Elliptic Calogero-Moser System From Two Dimensional Current Algebra

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Abstract

We show that elliptic Calogero-Moser system and its Lax operator found by Krichever can be obtained by hamiltonian reduction from the integrable hamiltonian system on the cotangent bundle to the central extension of the algebra of $sl_N(C)$ currents.
1 Two Dimensional Current Algebra

In this section we review relevant for us properties of two dimensional current algebra (all this can be found in [2] and in references therein). We consider the Lie algebra Maps(Στ, slN(C)) of slN(C) - valued currents on the elliptic curve Στ with modular parameter τ. This algebra has a remarkable central extension, which is defined via the $H^{(1,0)}(Στ)^*$-valued 2-cocycle:

$$c(X,Y) = \int_{Στ} \omega \wedge <X,dY>$$

(1.1)

$\omega \in H^{(1,0)}(Στ)$. Fix a holomorphic 1-differential $\omega$ with the periods along the $A$- and $B$-cycles on the $Στ$:

$$\int_A \omega = 1, \int_B \omega = τ$$

Let us take a cotangent bundle $T^* \hat{g}^{Στ}$, which is a space of 4-tuples

$$(φ, c, \bar{A}, κ), φ: Στ \to slN(C), \bar{A} \in Ω^{(0,1)}(Στ) \otimes slN(C), c, κ \in C$$

and the pairing between algebra $\hat{g}^{Στ}$ and its dual $(\hat{g}^{Στ})^*$ in this parameterization has a form

$$<(\bar{A}, κ); (φ, c)> = κ \cdot c + \int_{Στ} ω tr φ \bar{A}$$

(1.2)

On the cotangent bundle acts naturally a current group $SL_N(C)Στ$:

$$φ(z, \bar{z}) \to g(z, \bar{z})φ(z, \bar{z})g(z, \bar{z})^{-1}$$

$$\bar{A}(z, \bar{z})d\bar{z} \to g(z, \bar{z})\bar{A}(z, \bar{z})g(z, \bar{z})^{-1} + κg(z, \bar{z})\bar{∂}g(z, \bar{z})^{-1}$$

$$κ \to κ, c \to c + \int_{Στ} ω tr(φ\bar{∂}g)$$

(1.3)

This action preserves a symplectic form $Ω$ on $T^* \hat{g}^{Στ}$:

$$Ω = δc \wedge δκ + \int_{Στ} ω tr(δφ \wedge δ\bar{A})$$

(1.4)
The moment map has the form
\[ \mu = \kappa \bar{\partial} \phi + [\bar{A}, \phi] \] (1.5)

\section{Hamiltonian Reduction And Integrable Model}

For our purposes it is more convenient to consider slightly enlarged symplectic manifold, namely
\[ T^* \hat{g}^{\Sigma_r} \times O_{\nu}^- \]
where \( O_{\nu}^- = \mathbb{CP}^{N-1} \) with symplectic form \( \omega_{\nu} = -N\nu \Omega_{\text{Fubini-Shtudi}} \). We endow this symplectic manifold with the structure of \( SL_N(\mathbb{C})^{\Sigma_r} \) space. On the first factor the group acts as before, and on the second one acts the finite dimensional \( SL_N(\mathbb{C}) \) sitting at point \( 0 \in \Sigma_r \) in a usual way. We denote the homogenous coordinates on \( O_{\nu} \) as \( (f_1 : \ldots : f_N) \) Now we wish to apply a hamiltonian reduction at zero level of moment map. It amounts to
\[ \mu = i\nu(\text{Id} - f \otimes f^\ast) \frac{\delta(z, \bar{z})dz \wedge d\bar{z}}{\omega} \]

The generic element of \( \hat{g}^{\Sigma_r} \) by the action (1.3) can be transformed to semi-simple constant element of maximal torus of \( sl_N(\mathbb{C}) \). Let us make this transformation. It is defined ambiguously, once \( \bar{A} \) has a form \( \text{diag}(a_j) \), it can be transformed (by means of large gauge transformation) to \( \text{diag}(a_j + \omega_{m_jn_j}) \), \( \omega_{m_jn_j} = m_j + \tau n_j \), \( (m_j, n_j) \in \mathbb{Z}^2 \), so the real space of parameters, which distinguish the coadjoint orbits, is the moduli space of flat \( su_N \)-connections. After applying this gauge transformation we arrive to equation, written in terms of matrix elements:
\[ \kappa \bar{\partial} \phi_{ij} + a_{ij} \phi_{ij} = i\nu(\delta_{ij} - f_if_i^\ast) \]

here \( a_{ij} = a_i - a_j \), \( a_i \in \mathbb{C} \) are diagonal entries of \( \bar{A} \).

Diagonal component of this equation gives us two kinds of constraints:
\[ 1 - f_if_i^\ast = 0 \]
and

\[ \phi_{ii} = p_i = \text{const} \]

this comes from the fact that on the elliptic curve there are no meromorphic functions except constants with the only one pole of the first order. These condition imply that all \( O_\nu \)-like degrees of freedom are freezed (they become dynamical in the situation with group of currents, like in \( \mathfrak{g} \)).

Non-diagonal components have the following form:

\[ \phi_{ij} = \exp\left(\frac{a_{ij}(z - \bar{z})}{\kappa \tau_2}\right)\psi_{ij} \]

where \( \psi_{ij} \) is a section of meromorphic line bundle over \( \Sigma_\tau \) with one pole at the point 0 of the first order and the following monodromy properties:

\[ \psi_{ij}(z + 1) = \psi_{ij}(z) \]

\[ \psi_{ij}(z + \tau) = e^{-\frac{2\pi i a_{ij}}{\kappa}}\psi_{ij}(z) \]

The solution has the following form:

\[ \psi_{ij}(z) = \frac{\nu}{\kappa} \frac{\theta_{11}(z + \frac{a_{ij}}{\kappa})}{\theta_{11}(z)\theta_{11}(\frac{a_{ij}}{\kappa})} \]

(2.6)

It gives Krichever answer \( \mathfrak{g} \) for the Lax matrix. Invariants of the matrix \( \phi(z, \bar{z}) \) give us Hamiltonians for integrable model. The first non-trivial one is

\[ \text{tr} \phi(z, \bar{z})^2 = \sum_i \frac{1}{2} \beta_i^2 + \frac{\nu^2}{\kappa^2} \sum_{i<j} \phi\left(\frac{a_{ij}}{\kappa}\right) - \phi(z) \]

It is a Hamiltonian of elliptic Calogero-Moser model and as always due to the quantum corrections coupling constant \( \nu^2 \) gets shifted to \( \nu(\nu - 1) \). Note that elliptic Calogero-Moser system covers both periodical and nonperiodical Toda chains and lattices \( \mathfrak{h} \) which can be considered as a particular limits of the system above. To perform reduction to Toda theory one should introduce
new variables like $a_i = x_i + (j - 1) \frac{k}{n}$ rescale the coupling constant $\nu = \nu_0 e^{\frac{1}{n}}$
and take the limit $b \to \infty$.

3 Elliptic Deformation Of Two Dimensional
Yang-Mills Theory

We remind that trigonometric Calogero-Moser model (Sutherland model)
can be viewed as an effective theory for zero modes of the gauge field on a
cylinder with inserted Wilson line in representation $R_\nu$ ($N\nu$-th symmetric
power of the $N$-dimensional fundamental representation of $SU(N)$, which
corresponds to the orbit $O_\nu$ ([3],[4]).

In the same way we expect that some kind of elliptic deformation of
Yang-Mills theory should exist. We propose the following action for this
hypothetical theory:

$$S_\tau = \int_{\Sigma_\tau \times S} \omega \wedge \text{tr}(\phi \partial_t \bar{z} - \bar{z} \phi^2)$$ (3.7)

Here $F_{t\bar{z}} = \partial_t \bar{A} - \bar{\partial} A_t + [A_t, \bar{A}]$ is a component of the gauge field strenght,
$\omega = dz$ - holomorphic differential on $\Sigma_\tau$, $S$ is a time-like circle. In the limit
$Im \tau \to \infty$ only those modes of gauge fields survive which give rise to the
theory on a cylinder, rather then on a three-dimensional manifold. Note also
that if one adds Chern-Simons action to the Lagrangian above the system
of interacting particles in the external magnetic field immediately appears
according to the standard analysys.

We hope to get some information about wave functions and spectrum of
this model via this field theory language. Essentially what is needed here,
is the proper analogue of the decomposition of the regular representation of
the group $G$ as $G_L \times G_R$ - module:

$$L^2(G) = \bigoplus \alpha \otimes \alpha^*$$
for the loop group (these are the irreducible representations of the loop group which propagate along the torus, like in the situation with the affine Lie algebra (trigonometric case) \[7\], \[6\] irreducible representations of the finite-dimensional group propagated along the circle). From the results of \[4\] we understand that an appropriate replacement of this decomposition looks like

\[
\int_{\mathbb{C}^N} d\lambda M_{\lambda,\kappa} \otimes M^*_{\lambda,\kappa}
\]

where \(M_{\lambda,\kappa}\) is a Verma module with the highest weight \(\lambda\) and the level \(\kappa\).

Finally, in the trigonometric case states were enumerated by the invariants \(\text{Inv}(\alpha \otimes \alpha^* \otimes R_\nu)\) \[7\], \[6\], what is the same as the space of intertwiners \(\Phi: \alpha \rightarrow \alpha \otimes R_\nu\). Path integral (3.7) suggest that the same statement holds in the elliptic case, the only difference is that finite-dimensional representations \(\alpha\) of \(SU(N)\) (or \(SL_N(\mathbb{C})\)) are replaced by the Verma modules. We refer to \[4\] for elaborated treatment of this construction with intertwiners.

Let us mention one another important observation. We would like recall the elliptic Sklyanin algebras \[10\] which appeared in description of asymmetric spin systems with the nearest-neighbors interactions. This quadratic algebra has a particular representation in which generators can be realized as finite-difference operators with coefficients expressed in terms of elliptic functions. These operators act in finite dimensional representations with fixed spin \(j\). If one considers the limiting procedure described in \[10\] which results in differential operators with elliptic coefficients and consider Casimir in this representation then two particle elliptic Calogero-Moser Hamiltonian appears. Let us outline the difference between interpretation of spectral parameter in spin systems and Yang-Mills theory. In spin systems it is the coordinate on the parameter space while in YM situation it is nothing but the coordinate on the world sheet. The meaning of this difference as wee as the role of elliptic algebra in deformed YM theory definitely can shed additional light on the structure of the integrable systems in 3d. In particular the answer about the origin of the generalization of Yang-Baxter equation for
elliptic Calogero-Moser system should be found.

To conclude, we derived elliptic Calogero-Moser system starting from elliptic deformation of Yang-Mills theory. We expect to get elliptic Ruijsenaars’ models, which is the top system for the tower of integrable Hamiltonians for interacting particles, as a reduction of the integrable system on the cotangent bundle to current group in two dimensions (see [7]). We also postpone many quotations related with solitonic interpretation, finite-gap solutions of KP hierarchy and possible matrix model counterparts of our model.

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