We prove the existence of a close relationship between the generalized central series of Leibniz algebras. We also prove some analogs of the classical Schur and Baer group-theoretic theorems for Leibniz algebras.

1. Introduction

To reproduce the complete picture of problems encountered in our investigation, we first formulate some classical group-theoretical results. In 1951, Neumann [12] proved the so-called Schur theorem [10]: If a quotient group $G/\zeta(G)$ is finite, then the commutant $[G,G]$ is also finite and

$$[[G,G]] \leq t^{t^2+1},$$

where $t = |G/\zeta(G)|$. Later, Wiegold [17] and Wehrfritz [16] significantly improved this estimate. There are numerous analogs of the Schur theorem for groups (see, e.g., the survey [5]). Among these analogs, there are results connected with the groups of automorphisms.

Let $G$ be a group and let $A$ be a subgroup of the group of automorphisms $\text{Aut}(G)$. We set

$$C_G(A) = \{ g \in G \mid \alpha(g) = g \text{ for each } \alpha \in A \},$$

$$[G,A] = \langle [g,\alpha] = g^{-1}\alpha(g) \mid g \in G, \ \alpha \in A \rangle.$$

Subgroups $C_G(A)$ and $[G,A]$ are called the $A$-center and the $A$-commutator subgroup of the group $G$, respectively. In [6], Hegarty proved that if $A = \text{Aut}(G)$ and the quotient group $G/C_G(A)$ is finite, then the subgroup $[G,A]$ is also finite. A more general case was considered in [4], namely, $\text{Inn}(G) \leq A$ and the quotient group $A/\text{Inn}(G)$ is finite, where $\text{Inn}(G)$ is the group of inner automorphisms of the group $gG$. In this case, the following analog of the Schur theorem was proved: If the quotient group $G/C_G(A)$ is finite, then the subgroup $[G,A]$ is also finite. Moreover, an estimate for the order of the subgroup $[G,A]$ was established in terms of the orders $|G/C_G(A)|$ and $|A/\text{Inn}(G)|$. In particular, if $A = \text{Inn}(G)$ or $A = \text{Aut}(G)$, then we get the ordinary Schur and Hegarty theorems.

It is known that there exists close relationship between the Lie groups and algebras. There are numerous group-theoretic results established for Lie algebras, and vice versa. The investigated circle of problems is not an exception. Thus, in particular, an analog of the Schur theorem for Lie algebras is well known (see, e.g., [15]).

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In [14], Stitzinger and Turner proved a Lie analog of the Hegarty theorem: Let $L$ be an algebra over the field $F$ and let $\text{Der}(L)$ be an algebra of differentiations of the algebra $L$. According to [14], we set

$$H = \bigcap_{\alpha \in \text{Der}(L)} \text{Ker}(\alpha) \quad \text{and} \quad L^* = \sum_{\alpha \in \text{Der}(L)} \text{Im}(\alpha).$$

Thus, the subalgebras $H$ and $L^*$ are Lie analogs of the subgroups $C_G(\text{Aut}(G))$ and $[G, \text{Aut}(G)]$. It follows from the main result in [14] that if the quotient algebra $L/H$ is finite-dimensional, then the subalgebra $L^*$ is also finite-dimensional.

We now consider (left) Leibniz algebras. Let $L$ be an algebra over the field $F$ with binary operations $+$ and $[\cdot, \cdot]$. In this case, we say that $L$ is a left Leibniz algebra if

$$[[x, y], z] = [x, [y, z]] - [y, [x, z]]$$

for all $x, y, z \in L$ [2, 11]. Note that each Lie algebra $L$ is a Leibniz algebra. Moreover, the Lie algebras can be characterized as Leibniz algebras in which $[x, x] = 0$ for any $x \in L$.

As one of the directions of development of the theory of Leibniz algebras we can mention determination of the results similar to the facts from the theory of Lie algebras. At the same time, there is a noticeable difference between these types of algebras (see, e.g., the surveys [3, 7, 9]).

We now present some necessary definitions. A left (resp., right) center $\zeta^l(L)$ (resp., $\zeta^r(L)$) of the Leibniz algebra $L$ is defined according to the following rule:

$$\zeta^l(L) = \{x \in L \mid [x, y] = 0 \text{ for every } y \in L\} \quad \text{(resp., } \zeta^r(L) = \{x \in L \mid [y, x] = 0 \text{ for every } L}).$$

The left center is an ideal of the algebra $L$. However, for the right center, this is not the case. It is nothing but a subalgebra of the algebra $L$ and, in the general case, the left and right centers are different. Moreover, they may have even different dimensions (see Example 2.1 in [8]). The center $\zeta(L)$ of the algebra $L$ is the intersection of the left and right centers, i.e.,

$$\zeta(L) = \{x \in L \mid [x, y] = 0 = [y, x] \text{ for every } y \in L\}.$$

It is clear that $\zeta(L)$ is an ideal of the algebra $L$. Thus, we can consider the quotient algebra $L/\zeta(L)$.

In [8], the following modification of an analog of the Schur theorem was proved: If $L$ is a Leibniz algebra over the field $F$ and the codimensions $\text{codim}_F(\zeta^l(L)) = l$ and $\text{codim}_F(\zeta^r(L)) = r$ are finite, then $\dim_F([L, L]) \leq l(l + r)$.

In this connection, it is reasonable to pose the following natural question: Under the assumption that solely the codimension $\text{codim}_F(\zeta^l(L))$ is finite, is it true that the dimension $\dim_F([L, L])$ is finite? Example 3.1 in [8] gives negative answer to this question. However, we have a direct analog of the Schur theorem for Leibniz algebras: If $L$ is a Leibniz algebra over the field $F$ and the codimension $\text{codim}_F(\zeta(L)) = d$ is finite, then $\dim_F([L, L]) \leq d^2$ [8].

In view of the previous arguments, it is reasonable to consider analogs of the results obtained in [4] for Leibniz algebras. We first define analogs of the $A$-center and $A$-commutator subgroup for the Leibniz algebras. Let $L$ be a Leibniz algebra over the field $F$ and let $D$ be a subalgebra of the algebra $\text{Der}(L)$. We set

$$\text{Ann}_L(D) = \bigcap_{\alpha \in D} \text{Ker}(\alpha), \quad [L, D] = \sum_{\alpha \in D} \text{Im}(\alpha).$$
Let \( a \in L \). Consider a mapping \( l_a : L \to L \) defined by the rule \( l_a(x) = [a, x] \). It is known that \( l_a \) is the operation of differentiation of the algebra \( L \) and the set \( \text{Ad}^l(L) = \{ l_a \mid a \in L \} \) is an ideal of the algebra \( \text{Der}(L) \) (see, e.g., [7]).

In particular, the operation of differentiation of the algebra \( L \) and the set \( \text{Ad}^l(L) = \{ l_a \mid a \in L \} \) is an ideal of the algebra \( \text{Der}(L) \) (see, e.g., [7]).

Assume that \( \text{Ad}^l(L) \leq D \). Then

\[
\text{Ann}_L(D) \leq \text{Ann}_L(\text{Ad}^l(L)) = \zeta^r(L).
\]

Hence,

\[
\text{Ann}_L(D) \cap \zeta^i(L) \leq \text{Ann}_L(\text{Ad}^l(L)) \cap \zeta^i(L) = \zeta^r(L) \cap \zeta^i(L) = \zeta(L).
\]

In particular, \( \text{Ann}_L(D) \cap \zeta^i(L) \) is an ideal in \( L \).

We say that \( A_L(D) = \text{Ann}_L(D) \cap \zeta^i(L) \) is the \( D \)-center of the algebra \( L \). Note that if \( A_L(D) \leq \zeta(L) \) and \( D = \text{Ad}^l(L) \), then the \( D \)-center is an ordinary center of the algebra \( L \). We say that \( [L, D] \) is the \( D \)-derived subalgebra of the algebra \( L \). If \( D = \text{Ad}^l(L) \), then

\[
[L, D] = [L, \text{Ad}^l(L)].
\]

Let \( x \in [L, \text{Ad}^l(L)] \). Thus,

\[
x = [y, l_a] = l_a(y) = [a, y]
\]

for all \( a, y \in L \). This means that, in this analyzed case, the \( D \)-derived subalgebra \( [L, D] \) is an ordinary derived subalgebra \( [L, L] \) of the algebra \( L \).

The following theorem is the first main result of the present paper:

**Theorem A.** Suppose that \( L \) is a Leibniz algebra over the field \( F \), \( D \) is a subalgebra of the algebra \( \text{Der}(L) \) such that \( \text{Ad}^l(L) \leq D \), and the dimension \( \dim_F(D/\text{Ad}^l(L)) = k \) is finite. If the dimension \( \dim_F([L, A_L(D)]) = t \) is finite, then

\[
\dim_F([L, D]) \leq t(k + t).
\]

In [1], Baer generalized the Schur theorem as follows. He proved that if the quotient group \( G/\zeta_k(G) \) is finite, then the subgroup \( \gamma_{k+1}(G) \) is also finite. Here, \( \zeta_k(G) \) denotes the \( k \)-th term of the upper central series of the group \( G \) and \( \gamma_{k+1}(G) \) denotes the \( (k + 1) \)-th term of the lower central series of the group \( G \). An automorphic analog of this result was proved in [4]. In [13], Stewart proved that if \( L \) is a Lie algebra such that the quotient algebra \( L/\zeta_k(L) \) is finite-dimensional, then \( \gamma_{k+1}(L) \) is also a finite-dimensional subalgebra. Moreover, the upper bound for the dimension \( \dim_F(\gamma_{k+1}(L)) \) was found in [8]. In the cited work, an analog of the Baer theorem for Leibniz algebras was also proved.

Starting from \( A_L(D) \) and \( [L, D] \), where \( \text{Ad}^l(L) \leq D \), it is possible to construct the upper and lower \( D \)-central series for the Leibniz algebra \( L \). Let \( \zeta_1(L, D) = A_L(D) \). This enables us to find an increasing series of ideals \( \zeta_\omega(L, D) \) of the algebra \( L \), where

\[
\zeta_{\omega+1}(L, D)/\zeta_\omega(L, D) = \zeta_1(L/\zeta_\omega(L, D), D).
\]

As always, if \( \lambda \) is the limit ordinal number, then

\[
\zeta_\lambda(L, D) = \bigcup_{\mu<\lambda} \zeta_\mu(L, D).
\]
The last term $ζ_∞(L, D) = ζ_δ(L, D)$ of this series is called the upper $D$-hypercenter of the algebra $L$, while the number $δ$ is called the upper $D$-central length of the algebra $L$, which is denote by $zl(L, D)$.

The lower $D$-central series of the algebra $L$ is a decreasing series

$$L = γ_1(L, D) ≥ γ_2(L, D) ≥ ... ≥ γ_ν(L, D) ≥ γ_ν+1(L, D) ≥ ...,$$

where $γ_2(L, D) = [L, D]$ and, for each ordinal number $ν$, we set $γ_ν+1(L, D) = γ_ν(L, D), D]$. For the limit ordinal number $λ$, we set

$$γ_λ(L, D) = \bigcap_{μ<λ} γ_μ(L, D).$$

The following theorem is the second main result of the present paper:

**Theorem B.** Suppose that $L$ is a Leibniz algebra over the field $F$, $D$ is a subalgebra of the algebra $\text{Der}(L)$ such that $\text{Ad}^l(L) ≤ D$, and the dimension $\dim_F(D/\text{Ad}^l(L)) = k$ is finite. Let $Z$ be an upper $D$-hypercenter of the algebra $L$. Assume that the upper $D$-central length $zl(L, D) = m$ and that the dimension $\dim_F(L/Z) = t$ is finite. Then the subalgebra $γ_{m+1}(L, D)$ is finite-dimensional and there exists a function $f$ such that

$$\dim_F(γ_{m+1}(L, D)) ≤ f(k, m, t).$$

2. Proof of Theorem A

Since $\text{Ad}^l(L) ≤ D$, we have $A_L(D) ≤ ζ(L)$. This means that

$$\dim_F(L/ζ(L)) ≤ \dim_F(L/A_L(D)) = t.$$

Then the derived subalgebra $K = [L, L]$ is finite-dimensional and $\dim_F([L, L]) ≤ t^2$ [8].

We set $L_{ab} = L/K$. For any $α ∈ D$, we define a mapping $α_{ab} : L_{ab} → L_{ab}$ by the rule:

$$α_{ab}(x + K) = α(x) + K$$

for every $x ∈ L$. Let $x, y ∈ L$, $λ ∈ F$. Since

$$α_{ab}((x + K) + (y + K)) = α_{ab}((x + y) + K)$$

$$= α(x + y) + K = α(x) + α(y) + K$$

$$= (α(x) + K) + (α(y) + K) = α_{ab}(x + K) + α_{ab}(y + K)$$

and

$$α_{ab}(λ(x + K)) = α_{ab}(λx + K) = α(λx + K)$$

$$= λα(x) + K = λ(α(x) + K) = λα_{ab}(x + K),$$

$α_{ab}$ is an endomorphism of the algebra $L_{ab}$. 
Also let $x, y \in L$. Then

$$
\alpha_{ab}([x + K, y + K]) = \alpha_{ab}([x, y] + K)
= \alpha([x, y]) + K = [\alpha(x), y] + [x, \alpha(y)] + K
= ([\alpha(x), y] + K) + ([x, \alpha(y)] + K)
= [\alpha(x) + K, y + K] + [x + K, \alpha(y) + K]
= [\alpha_{ab}(x + K), y + K] + [x + K, \alpha_{ab}(y + K)],
$$
i.e., $\alpha_{ab} \in \text{Der}(L_{ab})$.

Let $\alpha \in D$. Consider a mapping $d(\alpha) : L_{ab} \to L_{ab}$ given by the rule

$$
d(\alpha)(x) = [x, \alpha_{ab}] = \alpha_{ab}(x), \quad x \in L_{ab}.
$$

Thus, we get

$$
d(\alpha)(x + y) = [x + y, \alpha_{ab}] = \alpha_{ab}(x + y)
= \alpha_{ab}(x) + \alpha_{ab}(y) = [x, \alpha_{ab}] + [y, \alpha_{ab}] = d(\alpha)(x) + d(\alpha)(y),
$$

$$
d(\alpha)(\lambda x) = [\lambda x, \alpha_{ab}] = \alpha_{ab}(\lambda x) = \lambda \alpha_{ab}(x) = \lambda [x, \alpha_{ab}] = \lambda d(\alpha)(x).
$$

In other words, $d(\alpha)$ is an endomorphism of the algebra $L_{ab}$.

Furthermore,

$$
\text{Im}(d(\alpha)) = [L_{ab}, \alpha_{ab}] \quad \text{and} \quad \text{Ker}(d(\alpha)) \supseteq A_{L_{ab}}(\alpha_{ab}).
$$

This yields

$$
\dim_F \left( \frac{L_{ab}}{\text{Ker}(d(\alpha))} \right) \leq \dim_F \left( \frac{L_{ab}}{A_{L_{ab}}(\alpha_{ab})} \right).
$$

Hence,

$$
[L_{ab}, \alpha_{ab}] = \text{Im}(d(\alpha)) \cong \frac{L_{ab}}{\text{Ker}(d(\alpha))}.
$$

If $x \in A_L(\alpha)$, then $\alpha_{ab}(x + K) = \alpha(x) + K = K$, i.e., $x + K \in A_{L_{ab}}(\alpha_{ab})$. Therefore,

$$
(A_L(\alpha) + K) / K \leq A_{L_{ab}}(\alpha_{ab}).
$$

It follows from the inclusion $A_L(D) \leq A_L(\alpha)$ that

$$
\dim_F \left( \frac{L}{A_L(\alpha)} \right) \leq \dim_F \left( \frac{L}{A_L(D)} \right) = t.
$$

Thus, $\dim_F \left( \frac{L_{ab}}{A_{L_{ab}}(\alpha_{ab})} \right) \leq t$ and, hence, $\dim_F ([L_{ab}, \alpha_{ab}) \leq t$ for any $\alpha \in D$. 

Let $B = \{x_1, \ldots, x_k\}$ be a basis of the quotient algebra $D/\text{Ad}^L(L)$ and let $\beta \in \text{Ad}^L(L) + \alpha$. Then $\beta = 1_z + \alpha$ for some $z \in L$. For any element $y \in L$, we get

$$[y, \beta] = \beta(y) = (1_z + \alpha)(y) = 1_z(y) + \alpha(y) = [z, y] + [y, \alpha].$$

This means that

$$[y, \beta] + K = [z, y] + [y, \alpha] + K = ([z, y] + K) + ([y, \alpha] + K) = [z + K, y + K] + ([y, \alpha] + K).$$

Since the quotient algebra $L/K$ is Abelian, $[z + K, y + K] = 0$, i.e., $[y, \beta] + K = [y, \alpha] + K$. On the other hand,

$$[x + K, \alpha_{ab}] = \alpha_{ab}(x + K) = \alpha(x) + K = [x, \alpha] + K.$$

This implies that $[L_{ab}, \alpha_{ab}] = ([L, \alpha] + K)/K$.

It is clear that

$$[L, D] = \langle [L, \beta] | \beta \in D \rangle = \langle [L, \beta] | \beta \in \text{Ad}^L(L) + \alpha_j, 1 \leq j \leq k \rangle.$$

Hence,

$$([L, D] + K)/K = \sum_{\beta \in \text{Ad}^L(L) + \alpha_j, 1 \leq j \leq k} ([L, \beta] + K)/K = \sum_{1 \leq j \leq k} ([L, \alpha_j] + K)/K = \sum_{1 \leq j \leq k} [L_{ab}, (\alpha_j)_{ab}].$$

This means that $\dim_F([L, D] + K)/K \leq tk$, whence it follows that

$$\dim_F([L, D]) \leq tk + t^2 = t(k + t).$$

Theorem A is proved.

If $D = \text{Ad}^L(L)$, then $A_L(D) = \zeta(L)$ and $[L, D] = [L, L]$. As a result, we get the following direct analog of the Schur theorem for Leibniz algebras:

**Corollary 2.1** [8]. Let $L$ be a Leibniz algebra over the field $F$. If the dimension $t$ of the quotient algebra $L/\zeta(L)$ is finite, then $\dim_F([L, L]) \leq t^2$.

In particular, if $L$ is a Lie algebra, then we get the following direct analog of the Schur theorem for Lie algebras:

**Corollary 2.2** [15]. Let $L$ be a Lie algebra over the field $F$. If the dimension $t$ of the quotient algebra $L/\zeta(L)$ is finite, then $\dim_F([L, L]) \leq t(t + 1)/2$.

Repeating (almost word by word) the proof of Theorem 1 in [14], we can show that if the dimension $\dim_F(L/A_L(\text{Der}(L)))$ is finite, then the algebra $\text{Der}(L)$ is also finite-dimensional. Thus, if $D = \text{Der}(L)$,
then we arrive at the following direct analog of the Hegarty theorem for Leibniz algebras:

**Corollary 2.3.** Let $L$ be a Leibniz algebra over the field $F$. If the dimension $t$ of the quotient algebra $L/A_L(\text{Der}(L))$ is finite, then

$$\dim_F([L, \text{Der}(L)]) \leq t(t + 1).$$

In particular, if $L$ is a Lie algebra, then we get the following direct analog of the Hegarty theorem for Lie algebras:

**Corollary 2.4** [14]. Let $L$ be a Lie algebra over the field $F$. If the dimension $t$ of the quotient algebra $L/A_L(\text{Der}(L))$ is finite, then

$$\dim_F([L, \text{Der}(L)]) \leq t(t + 1)/2.$$ 

### 3. Proof of Theorem B

Consider the upper $D$-central series of the algebra $L$:

$$\langle 0 \rangle = Z_0 \leq Z_1 \leq \ldots \leq Z_{m-1} \leq Z_m = Z.$$

We proceed by induction on $m$. If $m = 1$, then $Z_1 = A_L(D)$, $\dim_F(L/Z_1) = t$, and by virtue of Theorem A, the subalgebra $[L, D] = \gamma_2(L, D)$ is finite-dimensional and its dimension does not exceed $t(k + t)$.

Assume that the result is true for some $m-1$ and that $L$ is a Leibniz algebra satisfying the conditions of the theorem for $z_l(L, D) = m$. We set $U = L/Z_1$. For every $\alpha \in D$, we define the mapping $\alpha_U : U \to U$ according to the rule

$$\alpha_U(x + Z_1) = \alpha(x) + Z_1$$

for any $x \in L$. Since

$$\alpha_U((x + Z_1) + (y + Z_1)) = \alpha_U(x + Z_1) + \alpha_U(y + Z_1)$$

and

$$\alpha_U(\lambda(x + Z_1)) = \lambda \alpha_U(x + Z_1),$$

we conclude that $\alpha_U$ is an endomorphism of the algebra $U$. Moreover,

$$\alpha_U([x + Z_1, y + Z_1]) = [\alpha_U(x + Z_1), y + Z_1] + [x + Z_1, \alpha_U(y + Z_1)].$$

Hence, $\alpha_U \in \text{Der}(U)$.

Consider a mapping $\eta : D \to \text{Der}(U)$ specified by the rule $\eta(\alpha) = \alpha_U$. It is clear that $\eta$ is a homomorphism. Let $\alpha \in \text{Ad}^d(L)$, i.e., $\alpha = l_a$ for some $a \in L$. Then $\eta(\alpha) = \alpha_U = (l_a)_U$ and

$$(l_a)_U(x + Z_1) = l_a(x) + Z_1 = [a, x] + Z_1 = [a + Z_1, x + Z_1] = l_a + Z_1(x + Z_1),$$

whence it follows that $\eta(\text{Ad}^d(L)) = \text{Ad}^d(U)$. This means that

$$\text{Ad}^d(U) = \eta(\text{Ad}^d(L)) \leq \eta(D)$$
and the dimension $\eta(D)/\text{Ad}^l(U)$ is finite and does not exceed $k$. Moreover,

$$\langle 0 \rangle = Z_1/Z_1 \leq Z_2/Z_1 \leq \ldots \leq Z_{m-1}/Z_1 \leq Z_m/Z_1$$

is the upper $D$-central series of $L/Z_1$. Since $\text{ad}(L/Z_1, \eta(D)) = m - 1$, by the induction assumption, the subalgebra $\gamma_m(L/Z_1, \eta(D))$ is finite-dimensional and there exists a function $\beta(k, m - 1, t)$ such that

$$\dim_F(\gamma_m(L/Z_1, \eta(D))) \leq \beta(k, m - 1, t).$$

We have

$$\gamma_m(L/Z_1, D) = (\gamma_m(L, D) + Z_1)/Z_1.$$ 

Let

$$K/Z_1 = \gamma_m(L/Z_1, \eta(D)) = \gamma_m(L/Z_1, D).$$

We also note that the quotient algebra $K/Z_1$ is finite-dimensional and

$$\dim_F(K/Z_1) \leq \beta(k, m - 1, t) = r.$$ 

Hence, we can apply Theorem A to $K$. We conclude that the subalgebra $[K, D]$ is finite-dimensional and

$$\dim_F([K, D]) \leq r(k + r).$$

Finally, since

$$\gamma_{m+1}(L, D) = [\gamma_m(L, D), D] \leq [K, D],$$

the subalgebra $\gamma_{m+1}(L, D)$ is finite-dimensional and its dimension does not exceed

$$r(k + r) = \beta(k, m, t).$$

Theorem B is proved.

The function $\beta(k, m, t)$ is specified by the rule $\beta(k, 1, t) = t(k + t)$ and, in the general case,

$$\beta(k, m + 1, t) = \beta(k, m, t)(k + \beta(k, m, t)).$$

If $D = \text{Ad}^l(L)$, then $\zeta_k(L, D) = \zeta_k(L)$ and $\gamma_k(L, D) = \gamma_k(L)$. As a result, we obtain the following direct analog of the Baer theorem for Leibniz algebras:

**Corollary 3.1** [8]. Suppose that $L$ is a Leibniz algebra over the field $F$. If the dimension $t$ of the quotient algebra $L/\zeta_k(L)$ is finite, then

$$\dim_F(\gamma_{k+1}(L)) \leq 2^{k-1}k^{k+1}.$$ 

Finally, if $L$ is a Lie algebra, then we get the following direct analog of the Baer theorem for Lie algebras:

**Corollary 3.2** [8, 13]. Suppose that $L$ is a Lie algebra over the field $F$. If $L/\zeta_k(L)$ has the finite dimension $t$, then $\dim_F(\gamma_{k+1}(L)) \leq t^k(t + 1)/2$. 
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