The Discrete $Z_{2N_c}$ Symmetry And Effective Superpotential In SUSY Gluodynamics

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Abstract

We find an expression for the effective superpotential describing the $N_c$ vacua of $SU(N_c)$ SUSY gluodynamics. The superpotential reduces in some approximation to the Veneziano-Yankielowicz expression amended by the term restoring the discrete $Z_{2N_c}$ symmetry. Moreover, the superpotential, being restricted to one particular vacuum state, yields the expression which was derived recently to describe all the lowest-spin physical states of the theory. The corresponding scalar potential has no cusp singularities and can be used to study the domain walls interpolating between the chirally asymmetric vacua of the model.
Introduction

Supersymmetric gluodynamics, the theory of gluons and gluinos, seems to be an extremely useful testing ground for various nonperturbative phenomena occurring in conventional QCD. The Witten index of the $SU(N_c)$ SUSY gluodynamics equals to $N_c$ [1]. Thus, the ground state of the model consists of at least $N_c$ different vacua parametrized by the imaginary phase of a nonzero gluino condensate [1], [2]. The different vacua are related by discrete $Z_{2N_c}$ transformations of gluino fields. Once one of the $N_c$ vacua is chosen, the $Z_{2N_c}$ symmetry group spontaneously breaks down to the $Z_2$ subgroup. As a result of the discrete symmetry breaking one expects to find domain walls separating the $N_c$ vacua of the model.

Recently, Dvali and Shifman found that the $N = 1$ SUSY algebra admits some central extension if domain walls are present in the model [3], [4]. Thus, the domain walls saturating the BPS bound for the wall surface energy density might exist in the theory [3], [4].

There are yet another attractive arguments why BPS domain walls should be present in the model. Recently, $N = 1$ SUSY gluodynamics was realized [5] as a low-energy field theory emerging in a particular brane setup within the $M$ theory framework. In that picture the domain walls can be regarded as higher dimensional D-branes wrapped around some compactified dimensions [5]. The D-branes, being extended objects on which open strings can end in string theories [5], can also be viewed as BPS solitons in corresponding low-energy theories of supergravity [5]. Thus, in accordence with the Witten’s construction [5] the $N_c$ vacua of SUSY gluodynamics should be separated by the BPS domain walls on each of which color flux tubes (strings) can end [5]. This picture of the vacuum is quite attractive from the theoretical perspective as well as from the point of view of lattice simulations of SUSY Yang-Mills model where some indirect signatures of this construction could be observed [5].

The most straightforward way to study the vacuum structure would be to find explicitly the domain wall solutions (for recent reviews see Refs. [9], [10]). For that purpose one needs to have an effective action describing the $N_c$ vacua of the model.
The effective action for $N = 1$ SUSY Yang-Mills (SYM) was proposed by Veneziano and Yankielowicz (VY) \cite{VY}. The VY superpotential reproduces explicitly all the quantum anomalies of SUSY gluodynamics. However, it does not respect the discrete $Z_{2N_c}$ symmetry \cite{cusp} which is left once the chiral $U(1)_R$ invariance is broken by the axial anomaly.

In order to restore $Z_{2N_c}$ invariance the VY superpotential was amended by an additional term \cite{cusp}. The resulting expression is $Z_{2N_c}$ symmetric. However, the corresponding scalar potential possesses cusp singularities \cite{cusp}. These cusps are encountered in the field space as one interpolates between the $N_c$ vacua \cite{cusp}. For that reason the amended VY superpotential can not be used to describe the domain walls separating chirally asymmetric vacua \cite{cusp}. Moreover, considering SUSY YM with some heavy matter multiplets added (i.e. SUSY QCD with heavy flavors) and gradually integrating out those heavy states, one shows that the domain walls of the chirally asymmetric vacua cannot be found within the VY framework \cite{cusp}.

On the other hand, recent studies \cite{cusp} of the model which shares in the large $N_c$ limit some important features of SYM manifestly demonstrated the existence of BPS domain walls with the properties required in the brane construction \cite{cusp}.

Putting the whole set of arguments together one naturally concludes that it must be the VY framework which does not account adequately for all properties of the complicated ground state.

There is yet another reason to believe that the VY superpotential is not complete. In Refs. \cite{cusp}, \cite{cusp} it was shown that in order to account for all the lowest-spin excitations of the model, one necessarily needs to introduce an additional chiral superfield in the VY description \cite{cusp}.

The aim of this work is to use this additional chiral superfield to find an expression for the superpotential which would respect the $Z_{2N_c}$ invariance. The superpotential should lead as well to the scalar potential with no cusps. Moreover, in some approximation the superpotential should reduce to the known expression of Ref. \cite{cusp}. Finally, once restricted to some particular vacuum state it should yield the superpo-

\footnote{This assertion is valid modulo some assumptions on the vacuum structure and form of the Kähler potential, see discussions in Ref. \cite{cusp}.}
tential derived recently in Refs. [17], [18].

In the next section we show that such a superpotential can really be found.

1. The Discrete Symmetry and VY Superpotential

The classical action of $N = 1$ SYM theory is invariant under chiral, scale and superconformal transformations. Once quantum effects are taken into account, these symmetries are broken by the chiral, scale and superconformal anomalies respectively. Composite operators that appear in the expressions for the anomalies can be gathered into a composite chiral supermultiplet $\text{Tr} W^\alpha W_\alpha$ [19] (we use the notations of [20])

The effective action of the model can be a functional of the superfield

$$S \equiv \frac{\beta(g)}{2g} \langle \text{Tr} W^\alpha W_\alpha \rangle_Q \equiv A(y) + \sqrt{2} \theta \Psi(y) + \theta^2 F(y),$$

where the VEV is defined for nonzero value of an external (super)source $Q$ [21]. $\beta(g)$ stands for the SYM beta function which is known exactly [22]. The lowest component of the $S$ superfield $A$ is bilinear in gluino fields and has the quantum numbers of the scalar and pseudoscalar gluino-gluino bound states. The fermionic component in $S$ is related to the gluino-gluon composite and the $F$ component of the chiral superfield includes operators corresponding to both the scalar and pseudoscalar glueballs ($G_{\mu\nu}^2$ and $G_{\mu\nu} \tilde{G}^{\mu\nu}$ respectively) [11].

Assuming that the effective action (more precisely, the generating functional for one-particle-irreducible (1PI) Green’s functions [23]) of the model can be written in terms of the single superfield $S$, and requiring also that the effective action respects all the global continuous symmetries and reproduces the anomalies of the SYM theory, one derives the Veneziano-Yankielowicz effective superpotential [11]

$$W_{VY}(S) = \gamma S \ln \frac{S}{e\mu^3},$$

where $\gamma \equiv -(N_c g/16\pi^2 \beta(g)) > 0$, $\mu$ stands for the dimensionally transmuted scale of the model and $e \simeq 2.71$.

It was noticed in ref. [12] that the VY action does not respect the discrete $Z_{N_c}$.
symmetry – the nonanomalous remnant of anomalous $U(1)_{R}$ transformations. In order to make the action invariant under $Z_{N_{c}}$ transformations the VY superpotential was amended in Ref. [12] by the following term

$$\Delta W = i\gamma \frac{2\pi n}{N_{c}} S,$$

(2)

where $n$ enters in the action as an integer-valued Lagrange multiplier. The partition function of the theory should be regarded as a sum of path integrals where $n$ runs from $-\infty$ to $+\infty$ [12].

Thus, after the term (2) is included the action becomes $Z_{N_{c}}$ invariant [12]. The ground state of the model consists of at least $N_{c}$ different vacua labeled by different values of the phase of the gluino condensate. The resulting scalar potential which respects the discrete $Z_{N_{c}}$ symmetry can be written as [12], [13]

$$U(\phi) \propto (A^{\ast} A)^{2/3}\ln\left(\frac{A^{\ast}}{\mu^{3}} e^{2\pi n/N_{c}}\right)\ln\left(\frac{A}{\mu^{3}} e^{-i2\pi n/N_{c}}\right),$$

(3)

where

$$\frac{(2n - 1)\pi}{N_{c}} < \text{arg}(A) < \frac{(2n + 1)\pi}{N_{c}}.$$}

Thus, the complex plane of arg$(A)$ is divided into $N_{c}$ sectors. The potential is continuous in the plane, however it has cusps at arg$(A) = (2n + 1)\pi/N_{c}$ [12].

If one is restricting the superpotential to one particular vacuum state with some definite value of the phase of the gluino condensate, then the expression should account for all possible low-energy degrees of freedom of the theory. In Refs. [17], [18] it was argued that the VY Lagrangian should be modified further in order to include all the lowest-spin low-energy degrees of freedom of the $N = 1$ SUSY YM model. In fact, it was shown that to account for glueballs the effective superpotential should be defined in terms of two chiral supermultiplets [18]. The supermultiplet $S$ in that construction includes fields with quantum numbers of gluino-gluino “mesons” (along with the fermionic gluino-gluon state) while another chiral supermultiplet is needed

\[The actual discrete symmetry group of fermion field transformations is $Z_{2N_{c}}$. Since all the quantities below will be written in terms of fermion bilinears the symmetry reduces to $Z_{N_{c}}$.\]
to incorporate glueball states \[18\]

\[ W = W(S)_{VY} + W_1(S, \text{Another Chiral Superfield}). \]  \hspace{1cm} (4)

Thus, the second chiral superfield is needed to describe glueballs as excitations over one particular vacuum state \[18\]. In this respect, it would be nice to have that same superfield also restoring the discrete \( Z_{N_c} \) symmetry which is lost in the VY superpotential.

If this possibility is really realized, then the integer-valued Lagrange multiplier term (2) should be occurring once the new chiral superfield in (1) is integrated out. Some examples of this type were discussed in Ref. \[13\].

Hence, our goal is to find out an expression for the superpotential \( W \) as a function of two chiral superfields \( S \) and let us say \( X \) which would satisfy to the following requirements:

- The superpotential should be a homomorphic function of arguments;
- The superpotential should reproduce all the anomalies of the model, i.e. it should contain the VY superpotential as an ingredient \[11\];
- It should be invariant under the discrete \( Z_{N_c} \) transformations \[12\];
- The scalar potential should have at least \( N_c \) minima with broken chiral invariance;
- If the superfield \( X \) is integrated out, the superpotential should yield the expression (1) amended by the term (2);
- If the superpotential is restricted to one particular minimum with broken chiral symmetry, it should reproduce the generalized Veneziano-Yankielowicz superpotential derived in Refs. \[17\], \[18\].

We would like to argue that such a superpotential exists. The general form of the superpotential will be given in the next section. Here, we consider a simple expression. It can be obtained as a part of the general solution and should be regarded as a toy example used to elucidate the construction.
One defines
\[ W(S, X) \equiv \gamma S \ln \frac{S}{\epsilon \mu^3} + \gamma S \left( X - \frac{1}{N_c} \sinh(N_c X) \right), \] 
(5)

where the first term is nothing but the VY superpotential \([\text{I}]\). The second term is supposed to restore the discrete \(Z_{N_c}\) invariance of the VY superpotential. Notice that \(X\) is a dimensionless chiral superfield with zero \(R\) charge.

Let us now check whether the expression \((5)\) really satisfies to all the requirements listed above. First of all, the expression \((5)\) yields all the anomalies of the model; indeed, the first term in \((5)\) is just the VY superpotential which is designed to reproduce correctly the anomalies \([\text{II}]\). The second term does not contribute to the anomalies.

Consider the discrete \(Z_{N_c}\) transformations. The chiral superfield \(S\) transforms as
\[ S \rightarrow \exp\left(i \frac{2\pi k}{N_c}\right) S, \quad k = 0, 1, ..., N_c - 1. \]

As a result of this transformation the first term in the expression \((5)\) and its conjugate generate an additional term in the Lagrangian. This term has the form
\[ i \frac{2\pi k}{N_c} \gamma (S|_F - S^{|F^+}). \]

This expression can be eliminated by the following shift of the \(X\) superfield
\[ X \rightarrow X - i \frac{2\pi k}{N_c}. \] 
(6)

However, there is a restricted class of possible shifts which one is allowed to perform in the partition function of the model. The shifted fields, which along with the initial fields are being considered as physical ones, should also satisfy appropriate boundary conditions. In other words, the shifts we are discussing should be transforming the \(X\) field from one vacuum state to another. Anticipating the results of our discussions below, the vacuum values of the \(X\) field are just going to be multiples of \(i 2\pi/N_c\), thus the shifts \((6)\) do satisfy to the requirements set above. Hence, the superpotential \((5)\) really respects the discrete \(Z_{N_c}\) invariance. \(\text{†}\)

\(\text{†}\)One should also make sure that the Kähler potential of the model is invariant w.r.t. the shifts \((6)\). See the discussion of the Kähler potential in Section 3.
Let us now check what happens if one naively integrates out the $X$ field from the expression (5) (though, there is no physical reason to do that). The equation for $X$ minimizing the scalar potential is given as:

$$\frac{\partial W(S, X)}{\partial X} = \gamma S \left(1 - \cosh(N_c X)\right) = 0.$$  

Solving the equation for $X$ (at nonzero $S$ ) one finds

$$X_* = i\frac{2\pi n}{N_c}, \quad n = 0, \pm 1, \pm 2, \ldots, \pm \infty.$$  

Substituting this identity back into the superpotential (5) we derive

$$W(S, X_*) = \gamma S \ln S e^{\mu^3} + i\gamma \frac{2\pi n}{N_c} S.$$  

This expression is nothing but the VY superpotential (1) amended by the term (2) of Ref. [12]. Thus, the term (2) is obtained if the $X$ field is being integrated out. Later we will argue that the components of $X$ are related to glueballs. This excitations turn out to be lighter than the excitations described by the $S$ superfield [17], [18], so there is no physical reason to regard the components of the $X$ field as being integrated out. Thus, one should keep the $X$ field in the superpotential as a necessary ingredient.

The next step is to check whether the expression (5) produces the scalar potential with an appropriate $Z_{N_c}$ structure. Let us introduce the following notations

$$\Phi \equiv S^{1/3}, \quad Y \equiv X \Phi.$$  

Also, let us denote the components of the superfields $\Phi$, $X$ and $Y$ as $\phi$, $\phi_x$ and $\phi_y$ respectively. The superfields $\Phi$ and $Y$ have right physical dimensionality. The scalar potential $V$ can be written as a sum of two terms

$$V(\phi, \phi_y) = V_1(\phi, \phi_y) + V_2(\phi, \phi_y),$$

\footnote{In deriving the scalar potential one should actually switch to chiral superfields with an appropriate dimensionality $\Phi \equiv S^{1/3}$ and $Y \equiv X \Phi$ and calculate minima w.r.t. those superfields (in this particular case the answer is the same, see below).}

\footnote{The change of variables from $S$ to $\Phi$ is nonsingular in our case since we are dealing only with the phase of the theory where the VEV of the lowest component of $S$ is nonzero.}
where

\[ V_1(\phi, \phi_y) \propto |\frac{\partial W(\phi, \phi_y)}{\partial \phi_y}|^2 = 9\gamma^2 |\phi|^4 \left| \ln \frac{\phi^3}{\mu^3} + \phi_x - \frac{1}{N_c} \sinh(N_c \phi_x) - \frac{\phi_x}{3} [1 - \cosh(N_c \phi_x)] \right|^2, \]

\[ V_2(\phi, \phi_y) \propto \left| \frac{\partial W(\phi, \phi_y)}{\partial \phi_y} \right|^2 = \gamma^2 |\phi|^4 \left| 1 - \cosh(N_c \phi_x) \right|^2. \]

In these equations the substitution \( \phi_y = \phi_x \) is used.

After the potential is set one can list all the vacuum states of the model. All those configurations should satisfy to the equations \( V_1 = V_2 = 0 \). As one expects, there are \( N_c \) different vacua with broken chiral symmetry:

\[ |\phi| = \mu, \quad \text{Re} \phi_x = 0, \quad \text{Im} \phi_x = -3 \arg \phi = \frac{2\pi k}{N_c}, \quad k = 0, 1, \ldots N_c - 1. \]

These vacua differ from each other by the value of the phase of the gluino condensate

\[ \langle \lambda \lambda \rangle_k \propto \mu^3 \exp \left( i \frac{2\pi k}{N_c} \right), \quad k = 0, 1, \ldots, N_c - 1. \]

Interpolating from one vacuum state to another one no cusp singularities are encountered in (8) and (9). The presence of the chiral field \( X \) smooths out cusp singularities emerging in the case when \( X \) is being integrated out.

The next question we would like to elucidate is the physical interpretation of the new chiral superfield. The \( S \) superfield is related to the operator \( \text{Tr} W^2 \). The lowest component of \( S \) can be thought of as an interpolating field for a gluino-gluino bound state. The question is whether one can find analogous identifications for the components of the new chiral superfield. In order to clarify this question let us

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*In the expressions (8) and (9) the exact proportionality coefficients are set by the inverse metric defined as the second derivative of the Kähler potential (see Ref. [20]).

† One also finds that there is a vacuum state with the zero value of \( |\phi| \). The existence of the vacuum state with no gluino condensate was conjectured in Ref. [12]. The question whether this phase of the model can actually be realized in the fundamental theory is a subject of recent discussions [24], [23], [13]. In this work we concentrate on the vacua with the nonzero gluino condensate only.*
recall some results of Ref. [18]. In Ref. [18] an effective action describing physical excitations of the SUSY YM model in one of the $N_c$ vacua was constructed. The superpotential of Ref. [18] is also written in terms of two chiral superfields, $S$ and some chiral superfield $\chi$. The first superfield was shown to describe gluino-gluino and gluon-gluino bound states, while the second one was needed to include pure gluonic, glueball states into the description. We shall argue below that the superfield $X$ in (3) is also related to glueballs and is just a necessary ingredient of the effective superpotential (3). In other words, we are going to show here that the superpotential (3) reproduces the expression of Ref. [18] in the limit when one is restricted to some particular chirally asymmetric vacuum. To accomplish this task let us introduce the following notation:

$$\chi \equiv 16\gamma\left(X - \frac{1}{N_c}\sinh(N_cX)\right).$$

The $X$ field is a dimensionless chiral superfield and so is $\chi$. One rewrites the second term in the superpotential (5) in the following form:

$$\frac{1}{16} \chi S = \frac{1}{16N_c} \sum_{k=0}^{N_c-1} \chi \left( S - \langle S \rangle_k \right),$$

where

$$\langle S \rangle_k \equiv \mu^3 \exp\left(i \frac{2\pi k}{N_c}\right).$$

Using these identities the superpotential (3) can be presented in yet another very useful form

$$\mathcal{W}(S, \chi(X)) = \gamma S \ln\frac{S}{e\mu^4} + \frac{\chi(X)}{16N_c} \sum_{k=0}^{N_c-1} \left( S - \langle S \rangle_k \right).$$

The expression (13) makes it transparent that the superpotential we are discussing can be obtained as a sum of superpotentials defined for each particular $N_c$ vacuum state. Each of this vacua are labeled by the VEV of the gluino condensate with an appropriate phase. The initial $Z_{2N_c}$ symmetry is spontaneously broken down to $Z_2$ in each of these vacuum states. It is straightforward to determine how the expression (13) looks like when one restricts consideration to some particular vacuum state only. In that case one assumes that the VEV of the $S$ field takes a single value, let us say
for simplicity $\langle S \rangle_k = \mu^3$. Then the expression (12) reduces to the following formula

$$\frac{\chi(X)}{N_c} \sum_{k=0}^{N_c-1} \left( S - \langle S \rangle_k \right) \to \chi(X) \left( S - \mu^3 \right).$$

Substituting this expression back into Eq. (13) one derives the following superpotential

$$\mathcal{W}(S, \chi(X)) \equiv \gamma S \ln \frac{S}{e\mu^3} + \frac{1}{16} \chi(X) \left( S - \mu^3 \right). \quad (14)$$

This is exactly the superpotential obtained in Ref. [18]. It describes the vacuum state of the model with broken chiral invariance where the phase of the gluino condensate equals to zero. The initial $Z_{N_c}$ symmetry in this vacuum is broken down to $Z_2$.

Let us now return to our original question about the physical interpretation of the components of the $X$ field. The components of the $X$ field are related to the components of $\chi$ (Eq. (11)). On the other hand, the $\chi$ field is related to glueball excitations of the $N = 1$ SUSY YM model [18]. Thus, the components of the $X$ field should also be related to the VEV’s of the $G^2_{\mu\nu}$ and $G_{\mu\nu} \tilde{G}_{\mu\nu}$ composite operators [1].

Thus, if one is restricted to study physics about some particular vacuum state of the model, then the information about the whole $Z_{N_c}$ structure is lost. As a result, the $\chi$ field can be introduced in accordance with Eq. (11), and all the physical excitations about that ground state can be described in terms of components of the $\Phi$ and $\chi$ multiplets.

In our discussions we could have started from the generalized VY superpotential of Ref. [18] and derived the $Z_{N_c}$ symmetric superpotential (13) (and (3)). Indeed, the superpotential (14) describes physics of only one particular ground state with a

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‡‡The explicit derivation is cumbersome and can be done using some relations between $\chi$ and the tensor supermultimultiplet used in Refs. [17], [18]. For the completeness of discussions we present here simple approximation results of Refs. [17], [18]. For the lowest component $\phi_\chi$ $\partial^2 \text{Re} \phi_\chi \simeq -\frac{1}{8\mu^2} \frac{\beta(\alpha)}{2g} \langle G^2_{\mu\nu} \rangle_s$, $\partial^2 \text{Im} \phi_\chi \simeq \frac{1}{8\mu^2} \frac{\beta(\alpha)}{2g} \langle G_{\mu\nu} \tilde{G}_{\mu\nu} \rangle_p$, where the VEV’s are functions of some external fields $s$ and $p$. Thus, the real part of $\phi_\chi$ bears quantum numbers of a scalar $0^{++}$ pure gluonic state, while the imaginary part is associated with the pseudoscalar $0^{-+}$ glueball state [17], [18].

‡‡The statement that the components of the $\chi$ multiplet are related to glueballs assumes that the corresponding Kähler potential is a function of the sum $\chi + \chi^+$, see the detailed discussions below.
definite value of the phase of the gluino condensate \([18]\). If one would need to generalize that expression to include all the possible vacua, one should have summed the expression (14) w.r.t. the phase labeling all the vacua. The result of this summation is given in Eq. (12), the whole sum reduces to the quantity \(\chi \frac{S}{16}\). Thus, one would have arrived at the superpotential (4) written in terms of the \(\Phi\) and \(\chi\) fields (without any knowledge about the \(X\) superfield). Obviously, the question is how would one discover in this approach that there is a substructure relating the \(\chi\) field to the \(X\) field (as in Eq. (11))? The structure \(\chi X\) would emerge as a result of the \(Z_{N_c}\) symmetry and also supersymmetry itself. Indeed, if the superpotential (13) is written in terms of the \(\Phi\) and \(\chi\) fields alone, and if the \(\chi\) field is regarded as some fundamental field, then, one can check it explicitly that the resulting scalar potential would not produce a supersymmetric minimum with a nonzero value of the gluino condensate. Neither would it yield the correct \(Z_{N_c}\) invariance. Thus, in order to overcome these difficulties, one would postulate the relation analogous to (11) and declare \(X\) as the field with respect to which the variation of the superpotential should be taken.

Thus, one concludes that the relation (11) is a result of two symmetries: \(Z_{N_c}\) and supersymmetry.

Before we turn to the next section it is crucial to present yet another form of the superpotential (3). One can rewrite it as

\[
W(S, M(X)) \equiv \gamma S \ln \frac{S}{eM(X)},
\]

where the chiral superfield \(M(X)\) is defined in terms of \(X\)

\[
M(X) \equiv \mu^3 \exp \left( -X + \frac{1}{N_c} \sinh(N_c X) \right).
\]

The superpotential (15) has the VY form. Since the \(X\) field is a dimensionless field with zero \(R\) charge, the expression (15) is consistent with all the requirements of the VY construction (11). The only difference is that the scale parameter of the theory \(\mu\) is promoted into some chiral superfield \(M\). The relation between \(M\) and \(\mu\) is such that the \(X\) field can be regarded as a dynamical field setting the value of the phase

\[\text{Modulo the fact that both } \chi \text{ and } X \text{ are dimensionless fields and while deriving scalar potentials one should always be working in terms of rescaled fields with an appropriate dimensionality.}\]
of the gluino condensate. This is reminiscent to the case when the VEV of some dynamical field of a bigger theory can be regarded as a parameter in some effective theory approximation.

2. The General Solution

In this section we derive the general expression for the superpotential which satisfies to the properties listed in the previous section. Our ultimate goal would be to set an expression in the form \( \mathcal{W}(S, X) \), where the dependence of \( M \) on \( X \) would be given by a general function compatible with the conditions of the problem.

Thus, we are looking for a superpotential in the following form

\[
\mathcal{W}(S, X) = \gamma S \ln \left( \frac{S}{\varepsilon \mu^3} \right) + \gamma S \mathcal{F}(X).
\]

This expression can be written as \( S \) times some natural logarithm (as in \( (15) \)). The \( S \) superfield has \( R \) charge equal to 2 and the mass dimension equal to 3. Thus, the \( X \) field is a dimensionless field with zero \( R \) charge.

One requires that the superpotential is invariant under the discrete \( Z_{N_c} \) transformations. Under these transformations the \( VY \) part of the superpotential produces the term discussed in the previous section. The function \( \mathcal{F}(X) \) should be chosen in such a way that it would allow one to eliminate that term; i.e., there should exist a shift of the variable with the following property

\[
\mathcal{F}(X + \text{shift}) = \mathcal{F}(X) + \frac{i 2\pi k}{N_c}, \quad k = 0, 1, \ldots, N_c - 1.
\]

On the other hand, it should be possible to perform discrete shifts only; indeed, under any continuous \( U(1)_R \) transformation the superpotential will be producing the anomaly expression, this expression can not be eliminated. In other words, it should be allowed to “undo” the discrete transformations of \( S \) by shifting \( X \), however, the continuous transformations of \( S \) should not be possible to be eliminated. As we discussed it in the previous section not all of the shifts of variables are allowed. Shifted fields should satisfy appropriate boundary conditions. Thus, as before, the shifts are to be transforming values of the \( X \) field from one vacuum state to another.
We will make sure that this is the case here. Thus, the shifts we are looking for are of the form
\[
F(X + a \cdot k) = F(X) + i \frac{2\pi k}{N_c}, \quad k = 0, 1, \ldots N_c - 1, \tag{17}
\]
where \(a\) is some nonzero complex number.

The next constraint we are going to impose on the function \(F(X)\) is the following. One requires that there are \(N_c\) different minima of the potential. Thus, the equation
\[
\frac{\partial F(X)}{\partial X} \bigg|_{X_k} = 0, \tag{18}
\]
should have \(N_c\) different solutions for \(X_k\). The shifts in \((17)\) are supposed to transform these solutions into one another.

Once the \(X\) field is integrated out, the resulting additional term in the superpotential have to coincide with the term \((3)\) introduce in Ref. \([12]\). Thus, we get one more condition on the function \(F\):
\[
F(X_k) = i \frac{2\pi k}{N_c}, \quad k = 0, 1, \ldots N_c - 1. \tag{19}
\]

The solution of Eq. \((17)\) is a sum of its particular solution and a general solution of the corresponding homogeneous equation\(^\dagger\) (let us denote it as \(G(X)\)):
\[
F(X) = i \frac{2\pi}{N_c a} X + G(X). \tag{20}
\]
In terms of the function \(G(X)\) the expressions \((17) - (19)\) can be rewritten as
\[
G(X + a \cdot k) = G(X), \tag{21}
\]
\[
\frac{\partial G(X)}{\partial X} \bigg|_{X_k} = -i \frac{2\pi}{N_c a}, \tag{22}
\]
\[
G(X_k) = -i \frac{2\pi}{N_c a} X_k + i \frac{2\pi k}{N_c}. \tag{23}
\]
The solution to \((21)\) is a superposition of exponential functions which we choose to normalize as follows
\[
G(X) = - \frac{1}{N_c} \sum_{n=-\infty}^{+\infty} c_n \exp\left(\frac{i n}{a} X\right). \tag{24}
\]
\(^\dagger\)This statement is valid for infinitely differentiable analytic functions.
Substituting Eq. (24) into Eqs. (22) and (23) one derives respectively

\[ \sum_{n=-\infty}^{+\infty} n c_n \exp\left( in \frac{2\pi}{a} X_k \right) = 1, \]  
\[ \sum_{n=-\infty}^{+\infty} c_n \exp\left( in \frac{2\pi}{a} X_k \right) = i \frac{2\pi}{a} X_k - i2\pi k. \]  

Analyzing Eqs. (20), (24–26) it is convenient to introduce the following rescaling of the \( X \) field

\[ i \frac{2\pi}{N_c a} X \rightarrow X. \]

Under the shifts discussed above the new variable \( X \) transforms as

\[ X \rightarrow X + i \frac{2\pi}{N_c} k, \quad k = 0, 1, ... , N_c - 1. \]  

In terms of this variable the expression for the function \( \mathcal{F}(X) \) looks as

\[ \mathcal{F}(X) = X + \mathcal{G}(X), \]  

and the expressions (24–26) take the form

\[ \mathcal{G}(X) = -\frac{1}{N_c} \sum_{n=-\infty}^{+\infty} c_n \exp\left( nN_c X \right), \]  
\[ \sum_{n=-\infty}^{+\infty} n c_n \exp\left( nN_c X_k \right) = 1, \]  
\[ \sum_{n=-\infty}^{+\infty} c_n \exp\left( nN_c X_k \right) = N_c X_k - i2\pi k. \]

Let us now consider Eq. (30). This equation should have \( N_c \) different solutions for \( X_k \) describing the \( N_c \) vacua. It is convenient to introduce the notation: \( v \equiv \exp\left( N_c X_k \right) \).

In terms of \( v \) Eq. (30) could generically have an arbitrary big number of solutions. On the other hand, for each nonzero solution in terms of \( v \) there are \( N_c \) different solutions in terms of \( X \); indeed, if the expression \( v = |v| \exp(i \arg v) \) is a solution for \( v \), then using the relation \( v = \exp(N_c X_k) \), one finds \( N_c \) different solutions for the imaginary
part of $X_k$. Thus, in order to have only $N_c$ different solutions of Eq. (30) in terms of $X$, the algebraic Eq. (30) should have a single nonzero, multiply degenerate root for $v$. Let us denote this root as $v \equiv \alpha \exp(i\rho)$, with $\alpha$ and $\rho$ being some constants. Then, for the solutions $X_k$ one gets:

$$\text{Im}X_k = \frac{\rho}{N_c} + \frac{2\pi}{N_c}k, \quad \exp(N_c\text{Re}X_k) = \alpha.$$ 

One can check now that the shifts (6) really transform values of $X$ from one vacuum state to some another one. For simplicity of arguments, in what follows, it is convenient to choose $\alpha = 1$ and $\rho = 0$. This corresponds to some shifts of the $X$ complex coordinate system. In that case one derives the following relations

$$X_k = i\frac{2\pi}{N_c}k, \quad \sum_n n c_n = 1, \quad \sum_n c_n = 0. \quad (32)$$

The particular solution of the previous section corresponds to the case when $c_1 = 1/2$, $c_{-1} = -1/2$ and all other $c$’s are set to be equal to zero.

Summarizing, we can write down a general form of the superpotential as

$$W(S, M(X)) \equiv \gamma S \ln \frac{S}{eM(X)}, \quad (33)$$

where the field $M$ is given by the relation

$$M(X) \equiv \mu^3 \exp\left(-F(X)\right). \quad (34)$$

The function $F(X)$ is defined in accordance with Eqs. (28), (29–31) and the solutions for the vacua in terms of the rescaled variable are given in Eq. (32).

3. A Brief Comment on the Kähler Potential

So far we did not discuss what kind of Kähler potential $K(S+S, X^+, X)$ is supposed to be used in the effective action for $N = 1$ SUSY YM model [1]. There are no symmetry or anomaly arguments which would uniquely fix the form of $K$. However, 

Terms with derivatives of the superfields in the Kähler potential might lead to an unbounded from below potential in this case [2]. For that reason we consider $K$ as a function of the superfields only with no derivatives.
there is some piece of information one could still learn about the Kähler potential. We would like to elaborate on this point here.

In order to make the Kähler potential invariant under the shifts (6) one requires that \( \mathcal{K}(S^+, X^+, X) \) is actually a function of the sum of \( X^+ \) and \( X \), \( \mathcal{K}(S^+, X^+, X) = \mathcal{K}(S^+, X^++X) \). However, this is not the only form of the expression which is invariant w.r.t. the shift (6). For instance, the Kähler potential could also be a function of the sum of \( F = X + G(X) \) and its conjugate \( F^+(X^+) \). The sum \( X + X^+ \) does not change upon the shifts (6) and the function \( G \) itself is invariant under those transformations. Thus, one could write as a possibility

\[
\mathcal{K}(S^+, X^+, X) = \mathcal{K}(S^+, F(X) + F^+(X^+)).
\]

We would like to argue here that this type of dependence of the Kähler potential is in fact what is dictated by the physical particle content of the low-energy spectrum of the theory [17], [18].

First let us show that the same combination \( F(X) + F^+(X^+) \) appears in the expression for the superpotential we derived in the previous sections. The part of the superpotential containing the chiral superfield \( F(X) \) is written as \( \gamma S F(X) \). This can be presented in the following manner

\[
\frac{\gamma}{N_c} \sum_{k=0}^{N_c-1} F(X) \left( S - \langle S \rangle_k \right). \tag{35}
\]

Then one introduces a real superfield \( U_k \) \[20\]

\[
U_k = B + i\theta \chi - i\bar{\theta} \bar{\chi} + \frac{\theta^2}{16} (A^* - \langle S \rangle_k^*) + \frac{\bar{\theta}^2}{16} (A - \langle S \rangle_k) + \frac{\theta\sigma^\mu\bar{\theta}}{48} \varepsilon_{\mu\nu\alpha\beta} C^{\nu\alpha\beta} +
\]

\[
\frac{1}{2} \theta^2 \bar{\theta} \left( \frac{\sqrt{2}}{8} \Psi + \bar{\sigma}^\mu \partial_\mu \chi \right) + \frac{1}{2} \bar{\theta}^2 \theta \left( \frac{\sqrt{2}}{8} \Psi - \sigma^\mu \partial_\mu \bar{\chi} \right) + \frac{1}{4} \theta^2 \bar{\theta}^2 \left( \frac{1}{4} \Sigma - \partial^2 B \right), \tag{36}
\]

which is related to the superfield \( S \)

\[
S - \langle S \rangle_k = -4 \bar{D}^2 U_k. \tag{37}
\]

The \( F \) term of the chiral supermultiplet \( S \) is related to the fields \( \Sigma \) and \( C_{\mu\nu} \) in the following way\[17\]

\[
F = \Sigma + \frac{1}{6} \varepsilon_{\mu\nu\alpha\beta} \partial^\mu C^{\nu\alpha\beta},
\]

\[\text{In this notation } \Sigma \text{ is proportional to } G_{\mu\nu}^2 \text{ and } \varepsilon_{\mu\nu\alpha\beta} \partial^\mu C^{\nu\alpha\beta} \text{ is proportional to } G_{\mu\nu} \bar{G}^{\mu\nu} \[17\].\]
and \( A \) and \( \Psi \) are respectively the scalar and fermion components of the superfield \( S \).

One substitutes the expression for \( S \) in terms of \( U_k \) into Eq. (35). Then one replaces the \( \bar{D}^2 \) operator by the integration w.r.t. the \( \theta \) variable. Finally, putting the resulting expression together with its hermitian conjugate part one derives

\[
\frac{\gamma}{N_c} \sum_{k=0}^{N_c-1} \mathcal{F}(X) \left( S - \langle S \rangle_k \right) \Big|_F + \text{h.c.} = \frac{16\gamma}{N_c} (\mathcal{F}(X) + \mathcal{F}^+(X^+)) \sum_{k=0}^{N_c-1} U_k \Big|_D.
\]

Thus, all the terms in the Lagrangian of the model containing the chiral superfield \( \mathcal{F}(X) \) depend actually on the real combination \( \mathcal{F}(X) + \mathcal{F}^+(X^+) \). This combination can be integrated out using equations of motion \[26\], \[18\]. The equation of motion for the real superfield \( \mathcal{F}(X) + \mathcal{F}^+(X^+) \) leads to the following relation

\[
\frac{16\gamma}{N_c} \sum_{k=0}^{N_c-1} U_k = -\frac{\partial \mathcal{K}(S^+S, Z)}{\partial Z} \bigg|_{Z=\mathcal{F}(X) + \mathcal{F}^+(X^+)}.
\]

Thus, the whole Lagrangian can in principle be presented in terms of the degrees of freedom of the real tensor supermultiplet \( U_k \). Indeed, Eq. (37) sets how the components of \( S \) are related to some components of \( U_k \), and likewise, Eq. (38) gives the relation between the components of the chiral superfield \( \mathcal{F} \) (or \( X \)) and the components of the superfield \( U_k \). This is in agreement with the statement of Ref. [17] where it was shown that all the lowest-spin physical degrees of freedom of SUSY gluodynamics can be described by one real tensor supermultiplet \( U \equiv (\sum_{k=0}^{N_c-1} U_k)/N_c \).

One should notice that once the \( \mathcal{F} \) field, being appropriately rescaled, is considered as an independent fundamental field of the Lagrangian for which Eq. (38) is to be solved, the whole information on the \( Z_{N_c} \) vacuum structure is lost and one is simply dealing with some particular ground state. It is the definition of \( \mathcal{F} \) in terms of \( X \) that makes the \( Z_{N_c} \) structure feasible and Eq. (38) should actually be solved for the \( X \) field being appropriately rescaled. In other words, the information about the \( Z_{N_c} \) structure in this case is encoded in the relations between \( U_k \) and \( S \) and \( U_k \) and \( X \).

We conclude that the form of the Kähler potential which is dictated by the particle content of the model in some particular vacuum state [17], [18] is consistent with the symmetry requirements we have used to derive the effective superpotential in the previous sections.
Discussions

We derived the effective superpotential for SUSY gluodynamics which correctly reproduces the known properties of the complicated ground state structure of the model. The corresponding scalar potential is a smooth function of arguments and yields $N_c$ different vacua with the broken chiral invariance. The discrete $Z_{N_c}$ transformations shift one vacuum state into another one. The superpotential is given in terms of two chiral superfields. Once one superfield is integrated out, the superpotential reduces to the expression given in Ref. [12]. On the other hand, if one is restricted to study the excitations about some particular vacuum state with the nonzero gluino condensate only, then the superpotential reduces to the known expression of Ref. [18]. This last adequately describes all the lowest-spin degrees of freedom of the model [18]. The superpotential (5) can formally be brought to the original VY logarithmic form (see Eqs. (15) and (33)). In this case one could think of the VY superpotential where the scale parameter of the model $\mu$ is promoted into some dynamical chiral superfield. The VEV of the phase of that superfield would set the value of the phase of the gluino condensate. This superfield, as we have shown, is related to pure gluonic operators. Finally, as we mentioned before, the superpotential can be used to study the domain walls separating the chirally asymmetric vacua of the theory. The results of those studies will be reported elsewhere.

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