Some stochastic time-fractional diffusion equations with variable coefficients and time dependent noise

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Abstract
We prove the existence and uniqueness of mild solution for the stochastic partial differential equation
\[(\partial^{\alpha} - B) u(t, x) = u(t, x) \cdot \dot{W}(t, x),\]
where
\[\alpha \in (1/2, 1) \cup (1, 2);\]
\(B\) is an uniform elliptic operator with variable coefficients and \(\dot{W}\) is a Gaussian noise general in time with space covariance given by fractional, Riesz and Bessel kernel.

Keywords: Gaussian noisy environment, time fractional order spde, Fox H-functions, mild solutions, uniform elliptic operator, chaos expansion, Riesz kernel, Bessel kernel.

1 Introduction
In this article we prove the existence and uniqueness of the mild solution of the equation

\[
\begin{cases}
(\partial^{\alpha} - B) u(t, x) = u(t, x)\dot{W}(t, x), & t \in (0, T], x \in \mathbb{R}^d, \\
\left. \frac{\partial^k}{\partial t^k} u(t, x) \right|_{t=0} = u_k(x), & 0 \leq k \leq \lceil \alpha \rceil - 1, x \in \mathbb{R}^d, 
\end{cases}
\]  

(1.1)

with any fixed \(T \in \mathbb{R}^+, \alpha \in (1/2, 1) \cup (1, 2),\) where \(\lceil \alpha \rceil\) is the smallest integer not less than \(\alpha.\) Here we assume

- \(u_0(x)\) is bounded continuously differentiable. Its first order derivative bounded and Hölder continuous. The Hölder exponent \(\gamma > \frac{2-\alpha}{\alpha}\)
- \(u_1(x)\) is bounded continuous function(locally hölder continuous if \(d > 1)\)

In this equation, \(\dot{W}\) is a zero mean Gaussian noise with the following covariance structure

\[\mathbb{E}(\dot{W}(t, x)\dot{W}(s, y)) = \lambda(t - s)\Lambda(x - y),\]

where \(\lambda(\cdot)\) is nonnegative definite and locally intergrable and \(\Lambda(\cdot)\) is one of the following situations:
(i) Fractional kernel. $\Lambda(x) := \prod_{i=1}^{d} 2H_i(2H_i - 1)|x_i|^{2H_i - 1}$, $x \in \mathbb{R}^d$ and $1/2 < H_i < 1$.

(ii) Reisz kernel. $\Lambda(x) := C_{\alpha,d}|x|^{-\kappa}$, $x \in \mathbb{R}^d$ and $0 < \kappa < d$ and $C_{\alpha,d} = \Gamma(\frac{\kappa}{2})2^{-\alpha}\pi^{-d/2}/\Gamma(\frac{\alpha}{2})$.

(iii) Bessel kernel. $\Lambda(x) := C_{\alpha} \int_{0}^{\infty} \omega^{\frac{\alpha}{2} - 1} e^{-\omega|\frac{x}{\omega}|^2} d\omega$, $x \in \mathbb{R}^d$, $0 < \kappa < d$, and $C_{\alpha} = (4\pi)^{\alpha/2}\Gamma(\alpha/2)$.

$B := \sum_{i,j=1}^{d} a_{i,j}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{d} b_j(x) \frac{\partial}{\partial x_j} + c(x)$ is uniformly elliptic. Namely it satisfies the following conditions:

(i) $a_{i,j}(x), b_j(x)$ and $c(x)$ are bounded Hölder continuous functions on $\mathbb{R}^d$

(ii) $\exists a_0 > 0$, such that $\forall x, \xi \in \mathbb{R}^d$,

$$\sum_{i,j=1}^{d} a_{i,j}(x)\xi_i \xi_j \geq a_0 |\xi|^2.$$ 

The fractional derivative in time $\partial^{\alpha}$ is understood in Caputo sense:

$$\partial^{\alpha} f(t) := \begin{cases} \frac{1}{\Gamma(m - \alpha)} \int_{0}^{t} d\tau \frac{f^{(m)}(\tau)}{(t - \tau)^{\alpha+1-m}} & \text{if } m - 1 < \alpha < m, \\ \frac{d^m}{dt^m} f(t) & \text{if } \alpha = m. \end{cases}$$

Throughout this chapter, the initial conditions $u_k(x)$ are bounded continuous(Hölder continuous, if $d > 1$) functions. The study of the mild solution relies on the asymptote property of the Green’s function $Z, Y$ of the following deterministic equation.

$$\begin{cases} (\partial^{\alpha} - B) u(t, x) = f(t, x), & t > 0, x \in \mathbb{R}^d, \\ \frac{\partial^k}{\partial t^k} u(t, x) \bigg|_{t=0} = u_k(x), & 0 \leq k \leq \lfloor \alpha \rfloor - 1, x \in \mathbb{R}^d, \end{cases} \quad (1.2)$$

In [3] we cover the case $\alpha \in (1/2, 1)$. When $\alpha \in (1, 2)$, [6] showed that when $B$ is $\Delta$, Green’s function $Y$ of (1.2) the following:

$$Y(t, x) = C_d t^{\frac{\alpha}{2}(2-d)} f_{\frac{\alpha}{2}}(|x|^{2-d}; d - 1, \frac{\alpha}{2}(2 - d)),$$

where

$$f_{\frac{\alpha}{2}}(z; \mu, \delta) := \begin{cases} \frac{2}{\Gamma(\frac{\alpha}{2})} \int_{0}^{\infty} \phi(-\frac{\alpha}{2}, \delta; -zt)(t^2 - 1)^{\frac{\alpha}{2} - 1} dt, & \mu > 0, \\ \phi(-\frac{\alpha}{2}, \delta; -z), & \mu = 0; \end{cases}$$
\[ C_d = 2^{-n} \pi^{\frac{d-n}{2}} \] and the Wright’s function
\[ \phi\left(\frac{-\alpha}{2}, \delta; -z\right) := \sum_{n=0}^{\infty} \frac{z^n}{n! \Gamma(\delta - \frac{\alpha}{2} n)} \]

The solution of (1.2) has the following form:
\[ u(t, x) = J_0(t, x) + \int_0^t ds \int_{\mathbb{R}^d} dy f(s, y) Y(t - s, x - y), \quad (1.3) \]
where and throughout the chapter, we denote
\[ J_0(t, x) := \sum_{k=0}^{[\alpha]-1} \int_{\mathbb{R}^d} u_k(y) Z_{k+1}(t, x - y)dy. \quad (1.4) \]

For case of \( \alpha \in (1/2, 1) \), we use \( Z \) in place of \( Z_1 \). We have the following facts about \( Z_1(t, x) \), \( Z_2(t, x) \) and \( Y(t, x) \).

\[ Z_1(t, x) = D^{\alpha-1} Y(t, x); \quad Z_1(t, x) = \frac{\partial}{\partial t} Z_2(t, x) \]

As in [3], We first get the estimation of \( Y \), then use Wiener chaos expansion to obtain relation between the parameter \( \alpha, d, H_i \) and \( \kappa \) such that the mild solution exist.

The rest of the article is organized as follows. Section 2 gives more details about the solution of (1.1), estimation of \( Y \) for \( \alpha \in (1/2, 1) \) and some preliminaries about Wiener spaces. Section 3 gives the estimation of \( Y \) for \( \alpha \in (1, 2) \) and further estimations before proving the existence of the mild solution.

**Notation:** Throughout this chapter we denote
\[ p(t, x) := \exp\left\{-\sigma \left| \frac{x}{t^{\frac{2}{\alpha}}}\right|^{2-\alpha}\right\}, \]
where \( \sigma > 0 \) is a generic positive constant whose values may vary at different occurrence, so is \( C \).

## 2 Preliminary

We consider a Gaussian noise \( W \) on a complete probability space \( (\Omega, \mathcal{F}, P) \) encoded by a centered Gaussian family \( \{W(\varphi); \varphi \in L^2(\mathbb{R}^+ \times \mathbb{R}^d)\} \), whose covariance structure \( \lambda(s - t) \) is given by
\[ \mathbb{E}(W(\varphi)W(\psi)) = \int_{\mathbb{R}^2_+ \times \mathbb{R}^{2d}} \varphi(s, x)\psi(t, y)\lambda(s - t)\Lambda(x - y)dxdydsdt, \quad (2.1) \]
where \( \lambda : \mathbb{R} \rightarrow \mathbb{R}_+ \) is nonnegative definite and locally intergrable. Throughout the chapter, we denote
\[ C_t := 2 \int_0^t \lambda(s)ds, \quad t > 0. \quad (2.2) \]
\( \Lambda : \mathbb{R}^d \rightarrow \mathbb{R}_+ \) is a fractional, Reisz or Bessel kernel.
Definition 2.1. Let \( Z \) and \( Y \) be the fundamental solutions defined by (1.2) and (1.3). An adapted random field \( \{u = u(t, x) : t \geq 0, x \in \mathbb{R}^d\} \) such that \( \mathbb{E}[u^2(t, x)] < +\infty \) for all \((t, x)\) is a mild solution to (1.1), if for all \((t, x) \in \mathbb{R}_+ \times \mathbb{R}^d\), the process

\[
\{Y(t - s, x - y)u(s, y)1_{[0, t]}(s) : s \geq 0, y \in \mathbb{R}^d\}
\]
is Skorohod integrable (see (2.2)), and \( u \) satisfies

\[
u(t, x) = J_0(t, x) + \int_0^t \int_{\mathbb{R}^d} Y(t - s, x - y)u(s, y)W(ds, dy)
\]
almost surely for all \((t, x) \in \mathbb{R}_+ \times \mathbb{R}^d\), where \( J_0(t, x) \) is defined by (1.4).

We use a similar chaos expansion to the one used in chapter 3. To prove the existence and uniqueness of the solution we show that for all \((t, x)\),

\[
\sum_{n=0}^{\infty} n!\|f_n(\cdot, \cdot, t, x)\|^2_{\mathcal{H}^n} < \infty.
\]

(2.4)

3 Estimations of the Green’s functions

The fundamental solution of (1.2) is constructed by Levi’s parametrix method. We refer the reader to [2] for detail of this method. In this section \( x := (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d, \xi, \eta \) are defined the same way; \( t \in (0, T] \). We use \( \gamma \) to denote the Hölder exponents with respect to spatial variables. We can assume they are equal. For \( \alpha \in (1, 2) \), we assume

\[
\gamma > 2 - \frac{2}{\alpha}.
\]

For \( \alpha \in (\frac{1}{2}, 1) \), Chapter 4 gives the estimations the \( Z \) and \( Y \). For \( \alpha \in (1, 2) \), we need some lemmas before we can estimate \( Z_1, Z_2 \) and \( Y \).

From [5] we have

\[
Z_j(t, x - \xi) = Z_{j0}(t, x - \xi, \xi) + V_{Z_j}(t, x; \xi), \quad j = 1, 2.
\]

\[
Y(t, x - \xi) = Y_0(t, x - \xi, \xi) + V_Y(t, x; \xi).
\]

We refer the reader to [5] for the definitions of \( Z_{j0}(t, x - \xi, \xi), Y_0(t, x - \xi, \xi) \) and \( V_Y(t, x; \xi) \). Here we list their estimations which we use to get the estimations of \( Z_k \) and \( Y \) in section 3. These estimations are given in section 2.2 of [5] or Lemma 15 in [6].

Lemma 3.1.

\[
|Z_{10}(t, x - \xi, \eta)| \leq C t^{-\frac{ad}{2}} \mu_d(t^{-\frac{a}{2}}|x - \xi|)p(t, x - \xi),
\]

\[
|Z_{20}(t, x - \xi, \eta)| \leq C t^{-\frac{ad}{2}+1} \mu_d(t^{-\frac{a}{2}}|x - \xi|)p(t, x - \xi),
\]

where

\[
\mu_d(z) := \begin{cases} 1, & d = 1; \\
1 + |\log z|, & d = 2; \\
2^{-d}, & d \geq 3.
\end{cases}
\]

(3.1)
Lemma 3.2.
\[ |Y_0(t, x - \xi, \eta)| \leq C t^{\alpha - \frac{d}{2} - 1} \mu_d(t^{-\frac{d}{2}}|x - \xi|) p(t, x - \xi), \]
where
\[ \mu_d(z) := \begin{cases} 
1, & d \leq 3; \\
1 + |\log z|, & d = 4; \\
z^{4-d}, & d \geq 5.
\end{cases} \] (3.2)

The following estimations of \( V_{Z_1}, V_{Z_2} \) and \( V_Y \) are from Theorem 1 of [5], where \( \nu_1 \in (0, 1) \), such that \( \gamma > \nu_1 > 2 - \frac{2}{\alpha} \) and \( \nu_0 = \nu_1 - 2 + \frac{2}{\alpha} \).

Lemma 3.3.
\[ |V_{Z_1}(t, x; \xi)| \leq \begin{cases} 
C t^{(\gamma - 1)\frac{d}{2}} p(t, x - \xi), & d = 1; \\
C t^{\nu_0 - 1} |x - \xi|^{-d+\gamma - \nu_1 + 2 - \nu_0} p(t, x - \xi), & d \geq 2.
\end{cases} \] (3.3)

Lemma 3.4.
\[ |V_{Z_2}(t, x; \xi)| \leq \begin{cases} 
C t^{(\gamma - 1)\frac{d}{2} + 1} p(t, x - \xi), & d = 1; \\
C t^{\nu_0 - 1} |x - \xi|^{-d+\gamma - \nu_1 + 2 - \nu_0} p(t, x - \xi), & d \geq 2.
\end{cases} \] (3.4)

Lemma 3.5.
\[ |V_Y(t, x; \xi)| \leq \begin{cases} 
C t^{\alpha - 1 + (\gamma - 1)\frac{d}{2}} p(t, x - \xi), & d = 1; \\
C t^{\nu_0 - 1} |x - \xi|^{-d+\gamma - \nu_1 + 2 - \nu_0} p(t, x - \xi), & d \geq 2.
\end{cases} \] (3.5)

Based on the above three lemmas we have

Lemma 3.6. Let \( x \in \mathbb{R}^d, t \in (0, T] \). Then
\[ |Y(t, x - \xi)| \leq \begin{cases} 
C t^{-1 + \frac{d}{2}} p(t, x - \xi), & d = 1; \\
C t^{\alpha - \frac{d}{2} + \nu_0 - 2} |x - \xi|^{-d+\gamma - 2\nu_0 + \frac{2}{\alpha}} p(t, x - \xi), & d \geq 2.
\end{cases} \] (3.6)

Proof. We "add" together the estimation of \( Y_0 \) in Lemma 3.2 and \( V_y \) in Lemma 3.5 to get the estimation of \( Y \). We use the following inequality throughout the proof.
\[ a, b, \sigma > 0, \ \text{then} \ \exists \sigma, C > 0, \ \text{s.t.} \ \ x^a e^{-\sigma x^b} < C e^{-\sigma^a x^b}, \]

First when \( d = 1 \),
\[ |Y(t, x - \xi)| \leq |Y_0(t, x - \xi, \xi)| + |V_Y(t, x, \xi)| \leq C t^{\alpha - 1 + (\gamma - 1)\frac{d}{2}} p(t, x - \xi) + C t^{-1 + \frac{d}{2}} p(t, x - \xi) \leq C t^{-1 + \frac{d}{2}} p(t, x - \xi). \]

When \( d \geq 5 \), by the fact
\[ \nu_0 = \nu_1 - 2 + \frac{2}{\alpha} \ \text{and} \ \gamma > \nu_1 > 2 - \frac{2}{\alpha}, \]
we have
\[ 4 - \gamma + 2\nu_0 - \frac{2}{\alpha} = -\gamma + 2\nu_1 + \frac{2}{\alpha} \geq 0. \]

Therefore
\[
|Y_0(t, x - \xi, \xi)| \leq C t^{\alpha - \frac{d}{2} - 1} \left| \frac{x - \xi}{t^\theta} \right|^{4-d} p(t, x - \xi)
\]
\[
= C t^{\alpha - \frac{d}{2} - 1} \left| \frac{x - \xi}{t^\theta} \right|^{-d+\gamma-2\nu_0+\frac{2}{\alpha}} \left| \frac{x - \xi}{t^\theta} \right|^{4-\gamma+2\nu_0-\frac{2}{\alpha}} p(t, x - \xi)
\]
\[
\leq C t^{\alpha - \frac{d}{2} \gamma + \nu_0 \alpha - 2} |x - \xi|^{-d+\gamma-2\nu_0+\frac{2}{\alpha}} p(t, x - \xi).
\]

Furthermore because of the assumption
\[ \gamma > 2 - \frac{2}{\alpha}, \]
we have
\[ \alpha - \frac{\alpha}{2} \gamma + \nu_0 \alpha - 2 < \nu_0 \alpha - 1. \]

Therefore
\[
|Y(t, x - \xi)| \leq \left| Y_0(t, x - \xi, \xi) + V_y(t, x, \xi) \right|
\]
\[
\leq C t^{\alpha - \frac{d}{2} \gamma + \nu_0 \alpha - 2} |x - \xi|^{-d+\gamma-2\nu_0+\frac{2}{\alpha}} p(t, x - \xi)
\]
\[
+ C t^{\nu_0 \alpha - 1} |x - \xi|^{-d+\gamma-2\nu_0+\frac{2}{\alpha}} p(t, x - \xi)
\]
\[
\leq C t^{\alpha - \frac{d}{2} \gamma + \nu_0 \alpha - 2} |x - \xi|^{-d+\gamma-2\nu_0+\frac{2}{\alpha}} p(t, x - \xi).
\]

When \( d = 2 \) and \( d = 3 \), as in the previous cases, we first have
\[
|Y_0(t, x - \xi, \xi)| \leq C t^{\alpha - \frac{d}{2} - 1} p(t, x - \xi)
\]
\[
\leq C t^{\alpha - \frac{d}{2} \gamma + \nu_0 \alpha - 2} |x - \xi|^{-d+\gamma-2\nu_0+\frac{2}{\alpha}} p(t, x - \xi).
\]

Then as the last step in the case of \( d \geq 5 \), we have
\[
|Y(t, x - \xi)| \leq C t^{\alpha - \frac{d}{2} \gamma + \nu_0 \alpha - 2} |x - \xi|^{-d+\gamma-2\nu_0+\frac{2}{\alpha}} p(t, x - \xi).
\]

When \( d = 4 \), let’s first transform the estimation of \( Y_0 \) into the following form:
\[ t^{\xi \alpha} |x - \xi|^{\kappa_d} p(t, x - \xi). \]

We have
\[
|Y_0(t, x - \xi, \xi)| \leq C t^{\alpha - \frac{d}{2} - 1} \left\{ \left| \frac{x - \xi}{t^\theta} \right| + \left| \frac{t^\theta}{x - \xi} \right| \right\} p(t, x - \xi)
\]
\[
\leq C t^{\alpha - \frac{d}{2} - 1} \frac{t^\theta}{x - \xi} \left\{ \left| \frac{x - \xi}{t^\theta} \right|^{2\theta} + 1 \right\} p(t, x - \xi),
\]
for $\forall \theta > 0$.

If $|\frac{x - \xi}{t^{\frac{\alpha}{2}}}| \leq 1$, then

$$\left\{ \left| \frac{x - \xi}{t^{\frac{\alpha}{2}}} \right|^{2\theta} + 1 \right\} p(t, x - \xi) \leq 2p(t, x - \xi);$$

if $|\frac{x - \xi}{t^{\frac{\alpha}{2}}}| > 1$, then

$$\left\{ \left| \frac{x - \xi}{t^{\frac{\alpha}{2}}} \right|^{2\theta} + 1 \right\} p(t, x - \xi) \leq 2 \left| \frac{x - \xi}{t^{\frac{\alpha}{2}}} \right|^{2\theta} p(t, x - \xi) \leq Cp(t, x - \xi).$$

Therefore if we choose $\theta > 0$ such that

$$-\theta > -d + \gamma - 2\nu_0 + \frac{2}{\alpha},$$

we have

$$|Y_0(t, x - \xi, \xi)| \leq Ct^{\alpha - \frac{\alpha d}{2} - 1} \left| \frac{x - \xi}{t^{\frac{\alpha}{2}}} \right|^{-\theta} p(t, x - \xi) \leq Ct^{\alpha - \frac{\alpha d}{2} - 1} \left| \frac{x - \xi}{t^{\frac{\alpha}{2}}} \right|^{-d + \gamma - 2\nu_0 + \frac{2}{\alpha}} p(t, x - \xi) \leq Ct^{\alpha - \frac{\alpha}{2} \gamma + \nu_0 - 2} |x - \xi|^{-d + \gamma - 2\nu_0 + \frac{2}{\alpha}} p(t, x - \xi).$$

As in previous two cases, we end up with

$$|Y(t, x - \xi)| \leq Ct^{\alpha - \frac{\alpha}{2} \gamma + \nu_0 - 2} |x - \xi|^{-d + \gamma - 2\nu_0 + \frac{2}{\alpha}} p(t, x - \xi).$$

Let’s denote the estimation function of $Y$ by $t^{\zeta_d} |x - \xi|^{\kappa_d} p(t, x - \xi)$. For the estimation of integral (5.2) involving $Y$ and fractional kernel it more convenient to represent the estimation of $Y$ as the product of one dimensional functions. To this purpose, as in the case of $0 < \alpha < 1$, the estimation of $Y$ is represented as the product of one dimensional functions, which is shown in the following lemma.

**Lemma 3.7.** Let $x_i, \xi_i \in \mathbb{R}, t \in (0, T]$

$$|Y(t, x - \xi)| \leq C \prod_{i=1}^{d} t^{\zeta_d |x_i - \xi_i|^{\kappa_d} p(t, x_i - \xi_i)}, \quad (3.7)$$

where $\zeta_d$ and $\kappa_d$ are the powers of $t$ and $x - \xi$ in the estimation of $Y$, i.e.,

$$\zeta_d = \begin{cases} -1 + \frac{\alpha}{2}, & d = 1; \\ \alpha - \frac{\alpha}{2} \gamma + \nu_0 \alpha - 2, & d \geq 2. \end{cases} \quad (3.8)$$
and

\[
\kappa_d = \begin{cases} 
0, & d = 1; \\
-d + \gamma - 2\nu_0 + \frac{2}{\alpha}, & d \geq 2. 
\end{cases}
\]

Lemma 3.8.

\[
\sup_{t,x} \left| \int_{\mathbb{R}^d} Z_{k+1}(t, x - \xi) u_k(t, \xi) d\xi \right| \leq C \quad k = 0, 1.
\]

Proof. First recall that \(u_k(x)\) are bounded. Thanks to the following fact from [4]

\[
\int_{\mathbb{R}^d} Z_1^0(t, x, \xi)d\xi = 1 \quad \text{and} \quad \int_{\mathbb{R}^d} Z_2^0(t, x, \xi)d\xi = t,
\]

we only need to show

\[
\sup_{t,x} \int_{\mathbb{R}^d} V_{Z_i}(t, x, \xi)d\xi \leq C,
\]

since \(u_k\) are bounded.

Let’s consider the case \(d \geq 3\) and \(d = 2\) for \(V_{z_i}\) as examples. When \(d \geq 3\), by the estimation of \(V_{Z_1}\) in Lemma 3.1, we have

\[
\int_{\mathbb{R}^d} |V_{Z_1}(t, x, \xi)|d\xi \leq \int_{\mathbb{R}^d} C t^{-\frac{\alpha d}{2}} \mu_d(t^{-\frac{\alpha}{2}}|x - \xi|) p(t, x - \xi) dy \\
\leq \int_{\mathbb{R}^d} C t^{-\frac{\alpha d}{2} + d} \mu_d(z) p(t, x - \xi) dz \\
\leq C t^{-\frac{\alpha d}{2} + d} \\
\leq C,
\]

due to the fact \(t \in (0, T]\).

For the case \(d = 2, Z_1\), notice that

\[
\forall \theta > 0, \exists C > 0 \quad s.t. \quad (\log |z| + 1) < c|z|^{\theta},
\]

as shown in the case of \(d=4\) in the proof of 3.6. Then the above argument ends proof. The proof for the rest of the cases is almost the same, so we omit it.

\[
\square
\]

4 Miscellaneous estimations

For fractional kernel, we need the following estimation, which is immediate from Corollary 15 of [3].

Lemma 4.1. Let \(0 < r, s \leq T\) and

\[
2H_i + \frac{2\kappa_d}{d} > 0.
\]

(4.1)
Then for any $\rho_1, \tau_2 \in \mathbb{R}, \rho_1 \neq \tau_2$, we have

$$\int_{\mathbb{R}^2} |\rho_1 - \tau_1|^{2H_i - 2}|\rho_2 - \rho_1|^\kappa d|\tau_2 - \tau_1|^\frac{\alpha}{d} p(s, \rho_2 - \rho_1)p(r, \tau_2 - \tau_1) d\rho_1 d\tau_1 \leq C(s r)^{\theta_i},$$

where

$$\theta_i = \begin{cases} C(s r)^{\frac{H_i + \kappa d}{2d} - \frac{\alpha}{d}}, & 2H_i - 2 + \kappa_d/d \neq -1; \\ C(s r)^{\frac{d - \kappa_d + \kappa d}{2d} - \frac{\alpha}{d}}, & 2H_i - 2 + \kappa_d/d = -1. \end{cases}$$

Proof. In Corollary 15 of [3], let $\theta_1 = 2H_i - 2, \theta_2 = \kappa_d/d$. Then notice that for $0 < r \leq T$

$$\forall \epsilon < 0, \exists C > 0, s.t. \ \log r < Cr^\epsilon.$$

The proof is similar to Lemma 11 of [3].

**Lemma 4.2.** Let $-1 < \beta \leq 0, x \in \mathbb{R}^d$. Then, there is a constant $C$, dependent on $s, \alpha$ and $\beta$, but independent of $\xi$ and $s$ such that

$$\int_{\mathbb{R}^d} |x|^{\beta} p(s, x - \xi) dx \leq C s^{\frac{\alpha d}{2} + \frac{\beta d}{2}}.$$

For Bessel kernel, we need the following lemma.

**Lemma 4.3.** Assume $0 < s, r \leq T$ and $y_1, y_2, z_1, z_2 \in \mathbb{R}^d$, we have that

$$\int_{\mathbb{R}^{2d}} |Y(r, y_1 - y_2)Y(s, z_1 - z_2)| \int_{0}^{\infty} \omega^{-\frac{d}{2} - 1} e^{-\omega} e^{-\frac{|y_1 - \xi_1|^2}{4\omega}} d\omega dy_1 dz_1 \leq C \cdot (r s)^{\ell},$$

where

$$\ell := \zeta_d - \frac{\alpha}{4} + \frac{\alpha}{2} \kappa_d + \frac{\alpha}{2} d.$$

Proof. Recall that the estimation of $Y(t, x)$ in Lemma 9 of [3] and (3.6) has the following form:

$$C s^{\zeta_d} |x|^\alpha \kappa d p(t, x).$$

By substituting $Y$, we have

$$\int_{\mathbb{R}^{2d}} |Y(r, y_1 - y_2)Y(s, z_1 - z_2)| \int_{0}^{\infty} \omega^{-\frac{d}{2} - 1} e^{-\omega} e^{-\frac{|y_1 - \xi_1|^2}{4\omega}} d\omega dy_1 dz_1$$

$$\leq C \int_{\mathbb{R}^d} s^{\zeta_d} |z_2 - z_1|^\kappa d |p(s, z_2 - z_1)| \int_{0}^{\infty} I \cdot \omega^{-\frac{d}{2} - 1} e^{-\omega} d\omega dz_1,$$

where

$$I := \int_{\mathbb{R}^d} |y_2 - y_1|^\kappa d \exp \left\{-\sigma \left| \frac{y_2 - y_1}{r^\frac{\alpha}{2}} \right|^{\frac{2}{2 - \alpha}} \right\} \exp \left\{-\frac{|y_1 - z_1|^2}{4\omega} \right\} dy_1.$$
For $I$, we have two estimations:

\[
I \leq \int_{\mathbb{R}^d} |y_2 - y_1|^\kappa d \exp \left\{ -\sigma \left| \frac{y_2 - y_1}{r^{\frac{2}{\alpha}}} \right|^{\frac{2}{2-\alpha}} \right\} dy_1
\]

\[
\leq C r^{\frac{\kappa d}{2} + \frac{d}{2}},
\]

and

\[
I \leq \int_{\mathbb{R}^d} |y_2 - y_1|^\kappa d \exp \left\{ -\sigma \left| y_1 - z_1 \right|^2 \right\} dy_1
\]

\[
\leq C \omega^{\frac{\kappa d}{2} + \frac{d}{2}},
\]

by Lemma 4.2.

With the estimations of $I$, we have

\[
\int_0^\infty I \cdot \omega^{-\frac{\kappa}{2}} e^{-\omega} d\omega = \int_0^{r^\alpha} I \cdot \omega^{-\frac{\kappa}{2}} e^{-\omega} d\omega + \int_0^\infty I \cdot \omega^{-\frac{\kappa}{2}} e^{-\omega} d\omega
\]

\[
\leq r^{\frac{\kappa d}{2} + \frac{d}{2} - \frac{\kappa}{2}} + \int_0^\infty \omega^{\frac{\kappa d}{2} + \frac{d}{2}} \omega^{-\frac{\kappa}{2}} e^{-\omega} d\omega.
\]

For $\int_0^\infty \omega^{\frac{\kappa d}{2} + \frac{d}{2}} \omega^{-\frac{\kappa}{2}} e^{-\omega} d\omega$,

if $\frac{\kappa d}{2} + \frac{d}{2} - \frac{\kappa}{2} < 0$

\[
\int_0^\infty \omega^{\frac{\kappa d}{2} + \frac{d}{2}} \omega^{-\frac{\kappa}{2}} e^{-\omega} d\omega \leq \int_0^\infty \omega^{\frac{\kappa d}{2} + \frac{d}{2}} e^{-\omega} d\omega = C r^{\alpha \left(\frac{\kappa d}{2} + \frac{d}{2} - \frac{\kappa}{2}\right)};
\]

if $\frac{\kappa d}{2} + \frac{d}{2} - \frac{\kappa}{2} \geq 0$

\[
\int_0^\infty \omega^{\frac{\kappa d}{2} + \frac{d}{2}} \omega^{-\frac{\kappa d}{2} - \frac{\kappa}{2}} e^{-\omega} d\omega = \int_0^\infty \omega^{\frac{\kappa}{2} + \frac{d}{2}} e^{-\omega} d\omega = C
\]

\[
\leq C r^{\alpha \left(\frac{\kappa d}{2} + \frac{d}{2} - \frac{\kappa}{2}\right)}.
\]

Therefore we end up with

\[
\int_0^\infty I \cdot \omega^{-\frac{\kappa}{2}} e^{-\omega} d\omega \leq C r^{\alpha \left(\frac{\kappa d}{2} + \frac{d}{2} - \frac{\kappa}{2}\right)}.
\]

The estimation of integration with respect to $z_1$ is straightforward thank to fact that $C$ is independent of $z_1$.

We have

\[
\int_{\mathbb{R}^d} s^\kappa d |z_2 - z_1| p(s, z_2 - z_1) \nu^\kappa d \int_0^\infty I \cdot \omega^{-\frac{\kappa}{2}} e^{-\omega} d\omega dz_1
\]
\[
\leq C r^{\alpha\left(\frac{d}{2} + \frac{d}{2} - \frac{\alpha}{2}\right)} \cdot r^{\zeta_d} \cdot s^{\alpha\left(\frac{d}{2} + \frac{d}{2} - \frac{\alpha}{2}\right)} s^{\zeta_d},
\]
by Lemma 4.2.

By symmetry, we have
\[
\int_{\mathbb{R}^{2d}} |Y(r, y_1 - y_2)Y(s, z_1 - z_2)| \int_0^\infty \omega^{-\frac{\alpha}{2} - 1} e^{-\omega} e^{-\frac{|y_1 - z_1|^2}{4\omega}} d\omega dy_1 dz_1
\leq C s^{\alpha\left(\frac{d}{2} + \frac{d}{2} - \frac{\alpha}{2}\right)} \cdot r^{\alpha\left(\frac{d}{2} + \frac{d}{2} - \frac{\alpha}{2}\right)} r^{\zeta_d}.
\]

Combining the two estimations we get the estimation in the lemma.

The following lemma is Theorem 3.5 from [1].

**Lemma 4.4.** Let \( T_n(t) = \{ s = (s_1, \ldots, s_n) : 0 < s_1 < s_2 < \ldots < s_n < t \} \). Then
\[
\int_{T_n(t)} [(t - s_n)(s_n - s_{n-1}) \cdots (s_2 - s_1)]^h ds = \frac{\Gamma(1 + h)^n}{\Gamma(n(1 + h) + 1)} t^{n(1 + h)},
\]
if and only if \( 1 + h > 0 \).

## 5 Existence and uniqueness of the solution

**Theorem 5.1.** Assume the following conditions:

(1) \( \lambda(t) \) is a nonnegative definite locally integrable function;

(2) \( \alpha \in (1/2, 1) \cup (1, 2) \).

Then relation (2.4) holds for each \((t, x)\), if any of the following is true. Consequently, equation (1.1) admits a unique mild solution in the sense of Definition 2.1.

(i) \( \Lambda(x) \) is fractional kernel with condition:

\[
H_i > \begin{cases} 
\frac{1}{2}, & d = 1, 2, 3, 4 \\
1 - \frac{2}{d} - \frac{\gamma}{2d}, & d \geq 5, \alpha \in (0, 1) \\
1 - \frac{2}{d}, & d \geq 5, \alpha \in (1, 2)
\end{cases}
\]

and

\[
\sum_{i=1}^d H_i > d - 2 + \frac{1}{\alpha}.
\]

(ii) \( \Lambda(x) \) is the Reisz or Bessel kernel with condition:

\[
k < 4 - 2/\alpha;
\]
Proof. Fix \( t > 0 \) and \( x \in \mathbb{R}^d \).

Let

\[
(s, y, t, x) := (s_1, y_1, \ldots, s_n, y_n, t, x);
\]

\[
g_n(s, y, t, x) := \frac{1}{n!} Y(t - s_{\sigma(n)}, x - y_{\sigma(n)}) \cdots Y(s_{\sigma(2)} - s_{\sigma(1)}, y_{\sigma(2)} - y_{\sigma(1)}) ;
\]

\[
f_n(s, y, t, x) := g_n(s, y, t, x) J_0(s_{\sigma(1)}, x_{\sigma(1)}),
\]

where \( \sigma \) denotes a permutation of \( \{1, 2, \ldots, n\} \) such that \( 0 < s_{\sigma(1)} < \cdots < s_{\sigma(n)} < t \).

By iteration of \( u(t, x) \), we have

\[
n! \|f_n(\cdot, \cdot, t, x)\|_{H^\otimes n}^2
\]

\[
= n! \int_{[0,t]^2n} ds dr \int_{\mathbb{R}^{2nd}} dy dz f_n(s, y, t, x) f_n(r, z, t, x) \prod_{i=1}^n \Lambda(y_i - z_i) \prod_{i=1}^n \lambda(s_i - r_i), \quad (5.1)
\]

where \( dy := dy_1 \cdots dy_n \), the differentials \( dz, ds \) and \( dr \) are defined similarly. Set \( \mu(d\xi) := \prod_{i=1}^n \mu(d\xi_i) \).

Recall that \( J_0 \) is bounded, so we have

\[
n! \|f_n(\cdot, \cdot, t, x)\|_{H^\otimes n}^2
\]

\[
\leq C \left( \int_{[0,t]^2n} ds dr \int_{\mathbb{R}^{2nd}} dy dz g_n(s, y, t, x) g_n(r, z, t, x) \prod_{i=1}^n \Lambda(y_i - z_i) \prod_{i=1}^n \lambda(s_i - r_i) \right)^{1/2},
\]

Furthermore by Cauchy-Schwarz inequality,

\[
\int_{\mathbb{R}^{2nd}} dy dz g_n(s, y, t, x) g_n(r, z, t, x) \prod_{i=1}^n \Lambda(y_i - z_i)
\]

\[
\leq \left\{ \int_{\mathbb{R}^{2nd}} dy dz g_n(s, y, t, x) g_n(s, z, t, x) \prod_{i=1}^n \Lambda(y_i - z_i) \right\}^{1/2}
\]

\[
\cdot \left\{ \int_{\mathbb{R}^{2nd}} dy dz g_n(r, y, t, x) g_n(r, z, t, x) \prod_{i=1}^n \Lambda(y_i - z_i) \right\}^{1/2}
\]

(i) Let \( \Lambda(\cdot) = \varphi_H(\cdot) \) and use the estimation of \( Y \) in Lemma 3.6, we have

\[
\int_{\mathbb{R}^{2nd}} dy dz g_n(s, y, t, x) g_n(s, z, t, x) \prod_{i=1}^n \Lambda(y_i - z_i)
\]

\[
\leq C \prod_{i=1}^d \int_{\mathbb{R}^{2n}} \prod_{k=1}^n \varphi_{H_i}(y_{ik} - z_{ik}) \Theta_n(t, y_{ik}, s) \Theta_n(t, z_{ik}, s) dy_idz_i \quad (5.2)
\]
where
\[ \Theta_n(t, y_{ik}, s) := |s_{\sigma(k+1)} - s_{\sigma(k)}|^{\frac{\zeta_d}{d}} |y_{i\sigma(k+1)} - y_{i\sigma(k)}|^{\frac{\zeta_d}{d}} p(s_{\sigma(k+1)} - s_{\sigma(k)}, y_{i\sigma(k+1)} - y_{i\sigma(k)}) ; \]
\[ y_i = (y_{i1}, y_{i2}, \cdots , y_{ik}, \cdots , y_{in}), \quad z_i = (z_{i1}, z_{i2}, \cdots , z_{ik}, \cdots , z_{in}) ; \]
\[ dy_i := \prod_{k=1}^n dy_{ik} \quad dz_i := \prod_{k=1}^n dz_{ik} ; \]
and
\[ y_{\sigma(k+1)} = z_{\sigma(k+1)} := x_i ; \quad s_{\sigma(n+1)} = r_{\sigma(n+1)} := t. \]
Let’s first consider the case \( 2H_i - 2 + \kappa_d/d \neq -1 \). Applying Lemma 4.1 to
\[ \int_{\mathbb{R}^{2n}} \prod_{k=1}^n \varphi_{H_i}(y_{ik} - z_{ik}) \Theta_n(t, y_{ik}, s) \Theta_n(t, z_{ik}, s) dy_idz_i \] (5.3)
for \( dy_{\sigma(1)}dz_{\sigma(1)} \), we have
\[ \int_{\mathbb{R}^{2n}} \prod_{k=1}^n \varphi_{H_k}(y_{ik} - z_{ik}) \Theta_n(t, y_{ik}, s) \Theta_n(t, z_{ik}, s) dy_idz_i \]
\[ \leq C(s_{\sigma(2)} - s_{\sigma(1)})^{2\ell_i} \int_{\mathbb{R}^{2n}} \prod_{k=2}^n \varphi_{H_k}(y_{ik} - z_{ik}) \Theta_n(t, y_{ik}, s) \Theta_n(t, z_{ik}, s) dy_idz_i \]
where
\[ \ell_i = \frac{\zeta_d}{d} + \theta_i . \]
Applying Lemma 4.1 to (5.3) for \( dy_{\sigma(k)}dz_{\sigma(k)}, k = 2, \cdots , n, \) we have
\[ \prod_{i=1}^d \int_{\mathbb{R}^{2n}} \prod_{k=1}^n \varphi_{H_k}(y_{ik} - z_{ik}) \Theta_n(t, y_{ik}, s) \Theta_n(t, z_{ik}, s) dy_idz_i \leq \prod_{k=1}^n C^n(s_{\sigma(k+1)} - s_{\sigma(k)})^{2\ell} \]
where
\[ \ell = \sum_{i=1}^d \ell_i = \frac{\zeta_d}{2} + \frac{|H|\alpha}{2} + \frac{\kappa_d\alpha}{2} \quad \text{with} \quad |H| = \sum_{i=1}^d H_i. \] (5.4)
Due to the same argument, we have
\[ \prod_{i=1}^d \int_{\mathbb{R}^{2n}} \prod_{k=1}^n \varphi_{H_k}(y_{ik} - z_{ik}) \Theta_n(t, y_{ik}, r) \Theta_n(t, z_{ik}, r) dy_idz_i \leq \prod_{k=1}^n C^n(r_{\rho(k+1)} - r_{\rho(k)})^{2\ell} \]
Therefore
\[ \int_{\mathbb{R}^{2nd}} dydz \ g_n(s, y, t, x) g_n(r, z, t, x) \prod_{i=1}^n \Lambda(y_i - z_i) \leq C^n \left( \phi(s)\phi(r) \right)^{\ell}, \]
where
\[ \phi(s) := \prod_{i=1}^{n} (s_{\sigma(i+1)} - s_{\sigma(i)}), \quad \phi(r) := \prod_{i=1}^{n} (r_{\rho(i+1)} - r_{\rho(i)}), \]

with
\[ 0 < s_{\sigma(1)} < s_{\sigma(2)} < \ldots < s_{\sigma(n)} \quad \text{and} \quad 0 < r_{\rho(1)} < r_{\rho(2)} < \ldots < r_{\rho(n)}. \]

Hence,
\[ n! \| f_n(\cdot, \cdot, t, x) \|_{H^{0 \otimes n}}^2 \leq \frac{C_t^n}{n!} \int_{[0,t]^{2n}} \prod_{i=1}^{n} \lambda(s_i - r_i) (\phi(s)\phi(r))^\ell \, ds \, dr \]
\[ \leq \frac{C_t^n}{n!} \int_{[0,t]^{2n}} \prod_{i=1}^{n} \lambda(s_i - r_i) (\phi(s)^{2\ell} + \phi(r)^{2\ell}) \, ds \, dr \]
\[ = \frac{C_t^n}{n!} \int_{[0,t]^{n}} \prod_{i=1}^{n} \lambda(s_i - r_i) \phi(s)^{2\ell} \, ds \]
\[ = \frac{C_t^n}{n!} \int_{T_n(t)} \phi(s)^{2\ell} \, ds \]
\[ = \frac{C_t^n}{n!} \int_{T_n(t)} \phi(s)^{2\ell} \, ds \]
\[ = \frac{C_t^n}{n!} \frac{\Gamma(2\ell + 1)^n \ell^{(2\ell+1)n}}{\Gamma((2\ell + 1)n + 1)}, \]

where \( C_t = 2 \int_0^t \lambda(r) \, dr \). The last step is by Lemma 4.4.

Therefore,
\[ n! \| f_n(\cdot, \cdot, t, x) \|_{H^{0 \otimes n}}^2 \leq \frac{C_t^n}{n!} \frac{\Gamma(2\ell + 1)^n \ell^{(2\ell+1)n}}{\Gamma((2\ell + 1)n + 1)}, \]

and \( \sum_{n \geq 0} n! \| f_n(\cdot, \cdot, t, x) \|_{H^{0 \otimes n}}^2 \) converges if \( \ell > -1/2 \).

Next we need to show
\[ \ell > -1/2 \iff |H| > d - 2 + \frac{1}{\alpha}. \]

Firstly by definition of \( \ell \), (5.4)
\[ \ell > -1/2 \iff |H| > -\frac{1}{\alpha} - \kappa_d - \frac{2}{\alpha} \zeta_d. \]

Then using the definition of \( \zeta_d \) and \( \kappa_d \) in (4.2), (4.3) of [3] for \( 1/2 < \alpha < 1 \) and (3.8), (3.9) for \( 1 < \alpha < 2 \), we have:

when \( 1/2 < \alpha < 1 \),
\[ \frac{1}{\alpha} - \kappa_d - \frac{2}{\alpha} \zeta_d = \begin{cases} -1 + \frac{1}{\alpha}, & d = 1; \\ \frac{1}{\alpha} - 2, & d = 3 \text{ or } d \geq 5; \end{cases} \]
when \(1 < \alpha < 2\),

\[
\frac{1}{\alpha} - \kappa_d - \frac{2}{\alpha} \xi_d = \begin{cases} 
-1 + \frac{1}{\alpha}, & d = 1; \\
\frac{d - 2 + \frac{1}{\alpha}}{2}, & d \geq 2; 
\end{cases}
\]

For case \(2H_i - 2 + \kappa_d/d = -1\), applying Lemma 4.1 to (5.3), we have

\[
\prod_{i=1}^{d} \int_{\mathbb{R}^{2m}} \prod_{k=1}^{n} \varphi_{H_i}(y_{ik} - z_{ik}) \Theta_n(t, y_{ik}, s) \Theta_n(t, z_{ik}, s) dy_{ik} dz_{ik} \leq \prod_{k=1}^{n} C^n(s_{\sigma(k+1)} - s_{\sigma(k)})^{2\ell'},
\]

where

\[
\ell' = \xi_d + \frac{d\epsilon + \kappa_d + d}{4} \alpha \quad \text{with} \quad |H| = \sum_{i=1}^{d} H_i.
\]

Using the relation \(2H_i - 2 + \kappa_d/d = -1\), we have

\[
\ell' = \ell + \frac{d\alpha}{4} \epsilon.
\]

Since

\[
|H| > d - 2 + \frac{1}{\alpha} \implies \ell > -1/2,
\]

we can choose \(\epsilon\) big enough such that

\[
|H| > d - 2 + \frac{1}{\alpha} \implies \ell' > -1/2.
\]

Lastly, when \(\alpha \in (1/2, 1)\), for \(d \leq 4\), \(H_i > 1/2\) implies condition (4.1); for \(d > 4\), condition (4.1) is implied by

\[
H_i > 1 + \frac{2}{d} - \frac{\gamma}{2d}
\]

with \(\gamma_0\) sufficiently small; when \(\alpha \in (1, 2)\) for \(d = 1\), \(H_i > 1/2\) implies (4.1); for \(d \geq 2\), (4.1) is implied by

\[
H_i > 1 + \frac{2}{d}
\]

with \(\nu_0\) sufficiently small. This completes the proof of Theorem 5.1 for case of \(\Lambda(\cdot) = \varphi_H(\cdot)\)

(ii) Let \(x = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d\). For Reisz kernel, notice that

\[
|x|^{-\kappa} \leq C \prod_{i=1}^{d} |x_i|^{\xi_d},
\]

so this case is reduced to case (i) with \(H_i = (-\frac{\xi}{d} + 2)\frac{1}{2}, i = 1, 2, \cdots, d\). Correspondingly

\[
|H| > d - 2 + \frac{1}{\alpha}
\]

is

\[
\kappa < 4 - 2/\alpha,
\]
which also guarantees condition (4.1).

For Bessel kernel, applying Lemma 4.3 for $dy_{\sigma(i)}dz_{\sigma(i)}$ in the order of $i = 1, 2, \cdots, n$ to

$$\int_{\mathbb{R}^{2nd}} dydz g_n(s, y, t, x)g_n(s, z, t, x) \prod_{i=1}^{n} \Lambda(y_i - z_i)$$

yields

$$\int_{\mathbb{R}^{2nd}} dydz g_n(s, y, t, x)g_n(s, z, t, x) \prod_{i=1}^{n} \Lambda(y_i - z_i) \leq \prod_{k=1}^{n} C^n(s_{\sigma(k+1)} - s_{\sigma(k)})^{2\ell},$$

where

$$\ell := \zeta_d - \frac{\alpha}{4} \kappa + \frac{\alpha}{2} \kappa_d + \frac{\alpha}{2} d$$

As in case (i), $\sum_{n \geq 0} n!\|f_n(\cdot, \cdot, t, x)\|_{H^{\otimes n}}$ converges if $\ell > -1/2$. Then using the definition of $\zeta_d$ and $\kappa_d$ in (4.2), (4.3) of [3] for $1/2 < \alpha < 1$ and (3.8), (3.9) for $1 < \alpha < 2$, we have

$$\ell > -1/2 \iff \kappa < 4 - 2/\alpha.$$

This finishes the proof of the theorem. 

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