Branched Polymers, Complex Spins and the Freezing Transition

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Abstract. We show that by coupling complex three-state systems to branched-polymer like ensembles we can obtain models with $\gamma_{str}$ different from $\frac{1}{2}$. It is also possible to study the interpolation between dynamical and crystalline graphs for these models; we find that only when geometry fluctuations are completely forbidden is there a crystalline phase. These models share many of the properties of full two-dimensional quantum gravity.

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1 Introduction

Recent years have seen remarkable progress in the theory of two dimensional quantum gravity, due in large extent to the discretized approach (see [1], [2] for a review). We have a good understanding of the situation for $c < 1$, where $c$ is the central charge of the matter coupled to gravity, through the KPZ formula [3] and associated results. The situation changes for $c > 1$ where the KPZ formula gives meaningless predictions; numerical simulations [4] and analytical arguments [5], [6], [7] favour the view that quasi one-dimensional configurations called branched polymers dominate the statistical ensemble in this region (there is, however, no definite proof of this conjecture). For generic branched polymer ensembles the critical exponent $\gamma_{str} = \frac{1}{2}$.

Certain modifications of the weighting of different branched polymers lead to other positive values of $\gamma_{str}$ ([8], [9]).

The KPZ results show that the critical behaviour of spin systems (e.g. Ising spins) interacting with two-dimensional quantum gravity is modified from that observed on a regular lattice. This raises the question of what would happen in an extended theory which interpolates between the fixed lattice (or crystalline) model and the gravity model [10]. The only two-dimensional problem of this type that has been solved is the case of $R^2$ gravity (here $R$ is the scalar curvature density) [11]. If surfaces of large $R^2$ are suppressed the ensemble becomes flatter and flatter at short distance scales. However it is found that $\gamma_{str} = -\frac{1}{2}$, the pure gravity value, for all finite values of the $R^2$ coupling; only when all surfaces with non-minimal $R^2$ are completely forbidden does $\gamma_{str}$ change.

In this paper we will show that it is possible to change $\gamma_{str}$ for a branched polymer (BP) ensemble by introducing matter fields, rather than fiddling directly with the geometry, just as matter fields at criticality can change $\gamma_{str}$ in the two-dimensional models. Then we study the interpolation between “gravity” and “crystalline” phases.

As a first step, in Section 2 we classify the possible critical exponents obtainable from a generic class of quadratically coupled non-linear equations. In Section 3 we introduce and study the set of binary trees. Section 4 is concerned with the coupling of matter to binary trees. First we show that a normal Ising model coupled to binary trees yields the same results as pure trees. Introducing a more complicated model of $Z_3$ spins with a complex action leads to new behaviour, with $\gamma_{str}$ taking values of 2, $\frac{3}{2}$ and $\frac{2}{3}$ in addition to $\frac{1}{2}$. 
2 Categorization of Critical Behaviour

In what follows, we will be interested in the possible critical behaviour of systems of coupled equations of the form

\[ A_i = z + \alpha_i^{jk} A_j A_k. \]  

(1)

The \( A_i \) are grand canonical ensemble partition functions and \( z \) is related to the chemical potential \( \mu \) by \( z = e^{-\mu} \). The coefficients \( \alpha_i^{jk} \) depend on the coupling constants of the specific model and the repeated indices are summed over; we assume that the set of equations is irreducible in the sense that no linear combination of \( A_i \)'s decouples. At small \( z \) the \( A_i \) are power series in \( z \); as \( z \) increases the \( A_i \) become non-analytic at some critical value \( z_c \). Defining

\[ M_{ij} = \delta_{ij} - \alpha_i^{jk} A_k - \alpha_i^{jk} A_j \]  

(2)

and differentiating equation (1) w.r.t. \( z \) we obtain

\[ \frac{\partial A_i}{\partial z} = (M^{-1})_{ij} u^i \]  

(3)

where \( u^i = (1,...,1) \) and \( M^{-1} \) is the inverse of \( M \), \( M^{-1} = \frac{1}{\text{det}(M)} (\text{cof}(M))^T \). There are two ways in which the above expression can become non-analytic; either the \( A_i \) remain finite as \( z \uparrow z_c \) but their derivatives blow up, or the \( A_i \) themselves blow up in that limit. We analyse of these two cases separately.

If the \( A_i \) remain finite, but the \( \frac{\partial A_i}{\partial z} \) diverge, then we may write, with \( \zeta = z_c - z \),

\[ A_i(z) = R_i(\zeta) - \zeta^{1-\gamma_{str}} P_i(\zeta) \]  

(4)

where \( \gamma_{str} < 1 \) and \( R_i(\zeta) \) and \( P_i(\zeta) \) are regular functions at \( \zeta = 0 \) with \( P_i(0) \neq 0 \), \( R_i(0) \neq 0 \). Then

\[ \frac{\partial A_i}{\partial z} \sim \zeta^{-\gamma_{str}} \]  

(5)

Now, since at \( z = z_c \) the \( A_i \) remain finite, we must have \( \text{det}(M) = 0 \) for the derivatives to diverge. The general expression for the determinant is obtained by substituting (4) into (2) and expanding which gives

\[ \text{det}M = \zeta^{1-\gamma_{str}} P^{(1)}(\zeta) + \zeta^{2(1-\gamma_{str})} P^{(2)}(\zeta) + \ldots + \zeta^{n(1-\gamma_{str})} P^{(n)}(\zeta). \]  

(6)

The \( P^{(n)}(\zeta) \) depend on the \( \alpha_i^{jk} \) and therefore on the coupling constants.

We can now identify the different cases, starting with the most general; all the \( P^{(n)}(0) \) are nonzero. By comparing the leading powers in (6) and (5) we see that \( \gamma_{str} \) obeys

\[ -\gamma_{str} = -(1 - \gamma_{str}) \]  

(7)
and therefore $\gamma_{str} = \frac{1}{2}$. No constraints on the coupling constants are necessary for this case; so for example this value of $\gamma_{str}$ can hold for areas of the phase diagram of a system with two coupling constants.

The next case is that of $P^{(1)}(0) = 0$ but $P^{(2)}(0) \neq 0$. Then the expansion of the determinant (3) becomes

$$detM \sim \zeta^{2(1-\gamma_{str})} + O(\zeta^{2-\gamma_{str}}) + O(\zeta^{3(1-\gamma_{str})}) + ...$$ (8)

and the leading singularity of (8) is in the first term, leading to

$$- \gamma_{str} = -2(1 - \gamma_{str})$$ (9)

and to a value of $\gamma_{str} = \frac{2}{3}$. Notice that in order to make $P^{(1)}(0) = 0$ we are introducing a constraint in the coupling space. In the case of two coupling constants this leads to a line in the phase diagram.

Further values of $\gamma_{str}$ will be obtained if we introduce more constraints. If we demand $P^{(1)}(0) = P^{(2)}(0) = 0$ but allow $P^{(3)}(0) \neq 0$, the leading term will be the $O(\zeta^{3(1-\gamma_{str})})$ and we obtain $\gamma_{str} = \frac{3}{4}$; the two constraints necessary to achieve this situation mean that, in the case of two coupling constants, this value of $\gamma_{str}$ is confined to a point in the phase diagram. A higher number of constraints will lead to other values of $\gamma_{str}$, but the above instances are enough for our purposes. Note that not all of the possible exponents found in this way can arise from any set of equations; the larger the number of coupled equations the more exotic is the possible behaviour.

The second case is that the $A_i$ themselves become infinite as $z \uparrow z_c$; of course the derivatives will also blow up. In this case

$$A_i(z) = \zeta^{1-\gamma_{str}} P_i(\zeta) + R_i(\zeta)$$ (10)

with $\gamma_{str} > 1$. Again $detM$ must vanish at $z = z_c$. Putting (10) into (3) and expanding yields

$$detM \sim \sum_i \zeta^{(1-\gamma_{str})m_i} \zeta^{n_i} a_i$$ (11)

where $m_i, n_i$ are positive integers and $a_i$ are (coupling constant dependent) constants such that

$$m_i(1 - \gamma_{str}) + n_i > 0$$ (12)

so that we get

$$- \gamma_{str} = -(m_j(1 - \gamma_{str}) + n_j)$$ (13)

if $a_1 = a_2 = \ldots = a_{j-1} = 0$ and $a_j \neq 0$. By assumption $\gamma_{str} > 1$ so that we must always have $n_i > 0$. If $n_i = 1$ then $\gamma_{str} = 1$ (otherwise from (12) $m_i$ is negative,
Table 1: Constraints and $\gamma_{str}$

| $\gamma_{str}$ | Constraint       | Possible Values | $(n,m)$ | # of constraints |
|-----------------|------------------|-----------------|---------|-----------------|
| $\frac{3}{2}$   | $n_i - \frac{m_i}{2} > 0$ | $\frac{1}{2}$ | (1,1)   | 2               |
|                 |                  | $1$             | (2,2)   |                 |
|                 |                  | $\frac{3}{2}$   | (2,1)   |                 |
| $\frac{5}{4}$   | $n_i - \frac{m_i}{4} > 0$ | $\frac{1}{2}$ | (1,1)   | 4               |
|                 |                  | $\frac{1}{4}$   | (1,2)   |                 |
|                 |                  | $\frac{3}{4}$   | (1,3)   |                 |
|                 |                  | $1$             | (2,4)   |                 |
|                 |                  | $\frac{5}{4}$   | (2,3)   |                 |
| 2               | $n_i - m_i > 0$   | $1$             | (2,1)   | 1               |
|                 |                  | $2$             | (3,1)   |                 |
| 3               | $n_i - 2m_i > 0$  | $1$             | (3,1)   | 2               |
|                 |                  | $2$             | (4,1)   |                 |
|                 |                  | $3$             | (5,1)   |                 |
| 4               | $n_i - 3m_i > 0$  | $1$             | (4,1)   | 3               |
|                 |                  | $2$             | (5,1)   |                 |
|                 |                  | $3$             | (6,1)   |                 |
|                 |                  | $4$             | (7,1)   |                 |

which is forbidden); this implies logarithmic singularities in $A_i$ which are not possible from a set of polynomial equations so we conclude that $n_i > 1$. The question is now how many $a_i$ must be zero (i.e., how many constraints there are) to get a certain value of $\gamma_{str}$.

The number of constraints necessary to get different values of $\gamma_{str}$ can be obtained by considering equation (12). For instance, if we demand $\gamma_{str} = \frac{3}{2}$, (12) reduces to $n_i - \frac{m_i}{2} > 0$. We can then examine the possibilities for $n$ and $m$. Setting $m = 1$, we can put $n = 1$ (giving a value of $\frac{1}{2}$) and $n = 2$ (giving $\frac{3}{2}$); we can also put $n = m = 2$ giving a value of 1. Thus to obtain $\gamma_{str} = \frac{3}{2}$ we need a minimum of two constraints. The results of this analysis for some values of $\gamma_{str}$ are shown in Table 1.

## 3 Binary Trees

The properties of the ensemble of tree graphs are well known [13]. In this paper we will consider trees made of cubic vertices but modify the graphs slightly so that all the external lines except the root are attached to another line. This does not affect the recurrence relation generating the graphs which is shown in fig.1. Letting $T_N$ be
the number of graphs with $N$ external vertices (not counting the root) we have

$$T_N = \sum_{k=1}^{N-1} T_{N-k} T_k , \quad T_1 = 1$$

(14)

or

$$T(z) = \sum_{N=1}^{\infty} z^N T_N = \frac{1}{2} \left( 1 - \sqrt{1 - 4z} \right).$$

(15)

The exponent $\gamma_{str}$ for the ensemble of graphs with one marked point (the root in this case) is defined so that the generating function $G$ for the number of graphs of a given size has leading non-analytic behaviour

$$\frac{\partial G}{\partial z} = (z_{cr} - z)^{-\gamma_{str}}$$

(16)

so for the tree ensemble $\gamma_{str} = \frac{1}{2}$. Now we want to modify the problem so that one particular sort of graph is picked out and given a different weight; this is the ladder graph shown in fig. 2. For a given $N \geq 3$ there are precisely two of these, the one shown in fig. 2 and its mirror image. However, there is only one graph for each of $N = 1$ and $N = 2$ (fig. 3); we will define both of these to be ladders. All
the non-ladder graphs we will call “trees”; so in fact there are no trees for \( N < 4 \).

Letting \( L_N \) be the number of ladders with \( N \) external vertices we have

\[
L_1 = L_2 = 1 \\
L_{N \geq 3} = 2
\]

so that

\[
L(z) = \frac{2z}{1-z} - z - z^2 = z \left( \frac{1 + z^2}{1-z} \right)
\]

and the exponent \( \gamma_{str} \) takes the value 2 for a pure ladder ensemble. The number of trees \( T'_N \) satisfies

\[
T'_1 = T'_2 = T'_3 = 0 \\
T'_N = q \left\{ \sum_{k=1}^{N-1} (T'_{N-k} + L_{N-k})(T'_k + L_k) - L_{N-1} - \delta_{N3} \right\}, \quad N \geq 3.
\]

The factor \( q \) enables us to assign a different relative weight in the ensemble to trees and ladders. A typical tree with \( N \) external vertices gets a factor of \( q^{N-1} \); a tree which contains a ladder as a sub-graph has a smaller power of \( q \) associated with it.

For the generating function we find

\[
T'(z) = q \left\{ (T'(z) + L(z))^2 - zL(z) - z^3 \right\}
\]

which is easily solved to yield the generating function for the modified ensemble

\[
G(z) = T'(z) + L(z) = \frac{1}{2q} \left( 1 - \sqrt{1 - 4q ((1 - qz)L - qz^3)} \right).
\]

For \( q = 1 \) this just becomes the usual tree generating function \([19]\) with \( \gamma_{str} = \frac{1}{2} \) whereas at \( q = 0 \) it is equal to \( L(z) \). However for any positive non-zero value of \( q \) we find that \( \gamma_{str} = \frac{1}{2} \). This is easily seen by considering the behaviour of the argument of the square root as \( z \) is increased from zero; as \( z \) increases \( L(z) \) increases but before it diverges the argument of the square root must vanish (because it goes to \(-\infty \) if \( L(z) \) goes to \( \infty \)). Thus only at \( q = 0 \) exactly do we manage to “freeze out” the general trees and get a system which contains only ladders. This behaviour is very reminiscent of the \( R^2 \) model discussed in the Introduction.

Of course this phase structure is consistent with the general arguments we gave above. This is a system with one coupling constant, \( q \), and for generic values it has \( \gamma_{str} = \frac{1}{2} \); at one particular coupling, \( q = 0 \), it has \( \gamma_{str} = 2 \), which requires one constraint.
Figure 3: Coupling of spins systems to a binary tree. A circle contains graphs made of trees with $N$ external vertices excluding the root.

## 4 Matter coupled to binary trees

We can extend the model by coupling matter to the trees. Placing an Ising spin $\sigma_i = \pm 1$ at each of the vertices we obtain the recurrence relation shown in fig.3. The interaction for two neighboring spins is given by $e^{\beta \sigma_i \sigma_j}$. Using the identity $e^{\beta \sigma_i \sigma_j} = \cosh \beta [1 + t \sigma_i \sigma_j]$ where $t \equiv \tanh \beta$, we can take the link factor to be $1 + t \sigma_i \sigma_j$. Then

$$Z_N = \frac{1}{2^N} \sum_{q=1}^{N-1} \sum_{abcd} (1 + t \sigma_1 a)(1 + t \sigma_1 c)(1 + tbd)Z_{N-q}(a, \sigma_2, b)Z_q(c, d, \sigma_3).$$

(21)

In the absence of a magnetic field the dependence of $Z_N$ on the external configuration $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ must take the form

$$Z_N(\sigma) = A_N + \sigma_1 \sigma_2 B_N + \sigma_1 \sigma_3 B_N + \sigma_2 \sigma_3 C_N.$$  

(22)

Inserting (22) into the partition function (21) and equating coefficients depending on the same combination of spins, we find a system of coupled equations for the $A_N$, $B_N$, $C_N$ in terms of $A_{N-1}$, $A_{N-2}$, etc. Defining the grand-canonical partition functions

$$A(z, t) = \sum_{N=1}^{\infty} z^N A_N(t)$$

(23)

and similarly for $B(z, t)$, $C(z, t)$, we obtain the system of equations:

$$A = z + A^2 + t^3 B^2$$  \hspace{1cm} (24)

$$B = z + tAB + t^2 BC$$  \hspace{1cm} (25)

$$C = z + tC^2 + t^2 B^2.$$  \hspace{1cm} (26)
Singularities in any of the functions $A$, $B$ or $C$ (or in any of their derivatives) will signal a critical point in the full partition function (21).

The simplest situation is when the matter is decoupled, $t = 0$. Then $B = C = z$ and $A(z)$ is just the tree generating function $T(z)$. The system can also be solved exactly for $t = 1$. Then the equations (24)-(26) imply that $A - C = A^2 - C^2$ which only has one sensible solution, $A = C$ (the other, $A = 1 - C$ does not exhibit the correct behavior as $z \to 0$), which leads to $A = B$. Solving the remaining quadratic we find

$$A = B = C = \frac{1 - \sqrt{1 - 8z}}{4}. \quad (27)$$

Hence, $z_c = \frac{1}{8}$ and $\gamma_{str} = \frac{1}{2}$. It is straightforward to show that $\gamma_{str} = \frac{1}{2}$ for all $0 \leq t \leq 1$. Eliminating $A$ and $C$ from (24)-(26) gives

$$B = \frac{z}{1 - t + \frac{t}{2} \left[ \sqrt{1 - 4z - 4t^3B^2} + \sqrt{1 - 4zt - 4t^3B^2} \right]}. \quad (28)$$

Clearly $B$ itself cannot diverge as $z \uparrow z_c$; this is because the arguments of the square roots go negative when $B$ is still finite. Hence $\gamma_{str} < 1$. Differentiating with respect to $z$ we get

$$F \frac{\partial B}{\partial z} = \frac{B}{z} + \frac{B^2 t}{z} \left\{ \frac{1}{\sqrt{1 - 4z - 4t^3B^2}} + \frac{t}{\sqrt{1 - 4zt - 4t^3B^2}} \right\}. \quad (29)$$

where

$$F = 1 - \frac{2t^4B^3}{z} \left( \frac{1}{\sqrt{1 - 4z - 4t^3B^2}} + \frac{1}{\sqrt{1 - 4zt - 4t^3B^2}} \right) \quad (30)$$

The right hand side of (29) is clearly positive; the arguments of the square roots cannot vanish as $z \uparrow z_c$ because the coefficient of $\frac{\partial B}{\partial z}$ would vanish while they were still finite. So we are interested in the point where $F$ vanishes. $B$ is an increasing function of $z$ and grows faster than $z$ (by (28)). Hence $F$ is a monotonically decreasing function of $z$ at fixed $t$ and there is a value of $z$ for which $F = 0$ and $\frac{\partial B}{\partial z}$ diverges. Furthermore $\frac{\partial F}{\partial B}$ is a sum of negative definite terms; it follows that the generic value $\gamma_{str} = \frac{1}{2}$ applies everywhere and the cancelation necessary to get any other value of $\gamma_{str}$ cannot occur.

The critical structure of the theory is not changed by the addition of Ising spins. This result is expected because the Ising spins never have a diverging correlation length in less than two dimensions and so cannot affect the global properties of the geometry.

We now consider a generalised Ising model in which the spins take the values $1, e^{\pm \frac{2\pi}{3}}$. By allowing the partition function to include complex weights, we obtain
a richer phase structure than that of the 2-state case. The partition function is given by

\[ Z(\alpha, \beta) = \sum_s e^{\alpha \sum_{(ij)} (s_i s_j + s_i s_j^\dagger)} + \beta \sum_{(ij)} (s_i s_j - s_i s_j^\dagger) \]  

(31)

(note that in doing this it is necessary to assign an orientation to every link). The Boltzman factor for a link can be written in the form

\[ \tilde{L} = \lambda(1 + t(s_i s_j + s_i s_j^\dagger))(1 + \gamma(s_i s_j - s_i s_j^\dagger)) \]  

(32)

where,

\[ \lambda = \left[ (1 + \gamma^2)(1 - t^2)(1 + 2t) \right] \]  

\[ \gamma = \frac{\tan(\sqrt{3} \alpha)}{3} \]  

\[ t = \frac{e^{3\beta} - \cos(\sqrt{3} \alpha)}{2 \cos(\sqrt{3} \alpha) + e^{3\beta}}. \]  

(33)

We see from these expressions for that our model will be well defined for \( \lambda \) in the interval \([0, +\infty]\); however \( t \) must be restricted to \([0, 1]\) for \( \lambda \) to be analytic. We can thus drop \( \lambda \) and consider

\[ L = (1 + \mu S_i s_j^\dagger + \nu S_i^\dagger S_j) \]  

(34)

where \( \mu \equiv t + \gamma(1 - t) \) and \( \nu \equiv t - \gamma(1 - t) \) so that the partition function (31) becomes

\[ Z = \sum_{\text{trees}} Z_N = \sum_{\text{trees}} \sum_{\{S_i\}} \prod_{\text{links}} (1 + \mu S_i s_j^\dagger + \nu S_i^\dagger S_j). \]  

(35)

We can proceed as before to obtain a system of coupled equations which completely describe \( Z \). The most general form of \( Z_N \), taking into account its global \( \mathbb{Z}_3 \) symmetry, is

\[ Z_N(S_1, S_2, S_3) = A_N + S_1 S_2 B_N + S_1 S_3 \tilde{B}_N + S_1^\dagger S_2 D_N + S_1^\dagger S_3 \tilde{D}_N + S_2 S_3^\dagger C_N + S_3 S_2^\dagger E_N + S_1 S_2 S_3 F_N + S_1^\dagger S_2^\dagger S_3^\dagger G_N. \]  

(36)

By introducing this form into the partition function and then computing the sums leading to the grand canonical partition function, we arrive at the following set of equations:

\[ A = z + A^2 + \mu \nu (\mu + \nu) BD \]  

\[ B = z + \mu \tilde{B} A + \mu \nu BC + \nu^2 \mu DG \]  

\[ \tilde{B} = z + \mu \tilde{B} A + \mu^2 BC + \nu^3 DG \]
\[ C = z + \mu CE + \mu \nu \tilde{B} \tilde{D} + \nu^2 \mu FG \]
\[ D = z + \nu A \tilde{D} + \mu \nu DE + \mu^2 \nu BF \]
\[ \tilde{D} = z + \nu A \tilde{D} + \nu^2 DE + \mu^3 BF \]
\[ E = z + \nu CE + \mu \nu \tilde{B} \tilde{D} + \mu^2 \nu FG \]
\[ F = z + \mu (\mu + \nu) EF + \nu^2 \tilde{D}^2 \]
\[ G = z + \nu (\mu + \nu) CG + \mu^2 \tilde{B}^2. \] (37)

For some curves in the \((t, \gamma)\) plane this set of equations can be solved exactly. We will find that, taken together with the general discussion of section 1, this is sufficient to enable us to elucidate the entire phase diagram.

At \(t = \gamma = 0\), \(A\) is given as usual by the generating function of the Catalan numbers and all other quantities equal \(z\) so \(\gamma_{str} = \frac{1}{2}\). At \(\gamma = 0\) (this is the 3-state Potts model) we have that \(\mu = \nu = t\) and the equations can be reduced by elementary manipulations to the set

\[ A = z + A^2 + 2t^3 B^2 \]
\[ B = z + tBA + t^2 BC + t^3 BF \]
\[ C = z + tC^2 + t^2 BC + t^3 F^2 \]
\[ F = z + 2t^2 CF + t^2 B^2 \]
\[ \tilde{B} = B, \tilde{D} = D, E = C, G = F, \] (38)

which, like the Ising model (24)-(26), can be shown to have \(\gamma_{str} = \frac{1}{2}\) for all \(z\). At \(t = 1\), \(\mu = \nu = 1\) independently of \(\gamma\) and the system is again described by (38) having \(\gamma_{str} = \frac{1}{2}\) for all \(\gamma\); in this case the equations have elementary solutions

\[ A = \frac{1}{2} \left( 1 - \sqrt{\frac{5}{9} - \frac{4}{3} z + \frac{4}{9} \sqrt{1 - 12z}} \right) \]
\[ B = \sqrt{\frac{1 - 6z - \sqrt{1 - 12z}}{18}} \] (39)
\[ C = \frac{1}{4} \left( 1 - \sqrt{\frac{5}{9} - \frac{16}{3} z + \frac{4}{9} \sqrt{1 - 12z}} \right) \]
\[ F = C \]

and all the functions are non-analytic at \(z_{cr} = \frac{1}{12}\).

Along the line \(t = \frac{\gamma}{1+\gamma}, \nu = 0\) and the system of equations is greatly simplified and can be solved to give

\[ A = \frac{1}{2} \left( 1 - \sqrt{1 - 4z} \right) \]
\[
\begin{align*}
\tilde{B} &= \frac{z(1 - \mu z) + \mu^2 z^2}{1 - \mu z - \mu(1 - \mu z + \mu^2 z)^{1/2} \left(1 - \sqrt{1 - 4z}\right)} \\
B &= z + \mu \tilde{B}A \\
C &= \frac{z}{1 - \mu z} \\
D &= E = z \\
\tilde{D} &= z + \mu^3 BF \\
F &= \frac{z}{1 - \mu^2 z} \\
G &= z + \mu^2 \tilde{B}^2.
\end{align*}
\]
(40)

At small \(\mu\) the singularity is at \(z = \frac{1}{4}\) and \(A, B, \tilde{B}, G\) are non-analytic with \(\gamma_{str} = \frac{1}{2}\) while \(C, D, E, F\) are analytic. So for small \(\mu\) the \(\nu = 0\) line lies within the \(\gamma_{str} = \frac{1}{2}\) region and is not a phase transition. However as \(\mu\) increases we reach a point where the denominator of \(\tilde{B}\) vanishes for \(z < \frac{1}{4}\); this occurs at \(\mu \geq \mu_0\) where
\[
1 - \frac{3}{4} \mu_0 + \frac{\mu_0^2}{8} - \frac{\mu_0^3}{8} = 0
\]
(41)
which corresponds to \(\mu_0 \simeq 1.2633\), i.e. \(t \simeq 0.6316\). So at \(\mu = \mu_0\) the singularity is at \(z = \frac{1}{4}\) but now
\[
\tilde{B} \sim \frac{1}{\sqrt{1 - 4z}}
\]
(42)
and at this point \(\gamma_{str} = \frac{3}{2}\). For \(\mu > \mu_0\) we have that
\[
\tilde{B} \sim \frac{1}{z_{cr} - z}
\]
(43)
with \(z_{cr} < \frac{1}{4}\) so that \(A\) becomes analytic and \(\gamma_{str} = 2\). From our general considerations we expect that \(\gamma_{str} = \frac{3}{2}\) occurs at a point and \(\gamma_{str} = 2\) along a line in this two-dimensional coupling constant space. Hence for \(\mu > \mu_0\) the line \(\nu > 0\) separates two phases.

It is straightforward to show from (37) that for \(\mu \neq 0, \nu \neq 0\), all the potentials are finite (this is done by assuming that a potential diverges at the critical point and obtaining a contradiction). Thus the two phases which are separated by the \(\gamma_{str} = 2\) line must both have \(\gamma_{str} = \frac{1}{2}\), this being the only value which can exist on an area of the phase diagram of a system with two coupling constants. The nature of the phases is quite different. For \(t > t_0\) and \(\nu > 0\) the thermodynamic limit exists in the usual sense, e.g.
\[
\lim_{N \to \infty} \frac{1}{N} \log A_N = \text{const.}
\]
(44)
On the other hand when \(t > t_0\) and \(\nu < 0\) the potentials fluctuate in sign with increasing \(N\); \(z_{cr}\) has become complex. Of course this fluctuation is a remnant of
the complex (and sometimes negative) weights in the partition function. For \( t < t_0 \) the line \( \nu = 0 \) falls in the phase with the conventional thermodynamic limit as can be seen from the solutions \( \text{[10]} \). The line separating the two phases must be in the \( \nu < 0 \) region and, because all the potentials are finite, must therefore have \( \gamma_{\text{str}} = \frac{2}{3} \) thus completing the phase diagram shown in fig.4a.

\[
\begin{array}{c}
\text{a)} \\
1/2 & 3/2 \\
2/3 & 1/2
\end{array}
\]

\[
\begin{array}{c}
\text{b)} \\
1/2 & 3/2 \\
2/3 & 1/2
\end{array}
\]

Figure 4: The phase diagram for a) the complex Ising coupled to binary trees, and b) the complex Ising coupled to the ladder. Numbers indicate values of \( \gamma_{\text{str}} \).

Now consider the behaviour of the model when we weight ladder configurations differently from the rest of the ensemble as in section 3. At \( q = 0 \) only ladder configurations survive; in the thermodynamic limit they are effectively one-dimensional objects, and it is straightforward to compute the transfer matrix to obtain the phase diagram depicted in fig.4b. To elucidate the general structure of the phase diagram for the three coupling constants \( (\nu, \mu, q) \), one would in principle repeat the procedure that lead to equations \( \text{[17]}-\text{[20]} \). However it is clear that the general structure of the equations will be a system of coupled quadratic equations of the form discussed in section 2. Hence we can extend the analysis performed there to the present case.

It is clear that the plane \( q = 0 \) is exceptional; in this plane the coupled equations are in fact linear which is why \( \gamma_{\text{str}} \) takes integer values. Viewed as part of a three coupling constant \( (\nu, \mu, q) \) space, \( \gamma_{\text{str}} = 2 \) can only exist on a plane and \( \gamma_{\text{str}} = 3 \) only on a line. On the other hand of the values for \( \gamma_{\text{str}} \) which occur at \( q = 1 \), \( \gamma_{\text{str}} = \frac{1}{2} \) can exist on a volume, \( \gamma_{\text{str}} = \frac{3}{2} \) on a line, and \( \gamma_{\text{str}} = \frac{2}{3} \) on a plane. If we move from \( q = 0 \) to any positive \( q \), we cannot have \( \gamma_{\text{str}} = 2 \) on a volume and hence the two regions which have that value at \( q = 0 \) disappear for any positive \( q \); the same happens to the line \( \gamma_{\text{str}} = 3 \). The general form of the phase diagram for \( q = 1 \) is maintained for any positive non-zero value of \( q \).

This situation mirrors the case of no matter coupling closely: there is an exceptional plane \( q = 0 \) where, by virtue of the “freezing” of the configurations, the values \( \gamma_{\text{str}} \) takes are different from the ones at positive non-zero \( q \). As soon as one moves
away from that plane, one is in a region where all configurations are important for
the thermodynamics of the ensemble, and the results are qualitatively similar to
those obtained at $q = 1$ where there is no damping of non-ladder configurations.

5 Conclusions

In this paper we investigated models of matter coupled to binary trees. Coupling
the Ising model to such a system does not alter its critical properties, as might
be expected but by introducing a complex-action $Z_3$ model we showed that new
behaviour can arise, giving values of $\gamma_{str}$ different from $\frac{1}{2}$. Interestingly, for a specific
set of coupling constants $\nu = 0$, $t > t_0$, we find $\gamma_{str} = 2$ which is the behaviour
expected when one single type of linear configuration dominates the full partition
function; this is a sort of dimensional collapse driven by interaction with matter
fields, although of a different sort from that studied in [6]. On the other hand it is
also possible to formulate these models to study explicitly the transition between a
fixed configuration, the ladder, and the full fluctuating ensemble. We found that only
when fluctuations are totally suppressed do the critical exponents change relative to
the gravity case.

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