A Characterization of the Number of Roots of Linearized and Projective Polynomials in the Field of Coefficients

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Abstract

A fundamental problem in the theory of linearized and projective polynomials over finite fields is to characterize the number of roots in the coefficient field directly from the coefficients. We prove results of this type, of a recursive nature. These results follow from our main theorem which characterizes the number of roots using the rank of a matrix that is smaller than the Dickson matrix.

1 Introduction

Linearized polynomials with coefficients in a finite field $\mathbb{F}_{q^n}$ arise in many different problems and contexts. A fundamental problem in the theory of linearized polynomials is to characterize precisely the number of roots in $\mathbb{F}_{q^n}$ in terms of the coefficients of the given polynomial.

More precisely, let $L(x) = a_0x + a_1x^q + a_2x^{q^2} + \cdots + a_dx^{q^d}$ be a $q$-linearized polynomial with coefficients in $\mathbb{F}_{q^n}$. The roots of $L(x)$ that lie in the field $\mathbb{F}_{q^n}$ form an $\mathbb{F}_q$-vector space, which can have dimension anywhere between 0 and $d$. The fundamental problem of linearized polynomials is to somehow determine the dimension of this $\mathbb{F}_q$-vector space directly from the coefficients $a_0, a_1, \ldots, a_d$. The ultimate goal is to find necessary and sufficient conditions on the $a_i$ that tell when the dimension is equal to $k$, for each $k$ between 0 and $d$. In this paper we present results of this type, and our approach will

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generate further such results. Here is one example of our results (see Theorem 9 for the full statement which covers all possibilities).

**Theorem 1.** Let \( L(x) = ax + bx^q + cx^{q^2} \in \mathbb{F}_{q^n}[x] \). Then \( L(x) \) has \( q^2 \) roots in \( \mathbb{F}_{q^n} \) if and only if \( G_{n-1} = 0 \) and \( N(b/c)G_n = 1 \), where the quantities \( G_n \) and \( G_{n-1} \) are calculated directly from the coefficients \( a, b, c \), via a recursion.

For example, when \( n = 5 \) this theorem becomes the following.

**Theorem 2.** Let \( L(x) = ax + bx^q + cx^{q^2} \in \mathbb{F}_{q^5}[x] \). Let \( u = \frac{a^c}{b^{q+1}} \). Then \( L(x) \) has \( q^2 \) roots in \( \mathbb{F}_{q^5} \) if and only if \( b \neq 0 \), \( (1 - u)^{q^2+1} - u^q = 0 \), and \( N(1 - u) = N(c/b) \).

We have several results of this flavour, which are presented in Sections 4 and 5. Our proofs use a relationship between the number of roots (or the nullity) of \( L \) in \( \mathbb{F}_{q^n} \), and the rank of a matrix we denote \( A_L \), which can be calculated explicitly from the coefficients of \( L \). This relationship is established in Theorem 5 in Section 3 and is the main result of the paper. It is known that the nullity of a linearized polynomial is equal to the nullity of a Dickson matrix (see [11]). However, this matrix can be difficult to calculate and work with. In this paper we show that the nullity of a linearized polynomial is equal to the nullity of \( A_L - I \), a matrix which is different to and smaller than the Dickson matrix.

An interesting feature of our results is that they are recursive in \( n \), the degree of the extension of \( \mathbb{F}_q \). This means that all the expressions in the coefficients for \( \mathbb{F}_{q^n} \) that occur in our investigations (for example as entries in \( A_L \)) will also be used for \( \mathbb{F}_{q^{n+1}} \) and higher degree extensions.

Alongside linearized polynomials we also consider projective polynomials, which are polynomials of the form

\[
\sum_{i=0}^{d} a_i y^q \frac{q^i - 1}{q-1}, \quad a_i \in \mathbb{F}_{q^n}.
\]

We will prove similar results for projective polynomials, giving necessary and sufficient conditions on the coefficients for each possibility for the number of roots in \( \mathbb{F}_{q^n} \). The number of roots of a projective polynomial is related to the eigenvectors of our matrix \( A_L \), see Section 3.
A polynomial in $\mathbb{F}_{q^n}[x]$ is called a **permutation polynomial** if it induces a bijective function on $\mathbb{F}_{q^n}$. A linearized polynomial is a permutation polynomial if and only if its only root in $\mathbb{F}_{q^n}$ is 0. Therefore, giving if and only if conditions on the coefficients of a linearized polynomial to be a permutation polynomial can be seen as a special case of our results, namely, the case when the dimension of the vector space of roots in $\mathbb{F}_{q^n}$ is 0. For a detailed statement about $ax + bx^q + cx^{q^2}$ see Theorem 9 and for $ax + bx^q + cx^{q^2} + dx^{q^3}$ see Theorem 15.

Listing all previous work on this topic is impossible in a few paragraphs. We give a few references here, and others throughout the paper. Abhyankar [11] studied projective polynomials for a different reason (realizing Galois groups). Bluher [2] studied projective polynomials $x^{q+1} + ax + b$ and showed that the number of roots in the ground field is highly restricted. Helleseth and Kholosha [7] studied certain projective and linearized polynomials in even characteristic, and gave a criteria for the number of roots in terms of the coefficients. Our results generalize and extend theirs to all characteristics, and to higher degree. The number of roots of a projective polynomial was related to the eigenvectors of a certain matrix by von zur Gathen-Giesbrecht-Ziegler [6], however their paper was not constructive, and does not give criteria based on the coefficients.

The paper is laid out as follows. In Section 2 we give preliminaries and background for the paper. Then in Section 3 we present our main results relating the matrix $A_L$ to the number of roots. Section 4 applies the main results to the case of degree $q^2$, and Section 5 does the same for degree $q^3$. Finally, in Section 6 we outline some possibilities for future work.

### 2 Preliminaries

Throughout, $\mathbb{F}_q$ is a finite field, $\mathbb{F}_{q^n}$ an extension of degree $n$, and $\sigma$ an automorphism of $\mathbb{F}_{q^n}$ with fixed field $\mathbb{F}_q$; in other words $\sigma$ is a generator of the Galois group $\text{Gal}(\mathbb{F}_{q^n} : \mathbb{F}_q)$. We write $x^\sigma$ for the image of $x$ under $\sigma$. Note that $x^\sigma = x^{q^s}$ for some $s \in \{1, \ldots, n\}$ with $\gcd(n, s) = 1$. The reader may safely substitute $q$ for $\sigma$, however using $\sigma$ is more general. We use $N$ to denote the norm function from $\mathbb{F}_{q^n} \rightarrow \mathbb{F}_q$, i.e., $N(x) = x^{1+\sigma+\sigma^2+\cdots+\sigma^{n-1}}$ for $x \in \mathbb{F}_{q^n}$. 


2.1 Linearized Polynomials

A \( \sigma \)-linearized polynomial is a polynomial of the form

\[
L = a_0 x + a_1 x^\sigma + a_2 x^{\sigma^2} + \cdots + a_d x^{\sigma^d} \in \mathbb{F}_{q^n}[x].
\]  

(1)

If \( d \) is the largest integer such that \( a_d \neq 0 \), we call \( d \) the \( \sigma \)-degree of \( L \), and denote it \( \deg_\sigma(L) = d \). The \( \sigma \)-linearized polynomials are precisely those which define an \( \mathbb{F}_q \)-linear map from \( \mathbb{F}_{q^n} \) to itself. As an \( \mathbb{F}_q \)-linear map, a \( \sigma \)-linearized polynomial has a rank and a nullity, and as usual \( \text{rank}(L) + \text{null}(L) = n \).

Let \( L \) denote the set of all \( \sigma \)-linearized polynomials, and \( L_k \) the set of all \( \sigma \)-linearized polynomials of \( \sigma \)-degree at most \( k - 1 \). The set of \( \sigma \)-linearized polynomials with addition and composition form a ring, which we denote by \( (L, \circ) \). If \( M = b_0 x + b_1 x^\sigma + \cdots + b_e x^{\sigma^e} \), then

\[
L \circ M = \sum_{k=0}^{d+e} \left( \sum_i a_i b_{k-i}^{\sigma^i} \right) x^{\sigma^k}.
\]

If \( L = L_1 \circ L_2 \) for \( L_1, L_2 \in L \), we say that \( L_2 \) is a right component of \( L \), and \( L_1 \) is a left component of \( L \). We say that \( L \) is a left composition of \( L_2 \) and a left composition of \( L_1 \).

We define the greatest common right component and least common left composition of \( L, M \in L \) in the natural way, and denote them by \( \text{gcrc}(L, M) \) and \( \text{lclc}(L, M) \) respectively. This terminology follows [6].

**Remark 1.** The ring \( L \) is isomorphic to the skew-polynomial ring \( \mathbb{F}_{q^n}[t; \sigma] \); the non-commutative polynomial ring defined by \( t \alpha = \alpha^\sigma t \) for all \( \alpha \in \mathbb{F}_{q^n} \). Much of the following can also naturally be expressed in terms of skew-polynomials. Though we will not use this in this paper, we will make use of some basic facts from the theory of skew-polynomials.

**Remark 2.** In some literature, the symbol \( \otimes \) is used instead of \( \circ \), and the term symbolic divisor is used instead of component.

The following are useful well-known facts. We refer the reader to [8] and [16].

**Proposition 1.** Let \( L \) be a \( \sigma \)-linearized polynomial ring over \( \mathbb{F}_{q^n} \).

- The multiplicative identity of \( L \) is \( x \), and the centre of \( L \) is generated by \( \mathbb{F}_q x \) and \( x^{\sigma^n} \).
• $\mathcal{L}$ is a right-Euclidean domain with respect to $\sigma$-degree.

• The element $x^{\sigma^n} - x$ generates a maximal two-sided ideal in $\mathcal{L}$, and

$$\mathcal{L} / (x^{\sigma^n} - x) \cong M_{n \times n}(\mathbb{F}_q).$$

• The nullity (i.e. the $\mathbb{F}_q$-dimension of the set of zeros) of $L \in \mathcal{L}$ in $\mathbb{F}_q^n$ is equal to

$$\deg_\sigma(\gcd(L, x^{\sigma^n} - x)).$$

• The nullity of $L \in \mathcal{L}$ is equal to the nullity of its associated Dickson matrix

$$D_L := \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_{\sigma^{-1}} & a_0 & \cdots & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1^{-1}} & a_{2^{-1}} & \cdots & a_{0^{-1}} \end{pmatrix}$$

By $M \mod_\sigma L$ we mean the unique $R \in \mathcal{L}$ of $\sigma$-degree less than $\deg_\sigma(L)$ such that there exists $Q \in \mathcal{L}$ satisfying

$$M = Q \circ L + R.$$ 

If $x^\sigma = x^l$, then $M \mod_\sigma L$ turns out to be equal to the usual polynomial remainder $M \mod L$, although the quotients $Q, Q'$ are not equal; indeed if $M = Q' L + R, Q'$ is not in $\mathcal{L}$.

### 2.2 Projective Polynomials

Given $L \in \mathcal{L}$ of the form (I), we define its associated projective polynomial to be

$$P_L(y) = a_0 + a_1y + a_2y^2 + \cdots + a_dy^d,$$

where

$$y^{[i]} := y^{\sigma^{-i}}.$$ 

Note that $L(x) = xP_L(x^{\sigma-1})$, and so we say that $P_L$ is associated to $L$. We also say that $P_L$ has $\sigma$-degree $d$. 

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Projective polynomials were introduced by Abhyankar [1], where the coefficient field was the transcendental extension $F_q(t)$ of $F_q$. The original motivation was to find polynomials with a given Galois group. However they have also been studied over $F_{q^n}$, for example by Bluher [2], and in [6], due to their interesting possible number of roots, and their connection to calculating composition collisions. Projective polynomials with maximum number of roots have been used in [9] for attacking the discrete logarithm problem in cryptography; see Section 6 for further details.

The following is the most general known result for the number of roots of a projective polynomial, which is a generalisation of results of Bluher.

**Theorem 3 ([6]).** The number of roots of a projective polynomial of $\sigma$-degree $d$ over $F_{q^n}$ is of the form

$$\sum_i q^{a_i} - 1 \over q - 1$$

for some nonnegative integers $a_i$ with $\sum_i a_i = d$.

However this result does not give a way of determining the number of roots of a particular projective polynomial from its coefficients. In [7], Helleseth-Kholosha gave criteria for the case $d = 2$, $q$ even. In this paper we will extend this to $q$ odd and general $d$, and we give a way of determining the number of roots from the coefficients.

### 2.3 Companion Matrix

Let $L = a_0x + a_1x^\sigma + \cdots + a_dx^{\sigma d}$ be a $\sigma$-linearized polynomial in $F_{q^n}[x]$. Define the **companion matrix** of $L$ (and of $P_L$) as the $d \times d$ matrix

$$C_L = \begin{pmatrix}
0 & 0 & \cdots & 0 & -a_0/a_d \\
1 & 0 & \cdots & 0 & -a_1/a_d \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & -a_{d-1}/a_d
\end{pmatrix},$$

We define further the matrix

$$A_L = C_L C_L^\sigma \cdots C_L^{\sigma^{n-1}} , \quad (2)$$
and we denote the characteristic polynomial of $A_L$ by $\chi_L$. Matrices of this type appear also in the theory of curves over finite fields; see Section 6 for further details. In this paper we consider it for the following reason.

**Proposition 2.** The map

$$\psi : M \mapsto x^\sigma \circ M \mod L$$

defines a semilinear map on $L_d$ with associated matrix $C_L$ and automorphism $\sigma$.

The map

$$\phi : M \mapsto x^{\sigma^n} \circ M \mod L$$

defines a linear map on $L_d$ with associated matrix $A_L$.

**Proof.** Each map is well-defined, as $L$ is a right-Euclidean domain. Furthermore both are additive, as $L$ is a ring. Thus it suffices to show that $\psi(aM) = a^\sigma \psi(M)$ and $\phi(aM) = a\phi(M)$ for all $M \in L$, $a \in \mathbb{F}_{q^n}$. This follows from the fact that $x^\sigma \circ ax = a^\sigma x^\sigma$ for all $a \in \mathbb{F}_{q^n}$, and so $x^\sigma \circ (aM) = a^\sigma x^\sigma \circ M$. Thus $x^\sigma \circ (aM) \mod L = a^\sigma (x^\sigma \circ M \mod L)$, as claimed.

We choose the canonical basis $\{x, x^\sigma, \ldots, x^{\sigma^{d-1}}\}$ for $L_d$, and represent $M \in L_d$ as a column vector consisting of its coefficients. Now $x^\sigma \circ x^\sigma^i = x^\sigma^{i+1}$ for all $i$. Thus $\phi$ maps the $i$-th basis vector to the $(i+1)$-st for $i < d - 1$. Furthermore

$$x^\sigma \circ x^{\sigma^{d-1}} = x^{\sigma^d}$$

$$\equiv \frac{1}{a_d} (a_d x^{\sigma^d} - L) \mod L$$

$$= -\frac{1}{a_d} \left( a_0 x + a_1 x^{\sigma} + \cdots + a_{d-1} x^{\sigma^{d-1}} \right),$$

showing that $\phi$ has matrix $C_L$ with respect to the canonical basis.

Finally, as $\phi = \psi^n$, it follows that $\phi$ is a linear transformation of $L_d$, with associated matrix $A_L$ with respect to the canonical basis of $L_d$. $\square$

**Remark 3.** In [12] the characteristic polynomial of the matrix $A_L$ is referred to as the semi-characteristic polynomial of the semilinear transformation $\psi$. In [14] a similar matrix was used to construct new semifields and MRD codes.
Remark 4. Note that \( \det(A_L) = N\left(\det(C_L)\right) = N((-1)^d a_0/a_d) \). In fact, all the coefficients of \( \chi_L \) (the characteristic polynomial of \( A_L \)) are in \( \mathbb{F}_q \), as we now prove.

**Theorem 4.** The coefficients of the characteristic polynomial of \( A_L \) all lie in \( \mathbb{F}_q \).

**Proof.** The coefficients of the characteristic polynomial of \( A_L \) are the elementary symmetric polynomials in the eigenvalues of \( A_L \). Since the trace of \( A_L^i \) is equal to the sum of the \( i \)-th powers of the eigenvalues of \( A_L \), by Newton’s identities it is sufficient to show that the trace of \( A_L^i \) lies in \( \mathbb{F}_q \) for each \( i \geq 0 \).

We have that \( \text{Tr}(A_L^i) = \text{Tr}((C_L C_L^\sigma \cdots C_L^{\sigma^{n-1}})^i) \). By the well-known commutativity property of the trace function, we may move the left-most matrix \( C_L \) to the right-most position, and the trace is unchanged. Then applying the automorphism \( \sigma \) entry wise, it is clear that \( \text{Tr}(A_L^i) = \text{Tr}((A_L^i)^\sigma) = \text{Tr}(A_L^i) \), as claimed.

**Remark 5.** We note that the matrix \( A_L^i \) would be the matrix to use in place of \( A_L \) if we considered \( L \in \mathcal{L} \) on \( \mathbb{F}_{q^n} \). This is because \( C_L \) has entries in \( \mathbb{F}_q \), and so

\[
C_L C_L^\sigma \cdots C_L^{\sigma^{n-1}} = (C_L C_L^\sigma \cdots C_L^{\sigma^{n-1}})^i.
\]

### 2.4 A useful homomorphism

Given \( L = \sum a_i x^{\sigma^i} \in \mathcal{L} \) and \( \alpha \in \mathbb{F}_{q^n} \), define \( L_\alpha = \sum a_i \alpha^{[i]} x^{\sigma^i} \). It is clear that \( P_{L_\alpha}(y) = P_L(\alpha y) \). Furthermore it holds that

\[
L_\alpha \circ M_\alpha = (L \circ M)_\alpha,
\]

i.e. \( L \mapsto L_\alpha \) is a homomorphism on the ring of linearized polynomials with composition. Consequently, we have that

\[
\gcd(L, M)_\alpha = \gcd(L_\alpha, M_\alpha).
\]

**Remark 6.** This arises naturally in the skew-polynomial ring, where the map \( \ell(t) \mapsto \ell(\alpha t) \) is a ring-isomorphism.
3 Main Results

We present one Proposition before our main theorems.

3.1 Roots of $L$ and $P_L$

The following relates the roots of $P_L$ to the roots of the $L_\alpha$’s.

**Proposition 3.** The number of roots of $P_L(y)$ in $\mathbb{F}_{q^n}$ is equal to

$$\sum_{\lambda \in \mathbb{F}_q} \frac{\deg(\gcd(L(x), x^{\sigma^n} - \lambda x)) - 1}{q - 1}.$$ 

**Proof.** Suppose $y$ is a root of $P_L(y)$ with $N(y) = \lambda$. Let $\alpha$ be a fixed element of $\mathbb{F}_{q^n}$ with $N(\alpha) = \lambda$. Then $y = \alpha z^{\sigma^{-1}}$ for some $z \in \mathbb{F}_{q^n}$, and $z$ is unique up to $\mathbb{F}_q^\times$-multiplication (i.e., $y = \alpha z^{\sigma^{-1}} = \alpha w^{\sigma^{-1}}$ if and only if $z/w \in \mathbb{F}_q^\times$).

Hence $0 = P_L(y) = P_L(\alpha z^{\sigma^{-1}}) = P_{L_\alpha}(z^{\sigma^{-1}}) = L_\alpha(z)/z$, and thus the number of roots of $P_L(y)$ is equal to the number of one-dimensional $\mathbb{F}_q$-subspaces of $\ker(L_\alpha)$, which is

$$\frac{\deg(\gcd(L_\alpha(x), x^{q^n} - x)) - 1}{q - 1}.$$ 

Finally it suffices to show that $\deg(\gcd(L_\alpha(x), x^{q^n} - x)) = \deg(\gcd(L(x), x^{q^n} - \lambda x))$. Note that $(x^{q^n} - x)_{\alpha^{-1}} = N(\alpha^{-1})x^{q^n} - x = \lambda^{-1}x^{q^n} - x$. Hence if

$$D(x) = \gcd(L(x), x^{q^n} - \lambda x)$$
$$= \gcd(L(x), (x^{q^n} - x)_{\alpha^{-1}})$$
$$\Rightarrow D_\alpha(x) = \gcd(L_\alpha(x), x^{q^n} - x),$$

proving the claimed equality of degrees. Summing over $\lambda \in \mathbb{F}_q^\times$ gives the desired result.

Later we will use this to give a more useful formula for calculating the number of roots of $P_L(y)$. 

Note that the one-dimensional subspace of the kernel of $L(x)$ are in one-to-one correspondence with solutions of the system $P_L(y) = 0, N(y) = 1$, where $N(y) = y^{[n]}$ denotes the field norm from $\mathbb{F}_{q^n}$ to $\mathbb{F}_q$.

### 3.2 First Main Result

We are now ready for our main results, which relate the number of roots of $L$ and $P_L$ to properties of the matrix $A_L$. We emphasize again that the matrix $A_L$ is smaller than the Dickson matrix, although having the same property. For $L \in \mathcal{L}$ of $\sigma$-degree $d$, the Dickson matrix is $n \times n$ whereas $A_L$ is $d \times d$.

**Theorem 5.** Let $L \in \mathcal{L}$, and let $A_L$ be defined as in (2). Then for any $\alpha \in \mathbb{F}_q^*$, we have

$$\text{null}(L_\alpha) = \text{null}(A_L - \lambda I),$$

where $\lambda = N(\alpha)$. In particular, the nullity of $L$ is equal to the nullity of $A_L - I$.

**Proof.** Suppose $\deg_\sigma(L) = d$. By Proposition 2, $A_L$ is the matrix associated to the linear transformation $\phi$ of $\mathcal{L}_d$ defined by

$$\phi(M) = x^\sigma \circ M \mod r,$$

and so $A_L - \lambda I$ is the matrix associated to $\phi - \lambda I$, which is defined by

$$(\phi - \lambda I)(M) = (x^\sigma - \lambda x) \circ M \mod r.$$

Suppose $M \in \ker(\phi - \lambda I)$, which occurs if and only if $(x^\sigma - \lambda x) \circ M$ is a left composition of $L$. As $(x^\sigma - \lambda x)$ commutes with every element of $\mathcal{L}$, we have that $(x^\sigma - \lambda x) \circ M = M \circ (x^\sigma - \lambda x)$. Let $D = \text{gcrc}(L, x^\sigma - \lambda x)$, and let $A, B$ be such that $L = A \circ D$, $x^\sigma - \lambda x = B \circ D$. Note that $\text{gcrc}(A, B) = x$. Then

$$M \circ B \circ D \equiv 0 \mod r, A \circ D$$

$$\Leftrightarrow M \circ B \equiv 0 \mod r, A.$$

Thus $M \circ B$ is a left composition of both $A$ and $B$, and so is a left composition of $\text{lelc}(A, B)$, which we denote by $C$. Let $E, F$ be such that $C = E \circ A = F \circ B$. Then
there exists a unique $G$ such that $M \circ B = G \circ C$. Now $M \circ B = G \circ C = G \circ F \circ B$, and so $M = G \circ F$. Thus $M \in \ker(\phi - \lambda I)$ if and only if $M$ is a left composition of $F$ of $\sigma$-degree at most $d - 1$.

Now $\deg_{\sigma}(F) = \deg_{\sigma}(A)$, and $\deg(A) = d - \deg_{\sigma}(D)$, implying that $\deg(G) \leq \deg_{\sigma}(D) - 1$. Thus

$$\dim \ker(\phi - \lambda I) = \deg_{\sigma}(D),$$

which by Proposition 1 is equal to $\text{null}(L_\alpha)$.

### 3.3 Second Main Result

**Theorem 6.** The number of roots of $P_L$ in $\mathbb{F}_{q^n}$ is given by

$$\sum_{\lambda} \frac{q^{n_\lambda} - 1}{q - 1},$$

where $n_\lambda$ is the dimension of the eigenspace of $A_L$ with eigenvalue $\lambda$.

The number of roots of $L$ in $\mathbb{F}_{q^n}$ is given by $q^{n_1}$, i.e., the size of the eigenspace of $A_L$ corresponding to the eigenvalue $1$.

**Proof.** This follows immediately from Proposition 3 and Theorem 5.

**Remark 7.** In [6] it was shown that the number of roots of $P_L$ was related to the eigenspaces of a linear map acting on the vector space of roots of $L$ over the algebraic closure of $\mathbb{F}_{q^n}$. However there was no explicit way of obtaining this linear map directly from the coefficients of $L$, whereas in this paper $A_L$ does provide such a method.

### 3.4 Third Main Result

**Theorem 7.** Let $L \in \mathcal{L}$ have $\sigma$-degree $d$. Then $P_L$ has $\frac{q^d - 1}{q - 1}$ roots in $\mathbb{F}_{q^n}$ if and only if $A_L = \lambda I$.
for some $\lambda \in \mathbb{F}_q$. Also, $L$ has $q^d$ roots in $\mathbb{F}_{q^n}$ if and only if

$$A_L = I.$$  

**Proof.** By Theorem 6, $P_L$ has $\frac{q^d-1}{q-1}$ roots if and only if $A_L$ has an eigenspace of dimension $d$, which occurs if and only if $A_L$ is a multiple of the identity.

$L$ has $q^d$ roots in $\mathbb{F}_{q^n}$ if and only if it has nullity $d$, which by 5 occurs if and only if $A_L = I$. \qed

An immediate corollary is the following known result [10, Theorem 10], which was used in [13] to construct new maximum rank-distance codes.

**Corollary 1.** If $L$ is a $\sigma$-linearized polynomial of $\sigma$-degree $d$ such that $L$ has $q^d$ roots, then $N(a_0) = (-1)^nkN(a_d)$.

**Remark 8.** Note that if $x^\sigma = x^q$, then this theorem gives us criteria for when $L$ or $P_L$ split completely over $\mathbb{F}_{q^n}$.

In the next sections we will use our main results to develop explicit criteria for the number of roots of $L$ and $P_L$ in the case of $\sigma$-degree two and three. Our method will work for any $\sigma$-degree $d$, and any $n$, however the expressions become more complicated as $d$ increases.

### 4 Criteria for the number of roots for $\sigma$-degree 2

#### 4.1 Recursions for the matrix $C_k$

Let $L = ax + bx^\sigma + cx^\sigma^2$ with $a, b, c \in \mathbb{F}_{q^n}, c \neq 0$, and let

$$C = C_L = \begin{pmatrix} 0 & -a/c \\ 1 & -b/c \end{pmatrix}.$$  

We define $C_0$ to be the $k \times k$ identity matrix, and for $k \geq 1$ define

$$C_k = CC^\sigma \cdots C^\sigma^{k-1},$$
and note that \( C_n = A_L \). Observe that
\[
C_k = C_{k-1}C_{\sigma^{k-1}} = CC_{k-1}^\sigma.
\]
These two relations will allow us to obtain straightforward recursions for the entries of \( C_k \). We make the following substitutions in order to simplify the calculations. Let
\[
u = \frac{a\sigma c}{b^{\sigma+1}}
\]
and define matrices
\[
X = \begin{pmatrix} a/b & 0 \\ 0 & 1 \end{pmatrix}, \quad Z_k = \begin{pmatrix} (b/c)^{k-1} & 0 \\ 0 & (b/c)^k \end{pmatrix}
\]
and define \( Y_k \) by \( C_k = XY_kZ_k \).

**Lemma 1.** \( Y_k \) satisfies the recursions
\[
Y_k = Y_{k-1} \begin{pmatrix} 0 & -u \\ 1 & -1 \end{pmatrix}^{\sigma^{k-2}}, \quad Y_k = \begin{pmatrix} 0 & -1 \\ u & -1 \end{pmatrix}^{\sigma}Y_{k-1}^{\sigma}.
\]
Thus \( Y_k \) is a matrix each of whose entries is a polynomial in the indeterminate \( u \).

**Proof.** We have that
\[
Y_k = X^{-1}C_{k-1}Z_k^{-1} = X^{-1}C_{k-1}C_{\sigma^{k-1}}^{-1}Z_k^{-1} = X^{-1}(XY_{k-1}Z_{k-1})C_{\sigma^{k-1}}^{-1}Z_k^{-1},
\]
and so the proof of the first follows from a simple verification that \( Z_{k-1}C_{\sigma^{k-1}}Z_k^{-1} \) is the stated matrix \( \begin{pmatrix} 0 & -u \\ 1 & -1 \end{pmatrix}^{\sigma^{k-2}} \).

Similarly we have
\[
Y_k = X^{-1}CC_{k-1}^\sigma Z_k^{-1} = (X^{-1}CX^\sigma)Y_{k-1}^\sigma(Z_{k-1}^\sigma Z_k^{-1}Z_k^{-1})
\]
and the proof of the second follows from a verification that \( Z_{k-1}^\sigma Z_k^{-1} = (c/b)I \), and that \((c/b)X^{-1}CX^\sigma\) is the stated matrix \( \begin{pmatrix} 0 & -1 \\ u & -1 \end{pmatrix} \).
We will write the second column of $Y_k$ as $(F_k, G_k)^T$, where $F_k$ and $G_k$ are functions of $u$. In other words, we define $G_k$ to be the $(2, 2)$ entry of $Y_k$, and define $F_k$ to be the $(1, 2)$ entry of $Y_k$. Then $G_0 = 1$, $G_1 = -1$ and $G_2 = 1 - u$. It is clear from the first recursion in Lemma 1 that

$$Y_k = \begin{pmatrix} F_{k-1} & F_k \\ G_{k-1} & G_k \end{pmatrix},$$

and that

$$G_k + G_{k-1} + u^{\sigma^{k-2}}G_{k-2} = 0. \tag{3}$$

Furthermore, the second recursion gives us that $F_k = -G_{k-1}^\sigma$, and $G_k = uF_{k-1}^\sigma - G_{k-1}^\sigma$. Together this gives

$$G_k + G_{k-1}^\sigma + uG_{k-2}^\sigma = 0. \tag{4}$$

We observe that these recursions are essentially the same as those found by Helleseth-Kholosha in [7] in the special case $q$ even, although their method was very different from ours.

Therefore we can rewrite $Y_k$ in terms of the sequence $(G_k)$, as follows:

$$Y_k = \begin{pmatrix} -G_{k-2}^\sigma & -G_{k-1}^\sigma \\ G_{k-1} & G_k \end{pmatrix}.$$

Note that if $k = n$, (3) and (4) imply that

$$G_n^\sigma - G_n = G_{n-1}^\sigma - G_{n-1}^\sigma.$$

Thus, as $C_n = A_L$, by definition, we have $A_L = X Y_n Z_n$, which shows the following.

**Proposition 4.** Let $L = ax + bx^\sigma + cx^\sigma^2$ with $a, b, c \in \mathbb{F}_q$, $c \neq 0$, and let $G_k$ be as defined above. Then

$$A_L = N(b/c) \begin{pmatrix} -u^{-1}G_{n-2}^\sigma & -(a/b)G_{n-1}^\sigma \\ (c/b)^{\sigma^{-1}}G_{n-1} & C_n \end{pmatrix}.$$
To calculate the number of roots of $L$ and $P_L$, we need to know the characteristic polynomial of $A_L$. Directly from Proposition 4 we get

$$\det(A_L) = N(b/c)^2 u^{\sigma-1} (G_{n-1}^{\sigma+1} - G_n G_{n-2}^\sigma)$$
$$\tr(A_L) = N(b/c) (G_n - u^{\sigma-1} G_{n-2}^\sigma)$$
$$= N(b/c) (G_n + G_n^\sigma + G_{n-1}^\sigma).$$

Note that $\det(A_L) = N(\det(C))$ in general, and so

$$\det(A_L) = N(a/c).$$

Thus we have that

$$N(b/c)^2 u^{\sigma-1} (G_{n-1}^{\sigma+1} - G_n G_{n-2}^\sigma) = N(a/c),$$

and so

$$u^{\sigma-1} (G_{n-1}^{\sigma+1} - G_n G_{n-2}^\sigma) = N(u).$$

This proves the following. Recall that $\chi_L$ is the characteristic polynomial of $A_L$.

**Proposition 5.** Let $L = ax + bx^\sigma + cx^{\sigma^2}$ with $a, b, c \in \mathbb{F}_{q^n}$, $c \neq 0$, and let $G_k$ be as defined above. Let $\mu = N(b/c)$. Then

$$\chi_L(\mu x) = \mu^2 \left( x^2 - (G_n + G_n^\sigma + G_{n-1}^\sigma)x + N(u) \right).$$

### 4.2 Criteria for number of roots of $P_L$

If $\deg_\sigma(L) = 2$, then $A_L$ is a $2 \times 2$ matrix. Thus

$$\chi_L(x) = x^2 - \tr(A_L)x + \det(A_L).$$

By Theorem 6, the number of roots of $P_L$ in $\mathbb{F}_{q^n}$ is determined by the dimension of the eigenspaces of $A_L$ corresponding to eigenvalues in $\mathbb{F}_q$. Hence if $\text{char}(A_L)$ has two distinct roots in $\mathbb{F}_q$, then $P_L$ has two roots in $\mathbb{F}_{q^n}$. If $\text{char}(A_L)$ has no roots in $\mathbb{F}_q$, then $P_L$ has no roots in $\mathbb{F}_{q^n}$. If $\text{char}(A_L)$ has a double root in $\mathbb{F}_q$, then $P_L$ has either $1$ or $q + 1$ root(s) in $\mathbb{F}_{q^n}$; if $A_L$ is a scalar multiple of the identity then $P_L$ has $q + 1$ roots, otherwise it has $1$ root. Hence we have the following.
If $q$ is odd, let

$$\Delta_L = \text{Tr}(A_L)^2 - 4 \det(A_L)$$

$$= N(b/c)^2((G_n + G_n^\sigma + G_{n-1}^\sigma)^2 - 4N(u))$$

If $q$ is even, let

$$\Lambda_L = \frac{\det(A_L)}{\text{Tr}(A_L)^2}$$

and let $T_0$ denote the absolute trace, i.e. the trace from $\mathbb{F}_{q^n}$ to $\mathbb{F}_2$.

**Theorem 8.** Let $L(x) = ax + bx^\sigma + cx^\sigma^2 \in \mathbb{F}_{q^n}[x]$. Let $Z_L$ denote the number of solutions to $P_L(x) = a + bx + cx^{\sigma+1} = 0$ in $\mathbb{F}_{q^n}$. For any $q$, $Z_L = q + 1$ if and only if $G_{n-1} = 0$ and $G_n \in \mathbb{F}_q$.

If $q$ is odd and $b \neq 0$, then

- $Z_L = q + 1$ if and only if $\Delta_L = 0$ and $G_{n-1} = 0$.
- $Z_L = 1$ if and only if $\Delta_L = 0$ and $G_{n-1} \neq 0$.
- $Z_L = 2$ if and only if $\Delta_L$ is a non-zero square in $\mathbb{F}_q$.
- $Z_L = 0$ if and only if $\Delta_L$ is a non-square in $\mathbb{F}_q$.

If $q$ is even and $b \neq 0$, then

- $Z_L = q + 1$ if and only if $G_{n-1} = 0$ and $G_n \in \mathbb{F}_q$.
- $Z_L = 1$ if and only if $G_{n-1} \neq 0$ and $G_n \in \mathbb{F}_q$.
- $Z_L = 2$ if and only if $G_n \notin \mathbb{F}_q$ and $T_0(\Gamma_L) = 0$.
- $Z_L = 0$ if and only if $G_n \notin \mathbb{F}_q$ and $T_0(\Gamma_L) = 1$.

If $b = 0$ and $n$ is odd, then $Z_L = 1$ if $N(-a/c)$ is a nonzero square in $\mathbb{F}_q$, and $Z_L = 0$ otherwise.

If $b = 0$ and $n$ is even, then $Z_L = q + 1$ if $N_{q^n:q}(-a/c) \in \mathbb{F}_q$, and $Z_L = 0$ otherwise.
Proof. By Theorem 7, $P_L$ has $q + 1$ roots if and only if $A_L$ is a scalar multiple of the identity. By Proposition 4, $A_L$ is a diagonal matrix if and only if $G_{n-1} = 0$. Suppose $G_{n-1} = 0$. Then

$$G_n + u^\sigma G_{n-2} = 0 \quad \text{and} \quad G_n + uG_{n-2}^\sigma = 0.$$  

Thus we get that $G_n = G_n^\sigma$, and so $G_n \in \mathbb{F}_q$. Furthermore we have $-u^{\sigma n-1} G_{n-2}^\sigma = G_n^\sigma$, and so

$$A_L = N(b/c) \begin{pmatrix} G_n^\sigma & 0 \\ 0 & G_n \end{pmatrix}$$

Thus $A_L$ is a scalar multiple of the identity if and only if $G_{n-1} = 0$ and $G_n \in \mathbb{F}_q$, as claimed.

The remaining cases follow immediately from Theorem 6 and Proposition 5, together with the well-known criteria for the number of distinct roots of a quadratic polynomial.

4.3 Criteria for number of roots of $L$

We now present the analogous theorem for the linearized polynomial, which classifies all the possibilities for the number of roots.

**Theorem 9.** Let $L(x) = ax + b x^\sigma + cx^\sigma^2 \in \mathbb{F}_q[x]$. Then

(i) $L$ has $q^2$ roots in $\mathbb{F}_q$ if and only if $G_{n-1} = 0$ and $N(b/c)G_n = 1$;

(ii) $L$ has $q$ roots in $\mathbb{F}_q^*$ if and only if $N(b/c)^2 \left(1 - G_n - G_n^\sigma - G_n^\sigma^2 + N(u)\right) = 0$, and the conditions of (i) are not satisfied.

(iii) $L$ has 1 root in $\mathbb{F}_q^*$ if and only if $N(b/c)^2 \left(1 - G_n - G_n^\sigma - G_n^\sigma^2 + N(u)\right) \neq 0$.

**Proof.** By Theorem 8, $L$ has $q^2$ roots in $\mathbb{F}_q$ if and only if $A_L = I$. The same argument as in Theorem 8 shows that this occurs if and only if $G_{n-1} = 0$ and $N(b/c)G_n = 1$, as claimed.

$L$ has $q$ roots in $\mathbb{F}_q^*$ if and only if 1 is an eigenvalue of $\chi_L$ and $A_L \neq I$, and 1 root otherwise. \qed
4.4 Examples for small \( n \)

The sequence \((G_k)\) begins as follows, starting at \(G_0\).

\[
\begin{array}{|c|c|}
\hline
k & G_k \\
\hline
0 & 1 \\
1 & -1 \\
2 & 1 - u \\
3 & u^\sigma + u - 1 \\
4 & (1 - u)^{\sigma^2 + 1} - u^\sigma \\
5 & 1 - u^{\sigma^3 + 1} - (1 - u)^{\sigma^2 + 1} - (1 - u)^{\sigma^3 + \sigma} \\
\hline
\end{array}
\]

We proceed recursively to compute the \( G_k \) and \( A_L \). The conditions for the number of roots of \( P_L \) depend on \( \det(A_L) \) and \( \text{Tr}(A_L) \). Note that we always have that \( \det(A_L) = N(a/c) \).

We compute now the expression for \( \text{Tr}(A_L) \) for some small values of \( n \). We denote the trace function from \( \mathbb{F}_{q^n} \) to \( \mathbb{F}_q \) by \( \text{tr}_{q^n,q} \), and the trace function from \( \mathbb{F}_q \) to \( \mathbb{F}_q \) by \( \text{tr} \).

\( n = 4 \): For \( L \in \mathbb{F}_{q^4}[x] \), we have

\[
A_L = N(b/c) \begin{pmatrix}
-u^{\sigma^3}G_2^\sigma & -(a/b)G_4^\sigma \\
(c/b)^{\sigma^3}G_3 & G_4
\end{pmatrix}
\]

\[
= N(b/c) \begin{pmatrix}
u^{\sigma^3}(u - 1) & (a/b)(1 - u^\sigma - u^{\sigma^2}) \\
(c/b)^{\sigma^3}(u + u^\sigma - 1) & (1 - u)^{\sigma^2 + 1} - u^\sigma
\end{pmatrix}
\]

Furthermore we have

\[
\text{Tr}(A_L) = N(b/c)(1 - \text{tr}_{q^4,q}(u) + \text{tr}_{q^2,q}(u^{1+\sigma^2}))
\]

\[
\Delta_L = \text{Tr}(A_L)^2 - 4 \det(A_L)
\]

\[
= N(b/c)^2 \left( (G_4 - G_4^\sigma - G_3)^2 - 4u^{\sigma^3}G_3^{\sigma^2 + 1} \right).
\]

\( n = 5 \): For \( L \in \mathbb{F}_{q^5}[x] \), we have

\[
A_L = N(b/c) \begin{pmatrix}
-u^{\sigma^4}G_3^\sigma & -(a/b)G_5^\sigma \\
(c/b)^{\sigma^4}G_4 & G_5
\end{pmatrix}
\]

\[
\text{Tr}(A_L) = N(b/c)(\text{tr}(u - u^{1+\sigma^2}) - 1)
\]

\( n = 6 \): For \( L \in \mathbb{F}_{q^6}[x] \), we have

\[
\text{Tr}(A_L) = N(b/c)(1 - \text{tr}_{q^6,q}(u - u^{1+\sigma^2}) + \text{tr}_{q^3,q}(u^{1+\sigma^3}) - \text{tr}_{q^2,q}(u^{1+\sigma^2 + \sigma^4})).
\]
For $L \in \mathbb{F}_q^7[x]$, we have
$$\text{Tr}(A_L) = N(b/c)(\text{tr}(u - u^{\sigma^2 + 1} - u^{\sigma^3 + 1} + u^{\sigma^4 + \sigma^2 + 1}) - 1)$$

5 Criteria for the number of roots for $\sigma$-degree 3

5.1 Recursions for the matrix $C_k$

Define
$$C_L = \begin{pmatrix} 0 & 0 & -a/d \\ 1 & 0 & -b/d \\ 0 & 1 & -c/d \end{pmatrix},$$
and
$$C_k = C_L C_\sigma^k \cdots C_\sigma^{k-1}.$$

Note that
$$C_k = C_{k-1} C_\sigma^{k-1} = C_L C_\sigma^k.$$

We define
$$w = \frac{a^\sigma c}{b^{\sigma+1}} \quad \text{and} \quad z = \frac{b^\sigma d}{c^{\sigma+1}}.$$

We note the following identities:
$$wz = \frac{a^\sigma d}{bc^\sigma} \quad w^\sigma z^{\sigma+1} = \frac{a^{\sigma^2} d^{\sigma+1}}{c^{\sigma^2 + \sigma + 1}}.$$

Let us define
$$X = \begin{pmatrix} a/c & 0 & 0 \\ 0 & b/c & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad Z_k = \begin{pmatrix} (c/d)^{[k-2]} & 0 & 0 \\ 0 & (c/d)^{[k-1]} & 0 \\ 0 & 0 & (c/d)^{[k]} \end{pmatrix}$$
and define $Y_k$ by $C_k = X Y_k Z_k$. 

\[19\]
Lemma 2. $Y_k$ satisfies the recursions

$$Y_k = Y_{k-1} \begin{pmatrix} 0 & 0 & -w^\sigma z^{\sigma+1} \\ 1 & 0 & -z^\sigma \\ 0 & 1 & -1 \end{pmatrix}^{\sigma k-3}$$

$$Y_k = \begin{pmatrix} 0 & 0 & -1 \\ wz & 0 & -1 \\ 0 & z & -1 \end{pmatrix} Y_{k-1}^{\sigma}$$

Proof. The proof is analogous to Lemma 1. \qed

Thus $Y_k$ is a matrix each of whose entries is a polynomial in the two variables $w, z$. We will write the third column of $Y_k$ as $(F_k, G_k, H_k)^T$. In other words, we define $H_k$ to be the $(3, 3)$ entry of $Y_k$, with $H_0 = 1$, $H_1 = -1$ and $H_2 = 1 - z$.

It is clear from the first recursion that

$$Y_k = \begin{pmatrix} F_{k-2} & F_{k-1} & F_k \\ G_{k-2} & G_{k-1} & G_k \\ H_{k-2} & H_{k-1} & H_k \end{pmatrix},$$

and that

$$H_k + H_{k-1} + z^{\sigma k-2} H_{k-2} + w^{\sigma k-2} z^{\sigma k-2} H_{k-3} = 0.$$  

Furthermore, the second recursion gives us that $F_k = -H_{k-1}^\sigma$, $G_k = wzF_{k-1}^\sigma - H_{k-1}^\sigma$, and $H_k = zG_{k-1}^\sigma - H_{k-1}^\sigma$. Together this gives

$$H_k + H_{k-1}^\sigma + zH_{k-2}^\sigma + wz z^{\sigma+1} H_{k-3}^\sigma = 0.$$  

Note that if $k = n$, these imply that

$$H_n^\sigma - H_n = H_{n-1}^\sigma - H_{n-1}^\sigma.$$  

Therefore we can rewrite $Y_k$ in terms of the sequence $(H_k)$, as follows:

$$Y_k = \begin{pmatrix} -H_{k-3}^\sigma & -H_{k-3}^\sigma & -H_{k-3}^\sigma \\ -wz H_{k-4}^\sigma - H_{k-3}^\sigma & -wz H_{k-3}^\sigma - H_{k-2}^\sigma & -wz H_{k-2}^\sigma - H_{k-1}^\sigma \\ H_{k-2}^\sigma & H_{k-1}^\sigma & H_k \end{pmatrix}$$

Thus, as $C_n = A_L$ by definition, we have $A_L = XY_n Z_n$, which shows the following.
Furthermore, the coefficients of the characteristic polynomial \( A \) as defined above. \( A \) is given by \( A_L \).

\[
A_L = N(c/d) \begin{pmatrix}
H_n^{\sigma-2} + H_{n-1}^{\sigma-2} + z^{\sigma-2} H_{n-2} & -(a/b)z^{\sigma-1} H_{n-2} & -(a/c)H_n^{\sigma-1} \\
(d/c)z^{\sigma-2}(H_{n-1}^{\sigma-1} + H_{n-2}) & H_n^{\sigma-1} + H_{n-1} & -(b/c)(wzH_n^{\sigma-2} + H_{n-1}^{\sigma-1}) \\
(d/c)^{\sigma-1} + (d/c)^{\sigma-2} H_{n-2} & (d/c)^{\sigma-1} H_{n-1} & H_n 
\end{pmatrix}.
\]

Furthermore, the coefficients of the characteristic polynomial \( \chi = x^3 - \lambda_2 x^2 + \lambda_1 x - \chi_0 \) of \( A_L \) are given by

\[
\chi_0 = \det(A_L) = N(-a/d) = N(c/d)^3 N(-w) N(z)^2,
\]

and

\[
\chi_1 = N(c/d)^2(a_1 + a_2 + a_3)
\]

where

\[
a_1 = w^\sigma z^{\sigma+1}(H_{n-1}^{\sigma^3} + H_{n-2}^{\sigma^3} + z^{\sigma} H_{n-2}^{\sigma^3} + w^2 z^{\sigma+2} H_{n-3}^{\sigma^3})
\]

\[
a_2 = w^\sigma z^{\sigma+1}(H_{n}^{\sigma^2} H_{n-3}^{\sigma^3} + H_{n-1}^{\sigma^2} H_{n-2}^{\sigma^3})
\]

\[
a_3 = w^\sigma z^{\sigma+2}(H_{n}^{\sigma^2} H_{n-3}^{\sigma^4} + H_{n-1}^{\sigma^2} H_{n-2}^{\sigma^4}) + z^{\sigma}(H_n^{\sigma^2} H_{n-2}^{\sigma^3} + H_{n-1}^{\sigma^2} H_{n-2}^{\sigma^3})
\]

and

\[
\chi_2 = \text{Tr}(A_L) = N(c/d)(H_n^{\sigma^2} + H_n^2 + H_n + H_{n-1}^{\sigma^2} + H_{n-1}^2 + z H_{n-2}^{\sigma^2}).
\]

**Proof.** It can be easily verified using the matrix \( Y_k \) that \( A_L = XY_n Z_n \) has the given form. The proof of the coefficients \( \chi_1 \) and \( \chi_2 \) then follows by direct calculation, taking into account the two recursions and the fact that each coefficient is in \( \mathbb{F}_q \) by Theorem 4. For example, we have taken the trace of the matrix to the power of \( \sigma^2 \).

The constant term \( \chi_0 \) is equal to the determinant of \( A_L \), which is equal to the norm of the determinant of \( C_L \) (see Remark 4) and the determinant of \( C_L \) is \(-a/d\). \( \square \)

### 5.2 Criteria for number of roots of \( P_L \)

**Theorem 11.** If \( L = ax + bx^\sigma + cx^{\sigma^2} + dx^{\sigma^3} \) with \( d \neq 0 \), then the number of roots of \( P_L = a + bx + cx^{\sigma+1} + dx^{\sigma^2+\sigma+1} \) in \( \mathbb{F}_q \) is given by the eigenvalues and eigenvectors of \( A_L \), as follows.
Let $a$ denote the algebraic multiplicity and let $g$ denote the geometric multiplicity of an eigenvalue of $A_L$.

- $P_L$ has $q^2 + q + 1$ roots in $\mathbb{F}_{q^n}$ if and only if $A_L$ has one eigenvalue in $\mathbb{F}_q$ with multiplicity pair $(a, g) = (3, 3)$.
- $P_L$ has $q + 1$ roots in $\mathbb{F}_{q^n}$ if and only if $A_L$ has one eigenvalue in $\mathbb{F}_q$ with multiplicity pair $(a, g) = (3, 2)$.
- $P_L$ has 1 root in $\mathbb{F}_{q^n}$ if and only if $A_L$ has one eigenvalue in $\mathbb{F}_q$ with multiplicity pair $(a, g) = (3, 1)$.
- $P_L$ has $q + 2$ roots in $\mathbb{F}_{q^n}$ if and only if $A_L$ has one eigenvalue in $\mathbb{F}_q$ with multiplicity pair $(a, g) = (1, 1)$ and one eigenvalue in $\mathbb{F}_q$ with multiplicity pair $(a, g) = (2, 2)$.
- $P_L$ has 2 roots in $\mathbb{F}_{q^n}$ if and only if $A_L$ has one eigenvalue in $\mathbb{F}_q$ with multiplicity pair $(a, g) = (1, 1)$ and one eigenvalue in $\mathbb{F}_q$ with multiplicity pair $(a, g) = (2, 1)$.
- $P_L$ has 3 roots in $\mathbb{F}_{q^n}$ if and only if $A_L$ has three distinct eigenvalues in $\mathbb{F}_q$ each with multiplicity pair $(a, g) = (1, 1)$.
- $P_L$ has 0 roots in $\mathbb{F}_{q^n}$ if and only if $A_L$ has no eigenvalues in $\mathbb{F}_q$.

There are no other possibilities.

**Proof.** The follows from Theorem 6 and consideration of all possible eigenvalue-eigenvector behaviours. \(\square\)

Exact criteria for each case can be found in a similar way to the $\sigma$-degree 2 case. As an example, we give the exact criteria for the first case, when $P_L$ has the maximum number of roots in $\mathbb{F}_{q^n}$.

**Theorem 12.** If $L = ax + bx^\sigma + cx^\sigma^2 + dx^\sigma^3$ with $d \neq 0$, then $P_L = a + bx + cx^\sigma + 1 + dx^\sigma^2 + \sigma + 1$ has $q^2 + q + 1$ roots if and only if $H_{n-1} = H_{n-2} = 0$ and $H_n \in \mathbb{F}_q$.

**Proof.** By Theorem 7, $P_L$ has $q^2 + q + 1$ roots if and only if $A_L$ is a scalar multiple of the identity. If $A_L$ is a scalar multiple of the identity, then $H_{n-1} = H_{n-2} = 0$. Using the
second recursion, this implies that \(-wzH_{n-4}^{2} - H_{n-3}^{2} = 0\) also. Hence \(Y_n\) is a diagonal matrix if and only if \(H_{n-1} = H_{n-2} = 0\).

Suppose now \(H_{n-1} = H_{n-2} = 0\). Then

\[
H_{n} + w^{n-2}z^{n-2} + w^{n-3}H_{n-3} = 0 \\
H_{n} + w^{n}z^{1}H_{n-3} = 0.
\]

Thus we get that \(H_{n} = H_{n}^{2}\), so \(H_{n} \in \mathbb{F}_{q}\), and by Proposition 10 we get

\[
A_L = (c/d)^{[n]} \begin{pmatrix}
H_{n}^{2} & 0 & 0 \\
0 & H_{n}^{2} & 0 \\
0 & 0 & H_{n}
\end{pmatrix}.
\]

Hence \(C_n\) is a scalar multiple of the identity if and only if \(H_{n} \in \mathbb{F}_{q}\).

5.3 Criteria for number of roots of \(L\)

**Theorem 13.** For \(g \in \{0, 1, 2, 3\}\), if \(L = ax + bx^\sigma + cx^\sigma^2 + dx^\sigma^3\) with \(d \neq 0\), then \(L\) has \(q^g\) roots in \(\mathbb{F}_{q^n}\) if and only if \(1\) is an eigenvalue of \(A_L\) with geometric multiplicity \(g\).

**Proof.** The follows from Theorem 6.

One can calculate exact criteria for each case \(g = 0, 1, 2, 3\) from the matrix \(A_L\). As an example, we state the \(g = 3\) and \(g = 0\) cases here.

**Theorem 14.** If \(L = ax + bx^\sigma + cx^\sigma^2 + dx^\sigma^3\) with \(d \neq 0\), then \(L\) has \(q^3\) roots if and only if \(H_{n-1} = H_{n-2} = 0\) and \(H_{n}N(c/d) = 1\).

**Proof.** By the proof of Theorem 12 \(A_L\) is a scalar multiple of the identity if and only if \(H_{n-1} = H_{n-2} = 0\) and \(H_{n} \in \mathbb{F}_{q}\), in which case \(A_L\) is equal to the identity if and only if \(H_{n}N(c/d) = 1\).

**Theorem 15.** If \(L = ax + bx^\sigma + cx^\sigma^2 + dx^\sigma^3\) with \(d \neq 0\), then \(L\) is a permutation polynomial if and only if \(1 - \chi_2 + \chi_1 - \chi_0 \neq 0\), where \(\chi_i\) are given in Theorem 10.
Proof. \( L \) is a permutation polynomial if and only if 0 is its only root. This occurs if and only if 1 is not an eigenvalue of \( A_L \). The given expression is just the evaluation of the characteristic polynomial of \( A_L \) at 1, and so the result follows.

5.4 Example for \( n = 5 \)

Using the definitions and recursions, we see that the sequence \( (H_k)_k \) begins as follows.

| \( k \) | \( H_k \) |
|------|--------|
| 0    | 1      |
| 1    | −1     |
| 2    | 1 − \( z \) |
| 3    | \( z^\sigma + z - 1 - w^\sigma z^{1+\sigma} \) |
| 4    | 1 − \( z - z^\sigma - z^\sigma^2 + w^\sigma z^{\sigma+\sigma^2} + z^{1+\sigma^2} + w^\sigma z^{1+\sigma} \) |

Thus for a linearized or projective polynomial of \( \sigma \)-degree 3 to have maximum number of roots in \( \mathbb{F}_{q^5} \), we require that \( H_3 = H_4 = 0 \). Thus after some simple algebra we have that this occurs precisely when \( z^{\sigma^2+1} + z^\sigma - 1 = 0 \) and \( z^\sigma + z - 1 - w^\sigma z^{1+\sigma} = 0 \). When both of these are satisfied, we have \( H_5 = (1 − z − z^\sigma)(1 − z)^{\sigma^3} \). This immediately gives the following.

**Theorem 16.** Suppose \( L = ax + bx^\sigma + cx^{\sigma^2} + dx^{\sigma^3} \) has coefficients in \( \mathbb{F}_{q^5} \). Then \( P_L = a + bx + cx^{\sigma+1} + dx^{\sigma^2+\sigma+1} \) has \( q^2 + q + 1 \) roots in \( \mathbb{F}_{q^5} \) if and only if \( z^{\sigma^2+1} + z^\sigma - 1 = 0 \) and \( z^\sigma + z - 1 - w^\sigma z^{1+\sigma} = 0 \).

Also, \( L \) has \( q^3 \) roots in \( \mathbb{F}_{q^5} \) if and only if \( z^{\sigma^2+1} + z^\sigma - 1 = 0, z^\sigma + z - 1 - w^\sigma z^{1+\sigma} = 0, \) and \( N(z)N(c/d) = −1 \).

6 Applications

In this section we outline a few topics where our results could be applied.
6.1 MRD codes from Linearized Polynomials

A $\sigma$-linearized polynomial naturally defines an $\mathbb{F}_q$-linear map from $\mathbb{F}_{q^n}$ to itself. Conversely, for every such map there exist a unique $\sigma$-linearized polynomial of degree at most $n - 1$ representing it. By Proposition 1 every nonzero element of $L_{k-1}$ has rank at least $n - k + 1$, and $L_{k-1}$ has dimension $nk$ over $\mathbb{F}_q$. Such a set of maps is called a maximum rank-distance code (MRD code) and optimal with respect to dimension for a given minimum rank. This construction is known as a (generalised) Gabidulin code, though the construction with $a^\sigma = a^q$ is due to Delsarte. We refer to [13] for background.

The following result [10, Theorem 10] has been used in for example [13] and [15] to construct new families of MRD codes.

**Lemma 3.** If the nullity of a $\sigma$-polynomial $f(x)$ of $\sigma$-degree $k$ is equal to $k$, then $N(f_0) = (-1)^{nk}N(f_k)$.

Here $N(a) = a^1 + a^\sigma + \cdots + a^{n-1}$ denotes the field-norm from $\mathbb{F}_{q^n}$ to $\mathbb{F}_q$.

Further constructions have been obtained in the case $n = 6, 8$ [3], [5], again using properties of $\sigma$-linearized polynomials.

In order to improve on these results, or to classify such objects, more exact criteria for the rank of a $\sigma$-linearized polynomial are required. This is one of the motivations for this paper.

6.2 Hasse-Witt matrices

Let $C$ be a smooth projective curve of genus $g > 0$ that is defined over $\mathbb{F}_q$. The Hasse-Witt matrix $H$ represents the action of the Frobenius operator on the cohomology group $H^1(C, \mathcal{O}_C)$ where $\mathcal{O}_C$ is the structure sheaf. The Frobenius operator is a semilinear map analogous to $\psi$ in Proposition 2, and its $n$-th power is a linear map analogous to $\phi$ in Proposition 2.

The Hasse-Witt invariant of the curve $C$, also known as the $p$-rank of the curve, is the...
rank of the matrix

\[ HH^\sigma H^{\sigma^2} \cdots H^{\sigma^{n-1}} \]

for any \( n \geq g \). The similarity with our matrix \( A_L \) is clear. The Hasse-Witt invariant is therefore also equal to the rank of the Frobenius operator composed with itself \( g \) times.

There are papers in the literature that calculate the Hasse-Witt invariant for certain types of curves, by finding the rank of the matrix \( HH^\sigma H^{\sigma^2} \cdots H^{\sigma^{n-1}} \). Finding the rank and characteristic polynomial of these matrices therefore has applications in algebraic geometry. The results and methods of this paper are applicable to any curve whose Hasse-Witt matrix has the form of \( C_L \) in Section 2.3 i.e., the form of a companion matrix.

### 6.3 Cryptography

Recent work on the Discrete Logarithm Problem (DLP) relies on a supply of projective polynomials that split completely, i.e., they have all their roots in the ground field (see [9]). The projective polynomials \( x^{q+1} + Bx + B \) were used in setting new world records for the DLP because of the high number of \( B \)'s that exist where the polynomial splits completely.

This can be seen as a special case of our \( d = 2 \) results in Section 4 Theorem 8 which give precise if and only if conditions for a projective polynomial \( P_L(x) = a + bx + cx^{\sigma+1} \in \mathbb{F}_{q^n}[x] \) to split completely.

Moreover, Theorem 12 presents precise if and only if conditions for a projective polynomial \( P_L(x) = a + bx + cx^{\sigma+1} + dx^{\sigma^2+\sigma+1} \in \mathbb{F}_{q^n}[x] \) to split completely. It is possible that these projective polynomials could be used to speed up attacks on the DLP.

### 7 Comment

Some results of this paper, in particular the second statement of Theorem 7 were simultaneously and independently obtained by the authors of [4]. The results of this paper and those of [4] were each presented at the conference Combinatorics 2018 on June 7th 2018.
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