Dynamics of Density Patches in Infinite Prandtl Number Convection

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Abstract

This work examines the dynamics of density patches in the 2D zero-diffusivity Boussinesq system modified such that momentum is in a large Prandtl number balance. We establish the global well-posedness of this system for compactly supported and bounded initial densities, and then examine the regularity of the evolving boundary of patch solutions. For $k \in \{0, 1, 2\}$, we prove the global in time persistence of $C^{k+\mu}$-regularity, where $\mu \in (0, 1)$, for the density patch boundary via estimates of singular integrals. We conclude with a simulation of an initially circular density patch via a level-set method. The simulated patch boundary forms corner-like structures with growing curvature, and yet our analysis shows the curvature will be bounded for all finite times.

1. Introduction

In Earth’s mantle and many highly-pressurized gases, the rate of thermal diffusion $\kappa$ is negligible compared to the rate of momentum dissipation $\nu$. We will analyze the dynamics of density patches, modeling idealized plumes, in such fluids where the nondimensional Prandtl number $Pr = \nu/\kappa$ is large.

Many scenarios of convection, including those which concern the present study, are well-modeled by the Boussinesq system

$$\begin{cases}
\left( \frac{\partial}{\partial t} + u \cdot \nabla \right) \theta = \kappa \Delta \theta, \\
\left( \frac{\partial}{\partial t} + u \cdot \nabla \right) u = \nu \Delta u - \nabla \Pi + \theta e_2,
\end{cases} \quad (B)$$

$$\text{div } u = 0,$$
with initial temperature and velocity \((\theta_0, u_0) : \mathbb{R}^2 \to \mathbb{R} \times \mathbb{R}^2\). The unit vector \(e_2 = (0, 1)\) is the vertical direction antiparallel to gravity, and the pressure \(\Pi\) enforces incompressibility at each instance of time.

For the system \((B)\), global in time regularity has been well established for fixed positive Prandtl number with \(\kappa, \nu\) both positive (see e.g. [1]). The global regularity for \(\kappa, \nu\) both zero (degenerate Pr) remains unknown.

Known is the long time persistence of regularity for the partial viscosity scenarios—\(\kappa\) positive with zero \(\nu\), and \(\nu\) positive with zero \(\kappa\)—given \((\theta_0, u_0) \in H^m(\mathbb{R}^2)\) with integer \(m > 2\) [2,3]. For initial data of such regularity, [2] also provides the negative answer to XXI Century Problem 3 (see [4]) concerning the development of \(\nabla \theta\) singularities in the limit \(\kappa \to 0\) with fixed \(\nu\) positive (formally \(\text{Pr} \to \infty\)).

The question of well-posedness and global in time regularity for less regular initial data has been actively studied (e.g. [5–7]) for the zero diffusivity system

\[
\{\text{system (B) with } \kappa = 0 \text{ and } \nu \text{ positive}\}. 
\]

Without thermal diffusivity, convective plumes are idealized as density distributions initially of the form

\[
\theta_0 = 1_{P_0},
\]

where \(P_0 \subset \mathbb{R}^2\) is simply connected and bounded. If \(u_0\) is sufficiently regular, then the regularity results in [2] guarantee a unique solution to system \((B_1)\) of the form

\[
\theta(t) = 1_{P(t)},
\]

where \(u(t)\) is regular enough to define the flow map \(X(\cdot, t) : \mathbb{R}^2 \mapsto \mathbb{R}^2\) such that the evolving region \(P(t) = X(P_0, t)\) remains simply connected and bounded. However, inferring the regularity of the evolving boundary \(\partial P(t)\) in time is a classical problem. Given \(k \in \mathbb{Z}_+ = \{1, 2, \ldots\}\) and \(\mu \in (0, 1)\), we write \(\partial P(t) \in C^{k+\mu}\) if there exists some \(z : \mathbb{S}^1 \to \mathbb{R}^2\) such that

\[
\partial P(t) = \{z(\alpha) \mid \alpha \in \mathbb{S}^1\} \text{ and } z \in C^{k+\mu}(\mathbb{S}^1).
\]

The question of persistence of regularity for the patch boundary is precisely whether \(\partial P(t) \in C^{k+\mu}\) given \(\partial P(0) = \partial P_0 \in C^{k+\mu}\). This line of inquiry has its origin in the study of vortex patches in the two-dimensional incompressible Euler system. At some point, numerical studies like [8,9] were done which had conflicting conclusions on whether finite time contour singularities formed in the evolution of a vortex patch. Then, [10] showed global in time persistence of smoothness using paradifferential calculus and striated regularity methods to obtain the desired regularity of \(\nabla u\); soon after, [11] proved a similar result using geometric analysis techniques: the vortex patch boundary remains \(C^{1+\mu}\) for \(\mu \in (0, 1)\), given it was so initially. Another proof of this was given by [12].

More recently for the system \((B_1)\), the global persistence of \(C^{1+\mu}\)-regularity [13], \(C^{2+\mu}\)-regularity [14], and \(C^{k+\mu}\)-regularity for all \(k \in \mathbb{Z}_+\) [15] was shown for density patches. The \(k = 1\) and \(k \in \mathbb{Z}_+\) results made special use of striated regularity estimates in Besov spaces.
The present study connects this inquiry of contour singularity formation to the program of XXI Century Problem 3. In the limiting dynamics of large Prandtl number, the fluid velocity is at relative equilibrium in the thermal diffusive time scale \( \tau_\kappa = \frac{L^2}{\kappa} \), where \( L \) is the length scale. The nondimensional momentum equation is

\[
\frac{1}{\Pr} \left( \frac{\partial}{\partial t^*} + u \cdot \nabla^* \right) u = \Delta^* u - \nabla^* \Pi + Ra \theta e_2, \tag{*}
\]

where the starred derivatives have been rescaled by \( \tau_\kappa \) and \( L \), and the variables with underbars are the nondimensional counterparts to \((u, \theta, \Pi)\). The Rayleigh number \( Ra \) is independent to \( \Pr \). Thus formally the material derivative in (*) vanishes in the limit of large Prandtl number; specifically, in thermal diffusive time \( t^* = t/\tau_\kappa \), the fluid is in equilibrium dominated by viscosity. Accordingly, we consider the (dimensional) zero diffusivity system \((B_1)\) with the modified momentum equation

\[
-\Delta u + \nabla \Pi = \theta e_2. \tag{3}
\]

We thus consider patch dynamics in the system given by

\[
\begin{align*}
\left( \frac{\partial}{\partial t} + u \cdot \nabla \right) \theta &= 0, \\
-\Delta u + \nabla \Pi &= \theta e_2, \\
\text{div} u &= 0,
\end{align*} \tag{B^*}
\]

with initial data \( \theta_0 : \mathbb{R}^2 \to \mathbb{R} \). Above, the solution \( \theta(x, t) \) is an active scalar function of \( x \in \mathbb{R}^2 \) and \( t \in \mathbb{R} \), where the corresponding velocity \( u(t) : \mathbb{R}^2 \to \mathbb{R}^2 \) is given by the balance (3) at each instance of time.

The system \((B_*)\) is known also as transport-Stokes, and can model the sedimentation of inertialess particles in Stokes flow [16,17]. Let us briefly recount the literature concerning global well-posedness of this system in unbounded domains. Global well-posedness for \( L^\infty \) initial data holds on the infinite strip \( \mathbb{R} \times (0, 1) \) with no-slip boundary conditions and a flux condition imposed [18]. On the whole space \( \mathbb{R}^3 \), the analogue of system \((B_*)\) is globally well-posed for initial data in \( L^1 \cap L^\infty \) [19]; however, whether this is true in \( 2d \) without additional assumptions is unknown. Recent discussions of the well-posedness problem may be found in [20,21].

The question of regularity for an interface of density in the transport-Stokes system was first addressed in [22]. On bounded domains in \( \mathbb{R}^d \) with \( d \geq 2 \), [22] establishes global well-posedness for piecewise constant initial densities, where the boundary conditions are no-slip, and then shows that interfaces which are initially \( C^{1+\mu} \), will remain so for all time. More recently, [23] considers a contour dynamics equation derived from system \((B_*)\) on the horizontally periodic strip \( T \times \mathbb{R} \), wherein interfaces are shown to be locally well-posed in \( C^{1+\mu} \), even in the Rayleigh-Taylor unstable regime.

Our contributions concern the system \((B_*)\) on the whole space \( \mathbb{R}^2 \). We first establish the global well-posedness of the system for initial data of Yudovich-type,
meaning when \( \theta_0 \in L^1 \cap L^\infty \) is compactly supported. In particular, initial data (1) admit patch solutions (2) and we may study the regularity of \( \partial P(t) \) in time. Then, we prove the global persistence of \( C^{k+\mu} \) regularity for \( \partial P \) when \( k \in \{0, 1, 2\} \).

Further, we simulate on \( \mathbb{T}^2 \) the dynamics of an initially circular patch, and observe the interface develop corner-like structures with growing curvature. The results here rule out the possibility of a finite-time curvature singularity; however, corner formation at infinite time remains possible.

The rest of this paper is organized as follows: the main contributions are presented in Sect. 2. The key Lemmas 1 and 2 are proved, giving the details of the relevant singular integrals, in Sect. 3. The global well-posedness of the system \( (B_\ast) \), in particular Theorem 3 concerning the Yudovich class, is addressed in Sect. 4. In Sect. 5, the main Theorem 6 on the persistence of regularity for the evolving boundary \( \partial P(t) \) is finally proved. In Sect. 6, the numerical simulation in Fig. 1 is described in detail.

2. Main Results

We suppose an initial distribution (1) in the Cauchy problem for system \( (B_\ast) \) and show the ensuing solution velocity \( u(t) \) is regular enough such that the solution temperature has patch representation (2), where the evolving region

\[
P(t) = X(P_0, t)
\]

is given by the flow map \( X(\cdot, t) \) of our fluid velocity.

We start by introducing the streamfunction \( \psi \) such that \( u = \nabla^\perp \psi \) with \( \nabla^\perp = (-\partial_2, \partial_1) \), and then the momentum equation in \( (B_\ast) \) yields the following expression for vorticity \( \omega \equiv \nabla^\perp \cdot u \) at each instance of time:

\[
-\Delta \omega = \partial_1 \theta.
\]

The fundamental solution of the bilaplacian \( \Delta^2 \) in \( \mathbb{R}^2 \) is

\[
\frac{1}{8\pi} |z|^2 \left( \log |z| - 1 \right),
\]

where \( z \in \mathbb{R}^2 \). Given the fact \( \Delta \psi = \omega \) generally, and assuming \( \theta \) in equation (5) has enough decay at infinity, it follows \( \psi = -\Delta^{-1} \partial_1 \theta \) such that \( \psi = K \ast \theta \) with kernel

\[
K(z) = -\frac{z_1}{4\pi} \left( \log |z| - \frac{1}{2} \right).
\]

More precisely, we find the expression for vorticity

\[
\omega(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x_1 - y_1}{|x - y|^2} \theta(y) \, dy,
\]

and note that the integral converges absolutely for \( \theta \) in \( Y \equiv L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \). Denoting the integral as \( \omega = Q \ast \theta \), this is a convolution with a singular kernel
which is homogeneous of degree $-1$. Thus, the operator $(Q^*) : Y \to L^p$ giving vorticity from temperature is bounded for $2 < p \leq \infty$; we give a direct proof of this claim in Sect. 3.

Let $G(z) \equiv \nabla \perp K(z)$. Then the integral $u = G \ast \theta$, explicitly

$$u(x) = \int_{\mathbb{R}^2} G(x - y) \theta(y) \, dy,$$  

converges absolutely if $\theta \in Y$ is compactly supported. More quantitatively, we have

**Lemma 1.** Suppose $\theta \in L^\infty$ with $\text{spt} \, \theta \subset B(0, R)$. Let $u = G \ast \theta$. Then,

$$|u(x)| \leq C(1 + \log(|x| + R + 1)) \|\theta\|_Y.$$

From here, we find the gradient as $\nabla u = \nabla G \ast \theta$, and further, we compute the principal value of the second gradient of velocity $\nabla \nabla u(x)$. With the kernel $\nabla \nabla G : \mathbb{R}^2 \to \mathbb{R}^{2 \times 2 \times 2}$, we have

$$\nabla \nabla u(x) = \theta(x) E + \frac{1}{4\pi} \text{pv} \int_{\mathbb{R}^2} \theta(x-z) \nabla \nabla G(z) \, dz.$$  

(10)

The entries of the $E \in \mathbb{R}^{2 \times 2 \times 2}$, as well as $G$ and its gradients, $\nabla G$ and $\nabla \nabla G$, are given in Sect. 3. Of special note here is that the integral operator given by kernel $\nabla \nabla G$ is Calderón-Zygmund in each entry, thus bounded on $L^p$ for $p \in (1, \infty)$. Combining this with the previous observations, we deduce

**Lemma 2.** Suppose $\theta \in Y$. Let $\nabla u = \nabla G \ast \theta$. Then for $\mu \in (0, 1)$, there exists constant $C_\mu$ depending only on $\mu$ such that

$$\|\nabla u\|_{C^\mu} \leq C_\mu \|\theta\|_Y.$$

Initial data which satisfy the hypotheses of both Lemmas 1 and 2, namely compactly supported and bounded functions, are called Yudovich-type due to the classical result [24] establishing the global well-posedness of weak solutions within this class for the two-dimensional Euler system. This result allowed the study of long-time dynamics for vortex patches, in particular the program of vortex patch boundary regularity. For the system ($B_\ast$), the initial data of our concern are clearly Yudovich-type, and the Lemmas are powerful enough to establish the following:

**Theorem 3.** Let $\theta_0 \in L^1 \cap L^\infty$ and $\text{spt} \, \theta_0 \subset B(0, R_0)$. Then for arbitrary $T$, the system ($B_\ast$) has a unique weak solution

$$\theta \in L^\infty(0, T; L^1 \cap L^\infty).$$

Moreover, for $t \leq T$, the following estimates hold:

1. For $1 \leq p \leq \infty$,

$$\|\theta(t)\|_{L^p} \leq \|\theta_0\|_{L^p}$$
2. For \( R(0) = R_0 \), we have

\[ \text{spt} \, \theta(t) \subset B(0, R(t)) \]

with \( R(t) \) obeying the differential inequality

\[ \frac{dR}{dt}(t) \leq C(1 + \log(2R(t) + 1))\|\theta_0\|_Y. \]

3. For \( \mu \in (0, 1) \),

\[ \|\nabla u(t)\|_{C^\mu} \leq C_\mu \|\theta_0\|_Y \]

where \( u(t) = G \ast \theta(t) \).

**Corollary 4.** Let \( \theta_0 = 1_{P_0} \) where \( P_0 \subset \mathbb{R}^2 \) is simply connected and bounded. Then, the unique weak solution \( \theta \) to system \( (B_*) \) has the form

\[ \theta(t) = 1_{P(t)} \]

where \( P(0) = P_0 \) and \( P(t) \) is simply connected and bounded.

The corollary is due to an explicit construction: we represent the dynamics of the distribution \( 1_{P(t)} \) via the evolution of a compactly supported level-set function \( \varphi(t) \in C^k_c \). Specifically, we suppose \( \varphi_0 \in C^k_c \) depending on the regularity of the initial boundary (it is at least continuous) such that

\[ P_0 = \{ x \mid \varphi_0(x) > 0 \} \quad \text{and} \quad \partial P_0 = \{ x \mid \varphi_0(x) = 0 \}, \]

for \( P_0 \) in the claim. Then, we let \( \varphi \) be transported passively by \( u(t) = G \ast \theta(t) \) where \( \theta(t) \in Y \) is the unique weak solution with initial data \( 1_{P_0} \) guaranteed to exist by Theorem 3. Since \( u(t) \) is locally bounded and \( \nabla u(t) \in C^\mu \) for all time, we have \( \varphi \) as the unique global solution the Cauchy problem given by

\[ \left( \frac{\partial}{\partial t} + u \cdot \nabla \right) \varphi = 0, \]

with initial data \( \varphi_0 \). Further, we have explicitly \( \varphi(t) = \varphi_0 \circ X^{-1}(\cdot, t) \) where \( X \) is the well-defined flow map given by \( u \). It follows that the patch representation \( \theta(t) = 1_{P(t)} \) holds for all time, where \( P(t) = X(P_0, t) \) is defined by

\[ P(t) = \{ x \mid \varphi(x, t) > 0 \} \quad \text{and} \quad \partial P(t) = \{ x \mid \varphi(x, t) = 0 \}. \]

With global well-posedness of patches solutions to system \( (B_*) \) established, we may now examine the question of global in time regularity for the evolving boundary \( \partial P(t) \). We show first the local propogation of Hölder continuity as follows:

**Corollary 5.** Suppose that \( \partial P_0 \) is \( \mu \)-Hölder continuous at \( \tilde{x} \) with \( \mu \in (0, 1) \). Then, \( \partial P(t) \) is \( \mu \)-Hölder continuous at \( X(\tilde{x}, t) \) at any given time.
This result follows from a Lagrangian construction of \( \partial P(t) \). Rather than the construction (13), we consider some initial parametrization \( z_0 : S^1 \to \partial P_0 \). We observe that the flow map \( X(a, t) \) for fluid velocity \( u = G \ast 1_p \) then gives us the evolving patch boundary

\[
\partial P(t) = \{ X(z_0(\alpha), t) \mid \alpha \in S^1 \},
\]

such that

\[
z(t) = X(\cdot, t) \circ z_0
\]

parametrizes \( \partial P(t) \) for any and all time. We note that for all times the image of \( z(t) \) must be in \( B(0, R(t)) = \{ |x| \leq R(t) \} \) (16) wherein \( u(t) \) is \( C^{1+\mu}(B(0, R(t))) \), by Theorem 3. Thus if \( z_0 \) is \( \mu \)-Hölder continuous at \( \tilde{\alpha} \), then the composition of maps giving \( z(t) \) is \( \mu \)-Hölder continuous at \( \tilde{\alpha} \). Arranging \( z_0(\tilde{\alpha}) = \tilde{x} \), we have established Corollary 5.

Let us briefly demonstrate the difficulty in proving the persistence of higher regularity in the Lagrangian construction by examining the contour dynamics equation (CDE) for system \((B_\lambda)\), which we proceed to derive.

By its construction (15), \( z(t) \) obeys the evolution equation

\[
\frac{dz}{dt} = u(z(t), t).
\]

Suppose \( z_0 \) is a parametrization by arc length, such that \( z(t) \) is also. Examining the expression for velocity \( u(t) = \nabla \perp K \ast 1_p(t) \), we apply Green’s theorem to discover

\[
u(x, t) = \int_{P(t)} \nabla \perp K(x - y) \, dy = -\int_0^{2\pi} K(x - z(\sigma, t)) \frac{\partial z}{\partial \alpha}(\sigma, t) \, d\sigma,
\]

where \( z(t) \) is assumed to be clockwise oriented. Suppressing the time argument, we now have the contour dynamics equation

\[
\frac{\partial z}{\partial t}(\alpha) = -\int_0^{2\pi} K(z(\alpha) - z(\sigma)) \frac{\partial z}{\partial \alpha}(\sigma, t) \, d\sigma.
\]

(CDE)

So, we differentiate the above equation in \( \alpha \) to find the evolution equation

\[
\frac{\partial^2 z}{\partial t \partial \alpha}(\alpha) = -\int_0^{2\pi} \left[ \nabla K(z(\alpha) - z(\sigma)) \cdot \frac{\partial^2 z}{\partial \alpha}(\alpha) \right] \frac{\partial z}{\partial \alpha}(\sigma) \, d\sigma, \tag{19}
\]

where explicitly

\[
\nabla K(z) = \frac{1}{8\pi} \begin{pmatrix} -2 \log |z| & 0 \\ 0 & \frac{1}{|z|^2} \left( z_1^2 - z_2^2 \right) \end{pmatrix}.
\]

From here, we see that proving the global in time well-posedness of the CDE by itself is difficult, even more so is showing the global in time persistence of \( C^\mu \) regularity for \( \partial_\alpha z(t) \) evolving via (19). This challenge is typical of contour dynamics models.
Theorem 6. Let $k \in \{0, 1, 2\}$ and suppose that $\partial P_0 \in C^{k+\mu}$ with $\mu \in (0, 1)$. Then, $\partial P(t) \in C^{k+\mu}$ for all time.

The higher regularity results follow from an analysis of the Eulerian construction of $\partial P$ in Corollary 4. Recalling the level set function $\varphi$, we note the direction of the vector field $W = \nabla \perp \varphi$ is tangent to $\partial P = \partial P(t)$ in general. We then have the parametrization

$$z : S^1 \mapsto \partial P \quad \text{with} \quad \frac{\partial z}{\partial \alpha} = W \circ z,$$

(21)

given $W$ is bounded and non-vanishing everywhere on $\partial P$. Thus to guarantee $\partial P \in C^{1+\mu}$ for $\mu \in (0, 1)$, the relevant quantities to control are $|W(t)|_{L^\infty}$ and

$$|W|_{\text{inf}} := \inf_{x \in \partial P} |W(x)| = \inf_{x, \varphi(x) = 0} |W(x)|,$$

(22)

$$|W|_{\mu} := \sup_{x \neq x'} \frac{|W(x) - W(x')|}{|x - x'|^{\mu}}, \quad \mu \in (0, 1).$$

(23)

Differentiating (12), the evolution of $W(t)$ where $W_0 = \nabla \perp \varphi_0$ obeys

$$\left( \frac{\partial}{\partial t} + u \cdot \nabla \right) W = \nabla u \cdot W.$$

(24)

As transport preserves $|\theta(t)||_{L^p}$, the estimate of Lemma 2 gives immediately $|\nabla u(t)||_{L^\infty}$ is bounded uniformly in time by a positive constant $C_L$ depending only on the initial data. This is already sufficient to achieve $\partial P \in C^{1+\mu}$ with an $\exp(C_L \exp(C_L|t|))$ bound for $|W(t)|_{\mu}$ from Grönwall-type inequalities, see [11, Proposition 3].

With the geometrical insights of [11], we may improve this initial bound for $|W(t)|_{\mu}$ to an $\exp(C_L|t|)$ bound. Further more, these geometrical methods are used to prove persistence of $C^{2+\mu}$-regularity. Differentiating (24) yields an evolution equation,

$$\left( \frac{\partial}{\partial t} + u \cdot \nabla \right) \nabla W = [\nabla W, \nabla u] + \nabla \nabla u \cdot W,$$

(25)

where we have the tensor product commutator $[A, B] = A \cdot B - B \cdot A$. The term to control is the product $\nabla \nabla u \cdot W$, therein we observe the expression of $\nabla \nabla u$ for the patch $\theta = 1_P$ is

$$\nabla \nabla u(x) = 1_P(x)E + \frac{1}{4\pi} \text{pv} \int_{\mathbb{R}^2} 1_P(x - z) \nabla G(z) \, dz.$$  

(26)

Clearly estimating $|\nabla \nabla u|_{\mu}$ from this expression is difficult, but we use the fact that the vector field $W \in C^\mu(\mathbb{R}^2, \mathbb{R}^2)$ is divergence-free and tangent to $\partial P$ to reduce the problem to estimating $||\nabla \nabla u||_{L^\infty}$ (Corollary 23).

Beyond satisfying the cancellation property, the CZ kernel $\nabla \nabla G(z)$ has reflection symmetry such that we have $H : S^1 \mapsto \mathbb{R}^{2 \times 2 \times 2}$ and

$$\nabla \nabla G(z) = \frac{H(z)}{|z|^2} \quad \text{with} \quad H(-z) = H(z).$$

(27)
Because small neighborhoods of $\partial P$ look like half-circles, the reflection symmetry allows us to achieve a sufficient $L^\infty$ estimate on $\nabla^2 u(t)$ in Proposition 25. Ultimately, we show $|\nabla W(t)|_\mu$ may grow like $|t| \exp(C_3|t|)$, where $C_3$ depends only on $\mu$ and the initial data, and the global in time persistence of $C^{2+\mu}$ regularity for $\partial P(t)$ is proved. The complete details are given in Sect. 5.

While the simulations of [8,9] were initialized with two circular vortex patches of positive sign, a single circular patch of density is already unsteady in system $(B_\ast)$. Observe that for the initial data $\theta_0 = 1_{B(0,1)}$, the initial vorticity $\omega_0$ evaluated on the contour $z_0(\alpha) = (\sin \alpha, \cos \alpha)$,

$$
\omega_0 \circ z_0 = \int_0^{2\pi} \int_0^1 H(2r \sin(\alpha + \theta) - r^2) \cos \theta \, dr \, d\theta
$$

(28)

where $H$ is the Heaviside function, is not symmetric in $\alpha$. Thus the initially circular patch will not remain circular. More generally, for a nonnegative and nontrivial $\theta_0$, the center of mass

$$
q = \frac{1}{M} \int_{\mathbb{R}^2} \theta x \, dx,
$$

(29)

where $M = ||\theta_0||_{L^1}$, drifts upwards with vertical component $q_2$ increasing as

$$
\frac{dq_2}{dt} = -\frac{1}{M} \int_{\mathbb{R}^2} \text{div} (\theta u)x_2 \, dx = \frac{1}{M} \int_{\mathbb{R}^2} \theta u_2 \, dx > 0
$$

(30)

because $u_2 = G_2 * \theta$ is given by a positive operator on $\theta$. If the initial data has a horizontal reflection symmetry, the horizontal component $q_1$ is stationary.

Let us consider the system $(B_\ast)$ on $T^2$ such that the spatial domain is now computationally realizable. Observing that the problem is Galilean invariant, we choose a reference frame so that the mean velocity $\int_{T^2} u \, dx = \hat{u}(0)$ is zero. Then from integrating the momentum equation, we see that $\hat{u}(0)$ is indeed constant in time, and that we must have $\int_{T^2} \theta \, dx = 0$ for compatibility. Clearly, the arguments giving the main results, in particular Lemmas 1 and 2, hold readily on the domain $T^2$ wherein $L^\infty \subset L^1$.

**Corollary 7.** Let $\theta_0 \in L^\infty(T^2)$ such that

$$
\int_{T^2} \theta_0 \, dx = 0.
$$

Then, Theorem 3 holds for the system $(B_\ast)$ on $T^2$.

Our definition of patches must be modified to be mean zero in density. On the flat torus, the compatible patch data is

$$
\theta_0 = 1_{P_0} - \text{area}(P_0)
$$

(31)

where $P_0 \subset T^2$ is simply connected. For such initial data, the unique weak solution $\theta$ to system $(B_\ast)$ on $T^2$ then has the form

$$
\theta(t) = 1_{P(t)} - \text{area}(P_0),
$$

(32)
\[ \frac{\partial P(t = 0)}{t = 25} \quad t = 50 \quad t = 75 \quad t = 100 \]

**Fig. 1.** Snapshots of \( \partial P \) from a numerical patch solution \((N = 1024)\) to system \((B_*)\) on \(T^2\) for \(\partial P_0 = S^1(\frac{1}{2})\). For presentation, the axes are omitted and the vertical positions of curve \(\partial P(t)\) have been aligned between snapshots. Simulation details are given in Sect. 6.

where \( P(0) = P_0 \) and \( P(t) \) is simply connected. On this numerically tractable domain, we here implement a level-set method that approximates the evolution of the boundary \(\partial P(t)\) which we know is well-defined for all time.

**Corollary 8.** For the system \((B_*)\) on \(T^2\), if the initial data \(\theta_0\) has the form \((31)\), then the unique weak solution \(\theta\) in Corollary 7 has the form \((32)\).

The numerical method evolves a discrete level-set function \(\phi_{ij}(t)\) on a fixed \(N \times N\) uniform grid, discretizing \(T^2\), according to \((12)\). The scheme for solving \((12)\) is described in [25, Part II] and is first-order in space with respect to the uniform grid spacing \(h = 1/N\). The discrete fluid velocity \(u_{ij}\) is obtained from \((3)\) using a spectral collocation method, and time integration is done with Heun’s method, which is second-order and strong-stability preserving. Overall, the numerical solver for \((B_*)\) is second-order in time, first-order in space.

Using this algorithm, we compute the dynamics of the temperature patch which is initially circular. The initial curve \(\partial P_0\) is fixed as the embedded circle of radius one-half \(S^1(\frac{1}{2})\). The result of the simulation is presented in Fig. 1. We observe that the curve \(\partial P(t)\) forms corner-like structures in its evolution; however, our persistence of regularity results, Corollary 5 and Theorem 6, hold for \(\partial P(t)\) in this setting as well. Thus, if in light of Fig. 1 one asks whether there develops a curvature singularity in finite time, we provide proof that the curvature remains bounded for all time.

### 3. Proof of Lemmas 1 and 2

The expressions with singular integrals allow us to sufficiently control \(u\) and \(\omega\) via the bounds that follow.

**Proof of Lemma 1.** We write, explicitly,

\[
G(z) = \nabla K(z) = \frac{1}{8\pi} \left( \frac{2\hat{z}_1\hat{z}_2}{1 - 2 \log |z| - 2\hat{z}_1^2} \right),
\]

where \(\hat{z}_j = z_j/|z|\), and we observe that

\[
|G(x - y)| \leq C(1 + |\log |x - y||).
\]
It follows immediately that

\[
\int_{|x-y| \leq 1} |G(x-y)\theta(y)| \, dy \leq C \|\theta\|_{L^\infty}. \tag{35}
\]

Further if \(|x| \geq R + 1\), then \(1 \leq |x-y| \leq 2|x|\), and we can bound \(|G(x-y)|\) by \(C(1 + \log(2|x|))\). Now consider \(|x| \leq R + 1\) with \(|x-y| \geq 1\), we have instead the bound \(C(1 + \log(2R + 1))\). Combining these estimates, we deduce that

\[
\int_{|x-y| \geq 1} |G(x-y)\theta(y)| \, dy \leq C(1 + \log(|x| + R + 1))\|\theta\|_{L^1}. \tag{36}
\]

The result follows from (35) and (36).

**Proposition 9.** Suppose \(\theta \in Y\). Let \(\omega = Q * \theta\). Then for \(2 < p \leq \infty\),

\[
\|\omega\|_{L^p} \leq C_p \|\theta\|_Y.
\]

**Proof.** We split the integral

\[
-2\pi \omega(x) = \int_{|x-y| < 1} \frac{x_1 - y_1}{|x-y|^2} \theta(y) \, dy + \int_{|x-y| \geq 1} \frac{x_1 - y_1}{|x-y|^2} \theta(y) \, dy, \tag{37}
\]

and so deduce

\[
|\omega(x)| \leq \frac{\sqrt{2}}{2\pi} \left( \|\theta\|_{L^1} + 4\|\theta\|_{L^\infty} \right). \tag{38}
\]

The case \(p = \infty\) follows.

Let \(B \subset \mathbb{R}^2\) be the unit ball centered at the origin. Then we have

\[
\omega(x) = \omega 1_B(x) + \omega 1_{\mathbb{R}^2 \setminus B}(x) \tag{39}
\]

pointwise. From (38), it follows that

\[
\|\omega 1_B\|_{L^p} \leq C_p \|\theta\|_Y. \tag{40}
\]

and the convolution \(\omega = Q * \theta\) converges absolutely. Accordingly, we find

\[
\omega 1_{\mathbb{R}^2 \setminus B} = (Q 1_{\mathbb{R}^2 \setminus B}) * \theta, \tag{41}
\]

where the truncated kernel is

\[
Q 1_{\mathbb{R}^2 \setminus B}(z) = \begin{cases} Q(z), & \text{if } z \in \mathbb{R}^2 \setminus B, \\ 0, & \text{otherwise} \end{cases} \tag{42}
\]

Let \(2 < p < \infty\), such that \(|Q 1_{\mathbb{R}^2 \setminus B}|^p\) is integrable. We conclude

\[
\|\omega 1_{\mathbb{R}^2 \setminus B}\|_{L^p} \leq C_p \|\theta\|_{L^1}, \tag{43}
\]

from Young’s inequality for convolutions (e.g. see [26, Appendix A]). □
**Proposition 10.** Suppose $\theta \in Y$. Let $\nabla u = \nabla G \ast \theta$. Then for $2 < p \leq \infty$,

$$\|\nabla u\|_{L^p} \leq C_p \|\theta\|_Y.$$  

**Proof.** The singular kernel $G$ away from the origin has gradient $\nabla G$, with homogeneity of degree $-1$, such that, for integrable and bounded $\theta$, we have the absolutely convergent integral

$$\nabla u(x) = \int_{\mathbb{R}^2} \nabla G(x - y) \theta(y) \, dy,$$  

where we compute the kernel explicitly as

$$\nabla G(z) = -\frac{1}{4\pi |z|^4} \left( z_2(z_1^2 - z_2^2) \ z_3^3 + 3z_1z_2^2 \right) \left( z_1(z_2^2 - z_1^2) \ z_2(z_2^2 - z_1^2) \right).$$  

(45)

Each entry of $\nabla G$ has the same cancellation property and homogeneity as $Q$, so we conclude as in the proof of Proposition 9. 

**Proposition 11.** Suppose $\theta \in L^p$ for some $p \in (1, \infty)$. Let $u = G \ast \theta$. Then,

$$\|\nabla \nabla u\|_{L^p} \leq C_p \|\theta\|_{L^p}.$$  

**Proof.** For $\nabla \nabla u = (\partial_1 \nabla u, \partial_2 \nabla u)$, we differentiate carefully to resolve the strongly singular kernel. In particular, let $z = x - y$ such that

$$\partial_1 \nabla u(x) = \int_{\mathbb{R}^2} \nabla G(z) \partial_1 \theta(x - z) \, dz$$  

(46)

$$= \lim_{\epsilon \to 0} \int_{|z| \geq \epsilon} \frac{\partial}{\partial z_1} (\nabla G(z) \theta(x - z)) + \theta(x - z) \frac{\partial}{\partial z_1} \nabla G(z) \, dz. $$  

(47)

The first integral is

$$= \lim_{\epsilon \to 0} \int_{|z| = \epsilon} \nabla G(z) \theta(x - z)(-n_1) \cdot d\sigma$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{|z| = \epsilon} z_1 \nabla G(z) \theta(x - z) \, d\sigma$$

(48)

$$= \frac{1}{8} \begin{pmatrix} 0 & -3 \\ 1 & 0 \end{pmatrix} \theta(x).$$

We compute

$$\partial_1 \nabla G(z) = \frac{1}{4\pi} \frac{H_1(z)}{|z|^2} \quad \text{and} \quad \partial_2 \nabla G(z) = \frac{1}{4\pi} \frac{H_2(z)}{|z|^2}$$  

(49)

where each entry of $H : \mathbb{R}^2 \to \mathbb{R}^{2 \times 2}$, explicitly

$$H_1(z) = \frac{1}{|z|^4} \begin{pmatrix} 2z_1z_2(z_1^2 - 3z_2^2) z_4^4 + 6z_1^2z_2^2 - 3z_4^4 \\ z_4^4 - 6z_1^2z_2^2 + z_2^4 \end{pmatrix},$$

(50)
and
\[ H_2(z) = \frac{1}{|z|^4} \left( -z_1^4 + 6z_1^2z_2^2 - z_1^2 2z_2(z_1z_2^2 - z_1^3) \right) \]
(51)
is homogeneous of degree zero, mean zero on the unit sphere, and symmetric with respect to reflection. It follows
\[ \frac{\partial}{\partial_1} \nabla u(x) = \frac{1}{8} \left( \begin{array}{c} 0 -3 \\ 1 0 \end{array} \right) \theta(x) + \frac{1}{4\pi} \text{pv} \int_{\mathbb{R}^2} \frac{H_1(z)}{|z|^2} \theta(x - z) \, dz, \]
(52)
and
\[ \frac{\partial}{\partial_2} \nabla u(x) = \frac{1}{8} \left( \begin{array}{c} 1 0 \\ 0 -1 \end{array} \right) \theta(x) + \frac{1}{4\pi} \text{pv} \int_{\mathbb{R}^2} \frac{H_2(z)}{|z|^2} \theta(x - z) \, dz. \]
(53)
With the appropriate \( E \in \mathbb{R}^{2 \times 2 \times 2} \), we may write, compactly,
\[ \nabla \nabla u(x) = \theta(x) E + \frac{1}{4\pi} \text{pv} \int_{\mathbb{R}^2} \frac{H(z)}{|z|^2} \theta(x - z) \, dz. \]
(54)
We conclude by examining this expression as a sum of operators. The first summand is bounded like the identity, and the second is a Calderón–Zygmund operator.

Corollary 12. Suppose \( \theta \in Y \). Let \( \nabla u = \nabla G * \theta \). Then for \( p \in (2, \infty) \)
\[ \| \nabla u \|_{W^{1,p}} \leq C_p \| \theta \|_Y. \]

Proof of Lemma 2. To deduce the desired Hölder continuity of \( \nabla u \), we observe Corollary 12 and recall Morrey’s embedding: \( W^{1,p} \subset C^\mu \) for \( \mu = 1 - 2/p \) with \( p > 2 \).

4. Proof of Theorem 3

Let us first define what we mean by a (Yudovich) weak solution.

Definition 1. Let \( \theta_0 \in Y \), then \( \theta(x, t) \) is a weak solution of system \((B_u)\) given
\[ \theta \in L^\infty(0, T; L^1 \cap L^\infty) \]
for some \( T > 0 \), and that, for any \( \phi \in C^1(0, T; C^1_c) \), holds that
\[ \int_{\mathbb{R}^2} \theta(x, T) \phi(x, T) \, dx - \int_{\mathbb{R}^2} \theta_0 \phi(x, 0) \, dx = \int_0^T \int_{\mathbb{R}^2} \theta \left( \frac{\partial}{\partial t} + u \cdot \nabla \right) \phi \, dx \, dt, \]
where \( u(t) = G * \theta(t) \).
Notice from the definition that Lemma 2 plays a powerful role in proving the well-posedness of weak solutions with compactly supported initial data. If we can maintain that \( \theta(t) \) remains compactly supported in its evolution, then the characteristics for \( \theta \) have sufficient regularity to preserve \( L^p \) norms via the transport equation. With this observation, we obtain a sequence of smooth, compactly supported initial data \( \theta_0^\epsilon \) with regularization parameter \( \epsilon \). For such data, we obtain a family of global smooth solutions \( \theta^\epsilon \) with control of

\[
spt \theta^\epsilon(t) \subset B(0, R(t)) \quad \text{and} \quad u^\epsilon(t) \in C^{1+\mu}(B(0, R(t)))
\]  

(55)

for all time, depending only on \( \|\theta_0\|_Y \). Then, the limit \( \epsilon \to 0 \) yields a Yudovich weak solution via the classical arguments in [24]. The limiting solution \( \theta \) (possibly non-unique) inherits these uniform controls, and these controls are strong enough to achieve uniqueness. This section is dedicated to proving the claim of global well-posedness of classical solutions with uniform in \( \epsilon \) control of (55) (Proposition 18).

We begin by considering solutions to the sequence of linear equations

\[
\left( \frac{\partial}{\partial t} + u^n \cdot \nabla \right) \theta^{n+1} = 0,
\]

(56)

where \( u^n(t) := G * \theta^n(t) \) for positive integers \( n \), defined inductively. The initial data are fixed as \( \theta^n(0) = \theta_0 \) uniformly in \( n \), where

\[
\theta_0 \in H^1 \cap Y, \quad \text{and} \quad spt \theta_0 \subset B(0, R_0).
\]

(57)

For \( n = 0 \), take \( \theta^0(t) = \theta_0 \).

**Proposition 13.** The sequence of functions \( \theta^n \) described above is well defined. Moreover, the following uniform estimates hold.

1. For \( 1 \leq p \leq \infty \),

\[
\|\theta^n(t)\|_{L^p} \leq \|\theta_0\|_{L^p}.
\]

2. For \( R(0) = R_0 \), we have

\[
spt \theta^n(t) \subset B(0, R(t))
\]

with \( R(t) \) obeying the differential inequality

\[
\frac{dR}{dt}(t) \leq C(1 + \log(2R(t) + 1))\|\theta_0\|_Y.
\]

3. For \( \mu \in (0, 1) \),

\[
\|\nabla u^n(t)\|_{C^\mu} \leq C_\mu \|\theta_0\|_Y
\]

where \( u^n(t) = G * \theta^n(t) \).
Proof. For \( n = 0 \), all the estimates are immediate as \( \theta^0 = \theta_0 \). Now letting \( n \geq 0 \), suppose \( \theta^n \) satisfies the estimates of the theorem.

For the first estimate, the regularity of \( u^n \) in the linear transport equation (56) produces a unique global solution \( \theta^{n+1} \), and \( \|\theta^{n+1}\|_{L^\infty} \) is preserved along characteristics. Further, we may multiply (56) by \( \theta |\theta|^{p-2} \), for \( p \geq 2 \), and integrate by parts to discover

\[
\frac{d}{dt}\|\theta^{n+1}\|_{L^p} \leq 0, \quad (58)
\]

since \( u^n \) is divergence-free. We deduce \( \theta^{n+1} \) also satisfies Estimate 1, as desired.

For the second estimate, we observe \( u^n = G * \theta^n \) implies

\[
|u^n(x, t)| \leq C(1 + \log(|x| + R(t) + 1))\|\theta^n(t)\|_Y. \quad (59)
\]

Indeed \( \theta^n \in Y \), and the previous arguments produce

\[
\|\theta^{n+1}(t)\|_Y \leq \|\theta_0\|_Y. \quad (60)
\]

Moreover, \( \theta^{n+1} \) is transported by \( u^n \), which we showed is bounded. Let \( X(a, t) \) be the flow map generated by \( u^n \) with labels \( a \in \mathbb{R}^2 \), such that

\[
\frac{dX}{dt}(a, t) = u^n(X(a, t), t), \quad (61)
\]

where \( X(a, 0) = a \). Observe

\[
\text{spt} \ \theta^{n+1}(t) \subset \{ x \mid x = X(a, s), \ s \in [0, t], \ a \in B(0, R_0) \}, \quad (62)
\]

whereby the inequality (61) gives

\[
\frac{dR}{dt}(t) \leq \sup_{a \in B(0, R_0)} |u(X(a, t), t)| \leq C(1 + \log(2R(t) + 1))\|\theta^n(t)\|_Y, \quad (63)
\]

such that Estimate 1 implies \( \theta^{n+1} \) satisfies Estimate 2 in the Proposition.

For Estimate 3, we may simply apply Lemma 2 and the previous conclusions to deduce

\[
\|\nabla u^{n+1}(t)\|_{C^\mu} \leq C_{\mu}\|\theta_0\|_Y \quad (64)
\]

for \( \mu \in (0, 1) \), where \( u^{n+1}(t) = G * \theta^{n+1}(t) \). The results follow by induction. \( \square \)

**Proposition 14.** The sequence of functions \( \theta^n \) satisfies the uniform estimate

\[
\|\nabla \theta^n(t)\|_{L^p} \leq \|\nabla \theta_0\|_{L^p} \exp(C_L t)
\]

for \( 1 \leq p \leq \infty \).
**Proof.** Differentiating the transport equation (56) yields

\[ \left( \frac{\partial}{\partial t} + u^{n-1} \cdot \nabla \right) \nabla \perp \theta^n = (\nabla u^{n-1}) \nabla \perp \theta^n. \]  

(65)

We that multiply the above that \( \nabla \perp \theta^n \) \( \nabla \perp \theta^n \) for \( p \geq 2 \), and integrate by parts to find

\[ \frac{d}{dt} \| \nabla \perp \theta^n \|_{L^p}^p \leq p \int_{\mathbb{R}^2} |\nabla u^{n-1}| \| \nabla \theta^n \|_p \ dx. \]  

(66)

Therefore

\[ \frac{d}{dt} \| \nabla \theta^n \|_{L^p} \leq \| \nabla u^{n-1} \|_{L^\infty} \| \nabla \theta^n \|_{L^p} \]  

(67)

for \( 1 \leq p \leq \infty \), where the cases \( p = 1 \) and \( p = \infty \) are direct from (65). From here, Grönwall’s Lemma yields

\[ \| \nabla \theta^n(t) \|_{L^p} \leq \| \nabla \theta_0 \|_{L^p} \exp \left( \int_0^t \| \nabla u^{n-1}(s) \|_{L^\infty} \ ds \right). \]  

(68)

By the third estimate in Proposition 13, it follows \( \nabla u^{n-1} \) is bounded absolutely by a constant \( C_L \) which depends only on \( \| \theta_0 \|_Y \). \( \square \)

**Proposition 15.** For any \( t \), the sequence \( \theta^n(t) \) converges strongly in \( L^2 \) to some function \( \theta(t) \). Moreover, \( \theta \) obeys Estimates 1, 2 and 3 in Theorem 3.

**Proof.** We denote

\[ \vartheta^{n+1} = \theta^{n+1} - \theta^n \]  

(69)

and conclude by showing the sequence \( \vartheta^n(t) \) is summable in \( L^2 \) for any \( t \). Accordingly, we define \( v^n = G * \vartheta^n \) such that equation obeyed by \( \vartheta^{n+1} \) reads as

\[ \partial_t \vartheta^{n+1} + \langle u \rangle^n \cdot \nabla \vartheta^{n+1} + v^n \cdot \nabla \langle \theta \rangle^{n+1} = 0, \]  

(70)

where

\[ \langle u \rangle^n = \frac{1}{2} (u^n + u^{n-1}) \]  

(71)

and

\[ \langle \theta \rangle^{n+1} = \frac{1}{2} (\theta^{n+1} + \theta^n). \]  

(72)

Taking the \( L^2 \) inner product of (70) with \( \vartheta^{n+1} \) gives us

\[ \frac{d}{dt} \| \vartheta^{n+1}(t) \|_{L^2}^2 = -2 \int_{\mathbb{R}^2} v^n \cdot \nabla \langle \theta \rangle^{n+1} \vartheta^{n+1} \ dx \]  

(73)

after integration by parts, noting \( \langle u \rangle^n \) is divergence-free.
We estimate the convolution \( G \ast \vartheta^n \) using the fact that the supports are in the ball of radius \( R(t) \). We obtain

\[
|v^n(x, t)| \leq \|\vartheta^n(t)\|_{L^2} \left( \int_{|y| \leq R(t)} |G(x - y)|^2 \, dy \right)^{1/2}
\]

(74)

from the Cauchy–Schwarz inequality. Using Lemma 1 and evaluating the integral, we discover that

\[
|v^n(x, t)| \leq C (1 + \log(2R(t) + 1)) R(t) \|\vartheta^n(t)\|_{L^2}.
\]

(75)

Now the righthand side of (73) is bounded absolutely by

\[
C (1 + \log(2R(t) + 1)) R(t) \|\nabla \theta^{n+1}(t)\|_{L^2} \|\vartheta^n(t)\|_{L^2} \|\vartheta^{n+1}(t)\|_{L^2}.
\]

(76)

The estimate then becomes

\[
\frac{d}{dt} \Theta^{n+1}(t) \leq A_T \|\nabla \theta^{n+1}\|_{L^2} \Theta^n(t),
\]

(77)

where \( \Theta^n = \|\vartheta^n\|_{L^2} \), and \( A_T = C (1 + \log(2R(T) + 1)) R(T) \). Using Proposition 14 for \( p = 2 \), we write the integral expression, for all \( t \leq T \),

\[
\Theta^{n+1}(t) \leq A_T \|\nabla \theta_0\|_{L^2} \exp(C_L T) \int_0^t \Theta^n(s) \, ds,
\]

(78)

noting that \( \Theta^n(0) = 0 \) for all \( n \).

Denoting that \( C_T = A_T \|\nabla \theta_0\|_{L^2} \exp(C_L T) \), we compute \( n = 1 \):

\[
\Theta^2(t) \leq C_T \int_0^t \Theta^1(s) \, ds \\
\leq C_T \left( \sup_{t \leq T} \Theta^1(t) \right) t \\
\leq C_T C_1 t.
\]

(79)

there \( C_1 \) depends only on \( \|\theta_0\|_{L^2} \). Suppose now, for some \( n \), that

\[
\Theta^n(t) \leq C_1 \frac{(C_T t)^n}{n!}.
\]

(80)

Then we have

\[
\Theta^{n+1}(t) \leq C_1 C^{n+1}_n \int_0^t \frac{s^n}{n!} \, ds \\
\leq C_1 \frac{(C_T t)^{n+1}}{(n + 1)!}.
\]

(81)
It follows by induction that estimate (80) holds for all $n \geq 1$. Indeed,

$$
\sum_{n=1}^{\infty} \Theta^{n+1}(t) \leq C_1 \exp(CT t)
$$

such that $\Theta^n(t)$ is summable for all $t \leq T$, where $T$ is arbitrary. Therefore, the sequence $\Theta^n(t)$ converges strongly in $L^2$ to a function $\theta(t)$ for any $t$, and the uniform bounds in Proposition 13 are inherited by the limit.

In view of Lemma 1, the linear operator given by kernel $G$ is generally not bounded in $L^2$. Towards a convergence in more regular spaces, an immediate corollary of Proposition 13 is that $\|\nabla u^n\|_{L^2} \leq C\|\theta_0\|_Y$, uniformly in $n$. We further establish the higher regularity estimate as follows:

**Lemma 16.** Suppose $\theta \in Y \cap H^m$ for some positive integer $m$. Let $u = G * \theta$. Then,

$$
\|D^{m+2}u\|_{L^2} \leq \|\theta\|_{H^m}.
$$

**Proof.** We first observe that $u = G * \theta$ is a weak solution to

$$
-\Delta u + \nabla \Pi = \theta e_2,
$$

where $\Pi$ is given by $\Delta \Pi = \partial_2 \theta$. Then, we apply $D^m$ to the above equation and take the $L^2$ inner product with $D^m(-\Delta u)$ to arrive at

$$
\int_{\mathbb{R}^2} (D^m(\Delta u)) \cdot D^m(\Delta u) \, dx = \int_{\mathbb{R}^2} (D^m(\theta e_2)) \cdot D^m(\theta e_2) \, dx,
$$

whereby integration by parts twice gives

$$
\|D^{m+2}u\|_{L^2} = \|D^m\theta\|_{L^2}.
$$

For higher regularity initial data, we show the limit $\theta$ indeed solves our system. The precompactness for $\theta^n$ we require follows from $H^m$-energy estimates, where the $H^0$ estimate is immediate from Estimate 1 in Proposition 13.

**Proposition 17.** Let $\theta_0 \in Y \cap H^m$ for some integer $m > 2$. Then, the sequence $\theta^n$ satisfies the uniform estimate

$$
\|\theta^n(t)\|_{H^m} \leq C_m \|\theta_0\|_{H^m} \exp(C_m \exp(CL|t|)),
$$

where $C_m$ depends only on $m$ and $\theta_0$. 

Proof. Further, we apply the operator $D^\alpha$ with multi-index $\alpha$ to (56), multiply by $D^\alpha \theta^{n+1}$, integrate over $\mathbb{R}^2$, and sum over $|\alpha| \leq m$ to discover
\[
\frac{1}{2} \frac{d}{dt} \|\theta^{n+1}\|_{H^m}^2 = - \sum_{|\alpha| \leq m} \int_{\mathbb{R}^2} (D^\alpha (u^n \cdot \nabla \theta^{n+1})) D^\alpha \theta^{n+1} \, dx,
\]
which gives us that
\[
\frac{1}{2} \frac{d}{dt} \|\theta^{n+1}\|_{H^m}^2 = - \sum_{|\alpha| \leq m} \int_{\mathbb{R}^2} (D^\alpha (u^n \cdot \nabla \theta^{n+1})) - u^n \cdot \nabla (D^\alpha \theta^{n+1}) D^\alpha \theta^{n+1} \, dx,
\]
(86)
since $u^n$ is divergence-free. Using calculus inequalities (e.g. see [27, Lemma 3.4]) gives us that
\[
1 \frac{d}{dt} \|\theta^{n+1}\|_{H^m} \leq \sum_{|\alpha| \leq m} \|D^\alpha (u^n \cdot \nabla \theta^{n+1}) - u^n \cdot \nabla (D^\alpha \theta^{n+1})\|_{L^2} \leq C_m \left( \|\nabla u^n\|_{L^\infty} \|D^m \theta^{n+1}\|_{L^2} + \|D^m u^n\|_{L^2} \|\nabla \theta^{n+1}\|_{L^\infty} \right),
\]
(87)
with Proposition 14, Lemma 16, and the embedding $H^m \subset L^\infty$ for $m > 1$, we find
\[
\frac{d}{dt} \|\theta^{n+1}\|_{H^m} \leq C_m (\|D^m \theta^{n+1}\|_{L^2} + \exp(C_L|t|) \|\theta^n\|_{H^{m-1}}),
\]
(88)
where $C_m$ depends only on $m$ and $\theta_0 \in Y \cap H^m$. The result follows from iteration.

\[\square\]

Proposition 18. Let $\theta_0 \in Y \cap H^m$ for some integer $m > 2$, and suppose that $\text{spt} \, \theta_0 \subset B(0, R_0)$. Then, for arbitrary $T$, there exists a unique classical solution $\theta \in C^1(0, T; C^1_0)$ to system $(B_\bullet)$. Moreover, $\theta$ satisfies Estimates 1, 2 and 3 in Theorem 3.

Proof. Because of Proposition 17, the sequence $\theta^n(t)$ has a subsequence $\theta^{n_j}(t)$ which converges strongly in $H^m$ to the limit $\theta(t)$ in Proposition 15, for $0 \leq t \leq T$. As $m > 2$, the continuous embedding
\[
H^{m+k} \subset C^k_0 \equiv \{\text{space of } C^k \text{ functions vanishing at infinity}\}
\]
implies $\theta^{n_j}(t)$ converges strongly in $C^1_0$, and $\theta(t) \in C^1_0$ satisfies Estimates 1, 2 and 3.

Towards verifying $\theta$ is a solution, we define $u = G \ast \theta$ and observe that $\theta^{n_j} \in Y$ and $\theta \in Y$ are both supported in $B(0, R)$ where $R = R(t)$ is given by Estimate 2. At each instance of time, Lemma 1 implies
\[
|u^{n_j}(x) - u(x)| \leq C (1 + \log(|x| + R + 1)) \|\theta^{n_j} - \theta\|_Y.
\]
(90)
Then, we may conclude $u^{n_j} \to u$ pointwise via the continuous embedding $L^1_{\text{loc}} \subset H^1$. Estimate 3 and Azelà-Azcoli imply the convergence is uniform in $B(0, R)$. 


Similarly, we have also $\theta^{n_k-1}$ converging strongly to $\theta$ in $C^1_0$ for each fixed time, where $n_k - 1$ indexes a subsequence of $\theta^{n_j-1}$. It then follows that $u^{n_k-1}(t) \cdot \nabla \theta^{n_k}(t)$ converges uniformly. Pointwise in space and time, we have
\[
\frac{\partial_t \theta}{n_k} = -(u^{n_k-1}) \cdot \nabla \theta^{n_k},
\]
and thus $\partial_t \theta^{n_k}$ converges strongly to $-u \cdot \nabla \theta$ in $C(0, T; C_0)$. The distributional limit of $\partial_t \theta^{n_k}$ is accordingly $\partial_t \theta$; therefore $\theta$ is a classical solution to system $(B^*)$. \hfill \Box

**Proof of Theorem 3.** Existence with the estimates follows from Proposition 18 using the arguments in the classical paper of Yudovich [24], basically unmodified. Here, we prove uniqueness using the estimates from Lemma 2. Suppose with initial data $\theta_0 \in Y$ of compact support we have two weak solutions, $\theta_1$ and $\theta_2$. It follows that the difference $\vartheta = \theta_1 - \theta_2$ obeys the equations
\[
\left( \frac{\partial}{\partial t} + \langle u \rangle \cdot \nabla \right) \vartheta = -v \cdot \nabla \langle \theta \rangle
\]
in the sense of distributions, where $v(t) = G * \vartheta(t)$ and
\[
\langle u \rangle = \frac{1}{2}(G * \theta_1 + G * \theta_2),
\]
\[
\langle \theta \rangle = \frac{1}{2}(\theta_1 + \theta_2).
\]
At each instance of time, we have $\vartheta(t) \in Y$ and clearly
\[
spt \vartheta(t) \subset spt \theta_1(t) \cup spt \theta_2(t),
\]
since $\theta_1$ and $\theta_2$ obey the estimates. Through Lemma 2, the velocities $\langle u \rangle$ and $v$ have gradients bounded uniformly in time with constant depending only on $\|\theta_1\|_Y + \|\theta_2\|_Y$. We multiply equation (92) with $\vartheta$ and integrate by parts to discover
\[
\frac{d}{dt} \| \vartheta \|^2_{L^2} = 0,
\]
since $\langle u \rangle$ and $v$ are divergence-free and regular enough. Recalling that $\| \vartheta(0) \|^2_{L^2} = 0$, we have proved uniqueness. \hfill \Box

5. **Proof of Theorem 6**

We consider the dynamics of the patch solution $\theta(t) = 1_{P(t)}$ given by Corollary 4 and address here the question of regularity for the evolving boundary $\partial P(t)$.

First, we note that $\theta$ must necessarily satisfy the estimates in Theorem 3. The particle trajectories are volume-preserving such that the area of $P(t)$ is constant in time, and so we define the length scale
\[
L = \sqrt{\text{area}(P_0)} = \sqrt{\text{area}(P(t))}.
\]
Then, \( u = G * 1_P \) has the uniform bound in time
\[
\| \nabla u(t) \|_{C^\mu} \leq C_\mu C_L,
\]
where \( C_L = 1 + L^2 \) depends only on the initial data.

The regularity result for the distribution \( 1_P \) follows from analysis of gradients of the defining level-set function \( \varphi(t) \) satisfying (13), which is the unique global solution to the Cauchy problem
\[
\begin{aligned}
\left\{ \begin{array}{l}
\left( \frac{\partial}{\partial t} + u \cdot \nabla \right) \varphi = 0, \\
\varphi(x, 0) = \varphi_0(x),
\end{array} \right.
\end{aligned}
\]
where \( \varphi_0 \) satisfies (11). Above, the fluid velocity \( u = G * 1_P \) has the following convenient expression, using \( \varphi \):
\[
u(x, t) = \int_{\mathbb{R}^2} G(x - y) H(\varphi(y, t)) \, dy
\]
there \( H \) is the Heaviside function. By construction, \( P(t) \) is precisely the zero level-set of \( \varphi(t) \) such that
\[
1_P(t) = H \circ \varphi(t),
\]
and so our regularity results for the boundary rely on an analysis of \( \nabla \perp \varphi \), which is tangent to \( \partial P \). Recalling the evolution equation (24) for \( W(t) \), we must estimate the product \( \nabla u \cdot W \).

We provide an additional observation in estimating \( |\cdot|_{\mu} \): the singular kernel \( G \) away from the origin has gradient \( \nabla G \), with homogeneity of degree \(-1\), such that for each instance of time
\[
\nabla u(x) = \text{pv} \int_P \nabla G(x - y) \, dy,
\]
and so we write \( \nabla u = \nabla G * 1_P \). Since the degree in modulus is less than the dimension, integration by parts yields
\[
\text{pv} \int_P \nabla[G(x - y)] \cdot W(y) \, dy = 0
\]
for divergence-free \( W \) tangent to the boundary. Thus, we have commutative structure for the kernel \( \nabla G(z) \)
\[
\nabla u(x) \cdot W = (\nabla G \cdot W) * 1_P,
\]
and following estimate:

**Lemma 19.** Suppose \( u = G * 1_P \) and that \( W \in C^\mu(\mathbb{R}^2, \mathbb{R}^2) \) is divergence free and tangent to \( \partial P \). Then, there exists independent constant \( C_2 \) such that
\[
|\nabla u \cdot W|_{\mu} \leq C_2 \| \nabla u \|_{L^\infty} |W|_{\mu}.
\]
Using the particle trajectories of $u$, we derive from (24) pointwise estimates on equation $W$ (see [11, Proposition 3]) wherein Grönwall’s inequality yields bounds for quantities (22) and (23) which depend exponentially on
\[ \int_0^t \| \nabla u(s) \|_{L^\infty} \, ds \leq C_L |t|. \] (104)

The particle trajectories allow us to deduce the following regularity result:

**Proposition 20.** Suppose $P_0$ bounded and $\varphi_0 \in C^{1+\mu}(\mathbb{R}^2)$ in (98) such that $|W_0|_{\text{inf}} > 0$. Then, the unique global solution $\varphi$ has $W = \nabla \varphi$ which satisfies
\[ |W(t)|_\mu \leq |W_0|_\mu \exp((C_2 + \mu)C_L |t|), \]
\[ \|W(t)\|_{L^\infty} \leq \|W_0\|_{L^\infty} \exp(C_L |t|), \]
\[ |W(t)|_{\text{inf}} \geq |W_0|_{\text{inf}} \exp(-C_L |t|). \]

**Proof of Theorem 6 for $\partial P_0 \in C^{1+\mu}$.** Indeed, Proposition 20 guarantees the desired $C^{1+\mu}$ parametrization $z$ in (21) exists at each instance of time, if $\varphi_0$ in (12) is chosen such that $|W_0|_{\text{inf}} = |\nabla \varphi_0|_{\text{inf}}$ is non-vanishing. \hfill $\Box$

**Remark 21.** Lemma 19 is used only in the estimate of $|W|_\mu$ in Proposition 20 and thus unnecessary to deduce $\partial P \in C^{1+\mu}$. The infimum and supremum bounds are direct from (24) and $\nabla u \in C^\mu$ depends only on $\theta_0$ so that we may alternatively conclude with an $\exp(C_L \exp(C_L |t|))$ bound on $|W|_\mu$.

For $C^{2+\mu}$-regularity, we must examine the dynamics of $\nabla W$. The presence of $\nabla \nabla u \cdot W$ in equation (25) augments the Grönwall-type exponential bounds from particle trajectories. Inspecting expression (54) for $\nabla \nabla u$ in this context,
\[ \nabla \nabla u(x) = 1_P(x)E + \frac{1}{4\pi} \text{pv} \int_P \frac{H(x-y)}{|x-y|^2} \, dy, \] (105)
and we see that $|\nabla \nabla u|_\mu$ is difficult to estimate directly. From here, we recognize the expression for $\nabla \nabla u(x)$ has similar geometric properties to the strain tensor for vortex patches [11]. We adapt their arguments here.

**Proposition 22.** Suppose $u = G \ast 1_P$ and that $W \in C^\mu(\mathbb{R}^2, \mathbb{R}^2)$ is divergence free and tangent to $\partial P$. Then
\[ \nabla \nabla u(x) \cdot W = \frac{1}{4\pi} \text{pv} \int_P \frac{H(x-y)}{|x-y|^2} \cdot (W(x) - W(y)) \, dy. \]

**Proof.** Since $W$ is divergence free and tangent to $\partial P$, then
\[ \text{pv} \int_P \nabla[H(x-y)] \cdot W(y) \, dy \]
\[ = - \lim_{\delta \to 0} \int_{|x-y| = \delta, y \in P} \left( W(y) \cdot \left( \frac{x-y}{\delta} \right) \right) \nabla G(x-y) \, dy \]
\[ = -1_P(x)E \cdot W(x). \] (106)

The last equality follows from (48) in the proof of Proposition 11. \hfill $\Box$
Corollary 23. Suppose \( u = G \ast 1_P \) and that \( W \in C^\mu(\mathbb{R}^2, \mathbb{R}^2) \) is divergence free and tangent to \( \partial P \). Then, there exists independent constant \( C_3 \) such that

\[
|\nabla^2 u \cdot W|_\mu \leq C_3 \|\nabla u\|_{L^\infty} |W|_\mu.
\]

The estimation of \( \|\nabla u\|_{L^\infty} \) is consequence of the fact that small neighborhoods containing \( \partial P \) look like half-circles, and that the kernel \( H(z)/|z|^2 \) is reflection symmetric. To illustrate, consider the set of points \( x_0 \) with distance

\[
d(x_0) = \inf_{x \in \partial P} \{|x - x_0|\}
\]

less than a cutoff \( 0 < \delta \leq \infty \). This cutoff is explicit

\[
\delta^\mu = \frac{|\nabla \varphi|_{\inf}}{|\nabla \varphi|_\mu},
\]

given \( \varphi \) which satisfies (13). Such a choice ensures that the boundary \( \partial P \) can be straightened near the points \( x_0 \) where

\[
d(x_0) < \delta.
\]

Indeed, for such points \( x_0 \) the semicircle

\[
\Sigma(x_0) = \{z \mid |z| = 1, \nabla \varphi(\tilde{x}) \cdot z \geq 0\},
\]

where above \( \tilde{x} \in \partial P \) is any such that \( |\tilde{x} - x_0| = d(x_0) \) holds, is in fact well-approximated by the set of directions

\[
S_\rho(x_0) = \{z \mid |z| = 1, x_0 + \rho z \in P\}
\]

for every \( \rho \geq d(x_0) \). More quantitatively,

Lemma 24. (Geometric Lemma) Denote the symmetric difference as

\[
R_\rho(x_0) = (S_\rho(x_0) \setminus \Sigma(x_0)) \cup (\Sigma(x_0) \setminus S_\rho(x_0))
\]

and the Lebesgue measure on the unit circle as \( \mathcal{H}^1 \). Then,

\[
\mathcal{H}^1(R_\rho(x_0)) \leq 2\pi \left( (1 + 2^\mu) \frac{d(x_0)}{\rho} + 2^\mu \left( \frac{\rho}{\delta} \right)^\mu \right)
\]

for all \( \rho \geq d(x_0) \), \( \mu > 0 \) and \( x_0 \) such that \( d(x_0) < \delta = \left( \frac{|\nabla \varphi|_{\inf}}{|\nabla \varphi|_\mu} \right)^{1/\mu} \).

Proposition 25. Suppose \( u = G \ast 1_P \) and that \( \varphi \) satisfies (13) for \( P \). Then, there exists constant \( C_4 \) depending only on \( \mu, L, |W_0|_\mu \) and \( |W_0|_{\inf} \) such that

\[
\|\nabla^2 u(t)\|_{L^\infty} \leq C_4 (1 + |t|).
\]
Proof. We need only estimate the singular integral, which we split into $I_1$ and $I_2$. For δ given by (109), the latter has the bound

$$ \left| I_2(x_0) \right| = \left| \frac{1}{4\pi} \int_{\mathbb{R}^3 \cap |x_0 - y| \geq \delta} \frac{H(x_0 - y)}{|x_0 - y|^2} \, dy \right| \leq 1 + \log \left( \frac{\delta}{L} \right). $$ (112)

The remaining term,

$$ \left| I_1(x_0) \right| = \frac{1}{4\pi} \int_{\mathbb{R}^3 \cap |x_0 - y| < \delta} \frac{H(x_0 - y)}{|x_0 - y|^2} \, dy $$ (113)

vanishes for $d(x_0) > \delta$. We thus assume $d(x_0) < \delta$, and pass to polar coordinates centered at $x_0$ to find

$$ \left| I_1(x_0) \right| \leq \frac{1}{4\pi} \int_{d(x_0)}^{\delta} \frac{d\rho}{\rho} H^1(R_{\rho}(x_0)), $$ (114)

using the fact $\int_{\Sigma(x_0)} H(z) \, dH^1(z)$ vanishes by reflection symmetry. Now applying the Geometric Lemma, we integrate to discover

$$ \left| I_1 \right| \leq \frac{1}{2} \left( 1 + 2\mu + \frac{2\mu}{\mu} \right) $$ (115)

We then have our bound

$$ \| \nabla \nabla u \|_{L^\infty} \leq \left( 4 + \frac{1}{\mu} \right) \left( 1 + \log \left( \frac{\| \nabla \varphi \|_L^\mu}{\| \nabla \varphi \|_\text{inf}^\mu} \right) \right) $$ (116)

and use the estimates in Proposition 20 to conclude.

The proof of Corollary 23 and the Geometric Lemma may be found in the Appendix of [11]. It follows that the evolution equation (25) admits a Grönwall-type estimate.

Proposition 26. Suppose $P_0$ bounded and that $\varphi_0 \in C^{2+\mu}$ in (98). Then, the unique global solution $\varphi$ has $\nabla W = \nabla \nabla \perp \varphi$ which satisfies:

$$ |\nabla W(t)|_{\mu} \leq \exp((C_3 + \mu)C_L|t|) \left[ |\nabla W_0|_{\mu} + C_4 \int_0^t (1 + |s|) |W(s)|_{\mu} \, ds \right], $$

$$ \| \nabla W(t) \|_{L^\infty} \leq \exp(C_L|t|) \left[ \| \nabla W_0 \|_{L^\infty} + C_4 \int_0^t (1 + |s|) \| W(s) \|_{L^\infty} \, ds \right], $$

for positive $t$.

Proof of Theorem 6 for $\partial P_0 \in C^{2+\mu}$. With Proposition 20 and 26, we see that

$$ \| W(t) \|_{C^{1+\mu}} \leq C_S \| W_0 \|_{C^{1+\mu}} (1 + |t|) \exp(C_L|t|) $$ (117)

for some constant $C_S$ depending only on $\mu$, $L$ and $\varphi_0$, as desired.
6. Details of Fig. 1

6.1. Description of the Numerical Solver

In the compact domain \( \mathbb{T}^2 \), we develop a solver for the density patch problem in the system \((B_*)\) which resolves the dynamics of the patch boundary \( \partial P(t) \) in Corollary 8 using a level-set method that is second-order in time and first-order in space (for background, see [25]).

The algorithm begins with the expression

\[
\mathbf{u} = -\nabla \perp (\Delta^2)^{-1} \partial_1 \theta, \quad (118)
\]

where the scalar \( \theta : \mathbb{T}^2 \to \mathbb{R} \) has zero mean, so the operator \((\Delta^2)^{-1}\) on \( \partial_1 \theta \) is well-defined and the vector-field \( \mathbf{u} : \mathbb{T}^2 \to \mathbb{R}^2 \) has zero mean also. The discrete Fourier coefficients \( \hat{\mathbf{u}} \) are related explicitly to \( \hat{\theta} \)

\[
\hat{\mathbf{u}}(k) = \frac{k \perp k_1}{|k|^4} \hat{\theta}(k) \quad (119)
\]

for nonzero \( k \in 2\pi \mathbb{Z}^2 \) and \( k \perp = (-k_2, k_1) \). Accordingly, we set \( \hat{\mathbf{u}}(0) = 0 \).

With the spacing \( h = 1/N \), we discretize the space variable onto an \( N \times N \) lattice with coordinates \( x_{ij} = h(i - N/2, j - N/2) \) for \( i, j = 0, \ldots, N - 1 \). Note that we have fixed \( N \) to some power of two, and write \( \phi_{ij}(t) := \phi(x_{ij}, t) \) such that \( \theta_{ij} = H(\phi_{ij}) \). With this convention, we have

\[
\hat{\theta}(k) \approx h^2 \sum_{j=0}^{N-1} \sum_{i=0}^{N-1} H(\phi_{ij}) \exp \left( -i k \cdot x_{ij} \right), \quad (120)
\]

where \( i \) here is the imaginary unit. We thus resolve the advecting velocity with spectral accuracy:

\[
u_{ij} \approx \sum_{|k| \leq \pi N} \frac{k \perp k_1}{|k|^4} \hat{\theta}(k) \exp(i k \cdot x_{ij}). \quad (121)
\]

The advection operator \( F(\phi) = -\mathbf{u} \cdot \nabla \phi \) is discretized according to the monotone first-order upwinding scheme

\[
F(\phi_{ij}) \approx (u_{ij})^- \cdot D_{ij}^+ - (u_{ij})^+ \cdot D_{ij}^-, \quad (122)
\]

where the signed velocities \( (v)^\pm = \max(\pm v, 0) \) with \( v = (v)^+ - (v)^- \) product with the signed gradient

\[
D_{ij}^\pm = \pm \frac{1}{h} \left( \phi_{i \pm 1,j} - \phi_{ij} \right) \left( \phi_{i,j \pm 1} - \phi_{ij} \right). \quad (123)
\]

With our procedure given by (120-123), we integrate the system

\[
\frac{\partial \phi_{ij}}{\partial t} = F(\phi_{ij}) \quad (124)
\]

in time using the second-order SSRK (Heun’s) method where the time step is chosen such that \( CFL \leq 1/2 \), to complete the algorithm.
6.2. Verification of Convergence

Our verification of the numerical solver described in Sect. 6.1 addresses the implementation of the following three routines:

1. the level-set method for the transport equation,
2. the spectral-collocation method for the momentum equation,
3. the coupling of (1) and (2) to approximate patch solutions in the full system.

The implementation of first-order upwind to transport the level-set function \( \varphi \) was tested against various fixed divergence-free velocity fields on \( \mathbb{T}^2 \): the shear \( u(x) = (x_2^3, 0) \), and cellular flows \( u = \nabla \perp \psi \) where

\[
\psi(x) = \sin(k_1x_1) \sin(k_2x_2),
\]

for various mode numbers \( k \in 2\pi \mathbb{Z}^2 \).

The spectral solver for the momentum equation was tested on sinusoidal temperature distributions

\[
\theta(x) = \cos(k_1x_1 + k_2x_2)
\]

and it indeed recovers the exact solution

\[
u(x) = \frac{k_1 k_2^\perp}{|k|^4} \cos(k_1x_1 + k_2x_2)
\]

up to machine precision for any \( |k| \leq \pi N \), as expected by the Nyquist-Shannon sampling theorem.

These two solvers are coupled together in the full algorithm, so we verified that quantities like \( \| \nabla \nabla u \|_{L^\infty} \) and \( \| \nabla \nabla \varphi \|_{L^\infty} \) from simulations with the same initial data, but various \( N \), are converging for short time as we increase \( N \).

Finally, we examine if the numerical solver is appropriately handling the dynamics of the low regularity solution which has initial data \( \varphi_0(x) = \Phi(x) \). Consider the regularized system

\[
\begin{cases}
\left( \frac{\partial}{\partial t} + u \cdot \nabla \right) \varphi = 0, \\
-\Delta u + \nabla \Pi = \phi_\epsilon \ast H(\varphi)e_2, \\
\text{div } u = 0,
\end{cases}
\]

where we have \( \epsilon > 0 \) and the mollifier is the Gaussian

\[
\phi_\epsilon(r) = \frac{1}{\epsilon \sqrt{2\pi}} \exp \left( -\frac{r^2}{2\epsilon^2} \right).
\]

The system \( (\Phi_\epsilon) \) is solved with the algorithm given above for system \( (B_\epsilon) \), except the Gaussian filter with standard deviation \( \epsilon \) (convolution with \( \phi_\epsilon \)) is applied numerically to the points \( H(\varphi_{ij}) \) before the discrete Fourier transform in (120) is taken. We observe that the tendency towards curvature singularities observed in Fig. 1 is suppressed in this regularized system for any fixed \( \epsilon \). However, as we take epsilon to machine precision, the simulations recover the results of \( \epsilon = 0 \) (system \( (B_\epsilon) \)), in particular the picture in Fig. 1.
6.3. The Simulation in Fig. 1

The curve illustrated in the Figure is the zero level set of $\varphi_{ij}(t)$ whose initial condition for the algorithm was specified as $\varphi_0(x_{ij}) = \Phi(x_{ij})$, where

$$\Phi(x) = \cos(2\pi |x - \xi|), \quad \xi = \left(\frac{1}{2}, \frac{1}{2}\right)$$

has precisely the circle of radius one-half $S^1(\frac{1}{2})$ centered at $\xi$ as its zero level set. The patch moves upwards, as does the curve $\partial P(t)$. To depict the changing shape of $\partial P$ in time, the frame where the curve is drawn follows this movement.

While patch solutions to our problem maintain their area as they evolve in time, the sharpening of the $\partial P(t)$ in the fixed-grid simulations results in rapid decreases in the area once the variations of the curve are of the scale $h = 1/N$. Thus for the resolution given by $N = 1024$, we terminate the patch simulation at $t = 100$, before the relative error of the patch area is more than 0.01.

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