Inverted c-functions in thermal states

Matthias Kaminski
University of Alabama

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Entropic c-function in vacuum state

c-theorem: decreasing towards IR
Entropic c-function in thermal state

increasing towards IR

length scale
1. c-function & entanglement entropy
2. Gauge/gravity correspondence
3. $N=4$ Super-Yang-Mills (CFT)
   - c-function in vacuum state
   - c-function in thermal states
   - effect of the chiral anomaly
4. Discussion
1.1 c-Function

Zamolodchikov’s c-theorem in 2D

\[(c)_{UV} \geq (c)_{IR}\]

energy-momentum tensor:

\[\langle T^a_a \rangle = -\frac{c}{12} R\]

trace anomaly

[Zamolodchikov; JETP Lett.(1986)]

- IR/UV fixed points: c-function equals central charge of IR/UV CFT
- c-function measures degrees of freedom
- take a CFT: c-function constant

c-theorem in 4D (the a-theorem)

\[(a_4)_{UV} - (a_4)_{IR}\]

\[\langle T^b_b \rangle = \frac{c_{TT}}{16\pi^2} C^2 - \frac{a_4}{16\pi^2} \mathcal{E} - \frac{1}{4} F^2\]

[Komargodski,Schwimmer; (2011)]
[Cardy; Phys.Lett.B(1988)]
[Osborn; Phys.Lett.B(1988)]
1.2 **Entropic c-function**

\[ c_2 = 3\ell \frac{\delta S_a}{\delta \ell} \]

- **2D**

[Casini, Huerta; Phys. Lett. B (2004)]

\( \ell \): length scale (inverse energy scale)

\[ a_4 = \beta_4 \frac{\ell^3}{H^2} \frac{\partial S_a}{\partial \ell} \]

- **4D**

[Nishioka, Takayanagi; JHEP (2007)]

[Myers, Sinha; JHEP (2011)]

- **H**: IR-regulator
- **\( \beta_4 \)**: known constant

**c-function defined by entanglement entropy**
2. Holography: *theories & states*

[corresponding theories]

Conformal field theory (CFT) - *N=4 Super-Yang-Mills theory*

Gravitational theory - *Einstein-Maxwell-Chern-Simons theory*

**corresponding states**

Renormalization scale ↔ radial AdS coordinate

Vacuum state ↔ empty Anti de Sitter space

- A particular *metric* in gravity corresponds to a particular *state* in the CFT

[Maldacena; Adv.Theor.Math.Phys. (1998)]
2. Holography: *theories & states*

- **Conformal field theory (CFT)**
  - *N=4 Super-Yang-Mills theory*

- **Gravitational theory**
  - *Einstein-Maxwell-Chern-Simons theory*

**Corresponding theories**

**Corresponding states**

- Renormalization scale ➔ radial AdS coordinate
- Vacuum state ➔ empty Anti de Sitter space
- Thermal state ➔ black hole

**Boundary of Anti de Sitter space**

- A particular *metric* in gravity corresponds to a particular *state* in the CFT

*Maldacena; Adv.Theor.Math.Phys. (1998)*
2.1 Holographic entanglement entropy

Definition

\[ S_a = - \text{Tr} \rho_a \log \rho_a, \quad \rho_a = \text{Tr}_b |\psi\rangle \langle \psi| \]
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**Definition**

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**Holographically dual definition**  
[Ryu, Takayanagi; JHEP (2006)]

\[ S_a = \frac{1}{4G_5} A \]

\( G_5 \) is the 5-dimensional gravitational constant of Anti de Sitter spacetime

- Minimal surface area
- Holographic direction \( z \) (additional dimension)
- Strip length \( \ell \)
2.2 Gravity dual to $N=4$ SYM theory with magnetic field

Einstein-Maxwell-Chern-Simons action

$$S_{\text{grav}} = \frac{1}{2\kappa^2} \left[ \int_{\mathcal{M}} d^5x \sqrt{-g} \left( R + \frac{12}{L^2} - \frac{1}{4} F_{mn}F^{mn} \right) - \frac{\gamma}{6} \int_{\mathcal{M}} A \wedge F \wedge F \right]$$

5-dimensional Einstein-Maxwell action encodes $N=4$ Super-Yang-Mills theory with axial $U(1)$ gauge symmetry

5-dimensional Chern-Simons term encodes chiral anomaly

Einstein-Maxwell equations

$$R_{\mu\nu} + 4g_{\mu\nu} \frac{1}{2} \left( F_{\mu\alpha} F_{\nu}{}^{\alpha} - \frac{1}{6} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right)$$

$$\nabla_{\mu} F^{\mu\nu} = -\frac{\gamma}{8\sqrt{-g}} \epsilon^{\nu\alpha\beta\lambda\sigma} F_{\alpha\beta} F_{\lambda\sigma}.$$  

Solution: charged magnetic black brane metric  

[D’Hoker, Kraus; JHEP (2010)]

- magnetic extension of a (charged) Reissner-Nordstrom black brane

$$ds^2 = \frac{1}{z^2} \left( \frac{dz^2}{U(z)} - U(z) dt^2 + v(z)^2 (dx_1^2 + dx_2^2) + w(z)^2 (dx_3^2 + c(z) dt)^2 \right)$$

with numerically known solutions for $U$, $v$, $w$, $c$
2.3 Gravitational calculation

- calculate a geodesic in conformally deformed AdS metric

| Embedding Coordinates | Transverse $\chi^\mu = (z(\sigma), t(\sigma), x_1(\sigma), x_2, x_3)$ | Longitudinal $\chi^\mu = (z(\sigma), t(\sigma), x_1, x_2, x_3(\sigma))$
|---|---|---|
| Surface Coordinates | $\sigma^i = (\sigma, x_2, x_3)$ | $\sigma^i = (\sigma, x_1, x_2)$

Recall:

$$S_a = \frac{1}{4G_5} A$$

Entanglement entropy

$$S_a = \frac{1}{4G_5} V_{||} \int d\sigma \sqrt{\frac{v(z(\sigma))^4 \left(w(z(\sigma))^2 x_3'(\sigma)^2 + \frac{z'(\sigma)^2}{U(z(\sigma))}\right)}{z(\sigma)^6}}$$

with $V_{||} = \int_a^b \int_{-a}^{-b} dx_1 dx_3$ minimal surface area $A$

Reminder: metric is

$$ds^2 = \frac{1}{z^2} \left( \frac{dz^2}{U(z)} - U(z) dt^2 + v(z)^2 \left(dx_1^2 + dx_2^2\right) + w(z)^2 \left(dx_3^2 + c(z) dt^2\right) \right)$$
3.1 Entropic c-function in $N=4$ SYM vacuum state

zero temperature, no magnetic field, vanishing charge

$\Rightarrow$ c-function at all scales equal to central charge of $N=4$ SYM, which is a CFT
3.2 Entropic c-Function *increases* in **thermal** state

- **nonzero temperature**
- no magnetic field,
- vanishing charge

$\Rightarrow$ c-function increases
$\Rightarrow$ effect of the thermal state
3.3 Entropic c-Function \textit{increases} in \textit{thermal} state

now with temperature, magnetic field, charge, chiral anomaly

\begin{itemize}
    \item \textit{c}-function increases
    \item effect of the charged, magnetic, thermal state
    \item IR limit: thermal entropy
    \item proposal: measure of occupation number
\end{itemize}

\textcolor{red}{\textbf{Bold}}
3.4 Effect of the chiral anomaly

now with temperature, magnetic field, charge, chiral anomaly

⇒ maximal effect at 0.1
(thermal entropy has no maximum)
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now with temperature, magnetic field, charge, chiral anomaly

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4. Discussion

**c-function**

- *entropic* c-function defined from entanglement entropy
- entropic c-function in thermal states *increases* towards low energies (IR)
- proportional to *thermal* entropy in IR-limit (analytic result)
- *peaked* at intermediate scale of $B/T^2 \approx 0.1$

**Applications**

- heavy-ion-collisions: *measure for thermalization* via distribution of states
- quantum critical points (QCP): *detect QCP* via change in scaling behavior

[Yang et al.; Nature (2014)]
APPENDIX
Schematic picture: probing energy scales

- $U, v, w, c, E, P$
- Charged magnetic black brane geometry
- UV of QFT
- $AdS_5$
- IR of QFT
- Deformed $AdS_3 \times R^2$
- UV/IR crossover at $z_{\text{cross}}$
  - Depends on $B/T^2$
- Radial AdS coordinate $z_{\text{IR}}$
  - Parametrizing energy-scale
- Minimal surfaces
- $z = 0$
- $z = 1$
- $\ell_{UV}$
- $\ell_{IR}$
Numerical data confirming schematic picture
Thermal entropy density in the Einstein–Maxwell and Einstein–Maxwell–Chern–Simons theory

\[ s/T^3 \]

\[ \Delta s/T^3 \]

\[ \frac{\Delta s}{T^3} \]

\[ B/T^2 \]

\[ \mu/T = 1/5, \gamma = 0 \]

\[ \mu/T = 1/5, \gamma = \frac{2}{\sqrt{3}} \]

\[ \mu/T = 5, \gamma = 0 \]

\[ \mu/T = 5, \gamma = \frac{2}{\sqrt{3}} \]
Figure 13: The deviation $\delta c_4$ of the thermal c-function, evaluated in the thermal state dual to the AdS Schwarzschild black brane, from the central charge $a_4$, displayed as the blue points. The thermal subtraction $\beta_4 \ell^3 s$ is displayed as a black line. Its behavior precisely matches $\delta c_4$ in the IR. The bottom right inset graphic displays the same information only not in a log-log scale, the top left inset displays the difference as green points on a log-log scale.
Effect of chiral anomaly: entanglement entropy

Difference in entanglement entropy with and without Chern–Simons coupling

$S_{\text{EMGS}} - S_{\text{EM}}$

$B/T^2$

$S_{\text{EMGS}}$

$S_{\text{EM}}$