A new class of stable (2 + 1) dimensional thin shell wormhole

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Recently, Bañados, Teitelboim and Zanelli (BTZ) [1] has discovered an explicit vacuum solution of (2+1)-dimensional gravity with negative cosmological constant. It has been argued that the existence of such physical systems with an event horizon and thermodynamic properties similar to (3+1) dimensional black holes. These vacuum solutions of (2+1)-dimensional gravity are asymptotically anti-de Sitter and are known as BTZ black holes. We provide a new type of thin-shell stable wormhole from the BTZ black holes. This is the first example of stable thin shell wormhole in (2+1)-dimension. Several characteristics of this thin-shell wormhole have been discussed.

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I. INTRODUCTION

Pure gravity in (2+1) dimensions has renewed interest in recent years as gravity does not propagate out side the source, in other words, matter curves spacetime only locally. Recently, very interesting studies about (2 + 1) gravastars, (2 + 1) black holes, (2 + 1) wormholes and (2 + 1) dimensional stars have been developed [2–16].

In recent past, Morris and Thorne [17] have shown that general relativity admits that the presence of matter twists the geometric fabric of spacetime in such a way that a bridge can be formed between two asymptotic regions. This bridge can be interpreted as instantons describing a tunnelling between two distant regions. This geometrical structure is known as Wormhole. However, according Morris and Thorne [17], the construction of wormhole would require a very unusual form of matter, termed as exotic matter that characterized the stress energy tensor.

The necessary ingredients that supply fuel to construct wormholes remain an elusive goal for theoretical physicists. Several proposals have been proposed in literature [18–30]. Also some authors have used alternative theories of gravity to exclude exotic matter, such as Brans-Dicke theory, Brain world, C-field theory, Kalb-Ramond, Einstein-Maxwell theory etc [31–37].

To minimize the usage of exotic matter, Visser [38] has proposed a way, which is known as ‘Cut and Paste’ technique, to construct a wormhole in which the exotic matter is concentrated at the wormhole throat.

Visser’s tactics is more accessible because it minimizes the amount of exotic matter required. For this reason, Visser’s approach was adopted by various authors to construct thin shell wormholes [39–63].

Recently, Bañados, Teitelboim and Zanelli (BTZ) [1] has discovered an explicit vacuum solution of (2+1)-dimensional gravity with negative cosmological constant. It has been argued that the existence of such physical systems with an event horizon and thermodynamic properties similar to (3+1) dimensional black holes. These vacuum solution of (2+1)-dimensional gravity are asymptotically anti-de Sitter and are known as BTZ black holes. Motivated by the properties of this black hole solution, the authors have developed other interesting work about the geometry of the spinning black holes of standard Einstein theory in (2 + 1) dimension [3].

In this paper, we present a new type of thin-shell wormhole employing such a class of BTZ black holes by means of the cut-and-paste technique [38]. As far as our knowledge, this is the first example of stable thin shell wormhole in (2+1)-dimension. Various aspects of this thin-shell wormhole are analyzed, particularly the equation of state relating pressure and density. Also it has been discussed the attractive or repulsive nature of the wormhole. We also make a survey of stability of this wormhole.

II. THIN-SHELL WORMHOLE CONSTRUCTION

Bañados, Teitelboim and Zanelli (BTZ) [1] have discovered an explicit vacuum solution of (2+1)-dimensional gravity with negative cosmological constant as

$$ds^2 = -(M_0 - \Lambda r^2) dt^2 + (M_0 - \Lambda r^2)^{-1} dr^2 + r^2 d\theta^2.$$  \hspace{1cm} (1)

The parameter $M_0$ is the conserved mass associated with asymptotic invariance under time displacements.
mass is given by a flux integral through a large circle at space-like infinity.

From the BTZ black hole, we can take two copies of the region with \( r \geq a \):

\[
M^\pm = (x \mid r \geq a)
\]

and paste them at the junction surface

\[
\Sigma = \Sigma^\pm = (x \mid r = a)
\]

Here we take \( a > r_h \) to avoid horizon and this new construction produces a geodesically complete manifold \( M = M^+ \cup M^- \) with a matter shell at the surface \( r = a \), where the throat of the wormhole is located.

Here the junction surface is a one dimensional ring of matter. Let, \( \eta \) denotes the Riemann normal coordinate at the junction. We assume \( \eta \) be positive and negative in two sides of the junction surface.

Mathematically, we have \( x^\mu = (\tau, \theta, \eta) \) and the normal vector components \( \xi^\mu = (0, 0, 1) \) with the metric

\[
g_{\eta \eta} = 1, \quad g_{\eta i} = 0 \quad \text{and} \quad g_{ij} = \text{diag} \begin{pmatrix} -1, r^2 \end{pmatrix}.
\]

The second fundamental forms associated with the two sides of the shell \( \Sigma^+ \) are given by

\[
K^\pm_j = \frac{1}{2} g^{ik} \frac{\partial g_{kj}}{\partial \eta} \bigg|_{\eta = \pm 0} = \frac{1}{2} \frac{\partial r}{\partial \eta} \bigg|_{r=a} g^{ij} \frac{\partial g_{kj}}{\partial r} \bigg|_{r=a}.
\]

So, the discontinuity in the second fundamental forms is given as

\[
\kappa_{ij} = K^+_i - K^-_i \quad (3)
\]

Now, from Lanczos equation in (2+1) dimensional spacetime, one can obtain the surface stress energy tensor \( S_j^i = \text{diag}(\sigma, -v) \) where, \( \sigma \) and \( v \) are line energy density and line tension, respectively \cite{63} as

\[
\sigma = -\frac{1}{8\pi} \kappa^\theta, \quad (4)
\]

\[
v = -\frac{1}{8\pi} \kappa^\tau. \quad (5)
\]

To understand the dynamics of the wormhole, we assume the radius of the throat to be a function of proper time, or \( a = a(\tau) \). Also, overdot and prime denote, respectively, the derivatives with respect to \( \tau \) and \( a \).

Employing relevant information into Eqs. (1-5) and setting \( r = a \), we obtain

\[
\sigma = -\frac{1}{4\pi a} \left[ \sqrt{K^2a^2 - M_0 + \dot{a}^2} \right], \quad (6)
\]

\[
p = -v = \frac{1}{4\pi} \left[ \frac{K^2a}{\sqrt{K^2a^2 - M_0}} \right]. \quad (7)
\]

In the above equations, the overdot denotes, the derivatives with respect to \( \tau \). Note that line tension \( (v) \) is negative which implies that there is a line pressure \( (p) \) as opposed to a line tension. Here, we have used \( -\Lambda = K^2 \).

For a static configuration of radius \( a \), we obtain (assuming \( \dot{a} = 0 \) and \( \ddot{a} = 0 \)) from Eqs. (6) and (7)

\[
\sigma = -\frac{1}{4\pi a} \left[ \sqrt{K^2a^2 - M_0} \right], \quad (8)
\]

\[
p = -v = \frac{1}{4\pi} \left[ \frac{K^2a}{\sqrt{K^2a^2 - M_0}} \right]. \quad (9)
\]

Observe that the energy density \( \sigma \) is negative, however, the pressure \( p \) is positive. Moreover, on this shell, which is infinitely thin, the radial pressure is zero. One can note that \( \rho + \sigma, \sigma + 2p \) are positive. So the shell contains matter that violates only the null energy condition (NEC) and obeys the weak and strong energy condition. For the assumed case, variations of left hand side of the expressions of energy conditions have been shown in Figure 1.

\[
\text{FIG. 1: Variations of the expressions of energy conditions shown against } a.
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III. THE GRAVITATIONAL FIELD

Now, to study the nature of the gravitational field of the wormhole constructed from BTZ black hole, we calculate the observer’s three-acceleration \( a^\mu = u^\nu\rho u^\rho, \) where \( u^\nu = dx^\nu/d\tau = (1/\sqrt{K^2r^2 - M_0}, 0, 0) \). In view of the line

\[
\text{FIG. 1: Variations of the expressions of energy conditions shown against } a.
\]
element, Eq. (1), the only non-zero component is given by

\[ a^r = \Gamma^r_{tt} \left( \frac{dt}{d\tau} \right)^2 = K^2 r \]  

(10)

A radially moving test particle initially at rest obeys the equation of motion

\[ \frac{d^2 r}{d\tau^2} = -\Gamma^r_{tt} \left( \frac{dt}{d\tau} \right)^2 = -a^r. \]  

(11)

If \( a^r = 0 \), we obtain the geodesic equation.

However, we note that the wormhole is attractive since \( a^r > 0 \). This result is similar to the case of the thin shell wormhole constructed from Schwarzschild black hole [58].

**IV. TIME EVOLUTION OF RADIUS OF THE THROAT:**

Equations (8) and (9) can be written in the form

\[ v_0 = w(a) \sigma_0 \]  

(12)

where

\[ w(a) = \frac{K^2 a^2}{K^2 a^2 - M_0} \]  

(13)

[ suffix 0 indicates the static situation ]

Following Eiroa et al [43], we assume that the equation of state does not depend on the derivative of \( a(\tau) \) i.e. it is the same form as in the static one. Now putting \( \sigma \), \( v \) in place of \( \sigma_0 \), \( v_0 \) from (7) and (8) in (12), we get the following expression as

\[ \dot{a}(\tau) = \dot{a}^2(\tau_0) \left[ \frac{K^2 a^2(\tau_0) - M_0}{K^2 a^2(\tau) - M_0} \right] \]  

(15)

Here, \( \tau_0 \) is arbitrary fixed time.

After integration, this gives,

\[ a(\tau) \sqrt{K^2 a^2(\tau) - M_0} - a(\tau_0) \sqrt{K^2 a^2(\tau_0) - M_0} = \frac{M_0}{K} \left[ \cosh^{-1} \left( \frac{a(\tau) K}{\sqrt{M_0}} \right) - \cosh^{-1} \left( \frac{a(\tau_0) K}{\sqrt{M_0}} \right) \right] \]

\[ = 2a(\tau_0) \sqrt{K^2 a^2(\tau_0) - M_0} - M_0(\tau - \tau_0) \]  

(16)

The above implicit expression gives the time evolution of the radius of the throat.

The velocity and acceleration of the throat are

\[ \dot{a}(\tau) = \dot{a}(\tau_0) \sqrt{\frac{K^2 a^2(\tau_0) - M_0}{K^2 a^2(\tau) - M_0}} \]  

(17)

and

\[ \ddot{a}(\tau) = -\dot{a}^2(\tau_0) K^2 a(\tau) \left[ \frac{K^2 a^2(\tau_0) - M_0}{(K^2 a^2(\tau) - M_0)^2} \right] \]  

(18)

From the above two expressions indicate that sign of the initial velocity determines sign of the velocity and the acceleration is always negative. It is immaterial whether the initial velocity is positive or negative, the throat expands forever. This would imply that the equilibrium position is always unstable. However, if the initial velocity is zero, the velocity and acceleration of the throat would be zero i.e. throat be in static equilibrium position.

**V. THE TOTAL AMOUNT OF EXOTIC MATTER**

Now, we determine the total amount of exotic matter confined within the shell. This total amount of exotic matter can be quantified by the integral [44, 49, 51]

\[ \Omega_\sigma = \int [\rho + p_r] \sqrt{-g} \, d^2 x. \]  

(19)

By introducing the radial coordinate \( R = r - a \), we get

\[ \Omega_\sigma = \int_0^{2\pi} \int_{-\infty}^{\infty} [\rho + p_r] \sqrt{-g} \, dR \, d\theta. \]

Since the shell is infinitely thin, it does not exert any radial pressure. Moreover, energy density is located on the thin shell surface, \( \rho = \delta(R) \sigma(a) \). Then we have,

\[ \Omega_\sigma = \int_0^{2\pi} [\rho \sqrt{-g}]_{r=a} \, d\theta = 2\pi a \sigma(a) \]

\[ = -\frac{1}{2} \sqrt{K^2 a^2 - M_0} \]  

(20)

This NEC violating matter can be reduced by taking the value of \( a \) closer to \( r_h \), the location of the event horizon. The closer \( a \) is to \( r_h \), however, the closer the wormhole is to a black hole: incoming microwave background radiation would get blueshifted to an extremely high temperature [69]. The variation of the total amount of exotic matter with respect to the conserved mass and cosmological constant can best be seen graphically (Figure 2). Observe that the NEC violating matter on the thin shell can be reduced by increasing the conserved mass.

**VI. STABILITY**

Recall, if the initial velocity is zero, the velocity and acceleration of the throat would be zero i.e. throat be in static equilibrium position. Now, we shall study the stability of the configuration using two different approaches: (i) assuming Chaplygin gas equation of state on the thin shell and (ii) analyzing the stability to linearized radial perturbations around static solution situated at \( a_0 \).
A. Chaplygin gas equation of state

We analyze the stability of the shell taking Chaplygin gas equation of state at the throat.

We re-introduce an equation of state between the surface pressure $p$ and surface energy density $\sigma$ as

$$p = -\frac{A}{\sigma^\alpha}$$  \hspace{1cm} (21)

with $A$ and $\alpha$ are positive constants.

Putting equations (6) and (7) in equation (21), one gets the differential equation for the throat radius of thin shell wormholes threaded by Chaplygin gas-like exotic matter is

$$\frac{(-1)^\alpha}{(4\pi)^{\alpha+1}a^\alpha}(K^2a + \dot{a})(K^2a^2 - M_0 + \dot{a}^2)^{\alpha-1} + A = 0$$  \hspace{1cm} (22)

If the static solution exists (i.e. $\dot{a} = 0$, $\ddot{a} = 0$), then the throat radius ($a = a_0$) should satisfy the equation given by

$$\frac{(-1)^\alpha K^2a}{(4\pi)^{\alpha+1}a^\alpha}(K^2a^2 - M_0)^{\alpha-1} + A = 0$$  \hspace{1cm} (23)

Solution of equation (23) gives the radius of the throat for $\alpha < 1$. Assuming LHS of equation (23) as $H(a) = 0$ and we plot $H(a)$ vs. $a$. $H(a)$ cuts $a$ axis at some $a = a_0$, gives the radius of the throat. For different values of $\alpha$, we get different values of radius of the throat.

However, if we relax the restriction on $\alpha > 1$, one can also get the radius of the throat of the thin-shell wormholes constructed from the BTZ black holes (see figure 4).

Here $p$ and $\sigma$ obey the conservation equation

$$\dot{\sigma} + \frac{\dot{a}}{a}(p + \sigma) = 0.$$  \hspace{1cm} (24)

In the above equations, the overdot denotes, the derivative with respect to $\tau$. 
Using (21), from (24), we get
\[ \dot{\sigma} + \frac{\dot{a}}{a} \left( \frac{\sigma^{\alpha+1} - A}{\sigma^{\alpha}} \right) = 0. \] (25)

Solution of the above equation can be found as
\[ \sigma = \left[ A + \left( \sigma_0^{\alpha+1} - A \right) \left( \frac{a_0}{a} \right)^{\alpha+1} \right]^{\frac{1}{\alpha+1}} \] (26)
where, \( \sigma_0 = \sigma(a_0) \).

Equation (6) implies
\[ \dot{a}^2 + V(a) = 0. \] (27)

Here the potential \( V(a) \) is defined as
\[ V(a) = K^2 a^2 - M_0 - 16\pi^2 a^2 \left[ A + \left( \sigma_0^{\alpha+1} - A \right) \left( \frac{a_0}{a} \right)^{\alpha+1} \right]^{\frac{2}{\alpha+1}} \] (28)
Expanding \( V(a) \) around \( a_0 \), we obtain
\[ V(a) = V(a_0) + V'(a_0)(a - a_0) + \frac{1}{2} V''(a_0)(a - a_0)^2 + O \left[ (a - a_0)^3 \right], \] (29)
where the prime denotes the derivative with respect to \( a \). Since we are linearizing around \( a = a_0 \), we require that \( V(a_0) = 0 \) and \( V'(a_0) = 0 \). The configuration will be in stable equilibrium if \( V''(a_0) > 0 \). Using the conditions, \( V(a_0) = 0 \) and \( V'(a_0) = 0 \), we obtain from equation (28) as
\[ V''(a_0) = \frac{2(\alpha - 1)M_0K^2}{K^2a_0^2 - M_0} \] (30)
We find that the inequality \( V''(a_0) > 0 \) can only be satisfied if \( \alpha > 1 \). So we conclude that the wormhole is stable. However, for \( \alpha < 1 \), \( V''(a_0) < 0 \) and the wormhole is unstable.

One can also analysis the stability by means of the figures. The plot (figure 5) indicates that \( V(a) \) has a local minimum at some \( a \). In other words, it is stable for \( \alpha > 1 \). However, the plot (figure 6) indicates that \( V(a) \) has a local maximum at some \( a \). In other words, it is unstable for \( \alpha < 1 \). Thus for some specific choices of the parameter \( \alpha \), the thin-shell wormholes constructed from the BTZ black holes is stable.

**B. Linearized radial perturbations**

From equation (27), the potential \( V(a) \) can be written in terms of \( \sigma \) as
\[ V(a) = K^2 a^2 - M_0 - 16\pi^2 a^2 \sigma^2. \] (31)

The wormhole is stable.

The wormhole is unstable.

As above, expanding \( V(a) \) around \( a_0 \), we obtain
\[ V(a) = V(a_0) + V'(a_0)(a - a_0) + \frac{1}{2} V''(a_0)(a - a_0)^2 + O \left[ (a - a_0)^3 \right], \] (32)
where the prime denotes the derivative with respect to \( a \). Since we are linearizing around \( a = a_0 \), we require that \( V(a_0) = 0 \) and \( V'(a_0) = 0 \). The configuration will be in stable equilibrium if \( V''(a_0) > 0 \). The subsequent
analysis will depend on a parameter $\beta$, which is usually interpreted as the subluminal speed of sound and is given by the relation

$$\beta^2(\sigma) = \left. \frac{\partial p}{\partial \sigma} \right|_\sigma.$$

Recall conservation equation (24) which readily yields,

$$\sigma'' - \frac{1}{a^2}(p + \sigma)(2 + \beta^2) = 0. \quad (33)$$

Differentiating twice the potential $V(a)$, we get,

$$V''(a) = 2K^2 - 32\pi^2 \beta^2(p\sigma + \sigma^2) - 32\pi^2 p^2. \quad (34)$$

Replacing $\sigma''$ from equation (31), we find

$$V''(a) = 2K^2 + \frac{2}{a^2}M_0 \beta^2 - \frac{2K^4 a^2}{K^2 a^2 - M_0} \quad (35)$$

Now, for stability $V(a_0') > 0$ implies

$$\beta_0^2 < \frac{K^2 a_0^2}{K^2 a_0^2 - M_0} \quad (36)$$

There is a region of stability corresponding to $\beta_0^2 < \frac{K^2 a_0^2}{K^2 a_0^2 - M_0}$. Figure 7 shows a typical region of stability (below the curve) by choosing $M_0 = 3$ and $K^2 = 2$. However, if the radius of the throat is large enough, then the region of stability barely reaches down to $\beta_0^2 = 1$.

It is known that for ordinary matter, $\beta^2$ represents the velocity of sound, however, according to Poisson et al. [39] this interpretation of $\beta^2$ can be questioned as we are dealing with exotic matter and wormhole configurations with $\beta_0^2 > 1$ should not be ruled out. Although the small value of the radius of the throat $\beta_0^2$ greater than one, however, for large value of that, $\beta_0^2 \approx 1$. This implies the region of stability lies in region for which stable wormholes might be physically acceptable.

**VII. FINAL REMARKS**

In this paper, a new type of thin-shell wormhole has been developed using the BTZ black holes and the cut-and-paste technique. We provide an analysis of some aspects of the thin-shell wormhole pointing out the equation of state relating pressure and density, the attractive or repulsive nature of the gravitational field of the wormhole and its stability.

We obtained that the energy density $\sigma$ is negative and the pressure $p$ is positive. We get, $p + \sigma > 0$ and $\sigma + 2p > 0$, and we can conclude that the matter contained by the shell violates only null energy condition (NEC), but obeys the weak and strong energy conditions.

From the study of the gravitational field of the wormhole, we obtain that the wormhole is attractive since $a^r > 0$. We notice the similarity of this result with the one obtained in the case of the thin shell wormhole constructed from Schwarzschild black hole [58].

The stability of the wormhole is studied using two particular cases: (i) assuming Chaplygin gas equation of state on the thin shell and (ii) making an analysis of the stability to linearized radial perturbations around static solution situated at $a_0$. In the first case, we conclude that the thin wormhole which is constructed from BTZ black hole threaded by Chaplygin gas equation of state is stable. In the second case, of linearized radial perturbations, there is a region of stability corresponding to $\beta_0^2 < \frac{K^2 a_0^2}{K^2 a_0^2 - M_0}$.

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