Synchronization Problems in Automata without Non-trivial Cycles

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Abstract

In this paper, we study the computational complexity of various problems related to synchronization of weakly acyclic automata, a subclass of widely studied aperiodic automata. We provide upper and lower bounds on the length of a shortest word synchronizing a weakly acyclic automaton or, more generally, a subset of its states, and show that the problem of approximating this length is hard. We also show inapproximability of the problem of computing the rank of a subset of states in a binary weakly acyclic automaton and prove that several problems related to recognizing a synchronizing subset of states in such automata are NP-complete.

1 Introduction

The concept of synchronization is widely studied in automata theory and has a lot of different applications in such areas as manufacturing, coding theory, biocomputing, semigroup theory and many others [Vol08]. Let $A = (Q, \Sigma, \delta)$ be a deterministic finite automaton (which we simply call an automaton in this paper), where $Q$ is a set of states, $\Sigma$ is a finite alphabet and $\delta : Q \times \Sigma \to Q$ is a transition function. Note that our definition of an automaton does not include initial and accepting states. An automaton is called synchronizing if there exists a word that maps all its states to a fixed state $q \in Q$. Such word is called a synchronizing word. A state $q \in Q$ is called a sink state if all letters from $\Sigma$ map $q$ to itself.

In this paper synchronization of weakly acyclic automata is studied. A simple cycle in an automaton $A = (Q, \Sigma, \delta)$ is a sequence $q_1, \ldots, q_k$ of its states such that all the states in the sequence are different and there exist letters $x_1, \ldots, x_k \in \Sigma$ such that $\delta(q_i, x_i) = q_{i+1}$ for $1 \leq i \leq k - 1$ and $\delta(q_k, x_k) = q_1$. A simple cycle is a self-loop if it consists of only one state. An automaton is called weakly acyclic if all its simple cycles are self-loops. In other words, an automaton is weakly acyclic if and only if there exists an ordering $q_1, q_2, \ldots, q_n$ of its states such that if $\delta(q_i, x) = q_j$ for some letter $x \in \Sigma$, then $i \leq j$ (such ordering is called a topological sort [CLRS09]). Using topological sort, this class can be recognized in polynomial time. Weakly acyclic automata are called acyclic in [JM12] and partially
ordered in [BF80], where in particular the class of languages recognized by such automata is characterized.

Weakly acyclic automata arise naturally in synchronizing automata theory. Section 3 of this paper shows several examples of existing proofs where weakly acyclic automata appear implicitly in complexity reductions. Surprisingly, most of the computational problems that are hard for general automata remain very hard in this class despite of its very simple structure. Thus, investigation of weakly acyclic automata provides good lower bound on the complexity of many problems for general automata. An automaton is called aperiodic if for any word \( w \in \Sigma^* \) and any state \( q \in Q \) there exists \( k \) such that \( \delta(q, w^k) = \delta(q, w^{k+1}) \), where \( w^k \) is a word obtained by \( k \) concatenations of \( w \) [Tra07]. Obviously, weakly acyclic automata form a proper subclass of aperiodic automata, thus all hardness results hold for the class of aperiodic automata.

One of the most important questions in synchronizing automata theory is a famous Černý conjecture stating that any \( n \)-state synchronizing automaton has a synchronizing word of length at most \( (n-1)^2 \). The conjecture is proved for various special cases, including orientable, Eulerian, aperiodic and other automata (see [Vol08] for references), but is still open in general. The best upper bound achieved so far is \( \frac{n^3 - n}{6} \) [Pin83].

The concept of synchronization is often used as an abstraction of returning control over an automaton when there is no a priori information about its current state, but the structure of the automaton is known. If the automaton is synchronizing, we can apply a synchronizing word to it, and thus it will transit to a known state. If we want to perform the same operation when the current state is known to belong to some subset of states of the automaton, we come to the definition of a synchronizing set. A set \( S \subseteq Q \) of states of an automaton \( A \) is called synchronizing if there exists a word \( w \in \Sigma^* \) and a state \( q \in Q \) such that the word \( w \) maps each state \( s \in S \) to the state \( q \). The word \( w \) is said to synchronize the set \( S \). It follows from the definition that an automaton is synchronizing if and only if the set \( Q \) of all its states is synchronizing. Consider the problem Sync Set of deciding whether a given set \( S \) of states of an automaton \( A \) is synchronizing.

**Sync Set**

- **Input:** An automaton \( A \) and a subset \( S \) of its states;
- **Output:** Yes if \( S \) is a synchronizing set, No otherwise.

The Sync Set problem is PSPACE-complete [San05], even for binary strongly connected automata [Vor16] (an automaton is called binary if its alphabet has size two, and strongly connected if any state can be mapped to any other state by some word). In [Nat86] it is shown that the Sync Set problem is solvable in polynomial time for orientable automata, i.e. automata respecting some cyclic order. The problem of deciding whether the whole set of states of an automaton is synchronizing is also solvable in polynomial time [Vol08]. In [Ryz16] the complexity of finding a synchronizing set of maximum size in an automaton is investigated.

While there is a simple cubic bound on the length of a synchronizing word for the whole automaton, there exist examples of automata where the length of a shortest word synchronizing a subset of states is exponential in the number of states [Vor16]. On the other hand, a trivial upper bound \( 2^n - n - 1 \) on this length is known for \( n \)-state automata [Vor16]. In [Car14] Cardoso considers the length of a shortest word synchronizing a subset
of states in a synchronizing automaton.

We assume that the reader is familiar with the notions of an NP-complete problem (refer to the book by Sipser [Sip12]), an approximation algorithm and a gap-preserving reduction (for reference, see the book by Vazirani [Vaz01]).

Given an automaton $A$, the rank of a word $w$ with respect to $A$ is the number $|\{\delta(s, w) \mid s \in Q\}|$, i.e. the size of the image of $Q$ under the mapping defined in $A$ by $w$. More generally, the rank of a word $w$ with respect to a subset $S$ of states of $A$ is the number $|\{\delta(s, w) \mid s \in S\}|$. The rank of an automaton (resp. of a subset of states) is the minimum among the ranks of all words $w \in \Sigma$ with respect to the automaton (resp. to the subset of states).

In this paper we provide various results concerning computational complexity and approximability of the problems related to the subset synchronization in weakly acyclic automata. In Section 2 we prove some lower and upper bounds on the length of a shortest word synchronizing a weakly acyclic automaton or, more generally, a subset of its states. In Section 3 we investigate the computational complexity of finding such words. In Section 4 we give strong inapproximability results for computing the rank of a subset of states in binary weakly acyclic automata. In Section 5 we show that several other problems related to recognizing a synchronizing set in a weakly acyclic automaton are hard.

## 2 Bounds on the Length of Shortest Synchronizing Words

Each synchronizing weakly acyclic automaton has exactly one sink state and thus is a 0-automaton, which gives an upper bound $\frac{n(n-1)}{2}$ on the length of a shortest synchronizing word [Rys97]. The same bound can be deduced from the fact that each weakly acyclic automaton is aperiodic [Tra07]. However, for weakly acyclic automata a more accurate result can be obtained, showing that weakly acyclic automata of rank $r$ behave in a way similar to monotonic automata of rank $r$ (see [AV04]).

**Theorem 1.** Let $A = (Q, \Sigma, \delta)$ be a weakly acyclic automaton, and $w$ be a word of rank $r$ with respect to $A$. Then there exists a word of length at most $n - r$ and rank at most $r$ with respect to $A$.

**Proof.** Observe that the rank of a weakly acyclic automaton equals to the number of sink states in it. The conditions of the theorem implies that $A$ has at most $r$ sink states.

Consider a topological sort $q_1, \ldots, q_n$ of the set $Q$. Consider sets $S_1, \ldots, S_t$ constructed in the following way. Let $x_i, 1 \leq i \leq t$, be a letter mapping the state in $S_{i-1}$ with the largest index in the topological sort which is not a sink state to some other state, where $S_i = \{\delta(q, x_i) \mid q \in S_{i-1}\}, 1 \leq i \leq t$, and $S_0 = Q$. Since $A$ has at most $r$ sink states, word $w = x_1 \ldots x_t$ exists for any $t \leq n - r$ and has rank at most $r$ with respect to $A$. \hfill $\square$

The following simple example shows that the bound is tight. Consider an automaton $A = (Q, \Sigma, \delta)$ with states $q_1, \ldots, q_n$. Let each letter except some letter $x$ map each state to itself. For the letter $x$ define the transition function $\delta(q_i, x) = q_{i+1}$ for $1 \leq i \leq n - r$ and $\delta(q_n, x) = q_1$ for $n - r + 1 \leq i \leq n$. Obviously, $A$ has rank $r$ and a shortest word of rank $r$ with respect to $A$ has length $n - r$. 


Theorem 2. Let $S$ be a synchronizing set of states of size $k$ in a weakly acyclic $n$-state automaton $A = (Q, \Sigma, \delta)$. Then the length of a shortest word synchronizing $S$ is at most \( \frac{k(2n-k-1)}{2} \).

Proof. Consider a topological sort $q_1, \ldots, q_n$ of the set $Q$. Let $q_s$ be a state such that all states in $S$ can be mapped to it by some word $w = x_1 \ldots x_t$. We can assume that the images of all words $x_1 \ldots x_k$, $k \leq t$, are pairwise distinct, otherwise some letter in this word can be removed. Then a word $x_1 \ldots x_k$ maps at least one state of the set $\{\delta(q, x_1 \ldots x_{k-1}) \mid q \in S\}$ to some other state. Thus the maximum total number of letters in $w$ sending all states in $S$ to $q_s$ is at most \( (n-k)+(n-k+1)+\ldots+(n-1) = \frac{k(2n-k-1)}{2} \).

The following example shows that the bound is almost tight. Consider a binary automaton $A = (Q, \{0, 1\}, \delta)$ with $n$ states $q_1, \ldots, q_{k-1}, s_1, \ldots, s_\ell, t$, where $\ell = n-k$. Define $\delta(q_i, 0) = q_i$, $\delta(q_i, 1) = q_{i+1}$ for $1 \leq i \leq k-2$, $\delta(q_{k-1}, 1) = s_1$. Define also $\delta(s_i, 0) = s_{i+1}$ for $1 \leq i \leq \ell-1$, $\delta(s_i, 1) = t$ for $1 \leq i \leq \ell - 1$. Define both transitions for $s_\ell$ and $t$ as self-loops. Set $S = \{q_1, \ldots, q_{k-1}, s_\ell\}$. The shortest word synchronizing $S$ is $(10^{\ell-1})^{k-1}$ of length $(k-1)(n-k)$. The automaton in this example is binary weakly acyclic, and even has rank 2. Figure 1 gives the idea of the described construction.

3 Complexity of Finding Shortest Synchronizing Words

Now we proceed to the computational complexity of some problems, related to finding a shortest synchronizing word for an automaton. Consider first the following problem.

**Shortest Sync Word**

*Input:* A synchronizing automaton $A$;

*Output:* A shortest synchronizing word for $A$.

First, we note that the automaton in the construction of Berlinkov [Ber14] is weakly acyclic. Thus, the following theorem holds.

**Theorem 3.** For any $\gamma > 0$, the Shortest Sync Word problem for $n$-state weakly acyclic automata with alphabet of size at most $n^{1+\gamma}$ cannot be approximated in polynomial time within a factor of $d \log n$ for any $d < c_{sc}$ unless $P = NP$, where $c_{sc}$ is some constant.

In the Berlinkov’s reduction to the binary case, the automaton is no longer weakly acyclic. However, the binary automaton in Eppstein’s construction [Epp90] is weakly acyclic. Thus, the following theorem holds.
**Theorem 4.** Shortest Sync Word is NP-hard for binary weakly acyclic automata.

Consider now a more general problem of finding a shortest word synchronizing a subset of states.

**Shortest Set Sync Word**

**Input:** An automaton $A$ and a synchronizing subset $S$ of its states;

**Output:** A shortest word synchronizing $S$.

It follows from Theorem 2 that this problem is in NP for weakly acyclic automata, and thus it is reasonable to investigate its approximability.

**Theorem 5.** The Shortest Set Sync Word problem for $n$-state binary weakly acyclic automata cannot be approximated in polynomial time within a factor of $O(n^{1-\epsilon})$ for any $\epsilon > 0$ unless $P = NP$.

**Proof.** To prove this theorem, we construct a gap-preserving reduction from the Shortest Sync Word problem in $p$-state binary automata, which cannot be approximated in polynomial time within a factor of $O(p^{1-\epsilon})$ for any $\epsilon > 0$ unless $P = NP$ [GS15]. Let a binary automaton $A = (Q, \{0, 1\}, \delta)$ be the input of Shortest Sync Word. Let $Q = \{q_1, \ldots, q_p\}$. Construct a binary automaton $A' = (Q', \{0, 1\}, \delta')$ with the set of states $Q' = \{q_{i,j} | 1 \leq i \leq p, 1 \leq j \leq p^3+1\}$. Define $\delta'(q_{i,j}, x) = q_{i,(j+1)}$ for $1 \leq i \leq p, 1 \leq j \leq p^3$, $x \in \{0, 1\}$, where $k$ is such that $q_k = \delta(q_i, x)$. Define $\delta'(q_{i,(p^3+1)}, x) = q_{i,((p^3)+1)}$ for $1 \leq i \leq p$ and $x \in \{0, 1\}$. Take $S' = \{q_{i,1} | 1 \leq i \leq p\}$.

Observe that any word synchronizing $S'$ in $A'$ is a synchronizing word for $A$ because of the definition of $\delta'$. In the other direction, as a shortest synchronizing word for a $p$-state automaton has length at most $p^3$ [Pin83], a shortest synchronizing word for $A$ also synchronizes $S'$ in $A'$. Thus, the length of a shortest synchronizing word for $A$ equals to the length of a shortest word synchronizing $S'$ in $A'$, and we get a gap-preserving reduction with gap $O(p^{1-\epsilon}) = O(n^{1-\epsilon})$, as $A'$ has $O(p^4)$ states. It is easy to see that $A'$ is binary weakly acyclic, which concludes the proof. \qed

We note that Cerny’s conjecture implies that the inapproximability factor in the presented theorem is $O(n^{1/4-\epsilon})$.

4 Computing the Rank of a Subset of States

Assume that we know that the current state of the automaton $A$ belongs to a subset $S$ of its states. Even if it is not possible to synchronize $S$, it can be reasonable to minimize the size of the set of possible states of $A$, reducing the uncertainty of the current state as much as possible. One way to do it is to map $S$ to a set $S'$ of smaller size by applying some word to $A$. Recall that the size of the smallest such set $S'$ is called the rank of $S$. Consider the following problem of finding the rank of a subset of states in a given automaton.

**Set Rank**

**Input:** An automaton $A$ and a set $S$ of its states;

**Output:** The rank of $S$ in $A$. 

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The rank of an automaton, that is, the rank of the set of its states, can be computed in polynomial time [Rys92]. However, since the automaton in the proof of 
PSPACE-completeness of \textsc{Sync Set} in [Rys83] has rank 2 (and thus each subset of states in this automaton has rank either 1 or 2), it follows immediately that there is no polynomial c-approximation algorithm for the \textsc{Set Rank} problem for any \( c < 2 \) unless \( P = \text{PSPACE} \). It is possible to get much stronger bounds, as it is shown by the results of this section.

We shall need the \textbf{Chromatic Number} problem. A \textit{proper colouring} of a graph \( G = (V, E) \) is a colouring of the set \( V \) in such a way that no two adjacent vertices have the same colour. The chromatic number of \( G \), denoted \( \chi(G) \), is the minimum number of colours in a proper colouring of \( G \). A set of vertices in a graph is called \textit{independent} if no two vertices in this set are adjacent. A proper colouring of a graph can be also considered as a partition of the set of its vertices into independent sets.

\begin{center}
\textbf{Chromatic Number}
\begin{itemize}
  \item \textbf{Input}: A graph \( G \);
  \item \textbf{Output}: The chromatic number of \( G \).
\end{itemize}
\end{center}

This problem cannot be approximated within a factor of \( O(p^{1-\epsilon}) \) for any \( \epsilon > 0 \) unless \( P = \text{NP} \), where \( p \) is the number of vertices in the graph [Zuc07].

\textbf{Theorem 6.} The \textsc{Set Rank} problem for \( n \)-state weakly acyclic automata with alphabet of size \( O(\sqrt{n}) \) cannot be approximated within a factor of \( O(n^{\frac{1}{2}-\epsilon}) \) for any \( \epsilon > 0 \) unless \( P = \text{NP} \).

\textit{Proof.} We shall prove this theorem by constructing a gap-preserving reduction from the \textbf{Chromatic Number} problem. Given a graph \( G = (V, E) \), \( V = \{v_1, v_2, \ldots, v_p\} \), we construct an automaton \( A = (Q, \Sigma, \delta) \) as follows. The alphabet \( \Sigma \) consists of letters \( \tilde{v}_1, \ldots, \tilde{v}_p \) corresponding to the vertices of \( G \), together with a \textit{switching} letter \( \nu \). We use \( p \) identical \textit{synchronizing} gadgets \( T^{(k)}, 1 \leq k \leq p \), such that each gadget synchronizes a subset of states corresponding to an independent set in \( G \). Gadget \( T^{(k)} \) consists of a set \( \{s_i^{(k)}, t_i^{(k)} \mid 1 \leq i \leq p\} \cup \{f^{(k)}\} \) of states.

The transition function \( \delta \) is defined as following. For each gadget \( T^{(k)} \), for each \( 1 \leq i \leq p \), the state \( s_i^{(k)} \) is mapped to \( f^{(k)} \) by the letter \( \tilde{v}_i \). For each \( v_i v_j \in E \) the state \( s_i^{(k)} \) is mapped to \( t_i^{(k)} \) by the letter \( \tilde{v}_j \), and the state \( s_j^{(k)} \) is mapped to \( t_j^{(k)} \) by the letter \( \tilde{v}_i \). All yet undefined transitions corresponding to letters \( \tilde{v}_1, \ldots, \tilde{v}_p \) map a state to itself.

It remains to define the transitions corresponding to \( \nu \). For each \( 1 \leq k \leq p-1 \), \( \nu \) maps \( t_i^{(k)} \) and \( s_i^{(k)} \) to \( s_i^{(k+1)} \), and \( f^{(k)} \) to itself. Finally, \( \nu \) acts on all states in \( T^{(p)} \) as a self-loop.

Define \( S = \{s_i^{(1)} \mid 1 \leq i \leq p\} \). We shall prove that the rank of \( S \) is equal to the chromatic number of \( G \). Consider a proper colouring of \( G \) with the minimum number of colours and let \( I_1 \cup \ldots \cup I_{\chi(G)} \) be a partition of \( G \) into independent sets defined by this colouring. For each \( I_j \), consider a word \( w_j \) obtained by concatenating the letters corresponding to the vertices in \( I_j \) in some order. Consider now the word \( w_1 \nu w_2 \nu \ldots \nu w_{\chi(G)} \).

This word maps the set \( S \) to the set \( \{f^{(i)} \mid 1 \leq i \leq \chi(G)\} \), which proves that the rank of \( S \) is at most \( \chi(G) \).

In the other direction, note that after each reading of \( \nu \) all states except \( f^{(k)}, 1 \leq k \leq p-1 \), are mapped to the next synchronizing gadget (except the last gadget \( T^{(p)} \) which is mapped to itself). By definition of \( \delta \), only a subset of states corresponding to an
independent set of vertices can be mapped to some particular \( f^{(k)} \), and the image of \( S \) after reading any word is a subset of the states in some gadget together with some of the states \( f^{(k)}, 1 \leq k \leq p \). Hence, the rank of \( S \) is at least \( \chi(G) \).

Thus we have a gap-preserving reduction from the CHROMATIC NUMBER problem to the SET RANK problem with gap \( \Theta(p^{1-\varepsilon}) \) for any \( \varepsilon > 0 \). It is easy to see that \( n = \Theta(p^{2}) \), \( A \) is weakly acyclic and its alphabet has size \( O(\sqrt{m}) \), which finishes the proof of the theorem.

Using the classical technique of reducing the alphabet size (see [Vor16]), \( O(n^{\frac{1}{2}-\varepsilon}) \) inapproximability can be proved for binary automata. To prove the same bound for binary weakly acyclic automata, we have to refine the technique of the proof of the previous theorem.

**Theorem 7.** The Set Rank problem for \( n \)-state binary weakly acyclic automata cannot be approximated within a factor of \( O(n^{\frac{1}{2}-\varepsilon}) \) for any \( \varepsilon > 0 \) unless \( P = NP \).

**Proof.** To prove this theorem we construct a gap-preserving reduction from the Chromatic Number problem, extending the proof of the previous theorem.

Given a graph \( G = (V, E), V = \{v_1, v_2, \ldots, v_p\} \), we construct an automaton \( A = (Q, \{0,1\}, \delta) \). In our reduction we use two kinds of gadgets: \( p \) synchronizing gadgets \( T^{(k)} \), \( 1 \leq k \leq p \), and \( p \) waiting gadgets \( R^{(k)} \). Gadget \( T^{(k)} \) consists of a set \( \{v_{i,j}^{(k)} | 1 \leq i, j \leq p\} \) of states, together with a state \( f^{(k)} \), and \( R^{(k)}, 1 \leq k \leq p \), consists of the set \( \{u_{i,j}^{(k)} | 1 \leq i, j \leq p\} \).

For each \( i, j, k, 1 \leq i, j, k \leq p \), the transition function \( \delta \) is defined as:

\[
\delta(v_{i,j}^{(k)}, 0) = \begin{cases} 
    u_{i,j}^{(k+1)} & \text{if } i = j, \\
    v_{i+1,j}^{(k)} & \text{otherwise}
\end{cases}
\]

\[
\delta(v_{i,j}^{(k)}, 1) = \begin{cases} 
    u_{i,j}^{(k+1)} & \text{if there is an edge } v_i v_j \in E, \\
    v_{i+1,j}^{(k)} & \text{otherwise}
\end{cases}
\]

Here all \( v_{n+1,j}^{(k)}, 1 \leq j \leq p \), coincide with \( f^{(k)} \). We set \( \delta(u_{i,j}^{(k)}, x) = u_{i+1,j}^{(k)} \) for \( x \in \{0,1\} \). The states \( u_{i,j}^{(p+1)} \) are sink states: both letters 0 and 1 act on them as self-loops. Finally, we set \( S = \{v_{1,j}^{(1)} | 1 \leq j \leq p\} \). Figure 2 gives the idea of the construction for a graph with three vertices \( v_1, v_2, v_3 \) and one edge \( v_2 v_3 \).

The idea of the presented construction is similar to the construction in the proof of Theorem 6. A synchronizing gadget \( T^{(k)} \) synchronizes a set \( S^{(k)} \subseteq S \) of states corresponding to some independent set in \( G \). All the states corresponding to the vertices adjacent to vertices corresponding to \( S^{(k)} \) are mapped to the next waiting gadget \( R^{(k+1)} \), and get to the next synchronizing gadget \( T^{(k+1)} \) only after the states of \( S^{(k)} \) are synchronized (and thus mapped to \( f^{(k)} \)). Hence, the minimum size of a partition of \( V \) into independent sets equals to the rank of \( S \).

The number of states in \( A \) is \( O(p^3) \). Thus, we get \( O(n^{\frac{1}{2}-\varepsilon}) \) inapproximability. \( \square \)
Mycielski [Myc55] provided an example of a series of graphs which do not have three pairwise adjacent vertices, but have arbitrary large chromatic number. The reduction in Theorem 7 together with this example can be used to prove the following result showing that there is almost no connection between subset rank and pairwise synchronization of elements in this subset in binary weakly acyclic automata.

**Theorem 8.** There exists a pair of a binary weakly acyclic automaton $A$ and a subset $S$ of its states such that for any three states in $S$ at least two of them form a synchronizing subset, but the rank of $S$ is arbitrary large.

It might be interesting to investigate how large can be the rank of a set of states in an automaton such that any two states in this set form a synchronizing set (both in binary weakly acyclic and general automata).

## 5 Subset Synchronization

In this section, we obtain complexity results for several problems related to subset synchronization in weakly acyclic automata. Eppstein’s construction [Epp90] is a powerful
and flexible tool for such proofs. We shall need the following NP-complete SAT problem \[Sip12\].

**SAT**

*Input:* A set \(X\) of \(n\) boolean variables and a set \(C\) of \(m\) clauses;

*Output:* Yes if there exists an assignment of values to the variables in \(X\) such that all clauses in \(C\) are satisfied, No otherwise.

**Theorem 9.** The Sync Set problem in binary weakly acyclic automata is NP-complete.

**Proof.** Because of the polynomial upper bound on the length of a shortest word synchronizing a subset of states proved in Theorem 2, we can use such word as a certificate. Thus, the problem is in NP.

We reduce the SAT problem. Given \(X\) and \(C\), we construct an automaton \(A = (Q, \{0, 1\}, \delta)\). For each clause \(c_j\), we construct \(n + 1\) states \(y_i^{(j)}, 1 \leq i \leq n + 1\), in \(Q\). We introduce also a state \(f \in Q\). The transitions from \(y_i^{(j)}\) correspond to the occurrence of \(x_i\) into \(c_j\) in the following way: for \(1 \leq i \leq n\), \(\delta(y_i^{(j)}, a) = f\) if the assignment \(x_i = a, a \in \{0, 1\}\), satisfies \(c_j\), and \(\delta(y_i^{(j)}, a) = y_i^{(j)}\) otherwise. The transition function \(\delta\) also maps \(y_n^{(j)}\) to itself for all \(1 \leq j \leq m\) and both letters 0 and 1.

Let \(S = \{y_1^{(j)} \mid 1 \leq j \leq m\}\). The word \(w = a_1 a_2 \ldots a_n\) synchronizes \(S\) if \(a_i\) is the value of \(x_i\) in an assignment satisfying \(C\), and vice versa. Thus, the set is synchronizing if and only if all clauses in \(C\) can be satisfied by some assignment of binary values to the variables in \(X\).

The proof of Theorem 9 can be used to prove the hardness of a special case of the following problem, which is PSPACE-complete in general \[Koz77\].

**Finite Automata Intersection**

*Input:* Automata \(A_1, \ldots, A_k\) (with input and accepting states);

*Output:* Yes if there is a word which is accepted by all automata, No otherwise.

**Theorem 10.** Finite Automata Intersection is NP-complete when all automata in the input are binary weakly acyclic.

**Proof.** Observe first that if a word which is accepted by all automata exists then a shortest such word \(w\) has length at most linear in the total number of states in all automata. Indeed, for each automaton consider a topological sort of the set of its states. Each letter of \(w\) maps at least one state in some automaton to some other state, which has larger index in the topological sort of the set of states of this automaton. Thus, the considered problem is in NP.

For the hardness proof, we use the same construction as in Theorem 9. Having \(X\) and \(C\), define \(A\) in the same way as in Theorem 9. Define \(A_j = (Q_j, \{0, 1\}, \delta_j)\) as following. Take \(Q_j = \{y_i^{(j)}, 1 \leq i \leq n + 1\} \cup \{f\}\) and \(\delta_j\) to be the restriction of \(\delta\) to the set \(Q_j\). Set \(y_1^{(j)}\) to be the input state and \(f\) to be the only accepting state of \(A_j\). Then there exists a word accepted by all automata \(A_1, \ldots, A_m\) if and only if all the clauses in \(C\) are satisfiable by some assignment.

\[\Box\]
To obtain the next results, we shall need a modified construction of the automaton from the proof of Theorem 9, as well as some new definitions. A partial automaton is a triple \((Q, \Sigma, \delta)\), where \(Q\) and \(\Sigma\) are the same as in the definition of a finite deterministic automaton, and \(\delta\) is a partial transition function (i.e. the transition function which may be undefined for some argument values). Given an instance of the SAT problem, construct a partial automaton \(A_{\text{base}} = (Q, \{0, 1\}, \delta)\) as following. We introduce a state \(f \in Q\). For each clause \(c_j\), we construct \(n + 1\) states \(y_{i}^{(j)}, 1 \leq i \leq n + 1\), in \(Q\). For each \(c_j\), construct states \(z_i^{(j)}\) for \(h_i + 1 \leq i \leq n + 1\), where \(h_i\) is the smallest index of a variable occurring in \(c_j\). The transitions from \(y_{i}^{(j)}\) correspond to the occurrence of \(x_i\) in \(c_j\) in the following way: for \(1 \leq i \leq n\), \(\delta(y_{i}^{(j)}, a) = z_{i+1}^{(j)}\) if the assignment \(x_i = a, a \in \{0, 1\}\), satisfies \(c_j\), and \(\delta(y_{i}^{(j)}, a) = y_{i+1}^{(j)}\) otherwise. The transition function \(\delta\) also maps \(z_{n+1}^{(j)}\) to \(f\) for both letters 0 and 1.

A word \(w\) is said to carefully synchronize a partial automaton \(A\) if it maps all its states to the same state \(q\), and each mapping corresponding to a prefix of \(w\) is defined for each state. The automaton \(A\) is then called carefully synchronizing. We use \(A_{\text{base}}\) to prove the hardness of the following problem.

**Careful Synchronization**

*Input:* A partial automaton \(A\);

*Output:* Yes if \(A\) is carefully synchronizing, No otherwise.

For binary automata, Careful Synchronization is PSPACE-complete [Mar10]. We call a partial automaton aperiodic if for any word \(w \in \Sigma^*\) and any state \(q \in Q\) there exists \(k\) such that either \(\delta(q, w^k)\) is undefined, or \(\delta(q, w^k) = \delta(q, w^{k+1})\).

**Theorem 11.** Careful Synchronization is \(NP\)-hard for aperiodic automata over a three-letter alphabet.

*Proof.* We reduce the SAT problem. Given \(X\) and \(C\), we first construct \(A_{\text{base}}\). Then we add an additional letter \(r\) to the alphabet of \(A_{\text{base}}\) and introduce \(m\) new states \(s^{(m)}\). We define \(\delta(s^{(j)}, r) = y_{1}^{(j)}, \delta(y_{i}^{(j)}, r) = s^{(j)}, \delta(z_{i}^{(j)}, r) = s^{(j)}, \delta(f, r) = f, 1 \leq i \leq n\). All other transitions are left undefined. Let us call the constructed automaton \(A\).

The automaton \(A\) is carefully synchronizing if and only if all clauses in \(C\) can be satisfied by some assignment of binary values to the variables in \(X\). Moreover, the word \(w = rw_1w_2 \ldots w_n0\), is carefully synchronizing if \(w_i\) is the value of \(x_i\) in such an assignment.

Indeed, note that the first letter of \(w\) is necessarily \(r\), as it is the only letter defined for all the states. Moreover, each word starting with \(r\) maps \(Q\) to a subset of \(\{y_{i}^{(j)}, z_{i}^{(j)} \mid 1 \leq j \leq m + 1\}\) or to a single state \(f\). The only way for a word to map all states to \(f\) is to map them first to the set \(\{z_{n+1}^{(j)} \mid 1 \leq j \leq m\}\), because there are no transitions defined from any \(y_{n+1}^{(j)}\), except of the transitions defined by \(r\). But this exactly means that there exists an assignment satisfying \(C\).

The constructed automaton is aperiodic, because each cycle which is not a self-loop contains exactly one letter \(r\).

The following problem is closely related to careful synchronization.
Positive Matrix

Input: A set $M_1, \ldots, M_k$ of $n \times n$ binary matrices;
Output: Yes if there exists a sequence $M_{i_1} \times \ldots \times M_{i_k}$ of multiplications (possibly with repetitions) providing a matrix with all elements equal to 1, No otherwise.

By using the trick of adding one new letter from [GGJ16], the following result can be obtained from Theorem 11.

Theorem 12. Positive Matrix is NP-hard for two upper-triangular and two lower-triangular matrices.

Finally, we show the hardness of the following problem (PSPACE-complete in general [BV16]).

Subset Reachability

Input: An automaton $A = (Q, \Sigma, \delta)$ and a subset $S$ of its states;
Output: Yes if there exists a word $w$ such that $\{\delta(q, w) \mid q \in Q\} = S$, No otherwise.

Theorem 13. Subset Reachability is NP-complete for weakly acyclic automata.

Proof. Consider a topological sort of $Q$. Let $w$ be a shortest word mapping $Q$ to some reachable set of states. Then each letter of $w$ maps at least one state to a state with a larger index in the topological sort. Thus $w$ has length $O(|Q|^2)$, since the maximum total number of such mappings is $(n-1) + (n-2) + \ldots + 1 + 0$. Thus, the considered problem is in NP.

For the NP-hardness proof, we again reduce the SAT problem. Given an instance of SAT, construct $A_{\text{base}}$ first. Next, add a transition $\delta(y_{m+1}^{(j)}, a) = f$ for $a \in \{0, 1\}$, resulting in a deterministic automaton $A$.

Similar to the proof of Theorem 11, $C$ is satisfiable if and only if the set $\{z_j^{(n+1)} \mid 1 \leq j \leq m\} \cup \{f\}$ is reachable in $A$. \qed

6 Open Problems

The main open problem left by this paper is to improve the inapproximability bounds for the considered problems. It is also interesting to investigate the length of a shortest word synchronizing a subset of states in the class of aperiodic automata and its subclasses, such as monotonic automata. Classes of automata where the rank of a subset can be computed in polynomial time are also of a certain interest.

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