A non-periodic and two-dimensional example of elliptic homogenization

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Abstract

The focus in this paper is on elliptic homogenization of a certain kind of possibly non-periodic problems. A non-periodic and two-dimensional example is studied, where we numerically illustrate the homogenized matrix.

1 Introduction

Background. When studying the microscale behavior (beyond the reach of numerical solution methods) of physical systems, one is naturally lead to the concept of homogenization, i.e., the theory of the convergence of sequences of partial differential equations.

The homogenization of periodic structures using the two-scale convergence technique is well-established due to the pioneering work by Gabriel Nguetseng [10] and the further development work by Grégoire Allaire [1]. Generalizations of the two-scale convergence technique have been developed independently by, e.g., Maria Luísa Mascarenhas and Anca-Maria Toader [8] (scale convergence), Gabriel Nguetseng [11, 12] (Σ-convergence), and Anders Holmbom, Jeanette Silfver, Nils Svanstedt and Niklas Wellander [3, 6] (“generalized” two-scale convergence).

A simple but possibly powerful method of analyzing non-periodic structures is the $\lambda$-scale convergence technique introduced by Anders Holmbom and Jeanette Silfver [5]. $\lambda$-scale convergence is scale convergence in the special case of using the Lebesgue (i.e., $\lambda$) measure and test functions periodic in the second argument [5]. Homogenization techniques based on this approach are developed in the doctoral thesis [13] of Jeanette Silfver. These results are the point of departure for the main contributions in this paper.

Organization of the paper. In Section 2 we look at the convergence for sequences of functions. We start by stating the definition of the traditional notion of two-scale convergence as introduced by Gabriel Nguetseng [10]. Then we move on to generalizations of this convergence mode, namely “generalized”
two-scale convergence \[4, 6\] and scale convergence \[8\]. We conclude the section
by introducing \(\lambda\)-scale convergence and the important notion of asymptotically
uniformly distributed sequences \[5, 13\].

Section 3 deals with the convergence for sequences of partial derivatives. We
first look at how the two-scale convergence works for partial derivatives in the
periodic case, and then we consider the more general case of \(\lambda\)-scale convergence
of sequences of partial derivatives.

The concept of homogenization is introduced in Section 4 which begins
with stating the definition of H-convergence \[9\], i.e., the generalization of Sergio
Spagnolo’s G-convergence of sequences of symmetric matrices \[14, 15\]. We give a
theorem on the homogenization of a sequence of periodic matrices. We then in-
troduce the important “type-H\(\xi\)” property, which is employed in the end of the
section when formulating a theorem on the homogenization of \(\lambda\)-structures \[13\].

In Section 5 we specifically study a non-periodic and two-dimensional exam-
ple of a \(\lambda\)-structure with the property of not only having a periodic direction,
but also having oscillations with a monotonically increasing frequency in one,
non-periodic direction. We formulate and prove a series of proposition which
are needed in order to prove the main result of this paper, Theorem \[30\] in which
we present a homogenization result for the considered \(\lambda\)-structure in the form
of a homogenized matrix and a governing local problem.

The concluding Section 6 illustrates the results achieved in Section 5. We
numerically solve the local problem to obtain the non-constant homogenized
matrix, and we heuristically explain why it is isotropic on a line along the
periodic direction.

Notations. The following more or less handy notations are employed:

The \(N\)-tuple \((\xi_1, \ldots, \xi_N)\) in \(\mathbb{R}^N\) is denoted \((\xi_i)_{i=1, \ldots, N}\), or \(\xi\) whenever conve-
nient. Similarly, \((m_{ij})_{i,j=1, \ldots, N}\), or simply the mere majuscule \(M\) when handy,
denotes an \(N \times N\) matrix. A bold dot, i.e. \(\cdot\), represents a non-fixed variable,
e.g., \(\phi(\cdot, y)\) is the same as the function \(x \mapsto \phi(x, y)\) where \(y\) is held fixed as a
parameter. We will also allow expressions like, e.g., \(k \cdot\) meaning \(x \mapsto kx\). In this
paper, \(\Omega\) is always an open bounded non-empty subset of \(\mathbb{R}^N\) and, if nothing
else is stated, \(Y\) is the unit cube \((0, 1)^N\) in \(\mathbb{R}^N\). The \(N\)-tuple \(\left(\frac{\partial}{\partial x_i}\right)_{i=1, \ldots, N}\) of
partial derivative operators is denoted by the symbol \(\nabla\). Following, e.g., \[3\],
the function space \(W_{\text{per}}(Y)\) denotes the subspace of functions in \(H^1_{\text{per}}(Y)\) with
vanishing mean value.

2 Convergence for sequences of functions

The two-scale convergence method was introduced in 1989 by Gabriel Nguet-
seng \[10\], and a modern formulation is given by Definition \[4, 7\].

Definition 1. A sequence \(\{u^h\}\) in \(L^2(\Omega)\) is said to two-scale converge to the
limit \( u_0 \in L^2(\Omega \times Y) \) if, for any \( v \in L^2(\Omega; C_{\text{per}}(Y)) \),

\[
\lim_{h \to \infty} \int_{\Omega} u^h(x) v(x, hx) \, dx = \int_{\Omega} \int_{Y} u_0(x, y) v(x, y) \, dy \, dx.
\]

An important property of the two-scale convergence is given by Proposition 2 [7].

**Proposition 2.** If \( \{ u^h \} \) two-scale converges to \( u_0 \), and \( v \in L^2(\Omega; C_{\text{per}}(Y)) \), then

\[
u^h \rightharpoonup \int_Y u_0(\cdot, y) \, dy \quad \text{in} \quad L^2(\Omega)
\]

and

\[
v(\cdot, h \cdot) \rightharpoonup \int_Y v(\cdot, y) \, dy \quad \text{in} \quad L^2(\Omega).
\]

Analogous to a corresponding compactness result for weak convergence, we have Theorem 3 [7].

**Theorem 3.** Every bounded sequence in \( L^2(\Omega) \) has a subsequence which two-scale converges.

Introducing sequences of operators \( \tau^h \) defined below we may extend Definition 1 of two-scale convergence to a generalized version according to Definition 4 [4, 6].

**Definition 4.** Assume that \( Y \) is an open bounded subset of \( \mathbb{R}^M \). Let \( X \subset L^2(\Omega \times Y) \) be a linear space and

\( \tau^h : X \to L^2(\Omega) \)

linear operators. A sequence \( \{ u^h \} \) in \( L^2(\Omega) \) is said to two-scale converge to \( u_0 \in L^2(\Omega \times Y) \) with respect to \( \{ \tau^h \} \) if, for any \( v \in X \),

\[
\lim_{h \to \infty} \int_{\Omega} u^h(x)(\tau^h v)(x) \, dx = \int_{\Omega} \int_{Y} u_0(x, y) v(x, y) \, dy \, dx.
\]

In order to achieve a compactness result like Theorem 3 we need Definition 5.

**Definition 5.** Assume that \( Y \) is an open bounded subset of \( \mathbb{R}^M \). Let \( X \subset L^2(\Omega \times Y) \) be a normed space and

\( \tau^h : X \to L^2(\Omega) \)
linear operators. Then \( \{ \tau^h \} \) is two-scale compatible with respect to \( X \) if there exists \( C > 0 \) independent of \( h \) such that, for any \( v \in X \),

\[
\lim_{h \to \infty} \| \tau^h v \|_{L^2(\Omega)} \leq C \| v \|_{L^2(\Omega \times Y)}
\]

and

\[
\| \tau^h v \|_{L^2(\Omega)} \leq C \| v \|_X.
\]

Furthermore, \( X \) is called admissible with respect to \( \{ \tau^h \} \).

Using Definition 5, we have the compactness result according to Theorem 6 [4, 6].

**Theorem 6.** Assume that \( Y \) is an open bounded subset of \( \mathbb{R}^M \). Let \( \{ \tau^h \} \) be two-scale compatible with respect to \( X \), a separable Banach space dense in \( L^2(\Omega \times Y) \). Then every bounded sequence \( \{ u^h \} \) in \( L^2(\Omega) \) has a subsequence that two-scale converges with respect to \( \{ \tau^h \} \).

Using weak convergences, we can introduce a stronger version of two-scale compatibility than presented in Definition 5, see Definition 7.

**Definition 7.** Assume that \( Y \) is an open bounded subset of \( \mathbb{R}^M \). Let \( \{ \tau^h \} \) be two-scale compatible with respect to the linear space \( X \subset L^2(\Omega \times Y) \), and let \( \{ u^h \} \) be a bounded sequence in \( L^2(\Omega) \) two-scale converging to \( u_0 \). Then \( \{ \tau^h \} \) is strongly two-scale compatible if, for any \( v \in X \),

\[
u^h \rightharpoonup \int_Y u_0(\cdot, y) \, dy \quad \text{in} \quad L^2(\Omega)
\]

and

\[
\tau^h v \rightharpoonup \int_Y v(\cdot, y) \, dy \quad \text{in} \quad L^2(\Omega).
\]

Compare Definition 7 with the result of Proposition 4.

An alternative generalization of two-scale convergence is scale convergence introduced by Maria Luisa Mascarenhas and Anca-Maria Toader in 2001 [8]. The definition of scale convergence is according to Definition 8.

**Definition 8.** Assume that \( Y \) is a metrizable compact space, \( \mu \) a Young measure on \( \Omega \times Y \), and \( L^2_\mu(\Omega \times Y) \) is the space of all functions with \( \mu \)-integrable square. Furthermore, let \( \{ \alpha^h \} \) be a sequence of \( \mu \)-measurable functions \( \alpha^h : \Omega \to Y \). A sequence \( \{ u^h \} \) in \( L^2(\Omega) \) is said to scale converge to \( u_0 \in L^2_\mu(\Omega \times Y) \) with respect to \( \{ \alpha^h \} \) if, for any \( v \in L^2(\Omega; C(Y)) \),

\[
\lim_{h \to \infty} \int_{\Omega} u^h(x)v(x, \alpha^h(x)) \, dx = \int_{\Omega \times Y} u_0(x, y)v(x, y) \, d\mu(x, y).
\]
A special case of scale convergence is achieved by choosing the same measure and same class of admissible test functions as in two-scale convergence, namely the Lebesgue measure \( \lambda \) and \( L^2(\Omega; \mathbb{C}_{\text{per}}(Y)) \), respectively. This leads to Definition 9 [13, Definition 27] (see also [5, Definition 11]).

**Definition 9.** Assume that \( Y \) is the unit cube in \( \mathbb{R}^M \). Furthermore, let \( \{ \alpha^h \} \) be a sequence of functions \( \alpha^h : \Omega \rightarrow Y \). A sequence \( \{ u^h \} \) in \( L^2(\Omega) \) is said to \( \lambda \)-scale converge to \( u_0 \in L^2(\Omega \times Y) \) with respect to \( \{ \alpha^h \} \) if, for any \( v \in L^2(\Omega; \mathbb{C}_{\text{per}}(Y)) \),

\[
\lim_{h \to \infty} \int_{\Omega} u^h(x)v(x, \alpha^h(x)) \, dx = \int_{\Omega} \int_{Y} u_0(x, y)v(x, y) \, dy \, dx.
\]

A thorough treatment of how \( \{ \alpha^h \} \) could be chosen to obtain strong two-scale compatibility is found in the doctoral thesis [13, Subsection 2.4.2] of Jeanette Silfver (see also [5]). We reproduce the results below.

Following [13, Subsection 2.4.2], we omit cases where \( M \neq N \) letting \( \alpha^h : \mathbb{R}^N \rightarrow \mathbb{R}^N \) be a continuous bijection. Furthermore, \( \{ \bar{Y}^j \}_{j=1}^\infty \) is a covering of \( \mathbb{R}^N \) with unit cubes, and \( \{ \bar{Y}^j_k \}_{k=1}^N \) a covering of \( \bar{Y}^j \) with cubes with side lengths \( \frac{1}{h} \) in such a way that \( Y^j_k \) are \( Y \)-periodic repetitions of a cube \( Y_k \subset Y \). Assuming that \( \Omega \subset \mathbb{R}^N \) has a Lipschitz boundary, we define

\[
\Omega^h = (\alpha^h)^{-1}(\bar{Y}^j) \cap \Omega.
\]

We also assume, for each \( h \), that there exists a finite set \( q(h) \subset \mathbb{Z}_+ \) such that \( \Omega = \cup_{j \in q(h)} \Omega^h_j \). Similarly, introduce

\[
\Omega^h_{j,k} = (\alpha^h)^{-1}(\bar{Y}^j_k) \cap \Omega.
\]

Finally, it is assumed that \( \Omega^h_j \subset N_{r(h)}(x_{h,j}) \) for some ball \( N_{r(h)}(x_{h,j}) \), centered at some \( x_{h,j} \in \Omega^h_j \), with radius \( r(h) \to 0 \) as \( h \to \infty \). Given the setting above, Definition 10 makes sense [13, Definition 28] (see also [5, Definition 12]).

**Definition 10.** Suppose that for all cubes \( Y_k \subset Y \) and any \( \Omega^h_j, j \in q(h) \), such that \( (\alpha^h)^{-1}(Y^j) \cap \Omega \) does not intersect \( \partial \Omega \),

\[
\frac{\lambda(\Omega^h_{j,k})}{\lambda(\Omega^h_j)} - \lambda(Y_k) \left| < \epsilon \right. \tag{1}
\]

where \( \epsilon = \epsilon(h) \to 0 \) for \( h \to \infty \). Then \( \{ \alpha^h \} \) is said to be asymptotically uniformly distributed on \( \Omega \).

This is sufficient to obtain the strong two-scale compatibility as promised, see Proposition [13, Proposition 30] (see also [5, Proposition 15]).
**Proposition 11.** Assume that $\Omega$ has a Lipschitz continuous boundary. Let $\{\alpha^h\}$ be asymptotically uniformly distributed on $\Omega$. Then $\{\tau^h\}$ defined by
\[ \tau^h v = v(\cdot, \alpha^h(\cdot)) \]
is strongly two-scale compatible with respect to $L^2(\Omega; C_{\text{per}}(Y))$.

Note here that the admissible space in $L^2(\Omega \times Y)$ is in this case $L^2(\Omega; C_{\text{per}}(Y))$. As a consequence of Proposition 11 we have Corollary 12 [13, Corollary 31] (see also [5, Corollary 16]).

**Corollary 12.** Assume that $\Omega$ has a Lipschitz continuous boundary. Let $\{\alpha^h\}$ be asymptotically uniformly distributed on $\Omega$, and let $\{u^h\}$ strongly converge to $u$ in $L^2(\Omega)$. Then, up to a subsequence, $\{u^h\}$ $\lambda$-scale converges to $u$ with respect to $\{\alpha^h\}$.

Note that the second scale dependence of the $\lambda$-scale limit vanishes, just like how it works for two-scale limits in the case of strong convergence [7].

### 3 Convergence for sequences of N-tuples of partial derivatives

Since sequences of partial differential equations in the context of homogenization typically involve $N$-tuples of partial derivatives of solutions, i.e., $\nabla u^h$, we need to investigate the two-scale limits for these. Indeed, for traditional periodic two-scale convergence, we have Proposition 13 [1, 10].

**Proposition 13.** Let $\{u^h\}$ be a bounded sequence in $H^1(\Omega)$ such that the strong limit in $L^2(\Omega)$ is $u$. Then, up to a subsequence, there exists $u_1 \in L^2(\Omega; W_{\text{per}}(Y)^N)$ such that for any $v \in L^2(\Omega; C_{\text{per}}(Y)^N)$,
\[ \lim_{h \to \infty} \int_{\Omega} \sum_{i=1}^{N} \frac{\partial u^h}{\partial x_i}(x) v_i(x, hx) \, dx = \int_{\Omega} \int_{Y} \sum_{i=1}^{N} \left( \frac{\partial u}{\partial x_i}(x) + \frac{\partial u_1}{\partial y_i}(x, y) \right) v_i(x, y) \, dy \, dx. \]

It should be noted here that bounded sequences in $H^1(\Omega)$ strongly converge in $L^2(\Omega)$, and we know that strongly convergent sequences in $L^2(\Omega)$ have a subsequence with a two-scale limit with vanishing second scale.

A deciding step towards the homogenization of certain non-periodic problems is to prove the corresponding result for $\lambda$-scale convergence. We have Proposition 14 proved by Jeanette Silfver in [13, Proposition 35] (see also [5, Section 4]).
Proposition 14. Let $X$ be a Banach space for which
\[ \mathcal{D}(\Omega; C_\text{per}^\infty(Y)) \subset X \subset L^2(\Omega; L^2_\text{per}(Y)), \]
and $\{\alpha^h\}$ a sequence of functions
\[ \alpha^h : \mathbb{R}^N \to \mathbb{R}^N \]
which are continuous and bijective. We assume also that $\{\tau^h\}$ defined by
\[ \tau^h v = v(\cdot, \alpha^h(\cdot)) \]
is strongly two-scale compatible with respect to $X$. Furthermore, let $Z \subset L^2(\Omega; L^2_\text{per}(Y)^N)$ be a Banach space such that $Z \cap \mathcal{D}(\Omega; C_\text{per}^\infty(Y)^N)$ is dense in $Z$, and let $Z^\perp$ be its orthogonal complement in $L^2(\Omega; L^2_\text{per}(Y)^N)$. Assume also that, for any $v \in Z \cap \mathcal{D}(\Omega; C_\text{per}^\infty(Y)^N)$,
\[ \sum_{i=1}^N \sum_{j=1}^N \frac{\partial v_i}{\partial y_j}(\cdot, \alpha^h(\cdot)) \frac{\partial \alpha^h_j}{\partial x_i} \to 0 \quad \text{in } L^2(\Omega). \]
(2)

Then, for any bounded sequence $\{u^h\}$ in $H^1(\Omega)$ there exists a subsequence such that $u^h \to u$ in $L^2(\Omega)$ and, for any $v \in X$,
\[ \lim_{h \to \infty} \int_{\Omega} u^h(x)v(x, \alpha^h(x)) \, dx = \int_{\Omega} \int_Y u(x)v(x, y) \, dy \, dx, \]
(3)
and there exists $w_1 \in Z^\perp$ such that, for any $v \in X^N$,
\[ \lim_{h \to \infty} \int_{\Omega} \sum_{i=1}^N \frac{\partial u^h_i}{\partial x_i}(x) v_i(x, \alpha^h(x)) \, dx \]
\[ = \int_{\Omega} \sum_{i=1}^N \left( \frac{\partial u}{\partial x_i}(x) + w_{1,i}(x, y) \right) v_i(x, y) \, dy \, dx. \]
(4)

Proof. Property (3) is given by Corollary 12. By using Green’s formula twice, using assumption (2), and utilizing density, property (4) follows from orthogonality. \qed

Remark 15. The original proof of Jeanette Silfver with all details is found in [13, Subsection 2.4.2].

4 Elliptic homogenization

In this section we present a homogenization result for elliptic problems governed by the sequence $\{\alpha^h\}$ introduced earlier in this paper. Such results where first
published in [13], and in Sections 5 and 6 we present a special case in two dimensions and perform a numerical experiment, respectively.

The foundation upon which modern homogenization rests was erected in 1967-68 when Sergio Spagnolo developed the concept of G-convergence of sequences of symmetric matrices [14, 15], which has later been generalized by François Murat to H-convergence where the symmetry assumption is dropped, but at the cost of an imposed requirement on the sequence of flows [9]. We begin by introducing a space of matrix valued functions according to Definition 16.

**Definition 16.** If \( \infty > r \geq s > 0 \) and \( O \subset \mathbb{R}^N \) is open, the matrix valued function \( M \in L^\infty(O)^{N \times N} \) is said to belong to \( M_{r, s; O} \) if the system of structural conditions

\[
\begin{array}{ll}
\sup_{|\xi|=1} \left| \sum_{j=1}^{N} m_{ij} \xi_j \right|_{i=1,...,N} & \leq r, \quad \text{(bounded)} \\
\inf_{|\xi|=1} \sum_{i=1}^{N} \sum_{j=1}^{N} \xi_i m_{ij} \xi_j & \geq s, \quad \text{(coercive)}
\end{array}
\]

is satisfied a.e. in \( O \). Furthermore, the space \( M_{\text{per}}(r, s; Y) \) consists of those functions in \( M_{r, s; \mathbb{R}^N} \) which are \( Y \)-periodic.

We can now give Definition 17 defining the notion of H-convergence.

**Definition 17.** Let \( \{A^h\} \) be a sequence in \( M_{r, s; \Omega} \), and let \( B \in M_{r', s'; \Omega} \). Furthermore, assume that, for any \( f \in H^{-1}(\Omega) \), the sequence of solutions \( \{u^h\} \) in \( H^1_0(\Omega) \) to the sequence of problems

\[
\begin{aligned}
&-\sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial}{\partial x_i} \left( a_{ij}^h \frac{\partial u^h}{\partial x_j} \right) = f \quad \text{in } \Omega \\
&u^h = 0 \quad \text{on } \partial \Omega
\end{aligned}
\]

satisfies

\[
\begin{aligned}
&u^h \rightharpoonup u \quad \text{in } H^1_0(\Omega) \\
&\left( \sum_{j=1}^{N} a_{ij}^h \frac{\partial u^h}{\partial x_j} \right)_{i=1,...,N} \rightharpoonup \left( \sum_{j=1}^{N} b_{ij} \frac{\partial u}{\partial x_j} \right)_{i=1,...,N} \quad \text{in } L^2(\Omega)^N,
\end{aligned}
\]

where \( u \) is the unique solution to

\[
\begin{aligned}
&-\sum_{i=1}^{N} \sum_{j=1}^{N} \frac{\partial}{\partial x_i} \left( b_{ij} \frac{\partial u}{\partial x_j} \right) = f \quad \text{in } \Omega \\
u = 0 \quad \text{on } \partial \Omega
\end{aligned}
\]

Then \( \{A^h\} \) H-converges to \( B \).
Remark 18. In the literature, H-convergence is often called G-convergence. In this paper, we clearly separate the general notion of H-convergence from the special case of G-convergence which deals with symmetric matrices exclusively.

When an H-limit has been found, one has homogenized the sequence of partial differential equations (5), and (7) is the homogenized problem. For sequences of matrices which are periodic, we get Theorem 19 for the homogenization in this case [10, 1].

Theorem 19. Suppose \( A \in \mathcal{M}_{\text{per}}(r,s;Y) \). Then \( \{A(h \cdot)\} \) H-converges to \( B \) given by

\[
(b_{ij})_{i,j=1,...,N} = \left( \int_Y \sum_{k=1}^N a_{ik}(y) \left( \delta_{kj} + \frac{\partial z_j}{\partial y_k}(y) \right) dy \right)_{i,j=1,...,N},
\]

where \( z \in W_{\text{per}}(Y)^N \) uniquely solves the local problem

\[
- \left( \sum_{i=1}^N \sum_{k=1}^N \frac{\partial}{\partial y_i} \left( a_{ik} \left( \delta_{kj} + \frac{\partial z_j}{\partial y_k} \right) \right) \right)_{j=1,...,N} = 0 \quad \text{in } Y. \tag{8}
\]

We note that it is the term containing the local problem solution \( z \) which makes the H-limit \( B \) to deviate from the average of \( A \) over each periodicity cell \( Y \). It is due to the \( u_1 \) term in the two-scale convergence for partial derivatives and is obtained through a simple separation of variables of \( u_1 \) with \( \nabla u \) as the \( x \)-dependent part and \( z \) as the \( y \)-dependent part.

In the remainder of this paper, \( A^h \) in the sequence of problems (5) will be the composition \( A \circ \alpha^h \).

Below we present the result on non-periodic homogenization from [13] announced in the beginning of this section. We start by giving Definition [20, 13] Definition 51).

Definition 20. The sequence \( \{\alpha^h\} \) is of type \( H^\zeta_X \) if the following three conditions hold:

(i) The sequence \( \{\tau^h\} \) of operators \( \tau^h : X \to L^2(\Omega) \) defined by

\[
\tau^h v = v(\cdot, \alpha^h(\cdot)), \quad v \in X
\]

is strongly two-scale compatible with respect to a Banach space \( X \) for which

\[
D(\Omega; C^\infty_{\text{per}}(Y)) \subset X \subset L^2(\Omega; L^2_{\text{per}}(Y)).
\]

(ii) There exists a sequence \( \{p^h\} \) strongly convergent to zero in \( H^1(\Omega) \) such that

\[
p^h \left( \frac{\partial \alpha^h}{\partial x_i} \right)_{i,j=1,...,N} \to \left( \pi_{ij} \right)_{i,j=1,...,N} \quad \text{in } L^2(\Omega)^{N \times N},
\]

where \( \Pi \) is a diagonal matrix with

\[
(\pi_{ii})_{i=1,...,N} = (\zeta_i)_{i=1,...,N} \in L^\infty(\Omega)^N.
\]
(iii) There exists a Banach space \( Z \subset X \) for which \( Z \cap D(\Omega; C_{\text{per}}^0(Y))^N \) is dense in \( Z \), and such that, for any \( v \in Z \cap D(\Omega; C_{\text{per}}^0(Y))^N \),

\[
\sum_{i=1}^N \sum_{j=1}^N \frac{\partial v_i}{\partial y_j}(\cdot, \alpha^h(\cdot)) \frac{\partial \alpha^h_i}{\partial x_i} \rightarrow 0 \quad \text{in } L^2(\Omega).
\]

From Definition 20 and Proposition 14, we have the homogenization result of Theorem 21 [13, Theorem 52].

Theorem 21. Assume that \( \Omega \) has a Lipschitz continuous boundary and that

\( A \in C_{\text{per}}(Y)^{N \times N} \cap \mathcal{M}(r, s; \mathbb{R}^N) \).

Let \( \{\alpha^h\} \) be of type \( H^r_{L^2}(\Omega; C_{\text{per}}^0(Y)) \). Then \( \{A \circ \alpha^h\} \) \( H \)-converges to \( B \) given by

\[
\left( \sum_{j=1}^N b_{ij} \frac{\partial u}{\partial x_j} \right)_{i=1, \ldots, N} = \left( \int \sum_{j=1}^N a_{ij}(y) \left( \frac{\partial u}{\partial x_j} + w_{1,j}(\cdot, y) \right) \, dy \right)_{i=1, \ldots, N},
\]

\( u \in H^1_0(\Omega) \) being the weak limit of the sequence \( \{u^h\} \) of solutions to (5), if \( u \in H^1_0(\Omega) \) and \( w_1 \in Z^\perp \subset L^2(\Omega; L^2_{\text{per}}(Y))^N \) uniquely solve the homogenized problem

\[
\int_\Omega \int_Y \sum_{i=1}^N \sum_{j=1}^N a_{ij}(y) \left( \frac{\partial u}{\partial x_j}(x) + w_{1,j}(x, y) \right) \frac{\partial v_i}{\partial x_i} (x) \, dy \, dx = \int_\Omega f(x)v(x) \, dx \quad (9)
\]

for all \( v \in H^1_0(\Omega) \), and, for each \( x \in \Omega \), the local problem

\[
\sum_{i=1}^N \zeta_i(x) \int_Y \sum_{j=1}^N a_{ij}(y) \left( \frac{\partial u}{\partial x_j}(x) + w_{1,j}(x, y) \right) \frac{\partial v_i}{\partial y_i}(y) \, dy = 0 \quad (10)
\]

for all \( v \in W_{\text{per}}(Y) \).

Proof. First we make a weak formulation of the sequence of problems (5).

The homogenized problem (9) is obtained by passing the weak formulation to the limit, using Proposition 14.

The local problem (10) is derived by first choosing appropriate test functions, then utilizing the fact that \( \{\alpha^h\} \) is of type \( H^r_{L^2}(\Omega; C_{\text{per}}^0(Y))^r \), and finally employing the variational lemma. \( \square \)

Remark 22. The original proof of Jeanette Silfver with all details is found in [13, Subsection 3.5.1].
5 A non-periodic and two-dimensional example

In order to justify the concept of homogenization of \( \lambda \)-structures, we take a look at a non-periodic two-dimensional example (thus, fixing \( N = 2 \) from now on). Let \( \{ \alpha^h \} \) and \( \Omega \) be given by

\[
\left\{ \begin{array}{l}
\alpha^h_1(x) = hx_1 \\
\alpha^h_2(x) = hx_2|x_2|,
\end{array} \right. \quad x \in \mathbb{R}^2,
\tag{11}
\]

and \( \Omega = (a_1, b_1) \times (a_2, b_2) \), respectively, where \( b_i > a_i > 0, \ i = 1, 2 \). Thus, \( \Omega \) is an open interval in the first quadrant of \( \mathbb{R}^2 \). In Figure 1 we depict the behaviour of some entry of \( A \circ \alpha^h \) for some \( (0, 1) \)-periodic matrix \( A \). (To be specific, we have chosen an entry on the form \( 1 + |\sin \pi y_1 \sin \pi y_2|^{1/2} \).)

Note how the oscillation frequency increases with growing \( x_2 \), while the periodicity is preserved in the \( x_1 \) direction, as expected.

To homogenize the sequence \( \{ A \circ \alpha^h \} \), we must first check that \( \{ \alpha^h \} \) is asymptotically uniformly distributed on \( \Omega \). Indeed, we have Proposition 23.

**Proposition 23.** The sequence \( \{ \alpha^h \} \) given in (11) is asymptotically uniformly distributed on \( \Omega \).

**Proof.** We define \( Y^j = j + Y \) and \( Y^*_n = \frac{1}{n}(k + Y) \), where \( j \in \mathbb{Z}^2, n \in \mathbb{Z}_+ \) and

Figure 1: Some entry of \( A \circ \alpha^h, h = 1 \), in the first quadrant.
\( \hat{k} \in \{0, \ldots, n-1\}^2 \). Furthermore, \( Y^2_k = \hat{j} + Y_k \). Clearly, there exist bijections

\[
\begin{align*}
\mathcal{J} & : \{1, 2, \ldots\} \to \mathbb{Z}^2 \\
\mathcal{K} & : \{1, \ldots, n^2\} \to \{0, \ldots, n-1\}^2 
\end{align*}
\]

such that \( \{Y^\mathcal{J}(j)\}_{j=1}^\infty \) is a covering of \( \mathbb{R}^2 \), and \( \{Y^\mathcal{K}(k)\}_{k=1}^{n^2} \) is a covering of \( Y^\mathcal{J}(j) \). This merely shows that the way of enumerating the squares in this proof is equivalent to the way in Definition 10.

We have, for each \( \hat{j} = (j_1, j_2) \in \mathbb{Z}^2 \) and \( \hat{k} = (k_1, k_2) \in \{0, \ldots, n-1\}^2 \),

\[
\Omega^h_j = (\alpha^h)^{-1}(\hat{j}) \cap \Omega = \left( [a^h_{1,j}, b^h_{1,j}] \times [a^h_{2,j}, b^h_{2,j}] \right) \cap \Omega
\]

and

\[
\Omega^h_{j,k} = (\alpha^h)^{-1}(\hat{Y}^j) \cap \Omega = \left( [a^h_{1,j,k}, b^h_{1,j,k}] \times [a^h_{2,j,k}, b^h_{2,j,k}] \right) \cap \Omega,
\]

where

\[
\begin{align*}
a^h_{1,j} &= \frac{j_1}{h}, & b^h_{1,j} &= \frac{j_1 + 1}{h} \\
a^h_{2,j} &= \sgn(j_2)\sqrt{\frac{|j_2|}{h}}, & b^h_{2,j} &= \sgn(j_2 + 1)\sqrt{\frac{|j_2 + 1|}{h}}
\end{align*}
\]

and

\[
\begin{align*}
a^h_{1,j,k} &= \frac{j_1 + \frac{k_1}{n}}{h}, & b^h_{1,j,k} &= \frac{j_1 + \frac{k_1 + 1}{n}}{h} \\
a^h_{2,j,k} &= \sgn(j_2 + \frac{k_2}{n})\sqrt{\frac{|j_2 + \frac{k_2}{n}|}{h}}, & b^h_{2,j,k} &= \sgn(j_2 + \frac{k_2 + 1}{n})\sqrt{\frac{|j_2 + \frac{k_2 + 1}{n}|}{h}}.
\end{align*}
\]

In order to cover \( \Omega \), we can for each fixed \( h \) apparently do this with a finite union of \( \Omega^h_j \) sets. The smallest \( j_1 \) required for the covering will go like \( O(h) \) and the smallest \( j_2 \) like \( O(\sqrt{h}) \). For large \( h \), the diameter of the \( \Omega^h_j \) sets will go uniformly like \( O(\frac{1}{\sqrt{h}}) \), so we can fit them in balls of equal radius which tends to zero as \( h \to \infty \). We realize that what is left to check is (1).

For covering \( \Omega^h_j \) sets, let \( \hat{j} \) be chosen such that \( (\alpha^h)^{-1}(\hat{j}) \) does not intersect \( \partial \Omega \), i.e.,

\[
(a^h_{1,j}, b^h_{1,j}, a^h_{2,j}, b^h_{2,j}) \subset \Omega.
\]

For such \( \hat{j} \)'s we have the Lebesgue measures

\[
\lambda(\Omega^h_j) = \frac{1}{h\sqrt{h}} \left( \sqrt{j_2 + 1} - \sqrt{j_2} \right)
\]

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and
\[ \lambda(\Omega_{j,k}^h) = \frac{1}{h^{\frac{1}{2}}} \frac{1}{n} \left( \sqrt{j_2 + \frac{k_2 + 1}{n}} - \sqrt{j_2 + \frac{k_2}{n}} \right). \]

Thus,
\[ \frac{\lambda(\Omega_{j,k}^h)}{\lambda(\Omega_j^h)} = \frac{1}{n^2} \frac{\sqrt{j_2 + \frac{k_2 + 1}{n}} - \sqrt{j_2 + \frac{k_2}{n}}}{\sqrt{j_2 + \frac{k_2 + 1}{n}} + \sqrt{j_2 + \frac{k_2}{n}}}. \]

Since the smallest \( j_2 \) goes like \( O(\sqrt{h}) \), we must uniformly have that
\[ \frac{\lambda(\Omega_{j,k}^h)}{\lambda(\Omega_j^h)} \to \frac{1}{n^2} = \lambda(Y_k), \]
for any \( \hat{k} \in \{0, \ldots, n-1\}^2 \), as \( h \to \infty \). This implies
\[ \left| \frac{\lambda(\Omega_{j,k}^h)}{\lambda(\Omega_j^h)} - \lambda(Y_k) \right| < \epsilon, \]
where \( \epsilon = \epsilon(h) \to 0 \) as \( h \to \infty \).

In virtue of Proposition 11 we thus get Proposition 24.

**Proposition 24.** The sequence \( \{\tau^h\} \) defined by
\[ \tau^h v = v(\cdot, \alpha^h(\cdot)), \]
where \( \{\alpha^h\} \) is given by (11), is strongly two-scale compatible with respect to \( L^2(\Omega; C_{\text{per}}(Y)). \)

**Proof.** Use Proposition 23 together with Proposition 11.

Define the 2-tuple \( \zeta \) according to
\[ \begin{cases} \zeta_1(x) = 1 \\ \zeta_2(x) = 2x_2 \end{cases} \]
(12)
and define the diagonal matrix \( \Pi \) element-wise by
\[ (\pi_{ii}(x))_{i=1,2} = (\zeta_i(x))_{i=1,2}. \]
Proposition 25 below shows that \( \{\alpha^h\} \) is of right type to use in Theorem 21 for the homogenization to work out properly.
Proposition 25. The sequence \( \{\alpha^h\} \) given by (11) is of type \( H^1_{L^2(\Omega; C_{\text{per}}(Y))} \), where \( \zeta \) is given by (12).

Proof. We must check conditions (i)–(iii) of Definition 20. Condition (i) is an immediate consequence of Proposition 24. For condition (ii), we note that

\[
\begin{align*}
\frac{\partial \alpha^h_1}{\partial x_1}(x) &= h \\
\frac{\partial \alpha^h_2}{\partial x_2}(x) &= 2hx_2 \\
\frac{\partial \alpha^h_j}{\partial x_i}(x) &= 0, \quad i \neq j
\end{align*}
\]

and by choosing\( p^h(x) = \frac{1}{h} \), for which \( p^h \to 0 \) in \( H^1(\Omega) \), we get

\[
p^h \left( \frac{\partial \alpha^h_i}{\partial x_i} \right)_{i,j=1,2} = (\pi_{ij})_{i,j=1,2} \to (\pi_{ij})_{i,j=1,2} \quad \text{in } L^2(\Omega).
\]

We obviously have that \( \zeta \in L^\infty(\Omega)^2 \). Finally, we must check condition (iii) of Definition 20. In this context, it means that we must find a Banach space \( Z \subset L^2(\Omega; C_{\text{per}}(Y)^2) \) for which \( Z \cap D(\Omega; C^\infty_{\text{per}}(Y)^2) \) is dense in \( Z \), and such that, for any \( v \in Z \cap D(\Omega; C^\infty_{\text{per}}(Y)^2) \),

\[
h \sum_{i=1}^2 \frac{\partial v_i}{\partial y_i}(\cdot, \alpha^h(\cdot)) \zeta_i \to 0 \quad \text{in } L^2(\Omega), \quad (13)
\]

as \( h \to \infty \). If we define

\[
Z = \left\{ v \in L^2(\Omega; L^2_{\text{per}}(Y)^2) : \sum_{i=1}^2 \frac{\partial v_i}{\partial y_i} \zeta_i = 0 \right\}, \quad (14)
\]

we clearly have, for any \( v \in Z \cap D(\Omega; C^\infty_{\text{per}}(Y)^2) \), a satisfied weak convergence (13). What is left to prove is the density of \( Z \cap D(\Omega; C^\infty_{\text{per}}(Y)^2) \) in \( Z \), where \( Z \) is given by (14). We apparently have

\[
Z \cap D(\Omega; C^\infty_{\text{per}}(Y)^2) = \left\{ v \in D(\Omega; C^\infty_{\text{per}}(Y)^2) : \sum_{i=1}^2 \frac{\partial v_i}{\partial y_i} \zeta_i = 0 \right\}, \quad (15)
\]

and since \( D(\Omega; C^\infty_{\text{per}}(Y)^2) \) is dense in \( L^2(\Omega; L^2_{\text{per}}(Y)^2) \), it is easy to check that we have the desired density of \( Z \cap D(\Omega; C^\infty_{\text{per}}(Y)^2) \) in \( Z \).

Remark 26. We have silently used the fact that the vanishing divergence in the definition for \( Z \) and the expression for \( Z \cap D(\Omega; C^\infty_{\text{per}}(Y)^2) \) in (14) and (15), respectively, does not upset the inheritance of the density property for \( D(\Omega; C^\infty_{\text{per}}(Y)^2) \) in \( L^2(\Omega; L^2_{\text{per}}(Y)^2) \).
To obtain the orthogonal complement, we need Lemma 27 [10].

Lemma 27. Assume that $Y$ is a unit cube in $\mathbb{R}^N$. Let $f \in L^2_{\text{per}}(Y)^N$ be orthogonal to the space

$$\left\{ g \in C^\infty_{\text{per}}(Y)^N : \sum_{i=1}^N \frac{\partial g_i}{\partial y_i} = 0 \right\}$$

of divergence free functions. Then, for some $h \in W_{\text{per}}(Y)$,

$$(f_i)_{i=1,...,N} = \left( \frac{\partial h}{\partial y_i} \right)_{i=1,...,N}.$$

We are now ready to characterize the orthogonal complement.

Proposition 28. The orthogonal complement $Z^\perp \subset L^2(\Omega; L^2_{\text{per}}(Y)^2)$ of $Z$, defined in the proof of Proposition 25, is

$$Z^\perp = \left\{ \left( \zeta_i \frac{\partial u_1}{\partial y_i} \right)_{i=1,2} : u_1 \in L^2(\Omega; W_{\text{per}}(Y)) \right\}.$$

Proof. Suppose $v \in Z$ and $w_1 \in Z^\perp$. Then, by definition,

$$0 = \int_\Omega \int_Y \sum_{i=1}^2 v_i(x, y)w_1,i(x, y) \, dy \, dx$$

$$= \int_\Omega \int_Y \sum_{i=1}^2 v_i(x, y)\zeta_i(x) \frac{w_1,i(x, y)}{\zeta_i(x)} \, dy \, dx.$$

Since

$$\left( \frac{w_1,i(x, \cdot)}{\zeta_i(x)} \right)_{i=1,2} \in L^2_{\text{per}}(Y)^2 \quad \text{a.e. } x \in \Omega,$$

Lemma 27 implies that, for some $u_1(x, \cdot) \in W_{\text{per}}(Y)$ a.e. $x \in \Omega$,

$$\left( \frac{w_1,i(x, \cdot)}{\zeta_i(x)} \right)_{i=1,2} = \left( \frac{\partial u_1}{\partial y_i} \right)_{i=1,2} \quad \text{a.e. } x \in \Omega.$$

Hence, for some $u_1(x, \cdot) \in W_{\text{per}}(Y)$ a.e. $x \in \Omega$,

$$\left( w_1,i(x, \cdot) \right)_{i=1,2} = \left( \zeta_i(x) \frac{\partial u_1}{\partial y_i} \right)_{i=1,2} \quad \text{a.e. } x \in \Omega.$$

We know that $Z^\perp \subset L^2(\Omega; L^2_{\text{per}}(Y)^2)$, so $w_1 \in L^2(\Omega; L^2_{\text{per}}(Y)^2)$ holds. This implies

$$\|u_1\|_{L^2(\Omega; W_{\text{per}}(Y))} = \|\nabla_y u_1\|_{L^2(\Omega; L^2_{\text{per}}(Y)^2)}$$

$$\leq \max \left\{ 1, \frac{1}{2\sqrt{2}} \right\} \|w_1\|_{L^2(\Omega; L^2_{\text{per}}(Y)^2)}$$

$$< \infty,$$
where we have recalled that \( \Omega = (a_1, b_1) \times (a_2, b_2) \). Thus, \( u_1 \in L^2(\Omega; W_{\text{per}}(Y)) \), and we conclude that

\[
Z^\perp = \left\{ \left( \zeta_i \frac{\partial u_1}{\partial y_i} \right)_{i=1,2} : u_1 \in L^2(\Omega; W_{\text{per}}(Y)) \right\},
\]

and we are done. \( \square \)

We can now formulate a preliminary homogenization result in Proposition 29.

**Proposition 29.** Assume

\[
A \in C_{\text{per}}(Y)^{2 \times 2} \cap \mathcal{M}(r, s; \mathbb{R}^2)
\]

and let \( \{\alpha^h\} \) and \( \zeta \) be given by (11) and (12), respectively. Then \( \{A \circ \alpha^h\} \) \( H \)-converges to \( B \) given by

\[
\sum_{i=1}^{2} b_{ij} \frac{\partial u}{\partial x_j} \bigg|_{i=1,2} = \left( \int \sum_{j=1}^{2} a_{ij}(y) \left( \frac{\partial u}{\partial x_j} + \zeta_j \frac{\partial u_1}{\partial y_j}(\cdot, y) \right) dy \right)_{i=1,2}, \tag{16}
\]

\( u \in H^1_0(\Omega) \) being the weak limit of the sequence \( \{u^h\} \) of solutions to (10), if \( u \in H^1_0(\Omega) \) and \( u_1 \in L^2(\Omega; W_{\text{per}}(Y)) \) uniquely solve the homogenized problem

\[
\int_{\Omega} \int_{Y} \sum_{i=1}^{2} \sum_{j=1}^{2} a_{ij}(y) \left( \frac{\partial u}{\partial x_j}(x) + \zeta_j(x) \frac{\partial u_1}{\partial y_j}(x, y) \right) \frac{\partial v}{\partial x_i}(x) dy \, dx = \int_{\Omega} f(x) v(x) \, dx
\]

for all \( v \in H^1_0(\Omega) \), and, for each \( x \in \Omega \), the local problem

\[
\int_{Y} \sum_{i=1}^{2} \sum_{j=1}^{2} a_{ij}(y) \left( \frac{\partial u}{\partial x_j}(x) + \zeta_j(x) \frac{\partial u_1}{\partial y_j}(x, y) \right) \zeta_i(x) \frac{\partial v}{\partial y_i}(y) dy = 0 \tag{17}
\]

for all \( v \in W_{\text{per}}(Y) \).

**Proof.** This is an immediate consequence of Theorem 21 together with Proposition 25 and Proposition 28. \( \square \)

It is possible to improve Proposition 29 to yield an explicit homogenized matrix and a local problem of the same type as in Theorem 19. Indeed, we have Theorem 30.

**Theorem 30.** Assume

\[
A \in C_{\text{per}}(Y)^{2 \times 2} \cap \mathcal{M}(r, s; \mathbb{R}^2)
\]
and let \( \{ \alpha_h \} \) and \( \zeta \) be given by (11) and (12), respectively. Then \( \{ A \circ \alpha_h \} \) \( H \)-converges to \( B \) given by

\[
(b_{ij})_{i,j=1,2} = \left( \int_Y \sum_{k=1}^{2} a_{ik}(y) \left( \delta_{kj} + \zeta_{k} \frac{\partial z_{j}}{\partial y_{k}}(\cdot, y) \right) \, dy \right)_{i=1,2},
\]

where \( z \in L^\infty(\Omega; W_{\text{per}}(Y)^2) \) uniquely solves, for each \( x \in \Omega \), the local problem

\[
- \left( \sum_{i=1}^{2} \sum_{k=1}^{2} \zeta_{i}(x) \frac{\partial}{\partial y_{k}} \left( a_{ik} \left( \delta_{kj} + \zeta_{k}(x) \frac{\partial z_{j}}{\partial y_{k}}(x, \cdot) \right) \right) \right)_{j=1,2} = 0 \quad \text{in } Y. \tag{19}
\]

Proof. The proof will be performed in four steps, where the first step introduces an ansatz, the second and third steps derive the homogenized matrix (18) and the local problem (19), respectively. In the last step we prove the uniqueness of the solution to the local problem.

Step (i): Ansatz. Let \( u \in H_0^1(\Omega) \) and \( u_1 \in L^2(\Omega; W_{\text{per}}(Y)) \) solve the system of equations in Proposition 29 and make the ansatz

\[
u_1 = \sum_{j=1}^{2} \frac{\partial u}{\partial x_j} z_j,
\]

where \( z \in L^\infty(\Omega; W_{\text{per}}(Y)^2) \). We will see later why it is necessary that we must constrain ourselves to \( z(\cdot, y) \in L^\infty(\Omega)^2 \) a.e. \( y \in \mathbb{R}^2 \).

Step (ii): Homogenized matrix. Let us first derive the expression (18) for the homogenized matrix. From (16) and (20) we get

\[
\left( \sum_{j=1}^{2} b_{ij} \frac{\partial u}{\partial x_j} \right)_{i=1,2} = \left( \sum_{j=1}^{2} \int_Y \sum_{k=1}^{2} a_{ik}(y) \left( \delta_{kj} + \zeta_{k} \frac{\partial z_{j}}{\partial y_{k}}(\cdot, y) \right) \, dy \frac{\partial u}{\partial x_j} \right)_{i=1,2},
\]

which is satisfied if

\[
(b_{ij})_{i,j=1,2} = \left( \int_Y \sum_{k=1}^{2} a_{ik}(y) \left( \delta_{kj} + \zeta_{k} \frac{\partial z_{j}}{\partial y_{k}}(\cdot, y) \right) \, dy \right)_{i,j=1,2},
\]

where we note that \( B \in L^\infty(\Omega)^{2 \times 2} \), which requires \( z(\cdot, y) \in L^\infty(\Omega)^2 \) a.e. \( y \in \mathbb{R}^2 \).

Step (iii): Local problem. Next, let us derive the local problem (19). Fix some \( v \in W_{\text{per}}(Y) \) to be used in the local problem (17) in Proposition 29 whose
left-hand side becomes, for each \( x \in \Omega, \)

\[
\int_\gamma \sum_{i=1}^{2} \sum_{k=1}^{2} a_{ik}(y) \left( \frac{\partial u}{\partial x_k}(x) + \zeta_k(x) \frac{\partial}{\partial y_k} \sum_{j=1}^{2} \frac{\partial u}{\partial x_j}(x) z_j(x, y) \right) \zeta_i(x) \frac{\partial v}{\partial y_i}(y) \, dy
\]

\[
= \sum_{j=1}^{2} \int_\gamma \sum_{i=1}^{2} \sum_{k=1}^{2} \zeta_i(x) \frac{\partial v}{\partial y_i}(y) a_{ik}(y) \left( \delta_{kj} + \zeta_k(x) \frac{\partial z_j}{\partial y_k}(x, y) \right) \, dy \frac{\partial u}{\partial x_j}(x).
\]

This must be zero, which is the case if, for each \( x \in \Omega, \)

\[
\left( \int_\gamma \sum_{i=1}^{2} \sum_{k=1}^{2} \zeta_i(x) \frac{\partial v}{\partial y_i}(y) a_{ik}(y) \left( \delta_{kj} + \zeta_k(x) \frac{\partial z_j}{\partial y_k}(x, y) \right) \, dy \right)_{j=1,2} = 0.
\]

By partial integrating and using the divergence theorem, we obtain, for each \( x \in \Omega, \)

\[
\left( \int_\gamma \sum_{i=1}^{2} \sum_{k=1}^{2} n_i(y) \zeta_i(x) v(y) a_{ik}(y) \left( \delta_{kj} + \zeta_k(x) \frac{\partial z_j}{\partial y_k}(x, y) \right) \, dS \right)_{j=1,2}
\]

\[- \left( \int_\gamma v(y) \sum_{i=1}^{2} \sum_{k=1}^{2} \zeta_i(x) \frac{\partial}{\partial y_i} \left( a_{ik}(y) \left( \delta_{kj} + \zeta_k(x) \frac{\partial z_j}{\partial y_k}(x, y) \right) \right) \, dy \right)_{j=1,2} = 0,
\]

where \( n \) is the unit outward normal to \( \partial Y. \) Since \( v \) is \( Y \)-periodic, the surface integral vanishes, and we are left with, for each \( x \in \Omega, \)

\[- \left( \int_\gamma v(y) \sum_{i=1}^{2} \sum_{k=1}^{2} \zeta_i(x) \frac{\partial}{\partial y_i} \left( a_{ik}(y) \left( \delta_{kj} + \zeta_k(x) \frac{\partial z_j}{\partial y_k}(x, y) \right) \right) \, dy \right)_{j=1,2} = 0,
\]

which certainly is satisfied if, for each \( x \in \Omega, \)

\[- \left( \sum_{i=1}^{2} \sum_{k=1}^{2} \zeta_i(x) \frac{\partial}{\partial y_i} \left( a_{ik} \left( \delta_{kj} + \zeta_k(x) \frac{\partial z_j}{\partial y_k} \right) \right) \right)_{j=1,2} = 0,
\]

and we have shown (19).

**Step (iv): Uniqueness.** It remains to prove the uniqueness of the solution to the local problem. Fixing \( x \in \Omega, \) we can define a new, rescaled, \( y \)-variable \( \tilde{y}^{(x)} \) by letting

\[
(y_i^{(x)}(y))_{i=1,2} = \left( \frac{y_i}{\zeta_i(x)} \right)_{i=1,2}, \quad y \in Y.
\]

The new local problem can be written

\[- \left( \sum_{i=1}^{2} \sum_{k=1}^{2} \frac{\partial}{\partial y_i^{(x)}} \left( a_{ik}^{(x)} \left( \delta_{kj} + \frac{\partial z_j^{(x)}}{\partial y_k^{(x)}} \right) \right) \right)_{j=1,2} = 0 \quad \text{in } Y^{(x)},
\]

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effectively a “classical” local problem, where the $2 \times 2$ matrix $A^{(x)}$ and the 2-tuple $z^{(x)}$ are given according to

$$\begin{cases} A^{(x)} \circ y^{(x)} = A \\ z^{(x)} \circ y^{(x)} = z(x, \cdot) \end{cases},$$

and $Y^{(x)} = (0, 1) \times \left(0, \frac{1}{2x_2}\right)$. Since

$$A^{(x)} \in C_{\text{per}}(Y^{(x)})^{2 \times 2} \cap M(r, s; \mathbb{R}^2),$$

uniqueness is ensured just as in the “classical” case.

**Remark 31.** In Step (iv), the fact that $Y^{(x)}$, for each $x \in \Omega$, is a rectangle rather than the unit cube will, of course, not spoil our argumentation.

![Figure 2: The scalar factor of $A \circ \alpha^h$, $h = 3$.](image)

6 A numerical illustration

As an illustration of the theoretical results of Section 5, consider the sequence of problems (5) with $A^h = A \circ \alpha^h$ where $A$ is given as a product between a scalar factor and a unit matrix according to

$$(a_{ij}(y))_{i,j=1,2} = (1 + \frac{6}{10} \sin 2\pi y_1 \sin 2\pi y_2)\delta_{ij})_{i,j=1,2}, \quad y \in \mathbb{R}^2,$$

and let $\Omega = (\delta, 2)^2$ where $\delta \gtrsim 0$. Apparently,

$$A \in C_{\text{per}}(Y)^{2 \times 2} \cap M(\frac{19}{10}, \frac{19}{10}; \mathbb{R}^2),$$
so Theorem 30 is applicable. Furthermore, 
\[(a_{ij} \circ \alpha^h(x))_{i,j=1,2} = (1 + \frac{9}{10} \sin 2\pi h x_1 \sin 2\pi h x_2^2)(\delta_{ij})_{i,j=1,2}, \quad x \in (\delta, 2)^2,\]
see Figure 2. By using (19), the local problem for \(z\) becomes, for each \(x \in \Omega,\)
\[-\left(\sum_{i=1}^{2} \sum_{j=1}^{2} \frac{\partial}{\partial y_i} \left(c_{ij}(x, \cdot) \frac{\partial z_k}{\partial y_j}(x, \cdot) \right)\right)_{k=1,2} = (g_k(x, \cdot))_{k=1,2} \quad \text{in } Y, \quad (21)\]
where the matrix \(C\) and the 2-tuple \(g\) are given by
\[
\begin{align*}
c_{11}(x, y) &= (1 + \frac{9}{10} \sin 2\pi y_1 \sin 2\pi y_2) \\
c_{22}(x, y) &= 4x_2^2 \left(1 + \frac{9}{10} \sin 2\pi y_1 \sin 2\pi y_2\right) \\
c_{ij}(x, y) &= 0, \quad i \neq j
\end{align*}
\]
and
\[
\begin{align*}
g_1(x, y) &= \frac{9}{5} \pi \cos 2\pi y_1 \sin 2\pi y_2 \\
g_2(x, y) &= \frac{18}{5} \pi x_2^2 \sin 2\pi y_1 \cos 2\pi y_2
\end{align*}
\]
respectively. Solving (21) numerically (effectively we have a one-parameter family, with respect to \(x_2,\) of partial differential equations to solve) and then computing the homogenized matrix through (18), we get that \(B\) is diagonal with non-vanishing entries, functions with respect to \(x_2\) only, given according to Figure 3. Note the interesting feature that \(b_{11}|_{x_2=1/2} = b_{22}|_{x_2=1/2},\) where \(B\) obviously is proportional to the unit matrix, i.e., along the line \(x_2 = \frac{1}{2},\) the homogenized matrix is isotropic. The heuristic explanation to this is simple;
for large $h$ the mapped periodicity cells in the vicinity of the line $x_2 = \frac{1}{2}$ are near-perfect squares. Of course, it is crucial that $A$ is isotropic to begin with in order for the map $A \circ \alpha^h$ to exhibit a near-isotropy property on such mapped, near-perfect squares.

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