Supplementary Materials for

Generic generation of noise-driven chaos in stochastic time delay systems:
Bridging the gap with high-end simulations

Mickaël D. Chekroun et al.

Corresponding author: Mickaël D. Chekroun, michael-david.chekroun@weizmann.ac.il

Sci. Adv. 8, eabq7137 (2022)
DOI: 10.1126/sciadv.abq7137

The PDF file includes:

Notes S1 to S5
Figs. S1 to S10
Legends for movies S1 to S3
References

Other Supplementary Material for this manuscript includes the following:

Movies S1 to S3
Supplementary Note 1

Shear-induced chaos from Eq. (7) of the Main Text. As shown in Fig. 1, the stochastic Stuart-Landau (SL) equation Eq. (7) of the Main Text perturbed by the state-dependent random forcing $F(t, z)$ given by Eq. (6) of the Main Text, exhibits shear-induced chaos when the parameters $f_r, D$ and $\sigma$ involved in $F(t, z)$ are suitably chosen; cf. Fig. 1 (B and C). There, two snapshots of the pullback attractor, $A(t)$, are shown at times $t = 68$ and $t = 78$, exhibiting stretching and folding for $D = 60$ and $\sigma = 0.1$. In this figure, two nearby points are marked as red and cyan dots on the pullback attractor shown at $t = 68$ (Fig. 1B) which correspond to trajectories driven by the same noise path that have synchronised at that time (dashed vertical line in Fig. 1A prior $t \approx 70$). This synchronisation holds for a certain time-window before breaking down (while still driven by the same noise path) at a later time (dashed vertical line in Fig. 1A prior $t \approx 80$), illustrating the sensitive dependence on initial conditions (compare Fig. 1B with Fig. 1C), signature of chaotic dynamics. This on-off synchronisation behaviour repeats in the course of time as already visible in Fig. 1A; a feature also encountered for other types of noise-driven chaos as documented in e.g. [4, Fig. 7].

Supplementary Note 2

We recall briefly here the transport equation reformulation of the Koren-Tziperman-Feingold (KTF) model delay model [31,32], and also point out how the Stuart-Landau (SL) equation is obtained as the normal form of Galerkin–Koornwinder (GK) approximations of this model [32].

Transport equation reformulation. Recall the KTF model rewritten for the perturbed variable, $\tilde{H}(t) = h(t) - \tilde{h}$:

$$\dot{\tilde{H}} = -\tilde{H}(t) - \frac{2}{\mu} \tilde{h} \tilde{H}(t - \tau) - \frac{1}{\mu} \tilde{H}^2(t - \tau). \quad (S1)$$

Denote by $u(t, \theta)$ the historic part of $H$, that is $u(t, \theta) = H(t + \theta)$ for $\theta$ in $[-\tau, 0)$. Recall that the transport equation reformulation of Eq. (S1) is given by

$$\partial_t u = \partial_\theta u, \quad (S2)$$

subject to the boundary conditions

$$\partial_\theta u|_{\theta = 0} = -u(t, 0) - \frac{2}{\mu} \tilde{h} u(t, -\tau) - \frac{1}{\mu} u^2(t, -\tau). \quad (S3)$$

We introduce next the linear operator $A$ acting on $\Phi = (\phi, \theta(0))$ (with suitable regularity on the function $\phi$ over $[-\tau, 0]$; see [50])

$$[A\Phi](\theta) = \begin{cases} \frac{d\phi}{d\theta}, & \theta \in [-\tau, 0), \\ -\phi(0) - \frac{2}{\mu} \tilde{h} \phi(-\tau), & \theta = 0. \end{cases} \quad (S4)$$

By introducing finally the nonlinear operator $G$

$$[G\Phi](\theta) = \begin{cases} 0, & \theta \in [-\tau, 0), \\ -\frac{1}{\mu} \phi^2(-\tau), & \theta = 0, \end{cases} \quad (S5)$$

we can rewrite the boundary problem (S2)-(S3) as the following abstract evolution equation defined on a suitable Hilbert function space (see e.g. [50, Sec. 2])

$$\frac{du}{dt} = Au + G(u). \quad (S6)$$

GK approximations and Hopf normal form. Recall that the forcing—aimed at turning a limit cycle into a stochastic strange attractor—is designed from the SL equation, Eq. (4) of the Main Text. We succinctly recall below its derivation. Following [32], the SL equation is the Hopf normal form associated with the loss of stability of the unstable mode $e^N_1$ of an $N$-dimensional GK approximation of the KTF model, with $N$ sufficiently large.

Such Galerkin approximations are obtained from Eq. (S6) as

$$\frac{dy_N}{dt} = \Gamma_N(\tau)y_N + G_N(y_N), \quad y_N \in \mathbb{R}^N, \quad (S7)$$

subject to the initial condition $y(0) = y_0$.
in which the \( N \times N \) matrix \( \Gamma_N(\tau) \) (resp. \( G_N \)) results from the truncation of the operator \( A \) (resp. \( G(u) \)) to the \( N \)-dimensional subspace associated with the first \( N \) Koornwinder polynomials over \([-\tau, 0]\), while \( y_N \) provides the (time-dependent) coefficients of this Galerkin approximation of \( u(t, \theta) \) within this subspace; see [32, Eq. (45)-(48)] and [50, Sec. 5].

By rewriting the Gk system Eq. (S7) under the eigenmodes of \( \Gamma_N(\tau) \), we obtain

\[
\frac{d^2z}{dt^2} = \lambda_k^N z_k + \left( G_N \left( \sum_{j=1}^{N} z_j e_j^N, \bar{e}_k^N \right) \right), \quad k = 1, \ldots, N, \tag{S8}
\]

where \((\lambda_j^N, e_j^N)\) denotes the eigenelements of \( \Gamma_N(\tau) \) and \((\bar{\lambda}_j^N, \bar{e}_j^N)\) those of the transpose of \( \Gamma_N(\tau) \). The SL equation

\[
\frac{dz}{dt} = \lambda_1^N z - (\alpha_N + i\beta_N)|z|^2, \tag{S9}
\]

is finally obtained from Eq. (S8) by application of the approximation formulas of the center manifold following [51, Sec. 2.2]; see [32, Theorem III.1].

**Supplementary Note 3**

**Stochastic transport problem: Murmuration case**

We consider here the case when the forcing, \( F(t) \), is independent of the state itself i.e. corresponding to the murmuration case in the Main Text. Our goal is to make precise how to lift a stochastic forcing of the SL equation (Eq. (4) of the Main Text) into a stochastic forcing of the corresponding KTF transport equation formulation.

As recalled above, the SL equation (S9) is here derived from the loss of stability of the unstable mode \( e_1^N \) of a sufficiently high-dimensional GK approximation of the KTF model and provides, near criticality, a rigorous approximation of the unstable mode’s amplitude [32, Theorem III.1]. Thus the idea of "lifting" a forcing of the SL equation (S9) into a forcing of the unstable mode in the GK system, while expecting a similar dynamical response to such perturbations than in the case of the SL equation.

By adding a forcing \( F(t) \) to Eq. (S8) we consider therefore the following forced version of the GK system:

\[
\frac{d^2z}{dt^2} = \lambda_1^N z_1 + \left( G_N \left( \sum_{j=1}^{N} z_j e_j^N, \bar{e}_1^N \right) \right) + F(t), \tag{S10}
\]

and for \( k = 3, \ldots, N, \)

\[
\frac{d^2z_k}{dt^2} = (\lambda_k^N) z_k + \left( G_N \left( \sum_{j=1}^{N} z_j e_j^N, \bar{e}_k^N \right) \right) + F^*(t),
\]

where \((\lambda_1^N) \) denotes the conjugate of \( \lambda_1^N \). The reason is as follows. To proceed, recall that the GK system is placed after a Hopf bifurcation has occurred and hence the unstable mode \( e_1^N \) is associated with an unstable pair of complex eigenvalues. Thus forcing this mode leads to force its amplitude equation (the \( z_1 \)-equation) and the equation for its conjugate amplitude (the \( z_2 \)-equation).

Using \( y_N = \sum_{j=1}^{N} z_j e_j^N \), we can rewrite (S10) into the compact form [50, Sec. 5]

\[
\frac{dy_N}{dt} = \Gamma_N(\tau)y_N + G_N(y_N) + F_N(t), \tag{S11}
\]

where

\[
F_N(t) = F(t)e_1^N + F^*(t)e_2^N = F(t)e_1^N + \text{c.c.}, \tag{S12}
\]

with c.c. denoting the complex conjugate of the immediate term preceding the ‘+’ sign.

Denoting by \( K_1^N \) the (rescaled) Koornwinder polynomials used for the Galerkin approximation following [32, 50], the function \( u(t, \theta) = \sum_{j=1}^{N} y_N^j(t)K_1^N(\theta) \) approximates then the solution \( u(t, \theta) \) of the abstract evolution equation (S6); see [32, Sec. II.C]. The forcing \( F_N(t) \) translates in this format into the function

\[
f_N(t, \theta) = \sum_{j=1}^{N} F_0^j(t)K_1^N(\theta) = \sum_{j=1}^{N} F(t)e_1^N K_1^N(\theta) + \text{c.c.}, \tag{S13}
\]

in which \( F_0^j(t) \) (resp. \( e_1^N \)) denotes the \( j \)-th component (in \( \mathbb{R}^N \)) of the \( N \)-dimensional vector \( F_N(t) \) (resp. of the dominant eigenmode \( e_1^N \) of \( \Gamma_N(\tau) \)). In (S13), \( F_0^j(t) \) is also interpretable as the \( j \)-th coefficient of \( f_N(t, \theta) \) onto the Koornwinder basis.

Using the Koornwinder polynomials, the eigenvector, \( e_1^N \), has the following representation in the \( \theta \)-variable:

\[
\varphi_1^N(\theta) = \sum_{j=1}^{N} e_1^N K_1^N(\theta), \quad \theta \in [-\tau, 0], \tag{S14}
\]

and in particular \( f_N(t, \theta) \) simply becomes

\[
f_N(t, \theta) = F(t)\varphi_1^N(\theta). \tag{S15}
\]

We are then only left with determining the function towards which \( f_N(t, \theta) \) converges as \( N \to \infty \), in order to conclude about the forcing term to add to the right-hand side of (S6).

Denoting by \((\lambda_1, \varphi_1)\) the leading eigenpair of the linear operator \( A \), we have then that \((\lambda_1^N, \varphi_1^N)\) converges, as \( N \to \infty \), towards \((\lambda_1, \varphi_1)\), with

\[
\varphi_1^N \xrightarrow{L^2, N \to \infty} \varphi_1,
\]

where \( \xrightarrow{L^2} \) denotes the convergence for the norm associated with the inner product given by Eq. (16) of the Main Text.

The rigorous proof of this convergence goes beyond the scope of this article. Supplementary Figures 2–3 provide nevertheless numerical results supporting this convergence.

These results of numerical convergence of \( f_N(t, \theta) \) towards \( F(t)\varphi_1 + \text{c.c.} \) (in a mean-square sense) lead us to conclude that the limiting problem, as \( N \to \infty \), corresponds to the following forced evolution equation

\[
\frac{du}{dt} = (Au + G(u)) + \left( F(t)\varphi_1 + \text{c.c.} \right), \tag{S16}
\]
with \( A \) and \( G \) as defined in (S4) and (S5), respectively.

The latter equation rewrites as the following stochastic transport problem

\[
\partial_t u = \partial_\theta u + \left( F(t) \varphi_1 + \text{c.c.} \right), \quad \theta \in [-\tau, 0),
\]  

(S17)

subject to the boundary conditions

\[
\partial_t u|_{\theta=0} = -u(t, 0) - \frac{2}{\mu} \int u(t, -\tau) \, dt - \frac{1}{\mu} u^2(t, -\tau) + \left( F(t) \varphi_1(0) + \text{c.c.} \right).
\]

(S18)

Now by taking \( \theta = 0 \) in (S17) (via a continuity argument), one gets \( \partial_t u|_{\theta=0} = \partial_\theta u|_{\theta=0} + \left( F(t) \varphi_1(0) + \text{c.c.} \right) \) and by expressing \( \partial_t u|_{\theta=0} \) using (S18) one obtains

\[
\partial_t u|_{\theta=0} = -u(t, 0) - \frac{2}{\mu} \int u(t, -\tau) \, dt - \frac{1}{\mu} u^2(t, -\tau).
\]

(S19)

Eq. (S17) together with the boundary condition (S19) provides then the stochastic transport problem for the murmuration case analyzed in the Main Text. Supplementary Fig. 5 below shows the differences between the murmuration and the horseshoes cases as measured in terms of fluxes at the boundary and within the domain; see the Section ‘Noise transmission, fluxes, and murmuration cascade’ in the Main Text for more details.

Supplementary Fig. 9C shows a solution \( u(t, \theta) \) to such a stochastic transport problem (Egns. (9)-(10) in the Main Text), along with its time-dependent boundary value \( u(t, 0) \) at \( \theta = 0 \) (Supplementary Fig. 9D), providing a stochastic path \( H(t) \) (Eq. (11) in the Main Text).

**Supplementary Note 4**

For our calculations, as reported in the Main Text, the dominant eigenfunction \( \varphi_1 \) is approximated by \( \varphi_1^N \) according to (S14) with \( N = 20 \). Note that with \( N = 10 \) a high-precision (of order \( 10^{-8} \)) is already reached by evaluating the corresponding residual; see Supplementary Fig. 3. This is also supported by a simple energy distribution analysis onto the first few Koornwinder polynomials. In that respect, Supplementary Fig. 8 shows the energy distribution of the first eigenfunction \( \varphi_1 \) used to force the transport problem (S17)-(S19). From this distribution, one observes indeed that only the first two Koornwinder polynomials capture most of the energy contained in \( \varphi_1 \), which in other words corresponds to a large-scale forcing. Dynamically, this property means that the results of the Main Text are essentially unchanged qualitatively if only a low-order (Koornwinder) approximation of \( \varphi_1 \) is used in the corresponding experiments.

**Supplementary Note 5**

**Stochastic transport problem: Horseshoes case**

The derivation of the stochastic transport problem in the case where \( F \) is the state-dependent noise (horseshoes case), \( F(t, z) = i D f(t) |z|^2 + \sigma W(t) \) (see Eq. (6) of the Main Text) follows the same steps in which \( f_N(t, \theta) \) in (S15) is now replaced by

\[
f_N(t, \theta) = \sum_{j=1}^{N} F(t, (y_N, e_1^N)) e_1^N K_j + \text{c.c.},
\]

(S20)

where, we recall \( e_1^N \) denotes the first eigenvector of the transpose of \( \Gamma_N(\tau) \).

At a first glance, denoting by \( \hat{\varphi}_1 \) the first eigenvector of the adjoint operator \( A^* \), a formal limit as \( N \to \infty \) should lead to an analogue to (S16), namely

\[
\frac{du}{dt} = (Au + G(u)) + \left( F(t, (u, \hat{\varphi}_1)) \varphi_1 + \text{c.c.} \right).
\]

(S21)

Let \( \hat{\varphi}_1^N \) be the analogue of \( \varphi_1^N \) given by (S14) in which \( e_1^N \) is replaced by \( \hat{\varphi}_1^N \). To fully justify (S21), it is sufficient to have weak convergence of \( \hat{\varphi}_1^N \) towards \( \hat{\varphi}_1 \) in the sense that \( \langle u, \hat{\varphi}_1^N \rangle \) converges towards \( \langle u, \hat{\varphi}_1 \rangle \). Here again, such a convergence analysis goes beyond the scope of this work, and instead numerical evidences are provided in Supplementary Fig. 6A for the unforced KTF model.

Analytic expression of \( \hat{\varphi}_1 \) can be obtained for the KTF model; see (S23) below. In this case, the adjoint operator \( A^* \) is given for each \( \Phi = (\phi, x) \) for suitable function \( \phi \) on \( [-\tau, 0) \) and point-value \( x \) at \( \theta = 0 \) [88] by

\[
[A^*\Phi](\theta) = \left\{ \begin{array}{ll}
-\frac{d\phi}{d\theta}, & \theta \in [-\tau, 0), \\
-x + \phi(0), & \theta = 0.
\end{array} \right.
\]

(S22)

See e.g. [88, Theorem 5.2].

The first eigenfunction is then given by \( \hat{\varphi}_1 = (\hat{\varphi}_1^P, \hat{\varphi}_1^S) \), with

\[
\hat{\varphi}_1^P(\theta) = \alpha \exp(-\lambda_1^* \theta), \quad \theta \in [-\tau, 0),
\]

\[
\hat{\varphi}_1^S = -\frac{\alpha \mu}{2h} \exp(\lambda_1^* \tau),
\]

(S23)

where \( \alpha \) is an arbitrary non-zero complex-valued constant.

In general however analytic formulas for \( \hat{\varphi}_1 \) are not always accessible, and unlike for \( \varphi_1 \), (strong) approximation of \( \hat{\varphi}_1 \) by Koornwinder polynomials is not guaranteed to hold. For instance, \( \hat{\varphi}_1 \) exhibits typically discontinuities at the boundary \( \theta = 0 \) (see (S23) for the KTF model) causing spurious Gibbs phenomenon that manifests when attempting to approximate \( \hat{\varphi}_1 \) by smooth polynomials. This generic behaviour is tied to the domain \( D(A^*) \) of \( A^* \). Recall that like \( \varphi_1 \), the (adjoint) eigenmode \( \hat{\varphi}_1 \) is of the form \( \hat{\varphi}_1 = (\phi, x) \) with \( x \) a point-value. For \( \hat{\varphi}_1 \) to lie in \( D(A^*) \) it imposes that \( x \) is proportional to \( \phi(-\tau) \) which in general does not coincide with \( \phi(0) \), causing thus a discontinuity at \( \theta = 0 \) for \( \hat{\varphi}_1 \); see [88, Theorem 5.2].

We then need to rely on high-precision approximation schemes of \( A^* \) to compute \( \hat{\varphi}_1 \) and the corresponding inner product \( \langle u, \hat{\varphi}_1 \rangle \), involving quadrature techniques over a fine meshing, becomes cumbersome to compute at each time step. This constitutes a serious numerical constraint for computing snapshot attractors over a large ensemble of initial data from direct integration of (S21).

On the other hand, \( \varphi_1 \) can be approximated to a very high precision in terms of expansion over a few Koornwinder polynomials; see Supplementary Fig. 3 again for the KTF model. Such a low-dimensional approximation using a few Koornwinder polynomials (see (S14)) reduces the numerical burden of computing the inner product \( \langle u, \varphi_1 \rangle \), allowing for a coarser mesh to perform the required quadratures. Naturally,
these observations led us to ask what would be the consequences in dynamical terms of replacing the cumbersome term to compute, $\langle u, \phi_1 \rangle$ in (S21), by the term $\langle u, \varphi_1 \rangle$ and thus by its high-precision approximation $\langle u, \varphi_1^N \rangle$ with $N$ not too large?

Dynamically, such a substitution means that we replace the state-dependent part of the forcing, $i D f(t) \langle u, \varphi_1 \rangle |\langle u, \varphi_1 \rangle|^2$ by $i D f(t) \langle u, \varphi_1 \rangle |\langle u, \varphi_1 \rangle|^2$. Since the purpose of this state-dependent term is to enhance phase diffusion, it is reasonable to expect that the proposed replacement would not affect much the ability of the forced model to generate horseshoes if the phase of $\hat{z}_1 = \langle u, \varphi_1 \rangle$ and that of $z_1 = \langle u, \varphi_1 \rangle$ are relatively close to each other, i.e. if $z_1$ and $\hat{z}_1$ evolves in a nearly synchronous manner. To measure the level of coherence between these oscillating variables we employ the following complex order parameter [43]

$$r(t) e^{i \psi(t)} = \frac{e^{i \arg(z_1(t))} + e^{i \arg(\hat{z}_1(t))}}{2},$$

(S24)

where $r(t)$ in $[0, 1]$ is the modulus and $\psi(t)$ is the phase of the considered indicator. This indicator when computed for the unperturbed dynamics of the KTF model, reveals that $z_1$ and $\hat{z}_1$ are nearly synchronised as $r(t)$ remains very close to 1 in the course of time; see Supplementary Fig. 6B. We already reported abundantly in the Main Text that (S21) with $\langle u, \varphi_1 \rangle$ replaced by $\langle u, \varphi_1 \rangle$ led to the creation of horseshoes in the state-space for the right choice of parameters. Supplementary Fig. 7 below provides a few examples of such parameter-values leading to time-variability associated with stochastic horseshoes (Panels (C)-(D) and (E)-(F)).

Qualitatively, (S21) exhibits still horseshoes dynamics when $\langle u, \varphi_1 \rangle$ is used in the latter equation (see Supplementary Fig. 6C) supporting thus on a dynamical ground our choice motivated by the aforementioned numerical considerations regarding the use of $\langle u, \varphi_1 \rangle$ address state-dependence of the noise. We believe that this observation of preserving qualitatively the effects of a state-dependent forcing compared to another one when their phases are nearly synchronised is not limited to the KTF model and should hold for more general systems.
Supplementary Fig. 1: Shear-induced chaos from Eq. (7) of the Main Text. Here, $F$ is the state-dependent random forcing as in Eq. (6) of the Main Text while $\lambda^N$, $\alpha_N$, and $\beta_N$ are as given in the caption of Fig. 10 of the Main Text. The snapshot attractors shown in (B and C) correspond to the time frames marked by the dashed vertical lines in (A). The red dots in (B) and (C) correspond to the stochastic trajectory ($\mathbb{R}z(t)$) shown in red in panel A while the cyan dots to that shown in blue. These trajectories are driven by the same noise path with $D = 60$, $\sigma = 0.1$ and firing rate $f_r = 0.7$ in Eq. (6) of the Main Text.

Supplementary Fig. 2: Convergence, as $N$ increases, of the eigenvalues $\lambda^N_j$ (red asterisks) of the truncated operator $\Gamma_N(\tau)$ in (S7), towards those of the operator $A$ defined in (S4) (black circles). The reference eigenvalues are obtained by solving the characteristic equation $\lambda + 1 + \frac{2}{h} \exp(-\tau \lambda) = 0$ using the DDE-BIFTOOL Matlab package [87]. The top row corresponds to KTF model’s parameters, $\mu = 0.3$ and $\tau = 1$ (Regime A), while the bottom row corresponds to $\mu = 1.2$ and $\tau = 20$ (Regime B). Note that the spectral gap, along the real axis, between the unstable pair and the immediate stable one gets much smaller in Regime B than in Regime A. This phenomenon has consequences for application of the center manifold reduction techniques; see the Section ‘Response dependence to stochastic parameterization’ of the Main Text.
Supplementary Fig. 3: Residual error of $\varphi_N^1$ given by (S14) in approximating the first eigenfunction $\varphi_1$ of the linear operator $A$. Here, $\varphi_N^1$ is obtained according to (S14), from the first eigenmode $e_N^1$ of the $N \times N$ matrix $\Gamma_N(\tau)$ in (S7). To evaluate the error $\|A\varphi_N^1 - \lambda_1 \varphi_N^1\|$, we use the analytic formulas for the derivative of Koornwinder polynomials (see Proposition A.1 in [32]) allowing for reaching high-precision in evaluating $A\varphi_N^1$ while $\lambda_1$ is obtained as the dominant eigenvalue of $A$ using the DDE-BIFTOOL Matlab package [87]. Note that already with $N = 10$ Koornwinder polynomials a precision of order $10^{-8}$ is reached in approximating by $\varphi_N^1$ the genuine dominant mode, $\varphi_1$, of $A$. These convergence results are shown here for Regime A, i.e., $\mu = 0.3$ and $\tau = 1$, as an illustration.

Supplementary Fig. 4: Koornwinder polynomials and energy distribution of the first eigenfunction, $\varphi_1$, against the first few Koornwinder polynomials. (A) A few (normalized) Koornwinder polynomials. (B) Energy distribution of the first eigenfunction $\varphi_1$ used to force the transport problem (S17)-(S19), onto the first few Koornwinder polynomials; see also “Supplementary Note 4.” It shows that $\varphi_1$ corresponds to a large-scale perturbation in Eq. (10) of the Main Text.
Supplementary Fig. 5: Fluxes, $\delta(t, \theta) = \partial_t \partial_\theta u$, at the boundary and within the domain. See Main Text.

Supplementary Fig. 6: Weak convergence of $\hat{\varphi}^N_1$ towards $\hat{\varphi}_1$; order parameter $r(t)$; and a snapshot attractor exhibiting horseshoes from Eq. (S21). (A) Numerical illustration of the weak convergence of $\hat{\varphi}^N_1$ towards $\hat{\varphi}_1$. Here is shown the convergence of $\langle u(t), \hat{\varphi}^N_1 \rangle$ towards $\langle u(t), \hat{\varphi}_1 \rangle$ when $u(t)$ is the limit cycle of the KTF model, i.e. for the unperturbed dynamics. This convergence is quick: for $N = 6$ the precision is already of order $10^{-5}$.

(B) Order parameter $r(t)$ as computed from (S24). It shows that $\hat{z}_1 = \langle u, \hat{\varphi}_1 \rangle$ and $\hat{z}_1 = \langle u, \varphi_1 \rangle$ evolve in a synchronous fashion in the course of time; see Text.

(C) Snapshot attractor from Eq. (S21) in which $\langle u, \hat{\varphi}_1 \rangle$ has replaced $\langle u, \varphi_1 \rangle$ used in the Main Text. Here $3.4 \times 10^5$ initial data are used to produce this snapshot.
Supplementary Fig. 7: Parameter dependence in $\sigma$ for the mildly nonlinear regime, $\mu = 0.3$ and $\tau = 1$ (Regime A). The magenta curve in (B) corresponds to the power spectrum of the periodic orbit for $D = \sigma = 0$. The case $\sigma = 0$ and $D \neq 0$ corresponds to a random periodic case. The cases $\sigma = 0.05$ and $\sigma = 0.1$ to noise-driven solutions evolving on a random chaotic attractor exhibiting stretching and folding (horseshoes). The case $\sigma = 0.1$ is shown in Fig. 4 of the Main Text but is reproduced here for the sake of comparison (with the time series in (E) shown over a longer time interval).

Supplementary Fig. 8: Power spectra (linear scale) across a few Koornwinder coefficients together with the power spectrum of $f(t)$ shown in log-log scale. In each panel, the vertical dashed red line marks the location of the second bump’s dominant frequency as displayed by the noise-driven chaos time-series in Fig. 8B (red curve) of the Main Text.
Supplementary Fig. 9: Dynamics of the stochastic transport problem of the Main Text in a strongly nonlinear regime (Regime B). (A) 2-D projection onto the unstable mode $\varphi_1$ of the unperturbed KTF model’s limit cycle in Regime B (i.e. for $\mu = 1.2$ and $\tau = 20$). (B) Strange snapshot attractor in Regime B. Here the snapshot attractor $A(t)$, at a given time $t$, of the stochastic transport problem Eqs. (9)-(10) from the Main Text is shown in the reduced state space $(u(t, -\theta), u(t, 0))$. We observe stretch-and-fold features embedded in a “foggier” background than for the snapshot attractors shown in Figs. 3A and 3B of the Main Text. The parameter values of the noise terms are here $D = 40$, $f_r = 0.7$ and $\sigma = 0.1$, i.e. those used for Figs. 6E and 6F of the Main Text. Here $2.5 \times 10^5$ initial data are used to produce this snapshot. (C) shows a solution $u(t, \theta)$ to the associated stochastic transport problem Eqs. (9)-(10) from the Main Text, and its boundary value $u(t, 0)$ at $\theta = 0$, providing a stochastic path $H(t)$ shown in (D).

Supplementary Fig. 10: A snapshot of the vertical velocity field as modeled by the WRF model. The field is shown at the same specific time (January 18, 2020, UTC 12:40:00) for the same domain as shown in Fig. 1A of the Main Text. The magenta and cyan boxes correspond to (C) and (B) of Fig. 1 of the Main Text, respectively. See Main Text for more details.
Supplementary movie files

We provide three supplementary movie files to show the time-evolution of snapshot attractors for three different settings corresponding respectively to the snapshot attractors shown in Fig. 3 (A and B) of the Main Text, Fig. 3 (C and D) of the Main Text, and Supplementary Fig. 9B.

- **Supplementary Movie 1**: Time-evolution of snapshot attractors obtained from the stochastic transport problem (9)-(10) of the Main Text for Regime A (horseshoes case).

- **Supplementary Movie 2**: Time-evolution of snapshot attractors obtained from the stochastic transport problem (S17)-(S19) (murmuration case).

- **Supplementary Movie 3**: Time-evolution of snapshot attractors obtained from the stochastic transport problem (9)-(10) of the Main Text for Regime B; i.e. the case shown in Supplementary Fig. 9.
REFERENCES AND NOTES

1. B. Van der Pol, J. Van Der Mark, Frequency demultiplication. Nature, 120, 363–364 (1927).

2. L.-S. Young, Generalizations of SRB measures to nonautonomous, random, and infinite dimensional systems. J. Stat. Phys. 166, 494–515 (2017).

3. J. Guckenheimer, M. Wechselberger, L.-S. Young, Chaotic attractors of relaxation oscillators. Nonlinearity 19, 701–720 (2006).

4. M. D. Chekroun, E. Simonnet, M. Ghil, Stochastic climate dynamics: Random attractors and time-dependent invariant measures. Physica D 240, 1685–1700 (2011).

5. T. Bódai, G. Károlyi, T. Tél, A chaotically driven model climate: Extreme events and snapshot attractors. Nonl. Proc. Geophys. 18, 573–580 (2011).

6. Q. Wang, L.-S. Young. Strange attractors in periodically-kicked limit cycles and Hopf bifurcations. Commun. Math. Phys. 240, 509–529 (2003).

7. Q. Wang, L.-S. Young, Strange attractors with one direction of instability. Commun. Math. Phys. 218, 1–97 (2001).

8. Q. Wang, L.-S. Young. Toward a theory of rank one attractors. Ann. Math. 167, 349–480 (2008).

9. S. Wieczorek. Stochastic bifurcation in noise-driven lasers and Hopf oscillators. Phys. Rev. E 79, 036209, 2009.

10. K. C. A. Wedgwood, K. K. Lin, R. Thul, S. Coombes. Phase-amplitude descriptions of neural oscillator models. J. Math. Neurosci. 3, 2 (2013).

11. P. Ashwin, S. Coombes, R. Nicks. Mathematical frameworks for oscillatory network dynamics in neuroscience. J. Math. Neurosci. 6, 2 (2016).

12. K. K. Lin, L.-S. Young, Shear-induced chaos. Nonlinearity 21, 899 (2008).
13. K. Lu, Q. Wang, L.-S. Young. *Strange Attractors for Periodically Forced Parabolic Equations*, volume 224 of *Memoirs of the American Mathematical Society* (American Mathematical Society, Providence, RI, 2013).

14. G. A Bocharov, F. A. Rihan, Numerical modelling in biosciences using delay differential equations. *J. Comp. Appl. Math.* **125**, 183–199 (2000).

15. J. D. Murray, *Mathematical Biology: I. An Introduction* (Springer, 2002).

16. K. Bansal, J. O. Garcia, S. H. Tompson, T. Verstynen, J. M. Vettel, S. F. Muldoon. Cognitive chimera states in human brain networks. *Sci. Adv.* **5**, eaau8535 (2019).

17. A. Arenas, A. Díaz-Guilera, J. Kurths, Y. Moreno, C. Zhou, Synchronization in complex networks. *Phys. Rep.* **469**, 93–153 (2008).

18. M. C. Soriano, J. García-Ojalvo, C. R. Mirasso, I. Fischer. Complex photonics: Dynamics and applications of delay-coupled semiconductors lasers. *Rev. Mod. Phys.* **85**, 421–470 (2013).

19. N. Boers, M. D. Chekroun, H. Liu, D. Kondrashov, D.-D. Rousseau, A. Svensson, M. Bigler, M. Ghil, Inverse stochastic–dynamic models for high-resolution Greenland ice core records. *Earth Syst. Dyn.* **8**, 1171–1190 (2017).

20. A. Keane, B. Krauskopf, C. M. Postlethwaite, Climate models with delay differential equations. *Chaos* **27**, 114309 (2017).

21. S. K. J. Falkena, C. Quinn, J. Sieber, J. Frank, H. A. Dijkstra, Derivation of delay equation climate models using the Mori-Zwanzig formalism. *Proc. R. Soc. A* **475**, 20190075 (2019).

22. K. Gopalsamy, *Stability and Oscillations in Delay Differential Equations of Population Dynamics* (Springer Science & Business Media, 2013), vol. 74.

23. T. Erneux, J. Javaloyes, M. Wolfrum, S. Yanchuk, Introduction to focus issue: Time-delay dynamics. *Chaos* **27**, 114201 (2017).
24. H. Wernecke, B. Sándor, C. Gros, Chaos in time delay systems, an educational review. *Phys. Rep.* **824**, 1–40 (2019).

25. M. D. Chekroun, M. Ghil, J. D. Neelin, Pullback attractor crisis in a delay differential ENSO model, in *Advances in Nonlinear Geosciences*, A. Tsonis, Ed. (Springer, 2018), pp. 1–33.

26. T. Erneux, *Applied Delay Differential Equations* (Springer Science & Business Media, 2009), vol. 3.

27. K. D. Beheng, A parameterization of warm cloud microphysical conversion processes. *Atmos. Res.* **33**, 193–206 (1994).

28. J. W.-B. Lin, J. D. Neelin. Considerations for stochastic convective parameterization. *J. Atmos. Sci.* **59**, 959–975 (2002).

29. J. Berner, U. Achatz, L. Batte, L. Bengtsson, A. Cámara, H. M. Christensen, M. Colangeli, D. R. B. Coleman, D. Crommelin, S. I. Dolaptchiev, C. L. E. Franzke, P. Friederichs, P. Imkeller, H. Järvinen, S. Juricke, V. Kitsios, F. Lott, V. Lucarini, S. Mahajan, T. N. Palmer, Stochastic parameterization: Toward a new view of weather and climate models. *Bull. Am. Meteorol. Soc.* **98**, 565–588 (2017).

30. I. Koren, G. Feingold. Aerosol-cloud-precipitation system as a predator-prey problem. *Proc. Natl. Acad. Sci. U.S.A.* **108**, 12227–12232 (2011).

31. I. Koren, E. Tziperman, G. Feingold, Exploring the nonlinear cloud and rain equation. *Chaos* **27**, 013107 (2017).

32. M. D. Chekroun, I. Koren, H. Liu. Efficient reduction for diagnosing Hopf bifurcation in delay differential systems: Applications to cloud-rain models. *Chaos* **30**, 053130 2020.

33. H. Xue, G. Feingold, B. Stevens. Aerosol effects on clouds, precipitation, and the organization of shallow cumulus convection. *J. Atmos. Sci.* **65**, 392–406 (2008).
34. H. Wang, G. Feingold. Modeling mesoscale cellular structures and drizzle in marine stratocumulus. Part II: The microphysics and dynamics of the boundary region between open and closed cells. *J. Atmos. Sci.*, **66**, 3257–3275 (2009).

35. I. Koren, G. Feingold. Adaptive behavior of marine cellular clouds. *Sci. Rep.*, **3**, 2507 (2013).

36. S. Smale. Differentiable dynamical systems. *Bull. Am. Math. Soc.* **73**, 747–817 (1967).

37. J. Guckenheimer, P. Holmes, *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields* (Springer Science & Business Media, 2013), vol. 42.

38. A. Cavagna, A. Cimarelli, I. Giardina, G. Parisi, R. Santagati, F. Stefanini, M. Viale. Scale-free correlations in starling flocks. *Proc. Natl. Acad. Sci. U.S.A.* **107**, 11865–11870 (2010).

39. T. J. Schmit, P. Griffith, M. M. Gunshor, J. M. Daniels, S. J. Goodman, W. J. Lebair. A closer look at the ABI on the GOES-R series. *Bull. Am. Meteorol. Soc.* **98**, 681–698 (2017).

40. S. Lundersman, M. Morzfeld, F. Glassmeier, G. Feingold. Estimating parameters of the nonlinear cloud and rain equation from a large-eddy simulation. *Physica D* **410**, 132500 (2020).

41. W. C. Skamarock, J. B. Klemp, J. Dudhia, D. O. Gill, Z. Liu, J. Berner, W. Wang, J. G. Powers, M. G. Duda, D. M. Barker, X.-Y. Huang. A description of the advanced research WRF model version 4. *National Center for Atmospheric Research (NCAR) Technical Notes*, TN-556+STR, 2021.

42. A. Winfree, Biological rhythms and the behavior of populations of coupled oscillators. *J. Theor. Biol.* **16**, 15–42 (1967).

43. A. Winfree, *The Geometry of Biological Time* (Springer-Verlag, Berlin, 1980).

44. E. A. Coddington, N. Levinson, *Theory of Ordinary Differential Equations* (Tata McGraw-Hill Education, 1955).

45. J. Guckenheimer, Isochrons and phaseless sets. *J. Math. Biol.* **1**, 259–273 (1975).
46. E. M. Izhikevich, *Dynamical Systems in Neuroscience: The Geometry of Excitability and Bursting* (MIT Press, 2010).

47. M. Engel, J. S. W. Lamb, M. Rasmussen. Bifurcation analysis of a stochastically driven limit cycle. *Commun. Math. Phys.* **365**, 935–942 (2019).

48. T. Son Doan, M. Engel, J. S. W. Lamb, M. Rasmussen, Hopf bifurcation with additive noise. *Nonlinearity*, **31**, 4567–4601 (2018).

49. J. M. Ottino, *The Kinematics of Mixing: Stretching, Chaos, and Transport* (Cambridge Univ. Press, 1989), vol. 3.

50. M. D. Chekroun, M. Ghil, H. Liu, S. Wang. Low-dimensional Galerkin approximations of nonlinear delay differential equations. *Discrete Continuous Dyn. Syst.* **36**, 4133–4177 (2016).

51. M. D. Chekroun, H. Liu, J. C. McWilliams, Variational approach to closure of nonlinear dynamical systems: Autonomous case. *J. Stat. Phys.* **179**, 1073–1160 (2020).

52. A. Tantet, M. D. Chekroun, H. A. Dijkstra, J. D. Neelin. Ruelle-Pollicott resonances of stochastic systems in reduced state space. Part II: Stochastic Hopf Bifurcation. *J. Stat. Phys.* **179**, 1403–1448 (2020).

53. H. C. Rodean, *Stochastic Lagrangian Models of Turbulent Diffusion* (Springer, 1996), vol. 45.

54. S. N. Stechmann, J. D. Neelin. A stochastic model for the transition to strong convection. *J. Atmos. Sci.* **68**, 2955–2970 (2011).

55. A. J. Majda, S. N. Stechmann. Stochastic models for convective momentum transport. *Proc. Natl. Acad. Sci. U.S.A.* **105**, 17614–17619 (2008).

56. B. Khouider, A. J. Majda, M. A. Katsoulakis. Coarse-grained stochastic models for tropical convection and climate. *Proc. Natl. Acad. Sci. U.S.A.* **100**, 11941–11946 (2003).
57. J. Dorrestijn, D. T. Crommelin, A. P. Siebesma, H. J. J. Jonker, F. Selten. Stochastic convection parameterization with Markov chains in an intermediate-complexity GCM. *J. Atmos. Sci.* **73**, 1367–1382 (2016).

58. N. Chen, A. J. Majda, C. T. Sabeerali, R. S. Ajayamohan. Predicting monsoon intraseasonal precipitation using a low-order nonlinear stochastic model. *J. Climate*, **31**, 4403–4427 (2018).

59. A. Bellen, M. Zennaro, *Numerical Methods for Delay Differential Equations* (Oxford Univ. Press, 2013).

60. S. Maset. Numerical solution of retarded functional differential equations as abstract Cauchy problems. *J. Comput. Appl. Math.* **161**, 259–282 (2003).

61. A. Bellen, S. Maset. Numerical solution of constant coefficient linear delay differential equations as abstract Cauchy problems. *Numer. Math.* **84**, 351–374 (2000).

62. S. Yanchuk, G. Giacomelli. Spatio-temporal phenomena in complex systems with time delays. *J. Phys. A Math. Theor.* **50**, 103001 (2017).

63. F. T. Arecchi, G. Giacomelli, A. Lapucci, R. Meucci. Two-dimensional representation of a delayed dynamical system. *Phys. Rev. A*, **45**:R4225–R4228 (1992).

64. H. Aref. Stirring by chaotic advection. *J. Fluid Mech.*, **143**, 1–21 (1984).

65. J. M. Ottino, S. C. Jana, V. S. Chakravarthy. From Reynolds’s stretching and folding to mixing studies using horseshoe maps. *Phys. Fluids*, **6**:685–699, 1994.

66. K. Wójcik, P. Zgliczyński. Topological horseshoes and delay differential equations. *Discrete Continuous Dyn. Syst.* **12**, 827–852 (2005).

67. C. Feng, H. Zhao. Random periodic processes, periodic measures and ergodicity. *J. Differ. Equ.* **269**, 7382–7428 (2020).

68. O. E. Rossler, An equation for hyperchaos. *Phys. Lett. A* **71**, 155–157 (1979).
69. G. Baier, S. Sahle, Design of hyperchaotic flows. *Phys. Rev. E* **51**, R2712–R2714 (1995).

70. M. D. Chekroun, J. D. Neelin, D. Kondrashov, J. C. McWilliams, M. Ghil. Rough parameter dependence in climate models and the role of Ruelle-Pollicott resonances. *Proc. Natl. Acad. Sci. U.S.A.*, **111**, 1684–1690 (2014).

71. M. D. Chekroun, A. Tantet, H. A. Dijkstra, J. D. Neelin. Ruelle–Pollicott resonances of stochastic systems in reduced state space. Part I: Theory. *J. Stat. Phys.* **179**, 1366–1402 (2020).

72. T. Dror, V. Silverman, O. Altaratz, M. D. Chekroun, I. Koren, Uncovering the large-scale meteorology that drives continental, shallow, green cumulus through supervised classification. *Geophys. Res. Lett.* **49**, e2021GL096684 (2022).

73. R. Wood. Stratocumulus clouds. *Mon. Weather Rev.*, **140**:2373–2423, 2012.

74. A. Guillamon, G. Huguet. A computational and geometric approach to phase resetting curves and surfaces. *SIAM J. Appl. Dyn. Syst.* **8**, 1005–1042 (2009).

75. A. Mauroy, B. Rhoads, J. Moehlis, I. Mezic. Global isochrons and phase sensitivity of bursting neurons. *SIAM J. Appl. Dyn. Syst.* **13**, 306–338 (2014).

76. M. Detrixhe, M. Doubeck, J. Moehlis, F. Gibou. A fast Eulerian approach for computation of global isochrons in high dimensions. *SIAM J. Appl. Dyn. Syst.* **15**, 1501–1527 (2016).

77. M. J. Panaggio, D. M. Abrams. Chimera states: Coexistence of coherence and incoherence in networks of coupled oscillators. *Nonlinearity* **28**, R67 (2015).

78. J. M. Gomes, R. Weber dos Santos, E. M. Cherry. Alternans promotion in cardiac electrophysiology models by delay differential equations. *Chaos* **27**, 093915 (2017).

79. K. H. W. J. ten Tusscher, D. Noble, P.-J. Noble, A. V. Panfilov. A model for human ventricular tissue. *Am. J. Physiol. Heart Circ. Phys.* **286**, H1573–H1589 (2004).

80. J. E. Marsden, M. McCracken, *The Hopf Bifurcation and Its Applications* (Springer Science & Business Media, 2012), vol. 19.
81. T. Ma, S. Wang. *Phase Transition Dynamics* (Springer, 2014).

82. D. Kondrashov, M. D. Chekroun, P. Berloff. Multiscale Stuart-Landau emulators: Application to wind-driven ocean gyres. *Fluids* 3, 21, (2018).

83. M. D. Chekroun, H. Liu, J. C. McWilliams. Stochastic rectification of fast oscillations on slow manifold closures. *Proc. Natl. Acad. Sci. U.S.A.*, 118, e2113650118 (2021).

84. G. Thompson, T. Eidhammer. A study of aerosol impacts on clouds and precipitation development in a large winter cyclone. *J. Atmos. Sci.*, 71, 3636–3658 (2014).

85. S.-Y. Hong, Y. Noh, J. Dudhia. A new vertical diffusion package with an explicit treatment of entrainment processes. *Mon. Weather Rev.* 134, 2318–2341 (2006).

86. P. A. Jiménez, J. Dudhia, J. F. González-Rouco, J. Navarro, J. P. Montávez, E. García-Bustamante. A revised scheme for the WRF surface layer formulation. *Mon. Weather Rev.* 140, 898–918 (2012).

87. K. Engelborghs, T. Luzyanina, D. Roose. Numerical bifurcation analysis of delay differential equations using DDE-BIFTOOL. *ACM Trans. Math. Softw.* 28, 1–21 (2002).

88. R. B. Vinter, On the evolution of the state of linear differential delay equations in $M^2$: Properties of the generator. *IMA J. Appl. Math.* 21, 13–23, 1978.