Generalized Fibonacci zone plates

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We propose a family of zone plates which are produced by the generalized Fibonacci sequences and their axial focusing properties are analyzed in detail. Compared with traditional Fresnel zone plates, the generalized Fibonacci zone plates present two axial foci with equal intensity. Besides, we propose an approach to adjust the axial locations of the two foci by means of different optical path difference, and further give the deterministic ratio of the two focal distances which attributes to their own generalized Fibonacci sequences. The generalized Fibonacci zone plates may allow for new applications in micro and nanophotonics.

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Recently the Fibonacci sequence proposed by Italian mathematician Leonardo de Pisa has been successfully employed in the development of different photonic devices. Photonics is a potential field of applications for novel devices designed and constructed by using a Fibonacci sequence as a consequence of its unique properties. The focusing and imaging properties of Fibonacci optical elements, e.g., quasicrystals [1, 2], gratings [3-5], lenses [6-8], zone plates [9], etc., are studied in detail.

Focusing of soft x-ray [10] and extreme ultraviolet (EUV) has many applications in physical and life sciences, such as high-resolution microscopy, spectroscopy, and lithography [11]. Traditional Fresnel zone plates (TFZPs), which have inherent limitations [12, 13], can be used for this kind of focusing [14, 15]. Some aperiodic zone plates, generated with the fractal Cantor set, have been proposed to overcome some of these limitations [16, 17], another interesting mathematical generator of aperiodic zone plates is the aforementioned Fibonacci sequence. In mathematics, many mathematicians have extensively studied the Fibonacci sequence and its various generalizations [18-22] in the past decades.

In this letter, we introduce the aperiodic generalized Fibonacci sequences into the zone plates, and come to a conclusion that the generalized Fibonacci zone plates (GFZPs) can yield two equal axial intensity foci with adjustable location.

Let us review the standard Fibonacci sequence, which is defined by the following recurrence relation

\[ F_1 = F_2 = 1, \quad F_{n+2} = F_{n+1} + F_n \quad (n \in \mathbb{N}^+). \]  

(1)

Obviously, \( x_1 = (1+\sqrt{5})/2 \) and \( x_2 = (1-\sqrt{5})/2 \), which are associated with the classical geometrical problem of the golden section, are the characteristic roots of the characteristic equation \( x^2-x-1 = 0 \). The golden mean is defined as the limit of the ratio of two consecutive Fibonacci numbers:

\[ \varphi = \lim_{n \to \infty} F_{n+1}/F_n = (1+\sqrt{5})/2. \]  

(2)

Similarly, we can extend the standard Fibonacci sequence to the generalized Fibonacci sequences via the following initial seed elements

\[ F_j = a_j \quad (j, a_j \in \mathbb{N}^+). \]  

(3)

And the corresponding linear recursion relation of the generalized Fibonacci sequences can be written as

\[ F_n = \sum_{m=1}^{j} C_m F_{n-m} \quad (C_m \in \mathbb{R}, n, j \in \mathbb{N}, n > j \geq 2). \]  

(4)

![Fig. 1. Different optical path between zones under a plane wave incidence.](image)

We now retrospect the design of TFZPs based on the plane wave incidence as shown in Figure 1. All the designed rays are converged upon a single point, the expected focus \( P \). As known, the half wave zones of TFZPs can be determined by [14]

\[ \sqrt{r_m^2 + F} - \sqrt{r_{m+1}^2 + F} = \lambda/2, \]  

(5)

where \( r_m \) denotes the radius of the mth zone, \( F \) is the expected focal distance and \( \lambda \) is the incident wavelength. Suppose we alter the optical path difference between two
adjacent zones as \( K \lambda (K \in \mathbb{R}^+) \) instead of \( \lambda / 2 \), then one can further describe the radii of zones as

\[
    r_m = \sqrt{\left(mK\lambda\right)^2 + 2mK\lambda F},
\]

where \( K \) is the optical path difference parameter (OPDP) in this letter, whose influence on axial bifocal locations of GF\(Z\)Ps will be investigated later.

GF\(Z\)Ps can be generated similar to the process of TFZPs. Taking a generalized Fibonacci sequence into account, whose initial seed elements are \( F_1 = 1, F_2 = 2 \) and \( F_3 = 3 \) and the recursion relation is \( F_{n+3} = F_{n+2} + F_{n+1} + F_n \), after encoding three seed elements as \( (F^1, F^2, F^3) = (0, 01, 011) \), the six-order switching sequence \( F^6 \) is 011010110100111 while 1 denotes transparent zones and 0 denotes opaque ones. That means the number of total zones is 20 and the number of transparent zones is 11 shown in Figure 2.

![Flow chart of generation of GF\(Z\)Ps](image)

**Fig. 2.** Flow chart of generation of GF\(Z\)Ps.

Theoretically, the diffraction field of the GF\(Z\)Ps and the associated Fresnel zone plates can be numerically calculated by the Rayleigh-Sommerfeld diffraction integral formula under the condition of a plane wave incidence with unit amplitude [23-25]

\[
    U(x, y, z) = \int \int \iota(\xi, \eta, 0) \exp\left(ikR\right) \frac{1}{iR} \left(1 + \frac{i}{kR} \right) - \frac{\xi \eta}{R} d\xi d\eta, \quad (7)
\]

where \( \iota(\xi, \eta, 0) \) is the pupil function of GF\(Z\)Ps, \( i \) is the imaginary unit, \( k \) is the wave number, \( z \) is the axial distance from the pupil plane, and \( R \) denotes the distance between point \((\xi, \eta, 0)\) and point \((x, y, z)\).

To investigate the axial focusing performance of GF\(Z\)Ps, a typical kind of GF\(Z\)Ps of ten-order switching sequence \( F^{10} \) is shown in Figure 3(a), and the corresponding simulation parameters are as follows: \( \lambda = 632.8 \text{nm} \), \( (F^1, F^2, F^3) = (0, 01, 011) \), \( F = 4\text{cm} \), \( K = 0.5 \) and the corresponding linear recursion relation is \( F_{n+3} = F_{n+2} + F_{n+1} + F_n \), whose characteristic equation is \( x^3 - x^2 - x - 1 = 0 \), and the characteristic roots are 1.839, -0.420+0.606i, and -0.420-0.606i, respectively. For comparison, the TFZPs with the same resolution are represented in Figure 3(b). The number of transparent zones is 125 for GF\(Z\)Ps and 115 for TFZPs while each of them has a total of 330 zones. The axial normalized intensity computed for GF\(Z\)Ps and the associated TFZPs are shown in Figure 3(c). Obviously, in this case, the first focus of the GF\(Z\)Ps is located at \( f_1 = 3.086\text{cm} \) and the other one at \( f_2 = 5.682\text{cm} \) while the prime focal distance of TFZPs is 4.000cm. Thus, the ratio of the two focal distances satisfies \( f_2/f_1 = 1.841 \). However, because of the diversity of the GF\(Z\)Ps based on different encoded seed elements, they may present more than two foci under a plane wave illumination. But in this letter, we just consider the bifocal focusing properties.

![Intensity distribution along the optical axis produced by GF\(Z\)Ps and TFZPs](image)

**Fig. 3.** (a) GF\(Z\)Ps \((F^m)\) in this case; (b) TFZPs with the same resolution; (c) Normalized intensity distribution along the optical axis produced by GF\(Z\)Ps and TFZPs.

Apart from the value \( K \), the other parameters remain the same value above. In this case \( K \) is equal to 0.45, the two focal distances are changed to 2.777cm and 5.114cm shown in Figure 4(a). Other results are shown in Figure 4(b) ~ 4(e).

| OPDP (K) | Focal distances | f1(cm) | f2(cm) |
|----------|----------------|--------|--------|
| 0.45     | 2.777          | 3.086  | 5.682  |
| 0.50     | 2.777          | 3.086  | 5.682  |
| 0.85     | 2.777          | 3.086  | 5.682  |
| 1.54     | 2.777          | 3.086  | 5.682  |
| 2.38     | 2.777          | 3.086  | 5.682  |

From Table 1 we know that the ratio of the two focal distances remains the same while their absolute locations are adjustable due to the different OPDP. What’s important is that amplification or constriction factor of two focal distances is exactly equal to the ratio of the two values of OPDP, such as \( 2.777/5.252 \approx 0.45/0.85 \).
Besides, the two focal distances as a function OPDP are plotted in Figure 5. The two focal distances are directly proportional to the OPDP. The ratio of two OPDP is equal to that of two focal distances. Hence, in this way, one can regulate and control the two focal distances to a certain degree.

The Fibonacci zone plates can present two equal intensity foci and the ratio of the two focal distances approaches the golden mean \([7]\) which is a fixed value. But for GFZPs, the ratio may be adjustable due to the different generalized Fibonacci sequences.

For the purpose of adjustable ratio of the two axial focal distances, other four kinds of GFZPs are investigated. The parameters are as follows: \(\lambda = 632.8\text{nm}, F = 4\text{cm}, K = 0.5\). For two seed elements, they are \(F_1 = 2\) and \(F_2 = 3\) and coded as \((F^1, F^2) = (0, 01, 01)\); while for three seed elements, they are \(F_1 = 1, F_2 = 2\) and \(F_3 = 3\) and coded as \((F^1, F^2, F^3) = (0, 01, 010)\).

Figure 6 shows axial intensity distribution produced by GFZPs based on different linear recursion relations, which are \(F_{n+3} = -F_{n+2} + F_{n+1} - F_n, F_{n+2} = -F_{n+1} + F_n, F_{n+2} = 2F_{n+1} + 0.25F_n\) and \(F_{n+2} = -2F_{n+1} + 0.25F_n\), respectively. Here negative sign means complement operation. The ratio of the two focal distances changes from 1.841 to 1.618 and 2.116 due to the different generalized Fibonacci sequences.
usually applied, such as terahertz (THz) imaging [26] and x-ray microscopy [27], the proposed method of designing the structure of GFiZPs can offer reference for photon sieves [28-30].

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