On the monodromy of moduli spaces of sheaves on K3 surfaces

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Abstract

Let $S$ be a K3 surface and $\text{Aut} D(S)$ the group of auto-equivalences of the derived category of $S$. We construct a natural representation of $\text{Aut} D(S)$ on the cohomology of all moduli spaces of stable sheaves (with primitive Mukai vectors) on $S$. The main result of this paper is the precise relation of this action with the monodromy of the Hilbert schemes $S^{[n]}$ of points on the surface. A formula is provided for the monodromy representation, in terms of the Chern character of the universal sheaf. Isometries of the second cohomology of $S^{[n]}$ are lifted, via this formula, to monodromy operators of the whole cohomology ring of $S^{[n]}$.

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1 Introduction

A variety $X$, with an ample canonical or anti-canonical line bundle, is completely determined by the bounded derived category $D(X)$ of coherent sheaves on $X$. Moreover, $D(X)$ admits only the obvious auto-equivalences [BO]. In contrast, a projective variety $X$ with trivial canonical bundle, often admits interesting auto-equivalences. These symmetries have been used by Mukai [Mu3] and others to study moduli spaces of sheaves on abelian varieties and Calabi-Yau varieties. Equivalences of derived categories of K3 or abelian surfaces have been used to relate moduli spaces of sheaves on the surface $S$, to its Hilbert scheme. Any smooth and compact moduli space $M$, of stable sheaves on $S$, is deformation equivalent to the product $\text{Pic}^0(S) \times S^{[n]}$ of the identity component of the Picard group, with the Hilbert scheme $S^{[n]}$, of length $n$-subschemes [Y2, Y3]. In particular, $M$ and $\text{Pic}^0(S) \times S^{[n]}$ have isomorphic cohomology rings.

We concentrate in this paper on the case of a projective K3 surface $S$. Let $\text{Aut}D(S)$ be the group of auto-equivalences of $D(S)$. One may ask, to what extent $\text{Aut}D(S)$ acts on the collection of moduli spaces of stable sheaves on $S$? The most naive expectation fails; $\text{Aut}D(S)$ does not act via isomorphisms. We will show that $\text{Aut}D(S)$ does act on the level of cohomology of the moduli spaces. Moreover, the action is related to the monodromy representation of a fixed moduli space. We will consider a more general action (12) of a groupoid, whose morphisms are equivalences between derived categories of two, possibly non-isomorphic, K3 surfaces. These ideas are illustrated in section 1.2.1 in the simple case of moduli spaces of stable sheaves on an elliptic curve.

1.1 Statements of the results

1.1.1 Symmetries of moduli spaces of stable sheaves

The Mukai Lattice: Let $K_{top}(S)$ be the Grothendieck $K$-ring of topological complex vector bundles on $S$ [At]. The Euler characteristic $\chi : K_{top}(S) \to \mathbb{Z}$ is given by

$$\chi(v) = 2\text{rank}(v) + \int_S \left[ c_1(v)^2/2 - c_2(v) \right].$$

$K_{top}(S)$ is a free abelian group of rank 24. Any class $x \in K_{top}(S)$ is the difference $[E] - [F]$ of classes of complex vector bundles, and we denote by $x^\vee$ the class $[E^\vee] - [F^\vee]$ obtained from the dual vector bundles. The bilinear form on $K_{top}(S)$, given by

$$(x, y) := -\chi(x^\vee \otimes y),$$

(1)
is called the Mukai pairing. The pairing is symmetric, unimodular, of signature (4, 20), and the resulting lattice is called the Mukai lattice. We define a polarized weight 2 Hodge structure on $K^\text{top}(S)$, by setting $K^1(S) \subset K^\text{top}(S)$ to be the subspace of classes $v$ with $c_1(v)$ of type $(1,1)$, and $K^0(S)$ the subspace of classes $v$, orthogonal to $K^1(S)$, with $c_1(v)$ of type $(2,0)$. Note that the Chern character

$$ch : K^\text{top}(S) \to H^2(S, \mathbb{Z})$$

$$v \mapsto \text{rank}(v) + c_1(v) - c_2(v) + c_1(v)^2/2$$

is an isomorphism of free abelian groups preserving the ring structure ($ch(v)$ is integral, since the lattice $H^2(S, \mathbb{Z})$ is even, injectivity of $ch$ follows from Theorem 3.25 in [K], and surjectivity is easily verified). Caution: From section 3.2 onward, we will use Mukai’s normalization (30) of the isomorphism (2).

**Definition 1.1** A non-zero class $v \in K^1(S)$ will be called effective, if $(v, v) \geq -2$, rank$(v) \geq 0$, and the following conditions hold. If rank$(v) = 0$, then $c_1(v)$ is the class of an effective (or trivial) divisor on $S$. If both rank$(v)$ and $c_1(v)$ vanish, then $\chi(v) > 0$.

Note that if $v \in K^1(S)$, $v \neq 0$, and $(v, v) \geq -2$, then either $v$ or $-v$ is effective ([BHPV] chapter VIII Proposition 3.7). The class $v$ is primitive, if it is not a multiple of a class in $K^\text{top}(S)$ by an integer larger than 1. If $v$ is a primitive and effective class, then there almost always exists an ample line bundle $H$ on $S$, called $v$-suitable in Definition 3.4, such that the moduli space $\mathcal{M}_H(v)$, of $H$-stable coherent sheaves on $S$ with class $v$, is non-empty, smooth, connected, projective, and holomorphic symplectic of dimension $2 + (v, v)$ ([Mu1, OG1, Y2, Y3] and Theorem 1.7 below). The only exception, where a $v$-suitable ample line bundle $H$ need not exist, is the case rank$(v) = \chi(v) = 0$, and rank$(\text{Pic}(S)) > 1$ (see Condition 7.6 in that case). We use the $H$-stability of sheaves due to Gieseker, Maruyama, and Simpson [HL]. If a non-zero class $v$ is not effective, then every sheaf with class $v$ has an endomorphism algebra of dimension $\geq 1$ [Mu1]. In particular, there does not exist any $H$-stable sheaf with class $v$, for any polarization $H$. Caution: There may be an unstable such sheaf, so the term stably-effective is more accurate. We will use the shorter term.

**Universal classes**: A universal sheaf $\mathcal{E}_v$ exists over $\mathcal{M}_H(v) \times S$, if there exists a class $w \in K^1(S)$ satisfying $(v, w) = 1$ [Mu2]. In that case $\mathcal{E}_v$ is determined uniquely up to tensorization by a line bundle on $\mathcal{M}_H(v)$. We denote by

$$e_v \in K^\text{top}(\mathcal{M}_H(v) \times S)$$

the class of $\mathcal{E}_v$ in the topological $K$-ring. In general, a universal sheaf need not exist. Nevertheless, a universal class $e_v$ is constructed in a sequel to this paper [Ma4], where the integral structure of the cohomology of moduli spaces is studied (see also Definition 3.2 below). The class $e_v$ is canonical, up to tensorization with the pullback of the class of a topological line-bundle on $\mathcal{M}_H(v)$. In this introduction we state the results of this
paper using the existence of $e_v$, as it provides a simpler formulation of the results. The proofs we provide will use only the existence of the class $ch(e_v)$, which was introduced earlier in [Ma2] without relying on the existence of the integral class $e_v$ (see equations (25) and (28) and Lemma 3.14 below). 

Cohomological Fourier-Mukai transformations: Let $S_1$ and $S_2$ be projective $K3$ surfaces, $g : K_{top}(S_1) \to K_{top}(S_2)$ an isometry of Mukai lattices, and $v_1 \in K_{top}(S_1)$ a primitive and effective class. Set $v_2 := g(v_1)$ and assume that $v_2$ is effective. We do not assume that $g$ preserves the Hodge structures. Choose $v_i$-suitable polarizations $H_i$, $i = 1, 2$, and universal classes $e_{v_i}$ in $K_{top}(\mathcal{M}_H(v_i) \times S_i)$. The exterior product induces an isomorphism

$$K_{top}(\mathcal{M}_H(v_i)) \otimes K_{top}(S_i) \cong K_{top}(\mathcal{M}_H(v_i) \times S_i)$$

by the K"unneth Theorem ([At] Corollary 2.7.15). We can thus regard $(1 \otimes g)$ as a homomorphism from $K_{top}(\mathcal{M}_H(v_1) \times S_1)$ to $K_{top}(\mathcal{M}_H(v_1) \times S_2)$. Set $m := 2 + (v_1, v_1)$. Let $\pi_{ij}$ be the projection from $\mathcal{M}_H(v_1) \times S_2 \times \mathcal{M}_H(v_2)$ to the product of the $i$-th and $j$-th factors. We define a class in the middle cohomology $H^{2m}(\mathcal{M}_H(v_1) \times \mathcal{M}_H(v_2), \mathbb{Z})$ of the product:

$$\gamma(g, v_1) := c_m(\pi_{13}^{-1}(1 \otimes g)(e_{v_1})^\vee \otimes \pi_{23}^{-1}(e_{v_2})). \quad (3)$$

Above, $\pi_{ij}^i$ is the pull-back homomorphism of $K$-rings and $\pi_{ij}$ is the Gysin homomorphisms ([BFM], [K] Proposition IV.5.24). When the classes $e_{v_i}$ are defined in the algebraic $K$-group $K_{alg}(\mathcal{M}_H(v_i) \times S_i)$, the above formula yields the same class when $\pi_{ij}^i$ and $\pi_{ij}$ are the usual pull-back and push-forward homomorphisms of algebraic $K$-groups (see [Fu] or [Har] for their definition). Denote by

$$\gamma_{g,v_1} : H^*(\mathcal{M}_H(v_1), \mathbb{Z})_{free} \longrightarrow H^*(\mathcal{M}_H(v_2), \mathbb{Z})_{free} \quad (4)$$

the graded homomorphism, between the cohomology groups modulo their torsion subgroups, induced by the class $\gamma(g, v_1)$ using the K"unneth and Poincare-Duality theorems. Isometries between two Mukai lattices come in two flavors; orientation-preserving and orientation-reversing (Remark 4.3).

**Theorem 1.2** Let $g, S_i, v_i, H_i, e_{v_i}, i = 1, 2$, be as above.

1. $\gamma_{g,v_1}$ is an isomorphism of cohomology rings, which is independent of the choice of the universal classes $e_{v_i}$.

2. If $g$ is an isomorphism of Hodge structures, then so is $\gamma_{g,v_1}$. If $g$ is also orientation preserving, then $\gamma(g, v_1)$ is represented by a class in $K_{alg}([\mathcal{M}_H(v_1) \times \mathcal{M}_H(v_2)] \otimes \mathbb{Q}$, in general, and in $K_{alg}([\mathcal{M}_H(v_1) \times \mathcal{M}_H(v_2)])$, if universal sheaves exist over both moduli spaces.
3. The equality $\gamma_{g^{-1},v_2} = (\gamma_{g,v_1})^{-1}$ holds. Assume $S_3$ is a K3 surface, $f : K_{top}(S_2) \to K_{top}(S_3)$ an isometry, $v_3 := f(v_2)$ an effective class, and $H_3$ a $v_3$-suitable polarization. Then
\[ \gamma_{f,v_2} \circ \gamma_{g,v_1} = \gamma_{fg,v_1}. \]

4. There exists a topological line bundle $\ell$ on $M_{H_2}(v_2)$, depending uniquely (up to isomorphism) on the triple $(g,c_1(e_{v_1}),c_1(e_{v_2}))$, such that the following equality holds
\[ (\gamma_{g,v_1} \otimes \check{g})(ch(e_{v_1})) = ch(e_{v_2})ch(\ell), \]
where $\check{g} : H^*(S_1, \mathbb{Z}) \to H^*(S_2, \mathbb{Z})$ is the conjugate $ch \circ g \circ (ch)^{-1}$ of $g$ via the Chern character isomorphism (2).

5. Let $f : H^*(M_{H_1}(v_3), \mathbb{Z})_{free} \to H^*(M_{H_2}(v_3), \mathbb{Z})_{free}$ be an isomorphism of cohomology rings satisfying the analogue
\[ (f \otimes \check{g})(ch(e_{v_3})) = ch(e_{v_2})ch(\ell) \]
of equation (5), for some topological line-bundle $\ell$ on $M_{H_2}(v_2)$. Then $f = \gamma_{g,v_1}$. 

6. Let $c_k(M_{H_1}(v_1))$ be the image of $c_k(TM_{H_1}(v_1))$ in $H^{2k}(M_{H_1}(v_1), \mathbb{Z})_{free}$. Then the following equality holds:
\[ \gamma_{g,v_1}(c_k(M_{H_1}(v_1))) = c_k(M_{H_2}(v_2)). \]

**Relation with equivalences of derived categories:** An equivalence $\Phi : D(S_1) \to D(S_2)$, of the bounded derived categories of coherent sheaves, induces an isometry $\phi : K_{top}(S_1) \to K_{top}(S_2)$ (section 5.1). In some cases, the image $\Phi(F)$, of every $H_1$-stable sheaf $F$ on $S_1$ with class $v$, can be represented by an $H_2$-stable sheaf on $S_2$. In such cases, the auto-equivalence induces an isomorphism
\[ f : M_{H_1}(v) \to M_{H_2}(\phi(v)). \]
Furthermore, $(f_! \otimes \phi)(e_v)$ is a universal class on $M_{H_2}(\phi(v)) \times S_2$. Part 5 of Theorem 1.2 implies the equality $f_* = \gamma_{\phi,v}$ of isomorphisms of cohomology rings (section 5.3).

Consider instead the contravariant functor from $D(S_1)^{op}$ to $D(S_2)$ taking an object $F$ to the derived dual $\Phi(F)^{\vee}$ of $\Phi(F)$. Assume that $\Phi(F)^{\vee}$ can be represented by an $H_2$-stable sheaf on $S_2$, for every $H_1$-stable sheaf $F$ on $S_1$ with class $v$. Then we get an isomorphism $f : M_{H_1}(v) \to M_{H_2}(\phi(v)^{\vee})$. The induced isomorphism of $K$-rings $f_!$ satisfies the equation $(f_! \otimes \phi)(e_v) = (e_{\phi(v)^{\vee}})^{\vee}$, for a suitable choice of a universal class $e_{\phi(v)^{\vee}}$. The pushforward $f_*$ on the level of cohomology is identified as follows. Let
\[ D : K_{top}(S) \to K_{top}(S) \]
be the \textit{duality involution} sending \( w \) to \( w^\vee \). The analogue for the cohomology of a moduli space \( \mathcal{M}_H(v) \)
\[
D_{\mathcal{M}} : H^*(\mathcal{M}_H(v),\mathbb{Z})_{\text{free}} \rightarrow H^*(\mathcal{M}_H(v),\mathbb{Z})_{\text{free}}
\]
acts by \((-1)^i\) on \( H^{2i}(\mathcal{M}_H(v),\mathbb{Z}) \). \( D_{\mathcal{M}} \) is a ring isomorphism since the odd Betti numbers of \( \mathcal{M}_H(v) \) vanish (Theorem 1.7). Part 5 of Theorem 1.2 implies the equality
\[
f_* = D_{\mathcal{M}} \circ \gamma_{D_{\mathcal{M}}\phi,v}.
\]

Symmetries of the cohomology of moduli spaces of sheaves, arising from such contravariant functors, are analogous to symmetries of Grassmannians arising from outer-autormorphisms of \( GL(n) \) (see Example 4.16).

In most cases an equivalence sends some stable sheaves to complexes, and does not give rise to an isomorphism of moduli spaces. In many cases \( \Phi \) determines a birational isomorphism \( f : \mathcal{M}_H(v) \rightarrow \mathcal{M}_H(\phi(v)) \) depends, of course, also on the choice of polarizations. When \( \Phi \) is the identity, \( H_i, i = 1, 2 \), are two \( v \)-suitable polarization, and \( c_1(v) \) is a multiple of an ample class, the class \( \gamma(id, v) \) is Poincare-dual to a correspondence in \( \mathcal{M}_H(v) \times \mathcal{M}_H(v) \), which deforms to an isomorphism, under simultaneous small deformations of \( \mathcal{M}_H(v), i = 1, 2 \) (see Proposition 6.3 and Lemma 6.5).

**The moduli space \( \mathcal{M}_H(v) \) for \( v \) with negative rank:** The isometry \( \phi \), associated to an equivalence \( \Phi \), may send effective classes to non-effective classes. The shift auto-equivalence, for example, corresponds to the isometry \(-id\). If \( w \in K^1_{\text{top}}(S) \) is a primitive class and \((w,w) \geq -2\), but \( w \) is not effective, then the moduli space \( \mathcal{M}_H(w) \) is empty while \( \mathcal{M}_H(-w) \) is not, since \(-w \) is effective. Greater symmetry is obtained if we reset
\[
(\mathcal{M}_H(w), e_w) := (\mathcal{M}_H(-w), -e_{-w})
\]
(Definition 7.8). Then Theorem 1.2 extends (tautologically) relaxing the efficacy condition on \( v \), by the condition that \( v_i \) is primitive of type \((1,1)\).

**A groupoid representation:** Parts 1 and 3 of Theorem 1.2 may be reformulated as the construction of a representation for a groupoid
\[
\mathcal{G}.
\]
Recall, that a groupoid is a category all of which morphisms are isomorphisms. The objects of $\mathcal{G}$ are triples $(S,v,H)$, where $S$ is a projective $K3$ surface, $v \in K^{1,1}_\text{top}(S)$ is a primitive class, and $H$ is $v$-suitable ample line bundle. A morphism from $(S_1,v_1,H_1)$ to $(S_2,v_2,H_2)$ is an isometry $g : K^{1,1}_\text{top}(S_1) \to K^{1,1}_\text{top}(S_2)$ satisfying $g(v_1) = v_2$. If $(v_1,v_1) = (v_2,v_2)$, then $\text{Hom}_\mathcal{G}((S_1,v_1,H_1),(S_2,v_2,H_2))$ is a torsor for each of the two subgroups $\Gamma_{v_i} \subset OK^{1,1}_\text{top}(S_i)$ consisting of isometries stabilizing $v_i$ (Lemma 8.1). If $(v_1,v_1) \neq (v_2,v_2)$, then $\text{Hom}_\mathcal{G}((S_1,v_1,H_1),(S_2,v_2,H_2))$ is empty.

The representation constructed is the functor

$$\mathcal{H} : \mathcal{G} \to \mathcal{A}$$

(11)

into the category $\mathcal{A}$ of commutative $\mathbb{Z}$-algebras with a unit. The object $(S,v,H)$ is sent to the cohomology ring $H^*(\mathcal{M}_H(v),\mathbb{Z})_{\text{free}}$ (or to $(0)$ if $(v,v) < -2$). A morphism $g \in \text{Hom}_\mathcal{G}((S_1,v_1,H_1),(S_2,v_2,H_2))$ is sent to the isomorphism $\gamma_{g,v}$ given in (4).

Consider the analogous groupoid $\mathcal{D}$, with the same objects, whose morphisms $\Phi \in \text{Hom}_\mathcal{D}((S_1,v_1,H_1),(S_2,v_2,H_2))$ are equivalences $\Phi : D(S_1) \to D(S_2)$ of derived categories, sending $v_1$ to $v_2$. There is a natural functor $\mathcal{D} \to \mathcal{G}$, sending $\Phi$ to the associated isometry. We get the composite functor

$$\mathcal{D} \to \mathcal{G} \to \mathcal{A}$$

(12)

representing the groupoid $\mathcal{D}$.

Let $\Gamma$ be the isometry group of the Mukai lattice of a $K3$ surface $S$. Fix an effective class $v \in K_\text{top}(S)$ and a $v$-suitable polarization $H$. The subgroup $\Gamma_v \subset \Gamma$ stabilizing $v$ is also the automorphism group of the object $(S,v,H)$ of $\mathcal{G}$. As a corollary of Theorem 1.2 we get that $\Gamma_v$ acts on the cohomology of the corresponding moduli space:

**Corollary 1.3** (Corollary 3.13 and Lemma 5.2) The natural map

$$\gamma : \Gamma_v \longrightarrow \text{Aut}[H^*(\mathcal{M}_H(v),\mathbb{Z})_{\text{free}}]$$

(13)

$$g \mapsto \gamma_{g,v}$$

is group homomorphism. It is injective if $(v,v) \geq 2$.

Let $\mathcal{G}_n$ be the (full) subgroupoid of $\mathcal{G}$, whose objects consist of triples with class $v$ satisfying $2+(v,v) = 2n$. The restriction $\mathcal{H}_n$ of the functor $\mathcal{H}$ to $\mathcal{G}_n$ is determined by each of the representations (13), since any two objects of $\mathcal{G}_n$ are isomorphic in $\mathcal{G}_n$. In this sense, the functor $\mathcal{H}_n$ may be regarded as the “induced representation” $\text{Ind}_{\Gamma_v}^{\mathcal{G}_n} H^*(\mathcal{M}_H(v),\mathbb{Z})_{\text{free}}$.

### 1.1.2 The monodromy group of a moduli space

The second main result of this paper, is the relation between the action (13) of the stabilizer $\Gamma_v$, and the monodromy representation of the moduli space $\mathcal{M}_H(v)$ (Theorem 1.6). Roughly, the representation (13) acts via monodromy operators, modulo a sign change.
Definition 1.4 An **irreducible holomorphic symplectic manifold** is a simply-connected compact Kähler manifold $X$, such that $H^0(X, \Omega^2_X)$ is generated by an everywhere non-degenerate holomorphic two-form.

An irreducible holomorphic symplectic manifold of real dimension $4n$ admits a Riemannian metric with holonomy group $Sp(n)$ [B1]. Such a metric is called hyperkähler.

Definition 1.5 Let $M$ be an irreducible holomorphic symplectic manifold. An automorphism $g$ of the cohomology ring $H^*(M, \mathbb{Q})$ is called a **monodromy operator**, if there exists a family $\mathcal{M} \to B$ (which may depend on $g$) of irreducible holomorphic symplectic manifolds, having $M$ as a fiber over a point $b_0 \in B$, and such that $g$ belongs to the image of $\pi_1(B, b_0)$ under the monodromy representation. The **monodromy group** $\text{Mon}(M)$ of $M$ is the subgroup, of the automorphism group of the cohomology ring of $M$, generated by all the monodromy operators.

The monodromy group is interesting even when the moduli space is the Hilbert scheme $S^{[n]}$ of length $n$ subschemes on $S$. The Hilbert scheme is the moduli space $\mathcal{M}_H(v)$, where $v$ is the class of the ideal sheaf of a length $n$ subscheme. When $n \geq 2$, the Hilbert scheme $S^{[n]}$ admits deformations, which are not Hilbert schemes of points on any K3 surface. In fact, the moduli space, of irreducible holomorphic symplectic manifolds deformation equivalent to $S^{[n]}$, has dimension one larger than the corresponding moduli space of the surface $S$. Consequently, the monodromy group of $S^{[n]}$ is larger.

The isometry group $\Gamma$ of $K_{top}(S)$ has a natural character $\text{cov} : \Gamma \to \mathbb{Z}/2\mathbb{Z}$ (see (47)). This character sends reflections by $-2$ vectors to $0$ and reflections by $+2$ vectors to $1$. Note also that $\text{cov}(-1) = 0$ and $\text{cov}(D) = 1$, where $D$ is the duality involution (7). Isometries in the kernel of $\text{cov}$ are called orientation preserving. Modify the representation $\gamma$ in (13) to get the representation

$$mon : \Gamma_v \to \text{Aut}(H^*(\mathcal{M}_H(v), \mathbb{Z})_{\text{free}}).$$

where the duality involution $D_{\mathcal{M}_H(v)}$ is given in equation (8). This representation is faithful, if $(v, v) \geq 4$. If $(v, v) = 2$, then the kernel of $\text{mon}$ is $\{id, -\sigma_v\}$, where $\sigma_v$ is the reflection in $v$ (see section 7.4).

Theorem 1.6 Let $v$ be an effective and primitive class in $K_{top}(S)$ and $H$ a $v$-suitable polarization. Then the image $\text{mon}(\Gamma_v)$ is a normal subgroup of finite index\(^2\) in the monodromy group $\text{Mon}(\mathcal{M}_H(v))$.

\(^2\) $\text{Mon}(\mathcal{M}_H(v))$ is shown in [Ma4] to be the direct product $K \times \text{mon}(\Gamma_v)$, where $K$ is a finite abelian group of exponent $2$, which acts trivially on $H^2(\mathcal{M}_H(v))$. 

The $\Gamma_v$-invariance property (5) of the class $e_v$ no longer holds, if the representation $\gamma$ of $\Gamma_v$ is replaced by $\text{mon}$. A diagonal monodromy representation of $\Gamma_v$ decomposes a
canonical normalization $u_v$ of $e_v$ as a sum $u_v^+ + u_v^-$, of an invariant class $u_v^+$ and a class $u_v^-$, acted upon via the character $cov$ of $\Gamma_v$ (Lemma 4.15).

Sharper results are obtained about the weight 2 monodromy representation. We recall first the fundamental result about the second cohomology of a moduli space. Let $v$ and $H$ be as in Theorem 1.6 and $v^\perp \subset K_{top}(S)$ the sublattice orthogonal to $v$. Let $\pi_M$ and $\pi_S$ be the two projections from $\mathcal{M}_H(v) \times S$. Mukai introduced the natural homomorphism

$$\theta_v : v^\perp \rightarrow H^2(\mathcal{M}_H(v), \mathbb{Z})$$

$$\theta_v(x) = c_1[\pi_M((\pi_S(x^v) \otimes e_v)] .$$

The homomorphism $\theta_v$ is $\Gamma_v$-equivariant with respect to the representation $\gamma$ in (13) (see diagram (41)). Beauville constructed an integral symmetric bilinear form on the second cohomology of an irreducible holomorphic symplectic manifold [B1]. We will not need the intrinsic formula, but only the fact that it is monodromy invariant and its identification in the following theorem due to Mukai, Huybrechts, O'Grady, and Yoshioka:

**Theorem 1.7** ([OG1], [Y3] Theorem 8.1, and [Y4] Corollary 3.15) Let $v$ be a primitive and effective class and $H$ a $v$-suitable ample line-bundle.

1. $\mathcal{M}_H(v)$ is a smooth, non-empty, irreducible symplectic, projective variety of dimension $\dim(v) = \langle v, v \rangle + 2$. It is obtained by deformations from the Hilbert scheme of $\frac{\langle v, v \rangle}{2} + 1$ points on $S$.

2. The homomorphism (16) is an isomorphism of weight 2 Hodge structures with respect to Beauville’s bilinear form on $H^2(\mathcal{M}_H(v), \mathbb{Z})$ when $\dim(v) \geq 4$. When $\dim(v) = 2$, (16) factors through an isomorphism from $v^\perp / \mathbb{Z} \cdot v$.

**Note:** Most of the cases of Theorem 1.7 are proven in [Y3] Theorem 8.1 under the additional assumption that $\text{rank}(v) > 0$ or $c_1(v)$ is ample. Corollary 3.15 in [Y4] proves the remaining cases (rank($v$) = 0 and $c_1(v)$ not ample), under the assumption that $\mathcal{M}_H(v)$ is non-empty. The non-emptiness is proven in [Y4] Remark 3.4, under the assumption that $c_1(v)$ is nef and in complete generality in an unpublished note communicated to the author by K. Yoshioka.

Given an element $u$ of $H^2(\mathcal{M}_H(v), \mathbb{Z})$ with $(u, u)$ equal 2 or $-2$, let $\rho_u$ be the isometry given by $\rho_u(w) = \frac{-2}{(u,u)} w + (w, u)u$. Then $\rho_u$ is the reflection in $u$, when $(u, u) = -2$, and $-\rho_u$ is the reflection in $u$, when $(u, u) = 2$. Set

$$\mathcal{W} := \{\rho_u : u \in H^2(\mathcal{M}_H(v), \mathbb{Z}) \text{ and } (u, u) = \pm 2\}$$

(17)

to be the subgroup of $O(H^2(\mathcal{M}_H(v), \mathbb{Z}))$ generated by the elements $\rho_u$. Then $\mathcal{W}$ is a normal subgroup of finite index in $O(H^2(\mathcal{M}_H(v), \mathbb{Z}))$. The image of $\Gamma_v$, via the weight 2 monodromy representation of Theorem 1.6, is equal to $\mathcal{W}$, by Lemma 4.10.
Corollary 1.8 \( W \) is contained in the image \( \text{Mon}^2 \) in \( O(H^2(\mathcal{M}_H(v), \mathbb{Z})) \) of the whole monodromy group \( \text{Mon}(\mathcal{M}_H(v)) \). Assume \((v, v) \geq 2\). Then the representation (14) factors as the composition of the surjective homomorphism

\[
\Gamma_v \longrightarrow W \\
g \mapsto (-1)^{\text{cov}(g)} \cdot \theta_v \circ g \circ \theta_v^{-1}
\]

and a \( \text{Mon}(\mathcal{M}_H(v)) \)-equivariant injective homomorphism

\[
\mu : W \rightarrow \text{Mon}(\mathcal{M}_H(v)) \subset \text{Aut}[H^*(\mathcal{M}_H(v), \mathbb{Z})_{\text{free}}].
\]  

(18)

Above \( \text{Mon}(\mathcal{M}_H(v)) \) acts by conjugation on \( W \) and itself. The equivariance of \( \mu \) follows from the normality of \( W \) in \( OH^2(\mathcal{M}_H(v), \mathbb{Z}) \), the normality of \( \text{mon}(\Gamma_v) \) in Theorem 1.6, and the equivariance of the inclusion of the second cohomology in the total cohomology. Being monodromy equivariant, \( \mu \) is well defined for any compact Kähler manifold \( X \) deformation equivalent to \( \mathcal{M}_H(v) \).

In a sequel to this paper we prove the opposite inclusion \( \text{Mon}^2 \subset W \) [Ma4]. Consequently, \( W \) is equal to \( \text{Mon}^2 \). It follows, when \( n - 1 \) is not a prime power, that \( \text{Mon}^2 \) does not surject onto the quotient \( O(H^2(S[n], \mathbb{Z}))/\langle \pm 1 \rangle \), of the whole isometry group by its center. This, in turn, leads to a counter example to a version of the Generic-Torelli question: The weight 2 Hodge structure of a generic irreducible holomorphic symplectic manifold \( X \), deformation equivalent to \( S[n] \), does not determine the bimeromorphic class of \( X \), when \( n - 1 \) is not a prime power [Ma4]. The index of \( \text{Mon}^2 \) in \( O(H^2(S[n], \mathbb{Z}))/\langle \pm 1 \rangle \) is a lower bound for the number of bimeromorphic classes with the same generic weight 2 Hodge structure. This index is calculated in Lemma 8.3.

1.2 Related works

Let \( X \) be an irreducible holomorphic symplectic manifold deformation equivalent to the Hilbert scheme \( S[n] \). Corollary 1.8 describes the monodromy representation, of the finite index subgroup \( W \) of the isometry group \( O(H^2(X, \mathbb{Z})) \), in the automorphism group of the cohomology ring \( H^*(X) \). A related representation of \( SO(H^2(X, \mathbb{Z})) \), on the cohomology of a hyperkähler variety \( X \), was studied by Verbitsky via different techniques (see [Ve2] and Theorem 4.4 below, for Verbitsky’s result). Lemma 4.13 compares the two representations.

There are many parallels between i) the action (13), of the group \( \Gamma_v \), on the cohomology of \( \mathcal{M}_H(v) \), and ii) the action of a Weyl group \( W \), of a semisimple Lie group, on the cohomology of the cotangent bundle \( T^*\mathcal{B} \) of the flag variety \( \mathcal{B} \). The latter is of course the same as the cohomology of \( \mathcal{B} \). \( W \) is the reflection group of the root lattice, while the group \( \Gamma_v \) is equal to the reflection group of the sublattice \( v^\perp \) (Corollary 8.10). Both \( \mathcal{M}_H(v) \) and \( T^*\mathcal{B} \) have a holomorphic symplectic structure. Both actions are not realized by automorphisms. (The general Weyl group does act on the affine bundle \( G/T \rightarrow B \), which is a twisted version of \( T^*\mathcal{B} \). In both cases, the action is realized geometrically via
lagrangian correspondences. $W$ acts naturally, via the Steinberg correspondences, on the cohomology of $B$ (and, more generally, of any Springer fiber [CG]). In the Hilbert scheme case, lagrangian correspondences, analogous to the Steinberg variety, play a major role in the proof of Theorem 1.6 via local monodromy operators (Theorem 2.5).

Similar results were obtained by Nakajima, when the K3 surface is replaced with the resolution of a simple surface singularity [Na1]. A relationship with representations of affine Lie algebras, in the K3 case, is exhibited in [Na2].

### 1.2.1 Moduli spaces of stable sheaves on an elliptic curve

We verify the analogue of Theorem 1.2, about the symmetries of moduli spaces of stable sheaves, in the elliptic curve case. We also explain the analogue of Theorem 1.6, relating the above symmetries to the monodromy of a fixed moduli space.

Let $\Sigma$ be an elliptic curve. The analogue $H^*(\Sigma, \mathbb{Z})$, of the Mukai lattice, decomposes as the direct sum of $H^1(\Sigma, \mathbb{Z})$ with the even cohomology $H^{\text{even}}(\Sigma, \mathbb{Z}) := H^0(\Sigma, \mathbb{Z}) \oplus H^2(\Sigma, \mathbb{Z})$. The odd summand is endowed with the intersection pairing. The Mukai vector of a sheaf $F$ is $v(F) := (\text{rank}(F), c_1(F))$ and belongs to $H^{\text{even}}(\Sigma, \mathbb{Z})$. The Mukai pairing on the even summand is given by (31). Explicitly, it is the anti-symmetric bilinear form $((r', d'), (r'', d'')) = r''d' - r'd''$. The symmetry group $\Gamma_{\Sigma}$ of $H^*(\Sigma, \mathbb{Z})$ is thus $SL(2, \mathbb{Z}) \times SL(2, \mathbb{Z})$. This factorization of the symmetry group, separates the monodromy action from the action on $H^{\text{even}}(\Sigma, \mathbb{Z})$ of the group $\text{Aut}D(\Sigma)$ of auto-equivalences of the derived category. $\text{Aut}D(\Sigma)$ surjects onto the even $SL(2, \mathbb{Z})$ factor [Mu3]. Any auto-equivalence in $\text{Aut}D(\Sigma)$ maps a simple sheaf to a shift of a simple sheaf (Orlov’s determination of $\text{Aut}D(\Sigma)$ [Or2] reduces the check to the case of Mukai’s original Fourier-Mukai transform, in which it follows from [Mu3] Corollary 2.5). Over an elliptic curve, a sheaf is stable if and only if it is simple. $\text{Aut}D(\Sigma)$ acts on the collection of moduli spaces of stable sheaves (with a primitive Mukai vector) via isomorphisms, provided we use the convention in Definition 7.8, relating the moduli spaces $\mathcal{M}_\Sigma(r, d)$ and $\mathcal{M}_\Sigma(-r, -d)$.

Let $g : H^*(\Sigma_1, \mathbb{Z}) \to H^*(\Sigma_2, \mathbb{Z})$ be an isomorphism, compatible with the bilinear forms. Denote by $g = (g_0, g_1)$ the decomposition into even and odd factors. Define the class $\gamma(g, v)$ using the topological formula $\gamma(g, \mathcal{E}_v, \mathcal{E}_{g(v)})$ given below in equation (39). The class $\gamma(g, v)$ depends only on $v$, $g(v)$, and the odd factor $g_1$ of $g$ in the following sense. Let $\mathcal{M}_\Sigma(r, d)$ be a moduli space of stable sheaves on $\Sigma$, with a primitive vector $(r, d)$ with non-negative rank. Atiyah proved, that $\mathcal{M}_\Sigma(r, d)$ is naturally isomorphic to $\text{Pic}^d(\Sigma)$, via $F \mapsto \text{det}(F)$. Thus, the cohomologies of all moduli spaces $\mathcal{M}_\Sigma(r, d)$, with rank and degree coprime, are naturally identified with that of $\Sigma$. Under the above identifications, the class $\gamma(g, v)$ induces the unital ring isomorphism determined by $g_1 : H^1(\Sigma_1, \mathbb{Z}) \to H^1(\Sigma_2, \mathbb{Z})$. The proof is similar to that of Lemma 5.2.

The analogue of our main Theorem 1.6 is thus trivial in the elliptic curve case. Let $\Gamma_{\Sigma,v}$ be the subgroup of $\Gamma_{\Sigma}$ stabilizing a primitive vector $v$. Then $\Gamma_{\Sigma,v}$ is the product $\Gamma_{\Sigma,v}^{\text{even}} \times SL(2, \mathbb{Z})$, where the even factor is infinite cyclic. The even factor $\Gamma_{\Sigma,v}^{\text{even}}$ acts trivially on $H^*(\mathcal{M}_\Sigma(v), \mathbb{Z})$, while the odd $SL(2, \mathbb{Z})$ factor acts via the natural action on
$H^1(\Sigma, \mathbb{Z})$. We see that the image of $\Gamma_{\Sigma,v}$ in $\text{Aut}H^*(\mathcal{M}_\Sigma(v), \mathbb{Z})$ is the monodromy group.

Note, that we have broken Mirror Symmetry, which is supposed to interchange the two factors of $\Gamma_{\Sigma,v}$. The reason for this break of symmetry is, that we are considering only graded automorphisms of the cohomology ring of moduli spaces. This is expressed in formula (37), by the fact that the class $\gamma(g,v)$ is the middle dimensional Chern class of a complex (see also remark 5.5). The infinite cyclic even factor $\Gamma_{\Sigma,v}^{\text{even}}$ is generated by the endomorphism $\tau_v(w) := w + (w,v) \cdot v$. The endomorphism $\tau_v$ can be lifted to an auto-equivalence of the derived category, via transvections in stable sheaves parametrized by $\mathcal{M}_\Sigma(v)$ (these transvections are described in Theorem 7.4 under the name reflections).

Mirror symmetry would require the even factor $\Gamma_{\Sigma,v}^{\text{even}}$ to act non-trivially. Such transvections are mirror symmetric to Dehn twists [ST] in the odd monodromy factor of $\Gamma_{\Sigma}$.

### 1.3 Outline of the proofs

We reduce the proofs of Theorems 1.2 and 1.6 to smaller results, which are proven in the rest of the paper.

#### 1.3.1 Reduction of the proof of Theorem 1.2

The special case of the Theorem, where $(S_1, v_1, H_1) = (S_2, v_2, H_2) = (S, v, H)$ and $g$ is the identity, was proven in [Ma2] (Theorem 3.5 below). The class $\gamma(id,v)$ is Poincare dual to the class of the diagonal in $\mathcal{M}_H(v) \times \mathcal{M}_H(v)$. Consequently, $H^*(\mathcal{M}_{\Sigma}(v), \mathbb{Q})$ is generated by the K"unneth factors of the Chern classes of the universal class $e_v$ (Corollary 3.6). The general characterization of the homomorphism $\gamma_{g,v}$, in part 5 of Theorems 1.2, follows formally from the case $g = id$ and Corollary 3.6 (see Lemma 3.11). The multiplicative properties of the isomorphisms $\gamma_{g,v}$, listed in part 3 of Theorems 1.2, follow from this characterization of $\gamma_{g,v}$ and the $\gamma$-invariance of the universal class $e_v$ expressed in part 4 of Theorem 1.2 (Lemma 3.12).

We prove next parts 1 and 4 of Theorem 1.2 (restated in cohomological terms as Theorem 3.9). Let $\mathcal{G}$ be the groupoid (10). Denote by $\mathcal{G}_{OK} \subset \mathcal{G}$ the sub-groupoid, with the same objects, whose morphisms are those isometries $g \in \text{Hom}_\mathcal{G}((S_1, v_1, H_1), (S_2, v_2, H_2))$, for which $\gamma_{g,v_1}$ is a ring isomorphism, and $\gamma_{g,v_1} \otimes \bar{g}$ maps a universal class to a universal class (equation (5)). Lemma 3.12 implies that $\mathcal{G}_{OK}$ is indeed a sub-groupoid.

First we show that $\mathcal{G}_{OK}$ contains a sub-groupoid $\mathcal{G}_{\text{mon}} \subset \mathcal{G}$. We say that the two triples $(S_i, v_i, H_i)$, $i = 1, 2$, are deformation-equivalent, if the pairs $(S_i, v_i)$, $i = 1, 2$, are related by a deformation, satisfying mild conditions (Definition 6.2). The morphisms in $\text{Hom}_{\mathcal{G}_{\text{mon}}}((S_1, v_1, H_1), (S_2, v_2, H_2))$ are isometries $g : K_{\text{top}}(S_1) \rightarrow K_{\text{top}}(S_2)$, which arise as monodromy operators for such a deformation. The inclusion $\mathcal{G}_{\text{mon}} \subset \mathcal{G}_{OK}$ is proven in Lemma 6.5.

Next we show that $\mathcal{G}_{OK}$ contains another sub-groupoid $\mathcal{G}_{\text{Mukai}}$ of $\mathcal{G}$ (Lemma 5.6). The morphisms in $\text{Hom}_{\mathcal{G}_{\text{Mukai}}}((S_1, v_1, H_1), (S_2, v_2, H_2))$ arise in one of the following two ways (section 5.3):
1) There exists an equivalence functor $\Phi : D(S_1) \to D(S_2)$, taking $v_1$ to $v_2$, and $\Phi$ induces an isomorphism between the moduli spaces $M_{H_i}(v_1)$ and $M_{H_2}(v_2)$. We further assume that the functor $\Phi$ induces an orientation-preserving isometry $\phi$, i.e., that $\phi$ is in the kernel of the functor $\text{cov}$ defined below in (19). (The last assumption is automatically satisfied by Lemma 5.7, which depends on Theorems 1.2 and 1.6). In that case the isometry $\phi$ is in $\text{Hom}_{M_{\text{Mukai}}}((S_1, v_1, H_1), (S_2, v_2, H_2))$, by definition.

2) There exists an equivalence functor $\Phi : D(S_1) \to D(S_2)$, taking $v_1$ to $(v_2)^\vee$, and the composite functor $F \mapsto (\Phi(F))^\vee$ induces an isomorphism between the moduli spaces $M_{H_i}(v_1)$ and $M_{H_2}(v_2)$. We further assume that the functor $\Phi$ induces an orientation-preserving isometry $\phi$. (The last assumption is automatically satisfied by Lemma 5.7). In that case the isometry $g = D \circ \phi$ is in $\text{Hom}_{M_{\text{Mukai}}}((S_1, v_1, H_1), (S_2, v_2, H_2))$, by definition.

Two triples $(S, v_i, H_i)$, $i = 1, 2$, related by a morphism in $G_{\text{Mukai}}$, are said to be Mukai-equivalent. We conclude that $G_{\text{OK}}$ contains the sub-groupoid of $G$ generated by $G_{\text{mon}}$ and $G_{\text{Mukai}}$. By this we mean that any morphism in $G$, which is a composition of a finite set of morphisms in $G_{\text{mon}}$ and $G_{\text{Mukai}}$, is a morphism in $G_{\text{OK}}$.

Let $Ob(G)_n$ be the set of objects $(S, v, H)$ of $G$ satisfying the equality $2 + (v, v) = 2n$, $n \geq 0$. Yoshioka proved that any two triples $(S, v_i, H_i)$, $i = 1, 2$, in $Ob(G)_n$ are related by the equivalence relation generated by deformation and Mukai equivalences (Theorem 1.7). It follows that $\text{Hom}_{G_{\text{OK}}}(x, y)$ is non-empty, for any two objects $x$, $y$ in $Ob(G)_n$.

Identify the automorphisms in $G$ of $(S, v, H)$ with the subgroup $\Gamma_v$ of isometries of $K_{\text{top}}(S)$ which stabilize $v$. It remains to prove that $\text{Aut}_{G_{\text{OK}}}(S, v, H)$ contains $\Gamma_v$, for some object $(S, v, H) \in Ob(G)_n$, for each $n \geq 1$. This is done in two steps:

Step 1: We choose $v$ to be the class of the ideal sheaf of a length $n$ subscheme of $S$, so that $M_H(v)$ is the Hilbert scheme $S^{[n]}$ (regardless of $H$). We prove in this step that $\text{Aut}_{G_{\text{OK}}}(S, v, H)$ contains the index 2 subgroup $\Gamma_v^{\text{cov}} \subset \Gamma_v$, consisting of orientation preserving isometries (Definition 4.2).

Sub-step 1.1: The class $v$ belongs to the sublattice $U \subset K_{\text{top}}(S)$, consisting of classes $x$ with vanishing $c_1(x)$. $U$ is isometric to the rank 2 hyperbolic lattice, and $K_{\text{top}}(S)$ is the orthogonal direct sum $H^2(S, \mathbb{Z}) \oplus U$. Consequently, the isometry group $O(H^2(S, \mathbb{Z}))$ of $H^2(S, \mathbb{Z})$ is contained in the stabilizer $\Gamma_v$. Let $\Gamma_v^{\text{cov}}$ be the index two subgroup of $O(H^2(S, \mathbb{Z}))$ consisting of orientation preserving isometries. Every isometry in $\Gamma_v^{\text{cov}}$ is a monodromy operator for some deformation of the K3 surface $S$ (Corollary 6.7). This follows easily from the Torelli Theorem for K3 surfaces. Thus, $\Gamma_v^{\text{cov}}$ is contained in $\text{Aut}_{G_{\text{mon}}}(S, v, H)$ and in particular in $\text{Aut}_{G_{\text{OK}}}(S, v, H)$ (Theorem 6.1).

Sub-step 1.2: $\Gamma_v^{\text{cov}}$ is shown to be generated by $\Gamma_0^{\text{cov}}$ and reflections $\rho_u$ in $-2$ classes $u \in K_{\text{top}}(S)$ of topological line bundles with $c_1(u)$ a primitive class, and $c_1(u)^2 = 2n - 4$ (Proposition 8.6). Such a class $u$ belongs to $v^\perp$. Any two such reflections are conjugate under $\Gamma_0^{\text{cov}}$ (Lemma 8.1). The inclusion $\Gamma_v^{\text{cov}} \subset \text{Aut}_{G_{\text{OK}}}(S, v, H)$ would follow, once we find one such reflection in $\text{Aut}_{G_{\text{OK}}}(S, v, H)$. Proposition 7.1 exhibits such a reflection $\rho_u$ in $\text{Aut}_{G_{\text{OK}}}(S, v, H)$, for each $n$ (Theorem 7.7 exhibits another such reflection).

Step 2: It remains to prove that for each $n \geq 1$ there exists $(S, v, H) \in Ob(G)_n$ and $g \in \text{Aut}_{G_{\text{OK}}}(S, v, H)$, such that $\text{cov}(g) = 1$. This follows from Theorem 7.9 (see also
Theorem 7.10). This completes the proof of parts 1 and 4 of Theorem 1.2 (as well as Theorem 3.9).

Proof of part 2 of Theorem 1.2: Assume that \( g \in \text{Hom}_G((S_1, v_1, H_1), (S_2, v_2, H_2)) \) is an orientation-preserving Hodge isometry. Then \( g \) is induced by an auto-equivalence \( \Phi : D(S_1) \rightarrow D(S_2) \) (see [HLOY] or Theorem 5.1 below). Furthermore, \( \Phi \) is determined by an object \( F \) in the derived category of \( S_1 \times S_2 \) [Or1]. The class in \( K_{\text{alg}}[\mathcal{M}_{H_1}(v_1) \times \mathcal{M}_{H_2}(v)] \) representing \( \gamma(g, v_1) \) is constructed in terms of \( F \) in part 2 of Lemma 5.4.

Proof of part 6 of Theorem 1.2: The Chern classes \( c_k(TM_{H}(v)) \) are invariant under the duality involution \( D_{\mathcal{M}} \), given in (8), as they vanish for odd \( k \). Hence, part 6 holds for all morphisms \( g \) of \( \mathcal{G}_{\text{Mukai}} \). Part 6 holds for morphisms \( g \) of \( \mathcal{G}_{\text{mon}} \), as the Chern classes of \( TX \) determine global flat sections of the local system of integral cohomology over the base of any smooth deformation of a smooth and compact variety \( X \). Part 6 follows, since \( \mathcal{G}_{\text{Mukai}} \) and \( \mathcal{G}_{\text{mon}} \) generate \( G \). \( \square \)

Remark 1.9 The above proof shows that the groupoid \( \mathcal{G} \) is generated by the subgroupoids \( \mathcal{G}_{\text{mon}} \) and \( \mathcal{G}_{\text{Mukai}} \). This fact seems to be related to the conjecture that equivalences of derived categories of \( K3 \) surfaces induce orientation preserving isometries of Mukai lattices. We show that an equivalence of derived categories, which violates this conjecture, can not induce an isomorphism between any two moduli spaces of stable sheaves of dimension \( \geq 4 \) (Lemma 5.7).

1.3.2 Reduction of the proof of Theorem 1.6

We first prove that \( \text{mon}(\Gamma_v) \) is a subgroup of \( \text{Mon}(\mathcal{M}_H(v)) \). This follows from the proof of Theorem 1.2 as follows. The character \( \text{cov} \) of the isometry group of \( K_{\text{top}}(S) \), used in equation (14), extends to a functor

\[
\text{cov} : \mathcal{G} \rightarrow \mathbb{Z}/2\mathbb{Z},
\]  

(19)

where \( \mathbb{Z}/2\mathbb{Z} \) is the category with one object, whose automorphism group is \( \mathbb{Z}/2\mathbb{Z} \) (Remark 4.3). We use the functor \( \text{cov} \) to modify the functor \( \mathcal{H} \) in (11) and obtain a second functor

\[
\mathcal{H}_{\text{mon}} : \mathcal{G} \rightarrow \mathcal{A}.
\]  

(20)

Like \( \mathcal{H} \), the functor \( \mathcal{H}_{\text{mon}} \) sends the object \( (S, v, H) \) of \( \mathcal{G} \) to the algebra \( H^*(\mathcal{M}_H(v), \mathbb{Z})_{\text{free}} \). The isometry \( g \in \text{Hom}_\mathcal{G}((S_1, v_1, H_1), ((S_2, v_2, H_2)) \) is sent by \( \mathcal{H}_{\text{mon}} \) to the isomorphism

\[
(D_{\mathcal{M}_{H_2}(v_2)})^{\text{cov}(g)} \circ \gamma_{g,v_1} : H^*(\mathcal{M}_{H_1}(v_1), \mathbb{Z})_{\text{free}} \rightarrow H^*(\mathcal{M}_{H_2}(v_2), \mathbb{Z})_{\text{free}},
\]  

(21)

which is denoted \( \text{mon}(g) \). The representation \( \text{mon} \) given in (14) is thus the restriction of \( \mathcal{H}_{\text{mon}} \) to the automorphism group \( \text{Aut}_\mathcal{G}((S, v, H)) \). The monodromy operator \( \text{mon}(g) \) corresponds to the class

\[
\text{mon}(g, v_1) := \begin{cases} 
\gamma(g, v_1) & \text{if } \text{cov}(g) = 0, \\
\epsilon_m ( -\pi_{13} ; \{ \pi_{12} [ (1 \otimes Dg)(e_{v_1}) ] \otimes \pi_{23}^* (e_{v_2}) \}) & \text{if } \text{cov}(g) = 1,
\end{cases}
\]  

(22)
where we kept the notation used in (3).

The proof of Theorem 1.2 shows, that each isomorphism \( \text{mon}(g) \) in (21) is the composition of a finite set of isomorphisms, each either induced by an isomorphism of moduli spaces corresponding to a morphism in \( \mathcal{G}_{\text{Mukai}} \), or a monodromy operator corresponding to a morphism in \( \mathcal{G}_{\text{mon}} \). An automorphism of an irreducible holomorphic symplectic manifold induces a monodromy operator (Lemma 6.6). It follows that each isomorphism \( \text{mon}(g) \) in (21) is a monodromy operator.

We use results of Verbitsky to prove that \( \text{mon}(\Gamma_v) \) is a normal subgroup of finite index (Lemma 4.7).

\[ \square \]

### 1.3.3 The structure of the rest of the paper

We summarize the highlights of each of the following sections. More detailed summaries appear at the beginning of each section. In section 2 we define the notion of a local monodromy operator. We then relate and discuss two open questions: 1) a local Torelli type question 2.3, and 2) the question whether the monodromy operator \( \text{mon}(g) \) of Theorem 1.6 is local, whenever \( g \) is a Hodge-isometry. Partial results are summarized in Theorem 2.5. Section 3 begins with background material (sections 3.1, 3.2, and 3.3). We then restate Theorem 1.2, about symmetries of the collection of moduli spaces of sheaves on \( K^3 \) surfaces, without using the \( K \)-theoretic universal classes (Theorem 3.9). We also prove that the isomorphisms \( \gamma_{g,v} \) compose as stated in part 3 of Theorem 1.2 (Lemma 3.12). In section 4 we relate the representation \( \text{mon} \) of Theorem 1.6 to a Hodge-theoretic representation of Verbitsky (sections 4.2 and 4.6). We use Verbitsky’s results to show that the reflection group \( W \) of \( H^2(\mathcal{M}(v), \mathbb{Z}) \) maps via the homomorphism \( \mu \) of Corollary 1.8 onto a normal subgroup of finite index in the full monodromy group \( \text{Mon}(\mathcal{M}(v)) \) (Lemma 4.7). We bound the index \( [\mu(W) : \text{Mon}(\mathcal{M}(v))] \) in section 4.4 using an efficient set of generators for the cohomology ring \( H^*(\mathcal{M}(v)) \) and the relation of the Chern classes of \( \mathcal{M}(v) \) to these generators (Lemma 4.9). In section 5 we lift the topological formula (22) for the monodromy operator \( \text{mon}(g) \), to a Chow theoretic formula, whenever \( \text{mon}(g) \) maps the weight 2 Hodge structure of \( \mathcal{M}(v) \) to that of \( \mathcal{M}(g(v)) \) (Lemma 5.4). This lift is carried out using known results about equivalences of derived categories of K3 surfaces. In section 5.3 we define the groupoid \( \mathcal{G}_{\text{Mukai}} \), whose morphisms come from Fourier-Mukai functors preserving the stability of all sheaves in a moduli space. In section 6 we prove the main Theorems 1.2 and 1.6 in the case of monodromy operators of \( S^{[n]} \), arising from deformations of the surface \( S \). In section 7 we verify Theorems 1.2 and 1.6 in examples, which do not arise from deformations of the surface \( S \). The two key examples are Proposition 7.1 and Theorem 7.9, which suffice for the proof of Theorems 1.2 and 1.6. In section 8 we find a set of generators for the stabilizer \( \Gamma_v \) of the Mukai vector \( v \) of \( S^{[n]} \). This set of generators consist of isometries in \( \Gamma_v \), for which Theorems 1.2 and 1.6 were verified earlier.

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In the first submitted version of this paper, Theorem 1.2 was a conjecture and Theorem 1.6 was proven using local monodromy operators (Theorem 2.5). The proof relied on Theorems 7.7 and 7.10 below, which are both proven in [Ma3]. The current proofs of Theorems 1.2 and 1.6 are independent of Theorems 7.7 and 7.10 (and thus of the results of [Ma3]). We thank one of the referees for suggesting greater reliance on Yoshioka’s results.

2 Torelli questions and local monodromy operators

Let \( X \) be an irreducible holomorphic symplectic manifold deformation equivalent to the Hilbert scheme \( S^{[n]} \) of length \( n \) subschemes of a \( K3 \) surface \( S \). The homomorphism \( \mu \) in Corollary 1.8 lifts many isometries of \( H^2(X, \mathbb{Z}) \) to monodromy operators in \( \text{Aut}H^*(X, \mathbb{Z})_{\text{free}} \). When the isometry is of Hodge type, another such lift is the local monodromy operator (Definition 2.4), defined when an affirmative answer is known to a strong version of the local Torelli Question 2.3. We define this second lift, conjecture the equality of the two lifts, and describe examples where equality is known to hold (Theorem 2.5).

**Conjecture 2.1** The restriction homomorphism \( \text{Mon}(X) \to \text{Mon}^2(X) \) is injective.

The conjecture implies that the monodromy group \( \text{Mon}([\mathcal{M}_H(v)]) \) is equal to the image \( \text{mon}(\Gamma_v) \) of the homomorphism (14) (use also Corollary 1.8 and the containment \( \text{Mon}^2 \subset \mathcal{W} \) proven in [Ma4]). The conjecture holds when \( \dim(X) = 2 \) or 4, because then \( H^2(X, \mathbb{Q}) \) generates the cohomology ring of \( X \). An upper-bound for the order of the kernel of the homomorphism \( \text{Mon}(X) \to \text{Mon}^2(X) \), when \( \dim(X) > 4 \), is provided in section 4.4. Conjecture 2.1 is motivated by the following result of Beauville and Verbitsky:

**Proposition 2.2** Let \( X \) be an irreducible holomorphic symplectic manifold deformation equivalent to the Hilbert scheme \( S^{[n]} \), \( n \geq 1 \), of a \( K3 \) surface \( S \). A holomorphic automorphism of \( X \), which acts as the identity on \( H^2(X, \mathbb{Z}) \), is the identity automorphism.

**Proof:** The case \( X = S^{[n]} \) was proven by Beauville ([B3] Proposition 10). The statement follows, for deformations of \( S^{[n]} \), by Corollary 6.9 in [KV].

We describe below explicit examples of pairs, of a moduli space \( \mathcal{M}_H(v) \) and a monodromy operator \( \tilde{g} \in \text{Aut}[H^*(\mathcal{M}_H(v), \mathbb{Z})] \), inducing an isometry \( g \) in \( \mathcal{W} \subset \text{Mon}^2(\mathcal{M}_H(v)) \). In these examples the equality \( \tilde{g} = \mu(g) \), implied by Conjecture 2.1, is either not known to the author (in the very interesting Example 2.6), or is proven with further effort in [Ma3] (Theorems 7.7 and 7.10 below).
Let \( g \) be an isometry in the subgroup \( \mathcal{W} \subset OH^2(X, \mathbb{Z}) \) given in (17). Assume that \( g \) preserves the Hodge structure. Then it may be possible to lift \( g \) to a local monodromy operator in \( \text{Mon}(X) \). We formulate this statement as Question 2.3 below. Recall that the differential of the period map is an isomorphism \( H^1(X, TX) \to H^{2,0}(X)^* \otimes H^{1,1}(X) \) \([B1]\). Hence, \( g \) acts on the infinitesimal deformation space of \( X \). If \( g \) has finite order, then we can choose a \( g \)-invariant simply connected subset \( B \subset H^1(X, TX) \), open in the complex topology, and a local universal family \( \pi : \mathcal{X} \to B \), with special fiber \( X \) over the point \( 0 \in B \). If the order of \( g \) is infinite, we may not assume \( B \) to be \( g \)-invariant, but the discussion below generalizes, replacing \( B \) by the union \( B' \cup g(B') \) of an open subset \( B' \subset B \), containing 0, such that \( g(B') \) is contained in \( B \). We assume, for simplicity, that \( g \) has finite order. The local systems \( R^2_{\pi_*} \mathbb{Z} \) and \( g^* R^2_{\pi_*} \mathbb{Z} \) over \( B \) are trivial, as \( B \) is assumed simply connected. The trivializations conjugate any flat isomorphism \( R^2_{\pi_*} \mathbb{Z} \cong g^* R^2_{\pi_*} \mathbb{Z} \) to an automorphism of \( H^2(X, \mathbb{Z}) \). An affirmative answer to the following question would constitute a version of local Torelli, stronger than the known local Torelli theorem \([B1]\):

**Question 2.3** Is there always a \( g \)-invariant connected open neighborhood \( U \) of 0 in \( B \), a proper \( g \)-invariant analytic subset \( A \subset U \), and a fiber preserving isomorphism \( f : \mathcal{X}|_{U \setminus A} \to g^* \mathcal{X}|_{U \setminus A} \) between the restrictions of the families \( \mathcal{X} \) and \( g^* \mathcal{X} \) to \( U \setminus A \), inducing an isomorphism \( (R^2_{\pi_*} \mathbb{Z})|_{U \setminus A} \cong (g^* R^2_{\pi_*} \mathbb{Z})|_{U \setminus A} \), which is conjugated to the automorphism \( g \) of \( H^2(X, \mathbb{Z}) \) via the trivializations?

When the answer is affirmative we set \( U_0 := U \setminus (A \cup U^g) \), where \( U^g \) is the subset of fixed-points. We get the quotient family \( (\mathcal{X}|_{U_0})/f \to U_0/g \) and \( g \) generates its monodromy subgroup in \( \text{Mon}^2(X) \). The induced isomorphism of local systems \( f_* : (R^2_{\pi_*} \mathbb{Z})_{\text{free}} \to g^* (R^2_{\pi_*} \mathbb{Z})_{\text{free}} \) extends over the whole of \( U \), due to the triviality of both local systems. Denote by \( \tilde{g} \in \text{Aut}[H^*(X, \mathbb{Z})] \) the value of \( f \) at the fiber over the point \( 0 \in U \). Then \( \tilde{g} \) is a monodromy operator, which restricts to \( g \) on \( H^2(X, \mathbb{Z}) \) (Lemma 6.6). Conjecture 2.1 implies that \( \tilde{g} \) is equal to the monodromy operator \( \mu(g) \) in Corollary 1.8

\[
\tilde{g} = \mu(g). \tag{23}
\]

If the isomorphism \( f \) satisfying the conditions of Question 2.3 exists, then it is unique by Proposition 2.2. The uniqueness justifies the following:

**Definition 2.4** The automorphism \( \tilde{g} \in \text{Aut}[H^*(X, \mathbb{Z})] \) is the local monodromy operator, which restricts to \( g \) on \( H^2(X, \mathbb{Z}) \). Let \( \ell \) be an automorphism of \( H^*(X, \mathbb{Z}) \) and denote its restriction to \( H^2(X, \mathbb{Z}) \) by \( \ell_2 \). We call \( \ell \) a local monodromy operator, if the local monodromy operator, whose restriction to \( H^2(X, \mathbb{Z}) \) is \( \ell_2 \), exists and is equal to \( \ell \).

Let \( \mathcal{M}_H(v) \) be a moduli space as in Corollary 1.8. An isometry of \( H^2(\mathcal{M}_H(v), \mathbb{Z}) \) is called a surface monodromy operator, if it arises from deformations of \( \mathcal{M}_H(v) \) as a moduli space of sheaves on a \( K3 \) surface (Definition 6.2). When \( c_1(v) \) is a multiple of an ample class, then surface monodromy operators are in a natural bijection with signed isometries.
of $H^2(S, \mathbb{Z})$, leaving invariant the class $c_1(v)$ (Theorem 6.1). If $g$ is a surface monodromy operator, preserving the Hodge structure of $H^2(M_H(v), \mathbb{Z})$, then an affirmative answer to Question 2.3 follows from the Torelli Theorem for $K3$ surfaces, and the equality (23) follows from the proof of Theorem 1.6.

The examples in which equality (23) was proven suffice to conclude the following Theorem. Let $n \geq 2$, $W$ the reflection group (17) of $H^2(S^{[n]}, \mathbb{Z})$, and and $\mu : W \to \text{Mon}(S^{[n]})$ the homomorphism in Corollary 1.8.

**Theorem 2.5** [Ma3] The image $\mu(W)$ in $\text{Mon}(S^{[n]})$ is generated by monodromy operators, inducing reflections in $+2$ or $-2$ classes of $H^2(S^{[n]}, \mathbb{Z})$, each of which is a local monodromy operator for a suitable choice of a complex structure on the $K3$ surface $S$.

Theorem 2.5 follows from Theorems 7.7 and 7.10 below (both proven in [Ma3]) via Proposition 8.6 and Theorem 6.1 (see Steps 1 and 2 of the proof of Theorem 1.2).

O’Grady conjectured an affirmative answer to Question 2.3, when $g$ is a reflection (17) by a Chern class $c_1(L)$ of a line-bundle $L$ with $(c_1(L), c_1(L)) = 2$ [OG2]. He further conjectured, that after a small deformation of the pair $(X, L)$, the reflection $g$ is induced by a regular involution of $X$ (the I-conjecture). In dimension 2 the simplest example is the regular Galois involution of a double cover of $\mathbb{P}^2$, branched along a sextic.

Reflections by negative effective classes can not be induced by a regular involution of $X$. Following is a simple proof. Assume that the line bundle $L$ is effective, $(c_1(L), c_1(L)) < 0$, and the isometry $g$ is the reflection with respect to $c_1(L)$. Then $g(c_1(L)) = -c_1(L)$. If $\kappa$ is a Kähler class, then $(\kappa, c_1(L)) > 0$, because $c_1(L)$ is effective ([Hu] 1.11). Hence, $g(\kappa)$ is not a Kähler class and $g$ is not induced by a regular involution. A positive answer to Question 2.3 does imply, that the local monodromy $\tilde{g}$ is induced by a self-correspondence in $X \times X$ (see the proof of [Hu] Theorem 4.3). O’Grady’s I-Conjecture is replaced by the question:

**Question:** Describe the structure of the self-correspondence for a generic pair $(X_t, L_t)$, $t \in U^9$, deformation equivalent to $(X, L)$. Use the description to prove the equality (23).

We consider two sequences of reflections by negative Hodge classes, which are monodromy operators but not surface monodromy operators (Theorem 7.7 and Example 2.6).

**Example 2.6** Let $S$ be a $K3$ surface and $n$ an integer $\geq 2$. The second cohomology $H^2(S^{[n]}, \mathbb{Z})$ is isometric to $H^2(S, \mathbb{Z}) \oplus \text{span}_{\mathbb{Z}}\{\delta\}$, where $\delta$ is half the class of the big diagonal in $S^{[n]}$ and $(\delta, \delta) = 2 - 2n$. Let $v \in K_{top}(S)$ be the class of the ideal sheaf of a length $n$ subscheme and $D \in \Gamma_v$ the duality involution (7). Then the monodromy operator $\text{mon}(D)$, given in (14), acts on $H^2(S^{[n]}, \mathbb{Z})$ as the reflection $\rho_\delta(x) := x + [(x, \delta)/(n - 1)]\delta$. Let $\pi : S^{[n]} \to S^{(n)}$ be the Hilbert-Chow morphism from the Hilbert scheme to the symmetric product. The fiber product $Z := [S^{[n]} \times_S S^{[n]}] \subset [S^{[n]} \times S^{[n]}]$ is reducible but reduced of pure dimension $2n$ (part 1 of Lemma 2.7). $Z_*$ is the local monodromy operator restricting to $\rho_\delta$ on $H^2(S^{[n]})$ (Definition 2.4), by Lemma 2.7 part 2. Conjecture
2.1 implies the equality $Z_* = \text{mon}(D)$. Equivalently, $Z$ is Poincare dual to the class $\text{mon}(D, \nu)$ given in (22).

The equality $Z_* = \text{mon}(D)$ would follow (by part 5 of Theorem 1.2), independently of Conjecture 2.1, if one proves instead that $D_{S[n]} \circ Z_*$ and $D$ satisfy equation (6). Let $e$ be the class in $K_{\text{top}}(S^{[n]} \times S)$ of the ideal sheaf of the universal subscheme. Denote the dual class by $e^\vee$. Equation (6) translates in our case to the equality

$$(Z_1 \otimes 1)(e) \ = \ e^\vee,$$

where $Z_i \in \text{End}[K_{\text{top}}(S^{[n]})]$ is given by $Z_i(x) = \pi_2(\pi_i^1(x) \otimes \mathcal{O}_Z)$, and $\pi_i$ is the projection from $S^{[n]} \times S^{[n]}$ onto the $i$-th factor, $i = 1, 2$ (compare with the equality in part 2 of Theorem 7.7).

**Lemma 2.7**

1. The fiber product $Z := [S^{[n]} \times_S S^{[n]}]$ is reduced, of pure dimension $2n$, and its irreducible components are in bijection with ordered partitions of $n$.

2. $Z_*$ is the local monodromy operator restricting to $\rho_\delta$ on $H^2(S^{[n]})$ (Definition 2.4).

**Proof:** 1) Let $\lambda := (n_1 \leq n_2 \leq \ldots \leq n_k)$ be a partition of $n$ and $S_\lambda^{(n)}$ the corresponding locally closed subset of $S^{(n)}$. The dimension of $S_\lambda^{(n)}$ is $2k$. The fibers of $\pi$ over points of $S_\lambda^{(n)}$ are reduced and irreducible of dimension $n - k$, by Proposition 2.10 in [Hai]. Hence each partition contributes to the fiber product a $2n$-dimensional irreducible component. Each fiber of the two natural projections $p_\iota : Z \to S^{[n]}$ is isomorphic to a fiber of $\pi$ and is hence reduced and irreducible. $Z$ is reduced, since $p_\iota$ is a projective morphism with reduced fibers from $Z$ to the reduced scheme $S^{[n]}$.

2) The homomorphism $Z_* \in \text{End}(H^*(S^{[n]}, \mathbb{Z}))$ and the involution $\text{mon}(D)$ both act via the reflection $\rho_\delta$ on $H^2(S^{[n]}, \mathbb{Z})$. $\rho_\delta$ acts as an involution of the local deformation space $B$ of $S^{[n]}$, fixing the divisor $\text{Hilb} \subset B$ of deformations as Hilbert schemes of Kähler $K3$ surfaces. Let $\mathcal{X} \to B$ be the local universal family and $q : B \to B/\rho_\delta$ the quotient morphism. An affirmative answer to Question 2.3 is known in the case of $\rho_\delta$, since the Hilbert-Chow morphism is a contraction of type $A_1$ as in Namikawa’s work [Nam]. The quotient $B/\rho_\delta$ is the local deformation space of $S^{(n)}$, the generic fiber of the universal family $\overline{\mathcal{X}} \to B/\rho_\delta$ is smooth, and $\mathcal{X}$ is a resolution of the pullback of $\overline{\mathcal{X}}$ to $B$. It follows that $\mathcal{X}$ and $\rho_\delta^*\mathcal{X}$ restrict to isomorphic families over an open subset $U$ of $B$. The open subset $U$ is in fact the complement $B \setminus \text{Hilb}$ (possibly after replacing $B$ by a smaller open neighborhood of $0 \in B$, see the proof of Claim 3 in [Nam]). Let $Z \subset \mathcal{X} \times_B \rho_\delta^*\mathcal{X}$ be the closure of the graph of the isomorphism. The fiber $Z_0$ of $Z$ over $0 \in B$ is a subscheme of $Z$, since $Z$ is the reduced induced subscheme of the fiber product $[\mathcal{X} \times_S \overline{\mathcal{X}}] \rho_\delta^*\mathcal{X}$ and $Z$ is the fiber of the latter over $0$. $Z$ is reduced and hence isomorphic to $Z_0$. Consequently $Z_*$ is the local monodromy operator restricting to $\rho_\delta$ on $H^2(S^{[n]})$. 

\[\square\]
3 Automorphisms of the cohomology ring

We construct in section 3.1 a canonical normalization \( u_v \) of the Chern character of a universal (possibly twisted) sheaf over moduli spaces \( \mathcal{M}_H(v) \) of dimension \( \geq 4 \) (equation (27)). In section 3.2 we restate the relationship (Theorem 1.7), between the Mukai lattice of a K3 surface \( S \) and the weight 2 Hodge structure of moduli spaces of sheaves on \( S \), using the integral cohomology \( H^*(S, \mathbb{Z}) \), rather than the \( K \)-ring \( K_{top}(S) \). We recall in section 3.3, that the Künneth factors, of the Chern classes of a universal sheaf, are generators for the cohomology ring of a moduli space of sheaves on a K3 or abelian surface. In section 3.4 we restate Theorem 1.2, relating the cohomology rings of pairs of moduli spaces of sheaves on K3 surfaces. The new statement (Theorem 3.9) is in terms of the class \( u_v \) rather than the integral class \( e_v \). In section 3.5 we prove that the isomorphisms \( \gamma_{g,v} \) compose as stated in part 3 of Theorem 1.2.

3.1 Universal sheaves

Let \( S \) be a K3 surface, \( \mathcal{L} \) an ample line bundle on \( S \), and \( v \in K_{top}(S) \) an effective class (Definition 1.1). Let \( \mathcal{M} := \mathcal{M}(v) \) be the moduli space of \( \mathcal{L} \)-stable sheaves with class \( v \). We use the \( \mathcal{L} \)-stability of sheaves due to Gieseker, Maruyama, and Simpson [HL]. Recall the construction of semi-universal families over \( \mathcal{M} \times S \) ([Mu2] Theorem A.5). We start with an open covering \( \mathcal{U} := \{U_i\} \) of \( \mathcal{M} \), in the complex or étale topologies, local universal families \( \mathcal{E}_i \) over \( U_i \times S \), and isomorphisms

\[
g_{ij} : (\mathcal{E}_j)|_{U_{ij}\times S} \longrightarrow (\mathcal{E}_i)|_{U_{ij}\times S} \tag{24}
\]

over the intersections \( U_{ij} := U_i \cap U_j \). The co-boundary \( (\delta g)_{ijk} := g_{ij}g_{jk}g_{ki} \) consists of central automorphisms of \( (\mathcal{E}_i)|_{U_{ijk}\times S} \), since each \( \mathcal{E}_i \) is a family of stable, and hence simple, sheaves. It follows, that \( (\delta g)_{ijk} \) must be the multiplication by the pull-back \( \delta \) to \( \mathcal{M} \times S \) of a 2-cocycle \( \alpha \in Z^2(\mathcal{U}, \mathcal{O}_M) \). We get the direct image vector bundles \( V_i := p_*(\mathcal{E}_i \otimes \mathcal{L}^n) \) over \( U_i \), upon a choice of a sufficiently high power \( n \). The isomorphisms \( g_{ij} \) induce gluing transformations \( \psi_{ij} \) among the \( V_i \)'s and \( (\delta \psi)_{ijk} \) is multiplication by \( \alpha \). Note, that the \( \rho \)-th power of \( \alpha \) is the co-boundary \( \delta(\hat{\rho} \psi) \), where \( \rho \) is the common rank of all the \( V_i \). The families \( \mathcal{E}_i \otimes V_i^* \) glue naturally to a global semi-universal family \( \mathcal{F} \) of similitude \( \rho \). The vector bundles \( V_i^{\otimes \rho} \otimes (\hat{\rho} V_i^*) \) and their subbundles \( \mathrm{Sym}^\rho(V_i) \otimes (\hat{\rho} V_i^*) \) glue to a global vector bundle \( W \) and its subbundle \( W_+ \).

The projective bundles \( \mathbb{P}V_i \) glue naturally to a global \( \mathbb{P}^{\rho-1} \)-bundle \( p : \mathbb{P} \to \mathcal{M} \). We get a universal quotient sheaf \( \mathcal{E} \) of \( p^*\mathcal{F} \). The sheaf \( \mathcal{E} \) restricts as \( E \otimes \mathcal{O}(1) \) to the \( \mathbb{P}^{\rho-1} \times S \) fiber over \( E \in \mathcal{M} \). We have also a line-bundle \( \mathcal{O}_{\mathbb{P}}(\rho) \) over \( \mathbb{P} \), which restricts as \( \mathcal{O}(\rho) \) to each fiber of \( p \) and such that \( p_*\mathcal{O}(\rho) \) is \( W_+^\rho \). There exists a unique class

\[
ch(\mathcal{E}) \tag{25}
\]
in $H^*(\mathcal{M} \times S, \mathbb{Q})$ satisfying the following equality

$$(p \times id_S)^* ch(\mathcal{E}) = ch(\mathcal{E}) \exp[c_1(\mathcal{O}(-\rho))/\rho],$$

(26)

since the right hand side restricts to a trivial class on each fiber of $\mathbb{P} \times S$ over a point of $\mathcal{M} \times S$. The classes $ch(\mathcal{E})$ and $\mathcal{E}$ are canonical, up to a product by the pullback of a class $\exp(\ell) \in H^*(\mathcal{M}, \mathbb{Q})$, for $\ell \in H^{1,1}(\mathcal{M}, \mathbb{Q})$.

We introduce next a canonical normalization of the Chern character of a universal sheaf $\mathcal{E}$ over $\mathcal{M}_L(v) \times S$. Assume, that $dim \mathcal{M}_L(v) > 2$. Let $\eta_v \in H^2(\mathcal{M}_L(v), \mathbb{Q})$ be the class satisfying $(v, v) \cdot \eta_v = \theta_v(v)$, where $\theta_v$ is given in (16). Set

$$u_v := ch(\mathcal{E}) \cdot \pi_1^* \exp(\eta_v) \cdot \pi_2^* \sqrt{td_S}. \tag{27}$$

Then $u_v$ is a natural class in $H^*(\mathcal{M}_L(v) \times S, \mathbb{Q})$, which is independent of the choice of a universal sheaf. The appearance of the factor $\sqrt{td_S}$ is explained in section 3.2. If a universal sheaf $\mathcal{E}$ does not exist over $\mathcal{M}_L(v)$, replace $ch(\mathcal{E})$ in (27) by the class (25). Note, that $u_v$ is the normalization of the Chern character of the universal sheaf, which produces the equality $\theta_v(v) = 0$, when $u_v$ is substituted for $ch(\mathcal{E})$ in the cohomological translation (33) of the definition (16) of $\theta_v$ (because $(v, v) = -\chi(v^* \otimes v)$).

**Twisted sheaves**: The data $(\mathcal{E}_i, g_{ij})$ above is an example of a twisted sheaf.

**Definition 3.1** Let $X$ be a scheme or a complex analytic space, $\mathcal{U} := \{U_i\}_{i \in I}$ a covering, open in the complex or étale topology, and $\alpha \in Z^2(\mathcal{U}, \mathcal{O}_X^*)$ a Čech 2-cocycle. An $\alpha$-twisted sheaf consists of sheaves $\mathcal{E}_i$ of $\mathcal{O}_{U_i}$-modules over $U_i$, for all $i \in I$, and isomorphisms $g_{ij}$ as in (24) satisfying the conditions:

1. $g_{ii} = id$,
2. $g_{ij} = g_{ji}^{-1}$,
3. $g_{ij}g_{jk}g_{ki} = \alpha_{ijk} \cdot id$.

The $\alpha$-twisted sheaf is coherent, if the $\mathcal{E}_i$ are.

The class of $\alpha$-twisted sheaves, together with the obvious notion of homomorphism, is an abelian category, denoted by $\mathfrak{Mod}(X, \alpha)$, and referred to as the category of $\alpha$-twisted sheaves. The analogous category of $\alpha$-twisted coherent sheaves is denoted by $\mathfrak{Coh}(X, \alpha)$. The categories $\mathfrak{Mod}(X, \alpha)$ and $\mathfrak{Coh}(X, \alpha)$ depend, up to equivalence, only on the class of $\alpha$ in the cohomology group $H^2(X, \mathcal{O}_X^*)$, using the analytic or étale topology [Ca]. We will always assume, that $\alpha$ represents a class in the Brauer group of $X$, i.e., in the image of the connecting homomorphism $H^1(X, \mathbb{PGL}(n)) \to H^2(X, \mathcal{O}_X^*)$ of the short exact sequence

$$0 \to \mathcal{O}_X^* \to \mathbb{GL}(n) \to \mathbb{PGL}(n) \to 0,$$

for some $n \geq 2$. The class $[\alpha]$, of the cocycle $\alpha$ of the twisted universal sheaf, belongs to the Brauer group, since it is the image of the class of the projective bundle $\mathbb{P}$ defined above over the moduli space $\mathcal{M}$. The bounded derived category of complexes of $\alpha$-twisted sheaves on $X$ with coherent cohomology is denoted by $D_{coh}^b(X, \alpha)$ (see [Ca] for its construction).
The twisted universal sheaf defines an object in $D^b_{coh}(\mathcal{M}_H(v) \times S, \pi_1^* \alpha)$, which we denote by $\mathcal{E}_v$. We associate next to $\mathcal{E}_v$ a universal class $e_v$ in $K_{top}(\mathcal{M}_H(v) \times S)$. The $\alpha$-twisted universal sheaf $\mathcal{E}_v$ determines a class in the $K$-group of $\alpha$-twisted holomorphic vector bundles $K_{hol}(\mathcal{M}_H(v) \times S)_\alpha$. The latter, and its topological analogue $K_{top}(\mathcal{M}_H(v) \times S)_\alpha$, are defined as in the untwisted case. $K_{top}(\mathcal{M}_H(v) \times S)_\alpha$ depends only on the image $\delta(\alpha)$ of $\alpha$ in $H^3(\mathcal{M}_H(v) \times S, \mathbb{Z})$, under the connecting homomorphism of the exponential sequence. We proved that the cohomology $H^i(\mathcal{M}_H(v), \mathbb{Z})$ vanishes for odd $i$ [Ma4]. It follows that $\delta(\alpha)$ is trivial and $K_{top}(\mathcal{M}_H(v) \times S)_\alpha$ is isomorphic to the untwisted group $K_{top}(\mathcal{M}_H(v) \times S)$, canonically up to tensorization of the latter with a topological line-bundle.

**Definition 3.2** The universal class $e_v$ is the image of $\mathcal{E}_v$ under the composition

$$K_{hol}(\mathcal{M}_H(v) \times S)_\alpha \to K_{top}(\mathcal{M}_H(v) \times S)_\alpha \to K_{top}(\mathcal{M}_H(v) \times S).$$

**Remark 3.3** When $v$ is a primitive class, then the co-cycle $\alpha$ maps to a co-boundary, once we consider Čech cohomology with coefficients in the sheaf of complex valued smooth functions [Ma4]. Furthermore, the class of the semi-universal sheaf $\mathcal{F}$ in $K_{top}(\mathcal{M} \times S)$ is the product of a class $e_v$ and the pull-back of the class of a topological vector bundle $B$ on $\mathcal{M}$ of rank $\rho$ satisfying $W^* \cong B^{\otimes \rho} \otimes (\wedge B^*)$. The Chern character $ch(e_v)$ and the class $ch(\mathcal{E})$ in (25) are related by the equation

$$ch(\mathcal{E}) \exp[c_1(B)/\rho] = ch(e_v). \quad (28)$$

The proof uses the equation $[ch(\mathcal{F})/ch(\mathcal{E})]^\rho = ch(W^*)$ proven in section 3 of [Ma2].

### 3.2 The Mukai pairing on $H^*(S, \mathbb{Z})$

Let $S$ be a K3 surface and $\mathcal{L}$ an ample line bundle on $S$. The Todd class of $S$ is $1 + 2\omega$, where $\omega$ is the fundamental class in $H^4(S, \mathbb{Z})$. Its square root is $1 + \omega$. Given a coherent sheaf $F$ on $S$ of rank $r$, we denote by

$$v(F) := ch(F)\sqrt{td_S} = (r, c_1(F), \chi(F) - r)$$

its **Mukai vector** in

$$H^*(S, \mathbb{Z}) = H^0(S, \mathbb{Z}) \oplus H^2(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z}).$$

We used above the standard isomorphisms $H^0(S, \mathbb{Z}) \cong \mathbb{Z}$ and $H^4(S, \mathbb{Z}) \cong \mathbb{Z}$ to write the corresponding entries of $v(F)$ as integers. Mukai endowed the cohomology group $H^*(S, \mathbb{Z})$ with a weight 2 polarized Hodge structure. The bilinear form is

$$\langle (r', c', s'), (r'', c'', s'') \rangle = c' c'' - r' s'' - r'' s', \quad \text{or equivalently},$$

$$\langle \alpha, \beta \rangle = -\int_S \alpha \wedge \beta,$$
where $\alpha^v$ is the image of $\alpha$ under the duality operator

$$D : H^*(S,\mathbb{Z}) \rightarrow H^*(S,\mathbb{Z})$$

acting by $-1$ on the second cohomology (sending the Mukai vector $(r,c_1,s)$ to $(r,-c_1,s)$).

The Hodge filtration is induced by that of $H^2(S,\mathbb{Z})$. In other words, $H^{2,0}(S)$ is defined to be also the (2,0)-subspace of the complexified Mukai lattice. The isomorphism

$$K_{\text{top}}(S) \rightarrow H^*(S,\mathbb{Z}),$$

pulls back the Mukai pairing to the pairing in equation (1). Riemann-Roch’s Theorem yields the equation

$$\langle E, F \rangle := -\chi(E^\vee, F) := -\sum (-1)^i \dim \text{Ext}^i(E, F),$$

for the classes of coherent sheaves $E$ and $F$ on $S$.

Denote by $\mathcal{M}(v) := \mathcal{M}_L(v)$ the moduli space of $L$-stable sheaves with Mukai vector $v$ of non-negative rank $r(v) \geq 0$. Mukai constructed the natural homomorphism

$$\theta_v : v^\perp \rightarrow H^2(\mathcal{M}_L(v),\mathbb{Z})$$

given by

$$\theta_v(x) := \frac{1}{\rho} \left[ p_{\mathcal{M}_v}((ch(\mathcal{E}) \cdot \sqrt{td_S} \cdot \pi_S^*(x^v))) \right]_1,$$

where $\mathcal{E}$ is a quasi-universal family of similitude $\rho$. Note, that the homomorphism $\theta_v$ extends to the whole Mukai lattice, but the extension depends on the choice of $\mathcal{E}$.

The isometry (30) conjugates Mukai’s homomorphism (15) to the homomorphism (32). Indeed, the homomorphism $\theta_v$ does not change, if instead of the Chern character of a quasi-universal family $\mathcal{E}$ we let $ch(\mathcal{E})$ be the class (25), considered with similitude 1 (by the projection formula). The equality (28) allows us to further replace $ch(\mathcal{E})$ by $ch(e_v)$ when $v$ is primitive. The homomorphism $\theta_v$ in equation (32) does not change, if we replace $ch(\mathcal{E})\sqrt{td_S}$ in equation (33) by $u_v$.

The moduli space $\mathcal{M}_L(v)$, of $L$-stable sheaves with Mukai vector $v$, depends on the polarization $L$.

**Definition 3.4** An ample line bundle $L$ is said to be $v$-suitable, if every $L$-semi-stable sheaf with Mukai vector $v$ is $L$-stable.

**Throughout the paper, we will consider only moduli spaces $\mathcal{M}_L(v)$, where $L$ is $v$-suitable.**

We translate next the definition of the groupoid (10), replacing $K_{\text{top}}(S)$ by $H^*(S,\mathbb{Z})$, using the isomorphism (30). The objects of the groupoid

$$\mathcal{G}$$

(34)
are triples \((S, v, \mathcal{L})\) consisting of a \(K3\) surface \(S\), a primitive class \(v = (r, c, s)\) in \(H^*(S, \mathbb{Z})\) with \(c \in H^{1,1}(S, \mathbb{Z})\), and a \(v\)-suitable ample line bundle \(\mathcal{L}\) on \(S\). A morphism \(g \in \text{Hom}_G((S_1, v_1, \mathcal{L}_1)(S_2, v_2, \mathcal{L}_2))\) is an isometry \(g : H^*(S_1, \mathbb{Z}) \to H^*(S_2, \mathbb{Z})\) satisfying \(g(v_1) = v_2\).

### 3.3 The class of the diagonal

We will study automorphisms of the cohomology ring of a moduli \(\mathcal{M}_L(v)\) via correspondences in \(\mathcal{M}_L(v) \times \mathcal{M}_L(v)\). We have already dealt with the case of the identity automorphism in another paper. Let \(\pi_{ij}\) be the projection from \(\mathcal{M}_L(v) \times S \times \mathcal{M}_L(v)\) onto the product of the \(i\)-th and \(j\)-th factors.

**Theorem 3.5** ([Ma2]) Let \(\mathcal{E}'_v, \mathcal{E}''_v\) be any two universal families of sheaves over the \(m\)-dimensional moduli space \(\mathcal{M}_L(v)\). Assume, that \(\mathcal{L}\) is \(v\)-suitable. The class of the diagonal, in the Chow ring of \(\mathcal{M}_L(v) \times \mathcal{M}_L(v)\), is identified by

\[
c_m \left[ - \pi_{13, *} \left( \pi_{12}^*(\mathcal{E}'_v)^{\vee} \otimes \pi_{23}^*(\mathcal{E}''_v) \right) \right],
\]

where \(\pi_{13, *}\) is the \(K\)-theoretic push-forward and both the dual \((\mathcal{E}'_v)^{\vee}\) and the tensor product are taken in the derived category. Furthermore, the following vanishing holds

\[
c_{m-1} \left[ - \pi_{13, *} \left( \pi_{12}^*(\mathcal{E}'_v)^{\vee} \otimes \pi_{23}^*(\mathcal{E}''_v) \right) \right] = 0.
\]

An immediate corollary of the Theorem is:

**Corollary 3.6** The K"unneth factors of the Chern classes of any universal sheaf \(\mathcal{E}\) on \(\mathcal{M}_L(v) \times S\) generate the cohomology ring \(H^*(\mathcal{M}_L(v), \mathbb{Q})\).

**Remark 3.7** The geometric identification of the class (35) implies its independence of the choice of the universal families. This independence follows also formally from the vanishing (36) and Lemma 3.8. A universal family may not exist, in general, over the moduli space \(\mathcal{M}_L(v)\). Nevertheless, Theorem 3.5 holds in general, provided we replace the classes \(\text{ch}(\mathcal{E}_1)\) and \(\text{ch}(\mathcal{E}_2)\), in the topological translation (39) of (35), by the universal class in equation (25) ([Ma2] Section 3).

**Lemma 3.8** ([Ma4], Lemma 18) Let \(X\) be a topological space, \(x\) a class of rank \(r \geq 0\) in \(K_{\text{top}}(X)\), and \(L\) a complex line-bundle on \(X\). Then \(c_{r+1}(x \otimes L) = c_{r+1}(x)\) and \(c_{r+2}(x \otimes L) = c_{r+2}(x) - c_{r+1}(x)c_1(L)\).
3.4 The classes \( \gamma(g, v) \)

Given a projective variety \( M \), we denote by
\[
\ell : \oplus_i H^{2i}(M, \mathbb{Q}) \longrightarrow \oplus_i H^{2i}(M, \mathbb{Q})
\]
\[
(r + a_1 + a_2 + \cdots) \mapsto 1 + a_1 + (\frac{1}{2}a_1^2 - a_2) + \cdots
\]
the universal polynomial map, which takes the exponential Chern character of a complex of sheaves to its total Chern class.

Given two K3 surfaces \( S_1 \) and \( S_2 \) and two \( m \)-dimensional Mukai vectors \( v_i \in H^*(S_i, \mathbb{Z}) \), we denote by \( \pi_i \) the projection from \( \mathcal{M}(v_1) \times S_2 \times \mathcal{M}(v_2) \) on the \( i \)-th factor. Given classes \( \alpha_i \in H^*(\mathcal{M}(v_i) \times S_i, \mathbb{Q}) \) and a homomorphism \( g : H^*(S_1, \mathbb{Z}) \rightarrow H^*(S_2, \mathbb{Z}) \), set
\[
x := [\ell (\pi_{13*} (\{\pi_{12*} [(id \otimes g)(\alpha_1)]^\vee \cdot \pi_{23*} (\alpha_2)])])^{-1},
\]
\[
\gamma(g, \alpha_1, \alpha_2) := c_m(x),
\]
\[
\gamma'(g, \alpha_1, \alpha_2) := c_{m-1}(x).
\]

Then \( \gamma(g, \alpha_1, \alpha_2) \) is a class in \( H^{2m}(\mathcal{M}(v_1) \times \mathcal{M}(v_2), \mathbb{Q}) \).

If \( \mathcal{E}_i \) is a complex of sheaves on \( \mathcal{M}(v_i) \times S_i \), we set
\[
\gamma(g, \mathcal{E}_1, \mathcal{E}_2) := \gamma(g, ch(\mathcal{E}_1) \cdot \sqrt{td_{S_1}}, ch(\mathcal{E}_2) \cdot \sqrt{td_{S_2}}).
\]

When \( S_1 = S_2 \) and \( g = id \), Grothendieck-Riemann-Roch yields the equality \( \gamma(id, \mathcal{E}_1, \mathcal{E}_2) = c_m \left\{ -\pi_{13*} \left( \pi_{12*} (\mathcal{E}_1)^\vee \otimes \pi_{23*} (\mathcal{E}_2) \right) \right\} \)). See section 5.1 for a Chow-theoretic identification of \( \gamma(g, \mathcal{E}_1, \mathcal{E}_2) \), when \( g \) is the Hodge isometry of an auto-equivalence of the derived category of the surface. A conceptual interpretation of formula (37) is provided in section 5.3.

When \( (v, v) > 0 \), we set
\[
\gamma(g, v) := \gamma(g, u_v, u_{g(v)}),
\]
where \( u_v \) is given in equation (27). Given universal sheaves \( \mathcal{E}_v \) and \( \mathcal{E}_{g(v)} \) over the moduli spaces, we have the equality \( \gamma(g, v) = \gamma(g, \mathcal{E}_v, \mathcal{E}_{g(v)}) \). The latter equality, as well as the more general equality of the two definitions (3) and (40) of \( \gamma(g, v) \), is proven in Lemma 3.14 (using Theorem 3.9). When \( (v, v) = 0 \), set \( \gamma(g, v) := \gamma(g, \mathcal{E}_v, \mathcal{E}_{g(v)}) \), when universal sheaves \( \mathcal{E}_v \) and \( \mathcal{E}_{g(v)} \) exist. Use instead a universal class \( e_v \) or \( e_{g(v)} \) (Definition 3.2), in the absence of a universal sheaf. The existence of \( e_v \) is clear for two dimensional moduli spaces. The class \( \gamma(g, v) \) is independent of the choice of universal sheaves or classes in this case as well (Lemma 5.2).

We identify \( H^*(\mathcal{M}(v) \times \mathcal{M}(g(v)), \mathbb{Q}) \) with \( H^*(\mathcal{M}(v), \mathbb{Q})^* \otimes H^*(\mathcal{M}(g(v)), \mathbb{Q}) \) via the Künneth and Poincare-Duality theorems. The homomorphism corresponding to the class \( \gamma(g, v) \) is denoted by
\[
\gamma_{g,v} : H^*(\mathcal{M}(v), \mathbb{Q}) \longrightarrow H^*(\mathcal{M}(g(v)), \mathbb{Q}).
\]
We set $\gamma := \gamma_{g,v}$, when the context identifies the Mukai vector $v$.

We can now state parts 1 and 4 of Theorem 1.2 without assuming the existence of the universal class $e_v$:

**Theorem 3.9** Let $(S_i, v_i, \mathcal{L}_i), i = 1, 2$, be objects of the groupoid $\mathcal{G}$ given in (34) and $g \in \text{Hom}_g((S_1, v_1, \mathcal{L}_1), (S_2, v_2, \mathcal{L}_2))$ a morphism (which need not preserve the Hodge structures). Assume that $(v, v) > 0$.

1. The homomorphism
   $$\gamma_g : H^*(\mathcal{M}_{\mathcal{L}_1}(v_1), \mathbb{Z})_{\text{free}} \to H^*(\mathcal{M}_{\mathcal{L}_2}(v_2), \mathbb{Z})_{\text{free}}$$
   is an isomorphism of cohomology rings.

2. The isomorphism
   $$\left(\gamma_g \otimes g\right) : H^*(\mathcal{M}(v_1) \times S_1, \mathbb{Q}) \xrightarrow{\gamma_g} H^*(\mathcal{M}(v_2) \times S_2, \mathbb{Q})$$
   takes the class $u_{v_1}$ to the class $u_{v_2}$.

**Remark 3.10** Part 1 of the Theorem makes sense, even when the Mukai vectors $v_i$ are isotropic and $\mathcal{M}_{\mathcal{L}_i}(v_i)$ are K3 surfaces. In that case, part 1 follows easily from the work of Mukai (see Lemma 5.2).

Part 1 of the Theorem makes sense, even when we replace the K3 surface by an abelian surface or an elliptic curve; the two other Calabi-Yau varieties, for which Theorem 3.5 is known [B2, Ma2]. The abelian surface case seems very interesting, and merits further consideration. The elliptic curve case boils down to an old result of Atiyah. This is explained in section 1.2.1.

Part 2 of the Theorem implies the commutativity of the following diagram

\[
\begin{array}{ccc}
v_1^\perp & \xrightarrow{\theta_{v_1}} & H^2(\mathcal{M}_{\mathcal{L}_1}(v_1), \mathbb{Z}) \\
g \downarrow & & \downarrow \gamma_g \\
v_2^\perp & \xrightarrow{\theta_{v_2}} & H^2(\mathcal{M}_{\mathcal{L}_2}(v_2), \mathbb{Z}).
\end{array}
\]

(41)

The horizontal arrows are isomorphisms by Theorem 1.7.

### 3.5 Compositions

We show in Lemma 3.12 that the validity of Theorem 3.9 is closed under compositions. In particular, the homomorphisms $\gamma_{g,v}$ compose as expected, yielding a representation of the groupoid $\mathcal{G}$. We then discuss the cohomology ring of a fixed moduli space as a representation of the automorphism group of the corresponding object of $\mathcal{G}$.

We begin with a characterization of the class $\gamma(g, v)$. Let $\Delta_i$ be the diagonal in $\mathcal{M}_{\mathcal{L}_i}(v_i) \times \mathcal{M}_{\mathcal{L}_i}(v_i), i = 1, 2$. Denote by $[\Delta_i]$ the cohomology class Poincare dual to $\Delta_i$. 

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Lemma 3.11 Suppose that $f : H^*(\mathcal{M}_{L_1}(v_1), \mathbb{Q}) \to H^*(\mathcal{M}_{L_2}(v_2), \mathbb{Q})$ is a ring isomorphism, $g : H^*(S_1, \mathbb{Q}) \to H^*(S_2, \mathbb{Q})$ a linear homomorphism, and the pair $(f, g)$ satisfies

$$(f \otimes g)(u_{v_1}) = u_{v_2}. \quad (42)$$

Then

$$(1 \otimes f)([\Delta]) = \gamma(g, u_{v_1} \exp(\ell_1), u_{v_2} \exp(\ell_2)), \quad (43)$$

for any two classes $\ell_i \in H^2(\mathcal{M}_{L_i}(v_i), \mathbb{Q})$, $i = 1, 2$. In particular, given $g$, a ring isomorphism $f$ satisfying (42) is unique (if it exists). Furthermore, the class $\gamma'(g, u_{v_1}, u_{v_2})$, given in (38), vanishes.

Proof: We have the equalities

$$[\Delta] = \gamma(id, u_{v_2} \exp(f(\ell_1)), u_{v_2} \exp(\ell_2)) \overset{(42)}{=} \gamma(id, (f \otimes g)(u_{v_1} \exp(\ell_1)), u_{v_2} \exp(\ell_2)) = (f \otimes 1)[\gamma(g, u_{v_1} \exp(\ell_1), u_{v_2} \exp(\ell_2))].$$

Theorem 3.5 yields the first equality (see also Remark 3.7). The last equality is due to the fact, that evaluation of $(f \otimes 1)$ commutes with $\gamma$ because $f$ is a ring isomorphism. Evaluating $(f \otimes 1)^{-1}$ on both sides, we conclude the equality (43).

The vanishing of $\gamma'(g, u_{v_1}, u_{v_2})$ follows similarly from the corresponding vanishing (36) in Theorem 3.5 and Lemma 3.8. \qed

Lemma 3.12 Let $x_i := (S_i, v_i, L_i)$, $i = 1, 2, 3$, be objects of the groupoid $\mathcal{G}$, given in (34), $g \in \text{Hom}_G(x_1, x_2)$, and $f \in \text{Hom}_G(x_2, x_3)$. If $g$ and $f$ satisfy the statements of Theorem 3.9, then so do $g^{-1}$ and $f \circ g$. Moreover, the following equality holds

$$\gamma_{fg,v_1} = \gamma_{f,v_2} \circ \gamma_{g,v_1}. \quad (44)$$

Proof: The assumptions imply, that $\gamma_{g,v_1}$ is a ring isomorphism and $\gamma_{g,v_1} \otimes g$ maps the class $u_{v_1}$ to $u_{v_2}$. Similarly, the class $\gamma_{f,v_2}$ has the analogous properties. Hence, the composition $\phi := \gamma_{f,v_2} \circ \gamma_{g,v_1}$ is a ring isomorphism, and $\phi \otimes (fg)$ maps $u_{v_1}$ to $u_{v_3}$. Lemma 3.11 implies the equality $\phi = \gamma_{fg,v_1}$. The latter is precisely equality (44). The proof for $g^{-1}$ is similar. \qed

Note: The classes $\gamma(g, v)$ and $\gamma(g^{-1}, g(v))$ are equal, under the natural identification of $\mathcal{M}(v) \times \mathcal{M}(g(v))$ with $\mathcal{M}(g(v)) \times \mathcal{M}(v)$. A direct proof, without the hypothesis that $g$ satisfies the assumptions of Theorem 3.9, is not hard (see Lemma 4.4 part 3 in the eprint version math.AG/0305042 v2 of this paper). A second proof of the equality $\gamma(g, v)^{-1} = \gamma_{g^{-1}, g(v)}$ then follows, once one shows that $\gamma_{g,v}$ is a ring isomorphism.

Let $\Gamma$ be the isometry group of the Mukai lattice of a K3 surface $S$. As a corollary of Theorem 3.9 and lemma 3.12, we get that the stabilizer $\Gamma_v$, of a Mukai vector $v$, acts on the cohomology of the corresponding moduli space:
Corollary 3.13 Let \((S, v, \mathcal{L})\) be an object of the groupoid \(\mathcal{G}\). Assume, that \((v, v) \geq 2\). Then the natural map

\[
\gamma : \Gamma_v \rightarrow \text{Aut}[H^\ast(\mathcal{M}_\mathcal{L}(v), \mathbb{Z})_{\text{free}}]
\]

\[
g \mapsto \gamma_{g,v}
\]

is an injective group homomorphism.

Proof: The homomorphism \(\gamma\) is injective since diagram (41) is commutative.

Lemma 3.14 Let \(g \in \text{Hom}_g((S_1, v_1, \mathcal{L}_1), (S_2, v_2, \mathcal{L}_2))\). The two definitions (3) and (40) of the class \(\gamma(g, v_1)\) agree.

Proof: The two definitions are identical, when \((v_1, v_1) = 0\). Assume \((v_1, v_1) > 0\). Theorem 3.9 proves that the assumptions of Lemma 3.11 are satisfied for the operator \(f := \gamma_{g,v_1}\) corresponding to definition (40) of the class \(\gamma(g, v_1)\). The class \(\text{ch}(e_{v_1})\sqrt{\text{Id}_{S_1}}\) is equal to \(u_{v_1} \cdot \exp(\ell_i)\), for some class \(\ell_i\) in \(H^2(\mathcal{M}_{\mathcal{L}_i}(v_1), \mathbb{Q})\), \(i = 1, 2\), by equations (27) and (28). Lemma 3.11 implies the equality \(\gamma(g, u_{v_1}, u_{v_2}) = \gamma(g, \text{ch}(e_{v_1})\sqrt{\text{Id}_{S_1}}, \text{ch}(e_{v_2})\sqrt{\text{Id}_{S_2}})\).

The left hand side appears in the topological definition (40) of the class \(\gamma(g, v_1)\), while the right hand side is the topological translation of the K-theoretic definition (3) via Grothendieck-Riemann-Roch.

The set of universal classes, generating the cohomology ring of \(\mathcal{M}_\mathcal{L}(v)\), is invariant with respect to the representation (45). This is another corollary of the Theorem 3.9. Let \(B\) be the image of the Mukai lattice in \(H^\ast(\mathcal{M}_\mathcal{L}(v), \mathbb{Q})\), via the \(\Gamma_v\)-equivariant homomorphism

\[
H^\ast(S, \mathbb{Z}) \rightarrow H^\ast(\mathcal{M}_\mathcal{L}(v), \mathbb{Q})
\]

\[
x \mapsto -p_{\mathcal{M}_v}(u_v \cdot \pi_\mathcal{Y}(x))
\]

(compare with equation (33) defining \(\theta_v\)). The above homomorphism is the image of \(u_v\) under the identification of the Mukai lattice \(H^\ast(S, \mathbb{Z})\) with its dual. We use the Mukai pairing for the identification, rather than Poincare Duality, because the latter is not \(\Gamma\)-invariant. This identification sends \(x \in H^\ast(S, \mathbb{Z})\) to \(-\int_S x^\vee \cup (\bullet)\) in \(H^\ast(S, \mathbb{Z})^\ast\). Corollary 3.6 implies, that the cohomology ring \(H^\ast(\mathcal{M}_\mathcal{L}(v), \mathbb{Q})\) is generated by the projections \(B_i\) of \(B\) into \(H^{2i}(\mathcal{M}_\mathcal{L}(v), \mathbb{Q})\). Part 2 of Theorem 3.9 implies:

Corollary 3.15 Each of the vector subspaces \(B_i \subset H^{2i}(\mathcal{M}_\mathcal{L}(v), \mathbb{Q})\), generating the cohomology ring of \(\mathcal{M}_\mathcal{L}(v)\), is \(\Gamma_v\)-invariant.

Note, that \(B \otimes \mathbb{Q}\) consists of, at most, two irreducible \(\Gamma_v\) sub-representations: the trivial character and \(v^\perp \otimes \mathbb{Q}\), each with multiplicity \(\leq 1\).

The following lemma will be cited in subsequent sections. Denote by \(R[v]\) the weighted polynomial ring, formally generated by the vector spaces \(B_i\), \(1 \leq i \leq n - 1\), in Corollary 3.15, with vectors in \(B_i\) having degree \(2i\). Let \(h : R[v] \rightarrow H^\ast(S[v], \mathbb{Q})\) be the natural ring homomorphism and \(I^d\) its kernel in degree \(d\).
Lemma 3.16  (Lemma 10 in [Ma2])  $B_i$ is $23$-dimensional and $B_i$ is $24$-dimensional, for $2 \leq i \leq n/2$. If $n$ is odd, then $B_{(n+1)/2}$ is either $23$ or $24$ dimensional. The homomorphism $h$ is surjective. It is injective in degree $\leq n$. If $n$ is odd, then the dimension of $I^{n+1}$ is $\dim(B_{(n+1)/2}) - 23$.

4 The monodromy group

In section 4.1 we recall the orientation character of the isometry group of the Mukai lattice. This character arises in formula (14) for the monodromy representation. In sections 4.2 and 4.3 we show, that the monodromy subgroup $\mathcal{W}$, constructed in Corollary 1.8, maps onto a finite index subgroup of the monodromy group $\text{Mon}(S^{[n]}) \subset \text{Aut}(H^*(S^{[n]}, \mathbb{Z}))$. Furthermore, the subgroup $K$ of the monodromy group, which acts trivially on the second cohomology of $S^{[n]}$, is central, finite, and of exponent $2$. In section 4.4 we improve the upper bound for the order of $K$, using the monodromy-invariance of the Chern classes $c_2(TS^{[n]})$. In section 4.5 we provide several characterizations of the subgroup $\mathcal{W}$, of the monodromy group $\text{Mon}^2$, constructed in Corollary 1.8. In section 4.6 we compare the monodromy representation of Theorem 1.6 with a related representation constructed by Verbitsky (Lemma 4.13). In section 4.7 we study the equivariance properties, of the Chern character of the universal sheaf, with respect to the monodromy representation.

4.1 The orientation and covariance characters

Definition 4.1 Denote by $\Gamma$ the group of isometries of the Mukai lattice $H^*(S, \mathbb{Z})$ of a fixed $K3$ surface $S$. Let $G$ be the subgroup of (integral) Hodge-isometries of the Mukai lattice. Denote by $\Gamma_0$ the group of isometries of the second cohomology $H^2(S, \mathbb{Z})$ and let $G_0$ be the subgroup of integral Hodge-isometries of the weight 2 Hodge structure of $S$.

$\Gamma_0$ embeds in $O(3, -19)$ and $\Gamma$ embeds in $O(4, -20)$. The Hodge decompositions

$$H^2(S, \mathbb{C}) = [H^{2,0}(S) \oplus H^{0,2}(S)] \oplus H^{1,1}(S)$$
$$H^*(S, \mathbb{C}) = [H^{2,0}(S) \oplus H^{0,2}(S)] \oplus [H^{1,1}(S) \oplus H^0(S) \oplus H^4(S)]$$

imply that $G_0$ embeds naturally in $\text{Aut}(H^{2,0}) \times O(1, -19)$ and $G$ embeds naturally in $\text{Aut}(H^{2,0}) \times O(2, -20)$. The group $O(n, -m)$, $n, m > 1$, has, in addition to the determinant character, the orientation character

$$O(n, -m) \to \text{Aut}(H^{n-1}(S^{n-1}, \mathbb{Z})) \cong \mathbb{Z}/2\mathbb{Z}.$$  \hspace{1cm} (47)

The cone $C = \{v : (v, v) > 0\}$ is an $(\mathbb{R}^n \setminus \{0\})$-bundle over $\mathbb{R}^m$ and is hence homotopic to the sphere $S^{n-1}$. The character (47) is the action of $O(n, -m)$ on the top cohomology group of $C$.  

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In the case of $G_0$, the cone $C \subset H^{1,1}(S, \mathbb{R})$ has two components (the case $n = 1$) and a Hodge isometry is said to be a \textit{signed isometry} if it maps each component to itself. All automorphisms of $S$, as well as reflections along $-2$ curves on $S$, are signed isometries.

In the case of $G$, the cone $C \subset H^{1,1}(S, \mathbb{R}) \oplus H^0(S, \mathbb{R}) \oplus H^4(S, \mathbb{R})$ is homotopic to $S^1$. In this case, we will call (47) the \textit{covariance} character and denote it by

$$
cov : G \rightarrow \text{Aut}(H^{n-1}(C, \mathbb{Z})) \cong \mathbb{Z}/2\mathbb{Z}.
$$

(48)

We have the natural embedding of $G_0$ into $G$ via extension by the identity on $H^0 \oplus H^4$. This embedding pulls back the character $\text{cov}$ on $G$ to the same character on $G_0$. Similarly, the natural embeddings $G_0 \subset \Gamma_0 \subset \Gamma$ and $G_0 \subset G \subset \Gamma$ are all compatible with respect to the orientation character (47).

\textbf{Definition 4.2} Denote by $\Gamma^{\text{cov}}, \Gamma_0^{\text{cov}}, G^{\text{cov}},$ and $G_0^{\text{cov}},$ the respective kernels of the orientation character.

The covariance character sends the action of the Picard group, $\text{Aut}(S)$, and reflection in $(-2)$-vectors, to 0. The isometry $-\text{id}$, acting via multiplication by $-1$ on the whole Mukai lattice, is also covariant. Conjecturally, any auto-equivalence, of the derived category of a K3 surface, is sent by the covariance character to 0 (see [Sz, HLOY]). Many cases of this conjecture were proven in [HS] Proposition 5.5. On the other hand, the Duality Hodge isometry (29) is contravariant, $\text{cov}(D) = 1$. Reflections in $(+2)$-vectors are sent to 1 as well. This can be seen as follows. The determinant character is the product of the orientation character of the positive cone, with the orientation character of the negative cone. A reflection in a $(+2)$-vector has determinant $-1$, and it preserves the orientation of the negative cone.

\textbf{Remark 4.3} The positive cone in $H^{1,1}(S, \mathbb{R})$ has a distinguished component; the one containing the Kähler cone. Similarly, the positive cone $C$ in $H^2(S, \mathbb{R})$ has a distinguished \textit{orientation}; i.e., a generating class in $H^2(C, \mathbb{Z})$. Such an orientation is equivalent to the determination of an orientation of one 3-dimensional subspace of $H^2(S, \mathbb{R})$, to which the pairing restricts as a positive definite one. Let $\sigma$ be a holomorphic symplectic structure on $S$ and $\kappa$ a Kähler form. Then $\{\text{Re}(\sigma), \text{Im}(\sigma), \kappa\}$ is a basis for such a subspace. The orientation of $C$ is independent of the choices of $\kappa$ and $\sigma$. We can further assign a distinguished orientation to the Mukai lattice of $S$, by adding to the above basis the $+2$ Mukai vector $(1, 0, -1)$ as a fourth vector. These four vectors span a 4-dimensional positive definite sublattice of the Mukai lattice. This orientation defines an extension of the orientation character to a functor (19) from the groupoid $G$ to $\mathbb{Z}/2\mathbb{Z}$. Note that the Mukai vector $(1, 0, -1)$ is effective (Definition 1.1).

\section{4.2 Monodromy operators acting trivially on $H^2$}

Let $X$ be an irreducible holomorphic symplectic manifold. The theory of Lefschetz-modules, developed by Verbitsky and Looijenga-Lunts, introduces an action of the group
Spin\(H^2(X, \mathbb{R})\) on the cohomology of \(X\). The action is monodromy-equivariant and preserves the ring structure. We will use this theory to study the subgroup \(K \subset \text{Mon}(X)\) of monodromy operators, which act trivially on \(H^2(X, \mathbb{Z})\) (Corollary 4.6).

Let \(A_k \subset H^*(X, \mathbb{Z})_{\text{free}}, k \geq 0\), be the graded subalgebra generated by \(\oplus_{i=0}^{k} H^i(X, \mathbb{Z})_{\text{free}}\). Set \((A_k)^{\prime} := A_k \cap H^j(X, \mathbb{Z})_{\text{free}}\). Define \(A_k(Q)\) and \([A_k(Q)]^\prime\) using rational coefficients. The theory of Lefschetz-modules will enable us to construct a natural subspace \(C_k(Q)\) of \(H^k(X, \mathbb{Q})\), leading to a monodromy-invariant decomposition

\[
H^k(X, \mathbb{Q}) = [A_{k-2}(Q)]^k \oplus C_k(Q). \quad (49)
\]

Set \(d := \dim_C(X)\). Let \(h \in \text{End}[H^*(X, \mathbb{R})]\) be the linear endomorphism acting via multiplication by \((i - d)\) on \(H^i(X, \mathbb{R})\). Given a class \(a \in H^2(X, \mathbb{R})\), denote by \(e_a \in \text{End}[H^*(X, \mathbb{R})]\) the operator obtained by cup product with \(a\). The class \(a\) is called of Lefschetz type, if there exists \(f_a \in \text{End}[H^*(X, \mathbb{R})]\) satisfying the \(\mathfrak{sl}_2\) commutation relations

\[
[e_a, f_a] = h, \quad [h, e_a] = 2e_a, \quad [h, f_a] = -2f_a.
\]

Such \(f_a\) is unique, if it exists. The triple \((e_a, h, f_a)\) is called a Lefschetz triple. The set of classes \(a \in H^2(X, \mathbb{R})\) of Lefschetz type is a Zariski dense open subset.

Let \(\mathfrak{g}(X)\) be the graded Lie subalgebra of \(\text{End}[H^*(X, \mathbb{R})]\) generated by \(\{e_a, f_a\}\) for all \(a\) in \(H^2(X, \mathbb{R})\) of Lefschetz type. Denote by \(\mathfrak{g}_k(X)\) the subspace of grade \(k\). The primitive subspace \(\text{Prim}^k(X) \subset H^k(X, \mathbb{R})\) is the set of classes killed by \(\mathfrak{g}_{<0}(X)\) and \(\text{Prim}(X) := \oplus_k \text{Prim}^k(X)\). Set \(b_2 := \dim H^2(X, \mathbb{R})\). The following Theorem was proven by Verbitsky [Ve1], and refined in the above language by Looijenga and Lunts [LL], who proved also a more general version for all compact Kähler manifolds.

**Theorem 4.4** Let \(X\) be an irreducible holomorphic symplectic manifold.

1. ([LL] Proposition 4.5) \(\mathfrak{g}(X) \cong \mathfrak{so}(4, b_2-2, \mathbb{R})\). \(\mathfrak{g}(X)\) is defined over \(\mathbb{Q}\). The degree-zero summand \(\mathfrak{g}_0(X)\) is isomorphic to \(\mathfrak{so}(H^2(X, \mathbb{R})) \oplus \mathbb{R}h\). The homomorphism

\[
e : H^2(X, \mathbb{R}) \to \text{End}[H^*(X, \mathbb{R})],
\]

sending \(a\) to \(e_a\), is injective with image \(\mathfrak{g}_2(X)\). \(\mathfrak{g}_k(X) = 0\), for \(k \notin \{-2, 0, 2\}\). In particular, \(\mathfrak{g}_2(X)\) and \(\mathfrak{g}_{-2}(X)\) are abelian subalgebras.

2. ([Ve2]) The action of the semisimple part of \(\mathfrak{g}_0(X)\), which is isomorphic to \(\mathfrak{so}(H^2(X, \mathbb{R}))\), integrates to an action of \(\text{Spin}(H^2(X, \mathbb{R}))\) via ring isomorphisms. The action on the even cohomology factors through a representation

\[
\rho : SO(H^2(X, \mathbb{R})) \to \text{Aut}[H^{\text{even}}(X, \mathbb{R})]. \quad (50)
\]

3. ([LL] Proposition 1.6) \(\mathfrak{g}(X)\) preserves, infinitesimally, the Poincare pairing on \(H^*(X, \mathbb{R})\).
4. ([LL] Corollary 2.3) $H^*(X, \mathbb{R})$ is generated, as an $A_2$-module, by $\text{Prim}(X)$.

5. ([LL] Corollary 1.13) Let $W$ be an irreducible $\mathfrak{g}_0(X)$-submodule of $\text{Prim}^k(X)$. Then the $A_2$-submodule generated by $W$ is an irreducible $\mathfrak{g}(X)$-submodule. Conversely, all irreducible $\mathfrak{g}(X)$-submodules are of this type.

6. ([Ve2]) Let $I$ be the complex structure of $X$. Denote by $\text{ad}_I$ the semisimple endomorphism of $H^*(X, \mathbb{C})$, with $H^{p,q}(X)$ an eigenspace with eigenvalue $\sqrt{-1}(p-q)$. Then $\text{ad}_I$ is an element of $\mathfrak{g}_0(X) \otimes_{\mathbb{R}} \mathbb{C}$.

We could not determine directly from Theorem 4.4, how large is the intersection of the image of $\rho$ with the monodromy group. One seems to need some Torelli type result (see section 6.2). In that respect, Theorems 1.6 and 4.4 seem to complement each other nicely, in case $X = S^{[n]}$. Roughly, Theorem 1.6 implies, that the image of $\text{Mon} \to \text{Mon}^2$ is large (section 4.5), while Theorem 4.4 implies, that the kernel $K$ is small (Lemma 4.7). The action (14) is compared to Verbitsky’s (50) in Lemma 4.13.

Let $A_i'$ be the $\mathfrak{g}(X)$-submodule of $H^*(X, \mathbb{R})$ generated by $\text{Prim}(X) \cap A_i$. If $i \geq 2$, then $A_i'$ is the maximal $\mathfrak{g}(X)$-submodule of $H^*(X, \mathbb{R})$, which is contained in $A_i$, by parts 4 and 5 of the Theorem. Clearly, we have the equalities $A_2' = A_2$ and $(A_i')^k = H^k(X, \mathbb{R})$, for $k \leq i$.

**Lemma 4.5** $(A_{i-2})^i = (A_{i-2})^i$, for $i \geq 4$.

**Proof:** Let $x \in H^i(X, \mathbb{R})$. Then $x$ can be written in the form $x = \sum_{j=0}^{\lfloor \frac{i}{2} \rfloor} a_j x_j$, where $a_j \in (A_2)^{2j}$, $x_j \in \text{Prim}^{i-2j}(X)$, by part 4 of the Theorem. Note that $\text{Prim}^2(X) = 0$. For $j \geq 1$, $x_j$ belongs to $(A_{i-2})^{i-2j}$, by the equality $(A_{i-2})^{i-2j} = H^{i-2j}(X, \mathbb{R})$. Hence, $a_j x_j$ belongs to $(A_{i-2})^i$, for $j \geq 1$. Thus, $x$ belongs to $(A_{i-2})^i$ (respectively $(A_{i-2})^i$), if and only if $x_0$ belongs to $\text{Prim}^i(X) \cap (A_{i-2})^i$ (respectively $\text{Prim}^i(X) \cap (A_{i-2})^i$). But $\text{Prim}^i(X) \cap (A_{i-2})^i = \text{Prim}^i(X) \cap (A_{i-2})^i$, by definition of $A_{i-2}$. Hence, $x$ belongs to $(A_{i-2})^i$, if and only if it belongs to $(A_{i-2})^i$. \hfill \Box

The Poincare pairing restricts, as a non-degenerate pairing, to each irreducible $\mathfrak{g}(X)$-submodule, by part 3 of the Theorem and the fact that $\mathfrak{g}(X)$ is semi-simple. In particular, it restricts to a non-degenerate pairing on $A_i'$. Set $C_2 := H^2(X, \mathbb{Z})$, $C_3 := H^3(X, \mathbb{Z})_{\text{free}}$ and let

$$C_k \subset H^k(X, \mathbb{Z})_{\text{free}}, \quad k \geq 4,$$

be the weight $k$ summand of the orthogonal complement $(A_{i-2})^\perp$. Set $C_k(\mathbb{Q}) := C_k \otimes_{\mathbb{Z}} \mathbb{Q}$. Let $K \subset \text{Mon}(X)$ be the subgroup of monodromy operators, which act trivially on $H^2(X, \mathbb{Z})$.

**Corollary 4.6** 1. $H^k(X, \mathbb{Q})$ admits the monodromy-invariant and $\mathfrak{g}_0(X)$-invariant decomposition (49), for $k \geq 2$. 

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2. Assume, that each irreducible $g_0(X)$-submodule of $C_k(Q)$ appears in $C_k(Q)$ with multiplicity at most 1, for all $k \geq 2$. Then $K$ is a finite subgroup of exponent $\leq 2$.

3. Assume further, that none of the $g_0(X)$-modules $C_k(Q)$, $k \geq 2$, contains two irreducible $g_0(X)$-submodules with isomorphism classes, which are conjugate under the automorphism group of $g_0(X)$. Then $K$ is contained in the center of $Mon(X)$.

**Proof:**

1) $C_k(Q)$ and $[A'_k-2(Q)]^k$ are complementary, by definition, so (49) is a direct sum decomposition, by Lemma 4.5. The $g_0(X)$-invariance is clear. $g(X)$ is a $Mon(X)$-invariant Lie subalgebra of $End[H^*(X,R)]$, by its definition. Thus the decomposition (49) is $Mon(X)$-invariant as well.

2) Let $f$ be an element of $K$. Then $f$ commutes with $g_0(X)$, by definition of $g_0(X)$. Thus, $f$ acts on each irreducible $g_0(X)$-submodule of $C_k(Q)$ via multiplication by $\pm 1$, by the multiplicity assumption. The subspaces $C_k(Q)$, $k \geq 2$, generate $H^*(X,Q)$. Hence, $K$ is finite and has exponent $\leq 2$.

3) $Mon(X)$ acts on $g_0(X)$ via Lie algebra automorphisms. Each irreducible $g_0(X)$-submodule of $C_k(Q)$ is $Mon(X)$-invariant, by the assumed absence of $Aut g_0(X)$-conjugate $g_0(X)$-submodules. Thus $K$ is in the center of $Mon(X)$. \hfill \Box

The assumptions in parts 2 and 3 of Corollary 4.6 hold for the Hilbert schemes $S^{[n]}$, of a $K3$ surface $S$ (Lemma 4.8 below). These assumptions fail to hold for generalized Kummers of dimension $2n \geq 4$. Indeed, $K$ has exponent $> 2$ and is not abelian in that case. This can be seen as follows. The subgroup, of automorphisms of an abelian surface, generated by multiplication by $-1$ and by translations by points of order $n + 1$, acts trivially on the second cohomology of the generalized Kummer ([B3] proposition 9), but it acts faithfully on the higher cohomology [Sa]. Hence it injects into $K$.

### 4.3 A splitting of the monodromy group

In view of the local Torelli Theorem for irreducible holomorphic symplectic manifolds [B1], it is natural to compare the monodromy of the total cohomology ring of the Hilbert scheme $S^{[n]}$ and the monodromy for the weight 2 cohomology. Our model is the case $n = 1$; in which the monodromy group for K3 surfaces is an index 2 subgroup of the isometry group of the lattice $H^2(S,Z)$. It consists of isometries, which preserve the orientation of the positive cone. This identification of the monodromy is a consequence of the Torelli Theorem for K3 surfaces and the surjectivity of the period map (Corollary 6.7).

Let $Mon$ be the group of automorphisms of the ring $H^*(S^{[n]},Z)_{free}$, which is generated by monodromy (Definition 1.5). Denote by $Mon^2$ the image of $Mon$ in the isometry group of $H^2(S^{[n]},Z)$ and let $K$ be the kernel

$$0 \to K \to Mon \to Mon^2 \to 0.$$  \hfill (52)
The exact sequence (52) *splits naturally* in case \( n - 1 \) is a prime power, by Theorem 1.6. More generally, the pullback of (52), via the inclusion\(^3\) of \( \mathcal{W} \) in \( \text{Mon}^2 \), splits (see Corollary 1.8 for the inclusion \( \mathcal{W} \subset \text{Mon}^2 \)). The splitting is natural, since the image of \( \Gamma_v \) in Mon is a normal subgroup, which is isomorphic to \( \mathcal{W} \) (Lemmas 4.7 and 4.10). Conjecture 2.1 suggests that \( K \) is trivial. The rest of this section is dedicated to the study of \( K \). We start with a rough estimate of \( K \) in the following lemma.

**Lemma 4.7** \( K \) is contained in the center of Mon. Furthermore, \( K \) is isomorphic to a subgroup of \( (\mathbb{Z}/2\mathbb{Z})^{2n-4} \). Consequently, the image of the monodromy representation (14) of \( \Gamma_v \) is a normal subgroup of finite index in Mon.

Further constraints on \( K \) are discussed in section 4.4. Lemma 4.7 is a corollary of the following lemma. Let \( C_k \subset H^k(S[n], \mathbb{Z})_{free} \) be the Mon-submodule defined in (51). Then \( C_2, C_4, \ldots, C_{2n-2} \) generate the cohomology ring with rational coefficients (Lemma 3.16). Thus Mon is determined by its action on the \( C_i \)'s. Let \( \text{mon}^{2i} : \Gamma_v \to GL(C_{2i}(\mathbb{Q})) \) be the composition of \( \text{mon} \), given in (14), with the restriction homomorphism from Mon to \( GL(C_{2i}(\mathbb{Q})) \). Set \( \theta_{v,1} := \theta_v \), given in (32), and let \( \theta_{v,i} : H^*(S, \mathbb{Q}) \to C_{2i}(\mathbb{Q}), i \geq 2, \) be the composition of the homomorphism (46) with the projection \( H^*(S[n], \mathbb{Q}) \to C_{2i}(\mathbb{Q}) \).

**Lemma 4.8** \( C_{2i}(\mathbb{Q}) \) admits a Mon-invariant decomposition
\[
C_{2i}(\mathbb{Q}) = C'_{2i}(\mathbb{Q}) \oplus C''_{2i}(\mathbb{Q}).
\]

\( C'_i(\mathbb{Q}) \) either vanishes, or it is a one-dimensional character \( \chi'_i \) of Mon. The summand \( C''_i(\mathbb{Q}) \) either vanishes, or it is an irreducible Mon representation, which is isomorphic to \( H^2(S[n], \mathbb{Q}) \otimes \chi''_i \), for a character \( \chi''_i : \text{Mon} \to \{\pm 1\} \). The pullback of \( \chi'_i \) to \( \Gamma_v \), via (14), is the orientation character \( \text{cov} \), if \( C'_{2i}(\mathbb{Q}) \) does not vanish and \( i \) is odd. The pullback of \( \chi'_i \) to \( \Gamma_v \) is trivial, when \( i \) is even. The pullback of \( \chi''_i \) to \( \Gamma_v \) is the orientation character \( \text{cov} \), when \( C''_{2i}(\mathbb{Q}) \) does not vanish and \( i \) is even. The pullback of \( \chi''_i \) to \( \Gamma_v \) is trivial, when \( i \) is odd. The summand \( C''_i(\mathbb{Q}) \) does not vanish, for \( 2 \leq i \leq n/2 \), and \( C''_{2i}(\mathbb{Q}) \) does not vanish, for \( 0 < i \leq (n + 1)/2 \). Consequently, \( \theta_{v,i} \) is an isomorphism, for \( 1 \leq i \leq n/2 \), and
\[
\text{mon}^{2i}(g) = (-1)^{i \cdot \text{cov}(g)} [\theta_{v,i} \circ g \circ \theta_{v,i}^{-1}].
\]

**Proof of Lemma 4.7:** The kernel \( K \) of Mon \( \to \text{Mon}^2 \) acts on \( C''_{2i}(\mathbb{Q}) \) diagonally, via the character \( \chi''_i \). We conclude, that the action of \( K \) is determined by the \( 2n - 4 \) characters \( \{\chi'_i, \chi''_i\}, 2 \leq i \leq n - 1 \). The image of the monodromy representation (14), in the isometry group of \( H^2(S[n], \mathbb{Z}) \), is a normal subgroup of finite index in Mon (Lemma 8.3). Hence, the image of the monodromy representation (14) is a normal subgroup of finite index in Mon.

\(^3\)The equality \( \mathcal{W} = \text{Mon}^2 \) is proven in [Ma3]
**Proof of lemma 4.8:** The decomposition (53) of $C_{2i}(\mathbb{Q})$ is constructed in three steps: Step 1: Let us first determine the possible Hodge numbers of $C_{2i}(\mathbb{Q})$. The homomorphism $\theta_{v,i} : H^2(S,\mathbb{Z}) \to C_{2i}(\mathbb{Q})$ is surjective, by Corollary 3.6, and is $\Gamma_{v}^{\text{cov}}$-equivariant with respect to the action (14), by Theorem 1.6. Hence, $C_{2i}(\mathbb{Q})$ contains at most two irreducible $\Gamma_{v}^{\text{cov}}$-modules. The rank of $\theta_{v,i}$ can be 24, 23, 1, or 0. Moreover, $\theta_{v,i}$ is a rational Hodge structure homomorphism. Consequently, the possible Hodge numbers of $C_{2i}$ are as follows. $[h^{i-1,i+1}, h^{i,i}, h^{i+1,i-1}]$ is $(1, 22, 1), (1, 21, 1), (0, 1, 0),$ or $(0, 0, 0),$ and all the other $h_{k,2i-k}$ vanish.

Step 2: The decomposition (53) of $C_{2i}(\mathbb{Q})$ is defined to be the decomposition into irreducible $\mathfrak{g}_0(S^{[n]})$-submodules. If $h^{i-1,i+1}$ does not vanish, $C_{2i}(\mathbb{R})$ can not be a trivial $\mathfrak{so}(3,20,\mathbb{R})$-representation, because the Hodge structure is determined by the $\mathfrak{so}(3,20,\mathbb{R})$-module structure (Theorem 4.4 part 6). $\mathfrak{so}(3,20,\mathbb{R})$ does not have an irreducible representation of dimension 24. The minimal dimension of a non-trivial irreducible representation of $\mathfrak{so}(3,20,\mathbb{R})$ is the 23-dimensional standard representation. Summarizing, $C_{2i}(\mathbb{R})$ is a direct sum $C_{2i}(\mathbb{R}) \oplus C'_{2i}(\mathbb{R})$ with respect to Verbitsky’s $SO(H^2(S^{[n]},\mathbb{R}))$ action. The decomposition is defined over $\mathbb{Q}$, because $\mathfrak{so}(3,20,\mathbb{R})$ is defined over $\mathbb{Q}$, by Theorem 4.4 part 1. The summand $C'_{2i}$ is either zero or the trivial representation and $C''_{2i}$ is either zero or the standard representation. The non-vanishing of $C'_{2i}$ and $C''_{2i}$, in the stated ranges of $i$, follow from Lemma 3.16.

Step 3: When non-zero, $C'_{2i}(\mathbb{R})$ and $C''_{2i}(\mathbb{R})$ are irreducible subrepresentations of $\mathfrak{g}_0(S^{[n]})$ of different dimensions. Hence, each is $\text{Mon}$-invariant, and $K$ is finite, central, and of exponent $\leq 2$, by Corollary 4.6.

It remains to define the character $\chi''_i$. Consider the representation $H^2(S^{[n]},\mathbb{Q}) \otimes C''_{2i}(\mathbb{Q})$ of $\text{Mon}$. Restrict it to a representation of $(\Gamma_v)^{\text{cov}}$ via (14). This restriction is isomorphic to the tensor square of $H^2(S^{[n]},\mathbb{Q})$. It decomposes as a direct sum of three pairwise non-isomorphic and irreducible representations; the trivial character $V$ of $(\Gamma_v)^{\text{cov}}$, $\wedge^2 H^2(S^{[n]},\mathbb{Q})$, and another irreducible representation ([LL] Proposition 2.14). The image of $(\Gamma_v)^{\text{cov}}$ in $\text{Mon}$ is a normal subgroup, by Lemma 8.3 and the centrality of $K$. Hence, each of the three subrepresentations is $\text{Mon}$-invariant. $V$ determines the character $\chi''_i$. \(\square\)

### 4.4 The Chern classes of $TS^{[n]}$

We continue the study of the kernel $K$ of $\text{Mon} \to \text{Mon}^2$ and improve the upper bound for its order $|K|$, given in Lemma 4.7. We conjectured $K$ to be trivial (Conjecture 2.1). $K$ is embedded in $(\mathbb{Z}/2\mathbb{Z})^{2n-1}$ via the restriction of the characters $\chi'_i$ and $\chi''_i$, $2 \leq i \leq n - 1$ (Lemma 4.7). We prove in this section that the character $\chi'_i$ is trivial, for even $i$ (Lemma 4.9). The restriction of $\chi''_i$ to $K$ is trivial for $n \geq 4$, by part 1 of Theorem 34 of [Ma4]. The equality $\chi''_{2i} = \chi'_i$ is proven in part 3 of Theorem 34 of [Ma4] for integer values of $i$ in the range $1 \leq i \leq n/4$ mentioned in Lemma 4.8.

In case $n = 2$, the kernel is trivial (Lemma 4.7). In case $n = 3$, $|K|$ is bounded by 2, by the vanishing of $C'_{2i}$, for all $i$, and of $C''_{2i}$, for $i > 2$ (Example 14 in [Ma2]). Question
1 in [Ma2] asks, if \( C_{2i} \) and \( C_{2i}'' \) vanish, for \( i \) outside the corresponding ranges in Lemma 4.8. This suggested range of vanishing seems to be a close approximation of the truth [LS2]. When \( n = 4 \), the triviality of \( K \) follows from an affirmative answer to Question 1 in [Ma2] and Lemma 4.9.

Let \( v = (1, 0, 1 - n) \) be the Mukai vector of \( S^{[n]} \), \( \pi_M \) and \( \pi_S \) the two projections from \( \mathcal{M}(v) \times S \), and \( \mathcal{E}_v \) a universal sheaf. Consider the decomposition \( H^{4i}(\mathcal{M}(v), \mathbb{Q}) = (\mathcal{A}_{4i-2})^{4i}(\mathbb{Q}) \oplus C_{4i}'(\mathbb{Q}) \oplus C_{4i}''(\mathbb{Q}), i \geq 1 \), introduced in Lemma 4.8. Let \( E \) be a sheaf with Mukai vector \( v \).

**Lemma 4.9** The projections to \( C_{4i}'(\mathbb{Q}) \), of \( c_{2i}(TS^{[n]}) \) and \( 2c_{2i}[-\pi_M, (\mathcal{E}_v \otimes p_S^1(E^\vee))] \), are equal and span \( C_{4i}'(\mathbb{Q}) \). Consequently, we get:

1. If \( 4 \leq 2i \leq n/2 \), then \( c_{2i}(TS^{[n]}) \) does not belong to the subalgebra of \( H^*(S^{[n]}, \mathbb{Q}) \) generated by classes in \( H^k(S^{[n]}, \mathbb{Q}), k < 4i \).

2. The character \( \chi_{2i} \) is trivial, for all \( i \).

**Proof:** Denote the \( \Gamma_v \)-equivariant homomorphism (46) by \( u_v \) as well. The projection of \( ch_{2i}[-\pi_M, (\mathcal{E}_v \otimes p_S^1(E^\vee))] \) to \( C_{4i}' \) is equal to the projection of \( u_v(v) \) and hence spans \( C_{4i}' \). The coset \( c_{2i}(x) + (\mathcal{A}_{4i-2})^{4i}, x \in K_{top}(\mathcal{M}(v)) \), is a multiple of \( ch_{2i}(x) + (\mathcal{A}_{4i-2})^{4i} \) by a universal constant. Hence, it suffices to prove the equality of the projections to \( C_{4i}'(\mathbb{Q}) \) of \( ch_{2i}(TS^{[n]}) \) and \( 2u_v(v) \).

We express next the Chern classes of \( TS^{[n]} \) in terms of the Künneth factors of the Chern classes of the universal sheaf. The tangent bundle is isomorphic to the relative extension sheaf \( \mathcal{E}xt^1_{\pi_M}(\mathcal{E}_v, \mathcal{E}_v) \), while the sheaves \( \mathcal{E}xt^i_{\pi_M}(\mathcal{E}_v, \mathcal{E}_v) \), \( i = 0, 2 \), are the trivial line bundles. We get the equality of Chern polynomials

\[
ch(TM(v)) = c \left( -\pi_M, \left[ \mathcal{E}_v^\vee \otimes \mathcal{E}_v \right] \right).
\]

The class, on the right hand side, is independent of the choice of a universal sheaf. Grothendieck-Riemann-Roch yields the equality

\[
ch(TM(v) - 2\mathcal{O}_{\mathcal{M}(v)}) = -\pi_M (u_v^\vee \cup u_v).
\]

Let \( \Lambda := H^*(S, \mathbb{Z}) \) be the Mukai lattice and \( \{v_1, v_2, \ldots, v_{24}\} \) an orthonormal basis of \( \Lambda \otimes \mathbb{Q} \). We have the equalities \( u_v = \sum_i v_i \otimes u_v(v_i) \) and

\[
\pi_M (u_v^\vee \cup u_v) = -\sum_i \left( \sum_j (v_i, v_j) \cdot u_v(v_i) \cup u_v(v_j) \right) = -\sum_i (u_v(v_i) \cup u_v(v_i))^\vee.
\]

More invariantly, the class \(-\pi_M(u_v^\vee \cup u_v)\) is the image of the class \( q = \sum_i v_i \otimes v_i \), inverse to the Mukai pairing, in the tensor square of the Mukai lattice, under the composition

\[
\Lambda \otimes \Lambda \overset{u_v \otimes u_v}{\longrightarrow} H^*(\mathcal{M}(v)) \otimes H^*(\mathcal{M}(v)) \overset{1 \otimes D_{\mathcal{M}(v)}}{\longrightarrow} H^*(\mathcal{M}(v)) \otimes H^*(\mathcal{M}(v)) \overset{\cup}{\longrightarrow} H^*(\mathcal{M}(v)).
\]
Let \( u_{2k} : \Lambda \to H^{2k}(\mathcal{M}(v), \mathbb{Q}) \) be the composition of \( u_v \) with the projection onto the graded summand. Then \( ch_{2i}(T\mathcal{M}(v)) \) is the image of \( q \) under
\[
\sum_{j=0}^{2i} u_{2j} \cup D_{\mathcal{M}(v)} \circ u_{4i-2j} : \Lambda \otimes \Lambda \longrightarrow H^{4i}(\mathcal{M}(v), \mathbb{Q}). \tag{54}
\]
Let \( \bar{u}_{2k} : \Lambda \to H^{2k}(\mathcal{M}(v), \mathbb{Q})/[A_{2k-1}(\mathbb{Q})^{2k} + C''_{2k}(\mathbb{Q})] \cong C'_{2k}(\mathbb{Q}) \) be the composition of \( u_{2k} \) with the projection to \( C'_{2k}(\mathbb{Q}), k \geq 1 \). Set \( \bar{u}_0 := u_0 \). The projection of \( ch_{2i}(T\mathcal{M}(v)) \) to \( C'_{4i}(\mathbb{Q}) \) is the image of \( q \) under the composition
\[
\Lambda \otimes \Lambda \longrightarrow ([\Lambda/v^\perp] \otimes [\Lambda/v^\perp]) \oplus ([\Lambda/v^\perp] \otimes [\Lambda/v^\perp])^{(\bar{u}_4i \cup \bar{u}_0) + (u_0 \cup \bar{u}_4i)} C'_{4i}. \]
We ignored above the intermediate summands of (54), since their images are in \( A_{4i-2} \).

Assume, as we may, that \( n \geq 2 \), so that \( (v, v) \neq 0 \). The images of \( q \) and \( \frac{u\otimes u}{(v,v)} \) in \( [\Lambda/v^\perp] \otimes [\Lambda/v^\perp] \) are equal and \( u_0(v) = (v, v) \cdot 1 \in H^0(\mathcal{M}(v), \mathbb{Z}) \). Thus, the image of \( q \) is \( 2\bar{u}_{4i}(v) \). We conclude, that the image of \( ch_{2i}(T\mathcal{M}(v)) \) in \( C'_{4i}(\mathbb{Q}) \) is equal to \( 2\bar{u}_{4i}(v) \). \( \square \)

### 4.5 The monodromy subgroup of \( O(H^2(S^n, \mathbb{Z})) \)

We describe the image of \( \Gamma_v \) in \( O(H^2(S^n, \mathbb{Z})) \), under the monodromy representation of Theorem 1.6. Consider the homomorphism
\[
f : O(v^\perp) \longrightarrow O_+(v^\perp), \tag{55}\]
into the kernel of the orientation character (47) of \( O(v^\perp) \), sending \( g \) to \((-1)^{\text{cov}(g)} \cdot g \).

**Lemma 4.10** The following subgroups of \( O_+(v^\perp) \) are all equal.

1. The image via the monodromy representation \( \text{mon} \) of \( \Gamma_v \) in \( O(H^2(S^n, \mathbb{Z})) \), under the identification \( v^\perp \cong H^2(S^n, \mathbb{Z}) \) of Theorem 1.7.
2. The subgroup \( W(v^\perp) \) given in (17).
3. The subgroup of orientation preserving isometries, which can be extended to isometries of the whole Mukai lattice.
4. The image \( f(\Gamma_v) \), of the stabilizer \( \Gamma_v \) of \( v \) in the Mukai lattice, under the homomorphism (55).
5. The inverse image of \( \{1, -1\} \), under the natural homomorphism \( O_+(v^\perp) \rightarrow \text{Aut}((v^\perp)/v^\perp) \).

**Proof:** 1 = 4: Clear from the equality \( \theta_v \circ f(g) \circ \theta_v^{-1} = \text{mon}(g), g \in \Gamma_v \), where \( \theta_v \) is the isomorphism (33) used in Theorem 1.7.

3 = 4: The inclusion \( 4 \subset 3 \) is clear. We prove the reverse inclusion. Let \( \tilde{g} \) be an isometry of the whole Mukai lattice, which leaves \( v^\perp \) invariant and restricts to an
orientation preserving isometry \( g \) of \( v^\perp \). Then \( \tilde{g}(v) = v \) or \(-v\). If \( \tilde{g}(v) = v \), then \( \tilde{g} \) is an orientation preserving isometry in \( \Gamma_v \) and \( f(\tilde{g}) = g \). If \( \tilde{g}(v) = -v \), then \( \tilde{g} \) reverses the orientation of the positive cone of the Mukai lattice, since \( v \) belongs to that cone. Consequently, \(-\tilde{g} \) belongs to \( \Gamma_v \) and is orientation reversing. Hence, \( f(-\tilde{g}) = g \).

2 = 4: The inclusion \( \mathcal{W}(v^\perp) \subset f(\Gamma_v) \) is clear. The reverse inclusion follows from Corollary 8.10 and the fact, that \( f \) leaves invariant each reflection with respect to a \(-2\) vector, and multiplies by \(-1\) each reflection with respect to a \(+2\) vector.

4 = 5: Follows immediately from Lemma 8.3. \( \Box \)

When \( n - 1 \) is a prime power, \( \mathcal{W}(v^\perp) \) is the whole of \( O_+(v^\perp) \), by Lemma 8.3.

### 4.6 Comparison with Verbitsky’s representation

We study first the Zariski closures of the two actions of the group \( \Gamma_v \) in \( GL(H^*(S^{[n]}, \mathbb{C})) \) (Lemma 4.11). We then compare the monodromy action to Verbitsky’s related representation (Lemma 4.13).

The group \( O(n, m; \mathbb{R}) \), of a non-definite and non-degenerate bilinear form, has four connected components in the classical topology. Its Zariski closure \( O(n + m, \mathbb{C}) \), in the standard representation, has two connected components.

**Lemma 4.11** 1. The Zariski closure of the image \( \gamma(\Gamma_v) \) in \( GL(H^*(S^{[n]}, \mathbb{C})) \) is isomorphic to the group \( O(H^2(S^{[n]}, \mathbb{C})) \) of isometries of \( H^2(S^{[n]}, \mathbb{Z}) \) with respect to Beauville’s pairing.

2. The Zariski closure of \( \text{mon}(\Gamma_v) \) in \( GL(H^*(S^{[n]}, \mathbb{C})) \) is isomorphic to \( O(H^2(S^{[n]}, \mathbb{C})) \times \mathbb{Z}/2\mathbb{Z} \), if \( n \geq 3 \). It has four connected components.

3. Both the \( \gamma \) and the \( \text{mon} \) actions, of \( \Gamma_v \) on \( H^*(S^{[n]}, \mathbb{Z}) \), extend naturally to an action of the group \( O(H^2(S^{[n]}, \mathbb{Z})). O(H^2(S^{[n]}, \mathbb{Z})) \) acts on \( H^*(S^{[n]}, \mathbb{Q}) \) via ring automorphisms. The extended \( \gamma \)-action leaves invariant the class \( u_v \), given in (27).

**Proof:** 3) The character \( \text{cov} \) extends to the orientation character (47) of \( O(H^2(S^{[n]}, \mathbb{Z})). \)

Hence, an extension of \( \gamma \) yields an extension of \( \text{mon} \). We may restrict attention to the representation \( \gamma \). The group \( \Gamma_v \) is a normal subgroup, of finite index, in the group \( O(H^2(S^{[n]}, \mathbb{Z})) \) (see Lemma 8.3). Let \( \mathcal{R} \) be the weighted polynomial ring, generated by the subspaces of generators \( B_i \), introduced in Lemma 3.16. The invariance with respect to the representation \( \gamma \), of the subspaces \( B_i \), gives rise to a \( \Gamma_v \)-action on the ring \( \mathcal{R} \). Moreover, the surjective homomorphism \( q: \mathcal{R} \to H^*(S^{[n]}, \mathbb{Q}) \) is \( \Gamma_v \)-equivariant. Hence, the relations ideal in \( \mathcal{R} \) is \( \gamma(\Gamma_v) \)-invariant. The \( B_i \) are \( O(H^2(S^{[n]}, \mathbb{Z})) \) representations, via the isometry \( v^\perp \cong H^2(S^{[n]}, \mathbb{Z}) \). The \( \Gamma_v \) action on \( \mathcal{R} \) extends to an algebraic action of \( O(H^2(S^{[n]}, \mathbb{C})) \) on the complexification of \( \mathcal{R} \). \( \Gamma_v \) is Zariski dense in \( O(H^2(S^{[n]}, \mathbb{C})). \)

Hence, the relations ideal in \( \mathcal{R} \) is \( O(H^2(S^{[n]}, \mathbb{Z})) \)-invariant. The extension of the \( \gamma \)-action to an \( O(H^2(S^{[n]}, \mathbb{Z})) \)-action follows.
The invariance, of the class (27), with respect to the Zariski dense subgroup $\Gamma_v$, implies the invariance with respect to $O(H^2(S^{[n]}, \mathbb{C}))$ and thus with respect to $O(H^2(S^{[n]}, \mathbb{Z}))$ as well.

1) The subspaces $B_i$, generating $\mathcal{R}$, are defined in terms of the $\Gamma_v$-invariant class $u_v$. Hence the Zariski closure, of the image $\gamma(\Gamma_v)$ of $\Gamma_v$ in $GL(\mathcal{R} \otimes \mathbb{Q}, \mathbb{C})$, is isomorphic to $O(H^2(S^{[n]}, \mathbb{C}))$. The latter Zariski closure maps injectively into $GL(H^*(S^{[n]}, \mathbb{C}))$, via the homomorphism $q$. The Zariski closure in $GL(\mathcal{R} \otimes \mathbb{Q}, \mathbb{C})$ must map to the Zariski closure in $GL(H^*(S^{[n]}, \mathbb{C}))$, since the $\gamma$-action of $\Gamma_v$ on $H^*(S^{[n]}, \mathbb{C})$ factors through $q$.

2) When $n \geq 3$, the $\Gamma_v$-representation $v^\perp \otimes cov$ appears as a subrepresentation of $H^4(S^{[n]}, \mathbb{Q})$, by Lemma 4.8. Hence, the direct sum $v^\perp \oplus [v^\perp \otimes cov]$ appears as a subrepresentation of $H^2(S^{[n]}, \mathbb{Q}) \oplus H^4(S^{[n]}, \mathbb{Q})$. The Zariski closure, of the image of $\Gamma_v$ in the complexification of $v^\perp \oplus [v^\perp \otimes cov]$, is isomorphic to $O(H^2(S^{[n]}, \mathbb{C})) \times \mathbb{Z}/2\mathbb{Z}$. □

**Remark 4.12** Though natural, the extensions in part 3 of Lemma 4.11 are not unique, if $(n-1)$ is not a prime power. The quotient $O(H^2(S^{[n]}, \mathbb{Z}))/\Gamma_v$ is a finite abelian group of exponent 2 (section 8), and other extensions of the action are parametrized by characters of the quotient.

We may regard the monodromy representation $H^*(S^{[n]}, \mathbb{Z})$ as a representation of the subgroup $\mathcal{W}$, given in (17), since the representation $\text{mon}$ of $\Gamma_v$ factors through the injective homomorphism $\mu : \mathcal{W} \rightarrow \text{Mon}$ given in (18). Assume $n \geq 3$. Then the isomorphism $f : \Gamma_v \rightarrow \mathcal{W}$ conjugates the orientation character, of $\Gamma_v$, to the character

$$\phi : \mathcal{W} \rightarrow \mathcal{W}/[\mathcal{W} \cap \Gamma_v] \cong \mathbb{Z}/2\mathbb{Z}. \quad (56)$$

The unique extension of $g \in \mathcal{W}$ to $\Gamma$ sends $v$ to $\pm v$, where the sign is determined by $\phi(g)$. Note, that the image of $\mathcal{W}$ under the homomorphism (73) has order 2 and thus, the character $\phi$ is the restriction of (73). Hence, the character $\phi$ is precisely a character of the type mentioned in remark 4.12. Rewriting the formula (14) for the monodromy representation in terms of $\mathcal{W}$, we use the character $\phi$, instead of $cov$.

Let $\mathcal{SW} \subset \mathcal{W}$ be the index 2 subgroup $\mathcal{W} \cap SO(H^2(S^{[n]}, \mathbb{Z}))$. Let us compare the two representations of $\mathcal{SW}$ in $\text{Aut}[H^*(S^{[n]}, \mathbb{R})]$; the monodromy representation $\mu$ and the restriction of Verbitsky’s representation $\rho$, given in (50). Define $\eta : \mathcal{SW} \rightarrow \text{Aut}[H^*(S^{[n]}, \mathbb{R})]$ by $\eta(f) := \rho(f)^{-1}\mu(f)$.

**Lemma 4.13** Verbitsky’s representation $\rho$ agrees with the monodromy representation $\mu$ on the intersection of $\mathcal{SW}$ with the kernel of $\phi$. Furthermore, the map $\eta$ is a homomorphism from $\mathcal{SW}$ into the center of the subgroup of $\text{Aut}[H^*(S^{[n]}, \mathbb{R})]$, generated by $\text{Mon}$ and the image of $\rho$. The image of $\eta$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, if $n \geq 3$.

**Proof:** Given a monodromy operator $g \in \text{Mon}$, denote by $\bar{g}$ its restriction to $H^2(S^{[n]})$. The Mon-equivariance of $\rho$ means the equality $g\rho(f)g^{-1} = \rho(\bar{g}f\bar{g}^{-1})$, for $g \in \text{Mon}$ and $f \in SO(H^2(S^{[n]}, \mathbb{R}))$. Extend $\rho$ to a representation of $O(H^2(S^{[n]}, \mathbb{R}))$, by letting $-id$
act as the Duality operator $D_{S'^{(n)}}$ on $H^*(S'^{(n)})$ (see (14)). The extended $\rho$ remains $\text{Mon}$-equivariant. Then $\rho(\bar{g})^{-1}g$ commutes with $\rho(f)$, for all $g \in \text{Mon}$ and $f \in SO(H^2(S'^{(n)}, \mathbb{R}))$. The map $\eta(g) := \rho(\bar{g})^{-1}g$ is thus a representation of $\text{Mon}$.

$C_{2i}(\mathbb{R})$ and $C_{2i}'(\mathbb{R})$ are $\text{Mon}$-invariant and $\rho$-invariant, by Lemma 4.8. Thus, they are also $\eta$-invariant. Furthermore, for each $g \in \text{Mon}$, the element $\eta(g)$ must act on $C_{2i}(\mathbb{R})$ and $C_{2i}'(\mathbb{R})$ diagonally via a character of $\text{Mon}$ (since $\eta(g)$ is an intertwining operator of a zero or irreducible subrepresentation of $\rho$).

The character group $\text{Char}(SW)$ is generated by the restriction of $\phi$, by Corollary 8.10. $\text{Char}(SW)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, if $n \geq 3$, and $\text{Char}(SW)$ is trivial, if $n = 2$. Hence, the restriction of $\eta$ to the subgroup $\mu(\ker(\phi) \cap SW)$ of $\text{Mon}$ is trivial.

If $n \geq 3$, the non-triviality of $\eta$ follows, by comparing the Zariski closures of the images $\mu(SW)$ and $\rho(SW)$. The Zariski closure of $\mu(SW)$ in $G(\text{Hom}(S'^{(n)}, \mathbb{C}))$ has two connected components (Lemma 4.11). On the other hand, the Zariski closure of the image of $\rho$ has one connected component, because $\rho$ can be extended to a representation of $SO(H^2(S'^{(n)}, \mathbb{C}))$. $\square$

### 4.7 Monodromy-equivariance of the Chern character

The normalized Chern character $u_v$, given in (27), is invariant with respect to the diagonal action of the stabilizer $\Gamma_v$, provided we use on the moduli factor the representation $\gamma$, given in (45) (Theorem 3.9). We show in Lemma 4.15, that the class $u_v$ is not invariant, in general, if we replace $\gamma$ by the monodromy representation (14). Let us first identify the diagonal action of an appropriate subgroup of $O(v^\perp)$. Theorem 1.7 yields the isomorphism $v^\perp \cong H^2(\mathcal{M}(v), \mathbb{Z})$. Consequently, we can regard the class $u_v$ as an element of

$$[H^2(\mathcal{M}(v), \mathbb{Q}) \oplus \mathbb{Q} \cdot v] \otimes H^*(\mathcal{M}(v), \mathbb{Q}). \quad (57)$$

Let $\mathcal{W} := \mathcal{W}(v^\perp)$ be the subgroup of $O(v^\perp)$ described in Lemma 4.10. Consider the diagonal action of $\mathcal{W}$ on (57), regarding $\mathbb{Q} \cdot v$ as the trivial character, and identifying $H^2(\mathcal{M}(v), \mathbb{Q})$ with $v^\perp \otimes_\mathbb{Z} \mathbb{Q}$ via the isomorphism in Theorem 1.7.

The case of $S'^{(2)}$ is exceptional. Invariance holds for the monodromy action as well.

**Lemma 4.14** *When $v = (1, 0, -1)$ is the Mukai vector of $S'^{(2)}$, the normalized Chern character $u_v$ is invariant under the diagonal action of $\mathcal{W}$.*

**Proof:** Theorems 3.9 and 1.6 imply, that $u_v$ is invariant with respect to the monodromy representation of $(\Gamma_v)^{\text{cov}}$. It suffices to show, that $(\Gamma_v)^{\text{cov}}$ surjects onto $\mathcal{W}$. This follows from the fact, that the kernel of the homomorphism $f : \Gamma_v \to \mathcal{W}$, given in (55), contains (and is generated by) the orientation reversing isometry $-\sigma_v$. The Mukai vector $v = (1, 0, -1)$ of $S'^{(2)}$ is a $+2$ vector. $\Gamma_v$ surjects onto $O(v^\perp)$. The reflection $-\sigma_v$ is in $\Gamma_v$ and it restricts as $-id_{v^\perp}$. Hence, $f(-\sigma_v) = 1$. $\square$

We show next, that the class $u_v$ is not invariant, when $v = (1, 0, 1 - n)$ is the Mukai vector of $S'^{(n)}$ and $n \geq 3$. In that case, the homomorphism (14) induces an isomorphism
between $\Gamma_v$ and $\mathcal{W}$. It conjugates the orientation character of $\Gamma_v$ to the character $\phi$ of $\mathcal{W}$ given in (56). The kernel $\mathcal{W} \cap \Gamma_v$ of $\phi$ is precisely $\Gamma_v^{cov}$. The character $\phi$ is trivial if $(v, v) = 2$, but non-trivial if $(v, v) \geq 4$.

Set $u_v := u'_v + u''_v$ to be the decomposition of $u_v$ with respect to the direct sum decomposition of the left factor of (57), where $u'_v$ is in span$\{v\} \otimes H^*(S^{[n]}, \mathbb{Q})$. Let $u'_v := u'_v(\text{odd}) + u'_v(\text{even})$ be the decomposition of $u'_v$ according to the decomposition of the right hand factor $H^*(S^{[n]}, \mathbb{Q})$ into weights congruent to 2 modulo 4 (the odd case) and weights congruent to 0 modulo 4 (the even case). Let $u''_v := u''_v(\text{odd}) + u''_v(\text{even})$ be the analogous decomposition of $u''_v$. Set $u^+_v := u'_v(\text{even}) + u''_v(\text{odd})$ and $u^-_v := u'_v(\text{odd}) + u''_v(\text{even})$. Theorem 3.9 implies, that the class $u^+_v$ is invariant with respect to the diagonal action of $\mathcal{W} \cap \Gamma_v$. Furthermore, it implies the equivariance of $u^-_v$ and $u^+_v$ in the following Lemma.

**Lemma 4.15** The class $u_v$, given in (27), decomposes as a sum of the $\mathcal{W}$ invariant class $u^+_v$ and the class $u^-_v$, which is acted upon by the character $\phi$ of $\mathcal{W}$. The class $u^+_v$ does not vanish for $n \geq 1$. If $n \geq 3$, then the summand $u^-_v$ does not vanish as well.

**Proof:** Only the non-vanishing of $u^-_v$ remains to be proven. Let $u_v(2i)$ be the summand of degree $2i$, with respect to the weight decomposition of the right hand factor of (57). Lemma 3.16 implies, that $u''(2i)$ does not vanish for even weights in the range $0 < 2i \leq n + 1$. Similarly, $u'_v(2i)$ does not vanish for even weights in the range $4 \leq 2i \leq n$. It follows, that $u''_v(\text{even})$ does not vanish, for $n \geq 3$. Furthermore, $u'_v(\text{odd})$ does not vanish, for $n \geq 6$. \[
\]

**Example 4.16** The near-invariance, of the normalized Chern character $u_v$ of the universal sheaf, has the following simple analogue. Let $V$ be a $2n$-dimensional vector space, $n \geq 2$. The automorphism group $\text{Aut}(G(n, V))$ of $G(n, V)$ has two connected components $[C]$. The identity component is $\text{PGL}(V)$. The character $\text{Aut}(G(n, V)) \rightarrow \text{Aut}(G(n, V))/\text{PGL}(V)$ plays a role analogous to that of the character $cov$ in the monodromy representation (14). The Chern character $\text{ch}(\tau)$, of the tautological subbundle, is $\text{PGL}(V)$-invariant but not $\text{Aut}(G(n, V))$-invariant. Choose any isomorphism $f : V \rightarrow V^*$ and denote by $f$ also the induced isomorphism $G(n, V) \rightarrow G(n, V^*)$, sending a subspace $W$ to $f(W)$. Let $\iota : G(n, V^*) \rightarrow G(n, V)$ be the natural isomorphism. The composition $\iota \circ f$ is an automorphism of $G(n, V)$, given by $W \mapsto f(W)^\perp$. Clearly, $\iota^* \tau$ is isomorphic to the dual $q^*_{G(n, V^*)}$ of the universal quotient bundle $q_{G(n, V^*)}$. Hence, the pullback $(\iota \circ f)^* c_2(\tau) = f^*(c_2(q^*_{G(n, V^*)})) = c_2(q^*_{G(n, V)}) = c_2(q_{G(n, V)}) = c_1(\tau)^2 - c_2(\tau)$ is different from $c_2(\tau)$.

The proof of Theorem 7.10 relates the above example to the monodromy representation (14). A certain grassmannian bundle embeds in $S^{[n]}$ and plays an important role in the construction of the monodromy operator in Theorem 7.10.
5 Equivalences of Derived Categories

In section 5.1 we recall the language of Fourier-Mukai functors. In section 5.2 we use this language to provide a conceptual and motivic formula for the monodromy operator $\text{mon}(g)$, given in (21), associated to a Hodge-isometry $g$ of Mukai lattices (Lemma 5.4). The topological formula (39) for the class $\gamma(g, \mathcal{E}_v, \mathcal{E}_{g(v)})$ was motivated by the ideal case discussed in section 5.3. This is the case, where an auto-equivalence induces an isomorphism of moduli spaces.

5.1 Fourier-Mukai transformations

Given a projective variety $X$, denote by $D(X) := D^b_{\text{Coh}}(X)$ the bounded derived category of coherent sheaves on $X$. Given projective varieties $X_1$ and $X_2$, together with an object $F$ in $D(X_1 \times X_2)$, we get the functor

$$\Phi^F_{X_1 \to X_2} : D(X_1) \to D(X_2),$$

deﬁned by

$$\Phi^F_{X_1 \to X_2}(x) := R\pi_2\ast \left( F \otimes \pi_1^*(x) \right), \quad (58)$$

where $\pi_i : X_1 \times X_2 \to X_i$ is the projection, for $i = 1, 2$. If the functor is an equivalence, it is called a Fourier-Mukai transformation and the object $F$ is called its kernel. A theorem of Orlov implies, that every equivalence, of the derived categories of two projective varieties, is a Fourier-Mukai transformation. Moreover, the kernel is unique up to isomorphism in the derived category of their product [Or1].

Let $S_1$ and $S_2$ be K3 surfaces. An equivalence $\Phi : D(S_1) \to D(S_2)$ determines a class

$$Z_\Phi := \pi_1^* \sqrt{\text{td}_{S_1}} \cdot \text{ch}(F) \cdot \pi_2^* \sqrt{\text{td}_{S_2}}$$

in $H^*(S_1 \times S_2, \mathbb{Z})$, where $F$ is the isomorphism class of the kernel associated to $\Phi$ by Orlov’s Theorem. The integrality of the class $Z_\Phi$ was proven in [HLOY], using ideas of Mukai [Mu2].

For a K3 surface $S$, let $\text{Aut}D(S)$ be the group of auto-equivalences of its derived category. Denote by $G$ the group of Hodge isometries of the Mukai lattice of $S$. Results of Mukai and Orlov yield the following Theorem.

Theorem 5.1 ([HLOY] Theorem 1.6)

1. The class $Z_\Phi$ induces a Hodge isometry of Mukai lattices, denoted by $\text{ch}(\Phi) : H^*(S_1) \to H^*(S_2)$.

2. The resulting map

$$\text{ch} : \text{Aut}D(S) \to G \quad (59)$$

is a group homomorphism. Its image has index at most 2. Moreover, $G$ is generated by the image of $\text{ch}$ and the isometry $D$ of the Mukai lattice given in (29).
Let us prove part 1 of Theorem 3.9 for two dimensional moduli spaces.

**Lemma 5.2** Let \( v \) be a Mukai vector of a K3 surface \( S_1 \) and \( g : H^*(S_1, \mathbb{Z}) \to H^*(S_2, \mathbb{Z}) \) an isometry of Mukai lattices as in Theorem 3.9. Assume, in addition, that \( v \) is isotropic. Then part 1 of Theorem 3.9 holds.

**Proof:** The proof below assumes that universal families \( \mathcal{E}_v \) and \( \mathcal{E}_{g(v)} \) exist. The general proof is similar, using universal classes \( e_v \) and \( e_{g(v)} \) (Definition 3.2). The moduli spaces \( \mathcal{M}(v) \) and \( \mathcal{M}(g(v)) \) are K3 surfaces, by the work of Mukai. We note first a simplification, in the formula (37) for \( \gamma(g, \mathcal{E}_v, \mathcal{E}_{g(v)}) \), when the dimension \( m \) of the moduli spaces is 2. Given a class \( f \), in the \( K \)-group of \( \mathcal{M}(v) \times \mathcal{M}(g(v)) \), with \( c_1(f) = 0 \), then the following equation holds in \( H^4(\mathcal{M}(v) \times \mathcal{M}(g(v)), \mathbb{Z}) \).

\[
c_2(-f) = ch_2(f). \tag{60}
\]

This equality extends to an equality between the topological formula (37) and the one obtained from it by replacing \( c_2((\ell(\bullet))^{-1}) \) with \( (\bullet)_2 \), provided \( (\bullet)_1 = 0 \).

We claim, that the operator \( \gamma_{g,v} : H^*(\mathcal{M}(v), \mathbb{Q}) \to H^*(\mathcal{M}(g(v)), \mathbb{Q}) \), associated to the class \( \gamma(g, \mathcal{E}_v, \mathcal{E}_{g(v)}) \), is the grade 0 summand \( h_0 \), in the weight decomposition of the homomorphism

\[
h := ch(\Phi^{\mathcal{E}_v}) \circ (D \circ g \circ D) \circ ch(\Phi^{\mathcal{E}_v})^{-1}.
\]

Let \( \Delta \) be the diagonal in \( S_1 \times S_1 \). The above claim follows from the definition (39) of \( \gamma(g, \mathcal{E}_v, \mathcal{E}_{g(v)}) \), the vanishing of \( h_{-2} \) proven below, equation (60), the identity

\[
[ch(\Phi_{\mathcal{M}(v) \to S_1}^{\mathcal{E}_v}) = [ch(\Phi_{\mathcal{M}(v) \to S_1}^{\mathcal{E}_{g(v)}})]^{-1}]
\]

(Theorem 4.9 in [Mu2]), and the identity

\[
[(1 \otimes g)(ch(\mathcal{E}_v))]' = (D_M \otimes D)(1 \otimes g)(ch(\mathcal{E}_v)) = (1 \otimes DgD)(ch(\mathcal{E}_v')).
\]

The class \( \gamma(g, \mathcal{E}_v, \mathcal{E}_{g(v)}) \) is integral, since \( h \) is a composition of integral isometries. The homomorphism \( ch(\Phi^{\mathcal{E}_v}) : H^*(S, \mathbb{Z}) \to H^*(\mathcal{M}(v), \mathbb{Z}) \) maps the Mukai vector \( D(v) \) to the fundamental class \( w_v \) in \( H^4(\mathcal{M}(v), \mathbb{Z}) \), by Lemma 4.11 of [Mu2]. Consequently, \( h \) is an integral isometry of Mukai lattices, which maps \( w_v \) to the fundamental class \( w_{g(v)} \).

Considering orthogonal complements, we see that \( h \) maps the direct sum \( H^2(\mathcal{M}(v), \mathbb{Z}) \oplus H^4(\mathcal{M}(v), \mathbb{Z}) \) to \( H^2(\mathcal{M}(g(v)), \mathbb{Z}) \oplus H^4(\mathcal{M}(g(v)), \mathbb{Z}) \). In other words, the negative summands, in the weight decomposition of \( h \), vanish and \( h = h_0 + h_2 + h_4 \). We have the following equality

\[
(h_0(1), w_{g(v)}) = (h(1), w_{g(v)}) = (h(1), h(w_v)) = (1, \omega_v) = 1.
\]

Hence, \( h_0(1) = 1 \). Moreover, \( (h_0(x), h_0(y)) = (h(x), h(y)) = (x, y) \), for all \( x \) and \( y \) in \( H^2(\mathcal{M}(v)) \). It follows, that \( \gamma_{g,v} \) is a ring isomorphism. \( \square \)
5.2 A Chow-theoretic formula for monodromy operators

Given an object $F$ of $D(S_1 \times S_2)$ and a projective variety $M$, we construct the object

$$F_M := (\pi_{21}^* F)^L \otimes (\pi_{13}^* O_\Delta)$$

in $D(M \times S_1 \times M \times S_2)$, where $\Delta$ is the diagonal of $M \times M$ and $\pi_{ij}$ is the projection from $M \times S_1 \times M \times S_2$ onto the product of the $i$-th and $j$-th factors. Given a Fourier-Mukai transformation $\Phi^F_{S_1 \rightarrow S_2}$, we get a natural functor

$$\Phi^F_M : D(M \times S_1) \rightarrow D(M \times S_2),$$

using formula (58) with the kernel $F_M$. The functor $\Phi^F_M$ is an equivalence of derived categories as well [Or2].

**Definition 5.3** Let $\mathcal{M}(v)$ be a moduli space of stable sheaves on $S_1$ and $\mathcal{E}_v$ a universal sheaf over $\mathcal{M}(v) \times S_1$. We will refer to the class $\Phi^{F_{\mathcal{M}(v)}}(\mathcal{E}_v)$ as the relative Fourier-Mukai transform of the universal sheaf.

The construction of the Fourier-Mukai functor $\Phi^F_M$ generalizes to define an equivalence of derived categories of twisted sheaves $\Phi^F_M : D_{\text{coh}}(M \times S_1, \pi_M^* \alpha) \rightarrow D_{\text{coh}}(M \times S_2, \pi_M^* \alpha)$, for any class $\alpha$ in the Brauer group of $M$ (see section 3.1).

Let $(S_i, v_i, \mathcal{L}_i)$ be two objects of the groupoid $\mathcal{G}$. Let $p_i : \mathbb{P}_{v_i} \rightarrow \mathcal{M}_{\mathcal{L}_i}(v_i)$ be the projective bundle defined in section 3.1 and $\tilde{\mathcal{E}}_{v_i}$ the universal sheaf over $\mathbb{P}_{v_i}$ (see equation (26)). When a universal sheaf $\mathcal{E}_{v_i}$ exists over $\mathcal{M}_{\mathcal{L}_i}(v_i)$, we may assume that $p_i$ is the identity and $\tilde{\mathcal{E}}_{v_i} = \mathcal{E}_{v_i}.$

**Lemma 5.4**

1. The following equality holds in $H^*(\mathbb{P}_{v_1} \times S_2)$.

$$\text{ch} \left( \Phi^{F_{\mathcal{L}_1}}(\tilde{\mathcal{E}}_{v_1}) \right) \sqrt{td_{S_2}} = (\text{id}_{\mathcal{E}_{v_1}} \otimes \text{ch}(\Phi^F)) \left( \text{ch}(\tilde{\mathcal{E}}_{v_1}) \sqrt{td_{S_1}} \right).$$

2. Assume that $v_2 = \text{ch}(\Phi^F)(v_1)$. The following equality holds in the cohomology ring of $\mathbb{P}_{v_1} \times \mathbb{P}_{v_2}$.

$$(p_1 \times p_2)^* \gamma(\text{ch}(\Phi^F), v_1) = c_m \left[ - \pi_{13} \left( \pi_{12}^* (\Phi^{F_{\mathcal{L}_1}}(\tilde{\mathcal{E}}_{v_1}))^L \otimes \pi_{23}^* (\tilde{\mathcal{E}}_{v_2}) \right) \right],$$

where $m$ is the dimension of $\mathcal{M}(v_1)$ and $\pi_{ij}$ is the projection from $\mathbb{P}_{v_1} \times S_2 \times \mathbb{P}_{v_2}$.

3. If $v_2 = [\text{ch}(\Phi^F)(v_1)]^\vee$ and the isometry $\text{ch}(\Phi^F)$ is orientation-preserving, then

$$(p_1 \times p_2)^* \text{mon}(D \circ \text{ch}(\Phi^F), v_1) = c_m \left[ - \pi_{13} \left( \pi_{12}^* (\Phi^{F_{\mathcal{L}_1}}(\tilde{\mathcal{E}}_{v_1}))^L \otimes \pi_{23}^* (\tilde{\mathcal{E}}_{v_2}) \right) \right],$$

where $\text{mon}(g, v_1)$ is given in (22).
Remark 5.5 It is tempting to speculate, that the object
\[ R\pi_{13} \left( \pi_{12}^* (\Phi_{\mathcal{M}(v_1)} [\mathcal{E}_{v_1}])^\vee \otimes \pi_{23}^* (\mathcal{E}_{v_2}) \right), \]
on the right hand side of equation (61), may play a role in the study of equivalences of the derived category of moduli spaces of sheaves. One might even relax the assumption, that \( v_2 = ch(\Phi F)(v_1) \). We take in (61) the middle dimensional Chern class of the object, as we are studying graded automorphisms of the cohomology ring. See lemma 5.2, for two dimensional moduli spaces, and remark 3.10, for the elliptic curve case. It should be noted, however, that some adjustments to the above object would be necessary, when \( m > 2 \). When \( \Phi \) is the identity, \( v = v_i \), and \( \mathcal{E}_v = \mathcal{E}_{v_i} \), \( i = 1, 2 \), the above object fits in a distinguished triangle, involving also the relative extension sheaves \( E xt^i_{\pi_{13}} (\pi_{12}^* \mathcal{E}_v, \pi_{23}^* \mathcal{E}_v) \), \( i = 1, 2 \). The sheaf \( E xt^2_{\pi_{13}} (\pi_{12}^* \mathcal{E}_v, \pi_{23}^* \mathcal{E}_v) \) is the structure sheaf of the diagonal in \( \mathcal{M}(v) \times \mathcal{M}(v) \), while the sheaf \( E xt^1_{\pi_{13}} (\pi_{12}^* \mathcal{E}_v, \pi_{23}^* \mathcal{E}_v) \) is either 0, when \( \mathcal{M}(v) \) is two dimensional, or a torsion free sheaf of rank \( m - 2 \), when \( m > 2 \) [Mu2, Ma2]. One would need to eliminate the contribution of \( E xt^1_{\pi_{13}} (\pi_{12}^* \mathcal{E}_v, \pi_{23}^* \mathcal{E}_v) \). Theorem 3.5 is equivalent to the statement, that \( c_m(E xt^1_{\pi_{13}} (\pi_{12}^* \mathcal{E}_v, \pi_{23}^* \mathcal{E}_v)) = 1 - (m - 1)! \) times the class of the diagonal [Ma2].

Proof of lemma 5.4: 1) Let \( \iota : \Delta \hookrightarrow \mathbb{P}_{v_1} \times \mathbb{P}_{v_2} \), be the closed immersion of the diagonal and \( [\Delta] \) the cohomology class Poincare dual to the diagonal. Let \( \pi_{ij} \) be the projection from \( \mathbb{P}_{v_1} \times S_1 \times \mathbb{P}_{v_2} \times S_2 \) onto the product of the \( i \)-th and \( j \)-th factors.

\[ ch \left( \Phi_{\mathbb{P}_{v_1}} (\mathcal{E}_{v_1}) \right) \cdot \sqrt{td_{S_2}} = ch \left( \pi_{34}, \left\{ \pi_{12}^* \mathcal{E}_{v_1} \otimes \pi_{24}^* F \otimes \pi_{13}^* \mathcal{O}_{\Delta} \right\} \right) \cdot \sqrt{td_{S_2}} \]
\[ = \pi_{34}, \left( ch \left( \pi_{12}^* \mathcal{E}_{v_1} \otimes \pi_{24}^* F \otimes \pi_{13}^* \mathcal{O}_{\Delta} \right) \pi_{2}^* td_{S_1} \cdot \pi_{1}^* td_{\mathbb{P}_{v_1}} \right) \cdot \sqrt{td_{S_2}} \]
\[ = \pi_{34}, \left( ch(\mathcal{E}_{v_1}) \sqrt{td_{S_1}} \cdot \pi_{2}^* \sqrt{td_{S_1}} \cdot \pi_{1}^* \mathcal{O}_{\Delta} \right) \cdot \sqrt{td_{S_2}} \]
\[ = \left( ch(\Phi F) \otimes id_{\mathbb{P}_{v_1}} \right) \left( ch(\mathcal{E}_{v_1}) \right) \sqrt{td_{S_1}}. \]

The first equality is the definition of \( \Phi_{\mathbb{P}_{v_1}} \). The second follows from Grothendieck-Riemann-Roch. The third follows via Grothendieck-Riemann-Roch again, applied to compute the Chern character of the pushforward of \( \mathcal{O}_{\Delta} \) via the closed immersion \( \iota \).

\[ ch(\iota_4 \mathcal{O}_{\Delta}) = \iota_4 (td(N_{\Delta/\mathbb{P}_{v_1} \times \mathbb{P}_{v_1}})^{-1}) = [\Delta] \cup \pi_{1}^* td_{\mathbb{P}_{v_1}}^{-1} \]

([Fu] Example 15.2.15). The fourth equality follows from the projection formula and the definition of \( ch(\Phi F) \).

2) Set \( g := ch(\Phi F) \). Assume first that universal sheaves \( \mathcal{E}_{v_i} \), \( i = 1, 2 \), exist. Then \( \gamma(g, v_1) = \gamma(g, \mathcal{E}_{v_1}, \mathcal{E}_{v_2}) \), by Lemma 3.14, and part 2 follows from part 1 via Grothendieck-Riemann-Roch.

We sketch the proof of the general case. Let \( \gamma'(g, \mathcal{E}_{v_1}, \mathcal{E}_{v_2}) \) be the class obtained by taking instead the Chern class \( c_{m-1} \) on the right hand side of (61). Equations (26) and
The homomorphism $\gamma$ of the groupoid $\mathcal{G}$ given in (34). The objects of $\mathcal{G}_{Mukai}$ are the same as those of $\mathcal{G}$. Let $x_i := (S_i, v_i, H_i)$, $i = 1, 2$, be objects of $\mathcal{G}$. The morphisms in $\text{Hom}_{\mathcal{G}_{Mukai}}(x_1, x_2)$ come in two flavors and depend on a Fourier-Mukai transformation $\Phi^{F}_{S_1 \to S_2}$ with kernel $F$, satisfying the two conditions below.

1. The isometry $g := ch(\Phi^{F})$ is orientation-preserving (Definition 4.2), with respect to the orientation of the Mukai lattices in Remark 4.3, where $ch$ is the homomorphism (59).

2. One of the following holds for some integer $i$:

   (2a) $g(v_1) = v_2$ and $\Phi^{F}(E_1)$ is equivalent in $D(S_2)$ to the shift $E_2[i]$ of some $H_2$-stable sheaf $E_2$, for every $H_1$-stable sheaf $E_1$ with Mukai vector $v_1$.

   (2b) $g(v_1) = (v_2)^\vee$ and the dual $[\Phi^{F}(E_1)]^\vee$ is equivalent in $D(S_2)$ to the shift $E_2[i]$ of some $H_2$-stable sheaf $E_2$, for every $H_1$-stable sheaf $E_1$ with Mukai vector $v_1$.

If conditions (1) and (2a) hold, then $g$ belongs to $\text{Hom}_{\mathcal{G}_{Mukai}}(x_1, x_2)$. If conditions (1) and (2b) hold, then $D \circ g$ belongs to $\text{Hom}_{\mathcal{G}_{Mukai}}(x_1, x_2)$, where $D$ is the duality involution (29).

**Lemma 5.6** Let $x_i := (S_i, v_i, H_i)$, $i = 1, 2$, be objects of $\mathcal{G}$. If $f$ is a morphism in $\text{Hom}_{\mathcal{G}_{Mukai}}(x_1, x_2)$, then Theorem 3.9 holds for $f$. More specifically, $\gamma_{f,v_1}$ is an isomorphism of cohomology rings and $(\gamma_{f,v_1} \otimes f)(\alpha) = (\alpha_{v_2})$.

**Proof:** Let $\alpha$ be a class in the Brauer group of $\mathcal{M}_{H_1}(v_1)$ and $\mathcal{E}_{v_1}$ a $\pi_{\mathcal{M}(v_1)}^{*}\alpha$-twisted sheaf over $\mathcal{M}_{H_1}(v_1) \times S_1$. When condition (2a) holds, then the relative Fourier-Mukai transform $\Phi^{F,\mathcal{M}(v_1)}(\mathcal{E}_{v_1})$ is represented by a $\pi_{\mathcal{M}(v_1)}^{*}\alpha$-twisted sheaf $\mathcal{E}'$ on $\mathcal{M}_{H_1}(v_1) \times S_2$, which is...
a family of $H_2$-stable sheaves on $S_2$. This follows from a flatness result for Fourier-Mukai functors ([Mu4] Theorem 1.6). When condition (2b) holds, then $[\Phi^F_{M(v_1)}(E_{v_1})]'$ is represented by such a twisted sheaf $E'$. The classifying morphism

$$\kappa : M_{H_1}(v_1) \rightarrow M_{H_2}(v_2),$$

(63)

associated to the twisted sheaf $E'$, is an isomorphism. This is seen as follows. $\kappa$ pulls back the holomorphic symplectic 2-from of $M_{H_2}(v_2)$ to that of $M_{H_1}(v_1)$, by the construction of this 2-form in [Mu1]. Thus $\kappa$ is étale. It is surjective, since both moduli spaces are compact and irreducible (Theorem 1.7). It is injective, since $M_{H_2}(v_2)$ is simply-connected (Theorem 1.7).

Assume first condition (2a) with $g = f$. The pushforward $E_{v_2} := (\kappa \times \text{id}_{S_2})_*(E')$ is a $\pi^*_{M(v_2)}(k_*(a))$-twisted universal sheaf over $M(v_2) \times S_2$. Let $\mathbb{P}_{v_1}$ be the projective bundle over $M(v_1)$, as in Lemma 5.4, and $\tilde{k} : \mathbb{P}_{v_1} \rightarrow k_!\mathbb{P}_{v_1}$ the natural isomorphism. Then $(\kappa \times \text{id}_{S_2})_!(E')$ lifts to a universal sheaf $E'_{\tilde{k}}$ over $\mathbb{P}_{v_1}$. Lemma 5.4 part 1 yields the equality $(\text{id} \otimes g)\text{(ch}(E_{v_2})) = (\kappa \otimes \text{id}_{S_2})^*\text{(ch}(E_{v_2}))$ in $H^*(\mathbb{P}_{v_1} \times S_2)$. The equality $(\text{id}_{M(v_1)} \otimes g)(u_{v_1}) = (\kappa \times \text{id}_{S_2})^*(u_{v_2})$ follows from the latter, equations (26) and (27), and the injectivity of $p^*_1 : H^*(M(v_1)) \rightarrow H^*(\mathbb{P}_{v_1})$. Hence, $\kappa_\ast$ is a ring isomorphism satisfying $(\kappa \times \text{id})_*(u_{v_1}) = u_{v_2}$. The equality $\kappa_\ast = \gamma_{g,v_1}$ follows from Lemma 3.11.

Assume next condition (2b) with $g = D \circ f$. The equality $(D_{M(v_1)} \otimes D \circ g)(u_{v_1}) = (\kappa \times \text{id}_{S_2})^*(u_{v_2})$ follows from Lemma 5.4. Hence, $\kappa_\ast \circ D_{M(v_1)}$ is a ring isomorphism satisfying the equality $(\kappa_\ast \circ D_{M(v_1)} \otimes D \circ g)(u_{v_1}) = u_{v_2}$. Lemma 3.11 implies the equality $\gamma_{Dg,v_1} = \kappa_\ast \circ D_{M(v_1)}$.

Condition (1) above follows from (2), when $(v_1, v_1) > 0$. We prove the implication (2a)$\Rightarrow$(1) using Theorems 1.2 and 1.6. The proof of the implication (2b)$\Rightarrow$(1) is similar. Let $\Phi : D(S_1) \rightarrow D(S_2)$ be an equivalence, $v_1 \in K_{\text{top}}(S_1)$ a primitive and effective class satisfying $(v_1, v_1) > 0$, and $H_1$ a $v_1$-suitable polarization. Assume that the isometry $\phi : K_{\text{top}}(S_1) \rightarrow K_{\text{top}}(S_2)$ is orientation reversing (Remark 4.3). Set $v_2 := \phi(v_1)$.

**Lemma 5.7** For every $v_2$-suitable polarization $H_2$ and for every integer $i$, there exists an $H_1$-stable sheaf $E_1$ with Mukai vector $v_1$, such that $\Phi(E_1)[i]$ is not equivalent in $D(S_2)$ to any $H_2$-stable sheaf $E_2$ with Mukai vector $(-1)^i(v_2)$.

**Proof:** Assume otherwise that a pair $(H_2, i)$ exists, which satisfies condition (2a). Let $\kappa : M_{H_1}(v_1) \rightarrow M_{H_2}(v_2)$ be the isomorphism (63). Then $\kappa = \gamma_{\phi,v_1}$ by the proof of Lemma 5.6. On the other hand, the isomorphism $\text{mon}(\phi) := D_{M_{H_2}(v_2)} \circ \gamma_{\phi,v_1}$ given in (21) was shown to be a monodromy operator in the proof of Theorem 1.6. It follows that the automorphism $D_{M_{H_2}(v_2)}$ is a monodromy operator of $M_{H_2}(v_2)$. The lattice $H^2(M_{H_2}(v_2), \mathbb{Z})$ is isometric to $v_2^\perp$, by the assumption that $(v, v) \geq 2$ (Theorem 1.7). $D_{M_{H_2}(v_2)}$ acts on $H^2(M_{H_2}(v_2), \mathbb{Z})$ as multiplication by $-1$, which is orientation reversing. This contradicts the fact, that the distinguished orientation is monodromy invariant. □
6 The surface-monodromy representation

Let $S_1$ and $S_2$ be two $K3$ surfaces. A signed isometry from $H^2(S_1, \mathbb{Z})$ to $H^2(S_2, \mathbb{Z})$ is an isometry, which maps the distinguished orientation of the positive cone of $H^2(S_1, \mathbb{R})$ to that of $H^2(S_2, \mathbb{R})$ (Remark 4.3). The set of signed isometries of the second cohomologies of the two surfaces naturally embeds in $\text{Isom}[H^*(S_1, \mathbb{Z}), H^*(S_2, \mathbb{Z})]$. An isometry is extended to $H^i(S_1, \mathbb{Z})$, for $i = 0$ and 4, by sending the Poincare duals of a point and of $S_1$ to the corresponding classes in $H^i(S_2, \mathbb{Z})$. The main result of this section is the following special case of Theorem 3.9.

**Theorem 6.1** Let $x_i := (S_i, v_i, H_i)$, $i = 1, 2$, be two objects of the groupoid $G$. Assume that either each $c_1(v_i)$ is a non-zero multiple of an ample class, or that both $v_i$ are the Mukai vectors $(1, 0, 1 - n)$ of $S_i^{[n]}$, $n \geq 1$. Let $g \in \text{Hom}_G(x_1, x_2)$ be a morphism arising from a signed isometry of the second cohomologies of the two surfaces. In particular, both $v_i$ have the same rank and Euler characteristic. Then $\gamma_{g,v_1}$ is a monodromy operator. If $(v_i, v_i) > 0$, then $(\gamma_{g,v_1} \otimes g)(u_{v_1}) = u_{v_2}$.

The Hilbert scheme case of the Theorem is used in the reduction of the proof of Theorem 1.2 in section 1.3.1. The proof of Proposition 7.1 uses Theorem 6.1 for Mukai vectors $v_i$ with rank 0. We will prove that the morphism $g$ in Theorem 6.1 is a morphism of the sub-groupoid $G_{mon}$, used already in section 1.3.1 and defined below (Definition 6.2). We refer to the operator $\gamma_{g,v_1}$ in the Theorem as a surface-monodromy operator because it arises from deformations of the surface $S_1$ to $S_2$. The irreducible holomorphic symplectic manifold $S^{[n]}$ admits deformations, which do not arise from deformations of the $K3$ surface $S$. These deformations give rise to a larger monodromy group (Theorem 1.6).

The Hilbert schemes case of Theorem 6.1 is proven in section 6.2. The case $c_1(v_i) \neq 0$ of Theorem 6.1 is proven in section 6.3. The main ingredients are the Strong Torelli Theorem and the surjectivity of the period map for K3 surfaces.

6.1 Deformation equivalence

We prove next that the homomorphism $\gamma_{g,v_1} : H^*(\mathcal{M}_{H_1}(v_1), \mathbb{Q}) \to H^*(\mathcal{M}_{H_1}(v_1), \mathbb{Q})$ is a monodromy operator, whenever the moduli space $\mathcal{M}_{H_1}(v_1)$ on one $K3$ surface is related to a moduli space $\mathcal{M}_{H_2}(v_2)$ on another surface via a deformation of the surfaces (Lemma 6.5).

Let $T$ be a connected algebraic space, $p : \mathcal{S} \to T$ a smooth family of $K3$ surfaces, and $\mathcal{L}$ a line bundle on $\mathcal{S}$. Set $v := (r, d\mathcal{L}, s)$ to be the flat section of the local system $R^*p_*\mathbb{Z}$, with $\gcd(r, d, s) = 1$. Let $H$ be a section of $R^2p_*\mathbb{Z}$, consisting of ample classes, such that $H_t$ is $v_t$-suitable, for every $t \in T$. We do not assume $H$ to be continuous. Let $\text{Sp}_\mathcal{S}/T \to T$ be the moduli space of simple sheaves $E$ on $\mathcal{S}_t$, $t \in T$, with Mukai vector equal $v_t$ (see [AK] Theorem 7.4 for its existence). Assume, that there exists an algebraic
subspace $\mathcal{M}(v) \subset \text{Spl}_{S/T}$ and a smooth and proper morphism $\mathcal{M}(v) \to T$, such that for every point $t \in T$, $\mathcal{M}(v)_t$ is isomorphic to the moduli space $\mathcal{M}_{H_t}(v_t)$ of $H_t$-stable sheaves on $S_t$ with Mukai vector $v_t$.

We define next the subgroupoid $\mathcal{G}_{\text{mon}}$ of the groupoid $\mathcal{G}$ given in (34). The object of $\mathcal{G}_{\text{mon}}$ are the same. The morphisms arise from deformations, which we now describe. Let $S_i$ be a projective $K3$ surface with an ample line bundle $H_i$, $i = 1, 2$. Let $v_i := (r, dL_i, s)$ be an algebraic Mukai vector in $H^*(S_i, \mathbb{Z})$. Assume, that $H_i$ is $v_i$-suitable.

**Definition 6.2**

1. The objects $(S_1, v_1, H_1)$ and $(S_2, v_2, H_2)$ of $\mathcal{G}$ are said to be deformation-equivalent, if there exists data $(T, S, \mathcal{L}, H, \mathcal{M}(v))$ as above, and two points $t_1, t_2 \in T$, such that $(S_i, v_i, H_i) = (S_i, (r, d\mathcal{L}_t, s), H_i)$, $i = 1, 2$. If $v_i = (1, 0, 1 - n)$, $i = 1, 2$, $n \geq 1$, then we allow $H$ to be trivial on $T \setminus \{t_1, t_2\}$, $p : S \to T$ to be a smooth and proper analytic family of Kähler $K3$ surfaces over a connected analytic space $T$, and we let $\mathcal{M}(v) \to T$ be the relative Douady space of ideal sheaves of length $n$ zero-dimensional subschemes of the fibers of $p$.

2. A morphism $g \in \text{Hom}_G((S_1, v_1, H_1), (S_2, v_2, H_2))$ is a morphism in $\mathcal{G}_{\text{mon}}$, if the two objects are deformation-equivalent by data $(T, S, \mathcal{L}, H, \mathcal{M}(v))$ as above, and $g$ is a monodromy operator, corresponding to a homotopy class of paths in the base $T$ from $t_1$ to $t_2$.

$\mathcal{G}_{\text{mon}}$ is easily seen to be a sub-groupoid of $\mathcal{G}$.

**Proposition 6.3** ([Y3] Proposition 5.1) Assume that the base $T$, of the smooth family $p : S \to T$, is a smooth connected quasi-projective curve, the line bundle $\mathcal{L}$ is relatively ample, and $\text{Pic}(S_t)$ is generated by $\mathcal{L}_t$, for some $t \in T$. Then $\mathcal{L}_t$ is $v_t$-suitable for all but finitely many $t \in T$. Choose a section $H$ of $R^2_{p,*}\mathbb{Z}$ satisfying $H_t = \mathcal{L}_t$, for all but finitely many $t \in T$, and such that $H_t$ is $v_t$-suitable, for every $t \in T$. Then there exists an algebraic subspace $\mathcal{M}(v) \subset \text{Spl}_{S/T}$ and a smooth and proper morphism $\mathcal{M}(v) \to T$, such that for every point $t \in T$, $\mathcal{M}(v)_t$ is isomorphic to the moduli space $\mathcal{M}_{H_t}(v_t)$ of $H_t$-stable sheaves on $S_t$ with Mukai vector $v_t$.

If $H_i$, $i = 1, 2$, are two $v$-suitable polarizations and either $c_1(v)$ is ample or $-c_1(v)$ is ample, then the objects $(S, v, H_1)$ and $(S, v, H_2)$ are deformation-equivalent, by Proposition 6.3, and the identity isomorphy is a morphism in $\text{Hom}_{\mathcal{G}_{\text{mon}}}((S, v, H_1), (S, v, H_2))$. More generally, we have:

**Corollary 6.4** Let $(S_i, v_i, H_i)$, $i = 1, 2$, be two objects of $\mathcal{G}$ with Mukai vectors $v_i = (r, kA_i, s)$, with the same $r$, $s$, $k$. Assume that $c_1(A_1)^2 = c_1(A_2)^2$ and that both $A_i$ are ample and indivisible. Then $(S_i, v_i, H_i)$, $i = 1, 2$, are deformation equivalent.

**Proof:** The case where $H_i = A_i$ follows from the proof of [OG1] Proposition 2.3. Each $(S_i, v_i, H_i)$ is deformation equivalent to an object $(S'_i, v'_i, H'_i)$, where $v'_i = (r, kA'_i, s)$, and $H'_i = A'_i$, by Proposition 6.3.
Lemma 6.5  Let $g \in \text{Hom}_{\text{mon}}((S_1,v_1,H_1),(S_2,v_2,H_2))$ be a morphism. Then

$$\gamma_{g,v_1} : H^*(\mathcal{M}_{H_1}(v_1),\mathbb{Q}) \to H^*(\mathcal{M}_{H_2}(v_2),\mathbb{Q})$$

is a monodromy operator. Furthermore, $(\gamma_{g,v_1} \otimes g)(u_{v_1}) = u_{v_2}$.

Proof: We prove the case $(v_i,v_i) \geq 2$. The case $(v_i,v_i) \leq 0$ is similar. Let $\pi : \mathcal{M}(v) \times_T S \to T$ be the fiber product. There exists a 2-cocycle $\alpha$ of the sheaf $O^*_{\mathcal{M}(v)}$, defining a class in the Brauer group of $\mathcal{M}(v)$, and an $\alpha$-twisted universal sheaf $\mathcal{E}$ over $\mathcal{M}(v) \times_T S$ (by the proof of Theorem A.5 in [Mu2]). The normalization of the Chern character of a twisted universal sheaf, given in equations (25) and (27), produces a flat section $u_v$ of $R^*_\pi \mathbb{Q}$, with value $u_{v_t}$ over $t \in T$ given by (27).

Let $\eta$ be a homotopy class of paths in $T$ from $t_1$ to $t_2$. We get isomorphisms (which need not preserve Hodge structures)

$$g : H^*(S_1,\mathbb{Q}) \to H^*(S_2,\mathbb{Q})$$  \hspace{1cm} (64)
$$f : H^*(\mathcal{M}_{H_1}(v_1),\mathbb{Q}) \to H^*(\mathcal{M}_{H_2}(v_2),\mathbb{Q})$$

satisfying $g(v_1) = v_2$ and

$$(f \otimes g)(u_{v_1}) = u_{v_2}. \hspace{1cm} (65)$$

The equality $f = \gamma_{g,v_1}$ follows from Lemma 3.11. \hfill $\blacksquare$

6.2 Monodromy for Kähler K3 surfaces and Torelli

We prove the case $v_i = (1,0,1-n)$ of Theorem 6.1 in this section.

Let $\Gamma$ be a subgroup of the automorphism group of a lattice $L$, $\pi : S \to M$ a family of smooth compact complex varieties over a connected base $M$, and $\Phi : R^*_{\pi}\mathbb{Z} \xrightarrow{\Phi} L$ a trivialization of the weight $i$ local system of integral cohomology groups. Denote by $\phi_m$ the value of $\Phi$ at $m \in M$. Let $Ad_{\phi_m} : \Gamma \to \text{Aut}H^i(S_m,\mathbb{Z})$ be the homomorphism sending $\gamma \in \Gamma$ to $\phi^{-1}_m \gamma \phi_m$.

Lemma 6.6  Assume that the following conditions hold:

1. The automorphism group $\text{Aut}(S_m)$ acts faithfully on $H^i(S_m,\mathbb{Z})$ for every $m \in M$.

2. $\Gamma$ acts on the base $M$ in such a way, that $(S_m,\gamma \circ \phi_m)$ and $(S_{\gamma(m)},\phi_{\gamma(m)})$ are isomorphic, for every $\gamma \in \Gamma$. In other words, there exists a unique isomorphism $f : S_m \to S_{\gamma(m)}$ making the following diagram commutative

$$
\begin{array}{ccc}
H^i(S_m,\mathbb{Z}) & \xrightarrow{\phi_m} & L \\
\uparrow f^* & & \downarrow \gamma \\
H^i(S_{\gamma(m)},\mathbb{Z}) & \xrightarrow{\phi_{\gamma(m)}} & L.
\end{array}
\hspace{1cm} (66)
$$
3. The action of $\Gamma$ on $M$ is non-trivial. I.e., there exists a pair $\gamma \in \Gamma$ and $m \in M$, such that $\gamma(m) \neq m$.

Then the subgroup $Ad_{\phi_m}(\Gamma)$ of $Aut H^i(S, \mathbb{Z})$ consists of monodromy operators.

**Proof:** The existence of the trivialization $\Phi$ implies, that $\pi_1(M)$ acts trivially on $\mathbb{R}^n$. Let $m \in M$, $\gamma \in \Gamma$, and choose $\eta : [0,1] \to M$ a path from $m$ to $\gamma(m)$. As we deform $S_{\eta(t)}$ from $S_m$ to $S_{\gamma(m)}$, the class of the diagonal in $S_m \times S_m$ deforms flatly, through classes in the cohomology of $S_m \times S_m$, to a class $F$ inducing a homomorphism $F : H^i(S_{\gamma(m)}) \to H^i(S_m)$ satisfying $\phi_{\gamma(m)} = \phi_m \circ F$. Let $f : S_m \to S_{\gamma(m)}$ be the isomorphism (66) determined by the equality $\gamma \circ \phi_m \circ f^* = \phi_{\gamma(m)}$. We get the equality

$$(\phi_m^{-1} \circ \gamma \circ \phi_m) = F \circ (f^*)^*.$$ 

If $\gamma(m) = m$, then $F$ is the class of the diagonal, and the automorphism $\phi_m^{-1} \circ \gamma \circ \phi_m$ of $H^i(S_m, \mathbb{Z})$ is induced by the automorphism $f$. If $\gamma(m) \neq m$, we can use the isomorphism $f$ to glue the pulled back family $\eta^{-1}S$ to a family $(\eta^{-1}S)/f$ over a circle. The automorphism $\phi_m^{-1} \circ \gamma \circ \phi_m$ of $H^i(S_m, \mathbb{Z})$ is then the monodromy operator of the family $(\eta^{-1}S)/f$. Fix a point $m' \in M$ with a non-trivial $\Gamma$-orbit (assumption 3). Then $\Gamma$ is generated by the elements $\gamma$ satisfying $\gamma(m') \neq m'$. Hence, $Ad_{\phi_{m'}}(\Gamma)$ consists of monodromy operators. The same holds for every $m \in M$, since $M$ is connected. \hfill \Box

Denote by $L$ the $K3$ lattice

$$-E_8 \oplus -E_8 \oplus H \oplus H \oplus H,$$

where $H$ is the even rank 2 hyperbolic lattice. Set $L_C := L \otimes \mathbb{C}$. Let $Q \subset \mathbb{P}(L \otimes \mathbb{C})$ be the quadric defined by the pairing. Let $\Omega \subset Q$ be the analytic open subset $\{\omega : (\omega, \bar{\omega}) > 0\}$.

A marked triple $(S, \kappa, \phi)$ consists of a $K3$ surface $S$, a Kähler form $\kappa$, and a marking by an isometry $\phi : H^2(S, \mathbb{Z}) \to L$, mapping the distinguished orientation of the positive cone of $H^2(S, \mathbb{R})$ to a fixed orientation of that of $L \otimes \mathbb{R}$ (see Remark 4.3). The moduli of marked triples is

$$K\Omega^0 := \{([w], \kappa) \mid [w] \in \Omega, \kappa \in L \otimes \mathbb{R} \cap [\omega]^\perp, (\kappa, \kappa) > 0, \kappa, d) \neq 0 \forall d \in L \cap [\omega]^\perp \text{ with } (d, d) = -2\}$$

([BHPV] chapter VIII Theorems 12.3 and 14.1). $K\Omega^0$ is a connected 60-dimensional real manifold ([BHPV] chapter VIII Corollary 9.2). Let $\Gamma_{0}^{\text{cov}}$ be the group of orientation preserving isometries of $L$ (Definition 4.2). $K\Omega^0$ is a $\Gamma_{0}^{\text{cov}}$ invariant subset of $\mathbb{P}(L_C) \times L_C$. There exists a universal family $\pi : S \to K\Omega^0$ of $K3$ surfaces and a trivialization $\Phi : R^2_{\pi, \mathbb{Z}} \to L$, satisfying: (i) Each $\phi_{([w], \kappa)}$ is an isometry, (ii) $\phi_{([w], \kappa)}[H^{2,0}(S_{([w], \kappa)})] = [w]$, and (iii) $\phi_{([w], \kappa)}(\kappa)$ is a Kähler class in $H^{1,1}(S_{([w], \kappa)})$.

Note: we avoid taking the quotient of $K\Omega^0$ by $\Gamma_{0}^{\text{cov}}$ because some pairs $(S, \kappa)$ do have non-trivial automorphisms. The automorphism group is equal to the subgroup of $\Gamma_{0}^{\text{cov}}$
stabilizing ([w], κ). In particular, the universal family of K3’s does not descend to the quotient.

Proof of the case \( v_i = (1, 0, 1 - n) \) of Theorem 6.1: The hypothesis of Lemma 6.6 hold for K3 surfaces with \( i = 2, \Gamma = \Gamma_{0}^{\text{cov}} \), and \( M \) is the fine moduli space \( K\Omega_0^0 \) of marked triples. Assumptions 1 and 2 of Lemma 6.6 follow from the Strong Torelli Theorem. Assumption 3 is clear. Lemma 6.6 implies, that the isometry \( g \) in Theorem 6.1 is a monodromy operator for some family \( p : S \to T \) of K3 surfaces. (The total space \( S \) is assumed to be complex analytic and \( p \) is smooth and proper). There is a relative Douady family \( S[n] \to T \), parametrizing ideal sheaves of length \( n \) subschemes in the fibers of the family. Theorem 6.1 now follows from Lemma 6.5 in the Hilbert schemes case. \( \square \)

The special case \( v_1 = v_2 = (1, 0, 0) \) of Theorem 6.1 is the well known:

**Corollary 6.7** The group \( \Gamma_{0}^{\text{cov}} \) is equal to the monodromy group of a K3 surface \( S \).

**Proof:** The above proof shows that the monodromy group \( \text{Mon}(S) \) contains \( \Gamma_{0}^{\text{cov}} \). \( \text{Mon}(S) \) is contained in the index 2 subgroup \( \Gamma_{0}^{\text{cov}} \) of \( \Gamma_0 \), since the distinguished orientation of the positive cone is monodromy invariant (Remark 4.3). \( \square \)

### 6.3 Monodromy for polarized K3 surfaces and Torelli

We prove in this section the case of Theorem 6.1, where \( c_1(v_i) \) is a non-zero multiple of an ample class. The proof actually works when \( c_1(v_i) \) is zero, under the additional assumption that the isometry \( g \) in Theorem 6.1 maps some ample class of \( S_1 \) to an ample class of \( S_2 \). We already know, that the objects \( x_1 \) and \( x_2 \) are related by some morphism \( g \) in \( G_{\text{mon}} \), by Corollary 6.4. Furthermore, \( \gamma_{g,v_i} \) has the properties stated in Theorem 6.1, by Lemma 6.5. Hence, it suffices to prove the statement for automorphisms \( g \) of one object \((S, v, H)\) with \( v = (r, kA, s) \) and \( A \) ample, for each value of \( r, s, k \), and for \( c_1(A)^2 = 2d, d \geq 1 \). We need only consider automorphisms arising from orientation-preserving isometries of \( H^2(S, \mathbb{Z}) \) stabilizing \( A \), and prove that they belong to \( \text{Aut}_{G_{\text{mon}}}(S, v, H) \).

We keep the notation of section 6.2. Fix a primitive element \( h \in L \) satisfying \((h, h) = 2d, d \geq 1\). Let \( \Gamma_h \) be the subgroup of \( \Gamma_0 \) stabilizing \( h \) and set \( \Gamma_{h}^{\text{cov}} := \Gamma_h \cap \Gamma_{0}^{\text{cov}} \). Set \( \Omega'_h := \{ \omega \in \Omega : (\omega, h) = 0 \} \). \( \Omega'_h \) has two connected components, which are interchanged by complex conjugation. The two components are distinguished by the orientation of the positive cone of \( L_{\mathbb{R}} \) determined by the basis \( \{ \text{Re}(\omega), \text{Im}(\omega), h \} \) of the positive 3-dimensional subspace (see Remark 4.3). Choose one of the connected components of \( \Omega'_h \) and denote it by \( \Omega_h \). Then \( \Omega_h \) is \( \Gamma_{h}^{\text{cov}} \)-invariant. Set

\[
K\Omega_0^0_h := \{ \omega \in \Omega_h : (\omega, h) \in K\Omega_0^0 \}.
\]

\( K\Omega_0^0_h \) is a dense open subset of \( \Omega_h \). \( K\Omega_0^0_h \) admits a natural embedding as a closed subset of \( K\Omega_0^0 \).

\( \Gamma_{h}^{\text{cov}} \) acts on \( \Omega_h \) properly-discontinuously and hence the quotient \( \Omega_h/\Gamma_{h}^{\text{cov}} \) has a canonical structure of a normal analytic space. \( \Omega_h/\Gamma_{h}^{\text{cov}} \) is moreover a (normal) quasi-projective
variety [BB]. The complement $Y_{2d}$ in $\Omega_h/\Gamma_h^{cov}$, of the branch locus of the quotient map, is a Zariski dense smooth open subset of $\Omega_h/\Gamma_h^{cov}$ ([P] Proposition 2.2.2).

Let $\tilde{Y}_{2d}$ be the complement in $\Omega_h$ of the ramification locus of the quotient map. Then $\tilde{Y}_{2d}$ is an open and dense subset of $K\Omega_h^0$. The universal family of marked $K3$ surfaces over $K\Omega_h^0$ restricts to one over $\tilde{Y}_{2d}$, which we denote by $\tilde{\pi}: \tilde{S} \to \tilde{Y}_{2d}$. The natural section $h$ of $R^2_{\tilde{\pi}}Z$ corresponds to an ample line-bundle $L_t$ on the fiber $S_t$ ([BHPV] chapter VIII Corollary 3.9). Furthermore, the identity is the only automorphism of $S_t$ leaving the class of $L_t$ invariant. Hence, the family $\tilde{\pi}$ descends to $Y_{2d}$. The above construction is summarized in the following Proposition.

**Proposition 6.8** (A special case of [P] Theorem 3.4.1 and Corollary 3.4.2)

1. There exists a smooth and proper family of projective $K3$ surfaces $\pi: S \to Y_{2d}$, over the smooth quasi-projective variety $Y_{2d}$, whose pullback to $\tilde{Y}_{2d}$ is isomorphic to the universal family $\tilde{\pi}: \tilde{S} \to \tilde{Y}_{2d}$.

2. Choose a point $(\omega, h)$ in the subset $\tilde{Y}_{2d}$ of $K\Omega_h^0$. Let $t$ be the corresponding $\Gamma_h^{cov}$-orbit in $Y_{2d}$. We get an associated marking $\phi_{(\omega, h)}: H^2(S_t, \mathbb{Z}) \to L$. The monodromy group of the local system $R^2\pi_*\mathbb{Z}$ is conjugated, via the marking $\phi_{(\omega, h)}$, to the subgroup $\Gamma_h^{cov}$ of $O(L)$.

The construction of a smooth and proper relative moduli space of sheaves need not be possible over the whole family $\pi: S \to Y_{2d}$. We replace the above family by one, whose base is a smooth curve, for which Yoshioka’s construction (Proposition 6.3) applies. There exists a smooth connected quasi-projective curve $C \subset Y_{2d}$, for which the homomorphism $\pi_1(C) \to \pi_1(Y_{2d})$ is surjective. This follows from a version of Lefschetz’ Hyperplane Section Theorem ([De], see also [FL] Theorem 1.1). Denote by $p: S_{|C} \to C$ the restriction of the family $\pi: S \to Y_{2d}$. We may and do further assume, that $C$ passes through a point $y \in Y_{2d}$, corresponding to a $K3$ surface $S_y$ with a cyclic Picard group. The tautological section of $R^2_{\pi_*}\mathbb{Z}$ over $\tilde{Y}_{2d}$ comes from a section $h$ of $R^2_{\pi_*}\mathbb{Z}$, which restricts to a section $h_{|C}$ of $R^2_{p_*}\mathbb{Z}$.

We would like to apply Proposition 6.3 with the family $p: S_{|C} \to C$, the section $v := (r, k h_{|C}, s)$ of the local system of Mukai lattices, and a $v$-suitable choice $H$ of a section of $R^2_{p_*}\mathbb{Z}$. As in Proposition 6.3, the section $H$ is assumed to be equal to $h_{|C}$ away from finitely many points of $C$. We are missing however a relatively ample line bundle $L$ on $S_{|C}$ inducing the section $h_{|C}$. We can find instead an open étale covering $\{U_i\}$ of $C$, and such a line bundle $L_i$ over each $S_{|U_i}$. This follows from the algebraic construction of the moduli space of polarized $K3$ surfaces ([Vi] Theorem 1.13) and the existence of a universal family over the Hilbert scheme of polarized $K3$ surfaces ([Vi] section 1.7). Proposition 6.3 produces the moduli spaces $\mathfrak{M}(v)_i \to U_i$, which glue to the desired algebraic space $\mathfrak{M}(v)$, which is smooth and proper over $C$.

Let $v_t := (r, k \cdot h(t), s)$ be the value of $v$ at a point $t$ of $C$, and $\eta$ an element of $\pi_1(C, t)$ corresponding to a monodromy operator $g$ of $H^*(S_t, \mathbb{Z})$. Then $\eta$ induces a surface-monodromy element $g$ in $\text{Aut}_{\mathfrak{g}_\text{mon}}(S_t, v_t, H_t)$, by Definition 6.2. Every element of $\Gamma_h^{cov}$ is
obtained this way, by the surjectivity of the homomorphism $\pi_1(C, t) \to \pi_1(Y_{2d}, t)$ and by part 2 of Proposition 6.8. This completes the proof of Theorem 6.1. 

\section{Examples}

We verify in this section special cases of Theorem 1.6, inducing monodromy operators of the Hilbert scheme $S^{[n]}$, which do not come from deformations of the K3 surface $S$. In section 7.1 we consider reflections of the Mukai lattice of $S$ with respect to the class of a topological line-bundle on $S$. The automorphisms $\gamma_g$, associated to such reflections $g$, are shown to be monodromy involutions of the cohomology of Hilbert schemes $S^{[n]}$ (Proposition 7.1). In section 7.2 we review work of Seidel-Thomas, on the reflections of the derived category with respect to spherical objects. We then state an analogue of Proposition 7.1, when the reflection $g$ is with respect to the class of an algebraic line bundle on $S$ (Theorem 7.7). The automorphisms $\gamma_g$, associated to such reflections $g$, are shown to be local monodromy operators of $S^{[n]}$. The isometry $-1$ of the Mukai lattice is treated in section 7.3. In section 7.4 we verify cases of Theorem 1.6 for certain orientation reversing isometries (Theorems 7.9 and 7.10).

\subsection{Elliptic K3 surfaces}

Let $S$ be a K3 surface admitting an elliptic fibration $\pi : S \to \mathbb{P}^1$ with integral fibers and with a section. Denote by $\sigma$ and $f$ the classes of the section and the fiber in $H^2(S, \mathbb{Z})$. Let $v := (1, 0, 1-n)$ be the Mukai vector of the ideal sheaf of a length $n$ subscheme, $n \geq 2$. Fix a class $\beta \in H^2(S, \mathbb{Z})$, which is orthogonal to both $\sigma$ and $f$, and satisfies $(\beta, \beta) = 2n - 4$. We do not assume that $\beta$ is of Hodge-type $(1, 1)$. Then $v_0 := (1, \beta - f, n-1)$ is the Mukai vector in $v^\perp$ of a topological line-bundle and $(v_0, v_0) = -2$. Let $g$ be the reflection of $H^*(S, \mathbb{Z})$ with respect to $v_0$.

**Proposition 7.1** The endomorphism $\gamma_{g,v} \in \text{End}[H^*(S^{[n]}, \mathbb{Z})_{\text{free}}]$ is a monodromy operator and $(\gamma_{g,v} \otimes g)(u_v) = u_v$, where $u_v$ is the class $(27)$.

**Proof:** We consider below an auto-equivalence $\Phi$ of $D(S)$, inducing a Hodge isometry $\phi$ of $H^*(S, \mathbb{Z})$ as well as an isomorphism $\mathcal{M}(v) \cong \mathcal{M}_H(\phi(v))$, such that $\phi g \phi^{-1}$ is a surface-monodromy operator of $S$ and an element of $\text{Aut}_g(S, \phi(v), H)$. Theorem 6.1 proves the desired properties for $\gamma_{\phi g \phi^{-1}, \phi(v)}$, from which Proposition 7.1 then easily follows.

We recall first the set-up of Theorem 3.15 in [Y3]. Set $H := \sigma + kf$, $k \gg 0$. Then $H$ is a $(0, f, 0)$-suitable polarization, and the moduli space $\mathcal{M}_H(0, f, 0)$, which is a compactified relative Jacobian, is isomorphic to $S$. We choose a universal sheaf on $S \times \mathbb{P}^1, \mathcal{M}_H(0, f, 0)$ and regard it as a sheaf $\mathcal{P}$ on $S \times S$. Let $p_i : S \times S \to S$, $i = 1, 2$, be the two projections. The Fourier-Mukai functor $\Phi^P : D(S) \to D(S)$ with kernel $\mathcal{P}$, defined in (58), was studied in [Br]. Set $\Phi := [1] \circ \Phi^P$, where $[1]$ is the shift auto-equivalence. Then $\Phi(O_S)$ is represented by a line-bundle on the zero section. Replacing
\( P \) by \( P \otimes \mathcal{O}_P^i(1) \), for some \( i \in \mathbb{Z} \), we may assume that \( \chi(\Phi(\mathcal{O}_S)) = 1 \). Set \( e_1 := (1, 0, 0) \) and \( e_2 := (0, 0, 1) \).

**Lemma 7.2** The Hodge-isometry \( \phi \), associated to \( \Phi \), acts by \(-1\) on the sublattice of \( H^*(S, \mathbb{Z}) \) orthogonal to \( \Lambda := \text{span}_{\mathbb{Z}} \{ e_1, \sigma, f, e_2 \} \) and the matrix of the restriction of \( \phi \) to \( \Lambda \), in the given basis, is

\[
\begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & -1 & 0 & -1 \\
1 & -1 & 1 & 0
\end{pmatrix}
\]

**Proof:** When the Picard lattice of \( S \) is spanned by \( \sigma \) and \( f \), then \( \Lambda^\perp \) is the transcendental lattice in \( H^2(S, \mathbb{Z}) \). In this case \( \Lambda^\perp \) is \( \phi \) invariant. For a generic such \( S \), the only Hodge isometry of \( \Lambda^\perp \) is \( 1 \) or \(-1\). Hence, \( \phi \) acts on \( \Lambda^\perp \) by \( 1 \) or \(-1\). The latter conclusion follows, without the assumption on the Picard number, by a standard deformation argument. Columns 1, 3, and 4 of the above matrix were calculated in equation (3.18) in [Y3]. It is also shown there, that \( \phi \) has a \(-1\) eigenvector in \( \Lambda^\perp \) (the class \( \sigma - \tau + d(\tau)f \) in the notation of equation (3.18) in [Y3]). The second column of the matrix is determined by the other three, since \( \phi \) is an isometry.

\( \Phi \) induces an isomorphism

\[
\mathcal{M}_H(1, 0, 1 - n) \rightarrow \mathcal{M}_H(0, \sigma + nf, 1),
\]

by Theorem 3.15 in [Y3]. Set \( w := (0, \sigma + nf, 1) \). Consequently, \( \phi \) is a morphism from \( (S, v, H) \) to \( (S, w, H) \) in the groupoid \( \mathcal{G}_{\text{Mukai}} \) given in (62). The homomorphism \( \gamma_{\phi, v} \) is induced by the isomorphism (68) of the moduli spaces, by Lemma 5.6. Lemma 7.2 yields the equality \( \phi(1, \beta - f, n - 1) = (0, \alpha, 0) \), with \( \alpha := \sigma + (2 - n)f - \beta \). Note, that \((\alpha, \alpha) = -2\) and \((0, \alpha, 0)\) is orthogonal to \( w \).

We claim that \( \gamma_{\rho, w} \) is a monodromy operator and \( (\gamma_{\rho, w} \otimes \rho)(u_w)) = u_w \). When \( n \geq 3 \), then \( c_1(w) = \sigma + nf \) is ample and the claim follows from Theorem 6.1. When \( n = 2 \), \( c_1(w) \) is not ample. Nevertheless, \( c_1(w)^2 = 2 \) and \( c_1(w) \) is effective. These conditions imply that the object \( (S, w, H) \) is deformation equivalent to an object \( (S', w', H') \), with \( c_1(w')^2 = 2 \), where \( c_1(w') \) is ample and \( \text{Pic}(S') \) is cyclic generated by \( c_1(w') \) (this is proven by Yoshioka in the unpublished note proving Theorem 1.7 in the case \( \text{rank}(v) = 0 \) and \( c_1(v) \) not ample). Let \( \mathcal{G}_{\text{mon}} \) be the groupoid given in Definition 6.2. Choose any morphism \( \eta \) of \( \text{Hom}_{\mathcal{G}_{\text{mon}}}(\mathcal{M}_H^\perp(w), \mathcal{M}_H(w)) \). The conjugate \( \eta \circ \rho \circ \eta^{-1} \) of \( \rho \) is an element of \( \text{Aut}_{\mathcal{G}_{\text{mon}}}(S, w, H) \), by Theorem 6.1. We conclude that \( \rho \) is an element of \( \text{Aut}_{\mathcal{G}_{\text{mon}}}(S, w, H) \). Lemma 6.5 implies that \( \gamma_{\rho, w} \) is a monodromy operator in \( H^*(\mathcal{M}_H(w), \mathbb{Z})_{\text{free}} \) as claimed.

The equality \( g = \phi^{-1} \circ \rho \circ \phi \) implies the equality \( \gamma_{g, v} = \gamma_{\phi, v}^{-1} \circ \gamma_{\rho, w} \circ \gamma_{\phi, v} \), by Lemma 3.12. Hence, \( \gamma_{g, v} \) is a monodromy operator and \( (\gamma_{g, v} \otimes g)(u_v)) = u_v \). This completes the proof of Proposition 7.1.

### 7.2 Reflections by spherical objects

Fourier-Mukai functors, which induce \(-2\) reflections of the Mukai lattice, often arise from spherical objects on the K3 surface \( S \).
Definition 7.3 An object $x$ of $D(S)$ is spherical, if it satisfies the following conditions
\[ \text{Hom}^0(x, x) \cong \mathbb{C} \cong \text{Hom}^2(x, x), \]
\[ \text{Hom}^i(x, x) = 0, \text{ for } i \notin \{0, 2\}. \]

The Mukai vector $v_0$ of a spherical object, is a $-2$ vector $(v_0, v_0) = -2$. Examples of spherical objects include sheaves, which are both simple and rigid. In particular, any line bundle on $S$ is a spherical object. In addition, if $\iota : \mathbb{P}^1 \hookrightarrow S$ is an embedding, then the pushforward $\iota_*\mathcal{O}_{\mathbb{P}^1}(n)$, of any line bundle on $\mathbb{P}^1$, is a spherical object.

Given a complex $E_\bullet$ of locally free sheaves, the cone $C(E_\bullet)$ of $E_\bullet$ is the object in $D(S \times S)$ represented by the complex
\[
\pi_1^*(E_\bullet^\vee) \otimes \pi_2^*(E_\bullet) \xrightarrow{\text{ev}} \Delta_*\mathcal{O}_S,
\]
with $\Delta_*\mathcal{O}_S$ in degree 0. Above, $\Delta : S \hookrightarrow S \times S$ is the diagonal embedding and ev is the natural evaluation homomorphism. If we take, for example, $E_\bullet$ to be the trivial line bundle, then $C(\mathcal{O}_S)$ is the complex
\[
\mathcal{O}_{S \times S} \longrightarrow \Delta_*\mathcal{O}_S.
\]

Theorem 7.4 [ST] If the complex $E_\bullet$ represents a spherical object, then the Fourier-Mukai functor $\Phi_{C(E_\bullet)}^{S \rightarrow S}$ is an auto-equivalence of $D(S)$.

We will refer to the functor $\Phi_{C(E_\bullet)}^{S \rightarrow S}$ as the reflection functor with respect to $E_\bullet$. On the level of K-theory, it sends the class of a sheaf $F$, to
\[ F - \chi(E_\bullet, F) \cdot E_\bullet, \]
where $\chi(E_\bullet, F) := \sum (-1)^i \dim \text{Hom}^i(E_\bullet, F)$. On the level of the Mukai lattice, it induces the reflection $\tau_{v_0}$ with respect to the Mukai vector $v_0$ of $E_\bullet$
\[ \tau_{v_0}(v) := v + (v_0, v) \cdot v_0. \]

Yoshioka made extensive use of reflection functors in his study of the geometry of moduli spaces of sheaves on K3 and abelian surfaces [Y1, Y2, Y3].

Example 7.5 Assume, that the spherical object $E_\bullet$ is a simple sheaf $E_0$. Choose a moduli space $\mathcal{M}(v)$ of sheaves on $S$ with a universal sheaf $\mathcal{E}_v$ over $\mathcal{M}(v) \times S$. Then the relative Fourier-Mukai transform of $\mathcal{E}_v$ (Definition 5.3), with respect to the reflection functor of $E_0$, determines the class in K-theory, which is represented also by
\[
\tau(\mathcal{E}_v) := \mathcal{E}_v - \pi_0^*E_0 \otimes \pi_1^*(\pi_2^*E_0^\vee \otimes \mathcal{E}_v),
\]
where $\pi_i$ is the projection from $\mathcal{M}(v) \times S$. When $E_0$ is the trivial line bundle, we have
\[
\tau(\mathcal{E}_v) := \mathcal{E}_v - \pi_1^*R_{\pi_1,} \mathcal{E}_v.
\]
Assume, that \( v = (r, \mathcal{L}, -r) \), \( r \geq 1 \), \( \mathcal{L} \) satisfies condition 7.6 with respect to the ample line-bundle \( H \), and \( \gcd(r, \text{deg} \mathcal{L}) = 1 \) (so that there is a universal family \( \mathcal{E}_v \) over \( \mathcal{M}(v) \times S \)).

**Condition 7.6**

1. \( \mathcal{L} \) is an effective Cartier divisor with minimal degree in the sense that the subgroup \( H \cdot \text{Pic}(S) \) of \( H^4(S, \mathbb{Z}) \) is generated by \( c_1(H) \cdot c_1(\mathcal{L}) \). In particular, all curves in the linear system \( |\mathcal{L}| \) are reduced and irreducible.

2. The base locus of \( |\mathcal{L}| \) is either empty or zero-dimensional.

3. The generic curve in \( |\mathcal{L}| \) is smooth.

4. \( H^1(S, \mathcal{L}) = 0 \).

The condition implies, that the polarization \( H \) is \( v \)-suitable (Definition 3.4), by [Ma1] Lemma 3.5. Let \( v_0 = (1, 0, 1) \) be the Mukai vector of \( \mathcal{O}_S \). Denote by \( \mathcal{M}(v)^t \), \( t \geq 0 \), the subvariety of \( \mathcal{M}(v) \) consisting of sheaves \( F \) with \( h^1(F) \geq t \). Then \( \mathcal{M}(v) \setminus \mathcal{M}(v)^1 \) is a dense Zariski open subset and \( \mathcal{M}(v)^1 \) is a divisor, whose class is \( \theta(-v_0) \), where \( \theta \) is given in (32), by [Ma3] Lemma 4.11. Serre’s Duality yields the isomorphism \( H^1(S, F) \cong \text{Ext}^1(\mathcal{O}_S, F)^* \). Given \( F \in \mathcal{M}_H(v)^t \), we get a tautological extension

\[
0 \rightarrow H^1(S, F) \otimes \mathcal{O}_S \rightarrow E(F) \rightarrow F \rightarrow 0,
\]

and \( E(F) \) is \( H \)-stable with \( h^1(E(F)) = 0 \) ([Ma1] Lemma 3.7). Set \( w_t := (r+t, \mathcal{L}, -r+t) \). We get a regular morphism

\[
\mathcal{M}_H(v)^t \setminus \mathcal{M}_H(v)^{t+1} \rightarrow \mathcal{M}_H(w_t) \setminus \mathcal{M}_H(w_t)^1,
\]

which is a Grassmannian fibration with \( G(t, 2t) \) fibers [Ma1]. Note, that \( \dim \mathcal{M}(w_t) = \dim \mathcal{M}(v) - 2t^2 \) and thus \( \mathcal{M}(v)^t \) is non-empty, if and only if \( t^2 \leq (v, v)/2 + 1 \). We conclude, that the subset

\[
\mathcal{Z} := \{(F_1, F_2) : h^1(F_1) = h^1(F_2) \text{ and } E(F_1) \cong E(F_2)\}
\]

is of pure dimension equal to \( \dim(\mathcal{M}(v)) \). \( \mathcal{Z} \) is in fact a Zariski-closed subset, which we endow with the reduced subscheme structure, and each of its irreducible components is the closure of the fiber square of \( \mathcal{M}_H(v)^t \setminus \mathcal{M}_H(v)^{t+1} \) over \( \mathcal{M}_H(w_t) \), \( t \geq 0 \) [Ma3].

The Hilbert scheme \( S^{[n]} \) is a special case of \( \mathcal{M}_H(v) \), when \( r = 1, c_1(\mathcal{L})^2 = 2n - 4, n \geq 1 \), and the following choices are made to satisfy Condition 7.6: (a) \( n \geq 3 \) and \( H = \mathcal{L} \) is a generator of \( \text{Pic}(S) \). (b) \( n = 2, \phi : S \rightarrow \mathbb{P}^1 \) is an elliptic K3 with reduced and irreducible fibers, \( \mathcal{L} = \phi^*\mathcal{O}_{\mathbb{P}^1}(1) \), and \( H = \mathcal{O}_S(C) \otimes \phi^*\mathcal{O}_{\mathbb{P}^1}(k) \), where the curve \( C \) is either a section of \( \phi \) or a multi-section of minimal degree, and \( k \gg 0 \). (c) \( n = 1, \mathcal{L} = \mathcal{O}_S(\Sigma) \), where \( \Sigma \subset S \) is a smooth rational curve, and \( H \) restricts to \( \Sigma \) as a generator of the image of \( \text{Pic}(S) \) in \( \text{Pic}(\Sigma) \). The Brill-Noether loci \( (S^{[n]})^t := \mathcal{M}(v)^t \) parametrize ideal sheaves \( I_A \) of length \( n \) subschemes \( A \subset S \), such that \( h^1(I_A \otimes \mathcal{L}) \geq t \). These loci
admit a geometric description in terms of the morphism $\varphi : S \to |\mathcal{L}|^* \cong \mathbb{P}^{n-1}$, which is an embedding when $n \geq 4$. When $n \geq 2$, the stratum $(S^{[n]})^{t} \setminus (S^{[n]})^{t+1}$ consists of ideal sheaves $I_A$, such that $\varphi(A)$ spans a $\mathbb{P}^{n-1-t}$. When $n = 1$, then $(S^{[1]})^{1} = \Sigma$ and $Z \subset S \times S$ is the union $\Delta \cup [\Sigma \times \Sigma]$, where $\Delta$ is the diagonal.

The following Theorem is proven in [Ma3].

**Theorem 7.7**

1. The homomorphism $Z_{\ast} : H^{\ast}(\mathcal{M}_H(v), \mathbb{Z}) \to H^{\ast}(\mathcal{M}_H(v), \mathbb{Z})$, induced by the self correspondence $Z$, is the local monodromy operator inducing the reflection of $H^2(\mathcal{M}_H(v), \mathbb{Z})$ with respect to the $-2$ class of $\mathcal{M}_H(v)^1$ (Definition 2.4).

2. Set $Z_i := \delta_{2i}(\mathcal{M}_H(v)) \to K_{alg}(\mathcal{M}_H(v))$, where $\delta_i : Z \to \mathcal{M}_H(v)$, $i = 1, 2$, are the two projections. Then

$$Z_i(E_v - \pi_1(\mathcal{M}_H(v)^1 \times S)).$$

3. The isomorphism $Z_{\ast}$ and the reflection $\tau_{v_0}$ of $K_{top}(S)$ with respect to $O_S$ satisfy equation (6) with $\ell = O_{\mathcal{M}_H(v)}(\mathcal{M}_H(v)^1)$. Consequently, $Z$ is Poincare-Dual to the cohomology class $\gamma(\tau_{v_0}, v)$, the homomorphism $\gamma_{\tau_{v_0},v}$ is the monodromy operator $Z_{\ast}$, and $(\gamma_{\tau_{v_0},v} \otimes \tau_{v_0})(u_v) = u_v$.

### 7.3 The Hodge isometry $-id$

The isometry $-id$ fixes only the zero vector. Moreover, if a non-zero Mukai vector $v$ represents a sheaf, then $-v$ does not. Nevertheless, it is convenient to consider $-v$ as representing shifted complexes $F[1]$, where $F$ is a coherent sheaf with Mukai vector $v$.

**Definition 7.8** The moduli spaces $\mathcal{M}_E(v)$ and $\mathcal{M}_E(-v)$ are defined to be the same projective variety. A universal “sheaf” $\mathcal{E}_{-v}$ over $\mathcal{M}_E(-v) \times S$ is the $K$-theoretic class of the complex

$$\mathcal{E}_{-v} := \mathcal{E}_v[d], \ d \ odd,$$

where $\mathcal{E}_v$ is a universal sheaf over $\mathcal{M}_E(v) \times S$. We set $u_{-v} := -u_v$.

Let

$$\gamma_{-id} : H^{\ast}(\mathcal{M}(v), \mathbb{Z}) \to H^{\ast}(\mathcal{M}(-v), \mathbb{Z})$$

be the ring isomorphism, induced by the natural isomorphism $\mathcal{M}(v) \cong \mathcal{M}(-v)$. The identity $ch(\mathcal{E}_v[1]) = -ch(\mathcal{E}_v)$ translates to the equality

$$(\gamma_{-id} \otimes -id)_{\ast}(ch(\mathcal{E}_v)) = ch(\mathcal{E}_{-v}).$$

Moreover, Lemma 5.4 holds, with $\Phi$ being the shift auto-equivalence.
7.4 Stratified reflections with respect to $+2$ vectors

Let $u_0 = (1, 0, -1)$ be the $+2$ Mukai vector of the ideal sheaf of two points. Denote by $\sigma_{u_0}$ the reflection of the Mukai lattice with respect to $u_0$

$$\sigma_{u_0}(w) = w - (w, u_0)u_0. \quad (71)$$

Let $v_0$ be the $-2$ vector $(1, 0, 1)$ of the trivial line-bundle. The corresponding reflections, $\sigma_{u_0}$ and $\tau_{v_0}$, commute and satisfy the relation

$$-(\sigma_{u_0} \circ \tau_{v_0}) = D.$$

In particular, $\sigma_{u_0} = -(D \circ \tau_{v_0})$.

**Theorem 7.9** Let $(S, (r, L, s), H)$ be an object of the groupoid $G$, where the polarization $H$ and the line-bundle $L$ satisfy Condition 7.6. Assume that $r \geq 0$, $s \geq 0$, and

$$c_1(L)^2 < 2[r + s + rs]. \quad (72)$$

Then the composition $\Phi := ([1] \circ \Psi)^\vee$ of the reflection $\Psi : D(S) \rightarrow D(S)$ with respect to the spherical object $O_S$ (constructed in Theorem 7.4), the shift auto-equivalence, and the functor of taking dual, induces a regular isomorphism

$$f_{(r,L,s)} : \mathcal{M}_H(r, L, s) \rightarrow \mathcal{M}_H(s, L, r).$$

On the level of cohomology rings $f_{(r,L,s)}^* = D_{\mathcal{M}_H} \circ \gamma_{\sigma_{u_0},(r,L,s)}$. Furthermore, $f_{(r,L,s)} \circ f_{(s,L,r)} = id$.

**Proof:** The isomorphism $f_{(r,L,s)}$ is constructed in [Ma1] Theorem 3.21 (use also the inequality (72) above and [Ma1] Corollary 3.16 to conclude the emptiness of the Brill-Noether loci). Related results were proven independently in [Y1].

If $r = s$ then $f_{(r,L,r)}$ is a regular involution of the moduli space $\mathcal{M}_H(r, L, r)$. When the inequality (72) does not hold, the functor induces a non-regular birational isomorphism. The following theorem is proven in [Ma3].

**Theorem 7.10** Set $v := (1, L, 1)$. The statements of Theorem 3.9 holds for the automorphism $\sigma_{u_0}$ of the object $(S, v, H)$ of the groupoid (34), where the polarization $H$ and the line-bundle $L$ satisfy condition 7.6. Moreover, the class of the composition $D_{\mathcal{M}_H} \circ \gamma_{\sigma_{u_0},v}$ is the local monodromy operator restricting as $-\theta_v \circ \sigma_{u_0} \circ \theta_v^{-1}$ on $H^2(\mathcal{M}_H(v), \mathbb{Z})$ (Definition 2.4).

Note, that $\mathcal{M}(1, L, 1)$ is isomorphic to $S^{[n]}$, where $n := \frac{1}{2}c_1(L)^2$. Associated to the reflection $\sigma_{u_0}$ is a Lagrangian correspondence $Z \subset \mathcal{M}(v) \times \mathcal{M}(v)$ (see [Ma1] section 3.2). The class $\text{mon}(\sigma_{u_0}, v)$, given in (22), is Poincare-dual to the class of $Z$. This fact is proven in the course of the proof of Theorem 7.10. $Z$ is the graph of a regular involution, for $n = 1$ or 2 (Theorem 7.9). When $c_1(L)^2 = 2$ and $n = 1$, then $Z$ is the graph of the Galois involution of the double cover $\mathcal{M}(1, L, 1) \cong S \rightarrow |L|^* \cong \mathbb{P}^2$. $Z$ is reducible for $n \geq 3$. One of its irreducible components is the graph of a birational involution.
8 Generators for the stabilizer $\Gamma_v$

Let us study the structure of the stabilizer $\Gamma_v$ of the Mukai vector $v = (1, 0, -m)$ of the Hilbert scheme $S^{[m+1]}$, $m \geq 1$. Lemma 8.3 relates $\Gamma_v$ and $O(v^\perp)$. Proposition 8.6 states, that $\Gamma_v$ is generated by the subgroup $\Gamma_0$, in Definition 4.1, and reflections in Mukai vectors of line bundles. The character group of $\Gamma_v$ is calculated in Corollary 8.10.

The signature $(\ell_+, \ell_-)$ of $H^2(S, \mathbb{Z})$ is $(3, 19)$, however we will use in this section only the inequalities $\ell_+ \geq 3$ and $\ell_- \geq 3$, so that the results hold in case $S$ is an abelian surface (though $v$ is then the Mukai vector of $S^{[m]}$).

**Lemma 8.1** Let $\Lambda$ be an even unimodular lattice of signature $(\ell_+, \ell_-)$, satisfying $\ell_+ \geq 3$ and $\ell_- \geq 3$. 1) Let $M = \begin{bmatrix} 2a & b \\ b & 2d \end{bmatrix}$ be a symmetric matrix with $a, b, d \in \mathbb{Z}$, and $\lambda_1 \in \Lambda$ a primitive element with $(\lambda_1, \lambda_1) = 2a$. Then there exists a primitive rank 2 sublattice $\Sigma \subset \Lambda$, containing $\lambda_1$, and an element $\lambda_2 \in \Sigma$, such that $\{\lambda_1, \lambda_2\}$ is a basis for $\Sigma$, and $M$ is the matrix of the bilinear form of $\Sigma$ is this basis.

2) Assume that $\text{rank}(M) = 2$. Let $\{\lambda'_1, \lambda'_2\}$ be a basis for another primitive sublattice of $\Lambda$, having the same matrix $M$. Then there exists an isometry $g$ of $\Lambda$ satisfying $g(\lambda_i) = \lambda'_i$, $i = 1, 2$.

3) The kernel $O(\Lambda)^{cov}$ of the orientation character (47) acts transitively on the set of primitive integral classes $A \in \Lambda$, with fixed “squared-length” $(A, A)$.

**Proof:** 1) Let $\Sigma'$ be the rank 2 lattice with basis $\{e_1, e_2\}$, whose bilinear form is given by the matrix $M$. Then there exists a primitive embedding $\eta : \Sigma' \hookrightarrow \Lambda$, by Theorem 1.14.4 of [Ni], when $\text{rank}(M) = 2$, and by Proposition 1.17.1 of [Ni], when $\text{rank}(M) < 2$. Furthermore, there exists an isometry $g$ of $\Lambda$, satisfying $g(\eta(e_1)) = \lambda_1$, again by Theorem 1.14.4 of [Ni]. Take $\Sigma = (g \circ \eta)(\Sigma')$ and $\lambda_2 = g(\eta(e_2))$. 2) Follows from Theorem 1.14.4 of [Ni]. 3) We already explained the transitivity of the $O(\Lambda)$-action. The transitivity of $O(\Lambda)^{cov}$ would follow from that of $O(\Lambda)$, once we prove that the stabilizer of $A$ in $O(\Lambda)$ contains an orientation reversing isometry. It suffices to prove that $A^\perp$ contains a class $\lambda$ with $(\lambda, \lambda) = 2$, because the reflection with respect to $\lambda$ would be orientation reversing (see section 4.1). The existence of such $\lambda$ follows from part 1. □

A $-2$ vector $v_0$, whose reflection $\tau_{v_0}$ is in the stabilizer of $v$, must be in $v^\perp$. Consequently, $v_0$ has the form

$$v_0 = (r, \mathcal{L}, rm),$$

where $\mathcal{L}$ has self-intersection $2r^2m - 2$. The lattice $v^\perp$ is the direct sum $H^2(S, \mathbb{Z}) \oplus \text{span}_\mathbb{Z}\{(1, 0, m)\}$. It is even, but not unimodular; the primitive vector $(1, 0, m)$ sends $v^\perp$, via the Mukai pairing, to $2m \cdot \mathbb{Z}$. Observe, that the stabilizer $\Gamma_v$ embeds in the isometry group of $v^\perp$, but the latter is larger in general.
The quotient $(v^\perp)^*/v^\perp$ is isomorphic to $\mathbb{Z}/2m$. Embed $(v^\perp)^*$ in $v^\perp \otimes_{\mathbb{Z}} \mathbb{Q}$. We get the well-defined quadratic form $q : (v^\perp)^*/v^\perp \to \mathbb{Q}/2\mathbb{Z}$, determined by

$$q : \frac{(1,0,m)}{2m} \mapsto -1/2m,$$

satisfying

$$q \left( \frac{(r,0,rm)}{2m} \right) = -r^2/2m.$$

The isometry group $O[(v^\perp)^*/v^\perp]$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^\rho$, where the Euler number $\rho$ of $m$ is the number of distinct primes in the prime factorization $m = p_1^{e_1} p_2^{e_2} \cdots p_\rho^{e_\rho}$ (and $e_i$ are positive integers) (see [O]). We have the natural homomorphism

$$O(v^\perp) \to O[(v^\perp)^*/v^\perp]. \quad (73)$$

**Lemma 8.2** The homomorphism (73) is surjective.

**Proof:** Follows from [Ni] Theorem 1.14.2.

**Lemma 8.3** $\Gamma_v$ is the kernel of the natural surjective homomorphism (73). In particular, $\Gamma_v = O(v^\perp)$ if and only if $m = 1$. If $m$ is a prime power, then $\Gamma_v$ is a subgroup of $O(v^\perp)$ of index 2 and $O(v^\perp) = \Gamma_v \cup (-id_{v^\perp}) \cdot \Gamma_v$.

**Proof:** We prove only that $\Gamma_v$ is the kernel $N$ of (73). The rest follows from the description of $O[(v^\perp)^*/v^\perp]$.

There is a natural isomorphism between the residue groups span$_\mathbb{Z}\{v\}^*/\text{span}_\mathbb{Z}\{v\}$ and $(v^\perp)^*/v^\perp$, each being isomorphic to $H^*(S,\mathbb{Z})/[Zv+(v^\perp)]$. An element $\varphi \in O(v^\perp)$ extends to an element of $\Gamma$, if and only if the image $\tilde{\varphi}$ of $\varphi$ in $O[(v^\perp)^*/v^\perp]$ is contained in the image of $O[\text{span}_\mathbb{Z}\{v\}]$; i.e., if and only if $\tilde{\varphi}$ is the identity, or multiplication by $-1$ (Proposition 1.5.1 in [Ni]). The image $\tilde{\varphi}$ of $\varphi$ is determined by the image $\varphi(1,0,m)$ in $(v^\perp)^*/v^\perp$. Any isometry $\varphi \in O(v^\perp)$ satisfies $\varphi(1,0,m) = (r,c,rm)$ and $c \cdot c = 2m(r^2 - 1)$. We see, that an isometry $\varphi \in O(v^\perp)$ extends to an element of $\Gamma$, if and only if $r \equiv \pm 1 \pmod{2m}$. Any element of $\Gamma$, which leaves $v^\perp$ invariant, must send $v$ to $\pm v$. Hence, $N$ is contained in the subgroup $E$ generated by $\Gamma_v$ and $-id_{v^\perp}$. If $m = 1$, then $N = O(v^\perp)$, $v$ is a $+2$ vector, and the reflection $\sigma_v$ sends $v$ to $-v$ and acts as the identity on $v^\perp$. Hence, if $\varphi \in \Gamma$ extends an isometry in $O(v^\perp)$, then either $\varphi$ or $\sigma_v \circ \varphi$ is an extension in $\Gamma_v$. Thus $\Gamma_v = O(v^\perp)$.

If $m > 1$, then both $N$ and $\Gamma_v$ have index 2 in $E$, so it remains to show that $\Gamma_v$ is contained in $N$, i.e., that the additional condition $\varphi(v) = v$ implies, that $r \equiv 1 \pmod{2m}$. This is precisely equation (74), proven below. \[ Q.E.D. \]

Our next goal is to find a manageable set of generators for $\Gamma_v$ (Proposition 8.6).

**Lemma 8.4** The stabilizer $\Gamma_v$ is generated by $\Gamma_0$ (Definition 4.1) and reflections in $-2$ vectors in $v^\perp$. 61
Thus, \( r \) and consequently, straightforward calculation shows, that the composition \( g \) into \( \Gamma_0 \). We may thus assume that

\[
g(w) = (r, k\mathcal{L}, rm), \quad r, k \in \mathbb{Z} \text{ and } \mathcal{L} \text{ a primitive class.}
\]

We claim that the following equalities hold:

\[
\begin{align*}
k &= 2mc \text{ for some integer } c, \\
r - 1 &= 2m\rho \text{ for some integer } \rho, \\
c^2(\mathcal{L} \cdot \mathcal{L}) &= 2\rho + 2m\rho^2,
\end{align*}
\]

and consequently, \( g(w) = (1, 0, m) + 2m(\rho, c\mathcal{L}, m\rho) \).

The equalities \( g(1, 0, -m) = (1, 0, -m) \) and \( g(1, 0, m) = (r, k\mathcal{L}, rm) \) imply:

\[
g(2, 0, 0) = (r + 1, k\mathcal{L}, (r - 1)m).
\]

Thus, \( r \) is odd, \( k \) is even, and

\[
g(0, 0, m) = \left( \frac{r - 1}{2}, \frac{k}{2}, \frac{m(r + 1)}{2} \right).
\]

In particular, \( 2m \) divides \( r - 1 \) and \( k \). The equation (75) is equivalent to the statement, that \( (g(w), g(w)) = -2m \). As \( w \) is primitive, \( c \) and \( r = 1 + 2m\rho \) are relatively prime.

Let \( \Gamma'_v \) be the subgroup of \( \Gamma_v \) generated by \( \Gamma_0 \) and reflections in \(-2\) vectors in \( v^\perp \). We need to show, that the coset \( \Gamma'_v g \) is \( \Gamma'_v \). Given a \(-2\) vector \( u = (a, A, am) \in v^\perp \), we get

\[
\tau_u(g(w)) = g(w) + 2m[c(A \cdot \mathcal{L}) - ar] \cdot (a, A, am).
\]

Since \( \gcd\{c, r\} = 1 \), there are integers \( a, b \) such that \( bc - ar = 1 \). Choose \( u \) with such rank \( a \), satisfying \( A \cdot \mathcal{L} = b \) and with \( c\mathcal{L} + A \) primitive (possible by Lemma 8.1). Then the first Chern class of \( \tau_u(g(w)) \) is \( 2m(c\mathcal{L} + A) \) with \( c\mathcal{L} + A \) primitive. Thus, any coset of \( \Gamma'_v \) contains a representative \( g \) with \( c = 1 \). We may assume, that \( g \) is such a representative.

If \( \rho = -1 \), set \( v_0 := (-1, \mathcal{L}, -m) \). Equation (75) implies, that \( v_0 \) is a \(-2\) vector. A straightforward calculation shows, that the composition \( \tau_{v_0} \circ g \) fixes both \( v \) and \( w \). Hence, \( \tau_{v_0} \circ g \) comes from an isometry of \( \{v, w\}^\perp \), namely of \( H^2(S, \mathbb{Z}) \). Consequently, \( \Gamma'_v g = \Gamma'_v \).

When \( \rho \neq -1 \), we choose a \(-2\) vector \( u \) with \( a = 1 \) and \( A \in H^2(S, \mathbb{Z}) \) satisfying

\[
A \cdot \mathcal{L} = r - \rho - 1 = 2m\rho - \rho, \quad \text{and} \\
A \cdot A = 2m - 2
\]

(\( A \) exists, by Lemma 8.1, because \( \mathcal{L} \) is primitive). Equation (76) implies that

\[
(\tau_u \circ g)(w) = (1, 0, m) + 2m(-1, \mathcal{L} - (\rho + 1)A, -m).
\]

The invariant \( \rho \) of \( \tau_u \circ g \) is \(-1 \). Consequently, \( \Gamma'_v \tau_u g = \Gamma'_v \). But \( \Gamma'_v g = \Gamma'_v \tau_u g \). This completes the proof of Lemma 8.4. \( \square \)
Lemma 8.5  The stabilizing subgroup $\Gamma_v$ is generated by $\Gamma_0$ and reflections in $-2$ vectors $(a, A, am)$ in $v^\perp$, with $A$ a primitive class in $H^2(S, \mathbb{Z})$.

Proof: If $g \in \Gamma_v$ and $g(w) \neq -w$, simply apply the proof of Lemma 8.4. The point is that the reflection $\tau_u$ in (76) may be chosen with $u = (a, A, am)$ and $A$ primitive. If $g(w) = -w$, then $m = 1$ (see the proof of Lemma 8.4). Take $v_0 = (1, L, 1)$ with $L$ primitive and isotropic. Then $v_0$ is a $-2$ vector in $v^\perp$, $\tau_{v_0}(w) = -w$, and $\tau_{v_0}g$ belongs to $\Gamma_0$ $\square$

Proposition 8.6  $\Gamma_v$ is generated by $\Gamma_0$ and reflections in $-2$ vectors of the form $(1, -L, m)$, with $L$ a primitive class in $H^2(S, \mathbb{Z})$ of self-intersection $2m - 2$.

Note that $(1, -L, m)$ is the Mukai vector of the inverse of a line bundle $L$, if $c_1(L)^2 = 2m - 2$. We will need Lemmas 8.7 and 8.8 for the proof of Proposition 8.6.

Lemma 8.7  Let $L_1$ and $L_2$ be two classes in $H^2(S, \mathbb{Z})$ satisfying the following conditions:

$$
L_1 \cdot L_1 = 2a^2m - 2,
L_2 \cdot L_2 = 2b^2m - 2, \quad \text{and}
L_1 \cdot L_2 = 1 + 2abm.
$$

Then the Mukai vectors $v_1 := (a, -L_1, am)$, $v_2 := (b, -L_2, bm)$, and $v_0 := v_1 + v_2$ are all $-2$ vectors in $v^\perp$. The subgroup of $\Gamma_v$, generated by $\tau_{v_0}$, $\tau_{v_1}$, and $\tau_{v_2}$, is isomorphic to $\text{Sym}_3$ and is generated by any two out of the three reflections.

Proof: The equality $(v_1, v_2) = 1$ yields

$$
\begin{align*}
\tau_{v_1}(v_2) &= v_2 + (v_2, v_1) \cdot v_1 = v_0 \\
\tau_{v_2}(v_1) &= v_1 + (v_2, v_1) \cdot v_2 = v_0.
\end{align*}
$$

The class $L_0 := L_1 + L_2$ has self-intersection $2(a + b)^2m - 2$ and the vector $v_0 := v_1 + v_2$ is a $-2$ vector, whose reflection $\tau_{v_0}$ satisfies the relations:

$$
\begin{align*}
\tau_{v_0} &= \tau_{v_1} \circ \tau_{v_2} \circ \tau_{v_1} \\
\tau_{v_0} &= \tau_{v_2} \circ \tau_{v_1} \circ \tau_{v_2}.
\end{align*}
$$

Consequently, the subgroup of $\Gamma_v$, generated by $\tau_{v_0}$, $\tau_{v_1}$, and $\tau_{v_2}$, is isomorphic to $\text{Sym}_3$. Observe, that $(v_1, v_2) = (-v_0, v_1) = (-v_0, v_2) = 1$. The “triangle” with vertices $-v_0$, $v_1$, and $v_2$ is an extended Dynkin diagram of type $A_2$ $\square$.

Lemma 8.8  1. Given an even rank $r \geq 2$ and a class $L_0$ of self-intersection $2r^2m - 2$ (not necessarily primitive), we can decompose $L_0$ as a sum $L_0 = L_1 + L_2$ of two classes satisfying Condition (77) with $a = b = \frac{r}{2}$. We can further choose $L_1$ and $L_2$ primitive.
2. Fix a rank \( a \geq 1 \) and let \( \mathcal{L}_1 \) be a primitive class of self-intersection \( 2a^2m - 2 \). Then there exists a primitive class \( \mathcal{L}_2 \) of self-intersection \( 2m - 2 \), such that \( v_1 := (a, -\mathcal{L}_1, am) \) and \( v_2 := (1, -\mathcal{L}_2, m) \) satisfy condition (77).

Proof: Part 1) Say \( \mathcal{L}_0 = i\mathcal{L}'_0 \) with \( \mathcal{L}'_0 \) primitive. Set \( d := (\mathcal{L}'_0 \cdot \mathcal{L}'_0)/2 = r^2m - 1/i^2 \). Let \( \Sigma \subset H^2(S, \mathbb{Z}) \) be a rank 2 primitive sublattice containing \( \mathcal{L}'_0 \) as well as a class \( \mathcal{L}_1 \), such that \( \{ \mathcal{L}'_0, \mathcal{L}_1 \} \) is a basis for \( \Sigma \), and whose bilinear form has matrix \[
\begin{bmatrix}
2d & id \\
-id & (r^2m - 4)/2
\end{bmatrix}
\]
(Lemma 8.1). Set \( \mathcal{L}_2 := \mathcal{L}_0 - \mathcal{L}_1 \). Then \( \mathcal{L}_1, \mathcal{L}_2 \) are primitive and they satisfy condition (77) with \( a = b = r/2 \).

Part 2 follows immediately from Lemma 8.1 \( \square \)

Proof of Proposition 8.6: Let \( W_v \) be the subgroup of \( \Gamma_v \) generated by \( \Gamma_0 \) and reflections in \(-2\) vectors of the form \((1, -\mathcal{L}, m)\) with \( \mathcal{L} \) in \( H^2(S, \mathbb{Z}) \) a primitive class of self-intersection \( 2m - 2 \). Lemma 8.5 reduces the proof of the equality \( W_v = \Gamma_v \), to the proof, that \( W_v \) contains the reflection \( \tau_{v_0} \), for every \(-2\) vector \( v_0 = (r, -\mathcal{L}_0, rm) \), with \( \mathcal{L}_0 \) primitive. Since \( \tau_{v_0} = \tau_{-v_0} \), we may assume that the rank \( r \) is non-negative. We prove Proposition 8.6 by induction on the rank of a \(-2\) vector \( v_0 = (r, -\mathcal{L}_0, rm) \) in \( v^\perp \).

If \( r = 1 \) (and \( \mathcal{L}_0 \) is primitive), or if \( r = 0 \), then \( \tau_{v_0} \) is in \( W_v \) (by definition of \( W_v \)). Assume \( r \geq 2 \) and that any \(-2\) vector in \( v^\perp \), with primitive first Chern class and of rank \( 1 \leq a \leq r - 1 \), is in \( W_v \). If \( r \) is even, Part 1 of Lemma 8.8 implies, that \( \tau_{v_0} \) is in \( W_v \). If \( r \) is odd and \( \mathcal{L}_0 \) is primitive, then part 2 of Lemma 8.8 replaces \( v_0 \) by \( w_0 \), of rank \( r + 1 \), such that \( \tau_{v_0} \) is in \( W_v \) if and only if \( \tau_{w_0} \) is in \( W_v \). Part 1 implies that \( \tau_{w_0} \) is in \( W_v \). \( \square \)

Lemma 8.9 The set of \(-2\) vectors in \( v^\perp \) is the union of two \( O(v^\perp) \)-orbits (which are also \( \Gamma_v \)-orbits):

\[
A_+ := \{ v_0 = (r, \mathcal{L}, rm) : \mathcal{L} \cdot \mathcal{L} = 2r^2m - 2 \text{ and } \mathcal{L} \text{ is divisible by } 2 \} \quad \text{and} \\
A_- := \{ v_0 = (r, \mathcal{L}, rm) : \mathcal{L} \cdot \mathcal{L} = 2r^2m - 2 \text{ and } \mathcal{L} \text{ is not divisible by } 2 \}.
\]

The orbit \( A_+ \) is non-empty if and only if \( m \equiv 1 \pmod{4} \). Moreover, every vector \( v_0 \) in \( A_+ \) has odd rank.

Proof: \( A_+ \) is contained in the kernel of \( v^\perp \to (v^\perp)^*/2(v^\perp)^* \). Equivalently, if \( v_0 \) is in \( A_+ \), then the reflection \( \tau_{v_0} \) is in the kernel of \( \Gamma_v \to \text{Aut}(v^\perp/2v^\perp) \). Clearly, \( A_- \) contains elements which do not have this property.

Let \( v_0 := (r, b\mathcal{L}_0, rm) \) be a \(-2\) vector in \( v^\perp \), where \( b \) is an integer and \( \mathcal{L}_0 \) a primitive class. The equality

\[-1 = \frac{1}{2}(v_0, v_0) = b^2(\mathcal{L}_0, \mathcal{L}_0)/2 - r^2m\]

implies that \( b \) and \( rm \) are relatively prime.

If \( v_0 \in A_- \), then the lattice span\(\mathbb{Z}\{v, v_0\}\) is a primitive sub-lattice of the Mukai lattice. Indeed, given \( x, y \in \mathbb{Q} \), the vector \( xv + y(v_0 - rv) = (x, yb\mathcal{L}_0, 2myr - xm) \) is integral, if
and only if $x$, $yb$ and $2myr$ are integral. Since $v_0 \in A_-$, $b$ is odd and is thus relatively prime to $2mr$. Thus, $y$ must be an integer as well.

Given two $-2$ vectors in $A_-$, we get two primitive sublattices $\Delta_0 := \text{span}\{v, v_0\}$ and $\Delta_1 := \text{span}\{v, v_1\}$ of the Mukai lattice. The isomorphism $\Delta_0 \to \Delta_1$, sending the ordered basis $\{v, v_0\}$ to $\{v, v_1\}$, extends to a global isometry $g$ of the Mukai lattice (Lemma 8.1). Hence, $v_0$ and $v_1$ belong to the same $\Gamma_v$ orbit in $v^\perp$.

Let $v_0 := (r, bL_0, rm)$ belong to $A_+$. Then $b$ is even, $r$ is odd, and $\text{span}_\mathbb{Z}\{v, v_0\}$ is an index 2 sublattice of $\Delta_0 := \text{span}_\mathbb{Z}\{v, (0, b/2L_0, rm)\}$. It is easy to check that $\Delta_0$ is a primitive sublattice of the Mukai lattice and

$$\left\{ v, \frac{v_0 - v}{2} \right\} = \left\{ (1, 0, -m), \left( \frac{r - 1}{2}, \frac{b}{2}L_0, \frac{m(r + 1)}{2} \right) \right\}$$

is another basis of $\Delta_0$. The intersection form of $\Delta_0$ with respect to the latter basis is $\left( \begin{array}{cc} 2m & -m \\ -m & 2m - 2 \end{array} \right)$. We conclude that $m \equiv 1$ (mod 4). Conversely, if $m \equiv 1$ (mod 4), then $2m - 2 \equiv 0$ (mod 8) and there exists a class $L_0$ of self-intersection $\frac{2m - 2}{4}$. The vector $v_0 = (1, 2L_0, m)$ is then a $-2$ vector in $A_+$.

Given two vectors $v_0$ and $v_1$ in $A_+$, we get the two primitive sublattices $\Delta_0 := \text{span}\{v, \frac{1}{2}(v_0 - v)\}$ and $\Delta_1 := \text{span}\{v, \frac{1}{2}(v_1 - v)\}$. The isomorphism $\Delta_0 \to \Delta_1$, sending the basis $\{v, \frac{1}{2}(v_0 - v)\}$ to $\{v, \frac{1}{2}(v_1 - v)\}$, extends to an isometry of the Mukai lattice (Lemma 8.1). Hence, $v_0$ and $v_1$ belong to the same $\Gamma_v$ orbit in $v^\perp$. $\square$

**Corollary 8.10**

1. $\Gamma_v$ is generated by reflections in $+2$ and $-2$ vectors in $v^\perp$.

2. The character group $\text{Char}(\Gamma_v)$, of homomorphisms from $\Gamma_v$ to $\mathbb{C}^\times$, is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$.

**Proof:**

1) Wall proved, that $\Gamma_0$ is generated by reflections in $+2$ and $-2$ vectors in $H^2(S, \mathbb{Z})$ ([W] Theorem 4.8). The statement follows from Wall’s result and Lemma 8.4.

2) We construct first an isomorphism $\text{Char}(\Gamma_0) \cong \mathbb{Z}/2 \times \mathbb{Z}/2$. There is precisely one $\Gamma_0$-orbit of $-2$ vectors and one $\Gamma_0$-orbit of $+2$ vectors in $H^2(S, \mathbb{Z})$ (Lemma 8.1). Reflections in $-2$ vectors, in a fixed orbit, are all conjugate. Hence, every character is determined by its values on these two orbits. Let $v_0$ be a $-2$ vector, $u_0$ a $+2$ vector, and $\tau_{v_0}$ and $\sigma_{u_0}$ the reflections. Then $\det(\tau_{v_0}) = -1$ and $\det(\sigma_{u_0}) = 1$, while $\text{cov}(\tau_{v_0}) = 0$ and $\text{cov}(\sigma_{u_0}) = 1$. Hence, $\{\det, \text{cov}\}$ is a basis for $\text{Char}(\Gamma_0)$ as a $\mathbb{Z}/2$-module.

We prove next the equality $\text{Char}(\Gamma_v) = \text{Char}(\Gamma_0)$. Proposition 8.6 and Lemma 8.9 imply, that $\Gamma_v$ is generated by $\Gamma_0$ and reflections in $-2$ vectors $v_0$, whose $\Gamma_v$-orbit contains $-2$ vectors in $H^2(S, \mathbb{Z})$. Reflections in $-2$ vectors, in a fixed orbit of $\Gamma_v$, are all conjugate. Hence, the restriction $\text{Char}(\Gamma_v) \to \text{Char}(\Gamma_0)$ is injective. The restriction is surjective, since both $\text{cov}$ and $\det$ extend to $\Gamma_v$. $\square$
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