Compact support of spherically symmetric equilibria in non-relativistic and relativistic galactic dynamics

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Abstract

Equilibrium states in galactic dynamics can be described as stationary solutions of the Vlasov-Poisson system, which is the non-relativistic case, or of the Vlasov-Einstein system, which is the relativistic case. To obtain spherically symmetric stationary solutions the distribution function of the particles (stars) on phase space is taken to be a function \( \Phi(E,L) \) of the particle energy and angular momentum. We give a new condition on \( \Phi \) which guarantees that the resulting steady state has finite mass and compact support both for the non-relativistic and the relativistic case. The condition is local in the sense that only the asymptotic behaviour of \( \Phi \) for \( E \to E_0 \) needs to be prescribed, where \( E_0 \) is a cut-off energy above which no particles exist.

1 Introduction

Let \( f = f(t,x,v), \ t \in \mathbb{R}, \ x,v \in \mathbb{R}^3 \) be the density function on phase space of the stars in a galaxy. We assume that collisions among the stars are sufficiently rare to be neglected and that the stars interact only by the gravitational field which they create collectively. In a non-relativistic situation \( f \) obeys the Vlasov-Poisson system, in a general relativistic situation one obtains the Vlasov-Einstein system. We are interested in spherically symmetric equilibrium solutions of these systems, i. e., \( f \) is independent of time.
t, and $f(Ax,Av) = f(x,v)$, $A \in SO(3)$, $x,v \in \mathbb{R}^3$. The Vlasov-Poisson system then takes the form

$$v \cdot \nabla_x f - U'(x/r) \cdot \nabla_v f = 0,$$

(1.1)

$$U'(r) = \frac{4\pi}{r^2} \int_0^r s^2 \rho(s) ds,$$

(1.2)

$$\rho(r) = \rho(x) = \int f(x,v) dv.$$  

(1.3)

Here $r = |x|$, $'$ denotes derivative with respect to $r$, $U$ denotes the gravitational potential of the system and $\rho$ the spatial mass density induced by $f$; we assume that all the stars have mass one and note that due to spherical symmetry $U$ and $\rho$ depend only on $r$. The Vlasov-Einstein system takes the form

$$\frac{v}{\sqrt{1+v^2}} \cdot \nabla_x f - \sqrt{1+v^2} \mu' \frac{X}{r} \cdot \nabla_v f = 0,$$

(1.4)

$$e^{-2\lambda} (2r\lambda' - 1) + 1 = 8\pi r^2 \rho,$$

(1.5)

$$e^{-2\lambda} (2r\mu' + 1) - 1 = 8\pi r^2 p,$$

(1.6)

$$\rho(r) = \rho(x) = \int \sqrt{1+v^2} f(x,v) dv,$$

(1.7)

$$p(r) = p(x) = \int \left(\frac{x \cdot v}{r}\right)^2 f(x,v) \frac{dv}{\sqrt{1+v^2}}.$$  

(1.8)

Here $v^2 = v \cdot v$, and $\cdot$ denotes the Euclidean scalar product on $\mathbb{R}^3$. If $x = (r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta)$ then the space time metric is given by

$$ds^2 = -e^{2\mu} dt^2 + e^{2\lambda} dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right),$$

$\rho$ denotes the mass density and $p$ the radial pressure. As to the choice of coordinates on phase space, which leads to the above form of the static, spherically symmetric Vlasov-Einstein system, we refer to [14]. As boundary conditions we require asymptotic flatness, i. e.,

$$\lim_{r \to \infty} \mu(r) = \lim_{r \to \infty} \lambda(r) = 0,$$

(1.9)

and a regular center, i. e.,

$$\lambda(0) = 0.$$  

(1.10)

For the Vlasov-Poisson system we require

$$\lim_{r \to \infty} U(r) = 0,$$

(1.11)
and we will also need the radial pressure $p$ in the non-relativistic case, which is defined as
\[ p(r) = p(x) = \int \left( \frac{x \cdot v}{r} \right)^2 f(x,v) \, dv. \] (1.12)

There are essentially two approaches to construct solutions of these systems. The first is to observe that there exist invariants of the particle motion, namely the particle energy $E = E(x,v)$ and angular momentum squared $L = L(x,v)$; for the definition of these quantities cf. (2.1) and (2.2) respectively. The ansatz
\[ f(x,v) = \Phi(E,L) \] (1.13)
with some prescribed function $\Phi$ automatically satisfies the Vlasov equation, and it remains to solve the field equation(s) with the ansatz for $f$ substituted into the definitions for $\rho$ and $p$; these quantities become functionals of $U$ or $\mu$ since $E$ depends on $U$ or $\mu$ respectively. The main problem then is to show that the resulting steady state has finite (ADM) mass and compact support. In [1] this was done in the non-relativistic case for the so-called polytropic ansatz
\[ \Phi(E,L) = (E_0 - E)^k L^l, \]
where $E_0$ is some constant, $(\cdot)^+$ denotes the positive part, and $k > -1$, $l > -1$, $k + l + 1/2 > 0$, $k < 3l + 7/2$. In [10, 16] an existence result was established for the Vlasov-Einstein system, exploiting the fact that the Vlasov-Poisson system is the limit of the Vlasov-Einstein system as the speed of light tends to infinity, cf. [15]. (In the present paper the speed of light is set to unity.) This perturbation argument does not give good control over the class of models obtained, a fact which motivates the search for a better method. It should be noted that in order to obtain a steady state with finite mass $\Phi$ must vanish for energy values larger than some cut-off energy $E_0$.

The second approach is to define an energy-Casimir functional which has the property that its critical points are steady states, and then show that this functional has a minimizer over a certain set of phase space densities $f$. This approach was used in [6] for the Vlasov-Poisson system. It has the advantage that it provides a certain nonlinear dynamical stability property of the steady state obtained, and the resulting steady states are more general than the polytropic ones: Only certain growth conditions on $\Phi$ need to be prescribed. Nevertheless, the latter approach also makes use of global properties of the function $\Phi$, i. e., of growth conditions for $E$ close to $E_0$ and close to $-\infty$.

The present paper follows the first approach, but we give a new characterization of $\Phi$'s which lead to finite mass and compact support both in the
non-relativistic and in the relativistic case: Except for some mild technical assumptions we only require that

$$\Phi(E, L) = c(E_0 - E)^k L^l + O((E_0 - E)^{k+\delta}) L^l \text{ as } E \to E_0$$

where

$$k > -1, \quad l > -\frac{1}{2}, \quad k + l + \frac{1}{2} > 0, \quad k < l + \frac{3}{2}.$$ 

Thus the characterization is purely local: Only the asymptotic behaviour at $E = E_0$ needs to be controlled. Since such steady states together with their features of finite mass and compact support persist under perturbations of the ansatz function $\Phi$ as long as the form of the asymptotic expansion at $E_0$ is preserved, they might be called structurally stable. Another important point is that we obtain finite ADM mass and compact support in the relativistic case directly, i.e., without a perturbation argument as was used in [10, 16]. The assumptions are much more transparent than the smallness conditions required when starting from the Newtonian limit.

The main idea of the argument is as follows: If $R \in [0, \infty]$ denotes the radius of the support of a steady state then one can show that $E_0 - U(r) \to 0$ or $E_0 - e^\mu(r) \to 0$ as $r \to R^-$. One needs to show that $R < \infty$, and this is done by expanding all the relevant quantities in terms of $E_0 - U(r)$ or $E_0 - e^\mu(r)$, obtaining detailed estimates for the behaviour of the solution of the field equations as $r \to R^-$. This approach was strongly motivated by the paper of Makino [8] where a technique of this kind is used to prove finite mass and compact support for relativistic, spherically symmetric stellar models, i.e., for steady states of the Euler-Einstein system. The connection between the two situations is as follows. If a spherically symmetric steady state solution of the Vlasov-Einstein system corresponds to a choice of $\Phi$ which only depends on $E$, then the energy density $\rho$ and the pressure $p$ define a solution of the Euler-Einstein system describing a self-gravitating perfect fluid. (A similar relation between kinetic and fluid models holds in the non-relativistic case.) The asymptotic behaviour of $\Phi$ which is important in our results corresponds to the way in which $p$ depends on $\rho$ (equation of state) in the limit $\rho \to 0$.

Our paper proceeds as follows: In the next section we derive the reduced problems which are obtained by substituting the ansatz for $f$ into $\rho$ and $p$ in the field equation(s). We also show that a cut-off energy $E_0$ is necessary in order to obtain finite mass and compact support. This shows that except for the form of the dependence on $L$ our ansatz is quite general. The main result is then stated and proven in Section 3. In a last section we consider steady states which appear in the astrophysics literature and show
that our result applies to most of these and proves that these steady states have the physically very desirable properties of finite mass and compact support. Clearly, there are some polytropic steady states with finite mass and compact support which are not covered by our result: For the polytropes one needs \( k < 3l + 7/2 \) whereas we require \( k < l + 3/2 \). In the last section we also comment on this discrepancy from the viewpoint of the relation to static self-gravitating fluid bodies and from the viewpoint of recent stability results for the Vlasov-Poisson system; cf. [4, 11].

To conclude this introduction we mention some further references which are related to the present paper. Steady states of the Vlasov-Poisson system with axial symmetry are constructed in [11]. Spherically symmetric steady states with a vacuum region at the center are constructed in [12] both for the Vlasov-Poisson and the Vlasov-Einstein system. Among these there are examples of relativistic steady states violating Jeans’ Theorem, which holds for the Vlasov-Poisson system and says that all spherically symmetric steady states are obtained by an ansatz of the form (1.13), cf. [20]. In [13] steady states of the Vlasov-Poisson system are constructed where the matter is concentrated on a plane, using the variational approach mentioned earlier. Concerning the initial value problem for the time dependent systems we mention [7, 9, 19], where global existence of classical solutions to the Vlasov-Poisson system is established for general data. For the Vlasov-Einstein system the existence theory for the initial value problem is far less complete. We mention [14, 17] for the spherically symmetric, asymptotically flat case which is of interest here.

\section{The reduced problem}

Consider first the Vlasov-Poisson system. The particle energy and angular momentum squared

\[ E = E(x, v) = \frac{1}{2} v^2 + U(r), \quad L = L(x, v) = |x \times v|^2 \]  

are conserved along particle trajectories, i.e., are constant along solutions of the characteristic system

\[ \dot{x} = v, \quad \dot{v} = -U'(r) \frac{x}{r} \]

of (1.1); \( r = |x| \). If we make the ansatz (1.13) then upon substituting (1.13) and (2.1) into (1.3), \( \rho \) becomes a functional of \( U \), and it remains to solve the resulting nonlinear Poisson equation; for the moment we only require \( \Phi \)
to be a nonnegative, measurable function. Similarly, for the Vlasov-Einstein system one finds that

\[ E = E(x,v) = e^{\mu(r)} \sqrt{1 + v^2}, \quad L = L(x,v) = |x \times v|^2 \]  \hspace{1cm} (2.2)

are constant along solutions of

\[ \dot{x} = \frac{v}{\sqrt{1 + v^2}}, \quad \dot{x} = -\sqrt{1 + v^2} \mu'(r) \frac{x}{r}, \]

and it remains to solve the system (1.5) and (1.6) where \( \rho \) and \( p \) become functionals of \( \mu \) upon substituting (1.13) and (2.2) into (1.7) and (1.8). It is a simple computation to see that in the non-relativistic case substituting (1.13) into (1.3) yields

\[ \rho(r) = 2\pi r^2 \int_{U(r)}^{\infty} \Phi(E,L) dL dE \frac{dL dE}{\sqrt{2(E-U(r)) - L/2r^2}}. \]  \hspace{1cm} (2.3)

In the relativistic case we obtain

\[ \rho(r) = \frac{2\pi}{r^2} e^{-2\mu(r)} \int_{e^{\mu(r)}}^{\infty} \int_{0}^{r^2(E^2 - 2\mu(r) - 1)} \Phi(E,L) \frac{E^2 dL dE}{\sqrt{E^2 - e^{2\mu(r)}(1 + L/r^2)}}. \]  \hspace{1cm} (2.4)

and

\[ p(r) = \frac{2\pi}{r^2} e^{-2\mu(r)} \int_{e^{\mu(r)}}^{\infty} \int_{0}^{r^2(E^2 - 2\mu(r) - 1)} \Phi(E,L) \sqrt{E^2 - e^{2\mu(r)}(1 + L/r^2)} dL dE. \]  \hspace{1cm} (2.5)

Before proceeding further we demonstrate that \( \Phi \) has to vanish for large values of \( E \) if the resulting steady state is to have finite mass, i.e., if the quantity

\[ M = \int \rho(x) dx \]  \hspace{1cm} (2.6)

is finite; \( M \) is the total mass or the total ADM mass of the steady state respectively. Here and below it will be useful to recall that the solution of (1.5) satisfying (1.10) is given by

\[ e^{-2\lambda} = 1 - \frac{2m(r)}{r} \]  \hspace{1cm} (2.7)

where

\[ m(r) = 4\pi \int_{0}^{r} s^2 \rho(s) ds, \]  \hspace{1cm} (2.8)
at least as long as $2m(r) < r$. Using (2.7) one can rewrite (1.6) in the form

$$
\mu'(r) = e^{2\lambda} \left( \frac{m(r)}{r^2} + 4\pi r p(r) \right).
$$

(2.9)

Theorem 2.1 Let $\Phi : \mathbb{R}^2 \to [0, \infty]$ be measurable.

(a) Let $(f, U)$ be a solution of (1.1), (1.2), (1.3) in the sense that $f(x, v) = \Phi(E, L)$ and $U \in C^1([0, \infty])$ solves (1.2) with (2.3) substituted in. Let $M < \infty$. Then $U_\infty = \lim_{r \to \infty} U(r) < \infty$, and $\Phi(E, L) = 0$ a.e. for $E > U_\infty$, $L > 0$.

(b) Let $(f, \lambda, \mu)$ be a solution of (1.4), (1.5), (1.6), (1.7), (1.8) in the sense that $f(x, v) = \Phi(E, L)$ and $\lambda, \mu \in C^1([0, \infty])$ solve (1.5), (1.6) with (2.4), (2.5) substituted in. Let $M < \infty$. Then $\mu_\infty = \lim_{r \to \infty} \mu(r) < \infty$, and $\Phi(E, L) = 0$ a.e. for $E > e^{\mu_\infty}$, $L > 0$.

Proof: Let us first consider the non-relativistic case. Since

$$
0 \leq U'(r) \leq \frac{M}{r^2}, \quad r > 0,
$$

$U$ is increasing and has a finite limit as $r \to \infty$. By (2.3),

$$
M = 8\pi^2 \int_0^\infty \int_{U(r)}^\infty \int_0^{2\pi} (E - U(r)) \Phi(E, L) dL dE dr \sqrt{2(E - U(r) - L/2r^2)} 
\geq 8\pi^2 \int_0^\infty \int_{U_\infty}^\infty \Phi(E, L) \int_0^{\infty} \frac{dr}{\sqrt{2(E - U(r))}} \sqrt{2(E - U_\infty)} dL dE.
$$

The integral with respect to $r$ in the latter expression is infinite for any $E > U_\infty$ and $L > 0$, which implies that $\Phi$ has to vanish for such arguments.

Let us now consider the relativistic case. Since by (2.9) $\mu$ is increasing, the limit $\mu_\infty = \lim_{r \to \infty} \mu(r) \in ]-\infty, \infty]$ exists. Since $m(r) \leq M$ and thus $e^{2\lambda(r)} \leq 2$ for $r \geq 4M$, and $p \leq \rho$, we conclude that

$$
\mu'(r) \leq 2 \left( \frac{M}{r^2} + 4\pi r \rho(r) \right), \quad r \geq 4M,
$$

and

$$
\mu(r) \leq \mu(4M) + 2 \int_{4M}^\infty \frac{M}{r^2} dr + 4\pi \frac{1}{4M} \int_{4M}^\infty s^2 \rho(s) ds < \infty, \quad r \geq 4M
$$
which proves that $\mu_\infty < \infty$. Using (2.4) we obtain

$$
M = 8\pi^2 \int_0^\infty e^{-2\mu(r)} \int_0^\infty \int_0^r (e^{-2\mu(r)} - 1) \Phi(E,L) \frac{E^2 dLdE dr}{\sqrt{E^2 - e^{-2\mu(r)}(1 + L/r^2)}}
$$

$$
\geq 8\pi^2 e^{-2\mu_\infty} \int_0^\infty \int_0^\infty \int_0^\infty \Phi(E,L) \frac{E^2 dLdE dr}{\sqrt{E^2 - e^{-2\mu_\infty}(1 + L/r^2)}}
$$

and the assertion follows as in the non-relativistic case.

Although this will become important only in the next section where we prove our main result we restrict the ansatz (1.13) in the following way:

$$
L(t) = \phi(E) L^t
$$

with $t > -1/2$ and $\phi$ measurable. Observe now that in the non-relativistic case $E$ can attain any real value, whereas in the relativistic case $E > 0$. In order not to introduce unnecessary assumptions this needs to be reflected in the choice of domain of $\phi$, which we denote by $|E_{\text{min}}, \infty]$ with

$$
E_{\text{min}} = \begin{cases}
-\infty & \text{in the non-relativistic case,} \\
0 & \text{in the relativistic case.}
\end{cases}
$$

Since we are interested only in steady states with finite mass we assume that $\phi(E) = 0$ for all energies $E$ larger than some given $E_0 > E_{\text{min}}$. It will be necessary to have some information on the functional dependence of $\rho$ and $p$ on $U$ or $\mu$ respectively:

**Lemma 2.2** Let $\phi:]E_{\text{min}}, \infty[ \to \mathbb{R}$ be measurable, $E_0 > E_{\text{min}}$, and $k > -1$ such that on every compact subset $K \subset |E_{\text{min}}, \infty|$ there exists $C \geq 0$ such that

$$
0 \leq \phi(E) \leq C(E_0 - E)^{k_1}, \quad E \in K.
$$

Define

$$
g_m(u) = \int_u^\infty \phi(E)(E - u)^m dE, \quad u \in |E_{\text{min}}, \infty|,
$$

$$
h_m(u) = \int_u^\infty \phi(E)(E^2 - u^2)^m dE, \quad u \in |E_{\text{min}}, \infty|.
$$

If $m > -1$ and $k + m + 1 > 0$ then $g_m, h_m \in C(|E_{\text{min}}, \infty|)$. If $m > 0$ and $k + m > 0$ then $g_m, h_m \in C^1(|E_{\text{min}}, \infty|)$ with

$$
g_m' = -mg_{m-1}, \quad h_m' = -2mu h_{m-1}.
$$
Proof : The continuity assertions can be obtained using Lebesgue’s dominated convergence theorem. By the same tool one can show that the functions are left differentiable with the given derivatives. Since these left derivatives are again continuous the functions are differentiable as claimed. The details are fairly lengthy but largely technical, and are therefore omitted.

Using the lemma above we express $\rho$ and $p$ as functionals of $U$ or $\mu$ respectively. To this end we introduce for $a > -1$, $b > -1$ the constants

$$c_{a,b} = \int_0^1 s^a (1-s)^b ds = \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)},$$

(2.11)

where $\Gamma$ denotes the gamma function. It will be useful later to note that

$$\frac{c_{a,b-1}}{c_{a,b}} = \frac{a+b+1}{b}, \quad a > -1, \quad b > 0,$$

(2.12)

which follows from the functional relation $x\Gamma(x) = \Gamma(x+1)$. If $\phi$ is as in the lemma above and $f$ given by the ansatz (2.10) then in the non-relativistic case we obtain

$$\rho(r) = 2^{l+3/2} \pi c_{l,-1/2} r^{2l+1/2} g_{l+1/2}(U(r)),$$

(2.13)

$$p(r) = 2^{l+5/2} \pi c_{l,1/2} r^{2l+3/2} g_{l+3/2}(U(r)).$$

(2.14)

For the relativistic case we substitute $E^2 = (E^2 - e^2\mu) + e^2\mu$ into (2.4) and use the abbreviation

$$H_1(e^\mu) = e^{-(2l+4)\mu} h_{l+3/2}(e^\mu) + e^{-(2l+2)\mu} h_{l+1/2}(e^\mu)$$

(2.15)

to obtain

$$\rho(r) = 2\pi c_{l,-1/2} r^{2l} H_1(e^{\mu(r)}),$$

(2.16)

$$p(r) = 2\pi c_{l,1/2} r^{2l} e^{-(2l+4)\mu(r)} h_{l+3/2}(e^{\mu(r)}).$$

(2.17)

A steady state of the Vlasov-Poisson system is now obtained by solving the equation

$$U'(r) = \frac{2^{l+7/2} \pi^2 c_{l,-1/2}}{r^2} \int_0^r s^{2l+2} g_{l+1/2}(U(s)) ds, \quad r > 0,$$

(2.18)

which is (1.2) with (2.13) substituted in, in the case of the Vlasov-Einstein system one needs to solve

$$e^{-2\lambda(2r\lambda' - 1)} + 1 = 16\pi^2 c_{l,-1/2} r^{2l+2} H_1(e^\mu),$$

(2.19)

$$e^{-2\lambda(2r\mu' + 1)} - 1 = 16\pi^2 c_{l,1/2} r^{2l+2} e^{-(2l+4)\mu} h_{l+3/2}(e^\mu),$$

(2.20)
which is (1.5), (1.6) with (2.16) and (2.17) substituted in. It is known that these problems have solutions on $[0, \infty]$, and we state the corresponding result for further reference; the problem of interest of course is whether these solutions lead to steady states with compact support and finite mass.

**Theorem 2.3** Let $\phi$ be as in Lemma 2.2.

(a) Let $U_0 \in \mathbb{R}$. Then there exists a unique solution $U \in C^1([0, \infty[)$ of (2.18) with $U(0) = U_0$.

(b) Let $\mu_0 \in \mathbb{R}$. Then there exists a unique solution $(\lambda, \mu) \in C^1([0, \infty[)^2$ of (2.19), (2.20) with $\lambda(0) = 0$, $\mu(0) = \mu_0$.

**Proof**: In both cases local existence on some interval $[0, \delta]$ follows by a contraction argument, cf. [1, Thm. 3.6] for details in the non-relativistic case and [10, Thm. 3.2] for the relativistic case. In the non-relativistic case global existence is simple: Clearly, $U$ is increasing. Either $U \leq E_0$ on its maximal interval of existence, in which case $U$ exists globally in $r$, or $U(r) > E_0$, $r \geq R$ for some $R > 0$, in which case $\rho(r) = 0$, $r \geq R$, and again $U$ exists globally. For the relativistic case the inequality $2m(r) < r/2$ has to be controlled which makes the argument more involved than in the non-relativistic case, cf. [11, Thm. 3.4].

**Remark**: In the theorem above nothing is said about the boundary conditions at infinity. However, once we know that a steady state has finite mass then $U$ or $\mu$ has a finite limit at infinity, cf. Theorem 2.1. Subtracting this limit from $U$ or $\mu$ respectively and redefining $E_0$ accordingly gives a steady state with the same $f$, but which now satisfies the boundary condition at infinity; the boundary condition for $\lambda$ follows from (2.7) if the ADM mass is finite.

3 The main result

The following theorem is the main result of the present paper:

**Theorem 3.1** Let $k, l \in \mathbb{R}$ be such that

$$k > -1, \ l > -\frac{1}{2}, \ k + l + \frac{1}{2} > 0, \ k < l + \frac{3}{2}.$$  

Let $\phi : E_{\min}, \infty \rightarrow [0, \infty]$ be measurable and such that $\phi \in L^\infty_{\text{loc}}([E_{\min}, E_0])$, and

$$\phi(E) = c(E_0 - E)^k_+ + O((E_0 - E)^{k+\delta}_+) \quad \text{as} \ E \rightarrow E_0 -$$

for some $E_0 > E_{\min}$, $c > 0$, and $\delta > 0$. 

10
(a) Let \((f,U)\) be a steady state of the Vlasov-Poisson system in the sense that \(f(x,v) = \phi(E)L^1\) with \(E\) and \(L\) as defined in (2.1), and \(U \in C^1([0,\infty[)\) satisfies (1.3). Then the steady state has compact support and finite mass.

(b) Let \((f,\lambda,\mu)\) be a steady state of the Vlasov-Einstein system in the sense that \(f(x,v) = \phi(E)L^1\) with \(E\) and \(L\) as defined in (2.2), and \(\lambda,\mu \in C^1([0,\infty[)\) satisfy (1.5), (1.6). Then the steady state has compact support and finite ADM mass.

Clearly, if \(\phi\) is as in the theorem above then it satisfies the assumptions in Lemma 2.2 so that this lemma and Theorem 2.3 apply. The main tool in the proof of Theorem 3.1 is the following lemma, which is an adaptation of [8, Thm. 1] to our present situation:

**Lemma 3.2** Let \(x,y \in C^1([0,R[)\) be such that \(x,y > 0\) and

\[
rx' = \alpha(r)y - x + \frac{x + \gamma_1(r)y}{1 - \gamma_2(r)x}
\]
\[
ry' = y \left( c - \beta(r) \frac{x + \gamma_1(r)y}{1 - \gamma_2(r)x} \right)
\]

on \([0,R[\), where \(c > 0, \alpha,\beta,\gamma_1,\gamma_2 \in C([0,R[)\) with \(\alpha_0 = \inf_{r \in [0,R[} \alpha(r) > 0, \lim_{r \to R} \beta(r) = \beta_0 \in [0,c[, \gamma_1,\gamma_2 \geq 0, \text{ and } \lim_{r \to R} \gamma_1(r) = \lim_{r \to R} \gamma_2(r) = 0\). Also let \(1 - \gamma_2(r)x(r) > 0, \text{ for } r \in [0,R[\). Then \(R < \infty\).

**Proof**: As a first step we show that there exists \(r_* \in [0,R[\) such that \(x(r_*) > 1\). If not, then \(x(r) \leq 1, r \in [0,R[\). By assumption there exists \(r_0 \in [0,R[\) such that \(\beta(r) > 0, r \in [r_0,R[\). Choose \(K > 0\) such that \(K(1 + x(r_0)) - y(r_0) > 0\) and \(K\alpha_0 > 1 + c\). If for some \(r \geq r_0, K(1 + x(r)) - y(r) = 0\) then

\[
K(r(1+x) - y)'(r) = Krx'(r) - ry'(r)
\]
\[
\geq K\alpha_0 y(r) - Kx(r) - cy(r)
\]
\[
= (K\alpha_0 - c - 1)y(r) + K > 0.
\]

This implies that no such \(r\) exists, and

\[K(1 + x(r)) - y(r) > 0, r \in [r_0,R[.\]

By our assumption \(x \leq 1\), and

\[y(r) \leq K(1 + x(r)) \leq 2K, r \in [r_0,R[.\]
This implies that $R = \infty$, and
\[ ry' \geq y \left( c - \beta(r) \frac{1+2K\gamma_1(r)}{1-\gamma_2(r)} \right) \geq \frac{c - \beta_0}{2} y \]
for all $r \geq r_1$ sufficiently large, cf. the assumption on $\beta$. Integration of this inequality implies
\[ y(r) \geq y(r_1) (r/r_1)^{(c - \beta_0)/2} \rightarrow \infty, \quad r \rightarrow \infty, \]
a contradiction.

Thus we can assume that there exists some $r_* \in ]0, R]$ with $x(r_*) > 1$. Now
\[ rx' \geq -x + x^2 = x(x - 1) \quad (3.2) \]
which implies that $x(r) > 1$ for all $r \in [r_*, R]$, and upon integration of (3.2),
\[ x(r) \geq \left( 1 - \frac{x(r_*) - 1}{x(r_*)} \right)^{-1}, \quad r \in [r_*, R]. \]
Since the term in parenthesis vanishes for $r = r_* x(r_*) / (x(r_*) - 1)$ this implies that
\[ R \leq r_* \frac{x(r_*)}{x(r_*) - 1}. \]

\[ \square \]

**Proof of Theorem 3.1:**

**Step 1—The basic set-up:** Consider a solution of the reduced field equation(s) as given by Theorem 2.3. Consider the non-relativistic case first and define $[0, R]$ as the maximal interval on which $U < E_0$; we may assume that $U(0) < E_0$, or else the solution is trivial. If $R < \infty$ then $U(R) = E_0$. If $R = \infty$ then $U_\infty = \lim_{r \to \infty} U(r) \leq E_0$ exists by the monotonicity of $U$. Assume $U_\infty < E_0$. Then (2.13) and the monotonicity of $g_{l+1/2}$ imply that
\[ \rho(r) \geq 2^{l+3/2} \pi c_{l-1/2} r^{2l} g_{l+1/2}(U_\infty) = cr^2l, \quad r > 0, \]
where $c > 0$. Then (1.2) implies that $U'(r) \geq cr^{1+2l}$, $r > 0$, with a different positive constant $c$, and integrating this estimate implies that $U_\infty = \infty$, a contradiction. Thus we have the following

**Basic set-up:** There exists some $R \in ]0, \infty]$ such that $U$ exists on $[0, R]$ with $U < E_0$ on this interval, and
\[ \lim_{r \to R^-} U(r) = E_0. \]
Analogously, we have for the relativistic case the

**Basic set-up:** There exists some $R \in [0, \infty]$ such that $\lambda, \mu$ exist on $[0, R]$ with $e^\mu < E_0$ on this interval, and

$$\lim_{r \to R^-} e^\mu(r) = E_0.$$  

We may assume that $e^\mu(0) < E_0$, or else the solution is trivial. We choose $R$ maximal such that $e^\mu < E_0$ on $[0, R]$. The non-obvious case is $R = \infty$. By monotonicity, $\mu_\infty = \lim_{r \to R^-} \mu(r) \leq E_0$ exists. Assume that $\mu_\infty < E_0$. Then again $\rho(r) \geq cr^l$, $r > 0$, with a positive constant $c > 0$. By (2.7) and (2.9),

$$\mu'(r) = \left(1 - \frac{8\pi}{r} \int_0^r s^2 \rho(s) ds\right)^{-1} \left(\frac{4\pi}{r^2} \int_0^r s^2 \rho(s) ds + 4\pi r \rho'(r)\right) \geq \frac{c}{r^2} \int_0^r s^{2+2l} ds = cr^{1+2l}, \ r > 0.$$  

Integration of this inequality implies that $\mu_\infty = \infty$, a contradiction.

What we need to show in both cases is that $R < \infty$.

**Step 2—New variables:** We introduce new variables which bring the system into the form stated in Lemma 3.2. We define for the non-relativistic case

$$\eta(r) = E_0 - U(r), \quad (3.3)$$  

and for the relativistic case

$$\eta(r) = \ln E_0 - \mu(r); \quad (3.4)$$  

recall that $E_0 > 0$ in the relativistic case. Now define in both cases

$$x(r) = \frac{m(r)}{r\eta(r)} = \frac{4\pi}{r\eta(r)} \int_0^r s^2 \rho(s) ds, \quad (3.5)$$  

$$y(r) = 4\pi r^2 \rho^2(r) \frac{p(r)}{p'(r)} \quad (3.6)$$  

on the interval $]0, R[$ with $R$ from the previous step; note that $\eta, \rho, p > 0$ on that interval. In the non-relativistic case, $r\eta' = -rU' = -m/r$, whereas in the relativistic case by (2.9),

$$r\eta' = -r\mu' = -\frac{\eta x + yp^2/\rho^2}{1 - 2\eta x}.$$  

Thus in both cases

$$r\eta' = -\frac{x + \gamma_1(r)y}{1 - \gamma_2(r)x}, \quad (3.7)$$  

13
where in the non-relativistic case
\[ \gamma_1 = \gamma_2 = 0, \tag{3.8} \]
and in the relativistic case
\[ \gamma_1 = \frac{p^2}{\eta p^2}, \quad \gamma_2 = 2\eta. \tag{3.9} \]
Now
\[ rx' = \frac{4\pi r^2 \rho}{\eta} - x - x \frac{ry'}{\eta} = \alpha(r)y - x + x + \gamma_1(r)y \frac{x + \gamma_1(r)y}{1 - \gamma_2(r)x}, \]
where
\[ \alpha = \frac{p}{\eta p}. \]
In the non-relativistic case we find, using (2.13), (2.14), and (2.12), that
\[ \alpha = \frac{1}{l + 3/2} \frac{g_{l+1/2}(U)}{\eta_{l+1/2}(U)}. \tag{3.10} \]
In the relativistic case we find, using (2.16), (2.17), and (2.12), that
\[ \alpha = \frac{1}{2l + 3} \frac{h_{l+3/2}(e^\mu)}{\eta h_{l+3/2}(e^\mu) + e^{2\mu} \eta h_{l+1/2}(e^\mu)}. \tag{3.11} \]
Next consider the equation for \( ry' \). In both cases we find
\[ ry' = 2y + \frac{2 \rho'}{r} y - \frac{ry'}{p} y. \]
In the non-relativistic case Lemma 2.2 and (2.13), (2.14) imply
\[ \rho' = \frac{2l}{r} \rho - \frac{(l + 1/2)^2 c_{l,-1/2} r^2 g_{l-1/2}(U)}{U'}, \]
and
\[ p' = \frac{2l}{r} p - \rho U'. \tag{3.12} \]
With the definition of \( \eta \) and (3.7) this implies that
\[ ry' = y \left( (2l + 2) - \beta(r) \frac{x + \gamma_1(r)y}{1 - \gamma_2(r)x} \right), \]
where
\[ \beta = - \frac{\eta p}{p} + (2l + 1) \eta g_{l+1/2}(U) \frac{g_{l+1/2}(U)}{g_{l+1/2}(U)}. \tag{3.13} \]
In the relativistic case Lemma 2.2 and (2.16), (2.17) imply
\[ \rho' = \frac{2l}{r} \rho - 2\pi c_{l-1/2} r^{2l} \tilde{H}_l(e^\mu) \mu' \]
where
\[ \tilde{H}_l(e^\mu) = (2l + 4)e^{-(2l+4)\mu} h_{l+3/2}(e^\mu) + (4l + 5)e^{-(2l+2)\mu} h_{l+1/2}(e^\mu) \]
\[ + (2l + 1)e^{-2l\mu} h_{l-1/2}(e^\mu), \]
and
\[ p' = \frac{2l}{r} p - (p + \rho) \mu', \]
which is the Tolman-Oppenheimer-Volkov equation. With the definition of \( \eta \) and (3.7) this implies that
\[ ry' = y \left( (2l + 2) - \beta(r) \frac{x + \gamma_1(r) y}{1 - \gamma_2(r) x} \right), \]
where
\[ \beta = -\eta - \frac{\eta \rho}{p} + 2\eta \tilde{H}_l(e^\mu) \]
Thus both in the non-relativistic and in the relativistic case we obtain a system of the form which is stated in Lemma 3.2, and in the next two steps we will show that \( \gamma_1, \gamma_2, \alpha, \) and \( \beta \) satisfy the necessary assumptions.

**Step 3—Application of Lemma 3.2, the non-relativistic case:** In this case \( \gamma_1 = \gamma_2 = 0 \), cf. (3.8), so these functions satisfy the assumptions in Lemma 3.2. To investigate the asymptotic behaviour of \( \alpha \) and \( \beta \) we need to use the asymptotic expansion of \( \phi \); note that by Step 1, \( \eta(r) \to 0 \) for \( r \to R \).

First of all we may assume \( c = 1 \) in (3.1), since this factor cancels in \( \alpha \) and \( \beta \). A simple computation shows that
\[ \int_{u_0}^{E_0} (E_0 - E)^a (E - u)^b dE = c_{a,b} (E_0 - u)^{a+b+1}, u \leq E_0, a > -1, b > -1 \]
where \( c_{a,b} \) is defined by (2.11). Thus
\[ g_m(u) = c_{k,m} \eta^{k+m+1} + O(\eta^{k+m+1}), \eta = E_0 - u \to 0+. \]

Using (3.17) and (2.12) in (3.10) we find that
\[ \alpha(r) = \left( k + l + \frac{5}{2} \right)^{-1} + O(\eta(r) \delta), r \to R-. \]
Since we assume that $U(0) < E_0$,

$$\alpha(r) \rightarrow \frac{1}{l+3/2} \frac{g_{l+3/2}(U(0))}{(E_0 - U(0))g_{l+1/2}(U(0))} > 0, \ r \rightarrow 0^+,$$

and since $\alpha > 0$ on $]0,R[$ we have shown that

$$\inf_{r \in ]0,R[} \alpha(r) > 0$$

as required. Using (3.17) and (2.12) in (3.13) we see that

$$\beta(r) = -(k+l+5/2) + (2l+1) \frac{k+l+3/2}{l+1/2} + O(\eta(r)^{\delta}) \rightarrow k+l+\frac{1}{2}, \ r \rightarrow R^-,$$

and by our assumptions on $k$ and $l$ this limit lies in the interval $]0,2l+2[$ as required. Applying Lemma 3.2 we find that

$$\mathcal{R} < \infty,$$

and the proof of the theorem is complete in the non-relativistic case.

**Step 4—Application of Lemma 3.2, the relativistic case:** By (3.9) and since

$$\eta(r) \rightarrow 0 \text{ as } r \rightarrow R^-,$$

$\gamma_2$ is as required. We use (3.1) to compute the asymptotic behaviour of the various other quantities as $r \rightarrow R^-$. To this end we first observe that (3.1) implies that

$$\phi(E) = c'E_0^2 - E^2)^k + E O((E_0^2 - E^2)^{k+\delta}) \text{ as } E \rightarrow E_0^-,$$

which is more suitable for computing the integral $h_m$. Again we assume without loss of generality that $c' = 1$. Using the relation

$$\int_u^{E_0} E(E_0^2 - E^2)^a(E^2 - u^2)^b dE = \frac{1}{2} c_{a,b}(E_0^2 - u^2)^{a+b+1}, \ u \leq E_0, \ a > -1, \ b > -1,$$

we find that

$$h_m(\epsilon^\mu) = \frac{1}{2} c_{k,m}(E_0^2 - \epsilon^2)^{k+m+1} + O((E_0^2 - \epsilon^2)^{k+m+\delta+1})$$

$$= \frac{1}{2} c_{k,m} \epsilon^{k+m+1} + O(\epsilon^{k+m+\delta+1}), \ \epsilon = E_0^2 - \epsilon^2 \mu \rightarrow 0^+. \ (3.18)$$

In the following we need to observe that

$$\eta = \frac{1}{2E_0^2} \epsilon + O(\epsilon^2), \ \epsilon \rightarrow 0^+ \ (3.19)$$

and

$$E_0^2 e^{-2\mu(r)} = 1 + O(\epsilon(r)), \ r \rightarrow R^- \ (3.20)$$
Using (3.18), (3.19), (3.20), and (2.12) in (3.11) we find that
\[
\alpha(r) = \frac{1}{2l+3} 2E_0^2 e^{-2\mu(r)} \frac{c_{k,l+3/2}}{c_{k,l+1/2}} + O(\epsilon(r)^\delta) = \frac{1}{k+l+5/2} + O(\epsilon(r)^\delta), \quad r \to R^-.
\]
Since \(\alpha > 0\) on \([0,R]\) and \(\lim_{r \to 0^+} \alpha(r) > 0\) we find that
\[
\inf_{r \in [0,R]} \alpha(r) > 0
\]
as required. Since by (3.9) \(\gamma_1 = \alpha p/\rho\), (2.16) and (2.17) together with (3.18) imply that \(\gamma_1(r) \to 0\) as \(r \to R^-\).

It remains to examine the function \(\beta\). Using (3.18) in (2.15) and (3.14) we find
\[
H_l(e^\mu) = \frac{1}{2} c_{k,l+1/2} e^{-2l+2} \mu e^{k+l+3/2} + O(\epsilon^{k+l+\delta+3/2}),
\]
\[
\tilde{H}_l(e^\mu) = \frac{1}{2} c_{k,l-1/2} (2l+1) e^{-2l+1} \mu e^{k+l+1/2} + O(\epsilon^{k+l+\delta+1/2}),
\]
and using this together with (3.19), (3.20), and (2.12) in (3.15) we obtain
\[
\beta(r) = O(\epsilon(r)) - \alpha^{-1}(r) + 2(2l+1) \frac{e^{2\mu(r)} c_{k,l-1/2}}{2E_0^2 c_{k,l+1/2}} + O(\epsilon^\delta(r))
\]
\[
= -(k+l+5/2) + 2(k+l+3/2) + O(\epsilon^\delta(r))
\]
\[
\to k+l+1/2, \quad r \to R^-,
\]
and by our assumptions on \(k\) and \(l\) this limit lies in the required interval \([0,2l+2]\). Applying Lemma 3.2 completes the proof in the relativistic case.

\(\square\)

**Remark:** We note that the application of Lemma 3.2 provides an explicit upper bound on the radius \(R\) of the spatial support in terms of a point \(r^* > 0\) at which
\[
x(r^*) = \frac{m(r^*)}{r^* \eta(r^*)} > 1
\]
namely
\[
R \leq r^* \frac{x(r^*)}{x(r^*) - 1}.
\]
Such a point \(r^*\) must exist by the proof of Lemma 3.2, and this upper bound may be useful in numerical work on steady states.

We have shown that the spatial support of a steady state as in Theorem 3.1 is compact, but a bound on \(v\) over the support of \(f\) follows by the boundedness of \(E\) from above together with (2.1) or (2.2) and the boundedness of \(U\) or \(\mu\) respectively.
4 Examples and final remarks

The following is a list of types of static spherically symmetric models built of self-gravitating collisionless matter, non-relativistic and relativistic, to be found in the literature. In this list inessential multiplicative constants are omitted.

(NR1) Polytropic solutions of Vlasov-Poisson system. The distribution function is of the form \((E_0 - E)^k L^l\) for \(E < E_0\) and zero otherwise. For appropriate \(k\) and \(l\)—cf. the introduction—the existence of solutions with finite radius was proved in \([1]\). Theorem 3.1 applies, but only to a subclass of these, and we comment on this fact below.

(NR2) King models, cf. \([2, p. 232]\). The distribution function is of the form \(e^{E_0 - E - 1}\) for \(E < E_0\) and zero otherwise. Since \(e^{E_0 - E - 1} = (E_0 - E) + O((E_0 - E)^2), E \to E_0 -\), Theorem 3.1 applies to these models.

(NR3) Woolley-Dickens models, cf. \([3, p. 235]\). The distribution function is of the form \(e^{E_0 - E}\) for \(E < E_0\) and zero otherwise. Since \(e^{E_0 - E} = 1 + O((E_0 - E)), E \to E_0 -\), Theorem 3.1 applies.

(NR4) Wilson models, cf. \([4, p. 235]\). The distribution function is of the form \(e^{E_0 - E} - 1 - (E_0 - E)\) for \(E < E_0\) and zero otherwise. Since \(e^{E_0 - E} - 1 - (E_0 - E) = \frac{1}{2} (E_0 - E)^2 + O((E_0 - E)^3), E \to E_0 -\), this model is outside the range of the present approach.

(R1) Truncated Maxwell-Boltzmann models, cf. \([21, p. 59]\). The distribution function is given by an expression formally identical to that of the Woolley-Dickens models (of course \(E\) has a different definition in the two cases), and Theorem 3.1 applies.

(R2) Power-law models, cf. \([21, p. 68]\). The distribution function is given by \([E/E_0]^2 - \delta [1 - (E/E_0)^2]^\delta\) for \(E < E_0\) and zero otherwise. Since \([E/E_0]^2 - \delta [1 - (E/E_0)^2]^\delta = \frac{2\delta}{E_0^\delta} (E_0 - E)^\delta + O((E_0 - E)^{2\delta}), E \to E_0 -\), Theorem 3.1 applies if \(-1/2 < \delta < 3/2\).
Polytropic models, cf. [21, p. 68], [4]. The idea here is the following. Polytropic solutions of the Vlasov-Poisson system in the sense of (NR1) with \( l = 0 \) correspond to self-gravitating fluid models with \( p = \rho^{(n+1)/n} \), where \( \rho \) is the density and \( p \) is the pressure. This is due to the relation between kinetic and fluid models mentioned briefly in the introduction and discussed in more detail below. In general relativity it is also possible to look for a collisionless model with a given equation of state but there is no unique generalization of the polytropic case. In [22] Tooper considered different possibilities and their relationships. One possibility is to consider the equation of state which is formally identical to the non-relativistic polytropic one with \( \rho \) being interpreted as the energy density. If, on the other hand, \( \rho \) is interpreted as the mass density a one-parameter family of relations between pressure and energy density is obtained. Some authors, e.g. Fackerell[4], have considered the problem of producing corresponding distribution functions. Numerical calculations indicate that this is sometimes possible but not always. We can prove the relative statement that when it is possible (and the polytropic index \( n \) is restricted appropriately) the radius of the configuration is finite. Note that the in the case where \( \Phi \) depends on \( E \) alone the asymptotic behaviour of \( \Phi \) goes into the proof of Theorem 3.1 only via the asymptotic behaviour of the equation of state of the corresponding fluid model. In the first case the equation of state is directly in the correct form. In the second case the relation between pressure and energy density is \( p = C\rho^{n/(n+1)} + np \) for a positive constant \( C \). Then for small \( \rho \)

\[
p = C^{-(n+1)/n} \rho^{(n+1)/n} (1 + \rho^{1/n})
\]

Next we want to state two observations which may help to understand the relevance of the upper bound \( k < l + 3/2 \) that we required in our ansatz. We restrict ourselves to the non-relativistic case. The first observation is based on the correspondence of steady states of the Vlasov-Poisson system with steady states of the Euler-Poisson system. If \( (f,U) \) is a steady state of the Vlasov-Poisson system with \( f(x,v) = \phi(E) \) with \( \phi \) of the form (3.1), then \( (\rho,p,U) \) satisfy the Euler-Poisson system: Since the solution is static, i. e.,

\[
u(x) = \frac{1}{\rho(x)} \int v f(x,v) dv = 0, \quad x \in \mathbb{R}^3,
\]

the equation \( p' = -\rho U'' \) is all that remains of the Euler equations, and this is (3.12) in the isotropic case \( l = 0 \). Since the functions \( g_m \) in Lemma 2.2
are strictly decreasing on their support, $p$ can be written as a function of $\rho$, i.e., one obtains the equation of state

$$p = c_1 g_{1/2} \left( c_2 (g_{-1/2})^{-1}(\rho) \right) = c_3 \rho^{n+1} + O(\rho^{n+1}/r), \; \rho \to 0+$$

where $n = k + 3/2$, $c_1, c_2, c_3$ are positive constants which depend on the constants appearing in (2.13) and (2.14), and $\epsilon > 0$ depends on $\delta$. The critical value $k = 3/2$ corresponds to $n = 3$, and we now give an example of a steady state of the Euler-Poisson system, which has an equation of state of polytropic form with $3 < n < 5$ for $\rho$ small, and with unbounded support and infinite mass. It is based on a well-known explicit singular solution (cf. [3], p. 89).

**Example:** Let $n$ be a real number with $3 < n < 5$ and let $M$ be the set of functions $\rho(r)$ from $[0, \infty]$ to itself satisfying the following conditions:

1. $\rho$ is a smooth ($C^\infty$) function of $x = r^2$
2. $\rho'(r) < 0$ for $r > 0$
3. $\rho''(0) < 0$
4. $\rho(r) = r^{-\frac{(n+1)}{n-3}}$ for $r > 1$

This set of functions is convex. We claim that there is a function $\rho$ in $M$ with $\int_0^1 r^2 \rho(r) dr = \frac{n-1}{n-3}$. To see this, note first that the function $\rho_{S}(r) = r^{-\frac{n-1}{n-3}}$ satisfies this condition. However, it does not belong to $M$. Let $\rho_{-}$ be an element of $M$ which is everywhere less than or equal to $\rho_{S}$. It is clear that such a function exists. It satisfies the condition that $I_{-} = \int_0^1 r^2 \rho_{-}(r) dr < \frac{n-1}{n-3}$. It is also clear that there exist functions $\rho_{+} \in M$ such that $I_{+} = \int_0^1 r^2 \rho_{+}(r) dr$ is as large as desired. In particular $\rho_{+}$ can be chosen so that $I_{+} > \frac{n-1}{n-3}$. There exists a $\lambda$ in the interval $[0, 1]$ such that $\lambda I_{-} + (1 - \lambda) I_{+} = \frac{n+1}{n-3}$. Then $\rho = \lambda \rho_{-} + (1 - \lambda) \rho_{+}$ is the desired function.

Define $m(r) = 4\pi \int_0^r s^2 \rho(s) ds$, where $\rho$ is the function just constructed. Let $m_S$ be the function constructed in the corresponding way from $\rho_{S}$. By construction $\int_0^1 s^2 \rho(s) ds = \int_0^1 s^2 \rho_{S}(s) ds$. Thus $m(1) = m_S(1)$. It follows that $m(r) = m_S(r)$ for all $r > 1$. Now define $p(r) = \int_r^\infty s^{-2} m(s) \rho(s) ds$ and $p_S$ analogously. Then for $r > 1$ the functions $p$ and $p_S$ are equal. Since both $p$ and $p_S$ are strictly monotone for $r > 0$ it is possible to write $p = f(\rho)$ for a smooth function $f$ for $p$ in the interval $(0, \rho(0))$. Moreover, $df/d\rho$ is strictly positive. Both $\rho$ and $p$ are smooth functions of $x$ for $x \geq 0$. If it could be shown that $d\rho/dx$ and $dp/dx$ are non-vanishing at $x = 0$ then it would follow that $f$ has a
smooth extension to the interval \((0, \infty)\) with everywhere positive derivative. This is equivalent to showing that \(d^2 \rho / dr^2\) and \(d^2 p / dr^2\) are non-vanishing at \(r = 0\). The other of these two quantities is non-vanishing by assumption. The other can easy be computed to be equal to \(-\frac{4\pi}{3} \rho^2(0)\). The conclusion is that the functions \(\rho\) and \(p\) are related by an equation of state satisfying all the usual conditions, i.e. that for \(\rho > 0\) the conditions \(p > 0\) and \(dp / d\rho > 0\) are satisfied everywhere.

For \(r > 1\) the function \(m(r)\) can be calculated explicitly, as it is equal to \(m_S(r)\). The result is \(m(r) = 4\pi \frac{n-1}{n-3} r^{\frac{n-3}{n-1}}\). In the same way \(p(r)\) can be calculated explicitly for \(r > 1\) with the result \(p(r) = 4\pi \frac{(n-1)^2}{2(n+1)(n-3)} r^{\frac{2(n+1)}{n-1}}\). Comparing with the expression for \(\rho\) shows that the equation of state is of polytropic form with index \(n\) for \(\rho < \rho(1)\).

The example is not dependent on the condition \(n < 5\). That restriction was only made because that is the interesting case where it is known that there are some solutions with finite mass and radius, namely those with polytropic equation of state. Note also that it was shown in [18] that there exist some solutions of the Euler-Poisson and Euler-Einstein equations which have an equation of state which is asymptotically like a polytropic one with \(3 < n < 5\) (but not exactly polytropic) which have finite radius. This was done by a perturbation argument with the attendant disadvantages. The above example serves to show that the strategy applied by Makino [8] cannot be directly extended to the case \(n > 3\) and strongly suggests that the methods of this paper cannot be modified to cover cases with \(k > 3/2\). The polytropic steady states are in some sense structurally unstable.

The second observation is the following: Consider for \(k > 0\) the energy-Casimir functional

\[
\mathcal{D}(f) = \frac{k}{k+1} \int \int f^{1+1/k} dv dx + \frac{1}{2} \int \int v^2 f dv dx - \frac{1}{8\pi} \int |\nabla U_f|^2 dx,
\]

defined on the set

\[
\mathcal{F}_M = \{ f \in L^1 \cap L^{1+1/k}(\mathbb{R}^6) \mid f \geq 0, \int \int f dv dx = M, \int \int v^2 f dv dx < \infty \},
\]

where \(U_f\) denotes the potential induced by \(f\) and \(M > 0\) is a prescribed constant. One can show that \(\mathcal{D}\) is bounded from below on \(\mathcal{F}_M\) if \(k < 3/2\), whereas this functional is not bounded from below on this set if \(k > 3/2\). In the first case one can then show that the functional actually has a minimizer, at least if one restricts the set \(\mathcal{F}_M\) to spherically symmetric functions, and this minimizer is a steady state of the Vlasov-Poisson system of polytropic form which is dynamically stable in a well-defined sense, cf. [3, 4]. Again we
see that there seems to occur a loss of stability if one crosses the threshold \( k = 3/2 \), i.e., \( n = 3 \). In the Casimir part of the functional \( D \) the expression \( f^{1+1/k} \) can be replaced by more general functions \( Q(f) \), provided \( Q \) satisfies growth conditions both for small and for large values of \( f \), which have to satisfy the same restrictions as \( k \) above. This leads to stable steady states not necessarily of polytropic form.

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