Orthogonal Polynomials and Fourier Orthogonal Series on a Cone

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Abstract
Orthogonal polynomials and the Fourier orthogonal series on a cone in \( \mathbb{R}^{d+1} \) are studied. It is shown that orthogonal polynomials with respect to the weight function \((1-t)^\gamma (t^2-\|x\|^2)^{-\frac{1}{2}}\) on the cone \( \mathcal{V}^{d+1} = \{(x,t): \|x\| \leq t \leq 1\} \) are eigenfunctions of a second order differential operator, with eigenvalues depending only on the degree of the polynomials, and the reproducing kernels of these polynomials satisfy a closed formula that has a one-dimensional characteristic. The latter leads to a convolution structure on the cone, which is then utilized to study the Fourier orthogonal series. This narrative also holds, in part, for more general classes of weight functions. Furthermore, analogous results are also established for orthogonal structure on the surface of the cone.

Keywords Fourier orthogonal series · Orthogonal polynomials · PDE · Cone · Surface

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1 Introduction

Given an appropriate weight function \( \varpi \) defined on a domain \( \Omega \) in \( \mathbb{R}^d \), one can construct an orthogonal basis of polynomials in \( L^2(\Omega, \varpi) \) and study the Fourier series in orthogonal polynomials. For study beyond the abstract Hilbert space setting,
it is necessary to consider special weight functions on regular domains to ensure manageable structure for orthogonal polynomials.

The situation is best illustrated by the Fourier series in spherical harmonics, which are orthogonal with respect to the Lebesgue measure on the unit sphere $S^{d-1}$ of $\mathbb{R}^d$. The key ingredient for the Fourier analysis on the sphere is the addition formula of spherical harmonics (cf. [2,11]), which shows that the reproducing kernel of spherical harmonics of degree $n$ is equal to $Z_n(\langle x, y \rangle)$, where $Z_n$ is an ultraspherical polynomial of degree $n$. This closed-form expression of one-dimensional structure is utilized for just about every aspect of analysis on the unit sphere. Other most studied cases of orthogonal polynomials in $d$-variables are the classical ones in the following list [5]:

1. **Hermite**: $\varpi(x) = e^{-\|x\|^2}$ on $\mathbb{R}^d$;
2. **Laguerre**: $\varpi_\alpha(x) = x_1^\alpha \ldots x_d^\alpha e^{-|x|}$ on $\mathbb{R}^d +$;
3. **Ball**: $\varpi_\mu(x) = (1 - \|x\|^2)^\mu - \frac{1}{2}$ on $\mathbb{B}^d = \{ x : \|x\| \leq 1 \}$;
4. **Simplex**: $\varpi_\kappa(x) = x_1^{\kappa_1} \ldots x_d^{\kappa_d} (1 - |x|)^{\kappa_{d+1}}$ on $\mathbb{T}^d = \{ x : x_1, \ldots, x_d \geq 0, |x| \leq 1 \}$,

where $|x| = x_1 + \cdots + x_d$ and the parameters $\alpha \in \mathbb{R}^d$, $\mu \in \mathbb{R}$ and $\kappa \in \mathbb{R}^{d+1}$ are chosen so that the weight functions are integrable. In each of these cases, several orthogonal polynomial bases can be constructed explicitly and they are classical, some of them could be traced back to the work of Hermite [1]. The Fourier series of classical orthogonal polynomials have been studied in depth in recent years; see [2,5,13] and references therein. In the case of the ball and the simplex, the essential ingredient of the study is a closed-form expression for the reproducing kernels of orthogonal polynomials, which can be regarded as analogues of the addition formula of spherical harmonics.

The purpose of this paper is to study orthogonal polynomials and Fourier orthogonal series on a cone of revolution, which we assume to be

$$V^{d+1} := \left\{ (x, t) \in \mathbb{R}^{d+1} : \|x\| \leq t, \ 0 \leq t \leq b, \ x \in \mathbb{R}^d \right\},$$

and also on the surface of the cone

$$V_0^{d+1} := \left\{ (x, t) \in \mathbb{R}^{d+1} : \|x\| = t, \ 0 \leq t \leq b, \ x \in \mathbb{R}^d \right\}$$

for $d \geq 2$, where $b$ is either finite, say $b = 1$, or $b = +\infty$. For $d = 1$, the cone $V^2$ is a triangle on the plane. For $d \geq 2$, orthogonal structure on these domains have not been studied in the literature as far as we are aware. On the surface of the cone $V_0^{d+1}$, we will define orthogonality with respect to $\varphi(t) d\sigma(x,t)$, where $\sigma(x,t)$ is the surface measure of $V_0^{d+1}$. On the solid cone $V^{d+1}$ we will define the orthogonality with respect to an appropriate weight function $W$, which we choose to be of the form

$$W(x,t) = w(t)(t^2 - \|x\|^2)^{\mu - \frac{1}{2}}, \quad \mu > -\frac{1}{2}, \quad x \in \mathbb{B}^d, \quad 0 \leq t \leq b,$$

making use of the classical weight function $\varpi_\mu$ on the unit ball. For each $\varphi$ or $w$, we can construct orthogonal polynomials with respect to $W$ explicitly. We shall, however,
choose special $\varphi$ and $w$ so that the corresponding orthogonal polynomials possess two characteristic properties of classical orthogonal polynomials.

The first property is that there is a second order differential operator $D$, which is linear and has polynomial coefficients, such that orthogonal polynomials are the eigenfunctions of $D$ with eigenvalues depending only on the degree of the polynomials. For non-degenerate inner product on a solid domain in two-variables, a classification of orthogonal polynomials with such a property in [7] shows that, up to affine transformation and for positive definite integrals, there are only five families, and they are the four classical ones, (1)–(4), together with the product of Hermite-Laguerre on $\mathbb{R} \times \mathbb{R}_+$. For more than two-variables, no classification is known but the four classical families and some of their tensor products have this property [5]. As we shall show in Sect. 3, two families of orthogonal polynomials on the cone $\mathbb{V}_{d+1}$ for $d \geq 2$ possess this property, which are orthogonal with respect to $W(x, t)$ with $w(t) = (1 - t)^\gamma$ and with $w(t) = e^{-t}$. Furthermore, for spherical harmonics, the property is satisfied with $D$ being the Laplace–Beltrami operator on $S^{d-1}$. Our results give two families of orthogonal polynomials on the surface of the cone that possess this property, which are orthogonal with respect to $\varphi(t)d\sigma(x, t)$ with, somewhat surprisingly, $\varphi(t) = t^{-1}(1 - t)^\gamma$ and $\varphi(t) = t^{-1}e^{-t}$, both of which have a singularity at the apex of the cone. The second property is a closed-form expression for the reproducing kernel of orthogonal polynomials, akin to the addition formula of spherical harmonics. On the unit ball and on the simplex, the Fourier series in classical orthogonal polynomials have been studied actively in recent years (cf. [2,5,8,10] and the references therein), after such a formula was discovered about twenty years ago [14,15]. This property holds for orthogonal polynomials on compact domains. We consider the cone $\mathbb{V}_{d+1} = \{(x, t) : \|x\| \leq t, x \in \mathbb{R}^d, t \leq 1\}$, which is compact, and establish closed-form formulas for the reproducing kernel for orthogonal polynomials on $\mathbb{V}_{d+1}$ with $w(t) = t^\beta(1 - t)^\gamma$, $\beta \geq 0$ and $\gamma \geq -\frac{1}{2}$ and for $\varphi(t) = t^\beta(1 - t)^\gamma$ for $\beta \geq -1$ and $\gamma \geq -\frac{1}{2}$ on $\mathbb{V}_{d+1}$. In both cases, the closed-form formulas provide a one-dimensional structure for the kernel, which allows us to define a convolution structure on the cone. The projection operator of the Fourier orthogonal series of a function $f$ on the cone can be written as the convolution of $f$ and the Jacobi kernel of one-variable. As a consequence, many properties of the Fourier orthogonal series on the cone can be deduced from the Fourier–Jacobi series. As an example, we shall deduce sharp conditions for the convergence of the Cesàro means of the Fourier orthogonal series on the cone.

The paper is organized as follows. In the next section, we recall definitions of orthogonal polynomials of several variables and the Fourier orthogonal series, as well as several specific families of orthogonal polynomials that will be used later. The orthogonal structure on the solid cone will be studied in Sect. 3 and the closed formula for the reproducing kernel will be established in Sect. 4, which leads to a translation operator and a convolution structure that will be discussed in Sect. 5. These are then used in Sect. 6 to study the summability of the Fourier orthogonal series on the cone. Our development of orthogonal structure on the surface of the cone is parallel to that on the solid cone. Orthogonal structure on the surface will be studied in Sect. 7 and the closed formulas for the reproducing kernels will be derived in Sect. 8. The convolution structure on the surface and its application in the Fourier orthogonal series on the
surface will be discussed in Sect. 9. Finally, our results could be extended to more
general weight functions that include Dunkl’s reflection invariant weight function as
a factor, which will be briefly discussed in Sect. 10.

2 Preliminary

We provide general framework on orthogonal polynomials and the Fourier orthog-
onal series in the first subsection and review several families of specific orthogonal
polynomials that will be needed in several subsequent subsections.

2.1 Orthogonal Polynomials and Fourier Orthogonal Expansion

A general reference for this subsection is [5, Chapter 3]. Let $\mathcal{W}$ be a nonnegative weight
function defined on a domain $\Omega$ of positive measure in $\mathbb{R}^d$ so that $\int_\Omega \mathcal{W}(x)dx > 0$, and let $\langle \cdot, \cdot \rangle$ be an inner product defined by

$$\langle f, g \rangle = b_W \int_\Omega f(x)g(x)\mathcal{W}(x)dx,$$

where $b_W$ is a normalization constant such that $\langle 1, 1 \rangle = 1$. Denote by $\Pi^d$ the space of
polynomials of $d$-variables and by $\Pi^d_n$ the subspace of polynomials of degree at most
$n$ in $\Pi^d$. A polynomial $P \in \Pi^d_n$ is called orthogonal with respect to the inner product
$\langle \cdot, \cdot \rangle$ if $\langle P, Q \rangle = 0$ for all polynomials $Q \in \Pi^d_{n-1}$. Let $\mathcal{V}^d_n$ be the space of orthogonal
polynomials of degree $n$. It is known that

$$\dim \Pi^d_n = \binom{n + d}{n} \quad \text{and} \quad \dim \mathcal{V}^d_n = \binom{n + d - 1}{n}.$$  

Let $\mathbb{N}$ be the set of natural integers and let $\mathbb{N}_0 = \mathbb{N} \cap \{0\}$. The space $\mathcal{V}^d_n$ can have
many distinct bases. Let $\{P^n_k : |k| = n, k \in \mathbb{N}^d_0\}$ be an orthogonal basis of $\mathcal{V}^d_n$. For
$f \in L^2(\Omega)$, the Fourier orthogonal series is defined by

$$f = \sum_{n=0}^\infty \sum_{|k|=n} \hat{f}_k^n P^n_k = \sum_{n=0}^\infty \text{proj}_n f, \quad \text{where} \quad \hat{f}_k^n = \frac{\langle f, P^n_k \rangle}{\langle P^n_k, P^n_k \rangle},$$

and $\text{proj}_n : L^2 \mapsto \mathcal{V}^d_n$ denotes the orthogonal projection operator, which can be written
as an integral against the reproducing kernel $P_n$ of the space $\mathcal{V}^d_n$,

$$\text{proj}_n f(x) = \sum_{|k|=n} \hat{f}_k^n P^n_k = \int_\Omega f(y)P_n(x, y)\mathcal{W}(y)dy. \quad (2.1)$$
In terms of the basis \( \{ P_k^\alpha \mid n \} \), the reproducing kernel can be written as
\[
P_n(x, y) = \sum_{|k|=n} \frac{P^\alpha_k(x) P^\alpha_k(y)}{\langle P^\alpha_k, P^\alpha_k \rangle}.
\] (2.2)

The kernel, however, is uniquely defined and is independent of the choice of the orthogonal basis.

For our study of orthogonal structure in and on the cone, we shall need various properties of spherical harmonics, orthogonal polynomials on the unit ball in \( \mathbb{R}^d \) and on the triangle \( T^2 \), as well as the Jacobi polynomials and Gegenbauer polynomials on \([-1, 1]\). We shall collect what will be needed in subsequent subsections.

### 2.2 Classical Orthogonal Polynomials

We fix notations that will be used throughout this paper. For \( \alpha, \beta > -1 \), the Jacobi weight function is denoted by
\[
w_{\alpha, \beta}(x) = (1 - x)^\alpha (1 + x)^\beta, \quad -1 < x < 1.
\]

Its normalization constant \( c'_{\alpha, \beta} \), defined by \( c'_{\alpha, \beta} \int_{-1}^{1} w_{\alpha, \beta}(x) dx = 1 \), is given by
\[
c'_{\alpha, \beta} = \frac{1}{2^{\alpha+\beta+1} c_{\alpha, \beta}} \quad \text{with} \quad c_{\alpha, \beta} := \frac{\Gamma(\alpha + \beta + 2)}{\Gamma(\alpha + 1) \Gamma(\beta + 1)}.
\] (2.3)

The Jacobi polynomials \( P_{n}^{(\alpha, \beta)} \) are hypergeometric functions given by \([12, (4.21.2)]\)
\[
P_{n}^{(\alpha, \beta)}(x) = \frac{(\alpha + 1)_n}{n!} \binom{-n}{\alpha + \beta + 1/2} \binom{\alpha + 1}{\alpha + 1} x^n,
\]
and they are orthogonal polynomials with respect to \( w_{\alpha, \beta} \) satisfying
\[
c'_{\alpha, \beta} \int_{-1}^{1} P_{n}^{(\alpha, \beta)}(x) P_{m}^{(\alpha, \beta)}(x) w_{\alpha, \beta}(x) dx = h_{n}^{(\alpha, \beta)} \delta_{n,m},
\]
where
\[
h_{n}^{(\alpha, \beta)} = \frac{(\alpha + 1)_n (\beta + 1)_n (\alpha + \beta + n + 1)}{n!(\alpha + \beta + 2)_n (\alpha + \beta + 2n + 1)}.
\] (2.4)

The polynomial \( P_{n}^{(\alpha, \beta)} \) satisfies a differential equation \([12, (4.2.1)]\)
\[
(1 - x^2)u'' - (\alpha - \beta + (\alpha + \beta + 2)x)u' + n(n + \alpha + \beta + 1)u = 0.
\] (2.5)
We will use the Gegenbauer polynomials \( C_\lambda^\lambda \), also called ultraspherical polynomials when \( \lambda \) is half-integer. For \( \lambda > -\frac{1}{2} \), let \( w_\lambda \) be the weight function

\[
w_\lambda(x) = (1 - x^2)^{\lambda - \frac{1}{2}}, \quad -1 < x < 1.
\]

Its normalization constant \( c_\lambda \), defined by \( c_\lambda \int_{-1}^{1} w_\lambda(x) \, dx = 1 \), is given by

\[
c_\lambda := \frac{\Gamma(\lambda + 1)}{\Gamma(\frac{1}{2})\Gamma(\lambda + \frac{1}{2})}.
\]

(2.6)

The Genenbauer polynomials \( C_\lambda^\lambda \) are orthogonal with respect to \( w_\lambda \),

\[
c_\lambda \int_{-1}^{1} C_\lambda^\lambda(x) C_m^\lambda(x) w_\lambda(x) \, dx = h_\lambda^\lambda \delta_{n,m}, \quad h_\lambda^\lambda = \frac{\lambda}{n + \lambda} C_\lambda^\lambda(1),
\]

and are normalized so that \( C_\lambda^\lambda(1) = \frac{(2\lambda)_n}{n!} \). The polynomial \( C_\lambda^\lambda \) is a constant multiple of the Jacobi polynomial \( P_{\lambda - \frac{1}{2}, \lambda - \frac{1}{2}}^\lambda(n) \) and it is also related to the Jacobi polynomial by a quadratic transform [12, (4.7.1)]

\[
C_{2n}^\lambda(x) = \frac{(\lambda)_n}{(\frac{1}{2})_n} P_{\lambda - \frac{1}{2}, \lambda - \frac{1}{2}}^\lambda(2x^2 - 1).
\]

(2.8)

For convenience, see (2.13) below, we shall define

\[
Z^\lambda_n(t) := \frac{C_\lambda^\lambda(1) C_n^\lambda(t)}{h^\lambda_n} = \frac{n + \lambda}{\lambda} C_n^\lambda(t), \quad \lambda > 0, \quad Z^0_n(t) := \begin{cases} 2T_n(t), & n \geq 1 \\ 1, & n = 0 \end{cases},
\]

(2.9)

where \( T_n(x) = \arccos(\cos x) \) is the Chebyshev polynomial, and use this notation throughout the rest of this paper.

We will also need the Laguerre polynomials \( L_\alpha^n \). For \( \alpha > -1 \), these polynomials are orthogonal with respect to \( x^\alpha e^{-x} \) on \( \mathbb{R}_+ = [0, \infty) \),

\[
\frac{1}{\Gamma(\alpha + 1)} \int_0^{\infty} L_\alpha^n(x) L_m^\alpha(x) x^\alpha e^{-x} \, dx = \frac{(\alpha + 1)_n}{n!} \delta_{n,m}.
\]

(2.10)

### 2.3 Spherical Harmonics

There are many sources for this topic, we follow [2, Chapter 1]. Let \( P_\lambda^n \) denote the space of homogeneous polynomials of degree \( n \) in \( d \)-variables. A spherical harmonics \( Y \) of degree \( n \) is an element of \( P_\lambda^n \) that satisfies \( \Delta Y = 0 \), where \( \Delta \) is the Laplace operator of \( \mathbb{R}^d \). If \( Y \in P_\lambda^n \), then \( Y(x) = ||x||^n Y(x') \), \( x' = x/||x|| \in S^{d-1} \), so that \( Y \) is...
determined by its restriction on $S^{d-1}$. Let $\mathcal{H}_n^d$ denote the space of spherical harmonics of degree $n$. Then

$$\dim \mathcal{H}_n^d = \binom{n + d - 1}{n} - \binom{n + d - 3}{n - 2} = \binom{n + d - 2}{n} + \binom{n + d - 3}{n - 1},$$

(2.11)

and we also have $\dim P_n^d = \binom{n + d - 1}{n}$. Spherical harmonics of different degrees are orthogonal on the sphere. For $n \in \mathbb{N}_0$ let $\{Y^n_\ell : 1 \leq \ell \leq \dim \mathcal{H}_n^d\}$ be an orthonormal basis of $\mathcal{H}_n^d$ in this subsection; then

$$\frac{1}{\omega_d} \int_{S^{d-1}} Y^n_\ell(\xi) Y^m_\ell'(\xi) d\sigma(\xi) = \delta_{\ell, \ell'} \delta_{m, n},$$

where $\omega_d$ denotes the surface area $\omega_d = 2\pi^{\frac{d}{2}}/\Gamma\left(\frac{d}{2}\right)$ of $S^{d-1}$. Let $\Delta_0$ be the Laplace–Beltrami operator on the sphere, which is the restriction of $\Delta$ on the unit sphere. Spherical harmonics are eigenfunctions of this operator [2, (1.4.9)],

$$\Delta_0 Y = -n(n + d - 2)Y, \quad Y \in \mathcal{H}_n^d;$$

(2.12)

the eigenvalues depend only on the degree $n$ of $\mathcal{H}_n^d$. Moreover, spherical harmonics satisfy an addition formula [2, (1.2.3) and (1.2.7)],

$$\dim \mathcal{H}_n^d \sum_{\ell=1}^{\dim \mathcal{H}_n^d} Y^n_\ell(x) Y^n_\ell(y) = Z_n^{d-2}(\langle x, y \rangle), \quad x, y \in S^{d-1},$$

(2.13)

where $Z_n^k$ is defined in (2.9). For $f \in L^2(S^{d-1})$, its Fourier orthogonal series in spherical harmonics is defined by

$$f = \sum_{n=0}^{\infty} \sum_{\ell=0}^{\dim \mathcal{H}_n^d} \hat{f}_\ell^n Y^n_\ell, \quad \hat{f}_\ell^n = \frac{1}{\omega_d} \int_{S^{d-1}} f(y) Y^n_\ell(y) d\sigma,$$

where the orthogonal projection $\text{proj}_n: L^2(S^{d-1}) \to \mathcal{H}_n^d$ can be written as an integral

$$\text{proj}_n f(x) = \sum_{\ell=1}^{\dim \mathcal{H}_n^d} \hat{f}_{\ell, n} Y^n_\ell = \frac{1}{\omega_d} \int_{S^{d-1}} f(y) P_n(x, y) d\sigma(y)$$

in terms of the reproducing kernel $P_n(x, y) = \sum_{\ell} Y^n_\ell(x) Y^n_\ell(y)$. The addition formula (2.13) provides a closed formula for the reproducing kernel and shows that it has a one-dimensional structure. This formula plays an essential role in the study of Fourier orthogonal series on the sphere; see, for example, [2,11].
2.4 Classical Orthogonal Polynomials on the Unit Ball

Classical orthogonal polynomials on the unit ball $\mathbb{B}^d$ are orthogonal with respect to the inner product

$$\langle f, g \rangle_\mu = b_B^\mu \int_{\mathbb{B}^d} f(x) g(x) \sigma_\mu(x) \, dx,$$

where $\sigma_\mu$ is the weight function

$$\sigma_\mu(x) := (1 - \|x\|^2)^{\mu - \frac{1}{2}}, \quad \mu > -\frac{1}{2}, \quad \text{and} \quad b_B^\mu = \frac{\Gamma(\mu + \frac{d+1}{2})}{\pi^{\frac{d}{2}} \Gamma(\mu + \frac{1}{2})}$$

is the normalization constant so that $\langle 1, 1 \rangle_\mu = 1$. Let $\mathcal{V}_n^d(\sigma_\mu)$ be the space of orthogonal polynomials of degree $n$ with respect to $\sigma_\mu$. An orthogonal basis of $\mathcal{V}_n^d(\sigma_\mu)$ can be given explicitly in terms of the Jacobi polynomials and spherical harmonics. For $0 \leq m \leq n/2$, let $\{Y_{\ell,n-2m} : 1 \leq \ell \leq \dim \mathcal{H}_{n-2m}^d\}$ be an orthonormal basis of $\mathcal{H}_{n-2m}^d$.

Define

$$P_{n,\ell,m}^\mu(x) = P_m^{\mu - \frac{1}{2}, n-2m + \frac{d-2}{2}} \left( 2\|x\|^2 - 1 \right) Y_{\ell,n-2m}(x). \quad (2.15)$$

Then $\{P_{n,\ell,m}^\mu : 0 \leq m \leq n/2, 1 \leq \ell \leq \dim \mathcal{H}_{n-2m}^d\}$ is an orthogonal basis of $\mathcal{V}_n^d(\sigma_\mu)$.

For other explicit orthogonal bases of $\mathcal{V}_n^d(\sigma)$, see [5, Chapter 5].

As in the case of spherical harmonics, classical orthogonal polynomials on the unit ball are eigenfunctions of a second order differential operator: for $u \in \mathcal{V}_n^d(\sigma_\mu)$,

$$\left( \Delta - \langle x, \nabla \rangle^2 - (2\mu + d - 1)\langle x, \nabla \rangle \right) u = -n(n + 2\mu + d - 1)u. \quad (2.16)$$

Let $\mathbf{P}_n(\sigma_\mu; \cdot, \cdot)$ be the reproducing kernel of the space $\mathcal{V}_n^d(\sigma_\mu)$, as defined in (2.2). Then the kernel satisfies a closed formula [5, (5.2.7)] for $\mu \geq 0$,

$$\mathbf{P}_n(\sigma_\mu; x, y) = c_{\mu - \frac{1}{2}} \int_{-1}^1 Z_n^{\mu + \frac{d-1}{2}} \left( \langle x, y \rangle + t \sqrt{1 - \|x\|^2} \sqrt{1 - \|y\|^2} \right) \times (1 - t^2)^{\mu - 1} \, dt, \quad (2.17)$$

where $c_{\mu - \frac{1}{2}}$ is give by (2.6) and the identity holds when $\mu = 0$ under the limit

$$\lim_{\mu \to 0} c_{\mu - \frac{1}{2}} \int_{-1}^1 f(t)(1 - t^2)^{\mu - 1} \, dt = \frac{f(1) + f(-1)}{2}. \quad (2.18)$$
2.5 Jacobi Polynomials on the Triangle

In two variables, the cone \( \mathcal{V}^2 \) reduces to a triangle, but not the standard triangle \( \mathbb{T}^2 \). The classical orthogonal polynomials on the triangle are orthogonal with respect to the weight function

\[
\varpi_{\alpha, \beta, \gamma}(x_1, x_2) = x_1^\alpha x_2^\beta (1 - x_1 - x_2)^\gamma, \quad \alpha, \beta, \gamma > -1,
\]
on \( \mathbb{T}^2 \). Let \( \mathcal{V}^2_n(\varpi_{\alpha, \beta, \gamma}) \) denote the space of orthogonal polynomials with respect to \( \varpi_{\alpha, \beta, \gamma} \). Several bases of \( \mathcal{V}^2_n(\varpi_{\alpha, \beta, \gamma}) \) can be given explicitly in terms of the Jacobi polynomials. One of them consists of [5, Proposition 2.4.2]

\[
P^\alpha, \beta, \gamma_{\kappa, n}(x_1, x_2) = P_{2\kappa + \alpha + \beta + 1, \gamma}^\alpha n - \kappa (1 - 2x_1 - 2x_2)(x_1 + x_2)^{\kappa} P_{\beta, \alpha}^\kappa (x_1 - x_2 x_1 + x_2)^{\kappa},
\]
for \( 0 \leq \kappa \leq n \). The \( L^2 \) norms of these polynomials are given by

\[
h_{\kappa, n}(\alpha, \beta, \gamma) = \frac{c_{\alpha + \beta + 1, \gamma}}{c_{\alpha + \beta + 2k + 1, \gamma}} h_{\kappa, n}(\alpha + \beta + 2k + 1, \gamma),
\]
where \( c_{\alpha, \beta} \) is given by (2.3) and \( h_{\kappa, n}(\alpha, \beta) \) is given by (2.4).

In terms of this basis, the reproducing kernel \( P_n(\varpi_{\alpha, \beta, \gamma}; \cdot, \cdot) \) is given by

\[
P_n(\varpi_{\alpha, \beta, \gamma}; x, y) = \sum_{\kappa=0}^{n} P_{2\kappa + \alpha + \beta + 1, \gamma}^\alpha n - \kappa (1 - 2x_1 - 2x_2)(x_1 + x_2)^{\kappa} P_{\beta, \alpha}^\kappa (x_1 - x_2 x_1 + x_2)^{\kappa} \eta(x, y, t),
\]
where

\[
\eta(x, y, t) = \sqrt{x_1 y_1 t_1} + \sqrt{x_2 y_2 t_2} + \sqrt{(1 - x_1 - x_2)(1 - y_1 - y_2) t_3}.
\]

For further results and discussions on classical orthogonal polynomials on the ball and the triangle, see Sects. 5.2, 2.4 and 5.3 of [5].
3 Orthogonal Polynomials on the Cone

We consider orthogonal polynomials on the cone \( \mathbb{V}^{d+1} \) with respect to the weight function

\[
W(x, t) = w(t)(t^2 - \|x\|^2)^{\mu - \frac{1}{2}} \quad \|x\| \leq t, \quad 0 \leq t \leq b,
\]

for two families of \( w \). We call the first case Jacobi polynomials on the cone, for which \( w(t) = (1 - t)^\beta \), and call the second case Laguerre polynomials on the cone, for which \( w(t) = e^{-t} \). Each of these two families are eigenfunctions of a second order differential operator with eigenvalues depending only on the total degree of the polynomials.

3.1 Jacobi Polynomials on the Cone

We consider the weight functions

\[
W_{\mu, \beta, \gamma}(x, t) := (t^2 - \|x\|^2)^{\mu - \frac{1}{2}} t^\beta (1 - t)^\gamma, \quad \mu > -\frac{1}{2}, \beta > -1, \gamma > -1, \quad (3.1)
\]

defined on the solid cone

\[
\mathbb{V}^{d+1} = \{(x, t) : \|x\| \leq t \leq 1, \quad x \in \mathbb{B}^d\}
\]

and define the inner product

\[
\langle f, g \rangle_{\mu, \beta, \gamma} := b_{\mu, \beta, \gamma} \int_{\mathbb{V}^{d+1}} f(x, t)g(x, t)W_{\mu, \beta, \gamma}(x, t)dx \, dt,
\]

where \( b_{\mu, \beta, \gamma} \) is the constant chosen so that \( \langle 1, 1 \rangle_{\mu, \beta, \gamma} = 1 \). For \( (x, t) \in \mathbb{V}^{d+1} \), we let \( x = tu \) with \( u \in \mathbb{B}^d \), so that \( dx = t^d \, du \) and

\[
\int_{\mathbb{V}^{d+1}} f(x, t)W_{\mu, \beta, \gamma}(x, t)dx \, dt = \int_0^1 \int_{\|u\| \leq t} f(tu, t)t^{2\mu + \beta + d - 1}(1 - t)^\gamma \, dt \, du,
\]

which implies, in particular, that

\[
b_{\mu, \beta, \gamma} = c_{2\alpha, \gamma} b_{\mu}^B, \quad \alpha = \mu + \frac{\beta + d - 1}{2},
\]

where \( c_{\alpha, \beta} \) is defined in (2.3) and \( b_{\mu}^B \) is the normalized constant of \( \varpi_{\mu} \) on \( \mathbb{B}^d \).

Let \( \mathcal{V}_n^{d+1}(W_{\mu, \beta, \gamma}) := \mathcal{V}_n(\mathbb{V}^{d+1}, W_{\mu, \beta, \gamma}) \) be the space of orthogonal polynomials of degree exactly \( n \) with respect to the inner product \( \langle \cdot, \cdot \rangle_{\mu, \beta, \gamma} \).
Proposition 3.1 For \( m = 0, 1, 2, \ldots \), let \( \{ P_m^\alpha (\sigma, \mu) : |k| = m, k \in \mathbb{N} \} \) denote an orthonormal basis of \( \mathcal{V}_m^d(\sigma, \mu) \) on the unit ball \( \mathbb{B}^d \). Define \( \alpha := \mu + \frac{\beta + d - 1}{2} \) and

\[
Q_{m,k}(x, t) := P_{n-m}^{(2\alpha+2m, \gamma)}(1 - 2t)t^m P_k^m \left( \frac{\sigma, \mu}{t} \right). \tag{3.3}
\]

Then \( \{ Q_{m,k}(x, t) : |k| = m, 0 \leq m \leq n \} \) is an orthogonal basis of \( \mathcal{V}_n(\mathbb{B}^{d+1}, W_{\mu, \beta, \gamma}) \) and the norm of \( Q_{m,k} \) is given by

\[
H_{m,n}^{(\alpha, \gamma)} := \langle Q_{m,k}^n, Q_{m,k}^n \rangle_{\mu, \beta, \gamma} = \frac{c_{2\alpha, \gamma}}{c_{2\alpha+2m, \gamma}} h_{n-m}^{(2\alpha+2m, \gamma)}, \tag{3.4}
\]

where \( h_m^{(\alpha, \gamma)} \) is the norm square of the Jacobi polynomial in (2.4) and \( c_{\alpha, \beta} \) is in (2.3).

Proof It is clear that \( Q_{m,k} \) is a polynomial of degree \( n \) in \((x, t)\). Using (3.2) to separate variables, it is easy to see that \( \langle Q_{m,k}^n, Q_{m,k}^n \rangle_{\mu, \beta, \gamma} = H_{m,n} \delta_n \delta_m \delta_{\mu, \beta, \gamma} \). Furthermore, since \( P_k^m \) is orthonormal, we obtain

\[
\langle Q_{m,k}^n, Q_{m,k}^n \rangle_{\mu, \beta, \gamma} = c_{2\alpha, \gamma} \int_0^1 \left| P_{n-m}^{(2\alpha+2m, \gamma)}(1 - 2t) \right|^2 t^{2\alpha+2m}(1 - t)^\gamma \, dt
\]

\[
= \frac{c_{2\alpha, \gamma}}{c_{2\alpha+2m, \gamma}} h_{n-m}^{(2\alpha+2m, \gamma)},
\]

since \( h_n^{(\alpha, \gamma)} \) is defined with respect to the normalized weight and \( \frac{c_{\alpha, \beta}}{2\alpha+\beta+1} \) is the normalization constant of the Jacobi weight on \([-1, 1]\). Finally, the cardinality of the set \( \{ Q_{m,k}(x, t) : |k| = m, 0 \leq m \leq n \} \) is equal to

\[
\sum_{m=0}^n \dim \mathcal{V}_m^d(\sigma, \mu) = \sum_{m=0}^n \binom{m + d - 1}{m} = \binom{n + d}{d} = \dim \mathcal{V}_n(\mathbb{B}^{d+1}, W_{\mu, \beta, \gamma}),
\]

which verifies that the set is an orthonormal basis of \( \mathcal{V}_n(\mathbb{B}^{d+1}, W_{\mu, \beta, \gamma}) \). \( \square \)

It is not difficult to see that the structure of the orthogonal basis in (3.3) holds for a generic weight function \( w \) instead of the Jacobi weight; see [9]. However, it is the polynomials in (3.3) that share properties of classical orthogonal polynomials on the unit ball.

The weight function \( W_{\mu, \beta, \gamma} \) is a combination of the Jacobi weight on \([0, 1]\) and the classical weight function on the unit ball. From the point of view of differential equations satisfied by the classical orthogonal polynomials, the case \( \beta = 0 \) stands out. We shall define \( W_{\mu, \gamma} = W_{\mu, 0, \gamma} \). More explicitly,

\[
W_{\mu, \gamma}(x, t) = (t^2 - \|x\|^2)^{-\frac{1}{2}} (1 - t)^\gamma, \quad \mu > -\frac{1}{2}, \quad \gamma > -1. \tag{3.5}
\]

Our next theorem shows that \( \mathcal{V}_n(\mathbb{B}^{d+1}, W_{\mu, \gamma}) \) is an eigenspace of a second order differential operator.
Theorem 3.2 Let $\mu > -\frac{1}{2}$, $\gamma > -1$ and $n \in \mathbb{N}_0$. Then every $u \in \mathcal{V}_n(\gamma^{d+1}, W_{\mu,\gamma})$ satisfies the differential equation

$$D_{\mu,\gamma} u = -n(n + 2\mu + \gamma + d)u,$$  \hspace{1cm} (3.6)

where $D_{\mu,\gamma} = D_{\mu,\gamma}(x, t)$ is the second order linear differential operator

$$D_{\mu,\gamma} := (1 - t)\partial_t^2 + 2(1 - t)\langle x, \nabla_x \rangle \partial_t + \sum_{i=1}^{d} (t - x_i^2)\partial_{x_i}^2 - 2\sum_{i<j} x_i x_j \partial_{x_i} \partial_{x_j}$$

$$+ (2\mu + d)\partial_t - (2\mu + \gamma + d + 1)(\langle x, \nabla_x \rangle + t \partial_t),$$

where $\nabla_x$ and $\Delta_x$ denote the gradient and the Laplace operator in $x$-variable.

**Proof** By Proposition 3.1, it suffices to establish the result for $u = \frac{Q_n^m}{V_m}(x) \in (3.3)$. For simplicity, we write

$$u(x, t) = g(t) H(x, t), \quad g(t) = P_{n-m}^{(2\mu + 2m + d - (2\mu + \gamma + 2m + d + 1)t)}(1 - 2t), \quad H(x, t) = t^n P_{\frac{X}{t}}^m \left( \frac{X}{t} \right).$$

The differential equation for the Jacobi polynomial (2.5) shows that $g$ satisfies

$$t(1 - t)g''(t) + (2\mu + 2m + d - (2\mu + \gamma + 2m + d + 1)t)g'(t)$$

$$= -(n - m)(n + m + 2\mu + \gamma + d)g(t). \hspace{1cm} (3.7)$$

Since $H(x, t)$ is a homogeneous polynomial of degree $m$ in $(x, t)$, it satisfies the identity

$$\left( t \frac{\partial}{\partial t} + \langle x, \nabla_x \rangle \right) H(x, t) = m H(x, t). \hspace{1cm} (3.8)$$

Furthermore, let $h(x) := P_{\frac{X}{t}}^m(x)$; then $h \in \mathcal{V}_m^d(\mu)$ satisfies the differential equation (2.16). Since $H(x, t) = t^n h(X/t)$, it follows that

$$\left( t^2 \Delta_x - \langle x, \nabla_x \rangle^2 - (2\mu + d - 1)\langle x, \nabla_x \rangle \right) H = -m(m + 2\mu + d - 1)H. \hspace{1cm} (3.9)$$

Now, taking derivatives, a straightforward computation gives

$$t(1 - t)u_{tt} + 2(1 - t)\langle x, \nabla_x \rangle u_t + t(1 - t)g''(t)H(x, t)$$

$$+ 2(1 - t)g'(t) \left( t \frac{\partial}{\partial t} + \langle x, \nabla_x \rangle \right) H(x, t)$$

$$+ (1 - t)g(t) \left( 2\langle x, \nabla_x \rangle \frac{\partial}{\partial t} H(x, t) + t \frac{\partial^2}{\partial t^2} H(x, t) \right).$$
Using (3.8) on the second term in the righthand side, we obtain from (3.7)
\[
\begin{align*}
t(1-t)u_{tt} + 2(1-t)\langle x, \nabla_x \rangle u_t + \left(2\mu + d - (2\mu + \gamma + d + 1)t\right)u_t \\
= -(n-m)(n+m+2\mu + \gamma + d)u - (\gamma + 1)g(t)\frac{\partial}{\partial t} H(x,t) \\
+ (1-t)g(t) \left[ \frac{\partial^2}{\partial t^2} H(x,t) + (2\langle x, \nabla_x \rangle + 2\mu + d)\frac{\partial}{\partial t} H(x,t) \right].
\end{align*}
\]

(3.10)

Since \( \frac{\partial}{\partial t} H(x,t) \) is a homogeneous polynomial of degree \( m-1 \) in \( x, t \) variables, we deduce by (3.8) that
\[
t \frac{\partial^2}{\partial t^2} H(x,t) = (m-1) \frac{\partial}{\partial t} H(x,t) - \langle x, \nabla_x \rangle \frac{\partial}{\partial t} H(x,t),
\]
so that the square bracket in the righthand side of (3.10) becomes, applying (3.8) one more time,
\[
[\ldots] = (2\mu + m + d - 1 + \langle x, \nabla_x \rangle t) \frac{\partial}{\partial t} H(x,t) \\
= t^{-1} \left( -\langle x, \nabla_x \rangle^2 - (2\mu + d - 1)\langle x, \nabla_x \rangle + m(2\mu + m + d - 1) \right) H(x,t) \\
= -t \Delta_x H(x,t),
\]
where the last step follows from (3.9). With this identity and applying (3.8) again in the second term in the righthand side of (3.10), we obtain
\[
\begin{align*}
t(1-t)u_{tt} + 2(1-t)\langle x, \nabla_x \rangle u_t + \left(2\mu + d - (2\mu + \gamma + d + 1)t\right)u_t + t(1-t)\Delta_x u \\
= -(n-m)(n+m+2\mu + \gamma + d)u - m(\gamma + 1)u + (\gamma + 1)\langle x, \nabla \rangle u.
\end{align*}
\]

Now, adding (3.9) multiplied by \( g(t) \) to the above identity and using
\[
(n-m)(n+m+2\mu + \gamma + d) + m(m+2\mu + \gamma + d) = n(n+2\mu + \gamma + d),
\]
we conclude that
\[
[t(1-t)\partial_{tt} + 2(1-t)\langle x, \nabla_x \rangle \partial_t + t\Delta_x - \langle x, \nabla_x \rangle^2 + (2\mu + d)\partial_t \\
- (2\mu + \gamma + d + 1)\left(\langle x, \nabla_x \rangle + t\partial_t\right) + \langle x, \nabla_x \rangle]\ u = -n(n+2\mu + \gamma + d)\ u.
\]

Finally, using \( \langle x, \nabla_x \rangle^2 = \langle x, \nabla_x \rangle + \sum_{1 \leq i, j \leq d} x_i x_j \partial_{x_i} \partial_{x_j} \), it is easy to verify that the lefthand side of the above identity is exactly \( D_{\mu, \gamma} \). The proof is completed. \( \square \)

**Remark 3.1** When \( \beta \neq 0 \), the Jacobi polynomials \( Q_{m,k}^\beta \) on the cone with respect to \( W_{\mu, \beta, \gamma} \) also satisfy a differential equation, but the eigenvalues depend on both \( m \) and \( n \). In other words, \( V_n(\sqrt{d+1}, W_{\mu, \beta, \gamma}) \) is not an eigenspace of such a differential operator.
3.2 Laguerre Polynomials on the Cone

We consider the weight functions

\[ W_{\mu, \beta}(x, t) := (t^2 - \|x\|^2)^{\mu - \frac{1}{2}} t^{\beta} e^{-t}, \quad \mu > -\frac{1}{2}, \quad \beta > -1. \]

defined on the infinite solid cone

\[ \mathbb{V}^{d+1} = \{(x, t) : \|x\| \leq t, \quad t \in (0, \infty), \ x \in \mathbb{R}^d \} \]

and define the inner product

\[ \langle f, g \rangle_{\mu, \beta} := b_{\mu, \beta} \int_{\mathbb{V}^{d+1}} f(x, t) g(x, t) W_{\mu, \beta}(x, t) \, dx \, dt, \]

where \( b_{\mu, \beta} \) is the constant chosen so that \( \langle 1, 1 \rangle_{\mu, \beta} = 1 \). Similar to (3.2), we have

\[ \int_{\mathbb{V}^{d+1}} f(x, t) W_{\mu, \beta}(x, t) \, dx \, dt = \int_0^\infty \int_{\mathbb{B}^d} f(tu, t) t^{\beta + 2\mu + d - 1} e^{-t} dt (1 - \|u\|^2)^{\mu - \frac{1}{2}} du, \]

which implies, in particular, that

\[ b_{\mu, \beta} = cL^{2\mu + \beta + d - 1} b_B^{\alpha} \quad \text{with} \quad cL_\alpha = \frac{1}{\Gamma(\alpha + 1)}. \]

Let \( \mathcal{V}_n(\mathbb{V}^{d+1}, W_{\mu, \beta}) \) be the space of orthogonal polynomials of degree exactly \( n \) with respect to the inner product \( \langle \cdot, \cdot \rangle_{\mu, \beta} \). Similar to the Jacobi case, a basis for the Laguerre polynomials on the cone is given in the following:

**Proposition 3.3** For \( m = 0, 1, 2, \ldots \), let \( \{P_k^{m}(\alpha_{\mu}) : |k| = m, k \in \mathbb{N}_0^d\} \) denote an orthonormal basis of \( \mathcal{V}_m(\alpha_{\mu}) \) on the unit ball \( \mathbb{B}^d \). Let \( \alpha = \mu + \frac{\beta + d - 1}{2} \). Define

\[ L_{m, k}^n(x, t) := L_{n-m}^{(2\alpha+2m)}(t) P_k^m \left( \frac{x}{t} \right). \]

Then \( \{L_{m, k}^n(x, t) : |k| = m, \ 0 \leq m \leq n\} \) is an orthogonal basis of \( \mathcal{V}_n(\mathbb{V}^{d+1}, W_{\mu, \beta}) \) and the norm of \( L_{m, k}^n \) is given by

\[ H_{m,n} := \langle L_{m, k}^n, L_{m, k}^n \rangle_{\mu, \beta} = \frac{cL_{2\alpha}}{cL_{2\alpha+2m}} h_{n-m}^{(2\alpha+2m)}, \]

where \( h_m^{(\alpha)} \) denotes the norm of the Laguerre polynomial of degree \( m \).

When \( \beta = 0 \), as in the Jacobi case, the Laguerre polynomials on the cone are also eigenfunctions of a second order linear PDE with eigenvalues depending only on the degree of the polynomials.
Theorem 3.4 Let $\mu > -\frac{1}{2}$ and $n \in \mathbb{N}_0$. Then every $u \in \mathcal{V}_n(\mathbb{V}^{d+1}, W^L_{\mu,0})$ satisfies the differential equation

$$
\mathcal{D}_\mu u = -nu,
$$

where $\mathcal{D}_\mu = \mathcal{D}_\mu(x,t)$ is the second order linear differential operator

$$
\mathcal{D}_\mu := t \left( \Delta_x + \partial_t^2 \right) + 2\langle x, \nabla_x \rangle \partial_t - \langle x, \nabla_x \rangle + (2\mu + d - t)\partial_t.
$$

Proof The proof follows the same steps as in the Jacobi case, but simpler. We again write $u(x,t) = g(t)H(x,t)$, with $g(t) = L^{(2\mu+d+1+2m)}_{-m} (t)$ and $H$ being the same as before. Instead of (3.7), we now use the differential equation for the Laguerre polynomials [12, (5.1.2)], which shows that $g$ satisfies

$$
tg''(t) + (2\mu + 2m + d - t)g'(t) = -(n - m)g(t).
$$

(3.12)

As in the proof of Theorem 3.2, a straightforward computation shows that the above differential equation for $g$ leads to an analog of (3.10),

$$
tu_{tt} + 2\langle x, \nabla_x \rangle u_t + (2\mu + d - t)u_t = -(n - m)u - g(t)t \frac{\partial}{\partial t} H(x,t)
$$

$$
+ g(t) \left[ t \frac{\partial^2}{\partial t^2} H(x,t) + (2\langle x, \nabla_x \rangle + 2\mu + d) \frac{\partial}{\partial t} H(x,t) \right].
$$

(3.13)

The term in the square bracket, being exactly the same as in the proof of Theorem 3.2, is equal to $-t \Delta_x H(x,t)$. Hence, using (3.8) for $t \frac{\partial}{\partial t} H(x,t)$, we see that the righthand side of (3.13) becomes

$$
-(n - m)u - (m - \langle x, \nabla_x \rangle)u - t\Delta_x u = -nu + \langle x, \nabla_x \rangle u - t\Delta_x u.
$$

Moving the last two terms to the lefthand side of (3.13), we have proved (3.11). \hfill \Box

Remark 3.2 For $\beta \neq 0$, the Laguerre polynomials on the cone also satisfy a differential equation, but the eigenvalues depend on both $m$ and $n$. In other words, $\mathcal{V}_n(\mathbb{V}^{d+1}, W^L_{\mu,\beta})$ is not an eigenspace of such a differential operator.

4 Reproducing Kernel for the Jacobi Polynomials on the Cone

As the kernels of the orthogonal projection operators (2.1), reproducing kernels play an essential role for studying Fourier orthogonal expansions. In this section we describe a closed formula for the reproducing kernels of the Jacobi polynomials on the cone.
In terms of the orthogonal basis in Proposition 3.1, the reproducing kernel of 
\( V_n(V_{\nu+1}, W_{\mu, \beta, \gamma}) \) satisfies

\[
P_n(W_{\mu, \beta, \gamma}; (x, t), (y, s)) = \sum_{m=0}^{n} \sum_{|k|=m} \frac{Q_{n,m,k}(x, t) Q_{n,m,k}(y, s)}{H_{m,n}^{\alpha, \gamma}}.
\] (4.1)

To simplify our presentation, we consider \( \beta = 0 \) and \( \beta \neq 0 \) separately.

### 4.1 Jacobi Polynomials on the Cone with \( \beta = 0 \)

We consider the case \( \beta = 0 \), or the weight function \( W_{\mu, \gamma} \) in (3.5) first. We begin with the reproducing kernel on the triangle \( V_2 \).

For \( d = 1 \), the cone \( V_2 \) becomes a triangle \( V_2 = \{(x, t) \in \mathbb{R}^2 : |x| \leq t \leq 1\} \). It is related to the standard triangle \( T_2 \) by a change of variable \((x_1, x_2) \in T_2 \mapsto (x, t) \in V_2\),

\[
(x_1, x_2) = \left( \frac{t + x}{2}, \frac{t - x}{2} \right).
\]

Under this change of variables, the classical weight function \( \varpi_{\alpha, \beta, \gamma} \) on the triangle \( T_2 \) becomes \( 2^{-\alpha-\beta} \hat{\varpi}_{\alpha, \beta, \gamma} \), where

\[
\hat{\varpi}_{\alpha, \beta, \gamma}(x, t) := (x + t)^{\alpha}(t - x)^{\beta}(1 - t)^{\gamma}, \quad (x, t) \in V_2.
\]

In particular, we see that \( \hat{\sigma}_{\mu-\frac{1}{2}, \mu-\frac{1}{2}, \gamma} = W_{\mu, \gamma} \) when \( d = 1 \). For \( d \geq 2 \), we need \( \hat{\sigma}_{\alpha-\frac{1}{2}, \alpha-\frac{1}{2}, \gamma} \) for \( \alpha = \mu + \frac{d-1}{2} \). With a slight abuse of notation, we denote the reproducing kernel with respect to the weight function

\[
\hat{\varpi}_{\alpha, \gamma}(x, t) := \hat{\varpi}_{\alpha-\frac{1}{2}, \alpha-\frac{1}{2}, \gamma}(x, t) = (t^2 - x^2)^{\alpha-\frac{1}{2}}(1 - t)^{\gamma}
\]

on the triangle \( V_2 \) by

\[
P_n\left( \hat{\varpi}_{\alpha, \gamma}; (x, t), (y, s) \right) := P_n\left( \varpi_{\alpha-\frac{1}{2}, \alpha-\frac{1}{2}, \gamma}; \left( \frac{t + x}{2}, \frac{t - x}{2} \right), \left( \frac{s + y}{2}, \frac{s - y}{2} \right) \right),
\] (4.2)

where the righthand side is the reproducing kernel for \( V_n(T_2, \varpi_{\alpha-\frac{1}{2}, \alpha-\frac{1}{2}, \gamma}) \) on the triangle. Recall that \( Z_m^\alpha \) is defined in (2.9).

**Lemma 4.1** Let \( \alpha = \mu + \frac{d-1}{2} \). For \( (u, t) \in V_2 \) and \( 0 \leq s \leq 1 \),

\[
P_n\left( \hat{\varpi}_{\alpha, \gamma}; (u, t), (s, s) \right) = \sum_{m=0}^{n} \frac{h_m^{\alpha-\frac{1}{2}, \alpha-\frac{1}{2}}}{h_m^{\alpha-\frac{1}{2}, \alpha-\frac{1}{2}, \gamma}} P_{m+2\alpha, \gamma}^{n-m}(1 - 2t)
\times P_{m+2\alpha, \gamma}^{n-m}(1 - 2s) t^m s^m Z_m^\alpha \left( \frac{u}{t} \right),
\] (4.3)
where \( h_{m}^{(\alpha,\beta)} \) is the norm of the Jacobi polynomial in (2.4) and \( h_{m,n}^{\alpha,\beta,\gamma} \) is the norm of the Jacobi polynomial on the triangle in (2.20).

**Proof** By (4.2) and (2.21), we see that the kernel can be written as

\[
P_{n}(\tilde{\omega}_{\alpha,\gamma}; (u, t), (v, s)) = \sum_{k=0}^{n} \frac{P_{(2k+2\alpha,\gamma)}^{(2k+2\alpha,\gamma)}(1-2t)P_{(2k+2\alpha,\gamma)}^{(2k+2\alpha,\gamma)}(1-2s)}{h_{k,n}^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2},\gamma)}} \times t^{k}s^{k}P_{k}^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(\frac{u}{t})P_{k}^{(\alpha-\frac{1}{2},\alpha-\frac{1}{2})}(\frac{v}{s})
\]

(4.4)

for \((u, t) \in \mathbb{V}^{2}\) and \((v, s) \in \mathbb{V}^{2}\), where \( h_{k,n}^{(\mu-\frac{1}{2},\mu-\frac{1}{2},\gamma)} \) is given in (2.20). The Gegenbauer polynomials are special cases of the Jacobi polynomials. In particular [12, (4.7.1)],

\[
P_{m}^{(\mu-\frac{1}{2},\mu-\frac{1}{2})}(1)P_{m}^{(\mu-\frac{1}{2},\mu-\frac{1}{2})}(u) = \frac{C_{m}(1)C_{m}(u)}{h_{m}^{\alpha}} = \frac{m + \alpha}{\alpha}C_{m}(u) = \frac{Z_{m}^{\alpha}(u)}{Z_{m}^{\alpha}}.
\]

Hence, setting \( v = s \) in (4.4), we obtain (4.3). \(\Box\)

For \( d \geq 2 \), the reproducing kernel on the cone \( \mathbb{V}^{d+1} \) can be written as an integral of the reproducing kernel on \( \mathbb{V}^{2} \).

**Theorem 4.2** Let \( d \geq 2 \). For \( \mu \geq 0 \) and \( \gamma > -1 \), \((x, t), (y, s) \in \mathbb{V}^{d+1}\),

\[
P_{n}(W_{\mu,0,\gamma}; (x, t), (y, s)) = c_{\mu-\frac{1}{2}} \int_{-1}^{1} P_{n}(\tilde{\omega}_{\alpha,\gamma}; (\zeta(x, t, y, s; u), t), (s, s))(1-u^{2})^{\mu-1}du,
\]

(4.5)

where \( \alpha = \mu + \frac{d-1}{2} \) and

\[
\zeta(x, t, y, s; u) := \frac{(x, y)}{s} + \sqrt{t^{2} - \|x\|^{2} \sqrt{s^{2} - \|y\|^{2}} - u}.
\]

(4.6)

In the case \( \mu = 0 \), the identity (4.5) holds under the limit (2.18).

**Proof** In terms of the orthogonal basis \( Q_{m,k}^{n} \) in (3.3) of \( \mathcal{V}_{n}(\mathbb{V}^{d+1}) \), the reproducing kernel is given by

\[
P_{n}(W_{\mu,0,\gamma}; (x, t), (y, s)) = \sum_{m=0}^{n} \frac{P_{(2\alpha+2m,\gamma)}^{(2\alpha+2m,\gamma)}(1-2t)P_{(2\alpha+2m,\gamma)}^{(2\alpha+2m,\gamma)}(1-2s)}{H_{m,n}^{\alpha,\gamma}} \times s^{m}t^{m}P_{m}(\tilde{\omega}_{\mu}; \frac{x}{t}, \frac{y}{s}).
\]

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Using the closed formula of the reproducing kernel $P_m(\sigma, \cdot; \cdot)$ in (2.17) with

\[
\left( \frac{x}{t}, \frac{y}{s} \right) + \sqrt{1 - \frac{\|x\|^2}{t^2}} \sqrt{1 - \frac{\|y\|^2}{s^2}} u = \frac{\xi(x, t, y, s; u)}{t},
\]

we see that

\[
P_n(W_{\mu, 0, \gamma}; (x, t), (y, s)) = c_{\mu - \frac{1}{2}} \int_{-1}^{1} \sum_{m=0}^{n} \frac{P_{n-m}^{(2\alpha+2m, \gamma)} (1 - 2t) P_{n-m}^{(2\alpha+2m, \gamma)} (1 - 2s) H_{m,n}^{(\mu, \gamma)}}{H_{m,n}^{(\mu, \gamma)}} t^m s^m \\
\times Z_m \left( \frac{\xi(x, t, y, s; u)}{t} \right) (1 - u^2)^{\mu-1} du.
\]

(4.7)

The sum in the righthand side is comparable to the sum in the righthand side of (4.3) since, using the explicit formulas for the constants in (2.4), (2.20) and (3.4), we see that

\[
\frac{h_m^{(\alpha-\frac{1}{2}, \alpha-\frac{1}{2})}}{h_m^{(\alpha-\frac{1}{2}, \alpha-\frac{1}{2}, \gamma)} \pm \gamma}} = \frac{1}{H_{m,n}^{(\mu, \gamma)}} \quad \text{when} \quad \alpha = \mu + \frac{d-1}{2}.
\]

(4.8)

Putting the two identities together, we have proved (4.5). \hfill \Box

The reproducing kernel on the triangle satisfies a closed formula, given in (2.22), when the parameters are nonnegative. This allows us to derive a closed formula for the reproducing kernel on the cone from (4.5).

**Theorem 4.3** Let $d \geq 2$. For $\mu \geq 0$ and $\gamma \geq -\frac{1}{2}$, let $\alpha = \mu + \frac{d-1}{2}$; then

\[
P_n(W_{\mu, 0, \gamma}; (x, t), (y, s)) = c_{\mu - \frac{1}{2}} \int_{-1}^{1} \sum_{m=0}^{n} \frac{P_{n-m}^{(2\alpha+2m, \gamma)} (1 - 2t) P_{n-m}^{(2\alpha+2m, \gamma)} (1 - 2s) H_{m,n}^{(\mu, \gamma)}}{H_{m,n}^{(\mu, \gamma)}} t^m s^m \\
\times (1 - u^2)^{\mu-1} (1 - v_1^2)^{\gamma-1} (1 - v_2^2)^{-\frac{1}{2}} du dv.
\]

(4.9)

where $\xi(x, t, y, s; u, v) \in [-1, 1]$ is defined by

\[
\xi(x, t, y, s; u, v) = v_1 \sqrt{\frac{1}{2} \left( s t + (x, y) + \sqrt{t^2 - \|x\|^2} \sqrt{s^2 - \|y\|^2} u \right)} \\
+ v_2 \sqrt{1 - t \sqrt{1 - s}}.
\]

(4.10)

When $\mu = 0$ or $\gamma = -\frac{1}{2}$, the identity (4.9) holds under the limit (2.18).

**Proof** By (4.2), we use (2.22) with $(\alpha, \beta, \gamma) = (\mu + \frac{d-2}{2}, \mu + \frac{d-2}{2}, \gamma)$, $x_1 = (t + \xi)/2$, $x_2 = (t - \xi)/2$, $y_1 = s$ and $y_2 = 0$. Then, for $\eta$ defined by (2.23), we have

\[
\eta \left( \frac{t+\xi}{2}, \frac{t-\xi}{2}, (s, 0), v \right) = \sqrt{\frac{t+\xi}{2}} s v_1 + \sqrt{1 - t \sqrt{1 - s}} v_2.
\]
so that, by (4.2),
\[
P_n \left( \hat{\varpi}_{\alpha,\gamma}; (\zeta, t), (s, s) \right) = c_{\alpha - \frac{1}{2}} c_{\gamma} \\
\times \int_{-1}^{1} \int_{-1}^{1} Z_{2n}^{2\alpha + \gamma + 1} \left( \sqrt{\frac{t + \zeta}{2}} s v_1 + \sqrt{1 - t} \sqrt{1 - s} v_2 \right) \left( 1 - v_1^2 \right)^{\alpha - 1} \left( 1 - v_2^2 \right)^{\gamma - \frac{1}{2}} \, dv.
\]

(4.11)

Consequently, setting \( \zeta = \zeta(x, t, y, s; u) \) defined in (4.6), we obtain (4.9) from (4.2) and (4.5). Finally, since \( \|x\| \leq t \) and \( \|y\| \leq s \), Cauchy's inequality shows that
\[
|\langle x, y \rangle + \sqrt{t^2 - \|x\|^2} \sqrt{s^2 - \|y\|^2} u| \leq \|x\| \cdot \|y\| + \sqrt{t^2 - \|x\|^2} \sqrt{s^2 - \|y\|^2} \leq ts,
\]
form which it follows easily that \( \xi(x, t, y, s; u, v) \in [-1, 1] \).

In view of (4.1), the identity (4.9) can be regarded as an addition formula for the Jacobi polynomials on the cone.

### 4.2 Jacobi Polynomials on the Cone with \( \beta > 0 \)

For \( \beta > 0 \), the reproducing kernels of \( V_n (\mathbb{V}^{d+1}, W_{\mu, \beta, \gamma}) \) also satisfy a closed formula. The formula will be more involved as it requires an additional layer of complication as can be seen in the next theorem and its proof.

**Theorem 4.4** Let \( d \geq 2 \). For \( \mu \geq 0 \), \( \beta > 0 \) and \( \gamma > -1 \), let \( \alpha = \mu + \frac{\beta + d - 1}{2} \). Then, for \( (x, t), (y, s) \in \mathbb{V}^{d+1} \),
\[
P_n \left( W_{\mu, \beta, \gamma}; (x, t), (y, s) \right) = \hat{c}_{\mu, \beta} \\
\times \int_{[-1,1]}^{3} P_n \left( \hat{\varpi}_{\alpha,\gamma}; (\hat{\zeta}(x, t, y, s; z, u), t), (s, s) \right) \\
\times (1 - u^2)^{\mu - 1} (1 - z_1)^{\mu + \frac{d - 1}{2} + 1} (1 + z_1)^{\frac{\beta}{2} - 1} (1 - z_2^2)^{\frac{\beta - 1}{2}} \, du dz,
\]

(4.12)

where \( \hat{c}_{\mu, \beta} = c_{\mu - \frac{1}{2}} c_{\mu + \frac{d - 1}{2} + 1} \frac{\beta - 1}{2} \frac{c_{\beta}}{2} \) and
\[
\hat{\zeta}(x, t, y, s; z, u) := \frac{1 - z_1}{2} \left( \frac{\langle x, y \rangle}{s} + \sqrt{t^2 - \|x\|^2} \sqrt{s^2 - \|y\|^2} u \right) + \frac{1 + z_1}{2} z_2 t.
\]

(4.13)

In the case \( \mu = 0 \) or \( \beta = -\frac{1}{2} \), the identity (4.12) holds under the limit (2.18).
\textbf{Proof} Following the proof of Theorem 4.2, we see that (4.7) becomes

\begin{equation}
\mathbb{P}_n(W_{\mu, \beta, \gamma}; (x, t), (y, s)) = c_{\mu - \frac{d}{2}} \int_{[-1,1]^5} \frac{P_{n-m}^{(2\alpha+2m, \gamma)}(1-2t)P_{n-m}^{(2\alpha+2m, \gamma)}(1-2s)}{H_{m,n}}
\end{equation}

\begin{equation}
\times Z_m^{\alpha, \beta, \gamma} \left(\frac{1-z_1}{2} \xi(x, t, y, s; u) + \frac{1+z_1}{2} z_2 \right)
\end{equation}

\begin{equation}
\times (1-z_1)^{\lambda}(1+z_1)^{\sigma-1}(1-z_2^{2\gamma})^{\frac{\sigma-1}{2}} dz
\end{equation}

with \( \lambda = \mu + \frac{d-1}{2} \) and \( \sigma = \frac{\beta}{2} \), so that (4.14) becomes

\begin{equation}
\mathbb{P}_n(W_{\mu, \beta, \gamma}; (x, t), (y, s)) = \hat{c}_{\mu, \beta} \int_{[-1,1]^5} \frac{P_{n-m}^{(2\alpha+2m, \gamma)}(1-2t)P_{n-m}^{(2\alpha+2m, \gamma)}(1-2s)}{H_{m,n}}
\end{equation}

\begin{equation}
\times Z_m^{\alpha, \beta, \gamma} \left(\frac{1-z_1}{2} \xi(x, t, y, s; u) + \frac{1+z_1}{2} z_2 \right)
\end{equation}

\begin{equation}
\times (1-z_1)^{\lambda}(1+z_1)^{\sigma-1}(1-z_2^{2\gamma})^{\frac{\sigma-1}{2}} dz du.
\end{equation}

Comparing with (4.7), we see that the rest of the proof follows exactly as in the proof of Theorem 4.2. \( \square \)

We also have an analogue of Theorem 4.3 that shows \( \mathbb{P}_n(W_{\mu, \beta, \gamma}) \) also possesses a structure of one-dimension.

\textbf{Theorem 4.5} Let \( d \geq 2 \). For \( \mu \geq 0, \beta > 0 \) and \( \gamma \geq -\frac{1}{2} \), let \( \alpha = \mu + \frac{\beta+d-1}{2} \). Then

\begin{equation}
\mathbb{P}_n(W_{\mu, \beta, \gamma}; (x, t), (y, s)) = \hat{c}_{\mu, \beta} c_{\alpha - \frac{d}{2}} c_{\gamma} \int_{[1,1]^5} Z_{2n}^{\alpha+\gamma+1} \left(\hat{\xi}(x, t, y, s, \zeta, u, \rho) \right)
\end{equation}

\begin{equation}
\times (1-z_1)^{\lambda}(1+z_1)^{\sigma-1}(1-z_2^{2\gamma})^{\frac{\sigma-1}{2}} dz du.
\end{equation}
where \( \hat{\xi}(x, t, y, s; u, v) \in [-1, 1] \) is defined by

\[
\hat{\xi}(x, t, y, s; u, v) = v_2 \sqrt{1 - t} \sqrt{1 - s} + \frac{1}{2} v_1 \left( 2st + (1 - z_1) \left( \langle x, y \rangle + \sqrt{t^2 - \|x\|^2} \sqrt{s^2 - \|y\|^2} u \right) + (1 + z_1)z_2 st. \right)
\]

(4.17)

In the case \( \mu = 0 \) or \( \gamma = -\frac{1}{2} \), the identity (4.12) holds under the limit (2.18).

**Proof** The proof can be carried out as that of Theorem 4.4. By (4.2), the kernel in the righthand side of (4.12) becomes

\[
P_n \left( \sigma_{\alpha, \gamma}; (\hat{\zeta}, t), (s, s) \right) = P_n \left( \sigma_{\alpha - \frac{1}{2}, \alpha - \frac{1}{2}, \gamma}; \left( \frac{t + \hat{\zeta}}{2}, \frac{t - \hat{\zeta}}{2} \right), (s, 0) \right),
\]

where \( \hat{\zeta} = \hat{\zeta}(x, t, y, s; z, u) \) as in (4.13). Writing the righthand side as an integral by using (2.22), we obtain

\[
P_n \left( \sigma_{\alpha, \gamma}; (\hat{\zeta}, t), (s, s) \right) = c \int_{[-1,1]^3} Z_n^{2\alpha+\gamma+1} \left( \hat{\xi}(x, t, y, s, z, u, v)(1 - v_1)^{\alpha-1}(1 - v_2)^{\gamma-\frac{1}{2}} dv \right)
\]

where \( c = c_{\alpha - \frac{1}{2}} c_{\gamma} \) and, with \( \eta \) defined in (2.23),

\[
\hat{\xi}(x, t, y, s; z, u, v) = \eta \left( \left( \frac{t + \hat{\zeta}}{2}, \frac{t - \hat{\zeta}}{2} \right), (s, 0), v \right) = \sqrt{s} \frac{t + \hat{\zeta}}{2} v_1 + \sqrt{(1 - t)(1 - s)} v_2.
\]

Substituting this expression into (4.12), we then have (4.16).

Although the formula (4.16) is fairly complicated, what is of important is that it shows that the kernel has a one-dimensional structure. We will make use of this structure in the following section.

### 5 Convolution Structure on the Cone

The closed formula for the reproducing kernels suggests a convolution structure on the cone that is useful in the study of Fourier series in the Jacobi polynomials on the cone. For this development, what is important is the one-dimensional structure of the kernels, not the explicit formula of the closed formula itself.

We start with a definition suggested by the closed formula (4.16).
Definition 5.1 Let \( d \geq 2 \). For \( \mu \geq 0, \beta \geq 0 \) and \( \gamma \geq -\frac{1}{2} \), define \( \alpha = \mu + \frac{\beta+d-1}{2} \). For \( g \in L^1([-1, 1], w_{2\alpha+\gamma+1}) \), we define the operator \( T_{\mu, \beta, \gamma} \) on the cone \( \mathbb{V}^{d+1} \) by

\[
T_{\mu, \beta, \gamma} g((x, t), (y, s)) := c_{\mu, \beta} c_{\gamma} \int_{[-1, 1]^3} g(\hat{x}(x, t, y; z, u, v)) \times (1 - z_1)^{\mu + \frac{d-1}{2}}(1 + z_1)^{\frac{\beta}{2}}(1 - z_2)^{\frac{\beta-1}{2}} \times (1 - u^2)^{\mu - 1}(1 - v_1^2)^{\alpha - 1}(1 - v_2^2)^{\gamma - \frac{1}{2}} \, dz \, du \, dv,
\]

where \( \hat{x}(x, t, y; z, u, v) \) is defined by (4.17). When \( \mu = 0 \) or \( \beta = 0 \) or \( \gamma = -\frac{1}{2} \), the definition holds under the limit (2.18). If \( \beta = 0 \), the definition is simplified to

\[
T_{\mu, 0, \gamma} g((x, t), (y, s)) := c_{\mu} c_{\gamma} \int_{[-1, 1]^3} g(\hat{x}(x, t, y; z, u, v)) \times (1 - u^2)^{\mu - 1}(1 - v_1^2)^{\alpha - 1}(1 - v_2^2)^{\gamma - \frac{1}{2}} \, du \, dv,
\]

where \( \hat{x}(x, t, y; z, u, v) \) is defined by (4.10).

By the closed formulas of the reproducing kernel (4.9) and (4.16), we immediately obtain

\[
P_n(W_{\mu, \beta, \gamma}; (x, t), (y, s)) = T_{\mu, \beta, \gamma} Z_{2n}^{2\alpha+\gamma+1}((x, t), (y, s)),
\]

which is our motivation for the definition of \( T_{\mu, \beta, \gamma} \).

In the following, we consider the \( L^p \) norm \( \| \cdot \|_{L^p} \) for \( 1 \leq p \leq \infty \). We will always assume that the case \( p = \infty \) is the uniform norm over continuous functions. Because of (5.3) and the fact that \( Z_{2n}^\lambda \) is an even polynomial, we only need the action of \( T_{\mu, \beta, \gamma} \) on the function \( g \) that is even for the purpose of studying the Fourier orthogonal series.

Let \( \hat{g}_n^\lambda \) be the Fourier–Gegenbauer series of \( g \) defined by

\[
\hat{g}_n^\lambda = c_{\lambda} \int_{-1}^{1} g(u) \frac{C_n^\lambda(u)}{C_n^\lambda(1)}(1 - u^2)^{\lambda - \frac{1}{2}} \, du.
\]

Lemma 5.2 Let \( g \in L^1([-1, 1], w_{2\alpha+\gamma+1}) \) be an even function on \([-1, 1]\). Then

1. for each \( Q_n \in \mathcal{V}_n(\mathbb{V}^{d+1}, W_{\mu, \beta, \gamma}) \),

\[
b_{\mu, \beta, \gamma} \int_{\mathbb{V}^{d+1}} T_{\mu, \beta, \gamma} g((x, t), (y, s)) Q_n(y, s) W_{\mu, \beta, \gamma}(y, s) \, dy \, ds = \Lambda_n(g) Q_n(x, t),
\]

where \( b_{\mu, \beta, \gamma} \) is the normalization constant of \( W_{\mu, \beta, \gamma} \) and \( \Lambda_n(g) = \hat{g}_n^{2\alpha+\gamma+1} \).

2. for \( 1 \leq p \leq \infty \) and \((x, t) \in \mathbb{V}^{d+1}\),

\[
\| T_{\mu, \beta, \gamma} g((x, t), (\cdot, \cdot)) \|_{L^p(\mathbb{V}^{d+1}, W_{\mu, \beta, \gamma})} \leq \| g \|_{L^p([-1, 1], w_{2\alpha+\gamma+1})}.
\]

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**Proof** If $g$ is an even polynomial of degree at most $n$, then we can expand it in terms of the Gegenbauer polynomials of even degrees. That is, we can write

$$g(t) = \sum_{k=0}^{n} \Lambda_k Z_{2k}^{2\alpha+\gamma+1}(t),$$

where, by orthogonality and the fact that $h_k^\lambda = \frac{\lambda^k}{k!} C_k^\lambda(1)$,

$$\Lambda_k = \frac{C_{2\alpha+\gamma+1}}{C_{2k}^{2\alpha+\gamma+1}(1)} \int_{-1}^{1} g(t) C_{2k}^{2\alpha+\gamma+1}(t)(1-t^2)^{2\alpha+\gamma+\frac{1}{2}} dt = \frac{2n}{2n}.\]$$

Using the formula (5.3), we have

$$T_{\mu,\beta,\gamma} g((x, t), (y, s)) = \sum_{k=0}^{n} \Lambda_k P_k(W_{\mu,\beta,\gamma}; (x, t), (y, s)).$$

Consequently, by the reproducing property, for each $Q_k \in V_k(\mathbb{R}^{d+1}, W_{\mu,\beta,\gamma})$, $k \leq n$, we conclude

$$b_{\mu,\beta,\gamma} \int_{\mathbb{R}^{d+1}} T_{\mu,\beta,\gamma} g((x, t), (y, s)) Q_k(y, s) W_{\mu,\beta,\gamma}(y, s) dy ds = \Lambda_k Q_k(x, t).$$

This establishes the lemma when $g$ is a polynomial. The usual density argument then completes the proof of (1).

For $p = \infty$, the inequality (5.5) is evident. If $g$ is nonnegative, then $T_{\mu,\beta,\gamma} g$ is evidently nonnegative, so that $|T_{\mu,\beta,\gamma} g| \leq T_{\mu,\beta,\gamma} (|g|)$. Hence, applying (1) with $n = 0$ we see that the inequality (5.5) holds for $p = 1$. The case $1 < p < \infty$ follows from the Riesz–Thorin theorem. \(\square\)

For $(x, t) \in \mathbb{R}^{d+1}$, the operator $g \mapsto T_{\mu,\beta,\gamma} g((x, t), (\cdot, \cdot))$ defines a “translation” of $g$ by $(x, t)$. We use this operator to define a convolution structure with respect to $W_{\mu,\beta,\gamma}$ on the cone.

**Definition 5.3** Let $\mu \geq 0$, $\beta \geq 0$ and $\gamma \geq -\frac{1}{2}$, and let $\alpha = \mu + \frac{\beta}{2} + \frac{d-1}{2}$. For $f \in L^1(\mathbb{R}^{d+1}, W_{\mu,\beta,\gamma})$ and $g \in L^1([-1, 1], w_{2\alpha+\gamma+1})$, define the convolution of $f$ and $g$ on the cone by

$$(f *_{\mu,\beta,\gamma} g)(x, t) := b_{\mu,\beta,\gamma} \int_{\mathbb{R}^{d+1}} f(y, s) T_{\mu,\beta,\gamma} g((x, t), (y, s)) W_{\mu,\beta,\gamma}(y, s) dy ds.$$

The convolution on the cone satisfies Young’s inequality:

**Theorem 5.4** Let $p, q, r \geq 1$ and $p^{-1} = r^{-1} + q^{-1} - 1$. For $f \in L^q(\mathbb{R}^{d+1}, W_{\mu,\beta,\gamma})$ and $g \in L^r([-1, 1], w_{2\alpha+\gamma+1})$ with $g$ an even function,

$$\| f *_{\mu,\beta,\gamma} g \|_{L^p(\mathbb{R}^{d+1}, W_{\mu,\beta,\gamma})} \leq \| f \|_{L^q(\mathbb{R}^{d+1}, W_{\mu,\beta,\gamma})} \| g \|_{L^r([-1, 1], w_{2\alpha+\gamma+1})}. \quad (5.6)$$
Proof The standard proof applies in this setting. By Minkowski’s inequality,
\[
\|f *_{\mu, \beta, \gamma} g\|_{L^r(\mathbb{V}^{d+1}, W_{\mu, \beta, \gamma})} \leq b_{\mu, \beta, \gamma} \int_{\mathbb{V}^{d+1}} |f(x, t)|
\times \left(b_{\mu, \beta, \gamma} \int_{\mathbb{V}^{d+1}} |T_{\mu, \beta, \gamma} g((x, t), (y, s))|^r W_{\mu, \beta, \gamma}(y, s) dy ds\right)^{1/r} dx dt.
\]
By (5.5) in Lemma 5.2, we then conclude that
\[
\|f *_{\mu, \beta, \gamma} g\|_{L^r(\mathbb{V}^{d+1}, W_{\mu, \beta, \gamma})} \leq \|f\|_{L^1(\mathbb{V}^{d+1}, W_{\mu, \beta, \gamma})} \|g\|_{L^r([-1, 1], w_{2\alpha + \gamma + 1})}.
\]
Furthermore, by Hölder’s inequality and (5.5), we see that
\[
\|f *_{\mu, \beta, \gamma} g\|_{L^r(\mathbb{V}^{d+1}, W_{\mu, \beta, \gamma})} \leq \|f\|_{L^r(\mathbb{V}^{d+1}, W_{\mu, \beta, \gamma})} \|g\|_{L^r([-1, 1], w_{2\alpha + \gamma + 1})},
\]
where \(\frac{1}{r} + \frac{\gamma}{r} = 1\). The inequality (5.6) follows from interpolating the above two inequalities with \(\theta = r(1 - \frac{1}{p})\) by the Riesz–Thorin theorem.

The next proposition justifies calling \(f * g\) convolution. Recall that \(\hat{g}_{\mu}^{\gamma}\) denotes the Fourier–Gegenbauer series of \(g\). The projection operator \(\text{proj}_{\mu}^{\gamma}: L^2(\mathbb{V}^{d+1}, W_{\mu, \beta, \gamma}) \mapsto \mathcal{V}_n(\mathbb{V}^{d+1}, W_{\mu, \beta, \gamma})\) is defined by
\[
\text{proj}_{\mu}^{\beta, \gamma} f(x, t) = b_{\mu, \beta, \gamma} \int_{\mathbb{V}^{d+1}} f(y, s) \mathcal{P}_n(W_{\mu, \beta, \gamma}; (x, t), (y, s)) W_{\mu, \beta, \gamma}(y, s) dy ds.
\]

Proposition 5.5 For \(f \in L^1(\mathbb{V}^{d+1}, W_{\mu, \beta, \gamma})\) and \(g \in L^1([-1, 1], w_{2\mu + \gamma + \delta})\),
\[
\text{proj}_{\mu}^{\beta, \gamma} (f *_{\mu, \beta, \gamma} g) = \hat{g}_{\mu}^{2\mu + \beta + \gamma + \delta} \text{proj}_{\mu}^{\beta, \gamma} f.
\]

Proof For each \((x, t), \mathcal{P}_n(W_{\mu, \gamma}; (x, t), (\cdot, \cdot))\) is an element of \(\mathcal{V}_n(\mathbb{V}^{d+1}; W_{\mu, \gamma})\). Hence, by the identity (5.4) and the definition of \(f *_{\mu, \gamma} g\), we obtain
\[
\text{proj}_{\mu}^{\beta, \gamma} (f *_{\mu, \beta, \gamma} g)(x, t) = b_{\mu, \beta, \gamma} \int_{\mathbb{V}^{d+1}} (f *_{\mu, \beta, \gamma} g)(y, s)
\times \mathcal{P}_n(W_{\mu, \beta, \gamma}; (x, t), (y, s)) W_{\mu, \beta, \gamma}(y, s) dy ds
\]
\[
= \hat{g}_{\mu}^{2\mu + \beta + \gamma + \delta} b_{\mu, \beta, \gamma} \int_{\mathbb{V}^{d+1}} f(u, r) \mathcal{P}_n(W_{\mu, \beta, \gamma}; (x, t), (u, r)) du dr
\]
\[
= \hat{g}_{\mu}^{2\mu + \beta + \gamma + \delta} \text{proj}_{\mu}^{\beta, \gamma} f(x, t),
\]
where we have used the Fubini theorem in the second step.

By (4.9) and the definition of the convolution operator, we have the following:
Proposition 5.6  For $f \in L^1(V_d+1, W_{\mu,\beta,\gamma})$,

$$\text{proj}_{n}^{\mu,\beta,\gamma} f = f \ast_{\mu,\beta,\gamma} Z_{2n}^{2\mu+\beta+\gamma+d}.$$  

This shows that the convolution structure we defined can be used to study the Fourier series in the Jacobi polynomials on the cone, which we shall explore in the next section.

6 Fourier Series in the Jacobi Polynomials on the Cone

We now consider the Fourier orthogonal expansion with respect to the orthogonal polynomials on the cone.

The $n$-th partial sum operator $S_n(W_{\mu,\beta,\gamma}; f)$ is defined by

$$S_n(W_{\mu,\beta,\gamma}; f) = \sum_{k=0}^{n} \text{proj}_{k}^{\mu,\beta,\gamma} f,$$

which is the least square polynomial of degree $n$ in $L^2(V_d+1, W_{\mu,\beta,\gamma})$. Evidently, this operator can be written as an integral operator,

$$S_n(W_{\mu,\beta,\gamma}; f) = b_{\mu,\beta,\gamma} \int_{V_d+1} f(y, s) K_n(W_{\mu,\beta,\gamma}; (x, t), (y, s)) W_{\mu,\beta,\gamma}(y, s) dy ds,$$

where the kernel $K_n(W_{\mu,\beta,\gamma})$ is given by

$$K_n(W_{\mu,\beta,\gamma}; (x, t), (y, s)) = \sum_{k=0}^{n} P_k(W_{\mu,\beta,\gamma}; (x, t), (y, s)).$$

We first show that this operator can be written in terms of the $n$-th partial sum of the Fourier–Jacobi series. For $\alpha, \beta > -1$, and $f \in L^2(w_{\alpha,\beta}, [-1, 1])$, let $s_n(w_{\alpha,\beta}; f)$ denote the partial sum of the Fourier–Jacobi series defined by

$$s_n(w_{\alpha,\beta}; f, u) := c_{\alpha,\beta} \int_{-1}^{1} f(v) k_n(w_{\alpha,\beta}; u, v) w_{\alpha,\beta}(v) dv,$$

where the kernel $k_n(w_{\alpha,\beta})$ is defined by

$$k_n(w_{\alpha,\beta}; u, v) = \sum_{k=0}^{n} \frac{P_k^{(\alpha,\beta)}(u) P_k^{(\alpha,\beta)}(v)}{h_k^{(\alpha,\beta)}}.$$  

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Proposition 6.1 Let \( \mu \geq 0, \beta \geq 0 \) and \( \gamma \geq -\frac{1}{2} \), and let \( \alpha = \mu + \frac{\beta + d - 1}{2} \). For \((x, t), (y, s) \in \mathbb{V}^{d+1}\),

\[
K_n \left( W_{\mu, \beta, \gamma}; (x, t), (y, s) \right) = T_{\mu, \beta, \gamma} \left[ k_n \left( w_{2\alpha + \gamma + \frac{1}{2}, -\frac{1}{2}; 2\{\cdot\}^2 - 1, 1} \right) \right] ((x, t), (y, s)).
\] (6.1)

Furthermore, for \((y, s) \in \mathbb{V}^{d+1}\),

\[
K_n \left( W_{\mu, \beta, \gamma}; (0, 0), (y, s) \right) = k_n \left( w_{2\alpha, \gamma}; 1 - 2s, 1 \right). 
\] (6.2)

**Proof** The kernel \( K_n(W_{\mu, \beta, \gamma}) \) is a sum over \( k \) of \( P_k(W_{\mu, \beta, \gamma}) \). By (5.3), this requires summing over \( Z_{2k}^{2\alpha + \gamma + 1} \). The subindex 2k is undesirable for the sum. We use instead the relation

\[
Z_{2k}^{2\alpha + \gamma + 1}(u) = \frac{2k + \lambda}{\lambda} C_{2k}^{\alpha}(u) = \frac{P_k^{(\lambda - \frac{1}{2}, -\frac{1}{2})}(1) P_k^{(\lambda - \frac{1}{2}, -\frac{1}{2})}(2u^2 - 1)}{h_k^{(\lambda - \frac{1}{2}, -\frac{1}{2})}}
\]

that follows from the quadratic transform (2.8). This allows us to sum over \( k \) for \( 0 \leq k \leq n \) and leads to (6.1). Furthermore, by (4.17), \( \xi(0, 0, y, s; u, v) = v_2 \sqrt{1 - s} \), so that

\[
K_n \left( W_{\mu, \beta, \gamma}; (0, 0), (y, s) \right) = c_y \int_{-1}^{1} k_n \left( w_{2\alpha + \gamma + \frac{1}{2}, -\frac{1}{2}; 2(1 - s)v^2 - 1, 1} \right) (1 - v^2)^{\gamma - \frac{1}{2}} dv
\]

by the definition of \( T_{\mu, \beta, \gamma} \). Hence, (6.2) follows from the identity

\[
c_{\tau - \frac{1}{2}} \int_{-1}^{1} \frac{P_k^{(a, b)}(1) h_k^{(a, b)}(2su^2 - 1)}{P_k^{(a - \tau, b + \tau)}(1) h_k^{(a - \tau, b + \tau)}(1 - 2s)} (1 - u^2)^{\tau - 1} du = \frac{P_k^{(a - \tau, b + \tau)}(1) h_k^{(a - \tau, b + \tau)}(1 - 2s)}{h_k^{(a - \tau, b + \tau)}}
\]

with \( a = 2\alpha + \gamma + \frac{1}{2}, b = -\frac{1}{2}, \tau = \gamma + \frac{1}{2} \). The identity is an equivalent form of the Dirichlet-Mehler formula [12, (4.10.12)] for the Jacobi polynomials. \( \square \)

**Corollary 6.2** For \( \mu \geq 0, \gamma \geq -\frac{1}{2} \) and \( \alpha = \mu + \frac{\beta + d - 1}{2} \),

\[
S_n \left( W_{\mu, \beta, \gamma}; f \right) = f \ast_{\mu, \beta, \gamma} k_n \left( w_{2\alpha + \gamma + \frac{1}{2}, -\frac{1}{2}; 2\{\cdot\}^2 - 1, 1} \right).
\] (6.3)

The partial sum operator may not converge in \( L^p \) norm for \( p \neq 2 \), so we may need to study summability methods for the Fourier orthogonal expansions. The above corollary shows that the Fourier series in the Jacobi polynomials on the cone has a one-dimensional structure in terms of the Fourier–Jacobi series, from which its properties could be derived accordingly. We consider the Cesàro \((C, \delta)\) means as an example.
For $\delta > 0$, the Cesàro $(C, \delta)$ means $S_n^{\delta}(W_{\mu, \beta, \gamma}; f)$ of the Fourier series in the Jacobi polynomials on the cone is defined by

$$S_n^{\delta}(W_{\mu, \beta, \gamma}; f) := \frac{1}{(n+\delta)} \sum_{k=0}^{n} \binom{n-k+\delta}{n-k} \text{proj}_{k}^{\mu, \beta, \gamma} f,$$

which can be written as an integral of $f$ against the kernel $K_n^{\delta}(W_{\mu, \gamma}; \cdot, \cdot)$ defined by

$$K_n^{\delta}(W_{\mu, \beta, \gamma}; (x, t), (y, s)) := \frac{1}{(n+\delta)} \sum_{k=0}^{n} \binom{n-k+\delta}{n-k} P_k^{(\alpha, \beta)}(y) P_k^{(\alpha, \beta)}(v),$$

so that the Cesàro $(C, \delta)$ means $s_n^{\delta}(w_{\alpha, \beta}; g)$ of the Fourier-Jacobi series are given by

$$s_n^{\delta}(w_{\alpha, \beta}; g) = c_{\alpha, \beta} \int_{-1}^{1} g(v) k_n^{\delta}(w_{\alpha, \beta}; \cdot, v) w_{\alpha, \beta}(v) dv.$$

**Theorem 6.3** For $\mu \geq 0$ and $\gamma \geq -\frac{1}{2}$, define $\lambda_{\mu, \beta, \gamma} := 2\mu + \beta + \gamma + d$. Then, the Cesàro $(C, \delta)$ means for $W_{\mu, \beta, \gamma}$ on $\mathbb{R}^{d+1}$ satisfy

1. if $\delta \geq \lambda_{\mu, \beta, \gamma} + 1$, then $S_n^{\delta}(W_{\mu, \beta, \gamma}; f)$ is nonnegative if $f$ is nonnegative;
2. $S_n^{\delta}(W_{\mu, \beta, \gamma}; f)$ converge to $f$ in $L^1(\mathbb{R}^{d+1}, W_{\mu, \beta, \gamma})$ norm or $C(\mathbb{R}^{d+1})$ norm if $\delta > \lambda_{\mu, \beta, \gamma}$ and only if $\delta > \lambda_{\mu, \beta, \gamma}$ when $\gamma = -\frac{1}{2}$.

**Proof** Recall that $\alpha = \mu + \frac{\beta}{2} + \frac{d-1}{2}$, so that $\lambda_{\mu, \beta, \gamma} = 2\alpha + \gamma + 1$. From (6.1), it follows immediately that

$$K_n^{\delta}(W_{\mu, \beta, \gamma}; (x, t), (y, s)) = T_{\mu, \gamma} \left[ k_n^{\delta}(w_{2\alpha+\gamma+\frac{1}{2}, -\frac{1}{2}}; 2\cdot \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \right]((x, t), (y, s)).$$

Hence, the first assertion follows immediately from the fact [6] that $k_n^{\delta}(w_{a, b}; u, v) \geq 0$ if $\delta \geq a + b + 2$, which is $\delta \geq 2\alpha + \gamma + 2 = \lambda_{\mu, \beta, \gamma} + 1$ with $a = 2\alpha + \gamma + \frac{1}{2}$ and $b = -\frac{1}{2}$.

To prove the convergence of the second assertion, it is sufficient to show that

$$\max_{(x, t)} \int_{\mathbb{R}^{d+1}} \left| K_n^{\delta}(W_{\mu, \beta, \gamma}; (x, t), (y, s)) \right| W_{\mu, \beta, \gamma}(y, s) dy ds$$
is bounded. By (5.5), we see that this quantity is bounded by

\[
\int_{-1}^{1} k_n^\delta \left( w_{2a+\gamma + \frac{1}{2}}, -\frac{1}{2}; 2s^2 - 1, 1 \right) \left( 1 - s^2 \right)^{2\alpha + \gamma + 1} ds
\]

\[
= \frac{1}{2^{2\alpha + \gamma + 2}} \int_{-1}^{1} k_n^\delta \left( w_{2a+\gamma + \frac{1}{2}}, -\frac{1}{2}; u, 1 \right) \left( 1 - u \right)^{2\alpha + \gamma + 1} (1 + u)^{-\frac{1}{2}} du,
\]

where we have written the first integral as over \([0, 1]\), since the integrant is even, and then change the variable \(u = 2s^2 - 1\). That this term is bounded for \(\delta > 2\alpha + \gamma + 1\) follows from the fact that \(s_n^\delta (w_{a,b}; f)\) converges to \(f\) at \(t = 1\) if and only if \(\delta > \max\{a, b\} + \frac{1}{2}\) [12, Theorem 9.1.3]. Finally, using (6.1) and (3.2), we see that

\[
\int_{\mathbb{V}^{d+1}} |K_n^\delta (W_{\mu, \beta, \gamma}; (0, 0), (y, s))| W_{\mu, \beta, \gamma}(y, s) dy ds
\]

\[
= \frac{1}{b_{\mu}} \int_{0}^{1} \left| k_n^\delta (w_{2a, \gamma}; 1 - 2s, 1) \right| s^{2\alpha} (1 - s)^\gamma ds,
\]

which is bounded if and only if \(\delta > 2\alpha + \frac{1}{2}\) ([12, Theorem 9.1.3]), whereas when \(\gamma = -\frac{1}{2}, \lambda_{\mu, \beta, \gamma} = 2\alpha + \frac{1}{2}\). This completes the proof. \(\square\)

### 7 Orthogonal Structure on the Surface of a Cone

We now consider orthogonal polynomials on the surface of the cone \(\mathbb{V}^{d+1}\), which we denote by

\[
\mathbb{V}^{d+1}_0 := \{(x, t) \in \mathbb{V}^{d+1} : \|x\| = |t|, \ 0 \leq t \leq b\},
\]

with respect to a bilinear form defined by

\[
\langle f, g \rangle_w := b_w \int_{\mathbb{V}^{d+1}_0} f(x, t) g(x, t) w(t) d\sigma(x, t), \quad (7.1)
\]

where \(d\sigma(x, t)\) is the Lebesgue measure on \(\mathbb{V}^{d+1}_0\), \(w\) is a nonnegative function defined on \(\mathbb{R}_+\) such that \(\int_{\mathbb{R}} t^{-d-1} w(t) dt < \infty\) and \(b_w\) is a normalized constant so that \((1, 1) = 1\).

The bilinear form \(\langle \cdot, \cdot \rangle_w\) is an inner product on the space \(\mathbb{R}[x, t]/(\|x\|^2 - t^2)\), where \(\mathbb{R}[x, t]/(p)\) denotes the space of polynomials in \((x, t)\) variables modulo the polynomial idea \((p)\) generated by the polynomial \(p\). Let \(\mathcal{V}_n(\mathbb{V}^{d+1}_0, w)\) be the space of orthogonal polynomials with respect to the inner product \(\langle \cdot, \cdot \rangle_w\). Since \(||x||^2 - t\) is a quadratic polynomial, it is not difficult to see that

\[
\dim \mathcal{V}_n(\mathbb{V}^{d+1}_0, w) = \binom{n + d - 1}{n} + \binom{n + d - 2}{n - 1}.
\]
the same as the dimension of the space of spherical polynomials of degree \( n \) on \( S^d \).

As in the case of the cone, we consider two families of \( w \), the Jacobi weight and the Laguerre weight. The notation for some normalization constants may overlap with those already used in the previous sections, but they should cause little confusion since the values of these constants are of little substance.

### 7.1 Jacobi Polynomials on the Surface of the Cone

In this case, \( b = 1 \) in the definition of the cone so that \( \mathbb{V}_0^{d+1} = \{(x, t) : \|x\| = t, x \in \mathbb{R}^d, t \leq 1\} \) with \( d \geq 2 \). We choose \( w \) as the Jacobi weight

\[
\varphi_{\beta, \gamma}(t) = t^\beta (1 - t)^\gamma, \quad \beta > -d, \quad \gamma > -1.
\]

We then define the inner product on \( \mathbb{R}[x, t]/(\|x\|^2 - t^2) \) in terms of \( w = \varphi_{\beta, \gamma} \) by

\[
\langle f, g \rangle_{\beta, \gamma} = b_{\beta, \gamma} \int_{\mathbb{V}_0^{d+1}} f(x, t)g(x, t)t^\beta (1 - t)^\gamma d\sigma(x, t).
\]

The integral on the surface of the cone can be written as

\[
\int_{\mathbb{V}_0^{d+1}} f(x, t)d\sigma(x, t) = \int_0^1 \int_{\|x\|=t} f(x, t)d\sigma(x, t)
= \int_0^1 t^{d-1} \int_{S^{d-1}} f(t\xi, t)d\sigma(\xi)dt,
\]

which gives, in particular, that

\[
b_{\beta, \gamma} = \frac{1}{\omega_d} \frac{1}{t^{\beta+d-1}(1 - t)^\gamma} dt = \frac{1}{\omega_d} c_{\beta+d-1, \gamma},
\]

where \( \omega_d \) is the surface area of \( S^{d-1} \) and \( c_{\alpha, \gamma} \) is defined in (2.3).

A basis of \( \mathcal{V}_n(\mathbb{V}_0^{d+1}, \varphi_{\beta, \gamma}) \) can be given in terms of the Jacobi polynomials and spherical harmonics. Let \( \{Y_{m}^{\ell} : 1 \leq \ell \leq \dim \mathcal{H}_{m}^{d}\} \) denote an orthonormal basis of \( \mathcal{H}_{m}^{d} \). We define

\[
S_{m, \ell}^{n}(x, t) = P_{n-m}^{(2m+\beta+d-1, \gamma)}(1 - 2t)Y_{\ell}^{m}(x), \quad 0 \leq m \leq n, \quad 1 \leq \ell \leq \dim \mathcal{H}_{m}^{d}.
\]

Evidently, each \( S_{m, \ell}^{n} \) is a polynomial of degree \( n \) in \( (x, t) \).

**Proposition 7.1** For \( \beta > -d \) and \( \gamma > -1 \), the set \( \{S_{m, \ell}^{n} : 0 \leq m \leq n, \quad 1 \leq \ell \leq \dim \mathcal{H}_{m}^{d}\} \) is an orthogonal basis of \( \mathcal{V}_n(\mathbb{V}_0^{d+1}, \varphi_{\beta, \gamma}) \). More precisely,

\[
\langle S_{m, \ell}^{n}, S_{m', \ell'}^{n'} \rangle_{\beta, \gamma} = H_{m,n}^{\beta, \gamma} \delta_{n,n'} \delta_{m,m'} \delta_{\ell, \ell'},
\]
where, with $\alpha = \frac{\beta + d - 1}{2}$,

$$H_{m,n}^{\beta,\gamma} = \frac{c_{2\alpha+2m,\gamma}}{c_{2\alpha,\gamma}} h_{n-m}^{(2\alpha+2m,\gamma)}$$  \hspace{1cm} (7.5)

in terms of the norm of the Jacobi polynomial $h_{m}^{(a,b)}$ in (2.4).

**Proof** Since $Y_{m}^{\ell}$ are homogeneous polynomials,

$$Y_{m}^{\ell}(t \xi) = t^{m} Y_{m}^{\ell}(\xi).$$

For $(x,t) \in V_{d+1}$, we let $x = t \xi$ with $\xi \in S_{d-1}$ and use the integral (7.2). The proof follows exactly as in Proposition 3.1. In particular, the cardinality of the set \{\(S_{m,\ell} : 0 \leq m \leq n, 1 \leq \ell \leq \dim \mathcal{H}_{m}\)\} is

$$\sum_{m=0}^{n} \dim \mathcal{H}_{m} = \sum_{m=0}^{n} \left[ \binom{m+d-1}{m} + \binom{m+d-2}{m-1} \right] = \binom{n+d-1}{n} + \binom{n+d-2}{n-1},$$

which is equal to $\dim V_{n}(V_{d+1}, \varphi_{\beta,\gamma})$, so that the set is an orthonormal basis of $V_{n}(V_{d+1}, \varphi_{\beta,\gamma})$.

The polynomials $S_{m,\ell}$ defined in (7.4) are closely related to the polynomials $Q_{m,k}^{n}$ defined in (3.3) for $W_{\mu,\beta,\gamma}$ on $V_{d+1}$. Indeed, if we choose the orthogonal basis $P_{k}(\sigma_{m})$ in (3.3) as the basis defined in (2.15), then $Y_{k}^{m}$ is part of the basis and, consequently, $S_{m,\ell}$ is the restriction of the corresponding $Q_{m,k}^{n}$ on the surface $V_{d+1}$.

In particular, taking into account of Theorem 3.2, it is probably not surprising that there should be a differential operator that has $V_{n}(V_{d+1}, \varphi_{-1,\gamma})$ as an eigenspace. What is surprising, however, is that this holds only when $\beta = -1$ as shown in the following theorem.

**Theorem 7.2** Let $d \geq 2$. Every $u \in V_{n}(V_{d+1}, \varphi_{-1,\gamma})$ satisfies the differential equation

$$D_{-1,\gamma} u = -n(n + \gamma + d - 1) u,$$  \hspace{1cm} (7.6)

where $D_{-1,\gamma} = D_{-1,\gamma}(x,t)$ is the second order linear differential operator

$$D_{-1,\gamma} := t(1-t) \partial_{t}^{2} + (d - 1 - (d + \gamma)t) \partial_{t} + t^{-1} \Delta_{0}^{(x)},$$

where $\Delta_{0}^{(x)}$ is the Laplace–Beltrami operator in variable $x \in S_{d-1}$.

**Proof** We establish the result for $u = S_{m,\ell}^{n}$ in (7.3). For $x = t \xi$, we write

$$S_{m,\ell}^{n}(x,t) = g(t)t^{m} Y_{\ell}^{m}(\xi), \quad g(t) = P_{n-m}^{(\beta+2m+d-1,\gamma)}(1-2t).$$

The differential equation for the Jacobi polynomial shows that $g$ satisfies (3.7) with $2\mu$ replaced by $\beta$. Hence, a straightforward computation shows that the polynomial
\( f(t) = g(t)t^m \) satisfies the equation
\[
t(1-t)\frac{d^2}{dt^2} f(t) + ((\beta + d) - (\beta + \gamma + d + 1)t) \frac{d}{dt} f(t) = -n(n + \beta + \gamma + d) f(t) + m(m + \beta + d - 1)t^{-1} f(t).
\]

This identity also holds for \( u(x, t) = f(t) Y^m_\ell (\xi) \). When \( \beta = -1 \), the number in the last term, \( m(m + \beta + d - 1) \), is the eigenvalue of \( Y^m_\ell \) for \(-\Delta_0\). In particular,
\[
m(m + d - 2)t^{-1} f(t) Y^m_\ell = -t^{-1} f(t) \Delta^{(x)}_0 Y^m_\ell = -t^{-1} \Delta^{(x)}_0 S^m_{m, \ell}.
\]

Thus, \( u \) satisfies (7.6). The proof is complete. \( \square \)

**Remark 7.1** The proof shows, evidently, that the polynomial \( S^m_{m, \ell} \) satisfies a differential equation for \( \beta \neq -1 \), but the equation depends on both \( m \) and \( n \) so that the corresponding differential operator does not have \( \mathcal{V}_n(\mathbb{V}^{d+1}_0, \varphi_{\beta, \gamma}) \) as an eigenspace.

### 7.2 Laguerre Polynomials on the Surface of the Cone

In this case, \( b = +\infty \) in the definition of the cone so that \( \mathbb{V}^{d+1}_0 = \{(x, t) : \|x\| = t, x \in \mathbb{R}^d, t \geq 0\} \) with \( d \geq 2 \). We choose \( w \) as the Laguerre weight
\[
\varphi_{\beta}(t) = t^\beta e^{-t}, \quad \beta > -d.
\]

We then define the inner product on \( \mathbb{R}[x, t]/(\|x\|^2 - t^2) \) in terms of \( w = \varphi_{\beta} \) by
\[
(f, g)_{\beta} = b_\beta \int_{\mathbb{V}^{d+1}_0} f(x, t) g(x, t) t^\beta e^{-t} \, d\sigma(x, t)
\]
\[
= b_\beta \int_0^\infty \int_{\mathbb{S}^{d-1}} f(t\xi, t) g(t\xi, t) \, d\sigma(\xi) t^{\beta+d-1} e^{-t} \, dt.
\]

Let \( \{Y^m_\ell : 1 \leq \ell \leq \dim \mathcal{H}^d_m\} \) be an orthonormal basis of \( \mathcal{H}^d_m \). Then an orthogonal basis for \( \mathcal{V}_n(\mathbb{V}^{d+1}_0, \varphi_{\beta}) \) is given by
\[
S^m_{m, \ell}(x, t) = L^{2m+\beta+d-1}_{n-m}(t) Y^m_\ell (x), \quad 0 \leq m \leq n, \quad 1 \leq \ell \leq \dim \mathcal{H}^d_m, \quad (7.7)
\]
in terms of the Laguerre polynomials \( L^d_n \).

As in the case of Jacobi polynomials on the cone, these polynomials are eigenfunctions of a second order differential operator only when \( \beta = -1 \).

**Theorem 7.3** Let \( d \geq 2 \). Every \( u \in \mathcal{V}_n(\mathbb{V}^{d+1}_0, \varphi_{-1}) \) satisfies the differential equation
\[
D_{-1} u := \left( t \frac{\partial^2}{\partial t^2} + (d - 1 - t) \frac{\partial}{\partial t} + t^{-1} \Delta^{(x)}_0 \right) u = -nu, \quad (7.8)
\]
where \( \Delta^{(x)}_0 \) is the Laplace–Beltrami operator in variable \( x \in \mathbb{S}^{d-1} \).
Proof The proof is similar to that of (3.6). The polynomial $L_{n-m}^{2m+\beta+d-1}$ satisfies the differential equation (3.12) with $2\mu$ replaced by $\alpha$, from which follows easily that

$$L_{n-m}^{2m+\beta+d-1}(t)t^m$$

satisfies

$$t^2 \frac{d^2 f(t)}{dt^2} + (\beta + d - t) \frac{df(t)}{dt} = -n f(t) + m(m + \beta + d - 1)t^{-1} f(t).$$

Rest of the proof follows exactly as in the proof of Theorem 7.2. \qed

8 Reproducing Kernels for the Jacobi Polynomials on the Surface of the Cone

The reproducing kernel $P_n(\varphi_{\beta,\gamma})$ of the space $V_n(\mathbb{V}^{d+1}_0, \varphi_{\beta,\gamma})$ is uniquely defined by the property that, for all $Y \in V_n(\mathbb{V}^{d+1}_0, \varphi_{\beta,\gamma})$,

$$b_{\beta,\gamma} \int_{\mathbb{V}^{d+1}_0} P_n(\varphi_{\beta,\gamma}; (x, t), (y, s)) Y(y, s) \varphi_{\beta,\gamma}(s) \sigma(y, s) = Y(x, t).$$

The kernel can be written in terms of the orthogonal basis (7.3) as

$$P_n(\varphi_{\beta,\gamma}; (x, t), (y, s)) = \sum_{m=0}^{n} \sum_{\ell=1}^{\dim \mathcal{H}_m} \frac{S_n^{m,\ell}(x, t) S_n^{m,\ell}(y, s)}{H_{m,n}^{\beta,\gamma}}.$$

Similar to the case of the solid cone, we first give an expression of this kernel in terms of the reproducing kernel $P_n(\tilde{\varphi}_{\alpha,\gamma}; (x, t), (y, s))$, defined in (4.2), on the triangle $\mathbb{V}^2$.

Theorem 8.1 Let $d \geq 2$, $\beta \geq -1$, and $\gamma > -1$. For $(x, t), (y, s) \in \mathbb{V}^{d+1}_0$, let $y = sy'$ with $y' \in S^{d-1}$. Then, with $\alpha = \frac{\beta+d-1}{2}$,

$$P_n(\varphi_{\beta,\gamma}; (x, t), (y, s)) = c \int_{-1}^{1} \int_{-1}^{1} P_n(\tilde{\varphi}_{\alpha,\gamma}; \left(\frac{1-z_1}{2} (x, y') + \frac{1+z_1}{2} z_2 t, t\right), (s, s)) \times (1 - z_1)^{\frac{d-2}{2}} (1 + z_1)^{\frac{\beta-1}{2}} (1 - z_2^2)^{\frac{\beta}{2}} dz,$$

(8.1)

where $c = c_{\frac{d-2}{2}} c_{\frac{\beta-1}{2}} c_{\frac{d+1}{2}}$. In particular, for $\beta = -1$,

$$P_n(\varphi_{-1,\gamma}; (x, t), (y, s)) = P_n\left(\tilde{\varphi}_{\frac{d-2}{2},\gamma}; (x, y'), (s, s)\right).$$

(8.2)
The spherical harmonics $Y^m_\ell$ in the basis $S^m_{\ell,\ell}$ are chosen to be an orthonormal basis of $H^d_{m,\ell}$, so that they satisfy the addition formula (2.13). Thus, in terms of the orthogonal basis $S^m_{\ell,\ell}$ of (7.3), the reproducing kernel satisfies

$$P_n(\varphi_{\beta,y};(x,t),(y,s)) = \sum_{m=0}^n \frac{P^{(2\alpha_2+2m,y)}_{n-m}(1-2t)P^{(2\alpha_2+2m,y)}_{n-m}(1-2s)}{H^\beta_{m,n}} s^m t^m Z^\beta_m ((x',y')),$$

where $x = tx'$. Comparing (3.4) and (7.5), and using (4.3) and (4.8) with $\alpha = \frac{d-2}{2}$ or $\beta = -1$, we have proved (8.2), where we have used $\langle x',y' \rangle = \frac{(x,y)}{t}$. For $\beta > -1$, we follow the proof of Theorem 4.4 and use (4.15) to increase the index of $Z^\lambda_n$ form $\lambda = \frac{d-2}{2}$ to $\alpha = \lambda + \sigma$ with $\sigma = \frac{\beta+1}{2}$. This gives

$$P_n(\varphi_{\beta,y};(x,t),(y,s)) = c \int_{-1}^1 \int_{-1}^1 \sum_{m=0}^n \frac{P^{(2\alpha_2+2m,y)}_{n-m}(1-2t)P^{(2\alpha_2+2m,y)}_{n-m}(1-2s)}{H^\beta_{m,n}} s^m t^m Z^\alpha_m ((x',y')) + \left(1 - z_1\right)^{\frac{d-2}{2}} \left(1 + z_1\right)^{\frac{\beta+1}{2}} \left(1 - z_2\right)^{\frac{\beta}{2}} dz,$$

where $c = c_{d-2,\beta-1} c_{\beta+1/2}^\beta$. Hence, by (4.8), we can again use (4.3) to derive (8.1). □

Using the closed formula (4.11) of the reproducing kernels on the triangle, we can now derive closed formulas of the reproducing kernels on the surface of the cone.

**Theorem 8.2** Let $d \geq 2$, $\beta \geq -1$ and $\gamma \geq -\frac{1}{2}$. Let $\alpha = \frac{\beta+d-1}{2}$. Then, for $(x,t),(y,s) \in \mathbb{Y}_0^{d+1}$,

$$P_n(\varphi_{\beta,y};(x,t),(y,s)) = \hat{c}_{\beta,y} \int_{[-1,1]^\beta} Z_{2n}^{2\alpha+y+1} \left(\xi(x,t,y,s;z,v)\right) \times \left(1 - z_1\right)^{\frac{d-2}{2}} \left(1 + z_1\right)^{\frac{\beta+1}{2}} \left(1 - z_2\right)^{\frac{\beta}{2}} dz \left(1 - v_1^2\right)^{\alpha-1} \left(1 - v_2^2\right)^{-\frac{1}{2}} dv,$$  

where $\hat{c}_{\beta,y} = c_{d-2,\beta-1} c_{\beta+1/2}^\beta c_{y/2}^\beta$ and $\xi(x,t,y,s;z,v) \in [-1,1]$ is given by

$$\xi(x,t,y,s;z,v) = \frac{v_1}{2} \sqrt{2st + (1 - z_1)(x,y) + (1 + z_1)z_2st + v_2 \sqrt{1 - t} \sqrt{1 - s}}.$$  

(8.3)

In particular, if $\beta = -1$, then

$$P_n(\varphi_{-1,y};(x,t),(y,s)) = c \int_{[-1,1]^2} Z_{2n}^{\gamma+d-1} \left(v_1 \sqrt{st + (x,y)} + v_2 \sqrt{1 - t} \sqrt{1 - s}\right) \times \left(1 - v_1^2\right)^{\frac{d-4}{2}} \left(1 - v_2^2\right)^{-\frac{1}{2}} dv.$$  

(8.4)

These identities hold under the limit (2.18) when $\gamma = -\frac{1}{2}$, $\beta = -1$ and/or $d = 2$.  

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Proof Using (4.11) with $\zeta = \frac{1-z_1}{2} (x, y') + \frac{1+z_1}{2} z_2 t$, (8.3) is deduced from (8.1) by a straightforward substitution, just as in the proof of Theorem 4.3. \qed

Although $d$ is a positive integer in our statement, our identity holds for non-integer values of $d > 2$ by analytic continuation. In particular, this allows us to consider the limiting case of $d = 2$ in (8.3).

One particularly interesting case is the surface of cone in $\mathbb{R}^3$ with $\gamma = -\frac{1}{2}$. We state this limiting case of (8.5) when $d = 2$ and $\gamma = -\frac{1}{2}$ as a corollary.

Corollary 8.3 For $d = 2$, the reproducing kernel of $V_n(\mathbb{V}^3, \varphi)$ for $\varphi(t) = t^{-1} (1-t)^{-1/2}$ satisfies

$$P_n(\varphi_{-1, -\frac{1}{2}}; (x, t), (y, s)) = Z_{2n}^2 (\sqrt{s t(x, y)} + \sqrt{1 - t \sqrt{1 - s}}) + Z_{2n}^2 (\sqrt{st(x, y)} - \sqrt{1 - t \sqrt{1 - s}}) + Z_{2n}^2 (\sqrt{st(x, y)} - \sqrt{1 - t \sqrt{1 - s}}) + Z_{2n}^2 (\sqrt{st(x, y)} + \sqrt{1 - t \sqrt{1 - s}}).$$

(8.6)

Recall that $Z_n^2 = (2n + 1) P_n$, where $P_n$ is the Legendre polynomial. This identity for orthogonal polynomials on the surface of the cone in $\mathbb{R}^3$ is similar to the addition formula for spherical harmonics.

9 Convolution and Fourier Series on Surface of the Cone

We follow the development on the solid cone. All proofs can be carried out as in the case of solid cone, most with minuscule modification, and will be omitted.

Motivated by the closed formula (8.1), we define a translation operator.

Definition 9.1 Let $d \geq 2$. For $\beta \geq -1$ and $\gamma \geq -\frac{1}{2}$, let $\alpha = \frac{\beta + d - 1}{2}$. For $g \in L^1([-1, 1], w_{2\alpha+\gamma+1})$, we define the operator $T_{\beta, \gamma}$ on the surface of the cone $\mathbb{V}^{d+1}_0$ by

$$T_{\beta, \gamma} g((x, t), (y, s)) := \mathcal{C}_{\beta, \gamma} \int_{[-1, 1]^4} g(\xi(x, t, y, s; u, v)) \left(1 - z_1\right)_{\frac{d-2}{2}} (1 + z_1)_{\frac{\beta-1}{2}} \times (1 - z_2)_{\frac{\beta+1}{2}} dz_1 (1 - v_1)^{\alpha-1} (1 - v_2)^{\gamma-\frac{1}{2}} dv,$$

(9.1)

where $\xi(x, t, y, s; u, v)$ is defined by (8.4). When $\beta = -1$ or $\gamma = -\frac{1}{2}$, the definition holds under the limit (2.18).

By the closed formula of the reproducing kernel (8.1), we immediately obtain

$$P_n(\varphi_{\beta, \gamma}; (x, t), (y, s)) = T_{\beta, \gamma} Z_{2n}^{2\alpha+\gamma+1}((x, t), (y, s)).$$

(9.2)
The following lemma is the analogue of Lemma 5.2 with essentially the same proof.

**Lemma 9.2** Let \( d \geq 2 \). For \( \beta \geq -1 \) and \( \gamma \geq -\frac{1}{2} \), let \( \alpha = \frac{\beta + d - 1}{2} \). Let \( g \in L^1([-1, 1], w^{2\alpha + \gamma + 1}) \) be an even function on \([-1, 1]\). Then

1. for each \( Y_n \in \mathcal{V}_n(\mathbb{R}^{d+1}_0, \varphi_{\beta, \gamma}) \),

\[
b_{\beta, \gamma} \int_{\mathbb{R}^{d+1}_0} T_{\beta, \gamma} g((x, t), (y, s)) Y_n(y, s) \varphi_{\beta, \gamma}(s) d\sigma(y, s) = \Lambda_n(g) Y_n(x, t),
\]

where

\[
\Lambda_n(g) = c_{2\alpha + \gamma + 1} \int_{-1}^1 g(t) \frac{C_{2n}^{2\alpha + \gamma + 1}}{C_{2n}^{2\alpha + \gamma + 1}} (1 - t^2)^{2\alpha + \gamma + \frac{1}{2}} dt.
\]

2. for \( 1 \leq p \leq \infty \) and \((x, t) \in \mathbb{R}^{d+1}_0\),

\[
\|T_{\beta, \gamma} g((x, t), (\cdot, \cdot))\|_{L^p(\mathbb{R}^{d+1}_0, \varphi_{\beta, \gamma})} \leq \|g\|_{L^p([-1, 1], w^{2\alpha + \gamma + 1})}.
\]

We then define an analogue of the convolution structure on the surface of the cone.

**Definition 9.3** Let \( \beta \geq -1 \) and \( \gamma \geq -\frac{1}{2} \) and let \( \alpha = \frac{\beta + d - 1}{2} \). For \( f \in L^1(\mathbb{R}^{d+1}_0, \varphi_{\beta, \gamma}) \) and \( g \in L^1([-1, 1], w^{2\alpha + \gamma + 1}) \), define the convolution of \( f \) and \( g \) on the surface of the cone by

\[
(f \ast_{\beta, \gamma} g)(x, t) := b_{\beta, \gamma} \int_{\mathbb{R}^{d+1}_0} f(y, s) T_{\beta, \gamma} g((x, t), (y, s)) \varphi_{\beta, \gamma}(s) d\sigma(y, s).
\]

Again, the convolution on the cone satisfies Young’s inequality:

**Theorem 9.4** Let \( p, q, r \geq 1 \) and \( p^{-1} = r^{-1} + q^{-1} - 1 \). For \( f \in L^q(\mathbb{R}^{d+1}_0, \varphi_{\beta, \gamma}) \) and \( g \in L^r([\beta, \gamma + 1]; [-1, 1]) \) with \( g \) an even function,

\[
\|f \ast_{\beta, \gamma} g\|_{L^p(\mathbb{R}^{d+1}_0, \varphi_{\beta, \gamma})} \leq \|f\|_{L^q(\mathbb{R}^{d+1}_0, \varphi_{\beta, \gamma})} \|g\|_{L^r([-1, 1], w^{2\alpha + \gamma + 1})}.
\]

An analogue of the Proposition 5.5 also holds. The projection operator \( \text{proj}^\beta_{n, \gamma} : L^2(\mathbb{R}^{d+1}_0, \varphi_{\beta, \gamma}) \mapsto \mathcal{V}_n(\mathbb{R}^{d+1}_0, \varphi_{\beta, \gamma}) \) is defined by

\[
\text{proj}^\beta_{n, \gamma} f(x, t) = b_{\beta, \gamma} \int_{\mathbb{R}^{d+1}_0} f(y, s) \mathcal{P}_n(\sigma_{\beta, \gamma}; (x, t), (y, s)) \varphi_{\beta, \gamma}(s) d\sigma(y, s).
\]

By (8.2) and the definition of the convolution operator, we have

\[
\text{proj}^\beta_{n, \gamma} f = f \ast_{\beta, \gamma} Z^{2\alpha + \gamma + 1}_{2n}.
\]
Let $K_n(\varphi_{\beta,\gamma}; \cdot, \cdot)$ be the reproducing kernel of $\Pi_{n+1}^{d+1}$, which is the kernel of the partial sum $S_n(\varphi_{\beta,\gamma}; f)$ of the Fourier series in the Jacobi polynomials on $\mathbb{V}_0^{d+1}$.

**Proposition 9.5** Let $\beta \geq -\frac{1}{2}, \gamma \geq -\frac{1}{2}$ and $\alpha = \frac{\beta + d - 1}{2}$. For $(x, t), (y, s) \in \mathbb{V}_0^{d+1}$,

$$K_n(\varphi_{\beta,\gamma}; (x, t), (y, s)) = T_{\beta,\gamma} \left[ k_n \left( w_{2\alpha+\gamma+\frac{1}{2}, -\frac{1}{2}}; 2[\cdot]^2 - 1, 1 \right) \right] ((x, t), (y, s)).$$  

(9.6)

Furthermore, for $(y, s) \in \mathbb{V}_0^{d+1}$,

$$K_n(\varphi_{\beta,\gamma}; (0, 0), (y, s)) = k_n \left( w_{2\alpha,\gamma+1}; 1 - 2s, 1 \right).$$  

(9.7)

**Corollary 9.6** For $\beta \geq 0, \gamma \geq -\frac{1}{2}$ and $\alpha = \frac{\beta + d - 1}{2}$,

$$S_n(\varphi_{\beta,\gamma}; f) = f *_{\beta,\gamma} k_n \left( w_{2\alpha+\gamma+\frac{1}{2}, -\frac{1}{2}}; 2[\cdot]^2 - 1, 1 \right).$$  

(9.8)

Finally, we can deduce the convergence of the Cesàro $(C, \delta)$ means as follows.

**Theorem 9.7** For $\beta \geq -1$ and $\gamma \geq -\frac{1}{2}$, define $\lambda_{\beta,\gamma} := \beta + \gamma + d$. Then, the Cesàro $(C, \delta)$ means for $\varphi_{\beta,\gamma}$ on $\mathbb{V}_0^{d+1}$ satisfy

1. if $\delta \geq \lambda_{\beta,\gamma} + 1$, then $S^\delta_n(\varphi_{\beta,\gamma}; f)$ is nonnegative if $f$ is nonnegative;
2. $S^\delta_n(\varphi_{\beta,\gamma}; f)$ converge to $f$ in $L^1(\mathbb{V}_0^{d+1}, \varphi_{\beta,\gamma})$ norm or $C(\mathbb{V}_0^{d+1})$ norm if $\delta > \lambda_{\beta,\gamma}$ and only if $\delta > \lambda_{\beta,\gamma}$ when $\gamma = -\frac{1}{2}$.

**10 Generalizations to Reflection Invariant Weight Functions**

In our study so far, we have limited to the weight functions on the cone that arise from the classical weight functions on the unit ball and the unit sphere, since they are the most interesting cases and they already capture the essence of our study. In this last section, we point out further extensions.

**10.1 Spherical Harmonics with Reflection Group Invariance**

A profound extension of spherical harmonics is Dunkl’s theory of $h$-harmonics associated with reflection groups [3,5]. Let $G$ be a reflection group with a reduced root system $R$. Let $v \mapsto \kappa_v$ be a nonnegative multiplicity function defined on $R$ with the property that it is a constant on each conjugate class of $G$. Then the Dunkl operators [3] are defined by

$$D_i f(x) = \partial_i f(x) + \sum_{v \in R_+} \kappa_v \frac{f(x) - f(x\sigma_v)}{\langle x, \sigma_v \rangle} v_i, \quad i = 1, 2, \ldots, d,$$  

(10.1)
where \( x \sigma_v := x - 2 \langle x, v \rangle v / \| v \|^2 \), \( R_+ \) is a set of positive roots. These first order differential-difference operators commute in the sense that \( D_i D_j = D_j D_i \) for \( 1 \leq i, j \leq d \). The operator
\[
\Delta_h = D_1^2 + \cdots + D_d^2
\]
plays the role of the Laplace operator. An \( h \)-harmonic polynomial of degree \( n \) is a polynomial \( Y \in \mathcal{P}_n^d \) that satisfies \( \Delta_h f = 0 \). Such polynomials are orthogonal with respect to the inner product
\[
\langle f, g \rangle_\kappa = a_\kappa \int_{S^{d-1}} f(x)g(x)h_k^2(x)d\sigma(x), \quad h_k(x) = \prod_{v \in R_+} |\langle x, v \rangle|^\kappa_v, \quad (10.2)
\]
on the sphere, where \( a_\kappa \) is a normalization constant so that \( \langle 1, 1 \rangle_\kappa = 1 \). Let \( \mathcal{H}_n^d(h_k^2) \) be the space of \( h \)-harmonics of degree \( n \), which has the same dimension as that of \( \mathcal{H}_n^d \). Let \( \Delta_{h,0} \) be the restriction of \( \Delta_h \) on the sphere, which is the analog of the Laplace–Beltrami operator. Then \( \mathcal{H}_n^d(h_k^2) \) is the eigenspace of the operator \( \Delta_{h,0} \):
\[
\Delta_{h,0} Y_h = -n(n + 2\lambda_\kappa)Y_h, \quad \lambda_\kappa := |\kappa| + \frac{d-2}{2}, \quad (10.3)
\]
where \( |\kappa| = \sum_{v \in R_+} \kappa_v \). A linear operator, denoted by \( V_\kappa \), that satisfies the relations
\[
D_i V_\kappa = V_\kappa \partial_i, \quad 1 \leq i \leq d,
\]
is called an intertwining operator, which is uniquely determined if it also satisfies \( V_\kappa 1 = 1 \) and \( V_\kappa \mathcal{P}_n^d \subset \mathcal{P}_n^d \). Let \( P_n(h_k^2; \cdot, \cdot) \) be the reproducing kernel of \( \mathcal{H}_n^d(h_k^2) \). The kernel has a closed form in terms of the intertwining operator \( V_\kappa \),
\[
P_n(h_k^2; x, y) = V_\kappa \left[ Z_{n}^{h_k} (\cdot, y) \right] (x), \quad x, y \in S^{d-1}, \quad (10.4)
\]
where \( Z_{n}^{h_k} \) is defined in (2.9). The explicit formula of \( V_\kappa \) is known only in the case of \( G = \mathbb{Z}_2^d \) and the weight function
\[
h_k(x) = \prod_{i=1}^d |x_i|^{\kappa_i}, \quad \kappa_i \geq 0 \quad (10.5)
\]
involve under \( \mathbb{Z}_2^d \). In this case, the intertwining operator is a multiple beta integral, which gives in particular the closed formula for the reproducing kernel [14]
\[
P_n(h_k^2; x, y) = c_k^h \int_{[-1,1]^d} C_n^{h_k} (x_1 y_1 t_1 + \cdots + x_d y_d t_d) \prod_{i=1}^d (1 + t_i)(1 - t_i^2)^{\kappa_i - 1} dt,
\]
where \( c_k^h = c_{\kappa_1 - \frac{1}{2}} \cdots c_{\kappa_d - \frac{1}{2}} \) with \( c_{\lambda} \) defined by (2.6).
10.2 Orthogonal Structure on the Surface of the Cone

On the surface of the cone $\mathbb{V}^{d+1}_0$, we can extend the inner product (7.1) to

$$\langle f, g \rangle_{\kappa, w} = b_{\kappa} \int_{\mathbb{V}^{d+1}_0} f(x, t) g(x, t) h_{\kappa}^2(x) w(t) d\sigma(x, t)$$

with $h_{\kappa}$ in (10.2) invariant under a reflection group. Much of what we have done in Sect. 7 can be extended in this more general setting with $\alpha = \frac{1}{2}(\beta + d - 1)$ replaced by $\alpha = \frac{1}{2}(\beta + |\kappa| + d - 1)$.

For example, with the Jacobi weight function $w = \varphi_{\beta, \gamma}$, we consider

$$\Phi_{\kappa, \beta, \gamma}(t) = h_{\kappa}^2(x) t^\beta (1 - t)^\gamma, \quad (x, t) \in \mathbb{V}^{d+1}_0.$$

Similar to (7.3), we can give an orthogonal basis of $V_n(\mathbb{V}^{d+1}_0, \Phi_{\kappa, \beta, \gamma})$ by

$$S_{m, \ell}(x, t) = p^{(2m+2\alpha, \gamma)}_{n-m}(t) Y_{m, \ell}(x), \quad 0 \leq m \leq n, \quad 1 \leq \ell \leq \dim \mathcal{H}_m^d(h_{\kappa}^2), \quad (10.7)$$

where $\{Y_{m, \ell} : 1 \leq \ell \leq \dim \mathcal{H}_m^d(h_{\kappa}^2)\}$ denote an orthonormal basis of $\mathcal{H}_m^d(h_{\kappa}^2)$.

**Theorem 10.1** Let $d \geq 2$. Every $u \in V_n(\mathbb{V}^{d+1}_0, \Phi_{\kappa, -1, \gamma})$ satisfies the differential–difference equation

$$D_{\kappa, -1, \gamma} u = -n(n + |\kappa| + \gamma + d - 1) u,$$

where $D_{\kappa, -1, \gamma} = D_{\kappa, -1, \gamma}(x, t)$ is the second order linear differential-difference operator

$$D_{\kappa, -1, \gamma} := t(1 - t) \partial_t^2 + (|\kappa| + d - 1 - (|\kappa| + d + \gamma)t) + t^{-1} \Delta_{h, 0}^{(x)},$$

where $\Delta_{h, 0}^{(x)}$ is the operator in variable $x \in \mathbb{S}^{d-1}$.

**Proof** The proof follows that of Theorem 7.2 almost verbatim if we replace $\beta$ by $\beta + |\kappa|$ and use (10.3) instead of (2.12) in the last step. $\square$

An analog of Theorem 7.3 also holds. Furthermore, we can also state a closed formula for the reproducing kernel of $V_n(\mathbb{V}^{d+1}_0, \Phi_{\kappa, \beta, \gamma})$, which we state only for $h_{\kappa}$ in (10.5) invariant under $\mathbb{Z}_2^d$, because of the explicit formula (10.6).
Theorem 10.2 Let $d \geq 2$, $\kappa_i \geq 0$, $\beta \geq -1$ and $\gamma \geq -\frac{1}{2}$. Let $\alpha = \frac{1}{2}(1 + \beta + d - 1)$. Then, for $(x, t), (y, s) \in \mathbb{V}_0^{d+1}$,

$$P_n(\Phi_{\kappa, \beta, \gamma}; (x, t), (y, s)) = \hat{c}_{\beta, \gamma} \int_{[-1, 1]^{d+4}} Z_{2n}^{2\alpha+\gamma+1} \left( \xi(x, t, y, s; z, v, u) \right) \times \left(1 - z_1 \right)^{d-2} \left(1 + z_1 \right)^{\beta-1} \left(1 - z_2^2 \right)^{\beta} dz (1 - v_1^2)^{\alpha-1} (1 - v_2^2)^{\gamma-\frac{1}{2}} dv \times \prod_{i=1}^{d} (1 + u_i)(1 - u_i^2)^{\kappa_i-1} du,$$

(10.9)

where $\hat{c}_{\beta, \gamma} = c_{\kappa} c_{\beta-d-2} c_{\beta+1} c_{\gamma-\frac{1}{2}}$ and $\xi(x, t, y, s; z, v) \in [-1, 1]$ is given by

$$\xi(x, t, y, s; z, v) = \frac{v_1}{2} \sqrt{2st} + (1 - z_1) (x_1 y_1 u_1 + \cdots + x_d y_d u_d) + (1 + z_1) z_2 st + v_2 \sqrt{1-t} \sqrt{1-s},$$

and the identity holds under the limit (2.18) whenever $\beta = -1$, $\gamma = -\frac{1}{2}$ or $\kappa_i = 0$.

Although the closed formula (10.9) is complicated, its one-dimensional structure allows us to carry out the narrative that we developed so far for the more general weight functions on the surface $\mathbb{V}_0^{d+1}$. We could define the translation operator and the convolution in this general setting, and establish similar properties as those in Sect. 8. Instead of stating the results, which carries little additional difficulty, we shall state only a result for the convergence of the Cesàro means.

Theorem 10.3 For $\kappa_i \geq 0$, $\beta \geq -1$ and $\gamma \geq -\frac{1}{2}$, define $\lambda_{\kappa, \beta, \gamma} := \vert \kappa \vert + \beta + \gamma + d$. Then, the Cesàro $(C, \delta)$ means for $\Phi_{\kappa, \beta, \gamma}$ on $\mathbb{V}_0^{d+1}$ satisfy

1. if $\delta \geq \lambda_{\beta, \gamma} + 1$, then $S_{\delta}^{\beta}(\Phi_{\kappa, \beta, \gamma}; f)$ is nonnegative if $f$ is nonnegative;
2. $S_{\delta}^{\beta}(\Phi_{\kappa, \beta, \gamma}; f)$ converge to $f$ in $L^1(\mathbb{V}_0^{d+1}, \varphi_{\beta, \gamma})$ norm or $C(\mathbb{V}_0^{d+1})$ norm if $\delta > \lambda_{\beta, \gamma}$.

10.3 Orthogonal Structure on the Cone

On the unit ball $\mathbb{B}^d$ we can also consider the reflection invariant weight function

$$\omega_{\kappa, \mu}(x) = h_k^2(x) (1 - \|x\|^2)^{\mu+\frac{1}{2}}$$

for an $h_k$ invariant under a reflection group $G$, and study orthogonal polynomials with respect to $\omega_{\kappa, \mu}$ on the ball [5, Sect. 8.1]. In this setting, the space $\mathbb{V}_n^{d}(\omega_{\kappa, \mu})$ is the eigenspace of a second order differential-difference operator [5, Theorem 8.1.3]: for all $u \in \mathbb{V}_n^{d}(\omega_{\kappa, \mu})$,

$$ \left( \Delta_h - \langle x, \nabla \rangle^2 - 2\lambda_{\kappa, \mu} \langle x, \nabla \rangle \right) u = -n(n + 2\lambda_{\kappa, \mu}) u, \quad \lambda_{\kappa, \mu} = \vert \kappa \vert + \mu + \frac{d-1}{2}. $$

(10.10)
Furthermore, the reproducing kernel \( P(\sigma_{\kappa,\mu}; x, y) \) of \( \mathcal{V}_n^d(\sigma_{\kappa,\mu}) \) satisfies a closed form in terms of the intertwining operator \( V_\kappa \), which we state only for the case of \( G = \mathbb{Z}_2^d \).

For \( h_k \) in (10.5), we have [5, Theorem 8.1.6]

\[
P(\sigma_{\kappa,\mu}; x, y) = c_k h_k \int_{[-1,1]^{d+1}} Z_{d+1}^{\kappa,\mu}(x_1 y_1 t_1 + \cdots x_d y_d t_d + x_{d+1} y_{d+1} s)
\times \prod_{i=1}^d (1 + t_i)(1 - t_i^2)^{\kappa_i - 1}(1 - s^2)^{\mu - \frac{1}{2}} dt ds,
\]

where \( x_{d+1} = \sqrt{1 - \|x\|^2} \) and \( y_{d+1} = \sqrt{1 - \|y\|^2} \).

On the cone \( \mathcal{V}_0^{d+1} \), we can replace \( \sigma_\mu \) by \( \sigma_{\kappa,\mu} \) to study orthogonal structure for more general weight functions. For example, instead of the weight function (3.1), we can consider

\[
W_{\kappa,\mu,\beta,\gamma}(x, t) := h_k^2(x)(t^2 - \|x\|^2)^{\mu - \frac{1}{2}} t^{\beta(1 - t)} \gamma, \quad \mu > -\frac{1}{2}, \quad \gamma > -1,
\]

with \( h_k \) as in (10.2) invariant under a reflection group. Most of our results in Sect. 3 and 4 can be extended in this more general setting with \( \alpha = \mu + \frac{1}{2}(\beta + d - 1) \) replaced by \( \alpha = |\kappa| + \mu + \frac{1}{4}(\beta + d - 1) \). In particular, let \( \{ P_{m-n}^n(\sigma_{\kappa,\mu}; x) : |k| = n \} \) be an orthonormal basis of \( \mathcal{V}_n^{d} \) on the unit ball \( \mathbb{B}_d \). Then an orthogonal basis of \( \mathcal{V}_n(\mathcal{V}_0^{d+1}, W_{\kappa,\mu,\beta,\gamma}) \), analogous to that of (3.3), is given by

\[
Q_{m,k}^n(x,t) = P_{n-m}^m(\sigma_{\kappa,\mu}; x)(1 - 2t)^m P_k^n(\sigma_{\kappa,\mu}; x), \quad 0 \leq m \leq n, \quad |k| = n - m.
\]

In the case of \( \beta = 0 \), we have the following extension of Theorem 3.2:

**Theorem 10.4** For \( \mu > -\frac{1}{2}, \gamma > -1 \) and \( \kappa_i \geq 0 \), define \( \alpha = |\kappa| + \mu + \frac{d-1}{2} \). Then every \( u \in \mathcal{V}_n(\mathcal{V}_0^{d+1}, W_{\kappa,\mu,0,\gamma}) \) satisfies the differential–difference equation

\[
\mathcal{D}_{\kappa,\mu,\gamma} u = -n(n + 2\alpha + \gamma + 1)u,
\]

where \( \mathcal{D}_{\kappa,\mu,\gamma} = \mathcal{D}_{\kappa,\mu,\gamma}(x, t) \) is the second order linear differential–difference operator

\[
\mathcal{D}_{\kappa,\mu,\gamma} := t(1 - t)\partial_t + 2(1 - t)\langle x, \nabla_x \rangle \partial_t + t \Delta_h^{(x)} - \langle x, \nabla_x \rangle^2
+ (2\alpha + 1)\partial_t - (2\alpha + \gamma + 2)(\langle x, \nabla_x \rangle + t \partial_t) + \langle x, \nabla_x \rangle,
\]

where \( \nabla_x \) and \( \Delta_h^{(x)} \) denote the operators acting on the \( x \) variable.

**Proof** The proof follows that of Theorem 3.2 almost verbatim if we replace \( \mu \) by \( \mu + |\kappa| \) and use (10.10) instead of (2.16). We choose not, however, to carry out the last step of expanding \( \langle x, \nabla_x \rangle^2 \), which would lead to a final expression of \( \mathcal{D}_{\kappa,\mu,\gamma} \) that contains both \( \Delta_x \) and \( \Delta_h^{(x)} \).
We can also derive a closed formula for the reproducing kernel $P_n(W_{κ,μ,β,γ};⋯)$ of the space $V_{n}(V^{d+1};W_{κ,μ,β,γ})$, which we again state only for $h_{κ}$ in (10.5) invariant under $Z_{2}d$, using the explicit formula (10.6).

**Theorem 10.5** Let $d ≥ 2$. For $μ ≥ 0$, $β ≥ −\frac{1}{2}$, $γ ≥ −\frac{1}{2}$ and $κ_{i} ≥ 0$, let $α = |κ| + μ + \frac{β + d - 1}{2}$. Then

$$P_n(W_{κ,μ,β,γ}; (x, t), (y, s)) = \sum_{k,μ,β,γ} c_{k,μ,β,γ} c_{α−1,γ} \int_{1−1,1}^{d+4} Z^{2α+γ+1}(\xi(x, t, y, s; z, u, v))$$

$$\times \prod_{i=1}^{d} (1 + p_{i})(1 − p_{i}^{2})^{κ_{i}−1}(1 − z_{1})^{μ+\frac{d−1}{2}}(1 + z_{1})^{β−1}(1 − z_{2})^{−\frac{β−1}{2}}$$

$$\times (1 − u^{2})^{μ−1}du(1 − v_{1}^{2})^{α−1}(1 − v_{2}^{2})^{γ−1} \frac{d}{dpdz du dv}$$

where $\xi(x, t, y, s; u, v, p, q) ∈ [−1, 1]$ is defined by

$$\xi(x, t, y, s; u, v) = v_{2}\sqrt{1 − t}\sqrt{1 − s} + \frac{1}{2}v_{1}\sqrt{ρ(x, t, y, s, p, z)}.$$

in which

$$ρ(x, t, y, s, p, z) = 2st + (1 + z_{1})z_{2}st$$

$$+ (1 − z_{1})\left(p_{1}x_{1}y_{1} + ⋯ + p_{d}x_{d}y_{d} + \sqrt{t^{2} − \|x\|^{2}}\sqrt{s^{2} − \|y\|^{2}u}\right).$$

In the case $μ = 0$ or $β = −\frac{1}{2}$ or $γ = −\frac{1}{2}$ or $κ_{i} = 0$, the identity holds under the limit (2.18).

As in the previous subsection, we have no difficulty to carry out our narrative for the solid cone $V^{d+1}$ to more general weight functions $W_{κ,μ,β,γ}$, but choose not to state them. The corresponding result for the Cesàro means of the Fourier orthogonal series is given below:

**Theorem 10.6** For $κ ≥ 0$, $μ ≥ 0$, $β ≥ 0$ and $γ ≥ −\frac{1}{2}$, define $λ_{κ,μ,β,γ} := 2μ + |κ| + β + γ + d$. Then, the Cesàro $(C, δ)$ means for $W_{κ,μ,β,γ}$ on $V^{d+1}$ satisfy

1. if $δ ≥ λ_{κ,μ,β,γ} + 1$, then $S_{n}^{δ}(W_{κ,μ,β,γ}; f)$ is nonnegative if $f$ is nonnegative;
2. $S_{n}^{δ}(W_{κ,μ,β,γ}; f)$ converge to $f$ in $L^{1}(V^{d+1}, W_{κ,μ,β,γ})$ norm or $C(V^{d+1})$ norm if $δ > λ_{κ,μ,β,γ}$.

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