Time-Symmetrized Quantum Theory, Counterfactuals, and ‘Advanced Action’

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Abstract

Recent authors have raised objections to the counterfactual interpretation of the Aharonov-Bergmann-Lebowitz (ABL) rule of time symmetrized quantum theory (TSQT). I distinguish between two different readings of the ABL rule, counterfactual and non-counterfactual, and confirm that TSQT advocate L. Vaidman is employing the counterfactual reading to which these authors object. Vaidman has responded to the objections by proposing a new kind of time-symmetrized counterfactual, which he has defined in two different ways. It is argued that neither definition succeeds in overcoming the objections, except in a limited special case previously noted by Cohen and Hiley. In addition, a connection is made between TSQT and Price’s concept of ‘advanced action’, which further supports the special case discussed.

1. Introduction.

Time-Symmetrized Quantum Theory (hereinafter TSQT) was first introduced by Aharonov, Bergmann and Lebowitz (1964), and most recently defended by Vaidman (1996a, 1997a,b). The basic claim of TSQT (actually a novel interpretation of quantum theory) is that a fundamental time symmetry applies to the interval between two ideal measurements. According to TSQT, the maximally specified quantum state of a system between two measurements occurring at times $t_1$ and $t_2$ contains information based not only on the initial, or pre-selection measurement (yielding the state $|\psi_1(t_1)\rangle$) but also on the final, or post-selection measurement (yielding the state $|\psi_2(t_2)\rangle$). Ensembles of systems identified in this way are referred to as “pre- and post-selected ensembles.” Aharonov and Vaidman (1991) proposed that such ensembles be represented by a two-state vector, or “generalized state” $\Psi(t) = \langle\psi_2(t)|\psi_1(t)\rangle$, where $|\psi_1(t)\rangle = U(t,t_1)|\psi_1(t_1)\rangle$, and $\langle\psi_2(t)| = \langle\psi_2(t)|U(t_2,t)$. The propagator $U(t_2,t)$ represents a time-reversed evolution.

The novel philosophical claim of TSQT, as compared to most of the traditional interpretations of quantum mechanics— including interpretations as diverse as the Copenhagen interpretation, many worlds theories, and hidden variables theories such as Bohm’s theory—is that the ontological state of a quantum system has a causal dependence not only on past measurements, but on measurements to be made in its immediate future. In this respect, TSQT advocates have much in common with the philosophy of time of Price...
(1996), who argues that a fundamental time symmetry holds at the quantum level and that the world exhibits what he calls “advanced action,” essentially backwards causality. Indeed, Vaidman himself notes that the TSQT formalism “is not too far from the spirit of [Price’s] ‘advanced action’...” (1996, footnote 6). This connection between advanced action and TSQT will be discussed in more detail in Section 4.

Sections 2 and 3 of the paper will focus on the claim of Sharp and Shanks (1993) to have refuted TSQT. It will be argued that, despite Vaidman’s arguments to the contrary (1996a, 1997a,b), these authors have successfully refuted a broad claim of the theory: namely, the assertion that a probability rule derived by Aharonov, Bergmann and Lebowitz in their (1964)–hereinafter ‘the ABL rule’– can be universally applied to counterfactual measurements at times between the pre- and post-selection measurements. Section 4 discusses a special case, also noted in a slightly different context by Cohen (1995) and Cohen and Hiley (1996), in which the Sharp and Shanks criticism fails and the counterfactual usage of the ABL rule is valid.

2. TSQT and counterfactual claims.

In this section, I first review the ABL rule for calculating the probability of a given outcome of a measurement occurring between pre- and post-selection measurements. I then discuss proposed interpretations of the ABL rule. I distinguish between two readings, non-counterfactual and counterfactual. The non-counterfactual version is trivially correct; I argue that Vaidman is in fact employing the counterfactual version. Subsequently, I review the Sharp and Shanks proof that the counterfactual use of the ABL rule gives results inconsistent with quantum mechanics, and Vaidman’s counterarguments. I show that both of his arguments fail: one (henceforth called “counterargument I”) relies on the possibility of allowing for a counterfactual with true antecedent which I argue is irrelevant to the counterfactual usage under debate, and the other (henceforth called “counterargument II”) fails to solve (or even to take into account) the cotenability problem which is at the root of the general inapplicability of a counterfactual usage of the ABL rule.

The original ABL paper (1964) derived an expression for the probability of an outcome $c_j$ of a measurement of a non-degenerate observable $C$ with eigenvalues $\{c_i\}$ at a time $t$ between pre- and post-selection measurements at times $t_1$ and $t_2$ ($t_1 < t < t_2$). This expression, subsequently known as the ABL rule, is an uncontroversial result of quantum mechanics, applied to the situation in which the intervening $C$ measurement has actually occurred. In the case of zero Hamiltonian, which is sufficient for the purposes of this paper, the rule states that the probability of outcome $c_j$ in the case of a preselection for the state $|a\rangle$ and a post-selection for the state $|b\rangle$ is given by:

$$P_{ABL}(c_j|a,b) = \frac{|\langle b|c_j\rangle|^2|\langle c_j|a\rangle|^2}{\sum_i |\langle b|c_i\rangle|^2|\langle c_i|a\rangle|^2} \quad (1)$$

More recently, some authors (Albert, Aharonov and D’Amato, 1985; hereinafter AAD) began to make explicit use of a counterfactual interpretation of the ABL rule. Rather than just using the rule to retrodict the probability of an outcome $c_j$ for a measurement of $C$ which was actually carried out, AAD argued that the rule could be interpreted as “the probability...that a measurement of some complete set of observables $C$ within [the interval $(t_1, t_2)$], if it were carried out, would find that $C = c_j$” (1985, p. 5).
For clarity, I shall distinguish between two different readings of the ABL rule which, at first sight, both appear to be counterfactual. The first, non-counterfactual, reading asserts for a suitably pre- and post-selected system and noncommuting observables \( X, Y, \ldots \):

“If a measurement of \( X \) was performed at time \( t \), then the probability that result \( x \) was obtained is \( p(x) \); and if a measurement of \( Y \) was performed at time \( t \), then the probability that result \( y \) was obtained is \( p(y) \), and if...”

Such a non-counterfactual reading is discussed in Mermin (1997). It is essentially a conjunction of material conditionals, only one of which is non-vacuously true, because only one of the antecedents can be true at time \( t \).

A second reading of the ABL rule, the *bona fide* counterfactual usage, asserts that for any observable \( X \), the ABL probability gives the correct probability that the property associated with eigenvalue \( x \) is possessed by the system. Thus, the counterfactual usage makes the following counterfactual prediction for a given pre- and post-selected system:

“If I had performed a measurement of observable \( X \) on the system at time \( t \), then the probability of the result \( x \) would have been \( p(x) \), where \( X \) is any observable.”

This is the type of claim being made in AAD (1985) as noted above. The counterfactual reading is also used, for example, by Aharonov and Vaidman (1991) to argue that a single particle with suitably pre- and post-selected states can be found with certainty in \( N \) separate boxes, as well as in other examples of ‘curious’ quantum effects involving pre- and post-selected ensembles.\(^1\)

The possibility of the above two distinct readings of the ABL rule (1) arises because of quantifier ambiguity in the expression \( P_{ABL}(c_j|a,b) \). (In the following we omit the ABL subscript for brevity.) As originally derived, this expression is actually shorthand for \( P(c_j|a,b;C) \), where \( C \) is the observable actually measured at time \( t \), \( t_1 < t < t_2 \), in the selection of a particular system with generalized state \( \langle b||a \rangle \). Thus the explicitly characterized ABL rule is:

\[
P(c_j|a,b;C) = \frac{\langle b|c_j \rangle^2 \langle c_j|a \rangle^2}{\sum_i |\langle b|c_i \rangle|^2 |\langle c_i|a \rangle|^2}
\]  

(1')

Omission of the parameter \( C \) as in (1) creates an ambiguity in the variable \( c_j \): namely, whether or not it represents values of the observable actually measured. To make this ambiguity more evident, we will from now on use the parameter \( C \) to indicate the observable that is actually measured at time \( t \), and replace the variable \( c_j \) with the more general variable \( x_j \); i.e., \( x_j \) might represent a value from some observable \( O \) rather than \( C \). Thus the explicitly ambiguous form of the ABL rule can be written as:

\[
P(x_j|a,b;C) = \frac{\langle b|x_j \rangle^2 \langle x_j|a \rangle^2}{\sum_i |\langle b|x_i \rangle|^2 |\langle x_i|a \rangle|^2}
\]  

(1'')

An equation like (1'') is always understood as implicitly quantified. The two different readings correspond to two different quantifications. Specifically, the non-counterfactual reading asserts that equation (1'') holds for all \( a, b \in S \), all \( C \in O \), and all \( x_j \in R(C) \),

\(^1\)The argument crucially depends on the assumption that the given particle has legitimate counterfactual ABL probabilities for different possible intervening measurements.
where $S$ is the set of pure states, $O$ is the set of observables, and $R(C)$ is the range of values of $C$. (The possibility that no measurement was performed at time $t$ is covered if we include the identity $I$ in the set $O$.)

The counterfactual reading differs from the above in that equation (1′′) is asserted to hold for all $a, b \in S$, all $C \in O$, and all $x_j \in R(O)$, for all $O \in O$, including the cases where $O \neq C$. Thus the counterfactual reading allows $x_j$ to vary over the values of observables that were not actually measured at time $t$—and, of course, there is a fact of the matter as to whether or not a measurement was performed at time $t$, and if so, which observable was measured.

Vaidman (1996b) effectively employs the counterfactual reading by arguing in his discussion of a single particle inside three boxes that the outcomes receiving probability one according to the ABL rule correspond to ‘elements of reality’ for that particle:

“Elements of reality in the pre- and post-selected situations might be very peculiar. One such example is a single particle inside three boxes A, B, and C, with two elements of reality: ‘the particle is in box A’ and ‘the particle is in box B’ ...If in the intermediate time it was searched for in box A, it has to be there with probability one, and if, instead, it was searched for in box B, it has to be found there too with probability one.”

This usage clearly implies that the properties of being in box A or being in box B are considered as possessed by the same pre- and post-selected particle. Since a measurement of some observable $O$ was in fact performed at time $t$ between the pre- and post-selection of the particle (where $O$ is either the observable associated with the particle being in box A, or the observable associated with the particle being in box B), or no measurement was performed (in which case $O = I$), it is clear that Vaidman is allowing $x_j$ to vary over the values of observables $O' \neq O$ that were not actually measured between the pre- and post-selection of that particular particle.

However, Sharp and Shanks (1993, hereinafter S&S) present a proof that the counterfactual interpretation of the ABL rule leads to predictions incompatible with quantum mechanics. They consider an ensemble of spin-$\frac{1}{2}$ particles prepared at time $t_1$ in the state $|a_1\rangle$ (read as ‘spin up along direction a’). They then assume that this ensemble is subjected to a final post-selection spin measurement at time $t_2$ along direction $b$ (i.e., the observable $\sigma_b$ is measured). This measurement yields a mixture, which we shall call $M$, consisting of two subensembles $E_i, i = 1, 2$, described by the two-state vectors $\langle b_i|a_1\rangle$, where the subscript 2 denotes spin down. The weight of each subensemble $E_i$ is given by $|\langle b_i|a_1\rangle|^2$. (See Figure 1.)

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2Vaidman (1996b), p. 900. Vaidman uses box numbers instead of letters; I modify this for uniformity of notation.

3The phrase, “If, instead, it was searched for in box B” directly trades on the ambiguity concerning whether the $B$ measurement was actually performed in the selection of that particle. If this is the case, the left hand side of the ABL rule as formulated in (1′′) becomes $P(b|\psi_1, \psi_2; B)$ (with $\psi_1$ and $\psi_2$ being the pre- and post-selected states, respectively, and $b$ the eigenvalue corresponding to finding the particle in box B): the non-counterfactual reading. But if one can use the non-counterfactual reading for the $B$ ‘element of reality,’ then one is committed to the $B$ measurement being the one which was actually performed in the selection of the given particle. Then the proposition concerning the $A$ ‘element of reality’ uses the counterfactual reading, since the left hand side of (1′′) in this case would be $P(a|\psi_1, \psi_2; B)$, $a$ being the eigenvalue corresponding to finding the particle in box A; and $a$ is not in the range of values of $B$. The analogous argument applies to the case in which A is the observable actually measured.
Figure 1. No intermediate measurement is made. Two ensembles result: $E_1$, with two-state vector $\Psi_1 = \langle b_1 | a_1 \rangle$; and $E_2$, with two-state vector $\Psi_2 = \langle b_2 | a_1 \rangle$.

Now they consider each subensemble individually, asking the counterfactual question: If we had measured the spin of these particles along direction $c$ (i.e., observable $\sigma_c$) at a time $t$ between $t_1$ and $t_2$, what would have been the probability for outcome $c_1$?

They use the ABL rule to calculate the probability of outcome $c_1$ for each subensemble $E_i$ for such a counterfactual measurement. They then show that the total probability of outcome $c_1$ derived from the above calculation, taking into account the weights of the two subensembles $E_i$, in general disagrees with the quantum mechanical probability, which is given simply by $|\langle c_1 | a_1 \rangle|^2$.

The result obtained by S&S depends on the assumption that the intervening $\sigma_c$ measurement has definitely not occurred. In counterargument I, Vaidman (1996a) argues that it is permissible for his purposes to assume that the $\sigma_c$ measurement actually occurred because he wants to allow for the possibility of a counterfactual that is not necessarily ‘contrary to fact’. While Lewis (1973, p. 26) has argued cogently that one can have a counterfactual with a true antecedent, such an argument is of no use here because Lewis’ counterfactual with true antecedent simply reduces to a material conditional. This brings us back to the ordinary non-counterfactual reading of the ABL rule, which is not under dispute.

In counterargument II, Vaidman (1996a, 1997a) claims that the S&S calculation is not the right one. He shows that if one includes the dependence of the final measurement on the intervening measurement, then the resulting $P(c_1 | a_1)$ is in fact equal to $|\langle c_1 | a_1 \rangle|^2$.

However, what this ‘correction’ amounts to is assuming that the putative counterfactual measurement has actually occurred—just as in counterargument I—since the ensembles have been redefined to include the dependence on the intervening measurement. Thus, with the ensembles redefined in this way, it appears we are no longer considering a counterfactual
conditional of the form

\[ P \rightarrow Q \] (2)

("If it had been the case that P, then it would have been the case that Q") where \( P = \) “observable \( C \) is measured” and \( Q = \) “the probability of outcome \( c_1 \) is as given by the ABL rule (applied to systems appropriately pre- and post-selected according to the procedure of S&S)”, but instead the material conditional

\[ P \rightarrow Q. \] (3)

The basic conceptual problem is the following: in considering a counterfactual measurement of the spin along \( c \) (observable \( \sigma_c \)), we must take into account all the effects of that measurement on the system. Measurement of the observable \( \sigma_c \) results in a change in the mixture \( M \) of post-selected ensembles \( E_i \) described by the two-state vectors \( \langle b_1||a_1 \rangle \) and \( \langle b_2||a_1 \rangle \): were we to make the intervening measurement, we would not have the mixture \( M \) but rather the mixture \( M' \) consisting of the four subensembles \( E'_{jk} \) with weights

\[ |\langle b_k|c_j \rangle|^2 |\langle c_j|a_1 \rangle|^2, \]

where \( j, k = 1, 2 \). (See Figure 2.)

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**Figure 2**

\[ |b_1\rangle, \ E'_{11} : \ \Psi'_{11} = \langle b_1||a_1 \rangle^{(c_1)} \]

\[ |b_2\rangle, \ E'_{12} : \ \Psi'_{12} = \langle b_2||a_1 \rangle^{(c_1)} \]

\[ |b_1\rangle, \ E'_{21} : \ \Psi'_{21} = \langle b_1||a_1 \rangle^{(c_2)} \]

\[ |b_2\rangle, \ E'_{22} : \ \Psi'_{22} = \langle b_2||a_1 \rangle^{(c_2)} \]

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Figure 2: Intermediate measurement of \( \sigma_c \). Four sub-ensembles result: \( E'_{jk}, j, k = 1, 2 \), with associated two-state vectors \( \Psi'_{jk} = \langle b_k||a_1 \rangle^{(c_j)} \). The superscript \((c_j)\) indicates that the given sub-ensemble was obtained from systems yielding outcome \( c_j \) at time \( t \). We also define ensembles \( \eta_k = \sum_j E'_{jk} \) to compare the statistical weights of those ensembles having final outcome \( b_k \) with those of the \( E_i \). where \( i = k \).

This dependence of the actual mixture, \( M \) or \( M' \), obtaining at time \( t_2 \) on whether or not an intervening measurement is performed gives rise to a crucial difficulty which destroys the validity of the counterfactual claim. In order to clarify this point, and to lay the groundwork for a detailed critique of Vaidman’s counterargument II, we need some ideas from philosophical theories of counterfactuals.
The intuitive way to formulate truth-conditions for counterfactual statements is probably best exemplified by Goodman’s classic essay, “The Problem of Counterfactual Conditionals” (1947). Goodman’s approach, now commonly referred to as the metalinguistic account of counterfactuals, proposes that a counterfactual such as $P \rightarrow Q$ is true if and only if its antecedent $P$ implies (i.e., semantically entails) the consequent $Q$ when $P$ is conjoined with suitable background conditions $S$ and laws of nature $L$:

$$P \rightarrow Q \text{ is true iff } (P \& S \& L) \models Q. \quad (4)$$

In addition to (4), a further condition is necessary to specify what sorts of facts can be included in the collection of ‘suitable background conditions’ $S$. Some restrictions are no doubt required, because certain propositions—for instance $\neg P$—result in the conjunction in (4) being a contradiction, in which case $P \rightarrow Q$ is vacuously true (i.e., true for any $Q$). Metalinguistic accounts describe the restriction on membership in $S$ in terms of ‘cotenability’: the notion that the holding of any condition belonging to $S$ must be independent of the truth value of $P$. Thus condition (4) must be augmented by a further requirement:

The background conditions asserted by $S$ must be cotenable with $P$. \quad (4a).

For future reference, let us refer to the condition defined by (4) together with (4a) as ‘condition $\Gamma$’.

However (as discussed by Goodman, and more recently in Horwich 1988, Chapter 10), the cotenability notion can only be defined within the metalinguistic account in terms of the counterfactual $P \rightarrow S$; so the metalinguistic account faces a circularity problem in giving a satisfactory account of cotenability. Fortunately, one has recourse to more successful theories of counterfactuals in order to give a rigorous account of cotenability (for example, Lewis, 1973; and Stalnaker, 1968). While these theories are based on the notion of possible worlds, they can also be expressed in terms of cotenability, as Lewis (1973, p. 57) has shown. The failure of the counterfactual interpretation of the ABL rule can most easily be analyzed in terms of a failure of cotenability between the statement of background conditions, $S$, and the antecedent $P$. Therefore, for convenience, the analysis of truth-conditions for counterfactuals will first be discussed in terms of the metalinguistic account’s cotenability notion, and later related to the possible worlds formulation.

Returning now to the discussion following equation (3), the counterfactual usage of the ABL rule as studied in the S&S example faces a difficulty in that the mixture, $M$ or $M'$, obtaining at time $t_2$ depends on whether or not an intervening measurement has been performed at time $t$ (compare figures 1 and 2). Since the mixture must enter into the counterfactual calculation in the form of the weights of each of the component subensembles, with their associated two-state vectors used as inputs in the ABL formula (1), the characterization of the mixture obtaining at $t_2$ must be included in the background condition statement $S$ in (4). Thus, the statement of background conditions $S$ holding when $\neg P$ holds becomes false when $P$ holds and is therefore not cotenable with $P$ (see footnote 4). By this criterion alone, it is clear that the counterfactual usage of the ABL rule fails because condition $\Gamma$ is not met: (4a) fails.

The S&S proof goes further than this by demonstrating that the counterfactual ABL

\footnote{More precisely, the condition for cotenability of $S$ with $P$ in the metalinguistic account is the conjunction ($\neg P \& S$) $\& (P \rightarrow S)$.}
probability based on the statement of the actual background condition holding when $P$ is false (i.e., when $C$ is not measured and mixture M obtains) is incorrect. (Note that had this background condition statement been cotenable with $P$, the same procedure would yield a correct counterfactual probability. This is the case in the usual quantum mechanical counterfactual calculation of the probability of the outcome of a measurement at time $t$ given the quantum state at time $t'$. In this case the background condition is simply the quantum state at time $t'$, which does not depend on whether or not a subsequent measurement is performed).

We now describe in detail the steps employed in the S&S proof. First, consider the overall background conditions $S$ obtaining when $P$ is false. $S$ consists of the statement of contingent initial conditions $S_1$ concerning events at time $t_1$, as well as a statement concerning events at time $t_2$, which we shall call $S_2$. Thus, $S = S_1 \& S_2$. In the S&S proof, $S_1$ corresponds to the initial preselected ensemble of particles in state $|a_1\rangle$. $S_2$ states that mixture M obtains at time $t_2$. It is clear that $S_2$ follows from $S_1$ when $P$ does not hold; i.e., from the laws of quantum theory asserted by $L$ we have:

$$S_1 \& L \& \neg P \models S_2.$$  

(5)

S&S then consider a hypothetical case which differs from the actual one only in that the antecedent $P$ is true; i.e., $C$ is measured at $t$. They derive the consequence $R$ of the conjunction of the background conditions $S = S_1 \& S_2$, relevant quantum mechanical laws $L$, and $P$:

$$S \& L \& P \models R.$$  

(6)

$R$ states: “The probability of outcome $c_1$ of an intervening measurement of observable $C$ is as given by the ABL rule and the rule of total probability applied to systems in the mixture M.” Thus, $R$ is equivalent to $Q$ in (2), since the systems referred to in $Q$ are the systems obtaining when $P$ does not hold, which must be those in mixture M. The explicit calculation corresponding to (6) is:

$$P(c_1|a_1) = P_{ABL}(c_1|a_1\&b_1)P(b_1|a_1) + P_{ABL}(c_1|a_1\&b_2)P(b_2|a_1),$$  

(7)

where the left hand side of (7), $P(c_1|a_1)$, is the probability referred to in the proposition $R$. On the right hand side, the quantities $P_{ABL}(c_1|a_1\&b_k), k = 1, 2$ correspond to the component of $L$ asserting that the ABL rule gives the correct quantum mechanical probability for the given outcome of an intervening measurement for systems with pre- and post-selected states $\langle b_k|a_1\rangle$. The weighting factors $P(b_k|a_1), k = 1, 2$ correspond to the component of $L$ stating that the rule of total probability holds, together with background condition statement $S = S_1 \& S_2$, since the weights are those obtained from ensembles $E_i, i = 1, 2$. In terms of quantum theory, the quantities $P(b_k|a_1)$ are given by

$$P(b_k|a_1) = |\langle b_k|a_1\rangle|^2,$$  

(8)

which of course assumes that no intermediate measurement of $\sigma_c$ has been performed.
However, note that the left hand side of (6) is false because it is not possible (in general) for both \( P \) and \( S_2 \) to be true, given \( S_1 \) and \( L \) (this being the failure of cotenability discussed earlier). Therefore, the consequent \( R(= q) \) may well be false.

The S&S strategy is to show that \( R \) is indeed false, by comparing \( R\) to the true consequent \( Q' \) which results in the case in which \( P \) holds and the problematic statement of background conditions \( S \) has been replaced so as to ensure that the conjunction appearing on the left hand side of (6) is true. Thus, S&S next consider what would actually occur given the truth of \( P \). There are two equivalent ways to do this. What S&S do is simply to drop the inconsistent background condition statement \( S_2 \), and derive the entailment \( Q' \) of the conjunction of \( S_1 \), \( P \) and \( L \). Thus they perform the derivation:

\[
S_1 \& P \& L \models Q'.
\]  

(9) The proposition \( Q' \) states “The quantum mechanical probability for outcome \( c_1 \) for the given ensemble of particles (in this case, the pre-selected only ensemble) is as given by the usual quantum mechanical conditional probability.” Thus the explicit calculation corresponding to (9) is:

\[
P_{QM}(c_1|a_1) = |\langle c_1|a_1 \rangle|^2
\]  

(10)

Since the left hand side of (9) is now true, the consequent \( Q' \) must be true.

However, it may not be readily apparent that \( Q' \) really is the appropriate proposition to compare with \( R \), since it does not directly incorporate the ABL rule. In order to show that \( Q' \) is the correct proposition to compare with \( R \), we now consider the second, equivalent way of deriving the consequence of the conjunction of the correct background conditions, \( L \), and \( P \). This consists of deriving the analog of \( R \) in (6) for the case in which the background conditions are those obtaining when \( P \) is true. These correct background conditions, which we shall call \( S'_2 \), consist of the conjunction of \( S_1 \) and \( S'_2 \), where \( S'_2 \) states that subsequent to the intervening measurement \( P \) we have mixture \( M' \) consisting of the four subensembles \( E'_{jk} \) and their associated weights.

In order to facilitate the comparison of \( S_2 \) with \( S'_2 \), we define the ensembles \( \eta_k = \sum_j E'_{jk}, j, k = 1, 2 \), where \( \eta_k \) is the ensemble composed of all systems found in state \( |b_k \rangle \) at time \( t_2 \) for the case in which \( P \) holds (see Figure 2). Now, in direct analogy with (6), we can derive the consequence \( R' \) of the conjunction of \( S' \) with \( L \) and \( P \):

\[
S'_2 \& L \& P \models R'.
\]  

(6')

Note, just as in (9), that the left hand side of (6') is true, so the consequence \( R' \) must be true. The proposition \( R' \) states: “The probability for outcome \( c_1 \) of an intervening measurement of observable \( C \) is as given by the ABL rule and the rule of total probability applied to systems in the ensembles \( \eta_k = E'_{1k} + E'_{2k}, k = 1, 2 \), resulting when \( P \) holds.”

The explicit calculation corresponding to (6') is:

\[
5
\]
\[ P_{QM}(c_1|a_1) = P_{ABL}(c_1|a_1\&b_1)P_{\sigma_c}(b_1|a_1) + P_{ABL}(c_1|a_1\&b_2)P_{\sigma_c}(b_2|a_1) \]  

(12)

In (12), \( P_{ABL}(c_1|a_1\&b_k) \) is the probability given by the ABL rule using states \( |a_1\) and \( |b_k\) as inputs—just as in (7)—and \( P_{\sigma_c}(b_k|a_1) \) is the final probability of outcome \( b_k \) given that the intervening \( \sigma_c \) measurement was performed; i.e., the statistical weight of \( \eta_k \). Thus \( P_{\sigma_c}(b_k|a_1) \) is the analog of \( P(b_k|a_1) \) in (8) for the case in which the intervening measurement has been performed. Explicitly,

\[ P_{\sigma_c}(b_k|a_1) = P(b_k|c_1)P(c_1|a_1) + P(b_k|c_2)P(c_2|a_1) \]  

(13)

The right hand side of (13) is the weight of \( \eta_k \), the first term being the weight of subensemble \( E'_1 \) and the second term being the weight of subensemble \( E'_2 \). Using the rule of total probability and (13), it may easily be verified that equation (12) is identical to equation (10), as follows.

The right hand side of (12) can be rewritten, using (13), as

\[ P_{ABL}(c_1|a_1\&b_1)\left[P(b_1|c_1)P(c_1|a_1) + P(b_1|c_2)P(c_2|a_1)\right] + \\
P_{ABL}(c_1|a_1\&b_2)\left[P(b_2|c_1)P(c_1|a_1) + P(b_2|c_2)P(c_2|a_1)\right] \]  

(14)

Noting that the quantities \( P(b_k|c_j) \) are implicitly conditional also on \( a_1 \), i.e. they are equivalent to \( P(b_k|c_j\&a_1) \), and for simplicity dropping the ‘ABL’ subscript (which does not affect the calculation), (14) is equal to:

\[ P(c_1|a_1\&b_1)\left[P(b_1|c_1|a_1) + P(b_1|c_2|a_1)\right] + \\
P(c_1|a_1\&b_2)\left[P(b_2|c_1|a_1) + P(b_2|c_2|a_1)\right] \]  

(15)

Using the rule of total probability, the quantities in brackets can be simplified, so that (15) is equal to

\[ P(c_1|a_1\&b_1)\left[P(b_1|a_1)\right] + P(c_1|a_1\&b_2)\left[P(b_2|a_1)\right] \]  

(16)

Expression (16) can be rewritten as:

\[ P(c_1\&b_1|a_1) + P(c_1\&b_2|a_1) \]  

(17)

which, again applying the rule of total probability, yields finally

\[ P(c_1|a_1), \]  

which is exactly the quantity calculated in (10) using the conditional probability rule of quantum mechanics. Therefore, despite the fact that the statement of background condition \( S'_2 \) does not explicitly appear in (9), the proposition \( Q' \) derived from that formula is the exact analog of \( R \) for the case in which \( P \) holds; i.e., \( Q' \) is equivalent to \( R' \).

We have therefore established that \( Q' \) as derived in the S&S proof is indeed the correct proposition to compare with \( R(=Q \text{ of } (2)) \). Note that \( Q' \)—which must be true—will, in
general, differ from $R$ because the ABL rule referred to in $R(= Q)$ applies specifically to those systems in ensembles $E_i, i = 1, 2$, which actually obtain when the intervening measurement has not been performed, i.e., when $P$ does not hold; and in general, it will be the case that $E_i \neq \eta_k$ when $i= -k$. Therefore, in general $R$ will be false. (Further clarification of the relationship between $Q, Q', R$, and $R'$ will be made in the subsequent discussion of Lewis’ theory of counterfactuals and in the caption to Figure 3, which illustrates points made in that discussion.)

Now we are in a position to pinpoint the error in Vaidman’s counterargument II as stated in his (1997a,b). The core of counterargument II is the claim that the S&S proof (as well as other inconsistency proofs along similar lines) is flawed in that it assumes that no intervening measurement has occurred in the ‘counterfactual world’ (which corresponds to our hypothetical case of (6)). However, this claim is mistaken; it confuses derivation (5), performed to obtain the statement of actual background conditions $S_2$ entailed by the conjunction of $S_1, L$ and $\neg P$, with the derivation applying to the hypothetical case (6).

It might be objected that $S = S_1 \& S_2$ (in particular, $S_2$) is not the appropriate statement of background conditions for deriving consequence $R$ in (6), and that we should use instead $S'$ as in (6'). However, this move cannot be justified for several reasons.

First, it turns out that $S'$ is also not cotenable with $P$ in the given situation, though this failure of cotenability is less obvious than that of $S$. The failure of cotenability of $S'$ can best be seen through reference to Lewis’ theory of counterfactuals (1973, 1979). According to Lewis, a counterfactual is a ‘variably strict conditional’: a type of strict conditional requiring reference to what Lewis terms a set $\mathcal{S}_i$ of ‘spheres of accessibility’ of possible worlds defined with respect to the actual world $i$. Each sphere $\zeta_i^{(k)} \in \mathcal{S}_i$ contains only those possible worlds that are similar to the actual world $i$ according to a specified similarity relation for $\mathcal{S}_i$. The spheres are centered on $i$ and nested in a way that reflects their closeness, or degree of similarity, to $i$. In other words, $i$ is at the center of $\mathcal{S}_i$ with the smallest sphere $\zeta_i^{(1)}$ comprising worlds that are more similar to $i$ than are any of the worlds that are in spheres in $\zeta_i^{(k)}$ $i$, with $k > 1$, but not in $\zeta_i^{(1)}$. (And so on for $\zeta_i^{(2)}$, etc.)

According to Lewis, a counterfactual $P \rightarrow Q$ is vacuously true at $i$ if and only if no $P$-world (i.e., no possible world in which $P$ is true) belongs to any sphere in $\mathcal{S}_i$, and non-vacuously true at $i$ if and only if some sphere $\zeta_i^{(k)} \in \mathcal{S}_i$ contains at least one $P$-world and the material conditional $P \rightarrow Q$ holds at every world in $\zeta_i^{(k)}$ (1973, p. 16). $\zeta_i^{(k)}$ is also referred to as a ‘$P$-permitting sphere.’ As noted earlier, these truth-conditions can be shown to be fully equivalent to those in metalinguistic theories through reference to a natural definition of cotenability in terms of Lewis’ similarity relation between worlds. Thus, Lewis defines a statement $X$ as cotenable with $P$ at a world $i$ if and only if either (1) $X$ holds throughout $\mathcal{S}_i$ or (2) $X$ holds throughout some $P$-permitting sphere in $\mathcal{S}_i$. The truth conditions can now be stated in terms of cotenability: $P \rightarrow Q$ is true at $i$ iff $P$ and some auxiliary premise $X$, cotenable with $P$ at $i$, logically imply $Q$ (Lewis 1973, p. 57).

Clearly, the statement $S = S_1 \& S_2$ is not cotenable with $P$ at $i$ according to the above definition. But neither is $S' = S_1 \& S'_2$, since it does not hold throughout any $P$-permitting sphere in $\mathcal{S}_i$ (see Figure 3).  

---

6For example, Cohen (1995) and Miller (1996).
Therefore, the counterfactual (2) is still false if (6) is modified by substituting \( S' \) for \( S \). Though such a substitution would allow us to obtain the desired result that \( R = Q' \), this statement would now apply not to a counterfactual conditional but to a material conditional: the situation in which the measurement has been made in the actual world. This is in fact what Vaidman proposes in counterargument II, and it fails because, as noted earlier, it removes the counterfactual element from the S&S example.

There is a more intuitive reason, however, why we should insist that the background conditions \( S = S_1 \& S_2 \) are the correct ones for the counterfactual derivation (6) in this particular case. This is because the counterfactual reading of the ABL rule uses as input quantities on record at time \( t_3 \geq t_2 \). That is, the situation is one in which a pre- and post-selection measurement (but no intervening measurement) have already been performed; thus the experimenters are in possession of recorded results for measurement outcomes at both times \( t_1 \) and \( t_2 \). At that point, the question asked is a retrodictive question: “What would have happened had I measured observable \( O \) at time \( t, t_1 < t < t_2 \)” In view of the existence of actual results at \( t_2 \), such results are an indelible part of the history of

---

Figure 3. Neither \( S \) nor \( S' \) hold throughout any \( P \)-permitting sphere in \( S_i \). (It is assumed that \( L \) is true throughout \( S_i \).) Note also that although we have \( S' \& P \rightarrow R' \) in the overlap region, and equivalently \( S_1 \& P \rightarrow Q' \), we would need to have \( P \rightarrow Q \) throughout a \( P \)-permitting sphere in order for the counterfactual \( P \rightarrow Q \) to be true. But in general, \( Q' \neq Q \). The quantity \( Q' = R' \) in the overlap region merely gives the ordinary quantum mechanical probability for a measurement that has actually been performed. Note that neither \( R \) nor \( Q \), being false, hold anywhere in \( S_i \). (In exceptional cases where it turns out that \( S = S' \), we would have \( Q' = R' = R = Q \), so the counterfactual would be true. Such cases are discussed in Section 4.)
world \(i\) and cannot be disregarded. Note, furthermore, that such results are not limited to single-system measurement outcomes but could also be statistical data on a large number of measured systems, data which would clearly reflect the existence of ensembles \(E_i\).

2. Time-symmetrized counterfactuals.

In this section, I review and criticize two definitions given by Vaidman for a proposed ‘time-symmetrized counterfactual,’ which he claims is the only type of counterfactual for which the ABL rule can be considered to give correct counterfactual probabilities. I show that both definitions fail, in general, to establish a valid counterfactual use of the ABL rule. (In any case, it should be noted from the outset that neither Goodman’s, Stalnaker’s nor Lewis’ theories are restricted to time-asymmetric counterfactuals, i.e., counterfactuals whose consequent \(Q\) concerns events occurring at times later than, or whose background conditions are limited to events occurring at times earlier than, that of events that are the subject of the antecedent \(P\).)

Vaidman (1996a, 1997a,b) argues that discrepancies like that in the S&S proof arise because they make use of a time-asymmetric counterfactual, which he says is inappropriate for calculations involving the ABL rule. He proposes a new type of time-symmetrized counterfactual which he claims resolves these discrepancies. Vaidman has provided two definitions. The first, which I will call Definition 1, appears as ‘interpretation c’ in (1996a):

\[
\text{Definition 1. “Counterfactual probability [is] the probability for the results of a measurement if it has been performed in the world ‘closest’ to the actual world.”} \quad (1996, \text{p.9})
\]

Definition 1 is not new, since the idea of formulating truth-conditions for counterfactuals in terms of a ‘closest possible world’ has previously been proposed by Stalnaker (1968). The novelty of Definition 1 as proposed by Vaidman consists in the choice of ‘selection function’ (in Stalnaker’s terminology; cf. 1968, p. 120) that determines the closest possible world. The closest possible world \(j\) is defined by Vaidman as the one in which all measurements, except the intermediate measurement asserted by \(P\), have the same outcomes as in \(i\). This amounts to ignoring the inequivalence of the mixtures \(M\) and \(M'\) which gives rise to the cotenability problem discussed in section 2.

However, Vaidman’s argument for the legitimacy of ignoring the difference between the mixture \(M\) in world \(i\) and the mixture \(M'\) in world \(j\) is flawed. His claim is that, since it is postulated that the system is not disturbed between \(t\) and \(t_2\)–the period during which the mixtures differ–the difference between the mixtures has “no physical meaning.” (1996, p.10) However, this is not the case, as can be shown if one considers an experiment along the lines of the S&S example in which a large number \(N\) of instances is considered. In fact, I will now show that, due to the physical constraints imposed by quantum mechanics, in general there can be, for large \(N\), no such “closest possible world” \(j\).

The requirement for \(j\) is that all measurements, except the intervening one not occurring in \(i\), have the same outcomes as in \(i\). Consider each run of the experiment in which the spin-\(1/2\) particle is preselected as ‘up along \(a\)’ (\(|a_1\rangle\)). For each of these runs, when we find final outcome \(|b_1\rangle\) in \(i\) we must also find final outcome \(|b_1\rangle\) in \(j\) and likewise for final outcome \(|b_2\rangle\).

The experiment in \(i\) consists of the initial spin measurement along \(a\) and one subsequent spin measurement at an angle \(\theta_{ab}\) (between directions \(a\) and \(b\)), while in \(j\) it consists of
the initial a spin measurement and two subsequent measurements at relative angles \( \theta_{ac} \) and \( \theta_{cb} \).

So (assuming the spin directions are in the same plane) we must have

\[
\theta_{ab} = \theta_{ac} + \theta_{cb}.
\]  (19)

Now, the fraction \( F_i(b_1) \) of runs resulting in the particle being found up along \( \theta_{ab} \) in \( i \) is given by the weight of ensemble \( E_1 \), i.e.:

\[
F_i(b_1) = |\langle b_1|a_1 \rangle|^2 = \cos^2 \left( \frac{\theta_{ab}}{2} \right)
\]  (20)

The fraction \( F_j(b_1) \) of runs resulting in the particle being found up along \( \theta_{ab} \) in world \( j \) is given by the weight of ensemble \( \eta_1 \), i.e.:

\[
F_j(b_1) = |\langle b_1|c_1 \rangle|^2|\langle c_1|a_1 \rangle|^2 + |\langle b_1|c_2 \rangle|^2|\langle c_2|a_1 \rangle|^2 = \cos^2 \left( \frac{\theta_{cb}}{2} \right) \cos^2 \left( \frac{\theta_{ac}}{2} \right) + \sin^2 \left( \frac{\theta_{cb}}{2} \right) \sin^2 \left( \frac{\theta_{ac}}{2} \right)
\]  (21)

Since it is postulated that each run in \( j \) has the same outcome as in \( i \), we must have \( F_i(b_1) = F_j(b_1) \), i.e.:

\[
\cos^2 \left( \frac{\theta_{ab}}{2} \right) = \cos^2 \left( \frac{\theta_{cb}}{2} \right) \cos^2 \left( \frac{\theta_{ac}}{2} \right) + \sin^2 \left( \frac{\theta_{cb}}{2} \right) \sin^2 \left( \frac{\theta_{ac}}{2} \right)
\]  (22)

(In other words, the weight of ensemble \( E_1 \) must be equal to the weight of ensemble \( \eta_1 \)).

However, (22) is in general not satisfiable for \( \theta_{ab} = \theta_{ac} + \theta_{cb} \). For example, if we take \( \theta_{ac} = \theta_{cb} = \pi/4 \), we obtain:

\[
1/2 \sim .728 + .02 \sim .749
\]  (23)

Since the requirement (22) cannot in general be fulfilled, there is in general no closest possible world \( j \) to support the symmetrized counterfactual that Vaidman proposes.

Vaidman has proposed a second definition for the time-symmetrized counterfactual (1997a):

**Definition 2.** “If a measurement of an observable \( C \) were performed at time \( t \), then the probability for \( C = c_i \) would equal \( p_i \), provided that the results of measurements performed on the system at times \( t_1 \) and \( t_2 \) are fixed.” (1996a, p.9)

Definition 2 gives predictions consistent with quantum mechanics, if the phrase “the results of measurements performed on the system at times \( t_1 \) and \( t_2 \) are fixed.” is intended to stipulate that the ABL rule only applies to those individual systems that (i) are pre- and post-selected in the time-symmetrized state \( \langle b_k|a_1 \rangle, k = 1, 2 \), in the experiment actually performed (in the S&S example, no intervening measurement having been made) and (ii)

7Actually, (22) is a necessary but not sufficient condition for validity of the counterfactual use of the ABL rule. In certain cases, one may have the weights of \( E_i \) and \( \eta_k \) equal for \( i = k \), but the actual ensembles themselves cannot necessarily be identified: \( E_i \neq \eta_k \) (c.f. Bub and Brown, 1986, p. 2339). In showing, however, that (22) is not satisfied in general, we show that even this necessary condition does not hold and therefore this definition for a time-symmetrized counterfactual is, in general, untenable.
would have been pre- and post-selected in the same state if, counterfactually, a different measurement had been performed (in the S&S example, an intervening measurement having been made). Since such a system is stipulated to have the appropriate two-state vector for the counterfactual measurement as well as for the actual measurement (in this case a measurement of the identity $I$), the non-counterfactual reading of the ABL rule would in fact be applicable to such a system.

To clarify this point, let us augment the two-state vector notation with a subscript specifying which observable has been measured at time $t$ in the pre- and post-selection of system $K$. Thus, if system $K$ is preselected in state $|a_1\rangle$ and post-selected in state $|b_k\rangle$ via a measurement of observable $I$ at time $t$, the two-state vector $\Psi$ of system $K$ is $\Psi = \langle b_k|a_1 \rangle_I$. Let us say that system $K$ is “time-symmetrically fixed” in the sense of Definition 2 iff we can substitute a proposed counterfactually measured observable $O$ for the observable $I$ that was actually measured, and $K$ would still have the same two-state vector. Thus we can also assign to $K$ the two-state vector $\Psi' = \langle b_k|a_1 \rangle_O$. In virtue of the fact that $K$ can be described by $\Psi'$, we can then employ the non-counterfactual reading of the ABL rule to calculate the probability $P(x|a,b;O)$ where $x$ is in the range of $O$, even though $O$ is not the observable that was actually measured.

However, since we only exist and conduct experiments in one world, we cannot get to other counterfactual worlds and perform concurrent experiments there to find which systems do have the same post-selection outcomes for various intervening measurements. Since the counterfactual ABL probabilities only apply to the time-symmetrically fixed systems, and we have no way of identifying these, any information provided by a counterfactual interpretation of the ABL rule under Definition 2 would seem to have no clear physical meaning.

An illustration of this difficulty of Definition 2 is given in figure 4. In the actual world $i$, we conduct an experiment in which electrons are pre- and post-selected in the state $|z+\rangle$ (up along $z$). We perform an intervening measurement of $\sigma_\theta$, where $\theta = \frac{\pi}{2}$. If we start with 16 pre-selected electrons, a possible (ideal) distribution of measurement results is shown.
In the actual world \( i \), 16 systems preselected in the state \( |z_1\rangle \) are subjected to an intervening measurement of \( \sigma_{\theta} \), where \( \theta = \frac{\pi}{2} \) with respect to the z axis, and a final post-selection measurement of \( \sigma_z \). A possible (statistically ideal) distribution is shown.

In a counterfactual world \( j \), the same 16 systems are subjected to an intervening measurement of \( \sigma_{\phi} \), where \( \phi = \frac{\pi}{3} \) with respect to the z axis, and again a final post-selection \( \sigma_z \) measurement. A possible (statistically ideal) distribution is shown.

Only the three systems in bold-face, \( S_3 \), \( S_7 \), and \( S_{10} \), qualify as having their pre- and post-selection measurement outcomes ‘time-symmetrically fixed’ for state \( \langle z_1 || z_1 \rangle \). Applying the ABL rule for a counterfactual measurement of intervening observable \( \sigma_{\phi} \) to any other systems having the symmetrized state \( \langle z_1 || z_1 \rangle \) in world \( i \) would give incorrect probabilities for those systems.

Now, we ask the counterfactual question: “Consider a different angle \( \phi = \frac{\pi}{3} \). What would have been the probability of getting the result +1 were we to have measured \( \sigma_{\phi} \) at \( t \), instead of \( \sigma_{\theta} \)?” In order to apply the ABL rule only to time-symmetrically fixed systems, we have to imagine this measurement performed concurrently in a counterfactual world \( j \), and choose only those systems post-selected in state \( |z+\rangle \) in \( j \). However, not all of the systems post-selected in the actual world \( i \) via the \( \sigma_{\theta} \) measurement would necessarily be post-selected in \( j \) via the \( \sigma_{\phi} \) measurement. (See figure 4).

The only systems which are pre- and post-selected as \( |z+\rangle \) in both worlds in this example are \( S_3, S_7, \) and \( S_{10} \). These qualify as having their outcomes fixed in a time-symmetric sense. But since we are only actually performing the experiment in \( i \), we do not know which of our 8 systems with state \( \langle z+ || z+ \rangle \) in \( i \) have fixed outcomes for other possible experiments in other possible worlds such as \( j \). For example, were we to apply the counterfactual ABL reading to systems \( S_2, S_5, S_{12}, S_{15}, \) or \( S_{16} \), we would be using it incorrectly since these systems do not have outcome \( |z+\rangle \) in the counterfactual world \( j \) and therefore do not qualify as having fixed outcomes at both times, as is required for the symmetrized counterfactual. But there is no way to know this. Again, there is no way to make use of the information provided by the counterfactual ABL rule according to Definition 2; nor is it even clear that the notion of ‘time-symmetrically fixed’ is physically meaningful.
4. Limited counterfactual ABL interpretation and the ‘advanced action’ concept.

I will now discuss a special case in which Vaidman’s time-symmetrized counterfactual interpretation may be correctly applied to the ABL rule. This is the case in which the hypothetical intervening measurement is of either the pre-selection observable $A$ or the post-selection observable $B$. Before showing how this special case holds for Vaidman’s time-symmetrized counterfactual, I note that it falls under a class of counterfactuals identified by Cohen (1995) and Cohen and Hiley (1996) which they obtained by applying the consistency conditions of Griffiths (1984) in his consistent histories interpretation.

Cohen and Hiley found that a counterfactual interpretation of the ABL rule can be justified for a system preselected in state $|\psi_1(t_1)\rangle$ and postselected in $|\psi_2(t_2)\rangle$ if, for every pair $\alpha, \beta$ of eigenvalues of operator $C$,

$$\text{Re}\{\langle \psi_2(t) | P_\alpha | \psi_1(t) \rangle \langle \psi_1(t) | P_\beta | \psi_2(t) \rangle\} = 0.$$  

(24)

where $P_\nu$ is the projection operator onto the space of eigenstates with eigenvalue $\nu$ which are the possible outcomes of a measurement of $C$ at time $t$, $t_1 < t < t_2$. It can easily be seen that the above special case (i.e., with $(\alpha, \beta)$ from the set of eigenvalues either of observable $A$ or observable $B$—not both together) satisfies (24).

For example, if we choose $B$ as the counterfactually measured observable, and the pre- and post-selected states as $|a_1\rangle$ and $|b_1\rangle$ respectively, with $H = 0$, (24) becomes

$$\text{Re}\{\langle b_1 | b_1 \rangle \langle b_1 | a_1 \rangle \langle a_1 | b_2 \rangle \langle b_2 | b_1 \rangle\} = 0.$$  

(24a)

We see from the form of (24a) that any observable commuting with either the pre- or post-selection observable will give rise to at least one factor consisting of an inner product of orthogonal eigenvectors. Thus for such a case, (24) will always be satisfied.

Condition (24) is a powerful one because it provides a basis for assertions about the ontological properties of systems between measurements: pre- and post-selection experimental histories satisfying (24) permit inferences about the states of systems even at times when no measurement is made.

Let us now confirm that both of Vaidman’s definitions for a time-symmetrized counterfactual succeed for this case (as indeed they must, since even an ordinary time-asymmetric counterfactual reading of the ABL rule succeeds here). First, the existence of a closest possible world $j$ as postulated by Vaidman is permitted by condition (22) in this case. If the observable measured at time $t$ is the same as either the pre- or post-selection observable, we have either (i) $\theta_{ac} = 0$ or (ii) $\theta_{cb} = 0$. Putting each of these cases into (22), we obtain:

$$\cos^2\left(\frac{\theta_{ab}}{2}\right) = \cos^2\left(\frac{\theta_{cb}}{2}\right)$$  

(25i)

$$\cos^2\left(\frac{\theta_{ab}}{2}\right) = \cos^2\left(\frac{\theta_{ac}}{2}\right)$$  

(25ii)

---

8See also Griffiths (1996) and (1998).
9Equation (24) is derived under the assumption that the $P_\nu$ are components of a decomposition of the identity operator in terms of orthogonal projections satisfying the condition $P_\lambda P_\nu = \delta_{\lambda\nu} P_\nu$.
10More precisely, any observable satisfying the condition of footnote 9.
11For a detailed discussion of this interesting property, see Cohen (1995, p.4337).
By way of (19), we have also that
\[ \theta_{ab} = \theta_{cb} \text{ for } \theta_{ac} = 0, \quad \text{and} \]
\[ \theta_{ab} = \theta_{ac} \text{ for } \theta_{cb} = 0; \]
thus, (25i and ii) are correct in each case. So therefore there does exist a ‘closest possible world’ \( j \) for this special case, and Definition 1 is tenable.\(^{12}\)

In addition, Definition 2 will also be tenable for this case, since all appropriately pre-selected systems which are post-selected via no intervening measurement would also, with probability 1, be post-selected via an intervening measurement of either the pre- or post-selection observable. Thus all pre- and post-selected systems in world \( i \) can be considered to be time-symmetrically fixed. We can see that this is the case by referring again to condition (24). Consider a system preselected in state \( |a_1\rangle \) and post-selected in state \( |b_1\rangle \) via no intervening measurement. According to condition (24), were we to make an intervening measurement of observable \( B \) at time \( t \), the probability of outcome \( |b_1\rangle \) would be correctly given by the ABL rule; and that probability would be 1. Given that the system is not disturbed between times \( t \) and \( t_2 \), a post-selection measurement of \( B \) must therefore find the system in state \( |b_1\rangle \). So we can therefore say that if \( B \) had been measured at time \( t \), the same individual system would still have been post-selected.

In order to show that a pre- and post-selected system in state \( \langle b_1||a_1\rangle \) via no intervening measurement would still be pre- and post-selected via an intervening measurement of observable \( A \), we note that a counterfactual measurement of \( A \) at a time \( t_a, t_1 < t_a < t_2 \), would trivially confirm that the system is in state \( |a_1\rangle \). Then we again have to use (24) together with the ABL rule and a counterfactual \( B \) measurement at time \( t_b, t_1 < t_a < t_b < t_2 \), to show that the system can be considered to be in the state \( |b_1\rangle \) at a time \( t_b \) after \( t_a \) and before \( t_2 \), even though \( B \) has not actually been measured. If the system is in state \( |b_1\rangle \) prior to \( t_2 \), and is not disturbed up until \( t_2 \), then it must be post-selected in that state upon being subjected to a post-selection measurement of \( B \) at \( t_2 \). Therefore, in this case, the system qualifies as time-symmetrically fixed, and Definition 2 is tenable.

Additional justification for a limited ABL counterfactual interpretation, i.e. for the above special case, comes from Price’s concept of ‘advanced action’. This notion will be briefly described below, along with his arguments for the claim that we live in a world which exhibits advanced action.

Price (1996) raises an objection to the usual interpretations of quantum mechanics based on their explicit time-asymmetry, which assumes his contested \( \mu \)-innocence (micro-innocence) assumption: the notion that a quantum systems state depends on past interactions but not future ones. The objection consists in the claim, for which Price argues in his book, that there is no objective physical basis for such an asymmetry.

Price (p.187) discusses an example of a photon passing through two polarizers (a typical TSQT situation). He observes that our ordinary time-asymmetric intuition is to say that, with the history of the photon prior to the first polarizer held fixed (call this case (i)), the state of the photon between the first and second polarizers will not change if the

\(^{12}\)As an illustration of the point made in footnote 7, note that there is a case in which (22) is satisfied but (24) fails: the case in which the intervening measurement is of a spin direction orthogonal to both \( a \) and \( b \). (In this case, the spin directions are not coplanar.) This is a case in which the weights of \( E_i \) and \( \eta_k \) are equal for \( i = k \), but the ensembles themselves are not.
orientation of the future polarizer is changed. In terms of ensembles, this is the usual quantum mechanical way of defining states and ensembles in terms of a preselection measurement only. Now, in order to demonstrate the time-asymmetry of this intuition, Price asks us to consider the temporal mirror image, call it case (ii), of the above case, call it (i) (p. 187). Case (ii) is as follows: with the future of the photon after it passes the second polarizer held fixed, changes in the orientation of the first polarizer will not affect the state of the photon between the first and second polarizers. But we clearly reject this view, because we consider that the state of the photon after it passes the polarizer is dependent on the setting of that polarizer (p. 188).

Price argues that because standard quantum mechanics accepts case (i) but rejects case (ii), it implicitly endorses an additional time asymmetry above and beyond what can be accounted for by the usual time-asymmetric convention for assessing counterfactuals (which assumes that only the past is fixed). Since the physics of the microworld reflects no such asymmetry, Price views the above time-asymmetric residue as unacceptable, and therefore argues that the world must exhibit advanced action. If advanced action is admitted, then the asymmetry disappears because one rejects both cases above: the state of the photon in the region between the polarizers depends on the orientation of both the past and future polarizers. On such a view, it does not make sense to define an ensemble only in terms of a preselection measurement, because that single measurement does not yield the maximal information about the photon. In a world with advanced action, the photon has some additional predisposition based on the measurement to be made in its immediate future.

In exactly the same way, TSQT advocates argue that the full description of a quantum system, both in terms of its state description and any ensemble of which it would be considered a member, requires that one take into account such future measurements. Hence they argue that such systems should be labeled by a two-state vector rather than a one-state vector (Price himself does not, however, refer to two-state vectors \( \langle \Psi | \Phi \rangle \) in his discussion of the time-symmetry of quantum theory).

How can the above considerations support a counterfactual reading of the ABL rule for the special case described above? In a world with advanced action, the causal effects of measurements are fully time-symmetric. That means we should be able to consider an experiment of the kind discussed above (i.e., the photon passing two polarizers) from either the usual temporal direction or the time-reversed direction, and either way should correspond to a valid physical process.

Consider an experiment in which no observable (equivalently, the identity \( I \)) is measured at the intermediate time \( t \). Let our photon be described by the two-state vector \( \langle b_1 | a_1 \rangle \); i.e., it will be pre- and post-selected to begin with eigenvalue \( a_1 \) and to end up with eigenvalue \( b_1 \). Let the ABL probability of outcome \( x \) at time \( t \) be denoted by \( P(x; t) \). Under the assumption that the ABL rule is valid only for intervening measurements that have actually been performed (call this the ‘strict ABL assumption’), it is invalid to use the ABL rule to calculate \( P(b_1; t) \) for a counterfactual measurement of \( B \) in this situation. Yet, we also know that the photon is going to end up in state \( | b_1 \rangle \) at time \( t_2 \). The ABL rule gives the value 1 for \( P(b_1; t) \). If the system is not disturbed at all between \( t \) and \( t_2 \) (as is assumed), then this result seems fully consistent with that fact.

Now, one might argue that this merely reflects the arbitrary decision to postselect for the state \( | b_1 \rangle \) at \( t_2 \), and that the value of \( P(b_1; t) \) has nothing to do with the ontological
state of the photon at that time (or at any time prior to $t_2$). But this argument is weakened in a fully time-symmetric world with advanced action. This can be seen by viewing the same process from the time-reversed direction. (In the following, we denote ABL probabilities applying to the time-reversed direction by $P^R(x; t).$)

Suppose we have a system “starting out” at time $t_2$ in state $|b_1\rangle$, no measurement occurring at time $t$, and the system being found in state $|a_1\rangle$ at time $t_1$. Were we to calculate $P^R(b_1; t)$ and get the value 1, according to the strict ABL assumption we would be using the ABL rule incorrectly. Yet the result $P^R(b_1; t) = 1$ correctly describes the actual physical situation: we know that the photon must have the eigenvalue $b_1$ at time $t$, since it “started out” with that value (viewing the post-selection at $t_2$ as the starting point).

The fact that we view the “incorrect” ABL usage in a different light depending on the temporal direction considered should alert us to the fact that, as Vaidman or Price would put it, we are operating under a “time asymmetry prejudice.” If we truly want to rule out the “incorrect” use of the ABL rule, such a judgment should stand up to scrutiny when viewed from either temporal direction. Yet, as shown above, when viewed in the time-reversed direction, our argument for the rejection of the “incorrect” usage is considerably weaker. (A similar argument, though not explicitly in terms of the ABL rule, can be found in Griffiths (1984, pp. 238-9).)

5. Conclusion.

It has been argued that the counterfactual interpretation of the ABL rule is not valid in general. For the usual time-asymmetric counterfactual situation, as in the Sharp and Shanks proof, it gives results generally inconsistent with quantum mechanics. Counterarguments by Vaidman (1996a, 1997a,b) against this proof and against others like it (Cohen, 1995 and Miller, 1996) fail because they fail to take into account the fundamental cottenability problem posed by the counterfactual use of the ABL rule, and mistakenly identify as a flaw what is actually an appropriate application of background conditions for evaluation of the counterfactual statement. Proposed definitions for a new kind of time-symmetric counterfactual are problematic in that they rely either on the notion of a ‘closest possible world’ that in general does not exist, or on a time-symmetric fixing requirement that is generally impossible to fulfill.

However, there is an interesting special case in which the ABL rule may be correctly used in a counterfactual sense: that in which the hypothetical intervening measurement is that of an operator that commutes with either the pre- or post-selection operator. This case is a member of a class identified by Cohen and Hiley which satisfies a condition derived from Griffith’s consistent histories approach. In this case, both definitions of the time-symmetrized counterfactual are satisfied: there does exist a closest possible world, and the time-symmetrical fixing requirement can be fulfilled. Moreover, a counterfactual interpretation of the ABL probabilities in this case can be further supported by considering a physical picture that includes Price’s advanced action concept.

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\footnote{This particular case is addressed in this paper because advocates of the counterfactual interpretation of the ABL rule have cited it as an example of “curious properties” of pre- and post-selected ensembles.}
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