Multi-Photon Interference and Temporal Distinguishability of Photons

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A number of recent interference experiments involving multiple photons are reviewed. These experiments include generalized photon bunching effects, generalized Hong-Ou-Mandel interference effects and multi-photon interferometry for demonstrations of multi-photon de Broglie wavelength. The multi-photon states used in these experiments are from two pairs of photons in parametric down-conversion. We find that the size of the interference effect in these experiments, characterized by the visibility of interference pattern, is governed by the degree of distinguishability among different pairs of photons. Based on this discovery, we generalize the concept of multi-photon temporal distinguishability and relate it to a number of multi-photon interference effects. Finally, we make an attempt to interpret the coherence theory by the multi-photon interference via the concept of temporal distinguishability of photons.

Keywords: Interference; Distinguishability; Photon Counting.

1. Introduction and Historic Background

Interference of light, as a wave phenomenon, played a pivotal role in our understanding of light, from the early establishment of the classical wave theory to the formation of the modern coherence theory. Coming to the quantum age, it first provided a platform for conceptual understanding of quantum physics and then leads to fundamental test of quantum mechanics. More recently, it is associated with quantum information processing.

The most commonly-encountered interference phenomena are in the form of some beautiful interference fringes that have been well-studied by the classical coherence theory in Born and Wolf’s classic book Principle of Optics. In terms of the language of photon, these phenomena can be categorized as the single-photon interference effect and described by the famous Dirac’s statement:

Each photon ... only interferes with itself.
Interference between different photons never occurs.

However, with recent boom in quantum information science, interference with two and more photons came into focus. It plays an essential role in some quantum information protocols.
1.1. Early years of two-photon interference

The history of two-photon interference started at Pfleegor-Mandel experiment. After a dramatic demonstration of the interference effect between two independent lasers by Magyar and Mandel in 1963, a dark cloud was casted on the second part of the Dirac statement: is it true that no interference between different photons occurs? A true test would be to repeat the Magyar-Mandel experiment at single-photon level. This requires long time exposure. But it is well known that the phase of a laser diffuses in a long time period so that the interference fringes would be washed out. Under this circumstance, Pfleegor and Mandel designed an ingenious experiment to reveal the interference between independent lasers at single-photon level.

To overcome the problem of phase drift in long term, Pfleegor and Mandel employed the technique of intensity correlation between two detectors at separate locations. They discovered that there is a positive correlation when the detectors are one full fringe spacing apart but a negative correlation for half a fringe spacing separation, thus revealing the interference effect between independent lasers at single-photon level. This turns out to be the first two-photon interference phenomenon.

The Pfleegor-Mandel experiment can be easily explained in terms of classical wave theory. As is well-known, laser is the closest to a monochromatic wave with a phase that diffuses in a time period longer than coherence time. So if the observation time is short, as in the Magyar-Mandel experiment, both lasers have a steady phase even though they are operated independently, which gives rise to a steady interference fringe pattern observed by Magyar and Mandel:

\[ I(x) = I_0[1 + \cos(2\pi x/L + \Delta \varphi)], \]

where \( L \) is the fringe spacing and \( \Delta \varphi \) is the phase difference between the phases of the two lasers. On the other hand, when the observation time is long, as required in Pfleegor-Mandel experiment for acquiring data at single-photon level, the phases of the two independent lasers will diffuse independently leading to a random phase difference \( \Delta \varphi \). Thus intensity distribution will not show interference fringe. But for intensity correlation between two detectors at two locations, we have from Eq. (1):

\[ \langle I(x_1)I(x_2) \rangle_{\Delta \varphi} = I_0^2[1 + 0.5 \cos 2\pi(x_1 - x_2)/L]. \]

Immediately, we have

\[ \lambda_{12} \equiv \langle \Delta I(x_1)\Delta I(x_2) \rangle_{\Delta \varphi}/I_0^2 = 0.5 \cos 2\pi(x_1 - x_2)/L, \]

which equals +0.5 for \( x_1 - x_2 = L \) and −0.5 for \( x_1 - x_2 = L/2 \), consistent with the experimental results. Detailed study later confirmed Eq. (3).

But how can we understand the Pfleegor-Mandel experiment in terms of photons? Before we come to that, let us first examine the difference between intensity measurement and intensity correlation measurement. As seen in Fig.1, intensity measurement uses one detector and responses whenever there is one photon present.
whereas intensity correlation relies on two detectors and gives rise to a signal only when both detectors fire within the certain time window. So the intensity correlation requires the presence of two photons. In this way, we find that intensity measurement corresponds to one-photon detection and intensity correlation to two-photon detection.

Therefore Pfleegor-Mandel experiment is an interference effect involving two photons, for which Dirac’s statement on photon interference no longer applies. We need to generalize the Dirac statement to two-photon case as

$$A \text{ pair of photons only interferes with the pair itself in two-photon interference.}$$

With this generalized statement, let’s see how we can understand Pfleegor-Mandel experiment quantitatively in terms of photons. As discussed before, intensity correlation measurement involves two photons. The two detected photons have four possible ways to come from two lasers, as shown in Fig.2. Because of random phases of the two lasers, cases A and B give rise to no interference (If there were a steady phase difference between the two lasers, cases A and B would produce a phase dependent interference pattern). Using the generalized Dirac statement, we find that cases C and D will produce an interference fringe of

$$\frac{1 + \cos 2\pi(x_1 - x_2)/L}{L}.$$ (4)

Because of the randomness for the photons from a lasers, the four possibilities in Fig.2 have the same probability, assigned as $P_{20}$. Then the coincidence rate in the intensity correlation measurement is

$$R_2(x_1, x_2) \propto P_{20} + P_{20} + 2P_{20}[1 + \cos 2\pi(x_1 - x_2)/L]$$

which is in the exactly same form as Eq.(2). This indicates that the generalized Dirac statement indeed gives the correct prediction for Pfleegor-Mandel experiment. As a matter of fact, the generalized Dirac statement leads to interference between two wave amplitudes of two photons, i.e., two-photon waves. This picture of two-photon wave is indeed confirmed by the more rigorous quantum theory of light. The Dirac statement for two photons can be further generalized to arbitrary $N$ photons with the concept of $N$-photon waves.

Notice that the visibility in Pfleegor-Mandel experiment is only 50%, as shown in Eqs.(2 4). This is because of the existence of cases A and B. In fact, for all
classical sources of light, the probabilities for cases A and B are always larger than cases C and D, resulting in maximum visibility of 50% for classical fields. This is first pointed out by Mandel in 1983 and generally proved by Ou in 1988.

To obtain a visibility larger than 50% in two-photon interference, nonclassical fields with anti-bunching effect must be employed. Specifically with single-photon states in the two fields, the probabilities of cases A and B are zero, which leads to 100% visibility in two-photon interference. Indeed, more than 50% visibility in two-photon interference was observed by Ghosh and Mandel and by Ou and Mandel with two-photon state from parametric down-conversion.

Next, we consider another two-photon interference effect with two-photon state.

1.2. Hong-Ou-Mandel effect

Hong-Ou-Mandel effect is the most famous two-photon interference effect that has been widely applied in many fields. It is exploited in quantum information sciences and serves mainly as the fundamental process in the scheme of linear optical quantum computing.

A Hong-Ou-Mandel interferometer is made of a 50:50 beam splitter and two photons which enter the beam splitter from the opposite sides (Fig.3). For the two photons entering the beam splitter, there are four possible ways for them to come out. In Figs.3a and 3b, both photons go to either side of the beam splitter together. In Figs.3c and 3d, the two photons are either both transmitted or reflected, resulting them exit at separate ports. For the latter two cases, there is no way to tell them apart at the outputs. Therefore, quantum interference will occur. A detailed study of the phases shows that energy conservation in a lossless beam splitter leads to a 180° phase difference between the two cases. Since the amplitudes are the same in the two cases for a 50:50 beam splitter, there is a complete destructive interference between the two cases. So the probability is zero for the two photons exit at two separate sides. This is the Hong-Ou-Mandel effect.

Mathematically, we can calculate the probability of detecting two photons at two output ports of the beam splitter by finding the quantum state at the output. We may follow the procedure in Refs. and or more simply as follows. It is well known that the input-output relation in Heisenberg picture is given by

\[
\hat{b}_1 = \sqrt{T} \hat{a}_1 - \sqrt{R} \hat{a}_2, \\
\hat{b}_2 = \sqrt{T} \hat{a}_2 + \sqrt{R} \hat{a}_1.
\] (5)
Fig. 3. Four possible ways out for two photons entering a 50:50 beam splitter from two sides.

where $T, R$ are the transmissivity and reflectivity of the beam splitter, respectively. Or we may write in terms of the evolution operator $\hat{U}$ as

$$\hat{b}_1 = \hat{U}^\dagger \hat{a}_1 \hat{U}, \quad \hat{b}_2 = \hat{U}^\dagger \hat{a}_2 \hat{U}. \quad (6)$$

In Schrödinger picture, the output state is then

$$|\Phi_{out}\rangle = \hat{U} |\Phi_{in}\rangle. \quad (7)$$

With an input state of $|\Phi_{in}\rangle = |1_1, 1_2\rangle = \hat{a}_1^\dagger \hat{a}_2^\dagger |0\rangle$, we have

$$|\Phi_{out}\rangle = \hat{U} \hat{a}_1^\dagger \hat{a}_2^\dagger |0\rangle = \hat{U} \hat{a}_1^\dagger \hat{a}_2^\dagger \hat{U}^\dagger |0\rangle = (\hat{U} \hat{b}_2^\dagger \hat{U}^\dagger) (\hat{U} \hat{b}_2^\dagger \hat{U}^\dagger) |0\rangle,$$  

where we replaced $\hat{a}$’s with $\hat{b}$’s due to Schrödinger picture. By Eq. (6), we have

$$\hat{a}_1 = \hat{U} \hat{b}_2 \hat{U}^\dagger, \quad \hat{a}_2 = \hat{U} \hat{b}_2 \hat{U}^\dagger. \quad (9)$$

But from the reverse of Eq. (5), we then obtain

$$\hat{U} \hat{b}_1 \hat{U}^\dagger = \sqrt{T} \hat{b}_1 + \sqrt{R} \hat{b}_2,$$

$$\hat{U} \hat{b}_2 \hat{U}^\dagger = \sqrt{T} \hat{b}_2 - \sqrt{R} \hat{b}_1. \quad (10)$$

Furthermore, it is obvious that we have $\hat{U} |0\rangle = |0\rangle$, i.e., vacuum input leads to vacuum output. Substituting the above into Eq. (8), we finally obtain the output state as

$$|\Phi_{out}\rangle = (\sqrt{T} \hat{b}_1^\dagger + \sqrt{R} \hat{b}_2^\dagger)(\sqrt{T} \hat{b}_2^\dagger - \sqrt{R} \hat{b}_1^\dagger) |0\rangle$$

$$= (T - R) |1_1, 1_2\rangle - \sqrt{2TR} (|2_1, 0_2\rangle - |0_1, 2_2\rangle). \quad (11)$$

Therefore, we have $P_2(1, 1) = (T - R)^2$, which is zero for a 50:50 beam splitter, as we discussed according to the pictures in Fig.3.

The above analysis is for the single mode case. In practice, however, the fields are all in multiple modes. Under this circumstance, photons are described by wave packets. The interference effect described earlier only occurs when the two wave packets for the two photons coincide exactly at the beam splitter. On the other hand, when the two wave packets are well-separated arriving at the beam splitter, no interference occurs and we have a non-zero probability $P_2(1, 1)$ for detecting two photons at the opposite sides of the beam splitter. So as we scan the relative delay
between the two photon wave packets, the probability $P_2(1,1)$ will exhibit a drop and reach all the way to zero when the delay is zero. This is the Hong-Ou-Mandel dip. Fig.4 shows the set-up and the result.

When the two wave packets are well separated, the two photons behave independently as classical particles. So the overall probability of detecting two photons at the opposite sides is simply the sum of the probabilities of cases (c) and (d) in Fig.3. Then we easily obtain the classical probability $P_2(1,1)$ as

$$P_{cl}^2(1,1) = T^2 + R^2.$$  \hfill (12)

This corresponds to the flat lines on the two sides of the dip in Fig.4b.

1.3. Photon bunching and two-photon interference

Photon bunching effect, first discovered by Hanbury-Brown and Twiss\cite{17} in 1956, was the starting point for the field of quantum optics. The attempt to explain it leads to the development of the techniques in quantum optics.

Although it is a purely classical wave effect from Gaussian statistics of a thermal source and does not need quantum theory of light to understand it, explanation in terms of photons does need some imagination. It has long been argued that photon bunching effect is caused by Bosonian nature of photon\cite{18}. But this argument was soon overwhelmed by the success of the explanation by classical coherence theory\cite{19}. Later on, in his monumental work on quantum coherence, Glauber\cite{20} briefly discussed the connection between photon bunching effect and the interference of two-photon amplitudes, which is equivalent to our language of two-photon interference.

But to fully understand the photon bunching effect in terms of two-photon interference, we go back to the Hong-Ou-Mandel interferometer. Instead of the probability of $P_2(1,1)$, we are interested in $P_2(2,0)$, the probability of both photons exit in the same port. From Eq.\textcolor{red}{(11)}, we have

$$P_{2}^{HM}(2,0) = 2TR.$$  \hfill (13)
On the other hand, we may find the classical probability of detecting both photons at the same side of the beam splitter, i.e., case (a) or (b) in Fig.3. The result is simply the product of single-photon events due to independence:

\[ P_{cl}(2, 0) = P_{cl}(0, 2) = P_1(1, 0)P_1'(1, 0) = TR, \]  

(14)

where \( P_1'(1, 0) \) is the probability for one and the other input photons. From Eqs.(13, 14), we find that the quantum probability is twice the classical probability. This is exactly the same as the photon bunching effect. Its demonstration was first performed by Rarity and Tapster\(^{21}\), as shown in Fig.5.

To see that this is the result of two-photon interference, we consider Fig.6, which is the setup to measure \( P_2(2, 0) \). For two-photon detection, there are two possible ways to arrange the two photons (Figs.6a,6b). If the incoming two photons are well separated, they behave like classical particles and we add the probabilities of the two possibilities: \( P_{cl}^2 = |A|^2 + |A|^2 \). But if they are overlap at the beam splitter, we cannot distinguish the two possibilities and we add the amplitudes before taking the absolute value for overall probability: \( P_{qu}^2 = |A + A|^2 = 4|A|^2 = 2P_{cl}^2(2, 0) \). Note that the phases for the two cases are the same because the overall paths for the two photons in the two possibilities are the same due indistinguishability of the two photons. Therefore, it is constructive that is responsible for the photon bunching effect in Hong-Ou-Mandel interferometer. Scarcelli \textit{et al.}\(^{22}\) recently showed that it is also the fundamental principle behind the original photon bunching effect from a thermal source. This was consistent with the original view by Glauber\(^{20}\) who first explained photon bunching effect with an equivalent view of two-photon amplitudes.

2. Generalized Photon Bunching Effect and Constructive Multi-Photon Interference

The two-photon bunching effect discussed in the previous section can be generalized to the case of more photons.
2.1. Pair bunching effect

The first generalization is the pair bunching effect. In this case, two pairs of photons enter in two separate ports of a 50:50 beam splitter, as shown in Fig.7a. This scheme is similar to the Hong-Ou-Mandel interferometer of two photons but there are four photons here and unlike its predecessor, the middle term $|2, 2\rangle$ does not vanish, as seen in the output state derived in the same way as Eq.(11):

$$|\Phi^{(4)}_{\text{out}}\rangle = \left(\frac{1}{\sqrt{2}}\right)^2 \left(\frac{1}{\sqrt{2}}\right)^2 |0\rangle - \left(1/2\right) |2, 2\rangle.$$  (15)

But in this section, we are interested in the $|4_1, 0_2\rangle$-term. From Eq.(15), we find the probability for this is $P_4(4, 0) = 3/8$. However, if the four photons were classical particles, the classical probability is simply $P_4^{cl}(4, 0) = T^2 R^2 = 1/16$. So the ratio between quantum and classical prediction is $P_4^{\text{qu}}(4, 0)/P_4^{\text{cl}}(4, 0) = 6$. The six-fold ratio is a result of four-photon bunching effect. Experimentally, it was first demonstrated by Ou et al. The result is shown in Fig.7b.

The pair bunching effect is a result of constructive four-photon interference. As shown in Fig.8, there are six possible ways to arrange two pairs of photons (black and white circles) in four detectors. If the four photons are classical particles, the black and white pairs are distinguishable and we add the probabilities of these six possibilities, resulting in $P_4^{\text{cl}}(4, 0) = 6|A|^2$ with $A$ as the probability amplitude.
for each possibility. On the other hand, the four photons are quantum particles, there is no way to distinguish between the black and white pairs. We then add the amplitudes of each possibilities before obtaining the overall probability \( P_{cl}^{4}(4, 0) = |6A|^2 = 6P_{cl}^{4}(4, 0) \). The phases for all six possibilities are same due to the same path for the four photons. This leads to constructive four-photon interference. Note that this is similar to the two-photon bunching effect in Sect.1.3 except that there are four detectors here so that there are more possibilities.

2.2. Stimulated emission as a generalized photon bunching effect

As is well-known, stimulated emission was first proposed by Einstein\(^{24}\) to explain the spectrum of blackbody radiation. Phenomenologically, when a single photon interacts with an excited atom, it can stimulate the atom to emit. The atom can of course emit a photon spontaneously. From Einstein’s \( A \)- and \( B \)- coefficients, the rates of the stimulated emission and spontaneous emission are same and are denoted as \( R \). The overall rate is then \( 2R \). When there are \( N \) input photons, each photon may stimulate the atom and the overall rate is then \( (N + 1)R \).

The enhancement effect due to stimulated emission can be used to explain the photon bunching effect discussed in Sect.1.3. As shown in Fig.9, the two photons detected in two-photon coincidence measurement are from two excited atoms. In Fig.9a, the atoms in excited state independently emit photons due to spontaneous emission and two-photon detection probability in this case is simply the product of individual emission probability: \( P_{sp}^{2} = P_{1}^{2} = R \). In Fig.9b, the detected two photons are from stimulated emission, i.e., the photon spontaneously emitted from one atom stimulates the emission of another atom. Since the rate of stimulated emission is the same as that of spontaneous emission, we have \( P_{st}^{2} = R = P_{sp}^{2} = P_{1}^{2} \) so that the overall probability is

\[
P_{2} = P_{st}^{2} + P_{sp}^{2} = 2P_{1}^{2},
\]

which is exactly the ratio of photon bunching effect.

Although Einstein did not give any argument for stimulated emission, the quantum theory of light developed later fully explained it. In essence, it is from the Bosonic relation of \( \hat{a} \dagger |N \rangle = \sqrt{N+1} |N + 1 \rangle \). But this explanation relies on some complicated operator algebra. So is there a simpler physical principle underlying the phenomenon of stimulated emission?
Fig. 9. Photon bunching effect as a result of stimulated emission: two-photon coincidence from (a) spontaneous emission and (b) stimulated emission.

As discussed in Sect.1.3, the photon bunching effect can be understood as a constructive two-photon interference effect. Combining this with the above explanation in terms of stimulated emission, we may conclude that stimulated emission can also be explained by multi-photon interference.

To see the connection, we look at the two schemes in Fig.10. For the \((N+1)\)-photon interference scheme in Fig.10a, it can be shown, in a similar way to derive Eq.\((15)\), that the \((N+1)\)-photon detection probability is \(P_{N+1} = (N+1)/2^{N+1}\), which is \(N+1\) times the probability \(P^c_{N+1} = 1/2^{N+1}\) when the \(N+1\) photons were classical particles. The enhancement factor is exactly the same as the stimulated emission scheme in Fig.10b. This connection between stimulated emission and multi-photon interference was recently demonstrated by Sun et al.\cite{25}

3. Generalized Hong-Ou-Mandel Effect and Destructive Multi-Photon Interference

Although the two-photon Hong-Ou-Mandel effect cannot be generalized to the case of more photons with a symmetric beam splitter, as shown in Sect.2.1, we may consider its generalization with an asymmetric beam splitter with \(T \neq R\).
Fig. 11. Three possibilities for output of $|2_1, 1_2\rangle$: (a) all photons transmit; (b) and (c) one of the two photons from side 1 and the single photon from side 2 reflect.

3.1. *Three-photon Wang-Kobayashi interferometer*

The three-photon generalization of Hong-Ou-Mandel interferometer was proposed by Wang and Kobayashi\cite{26}, who considered an input state of $|2_1, 1_2\rangle$ to a beam splitter with $T \neq R$. With the method leading to Eqs. (11, 15), we may easily express the output state as

$$|\Phi_{\text{out}}^{(3)}\rangle = \sqrt{3}T^2 R |3_1, 0_2\rangle + \sqrt{3}T R^2 |0_1, 3_2\rangle + \sqrt{T}(T - 2R)|2_1, 1_2\rangle + \sqrt{R}(R - 2T)|1_1, 2_2\rangle. \quad (17)$$

Note that $P_3(2, 1) = T(T - 2R)^2$ and is equal to zero when $T = 2R = 2/3$. Under this condition, Eq. (17) becomes

$$|\Phi_{\text{out}}^{(3)}\rangle = \left(2 |3_1, 0_2\rangle + \frac{\sqrt{2}}{3} |0_1, 3_2\rangle - \frac{\sqrt{3}}{3} |1_1, 2_2\rangle \right)/3. \quad (18)$$

Fig. 11. Three possibilities for output of $|2_1, 1_2\rangle$: (a) all photons transmit; (b) and (c) one of the two photons from side 1 and the single photon from side 2 reflect.

Note that the disappearance of the $|2_1, 1_2\rangle$ is due to destructive three-photon interference. This complete cancellation of probability amplitude for the output of $|2_1, 1_2\rangle$ is similar to the two-photon Hong-Ou-Mandel effect and can be easily understood with the picture in Fig. 11, where there are three possible ways to obtain an output of $|2_1, 1_2\rangle$: (a) all three photons transmit through the beam splitter with a probability amplitude of $(2/3)^2$; (b) the single photon from side 2 and one of the two photons from side 1 are reflected with a probability amplitude of $-\sqrt{2/3}(\sqrt{1/3})^2$; and (c) the single photon from side 2 and the other one of the two photons from side 1 are reflected with a probability amplitude of $-\sqrt{2/3}(\sqrt{1/3})^2$. Thus the overall amplitude is $(2/3)^2 - 2\sqrt{2/3}(\sqrt{1/3})^2 = 0$.

This generalized Hong-Ou-Mandel three-photon interference effect was first demonstrated by Sanaka *et al.*\cite{27} in realizing a nonlinear phase gate.

3.2. *Fock state filtering effect*

The destructive three-photon interference effect in previous section can be easily generalized to an arbitrary input state of $|N_1, 1_2\rangle$. With a beam splitter of $T$, $R$, we can easily find the probability amplitude for an output of $|N_1, 1_2\rangle$:

$$A_{N+1}(N, 1) = \sqrt{T^{N-1}(T - NR)}, \quad (19)$$
which leads to the probability \( P_{N+1}(N, 1) = T^{N-1}(T - NR)^2 \), which equals to zero when \( T = NR = N/(N + 1) \).

This effect can be used as a Fock state filter when conditioned on the single-photon output at port 2. This idea was first proposed and demonstrated by Sanaka et al.\(^28\) and recently put into application for generating an entangled photon state\(^29\).

Consider an arbitrary state \( |\phi\rangle_1 = \sum_n c_n |n\rangle_1 \) input at port one and a single-photon state at port 2. The single photon at port 2 is often called the ancilla photon. We find from Eq.\(^{19}\) that when conditioned on the detection of a single photon at output port 2, the output state at port 1 then becomes

\[
|\phi'_\text{out}\rangle_1 = \mathcal{N} \sum_n c_n \sqrt{T^{n-1}(T - nR)} |n\rangle_1,
\]

where \( \mathcal{N} \) is the normalization factor. Notice that when we choose \( T, R \) so that \( R/T = n_0 = \text{integer} \), the state \( |n_0\rangle_1 \) disappears from the conditioned output state \( |\phi'_\text{out}\rangle_1 \). Thus, the Fock state \( |n_0\rangle_1 \) is filtered out.

### 3.3. Two-Pair Wang-Kobayashi Interferometer and Its Generalization

The idea in previous sections can be applied to two pairs of photons entering a beam splitter. For a beam splitter with arbitrary \( T, R \) and an input state of \( |2_1, 2_2\rangle \), we may use the method that leads to Eqs.\(^{11}\) and \(^{17}\) to obtain the output state as

\[
|\Phi_\text{out}^{(4)}\rangle = \sqrt{6TR} \left( |4_1, 0_2\rangle + |0_1, 4_2\rangle \right) + \left[ (T - R)^2 - 2TR \right] |2_1, 2_2\rangle \\
+ \sqrt{6TR(T - R)} \left( |3_1, 1_2\rangle - |1_1, 3_2\rangle \right).
\]

Obviously when \( T = R = 1/2, \) Eq.\(^{21}\) becomes Eq.\(^{15}\). But when \((T - R)^2 - 2TR = 0, \) or \( T = (3 \pm \sqrt{3})/6 \) and \( R = (3 \mp \sqrt{3})/6 \), the \( |2_1, 2_2\rangle \) term disappears from Eq.\(^{21}\), or \( P_4(2, 2) = 0. \) Again, this disappearance of \( |2_1, 2_2\rangle \) is a result of four-photon destructive interference, similar to the Hong-Ou-Mandel effect. Experimentally, this effect was first observed by Liu \textit{et al}\(^{30}\).

### 4. Multi-Photon de Broglie Wavelength and Phase-dependent Multi-Photon Interference

The interference effects discussed in Sects. 2 and 3 are phase independent. They are either constructive with 0° phase difference or destructive with 180° phase difference. In this section, we will allow the phase difference to change and reveal the more traditional interference pattern. In multi-photon interference, phases of individual photons may be adjusted independently and if the photons are spatially separated, we will have nonlocal phase correlation that exhibits some dramatic violation of local realism\(^{31}\). On the other hand, when all the photons are inseparable, they will sense the same phase shift \( \varphi \) resulting in an overall phase shift of \( N\varphi \) with \( N \) as the total photon number. This corresponds to the case when the \( N \) photons form
one entity and have an equivalent de Broglie wavelength of \( \lambda/N \) for the \( N \)-photon composite system, with \( \lambda \) as the single photon wavelength. Note that the key in this system is that all the photons stick together and behave like one single entity. For this type of states, if the photon number is large, the states will become a Schrödinger cat state, which is a superposition of macroscopic states.

4.1. **NOON state and Heisenberg limit**

One of these composite systems is the maximally entangled photon number state of two modes, which is also called the NOON state and has the form of

\[
|\text{NOON}\rangle = \left( |N,0\rangle + |0,N\rangle \right)/\sqrt{2}.
\]  

(22)

If we combine the two modes with a beam splitter and observe \( N \)-photon coincidence measurement (Fig. 12), it is straightforward to show that the \( N \)-photon coincidence is proportional to

\[
P_N = (1/2)(1 + \cos N \varphi).
\]  

(23)

The phase dependence in Eq. (23) may lead to a phase measurement precision of \( 1/N \), or the so-called Heisenberg limit, which is the ultimate quantum limit of phase measurement.\(^{37}\) In the following, we will see how to produce a phase dependence in Eq. (23) by quantum interference.

The NOON state is a special kind of superposition states with a total photon number as \( N \). It lacks the middle terms such as \( |(N - 1),1\rangle, |(N - 2),2\rangle \), etc. Otherwise, there would be terms of the form of \( \cos m\varphi \) with \( m < N \) in Eq. (22). This would not lead to Heisenberg limit in phase measurement.

4.2. **Generation of NOON state by quantum interference**

A NOON state cannot be generated by simply sending a Fock state of photon number \( N \) to a 50:50 beam splitter since the output state contains all the states mentioned above. On the other hand, Hong-Ou-Mandel interferometer\(^{12}\) provides method of how to produce a NOON state. From Eq. (11), we find that when \( T = R = 1/2 \), the output state from the beam splitter becomes

\[
|\Phi_2\rangle_{\text{out}} = \left( |2,0\rangle - |0,2\rangle \right)/\sqrt{2},
\]  

(24)
which is a two-photon NOON state. The disappearance of the $|1,1\rangle$ term is a result of two-photon interference. For the three- and four-photon cases, however, we see from Eqs. (17) and (21) that only one middle term can be set to zero and there still exist other unwanted middle terms.

To introduce extra degrees of freedom for cancellation of all the unwanted terms, we may use interference between a coherent state and a two-photon state. Consider the scheme in Fig. 13 where a weak coherent state $|\alpha\rangle \approx |0\rangle + \alpha|1\rangle + \frac{\alpha^2/\sqrt{2}}{\sqrt{2}}|2\rangle + \frac{\alpha^3/\sqrt{6}}{\sqrt{6}}|3\rangle + \ldots$ and a two-photon state $|\eta\rangle \approx |0\rangle + \eta|2\rangle + \ldots$ from spontaneous parametric down-conversion enter a 50:50 beam splitter from separate sides. Projecting to three-photon state, the input state is $\frac{\alpha^3}{\sqrt{6}}|3\rangle + \eta\alpha|1\rangle + |0\rangle$. With the method leading to Eqs. (11) and (17), we find the output state as

$$|\Phi_3\rangle_{\text{out}} = \frac{(\alpha^2 - \eta\sqrt{2})\alpha}{4}(|2,1\rangle + |1,2\rangle) + \frac{\alpha^2 + 3\eta\sqrt{2}\alpha}{4\sqrt{3}}(|3,0\rangle + |0,3\rangle).$$

(25)

When $\alpha^2 = \eta\sqrt{2}$, the coefficient of $|2,1\rangle$ and $|1,2\rangle$ are zero. The output state becomes a NOON state of three photons.

This method has been generalized to four-, five-, six-, and seven-photon cases by introducing more degrees of freedom. But the scheme becomes more and more complicated as photon number gets large. In 2004, Hofmann proposed an ingenious multi-photon interference method to cancel all the unwanted middle terms mentioned above for the production of the NOON state from $N$ independent single-photon states. The idea is based on the algebraic identity:

$$\prod_{n=1}^{N} (x - ye^{i\delta_n}) = x^N - y^N,$$

(26)

where $\delta_n = 2\pi(n - 1)/N$.

Consider the scheme in Fig. 14 to merge $N$ modes into one with $N - 1$ beam splitters. Each mode is in a single-photon state with a slight polarization twist:

$$|\delta_n\rangle = \frac{1}{\sqrt{2}}(|H\rangle_n - e^{i\delta_n}|V\rangle_n) = \frac{1}{\sqrt{2}}(\hat{a}_H^{\dagger}_n - e^{i\delta_n}\hat{a}_V^{\dagger}_n)|\text{vac}\rangle,$$

(27)
where $H, V$ denote the horizontal and vertical polarizations, respectively. By generalizing the method that leads to Eqs. (11) and (17) to $N-1$ beam splitter, we derive the output state, which is quite complicated. However, if we are only interested in the case when all $N$ photons are at the output port 1, the projected state can be found as

$$|\Phi_N(b_1)\rangle_{out} = \frac{1}{(2N)^{N/2}} \prod_n (\hat{b}_{H1} - e^{i\delta_n} \hat{b}_{V1}) |\text{vac}\rangle$$

$$= \sqrt{N!} (|N_H, 0_V\rangle_1 - |0_H, N_V\rangle_1), \quad (28)$$

where we used the identity in Eq. (26). Note that the state in Eq. (28) is not normalized due to projection and it norm $|||\Phi_N(b_1)\rangle_{out}|||^2 = 2(N!)/(2N)^N$ gives the projection probability.

This scheme was implemented by Mitchell et al.\cite{41} to produce a three-photon NOON state.

### 4.3. Demonstration of multi-photon de Broglie wavelength by projection measurement

The starting point for Hofmann’s scheme is single-photon state. However, current technology is not mature enough to produce single-photon states with consistent temporal profile.\cite{42} In this section, we will see how to demonstrate multi-photon de Broglie wavelength without the need of NOON states. The idea is to use some proper measurement schemes for projecting out the unwanted terms in realization of NOON states.

First consider the scheme in Fig.15a, which was first proposed by Wang and Kobayashi\cite{26}. It uses twice the three-photon Wang-Kobayashi interferometer, which was discussed in Sect.3.1. The first beam splitter takes out the $|2_1, 1_2\rangle$ term while the second beam splitter together with three detectors removes the $|1_1, 2_2\rangle$ term, leaving only the contribution from the NOON state part in Eq. (18). Wang and Kobayashi\cite{26} proved that with $T = 2/3$ for both beam splitters in Fig.15a, the three-photon coincidence rate is indeed proportional to $1 + \cos 3\varphi$, demonstrating...
three-photon de Broglie wavelength. Experimentally, this scheme is implemented by Liu et al.\textsuperscript{43}

Extension to four-photon case is straightforward: with an input state of $|2_1, 2_2\rangle$, we use choose $T = (3 \pm \sqrt{3})/6$ for the first beam splitter in Fig.15b to get rid of the $|2_2\rangle$ state and use a symmetric beam splitter as the second one. The four-photon coincidence measurement of $P_4(2, 2)$ at the output of the interferometer projects out the $|3, 1\rangle$ and $|1, 3\rangle$ terms as seen from Eq.(15), leaving only the NOON state contribution. Experimental implementation of the four-photon case was performed by Liu et al.\textsuperscript{30} with a demonstration of a four-photon de Broglie wavelength.

An even simpler interferometer with four-photon de Broglie wavelength can be formed with two symmetric beam splitters in the scheme of Fig.15b. In order to eliminate the contribution of the $|2_1, 2_2\rangle$ term in Eq.(15) with a second symmetric beam splitter in Fig.15b, we make a detection of $P_4(3, 1)$ in the output of the interferometer. Because of the symmetric beam splitter, $|2_1, 2_2\rangle$ won’t contribute to $P_4(3, 1)$, leaving only the contributions from the NOON state part in Eq.(15). This scheme was recently implemented by Nagata et al.\textsuperscript{44}

Although extension of the above simple scheme to six-photon case is possible, generalization to other photon numbers is difficult. However, a scheme proposed by Sun et al.\textsuperscript{45} can be easily generalized to arbitrary photon number. This scheme is a reverse process of the scheme proposed by Hofmann.\textsuperscript{40}

The scheme is sketched in Fig.16, where the input state is of an arbitrary form:

$$|\Psi_N\rangle_{in} = \sum_{n=0}^{N} c_n |N - n\rangle_H |n\rangle_V.$$  \hfill (29)

The fields at the detectors are related to the input fields as

$$\hat{b}_n = (\hat{a}_H - \hat{a}_V e^{i\delta_n})/N \sqrt{2} + ..., \hfill (30)$$

where we omit the fields in vacuum. The $N$-photon coincidence rate from all the detectors is proportional to

$$P_N \propto \left\langle \prod_{n=1}^{N} \hat{b}_n \prod_{m=1}^{N} \hat{b}_m \right\rangle = \left\langle (\hat{a}_H^{\dagger N} - \hat{a}_V^{\dagger N})(\hat{a}_H^{N} - \hat{a}_V^{N}) \right\rangle \Big/ 2^N N^{2N}$$
\[ P_\Psi(\text{NOON}) = |\langle \text{NOON} | \Psi_N \rangle|^2 = |c_0 - c_N|^2 \] (32)

is the NOON state projection probability. If there is a phase shift of $\varphi$ introduced between H and V fields and the two fields have equal strength, we have $c_N/c_0 = e^{iN \varphi}$. Then Eq. (31) becomes

\[ P_N \propto |c_0|^2 (1 - \cos N \varphi)/2^{N-2} N^{2N}, \] (33)

showing $N$-photon de Broglie wavelength.

This NOON state projection scheme was applied to a four-photon state from parametric down-conversion with four-photon de Broglie wavelength \cite{11} and to a weak coherent state with six-photon de Broglie wavelength \cite{17}, respectively.

5. Temporal Distinguishability of Photons

Complementary principle of quantum mechanics states that distinguishability inevitably degrades the interference effect. For traditional single-photon interference, distinguishability may only occur in paths. But for multi-photon interference, distinguishability among different particles may also lead to degradation of interference. Thus the multi-photon interference effect discussed in the previous section can be used to quantitatively characterize the degree of temporal distinguishability of photons.

5.1. Two-photon distinguishability and Hong-Ou-Mandel interference

As a matter of fact, the Hong-Ou-Mandel interference effect discussed in Sect.1.2 depends on the overlap between the two incoming photons. If the arrival times for the two photons at the beam splitter is quite different, temporal distinguishability between the two photons will diminish the interference effect.
Quantitatively, the visibility of the Hong-Ou-Mandel interference effect is related to the two-photon spectral wave function $\Phi_2(\omega_1, \omega_2)$ as

$$V_2^2 = \frac{\int d\omega_1 d\omega_2 \Phi_2^*(\omega_1, \omega_2) \Phi_2(\omega_2, \omega_1)}{\int d\omega_1 d\omega_2 |\Phi_2(\omega_1, \omega_2)|^2}, \quad (34)$$

where the two-photon spectral wave function $\Phi_2(\omega_1, \omega_2)$ is defined through an arbitrary two-photon state of one dimension:

$$|\Phi_2\rangle = \int d\omega_1 d\omega_2 \Phi_2(\omega_1, \omega_2) \hat{a}_1^\dagger(\omega_1) \hat{a}_2^\dagger(\omega_2) |\text{vac}\rangle. \quad (35)$$

We omit the spatial degree of freedom for simplicity of discussion. Its inclusion is straightforward.

From Eq. (34), we find that $V_2^2 = 1$, or the maximum interference effect when $\Phi_2(\omega_1, \omega_2) = \Phi_2(\omega_2, \omega_1)$, which is the condition for complete temporal indistinguishability of the two photons. But $V_2^2 = 0$, or no interference effect when

$$\int d\omega_1 d\omega_2 \Phi_2^*(\omega_1, \omega_2) \Phi_2(\omega_2, \omega_1) = 0, \quad (37)$$

which gives the criterion for complete temporal distinguishability of the two photons. It turns out that Eqs. (36, 37) can be generalized to arbitrary number of photons

### 5.2. Pair Distinguishability and Its Characterization

The first generalization is to two pairs of photons by Ou et al., who considered a two-pair bunching effect discussed in Sect. 2.1. When the two pairs are well separated in time, there is still some but less bunching effect. In fact, when the two pairs are distinguishable, the input state becomes

$$|\Phi_{\text{in}}^{(4)}\rangle' = |1_1, 1_2\rangle \otimes |1'_1, 1'_2\rangle, \quad (38)$$

instead of $|\Phi_{\text{in}}^{(4)}\rangle = |2_1, 2_2\rangle$. The output state is then

$$|\Phi_{\text{out}}^{(4)}\rangle' = (1/2)(|2_1, 0_2\rangle - |0_1, 2_2\rangle) \otimes (|2'_1, 0'_2\rangle - |0'_1, 2'_2\rangle), \quad (39)$$

from which we find the probability $P_4(4, 0) = 1/4$ and the ratio to the classical probability is then

$$P_4'(4, 0)/P_4^{\text{cl}}(4, 0) = 4. \quad (40)$$

This value is reduced from the maximum value of 6 in Sect. 2.1, when the two pairs are indistinguishable from each other. Thus, distinguishability results in degradation of the interference effect.

For partial distinguishability of the pairs, Tsujino et al. and de Riedmatten et al. attempted to describe it as a mixed state between the two extreme cases.
discussed above. However, this picture has some serious problem. A better description is given in Ref.\textsuperscript{52} in terms of the quantity $E/A$ with

$$A \equiv \int d\omega_1 d\omega_2 d\omega_1' d\omega_2' |\Phi_2(\omega_1, \omega_2) \Phi_2(\omega_1', \omega_2')|^2,$$

and

$$E \equiv \int d\omega_1 d\omega_2 d\omega_1' d\omega_2' \Phi_2(\omega_1, \omega_2) \Phi_2(\omega_1', \omega_2') \Phi_2^*(\omega_1, \omega_2) \Phi_2^*(\omega_1', \omega_2),$$

where $\Phi_2(\omega_1, \omega_2)$ is the two-photon wave function defined in Eq.(35).

For the two pairs of photons from parametric down-conversion, the four-photon state is given by \textsuperscript{52}

$$|\Phi_4\rangle = \int d\omega_1 d\omega_2 d\omega_1' d\omega_2' \Phi_2(\omega_1, \omega_2) \Phi_2(\omega_1', \omega_2') \hat{a}^{\dagger}_1(\omega_1) \hat{a}^{\dagger}_2(\omega_2) \hat{a}^\dagger_1(\omega_1') \hat{a}^\dagger_2(\omega_2') |\text{vac}\rangle.$$

So the four-photon wave function has the form of

$$\Phi_4(\omega_1, \omega_2, \omega_1', \omega_2') = \Phi_2(\omega_1, \omega_2) \Phi_2(\omega_1', \omega_2').$$

Hence the condition for indistinguishable pairs, i.e., $E = A$, can be rewritten in terms of the four-photon wave function as

$$\Phi_4(\omega_1, \omega_2, \omega_1', \omega_2') = \Phi_4(\omega_1', \omega_2, \omega_1, \omega_2') = \Phi_4(\omega_1, \omega_2', \omega_1', \omega_2).$$

Note that the permutation is between primed and unprimed variables, indicating permutation symmetry between two photons with one from each pair. Thus we have pair exchange symmetry.

On the other hand, for the condition for complete distinguishable pairs, i.e., $E = 0$, we have

$$\int d\omega_1 d\omega_2 d\omega_1' d\omega_2' \Phi_4(\omega_1, \omega_2, \omega_1', \omega_2') \Phi_4^*(\omega_1, \omega_2, \omega_1', \omega_2) = 0.$$

Both Eqs.(45) and (46) are extension of Eqs.(36) and (37) to the four-photon case of two pairs. These can further be generalized to arbitrary number of photons.

5.3. Description of photon temporal distinguishability of an $N$-photon state

With an $N$-photon state of arbitrary temporal profile in the form of

$$|\Phi_N\rangle = N^{-1/2} \int d\omega_1 d\omega_2 ... d\omega_N \Phi(\omega_1, ..., \omega_N) \hat{a}^{\dagger}(\omega_1) \hat{a}^{\dagger}(\omega_2) ... \hat{a}^{\dagger}(\omega_N) |0\rangle,$$

Eqs.(40) \textsuperscript{37} \textsuperscript{54} \textsuperscript{145} \textsuperscript{40} can be generalized to

$$\Phi(\omega_1, ..., \omega_N) = \Phi(P_{ij}\{\omega_1, ..., \omega_N\})$$

for indistinguishability between two photons labelled as $i$ and $j$ and

$$\int d\omega_1 d\omega_2 ... d\omega_N \Phi(\omega_1, ..., \omega_N) = \Phi(P_{ij}\{\omega_1, ..., \omega_N\}) = 0$$

for distinguishability between two photons labelled as $i$ and $j$. 

for distinguishability between two photons labelled as \(i\) and \(j\). Here \(P_{ij}\) is the permutation operation between the variables \(\omega_i\) and \(\omega_j\). \(N\) is the normalization coefficient and takes the form of

\[
N = \int d\omega_1 d\omega_2 ... d\omega_N \Phi^*(\omega_1, ..., \omega_N) \sum_P \Phi(P\{\omega_1, ..., \omega_N\}). \tag{50}
\]

Obviously, \(N\) has a maximum value of \(N!\) with

\[
I = \int d\omega_1 d\omega_2 ... d\omega_N |\Phi(\omega_1, ..., \omega_N)|^2, \tag{51}
\]

when condition in Eq.(48) is satisfied for all permutation. It has the minimum value of \(I\) when condition in Eq.(17) applies to all permutation. The former corresponds to the case when all the \(N\) photons are indistinguishable of each other (denoted as the \(N \times 1\) case) whereas the latter to the case when all the \(N\) photons are well separated from each other and become distinguishable (denoted as the \(1 \times N\) case).

Intermediate cases are described a combination of Eq.(50) for some photons and Eq.(51) for some other photons. For example, the situation for two separate pairs (denoted as \(2 \times 2\) case) is described by Eq.(46) for pair separation and by Eq.(36) for indistinguishability between photons within each pair.

### 5.4. Scheme of NOON state projection

Since distinguishability in photons will influence the effect of interference, we should be able to characterize the degree of photon distinguishability through the measurement of the visibility of interference, in a similar way as optical coherence from the visibility of Young’s double slit interference\(^2\). But here we need \(N\)-photon interference because of the involvement of \(N\) photons. For example, the two-photon Hong-Ou-Mandel effect provides a measure of temporal distinguishability of two photons, as in Eq.(34). The pair bunching effect can be used to characterize the distinguishability of two pairs of photons from parametric down-conversion, as in Sect.5.2.

Among the schemes of multi-photon interference, the NOON state projection measurement\(^46\) (Fig.16) easily applies to arbitrary number of photons. Let us consider this scheme first.

From Eqs.(31 32), we find the \(N\)-photon coincidence probability as

\[
P_N \propto |\langle \text{NOON} | \Psi_N \rangle|^2. \tag{52}
\]

For the input state of \(|\Psi_N\rangle = |k, N - k\rangle (k \neq 0, N)\), the coincidence probability is zero because of the orthogonality of \(\langle \text{NOON} | k, N - k \rangle = 0 \ (k \neq 0, N)\).

However, this is true only when all \(N\) photons are indistinguishable in time, due to complete destructive interference. When photons are distinguishable, the interference effect will deteriorate. The worst case is when the H-photons and V-photons are completely separate and there is no interference at all. The coincidence
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Figure 17. (a) The schematics for demonstrating three-photon temporal distinguishability; (b) the result of the experiment. Reproduced with permission from Ref.[48].

rate in this case sets a reference line to compare with. As the H-photons and V-photons overlap in time, the coincidence rate will drop below this reference, similar to the Hong-Ou-Mandel effect. In this way, we can define a visibility to describe the interference effect. The size of the visibility will depend on the degree of temporal distinguishability of the $N$ photons, thus providing a quantitative measure for the temporal distinguishability of the $N$ photons.

The dependence of the visibility on the scenarios of temporal distinguishability of the $N$ photons can be calculated, based on the conditions in Eqs.(45, 46). The form is very complicated in the NOON state projection measurement. For the arbitrary case of $M$ photons in H-polarization and $K$ photons in V-polarization, the visibility was derived and given in Ref.[54].

A simpler situation is for the input state of $|1_H, N_V\rangle$, for which the visibility is given by

$$V_{N+1}^{\text{NOON}} = m/N,$$

where $m$ is the number of V-photons that are indistinguishable from the single H-photon. The other $N - m$ V-photons are completely distinguishable from the $m + 1$ photons. Thus by scanning the relative delay between the single H-photon and the $N$ V-photons, we may observe a number of dips, corresponding to the overlap between the single H-photon and the groups of indistinguishable V-photons. The visibility of the dips in Eq.(53) gives the number of V-photons in the corresponding group.

The above situation was demonstrated by Liu et al.[55] for the case of $N = 2$ with three photons from two pairs of photons generated by two parametric down-conversion processes. Fig.17 shows the setup and the results of the experiment. Experimental demonstration of temporal distinguishability for the input states of $|2_H, 2_V\rangle$ and $|3_H, 3_V\rangle$ was performed by Xiang et al.[56]

5.5. Scheme of asymmetric beam splitter

Besides the scheme of NOON state projection, we find from Sect.3 that another interference scheme with asymmetric beam splitter in Sect.3.2 is a generalization of the Hong-Ou-Mandel interferometer. This scheme can then be used to characterize
the temporal distinguishability of $N$ photons in a way similar to the NOON state projection scheme.

For the simple situation of $|1_H, N_V\rangle$, we consider the arrangement in Fig.18, which is an equivalent to the scheme with an asymmetric beam splitter. The combination of a half wave plate (HWP) and a polarization beam splitter (PBS) gives rise to a beam splitter with variable transmissivity. For the scheme in Fig.18 and input state of $|1_H, N_V\rangle$, we find from Eq.(19) that the coincidence probability of $(N+1)$ detectors is zero when the rotation of the polarization due to the HWP is such that $\sin^2 2\theta = 1/(N+1)$. $\theta$ is the angle of rotation of the HWP. But this is true only when the $N+1$ photons are indistinguishable. The coincidence probability is nonzero if part of the $N+1$ photons become distinguishable. Coincidentally, it can be shown\cite{54} that the visibility in the scheme of Fig.18 is

$$V_{N+1}^{asy} = \frac{m}{N},$$

just like Eq.(53) for the NOON state projection scheme.

However, the similarity ends right here. For other input state of $|k_H, N_V\rangle$ with $k > 1$, the visibility\cite{54} is even more complicated than that of the NOON state projection scheme. This scheme is implemented by Ou et al.\cite{57} for $|1_H, 2_V\rangle$ state with a result similar to Fig.17b. The experiment with an input state of $|2_H, 2_V\rangle$ in this scheme was performed by Liu et al.\cite{55}

6. Optical Coherence as Interpreted as Photon Indistinguishability

Classical optical coherence theory, developed in the 1950s\cite{2}, was based on second-order or single-photon interference effect. In brief, the intensity distribution shows the interference fringes pattern as

$$I(x) \propto I_1 + I_2 + 2\sqrt{I_1 I_2} |\gamma| \cos 2\pi (x - x_0)/L,$$

where $I_1, I_2$ are the intensities of the two interfering fields, $L$ is the fringe spacing along the $x$-direction, and

$$\gamma = \langle E^*_1 E_2 \rangle / \sqrt{I_1 I_2}$$
is the degree of coherence between the two fields. Here \( x_0 \) in Eq. (55) is related to \( \text{Arg}(\gamma) \).

Quantum coherence theory was later constructed by Glauber \( ^{20} \) primarily along the same line as the classical theory but with quantum formulism of operators and quantum states. The physics was hidden beneath the complicated mathematical formula.

As discussed in previous section, distinguishability of photons lead to degradation of the visibility of interference. Thus the two should be related somehow to each other. In the following, we will make an initial attempt to reveal the connection.

In 1996, Javanainen and Yoo \( ^{58} \) showed that in a single realization, an interference fringe will form in the superposition region of two groups of photons of the same number \( N \), respectively, i.e., with a state of \( |N\rangle_1 |N\rangle_2 \). Later, the study was extended by Ou and Su \( ^{59} \) to the superposition of two groups of photons with different photon numbers \( n \) and \( m \), respectively, i.e., with a state of \( |n\rangle_1 |m\rangle_2 \). A quantum Monte Carlo simulation \( ^{59} \) shows that for the state of \( |n\rangle_1 |m\rangle_2 \), there is an interference fringe forming with a probability distribution of

\[
P(x) \propto n + m + 2\sqrt{nm} \cos 2\pi (x - x_0)/L, \tag{57}
\]

where \( x_0 \) is arbitrary and \( L \) is the fringe spacing. If we compare the above with Eq. (56), we find the normalized degree of coherence is simply \( \gamma = 1 \). This is not surprising in the sense that the photons in the quantum state \( |n, m\rangle \) belong to one wave function and are all indistinguishable in the superposition region.

On the hand, if there is partial indistinguishability among the photons, from the discussion in previous section we find that the visibility will drop. Assume that the input state is \( |N\rangle_1 |M\rangle_2 \) but only \( n \) photons among the \( N \) photons in mode 1 are indistinguishable from \( m \) photons among the \( M \) photons in mode 2. Therefore, only the \( n + m \) photons will give rise to an interference pattern, described by Eq. (57). The rest photons, i.e., \( N - n \) photons from mode 1 and \( M - m \) photons from mode 2, are distinguishable and produce no interference fringe. Thus the probability distribution in this case is given by

\[
P'(x) \propto N - n + M - m + n + m + 2\sqrt{nm} \cos 2\pi (x - x_0)/L \\
= N + M + 2\sqrt{nm} \cos 2\pi (x - x_0)/L. \tag{58}
\]

Comparing to Eq. (56), we have

\[
\gamma' = \sqrt{nm}/NM. \tag{59}
\]

Note that \( n/N \) and \( m/M \) are the percentages of indistinguishable photons in the two groups, respectively. Thus the degree of coherence is related to the percentage of indistinguishable photons among the photons involving in interference.

7. Summary and Discussion

This paper has discussed various interference effects involving multiple photons. Some are constructive interference effects (photon bunching); some are destruct-
tive interference effects (Hong-Ou-Mandel effect); and some are phase dependent interference effects (multi-photon de Broglie wavelength). We find that we need modify Dirac’s statement on photon interference to understand multi-photon interference effects. We also find that photon distinguishability leads to degradation in interference effect, confirming complementary principle of quantum mechanics in quantitative manner.

Although we only discussed the problem of temporal distinguishability between two pairs in one interference scheme in Sect.5.2, it has been shown\textsuperscript{54} that all the interference experiments involving two pairs of photons in Sects.2-4 can be explained in terms of the quantity \( \mathcal{E}/\mathcal{A} \), which describes the temporal distinguishability between the two pairs. This point can be generalized to all of the multi-photon interference schemes. Further study\textsuperscript{60} shows that the scheme with stimulated emission by \( N \) photons can be used to characterize quantitatively the temporal distinguishability of the incoming \( N \) photons, since it is a constructive multi-photon interference effect.

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