A new result for the local well-posedness of the generalized Camassa-Holm equations in critical Besov spaces $B^{\frac{1}{p}}_{p,1}$, $1 \leq p < +\infty$

Xi Tu$^*$ and Zhaoyang Yin$^{1,2} \dagger$ and Yingying Guo$^3 \ddagger$

$^1$School of Mathematics and Big Data, Foshan University, Foshan, 528000, China
$^2$Faculty of Information Technology, Macau University of Science and Technology, Macau, China
$^3$School of Mathematics and Big Data, Foshan University, Foshan, 528000, China

Abstract

This paper is devoted to studying the local well-posedness (existence, uniqueness and continuous dependence) for the generalized Camassa-Holm equations in critical Besov spaces $B^{\frac{1}{p}}_{p,1}$ with $1 \leq p < +\infty$, which improves the previous index $s > \max\{\frac{1}{2}, \frac{1}{p}\}$ or $s = \frac{1}{p}$, $p \in [1, 2]$, $r = 1$ in [34, 43]. The main difficulty is to prove the uniqueness, which need to use the Moser-type inequality. To overcome the difficulty, we use the Lagrange coordinate transformation to obtain the uniqueness.

Mathematics Subject Classification: 35Q53, 35B10, 35C05
Keywords: Local well-posedness, Generalized Camassa-Holm equations, Critical Besov spaces, Lagrangian coordinate transformation.

Contents

1 Introduction 1
2 Preliminaries 3
3 Local well-posedness 5
References 11

1 Introduction

In this paper we consider the Cauchy problem for the following generalized Camassa-Holm equation,

\[
\begin{cases}
  u_t - u_{txx} = \frac{1}{2}(3u_x^2 - 2u_x u_{xxx} - u_{xx}^2), & t > 0, \\
  u(0, x) = u_0(x),
\end{cases}
\]

(1.1)
which can be rewritten as

\[
\begin{align*}
    m &= u - u_{xx}, \\
    m_t - u_x m_x &= -\frac{1}{2} m^2 + um + \frac{1}{2} u_x^2 - \frac{1}{2} u_t^2, \quad t > 0, \\
    m(0, x) &= u(0, x) - u_{xx}(0, x) = m_0(x).
\end{align*}
\] (1.2)

The equation (1.1) was proposed recently by Novikov in [40]. He showed that the equation (1.1) is integrable by using as definition of integrability the existence of an infinite hierarchy of quasi-local higher symmetries [40] and it belongs to the following class [40]:

\[
(1 - \partial_x^2)u_t = F(u, u_x, u_{xx}, u_{xxx}),
\] (1.3)

which has attracted much interest, particularly in the possible integrable members of (1.3).

The most celebrated integrable members of (1.3) which have quadratic nonlinearity are the well-known Camassa-Holm (CH) equation [5] and the famous Degasperis-Procesi (DP) equation [23]:

\[
\begin{align*}
    (1 - \partial_x^2)u_t &= 3 uu_x - 2 u_x u_{xx} - uu_{xxx}, \\
    (1 - \partial_x^2)u_t &= 4 uu_x - 3 u_x u_{xx} - uu_{xxx}. 
\end{align*}
\] (1.4) (1.5)

Both the CH equation and the DP equation can be regarded as a shallow water wave equation [5,10,24]. They are completely integrable with a bi-Hamiltonian structure [7,22,27]. That means that the system can be transformed into a linear flow at constant speed in suitable action-angle variables (in the sense of infinite-dimensional Hamiltonian systems), for a large class of initial data [5,8,17,22]. It admits exact the single peakon solutions and and the multi-peakon solutions, which are orbitally stable [19]. It is worth mentioning that the peaked solitons present the characteristic for the traveling water waves of greatest height and largest amplitude and arise as solutions to the free-boundary problem for incompressible Euler equations over a flat bed, cf. [6,10,14,15,42]. Another remarkable feature of the CH equation and the DP equation is the so-called amplitude and arise as solutions to the free-boundary problem for incompressible Euler equations over a flat bed, cf. [6,10,14,15,42]. Another remarkable feature of the CH equation and the DP equation is the so-called wave breaking phenomena [9,13,33]. The main difference between DP equation and CH equation is that DP equation has short waves [38] and the periodic shock waves [26].

Concerning the local well-posedness and ill-posedness for the Cauchy problem of the CH equation in Sobolev spaces and Besov spaces, we refer to [11,12,20,29,37,41]. Global strong solutions to the CH equation were discussed in [9,11,12]. And the finite time blow-up strong solutions to the CH equation were proved in [9,11,12]. It was shown that there exist the global weak solutions to the CH equation [18,49] and the global conservative and dissipative solutions of CH equation [5,4].

The local well-posedness of the Cauchy problem of the DP equation in Sobolev spaces and Besov spaces were investigated in [28,30,53]. Similar to the CH equation, It was shown that there exist the global strong solutions [35,54,56] and finite time blow-up solutions [25,26,35,36,53,56] to the DP equation. The global weak solutions was established in [2,25,53,56].

The third celebrated integrable member of (1.3) which has cubic nonlinearity is the known Novikov equation [40]:

\[
(1 - \partial_x^2)u_t = 3 uu_x u_{xx} + u^2 u_{xxx} - 4 u_x^2 u_x.
\] (1.6)

It was showed that the Novikov equation is integrable, possesses a bi-Hamiltonian structure, and admits exact peakon solutions $u(t, x) = \pm \sqrt{c} e^{\pm x - ct}$ with $c > 0$ [31].

The local well-posedness for the Novikov equation in Sobolev spaces and Besov spaces was investigated in [37,48,50,51]. Wu and Yin proved the global existence of strong solutions under some sign conditions [37]. Yan, Li and Zhang studied the blow-up phenomena of the strong solutions [51]. The global weak solutions for the Novikov equation was established in [32,46].

Recently, the Cauchy problem of (1.1) in the Besov spaces $B^s_{p,r}$, $s > \max \{2 + \frac{1}{p}, \frac{3}{2}\}$ and the critical Besov space $B^\frac{3}{2,1}_p$ has been studied in [34,48]. The global weak solution of (1.1) was established in [44]. To our best knowledge, there is no paper concerning the Cauchy problem of (1.1) in the critical Besov space $B^\frac{3}{2,1}_p$, $1 \leq p < +\infty$, which is we shall investigate in this paper.
The main difficulty is to prove the uniqueness. For instance, one should use the following Moser-type inequality
\[ \|fg\|_{B^{s_1+s_2}_{p,1}} \leq C\|f\|_{B^{s_1}_{p,1}}\|g\|_{B^{s_2}_{p,1}}, \quad s_1, s_2 \leq \frac{d}{p}, s_1 + s_2 > \max\{0, \frac{2}{p} - 1\} \] (1.7)

to estimate (1.1). That is why one need the condition \( s > \max\{\frac{5}{2}, 2 + \frac{1}{p}\} \). To overcome the difficulty, we use the Lagrange coordinate transformation in this paper to investigate the uniqueness for the generalized Camassa-Holm equation. Indeed, combining with the estimation of the characteristic \( y(t, \xi) \), we will obtain the uniqueness without using (1.7).

The rest of our paper is as follows. In the second section, we introduce some preliminaries which will be used in the sequel. In the third section, we give the proof of Theorem 3.1 by using the Lagrangian coordinate transformation.

2 Preliminaries

In this section, we first recall some basic properties on the Littlewood-Paley theory, which can be found in [1].

Let \( \chi \) and \( \phi \) be a radical, smooth, and valued in the interval \([0, 1]\), belonging respectively to \( \mathcal{D}(\mathcal{B}) \) and \( \mathcal{D}(\mathcal{C}) \), where \( \mathcal{B} = \{ \xi \in \mathbb{R}^d : |\xi| \leq \frac{4}{3} \} \), \( \mathcal{C} = \{ \xi \in \mathbb{R}^d : \frac{2}{3} \leq |\xi| \leq \frac{4}{3} \} \). Denote \( \mathcal{F} \) by the Fourier transform and \( \mathcal{F}^{-1} \) by its inverse. For any \( u \in S'(\mathbb{R}^d) \), all \( j \in \mathbb{Z} \), define \( \Delta_j u = 0 \) for \( j \leq -2 \); \( \Delta_j u = \mathcal{F}^{-1}(\chi\mathcal{F}u) \); \( \Delta_j = \mathcal{F}^{-1}(\phi(2^{-j}.)\mathcal{F}u) \) for \( j \geq 0 \); and \( S_j u = \sum_{j' < j} \Delta_{j'} u \).

Let \( s \in \mathbb{R}, 1 \leq p, r \leq \infty \). The nonhomogeneous Besov space \( B^s_{p,r}(\mathbb{R}^d) \) is defined by
\[ B^s_{p,r} = \left\{ u \in S'(\mathbb{R}^d) : \|u\|_{B^s_{p,r}} = \| (2^j |\Delta_j u|_{L^r})_j \|_{l^p(\mathbb{Z})} < \infty \right\} \]
The nonhomogeneous Sobolev space is defined by
\[ H^s = \left\{ u \in S'(\mathbb{R}^d) : u \in L^2_{loc}(\mathbb{R}^d), \|u\|^2_{H^s} = \int_{\mathbb{R}^d} (1 + |\xi|^2)^s |\mathcal{F}u(\xi)|^2 d\xi < \infty \right\} \]
The nonhomogeneous Bony’s decomposition is defined by \( uv = Tu v + T_v u + R(u, v) \) with
\[ T_u v = \sum_j S_{j-1} u \Delta_j v, \quad R(u, v) = \sum_j \sum_{|j'-j| \leq 1} \Delta_j u \Delta_{j'} v. \]

Naturally, we introduce some properties about Besov spaces. For more details, see [1].

**Proposition 2.1.** [1,2,3] Let \( s \in \mathbb{R}, 1 \leq p, p_1, p_2, r, r_1, r_2 \leq \infty \).

1. \( B^s_{p,r} \) is a Banach space, and is continuously embedded in \( S' \).
2. If \( r < \infty \), then \( \lim_{j \to \infty} \|S_j u - u\|_{B^s_{p,r}} = 0 \). If \( p, r < \infty \), then \( C_0^\infty \) is dense in \( B^s_{p,r} \).
3. If \( p_1 \leq p_2 \) and \( r_1 \leq r_2 \), then \( B^s_{p_1, r_1} \hookrightarrow B^{s-d(\frac{1}{p} - \frac{1}{r_2})}_{p_2, r_2} \). If \( s_1 < s_2 \), then the embedding \( B^{s_2}_{p_2, r_2} \hookrightarrow B^{s_1}_{p_1, r_1} \) is locally compact.
4. \( B^s_{p,r} \hookrightarrow L^\infty \) \( \Leftrightarrow s > \frac{d}{p} \) or \( s = \frac{d}{p}, r = 1 \).
5. Fatou property: if \( (u_n)_{n \in \mathbb{N}} \) is a bounded sequence in \( B^s_{p,r} \), then an element \( u \in B^s_{p,r} \) and a subsequence \( (u_{n_k})_{k \in \mathbb{N}} \) exist such that
\[ \lim_{k \to \infty} u_{n_k} = u \text{ in } S' \quad \text{and} \quad \|u\|_{B^s_{p,r}} \leq C \liminf_{k \to \infty} \|u_{n_k}\|_{B^s_{p,r}}. \]
6. Let \( m \in \mathbb{R} \) and \( f \) be a \( S^m \)-multiplier (i.e. \( f \) is a smooth function and satisfies that \( \forall \alpha \in \mathbb{N}^d, \exists C = C(\alpha) \) such that \( |\partial^\alpha f(\xi)| \leq C(1 + |\xi|)^{m-|\alpha|}, \forall \xi \in \mathbb{R}^d \)). Then the operator \( f(D) = \mathcal{F}^{-1}(f\mathcal{F}) \) is continuous from \( B^s_{p,r} \) to \( B^{s-m}_{p,r} \).
Proposition 2.2. \[\text{Let } s \in \mathbb{R}, 1 \leq p, r \leq \infty.\]

\[
\begin{cases}
  B^s_{p,r} \times B^{-s}_{p',r'} \rightarrow \mathbb{R}, \\
  (u, \phi) \mapsto \sum_{|j-j'| \leq 1} \langle \Delta_j u, \Delta_j' \phi \rangle,
\end{cases}
\]

defines a continuous bilinear functional on \(B^s_{p,r} \times B^{-s}_{p',r'}\). Denote by \(Q^{-s}_{p',r'}\) the set of functions \(\phi\) in \(S'\) such that \(\|\phi\|_{B^{-s}_{p',r'}} \leq 1\). If \(u\) is in \(S'\), then we have

\[\|u\|_{B^s_{p,r}} \leq C \sup_{\phi \in Q^{-s}_{p',r'}} \langle u, \phi \rangle.\]

The useful interpolation inequalities are given as follows.

Proposition 2.3. \[\text{[135]}\]

1. If \(s_1 < s_2\), \(\lambda \in (0, 1)\) and \((p, r) \in [1, \infty]^2\), then we have

\[
\|u\|_{B^{\lambda s_1+(1-\lambda)s_2}_{p,r}} \leq \frac{C}{s_2-s_1} (1 + \frac{1}{1-\lambda}) \|u\|_{B^{\lambda s_1}_{p,r}} \|u\|_{B^{1-\lambda s_2}_{p,r}}.
\]

2. If \(s \in \mathbb{R}, 1 \leq p \leq \infty, \varepsilon > 0,\) a constant \(C = C(\varepsilon)\) exists such that

\[\|u\|_{B^s_{p,1}} \leq C\|u\|_{B^s_{p,\infty}} \ln \left(1 + \frac{\|u\|_{B^s_{p,\infty}}}{\|u\|_{B^s_{p,\infty}}}\right).\]

We now give the 1-D Moser-type estimates which we will use in the following.

Lemma 2.4. \[\text{[33]}\]
The following estimates hold:

1. For any \(s > 0\) and any \(p, r \in [1, \infty],\) the space \(L^\infty \cap B^s_{p,r}\) is an algebra, and a constant \(C = C(s)\) exists such that

\[
\|uv\|_{B^s_{p,r}} \leq C\|u\|_{L^\infty} \|v\|_{B^s_{p,r}} + \|u\|_{B^s_{p,r}} \|v\|_{L^\infty},
\]

\[\|u\partial_x v\|_{B^s_{p,r}} \leq C\|u\|_{B^{s+1}_{p,r}} \|v\|_{L^\infty} + \|u\|_{L^\infty} \|\partial_x v\|_{B^s_{p,r}}.\]

2. If \(1 \leq p, r \leq \infty, s_1 \leq s_2, s_2 > \frac{1}{p}(s_1 \geq \frac{1}{p} \text{ if } r = 1)\) and \(s_1 + s_2 > \max(0, \frac{2}{p} - 1)\), there exists \(C = C(s_1, s_2, p, r)\) such that

\[\|uv\|_{B^{s_1}_{p,1}} \leq C\|u\|_{B^{s_1}_{p,\infty}} \|v\|_{B^{s_2}_{p,\infty}}.\]

Here is the Gronwall lemma.

Lemma 2.5. \[\text{[11]}\]
Let \(m(t), a(t) \in C^1([0, T]), m(t), a(t) > 0.\) Let \(b(t)\) is a continuous function on \([0, T].\)

Suppose that, for all \(t \in [0, T],\)

\[
\frac{1}{2} \frac{d}{dt} m^2(t) \leq b(t) m^2(t) + a(t) m(t).
\]

Then for any time \(t\) in \([0, T],\) we have

\[m(t) \leq m(0) \exp \int_0^t b(\tau) d\tau + \int_0^t a(\tau) \exp \left(\int_\tau^t b(t') dt'\right) d\tau.\]

In the paper, we also need some estimates for the following 1-D transport equation:

\[
\begin{cases}
  f_t + v \partial_x f = g, \quad x \in \mathbb{R}, \quad t > 0, \\
  f(0, x) = f_0(x).
\end{cases}
\]
Lemma 2.6. \( \square \) Let \( 1 \leq p \leq \infty, \ 1 \leq r \leq \infty, \ \theta > -\min\left(\frac{1}{p}, \frac{1}{r}\right) \). Let \( f_0 \in B^\theta_{p,r}, \ g \in L^1([0,T]; B^\theta_{p,r}) \), and \( v \in L^p([0,T]; B^\theta_{\infty,\infty}) \) for some \( \rho > 1 \) and \( M > 0 \) such that

\[
\begin{align*}
\partial_x v \in L^1([0,T]; B^{\frac{1}{p}}_{p,\infty} \cap L^\infty), & \quad \text{if } \theta < 1 + \frac{1}{p}, \\
\partial_x v \in L^1([0,T]; B^{\frac{1}{p}}_{p,r} 1), & \quad \text{if } \theta > 1 + \frac{1}{p} \text{ or } (\theta = 1 + \frac{1}{p} \text{ and } r = 1).
\end{align*}
\]

Then the problem (2.1) has a unique solution \( f \) in
- the space \( C([0,T]; B^\theta_{p,r}) \), if \( r < \infty \),
- the space \( \left( \bigcap_{\theta' < \theta} C([0,T]; B^{\theta'}_{p,\infty}) \right) \bigcap C_w([0,T]; B^\theta_{p,\infty}) \), if \( r = \infty \).

Lemma 2.7. \( \square \) Let \( 1 \leq p \leq \infty, \ \theta > -\min\left(\frac{1}{p}, \frac{1}{\infty}\right) \). There exists a constant \( C \) such that for all solutions \( f \in L^\infty([0,T]; B^\theta_{p,r}) \) of (2.1), with initial data \( f_0 \in B^\theta_{p,r} \) and \( g \in L^1([0,T]; B^\theta_{p,r}) \),

\[
\|f(t)\|_{B^\theta_{p,r}} \leq \|f_0\|_{B^\theta_{p,r}} + \int_0^t \|g(t')\|_{B^\theta_{p,r}} dt' + \int_0^t V'(t') \|f(t')\|_{B^\theta_{p,r}} dt'
\]

or

\[
\|f(t)\|_{B^\theta_{p,r}} \leq e^{CV(t)} \left( \|f_0\|_{B^\theta_{p,r}} + \int_0^t e^{-C V(t')} \|g(t')\|_{B^\theta_{p,r}} dt' \right)
\]

with

\[
V'(t) = \begin{cases} \|\partial_x v(t)\|_{B^{\frac{1}{p}}_{p,\infty} \cap L^\infty}, & \text{if } \theta < 1 + \frac{1}{p}, \\ \|\partial_x v(t)\|_{B^{\frac{1}{p}}_{p,r} 1}, & \text{if } \theta > 1 + \frac{1}{p} \text{ or } (\theta = 1 + \frac{1}{p}, \ p < \infty, \ r = 1), \end{cases}
\]

and, if \( \theta = \frac{1}{p} - 1, \ 1 \leq p \leq 2, \ r = \infty \), \( V'(t) = \|\partial_x v(t)\|_{B^{\frac{1}{p}}_{p,1}} \).

If \( f = v \), then for all \( \theta > 0 \), \( V'(t) = \|\partial_x v(t)\|_{L^\infty} \).

Lemma 2.8. \( \square \) Let \( 1 \leq p < \infty \). Define \( \overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\} \). Suppose \( f \in L^1([0,T]; B^\frac{1}{p}_{p,1}) \) and \( a_0 \in B^\frac{1}{p}_{p,1} \). For \( n \in \overline{\mathbb{N}} \), denote by \( a^n \in C([0,T]; B^\frac{1}{p}_{p,1}) \) the solution of

\[
\begin{cases}
\partial_t a^n + A^n \partial_x a^n = f, \\
a^n(0,x) = a_0(x).
\end{cases}
\]  

(2.2)

Assume for some \( \beta \in L^1(0,T), \ \sup_{n \in \overline{\mathbb{N}}} \|A^n\|_{L^\frac{1}{p+1}_{p+1}} \leq \beta(t) \). If \( A^n \) converges to \( A^\infty \) in \( L^1([0,T]; B^\frac{1}{p+1}_{p+1}) \), then the sequence \( \{a^n\}_{n \in \overline{\mathbb{N}}} \) converges to \( a^\infty \) in \( C([0,T]; B^\frac{1}{p+1}_{p+1}) \).

3 Local well-posedness

In this section, we establish local well-posedness of (1.2) in the critical Besov space \( B^\frac{1}{p+1}_{p,1} \), \( 1 \leq p < +\infty \). Our main result can be stated as follows:

Theorem 3.1. Let \( u_0 \in B^\frac{2+\frac{1}{p}}{p+1}_{p,1}, \ m_0 = u_0 - u_{0xx} \in B^\frac{1}{p+1}_{p+1} \) with \( p \in [1, \infty) \). Then there exists a time \( T > 0 \) such that the generalized CH equation with the initial data \( u_0 \) is locally well-posed in the sense of Hadamard.

Proof. In order to prove Theorem 3.1, we proceed as the following five steps.

Step 1. Existence.

First, we construct approximate solutions which are smooth solutions of some linear equations. Starting for \( m_0(t, x) \triangleq m(0, x) = m_0 \), we define by induction sequences \( \{m_n\}_{n \in \mathbb{N}} \) by solving the following linear transport equations:

\[
\begin{align*}
\begin{cases}
\partial_t m_{n+1} - \partial_x u_n \partial_x m_{n+1} = & \frac{1}{2} (\partial_x u_n)^2 - \frac{1}{2} (u_n - m_n)^2 \\
= & u_n m_n + \frac{1}{2} (\partial_x u_n)^2 - \frac{1}{2} u_n^2 - \frac{1}{2} m_n^2 \\
& = F(m_n, u_n),
\end{cases}
\end{align*}

(3.1)

m_{n+1}(t, x)|_{t=0} = S_n m_0.
We assume that \( m_n \in L^\infty(0, T; B^{\frac{\beta}{p+1}}_{p,1}). \) Since \( B^{\frac{\beta}{p+1}}_{p,1} \) is an algebra and \( B^{\frac{\beta}{p+1}}_{p,1} \hookrightarrow L^\infty, \) we deduce that

\[
F(m_n, u_n) = \| \frac{1}{2}(\partial_x u_n)^2 - \frac{1}{2}(u_n - m_n)^2 \|_{B^{\frac{\beta}{p+1}}_{p,1}} \\
\leq \frac{1}{2}\|((\partial_x u_n)^2\|_{B^{\frac{\beta}{p+1}}_{p,1}} + \frac{1}{2}\|((u_n - m_n)^2\|_{B^{\frac{\beta}{p+1}}_{p,1}} \\
\leq \|\partial_x u_n\|_{B^{\frac{\beta}{p+1}}_{p,1}} \|\partial_x u_n\|_{L^\infty} + \|u_n - m_n\|_{L^\infty} \|u_n - m_n\|_{L^\infty} \\
\leq C\|m_n\|_{B^{\frac{\beta}{p+1}}_{p,1}}^2.
\]

(3.2)

which leads to \( F(m_n, u_n) \in L^\infty(0, T; B^{\frac{\beta}{p+1}}_{p,1}). \) Hence, from Lemma 2.6, the equation (3.1) has a global solution \( m_{n+1} \) which belongs to \( C([0, T]; B^{\frac{\beta}{p+1}}_{p,1}) \) for all positive \( T. \)

We define that \( U_n(t) \triangleq \int_0^t \| m_n(t')\|_{B^{\frac{\beta}{p+1}}_{p,1}} \, dt'. \) By Lemma 2.7, we infer that

\[
\|m_{n+1}\|_{B^{\frac{\beta}{p+1}}_{p,1}} \leq e^{\int_0^t \|\partial_x u(t')\|_{B^{\frac{\beta}{p+1}}_{p,1}} \, dt'} \left(\|S_{n+1}m_0\|_{B^{\frac{\beta}{p+1}}_{p,1}} + \int_0^t e^{CU_n(t')} \|F(m_n, u_n)\|_{B^{\frac{\beta}{p+1}}_{p,1}} \, dt'\right).
\]

(3.3)

Fix a \( T > 0 \) such that \( 2C^2T\|m_0\|_{B^{\frac{\beta}{p+1}}_{p,1}} < 1. \) Similar to the proof of Theorem 3.1 in [43], we obtain that

\[
\|m_{n}(t)\|_{B^{\frac{\beta}{p+1}}_{p,1}} \leq \frac{C\|m_0\|_{B^{\frac{\beta}{p+1}}_{p,1}}}{1 - 2C^2\|m_0\|_{B^{\frac{\beta}{p+1}}_{p,1}} T} \leq \frac{C\|m_0\|_{B^{\frac{\beta}{p+1}}_{p,1}}}{1 - 2C^2\|m_0\|_{B^{\frac{\beta}{p+1}}_{p,1}} T} \triangleq M, \quad \forall t \in [0, T].
\]

(3.4)

Therefore, \( \{m_n\}_{n \in \mathbb{N}} \) is uniformly bounded in \( L^\infty(0, T; B^{\frac{\beta}{p+1}}_{p,1}). \)

Then, we will use the compactness method for the approximating sequence \( \{m_n\}_{n \in \mathbb{N}} \) to get a solution \( m \) of (1.2). Since \( m_n \) is uniformly bounded in \( L^\infty([0, T]; B^{\frac{\beta}{p+1}}_{p,1}), \) we can deduce from (3.2) that \( \partial_t m_n \) is uniformly bounded in \( L^\infty([0, T]; B^{\frac{\beta}{p+1} - 1}_{p,1}). \) Thus,

\[
m_n \text{ is uniformly bounded in } C\left([0, T]; B^{\frac{\beta}{p+1}}_{p,1}\right) \cap C^\frac{1}{2}\left([0, T]; B^{\frac{\beta}{p+1} - 1}_{p,1}\right).
\]

Let \( \{\phi_j\}_{j \in \mathbb{N}} \) be a sequence of smooth functions with value in \([0, 1]\) supported in the ball \( B(0, j + 1) \) and equal to 1 on \( B(0, j). \) Notice that the map \( z \mapsto \phi_jz \) is compact from \( B^{\frac{\beta}{p+1}}_{p,1} \) to \( B^{\frac{\beta}{p+1} - 1}_{p,1} \) by Theorem 2.94 in [1]. Taking advantage of Ascoli’s theorem and Cantor’s diagonal process, there exists some function \( m_j \) such that for any \( j \in \mathbb{N}, \phi_jm_n \) tends to \( m_j. \) From that, we can easily deduce that there exists some function \( m \) such that for all \( \phi \in \mathcal{D}, \phi m_n \) tends to \( \phi m \) in \( C\left([0, T]; B^{\frac{\beta}{p+1} - 1}_{p,1}\right). \) Combining the uniform boundness of \( m_n \) and the Fatou property for Besov spaces, we really obtain that \( m \in L^\infty([0, T]; B^{\frac{\beta}{p+1}}_{p,1}). \) By virtue of the interpolation, we have \( \phi m_n \) tends to \( \phi m \in C\left([0, T]; B^{\frac{\beta}{p+1} - \varepsilon}_{p,1}\right) \) for any \( \varepsilon > 0. \) Next, it is a routine process to prove that \( m \) satisfies Eq. (1.2). Thanks to the right side of the Eq. (1.2), we get \( \partial_t m \in C\left([0, T]; B^{\frac{\beta}{p+1}}_{p,1}\right). \) In sum, we obtain \( m \) satisfies (1.2) and belongs to \( C\left([0, T]; B^{\frac{\beta}{p+1}}_{p,1}\right). \)
Step 2. Uniqueness.
Define $M(t, \xi) = m(t, y(t, \xi))$ and $U(t, \xi) = u(t, y(t, \xi))$, thus, $U_\xi(t, \xi) = u_x(t, y(t, \xi))y_\xi(t, y(t, \xi))$.
The associated Lagrangian scale of (1.2) is the following initial value problem

$$
\begin{cases}
\frac{d}{dt}y = -u_x(t, y(t, \xi)), & t > 0, \quad \xi \in \mathbb{R}, \\
y(0, \xi) = \xi,
\end{cases}
$$

(3.5)

Owing to (1.2) and (3.5), we can get

$$
y(t, \xi) = \xi - \int_0^t U_\xi(t, \xi) \, dt,
$$

(3.6)

$$
d_t y_\xi(t, \xi) = (M - U) y_\xi,
$$

(3.7)

$$
\frac{d}{dt} M(t, \xi) = \left[\frac{1}{2}m^2 + um + \frac{1}{2}u_x^2 - \frac{1}{2}u^2\right] \circ y
= \frac{1}{2} M^2 + UM + \frac{1}{2} \left(\frac{U_\xi}{y_\xi}\right)^2 - \frac{1}{2}u^2,
$$

(3.8)

$$
\frac{d}{dt} U(t, \xi) = \{G \ast [u_x^2 + \frac{1}{2}u_{xx}^2] - \frac{1}{2}u_x^2\} \circ y
= \frac{1}{2} \int_{-\infty}^{+\infty} e^{-|y(t, \xi) - x|\left(u_x^2 + \frac{1}{2}u_{xx}^2\right)} dx - \frac{1}{2}u_x^2 \circ y
= \frac{1}{2} \int_{-\infty}^{+\infty} e^{-|y(t, \xi) - y(t, \eta)|}\left|\left(U_x\right)\right|^2 + \frac{1}{2}(U - M)^2 \right| y_\eta \, d\eta - \frac{1}{2} \left(\frac{U_{\xi}}{y_\xi}\right)^2
$$

(3.9)

$$
\frac{d}{dt} U_\xi(t, \xi) = \{G \ast [u_x^2 + \frac{1}{2}u_{xx}^2] - u_x u_{xx}\} y_\xi \circ y
= \frac{1}{2} \int_{-\infty}^{+\infty} \text{sign}(y(t, \xi) - x) e^{-|y(t, \xi) - x|\left(u_x^2 + \frac{1}{2}u_{xx}^2\right)} dx - u_x u_{xx} y_\xi \circ y
= \frac{1}{2} \int_{-\infty}^{+\infty} \text{sign}(y(t, \xi) - y(t, \eta)) e^{-|y(t, \xi) - y(t, \eta)|\left|\left(U_x\right)\right|^2 + \frac{1}{2}(U - M)^2} y_\eta y_\xi \, d\eta
- U_\xi(U - M).
$$

(3.10)

Since $u$, $u_x$, $m$ is uniformly bounded in $C \left([0, T]; B^\frac{1}{2}_{p,1}\right) \to C \left([0, T]; L^p \cap L^\infty\right)$, we can easily deduce that $y_\xi$ is bounded in $L^\infty([0, T]; L^\infty)$ by the Gronwall inequality. So $M(t, \xi)$, $U(t, \xi)$ and $U_\xi(t, \xi)$ is bounded in $L^\infty([0, T]; L^\infty)$. Moreover, by (3.7) we deduce that $\frac{1}{2} \leq y_\xi \leq C_0$ for $T > 0$ small enough. Thus, we obtain that $U(t, \xi) \in L^\infty([0, T]; L^p \cap L^\infty)$, $y(t, \xi) - \xi \in L^\infty([0, T]; L^p \cap L^\infty)$ and $\frac{1}{2} \leq y_\xi(t, \xi) \leq C_0$ for any $t \in [0, T]$. Now we prove the uniqueness. Suppose that $m_1 = (1 - \partial_\xi^2)u_1, m_2 = (1 - \partial_\xi^2)u_2$ are two solutions to (1.2) then $M_i(t, \xi) = M_i(t, y_i(t, \xi))$, $U_i(t, \xi) = u_i(t, y_i(t, \xi))$, $U_{i\xi}(t, \xi) = u_{ix}(t, y_i(t, \xi))y_\xi(t, y(t, \xi))$ satisfies (3.6) - (3.10) for $i = 1, 2$. Hence, (1.2), (3.6) - (3.10) together with the Growall lemma yield

$$
\frac{d}{dt} (M_1 - M_2) = -\frac{1}{2} (M_1 - M_2) (M_1 + M_2) + U_1 (M_1 - M_2) + M_2 (U_1 - U_2)
+ \frac{1}{2} \frac{U_1 y_{\xi}}{y_{\xi}} (U_{\xi} - U_{2\xi}) + U_{2\xi} (y_{2\xi} - y_{1\xi})
- \frac{1}{2} (U_1 - U_2) (U_1 + U_2)
\leq C \left(\|M_1 - M_2\|_L^\infty + \|U_1 - U_2\|_L^\infty + \|U_{1\xi} - U_{2\xi}\|_L^\infty + \|y_{1\xi} - y_{2\xi}\|_L^\infty\right),
$$

(3.11)

$$
\frac{d}{dt} (U_1 - U_2) = \frac{1}{2} \int_{-\infty}^{+\infty} e^{-|y_1(t, \xi) - y_1(t, \eta)| - e^{-|y_2(t, \xi) - y_2(t, \eta)|}} \left|\frac{U_1}{y_{1\eta}}\right|^2 \, d\eta
+ \frac{1}{2} \int_{-\infty}^{+\infty} e^{-|y_2(t, \xi) - y_2(t, \eta)|} \left|\frac{U_2}{y_{2\eta}}\right|^2 - \left(\frac{U_{2\eta}}{y_{2\eta}}\right)^2 \, d\eta
$$

7
Hence, (3.11)-(3.16) together with the Growall lemma yield

\[
\begin{align*}
&\frac{1}{2} \int_{-\infty}^{+\infty} [e^{-|y_1(t,\xi)-y_1(t,\eta)|} - e^{-|y_2(t,\xi)-y_2(t,\eta)|}] (U_1 - M_1)^2 y_1 \eta \, d\eta \\
&+ \frac{1}{2} \int_{-\infty}^{+\infty} e^{-|y_2(t,\xi)-y_2(t,\eta)|} [(U_1 - M_1)^2 y_1 \eta - (U_2 - M_2)^2 y_2 \eta] \, d\eta \\
&+ \frac{1}{2} y_1 y_2 \xi \varepsilon (U_{1 \xi} \psi_{1 \xi} + U_{2 \xi} \psi_{2 \xi}) [y_{2 \xi} (U_1 - U_2) + U_{2 \xi} (y_2 \xi - y_1)] \\
&\leq C \|M_1 - M_2\|_{L^\infty} + \|U_1 - U_2\|_{L^\infty} + \|U_1 \xi - U_2 \xi\|_{L^\infty} + \|y_1 \xi - y_2 \xi\|_{L^\infty}.
\end{align*}
\]  

(3.12)

Since \(y_i (i = 1, 2)\) is monotonically increasing, then \(\text{sign}(y_i(\xi) - y_i(\eta)) = \text{sign}(\xi - \eta)\). Thus, we have

\[
\frac{d}{dt} (U_1 \xi - U_2 \xi) = \frac{1}{2} \int_{-\infty}^{+\infty} \left[ \text{sign}(\xi - \eta) e^{-|y_1(t,\xi)-y_1(t,\eta)|} - \text{sign}(\xi - \eta) e^{-|y_2(t,\xi)-y_2(t,\eta)|} \right] \\
\times \left( \frac{(U_{1 \eta})^2}{y_1 \eta} + \frac{1}{2} (U_1 - M_1)^2 \right) y_1 \eta y_1 \xi \eta \, d\eta \\
+ \frac{1}{2} \int_{-\infty}^{+\infty} \text{sign}(\xi - \eta) e^{-|y_2(t,\xi)-y_2(t,\eta)|} \\
\times \left( \frac{(U_{2 \eta})^2}{y_2 \eta} + \frac{1}{2} (U_2 - M_2)^2 \right) y_2 \eta y_1 \xi \eta \, d\eta \\
- \left\{ U_{1 \xi}[(U_1 - M_1) - (U_2 - M_2)] + (U_2 - M_2)(U_{1 \xi} - U_{2 \xi}) \right\} \\
= I_1 + I_2 + I_3.
\]  

(3.13)

If \(\xi > \eta\) (or \(\xi < \eta\)), then \(y_1(\xi) > y_1(\eta)\) (or \(y_1(\xi) < y_1(\eta)\)). Hence, we gain

\[
I_1 = - \int_{-\infty}^{\xi} e^{-(\xi - \eta)} \left( e^{-y_1(t,\xi) - y_1(t,\eta)} - e^{-y_2(t,\xi) - y_2(t,\eta)} \right) \\
\times \left( \frac{(U_{1 \eta})^2}{y_1 \eta} + \frac{1}{2} (U_1 - M_1)^2 \right) y_1 \eta y_1 \xi \eta \, d\eta \\
+ \int_{\xi}^{+\infty} e^{-(\xi - \eta)} \left( e^{y_1(t,\xi) - y_1(t,\eta)} - e^{y_2(t,\xi) - y_2(t,\eta)} \right) \\
\times \left( \frac{(U_{1 \eta})^2}{y_1 \eta} + \frac{1}{2} (U_1 - M_1)^2 \right) y_1 \eta y_1 \xi \eta \, d\eta \\
= \int_{-\infty}^{\xi} e^{-\xi - \eta} \left( e^{-}\int_{-\infty}^{\xi} \frac{U_{1 \eta}}{y_1 \eta} - \frac{U_{2 \eta}}{y_2 \eta} \right) d\tau \\
- \int_{\xi}^{+\infty} e^{\xi - \eta} \left( e^{-}\int_{\xi}^{+\infty} \frac{U_{1 \eta}}{y_1 \eta} - \frac{U_{2 \eta}}{y_2 \eta} \right) d\tau \\
\leq C \|U_1 \eta - U_2 \eta\|_{L^\infty} + \|y_1 \eta - y_2 \eta\|_{L^\infty} \left[ \int_{-\infty}^{\xi} e^{-\xi - \eta}\left( \frac{(U_{1 \eta})^2}{y_1 \eta} \right) \right. \\
+ \frac{1}{2} (U_1 - M_1)^2 y_1 \eta y_1 \xi \eta \, d\eta + \int_{\xi}^{+\infty} e^{\xi - \eta}\left( \frac{(U_{1 \eta})^2}{y_1 \eta} + \frac{1}{2} (U_1 - M_1)^2 \right) y_1 \eta y_1 \xi \eta \, d\eta \\
\leq C \|U_1 \eta - U_2 \eta\|_{L^\infty} + \|y_1 \eta - y_2 \eta\|_{L^\infty} \left[ \int_{-\infty}^{\xi} e^{-\xi - \eta} \left( \frac{(U_{1 \eta})^2}{y_1 \eta} \right) \right. \\
+ \frac{1}{2} (U_1 - M_1)^2 |y_1 \eta y_1 \xi| + \int_{\xi}^{+\infty} e^{\xi - \eta} \left( \frac{(U_{1 \eta})^2}{y_1 \eta} + \frac{1}{2} (U_1 - M_1)^2 \right) y_1 \eta y_1 \xi \, d\eta \\
\leq C \|U_1 \eta - U_2 \eta\|_{L^\infty} + \|y_1 \eta - y_2 \eta\|_{L^\infty}.
\]  

(3.14)

In the same way, we have

\[
I_2 \leq C \|M_1 - M_2\|_{L^\infty} + \|U_1 - U_2\|_{L^\infty} + \|U_1 \xi - U_2 \xi\|_{L^\infty} + \|y_1 \xi - y_2 \xi\|_{L^\infty} \\
I_3 \leq C \|M_1 - M_2\|_{L^\infty} + \|U_1 - U_2\|_{L^\infty} + \|U_1 \xi - U_2 \xi\|_{L^\infty}.
\]  

(3.15)

(3.16)

Hence, (3.11)-(3.16) together with the Growall lemma yield

\[
\|M_1 - M_2\|_{L^\infty} + \|U_1 - U_2\|_{L^\infty} + \|U_1 \xi - U_2 \xi\|_{L^\infty} + \|y_1 \xi - y_2 \xi\|_{L^\infty}
\]
3.1 depends continuously on the initial data. By 

\[ L \]

bounded in \( L \), we can immediately obtain that 

\[ \| u_1 - u_2 \|_{L^\infty} \leq C \| u_1 \circ y_1 - u_2 \circ y_1 \|_{L^\infty} \]

\[ \leq C \| u_1 \circ y_1 - u_2 \circ y_2 + u_2 \circ y_2 - u_2 \circ y_1 \|_{L^\infty} \]

\[ \leq C \| U_1 - U_2 \|_{L^\infty} + C \| u_{2x} \|_{L^\infty} \| y_1 - y_2 \|_{L^\infty} \]

\[ \leq C \| m_1(0) - m_2(0) \|_{B_{p,1}^\frac{1}{p}}. \] (3.17)

\[ \| u_{1x} - u_{2x} \|_{L^\infty} \leq C \| u_{1x} \circ y_1 - u_{2x} \circ y_1 \|_{L^\infty} \]

\[ \leq C \| u_{1x} \circ y_1 - u_{2x} \circ y_2 + u_{2x} \circ y_2 - u_{2x} \circ y_1 \|_{L^\infty} \]

\[ \leq C(\| U_{1x} \circ y_1 - U_{2x} \|_{L^\infty} + \| u_{2x} \|_{L^\infty} \| y_1 - y_2 \|_{L^\infty}) \]

\[ \leq C(\| U_{1x} - U_{2x} \|_{L^\infty} + \| y_1 - y_2 \|_{L^\infty} + \| u_{2x} \|_{L^\infty} \| y_1 - y_2 \|_{L^\infty}) \]

\[ \leq C \| m_1(0) - m_2(0) \|_{B_{p,1}^\frac{1}{p}}. \] (3.18)

So if \( m_1(0) = m_2(0) \), we can immediately obtain that \( u_1 = u_2, u_{1x} = u_{2x} \ a.e \ in \mathbb{R} \).

Set \( W = m_1 - m_2, u_1 = u_2 = v, u_{1x} = u_{2x} = v_x \). Hence, we obtain that

\[
\begin{aligned}
\partial_t W - v_x \partial_x W &= \frac{1}{4} W(2v - m_1 - m_2), \\
W|_{t=0} &= m_1(0) = m_2(0),
\end{aligned}
\] (3.19)

Applying Lemma 2.7 and Gronwall inequality, we deduce that

\[ \| m_1 - m_2 \|_{L^\infty} \leq C \| m_1(0) - m_2(0) \|_{B_{p,1}^\frac{1}{p}}. \] (3.20)

By the embedding \( L^\infty \hookrightarrow B_{\infty,\infty}^0 \), we get

\[ \| m_1 - m_2 \|_{B_{\infty,\infty}^0} \leq C \| m_1 - m_2 \|_{L^\infty} \leq C \| m_1(0) - m_2(0) \|_{B_{p,1}^\frac{1}{p}}. \]

**Step 3. The continuous dependence.** Then we prove the solution of (1.2) guaranteed by Theorem 3.1 depends continuously on the initial data.

Assume that \( m_0^n \) tends to \( m_0^\infty \) in \( B_{p,1}^{2+\frac{1}{p}} \), \( u_0^n \) tends to \( u_0^\infty \) in \( B_{p,1}^{2+\frac{1}{p}} \) and \( m^n, m^\infty \) are the solutions of (1.2) with the initial data \( m_0^n, m_0^\infty \) respectively. Similar to [72], we can find the solution of (1.2) with a common lifespan \( T \). By Step 1–Step 2, we have \( m^n, m^\infty \) are uniformly bounded in \( L^\infty([0,T];B_{p,1}^{2+\frac{1}{p}}) \), \( u^n, u^\infty \) are uniformly bounded in \( L^\infty([0,T];B_{p,1}^{2+\frac{1}{p}}) \) and

\[ \| (u^n - u^\infty)(t) \|_{B_{\infty,\infty}^0} \leq C \| m^n_0 - m^\infty_0 \|_{B_{p,1}^{\frac{1}{p}}}, \forall t \in [0,T]. \]

\[ \| (u^n_x - u^\infty_x)(t) \|_{B_{\infty,\infty}^0} \leq C \| m^n_0 - m^\infty_0 \|_{B_{p,1}^{\frac{1}{p}}}, \forall t \in [0,T]. \]
Taking advantage of the interpolation inequality, we see that

\[ u^n \to u^\infty, \text{in} C([0,T]; B^\frac{1}{p}_{p,1}), \]

\[ u^n_x \to u^\infty_x, \text{in} C([0,T]; B^\frac{1}{p}_{p,1}). \]

we next only need to prove \( m^n \to m^\infty \) in \( C([0,T]; B^\frac{1}{p}_{p,1}) \). Split \( m^n \) into \( w^n + z^n \) with \((w^n, z^n)\) satisfying

\[
\begin{aligned}
\begin{cases}
\partial_t w^n + \partial_x u^n \partial_x w^n = F(m^\infty, u^\infty, u^n), \\
w^n(0, x) = m^\infty_0
\end{cases}
\end{aligned}
\]

and

\[
\begin{aligned}
\begin{cases}
\partial_t z^n + \partial_x u^n \partial_x z^n = F(m^n, u^n) - F(m^\infty, u^\infty) \\
z^n(0, x) = v^n_0 - v^\infty_0 = m^n_0 - m^\infty_0.
\end{cases}
\end{aligned}
\]

(3.2) and Lemma 2.8 thus ensure that

\[ w^n \to w^\infty \quad \text{in} \quad C([0,T]; B^\frac{1}{p}_{p,1}). \] (3.21)

Thanks to (3.2), we have

\[
\| F(m^n, u^n) - F(m^\infty, u^\infty) \|_{B^\frac{1}{p}_{p,1}} \leq C(\| m^n - m^\infty \|_{B^\frac{1}{p}_{p,1}} + \| u^n - u^\infty \|_{B^\frac{1}{p}_{p,1}} + \| u^n - u^\infty \|_{B^\frac{1}{p}_{p,1}}).
\]

It follows that for all \( n \in \mathbb{N} \),

\[
\| z^n(t) \|_{B^\frac{1}{p}_{p,1}} \leq C \left( \| m^n_0 - m^\infty_0 \|_{B^\frac{1}{p}_{p,1}} + \int_0^t \| m^n - m^\infty \|_{B^\frac{1}{p}_{p,1}} + \| u^n - u^\infty \|_{B^\frac{1}{p}_{p,1}} + \| u^n - u^\infty \|_{B^\frac{1}{p}_{p,1}} \, dt \right);
\]

\[
\leq C \left( \| m^n_0 - m^\infty_0 \|_{B^\frac{1}{p}_{p,1}} + \int_0^t \| u^n_x - u^\infty_x \|_{B^\frac{1}{p}_{p,1}} + \| u^n - u^\infty \|_{B^\frac{1}{p}_{p,1}} + \| w^n - w^\infty \|_{B^\frac{1}{p}_{p,1}} + \| z^n \|_{B^\frac{1}{p}_{p,1}} \, dt \right).
\]

(3.22)

Using the facts that

- \( m^n_0 \) tends to \( m^\infty_0 \) in \( B^\frac{1}{p}_{p,1} \);
- \( u^n \) tends to \( u^\infty \) in \( C([0,T]; B^\frac{1}{p}_{p,1}) \);
- \( u^n_x \) tends to \( u^\infty_x \) in \( C([0,T]; B^\frac{1}{p}_{p,1}) \);
- \( w^n \) tends to \( w^\infty \) in \( C([0,T]; B^\frac{1}{p}_{p,1}) \),

and then applying the Gronwall lemma, we conclude that \( z^n \) tends to 0 in \( C([0,T]; B^\frac{1}{p}_{p,1}) \). By Lemma 2.6 2.7 we have \( z^\infty = 0 \) in \( C([0,T]; B^\frac{1}{p}_{p,1}) \).

Therefore,

\[
\| m^n - m^\infty \|_{L^\infty([0,T]; B^\frac{1}{p}_{p,1})} \leq \| w^n - w^\infty \|_{L^\infty([0,T]; B^\frac{1}{p}_{p,1})} + \| z^n - z^\infty \|_{L^\infty([0,T]; B^\frac{1}{p}_{p,1})}
\]

\[
\leq \| w^n - w^\infty \|_{L^\infty([0,T]; B^\frac{1}{p}_{p,1})} + \| z^n \|_{L^\infty([0,T]; B^\frac{1}{p}_{p,1})} \to 0 \quad \text{as} \ n \to \infty,
\]

that is

\[ m^n \to m^\infty \quad \text{in} \quad C([0,T]; B^\frac{1}{p}_{p,1}). \]

Hence, we prove the continuous dependence of (1.22) in critical Besov spaces \( C([0,T]; B^\frac{1}{p}_{p,1}) \) with \( p \in [1, +\infty) \).

Consequently, combining with \textbf{Step 1–Step 3}, we finish the proof of Theorem 3.1.
Acknowledgements. This work was partially supported by NNSFC (No. 11671407 and No. 11801076), FDCT (No. 0001/2013/A3), Guangdong Special Support Program (No. 8-2015) and the key project of NSF of Guangdong Province (No. 2016A030311004).

References

[1] H. Bahouri, J. Y. Chemin and R. Danchin, Fourier analysis and nonlinear partial differential equations, Grundlehren der Mathematischen Wissenschaften, 343, Springer, Heidelberg (2011).

[2] G. M. Coclite and K. H. Karlsen, On the well-posedness of the Degasperis-Procesi equation, J. Func. Anal., 233 (2006), 60–91.

[3] A. Bressan and A. Constantin, Global conservative solutions of the Camassa-Holm equation, Archive for Rational Mechanics and Analysis, 183 (2007), 215–239.

[4] A. Bressan and A. Constantin, Global dissipative solutions of the Camassa-Holm equation, Analysis and Applications, 5 (2007), 1–27.

[5] R. Camassa and D. D. Holm, An integrable shallow water equation with peaked solitons, Physical Review Letters, 71 (1993), 1661–1664.

[6] R. Camassa, D. Holm and J. Hyman, A new integrable shallow water equation, Advances in Applied Mechanics, 31 (1994), 1–33.

[7] A. Constantin, The Hamiltonian structure of the Camassa-Holm equation, Expositiones Mathematicae, 15(1) (1997), 53–85.

[8] A. Constantin, On the scattering problem for the Camassa-Holm equation, Proceedings of The Royal Society of London, Series A, 457 (2001), 953–970.

[9] A. Constantin, Existence of permanent and breaking waves for a shallow water equation: a geometric approach, Annales de l’Institut Fourier (Grenoble), 50 (2000), 321–362.

[10] A. Constantin, The trajectories of particles in Stokes waves, Inventiones Mathematicae, 166 (2006), 523–535.

[11] A. Constantin and J. Escher, Global existence and blow-up for a shallow water equation, Annali della Scuola Normale Superiore di Pisa - Classe di Scienze, 26 (1998), 303–328.

[12] A. Constantin and J. Escher, Well-posedness, global existence, and blowup phenomena for a periodic quasi-linear hyperbolic equation, Communications on Pure and Applied Mathematics, 51 (1998), 475–504.

[13] A. Constantin and J. Escher, Wave breaking for nonlinear nonlocal shallow water equations, Acta Mathematica, 181 (1998), 229–243.

[14] A. Constantin and J. Escher, Particle trajectories in solitary water waves, Bulletin of the American Mathematical Society, 44 (2007), 423–431.

[15] A. Constantin and J. Escher, Analyticity of periodic traveling free surface water waves with vorticity, Annals of Mathematics, 173 (2011), 559–568.

[16] A. Constantin and D. Lannes, The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi equations, Archive for Rational Mechanics and Analysis, 192 (2009), 165–186.

[17] A. Constantin and H. P. McKean, A shallow water equation on the circle, Comm. Pure Appl. Math., 55 (1999), 949–982.

[18] A. Constantin and L. Molinet, Global weak solutions for a shallow water equation, Communications in Mathematical Physics, 211 (2000), 45–61.

[19] A. Constantin and W. A. Strauss, Stability of peakons, Communications on Pure and Applied Mathematics, 53 (2000), 603–610.

[20] R. Danchin, A few remarks on the Camassa-Holm equation, Differential Integral Equations, 14 (2001), 953–988.
[21] R. Danchin, *A note on well-posedness for Camassa-Holm equation*, Journal of Differential Equations, **192** (2003), 429-444.

[22] A. Degasperis, D. D. Holm, and A. N. W. Hone, *A new integral equation with peakon solutions*, Theor. Math. Phys., **133** (2002), 1463–1474.

[23] A. Degasperis, M. Procesi, *Asymptotic integrability*, Symmetry and Perturbation Theory, **1**(1) (1999), 23-37.

[24] H. R. Dullin, G. A. Gottwald, and D. D. Holm, *On asymptotically equivalent shallow water wave equations*, Phys. D, **190** (2004), 1–14.

[25] J. Escher, Y. Liu and Z. Yin, *Global weak solutions and blow-up structure for the Degasperis-Procesi equation*, J. Funct. Anal., **241** (2006), 457–485.

[26] J. Escher, Y. Liu and Z. Yin, *Shock waves and blow-up phenomena for the periodic Degasperis-Procesi equation*, Indiana Univ. Math. J., **56** (2007), 87–177.

[27] A. Fokas and B. Fuchssteiner, *Symplectic structures, their B"acklund transformation and hereditary symmetries*, Physica D, **4**(1) (1981/82), 47–66.

[28] G. Gui and Y. Liu, *On the Cauchy problem for the Degasperis-Procesi equation*, Quart. Appl. Math., **69**, 445-464, (2011).

[29] Z. Guo, X. Liu, L. Molinet and Z. Yin, *Ill-posedness of the Camassa-Holm and related equations in the critical space*, J. Differential Equations, **266** (2019), 1698-1707.

[30] A. A. Himonas and C. Holliman, *The Cauchy problem for the Novikov equation*, Nonlinearity, **25** (2012), 449-479.

[31] A. N. W. Hone and J. Wang, *Integrable peakon equations with cubic nonlinearity*, Journal of Physics A: Mathematical and Theoretical, **41** (2008), 372002, 10pp.

[32] S. Lai, *Global weak solutions to the Novikov equation*, Journal of Functional Analysis, **265** (2013), 520-544.

[33] Y. A. Li and P. J. Olver. *Well-posedness and blow-up solutions for an integrable nonlinearly dispersive model wave equation*. J. Differential Equations, **162**(1) (2000), 27–63.

[34] B. Lin and Z. Yin, *The Cauchy problem for a generalized Camassa-Holm equation with the velocity potential*, Applicable Analysis, **96** (2017), 679–701.

[35] Y. Liu and Z. Yin, *Global Existence and Blow-up Phenomena for the Degasperis-Procesi Equation*, Commun. Math. Phys., **267** (2006), 801–820.

[36] Y. Liu and Z. Yin, *On the blow-up phenomena for the Degasperis-Procesi equation*, Int. Math. Res. Not. IMRN, **23** (2007), rnm117, 22 pp.

[37] J. Li and Z. Yin, *Remarks on the well-posedness of Camassa-Holm type equations in Besov spaces*, J. Differential Equations, **261** (2016), 6125-6143.

[38] H. Lundmark, *Formation and dynamics of shock waves in the Degasperis-Procesi equation*, J. Nonlinear. Sci., **17** (2007), 169–198.

[39] W. Luo and Z. Yin, *Local well-posedness and blow-up criteria for a two-component Novikov system in the critical Besov space*, Nonlinear Analysis. Theory, Methods Applications, **122** (2015), 1–22.

[40] V. Novikov, *Generalization of the Camassa-Holm equation*, J. Phys. A, **42** (2009), 342002, 14pp.

[41] G. Rodríguez-Blanco, *On the Cauchy problem for the Camassa-Holm equation*, Nonlinear Analysis. Theory Methods Application, **46** (2001), 309-327.

[42] J. F. Toland, *Stokes waves*, Topological Methods in Nonlinear Analysis, **7** (1996), 1-48.

[43] X. Tu and Z. Yin. *Blow-up phenomena and local well-posedness for a generalized Camassa–Holm equation in the critical Besov space[f]*, Monatshfte für Mathematik, **191** (2020), 801–829.

[44] X. Tu and Z. Yin. *The existence of global weak solutions for a generalized Camassa–Holm equation*, Applicable Analysis, (2020), 1-14.
[45] X. Tu and Z. Yin. *Global Weak Solution for a generalized Camassa-Holm equation*. Mathematische Nachrichten, **291**(2018), 2457–2475.

[46] X. Wu and Z. Yin, *Global weak solutions for the Novikov equation*, Journal of Physics A: Mathematical and Theoretical, **44** (2011), 055202, 17pp.

[47] X. Wu and Z. Yin, *Well-posedness and global existence for the Novikov equation*, Annali della Scuola Normale Superiore di Pisa. Classe di Scienze. Serie V, **11** (2012), 707–727.

[48] X. Wu and Z. Yin, *A note on the Cauchy problem of the Novikov equation*, Applicable Analysis, **92** (2013), 1116–1137.

[49] Z. Xin and P. Zhang, *On the weak solutions to a shallow water equation*, Communications on Pure and Applied Mathematics, **53** (2000), 1411–1433.

[50] W. Yan, Y. Li and Y. Zhang, *The Cauchy problem for the integrable Novikov equation*, J. Differential Equations, **253** (2012), 298–318.

[51] W. Yan, Y. Li and Y. Zhang, *The Cauchy problem for the Novikov equation*, Nonlinear Differential Equations and Applications NoDEA, **20** (2013), 1157–1169.

[52] W. Ye, W. Luo and Z. Yin, *The estimate of lifespan and local well-posedness for the non-resistive MHD equations in homogeneous Besov spaces*, arXiv preprint arxiv:2012.03489v1, (2020).

[53] Z. Yin, *On the Cauchy problem for an integrable equation with peakon solutions*, Ill. J. Math., **47** (2003), 649–666.

[54] Z. Yin, *Global existence for a new periodic integrable equation*, J. Math. Anal. Appl., **283** (2003), 129–139.

[55] Z. Yin, *Global weak solutions to a new periodic integrable equation with peakon solutions*, J. Funct. Anal., **212** (2004), 182–194.

[56] Z. Yin, *Global solutions to a new integrable equation with peakons*, Indiana Univ. Math. J., **53** (2004), 1189–1210.