FREE TIME MINIMIZERS FOR THE PLANAR THREE-BODY PROBLEM

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ABSTRACT. Free time minimizers of the action (called “semi-static” solutions by Mañe) play a central role in the theory of weak KAM solutions to the Hamilton-Jacobi equation [8]. We prove that any solution to Newton’s three-body problem which is asymptotic to Lagrange’s parabolic homothetic solution is eventually a free time minimizer. Conversely, we prove that every free time minimizer tends to Lagrange’s solution, provided the mass ratios lie in a certain large open set of mass ratios. We were inspired by the work of [4] who had shown that every free time minimizer for the N-body problem is parabolic, and therefore must be asymptotic to the set of central configurations. We exclude being asymptotic to Euler’s central configurations by a second variation argument. Central configurations correspond to rest points for the McGehee blown-up dynamics. The large open set of mass ratios are those for which the linearized dynamics at each Euler rest point has a complex eigenvalue.

1. INTRODUCTION AND RESULTS.

Lagrange’s parabolic homothetic solution (figure 1) to the three-body problem consists of an equilateral triangle expanding at the rate $t^{2/3}$. Analytically such a solution is described by equation (1) below with the $c_i$ there denoting the locations of the vertices of an equilateral triangle. Newton’s equations are Euler-Lagrange equations for a well-known action. Recently, Maderna and Venturelli [13] proved that the parabolic homothetic Lagrange solution (figure 1) is a very strong type of action minimizer called a “free time minimizer” (or “semi-static” by Mañe [15]) for this action. These minimizers are objects of central interest in the recently developed theory of weak KAM solutions [8].

Figure 1. The Lagrange parabolic homothetic solution: an equilateral triangle expanding at the rate $t^{2/3}$.
The parabolic homothetic central configuration solutions, of which Lagrange’s solution is an example, are given by:

\[
\gamma_c(t) = \alpha ct^{2/3}, \ c \text{ a central configuration, } \alpha = \alpha(c) \in \mathbb{R}
\]

Central configurations are special configurations which play an important role in the three-body problem. Modulo rigid motions and scaling there are exactly five central configurations for any given mass ratio. (See equation (11) in section 4 for the defining equation of central configurations.) Two are the Lagrange equilateral triangle configurations just described. The Lagrange central configurations count as two because there are two orientation types of labelled equilateral triangles, related to each other by reflection but not by rigid motion. The remaining three central configurations are the Euler configurations for which the three bodies lie on a line. We index the Euler configurations by which body lies between the other two.

As time \( t \to \infty \) in equation (1) the velocity of each body tends to zero. Solutions with this property are called “parabolic”. Although Chazy ([1, chapter III]) did not use this definition, his results show that any parabolic solution \( \gamma(t) \) asymptotes to one of the \( \gamma_c \) of equation (1) in the sense that

\[
\lim_{t \to \infty} t^{-2/3} \gamma(t) = c.
\]

(See subsection 4.4 for a proof of an equivalent statement.)

As stated above, the Lagrange solution is a FTM – a free time minimizer, and it is parabolic. Da Luz and Maderna [4] proved that every FTM is parabolic. These results beg us to ask three questions. Are the Euler parabolic homothetic solutions FTMs? To which of the five types of central configurations may a FTM be asymptotic? Among the parabolic solutions, which ones are FTMs? We give a complete answer to these questions for a large range of masses which we call the “spiraling range”, depicted in figure 2. See subsection 1.1 and Definition 5 for details on this range of masses.

**Theorem 1.** In the spiraling range of mass ratios, every free time minimizer \( \gamma(t) \) for the planar three-body problem tends to some Lagrange configuration: the limit \( c \) of equation (1) is an equilateral triangle. Equivalently, these orbits lie in the stable manifold of one of the Lagrange restpoints at infinity, as described in subsections 1.1 and 4.4.

We have the following converse to this theorem, valid for all mass ratios:

**Theorem 2.** Every parabolic solution \( \gamma(t), 0 \leq t < \infty \) asymptotic to a Lagrange configuration is a free time minimizer upon its restriction to a sub-interval \( T \leq t < \infty, T \) large enough.

1.1. McGehee blow-up and the spiraling range. The spiraling range condition of theorem 1 first arose in Siegel’s study of the triple collision singularity [27, 28]. At a triple collision in backward time with \( t \to 0^+ \), the three mutual distances tend to zero at the same \( t^{2/3} \) rate. (In fact the homothetic solutions (1) exhibit this behavior.) After rescaling by \( t^{2/3} \), these solutions also converge to central configurations. Certain eigenvalues computed by Siegel determine the rate of convergence and behavior of nearby noncollision orbits. For the case of an Eulerian central configuration, these eigenvalues are often complex which produces an oscillation.

In 1974, McGehee revolutionized celestial mechanics by a change of variables which partially compactifies phase space, adding a “collision manifold” [17, 18]. (We review McGehee’s coordinates in subsection 4.1 below.) Before compactification the flow has no fixed points. After compactification the flow admits fixed points on the collision manifold, indeed a pair of fixed points for each central configuration, so 10 fixed points in all, modulo rotation. Orbits converging to triple collision in forward or backward time now constitute the stable and unstable manifolds of these restpoints. The eigenvalues of these restpoints are the same, up to scaling, as those found by Siegel.
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Figure 2. The spiraling range of mass ratios, depicted in the mass simplex $m_1 + m_2 + m_3 = 1$. If $E_i$ denotes the collinear central configuration with mass $m_i$ in the middle, then the mass for which spiraling does not occur occupies the small shaded region near the corresponding vertex, a region in which where $m_i$ is much larger than the other masses. The spiraling range, where all three central configurations have nonreal eigenvalues, is represented by the large unshaded region of the simplex. The figure is based on results in [26].

A variation on McGehee’s method allows us to study parabolic orbits in a nearly identical way. We get a flow at infinity identical to that of the collision manifold flow. The restpoints – still parameterized by the central configurations, are now interpreted in terms of parabolic solutions. The eigenvalues of the linearized flow at these restpoints are those computed by Siegel. Their stable manifolds are comprised of the parabolic orbits. For the Euler central configurations there is an open connected range of mass ratios for which we get eigenvalues with nonzero imaginary parts. We will call this open set of mass ratios the “spiraling range” of mass ratios (see figure 2).

Remark. Following McGehee’s work, several authors, including one of us, have used the oscillatory phenomenon near triple collision to prove existence of interesting near-collision orbits [5, 29, 19, 20, 21, 22]. In these it was discovered that if the mass ratios are in the spiraling range then certain complicated chaotic behaviors, described by symbolic dynamics, are guaranteed to occur for the corresponding three-body problem.

2. The key lemmas

2.1. For theorem 1. In trying to understand free time minimizers, it is natural to consider the second variation of the action along a solution segment. If this second variation is negative then we call the corresponding solution “variationaly unstable”. The action of a variational unstable solution can be decreased by deforming the solution curve in the direction of the negative variation, so such solution curves cannot be free-time minimizers. Thus, Theorem 1 is an immediate corollary of

Lemma 1. (A) The Euler parabolic homothety solution is variationaly unstable if and only if the mass ratio associated to that Euler central configuration is spiraling (definition 3).
(B) For these same mass ratios, any parabolic solution asymptotic to that Euler parabolic homothety solution is variationally unstable.

(Part (B) applies to the solutions in the stable manifold of the corresponding Eulerian restpoints at infinity.)

For the proof of lemma 1 the interested reader may skip directly to section 5. This reader may need to refer back to section 4 for eigenvalue information.

2.2. For theorem 2. The key lemma for Theorem 2 is

Lemma 2. The stable manifold of a Lagrange restpoint \(c\) at infinity is an immersed Lagrangian submanifold which is a smooth embedded graph near \(c\).

Remark. We use McGehee blow-up coordinates (see subsection 4.1) to define what we mean here by “neighborhood of a restpoint \(c\) at infinity”. In those coordinates, a neighborhood for the restpoint \(c\) at infinity consists of the points \((u, c)\) of the form \(0 < u < \delta\), \(|s - c| < \delta\). See the definition towards the end of section 6.

The proof of theorem 2 combines this lemma with the theory of weak KAM solutions, a theory to be reviewed in the next section. The basic idea is as follows. Percino and Sanchez [24] constructed a weak KAM solution for each Lagrange parabolic homothetic solution. Where smooth, the graph of the differential of this solution forms a Lagrangian manifold. The main point of the proof is that this Lagrangian submanifold agrees with the stable manifold of lemma 2 near infinity. The full proof of the theorem is found in the last section of this article, section 6, which also contains the proof of lemma 2.

3. Free Time Minimizers, geodesics, and weak KAM.

3.1. Free Time Minimizers and Jacobi-Maupertuis metric. We define free time minimizers. We show they are minimizing geodesic rays for the zero-energy Jacobi-Maupertuis metric.

A ray in a Riemannian manifold is a geodesic whose domain is a half-line, and whose restriction to any closed sub-interval of the domain is a minimizing geodesic between its endpoints.

Mañé generalized the notion of rays and lines to Lagrangian dynamics defining “semi-static” curves. In the context of the N-body problem Maderna started calling semi-static curves free time minimizers and we stick to his terminology.

Consider the Newtonian N-body potential \(U : \mathbb{E} \to [0, \infty], \mathbb{E} = (\mathbb{R}^d)^N\) given by

\[
U(x) = \sum_{i<j} \frac{m_i m_j}{r_{ij}}
\]

for a configuration \(x = (r_1, \ldots, r_N) \in \mathbb{E}\) of \(N\) punctual positive masses \(m_1, \ldots, m_N\) with \(r_{ij} = |r_i - r_j|\), as well as the Lagrangian \(L : \mathbb{E} \times \mathbb{E} \to ]0, \infty]\)

\[
L(x, v) = K(v) + U(x) = \frac{1}{2} \sum_{i=1}^{N} m_i |v_i|^2 + U(x).
\]

The action of an absolutely continuous curve \(\gamma : [a, b] \to \mathbb{E}\) is given by

\[
A_L(\gamma) = \int_{a}^{b} L(\gamma(t), \dot{\gamma}(t)) dt
\]

Definition 1. A curve \(\gamma_* : [a, b] \to \mathbb{E}\) with domain a closed bounded interval is called free time minimizer if \(A(\gamma_*) \leq A(\gamma)\) for every other curve \(\gamma : [c, d] \to \mathbb{E}\) sharing its endpoints: \(\gamma_*(a) = \gamma(c), \gamma_*(b) = \gamma(d)\).

Note: \(d - c \neq b - a\) is allowed in the above definition.
Definition 2. A curve $\gamma$ with domain an unbounded interval $J$ (so $J$ is of the form $(-\infty, \infty)$ or $(-\infty, t_0]$ is a free time minimizer if, for each closed bounded sub-interval $[a, b] \subset J$, the restriction $\gamma|_{[a,b]}$ of $\gamma$ to $[a, b]$ is a free time minimizer in the sense above (def. 1).

Proposition 1 (Basic facts regarding free time minimizers).
(i) If $\gamma$ is a free time minimizer then its total energy $H = K - U$ is zero at each time.
(ii) The free time minimizers are precisely the minimizing geodesics for the zero energy Jacobi-Maupertuis [JM] metric

\[ \langle v, v \rangle_q = 4U(q)K(v) \]

on $E$, with these geodesics reparameterized so as to have 0 energy.

Proof. See [4] for a proof of fact (i). We now prove fact (ii). The inequality $a^2 + b^2 \geq 2ab$ with equality if and only if $a = b$ yields

$2\sqrt{U}\sqrt{K} \leq L = K + U$ with equality iff $H = 0$.

Consequently, for any absolutely continuous curve $\gamma : [a, b] \to E$ we have

$\ell_{JM}(\gamma) \leq A(\gamma)$ with equality iff $H = 0$ along $\gamma$

where $\ell_{JM}(\gamma)$ is the JM length functional. Now, if $\gamma$ is a free time minimizer, we have that $H = 0$ and so the Jacobi length and action are the same. Thus $\gamma$ must be a minimizing geodesic for the JM metric. \hfill $\square$

Note that it follows from the proposition that the FTM with domain $[T_0, \infty)$ are precisely the rays for the zero energy JM metric.

3.2. Weak KAM and Hamilton Jacobi. It follows from (ii) of proposition 1 that

\[ d_{JM}(q, q_0) = \inf_{\gamma : q_0 \to q} \int_{\gamma} L \]

is the JM distance between the points $q, q_0 \in Q$. Freezing $q_0$, we get the function $f(q) = d_{JM}(q, q_0)$ on $Q$. It is well known that on a Riemannian manifold the gradient of the distance function from a point (or a subvariety) is a unit vector wherever differentiable. Viewed in dual terms, this unit length gradient condition reads

\[ \|df\|_{JM} = 1 \]

where the subscript $K$ is the length of a covector relative to the dual to the metric $K$. In other words,

\[ \frac{1}{2}\|df\|_K^2 = U, \]

which is the Hamilton-Jacobi equation. Since

\[ df(q)(v) \leq \frac{1}{2}\|df\|_K^2 + K(v), \]

the relation (4) is equivalent to

\[ df(q)(v) \leq L(q, v) \] for all $q, v \in E$,

with equality realized for some $v$.

Now set $v = \dot{\gamma}$ for some curve $\gamma(t), t \geq 0$ and integrate:

\[ f(\gamma(a)) - f(\gamma(0)) \leq \int_{\gamma[0,a]} Ldt \]
A curve \( \gamma : [0, a] \to \mathbb{E} \) is called \( \text{calibrated by } f \) if
\[
(6) \quad f(\gamma(a)) - f(x) = \int_0^a L(\gamma, \dot{\gamma}) dt.
\]

**Definition 3.** A function \( f : \mathbb{E} \to \mathbb{R} \) which satisfies equation (5) for any curve \( \gamma : [0, a] \to \mathbb{E} \) is called \( \text{dominated}. \) If moreover for all \( x \in \mathbb{E} \) there exists a curve \( \gamma : [0, \infty) \to \mathbb{E} \) such that \( \gamma(0) = x \) and for all \( t > 0, \gamma([0, t]) \) is calibrated by \( f \), the function is called \( \text{(forward) weak KAM solution}. \)

It follows from the definitions that any curve \( \gamma \) calibrated by a dominated function is a free time minimizer.

**Remark 1.** Any weak KAM solution satisfies the classical Hamilton-Jacobi equation at differentiability points. (See Fathi [8] for details and a careful exposition.)

**Remark 2.** In a neighborhood of a point where \( f \) is smooth, the graph of it differential \( df \) forms a smooth Lagrangian graph sitting in the zero energy level of the cotangent bundle. This graph is invariant under the Hamiltonian flow and the solutions to Hamilton’s equation foliate the graph. If we write such a solution as \( (c(t), p(t)) \) then necessarily \( c \) is a piece of a calibrating curve, and \( p(t) = \dot{c}(t) \) is the Legendre transformation of the derivative of that calibrating curve.

**Remark 3.** There are also backwards weak KAM solutions. For these, the calibrated curves of equation (6) are parameterized as \( \gamma : (\infty, 0] \to \mathbb{E} \), and end at \( x \): \( \gamma(0) = x \).

Buseman found a nice way to construct a weak KAM solution out of a free time minimizer.

**Definition 4.** The Buseman function \( B_\gamma \) associated to a fixed free time minimizer \( \gamma : [0, +\infty) \to \mathbb{E} \) is
\[
B_\gamma(x) = \lim_{t \to +\infty} [d_{JM}(\gamma(0), \gamma(t)) - d_{JM}(x, \gamma(t))]
\]

Buseman’s inspiration came from horocycles in hyperbolic geometry. These horocycles are the level sets of the Buseman function associated to a hyperbolic geodesic \( \gamma \). Percino and Sanchez [24] re-expressed Buseman’s idea in the present language, as we have just done, and were able to prove

**Proposition 2.** [24] The Busemann function \( B_\gamma \) associated to any parabolic Lagrange solution \( \gamma_c \) is a weak KAM solution. Moreover, for any configuration \( x \), the corresponding calibrated curve of equation (6) will be asymptotic to \( c \).

This proposition will be central to proving Theorem 2

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### 4. Set-up

The three-body configuration space is three copies of the Euclidean plane. We write points, or “configurations”, as \( q = (q_1, q_2, q_3), q_i \in \mathbb{R}^2 \cong \mathbb{C} \) and we write velocities as \( v = (v_1, v_2, v_3), v_i \in \mathbb{R}^2 \cong \mathbb{C} \). The masses are \( m_i > 0 \). By a standard trick from introductory physics we can assume that the center of mass of the configuration is zero:
\[
(7) \quad m_1 q_1 + m_2 q_2 + m_3 q_3 = 0
\]
and that the total linear momentum is also zero:
\[
(8) \quad m_1 v_1 + m_2 v_2 + m_3 v_3 = 0.
\]
We write \( \mathbb{E} \cong \mathbb{R}^4 \subset \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \) for either four-dimensional linear subspace.

We introduce the mass metric:
\[
\langle v, w \rangle_m := m_1 v_1 \cdot w_1 + m_2 v_2 \cdot w_2 + m_3 v_3 \cdot w_3 = v \cdot M w
\]
where $M = \text{diag}(m_1, m_1, m_2, m_2, m_3, m_3)$ is the mass matrix, so that the usual kinetic energy is

$$K = \frac{1}{2} \langle v, v \rangle_m; v \in \mathbb{E}.$$ 

We will also make use of the moment of inertia of the configuration with respect to the origin

(9)  
$$I(q) = \langle q, q \rangle_m.$$ 

An alternative, translation invariant formula for the moment of inertia is

$$I(q) = \left( m_1 \ m_2 \ r_{12}^2 + m_1 \ m_3 \ r_{13}^2 + m_2 \ m_3 \ r_{23}^2 \right)/m \quad r_{ij} = |q_i - q_j| \quad m = m_1 + m_2 + m_3.$$ 

The Lagrangian for the three-body problem

$$L : \mathbb{E} \times \mathbb{E} \to [0, \infty]$$

is given by

$$L(q, v) = K(v) + U(q)$$

where

$$U(q) = \frac{m_1 m_2}{|q_1 - q_2|} + \frac{m_2 m_3}{|q_2 - q_3|} + \frac{m_1 m_3}{|q_1 - q_3|}$$

is the negative of the potential energy. Newton’s equations read

(10)  
$$\ddot{q} = \nabla_m U(q)$$

where the gradient is with respect to the mass-metric: $dU(q)(v) = \langle \nabla_m U(q), h \rangle_m$. The distance from the origin (which is triple collision) with respect to the mass metric is denoted by $r$:

$$r(q) = \sqrt{I(q)} = \sqrt{\langle q, q \rangle_m}.$$ 

Each configuration $q \in \mathbb{E}$ determines a unique normalized configuration $s = q/r$ with $r(s) = 1$.

A central configuration or CC is a point $q \in \mathbb{E}$ such that

(11)  
$$\nabla_m U(q) + \lambda q = 0.$$ 

Using the homogeneity of the potential it is easy to see that $\lambda = \frac{U(q)}{r(q)^2}$ which reduces to $\lambda = U(c)$ for a normalized central configuration. The homothetic parabolic motion associated to such a normalized central configuration $c \in \mathbb{E}$ is

(12)  
$$\gamma_c(t) = \left( \frac{9}{2} U(c) \right)^{\frac{1}{2}} ct^{2/3}$$

and is an exact solution to Newton’s equations.

4.1. McGehee coordinates. Following McGehee, define new variables

$$r = \sqrt{\langle q, q \rangle_m} \quad s = \frac{q}{r} \quad z = r^{\frac{1}{2}} \dot{q}.$$ 

The normalized configuration variable $s = (s_1, s_2, s_2)$ satisfies $I(s) = r(s)^2 = 1$.

Introduce the function

(13)  
$$v = \langle z, s \rangle_m$$

and define a new time variable $\tau$ by $d/d\tau = r^{\frac{1}{2}} d/dt$ and write $f' = df/d\tau$. Then the new, blown-up variables satisfy

(14)  
$$r' = vr$$

(15)  
$$s' = z - vs$$

(16)  
$$z' = \nabla_m U(s) + \frac{1}{2} vz.$$
In deriving these one uses the homogeneity of the potential to see that $U(q) = r^{-1}U(s)$ and $\nabla_m U(q) = r^{-2}\nabla_m U(s)$. The equations now make sense when $r = 0$: the triple collision singularity has been blown-up into the invariant manifold $\{r = 0\}$ and the differential equations for $(s, z)$ are independent of $r$.

In these coordinates, the eight-dimensional phase space $X = E \times E$ is given by the system of equations

\begin{align*}
(17) \quad m_1 s_1 + m_2 s_2 + m_3 s_3 &= 0, \quad m_1 z_1 + m_2 z_2 + m_3 z_3 = 0 \quad \text{and} \quad \langle s, s \rangle_m = 1.
\end{align*}

The flow preserves $X$ as well as the energy levels $H(q, p) = h$ which are now given by

\begin{align*}
H(s, z) = \frac{1}{2} \langle z, z \rangle_m - U(s) = rh.
\end{align*}

We will be especially interested in the case $h = 0$.

The rate of change of $v = \langle z, s \rangle_m$ satisfies

\begin{align*}
v' = \langle z, z \rangle_m - U(s) - \frac{1}{2} v^2 = \frac{1}{2} \langle z, z \rangle_m - \frac{1}{2} v^2 + rh.
\end{align*}

If $r = 0$ (triple collision) or $h = 0$ (zero energy), this simplifies to

\begin{align*}
v' = \frac{1}{2} (\langle z, z \rangle_m - v^2) \geq 0
\end{align*}

where the nonnegativity follows from the Cauchy-Schwarz inequality. Thus $v$ is a Lyapunov function on the triple collision ($r = 0$) and zero energy ($h = 0$) submanifolds.

When $h = 0$ we study the motions with $r \to \infty$ by replacing $r$ by $u = r^{-1}$. Then equation (14) is replaced by:

\begin{align*}
(18) \quad u' = -vu
\end{align*}

while equations (15), (16) and the energy equation $H(s, z) = 0$ remain unchanged. Now $\{u = 0\}$ is invariant and represents the dynamics at infinity for the zero energy problem.

4.2. Restpoints. A point $(s, z)$ is an equilibrium point for the differential equations (15), (16) if and only if

\begin{align*}
v^2 = 2U(s) \quad z = vs
\end{align*}

and

\begin{align*}
(19) \quad \nabla_m U(s) + U(s)s = 0
\end{align*}

which is exactly the equation (11) for a normalized central configuration.

Equation (19) can also be viewed in another way. The normalization condition $\langle s, s \rangle_m = 1$ defines a three-sphere $E \subset E$. Then (19) is the equation for critical points of the restriction of $U(s)$ to this sphere. In fact, the equations can be written $\tilde{\nabla} U(s) = 0$ where

\begin{align*}
(20) \quad \tilde{\nabla} U(s) = \nabla_m U(s) + U(s)s.
\end{align*}

is the gradient of the restriction of $U$ to the three-sphere with respect to the metric on this sphere induced by the mass metric. (The restricted gradient of a general smooth $f : E \to \mathbb{R}$ is $\tilde{\nabla} f(s) = (\nabla_m f(s))^T$, where $v^T$ is the orthogonal projection of the vector $v \in E$ to the tangent space at $s$ to the sphere. Using Euler’s identity for homogeneous functions, we find that if $f$ is homogeneous of degree $\alpha$ then $\tilde{\nabla} f(s) = \nabla_m f - \alpha f(s)s$, hence the expression for $\tilde{\nabla} U$.) Because of the rotational symmetry of the potential, there are actually five circles of critical points, one for each of the five central configuration shapes.
Each normalized central configuration \( s_0 \) determines two equilibrium points in the triple collision manifold \((0, s_0, z_0)\) where
\[
z_0 = v_0 s_0 \quad v_0 = \pm \sqrt{2U(s_0)}.
\]
For the zero energy problem we also get two equilibrium points at infinity with the same \((s_0, z_0)\) and \(u = 0\).

For a given normalized central configuration \(c\), its equilibria at collision and infinity are connected by the zero energy parabolic homothetic orbits of equation \((1)\), or, what is the same, equation \((12)\). These are precisely the zero energy solutions such that \(r\) changes while the \((s, z)\) remains at their equilibrium values \((21)\). The size \(r\) is given by
\[
R(\tau) = \exp(v_0 \tau) \quad \text{or} \quad u(\tau) = \exp(-v_0 \tau).
\]
Here \(\tau\) denotes the normalized time variable and \(v_0 = (s_0, z_0)\).

4.3. Stable and unstable manifolds. As just discussed the parabolic solutions are precisely the solutions in the stable manifold of one of the rest points at infinity. For the three-body problem, these rest points are all hyperbolic (after allowing for rotational symmetry) and their Lyapunov exponents will play an important role in what follows. These exponents have been calculated before \([27, 28]\) but we will present the results here (with some details relegated to an Appendix) for the sake of completeness and to correct some unfortunate typos which appeared in \([22]\).

Consider the variational equation of the blown-up differential equations \((14), (15), (16)\) at one of the equilibrium points \(p = (0, s, z)\). Differentiation and evaluation at \(r = 0\) gives the \(13 \times 13\) matrix:
\[
A = \begin{bmatrix}
v & 0 & 0 \\
0 & -v I - s z^t M & I - ss^t M \\
0 & D\nabla_m U(s) + \frac{1}{2} zs^t M & \frac{1}{2} v I + \frac{1}{2} zs^t M
\end{bmatrix}.
\]
For a restpoint at infinity, that is to say \(u = 0\), the only difference in \(A\) is that the upper left \(v\) so the two cases can be considered together.

Some words may be helpful regarding the terms \(sz^t M, ss^t M, zz^t M\) and \(zs^t M\). The term \(sz^t M\) in the 2-2 block of \(A\), for example, describes the linear operator taking \(\delta s\) to \(s(z, \delta s)_m\). All these terms arise from linearizing the quadratic function \(v = (s, z)_m = s^t M z\) which occurs as a factor in equations \((15), (16)\), and \((14)\).

Let \((\delta r, \delta s, \delta z) \in T_p X \subset \mathbb{R}^{13}\) denote a tangent vector to \(X\) at \(p\), where \(X\) is our eight-dimensional phase space defined by the normalization equations \((17)\). Linearizing the first and last of the normalization equations we find
\[
m_1 \delta s_1 + m_2 \delta s_2 + m_3 \delta s_3 = 0 \quad \text{and} \quad s^t M \delta s = 0
\]
which defines the three-dimensional tangent space to the sphere \(E \subset \mathbb{E}\) at \(s\). Since \(z = vs\) we also have
\[
z^t M \delta s = 0
\]
so we can ignore the terms involving \(z^t M\) in the second column of \(A\). In general, it is not true that \(s^t M \delta z = 0\), however, this equality does hold for vectors lying in \(T_p X\) and also tangent to the energy manifold \(H(s, z) = 0\), since for such vectors
\[
\delta H = z^t M \delta z - \nabla U(s) \cdot \delta s = vs^t M \delta z + U(s) s^t M \delta s = vs^t M \delta z.
\]
Thus vectors with \(\delta H = 0\) also have \(s^t M \delta z = 0\) and for these the third column of \(A\) also simplifies.

One easily checks that the vectors \((\delta r, \delta s, \delta z) = (1, 0, 0)\) and \((\delta r, \delta s, \delta z) = (0, 0, s)\) are eigenvectors in \(T_p X\) with eigenvalues \(\lambda_1 = v\) and \(\lambda_2 = v\). The first vector satisfies \(\delta H = 0\) while the
second vector satisfies \( \delta H = v \neq 0 \). The subspace \( \delta r = \delta H = 0 \) is a 6-dimensional subspace of \( T_pX \) invariant under \( A \). It follows that the other 6 eigenvalues of \( D \) must lie in this subspace. Dropping the \( z'M \) and \( s'M \) terms from \( A \) we find that the other eigenvectors are of the form \( (0, \delta s, \delta z) \) where \( (\delta s, \delta z) \) is an eigenvector of the 12 \( \times \) 12 matrix

\[
B = \begin{bmatrix}
-v I & I \\
D\nabla_m U(s) & \frac{1}{2} v I
\end{bmatrix}
\]

The following lemma (see [6] for the lemma’s origin) gives the eigenvectors and eigenvalues of \( B \) in terms of those of \( D\nabla_m U(s) \) or equivalently those of \( D\nabla_m U(s) \).

\[
(23)
\]

\[
D\nabla U(s) = D\nabla_m U(s) + U(s)I.
\]

**Lemma 3.** Let \( s \) be a normalized central configuration and \( z = vs \) where \( v^2 = 2U(s) \). If a vector \( \delta s \) satisfying \( (22) \) is an eigenvector of \( D\nabla U(s) \) with eigenvalue \( \alpha \) then the vectors \( (\delta s, k_1 \delta s) \) are eigenvectors of \( B \) with eigenvalues

\[
\lambda_\pm = -v \pm \sqrt{v^2 + 16\alpha} \\
k_\pm = v + \lambda_\pm
\]

**Proof.** The assumptions imply that

\[
D\nabla_m U(s)\delta s = (\alpha - U(s))\delta s = (\alpha - \frac{1}{2} v^2)\delta s.
\]

Then the product of \( B \) and \( (\delta s, k\delta s) \) is

\[
((-v + k)\delta s, (\alpha - \frac{1}{2} v^2 + \frac{1}{2} vk)\delta s).
\]

Setting this equal to \( \lambda(\delta s, k\delta s) \) leads to the equations

\[
k = v + \lambda \\
\lambda^2 + \frac{1}{2} v\lambda - \alpha = 0
\]

and solving for \( \lambda, k \) completes the proof. \( \square \)

Using the rotational symmetry, it is easy to guess one eigenvector of \( D\nabla U(s) \) satisfying \( (22) \). Let \( s^\perp = (s_1^\perp, s_2^\perp, s_3^\perp) \) denote the vector with each \( s_i \) rotated by 90° in the plane. Since \( U \) is rotationally invariant we have \( D\nabla U(s)s^\perp = 0 \) so \( \delta s = s^\perp \) is an eigenvector with \( \alpha = 0 \). There are two more eigenvectors satisfying \( (22) \) and they will determine what we will call the nontrivial eigenvalues of \( D\nabla U(s) \). These will be calculated in the appendix. For now, we just record the results.

**Proposition 3.** Let \( s \) be a normalized central configuration and \( p = (r, s, z) = (0, s, vs) \) one of the triple collision restpoints where either \( v = \sqrt{2U(s)} \) or \( v = -\sqrt{2U(s)} \). Let \( \alpha_1, \alpha_2 \) be the two nontrivial eigenvalues of \( D\nabla U(s) \). Then the eight Lyapunov exponents of the variational equations on \( T_pX \) are

\[
\lambda = v, -v, 0, -v \pm \sqrt{v^2 + 16\alpha_1}, -v \pm \sqrt{v^2 + 16\alpha_2}.
\]

The eigenvalues for an equilibrium at infinity \( p = (u, s, z) = (0, s, vMs) \) are the same except the first one becomes \( -v \).

The Lagrangian (equilateral) critical points form circles of local minima in \( E \). The corresponding nontrivial eigenvalues are both positive.

\[1\] From equation 20 we get that \( D\nabla U \) equals the expression of equation 23 plus the term \( \nabla U(s) \otimes s'M \) which we ignore since \( \langle s, \delta s \rangle_m = 0 \).
Proposition 4. The nontrivial eigenvalues of \( D\tilde{\nabla}U(s) \) at an equilateral central configuration are

\[
\alpha_1, \alpha_2 = \frac{3U(s)}{2} \left( 1 \pm \sqrt{k} \right)
\]

where

\[
k = \frac{(m_1 - m_2)^2 + (m_1 - m_3)^2 + (m_2 - m_3)^2}{2(m_1 + m_2 + m_3)^2}.
\]

The four corresponding nontrivial eigenvalues at one of the Lagrangian equilibrium points at triple collision or at infinity are

\[
\lambda = -\frac{\nu}{4} \left( 1 \pm \sqrt{12 + 13\sqrt{k}} \right)
\]

After allowing for the rotation, the Eulerian, collinear critical points are saddles with one positive and one negative nontrivial eigenvalue. Their values depend on the shape of the configuration. Consider the collinear central configuration with \( m_2 \) between \( m_1 \) and \( m_3 \). Instead of normalizing the configuration we can look for critical points of the translation and scale invariant function \( F(s) = I(s)U(s)^2 \) with no constraints. It is easy to see that if \( s \) is a critical point of \( F \) then the corresponding normalized configuration satisfies (19). Using \( F \), we may assume without loss of generality that

\[
s_1 = (0, 0) \quad s_2 = (r, 0) \quad s_3 = (1 + r, 0)
\]

where \( 0 < r < 1 \). This gives a function of one variable \( F(r) \) and setting \( F'(r) = 0 \) leads to the fifth degree equation

\[
g = (m_2 + m_3)r^5 + (2m_2 + 3m_3)r^4 + (m_2 + 3m_3)r^3
\]

\[
- (3m_1 + m_2)r^2 - (3m_1 + 2m_2)r - (m_1 + m_2) = 0.
\]

There is a unique root with \( 0 < r \) by Descartes’ rule of signs which will determine the shape of the collinear CC.

Proposition 5. The nontrivial eigenvalues of \( D\tilde{\nabla}U(s) \) at the collinear central configuration with \( m_2 \) between \( m_1, m_3 \) are

\[
\alpha_1, \alpha_2 = -U(s)\nu, U(s)(3 + 2\nu)
\]

where

\[
\nu = \frac{m_1(1 + 3r + 3r^2) + m_3(3r^3 + 3r^4 + r^5)}{(m_1 + m_3)r^2 + m_2(1 + r)^2(1 + r^2)}
\]

and \( r \) is the positive root of (24). The four corresponding nontrivial eigenvalues at the Eulerian equilibrium points at triple collision or at infinity are

\[
\lambda = -\frac{\nu}{4} \left( 1 \pm \sqrt{1 - 8\nu} \right), -\frac{\nu}{4} \left( 1 \pm \sqrt{25 + 16\nu} \right).
\]

The values at the other Eulerian restpoints are found by permuting the subscripts on the masses.

Note that the Eulerian restpoints have a pair of nonreal eigenvalues if and only if \( \nu > \frac{1}{8} \).

Definition 5. (i) We say that the Euler configuration with mass 2 in the middle is spiraling if \( \nu > \frac{1}{8} \) with \( \nu \) as in proposition 5.

(ii) If each of the three Euler configurations is spiraling then we say that the mass ratios \( m_1 : m_2 : m_3 \) are in the “spiraling range”.

Figure shows the masses for which \( \nu > \frac{1}{8} \) for each of the three Eulerian restpoints.
4.4. Parabolic motions tend to rest points at infinity. The qualitative study of parabolic solutions goes at least back to Chazy [1] (particularly chapter 3). We now show that the stable manifolds of the rest points at infinity are precisely the unions classical parabolic solutions, a fact well known to experts.

We will use the following weak form of the definition of “parabolic”.

**Definition 6.** A solution to the three-body problem is (future) parabolic if the solution’s domain contains a positive half line \([t_0, \infty)\) and if the Newtonian velocities of all three bodies tend to zero as (Newtonian) time tends to infinity.

**Remark.** Define “past parabolic” by letting time tend to negative infinity. We stick with future parabolic for simplicity.

**Proposition 6.** Any parabolic solution has energy 0 and lies in the stable manifold of one of the rest points at infinity. Conversely every solution in the stable manifold of a rest point at infinity and such that \(u > 0\) is a parabolic solution.

Alternatively: Let \(R_+\) denote the collection of rest points at infinity for which \(v > 0\). Then \(R_+\) is normally hyperbolic and its stable manifold \(W^s(R_+)\) is foliated by the stable manifolds \(W^s(c)\) tending to the rest point associated to the central configuration \(c\). Each parabolic solution lies in some \(W^s(c)\).

**Proof.** For the three-body problem, it seems that most of this result follows from Chazy’s work [1]. But he does not use our definition of parabolic. For completeness we will give a proof here using ideas from [16] [3].

For a parabolic motion, the kinetic energy \(K(v) \to 0\) as \(t \to \infty\). The energy equation \(K(v) - U(q) = h\) and the fact that \(U(q) > 0\) imply that \(h \leq 0\). To rule out the case \(h < 0\) we use the Lagrange-Jacobi identity \(\dot{I}(t) = 2K + 2h\). If \(K \to 0\) and \(h < 0\) then \(\dot{I}(t)\) has a negative upper bound for \(t\) sufficiently large which forces \(I(t) \to 0\) (total collapse) in finite time. Such a solution would not exist for large \(t > 0\).

It is also easy to see that \(r(t) \to \infty\) as \(t \to \infty\). Indeed the energy equation gives \(r(t)K(v) = r(t)U(q) = U(s)\). Now the normalized potential \(U(s)\) has a positive lower bound depending only on the choice of masses, so \(K(v) \to 0\) implies \(r(t) \to \infty\).

The main theorem in Marchal and Saari [16] describes the asymptotic behavior as \(t \to \infty\) for any solution of the n-body problem which exists for all \(t \geq 0\). For any such solution, either \(r(t)/t \to \infty\) or else all of the position vectors satisfy \(q_k = A_k t + O(t^{\frac{5}{2}})\) for some constant vectors \(A_k\), possibly zero. The second case implies that either \(r(t)/t \to L\) for some \(L > 0\) or else \(r(t) = O(t^{\frac{5}{2}})\) (the latter holding when all \(A_k = 0\)). We will show that in fact we have \(r(t) = O(t^{\frac{5}{2}})\) for parabolic orbits. To see this note that given any \(\epsilon > 0\) there is \(t_0\) such that \(\dot{I}(t) = 2K < \epsilon\) for \(t \geq t_0\). Then \(I(t) \leq I(t_0) + \dot{I}(t_0) (t - t_0) + \frac{1}{2} \epsilon (t - t_0)^2\). If \(r(t)/t \to L \in (0, \infty)\) then \(I(t)/t^2 \to L^2 \in (0, \infty)\) and we have the contradiction that \(0 < L^2 < \frac{1}{2} \epsilon\) for all \(\epsilon > 0\).

Next we show that the quantity \(v(t)\) appearing in the blown-up equations tends to a finite limit \(v(t) \to \bar{v} > 0\) as \(t \to \infty\). Recall that \(v(t)\) is non-decreasing since \(v\) is a Liapanov function on the zero energy surface. Also \(v(t) = r^{-1}r'(t) = \sqrt{r(t)} \dot{r}(t) = \frac{2}{3} \frac{d}{dt} r(t)^{\frac{3}{2}}\).

Since \(r(t) \to \infty\) we have \(v(t) > 0\) for \(t\) large so either \(v(t)\) approaches some \(\bar{v} > 0\) or else \(v(t) \to \infty\). But integration gives \(\frac{1}{t} \left( \frac{2}{3} r(t)^{\frac{3}{2}} - \frac{2}{3} r(0)^{\frac{3}{2}} \right) = \frac{1}{t} \int_0^t v(s) \, ds\).

If \(v(t) \to \infty\) as \(t \to \infty\) we would get \(r(t)^{\frac{3}{2}}/t \to \infty\) contradicting \(r(t) = O(t^{\frac{5}{2}})\).
Finally we can use the dynamics on the infinity manifold to finish the proof. First note that the estimate \( r(t) = O(t^{\frac{3}{2}}) \) shows that the rescaled time \( \tau \) with \( \dot{\tau}(t) = \tau^{\frac{3}{2}}(t) \) satisfies \( \tau(t) \to \infty \) as \( t \to \infty \), so a forward parabolic orbit exists for all \( \tau \geq 0 \) and we have \( u(\tau) = 1/r(\tau) \to 0 \) as \( \tau \to \infty \). We claim that the \( \omega \) limit set of our parabolic orbit consists of one of the restpoints in the manifold \( \{ u = 0 \} \).

Consider the subset \( S = \{ (u, s, z) : u = 0, H(s, z) = 0, v = \tilde{v} \} \) where \( \tilde{v} \) is the limit of \( v(\tau) \) for a certain parabolic solution. We will show that this solution avoids a neighborhood of the double collision singularities in \( S \). Since the solution exists for all \( t \geq 0 \) and \( \tau \geq 0 \) it does not actually have a double collision, but we want to avoid a whole neighborhood. Let \( w = s' = z - vs \), the component of \( z \) tangent to the ellipsoid \( E \). The energy equation can be written \( \frac{1}{2}v^{2} + \frac{1}{2}|w|^{2} = U(s) \). Since \( U(s) \to \infty \) at collision while \( v^{2} \) is bounded near \( S \), it follows that \( |w| \) is large near collision. Now \( v' = \frac{1}{2}(|z| - v^{2}) = \frac{1}{2}|w|^{2} \) while the arclength \( \sigma \) in \( E \) satisfies \( \sigma' = |s'| = |w| \). Hence the rate of change of \( v \) with respect to arclength is \( \frac{1}{2}|w| \). From this we see that there is some neighborhood \( U \) of the double collision singularities in \( S \) such that any initial condition in \( U \) crosses into the set \( v > \tilde{v} \). So our solution avoids \( U \).

We conclude that our parabolic solution must converge to the compact set \( S' = S \setminus U \) as \( \tau \to \infty \). Therefore it has a nonempty, compact \( \omega \) limit set contained in \( S' \). For orbits in the limit set we must have \( v'(\tau) = \frac{1}{2}|w(\tau)|^{2} = 0 \) for all \( \tau \) and this happens only at the restpoints, that is, at the points \( (u, s, z) = (0, c, vc) \) where \( c \) is a central configuration. For the three-body problem, the restpoints form five circles and the eigenvalue computations show that these are normally hyperbolic invariant manifolds. It follows that the omega limit set consists of just one of the restpoints.

The converse is easier. Assume that \( r(\tau) \to \infty \) and \( (s(\tau), z(\tau)) \to (c, \tilde{v}c) \) where \( \tilde{v} = \sqrt{2U(c)} \). Since \( K(z) = \sqrt{2}K(v) \to U(s) \) we have \( K(v) \to 0 \) as \( \tau \to \infty \). Inverting the change of timescale we find that \( t \to \infty \) as \( \tau \to \infty \) so the solution is parabolic.

\[ \square \]

5. PROOF OF LEMMA 1

In this section we revert to the classical variables and timescale but will make use of the eigenvalue computation of proposition 5.

5.1. Proof of part (A) of Lemma 1.

\textbf{Proof.} Consider perturbations \( \gamma' \) of a homothetic parabolic motion \( \gamma_c \) associated to a central configuration \( c \) as in (12), so \( r(c) = 1 \), \( \gamma_c(t) = \rho(t)c \) where \( \rho(t) = \left( \frac{9}{\tau}U(c) \right)^{\frac{3}{2}} t^{\frac{3}{2}} \). For any \( [a, b] \subset \mathbb{R}^+ \), \( v \in C^2([a, b], \mathbb{R}) \) such that \( v(a) = v(b) = 0 \), \( \langle c, v(t) \rangle_m = 0 \), we consider the variation of \( \gamma_c \) of the form

\[ \gamma'(t) = \rho(t)(c + \epsilon v(t)) \]

so that
\[
\frac{d\gamma'(t)}{d\epsilon} = \rho(t)v(t), \quad \frac{d^{2}\gamma'(t)}{d\epsilon^{2}} \bigg|_{\epsilon=0} = 0
\]

\[
A(\gamma'; a, b) = \frac{1}{2} \int_{a}^{b} \langle \gamma'(t), \gamma'(t) \rangle_m + \int_{a}^{b} U(\gamma'(t)) dt
\]

\[
\frac{dA(\gamma'; a, b)}{d\epsilon} = \int_{a}^{b} \langle \gamma'(t), \frac{d\gamma'(t)}{d\epsilon} \rangle_m dt + \int_{a}^{b} \langle \nabla_m U(\gamma'(t)), \frac{d\gamma'(t)}{d\epsilon} \rangle_m dt.
\]
We have
\[
\frac{d^2 A(\gamma^*; a, b)}{de^2} \bigg|_{e=0} = \int_a^b \left[ \left( \frac{d\gamma^*(t)}{de} \right)_m + \left( \frac{d\gamma^*(t)}{de} \right)_m, D\nabla_m U(\gamma(t)) \frac{d\gamma^*(t)}{de} \right]_m dt
\]
\[
= \int_a^b \left[ \rho(t)^2 (\dot{v}(t), \dot{v}(t))_m + \rho(t) \dot{\rho}(t) (v(t), v(t))_m + \dot{\rho}(t)^2 (v(t), v(t))_m \right] dt
\]
\[
+ \int_a^b \rho(t)^{-1} (v(t), D\nabla_m U(c)v(t))_m dt
\]
where in the last line we used that \( D\nabla_m U \) is homogeneous of degree \(-3\). We remark that the quantity \( \langle v, D\nabla_m U(c)v \rangle_m \) occurring in the last term is just the Hessian of \( U \) at \( c \), evaluated at the vector \( v \). See the remark at the end of this subsection. Integrate by parts and use that \( \ddot{\rho} = -U(c)/\rho^2 \) to get
\[
\int_a^b \left[ 2\rho(t) \dot{\rho}(t) (v(t), \dot{v}(t))_m + \dot{\rho}(t)^2 (v(t), v(t))_m \right] dt = \int_a^b \dot{\rho}(t) \frac{d}{dt} (\rho(t) (v(t), v(t))_m) dt
\]
\[
= -\int_a^b \ddot{\rho}(t) (v(t), v(t))_m dt = \int_a^b \rho(t)^{-1} U(c)(v(t), v(t))_m dt,
\]
so that
\[
\frac{d^2 A(\gamma^*; a, b)}{de^2} \bigg|_{e=0} = \int_a^b \rho(t)^2 (\dot{v}(t), \dot{v}(t))_m dt + \int_a^b \rho(t)^{-1} \langle v(t), D\nabla U(c)v(t) \rangle_m dt.
\]
where \( D\nabla U(c) = D\nabla U(c) + U(c)I \). This is exactly the quantity \( 23 \) which occurred in the computation of the eigenvalues in the last section.

Now recall from proposition \( 5 \) that \( D\nabla U(c) \) has an eigenvector, say \( \delta s = z \) with a negative eigenvalue \( \alpha_1 = -U(c)\nu \). Take \( v(t) = \varphi(t)z \) for the variation of equation \( 27 \) where \( \varphi \in C^2([a, b]) \) with \( \varphi(a) = \varphi(b) = 0 \). Plugging into equation \( 28 \) we find that
\[
\frac{d^2 A(\gamma^*; a, b)}{de^2} \bigg|_{e=0} = \int_a^b \rho(t)^2 (\dot{\varphi}(t)^2 + \alpha_1 \rho(t)^{-1} \varphi(t)^2) dt.
\]
This quadratic form in \( \varphi \) is positive definite i.e. \( Q(\varphi; a, b) \geq 0 \), for any \( [a, b] \subset (0, \infty) \) if and only if its Euler-Lagrange equation
\[
(\rho(t)^2 y')' - \alpha_1 \rho(t)^{-1} y = 0
\]
is disconjugate on \( (0, \infty) \) \( 11 \) Section XI.6. Plug in the expressions for \( \rho \) and \( \alpha_1 \) to find that this Euler-Lagrange equation reads
\[
t^2 y'' + \frac{4}{3} ty' + \nu y = 0
\]
which has solutions \( t^r \) where \( r \) a root of the indicial equation \( r^2 + \frac{1}{3} r + \frac{2}{9} \nu = 0 \). Equation \( 30 \) fails to be disconjugate if and only if \( r \) has an imaginary part, which is to say iff and only if the discriminant, \( \Delta = \frac{1 - 8 \nu}{9} \) is negative. \( \Delta \) is negative if and only if \( \nu > \frac{1}{8} \) in which case the solutions of \( 30 \) are
\[
y(t) = At^{-1/6} \cos(a \ln t) + Bt^{-1/6} \sin(a \ln t); \quad a^2 = \frac{1}{4} |\Delta| = \frac{1}{36} (8 \nu - 1),
\]
which has in fact infinitely many conjugate points on \( (0, \infty) \).

To finish the proof of part \( (A) \) just note that our instability condition \( \nu > \frac{1}{8} \) is precisely the condition for spiraling at the Eulerian repoint. \( \square \)
Remark on Hessians. Some words are in order regarding the term $D^2 U$ occurring in the formula and its relation to the Hessian $D^2 U$. If $f$ is any smooth function on a real vector space $\mathbb{E}$ then its Hessian $D^2 f(p)$ at $p \in \mathbb{E}$ is the coordinate independent bilinear symmetric form defined by $\frac{d^2}{d\epsilon^2} f(p + \epsilon v)|_{\epsilon=0} = (D^2 f(p))(v, v)$. If $\langle \cdot, \cdot \rangle$ is any inner product on $\mathbb{E}$, then $D^2 f(p)(v, v) = \langle D\nabla f(p)(v), v \rangle$ where $\nabla f$ is the gradient of $f$ with respect to the inner product, whereas the derivative $D$ of $D\nabla f$ is the usual derivative, or Jacobian, of vector fields $X$ on a vector space, given by $DX(p)(v) = \frac{d}{dt}|_{t=0} X(p + \epsilon v)$, and it is independent of the inner product.

5.2. Proof of Part (B) of Lemma [1]

Proof. We now consider a parabolic motion asymptotic to an Euler central configuration $c$. Shifting the origin of time, if necessary, such a solution can be written as $\alpha_0(t) = \rho(t)(c + \beta(t))$, $t \geq t_0 > 0$, where $\rho(t)c$ is the parabolic homothetic solution and $\beta(t) = O(t^{-d})$ as $t \to \infty$ for some $d > 0$.

A bit of explanation is in order regarding $\beta$’s rate of convergence to zero. If we represent $\alpha_0$ in McGehee (spherical) coordinates we get $|s(\tau) - c| \leq Ae^{-\mu \tau}$ for $\tau$ sufficiently large, where $\mu > 0$ is any number such that $-\mu$ is greater than all of the negative eigenvalues from proposition [3].

Our two representations of $\alpha_0$ are related by $s(t) = (c + \beta(t))/\sqrt{1 + |\beta(t)|^2}$. Integrating the relation $\tau s^2 = dt$ we have that $s = \frac{Ae^{3\mu(t)/2} + \eta}{2U(c)}$ where $\eta$ is exponentially small relative to the first term. Since $e^{-\mu \tau} \sim t^{-3/2}$ it follows that $\beta(t) \to 0$ at a rate $t^{-d}$ with $d = 2\mu/3v$.

Now take a variation $\alpha_\epsilon(t) = \rho(t)(c + \beta(t) + \epsilon v(t))$ with $v(t) = \varphi(t)z$ as before. So

$$d^2 A(\alpha_\epsilon; a, b) \frac{d}{d\epsilon} \left|_{\epsilon=0} = \rho(t)v(t), \frac{d^2 A(\alpha_\epsilon(t))}{d\epsilon^2} \right|_{\epsilon=0} = 0$$

$$= \int_a^b \rho(t)^2 \dot{\varphi}(t), \varphi(t)) \mathrm{d}t$$

$$+ \int_a^b \rho(t)^{-1} [U(c)\varphi(t), \varphi(t)] \frac{d^2 U(c + \beta(t))(\varphi(t), \varphi(t))] \mathrm{d}t$$

$$= Q(\varphi; a, b) + (D^2 U(c + \beta(t))(\varphi(t), \varphi(t)))$$

with $Q$ as per equation [20]. (We have written the Hessian of $U$ in the form $D^2 U$ as per the remark above on Hessians, rather than in the $D\nabla U$ form.) According to Part (A) there are $[a, b] \subset (0, \infty), \varphi_1 \in C^2([a, b])$ such that $\varphi_1(a) = \varphi_1(b) = 0$ and $Q(\varphi_1; a, b) < 0$.

Defining $\varphi_\lambda(t) = \varphi_1\left(\frac{t}{\lambda}\right)$ we have $\dot{\varphi}_\lambda(t) = \lambda^{-2} \varphi_1\left(\frac{t}{\lambda}\right)$

$$Q(\varphi_\lambda; \lambda a, \lambda b) = \int_{\lambda a}^{\lambda b} \lambda^{-\frac{5}{2}} \left[ \rho(\frac{t}{\lambda})^2 \varphi_1\left(\frac{t}{\lambda}\right)^2 - \mu \rho(\frac{t}{\lambda})^{-1} \varphi_1\left(\frac{t}{\lambda}\right)^2 \right] \mathrm{d}t = \lambda^{-\frac{3}{2}} Q(\varphi_1; a, b)$$

for $\lambda$ sufficiently large and $t \geq \lambda a$ we have that $|\beta(t)| \leq C_1 t^{-d}$ and so $||D^2 U(c + \beta(t)) - D^2 U(c)|| \leq C_2 t^{-d}$. Thus

$$\int_{\lambda a}^{\lambda b} ||D^2 U(c + \beta(t)) - D^2 U(c)|| \frac{\varphi_\lambda(t)^2}{\rho(t)^2} \mathrm{d}t \leq C_2 \int_{\lambda a}^{\lambda b} \frac{\varphi_\lambda(t)^2}{\rho(t)^2} \mathrm{d}t = C_2 \lambda^{\frac{1}{2} - d} \int_a^b \frac{\varphi_1(s)^2}{\rho(s)^2} \mathrm{d}s.$$  

Using $v(t) = \varphi_\lambda z$ we have

$$\frac{d^2 A(\alpha_\epsilon; \lambda a, \lambda b)}{d\epsilon^2} \bigg|_{\epsilon=0} = \lambda^{-\frac{5}{2}} \left( Q(\varphi_1; a, b) + C_2 \lambda^{\frac{1}{2} - d} \int_a^b \frac{\varphi_1(s)^2}{\rho(s)^2} \mathrm{d}s \right) < 0.$$  

for $\lambda$ sufficiently large. \qed
6. Symplectic Structure, Lagrangian submanifolds, and Proofs of Lemma 2 and Theorem 2

The differential equations of the three-body problem preserve the standard symplectic structure on $\mathbb{R}^{12}$

$$\omega = m_1 dq_1 \wedge dv_1 + m_2 dq_2 \wedge dv_2 + m_3 dq_3 \wedge dv_3.$$ 

Here, as usual, the wedge of vectors of one forms means adding the componentwise wedges so, for example, $(dx, dy) \wedge (du, dv) = dx \wedge du + dy \wedge dv$. The restriction of the flow to $X = \mathbb{E} \times \mathbb{E}$ preserves the restriction of $\omega$. The pullback of $\omega$ under the change of variables $q_i = r s_i, v_i = r^{-\frac{1}{2}} z_i$ is

$$\Omega_r = \sum_i m_i \left( r^{\frac{1}{2}} ds_i \wedge dz_i + r^{-\frac{1}{2}} dr \wedge s_i \cdot dz_i + \frac{1}{2} r^{-\frac{1}{2}} dr \wedge z_i \cdot ds_i \right).$$

If we use $u = 1/r$ instead we get

$$\Omega_u = \sum_i m_i \left( u^{-\frac{1}{2}} ds_i \wedge dz_i + u^{-\frac{1}{2}} s_i \cdot dz_i \wedge du + \frac{1}{2} u^{-\frac{1}{2}} z_i \cdot ds_i \wedge du \right).$$

In both cases we restrict to the eight dimensional subset $X$ of $\mathbb{R}^{13}$ where $r > 0, u > 0$ and where the normalizations (17) hold.

**Lemma 4.** Let $p(\tau)$ be any solution of the blown-up differential equations with $r(\tau) > 0, u(\tau) > 0$ and let vectorfields $a(\tau), b(\tau)$ be solutions of the variational equations along $p(\tau)$ which are tangent to an energy manifold. Then $\Omega_r(p(\tau))(a(\tau), b(\tau))$ and $\Omega_u(p(\tau))(a(\tau), b(\tau))$ are constant.

**Proof.** Let $\xi$ denote the vectorfield on $\mathbb{R}^{13}$ given by (14), (15), (16). Let $\eta = r^{-\frac{2}{3}} \xi$ be the same vectorfield without the change of timescale. Since $\eta$ is the pullback of the Hamiltonian field, it preserves the pullback form $\Omega_r$. In other words, the Lie derivative

$$L_\eta \Omega_r = \frac{d}{dt} \phi_t^* \Omega_r|_{t=0} = 0.$$

Since $\xi = f \eta$, where $f = r^{\frac{2}{3}}$, Cartan’s formula gives

$$L_\xi \Omega_r = d(\iota_\xi \Omega_r) + \iota_\xi d\Omega_r = d(f \iota_\eta \Omega_r) + 0 = d(df \Omega_r) = df \wedge dH.$$

Here we used the fact that $\iota_\eta \Omega_r = dH$ which is the pullback of the differential form version of Hamilton’s equations.

If $p, a, b$ are as in the statement of the lemma then

$$\frac{d}{d\tau} \Omega_r(p)(a, b) = L_\xi \Omega_r(p)(a, b) = (df \wedge dH)(p)(a, b) = 0$$

since $dH(p)(a) = dH(p)(b) = 0$. \qed

Using this lemma we can prove Lemma 2.

**Proposition 7.** Let $l = (u, s, z) = (0, s, vs)$ be one of the Lagrange restpoints at infinity with $v > 0$ and let $W^s_+(l)$ denote the part of the stable manifold with $u > 0$. Then $W^s_+(l)$ is a four-dimensional invariant manifold which is a Lagrangian submanifold of $X$. Similarly, at the restpoints with $v < 0$ the unstable manifold $W^u_+(l)$ is Lagrangian.

**Proof.** Propositions 3 and 4 give the eight eigenvalues of the variational equations at $l$. For positive masses, the quantity $k$ from proposition 4 satisfies $0 \leq k < 1$. It follows that if $v > 0$ then there are three positive eigenvalues

$$v, \frac{-v}{4} \left( 1 - \sqrt{13 \pm 12\sqrt{k}} \right)$$
four negative eigenvalues
\[-v, -v, \frac{-v}{4} \left( 1 + \sqrt{13 \pm 12\sqrt{k}} \right)\]
and one zero eigenvalue. The latter is due to the rotational symmetry. In fact \( l \) is part of a circle of equilibria. This circle is normally hyperbolic ([11], p.1) so each equilibrium has a four-dimensional stable manifold.

The first negative eigenvalue \(-v\) has eigenvector \((\delta u, \delta s, \delta z) = (1, 0, 0)\) and it follows that the stable manifold has an open subset \(W^s_+(l)\) with \( u > 0 \). Moreover, the other three stable eigenvectors are in the subspace \(\delta H = 0\). It follows that \(W^s_+(l)\) is contained in the energy manifold \(\{H = 0\}\). Using blown-up coordinates we need to show that the two-form \(\Omega_u\) vanishes on tangent vectors to \(W^s_+(l)\). Let \(a_0, b_0\) be two tangent vectors to \(W^s_+(l)\) at a point \(p_0 \in W^s_+(l)\). To show that \(\Omega_u(p_0)(a_0, b_0) = 0\) it suffices, by lemma 3, to show that \(\Omega_u(p(\tau))(a(\tau), b(\tau)) \to 0\) as \(\tau \to \infty\). For this we need estimates on the exponential decay of \(u\) and the components of \(a, b\).

Since \(u' = -vu\) and \(v(\tau)\) converges exponentially to the value \(v\) at the restpoint we have a lower bound \(u(\tau) \geq c \exp(-v\tau)\) for some constant \(c > 0\) which depends on the particular solution \(p(\tau)\) under consideration. To see this note that

\[u(\tau) \exp(v\tau) = u(0) \exp\left( \int_0^\tau (v - v(s)) \, ds \right)\]

and the integral is bounded above and below since the integrand tends to 0 exponentially. Since \(u(0) > 0\) we get a positive lower bound \(c\) as required. The lower bound on \(u\) gives upper bounds

\[u^{-\frac{1}{2}} \leq c_1 \exp\left( \frac{1}{2}v\tau \right) \quad u^{-\frac{2}{4}} \leq c_2 \exp\left( \frac{3}{2}v\tau \right)\]

for the coefficients in \(\Omega_u\).

A similar argument applies to the variational differential equations. We have \(\delta u' = -v(\tau)\delta u\). Since \(v(\tau) \to v\) exponentially, we have an estimate of the form \(|\delta u(\tau)| \leq c_3 \exp(-v\tau)\) for every solution of the variational equations. Since \(p(\tau) \to l\) exponentially, the other components \(\delta s, \delta z\) also decay at a rate governed by the eigenvalues at \(l\). The weakest of the attracting eigenvalues is

\[\lambda_w = \frac{-v}{4} \left( 1 + \sqrt{13 - 12\sqrt{k}} \right) < \frac{-v}{2}\]

so we will have upper bounds of the form

\[|\delta s_i(\tau)| \leq c_4 \exp(\lambda_w v\tau) \quad |\delta z_i(\tau)| \leq c_4 \exp(\lambda_w v\tau)\]

Substituting these estimates into the formula for \(\Omega_u\) gives

\[\Omega_u(p(\tau))(a(\tau), b(\tau)) \leq c_5 \exp\left( \frac{v}{2} + \lambda_w \right) \tau \to 0.\]

\(\square\)

**Proof of Lemma 2** From lemma 3 we see that the stable space of a Lagrange respoint at infinity is generated by the eigenvector \((1, 0, 0)\) and 3 eigenvectors \((0, \delta s_\alpha, k_\alpha - \delta s_\alpha)\) for eigenvectors \(\delta s_\alpha\) of the 3 eigenvalues \(\alpha\) of \(D\nabla U(s)\). Thus the projection of the stable space is the whole tangent space at \((0, c)\) of the configuration space. It follows from the implicit function theorem that \(W^s_+(l)\) is a graph near infinity. More precisely, there is a product neighborhood \(V\) of \((0, c)\) in the blown-up configuration space \([0, \infty) \times S^3\) and a smooth map \((u, s) \mapsto y(u, s)\) from \(V\) to the space \(E\) of blown up velocities such that the graph of this map coincides with the stable manifold of \(l\) in some neighborhood of \(l\). Now being Lagrangian does not make sense at \(u = 0\) since the symplectic structure explodes, so in the statement of lemma 2, when we say that \(W^s_+(l)\) is a “Lagrangian graph” we mean over \(V \setminus \{u = 0\}\). \(\square\)
Definition 7. By a “neighborhood of c at infinity” we mean a neighborhood of the form described in the end of the proof immediately above. When expressed in $\mathbb{E}$ such a neighborhood is a truncated open cone consisting of those points $q \in \mathbb{E}$ of the form $q = rs$ where $|s| = 1$, $|s - c| < \delta$ and $u = 1/r < \delta$.

Corollary 1. Let $B_c$ be the Buseman function (definition 4, proposition 2) associated to a homothetic parabolic Lagrange solution $\gamma_c$. Then there is a neighborhood of $c$ at infinity, $V \subset \mathbb{E}$ (see above definition 7), on which $B_c$ is smooth and such that
\[
\{(\gamma(t), \dot{\gamma}(t)^*): \text{$\gamma$ a curve calibrated by $B_c$, $\gamma(0) \in V$} = W^*_+(l) \cap (V \times \mathbb{E}^*), \quad l = (0, c, v c).
\]
(In this formula $v^*$ denotes the Legendre transform of the velocity $v$, which is its dual covector.)

The reader may wish to consult Remark 2 following Definition 3 for context here.

Proof. $W^*_+(l)$ is a Lagrangian graph near $c$ at infinity, and is the graph over some neighborhood $V \subset \mathbb{E}$ of infinity, as per the above terminology. Thus, being Lagrangian there is a differentiable function $f$ defined on $V$ such that $W^*_+(l) \cap (V \times \mathbb{E}) = \text{graph } df$. For any $x \in V$ there is a unique motion $\gamma(t)$ with $\gamma(0) = x$, $(x, \gamma(0)^*) \in W^*_+(l)$ and it is given by the solution to $\dot{\gamma}^* = df(\gamma)$ with $\gamma(0) = x$. Here $w^*$ denotes the dual of $w$ with respect to the mass metric – which is to say - the inverse Legendre transform of $w$ relative to our Lagrangian.

On the other hand for any $x \in V$ there is a $\alpha : [0, \infty) \to \mathbb{E}$ that starts at $x$ and calibrates $B_c$. For $t > 0$, $B_c$ is differentiable at $\alpha(t)$ and $dB_c(\alpha(t)) = \dot{\alpha}(t)^*$. Since $\alpha$ is asymptotic to $c$, we have that $(\alpha(t), \dot{\alpha}(t))^* \in W^*_+(l)$, so, by the graph description $\dot{\alpha}(t)^* = df(\alpha(t))$. Thus $dB_c(\alpha(t)) = df(\alpha(t))$, proving the corollary. (We also have that $B_c = f + k$ on $V$, $k$ a constant.)

Proof of Theorem 2. Let $\gamma(t)$ tend parabolically to a Lagrange central configuration $c$ and let $V$ be the neighborhood of $c$ at infinity as in Corollary 1 and lemma 2. Then, since $(\gamma(t), \dot{\gamma}(t)^*)$ lies on the stable manifold $W = W^*_+(l)$ we must have that $\gamma([T, \infty)) \subset V$ for $T$ large enough. Consequently for $t \geq T$ we have that $\dot{\gamma}(t)^* = dB_c(\gamma(t))$. The curve $\alpha : [0, \infty) \to \mathbb{E}$ calibrated by $B_c$ that starts at $\gamma(T)$ is also a solution of the differential equation $\dot{z}^* = dB_c(z)$. By uniqueness of solutions we have $\alpha(t) = \gamma(t + T)$. Then $\gamma : [T, \infty) \to \mathbb{E}$ is calibrated by $B_c$ and in particular it is a free time minimizer.

7. Appendix

The appendix provides the proofs of the propositions about eigenvalues used above.

Proof of Proposition 3. The first two eigenvalues in the list are from $(\delta r, \delta s, \delta z) = (1, 0, 0)$ and $(0, 0, s)$. The others come from the eigenvalues of $B$ found in the lemma. The eigenvalues $-v, 0$ come from the eigenvector $\delta s = s^+$ with $\alpha = 0$ and the other two come from the two nontrivial eigenvalues.

For the restpoints at infinity, the computation is the same except that the first eigenvalue on the list is now associated to $(\delta u, \delta s, \delta z) = (1, 0, 0)$ and has eigenvalue $-v$ instead of $v$.

To prove propositions 4 and 5 we need to find the nontrivial eigenvalues $\alpha_1, \alpha_2$ of $D\nabla U(s)$ for the equilateral and collinear central configurations of the three-body problem. These can be deduced from the work of Siegel but we will give a quick discussion here.

It is straightforward to calculate the $6 \times 6$ matrix $D\nabla U(s)$ with the result
\[
D\nabla U(s) = \begin{bmatrix}
D_{11} & D_{12} & D_{13} \\
D_{21} & D_{22} & D_{23} \\
D_{31} & D_{32} & D_{23}
\end{bmatrix}
\]
where the $2 \times 2$ blocks are

$$D_{ij} = \frac{m_i m_j}{r_{ij}^3} \left( I - 3 u_{ij} u_{ij}' \right), \quad u_{ij} = \frac{s_i - s_j}{r_{ij}} \quad \text{for } i \neq j$$

and

$$D_{ii} = - \sum_{j \neq i} D_{ij}.$$ 

It is more convenient to work with the matrix

$$P = \frac{I(s)}{U(s)} M^{-1} D \nabla U(s).$$

Since $P$ is invariant under scaling and translation, it can be computed without imposing the normalizations \[\text{(17)}\]. If $\beta$ is an eigenvalue of $P$ then $\alpha = U(s)(\beta + 1)$ is an eigenvalue of $D \nabla U(s)$ for the corresponding normalized $s$. So we are reduced to finding the nontrivial eigenvalues $\beta_1, \beta_2$ of $P$.

**Proof of Proposition 4.** Consider an equilateral triangle configuration $s$. Working with $P$ we can use the unnormalized configuration

$$s_1 = (1, 0) \quad s_2 = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \quad s_3 = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

for which

$$U(s) = \frac{m_1 m_2 + m_1 m_3 + m_2 m_3}{\sqrt{3}} \quad I(s) = \frac{3(m_1 m_2 + m_1 m_3 + m_2 m_3)}{m}.$$ 

Using these together with \[\text{(31)}\] gives

$$P = \frac{1}{4m} \begin{bmatrix} 5(m_2 + m_3) & 3 \sqrt{3}(m_3 - m_2) & -5m_2 & 3 \sqrt{3}m_2 & -5m_3 & -3 \sqrt{3}m_3 \\ 3 \sqrt{3}(m_3 - m_2) & -(m_2 + m_3) & 3 \sqrt{3}m_2 & m_2 & -3 \sqrt{3}m_3 & m_3 \\ -5m_1 & 3 \sqrt{3}m_1 & 5m_1 - 4m_3 & -3 \sqrt{3}m_1 & 4m_3 & 0 \\ 3 \sqrt{3}m_1 & m_1 & -3 \sqrt{3}m_1 & m_1 + 8m_3 & 0 & -8m_3 \\ -5m_1 & -3 \sqrt{3}m_1 & 4m_2 & 0 & 5m_1 - 4m_2 & 3 \sqrt{3}m_1 \\ -3 \sqrt{3}m_1 & m_1 & 0 & -8m_2 & 3 \sqrt{3}m_1 & -m_1 + 8m_2 \end{bmatrix}$$

One can guess 4 of the 6 eigenvalues of $P$. If $e_1 = (1, 0)$, $e_2 = (0, 1)$ then $(e_1, e_1, e_1)$ and $(e_2, e_2, e_2)$ are eigenvectors with eigenvalue $\beta = 0$. Also $s, s^\perp$ are eigenvectors with eigenvalues $\beta = 2, -1$ respectively. Since the trace of $P$ is 2, the remaining eigenvalues satisfy $\beta_1 + \beta_2 = 1$. Alternatively, the numbers $\gamma_i = \beta_i + 1$ which we really want, satisfy $\gamma_1 + \gamma_2 = 3$. We can also find the product $\gamma_1 \gamma_2$ as follows. We have

$$(\text{trace } P)^2 - \text{trace } P^2 = (1 + \beta_1 + \beta_2)^2 - (5 + \beta_1^2 + \beta_2^2) = 2\beta_1 \beta_2 - 2 = 2\gamma_1 \gamma_2 - 6.$$ 

With some computer assistance, this gives

$$\gamma_1 \gamma_2 = \frac{27(m_1 m_2 + m_1 m_3 + m_2 m_3)}{4(m_1 + m_2 + m_3)^2}$$

Solving the quadratic equation $\gamma^2 - 3\gamma + \gamma_1 \gamma_2 = 0$ gives the eigenvalues $\alpha_i = U(s) \gamma_i = v^2 \gamma_i/2$ of $D \nabla U(s)$ listed in the proposition. Then we get the nontrivial eigenvalues $\lambda$ of the equilibrium from proposition 3. \[\square\]
Proof of proposition 5. Consider a normalized collinear central configuration such that \( s_1 = (x_i, 0) \in \mathbb{R}^2 \). Then the unit vectors \( u_{ij} = (\pm 1, 0) \) so the \( 2 \times 2 \) matrices \( D_{ij} \) reduce to
\[
D_{ij} = \frac{m_im_j}{r_{ij}} \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix} \quad i \neq j.
\]
Rearranging the variables as \( q = (x_1, x_2, x_3, y_1, y_2, y_3) \) produces a block structure
\[
M^{-1}D\nabla U(s) = \begin{bmatrix} 2C & 0 \\ 0 & -C \end{bmatrix}
\]
where \( C \) is the \( n \times n \) matrix
\[
C = \begin{bmatrix}
m_1^2 + m_2^2 & -m_1^2 & -m_2^2 \\
-m_1^2 & m_1r_2r_3 & m_2r_1r_3 \\
-m_2^2 & m_2r_1r_3 & m_1^2 
\end{bmatrix}.
\]
An eigenvalue \( \mu \) of \( C \) determines two eigenvalues
\[
\alpha = -\mu + U(s), 2\mu + U(s)
\]
for \( D\nabla U(s) = M^{-1}D\nabla U(s) + U(s)I \).

It is possible to guess two eigenvectors of \( C \). First of all \( v_1 = (1, 1, 1) \) is an eigenvector with eigenvalue 0. Next, let \( v_2 = (x_1, x_2, x_3) \) be the vector of \( x \)-coordinates of the collinear central configuration. Then it is easy to see that
\[
Cv_2 = -M_0^{-1}\nabla_x U
\]
where \( \nabla_x \) is the partial gradient with respect to the \( x \)-coordinates and \( M_0 = \text{diag}(m_1, m_2, m_3) \).

Since \( s \) is a normalized central configuration, we have \( Cv_2 = U(s)v_2 \), so \( v_2 \) is also an eigenvector, with eigenvalue \( U(s) \). The remaining, nontrivial eigenvalue of \( C \) can now be found as \( \mu = \tau - U(s) \) where \( \tau = \text{trace}(C) \), i.e.,
\[
\tau = \left( \frac{m_1 + m_2}{r_{12}^2} + \frac{m_1 + m_3}{r_{13}^2} + \frac{m_2 + m_3}{r_{23}^2} \right).
\]
Therefore the nontrivial eigenvalues of \( D\nabla U(s) \) are
\[
\alpha = 2\tau - U(s), 2U(s) - \tau.
\]

To get the form shown in the proposition, let \( \nu \) be the translation and scale invariant quantity
\[
\nu = \frac{I(s)}{U(s)}\tau - 2.
\]
Then for the normalized configuration \( \alpha_1, \alpha_2 = -U(s)\nu, U(s)(3 + 2\nu) \) and it remains to show that \( \nu \) has the indicated form.

We just indicate a computer assisted way to prove it. Using the configuration \( s_1 = (0, 0), s_2 = (r, 0), s_3 = (1 + r, 0) \) we have
\[
r_{12} = r \quad r_{23} = 1 \quad r_{13} = 1 + r.
\]
Substituting these into the formulas for \( I(s), U(s), \tau \) expresses \( \nu = \frac{I(s)}{U(s)}\tau - 2 \) as a rational function \( \nu(r) \). Subtracting the expression \( (25) \) and factorizing the difference reveals that there is a factor of \( g(r) \) in the numerator, where \( g(r) \) is the fifth degree polynomial \( (24) \) giving the location of the central configuration. So \( \nu(r) \) is indeed given by \( (25) \) at the central configuration. \( \square \)

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