A parameter uniform method for two-parameter singularly perturbed boundary value problems with discontinuous data

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ABSTRACT

We consider two-parameter singularly perturbed problems of reaction-convection-diffusion type in one dimension. The convection coefficient and source term are discontinuous at a point in the domain. The problem is numerically solved using the upwind difference method on an appropriately defined Shishkin-Bakhvalov mesh. At the point of discontinuity, a three-point difference scheme is used. A convergence analysis is given and the method is shown to be first-order uniformly convergent with respect to the perturbation parameters. The numerical results presented in the paper confirm our theoretical results of first-order convergence. Summing up:

• The Shishkin-Bakhvalov mesh is graded in the layer region and uniform in the outer region as shown in the graphical abstract.
• The method presented here has uniform convergence of order one in the supremum norm.
• The numerical orders of convergence obtained in numerical examples with Shishkin-Bakhvalov mesh are better than those for Shishkin mesh.

Specifications Table

| Subject area                        | Mathematics       |
|-------------------------------------|-------------------|
| More specific subject area          | Numerical analysis|
| Name of your method                 | Finite difference method |
| Name(s) and reference(s) of original method | 1. P.A. Farrell, A.F. Hegarty, J.J.H. Miller, E. O'Riordan, & G.I. Shishkin (2004), Global maximum norm parameter-uniform numerical method for a singularly perturbed convection-diffusion problem with discontinuous convection coefficient, Math. Comput. Modelling, 40(11–12), 1375–1392. |
|                                    | 2. T. Linß(1999), An upwind difference scheme on a novel Shishkin-type mesh for a linear convection diffusion problem, J. Comput. Appl. Math. 110(1), 93–104.|
| Resource availability              | Matlab            |

Introduction

Many physical problems such as flows in chemical reactors, equations involving modeling of semiconductor devices, simulation of water pollution problems, and simulation of many fluid flows are modelled mathematically as singular perturbation problems (SPPs), see [1,9,10,19,22] for details. The solutions of these problems are characterized by presence of layers (narrow region of rapid change). Depending on the location of layers, these are called boundary layer or interior layer problems. In this article we will examine a SPP

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with two small perturbation parameters $\epsilon$ and $\mu$, multiplied to diffusion and convection term respectively. The convection coefficient and source term are discontinuous at a point in the domain.

Consider a singularly perturbed reaction-convection-diffusion problem, with a discontinuous source term and a convection coefficient.

$$
\begin{align*}
\mathcal{L}(y(x)) &= \epsilon y''(x) + \mu a(x)y'(x) - b(x)y(x) = f(x), \quad x \in (\Omega^- \cup \Omega^+), \\
y(0) &= y_0, \quad y(1) = y_1, \\
0 < \epsilon, \mu &\ll 1, a_1, a_2 \in \mathbb{R}, \text{ and } \Omega^- = (0, d), \Omega^+ = (d, 1), d \in \Omega = (0, 1). \text{ The coefficient } b(x) \geq \gamma > 0 \text{ is sufficiently smooth in } \tilde{\Omega}. \text{ The source term } f(x) \text{ and the convection coefficient } a(x) \text{ are discontinuous at the point } d. \text{ The coefficients } a(x), f(x) \text{ and their derivatives have a jump discontinuity at } d. \text{ Recall that the jump in any function } g \text{ at a point } a \text{ is defined as } [g](a) = g(a+) - g(a−). \text{ Also } a(x), f(x) \text{ are sufficiently smooth in } (\Omega^- \cup \Omega^+) \cup [0, 1]. \text{ Under the above assumptions, the BVP (1) admits a unique solution } y(x) \in C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+). \\
\end{align*}
$$

where $0 < \epsilon, \mu \ll 1, a_1, a_2 \in \mathbb{R}$, and $\Omega^- = (0, d), \Omega^+ = (d, 1), d \in \Omega = (0, 1)$. The coefficient $b(x) \geq \gamma > 0$ is sufficiently smooth in $\tilde{\Omega}$. The source term $f(x)$ and the convection coefficient $a(x)$ are discontinuous at the point $d$. The coefficients $a(x)$, $f(x)$ and their derivatives have a jump discontinuity at $d$. Recall that the jump in any function $g$ at a point $a$ is defined as $[g](a) = g(a+) - g(a−)$. Also $a(x)$, $f(x)$ are sufficiently smooth in $(\Omega^- \cup \Omega^+) \cup [0, 1]$. Under the above assumptions, the BVP (1) admits a unique solution $y(x) \in C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$.

The solution of above Eq. (1) has boundary layers at both boundaries due to the presence of small perturbation parameters $\epsilon$ and $\mu$. In addition, it has strong interior layers in the neighborhood of $d$ due to the discontinuity of $a$ and $f$ and the sign pattern of $a$ in the domain. The ratio $\epsilon/\mu^2$ is crucial in determining the width of boundary and interior layers. So the analysis of the above problems naturally splits into two cases: $\sqrt{\mu}/\epsilon \leq \sqrt{\rho}/\epsilon$ and $\sqrt{\mu} > \sqrt{\rho}/\epsilon$, where $\rho = \min_{x \in \Omega \setminus \{a\}} \left\{ \frac{b(x)}{a(x)} \right\}$ and $a = |\min(a_1, a_2)|$.

When $\mu = 1$, the problem is one parameter singularly perturbed problem with interior layers. Here, the solution has strong interior layers of width $O(\epsilon)$ in the neighbourhood of the point $x = d$. For work in this direction, see [2,6,7,12,14].

The study of two-parameter SPPs was initiated by O’Malley [15–17], who examined the asymptotic solution. He noted that the ratio $\sqrt{\rho}/\epsilon$ and $\mu$ is very important and decides the width of boundary layers. Some numerical methods for singularly perturbed two-parameter reaction-convection-diffusion equation with smooth data can be found in [8,18,23,25,26]. Physical significance of the singularly perturbed problems with interior layers due to discontinuous coefficients can be seen while, modelling one-dimensional stationary semiconductor device equations, see [13]. Assume that the semiconductor device has only one junction and that the doping profile has jump discontinuities at the junction, which give rise to interior layers corresponding to these discontinuities. In [13], Markowich discussed a finite difference scheme for this problem and proposed a finite difference method for the resolution of interior layers with reasonable number of grid points.

The study of numerical methods for singularly perturbed two-parameter problems with discontinuity in data is an open area of research with much to explore. In [24], Shanti et al. presented an almost first-order numerical technique for two-parameter singularly perturbed problem with a discontinuous source term. The method comprised of upwind difference scheme on an appropriately defined Shishkin mesh. This result was improved by Prabha et al. in [20]. They proposed an almost second-order method on Shishkin mesh comprising the central, mid-point, and upwind difference scheme. They used a five-point difference scheme at the point of discontinuity. An almost second-order method was given by Chandru et al. in [3] for a singularly perturbed two-parameter problem with a discontinuous source term. The method consisted of proper use of upwind, central, and mid-point upwind difference methods on a suitably chosen Shishkin mesh. A three-point scheme was used at the point of discontinuity. Prabha et al. in [21], examined two parameter SPP with discontinuous source and convection-coefficient. They proposed an upwind difference scheme layer adapted Shishkin mesh with a three-point difference scheme at the point of discontinuity. The method was proved to be almost first order convergent.

In this article, for Eq. (1), we have used upwind difference method on an appropriately defined Shishkin-Bakhvalov mesh. In this mesh, the layer part has graded mesh formed by inverting the boundary layer term. The outer region has a uniform mesh. The transition point is chosen as in Shishkin mesh. This mesh was first proposed by Linß for a one-parameter SPP in [11]. Shishkin-Bakhvalov mesh performs better than Shishkin mesh. In Shishkin mesh, the order of convergence is deteriorated due to a logarithmic factor, unlike here. At the point of discontinuity a three-point difference scheme is used. The proposed method is uniformly convergent of order one.

The main contribution of the present paper is uniform convergence of order one in the supremum norm. The orders of convergence obtained in numerical examples for the Shishkin-Bakhvalov mesh are better then those for the Shishkin mesh. This is an improvement of the result of Prabha et al. in [21]. Their method has uniform convergence of order one up to a logarithmic factor in the supremum norm.

Throughout this article, $\mathcal{C}$ denotes a generic positive constant independent of perturbation parameters, number of mesh points. Here, the supremum norm on the domain $\Omega$ is denoted by

$$
\|v\|_\Omega = \max_{x \in \Omega} |v(x)|.
$$

The structure of the paper is as follows. In Section “Apriori bounds”, a priori bounds on the solution are proved, followed by the decomposition of the solution and some derivative bounds in Section “Decomposition of the solution”. The numerical method is proposed in Section “Discrete problem”. Section “Error estimates” presents the error estimates for the difference method. Some numerical results are included in Section “Numerical results”, which verify the theoretical claims made. A summary of the main results is in Section “Conclusion”. 

Apriori bounds

In this section, we discuss the existence of a unique solution, the minimum principle, stability bound and the apriori bounds for the solution of Eq. (1).

**Theorem 1.** *The SPPs (1) has a solution* \( y(x) \in C^0(\Omega) \cap C^1(\Omega) \cap C^2(\Omega^− \cup \Omega^+) \).

**Proof.** The proof is by construction. Let \( u_1, a_2 \) be particular solutions to the differential equations

\[
eq u''(x) + \mu a_1(x)u'(x) - b(x)u_1(x) = f(x), \ x \in \Omega^-.
\]

and

\[
eq u''(x) + \mu a_2(x)u'(x) - b(x)u_2(x) = f(x), \ x \in \Omega^+.
\]

respectively. The convection coefficients \( a_1, a_2 \in C^2(\Omega) \) have the following properties:

\[
a_1(x) = a(x), \ x \in \Omega^-, \ a_1 < 0, x \in \Omega
\]

\[
a_2(x) = a(x), \ x \in \Omega^+, \ a_2 > 0, x \in \Omega.
\]

Consider the function

\[
y(x) = \begin{cases} u_1(x) + (y_0 - u_1(0))\phi_1(x) + A\phi_2(x), \ x \in \Omega^-, \\ u_2(x) + B\phi_1(x) + (y_1 - u_2(1))\phi_2(x), \ x \in \Omega^+, \end{cases}
\]

where \( A, B \) are constants chosen appropriately so that \( y \in C^1(\Omega) \) and \( \phi_1(x), \phi_2(x) \) are the solutions of the boundary value problems

\[
eq \phi''(x) + \mu a_1(x)\phi'(x) - b(x)\phi_1(x) = 0, \ x \in \Omega, \ \phi_1(0) = 1, \ \phi_1(1) = 0,
\]

and

\[
eq \phi''(x) + \mu a_2(x)\phi'(x) - b(x)\phi_2(x) = 0, \ x \in \Omega, \ \phi_2(0) = 0, \ \phi_2(1) = 1
\]

respectively.

We observe that the function \( y \) satisfies \( y(0) = y_0, y(1) = y_1 \), and \( cy''(x) + \mu a(x)y'(x) - b(x)y(x) = f(x), \ x \in \Omega^- \cup \Omega^+ \). Also on an open interval \((0,1)\), \( 0 < \phi_i < 1, i = 1, 2 \). So, \( \phi_1, \phi_2 \) cannot have an internal maximum or minimum, and hence

\[
\phi_i'(x) < 0, \ \phi_i''(x) > 0, \ x \in (0,1).
\]

For the existence of constants \( A \) and \( B \), we require that

\[
\begin{bmatrix} \phi_2(d) \\ \phi_2'(d) \\ -\phi_1(d) \\ -\phi_1'(d) \end{bmatrix} \neq 0.
\]

In fact, \( \phi_2'(d)\phi_1(d) - \phi_1'(d)\phi_2(d) > 0 \).

In the next result, we prove the minimum principle for the operator \( L \).

**Theorem 2. (Minimum Principle)** Suppose that a function \( z(x) \in C^0(\Omega) \cap C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+) \) satisfies \( z(0) \geq 0, \ z(1) \geq 0 \), \( Lz(x) \leq 0 \), \( \forall x \in \Omega^- \cup \Omega^+ \) and \([z'](d) \leq 0 \) then \( z(x) \geq 0 \ \forall \ x \in \Omega \).

**Proof.** See [21] for proof.

**Theorem 3.** Let \( y(x) \) be a solution of (1) then

\[
\|y\|_\Omega \leq \max\{|y(0)|, |y(1)|\} + \frac{1}{\gamma} \|f\|_{\Omega^- \cup \Omega^+}.
\]

**Proof.** Let \( \psi(x) = M \pm \gamma x \), where \( M = \max\{|y(0)|, |y(1)|\} + \frac{1}{\gamma} \|f\|_{\Omega^- \cup \Omega^+} \) and \( b(x) > \gamma > 0 \ \forall x \in \Omega \).

Now \( \psi''(0) \) and \( \psi''(1) \) are non negative. For each \( x \in \Omega^- \cup \Omega^+ \),

\[
L\psi(x) = \psi''(x) + \mu a(x)\psi'(x) - b(x)\psi(x) \leq 0.
\]

Since \( y \in C^0(\Omega) \cap C^1(\Omega) \)

\[
[y''(x)] = \pm[y'(x)] = 0 \text{ and } |y''(x)| = |y'(x)| = 0.
\]

It follows from the minimum principle that \( \psi(x) \geq 0, \forall x \in \Omega \), which implies

\[
\|y\|_\Omega \leq \max\{|y(0)|, |y(1)|\} + \frac{1}{\gamma} \|f\|_{\Omega^- \cup \Omega^+}.
\]
Theorem 4. If $y(x)$ is the solution of the Eq. (1) where $|y(0)| \leq C, |y(1)| \leq C$ then for $k = 1, 2$ it holds that

$$
\|y^{(k)}\|_{\Omega^{-} \cup \Omega^{+}} \leq \frac{C}{(\sqrt{e})^k} \left( 1 + \left( \frac{\mu}{\sqrt{e}} \right)^k \right) \max \{ \|y\|, \|f\| \},
$$

and

$$
\|y^{(3)}\|_{\Omega^{-} \cup \Omega^{+}} \leq \frac{C}{(\sqrt{e})^3} \left( 1 + \left( \frac{\mu}{\sqrt{e}} \right)^3 \right) \max \{ \|y\|, \|f\|, \|f^\prime\| \}.
$$

Proof. We first prove the result for the domain $\Omega^{-}$. The proof for $\Omega^{+}$ follows the same argument.

Given any point $x \in (0, d)$, we can construct a neighbourhood $N_p = (p, p + r)$ where $r > 0$ is such that $x \in N_p$ and $N_p \subset (0, d)$. As $y$ is differentiable in $N_p$ then the mean value theorem implies that there exists $q \in N_p$ such that

$$
y'(q) = \frac{y(p) - y(p)}{r}. \quad \Rightarrow |y'(q)| \leq \frac{|y(p) + r|}{r} \leq \frac{\|y\|}{r}.
$$

Also,

$$
y'(x) = y'(q) + \int_q^x y''(\xi) d\xi.
$$

Therefore, from the differential Eq. (1) and using integration by parts, we obtain

$$
y(x) = y'(q) + e^{-1} \int_q^x (f(\xi) + b(\xi)y(\xi) - \mu a(\xi)y'(\xi)) d\xi
= y'(q) + e^{-1} \int_q^x (f(\xi) + b(\xi)y(\xi) + \mu a(\xi)y(\xi)) d\xi - \mu e \frac{a(x)y(x) - a(q)y(q)}{\epsilon}.
$$

Using the fact that $x - q \leq r$ and taking modulus on both sides and after some simplifications, we arrive at the following bound

$$
|y'(x)| \leq C \left( \frac{1}{r} + \frac{r}{\epsilon} + \frac{\mu}{\sqrt{\epsilon}} \right) \max \{ \|y\|, \|f\| \}.
$$

If we choose $r = \sqrt{\epsilon}$ then the right-hand side of the above expression is minimized with respect to $r$ and we obtain the result for $k = 1$,

$$
\|y'\|_{\Omega^{-}} \leq \frac{C}{(\sqrt{e})^2} \left( 1 + \left( \frac{\mu}{\sqrt{e}} \right)^2 \right) \max \{ \|y\|, \|f\| \}, \quad x \in \Omega^{-}.
$$

For $k = 2$, the differential Eq. (1) gives,

$$
y^{(2)}(x) = \frac{1}{\epsilon} [f(x) + b(x)y(x) - \mu a(x)y'(x)]
|y^{(2)}(x)| \leq \frac{1}{\epsilon} \left( \|f\| + \|b\| \|y\| \right) + \frac{\mu}{\epsilon} \|a\| \left( \frac{C}{\sqrt{\epsilon}} \left( 1 + \frac{\mu}{\sqrt{\epsilon}} \right) \right) \max \{ \|y\|, \|f\| \}
\leq \frac{C}{\epsilon} \left( 1 + \frac{\mu}{\sqrt{\epsilon}} + \frac{\mu^2}{\epsilon} \right) \max \{ \|y\|, \|f\| \}.
$$

On simplifying we arrive at

$$
\|y^{(2)}\|_{\Omega^{-}} \leq \frac{C}{(\sqrt{e})^2} \left( 1 + \left( \frac{\mu}{\sqrt{\epsilon}} \right)^2 \right) \max \{ \|y\|, \|f\| \}.
$$

To obtain the required bounds for $k = 3$, we differentiate the Eq. (1) and arrive at

$$
y^{(3)}(x) = \frac{1}{\epsilon} [f''(x) + (b(x)y(x) - \mu a(x)y'(x))].
$$

Taking modulus on both sides and the bounds for $\|y''\|$ and $\|y''\|$ into consideration, we arrive at,

$$
|y^{(3)}(x)| \leq \frac{C}{\epsilon \sqrt{\epsilon}} \left( 1 + \mu + \sqrt{\epsilon} + \frac{\mu^2}{\sqrt{\epsilon}} + \frac{\mu^3}{\epsilon} \right) \max \{ \|y\|, \|f\|, \|f''\| \}.
$$

On simplifying, we arrive at

$$
\|y^{(3)}\|_{\Omega^{-}} \leq \frac{C}{(\sqrt{e})^3} \left( 1 + \left( \frac{\mu}{\sqrt{e}} \right)^3 \right) \max \{ \|y\|, \|f\|, \|f''\| \}.
$$

□
Decomposition of the solution

The bounds presented in the previous section are not sufficient for the error analysis of the discretization method for the singularly perturbed problems. Thus, to obtain sharp bounds, the solution $y(x)$ is decomposed as in [21] into layers and regular components as $y(x) = y^*(x) + u^*_1(x) + u^*_2(x)$. The regular component $y^*(x)$ is the solution of

$$
\begin{align*}
\mathcal{L}y^*(x) &= f(x), \quad x \in \Omega^- \cup \Omega^+, \\
y^*(0) &= y(0), \\
y^*(1) &= y(1), \\
y^*(d-) \text{ and } y^*(d+) &\text{ are chosen.}
\end{align*}
$$

(2)

The singular components $u^*_1(x)$ and $u^*_2(x)$ are the solutions of

$$
\begin{align*}
\mathcal{L}u^*_1(x) &= 0, \quad x \in \Omega^- \cup \Omega^+, \\
u^*_1(0) &= y(0) - v^*(0), \\
u^*_1(1) &= 0, \quad u^*_1(d-) \text{ and } u^*_1(d+) \text{ are chosen},
\end{align*}
$$

(3)

and

$$
\begin{align*}
\mathcal{L}u^*_2(x) &= 0, \quad x \in \Omega^- \cup \Omega^+, \\
u^*_2(0) &= 0, \\
u^*_2(1) &= y(1) - v^*(1), \quad u^*_2(d-) \text{ and } u^*_2(d+) \text{ are chosen}
\end{align*}
$$

(4)

respectively.

The regular and layer components are further decomposed as

$$
\begin{align*}
\nu^*(x) &= \begin{cases} 
u^*(x), & x \in \Omega^-, \\ \nu^*(x), & x \in \Omega^+. \end{cases} \\
u^*_1(x) &= \begin{cases} \nu^*_1(x), & x \in \Omega^-, \\ \nu^*_1(x), & x \in \Omega^+. \end{cases}
\end{align*}
$$

and

$$
\begin{align*}
u^*_2(x) &= \begin{cases} \nu^*_2(x), & x \in \Omega^-, \\ \nu^*_2(x), & x \in \Omega^+. \end{cases}
\end{align*}
$$

As $y \in C^1(\Omega)$, we have $|u^*_2(x)|d = -|v^*(x)|d - |u^*_1(x)|d$ and $|u^*_2(x)|d = -|v^*(x)|d - |u^*_1(x)|d$.

We will find the bounds on these components for case $\sqrt{a} \mu \leq \sqrt{c}$. First, let us decompose the regular part (similar to Prabha et al. [21]) as $v^*(x) = v^*_0(x) + \sqrt{c}e^*_1(x) + e^*_2(x)$, where $v^*_0(x), v^*_1(x)$ and $v^*_2(x)$ be the solution of the following problems:

$$
\begin{align*}
-b(x) v^*_0(x) &= f(x), \quad x \in \Omega^- \cup \Omega^+, \\
-b(x) v^*_1(x) &= -\frac{\mu}{\sqrt{c}} a(x) v^*_0(x) - \sqrt{c} e^*_1(x), \quad x \in \Omega^- \cup \Omega^+, \\
\mathcal{L} e^*_2(x) &= -\frac{\mu}{\sqrt{c}} a(x) v^*_1(x) - \sqrt{c} e^*_1(x) = F(x), \quad x \in \Omega^- \cup \Omega^+, \\
v^*_2(0) &= v^*_2(1) = 0, \quad v^*_2(d-) = v^*_2(d+) \text{ are chosen suitably,}
\end{align*}
$$

respectively.

Also, $v^*_2(x) \in C^0(\tilde{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$. 

**Theorem 5.** The regular component $v^*(x)$ and its derivatives up to order 3 satisfies the following bounds for $\sqrt{a} \mu \leq \sqrt{c}$

$$
\|v^*(k)\|_{L^2(\Omega^- \cup \Omega^+)} \leq C \left(1 + \frac{1}{(\sqrt{c})^{k-2}}\right), \quad k = 0, 1, 2, 3.
$$

**Proof.** To bound the regular component $v^*(x)$, we need to bound $v^*_0(x), v^*_1(x)$ and $v^*_2(x)$. With sufficient smoothness on the co-efficient $b(x)$ in $\tilde{\Omega}$ and $a(x), f(x)$ in $(\Omega^- \cup \Omega^+)$, we observed that $v^*_0(x), v^*_1(x)$ and its derivatives are bounded. To bound $v^*_2(x)$, Theorem 3 gives

$$
\|v^*_2(x)\|_{L^2(\Omega^- \cup \Omega^+)} \leq \frac{1}{\sqrt{c}} \left(\|v^*_1\| + \|v^*_0\|\right) \leq C.
$$

Now by Theorem 4

$$
\|v^*_2(x)\|_{L^2(\Omega^- \cup \Omega^+)} \leq \frac{C}{(\sqrt{c})^k} \left(1 + \left(\frac{\mu}{\sqrt{c}}\right)^k\right) \max \left\{\|v^*_1\|, \|F\|\right\}, \quad \|F\| \leq C \left(\|v^*_1\| + \|v^*_0\|\right)
$$

$$
\leq \left(\frac{C}{\sqrt{c}}\right)^k, \quad \text{for } k = 1, 2.
$$
Also \( \|v_2^{(3)}(x)\|_{Ω^-}\Delta t^* \leq \left( \frac{C}{\sqrt{\epsilon}} \right)^3 \left[ 1 + \left( \frac{\mu}{\sqrt{\epsilon}} \right)^3 \right] \max \{ \|v_0^*, F\|, \|F^{(1)}\| \} \leq \left( \frac{C}{\sqrt{\epsilon}} \right)^3. \)

Using the bounds for \( v_0^*, v_1^*, v_2^* \) and its derivatives in the expression for \( v^*(x) \), we have
\[
\|v^{(k)}(x)\|_{Ω^-}\Delta t^* \leq C \left( 1 + \frac{1}{(\sqrt{\epsilon})^{k-2}} \right), \quad k = 0, 1, 2, 3.
\]
\[\square\]

**Theorem 6.** Let \( \sqrt{\rho a} \leq \sqrt{\rho e} \). The singular components \( u^*_i(x) \) and \( w^*_i(x) \) and their derivatives up to order 3 satisfy the following bounds for \( k = 0, 1, 2, 3 \)
\[
\|u_i^{(k)}(x)\|_{Ω^-}\Delta t^* \leq \frac{C}{(\sqrt{\epsilon})^k} \begin{cases} e^{-\theta_1 x}, & x \in Ω^-, \\ e^{-\theta_1 (x-d)}, & x \in Ω^+, \end{cases}
\]
\[
\|w_i^{(k)}(x)\|_{Ω^-}\Delta t^* \leq \frac{C}{(\sqrt{\epsilon})^k} \begin{cases} e^{-\theta_2 (d-x)}, & x \in Ω^-, \\ e^{-\theta_2 (1-x)}, & x \in Ω^+, \end{cases}
\]
where,
\[
θ_1 = \frac{\sqrt{\rho a}}{2\sqrt{\epsilon}}.
\]

**Proof.** Consider a barrier function \( ξ^*_x(x) = Ce^{-\theta_1 x} \pm w_i^*(x), \quad x \in Ω^- \). For a large \( C \), \( ξ_+^*(0) \geq 0 \) and \( ξ^-_x(0) = Ce^{-\theta_1 d} \pm w_i^*(d^-) \geq 0 \). Now
\[
\mathcal{L} ξ^*_x(x) = Ce^{-\theta_1 x}(\epsilon \theta_1^2 - \mu a(x)\theta_1 - b(x))
\]
\[
\leq Ce^{-\theta_1 x}\left( \frac{\rho a}{4} + a(x) \frac{\mu \sqrt{\rho a}}{2\sqrt{\epsilon}} - b(x) \right)
\]
\[
\leq Ce^{-\theta_1 x}(\rho|a(x)| - b(x)) \leq 0.
\]
Therefore
\[
\|w_i^*\| \leq Ce^{-\theta_1 x}, \quad x \in Ω^-.
\]

Similarly choose a barrier function \( ξ^*_x(x) = Ce^{-\theta_1 (x-d)} \pm u_i^+(x), \quad x \in Ω^+ \) with large \( C \). Now \( ξ^*_x(d) \geq 0, ξ^*_x(1) \geq 0 \) with \( \mathcal{L} ξ^*_x(x) \leq 0 \) gives
\[
\|u_i^+\| \leq Ce^{-\theta_1(x-d)}, \quad x \in Ω^+.
\]

Using **Theorem 4** on \( Ω^- \) and \( Ω^+ \), we obtain the following bounds for the derivatives of \( u_i^* \) up to order 3,
\[
\|u_i^{(k)}\| \leq \frac{C}{(\sqrt{\epsilon})^k} \begin{cases} e^{-\theta_2 x}, & x \in Ω^-, \\ e^{-\theta_2 (x-d)}, & x \in Ω^+. \end{cases}
\]

Consider a barrier function \( ξ^*_x(x) = Ce^{-\theta_1 (d-x)} \pm u_i^+(x), \quad x \in Ω^- \). For any large \( C \), \( ξ^*_x(0) \geq 0 \) and \( ξ^*_x(d) \geq 0 \). Now
\[
\mathcal{L} ξ^*_x(x) = Ce^{-\theta_1 (d-x)}(\epsilon \theta_1^2 + \mu a(x)\theta_1 - b(x))
\]
\[
\leq Ce^{-\theta_1 (d-x)}\left( \frac{\rho a}{4} + a(x) \frac{\mu \sqrt{\rho a}}{2\sqrt{\epsilon}} - b(x) \right)
\]
\[
\leq Ce^{-\theta_1 (d-x)}\left( \frac{\rho a}{4} - \frac{\rho a(x)^2}{2} - b(x) \right) \leq 0.
\]
Therefore
\[
\|u_i^*\| \leq Ce^{-\theta_1 (d-x)}, \quad x \in Ω^-.
\]

For \( x \in Ω^+ = (d, 1) \), choose the barrier function \( ξ^*_x(x) = Ce^{-\theta_1 (1-x)} \pm u_i^+(x), \quad x \in Ω^+ \), with large \( C \). This gives \( ξ^*_x(d) \geq 0, ξ^*_x(1) \geq 0 \) and \( \mathcal{L} ξ^*_x(x) \leq 0, x \in Ω^+ \) gives
\[
\|u_i^+\| \leq Ce^{-\theta_1 (1-x)}, \quad x \in Ω^+.
\]
By **Theorem 4**, we have the following bounds for the derivatives of \( v^*_r \) of order up to 3,

\[
\|v^{(k)}_r\| \leq \frac{C}{(\sqrt{\alpha})^k} \left( e^{-\theta_2(1-k)} x \in \Omega^-, \right. \quad x \in \Omega^+.
\]

\]

Consider the case: \( s(\alpha u) > \sqrt{pe} \).

Let \( v^* \) be the regular component of the solution \( y \) of the Eq. (1). Let us decompose it as in [21]

\[ v^*(x) = v^*_0(x) + v^*_1(x) + \varepsilon_2 v^*_2(x), \]

where \( v^*_0(x), v^*_1(x) \) and \( v^*_2(x) \) are the solution of the following problems respectively:

\[
\mathcal{L}_r v^*_0(x) = \mu a(x) v^*_0(x) - b(x) v^*_0(x) = f(x), \quad x \in \Omega^- \cup \Omega^+, \quad v^*_0(0) = y(0), \quad v^*_0(1) = y(1),
\]

\[
\mathcal{L}_r v^*_1(x) = -v^*_1(x), \quad x \in \Omega^- \cup \Omega^+, \quad v^*_1(0) = v^*_1(1) = 0,
\]

\[
\mathcal{L}_r v^*_2(x) = -v^*_2(x), \quad x \in \Omega^- \cup \Omega^+, \quad v^*_2(0) = v^*_2(1) = 0,
\]

\( v^*_1(d-), v^*_2(d+) \) are chosen suitably, and \( v^*_0(x) \in C^0(\Omega) \cap C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+) \).

The proof of the next theorem follows the argument presented in [8, Section 3] closely.

**Theorem 7.** Let \( s(\alpha u) > \sqrt{pe} \). The regular component \( v^*(x) \) and its derivatives up to order 3 satisfies the following bounds

\[
\|v^{(k)}(x)\|_{\Omega^- \cup \Omega^+} \leq C \left( 1 + \left( \frac{e}{\mu} \right)^{(2-k)} \right), \quad k = 0, 1, 2, 3.
\]

**Proof.** For \( x \in \Omega^- \), the coefficient \( a < 0 \) and \( b > 0 \). Hence, we have that

\[
\mathcal{L}_r z(x)|_{(0,d)} \leq 0 \quad \text{and} \quad z(0) \geq 0, \quad \text{then} \quad z(x)|_{(0,d)} \geq 0.
\]

Also for \( x \in \Omega^+ \), the coefficients \( a > 0 \) and \( b < 0 \). We have the following result

\[
\mathcal{L}_r z(x)|_{(d,1)} \leq 0 \quad \text{and} \quad z(1) \geq 0, \quad \text{then} \quad z(x)|_{(d,1)} \geq 0.
\]

We further decompose the component \( v^*_0(x), x \in \Omega^- \cup \Omega^+ \), as follows,

\[
v^*_0(x) = s_0(x) + \mu s_1(x) + \mu^2 s_2(x) + \mu^3 s_3(x),
\]

where \( s_0(x) = -\frac{f(x)}{b(x)}, \quad s_1(x) = \frac{a(x)s_0(x)}{b(x)}, \quad s_2(x) = \frac{a(x)s_1(x)}{b(x)}, \quad \text{and} \]

\[
\mathcal{L}_r s_3(x) = -a(x)s_2(x), \quad x \in \Omega^- \cup \Omega^+, \quad s_3(0) = s_3(1) = 0.
\]

Assuming sufficient smoothness of the coefficients, the \( s_i, i = 0, 1, 2 \) and its derivatives are bounded independently of the perturbation parameter \( \mu \). In particular, if \( b \in C^7(\Omega), \alpha, f \in C^7(\Omega^- \cup \Omega^+) \) we have

\[
\|s^0_i\| \leq C, \quad 0 \leq i \leq 7,
\]

\[
\|s^1_i\| \leq C, \quad 0 \leq i \leq 6,
\]

\[
\|s^2_i\| \leq C, \quad 0 \leq i \leq 5.
\]

Using (5) and (6) we deduce that \( \|s_3\| \leq C \) and then from (7) we obtain

\[
\|s^i_0\| \leq \frac{C}{\mu}, \quad 0 \leq i \leq 5.
\]

We use these bounds for \( s_0(x), s_1(x), s_2(x) \) and \( s_3(x) \) to obtain

\[
\|v^{(i)}_0\|_{\Omega^- \cup \Omega^+} \leq \frac{C}{(\sqrt{\mu})^i}, \quad 0 \leq i \leq 5.
\]

Now to bound \( v^*_1(x) \) we decompose \( v^*_1(x), x \in \Omega^- \cup \Omega^+ \), as follows

\[
v^*_1(x) = \rho_0(x) + \rho_1(x) + \mu^2 \rho_2(x),
\]

where \( \rho_0(x) = v^*_0(x) / b(x), \quad \rho_1(x) = a(x) \rho_0(x) / b(x), \quad \text{and} \]

\[
\mathcal{L}_r \rho_2(x) = -a(x) \rho_1(x), \quad x \in \Omega^- \cup \Omega^+,
\]

\( \rho_2(0) = \rho_2(1) = 0 \).

Assuming sufficient smoothness of the coefficients, we have

\[
\|\rho^i_0\|_{\Omega^- \cup \Omega^+} \leq \frac{C}{(\sqrt{\mu})^i}, \quad 0 \leq i \leq 5
\]

\[
\]
and 
\[ \|\rho_1^{(i)}\|_{\Omega^- \cup \Omega^+} \leq \frac{C}{\mu^i}, \quad 0 \leq i \leq 4 \]

Using (5), (6) and (8) we obtain
\[ \|\rho_2^{(i)}\|_{\Omega^- \cup \Omega^+} \leq \frac{C}{\mu^{i+1}}, \quad 0 \leq i \leq 4. \]

We use these bounds for \( \rho_0(x), \rho_1(x) \) and \( \rho_2(x) \) to obtain
\[ \|v_1^{(i)}\|_{\Omega^- \cup \Omega^+} \leq C(1 + \mu^{i-1}), \quad 0 \leq i \leq 3. \]

To bound \( v_2^{(i)}(x), x \in \Omega^- \cup \Omega^+ \) we use the differential equation satisfied by it.
\[ \mathcal{L}v_2^{(2)}(x) = -v_1^{(1)}(x), \quad v_2^{(0)}(x) = v_2^{(1)}(x) = 0. \]

Application of Theorem 3 gives
\[ \|v_2^{(0)}\|_{\Omega^- \cup \Omega^+} \leq \max(|v_2^{(0)}(x)|, |v_2^{(1)}(x)|) + \frac{1}{\mu} \|v_1^{(2)}\| \leq \frac{C}{\mu^2}. \]

By Theorem 4 we have
\[ \|v_2^{(i)}\|_{\Omega^- \cup \Omega^+} \leq \frac{C}{\mu^i}, \quad \text{for } i = 1, 2. \]

Differentiating Eq. (9), we obtain
\[ \|v_2^{(i)}\|_{\Omega^- \cup \Omega^+} \leq \frac{C}{\mu^i}. \]

Substituting these bounds for \( v_0^{(i)}(x), v_1^{(i)}(x), v_2^{(i)}(x) \) and their derivatives into the equation for \( v^{(i)}(x) \) gives
\[ \|v^{(k)}\|_{\Omega^- \cup \Omega^+} \leq C \left( 1 + \left( \frac{\mu}{\mu} \right)^{(2-k)} \right), \quad k = 0, 1, 2, 3. \]

\[ \square \]

**Theorem 8.** Let \( \sqrt{\alpha} \mu > \sqrt{\beta} \varepsilon \). The singular components \( w_j^{(i)}(x) \) and \( w_j^{(i)}(x) \) satisfy the following bounds for \( k = 0, 1, 2, 3 \)

\[ \|w_j^{(i)}(x)\|_{\Omega^- \cup \Omega^+} \leq C \left\{ \begin{array}{ll} \left( \frac{1}{\mu} \right)^k e^{-\theta_2 x}, & x \in \Omega^-, \\ \left( \frac{\mu}{\varepsilon} \right)^k e^{-\theta_1 (x-d)}, & x \in \Omega^+, \end{array} \right. \]

\[ \|w_j^{(i)}(x)\|_{\Omega^- \cup \Omega^+} \leq C \left\{ \begin{array}{ll} \left( \frac{\mu}{\varepsilon} \right)^k e^{-\theta_1 (d-x)}, & x \in \Omega^-, \\ \left( \frac{1}{\mu} \right)^k e^{-\theta_1 (1-x)}, & x \in \Omega^+, \end{array} \right. \]

where
\[ \theta_1 = \frac{a \mu}{2 \varepsilon}, \quad \theta_2 = \frac{\rho}{2 \varepsilon}. \]

**Proof.** In region \( \Omega^- \), we will find the bound for the left and right layer term. For the left layer, consider a barrier function \( \xi_\pm(x) = Ce^{-\theta_2 x} \pm w_j^{(i)}(x), \quad x \in \Omega^- \) \( \Omega^- \), \( x \in \Omega^- \) \( x \in \Omega^- \). For a large \( C \), \( \xi_\pm(0) \geq 0 \) and \( \xi_\pm(d) \geq 0 \). Now
\[ \mathcal{L} \xi_\pm(x) = Ce^{-\theta_2 x} (\epsilon \theta_2^2 - \mu a(x) \theta_2 - b(x)) \]
\[ \leq Ce^{-\theta_2 x} \left( \frac{\alpha a}{4} + |a(x)| \frac{\beta}{2} - b(x) \right) \]
\[ \leq Ce^{-\theta_2 x} (\rho |a(x)| - b(x)) \leq 0. \]

therefore
\[ \|w_j^{(i)}\| \leq Ce^{-\theta_2 x}, \quad x \in \Omega^- \]

For the right layer term, consider a barrier function \( \xi_\pm(x) = Ce^{-\theta_1 (d-x)} \pm w_j^{(i)}(x), x \in \Omega^- \) \( x \in \Omega^- \). For any large \( C \), \( \xi_\pm(0) \geq 0 \) and \( \xi_\pm(d) \geq 0 \). Now
\[ \mathcal{L} \xi_\pm(x) = Ce^{-\theta_1 (d-x)} (\epsilon \theta_1^2 + \mu a(x) \theta_1 - b(x)) \]
\[ \leq Ce^{-\theta_1 (d-x)} \left( \frac{a^2 \mu^2}{4 \varepsilon} + |a(x)| \frac{\alpha \mu}{2 \varepsilon} - b(x) \right) \]
\[ \leq 0. \]
Therefore

\[ \|u^+_{r^+}\| \leq Ce^{-\theta_1(d-x)}, \; x \in \Omega^- . \]

In a similar way, we can prove the bounds for \( u^+_{r^+}(x) \) and \( u^+_r(x) \) in the region \( \Omega^+ \). The bounds for higher derivatives of \( u^+_{r^+} \) and \( u^+_r \) can be proved using the techniques given in [5, 18].

The unique solution \( y(x) \) of the problem (1) is now given by

\[
y(x) = \begin{cases} 
u^+(x) + u^+_{r^+}(x) + w^+_{r^+}(x), & x \in (0, d), \\ (\nu^+ - u^+_{r^+} + w^+_{r^+})(d-\varepsilon) = (\nu^+ + u^+_{r^+} + w^+_{r^+})(d+\varepsilon), & x = d, \\ \nu^+(x) + u^+_{r^+}(x) + w^+_{r^+}(x), & x \in (d, 1). \end{cases}
\]

Discrete problem

The differential Eq. (1) is discretized using the upwind finite difference method on a suitably constructed Shishkin-Bakhvalov mesh. The domain \( \Omega = [0, 1] \) is subdivided into six subintervals as follows

\[ \tilde{\Omega} = [0, \sigma_1] \cup [\sigma_1, d - \sigma_2] \cup [d - \sigma_2, d] \cup [d, d + \sigma_3] \cup [d + \sigma_3, 1 - \sigma_4] \cup [1 - \sigma_4, 1]. \]

Let \( \Omega_N = \{x_i\}_{0}^{N} \) denotes the mesh points with a point of discontinuity at the point \( x \frac{N}{2} = d \). The interior points of the mesh are denoted by \( \Omega_N = \{x_i : 1 \leq i \leq \frac{N}{2} - 1\} \cup \{x_{N} : \frac{N}{2} + 1 \leq i \leq N - 1\} \). Let \( \Omega_N^- = \{x_i : 1 \leq i \leq \frac{N}{2} - 1\} \) and \( \Omega_N^+ = \{x_i : \frac{N}{2} + 1 \leq i \leq N - 1\} \). The transition points in \( \tilde{\Omega} \) are

\[ \sigma_1 = \frac{4}{\theta_2} \ln N, \quad \sigma_2 = \frac{4}{\theta_1} \ln N, \quad \sigma_3 = \frac{4}{\theta_1} \ln N, \quad \sigma_4 = \frac{4}{\theta_2} \ln N. \]

On the sub-intervals \([0, \sigma_1], [d - \sigma_2, d], [d, d + \sigma_3] \) and \([1 - \sigma_4, 1]\) a graded mesh of \( \frac{N}{8} + 1 \) mesh points is constructed by inverting the layer function \( e^{-\theta_2} \) or \( e^{-\theta_1(x-d)} \) or \( e^{-\theta_1(d-x)} \) and \( e^{-\theta_1(1-x)} \) in the above sub-intervals respectively. On \([\sigma_1, d - \sigma_2] \) and \([d + \sigma_3, 1 - \sigma_4] \) a uniform mesh of \( \frac{N}{4} + 1 \) mesh points is taken. We assume that for the case \( \sqrt{\mu} \leq \sqrt{\frac{\mu}{\tau}} \) and \( \sqrt{\mu} > \sqrt{\frac{\mu}{\tau}} \), \( \max \{\varepsilon/\mu, \mu\} < N^{-4} \), otherwise the boundary layers could be resolved by standard uniform mesh.

The mesh points are given by

\[
x_i = \begin{cases} -\frac{8}{\theta_2} \log \left(1 + \frac{8i}{N} \left(\frac{1}{\sqrt{N}} - 1\right)\right), & 0 \leq i \leq \frac{N}{8}, \\ \sigma_1 + \frac{d - \sigma_1 - \sigma_2}{\frac{1}{2}}, & \frac{N}{8} \leq i \leq \frac{3N}{8}, \\ \frac{8}{\theta_1} \log \left(\frac{8i}{N} \left(1 - \frac{1}{\sqrt{N}}\right) + \frac{4}{\sqrt{N}} - 3\right), & \frac{3N}{8} \leq i \leq \frac{N}{2}, \\ \frac{8}{\theta_1} \log \left(\frac{8i}{N} \left(1 - \frac{1}{\sqrt{N}}\right) + 5 - \frac{4}{\sqrt{N}}\right), & \frac{N}{2} \leq i \leq \frac{5N}{8}, \\ \sigma_3 + \frac{(1 - d - \sigma_3 - \sigma_4)}{\frac{1}{2}}, & \frac{5N}{8} \leq i \leq \frac{7N}{8}, \\ 1 + \frac{8}{\theta_2} \log \left(\frac{8i}{N} \left(1 - \frac{1}{\sqrt{N}}\right) + \frac{8}{\sqrt{N}} - 7\right), & \frac{7N}{8} \leq i \leq N. \end{cases}
\]
The mesh generating function $\phi$, maps a uniform mesh $\xi$ onto a layer adapted mesh in $x$ by $x = \phi(\xi)$. The mesh in terms of the mesh generating function can be written as:

$$x_i = \phi(\zeta_i) = \begin{cases} \frac{8}{\theta_2} \phi_1(\zeta_i), & 0 \leq i \leq \frac{N}{8}, \\
\sigma_1 + \frac{(d - \sigma_1 - \sigma_2)(\zeta_i - \frac{1}{8})}{\frac{1}{4}}, & \frac{N}{8} \leq i \leq \frac{3N}{8}, \\
\frac{8}{\theta_3} \phi_2(\zeta_i), & \frac{3N}{8} \leq i \leq \frac{N}{2}, \\
\frac{8}{\theta_1} \phi_3(\zeta_i), & \frac{N}{2} \leq i \leq \frac{5N}{8}, \\
\frac{(1 - d - \sigma_1 - \sigma_2)(\zeta_i - \frac{5}{8})}{\frac{1}{4}}, & \frac{5N}{8} \leq i \leq \frac{7N}{8}, \\
1 - \frac{8}{\theta_2} \phi_4(\zeta_i), & \frac{7N}{8} \leq i \leq N, \end{cases}$$

with $\zeta_i = \frac{i}{N}$. The functions $\phi_1, \phi_3$ are monotonically increasing on $[0, \frac{1}{8}]$ and $[\frac{1}{2}, \frac{5}{8}]$ respectively. And $\phi_2, \phi_4$ are monotonically decreasing on $[\frac{1}{8}, \frac{1}{2}]$ and $[\frac{5}{8}, 1]$ respectively. These mesh generating functions $\phi_i$'s are defined with the help of corresponding mesh characterizing functions $\psi_i$'s as

$$\psi_i(\xi) = \exp(-\phi_i(\xi)), \quad i = 1, 2, 3, 4.$$

Lemma 1. We assume that the mesh-generating functions $\phi_1, \phi_2, \phi_3$ and $\phi_4$ satisfy the following conditions

$$\max_{\xi \in [0, \frac{1}{8}]} |\phi'_1(\xi)| \leq CN, \quad \max_{\xi \in [\frac{1}{4}, \frac{1}{2}]} |\phi'_2(\xi)| \leq CN,$$

$$\max_{\xi \in [\frac{1}{2}, 1]} |\phi'_3(\xi)| \leq CN, \quad \max_{\xi \in [\frac{1}{2}, 1]} |\phi'_4(\xi)| \leq CN$$

and

$$\int_0^{\frac{1}{8}} (\phi'_1(\xi))^2 d\xi \leq CN, \quad \int_{\frac{1}{2}}^{\frac{3}{8}} (\phi'_2(\xi))^2 d\xi \leq CN,$$

$$\int_{\frac{1}{2}}^{\frac{7}{8}} (\phi'_3(\xi))^2 d\xi \leq CN, \quad \int_{\frac{1}{2}}^{1} (\phi'_4(\xi))^2 d\xi \leq CN.$$

Proof. The mesh-generating functions $\phi_1(\xi) = -\log \left[ 1 - 8\xi \left(\frac{1}{\sqrt{N}} - 1\right)\right], \quad \xi \in [0, \frac{1}{8}].$

Therefore,

$$|\phi'_1(\xi)| \leq \frac{8\sqrt{N}}{\sqrt{N} + (1 - \sqrt{N})} \leq 8\sqrt{N} \leq CN.$$

Also mesh characterizing function

$$\psi_1(\xi) = \exp(-\phi_1(\xi)), \quad \xi \in [0, \frac{1}{8}],$$

$$\psi'_1(\xi) = \left(\frac{1}{\sqrt{N}} - 1\right)8\xi,$$

$$\Rightarrow |\psi'_1(\xi)| \leq 8, \quad \xi \in [0, \frac{1}{8}].$$

Similarly, we can prove the bounds for remaining functions in the intervals $[\frac{1}{8}, \frac{1}{2}], [\frac{1}{2}, \frac{7}{8}]$ and $[\frac{7}{8}, 1]$. \(\square\)

Using this Lemma 1 we see that for $0 \leq i \leq \frac{N}{8}$,

$$h_i = x_i - x_{i-1} = \frac{8}{\theta_2} \phi_1(\xi_i) - \phi_1(\xi_{i-1}) \leq \frac{8}{\theta_2} (\xi_i - \xi_{i-1}) \max_{\xi \in [0, \frac{1}{8}]} |\phi'_1(\xi)| \leq \frac{C}{\theta_2}.$$
Similarly, we can show that
\[
\begin{align*}
    h_i & \leq \left\{ \begin{array}{ll}
    \frac{g}{\theta_1} (\xi_i - \xi_{i-1}) \max_{\xi \in [\xi_i, \xi_{i+1}]} |\varphi_i'(\xi)| \leq \frac{C}{\theta_1}, & \frac{iN}{8} \leq i < \frac{N}{2} \\
    \frac{g}{\theta_1} (\xi_i - \xi_{i-1}) \max_{\xi \in [\xi_i, \xi_{i+1}]} |\varphi_i'(\xi)| \leq \frac{C}{\theta_1}, & \frac{N}{2} \leq i \leq \frac{7N}{8} \\
    \frac{g}{\theta_2} (\xi_i - \xi_{i-1}) \max_{\xi \in [\xi_i, \xi_{i+1}]} |\varphi_i'(\xi)| \leq \frac{C}{\theta_2}, & \frac{7N}{8} \leq i \leq N.
    \end{array} \right.
\end{align*}
\]

On the Shishkin-Bakhvalov mesh defined above, we use upwind finite difference method to discretize the differential Eq. (1). We define the difference scheme as: Find \( Y(x_i), \ \forall x_i \in \Omega_N \) such that:
\[
\begin{align*}
    L^N Y(x_i) & \equiv \varepsilon \delta^2 Y(x_i) + \mu a(x_i) D^+ Y(x_i) - b(x_i) Y(x_i) = f(x_i), \ x_i \in \Omega_N \\
    Y(0) &= y(0), \ Y(1) = y(1), \\
    D^y Y \left( \frac{x_N}{2} \right) &= D^+ Y \left( \frac{x_N}{2} \right). \quad (10)
\end{align*}
\]
where
\[
D^+ Y(x_i) = \frac{Y(x_{i+1}) - Y(x_i)}{x_{i+1} - x_i}, \quad D^- Y(x_i) = \frac{Y(x_i) - Y(x_{i-1})}{x_i - x_{i-1}},
\]
\[
D^y Y(x_i) = \begin{cases} 
    D^- Y(x_i), & i < \frac{N}{2}, \\
    \delta^2 Y(x_i), & \frac{N}{2} \leq i \leq \frac{N}{2}.
\end{cases}
\]

The following lemma demonstrates that the finite difference operator \( L^N \) has characteristics that are similar to those of the differential operator \( L \).

**Lemma 2** Discrete minimum principle. Suppose that a mesh function \( Y(x_i) \) satisfies \( Y(0) \geq 0, \ Y(1) \geq 0, \ L^N Y(x_i) \leq 0, \ \forall x_i \in \Omega_N, \) and \( D^+ Y(x_{\frac{N}{2}}) - D^- Y(x_{\frac{N}{2}}) \leq 0 \) then \( Y(x_i) \geq 0, \ \forall x_i \in \Omega_N. \)

**Proof.** We refer to [21] for proof. □

**Lemma 3.** If \( Y(x_i), x_i \in \tilde{\Omega}_N \) is a mesh function satisfying the difference scheme (10), then \( \|Y\|_{\Omega_N} \leq C. \)

**Proof.** Define the mesh function for \( x_i \in \tilde{\Omega}_N, \) as
\[
\omega^\pm(x_i) = M \pm Y(x_i),
\]
where \( M = \max \{ |Y(0)|, |Y(1)| \} + \frac{1}{\varepsilon} \|f\|_{\Omega_0, \Omega_1}. \) Now, \( \psi^\pm(0) \) and \( \psi^\pm(1) \) are non negative. For \( x_i \in \Omega_N, \)
\[
L^N \omega^\pm(x_i) = -b(x_i) M \pm L^N Y(x_i) = -b(x_i) M \pm f(x_i) \leq 0.
\]
Also
\[
D^+ \omega^\pm \left( \frac{x_N}{2} \right) - D^- \omega^\pm \left( \frac{x_N}{2} \right) = 0.
\]
It follows from the discrete minimum principle that \( \omega^\pm(x_i) \geq 0, \ \forall x_i \in \tilde{\Omega}_N, \) which implies
\[
\|Y\|_{\Omega_N} \leq C
\]
□

**Error estimates**

Let us denote the nodal error at each mesh point \( x_i \in \tilde{\Omega}_N \) by
\[
|e(x_i)| = |Y(x_i) - y(x_i)|,
\]
where \( Y \) and \( y \) are solutions of Eqs. (1) and (10) at a point \( x_i \) respectively.

We find the bounds for the nodal error \( |e(x_i)| \) in \( \Omega^- \) and \( \Omega^+ \) separately. To find the error bounds, we decompose the solution \( Y \) of the discrete problem (10) into regular, and layer parts as
\[
Y(x_i) = V^+(x_i) + W^+(x_i). \quad (11)
\]
We further split the regular and layer section into parts to the left and right of the discontinuity, i.e., in \( \Omega^- \) and \( \Omega^+. \)
Let $V^-(x_i)$ and $V^+(x_i)$ be mesh functions, which approximate $V(x_i)$ to the left and right sides of the point of discontinuity $x_{\frac{N}{2}} = d$ respectively, be defined as follows:

\[
V^+(x) = \begin{cases} 
V^-(x), & \text{for } 1 \leq i \leq \frac{N}{2} - 1, \\
V^+(x_i), & \text{for } \frac{N}{2} + 1 \leq i \leq N - 1.
\end{cases}
\]  

(12)

where $V^+(x)$ and $V^+(x)$ are, respectively, the solutions to the following discrete problems:

\[
\mathcal{L}^N V^-(x_i) = f(x_i), \quad 1 \leq i \leq \frac{N}{2} - 1, \quad V^+(0) = v^+(0), \quad V^+(x_{\frac{N}{2}}) = v^+(d-),
\]

\[
\mathcal{L}^N V^+(x_i) = f(x_i), \quad \frac{N}{2} + 1 \leq i \leq N - 1, \quad V^+(x_{\frac{N}{2}}) = v^+(d+), \quad V^+(1) = v^+(1).
\]

Similarly, we split the mesh function $W^+(x_i)$ into left and right layer components $W^+_l(x_i)$ and $W^+_r(x_i)$. We further decompose them into components on either side of the discontinuity, $x_{\frac{N}{2}} = d$.

The decomposition is as follows:

\[
W^+(x_i) = W^+_l(x_i) + W^+_r(x_i) = \begin{cases} 
W^+_l(x_i) + W^+_r(x_i), & \text{for } 1 \leq i \leq \frac{N}{2} - 1, \\
W^+_l(x_i) + W^+_r(x_i), & \text{for } \frac{N}{2} + 1 \leq i \leq N - 1.
\end{cases}
\]

where $W^+_l(x_i)$ and $W^+_r(x_i)$ are solutions of the following equations:

\[
\mathcal{L}^N W^+_l(x_i) = 0, \quad 1 \leq i \leq \frac{N}{2} - 1, \quad W^+_l(0) = w^+_l(0), \quad W^+_l(x_{\frac{N}{2}}) = w^+_l(d-),
\]

\[
\mathcal{L}^N W^+_r(x_i) = 0, \quad \frac{N}{2} + 1 \leq i \leq N - 1, \quad W^+_r(x_{\frac{N}{2}}) = w^+_r(d+), \quad W^+_r(1) = 0.
\]  

(13)

\[
\mathcal{L}^N W^+_r(x_i) = 0, \quad 1 \leq i \leq \frac{N}{2} - 1, \quad W^+_r(0) = 0, \quad W^+_r(x_{\frac{N}{2}}) = w^+_r(d-),
\]

\[
\mathcal{L}^N W^+_r(x_i) = 0, \quad \frac{N}{2} + 1 \leq i \leq N - 1, \quad W^+_r(x_{\frac{N}{2}}) = 0, \quad W^+_r(1) = w^+_r(1).
\]  

(14)

The unique solution $Y(x_i)$ of the problem (10) is defined by

\[
Y(x_i) = \begin{cases} 
(V^++W^+_l+W^+_r)(x_i), & 1 \leq i \leq \frac{N}{2} - 1, \\
(V^++W^+_l+W^+_r)(x_i) = (V^++W^+_l+W^+_r)(x_i), & i = \frac{N}{2},
\end{cases}
\]

\[
(V^++W^+_l+W^+_r)(x_i), \quad \frac{N}{2} + 1 \leq i \leq N - 1.
\]

The next lemma gives bounds on the discrete layer components.

**Lemma 4.** The layer components $W^+_l(x_i)$, $W^+_r(x_i)$, $W^+_r(x_i)$ and $W^+_r(x_i)$ satisfy the following bounds:

\[
|W^+_l(x_i)| \leq C_{Y^+_l}, \quad \gamma^+_l = \prod_{i=1}^l (1 + \theta_h k_i)^{-1}, \quad 1 \leq i \leq \frac{N}{2}, \quad \gamma^+_{l,0} = C_1,
\]

\[
|W^+_r(x_i)| \leq C_{Y^+_r}, \quad \gamma^+_r = \prod_{k=1}^l (1 + \theta h k_i)^{-1} \cdot \frac{N}{2} + 1 \leq i \leq N, \quad \gamma^+_{r,\frac{N}{2}} = C_1,
\]

\[
|W^+_r(x_i)| \leq C_{Y^+_r}, \quad \gamma^+_r = \prod_{k=1}^l (1 + \theta h k_i)^{-1}, \quad 1 \leq i \leq \frac{N}{2}, \quad \gamma^+_{r,\frac{N}{2}} = C_1,
\]

\[
|W^+_r(x_i)| \leq C_{Y^+_r}, \quad \gamma^+_r = C \prod_{k=1}^l (1 + \theta h_k)^{-1}, \quad \frac{N}{2} + 1 \leq i \leq N, \quad \gamma^+_{r,N} = C_1.
\]

**Proof.** Let us define the barrier function for the left layer term as

\[
\eta^+_l = \gamma^+_l \pm W^+_l(x_i), \quad 0 \leq i \leq \frac{N}{2}.
\]

For large enough $C$ and $C_1$, $\eta^+_{l,0} \geq 0$ and $\eta^+_{l,N/2} \geq 0$.

Consider,

\[
\mathcal{L}^N \eta^+_l = \mathcal{L}^N \eta^+_l \pm \mathcal{L}^N W^+_l(x_i)
\]

\[
= \gamma^+_{l+1} (2e\theta^2 \left( \frac{h}{h_{i+1} + h_i} \right) + 2e\theta^2 - \mu a(x_i)\theta_2 (1 + \theta h_{i+1}) - b(x_i)(1 + \theta h_{i+1})).
\]

\[
\leq \gamma^+_{l+1} (2e\theta^2 - \mu a(x_i)\theta_2 (1 + \theta h_{i+1}) - b(x_i)(1 + \theta h_{i+1}) as \frac{h_{i+1}}{h_{i+1} + h_i} - 1 \leq 0.
\]
For both the cases $\sqrt{a} \mu \leq \sqrt{\rho e}$ and $\sqrt{a} \mu > \sqrt{\rho e}$, on simplification, we get:

$$L^N \eta_{i,j} \leq \gamma_{i+1}^+ \left(2\epsilon_2^2 + \mu a(x_i) \theta_2 - b(x_i)\right) \quad \text{as} \quad -(\mu a(x_i) \theta_2 + b(x_i) \theta_2) h_{i+1} \leq 0$$

and

$$\leq \gamma_{i+1}^+ \left(\frac{pa}{2} + b[a(x_i)] - b(x_i)\right) \leq 0.$$

By discrete minimum principle for the continuous case [18], we obtain:

$$\eta_{i,j}^+ \geq 0 \Rightarrow W_{i}^{-+}(x_i) \leq C \prod_{k=1}^i (1 + \theta_2 h_k)^{-1}, \quad 1 \leq i \leq \frac{N}{2}.$$

For $\frac{N}{2} + 1 \leq i \leq N$, consider the barrier function for the left layer term as:

$$\eta_{i,j}^+ = \gamma_{i,j}^+ \pm W_{i}^{-+}(x_i), \quad \frac{N}{2} \leq i \leq N.$$

For large enough $C$ and $C_1$, $\eta_{N/2}^+ \geq 0$ and $\eta_{N}^+ \geq 0$.

Consider

$$L^N \eta_{i,j} = L^N \gamma_{i,j}^+ \pm L^N W_{i}^{-+}(x_i)$$

then

$$\leq \gamma_{i+1}^+ \left(2\epsilon_2^2 - \mu a(x_i) \theta_1 - b(x_i)(1 + \theta_1 h_{i+1})\right) \quad \text{as} \quad \frac{h_{i+1}}{h_{i+1} + h_i} - 1 \leq 0$$

and

$$\leq \gamma_{i+1}^+ \left(2\epsilon_2^2 - \mu a(x_i) \theta_1 - b(x_i)\right) \quad \text{as} \quad b(x_i) \theta_1 h_{i+1} \geq 0).$$

For case $\sqrt{a} \mu \leq \sqrt{\rho e}$, $\theta_1 = \frac{\sqrt{pa}}{2 \sqrt{e}}$, the above expression becomes,

$$L^N \eta_{i,j}^+ \leq \gamma_{i+1}^+ \left(\frac{pa}{2} - \mu a(x_i) \frac{\sqrt{pa}}{2 \sqrt{e}} - b(x_i)\right) \leq 0.$$

For the case $\sqrt{a} \mu > \sqrt{\rho e}$, $\theta_1 = \frac{\mu a}{2e}$, we obtain

$$L^N \eta_{i,j}^+ \leq \gamma_{i+1}^+ \left(\frac{\mu a^2}{2e} - \mu a(x_i) \frac{\mu a}{2e} - b(x_i)\right) \leq 0.$$

Hence by discrete minimum principle for continuous case [18], we obtain:

$$\eta_{i,j}^+ \geq 0 \Rightarrow W_{i}^{-+}(x_i) \leq C \prod_{k=1}^i (1 + \theta_1 h_k)^{-1}, \quad \frac{N}{2} + 1 \leq i \leq N.$$

Similarly, we define the barrier function for the right layer component as

$$\eta_{i,j}^- = \gamma_{i,j}^- \pm W_{i}^{-+}(x_i), \quad 0 \leq i \leq \frac{N}{2}.$$

For large enough $C$ and $C_1$, $\eta_{0}^- \geq 0$ and $\eta_{N/2}^- \geq 0$.

Consider

$$L^N \eta_{i,j}^- = L^N \gamma_{i,j}^- \pm L^N W_{i}^{-+}(x_i)$$

then

$$\leq \gamma_{i+1}^- \left(2\epsilon_2^2 + \mu a(x_i) \theta_1 - b(x_i)(1 + \theta_1 h_{i+1})\right) \quad \text{as} \quad \frac{h_{i+1}}{h_{i+1} + h_i} - 1 \leq 0 \quad \text{and} \quad b(x_i) \theta_1 h_i \leq 0.$$

For both the cases $\sqrt{a} \mu \leq \sqrt{\rho e}$ and $\sqrt{a} \mu > \sqrt{\rho e}$, on simplification, we get

$$L^N \eta_{i,j}^- \leq \frac{\gamma_{i+1}^-}{1 + \theta_1 h_i} (-b(x_i)) \leq 0.$$
By discrete minimum principle for the continuous case [18], we obtain

\[ \eta_{r,i}^N \geq 0 \Rightarrow W_{r,i}^{-N}(x_i) \leq C \prod_{k=i+1}^{N/2} (1 + \theta_2 h_k)^{-1}, \quad 1 \leq i \leq \frac{N}{2}. \]

Similarly, we prove the bound for \( W_{r,i}^{++} \) for \( \frac{N}{2} + 1 \leq i \leq N - 1. \) \( \square \)

**Lemma 5.** The error in the regular component satisfies the following error estimates for the mesh points, \( x_i \in \Omega_N \)

\[ |(V^* - v^*)(x_i)| \leq CN^{-i}, \]

where \( V^* \) and \( v^* \) are the regular part of the continuous and the discrete solution as defined by Eqs. (12) and (2), respectively.

**Proof.** The truncation error for the regular part of the solution \( y \) of the Eq. (1) for both the cases \( \sqrt{a} \mu \leq \sqrt{b} \) and \( \sqrt{a} \mu > \sqrt{b} \) is

\[ |L \cdot U(N(x_i) - f(x_i))| \leq \left| e \left( \frac{\partial^2}{\partial x^2} \right) v^*(x_i) + \mu |a(x_i)| \right| \left( D^* - \frac{d}{dx} \right) v^*(x_i) \]

\[ \leq CN^{-i}, \quad \text{for} \quad 1 \leq i \leq \frac{N}{2} - 1. \]

Similarly

\[ |L \cdot U(N(x_i) - v^*)| \leq CN^{-i}, \quad \text{for} \quad \frac{N}{2} + 1 \leq i \leq N - 1. \]

Define the barrier function

\[ \psi^*(x_i) = CN^{-i} \pm (V^* - v^*)(x_i), \quad 1 \leq i \leq \frac{N}{2} - 1. \]

For large \( C \), \( \psi^*(0) \geq 0 \), \( \psi^*(x_i) \geq 0 \) and \( L \cdot U(N(x_i) \leq 0 \). Hence using the approach given in [5], we get \( \psi^*(x_i) \geq 0 \) and

\[ |(V^* - v^*)(x_i)| \leq CN^{-i}, \quad 1 \leq i \leq \frac{N}{2} - 1. \] (15)

Similarly,

\[ |(V^* - v^*)(x_i)| \leq CN^{-i}, \quad \frac{N}{2} + 1 \leq i \leq N - 1. \] (16)

Combining the above results, we obtain

\[ |(V^* - v^*)(x_i)| \leq CN^{-i}, \quad \forall \; x_i \in \Omega_N. \]

\( \square \)

**Lemma 6.** The left singular component of the truncation error satisfy the following estimate at mesh point \( x_i \in \Omega_N \)

\[ |W_{l,i}^+ - w_{l,i}^+(x_i)| \leq CN^{-i}, \]

where \( W_{l,i}^+ \) and \( w_{l,i}^+ \) are the discrete and the continuous left layer components satisfying the Eq. (13) and Eq. (3), respectively.

**Proof.** In \( (\sigma_1, d) \) i.e., for \( \frac{N}{2} \leq i < \frac{N}{2} \), from Theorem 6, we obtain

\[ |w_{l,i}^+(x_i)| \leq C \exp^{-\theta_2 x_i} \leq C \exp^{-\theta_1 x_i} \leq CN^{-i}. \] (17)

Also from Lemma 4, we have that \( W_{l,i}^+ \) is a monotonically decreasing function, so

\[ |W_{l,i}^+(x_i)| \leq C \prod_{k=1}^{N/2} (1 + \theta_2 h_k)^{-1}, \quad \text{for} \quad \frac{N}{8} \leq i < \frac{N}{2}. \]

Now,

\[ |y_{l,i}^-| = \prod_{k=1}^{N/2} (1 + \theta_2 h_k)^{-1} \]

\[ \Rightarrow \log \left( y_{l,i}^- \right) = - \sum_{k=1}^{N/2} \log(1 + \theta_2 h_k). \]
Consider,
\[
\log \left( \prod_{k=1}^{N} (1 + \theta_k h_k) \right) \geq \sum_{k=1}^{N} \theta_k h_k - \sum_{k=1}^{N} \left( \frac{\theta_k h_k}{2} \right)^2, \text{ (as } \log(1 + t) \geq t - \frac{t^2}{2} \text{ for } t \geq 0) \\
= \theta_2 \sigma_1 - \sum_{k=1}^{N} \left( \frac{\theta_k h_k}{2} \right)^2 \text{ (as } \sum_{k=1}^{N} h_k = x \frac{N}{T}).
\]

Next, we calculate \( \sum_{k=1}^{N} \left( \frac{\theta_k h_k}{2} \right)^2 \).

For \( 1 \leq k \leq \frac{N}{8} \),
\[
h_k = x_k - x_{k-1} = \frac{8}{\theta_2} \left( \phi_1(\xi_k) - \phi_1(\xi_{k-1}) \right) = \int_{\xi_{k-1}}^{\xi_k} \phi_1'(\xi) d\xi, \quad \xi = k \frac{N}{8}
\]
\[
\frac{\theta_2 h_k}{8} = \int_{\xi_{k-1}}^{\xi_k} \phi_1'(\xi) d\xi \Rightarrow \left( \frac{\theta_2 h_k}{8} \right)^2 \leq (\xi_k - \xi_{k-1}) \int_{\xi_{k-1}}^{\xi_k} \phi_1'(\xi)^2 d\xi, \quad \text{by Holder's inequality}
\]
\[
\sum_{k=1}^{N} \left( \frac{\theta_k h_k}{2} \right)^2 \leq \sum_{k=1}^{N} (\xi_k - \xi_{k-1}) \int_{\xi_{k-1}}^{\xi_k} \phi_1'(\xi)^2 d\xi.
\]
\[
\leq N^{-1} \int_{0}^{1/2} \phi_1'(\xi)^2 d\xi 
\]
\[
\leq C. \quad \text{(from Lemma 4.1)}
\]

So
\[
\left| f_{i,T}^{+} - f_{i,T}^{-} \right| \leq CN^{-1}, \quad \left| W_{i}^{+}(x_i) \right| \leq CN^{-4}, \quad \text{for } \frac{N}{8} \leq i < \frac{N}{2}.
\]

Hence for all \( x_i \in [\sigma_1, d) \) we have

\[
\left| (W_{i,T}^{+} - W_{i,T}^{-})(x_i) \right| \leq \left| W_{i}^{+}(x_i) \right| + \left| W_{i}^{-}(x_i) \right| \leq CN^{-4}.
\]

For \( \sqrt{\alpha \mu} \leq \sqrt{\rho} \epsilon \), the truncation error for the left layer component in the inner region \((0, \sigma_1)\), i.e., for \( i = 1, 2, \ldots, \frac{N}{8} - 1 \), is

\[
\mathcal{E}^N(W_i^{+,-} - w_i^{+,-}(x_i)) \leq C \left[ \int_{x_{i-1}}^{x_{i+1}} |w_i^{+,-(3)}(x)| dx + \mu |a(x_i)| \int_{x_{i}}^{x_{i+1}} |w_i^{+,-(2)}(x)| dx \right]
\]
\[
\leq C \sqrt{\epsilon} \left[ \int_{x_{i-1}}^{x_{i+1}} e^{-\theta_2 x} dx + \int_{x_{i}}^{x_{i+1}} e^{-\theta_2 x} dx \right]. \quad \text{(from Theorem 3.2)}
\]
\[
\leq C \sqrt{\epsilon} \left[ \int_{x_{i-1}}^{x_{i+1}} e^{-8\phi_1(\xi)} \phi_1'(\xi) \frac{d\xi}{\theta_2} + \int_{x_{i}}^{x_{i+1}} e^{-8\phi_1(\xi)} \phi_1'(\xi) \frac{d\xi}{\theta_2} \right]
\]
\[
\leq C \sqrt{\epsilon} \left[ \int_{x_{i-1}}^{x_{i+1}} e^{-8\phi_1(\xi)} \phi_1'(\xi) |\phi_1'(\xi)| d\xi + \int_{x_{i}}^{x_{i+1}} e^{-8\phi_1(\xi)} |\phi_1'(\xi)| d\xi \right]
\]
\[
\leq CN^{-1} e^{7\theta_2 x_i} \max |\psi_1'| \leq CN^{-1} \quad \text{(as } \max |\psi_1'| \leq 8).\]

We choose the barrier function for the layer component as
\[
\psi^2(x_i) = CN^{-1} \pm (W_{i,T}^{+} - w_i^{+,-})(x_i), \quad i = 1, 2, \ldots, \frac{N}{8} - 1.
\]

For sufficiently large \( C \), we have \( \mathcal{E}^N \psi_i \leq 0 \). Hence by discrete maximum principle in [18], \( \psi_i \geq 0 \). So, by the comparison principle, we can obtain the following bounds:

\[
\left| (W_{i,T}^{+} - W_{i,T}^{-})(x_i) \right| \leq CN^{-1} \quad \forall 1 \leq i \leq \frac{N}{8} - 1.
\]

For \( \sqrt{\alpha \mu} > \sqrt{\rho} \epsilon \), the truncation error for the left layer component for \( i = 1, 2, \ldots, \frac{N}{8} - 1 \) is given by

\[
\mathcal{E}^N(W_i^{+,-} - w_i^{+,-}(x_i)) \leq C \left[ \int_{x_{i-1}}^{x_{i+1}} |w_i^{+,-(3)}(x)| dx + \mu |a(x_i)| \int_{x_{i}}^{x_{i+1}} |w_i^{+,-(2)}(x)| dx \right]
\]

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Choosing a barrier function for the layer component as

$$\psi^+(x_i) = CN^{-1} \pm (W^{+}_i - w^+_i)(x_i), \forall 1 \leq i \leq \frac{N}{8} - 1.$$ 

For sufficiently large $C$, we have $CN_i \psi_i \leq 0$. Using the discrete minimum principle in [18], we can obtain the following bounds:

$$|(W^+_i - w^+_i)(x_i)| \leq CN^{-1}, \forall 1 \leq i \leq \frac{N}{8} - 1.$$ 

Hence for the left layer component

$$|(W^+_i - w^+_i)(x_i)| \leq CN^{-1}, \forall 1 \leq i \leq \frac{N}{2} - 1. \quad (18)$$

By similar argument in the domains $(d, 1 - \sigma_d)$ and $(1 - \sigma_d, 1)$, we have

$$|(W^{++}_i - w^{++}_i)(x_i)| \leq CN^{-1}, \forall \frac{N}{2} + 1 \leq i \leq N - 1. \quad (19)$$

Combining the results (18) and (19), the desired result is obtained. \[\square\]

**Lemma 7.** The right singular component of the truncation error satisfies the following approximation for each mesh point, $x_i \in \Omega_N$

$$|(W^{r+}_i - w^{r+}_i)(x_i)| \leq CN^{-1},$$

where $W^{r+}_i$ and $w^{r+}_i$ are the discrete and the continuous right layer components satisfying the Eq. (14) and Eq. (4), respectively.

**Proof.** In $(0, d - \sigma_d)$, for $1 \leq i \leq \frac{3N}{8}$, the left layer component has the following bound from Theorem 8

$$|w^{r+}_i(x_i)| \leq Ce^{-\theta_1(d-x_i)} \leq Ce^{-\theta_1 \sigma_d} \leq CN^{-d}. \quad (20)$$

Also from Lemma 4, we see that $W^{r+}_r$ is increasing function. So

$$|W^{r+}_r(x_i)| \leq C \prod_{j=1}^{N} (1 + \theta_1 h_j)^{-1} \leq C|y^{r+}_r|$$

for $1 \leq i \leq \frac{3N}{8}$.

Now consider,

$$|y^{r+}_r| = \prod_{j=\frac{N}{2}+1}^{N} (1 + \theta_1 h_j)^{-1}$$

$$\log(y^{r+}_r) = -\sum_{j=\frac{N}{2}+1}^{N} \log(1 + \theta_1 h_j)$$

As $\log(1 + r^2) \geq t - \frac{t^2}{2}$ for $t \geq 0,$

$$\Rightarrow \sum_{j=\frac{N}{2}+1}^{N} \log(1 + \theta_1 h_j) \geq \sum_{j=\frac{N}{2}+1}^{N} \theta_1 h_j - \sum_{k=\frac{N}{2}+1}^{N} \left(\frac{\theta_1 h_j}{2}\right)^2,$$

as $\sum_{j=\frac{N}{2}+1}^{N} h_j = x_N$.

Now we calculate $\sum_{j=\frac{N}{2}+1}^{N} \left(\frac{\theta_1 h_j}{2}\right)^2$.

For $\frac{3N}{4} + 1 \leq j \leq \frac{N}{2}$,

$$h_j = x_j - x_{j-1} = d - \frac{8}{\theta_1} \phi_2(\xi) - \left(d - \frac{8}{\theta_1} \phi_2(\xi_{j-1})\right)$$

$$= \frac{8}{\theta_1} (\phi_2(\xi) - \phi_2(\xi_{j-1})) = \frac{8}{\theta_1} \int_{\xi_{j-1}}^{\xi_j} \phi_2'(\xi) d\xi \Rightarrow \frac{\theta_1 h_j}{8} = \int_{\xi_{j-1}}^{\xi_j} \phi_2'(\xi) d\xi$$

$$\left(\frac{\theta_1 h_j}{8}\right)^2 \leq (\xi_j - \xi_{j-1}) \int_{\xi_{j-1}}^{\xi_j} \phi_2'(\xi)^2 d\xi$$

(by Holder's inequality)
\[
\sum_{j=\frac{N}{8}+1}^{N} \left( \frac{\theta_j x_j}{\delta B} \right)^2 \leq N^{-1} \int_{\frac{1}{8}}^{\frac{7}{8}} \Phi^2(\xi)^2 d\xi \leq C \quad \text{(from Lemma 4.1)}.
\]

So
\[
\sum_{j=\frac{N}{8}+1}^{N} \log(1 + \theta_j x_j) \geq 4 \log N - C \Rightarrow |W_r^{+-}(x_j)| \leq C \gamma_{\frac{1}{\delta B}} \leq CN^{-4}, \quad \forall \ 1 \leq i \leq \frac{3N}{8}.
\]

Hence for all \( x_j \in (0, d - \sigma_2) \), we have
\[
|W_r^{+-} - w_r^{+-})(x_j)| \leq |W_r^{+-}(x_j)| + |w_r^{+-}(x_j)| \leq CN^{-4}.
\]

For \( \sqrt{\alpha_{\mu}} \leq \sqrt{\rho \kappa} \), the derivative bounds for right layer component \( w_r^{+-} \) in the inner region \( (d - \sigma_2, d) \) is given by Theorem 6. Truncation error for right layer component is given by,
\[
\begin{align*}
|\mathcal{L}^N(W_r^{+-} - w_r^{+-})(x_j)| & \leq C \left( \epsilon \int_{x_{j-1}}^{x_{j+1}} |u_r^{+-}(3)(x_j)| dx + \mu |a(x_j)| \right) \\
& \leq C \left( \epsilon \int_{x_{j-1}}^{x_{j+1}} |u_r^{+-}(3)(x_j)| dx + \mu |a(x_j)| \right) \\
& \leq C \frac{e^{-\theta_1(\xi-d)}}{\epsilon} \int_{x_{j-1}}^{x_{j+1}} e^{-\theta_1(\xi-d)} dx \\
& \leq C \frac{e^{-\theta_1(\xi-d)}}{\epsilon} \int_{x_{j-1}}^{x_{j+1}} e^{-7\theta_2(\xi)} |w_r^{+-}(\xi)| dx + \int_{x_{j-1}}^{x_{j+1}} e^{-7\theta_2(\xi)} |w_r^{+-}(\xi)| dx \\
& \leq C N^{-1} e^{\frac{\epsilon}{\epsilon} \theta_1(\xi-d)} \max |w_r^{+-}| \\
& \leq C N^{-1} \text{(as } \max |w_r^{+-}| \leq 8). 
\end{align*}
\]

By defining an appropriate barrier function and using the discrete minimum principle (in [18]), we can obtain the following bounds:
\[
|W_r^{+-} - w_r^{+-}(x_j)| \leq CN^{-1}, \quad \frac{3N}{8} < i < \frac{N}{2}.
\]

For case \( \sqrt{\alpha_{\mu}} > \sqrt{\rho \kappa} \), the derivative bounds for right layer component \( w_r^{+-} \) for \( \frac{3N}{8} < i < \frac{N}{2} \) are given by Theorem 8. Hence by using truncation error for the right layer component, we obtain,
\[
\begin{align*}
|\mathcal{L}^N(W_r^{+-} - w_r^{+-})(x_j)| & \leq C \left( \epsilon \int_{x_{j-1}}^{x_{j+1}} |u_r^{+-}(3)(x_j)| dx + \mu |a(x_j)| \right) \\
& \leq C_1 \int_{x_{j-1}}^{x_{j+1}} \left( \frac{\mu}{\epsilon} \right)^2 e^{-\theta_1(\xi-d)} dx + C_2 \int_{x_{j-1}}^{x_{j+1}} \left( \frac{\mu}{\epsilon} \right)^2 e^{-\theta_1(\xi-d)} dx \\
& \leq C \frac{\mu^2}{e} \left( \int_{x_{j-1}}^{x_{j+1}} e^{-7\theta_2(\xi)} |w_r^{+-}(\xi)| dx \right) + \int_{x_{j-1}}^{x_{j+1}} e^{-7\theta_2(\xi)} |w_r^{+-}(\xi)| dx \\
& \leq C \frac{\mu^2}{e} \left( \int_{x_{j-1}}^{x_{j+1}} e^{-\theta_1(\xi-d)} N^{-1} \max |w_r^{+-}| \right) \\
& \leq C \frac{\mu^2}{e} N^{-1} \text{(as } \max |w_r^{+-}| \leq 8). 
\end{align*}
\]

Choosing the barrier function for the layer component as
\[
\psi^2(x_j) = C_1 N^{-1} + C_2 N^{-1} \left( \frac{\mu}{\epsilon} \right) \frac{N_i}{\theta_1} \pm (W_r^{+-} - w_r^{+-})(x_j).
\]

For sufficiently large \( C_i \), by the application of the discrete minimum principle (in [18]) we obtain the following bounds:
\[
|W_r^{+-} - w_r^{+-}(x_j)| \leq C_1 N^{-1} + C_2 N^{-1} \left( \frac{\mu}{\epsilon} \right) \frac{N_i}{\theta_1} \leq CN^{-1}, \quad \text{for } \frac{3N}{8} < i < \frac{N}{2}.
\]

Hence the bound for the right layer component for \( x_j \in (d - \sigma_2, d) \) is
\[
|W_r^{+-} - w_r^{+-}(x_j)| \leq CN^{-1}. \tag{21}
\]

Similarly, we can prove the result for \( \frac{N}{2} + 1 \leq i \leq N\),
\[
|W_r^{+-} - w_r^{+-}(x_j)| \leq CN^{-1}. \tag{22}
\]

Combining the results (21) and (22) the final answer is obtained.
Lemma 8. Let \( y(x) \) and \( Y(x) \) be the solutions to the problems (1) and (10), respectively. The error \( e(x, \frac{N}{2}) \) estimated at the point of discontinuity \( x = \frac{N}{2} \) satisfies

\[
| (D^+ - D^-)(Y(x, N/2) - y(x, N/2)) | \leq \begin{cases}
\frac{C}{e^\theta_1}, & \sqrt{\mu} \leq \sqrt{\rho e}, \\
\frac{C\mu^2}{e^\theta_1}, & \sqrt{\mu} > \sqrt{\rho e}.
\end{cases}
\]

Proof. Consider

\[
|(D^+ - D^-)(Y(x, N/2) - y(x, N/2))| \leq |(D^+ - D^-)Y(x, N/2)|
\]

Since \( |(D^+ - D^-)Y(x, N/2)| = 0 \)

\[
|(D^+ - D^-)Y(x, N/2)| \leq \left| \left( \frac{d}{dx} - D^+ \right) y(x, N/2) \right| + \left| \left( \frac{d}{dx} - D^- \right) y(x, N/2) \right|
\]

\[
\leq C_1 h_{N+1} |y'| + C_2 h_N |y''|
\]

\[
\leq C\hat{h}|y''|
\]

\[
\leq \begin{cases}
\frac{C\hat{h}}{e}, & \sqrt{\mu} \leq \sqrt{\rho e}, \quad (\hat{h} = \max\{h_N, h_{N+1}\}) \\
\frac{C\hat{h}^2}{e^2}, & \sqrt{\mu} > \sqrt{\rho e}.
\end{cases}
\]

Using the fact that \( \hat{h} \leq C/\theta_1 \) in the given domain gives the lemma. \( \square \)

Theorem 9. Let \( \sqrt{e} < N^{-1} \) for \( \sqrt{\mu} \leq \sqrt{\rho e} \) and \( \max\left\{ \frac{\rho}{\mu}, \frac{\rho}{e} \right\} < N^{-1} \) for \( \sqrt{\mu} > \sqrt{\rho e} \). If \( y(x) \) and \( Y(x) \) be respectively the solutions of the problems (1) and (10) then,

\[
\| Y - y \| \leq CN^{-1},
\]

where \( C \) is a constant independent of \( e, \mu \) and discretization parameter \( N \).

Proof. For \( i = 1, 2, \ldots, N/2 - 1, N/2 + 1, \ldots, N - 1 \), from Lemma 5, Lemma 6, and Lemma 7, we have that

\[
\| Y - y \|_{\Omega_N} \leq CN^{-1}.
\]

Let \( \sqrt{\mu} \leq \sqrt{\rho e} \), to find error at the point of discontinuity \( x_i = x_N \), consider the discrete barrier function \( \psi_1(x_i) = \psi_1(x_i) \pm e(x_i) \) defined in the interval \( (d - \sigma_2, d + \sigma_3) \) where

\[
\psi_1(x_i) = CN^{-1} + \frac{C_1 \sigma}{eN(\log N)^2} \left\{ \begin{array}{ll}
\frac{x_i - (d - \sigma_2)}{d + \sigma_3 - x_i}, & x_i \in \Omega_N \cap (d - \sigma_2, d), \\
\frac{x_i - (d + \sigma_3)}{d - \sigma_2 - x_i}, & x_i \in \Omega_N \cap [d, d + \sigma_3]
\end{array} \right.
\]

and \( \sigma = \sigma_2 = \sigma_3 = \frac{4}{\theta_1} \log N \).

We have \( \phi_1(d - \sigma_2) \) and \( \phi_1(d + \sigma_3) \) are non-negative. And \( L^N \phi_1(x_i) \leq 0 \). \( x_i \in (d - \sigma_2, d + \sigma_3) \) \( |(D^+ - D^-)\phi_1(x_N/2)| \leq 0 \).

Hence by applying discrete minimum principle we get \( \phi_1(x_i) \geq 0 \).

Therefore, for \( x_i \in (d - \sigma_2, d + \sigma_3) \)

\[
| (Y - y)(x_i) | \leq C N^{-1} + \frac{C_2 \sigma^2}{eN(\log N)^2} \leq CN^{-1}.
\]

In second case \( \sqrt{\mu} > \sqrt{\rho e} \), consider the discrete barrier function \( \psi_1(x_i) = \psi_2(x_i) \pm e(x_i) \) defined in the interval \( (d - \sigma_2, d + \sigma_3) \) where

\[
\psi_2(x_i) = CN^{-1} + \frac{C_1 \sigma^2}{eN(\log N)^2} \left\{ \begin{array}{ll}
\frac{x_i - (d - \sigma_2)}{d + \sigma_3 - x_i}, & x_i \in \Omega_N \cap (d - \sigma_2, d), \\
\frac{x_i - (d + \sigma_3)}{d - \sigma_2 - x_i}, & x_i \in \Omega_N \cap [d, d + \sigma_3]
\end{array} \right.
\]

where \( \sigma = \sigma_2 = \sigma_3 = \frac{4}{\theta_1} \log N \). We have \( \phi_2(d - \sigma_2) \) and \( \phi_2(d + \sigma_3) \) are non-negative and \( L^N \phi_2(x_i) \leq 0 \). \( x_i \in (d - \sigma_2, d + \sigma_3) \) and \( |(D^+ - D^-)\phi_2(x_N/2)| \leq 0 \).

Hence by applying discrete minimum principle, we get \( \phi_2(x_i) \geq 0 \). Therefore, for \( x_i \in (d - \sigma_2, d + \sigma_3) \)

\[
| (Y - y)(x_i) | \leq C N^{-1} + \frac{C_2 \sigma^2 \mu^2}{e^2N(\log N)^2} \leq CN^{-1}.
\]

By combining the result (23) and (24) we obtain the desired result. \( \square \)
Numerical results

In this section, we have considered some singularly perturbed two-parameter boundary value problems with discontinuous convection coefficient and source term as test problems. The proposed scheme is used to solve these problems numerically.

Example 1.
\[
ey''(x) + \mu a(x)y'(x) - y(x) = f(x) \quad x \in (0, 1),
\]
y(0) = 2, \ y(1) = 1,
with
\[
a(x) = \begin{cases} 
-2, & 0 \leq x \leq 0.5, \\
2, & 0.5 < x \leq 1,
\end{cases}
\]
and \[f(x) = \begin{cases} 
-1, & 0 \leq x \leq 0.5, \\
1, & 0.5 < x \leq 1.
\end{cases}\]

Example 2.
\[
ey''(x) + \mu a(x)y'(x) - 2y(x) = f(x) \quad x \in (0, 1),
\]
y(0) = 0, \ y(1) = -1,
with
\[
a(x) = \begin{cases} 
-(1 + x), & 0 \leq x \leq 0.5, \\
(2 + x^2), & 0.5 < x \leq 1,
\end{cases}
\]
and \[f(x) = \begin{cases} 
-(14x + 1), & 0 \leq x \leq 0.5, \\
(2 - 2x), & 0.5 < x \leq 1.
\end{cases}\]

Since the exact solution for Example 1 and Example 2 is unknown, the maximum point-wise error and rate of convergence are computed using the double mesh principle (see [4], page 199). The double mesh difference is defined by
\[
E^N = \max_{x \in [0,1]} |Y^N(x) - Y^{2N}(x)|
\]
where \(Y^N(x_i)\) and \(Y^{2N}(x_i)\) represent the numerical solutions determined using \(N\) and \(2N\) mesh points respectively. The numerical rate of convergence is given by
\[
R^N = \frac{\log(E^N) - \log(E^{2N})}{\log 2}.
\]

Table 1 shows the results for various values of \(\mu\) and for \(\epsilon = 10^{-6}\) for Example 1. The order of convergence obtained approaches one as we increase the number of mesh points. In Table 2 the maximum point-wise error and order of convergence are given for Example 1 for varying values of \(\epsilon\) and keeping the value of \(\mu\) fixed.

Fig. 1 a and b represent the numerical solution and maximum point-wise error for Example 1 for the case \(\sqrt{a\mu} \leq \sqrt{\mu\epsilon}\) respectively with \(\epsilon = 10^{-8}, \mu = 10^{-6}\) and \(N = 256\). The numerical solution and maximum point-wise error for the case \(\sqrt{a\mu} > \sqrt{\mu\epsilon}\) for Example 1 for \(N = 256\) is given in Fig. 2a and b respectively with \(\epsilon = 10^{-12}, \mu = 10^{-4}\) and \(N = 256\).

![Fig. 1](image)

Fig. 1. (a) and (b): Numerical solution and errors for \(\epsilon = 10^{-8}, \mu = 10^{-6}\) when \(N = 256\) for Example 1.
Table 1  
Maximum point-wise error $E^N$ and approximate orders of convergence $R^N$ for Example 1 when $\epsilon = 10^{-6}$. 

| $\mu$   | Number of mesh points N |
|---------|-------------------------|
|         | 64          | 128      | 256       | 512       | 1024     |
| $10^{-1}$ | 3.3161e-01 | 2.1205e-01 | 1.2184e-01 | 6.5947e-02 | 3.4499e-02 |
| Order   | 0.64507 | 0.79490 | 0.88563 | 0.93474 |
| $10^{-3}$ | 3.0199e-01 | 1.8183e-01 | 9.9546e-02 | 5.2296e-02 | 2.6015e-02 |
| Order   | 0.73190 | 0.86918 | 0.92864 | 0.95830 |
| $10^{-4}$ | 2.9894e-01 | 1.7875e-01 | 9.7305e-02 | 5.0937e-02 | 2.6164e-02 |
| Order   | 0.74189 | 0.87739 | 0.93378 | 0.96113 |
| $10^{-5}$ | 2.9863e-01 | 1.7844e-01 | 9.7080e-02 | 5.0801e-02 | 2.6089e-02 |
| Order   | 0.74290 | 0.87823 | 0.93430 | 0.96142 |
| $10^{-6}$ | 2.9860e-01 | 1.7841e-01 | 9.7056e-02 | 5.0788e-02 | 2.6081e-02 |
| Order   | 0.74300 | 0.87831 | 0.93435 | 0.96145 |
| $10^{-7}$ | 2.9860e-01 | 1.7841e-01 | 9.7056e-02 | 5.0786e-02 | 2.6081e-02 |
| Order   | 0.74301 | 0.87832 | 0.93436 | 0.96145 |
| $10^{-10}$ | 2.9860e-01 | 1.7841e-01 | 9.7056e-02 | 5.0786e-02 | 2.6081e-02 |
| Order   | 0.74301 | 0.87832 | 0.93436 | 0.96145 |
| $256$    | 2.9860e-01 | 1.7841e-01 | 9.7056e-02 | 5.0786e-02 | 2.6081e-02 |
| Order   | 0.74301 | 0.87832 | 0.93436 | 0.96145 |

Table 2  
Maximum point-wise error $E^N$ and approximate orders of convergence $R^N$ for Example 1 when $\mu = 10^{-4}$. 

| $\epsilon$ | Number of mesh points N |
|------------|-------------------------|
|            | 64          | 128      | 256       | 512       | 1024     |
| $10^{-4}$  | 4.3793e-01 | 3.0942e-01 | 1.9296e-01 | 1.0991e-01 | 5.9142e-02 |
| Order      | 0.50113 | 0.68126 | 0.81198 | 0.89406 |
| $10^{-5}$  | 4.4915e-01 | 3.0223e-01 | 1.8274e-01 | 1.0226e-01 | 5.4505e-02 |
| Order      | 0.57151 | 0.72586 | 0.83750 | 0.90783 |
| $10^{-10}$ | 4.5302e-01 | 3.0188e-01 | 1.8160e-01 | 1.0136e-01 | 5.3951e-02 |
| Order      | 0.58557 | 0.73218 | 0.84130 | 0.99097 |
| $10^{-11}$ | 4.5349e-01 | 3.0186e-01 | 1.8149e-01 | 1.0127e-01 | 5.3895e-02 |
| Order      | 0.58716 | 0.73398 | 0.84170 | 0.99198 |
| $10^{-12}$ | 4.5353e-01 | 3.0186e-01 | 1.8148e-01 | 1.0126e-01 | 5.3889e-02 |
| Order      | 0.58732 | 0.73405 | 0.84174 | 0.91000 |
| $10^{-13}$ | 4.5354e-01 | 3.0186e-01 | 1.8148e-01 | 1.0126e-01 | 5.3888e-02 |
| Order      | 0.58734 | 0.73406 | 0.84175 | 0.91001 |
| $10^{-14}$ | 4.5354e-01 | 3.0186e-01 | 1.8147e-01 | 1.0126e-01 | 5.3863e-02 |
| Order      | 0.58733 | 0.73409 | 0.84170 | 0.91073 |
| $10^{-15}$ | 4.5355e-01 | 3.0182e-01 | 1.8147e-01 | 1.0114e-01 | 5.3746e-02 |
| Order      | 0.58753 | 0.73396 | 0.84334 | 0.91218 |
| $10^{-16}$ | 4.5347e-01 | 3.0154e-01 | 1.8095e-01 | 1.0082e-01 | 5.0923e-02 |
| Order      | 0.58862 | 0.73672 | 0.84375 | 0.98551 |
| $10^{-17}$ | 4.5270e-01 | 2.9936e-01 | 1.7931e-01 | 9.3085e-01 | 2.7602e-02 |
| Order      | 0.59666 | 0.73941 | 0.94586 | 1.7537 |

In Tables 3 and 4, maximum point-wise error and order of convergence are tabulated for Example 2. From these tables, we observe that the numerical order of convergence is consistent with the theoretical estimates presented in this paper.

For Example 2, Fig. 3a and b gives the numerical solution and maximum point-wise error for the case $\sqrt[\alpha]\mu \leq \sqrt[\epsilon]\mu$ respectively with $\epsilon = 10^{-8}$, $\mu = 10^{-6}$ and $N = 256$. The Fig. 4a and b show the numerical solution and maximum point-wise error for the case $\sqrt[\alpha]\mu > \sqrt[\epsilon]\mu$ respectively with $\epsilon = 10^{-12}$, $\mu = 10^{-4}$ and $N = 256$. From these figures, we observe that the maximum error is occurring at the point of discontinuity.
Fig. 2. (a) and (b): Numerical solution and errors for \( \epsilon = 10^{-12}, \mu = 10^{-4} \) when \( N = 256 \) for Example 1.

Table 3

Maximum point-wise error \( E^N \) and approximate orders of convergence \( R^N \) for Example 2 when \( \epsilon = 10^{-6} \).

| \( \mu \) | Number of mesh points \( N \) | 64 | 128 | 256 | 512 | 1024 |
|---|---|---|---|---|---|---|
| \( 10^{-4} \) | 5.3686e-01 | 3.4621e-01 | 1.2697e-01 | 4.696e-02 | 2.4601e-02 |
| Order | 0.63289 | 0.12697 | 1.4470 | 2.2109 |
| \( 10^{-5} \) | 5.5069e-01 | 3.7896e-01 | 1.5849e-01 | 4.7313e-02 | 1.0219e-02 |
| Order | 0.53919 | 1.2576 | 1.7440 |
| \( 10^{-6} \) | 5.5215e-01 | 3.8238e-01 | 1.6172e-01 | 4.961e-02 | 1.1616e-02 |
| Order | 0.53006 | 1.2414 | 2.0945 |
| \( 10^{-7} \) | 5.5230e-01 | 3.8272e-01 | 1.6204e-01 | 4.984e-02 | 1.1755e-02 |
| Order | 0.52915 | 1.2398 | 2.0839 |
| \( 10^{-8} \) | 5.5231e-01 | 3.8275e-01 | 1.6207e-01 | 4.9863e-02 | 1.1769e-02 |
| Order | 0.52905 | 1.2397 | 2.0829 |
| \( 10^{-9} \) | 5.5232e-01 | 3.8276e-01 | 1.6208e-01 | 4.9866e-02 | 1.1771e-02 |
| Order | 0.52905 | 1.2397 | 2.0827 |

With the use of the Shishkin-Bakhvalov mesh, we are able to improve the order of convergence to one, unlike the Shishkin mesh, where the order of convergence is deteriorated due to the presence of a logarithmic factor. In Table 5, we have compared the order of convergence obtained for the numerical method presented here on the Shishkin-Bakhvalov mesh and Shishkin mesh for Example 1.
Table 4
Maximum point-wise error $E_N$ and approximate orders of convergence $R_N$ for Example 2 when $\mu = 10^{-4}$.

| $\epsilon$  | Number of mesh points $N$ | 
|------------|---------------------------|
| $10^{-8}$  | 64: 5.9397e-01, 128: 4.4115e-01, 256: 2.8197e-01, 512: 1.6259e-01, 1024: 8.8048e-02 |
| Order      | 64: 0.42911, 128: 0.64574, 256: 0.79429, 512: 0.88487, 1024: 0.92381 |
| $10^{-9}$  | 64: 7.6541e-01, 128: 5.0315e-01, 256: 2.9775e-01, 512: 1.6443e-01, 1024: 8.7022e-02 |
| Order      | 64: 0.60902, 128: 0.75685, 256: 0.85662, 512: 0.91804, 1024: 0.92843 |
| $10^{-10}$ | 64: 8.0508e-01, 128: 5.1239e-01, 256: 2.9860e-01, 512: 1.6344e-01, 1024: 8.6241e-02 |
| Order      | 64: 0.65187, 128: 0.77903, 256: 0.86797, 512: 0.92381, 1024: 0.92843 |
| $10^{-11}$ | 64: 8.0927e-01, 128: 5.1325e-01, 256: 2.9858e-01, 512: 1.6344e-01, 1024: 8.6114e-02 |
| Order      | 64: 0.65746, 128: 0.78176, 256: 0.86932, 512: 0.92450, 1024: 0.92843 |
| $10^{-12}$ | 64: 8.0974e-01, 128: 5.1335e-01, 256: 2.9858e-01, 512: 1.6345e-01, 1024: 8.6022e-02 |
| Order      | 64: 0.65750, 128: 0.78179, 256: 0.86937, 512: 0.92459, 1024: 0.92843 |
| $10^{-13}$ | 64: 8.0976e-01, 128: 5.1336e-01, 256: 2.9858e-01, 512: 1.6345e-01, 1024: 8.6022e-02 |
| Order      | 64: 0.65749, 128: 0.78179, 256: 0.86937, 512: 0.92459, 1024: 0.92843 |
| $10^{-14}$ | 64: 8.0976e-01, 128: 5.1344e-01, 256: 2.9857e-01, 512: 1.6328e-01, 1024: 8.6538e-02 |
| Order      | 64: 0.65729, 128: 0.78210, 256: 0.87070, 512: 0.91597, 1024: 0.92607 |
| $10^{-15}$ | 64: 8.0948e-01, 128: 5.1325e-01, 256: 2.9734e-01, 512: 1.5930e-01, 1024: 9.4156e-02 |
| Order      | 64: 0.65734, 128: 0.78751, 256: 0.90032, 512: 0.75868, 1024: 0.92459 |
| $10^{-16}$ | 64: 8.0711e-01, 128: 5.1523e-01, 256: 2.8102e-01, 512: 1.4310e-01, 1024: 4.9626e-02 |
| Order      | 64: 0.64754, 128: 0.87454, 256: 0.97363, 512: 1.5278, 1024: 0.92459 |

Fig. 3. (a) and (b): Numerical solution and errors for $\epsilon = 10^{-8}, \mu = 10^{-6}$ when $N = 256$ for Example 2.

Fig. 4. (a) and (b): Numerical solution and errors for $\epsilon = 10^{-12}, \mu = 10^{-4}$ when $N = 256$ for Example 2.
Table 5

| $\mu$   | Mesh     | Number of mesh points N |
|---------|----------|-------------------------|
|         | 64       | 128                     | 256   | 512   |
| $10^{-1}$ | S-mesh   | 0.23077                 | 0.40876 | 0.57128 | 0.68814 |
|         | S-B mesh | 0.64598                 | 0.79977 | 0.88581 | 0.93482 |
| $10^{-4}$ | S-mesh   | 0.27379                 | 0.46997 | 0.63313 | 0.73471 |
|         | S-B mesh | 0.73267                 | 0.86950 | 0.92879 | 0.95837 |
| $10^{-7}$ | S-mesh   | 0.27851                 | 0.47689 | 0.64024 | 0.74010 |
|         | S-B mesh | 0.74265                 | 0.87771 | 0.93392 | 0.96120 |
| $10^{-10}$ | S-mesh  | 0.27904                 | 0.47767 | 0.64104 | 0.74070 |
|         | S-B mesh | 0.74377                 | 0.87863 | 0.93450 | 0.96152 |
| $10^{-13}$ | S-mesh  | 0.27904                 | 0.47767 | 0.64104 | 0.74070 |
|         | S-B mesh | 0.74377                 | 0.87864 | 0.93450 | 0.96152 |
| $10^{-16}$ | S-mesh  | 0.27904                 | 0.47767 | 0.64104 | 0.74070 |
|         | S-B mesh | 0.74377                 | 0.87864 | 0.93450 | 0.96152 |

Conclusion

In this article, a two-parameter SPP in one dimension with a discontinuous source term and convection coefficient is solved numerically by upwind difference method on a Shishkin-Bakvalov mesh. At the point of discontinuity, we consider a three-point difference scheme. The theoretical error estimates show that the proposed scheme is first-order convergent in the maximum norm. The use of the Shishkin-Bakvalov mesh helps in achieving the first-order convergence. The numerical results presented confirm the theoretical error estimates obtained. The numerical order of convergence approaches one as the number of mesh points increases. A comparison table between the numerical order of convergence obtained through the Shishkin mesh and the Shishkin-Bakvalov mesh shows the efficiency of the mesh used.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

CRediT authorship contribution statement

Nirmali Roy: Software, Validation, Formal analysis, Investigation, Resources, Data curation, Writing – original draft, Visualization. Anuradha Jha: Conceptualization, Methodology, Formal analysis, Investigation, Writing – original draft, Writing – review & editing, Visualization, Supervision, Project administration.

Data availability

No data was used for the research described in the article.

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