An $\mathcal{N} = 1$ Supersymmetric $G_2$-invariant Flow in $M$-theory

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Abstract

It was found that deformation of $S^7$ gives rise to renormalization group (RG) flow from $\mathcal{N} = 8$, $SO(8)$-invariant UV fixed point to $\mathcal{N} = 1$, $G_2$-invariant IR fixed point in four-dimensional gauged $\mathcal{N} = 8$ supergravity. Also BPS supersymmetric domain wall configuration interpolated between these two critical points. In this paper, we use the $G_2$-invariant RG flow equations for both scalar fields and domain wall amplitude and apply them to the nonlinear metric ansatz developed by de Wit, Nicolai and Warner some time ago. We carry out the $M$-theory lift of the $G_2$-invariant RG flow through a combinatoric use of the four-dimensional RG flow equations and eleven-dimensional Einstein-Maxwell equations. The non-trivial $r$ (that is the coordinate transverse to the domain wall)-dependence of vacuum expectation values makes the Einstein-Maxwell equations consistent not only at the critical points but also along the supersymmetric RG flow connecting two critical points. By applying an ansatz for an eleven-dimensional three-form gauge field with varying scalars, we discover an exact solution to the eleven-dimensional Einstein-Maxwell equations corresponding to the $M$-theory lift of the $G_2$-invariant RG flow.

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1 Introduction

By generalizing compactification vacuum ansatz to the nonlinear level, solutions of the 11-dimensional supergravity were obtained directly from constant scalar and pseudo-scalar expectation values at the various critical points of the $\mathcal{N} = 8$ supergravity potential [1]. They were able to reproduce all known Kaluza-Klein solutions of 11-dimensional supergravity except for $SU(3) \times U(1)$-invariant vacuum solution\footnote{The 11-dimensional embedding of $SU(3) \times U(1)$-invariant vacuum solution was recently found in [2] as a stretched 7-ellipsoid.}. round $S^7$, $SO(7)^-$-invariant parallelized $S^7$, $SO(7)^+$-invariant 7-ellipsoid, $SU(4)^-$-invariant stretched $S^7$, and $G_2$-invariant 7-ellipsoid. Among them the only round $S^7$ and $G_2$-invariant 7-ellipsoid are stable and supersymmetric. One of the important features of de Wit-Nicolai theory [3] is that four-dimensional spacetime when dimensionally reduced from 11-dimensional supergravity is warped [1, 4] by warp factor $\Delta(y)$. To be explicit, we have 11-dimensional metric

$$ds_{11}^2 = ds_4^2 + ds_7^2 = \Delta^{-1}(y) g_{\mu\nu}(x) dx^\mu dx^\nu + g_{mn}(y) dy^m dy^n. \quad (1.1)$$

Novelty of vacua with a non-trivial warp factor is that they correspond to inhomogeneous deformations of $S^7$. For $SO(7)^-$ and $SU(4)^-$ solutions, the function $\Delta(y)$ is a constant while for $SO(7)^+$ and $G_2$ solutions it is a non-trivial function of internal coordinate $y$. The existence of warp factor is crucial for understanding of the different relative scales of the 11-dimensional solutions corresponding to the critical points in $\mathcal{N} = 8$ gauged supergravity. The main result of [1] was that it is possible to write down the metric directly from the vacuum expectation values of the scalar and pseudo-scalar fields in 4-dimensional theory. Therefore both internal metric and warp factor depend on 4-dimensional coordinate $x$ as well as $y$. With explicit dependence on $x$ and $y$, the inverse metric on $S^7$, $g^{mn}(x, y)$, is given by [1, 4]

$$g^{mn}(x, y) = \Delta(x, y) \tilde{K}^{mIJ}(y) \tilde{K}^{nKL}(y) \left[ u_{ij}^{IJ}(x) + v_{ijIJ}(x) \right] \left[ \pi^{ij}_{KL}(x) + \pi^{ijKL}(x) \right] \quad (1.2)$$

where $\tilde{K}^{mIJ}(y)$ is the usual Killing vector on the unit round $S^7$(See also [3, 6]) and warp factor $\Delta(x, y)$ is defined by

$$\Delta(x, y) = \sqrt{\frac{\det g_{mn}(x, y)}{\det g_{mn}(y)}} \quad (1.3)$$

where the metric $\tilde{g}_{mn}(y)$ is that of the round $S^7$. The 28-beins $u_{ij}^{IJ}(x)$ and $v_{ijIJ}(x)$ fields are $28 \times 28$ matrices and are given in terms of scalar and pseudo-scalar fields in four-dimensional gauged supergravity. We denote the complex conjugations of $u_{ij}^{IJ}(x)$ and $v_{ijIJ}(x)$ by $\overline{u}_{ij}^{IJ}(x)$ and $\overline{v}_{ij}^{IJ}(x)$ respectively. Moreover, $SU(8)$ index pairs $[ij]$ and $SO(8)$ index pairs $[IJ]$ are antisymmetrized.
By analyzing scalar potential in the de Wit-Nicolai theory, it was found in \([5]\) that the deformation of \(S^7\) gives rise to renormalization group (RG) flow from \(\mathcal{N} = 8\), \(SO(8)\)-invariant ultraviolet (UV) fixed point to \(\mathcal{N} = 1\), \(G_2\)-invariant infrared (IR) fixed point, via AdS/CFT correspondence \([6, 7, 8]\). The flow interpolates between both fixed points. The critical points give four-dimensional \(AdS_4\) vacua and preserve \(G_2\) gauge symmetry in the supergravity side. Having established the holographic duals of both supergravity critical points and examined small perturbations around the corresponding fixed point field theories, one can proceed the supergravity description of the RG flow between the two fixed points. The supergravity scalars tell us that what relevant operators in the dual field theory would drive a flow to the fixed point in the IR. To construct the superkink corresponding to the supergravity description of the nonconformal RG flow connecting two critical points in \(d = 3\) conformal field theories, the form of a three-dimensional Poincare invariant metric but breaking full conformal group invariance takes the form

\[
g_{\mu\nu}(x) \, dx^\mu \, dx^\nu = e^{2A(r)} \eta_{\mu'\nu'} \, dx^{\mu'} \, dx^{\nu'} + \, dr^2, \quad \eta_{\mu'\nu'} = (-, +, +) \tag{1.4}\]

characteristic of spacetime with a domain wall where \(r\) is the coordinate transverse to the wall (interpreted as an energy scale) and \(A(r)\) is the scale factor in the four-dimensional metric. By minimization of energy-functional, in order to get BPS domain-wall solutions, one has to reorganize it into complete squares. By recognizing that de Wit-Nicolai theory has particular property, the scalar potential is written as the difference of two positive square terms, one can construct energy-functional in terms of complete squares. In \([9]\), we have found that the first order differential equations with respect to the \(r\)-direction for the varying scalar fields are the gradient flow equations of a superpotential defined on a restricted slice of complete scalar manifold.

In this paper, we find \(M\)-theory solutions that are holographic duals of flows of the maximally supersymmetric \(\mathcal{N} = 8\) theory in three-dimensions. By using the RG flow equations for scalar fields and \(A(r)\), implying that one can find the derivatives of these fields with respect to \(r\) explicitly, we generalize the scheme of \([1]\) and study several aspects of the embedding of gauged \(\mathcal{N} = 8\) supergravity into 11-dimensional supergravity. We will begin our analysis in Section 2 by summarizing relevant aspects of \([3]\) in four-dimensional gauged supergravity. In Section 3, we will investigate the lift of the four-dimensional solution to \(M\)-theory by solving 11-dimensional Einstein-Maxwell equations. We will start with the metric \((1.1)\) together with \((1.2)\), \((1.3)\) and \((1.4)\). Then we will apply an ansatz for 11-dimensional 3-form gauge field which is a natural extension of the Freund-Rubin parametrization \([10]\) but will be more complicated since we are dealing with non-constant vacuum expectation values. Recently, \(M\)-theory lift of the \(SU(3) \times U(1)\)-invariant RG flow was found in \([2, 11]\) by applying the RG flow equations given in \([12]\). Similar analysis in \(AdS_5\) supergravity and its lift to type IIB string theory were
given in [13]. In Section 4, we will summarize our results and will discuss about future direction.

Throughout this paper, we will be using the metric convention \((-+,\cdots,+)\). Our notation is that the \(d = 11\) coordinates with indices \(M,N,\cdots\) are decomposed into \(d = 4\) spacetime coordinates \(x\) with indices \(\mu,\nu,\cdots\) and \(d = 7\) internal space coordinates \(y\) with indices \(m,n,\cdots\). Denoting the 11-dimensional metric as \(g_{MN}\) and the antisymmetric tensor fields as \(F_{MNPQ} = 4 \partial_{[M}A_{NPQ]}\), the bosonic field equations are [14]

\[
R^N_M = \frac{1}{3} F_{MQR} F^{NQR} - \frac{1}{36} \delta^N_M F_{PQRS} F^{PQRS},
\]

\[
\nabla_M F^{MNPQ} = - \frac{1}{(4!)^2} E \epsilon^{NPRSTUVWXYZ} F_{RSTU} F_{VWXYZ},
\]

(1.5)

where the covariant derivative on \(F^{MNPQ}\) in the second relation is given by \(E^{-1} \partial_M (E F^{MNPQ})\) together with eilbein determinant \(E = \sqrt{-g_{11}}\). The 11-dimensional epsilon tensors with lower indices \(\epsilon_{NPQRSTUVWXY}\) are purely numerical.

2 The \(G_2\)-invariant holographic RG flow in 4 dimensions

The ungauged \(N = 8\) supergravity [13] has a local compact symmetry of the action \(H = SU(8)\) and a global non-compact symmetry of the equations of motion \(G = E_7(\pm 7)\), of which the subgroup \(L = SL(8,\mathbb{R})\) is a global symmetry of the action. An arbitrary element of the 133-dimensional Lie algebra of \(E_7(\pm 7)\) can be represented by a \(56 \times 56\) matrix(four \(28 \times 28\) block matrices)

\[
\begin{pmatrix}
\Lambda_{IJ}^{KL} & \Sigma_{IJPQ} \\
\Sigma_{MNKL} & \Lambda_{MN}^{PQ}
\end{pmatrix}
\]

where the indices \(I,J = 1,\cdots,8\) are antisymmetric in pairs. The \(H = SU(8)\) maximally compact subgroup of \(E_7(\pm 7)\) is generated by the 63-dimensional diagonal subalgebra

\[
D(\Lambda_I^J) = \begin{pmatrix}
\Lambda_{I}^J & 0 \\
0 & \Lambda_{I}^J
\end{pmatrix}, \quad \Delta = \Lambda_{I}^J P^Q = \delta_{I}^J \Lambda_{I}^J Q
\]

where \(\Lambda_I^J\) is an \(8 \times 8\), antihermitian trace-free generator of \(SU(8)\): \(\Lambda_I^J = -\Lambda_J^I, \Lambda_I^I = 0\). The 70 non-compact generators are parametrized by the complex, self-dual antisymmetric tensors \(\Sigma_{MNPQ}\) that satisfy

\[
\sum_{MNPQ} = (\Sigma_{MNPQ})^* = \frac{1}{24} \eta \epsilon^{IJKLMNPQ} \Sigma_{IJKL}
\]

where \(\eta = \pm 1\) is an arbitrary phase, chosen as +1. Then, \(L = SL(8,\mathbb{R})\) is the real subgroup of \(E_7(\pm 7)\) given by restricting the above 133 generators to the 28 generators of \(SO(8) \subset SU(8)\), \(\Lambda_{I}^J = \bar{\Lambda}_I^J\) plus the 35 real, self-dual antisymmetric tensors, \(\Sigma_{IJKL} = \bar{\Sigma}_{IJKL}\) (63 = 28 + 35).
It is well known that the 70 real, physical scalars of \( \mathcal{N} = 8 \) supergravity parametrize the coset space \( E_{7(+7)}/SU(8) \) (even though \( E_{7(+7)} \) symmetry is broken in the gauged theory) since 63 fields \((133 - 63 = 70)\) may be gauged away by an \( SU(8) \) rotation and can be represented by an element \( V(x) \) of the fundamental 56-dimensional representation of \( E_{7(+7)} \):

\[
V(x) = \exp \left( -\frac{1}{2\sqrt{2}} \lambda_{IJK}^{KL} \phi_{MNKL} - \frac{1}{2\sqrt{2}} \phi_{IJPQ} \right) = \left( \begin{array}{cc}
\lambda_{ij}^{KL} & \phi_{ijPQ} \\
\frac{1}{2\sqrt{2}} \lambda_{MN}^{PQ} & \frac{1}{2\sqrt{2}} \phi_{MN}^{PQ}
\end{array} \right)
\]

where \( SU(8) \) index pairs \([ij], \cdots\) and \( SO(8) \) index pairs \([IJ], \cdots\) are antisymmetrized. The 63 compact generators \( \Lambda \) can be set to zero by fixing an \( SU(8) \) gauge. Moreover \( \phi_{IJKL} \) is a complex self-dual tensor describing the 35 scalars \( \Re \phi_{IJKL} \) and 35 pseudo-scalar fields \( \Im \phi_{IJKL} \) of \( \mathcal{N} = 8 \) supergravity. The maximally supersymmetric vacuum with \( SO(8) \) symmetry, \( S^7 \), is where expectation values of both scalar and pseudo-scalar fields vanish. Let us denote self-dual and anti-self-dual tensors of \( SO(8) \) tensor as \( C_{IJKL}^+ \) and \( C_{IJKL}^- \). Turning on the scalar fields proportional to \( C_{IJKL}^+ \) yields an \( SO(7)^+ \)-invariant vacuum. Likewise, turning on pseudo-scalar fields proportional to \( C_{IJKL}^- \) yields \( SO(7)^- \)-invariant vacuum. Both \( SO(7)^\pm \) vacua are nonsupersymmetric. However, simultaneously turning on both scalar and pseudo-scalar fields proportional to \( C_{IJKL}^+ \) and \( C_{IJKL}^- \), respectively, one obtains \( G_2 \)-invariant vacuum with \( \mathcal{N} = 1 \) supersymmetry \([1]\). The most general vacuum expectation value of 56-bein preserving \( G_2 \)-invariance can be parametrized by

\[
\phi_{IJKL} = \frac{\lambda}{2\sqrt{2}} \left( \cos \alpha \ C_{IJKL}^+ + i \sin \alpha \ C_{IJKL}^- \right).
\]

Then two scalars \( \lambda \) and \( \alpha \) fields in the \( G_2 \)-invariant flow parametrize a \( G_2 \)-invariant subspace of the complete scalar manifold \( E_{7(+7)}/SU(8) \). The dependence of scalar 56-bein

\[
V(\lambda(x), \alpha(x)) = \left( \begin{array}{cc}
\lambda_{ij}^I & \phi_{ijKL} \\
\frac{1}{2\sqrt{2}} \lambda_{IJ}^+ & \frac{1}{2\sqrt{2}} \phi_{IJ}^+
\end{array} \right)
\]

on \( \lambda \) and \( \alpha \) was obtained in \([1, 5, 9]\) by exponentiating the vacuum expectation values \( \phi_{IJKL} \) of \( G_2 \)-singlet space. We refer to the Appendix A in \([9]\) for the explicit forms that one needs to know.

The structure of the scalar sector is encoded in the \( SU(8) \)-covariant \( T \)-tensor which is cubic in the 28-beins and antisymmetric in the indices \([ij]\):

\[
T_{ij}^{kl} = \left( \bar{u}^i_{IJ} + \bar{u}^i_{ijI} \right) \left( v_{lmJK} \bar{u}^{km}_{KI} - v_{lmJK} \bar{u}^{km}_{KL} \right).
\]

The superpotential which leads to \( G_2 \)-invariant flow is one of the eigenvalues of the symmetric tensor in \((ij)\):

\[
A_{ij} = -\frac{4}{21} T_{ij}^{jm}.
\]
Table 1: Summary of two supersymmetric critical points: symmetry group, vacuum expectation values of fields, cosmological constants and superpotential.

| Symmetry | $\lambda$ | $\alpha$ | $V$ | $W$ |
|-----------|-----------|-----------|-----|-----|
| $SO(8)$   | 0         | any       | $-6g^2$ | 1   |
| $G_2$     | $\sqrt{2} \sinh^{-1}\frac{2}{\sqrt{5}}(\sqrt{3} - 1)$ | $\cos^{-1}\frac{1}{2}\sqrt{3 - \sqrt{3}}$ | $-\frac{216\sqrt{2}}{25\sqrt{5}}\frac{3^{1/4}}{\sqrt{3}}g^2$ | $\sqrt{\frac{36\sqrt{2}3^{1/4}}{25\sqrt{5}}}$ |

which can be obtained by using some identities in $T$-tensor. It turned out [9, 1] that the $A_{1}^{ij}$ tensor has two distinct complex eigenvalues, $z_1$ and $z_2$, with degeneracies 7 and 1 respectively to have the following form:

$$A_{1}^{ij} = \text{diag}(z_1, z_1, z_1, z_1, z_1, z_1, z_1, z_2)$$

where the eigenvalues are functions of $\lambda$ and $\alpha$. In particular,

$$z_2(\lambda, \alpha) = p^7 + e^{7i\alpha}q^7 + 7\left(p^3q^4e^{4i\alpha} + p^4q^3e^{3i\alpha}\right)$$

and we denote $p, q$ by the following hyperbolic functions of $\lambda$:

$$p = \cosh\left(\frac{\lambda}{2\sqrt{2}}\right), \quad q = \sinh\left(\frac{\lambda}{2\sqrt{2}}\right).$$

The superpotential is given by the eigenvalues of $A_1$ tensor and the supergravity potential can be written in terms of superpotential:

$$W(\lambda, \alpha) = |z_2|,$$

$$V(\lambda, \alpha) = g^2\left[\frac{16}{7}(\partial_\lambda W)^2 + \frac{2}{7p^2q^2}(\partial_\alpha W)^2 - 6W^2\right].$$

(2.1)

The supersymmetric flow equations are

$$\partial_r \lambda = -\frac{8\sqrt{2}}{7}g\partial_\lambda W,$$

$$\partial_r \alpha = -\frac{\sqrt{2}}{7p^2q^2}g\partial_\alpha W,$$

$$\partial_r A = \sqrt{2}gW.$$  

(2.2)

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2It is easy to see that the eigenvalue $z_2$ in [9] becomes $z_1$ when we restrict to have $G_2$-invariant flow by taking $\lambda' = \lambda$ and $\phi = \alpha$. Moreover we denote here $z_2$ by $z_3$ in [9] on the restricted subspace. For the explicit form of $z_1$ in $A_1$ tensor, we refer to [9].

3The scalar potential can be written explicitly as

$$V(\lambda, \alpha) = 2g^2\left((7v^4 - 7v^2 + 3)c^3s^4 + (4v^2 - 7)v^5s^7 + c^5s^2 + 7v^3c^2s^5 - 3c^3\right)$$

where $c = \cosh\left(\frac{\lambda}{\sqrt{2}}\right)$, $s = \sinh\left(\frac{\lambda}{\sqrt{2}}\right)$ and $v = \cos \alpha$ by getting all the components of $A_1$ and $A_2$ tensors [9, 1].
Figure 1: Contour plots of the scalar potential \( V \) (left) and the superpotential \( W \) (right) in the 4-dimensional gauged supergravity. The axes \((\lambda, \alpha)\) are two vevs that parametrize the \(G_2\) invariant manifold in the 28-beins of the theory. The \(SO(8)\) UV critical point is located at an arbitrary point on the line \(\lambda = 0\), whereas the \(G_2\)-invariant IR critical point is located at \((\lambda, \alpha) = \left(\sqrt{2} \sinh^{-1} \sqrt{\frac{2}{5}} (\sqrt{3} - 1), \cos^{-1} \frac{1}{2} \sqrt{3 - \sqrt{3}} \right) \approx (0.732, 0.973)\). We have set the \(SO(8)\) gauge coupling \(g\) in the scalar potential as \(g = 1\). We also denote that the \(SO(7)^+\)-invariant IR critical point is located at \((\lambda, \alpha) = \left(\sqrt{2} \sinh^{-1} \sqrt{\frac{1}{2} \left(\frac{1}{\sqrt{3}} - 1\right)}, 0\right) \approx (0.569, 0)\) and the \(SO(7)^-\)-invariant IR critical point is located at \((\lambda, \alpha) = \left(\sqrt{2} \sinh^{-1} \frac{1}{2}, \frac{\pi}{2}\right) \approx (0.681, 1.57)\). Note that \(SO(7)^\pm\) critical points of a scalar potential are not those of a superpotential, observed by the above contour plots.

There exists a common supersymmetric critical point of both a scalar potential and a superpotential at \(\sinh \left(\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{2}{5}} (\sqrt{3} - 1), \cos \alpha = \frac{1}{2} \sqrt{3 - \sqrt{3}}\) where \(W = \sqrt{\frac{36\sqrt{2} 3^{1/4}}{20\sqrt{3}}}\). This implies the supersymmetry preserving vacua have negative cosmological constant because the scalar potential \(V\) at the critical points becomes \(V = -6g^2 W^2\) due to the stationary of the superpotential \(W\). This is the \(\mathcal{N} = 1\) supersymmetric critical point with \(G_2\)-symmetry. The flow we are interested in is the one starting at the \(\mathcal{N} = 8\) supersymmetric point specified by \(\lambda = 0\) and ends at the non-trivial \(\mathcal{N} = 1\) supersymmetric point. We summarize the properties of these critical points in Table 1 and draw the scalar potential and superpotential by contour plots in Figure 1.
3 The \( M \)-theory lift of the \( G_2 \)-invariant holographic RG flow

The \( G_2 \)-invariant RG flow discussed in the previous section is a supersymmetric domain wall solution in the 4-dimensional gauged supergravity. The nonlinear metric ansatz of de Wit-Nicolai-Warner [1] suggests that the RG flow solution in 4 dimensions can be lifted to a certain 11-dimensional solution in \( M \)-theory. In this section, we carry out the \( M \)-theory lift of the \( G_2 \)-invariant RG flow through a combinatoric use of the 4-dimensional RG flow equations and the 11-dimensional Einstein-Maxwell equations derived from the nonlinear metric ansatz of de Wit-Nicolai-Warner.

3.1 The 11-dimensional metric for the \( G_2 \)-invariant RG flow

The consistency under the Kaluza-Klein compactification of 11-dimensional supergravity, or \( M \)-theory, requires that the 11-dimensional metric for the \( G_2 \)-invariant RG flow is not simply a metric of product space but Eq. (1.1) for the warped product of \( AdS_4 \) space with a 7-dimensional compact manifold. Moreover, the 7-dimensional space becomes a warped-ellipsoidally deformed \( S^7 \) and its metric is uniquely determined through the nonlinear metric ansatz developed in [1, 4].

The 7-dimensional inverse metric is given by the formula (3.1):

\[
\begin{align*}
g^{mn} &= \frac{1}{2} \Delta \left[ \hat{K}^{mIJ} \hat{K}^{nKL} + (m \leftrightarrow n) \right] \left( u_{ij}^{IJ} + v_{ijIJ} \right) \left( \pi_{ij KL} + \pi^{ijKL} \right),
\end{align*}
\]

where \( \hat{K}^{mIJ} \) denotes the Killing vector on the round \( S^7 \) with 7-dimensional coordinate indices \( m, n = 5, \ldots, 11 \) as well as \( SO(8) \) vector indices \( I, J = 1, \ldots, 8 \). The \( u_{ij}^{IJ} \) and \( v_{ijIJ} \) are 28-beins in 4-dimensional gauged supergravity and are parametrized by the \( AdS_4 \) vacuum expectation values(vevs), \( \lambda \) and \( \alpha \), associated with the spontaneous compactification of 11-dimensional supergravity.

The 28-beins \( (u, v) \) or two vevs \( (\lambda, \alpha) \) are given by functions of the \( AdS_4 \) radial coordinate \( r = x^4 \). The metric formula (3.1) generates the 7-dimensional metric from the two input data of \( AdS_4 \) vevs \( (\lambda, \alpha) \). The \( G_2 \)-invariant RG flow subject to (2.2) is a trajectory in \( (\lambda, \alpha) \)-plane and is parametrized by the \( AdS_4 \) radial coordinate. Hereafter, instead of \( (\lambda, \alpha) \), we will use \( (a, b) \) defined by

\[
\begin{align*}
a &= \cosh \left( \frac{\lambda}{\sqrt{2}} \right) + \cos \alpha \sinh \left( \frac{\lambda}{\sqrt{2}} \right), \\
b &= \cosh \left( \frac{\lambda}{\sqrt{2}} \right) - \cos \alpha \sinh \left( \frac{\lambda}{\sqrt{2}} \right).
\end{align*}
\]

Let us first introduce the standard metric of a 7-dimensional ellipsoid. Using the diagonal
matrix $Q_{AB}$ given by

$$Q_{AB} = \text{diag} \left( 1, \ldots, 1, \frac{a^2}{b^2} \right),$$

the metric of a 7-dimensional ellipsoid with the eccentricity $\sqrt{1 - b^2/a^2}$ can be written as

$$ds^2_{EL(7)}(a,b) = dX^A Q_{AB}^{-1} dX^B - \frac{b^2}{\xi^2} \left( X^A \delta_{AB} dX^B \right)^2, \quad (3.3)$$

where the $R^8$ coordinates $X^A (A = 1, \ldots, 8)$ are constrained on the unit round $S^7$, $\sum_A (X^A)^2 = 1$, and $\xi^2 = b^2 X^A Q_{AB} X^B$ is a quadratic form on the ellipsoid. The standard metric (3.3) can be rewritten in terms of the 7-dimensional coordinates $y^m$ such that

$$ds^2_{EL(7)}(a,b) = a^{-2} \xi^2 d\theta^2 + \sin^2 \theta d\Omega_6^2, \quad (3.4)$$

where $\theta = y^7$ is the fifth coordinate in 11 dimensions and the quadratic form $\xi^2$ is now given by

$$\xi^2 = a^2 \cos^2 \theta + b^2 \sin^2 \theta, \quad (3.5)$$

which turns to 1 for the round $S^7$ with $(a,b) = (1,1)$. We will see that the 7-dimensional metric generated from the formula (3.1) is a “warped” metric of a 7-dimensional ellipsoid (3.3), where the geometric parameters $(a,b)$ for the 7-ellipsoid can be identified with the two vevs $(a,b)$ defined in [3.2]. It is the reason why we have introduced $(a,b)$ in (3.2) and prefer them rather than the original vevs $(\lambda, \alpha)$. In fact the Ricci tensor components generated from the 11-dimensional metric which we will derive in this subsection become much simpler in terms of $(a,b)$ (See Appendix B).

To calculate the 7-dimensional metric by using the formula (3.1), first we have to specify the Killing vector on the round $S^7$. It is given by

$$\overset{\circ}{K}_m^{IJ} = (\Gamma^{IJ})_{AB} L^2 \left( X^A \partial_m X^B - X^B \partial_m X^A \right), \quad (3.6)$$

where $X^A (A = 1, \ldots, 8)$ are the $R^8$ coordinates of the unit round $S^7$ and parametrized by the 7-dimensional coordinates $y^m$. The $8 \times 8$ matrices $\Gamma^{IJ}$ are the $SO(8)$ generators which rotate the $R^8$ indices $A, B$ into the $SO(8)$ indices $I, J$.

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4 Note that our assignment of $(a,b)$ in $Q_{AB}$ is different from (5.46) in [3]. The former is obtained by switching $a$ and $b$ and by reparametrizing the $R^8$ coordinates $X^A$ in the latter.

5 The $R^8$ coordinates $X^A$ are given in terms of the 7-dimensional coordinates $y^m$ by

$$X^1 = s_1 s_2 s_3 s_4 s_5 s_6 s_7, \quad X^2 = c_1 s_2 s_3 s_4 s_5 s_6 s_7, \quad X^3 = c_2 s_3 s_4 s_5 s_6 s_7, \quad X^4 = c_3 s_4 s_5 s_6 s_7,$$

$$X^5 = c_4 s_5 s_6 s_7, \quad X^6 = c_5 s_6 s_7, \quad X^7 = c_6 s_7, \quad X^8 = c_7,$$

where $s_m \equiv \sin y^m$ and $c_m \equiv \cos y^m (m = 1, \ldots, 7)$.

6 The $SO(8)$ matrices $\Gamma^{IJ}$ are defined by $\Gamma^{IJ} = \frac{1}{2} [\Gamma^I, \Gamma^J]$, $\Gamma^{11} = \Gamma^I$, where $\Gamma^I (I = 2, \ldots, 8)$ are the $SO(7)$ gamma matrices. We refer to Appendix B in [3] for further details about these matrices.
Applying the Killing vector (3.6) together with the 28-beins to the metric formula (3.1), we obtain a “raw” inverse metric \( g^{mn} \) including the warp factor \( \Delta \) not yet determined. Substitution of the raw inverse metric into the definition

\[
\Delta^{-1} \equiv \sqrt{\det(g^{mn} g_{np})},
\]

which is equivalent to (1.3), provides a self-consistent equation for \( \Delta \). For the \( G_2 \)-invariant RG flow, solving this equation yields the warp factor

\[
\Delta(x, y) = a^{-1} \xi^{-\frac{4}{3}}
\]

where \( \xi \) is given in (3.3). Then we substitute this warp factor into the raw inverse metric to obtain the 7-dimensional metric:

\[
ds_7^2 = g_{mn}(y) \, dy^m dy^n = (\Delta a)^{\frac{1}{2}} L^2 \left( a^{-2} \xi^2 \, d\theta^2 + \sin^2 \theta \, d\Omega_6^2 \right),
\]

where one can see that the standard 7-dimensional metric (3.4) is warped by a factor \((\Delta a)^{\frac{1}{2}}\). The nonlinear metric ansatz finally combines the 7-dimensional metric (3.7) with the four dimensional metric with warp factor to yield the warped 11-dimensional metric:

\[
ds_{11}^2 = \Delta^{-1} \left( dr^2 + e^{2A(r)} \eta_{\mu\nu} \, dx^\mu dx^\nu \right) + ds_7^2,
\]

(3.8)

where \( r = x^4 \) and \( \mu, \nu = 1, 2, 3 \) with \( \eta_{\mu\nu} = \text{diag}(-, +, +) \).

Note that two vevs \((\lambda, \alpha)\) as well as the domain wall amplitude \( A(r) \) have non-trivial \( r \)-dependence subject to the RG flow equations (2.2) so that the warp factor and the 7-dimensional metric also depend on \( r \) through the vevs. As we see in (2.2), \( \partial_r \lambda \) and \( \partial_r \alpha \) are proportional to the \( \lambda, \alpha \)-derivatives of \( AdS_4 \) superpotential, respectively. We also see that as a characteristic feature of supersymmetric RG flows both \( SO(8) \) and \( G_2 \) invariant critical points are simultaneous stationary points of the scalar potential and the superpotential in the 4-dimensional gauged supergravity(See Figure [4]). Therefore \((\partial_r \lambda, \partial_r \alpha) = (0, 0)\) at both \( SO(8) \) and \( G_2 \) invariant critical points and the scalar potential is related to the superpotential by \( V = -6 \, g^2 \, W^2 \) as shown in (2.1). On the other hand, \( \partial_r A \) is proportional to the superpotential itself and is relevant in both critical points. Thus one can say that the 11-dimensional geometries of critical point theories are completely determined by both vevs \((a, b)\) and scalar potential \( V \) without knowing the RG flow equations, that is, the \( r \)-dependence of the vevs.

Furthermore, the superkink amplitude \( A(r) \) asymptotically behaves as \( A(r) = r / r_{cr} \) near the critical points so that one can read \( 1 / r_{cr} \propto \sqrt{-V_{cr}} \) via the last equation in (2.2). Provided that the two vevs and the slope of the superkink \( r_{cr} \)(or the value of scalar potential \( V_{cr} \)) are known at each critical point, the 11-dimensional metric (3.8) reproduces the metric derived in [4]. The \( SO(8) \)-invariant critical point is specified by \((a, b) = (1, 1)\) and \( A(r) = 2r / L \) with the
AdS\textsubscript{4} radius $r_{ct} = L/2$. The metric (3.8) is so normalized for the maximally symmetric case as to generate the Ricci tensor:

$$R^N_M = \frac{6}{L^2} \text{diag} \left( -2, -2, -2, -2, 1, \ldots, 1 \right),$$

which fixes the round $S^7$ radius to be $L$, twice the AdS\textsubscript{4} radius, as expected. Since the superpotential satisfies $W = 1$ at the SO(8)-invariant critical point, this normalization fixes the SO(8) gauge coupling $g$ to be $\sqrt{2}L^{-1}$.

### 3.2 The Einstein-Maxwell equations for the $G_2$-invariant RG flow

The non-trivial $r$-dependence of vevs requires that the 11-dimensional Einstein-Maxwell equations become consistent not only at critical points but also along the supersymmetric RG flow connecting the two critical points. Even though Ref. [4] provides another formula involving both the metric and the 4-form gauge field strength, the analysis in mid eighties before AdS/CFT correspondence just focused on the critical point theories with constant vevs so that the $G_2$-invariant field strength ansatz in [1] is not quite general: they hold only for constant vevs. For solutions with varying scalars, the ansatz for the field strength will be a little more complicated. Although the results of [4] provide both the metric and the 4-form gauge field strength for nonconstant scalars, it is not very convenient to use. It is an open problem to identify our results here with those in [4]. In this subsection, we will apply the most general $G_2$-invariant ansatz for an 11-dimensional 3-form gauge field by acquiring the $r$-dependence of vevs and will derive the 11-dimensional Einstein-Maxwell equations corresponding to the $G_2$-invariant RG flow. Hereafter we use Greek indices for the 3-dimensional spacetime of a membrane world volume, whereas Latin indices for the $G_2$-invariant 6-dimensional unit sphere. Specifically, $\mu, \nu, \ldots = 1, 2, 3$ and $m, n, \ldots = 6, \ldots, 11$.

As a natural extension of the Freund-Rubin parametrization [10], the 3-form gauge field with 3-dimensional membrane indices may be defined by [2, 11]

$$A_{\mu\nu\rho} = -e^{3A(r)} \tilde{W}(r, \theta) \epsilon_{\mu\nu\rho}, \quad (3.9)$$

where $\tilde{W}(r, \theta)$ is a “geometric” superpotential [13] which will be relevant in search of a membrane probe moduli space (for example, [11]) and to be determined. This is simply geometric superpotential $\tilde{W}(r, \theta)$ times the volume form on the membrane measured using the four-dimensional metric (1.4). The exponential factor $e^{3A(r)}$ will be compensated by the same factor arising from the 11-dimensional metric (3.8) when we derive the geometric superpotential. The $\theta$-dependence of $\tilde{W}(r, \theta)$ is essential to achieve the $M$-theory lift of the RG flow.

The other components of 3-form gauge field have the indices on the round $S^6$ which is isomorphic to the coset space $G_2/SU(3)$ including the $G_2$-symmetry of our interest [17].
is explicitly performed in [17], one can geometrically construct the $G_2$-covariant tensors living on the round $S^6$ by using the imaginary octonion basis of $S^6$. Since one cannot construct any $G_2$-invariant vector through contracting those tensors, the $G_2$-covariance of the gauge field is achieved by using the $G_2$-covariant tensors only. Thus we arrive at the most general $G_2$-invariant ansatz:

$$
A_{4mn} = g(r, \theta) F_{mn},
$$
$$
A_{5mn} = h(r, \theta) F_{mn},
$$
$$
A_{mnp} = h_1(r, \theta) T_{mnp} + h_2(r, \theta) S_{mnp},
$$

(3.10)

where $m, n, p$ are the $S^6$ indices and run from 6 to 11. The almost complex structure on the $S^6$ is denoted by $F_{mn}$ and obeys $F_{mn} F^{nl} = -\delta^l_m$. The $S_{mnp}$ is the parallelizing torsion of the unit round $S^7$ projected onto the $S^6$, while the $T_{mnp}$ denotes the 6-dimensional Hodge dual of $S_{mnp}$. We refer to [17] for further details about the tensors. In the ansatz (3.10) in addition to the $r$-dependence of coefficients, the $A_{4mn}$ is new compared with the ansatz in previous works [1, 17] (See for example (3.4) in [17]). The above ansatz for gauge field is in fact the most general one which preserves the $G_2$-invariance and is consistent with the 11-dimensional metric (3.8).

Through the definition $F_{MNPQ} \equiv 4 \partial_{[M} A_{NPQ]}$, the ansatz (3.9) generates the field strengths

$$
F_{\mu\rho\sigma} = e^{3A(r)} W_r(r, \theta) \epsilon_{\mu\rho\sigma},
$$
$$
F_{\mu\nu\rho5} = e^{3A(r)} W_\theta(r, \theta) \epsilon_{\mu\nu\rho},
$$

(3.11)

while the ansatz (3.10) provides

$$
F_{mnpq} = 2 h_2(r, \theta) \epsilon_{mnpqrs} F^{rs},
$$
$$
F_{5mnp} = h_1(r, \theta) T_{mnp} + h_2(r, \theta) S_{mnp},
$$
$$
F_{4mnp} = h_3(r, \theta) T_{mnp} + h_4(r, \theta) S_{mnp},
$$
$$
F_{45mn} = h_5(r, \theta) F_{mn},
$$

(3.12)

where the coefficient functions which depend on both $r$ and $\theta$ are given by

$$
W_r = e^{-3A} \partial_r (e^{3A} \tilde{W}), \quad W_\theta = e^{-3A} \partial_\theta (e^{3A} \tilde{W}),
$$

\[7\]

In general, one cannot neglect $A_{\mu mn}, A_{\mu 4}, A_{\mu 5}$ and $A_{\mu 45}$ by the $G_2$-invariance only. However, except for $A_{\mu 45}$, such gauge fields provide the field strengths $F_{\mu mn}, F_{\mu mn4}, F_{\mu mn5}$ and $F_{\mu 45}$ which generate off-diagonal $(\mu m)$-, $(\mu 4)$-, $(\mu 5)$-components in the RHS of the Einstein equation. The Ricci tensor in the LHS, however, do not contain such off-diagonal elements and forbids the $A_{\mu mn}, A_{\mu 4}$ and $A_{\mu 5}$ to arise in the gauge field ansatz. The $A_{\mu 45}$ depends on $(r, \theta)$ only so that it does not affect field strengths at all. It can be zero as a gauge fixing.

\[8\]

Our convention of epsilon tensors is that the tensors with lower indices are purely numerical. The 6-dimensional epsilon tensor in (3.12) is the only exception and is defined as a constant tensor density on the round $S^6$, that is $\epsilon_{mnpqrs} = \sqrt{g_6} \epsilon_{mnpqrs}$, where $g_6$ is the determinant of the $S^6$ metric.
$$\tilde{h}_1 = \partial_b h_1 - 3 h, \quad \tilde{h}_2 = \partial_b h_2,$$
$$\tilde{h}_3 = \partial_c h_1 - 3 g, \quad \tilde{h}_4 = \partial_c h_2,$$
$$\tilde{h}_5 = \partial_c h - \partial g. \quad (3.13)$$

Comparing with the ansatz in the previous works [1, 17], the mixed field strengths $F_{\mu\nu\rho\delta}$, $F_{4mnp}$ and $F_{45mn}$ are new. In fact, they are not forbidden to arise by the $G_2$-invariance once we suppose that the 4-dimensional metric has the domain wall factor $e^{3A(r)}$ which breaks the 4-dimensional conformal invariance. At both $SO(8)$-invariant UV and $G_2$-invariant IR critical points, the 4-dimensional spacetime becomes asymptotically $AdS_4$ and the mixed field strengths should vanish there. Especially, the $F_{\mu\nu\rho\delta}$ and $F_{45mn}$ are proportional to $W_\theta$ and $\tilde{h}_5$, respectively, so that they must be subject to the non-trivial boundary conditions:

$$W_\theta = 0, \quad \tilde{h}_5 = 0, \quad (3.14)$$

at both UV and IR critical points. It is easy to see that the $F_{4mnp}$ goes to zero at both critical points without imposing any boundary condition since the field strength comes from the $r$-derivative of $A_{mnp}$ (note that $r = x^4$) and the vevs become $r$-independent at both critical points.

We also have the 11-dimensional Bianchi identity $\partial_{[M} F_{NPQR]} = 0$ which reads

$$\partial_\theta W_\mu - \partial_\mu W_\theta = 3 (\partial_\mu A) W_\theta, \quad \partial_\mu (\tilde{h}_1 + 3 h) = \partial_\mu (\tilde{h}_3 + 3 g), \quad \partial_\mu \tilde{h}_2 = \partial_\mu \tilde{h}_4, \quad (3.15)$$

and simply provides the integrability conditions for the gauge fields $e^{3A} \tilde{W}$, $h_1$ and $h_2$. The $W_\mu$ and $W_\theta$ obtained by solving the 11-dimensional Einstein-Maxwell equations should satisfy the first equation by imposing the RG flow equations.

Applying the field strength ansatz (3.11), (3.12) and the metric (3.8) to the 11-dimensional Maxwell equation in (3.3), we obtain (See (A.1-A.4) in Appendix A also)

$$a^3 D_\theta \tilde{h}_1 + L^2 D_r \tilde{h}_3 = 2 L \xi^{-2} (W_r \tilde{h}_2 - W_\theta \tilde{h}_4),$$
$$a^3 D_\theta \tilde{h}_2 + L^2 D_r \tilde{h}_4 = 2 L \xi^{-2} (W_\theta \tilde{h}_3 - W_r \tilde{h}_1) + \frac{12 a \xi^2 h_2}{\sin^2 \theta},$$
$$a^3 D_\theta W_\theta + L^2 D_r W_\theta = -\frac{8 a^3 \xi^4}{L^5 \sin^4 \theta} \left( h_1 \tilde{h}_4 - h_3 \tilde{h}_2 - 3 h_2 \tilde{h}_5 \right), \quad (3.16)$$

with $a^3 \tilde{h}_1$ and $L^2 \tilde{h}_3$ solved to be

$$a^3 \tilde{h}_1 = 2 L \xi^{-2} W_r h_2 - \frac{1}{4} L^2 a^2 \xi^{-2} \sin^2 \theta D_r \tilde{h}_5,$$
$$L^2 \tilde{h}_3 = -2 L \xi^{-2} W_\theta h_2 + \frac{1}{4} L^2 a^2 \xi^{-2} \sin^2 \theta D_\theta \tilde{h}_5. \quad (3.17)$$

9 We implicitly suppose that the field strength functions $\tilde{h}_1, \ldots, \tilde{h}_5$ are finally written as polynomials of vevs $(a,b)$ after the 11-dimensional Einstein-Maxwell equations are solved by imposing the RG flow equations. In Section 3.3, we will see that is the case.
Here we have defined
\[
(D_\theta, D_r) \equiv \hat{e}^{-1} (\partial_\theta, \partial_r) \hat{e}, \quad \hat{e} \equiv \xi^2 e^{3A},
\]
\[
(\bar{D}_\theta, \bar{D}_r) \equiv \bar{e}^{-1} (\partial_\theta, \partial_r) \bar{e}, \quad \bar{e} \equiv a^{-3} \xi^{-4} \sin^6 \theta,
\]
\[
(\bar{D}_\theta, \bar{D}_r) \equiv \bar{e}^{-1} (\partial_\theta, \partial_r) \bar{e}, \quad \bar{e} \equiv a^2 \xi^3 \sin^2 \theta.
\]

The first equation in (3.16) is trivially satisfied via substitution of (3.17). The second equation becomes
\[
a^3 D_\theta h_2' + L^2 D_r h_2 + a^3 \left[ 4 f^2 L^2 a^{-6} \xi^{-4} - \frac{12 a^{-2} \xi^2}{\sin^2 \theta} \right] h_2 \\
= \frac{1}{2} L a^{-1} \xi^{-4} \sin^2 \theta \left( a^3 W_\theta D_\theta + L^2 W_r D_r \right) \bar{h}_5,
\]
(3.18)
where the parameter \( f \) defined by
\[
f \equiv \sqrt{a^3 W_\theta^2 + L^2 W_r^2}
\]
(3.19)
corresponds to the Freund-Rubin parameter in each critical point theory. Note that (3.18) goes to (4.7) in [1] at both UV and IR critical points. To see that, one only has to remember that \( \bar{h}_5 = 0 \) and \( h_2 \) becomes \( r \)-independent at both critical points. Then one can see that, in (3.18), the RHS and the second term of the LHS vanish at both critical points to reduce the equation to (4.7) in [1]. To be more precise, one also has to show that the generalized Freund-Rubin parameter \( f \) becomes constant at both critical points. This can be seen from the boundary condition \( W_\theta = 0 \) and the 11-dimensional Einstein equation discussed below which certifies that the \( W_r \) becomes constant at the critical points.

The third equation becomes
\[
a^3 \bar{D}_\theta W_\theta + L^2 \bar{D}_r W_r = -\frac{16 \xi^2}{L^6 \sin^6 \theta} \left[ a^3 W_\theta h_2' h_2 + L^2 W_r h_2 h_2 - \frac{3}{2} L a^3 \xi^2 h_2 \bar{h}_5 \\
-\frac{1}{8} L a^2 \sin^2 \theta \left( a^3 h_2' D_\theta + L^2 h_2 D_r \right) \bar{h}_5 \right],
\]
(3.20)
which is new and is relevant only at the intermediate points of RG flow connecting the two critical points. One can easily see that the equation is satisfied at the two critical points (or end points of RG flow) by recalling the boundary condition (3.14) and that both \( h_2 \) and \( W_r \) become \( r \)-independent as well as two vevs \((a, b)\) there.

To summarize, the 11-dimensional Maxwell equations (3.16) reduce to the two independent equations (3.18) and (3.20) which consist of the \( G_2 \)-invariant 11-dimensional field equations together with the 11-dimensional Einstein equation discussed below. They should be satisfied

\[\text{We use the notation } h_2' \equiv \partial_\theta h_2 \text{ and } \bar{h}_2 \equiv \partial_r h_2.\]
everywhere along the supersymmetric RG flow if the \( M \)-theory solution corresponding to the \( G_2 \)-invariant RG flow exists. In other words, the equations must be consistent with each other provided that the \( r \)-dependence of field strengths is only through the vevs which are subject to the RG flow equations (2.2).

Via the field strengths ansatz (3.11), (3.12) and the warped metric (3.8), the 11-dimensional Einstein equation in (1.3) goes to (See (A.5-A.9) in Appendix A also)

\[
R_{1}^{1} = -\frac{4}{3} c_3 f^2 - \frac{1}{3} c_1 \left[ H + \frac{12 a \xi^2 h_3^2}{\sin^2 \theta} + c_2 \bar{h}_5^2 \right],
\]

\[
R_{4}^{4} = R_{1}^{1} + 2 c_3 L^{-2} a^3 W_{\theta}^2 + c_1 \left[ L^2 \left( \bar{h}_3^2 + \bar{h}_4^2 \right) + c_2 \bar{h}_5^2 \right],
\]

\[
R_{5}^{5} = R_{1}^{1} + 2 c_3 W_{r}^2 + c_1 \left[ a^3 \left( \bar{h}_1^2 + \bar{h}_2^2 \right) + c_2 \bar{h}_5^2 \right],
\]

\[
R_{6}^{6} = R_{1}^{1} + 2 c_3 f^2 + \frac{1}{2} c_1 \left[ H + \frac{16 a \xi^2 h_2^2}{\sin^2 \theta} + \frac{2}{3} c_2 \bar{h}_5^2 \right],
\]

\[
R_{5}^{4} = -2 c_3 W_{r} W_{\theta} + c_1 L^2 \left( \bar{h}_1 \bar{h}_3 + \bar{h}_2 \bar{h}_4 \right),
\]

(3.21)

where we have defined

\[
c_1 \equiv \frac{8 a^{-1} \xi^2}{L^8 \sin^6 \theta}, \quad c_2 \equiv \frac{3}{4} L^2 a^2 \xi^{-2} \sin^2 \theta, \quad c_3 \equiv a^{-4} \xi^{-\frac{16}{3}},
\]

and

\[
H \equiv a^3 \left( \bar{h}_1^2 + \bar{h}_2^2 \right) + L^2 \left( \bar{h}_3^2 + \bar{h}_4^2 \right).
\]

(3.22)

The Ricci tensor components in the LHS are generated by the warped 11-dimensional metric (3.8) as listed in Appendix B. Note that the off-diagonal (4,5), (5,4)-components arise due to \( \bar{h}_5 \), the \( \theta \)-dependence of \( \bar{W}(r, \theta) \) and the \( r \)-dependence of \( h_2 \). The boundary conditions (3.14) ensure that those off-diagonal components vanish and the diagonal \( R_{4}^{4} \) and \( R_{5}^{5} \) components degenerate to \( R_{1}^{1} \) and \( R_{6}^{6} \), respectively, at both critical points [1].

Even though one does not know an appropriate ansatz for \( h_2 \) and \( \bar{h}_5 \), the Einstein equations (3.21) can be solved with respect to certain combinations of field strength squares. For instance, the generalized Freund-Rubin parameter \( f \) can be solved to be

\[
f = \sqrt{\frac{R_{4}^{4} + R_{5}^{5} + 4 R_{1}^{1} + 6 R_{6}^{6}}{-2 c_3}}.
\]

(3.23)

One can also see that \( h_2, \bar{h}_5 \) and \( H \) obey the following equations

\[
\left( \frac{12 a \xi^2}{\sin^2 \theta} \right) \bar{h}_5^2 - c_2 \bar{h}_5^2 = \frac{3}{c_1} \left( R_{1}^{1} + 2 R_{6}^{6} \right),
\]

(3.24)

\[
H + 2 c_2 \bar{h}_5^2 = \frac{2}{c_1} \left( R_{1}^{1} + R_{4}^{4} + R_{5}^{5} + 3 R_{6}^{6} \right).
\]

(3.25)

\[\text{footnote}{\text{11 Other nonzero components are}}\]

\[
R_{2}^{2} = R_{4}^{4} = R_{1}^{1}, \quad R_{7}^{7} = R_{8}^{8} = R_{9}^{9} = R_{10}^{10} = R_{11}^{11} = R_{6}^{6}, \quad R_{5}^{5} = L^{-2} a^3 R_{5}^{4}.
\]
In order to determine all field strengths $W_r$, $W_\theta$, $h_2$ and $\tilde{h}_5$, separately, one has to specify a certain functional form of $\tilde{h}_5$ as an ansatz. Substitution of this ansatz into the Maxwell equation (3.18) may uniquely determine $\tilde{h}_5$. The other field strengths are automatically generated by substituting the $\tilde{h}_5$ into other 11-dimensional field equations and by invoking the RG flow equations. Suppose that there exists such a non-trivial solution to the $\tilde{h}_5$, there arises a non-trivial question whether the obtained field strengths are consistent with each other everywhere along the RG flow. It is the subject in Section 3.3 to answer this question.

3.3 The $M$-theory lift of the $G_2$-invariant RG flow: An exact solution in 11-dimensions

In Section 3.2 we have derived the 11-dimensional Einstein-Maxwell equations from the warped metric (3.8) and the field strength ansatz (3.11) and (3.12). The 11-dimensional field equations are closed within the field strengths $W_r$, $W_\theta$, $h_2$ and $\tilde{h}_5$ although they cannot be solved separately without imposing certain ansatz for them. Our final goal is to determine all the field strengths as polynomials of vevs $(a, b)$ by imposing their $r$-dependence controlled by the RG flow equations (2.2) and to confirm whether those field strengths satisfy the Maxwell equations (3.18) and (3.20) everywhere along the $G_2$-invariant RG flow. If this is the case, the field strengths $(W_r, W_\theta, h_2, h_5)$ and the 11-dimensional metric (3.8) consist of an exact $M$-theory solution provided that the $r$-dependence of vevs $(a, b)$ and the superkink amplitude $A(r)$ are subject to the 4-dimensional RG flow equations (2.2).

The key idea of the nonlinear metric ansatz in [1] is to recognize that a certain compactification of 11-dimensional supergravity goes to a critical point of $d = 4$, $\mathcal{N} = 8$ gauged supergravity after a truncation of the full 11-dimensional mass spectrum to its massless sector (including the scalar fields in 28-beins). For the supersymmetric critical points, such a truncation becomes consistent only if the supersymmetry transformation can be closed within the massless sector by imposing the nonlinear metric ansatz. This means that the metric formula (1.2) derived from the nonlinear metric ansatz is valid even for the scalar fields varying along the $AdS_4$ radial coordinates. On the other hand, the $G_2$-invariant RG flow is a supersymmetric domain wall solution with varying scalars interpolating two critical points with constant scalar vevs. With the aforementioned in mind, we have enough reason to believe the existence of an $M$-theory lift of the $\mathcal{N} = 1$, $G_2$-invariant RG flow.

As we see in the 11-dimensional metric (3.8), the two vevs $(\lambda, \alpha)$ arise in the metric only through the combinations of $(a, b)$ given in (3.2). The Ricci tensor calculated from the metric therefore contains $(a, b, A)$ and their $r$-derivatives only (See Appendix B). We have shown in Section 3.1 that the $(a, b)$ are in fact the geometric parameters which specify the warped ellipsoidal deformation of $S^7$ through $Q_{AB}$. With taking these facts into account, we prefer
to use \((a, b)\) and their flow equations rather than \((\lambda, \alpha)\) and the original flow equations (2.2). In terms of \((a, b)\) the flow equations (2.2) read in symmetric form

\[
\begin{align*}
\partial_r a &= -\frac{8}{7L} \left[ a^2 \partial_a W(a, b) + (ab - 2) \partial_b W(a, b) \right], \\
\partial_r b &= -\frac{8}{7L} \left[ b^2 \partial_b W(a, b) + (ab - 2) \partial_a W(a, b) \right], \\
\partial_r A &= \frac{2}{L} W(a, b),
\end{align*}
\]

(3.26)

where \(W(a, b)\) is the same \(AdS_4\) superpotential as in (2.1) but now is given by a polynomial of \((a, b)\):

\[
W(a, b) = \frac{1}{8} a^3 \sqrt{(a^2 + 7b^2)^2 - 112 (ab - 1)}.
\]

(3.27)

The \(SO(8)\) gauge coupling constant \(g\) in (2.2) has been replaced with \(\sqrt{2} L^{-1}\). Note that the derivatives of \(W(a, b)\) with respect to \(a\) and \(b\) do not vanish at the critical points. However, the \(r\)-derivatives of \(a\) and \(b\) do vanish at the critical points.

From the point of view of 11-dimensional supergravity, the flow equations (3.26) can be regarded as an ansatz for the \(r\)-dependence of the geometric parameters \((a, b)\) and the superkink amplitude \(A(r)\). If the \(G_2\)-invariant RG flow can be lifted to an \(M\)-theory solution, this ansatz must be correct and the 11-dimensional field equations can be solved by calculating all the \(r\)-derivatives with the flow equations (3.26).

As mentioned in Section 3.2, a certain ansatz for \(\tilde{h}_5\) is required to separate out each field strength from the Einstein equation (3.21). Calculating Ricci tensor components by invoking the flow equations (3.26), Eq. (3.24) can be written as

\[
\left( \frac{12 a \xi^2}{\sin^2 \theta} \right) h_2^2 - c_2 \tilde{h}_5^2 = 12 L^6 K_0(a, b) a \xi^{-2} \sin^6 \theta,
\]

(3.28)

where the polynomial \(K_0\) is given by

\[
K_0(a, b) = \frac{2 (ab - 1) (ab - 2) (a^2 - 7b^2)}{(a^2 + 7b^2)^2 - 112 (ab - 1)}.
\]

Eq. (3.28) immediately leads to the ansatz:

\[
\begin{align*}
h_2 &= L^3 \sqrt{K(a, b)} \xi^{-2} \sin^4 \theta, \\
\tilde{h}_5 &= 4L^2 \sqrt{K_5(a, b)} a^{-\frac{1}{2}} \sin^2 \theta,
\end{align*}
\]

(3.29)

where the polynomials \(K\) and \(K_5\) are subject to the constraint \(K - K_5 \equiv K_0\). Note that this ansatz corresponds to (4.13) in the previous work [1], except that \(\tilde{h}_5\) is identically zero from the beginning in [1] which focused on the critical point theories only.
The first Maxwell equation (3.18) involves \((W_r, W_\theta)\) and seems not to be closed within \((h_2, \tilde{h}_5)\). However, this is not true and by using \(H\) given by (3.25) the equation can be rewritten as a differential equation closed within \((h_2, \tilde{h}_5)\). Specifically, the equation turns to

\[
a^3 h_2 D_{\theta} h_2' + L^2 h_2 D_r \dot{h}_2 + a^3 \left[ 2 f^2 L^2 a^{-6} \xi^{-4} - \frac{12 a^{-2} \xi^2}{\sin^2 \theta} \right] h_2^2 = \frac{1}{2} \left[ a^3 \left( h_2' \right)^2 + L^2 \left( \dot{h}_2 \right)^2 - H \right] + \frac{1}{18} c_2^2 \left( \frac{1}{L^2} \left( D_\theta \tilde{h}_5 \right)^2 + \frac{1}{a^3} \left( D_r \tilde{h}_5 \right)^2 \right),
\]

(3.30)

where \(f\) and \(H\) are given by (3.23) and (3.25) and are directly determined by Ricci tensor components without knowing \((W_r, W_\theta)\).

Applying the ansatz (3.29) to the modified equation (3.30), both \(K\) and \(K_5\) are consistently determined as polynomials of \((a, b)\) by using the RG flow equations. They are

\[
K(a, b) = \frac{1}{4} b^2 (ab - 1),
\]

\[
K_5(a, b) = \frac{1}{4} \frac{(ab - 1) (-4a + a^2 b + 7b^2)^2}{(a^2 + 7b^2)^2 - 112 (ab - 1)}.
\]

(3.31)

Substituting \((h_2, \tilde{h}_5)\) in (3.29) with \((K, K_5)\) in (3.31) into the (44)- and (55)-components of the Einstein equation, one can obtain the equations which are closed in \(W_r\) and in \(W_\theta\), respectively. Solving these equations yields

\[
W_r = -\frac{L}{2} a^2 \left[ a^5 \cos^2 \theta + a^2 b (ab - 2) (4 + 3 \cos 2\theta) + b^3 (7ab - 12) \sin^2 \theta \right],
\]

\[
W_\theta = -\frac{a^3 \left[ 48 (1 - ab) + (a^2 - b^2) (a^2 + 7b^2) \right]}{8 W(a, b)} \sin \theta \cos \theta.
\]

(3.32)

The \((h_2, \tilde{h}_5)\) in (3.29) with \((K, K_5)\) in (3.31) and the \((W_r, W_\theta)\) in (3.32) consist of a \(G_2\)-invariant solution in 11-dimensional supergravity. The remaining thing we have to do is to check consistency of this solution along the \(G_2\)-invariant RG flow.

The generalized Freund-Rubin parameter \(f\) given by (3.23) should coincide with the \(f\) calculated from the solutions \((W_r, W_\theta)\) in (3.32) through (3.19). One can prove that this is the case by using the RG flow equations (3.26). Similarly, the \(H\) given by (3.25) is proved to be consistent with its definition (3.22) along the RG flow. All the 11-dimensional field equations (3.18), (3.20) and (3.21) are satisfied by substituting the 11-dimensional solution and imposing the RG flow equations (3.26).

We also have to check whether the 11-dimensional solution satisfies the boundary conditions (3.14) at both UV and IR critical points. The two input data of \((a, b)\) are

\[
a = 1, \quad b = 1,
\]

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for the $SO(8)$-invariant UV critical point, whereas
\[ a = \sqrt{\frac{6\sqrt{3}}{5}}, \quad b = \sqrt{\frac{2\sqrt{3}}{5}}, \]
for the $G_2$-invariant IR critical point. From the $K_5$ in (3.31) and the $W_\theta$ in (3.32), one can easily see that both $W_\theta$ and $\tilde{h}_5$ actually vanish at both critical points. Note that at the critical points $W_\theta$ vanishes so that $W_r$ is equal to the Freund-Rubin constant $f$. From the $K$ in (3.31) and the $W_r$ in (3.32), two field strengths $(f, h_2)$ read
\[ f = 3L^{-1}, \quad h_2 = 0, \]
at the UV critical point, whereas
\[ f = \frac{108}{25} \sqrt{\frac{2\sqrt{3}}{5}} L^{-1}, \quad h_2 = \frac{\sqrt{2\sqrt{3}}}{10} L^3 \xi^{-2} \sin^4 \theta, \]
at the IR critical point to reproduce the critical point theory results in [1].

Thus we have established that the solutions (3.29), (3.31) and (3.32) actually consist of an exact solution to the 11-dimensional supergravity, provided that the deformation parameters $(a, b)$ of the 7-ellipsoid and the domain wall amplitude $A(r)$ develop in the $AdS_4$ radial direction along the $G_2$-invariant RG flow (3.26).

Lastly, let us derive the geometric superpotential $\tilde{W}$ from the field strengths $(W_r, W_\theta)$. Recall that $W_r$ and $W_\theta$ are defined by (3.13) which is encoded to
\[ W_r = \partial_r \tilde{W} + 3 (\partial_r A) \tilde{W}, \quad W_\theta = \partial_\theta \tilde{W}. \tag{3.33} \]
By integrating $W_\theta$ with respect to $\theta$, the geometric superpotential is determined up to a polynomial of vevs such that
\[ \tilde{W} = \frac{a^3 \left[48(1 - ab) + (a^2 - b^2)(a^2 + 7b^2)\right]}{16 W(a, b)} \cos^2 \theta + X(a, b). \]
Substituting this equation into the first equation in (3.33) and using the RG flow equations (3.24), we find that the polynomial $X(a, b)$ should obey the equation
\[ \partial_r X + \frac{6}{L} WX = -\frac{1}{2L} a^2 b (-2a^2 + a^3 b - 12b^2 + 7ab^3). \]
To solve this equation, we require as an ansatz that the $X(a, b)$ should be chosen to make the $\tilde{W}(a, b, \theta)$ coincide with the 4-dimensional superpotential $W(a, b)$ up to a multiplicative constant when $\theta$ is fixed to some specific value. Then the $X(a, b)$ can be solved as
\[ X(a, b) = \frac{a^3 \left[8(1 - ab) + b^2(a^2 + 7b^2)\right]}{16 W(a, b)}. \]
which finally yields the geometric superpotential:

\[ \tilde{W}(a, b, \theta) = \frac{a^3 \left[ 48 (1 - ab) + (a^2 - b^2) (a^2 + 7b^2) \right] \cos^2 \theta + 8 (1 - ab) + b^2 (a^2 + 7b^2) \right]}{16 W(a, b)}. \] (3.34)

Note that the \( \tilde{W} \) coincides with half the superpotential \( W \) when \( \cos \theta = \sqrt{1/8} \). The similar thing happens in the geometric superpotential for the \( \mathcal{N} = 2, SU(3) \times U(1) \) flow found in [2, 11].

4 Discussions

In this paper, we have derived the 11-dimensional Einstein-Maxwell equations corresponding to the \( \mathcal{N} = 1, G_2 \)-invariant RG flow in the 4-dimensional gauged supergravity. The 11-dimensional metric generated from the nonlinear metric ansatz in [1, 4] is the same as the one in the critical point theories except for the non-trivial \( AdS_4 \) radial coordinate \( x^4 = r \) dependence of vevs \( (\lambda, \alpha) \) which can be encoded to the geometric parameters \( (a, b) \) for the 7-ellipsoid. Provided that the \( r \)-dependence of the vevs is controlled by the RG flow equations, we have found an exact solution to the 11-dimensional field equations. With this solution, one can say that the \( G_2 \)-invariant holographic RG flow found in [3, 9] can be lifted to an \( \mathcal{N} = 1 \) membrane flow in \( M \)-theory. In contrast to the \( \mathcal{N} = 2 \) membrane flow recently found in [3, 11], our solution does not have any Kähler structure but the almost complex structure on the round \( S^6 \) so that the field strength ansatz becomes rather complicated as shown in Section 3.2. Moreover, the field strengths \( (W_\theta, \tilde{h}_5) \) must be subject to the non-trivial boundary conditions (3.14) at both UV and IR critical points. Nevertheless, the geometric superpotential (3.34) we have found has the same property as in [3, 11], that is, the geometric superpotential becomes half the 4-dimensional superpotential for a specific angle of \( \theta \) \( (\cos \theta = \sqrt{1/8} \) in \( \mathcal{N} = 1 \) flow).

For future direction, it may be interesting to study a probe membrane moduli space for the \( \mathcal{N} = 1 \) membrane flow by optimizing the geometric superpotential obtained in this paper. It is well known that the exceptional group \( G_2 \) is an automorphism group of octonions [17]. The probe membrane study may shed light on the question how the \( G_2 \) symmetry or octonion algebra plays a role in \( d = 3, \mathcal{N} = 1 \) superconformal field theory on \( M \)-theory membranes.

Appendix A The 11-dimensional Einstein-Maxwell equations

In order to contrast 11-dimensional bosonic equations in this paper with the ones in the previous work [4], we will show the 11-dimensional Einstein-Maxwell equations without specifying the \( G_2 \)-covariant tensors on the round \( S^6 \). As in Section 3.2 we use Greek indices for the 3-
dimensional membrane world volume, whereas Latin indices for the $G_2$-invariant 6-dimensional unit sphere, say, $\mu, \nu, \ldots = 1, 2, 3$ and $m, n, \ldots = 6, \ldots, 11$.

The original form of the 11-dimensional Maxwell equation is given in (1.5), that is
\[
\nabla_M F^{MNPQ} = -\frac{1}{576} \sqrt{-g_{11}} \epsilon^{NPQRSTUVWXY} F_{RSTU} F_{VWXY},
\]
where $g_{11}$ stands for the determinant of the 11-dimensional metric with the signature $(-++\cdots +)$ and $\nabla_M$ denotes the covariant derivative in 11-dimensions. The epsilon tensor is normalized to be $\epsilon^{12\ldots11} = 1$ and $\epsilon^{12\ldots11} = g_{11}^{-1}$.

Via the field strength ansatz (3.11), the $(\nu\rho\lambda)$-components of the Maxwell equation become
\[
\partial_\nu (\tilde{e} W_\nu) + \partial_\rho (g_{44} g^{55} \tilde{e} W_\rho) = -\frac{1}{18L^7} 6^{mnpqrs} \left[ F_{4mnp} F_{5qrs} - \frac{3}{4} F_{45mn} F_{pqrs} \right], \quad \text{(A.1)}
\]
where we have defined $\tilde{e} \equiv \Delta^3 \sin^6 \theta$ as before in Section 3.2. The 6-dimensional epsilon tensor is the same as the one in Section 3.2. The equation (A.1) goes to the third equation in (3.16) and survives as the second Maxwell equation (3.20). Similarly, the $(npq)$-components of the Maxwell equation reduce to
\[
6^{m} \nabla_{m} F^{mnpq} + \nabla_{4} F^{4npq} + \nabla_{5} F^{5npq} = \frac{\Delta}{3L^7 \sin^6 \theta} 6^{npqrst} (W_{\theta} F_{4rst} - W_{r} F_{5rst}), \quad \text{(A.2)}
\]
where $6^{m}$ is the covariant derivative on the round $S^6$. The $T^{npq}$ and the $S^{npq}$ components of this equation turn to the first and the second equations in (3.16), respectively, and finally reduce to the first Maxwell equation (3.18). We also note that (A.2) corresponds to (3.16) in [1].

The $(np5)$-components of the Maxwell equation read
\[
6^{m} \nabla_{m} F^{5npq} - \nabla_{4} F^{45npq} = -\frac{\Delta}{12L^7 \sin^6 \theta} W_{r} 6^{npqrst} F_{qrst}, \quad \text{(A.3)}
\]
which provides the first equation in (3.17). Similarly, the $(np4)$-components read
\[
6^{m} \nabla_{m} F^{4npq} + \nabla_{5} F^{54npq} = \frac{\Delta}{12L^7 \sin^6 \theta} W_{\theta} 6^{npqrst} F_{qrst} \quad \text{(A.4)}
\]
which turns to the second equation in (3.17). Other remaining components of the Maxwell equation become identically zero and trivially satisfied.

In (1.3), we find the original form of the 11-dimensional Einstein equation:
\[
R_{MN} = -\frac{1}{36} \delta_{M}^{N} F^2 + \frac{1}{3} F_{MPQR} F^{NPQR},
\]
where $F^2 \equiv F_{PQRS} F^{PQRS}$. The 11-dimensional Plank length $l_{11}$ has been absorbed into the normalization of gauge field strengths. The useful relation is $L = (32 \pi^2 N)^{\frac{7}{8}} l_{11}$ where $L$ denotes the radius of the round $S^7$ and $N$ is the number of coincident $M$-theory membranes.
Applying the field strength ansatz (3.11) to the RHS of the Einstein equation, \( F^2 \) becomes

\[
F^2 = -24 \Delta^4 \left( W_r^2 + g_{44} g^{55} W_\theta^2 \right) + F_{mnpq} F^{mnpq} \\
+ 4 F_{4mnp} F^{4mnp} + 4 F_{5mnp} F^{5mnp} + 12 F_{5mn} F^{45mn},
\]

and we obtain

\[
R_{\mu}^\nu = \delta_{\mu}^\nu \left[ -\frac{1}{36} F^2 - 2 \Delta^4 \left( W_r^2 + g_{44} g^{55} W_\theta^2 \right) \right], \quad (A.5) \\
R_{4}^4 = -\frac{1}{36} F^2 - 2 \Delta^4 W_r^2 + \frac{1}{3} F_{4mnp} F^{4mnp} + F_{45mn} F^{45mn}, \quad (A.6) \\
R_{5}^5 = -\frac{1}{36} F^2 - 2 \Delta^4 g_{44} g^{55} W_\theta^2 + \frac{1}{3} F_{5mnp} F^{5mnp} + F_{54mn} F^{54mn}, \quad (A.7) \\
R_{m}^m = -\frac{1}{36} F^2 \delta_m^m + \frac{1}{3} F_{mpqr} F^{mpqr} + F_{4mnp} F^{4mnp} + F_{5mnp} F^{5mnp} + 2 F_{mp45} F^{mp45}, \quad (A.8) \\
R_{5}^4 = -2 \Delta^4 W_r W_\theta + \frac{1}{3} F_{5mnp} F^{4mnp}. \quad (A.9)
\]

At the critical points, (A.3) and (A.6) combine into (3.5) in [1]. Similarly, (A.7) and (A.8) degenerate into (3.6) in [1]. (A.9) has no corresponding equation in [1] and is characteristic of the domain wall metric. The off-diagonal components \( R_5^4 \) and \( R_4^5 \equiv g_{44} g^{55} R_5^4 \) also arise in \( \mathcal{N} = 2, SU(3) \times U(1) \)-invariant flow studied in [2, 11].

**Appendix B  The Ricci tensor in 11-dimensions**

The \( G_2 \)-invariant 11-dimensional metric (3.8) generated in Section 3.1 can be rewritten as

\[
ds^2_{11} = a \xi^2 \left( dr^2 + e^{2A(r)} \eta_{\mu\nu} dx^\mu dx^\nu \right) + L^2 \xi^{-\frac{3}{2}} \left( a^{-2} \xi^2 \ d\theta^2 + \sin^2 \theta \ d\Omega_6^2 \right), \quad (B.1)
\]

with the quadratic form \( \xi^2 = a^2 \cos^2 \theta + b^2 \sin^2 \theta \) in (3.3). Imposing \( r \)-dependence to two vevs \( (a, b) \), the warped metric (B.1) generates Ricci tensor components:

\[
R_1^1 = \frac{1}{6L^2} a^{-3} \xi^{-\frac{4}{3}} \left[ 3L^2 \xi^2 \left[ a^2 - 3a \tilde{A} - a \ddot{\tilde{A}} - 2a^2 \left( \ddot{\tilde{A}} + 3 \tilde{A}^2 \right) \right] \\
+ 4a^2 \left( a^2 \xi' - 2L^2 \dot{\xi} \right) - 4a^2 \xi \left[ a^3 \left( \xi' + 6 \xi \cot \theta \right) + L^2 \left( \ddot{\xi} + 3 \dot{\xi} \tilde{A} \right) \right] \right], \quad (B.2)
\]

\[
R_4^4 = \frac{1}{6L^2} a^{-3} \xi^{-\frac{4}{3}} \left[ -3L^2 \xi^2 \left[ 2a^2 + 3a \ddot{\tilde{A}} + a \ddot{\tilde{A}} + 6a^2 \left( \ddot{\tilde{A}} + \ddot{\tilde{A}} \right) \right] \\
+ 4a^2 \left( a^2 \xi' - 2L^2 \dot{\xi} \right) - 4a^2 \xi \left[ a^3 \left( \xi' + 6 \xi \cot \theta \right) + L^2 \left( \ddot{\xi} + 3 \dot{\xi} \tilde{A} \right) \right] \right], \quad (B.3)
\]

\[
R_5^5 = \frac{1}{3L^2} a^{-3} \xi^{-\frac{4}{3}} \left[ -3L^2 \xi^2 \dot{a}^2 + 3L^2 a \xi^2 \left( \ddot{a} + 3a \ddot{\tilde{A}} \right) \\
+ a^5 \left[ 18\xi^2 - 4\xi' - 2\xi \left( \ddot{\xi} + 6 \xi \cot \theta \right) \right] - 2L^2 a^2 \left[ -\dot{\xi}^2 + \xi \left( \ddot{\xi} + 3 \dot{\xi} \tilde{A} \right) \right] \right], \quad (B.4)
\]

\[
R_6^6 = \frac{1}{3L^2} a^{-1} \xi^{-\frac{4}{3}} \left[ \frac{15a^4 \xi^4}{\sin^2 \theta} - a^3 \left[ \frac{3 \left( 2 + 3 \cos 2\theta \right) \xi^2}{\sin^2 \theta} + \xi' - \xi \left( \ddot{\xi} + 6 \xi \cot \theta \right) \right] \right] \]

\[
+ \frac{1}{3L^2} a^{-3} \xi^{-\frac{4}{3}} \left[ \frac{15a^4 \xi^4}{\sin^2 \theta} - a^3 \left[ \frac{3 \left( 2 + 3 \cos 2\theta \right) \xi^2}{\sin^2 \theta} + \xi' - \xi \left( \ddot{\xi} + 6 \xi \cot \theta \right) \right] \right], \quad (B.6)
\]

\[
R_7^7 = \frac{1}{6L^2} a^{-3} \xi^{-\frac{4}{3}} \left[ -3L^2 \xi^2 \dot{a}^2 + 3L^2 a \xi^2 \left( \ddot{a} + 3a \ddot{\tilde{A}} \right) \\
+ a^5 \left[ 18\xi^2 - 4\xi' - 2\xi \left( \ddot{\xi} + 6 \xi \cot \theta \right) \right] - 2L^2 a^2 \left[ -\dot{\xi}^2 + \xi \left( \ddot{\xi} + 3 \dot{\xi} \tilde{A} \right) \right] \right], \quad (B.7)
\]

\[
R_8^8 = \frac{1}{3L^2} a^{-1} \xi^{-\frac{4}{3}} \left[ \frac{15a^4 \xi^4}{\sin^2 \theta} - a^3 \left[ \frac{3 \left( 2 + 3 \cos 2\theta \right) \xi^2}{\sin^2 \theta} + \xi' - \xi \left( \ddot{\xi} + 6 \xi \cot \theta \right) \right] \right] \]

\[
+ \frac{1}{3L^2} a^{-3} \xi^{-\frac{4}{3}} \left[ \frac{15a^4 \xi^4}{\sin^2 \theta} - a^3 \left[ \frac{3 \left( 2 + 3 \cos 2\theta \right) \xi^2}{\sin^2 \theta} + \xi' - \xi \left( \ddot{\xi} + 6 \xi \cot \theta \right) \right] \right], \quad (B.8)
\]
\[ + L^2 \left[ - \ddot{\xi}^2 + \xi \left( \dddot{\xi} + 3 \dot{A} \dddot{A} \right) \right], \]  
\[ R^4_5 = a^{-2} \xi^{-\frac{B}{4}} \left[ 6 \left( a \xi \dot{\xi} - \dot{a} \xi^2 \right) \cot \theta - 2a \dot{\xi} \dot{\xi}' \right], \]  
with identities:

\[ R^2_2 = R^3_3 = R^1_1, \quad R^7_7 = R^8_8 = R^9_9 = R^{10}_{10} = R^{11}_{11} = R^6_6, \quad R^5_4 = L^{-2} a^3 R^4_5. \]

We have used the abbreviation such as \( \dot{a} \equiv \partial_r a, \dot{\xi} \equiv \partial_r \xi, \dot{\xi}' \equiv \partial_\theta \xi \) and so on. By using for example the relations

\[ \xi \dot{\xi}' = (b^2 - a^2) \cos \theta \sin \theta, \quad \xi \ddot{\xi} = a \ddot{a} \cos^2 \theta + b \ddot{b} \sin^2 \theta, \]

the Ricci tensor components are given by \((a, b, A)\) and their derivatives only.

Applying the RG flow equations (3.26), all the \( r \)-derivatives in the Ricci tensor components can be replaced with polynomials of \((a, b)\). As a simple calculation, let us derive the polynomial \( K_0 \) in Section 3.3. The linear combination in the RHS of (3.24) can be evaluated as

\[ R^1_1 + 2R^6_6 = \frac{1}{2L^2} a^{-3} \xi^{-\frac{B}{4}} \left[ 4a^5 + 20a^3 b^2 + L^2 a^2 - L^2 a \left( \ddot{a} + 3 \dot{a} \dot{A} \right) - 2L^2 a^2 \left( \dddot{A} + 3 \dot{A} \dddot{A} \right) \right]. \]

Iterative use of the RG flow equations in the RHS yields

\[ R^1_1 + 2R^6_6 = \frac{64 (ab - 1) (ab - 2) (a^2 - 7b^2)}{(a^2 + 7b^2)^2 - 112 (ab - 1)} L^{-2} \xi^{-\frac{B}{4}}. \]

from which one can read the polynomial \( K_0 \) in (3.28) as in the text.

**Acknowledgments**

This research was supported by grant 2000-1-11200-001-3 from the Basic Research Program of the Korea Science & Engineering Foundation. We are grateful to H. Nicolai for correspondence.

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