbolic case,

$$(2g, ix) = \sum_{n=-\infty}^{\infty} \frac{1}{(ix + n)^{2g}} = (-1)^g \pi^{2g} \sum_{k=1}^{g} A_{2g,2k} \cosech^{2k} \pi x .$$

(15)
The explicit form of the recurrence is given by Ling, [6], and no doubt elsewhere after one hundred and sixty years. It is

\[
A_{2g+2,2k} = \frac{1}{2g(2g+1)}((2k-1)(2k-2)A_{2g,2k-2} + 4k^2 A_{2g,2k}) \tag{16}
\]

and trivially \( A_{2g,2g} = 1 \).

Now set \( x = m\mu/\pi \) and sum again over \( m \) referring to the definition (7). One finds,

\[
\sum_{m,n=-\infty}^{\infty} \frac{1}{(i\mu n/\pi + n)^{2g}} - 2\zeta_R(2g) = -2\pi^{2g} \sum_{k=1}^{g} A_{2g,2k} Q_k(\mu), \tag{17}
\]

and we have arrived at an Eisenstein series.

Equation (17) can be used to determine \( Q_k \) recursively in terms of the left hand side which we have shown in [1] is given in elliptic function terms. (This is a standard result.) Such is Ling’s method of finding the sums \( Q_k \) and is a little roundabout. A further instalment is given in [10].

The energy (12) can then be determined. Again one has to combine the Eisenstein series values and the calculation is actually not so different from our earlier one. The only novelty is that there is now no explicit mention of the mode degeneracies. They are, however, lurking in the analysis as the following shows.

An alternative way of arranging the evaluation of the \( Q_g \) is to rewrite the definition (7) as a \( q \)-series as done by Zucker, [7]. We re-express his approach somewhat to fit our requirements. Resummation produces

\[
Q_g(\mu) = 2^{2g} \sum_{n=0}^{\infty} \binom{n + 2g - 1}{n} \frac{q^{2(n+g)}}{1 - q^{2(n+g)}} = \frac{2^{2g}}{(2g-1)!} \sum_{n=0}^{\infty} \left( (n^2 - (g-1)^2) \ldots n \right) \frac{q^{2n}}{1 - q^{2n}}, \tag{18}
\]

where \( q = e^{-\mu} \), and, not surprisingly, we see here the scalar degeneracies making an effective appearance. The full sphere degeneracies emerge on combining the two terms in (12) using (18).

Since they do not occur in our earlier work on the full sphere we now discuss the \( \text{sech} \) sums, (8) which follow from a translation of the argument.

To begin with, quite easily from (14) and (15),

\[
(2g, \frac{1}{2} - ix) = 2^{2g} \sum_{n=-\infty}^{\infty} \frac{1}{(i2x + 2n + 1)^{2g}} = \pi^{2g} \sum_{k=1}^{g} (-1)^{k+g} A_{2g,2k} \text{sech}^{2k} \pi x, \tag{19}
\]
and, corresponding to (17)

\[ 2^{2g} \sum_{m,n=-\infty}^\infty \frac{1}{(i2m\mu/\pi + 2n + 1)^{2g}} - 2\zeta_R(2g, 1/2) = 2\pi^{2g} \sum_{k=1}^{g} (-1)^{k+g} A_{2g,2k} R_k(\mu), \]

which contains an ‘even-odd’ Eisenstein series. Reference to the series in Glaisher [11] shows that the Jacobi function \( zn \) is involved this time. This has the expansion

\[ \hat{K} zn u = -\tan x - 4 \sum_{n=1}^{\infty} (-1)^{n-1} \frac{q^{2n}}{1 - q^{2n}} \sin 2nx, \]

where \( x = \pi u/2K \) and \( \hat{K} = 2K/\pi \), which can also be obtained from that for \( \hat{K} zs u \) by adding \( \pi/2 \) to \( x \). Here, as usual, \( q = e^{-\mu} = e^{-\pi K'/K} \), and \( K, K' \) are given by

\[ K(k) = \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}, \quad K' = K(k'), \]

where \( k^2 + k'^2 = 1 \).

Equivalently the \( q \)-series can be constructed, as for cosech,

\[ R_g(\mu) = 2^{2g} \sum_{n=0}^{\infty} (-1)^n \binom{n + 2g - 1}{n} \frac{q^{2(n+g)}}{1 - q^{2(n+g)}} = (-1)^g \frac{2^g}{(2g-1)!} \sum_{n=0}^{\infty} ((n^2 - (g - 1)^2) \ldots n)(-1)^n \frac{q^{2n}}{1 - q^{2n}}, \]

cf Zucker, [7], who uses the expansion of \( nc^2 \) to extract the Lambert series.

As has been mentioned, our previous analysis [1] expanded the degeneracies in powers of \( n^2 \), \( n \) being the mode label, and treated each power separately making use of classic elliptic facts. This is the same as Zucker’s method and he, more or less, reproduces the degeneracy expansion obtained, e.g. by Cahn and Wolf, [12], as well as the elliptic expansions to be found in Glaisher, [11]. Zucker’s general method of finding closed forms for the sums \( Q_g(\mu) \) when \( \mu \) corresponds to a singular modulus is thus entirely equivalent to our own programme on the sphere, [1]. By the same token, the energy on the periodic half–sphere, (6), can also be expressed in finite terms using the \( S_{g,f} \), reduced to the \( Q_g \) and \( R_1 \). It does not seem possible to repeat this statement for the remaining tilings, (2), at least not obviously.

The rather negative moral of this calculation is that, in order to evaluate the summations that occur for the sphere expressions, (2), in elliptic terms it seems
necessary to re-introduce the degeneracies in one way or another and to treat the
individual terms, as in our earlier work. The conclusion is that one might as well
employ the mode-degeneracy expressions in the first place, if these are known,
particularly since, in addition, some manipulation was required to obtain (12) and
(13).

Therefore, although the expressions (2) are particularly compact for all the
tilings, and provide adequately convergent numerical forms, the earlier piecemeal
approach has elliptic advantages and so we turn to the degeneracy–mode approach
for the tilings, knowing that an elliptic formulation is possible for the quarter–sphere
case at least. We would anticipate that this should be true for all the odd sphere
‘periodic’ tilings because the heat–kernel expansion terminates, just as it does on
the full odd sphere. If this is so, then we could turn the calculation around and
produce exact forms for the ‘new’ hyperbolic summations in (2). This will not be
attempted here.

4. Mode forms applied to the quarter–sphere.

In this section we detail the eigenvalue form of the thermodynamical quantities
and, as a typical example, consider the specific case of the quarter-sphere, which
has been treated previously [13] in another connection.

The Laplacian eigenvalues can be written in the form \( \lambda = \omega^2/a^2 \) with, [4],

\[ \omega_{m,n} = (\alpha + 2n + m), \quad n, m = 0, 1, \ldots \]

and the degeneracies are, \( (d > 1) \),

\[ \binom{m + d - 2}{d - 2}. \]

The parameter \( \alpha = (d + 3)/2 \) for Dirichlet (D) conditions, and \( a = (d - 1)/2 \) for
Neumann (N). The D and N expressions are added to get the ‘periodic’ (P) form.

For odd spheres we set \( d = 2r + 1 \). For \( d \geq 5 \) we shift the \( m \) label by \( r + 1 \)
in the D part and by \( r - 1 \) in N, so that the eigenvalues and periodic degeneracies,
\( g_{m,n} \), read, effectively,

\[ \omega_{m,n} = (2n + m + 1), \quad n = 0, 1, \ldots, \quad m = 1, 2, \ldots \]

\[ g_{m,n} = \binom{m + r - 2}{2r - 1} + \binom{m + r}{2r - 1} \]

\[ \equiv g_{m}. \]
We are allowed to extend and adjust the range of \( m \) by the vanishing of the binomials. For \( d = 3 \) attention is needed because the label \( m \) runs over 0,1,.... We continue with \( d \geq 5 \) and for \( d = 3 \) we only have to add the contribution from the \( m = 0 \) mode at the end.

The degeneracies are made more explicit by remarking that \( g_m \) is polynomial in odd powers of \( m \). Precisely,

\[
g_m = m \sum_{k=0}^{r-1} c_k m^{2k}.
\]

These are of course related to Stirling polynomials, but we will leave them as they are for the time being.

To proceed with the summation, residue classes mod 2 are introduced so that \( m = 2l + p \) with \( 0 \leq l \leq \infty \) and \( p = 0,1, \) i.e. \( m \) is even (including 0) or odd. The two \( p \) values are treated separately at first. Then

\[
\omega_{m,n} = 2(n + l) + p + 1 = 2N + p + 1,
\]

where a further advantageous relabelling has been made to the quadrant coordinates,

\[
N = l + n, \quad \text{and} \quad \nu = l - n,
\]

so that \( m = N + \nu + p \) with \( N = 0,1,\ldots \) and \( -N \leq \nu \leq N \). Note that for \( N \) odd \( \nu \) only takes odd values and for \( N \) even \( \nu \) only takes even values.

We now write down the particular spectral quantity in which we are interested; the internal energy,

\[
aE = E = E_0 + \sum_{m,n=0}^{\infty} g_m \omega_{m,n} \frac{q^{2\omega_{m,n}}}{1 - q^{2\omega_{m,n}}},
\]

where the \( m \) label is extended to 0 which is allowed because \( g_0 = 0 \) and we introduced \( q = e^{-\pi/\xi}, \xi = (2\pi a)/\beta \). The sums are reorganised to allow for the \( \nu \) independence of the eigenvalues. Firstly

\[
\sum_{m,n=0}^{\infty} = \sum_{p=0,1} \sum_{l,n=0}^{\infty},
\]

and then

\[
\sum_{l,n=0}^{\infty} = \sum_{N=0}^{\infty} \sum_{\nu=-N}^{N} + \sum_{N=1}^{\infty} \sum_{\nu=-N}^{N}.
\]
This gives

\[
\overline{E} = E_0 + \sum_{p=0,1} \left( \sum_{N=0}^{\infty} \sum_{\nu=-N}^{N} \right) g_{N+\nu+p}(2N+p+1) \frac{q^{2(2N+p+1)}}{1 - q^{2(2N+p+1)}}. \tag{23}
\]

The summation ranges, \(N\) even and \(N\) odd, are rewritten by \(N \rightarrow 2N\) and \(N \rightarrow 2N + 1\) respectively. The relevant ‘degeneracies’ are then

\[
g_{\text{even}} = \sum_{\nu=-N}^{N} g_{2N+2\nu+p} = \sum_{k=0}^{r-1} c_k \sum_{\nu=-N}^{N} (2N+p+2\nu)^{2k+1}
\]

\[
= \sum_{k=0}^{r-1} c_k 2^{2k+1} \sum_{\nu' = p/2}^{2N+p/2} \nu'^{2k+1}
\]

\[
= \sum_{k=0}^{r-1} c_k 2^{2k} k + 1 \left( B_{2k+2}(2N + 1 + p/2) - B_{2k+2}(p/2) \right)
\]

and

\[
g_{\text{odd}} = \sum_{\nu=-N-1}^{N} g_{2N+2\nu+p+2} = \sum_{k=0}^{r-1} c_k \sum_{\nu=-N-1}^{N} (2N+p+2+2\nu)^{2k+1}
\]

\[
= \sum_{k=0}^{r-1} c_k 2^{2k+1} \sum_{\nu' = p/2}^{2N+p+2} \nu'^{2k+1}
\]

\[
= \sum_{k=0}^{r-1} c_k 2^{2k} k + 1 \left( B_{2k+2}(2N + 2 + p/2) - B_{2k+2}(p/2) \right).
\]

Adding up all contributions, we have

\[
\overline{E} = E_0 + \sum_{p=0,1} \sum_{N=0}^{\infty} \sum_{k=0}^{r-1} \frac{c_k 2^{2k}}{k + 1} \times \left( \left( B_{2k+2}(2N + 1 + p/2) - B_{2k+2}(p/2) \right)(4N + p + 1) \frac{q^{2(4N+p+1)}}{1 - q^{2(4N+p+1)}} \right.
\]

\[+ \left. \left( B_{2k+2}(2N + 2 + p/2) - B_{2k+2}(p/2) \right)(4N + p + 3) \frac{q^{2(4N+p+3)}}{1 - q^{2(4N+p+3)}} \right).
\]

Combining \(p = 0\) and \(p = 1\),

\[
\overline{E} = E_0 + \sum_{k=0}^{r-1} \frac{c_k 2^{2k}}{k + 1} \times \left( \sum_{n=0}^{\infty} B_{2k+2}((n + 1)/2) \frac{n q^{2n}}{1 - q^{2n}} - B_{2k+2}(0) \frac{(2n - 1) q^{2(2n-1)}}{1 - q^{2(2n-1)}} \right.
\]

\[+ \left. B_{2k+2}(1/2) \frac{2n q^{4n}}{1 - q^{4n}} \right).
\]
In order to connect this to elliptic functions we want an explicit expansion of the Bernoulli polynomials in powers of the summation index $n$. To this end we apply
\[ B_j(x + 1/2) = 2^{1-j}B_j(2x) - B_j(x) = \sum_{k=0}^j \binom{j}{k} (2^{1-k} - 1)B_k x^{j-k}. \]
Noting that $B_{2k+1} = 0$ for $k = 1, 2, ..., $ and, combining the various contributions, the final answer for the internal energy on the D+N quarter sphere (equivalently the ‘periodic hemisphere’) can be cast into the general form,
\[
\overline{E}_d = \overline{E}_{d,0} + \sum_{l=0}^{r-1} \sum_{n=1}^{\infty} n^{2l+3} \frac{q^{2n}}{1 - q^{2n}} + J_r \sum_{n=0}^{\infty} (2n + 1) \frac{q^{2(2n+1)}}{1 - q^{2(2n+1)}},
\]
where the constants $I_{l,r}$ and $J_r$ are
\[
I_{l,r} = \sum_{j=0}^{r-1-l} \frac{c_{l+j}}{2(l+j+1)} \left(2 + 2l + 2j\right) \frac{2}{2j} (1 - 2^{2j-1})B_{2j},
\]
\[
J_r = \sum_{k=0}^{r-1} \frac{c_k}{2(k+1)} (1 - 2^{2k+2})B_{2+2k}.
\]
The zero temperature value, $\overline{E}_{d,0}$, is a generalised Bernoulli coefficient, [4].

This form of the energy, for all dimensions $d$, is then related to the Fourier expansions of $z s u$ and $n s u$.

In $d = 5$ dimensions the answer reads explicitly,
\[
\overline{E}_5 = \overline{E}_{5,0} + \frac{1}{24} \sum_{n=1}^{\infty} n^5 \frac{q^{2n}}{1 - q^{2n}} + \frac{1}{12} \sum_{n=1}^{\infty} n^3 \frac{q^{2n}}{1 - q^{2n}}
\]
\[
- \frac{1}{8} \sum_{n=0}^{\infty} (2n + 1) \frac{q^{2(2n+1)}}{1 - q^{2(2n+1)}},
\]
which, equivalently from (13), is to be compared with,
\[
\overline{E}_5 = \overline{E}_{5,0} + \frac{1}{26}(7Q_2 + 5Q_3 + Q_1 - R_1).
\]
Numerical comparison has also been performed as a check of the calculation.
As mentioned, in $d = 3$ dimensions we have to add the $m = 0$ contribution to the result [24] to get the correct answer. Then one finds,

$$E_3 = E_{3,0} + \frac{1}{2} \sum_{n=1}^{\infty} n^3 \frac{q^{2n}}{1-q^{2n}} + \frac{1}{2} \sum_{n=0}^{\infty} (2n+1) \frac{q^{2(2n+1)}}{1-q^{2(2n+1)}}, \quad (27)$$

with $E_{3,0} = 11/480$.

Alternatively, according to (13),

$$E_3 = E_{3,0} + \frac{1}{16} (3Q_1 + 3Q_2 + R_1) \quad (28)$$

and, again, agreement can be shown since, from (18),

$$\frac{1}{16} (3(Q_1 + Q_2) + R_1) = \frac{1}{4} \sum_{n=1}^{\infty} (2n^3 + (1-(-1)^n)n) \frac{q^{2n}}{1-q^{2n}}.$$

For comparison, the energies on the full spheres are

$$E_3\big|_{full} = \epsilon_2$$

$$E_5\big|_{full} = \frac{1}{12} (\epsilon_3 - \epsilon_2)$$

in terms of the partial energies, $\epsilon_t$, defined in [1] as,

$$\epsilon_t(\xi) = -\frac{B_{2t}}{4t} + \sum_{n=1}^{\infty} \frac{n^{2t-1}q^{2n}}{1-q^{2n}}. \quad (29)$$

As we have mentioned, the computation via the hyperbolic summations is an unnecessary detour only in that, to evaluate elliptically, it seems that the degeneracies have to be reintroduced.

5. Temperature inversion.

Apart from the final term, the structure of the energy, (24), is similar to that on the full sphere and so, for this part, the conclusions will be the same as in our previous paper, [1]. The differences are due to the effect of the polar singularities which also account for the final term in $E$. Because of this term, the recursion formula for the $\epsilon_t$ employed in (1) now shows that any energy can be expressed, as a polynomial in $\epsilon_2$ and $\epsilon_3$, plus a multiple of this last term.
The final term in (24) also affects the behaviour under temperature inversion, \( \xi \to 1/\xi \). To investigate this, it is rewritten
\[
\sum_{n=0}^{\infty} (2n + 1) \frac{q^{2(2n+1)}}{1 - q^{2(2n+1)}} = \sum_{n=0}^{\infty} n \frac{q^{2n}}{1 - q^{2n}} - \sum_{n=0}^{\infty} 2n \frac{q^{4n}}{1 - q^{4n}},
\]
and, while the first term on the right has a well defined behaviour under \( \xi \to 1/\xi \), described in [1], the second apparently does not, as it corresponds to a different temperature. The conclusion is that the inversion properties on the sphere do not carry over to its orbifold factors.

6. The specific heat and modular covariant derivatives.

In [1] it was shown how the classic elliptic recursions allow the internal energies on all odd spheres to be determined in terms of just two parameters. The same statement holds for the specific heat, as we demonstrate. From now on we revert to the full sphere.

Before Weierstrass, Eisenstein introduced the double series
\[
E_n(z) = \sum_{m_1, m_2 = -\infty}^{\infty} \frac{1}{(z + m_1 \omega_1 + m_2 \omega_2)^n},
\]
denoting them by \((n, z)\). If \(n > 2\) they are the higher derivatives of the Weierstrass \(\wp\)-function. For the time being we adhere to Eisenstein. When \(n\) equals 1 or 2, the sums are not absolutely convergent and Eisenstein defines them by a limiting procedure which depends on the periods, \(\omega_1, \omega_2\). We will not expound this here (see Weil [8]). It is similar to the later procedures of Glaisher and Hurwitz.

The basic Eisenstein function, \(E_n(z)\), has the power series expansion
\[
E_n(z) = \frac{1}{z^n} + (-1)^n \sum_{t=1}^{\infty} \left( \frac{2t-1}{n-1} \right) G_t(\omega_1, \omega_2) z^{2t-n} \tag{30}
\]
in terms of the more common Eisenstein series,
\[
G_t(\omega_1, \omega_2) = \sum_{m_1, m_2}^{\infty} \frac{1}{(m_1 \omega_1 + m_2 \omega_2)^{2t}}, \tag{31}
\]
by
\[
\overline{G}_t(\omega_1, \omega_2) = G_t(\omega_1, \omega_2), \quad t \geq 2
\]
and
\[
\mathcal{G}_1(\omega_1, \omega_2) = G_1(\omega_1, \omega_2) + \frac{\pi i}{\omega_1 \omega_2}.
\] (32)

\(G_1\) is the quantity that behaves homogeneously under modular transformations of the periods, e.g. [14].

The \(\mathcal{G}_t\) are related to the partial energies by,
\[
\epsilon_t(\xi) = (-1)^t \frac{(2t-1)!}{2(2\pi)^{2t}} G_t(1, i/\xi),
\] (33)

for all \(t\).

By paralleling trigonometric theory, Eisenstein derived a number of basic identities for the \(E_n\) including, effectively, the first order differential equation for \(\wp\), some years before Weierstrass. This equation is the origin of the recursion used in [1] and which is repeated here,
\[
\frac{2\pi i}{\omega_1} \frac{\partial \mathcal{E}_1}{\partial \omega_2} = \mathcal{E}_3 - \mathcal{E}_1 \mathcal{E}_2.
\] (35)

From this, one can obtain an equation for a ‘partial specific heat’, \(\sigma_t\), defined by
\[
\sigma_t = \left(\frac{2\pi}{\xi}\right)^2 D \epsilon_t
\]
in terms of the more convenient quantities,
\[
D \epsilon_t = q^2 \frac{d\epsilon_t}{dq^2} = -\frac{d \epsilon_t(\xi)}{2\pi d(1/\xi)} = \frac{1}{2\pi i} \frac{d \epsilon_t(\tau)}{d\tau} = -a \frac{d \epsilon_t(\beta)}{d\beta}, \quad t \geq 1.
\] (36)

Our notation is that, for example, \(\epsilon = \epsilon(\xi) = \epsilon(\tau) = \epsilon(\beta)\), depending on whether we wish to use \(\xi\), \(\tau\) or \(\beta\) as the variable.

By choosing \(\omega_1 = 1\), \(\omega_2 = i/\xi = \tau = i\beta/2\pi a\) and substituting the power series into (35) one easily finds
\[
D \epsilon_1 = -2 \epsilon_1^2 + \frac{5}{6} \epsilon_2
\]
\[
D \epsilon_t = -4t \epsilon_1 \epsilon_t + \frac{2t + 3}{2(2t + 1)} \epsilon_{t+1} - \sum_{k=2}^{t-1} \left(\frac{2t}{2k - 1}\right) \epsilon_k \epsilon_{t-k+1}, \quad t > 1,
\] (37)
displaying a dependence on $\epsilon_1$ which is vital for maintaining the proper inversion behaviour under $\xi \rightarrow 1/\xi$, which is,

$$\epsilon_t(1/\xi) = (-1)^t \frac{1}{\xi^{2t}} \epsilon_t(\xi), \quad t \geq 2. \quad (38)$$

A formal way of describing the situation is to define a ‘modular covariant derivative’, $\mathcal{D}$, by

$$\mathcal{D} \epsilon_1 \equiv (D + 2\epsilon_1)\epsilon_1$$

$$\mathcal{D} \epsilon_t \equiv (D + 4t\epsilon_1)\epsilon_t, \quad t > 1, \quad (39)$$

so that $\mathcal{D} \epsilon_t$ transforms under inversion, $\xi \rightarrow 1/\xi$, like $\epsilon_{t+1}$. Note that, as usual, the covariant derivative depends on the behaviour of its operand and we shall extend $\mathcal{D}$ to any modular form of weight $2t$. (The terminology is that the homogeneous form, $G_t(\omega_1, \omega_2)$, is said to have dimension $-2t$ while the inhomogeneous, holomorphic form, $G_t(1, \tau)$, has weight $2t$.) Although not a modular form, $\epsilon_1$ can be said to have weight $2$. Note that $\mathcal{D} \epsilon_1$ differs from the general structure of $\mathcal{D} \epsilon_t$. This definition is not that used by Lang, [15], who employs a covariant derivative, $\partial$, in a discussion of mod $p$ modularity.

The general modular form of weight $k$ is an isobaric polynomial in the two invariants, $g_2$ and $g_3$, of elliptic function theory. This follows essentially from the recursion (34). (Equivalent to $(g_2, g_3)$ are $(\epsilon_2, \epsilon_3)$, $(G_2, G_3)$ and Ramanujan’s $(M, N)$, defined later.) Thus the general form, $\mathcal{F}$, is written

$$\mathcal{F} = \sum_{a, b: 4a + 6b = k} \mathcal{F}_{a, b} g_2^a g_3^b. \quad (40)$$

In this setting, the combination in $\mathcal{D}$ appears in Ogg, [16], pp.17,18. Ogg uses the modular transformation behaviour of $\epsilon_1$ to show that $\mathcal{D}$ increases the weight by 2 but we could in fact determine this behaviour immediately from (37). Hurwitz, [17] §4 employs this device.

Higher derivatives can be constructed simply since the usual rules apply. For example (37) and its derivative read, for all $t$,

$$\mathcal{D} \epsilon_t = \frac{2t + 3}{2(2t + 1)} \epsilon_{t+1} - \sum_{k=2}^{t-1} \binom{2t}{2k - 1} \epsilon_k \epsilon_{t-k+1} \quad (40)$$

$$\mathcal{D}^2 \epsilon_t = \frac{2t + 3}{2(2t + 1)} \mathcal{D} \epsilon_{t+1} - 2 \sum_{k=2}^{t-1} \binom{2t}{2k - 1} \mathcal{D} \epsilon_k \epsilon_{t-k+1}, \quad (40)$$

and so on. An equivalent set of equations is given by Ramanujan [18] pp.165,166, but not using a covariant derivative.
Before drawing any consequences from these equations, some relevant elliptic facts are interpolated.

The dependence on $\epsilon_1$ means that the complete elliptic function of the second kind, $E$, now makes an appearance via,

$$
\epsilon_1 = -\frac{1}{24} \left( \frac{2K}{\pi} \right)^2 \left( \frac{3E}{K} + k^2 - 2 \right).
$$

(41)

Zucker, [7], from results of Ramanujan, says that $E$, as well as $K$, can be expressed in terms of algebraic numbers and Gamma functions at singular moduli.

(41) can be transformed by use of the standard differential relation between $K$ and $E$ (e.g. Fricke [19] I,p.46)

$$
2k^2k'^2 \frac{dK}{dk^2} = E - k'^2 K
$$

to which we can also add,

$$
D = \frac{1}{2} \left( \frac{Kk'}{\pi} \right)^2 \frac{d}{dk^2}, \quad \frac{K}{\pi} = \frac{2K}{\pi},
$$

(42)

which ellipticises $D$. In fact, direct manipulation with $q$-series gives,

$$
\epsilon_1 = -\frac{1}{6} D \log \left( k k' R^3 \right),
$$

(43)

which connects with the alternative expression for $\epsilon_1$,

$$
\epsilon_1 = -D \log \eta,
$$

(44)

in terms of Dedekind’s $\eta$-function,

$$
\eta(\tau) = q^{1/12} \prod_j (1 - q^{2j}), \quad q = e^{i\pi\tau} = e^{-\pi/\xi}.
$$

(45)

This includes both the Casimir (zero temperature) contribution and the statistical mode sum.

We have thus more or less arrived at Jacobi’s result

$$
\eta^{24} = \frac{1}{(2\pi)^{12}} \left( g_2^3 - 27g_3^2 \right) = \frac{1}{28} k^4 k'^4 R^{12},
$$

(46)

another proof of which occurs shortly.
Returning to (40), it is apparent from the standard recursions, (34), (40), that the ordinary derivative, $D \epsilon_t$, can be expressed as a polynomial in $\epsilon_1$, $\epsilon_2$ and $\epsilon_3$ (linear in $\epsilon_1$, if $t > 1$) and the conclusion is that, at temperatures corresponding to singular moduli, not only the internal energy but also the specific heat can be expressed in finite terms in algebraic numbers and gamma functions. Furthermore, the same statement holds for all higher derivatives at singular moduli, as follows by iteration of (40) and elimination. Incidentally, it is, of course, possible to derive (37) in Weierstrassian vein, and some might prefer this. Such a version is detailed by van der Pol [20].

Kaneko and Zagier, [21], refer to the algebra generated by $\epsilon_1$, $\epsilon_2$ and $\epsilon_3$ as the algebra of quasi–modular forms.

For the circle, the three-sphere and the five-sphere we find for the total specific heats for conformal scalars,

$$\sigma^{(1)}(\xi) = \left(\frac{2\pi}{\xi}\right)^2 \left[-4\epsilon_1^2 + \frac{5}{3} \epsilon_2\right]$$

$$\sigma^{(3)}(\xi) = \left(\frac{2\pi}{\xi}\right)^2 \left[-8\epsilon_1 \epsilon_2 + \frac{7}{10} \epsilon_3\right]$$

$$\sigma^{(5)}(\xi) = \left(\frac{2\pi}{\xi}\right)^2 \left[-\epsilon_1 \epsilon_3 + \frac{100}{21} \epsilon_2^2 + \frac{2}{3} \epsilon_1 \epsilon_2 - \frac{7}{120} \epsilon_3\right].$$ (47)

As a particular numerical case, at the lemniscate point, $\xi = 1$, (see [1]), one finds

$$\sigma^{(1)}(1) = \frac{1}{1152\pi^4} \left(\Gamma^8(1/4) + 24\pi^2 \Gamma^4(1/4) - 288\pi^4\right) \approx 0.380810377.$$

Other examples we leave to the reader.

Although equation (37) provides a systematic method, another means of finding the derivative in (36) is to apply it to the explicit forms of the partial energies, $\epsilon_t$, in terms of $k$ and $K$. This is done by Glaisher [11], §§78,79, who gives the necessary rules. One can easily see that the general conclusion will be the same. The structure of $\epsilon_t$ is a polynomial in $k^2$ multiplied by a power of $K$. The ellipticised form of $D$, (42), shows that $\epsilon_1$ will appear through the action of $d/dk^2$ on the $K$ factor.

Yet another method is to write the Eisenstein function, $\mathcal{E}_n$, in terms of the $\theta_1$–function, and then use the differential (heat) equation that this satisfies, or use standard theta–function identities. In modern parlance this is an approach via Jacobi forms, which are functions of two variables. We remark that instead of the $\theta$–functions one can use the Jacobi elliptic functions, $sn u$, etc. The physical
significance of the $u$ variable could be the radial separation of source and field point on the sphere (e.g. Allen et al [22]).

We now make some further remarks on the recursions and their history. Because of the recursions, it is sufficient to give the derivatives of just $\epsilon_1$, $\epsilon_2$ and $\epsilon_3$, and, in order to express these with reasonable factors, it is convenient, and conventional, to renormalise them. We will use Ramanujan’s $L$, $M$ and $N$ notation where $L = -24\epsilon_1$, $M = 240\epsilon_2$ and $N = -504\epsilon_3$. Then from Eisenstein’s equation, (37), one rapidly finds the relations often employed by Ramanujan,

$$DL = \frac{1}{12}(L^2 - M), \quad \text{or} \quad D L = -\frac{1}{12}M,$$
$$DM = \frac{1}{3}(LM - N), \quad \text{or} \quad D M = -\frac{1}{3}N,$$
$$DN = \frac{1}{2}(LN - M^2), \quad \text{or} \quad D N = -\frac{1}{2}M^2. \quad \text{(48)}$$

An important result now follows easily on noting that from (48)

$$D\log(M^3 - N^2) = L, \quad \text{(49)}$$

and so, from (44),

$$D\log(M^3 - N^2) = 24D\log\eta,$$

whence, again, Jacobi’s relation between the $\eta$–function and the discriminant, $\Delta = g_3^3 - 27g_2^2$,

$$\eta^{24} = \frac{1}{1728} (M^3 - N^2) = \frac{1}{(2\pi)^{12}} \Delta, \quad \text{(50)}$$

the 1728 being the coefficient of $q^2$ in $M^3 - N^2$.

The Weierstrassian way of deriving this relation is somewhat more involved, e.g. [23], [24]. Even Weil’s reconstruction of Eisenstein’s approach mimics this $\wp$ method, [8], p.33. The modular form proof, [24–27], is quite different and depends on the uniqueness of cusp forms of weight 12.

Equation (49) is equivalent to the statement that $\Delta$ is covariant constant, (put $2t = 12$ in (39)),

$$D\Delta = (D - L)\Delta = 0, \quad \text{(51)}$$

which highlights the special connecting role played by $L$. This condition also follows, as noted by Tuite, [28], from the fact that there are no (elliptic) cusp forms of weight 14 and so works the argument in reverse.

Equation (51) is a disguised version of the statistical mechanical relation, $E = -\partial\log Z/\partial\beta$, on the circle.
Incidentally, the use of $D$ and $\mathcal{D}$ allows one to give a more general and systematic treatment of Ramanujan’s Notebook Entry 45 (p.352), see [29] part V, p.484. Also, as a side comment, we note that equations (48) and (49) allow derivations of the nonlinear differential equations satisfied by $M$, $N$ and $\Delta$, e.g. [30,20], relatively easily.

The relations (34) and (37) were first obtained by Eisenstein. Later derivations have been given by Ramanujan, [18], as mentioned, van der Pol [20], Rankin, [31], and Skoruppa [32]. Rankin and Skoruppa use a number theoretic approach, aspects of which are reminiscent of Glaisher’s work.

7. **Elliptic formulation of the free energy on the circle.**

Equation (44) allows one to obtain an expression for a ‘partial free energy’, $f_1$, defined by,

$$\epsilon_1 = \frac{\partial}{\partial \xi} \left( \frac{1}{\xi} f_1 \right)$$

i.e.

$$f_1 = \frac{\xi}{2\pi} \log \eta,$$ \hspace{1cm} (52)

On the circle the total (scaled) free energy is the well known equation,

$$F_1 = 2f_1 - \frac{\xi}{2\pi} \log \xi = \frac{\xi}{2\pi} \log (\eta^2/\xi).$$ \hspace{1cm} (53)

There is a degeneracy factor of 2, and the log $\xi$ term is the zero mode contribution.

Relation (50) now enables one to give an elliptic interpretation to the partial free energy, (52), by

$$f_1 = \frac{\xi}{48\pi} \log (\Delta/c), \hspace{1cm} c = (2\pi)^{12}.$$ \hspace{1cm} (54)

Thus the free energy on the circle can be thought of as a function, but not an algebraic one, of the elliptic modulus, $k$, remembering that $\xi = K/K'$. See the explicit form in (43), (46).

The choice of constants, in (52), is governed by the desire to satisfy Nernst’s theorem i.e. to make the entropy vanish at absolute zero, $\xi = 0$. A partial entropy on the circle can be defined by $s_1 = (f_1 - e_1)/\xi = -df_1/d\xi$ which is again elliptic. It should be said that, when we define these ‘partial quantities’, we are leaving the zero mode contribution in (53) aside as it causes problems with Nernst’s theorem which have to be addressed separately.
8. Higher sphere thermodynamics.

The question is now whether the higher sphere statistical mechanics can be ‘ellipticised’ in the same way as on the circle. The internal energy has already been treated in [1] but the problem is the free energy, and thence the entropy, which demands an effective integration.

It must be remarked however that Zucker, [33], following Selberg and Chowla, uses (46) to derive closed form values for $K$ at singular moduli from computed values of $\eta$ via the Epstein $\zeta$–function. If this were the only method, then the elliptic character of the circle free energy would be secondary. However there are other ways of finding the singular $K$ which do not involve Kronecker’s limit formula. We can still therefore maintain the attitude of our earlier work, [1], and regard the question of the possible ellipticisation of the free energy on all odd spheres as motivation for further investigation. This will be undertaken in a further more technical communication.

8. Summary.

It has been shown that the temperature inversion properties of the internal energy on odd spheres does not carry through to their orbifold factors. The specific case of the quarter sphere is considered in detail. During the analysis, hyperbolic summations arise that have not so far been encountered.

It is also proved that the specific heat on any odd (full) sphere can be expressed in terms of just the three partial energies, $\epsilon_1$, $\epsilon_2$ and $\epsilon_3$, the novelty being the appearance of $\epsilon_1$, related to the discriminant The derivation makes use of the notion of modular covariant derivative with $\epsilon_1$ as a ‘connection’. We note, incidentally, that the discriminant is covariant constant.

Finally we have demonstrated that the free energy on the circle can be ellipticised but have left the corresponding statement for all (odd) spheres open and subject to later analysis.

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