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Classification of knots in $T \times I$ with at most 4 crossings

We compose the table of knots in the thickened torus $T \times I$ having diagrams with $\leq 4$ crossings. The knots are constructed by the three-step process. First we list regular graphs of degree 4 with $\leq 4$ vertices, then for each graph we enumerate all corresponding knot projections, and after that we construct the corresponding minimal diagrams. Several known and new tricks made it possible to keep the process within reasonable limits and offer a rigorous theoretical proof of the completeness of the table. For proving that all knots are different we use a generalized version of the Kauffman polynomial.

*Key words:* knot, thickened torus, knot table.

**Introduction**

The interest in knots in manifolds of type $F \times I$, where $F$ is closed orientable surface, has increased in recent years. The torus $T = S^1 \times S^1$ is the most simple closed orientable surface after $S^2$. So the theory of knots in $T \times I$ is a natural generalization of the theory of knots in $S^2 \times I$, which is equivalent to the theory of knots in $S^3$. Knots in $T \times I$ can be represented by diagrams similar to spherical diagrams of classical knots. The Reidemeister moves play the same role: they implement knot isotopies.

First tables of knots had been composed by P. Tait in 1876 [1]. Then these tables had been enlarged ([2, 3]). Now there exist tables of knots in $S^3$ having diagrams with $\leq 16$ and even $\leq 22$ crossings [4, 5]. On the other hand, there are only a few papers on tabulation of knots in thickened surfaces, see [6,7] for knots in $RP^2 \times I$, which is the punctured projective space. An efficient method for tabulating tangles is described in [8]. Links in the thickened torus had been studied in [9, 10]. See also [11].

This paper is devoted to tabulating knots in the thickened torus $T \times I$ having diagrams with $\leq 4$ crossings. The knots are constructed by the three-step process. First we list regular graphs of degree 4 with $\leq 4$ vertices, then for each graph we enumerate all corresponding knot projections, and after that we construct the corresponding minimal diagrams. Several known and new tricks made it possible to keep the process within reasonable limits and offer a rigorous theoretical proof of the completeness of the table. For proving that all knots are different we use a generalized version of the Kauffman polynomial [12], see also [13].

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1Both authors are supported by RFBR, project no. 12-01-00748, Scientific School Grant no. 1414.2012.1, and the joint research project 12-C-1-1018-1 of Ural and Siberian branches of RAS
§1. The main result

**Definition 1.** Let \( T = S^1 \times S^1 \) be the two-dimensional torus and let \( I \) be the interval \([0, 1]\). A knot in \( T \times I \) is an arbitrary simple closed curve \( K \subset T \times I \). Two knots \( K, K' \subset T \times I \) are equivalent if the pairs \((T \times I, K), (T \times I, K')\) are homeomorphic.

Knots in \( T \times I \), as well as classical knots, can be represented by projections and diagrams. By a projection of a knot in \( T \times I \) we mean a regular graph \( G \subset T \) of degree 4 such that the “straight ahead” rule determines a cycle composed of all the edges of \( G \). This cycle can be converted into a knot diagram by breaking it in each crossing point to show which strand is going over the other. Two projections \( G, G' \) are called equivalent if the pairs \((T, G), (T, G')\) are homeomorphic. The diagram equivalence has the same meaning. In addition we allow simultaneous crossing change at all crossings.

**Definition 2.** A diagram of a knot \( K \subset T \times I \) is called minimal if its complexity (the number of crossings) is no more than the complexity of any knot equivalent to \( K \). A projection \( G \subset T \) is minimal if at least one of the corresponding knot diagrams is minimal.

We say that a knot \( K \subset T \times I \) is local if it is contained in a ball \( V \subset T \times I \) and composite, if there is a ball \( V \subset T \times I \) such that \( \partial V \) decomposes \( K \) into two nontrivial arcs in \( V \) and the complement of \( V \). Our table consists of knots which are prime, that is, neither local nor composite.

**Theorem 1.** There exist exactly 64 different prime knots in \( T \times I \) having \( \leq 4 \) crossings. Diagrams of those knots are shown in Fig. 1.

The proof of Theorem 1 consists of three steps. First we list all abstract regular graphs with \( \leq 4 \) vertices and classify all prime projections in \( T \). Then we enumerate all corresponding diagrams. By performing this step we use different tricks for removing duplicates (i.e. diagrams representing equivalent knots). At the last step we use a generalized version of the Kauffman polynomial for proving that all knots thus obtained are different.

§2. The enumeration of graphs and projections

**Lemma 1.** Any regular graph \( G \) with \( n \leq 4 \) vertices contains a loop or a multiple edge.

*Proof.* To the contrary, suppose that \( G \) contains no loops and multiple edges. Denote by \( N \) the number of edges of \( G \). Then \( N \) is at most \( C_n^2 = n(n - 1)/2 \). On the other hand, we have the equality \( N = 2n \), because \( G \) is regular. This contradicts the assumption \( n \leq 4 \). \[ \]

Note that prime projections have no trivial loops, since any trivial loop in a knot diagram can be removed by the first Reidemeister move. Moreover, any knot projection cannot contain more than two nontrivial loops (otherwise the projection would be disconnected).
**Lemma 2.** There exist exactly 15 regular graphs with $\leq 4$ vertices and $\leq 2$ loops, including the circle without vertices. See Fig. 2.

![Regular graphs with $\leq 4$ vertices and $\leq 2$ loops](image)

**Proof.** It follows from Lemma 1 that all regular graphs with $n \leq 4$ vertices can be obtained from the circle by $n$ operations of the following two types: 1) insertion of a loop, and 2) identification of a point on a loop with a point on another edge. It remains to construct all graphs which can be obtained from the circle by 1, 2, 3 or 4 such operations and remove all duplicates and graphs with $\geq 3$ loops.

**Definition 3.** Suppose that a projection $G$ and a disk $D$ in $T$ are chosen so that $G \cap D$ consists of two disjoint proper arcs $l_1, l_2 \subset D$. Let $l'_1, l'_2 \subset D$ be two new arcs such that they have the same endpoints and $l'_1 \cap l'_2$ consists of two transverse points. Then the biangle addition consists in replacing $l_1, l_2 \subset D$ by $l'_1, l'_2 \subset D$.

For performing this operation it suffices to choose a simple arc $\alpha \subset T$ connecting two non-vertex points of $G$ and a regular neighborhood $D \subset T$ of $\alpha$, see Fig. 3. The inverse operation is called the biangle removal.

Let $G \subset T$ be a projection. By analogy with knot diagrams we will say that $G$ is local if it is contained in a disk in $T$, and composite if there is a disk $D \subset T$ such that neither $D \cap G$ nor $(T \setminus \text{Int}D) \cap G$ are trivial arcs. In particular, prime projections have no trivial loops.

**Theorem 2.** There exist exactly 36 different prime projections in $T$ with $\leq 4$ crossings, see Fig. 4. All these projections are minimal.

**Proof.** It is easy to see that any projection without vertices (i.e. any nontrivial circle in $T$) is equivalent to the projection $0_1$ in Fig. 4. Let $G$ be a prime
Figure 3: Biangle addition

Figure 4: Projections in $T$, which is represented as a square with identified opposite sides
projection corresponding to one of the graphs \(b, c, e, \) and \(h\). We cut \(G\) at all vertices so as to produce \(n + 1\) disjoint circles, where \(n\) is the number of vertices of \(G\). Circles corresponding to the loops of \(G\) are nontrivial. It follows that they are parallel in \(T\). Each of the remaining circles may be either trivial or not. We list all combinations of their types and obtain projections \(1_1, 2_1 - 2_2, 3_1 - 3_3, 4_1 - 4_6\) for \(n = 1, 2, 3, 4\) respectively.

Let us consider a prime projection \(G\) of type \(d\). It is the union of two circles with two common points. With respect to the way how \(G\) lies in \(T\), exactly one of these points is transverse, since otherwise we would have a link of two components. Let us cut \(G\) at the non-transverse point so as to get two circles \(\mu, \lambda\) with one transverse crossing point. The complement to the circles in \(T\) is a disk. It follows that there is only one way to perform the inverse operation, i.e. identification of a point of \(\mu\) with a point of \(\lambda\). The identification produces a projection equivalent to \(2_3\).

Let us prove that there are no prime projections corresponding to the graphs \(f, i, j, k\). Indeed, every such projection \(G\) can be obtained from the projection \(2_3\) by attaching one or two loops. The complement to \(2_3\) in \(T\) consists of two disks. It follows that the loops are trivial, in contradiction with our assumption that \(G\) is prime.

Let us enumerate projections corresponding to the graph \(g\), which consists of three circles such that every circle has one common point with each of the two other circles. Suppose that the common point of two circles (denote them \(\mu, \lambda\)) is transverse. As before, the complement to \(\mu \cup \lambda\) in \(T\) consists of two disks. Therefore there is only one way to place the third circle into \(T\). We get the projection \(3_7\).

Suppose that in all three common points of the circles the intersection is non-transverse. Each circle can be either trivial or not. Investigation of all possibilities shows that one can get a projection of a knot only in the case when the number of trivial circles is odd \((3 \text{ or } 1)\). Moreover, in the first case \(G\) must be contained in an annulus in \(T\) having connected complement. This gives the projection \(3_5\). In the second case we get projections \(3_4\) and \(3_6\), depending on how a nontrivial circle and the trivial one are approaching to the second nontrivial circle: from the same side or not.

Let us list all projections of type \(l\). Each of them can be obtained from some projection \(G\) of type \(g\) by inserting a loop, which must be nontrivial. Suppose that \(G\) is prime. Hence it is one of the projections \(3_4, 3_5\). Since the complements to \(3_6\) and \(3_7\) consist of disks, one cannot insert a nontrivial loop. There is only one way to add a nontrivial loop to \(3_4\) as well as to \(3_5\). Doing so we get projections \(4_7\) and \(4_8\).

Suppose that \(G\) is not prime. Then it is either the standard projection of a local trefoil, or a projection composed from a local trefoil and a nontrivial circle embedded in \(T\). In the first case inserting a loop gives us a composite projection, in the second one we get two projections \(4_9\) and \(4_{10}\) of type \(l\).

We construct projections of type \(m\) in the same way as the projections of type \(g\). Any projection of type \(m\) consists of four circles such that each circle has two common point with the other three circles. Suppose that at least one
of the common point of two circles (denote them \( \mu, \lambda \)) is transverse. As before, the complement to \( \mu \cup \lambda \) in \( T \) is a disk. Therefore there is only one way to place two other circles into \( T \). We get the projection 4\(_{11}\).

Suppose that at all four common points of the circles the intersection is non-transverse. Each circle can be either trivial or not. As in the case of graph \( g \), investigation of all possibilities shows that one can get a projection of a knot only in the case when the number of trivial circles is odd (1 or 3). We get the projection 4\(_{12}\) in the first case and the projection 4\(_{13}\) in the second one.

Note that any projection \( G \) of type \( n \) contains two triple edges. Then the projection \( G' \subset T \) obtained from \( G \) by removing two edges \( e_1, e_2 \) from one of those triple edges is of type \( d \) and thus is equivalent to the projection 2\(_3\). The complement to \( G' \) in \( T \) consists of two disks. In order to restore \( G \) we should add \( e_1, e_2 \) to \( G' \) so that they have common endpoints on an edge \( e \) of \( G' \). One of those two edges must approach to \( e \) from the same side while the other from the different sides. Otherwise we obtain a projection of a link. Therefore there is only one way to add \( e_1, e_2 \) to \( G', \) which gives the projection 4\(_{14}\).

Now we consider a projection \( G \) of the last type \( o \). Suppose that one of the faces of \( G \) is a biangle. Let us remove this biangle by the operation in Fig. 3, see Definition 2. We get the projection which has two vertices and thus corresponds to one of the graphs \( c \) or \( d \). It follows that one can get \( G \) from a projection of type \( c \) or \( d \) by the biangle addition. The idea of the next part of the proof is to list all projections of types \( c \) and \( d \) and to look along which arcs one can add biangles to them so as to get prime projections of type \( o \). In the case of the type \( c \) projection we may consider only arcs connecting loops.

All projections of types \( c \) and \( d \) with arcs producing prime projections of type \( o \) are represented in Fig. 5. In the cases 1 – 6 we get the projections 4\(_{15} - 4_{20}\). In cases 7 – 10 we get two new projections of type \( o \): 4\(_{21}\) in case 7 and 4\(_{22}\) in case 8. Cases 9, 10 give projections 4\(_{19}\) and 4\(_{20}\) obtained earlier.

Figure 5: How one can obtain all projections of type \( o \)

Now suppose that \( G \) has no biangle faces. Then its double edges define disjoint nontrivial circles in \( T \). Each of the circles contains two vertices and their union decomposes \( T \) into two annuli. The remaining four edges must be contained in these annuli (\( m \) in the first annulus, \( n \) in the second one) and
connect every vertex of the first circle with every vertex of the second one. There are three possibilities: \((m, n) = (0, 4), (1, 3)\) and \((2, 2)\). We get the projections \(4_{23}, 4_{24}\) in the first and second cases, and a link projection in the third one.

Let us prove that all projections in Fig. 4 are different. To this end we count the angles in each disc or annular face of the projection. It turns out that the sets of numbers thus obtained (with specifying the numbers corresponding to annular faces) are different for all projections in Fig. 4 except projections \(4_{11}, 4_{21}\). These projections have the same set \(\{2, 2, 4, 8\}\) and are different since the biangle faces have a common vertex in the first case and no common vertices in the second one. Also all 36 projections are minimal, see the last sentence of the proof of Theorem 1.

§3. Proof of Theorem 1

We recover all prime knots having projections with \(n \leq 4\) crossing points (see Theorem 2 and Fig. 5) by indicating the types of crossings using all \(2^n\) possible ways. However one can essentially reduce this procedure by using the following ideas.

1. The simultaneous crossing changes in all crossings converts any diagram to an equivalent one. Therefore for any projection with \(n\) vertices it suffices to consider \(2^{n-1}\) possibilities.

2. Let \(x, y\) be the vertices of a biangle face of a given projection. Then there are only two ways of indicating the crossing types at \(x, y\) such that one cannot remove the biangle by the second Reidemeister move. Therefore every biangle reduces the procedure in two times.

3. There are only two ways to indicate the under- and over-crossings for the fragment shown in the center of Fig. 6. All other ways give us non-minimal diagrams.

4. Let a given projection contain a triangle face. Suppose that the types of its vertices are chosen so that we can perform the third Reidemeister move and get a diagram constructed earlier. Then one may drop this choice.

Those ideas are enough for removal almost all duplicates. The only exception is the diagram in Fig. 7 (left). It can be transformed into the diagram \(4_{31}\) (right) we have constructed before.
All knots in Fig. 1 are distinct. This can be proved by using the generalized Kauffman polynomial, which is slightly different from the usual normalized Kauffman bracket \([9, 10]\). We use two variables \(a\) and \(x\). These two variables are needed for taking into account the number and the types (trivial or not) of the circles in \(T\) which we obtain after resolving all crossings. In addition we use a non-standard normalization: we set the polynomial of the trivial knot to be \((-a^2 - a^{-2})\), not 1 as usual. This is because quite often resolving all crossings produce no trivial circles. In those cases we would be forced to divide the polynomials we have obtained by \((-a^2 - a^{-2})\), which is inconvenient. The exact formula is the following:

\[
X(K) = (-a)^{-3w(K)} \sum_s a^\alpha(s) - \beta(s) (-a^2 - a^{-2}) \gamma(s) x^\delta(s),
\]

where \(\alpha(s)\) and \(\beta(s)\) are the numbers of markers \(A\) and \(B\) in a given state \(s\), and \(\gamma(s)\), \(\delta(s)\) are the numbers of trivial and nontrivial circles in \(T\) obtained by resolving all crossing points. Just as for the original Kauffman polynomial, the sum is taking over all states. Of course, \(w(K)\) is the writhe of the diagram. The next table was obtained by direct calculations. One can see that all polynomials are distinct. Therefore all knots in Fig. 1 are also distinct. It follows that all 36 projections in Fig. 2 are minimal.

\[
\begin{align*}
0_1: & \quad x \\
1_1: & \quad -(x^2 a^{-4} - a^{-4} - 1) \\
2_1: & \quad x^3 a^{-8} + x(-2a^{-8} - a^{-4}) \\
2_2: & \quad x^3 + x(-a^{-4} - 1 - a^4) \\
2_3: & \quad x^2(a^{-6} - a^{-10}) - a^{-6} - a^{-2} \\
2_4: & \quad x(a^{-8} + a^{-6} - a^{-2}) \\
3_1: & \quad -(x^4 a^{-12} + x^2(-a^{-8} - 3a^{-12}) + a^{-8} + a^{-12}) \\
3_2: & \quad -(x^4 a^{-4} + x^2(-a^{-8} - 2a^{-4} - 1) + a^{-8} + a^{-4}) \\
3_3: & \quad -(x^4 a^{-4} + x^2(-2 - a^{-4} - a^{-8}) + 1 + a^4) \\
3_4: & \quad -(x^3(a^{-10} - a^{-14}) + x(-a^{-10} - a^{-6} + a^{-14})) \\
3_5: & \quad -(x^3(a^{-2} - a^{-6}) + x(a^{-10} - a^{-2} - a^2)) \\
3_6: & \quad -(x^2(a^{-16} + a^{-8} - a^{-12}) - a^{-8} - a^{-4})
\end{align*}
\]
\[3_7: -(x^2(a^{-8} - a^{-4}) + a^{-4} + a^{-16})\]
\[3_8: -(x^2a^{-12} - a^{-12} - 1)\]
\[3_9: -(x^2(a^{-8} - a^{-4} + a^{-12}) - a^{-8} - 2a^{-12} + a^{-4})\]
\[3_{10}: -(x^2(2a^{-4} - 1) - 2a^{-4} - a^{-8} + a^4)\]
\[3_{11}: -x(-a^-6 - a^2 - a^{-8} + a^{-4} + a^{-2})\]
\[4_1: x^5 + x^3(-1 - 2a^4 - 2a^{-4}) + x(1 + a^8 + a^4 + a^{-4} + a^{-8})\]
\[4_2: x^5a^{-16} - x^3(a^{-10} + 4a^{-16}) + x(2a^{-12} + 3a^{-16})\]
\[4_3: x^5a^{-8} - x^3(a^{-12} + 3a^{-8} + a^{-4}) + x(a^{-4} + 2a^{-12} + 2a^{-8})\]
\[4_4: x^5a^{-8} + x^3(-2a^{-4} - 2a^{-8} - a^{-12}) + x(a^{-12} + 2a^{-4} + a^{-8} + 1)\]
\[4_5: x^5 + x^3(-2a^4 - 2 - a^{-4}) + x(a^8 + 2a^4 + 2)\]
\[4_6: x^5 - x^3(a^4 + 3 + a^{-4}) + x(a^4 + 3 + a^{-4})\]
\[4_7: x^4(a^{-2} + a^4) - x^2(a^6 - 2a^6 + a^2) - a^{-10} - a^{-6}\]
\[4_8: x^4(a^{-14} - a^{-18}) - x^2(-2a^{-10} - 2a^{-18} + a^{-14}) - a^{-18} - a^{-14}\]
\[4_9: x^4(a^{-10} + a^{-6}) + x^2(-a^{-2} + a^{-14} + a^{-10} - a^{-6}) - a^{-14} - a^{-10}\]
\[4_{10}: x^4(-a^{-18} + a^{-14}) + x^2(-a^{-10} + 2a^{-18} - 2a^{-14}) + a^{-10} + a^{-14}\]
\[4_{11}: x^4(-a^{-2} + a^2) + x^2(-a^6 + a^6 + a^2 - 2a^2) + a^6 + a^2\]
\[4_{12}: x^4(a^{-6} - a^{-10}) + x^2(-a^{-6} + a^{-14} - a^{-2}) + a^{-6} + a^{-10}\]
\[4_{13}: x^4(a^{-6} - a^{-10}) + x^2(-a^{-4} + 2a^{-4} + a^{-2}) + a^2\]
\[4_{14}: x^4(2 - a^{-4} - a^4) + x(-1 + a^8 + a^{-8})\]
\[4_{15}: x^4(a^{-12} - 2a^{-16} + a^{-20}) + x(-a^{-8} + 2a^{-16})\]
\[4_{16}: x^4(a^{-12} - a^{-16} + a^{-20}) + x(-a^{-8} - a^{-12} - a^{-20})\]
\[4_{17}: x^4(a^{-4} - a^{-8} + a^{-12}) + x(a^{-10} + a^4 + a^{-16})\]
\[4_{18}: x^4(a^{-10} - a^{-14} + a^{-18} - a^{-22}) - a^{-6} - a^{-10}\]
\[4_{19}: x^3(a^{-12} - a^{-8}) + x(-a^{-16} - a^{-20} + a^{-4})\]
\[4_{20}: x^3(a^{-4} - 1) + x(2 - a^{-4})\]
\[4_{21}: x^3a^{-16} - x(2a^{-16} + a^{-8})\]
\[4_{22}: x^3a^{-8} - x(a^{-12} + a^{-8} - 1 + a^4 + a^{-4})\]
\[4_{23}: x^3(a^{-14} - a^{-30} - a^{-10} + a^{-6}) - a^{-2} - a^{-24}\]
\[4_{24}: x^3(a^{-6} + a^2 - 2a^{-2} - a^{-10}) + a^{-14} + a^{-2}\]
\[4_{25}: x(a^{-20} + a^{-18} - a^{-16} - a^{-14} + a^{-12} + a^{-10} - a^{-6})\]
\[4_{26}: x^3(a^{-10} - a^{-6}) + x(2a^{-6} - 2a^{-10} + a^{-16})\]
\[4_{27}: x^3(a^{-6} - a^{-2}) + x(a^2 + a^{-8} - a^{-10})\]
\[4_{28}: x^3(a^{-2} - a^2) + x(a^6 + a^2 + 1 - a^{-2} - a^{-6})\]
\[4_{29}: x(a^{-16} - a^{-2} + a^{-6} + a^{-14} - a^{-38})\]
\[4_{30}: x(a^{-4} + a^{-6} + a^{-16} - 2a^{-8} - 2a^{-10} + 2a^{-14})\]
\[4_{31}: x(a^{-2} - a^6 - a^4 + a^2 + 3 - a^{-4} - a^{-6})\]
\[4_{32}: x^3a^{-8} - x(a^{-2} + a^{-4} - a^{-6} + a^{-8} + a^{-12})\]
§4. Final remarks

1. The degrees of \( x \) in every polynomial have the same parity. This parity depends only on the type (trivial or nontrivial) of the element \([K]\) of \( H_1(T; \mathbb{Z}_2)\) corresponding to the given knot \( K \). One can easily see that \([K] = 0\) if and only if any diagram of \( K \) crosses each side of the square in an even number of points.

2. The table contains exactly 10 homologically trivial knots: 2₅, 3₇, 4₇, 4₉, 4₁₈, 4₂₃, 4₃₉, 4₄₅, 4₄₆.

3. The table contains exactly 23 alternating diagrams: 1₁, 2₂, 2₃, 3₃, 3₅, 3₆, 3₁₀, 4₁, 4₇, 4₁₃, 4₁₄, 4₁₇, 4₁₉, 4₂₂, 4₂₄, 4₃₆, 4₃₇, 4₃₉, 4₄₅. As it should be, each of the corresponding knots has exactly one minimal diagram.

4. About a half of 36 projections from Theorem 2 determine only one knot each. The projection \( 4₁ \) determine the maximal number of knots (6). The average number of knots for one projection is about 1.8.

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