Bosonization Approach and Novel Traveling Wave Solutions of the Superfield Gardner Equation

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Abstract

In this paper, the bosonization of the superfield Gardner equation is presented and novel traveling wave solutions are extracted from the coupled bosonic equations by using the mapping and deformation relations. In the case of two-fermionic-parameter bosonization procedure, we provide a special solution in the form of Jacobian elliptic functions. Meanwhile, we discuss and formally derive traveling wave solutions of N fermionic parameters bosonization procedure. This technique can also be applied to treat the N = 1 supersymmetry KdV and mKdV systems which are obtained in two limiting cases.

Keywords

Supersymmetry, Superfield Gardner Equation, Bosonization, Traveling Wave Solutions

1. Introduction

The supersymmetry (SUSY), applied to treat fermions and bosons in a unified way in elementary particle physics since the concept first arose in 1971 by Ramond, Goland and Likhtman, has been researched extensively during past four decades [1]-[6]. The starting point of SUSY is the supersymmetric versions of well known KdV equation first by Kupershmidt in 1984 (a simple fermionic but not supersymmetric extension) and later found independently of the work of Manin-Radual on super KP hierarchy [7] [8] [9] [10]. It was pointed out afterwards that the latter is indeed a truly N = 1 (N refers to the number of super-
symmetries, for \( N = 1 \) standard, for \( N > 1 \) extended) sKdV equation which is invariant under supersymmetric transformation [8]. Since then, various properties have been established for its supersymmetric versions, such as Lax representation, bi-Hamiltonian structures, Backlund transformation (BT), Painleve analysis, N-soliton solutions, (non) local conserved quantities, etc. [9]-[15].

The integrability of sKdV equation can be established in the way to supersymmetrize the unique Gardner transformation. For the well known KdV equation

\[
6u_x + uu_{tt} + u_{xxx} = 0
\]

we extend the classical spacetime \( x,t \) to a super-spacetime \( x,t,\theta \), where \( \theta \) is a Grassmannian odd number \( \theta^2 = 0 \). Now we write the \( N = 1 \) supersymmetric KdV (sKdV) equation accompanied with a fermionic super-variable \( \Phi(x,t,\theta) \) under the compact form

\[
\Phi_t - 3D^2(\Phi D\Phi) + D^4\Phi = 0
\]

Here the covariant super-derivative \( D \) is defined by \( D = \partial + \theta\partial \). Mathieu found that a unique extension

\[
\Phi = \chi + \epsilon\chi + \epsilon^2\chi(D\chi)
\]

of establishing the integrability maps a solution of the superfield Gardner equation

\[
\chi_t - 3D^2(\chi D\chi) - 3\epsilon^2(D\chi)D^2(\chi D\chi) + D^4\chi = 0
\]

into a solution of the sKdV equation [8] [9] [15]. The Gardner equation is also called the extended KdV equation with the variable-sign cubic non-linear term or the combined KdV and mKdV (KdV-mKdV) equation. It is widely used in various branches of physics, such as plasma physics, fluid physics, nonlinear phenomena and quantum field theory, etc., and it also describes a variety of wave phenomena in plasma and solid state [16] [17] [18] [19] [20].

This map was also used to recover an infinite number of conservation laws for the sKdV equation, and construct interesting BT [6]. It is easy to show that such super equation is invariant under the supersymmetry transformation: \( x \rightarrow x - \eta \theta \), \( \theta \rightarrow \theta + \eta \) (\( \eta \) is an anticommuting parameter.). The component form of the above equation with the superfield \( \chi = \xi + \theta u \) can be rewritten as

\[
u_t + u_{xxx} - 6uu_x + 3\xi\xi_{xx} - \epsilon^2[6u_x u_{xx} - 3(\xi\xi_x)]=0
\]

(5a)

\[
\xi_t + \xi_{xxx} - 3(\xi u)_x - 3\epsilon^2(\xi u)_{xx} = 0
\]

(5b)

Note that \( \xi \) and \( u \) are new setted fermionic and bosonic functions, the usual Gardner equation is recovered by setting the fermionic variable to be absent and the sKdV equation is the limiting case where \( \epsilon \rightarrow 0 \).

Nonlinear partial differential equations play an important role in nonlinear physics, even nonlinear science. Various effective methods have been proposed to derive explicit or formal solutions. Recently, a simple but powerful bosonization approach, which main idea is to consider fields in a Grassmannian algebra
and rewrite a system in a basis of this algebra to arrive at a system of ordinary (commutative) evolution equations, can effectively simplify such systems containing anti-communicating fermionic fields [7] [21] [22]. In [23], B. Ren et al. used this approach in the \( N = 1 \) supersymmetric Burgers (SB) system, and the exact solutions of the usual pure bosonic systems are obtained with the mapping and deformation method and Lie point symmetries theory. In [24], the Lie point symmetries of the supersymmetric KdV-a system are considered and similarity reductions of it are conducted. Several types of similarity reduction solutions of the coupled bosonic equations are also simply obtained. The motivation and purpose of this paper is to show this procedure and outcome of the method by taking the superfield Gardner equation and to acquire novel traveling wave solutions of this equation.

The rest of this paper is organized as follows. Section 2 and 3 are brief reminders of fairly basic illustrations of the bosonization approach of the superfield Gardner equation with two and three fermionic parameters. In Section 4 we present the \( N \) fermionic parameters bosonization case. In Section 5 we give the \( N = 1 \) supersymmetric KdV and mKdV equations using parallel procedure in two particular cases, and we will also give a short summary.

2. Two-Fermionic-Parameter Bosonization

To get direct comprehension and fixed notations of the bosonization approach for superfield Gardner equation with multi-fermionic parameters, we first concentrate on a linear space \( G(V) \). Mathematically, such uncomplicated method used for vanishing fermionic fields is based on direct sum of superspace \( G(V) = \Lambda_0 \oplus \Lambda_1 \), where \( \Lambda_0 \) and \( \Lambda_1 \) represents subspace containing even and odd elements, respectively [25]. Here and in the following we omit the exterior algebra sign \( \wedge \) and denote it briefly by ordinary multiplication. Moreover, relevant solutions may involve rich (super) symmetries physically. An exact example is the cases of two and three fermionic parameters bosonization, thereby we can directly derive the usual Gardner equation and coupled equations appear below. For the case of two fermionic parameters \( \theta_1 \) and \( \theta_2 \) with \( \theta_1^2 = \theta_2^2 = 0 \), let the two component fields \( u \) and \( \xi \) be expanded as

\[
\begin{align*}
\theta_0 + u_0 \theta_1 \theta_2 , \quad \xi = v_0 + v_2 \theta_2 ,
\end{align*}
\]

here \( u_i = u_i(x,t)(i = 0,1) \), \( v_i = v_i(x,t)(i = 1,2) \) are all usual bosonic functions with respect to spacetime variable \( x \) and \( t \), thus we get nonlinear Partial Differential Equations (PDEs) in the component form by using (5)

\[
\begin{align*}
(7a) & \quad u_{00} + u_{0xxx} - 6u_0 u_0 f(u_0) = 0 \\
(7b) & \quad v_{00} + v_{0xxx} - 3(v_0 u_0)_x f(u_0) = 0 \\
(7c) & \quad u_{tt} + u_{xxx} - 6(u_0 u_1)_x f(u_0) - 6\epsilon^2 u_0 u_2 u_1 + F_1 = 0
\end{align*}
\]

where \( f(u_0) = 1 + \epsilon u_0 \) and \( F_1 = 3\left[ f(u_0) (v_1 v_2 - v_1 v_1) \right]_x \).

Next we introduce the traveling wave variable \( X = kx + \omega t + x_0 \) along with
the constants of wavenumber $k$, angular frequency $\omega$ and phase $x_0$, therefore, above equations would be changed to a system consisting of ordinary differential equations (ODEs):

$$\omega u_{0x} + k^3 u_{0xxx} - 6k u_x u_{0x} f(u_0) = 0 \quad (8a)$$

$$\omega v_{ix} + k^3 v_{ixxx} - 3k (v_i u_0)_x f(u_0) = 0 \quad (i = 1, 2) \quad (8b)$$

$$\omega u_{1x} + k^3 u_{1xxx} - 6k (u_0 u_1)_x f(u_0) - 6e^k u_0 u_{0x} u_1 + F_i(X) = 0. \quad (8c)$$

Note that we denote $F_i(X) = 3k^2 \left[ f(u_0) (v_{i2x} - v_{2i1}) \right]_X$ in here.

The traveling waves we discuss are only in the usual spacetime $x, t$ but not in the super-spacetime $x, t, \theta$, for example, $\chi(x, t, \theta) = \chi(X + \theta \zeta)$ with Grassmannian constant $\zeta$ different from those in the usual spacetime. In addition to a directly integrable ODE in $u_0$, the solution of the residual system which are related to third-order linear (non)homogeneous ODEs in $u_1, v_1$ and $v_2$, can be obtained through the variable transformation from ordinary coordinates space to phase space on the base of periodic wave solutions of usual Gardner equation. We first solve out $u_{0x}$ from Equation (8a), and the result reads

$$u_{0x} = \frac{a_0}{k^{2}} \sqrt{k \lambda}(z), \quad \lambda(z) = k e^z z^4 + 2kz^3 - \omega z^2 - 2C_1 z - C_2 \quad (9)$$

where $a_0^2 = 1$, $C_1$ and $C_2$ are two arbitrary integral constants, here and below the new variable $z$ stands for function $u_0$.

To get the mapping relations of $u_1, v_1$ and $v_2$, we introduce the variable transformations as follows $u_i(X) = P_i(u_0)$, $v_i(X) = Q_i(u_0)$ and $v_2(X) = Q_2(u_0(X))$. Applying the transformation via Equation (8a), the linear ODEs (8b)-(8c) are reduced to mapping and deformation relations between the traveling wave solutions of the classical Gardner equation and its supersymmetric equation by exploiting the known solutions of classical Gardner equation

$$K_e(P_i) + R_i(z) = 0 \quad (10a)$$

$$K_e(Q_i) = 0, \quad i = 1, 2 \quad (10b)$$

where linear operators read

$$K_e = \lambda(z) \frac{d^2}{dz^2} + (2k e^z z^3 + 3k^2 z^2 - \omega z^2 - C_1) \frac{d}{dz} + (\omega - 6k e^z z^2 - 6kz) \quad (11a)$$

$$K_o = \lambda(z) \frac{d^3}{dz^3} + (6k e^z z^3 + 9k^2 z^2 - 3\omega z^2 - 3C_1) \frac{d^2}{dz^2} + 3kzf(z) \frac{d}{dz} - 3kf(z) \quad (11b)$$

and the nonhomogeneous term $R_i(z)$ is

$$R_i(z) = -3a_0 f(z) \left( Q_i \frac{dQ_i}{dz} - Q_2 \frac{dQ_2}{dz} \right) \sqrt{k \lambda(z)} - A_{i,1}. \quad (12)$$

while $A_{i,1}$ is just an arbitrary integral constant. On this basis, the mapping and deformation relations are obtained as

$$P_i = \sqrt{\lambda(z)} \left[ A_{i,3} + \int \frac{A_{i,2}}{\lambda(y)^{3/2}} dy + \int \frac{R_i(y_i) dy_i}{\lambda(y)^{5/2}} \right] \quad (13a)$$
\[ Q_i = B_{i,3}z + B_{i,2}\sqrt{\eta(z)}\sin[H(z) + B_{i,1}], i = 1, 2. \]  

(13b)

where \( A_{i,2} \) and \( B_{i,j} (i, j = 1, 2) \) are arbitrary constants, and

\[
H(z) = \int_{\tau}^{\tilde{\tau}} \frac{\tau f(y)\sqrt{\eta(y)}}{\eta(y)} dy \text{ with auxiliary function }
\eta(z) = -\tau_1 + \epsilon \omega z^2 + k_2 - \epsilon^4 C_2 \cdot \epsilon^4 C_2. \]

Coefficients for used are defined as
\[
\tau_1 = C_1 - \epsilon \omega C_2, \quad \tau_2 = -\epsilon^4 C_1 + k C_2 + \omega C_1 \quad \text{and} \quad \tau = \sqrt{\tau_1 \tau_2}.
\]

Thus, we have constructed the general two-fermionic-parameter traveling wave solutions of the supersymmetric version of Gardner system

\[ u = u_0 + \sqrt{\lambda(u_0)} \left[ A_{1,2} + \int_0^{\lambda(y)} \frac{R_i(y)dy_i}{\lambda(y)^2} \right] \theta_1 \theta_2 \]  

(14a)

\[ \xi = \sum_{i=1}^{2} \left( B_{i,0}u_0 + B_{i,2}\sqrt{\eta(u_0)}\sin[H(u_0) + B_{i,1}]\theta_1 \right) \]  

(14b)

with the known solution \( u_0 \) or Equation (8a) of the usual Gardner equation which have been extensively studied in many literatures [18] [26]. For a special case which takes \( B_{i,2} = B_{i,3} = 0 \ (i = 1, 2) \) to eliminate nonhomogeneous terms, the above traveling wave solution becomes

\[ \chi = B_{1,1}u_0 \theta_1 + B_{2,2}u_0 \theta_2 + \theta(u_0 + A_{1,2}u_0, \theta_1 \theta_2) \]  

(15)

It is interesting to see that the expression (9) is a trivial type of the symmetries or conservation quantity of standard Gardner equation
\[ \sigma = A\sigma_0, \alpha = x, t; \quad T = \sigma, \quad \rho = \sigma_{x, t} - 6u_0, \sigma f(u_0). \]

In fact, for any given \( u_0(x, t) \) being a solution of the usual Gardner equation, a certain type of solutions of the bosonic-looking equation can be constructed as follows: \( v_1 = B_{i,0}u_0, \quad v_2 = B_{i,2}u_0, \quad u_i = \sigma(u_0), \) where \( \sigma(u_0) \) represents any symmetry of the usual Gardner equation, and we have much freedom to choose \( u_0 \) such that it can construct solutions without restricting to the traveling wave type solutions and infinitely many symmetries of the superfield Gardner equations.

It is clear that the solution (9) can be expressed by the form of the Jacobian elliptic sine functions, i.e.,

\[ u_0 = -\frac{1}{2e^2} + \frac{a_{k,lm}}{e} sn(X_1, m), \]  

(16)

where \( X_1 = lX \), modulus \( m \) and constants \( C_i (i = 1, 2) \) are related to other known constants through following relations:

\[
m = \frac{a_0\sqrt{2k(2e^2\omega + 3k - 2e^2k\tau^2)}}{2k^2\tau e}, \quad C_1 = \frac{e^2\omega + k}{2e^2}, \]

\[ C_2 = \frac{(4e^2k^2\tau^2 - 4e^2\omega - 5k)(4e^2k^2\tau^2 - 1)}{16e^5}. \]  

(17)

Therefore, we derive special type solutions of the superfield Gardner equation:

\[ u = \frac{1}{2e^2} + \frac{a_{k,lm}}{e} S + \theta_1 \theta_2 cn(X_1, m) dn(X_1, m)(A_i + u_i) \]  

(18)
where
\[ u_i' = \frac{1}{C\sqrt{1-m^2S^2}} \left\{ A_i \left[ C((E+F)m^2 + E - F)\sqrt{1-m^2S^2} \right. \right. \]
\[ \times \left( S^2 - 1 - m^2C^2 \right) S \left. \left. \right] + A_i \left[ C((E+F)m^2 + E - F)\sqrt{1-m^2S^2} \right. \right. \]
\[ + m^4SC^2 - 4imkm \left( S^2 - \frac{1}{2} \right) + S'm^2 + 2lmke - S \right\} \right\} \] (19)

\( A_i (i=1,2,3) \) and \( B_i (i=1,2) \) are all arbitrary integral constants, the Jacobian elliptic functions are denoted as \( S = sn(X_i,m) \) and \( C = cn(X_i,m) \), the incomplete elliptic integrals are denoted as \( E = E(S,m) \) and \( F = F(S,m) \).

We also note that for the critical modulus \( m = 1 \), above solutions read:
\[ u = \frac{1}{2\epsilon^2} + \frac{a_{klm}}{\epsilon}\tanh X_i + \theta_i \theta_j u_i \]
\[ \xi = (B_i \theta_i + B_j \theta_j) \left( -\frac{1}{2\epsilon^2} + \frac{a_{klm}}{\epsilon}\tanh X_i \right) \] (20)

where
\[ u_i = \text{sech}^2 X_i \left\{ A_i + A_i \left[ \frac{2}{3} \cosh^3 X_i + \cosh X_i \right] \sinh X_i + X_i \right\} \]
\[ + A_i \left[ -\frac{4}{3} \epsilon kl \cosh^4 X_i + \frac{1}{3} \sinh(2X_i) \cosh^2 X_i + X_i + \frac{1}{2} \sinh(2X_i) \right] \} \] (21)

3. Three-Fermionic-Parameter Bosonization

For the case of three fermionic parameters \( \theta_i (i=1,2,3) \) with \( \theta_j = 0 (i=1,2,3) \), let the two component fields \( u \) and \( \xi \) be expressed as
\[ u = u_0 + \sum_{i,j=1}^{4} u_i \theta_j \theta_k, \quad \xi = \sum_{i=1}^{4} v_i \theta_j, \quad (\theta_j = \theta_2 \theta_3) \] (22)

here \( u_i = u_i(x,t) (i=0,1,2,3) \) and \( v_i = v_i(x,t) (i=1,2,3,4) \) are all usual bosonic functions with respect to spacetime variable \( x \) and \( t \). The symbol \( \epsilon_{ijk} \) is the third-order Levi-Civita tensor. Then from (5), we get PDEs in component form as
\[ u_{00} + u_{0xxx} - 6u_0u_{0x}f(u_0) = 0 \] (23a)
\[ u_{ii} + u_{ixxx} - 6(u_i u_0), f(u_0) - 6\epsilon^2 u_0 u_{ii} u_i + F_i = 0, i = 1,2,3 \] (23b)
\[ v_{ii} + v_{ixxx} - 3(v_i u_0), f(u_0) = 0, i = 1,2,3 \] (23c)
\[ v_{ii} + v_{4xxx} - 3(v_i u_0), f(u_0) + G_4 = 0 \] (23d)

where the somewhat complex nonhomogeneous terms read
\[ F_i = 3 \left[ f(u_0) \left( v_i \nu_{xx} - v_i v_{xx} \right) \right], \epsilon_{ijk} = 1 \] (24a)
\[ G_4 = -3 \left[ f(u_0) \sum_{i=1}^{3} u_i v_i \right] - 3\epsilon^2 u_0 \sum_{i=1}^{3} u_i v_i \] (24b)
Introducing the traveling wave variable $X = kx + \omega t + x_0$ with constants $k$, $\omega$ and $x_0$ and variable transformations $u_i = P_i(u_0(X))$ ($i = 0, 1, 2, 3$) and $v_i = Q_i(u_0(X))$ ($i = 1, 2, 3, 4$), similar to the previous case, the above bosonization system becomes following ODEs

\begin{align}
K_u(P_i) + R_i &= 0, i = 1, 2, 3 \\
K_u(Q_i) &= 0, i = 1, 2, 3 \\
K_v(Q_i) + G_i' &= 0
\end{align}

where nonhomogeneous terms are

\begin{align}
R_i &= -3a_0f(z)\sqrt{k\lambda(z)}\left(Q_i\frac{dQ_i}{dz} - Q_i\frac{dQ_i}{dz}\right) - A_{i,1}, e_{ii} = 1 \\
G_i' &= -3k\frac{d}{dz}\left[f(z)\sum_{i=1}^{3}u_iv_i\right] - 3\epsilon^{2}kz^2\sum_{i=1}^{3}u_i\frac{dv_i}{dz}
\end{align}

with arbitrary integral constants $A_{i,j}$ ($i, j = 1, 2, 3$). Mapping and deformation relations with respect to $u_0$ via traveling wave variable are constructed as

\begin{align}
P_i &= \sqrt{\lambda(z)}\left[A_{i,3} + \int_{y}^{z} R_1(y, \eta)dy_1\right] (i = 1, 2, 3) \\
Q_i &= B_{i,1}z + B_{i,2}\sqrt{\eta(z)}\sin[H(z) + B_{i,3}]
\end{align}

\begin{align}
Q_i &= \frac{\epsilon^2}{r_1} E_1(y)dy + \sqrt{\eta(z)}\left[\frac{(ky^2 - C_i)f(y)}{\tau^{2}\sqrt{\lambda(y)}}\sin[H(z) - H(y)]\right]
\end{align}

where $A_{i,j}$ ($i, j = 1, 2, 3, j = 2, 3$) and $B_{i,j}$ ($i, j = 1, 2, 3$) are some arbitrary integral constants, and function $E_1(z) = \int_{y}^{z} G_2(y)dy + r_2$ with integral constant $r_2$.

Therefore, we have obtained the three-fermionic-parameter traveling wave solutions of the superfield Gardner system. While one of the Grassmann numbers $\theta_i$ ($i = 1, 2, 3$) tends to zero, the solution turns back to above section. Similar to the two-fermionic-parameter case, we write a special type solution

\begin{align}
u &= u_0 + \sigma_1(u_0)\theta_2\theta_3 + \sigma_2(u_0)\theta_1\theta_3 - d_1\left[2\sigma_1(u_0) + d_2\sigma_2(u_0)\right]\theta_2 \theta_3
\
\xi &= (d_1\theta_1 + d_2\theta_2 + d_3\theta_3 + d_4\theta_1\theta_2\theta_3)u_0
\end{align}

where $d_i$ ($i = 1, 2, 3, 4$) are some constants, $\sigma_i(u_0)$ ($i = 1, 2$) are arbitrary symmetries of the usual Gardner equation, and $u_0$ is an arbitrary solution of the usual Gardner equation.

### 4. N-Fermionic-Parameter Bosonization

Motivated by above sections, we repeat same procedure to get traveling wave solutions of the superfield Gardner equation via bosonizaion approach with N fermionic parameters here. The component fields $u$ and $\xi$ can be expressed as
Here and below we denote by \( M_i \) the set of multi-indices which satisfies \( 1 \leq i_1 < i_2 < \cdots < i_q \leq N \), \( N_1 = \left\lceil \frac{N}{2} \right\rceil \) and \( N_2 = \left\lceil \frac{N + 1}{2} \right\rceil \) the upper bound of summations. The elements \( u_{i_1 \ldots i_q} \) and \( v_{i_1 \ldots i_q} \) are bosonic smooth real or complex valued functions defined in the commutative algebra. Thus the super Gardner model (5) is transformed to a new pure bosonic-looking system with \( 2^N \) coupled nonlinear PDEs:

\[
\begin{align*}
    u_{00} + u_{0xxx} - 6u_x^2 u_{0x} - 6e^2 u_0^2 u_{0x} &= 0 \\
    L_x(u_{i_1 \ldots i_q}) + F_{i_1 \ldots i_q} &= 0 \\
    L_x(v_{i_1 \ldots i_{q+1}}) + G_{i_1 \ldots i_{q+1}} &= 0
\end{align*}
\]

and nonhomogeneous terms are

\[
F_{i_1 \ldots i_q} = \begin{cases} 
    3T[\xi, \bar{\xi}], & \text{for } n = 1, \\
    -3T[u, u], + 3T[\bar{\xi}, \xi], -6e^2 \left( u_x T[u, u] \right) + 2u_x T[u, \bar{\xi}] + T[u, u, \bar{\xi}], & \text{for } n = 2, \cdots, N_1, \\
    0, & \text{for } n = 1, \\
    -3T[\xi, u] - 3e^2 \left( u_x T[u, u] \right) + 2u_x T[u, \bar{\xi}] + T[u, u, \bar{\xi}], & \text{for } n = 2, \\
    -3T[\xi, u] - 3e^2 \left( u_x T[u, u] \right) + T[u, u, \bar{\xi}] + 2u_x T[u, \bar{\xi}], & \text{for } n = 3, \cdots, N_2.
\end{cases}
\]

We have also used the shorthand notations

\[
T[x_1, \cdots, x_s] = \sum_{j} e_{\mu_j} \prod_{i=1}^{s} x_i^{\mu_j_i}
\]

and

\[
J = \left\{ (j_1, \cdots, j_s) | 1 \leq j_{k+1} < \cdots < j_{k+1} \leq m, n_i \leq k_i \leq n_i', \sum_{i=1}^{s} k_i = m, j_{h_i} \neq j_{h_i} (h_i \neq h_i) \right\},
\]

with

\[
\mu_j = i_{j_1+1} \cdots i_{j_{k+1}}, \quad n_i = 2 - |x_i|, \quad n_i' = m - \sum_{h=1,h \neq i}^{s} n_{h_i}
\]

and \( m = 2n \) for even case; \( m = 2n - 1 \) for odd case. The parity function \( |J| \) is defined in [25]. The symbol \( e_{\mu_j} \) is the generalized Levi-Civita an-
ti-symmetric tensor in $m$ dimensions. Equation (30a) apparently suggests that it is the standard Gardner equation. Each solution in the case of $n-1$ fermionic parameters is a portion of solutions of the system consisting of $n$ homologous Grassmannian variables. Introducing the traveling wave variable and variable transformations, mapping transformations are arranged as follows:

1) Equation of $u_0$: $u_{0x} = \frac{a_0}{k^2} \sqrt{k \lambda(z)}$ (33a)

2) Equations of $u_{i \to 2a}$: $K_s(P) - R_{i \to 2a} = 0$ (33b)

3) Equations of $v_{i \to 2a-1}$: $K_s(Q) + G'_{i \to 2a-1} = 0$ (33c)

Here the nonhonogenerous terms are $G'_{i \to 2a-1}(z) = u_0^{-1} G_{i \to 2a-1}(z)$, each usual bosonic functions $u_{i \to 2a}$ and $v_{i \to 2a-1}$ are represented by $P = P(u_0(X))$ and $Q = Q(u_0(X))$, respectively.

Finally, the mapping and deformation relations of $u_{i \to 2a}$ and $v_{i \to 2a-1}$ with $N$ fermionic parameters can be formally rewritten as

$$P = \sqrt{\lambda(z)} \left[ A_{i \to 2a,3} + \int z A_{i \to 2a,2} + \int y R_{i \to 2a}(y) dy \right]$$ (34a)

$$Q = B_{i \to 2a-1, a} z + B_{i \to 2a-1, a} \sqrt{\eta(z)} \sin[H(z) + B_{i \to 2a-1, a}]$$

$$+ \frac{\varepsilon^2 z}{\tau_1} \int E_{i \to 2a-1}(y) dy + \sqrt{\eta(z)} \times \int \frac{E_{i \to 2a-1}(y)}{\sqrt{\eta(y)}} \left\{ \left[ ky^2 - C_j \right] f(y) \right\} \frac{1}{\tau \sqrt{\lambda(y)}} dy$$ (34b)

$$\times \sin[H(z) - H(y)] - \frac{\varepsilon^2 y}{\tau_1} \cos[H(z) - H(y)] \right\} dy$$

in which auxiliary functions are

$$R_{i \to 2a}(z) = -\int z F_{i \to 2a}(y) dy - A_{i \to 2a,3}$$ (35a)

$$E_{i \to 2a-1}(z) = -\int z G_{i \to 2a-1}(y) dy + r_{i \to 2a-1}$$ (35b)

$A_{i \to 2a, j}$, $B_{i \to 2a-1, j}$ ($j = 1, 2, 3$) and $r_{i \to 2a-1}$ are some arbitrary integral constants.

5. Discussion and Summary

With relationships at hand, in case of preserving the square and cubic non-linear terms in Equations (5) or taking limitation $\varepsilon \to 0$, one can conveniently acquire the corresponding result of the integrable $N = 1$ sKdV regardless of some integral constants. Similarly to Section 4, the general traveling wave solutions of the $N = 1$ sKdV Equation (2) with $N$ fermionic parameters are

$$u_{i \to 2a} = \sqrt{2ku_0^2 - \omega u_0^2 - 2C_1 u_0 - C_2} \left[ A_{i \to 2a,3} + \int z A_{i \to 2a,2} + \int y R_{i \to 2a}(y) dy \right]$$ (36a)
\[\xi_{\xi_0 \rightarrow \xi_{1}} = B_{\xi_0 \rightarrow \xi_{1}} \xi_{0} + B_{\xi_0 \rightarrow \xi_{1}} \xi_{1} \sin \left[ H(u) + B_{\xi_0 \rightarrow \xi_{1}} \xi_{1} \right] \]
\[+ \sqrt{ku_0 - C_1} \int_{u_0}^{u_1} \sin \left[ H(u) - H(y) \right] E_{\xi_0 \rightarrow \xi_{1}} \sqrt{k} \frac{\sin y}{\sqrt{C_1}} \frac{(2ky^2 - \omega y^2 - 2C_1y - C_2)}{dy}, \] 

where

\[H(z) = \int_{z}^{y} \frac{\sqrt{C_1(kC_2 + \omega C_1)}}{(ky^2 - C_1 \sqrt{2ky^2 - \omega y^2 - 2C_1y - C_2})} dy\].

Due to the case of taking limitation \(\varepsilon \rightarrow \pm 1\) and the absence of non-linear quadratic term \(D^2(\chi \partial \chi)\) in Equation (4), traveling wave solutions of the usual integrable \(\text{N} = 1\) smKdV Equation [10] [27]

\[\Phi \xi - 3(D\Phi) D^2(\Phi D\Phi) + D^4\Phi = 0 \tag{37}\]

with \(N\)-fermionic-parametric Bosonization procedure can be derived as

\[\Phi = \xi + \theta u = \sum_{n=1}^{N} v_{n \rightarrow \xi_{1}} \theta_{n} \cdots \theta_{n_{2-1}} + \theta \left( u_0 + \sum_{n=1}^{N} \sum_{n_{2n}} u_{n \rightarrow \xi_{1}} \theta_{n} \cdots \theta_{n_{2n}} \right) \tag{38}\]

where

\[u_{n \rightarrow \xi_{2n}} = \sqrt{k_2 - \omega_2 y^2 - 2C_2y - C_2} \left[ A_{n \rightarrow \xi_{2n}} + \int \frac{A_{n \rightarrow \xi_{2n}}}{(ky^2 - \omega y^2 - 2C_1y - C_2)} dy \right] \tag{39a}\]

\[v_{n \rightarrow \xi_{2n}} = B_{n \rightarrow \xi_{2n}} \xi_{0} + B_{n \rightarrow \xi_{2n}} \xi_{1} \sinh \left[ H(u) + B_{n \rightarrow \xi_{2n}} \xi_{1} \right] \]
\[+ \frac{\xi_{0}}{C_2} \int_{u_0}^{u_1} E_{n \rightarrow \xi_{2n}} \left\{ \cosh \left[ H(u) - H(y) \right] \right\} \frac{\sqrt{C_2}}{\sqrt{\omega y^2 + 2C_1y + C_2 - ky^2}} \sinh \left[ H(u) - H(y) \right] \right\} dy, \tag{39b}\]

with

\[F_{n \rightarrow \xi_{2n}} = \begin{cases} 3(u_{0}T[\xi_0, \xi_1]), & \text{for } n = 1, \\ -6(u_{0}T[u_0,u_0])_{xx} - 6T[u_0,u_0]_{x} + 3u_{0}T[\xi_0, \xi_1] + 3T[\xi_0, \xi_1, u], & \text{for } n = 2, \cdots, N_2; \end{cases} \tag{40a}\]

\[G_{n \rightarrow \xi_{2n+1}} = \begin{cases} 0, & \text{for } n = 1, \\ -3T[u_0,u_0 \xi] - 6u_0 T[u_0, \xi], & \text{for } n = 2, \\ -3T[u_0,u_0 \xi] - 3T[u_0,u_0 \xi] - 6u_0 T[u_0, \xi], & \text{for } n = 3, \cdots, N_2, \end{cases} \tag{40b}\]

and

\[H(z) = \int_{z}^{y} \frac{1}{y \sqrt{\omega y^2 + 2C_1y + C_2 - ky^2}} dy. \]

In summary, the bosonization approach with multi-fermionic parameters to
deal with supersymmetric systems is developed in the super Gardner equation with the role of traveling wave solution. The procedure and technique are also available for \( N = 1 \) sKdV and smKdV equations derived from two particular cases. We expect this procedure exhibited in our paper could be successfully applied or formulated in the \( N = 1 \) supersymmetric sine-Gordon equation, especially in the \( N = 2 \) version of KdV (SKdVa) equations \[8\] \[25\] \[28\] \[29\] \[30\]. For example, in the case of two fermionic parameters, letting 
\[
\phi = \phi_1 s_1 + \phi_2 s_2, \quad \psi = \psi_1 s_1 + \psi_2 s_2
\]
in bosonic field 
\[
\Phi = \frac{u}{2} + \theta \phi + \theta \psi - \xi \theta \sin \frac{u}{2},
\]
the simple traveling wave solutions of the \( N = 1 \) supersymmetric Sine-Gordon equation 
\[
D_x D_t \Phi = \sin \Phi
\]
are
\[
u = u_0 + \sqrt{E + \cos u_0} \left[ \frac{4D \cos \frac{u_0}{2} + b_1}{2(E + \cos u_0)^2} du + b_2 \right] s_1 s_2
\]
\[
S = \sum_{i=1}^{2} \left[ A_i \sin \frac{u_0}{2} + A_j \cos \left( \frac{2 \sin \frac{u_0}{2}}{\sqrt{2(E + \cos u_0)}} \right) \right] s_i
\]
where \( D_x = \partial_x + \theta \partial_x \) and \( D_t = \partial_t + \theta \partial_t \) are the usual super derivatives, \( S \) represents \( \phi \) and \( \psi \), \( E, B_1, B_2, A_1 \) and \( A_2 \) are integral constants, \( u_0 \) is a solution of the standard sine-Gordon equation.

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**Conflicts of Interest**

The authors declare no conflicts of interest regarding the publication of this paper.

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