Higher Loop Results for the Plaquette, Using the Clover and Overlap Actions∗

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We calculate the perturbative value of the free energy in QCD on the lattice. This quantity is directly related to the average plaquette.

Our calculation is done to 3 loops using the clover action for fermions; the results are presented for arbitrary values of the clover coefficient, and for a wide range of fermion masses.

In addition, we calculate the 2 loop result for the same quantity, using the overlap action.

1. INTRODUCTION

In this work, we compute the perturbative expansion of the average plaquette, in SU(N) gauge theory with Nf fermion flavours. We present separate calculations using the clover action (3-loops), and the overlap action (2-loops) [1]. The simpler case of Wilson fermions was performed in [2].

The average plaquette can be related to the perturbative free energy of lattice QCD, as well as to the expectation value of the action. The results can be used: a) In improved scaling schemes, using an appropriately defined effective coupling. b) In long standing efforts, starting with [3], to determine the value of the gluon condensate. c) In studies of the interquark potential [4]. d) As a test of perturbation theory, at its limits of applicability.

In standard notation, the action consists of a gluonic and a fermionic part:

\[ S = S_G + S_F, \]

\[ S_G = \beta \sum_P E_G(P) \]

with

\[ E_G(P) = 1 - \text{Re Tr}(P)/N \]

The sum runs over all 1×1 plaquettes P. The fermionic action S_F, in the clover case, contains the Wilson term with bare fermionic mass m, and the standard clover term multiplied by the coefficient c_{SW}, which is a free parameter in the present work; it is normally tuned in a way as to minimize O(a) effects.

The average value of the action density, S/V, is directly related to the average plaquette; in particular, for the gluonic part we have:

\[ \langle S_G/V \rangle = 6 \beta \langle E_G(P) \rangle. \]  (3)

As for \( \langle S_F/V \rangle \), it is trivial in any action which is bilinear in the fermion fields [2], and leads to:

\[ \langle S_F/V \rangle = -4 NN_f \]  (4)

We will calculate \( \langle E_G \rangle \) in perturbation theory:

\[ \langle E_G \rangle = c_1 g^2 + c_2 g^4 + c_3 g^6 + \cdots \]  (5)

The n-loop coefficient can be written as \( c_n = c_n^G + c_n^F \) where \( c_n^G \) is the contribution of diagrams without fermion loops and \( c_n^F \) comes from diagrams containing fermions. The coefficients \( c_n^G \) have been known for some time up to 3 loops [52]. The coefficients \( c_n^F \) are also known to 3 loops for Wilson fermions [4]; in the present work we extend this computation to clover fermions.

The calculation of \( c_n \) proceeds most conveniently by computing first the free energy \(-\ln(Z)/V\), where Z is the full partition function

\[ Z \equiv \int [DU D\bar{\psi}i D\psi] \exp(-S). \]  (6)

The average of \( E_G \) is then extracted as follows

\[ \langle E_G \rangle = -\frac{1}{6} \frac{\partial}{\partial\beta} \left( \frac{\ln Z}{V} \right). \]  (7)
In particular, the perturbative expansion of \((\ln Z)/V\):

\[
(\ln Z)/V = d_0 - \frac{3(N^2 - 1)}{2} \ln \beta + \frac{d_1}{\beta} + \frac{d_2}{\beta^2} + \cdots \tag{8}
\]

leads immediately to the relations:

\[ c_2 = d_1/(24 N^2), \quad c_3 = d_2/(24 N^3). \]

2. CLOVER FERMIONS

Up to 3 loops, a total of 62 diagrams contribute, of which 26 involve fermionic loops. The involved algebra of the lattice perturbation theory was carried out using our computer package in Mathematica. The value for each diagram is computed numerically for a sequence of finite lattice sizes. Diagrams must be grouped in several infrared-finite sets, before extrapolating their values to infinite lattice size; extrapolation leads to a (small) systematic error, which is estimated quite accurately.

The pure gluonic contributions are already known:

\[
c_1^G = \frac{N^2 - 1}{8 N}, \quad c_2^G = (N^2 - 1) \left( 0.0051069297 - \frac{1}{128 N^2} \right), \quad c_3^G = (N^2 - 1) \left( \frac{-0.00265487(17)}{N} + \frac{0.00794223(19)}{N} \right). \tag{9}
\]

Fermionic contributions take the form:

\[
c_1^F = 0, \quad c_2^F = (N^2 - 1) h_2 \frac{N_f}{N}, \quad c_3^F = (N^2 - 1) \left( h_{30} N_f + h_{31} \frac{N_f}{N^2} + h_{32} \frac{N_f^2}{N} \right). \tag{10}
\]

The coefficients \(h_2, h_{30}, h_{31}, h_{32}\) depend polynomially on the clover parameter \(c_{SW}\):

\[
h_2 = h_2^{(0)} + h_2^{(1)} c_{SW} + h_2^{(2)} c_{SW}^2, \quad h_3 = h_3^{(0)} + h_3^{(1)} c_{SW} + h_3^{(2)} c_{SW}^2 + h_3^{(3)} c_{SW}^3 + h_3^{(4)} c_{SW}^4. \tag{11}
\]

We have calculated the values of \(h_2^{(j)}, h_3^{(j)}\) for typical values of the bare mass \(m\) (see Ref. [1] for detailed numerical values).

We list below some typical examples of values for \(\langle E_G \rangle\), setting \(N = 3\). For \(N_f = 0\) we have:

\[
(1/3) g^2 + 0.0339109931(3) g^4 + 0.0137063(2) g^6 \tag{12}
\]

For \(N_f = 2\) and \(m = -0.518106\) (corresponding to \(\kappa = (8 + 2 m)^{-1} = 0.1436\)):

\[
c_{SW} = 0.0 : (1/3) g^2 + 0.02618520(3) g^4 + 0.0119649(3) g^6, \]

\[
c_{SW} = 2.0 : (1/3) g^2 + 0.03663456(3) g^4 + 0.0110200(13) g^6. \tag{13}
\]

For \(N_f = 2\) and \(m = 0.038\):

\[
c_{SW} = 0.0 : (1/3) g^2 + 0.030438866(3) g^4 + 0.0138181(2) g^6, \]

\[
c_{SW} = 1.3 : (1/3) g^2 + 0.025219798(9) g^4 + 0.0129659(5) g^6, \tag{14}
\]

\[
c_{SW} = 2.0 : (1/3) g^2 + 0.01800170(1) g^4 + 0.012948(1) g^6. \]

For \(N_f = 3\) and \(m = 0.038\):

\[
c_{SW} = 0.0 : (1/3) g^2 + 0.028702803(5) g^4 + 0.0139032(2) g^6, \]

\[
c_{SW} = 1.3 : (1/3) g^2 + 0.02087420(1) g^4 + 0.0128495(8) g^6, \tag{15}
\]

\[
c_{SW} = 2.0 : (1/3) g^2 + 0.0100476(2) g^4 + 0.013547(1) g^6. \]

3. OVERLAP FERMIONS

The fermionic action now reads [3]:

\[
S_f = \sum_f \sum_{x,y} \bar{\psi}_x^f D_N(x,y) \psi_y^f. \tag{16}
\]

with: \(D_N = M_0 [1 + X (X^\dagger X)^{-1/2}]\), and: \(X = D_W - M_0\); the sum on \(f\) runs over all flavors. Here, \(D_W\) is the massless Wilson-Dirac operator with \(r = 1\), and \(M_0\) is a free parameter whose value must be in the range \(0 < M_0 < 2\), in order to guarantee the correct pole structure of \(D_N\).

Fermionic vertices are obtained by separating the Fourier transform of \(D_N\) into a free part (inverse propagator \(D_0\)) and an interaction part \(\Sigma\) [7,8]:

\[
\frac{1}{M_0} D_N(q,p) = D_0(p) (2\pi)^4 \delta^4(q-p) + \Sigma(q,p). \tag{17}
\]
\[ \Sigma(q,p) = \frac{1}{\omega(p)+\omega(q)}X_1(q,p) \]
\[ -\frac{1}{\omega(p)\omega(q)}X_0(q)X_1^\dagger(q,p)X_0(p) \]
\[ +\frac{1}{\omega(p)+\omega(q)}X_2(q,p) \]
\[ -\frac{1}{\omega(p)\omega(q)}X_0(q)X_2^\dagger(q,p)X_0(p) \]
\[ + \int \frac{d^4k}{(2\pi)^4} \frac{1}{\omega(p)\omega(q)\omega(k)} \frac{1}{\omega(q)\omega(k)} \times \]
\[ -X_0(q)X_1^\dagger(q,k)X_1(k,p) \]
\[ -X_1(q,k)X_0^\dagger(k)X_1(k,p) \]
\[ -X_1(q,k)X_0^\dagger(k,p)X_0(p) \]
\[ + \frac{\omega(p)+\omega(q)+\omega(k)}{\omega(p)\omega(q)\omega(k)} \times \]
\[ X_0(q)X_1^\dagger(q,k)X_1(k,p)X_0(p) \right] + \mathcal{O}(g^3) \]

where \( X_0, X_1, X_2 \) denote the parts of the Dirac-Wilson operator with 0, 1, 2 gluons (of order \( \mathcal{O}(g^0) \), \( \mathcal{O}(g^1) \), \( \mathcal{O}(g^2) \), respectively), and:
\[ \omega(p) = (\sum_\mu \sin^2 p_\mu + [\sum_\mu (1-\cos p_\mu) - M_0^2]^2)^{1/2} \] \hspace{1cm} (19)

From the form of \( \Sigma(q,p) \), we see that the 2-gluon vertex splits into two different types of contributions:

Only two fermionic diagrams contribute to 2-loops, leading to:
\[ c_1^F = 0, \quad c_2^F = (N^2 - 1) h_2 \frac{N_f}{12N} \] \hspace{1cm} (20)

The quantity \( h_2 \) now depends only on the parameter \( M_0 \); our result is shown in the graph below.

It is worth noting that the dependence on \( M_0 \) is smooth all the way to the endpoint values \( M_0 = 0, 2 \), despite the change in propagator poles at these values.

Typical values of \( \langle E_W \rangle \) : Setting \( N = 3, N_f = 2 \), we obtain:
\[ M_0 = 0.01 : \quad \langle E_W \rangle = 0.029372693(2) g^4, \]
\[ M_0 = 1.99 : \quad \langle E_W \rangle = 0.01975396(10) g^4, \]
\[ N_f = 0 : \quad \langle E_W \rangle = 0.03391099316 g^4. \]

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