Biliaison classes of curves in \( \mathbb{P}^3 \)

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Abstract. We characterize the curves in \( \mathbb{P}^3 \) which are minimal in their biliaison class. Such curves are exactly the curves which do not admit an elementary descending biliaison. As a consequence we have that every curve in \( \mathbb{P}^3 \) can be obtained from a minimal one by means of a finite sequence of ascending elementary billiaisons.

0. Introduction. Let \( C \) be a curve (i.e. a locally C.M. subscheme equidimensional of dimension 1) in \( \mathbb{P} = \mathbb{P}^3_k \) over an algebraically closed field \( k \). Let \( J_C \) be its ideal sheaf in \( \mathcal{O}_\mathbb{P} \) and \( I_C = H^0_*(J_C) \) be the homogeneous ideal of \( C \) in the polynomial ring \( R = k[x_0, \ldots, x_3] = H^0_*(\mathcal{O}_\mathbb{P}) \). Denote by

\[
\begin{align*}
    s(C) &= \inf\{n \in \mathbb{Z} \mid h^0(J_C(n)) \neq 0\}; \\
    e(C) &= \sup\{n \in \mathbb{Z} \mid h^1(\mathcal{O}_C(n)) \neq 0\}.
\end{align*}
\]

In [R] P. Rao introduced the notion of billiaison (double linkage) class of curves and proved that two curves \( C \) and \( C' \) are in the same class if and only if \( H^1_*(J_C) \cong H^1_*(J_{C'})(h) \) for some \( h \in \mathbb{Z} \). In [LR] R. Lazarsfeld and P. Rao proved that curves with \( s(C) \geq e(C) + 4 \) are minimal in their billiaison class. In [MDP] M. Martin-Deschamps and D. Perrin gave a construction of the minimal curve in each billiaison class starting from the minimal free graded resolution of the Rao module \( M(C) = H^1_*(J_C) \).

In this paper, by completing the result of Lazarsfeld and Rao, we give a characterization of minimal non ACM curves in the case \( s(C) \leq e(C) + 3 \); these curves are those satisfying the following condition: (*) for every \( s(C) \leq s \leq e(C) + 3 \) there is a form \( H_s \) of degree \( <s \) dividing every element in \( H^0(J_C(s)) \) and such that \( H_s \cdot H^0(\omega_C(3 - s)) = 0 \).

We see also that these curves are exactly the curves that do not admit any elementary descending billiaison; from this one can deduce the following theorem that improves a result obtained by M. Martin-Deschamps and Perrin [MDP] and by E. Ballico, G. Bolondi and J. Migliore [BBM]:

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Every curve in \( \mathbf{P}^3 \) can be obtained from a minimal one by means of a finite number of ascending elementary biliaisons.

The paper is organized as follows: in Sections 1 and 2 we prove the main result (Theorem 1) and precisely in Section 1 we prove that a non ACM curve with \( s(C) \leq e(C) + 3 \) that do not admit any elementary descending biliaison satisfy condition (*), while in Section 2 we prove that such curve is minimal.

In Section 3 we deduce the above result regarding the structure of a given biliaison class together with some other interesting consequences.

We refer to [MDP] for all the notations and in particular for the basic results concerning the biliaison.

By abuse of notation we will denote with the same symbol an homogeneous polynomial in \( R \) and the surface of \( \mathbf{P}^3 \) that it defines.

1. In this section first we state the main result of the paper. The notations are as in the introduction. In order to include all possible curves in \( \mathbf{P}^3 \) we consider a line as a minimal curve in the biliaison class of the ACM curves in \( \mathbf{P}^3 \).

**Theorem 1.** For a curve \( C \subset \mathbf{P}^3 \) the following are equivalent:

1) \( C \) is minimal in its biliaison class
2) \( C \) does not admit any elementary descending biliaison
3) \( C \) is of one (and only one) of the following types:

   a) a line
   b) \( s(C) \geq e(C) + 4 \)
   c) \( s(C) \leq e(C) + 3 \) and (*) for every \( s(C) \leq s \leq e(C) + 3 \) there is a form \( H_s \) of positive degree \( < s \) dividing every element in \( H^0(J_C(s)) \) and such that \( H_s \cdot H^0(\omega_C(3-s)) = 0 \).

**Proof.** In the rest of this Section we prove that 2) implies 3). First we prove some preliminary result; recall that, for every \( Q \in H^0(J_C(s)) \) and for every \( m \in \mathbb{Z} \) we have an exact sequence:

\[
0 \to H^0(\mathcal{O}_Q(m)) \to \text{Hom}(J_{C/Q}, \mathcal{O}_Q(m)) \to H^0(\omega_C(4-s+m)) \to 0
\]

in particular for \( m = -1 \) we have an isomorphism : \( \text{Hom}(J_{C/Q}, \mathcal{O}_Q(-1)) \cong H^0(\omega_C(3-s)) \).

**Proposition 1.** Let \( s(C) \leq s \leq e(C) + 3 \), \( Q \in H^0(J_C(s)) \), \( \xi \in H^0(\omega_C(3-s)) \), \( \xi \neq 0 \) and let \( \eta : J_{C/Q} \to \mathcal{O}_Q(-1) \) the morphism of \( \mathcal{O}_Q \)-modules corresponding to \( \xi \). Then \( \eta \) is not injective if and only if there is a form \( H \) of degree \( < s \) dividing \( Q \) and such that \( H \cdot \xi = 0 \).

**Proof of Proposition 1.** The proof is similar to that given in [S] Proposition 2.2.3; see also [MDP] Proposition III.2.6.

Assume that \( \eta \) is not injective; by [S] Proposition 2.2.3 its kernel and its image are respectively of the form \( J_{D/Q} \) and \( J_{E/Q}(-1) \), where \( D,E \) are subschemes of \( Q \) containing respectively surfaces \( H, K \) of degree \( < s \) such that \( H \cdot K = Q \). Since \( J_{E/Q}(-1) \subseteq J_{K/Q}(-1) \) and \( H \) kills \( J_{K/Q}(-1) \) we have \( H \cdot \eta = 0 \) and hence \( H \cdot \xi = 0 \).
Conversely assume that there is a form $H$ of degree $< s$ dividing $Q$ and such that $H \cdot \xi = 0$ and let denote by $u : I_{C/Q} \to R_{Q(-1)}$ the (degree zero) homomorphism of graded $R$-modules corresponding to $\eta$; from the exact sequence

$$0 \to H^0(\mathcal{O}_Q(h - 1)) \to \text{Hom}(\mathcal{J}_{C/Q}, \mathcal{O}_Q(-1))(h) \to H^0(\omega_C(3 - s + h)) \to 0$$

where $h$ is the degree of $H$, we see that $H \eta \in H^0(\mathcal{O}_Q(h - 1))$. It follows that, if $Q' \in I_C$ is a surface without common components with $Q$, we have $Hu(Q') = SQ'(\text{mod } Q)$, where $S$ is a form of degree $h - 1$; since $H|Q$ we have $H|SQ'$; but $H$ and $Q'$ do not have common components and $\deg S < \deg H$. It follows $S = 0$, hence $H \eta = 0$ and hence the image of $\eta$ is of the form $\mathcal{J}_{E/Q}(-1)$, where $E \subset Q$ contains the surface $K$ where $H \cdot K = Q$. By [MDP] Proposition III.2.6 $\eta$ is not injective. □

**Remark 1.** Fix $s$ and $\xi \neq 0$. For $Q \in H^0(\mathcal{J}_C(s))$, if the corresponding morphism $\eta : \mathcal{J}_{C/Q} \to \mathcal{O}_Q(-1)$ is not injective we will denote by $H_Q$ the surface contained in $Q$ defined as the 2-dimensional component of the subscheme $D \subset Q$ whose ideal sheaf is $\ker \eta$; we note that $H_Q$ can be characterized as follows: $H_Q|Q$, $H_Q \cdot \xi = 0$ and for every $H'$ s.t. $H'|Q$ and $H' \cdot \xi = 0$ is $H_Q|H'$. In fact from above result we have $H' \cdot \eta = 0$ and hence the image of $\eta$ is of the form $\mathcal{J}_{E/Q}(-1)$, where $E \subset Q$ contains the surface $K'$ whith $H' \cdot K' = Q$. Since $K$ is the 2-dimensional component of $E$ we have $K'|K$ and hence $H = H_Q$ divides $H'$.

**Proposition 2.** Let $s(C) \leq s \leq e(C) + 3$, $Q, Q' \in H^0(\mathcal{J}_C(s))$, $\xi \in H^0(\omega_C(3 - s))$, $\xi \neq 0$ and let $\eta : \mathcal{J}_{C/Q} \to \mathcal{O}_Q(-1)$ and $\eta' : \mathcal{J}_{C/Q} \to \mathcal{O}_Q(-1)$ the morphisms corresponding to $\xi$. Assume $\eta$ and $\eta'$ are not injective and let $H_Q$ and $H_{Q'}$ the surfaces defined above. Then $H_Q = H_{Q'}$.

**Proof of Proposition 2.** Put $H = H_Q$ and $H' = H_{Q'}$. Assume they have degrees $0 < h \leq h' < s$. Since $H \cdot \xi = 0$ and $H' \cdot \xi = 0$ by remark 1 it is enough to prove $H|H'$.

Since $H' \cdot \xi = 0$ we have $H' \cdot u(Q_i) = SQ_i(\text{mod } Q)$ for all $Q_i \in I_C$, where $S$ is a surface of degree $h' - 1$ and $u$ is as before; from this we get $K|S$ since the image $u(I_{C/Q})$ is contained in the ideal $I_{E/Q}(-1)$ and hence all $u(Q_i)$ are multiple of $K$; in particular for $Q_i = Q'$, putting $u(Q) = KT$, $S = KV$ we have $H'T = H'K'V(\text{mod } H)$ i.e. $H'T = H'K'V + ZH$ with $Z$ of degree $h' - 1$. It follows that $H'$ divides $ZH$ and hence $H'$ have a common factor of positive degree with $H$.

Let $F = GCM(H, H')$; we will show that $F = H$; assume the contrary, i.e. $\deg F < h$ and put $H = F \cdot \overline{H}$, $Q = F \cdot \overline{Q} = F \cdot \overline{H} \cdot K$ and similarly $H' = F \cdot \overline{H}'$, $Q' = F \cdot \overline{Q}' = F \cdot \overline{H}' \cdot K'$. Now let $Y$ the residue curve of $C$ with respect to $F$ i.e. the curve whose homogeneous ideal is $(I_{C : F})$: it is easy to see that $(I_{C : F})$ is a saturated ideal and that $Y \subset C$ is a curve i.e. does not have zero-dimensional components (isolated or embedded); moreover $Y$ is no empty: in fact if it where $F \in I_C$ we would have $F \cdot \xi = 0$ and this is contrary to the fact that $H$ is the surface of minimal degree contained in $Q$ such that $H \cdot \xi = 0$.

From the exact sequence

$$0 \to \mathcal{O}_Y(-f) \to \mathcal{O}_C \to \mathcal{O}_{C \cap F} \to 0$$

we have an exact sequence of sheaves on $C$

$$0 \to \omega_{C \cap F} \to \omega_C \to \omega_Y(f)$$

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where $f = \deg F$. From the above we obtain an exact sequence:

$$0 \to H^0(\omega_{C \cap F}(3-s)) \to H^0(\omega_C(3-s)) \to H^0(\omega_Y(3-s+f))$$

and let $\xi$ be the image of $\xi$ in $H^0(\omega_Y(3-s+f))$. Hence we have:

1) $\xi \neq 0$: In fact if it were zero, $\xi$ would be in $H^0(\omega_{C \cap F}(3-s))$ and hence $F \cdot \xi = 0$; it follows $3-s+f \geq -e(Y)$.
2) $Y \subset \overline{Q}$ and $Y \subset \overline{Q'}$: it follows $s(Y) \leq s-f$.
3) $\overline{H}$ divides $\overline{Q}$ and has degree $0 < h-f < s-f$; $\overline{H'}$ divides $\overline{Q'}$ and has degree $0 < h'-f < s-f$.
4) $\overline{H} \cdot \overline{e} = 0$ and $\overline{H'} \cdot \overline{e} = 0$: In fact $H^0(\omega_Y(3-s+f))$ embeds in $H^0(\omega_C(3-s+f))$ and the composition $H^0(\omega_C(3-s)) \to H^0(\omega_Y(3-s+f)) \to H^0(\omega_C(3-s+f))$ is the multiplication by $F$. Hence $H \cdot \xi = \overline{H} \cdot F\xi = 0$ implies that $\overline{H} \cdot \xi = 0$ and similarly for $\overline{H'}$.

We have hence reproduced for $Y$ the hypotheses of Proposition 2 and by the first part of proof $\overline{H}$ and $\overline{H'}$ have a common factor of positive degree and this is absurd since they were coprime. $\square$

Now we conclude the proof that 2) implies 3) in Theorem 1. Assume that $C$ does not admit any elementary descending biliaison of height -1 and $s(C) \leq e(C) + 3$. Using [MDP] Proposition III.2.6 three cases are possible.

i) There is an $s$, $s(C) \leq s \leq e(C) + 3$, a surface $Q$ of degree $s$ containing $C$ and a surjective homomorphism $J_{C/Q} \to \mathcal{O}_Q(-1)$; in this case $C$ is the section of $Q$ with a plane $S$ and hence we can make a descending elementary biliaison of $C$ on $S$ with a line.

ii) There is an $s$, $s(C) \leq s \leq e(C) + 3$, a surface $Q$ of degree $s$ containing $C$ and an injective non surjective homomorphism $J_{C/Q} \to \mathcal{O}_Q(-1)$; in this case $C$ admit an elementary descending biliaison on $Q$.

iii) for all $s$, $s(C) \leq s \leq e(C) + 3$ and all $Q$ of degree $s$, every non zero homomorphism $J_{C/Q} \to \mathcal{O}_Q(-1)$ is not injective. In this case, by Proposition 2, for every $s$ and every non zero $\xi \in H^0(\omega_C(3-s))$ there is a form of degree $< s$ which divides every element in $H^0(J_{C}(s))$ and such that $H\xi = 0$.

The last step is to make $H$ independent of $\xi$. Since for fixed $H$ the set of $\xi \in H^0(\omega_C(3-s))$ s. t. $H\xi = 0$ is a $k$-subspace of $H^0(\omega_C(3-s))$ and since the set of forms of degree $< s$ dividing the GCD of $H^0(J_{C}(s))$ is finite, we have the result. This will finish our proof.

**Remark 2.** We note that $H$ divides all $H^0(J_{C}(i))$ for all $i = s(C), \ldots, s$ and kills all $H^0(\omega_C(j))$ for $j = -e(C), \ldots, 3-s$. This last statement follows from the fact that there is a linear form in $R$ which is not a zero divisor for $H_*(\omega_C)$.

**Remark 3.** If $C$ is a curve satisfying condition 3) c) of Theorem 1, then $C$ is not ACM. In fact an ACM curve has all its minimal generators in degree $\leq e(C) + 3$ but for the condition 3) c) all these generators have a common factor of positive degree. As a consequence we have that an ACM curve of degree $> 1$ always admits a descending elementary biliaison.

2. In this Section we prove the implication 3) $\Rightarrow$ 1) of Theorem 1. Let $C$ be a curve satisfying condition 3) c); we want to prove that it is minimal in its biliaison class.
Let 

$$0 \to P \to N_1 \to I_C \to 0$$

be a minimal $N$-resolution of $I_C$ (see [LR] or [MDP]); we know that $P$ is free graded $R$-module. Now let $C'$ another curve in the same biliason class of $C$; by adding a free graded $R$-module $L$ we can assume that $C, C'$ have $N$-resolutions, (not necessarily minimal) of the form:

$$0 \to A \xrightarrow{\alpha} N \xrightarrow{\tau} I_C \to 0$$

$$0 \to B \xrightarrow{\beta} N \xrightarrow{\sigma} I_{C'}(d) \to 0$$

where

$$A = \bigoplus_{i=1}^{n} R(-a_i) = P \oplus L, \ a_1 \leq a_2 \leq \cdots \leq a_n$$

$$B = \bigoplus_{i=1}^{n} R(-b_i), \ b_1 \leq b_2 \leq \cdots \leq b_n$$

as in [LR] it is enough to prove that for all $i = 1 \cdots n$ is $a_i \leq b_i$; as in [LR] it is enough to prove that, for all $t \in \mathbb{Z}$ is rank$A_{\geq t} \geq$ rank$B_{\geq t}$; we consider three cases:

1) $t \leq -(e(C) + 4)$ in this case we use the same proof as in [LR];

2) $t > -s(C)$ in this case we use the same proof as in [LR];

3) let $-s(C) \geq t > -(e(C) + 4)$ and assume that there exists $t$ such that rank$A_{\geq t} <$ rank$B_{\geq t}$. We put $s = t$ and we have $s(C) \leq s \leq e(C) + 3$ and rank$A_{\geq s} <$ rank$B_{\geq s}$.

We assume also that $s$ is minimal with this property.

Put $B = B_1 \oplus B_2$ where $B_1 = \bigoplus_{b_i \leq s} B(-b_i), B_2 = \bigoplus_{b_i > s} B(-b_i)$ with ranks $r, n - r$ and similarly $A = A_1 \oplus A_2$ where $A_1 = \bigoplus_{a_i \leq s} A(-a_i), A_2 = \bigoplus_{a_i > s} A(-a_i)$ with ranks $r', n - r'$ and $r > r'$.

Let $H = H_s$ the form of degree $h < s$ given in 3c) of Theorem 1; put $A'_2 = A_2(h), \ A' = A_1 \oplus A'_2 = A_1 \oplus A_2(h), \ A = A_1 \oplus A_2(h), \ \phi = 1 \oplus H$.

Having fixed notations we prove some Propositions.

**Proposition 3.** Under the above hypotheses and notations there is an exact sequence $0 \to A_1 \oplus A'_2 \xrightarrow{\alpha'} A'_2 \oplus M \xrightarrow{\tau'} I_C \to 0$ and a commutative diagram

$$\begin{array}{ccc}
0 & \to & A \\
\downarrow \phi & & \downarrow \psi \\
0 & \to & A_1 \oplus A'_2 \\
& & \xrightarrow{\alpha'} A'_2 \oplus M \\
& & \xrightarrow{\tau'} I_C \\
& & \to 0
\end{array}$$

with $M$ torsion free graded $R$-module of rank $r' + 1$ and where the map $\alpha'$ is the direct sum of the identity on $A'_2$ and an inclusion $\theta : A_1 \to M$.

**Proof of Proposition 3.** The Proposition follows from the fact that, if $s < p$, then $\text{Ext}^1(I_C, R(-p))^0 \cong H^0(\omega_C(4-p))$ with $3 - s \geq 4 - p$ is killed by $H$. More in detail let $E \in \text{Ext}^1(I_C, A)^0$ be the extension $0 \to A \to N \to I_C \to 0$ and let $\phi E \in \text{Ext}^1(I_C, A')^0$ the composite extension (see [McL], Ch.III, Lemma 1.4); $\phi E$ has the form

$$0 \to A_1 \oplus A_2(h) \to N' \to I_C \to 0.$$
In order to prove our Proposition we prove that there is a map $N' \to A_2(h)$ such that
the composition $A_2(h) \to N' \to A_2(h)$ is the identity. In fact let $E \in \text{Ext}^1(I_C, A_2)^0$ the
extension $0 \to A_2 \to N \to I_C \to 0$ induced by $E$, where $N$ is the quotient of $N$ by $A_1$;
we see easily that the extension $H \cdot E$ has the form:

$$0 \to A_2(h) \to N'/A_1 \to I_C \to 0$$

and by the above observation it splits; hence there is a map $N'/A_1 \to A_2(h)$ and the
composition $A_2(h) \to N' \to N'/A_1 \to A_2(h)$ is the identity. Moreover $A_1$ is contained in
the kernel $M$ of the map $N' \to A_2(h)$. \square

By Proposition 3 we have hence the following exact sequences and commutative
diagrams:

$$0 \to A \xrightarrow{\alpha} N \xrightarrow{\tau} I_C \to 0$$

$$0 \to A' \xrightarrow{\alpha'} N' \xrightarrow{\tau'} I_C \to 0$$

where $N' = A'_2 \oplus M$,

$$0 \to A_2 \xrightarrow{\alpha_2} N \xrightarrow{\rho} M \to 0$$

$$0 \to A'_2 \xrightarrow{\alpha'_2} N' \xrightarrow{\rho'} M \to 0$$

where $\rho'$ is the projection, $\rho = \rho' \circ \psi$, $\alpha_2, \alpha'_2$ the restrictions of $\alpha, \alpha'$ respectively.

Moreover we have an exact sequence

$$0 \to A_1 \xrightarrow{\theta} M \xrightarrow{\lambda} I_C \to 0$$

where $\lambda = \tau'|M$.

**Proposition 4.** Under the above hypotheses and notations the composition

$$B_1 \xrightarrow{\beta_1} N \xrightarrow{\psi} N' \xrightarrow{\pi} A_2(h)$$

is zero, where $\beta_1 = \beta|B_1$ and $\pi$ is the projection.

**Proof of Proposition 4.** A summand of $B_1$ is of the form $R(-b)$ with $b \leq s$; since
the terms in $A_2(h)$ have the form $R(-p + h)$, $s - h < p - h$, hence $-p + h + b < h$, the map
$R(-b) \to R(-p + h)$ will be zero if we will prove that it factors through the multiplication
by $H$ or equivalently that restricted to the surface $H$ it is zero. To this end we first restrict
to the affine open set $U = \mathbf{P}^3 \setminus S$ where $S$ is a surface containing $C$ and without common components with $H$. Since $J_{C,U} \cong \mathcal{O}_U$ the induced sequences

$$0 \to A_U \to N_U \to J_{C,U} \to 0$$

$$0 \to A_{1,U} \oplus A_{2}(h)_U \to A_{2}(h)_U \oplus M_U \to J_{C,U} \to 0$$

split, hence they remain exact when restricted to $H \cap U$. Now the map $\mathcal{O}(-b)_U \to J_{C,U}$
induced by $R(-b) \to I_C$ is zero when restricted to $H \cap U$, since $b \leq s$ and $H$ divides all
$H^0(J_{C}(b))$ for $b \leq s$; so the map $\mathcal{O}(-b)_U \to \mathcal{O}(-p + h)_U$ factors (mod $H \cap U$) through
$A_U$, but the map $A_U \to \mathcal{O}(-p + h)_U$ is zero (mod $H \cap U$). \square

From the Proposition 4 follows that the map $\phi \circ \beta_1 : B_1 \to N'$ factors though
the inclusion $M \to N'$ and since $\text{rk}B_1 = r > r'$ and $\text{rk}M = r' + 1$ it follows that
$\text{rk}B_1 = \text{rk}M = r = r' + 1$. 

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Proposition 5. Under the above hypotheses and notations the composition map

\[ N \xrightarrow{\sigma} I_{C'}(d) \xrightarrow{H} I_{C'}(d+h) \]

factors through the map \( \psi : N \to N' \).

Proof of Proposition 5. We see easily that \( \text{coker}\psi \) is isomorphic to \( \text{coker}\phi \) hence it is annihilated by~\( H \). From the exact sequence

\[ \text{Hom} (N', I_{C'}(d)) \to \text{Hom} (N, I_{C'}(d)) \to \text{Ext}^1 (\text{coker}\psi, I_{C'}(d)) \]

we get the result since \( H \) annihilates \( \text{Ext}^1 (\text{coker}\psi, I_{C'}(d)) \). \( \square \)

Now let \( \sigma' : N' \to I_{C'}(d+h) \) be the map given by Proposition 5.

Proposition 6. Under the above hypotheses and notations the restriction \( \sigma' |_M : M \to I_{C'}(d+h) \) is zero.

Proof of Proposition 6. In fact the kernel contains \( B_1 \) hence has rank \( r = \text{rk}M \) and \( I_{C'}(d+h) \) is torsion free. \( \square \)

Proposition 7. Under the above hypotheses and notations the composition map \( \gamma : A_1 \xrightarrow{\alpha_1} N \xrightarrow{\sigma} I_{C'}(d) \) is zero.

Proof of Proposition 7. In fact the composition of \( \gamma \) with the multiplication map \( I_{C'}(d) \xrightarrow{H} I_{C'}(d+h) \) factors through the map \( \sigma' |_M : M \to I_{C'}(d+h) \) hence it is zero and also \( \gamma \) is zero. \( \square \)

As a conclusion the map \( \alpha_1 : A_1 \to N \) factors as \( \beta_1 \circ \zeta \) where \( \zeta : A_1 \to B_1 \) and \( \beta_1 : B_1 \to N \); it follows \( \text{rk}A_1 \leq \text{rk}B_1 \). On the other hand remember that \( s \) was minimum such that \( \text{rk}A_{\geq-s} < \text{rk}B_{\geq-s} \). Hence we have:

a) for \( t < s \) the map \( \zeta \) induces an isomorphism \( A_{\geq-t} \cong B_{\geq-t} \)

b) for \( t = s \) we can change basis in \( \bigoplus_{b_i=s} B(-b_i) \) such that \( \bigoplus_{b_i=s} B(-b_i) = \bigoplus_{a_i=s} A(-a_i) \oplus R(-s) \) and \( \zeta \) is the identity on \( \bigoplus_{a_i=s} A(-a_i) \).

We can factor out \( N \) by \( A_1 \) and we can assume in the sequel \( A_1 = 0 \), \( B_1 = R(-s) \). Let now \( Q = H \cdot K \) be the image of the generator of \( R(-s) \) in \( I_C \); as in the proof of Proposition 4 we restrict to the affine open set \( U = \mathbb{P}^3 \setminus S \) where \( S \) is a surface containing \( C \) and without common components with \( Q \). Moreover we still denote by \( Q, H, K \) the local equations of the surfaces \( Q, H, K \) on \( U \).

Since \( J_{C,U} \cong \mathcal{O}_U \) the induced sequences \( 0 \to A_U \to N_U \to J_{C,U} \to 0 \) and \( 0 \to A'_U \to N'_U \to J_{C,U} \to 0 \) split. Hence \( N_U \cong N'_U \cong \mathcal{O}_{U}^\oplus n+1 \). We write down the matrix of the map \( \psi_U : \mathcal{O}_U^\oplus n+1 \to \mathcal{O}_U^\oplus n+1 \) induced by \( \psi \). It has the form

\[
M = \begin{pmatrix}
A_2 & I \\
H I_n & C_n \\
0 & 1
\end{pmatrix}
\]
where $I_n$ is the unit matrix and $C_n$ is a one column matrix.

We also consider the matrix induced by the map $\beta : B \to N$; it has the form

$$P = A_2 \begin{pmatrix} B_1 & B_2 \\ \overline{D_n} & * \\ \overline{Q} & * \end{pmatrix} I$$

where $D_n$ is a one column matrix.

The product $M \cdot P$ is associated to the map $\tilde{\phi}_U \circ \tilde{\beta}_U : B_U \to N'_U \cong A_2(h)_U \oplus J_U$ and has the form

$$M \cdot P = A_2(h) \begin{pmatrix} B_1 & B_2 \\ \overline{HD_n + QC_n} & * \\ \overline{Q} & * \end{pmatrix} I$$

By Proposition 4 we have $HD_n + QC_n = 0$, hence $D_n = -KC_n$ and the matrix $P$ has the form

$$P = A_2 \begin{pmatrix} B_1 & B_2 \\ \overline{-KC_n} & * \\ \overline{Q} & * \end{pmatrix} I$$

We finish by observing that all $n \times n$ minors of $P$ are multiple of $K$ and this is contrary to Hilbert-Burch Theorem.

3. In this section we deduce some consequences from Theorem 1. The first consequence, whose proof is trivial, is an improvement of the Lazarsfeld-Rao property.

**Theorem 2.** Every curve in $\mathbb{P}^3$ can be obtained from a minimal one by means of a finite number of ascending elementary biliaisons.

Next we recall that in [MR] Maggioni and Ragusa determined bounds for the Betti numbers of a curve $C \subset \mathbb{P}^3$ that can be obtained as in Theorem 2, so the results of [MR] apply to every curve in $\mathbb{P}^3$. In particular we have the following.

**Theorem 3.** Let $C$ any curve in $\mathbb{P}^3$ and let $C_0$ be a minimal curve in the biliaison class of $C$. Let us denote by $\nu(C), \nu(C_0)$ the minimal number of generators of $I_C, I_{C_0}$ respectively. Then we have

$$\nu(C) \leq \nu(C_0) + s(C) - s(C_0).$$

**Proof.** The proof is very easy; if $C$ is obtained from $C'$ by means of an elementary biliaison of type $(1, s)$ on a surface $Q$ of degree $s$, two cases are possible:
a) $s = s(C')$; in this case $s(C) = s(C')$ and $C, C'$ have the same number of minimal generators;  

b) $s > s(C')$; in this case $s(C) = s(C') + 1$ and $C$ has at most one more minimal generator than $C'$, i.e. the surface $Q$. 

We apply Theorem 3 to the following cases:

1) $C$ is a ACM curve. In this case $C_0$ is a line and we get the well known bound

$$\nu(C) \leq s(C) + 1.$$ 

2) $C$ is an Arithmetically Buchsbaum curve of diameter $\mu$. In this case it is $s(C_0) = 2\mu$ and $\nu(C_0) = 3\mu + 1$ (see e.g. [MDP] ) and we find the Amasaki bound $\nu(C) \leq s(C) + \mu + 1$.

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