STATEMENT FROM THE AUTHOR

This paper is wrong. The main idea is probably interesting but the statement (and obviously, the proof) are wrong. I thank Y. Genzmer for pointing out the issues which led me to realizing the mistake.

I am leaving the contents as I wrote them in case someone finds them relevant, even if only for avoiding the same mistakes.

Essentially: the Weierstrass derivation is wrongly computed everywhere. The simplest example (Genzmer) is a homogeneous cubic $x^3 + xy^2 + y^3$.

Pedro Fortuny Ayuso, June 5 2021.
IRREDUCIBILITY CRITERION FOR SINGULAR HYPER SURFACES OF \((\mathbb{C}^n,0)\)

P. FORTUNY AYUSO

Abstract. Using vector fields we obtain an irreducibility criterion for hypersurfaces. It only requires the Weierstrass division.

1. Introduction and Notation

The problem of deciding whether a plane curve (in implicit form, obviously) is irreducible is of paramount importance for studying its singularity (see [1, 2, 3, 7, 5] just for several relevant examples, without any aim to completeness). The usual ways to solve it in characteristic 0 are by means of the Newton diagram (and Puiseux series) or using Approximate Roots, following Abhyankar and Moh. Both ways require “going further” than the first Puiseux exponent, and a very delicate analysis of the singularity.

In this short note we provide a criterion which, using elementary vector fields and Weierstrass division decides whether a germ of (reduced) hypersurface \(f = 0\) is or not irreducible. The result is based on the fact that vector fields can have “bad order of tangency” with a hypersurface if and only if this is reducible, because the contact structure associated to vector fields is essentially different in the reducible and irreducible cases.

2. Irreducibility criterion for hypersurface singularities

Let \(f : (\mathbb{C}^n,0) \to (\mathbb{C},0)\) define a reduced germ of hypersurface \(f = 0\). For simplicity, we denote \((x,\overline{y})\) a system of coordinates in \((\mathbb{C}^n,0)\) such that \(f(x,\overline{y})\) is in Weierstrass form:

\[
f(x,\overline{y}) = x^k + \sum_{i=0}^{k-1} F_i(y)x^i.
\]

where \(k\) is the multiplicity of \(f\). From here on, we assume that \(k > 2\) (the case \(k = 2\) is trivial) and \(F_0(\overline{y}) \neq 0\) (the hypersurface does not contain \(x = 0\)). Let \(df\) denote the differential form of \(f\) and \(X \in \mathfrak{X}(\mathbb{C}^2,0)\) a germ of holomorphic vector field in \((\mathbb{C}^2,0)\).

Definition 1. The tangency function of \(X\) with \(f(x,\overline{y}) = 0\) (with respect to the specific coordinate function) is the remainder \(R(x,\overline{y})\) of the Weierstrass division:

\[
df(X) = Q(x,\overline{y})f(x,\overline{y}) + R(x,\overline{y}).
\]

The tangency order of \(X\) with \(f(x,\overline{y})\) in the \(x\)-direction is the order of \(R(x,\overline{y})\) as a power series in \(x\).

The tangency function measures, in some sense, “how” \(X\) fails to be tangent to \(f(x,\overline{y}) = 0\) [6, 3, 4], the tangency order the order of that “failure”. We shall omit the qualifier “in the \(x\)-direction” because it is unnecessary in what follows.

The irreducibility criterion is the following (recall that \(x\) does not divide \(f(x,\overline{y})\)):
Theorem 1. With the notations and hypotheses above (recall that $f$ is reduced and $k > 2$), then $f(x, \overline{y}) = 0$ is reducible if and only if there exists an integer $2 \leq r < k$ and a vector field

$$X = x^r \frac{\partial}{\partial x}$$

whose tangency order with $f(x, \overline{y})$ is 0.

Proof. Assume $f(x, \overline{y}) = 0$ is reducible and set $f(x, \overline{y}) = f_1(x, \overline{y})f_2(x, \overline{y})$. As $f$ is in Weierstrass form, we can assume $f_1(x, \overline{y})$ is too and $f_2(x, \overline{y})$ is almost:

$$f_1(x, \overline{y}) = x^{a_1} + \sum_{i=0}^{a_1} F_i^1(\overline{y})x^i, \quad f_2(x, \overline{y}) = u(x, \overline{y})\left(x^{a_2} + \sum_{i=0}^{a_2} F_i^2(\overline{y})x^i\right).$$

Let $X$ be the vector field

$$X = x^{a_1+1} \frac{\partial}{\partial x}.$$

As $f_1(x, \overline{y})$ has degree $a_1$ in $x$, performing the Weierstrass division, we obtain $x^{a_1+1} = x^a f_1(x, \overline{y}) + R(x, \overline{y})$ where $R(x, \overline{y}) = -xF_0^1(\overline{y}) + x^y S(x, \overline{y})$ with $S(x, \overline{y})$ a holomorphic function. From this:

$$df(X) = \left((f_2 \frac{\partial f_1}{\partial x} + f_1 \frac{\partial f_2}{\partial x}) dx + (\cdots)\right)(x f_1 + R) \frac{\partial}{\partial x},$$

where the dots denote an irrelevant holomorphic 1-form. This gives:

$$df(X) = x f_1 f_2 \frac{\partial f_1}{\partial x} + x f_2 \frac{\partial f_2}{\partial x} + x(-F_0^1(\overline{y}) + x S(x, \overline{y})) \left(f_1 \frac{\partial f_2}{\partial x} + f_2 \frac{\partial f_1}{\partial x}\right).$$

The first term is a multiple of $f$. The second one is:

$$x f_2^2 \left(\frac{\partial (f_2/u)}{\partial x} + (f_2/u) \frac{\partial u}{\partial x}\right) = v(x, \overline{y})\left(x^{2a_1+a_2} + \sum_{i=1}^{2a_1+a_2-1} h_i(\overline{y})x^i\right)$$

(for a unit $v(x, \overline{y})$). This term is not a multiple of $f$ because $f$ is reduced. Notice the indices starting at 1. The last term in (2) has degree $a_1 + a_2$ in $x$ as a Weierstrass polynomial (except for a unit). Performing the Weierstrass division, we obtain:

$$df(X) = Q(x, \overline{y}) f(x, \overline{y}) - w(x, \overline{y}) F_0^1(\overline{y})^2 F_2^2(\overline{y}) + x T(x, \overline{y})$$

for a unit $w(x, \overline{y})$ and some holomorphic function $T(x, \overline{y})$. Thus, the contact order of $X$ with $f = 0$ is 0.

The reciprocal: If $f(x, \overline{y})$ is irreducible of multiplicity $n$ then, for any $X = x^r \frac{\partial}{\partial x}$ with $2 \leq r < n$:

$$df(X) = \left(\frac{\partial f}{\partial x} dx + \cdots\right)x^r \frac{\partial f}{\partial x} = x^r \frac{\partial f}{\partial x},$$

whose contact order with $f$ is exactly $r - 1$ (as $x$ does not divide $f$). □

References

[1] S.S. Abhyankar. Irreducibility criterion for germs of analytic functions of two complex variables. Adv. in Math., (74):190–257, 1989.

[2] S.S. Abhyankar and T.T. Moh. Newton puiseux expansion and generalized tshianhausen transformation ii. J. reine und angew Math, (261):29–54, 1973.

[3] P. Fortuny Ayuso. Vector flows and the analytic moduli of singular plane branches. Rev. R. Acad. Cienc. Exactas. RACSAM, 113(4):4107–4118, 2019.

[4] P. Fortuny Ayuso. Normal forms for hypersurface singularities. arXiv:2106.00562, 2021.

[5] E. R. García Barroso and J. Gwoździewicz. A discriminant criterion of irreducibility. Kodai Math. J., (35):405–414, 2012.

[6] P. Fortuny Ayuso and J. Ribón. The action of a plane singular holomorphic flow on a non-invariant branch. Canadian Journal of Mathematics, 2019.

[7] T.C. Kuo and Y.C. Lu. On analytic function germs of two complex variables. Topology, (16):299–310, 1977.
[8] T.T. Moh. On approximate roots of a polynomial. *J. reine und angew. Math.*, (278):301–306, 1975.

Dpto. Matemáticas, Universidad de Oviedo. Oviedo, Spain.

Email address: fortunypedro@uniovi.es