An Elementary and Real Approach to
Values of the Riemann Zeta Function*

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An elementary approach for computing the values at negative integers of the Riemann zeta function is presented. The approach is based on a new method for ordering the integers and a new method for summation of divergent series. We show that the values of the Riemann zeta function can be computed, without using the theory of analytic continuation and functions of complex variable.

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1 Introduction

The Riemann zeta function \(\zeta(s)\) is one of the most important objects in the study of number theory (if not in the whole modern mathematics). It is a classical and well-known result that \(\zeta(s)\), originally defined on the half plane \(\Re(s) > 1\), can be analytically continued to a meromorphic function on the entire complex plane with the only pole at \(s = 1\), which is a simple pole with residue 1.

One of the main reasons of interest to \(\zeta(s)\) is that the special values of \(\zeta(s)\) at integers have been proved or conjectured to have significant arithmetic meanings. For instance, Zagier’s [40, 41] conjecture concerning the relation between \(\zeta(n)\) and the \(n\)-logarithms for \(n \geq 2\) and Lichtenbaum’s [26] conjecture connecting \(\zeta(n)\) with motivic cohomology.

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Although the Riemann zeta function is considered as being primarily relevant to the "purest" of mathematical disciplines, number theory, it also occurs, for instance, in applied statistics (Zipf’s law, Zipf-Mandelbrot law), physics, cosmology [9, 10, 11]. For example, its special values $\zeta(3/2)$ is employed in calculating the critical temperature for a Bose-Einstein condensate, $\zeta(4)$ is used in Stefan-Boltzmann law and Wien approximation. The Riemann zeta function appears in models of quantum chaos [4, 5, 22, 31] and shows up explicitly in the calculation of the Casimir effect [27, 28].

The method of zeta function regularization [12], which is based on the analytic continuation of the zeta function in the complex plane, is used as one possible means of regularization of divergent series in quantum field theory [1, 8, 42, 43].

So, the Riemann zeta function is an interesting object for study not only for mathematicians but for physicists as well.

# 2 Zeta Function and its Values at Negative Integers

The zeta function was first introduced by Euler and is defined by

$$\zeta(s) = \sum_{u=1}^{\infty} \frac{1}{u^s}.$$  

The series is convergent when $s$ is a complex number with $\Re(s) > 1$.

In 1859 Riemann defined $\zeta(s)$ for all complex numbers $s$ by analytic continuation. Several techniques permit to extend the domain of definition of the zeta function (the continuation is independent of the technique used because of uniqueness of analytic continuation). One can, for example, consider the zeta alternating series, so-called the Dirichlet eta function

$$\eta(s) = \sum_{u=1}^{\infty} \frac{(-1)^{u-1}}{u^s},$$

defining an analytic function for $\Re(s) > 0$. When the complex number $s$ satisfies $\Re(s) > 1$, we have

$$\eta(s) = \sum_{u=1}^{\infty} \frac{1}{u^s} - \sum_{u=1}^{\infty} \frac{2}{(2u)^s} = \zeta(s) - \frac{2}{2^s} \zeta(s)$$
or
\[ \zeta(s) = \frac{\eta(s)}{1 - 2^{1-s}}, \quad \Re(s) > 1. \]

Since \( \eta(s) \) is defined for \( \Re(s) > 0 \), this identity permits to define the zeta function for all complex numbers \( s \) with positive real part, except for \( s = 1 + 2n\pi i / \log 2 \) where \( n \) is an integer. For \( n = 0 \), we have a pole at \( s = 1 \), and for \( n \neq 0 \) one can use the derivative of the \( \eta \)-function since it is known that \( \eta \) has zeros at these isolated points on the line \( \Re(s) = 1 \) [25, 32, 34, 44].

So, the zeta alternating series is linked with the original series (the zeta function) by the simple relation

\[ (1) \quad \tilde{\zeta}(s) = (1 - 2^{1-s})\zeta(s) \]

where we have put \( \tilde{\zeta}(s) = \eta(s) \).

Around 1740, Euler [13, 14, 15] discovered a method of calculating the values of the divergent series

\[ 1 + 1 + 1 + 1 + ... = -\frac{1}{2} \]
\[ 1 + 2 + 3 + 4 + ... = -\frac{1}{12} \]
\[ 1 + 4 + 9 + 16 + ... = 0 \]
\[ 1 + 8 + 27 + 64 + ... = \frac{1}{120} \]

etc.

In modern terms, these are the values at non-positive integer arguments of the Riemann zeta function \( \zeta(s) \), which, as it was said above, defined by the series, absolutely convergent in \( \Re(s) > 1 \)

\[ \zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + ... \]

To find these values, Euler considered the alternating series

\[ (2) \quad 1^m - 2^m + 3^m - 4^m + 5^m - 6^m + ... = \frac{1}{\zeta(s)}. \]

He observed that the value (2) is obtained as a limit of the power series

\[ 1^m - 2^m x + 3^m x^2 - 4^m x^3 + ... \]
as \( x \to 1 \), since, although the series itself converges only for \( |x| < 1 \), it has an expression as a rational function (analytic continuation, as we now put it), finite at \( x = 1 \), which is obtained by a successive application of multiplication by \( x \) and differentiation (equivalently, applying the Euler operator \( x \cdot d/dx \) successively after once multiplied by \( x \)) to the geometric series expansion

\[
\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + \ldots \quad (|x| < 1)
\]

For instance, if we substitute \( x = 1 \) in (3), we find formally

\[
\frac{1}{2} = 1 - 1 + 1 - 1 + 1 - 1 + \ldots = \tilde{\zeta}(0)
\]

and hence, in view of (1), we have \( \zeta(0) = -\frac{1}{2} \). More examples are

\[
\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \ldots
\]

\[
\frac{1-x}{(1+x)^3} = 1 - 2^2x + 3^2x^2 - 4^2x^3 + 5^2x^4 - \ldots
\]

\[
\frac{1-4x+x^2}{(1+x)^4} = 1 - 2^3x + 3^3x^2 - 4^3x^3 + 5^3x^4 - \ldots
\]

which give us

\[
\tilde{\zeta}(-1) = \frac{1}{4}, \quad \tilde{\zeta}(-2) = 0, \quad \tilde{\zeta}(-3) = -\frac{1}{8}
\]

and in turn

\[
\zeta(-1) = -\frac{1}{12}, \quad \zeta(-2) = 0, \quad \zeta(-3) = \frac{1}{120}.
\]

Euler found these values of divergent series, having no notion of the analytic continuation and functions of complex variable.

In modern concepts this method provides no rigorous way for obtaining the values of \( \zeta(s) \) at non-positive integers because it is commonly considered that the values of \( \zeta(-m) \) should be established as values of the analytically continued function \( \zeta(s) \) at \( s = -m \).

There are different approaches to evaluation of the values of \( \zeta(-m) \). These values can be computed by using the functional equation satisfied by this function [35], by using the Euler-Maclaurin formula [18, 33], in [30] the values of \( \zeta(-m) \) are derived from a particular series involving \( s\zeta(s+1) \), another
method is presented in [29]. Other approaches are based on the so-called $q$-extensions ($q$-analogs) of Riemann zeta function [21, 23, 24].

However, in this paper we find in a mathematically rigorous way a new approach to the values of $\zeta(s)$ at non-positive integer points, without addressing the theory of analytic continuation and functions of complex variable.

3 Definitions and Preliminaries

We shall deal with two series

$$\tilde{\zeta}(s) = \sum_{u=1}^{\infty} \frac{(-1)^{u-1}}{u^s} \quad \text{(Dirichlet eta function)}$$

and

$$\zeta(s) = \sum_{u=1}^{\infty} \frac{1}{u^s} \quad \text{(Riemann zeta function)}$$

and aim to evaluate $\zeta(s)$ and $\tilde{\zeta}(s)$ at non-positive integers ($\tilde{\zeta}(-m), \zeta(-m)$) in elementary and real way, without using complex-analytical techniques.

As a starting point we take a new method for ordering the integers [37, 38], which provides very well not only a real and elementary approach to computing the values of $\zeta(s)$ at negative integers but also a potentially new field of research [6, 36, 38].

In this paper we restrict ourselves to considering the values of $\zeta(s)$ and $\tilde{\zeta}(s)$ at non-positive integers and leave the issue of computing the values of $\zeta(s)$ and $\tilde{\zeta}(s)$ at positive integers for our next work [7].

To make paper self-contained, we introduce some basic definitions and propositions [38] (proofs can be found therein), necessary for the aim of this paper.

To be more specific, let us consider the set of all integer numbers $\mathbb{Z}$.\footnote{We denote by $a, b, c$ integer numbers and by $n, m, k$ natural numbers}

**Definition 1** We say that $a$ precedes $b$, $a, b \in \mathbb{Z}$, and write $a \prec b$, if the inequality $\frac{1}{a} < \frac{1}{b}$ holds; $a \prec b \iff \frac{1}{a} < \frac{1}{b}$.\footnote{Assuming by convention $0^{-1} = \infty$}

This method of ordering, obviously, gives that any positive integer number (including zero) precedes any negative integer number, and the set $\mathbb{Z}$ has zero as the first element and $-1$ as the last element, i. e. we have $\mathbb{Z} =$
[0, 1, 2, ... − 2, −1]. In addition, the following two necessary conditions of axioms of order hold:

1. either $a \prec b$ or $b \prec a$

2. if $a \prec b$ and $b \prec c$ then $a \prec c$

**Definition 2** A function $f(x)$, $x \in \mathbb{Z}$, is called regular if there exists an elementary function $F(x)$ such that $F(z + 1) - F(z) = f(z)$, $\forall z \in \mathbb{Z}$. The function $F(x)$ is said to be a generating function for $f(x)$.

**Remark 3** If $F(x)$ is a generating function for $f(x)$, then the function $F(x) + C$, where $C$ is a constant, is also a generating function for $f(x)$. So, any function $F(x)$ which is generating for $f(x)$ can be represented in the form $F(x) + C(x)$, where $C(x)$ is a periodic function with the period 1.

Suppose $f(x)$ is a function of real variable defined on $\mathbb{Z}$ and $\mathbb{Z}_{a,b}$ is a part of $\mathbb{Z}$ such that $\mathbb{Z}_{a,b} = [a, b]$ if $a \leq b$ and $\mathbb{Z}_{a,b} = \mathbb{Z} \setminus (b, a)$ if $a > b$, where $\mathbb{Z} \setminus (b, a) = [a, -1] \cup [0, b]$.

**Definition 4** For any $a, b \in \mathbb{Z}$

\[
\sum_{u=a}^{b} f(u) = \sum_{u \in \mathbb{Z}_{a,b}} f(u) \tag{4}
\]

This definition satisfies the condition of generality and has a real sense for any integer values of $a$ and $b$ ($a \geq b$). The definition \(^4\) extends the classical definition of sum $\sum_{a}^{b} f(n)$ to the case $b < a$.

We introduce for regular functions the following quite natural conditions:

1. If $S_n = \sum_{u=a}^{n} f(u)$, then $\lim_{n \to \infty} S_n = \sum_{u=a}^{\infty} f(u)$.

2. If $S_n = \sum_{u=1}^{n/2} f(u)$, then $\lim_{n \to \infty} S_n = \sum_{u=1}^{\infty} f(u)$.

3. If $\sum_{u=a}^{\infty} f(u) = S$, then $\sum_{u=a}^{\infty} af(u) = aS$, $a \in \mathbb{R}$.

\(^3\) geometrically, the set $\mathbb{Z}$ can be represented as cyclically closed (closed number line)
\(^4\) observing the established order of elements $\mathbb{Z}_{a,b}$
\(^5\) $n \to \infty$ means that $n$ unboundedly increases, without changing the sign
4. If \( \sum_{u=a}^{\infty} f_1(u) = S_1 \) and \( \sum_{u=a}^{\infty} f_2(u) = S_2 \), then \( \sum_{u=a}^{\infty} (f_1(u) + f_2(u)) = S_1 + S_2 \).

5. For any \( a \) and \( b \), \( a \leq b \): \( F(b + 1) - F(a) = \sum_{u=a}^{b} f(u) \).

6. If \( G = [a_1, b_1] \cup [a_2, b_2] \), \([a_1, b_1] \cap [a_2, b_2] = \emptyset \), then
\[
\sum_{u \in G} f(u) = \sum_{u=a_1}^{b_1} f(u) + \sum_{u=a_2}^{b_2} f(u).
\]

The conditions (1)-(6) define a method of summation of infinite series, which is regular due to (1).

**Proposition 5** If \( f(x) \) is a regular function and \( a \in \mathbb{Z} \) is a fixed number, then
\[
(5) \quad \sum_{u=a}^{a-1} f(u) = \sum_{u \in \mathbb{Z}} f(u)
\]

**Proposition 6** For any numbers \( m \) and \( n \) such that \( m < n \)
\[
(6) \quad \sum_{u=m}^{n} f(u) = \sum_{u=-n}^{-m} f(-u)
\]

**Proposition 7** Let \( f(x) \) be a regular function and let \( a, b, c \) be any integer numbers such that \( b \in \mathbb{Z}_{a,c} \). Then
\[
(7) \quad \sum_{u=a}^{c} f(u) = \sum_{u=a}^{b} f(u) + \sum_{u=b+1}^{c} f(u)\text{ }
\]

**Proposition 8** Suppose \( f(x) \) is a regular function. Then
\[
(8) \quad \sum_{u=a}^{a-1} f(u) = 0 \quad \forall a \in \mathbb{Z}
\]
or, which is the same in view of (3)
\[
(9) \quad \sum_{u \in \mathbb{Z}} f(u) = 0
\]

\(^{6}\)the prime on the summation sign means that \( \sum_{u=b+1}^{c} f(u) = 0 \) for \( b = c \)
**Proposition 9** For any regular function $f(x)$

\[
\sum_{u=a}^{b} f(u) = - \sum_{u=b+1}^{a-1} f(u) \quad \forall a, b \in \mathbb{Z}
\]

From (10), letting $a = 0$ and $b = -n$, we have

\[
\sum_{u=0}^{-n} f(u) = - \sum_{u=-n+1}^{-1} f(u)
\]

and using (6), we get

\[
\sum_{u=0}^{-n} f(u) = - \sum_{u=1}^{n-1} f(-u)
\]

and

\[
\sum_{u=1}^{-n} f(u) = - \sum_{u=0}^{n-1} f(-u)
\]

Using (11), we obtain

**Theorem 10** For any even regular function $f(x)$

\[
\sum_{u=1}^{\infty} f(u) = - \frac{f(0)}{2}
\]

independently on whether the series is convergent or not in a usual sense.

**Example 11** Consider some examples of both convergent and divergent series.

1. Convergent series

\[
\sum_{u=1}^{\infty} \frac{1}{4u^2 - 1} = \frac{1}{2}
\]

2. Divergent series

\[
\sum_{u=1}^{\infty} \frac{2u^2 + 1/2}{(2u^2 - 1/2)^2} = -1
\]

\[
\sum_{u=1}^{\infty} \frac{(4^u - 1)(u - 1/2) - 1}{2u^2 + u + 1} = \frac{1}{4}
\]
\[
\sum_{u=1}^{\infty} \frac{(u^2 + 1/4) \tan(1/2) \cos u - u \sin u}{(4u^2 - 1)^2} = -\frac{\tan(1/2)}{8}
\]

with the generating functions, respectively

\[
F(n) = \frac{-1}{2(2n - 1)}, \quad F(n) = \frac{(-1)^{n-1}}{(2n - 1)^2},
\]

\[
F(n) = \frac{-(n - 1/2)}{2^{n^2 - n + 1}}, \quad F(n) = \frac{\sin(n - 1/2)}{8(2n - 1)^2 \cos(1/2)}
\]

2. Divergent series

\[
1 - 1 + 1 - 1 + ... = \frac{1}{2}
\]

\[
1 + 1 + 1 + 1 + ... = -\frac{1}{2}
\]

\[
1^{2k} + 2^{2k} + 3^{2k}... = 0 \quad \forall k
\]

\[
1^{2k} - 2^{2k} + 3^{2k} - ... = 0 \quad \forall k
\]

with the generating functions, respectively

\[
F(n) = \frac{(-1)^n}{2}, \quad F(n) = n - 1, \quad F(n) = B_{2k}(n - 1)
\]

\[
F(n) = \frac{(-1)^n}{2k + 1} \sum_{u=1}^{2k+1} (2^u - 1) \binom{2k + 1}{u} B_u(n - 1)^{2k+1-u}
\]

**Theorem 12** For any polynomial \( f(x), x \in R \)

\[
(13) \quad \lim_{n \to \infty} (-1)^n f(n) = 0
\]

\[
(14) \quad \lim_{n \to \infty} f(n) = \int_{-1}^{0} f(x)dx
\]
4 Main Results

Using (13) we immediately obtain

**Theorem 13** Let \( \alpha(x) \) and \( \beta(x) \) be elementary functions defined on \( \mathbb{Z} \) and satisfying the condition \( \alpha(x) - \beta(x) = f(x) \), where \( f(x) \) is a polynomial. Suppose that \( \mu(x) \) is a function such that \( \mu(x) = \alpha(x) \) if \( 2 \mid x \) and \( \mu(x) = \beta(x) \) if \( 2 \not\mid x \). Then

\[
\lim_{n \to \infty} \mu(n) = \frac{1}{2} \lim_{n \to \infty} (\alpha(n) + \beta(n)).
\]

From Theorems 12 and 13 we get the following

**Theorem 14** Let \( a_u = a_1 + (u - 1)d, \ d \geq 0, \) is an arithmetic progression. Then

1) \[
\sum_{u=1}^{\infty} a_u = \frac{5d - 6a_1}{12}
\]

2) \[
\sum_{u=1}^{\infty} (-1)^{u-1} a_u = \frac{2a_1 - d}{4}
\]

**Example 15**

\[
\sum_{u=1}^{\infty} 1 = 1 + 1 + 1 + ... = \frac{1}{2}, \quad (d = 0)
\]

\[
\sum_{u=1}^{\infty} u = 1 + 2 + 3 + ... = \frac{1}{12}, \quad (d = 1)
\]

\[
\sum_{u=1}^{\infty} (2u - 1) = 1 + 3 + 5 + ... = \frac{1}{3}, \quad (d = 2)
\]

etc.

\[
\sum_{u=1}^{\infty} (-1)^{u-1} = 1 - 1 + 1 - 1 + ... = \frac{1}{2}, \quad (d = 0)
\]

\[
\sum_{u=1}^{\infty} (-1)^{u-1} u = 1 - 2 + 3 - 4 + ... = \frac{1}{4}, \quad (d = 1)
\]

\[
\sum_{u=1}^{\infty} (-1)^{u-1} (2u - 1) = 1 - 3 + 5 - 7 + ... = 0, \quad (d = 2)
\]

etc.
Let us now consider the Bernoulli polynomials. Bernoulli polynomials can be defined, in a simple way, by the symbolic equality $B_n(t) = (B + t)^n$, where the right hand side members should be expanded by the binomial theorem, and then each power $B^n$ should be replaced by $B_n$, and $B_n$ are the Bernoulli numbers (analogously, the symbolic equality $B_n = (B + 1)^n$ can be written for the Bernoulli numbers).

Bernoulli numbers play an important role in many topics of mathematics like analysis, number theory, differential topology, and in many other areas. These numbers were first introduced by Jacob Bernoulli (1654-1705) and appeared in *Ars Conjectandi*, his famous treatise published posthumously in 1713, when he studied the sums of powers of consecutive integers $1^k + 2^k + 3^k + ... + n^k$. They often occur when expanding some simple functions in a power series. For instance, in the series

\[
\cot(x) = \frac{1}{x} - \frac{2^2 B_2}{2!} x + \frac{2^4 B_4}{4!} x^3 - ... - (-1)^k \frac{2^{2k} B_{2k}}{2k!} x^{2k-1} + ... \quad 0 < |x| < \pi
\]

which appeared in astronomical works of J. Bernoulli.

Let

\[
B_k(n) = \frac{1}{k+1} \sum_{u=0}^{k} \binom{k+1}{u} B_u n^{k+1-u} = \sum_{u=1}^{k} u^k
\]

be the Bernoulli polynomial.

We derive two well-known equalities, without invoking the complex-analytical notions.

Since $B_k(n) - B_k(n - 1) = n^k$, the function $f(x) = x^k$ is regular. Then, in view of (11), we have

\[
(15) \quad B_k(-n) = (-1)^{k-1} B_k(n - 1)
\]

On the one hand, according to (15)

\[
\frac{1}{k} B_k(-1) = \frac{-1}{k(k+1)} \sum_{u=1}^{k} (-1)^{u+k} \binom{k+1}{u} B_u = \frac{(-1)^{k-1}}{k} \sum_{u=0}^{0} u^k = 0
\]

On the other hand, in view of (14)

\[
\lim_{n \to \infty} B_{k-1}(n) = \int_{-1}^{0} B_{k-1}(x) dx = \frac{1}{k(k+1)} \sum_{u=0}^{k-1} (-1)^{u+k} \binom{k+1}{u} B_u = \sum_{u=1}^{\infty} u^{k-1}
\]
Adding these two, we get

\[ \sum_{u=1}^{\infty} u^{k-1} = -\frac{B_k}{k} \quad \forall k \]  

(16)  

Now taking the formula

\[ \frac{(2^k - 1)}{k} B_k - \frac{(-1)^n}{k} \sum_{u=1}^{k} (2^u - 1) \binom{k}{u} B_u n^{k-u} = \sum_{u=1}^{n} (-1)^{u-1} u^{k-1} \quad \forall k \]

which holds for any natural number \( n \), and then passing to the limit (letting \( n \to \infty \)) and taking into account (13), we obtain

\[ \sum_{u=1}^{\infty} (-1)^{u-1} u^{k-1} = \frac{(2^k - 1)}{k} B_k \quad \forall k \]  

(17)  

So, we have obtained the equalities (16) and (17), and all the above propositions solely within the framework of real analysis.

Combining (16), (17), Theorems 10 and 14, and putting \( k - 1 = m \), we elementarily and immediately arrive at

**Proposition 16**  
For each non-negative integer \( m \), we have

(i) for Riemann zeta function

\[ 1^m + 2^m + 3^m + 4^m + ... = \zeta(-m) = -\frac{B_{m+1}}{m+1} \]

(ii) for Dirichlet eta function

\[ 1^m - 2^m + 3^m - 4^m + ... = \tilde{\zeta}(-m) = \frac{2^{m+1} - 1}{m+1} B_m \]

Using (15), we can also show that all odd Bernoulli numbers \( B_k \) (except for \( B_1 = 1/2 \)) are equal to zero. Indeed,

\[ B_k(n) - (-1)^{k-1} B_k(-n) = \frac{2}{k+1} \sum_{u=0}^{\lfloor (k-1)/2 \rfloor} \binom{k+1}{2u+1} B_{2u+1} n^{k-2u} = n^k \]

whence we immediately obtain \( B_1 = 1/2 \) and \( B_{2u-1} = 0, u = 1, 2, ... \).
5 Conclusion

In conclusion, we would like to emphasize that, unlike of the many other methods that make use of either contour integration or analytic continuation of $\zeta(s)$, our approach, elaborated within the new theoretical direction [38], has allowed us to obtain the values of the Riemann zeta function (and also the Dirichlet eta function) at non-positive integers in a purely real way, without any notions of complex analysis and analytic continuation, and it follows from one single concept. It would be interesting to apply the similar reasonings and techniques, within the setting of [38], to the some zeta related functions and other special functions. We plan to study this in our subsequent works.

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