On isometric minimal immersion of a singular non-CSC extremal Kähler metric into 3-dimensional space forms

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Abstract

On any compact Riemann surface there always exists a singular non-CSC (constant scalar curvature) extremal Kähler metric which is called a non-CSC HCMU (the Hessian of the Curvature of the Metric is Umbilical) metric. In this paper, by moving frames, we show that any non-CSC HCMU metric cannot be isometrically minimal immersed into 3-dimensional real space forms even locally. In general, any non-CSC HCMU metric cannot be isometrically immersed into 3-dimensional real space forms with constant mean curvature (CMC).

1 Introduction

Since Calabi proposed the famous Calabi conjecture, Kähler-Einstein metric is one of the hot topics in geometry. For the existence of Kähler-Einstein metrics, one can refer to [15, 16, 17]. In 1982, Calabi [1] replaced Kähler-Einstein metric with extremal Kähler metric. In a fixed Kähler class, an extremal Kähler metric is the critical point of the following Calabi energy functional

$$C(g) = \int_M R^2 dg,$$

where $R$ is the scalar curvature of the metric $g$ in the given Kähler class. The Euler-Lagrange equations of $C(g)$ are $R_{\alpha\beta} = 0$ for all indices $\alpha, \beta$, where $R_{\alpha\beta}$ is the second-order $(0, 2)$ covariant derivative of $R$. When $M$ is a compact Riemann surface, Calabi in [1] proved that an extremal Kähler metric is a CSC (constant scalar curvature) metric.

A natural question is whether or not an extremal Kähler metric with singularities on a compact Riemann surface is still a CSC metric. In [3], X.X.Chen first gave an example of a non-CSC extremal Kähler metric with singularities. We often call a non-CSC extremal Kähler metric with finite singularities on a compact Riemann surface a non-CSC HCMU (the Hessian of the Curvature of the Metric is Umbilical) metric. In [7, 8], Q.Chen, B.Xu and Y.Y.Wu reduced the existence of a non-CSC HCMU metric to the existence of a meromorphic 1-form on the underlying Riemann surface. It is interesting that on any compact Riemann surface there always exists a non-CSC HCMU metric. For more properties of non-CSC HCMU metrics, one can refer to [4, 5, 6, 9, 12, 13] and the references cited in these papers.

Recently, isometric immersions of a non-CSC HCMU metric into some “good” higher dimensional spaces have been studied. In [10], C.K.Peng and Y.Y.Wu proved that any non-CSC HCMU metric can be locally isometric immersed into 3-dimension Euclidean space $\mathbb{R}^3$. They got a one-parameter family of isometric immersions from a compact Riemann surface with a singular non-CSC extremal Kähler metric to $\mathbb{R}^3$, each of whom is a Weingarten surface. In [14], we...
proved that any non-CSC HCMU metric can be locally isometric immersed into 3-dimensional space forms. As an application, we proved that any non-CSC HCMU metric can be locally isometric immersed into complex projective space \( \mathbb{C}P^n(n \geq 3) \) with Fubini-Study metric.

In this manuscript, we consider the following question: Suppose \( g \) is a non-CSC HCMU metric on a compact Riemann surface \( M \); For any point \( P \in M \), whether or not there exist an open neighborhood \( U \) of \( P \) and an isometric minimal immersion \( F : U \rightarrow \mathbb{Q}_c^3 \), where \( \mathbb{Q}_c^3 \) denotes the 3-dimensional space form with section curvature \( c \). The following theorem is our main result.

**Theorem 1.1.** Let \( g \) be a non-CSC HCMU metric on a compact Riemann surface \( M \) with the character 1-form \( \omega \). Denote \( M^* = M \setminus \{ \text{zeros and poles of } \omega \} \). Then for any point \( P \in M^* \), and any open neighborhood \( U \subseteq M^* \) of \( P \), there doesn’t exist an isometric minimal immersion \( F : U \rightarrow \mathbb{Q}_c^3 \).

Furthermore, we can prove the following theorem in a similar way.

**Theorem 1.2.** Let \( g \) be a non-CSC HCMU metric on a compact Riemann surface \( M \) with the character 1-form \( \omega \). Denote \( M^* = M \setminus \{ \text{zeros and poles of } \omega \} \). Then for any point \( P \in M^* \), and any open neighborhood \( U \subseteq M^* \) of \( P \), there doesn’t exist an isometric immersion \( F : U \rightarrow \mathbb{Q}_c^3 \) of constant mean curvature.

## 2 Preliminaries

### 2.1 Non-CSC HCMU metric

**Definition 2.1** ([11]). Let \( M \) be a Riemann surface, \( P \in M \). A conformal metric \( g \) on \( M \) is said to have a conical singularity at \( P \) with the singular angle \( 2\pi \alpha(\alpha > 0, \alpha \neq 1) \) if in a neighborhood of \( P \)

\[
g = e^{2\varphi}|dz|^2, \tag{1}
\]

where \( z \) is a local complex coordinate defined in the neighborhood of \( P \) with \( z(P) = 0 \) and

\[
\varphi - (\alpha - 1) \ln |z|
\]

is continuous at 0.

**Definition 2.2** ([8]). Let \( M \) be a Riemann surface, \( P \in M \). A conformal metric \( g \) on \( M \) is said to have a cusp singularity at \( P \) if in a neighborhood of \( P \)

\[
g = e^{2\varphi}|dz|^2, \tag{2}
\]

where \( z \) is a local complex coordinate defined in the neighborhood of \( P \) with \( z(P) = 0 \) and

\[
\lim_{z \to 0} \frac{\varphi + \ln |z|}{\ln |z|} = 0.
\]

**Definition 2.3** ([4]). Let \( M \) be a compact Riemann surface and \( P_1, \ldots, P_N \) be \( N \) points on \( M \). Denote \( M \setminus \{ P_1, \ldots, P_N \} \) by \( M^* \). Let \( g \) be a conformal metric on \( M^* \). If \( g \) satisfies

\[
K_{,zz} = 0, \tag{3}
\]

where \( K \) is the Gauss curvature of \( g \), we call \( g \) an HCMU metric on \( M \).
In this paper, we always consider non-CSC HCMU metrics with finite area and finite Calabi energy, that is,
\[ \int_{M^*} dg < +\infty, \quad \int_{M^*} K^2 dg < +\infty. \] (4)

From [2], [9], [12], we know that each singularity of a non-CS C HCMU metric is conical or cusp if it has finite area and finite Calabi energy.

We now list some results of non-CSC HCMU metrics, which will be used in this paper. For more results one can refer to [5], [8] and the references cited in it.

First the equation (3) is equivalent to
\[ \nabla K = \sqrt{-1} e^{-2\tau} K \frac{\partial}{\partial z}, \]
which is a holomorphic vector field on \( M^* \). In [4], [9], the authors proved that the curvature \( K \) can be continuously extended to \( M \) and there are finite smooth extremal points of \( K \) on \( M^* \).

In [5], [8], the authors proved the following fact: each smooth extremal point of \( K \) is either the maximum point of \( K \) or the minimum point of \( K \), and if we denote the maximum of \( K \) by \( K_1 \) and the minimum of \( K \) by \( K_2 \) then if all the singularities of \( g \) are conical singularities,
\[ K_1 > 0, \quad K_1 > K_2 > -(K_1 + K_2); \]
if there exist cusps in the singularities,
\[ K_1 > 0, \quad K_2 = -\frac{1}{2} K_1. \]

In [9], C.S.Lin and X.H.Zhu proved that \( \nabla K \) is actually a meromorphic vector field on \( M^* \). In [7], Q.Chen and the second author defined the dual 1-form of \( \nabla K \) by \( \omega(\nabla K) = \frac{\sqrt{-1}}{4} \). They call \( \omega \) the character 1-form of the metric. Denote \( M^* \{ \text{smooth extremal points of } K \} \) by \( M' \).

Then on \( M' \)
\[ \begin{align*}
\frac{dK}{3(K - K_1)(K - K_2)(K + K_1 + K_2)} &= \omega + \bar{\omega}, \\
g &= -\frac{4}{3}(K - K_1)(K - K_2)(K + K_1 + K_2)\omega\bar{\omega}.
\end{align*} \] (5)

By (5), some properties of \( \omega \) are got in [7]:

- \( \omega \) only has simple poles,
- at each pole, the residue of \( \omega \) is a non-zero real number,
- \( \omega + \bar{\omega} \) is exact on \( M \setminus \{ \text{poles of } \omega \} \).

Conversely, if a meromorphic 1-form \( \omega \) on \( M \) which satisfies the properties above, then we pick two real numbers \( K_1, K_2 \) such that \( K_1 > 0, K_1 > K_2 > -(K_1 + K_2) \) or \( K_1 > 0, K_2 = -\frac{1}{2} K_1 \), and consider the following equation on \( M \setminus \{ \text{poles of } \omega \} \)
\[ \begin{align*}
\frac{dK}{3(K - K_1)(K - K_2)(K + K_1 + K_2)} &= \omega + \bar{\omega}, \\
K(P_0) &= K_0,
\end{align*} \] (6)
where \( P_0 \in M \setminus \{ \text{poles of } \omega \} \) and \( K_2 < K_0 < K_1 \). We get that (6) has a unique solution \( K \) on \( M \setminus \{ \text{poles of } \omega \} \) and \( K \) can be continuously extended to \( M \). Furthermore, we define a metric \( g \) on \( M \setminus \{ \text{poles of } \omega \} \) by
\[ g = -\frac{4}{3}(K - K_1)(K - K_2)(K + K_1 + K_2)\omega\bar{\omega}, \]
where $K$ is the solution of (6). Then it can be proved that $g$ is a non-CSC HCMU metric, $K$ is the Gauss curvature of $g$ and $\omega$ is the character 1-form of $g$.

It is interesting that on any compact Riemann surface there always exists a meromorphic 1-form satisfying the properties (see [5]). So there always exists a non-CSC HCMU metric on a compact Riemann surface.

2.2 Riemannian submanifolds

In this section, we recall some facts of Riemannian submanifolds. For more results, one may consult [18] and references cited in it.

Let $F : M^n \to M^n+p$ be an immersion of a smooth manifold $M$ of dimension $n$ into a smooth manifold $\overline{M}$ of dimension $n + p$. The number $p$ is called the codimension of $F$. If $\langle \cdot, \cdot \rangle_{M}$ is a Riemannian metric on $M$, for every point $P \in M$ and any $X, Y \in T_PM$, define $\langle X, Y \rangle_{M} = \langle F^*X, F^*Y \rangle_{\overline{M}}$. Then $\langle \cdot, \cdot \rangle_{M}$ is a Riemannian metric on $M$. In this case, $F$ becomes an isometric immersion of $M$ into $\overline{M}$. We will often drop the subscript and denote a Riemannian metric simply by $\langle \cdot, \cdot \rangle$, assuming that the underlying manifold will be clear from the context.

Let $F : M^n \to M^n+p$ be an isometric immersion. Since $F$ is an immersion, then, for each point $P \in M$, there exists a neighborhood $U \subseteq M$ of $P$ such that $F : U \to \overline{M}$ is an imbedding. Therefore, we may identity $U$ with $F(U)$. Hence, the tangent space of $M$ at $P$ is a subspace of $\overline{M}$ at $P$. Then we have

$$T_PM = T_PM \oplus T^\perp_PM,$$

where $T^\perp_PM$ is the orthogonal complement of $T_PM$ in $T_PM$. In this way, we obtain a vector bundle

$$T^\perp M = \bigcup_{P \in M} T^\perp_PM,$$

which is called the normal bundle of $M$.

Let $\nabla, \overline{\nabla}$ be the Levi-Civita connections of $M, \overline{M}$, respectively. Denote the sets of smooth vector fields and smooth normal vector fields on $M$ by $\chi(M), \chi^\perp(M)$, respectively. Then for any two smooth vector fields $X, Y \in \chi(M)$, by (7), we obtain the Gauss formula

$$\overline{\nabla}_XY = \nabla_XY + B(X, Y),$$

where $B : TM \times TM \to T^\perp M$ is called the second fundamental form of $F$.

Similarly, for any $X \in \chi(M), \xi \in \chi^\perp(M)$, by (7), we obtain the Weingarten formula

$$\nabla_X\xi = -A_\xi X + \nabla^\perp_X\xi,$$

where $A_\xi : TM \to TM$ is called the shape operator of $f$ with respect to $\xi$, and $\nabla^\perp$ is called the normal connection of $F$. By the Gauss and Weingaren formulas, $B$ and $A_\xi$ satisfy

$$\langle A_\xi X, Y \rangle = \langle B(X, Y), \xi \rangle.$$ (8)

If the codimension $p = 1$, we call the isometric immersion $F : M^n \to \overline{M}^{n+1}$ is a hypersurface of $\overline{M}$. Let $F : M^n \to \overline{M}^{n+1}$ be an orientable hypersurface. Choosing a local smooth unit normal vector field $\xi$ along $F$ and a local smooth orthonormal tangential frame $e_1, \ldots, e_n$, then the mean curvature vector $H$ of $F$ is defined by

$$H = \frac{1}{n} \sum_{i=1}^{n} B(e_i, e_i).$$
Denote $A = A_\xi$, then, by (3),
\[
H = \frac{1}{n} \left( \sum_{i=1}^{n} < Ae_i, e_i > \right) \xi.
\]

If $H \equiv 0$, the isometric immersion $F$ is called a minimal immersion. Generally, $F$ is called a constant mean curvature immersion if $\|H\|$ is a constant.

### 2.2.1 Basic equations

Using the Gauss and Weingarten formulas, the basic equations of isometric immersion $F : M^n \to \overline{M}^{n+p}$ can be written as follows.

**Gauss-equation**
\[
R(X, Y, Z, W) = \overline{R}(X, Y, Z, W) + < B(X, Z), B(Y, W) > - < B(X, W), B(Y, Z) >;
\]

**Codazzi-equation**
\[
(\overline{R}(X, Y)Z) = (\nabla^{\perp} X)B(Y, Z) - (\nabla^{\perp} Y)B(X, Z);
\]

**Ricci-equation**
\[
(\overline{R}(X, Y)\xi) = R^{\perp}(X, Y)\xi + B(A_\xi X, Y) - B(X, A_\xi Y),
\]

where $X, Y, Z, W \in \chi(M), \xi \in \chi^{\perp}(M), R^{\perp}$ denotes the curvature tensor of the normal bundle $T^{\perp}M$ and $\overline{R}$, $\overline{\overline{R}}$ are Riemannian curvature tensors of $M, \overline{M}$, respectively.

In particular, if $\overline{K}(X, Y) = \overline{R}(X, Y, X, Y)$ and $K(X, Y) = R(X, Y, X, Y)$ denote the sectional curvatures in $\overline{M}$ and $M$ of the plane generated by the orthonormal vectors $X, Y \in T_P M$, the Gauss-equation becomes
\[
K(X, Y) = \overline{K}(X, Y) + < B(X, X), B(Y, Y) > - < B(X, W), B(Y, Z) >.
\]

In the case of a hypersurface $F : M^n \to \overline{M}^{n+1}$, the Gauss-equation can be written as
\[
R(X, Y, Z, W) = \overline{R}(X, Y, Z, W) - < AX, W > < AY, Z > + < AX, Z > < AY, W >.
\]

The Codazzi-equation becomes
\[
(\overline{R}(X, Y)\xi) = (\nabla^{\perp} A)(X) - (\nabla^{\perp} A)Y,
\]

where
\[
(\nabla^{\perp} A)X = \nabla^{\perp} AX - A\nabla^{\perp} X.
\]

Moreover, if $\overline{\overline{M}}^{n+1}$ has constant section curvature $c$, then the basic equations reduce, respectively, to

**Gauss-equation**
\[
R(X, Y)Z = c(X \wedge Y)Z + (AX \wedge AY)Z,
\]

where $(X \wedge Y)Z = < Y, Z > < X, Z > < Y$.

**Codazzi-equation**
\[
(\nabla^{\perp} A)X = (\nabla A)Y.
\]

**Remark 2.1.** In the case of hypersurfaces, the Ricci-equation is identity.
We now, using moving frames, give the basic equations of the hypersurface \( F : M^n \to \overline{M}^{n+1} \). We will make use of the following convention on the ranges of indices:

\[
1 \leq A, B, C, \ldots \leq n + 1,
\]

\[
1 \leq i, j, k, \ldots \leq n,
\]

and we shall agree that repeated indices are summed over the respective.

Let \( e_1, \ldots, e_n, e_{n+1} \) be a local orthonormal frame of \( M \), such that \( e_1, \ldots, e_n \) are tangential to \( M \), then \( e_{n+1} \) is perpendicular to \( M \). Let \( \theta^1, \ldots, \theta^n, \theta^{n+1} \) be its dual coframe. Then the structure equations of \( M \) can be written as follows:

\[
\begin{aligned}
\{ d\theta^A &= -\theta^A_B \wedge \theta^B + \theta^A_C \Phi^A = 0, \\
\{ d\theta^A_B &= -\theta^A_C \wedge \theta^C_B + \Phi^A_B, \Phi^A = \frac{1}{2} \overline{R}^{A}_{BCD} \theta^C \wedge \theta^D,
\end{aligned}
\]

where \( \theta^A_B \) and \( \Phi^A_B \) are connection forms and curvature forms of \( M \).

Set \( F^* \theta^A = \omega^A, F^* \theta^A_B = \omega^A_B \), then the structure equations of \( M \) are

\[
\begin{aligned}
\{ d\omega^i &= -\omega^i_j \wedge \omega^j, \omega^i_j + \omega^j_i = 0, \\
\{ d\omega^i_j &= -\omega^i_k \wedge \omega^j_k + \Omega^i_j, \Omega^i_j = \frac{1}{2} R^i_{jkl} \omega^j \wedge \omega^l.
\end{aligned}
\]

The basic equations are

\[
\text{(Gauss-equation)} \quad R^i_{jkl} = \overline{R}^i_{jkl} + (h^{n+1}_{ik} h^{n+1}_{jl} - h^{n+1}_{il} h^{n+1}_{jk}),
\]

\[
\text{(Codazzi-equation)} \quad \overline{R}^n_{ijk} = h^{n+1}_{ijk} - h^{n+1}_{ijk},
\]

where \( \omega^i_{n+1} = h^{n+1}_{ij} \omega^j, h^{n+1}_{ij} = h^{n+1}_{ji}, h^{n+1}_{ij} \omega^k = dh^{n+1}_{ij} - h^{n+1}_{ik} \omega^j - h^{n+1}_{kj} \omega^i \). In fact, by \( \frac{8}{8} \), we have

\[
A(e_i) = \sum_{j=1}^{n} h^{n+1}_{ij} e_j.
\]

If the section curvature of \( M \) is a constant \( c \), then the basic equations become

\[
\begin{aligned}
\{ R_{ijkl} &= c(\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) + h_{ik} h_{jl} - h_{il} h_{jk} \quad \text{(Gauss-equation)}, \\
h_{ikj} &= h_{ijk} \quad \text{(Codazzi-equation)},
\end{aligned}
\]

(9)

where \( h_{ij} = h^{n+1}_{ij} \).

2.2.2 The Fundamental Theorem of Hypersurfaces

From now on, let \( \overline{M}^{n+1} = \mathbb{Q}_c^{n+1} \), where \( \mathbb{Q}_c^{n+1} \) denotes \((n+1)-dimension space form with constant sectional curvature \( c \). Then the fundamental theorem of hypersurfaces can be written as follows.

**Theorem 2.1** \([18]\). Let \( M^n \) be a simply connected Riemannian manifold, and let \( A \) be a symmetric section of \( \text{End}(TM) \) satisfying the Gauss and Codazzi equations. Then there exist an isometric immersion \( F : M^n \to \mathbb{Q}_c^{n+1} \) and a unit normal vector field \( \xi \) such that \( A \) coincides with the shape operator \( A_\xi \) of \( F \) with respect to \( \xi \).
3 Proof of Theorem 1.1

3.1 Reduce the existence of the isometric minimal immersion $F: U \to \mathbb{Q}_c^3$ to the existence of some kind of 1-forms

By the theorem 2.1 and the basic equations (9), one can easily prove the following lemma.

**Lemma 3.1.** Let $M$ be a simply connected Riemann surface. Let $g = (\omega^1)^2 + (\omega^2)^2$ be a Riemannian metric of $M$, and $\omega^1_2$ be the connection form of $g$, then there exists an isometric minimal immersion $F: M \to \mathbb{Q}_c^3$ if and only if there exist two 1-forms

$$\begin{align*}
\omega^3_1 &= h_{11}\omega^1 + h_{12}\omega^2, \\
\omega^3_2 &= h_{21}\omega^1 + h_{22}\omega^2,
\end{align*}$$

which satisfy

$$\begin{align*}
h_{11} &= -h_{22}, \\
h_{12} &= h_{21},
\end{align*}$$

and

$$\begin{align*}
d\omega^1_1 &= -\omega^3_1 \wedge \omega^3_2 - \omega^1 \wedge \omega^2 \quad \text{(Gauss-equation)}, \\
d\omega^1_2 &= \omega^1_2 \wedge \omega^3_2 \quad \text{(Codazzi-equation)}, \\
d\omega^1_3 &= -\omega^2_2 \wedge \omega^3_1 \quad \text{(Codazzi-equation)}.
\end{align*}$$

3.2 Proof of Theorem 1.1

**Lemma 3.2.** Let $M$ be a compact Riemann surface, and $g$ be a non-CSC HCMU metric on $M$. Suppose $\omega$ and $K$ are the character 1-form and the Gauss curvature of $g$. Suppose the maximum and the minimum of $K$ are $K_1, K_2$ respectively. Denote $M \ {\text{zeros and poles of } \omega}$ by $M^*, \sqrt{-\frac{4}{3}(K - K_1)(K - K_2)(K + K_1 + K_2)}$ by $\mu = \mu(K)$. If for any point $P \in M^*$, there exist an open neighborhood $P \in U \subseteq M^*$ and an isometric minimal immersion $F: U \to \mathbb{Q}_c^3$, then there is a complex value function $h$ such that

$$\begin{align*}
K &= c - \frac{4|h|^2}{\mu^2}, \\
b &= -\frac{\mu'h}{4}, \\
a &= \frac{\mu'h}{4} + \frac{\mu^2}{4(K-\mu)}.
\end{align*}$$

where $dh = a\omega + b\overline{\omega}$.

**Proof.** Set

$$\begin{align*}
\omega^1 &= \frac{\omega + \overline{\omega}}{2} - \mu, \\
\omega^2 &= \frac{\omega - \overline{\omega}}{2\sqrt{-1}} - \mu.
\end{align*}$$

Then, by (5),

$$g = \mu^2\omega \overline{\omega} = (\omega^1)^2 + (\omega^2)^2,$$

and

$$dK = \frac{\mu^2}{4}(\omega + \overline{\omega}).$$

Since

$$d\omega^1 = \mu'(K)dK \wedge \frac{\omega + \overline{\omega}}{2} = 0,$$
\[ d\omega^2 = \mu' dK \wedge \frac{\omega - \overline{\omega}}{2\sqrt{-1}} = \frac{\mu'}{2} \omega^1 \wedge \omega^2, \]
then the connection 1-form of \( g \) is
\[ \omega_1^2 = \frac{\mu'}{2} \omega^2. \]

By Lemma 3.1, there exist two 1-forms
\[ \begin{cases} \omega_1^3 = h_{11} \omega^1 + h_{12} \omega^2, \\ \omega_2^3 = h_{21} \omega^1 + h_{22} \omega^2, \end{cases} \]
satisfying
\[ \begin{cases} h_{11} = -h_{22}, \\ h_{12} = h_{21}, \end{cases} \quad (10) \]
and
\[ \begin{cases} d\omega_1^2 = -\omega_1^3 \wedge \omega_2^3 - \omega_1^1 \wedge \omega^2 \text{(Gauss-equation)}, \\ d\omega_1^3 = \omega_1^2 \wedge \omega_2^3 \text{(Codazzi-equation)}, \\ d\omega_2^3 = -\omega_1^2 \wedge \omega_1^3 \text{(Codazzi-equation)}. \]

Assume
\[ \begin{cases} \omega_1^3 = f \omega + \overline{f} \omega, \\ \omega_2^3 = h \omega + \overline{h} \omega. \end{cases} \]

Then
\[ h_{11} = \frac{f + \overline{f}}{\mu}, h_{12} = \frac{\sqrt{-1}(f - \overline{f})}{\mu}, h_{21} = \frac{h + \overline{h}}{\mu}, h_{22} = \frac{\sqrt{-1}(h - \overline{h})}{\mu}. \]

So, by (10),
\[ f = -\sqrt{-1} h. \]

Therefore,
\[ \begin{cases} \omega_1^3 = -\sqrt{-1}(h \omega - \overline{h} \omega), \\ \omega_2^3 = h \omega + \overline{h} \omega. \end{cases} \]

Since
\[ \begin{cases} d\omega_1^2 = -K \omega_1^1 \wedge \omega^2, \\ \omega_1^1 \wedge \omega_2^3 = \frac{-4|h|^2}{\mu^2} \omega^1 \wedge \omega^2, \end{cases} \]
then the Gauss-equation becomes
\[ K = c - \frac{4|h|^2}{\mu^2}. \]

Let \( dh = a \omega + b \overline{\omega} \), then \( d\overline{h} = \overline{a} \omega + b \overline{\omega} \), and
\[ \begin{cases} d\omega_1^3 = \sqrt{-1} (b + \overline{b}) \omega \wedge \overline{\omega}, \\ d\omega_2^3 = (\overline{b} - b) \omega \wedge \overline{\omega}. \end{cases} \]

Since
\[ \begin{cases} \omega_1^2 \wedge \omega_2^3 = \frac{\mu'}{4\sqrt{-1}} (h + \overline{h}) \omega \wedge \overline{\omega}, \\ \omega_1^3 \wedge \omega_2^3 = \frac{-\mu'}{4} (h - \overline{h}) \omega \wedge \overline{\omega}, \end{cases} \]
then the Codazzi-equation becomes
\[ \begin{cases} b + \overline{b} = \frac{-\mu'}{4} (h + \overline{h}), \\ b - \overline{b} = \frac{-\mu'}{4} (h - \overline{h}), \end{cases} \]
i.e.,

\[ b = -\frac{\mu' \mu}{4} h. \]

To sum up, we get

\[
\begin{aligned}
K &= c - \frac{4h^2}{\mu^2} \quad \text{(Gauss-equation)}, \\
b &= -\frac{\mu' \mu h}{4} \quad \text{(Codazzi-equation)}.
\end{aligned}
\]

Differentiating two sides of the Gauss-equation, we get

\[ a h + bh = -\frac{\mu^2[2\mu' \mu (K - c) + \mu^2]}{16}. \]

Since \( b = -\frac{\mu' \mu}{4} h \), then

\[ a = -\frac{\mu^2[3\mu' \mu (K - c) + \mu^2]}{16h} = \frac{3\mu' \mu h}{4} + \frac{\mu^2 h}{4(K - c)}. \]

**Lemma 3.3.** There does not exist a function \( h \) satisfying the conditions in Lemma 3.2.

**Proof.** Since \( dh = a \omega + b \overline{\omega} \), so

\[ d^2h = d(a \omega + b \overline{\omega}) = da \wedge \omega + db \wedge \overline{\omega} = 0. \]

Since

\[
\begin{aligned}
da &\equiv \frac{\mu^3 h}{16} [3\mu'' + \frac{\mu'}{K - c} - \frac{\mu}{(K - c)^2}] \overline{\omega} \quad \text{(mod } \omega), \\
db &\equiv -\frac{\mu^2 h}{16} [\mu'' \mu + 4(\mu')^2 + \frac{\mu' \mu}{K - c}] \omega \quad \text{(mod } \overline{\omega}), \\
da \wedge \omega &\equiv \frac{\mu^3 h}{16} [3\mu'' + \frac{\mu'}{K - c} - \frac{\mu}{(K - c)^2}] \overline{\omega} \wedge \omega, \\
db \wedge \overline{\omega} &\equiv \frac{\mu^2 h}{16} [\mu'' \mu + 4(\mu')^2 + \frac{\mu' \mu}{K - c}] \overline{\omega} \wedge \omega,
\end{aligned}
\]

so

\[ da \wedge \omega + db \wedge \overline{\omega} = \frac{\mu^2 h}{16(K - c)^2} [4\mu'' \mu (K - c)^2 + 4(\mu')^2 (K - c)^2 + 2\mu' \mu (K - c) - \mu^2] \overline{\omega} \wedge \omega = 0. \]

Thus

\[ 4\mu'' \mu (K - c)^2 + 4(\mu')^2 (K - c)^2 + 2\mu' \mu (K - c) - \mu^2 = 0. \]  \hspace{1cm} (11)

Suppose

\[ \mu = \sqrt{-\frac{4}{3}(K - K_1)(K - K_2)(K + K_1 + K_2)} = (-\frac{4}{3}K^3 + \lambda_1 K + \lambda_2)^{1/2}, \]

where \( \lambda_1 = \frac{4}{9}(K_1^2 + K_2^2 + K_1 K_2), \lambda_2 = \frac{4}{9}K_1 K_2(K_1 + K_2) \), then

\[ \mu' = \frac{1}{2}(-\frac{4}{3}K^3 + \lambda_1 K + \lambda_2)^{-1/2}(-4K^2 + \lambda_1), \]

\[ \mu'' = -\frac{1}{4}(-\frac{4}{3}K^3 + \lambda_1 K + \lambda_2)^{-3/2}(-4K^2 + \lambda_1)^2 - 4K(-\frac{4}{3}K^3 + \lambda_1 K + \lambda_2)^{-1/2}, \]
\[
\begin{align*}
\mu \mu'' &= -\frac{1}{4}(-\frac{4}{3}K^3 + \lambda_1 K + \lambda_2)^{-1}(-4K^2 + \lambda_1)^2 - 4K, \\
(\mu')^2 &= \frac{1}{4}(-\frac{4}{3}K^3 + \lambda_1 K + \lambda_2)^{-1}(-4K^2 + \lambda_1)^2, \\
\mu \mu' &= \frac{1}{2}(-4K^2 + \lambda_1), \\
\mu \mu'' + (\mu')^2 &= -4K,
\end{align*}
\]

So
\[
4\mu''\mu(K-c)^2 + 4(\mu')^2(K-c)^2 \neq \mu^2 - 2\mu'\mu(K-c),
\]
that is the identity (11) is not true.

The proof of Theorem 1.1 obtains from Lemmas 3.2, 3.3.

References

[1] E. Calabi, “Extremal Kähler metrics” in Seminar on Differential Geometry, Ann. of Math. Stud. 102, Princeton Univ. Press, Princeton, 259-290 (1982)

[2] X.X. Chen, Weak limits of Riemannian metrics in surfaces with integral curvature bound, Calc. Var. 6, 189-226 (1998)

[3] X.X. Chen, Extremal Hermitian metrics on Riemann surfaces, Calc. Var. Partial Differential Equations 8, no. 3, 191-232 (1999)

[4] X.X. Chen, Obstruction to the Existence of Metric whose Curvature has Umbilical Hessian in a K-Surface, Comm. Anal. Geom. 8, no. 2, 267-299 (2000)

[5] Q. Chen, X.X. Chen and Y.Y. Wu, The Structure of HCMU Metric in a K-Surface, Int. Math. Res. Not. 2005, no. 16, 941-958 (2005)

[6] Q. Chen and Y.Y. Wu, Existences and Explicit Constructions of HCMU metrics on \(S^2\) and \(T^2\), Pac. J. Math. 240, no. 2, 267-288 (2009)

[7] Q. Chen and Y.Y. Wu, Character 1-form and the existence of an HCMU metric, Math. Ann. 351, no. 2, 327-345 (2011)

[8] Q. Chen, Y.Y. Wu and B. Xu, On One-dimensional and singular calabi’s extremal metrics whose gauss curvatures have nonzero umbilical Hessians, Isr. J. Math. 208, 385-412 (2015)

[9] C.S. Lin and X.H. Zhu, Explicit construction of extremal Hermitian metric with finite conical singularities on \(S^2\), Comm. Anal. Geom. 10, no. 1, 177-216 (2002)

[10] C.K. Peng, Y.Y. Wu, A one-dimensional singular non-CSC extremal Kähler metric can be imbedded into \(\mathbb{R}^3\) as a Weingarten surface. Results Math. 75, 133 (2020)

[11] M. Troyanov, Prescribing curvature on compact surface with conical singularities. Tran. Am. Math. Soc. 324(2), 793-821 (1991)

[12] G.F. Wang and X.H. Zhu, Extremal Hermitian metrics on Riemann surfaces with singularities, Duke Math. J. 104, 181-210 (2000)
[13] Z.Q.Wei and Y.Y.Wu, Multi-valued holomorphic functions and non-CSC extremal Kähler metrics with singularities on compact Riemann surfaces, Differ. Geom. Appl. 60(10), 66-79 (2018)

[14] Z.Q.Wei and Y.Y.Wu, Local isometric imbedding of a compact Riemann surface with a singular non-CSC extremal Kähler metric into 3-dimension space forms, J.Geom.Anal 32,27 (2022)

[15] S.T.Yau, Calabi’s conjecture and some new results in algebraic,Proceedings of the National Academy of science of the united states of America, 74(5):1798-1799(1977)

[16] S.T.Yau, On the Ricci Curvature of a compact Kähler manifold and the complex Monge-Ampère equation I, Communications on Pure and Applied Mathematics, 31(3): 339-411(1978)

[17] T.Aubin, Nonlinear Analysis on Manifolds, Monge Ampère Equations, Grundlehren der Mathematicschen Wissenchaften, 252. Spring-Verlag, New York, 1982

[18] Marcos Dajczer, Submanifolds and Isometric Immersions, Houston,Texas, (1990) Addison-Wesley, 1957

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