On the Metric Independent Exotic Homology.

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Abstract

Different types of nonstandard homology groups based on the various subcomplexes of differential forms are considered as a continuation of the recent authors works. Some of them reflect interesting properties of dynamical systems on the compact manifolds. In order to study them a Special Perturbation Theory in the form of Spectral Sequences is developed. In some cases a convenient fermionic formalism of dealing with differential forms is used originated from the work of Witten in the Morse Theory (1982) and the authors work where some nonstandard analog of Morse Inequalities for vector fields was found (1986).

In the work [10] we invented some sort of exotic homology of the first and second kind. Let us remind here that the second kind exotic homology defined in this work are constructed in the following natural way: for every linear space $L$ with operator $d' : L \to L$ we can define homology on any $d$-invariant subspace $T \subset L$ such that $(d')^2 : T \to 0$. So we have by definition $H_T = \text{Ker}(d')/\text{Im}(d')$. Our main example was based on the De-Rham complex $\Lambda^*(M) = \sum_{i=0}^{n} \Lambda^i$ of real-valued differential forms for any $C^\infty$-manifold $M$. We considered a family of operators $d' = d + \lambda \omega^*$ such that $d'(a) = da + \lambda \omega \wedge a$. Here $\omega$ is an one-form (maybe non closed), and $a$ is any $C^\infty$-form. A lot of work was done since 1986 (see, for example, [1, 2]) for the case of the closed one-form $\omega$ where $(d')^2 = 0$. We do not discuss this case here.

I. Perturbation of $d$ by the nonclosed odd form.
Consider the standard real-valued nonclosed forms \( \omega \). Our operators \( d' \) are defined for all values of \( \lambda \in R \) on the subspace \( T \subset \Lambda^* \) such that \( a \in T \) if and only if \( (\Omega) \wedge a = (\Omega)^*(a) = 0 \) where \( \Omega = d\omega \). Here \( R \) is a field of ordinary numbers like \( R \) or \( C \). We especially mention that here because later on (see the part II) this ring will became a \( \mathbb{Z}_2 \)-graded supercommutative ring with unit. Let us concentrate first on the special case when \( \lambda = 0 \) but the standard De Rham operator \( d' = d \) is considered on the more general class of subspaces \( T \subset \Lambda^*(M) \) such that \( a \in T \) if and only if \( a \wedge \Omega = 0 \). Here \( \Omega \) is any fixed closed differential form \( d\Omega = 0 \). In the example above we have \( \Omega = d\omega \), so this form is exact. Anyway, the operator \( d \) commutes with the multiplication operator 
\[
\Omega : u \mapsto u \wedge \Omega, d\Omega = \Omega d
\]
We take now an odd-dimensional \((2n+1)\)-manifold \( M \) which is a nondegenerate energy level in the symplectic manifold \( N \) with symplectic 2-form \( w \). We take a \( 2n \)-form \( \Omega = w^n \) restricted on the \( 2n+1 \)-submanifold \( M \). It turns out that the exotic homology here are associated with Hamiltonian System (1-foliation) on the submanifold \( M \). We denote the subspace \( T \) here as \( T_\Omega \) and homology \( \text{Ker}(d)/\text{Im}(d) \) in this subspace as \( H^*_\Omega \). The form \( w \) restricted on \( M \) is degenerate along the Hamiltonian vector field \( X \) only, i.e. \( w(X,Z) = 0 \) for every vector field \( Z \). People in Symplectic Geometry call \( X \) a "Reeb Vector Field".

**Lemma 1** \( C^\infty \)-differential k-form \( a \in \Lambda^*(M) \) belongs to the subspace \( T^k_\Omega \subset \Lambda^k \) if and only if

a. It is equal to zero for \( k = 0 \);

b. For \( k = 1 \) its values on the vector field \( X \) is identically equal to zero;

c. For \( k \geq 2 \) the subspace \( T^k_\Omega \) is equal to the whole space \( \Lambda^k(M) \)

This statement was proved in the work [10] for \( n = 1 \) in the slightly different terminology. For the case \( k = 0 \) our statement is obvious. There is no difference for the case \( k = 1, n > 1 \). Let us remind this proof here. According to the Darboux theorem, there exist local coordinate system \( p_i, q_i, r, i = 1, 2, \ldots, n \) such that \( w = \sum_i dp_i \wedge dq_i \). So the direction of the Reeb vector field exactly corresponds to the variable \( r \). We have \( w^n = (\text{const}) \prod dp_1 \wedge \ldots \wedge dq_n \), so the equality \( w^n \wedge a = 0 \) implies locally \( a = \sum(c, dp_i + l, dq_i) \) without \( dr \). Therefore we have \( (a,X) = 0 \) because \( (dp_i, X) = (dq_i, X) = 0 \). For \( k = 1 \)
our lemma is proved. For $k > 1$ it is obvious. Consider now the homology of this complex

$$0 \to T^1_\Omega \to \Lambda^2(M) \to \Lambda^3(M) \to \ldots \Lambda^n(M) \to 0$$

with standard boundary operator $d$.

**Theorem 1** For the compact smooth manifold $M$ a natural exact sequence is well-defined:

$$0 \to \mathbb{R} \to \text{Ker}(\nabla_X) \to H^1_\Omega \to H^1(M) \to C^\infty(M)/\nabla_X(C^\infty(M)) \to H^2_\Omega \to H^2(M) \to 0$$

where

$$\nabla_X : C^\infty(M) \to C^\infty(M)$$

is a derivative of function $f$ along vector field $X$

$$\nabla_X(f) = X^i \partial_i f, X = (X^i)$$

Remark: I clarified this question after the very useful discussion with D.Dolgopyat.

Let us construct all homomorphisms and prove exactness of this sequence for compact manifolds. We start from the left part.

For every function $f \in C^\infty(M)$ we have a form $df$. The equality $\nabla_X f = X^i \partial_i f = 0$ implies $(df, X) = 0$ by definition. So we have a map $\text{Ker}(\nabla_X) \to H^1_\Omega$ whose kernel is exactly constant functions. The image of this map in the group $H^1_\Omega$ consists of all exact forms such that $(u, X) = 0$. Therefore a factor-group by this image lies in the first homology group $H^1(M)$. Therefore we constructed a second map $H^1_\Omega \to H^1(M)$ and proved exactness of the sequence in the term $H^1_\Omega$.

For the construction of the next map, we simply take $u \to (u, X)$. This function belongs to $C^\infty(M)$. Varying the closed 1-form $u$ in the homology class $u + df$ we have a correct result in the factor $C^\infty(M)/\nabla_X(C^\infty(M))$. Its kernel contains exactly all closed 1-forms $u$ such that there exists a function $f$ for which we have $(u + df, X) = 0$. So we proved exactness in this term.

By definition, the next group $H^2_\Omega$ consists of all closed 2-forms $dv = 0$ modulo $du$ where $(u, X) = 0$. Let us consider part $H^2_\Omega \subset H^2_\Omega$ of this group represented by the exact 2-forms $v = dz$ but maybe $(z, X) \neq 0$. The set of
the quantities \((z, X) \in C^\infty(M)\) can be identified with all space \(C^\infty(M)\). So we have a map

\[ C^\infty(M) \to H^{2,0}_\Omega \subset H^2_\Omega \]

whose kernel is exactly represented by the projections of the closed 2-forms \((u, X) \in C^\infty(M)\). This projection is well-defined modulo terms like \(\nabla_X f\). So this map is also constructed. Our sequence is exact in this term as well.

The next map \(H^2_\Omega \to H^2(M)\) is natural. Its kernel obviously is equal to \(H^{2,0}_\Omega\) by definition. It is an epimorphism.

So our theorem is proved. By definition, for the case \(\lambda = 0\) we have

\[ H^i_\Omega = H^i(M), i \geq 3 \]

**Question:** We defined our exact sequence formally for smooth functions only. Is it possible to make a proper completion of it to the Hilbert spaces such that all homology will remain the same as in the Hodge Theory? Our complex is nonelliptic here. Is it true that after the proper ergodicity requirements our homology will be finite-dimensional even after the completion?

Let us consider now the case \(\lambda \neq 0\) where \(\Omega = d\omega\). We are dealing with the same subspace in the space of smooth differential forms, but \(Z\)-grading is lost: it should be replaced by the \(Z_2\)-grading \(H^{odd}_{\Omega,\lambda}\omega\) and \(H^{even}_{\Omega,\lambda}\omega\) where \(H^{*}_{\Omega,\lambda=0} = H^{*}_\Omega\). Acting by the operator \(d' = d + \lambda\omega^*\) on the odd forms \(v_1 + v_3 + v_5 + \ldots = v + \lambda v' + \lambda^2 v'' + \ldots \in \Lambda^{odd}(M)\) and even forms \(u_2 + u_4 + \ldots = u + \lambda u' + \lambda^2 u'' + \ldots \in \Lambda^{even}(M)\), we are coming along the line of the work [1] to the spectral sequence: its first differential is determined by the zero order in \(\lambda\), so it is ordinary \(d\). Its second differential (determined by the first order approximation in the variable \(\lambda\)) is generated by the multiplication on the form \(\omega\). In this simple case this spectral sequence has only two nontrivial differentials \(d_1 = d, d_2 = \omega^*\) for all dimensions \(2n + 1 > 3\):

**Lemma 2** Following differential is well-defined on the group \(H^*_\Omega = E_2:\)

\[ d_2 = \omega^*: H^{odd}_{\Omega,\lambda=0} \to H^{even}_{\Omega,\lambda=0} \to H^{odd}_{\Omega,\lambda=0}, d_2^2 = 0 \]

such that

\[ d_2(u) = \omega^*(u) = \omega \wedge u \]

for the even forms, and

\[ d_2(v) = \omega^*(v) = \omega \wedge v \]
for the odd forms, where \( d_u = d_v = 0 \) are the even and odd representative cocycles correspondingly.

Remark: The groups \( \text{Ker}(\omega^*)/\text{Im}(\omega^*) \) in good cases are equal to \( H_{\Omega,\lambda\omega}^* \) for odd and even cases for all odd dimensions \( n > 3, \lambda \neq 0 \) and small enough. However, in order to prove this statement we need to perform more serious analysis of this complex taking into account its nonellipticity.

For the special case \( n = 3 \) our spectral sequence is more complicated: we define "Massey Products" generated by the form \( \omega \): By definition, we call by the first Massey product a multiplication operator \( \omega^* : u \rightarrow \omega \wedge u \). The second Massey product is a following operator: let \( \omega \wedge u = dv \) in our complex.

We put

\[
d_3(u) = \{\omega, \omega, u\} = \omega \wedge v
\]

This operation is well-defined on the homology group \( E_3 \) determined by the operator \( d_2 = \omega^* \) on the group \( E_2 = H_{\Omega,\lambda\omega}^* \) with natural \( \mathbb{Z}_2 \)-grading. By induction, we define the next Massey product \( d_{l+1} \) acting on the homology group \( E_{l+1} = H(E_l, d_l) \) of the operator \( d_l \) acting on \( E_l \) with natural \( \mathbb{Z}_2 \)-grading as a map reversing this grading by the formula:

\[
d_{l+1}(u) = \{\omega, \ldots, \omega, u\} = \omega \wedge v
\]

( here \( \omega \) enters \( l \) times in the \( l \)-th Massey Product equal to the differential \( d_{l+1} \), such that \( dv = d_l(u) \), and all \( d_j(u) \) are equivalent to 0 for \( j \leq l \). Let us mention that we are speaking here about the class of \( u \) in the group \( E_l \).

So we consider any compact odd-dimensional manifold \( M = M^{2n-1} \) in the symplectic (noncompact) manifold \( N = N^{2n+2} \) with exact symplectic 2-form \( d\omega \), and operator \( d' = d + \lambda\omega \) on \( M \) acting in the kernel of the multiplication operator by the form \( \Omega = (d\omega)^n \).

**Theorem 2** For \( 2n + 1 = 3 \) the sequence on Massey products \( \{\omega, \ldots, \omega, u\} \) determines a well-defined spectral sequence of groups and differentials \( E_l = E_l^{\text{odd}} \oplus E_l^{\text{even}}, d_l \) where differentials are grading reversing. The group \( E_1 \) are equal to the kernel of multiplication operator by the form \( \Omega = d\omega \) in the space of all smooth differential forms \( \Lambda^*(M) \), and \( d_1 = d \). For all odd dimensions \( 2n + 1 > 3 \) all differentials \( d_i \) are equal to zero for \( i > 2 \).

Of course, this Spectral Sequence is a partial case of the more general class of spectral sequences generated by the pair of two anticommuting differentials \( d' = d_1 + d_2, (d')^2 = d_1^2 + d_2^2 + d_1d_2 + d_2d_1 = 0 \), but in our cases
all spectral sequence can be expressed in the form of Massey Products. This analytically defined spectral sequence and its calculation through the Massey Products appeared first time in the work [1], for the closed 1-form $\omega$ where we were dealing with the elliptic complex of all forms. As M.Farber pointed out to me, this construction solved the problem of calculation of differentials in the purely homological spectral sequence already invented by him purely topologically in the work [7], developing an old work of J.Milnor [6], so our analytical approach very easily led to the results unclear from the purely topological point of view. The general existence theorem of the analytical ”perturbation spectral sequence”, was formulated and proved for the deformations of all elliptic complexes in [8]. We studied in details some specific very interesting example in the work [10]. In the present work we are dealing with nonelliptic complexes, so we may have difficulties of the functional type. We cannot identify at the moment the groups $E_\infty$ for this spectral sequence. As B.Mityagin pointed out to the author, John von Neumann in 1930s performed some considerations looking like construction of our perturbation spectral sequence in the theory of operators, but there was no notions of the homological algebra at that time for the proper understanding what is going on. This comparison was done first time in the work [10].

More general exotic homology can be naturally defined for the differential forms $\Lambda^*(M, R)$ with coefficients in the $\mathbb{Z}_2$-graded skew commutative (supercommutative) associative ring with unit $R = R^+ \oplus R^-$ instead of ordinary numbers: they are based on the operator

$$d' = d + \sum_{i \geq 1} \lambda_i \omega^i$$

where dimension of the form $\omega^i$ is equal to $i \geq 1$, and $\lambda_i \in R^+$ if $i = 2j + 1$, $\lambda_i \in R^-$ if $i = 2j$. We have $(\sum_i \lambda_i \omega^i)^2 = 0$.

These exotic homology groups are defined on the subspace $T_\Omega \subset \Lambda^*(M)$ where $\Omega = \sum_i \lambda_i (d\omega^i)^*$. We have

$$T_\Omega = T_\Omega^{\text{even}} \bigoplus T_\Omega^{\text{odd}}$$

$$d' : T_\Omega^{\text{even}} \rightarrow T_\Omega^{\text{odd}} \rightarrow T_\Omega^{\text{even}}$$

So we have finally $H_\Omega^{\text{odd}}$ and $H_\Omega^{\text{even}}$. Many previous statements can be easily extended to this case.
II. Perturbation of $d$ by the contraction operators.

We can define another class of exotic homology based on the perturbation of the previously defined operators by the following tensor contraction operators on the spaces of differential forms with coefficients in the supercommutative $\mathbb{Z}_2$-graded ring $R$: consider the operator $d'$

$$d' = d + \lambda_i \sum_{i \geq 1} \omega^i \star + \sum_j \mu_j \hat{X}_j$$

where the last operators $\hat{X}_j$ are equal to the action of some selected skew symmetric tensors $X_j$ with $j$ upper indices on differential forms by the purely algebraic standard tensor contraction. By definition, the action of operator $\hat{X}_j$ on the forms of dimension less than $j$ is equal to zero, coefficients $\lambda_i \in R^+$ belong to the even part of the ring $R$ and $\mu \in R^-$ belong to the odd part of $R = R^+ \oplus R^-$. The simplest case here is an operator

$$d' = d + \lambda \omega^1 \star + \mu \hat{X}_1$$

where $\lambda, \mu$ are the ordinary numbers. Let us remind here that there exists a natural external product of skew symmetric tensors with upper indices as well as for differential forms. Following [3, 4, 5, 9], we use an algebraic language of fermionic creation and annihilation operators very convenient for the calculations with differential forms and other skew-symmetric tensors. In the given system of local coordinates $x^i$ we introduce operators $a^i, a^+_j$ with standard commutation relations

$$a^i a^+_j + a^+_j a^i = \delta^i_j, a^+_j a^+_i = -a^+_i a^+_j, a^i a^i = -a^i a^i$$

No metric on the manifold is given, so we do not consider annihilation operators as conjugated to the creation operators–no conjugation operation can be defined without metric. The local basis of differential $k$-forms is given in this ”Dirac-Fock Space” by the creation operators applied to the ”vacuum vector”

$$dx^{i_1} \wedge \ldots \wedge dx^{i_k} = a^{i_1} a^{i_2} \ldots a^{i_k} \Phi_0$$

”Vacuum” vector represents a constant function 1, i.e. $(\Phi_0 = 1)$. It satisfies to the relations $a^+_j \Phi_0 = 0$. The operators $\hat{X}_{i_1\ldots i_k}$ corresponding to the contraction with skew symmetric tensor fields $X^{i_1\ldots i_k}(x)$ with $k$ upper indices are given by the operators

$$\hat{X}_{i_1\ldots i_k} = X^{i_1\ldots i_k}(x) a^+_i \ldots a^+_k$$
acting in this Fock space. Let $\omega_1 = \omega = \sum_i \omega_i dx^i$ and $X_1 = X = (X^i(x))$. Denoting $\omega^1$ by $\omega$ and $X_1$ by $X$, we have:

$$d = a^i \partial_i, \omega^* = \omega_i(x) a^i, \dot{X}_1 = X(x)a_i^+$$

We recommend to prove following (known) geometrical identities using this language in order to understand how useful it is:

**Lemma 3** The anticommutators can be computed by the following formulas:

$$d\dot{X} + \dot{X} d = \nabla_X$$

$$d\omega^* + \omega^* d = (d\omega)$$

$$\omega^* \dot{X} + \dot{X} \omega^* = \omega(X) = (\omega_i X^i(x))$$

where $X$ is a vector field, $\nabla_X$ is a Lie derivative of the differential forms along the vector field $X$ and $\omega(X)$ is a value of 1-form on the vector field.

The proof easily follows from the direct elementary calculation with fermions. It is really much more convenient to calculate everything with differential forms using the fermionic language.

So we have

$$(d')^2 = (d + \lambda \omega^* + \mu \dot{X})^2 = \lambda(d\omega) + \mu \nabla_X + \lambda \mu \omega(X)$$

**Lemma 4** For the case $\lambda = 0$ kernel of the operator $\nabla_X$ is equal to the complex of differential forms invariant under the one-parametric diffeomorphism group generated by the vector field $X$. For $\mu = 0$ corresponding homology group $H^*_{X,\mu=0}$ is $\mathbb{Z}$-graded and isomorphic to the homology of $\nabla_X$-invariant De Rham Complex $H^*_{\text{inv}}$. In particular, for the case of isometry on the compact manifold $M$ the homology of this complex coincide with the standard $H^*(M)$.

In order to calculate the homology of perturbed $\mathbb{Z}_2$-graded complex with operator $d' = d + \mu \dot{X}$ we construct a spectral sequence $E_l, d_l$ similar to the previous cases, using the decomposition in the variable $\mu$. In the zero term we have as before a $\mathbb{Z}$-graded groups

$$E_2 = H^*_{X,\mu=0} = \sum_{i \geq 0} H^i_{\text{inv}}, d_2(u) = \dot{X}(u)$$
for the representative closed forms $u$. All higher differentials $d_l$ can be constructed easily: they are based on the action of the operator $\hat{X}$ instead of multiplication operator. This operator is purely algebraic (non differential). Last property allows us to call all higher differentials the analogs of Massey products.

**Lemma 5** There exists a natural spectral sequence $E_l, d_l$ where

$$E_2 = \sum_{i \geq 0} H^i_{\text{inv}}, d_2 = \hat{X} : H^i_{\text{inv}} \to H^{i-1}_{\text{inv}}$$

$$d_l : E^k_l \to E^{k-2l+3}_l, l \geq 2$$

Here $d_2$ is given by the contraction of the closed invariant form with vector field $X$, i.e. $d_2(u) = \hat{X}(u)$. The higher differentials $d_{l+1}, l + 1 \geq 3$, are defined by the formula $d_{l+1}(u) = \hat{X}(v)$ where $d_l(u) = dv$ for the $k$-cocycle $u$ presenting the corresponding element of the group $E_{l+1} = \sum_{k \geq 0} E^k_{l+1}$. All $d_j$ with $j \leq l$ annihilate this element, and $d_{l+1}(u)$ is presented by the $k - 2l - 1$-form in the class $d_{l+1}(u) \in E^{k-2l-1}_l$.

**Remark:** As before, we expect that in this case the group $E_\infty$ for this spectral sequence is isomorphic to the perturbed homology group of the operator $d + \mu \hat{X}$ for the small $\mu \neq 0$. No problem to prove this fact for the case of isometry group (i.e. if the closure of this one-parametric group in the diffeomorphism group is compact). This case is similar to the elliptic complexes. However, for the general case we will have functional difficulties.

Let us consider the special second differential

$$d_2 = \hat{X} : H^1_{\text{inv}} \to H^0_{\text{inv}} = R$$

Following the old works of Arnold and others of early 1960s, we consider an analog of the "rotation number" for the vector field $X$ on the compact manifold $M$. Starting from the fixed point $x$ belonging to the topological closure of trajectory $x(t)$ of this dynamical system, we define a limiting cycle $a \in H_1(M, R)$: consider a sequence $t_k \to +\infty$ such that $x(t_k) \to x$. These pieces of trajectory $[x(t_k), x(t_{k+1})]$ can be transformed into the closed curves joining their ends by the small paths. We are coming to the cycles $a_k \in H_1(M)$ made almost completely out of pieces of the trajectory $x(t)$. There exists a limit

$$a_k / (t_{k+1} - t_k) \to a \in H_1(M), k \to +\infty$$

independent of trajectory.
Lemma 6  Let \( a = 0 \). Then the special differential \( d_2 = \hat{X} : H_{inv}^1(M) \to R \) is equal to zero.

Proof of this lemma easily follows from the fact that the value of the invariant closed 1-form \( u_1(X) \) on the trajectory is constant. Therefore we have \( d_2(u_1) = u_1(X) = (u_1, a) = 0 \). Lemma is proved.

Remark: If \( a \neq 0 \) and \( X \) generates an isometry in some Riemannian metric, we can prove an inverse statement.

As a conclusion, let us mention that there are no natural differential operators on the spaces of tensors independent on Riemannian metric or other similar additional structures except \( d \) on the spaces of differential forms. However, there is a natural metric independent class of purely algebraic operators described above. All of them may be combined with \( d \). As it was demonstrated in the examples above, these exotic differentials and complexes may lead to the interesting geometrical and analytical objects. One should not expect, however, that this sort of quantities never appeared in topology before:

Example: The Equivariant Homology. Consider the operators \( d' = d + a_i \hat{X}_i \) for the polynomial generators \( a_i \) in the \( Z_+ \)-graded algebra \( R = R^+ = Z(a_1, \ldots, a_m) \) with 2-dimensional polynomial generators \( a_i \). This complex was invented by M. Atiyah and others for the definition of the ”equivariant homology”. A lot of people studied them since that. Here \( X_i \) are vector fields (the infinitesimal isometries) on the manifold \( M \) generating the action of compact commutative group \( G = T^m \): \( [\nabla_{X_i}, \nabla_{X_j}] = 0 \). The operator \( d' \) acting on the \( G \)-invariant \( R \)-valued forms can be naturally considered as a perturbation of the ordinary \( d \) in the subspace \( \text{Ker}(d')^2 = \Lambda_{inv}^* \subset \Lambda^*(M, R) \), so it is a part of our general scheme: the spectral sequence for calculation of these homology can be naturally considered as a by-product of perturbations similar to the spectral sequence of Massey products.

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