THE STRUCTURE OF ONE-RELATOR RELATIVE PRESENTATIONS AND THEIR CENTRES

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Suppose that $G$ is a nontrivial torsion-free group and $w$ is a word in the alphabet $G \cup \{x_1^{\pm 1}, \ldots, x_n^{\pm 1}\}$ such that the word $w' \in F(x_1, \ldots, x_n)$ obtained from $w$ by erasing all letters belonging to $G$ is not a proper power in the free group $F(x_1, \ldots, x_n)$.

We show how to reduce the study of the relative presentation $\hat{G} = \langle G, x_1, x_2, \ldots, x_n \mid w = 1 \rangle$ to the case $n = 1$. It turns out that any such “$n$-variable” group $\hat{G}$ can be obtained from similar “one-variable” groups by using an explicit construction similar to wreath product. As an illustration, we prove that, for $n \geq 2$, the centre of $G$ is always trivial. For $n = 1$, the centre of $\hat{G}$ is also almost always trivial; there are several exceptions, and all of them are known.

Key words: relative presentations, one-relator groups, centre, asphericity.

0. Introduction

Let $G$ be a group. A group given by a one-relator relative presentation over $G$ is

$$\hat{G} = \langle G, x_1, x_2, \ldots, x_n \mid w = 1 \rangle \overset{\text{def}}{=} G * F(x_1, x_2, \ldots, x_n) / \langle \langle w \rangle \rangle.$$ 

Here $x_1, \ldots, x_n$ are some letters (not belonging to $G$) and $w$ is a word in the alphabet $G \cup \{x_1^{\pm 1}, \ldots, x_n^{\pm 1}\}$ (such a word can be considered as an element of the free product $G * F(x_1, x_2, \ldots, x_n)$ of $G$ and the free group with basis $x_1, x_2, \ldots, x_n$). In other words, the presentation of the group $\hat{G}$ is obtained from a presentation $G = \langle A \mid R \rangle$ of $G$ by adding several new generators and one new relator: $\hat{G} = \langle A \cup \{x_1, x_2, \ldots, x_n\} \mid R \cup \{w\} \rangle$.

Such groups $\hat{G}$ are natural generalisations of one-relator groups and have been studied by many authors (see, e.g., [How87], [BoP92], [DuH93], [Met01], [Kl06b] and the references therein). To obtain any meaningful result, it is necessary to impose some constraints on the group $G$ and/or relation $w$. In this paper, we assume only two restrictions:

(a) the group $G$ is torsion-free;
(b) the word $w \in G * F(x_1, x_2, \ldots)$ is such that the word $w' \in F(x_1, x_2, \ldots)$ obtained from $w$ by erasing coefficients belonging to $G$ is not a proper power\(^*\) in the free group $F(x_1, x_2, \ldots)$.

The same situation was considered in [Kl06a], [Kl06b], and [Kl07] (and also in [Kl93], [FeR96], [CR01], [Kl05], and [FoR05] in the case $n = 1$).

The main result of this paper (Theorem 3) shows how to reduce studying the groups $\hat{G}$ to the case $n = 1$. It turns out that the “$n$-variable” group $\hat{G}$ can be obtained from similar “one-variable” groups by using an explicit construction that involves free iterated amalgamated products (see Section 3) and amalgamated semidirect products (see Section 4).

Probably, this structural theorem (Theorem 3) may have many applications, one of which is considered in this paper. Namely, we study the centre of the group $\hat{G}$.

First, we consider the case $n = 1$, i.e., the following situation. Let $G$ be a torsion-free group. Suppose that the group $\hat{G}$ is obtained from $G$ by adding one generator and one unimodular relator, i.e., a relator with exponent sum one:

$$\tilde{G} = \langle G, t \mid w = 1 \rangle \overset{\text{def}}{=} (G * \langle t \rangle_\infty) / \langle \langle w \rangle \rangle,$$

where $w \equiv g_1 t^{\varepsilon_1} \ldots g_q t^{\varepsilon_q}$, $g_i \in G$, $\varepsilon_i \in \mathbb{Z}$, and $\sum \varepsilon_i = 1$.

In this case, we say that the group $\tilde{G}$ is given by a unimodular relative presentation over $G$. It is known that $\tilde{G}$ inheres some properties of the initial group $G$. In particular,

- the abelianisations of these groups are isomorphic: $G/\langle G, [G, G] \rangle \cong \tilde{G}/\langle \tilde{G}, [\tilde{G}, \tilde{G}] \rangle$;
- $G$ embeds (naturally) into $\tilde{G}$ [Kl93] (see also [FeR96]); therefore, $\tilde{G}$ is nontrivial if $G$ is nontrivial, $\tilde{G}$ is nonabelian if $G$ is nonabelian, etc.*
- $\tilde{G}$ (as well as $G$) is torsion-free [FoR05];
- $\tilde{G}$ is nonsimple if $G$ is nonsimple [Kl95];
- the Tits alternative holds for $G$ (i.e., $\tilde{G}$ either contains a nonabelian free subgroup or is virtually solvable) provided that it holds for $G$ [Kl07].

In this paper, we establish yet another property of this kind:

- the centre of $\tilde{G}$ is either trivial or isomorphic to the centre of the initial group $G$.

More precisely, we prove the following theorem.

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\* We say that an element $h$ of a group $H$ is a proper power if there exists an element $h' \in H$ and an integer $k \geq 2$ such that $h = (h')^k$. In particular, the identity element is a proper power: $1 = 1^2$.

\** However, the natural mapping $G \to \tilde{G}$ is never surjective, except in the case when $w \equiv gt$ [CR01].
Theorem 1. If a group $G$ is torsion-free and a word $w \in G * \langle t \rangle$ is unimodular, then the centre of the group $\hat{G} = \langle G, t \mid w = 1 \rangle$ is trivial, except in the following two cases:

1) $w \equiv gtg'$, where $g, g' \in G$ (and so $\hat{G} \simeq G$), and the centre of $G$ is nontrivial;
2) the group $G$ is cyclic and $\hat{G}$ is a one-relator group with nontrivial centre.

One-relator groups with nontrivial centre have been well studied ([Mu64], [BaTa68], [Pi74]). The centre of each such group is infinite cyclic (except in the case when the entire group is free abelian of rank 2; this case is impossible in our situation). The simplest nonobvious example of a unimodular presentation with nontrivial centre is the braid group on three strands $\hat{G} = \langle g, t \mid gtg = t^3g \rangle$. The centre of this group is generated by $(gt)^3$.

In the general case, any calculations in the group $\hat{G}$ are difficult, because the word problem in this group is not solved (so far). For example, the natural method of finding the centre by the formula

$$\text{centre of } \hat{G} = \langle \text{centraliser of } G \rangle \cap \langle \text{centraliser of } t \rangle$$

does not work, because we can calculate none of these centralisers, so to find the centre, we need some manoeuvre. Actually, the proof of Theorem 1 is not long, but it is heavily based on some results from [Kl05].

Next, we pass to the “multivariable” case. Actually, we consider even a more general situation. In [Kl06a], we suggested a generalisation of the notion of unimodularity to the case when the word $w$ is an element of the free product of a group $G$ and any (i.e., not necessarily cyclic) group $T$. In this paper, we need an even more general definition. We say that a word $w \equiv g_1t_1 \ldots g_qt_q \in G * T$ is generalised unimodular if

1) $\prod t_i \neq 1$, and the group $T$ is torsion-free;
2) the cyclic subgroup $\langle \prod t_i \rangle$ of $T$ is a free factor of some normal subgroup $R = \langle \prod t_i \rangle * S$ of $T$;
3) the quotient group $T/R$ is a group with the strong unique-product property.

Recall that a group $H$ is called a UP-group, or a group with the unique-product property, if the product $XY$ of any two finite nonempty subsets $X, Y \subseteq H$ contains at least one element which decomposes uniquely into the product of an element from $X$ and an element from $Y$. Some time ago, there was the conjecture that any torsion-free group is UP (the converse is, obviously, true). However, it turned out that there exist counterexamples ([Pr88], [RS87]).

We say that a group $H$ has the strong unique product property if the product $XY$ of any two finite nonempty subsets $X, Y \subseteq H$ such that $|Y| \geq 2$ contains at least two uniquely decomposable elements $x_1y_1$ and $x_2y_2$ such that $x_1, x_2 \in X$, $y_1, y_2 \in Y$, and $y_1 \neq y_2$.

As far as we know, all known examples of UP-groups have the strong UP-property. In particular, all right orderable groups, locally indicable groups, and diffuse groups in the sense of Bowditch have the strong UP property.

The main examples of generalised unimodular presentations are groups of the form

$$\hat{G} = \langle G, x_1, x_2, \ldots, x_n \mid w = 1 \rangle,$$

where the word $w \in G * F(x_1, x_2, \ldots)$ is such that the word $w' \in F(x_1, x_2, \ldots)$ obtained from $w$ by erasing all coefficients belonging to $G$ is not a proper power in the free group $F(x_1, x_2, \ldots)$.

Indeed, suppose that the word $w$ has the form $w \equiv g_1x_1^{q_1}g_2x_2^{q_2} \ldots g_qx_q^{q_q}$ and $w' \in F(x_1, \ldots, x_n)$ is obtained from $w$ by erasing all coefficients: $w' = x_1^{q_1}x_2^{q_2} \ldots x_n^{q_n}$. Consider the groups

$$T = F(x_1, \ldots, x_n) \quad \text{and} \quad T_1 = \langle x_1, \ldots, x_n \mid w' = 1 \rangle = T/\langle w' \rangle.$$

By the Brodiskii theorem [B84], if $w'$ is not a proper power in the free group $F(x_1, \ldots, x_n)$, then the group $T_1$ is locally indicable and, hence, has the strong UP property. By the Cohen–Lyndon theorem [CoLy63], the element $w'$ is a primitive element of the free subgroup $\langle w' \rangle$ of $T$. Thus, the word $w$, considered as an element of the free product $G*T$, is generalised unimodular.

In Section 5, we prove our main result, Theorem 3. As a corollary of this structural theorem, in Section 6, we obtain the following fact generalising Theorem 1.

Theorem 2. Suppose that $G$ and $T$ are torsion-free groups and a cyclically reduced word $w = g_1t_1 \ldots g_qt_q \in G * T$ is generalised unimodular. Then

1) the natural mapping $G \rightarrow \hat{G} = \langle G, T \mid w = 1 \rangle$ def $(G * T)/\langle w \rangle$ is injective;
2) if the centre of $\hat{G}$ is nontrivial and $G$ is noncyclic, then $q = 1$ and either $t_1 \in Z(T)$ and $\langle g_1 \rangle \cap Z(G) \neq 1$ (in this case, $\hat{G} = G * T$ is a free product of $G$ and $T$ with amalgamated cyclic subgroups) or the group $T$ is cyclic (in this case, $T = \langle t_1 \rangle$ and $\hat{G} \simeq G$).

This theorem implies a multivariable analogue of Theorem 1.
Corollary 1. Suppose that $G$ is a nontrivial torsion-free group and a word $w \in G * F(x_1,x_2,\ldots)$ is such that the word $w' \in F(x_1,x_2,\ldots)$ obtained from $w$ by erasing all coefficients belonging to $G$ is not a proper power in the free group $F(x_1,x_2,\ldots)$. Then

1) \cite{Kl06a} the natural mapping $G \to \hat{G} = \langle G,x_1,x_2,\ldots,x_n \mid w = 1 \rangle$ is injective;
2) if $n \geq 2$, then the centre of $\hat{G}$ is trivial.

Proof. If the group $G$ is noncyclic, the assertion follows immediately from Theorem 2. If $G$ is cyclic, then $\hat{G}$ is a one-relator group with at least three generators; the triviality of the centres of such groups is well known \cite{Mu64}.

These results on the centre of $\hat{G}$ are not surprising. However, they easily implies the Kervaire–Laudenbach conjecture for torsion-free groups \cite{Kl93}, i.e., the nontriviality of each group of the form
\[ \langle H, t \mid w = 1 \rangle, \]
where $H$ is a nontrivial torsion-free group and $w$ is any word in the alphabet $H \cup \{ t^\pm 1 \}$.

Indeed, if the group $\hat{H} = \langle H, t \mid w = 1 \rangle$ is trivial, then the word $w$ must be unimodular (otherwise, $\hat{H}$ admits an epimorphism onto a nontrivial cyclic group). Therefore, the group $H = \langle H, t, x \mid w = 1 \rangle = \hat{H} * \langle x \rangle_{\infty}$ is centreless. This can be derived from either Theorem 1 (by setting $G = H * \langle x \rangle_{\infty}$) or Corollary 1(2) (by setting $G = H$). Clearly, the triviality of the centre of $\hat{H}$ implies the nontriviality of $\hat{H}$. Thus, both Theorem 1 and Corollary 1(2) can be considered as strengthenings of the main result of \cite{Kl93}.

Other theorems on the centres of one-relator relative presentations can be found in \cite{BS81} and \cite{Met01}. A different “strong-noncommutativity” property of such presentations was proven in \cite{Kl06b}: if $G$ is a nontrivial torsion-free group and $n \geq 2$, then the group $\langle G, x_1, x_2, \ldots, x_n \mid w = 1 \rangle$ is always SQ-universal.

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Notation which we use is mainly standard. Note only that if $k \in \mathbb{Z}$, $x$ and $y$ are elements of a group, and $\varphi$ is a homomorphism from this group into another group, then $x^k$, $x^{ky}$, $x^{-k}$, $x^{-ky}$, and $x^{-\varphi}$ denote $y^{-1}xy$, $y^{-1}x^{ky}$, $y^{-1}x^{-1}y$, $\varphi(x)$, $\varphi(x^k)$, and $\varphi(x^{-1})$, respectively; the commutator $[x,y]$ is understood as $x^{-1}y^{-1}xy$. If $X$ is a subset of a group, then $\langle X \rangle$, $\langle \langle X \rangle \rangle$, and $C(X)$ denote the subgroup generated by $X$, the normal subgroup generated by $X$, and the centraliser of $X$, respectively. The centre of a group $G$ is denoted by $Z(G)$. The symbol $|X|$ denotes the cardinality of a set $X$. The letters $\mathbb{Z}$ and $\mathbb{N}$ denote the set of integers and the set of positive integers, respectively.

1. Howie diagrams

In this section, we recall (following \cite{Kl05}) some facts concerning diagrams introduced in \cite{How83}. The only new result of this section is Lemma 3.

Throughout this paper, the term “surface” means “closed oriented two-dimensional surface”.

A map $M$ on a surface $S$ is a finite set of continuous mappings $\{ \mu_i : D_i \to S \}$, where $D_i$ is a compact oriented two-dimensional disk, called the $i$th face, or cell, of the map; the boundary of each face $D_i$ is partitioned into finitely many intervals $\psi_{ij} \subset \partial D_i$, called the pre-edges of the map, by a nonempty set of points $c_{ij} \in \partial D_i$, called the corners of the map. The images of the corners $\mu_i(c_{ij})$ and pre-edges $\mu_i(\psi_{ij})$ are called the vertices and edges of the map, respectively. It is assumed that

1) the restriction of $\mu_i$ to the interior of each face $D_i$ is a homeomorphic embedding preserving orientation; the restriction of $\mu_i$ to each pre-edge is a homeomorphic embedding;
2) different edges do not intersect;
3) the images of the interiors of different faces do not intersect;
4) $\bigcup \mu_i(D_i) = S$.

Sometimes, we interpret a map $M$ as a continuous mapping $M : \bigsqcup D_i \to S$ from a discrete union of disks onto the surface.

The union of all vertices and edges of a map is a graph on the surface, called the 1-skeleton.

We say that a corner $c$ is a corner at a vertex $v$ if $M(c) = v$. There is a natural cyclic order on the set of all corners at a vertex $v$; we call two corners at $v$ adjacent if they are neighboring with respect to this order.

By abuse of language, we say that a point or a subset of the surface is contained in a face $D_i$ if it lies in the image of $\mu_i$. Similarly, we say that a face $D_i$ is contained in some subset $X \subseteq S$ of the surface $S$ if $M(D_i) \subseteq X$.

Figure 1 presents a map on the sphere with 5 faces — $A$, $B$, $C$, $D$, and $E$, 18 corners — $a_1$, $b_1$, $c_1$, $d_1$, and $e_1$, 6 vertices, 9 edges, and 18 pre-edges. Note that the number of corners always equals the number of pre-edges and is twice the number of edges, and the value
\[ e(S) \overset{\text{def}}{=} (\text{the number of vertices}) - (\text{the number of edges}) + (\text{the number of faces}) \]
does not depend on the choice of a map on the surface $S$ and is called the Euler characteristic of this surface. The Euler characteristic of the sphere (the only surface of our real interest in this paper) is two.
Suppose that we have a map $M$ on a surface $S$, the corners of the map are labeled by elements of a group $H$, and the edges are oriented (in the figures, we draw arrows on the edges) and labelled by elements of a set $\{t_1, t_2, \ldots\}$ disjoint from the group $H$. The label of a corner or an edge $x$ is denoted by $\lambda(x)$.

The label of a vertex $v$ of such a map is defined by the formula

$$\lambda(v) = \prod_{i=1}^{k} \lambda(c_i),$$

where $c_1, \ldots, c_k$ are all corners at $v$ listed clockwise. The label of a vertex is an element of the group $H$ determined up to conjugacy.

For instance, the label of the uppermost vertex in Fig. 1 is $\lambda(b_3)\lambda(c_2)\lambda(d_1)$.

The label of a face $D$ is defined by the formula

$$\lambda(D) = \prod_{i=1}^{k} (\lambda(M(e_i)))^{\varepsilon_i} \lambda(c_i),$$

where $e_1, \ldots, e_k$ and $c_1, \ldots, c_k$ are all pre-edges and all corners of $D$ listed anticlockwise, the endpoints of $e_i$ are $c_{i-1}$ and $c_i$ (subscripts are modulo $k$), and $\varepsilon_i = \pm 1$ depending on whether the homeomorphism $e_i \xrightarrow{M} M(e_i)$ preserves or reverses orientation. Simply speaking, to obtain the label of a face, we should go around its boundary anticlockwise, writing out the labels of all corners and edges we meet, the label of an edge traversed against the arrow should be raised to the power $-1$.

The label of a face is an element of the group $H \ast F(t_1, t_2, \ldots)$ (the free product of $H$ and the free group with basis $\{t_1, t_2, \ldots\}$) determined up to a cyclic permutation. More precisely, the right-hand side of our formula for $\lambda(D)$ is called the label of the face $D$ written starting with the pre-edge $e_1$. 

Fig. 1
For instance, if the label of each edge in Fig. 1 is \( t \), then the label of the face \( B \) written starting with the pre-edge \( \alpha \) is
\[
t\lambda(b_4)t\lambda(b_5)t^{-1}\lambda(b_6)\lambda(b_1)t^{-1}\lambda(b_2)t\lambda(b_3).
\]

Such a labelled map is called a Howie diagram (or simply diagram) over a relative presentation
\[
K = \langle H, t_1, t_2, \ldots \mid w_1 = 1, w_2 = 1, \ldots \rangle
\]
if
1) some vertices and faces are separated out and called exterior; the remaining vertices and faces are called interior;
2) the label of each interior face is a cyclic permutation of one of the words \( w_i^{\pm 1} \);
3) the label of each interior vertex is the identity element of \( H \).

A diagram is said to be reduced if it contains no such edge \( e \) that both faces containing \( e \) are interior, these faces are different and their labels written starting with the M-preimages of \( e \) are mutually inverse; such a pair of faces with a common edge is called a reducible pair. For example, the faces \( C \) and \( E \) in Fig. 1 form a reducible pair if \( \lambda(c_0) = \lambda(e_0) \), \( \lambda(c_1) = \lambda(e_2) \), \( \lambda(c_2) = \lambda(e_1) \) and all edges have the same label.

The following lemma is an analogue of the van Kampen lemma for relative presentations.

**Lemma 1** [H83]. The natural mapping from a group \( H \) to the group with relative presentation (**) is noninjective if and only if there exists a spherical diagram over this presentation with no exterior faces and a single exterior vertex whose label is not \( 1 \) in \( G \). A minimal (with respect to the number of faces) such diagram is reduced.

If this natural mapping is injective, then we have the equivalence: the image of an element \( u \in H * F(t_1, t_2, \ldots) \setminus \{1\} \) is \( 1 \) in the group (**) if and only if there exists a spherical diagram over this presentation without exterior vertices and with a single exterior face with label \( u \). A minimal (with respect to the number of faces) such diagram is also reduced.

Diagrams on the sphere with a single exterior face and no exterior vertices are also called disk diagrams, the boundary of the exterior face of such a diagram is called the contour of the diagram.

Let \( \varphi : P \to P^\varphi \) be an isomorphism between two subgroups of a group \( H \). A relative presentation of the form
\[
\langle H, t \mid \{ p^t = p^\varphi ; \; p \in P \setminus \{1\} \}, w_1 = 1, \; \ldots \rangle
\]
is called a \( \varphi \)-presentation. A diagram over a \( \varphi \)-presentation (**) is called \( \varphi \)-reduced if it is reduced and different interior cells with labels of the form \( p^t p^{-\varphi} \), where \( p \in P \), have no common edges.

**Lemma 2** [Kl05]. A minimal (with respect to the number of faces) diagram among all spherical diagrams over a given \( \varphi \)-presentation without exterior faces and with a single exterior vertex with nontrivial label is \( \varphi \)-reduced. If no such diagrams exists, then a minimal diagram among all disk diagrams with a given label of contour is \( \varphi \)-reduced. In other words, the complete \( \varphi \)-analogue of Lemma 1 is valid.

The idea of the proof is shown in Fig. 2.

![Fig. 2](image)

A relative presentation (\( \varphi \)-presentation) over which there exists no reduced (respectively, \( \varphi \)-reduced) spherical diagrams with no exterior faces and a single exterior vertex are called aspherical (respectively, \( \varphi \)-aspherical).

**Lemma 3.** Suppose that \( H \) is a group, a word \( v \in H * F(t_1, t_2, \ldots) \) is not a proper power in \( H * F(t_1, t_2, \ldots) \), and a positive integer \( l \) is such that \( v^l \) is not conjugate in \( H * F(t_1, t_2, \ldots) \) to elements of the set \( H \cup \{w_i^{\pm 1}\} \) and the presentation
\[
L = \langle H, t_1, t_2, \ldots \mid v^l = 1, w_1 = 1, w_2 = 1, \ldots \rangle
\]

obtained from presentation (**) by adding the relation \( v^l = 1 \) is aspherical (or \( \varphi \)-aspherical, if the initial presentation (**) is a \( \varphi \)-presentation). Then
1) in the group \( K \) with presentation (**), the centraliser of the element \( v^k \) coincides with the cyclic group \( \langle v \rangle \) for any positive integer \( k \);
2) if the group \( H \) is nontrivial, then the centre of the group \( K \) is trivial.

**Proof.** The first assertion is proven by standard argument. First, we can assume that \( k = l \), because \( C(v^k) \subseteq C(v^{kl}) \) and the asphericity of presentation \( L \) implies that of the presentation

\[
L_k = \langle H, t_1, t_2, \ldots \mid v^{kl} = 1, w_1 = 1, w_2 = 1, \ldots \rangle
\]

for each positive integer \( k \) (because a cell with label \( w^k \) can be transformed into \( k \) cells with labels \( w \); see [BoP92]).

Consider a word \( u \) commuting with \( v^l \) in the group \( K \) and a disk diagram over presentation \( (\ast) \) with contours labelled by \( [u, v^l] \). Let us glue together segments of the contour of this diagram to obtain an annulus \( A \) with contours labelled by \( v^l \) and \( v^{-l} \) (if necessary, we add cells, whose boundary labels equal 1 in \( H \ast F(t_1, t_2, \ldots) \)). Attaching two new cells \( \Gamma_+ \) and \( \Gamma_- \) to the annulus \( A \) along the contours, we obtain a spherical diagram \( D \) over presentation \( L \) without external vertices and faces. This diagram has the following properties (Fig.3):

- a) the label of the face \( \Gamma_+ \) written starting with some point \( p_+ \in \partial \Gamma_+ \) is \( v^\pm 1 \);
- b) the labels of the other faces belong to \( \{w_\pm^1\} \cup \{1\} \);
- c) the points \( p_+ \) and \( p_- \) are joined by a path \( \pi \) with label \( u \);
- d) the point \( p_+ \) is joined with some point \( p'_- \in \partial \Gamma_- \) by a path \( \pi' \) whose label equals 1 in the free product \( H \ast F(t_1, t_2, \ldots) \), and the label of the cell \( \Gamma_- \) written starting with the point \( p'_- \) is \( v^{-l} \).

The last property follows from the asphericity of the presentation \( L \): in the diagram \( D \) and in all diagrams obtained from \( D \) by reductions (eliminations of reducible pairs), the cell \( \Gamma_+ \) can form a reducible pair only with the cell \( \Gamma_- \).

![Fig. 3](image)

Properties a) and d) and the condition that \( v \) is not a proper power imply that the segment \( \sigma \) of the boundary of \( \Gamma_- \) between the points \( p_- \) and \( p'_- \) has label \( v^l \). It remains to note that the path \( \pi \sigma \) is homotopic in the annulus \( A = D \setminus \{\Gamma_\pm\} \) to a path of the form \( \pi' \delta \), where \( \delta \) is a path with label \( v^{-l} \) around the cell \( \Gamma_- \) starting and ending at \( p'_- \). This homotopy implies that the label \( u \) of the path \( \pi \) equals \( v^{-l} \) in the group \( K \). This proves the first assertion of the lemma.

Let us prove the second assertion. According to assertion 1), the centre of \( K \) must be contained in the cyclic group \( \langle v \rangle \). Applying once again assertion 1) to a hypothetical central element of the form \( w^k \), we obtain \( K = \langle v \rangle \). Therefore, \( v^k = h \) for some \( s \in \mathbb{N} \) and \( k \in H \). This contradicts the \((\varphi)\)-asphericity of the presentation \( L \).

2. **Proof of Theorem 1**

In [Kl05], we showed that the group \( \tilde{G} \) always (with some obvious exceptions) admits a relative \((\varphi)\)-presentation, which is \((\varphi)\)-aspherical and remains such after adding some additional relation. By virtue of Lemma 3, this implies Theorem 1. More precisely, the proof decomposes into two cases.

**Case 1: the word \( w \) has the form** \( \text{ct} \prod_{i=0}^m (b_i a_i) = 1 \), **where** \( c, a_i, b_i \in G \) (i.e., the complexity of \( w \) does not exceeds one in terminology of [FoR05]).

**Lemma 4** ([Kl05], Lemma 23). If \( G \) is a torsion-free group and \( m \geq 0 \), then there exists a \( d \in \{2, 3\} \) for which the presentation

\[
\left\langle G, t \mid \text{ct} \prod_{i=0}^m (b_i a_i)^2 = 1, (a^d b)^4 = 1 \right\rangle
\]
Lemma 6. If $G$ is a torsion-free group and $\langle G^2 \rangle$ is a cyclic group, then the group $G$ is cyclic itself.

Case 2: the word $w$ is not conjugate to a word of the form $ct \prod_{i=0}^{m}(b_i a_i') = 1$ (i.e. the complexity of $w$ is higher than one in terminology of [FoR05]).

In this case, the assertion of Theorem 1 follows immediately from Lemma 3 and the following two lemmata.

Lemma 6 ([Kl05], Lemma 2; see also [Kl93], [Fer96]). The group $G$ has a relative presentation of the form

$$\tilde{G} \simeq \left\langle H, t \mid \{p^i = p^e, \ p \in P \setminus \{1\}, \ ct \prod_{i=0}^{m}(b_i a_i') = 1\right\rangle,$$

(1)

where $a_i, b_i, c \in H$, $P$ and $P^e$ are isomorphic subgroups of $H$, and $\varphi: P \to P^e$ is an isomorphism. The groups $H$, $P$ and $P^e$ are free products of finitely many isomorphic copies of $G$. If the word $w$ is not conjugate in $G*\langle t \rangle_\infty$ to a word of the form $ct \prod_{i=0}^{m}(b_i a_i')$, then the groups $P$ and $P^e$ are nontrivial.

Lemma 7 ([Kl05], Lemma 10). If $G$ is a noncyclic torsion-free group and $P \neq \{1\}$ in presentation (1), then there exist elements $a, b \in H$ such that the presentation

$$\tilde{G}/\langle \langle a^2 b \rangle \rangle \simeq \left\langle H, t \mid \{p^i = p^e, \ p \in P \setminus \{1\}, \ ct \prod_{i=0}^{m}(b_i a_i') = 1, \ a^2 b = 1\right\rangle$$

obtained from presentation (1) by adding the relator $a^2 b = 1$ is $\varphi$-aspherical.

Theorem 1 is proven. In the rest of this paper, we study generalised unimodular presentations.

3. Free iterated amalgamated products

This section is an extended version of a similar section of [Kl06b].

Let $\{M_j : j \in J\}$ be a family of groups. We define a group $M_J$, a [strict] free iterated amalgamated product (FIAP) of the groups $M_j$, by induction as follows:

- if $J = \emptyset$, then we set $M_J = \{1\}$;
- if $J$ is a finite nonempty set, then a [strict] FIAP of the family $\{M_j : j \in J\}$ is any amalgamated free product of the form

$$M_J = M_{j_0} \ast_{H = H^e} M_{J \setminus \{j_0\}},$$

where $j_0$ is an element of the set $J$, $M_{J \setminus \{j_0\}}$ is a [strict] FIAP of the family $\{M_j : j \in J \setminus \{j_0\}\}$, and $\varphi : H \to H^e$ is an isomorphism between a [proper] subgroup $H \subseteq M_{j_0}$ and some subgroup $H^e \subseteq M_{J \setminus \{j_0\}}$;
- if the set $J$ is infinite, then a [strict] FIAP of the family $\{M_j : j \in J\}$ is the direct limit

$$M_J = \lim \{M_K; \ K \text{ is a finite subset of } J\},$$

where $M_K$ is a [strict] FIAP of the family groups $\{M_j : j \in K\}$; for each pair of finite subsets $K \subset K' \subset J$, there is a homomorphism $M_K \to M_{K'}$ which is identity on the groups $M_j$, where $j \in K'$; the direct limit is taken over this family of homomorphisms.

Remark. The definition of strict FIAP requires that the amalgamated subgroup $H$ is proper only in one factor, in $M_{j_0}$. Therefore, any nontrivial group $G$ can be decomposed into a strict FIAP: $G = \{1\} * G$ (while the trivial group is a strict FIAP of the empty family of groups). Similarly, if a group $G$ is a union of a strictly increasing chain of subgroups, i.e., $G = \bigcup G_i$, where $G_1 \subset G_2 \subset \ldots$, then $G$ decomposes into a strict FIAP of the groups $G_i$.

Let $I$ be a set, and let $\Omega$ be a family of subsets of $I$. For each $i \in I$, let $G_i$ be a group, and for each $\omega \in \Omega$, let $G_\omega$ be a quotient of the free product $\ast_{i \in \omega} G_i$:

$$G_\omega = \left(\ast_{i \in \omega} G_i\right)/N_\omega.$$

The natural question arises: under what conditions are the natural mappings

$$\varphi_\omega : G_\omega \to G_I \overset{\text{def}}{=} \left(\ast_{i \in I} G_i\right)/\left\langle \bigcup_{\omega \in \Omega} N_\omega \right\rangle$$

injective? Or under what conditions is the group $G_I$ a free iterated amalgamated product of the groups $G_\omega$?

The following proposition gives some sufficient condition for this question to have a positive answer.
Proposition 1. Suppose that
\[ N_\omega \cap \left( \bigstar_{j \in \omega \setminus \{i\}} G_j \right) = \{1\} \]  
for each \( \omega \in \Omega \) and each \( i \in \omega \setminus (\bigcap \Omega) \). Suppose also that, for each finite subfamily \( F \subseteq \Omega \) with \( |F| \geq 2 \), there exist elements \( \min, \max \in \bigcup F \) such that
1) the element \( \min \) belongs to precisely one set \( \omega_{\min} \in F \);
2) the element \( \max \) belongs to precisely one set \( \omega_{\max} \in F \);
3) \( \omega_{\min} \neq \omega_{\max} \).
Then all of the natural mappings \( \varphi_\omega : G_\omega \to G_I \) are injective and the group \( G_I \) is a free iterated amalgamated product of groups \( G_\omega \). If, in addition,
\[ \left( \bigstar_{j \in \omega \setminus \{i\}} G_j \right) N_\omega \not\supseteq G_i \]
for each \( \omega \in \Omega \) and each \( i \in \omega \setminus (\bigcap \Omega) \), then the free iterated amalgamated product is strict.

Example. Suppose that \( I = \{a, b, c, d, e, f\} \) and \( \Omega = \{\{a, b, d, e\}, \{b, c, e, f\}, \{d, e, f\}\} \).

Let \( A, \ldots, F \) be the corresponding six groups \( G_i \), and let \( ABDE, BCEF \), and \( DEF \) be the three groups \( G_\omega \). It is easy to see that conditions 1), 2), and 3) hold for the family \( \Omega \) and each of its two-set subfamilies. Suppose that condition (***), holds too. Then the validity of Proposition 2 (for this example) is implied by the following decomposition of \( G_I \) into an amalgamated free product:
\[ G_I = \left( \left( DEF \ast B \right) \ast ABDE \right) \ast BCEF. \]

To prove Proposition 1 in the general case, we need a lemma.

Lemma 8 ([KL06b], Lemma 1). Suppose that the conditions of Proposition 1 hold, \( \Omega' \) is a finite subfamily of \( \Omega \), \( \omega \in \Omega \), and \( \alpha \subseteq \omega \cap (|\Omega'|) \) is a proper subset of \( \omega \) contained in \( \bigcup \Omega' \) and containing \( \bigcap \Omega \). Then the natural mapping
\[ \bigstar_{i \in \alpha} G_i \to G_{\Omega'} \defeq \left( \bigstar_{i \in \Omega'} G_i \right) / \left\langle \bigcup_{\omega' \in \Omega'} N_{\omega'} \right\rangle \]
is injective.

Proof.

Case 1: \( \omega \in \Omega' \). Let us use induction on the cardinality of \( \Omega' \). If \( |\Omega'| = 1 \) (i.e., \( \Omega' = \{\omega\} \)), then the assertion of Lemma 8 is true by condition (***). Suppose that \( |\Omega'| \geq 2 \). In this case, according to conditions 1), 2), and 3), the family \( F = \Omega' \) contains a set \( \omega' \neq \omega \) that has an element \( m \in \omega' \) not belonging to \( \bigcup (\Omega' \setminus \{\omega'\}) \).

By the inductive hypothesis (applied to the set \( \omega' \) as \( \omega \) and the family \( \Omega' \setminus \{\omega'\} \) as \( \Omega' \)), the groups
\[ G_i \text{ with } i \in \beta \defeq \omega' \cap \left( \bigcup (\Omega' \setminus \{\omega'\}) \right) \]
freely generate their free product in the group \( G_{\Omega' \setminus \{\omega'\}} \). But according to condition (***) the same groups \( G_i \) with \( i \in \beta \) freely generate their free product in the group \( G_{\omega'} \) (because \( \omega' \) contains an element \( m \) not belonging to \( \beta \)). Therefore, the group \( G_{\Omega'} \) decomposes into the amalgamated free product of \( G_{\Omega' \setminus \{\omega'\}} \) and \( G_{\omega'} \) with amalgamated subgroup \( \bigstar_{i \in \beta} G_i \). The groups \( G_i \) with \( i \in \alpha \) lie in the factor \( G_{\Omega' \setminus \{\omega'\}} \). Therefore, the assertion of Lemma 8 follows from the inductive hypothesis applied to the set \( \omega \) and the family \( \Omega' \setminus \{\omega'\} \) as \( \Omega' \).

Case 2: \( \omega \notin \Omega' \). In this case, the proof is similar. We again use induction on the cardinality of \( \Omega' \). If \( \Omega' = \emptyset \), then we have nothing to prove. Suppose that \( |\Omega'| \geq 1 \). In this case, according to conditions 1), 2), and 3), the family \( F = \Omega' \cup \{\omega\} \) contains a set \( \omega' \neq \omega \) with an element \( m \in \omega' \) not lying in \( \bigcup (F \setminus \{\omega'\}) \) (see Fig.4).
By the inductive hypothesis (applied to the set $\omega'$ as $\omega$ and the family $\Omega' \setminus \{\omega'\}$ as $\Omega'$), the groups

$$G_i \text{ with } i \in \beta \overset{\text{def}}{=} \omega' \cap \left(\bigcup (\Omega' \setminus \{\omega'\})\right)$$

freely generate their free product in the group $G_{\Omega' \setminus \{\omega'\}}$. Therefore, the groups

$$G_i \text{ with } i \in \gamma \overset{\text{def}}{=} \beta \cup (\omega \cap \omega') = \omega' \cap \left(\bigcup (\Omega' \cup \omega \setminus \{\omega'\})\right)$$

freely generate their product in the group

$$H = \left( \bigstar_{j \in (\omega \cap \omega') \setminus \beta} G_j \right) * G_{\Omega' \setminus \{\omega'\}}.$$

But condition (****) implies that the same groups $G_i$ with $i \in \gamma$ freely generate their free product in $G_{\omega'}$ (because $\omega'$ contains an element $m$ not belonging to $\gamma$). Therefore, the group $G_{\Omega'}$ decomposes into the amalgamated free product of the groups $H$ and $G_{\omega'}$:

$$G_{\Omega'} = H \bigstar_{G_i : i \in \gamma} G_{\omega'}.$$

The groups $G_i$ with $i \in \alpha$ lie in the factor $H$. Therefore, by the inductive hypothesis applied to the set $\omega$ and the family $\Omega' \setminus \{\omega'\}$ as $\Omega'$, the groups $G_i$ with $i \in \alpha \cap (\bigcup (\Omega' \setminus \{\omega'\})$ freely generate their free product in $G_{\Omega' \setminus \{\omega'\}}$. This immediately implies that the groups $G_i$ with subscripts $i \in \alpha$ freely generate their free product in $H$ and, hence, in the group $G_{\Omega'}$, which contains $H$ as a subgroup. Lemma 8 is proven.

**Proof of Proposition 1.** Clearly, it is sufficient to prove Proposition 1 for a finite family $\Omega$ of cardinality larger than one. In this case,

$$G_I = G_\Omega * \left( \bigstar_{i \notin \bigcup \Omega} G_i \right),$$

and $G_\Omega$ decomposes into the amalgamated free product:

$$G_\Omega = G_{\omega_{\min}} \bigstar_{K} G_{\Omega \setminus \{\omega_{\min}\}},$$

where the amalgamated subgroup $K$ is (by virtue of Lemma 8) the free product of the groups $G_i$ with $i \in \omega_{\min} \cap \bigcup (\Omega \setminus \{\omega_{\min}\})$. An obvious inductive argument completes the proof.

In what follows, we need a property of free iterated amalgamated products.
Proposition 2. Suppose that a group $M_J$ is a strict free iterated amalgamated product of finitely generated groups $M_j$, where $j \in J$. Then
1) in the group $M_J$, all the subgroups $M_j$, where $j \in J$, are pairwise different;
2) if an element $h \in M_J$ permutes the groups $M_j$, i.e., for each $j \in J$, there exists a $k \in J$ such that $M_J^h = M_k$, then $h$ lies in one of the groups $M_i$, where $i \in J$; in particular, each central element of $M_J$ is contained in the centre of one of the groups $M_i$.

Proof. Let us prove the first assertion. Suppose that $M_i = M_k$, where $i$ and $k$ are different elements of the set $J$. The group $M_i$ is finitely generated; hence, the equality $M_i = M_k$ holds in a FIAP $M_P$ of a finite family of groups whose direct limit is the group $M_J$. Here $P$ is a finite set containing $i$ and $k$. Now, let us use induction on the cardinality of $P$. The group $M_P$ decomposes into an amalgamated product

$$M_P = M_{p_0} \ast_H M_{P\setminus\{p_0\}}.$$  

The subgroup $H$ is a proper subgroup of $M_{p_0}$ by the definition of the strict FIAP. Therefore, $M_{p_0}$ can coincide with no subgroup of $M_{P\setminus\{p_0\}}$. In particular, $i \neq p_0 \neq k$. But then, the equality $M_i = M_k$ holds in the group $M_{P\setminus\{p_0\}}$. Applying the inductive hypothesis, we complete the proof of the first assertion.

The second assertion is proven similarly. Suppose that $h \in \langle M_{j_1}, \ldots, M_{j_l} \rangle$. Consider the equalities

$$M_{j_1}^h = M_{k_1}, \ldots, M_{j_l}^h = M_{k_l}.$$  

The groups $M_i$ are finitely generated; hence, these equalities hold in a FIAP $M_P$ of a finite family of groups whose limit is $M_J$. Here, $P$ is a finite set containing $j_1, \ldots, j_l$ and $k_1, \ldots, k_l$. Let us use induction on the cardinality of $P$. The group $M_P$ decomposes into an amalgamated product

$$M_P = M_{p_0} \ast_H M_{P\setminus\{p_0\}}.$$  

The subgroup $H$ is a proper subgroup in $M_{p_0}$ by the definition of strict FIAP. Moreover, we can assume that $H$ is a proper subgroup of $M_{P\setminus\{p_0\}}$, because otherwise the element $h$ belongs to $M_{p_0}$ and we have nothing to prove.

If $p_0 \notin \{j_1, \ldots, j_l, k_1, \ldots, k_l\}$, then $h \in M_{P\setminus\{p_0\}}$, the above equalities hold in the group $M_{P\setminus\{p_0\}}$, and the assertion is proven by the inductive hypothesis.

If $p_0 = j_l$, then the group $(M_{p_0})^h$ either lies in the factor $M_{P\setminus\{p_0\}}$ or coincides with $M_{p_0}$ (depending on whether or not $k_l$ and $j_l$ are equal). Standard properties of free amalgamated products implies that the first case is impossible, and the second case is possible only if $h \in M_{p_0}$.

The case $p_0 = k_l$ is considered similarly: we repeat the argument of the preceding paragraph replacing $h$ by $h^{-1}$.

Proposition 2 is proven.

Remark. It can be shown that Proposition 2 becomes false if we remove either of the condition: the strictness of the FIAP or the finite generatedness of the factors.

4. Amalgamated semidirect products

Suppose that a group $A$ acts on a group $B$ by automorphisms $\varphi: A \to \text{Aut} B$, and $N \triangleleft A$ and $N^\psi \subseteq B$ are isomorphic subgroups of $A$ and $B$ such that $N$ is normal in $A$ and the isomorphism $\psi: N \to N^\psi$ is consistent with the action $\varphi$, i.e.,

$$(n^a)^\psi = (n^\psi)^{a^\varphi} \quad \text{and} \quad b^{n^\psi} = b^n \quad \text{for all} \ a \in A, \ b \in B, \ \text{and} \ n \in N.$$  

Consider the semidirect product $A \ltimes B$ corresponding to the action $\varphi$, i.e., the group consisting of the formal products $ab$ multiplied by the formula $(ab)(a_1b_1) = (aa_1)(b^a_1b_1)$. The elements of the form $nn^{-\psi}$, where $n \in N$, constitute a subgroup in the semidirect product $A \ltimes B$:

$$\begin{align*}
(nn^{-\psi})(n_1n_1^{-\psi}) &= (n_1)(n^{-\psi})n_1^{-1}n_1^{-\psi} = (n_1)(n^{-\psi}n_1^{-\psi}) = (n_1)(n^{-\psi}n_1^{-1})^{-1} = (n_1)(n_1)^{-1}; \\
(nn^{-\psi})^{-1} &= n^{-\psi}n^{-1} = n^{-1}(n^{-\psi})^{-1} = n^{-1}(n^{-1})^{-1} = n^{-1}n^{-1} = n^{-1}n^{-\psi}. 
\end{align*}$$

This subgroup is normal:

$$\begin{align*}
(nn^{-\psi})^a &= a(n^{-\psi})^a = a(n^{-a})^{-\psi} = a(n^a)^{-\psi}; \\
(nn^{-\psi})^b &= b^{-1}(nn^{-\psi})b = nb^{-n^\psi}n^{-\psi}b = nb^{-n^\psi}n^{-\psi}b = nn^{-\psi}. 
\end{align*}$$

The semidirect product of the groups $B$ and $A$ (with respect to the action $\varphi$) with amalgamated (by means of the isomorphism $\psi$) subgroups $N$ and $N^\psi$ is the quotient group

$$A \ltimes B \overset{\text{def}}{=} (A \ltimes B)/(\langle nn^{-\psi} : n \in N \rangle).$$

Clearly, this group contains $A$ and $B$ as subgroups and is the product of these subgroups; the subgroup $B$ is normal and $A \cap B = N$. 

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5. The structure of the group $\hat{G}$

Let $G$ and $T$ be torsion-free groups and let $w = g_1t_1\ldots g_qt_q \in G \ast T$ be a cyclically reduced generalised unimodular word, which means that

$$\prod t_i = t \ast (t)_\infty \ast S = R \triangleleft T$$

and $T/R$ is a group with strong UP property.

We study the group

$$\hat{G} = \langle G, T \mid w = 1 \rangle \overset{\text{def}}{=} (G \ast T) / \langle \langle \prod g_i t_i \rangle \rangle .$$

First, we describe the main idea of our approach. The group $G \ast T$ decomposes into the amalgamated semidirect product

$$G \ast T = T \triangleleft \langle \langle G, R \rangle \rangle .$$

The normal closure $\langle \langle G, R \rangle \rangle$ of $G \ast R$ in $G \ast T$ decomposes, in its turn, into the free product

$$\langle \langle G, R \rangle \rangle = \langle * y \in T/R \rangle G \ast (t)_\infty \ast S \ast \langle t \rangle_\infty ,$$

and

$$w \in L_1 = \langle * G^y \rangle \ast S \ast (t)_\infty , \quad \text{where } X_1 \text{ is a finite subset of } T/R .$$

The group $\langle \langle G, R \rangle \rangle$ can be considered as a FIAP of the groups $L_1^x$, where $x \in T/R$. The generalised unimodularity guarantees that a similar decomposition remains valid for the quotient by the normal closure of $w$. The reason is that the quotient of $L_1^x$ by the normal closure of $w^x$ has an (ordinary) unimodular relative presentation over the group

$$H_x = \langle * G^y \rangle \ast S .$$

The details of these arguments are as follows.

Let us decompose $T$ into a union of cosets with respect to the subgroup $R$:

$$T = \bigoplus_{x \in T/R} c_x R , \quad \text{where } c_1 = 1 ,$$

and write the relation $\prod t_i g_j = 1$ in the form

$$\prod_{i=1}^q g_i^{c_{x_i} r_i} = 1 , \quad (2)$$

where $r_i \in R$, $x_i = t_i t_{i+1} \ldots t_q R$, and $c_{x_i} r_i = t_i t_{i+1} \ldots t_q$. Let $X_1$ be the set of all $x_i \in T/R$ that occurs in relation (2). For each $x \in T/R$, consider an isomorphic copy $G^{(c_x)}$ of the group $G$ (assuming that the isomorphism maps each element $g \in G$ to the element $g^{(c_x)} \in G^{(c_x)}$). Take also an isomorphic copy $R = \langle \langle t \rangle \rangle_\infty \ast S$ of the group $T = \langle \langle t \rangle \rangle_\infty * S$.

We set

$$H_1 = \langle * G^{(c_y)} \rangle$$

and consider the unimodular relative presentation

$$\tilde{H}_1 = \langle H_1, T \mid \prod_i (g_i^{(c_{x_i})} \tilde{t}_i) = 1 \rangle$$

over the group $H_1$. The group $\tilde{H}_1$ is the quotient of the group

$$L_1 = H_1 * \langle \langle \tilde{t} \rangle \rangle_\infty = R * \langle * G^{(c_y)} \rangle$$

by the normal subgroup $N_1 = \langle \langle \prod g_i^{(c_{x_i} \tilde{t}_i)} \rangle \rangle$. Now, consider the free product

$$L = R * \langle * G^{(c_y)} \rangle .$$
and the right action \( \varphi : T \to \text{Aut} L \) of the group \( T \) on \( L \):

\[
(\tau)^{xy} = \tau^{x}, \quad (g^{(cy)})^{xw} = (g^{(cy)})^{x} \tau^{y},
\]

where \( x \in T, y \in T/R, \) and the element \( a \in R \) is uniquely determined by the equality \( c_{y}x = c_{y}a \).

For each \( x \in T/R, \) we set

\[
X_x = X_1 x \subseteq T/R, \quad H_x = \mathfrak{S} \ast \left( \bigodot_{y \in X_x} G^{(cy)} \right), \quad L_x = L_1^{x} = H_x \ast \langle T \rangle \infty = \mathfrak{R} \ast \left( \bigodot_{y \in X_x} G^{(cy)} \right),
\]

\[
N_x = N_1^{\chi} < L_x, \quad \tilde{H}_x = L_x / N_x,
\]

where \( \chi \in T \) is any representative of the element \( x \in T/R \).

**Proposition 3.** The groups \( \tilde{H}_x \) have the following properties:

1) all of them are isomorphic; an isomorphism \( \tilde{H}_1 \to \tilde{H}_x \) acts as follows: \( \tilde{t} \mapsto \tilde{t}^{x}, \quad \tilde{r} \mapsto \tilde{r}^{y} \), \( g^{(cy)} \mapsto (g^{(cy)})^{x} \tau^{y} \), where the element \( r \in R \) is uniquely determined by the equality \( c_{y}x = c_{y}r \);

2) The group \( \tilde{H}_x \) has a unimodular relative presentation over the group isomorphic to \( H_1 \), which is the free product of the group \( S \) and \( p \) isomorphic copies of the group \( G \), where \( p = |X_x| = |X_1| \) equals the number of different cosets among \( R, t_1t_2 \ldots t_qR, t_3t_4 \ldots t_qR, \ldots, t_qR \);

3) in the group \( \tilde{H}_x \), we have the decomposition

\[
\langle \mathfrak{R}, \{ G^{(cy)} : y \in Y \} \rangle = \mathfrak{R} \ast \left( \bigodot_{y \in Y} G^{(cy)} \right)
\]

for each proper subset \( Y \subset X_x \); in particular, the natural mapping \( \mathfrak{R} \to \tilde{H}_x \) is injective if \( w \notin T \);

4) the natural mapping

\[
\tilde{H}_x = L_x / N_x \to K \overset{\text{def}}{=} L \bigg/ \left( \bigcup_{y \in T/R} N_y \right)
\]

is injective, and the group \( K \) is a free iterated amalgamated product of the groups \( \{ \tilde{H}_x : x \in T/R \} \);

5) the FIAP \( K \) is strict if the group \( G \) is noncyclic.

**Proof.**

1) The isomorphism \( \tilde{H}_1 \to \tilde{H}_x \) is the action \( \varphi \) of the element \( c_x \) on \( L \). It maps the group \( L_1 \) onto the group \( L_x \). The subgroup \( N_1 < L_1 \) is mapped onto the subgroup \( N_x < L_x \), which implies the isomorphism of the quotients \( H_1 = L_1 / N_1 \) and \( H_x = L_x / N_x \).

2) The group \( \tilde{H}_1 \) has the required property by definition. The fulfillment of this property for the other groups \( \tilde{H}_x \) follows from the isomorphism \( \tilde{H}_1 \to \tilde{H}_x \) from assertion 1).

3) The group \( \tilde{H}_1 \) has the required property because of the following lemma.

**Lemma 9** ([Kl06a], Lemma 1). Suppose that a group \( H \) is torsion-free, a subgroup \( P \subseteq H \) is a free factor of \( H \), and a word \( v \in H \ast \langle z \rangle \infty \) is unimodular and nonconjugate in \( H \ast \langle z \rangle \infty \) to elements of the subgroup \( P \ast \langle z \rangle \infty \). Then \( \langle v \rangle \cap (P \ast \langle z \rangle \infty) = \{ 1 \} \). In other words, the element \( z \) is transcendental over \( P \) in \( H = \langle H, z \mid v = 1 \rangle \).

The other groups \( \tilde{H}_x \) have property 3) because of the isomorphism \( \tilde{H}_1 \to \tilde{H}_x \) from assertion 1).

4) Let us show that the family of subproducts \( \{ L_x : x \in T/R \} \) of the free product \( L \), together with the subgroups \( N_x < L_x \), satisfies the conditions of Proposition 1. Indeed, conditions 1), 2), and 3) of this proposition follow immediately from the strong UP property of the group \( T/R \). Condition (***) is implied by decomposition (4). Thus, property 4) is a corollary of Proposition 1.

5) According to Proposition 1, it is sufficient to show that \( G^{(cy)} \not\subseteq \langle \mathfrak{R}, \{ G^{(cy)} : z \in X_x \setminus \{ y \} \} \rangle \) for each \( x \in T/R \) and each \( y \in X_x \) in the group \( \tilde{H}_x \). By virtue of the isomorphism \( \tilde{H}_x \simeq \tilde{H}_1 \), we can put \( x = 1 \) without loss of generality. So, we have to prove the impossibility of the inclusion

\[
G^{(cy)} \not\subseteq \langle \mathfrak{R}, \{ G^{(cy)} : z \in X_1 \setminus \{ y \} \} \rangle \quad \text{in the group} \quad \tilde{H}_1 = \left< H_1, t \prod_i (g_i^{(c_{yi})})^{\tau} = 1 \right>.
\]
Consider the quotient $U = \tilde{H}_1/\langle x \rangle \langle G^{(c_x)} ; z \in X_1 \setminus \{ y \} \rangle$. The group $U$ has a unimodular relative presentation over the group $G^{(c_x)}$. Therefore, the natural mapping $G^{(c_x)} \to U$ is injective [K193]. So, inclusion (5) implies that the group $G^{(c_x)}$, which is isomorphic to $G$, lies in the cyclic subgroup $(\tilde{T})$ of $U$. Hence, the group $G$ is cyclic, which contradicts the assumption. Proposition 3 is proven.

The action $\varphi$ of the group $T$ on $L$ descends to an action on $K$. We denote this action by the same letter $\varphi : T \to \text{Aut} K$. This action is consistent with the isomorphism $T \cong R \cong \tilde{R} \subseteq K$. Indeed, formulae (3) imply that

$$\langle \tilde{t} \rangle \varphi^r = \langle \tilde{t} \rangle^r \quad \text{and} \quad \langle g(c_x) \rangle \varphi^r = \langle g(c_x) \rangle^r$$

for all $x \in T$, $y \in T/R$ and $r \in R$.

**Theorem 3.** The amalgamated semidirect product $P = T \rtimes K$ corresponding to the action $\varphi$ and the isomorphism $r \mapsto \tilde{t}$ is isomorphic to the group $\tilde{G}$. The isomorphism $P \to \tilde{G}$ is identity on $T$ and maps the subgroup $G^{(1)} \subseteq P$ onto $\tilde{G}$ and the subgroup $K \triangleleft P$ onto $\langle G(S) \rangle = \langle G(R) \rangle \triangleleft \tilde{G}$.

**Proof.** The group $G$ embeds in $P$ as a subgroup: $G = G^{(1)} \subseteq K \subseteq P$. According to the definition of the action, we have $G^{(c_x)} = G^{c_x}$. Thus, the relation of the group $\tilde{H}_1$, which is valid in $K$, and the equality $R = \tilde{R}$ in the group $P$ give relation (2). So, the mapping defined by the formulae $G \ni g \mapsto g^{(1)} \in G^{(1)}$ and $T \ni x \mapsto x$ is a homomorphism from $\tilde{G}$ into $P$. The inverse homomorphism has the form $G^{(c_x)} \ni g^{(c_x)} \mapsto g^{c_x}$ and $T \ni x \mapsto x$.

**Remark.** Theorem 3 and Proposition 3(3) imply that the natural mapping $T \to \tilde{G}$ is injective if $w \notin T$. In the case when the group $T$ is free, this fact was first proven by S. V. Ivanov, who used geometrical methods (unpublished).

### 6. Proof of Theorem 2

The first assertion of the theorem follows immediately from Theorem 3. To prove the second assertion, consider a cyclically reduced generalised unimodular word $w = g_1 t_1 \ldots g_q t_q$ and put $t = t_1 t_2 \ldots t_q$. By virtue of Theorem 1, we can assume without loss of generality that the group $T$ is not cyclic.

**Case 1:** $q = 1$ and $g_1 = 1$. In this case, the centre of the group $\tilde{G} \simeq G \ast \langle T/\langle t \rangle \rangle$ is trivial, if $\langle t \rangle \neq 1$. If $\langle t \rangle = 1$, then $R = T$, $S = \{ 1 \}$ (see the definition of unimodularity), and $T = R = \langle t \rangle \ast S = \langle t \rangle$ which contradicts the assumption that the group $T$ is noncyclic.

**Case 2:** $q = 1$ and $g_1 \neq 1$. In this case, $T \neq \langle t \rangle$ (because the group $T$ is assumed to be noncyclic),

$$\tilde{G} \simeq G \ast \langle t \rangle_{g_1 = t^{-1}}$$

and the assertion of Theorem 2 is a corollary of the following well-known simple fact (see, e.g., [LS77]):

**Lemma 10.** The centre of a free product with amalgamated subgroup which is proper in both factors coincides with the intersection of the centres of the factors.

**Case 3:** $q > 1$ and the group $\langle t_1, \ldots, t_q \rangle$ is cyclic (therefore, this group is generated by the element $t = \prod t_i$ by virtue of unimodularity). In this case, the group $\tilde{G}$ is an amalgamated free product:

$$\tilde{G} \simeq \langle G, t \mid w = 1 \rangle \ast \langle t \rangle$$

By virtue of Lemma 10, the centre of $\tilde{G}$ can be nontrivial only if the centre of $\tilde{G} = \langle G, t \mid w = 1 \rangle$ nontrivially intersects $\langle t \rangle$. By Theorem 1, the nontriviality of the centre of $\tilde{G}$ implies that the group $G$ is cyclic, which contradicts the conditions of Theorem 2.

**Case 4:** the group $\langle t_1, \ldots, t_q \rangle$ is noncyclic. In this case, we can assume without loss of generality that $T = \langle t_1, \ldots, t_q \rangle$. We assume also that the group $G$ is noncyclic; we have to prove that the centre of $\tilde{G}$ is trivial.

First, note that it is sufficient to prove the assertion for finitely generated group $G$. Indeed, the group $\tilde{G}$ decomposes into an amalgamated free product

$$\tilde{G} = G \ast \langle g_1, \ldots, g_q \rangle \ast T \mid g_1 t_1 \ldots g_q t_q = 1$$

If $G \neq \langle g_1, \ldots, g_q \rangle$, then this decomposition implies (by Lemma 10) that the centre of $\tilde{G}$ is contained in $G$. On the other hand, consider the decomposition $\tilde{G} = T \rtimes K$ from Theorem 3. According to the description of the group $K$
(see Section 5), $G$ is contained in the subgroup $H_1$ of $K$, which is the free product of the group $S$ and several copies of the group $G$:

$$G = G^{(1)} \subseteq H_1 = S \ast \bigast_{y \in Y_1} G^{(c_y)} \subset K.$$ 

The centre of $H_1$ can be nontrivial only if $S = \{1\}$ and $|X_1| = 1$. This means (by Proposition 3) that $T = R = \langle t \rangle$, i.e., the group $T$ is cyclic, which contradicts the assumption.

In what follows, we assume that $G$ is a finitely generated noncyclic group and $q \geq 2$. We have to prove that the centre of $\hat{G}$ is trivial. Let us use Theorem 3 again. Suppose that $fy$ is a central element of the group $\hat{G} = T \wr K$, where $y \in T$ and $f \in K$. According to Proposition 3, the group $K$ is a strict free iterated amalgamated product of the groups $\{H_x; x \in T/R\}$. The element $f \in K$ permutes the factors:

$$\hat{H}_f^\alpha = \hat{H}_x^{y^{-1}} = \hat{H}_{xy^{-1}}.$$ 

Therefore, $f \in \hat{H}_z$ for some $z \in T/R$ (by Proposition 2). Thus, the centrality of the element $fy$ means that

$$\hat{H}_z = \hat{H}^{fy} = \hat{H}_y^z = \hat{H}_{zy}.$$ 

According to Proposition 2, this equality can hold only for $z = zy \in T/R$. This implies that $y \in R$ and $fy \in Z(K)$. Applying Proposition 2 once again, we see that $fy \in Z(\hat{H}_z)$ for some $z \in T/R$. To complete the proof, it remains to recall that, according to Theorem 1, the centre of $H_z \simeq \hat{H}_1$ can be nontrivial only if $H_1$ is cyclic, which contradicts the assumption about the noncyclicity of the group $G \simeq G^{(1)} \subseteq H_1$.

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