Abstract

In this paper, we link and extend two approaches to estimating time-varying treatment effects on repeated continuous outcomes—time-varying Difference in Differences (DiD; see Roth et al. [2023] and De Chaisemartin and d’Haultfoeuille [2023] for reviews) and Structural Nested Mean Models (SNMMs; see Vansteelandt and Joffe [2014] for a review). In particular, we show that SNMMs, which were previously only known to be nonparametrically identified under a no unobserved confounding assumption, are also identified under a generalized version of the parallel trends assumption typically used to justify time-varying DiD methods. Because SNMMs model a broader set of causal estimands, our results allow practitioners of existing time-varying DiD approaches to address additional types of substantive questions under similar assumptions. SNMMs enable estimation of time-varying effect heterogeneity, the lasting effects of a “blip” of treatment at a single time point, effects of sustained interventions (possibly on continuous or multi-dimensional treatments) when treatment repeatedly changes value in the data, controlled direct effects, effects of dynamic treatment strategies that depend on covariate history, and more. Our results also allow analysts who apply SNMMs under the no unobserved confounding assumption to estimate some of the same causal effects under alternative
identifying conditions and thus potentially to triangulate evidence. We provide a method for sensitivity analysis to violations of our parallel trends assumption. We further explain how to estimate optimal treatment regimes via optimal regime Structural Nested Mean Models under parallel trends assumptions plus an assumption that there is no effect modification by unobserved confounders. Finally, we illustrate our methods with real data applications estimating effects of Medicaid expansion on uninsurance rates, effects of floods on flood insurance take-up, and effects of sustained changes in temperature on crop yields.

1 Introduction

In this paper, we link and extend two approaches to estimating time-varying treatment effects on repeated continuous outcomes—time-varying Difference in Differences (DiD; see Roth et al. [2023] and De Chaisemartin and d’Haultfoeuille [2023] for reviews) and Structural Nested Mean Models (SNMMs; see Vansteelandt and Joffe [2014] for a review). In particular, we show that SNMMs, which were previously only known to be nonparametrically identified under a no unobserved confounding assumption, are also identified under parallel trends assumptions similar to those typically used to justify time-varying DiD methods. Because SNMMs model a broader set of causal estimands, our results allow practitioners of time-varying DiD approaches to address additional types of substantive questions (such as time-varying effect heterogeneity, the lasting effects of a “blip” of treatment at a single time point, effects of sustained interventions on possibly continuous valued treatment variables, effects of dynamic treatment rules that depend on covariate history, and others) under similar assumptions. Our results also allow analysts who apply SNMMs under the no unobserved confounding assumption to estimate some of the same causal effects under alternative identifying conditions and thus potentially to triangulate evidence.

SNMMs [Robins, 1994, 1997, 2004] are models of time-varying effect heterogeneity in the treated. In the so-called ‘staggered adoption’ setting (in which all units start off untreated and then initiate treatment in a staggered fashion) that is the focus of most time-varying DiD literature, SNMMs can model conditional effects of first treatment initiation in the initiators as a function of time-varying covariate history up to the time of initiation. For example, an SNMM could model how the effect of Medicaid expansion varies with unemployment rate prior to expansion. (See our real data application in Section 6.1.) Alternative DiD approaches cannot characterize such time-varying effect heterogeneity. Marginal effects of initiation at each time (i.e. not conditional on time-varying covariate history) targeted by standard DiD approaches are also identified given SNMM parameters by simply averaging the conditional
When treatment is not an absorbing state and may repeatedly change value over time in the data, investigators may be interested in effects of interventions setting treatment at all future time points, rather than merely at the time of initiation. Under an appropriate parallel trends assumption, SNMMs can model the conditional effects of initiating and sustaining a treatment in those who initiated it, even if some initiators later discontinue in the data. Such effects have caused some consternation in the DiD literature (see Section 3.4 of [Roth et al. 2023]). As an example with a continuous valued treatment, in Section 6.3 we use an SNMM to estimate the average effects of sustained changes to ‘growing degree days’ starting in year $m$ on future county-level crop yields, conditional on past growing degree days.

SNMMs also directly model the (conditional) lasting effects of a final ‘blip’ of treatment. For example, one might employ an SNMM to model the (conditional) lasting effects on student test scores of a single grade of school with a highly rated teacher followed by teachers with average ratings in subsequent grades. In Section 6.2, we estimate the lasting impact of one final local flood compared to no further floods on county flood insurance take-up given flood history. Such contrasts could not be targeted by standard DiD methods.

If one is willing to assume a parallel trends assumption specific to an arbitrary treatment strategy $g$, we show that the general regime SNMM [Robins 2004] for strategy $g$ is identified and along with it the expected outcome trajectory had everybody in the population followed strategy $g$. In particular, $g$ can even be a dynamic regime in which treatment assignment at each time depends on covariate and treatment history. For example, $g$ might state ‘administer vasopressors whenever mean arterial blood pressure falls below 65 mmHG’. Effects of such complex strategies cannot be estimated via standard DiD approaches.

Another advantage of SNMMs is that they straightforwardly admit multi-dimensional treatments with continuous valued and/or discrete components. The problem of estimating effects of time-varying continuous valued treatments when units do not share a baseline treatment value has not yet been solved in the DiD literature [de Chaisemartin et al. 2024a] but is straightforward to solve using SNMMs (as in the growing degree days and crops example in Section 6.3, mentioned above). Moreover, multidimensional treatments enable estimation of controlled direct effects [Robins and Greenland 1992], such as effects of Medicaid expansion on bankruptcy rates barring concurrent or future minimum wage increases.

We further show that under the much stronger assumptions of parallel trends under all treatment strategies and no effect modification by unobserved confounders (where the second assumption is essen-
tially implied by the first, as we demonstrate in Appendix [C], it is possible to identify the parameters of an optimal regime SNMM [Robins, 2004] and thus estimate optimal dynamic treatment strategies. This last result might be considered a contribution to the reinforcement learning literature, where estimation of optimal regime SNMMs is sometimes called ‘A-learning’ [Schulte et al., 2014].

We note that throughout we are assuming availability of panel data, i.e. data on the same units over time, with repeated outcome measures. (Of course, units might be geographic and their outcomes composites of individual residents, such as unemployment rate in a state or county. In this case, our method would not require that the same individual residents are surveyed at each time point to ascertain the geographic unit level outcomes.) We also stress that while many SNMM studies consider settings with an outcome measured only once at the end of follow up, the methods in this paper would not be applicable to these studies.

The organization of the paper is as follows. In Section 2 we introduce notation, state and discuss assumptions, and describe causal contrasts that can be targeted by additive SNMMs. In Section 3 we establish identification and provide estimators for the parameters of an additive SNMM under our parallel trends assumption. In Section 4 we introduce SNMMs for multiplicative effects and extend our results to this setting. In Section 5 we discuss estimation of the optimal regime via an optimal regime SNMM under parallel trends assumptions for all regimes plus an additional ‘no effect modification by unobserved confounders’ assumption. In Section 6 we present several applications to real data, including effects of Medicaid expansion on uninsurance rates, effects of floods on flood insurance take-up, and effects of temperature on crop yields. In Section 7 we discuss sensitivity analysis for violations of conditional parallel trends. In Section 8 we discuss the contributions of this paper within the time-varying DiD literature. We explain why our parallel trends assumption (which was concurrently proposed by Renson et al., 2023), might be more plausible than alternative assumptions common in the DiD literature (e.g. Callaway and Sant’Anna, 2021) in the presence of time-varying confounders. In Section 9 we conclude.

The theme of the paper is that it can be fruitful to use SNMMs under assumptions similar to those that have traditionally justified DiD.
2 Notation, SNMMs, and Assumptions

2.1 Notation

Suppose we observe a cohort of \( N \) subjects indexed by \( i \in \{1, \ldots, N\} \). Assume that each subject is observed at regular intervals from baseline time 0 through end of follow-up time \( K \), and there is no loss to follow-up. At each time point \( m \), the data are collected on \( O_m = (Z_m, Y_m, A_m) \) in that temporal order. \( A_m \) denotes the (possibly multidimensional with discrete and/or continuous components) treatment received at time \( m \), \( Y_m \) denotes the outcome of interest at time \( m \), and \( Z_m \) denotes a vector of covariates at time \( m \) excluding \( Y_m \). Hence, \( Z_0 \) constitutes the vector of baseline covariates other than \( Y_0 \). For arbitrary time varying variable \( X \): we denote by \( \bar{X}_m = (X_0, \ldots, X_m) \) the history of \( X \) through time \( m \); we denote by \( X_m = (X_m, \ldots, X_K) \) the future of \( X \) from time \( m \) through time \( K \); and, unless otherwise stated, whenever the negative index \( X_{-1} \) appears it denotes the null value with probability 1. Define \( \bar{L}_m \) to be \((\bar{Z}_m, \bar{Y}_{m-1})\), the covariate and outcome history through time \( m \) excluding \( Y_m \). Hence, \( L_0 = Z_0 \).

(As no outcomes are collected subsequent to treatment \( A_K \), note that our estimators will not make use of \( A_K \) or \( Z_K \).)

We adopt the counterfactual framework for time-varying treatments (Robins, 1986) which posits that corresponding to each time-varying treatment regime \( \bar{a}_m \), each subject has a counterfactual or potential outcome \( Y_{m+1}(\bar{a}_m) \) that would have been observed had that subject received treatment regime \( \bar{a}_m \). More generally, let \( g \) denote an arbitrary, possibly dynamic treatment strategy, where \( g \equiv (g_0, g_1, \ldots, g_K) \) is a vector of functions \( g_t : (\bar{L}_t, \bar{A}_{t-1}) \rightarrow a_t \) that determine treatment values at each time point given observed history. Let \( Y_k(g) \) then denote the counterfactual value of the outcome at time \( k \) under \( g \). We will adopt the convention that \( Y_k(\bar{a}_m, g_{m+1}) \) denotes the counterfactual outcome under a regime assigning treatments \( \bar{a}_m \) through time \( m \) and then following \( g \) thereafter.

2.2 SNMMs

SNMMs are models of time-varying effect heterogeneity. Formally, they are directly concerned with the causal contrasts

\[
\gamma_{g} \quad (\bar{l}_m, \bar{a}_m) = E[Y_k(\bar{a}_m, g_{m+1}) - Y_k(\bar{a}_{m-1}, g_{m}) | \bar{A}_m = \bar{a}_m, \bar{L}_m = \bar{l}_m]
\]

(1)
for all \( k > m \). \( \gamma_{mk}^g(\bar{l}_m, \bar{a}_m) \) is the average effect at time \( k \) among units with history \((L_m = \bar{l}_m, A_m = \bar{a}_m)\) of receiving treatment \( a_m \) at time \( m \) and then following \( g \) thereafter compared to following \( g \) at time \( m \) and thereafter. We will sometimes suppress the dependence of \( \gamma_{mk}^g(\bar{l}_m, \bar{a}_m) \) on \( g \) and write \( \gamma_{mk}^*(\bar{l}_m, \bar{a}_m) \) for notational simplicity.

An important special case occurs when \( g = \bar{0} \) is the ‘untreated’ regime. (In fact, historically, SNMMs were originally developed for this special case [Robins, 1994] and were only later extended to general \( g \) [Robins, 2004].) In the \( g = \bar{0} \) setting, the contrasts in (1) represent the conditional effects of one last ‘blip’ of treatment at time \( m \) and at level \( a_m \). Hence (1) are accordingly often referred to as ‘blip functions’.

For example, in Section 6.2, we consider a blip function representing the lasting conditional effects on flood insurance take-up of a single local flood followed by no further floods compared to no further floods at all, given flood history.

A parametric SNMM imposes functional forms on the blip functions \( \gamma_{mk}^g(\bar{l}_m, \bar{a}_m) \) for each \( k > m \), i.e.

\[
\gamma_{mk}^g(\bar{l}_m, \bar{a}_m) = \gamma_{mk}^g(\bar{l}_m, \bar{a}_m; \psi^*),
\]

where \( \psi^* \) is the true value of an unknown finite dimensional parameter vector and \( \gamma_{mk}^g(\bar{l}_m, \bar{a}_m; \psi) \) is a known function equal to 0 whenever \( \psi = 0 \) or \( a_m = g(\bar{l}_m, \bar{a}_{m-1}) \).

**Remark 1. Effects of Initiation** As we mentioned in the introduction, most applications of time-varying DiD methods have been in staggered adoption settings where all units start off untreated and then begin treatment at different times. Interest centers on the average effect of the observed treatment trajectory compared to never initiating treatment in those that actually initiated treatment at level \( a_m \) at time \( m \) (and then possibly changed their treatment value afterward under the observational regime). If we let \( g = \bar{0} \) and code exposure such that it takes a non-zero value only at the time of first treatment and 0 at all other times, then (1) represents the conditional effects of treatment initiation in the treated. We utilize this treatment coding when analyzing effects of Medicaid expansion in Section 6.1. SNMMs for initiation effects have been called ‘coarse SNMMs’ and studied in previous work [Robins, 1998, Lok and DeGruttola, 2013].

**Remark 2. Effects of Sustained Interventions** A simple but important example of a dynamic regime \( g \) is the ‘maintain previous treatment value’ regime

\[
g_{\text{sus}}(\bar{l}_m, \bar{A}_{m-1}) = A_{m-1}.
\]
This regime facilitates estimation of effects of sustained interventions. For example,

$$
\gamma_{mk}^*(\bar{1}_m, (\bar{0}_{m-1}, 1)) = E[Y_k(\bar{0}_{m-1}, 1) - Y_k(0)|A_m = (\bar{0}_{m-1}, 1), L_m = \bar{1}_m]
$$

is the conditional effect of starting and continuing treatment at time $m$ compared to never starting in those who did start at $m$. In Section 6.3 we apply regime 3 to study the conditional effects of a sustained change in (continuous valued) ‘growing degree days’ on crop yields given growing degree days prior to the change.

**Remark 3. Controlled Direct Effects** Multidimensional treatments enable the contrasts in 1 to straightforwardly represent controlled direct effects (CDEs) [Robins and Greenland, 1992]. Suppose $A_m = (A_{m1}, A_{m2})$ is a two dimensional treatment and $g = \bar{0}$ is the untreated regime. For example, suppose $A_{m1}$ denotes initial Medicaid expansion (i.e. taking the value 1 in the first year of expansion and 0 otherwise, as in Remark 2) and $A_{m2}$ denotes a minimum wage increase in year $m$. $\gamma_{mk}^*(\bar{a}_m, \bar{1}_m)$ with $a_m = (1, 0)$ is then the conditional effect of Medicaid expansion and no future minimum wage increase in year $m$.

**Remark 4. Derived quantities** Given knowledge of $\gamma_{mk}^*(\bar{1}_m, \bar{a}_m)$, additional derived quantities of interest are identified under the Consistency assumption 4 stated in the following subsection [Robins, 1994, 2004]. For any subject history $(\bar{L}_m = \bar{1}_m, \bar{A}_m = \bar{a}_{m-1})$ of interest, expected conditional counterfactual outcomes under $g_m$, i.e. $E[Y_k(a_{m-1}, g_m)|\bar{L}_m = \bar{1}_m, \bar{A}_{m-1} = \bar{a}_{m-1}]$ for $k > m$, would also be identified as $E[Y_k - \sum_{j=m}^{k-1} \gamma_{jk}^*(\bar{I}_j, \bar{A}_j)|\bar{L}_m = \bar{1}_m, \bar{A}_{m-1} = \bar{a}_{m-1}]$. Quantities that further condition on treatment at $m$, i.e. $E[Y_k(a_{m-1}, g_m)|\bar{L}_m = \bar{1}_m, \bar{A}_m = \bar{a}_m]$ for $k > m$ would be identified as $E[Y_k - \sum_{j=m}^{k-1} \gamma_{jk}^*(\bar{L}_j, \bar{A}_j)|\bar{L}_m = \bar{1}_m, \bar{A}_{m-1} = \bar{a}_{m-1}]$. Marginalizing over $\bar{L}_m$ then identifies $E[Y_k(a_{m-1}, g_m)|\bar{A}_{m-1} = \bar{a}_{m-1}]$ and $E[Y_k(a_{m-1}, g_m)|\bar{A}_m = \bar{a}_m]$ for $k > m$. In staggered adoption settings with treatment coded as described in Remark 2 and $g = \bar{0}$, $E[\gamma_{mk}^*(\bar{I}_m, \bar{a}_m)|\bar{A}_m = \bar{a}_m]$ are the same causal estimands targeted by standard DiD approaches such as [Callaway and Sant’Anna, 2021] if $a_m = 1$. The derived quantities $E[Y_k(g)]$ for each $k > 0$, i.e. the expected counterfactual outcome trajectory under no treatment, would also be identified by $E[Y_k - \sum_{j=0}^{k-1} \gamma_{jk}^*(\bar{L}_j, \bar{A}_j)].$
2.3 Assumptions

We make the standard Consistency assumption

\[ Y_m(\bar{A}_{m-1}) = Y_m \forall m \leq K \]  

(4)

stating that observed outcomes are equal to counterfactual outcomes corresponding to observed treatments. Throughout, we will also assume, for regime \( g \) of interest,

\[ f_{A_m|\bar{L}_m,A_{m-1}}(g(\bar{L}_m,\bar{a}_{m-1})|\bar{L}_m,\bar{a}_{m-1}) > 0 \]  

whenever \( f_{L_m,A_{m-1}}(\bar{L}_m,\bar{a}_{m-1}) > 0 \).  

(5)

However, we note that under parametric models (2) positivity is not strictly necessary.

Robins [1994, 1997, 2004] has shown that \( \gamma^* = (\gamma_{01}^*, \ldots, \gamma_{(K-1)K}^*) \), the vector of all blip functions \( \gamma^*_{mk}(\bar{l}_m,\bar{a}_m) \) with \( m < k \leq K \), is nonparametrically identified and described how to consistently estimate the parameter \( \psi^* \) of a parametric SNMM (2) under the assumption that there are no unobserved confounders, i.e.

\[ Y_k(\bar{a}_{m-1},\bar{g}_m) \perp \perp A_m|\bar{A}_{m-1} = \bar{a}_{m-1},\bar{L}_m \forall k > m, \bar{a}_{m-1}. \]  

In this paper, we will instead make the parallel trends assumption

\[ E[Y_k(\bar{a}_{m-1},\bar{g}_m) - Y_{k-1}(\bar{a}_{m-1},\bar{g}_m)|\bar{A}_m = \bar{a}_{m-1},\bar{L}_m] = \]  

\[ E[Y_k(\bar{a}_{m-1},\bar{g}_m) - Y_{k-1}(\bar{a}_{m-1},\bar{g}_m)|\bar{A}_m = (\bar{a}_{m-1},g(\bar{L}_m,\bar{a}_{m-1})),\bar{L}_m] \]  

\( \forall k > m. \)  

(6)

When \( k = m + 1 \), this assumption reduces to

\[ E[Y_{m+1}(\bar{a}_{m-1},g_m) - Y_m|\bar{A}_m = \bar{a}_m,\bar{L}_m] = E[Y_{m+1}(\bar{a}_{m-1},g_m) - Y_m|\bar{A}_m = (\bar{a}_{m-1},g_m(\bar{L}_m,\bar{a}_{m-1})),\bar{L}_m]. \]  

(7)

The time-varying conditional parallel trends assumption states that, conditional on observed covariate history through time \( m \) and treatment history through time \( m - 1 \), the expected counterfactual outcome trends under strategy \( g \) from time \( m \) onwards do not depend on whether the treatment actually received at time \( m \) would have been assigned under \( g \). Note that the substantive interpretation of this assumption depends on the particular strategy \( g \) for which it is made. Most DiD literature (with the exception of
Renson et al. [2023] makes parallel trends assumptions relative to the untreated $g = 0$ regime. We compare our parallel trends assumption to others in the DiD literature in Section 8.

Remark 5. **No conditioning on immediately prior outcomes** Note that we have defined $\bar{L}_m$ to not include $Y_m$. If it did, then when $k = m + 1$ the parallel trends assumption would imply that there is no unobserved confounding, which we do not wish to assume as then the estimators we introduce would not be needed. This means we cannot adjust for the most recent outcome or estimate effect heterogeneity conditional on the most recent outcome with the methods in this paper. However, $\bar{Y}_{m-1}$ is included in $\bar{L}_m$.

Remark 6. **Parallel trends and CDEs** The parallel trends assumption may become more plausible under a CDE intervention, which is a possible motivation for considering CDEs. Suppose an investigator is interested in effects of Medicaid expansion on some outcome that is impacted both by Medicaid and the minimum wage. Given that states that expand Medicaid in a given year are also more likely to later increase the minimum wage, future minimum wage increases can lead to violations of the parallel trends assumption in an analysis where Medicaid is the sole treatment. However, by estimating the joint effect of Medicaid expansion and no future minimum wage expansion compared to never expanding Medicaid or increasing the minimum wage (i.e. the controlled direct effect of Medicaid expansion setting minimum wage increases to 0), the particular threat to the parallel trends assumption posed by future minimum wage increases is eliminated.

3 Identification and g-Estimation of SNMMs Under Parallel Trends

3.1 Nonparametric Identification

For any $m = 0, ..., K - 1$, let $H_m(\gamma)$ denote a vector with components

$$H_{mk}(\gamma) = Y_k - \sum_{j=m}^{k-1} \gamma_{jk}(\bar{L}_j, \bar{A}_j)$$

for $k = m + 1, ..., K$. Here, the $\gamma_{jk}(\bar{L}_j, \bar{A}_j)$ are arbitrary functions of covariate and treatment history not necessarily equal to the true blip functions $\gamma_{jk}^*(\bar{L}_j, \bar{A}_j)$. Robins [1994, 2004] showed that for every $m = 0, ..., K - 1$, $H_m(\gamma^*)$ (i.e. $H_m(\gamma)$ evaluated at the true blip function) has the following important
property:

\[ E[H_m(\gamma^*)|\bar{L}_m, \bar{A}_m] = E[\sum_{m+1}^{m} (\bar{A}_{m-1}, \bar{g}_m)|L_m, \bar{A}_m]. \] \hspace{1cm} (9)

By (9) and the time-varying conditional parallel trends assumption (6), it follows that for all \( m = 0, ..., K - 1 \) and \( k = m + 1, ..., K \),

\[ E[H_{mk}(\gamma^*) - H_{m,k-1}(\gamma^*)|\bar{A}_m = \bar{a}_m, \bar{L}_m] = E[H_{mk}(\gamma^*) - H_{m,k-1}(\gamma^*)|\bar{A}_m = (\bar{a}_{m-1}, \bar{g}_m(\bar{L}_m, \bar{a}_{m-1})), \bar{L}_m]. \] \hspace{1cm} (10)

That is, given the true blip function, the observable quantity \( H_{mk}(\gamma^*) \) behaves like the counterfactual quantity \( Y_k(\bar{A}_{m-1}, \bar{g}_m) \) in that its conditional trend does not depend on \( A_m \). We can exploit this property to establish identification and construct multiply robust estimating functions for \( \gamma^* \).

**Theorem 1.** Under (4), (5), and (6),

(i) \( \gamma^* \) is identified from the joint distribution of \((\bar{L}_{K-1}, \bar{A}_{K-1}, Y_K)\) as the unique solution to

\[ E \left\{ \sum_{m=0}^{K-1} \sum_{k=m+1}^{K} [s_{mk}(\bar{L}_m, \bar{A}_m) - E\{s_{mk}(\bar{L}_m, \bar{A}_m)|\bar{L}_m, \bar{A}_{m-1}\}] \{H_{mk}(\gamma) - H_{m,k-1}(\gamma)\} \right\} = 0 \] \hspace{1cm} (11)

where \( s_{mk}(\bar{L}_m, \bar{a}_m) \) is an arbitrary function of \((\bar{L}_m, \bar{a}_m)\) for \( m = 0, ..., K \) and \( k = m + 1, ..., K \).

(ii) Let \( \tilde{E}\{s_{mk}(\bar{L}_m, \bar{A}_m)|\bar{L}_m, \bar{A}_{m-1}\} \) be a conditional expectation taken with respect to some density/mass function \( \tilde{f}(A_m|\bar{L}_m, \bar{A}_{m-1}) \) not necessarily equal to the true density \( f(A_m|\bar{L}_m, \bar{A}_{m-1}) \). Similarly, for all \( m = 0, ..., K - 1 \) and \( k = m + 1, ..., K \), let \( \tilde{E}\{H_{mk}(\gamma) - H_{m,k-1}(\gamma)|\bar{L}_m, \bar{A}_{m-1}\} \) be some function not necessarily equal to \( E\{H_{mk}(\gamma) - H_{m,k-1}(\gamma)|\bar{L}_m, \bar{A}_{m-1}\} \).

Then \( \gamma^* \) satisfies

\[ E \left( \sum_{m=0}^{K-1} \sum_{k=m+1}^{K} [s_{mk}(\bar{L}_m, \bar{A}_m) - \tilde{E}\{s_{mk}(\bar{L}_m, \bar{A}_m)|\bar{L}_m, \bar{A}_{m-1}\}] \times [H_{mk}(\gamma^*) - H_{m,k-1}(\gamma^*) - \tilde{E}\{H_{mk}(\gamma^*) - H_{m,k-1}(\gamma^*)|\bar{L}_m, \bar{A}_{m-1}\} \right) = 0 \] \hspace{1cm} (12)

for any arbitrary \( s_{mk}(\bar{L}_m, \bar{a}_m) \), if for any \( m = 0, ..., K - 1 \) and \( k = m + 1, ..., K \), either

(a) \( \tilde{E}\{H_{mk}(\gamma^*) - H_{m,k-1}(\gamma^*)|\bar{L}_m, \bar{A}_{m-1}\} = E\{H_{mk}(\gamma^*) - H_{m,k-1}(\gamma^*)|\bar{L}_m, \bar{A}_{m-1}\} \), or

(b) \( \tilde{f}(A_m|\bar{L}_m, \bar{A}_{m-1}) = f(A_m|\bar{L}_m, \bar{A}_{m-1}) \).
Thus, (12) is a multiply robust estimating function.

The proof can be found in Appendix A.

3.2 Estimation

Theorem 1 establishes nonparametric identification of time-varying heterogeneous effects (1) under a parallel trends assumption conditional on time-varying covariates. In practice, estimation would proceed by first specifying a model (2) with finite dimensional parameter \( \psi^* \).

To construct an estimator of \( \psi^* \), we shall need to estimate multiple unknown nuisance functions. In what follows, for any \( m = 0, \ldots, K \) we define the vector \( H^\dagger_{m,k}(\gamma) \), which has components:

\[
H^\dagger_{m,k}(\psi) \equiv H_{m,k}(\psi) - H_{m,k-1}(\psi).
\]

Also, for all \( m = 0, \ldots, K - 1 \), let

\[
\nu^*_{m}(\bar{l}_m, \bar{a}_m) \equiv E\{H^\dagger_{m}(\psi^*)|\bar{L}_m = \bar{l}_m, \bar{A}_m = \bar{a}_m\}.
\]

We will assume from here on that either \( A_m \) is Bernoulli or \( s_{mk}(\bar{l}_m, \bar{a}_m) \) from the previous section is linear in \( a_m \); see Remark 9 for a discussion of the general case. Then we must estimate \( \nu^*_{m}(\bar{l}_m, \bar{a}_m) \) and \( \pi^*_{m}(\bar{l}_m, \bar{A}_m) \equiv E(A_m|\bar{L}_m, \bar{A}_m) \) (hereafter nuisance functions). We will consider state-of-the-art cross-fit doubly robust machine learning (DR-ML) estimators of \( \psi^* \) in which the nuisance functions are estimated by arbitrary machine learning algorithms chosen by the analyst [Chernozhukov et al., 2018, Smucler et al., 2019].

In what follows, \( P_N \) denotes the sample average and \( P \) is the true data generating law. If not stated otherwise, the expectation \( E[\cdot] \) is evaluated at \( P \). For any function \( f(O) \), we let \( \|f\| \) denote \( \{\int f(O)^2dP(O)\}^{1/2} \); we also use \( \|\cdot\|_2 \) to denote the Euclidean norm.

Algorithm 1 computes our cross fit estimator \( \hat{\psi}^{cf} \), where \( s = (s_0, \ldots, s_{K-1}) \) is a user chosen vector of conformable vector functions. Note that while we have described the algorithm for two folds, it is straightforward to generalize to greater than two.

Remark 7. Handling nuisance functions that contain the parameter of interest Because step 2 of the cross-fitting procedure estimates conditional expectations of functions of \( \psi \), the estimator \( \hat{\psi}^{(1)} \) must, in general, be solved iteratively and might be difficult to compute. Remark 20 in Appendix 5 of
Algorithm 1: Implementation of a cross-fit DR-ML estimator

1. Randomly split the $N$ study subjects into two parts: an estimation sample of size $n$ and a training (nuisance) sample of size $n_{tr} = N - n$ with $n/N \approx 1/2$. Without loss of generality we shall assume that $i = 1, \ldots, n$ corresponds to the estimation sample.

2. Applying machine learning methods to the training sample data, construct estimators $\hat{\pi}^{(-1)}$ and $\hat{\nu}^{(-1)}$ of $\pi^*$ and $\nu^*$, where $\pi^* \equiv (\pi_0, \ldots, \pi_{K-1})$, $\mu^* \equiv (\mu_0, \ldots, \mu_{K-1})$ and likewise for $\hat{\pi}^{(-1)}$ and $\hat{\nu}^{(-1)}$.

3. Consider the estimating function

$$U_m(O; \psi, s_m, \pi, \nu) = s_m(T_m, A_{m-1}) \{A_m - \pi_m(T_m, A_{m-1})\} \{H_m^\dagger(\psi) - \nu_m(T_m, A_{m-1})\}$$

where $s_m(T_m, A_{m-1})$, $m = 0, \ldots, K$ is an arbitrary $p \times (K - m)$-dimensional function of the covariate and treatment history. Compute $\hat{\psi}^{(1)}$ from the $n$ subjects in the estimation sample as the (assumed unique) solution to vector estimating equations

$$\mathbb{P}_N \left\{ \sum_{m=0}^K U_m(\psi, s_m, \hat{\pi}^{(-1)}, \hat{\nu}^{(-1)}) \right\} = 0.$$

4. Next, compute $\hat{\psi}^{(2)}$ just as $\hat{\psi}^{(1)}$, but with the training and estimation samples reversed.

5. Finally, the cross fit estimate $\hat{\psi}^{cf}$ is $(\hat{\psi}^{(1)} + \hat{\psi}^{(2)})/2$.

Liu et al. [2021] discusses strategies for reducing the computational burden. If parameters $\psi_{mk}$ and $\psi_{m'k}$ of blip functions $\gamma_{mk}^\alpha$ and $\gamma_{m'k}^\alpha$ are variation independent for $m \neq m'$, then it is possible to backwards recursively estimate $\psi_{mk}$ given estimates $\hat{\psi}_{m'k}$ for $m' > m$. If, for each $j$ and $k$, $\gamma_{jk}(\tilde{L}_j, \tilde{A}_j; \psi_{jk})$ is further assumed to be linear in $\psi_{jk}$, i.e. $\gamma_{jk}(\tilde{L}_j, \tilde{A}_j; \psi_{jk}) = \psi_{jk}^T R_{jk}$ for a given vector transformation $R_{jk}(\tilde{L}_j, \tilde{A}_j)$, then it is possible to estimate $E[H_{mk}(\psi^*) - H_{m,k-1}(\psi^*)|\tilde{L}_m, \tilde{A}_{m-1}]$ noting that

$$E[H_{mk}(\psi_{mk}; \hat{\psi}_{m+1,k}) - H_{m,k-1}(\psi_{mk}; \hat{\psi}_{m+1,k-1})|\tilde{L}_m, \tilde{A}_{m-1}] = E[(Y_k - \sum_{j=m+1}^k \gamma_{jk}^T (\tilde{L}_j, \tilde{A}_j; \hat{\psi}_{jk})) - (Y_{k-1} - \sum_{j=m+1}^{k-1} \gamma_{jk-1}^T (\tilde{L}_j, \tilde{A}_j; \hat{\psi}_{jk-1}))|\tilde{L}_m, \tilde{A}_{m-1}]$$

$$- \psi_{mk}^T E[R_{mk}|\tilde{L}_m, \tilde{A}_{m-1}] + \psi_{mk-1}^T E[R_{mk-1}|\tilde{L}_m, \tilde{A}_{m-1}]$$

where $\hat{\psi}_{m+1,k}$ denotes $(\hat{\psi}_{m+1,1}, \ldots, \hat{\psi}_{m+1,k})$; in practice, population expectations would need to be replaced by estimates. The resulting estimator never requires estimation of a conditional expectation of a function of a yet to be estimated $\psi_{mk}$ and is thus straightforward to compute. Let $\hat{\psi}^{(1)}$ denote the parameter estimate obtained from the cross-fitting procedure when $\psi_{mk}(\tilde{l}_m, \tilde{a}_{m-1}; \psi^*)$ is estimated as described above. To improve efficiency, one can then directly apply machine learning methods to construct an estimator of the nuisance pseudo-outcome regression function $\hat{v}_m(k, \tilde{l}_m, \tilde{a}_{m-1}; \hat{\psi}^{(1)})$ with $\hat{\psi}^{(1)}$ plugged in to construct the
pseudo-outcomes, then construct a new cross-fit estimator \( \tilde{\psi}^{(1)} \) using \( \hat{v}_m(k, \hat{\lambda}_m, \hat{\pi}_{m-1}; \tilde{\psi}^{(1)}) \). See Lewis and Syrgkanis [2020] for a related approach to estimation enabling lasso estimation of sparse high dimensional blip function parameters. If there is insufficient data to estimate variation independent blip function parameters at each time point, analysts might consider adding additional parametric restrictions, which we discuss in Remarks 8 and 10 below.

The following theorem provides the basis of inference of \( \psi^* \)

\[ \text{Theorem 2. Suppose that the following conditions hold:} \]

1. The map \( \psi \mapsto \mathbb{P} \left\{ \sum_{m=0}^{K} U_m(\psi, s_m, \pi, \nu) \right\} \) is differentiable at \( \psi^* \) uniformly in \( (\pi, \nu) \).

2. The matrix

\[
V(\psi^*, s, \pi^*) = -E \left\{ \sum_{m=0}^{K} s_m(\mathcal{L}_m, \mathcal{A}_{m-1}) \{A_m - \pi^*_m(\mathcal{L}_m, \mathcal{A}_{m-1})\} \frac{\partial}{\partial \psi} H^\dagger_m(\psi) \big|_{\psi = \psi^*} \right\}
\]

is nonsingular.

3. For \( m = 0, \ldots, K \) and for a selected \( p \times (K - m) \) function \( s_m(\mathcal{L}_m, \mathcal{A}_{m-1}) \), the maximum absolute value of any element in \( s_m(\mathcal{L}_m, \mathcal{A}_{m-1}) \) is upper bounded by \( C \) with probability one, where \( C < \infty \).

4. \( \hat{\psi}^{(s)} - \psi^* = o_P(1), \quad \| \hat{\pi}^{(-s)}_m - \pi^*_m \| = o_P(1) \) and \( \| \hat{\nu}^{(-s)}_{mk} - \nu^*_{mk} \| = o_P(1) \) for \( m = 0, \ldots, K, \quad k = m + 1, \ldots, K \) and \( s = 1, 2 \).

5. For \( m = 0, \ldots, K - 1, \quad k = m + 1, \ldots, K \) and \( s = 1, 2, \)

\[
\| \hat{\pi}^{(-s)}_m - \pi^*_m \| \quad \| \hat{\nu}^{(-s)}_{mk} - \nu^*_{mk} \| = o_P(N^{-1/2}).
\]

Then \( \hat{\psi}_{cf} \) satisfies the expansion

\[
\sqrt{N}(\hat{\psi}_{cf} - \psi^*) = V(\psi^*, s, \pi^*)^{-1} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \sum_{m=0}^{K} U_m(O_i; \psi^*, s_m, \pi^*, \nu^*) + o_P(1).
\]

Hence we have shown that under fairly standard conditions in semiparametric causal inference, \( \hat{\psi}_{cf} \) is an asymptotically linear estimator of \( \psi^* \); furthermore, it can be straightforwardly shown that its asymptotic variance is equal to the variance of its influence function.

It follows from (9) and Theorems 3 and 4, that we can also construct regular and asymptotically linear (RAL) plug-in estimators of other quantities of interest using \( \gamma(\hat{\psi}_{cf}) \). \( \mathbb{P}_N[H_{0k}(\hat{\psi}_{cf})] \) is a RAL
estimator for $E[Y_k(g)]$. $\mathbb{P}_N[H_{mk}(\hat{\psi}^{cf})]$ is a RAL estimator for $E[Y_k(\hat{A}_{m-1}, \hat{g}_m)]$. $\mathbb{P}_N^B[H_{mk}(\hat{\psi}^{cf})]$ is a RAL estimator for $E[Y_k(\hat{A}_{m-1}, \hat{g}_m) | (\hat{A}_{m-1}, \hat{L}_m) \in B]$ where $\mathbb{P}_N^B[\cdot]$ denotes sample average among subjects with $(\hat{A}_{m-1}, \hat{L}_m) \in B$ and $B$ an event in the sample space of treatment and covariate history with positive probability. For example, $B$ could be the event that there was no treatment through $m-1$ and the average value of another covariate through time $m$ exceeds some threshold. Of course, we can estimate the effect in the treated for a particular history $(\hat{A}_m, \hat{L}_m) = (\hat{a}_m, \hat{l}_m)$ directly by $\gamma_{mk}(\hat{I}_m, \hat{a}_m; \hat{\psi}^{cf})$.

**Remark 8. Parametric nuisance models** Alternatively, one can specify parametric nuisance models for $\pi_m^*(\bar{L}_m, \bar{A}_{m-1})$ and $v_m^*(\bar{L}_m, \bar{A}_{m-1}; \psi^*)$, respectively, and forego cross fitting. Under the assumptions of Theorem 2, if at each $m = 0, \ldots, K - 1$ and $k = m+1, \ldots, K$, either a model for $\pi_m^*(\bar{L}_m, \bar{A}_{m-1})$ or a model for $v_m^*(\bar{L}_m, \bar{A}_{m-1}; \psi^*)$ is correctly specified, then the corresponding estimator based on the aforementioned estimating functions is consistent and asymptotically normal under standard regularity conditions [Newey and McFadden 1994] Moreover, confidence intervals for $\psi^*$ and other derived quantities of interest can be computed via the nonparametric bootstrap.

**Remark 9. Non-linear effects** In the general case where $A_m$ is not necessarily binary or the treatment effect is non-linear in $A_m$, multiply robust estimation remains possible under a model for the conditional density/mass function $f(A_m|\bar{L}_m, \bar{A}_{m-1})$. Moreover for cases where the treatment effect is linear in some vector of transformations of $A_m$, then one can model the conditional mean of each component of the vector.

**Remark 10. Closed form estimation for linear models** Suppose the blip model is linear in $\psi$ (i.e. $\gamma_{mk}(\bar{a}_m, \bar{l}_m; \psi^*) = a_m \psi^T R_{mk}(\bar{a}_m-1, \bar{l}_m)$ for $R_{mk}(\bar{a}_m-1, \bar{l}_m)$ some transformation of history through time $m$ in the dimension of $\psi^*$) and the nuisance model $v_{mk}(\bar{l}_m, \bar{a}_m-1; \phi)$ is specified to be linear in $\phi$ (i.e. $E[H_{mk}(\psi^*) - H_{mk-1}(\psi^*)|\bar{L}_m, \bar{A}_{m-1}] = \phi^TD_{mk}(\bar{a}_m-1, \bar{l}_m)$ for $D_{mk}(\bar{a}_m-1, \bar{l}_m)$ some transformation of history through time $m$ in the dimension of $\phi$). Then the doubly robust estimator $(\hat{\psi}, \hat{\phi})$ is available in closed form (without cross-fitting) as

$$
(\hat{\psi}, \hat{\phi})^T = \left[ \sum_i \sum_{k>m} (Y_{ik} - Y_{ik-1}) \begin{pmatrix} s_{im}X_{im} \\ D_{imk} \end{pmatrix} \right] \left[ \sum_i \sum_{k>m} (V_{imk} - V_{imk-1}, D_{imk}) \begin{pmatrix} s_{im}X_{im} \\ D_{imk} \end{pmatrix} \right]^{-1},
$$

where $V_{imk} = \sum_{j=m}^{k-1} A_{ij} R_{jk}$, $X_{im} = A_{im} - \bar{\pi}_{im}(\bar{l}_m, \bar{A}_{im-1})$, and $s_{im}$ is the usual index function with dimension equal to the dimension of $\psi$. 

14
4 Multiplicative SNMMs

The SNMM framework also readily handles multiplicative effects when the parallel trends assumption is assumed to hold on the additive scale, a scenario that has been discussed in the DiD literature [Ciani and Fisher, 2019]. Define the multiplicative causal contrasts

\[ e^{\gamma_{mk}^g}\bar{l}_m, \bar{a}_m} = \frac{E[Y_k(\bar{a}_m, \bar{g}_m)|A_m = \bar{a}_m, L_m = \bar{l}_m]}{E[Y_k(\bar{a}_{m-1}, \bar{g}_m)|A_m = \bar{a}_m, L_m = \bar{l}_m]} \] (14)

for \( k > m \). \( \gamma_{mk}^g\bar{l}_m, \bar{a}_m) \) is the average multiplicative effect at time \( k \) among units with history \((\bar{A}_m = \bar{a}_m, \bar{L}_m = \bar{l}_m)\) of receiving treatment \( a_m \) at time \( m \) and then following \( g \) thereafter compared to following \( g \) at time \( m \) and thereafter.

A parametric multiplicative SNMM imposes functional forms on the multiplicative blip functions \( \gamma_{mk}^g(\bar{A}_m, \bar{L}_m) \) for each \( k > m \), i.e.

\[ e^{\gamma_{mk}^g}\bar{l}_m, \bar{a}_m) = e^{\gamma_{mk}^g}(\bar{a}_m, \bar{l}_m; \psi_*^g) \] (15)

where \( \psi_*^g \) is an unknown parameter vector and \( \gamma_{mk}^g(\bar{a}_m, \bar{l}_m; \psi_*^g) \) is a known function.

We make the same parallel trends assumption [6] as in the additive setting. Let

\[ H_{mk}^X(\gamma) \equiv Y_k \exp\{- \sum_{j=m}^{k-1} \gamma_{jk}(\bar{A}_j, \bar{L}_j)\} \text{ for } k > m \text{ and } H_{tt}^X \equiv Y_t. \] (16)

Robins [1994, 2004] showed that \( H_{mk}^X(\gamma_{g}^{g\times}) \) evaluated at the true blip function has the following important properties for all \( k > m \):

\[ E[H_{mk}^X(\gamma_{g}^{g\times})|\bar{L}_m, \bar{A}_m] = E[Y_k(\bar{A}_{m-1}, \bar{g}_m)|\bar{L}_m, \bar{A}_m] \] (17)

\[ E[H_{ot}^X(\gamma_{g}^{g\times})] = E[Y_k(\bar{g})]. \] (18)

By [17] and the time-varying conditional parallel trends assumption [6], it follows that

\[ E[H_{mk}^X(\gamma_{g}^{g\times}) - H_{mk-1}^X(\gamma_{g}^{g\times})|\bar{A}_m = \bar{a}_m, \bar{L}_m] = \]

\[ E[H_{mk}^X(\gamma_{g}^{g\times}) - H_{mk-1}^X(\gamma_{g}^{g\times})|\bar{A}_m = (\bar{a}_{m-1}, \bar{g}(\bar{L}_{m-1}, \bar{a}_{m-1})), \bar{L}_m] \] (19)

\( \forall k > m. \)
That is, given the true multiplicative blip function, the quantity $H_{mk}^\times (\gamma^g \times \ast)$ behaves like the counterfactual quantity $Y_k(\bar{A}_{m-1}, g, \bar{g})$ in that its conditional expected trend does not depend on $A_m$. We can again exploit this property to identify and construct doubly robust estimating equations for $\gamma^g \times \ast$ and various derivative quantities of interest. Identification and estimation theorems and proofs are identical to Section (3) with $H_{mk}^\times$ in place of $H_{mk}$, $\gamma^g \times$ in place of $\gamma^g$, $\psi \times$ in place of $\psi$, and (17) in place of (6).

5 Optimal Regime SNMMs

In this section, we assume that parallel trends (6) holds for all $g \in \mathcal{G}$, for $\mathcal{G}$ the set of all treatment rules. Let $\bar{U}_m$ denote a time-varying unobserved confounder, possibly multivariate and containing continuous and/or discrete components. Suppose the causal ordering at time $m$ is $(Y_m, U_m, L_m, A_m)$. We assume that were we able to observe $\bar{U}_m$ in addition to the observed variables we could adjust for all confounding, i.e.

\[ U \perp \perp Y_{m+1}(\bar{a}_{K-1}) | \bar{L}_m, \bar{A}_m = \bar{a}_{m-1}, \bar{U}_m \forall m < K, \bar{a}_{K-1}. \] (20)

We further assume

No Additive Effect Modification by $U$:

\[ \gamma_{mk}^g(\bar{l}_m, \bar{a}_m) = \gamma_{mk}^g(\bar{l}_m, \bar{a}_m, \bar{u}_m) \equiv E[Y_k(\bar{a}_m, g_{m+1}) - Y_k(\bar{a}_{m-1}, g_m)|\bar{A}_m = \bar{a}_m, L_m = \bar{l}_m, \bar{U}_m = \bar{u}_m] \] (21)

for all regimes $g \in \mathcal{G}$. In Appendix C, we show that (21) practically, though not strictly mathematically, follows from (6) holding for all $g \in \mathcal{G}$.

Suppose we want to maximize the expectation of some utility $Y = \sum_{k=0}^{K} \tau_k Y_k$ which is a weighted sum of the outcomes we observe at each time step with weights $\tau_k$. Let $Y(g) = \sum_{k=0}^{K} \tau_k Y_k(g)$ denote the counterfactual value of the utility under treatment regime $g$. We want to find $g^{opt} = \arg \max_{g} E[Y(g)]$. Under (20) and (21), it follows from results in Robins [2004] that $g^{opt}$ is given by the following backward recursion:

\[ g_{K-1}^{opt}(\bar{L}_{K-1}, \bar{A}_{K-2}) = \arg \max_{a_{K-1}} E[Y_K(\bar{A}_{K-2}, a_{K-1})|\bar{L}_{K-1}, \bar{A}_{K-2}] \]
\[ g_{m-1}^{opt}(\bar{L}_{m-1}, \bar{A}_{m-2}) = \arg \max_{a_{m-1}} \sum_{j > m} \tau_j E[Y_j(\bar{A}_{m-2}, a_{m-1}, g_{m}^{opt})|\bar{L}_{m-1}, \bar{A}_{m-2}] \]

That is, the optimal treatment rule at each time step is the rule that maximizes the weighted sum of expected
future counterfactual outcomes assuming that the optimal treatment rule is followed at all future time steps. Define the optimal regime blip function

\[ \gamma_{mk}^{opt}(\bar{l}_m, \bar{a}_m) \equiv E[Y_k(\bar{a}_m, g_{m+1}^{opt}) - Y_k(\bar{a}_{m-1}, 0, g_{m+1}^{opt}) | \bar{A}_m = \bar{a}_m, \bar{L}_m = \bar{l}_m]. \]  

(22)

This is the conditional effect of treatment level \( a_m \) followed by the optimal regime compared to treatment level 0 followed by the optimal regime in those receiving treatment level \( a_m \). Under the assumption that unobserved confounders are not effect modifiers, (22) does not depend on \( A_m \), i.e.

\[ \gamma_{mk}^{opt}(\bar{l}_m, \bar{a}_m) = E[Y_k(\bar{a}_m, g_{m+1}^{opt}) - Y_k(\bar{a}_{m-1}, 0, g_{m+1}^{opt}) | \bar{A}_m = \bar{a}_{m-1}, \bar{L}_m = \bar{l}_m]. \]  

(23)

In terms of this blip function, then,

\[ g_{mk}^{opt}(L_m, \bar{A}_{m-1}) = \arg\max_{a_m} \sum_{j > m} \tau_j \gamma_{mj}^{opt}(L_m, \bar{A}_{m-1}, a_m). \]  

(24)

Under parametric model \( \gamma_{mj}^{opt}(\bar{L}_m, \bar{A}_{m-1}, a_m) = \gamma_{mj}^{opt}(\bar{L}_m, \bar{A}_{m-1}, a_m; \psi_{g_{opt}}^{*}) \), we can estimate \( \psi_{g_{opt}}^{*} \) via g-estimation as follows. First, define

\[ H_{mk}^{opt}(\psi_{g_{opt}}) \equiv Y_k + \sum_{j = m}^{K} \tau_j \gamma_{jk}^{opt}(L_j, \bar{A}_{j-1}, \arg\max_{a_j} \sum_{r = j}^{K} \tau_r \gamma_{jr}^{opt}(L_j, \bar{A}_{j-1}, a_j; \psi_{g_{opt}}^{*}) - \gamma_{jk}^{opt}(\bar{L}_j, \bar{A}_{j-1}, A_j; \psi_{g_{opt}}^{*})). \]

Given (23), Lemma 3.2 of Robins 2004 implies that, evaluating at the true parameter value,

\[ E[H_{mk}^{opt}(\gamma_{g_{opt}}^{*}) | \bar{L}_m, \bar{A}_m] = E[Y_k(\bar{a}_{m-1}, g_{opt}) | \bar{L}_m, \bar{A}_m]. \]  

(25)

Intuitively, this is because the effect of observed treatment compared to 0 treatment (followed by the optimal regime) is subtracted off, then the effect of optimal treatment relative to 0 treatment (again followed by the optimal regime) is added back on. Once again, we have the crucial property that \( H_{mk}^{opt}(\psi_{g_{opt}}^{*}) \) behaves in conditional expectation like the counterfactual \( Y_k(\bar{a}_{m-1}, g_{opt}) \). Therefore, in particular, \( H_{mk}^{opt}(\psi_{g_{opt}}^{*}) \) satisfies parallel trends, i.e.

\[ E[H_{mk}^{opt}(\psi_{g_{opt}}^{*}) - H_{mk-1}^{opt}(\psi_{g_{opt}}^{*}) | \bar{A}_m = \bar{a}_m, \bar{L}_m] = 0 \]

\[ E[H_{mk}^{opt}(\psi_{g_{opt}}^{*}) - H_{mk-1}^{opt}(\psi_{g_{opt}}^{*}) | \bar{A}_{m-1} = \bar{a}_{m-1}, \bar{L}_m]. \]

Thus, again applying the logic of g-estimation, we can construct consistent and asymptotically normal estimators of \( \psi_{g_{opt}}^{*} \) as in Section 3.2 with \( H_{mk}^{opt}(\psi_{g}) \) in place of \( H_{mk}(\psi_{g}) \). We can further consistently estimate the optimal
treatment rule as
\[
g^{\text{opt}}(\bar{L}_m, \bar{A}_{m-1}) = \underset{\gamma_m \in \mathbb{R}}{\text{argmax}} \sum_{j \geq m} \tau_j \gamma_j \psi(g^{\text{opt}}_{m}, \bar{L}_m, \bar{A}_{m-1}, a_m; \hat{\psi}_{\text{opt}}).
\] (26)

We can also consistently estimate the expected value of the counterfactual utility \(E[Y(g^{\text{opt}})]\) under the optimal regime by \(\hat{E}[Y(g^{\text{opt}})] = P \sum_{k=0}^{K} \tau_k P[H_{\text{opt}}^k(\hat{\psi}_{\text{opt}})]\).

6 Real Data Applications

We illustrate our approach with applications to real data. We first fit an SNMM to model conditional effects of Medicaid expansion on county level uninsurance rates given past unemployment rates. This is a staggered adoption application highlighting the ability of SNMMs to characterize time-varying effect heterogeneity. We also fit an SNMM to model effects of floods on flood insurance take-up using data previously analyzed by [Gallagher 2014]. In this application, the blip function itself is directly of interest, as we estimate lasting effects of a ‘blip’ of exposure, i.e. a flood. Finally, we use an SNMM to estimate the effects of sustained changes in ‘growing degree days’ on crop profitability at the county level using data previously analyzed by [Deschênes and Greenstone 2012]. This example highlights the ability of SNMMs to estimate effects of sustained interventions and effects of continuous valued treatments when there is no shared baseline. We specified parametric nuisance models for all of these applications to illustrate simple implementations. Code and data for these analyses can be found at https://github.com/zshahn/did_snmm.

6.1 Impact of Medicaid Expansion on Uninsurance Rates

One of the Affordable Care Act’s primary goals was to provide increased insurance coverage through the expansion of Medicaid eligibility. Originally a nationwide policy, the choice to expand Medicaid using federal subsidies was ultimately left to individual states, leading to its staggered adoption beginning in 2014. We estimate how the effect of Medicaid expansion on county level uninsurance rates depends on county level unemployment rate and population. Table 1 summarizes treatment timing over the period 2014-2019 considered in the study. Because

| Year | # Counties in States First Expanding Medicaid |
|------|---------------------------------------------|
| 2014 | 1216                                        |
| 2015 | 113                                         |
| 2016 | 174                                         |
| 2017 | 64                                          |
| 2018 | 0                                           |
| 2019 | 151                                         |
| Never | 1421                                        |

Table 1: Staggered Medicaid adoption by county
this is a staggered adoption setting, we coded treatment as in Remark 1 with $A_m = 1$ only if a county’s state first expanded Medicaid in year $m$ and $A_m = 0$ otherwise. We also took $g = 0$ to be the untreated regime.

We specified the SNMM

$$E[Y_k - Y_k(0)|\bar{A}_m = (0_{m-1}, 1), \bar{L}_m] = \psi_{0mk} + \psi_{pop L_{mk}^{pop}} + \psi_{unemp L_{mk}^{unemp}},$$

which posits that the effect varies flexibly with time of expansion and time since expansion through parameters $\psi_{0mk}$ and linearly with the previous year’s unemployment rate and 2013 log population through parameters $\psi_{pop}$ and $\psi_{unemp}$. The parallel trends assumption that $E[Y_k(0) - Y_{k-1}(0)|\bar{A}_m = (0_{m-1}, 1), L_m] = E[Y_k(0) - Y_{k-1}(0)|\bar{A}_m = 0, \bar{L}_m]$ for all $m$ and $k$ states that future expected uninsurance rate trends under no expansion are similar for counties with similar covariate history regardless of whether their state expanded in year $m$. We specified nuisance models $logit^{-1}(Pr(A_m = 1|\bar{A}_{m-1} = 0, \bar{L}_m)) = \beta_{0m} + \beta_{1m}L_{mk}^{unemp} + \beta_{2m}L_{mk}^{pop} + \beta_{3m}L_{mk}^{unemp2} + \beta_{4m}(L_{mk}^{unemp})^2 + \beta_{5m}(L_{mk}^{pop})^2$ and $E[H_{mk}(\psi) - H_{m,k-1}(\psi)|\bar{A}_{m-1} = 0, \bar{L}_m] = \lambda_{0mk} + \lambda_{1mk}^{T}$. We obtained point estimates for the SNMM parameters using the closed form linear estimator from Remark 10 and estimated standard errors via bootstrap.

Figure 4 displays the expected time-varying heterogeneous effects of Medicaid expansion on uninsurance rates for expanding counties of median log population (10.2) at the first quartile versus the third quartile unemployment rate. Expansion was estimated to be more impactful in counties with higher unemployment rates. The estimated positive coefficient of log population $\psi_{pop} = 0.80$ (95% CI = [0.56, 1.04]) suggests that the impact of Medicaid expansion on uninsurance was greater (i.e. ‘more negative’) in areas with lower population.

### 6.2 Impact of Floods on Flood Insurance

Gallagher [2014] used a fixed effects regression model to look at effects of floods on flood insurance coverage at the county level. He argued that each county’s flood risk is constant over time. We fit a SNMM under the assumption of parallel trends in insurance coverage absent future floods in counties with similar flood history from 1958. We specified a parametric linear blip model $\gamma_m^*(\bar{L}_m, \bar{a}_m) = a_m(1, m - 1980, k - m, (k - m)^2, rate_{m-1})^T$ for $g = 0$, where $rate_{m-1}$ denotes the county’s proportion of flood years since 1958. We specified nuisance models $E[A_m|\bar{A}_{m-1}, \bar{L}_m] = \beta_{0m} + \beta_{1m}rate_{m-1}$ and $E[H_{mk}(\psi) - H_{m,k-1}(\psi)|\bar{A}_{m-1}, \bar{L}_m] = \lambda_{0mk} + \lambda_{1m}rate_{m-1}$. We obtained blip model parameter estimates via the closed form linear estimators of Remark 10 and estimated standard errors via bootstrap. Each line in Figure 2 depicts the estimated effect of a flood occurring at its leftmost time point followed by no further floods on flood insurance uptake over the subsequent 15 years for a county with the median historical flood rate, i.e. $\gamma_m^*(rate_{m-1} = rate_{median}, a_m = 1; \hat{\psi})$ for $k \in m, \ldots, m + 15$. These quantities were directly extracted from our blip function estimates. We see that there is an initial surge in uptake followed by a steep decline, and the estimated initial surge is larger for more recent floods. We did
not find statistically significant effect heterogeneity as a function of historical county flood rate. Gallagher (2014) obtained qualitatively similar results and argued that most of the decline in the effect of a flood is due to residents forgetting about it as opposed to migration. It might be interesting to explore other blip model specifications, perhaps conditioning on further aspects of flood history such as years since previous flood or on average flood insurance premiums in the area.

6.3 Effects of Sustained Temperature Changes on Crop Profitability

Deschênes and Greenstone [2012] analyzed the relationship between growing degree days (GDD) and crop profits per acre at the county level using two way fixed effects regression. (GDD is essentially a composite measure of temperature, truncated at an upper limit each day over the growing season.) They used data from 2,342 counties collected every five years from 1987 through 2002. Two way fixed effects is now known to impose unwanted effect homogeneity assumptions [De Chaisemartin and d’Haultfoeuille, 2020], but de Chaisemartin et al. [2024b] point out that parallel trends assumptions from the existing DiD literature do not easily accommodate the continuous treatment in this application because counties do not share a baseline level. Furthermore, the exposure changes value repeatedly, another challenge in the DiD literature. (See Figure 3 for illustration.) We reanalyze the data to estimate effects of a sustained change in GDD on crop profits given history of growing degree days using an SNMM under our conditional parallel trends assumption.

Letting $A_m$ denote GDD at measurement $m$. We specify a parametric general regime SNMM

$$
\gamma_{m,k}(\bar{a}_m) = E[Y_k(\bar{a}_m, g) - Y_k(\bar{a}_{m-1}, g)|\bar{A}_m = \bar{a}_m]
$$

for $g$ the ‘keep treatment fixed’ regime \[3\]. \(27\) is a model for the effect of a sustained change in GDD to level $a_m$ starting in year $m$ on crop profits in year $k$ compared to permanently fixing GDD at $a_{m-1}$, given the history of GDD through time $m$. The parametric model allows this effect to depend on the new GDD level $a_m$, the prior GDD level $a_{m-1}$, the time $k - m$ since the change, and the time $m$ of the change. It imposes linearity assumptions and the constraint that the effect of the GDD change does not depend on GDD history beyond its prior level. Of course, alternative parametric models could be specified, and we note that the model specification can become arbitrarily flexible as sample size increases.

We make the parallel trends assumption \[6\] that for each time point $m$, in counties with similar GDD histories through $m - 1$, future counterfactual crop profit trends fixing GDD at its time $m - 1$ level are mean independent of actual GDD at $m$. We specified parametric nuisance models $E[A_m|\bar{A}_{m-1}] = \beta_{m}^T (1, A_0, A_{m-1}, A_0A_{m-1})$ and $E[H_{mk}(\psi) - H_{m,k-1}(\psi)|\bar{A}_{m-1}] = \lambda_{mk0} + \lambda_1 A_0 + \lambda_2 A_{m-1} + \lambda_3 A_0A_{m-1}$. We then estimated the blip model parameters using the closed form estimator from Remark \[10\] and estimated standard errors by bootstrap.
Under our assumptions, we can directly query our estimated SNMM to obtain effect estimates of interest. Table 2 contains estimated effects of a sustained increase of 1000 GDD in 1992 compared to keeping GDD at 1987 levels on crop profitability in 1997 and 2002 in counties with various levels of 1987 GDD. Effects were estimated to be larger when the increase was sustained for longer and in counties with higher pre-increase GDD levels.

| GDD_{1987} | \gamma_{1992,k}(a_{1992} = GDD_{1987} + 1000, a_{1987} = GDD_{1987}; \psi) | k=1997 | k=2002 |
|------------|--------------------------------------------------------------------------------|--------|--------|
| 1000       | 6.1 (-4.9, 17.2)                                                               | 3.7 (-9.8, 17.2) |
| 2000       | 6.3 (-0.7, 13.2)                                                               | 7.6 (-1.4, 16.5) |
| 3000       | 6.4 (2.0, 10.8)                                                                | 11.4 (4.6, 18.2) |
| 4000       | 6.5 (0.4, 12.6)                                                                | 15.3 (6.4, 24.1) |

Table 2: SNMM estimated effects of a sustained increase of 1000 GDD from 1987 to 1992 on crop profits per acre in 1997 and 2002 for various baseline GDD levels in 1987.

7 Sensitivity Analysis

Conditional parallel trends assumptions are strong and untestable, and sensitivity analysis for violations of the parallel trends assumption is therefore desirable. We adapt the approach to sensitivity analysis for unobserved confounding in SNMMs of Robins et al. [2000] and Robins [2004] to sensitivity analysis for non-parallel trends. We describe a general class of bias functions characterizing deviations from parallel trends given covariate history. For any particular bias function from this class, we provide a corresponding unbiased estimate of SNMM parameters assuming that the bias function is correctly specified. An analyst can then execute a sensitivity analysis by specifying a plausible range of bias functions (e.g. a grid of parameters covering a plausible range within a parametric subclass of bias functions) and producing the corresponding range of plausible effect estimates. This approach to sensitivity analysis is complementary to that developed by Rambachan and Roth [2023], as it allows deviations to depend on time-varying covariates and allows for sensitivity analysis of all SNMM parameters (e.g. those characterizing effect heterogeneity) and derived quantities.

Define

\[ c_{mk}(\bar{l}_m, \bar{a}_m) = \]
\[ E[Y_k(\bar{a}_{m-1}, g_{\bar{a}_m}) - Y_{k-1}(\bar{a}_{m-1}, g_{\bar{a}_m})]\mid \bar{l}_m = \bar{l}_m, \bar{A}_m = \bar{a}_m] - \]
\[ E[Y_k(\bar{a}_{m-1}, g_{\bar{a}_m}) - Y_{k-1}(\bar{a}_{m-1}, g_{\bar{a}_m})]\mid \bar{l}_m = (\bar{a}_{m-1}, g(\bar{l}_m, \bar{a}_{m-1}))]. \tag{28} \]

\[ c_{mk}(\bar{l}_m, \bar{A}_m) \] characterizes the magnitude of deviation from the parallel trends assumption \([6]\). It is a general function in that it allows deviations to depend on treatment and covariate history, treatment time \(m\), and outcome time \(k\). If parallel trends holds or \(a_m = g(\bar{l}_m, \bar{a}_{m-1})\), then \(c_{mk}(\bar{l}_m, \bar{a}_m) = 0\).
Given a bias function \( (28) \), define the bias adjusted version of the ‘blipped down’ quantity \( (8) \) as

\[
H_{mk}^{a}(\gamma^g) \equiv H_{mk}(\gamma^g) - c_{mk}^{g}(\bar{L}_m, \bar{A}_m),
\]

where \( H_{mk} \) is defined in \( (8) \).

**Lemma 1.** If \( (28) \) is correctly specified, then

\[
E[H_{mk}^{a}(\gamma^g) - H_{mk-1}(\gamma^g)|\bar{L}_m, \bar{A}_m] = E[H_{mk}^{a}(\gamma^g) - H_{mk-1}(\gamma^g)|\bar{L}_m, \bar{A}_{m-1}].
\]

**Proof.** See Appendix D.

Lemma 1 states that under correct specification of the bias function \( (28) \), the conditional expectation of \( H_{mk}^{a}(\gamma^g) - H_{mk-1}(\gamma^g) \) does not depend on \( A_m \). This is the same crucial property satisfied by \( H_{mk}(\gamma^g) - H_{mk-1}(\gamma^g) \) in \( (10) \) that enabled identification of \( \gamma^g \). Thus, it follows that identification and estimation of \( \gamma^g \) under bias function \( (28) \) may proceed exactly as identification and estimation of \( \gamma^g \) under the parallel trends assumption \( (6) \) except substituting \( H_{mk}^{a}(\gamma^g) - H_{mk-1}(\gamma^g) \) for \( H_{mk}(\gamma^g) - H_{mk-1}(\gamma^g) \). Future work should explore suitable parameterizations of the bias function for tractable and informative sensitivity analysis.

### 8 Relation to Other Work

#### 8.1 Estimands

Most time-varying DiD work emphasizes effects of so-called ‘staggered adoption’ strategies. In the staggered adoption setting, all units start at a common baseline level of treatment and then deviate from that baseline level at different times in a staggered fashion. In the simplest and most commonly considered case, the baseline treatment level is 0 or ‘untreated’, and units then initiate a binary treatment at different times. For times \( k > m \), interest centers on \( E[Y_k - Y_k(0)|\bar{A}_m = (0_{m-1}, 1)] \), the effect of starting treatment at time \( m \) compared to never starting treatment on the outcome at time \( k \) among units that actually started treatment at time \( m \). Callaway and Sant’Anna [2021] call these effects \( ATT(m, k) \). We mentioned in Remark 4 that these estimands are identified in terms of SNMM parameters as \( E[\gamma_{mk}^{g}(\bar{A}_m = (0_{m-1}, 1), \bar{L}_m)|\bar{A}_m = (0_{m-1}, 1)] \) for \( g = 0 \) and treatment coded as in Remark 1. Thus, SNMMs can target the same \( ATT(m, k) \) estimands as standard DiD approaches and also model their heterogeneity as a function of time-varying covariates.

Some work in the time-varying DiD literature has studied effects of interventions beyond initiation of a binary treatment. de Chaisemartin et al. [2024a] call these effects \( ATT(m, k) \). For example, consider effects of initial changes in treatment level when not all units begin at the same baseline treatment and treatment may be non-binary. Callaway and Sant’Anna [2021] call these effects \( ATT(m, k) \).
et al. [2024] consider effects of initiation of continuous valued treatments from a shared baseline treatment. SNMMs of continuous valued treatments under parallel trends assumptions conditional on treatment history can straightforwardly model these effects as well.

When treatment changes value repeatedly (e.g. switches on and off) in the data, ATT($m, k$) effects are marginal over post-initiation treatment patterns under the observational regime, similar to intention to treat effects in randomized trials with imperfect compliance. Just as per-protocol effects are of interest in trials with non-compliance, effects of sustained interventions are of interest in DiD settings. However, apart from [Renson et al. 2023], we are not aware of methods estimating these effects under parallel trends assumptions (see Section 3.4 of [Roth et al. 2023]). Effects of dynamic treatment strategies that may depend on covariates have also not been addressed under parallel trends except by [Renson et al. 2023].

In concurrent and cross-citing work, [Renson et al. 2023] showed that a variation on the g-formula identifies $E[Y(g)]$ under parallel trends assumption (6). They proposed inverse probability weighted, outcome-regression based, and doubly robust estimators for this functional distinct from our own. The primary advantage offered by SNMMs compared to their estimators is the ability to model time-varying effect heterogeneity. SNMMs further enable simultaneous estimation of effects of a range of interventions. For example, in our crop yield application in Section 6.3 we are able to jointly model the effects of a sustained change from any continuous valued GDD level to any other at any time assuming parallel trends (6) under the ‘keep treatment fixed’ regime (3). [Renson et al. 2023], however, only provide methods to estimate the effect of a single GDD trajectory at a time. The primary drawback of SNMMs relative to the methods of [Renson et al. 2023] is the requirement in practice to specify a parametric SNMM.

In other near concurrent work cross-citing our own, [Blackwell et al. 2024] consider estimation of controlled direct effects under parallel trends. However, they consider a different data structure lacking an intermediate outcome measurement and accordingly require different assumptions.

Finally, our work is the only work we are aware of that leverages parallel trends assumptions to estimate optimal dynamic regimes as in Section 5. While we also require the extremely strong assumption of no effect modification by unobserved confounders, the method is one of the few [Zhang and Tchetgen 2024, Han 2021] to enable reinforcement learning in the presence of unobserved confounding.

8.2 Parallel Trends Assumptions

Standard DiD parallel trends assumptions in the staggered adoption setting [Callaway and Sant’Anna 2021, de Chaisemartin et al. 2024a] are made relative to control groups defined in terms of their observed future treatment trajectories (most commonly, the never treated). Suppose larger values of a time-varying variable make treatment more likely to be initiated at the following time point. Then the future observed values of this variable in the never treated group will tend to be lower than the (counterfactual untreated) future values of the same
variable in the group initiating treatment at time \( m \), even if the observed trajectory of the variable is similar in the two groups through time \( m \). If the variable is further associated with trends in the counterfactual untreated outcome, those trends would therefore also differ between the never treated and initiating-at-\( m \) groups, thus violating the parallel trends assumption with respect to the never treated group. (Similar logic would apply to other control groups defined based on observed future treatment trajectories such as the ‘not yet treated’ [by time of outcome measurement] group. \cite{Callaway+SantAnna2021}.)

Our parallel trends assumption, by contrast, is not relative to a control group defined in terms of future observed treatments. In the staggered adoption setting when treatment is coded as in Remark 1, our parallel trends assumption\( (6) \) states that initiators at time \( m \) have the same expected future counterfactual untreated outcome trends as those who did not yet initiate at time \( m \) if they have the same covariate history through \( m \). Unlike the previous assumptions in the DiD literature (with the exception of \cite{Renson+2023}), this parallel trends assumption might hold even if time-varying covariates influence both treatment and trends. Thus, our parallel trends assumption will allow identification in some additional circumstances compared to standard time-varying DiD methods for initiation effects.

When we are interested in effects of interventions beyond initiation (e.g. blip effects, effects of sustained interventions, or CDEs) and the treatment might switch values multiple times, then our parallel trends assumption\( (6) \) imposes additional restrictions compared to the setting when treatment is coded as in Remark 1. Future expected trends are required to be equal in the treated and untreated given covariate and treatment history \textit{at all times}, not only when there has been no prior treatment. However, this is a natural generalization of the staggered adoption setting, as ‘no prior treatment’ is just a particular treatment history. The more general assumption is required for identification and estimation of effects of interventions involving treatments following initiation.

While the interpretation of the parallel trends assumption\( (6) \) depends on \( g \), it may be difficult to justify the assumption for one regime over another, especially if the regimes in question are complex. If a practitioner invokes\( (6) \) for an arbitrary complex regime of interest, a critic might argue that the practitioner is making the assumption out of convenience and effectively making the assumption for all \( g \). However, assuming parallel trends for \textit{all} regimes is a much stronger assumption than is usually made in the DiD literature and imposes restrictions on effect heterogeneity. In Appendix C, we show that parallel trends under all regimes (practically) implies that there is no effect modification on the additive scale by unobserved confounders.

9 Conclusion

To summarize, we have shown that SNMMs expand the set of causal questions that can be addressed under parallel trends assumptions. In particular, we have shown that additive and multiplicative general regime SNMMs are identified under time-varying conditional parallel trends assumptions, and we have provided estimators. Using SNMMs, it is possible to do many things that were not possible with standard DiD approaches, such as:
characterize effect heterogeneity as a function of time-varying covariates (e.g. unemployment in the Medicaid expansion analysis of Section 6.1), estimate the effect of one final blip of treatment (as in the flood analysis of Section 6.2), and other derived contrasts, estimate controlled direct effects, estimate effects of sustained interventions when treatment changes value in the data (as in the crop yields example of Section 6.3), and estimate time-varying effects of interventions on continuous valued treatments that do not share a baseline level (the crop example again). We have also explained how to estimate counterfactual expectations under a possibly complex dynamic regime as long as parallel trends holds with respect to that regime. Under the stronger assumption that unobserved confounders are not effect modifiers on the relevant scale, we have further shown that optimal dynamic treatment regimes are identified via optimal regime SNMMs. We hope these new capabilities can be put to use in a wide variety of applications.

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Appendix

A Proof of Theorem 1

Part (i)

\[ E[U_{mk}(s_m, \gamma^*)] = E[E[U_{mk}(s_m, \gamma^*)|\bar{L}_m, \bar{A}_m]] \]
\[ = E[E[H_{mk}(\gamma^*) - H_{mk-1}(\gamma^*)|\bar{L}_m, \bar{A}_m-1](s_m(k, \bar{L}_m, \bar{A}_m) - E[s_m(k, \bar{L}_m, \bar{A}_m)|\bar{L}_m, \bar{A}_m-1]) | \bar{A}_m = \bar{a}_m, \bar{L}_m] \]
\[ = E[E[H_{mk}(\gamma^*) - H_{mk-1}(\gamma^*)|\bar{L}_m, \bar{A}_m-1] \times \]
\[ E[s_m(k, \bar{L}_m, \bar{A}_m) - E[s_m(k, \bar{L}_m, \bar{A}_m)|\bar{L}_m, \bar{A}_m-1]|\bar{A}_m-1, \bar{L}_m] | \bar{A}_m = (\bar{a}_m - 1, g), \bar{L}_m] \]
\[ \text{by nested expectations} \]
\[ = 0 \]

The above establishes that the true blip functions are a solution to these equations. The proof of uniqueness follows from the two Lemmas below.

Lemma 2. Any functions \( \gamma \) that satisfy (11) for all \( s_m(k, \bar{l}_m, \bar{a}_m) \) also satisfy (10), i.e.

\[ E[H_{mk}(\gamma) - H_{mk-1}(\gamma)|\bar{A}_m = \bar{a}_m, \bar{L}_m] = E[H_{mk}(\gamma) - H_{mk-1}(\gamma)|\bar{A}_m = (\bar{a}_m - 1, g), \bar{L}_m] \forall k > m. \]

Proof. Since (11) must hold for all \( s_m(k, \bar{l}_m, \bar{a}_m) \), in particular it must hold for

\[ s_m(k, \bar{l}_m, \bar{a}_m) = E[H_{mk}(\gamma) - H_{mk-1}(\gamma)|\bar{A}_m, \bar{L}_m]. \]
Plugging this choice of $s_m(k, l_m, a_m)$ into $U_{mk}$, we get

$$E[\{H_{mk}(\gamma) - H_{mk-1}(\gamma)\} \times$$

$$\{E[H_{mk}(\gamma) - H_{mk-1}(\gamma)|\bar{A}_m, \bar{L}_m] - E[H_{mk}(\gamma) - H_{mk-1}(\gamma)|\bar{L}_m, \bar{A}_{m-1}]\} = 0$$

$$\implies$$

$$E[H_{mk}(\gamma) - H_{mk-1}(\gamma)|\bar{A}_m, \bar{L}_m] = E[H_{mk}(\gamma) - H_{mk-1}(\gamma)|\bar{L}_m, \bar{A}_{m-1}],$$

which proves the result.

Lemma 3. \(\gamma^*\) are the unique functions satisfying (10).

Proof. We proceed by induction. Suppose for any \(k\) there exist other functions \(\gamma'\) in addition to \(\gamma^*\) satisfying (10) and satisfying \(\gamma'_{mk}(l_m, (a_{m-1}, g(l_m, a_{m-1})) = 0\). Then

$$E[H_{(k-1)k}(\gamma^*) - H_{(k-1)(k-1)}(\gamma^*)|\bar{A}_{k-1}, \bar{L}_{k-1}] = E[H_{(k-1)k}(\gamma) - H_{(k-1)(k-1)}(\gamma)|(\bar{A}_{k-2}, g), \bar{L}_{k-1}]$$

$$E[H_{(k-1)k}(\gamma') - H_{(k-1)(k-1)}(\gamma')|\bar{A}_{k-1}, \bar{L}_{k-1}] = E[H_{(k-1)k}(\gamma') - H_{(k-1)(k-1)}(\gamma')|(\bar{A}_{k-2}, g), \bar{L}_{k-1}].$$

Differencing both sides of the above equations using the definition of \(H_{mk}(\gamma)\) yields that

$$E[\gamma_{(k-1)k}(\bar{L}_{k-1}, \bar{A}_{k-1}) - \gamma'_{(k-1)k}(\bar{L}_{k-1}, \bar{A}_{k-1}|\bar{A}_{k-1}, \bar{L}_{k-1}]$$

$$= \gamma^*_{(k-1)k}(\bar{L}_{k-1}, \bar{A}_{k-1}) - \gamma'_{(k-1)k}(\bar{L}_{k-1}, \bar{A}_{k-1})$$

$$= E[\gamma^*_{(k-1)k}(\bar{L}_{k-1}, \bar{A}_{k-1}) - \gamma'_{(k-1)k}(\bar{L}_{k-1}, \bar{A}_{k-1})|(\bar{A}_{k-2}, g), \bar{L}_{k-1}]$$

$$= 0,$$

where the last equality follows from the assumption that \(\gamma'_{mk}(a_m, l_m) = 0\) when \(a_m = g(l_m, a_{m-1})\). This establishes that \(\gamma^*_{(k-1)k} = \gamma'_{(k-1)k}\) for all \(k\). Suppose for the purposes of induction that we have established that \(\gamma^*_{(k-j)k} = \gamma'_{(k-j)k}\) for all \(k\) and all \(j \in \{1, \ldots, n\}\). Now consider \(\gamma^*_{(k-(n+1))k}\) and \(\gamma'_{(k-(n+1))k}\). Again
we have that

\[
E[H_{(k-(n+1))k}(\gamma^*) - H_{(k-(n+1))(k-1)}(\gamma^*)|A_{k-(n+1)}, L_{k-(n+1)}] = \\
E[H_{(k-(n+1))k}(\gamma^*) - H_{(k-(n+1))(k-1)}(\gamma^*)|(A_{k-(n+2)}, 0), L_{k-(n+1)}]
\]

and

\[
E[H_{(k-(n+1))k}(\gamma') - H_{(k-(n+1))(k-1)}(\gamma')|A_{k-(n+1)}, L_{k-(n+1)}] = \\
E[H_{(k-(n+1))k}(\gamma') - H_{(k-(n+1))(k-1)}(\gamma')(A_{k-(n+2)}, g), L_{k-(n+1)}].
\]

And again we can difference both sides of these equations plugging in the expanded definition of

\[H_{(k-(n+1))k}(\gamma)\] to obtain

\[
E[\{Y_k - \sum_{j=k-(n+1)}^{k-1} \gamma^*_j(A_j, L_j)\} - \{Y_k - \sum_{j=k-(n+1)}^{k-1} \gamma'_j(A_j, L_j)\}|A_{k-(n+1)}, L_{k-(n+1)}] = \\
E[\{Y_k - \sum_{j=k-(n+1)}^{k-1} \gamma^*_j(A_j, L_j)\} - \{Y_k - \sum_{j=k-(n+1)}^{k-1} \gamma'_j(A_j, L_j)\}|(A_{k-(n+2)}, g), L_{k-(n+1)}].
\]

Under our inductive assumption, (32) reduces to

\[
E[\gamma^*_k(A_{k-(n+1)}, L_{k-(n+1)}) - \gamma'_k(A_{k-(n+1)}, L_{k-(n+1)})|A_{k-(n+1)}, L_{k-(n+1)}] = \\
\gamma^*_k(A_{k-(n+1)}, L_{k-(n+1)}) - \gamma'_k(A_{k-(n+1)}, L_{k-(n+1)}) = \\
E[\gamma^*_{(k-(n+1))k}(L_{k-(n+1)}, A_{k-(n+1)}) - \gamma'_{(k-(n+1))k}(L_{k-(n+1)}, A_{k-(n+1)})|A_{k-(n+2)}, g), L_{k-(n+1)}] = \\
0,
\]

proving that \(\gamma^*_{(k-(n+1))k} = \gamma'_{(k-(n+1))k}\) for all \(k\). Hence, by induction, the result follows. \(\Box\)

Uniqueness is a direct corollary of Lemmas 2 and 3.
Part (ii)

\[ E[U_{mk}^+(s_m, \gamma_m^*, \hat{v}_m, \hat{\pi}_m)] = E[E[U_{mk}^+(s_m, \gamma_m^*, \hat{v}_m, \hat{\pi}_m)|\tilde{L}_m, \tilde{A}_m]] \\
= E[ \\
E[H_{mk}(\gamma^*) - H_{mk-1}(\gamma^*) - \hat{v}_m(k, \tilde{L}_m, \tilde{A}_{m-1}, \gamma^*)|\tilde{L}_m, \tilde{A}_{m-1}]q(\tilde{L}_m, \tilde{A}_{m-1}) \times \\
(s_m(k, \tilde{T}_m, \tilde{A}_m) - E_{\tilde{m}}[s_m(k, \tilde{T}_m, \tilde{A}_m)|\tilde{L}_m, \tilde{A}_{m-1}]) \\
] \\
\text{by (10)} \\
= E[ \\
E[E[H_{mk}(\gamma^*) - H_{mk-1}(\gamma^*) - \hat{v}_m(k, \tilde{L}_m, \tilde{A}_{m-1}, \gamma^*)|\tilde{L}_m, \tilde{A}_{m-1}]q(\tilde{L}_m, \tilde{A}_{m-1}) \times \\
E[s_m(k, \tilde{T}_m, \tilde{A}_m) - \bar{E}[s_m(k, \tilde{T}_m, \tilde{A}_m)|\tilde{L}_m, \tilde{A}_{m-1}]]|\tilde{L}_m, \tilde{A}_{m-1}] \\
] \\
\text{by (33)}
\]

Now the result follows because if \( \hat{v}_m(k, \tilde{L}_m, \tilde{A}_{m-1}, \gamma^*) = v_m^*(k, \tilde{L}_m, \tilde{A}_{m-1}, \gamma^*) \), then

\[ E[H_{mk}(\gamma^*) - H_{mk-1}(\gamma^*) - \hat{v}_m(k, \tilde{L}_m, \tilde{A}_{m-1}, \gamma^*)|\tilde{L}_m, \tilde{A}_{m-1}] = 0 \]

and if \( \bar{E}[s_m(\tilde{L}_m, \tilde{A}_m)|\tilde{L}_m, \tilde{A}_{m-1}] = E[s_m(\tilde{L}_m, \tilde{A}_m)|\tilde{L}_m, \tilde{A}_{m-1}] \) then

\[ E[s_m(k, \tilde{T}_m, \tilde{A}_m) - \bar{E}[s_m(k, \tilde{T}_m, \tilde{A}_m)|\tilde{L}_m, \tilde{A}_{m-1}]]|\tilde{L}_m, \tilde{A}_{m-1}] = 0. \]

**B Proof of Theorem 2**

*Proof.* Along the lines of Proposition 1 in [Kennedy 2022] and Lemma 3 in [Kennedy et al. 2023], we have that

\[
\sqrt{N}(\hat{\psi}^f - \psi^*) = V(\psi^*, s, \pi^*)^{-1}\sqrt{N}(\mathbb{P}_N - \mathbb{P}) \left\{ \sum_{m=0}^{K} U_m(\psi^*, s_m, \pi^*, \nu^*) \right\} \\
+ \sqrt{N}O_p \left( \sum_{s=1}^{2} \left( \frac{n_s}{N} \right) \mathbb{P} \left\{ \sum_{m=0}^{K} U_m(\psi^*, s_m, \pi^{(-s)}, \nu^{(-s)})) \right\} \right) + o_p(1).
\]
In order to apply Lemma 3, we require Assumptions 1, 2 and 4. Then the main result follows so long as we can show that

$$\sqrt{N}O_p \left( \sum_{s=1}^{2} \left( \frac{N_s}{N} \right) \mathbb{P} \left\{ \sum_{m=0}^{K} U_m(\psi^*, s_m, \hat{\pi}^{(-s)}, \hat{\nu}^{(-s)}) \right\} \right) = o_p(1). \quad (34)$$

Fix $s = 1$. We will proceed first by showing that each component of

$$\mathbb{P} \left\{ \sum_{m=0}^{K} U_m(\psi^*, s_m, \hat{\pi}^{(-1)}, \hat{\nu}^{(-1)}) \right\} 

(35)$$
is $o_p(N^{-1/2})$. For any $m = 0, ... K$,

$$\sum_{m=0}^{K} U_m(O; \psi^*, s_m, \hat{\pi}^{(-1)}, \hat{\nu}^{(-1)}) = \left\{ \sum_{m=0}^{K-1} \sum_{k=0}^{K} s_{mk1}(\bar{L}_m, \bar{A}_{m-1}) \left\{ A_m - \hat{\pi}^{(-1)}(\bar{L}_m, \bar{A}_{m-1}) \right\} \{ H_{mk}(\psi^*) - H_{m,k-1}(\psi^*) - \hat{\nu}_{mk}^{(-1)}(\bar{L}_m, \bar{A}_{m-1}) \} \right\} \right.$$  

$$\vdots$$  

$$\left\{ \sum_{m=0}^{K-1} \sum_{k=0}^{K} s_{mkp}(\bar{L}_m, \bar{A}_{m-1}) \left\{ A_m - \hat{\pi}^{(-1)}(\bar{L}_m, \bar{A}_{m-1}) \right\} \{ H_{mk}(\psi^*) - H_{m,k-1}(\psi^*) - \hat{\nu}_{mk}^{(-1)}(\bar{L}_m, \bar{A}_{m-1}) \} \right\}$$

where $s_{mkj}(\bar{L}_m, \bar{A}_{m-1})$ is the $(j, k)$th element of $s_m(\bar{L}_m, \bar{A}_{m-1})$. Then

$$\left| \mathbb{P} \left\{ \sum_{m=0}^{K-1} \sum_{k=0}^{K} s_{mk1}(\bar{L}_m, \bar{A}_{m-1}) \left\{ A_m - \hat{\pi}^{(-1)}(\bar{L}_m, \bar{A}_{m-1}) \right\} \{ H_{mk}(\psi^*) - H_{m,k-1}(\psi^*) - \hat{\nu}_{mk}^{(-1)}(\bar{L}_m, \bar{A}_{m-1}) \} \right\} \right|$$  

$$\leq \mathbb{P} \left\{ \sum_{m=0}^{K-1} \sum_{k=0}^{K} s_{mk1}(\bar{L}_m, \bar{A}_{m-1}) \left\{ A_m - \hat{\pi}^{(-1)}(\bar{L}_m, \bar{A}_{m-1}) \right\} \{ H_{mk}(\psi^*) - H_{m,k-1}(\psi^*) - \hat{\nu}_{mk}^{(-1)}(\bar{L}_m, \bar{A}_{m-1}) \} \right\}$$  

$$\leq \mathbb{P} \left\{ \sum_{m=0}^{K-1} \sum_{k=0}^{K} |s_{mk1}(\bar{L}_m, \bar{A}_{m-1})| \left\{ A_m - \hat{\pi}^{(-1)}(\bar{L}_m, \bar{A}_{m-1}) \right\} \{ H_{mk}(\psi^*) - H_{m,k-1}(\psi^*) - \hat{\nu}_{mk}^{(-1)}(\bar{L}_m, \bar{A}_{m-1}) \} \right\}$$  

$$= \mathbb{P} \left\{ \sum_{m=0}^{K-1} \sum_{k=0}^{K} \left| A_m - \hat{\pi}^{(-1)}(\bar{L}_m, \bar{A}_{m-1}) \right| \left| H_{mk}(\psi^*) - H_{m,k-1}(\psi^*) - \hat{\nu}_{mk}^{(-1)}(\bar{L}_m, \bar{A}_{m-1}) \right| \right\}$$  

$$\leq \mathbb{P} \left\{ \sum_{m=0}^{K-1} \sum_{k=0}^{K} \left| A_m - \hat{\pi}^{(-1)}(\bar{L}_m, \bar{A}_{m-1}) \right| \left| H_{mk}(\psi^*) - H_{m,k-1}(\psi^*) - \hat{\nu}_{mk}^{(-1)}(\bar{L}_m, \bar{A}_{m-1}) \right| \right\}$$  

$$= O_p \left( \sum_{m=0}^{K-1} \sum_{k=0}^{K} \left| \hat{\pi}_m - \pi^*_m \right| \left| \hat{\nu}_{mk} - \nu^*_k \right| \right)$$  

$$= o_p(N^{-1/2}).$$

where we repeatedly use the triangle inequality plus Assumptions 3 and 5. The same reason holds with $s = 2$. Then because $n_1/N$ and $n_2/N$ both tend to a constant by assumption, the above results imply $[]$. 

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C Relationship Between Universal Parallel Trends and No Additive Effect Modification by Unobserved Confounders Assumptions

To identify the counterfactual expectation $E[Y_k(g)]$ for a given $g$, we assumed parallel trends under $g$. One might argue, however, that if one assumes that parallel trends happen to hold for a particular regime of interest, one is really in effect assuming that parallel trends holds for all regimes $g \in G$. To identify optimal treatment strategies via optimal regime SNMMs, we needed to make the additional assumption (21) that there is no additive effect modification by unobserved confounders. Below, we sketch an argument that parallel trends under all regimes effectively, though not strictly mathematically, implies no effect modification by unobserved confounders.

We will attempt to show the contrapositive of our result of interest, i.e. we will try to show that if there is additive effect modification by an unobserved confounder then parallel trends under all regimes cannot hold. Suppose that $\bar{L}, \bar{U}$ is a minimal set satisfying sequential exchangeability. And suppose that $\bar{U}$ is an effect modifier, i.e. for some $m, k$, and $g$

$$E[Y_k(\bar{a}_{m-1}, g_m) - Y_k(\bar{a}_{m-1}, 0, g_{m+1})|\bar{A}_m, \bar{L}_m, \bar{U}_m = \bar{u}_m] \quad (36)$$

depends on $\bar{u}_m$. If parallel trends does hold for all regimes, in particular it would hold for $g_m$ and $(0, g_{m+1})$, i.e.

$$E[Y_k(\bar{a}_{m-1}, g_m) - Y_k(\bar{a}_{m-1}, 0, g_{m})|\bar{A}_m, \bar{L}_m] \quad (37)$$

would not depend on $A_m$ and

$$E[Y_k(\bar{a}_{m-1}, 0, g_{m+1}) - Y_{k-1}(\bar{a}_{m-1}, 0, g_{m+1})|\bar{A}_m, \bar{L}_m] \quad (38)$$

would not depend on $A_m$. Therefore, the difference of the above two quantities also would not depend on $A_m$, i.e.

$$E[Y_k(\bar{a}_{m-1}, g_m) - Y_k(\bar{a}_{m-1}, 0, g_{m+1})|\bar{A}_m, \bar{L}_m] - E[Y_{k-1}(\bar{a}_{m-1}, g_m) - Y_{k-1}(\bar{a}_{m-1}, 0, g_{m+1})|\bar{A}_m, \bar{L}_m] \quad (39)$$

would not depend on $A_m$. But $U_m$ being an effect modifier (i.e. (36) depending on $u_m$) and the assumption that $(\bar{L}_m, \bar{U}_m)$ is a minimal sufficient set together imply that the first conditional expectation in (39) does depend on $A_m$. Now, it is possible for the full difference in (39) to still not depend on $A_m$ if the second conditional expection also depends on $A_m$ and in such a way as to cancel out the dependence on $A_m$ of the first conditional expectation. While this is technically possible, the type of cancelling out required is not plausible. So by reductio ad absurdity (though not the formal Latin absurdum indicating a true contradiction), parallel trends for all
regimes “effectively” implies no additive effect modification by unobserved confounders.

D Proof of Lemma 1, enabling sensitivity analysis

Recall that we define bias function

\[
c_{mk}^g(\bar{l}_m, \bar{a}_m) =
\]

\[
E[Y_k(\bar{a}_{m-1}, g_m) - Y_{k-1}(\bar{a}_{m-1}, g_m)|\bar{l}_m = \bar{l}_m, \bar{A}_m = \bar{a}_m] -
\]

\[
E[Y_k(\bar{a}_{m-1}, g_m) - Y_{k-1}(\bar{a}_{m-1}, g_m)|\bar{l}_m = \bar{l}_m, \bar{A}_m = (\bar{a}_{m-1}, g(\bar{l}_m, \bar{a}_{m-1}))].
\]

We can then prove Lemma 1 as follows.

\[
E[H_{mk}^\alpha(\gamma^*) - H_{m,k-1}(\gamma^*)|\bar{l}_m, \bar{A}_m]
\]

\[
= E[H_{mk}^\alpha(\gamma^*) - H_{m,k-1}(\gamma^*) - c_{mk}(\bar{A}_m, \bar{l}_m)|\bar{l}_m, \bar{A}_m]
\]

\[
= E[Y_k(\bar{A}_{m-1}, g) - Y_{k-1}(\bar{A}_{m-1}, g) - c_{mk}(\bar{A}_m, \bar{l}_m)|\bar{l}_m, \bar{A}_m]
\]

\[
= E[E[Y_k(\bar{A}_{m-1}, g) - Y_{k-1}(\bar{A}_{m-1}, g)|\bar{l}_m, \bar{A}_{m-1}, A_m = g(\bar{l}_m, \bar{A}_{m-1})]|\bar{l}_m, \bar{A}_m]
\]

\[
= E[Y_k(\bar{A}_{m-1}, g) - Y_{k-1}(\bar{A}_{m-1}, g)|\bar{l}_m, \bar{A}_{m-1}, A_m = g(\bar{l}_m, \bar{A}_{m-1})]
\]

\[
= E[H_{mk}^\alpha(\gamma^*) - H_{m,k-1}(\gamma^*) - c_{mk}(\bar{A}_m, \bar{l}_m)|\bar{l}_m, \bar{A}_{m-1}, A_m = g(\bar{l}_m, \bar{A}_{m-1})]
\]

\[
= E[H_{mk}^\alpha(\gamma^*) - H_{m,k-1}(\gamma^*)|\bar{l}_m, \bar{A}_{m-1}, A_m = g(\bar{l}_m, \bar{A}_{m-1})]
\]

where the first equality is by the definition of \(H_{mk}^\alpha(\gamma^*)\), the second equality is by \(g\), the third equality is by the definition of \(c_{mk}(\bar{A}_m, \bar{l}_m)\), the fourth equality is by nested expectations, the fifth equality is again by \(g\) and because \(c_{mk}(\bar{A}_m, \bar{l}_m) = 0\) when \(A_m = g(\bar{l}_m, \bar{A}_{m-1})\), and the final equality is again because \(c_{mk}(\bar{A}_m, \bar{l}_m) = 0\) when \(A_m = g(\bar{l}_m, \bar{A}_{m-1})\). This completes the proof.
Figure 1: Estimated expected heterogeneous effects (and 95% confidence intervals) of Medicaid expansion on uninsurance rates in an expanding median population county. Each line represents expected ongoing effects of an expansion at the year the line originates in counties with pre-expansion unemployment rate corresponding to the color of the line. Red lines (effects of expansions in higher unemployment counties) indicate greater uninsurance reduction effects than blue lines (effects of expansions in lower unemployment counties) originating in the same year. The graph also shows how estimated effects vary by year of expansion and years since expansion.
Figure 2: Each line depicts the estimated effect of a flood occurring at its leftmost time point followed by no further floods on flood insurance uptake over the subsequent 15 years for a county with the median historical flood rate, i.e. $\gamma_{mk}(rate_{m-1} = rate_{median}, a_m = 1; \psi)$ for $k \in m, \ldots, m + 15$. These quantities were directly extracted from our blip function estimates.

Figure 3: GDD trajectories for 100 random counties from 1987 through 2002. There is not a shared baseline level and values jump repeatedly over time.