Cardinal arithmetic and Woodin cardinals

Ralf Schindler

Institut für Formale Logik, Universität Wien, 1090 Wien, Austria

rds@logic.univie.ac.at

http://www.logic.univie.ac.at/~rds/

Abstract

Suppose that there is a measurable cardinal. If $2^{\aleph_1} < \aleph_\omega$ but $\aleph_0^{\aleph_0} > \aleph_\omega$, then there is an inner model with a Woodin cardinal. This essentially answers a question of Gitik and Mitchell (cf. [GiMi96, Question 5, p. 315]).

We refer the reader to AbMax for an introduction to cardinal arithmetic and to Shelah’s pcf theory (cf. also BuMa90). The perhaps most striking result of Shelah’s in cardinal arithmetic is that if $\aleph_\omega$ is a strong limit cardinal then

$$2^{\aleph_\omega} < \min(\aleph_{(2^{\aleph_0})^+}, \aleph_{\omega_4}).$$

Magidor was the first one to produce a model of set theory in which the GCH holds below $\aleph_\omega$, but $2^{\aleph_\omega} = \aleph_{\omega+2}$ (cf. Ma77a, Ma77b). It is now known how to produce models in which there are arbitrarily large countable gaps between $\aleph_\omega$ and $2^{\aleph_\omega}$, while the GCH holds below $\aleph_\omega$ (cf. for instance GiMa92). A strong cardinal is more than enough for this purpose. In fact, many equiconsistencies are known. We refer the reader to Gi∞ and Mi∞ and to the references given there.

It is, however, open if it is possible to have that $\aleph_\omega$ is a strong limit cardinal, but $2^{\aleph_\omega} > \aleph_{\omega_1}$ (cf. Gi∞, Section 7, Problem 1). This problem is just one of the key open problems of pcf theory in disguise. Gitik and Mitchell have shown that a strong cardinal is not enough for producing such a model:

**Theorem 0.1** ([GiMi96, Theorem 5.1]) If $2^{\aleph_0} < \aleph_\omega$ and $\aleph_\omega^{\aleph_0} > \aleph_{\omega_1}$ then there is a sharp for a model with a strong cardinal.

The purpose of this note is to prove the following theorem which in a certain sense improves Theorem 0.1. It will say that you will need at least a Woodin cardinal in order to produce a model in which $\aleph_\omega$ is a strong limit cardinal, but $2^{\aleph_\omega} > \aleph_{\omega_1}$.

As far as we know this is the first statement in cardinal arithmetic which is known to practically imply the consistency of a Woodin cardinal.

**Theorem 0.2** If $2^{\aleph_1} < \aleph_\omega$, $\aleph_\omega^{\aleph_0} > \aleph_{\omega_1}$, and there is a measurable cardinal then there is an inner model with a Woodin cardinal.
Our proof of Theorem 0.2 will make use of Shelah’s pcf theory. Specifically, we’ll need the following theorems which are due to Shelah. Recall that if \( a \) is a set of regular cardinals then \( \text{pcf}(a) \) is the set of all possible cofinalities of \( \prod a/U \) where \( U \) is an ultrafilter on \( a \).

**Theorem 0.3** ([BuMa90, Theorem 5.1]) Let \( 2^{\aleph_0} < \aleph_\omega \). Then \( \text{pcf}(\{\aleph_n; n < \omega\}) = \{\kappa < \aleph_\omega^\omega; \kappa \text{ is regular} \} \).

**Theorem 0.4** ([BuMa90, Theorem 6.10]) Let \( 2^{\aleph_0} < \aleph_\omega \). Let \( d \subseteq \text{pcf}(\{\aleph_n; n < \omega\}) \) and \( \mu \in \text{pcf}(d) \). There is then some \( d' \subset d \) such that \( \text{Card}(d') = \aleph_0 \) and \( \mu \in \text{pcf}(d') \).

We shall also need the following simple “combinatorial” fact, Lemma 0.5. Let \( \kappa \) and \( \lambda \) be cardinals with \( \lambda \leq \kappa \). \( H_\kappa \) is the set of all sets which are hereditarily smaller than \( \kappa \), and \( [H_\kappa]^\lambda \) is the set of all subsets of \( H_\kappa \) of size \( \lambda \). Recall that \( S \subseteq [H_\kappa]^\lambda \) is called stationary if and only if for all models \( M = (H_\kappa; \ldots) \) of finite type and with universe \( H_\kappa \), there is some \( X \subseteq S \) such that \( X \) is the universe of an elementary submodel of \( M \), i.e., \( (X; \ldots) \prec M \). We say that \( S \subseteq [H_\kappa]^\lambda \) is \( \ast \)-stationary if and only if \( \forall \) \( S \cap \{x \subseteq x\} \) is stationary, i.e., if for all models \( M = (H_\kappa; \ldots) \) of finite type and with universe \( H_\kappa \), there is some \( X \subseteq S \) such that \( \omega X \subset X \) and \( X \) is the universe of an elementary submodel of \( M \). We let \( NS_{\omega_1} \) denote the non-stationary ideal on \( \omega_1 \).

**Lemma 0.5** Let \( \kappa \geq 2^{\aleph_1} \) be regular, and let \( \Phi: [H_\kappa]^{2^{\aleph_1}} \to NS_{\omega_1} \). There is then a pair \((C, S)\) such that \( C \) is a closed unbounded subset of \( \omega_1 \), \( S \) is \( \ast \)-stationary in \([H_\kappa]^{2^{\aleph_1}}\), and \( C \cap \Phi(X) = \emptyset \) for all \( X \subseteq S \).

**PROOF.** Suppose that for every club \( C \subseteq \omega_1 \) the set
\[
\{X \in [H_\kappa]^{2^{\aleph_1}}; C \cap \Phi(X) = \emptyset\}
\]
is not \( \ast \)-stationary. This means that for every club \( C \subseteq \omega_1 \) there is a model \( M_C \) of finite type and with universe \( H_\kappa \) such that for every \( (X; \ldots) \prec M_C \) with \( \omega X \subset X \) and \( \text{Card}(X) = 2^{\aleph_1} \), we have that \( C \cap \Phi(X) \neq \emptyset \). As there are only \( 2^{\aleph_1} \) many subsets of \( \omega_1 \), there is a model \( M \) of finite type and with universe \( H_\kappa \) such that for every club \( C \subseteq \omega_1 \), if \( (X; \ldots) \prec M \) is such that \( 2^{\aleph_1} \subset X \) then \( (X; \ldots) \prec M_C \) for every club \( C \subseteq \omega_1 \). Pick \( (X; \ldots) \prec M \) with \( \omega X \subset X \), \( \text{Card}(X) = 2^{\aleph_1} \), and \( 2^{\aleph_1} \subset X \). We shall have that \( C \cap \Phi(X) \neq \emptyset \) for every club \( C \subseteq \omega_1 \), which means that \( \Phi(X) \) is stationary. Contradiction! \( \square \)

Our proof of Theorem 0.2 will use the core model theory of [St96]. The basic idea for its proof will be the following. We shall first use Theorems 0.3 and 0.4 as
well as Lemma 0.5 for isolating a “nice” countable set $d' \subset \{ \aleph_{\alpha+1} : \alpha < \omega_1 \}$ with $\aleph_{\omega_1+1} \in \pcf(d')$. We shall then use a covering argument to prove that $\pcf(d') \subset \aleph_{\omega_1}$ yielding the desired contradiction. However, the covering argument plays the key role in choosing the “nice” $d'$ we start with.

**Proof** of Theorem 0.2. Suppose not. Let $\Omega$ be a measurable cardinal, and let $K$ denote Steel’s core model of height $\Omega$ (cf. [St96]). Let $\kappa > \aleph_{\omega_1}$ be a regular cardinal.

Fix $X \prec H_\kappa$ for a while, where $\omega X \subset X$. Let $\pi : H_X \cong x \prec H_\kappa$ be such that $H = H_X$ is transitive. Let $\bar{K} = K_X = \pi^{-1}(K||\aleph_{\omega_1})$. We know by [MiScSt97] that there is an $\omega$-maximal normal iteration tree $T$ on $K$ (of successor length) such that $\mathcal{M}_\kappa^T \not\succeq \bar{K}$. Let $T = T_X$ be the shortest such tree.

If $E$ is an extender then we shall write $\nu(E)$ for the natural length of $E$ (cf. [MiSt94, p. 6]). We shall set

$$\Phi(X) = \{ \alpha < \omega_1 : \exists \beta + 1 \in (0, \infty) \upharpoonright T \ (\text{crit}(E^T_\beta) < \aleph^H_\alpha \leq \nu(E^T_\beta)) \}.$$  

We aim to apply Lemma 0.5 to $\Phi$.

**Claim 1.** $\Phi(X)$ is a non-stationary subset of $\omega_1$.

**Proof.** Suppose that $S_0 = \Phi(X)$ is stationary. Let $S$ be the set of all limit ordinals of $S_0$. $S$ is stationary, too. Let $F : S \to \text{OR}$ be defined by letting $F(\alpha)$ be the least $\bar{\alpha} < \alpha$ such that $\text{crit}(E^T_{\bar{\alpha}}) < \aleph^H_\alpha$, where $\beta + 1 \in (0, \infty) \upharpoonright T$ is unique such that $\text{crit}(E^T_{\bar{\alpha}}) < \aleph^H_\alpha \leq \nu(E^T_{\bar{\alpha}})$. Let $\bar{S} \subset S$ be stationary such that $F \upharpoonright \bar{S}$ is constant. For $\alpha \in \bar{S}$ let $\beta(\alpha)$ be the unique $\beta$ such that $\text{crit}(E^T_{\bar{\alpha}}) < \aleph^H_\alpha \leq \nu(E^T_{\bar{\alpha}})$.

Using the initial segment condition ([MiSt94, Definition 1.0.4 (5)] (cf. also [SchStZa, Definition 2.4]) it is easy to see that we must have $\text{crit}(E^T_{\bar{\alpha}(\alpha)}) = \text{crit}(E^T_{\bar{\alpha}(\alpha')})$ whenever $\{\alpha, \alpha'\} \subset \bar{S}$. Hence $E^T_{\bar{\alpha}(\alpha)} = E^T_{\bar{\alpha}(\alpha')}$ whenever $\{\alpha, \alpha'\} \subset \bar{S}$. Let us write $E$ for this unique extender. We’ll have to have $\nu(E) \geq \aleph^H_\omega$, so that $E$ cannot have been used in $T$. Contradiction! $\square$

We shall now define $\Phi : [H_\kappa]^{<\omega_1} \to NS_{\omega_1}$ as follows. Let $X \in [H_\kappa]^{<\omega_1}$. If $X \prec H_\kappa$ is such that $\omega X \subset X$ then we let $\Phi(X)$ be defined as above. Otherwise we set $\Phi(X) = \emptyset$. Let $(C, S)$ be as given by Lemma 0.3. We let

$$d = \{ \aleph_{\alpha+1} : \alpha \in C \}.$$  

We know that $\aleph_{\omega_1+1} \in \pcf(d)$, by [BuMa90, Remark 1.8]. By Theorems 0.3 and 0.4 there is a countable $d' \subset d$ with $\aleph_{\omega_1+1} \in \pcf(d')$. We shall now derive a contradiction by showing that $\pcf(d') \subset \sup(d')^+ + 1$.

\[\text{Note: The lift-up arguments which are to follow will be simplified by the assumption, which we may make without loss of generality, that every element of } C \text{ is a limit ordinal.}\]
Let $\lambda = \sup(d')$.

**Main Claim.** For every $f \in \prod d'$ there is some $g \in K$, $g: \lambda \to \lambda$ such that $g(\mu) > f(\mu)$ for all $\mu \in d'$.

Suppose that the Main Claim holds. Then certainly

$$\{[g \upharpoonright d']_U: g \in K \land g: \lambda \to \lambda\}$$

is cofinal in $\prod d'/U$. But there are only $\leq \lambda^{+}$ many $g \in K$, $g: \lambda \to \lambda$, so that we must certainly have $\text{cf}(\prod d'/U) \leq \lambda^{+}$. This contradiction proves Theorem 0.2.

It therefore suffices to prove the Main Claim. Fix $f \in \prod d'$ for the rest of this proof. By the choice of $(C, S)$ there is some $X < H_\kappa$ with $\text{Card}(X) = 2^{\aleph_1}$, $\omega X \subset X$, $C \cap \Phi(X) = \emptyset$, and $f \in X$. Let $\pi: H = H_\kappa \cong X < H_\kappa$, and let $T = T_X$ be as defined above. Then if $\beta + 1 \in (0, \infty]_T$ and $\aleph_{\alpha + 1} \in d'$ we do not have that $\text{crit}(E_\beta^T) < \aleph_\alpha^H \leq \nu(E_\beta^T)$.

Let $(D^T \cap (0, \infty]_T) \cup \{lh(T)\} = \{\alpha_0 + 1 < \ldots < \alpha_N + 1\}$, where $0 \leq N < \omega$. Let $\alpha_{n}^T$ be the $T$-predecessor of $\alpha_{n+1}$ for $0 \leq n \leq N$. Let us pretend that $N > 0$ and $\alpha_0^T = 0$, i.e., that we immediately drop on $(0, \infty]_T$. Notice that $\text{crit}(\pi) \leq \pi^{-1}(2^{\aleph_1})^+$. Let us assume without loss of generality that $\alpha \in C \Rightarrow \aleph_\alpha > (2^{\aleph_1})^+$.

For each $\alpha \in C$ there is a least $\beta(\alpha) \in [0, \infty]_T$ such that $M_{\beta(\alpha)}^{T}|\aleph_\alpha^H = \bar{K}|\aleph_\alpha^H$. Let $n(\alpha)$ be the unique $n < N$ such that $\alpha_{n-1}^T < \beta(\alpha) \leq \alpha_n^T$. Then $D^T \cap (\beta(\alpha), \alpha_{n(\alpha)}^T]_T = \emptyset$ and $\rho_{\omega}(M_{\alpha_{n(\alpha)}^T}^{T}) \leq \aleph_\alpha^H$. Let $\eta(\alpha)$ be the least $\eta$ such that $\rho_{\omega}(M_{\alpha_{n(\alpha)}^T}^{T}) \leq \aleph_\alpha^H$ (if $\beta(\alpha) < \alpha_n^T$ then $\eta(\alpha) = M_{\alpha_n^T}^{T} \cap \text{OR}$). Let $m(\alpha)$ be the unique $m < \omega$ such that $\rho_{m+1}(M_{\alpha_{n(\alpha)}^T}^{T}) \leq \aleph_\alpha^H < \rho_{m}(M_{\alpha_{n(\alpha)}^T}^{T}) \leq \aleph_\alpha^H$. The following is easy to verify.

**Claim 2.** We may partition $C$ into finitely many sets $C_0, \ldots, C_k$, $0 \leq k < \omega$, such that $\sup(C_l) \leq \min(C_{l+1})$ whenever $l < k$ and such that for all $l \leq k$, if $\{\alpha, \alpha'\} \subset C_l$ then $n(\alpha) = n(\alpha')$, $\eta(\alpha) = \eta(\alpha')$, and $m(\alpha) = m(\alpha')$.

In order to finish the proof of the Main Claim it therefore now suffices to find, for an arbitrary $l \leq k$, some $g \in K$, $g: \lambda \to \lambda$ such that $g(\aleph_{\alpha+1}) > f(\aleph_{\alpha+1})$ for all $\alpha \in C_l$.

Let us fix $l \leq k$. Let us write $n$, $\eta$, and $m$ for $n(\alpha)$, $\eta(\alpha)$, and $m(\alpha)$, where $\alpha$ is any member of $C_l$. Let $\mathcal{M} = M_{\alpha_{n(\alpha)}^T}^{T}|\eta$.

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\textsuperscript{4}I.e., in this note we simply ignore the possibility that we might have $\alpha_0^T > 0$. In any event, it can be shown that if we were to use Friedman-Jensen premise rather than Mitchell-Steel premise then the case that $\alpha_0^T > 0$ would not come up by an argument of Sch02.
Let $\lambda_l = \sup(\{\aleph^n_\alpha : \alpha \in C_l\})$. We may define
\[ \tilde{\pi} : \mathcal{M} \rightarrow \tilde{\mathcal{M}} = Ulm(\mathcal{M}; \pi \upharpoonright \lambda_l). \]
By the argument of [MiScSt97], we shall have that $\tilde{\mathcal{M}} \preceq K$. In particular, $\tilde{\mathcal{M}} \in K$.

Let us write $\gamma^-$ for the cardinal predecessor of $\gamma$ if $\gamma$ is a successor cardinal (otherwise we let $\gamma^- = \gamma$). Let $\tilde{\lambda}_l = \sup(\pi^n \lambda_l)$. Let us define $g : \tilde{\lambda}_l \rightarrow \tilde{\lambda}_l$ as follows.
Let $\gamma < \tilde{\lambda}_l$, and let us write $\gamma^-$ for $(\gamma^-)^{\mathcal{M}}$, i.e., $\gamma^-$ in the sense of $\mathcal{M}$. Let
\[ \sigma_{\gamma} : \tilde{\mathcal{M}}_{\gamma} \cong h^\mathcal{M}_m(\gamma^- \cup \{p_{\tilde{\mathcal{M}}}\}) \prec_m \tilde{\mathcal{M}}, \]
where $\tilde{\mathcal{M}}_{\gamma}$ is transitive. We let
\[ g(\lambda) = (\gamma^-)^{\tilde{\mathcal{M}}_{\gamma}}. \]
Notice that $g \in K$. We are left with having to verify that $g(\aleph_{\alpha+1}) > f(\aleph_{\alpha+1})$ for all $\alpha \in C_l$.

Fix $\alpha \in C_l$. Let us assume without loss of generality that $\beta(\alpha) < \alpha^*_n$, the other case being easier. Then $\eta = \mathcal{M}^{T^\alpha}_n \cap OR$, i.e., $\mathcal{M} = \mathcal{M}^{T^\alpha}_n$. Consider
\[ \pi^{T^\alpha}_n(\alpha) : \mathcal{M}^{T^\alpha}_n \rightarrow \mathcal{M}^{T^\alpha}_n. \]
It is easy to verify that
\[ \tilde{\mathcal{M}}_{\aleph_{\alpha+1}} = Ulm(\mathcal{M}^{T^\alpha}_n; \pi \upharpoonright \pi^{-1}(\aleph_{\alpha})), \]
and that there is a map
\[ \varphi_\alpha : \tilde{\mathcal{M}}_{\aleph_{\alpha+1}} \rightarrow \tilde{\mathcal{M}} \]
which is defined by
\[ \tau^{\mathcal{M}_{\aleph_{\alpha+1}}}(\vec{\xi}, p_{\mathcal{M}_{\aleph_{\alpha+1}}}) \mapsto \tau^{\tilde{\mathcal{M}}}(\vec{\xi}, p_{\tilde{\mathcal{M}}}), \]
where $\vec{\xi} < \aleph_\alpha$ and $\tau$ is an appropriate term, and which is just the inverse of the collapsing map obtained from taking $h^\mathcal{M}_m(\aleph_\alpha \cup \{p_{\mathcal{M}}\})$. I.e., $\varphi_\alpha = \sigma_{\aleph_{\alpha+1}}$.

We now use the fact that $\alpha \in C$, i.e., that if $\beta + 1 \in [0, \infty]_T$ then we do not have that $\text{crit}(E^\beta_T) < \aleph^H_\alpha \leq \nu(E^\beta_T)$. This implies that $\pi^{T^\alpha}_n(\alpha)$ is the inverse of the collapsing map obtained from taking $h^\mathcal{M}_m(\aleph^H_\alpha \cup \{p_{\mathcal{M}}\})$, and that $\aleph^H_{\alpha+1} \subset \mathcal{M}^{T^\alpha}_n(\alpha)$. In fact, $\aleph^H_{\alpha+1} = \aleph^{K}_{\alpha}$, by [MiScSt97], $= \aleph_{\alpha}^{+\mathcal{M}^{T^\alpha}_n(\alpha)}$. We therefore have that
\[ f(\aleph_{\alpha+1}) < \sup(\pi^{-\aleph^H_{\alpha+1}}) = \aleph^{+\tilde{\mathcal{M}}_{\aleph_{\alpha+1}}}_{\alpha}. \]
We have shown that $g(\aleph_{\alpha+1}) > f(\aleph_{\alpha+1})$. \hfill \Box

We would like to prove that if $2^{\aleph_1} < \aleph_\omega$, but $\aleph^H_\omega > \aleph_\omega$, then Projective Determinacy (or even $AD^{L(R)}$) holds, using Woodin’s core model induction. This, however, would require a solution of problem #5 of the list [SchSt].
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