Spatially Inhomogeneous Bernstein’s Problem and De Giorgi’s Conjecture

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Abstract. In this note, we propose Bernstein’s problem and De Giorgi’s conjecture for spatially inhomogeneous equations, as well as De Giorgi’s conjecture for system of reaction-diffusion equations.

1 Bernstein’s problem and De Giorgi’s conjecture

Consider

\[ u_t = \Delta u + \frac{1}{\varepsilon^2} (u - u^3), \quad \mathbf{x} = (x', x_n) \in \mathbb{R}^n, \quad t > 0, \]

where \( x' = (x_1, \cdots, x_{n-1}) \). It was shown in [1, 6] (and reference therein) that, as \( 0 < \varepsilon \ll 1 \), a sharp internal layer (interface) of \( u \) develops between the two regions \( \{ u \approx 1 \} \) and \( \{ u \approx -1 \} \) in time scale \( \tau = t/\varepsilon^2 \). In time scale \( t \), the interface propagates according to the following mean curvature flow equation:

\[ V = -(n - 1) \kappa, \quad \mathbf{x} \in \Gamma_t, \quad t > 0, \]

where \( \Gamma_t := \{ \mathbf{x} \in \mathbb{R}^n \mid u(\mathbf{x}, t) = 0 \} \) is a hypersurface, \( V \) denotes the normal velocity of \( \Gamma_t \) and \( \kappa \) denotes the mean curvature.

The well known Bernstein’s problem is about the steady state of (2):

**Bernstein’s Problem:** Let \( h : \mathbb{R}^{n-1} \to \mathbb{R} \) be a \( C^2 \) function, \( \mathcal{G}(h) \subset \mathbb{R}^n \) be the graph of \( h \). If \( \kappa = 0 \) on \( \mathcal{G}(h) \), then \( \mathcal{G}(h) \) is a hyperplane.

The answer was shown to be positive for \( n \leq 8 \) (E. De Giorgi, Almgren, Simons) and negative for \( n \geq 9 \) (E. Bombieri - De Giorgi - E. Giusti). Clearly, Berstein’s problem is related to the shape of the stationary solution of (1). For which, De Giorgi [2] proposed the following conjecture:
De Giorgi’s Conjecture: Assume that \( u \) is a stationary, entire solution (defined for all \( x \in \mathbb{R}^n \)) of (1), that it satisfies \( |u| \leq 1 \) and \( \partial u / \partial x_n > 0 \). Then, at least for \( n \leq 8 \), the level sets of \( u \) must be hyperplanes.

This conjecture was partially proved recently in [3], [8] and references therein.

2 Spatially Inhomogeneous Bernstein’s Problem and De Giorgi’s Conjecture

In the last two decades, many authors studied the following spatially inhomogeneous reaction-diffusion equation

\[
(3) \quad u_t = \nabla (A(x) \nabla u) + \frac{1}{\varepsilon^2} B(x) u \left( Z^2(x) - u^2 \right), \quad x \in \mathbb{R}^n, \ t > 0,
\]

where \( A, B \) and \( Z \) are bounded, smooth functions with positive infimums. A special case is \( Z \equiv 1 \) (typical double-well potential with equal-well-depth).

It is easily seen that, when \( 0 < \varepsilon \ll 1 \), (3) has exactly three stationary solutions: \( Z_+, 0 \) and \( Z_- \) with \( Z_{\pm} \approx \pm Z \). As the homogeneous case, it was shown in [4, 5, 6] that, in time scale \( t \), the law of the propagation of the level set \( \Gamma_t := \{ x \mid u(x, t) = 0 \} \) is

\[
(4) \quad V = - (n - 1) a(x) \kappa + c(x) \nabla d(x) \cdot n \quad \text{for} \quad x \in \Gamma_t
\]

where \( n \) is the normal direction to \( \Gamma_t \), \( V \) and \( \kappa \) are as above, \( a, c, d \) are bounded, smooth functions with \( \inf a > 0 \). Similar to the homogeneous case, we propose the following problem:

Spatially Inhomogeneous Bernstein’s Problem: Let \( h : \mathbb{R}^{n-1} \rightarrow \mathbb{R} \) be a \( C^2 \) function, \( \mathcal{S}(h) \subset \mathbb{R}^n \) be the graph of \( h \). Assume that

\[
(5) \quad - (n - 1) a(x) \kappa + c(x) \nabla d(x) \cdot n = 0 \quad \text{for} \quad x \in \mathcal{S}(h).
\]

Then there exists \( h_0 : \mathbb{R}^{n-1} \rightarrow \mathbb{R} \), whose graph is a hyperplane, such that

(i) \( h - h_0 \) is quasi-periodic if \( a, c, d \) are periodic;

(ii) \( h - h_0 \) is almost periodic if \( a, c, d \) are almost periodic.

In almost periodic case (ii), a primary analysis indicates that an additional condition maybe also needed: For any ball \( B \subset \mathbb{R}^n \),

\[
(6) \quad \int_B [a(x) - a_0] dx, \quad \int_B [c(x) - c_0] dx, \quad \int_B [d(x) - d_0] dx
\]
are bounded by $M$ (independent of $B$), where $a_0, c_0$ and $d_0$ are the averages of $a, c$ and $d$, respectively.

**Spatially Inhomogeneous De Giorgi's Conjecture.** Assume that $u$ is an entire, stationary solution of (3), that it satisfies

\[ Z_-(x) \leq u(x) \leq Z_+(x), \quad \lim_{x_n \to \pm \infty} u(x) = Z_{\pm}(x). \]

Then at least for $n \leq 8$, the 0-level set of $u$: $\Gamma := \{ x \mid u(x) = 0 \}$ has the following properties:

(i) $\Gamma$ is the graph of a function $x_n = h(x')$ and $h(x')$ is quasi-periodic if $A, B, Z$ are periodic;

(ii) $\Gamma$ is the graph of a function $x_n = h(x')$ and $h(x')$ is almost periodic if $A, B, Z$ are almost periodic and if they satisfy similar conditions as in (6).

In the periodic case, denote by $X$ the period of $A, B$ and $Z$ in $x_n$-direction. One see that $\frac{\partial u}{\partial x_n} > 0$ as in De Giorgi's conjecture may be not always true, instead,

\[ u(x', x_n) < u(x', x_n + X), \quad x \in \mathbb{R}^n \]

may play a similar role and be used as an additional condition for the conjecture.

3 Systems of Reaction-diffusion Equations

It is well known that the solution $u$ of FitzHugh-Nagumo equation

\[ \begin{cases} u_t = \Delta u + \frac{1}{\varepsilon^2}(u - u^3 - v), & x \in \mathbb{R}^n, \quad t > 0, \\ v_t = \Delta v + u - v, & x \in \mathbb{R}^n, \quad t > 0 \end{cases} \]

behaviors as that of (1). From (8) one also derives the law of the propagation of the level set of $u$, which is similar to (2) (3 6). For such reasons, we propose an analogue of De Giorgi’s conjecture for the following spatially inhomogeneous FitzHugh-Nagumo equations:

\[ \begin{cases} u_t = \nabla(A(x) \nabla u) + \frac{1}{\varepsilon^2}B(x)u(Z^2(x) - u^2) - D(x)v, & x \in \mathbb{R}^n, \quad t > 0, \\ v_t = \nabla(\alpha(x) \nabla v) + \beta(x)u - \gamma(x)v, & x \in \mathbb{R}^n, \quad t > 0. \end{cases} \]

**De Giorgi Type Conjecture for FitzHugh-Nagumo Equations.** Let $A, B, D, Z, \alpha, \beta, \gamma$ be positive, almost periodic functions, $0 < \varepsilon \ll 1$. Assume that $(u, v)$ is an entire stationary solution of (9), and that

\[ Z_-(x) \leq u(x) \leq Z_+(x), \quad \lim_{x_n \to \pm \infty} u(x) = Z_{\pm}(x). \]
Then at least for \( n \leq 8 \), the 0-level set of \( u \): \( \Gamma := \{ x \mid u(x) = 0 \} \) must be the graph of a function \( x_n = h(x') \).

Moreover, function \( h(x') \) is a constant (resp. quasi-periodic, almost periodic) when \( A, B, D, Z, \alpha, \beta, \gamma \) are constant (resp. periodic, almost periodic and satisfy similar conditions as in (6)).

Remark. Both Bernstein’s problem and De Giorgi’s conjecture listed above can be extended to more general cases. For example, \( a = a(x, n) \), \( B(x) = B(x, u, \nabla u) \), etc.. One can consider other systems, for example, a three-component reaction-diffusion system arising in gas discharge phenomena (cf. [7]). Finally, one can consider spatially homogeneous and/or inhomogeneous analogue of the above Bernstein’s problem and De Giorgi’s conjecture in various manifolds.

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