Reconstruction of the dark sectors’ interaction: A model-independent inference and forecast from GW standard sirens

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ABSTRACT

Interacting dark matter (DM) - dark energy (DE) models have been intensively investigated in the literature for their ability to fit various data sets as well as to explain some observational tensions persisting within the ΛCDM cosmology. In this work, we employ the Gaussian processes (GP) algorithm to perform a joint analysis by using the geometrical cosmological probes such as Cosmic chronometers, Supernova Type Ia, Baryon Acoustic Oscillations, and the H0LiCOW lenses sample to infer a reconstruction of the coupling function between the dark components in a general framework, where the DE can assume a dynamical character via its equation of state. In addition to the joint analysis with these data, we simulate a catalogue with standard siren events from binary neutron star mergers, within the sensitivity predicted by the Einstein Telescope, to reconstruct the dark sector coupling with more accuracy in a robust way. We find that the particular case, where \( w = -1 \) is fixed on the DE nature, has a statistical preference for an interaction in the dark sector at late times. In the general case, where \( w(z) \) is analysed, we find no evidence for such dark coupling, and the predictions are compatible with the ΛCDM paradigm. When the mock events of the standard sirens are considered to improve the kernel in GP predictions, we find a preference for an interaction in the dark sector at late times.

Key words: cosmological parameters – cosmology: observations – dark energy – dark matter

1 INTRODUCTION

Up-to-date observational data suggest that our universe is mainly driven by a pressure-less or cold dark matter (CDM) and a dark energy (DE) fluid where around 96 per cent (~28 per cent DM + 68 percent DE) of the total energy budget of the universe is occupied by this joint dark fluid (Planck Collaboration et al. 2020). The fundamental nature of these fluids, such as, their origin and dynamics are yet to be known even after a series of astronomical missions. Therefore, understanding the dark picture of the universe has remained one of the greatest challenges in cosmology. In order to reveal the physics of the dark sectors, various cosmological models have been proposed and investigated over the last several years (Copeland, Sami & Tsujikawa 2006; Sotiriou and Faraoni 2010; Cai et al 2010; De Felice S. Tsujikawa 2010; Capozziello and De Laurentis 2011; Clifton et al 2012; Bamba et al 2012; Cai et al 2016; Nojiri, Odintsov & Oikonomou 2017; Bahamonde et al 2021). The standard cosmological model ΛCDM is one of the simplest cosmological models which fits excellently to most of the observational probes. However, the physics of the dark fluids is not clear in this model as well – the cosmological constant problem, for instance, is a serious issue (Weinberg 1998). Additionally, in this canonical picture of the universe, several anomalies and tensions between different cosmological probes may indicate a revision of the ΛCDM cosmology, see refs. Perivolaropoulos & Skara (2021); Schöneberg et al (2021). In the ΛCDM model, we assume the simplest possibility for its ingredients – the independent evolution of DM and DE. As the physics of the dark sector is not yet clear, there should not be any reason to exclude the possibility of an interaction between these components. By allowing the interaction or the energy exchange between DM and DE, one naturally generalizes the non-interacting scenarios.

The theory of the dark sector interaction did not appear suddenly in the literature. The limitations or issues of the standard cosmological model at the fundamental level motivated to relax the independent evolution of DM and DE. For instance, an interaction in the dark sector can provide a possible/promising explanation/solution to the cosmic coincidence problem (del Campo, Herrera, & Pavón (2009); Velten, vom Martens, & Zimdahl (2014)), \( H_0 \) tension (Di Valentinio, Melchiorri, & Mena 2017; Kumar & Nunes 2017; Yang et al 2018; Pan et al 2019; Di Valentinio et al 2020; Lucca & Hooper 2020; Kumar 2021; Di Valentinio et al 2021; Anchordoqui et al 2021; Renzi, Hogg, & Giarrè 2021; Akarsu et al 2021; Theodoropoulos & Perivolaropoulos 2021; Di Valentinio 2021) that arises between the CMB measurements by Planck satellite within the ΛCDM cosmology (Planck Collaboration et al. 2020) and SH0ES (Riess et al. 2019, 2021).
2 METHODOLOGY, DATA SETS AND THE THEORETICAL BACKGROUND

This section is divided into the following three parts: the Gaussian process, the observational data, and a basic framework of the theory, which we are going to test in this article using the observational data following the model-independent Gaussian approach.

2.1 Gaussian process

In a nutshell, the GP in cosmology allows us, given an observational data set \( f(z) \pm \sigma_f \), to obtain a function \( f(z) \) without the need to assume a parametrization or physical model about the dark nature of the main components of the universe. The GP method adequately describes the observed data based on a distribution over functions. The reconstructed function \( f(z) \) (and its derivatives \( f'(z), f''(z), \ldots \)) have a Gaussian distribution with mean and Gaussian error at each data point \( z \). The functions at different points \( z \) and \( z' \) are related by a covariance function \( k(z, z') \), which only depends on a set of kernels with hyperparameters \( l \) and \( \sigma_f \), describing the strength and extent of the correlations among the reconstructed data points, respectively. Thus, \( l \) gives a measure of the coherence length of the correlation in the x-direction, and \( \sigma_f \) denotes the overall amplitude of the correlation in the y-direction. In general, the hyperparameters are constant since their values point to a good fit of the function rather than a model that mimics this behavior, which means that the GP optimizes both concerning the observed data. Finally, the GP method is model-independent in a physical model and assumes a particular statistical kernel that determines the correlation between the reconstructed data points. The entire methodology used in this work is described in detail in section II of Ref. Bonilla, Kumar, & Nunes (2021).
2.2 Observational data sets

In this section we shall describe the geometrical probes in detail that we have used to trace the interaction in the dark sector.

- Cosmic Chronometers (CC): The CC approach is very powerful to detect the expansion history of the universe that comes through the measurements of the Hubble parameter. Here we take into consideration 30 measurements of the Hubble parameter distributed over a redshift interval $0 < z < 2$ as in Ref. MoreSCO et al. (2016).

- Supernovae Type Ia (SNe): The first astronomical data probing the accelerating expansion of our universe are SNs. Certainly, SNs are very important astronomical probes in analysing the properties of DE and the expansion history of the universe. The latest compilation of SN data (Pantheon sample) that we have used in this work, consists of 1048 SN data points in the redshift range $0.01 < z < 2.3$ (Scolnic et al. 2018). In the context of a universe with zero curvature, the entire Pantheon sample can be summarized in terms of six model-independent $E(z)^{-1}$ data points (Riess et al. 2018). Here we use the six data points as reported in ref. Haridasu et al. (2018) in the form of $E(z)$ taking into account the theoretical and statistical considerations for its implementation.

- Baryon Acoustic Oscillations (BAO): Another important cosmological probe is the BAO data. With the use of BAO, the expanding spherical wave produced by baryonic perturbations of acoustic oscillations in the recombination epoch can be traced through the correlation function of the large-scale structure displaying a peak around $150h^{-1}$Mpc. Here we have used BAO measurements from various astronomical surveys: (i) measurements from the Sloan Digital Sky Survey (SDSS) III DR-12 which report three effective binned redshifts $z = 0.38, 0.51$ and 0.61 (Alam et al. 2017), (ii) measurements from the clustering of the SDSS-IV extended Baryon Oscillation Spectroscopic Survey DR14 quasar sample reporting four effective binned redshifts $z = 0.98, 1.23, 1.52$ and 1.94, as in Zhao et al. (2019), (iii) measurements from the high-redshift Lyman-α survey reporting two effective binned redshifts at $z = 2.33$ du Mas des Bourboux et al. (2020) and $z = 2.4$ du Mas des Bourboux et al. (2017). All the measurements are presented in terms of $H(z) \times (r_d/r_{d, fid})$ km s$^{-1}$Mpc$^{-1}$, where $r_d$ denotes the co-moving sound horizon and $r_{d, fid}$ is the fiducial input value provided in the above surveys.

- H0LiCOW sample: Finally, we use the sample from the $H_0$ Lenses in COSMOGRAIL’s Wellspring program, another geometrical probe in this list which measures the Hubble constant in a direct way (without assuming any model in the background). The H0LiCOW collaboration has measured six lens systems, determined by measurements of time-delay distances, $D_{\Delta t}$, between multiple images of strong gravitational lens systems due to elliptical galaxies (Wong et al. 2020). The entire information is encapsulated in the time-delay distance $D_{\Delta t}$. Along with these six systems of strongly lensed quasars, the angular diameter distance to the lens $D_l$ also offers some additional information in terms of four more data points. Therefore, in total, one can employ 10 data points and we have used them in this work (we refer to Birrer et al. (2019); Pandey, Raveri, & Jain (2020) for more details in this context).

2.3 Theoretical framework

For a model-independent theoretical description of the dark sectors’ interaction, in this work, we follow the similar methodology as in ref. Yang, Guo, & Cai (2015). In the context of a Friedmann–Lemaître–Robertson–Walker universe, we assume that the total energy density of the universe is comprised by DE and DM only where both of them are coupled through a non-gravitational interaction. Thus, the conservation equations for DM and DE are modified as

\begin{align}
\dot{\rho}_{DM} + 3H\rho_{DM} &= -Q(t), \\
\dot{\rho}_{DE} + 3H\rho_{DE}(1 + w) &= Q(t),
\end{align}

where $w = p_{DE}/\rho_{DE}$ is the equation of state of DE ($p_{DE}$ denotes the pressure of the DE fluid), $H = \dot{a}/a$ is the expansion rate of the universe and it is related to the total energy density of the universe as $3H^2 = \rho_{DM} + \rho_{DE}$ (in the units where $8\pi G = 1$). The function $Q(t)$ describes the interaction between DM and DE, and usually it is taken to be a function of the energy densities of DM and DE. For $Q(t) = 0$ with $w = -1$, the standard $\Lambda$CDM cosmology is recovered. Now, combining the conservation equations (1) and (2) with expansion rate of the universe $H(z)$, we obtain (Yang, Guo, & Cai 2015):

\begin{equation}
-wq = 2\left(EE'' + E^2E'' - \frac{w'}{w}E^2E'\right)(1 + z)^2 - 2\left(5 + 3w\right)E^2E' - 3\frac{w''}{w}E^3\right)(1 + z) + 9(1 + w)E^3,
\end{equation}

where for convenience, we have used a dimension-less variable $q = Q(t)/H_0^2$ to characterize the interaction, $E(z) = H(z)/H_0$ is the normalized Hubble rate and the prime denotes the differentiation with respect to the redshift $z$. Let us note that the symbol $q$ is usually used to represent the deceleration parameter in the literature, but here this symbol has a different meaning as defined above. The detailed derivation of equation (3) is given in Appendix A. Now, using the normalized co-moving distance,

\begin{equation}
D = \frac{H_0}{c} \left(\frac{1}{1 + z}\right) d_L(z),
\end{equation}

where $d_L(z)$ represents the luminosity distance at redshift $z$, eq.(3) can be expressed alternatively as

\begin{equation}
-wq = 2\left( \frac{3D''''}{D''} - \frac{D'''''}{D''} + \frac{w'D'''}{w'D''} \right)(1 + z)^2 + 2\left(5 + 3w\right)\frac{D'''}{D''} + \frac{3w''}{wD''^2}\right)(1 + z) + \frac{9(1 + w)}{D^5}.
\end{equation}

The above methodology represents a general framework to reconstruct the coupling function with minimal assumption. The only assumption of fact is the validity of the cosmological principle, and a possible coupling between DM and DE has been assumed as a theoretical prior, where this second assumption must be tested with the observational data.

In what follows, we will test this theoretical framework.
3 RESULTS AND DISCUSSIONS

In this section, we present and discuss the results of our analyses considering two separate cases. First, we fix the EoS of DE to $w = -1$, and reconstruct the interaction function $q = Q(t)/H_0^3$. This possibility characterizes a very well-known sub-class of the interaction scenario in the dark sector, known by interacting vacuum energy. Secondly, we consider a very general possibility assuming $w$ as a free and dynamical function, and similarly reconstruct the interaction function $q$. Thus, having both the possibilities, we explore a very general description of the dark coupling in a model-independent approach. Before we enter into the main results, we rescale the function $q$ (see eq. (5)) to $\delta(z) = q(1 + z)^{-5}$. Such a pre-factor is just considered as a scale transformation with respect to $z$, introduced to better expose the results in the graphical description. Thus, here onwards the function $\delta(z)$ will characterize the coupling function.

We proceed further considering the above two scenarios of coupling in the dark sector. To reconstruct $\delta(z)$, we use $M_{\theta/2}$ kernel in all the analyses performed in this work. In the case where we assume $w(z)$ to be a free function, we follow the same methodology as presented in Bonilla, Kumar, & Nunes (2021). For this purpose, we have used modified versions of some numerical routines available in the public GAPP (Gaussian Processes in Python) code Seikel, Clarkson, & Smith (2012). In all of our analyses, we employ GP to perform a joint analysis using the minimal data set combination CC+SN+BAO, which to our knowledge has not been investigated previously in the literature. We now present and discuss our main results.

3.1 Interacting Vacuum Energy

In Fig. 1, we have shown the reconstruction of $\delta(z)$ using the data combinations CC+SN+BAO and CC+SN+BAO+H0LiCOW. In both analyses, we note that for $z > 0.5$ the dynamical coupling function $\delta(z)$ between the dark components is statistically well compatible with $\delta(z) = 0$. It is interesting to note that the GP mean predicts a possible oscillation in $\delta(z)$, where we can note an oscillation between positive and negative values in the analysed range of $z$. This result strengthens some earlier interaction models having a sign changeable property, see for instance Pan et al. (2019); Pan, Yang, & Paliahanasis (2020); Yang et al. (2021). For the present scenario, at late cosmic time, i.e. for $z < 0.5$ ($z < 0.25$), we find a trend towards $\delta < 0$ for CC+SN+BAO+H0LiCOW data. When evaluated at present moment, we find $\delta(z = 0) = -0.37 \pm 0.24$ ($-0.76 \pm 0.12$) at 1$\sigma$ CL from CC+SN+BAO and CC+SN+BAO+H0LiCOW data. This suggests an interaction in the dark sector at more than 3$\sigma$ CL from the CC+SN+BAO+H0LiCOW joint analysis. It is important to emphasize that these constraints on $\delta(z)$ are subject to the condition $w = -1$. Also, we notice that the combined analysis with several
data sets offers a more stringent bound on the interaction function compared to Yang, Guo, & Cai (2015), where only the SN Ia Union 2.1 data set Suzuki et al. (2012) was employed.

3.2 General interaction scenario in the dark sector

In the previous subsection, we analysed a particular interaction case, namely the interacting vacuum-energy ($w = -1$) to get the constrains on $\delta(z)$. As a second round of analysis, we relax these conditions by assuming $w(z)$ to be a free function. This possibility allows us to reconstruct the coupling in the dark sector in a general way, because in this case, no physical assumption is considered on the EoS of DE. In Fig. 2, we show the reconstruction of $\delta(z)$ from CC+SN+BAO and CC+SN+BAO+H0LiCOW data combinations. Since in this scenario, we have an additional free parameter $w$ to propagate errors compared to the previous case with $w = -1$, it is expected that large error bars might be imposed on the reconstructed $\delta(z)$ compared to the case with $w = -1$. As a general feature of the GP mean, we can note a flux of energy from DM to the DE at high $z$, and as cosmic time evolves up to approximately $z < 0.5$, the coupling function $\delta(z)$ reverses its sign, at low $z$ (late time). This again goes in support of some phenomenological models of the interaction (Pan et al. 2019; Pan, Yang, & Paliathanasis 2020). In this general framework, we find $\delta(z = 0) = -0.31 \pm 0.77$ at 1σ CL from CC+SN+BAO data and $\delta(z = 0) = -0.64 \pm 0.43$ at 1σ CL from CC+SN+BAO+H0LiCOW data. These predictions are compatible with the $\Lambda$CDM cosmology, i.e., $\delta = 0$.

4 FORECAST FROM GRAVITATIONAL WAVE STANDARD SIRENS

To impose more robust and accurate constraints on the $\delta(z)$ function, we optimize the covariance function using mock gravitational waves (GW) data generated by assuming the $\Lambda$CDM model as a fiducial one. As argued in Seikel & Clarkson (2013), for non-parametric regression method such as GP, we aim to generate confidence limits such that the true function is trapped appropriately. But it can be problematic when evaluating functions as $w$ and $\delta(z)$ because they are quantities that depend on the second and third order derivatives of the cosmological observable. We can avoid the problem in identifying an appropriate covariance function which can reproduce expected models accurately by adding simulated data. To accomplish this object, we create a standard sirens mock catalogue, the gravitational wave analogue of the astronomical standard candles, which can provide powerful information about the dynamics of the universe. For a given GW strain signal, $h(t) = A(t) \cos(\Phi(t))$, the stationary-phase approximation can be used for the orbital phase of inspiraling binary system to obtain its Fourier transform $\tilde{h}(f)$. For a coalescing binary system of masses $m_1$ and $m_2$,

$$\tilde{h}(f) = QA f^{-7/6} e^{i\Phi(f)}.$$

Here $A = 1/d_L$ is the luminosity distance to the merger’s redshift, and $\Phi(f)$ is the binary system’s inspiral phase. For more details on the post-Newtonian coefficients and waveforms, one may refer to D’Agostino & Nunes (2019) and Appendix A therein. After defining the GW signal, for a high enough signal-to-noise ratio (SNR), one may obtain upper bounds on the free parameters of the GW signal $\tilde{h}(f)$ by using the Fisher information analysis. Estimating $d_L(z)$ from GW standard sirens mock data is well established approach, see D’Agostino & Nunes (2019) and references therein. In what follows, we briefly describe our methodology that is used to generate the standard sirens mock catalogue.

In order to generate the mock standard siren catalogue, we consider the ET power spectral density noises. The ET is a third-generation ground detector, and covers frequencies in the range $1 - 10^4$ Hz. The ET is sensitive to signal amplitude, which is expected to be ten times larger than the current advanced ground-based detectors. The ET conceptual design study predicts BNS detection of an order of $10^3 - 10^2$ per year. However, only a small fraction ($\sim 10^{-3}$) of them is expected to be accompanied by a short $\gamma$-ray burst observation. Assuming a detection rate of $O(10^3)$, the events with short $\gamma$-ray bursts will be $O(10^4)$ per year. In our simulations, 1000 BNS mock GW standard sirens merger events up to $z = 2$ are considered. In the mock catalogue, we have used the input values $H_0 = 67.4 \text{ km} \text{s}^{-1} \text{Mpc}^{-1}$ and $\Omega_{\text{m0}} = 0.31$ for the Hubble constant and matter density parameter, respectively, in agreement with the most recent Planck CMB data (within the $\Lambda$CDM paradigm) Planck Collaboration et al. (2020). We have estimated the measurement error on the luminosity distance for each event using the Fisher matrix analysis on the waveforms (see ref. D’Agostino & Nunes (2019) for details).

We have calculated the SNR of each event, and confirmed that it is a GW detection provided SNR > 8. In what follows, we describe the evolution of the interaction function with the inclusion of the mock GW standard sirens with the standard cosmological probes.

In Fig. 3, we show the reconstructed interaction function $\delta(z)$ from CC+SN+BAO+GW and CC+SN+BAO+H0LiCOW+GW data combinations for the simple interaction scenario with $w = -1$. When evaluated at the present moment, we find $\delta(z = 0) = -0.70 \pm 0.14$ at 1σ CL for CC+SN+BAO, and $\delta(z = 0) = -0.83 \pm 0.01$ at 1σ CL for CC+SN+BAO+H0LiCOW under the interacting vacuum-energy assumption. Analysing the behavior of the $\delta$ function in the range $z \in [0, 2.5]$, we find an evidence for a sign transition in $\delta$ that quantifies the interaction between the dark components. We clearly notice a preference for $\delta < 0$ at late times. In Fig. 4, we show the reconstructed interaction function $\delta(z)$ from CC+SN+BAO+GW and CC+SN+BAO+H0LiCOW+GW data combinations, under the general assumption where $w(z)$ is a free function of $z$. In this case, we find $\delta(z = 0) = -0.49 \pm 0.69$ at 1σ CL from CC+SN+BAO+GW and $\delta(z = 0) = -0.705 \pm 0.066$ at 1σ CL from CC+SN+BAO+H0LiCOW+GW. In our analysis, even including GW mock data in CC+SN+BAO, we note that $\delta$ is compatible with $\Lambda$CDM. On the other hand, from CC+SN+BAO+H0LiCOW+GW data, we can notice a prediction for $\delta < 0$ at late times.

It is important to note that the presence of GWs mock catalogue (assuming a fiducial $\Lambda$CDM model) is used for the purpose of optimizing the covariance function, as argued previously. The results summarized in Fig. 4 are the most realistic, being the ones for the most general case.

5 CONCLUSIONS

In this work, we have presented some generalized aspects of dark sectors’ interaction that may be of interest to the community: (i) We have investigated the case where DE can assume a dynamical character through its equation of state along with the simplest vacuum-energy case $w = -1$. (ii) We have studied joint analyses of the dark sector interaction with several geometrical probes following a minimally model-dependent way of GP (iii) We have optimized the covariance function using mock GW standard sirens to better reconstruct the function $\delta(z)$. In short, all these pieces of investigation have led to more general and robust results which could be helpful in order to
have a deeper understanding of the physics of the dark sector. Our observations are as follows: We find, for both interacting vacuum and general scenario, that, $\delta(z)$ exhibits a transient nature according to the analyses from CC+SN+BAO and CC+SN+BAO+H0LiCOW data, and at very late time, $\delta(z)$ enters into the negative region (see Figs. 1 and 2). This conclusion remains unaltered when we include the 1000 mock GW standard sirens to the above two combined data sets (see Figs. 3 and 4). However, the indication of a late interaction is strongly pronounced in the context of the interacting vacuum scenario where we find that for the CC+SN+BAO, $\delta(z = 0) \neq 0$ at more than 1σ CL, but $\delta(z = 0) \neq 0$ remains valid at more than 3σ CL for CC+SN+BAO+H0LiCOW data. This is an interesting result in this work because the transfer of energy among the dark sector components, which has been observed in some phenomenological models, is not ruled out in light of the model-independent analysis, and also for the combined analyses that we have performed during the reconstruction. However, concerning the general interacting picture, we see that $\delta(z = 0) = 0$ is compatible within 1σ but for CC+SN+BAO+H0LiCOW data, we find a strong preference of an interaction at several standard deviations. Summarizing the results, we find that the model-independent analyses indicate for a possible interaction in the dark sector which is strongly preferred for the scenario with $w = -1$. Based on the findings of this study, we believe that it will be worthwhile to investigate, in future communications, various statistical techniques for reconstructing the function $\delta(z)$, such as the use of neural networks, principal component analysis, and others, which may provide statistical improvements over the standard GP method that we use in the study of cosmological parameters.

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APPENDIX A: DERIVATION OF THE KEY EQUATION

Here we show the derivation of eqn. (3) which is the heart of this work. Now, subtracting eqn.(2) from eqn.(1) one can obtain

\[ \rho_{\text{DM}} - \rho_{\text{DE}} + 3H \left( \rho_{\text{DM}} - (1 + w) \rho_{\text{DE}} \right) = -2Q(t). \]  

(\textit{A1})

Now, using the Friedmann’s equations \( 3H^2 = \rho_{\text{DM}} + \rho_{\text{DE}} \) and \( 2H + 3\dot{H} = -\rho_{\text{DE}} = -w\rho_{\text{DE}} \), one can express \( \rho_{\text{DM}} \) and \( \rho_{\text{DE}} \) respectively as

\[ \rho_{\text{DM}} = 3H^2 + \frac{2\dot{H}}{w} + \frac{3H^2}{w}, \]  

(\textit{A2})

and

\[ \rho_{\text{DE}} = \frac{2\dot{H}}{w} - \frac{3H^2}{w}. \]  

(\textit{A3})

Now differentiating (\textit{A2}) and (\textit{A3}) with respect to the cosmic time one can write down:

\[ \dot{\rho}_{\text{DM}} = 6HH + 2\frac{\ddot{H}}{w} - 2\frac{\dot{H}^2}{w^2} + 6\frac{\dot{H}H}{w} - \frac{3\dot{H}^2}{w^2}, \]  

(\textit{A4})

and

\[ \dot{\rho}_{\text{DE}} = -\frac{2\ddot{H}}{w} + 2\frac{\dot{H}^2}{w} - 6\frac{\dot{H}H}{w} + \frac{3\dot{H}^2}{w^2}. \]  

(\textit{A5})

Notice that, by changing \( \frac{d}{dt} \rightarrow \frac{d}{dz} \) where \( z \) denotes the redshift, one can express

\[ \dot{w} = -\frac{wH}{a}, \quad \dot{H} = -(1 + z)H'H', \]  

(A6)

where prime denotes the derivative with respect to \( z \). Consequently, one can express \( \dot{H} \) as

\[ \dot{H} = (1 + z)^2H'H' + (1 + z)^2h^2H'' + (1 + z)h^2H' \]  

(A7)

Now, inserting eqns. (\textit{A4}), (\textit{A5}) into eqn. (\textit{A1}) and using the derivatives of cosmological parameters with respect to \( z \), and doing some simple algebraic calculations, one arrives at

\[ -wQ = 2 \left( H^2 + \dot{H}^2 H'' - \frac{w'}{w}H^2H' \right) (1 + z)^2 \]

\[ -2 \left( 5 + 3w \right) \frac{H^2H'}{w} - 3 \frac{w'H}{w} \]

\[ +9(1 + w)H^2, \]  

(A8)

which if divided by \( H_0^3 \), can be expressed as

\[ -wq = 2 \left( E + E' \right) \left( 2E + E'' \right) - \frac{w'}{w} \left( E' + E'' \right) (1 + z)^2 \]

\[ -2 \left( 5 + 3w \right) E + E' - 3 \frac{w'}{w} E \]

\[ +9(1 + w)E, \]  

(A9)

where \( q = Q/H_0^3 \) and \( E = H/H_0 \).