A large number of $m$-coloured complete infinite subgraphs

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Abstract

Given an edge colouring of a graph with a set of $m$ colours, we say that the graph is $m$-coloured if each of the $m$ colours is used. For an $m$-colouring $\Delta$ of $\mathbb{N}^{(2)}$, the complete graph on $\mathbb{N}$, we denote by $F_\Delta$ the set all values $\gamma$ for which there exists an infinite subset $X \subset \mathbb{N}$ such that $X^{(2)}$ is $\gamma$-coloured. Properties of this set were first studied by Erickson in 1994. Here, we are interested in estimating the minimum size of $F_\Delta$ over all $m$-colourings $\Delta$ of $\mathbb{N}^{(2)}$. Indeed, we shall prove the following result. There exists an absolute constant $\alpha > 0$ such that for any positive integer $m \neq \{ (\frac{n}{2}) + 1, (\frac{n}{2}) + 2 : n \geq 2 \}$, $|F_\Delta| \geq (1 + \alpha)\sqrt{2m}$, for any $m$-colouring $\Delta$ of $\mathbb{N}^{(2)}$, thus proving a conjecture of Narayanan. This result is tight up to the order of the constant $\alpha$.

1 Introduction

Frank Ramsey [8] proved in the 30’s that whenever the edges of a complete infinite countable graph are finitely coloured there exists a monochromatic infinite complete subgraph. Since then, numerous generalisations of this result have been proved and we shall refer the reader to the book of Graham, Rothschild and Spencer [5] for an overview of more recent results.

In Ramsey theory one is usually concerned with finding large monochromatic substructures in any finite colouring of a ‘rich’ structure. Perhaps the most fundamental problem in the area is concerned with finding good estimates for the well known diagonal ramsey numbers. Unfortunately, the current best lower and upper bounds are still quite far apart (see [11, 1, 2, 9]).

In 1975, Erdős, Simonovits and Sós [3], started a new line of research, commonly known as Anti-Ramsey theory. The problems in this area lie at the opposite end of Ramsey theory, in here one is interested in finding large totally multicoloured (rainbow) substructures.

A natural problem is to investigate what happens in between these two extremes. We say $\Delta$ is an $m$-colouring of the edges of $\mathbb{N}^{(2)}$ if $\Delta$ uses exactly $m$ colours, or in other words $\Delta : \mathbb{N}^{(2)} \rightarrow [m]$ is a surjective map. (As usual, we write $X^{(r)}$ for the set of $r$-subsets of a set $X$, and identify $X^{(2)}$ with the complete graph with vertex set $X$.) Let $\Delta : \mathbb{N}^{(2)} \rightarrow [m]$ be an edge colouring of the complete graph on $\mathbb{N}$, for a subset $X \subset \mathbb{N}$, we define $\gamma_\Delta(X)$ or $\gamma(X)$ to be the size of the set $\Delta(X^{(2)})$. Moreover, we let

$$F_\Delta = \{ \gamma(X) : X \subset \mathbb{N} \text{ such that } X \text{ is infinite} \}.$$

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Our main goal in this paper is to study the size of $\mathcal{F}_\Delta$, for finite colouring $\Delta$ of $\mathbb{N}^{(2)}$. This line of research has been pursued first by Erickson, later by Stacey and Weidl and recently by Narayanan and Narayanan and Kittipassorn.

Indeed, in 1996, Erickson [4] noted that for any $m$-colouring $\Delta$ ($m \geq 2$) of the complete graph on $\mathbb{N}$, the set $\{1, 2, m\}$ is always contained in $\mathcal{F}_\Delta$. Observe that Ramsey’s Theorem is equivalent to saying that $1 \in \mathcal{F}_\Delta$, and by the assumption on $\Delta$, one trivially has that $m \in \mathcal{F}_\Delta$. In the same paper, Erickson conjectured that the only values which are guaranteed to be in $\mathcal{F}_\Delta$, for any $m$-colouring $\Delta$ are 1, 2 and $m$. More precisely, he stated the following conjectured.

**Conjecture 1.1** (Erickson). Let $m > k > 2$ be positive integers. Then there exists an $m$-colouring $\Delta : \mathbb{N}^{(2)} \rightarrow [m]$ such that $k \notin \mathcal{F}_\Delta$.

This conjecture was partially settled by Stacey and Weidl [10]. They showed that for every $k$, there exists $C_k$ such that for any $m \geq C_k$ there is an $m$-colouring $\Delta$ of $\mathbb{N}^{(2)}$ for which $k \notin \mathcal{F}_\Delta$. This result implies it is a hopeless task to try to find particular values which appear in $\mathcal{F}_\Delta$, for every finite colouring $\Delta$. Therefore, one may turn to the question of estimating how small the set $\mathcal{F}_\Delta$ can be, for an arbitrary $m$-colouring $\Delta$ of $\mathbb{N}^{(2)}$. To do so, let us define a function $\psi$ on the naturals where

$$\psi(m) = \min_{\Delta : \mathbb{N}^{(2)} \rightarrow [m]} |\mathcal{F}_\Delta|.$$

This function $\psi$ was introduced by Narayanan [6] in 2014. Indeed, he showed the following theorem.

**Theorem 1.2** (Narayanan). Let $n \geq 2$ be the largest natural number such that $m \geq \binom{n}{2} + 1$. Then $\psi(m) \geq n$.

It is not hard to see that this theorem is tight for every integer $m$ of the form $\binom{n}{2} + 1, n \geq 2$. This can been seen by taking a rainbow complete graph on the first $n$ integers and colouring the remaining edges of $\mathbb{N}^{(2)}$ with a colour that has not appeared before. It is easy to check that for this colouring of $\mathbb{N}^{(2)}$, the set $\mathcal{F}_\Delta = \{\binom{t}{2} + 1 : t \leq n\}$. Note that Theorem 1.2 shows that $\lceil \sqrt{2m} \rceil \leq \psi(m)$, for every positive integer $m \geq 2$. Turning now to upper bounds for $\psi$, Narayanan proved that there is a subset $A \subset \mathbb{N}$ with ‘density’ 1, where $\psi(m) = o(m)$, for every $m \in A$ as $m \to \infty$. Unfortunately, it still an open whether $\psi(m) = o(m)$ as $m \to \infty$.

Indeed, the behaviour of $\psi$ inside intervals of the form $[\binom{n}{2} + 2, \binom{n+1}{2}]$ is far from being understood. There is one value, however, inside such intervals, for which we know the exact value of $\psi$, namely when $m = \binom{n}{2} + 2$. As a consequence of a slightly stronger version of Theorem 1.2 proved in [6], we know that $\psi(\binom{n}{2} + 2) \geq n + 1$. Moreover, by considering a similar colouring $\Delta$ as the one described above, we have equality. To see this, take as before, a rainbow colouring on the first $n$ positive integers and colour the edges between any two integers in $[n + 1, \infty]$ with a new colour, finally colour every edge between an integer in $\{1, 2, \ldots, n\}$ and the remaining integers with another distinct colour from those used before. It is clear this colouring uses $\binom{n}{2} + 2$ colours and it is easy to check that in this case $\mathcal{F}_\Delta = \{1\} \cup \{\binom{t}{2} + 2 : t \leq n\}$.

Rather surprisingly, it is not even known whether $\psi$ is a monotone function. Narayanan, however, conjectured $\psi$ is far from being monotone and that it actually ‘suffers’ huge jumps. In [6], he stated the following beautiful conjecture.

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**Note:** The above text is a reproduction of the content from the original document, with minimal adjustments to improve readability and coherence. The document discusses the study of the size of $\mathcal{F}_\Delta$ for finite colourings, focusing on Conjecture 1.1 by Erickson and Theorem 1.2 by Narayanan, along with related results and conjectures by Erickson, Stacey and Weidl, and Narayanan. The text also mentions a function $\psi(m)$ and its properties, including upper and lower bounds, and a conjecture about its monotonicity and behavior inside intervals.
**Conjecture 1.3.** There is an absolute constant $c > 0$ such that $\psi\left(\binom{n}{2} + 3\right) > (1 + c)n$, for all natural numbers $n \geq 2$.

Our main aim in this paper is to prove this conjecture in the following slightly more general form.

**Theorem 1.4.** There exists a constant $\alpha > 0$ such that $\psi(m) \geq (1 + \alpha)\sqrt{2m}$, for every integer $m \notin \left\{\binom{n}{2} + 1, \binom{n}{2} + 2 : n \geq 2\right\}$.

Observe that it follows immediately from this result that $\psi$ is not a monotone function as $\psi\left(\binom{n}{2} + 3\right)$ is much bigger than $\psi\left(\binom{n+1}{2}\right)$. Moreover, by making a small variation on the colourings described above, it is possible to construct $(\binom{n}{2} + 3)$-colourings $\Delta$ of $\mathbb{N}^{(2)}$ (for all $n \geq 4$) for which $|F_\Delta| \leq (1 + \frac{1}{2})n = \frac{3}{2}\sqrt{2m} + O(1)$, thus implying that our result is tight up to the order of the constant $\alpha$. To this end, take a rainbow colouring on the first $n$ integers and colour every edge between any two integers greater than $n$ with a colour not used before, say colour 1. Finally, colour the remaining edges with an endpoint in $\{1, 2, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\}$ with a distinct colour, say colour 2 and colour the rest of edges (those incident with an integer in $\{\left\lceil \frac{n}{2} \right\rceil, \left\lceil \frac{n}{2} \right\rceil + 1, \ldots, n\}$) with yet a new colour, distinct from all colours we have used before.

### 2 Notation

Our notation is standard. As usual, we write $[n]$ for $\{1, 2, \ldots, n\}$, the set of the first $n$ natural numbers. We denote a surjective map $f$ from a set $X$ to another set $Y$ by $f : X \to Y$. In this paper, by a *colouring* of a graph we mean a colouring of the edges of the graph, unless stated otherwise. Moreover, as expressed above, by an *$m$-colouring* of a graph, we mean a colouring which uses *exactly* $m$. Indeed, given a colouring $\Delta : \mathbb{N}^{(2)} \to C$ of the complete graph on $\mathbb{N}$, a subset $X$ of $\mathbb{N}$ is $m$-coloured if $\Delta(X^{(2)})$, the set of values attained by $\Delta$ on the edges with both endpoints in $X$, has size exactly $m$ colours. Now, the colouring $\Delta$ is called a *finite* colouring if $C$ has finite order and an *infinite* colouring, otherwise. As usual given a colouring of $X^{(2)}$, we say $X$ is *monochromatic* if all edges of $X^{(2)}$ have the same colour and $X$ is *rainbow* if every edge has a *distinct* colour. We shall denote by $(A, B)$ the complete bipartite graph between the sets $A$ and $B$.

### 3 Framework and preliminary results

To prove Theorem 1.4, we shall first show how we can pass from a $m$-colouring of $\mathbb{N}^{(2)}$ to a colouring of a finite complete graph using exactly $m$ colours and establish a correspondence between $\gamma$-coloured infinite subsets of $\mathbb{N}$ and $\gamma$-coloured subgraphs of the complete graph. To see this, suppose $\Delta : \mathbb{N}^{(2)} \to [m]$ is an $m$-colouring of $\mathbb{N}^{(2)}$. By Ramsey’s Theorem, there exists an infinite set $X_1 \subset \mathbb{N}$ all of whose edges are coloured with the same colour. Now, we claim we can find an infinite subset $X' \subset X_1$ together with a finite set $X \subset \mathbb{N} \setminus X'$ such that all $m$ colours appear in $\Delta((X \cup X')^{(2)})$ and more importantly, for every vertex $x \in X$, every edge between $x$ and $X'$ has the same colour.
Lemma 3.1. Let \( m \geq 2 \) be a positive integer and let \( \Delta : \mathbb{N}^{(2)} \to [m] \) be a \( m \)-colouring of \( \mathbb{N}^{(2)} \). Then, we can find an infinite set \( X' \subset \mathbb{N} \) and a finite set \( X \subset (\mathbb{N} \setminus X') \) satisfying the following three properties.

1. \( X' \) is a monochromatic infinite complete graph;
2. For every vertex \( x \in X \), every edge from \( x \) to \( X' \) has the same colour;
3. \( \Delta((X \cup X')^{(2)}) = [m] \).

Proof. To prove this lemma we shall consider the following iterative procedure. We start with \( X' = X_1 \) and \( X = \emptyset \). If \( \Delta((X' \cup X)^{(2)}) \) spans all \( m \) colours, we shall stop, otherwise, there must exist some edge \( e = (a, b) \) of colour \( m' \notin \Delta((X' \cup X)^{(2)}) \). First, suppose neither \( a \) nor \( b \) belong to \( X' \cup X \). In this case, we add both \( a \) and \( b \) to \( X \) and we pass to an infinite subset \( X'' \subset X' \) in which every edge from \( a \) to \( X'' \) has the same colour and similarly, every edge from \( b \) to \( X \) has the same colour. We then set \( X = X \cup \{a, b\} \) and \( X' = X'' \). Note that we have added the new colour \( m' \) to \( \Delta((X \cup X')^{(2)}) \) without affecting the presence of colours that already appeared before.

Suppose now exactly one of vertex, say \( a \), belongs to \( X \cup X' \). If \( a \in X \), we add \( b \) to \( X \) and, as before, we pass to an infinite subset \( X'' \subset X' \) where every edge from \( b \) to \( X'' \) has the same colour. If \( a \in X' \), then we remove \( a \) from \( X' \) and add both \( a \) and \( b \) to \( X \). As before, we pass to an infinite subset of \( X'' \subset X' \) in which every edge from \( b \) to \( X'' \) has the same colour. Observe that by assumption on \( X_1 \), \( a \) sends colour 1 to every vertex in \( X'' \). Hence, by setting \( X = X \cup \{a, b\} \) and \( X' = X'' \), we enlarge the set \( \Delta((X \cup X')^{(2)}) \) by at least one new colour. As the total number of colours is finite, this process must stop and we have find our desired sets \( X \) and \( X' \).

Having the sets \( X \) and \( X' \), as in Lemma 3.1, in mind, we may now construct a coloured complete graph \( G(X', X) = G \) on a vertex set of size \( |X| + 1 \) as follows. Set \( V(G) = X \cup \{x'\} \), where \( x' \) is the special vertex of \( G \) obtained by identifying all vertices of \( X' \). The colouring \( \Delta' \) of the edges of \( G \) is obtained from the colouring \( \Delta \) induced on \( X \cup X' \). Finally, we colour the special vertex \( x' \) with the unique colour of the edges in \( X' \).

We shall always denote by \( \Delta' \) the special colouring of \( G(X, X') \) which includes the colouring of the edges of \( G \) together with the colouring of the special vertex \( x' \).

Given a complete graph \( G \) with a special vertex \( x' \) and a special colouring \( \Delta' \), we say with a slight abuse of notation that \( \Delta' \) is an \( m \)-colouring of \( G \) if the total number of colours used by \( \Delta' \) is \( m \) (including the colour of the special vertex). For a subset \( S \subseteq V(G) \) with \( x' \in S \), we let \( \gamma(S) \) or \( \gamma_{\Delta'}(S) = \{\Delta'(S^{(2)}) \cup \{\Delta'(x')\}\} \). Finally, we define \( \mathcal{G}_{\Delta'}(G) = \{\gamma(S) : S \subset V(G) \text{ and } x' \in S\} \).

To conclude the reduction of our problem from a finite colouring of \( \mathbb{N}^{(2)} \) to a colouring of a complete graph \( G \), we need to establish a correspondence between coloured complete subgraphs of \( G \) and coloured infinite subgraphs of \( \mathbb{N}^{(2)} \).

Proposition 3.2. Let \( m \geq 2 \) be a positive integer and \( \Delta : \mathbb{N}^{(2)} \to [m] \) an \( m \)-colouring of \( \mathbb{N}^{(2)} \). Let \( X, X' \subset \mathbb{N} \) be two subset of \( \mathbb{N} \), as in Lemma 3.1. Finally, let \( G = G(X, X') \) be a coloured complete graph as constructed above with a special colouring \( \Delta' \). Then, \( \mathcal{G}_{\Delta'}(G) \subseteq \mathcal{F}_\Delta \).
The proof of this proposition follows from the definition of $G(X, X')$.

This allows us to restrict our attention to coloured finite complete graphs with a coloured special vertex, in order to find a lower bound for the size of $F_\Delta$.

We remark that given a special $m$-colouring $\Delta'$ of a finite complete graph $G = K_{n+1}$ with a special vertex $x'$, we may construct a colouring $\Delta$ of $\mathbb{N}^{(2)}$ with $F_\Delta \subseteq G'_\Delta$. To see this, consider a copy of the coloured complete graph $G \setminus \{x'\}$ with vertex set the first $n$ natural numbers and colour every edge between any two naturals in $[n+1, \infty]$ with colour $\Delta'(x)$. For every $i \in [n]$, we colour every edge between $i$ and any natural in $[n+1, \infty]$ with the colour of the edge between the vertex corresponding to $i$ in $G \setminus \{x'\}$ and the special vertex $x'$. This observation implies that

$$\psi(m) = \min_{G,m\text{-colouring } \Delta'} |G'_\Delta(G)|,$$

where the minimum is taken over every finite complete graph $G$ with a special $m$-colouring $\Delta'$.

It is clear that this minimum can actually be taken over bounded size complete graphs, where the bound depends only on $m$.

From now on, we shall only be considering complete graphs $G$ with a special $m$-colouring $\Delta'$. Our aim is to bound the size of $G'_\Delta(G)$.

**Theorem 3.3.** Let $m, n \geq 2$ be positive integers. Let $G = K_{n+1}$ be a complete graph on $n$ vertices with a special vertex $x'$ and let $\Delta'$ be a special $m$-colouring of $G$. Then we can find a sequence of pairwise disjoint subsets of $V(G)$, $A_1, A_2, \ldots, A_l$, for some $l \in [n]$, satisfying the following properties.

1. $A_1 = \{x'\}$;
2. $A_i$ has either one or two elements, for all $i \in [l]$.
3. $\{\gamma_{\Delta'}(\bigcup_{i=1}^{j} A_i)\}_{j \in [l]}$ forms a strictly increasing sequence and all $m$ colours appear in $\gamma_{\Delta'}(\bigcup_{j=1}^{l} A_j)$;
4. Whenever $A_l$ has two vertices, the set of colours appearing on the edges of the bipartite graph $(\bigcup_{i=1}^{l-1} A_i, A_l)$ is contained in the set $\gamma_{\Delta'}(\bigcup_{i=1}^{l-1} A_i)$, for every $l \geq j$.

Moreover, the sets $A_1, \ldots, A_l$ can be chosen as to maximize the number of new colours that appear on each step.

(5)* For every $i < j$ and for any vertices $v \in A_i$, $w \in A_j$, the size of $\gamma_{\Delta'}(\bigcup_{i=1}^{j} A_i) \setminus \gamma_{\Delta'}(\bigcup_{i=1}^{j-1} A_i)$ is greater or equal than the size of $\gamma_{\Delta'}(\bigcup_{i=1}^{j} A_i) \setminus \gamma_{\Delta'}(\bigcup_{i=1}^{j-1} A_i)$.

**Proof.** Our proof of this theorem is similar in spirit to the proof of Lemma 3.1. We proceed interatively, finding at step $i$ a new set $A_i$. One difference is that here, we want to avoid as much as possible the choice of sets of order 2, i.e. if we can find a set $A_i$ of order 1, for which the sequence $A_1, \ldots, A_i$ satisfies all properties (1) – (4), then we shall choose that set.

To begin this procedure, we set $A_1 = \{x'\}$. Suppose we have found already found the first $t$ sets $A_1, \ldots, A_t$. If $\gamma(\bigcup_{i=1}^{t} A_i)$ contains all the $m$ colours, we stop. Otherwise, there is an edge $e = (a, b)$ whose colour does not appear in $\gamma(\bigcup_{i=1}^{t} A_i)$. 

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If there is such an edge $e = (a, b)$ with exactly one endpoint, say $a$, belonging to the set $\bigcup_{i=1}^{t} A_i$, then we set $A_{t+1} = \{b\}$. Note that $A_{t+1}$ has order 1 and the set $\gamma_{\Delta'}(\bigcup_{i=1}^{t+1} A_i)$ strictly contains $\gamma_{\Delta'}(\bigcup_{i=1}^{t} A_i)$. Hence, the properties (1) – (3) hold for the sequence $A_1, A_2, \ldots, A_{t+1}$. Finally we need to check the sequence $A_1, A_2, \ldots, A_{t+1}$ also satisfy property (4). To see this, suppose $A_i$ has two elements, for some $i \leq t$, then the set of colours that appear on the edges of the bipartite graph $(b, \bigcup_{j=1}^{i} A_j)$ must be same as the set of colours that appear on the edges of $(A_i, \bigcup_{j=1}^{i-1} A_j)$, since otherwise we would have chosen $A_i$ to be $\{b\}$. This proves (4) also holds.

Suppose now that for every edge $e = (a, b)$ with a colour not in $\gamma_{\Delta'}(\bigcup_{i=1}^{t} A_i)$, neither of the endpoints if $e$ belong to $\bigcup_{i=1}^{t} A_i$. Then we set $A_{t+1} = \{a, b\}$. As before, it is easy to see all properties (1) – (4) are satisfied by the new sequence $A_1, A_2, \ldots, A_{t+1}$. Observe this process must stop as the number of colours is finite.

Furthermore, note that we can choose at each step a set $A_i$ as to maximize the number of new colours appearing on the edges between $A_i$ and $\bigcup_{j=1}^{i-1} A_j$, implying the sets $A_1, A_2, \ldots, A_t$ may be chosen to satisfy (5)*.

\[ \sum_{i=1}^{j} c(i) \leq \left( \frac{j}{2} \right) + 1 \quad (2) \]

Indeed, if $A_{j+1}$ has order two, from property (3) we deduce that $c(j+1) = c(j+1) + 1 \leq \left( \frac{j}{2} \right) + 1 + 1 \leq \left( \frac{j+1}{2} \right) + 1$. Suppose now $A_{j+1} = \{w\}$ has order one. Let $2 \leq t \leq j - 1$ be largest index for which $A_t$ has order two (assume $t$ is well defined), then by property (4), the number of new colours in the bipartite graph $(w, \bigcup_{i=1}^{j} A_i)$ is the same as the number of new colours in $(w, \bigcup_{i=t}^{j} A_i)$, which implies $c(j+1) \leq c(j) + 2 + (j - t) \leq j \leq \left( \frac{j+1}{2} \right) + 1$. If no set has order 2, then $c(j+1) \leq c(j) + j \leq \left( \frac{j+1}{2} \right) + 1$, as claimed. Therefore, $l$ must be at least $n$. Hence, for every $k \in [l]$, the subgraphs $G[\bigcup_{i=1}^{k} A_i]$ are $\gamma_k$-coloured where $\gamma_k \neq \gamma_{k'}$ if $k \neq k'$. Note that we have shown that $\gamma_{k+1} \leq \gamma_k + k$, for every $k \in [l]$.

The second consequence is the following result which was proved by Narayanan and Kittipassorn [7] and which answered a question of Narayanan from [6].

**Theorem 3.4.** Let $m \geq 2$ be a positive integer and let $\Delta : \mathbb{N}^{(2)} \to [m]$ be an $m$-colouring of the complete graph on $\mathbb{N}$ and suppose $n$ is a natural number such that $m > \left( \frac{n}{2} \right) + 1$. Then,

\[ \mathcal{F}_\Delta \cap \left( \left( \frac{n}{2} \right) + 1, \left( \frac{n+1}{2} \right) + 1 \right) \neq \emptyset. \]
Sketch of the proof. As described before, we create a special $m$-colouring $\Delta'$ of a complete graph $G$ associated with the colouring $\Delta$. We now apply Theorem 3.3 to $G$ and denote by $A_1, \ldots, A_t$ the sets given by Theorem 3.3 satisfying (5)*. Denote by $\gamma_k$ the order of the set $\gamma_{\Delta'}(\bigcup_{i=1}^k A_i)$, for every $k \in [t]$.

Now, let $t \in [t]$ be the largest index for which $\gamma_t = |\gamma_{\Delta'}(\bigcup_{i=1}^t A_i)| \leq \binom{n+1}{2} + 1$. If $\gamma_t \in (\binom{n}{2} + 1, \binom{n+1}{2} + 1]$, we are done. So we may assume $\gamma_t \leq \binom{n}{2} + 1$. Note that equation (2) tells us that $t \geq n + 1$.

We claim that $\gamma_{t+1} \leq \gamma_t + n$, which contradicts the maximality of $t$.

First, suppose that every set $A_j$ has order 1, for every $j \leq t$. If $A_{t+1}$ has two elements then by property (4), $\gamma_{t+1} = \gamma_t + 1$, as claimed.

Suppose the claim is false, then we have $\gamma_{t+1} \geq \gamma_t + n + 1$. By property (5)*, we deduce there exist an increasing sequence $i_1 < \ldots < i_n$ in $\{1, 2, \ldots, t\}$, where $c(i_k) \geq k$. Hence, $\gamma_t$ is at least $\left(\binom{n+1}{2}\right) + (t - (n + 1))$, which is a contradiction, and the claim follows.

Secondly, suppose some set $A_k$ has order two, for some $k \leq t$. We may apply the same reasoning as we just did to the sequence $A_k, A_{k+1}, \ldots, A_t$, where $k \leq t$ is the largest index for which $A_k$ has order two. \hfill \qed

4 Proof of the main theorem

The aim of this section is to prove Theorem 1.4. As we saw in Section 3, in order to prove Theorem 1.4, it suffices to show that for every $m \notin \{(\binom{n}{2} + 1, \binom{n}{2} + 2 : n \geq 2\}$ and every special $m$-colouring $\Delta'$ of a complete graph $G$ with a special vertex $x'$,

$$|G'_{\Delta}(G)| \geq (1 + \alpha)\sqrt{2m}.$$

To prove this inequality, we need to introduce a few more definitions. We shall assume from now on that the special vertex $x'$ has colour 1.

Moreover, given a vertex $v \in G \setminus \{x'\}$, we let $\rho(v)$ be the number of distinct colours which appear only on the edges incident with $v$, or in other words $\rho(v)$ counts the number of colours which cease to exist when the vertex $v$ is deleted. In particular, the colour 1 never contributes to $\rho(v)$. We also denote by $G^{\geq 1} \subseteq G$ the complete subgraph induced on the set of vertices $\{v \in G' : \rho(v) \geq 1\} \cup \{x'\}$ and we let $G' = G \setminus \{x'\}$. We shall call an edge $e \in G^{(2)}$ uniquely coloured if $\Delta'(e) \neq \{1\}$ and no other edge shares the same colour as $e$.

Finally, we say that $\Delta'$ is a bad (special) colouring of a complete graph $G = K_{n+1}$ (for some $n \geq 1$), if satisfies one of the following two types.

(i) $G$ is a rainbow complete graph and none of the edges in $G$ have colour 1;

(ii) $G \setminus \{x'\} = G'$ forms a rainbow complete graph where every edge has a colour distinct from 1. Moreover, every edge incident with $x'$ has either colour 1 or colour 2 and colour 2 does not appear in $G'$.

Observe that if $m \notin \{(\binom{n}{2} + 1, \binom{n}{2} + 2 : n \geq 2\}$ then no $m$-colouring can be bad.

We shall deduce Theorem 1.4 from the following stronger result.
**Theorem 4.1.** There exists an absolute constant \( \alpha' > 0 \) such that if \( G = K_{n+1} \) is a complete graph with a special \( m \)-colouring \( \Delta' \) and \( G = G^{\geq 1} \), then at least one of the following holds.

- \( \Delta' \) is a bad colouring;
- \(|G'_{\Delta'}(G)| \geq (1+\alpha')\sqrt{2m} \).

**Deduction of Theorem 1.4 from Theorem 4.1.** Let \( \Delta \) be an \( m \)-colouring of \( \mathbb{N}^{(2)} \). Construct a coloured complete graph \( G \) with a special vertex \( x' \) and with a special \( m \)-colouring \( \Delta' \), as seen before. Consider the subgraph \( G^{\geq 1} \subseteq G \). Note that the colouring induced on \( G^{\geq 1} \) can not be bad, by the assumption on \( m \). The conclusion of the theorem now follows from Proposition 3.2 and Theorem 4.1.

The rest of this section will be devoted to prove Theorem 4.1.

Firstly, we state the following lemma which will be useful later in our proof. It asserts that Theorem 4.1 holds when the special colouring of \( G \) satisfies some properties.

**Lemma 4.2.** Let \( n \geq m \geq 15 \) be positive integers and let \( \rho \geq 6 \). Let \( \Delta' : G^{(2)} \rightarrow [m] \) be a special \( m \)-colouring of a complete graph \( G = K_{n+1} \) with special vertex \( x' \) of colour 1. Suppose \( \Delta' \) is not a bad colouring and every edge in \( G^{(2)} = (G \setminus \{x'\})^{(2)} \) is uniquely coloured. Moreover, suppose \( \rho(v) = \rho \), for every vertex in \( G' \). Then,

\[
|G'_{\Delta'}(G)| \geq \frac{5}{4}\sqrt{2m}.
\]

**Proof.** Observe that \( \rho \geq n-1 \), since every edge in \( G^{(2)} \) is uniquely coloured. If \( \rho = n \), then the colouring \( \Delta' \) would be a bad colouring of type (i). Hence, we may assume \( \rho = n-1 \). Let \( A = \{a_1, a_2, \ldots, a_k\} \subseteq [m] \) be the set of distinct colours (including colour 1) appearing on the edges incident with \( x' \). As \( \Delta' \) is not a bad colouring, \( k \geq 2 \), if \( 1 \notin A \) and \( k \geq 3 \), otherwise.

We partition now \( V(G') \) into \( k \) sets, \( V_1, V_2, \ldots, V_k \) such that for every \( v \in V_i \) the colour of the edge \((v, x')\) is \( a_i \). Note that since \( \rho = n-1 \), each \( V_i \) must have size at least 2, if \( a_i \neq 1 \). Clearly, we may assume the size \( V_k \) is smaller or equal than \( \frac{n}{2} \). Let \( x' = v_1 < v_2, \ldots < v_{n+1} \) be an ordering of the vertices of \( G \) with the property that every vertex of \( V_j \) comes before every vertex of \( V_j \), for every \( i < j \in [k] \). Furthermore, denoting by \( T_i \) the set of the first \( i \) vertices in this ordering, then we have \( \gamma_{\Delta'}(T_i) < \gamma_{\Delta'}(T_j) \), for every \( i < j \).

Consider the set \( S \subseteq [n+1] \) containing every \( s \in [n+1] \) with the property that \( v_s \in V_l \), for some \( l < k \) and \( v_s \) is not the first vertex of \( V_j \) in the above ordering. Furthermore, let \( T^*_s = T_s \setminus \{v_s\} \cup \{v_k\} \), for every \( s \in S \). As before, it is easy to see that \( \gamma_{\Delta'}(T^*_s) < \gamma_{\Delta'}(T^*_p) \), for every \( s < p \in S \). Finally, note that \( \gamma_{\Delta'}(T^*_s) \neq \gamma_{\Delta'}(T^*_p) \), for every \( s \in S \) and \( p \in [n+1] \). This holds since by construction \( \gamma_{\Delta'}(T^*_s) = \gamma_{\Delta'}(T_s) + 1 \), for every \( s \in S \), and the difference between \( \gamma_{\Delta'}(T_s) \) and \( \gamma_{\Delta'}(T_s+1) \) is greater than 1, for every \( s \geq 2 \). Hence,

\[
|G'_{\Delta'}(G)| \geq n + |S| \geq n + \left( \frac{n-|V_k|}{2} \right) \geq \frac{5}{4}n \geq \frac{5}{4}\sqrt{2m}.
\]
Proof of Theorem 4.1. Let $\alpha' \ll 1$, where $\alpha'$ is an absolute constant to be determined at the end. The proof will be by induction on $m$, the number of colours used by $\Delta'$. For $m \leq 15$ the statement follows immediately since $6 = \lceil \sqrt{30} \rceil \geq (1 + \alpha')\sqrt{30}$ which holds for $\alpha' \leq \frac{1}{20}$.

We may assume that the statement of the theorem holds for every integer $m' < m$ and we want to show it also holds for $m$. Let $G = K_{n+1}$ be a complete graph with a special $m$-colouring $\Delta'$ and with the special vertex $x'$. From now on, we shall assume, for contradiction, that the colouring $\Delta'$ does not satisfy the conclusions of Theorem 4.1. We may deduce the following series of claims.

Claim 4.3. For every $v \in V(G')$, $\rho(v) \leq \lceil (1 + \alpha')\sqrt{2m} \rceil$.

Suppose this claim is false. Let $v$ be a vertex with $\rho(v) \geq \lceil (1 + \alpha')\sqrt{2m} \rceil$. Moreover, let $w_1, w_2, \ldots, w_{\lceil \alpha'\sqrt{2m} \rceil}$ be some neighbours of $v$ for which the colours of the edges $(v, w_i)$ have distinct colours for distinct $i$ and these colours only appear on edges incident with $v$. We define the sets $W_i = \{v\} \cup \{w_{j} \mid j \neq i\} \cup \{x'\}$, for every $i \in \{1, 2, \ldots, \lceil \alpha'\sqrt{2m} \rceil\}$. It follows from the definition of $\rho$ that $|\gamma_{\Delta'}(W_j)| > |\Delta'(W_i)|$, for every $j \geq i$, which is a contradiction.

Claim 4.4. For no two distinct vertices $v, w \in G'$ we have $\rho(v) \neq \rho(w)$.

Suppose the claim is false and let $v, w$ be two distinct vertices with $\rho(v) > \rho(w)$. We may assume $\rho(v) = \max\{\rho(v) : v \in G'\} = \rho$. Consider now the induced complete subgraph $W = G \setminus \{v\}$ obtained by removing the vertex $v$. Note that there might exist a vertex $z \in W$ such that $\rho_W(z) = 0$. However, we may keep deleting such vertices until the remaining subgraph $W_0$ satisfies $V(W_0^{2}) = V(W_0)$. Moreover, note that this procedure never deletes all edges of some colour present on $W$, therefore $\gamma_{\Delta'}(W) = \gamma_{\Delta'}(W_0)$. Denote by $B$ the set of deleted vertices, i.e. $B = W \setminus W_0$.

Since $\rho(v) \geq 1$, we may apply induction to $W_0$. Suppose the colouring on $W_0$ is not a bad colouring. Then, by Claim 4.3, we have that the total number of colours appearing in $W_0$ is at least $m - (1 + \alpha')\sqrt{2m}$. By induction, we obtain that $|G'_{\Delta'}(W_0)| \geq (1 + \alpha')\sqrt{2m - 2(1 + \alpha')\sqrt{2m}}$. We denote by $\gamma$ the size of $\gamma_{\Delta'}(W)$. Observe, clearly that $\gamma$ is the largest value in $G'_{\Delta'}(W_0)$.

Consider now the subgraph $W_2 \setminus \{v\} \cup \{v\}$ and observe that the size of $\gamma_{\Delta'}(W \setminus \{v\} \cup \{v\})$, which we denote by $\gamma_1$, is greater than $\gamma$, because $\rho(v) > \rho(w)$.

Finally, by considering the entire graph $G$, we obtain yet another value $\gamma_2 \notin G'_{\Delta'}(W)$, where $\gamma_2 = \gamma_{\Delta'}(G)$, as $\rho(w) \geq 1$. We have just proved that $\gamma_2 > \gamma_1 > \gamma$. Therefore,

$$|G'_{\Delta'}(G)| \geq 2 + (1 + \alpha')\sqrt{2m - 2(1 + \alpha')\sqrt{2m}} \geq (1 + \alpha')\sqrt{2m}.$$ 

The second inequality holds, by a simple calculation, for every $m \geq 3$, as long as $\alpha' \leq \frac{1}{10}$.

Suppose now the colouring induced by $\Delta'$ on $W_0$ is a bad colouring. We split our analysis into two cases, depending on the type of the bad colouring.

Case 1. The colouring induced on $W_0$ is a bad colouring of type (i).

First, suppose that $W_0 = W$. Since $W_0$ is a rainbow complete graph, $\rho = \rho(v)$ must be equal to $n$. However, in this case the colouring $\Delta'$ on $G$ would also be of type (i), which is a contradiction.
Recall $B = W \setminus W_0$ and we denote by $b$ its size. We know $B \neq \emptyset$. Note that $B$ is the set of vertices $z \in G'$, where $\rho(z) = 1$ and the edge $(v, z)$ is uniquely coloured in $G$. Denoting by $k$ the size of $W_0$, the following holds.

\[
\binom{k}{2} + \rho(v) + 1 \geq m \implies k \geq \sqrt{2m - 2(1 + \alpha')\sqrt{2m}}. \tag{3}
\]

Let us write $V(W_0) = \{x_1 = x', x_2, \ldots, x_k\}$. Note that $G'_\Delta(G)$ contains every integer of the form $\binom{j}{2} + 1$, for every $l \in \{1, 2, \ldots, k\}$.

First, suppose there exists a vertex $u \in W_0$ such that the edge $(v, u)$ does not have colour 1. There are now two cases we need to consider. The first case is when $W_0 \cup \{v\}$ forms a rainbow complete subgraph (not using colour 1). Let $a$ be a vertex in $B$. By definition $B$, the colour of every edge $(a, x_1)$, where $x_1 \in W_0$, belongs to the set $\gamma_\Delta(W_0)$. We may assume, the colour of $(a, x_1)$ (possibly colour 1) appears in the set $\gamma_\Delta(\{x', x_2, x_3\})$. Set now

\[
X_0 = \{x', v, a, x_2, x_3\} \text{ and } X_j = X_{j-1} \cup \{x_{j+3}\},
\]

for every $j \in \{1, \ldots, k - 3\}$. It is easy to see that $\binom{j+4}{2} + 2 \leq \gamma_\Delta(X_j) < \binom{j+5}{2}$, since $X_j$ has order $j + 5$ and it is not rainbow and the edge $(a, v)$ is uniquely coloured. This implies $G'_\Delta(G)$ has order at least $2k - 4 \geq 2\sqrt{2m - 2(1 + \alpha')\sqrt{2m}} - 4 \geq (1 + \alpha')\sqrt{2m}$, which is a contradiction. If $W_0 \cup \{v\}$ does not form a rainbow complete subgraph, then there is some vertex $u' \in W_0$ for which the colour of the edge $(v, u')$ already belongs to $\gamma_\Delta(W_0)$. We may assume that the colour of $(v, u')$ belongs to $\gamma_\Delta(\{x', x_2, x_3\})$ and also that the colour of the edge $(v, u)$ does not belong to $\gamma_\Delta(\{x', x_2, x_3, \ldots, x_{k-1}\})$. In this case, set

\[
X_0 = \{x', v, u, x_2, x_3\} \text{ and } X_j = X_{j-1} \cup \{x_{j+3}\},
\]

for every $j \in \{1, 2, \ldots, k - 4\}$. As before, we have that $\binom{j+4}{2} + 2 \leq \gamma_\Delta(X_j) < \binom{j+5}{2}$, since $X_j$ is not rainbow and the colour of edge $(u, v)$ does not belong to $\gamma_\Delta(\{x', x_2, \ldots, x_{k-1}\})$. Therefore, the size of $G'_\Delta(G)$ is at least $2k - 5 \geq 2\sqrt{2m - 2(1 + \alpha')\sqrt{2m}} - 5 \geq (1 + \alpha')\sqrt{2m}$, which is a contradiction.

Thus, every edge from $v$ to $W_0$ has colour 1. By the same argument, we must have that every edge from $a$ to $W_0$ has colour 1. Finally, set

\[
X_0 = \{x', v, a\} \text{ and } X_j = X_{j-1} \cup \{x_{j+1}\},
\]

for every $j \in \{1, 2, \ldots, k - 1\}$. Observe that $\gamma_\Delta(X_j) = \binom{j+1}{2} + 2$, since $X_j$ contains $j + 1$ vertices from $W_0$ contributing to $\binom{j+1}{2}$ colours and $(a, v)$ is uniquely coloured. Therefore, the order of $G'_\Delta(G)$ is at least $2k - 1 \geq 2\sqrt{2m - 2(1 + \alpha')\sqrt{2m}} - 1 \geq (1 + \alpha')\sqrt{2m}$, which is a contradiction. This completes the analysis of Case 1.

**Case 2. The colouring induced on $W_0$ is a bad colouring of type (ii).**

Observe that if $W_0$ is strictly contained in $W$, an almost identical reasoning as we did in Case 1 would imply $|G'_\Delta(G)| \geq (1 + \alpha')\sqrt{2m}$, obtaining a contradiction.

Hence, we have that $W_0 = W$. Recall that the only colours present on the edges incident with the special vertex $x'$ are colours 1 or 2 and colour 2 does not appear in $\gamma_\Delta(W')$. As
\( \rho = \rho(v) = \max \{ \rho(z) : z \in G \} \), it is easy to check that \( \rho \) must be at least \( n - 1 \). We now claim the following.

**Claim 4.5.** The colour of the edge \((v, x')\) is not 1. Moreover, if there is an edge incident with \( x' \) of colour 2 in \( W^{(2)} \), then \((v, x')\) does not have colour 2.

Suppose the claim does not hold. Then, \( \rho = n - 1 \) implying \( G' \) forms a rainbow complete subgraph, no edge of which is coloured with 1 or 2. Therefore, \( \rho(w) \geq n - 1 = \rho(v) \), which is a contradiction. This concludes the proof of the claim.

Now, since \( \rho(v) \geq n - 1 \), there exists a subset \( W^* \subseteq W' \) of order at least \( n - 2 \) such that \( W^* \cup \{ v \} \) forms a rainbow complete subgraph and neither colour 1 nor colour 2 belongs to \( \bigcup \gamma_{\Delta'}(W^* \cup \{ v \}) \). Denoting by \( k \) the order of \( W^* \), we have that \( k \geq \sqrt{2m - 3} \), as \( n + 1 \geq \sqrt{2m} \).

Let us now split \( W^* \) into two sets \( W_1 \cup W_2 = W^* \) such that every edge of \((x', W_1)\) has colour 1 and every edge of \((x', W_2)\) has colour 2. Clearly, one of the sets has size at least \( \frac{\sqrt{2m - 3}}{2} \), we shall assume, without loss of generality, it is \( W_1 \). We write \( W_1 = \{ x_1, x_2, \ldots , x_{k_1} \} \) and \( W_2 = \{ x_{k_1+1}, \ldots , x_k \} \), where \( k_1 \geq \frac{\sqrt{2m - 3}}{2} \). Observe that \( \bigcup \gamma_{\Delta'}(G) \) contains the set \( \{ (\ell) : 1 \leq \ell \leq k_1 \} \) and the set \( \{ (\ell) + 1 : k_1 + 1 \leq l \leq k \} \). We set
\[
X_0 = \{ x', v \} \quad \text{and} \quad X_j = X_{j-1} \cup \{ x_j \},
\]
for every \( j \in \{ 1, 2, \ldots , k_1 \} \). As the colour of the edge \((v, x')\) does not appear in \( \bigcup \gamma_{\Delta'}(W^* \cup \{ v \}) \), it is easy to see that \( \bigcup \gamma_{\Delta'}(X_j) = \binom{\ell+1}{2} + 1 \), for every \( 1 \leq j \leq k_1 \). Therefore, \( |\bigcup \gamma_{\Delta'}(G)| \geq \frac{3}{2} \sqrt{2m - 3} \geq (1 + \alpha') \sqrt{2m} \), which is a contradiction.

This concludes the analysis of Case 2. and the proof of Claim 4.4.

From now on, we shall assume \( \rho(v) = \rho \), for every \( v \in V(G') \). We claim that \( \rho \) can not be ‘too’ small.

**Claim 4.6.** Let \( \beta = (2 + 3\alpha') - \frac{2}{1 + 3\alpha'} \). Then \( \rho \geq (1 - \beta) \sqrt{2m} \).

Suppose \( \rho < (1 - \beta) \sqrt{2m} \). Let \( A_1, A_2, \ldots , A_t \) be the \( t \) sets obtained by applying Theorem 3.3 to \( G \). Our aim is to show that \( t \) must be at least \( (1 + \alpha') \sqrt{2m} \), obtaining this way a contradiction. To do so, we shall need to bound \( n \), the number of vertices of \( G \). Let \( m_0 \leq m \) be the number of colours which appear in exactly one edge of \( G \). Note that \( \frac{m_0}{2} \geq m_0 \). Moreover, each of the remaining colours must appear in at least \( 2 \) edges of \( G \). Hence, we have the following.
\[
m_0 + 2(m - m_0) \leq \binom{n + 1}{2} \quad \Rightarrow \quad 2m - \frac{np}{2} \leq \binom{n + 1}{2} \quad \Rightarrow \quad 4m \leq \binom{n + 1}{2} (1 - \beta) \sqrt{2m} n.
\]
Since \( \beta = (2 + 3\alpha') - \frac{2}{1 + 3\alpha'} \), we have that \((1 - \beta) + (1 + 3\alpha')(1 + 3\alpha') \leq 2 \), implying \( n \) must be at least \( (1 + 3\alpha') \sqrt{2m} \).

For every \( j \in [t] \), recall \( c(j) \) denotes the order of the set \( \bigcup_{i=1}^{j} A_j \setminus \bigcup_{i=1}^{j-1} A_i \). Now, let \( T_1 \subseteq [t] \) be the set of indices for which \( A_j \) has size 1. Define, similarly, \( T_2 \subseteq [t] \) to be the set of indices for which \( A_j \) has size \( 2 \). Denote by \( t_1 \) and \( t_2 \), the sizes of \( T_1 \) and \( T_2 \), respectively. Suppose \( t_2 \leq 2\alpha' \sqrt{2m} \). Observe that \( \bigcup_{i=1}^{j} V(A_i) = V(G) \), as \( \rho \geq 1 \). Therefore,
\[
n + 1 \leq t_1 + 2 \cdot t_2 \Rightarrow t_1 + t_2 \geq (1 + \alpha') \sqrt{2m},
\]
11
which is a contradiction.

We must have that $t_2 \geq 2\alpha' \sqrt{2m}$. Suppose $t \leq (1 + \alpha') \sqrt{2m}$. Let $\tau : [t] \to [t]$ be a function defined as $\tau(i) = \{i \leq j : i \in T_2\}$. Observe that $c(j) = 1$ whenever $|A_j| = 2$. Moreover, it is easy to see using property (4), that $c(j) \leq j - \tau(j)$, for every $j \in [t]$. The following holds.

$$m \leq \sum_{i=1}^{t} c(j) \leq \sum_{i \in T_1} (i - \tau(i)) + t_2 \leq \left(\frac{t_1}{2}\right) + t_2 = \frac{t_1}{2} + t_2$$

$$\implies 2m \leq t_1^2 + 2t_2 = (t - t_2)^2 + 2t_2 \implies 2m \leq 2m(1 - \alpha')^2 + 4\alpha' \sqrt{2m} \leq 2m - \alpha' \cdot 2m + 4\alpha' \sqrt{2m},$$

which is impossible, as $\alpha' \cdot 2m > 4\alpha' \sqrt{2m}$, for every $m \geq 10$. Hence, $t \geq (1 + \alpha') \sqrt{2m}$ and we obtain again a contradiction. This concludes the proof of Claim 4.6.

Claim 4.6 easily implies the number of vertices in $G$ can not be too large.

Claim 4.7. $|V(G')| = n \leq (1 - \beta)^{-1} \sqrt{2m}$.

Clearly, there are at least $\frac{n^2}{2}$ distinct colours in $G$, hence $m \geq \frac{n^2}{2}$. Therefore, $n \leq (1 - \beta)^{-1} \sqrt{2m}$, by the lower bound we have on $\rho$.

Claim 4.8. There must exist a triangle $w,y,z \in G'$ such that both edges $(w,y),(w,z)$ are uniquely coloured but the edge $(y,z)$ is not.

We shall prove this claim by looking at the graph $G^* \subset G'$ spanned by the uniquely coloured edges. Indeed, observe first we may assume $G^*$ is not a complete graph, since otherwise Theorem 4.1 would follow by Lemma 4.2, which is a contradiction.

Secondly, we observe that the degree of any vertex in $G^*$ is at least $\frac{n}{2}$. To see this, let $un(v)$ be the number of uniquely coloured edges incident with a vertex $v$. Since $\rho \geq (1 - \beta) \sqrt{2m}$, $v$ must be incident with at least $2((1 - \beta) \sqrt{2m} - un(v)) + un(v) = 2(1 - \beta) \sqrt{2m} - un(v)$ vertices.

Suppose $un(v) \leq n/2$, then $un(v) \leq \frac{1}{2}(1 - \beta)^{-1} \sqrt{2m}$, implying $n \geq 2(1 - \beta) \sqrt{2m} - \frac{1}{2}(1 - \beta)^{-1} \sqrt{2m} > (1 - \beta)^{-1} \sqrt{2m}$, which is a contradiction by Claim 4.7. (The last inequality holds if $\beta \leq \frac{1}{10}$). Now, there must exist an induced path of length 2, $yuvz$, in $G^*$ because the graph $G^*$ is not the union of disjoint complete subgraphs. We have thus found our desired triangle, namely $y,w,z$, concluding the proof of Claim 4.8.

Let $w,y,z$ be such a triangle as in Claim 4.8.

Consider the complete subgraph $W = V(G) \setminus \{z,y\}$. As $\rho > 2$ (by Claim 4.6), $W^{\geq 1} = W$. In the same way as in the proof of Claim 4.4, we consider two cases. Firstly, we consider the case when the colouring induced by $\Delta'$ on $W$ is not a bad colouring.

**Suppose the colouring induced on $W$ is not a bad colouring.**

Our goal is to show the existence of least three distinct values $\gamma_1, \gamma_2, \gamma_3$ which belong to $G^*_\Delta(G)$ but are not in $G^*_\Delta(W)$. Denote by $\gamma$ be the largest value in $\gamma_\Delta(W)$. Now, consider the following three complete subgraphs:

- $W_1 = (W \setminus \{w\}) \cup \{z\}$;
- $W_2 = (W \setminus \{w\}) \cup \{z,y\}$;
\[ W_3 = G. \]

Observe that \( \gamma(W_1) = \gamma + 1, \gamma(W_2) = \gamma + \rho \) and \( \gamma(W_3) = \gamma(G) \). So these three values are distinct and are strictly bigger than \( \gamma \). By induction hypothesis we have that:

\[
|G'_{\Delta}(G)| \geq (1 + \alpha') \sqrt{2m - 4(1 + \alpha' \sqrt{2m}) + 3} > (1 + \alpha') \sqrt{2m}.
\]

The last inequality holds for every \( m \geq 15 \) and \( \alpha' \leq \frac{1}{100} \).

Suppose now the colouring induced on \( W \) is a bad colouring. It is easy to see that the bad colouring on \( W \) must be of type (ii). Suppose not. Then, \( W \) is a rainbow complete graph. As \( \rho(v) = \rho \), for every \( v \in G' \), \( \rho \) must be \( n \) implying the colouring on \( G \) would also be a bad colouring of type (i), which is a contradiction. Hence, we may assume the colouring on \( W \) is of type (ii). Note first that \( \rho \) must be \( n - 1 \). Moreover, since the edge \((y, z)\) is not uniquely coloured, \( W' \cup \{z\} \) must form a rainbow complete graph. Finally, note that the edge \((z, x')\) can not have colour 1 (or colour 2, if colour 2 already appears in \( W^{(2)} \)) and its colour can only be shared by the edge \((y, z)\), as otherwise \( \rho(z) < n - 1 \). We may then argue as we did in Case 2., by considering the rainbow graph \( W \cup \{z\} \), this way deducing \( |G'_{\Delta}(G)| \geq (1 + \alpha') \sqrt{2m} \), which is a contradiction.

To conclude the proof, we may take \( \alpha' = \frac{1}{100} \). Observe that for this choice of \( \alpha' \), \( \beta \leq \frac{1}{10} \).

This completes the proof of Theorem 4.1.

5 Concluding remarks

The first remark we would like to mention concerns the size of \( \alpha' \) in the main theorem. Although we are confident our methods could be sharpened to get a larger constant \( \alpha' \), we doubt our approach could be improved to get its correct order, even when \( m = \left( \frac{n}{2} \right) + 3 \), for some integer \( n \geq 2 \).

Secondly, we remark that our theorem is probably only sharp (up to the order of \( \alpha' \)) for values of \( m \) in the frontier of the intervals of the form \( \left[ \left( \frac{n}{2} \right) + 3, \left( \frac{n+1}{2} \right) \right] \) (for some \( n \geq 3 \)). Indeed, we conjecture \( \psi \) changes its behaviour as \( m \) moves further away from the endpoints of such intervals.

**Conjecture 5.1.** For every positive integer \( k \), there exists a positive constant \( C_k \) where \( C_k \to \infty \) as \( k \to \infty \) such that the following holds. Let \( m \geq \left( \frac{k+2}{2} \right) \) be a positive integer where \( \left( \frac{n}{2} \right) + k \leq m \leq \left( \frac{n+1}{2} \right) - k \), then \( C_k \sqrt{2m} \leq \psi(m) \).

Recall that it is still unknown whether \( \psi(m) = o(m) \) as \( m \to \infty \). We remind the reader of Narayanan’s conjecture.

**Conjecture 5.2** (Narayanan). \( \psi(m) = o(m) \) as \( m \to \infty \).

It seems plausible that the function \( \psi \) is unimodal inside those intervals. Finally, we remark that it is possible to deduce from our theorems a characterization of the \( m \)-colourings \( \Delta \) of \( \mathbb{N}^{(2)} \).
for which $|F_\Delta| = \psi(m)$, for every $m \in \{ \binom{n}{2} + 1, \binom{n}{2} + 2 : \text{for } n \text{ sufficiently large}\}$. We are able to prove $\Delta$ must be (up to a permutation of the naturals) one of the two colourings described in the Introduction.

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