THE SPREADING FRONTS IN A MUTUALISTIC MODEL WITH ADVECTION

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Abstract. This paper is concerned with a system of semilinear parabolic equations with two free boundaries, which describe the spreading fronts of the invasive species in a mutualistic ecological model. The advection term is introduced to model the behavior of the invasive species in one dimension space. The local existence and uniqueness of a classical solution are obtained and the asymptotic behavior of the free boundary problem is studied. Our results indicate that for small advection, two free boundaries tend monotonically to finite limits or infinities at the same time, and a spreading-vanishing dichotomy holds, namely, either the expanding environment is limited and the invasive species dies out, or the invasive species spreads to all new environment and establishes itself in a long run. Moreover, some rough estimates of the spreading speed are also given when spreading happens.

1. Introduction. The spreading of species from their native habitats to alien environments is a serious threat to biological diversity [22], and mathematical investigation on the spreading of population has been attracting much attention.

In the pioneering works of Fisher [11] and Kolmogorov et al [17], the diffusive logistic equation over the entire space \( \mathbb{R} \):

\[
\frac{du}{dt} - du_{xx} = u(a - bu), \ x \in \mathbb{R}, \ t > 0
\]  

was studied, traveling wave solutions have been found for (1): For any \( c \geq c^* := 2\sqrt{ad} \), there exists a solution \( u(x, t) := W(x - ct) \) such that \( W'(y) < 0 \) for \( y \in \mathbb{R}^1 \), \( W(-\infty) = a/b \) and \( W(+\infty) = 0 \); no such solution exists if \( c < c^* \). The number \( c^* \) is then called the minimal speed of the traveling waves. These results have been further developed in [1, 2, 15, 26, 27, 28] and the references therein.

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To describe the spreading process of invasive species and the front of the expanding habitat, Du and Lin [6] studied the following diffusive logistic problem,

\[
\begin{cases}
    u_t - d u_{xx} = u(a - bu), & 0 < x < h(t), \ t > 0, \\
    u_x(0, t) = u(h(t), t) = 0, & t > 0, \\
    h'(t) = -\mu u_x(h(t), t), & t > 0, \\
    u(x, 0) = u_0(x), & 0 \leq x \leq h_0,
\end{cases}
\]

where the unknown \( u(x, t) \) stands for the population density of an invasive species over a one-dimensional habitat, and the unknown \( x = h(t) \) is the free boundary and is used to describe the expanding front. The free boundary condition is given by \( h'(t) = -\mu u_x(h(t), t) \), which means that the spreading front expands at a speed that is proportional to the population gradient at the front, the positive constant \( \mu \) measures the ability of the invasive species to transmit and diffuse in the new habitat, see [20] for details.

A spreading-vanishing dichotomy was first presented in [6] for problem (2), namely, as time \( t \to \infty \), the population \( u(x, t) \) either successfully establishes itself in the new environment (called spreading), in the sense that \( h(t) \to \infty \) and \( u(x, t) \to a/b \), or the population fails to establish and vanishes eventually (called vanishing), namely \( h(t) \to h_\infty \leq \sqrt[2]{a/b} \) and \( u(x, t) \to 0 \). It was also shown that if spreading occurs, for large time, the spreading speed approaches a positive constant \( k_0 \), i.e., \( h(t) = [h_0 + o(1)]t \) as \( t \to \infty \). \( k_0 \) is then called the asymptotic spreading speed, which is uniquely determined by an auxiliary elliptic problem induced from (2). Furthermore, they found that \( k_0 < c^* \), where \( c^* := 2\sqrt{ad} \) is the minimal speed of the traveling waves [7]. Hereafter, Du and Guo [4] extended the free boundary problem (1.2) to a higher dimension domain.

Since the work of Du and Lin [6], there have been many theoretical developments on the free boundary problem in homogeneous environment. For example, Du and Lou [9] considered a two free boundaries problem with a general nonlinear term. In [12, 13], Gu, Lin and Lou studied how advection term \( (\beta u_x) \) affects the asymptotic spreading speeds when spreading occurs. See also [16] for a free boundary problem with a general nonlinear term, [18, 31] for diffusive logistic model in heterogeneous environment, [5] for diffusive logistic model in time-periodic environment, [23] for diffusive logistic model with seasonal succession, [19] for information diffusion in online social networks, [25, 29] for Lotka-Volterra type prey-predator model and [8, 14] for Lotka-Volterra type competition model.

In this paper, we consider a two-species mutualistic model, which was proposed by May [21] in 1976, and the model is described by the following coupled O.D.E. system:

\[
\begin{cases}
    \dot{x}(t) = r_1 x(1 - \frac{x}{K_1 + a_1y} - \varepsilon_1 x), \\
    \dot{y}(t) = r_2 y(1 - \frac{y}{K_2 + a_2x} - \varepsilon_2 y),
\end{cases}
\]

where \( r_i, K_i, a_i, \varepsilon_i (i = 1, 2) \) are positive constants. Linearization and spectrum analysis show that the unique positive equilibrium is locally asymptotically stable, and moreover, it is globally asymptotically stable in the positive quadrant by constructing Lyapunov functional.

Considering the spatial spreading, we assume that one species is native and the other is invasive. Inspired by the former work, we will study how invasive species are spreading spatially over further to larger area, especially, small advection will be introduced to consider the long time behavior of the native species and the invasive
species. For simplicity, assume that the native species lives in the whole habitat \((-\infty, \infty)\), and the environment in \(g(t) < x < h(t)\) is occupied by the invasive species, whose density is denoted by \(u(x,t)\) and density of the native is denoted by \(v(x,t)\). Assume that \(h(t)\) grows at a rate that is proportional to population gradient of the invasive species at the front \([20]\), then the conditions on the right front is

\[
u(h(t), t) = 0, \hspace{1em} -\frac{\partial u}{\partial x} (h(t), t) = h'(t).\]

Similarly, the conditions on the left front is

\[
u(g(t), t) = 0, \hspace{1em} -\frac{\partial u}{\partial x} (g(t), t) = g'(t).\]

In such a case, we have the problem for \(u(x,t)\) and \(v(x,t)\) with free boundaries \(x = g(t)\) and \(x = h(t)\) such that

\[
\begin{align*}
\frac{\partial u(x,t)}{\partial t} &= \frac{\partial^2 u}{\partial x^2} - \beta u + \alpha_1 u(1 - \frac{u}{K_1 + \alpha_2 v} - \varepsilon_1 u), \hspace{1em} g(t) < x < h(t), \hspace{1em} t > 0, \\
\frac{\partial v(x,t)}{\partial t} &= \frac{\partial^2 v}{\partial x^2} + \beta v(1 - \frac{v}{K_2 + \alpha_2 u} - \varepsilon_2 v), \hspace{1em} -\infty < x < \infty, \hspace{1em} t > 0, \\
u(x, t) &= 0, \hspace{1em} x \leq g(t) \text{ or } x \geq h(t), \hspace{1em} t > 0, \\
g(0) &= -h_0, \hspace{1em} g'(t) = -\theta \frac{\partial u}{\partial x} (g(t), t), \hspace{1em} t > 0, \\
h(0) &= h_0, \hspace{1em} h'(t) = -\theta \frac{\partial u}{\partial x} (h(t), t), \hspace{1em} t > 0, \\
u(x, 0) &= u_0(x), \hspace{1em} v(x, 0) = v_0(x), \hspace{1em} -\infty < x < \infty,
\end{align*}
\]

where \(x = g(t)\) and \(x = h(t)\) are the moving left and right boundaries to be determined, \(h_0, \mu\) and \(\beta\) are positive constants, and the initial functions \(u_0\) and \(v_0\) are nonnegative and satisfy

\[
\begin{align*}
u_0 &\in C^2([-h_0, h_0]), \hspace{1em} u_0(\pm h_0) = 0 \text{ and } 0 < u_0(x), x \in (-h_0, h_0), \\
v_0 &\in C^2(-\infty, \infty) \cap L^\infty(-\infty, \infty) \text{ and } 0 < u_0(x), x \in (-\infty, -h_0) \cup [h_0, \infty).
\end{align*}
\]

In this paper, we assume that \(\beta < 2\sqrt{r_1d}\), it is well known that \(2\sqrt{r_1d}\) is the minimal speed of the traveling waves to the Cauchy problem

\[
u_t - \Delta u - r_1 (1 - b u), \hspace{1em} t > 0, \hspace{1em} x \in \mathbb{R}
\]

and it is also the maximal asymptotic spreading speed of the free boundary to problem (4) with \(\nu \equiv 0\) ([6]).

The remainder of this paper is organized as follows. In the next section, the global existence and uniqueness of the solution to problem (4) are proved by using a contraction mapping theorem. Section 3 is devoted to sufficient conditions for the invasive species to vanish or spread, a spreading-vanishing dichotomy will be given. Some rough estimates of the spreading speed are also given in Section 4.

2. Existence and uniqueness. In this section, we first present the following local existence and uniqueness result by the contraction mapping theorem and then give the global existence by using suitable estimates.

**Theorem 2.1.** For any given \((u_0, v_0)\) satisfying (5), and any \(\alpha \in (0, 1)\), there is a \(T > 0\) such that problem (4) admits a unique solution

\[
u, v, g, h \in C^{1+\alpha, (1+\alpha)/2}(D_T) \times C^{1+\alpha, (1+\alpha)/2}(D^\infty_T) \times C^{1+\alpha/2}([0, T]),
\]

moreover,

\[
\|u\|_{C^{1+\alpha, (1+\alpha)/2}(D_T)} + \|v\|_{C^{1+\alpha, (1+\alpha)/2}(D^\infty_T)} + \|g\|_{C^{1+\alpha/2}([0, T])} + \|h\|_{C^{1+\alpha/2}([0, T])} \leq C,
\]

where \(D_T = \{(x, t) \in \mathbb{R}^2 : x \in [g(t), h(t)], t \in [0, T]\}, \)

\(D^\infty_T = \{(x, t) : x \in \mathbb{R}, t \in [0, T]\}, \)

\(C\) and \(T\) only depend on \(h_0, \alpha, \|u_0\|_{C^2([-h_0, h_0])}, \|v_0\|_{C^2([-h_0, h_0])}\) and \(\|v_0\|_{L^\infty(-\infty, \infty)}\).
Proof. As in [30], we first straighten the double free boundary fronts by making the following change of variable:

\[ y = \frac{2h_0x}{h(t) - g(t)} - \frac{h_0(h(t) + g(t))}{h(t) - g(t)}, \quad w(y, t) = u(x, t), \quad z(y, t) = v(x, t). \]

Then problem (4) can be deduced to

\[
\begin{aligned}
    w_t &= (A - \sqrt{B} \beta)w_y + dBw_{yy} + r_1w(1 - \frac{w}{K_1 + v_1w} - \varepsilon_1w), & t > 0, \quad -h_0 < y < h_0, \\
    z_t &= AZ_y + eBu_{yy} + r_2z(1 - \frac{w}{K_2 + v_2w} - \varepsilon_2z), & t > 0, \quad -\infty < y < \infty, \\
    w &= 0, \quad h'(t) = -\frac{2h_0}{A(t) - A_0(t)} \frac{\partial w}{\partial y}, & t > 0, \quad y \geq h_0, \quad (8) \\
    w &= 0, \quad g'(t) = -\frac{2h_0g}{A(t) - A_0(t)} \frac{\partial w}{\partial y}, & t > 0, \quad y \leq -h_0, \\
    h(0) = h_0, \quad g(0) = -h_0, \\
    w(y, 0) = w_0(y) := w_0(y), \quad z(y, 0) := z_0(y) := v_0(y), & -\infty \leq y \leq \infty,
\end{aligned}
\]

where \( A = A(h, g, y) = y \frac{h'(t) - g(t)}{h(t) - g(t)} + h_0 \frac{h'(t) + g'(t)}{h(t) - g(t)} \) and \( B = B(h, g) = \frac{4h_0^2}{(h(t) - g(t))^2} \).

This transformation changes the free boundaries \( x = h(t) \) and \( x = g(t) \) to the fixed lines \( y = h_0 \) and \( y = -h_0 \) respectively, and the equations become more complex, since now the coefficients in the first and second equations of (8) contain unknown functions \( h(t) \) and \( g(t) \).

The rest of the proof is by the contraction mapping argument as in [6, 30] with suitable modifications, we omit the details here. \( \square \)

To show the global existence of the solution, we need the following estimates.

**Lemma 2.2.** Let \((u, v; g, h)\) be a solution to problem (4) defined for \( t \in [0, T_0) \) for some \( T_0 \in (0, +\infty) \). Then there exist constants \( C_1 \) and \( C_2 \) independent of \( T_0 \) such that

\[
0 < u(x, t) \leq C_1 \text{ for } g(t) < x < h(t), \ t \in (0, T_0], \\
0 < v(x, t) \leq C_2 \text{ for } -\infty < x < \infty, \ t \in (0, T_0].
\]

Proof. The positivity of \( u \) and \( v \) are obvious since that the initial values are non-trivial and nonnegative, and the system is quasi-increasing. Considering its upper bounds, it is easy to show that

\[
u \leq \max\{\|u_0\|_{C([-h_0, h_0])}, 1/\varepsilon_1\} \text{ for } g(t) < x < h(t), \ t \in (0, T_0], \\
v \leq \max\{\|v_0\|_{L^\infty(-\infty, \infty)}, 1/\varepsilon_2\} \text{ for } -\infty < x < \infty, \ t \in (0, T_0]
\]

since that

\[
\frac{\partial u(x, t)}{\partial t} \leq d \frac{\partial^2 u}{\partial x^2} - \beta u_x + r_1u(1 - \varepsilon_1u), \ g(t) < x < h(t), \ 0 < t \leq T_0, \\
\frac{\partial v(x, t)}{\partial t} \leq e \frac{\partial^2 v}{\partial x^2} + r_2v(1 - \varepsilon_2v), \ -\infty < x < \infty, \ 0 < t \leq T_0.
\]

The next lemma shows that the left free boundary for problem (4) is strictly monotone decreasing and the right boundary is increasing.

**Lemma 2.3.** Let \((u, v; g, h)\) be a solution to problem (4) defined for \( t \in (0, T_0] \) for some \( T_0 \in (0, +\infty) \). Then there exists a constant \( C_3 \) independent of \( T_0 \) such that

\[
0 < -g'(t), \ h'(t) \leq C_3 \text{ for } t \in (0, T_0].
\]
Proof. Using the strong maximum principle to the equation of $u$ gives that
\[
u_{x}(h(t), t) < 0 \quad \text{for } 0 < t \leq T_0.
\]
Hence $h'(t) > 0$ for $t \in (0, T_0]$ by using the free boundary condition in (4). Similarly, $g'(t) < 0$ for $t \in (0, T_0]$. It remains to show that $-g'(t), h'(t) \leq C_3$ for $t \in (0, T_0]$ and some $C_3$. The proof is similar as that of Lemma 2.2 in [6], but we sketch it here since we found that the advection term have different effects to the left and right free boundaries.

Set
\[
\Omega = \{(x, t) : h(t) - \frac{1}{M} < x < h(t), \ 0 < t \leq T_0\}
\]
and constitute an auxiliary function
\[
w(x, t) = C_1[2M(h(t) - x) - M^2(h(t) - x)^2].
\]
In the following proof, we will choose $M$ such that $w(x, t)$ is the supersolution of $u(x, t)$ in $\Omega$.

Straightforward computation show that
\[
w_t = 2C_1MH'(t)(1 - M(h(t) - x)) \geq 0,
\]
\[-w_{xx} = 2C_1M^2,
\]
\[0 \geq w_x = 2C_1M[-1 + M(h(t) - x)] \geq -2C_1M,
\]
\[r_1 u(1 - \frac{u}{K_1 + \alpha v} - \varepsilon u) \leq r_1 C_1.
\]
It follows that
\[
w_t - dw_{xx} + \beta w_x \geq 2dC_1M^2 - 2\beta C_1 M \geq r_1 C_1 \geq r_1 u(1 - \frac{u}{K_1 + \alpha v} - \varepsilon u)
\]
if $M^2 \geq \frac{r_1 + 2\beta M}{2d}$. On the other hand,
\[
w(h(t) - \frac{1}{M}, t) = C_1 \geq u(h(t) - \frac{1}{M}, t),
\]
\[w(h(t), t) = 0 = u(h(t), t).
\]
Recalling that $w(h_0, 0) = u_0(h_0) = 0$ gives that $w(x, 0) \geq u_0(x)$ in $[h_0 - \frac{1}{M}, h_0]$ if $M \geq \frac{4||u_0||_{C^1([h_0, h_0])}}{3C_1}$. Making use of the comparison principle yields $u(x, t) \leq w(x, t)$ in $\Omega$ if we take
\[
M = \max \left\{\frac{\beta}{d} + \sqrt{\frac{r_1}{2d}}, \frac{4||u_0||_{C^1([h_0, h_0])}}{3C_1}, \frac{1}{2h_0}\right\}.
\]
Noticing that $u(h(t), t) = w(h(t), t) = 0$, we have
\[
u_x(h(t), t) \geq w_x(h(t), t) = -2MC_1.
\]
Recollecting the free boundary condition in (4) deduces
\[
0 < h'(t) \leq 2\mu MC_1 := C_{31}, \quad 0 < t \leq T_0,
\]
where $C_{31}$ is independent of $T_0$. Analogously, we can define
\[
w(x, t) = C_1[2M(x - g(t)) - M^2(x - g(t))^2]
\]
over the region
\[
\Omega' = \{(x, t) : g(t) < x < g(t) + \frac{1}{M}, \ 0 < t \leq T\}
\]
get that
\[ 0 < -g'(t) \leq 2\mu M_1 C_1 := C_{32}, \quad 0 < t \leq T, \]
where
\[ M_1 = \max \left\{ \sqrt{\frac{r_1}{2d}}, 4\|u_0\|_{C^1([-h_0,h_0])}, \frac{1}{2h_0} \right\} \]
and \( C_{32} \) is independent of \( T_0 \) or \( \beta \).

Since that \( u, v \) and \( g', h' \) are bounded by constants independent of \( T_0 \), the global solution is guaranteed.

**Theorem 2.4.** The solution of problem (4) exists and is unique for all \( t \in (0, \infty) \).

In what follows, we exhibit the comparison principle, which can be proved similarly as Lemma 3.5 in [6].

**Lemma 2.5.** *(The Comparison Principle)* Assume that \( \overline{g}, \overline{h} \in C^1([0, +\infty)), \overline{\sigma}(x,t) \in C([\overline{\sigma}(t), \overline{h}(t)] \times [0, +\infty)) \cap C^{2,1}((\overline{\sigma}(t), \overline{h}(t)) \times (0, +\infty)), \overline{v}(x,t) \in C^{2,1}((-\infty, \infty) \times (0, +\infty)), \) and

\[
\begin{align*}
\frac{\partial \overline{v}}{\partial t} & \geq d \frac{\partial^2 \overline{v}}{\partial x^2} - \beta \overline{v} + r_1 \overline{v}(1 - \frac{\overline{v}}{K_1 + \alpha_1 \overline{u}} - \varepsilon_1 \overline{u}), \quad \overline{g}(t) < x < \overline{h}(t), \quad t > 0, \\
\frac{\partial \overline{u}}{\partial t} & \leq -\mu \frac{\partial \overline{u}}{\partial x} + r_2 \overline{v}(1 - \frac{\overline{v}}{K_2 + \alpha_2 \overline{u}} - \varepsilon_2 \overline{v}), \quad \overline{g}(t) < x < \overline{h}(t), \quad \overline{u}_0(0) < 0, \\
\overline{u}(0) & \leq 0, \quad \overline{g}(t) \leq -\mu \frac{\partial \overline{u}}{\partial x}(\overline{g}(t), t), \quad t > 0, \\
\overline{h}(0) & \geq 0, \quad \overline{h}(t) \geq -\mu \frac{\partial \overline{u}}{\partial x}(\overline{h}(t), t), \quad t > 0, \\
\overline{v}(x,0) & \geq 0, \quad \overline{v}(x,0) \geq \overline{v}_0(x), \quad \overline{u}(x,0) \geq 0, \quad \overline{h}(x,0) \geq 0,
\end{align*}
\]

then the solution \((u, v; g, h)\) to the free boundary problem (4) satisfies

\[ h(t) \leq \overline{h}(t), \quad g(t) \geq \overline{g}(t), \quad t \in [0, +\infty), \]
\[ u(x,t) \leq \overline{u}(x,t), \quad (x,t) \in [g(t), h(t)] \times [0, +\infty). \]
\[ v(x,t) \leq \overline{v}(x,t), \quad (x,t) \in (-\infty, \infty) \times [0, +\infty). \]

**Remark 1.** The pair \((\overline{u}, \overline{v}, \overline{g}, \overline{h})\) in Lemma 2.5 is usually called an upper solution of problem (4). We can define a lower solution by reversing all the inequalities in the obvious places. Moreover, one can easily prove an analogue of Lemma 2.5 for lower solution.

We next fix \( v_0 \) and \( \mu \), let \( u_0 = \sigma \phi(x) \) and examine the dependence of the solution on \( \sigma \), and we write \((u^\sigma, v^\sigma; g^\sigma, h^\sigma)\) to emphasize this dependence. As a corollary of Lemma 2.5, we have the following monotonicity:

**Corollary 1.** Let \( u_0 = \sigma \phi(x) \). For fixed \( v_0 \) and \( \mu \). If \( \sigma_1 < \sigma_2 \). Then \( u^{\sigma_1}(x,t) \leq u^{\sigma_2}(x,t) \) and \( v^{\sigma_1}(x,t) \leq v^{\sigma_2}(x,t) \) in \([g^{\sigma_1}(t), h^{\sigma_1}(t)] \times (0, \infty), \quad g^{\sigma_1}(t) \geq g^{\sigma_2}(t) \) and \( h^{\sigma_1}(t) \leq h^{\sigma_2}(t) \) in \((0, \infty)\).

3. **Spreading-vanishing dichotomy.** It follows from Lemma 2.3 that \( h(t) \) and \( g(t) \) are monotone, and therefore there exists \( h_\infty, g_\infty \in (0, +\infty) \) such that

\[ \lim_{t \to +\infty} h(t) = h_\infty \quad \text{and} \quad \lim_{t \to +\infty} g(t) = g_\infty. \]

Thus, we have four cases: (I) : \( h_\infty = \infty = g_\infty \) \((II) : h_\infty < \infty, g_\infty > -\infty \), (III) : \( h_\infty < \infty, g_\infty = -\infty \) and (IV) : \( h_\infty = \infty, g_\infty > -\infty \). The next lemma shows that the last two cases do not happen, both \( h_\infty \) and \( g_\infty \) are finite or infinite simultaneously.
Lemma 3.1. If \( h_\infty < \infty \) or \( g_\infty > -\infty \), then both \( h_\infty \) and \( g_\infty \) are finite and
\[
h_\infty - g_\infty \leq \frac{2\pi d}{\sqrt{4r_1 d - \beta^2}}, \quad \lim_{t \to \infty} \|u(\cdot, t)\|_{C([g(t), h(t)])} = 0,
\]
\[
\lim_{t \to \infty} v(\cdot, t) = \frac{K_2}{1 + K_2 \varepsilon_2} \text{ locally uniformly for } x \in (-\infty, \infty).
\]

Proof. Without loss of generality, we assume that \( h_\infty < \infty \), and divide the proof in the following steps.

Step 1. We prove that \( h_\infty - g_\infty \leq \frac{2\pi d}{\sqrt{4r_1 d - \beta^2}} \).

This step can be obtained by the similar argument in Lemma 5.2 of [6]. But much difficulty is induced by the introduce of \( \beta \), we sketch it here to see the effect of \( \beta \). By contradiction, if
\[
h_\infty - g_\infty \geq \frac{2\pi d}{\sqrt{4r_1 d - \beta^2}} > 0,
\]
then one can choose \( T > 0 \) large and \( l > 0 \) such that
\[
h(t) - g(t) > 2l > \frac{2\pi d}{\sqrt{4r_1 d - \beta^2}} \text{ for } t \geq T.
\]
It follows that there exists \( x_0 \in (g(T), h(T)) \) such that
\[
g(t) \leq g(T) < x_0 - l < x_0 < x_0 + l < h(T) \leq h(t) \text{ for } t \geq T.
\]

Now, consider the eigenvalue problem
\[
d\phi'' + (\beta - \varepsilon_3)\phi' - r_1 \phi = \lambda \phi \quad \text{in} \quad (x_0 - l, x_0 + l), \quad \phi(x_0 \pm l) = 0. \tag{9}
\]
Since \( l > \frac{\pi d}{\sqrt{4r_1 d - \beta^2}} \), for small \( \varepsilon_3 > 0 \), the principal eigenvalue of problem (9), denoted by \( \lambda_1 \), satisfies \( \lambda_1 < 0 \), and the following logistic-type problem
\[
\begin{cases}
-dw'' + (\beta - \varepsilon_3)w' - w = \frac{w}{r_1} - \varepsilon_1 w, & x_0 - l < x < x_0 + l, \\
w(x_0 - l) = w(x_0 + l) = 0
\end{cases} \tag{10}
\]
admits a unique positive solution satisfies \( 0 < w < \frac{K_1 r_1}{1 + K_1 \varepsilon_1} \) in \((x_0 - l, x_0 + l)\). By the maximum principle, one sees \( w'(x_0 + l) < 0 < w'(x_0 - l) \), so \( w' \) has zero point in \((x_0 - l, x_0 + l)\). Let the one that is closest to \( x_0 + l \) be \( x_1 \in (x_0 - l, x_0 + l) \). Then \( w'(x) < 0 \) in \((x_1, x_0 + l)\),
which is very crucial in constructing a sub-solution of \( u \) in \([x_1, x_0 + l] \times [T, +\infty)\) later.

Set \( \varepsilon_3 = 0 \) in problem (9), and denote the corresponding principal eigenvalue-eigenfunction by \((\lambda_1^0, \phi_1^0)\). It is easy to check that \( \varepsilon_4 \phi_1^0 \) is a lower solution of \( u \) in \([x_0 - l, x_0 + l] \times [T, +\infty) \) provided \( \varepsilon_4 \) is small (\( \varepsilon_4 \) depends on \( T \)). Therefore,
\[
u(x_1, t) \geq \varepsilon_4 \phi_1^0(x_1) > 0 \text{ for } t \geq T.
\]

We now construct a lower solution of \( u(x, t) \) in \([x_1, h(t)] \times [T, \infty)\). Define
\[
z(x, t) = \varepsilon_5 w(x_1 + \frac{x_0 + l - x_1}{h(t) - x_1} (x - x_1)), \quad x_1 \leq x \leq h(t), \ t \geq T,
\]
where \( \varepsilon_5 \) is a constant to be determined later.
and calculate

\[ z_t - d_{xx} + \beta z_x = \frac{(x_0 + l - x_1)^2}{(h(t) - x_1)^2} h'(t) \varepsilon_5 w' - \frac{d(x_0 + l - x_1)^2}{(h(t) - x_1)^2} \varepsilon_5 w'' + \beta_r \frac{d(x_0 + l - x_1)}{h(t) - x_1} \varepsilon_5 w' \]

\[ = \frac{(x_0 + l - x_1)^2}{(h(t) - x_1)^2} \varepsilon_5 \left[ -dw'' - \frac{(h(t) - x_1)}{x_0 + l - x_1} h'(t)w' + \beta \frac{(h(t) - x_1)}{x_0 + l - x_1} w'' \right]. \]

Using the fact that \( h'(t) \to 0 \) as \( t \to +\infty \), we can find \( T_0 > T \) such that \( h'(t) \leq \frac{x_0 + l - x_1}{h_\infty - x_1} \varepsilon_3 \), and then

\[ z_t - d_{xx} + \beta z_x \leq \frac{(x_0 + l - x_1)^2}{(h(t) - x_1)^2} \varepsilon_5 \left[ -dw'' - \varepsilon_3 w' + \beta w' \right] \]

since that \( w' \leq 0 \) and \( \frac{(h(t) - x_1)}{x_0 + l - x_1} \geq 1 \) for \( (x, t) \in [x_1, x_0 + l] \times [T_0, +\infty) \). Furthermore, owing to \(-dw'' - \varepsilon_3 w' + \beta w' = w(r_1 - \frac{w}{K_1} - \varepsilon_1) \geq 0 \) and \( \frac{(x_0 + l - x_1)}{h(t) - x_1} \leq 1 \), we then have

\[ z_t - d_{xx} + \beta z_x \leq \varepsilon_5 w(r_1 - \frac{w}{K_1} - \varepsilon_1) \leq z(r_1 - \frac{z}{K_1} - \varepsilon_1 z) \]

for \( (x, t) \in [x_1, h(t)] \times [T_0, +\infty) \).

Next we choose \( \varepsilon_5 \) small so that \( \varepsilon_5 w(x_1) \leq \varepsilon_4 \phi_1^0(x_1) \) and \( \varepsilon_5 w(x_1 + \frac{x_0 + l - x_1}{h(t) - x_1} (x - x_1)) \leq u(x, T_0) \) in \( [x_1, h(T_0)] \). Then \( z(x, t) \) satisfies

\[
\begin{cases}
  z_t - d_{xx} + \beta z_x \leq zr_1(1 - \frac{x}{K_1} - \varepsilon_1), & x_1 < x < h(t), \ t \geq T_0, \\
  z(x_1, t) = \varepsilon_5 w(x_1) \leq u(x_1, t), & z(h(t), t) = 0, \ t \geq T_0, \\
  z(x, T_0) = \varepsilon_5 w(x_1 + \frac{x_0 + l - x_1}{h(T_0) - x_1} (x - x_1)) \leq u(x, T_0), & x_0 \leq x \leq h(T_0). 
\end{cases}
\]

Hence we can apply the comparison principle to conclude that

\[ z(x, t) \leq u(x, t) \text{ for } x \in [x_1, h(t)], \ t \geq T_0. \]

It follows that

\[ u_x(h(t), t) = z_x(h(t), t) = \varepsilon_5 \frac{x_0 + l - x_1}{h(t) - x_1} \phi'(x_0 + l) \to \varepsilon_5 \frac{x_0 + l - x_1}{h_\infty - x_1} \phi'(x_0 + l) < 0. \]

On the other hand, we have

\[ u_x(h(t), t) = -\frac{1}{\mu} h'(t) \to 0 \text{ as } t \to \infty. \]

This contradiction proves that \( h_\infty - g_\infty \leq \frac{2\pi \varepsilon}{\sqrt{4\pi d - \beta^2}} \).

**Step 2.** \( \lim_{t \to +\infty} ||u(\cdot, t)||_{C([g(t), h(t)])} = 0. \)

Let \( \tilde{u}(x, t) \) denote the unique solution of the problem

\[
\begin{cases}
  \tilde{u}_t - d_{xx} + \beta \tilde{u}_x = r_1 \tilde{u} - r_1 \varepsilon_1 \tilde{u}^2, & g_\infty < x < h_\infty, \ t > 0, \\
  \tilde{u}(g_\infty, 0) = 0, & \tilde{u}(h_\infty, 0) = 0, \ t > 0, \\
  \tilde{u}(x, 0) = \tilde{u}_0(x), & g_\infty < x < h_\infty, 
\end{cases}
\]

with

\[ \tilde{u}_0(x) = \begin{cases} u_0(x) & 0 \leq x \leq h_0, \\
 0 & \text{otherwise}. \end{cases} \]

The comparison principle gives \( 0 \leq u(t, x) \leq \tilde{u}(t, x) \) for \( t > 0 \) and \( x \in [g(t), h(t)] \).

On the other hand, let \( U(x, t) = e^{-\frac{\beta}{\mu} t} \tilde{u}(x, t) \), it is easy to check that \( U(x, t) \) solves

\[
\begin{cases}
  U_t - d_{xx} = (r_1 - \frac{\beta}{\mu} t) U - r_1 \varepsilon_1 e^{-\frac{\beta}{\mu} t} U^2, & g_\infty < x < h_\infty, \ t > 0, \\
  U(g_\infty, 0) = 0, & U(h_\infty, 0) = 0, \ t > 0, \\
  U(x, 0) = \tilde{u}_0(x), & g_\infty < x < h_\infty, 
\end{cases}
\]

\[ U(x, t) = U(x, 0) e^{-\frac{\beta}{\mu} t}, \]

\[ U(x, t) = U(x, 0) e^{-\frac{\beta}{\mu} t}. \]

\[ U(x, t) = U(x, 0) e^{-\frac{\beta}{\mu} t}. \]
Since $h_\infty - g_\infty \leq \frac{2\pi d}{\sqrt{4r_1d - g_2}}$, it follows from a well-known conclusion ([3]) on the logistic problem ([12]) that $U(x, t) \to 0$ uniformly for $x \in [g_\infty, h_\infty]$ as $t \to \infty$. Thus \( \lim_{t \to +\infty} ||u(\cdot, t)||_{C([g(t), h(t))]} = 0 \).

**Step 3.** $\lim_{t \to -\infty} v(\cdot, t) = \frac{K_2}{1 + \varepsilon_2 K_2}$ locally uniformly for $x \in (-\infty, \infty)$.

Since that $\lim_{t \to -\infty} ||u(\cdot, t)||_{C([g(t), h(t))]} = 0$ and $u \equiv 0$ for $x < g(t)$ or $x > h(t)$, therefore for any $\varepsilon > 0$, there exists a $T_1$ such that $0 \leq u(x, t) \leq \varepsilon$ for $-\infty < x < \infty$, $t \geq T_1$, we then have

$$r_2 v(1 - \frac{v}{K_2} - \varepsilon_2 v) \leq v_t - ev_{xx} \leq r_2 v(1 - \frac{v}{K_2 + \alpha_2 \varepsilon} - \varepsilon_2 v), \quad -\infty < x < \infty, \ t \geq T_1.$$  

Let $\overline{v}$ be the unique solution to

$$\begin{cases} 
\overline{v}_t - e \overline{v}_{xx} = r_2 \overline{v}(1 - \frac{\overline{v}}{K_2 + \alpha_2 \varepsilon} - \varepsilon_2 \overline{v}), & -\infty < x < \infty, \ t \geq T_1, \\
\overline{v}(x, T_1) = v(x, T_1), & -\infty < x < \infty,
\end{cases}$$

it is well known ([10]) that $\lim_{t \to -\infty} \overline{v}(\cdot, t) = \frac{1}{\frac{1}{\varepsilon_2} + \frac{K_2}{\alpha_2 \varepsilon}}$ locally uniformly for $x \in (-\infty, \infty)$. Similarly, let $v$ be the unique solution to

$$\begin{cases} 
\overline{v}_t - e \overline{v}_{xx} = r_2 \overline{v}(1 - \frac{\overline{v}}{K_2} - \varepsilon_2 \overline{v}), & -\infty < x < \infty, \ t \geq T_1, \\
\overline{v}(x, T_1) = v(x, T_1), & -\infty < x < \infty,
\end{cases}$$

and then $\lim_{t \to -\infty} v(\cdot, t) = \frac{1}{\frac{1}{\varepsilon_2} + \frac{K_2}{\alpha_2 \varepsilon}}$ locally uniformly for $x \in (-\infty, \infty)$. Moreover, the comparison principle implies that $\overline{v}(t) \leq v(x, t) \leq \overline{v}(x, t)$ for $(x, t) \in (-\infty, \infty) \times [T_1, \infty)$, the desired limit holds since that $\varepsilon$ is arbitrary small.

Next, we consider the asymptotic behavior of the solution to problem (4) when the spreading occurs.

**Theorem 3.2.** If $h_\infty = -g_\infty = +\infty$, then the solution of free boundary problem (4) satisfies $\lim_{t \to +\infty} (u(x, t), v(x, t)) = (u^*, v^*)$ uniformly in any bounded subset of $(-\infty, \infty)$, where $(u^*, v^*)$ is the unique positive equilibrium of the corresponding ODE systems:

$$\begin{cases} 
u'(t) = r_1 u(1 - \frac{u}{K_1 + \alpha_1 v} - \varepsilon_1 u), & t > 0, \\
u'(t) = r_2 v(1 - \frac{v}{K_2 + \alpha_2 u} - \varepsilon_2 v), & t > 0.
\end{cases} \tag{13}$$

**Proof.** We divide the proof in three parts.

(1) The limit superior of the solution

It follows from the comparison principle that $(u(x, t), v(x, t)) \leq (\overline{u}(t), \overline{v}(t))$ for $(x, t) \in (-\infty, \infty) \times (0, \infty)$, where $(\overline{u}(t), \overline{v}(t))$ is the solution of the problem

$$\begin{cases} 
\overline{u}'(t) = r_1 u(1 - \frac{u}{K_1 + \alpha_1 v} - \varepsilon_1 u), & t > 0, \\
\overline{v}'(t) = r_2 v(1 - \frac{v}{K_2 + \alpha_2 u} - \varepsilon_2 v), & t > 0, \\
\overline{u}(0) = ||u_0||_{L^\infty([-h_0, h_0])}, \overline{v}(0) = ||v_0||_{L^\infty([-h_0, h_0])}.
\end{cases} \tag{14}$$

It is well known that the unique positive equilibrium $(u^*, v^*)$ is globally stable for the ODE system (14) and $\lim_{t \to +\infty} (\overline{u}(t), \overline{v}(t)) = (u^*, v^*)$; therefore we deduce

$$\lim_{t \to +\infty} \sup_{t \to +\infty} (u(x, t), v(x, t)) \leq (u^*, v^*) \tag{15}$$

uniformly for $x \in (-\infty, \infty)$.

(2) The lower bound of the solution for a large time.
Note that $\beta < 2\sqrt{dL_1}$ by assumption, there is $L_0$ such that
\[ r_1 > \frac{\pi^2d}{4L_0^2} + \frac{\beta^2}{4d} \quad \text{and} \quad r_2 > \frac{\pi^2d}{4L_0^2}. \]

This implies that the principal eigenvalues $\lambda_1^*$ and $\mu_1^*$ of
\[
\begin{cases}
-d\phi_{xx} + \beta \phi_x = \lambda_1^* \phi, \quad x \in (-L_0, L_0), \quad \phi(\pm L_0) = 0, \\
-d\psi_{xx} = \mu_1^* \psi, \quad x \in (-L_0, L_0), \quad \psi(\pm L_0) = 0,
\end{cases}
\]
satisfy
\[ \lambda_1^* = \frac{\pi^2d}{4L_0^2} + \frac{\beta^2}{4d} < r_1, \quad \mu_1^* = \frac{\pi^2d}{4L_0^2} < r_2, \]
and the corresponding eigenfunctions can be choose as $\phi(x) = e^{\frac{\beta}{\pi}x} \cos \frac{\pi}{2L_0} x$ and $\psi(x) = \cos \frac{\pi}{2L_0} x$.

Since $h_\infty = -g_\infty = +\infty$, for any $L \geq L_0$, there exists $t_L > 0$ such that $g(t) \leq -L$ and $h(t) \geq L$ for $t \geq t_L$.

Letting $u = \delta \phi$ and $v = \delta \psi$, we can choose $\delta$ sufficiently small such that $(u, v)$ satisfies
\[
\begin{cases}
u_t \leq dv_{xx} - \beta v_x + r_1 v(1 - \frac{v}{K_1 + \alpha_1 v} - \varepsilon_1 v), \quad -L_0 < x < L_0, \quad t > t_{L_0}, \\
u_t \leq dv_{xx} + r_2 v(1 - \frac{v}{K_2 + \alpha_2 v} - \varepsilon_2 v), \quad -L_0 < x < L_0, \quad t > t_{L_0}, \\
u(x, t) = v(x, t) = 0, \quad x = \pm L_0, \quad t > t_{L_0}, \\
u(x, t_{L_0}) \leq u(x, t_{L_0}), \quad v(x, t_{L_0}) \leq v(x, t_{L_0}), \quad -L_0 \leq x \leq L_0,
\end{cases}
\]
which means that $(u, v)$ is a lower solution of the solution $(u, v)$ in $[-L_0, L_0] \times [t_{L_0}, \infty)$. We then have $(u, v) \geq (\delta \phi, \delta \psi)$ in $[-L_0, L_0] \times [t_{L_0}, \infty)$, which implies that the solution cannot decay to zero, this result will be used in the next part.

(3) The limit inferior of the solution.

We extend $\phi(x)$ to $\phi_{L_0}(x)$ by defining $\phi_{L_0}(x) := \phi(x)$ for $-L_0 \leq x \leq L_0$ and $\phi_{L_0}(x) := 0$ for $x < -L_0$ or $x > L_0$, similarly, $\psi_{L_0}(x) := \psi(x)$ for $-L_0 \leq x \leq L_0$ and $\psi_{L_0}(x) := 0$ for $x < -L_0$ or $x > L_0$. Now for $L \geq L_0$, $(u, v)$ satisfies
\[
\begin{cases}
u_t = dv_{xx} - \beta v_x + r_1 v(1 - \frac{v}{K_1 + \alpha_1 v} - \varepsilon_1 v), \quad g(t) < x < h(t), \quad t > t_L, \\
u_t = dv_{xx} + r_2 v(1 - \frac{v}{K_2 + \alpha_2 v} - \varepsilon_2 v), \quad g(t) < x < h(t), \quad t > t_L, \\
u(x, t) = v(x, t) = 0, \quad x = g(t) \text{ or } x = h(t), \quad t > t_L, \\
u(x, t_L) \geq \delta \phi_{L_0}, \quad v(x, t_L) \geq \delta \psi_{L_0}, \quad -L \leq x \leq L,
\end{cases}
\]
therefore, we have $(u, v) \geq (w, z)$ in $[-L, L] \times [t_L, \infty)$, where $(w, z)$ satisfies
\[
\begin{cases}	w_t = dw_{xx} - \beta w_x + r_1 w(1 - \frac{w}{K_1 + \alpha_1 w} - \varepsilon_1 w), \quad g(t) < x < h(t), \quad t > t_L, \\	z_t = dz_{xx} + r_2 z(1 - \frac{z}{K_2 + \alpha_2 z} - \varepsilon_2 z), \quad g(t) < x < h(t), \quad t > t_L, \\
w(x, t) = z(x, t) = 0, \quad x = \pm L, \quad t > t_L, \\
w(x, t_L) = \delta \phi_{L_0}, \quad z(x, t_L) = \delta \psi_{L_0}, \quad -L \leq x \leq L.
\end{cases}
\]
The system (18) is quasimonotone increasing; therefore, it follows from the upper and lower solution method and the theory of monotone dynamical systems ([24], Corollary 3.6) that $\lim_{t \to +\infty} (w(x, t), v(x, t)) \geq (w_L(x), z_L(x))$ uniformly in $[-L, L]$. 

\[ \]
where \((w_L, z_L)\) satisfies
\[
\begin{aligned}
-\frac{d}{dx}w''_L + \beta w'_L &= r_1 w_L \left(1 - \frac{w_L}{K_1 + \alpha_1 z_L} - \varepsilon_1 w_L\right), \quad -L < x < L, \\
-\varepsilon z''_L &= r_2 z_L \left(1 - \frac{z}{K_2 + \alpha_2 w_L} - \varepsilon_2 z_L\right), \quad -L < x < L, \\
w_L(x) &= z_L(x) = 0, \quad x = \pm L
\end{aligned}
\]
and is the minimum upper solution over \((\delta \phi_{L,0}, \delta \psi_{L,0})\).

The pair \((w_L(x), z_L(x))\) depends on \(L\), it increases with \(L\), that is, if \(0 < L_1 < L_2\), then \((w_{L_1}(x), z_{L_1}(x)) \leq (w_{L_2}(x), z_{L_2}(x))\) in \([-L_1, L_1]\). The result is derived by comparing the boundary conditions and initial conditions in (18) for \(L = L_1\) and \(L = L_2\).

Let \(L \to \infty\). By classical elliptic regularity theory and a diagonal procedure, it follows that \((w_L(x), z_L(x))\) converges uniformly on any compact subset of \((-\infty, \infty)\) to \((w_{\infty}, z_{\infty})\) that is continuous on \((-\infty, \infty)\) and satisfies
\[
\begin{aligned}
-\frac{d}{dx}w''_{\infty} + \beta w'_{\infty} &= r_1 w_{\infty} \left(1 - \frac{w_{\infty}}{K_1 + \alpha_1 z_{\infty}} - \varepsilon_1 w_{\infty}\right), \quad -\infty < x < \infty, \\
-\varepsilon z''_{\infty} &= r_2 z_{\infty} \left(1 - \frac{z_{\infty}}{K_2 + \alpha_2 w_{\infty}} - \varepsilon_2 z_{\infty}\right) \quad -\infty < x < \infty, \\
w_{\infty}(x) &\geq \delta \phi_{L_0}, \quad z_{\infty}(x) \geq \delta \psi_{L_0}, \quad -\infty < x < \infty.
\end{aligned}
\]

Next, we observe that \((w_{\infty}(x), z_{\infty}(x)) \equiv (u^*, v^*)\), which can be derived by considering the corresponding ODE system, whose solution tends to the unique constant solution \((u^*, v^*)\).

Now for any given \([-M, M]\) with \(M \geq L_0\), since that \((w_L(x), z_L(x)) \to (u^*, v^*)\) uniformly in \([-M, M]\), which is the compact subset of \((-\infty, \infty)\), as \(L \to \infty\), we deduce that for any \(\varepsilon > 0\), there exists \(L^* > L_0\) such that \((w_{L^*}(x), z_{L^*}(x)) \geq (u^* - \varepsilon, v^* - \varepsilon)\) in \([-M, M]\). As above, there exists \(t_{L^*}\) such that \([g(t), h(t)] \supseteq [-L^*, L^*]\) for \(t \geq t_{L^*}\). Therefore,
\[
(u(x, t), v(x, t)) \geq \max \left\{w(x, t), z(x, t)\right\} \text{ in } [-L^*, L^*] \times [t_{L^*}, \infty),
\]
and
\[
\lim_{t \to +\infty} \max \left\{w(x, t), z(x, t)\right\} \geq \max \left\{w_{L^*}(x), z_{L^*}(x)\right\} \text{ in } [-L^*, L^*].
\]
Using the fact that \((w_{L^*}(x), z_{L^*}(x)) \geq (u^* - \varepsilon, v^* - \varepsilon)\) in \([-M, M]\) gives
\[
\lim_{t \to +\infty} \max \left\{u(x, t), v(x, t)\right\} \geq \max \left\{u^* - \varepsilon, v^* - \varepsilon\right\} \text{ in } [-M, M].
\]
Since \(\varepsilon > 0\) is arbitrary, we then have \(\lim_{t \to +\infty} u(x, t) \geq u^*\) and \(\lim_{t \to +\infty} v(x, t) \geq v^*\) uniformly in \([-M, M]\), which together with (15) imply that \(\lim_{t \to +\infty} u(x, t) = u^*\) and \(\lim_{t \to +\infty} v(x, t) = v^*\) uniformly in any bounded subset of \((-\infty, \infty)\).

Combing Lemma 3.1 and Theorem 3.2, we obtain the following spreading-vanishing dichotomy for problem (4):

**Theorem 3.3.** Let \((u(x, t), v(x, t); g(t), h(t))\) be the solution of the free boundary problem (4). Then the following alternative holds:

Either

(i) Spreading: \(h_\infty = g_\infty = +\infty\) and \(\lim_{t \to +\infty} (u(x, t), v(x, t)) = (u^*, v^*)\) uniformly for \(x\) in any bounded set of \(\mathbb{R}^1\), where \((u^*, v^*)\) is the unique positive equilibrium of problem (13);

or
(ii) Vanishing: $h_\infty - g_\infty \leq \frac{2\pi d}{\sqrt{4r_1d - \beta^2}}$ and $\lim_{t \to +\infty} \| u(\cdot, t) \|_{C([g(t), h(t)])} = 0$ and $\lim_{t \to +\infty} \nu(x, t) = \frac{K_2}{1+\varepsilon K_2}$ uniformly for $x$ in any bounded set of $\mathbb{R}^1$.

Moreover, Lemma 3.1 implies the following result.

**Theorem 3.4.** If $h_0 \geq \frac{\pi d}{\sqrt{4r_1d - \beta^2}}$, then $h_\infty = -g_\infty = +\infty$ and spreading happens.

We can now prove the following sharp criteria governing the spreading-vanishing dichotomy.

**Theorem 3.5.** *(Sharp threshold)* Fix $\mu_0$ and $(\phi, v_0)$ satisfying (5). Let $(u, v; g, h)$ be a solution of (4) with $(u_0, v_0) = (\sigma \phi(x), v_0(x))$ for some $\sigma > 0$. Then there exists $\sigma^* = \sigma^*(\phi, v_0(x)) \in [0, \infty)$ such that spreading occurs when $\sigma > \sigma^*$, and vanishing occurs when $0 < \sigma \leq \sigma^*$.

**Proof.** It follows from Theorem 3.4 that spreading always occurs if $h_0 \geq \frac{\pi d}{\sqrt{4r_1d - \beta^2}}$.

Hence, in this case we have $\sigma^*(\phi, v_0) = 0$ for any $\phi(x)$ and $v_0(x)$.

For the remaining case

$$h_0 < \frac{\pi d}{\sqrt{4r_1d - \beta^2}},$$

define

$$\sigma^* := \sup\{\sigma_0 : h_\infty(\sigma \phi, v_0) - g_\infty(\sigma \phi, v_0) < \infty \text{ for } \sigma \in (0, \sigma_0]\}.$$

Firstly, we show that in this case vanishing occurs for all small $\sigma > 0$; therefore, $\sigma^* \in (0, \infty]$.

In fact, since that $(u(x, t); h(t), g(t))$ satisfies

\[
\begin{align*}
\frac{\partial u(x, t)}{\partial t} &\leq d\frac{\partial^2 u}{\partial x^2} - \beta u_x + r_1 u(1 - \varepsilon_1 u), & g(t) < x < h(t), & t > 0, \\
u(x, t) &\geq 0, & x \leq g(t) \text{ or } x \geq h(t), & t > 0, \\
g(0) &\equiv -h_0, & g'(t) &\equiv -\mu \frac{\partial g}{\partial x}(g(t), t), & t > 0, \\
h(0) &\equiv h_0, & h'(t) &\equiv -\mu \frac{\partial h}{\partial x}(h(t), t), & t > 0, \\
u(x, 0) &\equiv u_0(0), & \neg \infty < x < \infty, &
\end{align*}
\]

we need to construct a suitable upper solution $(\overline{u}(x, t); \overline{h}(t), \overline{g}(t))$ such that it satisfies

\[
\begin{align*}
\frac{\partial \overline{u}(x, t)}{\partial t} &\geq d\frac{\partial^2 \overline{u}}{\partial x^2} - \beta \overline{u}_x + r_1 \overline{u}(1 - \varepsilon_1 \overline{u}), & \overline{g}(t) &< x < \overline{h}(t), & t > 0, \\
\overline{u}(x, t) &\equiv 0, & x \leq \overline{g}(t) \text{ or } x \geq \overline{h}(t), & t > 0, \\
\overline{g}(0) &\equiv -h_0, & \overline{g}'(t) &\equiv -\mu \frac{\partial \overline{u}}{\partial x}(\overline{g}(t), t), & t > 0, \\
\overline{h}(0) &\equiv h_0, & \overline{h}'(t) &\equiv -\mu \frac{\partial \overline{h}}{\partial x}(\overline{h}(t), t), & t > 0, \\
\overline{u}(x, 0) &\equiv u_0(0), & \neg \infty < x < \infty, &
\end{align*}
\]

By the assumption that $h_0 < \frac{\pi d}{\sqrt{4r_1d - \beta^2}}$, there exists a small $\delta > 0$ such that

$$\frac{\pi^2 d^2}{h_0^2(1 + \delta)^2} \geq 4r_1d - \beta^2 - \beta h_0 \delta^2 + 4\delta.$$

Similarly as in [6, 12], we set

$$\overline{h}(t) = -\overline{g}(t) = h_0(1 + \delta - \frac{\delta}{2} e^{-\delta t}), & t \geq 0,$$

and

$$\overline{u}(x, t) = \varepsilon e^{-\delta t} e^{\frac{\pi x}{2\overline{h}(t)}} \cos\left(\frac{\pi x}{2\overline{h}(t)}\right), & \overline{h}(t) \leq x \leq \overline{h}(t), & t \geq 0.$$
\( \bar{u}(x, t) = 0, \ x < -\bar{h}(t) \) or \( x > \bar{h}(t), \ t \geq 0. \)

Direct computations yield that we can chose \( \varepsilon \) sufficiently small such that the first four inequalities or equation in (20) hold, then if \( u_0(x) \leq \bar{u}(x, 0) \) in \([-h_0, h_0]\), we can apply Lemma 2.5 to conclude that \( g(t) \geq -\bar{h}(t) \) and \( h(t) \leq h(t) \) for \( t > 0 \).

It follows that \( h_\infty - g_\infty \leq \lim_{t \to \infty} 2\bar{h}(t) = 2h_0(1+\delta) < \infty \), and therefore, vanishing happens by Lemma 3.2 if \( \delta \) is small enough.

Secondly, as Lemma 5.1 in [18], we can construct a vector \( (w;g,h) \) such that \( u \geq \bar{u} \) in \([g(t), \bar{h}(t)] \times [0, T_0] \) for some \( T_0 > 0 \), and also \( g(t) \leq \bar{g}(t) \), \( h(t) \leq \bar{h}(t) \) in \([0, T_0] \).

Moreover, we can choose \( T_0 \) such that \( h(T_0) - g(T_0) > \frac{2\pi d}{\sqrt{4r_1 d - \beta^2}} \), then \( h_\infty - g_\infty = \infty \) by Lemma 3.1, which implies that spreading occurs for all large \( \sigma \).

Therefore, \( \sigma^* \in (0, \infty) \), spreading occurs when \( \sigma > \sigma^* \), and vanishing occurs when \( 0 < \sigma < \sigma^* \) by Corollary 1.

Lastly, we claim that vanishing occurs when \( \sigma = \sigma^* \). Otherwise \( h_\infty - g_\infty = \infty \) for \( \sigma = \sigma^* \). Then there exists \( T_1 > 0 \) such that \( h(T_1) - g(T_1) > \frac{2\pi d}{\sqrt{4r_1 d - \beta^2}} \).

By the continuous dependence of \( (u,v;g,h) \) on its initial values, we can find \( \varepsilon > 0 \) sufficiently small so that the solution of (4) with \( (u_0,v_0) = ((\sigma^* - \varepsilon)\phi(x), v_0(x)) \), denoted by \( (u_\varepsilon,v_\varepsilon;g_\varepsilon,h_\varepsilon) \) satisfies \( h(T_1) - g(T_1) > \frac{2\pi d}{\sqrt{4r_1 d - \beta^2}} \). This implies that spreading occurs for \( (u_\varepsilon,v_\varepsilon;g_\varepsilon,h_\varepsilon) \), contradicting the definition of \( \sigma^* \). This completes the proof. \( \square \)

Similarly, if we consider \( \mu \) instead of \( u_0 \) as a varying parameter, the following result holds, see also Theorem 4.4 in [8].

**Theorem 3.6.** (Sharp threshold) Fix \( h_0, u_0 \) and \( v_0 \). Then there exists \( \mu^* \in (0, \infty) \) such that spreading occurs when \( \mu > \mu^* \), and vanishing occurs when \( 0 < \mu < \mu^* \).

4. **Estimates of spreading speed.** To derive the estimates of the asymptotic spreading speed, we first recall the known result for (4) with \( \beta = 0 \) and \( \alpha_1 = 0 \), see Proposition 1 in [7].

**Theorem 4.1.** Let \( (u,v;g,h) \) be the unique solution of (4) with \( \beta = \alpha_1 = 0 \). If \( h_\infty = -g_\infty = \infty \). Then

\[
\lim_{t \to \infty} \frac{g(t)}{t} = \lim_{t \to \infty} \frac{h(t)}{t} = c_0,
\]

where \( (c_0,q(x)) \) is the unique positive solution of the problem

\[
\begin{cases}
-\rho'' + c_0q' = r_1q[1 - (\frac{q}{K_1} + \varepsilon_1)q], & x > 0, \\
q(0) = 0, \ q(\infty) = \frac{K_1}{1+\varepsilon_1K_1}, \ q(x) > 0, & x > 0, \\
\mu q'(0) = c_0.
\end{cases}
\]

(21)

Theorem 4.1 shows that if there is no advection, the asymptotic spreading speed of the left frontier and that of the right frontier are the same when invasive species is spreading.

To address the change induced by an advection term, we first study the following problem:

\[
\begin{cases}
dq'' - (c-\beta)q' + r_1q[1 - \omega q] = 0 & \text{for } x \in (0, \infty), \\
q(0) = 0, \ q(\infty) = \frac{1}{\beta}, \ q(x) > 0 & \text{for } x \in (0, \infty),
\end{cases}
\]

(22)

where \( \omega > 0 \). Usually, \( q(z) \) is called a semi-wave with speed \( c \). We will derive the rightward spreading speed by this semi-wave. Consequently, for the leftward
spreading speed, the corresponding semi-wave is governed by
\[
\begin{aligned}
&dq'' - (c + \beta)q' + r_1 q[1 - \omega q] = 0 \quad \text{for } x \in (0, \infty), \\
&q(0) = 0, \quad q(\infty) = \frac{1}{r_1}, \quad q(z) > 0 \quad \text{for } x \in (0, \infty).
\end{aligned}
\]  
(23)

We now present the properties of the semi-waves, see Propositions 2.2, 2.4 and 2.5 in [12].

**Proposition 1.** The following conclusions hold.

(i) Problem (22) has exactly one solution \((c, q) = (c^*_r, q^*_r)\) such that
\[
\mu(q^*_r)'(0) = c_r^*.
\]  
(24)

Moreover, \(c_r^* := c_r^*(\beta, d, r_1, \omega) \in (0, 2\sqrt{r_1d} + \beta)\);

(ii) Problem (23) has exactly one solution \((c, q) = (c^*_l, q^*_l)\) such that
\[
\mu(q^*_l)'(0) = c_l^*.
\]  
(25)

Moreover, \(c_l^* := c_l^*(\beta, d, r_1, \omega) \in (0, 2\sqrt{r_1d} - \beta)\);

(iii) \(0 < c_r^* < c^* < c_l^*\), where \(c^*\) is the speed in (22) (or (23)) with \(\beta = 0\);

(iv) \(c_l^*\) and \(c_r^*\) depend continuously on the parameter \(\omega\) and are strictly strictly decreasing in \(\omega\), that is, for any \(\omega_0 > 0\) and \(\omega_1 > \omega_2 > 0\), we have
\[
c_r^*(\omega_1) < c_r^*(\omega_2), \quad \lim_{\omega \to \omega_0} c_r^*(\omega) = c_r^*(\omega_0),
\]
\[
c_l^*(\omega_1) < c_l^*(\omega_2), \quad \lim_{\omega \to \omega_0} c_l^*(\omega) = c_l^*(\omega_0).
\]

Next we give some rough estimates on the spreading speed when spreading happens.

**Theorem 4.2.** If \(h_\infty = -g_\infty = +\infty\), then
\[
c_r^*(\beta, d, r_1, \omega_1) \leq \liminf_{t \to +\infty} \frac{h(t)}{t} \leq \limsup_{t \to +\infty} \frac{h(t)}{t} \leq c_r^*(\beta, d, r_1, \omega_2),
\]
\[
c_l^*(\beta, d, r_1, \omega_1) \leq \liminf_{t \to +\infty} \frac{-g(t)}{t} \leq \limsup_{t \to +\infty} \frac{-g(t)}{t} \leq c_l^*(\beta, d, r_1, \omega_2),
\]
where \(\omega_1 = \frac{1}{R_1} + \epsilon_1\) and \(\omega_2 = \frac{1}{R_1 + \alpha_{1\nu}} + \epsilon_1\).

**Proof.** We first estimate the rightward asymptotic spreading speed.

By Theorem 3.2, we have \(\lim_{t \to \infty} (u(\cdot, t), v(\cdot, t)) = (u^*, v^*)\) locally uniformly in \(\mathbb{R}\). In particular,
\[
u(x, t) \geq \frac{1}{\omega_1} \quad \text{for } 0 \leq x \leq h_0, \quad t > T
\]
for some \(T > 0\) since that \(u^* > \frac{1}{\omega_1}\).

Define
\[
\hat{u}(x, t) := q^*_r(c^*_r t - x), \quad x \in [0, c^*_r t],
\]
where \(c^*_r = c^*_r(\beta, d, r_1, \omega_1)\), and \((c^*_r, q^*_r(z))\) satisfies
\[
\begin{aligned}
&dq'' - (c - \beta)q' + r_1 q[1 - \omega_1 q] = 0 \quad \text{for } z \in (0, \infty), \\
&q(0) = 0, \quad q(\infty) = \frac{1}{\omega_1}, \quad q'(0) = \frac{1}{\omega_1} \text{ and } q'(z) > 0 \quad (z > 0),
\end{aligned}
\]
we have
\[
\hat{u}(x, t) \leq \frac{1}{\omega_1}, \quad \hat{u}_t - d\hat{u}_{xx} + \beta \hat{u}_x = r_1 \hat{u}[1 - \omega_1 \hat{u}] \quad \text{for } x \in [0, c^*_r t], \quad t > 0,
\]
and
\[
\hat{u}(c^*_r t, t) = 0, \quad c^*_r = -\mu \hat{u}_x(c^*_r t, t) \quad \text{for } t \geq 0.
\]
Then (\(\tilde{u}(x,t), c^*_t(t)\)) is a lower solution of \((u(x,t + T), h(t + T))\) on \(\{(x,t) \mid x \in [0, c^*_t(t), t > 0]\}\) by comparison principle (Lemma 2.5 with \(v \equiv 0\), and
\[c^*_t(t) \leq h(t + T), \quad \tilde{u}(x,t) \leq u(x,t + T) \quad \text{in} \quad \{(x,t) \mid x \in [0, c^*_t(t), t > 0]\}\]
This implies that
\[
\liminf_{t \to \infty} \frac{h(t)}{t} \geq c^*_t(\beta, d, r_1, \omega_1). \tag{27}
\]
Next we estimate the upper bound of the rightward spreading speed. Consider the problem
\[
\begin{cases}
\dot{\xi}(t) = r_1 \xi(1 - \frac{\xi}{K_1 + \alpha_1 v} - \varepsilon_1 \xi), & \xi(0) = \|u_0\|_\infty + 1, \\
\dot{\eta}(t) = r_2 \eta(1 - \frac{\eta}{K_2 + \alpha_2 v} - \varepsilon_2 \eta), & \eta(0) = \|v_0\|_\infty + 1.
\end{cases}
\]
A simple comparison shows that
\[u(x,t) \leq \xi(t), \quad v(x,t) \leq \eta(t) \quad \text{for} \quad x \in [g(t), h(t)], \quad t > 0.\]
Moreover, it is well known that \(\lim_{t \to \infty} (\xi(t), \eta(t)) = (u^*, v^*)\), therefore for any small \(\varepsilon > 0\), there exists \(\hat{T} > 0\) such that
\[u(x,t) \leq u^* + \varepsilon/2, \quad \frac{1}{K_1 + \alpha_1 v} + \varepsilon_1 \geq \frac{\omega_2}{1 + \omega_2 \varepsilon} = \frac{1}{u^* + \varepsilon}
\]
for \(x \in [g(t), h(t)], t \geq \hat{T}\). Note that \(u(x,t)\) satisfies
\[
\begin{cases}
\frac{\partial u(x,t)}{\partial t} - \frac{d^2 u}{d x^2} + \beta u_x \geq r_1 u(1 - \frac{1}{u^* + \varepsilon} u), & g(t) < x < h(t), \quad t > \hat{T}, \\
\dot{u}(x,t) = 0, & x \leq g(t) \text{ or } x \geq h(t), \quad t > \hat{T}, \\
h(0) = h_0, \quad h'(t) = -\mu \frac{\partial u}{\partial x}(h(t),t), & t > 0, \\
\dot{u}(x,0) = u_0(x), & -\infty < x < \infty.
\end{cases}
\]
Let \((\tilde{c}^*_t, \tilde{q}^*_t(z))\) be the solution of the problem
\[
\begin{cases}
dq' - (\varepsilon - \beta)q' + r_1 q[1 - \frac{1}{u^* + \varepsilon}] = 0 \quad \text{for} \quad z \in (0, \infty), \\
q(0) = 0, \quad q(\infty) = u^* + \varepsilon, \quad q'(z) = \tilde{q}^*_t(z) \quad \text{and} \quad q'(z) > 0 \quad (z > 0),
\end{cases}
\]
where \(\tilde{c}^*_t = c^*_t(\beta, d, r_1, \frac{1}{u^* + \varepsilon})\), therefore, there exists \(\tilde{x} > h(\hat{T})\) large such that
\[u(x,\hat{T}) \leq u^* + \varepsilon/2 < \tilde{q}^*_t(\tilde{x} - x) \quad \text{for} \quad x \in [0, h(\hat{T})].
\]
Define
\[\tilde{u}(x,t) := \tilde{q}^*_t(\tilde{c}^*_t(t) + \tilde{x} - x) \quad \text{for} \quad x \in [0, \tilde{c}^*_t(t) + \tilde{x}], \quad t > 0.
\]
Then \((\tilde{u}, \tilde{c}^*_t(t) + \tilde{x})\) is an upper solution of \((u(x,t + \hat{T}), h(t + \hat{T}))\) on \(\{(x,t) \mid x \in [0, h(t + \hat{T})], \quad t > 0\}\), and by the comparison principle (Lemma 2.5 with \(v \equiv 0\)) we have
\[h(t + \hat{T}) \leq \tilde{c}^*_t(t) + \tilde{x}, \quad u(x,t + \hat{T}) \leq \tilde{u}(x,t) \quad \text{for} \quad x \in [0, h(t + \hat{T})] \text{ and } t > 0,
\]
which implies that
\[
\limsup_{t \to \infty} \frac{h(t)}{t} \leq \tilde{c}^*_t = c^*_t(\beta, d, r_1, \frac{1}{u^* + \varepsilon}). \tag{28}
\]
Recalling that \( \frac{1}{\omega} = \omega_2 \) and letting \( \epsilon \to 0 \) give that

\[
\limsup_{t \to \infty} \frac{h(t)}{t} \leq c^*_r(\beta, d, r_1, \omega_2)
\]

by Proposition 1 (iv).

The leftward spreading speed can be discussed similarly.

\[\square\]

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