ESTIMATION OF RECURRENCE FOR NILPOTENT GROUP ACTION

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Abstract. We estimate size of recurrence of an action of a nilpotent group by homeomorphisms of a compact space for polynomial mappings into a nilpotent group form the partial semigroup \((P_f(N), \circ)\). To do this we have used algebraic structure of the Stone-Čech copactification partial semigroup and that of the given nilpotent group.

1. Introduction

One of the earliest Ramsey theoretic result is celebrated van der Waerden theorem on arithmetic progressions, which states that if the set of integers is partitioned into finitely many classes then at least one of the classes contains arbitrarily long arithmetic progressions.

**Theorem 1.1.** Whenever we partition the set of integers into finitely many cells then one of the cell will contain arbitrarily long arithmetic progression.

Furstenberg and Weiss [FW] offered a new approach, based on the methods of topological dynamics, to results of this type.

**Theorem 1.2.** Let \((X, d)\) be a compact metric space and let \(T_1, T_2, \ldots, T_k\) be commuting self-homeomorphisms of \(X\): Then for any \(\epsilon > 0\) there exist \(x \in X\) and \(n \in \mathbb{N}\) such that \(d(T_i^n x, x) < \epsilon\) for all \(i = 1, 2, \ldots, k\).

An IP version of the above theorem was also presented in [FW].

**Theorem 1.3.** Let \(T^{(1)}_{\alpha \in \mathcal{F}}, T^{(2)}_{\alpha \in \mathcal{F}}, \ldots, T^{(k)}_{\alpha \in \mathcal{F}}\) be an IP-systems in \(G\) of commuting self-homeomorphisms of a compact metric space \((X, d)\). Then for any \(\epsilon > 0\) there exists \(x \in X\) and \(n \in \mathbb{N}\) such that \(d(T^{(i)}_{\alpha \in \mathcal{F}} x, x) < \epsilon\) for all \(i = 1, 2, \ldots, k\).

A combinatorial version of the above theorem is the following.

**Theorem 1.4.** Let \(G\) be an abelian group, and let \(T^{(1)}_{\alpha \in \mathcal{F}}, T^{(2)}_{\alpha \in \mathcal{F}}, \ldots, T^{(k)}_{\alpha \in \mathcal{F}}\) be IP-systems in \(G\). For any finite coloring of \(G\) there exist \(h \in G\) and a nonempty \(\alpha \in \mathcal{F}\) such that the elements \(hT^{(1)}_{\alpha \in \mathcal{F}}, hT^{(2)}_{\alpha \in \mathcal{F}}, \ldots, hT^{(k)}_{\alpha \in \mathcal{F}}\) all have the same color.

In [B13] Theorem 4.1] Bergelson and Leibman established a nil-IP-multiple recurrence theorem which extended all the abelian results mentioned above to a nilpotent setup.

**Theorem 1.5.** Let \(G\) be a nilpotent group of self-homeomorphisms of a compact metric space \((X, d)\) and let \(P_1, \ldots, P_k : \mathcal{F} \rightarrow G\) be polynomial mappings satisfying \(P_i(0) = 1_G\) for all \(i = 1, 2, \ldots, k\). Then for any \(\epsilon > 0\) there exist \(x \in X\) and \(\alpha \in \mathcal{F}\) such that \(d(P_i(\alpha)x, x) < \epsilon\) for all \(i = 1, 2, \ldots, k\).
In the present work we will prove that the collection of $\alpha$’s in the partial semigroup (to be defined in the next section) $(P_f(\mathbb{N}), \psi)$ will be an IP’-set. Our machinery in the proof will be using algebraic structure of Stone-Čech compactification of discrete semigroup. In fact we are inspired by [H] and [HM].

2. Preliminaries

For our purpose let us first introduce a brief algebraic structure of $\beta S$ for a discrete semigroup $(S,+)$. We take the points of $\beta S$ to be the ultrafilters on $S$, identifying the principal ultrafilters with the points of $S$ and thus pretending that $S \subseteq \beta S$. Given $A \subseteq S$ let us set,

$$\overline{A} = \{ p \in \beta S : A \in p \}.$$ 

Then the set $\{ \overline{A} : A \subseteq S \}$ is a basis for a topology on $\beta S$. The operation $+$ on $S$ can be extended to the Stone-Čech compactification $\beta S$ of $S$ so that $(\beta S, +)$ is a compact right topological semigroup (meaning that for any $p \in \beta S$, the function $\rho_p : \beta S \to \beta S$ defined by $\rho_p(q) = q + p$ is continuous) with $S$ contained in its topological center (meaning that for any $x \in S$, the function $\lambda_x : \beta S \to \beta S$ defined by $\lambda_x(q) = x + q$ is continuous). Given $p, q \in \beta S$ and $A \subseteq S$, $A \in p + q$ if and only if $\{ x \in S : -x + A \in q \} \in p$, where $-x + A = \{ y \in S : x + y \in A \}$.

A nonempty subset $I$ of a semigroup $(T, +)$ is called a left ideal of $T$ if $T + I \subseteq I$, a right ideal if $I + T \subseteq I$, and a two sided ideal (or simply an ideal) if it is both a left and right ideal. A minimal left ideal is the left ideal that does not contain any proper left ideal. Similarly, we can define minimal right ideal and smallest ideal.

Any compact Hausdorff right topological semigroup $(T, +)$ has a smallest two sided ideal

$$K(T) = \bigcup \{ L : L \text{ is a minimal left ideal of } T \} = \bigcup \{ R : R \text{ is a minimal right ideal of } T \}$$

Given a minimal left ideal $L$ and a minimal right ideal $R$, $L \cap R$ is a group, and in particular contains an idempotent. An idempotent in $K(T)$ is called a minimal idempotent. If $p$ and $q$ are idempotents in $T$, we write $p \leq q$ if and only if $p + q = q + p = p$. An idempotent is minimal with respect to this relation if and only if it is a member of the smallest ideal. See [HS] for an elementary introduction to the algebra of $\beta S$ and for any unfamiliar details.

**Definition 2.1.** (Partial semigroup) Let $G$ be a set, let $X \subseteq G \times G$ and let $* : X \to G$ be an operation. The triple $(G, X, *)$ is a partial semigroup if it satisfies the following properties:

For any $x, y, z \in G$, then $(x * y) * z = x * (y * z)$ in the sense that either both sides are undefined or both are defined and equal. For any $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in G$ there exists $y \in G \setminus \{ x_1, \ldots, x_n \}$ such that $x_1 * y$ is defined. Observe that the second condition implies that $G$ is infinite. A partial semigroup is commutative if $x * y = y * x$ for every $(x, y) \in X$.

**Example 2.2.** Let $G = \mathcal{F}(\mathbb{N})$, let $X := \{ (\alpha, \beta) \in G^2 : \alpha \cap \beta = \emptyset \}$ be the family of all pairs of disjoint sets, and let $* : X \to G$ be the union. It is easy to check that this is a commutative partial semigroup. We shall denote this partial semigroup as $(\mathcal{F}(\mathbb{N}), \psi)$.
To define a nil IP-polynomial we need to recall some facts. Let us denote the collection of finite subsets of \( \mathbb{N} \) be \( \mathcal{F} \).

**Definition 2.3.** For a semigroup \( G \), an IP-system is a mapping \( \mathcal{F} \to G \) defined as \( \alpha \to g_\alpha \), so that \( g_{\alpha \cup \beta} = g_\alpha g_\beta \) provided that \( \alpha \cap \beta = \emptyset \).

However if the above \( G \) is commutative in nature, then we can define a IP-polynomial inductively as given below.

**Definition 2.5.** For a nilpotent group \( G \), an IP-polynomial defined as an IP-polynomial inductively as given below.

**Definition 2.4.** Assuming a IP-polynomial of degree 0 be constant, we say that an IP-polynomial defined as \( P : \mathcal{F} \to G \) of degree \( d \) if for any \( \beta \in \mathcal{F} \), there exists a polynomial mapping \( D_\beta P : \mathcal{F}(\mathbb{N} \setminus \beta) \to G \) of degree \( d \leq d - 1 \) such that

\[
P(\alpha \cup \beta) = P(\alpha) + (D_\beta P)(\alpha) \quad \forall \alpha \in \mathcal{F}(\mathbb{N} \setminus \beta).
\]

In the case of abelian \( G \), it was proved in \cite{BL1} Theorem 8.1, that the necessary and sufficient condition for a mapping \( P : \mathcal{F} \to G \) to be an IP-polynomial is that there must exists a \( d \in \mathbb{N} \) and a family \( \{g(j_{12},...,j_{kd})\}_{(j_1,j_2,...,j_d) \in \mathbb{N}^d} \) of elements in \( G \), such that for any \( \alpha \in \mathcal{F} \), one has

\[
P(\alpha) = \prod_{(j_1,j_2,...,j_d) \in \mathbb{N}^d} g(j_{1j_2},...,j_{kd}).
\]

It is a characterization of commutative IP-polynomials which can be generalized in case of nilpotent groups.

**Definition 2.5.** For a nilpotent group \( G \), we define a mapping \( P : \mathcal{F} \to G \) to be a nil IP-polynomial if for some \( d \in \mathbb{N} \) and a linear order \( \prec \) on \( \mathbb{N}^d \) such that for any \( \alpha \in \mathcal{F} \) we have,

\[
P(\alpha) = \prod_{(j_1,j_2,...,j_d) \in \mathbb{N}^d} g(j_{1j_2},...,j_{kd}).
\]

V. Bergelson and Leibman in \cite{BL2} Theorem 4.1 proved the following result:

**Theorem 2.6.** Let \( G \) be a nilpotent group of self-homeomorphisms of a compact metric space \((X, \rho)\). For any weight (defined in next section) \( \omega \in \mathcal{W} \), any \( k \in \mathbb{N} \), and any \( \epsilon > 0 \), \( \exists N \in \mathbb{N} \) such that if \( S \) is a set of cardinality \( \geq N \), and \( A \) is a system of \( k \) polynomial mappings \( \mathcal{F}(S) \to G \) satisfying \( \omega(P) \leq \omega \) and \( P(\emptyset) = 1_G \), \( P \in A \), there exists a point \( x \in X \) and a nonempty \( \alpha \in \mathcal{F}(S) \) such that \( \rho(P(\alpha)x, x) < \epsilon \) \( \forall P \in A \).

In this paper we will prove the following theorem which is algebraic version of the above theorem:

**Theorem 2.7.** Let \( \mathcal{R} \) be a system and \( v = v + v \in \beta \mathcal{F} \) and let \( A \) be a picewise syndetic subset of \( G \) and let \( L \) be a minimal left ideal of \( \beta G \) such that \( A \cap L \neq \emptyset \). Then,

\[
\{ \alpha \in \mathcal{F} : A \cap L \cap \rho \in \mathcal{R} (\mathcal{P}(\alpha)^{-1}A) \neq \emptyset \} \in v.
\]

And we will apply this theorem present the following theorem which is the more refined version of Nilpotent PHJ given in \cite{BL3} \cite{PZK}

**Theorem 2.8.** Let \( X \) be a compact metric space and \( G \) be the nilpotent group of self homeomorphisms acting on \( X \). Let \( A \) be a system consisting of polynomials \( \{P_1, P_2, \ldots, P_k\} \). Then for every \( x \in X, \epsilon > 0 \), there exists some \( a \in G \) such that

\[
\{ \alpha \in \mathcal{F} : \rho(T^{P_i}(a)x, ax) < \epsilon \}
\]

is an IP*- set in the partial semigroup \((\mathcal{P}(\mathbb{N}), \cup)\).
3. POLYNOMIAL MAPPINGS AND TRIANGULAR MONOMIALS

In this section we will follow [BL3]. Let $G_1 \geq G_2 \geq \ldots \geq G_l \geq G_{l+1} \geq \ldots$ be the lower central series of a nilpotent group $G$, and as $G$ is nilpotent there must exist some $n \in \mathbb{N}$ such that $G_n = \{1_G\}$.

Let $\mathcal{F}^{\leq d}(S)$ be the set of all subsets of $S$ of cardinality $\leq d$.

**Definition 3.1.** Let $S$ be a nonempty set. For a Nilpotent Group $G$, Monomial of degree $d$ on $S$ with values in $G$ is a pair $(u, \prec)$, where $u : S^d \to G$, is a mapping and $\prec$ is a linear order on $S^d$. The monomial $(u, \prec)$ induces a monomial mapping $P_u : \mathcal{F}(S) \to G$ by the rule $P_u(\alpha) = \prod_{s \in \alpha}^\prec u(s)$.

The level of a monomial $(u, \prec)$ of degree $d$ is defined to be the positive integer $l$ such that, $u(S^d) \subseteq G_l \setminus G_{l+1}$. For the nilpotency class $q$ of $G$, we may define the level of the identity monomial $u(S^d) = 1_G$, is $q + 1$. We can endow the set of weight $W$ of monomials, by lexicographic ordering.

**Definition 3.2.** A polynomial mapping $P : \mathcal{F}(S) \to G$ is defined to be the product of finitely many monomial mappings as $P(\alpha) = P_{u_1}(\alpha)P_{u_2}(\alpha)\cdots P_{u_m}(\alpha)$, $\alpha \in \mathcal{F}(S)$ where $P_{u_1}, P_{u_2}, \ldots, P_{u_m}$ are monomial mappings corresponding to $(u_1, \prec_1)$, $(u_2, \prec_2)$, …, $(u_m, \prec_m)$. And the weight $w(P)$ is taken to be minimum over the possible all representations of $P$ as the product $P = P_{u_1}P_{u_2}\cdots P_{u_m}$ of monomial mappings.

As, it is not guaranteed for a polynomial $P$ of weight $(l, d)$, $P(\mathcal{F}(S)) \subseteq G_l \setminus G_{l+1}$, $(P(\mathcal{F}(S))) \subseteq G_l$ but not guaranteed in $G_{l+1}$ the triangular monomial was introduced.

**Definition 3.3.** If $v : \mathcal{F}^{=d}(S) \to G$ is a mapping and $\prec$ is a linear order on $\mathcal{F}^{=d}(S)$, then the pair $(v, \prec)$ is called a triangular monomial of degree $d$. This induces a polynomial mapping $P : \mathcal{F}(S) \to G$ by the rule

$$P_v(\alpha) = \prod_{t \in \mathcal{F}^{=d}(\alpha)}^\prec v(t).$$

Now as [BL3] we can represent a polynomial mapping $P$ in the form $P = P_0P_1P_{d-1}\cdots P_0Q$, where each $P_i$, $i = 0, \ldots, d$ is a polynomial mapping induced by the triangular monomial $v_i$, $i = 0, \ldots, d$ and $Q$ is the polynomial mappings of higher degree.

Now using the triangular monomials in [BL3, Section 3.1], the weight of a polynomial is defined as:

**Definition 3.4.** Let $P : \mathcal{F}(S) \to G$ be a polynomial mapping. The weight of $P$ is defined to be the pair $(l, d)$, whenever $P$ has a representation $P = P_vQ$, where $P_v$ is the monomial mapping induced by the triangular monomial $v$, $w(v) = (l, d)$ and $w(Q) < (l, d)$. If $\varphi : G_l \to G_l/G_{l+1}$, then we call $\varphi \circ v : S^d \to G_l/G_{l+1}$ the principal part of $P$ and denote it by $M(P)$. We define the $\sim$ relation as, $P \sim P'$ iff $w(P) = w(P')$ and the principal parts coincides.

However the following definitions and the reduction of weight of a system follows from [BL3].

**Definition 3.5.** Let us denote by $W$ the set of weights of polynomials $\mathcal{F}(S) \to G$; that is, the set of pairs $(l, d)$ with $l, d \in \mathbb{Z}$, $1 \leq l \leq q$, $d \geq 0$. Let $A$ be a system;
the weight vector $w(A)$ of $A$ is a function $w(A) : W \to \{0, 1, 2, \ldots\}$ defined by the number of equivalence classes of polynomial mappings of weight $w$ having its representatives in $A$.

We order the weight vector lexicographically: $w(A) < w(A')$ if or some $w \in W$ one has $w(A)(w) < w(A')(w)$ and $w(A)(w') = w(A')(w')$ for all $w' > w$. We say that a system $A$ precedes a system $A'$ if $w(A) < w(A')$.

For any nil IP polynomial $P : F(S) \to G$, if the weight $w(P) = (l, d)$ then for any $g \in G$, as $g^{-1}Pg = P[P, g]$, we have $w(g^{-1}Pg) \leq w(P)$

**Definition 3.6.** Let $\gamma \in F(S)$, let $P$ be a mapping $F(S) \to G$; We define $U_{\gamma}P : F(S \setminus \gamma) \to G$ by

$$U_{\gamma}P(\alpha) = P(\gamma \cup \alpha).$$

Now we state the following theorems from [BL3] of our interest:

**Theorem 3.7.** Let $S$ be a set, let $G$ be a nilpotent group and $A$ be a system of polynomial mappings $F(S) \to G$. Then the following holds:

1. $w(P^{-1}U_{\gamma}P) \leq w(P)$.
2. If $\gamma \in F(S)$ and a system $A$ of polynomial mappings $F(S \setminus \gamma) \to G$ is such that each element of $A'$ is equivalent to $P_{F(S \setminus \gamma)}$ for some $P \in A$, then $w(A') \leq w(A)$.
3. If $A'$ consists of polynomial mappings of the form $PQ$ and $Q$ where $P \in A$ and $Q$ is a polynomial mapping $F(S) \to G$ then $w(A') \leq w(A)$.
4. If $A'$ consists of polynomial mappings of the form $PQ$ and $QP$ where $P \in A$ and $Q$ is a polynomial mapping $F(S) \to G$ with $w(Q) < w(P)$ then $w(A') \leq w(A)$.
5. Let $Q \in A$ be a nontrivial polynomial mapping with $w(Q) \leq w(P)$ for $P \in A$. If $A'$ is a system of polynomial mappings of the form $Q^{-1}P$ and $P^{-1}Q$, then $w(A') < w(A)$.

A parallel version of polynomial system in nilpotent group named as VIP$(G_\ast)$ polynomial which capture a polynomial as a prefiltration of the group $G$ and it has a weight defined in another way presents in [PZK]. Though we are using the technique to handle polynomials as it in [BL3] but it can be similarly done by VIP$(G_\ast)$ polynomials and the theorems 2.5, 2.6 hold for those polynomials also.

**Proof of Theorem 2.7.** If possible the Theorem is not true. Let $R$ be the minimal among all counterexamples. Then

$$D = F \setminus \{\alpha \in F : \overline{A} \cap L \cap_{p \in R} P(\alpha)^{-1}A \neq \emptyset\} \in \nu.$$ 

Due to piecewise syndeticity of $A$, we can choose $q_0 \in L$, such that $A \in q_0$. Then $B = \{\gamma \in F : \gamma A \in q_0\}$ is syndetic. Therefore we can find a finite $H \in F(G)$, such that $G \subseteq \cup_{t \in H} t^{-1}B$. Let us pick $t_0 \in H$ such that $C_0 = t_0^{-1}B \in q_0$. Pick, $Q \in R$ is of minimal degree polynomial. And take $A = \{t_0^{-1}PQ^{-1}t_0 : P \in R\}$. Then $w(A) < w(R)$. Let us set

$$E_0 = \{\alpha \in F : \overline{C_0} \cap L \cap_{R \in A} R(\alpha)^{-1}C_0 \neq \emptyset\} \in \nu.$$ 

Since $w(A) < w(R)$ and $R$ is minimal among the counterexamples, so $E_0 \in \nu$.

Let us pick $\delta_1 \in E_0 \cap D^*$ and pick

$$r_1 \in \overline{C_0} \cap L \cap_{R \in A} R(\delta_1)^{-1}C_0.$$ 

Let us take, $q_1 = ([t_0, Q^{-1}(\delta_1)]Q(\delta_1))^{-1}r_1$. 

(3.1)
And so from (3.2), \( R(\delta_1)^{-1}C_0 \in r_1 \) so that \( (t_0^{-1}PQ^{-1}(\delta_1)t_0)^{-1}C_0 \in r_1 \). This implies that
\[
\begin{align*}
t_0^{-1}Q(\delta_1)P(\delta_1)^{-1}t_0C_0 & \in r_1 \\
\implies t_0^{-1}Q(\delta_1)P(\delta_1)^{-1}t_0C_0 & \in r_1 \\
\implies t_0^{-1}Q(\delta_1)P(\delta_1)^{-1}t_0^{-1}B & \in r_1 \\
\implies t_0^{-1}Q(\delta_1)P(\delta_1)^{-1}t_0^{-1}B & \in r_1 \\
\implies [t_0, Q^{-1}(\delta_1)]Q(\delta_1)t_0^{-1}P(\delta_1)^{-1}B & \in r_1 \\
\implies \cap_{p \in R} t_0^{-1}P(\delta_1)^{-1}B & \in q_1.
\end{align*}
\]

Now choose
\[ (3.3) \quad t_1^{-1}B \in q_1. \]

Also choose,
\[
B = \{ t_1^{-1}PQ^{-1}t_1, t_0^{-1}P(\delta_1)^{-1}(U_\delta, P)t_0t_1^{-1}Q^{-1}t_1 : P \in R \}
\]
and so, \( w(B) < w(R). \)

Choose
\[ (3.4) \quad C_1 = t_1^{-1}B \bigcap_{p \in R} t_0^{-1}P(\delta_1)^{-1}B \]

And from (3.3) and (3.4) \( C_1 \in q_1 \).

Let
\[
E_1 = \{ \alpha \in F : \overline{C_1} \cap L \cap_{R \in B} \overline{R(\alpha)^{-1}C_1} \neq \emptyset \} \in \nu.
\]

Since \( w(B) < w(R) \). And choose
\[ (3.5) \quad \delta_2 \in D^* \cap (-\delta_1 \cup D^*) \cap E_1 \]

So, \( r_2 \in \overline{C_1} \cap L \cap_{R \in B} \overline{R(\delta_2)^{-1}C_1} \) and let, \( q_2 = ([t_1, Q^{-1}(\delta_1)]Q(\delta_2)^{-1}r_2. \)

Now \( t_1^{-1}PQ^{-1}t_1 \in B \) implies that
\[
\begin{align*}
(t_1^{-1}PQ^{-1}(\delta_2)t_1)^{-1}C_1 & \in r_2 \\
\implies (t_1^{-1}Q(\delta_2)P(\delta_2)^{-1}t_1)^{-1}t_1^{-1}B & \in r_2 \\
\implies t_1^{-1}Q(\delta_2)P(\delta_2)^{-1}B & \in r_2 \\
\implies [t_1Q(\delta_2)^{-1}Q(\delta_2)t_1^{-1}P(\delta_2)^{-1}B & \in r_2 \\
\implies t_1^{-1}P(\delta_2)^{-1}B & \in ([t_1, Q^{-1}(\delta_2)]Q(\delta_2)^{-1}r_2 = q_2 \\
\implies \cap_{p \in R} t_1^{-1}P(\delta_2)^{-1}B & \in q_2
\end{align*}
\]

Again
\[
\begin{align*}
t_0^{-1}P(\delta_1)^{-1}(U_\delta, P)t_0t_1^{-1}Q^{-1}t_1 & \in B \\
\implies ([t_0^{-1}P(\delta_1)^{-1}(U_\delta, P)t_0t_1^{-1}Q^{-1}t_1](\delta_2))^{-1}C_1 & \in r_2 \\
\implies t_1^{-1}Q(\delta_2)t_1t_0^{-1}P(\delta_1 \cup \delta_2)^{-1}P(\delta_1)t_0t_1^{-1}P(\delta_1)^{-1}B & \in r_2 \\
\implies t_1^{-1}Q(\delta_2)t_1t_0^{-1}P(\delta_1 \cup \delta_2)^{-1}B & \in r_2 \\
\implies [t_1, Q^{-1}(\delta_1)]Q(\delta_2)t_0^{-1}P(\delta_1 \cup \delta_2)^{-1}B & \in r_2 \\
\implies t_0^{-1}P(\delta_1 \cup \delta_2)^{-1}B & \in q_2 \\
\implies \cap_{p \in R} t_0^{-1}P(\delta_1 \cup \delta_2)^{-1}B & \in q_2
\end{align*}
\]

Now we can choose iteratively,
\[ (1) \ t_j^{-1}B \in q_j, j \in \{0, 1, \ldots, m\} \]
\[ (2) \ \delta_l \cup \delta_{l+1} \cup \ldots \cup \delta_m \in D^* \ \text{for} \ l \in \{0, 1, \ldots, m\} \]
(3) \( t^{-1}_l P(\delta_l \cup \delta_{l+1} \cup \ldots \cup \delta_m)^{-1}B = q_m \forall P \in R \) and \( l \in \{0, 1, \ldots, m-1\} \)

Now, since, \( H \in P_f(G) \) so we may choose \( l < m \) and \( t_i = t_m \) and put \( \delta = \delta_l \cup \delta_{l+1} \cup \ldots \cup \delta_m \in D^* \).

Take \( a \in t^{-1}_m B \cap \cap_{p \in R} t^{-1}_m P(\delta)^{-1}B \in q_m \).

This implies \( t_m a \in B \cap \cap_{p \in R} P(\delta)^{-1}B \).

Now, \( t_m a \in B \Rightarrow (t_m a)^{-1} A \in q_0 \Rightarrow A \in t_m a q_0 \)

And, \( t_m a \in P(\delta)^{-1} B \Rightarrow P(\delta) t_m a \in B \Rightarrow (P(\delta) t_m a)^{-1} A \in q_0 \Rightarrow P(\delta)^{-1} A \in t_m a q_0 \)

From \( (3.8) \) and \( (3.9) \) it is clear that \( t_m a q_0 \in \overline{A} \cap L \cap_{p \in R} \overline{P(\delta)^{-1} A} \) but this contradicts \( \delta \in D \).

Hence the theorem is proved. \( \square \)

Now we are in situation to prove the theorem 2.8.

Proof of theorem \[ \text{2.8} \] Let \( X \) be a compact metric space with action of \( G \), a nilpotent group.

Let us choose an \( \varepsilon > 0 \), and \( x \in X \). Let \( Y = \overline{Gx} \) be the orbit closure of \( x \) in \( X \).

Then \( Y \) is itself closed and hence compact.

Let us take \( V_1, V_2, \ldots, V_m \) be balls of radius \( \varepsilon \) covering \( Y \). Let us give those \( g \)'s color \( i \) for which \( gx \in V_i \). This gives a partition of \( G \) as \( G = \cup_{r=1}^m C_j \) \( (r \leq m) \).

Take some \( C_j \) which is piecewise syndetic.

Then using theorem \[ \text{2.7} \] we obtain that there exists some \( a \in G \), such that \( \{a, P_1(a), \ldots, P_k(a)\} \subset V_j \), where for all such \( a \in G \), the collection of \( a \) is IP*. Therefore we have that

\[
\text{diam}\{T^{a x}, T^{P_1(a) x}, \ldots, T^{P_k(a) x}\} < \varepsilon.
\]

Then we see that, \( \rho(T^{P_i(a)} ax, ax) < \varepsilon \) for all \( i \in \{1, 2, \ldots, k\} \) where the collection of \( a \) is IP*. So for every \( x \in X \), \( \varepsilon > 0 \) we get \( \rho(T^{P_i(a)} ax, ax) < \varepsilon \) and the collection of \( a \) is an IP* set. \( \square \)

We now state two corollaries of the theorem \[ \text{2.7} \] which can be obtained as in \[ \text{[H]HM}. \]

Corollary 3.8. Let \( (G, \cdot) \) be a nilpotent group, let \( R \in \mathcal{R} \), and let \( \langle \alpha_n \rangle_{n=1}^\infty \) be a sequence in \( \mathcal{F} \) such that \( \alpha_n < \alpha_{n+1} \forall n \).

If \( A \) be a piecewise syndetic subset of \( G \), then there exists \( r \in \bar{A} \cap K(\beta G) \) and \( \beta \in FU(\langle \alpha_n \rangle_{n=1}^\infty) \) such that \( \{p(\beta) r : p \in R\} \subseteq \bar{A} \).

The second one is the following:

Corollary 3.9. Let \( (G, \cdot) \) be a nilpotent group, let \( R \in \mathcal{R} \), and let \( \langle \alpha_n \rangle_{n=1}^\infty \) be a sequence in \( \mathcal{F} \) such that \( \alpha_n < \alpha_{n+1} \forall n \).

If \( A \) be a piecewise syndetic subset of \( G \), then there exists \( \beta \in FU(\langle \alpha_n \rangle_{n=1}^\infty) \) such that \( \{a \in A : \{p(\beta) a : p \in R\} \subseteq A\} \) is piecewise syndetic.
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