LOCALIZED FRAMES AND COMPACTNESS

FAWWAZ BATAYNEH AND MISHKO MITKOVSKI†

Abstract. We introduce the concept of weak-localization for generalized frames and use this concept to define a class of weakly localized operators. This class contains many important operators, including: Short Time Fourier Transform multipliers, Calderon-Toeplitz operators, Toeplitz operators on various functions spaces, Anti-Wick operators, and many others. In this paper, we study the boundedness and compactness of weakly localized operators. In particular, we provide a characterization of compactness for weakly localized operators in terms of the behavior of their Berezin transform.

1. Introduction and Preliminaries

The main goal of this paper is to develop a general setting in which boundedness and compactness of a large class of operators can be determined by their behavior on a very restricted special class of elements in the underlying Hilbert space. The usual setting for this type of questions are the classical function spaces of Bergman type. It is well known that in these spaces the boundedness and compactness of a wide variety of Hankel and Toeplitz operators can be determined solely in terms of their behavior on the reproducing kernels [1,6,10,19–22,27,30]. In this paper, we show that results of this kind are not special to Bergman type spaces nor to Toeplitz and Hankel operators. It turns out that such results hold in a much greater generality. Indeed, our only requirement for the underlying Hilbert space is to possess a generalized weakly localized frame whose members are indexed by some locally compact Hausdorff group and whose topology is second countable. We show that on such spaces the norm and the essential norm of any operator which is localized in a certain sense is very much determined by the behavior of the operator on the elements of the generalized frame.

1.1. Frame families. As mentioned above, we assume that the underlying Hilbert space \( \mathcal{H} \) possesses a generalized frame indexed by a locally compact Hausdorff group \( G \) whose topology is second countable (or equivalently first countable and \( \sigma \)-compact). Under these assumptions, by a theorem of Struble [28], there exists a metric \( d \) on \( G \) which is left-invariant (\( d(zx,zy) = d(x,y) \) for all \( x,y,z \in G \)), proper, and generates the topology of \( G \). Here by proper we just mean that all the closed balls (with respect to the metric \( d \)) are compact. Since all of the left-invariant Haar measures on \( G \) are multiples of each other, we will pick and fix one of them and denote it by \( \lambda \). Finally, we will assume that this measure \( \lambda \) is doubling, i.e., there exists a constant \( c > 1 \), so that, for all \( x \in G \) and \( r > 0 \), we have \( \lambda(D(x,2r)) \leq c\lambda(D(x,r)) \).

Definition 1.1. Let \( \{f_x\}_{x \in G} \) be a family of vectors in a Hilbert space \( \mathcal{H} \). We say that \( \{f_x\}_{x \in G} \) forms a frame family if the following two properties hold.

2000 Mathematics Subject Classification. 42B35, 43A22, 47B35, 47B38.
Key words and phrases. Localized frames, Localized operator, Multipliers, Toeplitz operator.
† Research supported in part by National Science Foundation DMS grant # 1101251.
(i) the mapping \( x \to f_x \) is continuous (and hence measurable),
(ii) there exist constants \( c, C > 0 \) such that for every \( f \in \mathcal{H} \) the following inequalities hold
\[
c \|f\|^2 \leq \int_G |\langle f, f_x \rangle|^2 d\lambda(x) \leq C \|f\|^2.
\]

**Remark 1.2.** Most often only weak measurability is required in (i). We impose a stronger condition just to make sure that Lemma 4.1 below holds. In all other places weak measurability is sufficient. This slightly stronger assumption doesn’t exclude any of the important examples of frame families.

Notice that for \( G = \mathbb{Z} \) this definition reduces to the usual definition of frames. As in the classical case we will say that the frame family is a Parseval frame family if \( c = C = 1 \). If only the right side of the frame inequality holds then we will say that \( \{f_x\}_{x \in G} \) forms a Bessel family.

For a given frame family \( \{f_x\}_{x \in G} \), as usual, we can define the analysis and synthesis operators, and use them to define the frame operator and the dual frame family which we will denote by \( \{\tilde{f}_x\}_{x \in G} \). It is easy to see that in case of Parseval frame families its dual frame family coincides with the original one, i.e., \( f_x = \tilde{f}_x \).

We will say that the frame family \( \{f_x\}_{x \in G} \) is bounded if \( \sup_{x \in G} \|f_x\| < \infty \). Clearly, the frame family \( \{f_x\}_{x \in G} \) is bounded if and only if the dual frame family \( \{\tilde{f}_x\}_{x \in G} \) is bounded.

Any Hilbert space \( \mathcal{H} \) equipped with a frame family \( \{f_x\}_{x \in G} \) can be embedded into \( L^2(G) \) as a closed subspace with equivalent norm in the following natural way (here and elsewhere \( L^2(G) \) is assumed to be equipped with the left Haar measure \( \lambda \)). To any \( f \in \mathcal{H} \) we associate \( F \in L^2(G) \) by \( F(x) = \langle f, f_x \rangle \). This is clearly an embedding. The frame family condition implies that the norms on \( L^2(G) \) and \( \mathcal{H} \) are equivalent, and in the case of Parseval frame families they are equal. In the later case the orthogonal projection is given by the formula
\[
P(F) = \int_G F(x)f_xd\lambda(x).
\]

Our continuity assumption on \( x \to f_x \) also implies that for any \( f \in \mathcal{H} \) the function \( F(x) = \langle f, f_x \rangle \) is continuous. In the case of a bounded frame family we moreover have that such a function \( F \) is in \( C_0(G) \).

### 1.2. Weakly localized frames

Let \( \mathcal{F} = \{f_x\}_{x \in G} \) and \( \mathcal{G} = \{g_x\}_{x \in G} \) be two families of vectors in \( \mathcal{H} \) indexed by the group \( G \).

The pair \( (\mathcal{F}, \mathcal{G}) \) is said to be localized (or \( L^1 \)-localized) if there exists a function \( a \in L^1(G) \) such that \( |\langle f_x, g_y \rangle| \leq a(x^{-1}y) \) for all \( x, y \in G \). This concept was essentially introduced by Gröchenig [14] for special (but important) class of functions \( a \). In the general form as above it was studied by Balan et al [2, 3] and Futamura [12]. In this paper we introduce a slightly weaker notion which allows us to treat yet another important function space, the Bergman space.

**Definition 1.3.** Given a positive measurable function \( p : G \to (0, \infty) \) we say that the pair \( (\mathcal{F}, \mathcal{G}) \) is \( p \)-weakly localized if
\[
\int_G |\langle f_x, g_y \rangle| p(y)d\lambda(y) \lesssim p(x), \quad \int_G |\langle f_x, g_y \rangle| p(x)d\lambda(x) \lesssim p(y),
\]

(1.1)
and

\[
\lim_{R \to \infty} \sup_{x \in G} \frac{1}{p(x)} \int_{D(x,R)^c} |\langle f_x, g_y \rangle| p(y) d\lambda(y) = 0, \quad \lim_{R \to \infty} \sup_{y \in G} \frac{1}{p(y)} \int_{D(y,R)^c} |\langle f_x, g_y \rangle| p(x) d\lambda(x) = 0.
\]

(1.2)

Notice that this definition is symmetric in the sense that \((F, G)\) is \(p\)-weakly localized if and only if \((G, F)\) is \(p\)-weakly localized. We will say that a frame family \(\{f_x\}_{x \in G}\) is \(p\)-weakly localized if the pair \((\tilde{F}, F)\) is \(p\)-weakly localized, where \(\tilde{F} = \{\tilde{f}_x\}_{x \in G}\) is the dual frame family of \(F\).

The term localized is related to the equalities (1.2). To measure this localization more precisely we use the following function

\[
\rho(\epsilon) := \inf \left\{ R > 0 : \int_{D(x,R)^c} |\langle f_x, g_y \rangle| p(y) d\lambda(y) \leq \epsilon p(x), \int_{D(y,R)^c} |\langle f_x, g_y \rangle| p(x) d\lambda(x) \leq \epsilon p(y) \right\}.
\]

(1.3)

Clearly, if the pair \((F, G)\) is \(p\)-weakly localized, then \(\rho(\epsilon)\) is a decreasing function such that \(\rho(\epsilon) < \infty\) for all \(\epsilon > 0\). We are mostly interested in the behavior of this function near 0, i.e., when \(\epsilon > 0\) is small. For well localized frames \(\rho(\epsilon)\) decays faster as \(\epsilon\) grows away from 0. We will use this function to estimate the norm and the essential norm of weakly localized operators.

1.3. Weakly localized operators. Let \(T : \mathcal{H} \to \mathcal{H}\) be a linear operator. We will say that the operator \(T\) is \(p\)-weakly localized with respect to the pair \((F, G)\) if the pair \((T, G)\) is \(p\)-weakly localized, where \(T = \{Tf_x\}_{x \in G}\). Notice that \(p\)-weak localization with respect to \((F, G)\) is in general different from \(p\)-weak localization with respect to \((G, F)\). In Section 2 we will prove that the class of \(p\)-weak localized operators forms an algebra which can be viewed as an analog of the result of Futamura [12] in a more general situation.

The most important subclass of the class of \(p\)-weakly localized operators is the class of so called multipliers [4, 5]. They are defined in the following way. Let \(F = \{f_x\}_{x \in G}\) and \(G = \{g_x\}_{x \in G}\) be two Bessel families in \(\mathcal{H}\) such that the pair \((F, G)\) is \(p\)-weakly localized. For every \(u \in L^\infty(G)\) let \(T_u : \mathcal{H} \to \mathcal{H}\) be the linear operator defined by

\[
T_u f = \int_{G} u(x) \langle f, g_x \rangle f_x d\lambda(x).
\]

For obvious reasons, we will call all such operators multiplier operators or just multipliers. This class includes Short Time Fourier Transform (STFT) multipliers, Toeplitz operators on various functions spaces, and many others (more details are given in Section 5). The results in this paper provide norm and essential norm estimates for general \(p\)-weakly localized operators. In particular, we give a characterization of compactness for \(p\)-weakly localized operators solely in terms of their behavior on the frame family \(F\) and/or its dual. The techniques that we use are in essence similar to the ones used in [20]. However, in this level of generality our results show that the results that were seemingly very much special to the class of Toeplitz operators acting on Bergman-type spaces can actually be extended to a much bigger class of seemingly unrelated operators.

The paper is organized as follows. In the next section we give some simple preliminary results regarding the class of \(p\)-weakly localized operators. The main results are proved in
Sections 3 and 4. In the last section we give some concrete examples where our results can be applied.

2. Basic properties of weakly localized operators

The following lemma will be used to show few basic properties of weakly localized operators.

**Lemma 2.1.** Let $\mathcal{F}^i = \{f^i_x\}_{x \in G}, \mathcal{G}^i = \{f^i_x\}_{x \in G}, i = 1, 2$ be four families in $\mathcal{H}$ such that the pairs $(\mathcal{F}^1, \mathcal{G}^1)$ and $(\mathcal{F}^2, \mathcal{G}^2)$ are $p$-weakly localized. If

$$\langle k_y, l_z \rangle \lesssim \int_G \langle f^1_y, g^1_x \rangle \langle g^2_x, f^2_z \rangle \, d\lambda(x),$$

for all $y, z \in G$, then the pair $(\mathcal{K}, \mathcal{L})$ is $p$-weakly localized, where $\mathcal{K} = \{k_x\}_{x \in G}, \mathcal{L} = \{l_x\}_{x \in G}$.

**Proof.** Since the pairs $(\mathcal{F}^i, \mathcal{G}^i)$ are both $p$-weakly localized we easily obtain

$$\int_G \langle k_y, l_z \rangle |p(y)\, d\lambda(y) \lesssim \int_G \int_G \langle f^1_y, g^1_x \rangle |p(y)\, d\lambda(y) \langle g^2_x, f^2_z \rangle \, d\lambda(x) \lesssim p(z).$$

Similarly, $\int_G \langle k_y, l_z \rangle |p(z)\, d\lambda(z) \lesssim p(y)$.

It remains to show that the equalities (1.2) hold for the pair $(\mathcal{K}, \mathcal{L})$. We show one of them, the other one being similar. Denote

$$I(x, y) := |\langle f^1_y, g^1_x \rangle| \int_{D(y, R)}^c |\langle g^2_x, f^2_z \rangle| \, p(z)\, d\lambda(z).$$

Then

$$\int_{D(y, R)}^c |\langle k_y, l_z \rangle| p(z)\, d\lambda(z) \lesssim \int_G I(x, y)\, d\lambda(x) = \left( \int_{D(y, R/2)}^c I(x, y)\, d\lambda(x) + \int_{D(y, R/2)}^c I(x, y)\, d\lambda(x) \right).$$

We estimate each of the last two integrals separately. For the first integral, notice first that for every $x \in D(y, R/2)$ we have $D(y, R)^c \subseteq D(x, R/2)^c$. Therefore,

$$\int_{D(y, R/2)}^c I(x, y)\, d\lambda(x) \leq \int_{D(y, R/2)}^c |\langle f^1_y, g^1_x \rangle| \int_{D(x, R/2)^c} |\langle g^2_x, f^2_z \rangle| \, p(z)\, d\lambda(z)\, d\lambda(x).$$

Denote

$$C_1(r) := \sup_{x \in G} \frac{1}{\mathcal{P}(x)} \int_{D(x, r)^c} |\langle g^2_x, f^2_z \rangle| \, p(z)\, d\lambda(z).$$

Since the pair $(\mathcal{F}^2, \mathcal{G}^2)$ is weakly localized, we have that $C_1(r) \to 0$ as $r \to \infty$. Therefore,

$$\int_{D(y, R/2)}^c I(x, y)\, d\lambda(x) \leq C_1(R/2) \int_{D(y, R/2)}^c |\langle f^1_y, g^1_x \rangle| \, d\lambda(x) \lesssim C_1(R/2) \to 0, \text{ as } R \to \infty.$$

The other integral is even easier to estimate. Indeed,

$$\int_{D(y, R/2)}^c I(x, y)\, d\lambda(x) \leq \int_{D(y, R/2)}^c |\langle f^1_y, g^1_x \rangle| \int_{D(y, R)^c} |\langle g^2_x, f^2_z \rangle| \, p(z)\, d\lambda(z)\, d\lambda(x) \lesssim C_2(R/2) \to 0,$$

where similarly as before,

$$C_2(r) := \sup_{y \in G} \frac{1}{\mathcal{P}(y)} \int_{D(x, r)^c} |\langle f^1_y, g^1_x \rangle| \, p(x)\, d\lambda(x).$$

$\square$
For a given frame family \( \{ f_x \}_{x \in G} \), as mentioned above, it was essentially proved in [12] that the collection of all \( L^1 \) localized operators with respect to \((\tilde{\mathcal{F}}, \mathcal{F})\) forms an algebra. We prove that a similar result holds for \( p \)-weakly localized operators.

**Proposition 2.2.** Let \( p : G \to (0, \infty) \) be a positive measurable function and let \( \{ f_x \}_{x \in G} \) be a \( p \)-weakly localized frame family in \( \mathcal{H} \). Let \( \mathcal{L} \) be the collection of all \( p \)-weakly localized operators with respect to \((\tilde{\mathcal{F}}, \mathcal{F})\). Then \( \mathcal{L} \) forms an algebra. Moreover, if \( \mathcal{F} \) is a Parseval frame family, then \( \mathcal{L} \) is a *-algebra.

**Proof.** It is easy to see that \( \mathcal{L} \) is closed under addition and scalar multiplication. Let \( A, B \in \mathcal{L} \). Using the expansion formula for frame families we obtain

\[
\left| \left\langle AB\tilde{f}_y, f_z \right\rangle \right| \leq \int_G \left| \left\langle B\tilde{f}_y, f_x \right\rangle \right| \left| \left\langle A\tilde{f}_x, f_z \right\rangle \right| d\lambda(x).
\]

Therefore, by Lemma 2.1 we obtain that \( AB \in \mathcal{L} \).

The next proposition shows that every multiplier operator is \( p \)-weakly localized.

**Proposition 2.3.** Let \( \mathcal{F} = \{ f_x \}_{x \in G} \) and \( \mathcal{G} = \{ g_x \}_{x \in G} \) be two Bessel families in \( \mathcal{H} \) such that the pair \((\mathcal{F}, \mathcal{G})\) is \( p \)-weakly localized. Then for every \( u \in L^\infty(G) \), the multiplier operator \( T_u \) is \( p \)-weakly localized with respect to \((\mathcal{F}, \mathcal{G})\).

**Proof.** First, notice that since \( \mathcal{F} \) and \( \mathcal{G} \) are Bessel families, the operator \( T_u \) is well defined for all \( u \in L^\infty(G) \). We need to prove that the pair \((\mathcal{T}, \mathcal{G})\) is weakly localized, where as before \( \mathcal{T} = \{ Tf_x \}_{x \in G} \). This is a simple consequence of Lemma 2.1 due to

\[
|\langle Tf_y, g_z \rangle| \leq \|u\|_\infty \int_G \left| \left\langle f_y, g_x \right\rangle \right| \left| \left\langle f_x, g_z \right\rangle \right| d\lambda(x).
\]

### 3. Norm estimates of weakly localized operators

It is clear that each \( p \)-weakly localized operator \( T \) must be bounded. In this section we provide norm estimates for such operators \( T \) in terms of the size of \( \sup_{x \in G} \|Tf_x\| \). To do this we will exploit our assumption that our Haar measure \( \lambda \) is doubling. The only reason that we imposed this condition was to have the following useful covering lemma.

**Lemma 3.1.** There exists an integer \( N > 0 \) (depending only on the doubling constant of the measure \( \lambda \)) such that for any \( r > 0 \) there is a covering \( \mathcal{D}_r = \{ F_j \} \) of \( G \) by disjoint Borel sets satisfying

- \( (1) \) every point of \( G \) belongs to at most \( N \) of the sets \( G_j := \{ x \in G : d(x, F_j) \leq r \} \),
- \( (2) \) \( \text{diam} F_j \leq 4r \) for every \( j \).

The proof of this result can be found in [21].

Let \( \mathcal{F} = \{ f_x \}_{x \in G} \) be a fixed frame family and let \( T \) be a \( p \)-weakly localized operator with respect to the pair \((\tilde{\mathcal{F}}, \mathcal{F})\). We next show that each decomposition \( \mathcal{D}_r = \{ F_j \} \) of \( G \) defines a sequence of operators \( \{ T_j \} \) that, loosely speaking, gives an approximate decomposition of the operator \( T \). These operators \( T_j : \mathcal{H} \to \mathcal{H} \) are given by

\[
T_j f := \int_{F_j} \int_{G_j} \langle f, f_x \rangle \langle T\tilde{f}_y, f_x \rangle \tilde{f}_y d\lambda(x) d\lambda(y).
\]
**Proposition 3.2.** If the operator $T$ satisfies (1.1) with respect to the pair $(\tilde{F}, F)$, then the corresponding series $\sum_{j=1}^{\infty} T_j$ converges in the strong operator topology.

**Proof.** Let $f \in \mathcal{H}$. It is enough to show that the partial sums $S_n f = \sum_{j=1}^{n} T_j f$ form a Cauchy sequence. By the frame condition on $\tilde{F}$ we first have

$$\| (S_n - S_m) f \|^2 \lesssim \sum_{j=m+1}^{n} \int_{F_j} \int_{G_j} \langle f, f_x \rangle \langle T \tilde{f}_x, f_y \rangle d\lambda(x) \, d\lambda(y).$$

Using the fact that $T$ satisfies (1.1), the classical Schur test applied to the integral operator $Rf(y) := \int_{G_j} \langle T \tilde{f}_x, f_y \rangle f(x) d\lambda(x)$ implies:

$$\| (S_n - S_m) f \|^2 \lesssim \sum_{j=m+1}^{n} \int_{F_j} |\langle f, f_y \rangle|^2 d\lambda(y).$$

The last term converges to 0 as $m, n \to \infty$ since

$$\sum_{j=1}^{\infty} \int_{F_j} |\langle f, f_y \rangle|^2 d\lambda(y) = \int_{G} |\langle f, f_y \rangle|^2 d\lambda(y) \lesssim \| f \|^2.$$

$\square$

Define $A := \sum_{j=1}^{\infty} T_j$, where the series is taken in the strong operator topology (the convergence is established in the previous proposition). Notice that since all of the operators $T_j$ depend on the decomposition $\mathcal{D}_r$, the operator $A$ also depends on $\mathcal{D}_r$. We next show that $T$ can be approximated arbitrarily well by the operator $A$ with an appropriate choice of $\mathcal{D}_r$. How large this $r > 0$ should be chosen depends on the localization function $\rho(\epsilon)$ introduced in (1.3).

**Proposition 3.3.** Let $\mathcal{F}$ be a frame and let $T$ be a $p$-weakly localized operator with respect to the pair $(\tilde{F}, F)$. For any $\epsilon > 0$ and any $r > \rho(\epsilon)$ the operator $A$ induced by the corresponding decomposition $\mathcal{D}_r$ satisfies $\| T - A \| < \epsilon$.

**Proof.** Let $\epsilon > 0$. Consider the integral operator $R_j f(y) := \int_{G} K_j(x, y) f(x) d\lambda(x)$ with kernel defined by $K_j(x, y) = 1_{G_j}(x) 1_{F_j}(y) \langle T \tilde{f}_x, f_y \rangle$. By definition of $F_j$ and $G_j$ we have that $K_j(x, y) = 0$ unless $d(x, y) \geq r$. Using the fact that $T$ is a $p$-weakly localized we have that $\rho(\epsilon) < \infty$. Let $r > \rho(\epsilon)$. By the classical Schur test the integral operator $R_j$ induced by the corresponding decomposition $\mathcal{D}_r$ has norm no greater than $\epsilon$ (as an operator from $L^2(G)$ into itself).

Consider now the operator $A$ induced by this decomposition $\mathcal{D}_r$. Observe that for any $f \in \mathcal{H}$ we have

$$T f = \int_{G} \langle T f, f_y \rangle \tilde{f}_y d\lambda(y) = \sum_{j=1}^{\infty} \int_{F_j} \langle T f, f_y \rangle \tilde{f}_y d\lambda(y)$$

$$= \sum_{j=1}^{\infty} \int_{F_j} \int_{G} \langle f, f_x \rangle \langle T \tilde{f}_x, f_y \rangle \tilde{f}_y d\lambda(x) d\lambda(y).$$
Therefore
\[
(T - A)f = \sum_{j=1}^{\infty} \int_{F_j} \int_{G_j} \langle f, f_x \rangle \left\langle T \tilde{f}_x, f_y \right\rangle \tilde{f}_y d\lambda(x) d\lambda(y).
\]

We then have
\[
\| (T - A)f \|_2^2 \leq \frac{1}{C^2} \sum_{j=1}^{\infty} \int_{F_j} \left| \int_{G_j} \langle f, f_x \rangle \left\langle T \tilde{f}_x, f_y \right\rangle d\lambda(x) \right|^2 d\lambda(y),
\]
where \( C > 0 \) is the upper frame constant. Using the uniform norm estimate for the integral operator \( R_j \), we obtain
\[
\| (T - A)f \|_2^2 \leq \frac{1}{C^2} \sum_{j=1}^{\infty} \int_{F_j} \epsilon^2 |\langle f, f_y \rangle|^2 d\lambda(y) \leq \epsilon^2 \| f \|_2^2.
\]

The following results provide operator norm estimates of weakly localized operators in terms of their behavior on the frame family \( F \) and its dual.

**Theorem 3.4.** Let \( F \) be a frame family and \( T \) be a weakly localized operator with respect to the pair \((\tilde{F}, F)\). Then for any \( 0 < \epsilon < 1 \) and \( r > \rho(\epsilon \| T \|) \) we have the following estimate
\[
\| T \| \leq \frac{\sqrt{N\lambda(D(\epsilon, 4r))}}{1 - \epsilon} \sup_{y \in G} \left( \int_{D(y, 3r)} |\left\langle T \tilde{f}_x, f_y \right\rangle|^2 d\lambda(x) \right)^{1/2},
\]
where \( e \in G \) denotes the identity element and \( N \) is the covering constant from Lemma 3.1.

**Proof.** If \( T = 0 \) there is nothing to prove. Otherwise, let \( 0 < \epsilon < 1 \) and \( r > \rho(\epsilon \| T \|) \). Consider the operator \( A \) defined in Proposition 3.3 that corresponds to this \( r \). We have \( \| T \| \leq \| T - A \| + \| A \| < \epsilon \| T \| + \| A \| \), and hence \( \| T \| \leq \frac{1}{1 - \epsilon} \| A \| \).

We next estimate the norm of \( A \). First, observe that
\[
\| Af \|_2^2 \leq \frac{1}{C^2} \sum_{j=1}^{\infty} \int_{F_j} \left| \int_{G_j} \langle f, f_x \rangle \left\langle T \tilde{f}_x, f_y \right\rangle d\lambda(x) \right|^2 d\lambda(y)
\]
\[
\leq \sum_{j=1}^{\infty} \int_{F_j} C(y, 3r) d\lambda(y) \int_{G_j} |\langle f, f_x \rangle|^2 d\lambda(x),
\]
where \( C > 0 \) is the upper frame constant and
\[
C(y, r) := \int_{D(y, r)} |\left\langle T \tilde{f}_x, f_y \right\rangle|^2 d\lambda(x).
\]

By Lemma 3.1 and the invariance of the Haar measure \( \lambda \) we have \( \lambda(F_j) \leq \lambda(D(e, 4r)) \), and hence
\[
\int_{F_j} C(y, 3r) d\lambda(y) \leq \lambda(D(e, 4r)) \sup_{y \in F_j} C(y, 3r)
\]
for all \( j \).
Therefore,
∥Af∥^2 \leq \frac{1}{C^2} \lambda(D(e,4r)) \sup_{y \in G} C(y,3r) \sum_{j=1}^{\infty} \int_{G_j} |\langle f, f_x \rangle|^2 d\lambda(x).

The finite overlap property of \{G_j\} implies that

∥Af∥^2 \leq N\lambda(D(e,4r)) \sup_{y \in G} C(y,3r) ∥f∥^2,

where \(N\) is the constant from Lemma 3.1.

□

As a simple consequence of Theorem 3.4 we obtain the following corollary.

Corollary 3.5. Let \(\mathcal{F}\) be a frame family and \(T\) be a weakly localized operator with respect to the pair \((\tilde{\mathcal{F}}, \mathcal{F})\). Then for any \(0 < \epsilon < 1\) and \(r > \rho(\|T\|)\) we have the following estimate

∥T∥ \leq \frac{\sqrt{N\lambda(D(e,4r))}}{1 - \epsilon} \sup_{x \in G} ∥T^* f_x∥,

where \(N\) is the constant from Lemma 3.1. Moreover, if \(\mathcal{F}\) is a Parseval frame family, then

∥T∥ \leq \frac{\sqrt{N\lambda(D(e,4r))}}{1 - \epsilon} \sup_{x \in G} ∥T f_x∥.

4. Compactness of weakly localized operators

In this section we provide several criteria for compactness of weakly localized operators. Again, a crucial role will be played by the operators \(T_j\) defined in (3.1). We will need the following crucial property of these operators.

Lemma 4.1. Each operator \(T_j\) is compact.

Proof. Notice that since the metric \(d\) is proper, by Lemma 3.1, we have that the closure \(\overline{F_j}\) is compact. To any element \(h \in \mathcal{H}\), we associate a function \(a \in C(\overline{F_j})\) defined by

\[ a(y) = \int_{G_j} \langle h, f_x \rangle \langle T_j \tilde{f}_x, f_y \rangle d\lambda(x). \]

Let \(\{h_n\}\) be a sequence in \(\mathcal{H}\) bounded by 1, and let \(\{a_n\}\) be the corresponding sequence of functions in \(C(\overline{F_j})\). It is easy to see that \(|a_n(y)| \lesssim ∥f_y∥\) and \(|a_n(y) - a_n(z)| \lesssim ∥f_y - f_z∥\) with implied constants independent of \(h_n\). These inequalities imply, by the Arzela-Ascoli criterion, that \(\{a_n\}\) has a convergent subsequence \(\{a_{n_k}\}\). We need to show that for the corresponding subsequence \(\{h_{n_k}\}\) in \(\mathcal{H}\) we have that \(\{T_j h_{n_k}\}\) converges. But this is clear since for every \(g \in \mathcal{H}\) with \(∥g∥ \leq 1\) we have

\[ |\langle T_j h_{n_k} - T_j h_{n_1}, g \rangle| = \int_{F_j} |a_{n_k}(y) - a_{n_1}(y)| \left| \langle \tilde{f}_y, g \rangle \right| d\lambda(y) \leq \|a_{n_k} - a_{n_1}\|_{\infty} \sqrt{\lambda(F_j)}. \]

□

Again, let \(A = \sum_{j=1}^{\infty} T_j\), where the series is taken in the strong operator topology. Notice that even though all the partial sums in this series are compact the whole series, i.e., \(A\) may not be compact.

To prove our characterization of compactness we first estimate the essential norm of weakly localized operators.
Theorem 4.2. Let $\mathcal{F}$ be a frame family and $T$ be a weakly localized operator with respect to the pair $(\tilde{\mathcal{F}}, \mathcal{F})$. Then for $r > 0$ large enough we have the following estimate

$$
\|T\|_{\text{ess}} \leq \frac{\sqrt{N\lambda(D(e,4r))}}{1 - \epsilon} \limsup_{d(y,e)\to\infty} \left( \int_{D(y,3r)} \left| \langle T \tilde{f}_x, f_y \rangle \right|^2 d\lambda(x) \right)^{1/2},
$$

where $N$ is the constant from Lemma 3.1.

Proof. If $\|T\|_{\text{ess}} = 0$ there is nothing to prove. Otherwise, let $0 < \epsilon < 1$ and $r > \rho(\epsilon \|T\|_{\text{ess}})$. Consider the operator $A$ from Proposition 3.3 that corresponds to this $r$. For every $n \in \mathbb{N}$, we have

$$
\|T\|_{\text{ess}} \leq \left\| T - \sum_{j=1}^{n} T_j \right\| \leq \|T - A\| + \|A_n\| < \epsilon \|T\|_{\text{ess}} + \|A_n\|,
$$

where $A_n = \sum_{j=n+1}^{\infty} T_j$ with the limit in the sum taken in the strong operator topology.

Therefore, for every $n \in \mathbb{N}$, we have $\|T\|_{\text{ess}} \leq \frac{1}{1 - \epsilon} \|A_n\|$. We next estimate the norm of the tails $A_n$.

First observe that

$$
\|A_n f\|^2 \leq \frac{1}{C^2} \sum_{j=n+1}^{\infty} \int_{F_j} \left( \frac{1}{C \lambda(D(e,4r))} \int_{G_j} \langle f, f_x \rangle |T \tilde{f}_x, f_y\rangle^2 d\lambda(x) \right)^2 d\lambda(y)
$$

$$
\leq \frac{1}{C^2} \sum_{j=n+1}^{\infty} \int_{F_j} C(y,3r)d\lambda(y) \int_{G_j} |\langle f, f_x \rangle|^2 d\lambda(x),
$$

where

$$
C(y,r) = \int_{D(y,r)} |\langle T \tilde{f}_x, f_y\rangle|^2 d\lambda(x).
$$

By Lemma 3.1 and the invariance of the Haar measure $\lambda$ we have $\lambda(F_j) \leq \lambda(D(e,4r))$, and hence

$$
\int_{F_j} C(y,3r)d\lambda(y) \leq \lambda(D(e,4r)) \sup_{y \in F_j} C(y,3r)
$$

for all $j$. Therefore,

$$
\|A_n f\|^2 \leq \frac{1}{C^2} \lambda(D(e,4r)) \sup_{y \in \bigcup_{j \geq n+1} F_j} C(y,3r) \sum_{j=n+1}^{\infty} \int_{G_j} |\langle f, f_x \rangle|^2 d\lambda(x).\n$$

The finite overlap property of $\{G_j\}$ implies that

$$
\|A_n f\|^2 \leq N\lambda(D(e,4r)) \sup_{y \in \bigcup_{j \geq n+1} F_j} C(y,3r) \|f\|^2,
$$

where the constant $N$ is the one from Lemma 3.1. By taking the infimum on both sides we obtain the desired estimate.

$$
\|T\|_{\text{ess}} \leq \frac{1}{1 - \epsilon} \inf_{n} \|A_n\| \leq \frac{\sqrt{N\lambda(D(e,4r))}}{1 - \epsilon} \limsup_{d(y,e)\to\infty} \left( \int_{D(y,3r)} \left| \langle T \tilde{f}_x, f_y \rangle \right|^2 d\lambda(x) \right)^{1/2}.
$$

□
Corollary 4.3. Let $\mathcal{F}$ be a bounded frame family and $T$ be a weakly localized operator with respect to the pair $(\tilde{\mathcal{F}}, \mathcal{F})$. Then $T$ is compact if and only if
\[
\lim_{d(x,e) \to \infty} \|T^* f_x\| = 0.
\]

Proof. The only if part is easy and it doesn’t even require $T$ to be a weakly localized operator. Namely, since $\mathcal{F}$ is a bounded frame family we have $f_x \to 0$ weakly as $d(x,e) \to \infty$. Therefore $\|T^* f_x\| \to 0$ as $d(x,e) \to \infty$ since $T^*$ is compact.

The other direction is a trivial consequence of the estimate in the previous theorem. □

Corollary 4.4. Let $\mathcal{F}$ be a bounded Parseval frame family and $T$ be a weakly localized operator with respect to the pair $(\tilde{\mathcal{F}}, \mathcal{F})$. Then $T$ is compact if and only if
\[
\lim_{d(x,e) \to \infty} \|T f_x\| = 0.
\]

Proof. As above, the only if part is a general statement valid for all bounded operators $T$. For the other direction, notice that since $\mathcal{F}$ is a Parseval frame family we have that $\tilde{T}$ is weakly localized as well. Therefore, by the previous corollary, we have that $T^*$ is compact which implies that $T$ is compact. □

4.1. Berezin transform and compactness. In what follows we will finally exploit the group structure of $G$. From now on, besides assuming that the mapping $x \to f_x$ is continuous, we also assume the following invariance of the inner product:
\[
\langle \tilde{f}_x, f_y \rangle = \langle \tilde{f}_{x}, f_{2y} \rangle, \quad \text{for all } x, y, z \in G.
\]

This invariance assumption allows us to introduce the following interesting class of “translation” operators. For each $y \in G$ we define $U_y : \mathcal{H} \to \mathcal{H}$ by
\[
U_y h = \int_G \langle h, \tilde{f}_x \rangle f_y d\lambda(x).
\]

The frame condition of $\mathcal{F} = \{f_x\}_{x \in G}$ assures that the integral converges and $\|U_y\| \simeq 1$ with the implied constants only depending on the frame constants. The left invariance of $\lambda$ implies that $U_y f_z = f_{yz}$ for all $z \in G$. In addition, it is easy to see that the adjoint $U_y^*$ is given by
\[
U_{y}^{*}h = \int_G \langle h, f_x \rangle \tilde{f}_{y^{-1}x} d\lambda(x).
\]

For $U_y^*$ we have similar formulas $U_{y}^{*} \tilde{f}_x = \tilde{f}_{y^{-1}x}$ and $\|U_{y}^{*}\| \simeq 1$ with the implied constants independent of $y \in G$.

Let $\mathcal{F} = \{f_x\}_{x \in G}$ be a bounded frame family in $\mathcal{H}$. For a given linear operator $T : \mathcal{H} \to \mathcal{H}$ we define the Berezin transform of $T$ to be the function $B(T) : G \to \mathbb{C}$ given by $B(T)(x) = \langle T \tilde{f}_x, f_x \rangle$. It is clear that $B : \mathcal{B}(\mathcal{H}) \to C(G)$ is linear and bounded. Our next result shows that under the assumption that $B$ is injective we have that the Berezin transform $B$ restricted on the class of $(\tilde{\mathcal{F}}, \mathcal{F})$ weakly localized operators maps the class of compact operators into the class $C_0(G)$.

Theorem 4.5. Let $\mathcal{F} = \{f_x\}_{x \in G}$ be a bounded frame family in $\mathcal{H}$. Assume that the Berezin transform $B : \mathcal{B}(\mathcal{H}) \to C(G)$ is injective. Then a bounded linear operator $T$ which is weakly localized with respect to $(\tilde{\mathcal{F}}, \mathcal{F})$ is compact if and only if $\lim_{d(x,e) \to \infty} B(T)(x) = 0$. 
Proof. The if part is easy and is a direct consequence of Theorem 4.3. To prove the only if part, seeking contradiction, assume that $B(T) \in C_0(G)$ but $T$ is not compact. In this case, by Theorem 4.3 there exists $r > 0$ large enough such that

$$0 < 1 \lesssim \limsup_{d(y,e) \to \infty} \left( \int_{D(y,3r)} \left| \leftlangle T\tilde{f}_x, f_y \rightrangle \right|^2 \, d\lambda(x) \right)^{1/2}.$$ 

There exist a sequence $\{y_n\} \subseteq G$ with $d(y_n, e) \to \infty$ such that for all $n$

$$1 \lesssim \int_{D(y_n,3r)} \left| \leftlangle T\tilde{f}_x, f_{y_n} \rightrangle \right|^2 \, d\lambda(x).$$

Changing variables we obtain

$$1 \lesssim \int_{D(y_n,3r)} \left| \leftlangle TU_{y_n}^* \tilde{f}_{y_n^{-1} x}, U_{y_n} f_e \rightrangle \right|^2 \, d\lambda(x) = \int_{D(e,3r)} \left| \leftlangle U_{y_n}^* T U_{y_n}^* \tilde{f}_x, f_e \rightrangle \right|^2 \, d\lambda(x).$$

Since $\left\| U_{y_n}^* T U_{y_n}^* \right\| \simeq 1$ the sequence of operators $\{U_{y_n}^* T U_{y_n}^*\}$ has a subsequence which converges in the weak operator topology. To avoid cumbersome notation we will keep denoting this subsequence by $\{U_{y_n}^* T U_{y_n}^*\}$. Denote the limit of this subsequence by $T_0$. Using the fact that the frame family $\mathcal{F}$ is bounded and the dominated convergence theorem we obtain

$$\int_{D(e,3r)} \left| \leftlangle T_0\tilde{f}_x, f_e \rightrangle \right|^2 \, d\lambda(x) > 0. \tag{4.1}$$

On the other hand, for every $x \in G$ we have

$$B(T_0)(x) = \left\langle T_0\tilde{f}_x, f_x \right\rangle = \lim_{n \to \infty} \left\langle U_{y_n}^* T U_{y_n}^* \tilde{f}_x, f_{y_n} \right\rangle = \lim_{n \to \infty} \left\langle T\tilde{f}_{y_n x}, f_{y_n x} \right\rangle = 0.$$ 

The last equality is due to the fact that $d(y_n x, e) = d(x, y_n^{-1}) \to \infty$, which follows from $d(e, y_n^{-1}) = d(y_n, e) \to \infty$ by triangle inequality.

Now, since the Berezin transform is injective by assumption we obtain that $T_0$ is a zero operator which obviously contradicts (4.1). This finishes the proof.

\[\square\]

Remark 4.6. It should be mentioned that our norm and essential norm estimates show that all of the results above hold true not just for $p$-weakly localized operators but also for operators belonging in the (operator norm) closure of the algebra of $p$-weakly localized operators.

5. Examples

We give several examples where our results apply. A more thorough treatment of these examples from a point of view which is more or less similar to ours can be found in [11,16,17].

To avoid confusion we specify the notation for the basic unitary operators on $L^2(\mathbb{R})$:

(i) Translation: $T_af(x) = f(x - a),$ 
(ii) Modulation: $M_af(x) = e^{2\pi i ax} f(x),$ 
(iii) Dilation: $D_af(x) = \frac{1}{\sqrt{a}} \tilde{f}\left(\frac{x}{a}\right),$ $a > 0.$
5.1. Time-frequency (Gabor) frame families. In this case the group is \( G = \mathbb{R} \times \mathbb{R} \) with the addition, and equipped with the usual \( \mathbb{R}^2 \)-topology. The corresponding Haar measure is just the Lebesgue measure on \( \mathbb{R}^2 \) and the corresponding invariant metric is the Euclidean metric on \( \mathbb{R}^2 \). The Hilbert space \( \mathcal{H} \) is \( L^2(\mathbb{R}) \) (for simplicity we give the example with \( \mathbb{R} \) instead of \( \mathbb{R}^n \)). For arbitrary fixed \( \psi \in L^2(\mathbb{R}) \) with norm 1 (usually referred to as the window function) and \( x = (a, b) \in \mathbb{R} \times \mathbb{R} \) define \( f_x = M_a T_b \psi \). It is well known that this family forms a Parseval frame family in the sense of Definition 1.1. In this case the frame coefficient \( \langle f, M_a T_b \psi \rangle \) defines the so called the short-time Fourier transform of \( f \in L^2(\mathbb{R}) \) with respect to the window \( \psi \). The most classical window function is the normalized Gaussian \( \psi(x) = \frac{1}{\sqrt{\pi}} e^{-x^2/2} \) in which case the corresponding frame family is just the Gabor frame family.

The multiplier operators in this case correspond to the family of so called time-frequency localization operators. They are closely related to the class of pseudo-differential operators (see [15]). Their properties have been thoroughly studied (see for example [8, 9, 29] and the references therein). When the window function \( \psi \) is sufficiently localized, the corresponding frame family is localized in which case our results provide estimates for the norm and the essential norm of the corresponding time-frequency localization operators.

5.2. Wavelet (Caledron) frame families. In this case the group is the “ax+b”-group \( G = \mathbb{R}^+ \times \mathbb{R} \) equipped with the operation \((a_1, b_1) * (a_2, b_2) = (a_1 a_2, a_1 b_2 + b_1)\). The corresponding Haar measure is \( \frac{da db}{a^2} \) and the corresponding left-invariant metric is the Riemannian metric given by the length element \( ds^2 = \frac{da^2 + db^2}{a^2} \). The underlying Hilbert space \( \mathcal{H} \) is \( L^2(\mathbb{R}) \) (again for simplicity we give the example with \( \mathbb{R} \) instead of \( \mathbb{R}^n \)). Let \( \psi \in L^2(\mathbb{R}) \) be such that

\[
\int_{\mathbb{R}} \frac{|\hat{\psi}(\xi)|^2}{|\xi|} d\xi = 1. 
\]

For \( x = (a, b) \in \mathbb{R}^+ \times \mathbb{R} \) define \( f_x = D_a T_b \psi \). The well known Calderon reproducing formula shows that this family forms a Parseval frame family. In this case the frame coefficient \( \langle f, D_a T_b \psi \rangle \) defines the so called continuous wavelet transform of \( f \in L^2(\mathbb{R}) \). The multiplier operators in this case correspond to the family of Calderon-Toeplitz operators. Their properties have been studied in [18, 23, 25] for example. When the wavelet function \( \psi \) is sufficiently localized, the corresponding wavelet frame family is localized and our results provide estimates for the norm and the essential norm of the corresponding Calderon-Toeplitz operators.

5.3. Bergman space frame families. In this case the group \( G \) is the unit disc \( \mathbb{D} \) equipped with the operation \( z * w = \frac{z+w}{1+zw} \). If \( \varphi_z \) denotes the Mobius transformation of the disc \( \mathbb{D} \) (indexed by the point \( z \in \mathbb{D} \)), then \( \varphi_z \circ \varphi_w = \varphi_{zw} \) up to a unimodular multiplicative constant. This property reveals easily that the corresponding Haar measure is the invariant hyperbolic measure \( d\lambda(z) = \frac{dA(z)}{(1-|z|^2)^2} \), where \( dA(z) \) is just the area measure on \( \mathbb{D} \), and the induced invariant metric \( d \) is the Bergman metric. The Hilbert space \( \mathcal{H} \) is the standard Bergman space \( \mathcal{A}^2(\mathbb{D}) \). For \( x = z \in \mathbb{D} \) define \( f_x = k_z \), where \( k_z \) is the normalized reproducing kernel in the Bergman space. The Bergman space reproducing formula shows that this family forms a Parseval frame family in the sense of Definition 1.1. It is clear that the multiplier operators in this case correspond to the family of Toeplitz operators. Their properties have been studied in [1, 20, 27]. It is important to note that the Parseval frame family \( \{k_z\} \) is \( p \)-weakly localized with respect to the weight \( p(z) = (1-|z|^2)^a \) for any \( a < 0 \) (but it isn’t \( L^1 \)-localized). Therefore, our results apply in this case.
5.4. Bargmann-Fock space frame families. In this case the group is $G = \mathbb{C}$ equipped with addition. The corresponding Haar measure is the Lebesgue (area measure) on $\mathbb{C}$, and the induced invariant metric $d$ is the Euclidean metric. The Hilbert space $\mathcal{H}$ is the Bargmann-Fock space $\mathcal{F}^2(\mathbb{C})$. For $x \in \mathbb{C}$ define $f_x = k_z$, where $k_z$ is the normalized reproducing kernel in the Bargmann-Fock space. The usual reproducing formula shows that this family forms a Parseval frame family. Note that there is a well-known 1-1 unitary correspondence (Bargmann correspondence) between these frame families and the Gabor frame families. It is important to note that this Parseval frame family $\{k_z\}$ is weakly-located with respect to the weight $p(z) \equiv 1$ (it is even $L^1$-located). The multiplier operators in this case correspond to the family of Toeplitz operators. Their properties have been studied in [6, 20, 30]. It is easy to see that the Parseval frame family $\{k_z\}$ in this space is $L^1$ localized. Therefore, all our results apply in this case.

5.5. de Branges space frame families. In this case the group is $G = \mathbb{R}$ equipped with addition. The corresponding Haar measure is the Lebesgue measure on $\mathbb{R}$, and the induced invariant metric $d$ is the Euclidean metric. The Hilbert space $\mathcal{H}$ is the de Branges space $\mathcal{B}_E$. For $x \in \mathbb{R}$ define $f_x = k_x^\Theta$, where $k_x^\Theta$ is the normalized reproducing kernel in $\mathcal{B}_E$. In the case when the phase function $\phi$ satisfies $0 < c < \phi'(x) < C < \infty$ we have that this family forms a frame family. The most classical space among de Branges spaces is the Paley-Wiener space. Unfortunately, the normalized reproducing kernels in the Paley-Wiener space are not weakly localized no matter which weight $p$ we choose. Compactness of Toeplitz operators on Paley-Wiener spaces was studied in [24, 26].

5.6. Model spaces frame families. In this case the group is again $G = \mathbb{R}$ equipped with addition. The corresponding Haar measure is the Lebesgue measure on $\mathbb{R}$, and the induced invariant metric $d$ is the Euclidean metric. The Hilbert space $\mathcal{H}$ is the model space $\mathcal{K}_\Theta$, where $\Theta$ is a meromorphic inner function. For $x \in \mathbb{R}$ define $f_x = k_x^\Theta$, where $k_x^\Theta$ is the normalized reproducing kernel in $\mathcal{K}_\Theta$. In the case when $0 < c < |\Theta'(x)| < C < \infty$ this family forms a Parseval frame family in the sense of Definition 1.1.

The multiplier operators in this case correspond to the family of truncated Toeplitz operators. Some of their properties, including boundedness and compactness, have been studied in [7] for example (see also [13]).

5.7. Hardy space frame families. The group $G = \mathbb{R}^+ \times \mathbb{R}$ is again the “$ax + b$”-group but now usually viewed as the upper half-plane $\mathbb{C}_+$. The underlying Hilbert space is $\mathcal{H}$ is the classical Hardy space $\mathcal{H}^2(\mathbb{C}_+)$ and the wavelet function is $\psi(x) = \frac{1}{(x+1)^2}$. For $w = b + ia \in \mathbb{C}_+$ define $f_w = D_0 T_b \psi = \frac{(3w)^{3/2}}{(z-w)^2}$. Notice that in this case $\{f_w\}_{w \in \mathbb{C}_+}$ do not correspond to normalized reproducing kernels of the Hardy space. The frame coefficient $\langle f, f_w \rangle$ is equal to $2\pi i (3w)^{3/2} f'(w)$.

REFERENCES

[1] S. Axler and D. Zheng, Compact operators via the Berezin transform, Indiana Univ. Math. J. 47 (1998), no. 2, 387–400. ↑1, 12
[2] R. Balan, P. G. Casazza, C. Heil, and Z. Landau, Density, overcompleteness, and localization of frames. I. Theory, Journal of Fourier Analysis and Applications 12 (2006), no. 3, 307–344. ↑2
[3] ———, Density, overcompleteness, and localization of frames. I. Theory, Journal of Fourier Analysis and Applications 12 (2006), no. 2, 105–143. ↑2
Localization and Berezin transform on the Fock space

[30] J. Xia and D. Zheng, Wavelet transforms and localization operators

[29] M. Wong, The essential norm of operators in the Toeplitz algebra on

[28] R. A. Struble, The essential norm of operators on

[27] D. Suarez, Toeplitz and Hankel operators on the Paley-Wiener space

[26] M. Smith, On Calderón-Toeplitz operators

[25] K. Nowak, Localization of frames, Banach frames, and the invertibility of the frame operator, J. Fourier Anal. Appl. 10 (2004), 105–132. ↑12

[24] K. Gröchenig and T. Strohmer, Pseudodifferential operators on locally compact abelian groups and Sjöstrand’s symbol class, Journal für die reine und angewandte Mathematik 2007 (2007), no. 613, 121–146. ↑12

[23] M. Mitkovski and B. D. Wick, A Reproducing Kernel Thesis for Operators on Bergman-type Function Spaces, J. Funct. Anal. 267 (2014), 2028–2055. ↑1, 5

[22] M. Mitkovski, D. Suárez, and B. D. Wick, The Essential Norm of Operators on $A^p(B_n)$, Integral Equations Operator Theory 75 (2013), no. 2, 197–233. ↑1

[21] J. Isralowitz, Localization and Compactness in Bergman and Fock Spaces (2014), 1–18, to appear in Indiana Univ. Math. J. ↑1, 3, 12, 13

[20] J. Isralowitz, M. Mitkovski, and B. D. Wick, Localization and Compactness in Bergman and Fock Spaces (2013), 1–28 pp., to appear in Journal of Operator Theory, available at http://arxiv.org/abs/1305.7475. ↑1

[19] J. Isralowitz, M. Mitkovski, and B. D. Wick, Localization and Compactness in Bergman and Fock Spaces (2014), 1–18, to appear in Indiana Univ. Math. J. ↑1, 3, 12, 13

[18] M. Mitkovski and B. D. Wick, A Reproducing Kernel Thesis for Operators on Bergman-type Function Spaces, J. Funct. Anal. 267 (2014), 2028–2055. ↑1, 5

[17] M. Mitkovski, D. Suárez, and B. D. Wick, The Essential Norm of Operators on $A^p(B_n)$, Integral Equations Operator Theory 75 (2013), no. 2, 197–233. ↑1

[16] A Grossman, J Morlet, and T Paul, Transforms associated to square integrable group representations I, J. Math. Phys. 26 (1985), 2473–2479. ↑11

[15] A. Grossman, J. Morlet, and T. Paul, Transforms associated to square integrable group representations II, Ann. Inst. Henri Poincare, Phys. Theorique 45 (1986), 293–309. ↑11

[14] K. Gröchenig, Localization of frames, Banach frames, and the invertibility of the frame operator, J. Fourier Anal. Appl. 10 (2004), 105–132. ↑12

[13] F. Futamura, Localizable operators and the construction of localized frames, Proc. Amer. Math. Soc. 137 (2009), no. 12, 4187–4197. ↑2, 3, 5

[12] S. R. Garcia and W. T. Ross, Recent progress on truncated Toeplitz operators, Blaschke products and their applications, 2013, pp. 275–319. ↑13

[11] K Gröchenig and T. Strohmer, Pseudodifferential operators on locally compact abelian groups and Sjöstrand’s symbol class, Journal für die reine und angewandte Mathematik 2007 (2007), no. 613, 121–146. ↑12

[10] E. Cordero and K. Gröchenig, Time–frequency analysis of localization operators

[9] E. Cordero and K. Gröchenig, Necessary conditions for Schatten class localization operators

[8] Cordero E. and K. Gröchenig, Time–frequency analysis of localization operators

[7] A. Baranov, I. Chalendar, E. Fricain, J. Mashreghi, and D. Timotin, Localization of frames, Banach frames, and the invertibility of the frame operator

[6] W. Bauer and J. Isralowitz, Compactness characterization of operators in the Toeplitz algebra of the Fock space $F_p^n$, J. Funct. Anal. 263 (2012), no. 5, 1323–1355. ↑1, 13

[5] P. Balazs, Basic definition and properties of Bessel multipliers

[4] P. Balazs, D. Bayer, and A. Rahimi, Multipliers for continuous frames in Hilbert spaces, Journal of Physics A: Mathematical and Theoretical 45 (2012), no. 24, 244023. ↑13

[3] P. Balazs, Basic definition and properties of Bessel multipliers, Journal of Mathematical Analysis and Applications 325 (2007), no. 1, 571–585. ↑13

[2] W. Bauer and J. Isralowitz, Compactness characterization of operators in the Toeplitz algebra of the Fock space $F_p^n$, J. Funct. Anal. 263 (2012), no. 5, 1323–1355. ↑1, 13

[1] A. Baranov, I. Chalendar, E. Fricain, J. Mashreghi, and D. Timotin, Bounded symbols and Reproducing Kernel Thesis for truncated Toeplitz operators, J. Funct. Anal. 259 (2010), no. 10, 2673-2701. ↑13
Fawwaz Batayneh, Department of Mathematical Sciences, Clemson University, O-110 Martin Hall, Box 340975, Clemson, SC USA 29634
E-mail address: fbatayn@g.clemson.edu

Mishko Mitkovski, Department of Mathematical Sciences, Clemson University, O-110 Martin Hall, Box 340975, Clemson, SC USA 29634
E-mail address: mmitkov@clemson.edu
URL: http://people.clemson.edu/~mmitkov/