Maximum average with minimum cost method for solving assignment problem and comparative analysis of traditional method

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Abstract
Assignment problem is a special case of transportation problem in linear programming. In this article to introduce a new approach to solve the Assignment problem namely “Maximum Average with Minimum Cost Method (MAMCM)”. By using this method the optimum solution is reached with less calculation and this method is very easy to understand and apply it. A numerical illustration using MAMC method and existing method have been solved and compared. Some special cases of assignment problem have been discussed in this paper. MAMCM is different and very simple from the traditional method to apply in the assignment problems.

Keywords
Assignment Problem, Transportation Problem, Balanced and Unbalanced Assignment Problem, Hungarian Algorithm, Average cost method.

AMS Subject Classification
05C50.

1. Introduction
The assignment problem is a particular case of transportation problem where the objective is to assign a number of resources to equal number of activity on one to one basis. The main concept of Assignment Problem is to find the optimum allocation of a number of resources to an equal number of destination points. Application of Assignment Problem are various real world allocation problem deal with such as Production planning, Telecommunication, Vehicle routes, Economics, Several Construction Sites, Spectrum allocation etc. In this paper the MAMCM solve the Assignment Problem in both the case balanced and unbalanced. Especially this method used to select assignment is very easy for unbalanced Assignment Problem comparatively with other traditional method of Assignment Problem. The Assignment Problem deal with how to allocate \( n \) jobs to “\( m \)” machine. In an one-to-one fashion in the best possible way. The main aim of Assignment Problems is to allocate the resources in the suitable destination point with optimum cost. The first specialized method for the Assignment Problem was given by Kuhn \cite{1} in 1955 and was subsequently extended for the solution of much more
general network flow problems. In 1981 Bertsekas [2] studied the problem who gave new Algorithm for the solution of the problem. In 1980 Hung and Rome [3] solve the assignment problem by relaxation method. Different methods have been presented for Assignment Problem and various articles have been published in 2007 Anshuman and Rudrajit [4] solve Assignment Problem using Genetic Algorithm, and Simulated Annealing, the exact method [5] and also [6-8] for the history of these methods. Assignment problem refers to the analysis on how to assign objects to activity in the best possible way (optimal way) [9, 10]. In this article developed an optimum solution method for Assignment Problem. The corresponding method has been formulated and numerical example has been considered to illustrate the method. Finally the optimal solutions among MAMCM and traditional methods have been compared.

### 2. Mathematical Formulation of the Assignment Problem

Given \( n \) resources (or facilities) and \( n \) activities (or jobs) and effectiveness (in terms of cost, profit, time, etc.), of each resource (facility) for each activity (job), the problem lies in assigning each resource to one and only one activity (job) so that the given measure of effective is optimized. The data matrix for this problem is shown in Table 1.

### Table 1

| Resources (Workers) | Activity(jobs) | Supply |
|--------------------|---------------|--------|
| \( W_1 \)         | \( C_{11} \)   | 1      |
| \( W_2 \)         | \( C_{12} \)   | 1      |
| \( \vdots \)      | \( \vdots \)   | \( \vdots \) |
| \( W_n \)         | \( C_{1n} \)   | 1      |
| Demand            | 1             | \( n \) |

Let \( x_{ij} \) denote the assignment of facility \( i \) to job \( j \) such that:

\[
x_{ij} = \begin{cases} 
1, & \text{if } j^{th} \text{ job is assigned to } i^{th} \text{ machine} \\
0, & \text{otherwise}
\end{cases}
\]

Then the mathematical model of the assignment problem can be stated as:

**Minimize** \( Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}x_{ij} \)

**Subject to the conditions**

\[
\sum_{j=1}^{n} x_{ij} = 1, \quad \text{for all } i \text{ (resource available)}
\]

\[
\sum_{i=1}^{n} x_{ij} = 1, \quad \text{for all } j \text{ (activity requirement)}
\]

and \( x_{ij} = 0 \) or 1 for all \( i \) and \( j \); where \( c_{ij} \) represent the cost of assignment of resource \( i \) to activity \( j \).

### 3. Unbalanced Assignment Models

If the number of rows is not equal to the number columns in the cost matrix of the given assignment problem, then the given assignment problem is said to be unbalanced assignment problem.

In this type of assignment problem first convert it into balanced one by adding dummy rows (or) dummy columns with zero cost elements in the cost matrix depending upon whether \( m < n \) (or) \( m > n \).

**Case (i):** \( m < n \). In this case the number of rows is less than the number columns so adding dummy rows with zero cost elements. Now the number of rows is equal to the number columns in the cost matrix that is \( m = n \) and then solve by usual balanced method.

**Case (ii):** \( m > n \). In this case the number of rows is greater than the number columns so adding dummy columns with zero cost elements. Now the number of columns is equal to the rows in the cost matrix that is \( m = n \) and then solve by usual balanced method.

### 4. A Novel Approach for solving Assignment Problem

The stepwise procedure of proposed method is carried out as follows.

**Step 1:** Construct the assignment matrix (or) table for the given assignment problem and then, convert in to balanced one if not.

**Step 2:** Find the average value in each row and each column, it is denoted by \( u_i \) and \( v_j \); defined by

\[
u_i = \frac{1}{N} \sum_{j=1}^{n} c_{ij}, \quad 1 \leq i \leq n,
\]

and

\[
v_j = \frac{1}{N} \sum_{i=1}^{n} c_{ij}, \quad 1 \leq j \leq n
\]

where \( c_{ij} \)’s are cost element of assignment problem and \( N \) is number of element in its rows(columns).

**Step 3:** Identify the largest value of \( u_i \) and \( v_j \) and make an assignment to the smallest value in the selected row (or) column by marking a circle (O) around it and also cross out assigned row and its
column. If the selected minimum Value is tie, then choose the minimum value with largest value of $u_i$ and $v_j$.

**Step 4** Repeat Step 2 and 3 until the assignment reduce in to $2 \times 2$ matrix form. That is

\[
\begin{array}{cc}
  c_{11} & c_{12} \\
  c_{21} & c_{22}
\end{array}
\]

**Condition I:** If $c_{11} + c_{22} \leq c_{12} + c_{21}$ then assign $c_{11}$ and $c_{22}$ by marking a circle $(O)$ around it and also cross $(\times)$ out $c_{12}$ and $c_{21}$.

**Condition II:** If $c_{11} + c_{22} > c_{12} + c_{21}$ then assign $c_{12}$ and $c_{21}$ by marking a circle $(O)$ around it and also cross $(\times)$ out $c_{11}$ and $c_{22}$.

**Step 5:** Test for Optimality: Now each row and each column has one and only one assignment.

1. All the assigned value is less than unassigned value in its row and column. The current solution is optimum. Go to step 6.

2. Suppose the assigned value is less than or equal to unassigned value in its row and column. The current solution is optimum and Alternate optimum solution exists. Go to step 6.

Suppose not go to improved optimum solution.

**Improved optimum solution:** If at least one assigned value is greater than unassigned value then select the unassigned value and draw a closed loop Start and end to the selected unassigned cost and connect the assigned cost in its row and column. Now the closed loop has two assigned value and two unassigned values.

**Case (i):** If the sum of assigned values less than or equal to sum of unassigned values then the current solution is optimum go to step 6.

**Case (ii):** If the sum of assigned values greater than sum of unassigned values then the current solution is not optimum. Rearrange the allocation cross $(\times)$ the current assigned cost and make an assignment for unassigned cost in the closed loop and go to step 6.

**Step 6:** Finally to find the optimum solution of total assignment cost (or) Optimum value

Minimize $Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij}x_{ij}$.

### 5. Numerical Illustration

#### 5.1 Example 5.1

Five men are obtainable to do five different jobs from past records, the time (in hours) that each man takes to each job is known and is given in the following table

| Men | $J_1$ | $J_2$ | $J_3$ | $J_4$ |
|-----|------|------|------|------|
| $A$  | 5    | 7    | 11   | 6    |
| $B$  | 8    | 5    | 9    | 6    |
| $C$  | 4    | 7    | 10   | 7    |
| $D$  | 10   | 4    | 8    | 3    |

Find out how should be assigned the Jobs in way that will minimize the total time taken.

#### 5.2 Solution of Example 5.1

Using MAMCM Step 1:

| Men | Jobs   | $u_i$ | $v_j$ |
|-----|--------|------|------|
| $A$ | $J_1$  | 7    | 6.0  |
| $B$ | $J_2$  | (∞)  | 6.3  |
| $C$ | $J_3$  | 10   | 6.0  |
| $D$ | $J_4$  | 8    | –    |

Test for Optimality: Now each row and each column has one and only one assignment.

#### Step 2 and 3:

| Men | Jobs | $u_i$ |
|-----|------|------|
| $A$ | $J_1$ | 7    |
| $B$ | $J_2$ | (∞)  |
| $C$ | $J_3$ | 10   |
| $D$ | $J_4$ | 8    |

#### Step 4:

| Men | Jobs |
|-----|------|
| $A$ | $J_3$ |
| $D$ | (∞)  |

#### Step 5:

| Men | Jobs   | $u_i$ |
|-----|--------|------|
| $A$ | $J_1$  | 7    |
| $B$ | $J_2$  | (∞)  |
| $C$ | $J_3$  | 10   |
| $D$ | $J_4$  | 8    |

Step 6: The optimum assignment schedule is given by

Men $A \rightarrow$ Job 4, Men $B \rightarrow$ Job 2, Men $C \rightarrow$ Job 1, Men $D \rightarrow$ Job 3.

The optimum Assignment cost Minimize $z = Rs. 23$

**Note**

1. For the above problem, The alternative optimum exists therefore optimum assignment schedule is

Men $A \rightarrow$ Job 1, Men $B \rightarrow$ Job 2, Men $C \rightarrow$ Job 3, Men $D \rightarrow$ Job 4.

The optimum Assignment cost Minimize $z = Rs. 23$.

2. The optimum assignment schedule is Men $A \rightarrow$ Job 4, Men $B \rightarrow$ Job 3, Men $C \rightarrow$ Job 1, Men $D \rightarrow$ Job 2. The optimum cost Minimize $z = Rs. 23$. 
5.3 Example 5.2
Assign four trucks \(T_1, T_2, T_3\) and \(T_4\) to vacant spaces \(A, B, C, D, E, F\), so that the distance travelled is minimized. The table below shows the distance.

| Trucks | \(A\) | \(B\) | \(C\) | \(D\) | \(E\) | \(F\) |
|--------|-------|-------|-------|-------|-------|-------|
| \(T_1\) | 4     | 7     | 3     | 7     | 6     | 8     |
| \(T_2\) | 8     | 2     | 5     | 5     | 6     | 8     |
| \(T_3\) | 4     | 9     | 6     | 9     | 3     | 5     |
| \(T_4\) | 7     | 5     | 4     | 8     | 6     | 7     |

5.4 Solution of Example 5.2
Using MAMCM

Step 1:

| Trucks | \(A\) | \(B\) | \(C\) | \(D\) | \(E\) | \(F\) |
|--------|-------|-------|-------|-------|-------|-------|
| \(T_1\) | 4     | 7     | 3     | 7     | 0     | 0     |
| \(T_2\) | 8     | 2     | 5     | 5     | 0     | 0     |
| \(T_3\) | 4     | 9     | 6     | 9     | 0     | 0     |
| \(T_4\) | 7     | 5     | 4     | 8     | 0     | 0     |
| \(T_5\) | 6     | 3     | 5     | 4     | 0     | 0     |
| \(T_6\) | 6     | 8     | 7     | 3     | 0     | 0     |

The given Assignment problem is unbalanced. Convert in to balanced to introduce dummy columns \(T_5\) and \(T_6\) with zero distance.

Step 2 and 3:

| Jobs | Men | \(u_i\) | \(v_j\) |
|------|-----|---------|---------|
| \(A\) | 4   | 3.5    | 5.8     |
| \(B\) | 8   | 3.3    | 5.8     |
| \(C\) | 4   | 3.1    | 5.8     |
| \(D\) | 7   | 2.8    | 5.8     |
| \(E\) | 6   | 2.5    | 5.8     |
| \(F\) | 6   | 1      | 5.8     |

Step 4:

Step 5:

The optimum assignment schedule is given by \(A \rightarrow M_1, B \rightarrow M_2, C \rightarrow M_3, D \rightarrow M_4\).

The optimum Assignment cost Minimize \(Z = 12\)

5.5 Example 5.3
A company has four machines to do three jobs. Each job can be assigned to one and only one machine. The cost of each job in each machine is given in the following table.

| Machines | \(M_1\) | \(M_2\) | \(M_3\) | \(M_4\) |
|----------|--------|--------|--------|--------|
| \(A\)    | 18     | 24     | 28     | 32     |
| \(B\)    | 8      | 13     | 17     | 19     |
| \(C\)    | 10     | 15     | 19     | 22     |

5.6 Solution of Example 5.3
Using MAMCM

Step 1:

| Machines | \(M_1\) | \(M_2\) | \(M_3\) | \(M_4\) |
|----------|--------|--------|--------|--------|
| \(A\)    | 18     | 24     | 28     | 32     |
| \(B\)    | 8      | 13     | 17     | 19     |
| \(C\)    | 10     | 15     | 19     | 22     |

The given assignment problem is unbalanced. Convert into balanced one add a dummy job \(D\) with zero cost elements. The balanced assignment problem is

Step 2 and 3:
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| Jobs | Men | \( M_1 \) | \( M_2 \) | \( M_3 \) | \( M_4 \) | \( u_i \) |
|------|-----|--------|--------|--------|--------|--------|
| A    | 18  | 24     | 28     | 32     | 25.5   | 23.3   |
| B    | 8   | 13     | 17     | 19     | 14.3   | 12.7   |
| C    | 10  | 15     | 19     | 22     | 16.5   | 14.7   |
| D    | 0   | 0      | 0      | 0      | 0      | –      |
| \( v_j \) |      | 9.0    | 13     | 16     | 18.3   |
|      |      | 12.0   | 17.3   | 21.3   | –      |

Step 4:

| Machines | Jobs | \( M_1 \) | \( M_2 \) |
|----------|------|--------|--------|
| B        | 13   | ZZ     | 17     |
| C        | ZZ   | 15     | 19     |

Step 5:

| Machines | Jobs | \( M_1 \) | \( M_2 \) | \( M_3 \) | \( M_4 \) |
|----------|------|--------|--------|--------|--------|
| A        | 18   | 24     | 28     | 32     |
| B        | 8    | 13     | 17     | 19     |
| C        | ZZ   | ZZ     | 19     | 22     |
| D        | A    | A      | A      | (0)    |

Step 6:

The optimum Assignment schedule is given by \( A \rightarrow M_1, B \rightarrow M_2, C \rightarrow M_3, D \rightarrow M_4 \).

The optimum Assignment cost Minimize \( z = 50 \) units of cost.

Note: For the above problem have the alternative optimum solution. Therefore the optimum assignment schedule is Job \( A \rightarrow M_1, B \rightarrow M_3, C \rightarrow M_2, D \rightarrow M_4 \) with the same optimum cost Minimize \( z = 50 \) units of cost.

6. Results and Discussion

After obtaining an optimum solution by the proposed “Maximum Average with Minimum Cost Method” (MAMCM), the result is compared with the results obtained by Hungarian Method are shown in Table 2.

| Example | No. of Steps | Optimum solution |
|---------|--------------|------------------|
|         | Hungarian    | MAMCM            | Hungarian | MAMCM |
| 5.1     | 10           | 6                | 9         | 9      |
| 5.2     | 10           | 6                | 12        | 12     |
| 5.3     | 10           | 6                | 50        | 50     |

As observed from Table 2, the MAMCM provides comparatively a best Optimum solution than the results obtained by the traditional algorithm by Hungarian Method. Efficiency of MAMCM has also been tested by solving several number of time minimizing balanced and unbalanced Assignment problems and it is found that the MAMCM yields comparatively a best result with minimum calculation.

7. Conclusion

A new algorithm MAMCM is developed to acquire an optimal solution with minimum steps for the Assignment problems. In a real life situation, such that the changing fiscal condition, allocation problems is one of the important factor for decision makers. The proposed algorithm is used for balanced and unbalanced types of assignment problems. Especially this method is very convenient to select the minimum cost for unbalanced assignment problems. Compare to existing method the efficiency of MAMCM showed by numerical example and also the optimum solution of MAMCM is same as obtainable method. It can be conclude that MAMCM is different approach to solve the Assignment problems with less calculation and time. It is very helpful for decision makers who are dealing with resource allocation problem.

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