On automatic bias reduction for extreme expectile estimation
Antoine Usseglio-Carleve, Stéphane Girard, Gilles Stupfler

To cite this version:
Antoine Usseglio-Carleve, Stéphane Girard, Gilles Stupfler. On automatic bias reduction for extreme expectile estimation. CMStatistics 2020 - 13th International Conference of the ERCIM WG on Computational and Methodological Statistics, Dec 2020, London / Virtual, United Kingdom. hal-03087164

HAL Id: hal-03087164
https://hal.science/hal-03087164v1
Submitted on 23 Dec 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
On automatic bias reduction for extreme expectile estimation

Joint work with
Stéphane Girard
Gilles Stupfler

Antoine Usseglio-Carleve
Introduction

Expectile estimation

Bias reduction

Simulation study and real data example
**Expectiles**

Quantiles [Koenker and Bassett, 1978] have been recently criticized [Acerbi, 2002], [Artzner et al., 1999] for not being a coherent risk measure.

\[
q(\alpha) \in \arg\min_{t \in \mathbb{R}} \mathbb{E} [\rho_{\alpha}(Y - t) - \rho_{\alpha}(Y)],
\]

where \( \rho_{\alpha}(y) = |\alpha - 1_{\{y \leq 0\}}| |y| \). Some authors thus proposed expectiles [Newey and Powell, 1987] as an alternative:

\[
e(\alpha) = \arg\min_{t \in \mathbb{R}} \mathbb{E} [\eta_{\alpha}(Y - t) - \eta_{\alpha}(Y)],
\]

where \( \eta_{\alpha}(y) = |\alpha - 1_{\{y \leq 0\}}| y^2 \).
Expectiles

Figure: Quantile (red) and expectile (blue) loss functions for $\alpha = 0.05$, 0.5 and 0.95.
Expectiles

• According to [Jones, 1994], $e(\alpha)$ is solution of

$$\bar{E}(y) = \frac{E [(Y - y)1_{\{Y > y\}}]}{2E [(Y - y)1_{\{Y > y\}}] + (y - E[Y])} = 1 - \alpha.$$  

• According to [Bellini et al., 2014], if $\bar{F}(y) = y^{-1/\gamma} \ell(y)$, then

$$\lim_{\alpha \to 1} \frac{\bar{F}(e(\alpha))}{1 - \alpha} = \gamma^{-1} - 1; \quad \lim_{\alpha \to 1} \frac{e(\alpha)}{q(\alpha)} = (\gamma^{-1} - 1)^{-\gamma},$$

for $\gamma < 1$. 

Introduction

Expectile estimation

Bias reduction

Simulation study and real data example
Intermediate expectiles estimation

$Y_1, \ldots, Y_n$ are i.i.d. realizations of $Y$. If $\alpha_n \ll 1 - 1/n$ (or equivalently $n(1 - \alpha_n) \to \infty$ as $n \to \infty$) is an intermediate sequence, two approaches have been considered for expectile estimation:

- The first one, used in [Daouia et al., 2018], directly derives from the definition of expectiles:

$$\hat{e}_n(\alpha_n) = \operatorname{arg\,min}_{\theta \in \mathbb{R}} \sum_{i=1}^{n} \eta_{\alpha_n}(Y_i - \theta).$$

- The second one, introduced in [Girard et al., 2020], uses the property of [Jones, 1994]:

$$\hat{e}_n(\alpha_n) = \inf \left\{ y \in \mathbb{R} \mid \hat{E}_n(y) \leq 1 - \alpha_n \right\},$$

with

$$\hat{E}_n(y) = \frac{\sum_{i=1}^{n} (Y_i - y) \mathbb{1}\{Y_i > y\}}{\sum_{i=1}^{n} |Y_i - y|}.$$
Extreme expectiles estimation

Let us assume

$$\lim_{t \to \infty} \frac{F(ty)}{F(t)} = y^{-1/\gamma}.$$  

In this context, extreme quantiles may be estimated using the Weissman estimator. If $\beta_n >> 1 - 1/n$ and $\alpha_n << 1 - 1/n$,

$$\frac{q(\beta_n)}{q(\alpha_n)} \approx \left( \frac{1 - \beta_n}{1 - \alpha_n} \right)^{-\gamma} \Rightarrow \hat{q}^*_n(\beta_n) = \hat{q}_n(\alpha_n) \left( \frac{1 - \beta_n}{1 - \alpha_n} \right)^{-\hat{\gamma}}.$$  

Since quantiles and expectiles are asymptotically proportional, the same approximation holds for extreme expectiles, and [Daouia et al., 2018] introduced

$$\hat{e}^*_n(\beta_n) = \hat{e}_n(\alpha_n) \left( \frac{1 - \beta_n}{1 - \alpha_n} \right)^{-\hat{\gamma}} \quad \text{or} \quad \tilde{e}^*_n(\beta_n) = \tilde{q}^*_n(\beta_n) \left( \hat{\gamma}^{-1} - 1 \right)^{-\hat{\gamma}}.$$
Tail index estimation

Let us consider the second order assumption \((C_2)\):

\[
\forall y > 0, \lim_{t \to \infty} \frac{1}{A(1/F(t))} \left( \frac{F(ty)}{F(t)} - y^{-1/\gamma} \right) = y^{-1/\gamma} \frac{y^{\rho/\gamma} - 1}{\gamma \rho}.
\]

The most widespread estimator of the tail index \(\gamma\) is the Hill estimator:

\[
\hat{\gamma}_{k_n}^H = \frac{1}{k_n} \sum_{i=1}^{k_n} \log \frac{Y_{n-i+1,n}}{Y_{n-k_n,n}},
\]

where \(k_n \to \infty\) and \(k_n/n \to 0\) as \(n \to \infty\). Under \((C_2)\), and if \(\sqrt{k_n}A(n/k_n) \to \lambda \in \mathbb{R}\), then

\[
\sqrt{k_n} (\hat{\gamma}_{k_n}^H - \gamma) \to \mathcal{N} \left( \frac{\lambda}{1 - \rho}, \gamma^2 \right).
\]
Tail index estimation

Using the asymptotic relationship between quantiles and expectiles, we can introduce the following tail index estimator (see [Girard et al., 2020]):

$$\hat{\gamma}_{kn}^E = \left(1 + \frac{n\hat{F}_n(\hat{e}_n(1 - k_n/n))}{k_n}\right)^{-1}.$$ 

Under \((C_2)\) with \(0 < \gamma < 1/2\), and if \(\sqrt{k_n}A(n/k_n) \to \lambda_1 \in \mathbb{R}\) and \(\sqrt{k_n}q(1 - k_n/n)^{-1} \to \lambda_2 \in \mathbb{R}\), then

$$\sqrt{k_n} \left(\hat{\gamma}_{kn}^E - \gamma\right) \to \mathcal{N} \left( \frac{\gamma (\gamma^{-1} - 1)^{1-\rho}}{1 - \rho - \gamma} \lambda_1 + \gamma^2 (\gamma^{-1} - 1)^{\gamma+1} \mathbb{E}[Y]\lambda_2, \frac{\gamma^3 (1 - \gamma)}{1 - 2\gamma} \right).$$
Tail index estimation

Asymptotic variances

Figure: Asymptotic variances of $\tilde{\gamma}_{\eta n}^H$ (black curve) and $\tilde{\gamma}_{\eta n}^E$ (red curve) as functions of $\gamma \in (0, 1)$. 
Back to extreme expectiles estimation

Figure: Mean estimates of 1,000 estimates $\hat{e}_n^*(\beta_n)$ using $\tilde{\gamma}_{k_n}^H$ (black) and $\tilde{\gamma}_{k_n}^E$ (blue) for $k_n = 1, \ldots, 100$ in the case of a Burr distribution with $\gamma = 0.25$, $\rho = -5$, $n = 1,000$ and $\beta_n = 1 - 5/n = 0.995$.

Why so much bias?
Introduction

Expectile estimation

Bias reduction

Simulation study and real data example
Bias reduction of the tail index estimators

By doing the assumption that $A(t) = b \gamma t^\rho$, the following bias-reduced version of $\tilde{\gamma}_{k_n}^H$ is proposed in [Gomes and Martins, 2002]:

$$
\tilde{\gamma}_{k_n}^H = \hat{\gamma}_{k_n}^H \left(1 - \frac{\bar{b}}{1 - \bar{\rho}} \left(\frac{n}{k_n}\right)^{\bar{\rho}}\right).
$$

We thus propose a similar approach for $\tilde{\gamma}_{k_n}^E$. For that purpose, we notice $\bar{F}(e(\alpha))/(1 - \alpha) = (\gamma^{-1} - 1)(1 + r(\alpha))$, where $1 + r(\alpha) =

$$
\left(1 - \frac{\mathbb{E}[Y]}{e(\alpha)}\right) \frac{1}{2\alpha - 1} \left(1 + A \left(\frac{1}{\bar{F}(e(\alpha))}\right) \frac{1}{\gamma(1 - \gamma - \rho)(1 + o(1))}\right)^{-1}
$$

as $\alpha \uparrow 1$. 

Antoine Usseglio-Carleve - On automatic bias reduction for extreme expectile estimation

14 of 25
Bias reduction of the tail index estimators

We thus introduce the following bias-reduced estimator:

$$\tilde{\gamma}_{k_n}^E = \left(1 + \frac{n\hat{F}_n(\hat{e}_n(1 - k_n/n))}{k_n} \frac{1}{1 + \bar{r}(1 - k_n/n)} \right)^{-1},$$

where $1 + \bar{r}(1 - k_n/n) =

$$\left(1 - \frac{\bar{Y}_n}{\hat{e}_n(1 - k_n/n)}\right) \frac{1}{1 - 2k_n/n} \left(1 + \frac{b[\hat{F}_n(\hat{e}_n(1 - k_n/n))]^{-\bar{\rho}}}{1 - \bar{\gamma} - \bar{\rho}} \right)^{-1}.$$

Under some conditions concerning $\bar{\rho}$ and $\bar{b}$, we can prove

$$\sqrt{k_n} \left(\tilde{\gamma}_{k_n}^E - \gamma\right) \to \mathcal{N} \left(0, \frac{\gamma^3(1 - \gamma)}{1 - 2\gamma}\right).$$
Bias reduction of the extrapolation step

We can find some bias reduction approaches for extreme quantile estimators (see for instance [Gomes and Pestana, 2007]). The second order condition $C_2$ giving

$$q(\beta_n) = q \left(1 - \frac{k_n}{n}\right) \left(\frac{n(1 - \beta_n)}{k_n}\right)^{-\gamma} \left(1 + \frac{\left(\frac{n(1 - \beta_n)}{k_n}\right)^{-\rho}}{\rho} - 1 A \left(\frac{n}{k_n}\right) (1 + o(1))\right),$$

we easily deduce, with $A(t) = b\gamma t^\rho$,

$$\hat{q}_n^{*, RB}(\beta_n) = \hat{q}_n^*(\beta_n) \left(1 + \frac{[n(1 - \beta_n)/k_n]^{-\bar{\rho}}}{\bar{\rho}} - 1 \overline{b\gamma}(n/k_n)^{\bar{\rho}}\right).$$
Bias reduction of the extrapolation step

The bias reduction of extreme expectiles is less obvious, and 3 bias terms have to be eliminated, hence

\[
\begin{align*}
\hat{e}^{*, RB}_n (\beta_n) &= \hat{e}^*_n (\beta_n)(1 + \overline{B}_{1,n})(1 + \overline{B}_{2,n})(1 + \overline{B}_{3,n}) \\
\tilde{e}^{*, RB}_n (\beta_n) &= \tilde{e}^*_n (\beta_n)(1 + \overline{B}_{1,n})(1 + \overline{B}_{3,n})
\end{align*}
\]

where

\[
\begin{align*}
1 + \overline{B}_{1,n} &= 1 + \frac{[n(1-\beta_n)/k_n]^{-\rho} - 1}{\rho} \gamma (n/k_n)^{\rho} \\
1 + \overline{B}_{2,n} &= (1 + \overline{r} (1 - \frac{k_n}{n}))^{\gamma} \left(1 + \frac{(\gamma^{-1} - 1)^{-\rho} (1 + \overline{r}(1 - \frac{k_n}{n}))^{-\rho} - 1}{\rho} \gamma (n/k_n)^{\rho} \right)^{-1} \\
1 + \overline{B}_{3,n} &= (1 + \overline{r}^* (\beta_n))^{\gamma} \left(1 + \frac{(\gamma^{-1} - 1)^{-\rho} (1 + \overline{r}^*(\beta_n))^{-\rho} - 1}{\rho} \gamma (1 - \beta_n)^{\rho} \right)
\end{align*}
\]

Note that these bias reduced estimators are proposed in the R package Expectrem.
Introduction

Expectile estimation

Bias reduction

Simulation study and real data example
Simulation study

• We simulate $n = 1,000$ independent realizations $Y_1, \ldots, Y_n$ of a Burr distribution:

$$
\bar{F}(y) = (1 + y^{-\rho/\gamma})^{1/\rho}.
$$

• We consider $\rho = -5, -1$ and $-0.5$, and $\gamma = 0.1, 0.2, 0.3$ and $0.4$.
• For each case, we estimate the expectile of level $\beta_n = 1 - 5/n = 0.995$.
• How to choose $k_n$?
Simulation study

For the Hill based estimators, we choose the $k_n$ which minimizes the following AMSE with $A(t) = b\gamma t^\rho$:

$$A \left( \frac{n}{k_n} \right)^2 \left( \frac{1}{1 - \rho} \right)^2 + \frac{\gamma^2}{k_n},$$

hence

$$k_n^H = \left( \frac{1 - \rho}{-2\rho b^2} \right)^{1/(1 - 2\rho)} n^{-2\rho/(1 - 2\rho)}.$$

For $\hat{\gamma}_{k_n}^E$, we minimize the following Partial AMSE:

$$\left[ \frac{\gamma(\gamma^{-1} - 1)^{1-\rho}}{1 - \gamma - \rho} A(n/k_n) \right]^2 + \frac{\gamma^3(1 - \gamma)}{1 - 2\gamma} \times \frac{1}{k_n},$$

hence

$$\hat{k}_n^E = \min \left( \left[ \left( \frac{(\gamma^{-1} - 1)^{2\rho-1}(1 - \gamma - \rho)^2}{-2\rho b^2 (1 - 2\gamma)} \right)^{1/(1 - 2\rho)} \right], \left\lfloor \frac{n}{2} \right\rfloor - 1 \right).$$
Simulation study
Reinsurance example

Let $Y$ be a claim amount, and $R$ a retention level.

- Insurer pays $\min(Y, R)$.
- Reinsurer pays $\max(Y - R, 0)$.

The reinsurance premium $\Pi(R)$ may be calculated using the distortion principle:

$$\Pi_g(R) = \int_R^\infty g(\bar{F}(x)) \, dx,$$

where $g : [0, 1] \to [0, 1]$ is a nondecreasing concave function.
Reinsurance example

If $Y$ is heavy-tailed and $g(1/.)$ regularly varying with index $\delta < -\gamma$, then

$$\lim_{\beta \uparrow 1} \frac{\Pi g(e(\beta))}{e(\beta) g(1 - \beta)} = \frac{(\gamma^{-1} - 1)^{-\delta}}{-\delta/\gamma - 1},$$

by taking an extreme expectile as retention level. We thus use our expectile estimators to approximate the reinsurance premium, and compare our results with those obtained with the data set secura in [Vandewalle and Beirlant, 2006], with two distortion functions:

$$\left\{ \begin{array}{l} g(x) = x \text{ (Net premium principle)} \\ g(x) = 1 - (1 - x)^{\kappa} \text{ (Dual-power principle)} \end{array} \right.$$
Reinsurance example

![Graph of \( \Pi_g(R) \) as function of \( R = e(\beta) \) for \( \beta \) ranging from \( 1 - 10/n \approx 0.973 \) to \( 1 - 1/(8n) \approx 0.9997 \) (here \( n = 370 \)). The premiums are estimated using \( \tilde{\gamma}^H_{k_n} \) (solid blue curve), \( \tilde{\gamma}^E_{k_n} \) (solid red curve) and the naïve estimator (dotted blue curve). The black curve is constructed by linear interpolation using the estimates found in [Vandewalle and Beirlant, 2006]. \( \kappa = 1.366 \).]
Materials

- R package Expectrem, available at https://github.com/AntoineUC/Expectrem.
- Girard, S., Stupfler, G. and Usseglio-Carleve, A. (2020) On automatic bias reduction for extreme expectile estimation, preprint.
Acerbi, C. (2002). Spectral measures of risk: A coherent representation of subjective risk aversion. *Journal of Banking & Finance*, 26(7):1505–1518.

Artzner, P., Delbaen, F., Eber, J., and Heath, D. (1999). Coherent measures of risk. *Mathematical Finance*, 9(3):203–228.

Bellini, F., Klar, B., Muller, A., and Gianin, E. R. (2014). Generalized quantiles as risk measures. *Insurance: Mathematics and Economics*, 54:41–48.

Daouia, A., Girard, S., and Stupfler, G. (2018). Estimation of tail risk based on extreme expectiles. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 80(2):263–292.
Girard, S., Stupfler, G., and Usseglio-Carleve, A. (2020+). Nonparametric extreme conditional expectile estimation. *Scandinavian Journal of Statistics*.

Gomes, M. I. and Martins, M. J. (2002). “Asymptotically unbiased” estimators of the tail index based on external estimation of the second order parameter. *Extremes, 5*(1):5–31.

Gomes, M. I. and Pestana, D. (2007). A sturdy reduced-bias extreme quantile (VaR) estimator. *Journal of the American Statistical Association, 102*(477):280–292.

Jones, M. (1994). Expectiles and M-quantiles are quantiles. *Statistics & Probability Letters, 20*:149–153.
Koenker, R. and Bassett, G. J. (1978). Regression quantiles. *Econometrica*, 46(1):33–50.

Newey, W. and Powell, J. (1987). Asymmetric least squares estimation and testing. *Econometrica*, 55(4):819–847.

Vandewalle, B. and Beirlant, J. (2006). On univariate extreme value statistics and the estimation of reinsurance premiums. *Insurance: Mathematics and Economics*, 38(3):441–459.