Exact solutions and internal waves for the Antarctic Circumpolar Current in spherical coordinates

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Funding information
Vienna Science and Technology Fund; Austrian Science Fund

Abstract
This paper is concerned with some analytical aspects pertaining to the Antarctic Circumpolar Current. We use spherical coordinates in a rotating frame to derive a new exact and partially explicit solution to the governing equations of geophysical fluid dynamics for an inviscid and incompressible azimuthal flow with a discontinuous density distribution and subjected to forcing terms. The latter are of paramount importance for the modeling of realistic flows, that is, flows that are observed in an averaged sense in the ocean. The discontinuous density triggers the appearance of an interface that plays the role of an internal wave. Although the velocity and the pressure are determined explicitly, we use functional analytical techniques that uniquely render the surface and interface defining functions in an implicit way as soon as a small enough pressure is applied on the free surface. Additionally, we consider a particular example, where the interface can be determined explicitly. We conclude our discussion by setting out monotonicity relations between the surface pressure and its distortion that concur with the physical...
Undoubtedly, the Antarctic Circumpolar Current (ACC) is one of the most significant currents in the Earth’s oceans. To name one of its many unique features, we say that it is the only current that completely encircles the polar axis, occupying a tremendous area: indeed, ACC flows eastward through the southern regions of Atlantic, Indian, and Pacific Oceans along 23,000 km, and extends in places over 2000 km in width, cf.\textsuperscript{1–4}. Its massiveness is also reflected by the huge volumes of water it transports estimated to be between 165 million and 182 million cubic meters of water every second, cf.\textsuperscript{5}, which represents more than 100 times the flow of all the rivers on Earth.

Among the many factors that shape the complex behavior of ACC is the presence of stratification that accommodates observed sharp changes in water density (due to variations in temperature and salinity, cf.\textsuperscript{6–11}). So-called \textit{fronts} or \textit{jets} arise due to stratification in the meridional direction, cf.\textsuperscript{12}; the two main fronts of the ACC are the Subantarctic Front to the north and the Polar Front further south, see Figure 1. Stratification in the vertical direction can cause internal waves; for the underpinning mechanics of internal wave generation, we refer the reader to Refs.\textsuperscript{13, 14, and 15}. However, allowing for (discontinuous) stratification complicates the analysis of an already very demanding analytical problem. Certain progress for two-dimensional stratified flow was made quite recently, cf.\textsuperscript{7,16–27}. On a related note, we would like to point out the very recent results by Escher et al.\textsuperscript{28} on stratified water flows with singular density gradients.

Here we address, from a mathematical perspective, the topic of stratified geophysical water flows exhibiting vertical structure, internal waves (arising as a result of the discontinuous stratification), and a preferred propagation direction. In doing so, we derive and, subsequently analyze an exact, partially explicit solution to the geophysical water wave equations written in spherical coordinates and in a rotating frame. This solution portrays an incompressible, inviscid, stratified, steady flow moving purely in the azimuthal direction, that is, the velocity profile and the pressure are described below and up to the free surface as a function of depth and the angle of latitude. As such, this solution is suitable for a depiction of ACC. Our contribution here follows a line of work initiated by Constantin and Johnson\textsuperscript{6,29–31} and Constantine and Johnson\textsuperscript{7,8,32,33} on the derivation of explicit and exact solutions to the governing equations of geophysical fluid dynamics (GFD) that describe surface waves and their interactions with the underlying currents that are ubiquitous in Earth’s ocean basins. For a selective list of recent references, in this direction, we point the reader to Refs.\textsuperscript{34, 35, 36, 37, 38, 39, 40, 41, 42, and 43}. We would like to emphasize that the availability of exact and/or explicit solutions in fluid mechanics is special and quite rare—due to the complexity of the governing equations. However, once available, they provide new avenues of investigation of physically realistic flows, by means of asymptotic\textsuperscript{44}, or multiple-scale methods.\textsuperscript{7}
The new aspect of our investigation—compared to the existing mathematical literature on ACC (e.g., Refs. 33, 45, 46, 47, 48, 49, 50, and 51)—is the presence of a discontinuous density stratification. More precisely, we allow a vertical layering of the flow, with two layers of different, nonconstant densities, where the denser layer sits below the less dense one (stable stratification): thus, an interface, which plays the role of an internal wave, arises. Although, in general, the interface is given implicitly, we are able to devise a scenario where it arises as the solution of an explicitly solvable differential equation. As an added bonus we prove interesting regularity properties for the interface. Besides stratification, our solution accounts for forcing terms being necessary for the dynamical balance of the ACC, cf.33,52. In line with these ideas, we point to Refs. 53 and 54 for the relevance of stratification to maintain the equilibrium of ACC: baroclinic instability (arising from stratification) generates eddy-induced cells (acting to flatten the isopycnals) that counterbalance the wind-driven Ekman cell (acting to steepen isopycnals).
The layout of the paper is as follows: we introduce in Section 2 the governing equations (in spherical coordinates) and their boundary conditions for geophysical flows. Thereafter, we derive in Section 3 explicit solutions for the velocity field and the corresponding pressure function in the two layers of the fluid domain. From the dynamic boundary condition, we find an implicit relation between the imposed pressure and the resulting surface distortion. The interface defining function also appears implicitly as a condition expressing the balance of forces at the interface. The two implicit relations are then subjected to the implicit function theorem; this way we are able to prove that any small enough perturbation of the pressure required to preserve an undisturbed free surface (following the Earth’s curvature) triggers unique functions, describing the surface and the interface, respectively. The last section of the paper is devoted to proving that the solution we derived displays expected physical properties: a decay of the surface height occurs as soon as the pressure along the free surface increases. A regularity property of the interface defining function is also proved, followed by a particular explicit example.

2 PHYSICAL PROBLEM AND GOVERNING EQUATIONS

In this section, we provide the governing equations for geophysical flows written in spherical coordinates to accommodate the shape of the Earth, together with the boundary conditions for the free surface and a rigid bed.

We will work in a system of right-handed coordinates \((r, \theta, \varphi)\) where \(r\) denotes the distance to the center of the ball, \(\theta \in [0, \pi]\) is the polar angle (the convention being that \(\pi/2 - \theta\) is the angle of latitude), and \(\varphi \in [0, 2\pi]\) is the azimuthal angle (angle of longitude). Although in this coordinate system, the North and South poles are located at \(\theta = 0\) and \(\theta = \pi\), respectively, the Equator sits on \(\theta = \pi/2\), the ACC is situated around \(\theta = 3\pi/4\). The unit vectors in this system are

\[
\begin{align*}
\mathbf{e}_r &= (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta), \\
\mathbf{e}_\theta &= (\cos \theta \cos \varphi, \cos \theta \sin \varphi, -\sin \theta), \\
\mathbf{e}_\varphi &= (-\sin \varphi, \cos \varphi, 0),
\end{align*}
\]

with \(\mathbf{e}_\varphi\) pointing from West to East and \(\mathbf{e}_\theta\) from North to South, cf. Figure 2.

Throughout this paper, we make the following simplifying assumption on ACC’s location. The angle of latitude \(\theta\) is assumed to lie in the compact interval \(I_\theta\):

\[
\theta \in I_\theta := \left[\frac{3\pi}{4} - \frac{\pi}{18}, \frac{3\pi}{4} + \frac{\pi}{18}\right].
\]

We are guided in our study by the observations made in Maslowe, asserting that the Reynolds number is, in general, extremely large for oceanic flows. Accordingly, we will consider incompressible and inviscid flows. For \(0 < r_2 < r_1 \ll R\) and \(R_j := R + r_j, j = 1, 2\), we consider the two fluid layers \(D_j\) separated by an interface and bounded by the bottom and a free surface, which are described by the graphs of the functions \(h, d,\) and \(k\), respectively, cf. Figure 3:
FIGURE 2  The spherical coordinate system: $\theta$ is the polar angle, $\varphi$ is the azimuthal angle (angle of longitude), and $r$ represents the distance to the origin.

FIGURE 3  A schematic illustration of the fluid domain in the flattened $(r, \theta)$-plane being bounded by the prescribed ocean bottom at $r = d$ and the free surface at $r = R_1 + k$. The fluid layers $D_1$ and $D_2$ are separated by the internal wave at $r = R_2 + h$. The boundaries of the intervals $(R_2 + h , R_2 + h)$ and $(R_1 + k, R_1 + k)$, which contain the interface and the free surface, respectively, as well as the deepest depth $d$ are indicated by dashed lines.
\[ D_1 := \{(r, \theta, \varphi) : R_2 + h(\theta, \varphi) < r < R_1 + k(\theta, \varphi)\}, \]
\[ D_2 := \{(r, \theta, \varphi) : d(\theta, \varphi) < r < R_2 + h(\theta, \varphi)\}. \]

We associate \( R \) with the Earth’s radius. The given function \( d \approx R \) describes the bottom topography, whereas \( h \) and \( k \) describe the unknown deviations of the interface and the free surface from their unperturbed locations at \( R_2 \) and \( R_1 \), respectively. We assume that \( h \) is restricted to some interval \((h_-, h_+)\) for all \((\theta, \varphi) \in I_\theta \times [0, 2\pi]\); likewise, \( k \in (k_-, k_+) \) by assumption. To rule out mixing of the two layers, we furthermore assume that \( d < R_2 + h_- \) and \( R_2 + h_+ < R_1 + k_- \). The prescribed density \( \rho \) is assumed to satisfy \( \rho (r, \theta, \varphi) = \rho (r) = \rho_j + \varepsilon(r) \) in \( D_j \) for positive constants \( \rho_1 < \rho_2 \), and a slight depth depending smooth variation \( \varepsilon : (d_- + R_1 + k_+) \to \mathbb{R} \), which satisfies \( \varepsilon \equiv 0 \) in \((R_2 + h_-, R_2 + h_+)\); here \( d_- := \min_{(\theta, \varphi) \in I_\theta \times [0, 2\pi]} d(\theta, \varphi) \). In particular, \( \rho \) is a discontinuous depth-dependent function with a jump of height \( \rho_2 - \rho_1 \) at the interface \( R_2 + h \). The function \( |\varepsilon(r)| < \rho_2 - \rho_1 \) accounts for comparably small density gradients away from the interface due to, for example, slight changes in salinity or temperature. By writing \( \rho_j(r) \), we indicate that \( r \) is taken from the layer \( D_j \), \( j = 1, 2 \), whereas \( \rho_j \) always refers to the beforementioned constants.

Let
\[ u_j := w_j e_r + v_j e_\theta + u_j e_\varphi, \quad j = 1, 2 \]
denote the velocity field within the fluid layer \( D_j \). The Euler equations for \( (w_j, v_j, u_j) \) in the rotating frame are given by
\[ w_{j,t} + w_j w_{j,r} + \frac{v_j}{r} w_{j,\theta} + \frac{u_j}{r \sin \theta} w_{j,\varphi} - \frac{v_j^2 + u_j^2}{r} - 2\Omega u_j \sin \theta - r\Omega^2 \sin^2 \theta = -\frac{p_{j,r}}{\rho} + \mathbf{B}_j^r, \]
\[ v_{j,t} + w_j v_{j,r} + \frac{v_j}{r} v_{j,\theta} + \frac{u_j}{r \sin \theta} v_{j,\varphi} + \frac{w_j v_j - u_j^2 \cot \theta}{r} - 2\Omega v_j \cos \theta - r\Omega^2 \sin \theta \cos \theta = -\frac{p_{j,\theta}}{r \rho} + \mathbf{B}_j^\theta, \]
\[ u_{j,t} + w_j u_{j,r} + \frac{v_j}{r} u_{j,\theta} + \frac{u_j}{r \sin \theta} u_{j,\varphi} + \frac{w_j u_j + v_j u_j \cot \theta}{r} + 2\Omega w_j \sin \theta + 2\Omega v_j \cos \theta = -\frac{p_{j,\varphi}}{r \rho \sin \theta} + \mathbf{B}_j^\varphi, \]
which incorporate both Coriolis effects and centripetal acceleration \((\Omega \approx 7.29 \times 10^{-5} \text{ rad s}^{-1} \) refers to the constant rotation speed of the Earth\), cf.\( ^{32} \). Here, \( p_j(r, \theta, \varphi) \) denotes the pressure field and \( \mathbf{B}_j = (\mathbf{B}_j^r e_r, \mathbf{B}_j^\theta e_\theta, \mathbf{B}_j^\varphi e_\varphi) \) is the prescribed body-force vector. Additionally to (8), the equation of mass conservation is supposed to be satisfied:
\[ \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \rho w_j) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\rho v_j \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \varphi} (\rho u_j) = 0. \]
The GFD governing equations (8) and (9) are supplemented with the following boundary conditions. At the free surface \( r = R_1 + k(\vartheta, \varphi) \), we require the dynamic boundary condition

\[ p_1 = P_1(\vartheta, \varphi) \]  

(for a prescribed function \( P_1 \)) and the kinematic boundary condition

\[ w_1 = \frac{v_1}{r} \frac{\partial k}{\partial \vartheta} + \frac{u_1}{r \sin \vartheta} \frac{\partial k}{\partial \varphi} \]  

(11)

to be satisfied. At the interface \( r = R_2 + h(\vartheta, \varphi) \), we require the normal components of the velocity fields \( u_j \) to be equal:

\[
(w_1 e_r + v_1 e_\vartheta + u_1 e_\varphi) \cdot \left( e_r - \frac{h_\vartheta}{r} e_\vartheta - \frac{h_\varphi}{r \sin \vartheta} e_\varphi \right) = (w_2 e_r + v_2 e_\vartheta + u_2 e_\varphi) \cdot \left( e_r - \frac{h_\vartheta}{r} e_\vartheta - \frac{h_\varphi}{r \sin \vartheta} e_\varphi \right).
\]  

(12)

Moreover, to ensure a balance of forces, we require that

\[ p_1 = p_2 \quad \text{on} \quad R_2 + h(\vartheta, \varphi). \]  

(13)

At the rigid ocean bottom \( r = d(\vartheta, \varphi) \), it holds that

\[ w_2 = \frac{v_2}{r} \frac{\partial d}{\partial \vartheta} + \frac{u_2}{r \sin \vartheta} \frac{\partial d}{\partial \varphi}. \]  

(14)

3 EXACT EXPLICIT AND IMPLICIT SOLUTIONS

We seek for a steady flow governed by (8) with \( \mathbf{f}_j(r, \vartheta) := (-g, G(r, \vartheta), 0) \), where \( g \) is the gravity of Earth and \( G \) denotes a general body force vector in \( \vartheta \)-direction, and (9) together with (10)–(14), which propagates purely in the azimuthal direction and does not depend on \( \varphi \). Therefore, the velocity field satisfies \( w_j = v_j = 0 \) and \( u_j = u_j(r, \vartheta) \), \( p_j = p_j(r, \vartheta) \), \( h = h(\vartheta) \), \( k = k(\vartheta) \), and for consistency \( d = d(\vartheta) \). Then (9) and (10)–(14) are automatically satisfied, while the Euler equations reduce to

\[
\begin{align*}
-u_j^2 & - \frac{u_j}{r} - 2\Omega u_j \sin \vartheta - r\Omega^2 \sin^2 \vartheta = -\frac{p_{j,r}}{\rho} - g, \\
-\frac{u_j^2}{r} \cot \vartheta & - 2\Omega u_j \cos \vartheta - r\Omega^2 \sin \vartheta \cos \vartheta = -\frac{p_{j,\vartheta}}{\rho r} + G(r, \vartheta), \\
0 & = p_{j,\varphi},
\end{align*}
\]  

(15)
which can be rewritten as

\[
\begin{align*}
\rho \left( u_j + \Omega r \sin \theta \right)^2 &= p_{j,r} + g \rho \\
\rho (u_j + \Omega r \sin \theta)^2 \cot \theta &= p_{j,\theta} - \rho r G(r, \theta).
\end{align*}
\]  

(16)

Thus, \( p \) can be eliminated, and the flow \((0, 0, u_j)\) satisfies

\[
\partial_\theta \left( \frac{\rho(r)(u_j(r, \theta) + \Omega r \sin \theta)}{r} \right)^2 = \partial_r \left( \rho(r) \left[ (u_j(r, \theta) + \Omega r \sin \theta)^2 \cot \theta + r G(r, \theta) \right] \right)
\]  

(17)

in \( D_j, j = 1, 2 \). This equation can be explicitly solved by employing the method of characteristics (cf. 48):

\[
u_j(r, \theta) = -\Omega r \sin \theta + \sqrt{\frac{F_j(r \sin \theta) + r \sin \theta \int_0^{f(\theta)} H_{j,r}(\tilde{r}(s), \tilde{\theta}(s)) \, ds}{\rho_j(r)}},
\]

(18)

where \( x \mapsto F_j(x), j = 1, 2 \), are arbitrary continuously differentiable functions and

\[
f(\theta) := \frac{1}{2} \log \frac{1 - \cos \theta}{1 + \cos \theta}, \quad H_j(r, \theta) := r \rho_j(r) G(r, \theta)|_{r \in D_j},
\]

(19)

\[
\tilde{r}(s) := r \sin \theta e^s + e^{-s} \frac{2}{2}, \quad \tilde{\theta}(s) := \arccos \frac{1 - e^{2s}}{1 + e^{2s}}.
\]

Plugging (18) into (16) yields that

\[
p_{j,r}(r, \theta) = -g \rho_j(r) + \frac{F_j(r \sin \theta) + r \sin \theta \int_0^{f(\theta)} H_{j,r}(\tilde{r}(s), \tilde{\theta}(s)) \, ds}{r},
\]

(20)

\[
p_{j,\theta}(r, \theta) = H_j(r, \theta) + \cot \theta \left[ F_j(r \sin \theta) + r \sin \theta \int_0^{f(\theta)} H_{j,r}(\tilde{r}(s), \tilde{\theta}(s)) \, ds \right].
\]

(21)

Integrating (20) in the lower layer \((j = 2)\) with respect to \( r \) yields that

\[
p_2(r, \theta) = \int_{d(\theta) \sin \theta}^{r \sin \theta} \left[ \frac{F_2(y)}{y} + L_2(y, \theta) \right] dy - g \int_{d(\theta)}^{r} \rho_2(r') \, dr' + f_2(\theta),
\]

(22)

where for \( j = 1, 2 \), we set

\[
L_j(y, \theta) := \int_0^{f(\theta)} H_{j,r} \left( \frac{y e^s + e^{-s}}{2}, \tilde{\theta}(s) \right) \, ds,
\]

(23)
which by (19) satisfies
\[ L_{j,\theta}(y, \theta) = \csc \theta H_{j,r}(y \csc \theta, \theta), \]  

thus
\[ \int_{d(\theta) \sin \theta}^{r \sin \theta} L_{j,\theta}(y, \theta) \, dy = \int_{d(\theta)}^{r} H_{j,r}(r', \theta) \, dr', \]  

and therefore, \( f_2 \) is given by
\[ f_2(\theta) := \int_{d(\theta)}^{d(3\pi/4)} \left[ \frac{F_2(y)}{y} + L_2(y, \theta) \right] \, dy \]
\[ + \int_{3\pi/4}^{\theta} H_2 \left( \frac{d(3\pi/4)}{\sin \theta'}, \theta' \right) \, d\theta' - g \int_{d(3\pi/4)}^{d(\theta)} \rho_2(r') \, dr'. \]  

In the upper layer, we obtain
\[ p_1(h, r, \theta) = \int_{(R_2 + h(\theta)) \sin \theta}^{r \sin \theta} \left[ \frac{F_1(y)}{y} + L_1(y, \theta) \right] \, dy - g \int_{R_2 + h(\theta)}^{r} \rho_1(r') \, dr' + f_1(h, \theta). \]  

for the function
\[ f_1(h, \theta) := \]
\[ \int_{3\pi/4}^{\theta} L_1 \left( (R_2 + h(\theta')) \sin \theta', \theta' \right) (h'(\theta') \sin \theta' + (R_2 + h(\theta')) \cos \theta') \, d\theta' \]
\[ + \int_{3\pi/4}^{\theta} H_1 \left( R_2 + h(\theta'), \theta' \right) \, d\theta' - g \int_{3\pi/4}^{\theta} \rho_1(R_2 + h(\theta')) h'(\theta') \, d\theta' \]
\[ + \int_{3\pi/4}^{\theta} F_1 \left( (R_2 + h(\theta')) \sin \theta' \right) \left[ \cot \theta' + \frac{h'(\theta')}{R_2 + h(\theta')} \right] \, d\theta'. \]  

At this point, we note that without loss of generality, it can be assumed that
\[ h\left(\frac{3\pi}{4}\right) = 0; \]  

otherwise, one may consider \( \tilde{h}(\theta) := h(\theta) - h(3\pi/4) \). Then (27)–(28) hold true for \( \tilde{h} \) and \( \tilde{h}(3\pi/4) = 0 \).

From (10) and (27), we obtain that
\[ P_1(\theta) = \int_{(R_2 + h(\theta)) \sin \theta}^{(R_1+k(\theta)) \sin \theta} \left[ \frac{F_1(y)}{y} + L_1(y, \theta) \right] \, dy - g \int_{R_2 + h(\theta)}^{R_1 + k(\theta)} \rho_1(r') \, dr' + f_1(h, \theta). \]
Letting $P^0_1$ denote the surface pressure related to the undisturbed interface and free surface (i.e., $h = k \equiv 0$), Equation (30) yields that

$$P^0_1(\theta) = \int_{R_2 \sin \theta}^{R_1 \sin \theta} \left[ \frac{F_1(y)}{y} + L_1(y, \theta) \right] dy - g \int_{R_2}^{R_1} \rho_1(r') dr'$$

$$+ \int_{\frac{3\pi}{4}}^{\theta} \left[ F_1(R_2 \sin \theta') \cot \theta' + H_1(R_2, \theta') + L_1(R_2 \sin \theta', \theta') R_2 \cos \theta' \right] d\theta'. \quad (31)$$

**Assumption 1.** The average azimuthal velocity component of the ACC (in the Eastern direction) is nonnegative and does not exceed $1 \text{ m s}^{-1}$. Measurements concerning the density distribution and internal wave energy in the ACC, the latter peaks at depths between 1000 and 1500 m, suggest a density increase of $0.2–0.5 \text{ kg m}^{-3}$ in the region of internal waves, see Ref. 15. Therefore, we make the following assumptions on $u$ and $\rho$.

(i) The azimuthal velocity $u$ satisfies

$$0 \leq u \leq 1 \text{ m s}^{-1} \quad \text{in} \quad \overline{D_1} \cup \overline{D_2}. \quad (32)$$

(ii) In the region of the interface at $1000 \text{ m} \leq R_1 - R_2 \leq 1500 \text{ m}$ depth, the prescribed background density satisfies

$$\rho_1 \approx 1026 \text{ kg m}^{-3}, \quad \rho_2 = \rho_1(1 + \sigma), \quad 0.2 \text{ kg m}^{-3} \leq \sigma \rho_1 \leq 0.5 \text{ kg m}^{-3}. \quad (33)$$

**Remark 1.** From the assumptions (i)–(ii), the fact that $R_1 + h \approx 6.371 \text{ km}$ and $\Omega \approx 7.29 \times 10^{-5} \text{ rad s}^{-1}$, we infer strict positivity of the functions

$$(r, \theta) \mapsto F_j(r \sin \theta) + r \sin \theta \int_0^{f(\theta)} H_{1,r}(\tilde{\rho}(s), \tilde{\theta}(s)) \, ds \quad (34)$$

in (18) for all $\theta \in I_\theta$ and $r$ within the respective fluid layer. Particularly, the square root in (18) is well defined.

**Remark 2.** The nonlinear advection terms in (15) are small perturbations of the linear flow $u_0$ governed by

$$-2\Omega u_0 \sin \theta = -\frac{1}{r} \tilde{p}_r,$$

$$-2\Omega u_0 \cos \theta = -\frac{1}{r \rho} \tilde{p}_\theta + G(r, \theta), \quad (35)$$

where $\tilde{p} = p + \rho gr - \frac{1}{2} \rho r^2 \Omega^2 \sin^2 \theta$ absorbs the centripetal and gravitational acceleration; cf. the discussion in Constantin and Johnson.\(^{33}\) As $\frac{1}{r \rho} \tilde{p}_\theta$ is relatively small, we infer from (35) in combination with Assumption 1 that

$$G(r, \theta) \approx -2\Omega u_0 \cos \theta \geq 0. \quad (36)$$
3.1 Implicit description of interface and surface

In the next step, we employ dimensionless variables to obtain an implicit formulation for the interface. Let

\[ h(\theta) := \frac{h(\theta)}{R_2}, \]  

and consider the nonlinear operator \( G \) on the space of dimensionless interfaces \( h \) defined by

\[ G(h)(\theta) := \frac{p_2((1 + h(\theta))R_2, \theta) - f_1(h, \theta)}{P_{atm}}. \]  

(38)

Noting that \( f_1(h, \theta) = p_1((1 + h(\theta))R_2, \theta) \) by (27) and (28), where \( f(h, \theta) \) is expressed in terms of \( h \) according to (37), we find that

(13) is satisfied if and only if \( G(h) = 0 \).

(39)

To infer an implicit formulation for the free surface, we set

\[ k(\theta) := \frac{k(\theta)}{R_1}, \quad p_1(\theta) := \frac{P_1(\theta)}{P_{atm}}, \]  

and define

\[ F(h, k, p_1)(\theta) := \frac{p_1(h, (1 + k(\theta))R_1, \theta)}{P_{atm}} - p_1(\theta), \]  

(41)

where \( p_1(h, \cdot, \cdot) \) is expressed by means of \( h \) via (37). Then

(10) is satisfied if and only if \( F(h, k, p_1) = 0 \),

(42)

and by setting \( P_1^0 := P_{atm}^{-1}P_1^0 \), we infer from (31) and (41) that

\[ F(0, 0, P_1^0) = 0. \]

(43)

Generally, the dimensionless interface \( \bar{h} \) and surface \( \bar{k} \) are implicitly given by the abstract operator equation

\[ (G(\bar{h}), F(\bar{h}, \bar{k}, p_1)) = 0. \]

(44)

We aim to solve (44) locally around the undisturbed state with the help of the implicit function theorem. For this purpose, we calculate the derivatives of \( G \) and \( F \) at 0 and \((0, 0, P_1^0)\), respectively, with respect to both \( \bar{h} \) and \( \bar{k} \). As \( G \) does not depend on \( \bar{k} \), it holds that \( G_{\bar{k}}(\bar{h}) = 0 \) for arbitrary \( \bar{h} \). Furthermore, a direct calculation using (24) and (29) confirms that

\[ P_{atm} \lim_{s \to 0} \frac{G(s\bar{h}) - G(0)(\theta)}{s} = (F_2(R_2 \sin \theta) - F_1(R_2 \sin \theta) - gR_2(\rho_2 - \rho_1) \]

\[ + [L_2(R_2 \sin \theta, \theta) - L_1(R_2 \sin \theta, \theta)]R_2 \sin \theta\bar{h}(\theta), \]

(45)
hence,

\[
(G_n(0)\hat{h})(\theta) = P_{\text{atm}}^{-1}(F_2(R_2 \sin \theta) - F_1(R_2 \sin \theta) - gR_2(\rho_2 - \rho_1) + [L_2(R_2 \sin \theta, \theta) - L_1(R_2 \sin \theta, \theta)]R_2 \sin \theta)\hat{h}(\theta).
\]

(46)

**Remark 3.** For \( j = 1, 2 \), let \( \tilde{F}_j(\theta) := F_j(R_2 \sin \theta) \), \( \tilde{L}_j := L_j(R_2 \sin \theta, \theta)R_2 \sin \theta \) and \( \tilde{u}_j(\theta) := u_j(R_2, \theta) \). Due to (18) and Assumption 1, we infer that

\[
(\tilde{F}_2 - \tilde{F}_1 + \tilde{L}_2 - \tilde{L}_1)(\theta) = \rho_1((u_2^2 - u_1^2)(\theta) + 2\Omega R_2(\tilde{u}_2 - \tilde{u}_1)(\theta) \sin \theta) + \sigma \rho_1(\tilde{u}_2(\theta) + \Omega R_2)^2 \leq R_2 \left( \frac{\rho_1 \tilde{u}_2^2(\theta)}{R_2} + 2\Omega \tilde{u}_2(\theta) + \sigma \rho_1 \left( \tilde{u}_2(\theta) + \Omega R_2 \right)^2 \right) < 0.02R_2 \text{ kg m}^{-1} \text{ s}^{-2}
\]

(47)

for all \( \theta \in I_\theta \). On the other hand,

\[
gR_2(\rho_2 - \rho_1) = gR_2\sigma \rho_1 > 1.9R_2 \text{ kg m}^{-1} \text{ s}^{-2}.
\]

(48)

Consequently, there exists a constant \( \alpha < 0 \) such that for all \( \theta \in I_\theta \),

\[
(\tilde{F}_2 - \tilde{F}_1 + \tilde{L}_2 - \tilde{L}_1)(\theta) - gR_2(\rho_2 - \rho_1) \leq \alpha.
\]

(49)

Let \( C(I_\theta) \) denote the Banach space of continuous functions on the compact interval \( I_\theta \) equipped with the supremum norm \( \| \cdot \|_{\infty} \), that is, \( \| f \|_{\infty} := \sup_{\theta \in I_\theta} |f(\theta)| \); cf. the definition of \( I_\theta \) in (4). Employing Remark 3 yields that the operator norm of \( G_n(0) : C(I_\theta) \to C(I_\theta) \) satisfies

\[
\| G_n(0) \| \geq |\alpha| P_{\text{atm}}^{-1} > 0,
\]

(50)

hence \( G_n(0) \) induces a linear topological automorphism of \( C(I_\theta) \). Likewise, \( G_n(0) \) induces a linear topological automorphism of the Banach space

\[
C^1(I_\theta) := \{ f : I_\theta \to \mathbb{R} \mid f \text{ is continuously differentiable and } \| f \|_{C^1(I_\theta)} < \infty \},
\]

(51)

which is equipped with the norm \( \| \cdot \|_{C^1(I_\theta)} \) given by \( \| f \|_{C^1(I_\theta)} := \| f \|_{\infty} + \| f' \|_{\infty} \).

Next, we compute for \( \theta \in I_\theta \) that

\[
P_{\text{atm}}(F_n(0, 0, P_1^0)\hat{h})(\theta)
= \lim_{s \to 0} \frac{p_1(sh, R_1, \theta) - p_1(0, R_1, \theta)}{s}
= (F_1(R_2 \sin \theta) + L_1(R_2 \sin \theta, \theta)R_2 \sin \theta - g\rho_1R_2
- F_1(R_2 \sin \theta) - L_1(R_2 \sin \theta, \theta)R_2 \sin \theta + g\rho_1R_2)\hat{h}(\theta) = 0.
\]

(52)
Furthermore, for all \(\theta \in \mathcal{I}_\theta\),

\[
P_{\text{atm}}(F_h(0, 0, P_1^0)k)(\theta) = \lim_{s \to 0} \frac{P_1(0, (1 + sk)R_1, \theta) - P_1(0, R_1, \theta)}{s}
= (F_1(R_1 \sin \theta) + L_1(R_1 \sin \theta, \theta)R_1 \sin \theta - gR_1 \rho(R_1))h(\theta).
\] (53)

**Remark 4.** With the help of (18) and Assumption 1, we obtain that

\[
F_1(R_1 \sin \theta) + L_1(R_1 \sin \theta, \theta)R_1 \sin \theta - gR_1 \rho(R_1)
= \rho(R_1)\left[(u_1(R_1, \theta) + \Omega R_1 \sin \theta)^2 - gR_1\right]
\leq \rho(R_1)\left[(u_1(R_1, \theta) + \Omega R_1)^2 - gR_1\right]
< -\rho(R_1) \cdot 6 \cdot 10^7 \text{ kg m}^{-1} \text{ s}^{-2}
\] (54)

for all \(\theta \in \mathcal{I}_\theta\).

Remark 4 yields a lower bound for the operator norm of \(F_h(0, 0, P_1^0) : C(I_\theta) \to C(I_\theta)\):

\[
\|F_h(0, 0, P_1^0)\| \geq \beta > 0
\] (55)

for some \(\beta > 0\). Hence, \(F_h(0, 0, P_1^0)\) is a linear topological automorphism of \(C(I_\theta)\).

We conclude that

\[
(G, F)_{(\theta, k)}(0, 0, P_1^0) = \begin{pmatrix}
G_h(0) & G_k(0) \\
F_h(0, 0, P_1^0) & F_k(0, 0, P_1^0)
\end{pmatrix} = \begin{pmatrix}
G_h(0) & 0 \\
0 & F_h(0, 0, P_1^0)
\end{pmatrix}
\] (56)

constitutes a linear topological automorphism of \(C^1(I_\theta) \times C(I_\theta)\). An application of the implicit function theorem yields the following result.

**Theorem 1.** Let Assumption 1 be satisfied and let \(F, G\) be continuously differentiable in each fluid layer. Then for every small enough perturbation \(P_1 \in C(I_\theta)\) of \(P_1^0\), there exist unique functions \(\tilde{h} \in C^1(I_\theta)\) and \(\tilde{k} \in C(I_\theta)\) obeying (44). In this case, \(P_1\) equals the dimensionless version of (30). Furthermore, there exists a unique continuously differentiable implicit map \(\mathbb{I} : P_1 \mapsto (\tilde{h}, \tilde{k})\) defined on a local neighborhood of \(P_1^0\) such that

\[
(\tilde{h}, \tilde{k}) = \mathbb{I}(P_1) \text{ if and only if } (44) \text{ holds.}
\] (57)

If \(F, G\) are \(n \geq 2\) times continuously differentiable or infinitely differentiable, then the local map \(\mathbb{I}\) is \(C^n\) or \(C^\infty\), respectively.

### 4 QUALITATIVE ANALYSIS OF SOLUTIONS

This final section presents qualitative results for the interface and the free surface, and a specific example allowing for an explicit description of the interface.
Theorem 2 (Smoothness of the interface). Let Assumption 1 be satisfied, let $F, G$ be infinitely differentiable in each fluid layer, and according to Theorem 1, let $h \in C^1(I_0)$ be the interface corresponding to a given small enough perturbation $P_1$ of $P_0$. Then $h \in C^\infty(I_0)$; if $F, G$ are $n$ times continuously differentiable for $n \geq 1$, then $h \in C^{n+1}(I_0)$.

Proof. By Theorem (1), it holds that $(Q(h))(\theta) = 0$ for all $\theta \in I_0$. Hence, via differentiation with respect to $\theta$, we infer that

$$(A_1(\theta) + A_2)h'(\theta) + B(\theta) = 0$$

for

$$A_1(\theta) := \frac{\bar{F}_2(\theta) - \bar{F}_1(\theta) + \bar{L}_2(\theta) - \bar{L}_1(\theta)}{1 + h(\theta)},$$

$$A_2 := -gR_2(\rho_2 - \rho_1),$$

$$B(\theta) := [\bar{F}_2(\theta) - \bar{F}_1(\theta) + \bar{L}_2(\theta) - \bar{L}_1(\theta)] \cot \theta + \bar{H}_2(\theta) - \bar{H}_1(\theta),$$

$$F_j(\theta) := F_j((1 + h(\theta))R_2 \sin \theta),$$

$$\bar{H}_j(\theta) := H_j((1 + h(\theta))R_2, \theta),$$

$$\bar{L}_j(\theta) := L_j((1 + h(\theta))R_2 \sin \theta, \theta)(1 + h(\theta))R_2 \sin \theta.$$  

As $F$ and $G$ are $C^\infty$, respectively, $C^n$, both $A := A_1 + A_2$ and $B$ inherit the regularity from $h$, that is, $A, B \in C^1(I_0)$. Below we show that

$$A(\theta) \leq \gamma \quad \text{for all} \quad \theta \in I_0 \quad \text{for some} \quad \gamma < 0,$$

from which we deduce that $h' = -B/A \in C^1(I_0)$, hence $h \in C^2(I_0)$. By induction, it follows that $h \in C^\infty(I_0)$, respectively, $h \in C^{n+1}(I_0)$.

To see that $A(\theta) < 0$ for all $\theta \in I_0$, we first recall from (48), which is based on Assumption 1, that

$$A_2 = -gR_2\sigma\rho_1 < -1.9 \text{ kg m}^{-1} \text{ s}^{-2}.$$  


Furthermore, by setting $\bar{u}_j(\theta) := u_j((1 + h(\theta))R_2, \theta)$, an application of (18) and Assumption 1 yields—similarly as in Remark 3—the following estimate:

$$A_1(\theta) = \rho_1 \left( \frac{(\bar{u}_2^2 - \bar{u}_1^2)(\theta) + 2\Omega R_2 (1 + h(\theta)) (\bar{u}_2 - \bar{u}_1)(\theta) \sin \theta}{1 + h(\theta)} + \frac{\rho_1 \sigma (\bar{u}_2(\theta) + \Omega R_2 (1 + h(\theta)) \sin \theta)^2}{1 + h(\theta)} \right) \leq 2R_2 \left( \frac{\rho_1 \bar{u}_2(\theta)}{R_2} + 2\Omega R_2 \bar{u}_2(\theta) + \frac{\rho_1 \sigma (\bar{u}_2(\theta) + \Omega R_2)^2}{R_2} \right)$$

$$< 0.04R_2 \text{ kg m}^{-1} \text{s}^{-2}.$$

Comparing (66) and (67) shows (65), which finishes the proof.

**Example 1.** We present now an explicit solution (in terms of the velocity field, the pressure, and the free surface and the interface) for an a priori given explicit density. More precisely, the scenario that we consider assumes that the density $\rho$ is as follows: throughout the bottom layer $\{(r, \theta, \varphi) : d(\theta) \leq r \leq R_2 + h(\theta)\}$, the density is constant and of the size indicated in Assumption 1. For the upper layer $\{(r, \theta, \varphi) : R_2 + h(\theta) \leq r \leq R_1 + k(\theta)\}$, we set

$$\rho_1(r) := \rho_2 - a_1 r,$$  \hspace{1cm} (68)

where $a_1$ is a constant such that it allows for the density as indicated in Assumption 1. Consequently, recalling the notation from the proof of Theorem 2, we have that

$$A_2 = -gR_2 (\rho_2 (R_2 + h(\theta)) - \rho_1 (R_2 + h(\theta))) = -g a_1 R_2^2 (1 + h(\theta)).$$  \hspace{1cm} (69)

To indicate the velocity field, we will particularize the functions $F_1$ and $F_2$ from (18). That is, we set

$$F_1(x) := \alpha_1 \rho_2 \Omega^2 x^2, \quad F_2(x) := \alpha_2 \rho_2 \Omega^2 x^2,$$  \hspace{1cm} (70)

where $\alpha_1, \alpha_2$ are dimensionless constants satisfying $\alpha_2 > \alpha_1$. Hence,

$$F_2(\theta) - F_1(\theta) = (\alpha_2 - \alpha_1) \rho_2 \Omega^2 R_2^2 (1 + h(\theta))^2 \sin^2 \theta.$$  \hspace{1cm} (71)

Moreover, we need to specify the functions $H_1, H_2$ given by means of formula (19). To this end, we assume the forcing term $G$ to be given as in formula (36). Therefore,

$$H_1(r, \theta) = -2\Omega u_0 (\rho_2 r - a_1 r^2) \cos \theta, \quad H_2(r, \theta) = -2\Omega u_0 \rho_2 r \cos \theta,$$  \hspace{1cm} (72)

and

$$\bar{H}_2(\theta) - \bar{H}_1(\theta) = -2 a_1 \Omega u_0 R_2^2 (1 + h(\theta))^2 \cos \theta.$$  \hspace{1cm} (73)
Utilizing now formula (23), we find
\[ L_2(r \sin \theta, \theta) - L_1(r \sin \theta, \theta) = -4a_1 \Omega u_0 r (\sin \theta - 1) \] (74)
and thus,
\[ \bar{L}_2(\theta) - \bar{L}_1(\theta) = -4a_1 \Omega u_0 R_2^2 (1 + \bar{h}(\theta))^2 \sin \theta (\sin \theta - 1). \] (75)
Hence, availing of the previous formulas, we find that Equation (58) becomes
\[
\begin{align*}
\left[(\alpha_2 - \alpha_1) \rho_2 \Omega^2 \sin \theta \cos \theta - 4a_1 u_0 \Omega \cos \theta (\sin \theta - 1) - 2a_1 u_0 \Omega \cos \theta \right] (1 + \bar{h}(\theta)) \\
+ \left[(\alpha_2 - \alpha_1) \rho_2 \Omega^2 \sin^2 \theta - 4a_1 u_0 \Omega \sin \theta (\sin \theta - 1) - ga_1 \right] \bar{h}'(\theta) = 0,
\end{align*}
\] (76)
which can be written as
\[ \frac{\bar{h}'(\theta)}{(1 + \bar{h}(\theta))} = -\frac{E(\theta)}{E(\theta)}, \] (77)
where
\[
\begin{align*}
E(\theta) &:= (\alpha_2 - \alpha_1) \rho_2 \Omega^2 \sin \theta \cos \theta - 4a_1 u_0 \Omega \cos \theta (\sin \theta - 1) - 2a_1 u_0 \Omega \cos \theta, \\
E(\theta) &:= (\alpha_2 - \alpha_1) \rho_2 \Omega^2 \sin^2 \theta - 4a_1 u_0 \Omega \sin \theta (\sin \theta - 1) - ga_1.
\end{align*}
\] (78)
Remarking that \( E'(\theta) = 2E'(\theta) \), we obtain from (77) that
\[ 1 + \bar{h}(\theta) = \sqrt{\frac{E(3\pi/4)}{E(\theta)}}. \] (79)

**Theorem 3** (Monotonicity relations). Let Assumption 1 be satisfied, let \( F, G \) be continuously differentiable in each fluid layer, and according to Theorem 1, let \( \bar{h} \in C(I_\theta) \) be the free surface corresponding to a given small enough perturbation \( P_1 \in C^1(I_\theta) \) of \( P_1^0 \). Then, in fact, \( \bar{h} \) belongs to \( C^1(I_\theta) \) and for arbitrary \( \hat{\theta} \in I_\theta \), it holds that
\[ P_1'(\hat{\theta}) < 0 \quad \text{if} \quad \bar{h}'(\hat{\theta}) \geq 0 \] (80)
and
\[ \bar{h}'(\hat{\theta}) < 0 \quad \text{if} \quad P_1'(\hat{\theta}) \geq 0. \] (81)

**Proof.** First of all, the assertion of Theorem 1 can easily be restricted to setting \((\bar{h}, \bar{h}, P_1) \in C^1(I_\theta) \times C^1(I_\theta) \times C^1(I_\theta)\), because in this case, \((G, F)_{(\bar{h}, \bar{h}, P_1)}(0, 0, P_1^0)\) constitutes a linear topological automorphism of \( C^1(I_\theta) \times C^1(I_\theta) \). Now it is ensured that the equation \( F(\bar{h}, \bar{h}, P_1) = 0 \) with \( F \) being
given by (41) can be differentiated with respect to \( \theta \), which under the application of (18) results in
\[
P_{\text{atm}} \beta_1^\prime(\theta) = \left[ \frac{(\bar{u}_1(\theta) + \Omega R_1(1 + k(\theta)) \sin \theta)^2}{1 + k(\theta)} - g R_1 \right] \rho (R_1(1 + k(\theta))) k' (\theta) + B(\theta) \tag{82}
\]
for \( \theta \in I_\theta \), where \( \bar{u}_1(\theta) := u_1((1 + k(\theta)) R_1, \theta) \) and
\[
B(\theta) := L_1 R_1(1 + k(\theta)) \sin \theta, \theta (1 + k(\theta)) R_1 \cos \theta \tag{83}
\]
\[
+ F_1 (R_1(1 + k(\theta)) \sin \theta) \cot \theta + H_1 (R_1(1 + k(\theta)), \theta).
\]

Similarly as in the proof of Theorem 2 and thanks to Assumption 1, there exists a constant \( \eta < 0 \) such that for all \( \theta \in I_\theta \),
\[
\frac{(\bar{u}_1(\theta) + \Omega R_1(1 + k(\theta)) \sin \theta)^2}{1 + k(\theta)} - g R_1 \leq \eta. \tag{84}
\]

Consequently, we infer (80) and (81) from (82) in combination with (84) and the Remarks 1 and 2 that \( B(\theta) \) is strictly positive for all \( \theta \in I_\theta \).

**ACKNOWLEDGMENTS**

The authors want to thank an anonymous referee for valuable comments and suggestions. C. I. Martin would like to acknowledge the support of the Austrian Science Fund (FWF), grant P 33107-N; R. Quirchmayr acknowledges the support of Vienna Science and Technology Fund (WWTF), grant MA16-009, and the FWF grant J 4339-N32.

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**How to cite this article:** Martin CI, Quirchmayr R. Exact solutions and internal waves for the Antarctic Circumpolar Current in spherical coordinates. *Stud Appl Math*. 2022;148:1021–1039. https://doi.org/10.1111/sapm.12467