On the recursion operators for integrable equations

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Abstract
It is widely known that the recursion operator is a very important component of integrability. It allows one to describe in a compact form both hierarchies of the generalized symmetries and infinite series of the local conservation laws. In the literature, we can find several methods for constructing recursion operators, some of them use the Hamiltonian approach and the others are based on the Lax representation of the equation. In the present article we discuss an alternative method, suggested in Habibullin and Khakimova (2018 Theor. Math. Phys. 196 1200–16), which is connected only with the first several generalized symmetries of the given equation. Efficiency of the method is illustrated with the examples of Korteweg–de Vries, Krichever–Novikov and Kaup–Kupershmidt equations and discrete autonomous and non-autonomous models.

Keywords: generalized symmetry, recursion operator, integrable equations, Krichever–Novikov equation, KdV equation, Volterra lattice, Kaup–Kupershmidt equation

1. Introduction

In a series of our articles [1–5] we studied the properties and applications of the generalized invariant manifolds (GIM) for nonlinear equations. For a given differential (or discrete) equation GIM is a differential (or, respectively, difference) equation compatible with its linearization. We have shown that an appropriately chosen GIM generates the recursion operator, describing the hierarchy of the higher symmetries for the equation under consideration and also allows to construct the corresponding Lax pair. In the over mentioned articles for constructing the recursion operator we used a rather labor-consuming computational algorithm. In the present article we have significantly changed and improved the algorithm making it more convenient for use.
The problem of constructing the recursion operators for the integrable equations has been investigated by many authors. Several methods are worked out to study the task. Some of them use the Lax representation (see, for instance, [6–11]). This way is very effective when the Lax pair is known. If it is not the case then it is reasonable to study directly the defining equation (see, equation (2.11) below). To solve the equation the most authors use the multi-Hamiltonian approach (see [12–16]). Their basic goal is to find two Hamiltonian operators $H_1$ and $H_2$ to the given equation. Then the recursion operator is given by the following formula $R = H_2 H_1^{-1}$. However, usually the operators $H_1$ and $H_2$ are nonlocal and this causes some difficulties in applying this method.

In the present article we concentrate on the alternative method for constructing the recursion operator which is based on the symmetry approach (partly it is announced in [5]). The recursion operator provides an effective tool for evaluating higher symmetries for integrable equations. We assume that the recursion operator can be represented as a weakly nonlocal pseudo-differential operator of the form [17]

$$R = R_0 + \sum_{i=1}^{m} g^{(i)} D^{-1} h^{(i)}, \quad (1.1)$$

where $R_0$ is a differential operator. The non-local part consists of the combinations of the generators of the symmetries $u_\tau = g^{(j)}$ and the variational derivatives $h^{(j)}$ of the conserved densities. Appearance of symmetry generators in (1.1) is easy to explain. Indeed, by applying the operator $R$ to the generator of trivial symmetry $u_m = 0$ we find an equation of the form

$$u_\tau = c_1 g^{(1)} + c_2 g^{(2)} + \cdots + c_m g^{(m)},$$

which is a symmetry of the given equation for any choice of the constants $c_1, c_2, \ldots, c_m$. Therefore each of the equations $u_\tau = g^{(j)}$, $j = 1, \ldots, m$ also defines a symmetry. In a similar way we can prove the statement about the factors $\{h^{(j)}\}$. Here we have to use the well-known fact that the conjugate operator $R^*$ converts the variational derivative of a conserved density into the variational derivative of the other conserved density. For more details see [18, 19].

It was observed earlier that in the most cases the recursion operator can be reduced to the form (1.1). Note that the mentioned above symmetries $u_\tau = g^{(j)}$ are the members of the hierarchy of the symmetries started with the classical ones $g^{(1)} = u_\tau$, $g^{(2)} = u_\tau$ and followed by the higher symmetries arranged in ascending order

$$g^{(1)} = u_\tau, \quad g^{(2)} = u_\tau, \quad g^{(3)} = u_\tau, \ldots, \quad g^{(m)} = u_\tau, \ldots, \quad (1.2)$$

such that the orders of the symmetries satisfy the inequalities $\text{ord} g^{(j)} < \text{ord} g^{(j+1)}$. We refer to the set $S = \{g^{(1)}, g^{(2)}, \ldots, g^{(m)}\}$ of the generators in (1.1) as the set of seed symmetries. In the simplest case $m = 1$ exemplified by the Korteweg–de Vries equation $S$ contains the only generator $g^{(1)} = u_\tau$. For the Kaup–Kupershmidt equation [20]

$$u_\tau = u_\tau + 10 u u_3 + 25 u_1 u_2 + 20 u^2 u_1$$

we have $m = 2$ and $S$ consists of two generators $g^{(1)} = u_\tau$ and $g^{(2)} = u_\tau + 10 u u_3 + 25 u_1 u_2 + 20 u^2 u_1$. For the Krichever–Novikov equation [21]

$$u_\tau = u_{xxx} - \frac{3}{2} u_{xx}^2 + \frac{P(u)}{u_x} \quad \text{with} \quad P^{(m)}(u) = 0 \quad (1.3)$$

there are two different recursion operators $R_1$ and $R_2$ (see [19]). For the first one we have $m = 2$ and the seed symmetries are $g^{(1)} = u_\tau$ and $g^{(2)} = u_{xxx} - \frac{3}{2} u_{xx}^2 + \frac{P(u)}{u_x}$. For the second
recursion operator $m = 3$ and $S$ contains in addition to these two generators also the generator of the fifth order symmetry of the Krichever–Novikov equation

$$g^{(5)} = u_5 - 5\frac{u_2 u_3}{u_1} - 5 u_1^2 + 25\frac{u_2 u_4}{u_1^2} - 45\frac{u_2^2}{u_1^2} - \frac{5}{18} P^2(u) + \frac{5}{3} P^2(u) \left( \frac{u_3}{u_1} \right) P(u) - \frac{5}{9} u_1 P''(u).$$

Recall that for the Harry Dym equation $u_t = u^3 u_{xxx}$ the recursion operator $R = u^3 D_x u D_x^{-1} \frac{1}{u}$ found in [22] is rewritten as follows

$$R = u^4 D_x^2 \frac{1}{u^2} + 3 u^2 u_1 D_x \frac{1}{u^2} + 3 u u_2 + u D_x^{-1} \frac{1}{u^2}.$$  

Thus for this equation we have the only seed symmetry coinciding with the right hand side of the equation, $S = \{ g^{(2)} = u^3 u_{xxx} \}$, hence $m = 1$.

Let us denote through $L_1$ a differential operator of the order $m$ which annihilates all of the functions $g^{(i)}$, $i = 1, 2, \ldots, m$, in the set $S$. Theorem 1, presented below, asserts that then the operator $L_2 = L_1 R$ is also a differential operator. Thus we come up to the following representation of the recursion operator as the ratio of two differential operators

$$R = L_1^{-1} L_2.$$  \hspace{1cm} (1.4)

It is remarkable that this representation provides an effective tool for solving the defining equation (2.11).

**Remark.** Number $m$ of the non-localities in (1.1) is closely connected with the action of the recursion operator on the hierarchy (1.2), more precisely we have

$$R g^{(j)} = g^{(j+m)}.$$  \hspace{1cm} (1.5)

Therefore one can recover the whole hierarchy of the higher symmetries by the iterated application of $R$ to the set $S$.

Using the representation (1.4), we can immediately find the Lax pair for the corresponding integrable equation (see below (2.18) and (2.19)), which, however, does not coincide with the usual one.

The article is organized as follows. In section 2 we derive the formula (1.4) and explain how to use it to solve effectively the defining equation (2.11). The application of the symmetry algorithm for constructing the recursion operator is illustrated with the examples of the KdV, Kaup–Kupershmidt and Krichever–Novikov equations. In section 3 the algorithm is adopted to the integrable lattices. It is explained with the example of the Volterra chain. A rather complicated example of a non-autonomous lattice of the relativistic Toda type with the periodic coefficients is studied in section 4.

### 2. Recursion operators for integrable PDE

#### 2.1. Factorization of the recursion operators with weak non-localities

Let us consider an integrable equation of the form

$$u_t = f(u, u_1, u_2, \ldots, u_k), \quad \frac{\partial f}{\partial u_k} \neq 0,$$  \hspace{1cm} (2.1)
where \( u_j = D_j u \) and \( D_j \) is the operator of the total derivative with respect to the variable \( x \). The order \( k \) of the highest order derivative of \( u \) in the rhs of the equation (2.1) is called the order of the equation (2.1). An evolutionary type equation of the order \( r \)

\[
u_x = g(t, x, u, u_1, \ldots, u_r)
\]

(2.2)
is called a symmetry of the equation (2.1) if the flows defined by these two equations commute, i.e. the equation

\[
D_x g - D_x f = 0,
\]

(2.3)
where all of the derivatives with respect to \( t \) and \( \tau \) are expressed by means of the equations (2.1) and (2.2) is satisfied identically (see [18, 19]). It is well known that any integrable equation (2.1) possesses an infinite hierarchy of the symmetries which is completely described in terms of the recursion operator \( R \). Below in this section we assume that equation (2.1) admits the recursion operator of the form (1.1).

**Theorem 1.** For any weakly non-local pseudo-differential operator \( R \) there exists a pair of the differential operators \( L_1 \) and \( L_2 \) such that the following equation is satisfied

\[
L_1 R = L_2.
\]

(2.4)

**Proof.** Let us define the differential operator

\[
L_1 = D_x^m + \alpha_1 D_x^{m-1} + \cdots + \alpha_m
\]

(2.5)
such that \( L_1 g^{(i)} = 0 \) for all \( i = 1, 2, \ldots, m \). Evidently such a kind of operator exists. Now we have to show that \( L_1 R \) is also a differential operator. At first we prove that the composition of the form \( L_1 g^{(i)} D_x^{-1} h^{(i)} \) is a differential operator. Indeed, due to the Leibniz rule we have

\[
D_x \left( g^{(i)} D_x^{-1} h^{(i)} \right) = D_x \left( g^{(i)} \right) D_x^{-1} h^{(i)} + \sum_{k=1}^{j} c^j_k D_x^{j-k} \left( g^{(i)} \right) D_x^{-k} h^{(i)},
\]

(2.6)
where \( c^j_k, k = 1, 2, \ldots, j \) are the binomial coefficients. It is easily seen that only the first summand in (2.6) might contain the nonlocal term \( D_x^{-1} h^{(i)} \). Therefore we can write

\[
D_x \left( g^{(i)} D_x^{-1} h^{(i)} \right) = D_x \left( g^{(i)} \right) D_x^{-1} h^{(i)} + H_{ij},
\]

(2.7)
where for any \( i \) and \( j \) the term \( H_{ij} \) defines a local differential operator. Now we apply \( L_1 \) to \( g^{(i)} D_x^{-1} h^{(i)} \) and due to the reasonings above we get

\[
L_1 g^{(i)} D_x^{-1} h^{(i)} = L_1 \left( g^{(i)} \right) D_x^{-1} h^{(i)} + \sum_{ij=1}^{m} \alpha_{m-j} H_{ij}.
\]

Since \( L_1 \left( g^{(i)} \right) = 0 \) we find the relation

\[
L_1 g^{(i)} D_x^{-1} h^{(i)} = \sum_{ij=1}^{m} \alpha_{m-j} H_{ij}
\]
which implies
Therefore $L_2$ is also a differential operator.

2.2. Construction of the operators $L_1, L_2$ and $R$

In this section we assume that equation (2.1) admits a recursion operator with weak non-localities and discuss on the way to find it.

Examples presented in the Introduction convince us that usually the set $S$ of the seed symmetries contains one or two classical symmetries $g^{(1)} = u_x$ and/or $g^{(2)} = u_t$ and only for the most complicated Krichever–Novikov equation the second recursion operator contains three generators $g^{(1)} = u_x$, $g^{(2)} = u_t$ and $g^{(3)} = u_\tau$. Hence when we look for the recursion operator we have to examine at least all of these possible choices of the set $S$:

$$S_1 = \{ u_x \}, \quad S_2 = \{ u_t \}, \quad S_3 = \{ u_x, u_t \}, \quad S_4 = \{ u_x, u_t, u_\tau \}, \quad (2.8)$$

where $u_\tau = g^{(3)}$ is the closest higher symmetry of the equation (2.1).

For the chosen $S$ we take the operator $L_1 = L_1(S)$ as the minimal order differential operator which annulates all of the generators in $S$. Obviously such an operator is not unique, it is defined up to multiplication by an arbitrary function. We can represent $L_1$ in an explicit form through the following determinant

$$L_1 U = \rho \begin{vmatrix} D_x^m g^{(1)} & D_x^{m-1} g^{(1)} & \cdots & g^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ D_x^m g^{(m)} & D_x^{m-1} g^{(m)} & \cdots & g^{(m)} \\ D_x^m U & D_x^{m-1} U & \cdots & U \end{vmatrix}, \quad (2.9)$$

where $\rho$ is a function of the dynamical variables $u, u_1, \ldots$. As soon as $L_1$ is found we can start to search $L_2$. As it was mentioned in the remark above (see Introduction) the order $m_2$ of the operator $L_2$ is easily determined by

$$m_2 = m + \text{ord } g^{(m+1)} - \text{ord } g^{(1)}. \quad (2.10)$$

In order to find $L_2$ we use the factorized representation $R = L_1^{-1}L_2$. We substitute it into the defining equation for $R$ (see [6, 23])

$$\frac{d}{dt} R = [F^*, R]. \quad (2.11)$$

Here $F^*$ is the linearization operator of the equation (2.1):

$$F^* = \frac{\partial f}{\partial u} + \frac{\partial f}{\partial u_1} D_x + \frac{\partial f}{\partial u_2} D_x^2 + \cdots + \frac{\partial f}{\partial u_k} D_x^k. \quad (2.12)$$

Let us substitute $R = L_1^{-1}L_2$ into the equation (2.11) and after some simple transformations obtain

$$\frac{d}{dt} (L_2)L_1^{-1} + L_2F^*L_1^{-1} = \frac{d}{dt} (L_1)L_1^{-1} + L_1F^*L_1^{-1} =: A. \quad (2.13)$$
It immediately follows from (2.13) that the operators $L_1$ and $L_2$ solve one and the same equation
\begin{equation}
\frac{d}{dt}(L_j) = AL_j - L_j F^*, \quad j = 1, 2.
\end{equation}
(2.14)

It is easily checked that for any choice of $S$ the operator $A$ defined by (2.13) is a differential operator. Indeed, due to the construction the following equation
\begin{equation}
L_j U = 0
\end{equation}
(2.15)
is compatible with the linearized equation
\begin{equation}
\frac{d}{dt} U = F^* U.
\end{equation}
(2.16)

By applying the operator $\frac{d}{dt}$ to the equation (2.15) we obtain due to (2.16) that
\begin{equation}
\left(\frac{d}{dt} L_1 + L_1 F^*\right) U = 0
\end{equation}
(2.17)
for any solution $U$ of the equation (2.15). Hence the kernels of the operators $L_1$ and $\frac{d}{dt} L_1 + L_1 F^*$ satisfy the relation $\ker L_1 \subset \ker\left(\frac{d}{dt} L_1 + L_1 F^*\right)$. Therefore the latter operator is divided by the former one, i.e. there exists a differential operator $A$ such that $\frac{d}{dt} L_1 + L_1 F^* = AL_1$.

We begin with $S_1$, determine $L_1$ and then look for $A$ as discussed above. As soon as the operator $A$ is found we can use (2.14) with $j = 2$ to construct operator $L_2$ of the order $m_2$. If $L_2$ is found then we can define $R = L_1^{-1} L_2$, if such $L_2$ does not exist, we pass to the next choice of $S$.

For the operators $L_1$ and $L_2$ found we can easily determine $R = L_1^{-1} L_2$. A convenient way to do this is to substitute the explicit expressions of the operators $L_1$, $L_2$ and the ansatz (1.1) into the equation $L_1 R = L_2$ and then look for the unknown operator $R_0$ and the unknown coefficients $h^{(j)}$ in (1.1) by comparison of the coefficients in front of the powers of $D_x$.

It is worth mentioning that the operators $L_1$ and $L_2$ allow one to derive the Lax pair for the equation (2.1):
\begin{equation}
L_2 \psi = \lambda L_1 \psi,
\end{equation}
(2.18)
\begin{equation}
\frac{d}{dt} \psi = F^* \psi,
\end{equation}
(2.19)
where $\lambda$ is the spectral parameter. The Lax pair (2.18) and (2.19) is not ‘fake’ since it is generated by the recursion operator. About ‘fake’ Lax pairs see [24].

2.3. Examples

Example 1. As an illustrative example we take the well studied Kortweg–de Vries equation
\begin{equation}
u_t = u_3 + uu_1,
\end{equation}
(2.20)

Among the potential values of the parameter $m$ we first choose $m = 1$ and take $S = \{u_1\}$. Thus the first pretender for $L_1$ is
\begin{equation}
L_1 U = \rho \begin{vmatrix}
  u_2 \\
  u_1
\end{vmatrix}
\begin{vmatrix}
  \rho \\
  u_1
\end{vmatrix}.
\end{equation}
(2.21)
For the sake of simplicity we put $\rho = -1$, then
\[ L_1 = u_1 D_x - u_2. \]  
(2.22)

Direct computations convince us that the operator
\[ A = \frac{d}{d t} \left( L_1 \right) L_1 - 1 + L_1 F^* L_1^{-1} \]
where
\[ F^* = D_3^3 + u D_x + u_1 \]  
(2.23)
is a third order differential operator of the form
\[ A = D_3^3 - \frac{3u_2}{u_1} D_x^2 + \left( u + \frac{3u_2^2}{u_1^2} \right) D_x + 3 \left( \frac{u_4}{u_1} + u_1 - \frac{u_3 u_1}{u_1^2} \right). \]  
(2.24)

Due to the formula (2.10) the corresponding operator $L_2$ should be of the third order
\[ L_2 = \beta(3) D_3^3 + \beta(2) D_x^2 + \beta(1) D_x + \beta(0). \]  
(2.25)

The unknown coefficients $\beta(i)$ are determined from the equation
\[ \frac{d}{d t}(L_2) = A L_2 - L_2 F^*. \]  
(2.26)

To this end, we derive some linear algebraic equations by collecting the coefficients in front of the powers of $D_x$. Solving these equations, we obtain:
\[ L_2 = u_1 D_3^3 - u_2 D_x^2 + \frac{2}{3} u u_1 D_x + u_1^2 - \frac{2}{3} u u_2. \]  
(2.27)

Since the requested $L_2$ is found we pass to the final stage. Having the operators $L_1$ and $L_2$ it is very easy to find the recursion operator $R = L_1^{-1} L_2$. Since $m = 1$ we can conclude that $R$ has the only nonlocal term. Also we can find order of the operator $R_0$. It coincides with the difference between $m_2$ and $m$ and equals two in this case. Therefore,
\[ R = r^{(2)} D_x^2 + r^{(1)} D_x + r^{(0)} + u_1 D_x^{-1} h. \]  
(2.28)

We substitute $L_1$, $L_2$ and $R$ into the equation $L_2 = L_1 R$ and get
\[ u_1 D_3^3 - u_2 D_x^2 + \frac{2}{3} u u_1 D_x + u_1^2 - \frac{2}{3} u u_2 = \left( u_1 D_x - u_2 \right) \left( r^{(2)} D_x^2 + r^{(1)} D_x + r^{(0)} + u_1 D_x^{-1} h \right). \]  
(2.29)

From the latter we find $r^{(2)} = 1$, $r^{(1)} = 0$, $r^{(0)} = \frac{2}{3} u$, $h = 1$ and therefore we have (see also [8])
\[ R = D_x^2 + \frac{2}{3} u + \frac{1}{3} u_1 D_x^{-1}. \]  
(2.30)

To compare we recall the usual factorized representation of $R$ through the Hamiltonian operators $H_1 = D_3$, $H_2 = D_3^3 + 4u D_x + 2u_1$
\[ R = H_3 H_1^{-1}. \]  
(2.31)
**Example 2.** As the second illustrative example we consider the Kaup–Kupershmidt equation

\[ u_t = u_5 + 10u u_3 + 25u_1 u_2 + 20u^2 u_1. \]  

(2.32)

Below we will use its linearization operator

\[ F^* = D_3^2 + 10u D_3 + 25u_1 D_2 + (25u_2 + 20u^2) D_1 + 10u_3 + 40u u_1. \]  

(2.33)

In order to find the set \( S \) of the seed symmetries for the equation (2.32) we have to examine the possible cases \( S_1 = \{ u_5 \}, S_2 = \{ u_6 \}, S_3 = \{ u_7 \}, S_4 = \{ u_5, u_5, u_7 \} \), where \( u_7 = g^{(3)} \) is the next symmetry of the Kaup–Kupershmidt equation, it is of the seventh order.

We started with the case \( S = S_1 \). For the corresponding \( L_1 \) we found \( A \) from the equation (2.13), then since due to (2.10) \( m_2 = 1 + 5 - 1 = 5 \) we searched a fifth order differential operator \( L_2 \) satisfying the equation \( \frac{d}{du} L_2 = AL_2 - L_2 F^* \) and observed that such operator does not exist. In a similar way we have verified that the case \( S = S_2 \) does not fit, as well.

Then we passed to the case \( S = S_3 \) and succeeded. Operator \( L_1 \) is found from the relation

\[ L_1 U = \begin{vmatrix} u_3 & u_2 & u_1 \\ u_2 & u_1 & u_t \\ U_2 & U_1 & U \end{vmatrix} \]  

(2.34)

and is of the form

\[ L_1 = \alpha D_3^2 + \beta D_3 + \gamma, \]  

(2.35)

where

\[
\begin{align*}
\alpha &= u_2 u_5 + 10u_3 u_3 - u_1 u_6 - 35u_2^2 u_3 - 10u_3 u_4 - 40u u_1^3, \\
\beta &= 10u_1 u_3 - u_3 u_7 + 45u_3^2 u_4 - 10u_4 u_2 + 120u_2 u_5 \\
&\quad + 60u_1 u_2 u_3 + 40u_4^2 + u_1 u_7, \\
\gamma &= -120u_2 u_2 u_3 - 10u_1 u_2 u_4 + u_4 u_3 + 35u_2 u_4 - 60u_2^2 u_3 \\
&\quad - 40u_1 u_2 u_4 + u_3 u_4 + 10u_4 u_2 u_4 - 45u_1 u_2 u_4.
\end{align*}
\]

Then we look for the operator

\[ A = \sum_{j=0}^{5} A^{(j)} D_j \]  

(2.36)

from the equation:

\[ \frac{d}{du} (L_1) = AL_1 - L_1 F^*. \]  

(2.37)

We note that the operators \( A \) and \( F^* \) are gauge equivalent and therefore they are very similar to each other, at least they have one and the same order. We have solved the equation (2.37) and found all of the coefficients \( A^{(j)} \), however some of them turned out to be huge and hence we give in an explicit form only the simplest three:

\[
\begin{align*}
A^{(5)} &= 1, & A^{(4)} &= -5 \frac{\alpha}{\alpha^2}, \\
A^{(3)} &= 20 \frac{\alpha^2}{\alpha^2} + 5 \frac{\beta \alpha}{\alpha^2} - 5 \frac{\beta}{\alpha} - 10 \frac{\alpha x}{\alpha} + 10 u.
\end{align*}
\]
At the next step we look for the eighth order differential operator \( L_2 \) (recall that \( m_2 = m + \text{ord } g^{(m+1)} - \text{ord } g^{(1)} = 2 + \text{ord } g^{(3)} - \text{ord } g^{(1)} = 2 + 7 - 1 = 8 \))

\[
L_2 = \sum_{k=0}^{8} b^{(k)} D^k
\]

from the equation \( \frac{d}{dt}(L_2) = AL_2 - L_2 F^x \). In this case the equation is successfully solved, here we present the list of the coefficients in (2.38):

\[
\begin{align*}
b^{(8)} &= \alpha, \\
b^{(7)} &= \beta, \\
b^{(6)} &= 12\alpha + \gamma, \\
b^{(5)} &= 60 u_1 \alpha + 12 u \beta, \\
b^{(4)} &= (133 u_2 + 36 u^2) \alpha + 48 u_1 \beta + 12 u \gamma, \\
b^{(3)} &= (169 u_3 + 264 u u_2) \alpha + (85 u_2 + 36 u^2) \beta + 36 u_1 \gamma, \\
b^{(2)} &= (132 u_4 + 394 u u_3 + 381 u^2 + 32 u^3) \alpha + (84 u_3 + 192 u u_2) \beta + (49 u_2 + 36 u^2) \gamma, \\
b^{(1)} &= (63 u_5 + 304 u u_4 + 852 u u u_3 + 240 u^2 u_2) \alpha \\
&\quad + (48 u_4 + 202 u u_3 + 189 u^2 + 32 u^3) \beta + (35 u_3 + 120 u u_2) \gamma, \\
b^{(0)} &= (17 u_6 + 122 u u_5 + 444 u u u_4 + 324 u^2 + 192 u u_2 + 368 u u u_2) \alpha \\
&\quad + (15 u_5 + 102 u u_4 + 272 u u u_3 + 144 u^2 u_2) \beta + (13 u_4 + 82 u u_3 + 69 u^2 + 32 u^3) \gamma.
\end{align*}
\]

Let us find now the required recursion operator \( R \). Since the order of its differential part is 6 = \( m_2 - m \) and it has two non-local terms, then \( R \) must be of the form:

\[
R = \sum_{j=0}^{6} r^{(j)} D^j + u_1 D^{-1} h^{(1)} + u_1 D^{-1} h^{(2)}.
\]

The relation \( L_1 R = L_2 \) allows us to find all of the functional parameters in (2.39):

\[
\begin{align*}
r^{(6)} &= 1, \\
r^{(5)} &= 0, \\
r^{(4)} &= 12 u, \\
r^{(3)} &= 36 u_1, \\
r^{(2)} &= 49 u_2 + 36 u^2, \\
r^{(1)} &= 35 u_3 + 120 u u_1, \\
r^{(0)} &= 13 u_4 + 82 u u_2 + 69 u^2 + 32 u^3, \\
h^{(1)} &= 2 u_5 + 8 u^2 \gamma, \\
h^{(2)} &= 2.
\end{align*}
\]

So we have the final form of \( R \) coinciding with that found earlier in [7]:

\[
R = D_6^6 + 12 u D_5^4 + 36 u_1 D_4^3 + (49 u_2 + 36 u^2) D_3^2 + (35 u_3 + 120 u u_1) D_2 + 13 u_4 + 82 u u_2 + 69 u^2 + 32 u^3 + u_1 D^{-1}_x (2 u_2 + 8 u^2) + 2 u_1 D^{-1}_x
\]

Having the operators \( L_1 \) and \( L_2 \) we can immediately write down the Lax pair of the form (2.18) and (2.19), which does not coincide with the known one found years ago in [20].

**Example 3.** The next example is connected with the famous Krichever–Novikov equation [21]

\[
u_t = u_3 - \frac{3}{2} u^2 + \frac{P(u)}{u_1} \quad \text{with} \quad P'''(u) = 0.
\]

It is the most interesting representative of the class of the third order integrable equations of the form (2.1). All the other equations of that class can be derived from (2.41) by appropriate manipulations like the limit procedure or the differential substitutions. The algebraic structures related to (2.41) are the most generic and complicated. The linearization operator of the equation (2.41) is given by
\[ F^* = D_x^3 - \frac{3u_2}{u_1} D_x^2 + \frac{3u_2^2 - 2P(u)}{2u_1^2} D_x + \frac{P'(u)}{u_1}. \] (2.42)

The set \( S \) for the equation (2.41) consists of the generators of two classical symmetries \( g^{(1)} = u_t \) and \( g^{(2)} = u_t \) hence \( m = 2 \) and the operator \( L_1 \) is defined by

\[
L_1 U = \begin{vmatrix}
    u_3 & u_2 & u_1 \\
    u_{2t} & u_t & u_t \\
    U_2 & U_1 & U
\end{vmatrix}. \tag{2.43}
\]

Coefficients of the operator \( L_1 \)

\[ L_1 = \alpha D_x^2 + \beta D_x + \gamma, \tag{2.44} \]

are as follows

\[
\alpha = 4u_2u_3 - u_1u_4 - \frac{3u_2^3}{u_1} + \frac{2P(u)u_2}{u_1} - P'(u)u_1,
\]

\[
\beta = u_1u_5 - 3u_2u_4 + \frac{9u_2^2u_3}{u_1} - 4u_3^2 - \frac{2P(u)(u_1u_1 - u_2^2)}{u_1^2} - P'(u)u_2 - \frac{P''(u)u_1^2}{u_1^3} - \frac{3u_2^4}{u_1^4},
\]

\[
\gamma = \left(u_3 + \frac{u_2^2}{u_1}\right) P'(u) + \left(u_1 + \frac{3u_2^2}{u_1}\right) u_4 - u_2u_5 - \frac{6u_2^3u_3}{u_1^2} - P''(u)u_1u_2 - \frac{2P(u)}{u_1^2} + \frac{3u_2^3}{u_1^3}.
\]

The corresponding operator \( A \) is of the third order

\[
A = \sum_{j=0}^{3} A^{(j)} D_x^j, \tag{2.45}
\]

it is found from the equation (2.14) with \( j = 1 \)

\[
A^{(3)} = 1, \quad A^{(2)} = -\frac{3\alpha_2}{\alpha} - \frac{3u_2}{u_1},
\]

\[
A^{(1)} = \frac{6u_3}{u_1} + \frac{6\alpha_2^2}{\alpha} - \frac{3(2\alpha u_2 + \beta u_1)\alpha_2}{\alpha^2u_1} + \frac{15u_2^2}{2u_1^2} - \frac{3\alpha x}{\alpha} - \frac{3\beta_x}{\alpha} - \frac{P(u)}{u_1^2},
\]

\[
A^{(0)} = \frac{3u_4}{u_1} + \frac{3(5\alpha u_2 + \beta u_1 + 2\alpha u_1 u_3)}{\alpha^2u_1} + \frac{\alpha x}{\alpha} + \frac{3(\alpha u_2 + 2\alpha u_1 + \beta u_1)\alpha x}{\alpha^2u_1} - \frac{6u_3}{\alpha^3} - \frac{3(2\alpha u_2 + 3\beta u_1)\alpha_2}{\alpha^2u_1} + \left(\frac{9\beta_2}{\alpha^2} - \frac{3\beta_2^2}{\alpha^3} - \frac{6\beta u_2}{\alpha^2 u_1} + \frac{P(u)}{\alpha^2 u_1} - \frac{15u_2^2}{2u_1^2} + \frac{3\gamma}{\alpha^2}\right)\alpha x
\]

\[
- \frac{3\beta_x}{\alpha} + \frac{3(2\alpha u_2 + \beta u_1)\beta_2}{\alpha^2 u_1} - \frac{P'(u)}{u_1^2} + \frac{\alpha_2}{\alpha} - \frac{3\gamma_2}{\alpha} - \frac{3\beta_2^2}{\alpha^2 u_1} + \frac{4P(u)u_2}{u_1^3} - \frac{12u_3}{u_1^3}.
\]

Let us evaluate the order of the operator \( L_2 \). According to the formula (2.10) we have \( m_2 = m + 5 - 1 = 6 \). The coefficients of the operator \( L_2 \)

\[
L_2 = \sum_{k=0}^{6} b^{(k)} D_x^k \tag{2.46}
\]

are found from (2.14) with \( j = 2 \). They are of the form
\[ b^{(6)} = \alpha, \quad b^{(5)} = \beta - 4 \mu_2 \alpha \quad b^{(4)} = \left( \frac{14 \mu_2}{u_1} - \frac{10 \alpha}{u_1} - \frac{4 P(u)}{3 \mu_1} \right) \alpha - \frac{4 \mu_2}{u_1} \beta + \gamma, \]
\[ b^{(3)} = \left( \frac{48 \mu_2 u_3}{u_1^2} - \frac{10 u_3}{u_1} - \frac{38 \mu_2^2}{u_1^2} + \frac{28 \mu_2 (u_2 u_3 - 10 \mu_2 (u_2 u_3 - 10 P(u))}{3 \mu_1^2} \right) \alpha \]
\[ + \frac{6 \mu_2}{u_1} - \frac{10 \alpha}{u_1^2} + \frac{4 P(u)}{3 \mu_1^2} \beta - \frac{4 \mu_2}{u_1} \gamma, \]
\[ b^{(2)} = \left( \frac{28 \mu_2^2}{u_1^4} - \frac{5 u_3}{u_1^2} - \frac{4 P(u)^2}{9 u_1^4} + \frac{14}{9} P'(u) + \frac{32 P'(u) u_3}{3 \mu_1^2} + \frac{32 P'(u) u_3}{3 \mu_1^2} + \frac{32 u_3 u_4}{u_1^2} \right) \alpha \]
\[ + \frac{69 u_3^4}{u_1^6} - \frac{92 P(u) u_3^2}{3 u_1^2} + \frac{124 u_3 u_4}{u_1^4} \frac{1}{u_1^4} \alpha - \left( \frac{2 u_3}{u_1} - \frac{6 \mu_2}{u_1} + \frac{4 P(u)}{3 \mu_1^2} \right) \gamma \]
\[ - \frac{4 \alpha}{u_1} - \frac{22 \alpha u_3}{u_1^2} + \frac{18 u_3}{u_1^2} - \frac{20 P(u) u_3}{u_1^2} - \frac{2 P'(u)}{u_1} \beta, \]
\[ b^{(1)} = \left( P'''(u) u_1 - \frac{70 u_3 u_4}{u_1^2} + \frac{17 P'(u) u_3}{u_1^2} + \frac{8 u_3 u_4}{u_1^2} - \frac{8 P'(u) u_3^2}{3 \mu_1^2} + \frac{48 P(u) u_3}{u_1^2} \right) \alpha \]
\[ + \frac{19 u_3 u_4}{u_1^2} - \frac{14 P(u) u_3}{u_1^2} + \frac{u_3}{u_1^2} + \frac{43 u_3 u_4}{u_1^2} + \frac{15 u_3 u_4}{u_1^2} - \frac{38 P(u) u_3 u_4}{3 \mu_1^2} - \frac{66 u_3^2}{u_1^2} \frac{u_4}{u_1^2} - \frac{14 P(u) u_3}{u_1^2} \]
\[ + \frac{6 u_3^4}{u_1^6} - \frac{15 u_3^4}{u_1^6} + \frac{32 P(u) u_3}{u_1^2} + \frac{8 P(u) u_3}{u_1^2} - \frac{4 P(u) u_3}{u_1^2} \]
\[ - \frac{2 u_3}{u_1} - \frac{8 u_3 u_4}{u_1^2} + \frac{6 u_3^2}{u_1^4} + \frac{4 P(u) u_3}{u_1^2} + \frac{2 P'(u)}{u_1} \gamma, \]
\[ b^{(0)} = \left( \frac{4 P(u)^2 u_2^2}{3 u_1^4} + \frac{23 P'(u) u_2 u_3}{u_1^4} - \frac{2 P(u) u_2 u_4}{u_1^4} + \frac{4 P(u) P'(u) u_2}{u_1^4} - \frac{5 P'(u)^2}{9 u_1^4} \right) \alpha \]
\[ + \frac{34 P(u) u_3^2}{3 u_1^4} + \frac{4 P(u) P''(u)}{9 u_1^4} - \frac{7 u_3 u_4}{u_1^4} + \frac{26 P'(u) u_3^2}{3 \mu_1^2} - \frac{14 P(u) u_3^2}{3 \mu_1^2} \frac{u_4}{u_1^2} \]
\[ + \frac{10 P''(u) u_3^2}{3 \mu_1^2} - \frac{15 u_3 u_4}{u_1^4} + \frac{u_3 u_4}{u_1^4} + \frac{4 P(u) u_3^2}{u_1^4} - \frac{4 P''(u) u_3}{3 \mu_1^2} \frac{u_4}{u_1^2} \]
\[ + \frac{3 u_3 u_4}{u_1^4} + \frac{2 u_3 u_4}{u_1^4} - \frac{5 P'(u) u_4}{u_1^4} + \frac{21 u_3^2}{u_1^4} \frac{u_4}{u_1^4} - \frac{8 P(u)^2 u_3}{9 u_1^4} - \frac{P''(u) u_2}{u_1^4} \]
\[ - \frac{6 u_3^4}{u_1^6} + \frac{3 u_3^4}{u_1^6} + \frac{5 P'''(u) u_3}{u_1^4} \alpha + \frac{5}{9} P'''(u) u_1 - \frac{6 u_3 u_4}{u_1^4} - \frac{P'(u) u_2}{u_1^4} \]
\[ + \frac{u_3 u_4}{9 u_1^4} + \frac{2 P(u) u_3}{u_1^2} + \frac{4 P(u) P'(u)}{u_1^2} - \frac{P''(u) u_2}{u_1^4} \]
\[ + \frac{2 P(u) u_4}{u_1^4} + \frac{5 u_3 u_4}{u_1^4} + \frac{15 u_3 u_4}{u_1^4} - \frac{14 P(u) u_2 u_3}{3 \mu_1^2} - \frac{16 P(u) u_3^2}{3 \mu_1^2} \frac{u_4}{u_1^2} + \frac{6 u_3^5}{u_1^8} \beta \]
\[ + \frac{8 P'(u) u_2}{u_1^4} + \frac{10}{9} P''(u) \gamma, \]
Having the set $S$ we can easily write down the ansatz for the recursion operator $R$:

$$R = \sum_{j=0}^{4} r^{(j)} D_x^j + u_1 D_x^{-1} h^{(1)} + u_2 D_x^{-1} h^{(2)}.$$

(2.47)

The relation $L_1 R = L_2$ allows one to find all of the functional parameters in (2.47):

$$r^{(4)} = 1, \quad r^{(3)} = -\frac{4u_2}{u_1}, \quad r^{(2)} = \frac{2u_3}{u_1} + \frac{6u_2^2}{u_1^2} - \frac{4P(u)}{3u_1^3},$$

$$r^{(1)} = -\frac{2u_4}{u_1} + \frac{8u_2u_3}{u_1^2} - \frac{6u_2^2}{u_1^2} + \frac{4P(u)u_2}{u_1^2} - \frac{2P'(u)}{u_1},$$

$$r^{(0)} = \frac{u_5}{u_1} - \frac{4u_2u_4}{u_1^2} + \frac{2u_3^2}{u_1^2} + \frac{8u_2^2u_3}{u_1^3} - \frac{3u_2^3}{u_1^3} + \frac{4P(u)u_2^2}{3u_1^4}$$

$$+ \frac{4P(u)^2}{9u_1^4} - \frac{8P'(u)u_2}{3u_1^5} + \frac{10}{9} P''(u),$$

$$h^{(1)} = -\frac{u_6}{u_1} + \frac{6u_2u_5}{u_1^2} - \frac{5}{9} P'''(u) + \frac{5P''(u)u_2}{3u_1^3} - \frac{10P(u)u_2}{9u_1^4}$$

$$- \frac{10u_2(4u_1u_3 - 5u_2^2)P(u)}{3u_1^5} - \frac{15u_2(2u_1u_3 - u_2^2)(2u_1u_3 - 3u_2^2)}{2u_1^7}$$

$$+ \frac{5}{18} u_4 (3u_4 + P'(u)),$$}

$$h^{(2)} = -\frac{u_6}{u_1} + \frac{4u_2u_3}{u_1^2} - \frac{3u_2^3}{u_1^3} - \frac{P'(u)}{u_1^2} + 2P(u)u_2 - \frac{2P(u)u_2}{u_1^2}.

Thus we obtain the final form of the operator $R$

$$R = D_x^4 - \frac{4u_2}{u_1} D_x^3 + \left( \frac{6u_2^2}{u_1^2} - \frac{2u_3}{u_1} + \frac{4P(u)}{3u_1^3} \right) D_x^2$$

$$+ \left( \frac{8u_2u_3}{u_1^2} - \frac{2u_4}{u_1} - \frac{6u_2^2}{u_1^2} + \frac{4P(u)u_2}{u_1^2} - \frac{2P'(u)}{3u_1} \right) D_x + \frac{u_5}{u_1} - \frac{4u_2u_4}{u_1^2}$$

$$- \frac{2u_3^2}{u_1^2} + \frac{8u_2^2u_3}{u_1^3} - \frac{3u_2^3}{u_1^3} + \frac{4P(u)u_2^2}{3u_1^4} - \frac{8P'(u)u_2}{9u_1^4} + \frac{10}{9} P''(u)$$

$$+ u_1 D_x^{-1} \left( \frac{6u_2u_5}{u_1^2} - \frac{u_6}{u_1} - \frac{5}{9} P'''(u) + \frac{5P''(u)u_2}{3u_1^3} - \frac{10P(u)u_2}{9u_1^4} \right)$$

$$- \frac{10u_2(4u_1u_3 - 5u_2^2)P(u)}{3u_1^5} - \frac{15u_2(2u_1u_3 - u_2^2)(2u_1u_3 - 3u_2^2)}{2u_1^7}$$

$$+ \frac{5}{18} u_4 (3u_4 + P'(u))$$

$$+ u_1 D_x^{-1} \left( \frac{4u_2u_3}{u_1^2} - \frac{u_4}{u_1^2} - \frac{3u_2^3}{u_1^3} - \frac{P'(u)}{u_1^2} + \frac{2P(u)u_2}{u_1^2} \right).$$

(2.48)

It coincides with $R$ found earlier in [11].
The second recursion operator \( R_2 \) for the Krichever–Novikov equation can also be evaluated by using the symmetry method. The corresponding operators \( L_1 \) and \( L_2 \) have the orders \( m = 3 \) and, respectively, \( m_2 = 9 \). We do not give explicit expressions neither for \( R_2 \) nor for \( L_1, L_2 \), since they are rather large. For more information about the \( R_2 \) operator, we refer the reader to [19].

3. Construction of the recursion operators for the integrable lattices

In this section we concentrate on the integrable lattices of the form

\[
\begin{align*}
\frac{df}{du_{n+k}} + \frac{df}{du_{n-k}} &\neq 0
\end{align*}
\] (3.1)

where the sought function \( u = u_n(t) \) depends on the integer \( n \) and real \( t \). The non-negative integer \( k \) in (3.1) is called the order of the equation (3.1). A lattice of the same form

\[
\begin{align*}
\frac{df}{du_{n+m}} + \frac{df}{du_{n-m}} &\neq 0
\end{align*}
\] (3.2)

is a symmetry of the lattice (3.1) if the following condition is satisfied identically on the dynamical variables \( u_n, u_{n\pm 1}, \ldots \).

Stress that in

\[
\begin{align*}
D_t g - D_{\tau f} & = 0
\end{align*}
\] (3.3)

all of the derivatives with respect to \( t \) and \( \tau \) are replaced due to the equations (3.1) and (3.2).

Since the lattice (3.1) is assumed to be integrable then it admits an infinite hierarchy of the symmetries

\[
\begin{align*}
\{ u_{\tau_1} = g^{(1)}, u_{\tau_2} = g^{(2)}, \ldots, u_{\tau_j} = g^{(j)}, \ldots \}
\end{align*}
\] (3.4)

First several members of the hierarchy are the classical symmetries and the others are generalized ones. The hierarchy is effectively described by the recursion operator. Below we assume that the recursion operator \( R \) is a pseudo-difference operator with weak non-localities, i.e. it can be represented as

\[
\begin{align*}
R = R_0 + \sum_{j=1}^{m} g^{(j)}(D_n - 1)^{-1} h^{(j)},
\end{align*}
\] (3.5)

where \( D_n \) is the shift operator acting due to the rule \( D_nq_n = q_{n+1}, R_0 \) is a difference operator

\[
\begin{align*}
R_0 = \gamma^{(s)} D_s^n + \gamma^{(s-1)} D_s^{n-1} + \cdots + \gamma^{(-s)} D_s^{-s}, \quad s > 0,
\end{align*}
\] (3.6)

the coefficients \( h^{(j)}, j = 1, \ldots, m \) are functions of the variables \( u_n, u_{n\pm 1}, \ldots \) and the set of coefficients \( S = \{g^{(1)}, g^{(2)}, \ldots, g^{(m)}\} \) is a subset of the generators of the symmetries from the hierarchy (3.4). It is supposed that the generators \( \{g^{(1)}, g^{(2)}, \ldots, g^{(m)}\} \) constitute a linearly independent set over the field of the complex numbers.

**Theorem 2.** Let \( R \) be a weakly nonlocal difference operator. Then there exists a pair of difference operators \( L_1 \) and \( L_2 \) of the form

\[
\begin{align*}
L_1 = \alpha^{(0)} D_n^m + \alpha^{(1)} D_n^{m-1} + \cdots + \alpha^{(m)},
\end{align*}
\] (3.7)

\[
\begin{align*}
L_2 = \beta^{(p)} D_n^p + \beta^{(p-1)} D_n^{p-1} + \cdots + \beta^{(-q)} D_n^{-q}, \quad p > m, \quad q > 0
\end{align*}
\] (3.8)

such that the following condition is satisfied.
\[ L_1 R = L_2. \] (3.9)

**Proof.** We first determine the difference operator \( L_1 \) satisfying the condition \( \forall g \in S \) \( L_1 g = 0 \), i.e. the kernel of \( L_1 \) coincides with \( S \). Then obviously the operator is given by the following explicit formula:

\[
L_1 U = \rho \begin{bmatrix} g_1^{(1)} & g_1^{(2)} & \cdots & \cdots & g_1^{(m)} \\ g_{m-1}^{(1)} & g_{m-1}^{(2)} & \cdots & \cdots & g_{m-1}^{(m)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ g_1^{(1)} & g_1^{(2)} & \cdots & \cdots & g_1^{(m)} \\ U_{m-1} & U_{m-2} & \cdots & \cdots & U_1 \end{bmatrix},
\] (3.10)

where \( g_m^{(j)} := D_m^n g^{(j)} \), \( U_j = D_n^j U \) and \( \rho \) is an arbitrary function of the dynamical variables \( u_n, u_{n\pm 1}, \ldots \). Let us prove now that for this choice of \( L_1 \) the operator \( L_2 := L_1 R \) is also a difference operator, i.e. \( L_2 \) does not contain any nonlocal term. In fact, it suffices to verify that the product \( L_1 g(D_n - 1)^{-1} h \) is a difference operator whenever \( L_1 g = 0 \). Because of (3.7), this product can be written as follows

\[
\begin{align*}
\alpha^{(0)} & g_m(D_n - 1)^{-1} D_n^m h + \alpha^{(1)} g_{m-1}(D_n - 1)^{-1} D_n^{m-1} h + \cdots + \alpha^{(m)} (D_n - 1)^{-1} h \\
&= \alpha^{(0)} g_m(D_n - 1)^{-1} (D_n^m - 1) h + \alpha^{(1)} g_{m-1}(D_n - 1)^{-1} (D_n^{m-1} - 1) h \\
&\quad + \alpha^{(m-1)} h + L_1(g)(D_n - 1) h,
\end{align*}
\] (3.11)

where \( g_m := D_n^m g \). Let us observe that the last term in (3.11) vanishes since \( L_1(g) = 0 \). All of the other terms in (3.11) contain linear combinations of the products \((D_n - 1)^{-1} (D_n^j - 1)\) which do not produce nonlocalities for any integer \( j \geq 0 \).

For the known discrete integrable equation (3.1) the set \( S \) is of one of the forms

\[
S_1 = \left\{ g^{(0)} \right\}, \quad S_2 = \left\{ f \right\}, \quad S_3 = \left\{ g^{(0)}, f \right\}, \quad S_4 = \left\{ f, g^{(1)} \right\},
\]

\[
S_5 = \left\{ g^{(0)}, f, g^{(1)} \right\}, \quad S_6 = \left\{ f, g^{(1)}, g^{(2)} \right\}.
\] (3.12)

Where \( u_{S_5} = g^{(0)} \) is a zero order classical symmetry of the equation (3.1), in the case when the lattice admits such a symmetry, \( u_t = f \) is the equation (3.1) itself, \( u_{S_6} = g^{(1)} \) is the closest to \( u_t \) member of the symmetry hierarchy and \( u_{S_6} = g^{(2)} \) is the next one. The set \( S_1 \) defines a set of seed symmetries for the coupled lattice discussed in [5], the case \( S_2 \) corresponds to the Volterra chain (see, example below), the set \( S_3 \) is related to the example studied below in the section 4. Note that it can be verified that the set of the form \( S_4 \) corresponds to the Yamilov discretization of the Krichever–Novikov equation (YdKN) (see [25, 26]) and the case \( S_5 \) corresponds to the second recursion operator for the YdKN. Explicit forms of these operators are found in [27, 28].

Let us suppose now that the lattice (3.1) admits a weakly non-local recursion operator and discuss on the method of its finding.

We have to determine the appropriate set of the seed symmetries. To this end we should examine the possible sets listed in (3.12). Let us begin with the simplest one \( S_1 \) (or \( S_2 \) if the lattice does not admit any zero order symmetry). Then we construct the operator \( L_1 \) due to the determinant representation (3.10) and look for the operator \( L_2 \) such that the ratio \( R = L_1^{-1} L_2 \) solves the defining equation for the recursion operator.
\[ \frac{d}{dt} R = [F^*, R], \]  

where \( F^* \) is the Frechet derivative (or the linearization operator) for the equation (3.1):  
\[
F^* = \frac{\partial f}{\partial u_{n+k}} D_n^k + \frac{\partial f}{\partial u_{n+k-1}} D_n^{k-1} + \cdots + \frac{\partial f}{\partial u_{n-k}} D_n^{-k}.
\]  

(3.13)  

The natural numbers \( m, p \) and \( q \) from the formulas (3.7) and (3.8) are related to each other by the formulas  
\[
p = m + l, \quad q = l, \quad l = \text{ord}_g(m+1) - \text{ord}_g(1).
\]  

(3.14)  

For the operator \( L_1 \) found in virtue of the formula (3.10) we determine the operator \( A \) from the equation  
\[
\frac{d}{dt} L_1 = A L_1 - L_1 F^*.
\]  

(3.15)  

In the next stage we examine the equation  
\[
\frac{d}{dt} L_2 = A L_2 - L_2 F^*.
\]  

(3.16)  

and clarify whether it has a solution \( L_2 \) of the form (3.8) with the parameters \( p \) and \( q \) given by (3.15). If the answer is positive then we can conclude that \( R = L_1^{-1} L_2 \) is the recursion operator for the lattice (3.1). Otherwise we pass to the next member of the sequence (3.12) and so on.  

As it ensues from the theorem 2 if the lattice (3.1) admits a weakly nonlocal recursion operator then it can be found by using the suggested procedure. In other words for some choice of the set of seed symmetries the operator \( L_2 \) obligatorily exists satisfying the equation (3.17). Assume that such \( L_2 \) is found, then to specify the undetermined coefficients in (3.5) we use the equation \( L_2 = L_1 R \). Let us write it down in an expanded form  
\[
\beta(p) D^p_n + \beta(p-1) D^{p-1}_n + \cdots + \beta(-q) D^{-q}_n = (\alpha^{(0)} D^m_n + \alpha^{(1)} D^{m-1}_n + \cdots + \alpha^{(m)}) (\gamma^{(0)} D^q_n + \cdots + \gamma^{(-q)} D^{-q}_n) \times (g^{(1)}(D_n - 1)^{-1} h^{(1)} + \cdots + g^{(m)}(D_n - 1)^{-1} h^{(m)}).
\]  

(3.17)  

Indeed the comparison of the coefficients in front of the different powers of \( D_n \) in (3.18) allows one to find effectively the coefficients \( h^{(j)} \) and \( \gamma^{(j)} \).  

3.1. Evaluation of the recursion operator for the Volterra chain  

As an illustrative example of the application of the algorithm above we take the well-known Volterra chain  
\[
\frac{d}{dt} u_n = u_n (u_{n+1} - u_{n-1}).
\]  

(3.18)  

Let us find its linearization  
\[
\frac{d}{dt} U_n = F^* U_n, \quad \text{where} \quad F^* = u_n D_n + (u_{n+1} - u_{n-1}) - u_n D^{-1}_n.
\]  

(3.19)
Since the Volterra chain does not admit any zero order autonomous symmetry we start with the set $S_2$. Let us define the operator $L_1$ through the determinant representation (3.10)

$$L_1U_n = \frac{1}{u_{n+1,l}u_{n+1,j}} \begin{vmatrix} u_{n+1,l} & u_{n,l} \\ U_{n+1} & U_n \end{vmatrix}$$

(3.21)

or, the same

$$L_1 = (D_n - 1) \frac{1}{u_{n,l}}.$$  

(3.22)

Then we look for the operator $A$ of the form

$$A = A^{(1)}D_n + A^{(0)} + A^{(-1)}D_n^{-1}$$

(3.23)

from the equation (3.16):

$$\frac{1}{(u_{n+1,l})^2}D_n - \frac{1}{(u_{n,l})^2} = \left(A^{(1)}D_n + A^{(0)} + A^{(-1)}D_n^{-1}\right) \left(\frac{1}{u_{n+1,l}}D_n - \frac{1}{u_{n,l}}\right)$$

$$- \left(\frac{1}{u_{n+1,l}}D_n - \frac{1}{u_{n,l}}\right)(u_{n}D_n + (u_{n+1} - u_{n-1}) - u_{n}D_n^{-1}).$$  

(3.24)

By comparison of the coefficients at the powers of the operator $D_n$ in the last equation we obtain

$$A^{(1)} = \frac{u_{n+1}u_{n+2,l}}{u_{n+1,l}}, A^{(0)} = \frac{u_{n+1}u_{n+2,l}}{u_{n+1,l}} - \frac{u_{n}u_{n+1,l}}{u_{n,l}}, A^{(-1)} = \frac{-u_{n+1}u_{n-1,l}}{u_{n,l}}.$$  

(3.25)

Due to the formula (3.15) the corresponding operator $L_2$ should be of the form

$$L_2 = b^{(2)}D_n^2 + b^{(1)}D_n + b^{(0)} + b^{(-1)}D_n^{-1}. $$  

(3.26)

Let us substitute the ansatz (3.26) into the equation (3.17) and find

$$b^{(2)}D_n^2 + b^{(1)}D_n + b^{(0)} + b^{(-1)}D_n^{-1} = \left(A^{(1)}D_n + A^{(0)} + A^{(-1)}D_n^{-1}\right)\left(b^{(2)}D_n^2 + b^{(1)}D_n + b^{(0)} + b^{(-1)}D_n^{-1}\right)$$

$$- \left(b^{(2)}D_n^2 + b^{(1)}D_n + b^{(0)} + b^{(-1)}D_n^{-1}\right)\left(u_{n}D_n + (u_{n+1} - u_{n-1}) - u_{n}D_n^{-1}\right).$$  

(3.27)

By collecting the coefficients at the different powers of $D_n$ we can derive some equations for the factors $b^{(1)}$. We will skip the details of the calculation and present only the answers

$$b^{(2)} = \frac{c_1}{u_{n+2} - u_{n}}, b^{(1)} = \frac{c_3 + c_1(u_{n+2} + u_{n+1})}{u_{n+1}(u_{n+2} - u_{n})} - \frac{c_1}{u_{n+1} - u_{n-1}},$$

$$b^{(0)} = \frac{c_1}{u_{n+2} - u_{n}} - \frac{c_3 + c_1(u_{n} + u_{n-1})}{u_{n}(u_{n+1} - u_{n-1})}, b^{(-1)} = -\frac{c_1}{u_{n+1} - u_{n-1}}.$$  

(3.28)

Without loss of generality, we can put $c_1 = 1$ and $c_3 = 0$ and get

$$L_2 = \frac{1}{u_{n+2} - u_{n}}D_n^2 + \left(\frac{u_{n+2} + u_{n+1}}{u_{n+1}(u_{n+2} - u_{n})} - \frac{1}{u_{n+1} - u_{n-1}}\right)D_n$$

$$+ \frac{1}{u_{n+2} - u_{n}} - \frac{u_{n} + u_{n-1}}{u_{n}(u_{n+1} - u_{n-1})} = \frac{1}{u_{n+1} - u_{n-1}}D_n^{-1}.$$  

(3.29)
Thus both operators are found and we are ready to construct the recursion operator:

\[ R = r^{(1)} D_n + r^{(0)} + r^{(-1)} D_n^{-1} + u_{nt}(D_n - 1)^{-1} h. \] (3.30)

Let us substitute into the equation \( L_2 = L_1 R \) the detailed representations of the operators given by (3.22), (3.26) and (3.30). As a result we get the equation

\[
\begin{align*}
&b^{(2)} D_n^2 + b^{(1)} D_n + b^{(0)} + b^{(-1)} D_n^{-1} \\
&= \left( \frac{1}{u_{n+1,t}} D_n - \frac{1}{u_{nt}} \right) \left( r^{(1)} D_n + r^{(0)} + r^{(-1)} D_n^{-1} + u_{nt}(D_n - 1)^{-1} h \right),
\end{align*}
\] (3.31)

which is easily solved

\[
\begin{align*}
r^{(1)} &= u_n, \\
r^{(0)} &= (u_{n+1} + u_n), \\
r^{(-1)} &= u_n, \\
h &= \frac{1}{u_n}.
\end{align*}
\] (3.32)

Therefore the recursion operator reads as follows

\[ R = u_n D_n + (u_{n+1} + u_n) + u_{nt} D_n^{-1} + u_{nt}(D_n - 1)^{-1} \frac{1}{u_n}. \] (3.33)

It coincides with \( R \) found earlier in [6, 15]. Note that in [15] this operator is represented as a ratio \( R = H_2 H_1^{-1} \) of two Hamiltonian operators

\[ H_1 = u_n(D - D^{-1}) u_n, \quad H_2 = u_n(D u_n D + u_n D + D u_n - u_n D^{-1} - D^{-1} u_n - D^{-1} u_n D^{-1}) u_n. \]

### 4. A non-autonomous example

Nowadays non-autonomous integrable lattices are intensively studied. Symmetry integrability conditions for this class of equations has been formulated in [29]. The symmetry algorithm for constructing the recursion operator can be effectively applied to the non-autonomous lattices as well. Let us consider the following non-autonomous lattice of the relativistic Toda type

\[ u_{n,t} = h_n h_{n-1} (a_n u_{n+2} - a_{n-1} u_{n-2}), \] (4.1)

where \( h_n = u_{n+1} u_n - 1 \) and the coefficient \( a_n \) is an arbitrary periodic function of the period 2, i.e. \( a_{n+2} = a_n \). This lattice has been found within the frame of the integrable classification of the discrete equations on the quadratic graphs. Namely, it is a symmetry in the direction \( n \) of the equation

\[ u_{n+1, m+1} (u_{n, m} - u_{n, m+1}) - u_{n+1, m} (u_{n, m} + u_{n, m+1}) + 2 = 0. \] (4.2)

Notice that equations (4.1) and (4.2) were found by Garifullin and Yamilov in [30]. In [31] the recursion operator for the non-autonomous lattice (4.1) is constructed in an unusual way. The authors observed that the lattice (4.1) can be reduced to an autonomous system found by Tsuchida in [32], for which the recursion operator has already been found in [27]. In [31] the known recursion operator was recalculated to the scalar form in an appropriate way.

In this article, we derive the recursion operator directly using the symmetry algorithm discussed above.

As it has been observed in [31] the lattice (4.1) possesses a rather large hierarchy of the symmetries. Since the lattice is not autonomous then the set \( S \) of the seed symmetries obviously might also contain non-autonomous symmetries. The first two members of the symmetry hierarchy are
\[ u_{n+1} = (-1)^n u_n, \quad (4.3) \]
\[ u_{n+2} = h_n h_{n-1} (c_n u_{n+2} - c_{n-1} u_{n-2}), \quad c_{n+2} = c_n, \quad (4.4) \]
where \( c_n \) is an arbitrary periodic function of \( n \) with period equal to two.

As potential sets of seed symmetries, consider the following three sets:
\[ S_1 = \{ u_{n, \tau_1} \}, \quad S_2 = \{ u_{n, \tau_2} \}, \quad S_3 = \{ u_{n, \tau_1}; u_{n, \tau_2} \}. \quad (4.5) \]
We checked that the first two sets do not fit, but the latest is surely the required set of seed symmetries.

The operator \( L_1 \) corresponding to \( S_3 \) is given by
\[ L_1 U_n = \begin{vmatrix} D_n^2(u_{n, \tau_1}) & D_n(u_{n, \tau_1}) & u_{n, \tau_1} \\ D_n^2(u_{n, \tau_2}) & D_n(u_{n, \tau_2}) & u_{n, \tau_2} \\ U_{n+2} & U_{n+1} & U_n \end{vmatrix}. \quad (4.6) \]
As a result we have
\[ L_1 = \alpha D_n^2 + \beta D_n + \gamma, \quad (4.7) \]
where
\[ \alpha = (-1)^{n+1} h_n (u_{n+1} h_{n-1} (c_n u_{n+2} - c_{n-1} u_{n-2}) + u_{n+1} h_{n+1} (c_{n+2} u_{n+3} - c_{n+1} u_{n-1})), \]
\[ \beta = (-1)^n (u_{n+2} h_{n-1} (c_n u_{n+2} - c_{n-1} u_{n-2}) - u_{n+2} h_{n+1} (c_{n+2} u_{n+3} - c_{n+1} u_{n-1})), \]
\[ \gamma = (-1)^n h_{n+1} (u_{n+1} h_{n+2} (c_n u_{n+4} - c_{n-1} u_{n}) + u_{n+1} h_{n+2} (c_{n+2} u_{n+3} - c_{n+1} u_{n-1})). \]
Obviously the linearization of (4.1) is
\[ U_{n, t} = F^* U_n, \quad (4.8) \]
with
\[ F^* = a_n h_n h_{n-1} D_n^2 + u_{n+1} h_{n-1} (a_n u_{n+2} - a_{n-1} u_{n-2}) D_n \]
\[ + (u_{n+1} h_{n-1} + u_{n-1} h_n) (a_n u_{n+2} - a_{n-1} u_{n-2}) \]
\[ + u_n h_n (a_n u_{n+2} - a_{n-1} u_{n-2}) D_n^{-1} - a_{n-1} h_n h_{n-1} D_n^{-2}. \]

In order to find the operator
\[ A = A^{(2)} D_n^2 + A^{(1)} D_n + A^{(0)} + A^{(-1)} D_n^{-1} + A^{(-2)} D_n^{-2} \quad (4.9) \]
we solve the equation
\[ \frac{d}{dt} L_1 = A L_1 = L_1 F^*, \quad (4.10) \]
which is equivalent to
\[ \alpha D_n^2 + \beta D_n + \gamma, \]
\[ = (A^{(2)} D_n^2 + A^{(1)} D_n + A^{(0)} + A^{(-1)} D_n^{-1} + A^{(-2)} D_n^{-2})(\alpha D_n^2 + \beta D_n + \gamma) \]
\[ - (\alpha D_n^2 + \beta D_n + \gamma) (a_n h_n h_{n-1} D_n^2 + u_{n+1} h_{n-1} (a_n u_{n+2} - a_{n-1} u_{n-2}) D_n) \]
\[ + (u_{n+1} h_{n-1} + u_{n-1} h_n) (a_n u_{n+2} - a_{n-1} u_{n-2}) \]
\[ + u_n h_n (a_n u_{n+2} - a_{n-1} u_{n-2}) D_n^{-1} - a_{n-1} h_n h_{n-1} D_n^{-2}). \quad (4.11) \]

It is easily checked that (4.11) implies
\[ A^{(2)} = a_n b_{n+1} (u_{n+1} S_n + u_{n+2} S_{n+1}) / u_{n+3} S_{n+2} + u_{n+2} S_{n+3}, \]
\[ A^{(1)} = a_{n-1} h_n + a_{n+1} (u_n h_n S_{n+2} + u_n^2 u_{n+2} S_{n+3} + u_n S_{n+2}), \]
\[ \frac{d}{dt} \ln (u_{n+1} S_n + u_{n+2} S_{n+1}) + \left( \frac{u_{n+1} h_n a_n S_n - u_{n+1} a_n + 1 h_n + u_{n+2} S_{n+2}}{u_{n+1} S_n + u_{n+2} S_{n+1}} \right) a_{n-1} \]
\[ A^{(-1)} = - a_n h_n (u_{n+1} S_n + u_{n+2} h_n S_{n+2} + u_n^2 u_{n+2} S_{n+3}) / u_{n+1} S_n + u_{n+2} S_{n+1} \]
\[ + a_{n-1} h_n (u_{n+1} S_n + u_{n+2} S_{n+1}) (u_{n-2} S_n + u_{n-1} h_n S_{n+2} + u_{n-2} u_{n+2}^2 S_{n+3}) / (u_{n+1} S_n + u_{n+2} S_{n+1}), \]
\[ A^{(-2)} = - a_{n-1} h_n h_n - (u_{n+1} S_n + u_{n+1} h_n S_{n+2}) / u_{n-1} S_n + u_{n-2} S_{n+1}, \] (4.12)

where \( g_n = h_n h_n - (c_m u_{n+2} - c_n - u_{n-2}) \).

Next we look for the operator \( L_2 \) which has to be of the form
\[ L_2 = b^4 D^4 + b^3 D^3 + b^2 D^2 + b^1 D + b^0 + b^{-1} D^{-1} + b^{-2} D^{-2}. \] (4.13)

Indeed in (3.8) due to (3.15) we have \( p = m + l, q = l \) where \( m = 2, l = 2 \).

Therefore the defining equation
\[ \frac{d}{dt} L_2 = A L_2 - L_2 F^* \]
takes the form:
\[ b^4 D^4 + b^3 D^3 + b^2 D^2 + b^1 D + b^0 + b^{-1} D^{-1} + b^{-2} D^{-2} \]
\[ = \left( A^{(2)} D^2 + A^{(1)} D + A^{(0)} + A^{(-1)} D^{-1} + A^{(-2)} D^{-2} \right) \]
\[ \times \left( b^4 D^4 + b^3 D^3 + b^2 D^2 + b^1 D + b^0 + b^{-1} D^{-1} + b^{-2} D^{-2} \right) \]
\[ - \left( b^4 D^4 + b^3 D^3 + b^2 D^2 + b^1 D + b^0 + b^{-1} D^{-1} + b^{-2} D^{-2} \right) \]
\[ \times \left( a_n h_n h_{n-1} D^2 + a_n h_n - (a_n u_{n+2} - a_n - u_{n-2}) D_n \right) \]
\[ + (u_n h_n h_{n-1} + u_n - 1 h_n) (a_n u_{n+2} - a_n - u_{n-2}) \]
\[ + a_n h_n (a_n u_{n+2} - a_n - u_{n-2}) D_n^{-1} - a_n - 1 h_n h_{n-1} D_n^{-2}. \] (4.14)
By comparison of the coefficients in front of the powers of $D_n$ we get several equations which imply

$$b^{(4)} = (-1)^{n+1} \mu c_n h_{n-1} h_{n+2} (u_{n+1} g_n + u_n g_{n+1}),$$

$$b^{(3)} = (-1)^{n+1} \mu c_{n-1} h_n h_{n+1} (u_{n+2} g_n - u_n g_{n+2}) + (-1)^{n+1} \mu b_{n+1} (u_{n+1} g_{n+2} + u_n g_{n+1})$$

$$+ c_{n-1} (h_n h_{n+2} - 1) + (-1)^n \mu u_{n+2} g_{n+2} + u_n g_{n+1}),$$

$$b^{(2)} = (-1)^{n+1} \mu (u_n g_n + u_n g_{n+1}) \left( \frac{u_{n+1} g_{n+1} - u_n g_n}{h_n} + c_n (h_{n-1} h_{n+1} - 1) \right)$$

$$+ c_{n-1} (h_n h_{n+2} - 1) + (-1)^n \mu u_{n+1} g_{n+2} + u_n g_{n+1}),$$

$$b^{(1)} = (-1)^{n+1} \mu (u_{n+1} g_{n+1} + u_{n+2} g_{n+2}) - c_{n-1} (h_{n+1} h_n - 1) - (s_n^{(1)} + (-1)^n s_n^{(2)})) (u_{n+2} g_{n+2} - u_n g_{n+1}),$$

$$b^{(0)} = (-1)^{n+1} \mu (u_n g_{n+2} - u_n g_n) \left( \frac{u_{n+1} g_{n+1} + u_{n+2} g_{n+2}}{h_n} + \frac{u_n g_{n+1} + u_n g_{n+2}}{h_{n-1}} \right)$$

$$+ c_{n-1} (h_{n-1} h_n - 1) + (-1)^n \mu (u_{n+1} g_{n+2} + u_{n+2} g_{n+1}) - c_{n-1} (h_{n-1} h_n - 1) - (s_n^{(1)} + (-1)^n s_n^{(2)}),$$

$$b^{(-1)} = (-1)^{n+1} \mu c_n h_{n-1} h_{n+1} (u_{n+2} g_n - u_{n+1} g_{n+1}) + (-1)^{n+1} \mu u_{n+1} g_{n+2} + u_n g_{n+1}),$$

$$b^{(-2)} = (-1)^{n+1} \mu c_n h_{n-1} h_{n+1} (u_{n+1} g_{n+2} + u_n g_{n+1}),$$

Here $\mu$ is an arbitrary constant, the factors $s_n^{(1)}$, $s_n^{(2)}$ are arbitrary periodic functions with the period 2. By substituting the coefficients into (4.13) one finds the searched operator $L_2$. Thus we have both operators $L_1$, $L_2$. We can conclude that the recursion operator has the form

$$R = r^{(2)} D_n^2 + r^{(1)} D_n + r^{(0)} + r^{(-1)} D_n^{-1} + r^{(-2)} D_n^{-2}$$

$$+ u_{n, r_1} (D_n - 1)^{-1} \tilde{q} + u_{n, r_2} (D_n - 1)^{-1} \tilde{p}.$$  

(4.15)

with undetermined factors $\tilde{p}$, $\tilde{q}$ and $r^{(i)}$, $i = 2, -\frac{3}{2}$. In order to find these factors we solve the equation $L_2 = L_1 R$ or the same

$$b^{(4)} D_n^2 + b^{(3)} D_n^3 + b^{(2)} D_n^2 + b^{(1)} D_n + b^{(0)} + b^{(-1)} D_n^{-1} + b^{(-2)} D_n^{-2} =$$

$$= (\alpha D_n^2 + \beta D_n + \gamma) \left( r^{(2)} D_n^2 + r^{(1)} D_n + r^{(0)} + r^{(-1)} D_n^{-1} + r^{(-2)} D_n^{-2} + u_{n, r_1} (D_n - 1)^{-1} \tilde{q} + u_{n, r_2} (D_n - 1)^{-1} \tilde{p} \right).$$  

(4.16)

By comparing the coefficients of the powers of the operator $D_n$, we find:
Finally by taking $\mu = 1$, we find the recursion operator

$$R = c_n h_n h_{n-1} D_n^2 + \frac{u_{n+1} g_n - u_n g_{n-1}}{h_n} + c_n h_{n-1} h_{n+1}$$

$$+ c_{n-1} h_{n-2} + s_n = \frac{u_{n+1} g_n - u_n g_{n-1}}{h_n} + c_n h_{n-1} h_{n+1} D_n^2$$

$$+ u_{n+1}(D_n-1)^{-1}( - (-1)^{n+1} \frac{g_n - g_{n-1}}{h_n} + u_{n+1} g_{n+1} + \frac{g_{n+1}}{h_n})$$

(4.17)

where $g_n = h_n h_{n-1} (c_n u_{n+2} - c_{n-1} u_{n-2})$, the factors $u_{n,1}$ and $u_{n,2}$ are defined by (4.3) and, respectively, by (4.4). We denoted $s_n = (-1)^{n+1} s_n^{(1)} - s_n^{(2)}$. Obviously in the formula for $R$ function $s_n$ is considered as an arbitrary function satisfying the periodicity condition $s_n = s_{n+2}$. Recursion operator found in [31] is reduced to (4.17) with $s_n = 0$.

5. Conclusions

The recursion operators are of decisive importance in the theory of integrability. There are several methods for finding the recursion operator for a given integrable equation. In fact, they are all based either on the Lax representation, or on the multi-Hamiltonian approach. Here we proposed an alternative method based solely on the concept of symmetry. As a rule, the recursion operator is used to construct symmetries and conserved densities. We noticed that the recursion operator can be effectively obtained from the set of several first symmetries, which we called the set of seed symmetries. We look for the recursion operator as the ratio of two differential operators $R = L_1^{-1} L_2$. Using the operators $L_1$ and $L_2$, one can easily define the Lax pair for the integrable nonlinear equation under consideration (see formulas (2.18) and (2.19)).

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