Super-critical and sub-critical Hopf bifurcations in two and three dimensions

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Abstract Hopf bifurcations have been studied perturbatively under two broad headings, viz., super-critical and sub-critical. The criteria for occurrences of such bifurcations have been investigated using the renormalization group. The procedure has been described in detail for both two and three dimensions and has been applied to several important models, including those by Lorenz and Rossler.

Keywords Renormalization group · Hopf bifurcations · Lorenz model · Rossler model

1 Introduction

Hopf bifurcations, introduced quantitatively in the next section, have played a pivotal role in the development of the theory of dynamical systems in different dimensions [1–10]. The uniqueness of such bifurcations lies in two aspects: unlike other common types of bifurcations (viz., pitchfork, saddle-node, or transcritical) Hopf bifurcations cannot occur in one dimension. The minimum dimensionality has to be two. The other aspect is that Hopf bifurcations deal with birth or death of a limit cycle (LC) when a LC emanates from or shrinks onto a fixed point, the focus. Thus, unlike the other kinds of bifurcations which mostly deal with stability properties of (fixed) points, Hopf bifurcations deal with points as well as isolated phase orbits, the LCs. Two types of Hopf bifurcations are common and they go under the broad headings: super-critical (forward) and sub-critical (backward) [4–8,11,12] (defined in Sect. 2). In this paper the criteria for occurrences of these two types of Hopf bifurcations have been studied using renormalization group (RG) [13–23], the operational aspects of which have been elaborately explained both for two and three dimensions. There exists a criterion [8,23] that deals with such aspects in two-dimensional dynamical systems. However for dimension greater than two, apart from the work of Martin and Mclaughlin [24] for the Lorenz model, all other prescriptions for finding the forward or backward nature of Hopf bifurcations in dynamical systems rely on the construction of the Poincare map [25]. Here we propose a systematic and analytic perturbative method involving the RG developed by Chen et al. [15] a couple of decades back. We show in this paper that this method can be used to address the issue of forward or backward Hopf bifurcations in three dimensions if at the instability point the eigenvalues are all negative except for a pair of imaginary numbers.

The paper has been organized as follows: in Sect. 2 we rederive the well-known criterion that is commonly used to discriminate super-critical and sub-critical Hopf bifurcations, using the RG. We also explain through an example how the predictions made by the
RG have certain advantages. In Sect. 3 we develop the RG-procedure for three dimensions. Sections 4 and 5 are devoted to detailed analyses of Hopf bifurcations in the Lorenz and Rossler models [23,26–28], respectively, where the formalism developed in Sect. 3 has been extensively applied. The paper has been summarized in Sect. 6.

2 RG in 2D Hopf-bifurcations

In this section we first introduce Hopf-bifurcations briefly for two-dimensional dynamical systems followed by a detailed analysis of how the amplitude equation (derived from the RG) can be used to understand its super-critical or sub-critical nature.

A 2D-dynamical system, which in polar form looks like

\[ \dot{r} = \xi r - \lambda r^3 \]

\[ \dot{\theta} = \omega \] (1)

undergoes Hopf-bifurcation when the co-efficient of the linear term of Eq. (1), i.e., \( \xi \), becomes zero. The bifurcation is super critical if \( \lambda > 0 \) and subcritical if \( \lambda < 0 \). When \( \xi > 0 \) the origin is an unstable spiral. For an arbitrary two-dimensional system it is non-trivial to establish whether a Hopf bifurcation is forward or backward. There is a well-established criterion [23] that decides which way the system will go and the method used to arrive at it uses center manifold theory. Here we shall see how the RG comes to our help in deciding whether for a generalized 2D system undergoing Hopf bifurcation, it will be super critical or sub critical. The result that we arrive at uses perturbation theory. Hence, this section serves as a good rehearsing ground for applying the RG-technique, which has been employed to study Hopf bifurcations in 3D in the next section. Therefore let us start out with a time-scaled \((\tau = \omega t)\) 2D-dynamical system [8],

\[ \dot{x} = \xi x - y + \lambda f(x, y), \]

\[ \dot{y} = x + \xi y + \lambda g(x, y), \] (3)

where to effect a perturbation analysis, we have taken the nonlinear parts included in the functions \( f(x, y) \) and \( g(x, y) \) as small, \( \lambda \) being the perturbation parameter. Here \( \xi \) is the bifurcation parameter so that \( \xi = 0 \) is the point of Hopf bifurcation. The polynomial structures of these nonlinear functions can be written as

\[ f(x, y) = \sum_{i,j} f_{ij} x^i y^j \quad (i + j \geq 2) \] (5)

and

\[ g(x, y) = \sum_{i,j} g_{ij} x^i y^j \quad (i + j \geq 2). \] (6)

Differentiating Eq. (3) with respect to time, we get

\[ \ddot{x} = \xi \dot{x} - \dot{y} + \lambda \left[ \dot{x} \frac{df}{dx} + \dot{y} \frac{df}{dy} \right] \]

\[ \Rightarrow \ddot{x} + (1 - \xi^2)x = -2\mu y \]

\[ + \lambda \left[ \xi f(x, y) - g(x, y) + \xi x \frac{df}{dx} \right. \]

\[ - y \frac{df}{dx} + x \frac{df}{dy} - \xi y \frac{df}{dy} \]

\[ + \lambda^2 \left[ f(x, y) \frac{df}{dx} + g(x, y) \frac{df}{dy} \right]. \] (7)

We see that Hopf bifurcation occurs right at the point \( \xi = 0 \) and that the origin is unstable for \( \xi > 0 \). To analyze the role of the lowest nonlinear term in driving the system at that point, we put \( \xi = 0 \) in the above equation to obtain

\[ \ddot{x} + x = \lambda \left[ -g(x, y) - y \frac{df}{dx} + x \frac{df}{dy} \right]. \] (8)

Here \( \lambda \) being a perturbation parameter we can expand \( x \) and \( y \) perturbatively as

\[ x = x_0 + \lambda x_1 + \lambda^2 x_2 + \cdots \]

\[ y = y_0 + \lambda y_1 + \lambda^2 y_2 + \cdots. \] (9)

The RG-technique, which we apply here to derive the amplitude and phase equations has been discussed in details in [13,14,21]. The central idea lies in ‘cutting-off’ the secular divergences arising from integration of the resonant terms, by introducing a flexible origin of the time scale. This flexibility in the choice of the origin leads to the RG-flow equations, which appear in the guise of the amplitude and phase equations of the problem. The result is that at the \( n \)th order of perturbation, the equation

\[ \dot{x}_n + \omega^2 x_n = P_n(a) \sin(\omega t + \theta) + Q_n(a) \cos(\omega t + \theta) \]

\[ + \text{other regular (non-resonant) terms of lower orders in perturbation,} \] (11)

where \( P_n(a) \) and \( Q_n(a) \) are functions of the amplitude \( a \), leads to the amplitude and phase equations as,

\[ \frac{da}{dt} = -\frac{\lambda^n P_n}{2\omega} + \text{lower order terms in } \lambda, \] (12)
\[
\frac{d\theta}{dr} = -\frac{\lambda^0 Q \theta}{2a_0^0} + \text{lower order terms in } \lambda. \tag{13}
\]

This result may seem similar to that derived by standard perturbative techniques like averaging or multipletime-scale analysis, there are subtle differences [15] between these methods and the RG, which, however, will not concern us in the discussions to follow. Our objective here will be to write the amplitude equation for Eq. (8) up to a relevant order of perturbation so that we can understand the role of the \(a^3\)-term (lowest nonlinear power of \(a\)), in governing the dynamics. By ‘relevant order’ we mean, that, beyond that order of perturbation there cannot be any \(a^3\)-term, in the amplitude equation. Therefore, in what follows, our quest will be to identify the \(a^3 \sin(t + \theta)\) terms from the RHS of Eq. (8). With the lowest power of \(x\) and \(y\) in \(f(x, y)\) and \(g(x, y)\) as 2, it is to understand that third and higher orders of perturbation will not contain \(a^3\)-terms. That is why perturbative calculations up to second order suffice our purpose. Here we state the main expressions only and have shown all the steps in the Appendix.

Returning to Eq. (8), we can Taylor-expand the functions on the RHS by involving the perturbation expansions Eqs. (13) and (14) up to order \(O(\lambda^2)\) to get the following equations:

\[
\begin{align*}
\ddot{x}_0 + x_0 &= 0, \tag{14} \\
\ddot{x}_1 + x_1 &= -g(x_0, y_0) - y_0 \frac{df}{dx}(x_0, y_0) \\
&+ x_0 \frac{dy}{dx}(x_0, y_0), \tag{15} \\
\ddot{x}_2 + x_2 &= -x_1 \frac{dg}{dx}(x_0, y_0) + x_1 \frac{dy}{dx}(x_0, y_0) \\
&- y_1 \frac{dg}{dy}(x_0, y_0) - y_1 \frac{df}{dy}(x_0, y_0) \\
&+ x_0 x_1 \frac{\partial^2 f}{\partial x^2}(x_0, y_0) - y_0 y_1 \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) \\
&+ x_0 x_1 \frac{\partial^2 g}{\partial x \partial y}(x_0, y_0) + x_0 y_1 \frac{\partial^2 g}{\partial y^2}(x_0, y_0) \\
&+ f(x_0, y_0) \frac{\partial f}{\partial x}(x_0, y_0) + g(x_0, y_0) \frac{\partial f}{\partial y}(x_0, y_0). \tag{16}
\end{align*}
\]

First-order calculations lead to

\[
\begin{align*}
\ddot{x}_1 + x_1 &= \text{regular terms} - \frac{a^3}{8}(g_{xx} + g_{yy}) \\
&= f_{xx} + f_{yy} \sin \phi. \tag{19}
\end{align*}
\]

In the second order calculations (Eq. 16) we have to know the renormalized expressions of \(x_1\) and \(y_1\). For this, it is important to realize from Eq. (15) and from \(y_1 = -\dot{x}_1 + f(x_0, y_0)\) that the lowest power of \(a\) in \(x_1\) and \(y_1\) is 2. Therefore the regular (non-secular) \(a^2\)-terms on the RHS of Eq. (15) have to be identified in order to get the \(a^3\)-terms of Eq. (16) that contribute to the amplitude equation. Considerations of this kind lead to the following (details are done in the Appendix):

\[
\begin{align*}
x_1 &= -a^2 \left[ \frac{1}{4}(g_{xx} + g_{yy}) + \frac{1}{12}(4f_{xy} - g_{xx} + g_{yy}) \\
&\times \cos 2\phi + \frac{1}{6}(-f_{xx} + f_{yy} - g_{xy}) \sin 2\phi \right] \\
&+ \text{higher powers of } a. \tag{21}
\end{align*}
\]

Accordingly, from Eq. (20), we get the renormalized \(y\) as

\[
\begin{align*}
y_1 &= a^2 \left[ \frac{1}{4}(f_{xx} + f_{yy}) + \frac{1}{12}(-f_{xx} + f_{yy} - 4g_{xy}) \\
&\times \cos 2\phi + \frac{1}{6}(-f_{xy} + g_{xx} - g_{yy}) \sin 2\phi \right] \\
&+ \text{higher powers of } a. \tag{22}
\end{align*}
\]

This leads to

\[
\begin{align*}
\ddot{x}_2 + x_2 &= \frac{a^3}{8} \sin \phi \left[ g_{xy}(g_{xx} + g_{yy}) - f_{xy}(f_{xx} + f_{yy}) \\
&+ (f_{xx} g_{xx} - f_{yy} g_{yy}) \right] + \text{regular terms.} \tag{23}
\end{align*}
\]

Finally, using the general result of Eq. (12), we get the amplitude equation up to second order as [combining Eqs. (19) and (23)],

\[
\frac{da}{dt} = \frac{a^3}{16} \left[ \lambda(f_{xxx} + f_{xxy} + g_{xxy} + g_{yy}) \\
+ \lambda^2(f_{xy}(f_{xx} + f_{xy}) - g_{xy}(g_{xx} + g_{yy}) \\
- f_{xx}g_{xx} + f_{yy}g_{yy}) \right] + \text{higher powers of } a. \tag{24}
\]

The sign of the quantity within the [ ] brackets dictates the dynamics right at the point of the Hopf bifurcation \([\xi = 0\) in Eqs. (3) and (4)]. If the sign of this quantity be negative, then the nonlinear amplitude term of lowest
power (here \(a^3\)) drives the system towards the origin and we get a super-critical Hopf bifurcation. On the contrary, when the sign is positive, this nonlinear term drives the system away from the origin which is the case of sub-critical Hopf bifurcation. As an example, let us consider the Van der Pol equation,

\[
\begin{align*}
\dot{x} &= -y, \\
\dot{y} &= x - (x^2 - 1)y.
\end{align*}
\]

(25) (26)

This is a Lienard system where the function \(f(x, y)\) (Eq. 3) is zero (and also \(\mu = 0\)). The criterion of Eq. (24) (with \(\lambda = 1\)) evaluates to

\[
\frac{da}{dr} = \frac{a^3}{16}(-1)
\]

(27)

with all the partial derivatives evaluated at \(x = y = 0\). This implies supercritical Hopf bifurcation at \(\xi = 0\) which means, the lowest-power nonlinear term drives the system towards the origin.

Now, let us apply the same principle to the oscillator

\[
\begin{align*}
\dot{x} &= -y, \\
\dot{y} &= x - (x^2 - \alpha)(x^2 - \beta)y^3.
\end{align*}
\]

(28) (29)

In this case, the criterion of Eq. (24) tells us that

\[
\frac{da}{dr} = -\frac{a\beta}{16}
\]

(30)

which means that for \((\alpha, \beta)\) of the same sign, the bifurcation will be supercritical and for \((\alpha, \beta)\) of opposite signs it will be subcritical. But, this is just the reverse to what one sees in the numerical phase plot. Numerically, it is seen that when \((\alpha, \beta)\) both are positive then there is a stable origin girdled by an unstable LC which in turn is surrounded by a stable LC. As \(\beta\) is made zero, the inner LC gradually engulfs the stable origin rendering it unstable, which is clearly a case of subcritical Hopf bifurcation, as opposed to the super-critical case predicted by Eq. (24). This is because the fixed point \(x = y = 0\) is not an unstable spiral but a center for all \(\alpha\) and \(\beta\) and hence Eq. (24) is not applicable. Perturbative RG can still be employed by combining Eq. (28) and Eq. (29) as

\[
\ddot{x} + x = -\lambda(x^2 - \alpha)(x^2 - \beta)\dot{x}^3,
\]

(31)

(where \(\lambda\) is a perturbation parameter) and then, just by identifying the coefficients of \(\sin(t + \theta)\) from the RHS of Eq. (31), we arrive at the amplitude equation

\[
\frac{da}{dr} = -\frac{a^3}{128} \left[ 48\alpha\beta - 8(\alpha + \beta)a^2 + 3a^4 \right]
\]

(32)

\[
\frac{1}{x} \left( \frac{d}{dt} \right)^2 + a^2 + \alpha x^2 + \beta x^2 = 0
\]

(33)

which has the fixed points at \(a = 0\) and

\[
a = \left[ \frac{4}{3} \left( (\alpha + \beta) \pm \sqrt{(\alpha + \beta)^2 - 9\alpha\beta} \right) \right]^{\frac{1}{2}}.
\]

From Eqs. (32) and (33), it is easy to see that for \((\alpha, \beta > 0)\), the origin is stable surrounded by two LCs, inner unstable and outer stable. As \(\beta \to 0\), the system undergoes a subcritical Hopf bifurcation corroborated accurately by the plot of Figs. 1 and 2. With \(\alpha > 0\) and \(\beta < 0\), Eqs. (32) and (33) tell us that there is an unstable...
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Fig. 3 Phase plots of the system $\ddot{x} + x + x^3(x^2 - \alpha)(x^2 - \beta)$ with $\beta = -1$, $\alpha = -0.1$ makes the origin unstable and gives one stable LC at $a = 0.72$, which shrinks to 0 at $\alpha = 0$ in a supercritical Hopf bifurcation. The locations of the respective LCs are found exactly as predicted by the fixed points of the amplitude equation (Eq. 65)

Fig. 4 Phase plots of the system $\ddot{x} + x + x^3(x^2 - \alpha)(x^2 - \beta)$ with $\alpha = 0.1$ giving a stable origin with no LC around. The locations of the respective LCs are found exactly as predicted by the fixed points of the amplitude equation (Eq. 65)

3 Renormalization group in three dimensions

In this section we explicitly show how the RG-procedure works in 3D. Before going into the specific systems (Lorenz and Rossler models) let us consider a differential equation of the form

$$f(D)u = R \cos \omega t + S \sin \omega t,$$

where $f(D)$ is some cubic polynomial of the differential operator $D \equiv \frac{d}{dt}$ and is factorizable as

$$f(D) = (D^2 + \omega^2)(D + \alpha).$$

Here $\alpha$ is some number and $\omega$ is the same frequency that occurs in the resonant terms on the RHS of Eq. (34). As we shall see in the next two sections, that a differential equation of the form of Eq. (34) emerges naturally in the study of dynamical systems like the Lorenz or Rossler attractors. On integrating Eq. (34) we get at the first stage,

$$(D^2 + \omega^2)u = \left[ \frac{R}{-\omega^2 - \alpha^2} \cos \omega t + \frac{S}{-\omega^2 - \alpha^2} \sin \omega t \right]$$

$$= \left[ P \cos \omega t + Q \sin \omega t \right].$$

where in $P$ and $Q$, the co-efficients of $\cos \omega t$ and $\sin \omega t$ from the RHS of Eq. (34) get mixed up (unlike the 2D case) as,

$$P = (R\alpha + S\omega)/(\omega^2 + \alpha^2),$$

$$Q = (R\omega + S\alpha)/(\omega^2 + \alpha^2).$$

Another integration yields

$$u = \frac{1}{4\omega^2} \left[ P \cos \omega t + Q \sin \omega t \right]$$

$$+ \frac{t}{2\omega} \left[ P \sin \omega t - Q \cos \omega t \right].$$

The $t$-divergences in the last two terms of Eq. (39) are physically unacceptable in a perturbation theory where $u$ plays the role of some $x_n$ in a perturbative expansion of the form of Eq. (9), (viz., $x = x_0 + \lambda x_1 + \lambda^2 x_2 + \cdots$) built around some purely oscillatory form of $x_0$ given by $x_0 = A \cos \omega t + B \sin \omega t$. Then the RG-method consists of expanding $A$ and $B$ perturbatively and defining renormalization constants $Z_n^A$ and $Z_n^B$ so as to cut-off the secular divergences order by order. Thus,

$$A = A(\mu) \left[ 1 + \lambda Z_1^A + \lambda^2 Z_2^A + \cdots \right],$$

$$B = B(\mu) \left[ 1 + \lambda Z_1^B + \lambda^2 Z_2^B + \cdots \right],$$

where $\mu$ is a new arbitrary time origin introduced to sieve out a regularized part from the $t$-divergent terms of Eq. (39) as, $t \sin \omega t = (t - \mu + \mu) \sin \omega t$, and a similar split-up for the $t \cos \omega t$ term. If we are working at the $n$th order of perturbation, then this split up along with Eqs. (40) and (41) allows us to define the renormalization constants in such a way as to nullify the divergent $\mu \sin \omega t$ and $\mu \cos \omega t$ terms order by order. Thus, at the $n$th order of perturbation we define

$$Z_n^A = \frac{\mu}{A(\mu)} \frac{Q}{2\omega}$$

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and
\[ Z_n^{(b)} = -\frac{\mu}{B(\mu)} \frac{P}{2\omega}, \]
where, it is presumed that the divergences up to \((n - 1)\)th order of perturbation have already been similarly renormalized.

This leaves us with the result
\[ u = A(\mu) \cos \omega t + B(\mu) \sin \omega t \]
+ renormalized lower order terms
+ \(\lambda^n \left( \frac{t - \mu}{2\omega} \frac{P}{\sin \omega t} - (t - \mu) \frac{Q}{2\omega} \cos \omega t \right) \]
+ other non-divergent terms.

Now, since the time origin \(\mu\) was chosen as arbitrary, therefore the dynamics should be independent of \(\mu\). Thus,
\[
\frac{da}{d\mu} = 0 = \frac{dA}{d\mu} \cos \omega t + \frac{dB}{d\mu} \sin \omega t
+ \lambda^n \left( -\frac{P}{2\omega} \sin \omega t + \frac{Q}{2\omega} \cos \omega t \right)
\]
whence, equating the co-efficients of \(\cos \omega t\) and \(\sin \omega t\) terms give the RG-flow equations at the \(n\)th order as
\[
\frac{dA}{d\mu} = -\lambda^n \frac{Q}{2\omega} \quad \text{and lower order terms in } \lambda,
\]
\[
\frac{dB}{d\mu} = \lambda^n \frac{P}{2\omega} \quad \text{and lower order terms in } \lambda.
\]

Finally, exploiting the arbitrariness of \(\mu\), we substitute \(\mu = t\) in Eq. (44) to obtain the renormalized \(u(t)\) up to the \(n\)th order. Thus, the central results transpiring from the divergent terms of Eq. (39) are
\[
\frac{dA}{dt} = \lambda^n \frac{1}{2\omega} \left[ \text{coefficient of } t \cos \omega t \right],
\]
\[
+ \text{lower order terms in } \lambda.
\]
\[
\frac{dB}{dt} = \lambda^n \frac{1}{2\omega} \left[ \text{coefficient of } t \sin \omega t \right],
\]
\[
+ \text{lower order terms in } \lambda.
\]

We shall require these results in the following two sections devoted to the study of super-critical and sub-critical Hopf bifurcations in the Lorenz and Rossler systems.

### 4 Lorenz model

In this section we shall use the RG-method described in Sect. 3 to study super-critical and sub-critical Hopf bifurcations in one of the most important and historically famous 3D-dynamical systems, viz. the Lorenz attractor, described by the equations,
\[
\dot{x} = \sigma (-x + y),
\]
\[
\dot{y} = -xz + rx - y,
\]
\[
\dot{z} = xy - bz,
\]
where \(b, r, \sigma\) are controllable parameters, all positive. This system, has its fixed point at \(x_0 = y_0 = \sqrt{b(r - 1)}\) and \(z_0 = r - 1\). Therefore, shifting the origin to the fixed point we move to a new set of coordinates \(X = x - \alpha, Y = y - \alpha, \text{and } Z = z - (r - 1)\) where
\[
\alpha = \sqrt{b(r - 1)}.
\]
This leads to a new set of equations in the shifted coordinates as
\[
(D + \sigma)X = \sigma Y,
\]
\[
(D + 1)Y = -ZX + X - \alpha Z,
\]
\[
(D + b)Z = XY + \alpha X + \alpha Y,
\]
where, as before, \(D \equiv \frac{d}{dt}\). From this point, our focus will be to cast this system in the form of Eqs. (34) and (35) of the last section. In doing so, we note that Eq. (55) can be written as,
\[
Z = -\frac{1}{\alpha} ((D + 1)Y + ZX - X)
\]
which when placed in Eq. (56) yields
\[
(D + b)(D + 1)Y + ZX - X = -\alpha XY - \alpha^2 X - \alpha^2 Y.
\]
Substituting \(Y\) from Eq. (54) in Eq. (58) we have, after some algebra and rearrangements,
\[
D^3 X + (1 + \sigma + b)D^2 X + \omega_0^2 DX + \omega_0^2 (1 + \sigma + b)X
= -\sigma (D + b)XZ - \alpha \sigma XY + (r_0 - r)bX
+ 2\sigma b(r_0 - r)X,
\]
where
\[
r_0 = \frac{\sigma + b + 3}{\sigma - b - 1},
\]
\[
\omega_0^2 = \frac{2\sigma (1 + \sigma)}{\sigma - b - 1}.
\]
We note that the LHS of Eq. (59) is already in the form of that of Eq. (34). The operator on the LHS of Eq. (59) is factorizable as
\[
LX = (D^2 + \omega_0^2)(D + \sigma + b + 1)X.
\]
From this point to make progress, we take a perturbative approach by tagging the RHS of Eq. (59) with some perturbation parameter $\varepsilon$ and invoking the perturbative expansions in $X$, $Y$, and $Z$ as

$$X = X_0 + \varepsilon X_1 + \varepsilon^2 X_2 + \cdots .$$  \hspace{1cm} (63)

and similar expansions for $Y$ and $Z$. In the zeroth order the equation

$$LX_0 = 0$$  \hspace{1cm} (64)

yields three independent solutions: two trigonometric (oscillatory) and one exponentially decaying. Since we are only interested in long time behavior we omit the latter from our considerations and with appropriate choice of initial conditions, continue working with the solution

$$X_0 = A \cos \omega_0 t .$$  \hspace{1cm} (65)

Using this in Eq. (54) gives the zeroth order for $Y$ as

$$Y_0 = X_0 + \frac{1}{\sigma} \dot{X}_0$$

$$= A \cos \omega_0 t - \frac{\omega_0}{\sigma} A \sin \omega_0 t .$$  \hspace{1cm} (66)

Similarly, from the linear terms of Eq. (55) we get

$$Z_0 = \frac{1}{\alpha} (X_0 - Y_0 - \dot{Y}_0)$$

$$= \frac{1}{\alpha} \left[ \left( \frac{1}{\alpha} + 1 \right) \omega_0 A \sin \omega_0 t + \frac{\omega_0^2}{\sigma} \cos \omega_0 t \right] .$$  \hspace{1cm} (67)

The $XZ$ term in Eq. (55) is nonlinear and hence does not participate at this order of calculation. Its explicit presence in the RHS of Eq. (59) begins to be felt at the first order of perturbation, viz.,

$$LX_1 = -\sigma (D + b) X_0 Z_0 - \alpha \sigma X_0 Y_0 + b (r_0 - r) \dot{X}_0 + 2 \alpha b (r_0 - r) X_0 .$$  \hspace{1cm} (68)

On using the expressions for $X_0$, $Y_0$, and $Z_0$, Eq. (68) becomes,

$$LX_1 = -\frac{A^2}{2\alpha} (\alpha^2 \sigma + \omega_0^2 b)$$

$$-\frac{A^2}{2\alpha} \left[ 2 \omega_0^2 (1 + \sigma) + \alpha^2 \sigma + \omega_0^2 b \right] \cos \omega_0 t$$

$$+ \frac{A^2 \omega_0}{2\alpha} \left[ \alpha^2 - b (1 + \sigma) + 2 \omega_0^2 \right] \sin \omega_0 t$$

$$- 2 \sigma b \Delta A \cos \omega_0 t + b \omega_0 \Delta r A \sin \omega_0 t .$$  \hspace{1cm} (69)

In evaluating $X_1$ from the above equation, we note that the first three terms are regular while the last two terms are secular (resonant). On integrating the secular part, we get divergent terms as well as regular $\cos \omega_0 t$ and $\sin \omega_0 t$ terms (Eq. 39). Incidentally, these two regular terms do not spawn any further secular terms in the second order calculations which we come to shortly. Hence, for now, we work with the regular part of Eq. (69) and stack the secular ones to be dealt with along with second-order terms later. Introducing constants as

$$\beta = \sigma + b + 1 ,$$  \hspace{1cm} (70)

$$P_1 = -\frac{A^2}{2\alpha} (\alpha^2 \sigma + b \omega_0^2) ,$$  \hspace{1cm} (71)

$$P_2 = -\frac{A^2}{2\alpha} (\alpha^2 \sigma + 2 \omega_0^2 (1 + \sigma)) ,$$  \hspace{1cm} (72)

$$P_3 = \frac{A^2 \omega_0}{2\alpha} \left[ \alpha^2 - b (1 + \sigma) + 2 \omega_0^2 \right] .$$  \hspace{1cm} (73)

Equation (69) takes the form (with no secular terms in RHS)

$$(D^2 + \omega_0^2)(D + \beta) X_1 = P_1 + P_2 \cos 2 \omega_0 t + P_3 \sin 2 \omega_0 t .$$  \hspace{1cm} (74)

We have (on integration)

$$X_{1F} = -\frac{A^2}{2\alpha} \left( \frac{1}{2} + \frac{b}{\beta} \right) + \frac{2 \omega_0 P_3 - \beta P_2}{3 \omega_0^2 (4 \omega_0^2 + \beta^2)} \cos 2 \omega_0 t$$

$$- \frac{\beta P_3 + 2 \omega_0 P_2}{3 \omega_0^2 (4 \omega_0^2 + \beta^2)} \sin 2 \omega_0 t$$  \hspace{1cm} (75)

which, on using Eqs. (70)–(73) leads to

$$X_{1F} = \frac{A^2}{2\alpha} \left[ -\alpha_1 + \alpha_2 \cos 2 \omega_0 t + \alpha_3 \sin 2 \omega_0 t \right] ,$$  \hspace{1cm} (76)

where

$$\alpha_1 = \frac{1}{2} + \frac{b}{\beta} ,$$  \hspace{1cm} (77)

$$\alpha_2 = \frac{1}{4 \omega_0^2 + \beta^2} \left[ \frac{\beta^2}{2} + \frac{1}{3} (1 + \sigma) \beta + \frac{4}{3} \omega_0^2 \right]$$

$$+ \frac{2}{3} \omega_0 \frac{1 + b}{\sigma} ,$$  \hspace{1cm} (78)

$$\alpha_3 = \frac{\omega_0}{4 \omega_0^2 + \beta^2} \left[ \frac{\beta^2}{3} + \frac{1}{3} (1 + \sigma) - \frac{\beta (1 + b)}{3 \sigma} \right] .$$  \hspace{1cm} (79)

Here, by $X_{1F}$ we mean the finite part of $X_1$ coming from the regular terms of Eq. (69) only. Using this in Eq. (54), we get the finite part of $Y_1$ as

$$Y_{1F} = \frac{A^2}{2\alpha} \left[ -\alpha_1 + \beta_2 \cos 2 \omega_0 t + \beta_3 \sin 2 \omega_0 t \right] .$$  \hspace{1cm} (80)
where
\[ \beta_2 = \alpha_2 + \frac{2\alpha_0\alpha_3}{\sigma}, \]
and
\[ \beta_3 = \alpha_3 - \frac{2\alpha_0\alpha_2}{\sigma}. \]  
(81)  
(82)

Similarly, using Eqs. (76), (77) along with Eqs. (65) and (67) in Eq. (55) (this time, considering the nonlinear XZ term) we get
\[ Z_{1F} = \frac{1}{\alpha} [X_{1F} - (D + 1)Y_{1F} - X_0Z_0] \]
\[ = \frac{A^2}{2\alpha} \left[ -\gamma_1 + \gamma_2 \cos 2\omega_0 t + \gamma_3 \sin 2\omega_0 t \right] \]  
(83)

where
\[ \gamma_1 = \frac{\omega_0^2}{\sigma} \]  
(84)
\[ \gamma_2 = -\left( \frac{\omega_0^2}{\sigma} + \frac{2\omega_0\alpha_3}{\sigma} + \frac{2\alpha_0}{\alpha} (\alpha_3 - \frac{2\omega_0\alpha_2}{\sigma}) \right) \]  
(85)
\[ \gamma_3 = -\frac{\omega_0}{\alpha} \left( \frac{1}{\sigma} + \frac{2\sigma_2\omega_0}{\alpha\sigma} \right) + \frac{2\omega_0}{\alpha} \left( \alpha_2 + \frac{2\alpha_3\omega_0}{\sigma} \right). \]  
(86)

This completes our calculation for the finite parts of X, Y and Z at the first-order of perturbation.

We now proceed to the second-order by writing the equation
\[ LX_2 = -\sigma (X_0 Y_{1F} + Y_0 X_{1F}) + \sigma (D + b)(X_0 Z_{1F} + X_{1F} Z_0) - 2\sigma b \Delta r A \cos \omega_0 t + b\omega_0 \Delta r A \sin \omega_0 t \]  
(87)
from Eq. (59) above. The last two secular terms on the RHS of Eq. (87) have been borrowed from Eq. (69), where we had intentionally suppressed these terms and studied only the finite contributions coming from the regular terms only. The reason for this [explained earlier in the discussion following Eq. (69)] becomes more succinct now. Had we considered these secular terms in the first order then the \( \cos \omega_0 t \) and \( \sin \omega_0 t \) terms coming by integrating them [Eq. (39) above] would not have given any new secular terms when multiplied with \( X_0 \) and \( Y_0 \) in Eq. (87). Therefore, shifting these secular terms from the first order equation Eq. (69) to the second order equation Eq. (87) does not affect the structure of the RG-flow equations evaluated up to the second-order. Our present focus is only on the secular terms in the RHS of Eq. (87).

Using expressions of \( X_0, Y_0, \) and \( Y_0 \) [from Eqs. (65)–(67)] along with those of \( X_1, Y_1, \) and \( Z_1 \) [from Eqs. (76), (80), and (69)] we evaluate the following:

(i) \[ X_0 Y_{1F} + X_{1F} Y_0 \Rightarrow \frac{A^3}{2\alpha} \left[ \mu_1 \cos \omega_0 t + \mu_2 \sin \omega_0 t \right], \]  
(88)

where \( \Rightarrow \) means ‘secular terms only’ and
\[ \mu_1 = -2\alpha_1 + \frac{\alpha_2 + \beta_2}{2} - \frac{\omega_0\alpha_3}{\sigma}, \]
\[ \mu_2 = \frac{\alpha_3 + \beta_3}{2} + \frac{\omega_0}{\sigma} \left( \frac{\alpha_1 + \alpha_2}{2} \right). \]  
(89)  
(90)

(ii) \[ X_0 Z_{1F} + X_{1F} Z_0 \Rightarrow \frac{A^3}{2\alpha} \left[ \lambda_1 \cos \omega_0 t + \lambda_2 \sin \omega_0 t \right], \]  
(91)

with
\[ \lambda_1 = -\gamma_1 + \frac{\gamma_2}{2} + \frac{1}{2\alpha} \left( \frac{1}{\sigma} + 1 \right) \omega_0 \alpha_3 \]
\[ -\frac{\omega_0^2}{\alpha\sigma} \left( \frac{\alpha_1 - \alpha_2}{2} \right), \]
\[ \lambda_2 = \frac{\gamma_3}{2} - \frac{\omega_0\alpha_1}{\alpha} \left( \frac{1}{\sigma} + 1 \right) - \frac{1}{2\alpha} \left( \frac{1}{\sigma} + 1 \right) \omega_0 \alpha_2 \]
\[ +\frac{\omega_0^2\alpha_3}{2\sigma\alpha}. \]  
(92)  
(93)

Putting all this back in Eq. (87) we get the secular terms in the RHS of that equation as,
\[ L X_2 = -\frac{A^3}{2\alpha} C_1 \cos \omega_0 t - \frac{A^3}{2\alpha} C_2 \sin \omega_0 t - 2\sigma b \Delta r A \cos \omega_0 t + b\omega_0 \Delta r A \sin \omega_0 t + \text{non-resonant terms}, \]  
(94)

where
\[ C_1 = \alpha \sigma \mu_1 + \sigma b \lambda_1 + \sigma \omega_0 \lambda_2, \]
\[ C_2 = \alpha \sigma \mu_2 + \sigma b \lambda_2 - \sigma \omega_0 \lambda_1. \]  
(95)  
(96)

Having done all the necessary calculations, we have cast the RHS of Eq. (94) is the generic form of Eq. (34) (Sect. 3). Therefore, now we can directly write down the RG-equation by using the results of Sect. 3. We shall have just one RG-equation (for \( A \)) here, because, in writing the solution of \( X_0 \) in Eq. (65) we had chosen the appropriate initial conditions accordingly. This, obviously, is not any simplification, but saves cumbersome algebra. Thus, our sought after RG-flow equation for the amplitude \( A \) is obtained by combining Eqs. (38), (39) and (46) as,
where we have used the form of the operator $L$ as in Eq. (62) with $\beta = \sigma + b + 1$ [compare this with Eq. (35)]. Also, we have associated the terms on the RHS of Eq. (97) with the appropriate powers of $\varepsilon$, to lay bare the orders of perturbation from which they have come. The structure of the above amplitude equation (Eq. 97) is reminiscent of the general discussions on Hopf bifurcations we made at the beginning of Sect. 2 [see the discussions following Eq. (2)]. In those lines, it is clear from the RHS of Eq. (97) that Hopf bifurcation occurs right at the point where the co-efficient of the linear term (in $A$) vanishes i.e., when
\[ \Delta r = 0 \Rightarrow r = r_0 \text{ (Hopf point)}. \]

This ensures that the $A^3$ term comes to the forefront as the only player to lead the system towards a supercritical or a subcritical Hopf bifurcation, depending on whether its co-efficient is negative or positive respectively. To understand these bifurcations, we first go to one of the extremes and take $\sigma$ very large. Then from Eqs. (60), (69), and (70) we have,
\[ r_0 \approx \sigma, \quad \omega_0^2 \approx 2b\sigma, \quad \text{and} \quad \beta \approx \sigma. \]

Accordingly from Eqs. (77)–(79) we get
\[ \alpha_1 = \frac{1}{2} + \frac{b}{\beta} \approx \frac{1}{2} + \frac{b}{\sigma} \approx \frac{1}{2}, \]
\[ \alpha_2 \approx \frac{1}{(8b\sigma + \sigma^2)} \left[ \frac{\sigma^2}{2} + \frac{1}{3}(1 + \sigma)\sigma + \frac{8}{3}b\sigma \right] + \frac{4}{3}b\sigma \frac{1 + b}{\sigma} \approx \frac{1}{\sigma^2} \left[ \frac{\sigma^2}{2} + \frac{\sigma^2}{3} \right] = \frac{5}{6}, \]
\[ \alpha_3 \approx \frac{\omega_0}{8b\sigma + \sigma^2} \left[ \frac{\sigma}{3} + \frac{2}{3}(1 + \sigma) + \frac{1 + b}{3} \right] \approx \frac{\omega_0}{\sigma^2} \left( \frac{\sigma}{3} + \frac{2}{3} \right) = \frac{\omega_0}{\sigma}. \]

Similar approximations (with $\sigma \to \infty$) in Eqs. (81), (82), (84)–(86), (89), (90), (92), (93) leads to
\[ \beta_2 \approx \frac{5}{6}, \quad \beta_3 \approx \frac{2}{3} \frac{\omega_0}{\sigma}, \quad \gamma_1 \approx \frac{2}{3} b, \quad \gamma_2 \approx \frac{2}{3} b, \quad \gamma_3 \approx \frac{2}{3} \frac{\omega_0}{\sigma} \]
\[ \mu_1 \approx -\frac{1}{6}, \quad \mu_2 \approx \frac{13}{12} \frac{\omega_0}{\sigma}, \quad \lambda_1 \approx -\frac{5}{6} b, \quad \lambda_2 \approx -\frac{7}{12} \frac{\omega_0}{\sigma}. \]

Putting Eqs. (100)–(103) into the Eqs. (95) and (96) we get finally the sign of the co-efficient of $A^3$ in the amplitude equation (Eq. 97) from the sign of
\[ C_1 + C_2 \frac{\beta}{\omega_0} \approx -7 b \frac{\sigma^2}{6} + \frac{1}{4} \frac{\sigma_0 b}{\alpha} \frac{\sigma}{\omega_0} \approx -11 b \frac{\sigma^2}{12} < 0 \]
which is negative, vindicating the fact that for very large $\sigma$ we have $(r = r_0)$ as the point of supercritical Hopf bifurcation. But that is not always the case. For a moderate value of the Prandtl number as $\sigma = 10$ and the parameter $b = \frac{2}{3}$ (these are precisely the values that Lorenz used in his original simulation [8]) we get from Eq. (60) the values $r_0 = 24.74, \omega_0^2 = 92.63$ and accordingly the following set of values for the various constants follow:
\[ \alpha_1 = 0.7, \quad \alpha_2 = 0.5, \quad \alpha_3 = 0.18, \]
\[ \beta_2 = 1.0, \quad \beta_3 = -0.7, \quad \gamma_1 = 1.18, \quad \gamma_2 = 0.46, \quad \gamma_3 = 1.6, \]
\[ \mu_1 = -0.74, \quad \mu_2 = 0.65, \quad \lambda_1 = -1.36, \quad \lambda_2 = -0.4. \]

All these lead to $C_1 = -133.66$ and $C_2 = 171.96$ and hence the sign of the co-efficient of $A^3$ in Eq. (97) is obtained from the sign of
\[ C_1 + C_2 \frac{\beta}{\omega_0} = 110.52 > 0 \]

which is positive thus signaling at a subcritical Hopf bifurcation. Therefore, there is a critical value of the Prandtl number $(\sigma)$ below which the Hopf bifurcation is subcritical and above which supercritical. This is the information we extract from the amplitude equation Eq. (97) derived using RG.

The results that we have obtained are in agreement with all available numerical data and a specific $\sigma = 10$ calculation of [24]. For $\sigma = 10$, (Fig. 5) the Hopf bifurcation is known to be subcritical and for $\sigma = 50$, [Springer]
Fowler and McGuinness [29] found a periodic orbit for \( r > r_0 \).

We close this section by mentioning that there is a lower bound on the Prandtl number \( \sigma \), for the above analysis to make sense. That is obtained trivially as \( \sigma > b + 1 \) from the definitions of \( r_0 \) and \( \omega_0 \) in Eqs. (60) and (61) and the requirement that all parameters of the Lorenz system are positive, failing which, the fixed points \( x_0 = y_0 = \pm \sqrt{b(r-1)} \) and \( z_0 = r - 1 \) are stable for all \( r > 1 \) thus obscuring any Hopf bifurcation.

5 Rossler model

In this section we study plausible Hopf bifurcations in the Rossler model. Having detailed the methodology and its application for the Lorenz model in the last two sections, our discussion in this section will be brief as because the main line of approach remains the same. The Rossler systems is given by the three equations

\[
\begin{align*}
Dx &= -y - z, \quad (107) \\
Dy &= x + ay, \quad (108) \\
Dz &= b + xz - cz, \quad (109)
\end{align*}
\]

where \( D \equiv \frac{d}{dt} \) and \( a, b, c \) are adjustable parameters. These three equations can be combined to give a single variable equation in \( y \) as

\[
D^3y + (c - k)D^2y + (1 + kc)Dy + cy \\
= \dot{y}\ddot{y} - k\dot{y}^2 + k^2y\ddot{y} + y\dot{y} - ky^2 - k, \quad (110)
\]

where \( a = b = k \) has been used. This is not any essential restriction and numerous numerical experiments can be carried out with various values of \( a, b, \) and \( c \). But for the sake of algebraic simplicity we stick to equal values of \( a \) and \( b \) here and focus on Hopf bifurcations as the parameter \( c \) is varied.

The fixed points of the Rossler system Eqs. (107)–(109) are obtained at

\[
\begin{align*}
x_0 &= z_0 = \frac{d}{2}, \quad (111) \\
y_0 &= -\frac{d}{2a}, \quad (112)
\end{align*}
\]

with

\[
d = c \pm \sqrt{c^2 - 4ab}. \quad (113)
\]

Shifting our origin to one of the fixed points as \( u_1 = x - x_0, u_2 = y - y_0 \) and \( u_3 = z - z_0 \), we can recast Eq. (110) as

\[
\begin{align*}
\ddot{u} + (c - k + k\gamma_0)\dot{u} + (1 - kc - k^2\gamma_0 - y_0)\dot{u} \\
+ (c + 2k\gamma_0)u \\
= \ddot{u} - ku\ddot{u} - ku^2 + (k^2 + 1)u\dot{u} - ku^2, \quad (114)
\end{align*}
\]

where to simplify notation, \( u \) has been written in place of \( u_2 \) in the above equation. From this point our focus will be to derive an amplitude equation where a quantity like

\[
\Delta c = c - c_0 \quad (115)
\]

appears whose zero value corresponds to the Hopf-bifurcation point (this role was played by the parameter \( r \) the Lorenz system). If there is a Hopf bifurcation in the Rossler system, then we expect \( \Delta c \) to appear in the co-efficient of the linear term (in amplitude) in the amplitude equation. Only then can we infer that a Hopf bifurcation occurs at \( c = c_0 \).

Expanding \( y_0 \) in Eq. (112) about \( c_0 \) we get (with \( \Delta c = c - c_0 \))

\[
y_0 = \alpha_1 + \alpha_2\Delta c, \quad (116)
\]

where

\[
\begin{align*}
\alpha_1 &= -\frac{c_0 \pm \sqrt{c_0^2 - 4ab}}{2a}, \quad (117) \\
\alpha_2 &= -\frac{1}{2a} \pm \frac{c_0}{2a\sqrt{c_0^2 - 4ab}}. \quad (118)
\end{align*}
\]

Inserting these in Eq. (114), allows us to write that equation as,

\[
Lu = \frac{1}{2a} \left( \frac{d}{dt} (u^2) - k \frac{d}{dt} (u\dot{u}) + \frac{1 + k^2}{2} \frac{d}{dt} (u^2) - ku^2 - \Delta c (1 + \alpha_2)\dot{u} - (k + (1 + k^2)\alpha_2)u \\
+ (1 + 2k\alpha_2)u \right), \quad (119)
\]
where the operator $L$ is product separable as

$$Lu = (D^2 + \omega_0^2)(D + \sigma)u$$  \hspace{1cm} (120)

with $\omega_0$ and $\sigma$ given by

$$\omega_0^2 = \frac{c_0 - 2x_0}{c_0 - x_0 - k},$$  \hspace{1cm} (121)

$$\sigma = c_0 - x_0 - k.$$  \hspace{1cm} (122)

These values of $\omega_0$ and $\sigma$ are easily obtainable by comparing the cubic operator on the RHS of Eq. (120) with the cubic characteristic-equation of the linearized Rossler model.

Now, as was done in studying the Lorenz model, we invoke the perturbation expansion in $u$ as

$$u = u_0 + \lambda u_1 + \lambda^2 u_2$$  \hspace{1cm} (123)

with $\lambda$ as the perturbation parameter. Using this expansion in Eq. (119) we can easily segregate the terms of different orders and obtain the equations for zeroth and first orders as

$$Lu_0 = 0,$$  \hspace{1cm} (124)

$$Lu_1 = \beta_1 + \beta_2 \sin 2\omega_0 t + \beta_3 \cos 2\omega_0 t + \Delta c(\gamma_1 \cos \omega_0 t + \gamma_2 \sin \omega_0 t),$$  \hspace{1cm} (125)

where we have used the following abbreviations:

$$\beta_1 = -\frac{k A^2}{2}, \hspace{0.5cm} \beta_2 = \frac{1}{2} \omega_0^2 A^2 - \frac{1}{2} (1 + k^2) \omega_0 A^2,$$

$$\beta_3 = k \omega_0^2 A^2 - \frac{k}{2} A^2,$$

$$\gamma_1 = A \omega_0^2 (1 + \alpha_2) - A(1 + 2 \alpha_2),$$

$$\gamma_2 = -\left[ k + (1 + k^2) \alpha_2 \right] A \omega_0.$$  \hspace{1cm} (126)

Emulating our approach for the Lorenz model, we write the solution of Eq. (124) as

$$u_0 = A \cos \omega_0 t.$$  \hspace{1cm} (127)

For the first order equation, the solution for the regular (non-resonant) part is

$$u_0 = u_{1F} = \delta_1 + \delta_2 \sin 2\omega_0 t + \delta_3 \cos 2\omega_0 t,$$  \hspace{1cm} (128)

where $u_{1F}$ represents the finite (non-divergent) part of $u_1$ and

$$\delta_1 = \frac{\beta_1}{\sigma}, \hspace{0.5cm} \delta_2 = \frac{\beta_2 \sigma + 2 \beta_3 \omega_0}{4 \omega_0^2 + \sigma^2},$$

$$\delta_3 = \frac{\beta_3 \sigma - 2 \beta_2 \omega_0}{4 \omega_0^2 + \sigma^2}.$$  \hspace{1cm} (129)

The divergent (resonant) terms on the RHS of Eq. (125) can be (as was done in the Lorenz model) stacked with the divergent terms of the second order equation,

$$Lu_2 = \frac{1}{2} \frac{d}{dt}(u_0 \dot{u}_1 F) - k \frac{d}{dt}(u_0 \dot{u}_1 F + u_1 F \dot{u}_0) + \frac{1 + k^2}{2} \frac{d}{dt}(2u_0 \dot{u}_1 F - 2k u_0 \dot{u}_1 F + \Delta c(\gamma_1 \cos \omega_0 t + \gamma_2 \sin \omega_0 t))$$  \hspace{1cm} (130)

because [see explanation following Eq. (87)], had we integrated the resonant terms in the first order, then the regular $\cos \omega_0 t$ and $\sin \omega_0 t$ terms coming from there, would not have product any new secular terms in the different produced appearing on the RHS of Eq. (130).

Identifying the secular terms from the RHS of Eq. (130) we find,

$$Lu_2 = A^3 C_1 \cos \omega_0 t + A^3 C_2 \sin \omega_0 t + A \Delta c \gamma_1 \cos \omega_0 t + A \Delta c \gamma_2 \sin \omega_0 t,$$

$$+ \text{regular terms}$$  \hspace{1cm} (131)

where

$$C_1 = \frac{1}{A^2} \left[ \left( \frac{\omega_0^3}{2} + \frac{1 + k^2}{2} \omega_0 \right) \delta_2 \right. + \left. \left( \frac{\omega_0^2 k}{2} - k \right) \delta_3 + (\omega_0^2 k - 2k) \delta_1 \right],$$

$$C_2 = \frac{1}{A^2} \left[ -\left( \frac{\omega_0^3}{2} + \frac{1 + k^2}{2} \omega_0 \right) \delta_3 + (\omega_0^2 k - 2k) \delta_2 - \frac{1 + k^2}{2} \omega_0 \delta_1 \right].$$  \hspace{1cm} (132)

The solution of Eq. (131) is obtained as

$$u_2 = -\frac{A \cdot \Delta c \cdot (\gamma_1 \omega_0 + \gamma_2 \sigma) + A^3 (C_1 \omega_0 + C_2 \sigma)}{2 \omega_0 (\omega_0^2 + \sigma^2)} \times t \cos \omega_0 t$$

$$+ \frac{2 \omega_0 (\omega_0^2 + \sigma^2)}{A \cdot \Delta c \cdot (\gamma_1 \sigma - \gamma_2 \omega_0) + A^3 (C_1 \sigma - C_2 \omega_0)} \times t \sin \omega_0 t$$

$$+ \text{regular part}.$$  \hspace{1cm} (133)

Going by the methodology developed in Sect. 3 we obtain the RG-equation for the amplitude by combining Eqs. (38), (39), and (46) as

$$\frac{dA}{dt} = -\frac{A}{2 \omega_0 \omega_0^2 + \sigma} \left[ \lambda (\gamma_1 \omega_0 + \gamma_2 \sigma) \Delta c + \lambda^2 \left( C_1 + C_2 \frac{\sigma}{\omega_0} \right) A^2 \right].$$  \hspace{1cm} (134)
This bears resemblance with Eq. (97), (i.e., the amplitude equation for the Lorenz model) in that, the coefficient of the linear term $A$ has a $\Delta c$, which becomes zero at the Hopf-point. Thus,

$$c = c_0 \quad \text{(Hopf point)}$$  \hspace{1cm} (135)

is the point in parameter space where the system undergoes Hopf bifurcation.

To illustrate, we consider two distinct points in parameter space,

(i) $a_0 = b_0 = 0.2, \quad c_0 = 5.7$ (Fig. 6)

(ii) $a_0 = b_0 = 0.1, \quad c_0 = 14.0$ (Fig. 7)

These values are well-known in numerical experiments done with the Rossler systems in context of Hopf bifurcations [8]. For case (i), we have the values from Eqs. (118), (121), (122), (126), (132) as:

$$a = b = 0.2; \quad c = 5.7$$

$$\omega_0 = 5.43; \quad \alpha_2 = -5.0062; \quad \sigma = -0.193$$

$$\beta_1 = -0.1A^2; \quad \beta_2 = 77.1276A^2; \quad \beta_3 = 5.7922A^2$$

$$\delta_1 = 0.5181A^2; \quad \delta_2 = 0.4071A^2; \quad \delta_3 = -7.1121$$

$$\gamma_1 = -109.014A; \quad \gamma_2 = 27.1739A$$

$$C_1 = -2.9664; \quad C_2 = 588.3718.$$  \hspace{1cm} (136)

Putting these values in the amplitude equation we obtain

$$\frac{dA}{dt} = \frac{1}{22(\omega_0^2 + \sigma^2)}(23.89)A^3$$  \hspace{1cm} (137)

at the Hopf-bifurcation point. Here the coefficient of $A^3$ being positive we understand that a subcritical Hopf bifurcation occurs at $a_0 = b_0 = 0.2$ and $c_0 = 5.7$.

For case (ii) on the other hand, we obtain the following set of values:

$$a = b = 0.1; \quad c = 14$$

$$\omega_0^2 = 140 \Rightarrow \omega_0 = \pm11.83; \quad \alpha_2 = -10$$

$$\beta_1 = -0.05A^2; \quad \beta_2 = 821.837A^2; \quad \beta_3 = 13.95A^2$$

$$\delta_1 = 0.5A^2; \quad \delta_2 = 0.443A^2; \quad \delta_3 = -3.46A^2$$

$$\gamma_1 = -1241A; \quad \gamma_2 = 118.3A$$

$$C_1 = 386.321; \quad C_2 = 2880.55.$$  \hspace{1cm} (138)

which yield the amplitude equation (at the Hopf-point)

$$\frac{dA}{dt} = -1.3A^3.$$  \hspace{1cm} (139)

This co-efficient being negative, we understand that a super-critical Hopf bifurcation occurs at the point $a_0 = b_0 = 0.1$ and $c_0 = 14$ of the point parameter space.

### 6 Summary

In this paper the criteria for occurrences of super-critical and sub-critical Hopf bifurcations have been studied for dynamical systems in two as well as three dimensions. In doing so we have employed the RG for perturbatively deriving the corresponding amplitude equation up to some relevant order of the amplitude, where, putting off the linear term in amplitude, the sign of the coefficient of the lowest nonlinear power guides us correctly to the kind of Hopf bifurcation the system shows. This strategy has been successfully applied to first rederive the well-known criterion (for two dimensions) which tells one type of Hopf bifurcation from the
other and then show some limitations of that criterion through examples where the RG works authentically. The extension of this RG formalism to three dimensions, although nontrivial, has been done and applied to the highly important models of Lorenz and Rossler. The emphasis of the study for these systems has been laid on identifying regions in parameter space where super-
or sub-critical Hopf bifurcations can occur. Calculations to second order in perturbation have been done elaborately.

**Appendix: Derivation of Eq. (24)**

In this Appendix we show how to derive Eqs. (19) and (23) using Eqs. (15) and (16). At first we analyze the terms on the RHS of Eq. (16) one by one. The ‘\(g(x_0, y_0)\)’ series (Eq. 6) gives \(a^3 \sin \phi\) from two terms. They are \(x_0^2 y_0\) and \(y_0^2\), as is obvious from Eqs. (17), (18). The former one yields \(a^2 \cos^2 \phi \cdot a \sin \phi \Rightarrow a^3 \sin \phi\) as the relevant part. These terms give secular divergence as has been discussed earlier (Eqs. 11 and 12). Now for the co-efficients \(g_{ij}\). In the format of Eq. (6) the co-efficient of \(x_0^2 y_0\) is \(g_{21}\) and of \(y_0^3\) is \(g_{03}\). These two co-efficients can be sieved out of the series Eq. (6) by taking the Taylor-derivatives as

\[
g_{21} = \frac{1}{2} g_{xx}y, \tag{140}\]

and

\[
g_{03} = \frac{1}{6} g_{yy}, \tag{141}\]

where the subscripts mean partial derivatives at \((x = 0)\) and \((y = 0)\). Therefore, the co-efficient of the secular \(\sin \phi\) terms coming from the term ‘\(-g(x_0, y_0)\)’ of Eq. (15) is

\[
-g(x_0, y_0) \Rightarrow -\frac{a^3}{8}(g_{xx}y + g_{yy}), \tag{142}\]

where the ‘\(\Rightarrow\)’ means ‘relevant secular contribution’.

The other two terms of Eq. (15) can be similarly analyzed. The second term on the RHS of Eq. (15) is

\[
-\frac{\partial f}{\partial x}(x_0, y_0) = -\sum_{i,j} f_i j f_{ij} x_0^{i-1} y_0^{j+1}. \tag{143}\]

Here the relevant \(x_0^2 y_0\) and \(y_0^3\) terms are obtained from the combinations \((i = 3, j = 0)\) and \((i = 1, j = 2)\) respectively. For the former combination we have \(-\frac{3}{4} a^3 f_{30} = -\frac{1}{5} a^3 f_{xxx}\) as the relevant secular contribution and for the latter combination we have \(-\frac{3}{4} a^3 f_{12} = -\frac{1}{8} a^3 f_{xyy}\) as the relevant secular contribution, thus allowing us to write

\[
-\frac{\partial f}{\partial x}(x_0, y_0) \Rightarrow -\frac{a^3}{8} (f_{xxx} + 3 f_{xyy}). \tag{144}\]

Similarly, for the third term on the RHS of Eq. (15) we have,

\[
\frac{\partial f}{\partial y}(x_0, y_0) = \sum_{i,j} f_{ij} j f_{j0} y_0^{i+1} y_0^{j-1}, \tag{145}\]

which gives only a \(y_0^2\) term \((i = 1, j = 2)\) as the relevant one for our purpose, but no \(y_0^3\) term \((as i \geq 2)\). This term has \(2 \cdot 12 \frac{a^3}{4}\) as the co-efficient of \(\sin \phi\) which hence leads to

\[
\frac{\partial f}{\partial y} \Rightarrow \frac{a^3}{4} f_{xyy}. \tag{146}\]

Adding up Eqs. (142), (144), (146) we have the co-efficient of \(\sin(t + \theta)\) = \(\sin \phi\) from the first order terms of Eq. (19).

Now the regular \(a^2\)-terms on the RHS of Eq. (15) are as follows:

\[
-g(x_0, y_0) \Rightarrow -g_{20} x_0^2 - g_{11} x_0 y_0 - g_{02} y_0^2 - \cdots
\]

\[
= -\frac{a^2}{2} \left((g_{20} + g_{02}) + (g_{20} - g_{02}) \cos 2\phi + g_{11} \sin 2\phi\right), \tag{147}\]

\[
-\frac{\partial f}{\partial x}(x_0, y_0) \Rightarrow -\frac{a^2}{2} \left[f_{11} - f_{11} \cos 2\phi + 2 f_{20} \sin 2\phi\right], \tag{148}\]

\[
\frac{\partial f}{\partial y}(x_0, y_0) \Rightarrow \frac{a^2}{2} \left[f_{11} + f_{11} \cos 2\phi + 2 f_{20} \sin 2\phi\right], \tag{149}\]

where ‘\(\Rightarrow\)’ means ‘relevant regular term’. Adding up the terms we get the regular \(a^2\)-terms on the RHS of Eq. (15) as,

\[
\dot{x}_1 + x_1
\]

\[
= \frac{a^2}{4} \left[-(g_{xx} + g_{yy}) + (4 f_{xy} - g_{xx} + g_{yy}) \cos 2\phi + 2(-f_{xx} + f_{yy} - g_{xy}) \sin 2\phi\right] + \text{higher powers of} \ a \tag{150}\]

which, on integration, gives the renormalized \(x_1\) of Eq. (21).

Now we turn to Eq. (16) for the \(a^2 \sin \phi\) terms that come in the second-order of perturbation. There are three types of terms on the RHS of Eq. (16): (i) product of \(x_1\) (or \(y_1\)) with the first derivatives of \(f(x, y)\) or \(g(x, y)\), (ii) product of \(x_0\) (or \(y_0\), \(x_1\) (or \(y_1\)) and the
second derivatives of \( f \) or \( g \), (iii) product of the functions and their first derivatives. Calculations are easy and, in order to illustrate, we pick up one term from each of the above three categories.

Among the four terms of type (i), viz., \(-x_1 \frac{\partial g}{\partial x}(x_0, y_0), x_1 \frac{\partial f}{\partial y}(x_0, y_0), -y_1 \frac{\partial g}{\partial y}(x_0, y_0)\), and \(-y_1 \frac{\partial f}{\partial x}(x_0, y_0)\), we work out the case of \(-x_1 \frac{\partial g}{\partial x}\) here, and state the results for the other three terms. For the term \(-x_1 \frac{\partial g}{\partial x}\), it is clear that the \(a^2\)-terms from \(x_1\) and the linear terms from \(\frac{\partial g}{\partial x}\) combine to give the required \(a^3\)-terms. Among these \(a^3\)-terms, the constant and cos \(2\phi\) terms from \(x_1\) (Eq. 21) combine with \(y_0(=a \sin \phi)\) of \(\frac{\partial g}{\partial x}(x_0, y_0)\) to give our sought after secular term \(a^3 \sin \phi\). Also, the sin \(2\phi\) term of \(x_1\) combine with \(x_0(=a \cos \phi)\) term of \(\frac{\partial g}{\partial x}(x_0, y_0)\) to give the same.

Thus considering only \(a^3\)-terms, we have,

\[
-x_1 \frac{\partial g}{\partial x}(x_0, y_0) \Rightarrow -x_1 [2g_{20}x_0 + g_{11}y_0] = -x_1 [g_{xx}x_0 + g_{xy}y_0]
\]

which, on using Eqs. (17), (18), (21) leads to the expression

\[
\Rightarrow a^3 \sin \phi \left[ -\frac{1}{6} (f_{yy}g_{xx} + f_{xy}g_{xy}) - \frac{5}{24} (g_{xx} + g_{yy})g_{xy} - \frac{1}{8} f_{xy}g_{xy} \right].
\]

(152)

For the other three terms of type (i) we have:

\[
x_1 \frac{\partial f}{\partial y}(x_0, y_0) \Rightarrow x_1 [f_{11}x_0 + 2f_{02}y_0]
\]

\[
\Rightarrow a^3 \sin \phi \left[ -\frac{1}{12} f_{xy}(f_{xx} + f_{yy} + g_{xy}) - \frac{1}{24} f_{yy}(7g_{xx} + 5g_{yy}) \right].
\]

(153)

\[
y_1 \frac{\partial g}{\partial y}(x_0, y_0) \Rightarrow y_1 [g_{11}x_0 + 2g_{02}y_0]
\]

\[
\Rightarrow a^3 \sin \phi \left[ -\frac{1}{12} g_{xy}(f_{xy} - g_{xx} - g_{yy}) - \frac{1}{24} g_{yy}(7f_{xx} + 5f_{yy}) \right].
\]

(154)

and

\[
y_1 \frac{\partial f}{\partial x}(x_0, y_0) \Rightarrow -y_1 [2f_{20}x_0 + f_{11}y_0]
\]

\[
\Rightarrow a^3 \sin \phi \left[ -\frac{1}{12} (g_{yy} - g_{xx})f_{xx} - \frac{5}{24} (f_{xx} + f_{yy})f_{xy} - \frac{1}{6} f_{xy}g_{xy} \right].
\]

(155)

Adding up Eqs. (151) to (155), we get the relevant secular contribution from the type-(i) terms as,

\[
-x_1 \frac{\partial g}{\partial x}(x_0, y_0)x_1 \frac{\partial f}{\partial y}(x_0, y_0) + (x_0, y_0)
\]

\[
y_1 \frac{\partial g}{\partial y}(x_0, y_0) - y_1 \frac{\partial f}{\partial x}(x_0, y_0)
\]

\[
\Rightarrow a^3 \sin \phi \left[ -\frac{1}{6} (f_{xx}g_{xx} + f_{xy}g_{xy}) - \frac{5}{24} (f_{yy}g_{xx} + 2f_{yy}g_{yy} + f_{xx}g_{yy}) + \frac{1}{8} g_{xy}(g_{xx} + g_{yy}) - \frac{1}{8} f_{xy}(f_{xx} + f_{xy}) \right].
\]

(156)

The type-(ii) terms of Eq. (16) are \(-y_0x_1 \frac{\partial^2 f}{\partial x^2}(x_0, y_0), -y_0y_1 \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0), -x_0x_1 \frac{\partial^2 f}{\partial x^2}(x_0, y_0), \) and \(x_0y_1 \frac{\partial^2 f}{\partial y^2}(x_0, y_0), \) of which we elucidate only the first one. These terms are simpler than those of type (i). In the term \(-y_0x_1 \frac{\partial^2 f}{\partial x^2}(x_0, y_0), \) the \([y_0x_1]\) part gives \(a^3\).

It is only the \(f_{20}\) term [co-efficient of \(x_0^2\) in Eq. (5)] from the second derivative that participates to yield our relevant secular term \(a^3 \sin \phi\). Thus,

\[
y_0x_1 \frac{\partial^2 f}{\partial x^2}(x_0, y_0) \Rightarrow [-a \sin \phi] \cdot [x_1] 2f_{20}
\]

\[
\Rightarrow a^3 \sin \phi \left[ \frac{1}{24} f_{xx}(7g_{xx} + 5g_{yy} - 4f_{xy}) \right].
\]

(157)

Similarly, for the other three terms of type-(ii) we have:

\[
y_0y_1 \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) \Rightarrow [-a \sin \phi] \cdot [y_1] f_{11}
\]

\[
\Rightarrow a^3 \sin \phi \left[ -\frac{1}{24} f_{xy}(7f_{xx} + 5f_{yy} + 4g_{xy}) \right]
\]

(158)

\[
x_0x_1 \frac{\partial^2 f}{\partial x^2}(x_0, y_0) \Rightarrow [a \cos \phi] \cdot [x_1] f_{11}
\]

\[
\Rightarrow a^3 \sin \phi \left[ \frac{1}{12} f_{xy}(f_{xx} - f_{yy} + g_{xy}) \right]
\]

(159)

and

\[
x_0y_1 \frac{\partial^2 f}{\partial y^2}(x_0, y_0) \Rightarrow [a \cos \phi] \cdot [y_1] 2f_{02}
\]

\[
\Rightarrow a^3 \sin \phi \left[ \frac{1}{12} f_{xy}(-f_{xy} + g_{xx} - g_{yy}) \right].
\]

(160)

Again, adding up Eqs. (157) to (160) we get the relevant secular contributions from the type (ii) terms as,
\[
- y_0 x_0 \frac{\partial^2 f}{\partial x^2}(x_0, y_0) - y_0 y_1 \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) \\
- x_0 y_1 \frac{\partial^2 f}{\partial x \partial y}(x_0, y_0) + x_0 y_0 \frac{\partial^2 f}{\partial y^2}(x_0, y_0) \\
\Rightarrow a^3 \sin \phi \left[ \frac{1}{24} f_{xxx}(7g_{xx} + 5g_{yy}) - \frac{3}{8} f_{xy}(f_{xx} + f_{yy}) \\
+ \frac{1}{12} (f_{yy}g_{xx} - f_{xy}g_{xy} - f_{xy}g_{yy}) \right].
\]

The last two terms of Eq. (16) consists of products of the nonlinear functions and their first derivatives. We elucidate the term \[ f(x_0, y_0) \frac{\partial f}{\partial x}(x_0, y_0) \] and the other one follows similarly.

Writing
\[
f(x_0, y_0) \frac{\partial f}{\partial x}(x_0, y_0) = \sum_{i,j,k,l} k f_{ij} f_{kl} x_0^{i+k-1} y_0^{j+l}
\]
we again search for terms \[ x_0^2 y_0 \] and \[ y_0^3 \] which give the secular \( a^3 \sin \phi \). Since \( i + j \geq 2 \) as well as \( k + l \geq 2 \), for the term \[ x_0^2 y_0 \] (for which \( i + k = 3 \) and \( j + l = 1 \)), the permissible \( (i \ j \ k \ l) \) combinations are \( (2 \ 0 \ 1 \ 1) \) and \( (1 \ 1 \ 2 \ 0) \) which add up to yield
\[
\Rightarrow (1.2f_{00} f_{11} + 2 f_{11} f_{20}) x_0 y_0, \\
\Rightarrow a^3 \sin \phi \frac{3}{8} f_{xx} f_{xy}.
\]

For the \[ y_0^3 \] term (where, from Eq. (162), we have \( i + k = 1 \) and \( j + l = 3 \) the allowed \( (i \ j \ k \ l) \) combinations are \( (0 \ 2 \ 1 \ 1) \) and \( (1 \ 1 \ 0 \ 2) \) which add up to give
\[
\Rightarrow (1. f_{02} f_{11} + 0. f_{11} \cdot f_{02}) y_0^3, \\
\Rightarrow a^3 \sin \phi \frac{3}{8} f_{yy} f_{xy}
\]
Adding Eqs. (163) and (164) we get the relevant secular contribution from this term as
\[
f(x_0, y_0) \frac{\partial f}{\partial x}(x_0, y_0) \Rightarrow a^3 \sin \phi \left[ \frac{3}{8} f_{xy}(f_{xx} + f_{yy}) \right].
\]

Similarly, for the term
\[
g(x_0, y_0) \frac{\partial g}{\partial y}(x_0, y_0) = \sum_{ij} \sum_{kl} l f_{kig} x_0^{i+k} y_0^{j+k-1}
\]
we get \[ x_0^2 y_0 \] from the \( (i \ j \ k \ l) \) combinations given by \( (2 \ 0 \ 0 \ 2) \) and \( (1 \ 1 \ 1 \ 1) \) while the \[ y_0^3 \] term is obtained from the combination \( (0 \ 2 \ 0 \ 2) \). Adding them we get the \[ a^3 \sin \phi \] term as
\[
g(x_0, y_0) \frac{\partial g}{\partial y}(x_0, y_0) \Rightarrow a^3 \sin \phi \left[ \frac{1}{8} f_{yy}(g_{xx} + 3g_{yy}) + \frac{1}{4} f_{xy} g_{xy} \right].
\]

Thus the secular \[ a^3 \sin \phi \] terms at second order Eq. (16) can be obtained by adding Eqs. (156), (161), (165), and (167) to yield Eq. (23), and hence Eq. (163).

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