Polynomial interpolation of modular forms for
Hecke groups

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Abstract
For $m = 3, 4, ...$, let $\lambda_m = 2 \cos \pi / m$ and let $J_m(m = 3, 4, ...)$ be triangle functions for the Hecke groups $G(\lambda_m)$ with Fourier expansions $J_m(\tau) = \Sigma_{n=-1}^{\infty} a_n(m) q^n$, where $q_m(\tau) = \exp 2 \pi i \tau / \lambda_m$. (When normalized appropriately, $J_3$ becomes Klein’s $j$-invariant $j(\tau) = 1 / \theta^{2 \pi i} + 744 + ...$)
For $n = -1, 0, 1, 2$ and 3, Raleigh gave polynomials $P_n(x)$ such that $a_{-1}(m)^n q_{m+2} a_n(m) = P_n(m)$ for $m = 3, 4, ...$, and conjectured that similar relations hold for all positive integers $n$. This was proved by Akiyama.
We apply work of Hecke to study experimentally similar polynomial interpolations of the $J_m$ Fourier coefficients and the Fourier coefficients of other, positive weight, modular forms for $G(\lambda_m)$. We connect these polynomials (again, only empirically) with variants of Dedekind’s eta function, with the Fourier expansions of some standard Hauptmoduln, and, in the case of analogues of Eisenstein series for $\text{SL}(2, \mathbb{Z})$, with certain divisor sums.

1 Introduction

1.1 An example
Here is an example of a sequence $\{P_n(x)\}$ from $\mathbb{Q}[x]$ and a corresponding sequence of modular forms $\{f_m(x)\}$ having the relationship we examine in this article. Let $T_m :=$ the cyclic subgroup of $\text{SL}(2, \mathbb{R})$ generated by
$$
\begin{pmatrix}
1 & 2\pi / m \\
0 & 1
\end{pmatrix},
$$
let $f_m(x) := \sin(mx)$, and let
$$
Q_n(x) := (-1)^{(n-1)/2} x^n / n!.
$$
Furthermore let $P_n(x) = Q_n(x)$ if $n$ is odd and $P_n(x) = 0$ if $n$ is even. Members of $\text{SL}(2, \mathbb{R})$ act on $\mathbb{R}$ as follows. If
$$
M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
$$

and $x$ is real, we set

$$M(x) := \frac{ax + b}{cx + d}.$$ 

Thus the $T_m$ act on $\mathbb{R}$ by translation. From the periodicity and Taylor series of sine, we know that $f_m(x)$ is invariant (weight-0 modular) with respect to to the action of the $T_m$ and equal to $\sum_{n=0}^{\infty} P_n(m)x^n$. We say that the elements of $(f_m)$ are interpolated by the sequence of polynomials $(P_n(x))_{n=0,1,...}$.

1.2 First sketch of the background.

Let $\mathbb{Z}, \mathbb{Q}, \mathbb{C}$ and $\mathbb{H}$ denote, respectively, the set of rational integers, the set of rational numbers, the set of complex numbers, and the set of complex numbers with positive imaginary parts. (We will reserve the letter $\tau$ for elements of the upper half-plane, and $z$ for generic complex numbers.) We write $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{i\infty\}$, and we equip $\mathbb{H}^*$ with the Poincaré metric. Figures $T$ made by three geodesics of $\mathbb{H}^*$ are called hyperbolic or circular-arc triangles. Let $\lambda_m = 2\cos \pi/m$. For $m = 3, 4, ..., $ we define the Hecke group $G(\lambda_m)$ as the discrete group generated by the maps $z \to -1/z$ and $z \to z + \lambda_m$. The full modular group $SL(2, \mathbb{Z})$ is identical to $G(\lambda_3)$.

To define modular forms for the Hecke groups, we preview a definition from Berndt [7], which we will quote again in a later section. (We depart occasionally from Berndt’s choices of variable to avoid clashes with some of our other notation.)

We say that $f$ belongs to the space $M(\lambda, k, \gamma)$ if

1. $f(\tau) = \sum_{n=0}^{\infty} a_n e^{2\pi in\tau/\lambda},$

where $\lambda > 0$ and $\tau \in \mathbb{H}$, and

2. $f(-1/\tau) = \gamma(\tau/i)^k f(\tau)$, where $k > 0$ and $\gamma = \pm 1$.

We say that $f$ belongs to the space $M_0(\lambda, k, \gamma)$ if $f$ satisfies conditions 1 and 2 and if $a_n = O(n^c)$ for some real number $c$, as $n$ tends to $\infty$.

Members of $M(\lambda, k, \gamma)$ are known as modular forms for $G(\lambda)$ of weight $k$. Condition 1 tells us that they are invariant under translations $\tau \mapsto \tau + \lambda$. Next we preview Berndt’s definition of cusp forms for Hecke groups. If $f \in M(\lambda, k, \gamma)$ and $f(i\infty) = 0$, then we call $f$ a cusp form of weight $k$ and multiplier $\gamma$ with respect to $G(\lambda)$. For cusp forms, the constant terms of condition 1 vanish. We denote by $C(\lambda, k, \gamma)$ the vector space of all cusp forms of this kind.

For our purposes, Schwarz triangles $T$ are hyperbolic triangles in $\mathbb{H}^*$ with certain restrictions on the angles at the vertices. From a Euclidean point of view,
Let finite. Recall that \( G \) is \( \pi \)-function the hyperbolic triangle with vertices \( z \) and \( \lambda \). This means that for compact \( 1 \) there is a conformal, onto map \( \phi \) these transformations is a triangle group. By the Riemann Mapping Theorem is associated to a collection of Möbius transformations. The group generated by we get another Schwarz triangle. The reflection between two triangles in \( H^* \) is effected by a Möbius transformation, so the orbit of \( T \) under repeated reflections is associated to a collection of Möbius transformations. The group generated by these transformations is a triangle group. By the Riemann Mapping Theorem there is a conformal, onto map \( \phi : T \rightarrow H^* \) called a triangle function.

Hecke groups are triangle groups \( H \) that act properly discontinuously on \( H^* \). This means that for compact \( K \subset \mathbb{H} \), the set \( \{ \mu \in H \text{ s.t. } K \cap \mu(K) \neq \emptyset \} \) is finite. Recall that \( G(\lambda_m) \) is the Hecke group generated by the maps \( z \mapsto -1/z \) and \( z \mapsto z + \lambda_m \). Hecke established in \([23]\) that \( G(\lambda_m) \) has the structure of a free product of cyclic groups \( C_2 \ast C_m \), generalizing the relation \([39, 13]\) \( SL(2, \mathbb{Z}) = C_2 \ast C_3 \).

Let \( \rho = -\exp(-\pi i/m) = -\cos(\pi/m) + i\sin(\pi/m) \), and let \( T_m \subset \mathbb{H}^* \) denote the hyperbolic triangle with vertices \( \rho, i, \) and \( i\infty \). The corresponding angles are \( \pi/m, \pi/2 \) and \( 0 \) respectively. Let \( \phi_{\lambda_m} \) be a triangle function for \( T_m \). The function \( \phi_{\lambda_m} \) has a pole at \( i\infty \) and period \( \lambda_m \). For \( P, Q \in \mathbb{H}^* \), let us write \( P \equiv_H Q \) when \( \mu \in H \) and \( Q = \mu(P) \). Then \( \phi_{\lambda_m} \) extends to a function \( J_m : \mathbb{H}^* \rightarrow \mathbb{H}^* \) by declaring that \( J_m(P) = J_m(Q) \) if and only if \( P \equiv_H Q \). \( J_m \) is a modular function for \( G(\lambda_m) \).

Schwarz, Lehner, Raleigh and others studied Schwarz triangle functions, which map hyperbolic triangles \( T \) in the extended upper half \( z \)-plane onto the extended upper half \( w \)-plane. For certain \( T = T_m \), a triangle function \( \phi_{\lambda_m} : T \rightarrow \mathbb{H}^* \) extends to a map \( J_m : \mathbb{H}^* \rightarrow \mathbb{H}^* \) invariant under modular transformations from \( G(\lambda_m) \). Suitably normalized, the \( J_m \) become analogues \( j_m \) of the normalized Klein’s modular invariant

\[
j(\tau) = 1/q + 744 + 196884q + \ldots
\]

where \( q = q(\tau) = \exp(2\pi i \tau) \) and \( j_3(\tau) = j(\tau) \). The \( j_m \) are studied in conjecture 1 below.

With \( \lambda_m = 2\cos \pi/m \) and \( q_m(\tau) = \exp(2\pi i \tau/\lambda_m) \), the original \( J_m \) have Fourier series \( J_m(\tau) = \sum_{n \geq 1} a_n(m)q_m(\tau)^n \). For \( n = -1, 0, 1, 2 \) and \( 3 \), Raleigh gave polynomials \( P_n(x) \) such that \( a_{-1}(m)q_m^{2n+2}a_0(m) = P_n(m) \) for \( m = 3, 4, \ldots \), and conjectured that similar relations hold for all positive integers \( n \). Akiyama proved this conjecture in the passage after his (Akiyama’s) equation (6) \([1]\).

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1 This means that for compact \( K \subset \mathbb{H} \), the set \( \{ \mu \in H \text{ s.t. } K \cap \mu(K) \neq \emptyset \} \) is finite. Recall that \( G(\lambda_m) \) is the Hecke group generated by the maps \( z \mapsto -1/z \) and \( z \mapsto z + \lambda_m \). Hecke established in \([23]\) that \( G(\lambda_m) \) has the structure of a free product of cyclic groups \( C_2 \ast C_m \), generalizing the relation \([39, 13]\) \( SL(2, \mathbb{Z}) = C_2 \ast C_3 \).

2\( [23] \)

3 For \( j_m \), see \([39]\), Chapter VII, equation (23).
Hecke built families of modular forms \( f_m \) for \( G(\lambda_m) \) sharing particular properties. Earlier authors, whose work we will also describe, had already built modular functions (meromorphic functions invariant under the action of \( G(\lambda) \), thus, of weight zero) from triangle functions.

1.3 Plan of the article.

The plan of the article is as follows. (1) An elaboration of the preceding discussion to establish a basis for the code in our experiments\(^5\). (2) Conjectures on polynomials in \( \mathbb{Q}[x] \) interpolating the coefficients in Fourier expansions of triangle functions for \( G(\lambda_m) \). (3) A survey of Hecke’s theory of modular forms for \( G(\lambda_m) \), especially, the construction of modular forms from modular functions. (4) Several conjectures about polynomials in \( \mathbb{Q}[x] \) interpolating the coefficients in Fourier expansions of Hecke modular forms on \( G(\lambda_m) \). (5) Several data plots and tables. Tables at the end of the article focus on the triangle functions, since they are the basis of our construction of positive-weight modular forms, but more extensive collections of plots and tables are available within the Sagemath and Mathematica notebooks on \([10]\).

1.4 Methods.

Our conjectures are based on numerical experiments; here is a little more detail on the way we arrive at them. We begin with a list of modular functions or modular forms \( f_m \) for \( G(\lambda_m), m = 3, 4, ... \) sharing certain properties picked out by Hecke’s theory. Then we make tables of polynomials \( Q_n(x) \) generated by Lagrangian interpolation from the values of the coefficient \( k_m(n) \) in Fourier expansions \( f_m = \sum_n k_m(n) X_m^n \), where \( X_m \) is a variable related to \( q_m(\tau) \). Thus, we are seeking \( Q_n(x) \) such that

\[
Q_n(m) = k_m(n)
\]

for \( m = 3, 4, ... \). If the degrees of the \( Q_n(x) \) we obtain are linear in \( n \), we take this to be evidence that the \( Q_n(x) \) do satisfy equation (1) for all integers \( m \) greater than two. (Typically, the alternative outcome is that the degree of every polynomial \( Q_n(x) \) that we generate in a given table is equal to the size of the data set we are trying to interpolate.)

\(^5\) We have documentation in the data repository \([10]\). Mathematica notebook names end in the suffix “.nb”, and SageMath notebook names end in the suffix “.ipynb”. Numerical data files named in the notebooks is stored in the folder “data” on \([10]\). A green “Code” button on the top page of the repository contains a drop-down menu with a download option. A Mathematica notebook (“mf25.nb”) in the repository is a searchable library of functions that may not be defined explicitly within our other notebooks. We used SageMath release 9.1.
1.5 Work of Lehner, Raleigh and Leo.

The earliest computer code we located for calculating Fourier expansions of
triangle functions for Hecke groups is that of Leo[4]; it is based on Lehner’s
construction. Leo also calculates the Fourier coefficients of weight 4 and weight
6 Hecke-analogues of classical Eisenstein series in Chapter 4 of [28]. Our code
for triangle functions, which is based on Leo’s, comes from the papers of Lehner
and Raleigh. J. Jermann’s package is also concerned with modular forms of
triangle groups for Hecke groups, but we did not make use of it.

1.6 Disclaimer.

The article describes experiments and states conjectures. It contains no theo-
rems except ones that we quote from the existing literature.

2 A glossary

Some special functions in this list are related; different notations for similar
objects are used by Lehner and Raleigh, and we included all of them.

1. The digamma function \( \psi(z) := \Gamma'(z)/\Gamma(z) \).

2. The Schwarzian derivative

\[
\{ w, z \} = \frac{2w'w''w'' - 3w''^2}{2w'^2}
\]  

for \( w = w(z) \). (In section 3 below, we discuss Caratheodory’s presentation
of a well-known theorem of Schwarz; Carathéodory writes the left side of
our equation (4) as \( \{ w, z \} = \frac{w' w'' - 3w''^2}{w'^2} \), but we infer that the
Schwarzian derivative \( \{ w, z \} \) is intended from the automorphy property of
that theorem’s clause 2.)

3. The Pochhammer symbol

\( (a)_0 := 1 \) and, for \( n \geq 1 \), \( (a)_n := a(a+1)\ldots(a+n-1) = \Gamma(a+n)/\Gamma(a) \).

4. The function \( c_\nu \) given by

\[
c_\nu = c_\nu(\alpha, \beta, \gamma) := \frac{(\alpha)^\nu (\beta)^\nu}{\nu!(\gamma)^\nu}, \nu \geq 0.
\]
To facilitate comparison with Raleigh’s equation (9.1) \cite{15}, we remark that

\[ c_\nu = \frac{\Gamma(\alpha + \nu)}{\Gamma(\alpha)} \cdot \frac{\Gamma(\beta + \nu)}{\Gamma(\beta)} \cdot \frac{\Gamma(1)}{\Gamma(1 + \nu)} \cdot \frac{\Gamma(\gamma)}{\Gamma(\gamma + \nu)}. \]  

(3)

In the terms of this article’s Theorem 1 below, Raleigh is treating the case \( \lambda = 0 \), for which (equation (7) below) \( \gamma = 1 \) and the expression on the right side of (3) becomes, as in Raleigh,

\[ \frac{\Gamma(\alpha + \nu)\Gamma(\beta + \nu)}{\Gamma(\alpha)\Gamma(\beta)(\nu!)^2}. \]

5. The function \( e_\nu \) given by \cite{13}

\[ e_\nu = e_\nu(\alpha, \beta) := \sum_{p=0}^{\nu-1} \left( \frac{1}{\alpha + p} + \frac{1}{\beta + p} - \frac{2}{1 + p} \right). \]

Here, we are dealing with the same ambiguity present in the definition of \( c_\nu \): this is a specialization to the case \( \gamma = 1 \) of the \( e_\nu \) for \( \nu \geq 1 \) given by \cite{14}

\[ e_\nu = e_\nu(\alpha, \beta, \gamma) := \sum_{p=0}^{\nu-1} \left( \frac{1}{\alpha + p} + \frac{1}{\beta + p} - \frac{2}{\gamma + p} \right). \]

Unless it is explicitly indicated to be otherwise, we intend the former (Raleigh’s) definition.

6. (a) \cite{15} Gauss’s hypergeometric series

\[ F(\alpha, \beta, \gamma; z) := \sum_{\nu=0}^{\infty} c_\nu(\alpha, \beta, \gamma) z^\nu. \]

\( F \) is occasionally written in \cite{15} as \( \phi_1 \) (for example, on, p. 152.)

(b) \cite{16}

\[ F_1(\alpha, \beta, \gamma; z) := F(\alpha, \beta, \gamma + 1; z). \]

(c) \cite{17} Alternatively, dropping \( \gamma \):

\[ F_1(\alpha, \beta; z) := \sum_{\nu=1}^{\infty} \frac{(\alpha_k(\beta))_\nu}{(\nu!)^2} c_\nu(\alpha, \beta). \]

It is in the latter form, defined more cryptically in \cite{28}, p. 244, that we will use \( F_1 \); to establish his series for the triangle functions, which
we will apply below, Lehner uses this definition of $F_1$, as well as certain theorems from Fricke. Referring to item 4, we see that

$$F_1(\alpha, \beta; z) = \sum_{\nu=1}^\infty c_\nu(\alpha, \beta, 1)e_\nu(\alpha, \beta).$$

We will derive another form of $F_1(\alpha, \beta; z)$ in item 7.

7. With $F = F(\alpha, \beta, \gamma; z)$, a special function

$$F^*(\alpha, \beta, \gamma; z) := \frac{\partial F}{\partial \alpha} + \frac{\partial F}{\partial \beta} + 2 \frac{\partial F}{\partial \gamma};$$

$F^*$ may be written

$$F^*(\alpha, \beta, \gamma; z) = \sum_{\nu=1}^\infty c_\nu(\alpha, \beta, \gamma)e_\nu(\alpha, \beta, \gamma)z^{\nu}.$$  

It follows that $F^*(\alpha, \beta, 1; z) = F_1(\alpha, \beta; z)$.

8. A special function $\phi_2^*(z)$ is defined as a certain limit but is immediately reduced to

$$\phi_2^*(z) = F(\alpha, \beta, 1; z) \log z + F^*(\alpha, \beta, 1; z).$$

9. The set $\mathcal{Q} = \{2, 5, 6, 8, 10, 11, 14, 15, 17, 18, 20, 22, 23, \ldots\}$ of positive integers not represented by the quadratic form $x^2 + xy + y^2$. B. Cloitre asserts on the cited page that $\mathcal{Q}$ is also the set of non-negative integers $n$ such that $\delta(n)$ is non-zero, where $\eta$ is Dedekind’s eta function and

$$\sum_n \delta(n)x^n = \eta(x^3)/\eta(x)^3.$$  

10. The McKay-Thompson series of class 4A, $\{1, 24, 276, 2048, \ldots\}$, which is the sequence of coefficients in the $q$-series of a certain Hauptmodul discussed in [30]. We identified it with the sequence $\{\phi_n\}$ of our conjecture 1 after finding it in on [46].

11. As usual, the cardinality of a finite set $S$ is written $\# S$, the $n^{th}$ prime number is denoted by $p_n$, the number of primes less than or equal to $x$ is written $\pi(x)$, and $\sigma_k(n) := \sum_{d \mid n} d^k$.

\section{Calculation of Schwarz’s inverse triangle function}

Schwarz proved

\cite{19\,19,20\,20,21\,21} equation (387.4) on p. 153

\cite{19\,19,20\,20,21\,21} p. 152, equations 386.2 and 386.3
Theorem 1.  

1. Let the half-plane \( \Im z > 0 \) be mapped conformally onto an arbitrary circular-arc triangle whose angles at its vertices \( A, B, \) and \( C \) are \( \pi \lambda, \pi \mu, \) and \( \pi \nu, \) and let the vertices \( A, B, C \) be the images of the points \( z = 0, 1, \infty, \) respectively. Then the mapping function \( w(z) \) must be a solution of the third-order differential equation

\[
\{w, z\} = \frac{1 - \lambda^2}{2z^2} + \frac{1 - \mu^2}{2(1 - z^2)} + \frac{1 - \lambda^2 - \mu^2 + \nu^2}{2z(1 - z)}. \tag{4}
\]

2. If \( w_0(z) \) is any solution of equation (4) that satisfies \( w'_0(z) \neq 0 \) at all interior points of the half-plane, then the function

\[
w(z) = \frac{aw_0(z) + b}{cw_0(z) + d} \quad (ad - bc \neq 0)
\]

is likewise a solution of equation 3.

3. Also, every solution of equation (4) that is regular and non-constant in the half-plane \( \Im z > 0 \) represents a mapping of this half-plane onto a circular-arc triangle with angles \( \pi \lambda, \pi \mu, \) and \( \pi \nu. \)

(In Carathéodory’s lexicon, a regular function is one that is differentiable on an open connected set.)

Let us write

\[
\alpha = \frac{1}{2}(1 - \lambda - \mu + \nu), \tag{5}
\]

\[
\beta = \frac{1}{2}(1 - \lambda - \mu - \nu), \tag{6}
\]

and

\[
\gamma = 1 - \lambda. \tag{7}
\]

The solutions \( w \) of equation (4) are inverse to triangle functions; they are quotients of arbitrary solutions of

\[
u'' + p(z)u' + q(z)u = 0 \tag{8}
\]

when

\[
p = \frac{1 - \lambda}{z} - \frac{1 - \mu}{1 - z}
\]

and

\[
q = -\frac{\alpha \beta}{z(1 - z)}.
\]

\begin{footnotes}
\footnotetext[22]{$^1$}{\S374}
\footnotetext[23]{$^2$}{p. 124}
\footnotetext[24]{$^3$}{p. 136, equation (376.4)}
\end{footnotes}
Equation (8) reduces to the hypergeometric differential equation
\[ z(1-z)u'' + (\gamma - (\alpha + \beta + 1)z)u' - \alpha \beta u = 0. \quad (9) \]
As long as \( \gamma \) is not a non-positive integer, \( u = F(\alpha, \beta, \gamma; z) \) is a solution of equation (9); it is the only solution regular at \( z = 0 \), and it satisfies \( F(\alpha, \beta, \gamma; 0) = 1 \).

In [15], we find that when \( \gamma = 1 \) and \( \lambda = 0 \), another, linearly independent, solution of equation (8) is \( \phi^* \). Section 394, pp. 165 - 167 of [15] is devoted to the case \( \lambda = 0 \). There we find that the mapping function \( w \) of Theorem 1 satisfies
\[ w = \frac{1}{\pi i} \left[ \phi^*_1 - (2\psi(1) - \psi(1-\alpha) - \psi(1-\beta)) \right] + i \frac{\sin \pi \mu}{\cos \pi \mu + \cos \pi \nu}. \quad (10) \]

4 Inversion of Schwarz’s inverse triangle function

Following Lehner and Raleigh, we consider the Schwarz triangle \( T_m \) with vertices at \( \rho = -\exp(-\pi i/m), i, \) and \( i\infty \). In terms of Theorem 1, \( T_m \) has \( \lambda = 0 \) (an angle 0 at the vertex \( i\infty \)), \( \mu = 1/2 \) (an angle \( \pi/2 \) at \( i \)), and \( \nu = 1/m \) (an angle \( \pi/m \) at \( \rho \)). In this situation, \( \gamma = 1 \).

Let \( J_m \) be automorphic for \( G(\lambda_m) \) with \( J_m(\rho) = 0, J_m(i) = 1, \) and \( J_m(i\infty) = \infty \). In terms of Theorem 1, \( w \) and \( J_m \) are inverse functions. We are going to write down the Fourier expansion \( \sum_{n=-1}^{\infty} a_n q_m(\tau)^n \) of \( J_m \).

By clause 2 of Theorem 1, if \( w \) satisfies equations (4) and (10), so does \( \tau = \tau(z) = \lambda_m w(z)/2 \), and therefore
\[ 2\pi i/\lambda_m = \frac{\phi^*_1}{\phi_1} - (2\psi(1) - \psi(1-\alpha) - \psi(1-\beta)) - \pi \sec(\pi/m). \]

Let us write \( \log A_m = -2\psi(1) + \psi(1-\alpha) + \psi(1-\beta) - \pi \sec(\pi/m) \). In general, \( A_m = a_{-1}(m) \). Recalling the definitions of \( \phi_1 \) and \( \phi^*_1 \) from our glossary items 6 and 8, we find (abbreviating \( J_m(\tau) \) as \( J_m \)) that
\[ 2\pi i/\lambda_m = -\log J_m + F^*(\alpha, \beta, 1; J_m) + \log A_m. \quad (11) \]

Equation (11) is equation (6) of [35], but Raleigh suppresses the subscripts. He also writes \( \exp 2\pi i/\lambda_m \) as \( x_m \), so that (in our earlier notation) \( x_m = q_m(\tau) \).

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25 [15], p. 137, equations 376.5-7
26 Final paragraph of [15], §377, p. 138.
27 §§386-388 (pp. 151-158) [15], p. 166, equation 394.4
28 [15], p. 166, equation 394.4
29 For example two lines below equation (13).
In Raleigh’s notation, after taking exponentials,

\[ x_m/A_m = \frac{1}{J_m} \exp \frac{F^*(\alpha, \beta, 1; 1/J_m)}{F(\alpha, \beta, 1; 1/J_m)}, \]  \hspace{1cm} (12)

the right side of which has a power series in \( J_m \) with rational coefficients. Writing \( X_m = x_m/A_m \) we can regard \( X_m \) as a power series in \( J_m \) with rational coefficients. Following \[27\] and \[35\], we inverted this power series to obtain one for the modular function \( J_m \), also with rational coefficients. By construction, the Fourier expansion of \( J_m \) in \( X_m \) is normalized so that the coefficient of \( 1 \) is 1.

Let \( I \) be a formal operation taking a power series \( \sigma_p v^q \) to its inverse; that is, if \( u = \sigma_p v^q \) then \( v = I_p \sigma_p u^q \). Let \( Y_m \) be a power series such that

\[ Y_m(J_m) = J_m \exp \frac{F^*(\alpha, \beta, 1; J_m)}{F(\alpha, \beta, 1; J_m)} = X_m(1/J_m) \]

and hence

\[ Y_m(1/J_m) = \frac{1}{J_m} \exp \frac{F^*(\alpha, \beta, 1; 1/J_m)}{F(\alpha, \beta, 1; 1/J_m)} = X_m(J_m), \]

so that \( I_p (Y_m)(X_m(J)) = 1/J_m \) and, therefore, \( J_m = 1/I_p (Y_m)(X_m) \).

Remark 1. We noticed several typos in \[35\]. Four of Raleigh’s equations—(I), (10), and the two equations on p. 109 that begin “a...q” (where Raleigh’s q is our m)—are pairwise contradictory. From the second paragraph on Raleigh’s p. 110, we expect that \( A_3 = a_{-1}(3) = 1/1728, A_4 = a_{-1}(4) = 1/256, \) and \( A_6 = a_{-1}(6) = 1/108 \). These values are consistent with Raleigh’s equation (10), but not with the others. We infer that all of them except (10) are incorrect. Thus, following Raleigh by writing \( \psi \) for the digamma function, \( \alpha(m) \) for \( (1/2 - 1/m)/2 \), and \( \beta(m) \) for \( (1/2 + 1/m)/2 \),

\[ a_{-1}(m) = \exp (-2\psi(1) + \psi(1-\alpha(m)) + \psi(1-\beta(m)) - \pi \sec(\pi/m)). \]

5 Raleigh’s polynomials for triangle functions

Let \( X_m \) be the variable from the previous section. We define some operators on infinite series in \( X_m \).

Definition 1. Let \( f = \sum_{n=a}^\infty k_n X_m^n \) where \( k_n \) is a rational number for \( n = a, a + 1, \ldots \), and \( k_a \neq 0 \).

1. Let \( g = \sum_{n=a}^\infty k_n (2^6 m^3 X_m)^n = \sum_{n=a}^\infty \tilde{k}_n X_m^n \) (say). Then

\[ \tilde{f} := g/\tilde{k}_a. \]

\[ \text{[30]} \text{[35]}, \text{equation (12)}. \]
\[ \text{[31]} \text{The substitution involved appears in 28}. \]
2. \[ f^* := \frac{1}{k_a} \sum_{n=a}^{\infty} k_n X_m^{n-a}. \]

Recall, from the passage following equation (12) in the previous section, that the Fourier expansion of \( J_m \) in \( X_m \) has the form

\[ J_m(\tau) = 1/X_m + \sum_{n=0}^{\infty} a_n(m)X_m^n. \]

**Definition 2.** For the present purpose, we regard \( J_m \) as a Laurent series in \( X_m \) and write

\[ j_m := J_m. \]

**Conjecture 1.**

1. For each integer \( n \) greater than \(-2\), there exists a polynomial \( C_n(x) \in \mathbb{Q}[x] \) that satisfies the relation \( c_m(n) = C_n(m) \) for \( m = 3, 4, \ldots \).

2. Let \( \{\phi_n\} \) be as in item 10 of our glossary. For some degree \( 2n \), irreducible, monic polynomial \( \gamma_n(x) \) in \( \mathbb{Q}[x] \):

\[ C_n(x) = \phi_n \cdot (x - 2)(x + 2)x^{n+1}\gamma_n(x). \]

3. \( j_3 \) is the modular function on \( SL(2, \mathbb{Z}) \) usually denoted \( j \).

4. The complex roots of \( \gamma_n(x) \) lie in the disk with center zero and radius \( n/\log(n) \).

5. Let \( G_n \) be the Galois group of \( \gamma_n(x) \) over the rationals. The size of \( G_n \) is \( 2^n! \) and (if \( n \) is greater than two) \( G_n \) is isomorphic to a permutation group on \( 2n \) elements \( \{e_1, \ldots, e_{2n}\} \) with three generators: a transposition \( (e_j, e_k) \), a product \( (e_j, e_{j'}) (e_k, e_{k'}) \), and a product \( \Gamma_1 \Gamma_2 \) of disjoint cycles \( \Gamma_1 \) and \( \Gamma_2 \), each of length \( n \), such that \( \Gamma_1 \) sends \( e_j \) to \( e_{j'} \) and \( \Gamma_2 \) sends \( e_k \) to \( e_{k'} \).

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32Some code for \( j_m \) Fourier expansions appearing in SageMath notebooks cited below was generated in 'j from scratch.ipynb' \[^{10} \] (which employs a “dictionary” (the definitions at the top of the notebook) distinct from the corresponding dictionaries in the notebooks where it is reproduced).

[^10]: Notebook “conjecture 1.nb”, Notebook “conjecture 1 clause 2.ipynb”, Notebook “conjecture 1 clause 3.nb”, Notebooks “conjecture1clause4.nb”, “conjecture1clause4d.nb”, “conjecture 1 clause 4 no2.ipynb”, “conjecture 1 clause 4 no3.ipynb”, “conjecture 1 clause 4 no4.ipynb”, “conjecture 1 clause 4 no5.ipynb”, and “conjecture 1 clause 4 no6.ipynb”, Folder “conjecture1clause5”.
Let \( n \) be larger than one and let \( \pi_n \) be the set of prime numbers dividing the denominator of at least one non-zero coefficient of \( C_n(x) \) in its unfactored form. Then

(a) \( \pi_2 = \{3\} \) and \( \pi_3 \) is empty.

(b) If \( \pi_n \) is ordered by size, it contains no gaps. That is, if \( p \) and \( p' \) are consecutive elements of \( \pi_n \) with \( p = p_k \) and \( p' = p_j \), then \( j = k + 1 \).

(c) If \( n \) is an odd prime other than 3, then

\[
\pi_n = \{3, ..., k, ..., p\}_{\text{prime}}
\]

where \( p \) is the greatest prime less than \( n \).

(d) If \( n \) is composite and \( n + 1 \) is prime, then

\[
\pi_n = \{3, ..., k, ..., n + 1\}_{\text{prime}}.
\]

(e) If \( n \) and \( n + 1 \) are both composite, then

\[
\pi_n = \{3, ..., k, ..., p\}_{\text{prime}}
\]

where \( p \) is the greatest prime less than \( n \).

Clause 2 implies that, for \( m \) greater than or equal to three, \( c_n(m) \) is nonzero. It is already known that, for all integers \( n \geq -1 \), the \( n^{th} \) Fourier coefficient of \( j = j_3 \), namely \( c(n) = c_n(3) \), is positive.\(^{39}\) We tested clause 4 in several ways. We approximated the roots of the \( \gamma_n(x) \) with root-finding routines and compared their complex moduli with \( n/\log(n) \). We used the argument principle to count the zeros in central disks of radius \( n/\log(n) \). We superimposed plots of the roots of \( \gamma_n(x) \) against plots of circles with radius \( n/\log(n) \) and center at the origin. An example is depicted in Figure 1.\(^{40}\) For clause 5, we computed the Galois groups in Magma. For clause 6, some sequences we generated in the analysis were identified in \[40\] and \[48\].

**Conjecture 2.**\(^{41}\) Let the Fourier expansion of \( J_m(\tau) \) be

\[
J_m = \sum_{n=-\infty}^{\infty} a_m(n)X_m^n.
\]

1. \(^{42}\) We have

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38Notebook “conjecture 1 clause 6.ipynb”, \[10\].
39See, for example, page 199 in \[36\].
40Notebook “conjecture1clause4d.nb”, \[10\].
41Relevant documents in \[10\] are notebooks “conjecture 2.nb”, “conjecture2no1.ipynb”, “capital-J make data file1jun2T.ipynb” and associated data files.
42For clause 1, see “conjecture 2.nb”, “conjecture 2 clause 1b.ipynb”, , and “conjecture 2 clause 1b no2.ipynb”, \[10\].
(a) There exist polynomials $A_n(x)$ such that $A_{-1}(x) \equiv 1$, $A_0(x) = 3x^2 + 4$, $A_1(x) = 69x^4 - 8x^2 - 48$, and $A_n(m) = m^{2n+2}a_m(n)$ for $m = 3, 4, \ldots$.

(b) Let $C_n(x)$ be as in conjecture 1. We have:
$$A_n(x) = 2^{-6n-6}x^{-n-1}C_n(x).$$

2. Let $\pi_n$ be the set of prime numbers dividing the denominator of at least one non-zero coefficient of $A_n$. Then
   (a) $\pi_2 = \{3\}$.
   (b) If $\pi_n$ is ordered by size, it contains no gaps. That is, if $p$ and $p'$ are consecutive elements of $\pi_n$ with $p = p_k$ and $p' = p_j$, then $j = k + 1$.
   (c) If $n$ is an odd prime, then
   $$\pi_n = \{2, \ldots, k, \ldots, p\}_{k \text{ prime}}$$
   where $p$ is the greatest prime less than $n$.
   (d) If $n$ is composite and $n + 1$ is prime, then
   $$\pi_n = \{2, \ldots, k, \ldots, n + 1\}_{k \text{ prime}}.$$
   (e) If $n$ and $n + 1$ are both composite, then
   $$\pi_n = \{2, \ldots, k, \ldots, p\}_{k \text{ prime}}$$
   where $p$ is the greatest prime less than $n$.

The existence statement in clause 1a of conjecture 2 is equivalent up to some changes of variable, obviously, to Raleigh’s conjecture. We identified the leading numerical term in clause 1b of conjecture 2 after looking at [49]. Clause 2 of conjecture 2 is only a slight refinement of [1], proposition 2.

6 Survey of Hecke’s theory of modular forms

When the $w$-image of $\mathbb{H}^*$ is $T_m$, the inverse of $w$ is $\phi_{\lambda_m}$. The extension by modularity $J_m$ of $\phi_{\lambda_m}$ to $\mathbb{H}^*$, is periodic with period $\lambda_m$ and maps $\rho$ to 0, $i$ to 1, and $i\infty$ to $\infty$. These mapping properties allow us, following Berndt’s exposition of Hecke, to construct positive weight modular forms for $G(\lambda_m)$ from $J_m$. This section describes results of Hecke that are perhaps most easily accessible for the classical case $m = 3$ in Schoeneb and, for the general case, in Berndt. [49]
6.1 The case \( m = 3 \).

By keeping track of the weights, zeros and poles of the constituent factors in the numerator and denominator of the fraction defining

\[
J_{a,b,c} = \frac{f^a}{J(J-1)c},
\]

Schoeneberg demonstrates that \( f_{a,b,c} \) is an entire modular form of weight 2 for \( SL(2, \mathbb{Z}) \) if \( a \geq 2, 3c \leq a, 3b \leq 2a, b+c \geq a \) and \( a, b, c \) are integers. (Schoeneberg speaks of “dimension \(-2a\).”) Thus he is able to write down a weight 4 entire modular form \( E_{4}^* = f_{2,2,1} \) for \( SL(2, \mathbb{Z}) \) with a zero of order \( 3 \) at \( \rho = e^{2\pi i/3} \) and a weight 6 entire modular form \( E_{6}^* = f_{3,2,1} \) for \( SL(2, \mathbb{Z}) \) with a zero of order \( 2 \) at \( i \). (Schoeneberg writes \( G_{4}^*, G_{6}^* \).) It is well known that the (vector space) dimension of the spaces of weight 4 and 6 entire modular forms for \( SL(2, \mathbb{Z}) \) is equal to one, so \( E_{4}^* \) and \( E_{6}^* \) may be identified with the usual weight 4 and weight 6 Eisenstein series, up to a normalization. Finally, Schoeneberg defines the weight 12 cusp form \( \Delta^* = E_{4}^* - E_{6}^* \) with a zero of order 1 at \( i \infty \). It is a multiple of \( \Delta \).

6.2 The case \( m \geq 3 \).

We quote statements from Berndt, which is an exposition of Hecke. We depart occasionally from Berndt’s choices of variable to avoid clashes with our earlier notation.

**Definition 3.** \(^{52}\) We say that \( f \) belongs to the space \( M(\lambda, k, \gamma) \) if

1. \[
f(\tau) = \sum_{n=0}^{\infty} a_n e^{2\pi i n / \lambda},
\]
   where \( \lambda > 0 \) and \( \tau \in \mathbb{H} \), and

2. \( f(-1/\tau) = \gamma \cdot (\tau/i)^k f(\tau) \), where \( k > 0 \) and \( \gamma = \pm 1 \).

We say that \( f \) belongs to the space \( M_0(\lambda, k, \gamma) \) if \( f \) satisfies conditions 1 and 2, and if \( a_n = O(n^c) \) for some real number \( c \), as \( n \) tends to \( \infty \).

After defining the notion of a fundamental region in the usual way and defining as \( G(\lambda) \) the group of linear fractional transformations generated by \( \tau \mapsto -1/\tau \) and \( \tau \mapsto \tau + \lambda \), Berndt states (for \( \tau = x + iy \))

**Theorem 2.** \(^{53}\) Let \( B(\lambda) = \{ \tau \in \mathbb{H} : x < \lambda/2, |\tau| > 1 \} \). Then if \( \lambda \geq 2 \) or if \( \lambda = 2 \cos(\pi/m) \), where \( m \geq 3 \) is an integer, \( B(\lambda) \) is a fundamental region for \( G(\lambda) \).

\(^{50}\) Theorem 16, p.45
\(^{51}\) and other writings.
\(^{52}\) Definition 2.2
\(^{53}\) Theorem 3.1
Definition 4. Let \( T_A = \{ \lambda : \lambda = 2 \cos(\pi/m), m \geq 3, m \in \mathbb{Z} \} \).

Berndt states in his Theorem 5.4 that \( G(\lambda) \) is discrete if and only if \( \lambda \) belongs to \( T_A \). This discreteness is the premise of the theory of automorphic functions generally. He embeds within the proof of his Lemma 3.1 (which we omit), the

Definition 5. \( \tau_\lambda \) denotes the intersection in \( \mathbb{H} \) of the line \( x = -\lambda/2 \) and the unit circle \( |\tau| = 1 \).

(Berndt remarks at the top of page 35 that \( \tau_\lambda \) is the lower left corner of \( B(\lambda) \)). and that \( \pi \theta = \pi - \arg(\tau_\lambda) \), so that \( \cos(\pi \theta) = \lambda/2 \).

To characterize Eisenstein series, we need to keep track of some analytical properties. The next definition summarizes the second paragraph of Berndt’s Chapter 5. (Throughout his Chapter 5, \( \lambda < 2 \)).

Definition 6. Let \( f \in M(\lambda, k, \gamma) \), \( f \) not identically zero.

1. \( N = N_f \) counts the zeros of \( f \) on \( \overline{B(\lambda)} \) with multiplicities.
2. \( N_f \) does not count zeros at \( \tau_\lambda \), at \( \tau_\lambda + \lambda \), at \( i \), or at \( i\infty \).
3. If \( \tau_0 \in \overline{B(\lambda)} \), \( f(\tau_0) = 0 \) and \( \Re(\tau_0) = -\lambda/2 \), then \( f(\tau_0 + \lambda) = 0 \) and \( N_f \) counts only one of the two zeros.
4. If \( \tau_0 \in \overline{B(\lambda)} \), \( f(\tau_0) = 0 \), and \( |\tau_0| = 1 \), then, \( f(-1/\tau_0) = 0 \), and \( N_f \) counts only one of these two zeros.
5. The numbers \( n_\lambda, n_i, \) and \( n_{i\infty} \) are the orders of the zeros of \( f \) at \( \tau_\lambda, i \) and \( i\infty \), respectively. The order \( n_{i\infty} \) is measured in terms of \( \exp(2\pi i \tau/\lambda) \).

The multiplier \( \gamma \) is given by

Theorem 3. Let \( f \in M(\lambda, k, \gamma) \) and let \( n_i \) be the order of the zero of \( f \) at \( \tau = i \). Then

\[
\gamma = (-1)^{n_i}.
\]

The next two results tell us that the only nontrivial case in this theory is the one that we are interested in.

Theorem 4. If \( \dim M(\lambda, k, \gamma) \neq 0 \),

\[
N_f + n_{i\infty} + \frac{1}{2} n_i + \frac{n_\lambda}{m} = \frac{1}{2} k \left( \frac{1}{2} - \theta \right).
\]

By Berndt’s equation (5.16), if \( m \geq 3 \) then the right side can be written as \( k(m-2)/4m \).

\( ^{55} \) Definition 3.4
\( ^{56} \) Corollary 5.2
\( ^{57} \) Lemma 5.1
Theorem 5. \[57\] If \( \dim M(\lambda, k, \gamma) \neq 0 \), then \( \theta = 1/m \) where \( m \geq 3 \) and \( m \in \mathbb{Z} \).

We are concerned with \( \lambda \in T_A \). This makes \( \lambda < 2 \) as in all the results of Berndt’s Chapter 5.

One estimate for \( \dim M(\lambda, k, \gamma) \) is

Theorem 6. \[58\] If \( \lambda \not\in T_A \), then \( \dim M(\lambda, k, \gamma) = 0 \). If \( \lambda = 2 \cos(\pi/m) \in T_A \), then for nontrivial \( f \in M(\lambda, k, \gamma) \), the weight \( k \) has the form

\[
k = \frac{4h}{m - 2} + 1 - \gamma,
\]

where \( h \geq 1 \) is an integer. Furthermore,

\[
\dim M(\lambda, k, \gamma) = 1 + \left\lfloor \frac{h + (\gamma - 1)/2}{m} \right\rfloor.
\]

Eliminating \( h \), we find that

\[
\dim M(\lambda, k, \gamma) = 1 + \left\lfloor k \left( \frac{1}{4} - \frac{1}{2m} \right) + \frac{\gamma}{4} - \frac{1}{4} \right\rfloor. \quad (13)
\]

Berndt proves that the dimension formula above holds also when \( h = 0 \). \[59\]

The existence of certain modular forms is provided by

Theorem 7. \[60\] Let \( \lambda \in T_A \). Then there exist functions \( f_{\lambda}, f_i, \) and \( f_{\infty} \in M(\lambda, k, \gamma) \) such that each has a simple zero at \( \tau_{\lambda}, i, \) and \( i\infty \), respectively, and no other zeros. Here, \( \gamma \) is given by Theorem 3 of the present article, and \( k \) is determined in each case from Theorem 4 of the present article. Thus, \( f_{\lambda} \in M(\lambda, 4/(m - 2), 1), f_i \in M(\lambda, 2m/(m - 2), -1), \) and \( f_{\infty} \in M(\lambda, 4m/(m - 2), 1). \)

Remark 2. \[61\] By the Riemann mapping theorem there exists a function \( g(\tau) \) that maps the simply connected region \( B(\lambda) \) one-to-one and conformally onto \( \mathbb{H} \). If we require that \( g(\tau_{\lambda}) = 0, g(i) = 1, \) and \( g(i\infty) = \infty \), then \( g \) is determined uniquely.

Now we can write down \( f_{\lambda}, f_i, \) and \( f_{\infty} \) explicitly. The next theorem is extracted from the proof of Theorem 7. \( f_{\lambda} \) and \( f_i \) correspond to Eisenstein series and \( f_{\infty} \) to a cusp form. In our code, we take \( g \) to be a normalized form of \( J_m \).

Theorem 8. \[62\]

\[
f_{\lambda}(\tau) = \left\{ \frac{g'(\tau)^2}{g(\tau)(g(\tau) - 1)} \right\}^{1/(m-2)},
\]
In our applications to Lehmer’s problem, we will be interested in the dimensions of the weight 12 cusp spaces for $\lambda = \lambda_m = 2 \cos \pi / m$.

**Definition 7.** If $f \in M(\lambda, k, \gamma)$ and $f(i \infty) = 0$, then we call $f$ a cusp form of weight $k$ and multiplier $\gamma$ with respect to $G(\lambda)$. We denote by $C(\lambda, k, \gamma)$ the vector space of all cusp forms of this kind.

**Remark 3.**
\[
\dim C(\lambda, k, \gamma) \geq \dim M(\lambda, k, \gamma) - 1.
\]

**Remark 4.** In view of (i) Theorem 6, (ii) equation (12), (iii) Remark 2, and (iv) the fact that $\gamma = \pm 1$, we see that $\dim C(\lambda_m, 12, \gamma) > 1$ when $m$ is greater than or equal to 12.

### 7 Modular forms studied in our experiments

We are going to write down versions of the functions from Theorem 8 such that, at $m = 3$, they reduce to corresponding functions in the classical theory. Some have fixed weights (four, six and twelve) and others have weights that vary with $m$. The classical objects (in Serre’s notation [39]), are Klein’s $j$-invariant, the weight four Eisenstein series $E_2$, the weight six Eisenstein series $E_3$, and the generating function of Ramanujan’s tau function, namely the normalized weight twelve cusp form $\Delta$. They all belong to one-dimensional vector spaces of modular forms and the number of zeros each one has in a given fundamental region is small, so the identifications follow by comparison of the initial Fourier coefficients.

Corresponding to $f_\lambda$, we have

**Definition 8.**
1. $H_{\lambda,m}(\tau) := \left\{ \frac{g'(\tau)^m}{g(\tau)^{m-1}(g(\tau) - 1)} \right\}^{1/(m-2)}$.

2. $H_{\lambda,4,m}(\tau) := H_{\lambda,m}(\tau)^{m-2}$.

Corresponding to $f_i$, we state

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63 [7], Definition 5.2
64 [7], equation (5.25)
65 [39], Chapter VII, equations (20-21)
Definition 9. 1. 
\[ H_{1,m}(\tau) := \left\{ \frac{J_m'(\tau)^m}{J_m(\tau)^{m-1}(J_m(\tau) - 1)} \right\}^{1/(m-2)}. \]

Definition 10. 1. Corresponding to \( f_x \), we have
\[ \Delta_{x,m}(\tau) := \left\{ \frac{J_m'(\tau)^{2m}}{J_m(\tau)^{2m-2}(J_m(\tau) - 1)^m} \right\}^{1/(m-2)}. \]

2. \( \Delta_m^\circ := H_{\lambda,m}^3/J_m \).

3. \( \Delta_{12,m}^\circ := H_{\lambda,4,m}^3/J_m \).

Remark 5. By Berndt’s theorem 7 above, we have the following table of weights:

| \( H_{\lambda,m} \) | \( H_{\lambda,4,m} \) | \( H_{1,m} \) | \( \Delta_m^\circ \) | \( \Delta_{12,m}^\circ \) | \( \Delta_{x,m} \) |
|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|
| \( 4/(m-2) \)     | 4                 | \( 2m/(m-2) \)   | \( 12/(m-2) \)   | 12                | \( 4m/(m-2) \)   |

8 Interpolation by polynomials

In this section, we state conjectures about polynomials interpolating coefficients of modular forms for Hecke groups. Conjectures 6 and 7 bear on Lehmer’s question about the existence of zeros of Ramanujan’s tau function.

Berndt’s (Hecke’s) theorems 7 and 8 above make it clear that Akiyama’s theorem proving Raleigh’s conjecture on the interpolation of the coefficients of the Fourier expansions of Hecke triangle functions extends in some way to the modular forms defined in the previous section. We did experiments to explore the details; our observations are summarized in the conjectures below.

8.1 Analogues of \( SL(2,\mathbb{Z}) \) Eisenstein series.

We found the sequence \( \{e_{4,n}\} \) mentioned below on \[41\].

Conjecture 3. Let the Fourier expansion of \( H_{\lambda,4,m}(\tau) \) be
\[ H_{\lambda,4,m}(\tau) = \sum_{n=0}^{\infty} \beta_{4,m}(n)X^n_m. \]

1. \( H_{\lambda,4,3}(\tau) \) reduces to Serre’s weight-4 Eisenstein series \( E_2 \) in the sense that \( \beta_{4,3}(n) = 240\sigma_3(n) \) for \( n = 1, 2, 3, \ldots \).

2. For each \( n \) there is a polynomial \( B_{4,n}(x) \) with rational coefficients such that \( m^3\beta_{4,m}(n) = B_{4,n}(m) \) for \( m = 3, 4, \ldots \).
3. If \( n \) is positive, then the degree of \( B_{4,n}(x) \) is \( 6n \).

4. \( B_{4,0}(x) \equiv 1 \) and, if \( n \) is positive, then
\[
B_{4,n}(x) = e_{4,n}(x^2 - 4)x^{4n}b_{4,n}(x),
\]
where \( e_{4,n} = 16 \sum_{\nu|n} (-1)^{\nu}n^{\nu} \) and \( b_{4,n}(x) \) is a monic irreducible polynomial in \( \mathbb{Q}[x] \).

**Conjecture 4.** Let the Fourier expansion of \( H_{\lambda,m} \) be
\[
H_{\lambda,m} = \sum_{n=0}^{\infty} \beta_m(n)X_m^n.
\]

1. For each \( n \) there is a polynomial \( B_n(x) \) with rational coefficients such that \( \beta_m(n) = B_n(m) \) for \( m = 3, 4, \ldots \).

2. If \( n \) is positive, then the degree of \( B_n(x) \) is \( 3n - 1 \).

3. \( B_0(x) = 1 \) and \( B_1(x) = 16x(x + 2) \).

4. Let \( \mathcal{P} \) be as in item 9 of our glossary and let \( e_n = 16(-1)^{n+1} \sum_{\nu|n} 1/\nu \).
   If \( n \) is greater than 2 and belongs to \( \mathcal{P} \), then
   \[
   B_n(x) = e_n(x^2 - 4)(x - 6)x^n b_n(x),
   \]
   where \( b_n(x) \) is a monic irreducible polynomial. Otherwise (for \( n \) greater than one) \( B_n(x) = e_n(x^2 - 4)x^n b_n(x) \) where, again, \( b_n(x) \) is a monic irreducible polynomial in \( \mathbb{Q}[x] \).

5. \( H_{\lambda,3} \) reduces to \( E_2 \) in the same sense as in conjecture 3.1.

(We identified the \( e_n \) after reading [44] and [45].)

Thus, in the range of our observations (\( 3 \leq m \leq 302, 0 \leq n \leq 100 \)), the only integer value of \( m \) such that \( H_{\lambda,m} \) has any vanishing coefficients is six, and \( \beta_n(6) \) is zero just if \( n \) is in \( \mathcal{P} \).

**Conjecture 5.** Let the Fourier expansion of \( H_{i,m} \) be
\[
H_{i,m} = \sum_{n=0}^{\infty} \delta_m(n)X_m^n.
\]

1. For each non-negative integer \( n \), there is a polynomial \( D_n(x) \) in \( \mathbb{Q}[x] \) such that

---

\(^{68}\) Notebooks “conjecture 4.1-4.3.ipynb, conjecture 4.4a.ipynb, conjecture 4.4b.ipynb”, conjecture 4.5.ipynb. N.B.: Contrary to appearances, the function denoted “H4” in these SageMath notebooks is not the function covered in the previous conjecture. “H4” is \( H_{\lambda,m} \).

\(^{69}\) In [10] notebook “conjecture 5.ipynb”.

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19
(a) \( D_n(m) = \delta_n(n) \) for \( n = 0, 1, \ldots \) and \( m = 3, 4, \ldots \).

(b) The degree of \( D_n \) is \( 3n \).

(c) \( D_n(x) = a \) rational number \( d_n \times \) a product of monic irreducible polynomials.

(d) \( d_0 = 1 \) and, for \( n \) a positive integer, \( d_n = 24(-1)^n \sum^* \nu \). Again, the asterisk means that the sum is taken over the odd positive divisors of \( n \).

2. \( D_n(m) = (-1)^m \delta_n(m) \) for \( m = 3, 4, \ldots \).

3. \( D_0(x) = 1 \) identically, \( D_1(x) = -24(x - 2/3)x^2 \), and \( D_2(x) = 24(x - 2/3)(x - 2)x^3(x - 14) \).

4. For \( n \) larger than two, \( D_n(x) = d_n(x - 2)(x - 2/3)x^{n+1}\epsilon_n(x) \) where \( \epsilon_n(x) \) is a monic irreducible polynomial in \( \mathbb{Q}[x] \).

5. \( \delta_3 \) reduces to Serre’s weight-6 Eisenstein series \( \tilde{E}_3 \) in the sense that \( \delta_3(0) = 1 \) and \( \delta_3(n) = -504\sigma_5(n) \) for \( n = 1, 2, 3, \ldots \).

8.2 Analogues of \( SL(2, \mathbb{Z}) \) cusp forms.

Let \( \Delta \) be usual normalized discriminant, a weight 12 cusp form for \( SL(2, \mathbb{Z}) = G(\lambda_3) \) with integer coefficients. Its Fourier expansion is written

\[
\Delta(\tau) = \sum_{n=1}^{\infty} \tau(n)q^n
\]

where \( q = e^{2\pi i \tau} \) and \( \tau(n) \) is Ramanujan’s function. (The reader will not confuse the complex number \( \tau \) with Ramanujan’s function \( \tau(n) \) or any of its relatives defined below.) Whether or not the equation \( \tau(n) = 0 \) has any solutions is, of course, an open question. Several authors have eliminated various classes of integers as values of \( \tau \). It will be apparent that each of the conjectures about cusp-form analogues implies that \( \tau \) has no zeros.

From definition 10.2,

\[
\Delta_{x,m}(\tau)^m = \frac{J_m'(\tau)^{2m}}{J_m(\tau)^{2m-2}(J_m(\tau) - 1)^m}
\]

and, by Theorem 7 in our sketch of Hecke’s theory, its weight is \( 4m \). Since it raises a cusp form beginning with an \( X^1 \) term to high powers, we will use the star operator (definition 1.2) to state the following conjecture.

\footnote{These results are summarized in \[25\]. Relevant citations are \[3, 1, 5, 6, 17, 29, 32\]. and \[25\] itself.}
Conjecture 6. Let the Fourier expansion of \((\Delta_{x,m}(\tau)^{m-2})\) be written as 
\[
(\Delta_{x,m}(\tau)^{m-2}) = \sum_{n=0}^{m^2} \tau_{m}(n)X_m.
\]
1. \(\tau_{3}(n-1) = \tau(n)\) for \(n = 1, 2, \ldots\).
2. There is a set of polynomials \(T_n(x), n = 1, 2, 3, \ldots\) such that, for each \(n\), \(T_n(m) = \tau_{m}(n)\).
3. \(T_n(x) = (-8)^n(x - 2)^3x t_n(x)/n!\) where \(t_n\) is a polynomial with rational coefficients that is irreducible over \(\mathbb{Q}[x]\).

Conjecture 7. Let the Fourier expansion of \(\Delta_{x,m}(\tau)\) be 
\[
\Delta_{x,m}(\tau) = \sum_{n=1}^{\infty} \tau_{x,m}(n)X_m^n.
\]
1. \(\tau_{x,3}(n) = \tau(n)\) for \(n = 1, 2, 3, \ldots\).
2. There is a set of polynomials \(T_{x,n}(x)\) with coefficients in \(\mathbb{Q}\) such that \(\tau_{x,m}(n) = T_{x,n}(m)\).
3. \(T_{x,1}(x) \equiv 1\) identically, and, if \(n\) is greater than one,
   (a) \(T_{x,n}(x) = s_{x,n}(x - 2)^3 x t_{x,n}(x)\), where \(t_{x,n}(x)\) is a monic irreducible polynomial over \(\mathbb{Q}\) of degree \(2n - 4\) and
   (b) \(s_{x,n}\) is (in the notation of [16], Chapter 7, Theorem 7) the coefficient of \(q^n\) in the Fourier expansion of \(\Delta_{x}(z)\).
   (c) Also,
   \[
s_{x,n} = (-1)^{n+1} \sum_{\nu \mid n \atop \nu \text{ odd}} \nu^3.
\]
   This sum is the coefficient of \(q^n\) in the Fourier expansion of \(E_{x,4}\), the unique normalized weight-4 modular form for \(\Gamma_0(2)\) with simple zeros at \(\infty\); it is also the number of representations of \(n - 1\) as a sum of 8 triangular numbers.
   (d) Finally, \(s_{x,n}\) is the coefficient of \(q^n\) in the expansion of \(\eta(2z)^{16}/\eta(z)^{-8}\) where \(\eta(z)\) is Dedekind’s function ([11], equation (2-16).

Conjecture 8. Let the Fourier expansion of \(\Delta_{m}(\tau)\) be 
\[
\Delta_{m}(\tau) = \sum_{n=1}^{\infty} \tau_{m}(n)X_m^n.
\]
1. $\tau_3^n(n) = \tau(n)$ for $n = 1, 2, 3, \ldots$.

2. There is a set of polynomials $T_n^\circ(x)$ with coefficients in $\mathbb{Q}$ such that $\tau_n^\circ(n) = T_n^\circ(m)$.

3. $T_1^\circ(x), T_2^\circ(x),$ and $T_3^\circ(x)$ are irreducible polynomials over $\mathbb{Q}$ of degrees 3, 6, and 9, respectively.

4. If $n$ is greater than 3, $T_1^\circ(x) = s_n^\circ(x-2)x^{n-1}$, where $s_n^\circ$ is a rational number and $T_n^\circ(x)$ is a monic polynomial, irreducible over $\mathbb{Q}$, of degree $2n - 3$. Furthermore,
   
   (a) $\sum_{n=0}^{\infty} s_n^\circ q(\tau)^n = \prod_{n \text{ odd}} (1 - q(\tau)^n)^{24} \times \prod_{n=2(4)} (1 - q(\tau)^n)^{-24} = \eta^{24}(\tau)\eta^{24}(4\tau)\eta^{-48}(2\tau)$.

   (b) $s_n^\circ = (-1)^{n+1} \times \text{the coefficient of } q(\tau)^n \text{ in } (\eta(2\tau)/\eta(\tau))^{24}$.

5. There is no corresponding set of interpolating polynomials for $\Delta_3^\circ$.

The product decomposition in clause 3(a) above is a guess based on 43 terms of the series using Euler’s method.

**Conjecture 9.**

Let the Fourier expansion of $\overline{\Delta_{12,m}^\circ}(\tau)$ be

$$\Delta_{12,m}^\circ(\tau) = \sum_{n=1}^{\infty} \tau_{12,m}^\circ(n)X_m^n.$$

1. $\tau_{12,3}^\circ(n) = \tau(n)$ for $n = 1, 2, 3, \ldots$.

2. There is a set of polynomials $T_{12,n}^\circ(x), n = 1, 2, \ldots$ of degree $3n - 3$ with coefficients in $\mathbb{Q}$ such that $\tau_{12,m}^\circ(n) = T_{12,n}^\circ(m)$ for each $m = 3, 4, \ldots$.

3. For each $n$, there are zeros of $T_{12,n}^\circ(x)$ on both axes of the complex plane, and there are no other complex zeros (Figures 2 and 3 illustrate this for $n = 11$ and 24.)

4. $T_{12,n}^\circ(x) = (-1)^{n+1}\tau(n)x^{n-1}t_{12,n}^\circ(x)$, where $t_{12,n}^\circ(x)$ is a monic irreducible polynomial over $\mathbb{Q}$.

---

79 (Theorem 14.8); English-language version of [19] in [34]; and [47]. The second decomposition appears in [43].

80 Notebook “conjecture 9.ipynb”, [10].

81 Notebook “conjecture 9.nb” [10] contains plots of the complex zeros for $n$ between 1 and 24.
9 Lehmer’s question

Remark 6. By clause 4 of conjecture 9, for \( m = 3, 4, \ldots \): \( \tau_{12,m}^2(n) = 0 \) if and only if \( \tau(n) = 0 \).

More generally, we have

Conjecture 10. Letting \( T_n(x) \) and \( \tau_m \) stand for the various polynomials and Fourier coefficients in conjectures 6 through 9, none of the \( T_n(x) \) has an integer root greater than two; consequently, none of the \( \tau_m \) vanishes for \( m = 3, 4, \ldots \).

Let \( d(m, n) \) be the minimum Euclidean distance to \( m \) of any complex root of \( T_n(x) \). We have (in effect) conjectured above that in each case \( T_n(3) = \tau(n) \), so the behavior of \( d(3, n) \) measures how closely we can come to the assertion that \( \tau(n) = 0 \) for some \( n \).

Conjecture 11. For any positive real number \( r \), \( d(3, n) \) is less that \( e^{-rn} \) for sufficiently large \( n \).

10 Other questions

1. Like \( G_n \) in clause 5 of conjecture 1, the index-\( n \) hyperoctahedral group has size \( 2^n n! \). Are they isomorphic?

2. In conjectures 1–9, the \( n^{th} \) interpolating polynomial is written as a product of a numerical term and several monic polynomials belonging to \( \mathbb{Q}[x] \). In each case, all but one of the monic factors is given explicitly, i.e., in terms of \( n \), but without reference to the Fourier expansion of the underlying modular form. The “inexplicit” factor can, of course, be written in terms of the first \( n \) of these coefficients, but can it be expressed in the same way as the other factors: without reference to the Fourier coefficients?

3. While checking our calculations, we compared the Fourier expansion of \( H_{\lambda, 4}(x/A_4) \) (abusing notation in the obvious way) with Leo’s expansion of the weight 4 Eisenstein series at \( m = 4 \). (Recall that \( A_4 = 1/256 \).) Within the range of our observations, they do coincide. The expansions (in our own notation) both begin

\[ 1 + 48q + 624q^2 + 1344q^3 + \ldots \]

Let

\[ E_{\gamma, 2} = 1 + 24 \sum_{\nu=1}^{\infty} \sum_{\nu \mid \nu \text{ odd}} \mu \nu q^n. \]

---

For this proposal, we depend on graphical evidence which we sample figures 10 – 17. More extensive collections of plots are in notebooks “conjecture 6.1.nb”, “conjecture 6.2.nb”, “conjecture 7.nb”, and “conjecture 8.nb”. [10].

[10] [28] p.54

[23]
Sloane comments that the sequence $\{1, 48, 624, \ldots\}$ is the same as that of the coefficients of $E_{7,2}$. $E_{7,2}$ is a weight 4, level 2 modular form, that is, a weight 4 modular form for the $SL(2, \mathbb{Z})$ subgroup $\Gamma_0(2)$. We propose in conjecture 7 (c) above that $s_{E, n}$ is the coefficient of $q^n$ in the Fourier expansion of $E_{3,4}$, the unique normalized weight-4 modular form for $\Gamma_0(2)$ with simple zeros at $ix$. We have also proposed in conjectures 1, 2, 7 and 8 that interpolating polynomials are products of monic polynomials with rational numbers equal or related to Fourier coefficients of other classical Hauptmoduln. What is the relationship between modular forms for subgroups of $SL(2, \mathbb{Z})$ and modular forms for the other $G(\lambda_m)$?

4. Both $J_m$ and $\overline{J_m}$ (that is, $j_m$) appear to be interpolated by polynomials. On the other hand, $\overline{\Delta_m}$ appears to be interpolated by polynomials, but $\Delta_m$ does not. Why are the situations different?

11 Figures

11.1 Figure 1.
Fourier expansion of $j_m(\tau)$ (conjecture 1.)

11.2 Figure 2.

Notebook “conjecture1clause4d.nb”, [10].

Notebook “conjecture 6Laptop.nb”, [10].

Roots of $T_{17}$ (conjecture 6.)

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87 Notebook “conjecture1clause4d.nb”, [10].
88 Notebook “conjecture 6Laptop.nb”, [10].
11.3 Figure 3.

Roots of $T_{35}$ (conjecture 6) \[89\]

11.4 Figure 4.

Roots of $T_{35,20}$ (conjecture 7.) \[90\]

\[89\textit{ibid.}\]

\[90\text{Notebook “conjecture 7.nb”, [10]}\]
11.5 Figure 5.

Roots of $T_{x,50}$ (conjecture 7.)

11.6 Figure 6.

Roots of $T_{19}$ (conjecture 8.)

ibid. Notebook “conjecture 8.nb”.
11.7 Figure 7.

Roots of $T_{50}^\infty$ (conjecture 8.)
11.8 Figure 8.

Roots of $T_{12,11}^5$ (Conjecture 9.)

11.9 Figure 9.

Roots of $T_{12,24}^5$ (Conjecture 9.)

\[^{94}\text{Notebook “Conjecture 9.nb”, [10].}\]

\[^{95}\text{ibid.}\]
11.10 Figure 10.

$y = \min \log |\text{root} - 3|$ for polynomial $n$ vs $y = -\pi n$

$y = \log(\text{minimum distance of roots of } T_n \text{ from 3})$ in blue vs $y = -\pi n$ in red; Conjectures 6 and 11.

Notebook “conjecture 6Laptop.nb”, [10].
11.11 Figure 11.

\[ y = \min \log |\text{root} - 3| \text{ for polynomial } n \text{ vs } y = -4n \]

\[ y = \log(\text{minimum distance of roots of } T_n \text{ from 3}) \text{ in blue vs } y = -4n \]

in red; conjectures 6 and 11. \textsuperscript{97}

\textsuperscript{97} ibid.
11.12 Figure 12.

\[ y = \min \log |\text{root} - 3| \text{ for polynomial } n \text{ vs } y = -\pi n \]

Notebook “conjecture 7.nb”, [10]
11.13 Figure 13.

\[ y = \log(\text{minimum distance of roots of } T_{x,n} \text{ from 3}) \text{ in blue vs } y = -4n \text{ in red; conjectures 7 and 11.} \]

\[ ^{99} \text{ibid.} \]
11.14 Figure 14.

\[ y = \min \log |\text{root} - 3| \text{ for polynomial } n \text{ vs } y = -\pi n \]

\( y = \log(\text{minimum distance of roots of } T_n^\circ \text{ from 3}) \text{ in blue vs } y = -\pi n \) in red; conjectures 8 and 11. \[100\]

\[100\] Notebook “conjecture 8.nb” \[10\].
y = log(minimum distance of roots of $T_n^3$ from 3) in blue vs $y = -4n$ in red; conjectures 8 and 11. \[\text{ibid.}\]
Figure 16.

\[ y = \min \log |\text{root} - 3| \text{ for polynomial } n \text{ vs } y = -\pi n \]

\[ y = \log(\text{minimum distance of roots of } T_{12,n}^\circ \text{ from 3}) \text{ in blue vs } y = -\pi n \text{ in red; conjectures 9 and 11.} \]

\[ ^{102}\text{Notebook “Conjecture 9.nb”.} \]
$y = \log(\text{minimum distance of roots of } T_{12,n}^{\circ} \text{ from } 3)\text{ in blue vs } y = -4n$ in red; conjectures 9 and 11.\textsuperscript{103}

\textsuperscript{103}ibid.
### Table 1

| c_m | p_n | q (conjecture 1) |
|-----|-----|------------------|
| 104 | 104 |                  |

Notebook “conjecture 1 tables.ipynb”.
Table 2.

\[
\begin{array}{l}
\text{Polynomials } C_n(x) \text{ (conjecture 1.)} \\
\end{array}
\]

\[
\begin{array}{l}
\text{ni } \emptyset \times 3 + 32x \\
\text{n1 } 27x^6 + 32x^6 - 192x^2 \\
\text{n2 } 2(34x^2 - 239x/72 + \sqrt{2731/3}x^2 + 3267/372/393x^3 + 13107/27x^3) \\
\text{n3 } 1122x^2 + 12 - 1222x^2 + 10 + 3336x^2 + 10 - 517x^2 + 16 - 5513x^2 \\
\text{n5 } 4012x^2 + 15 + 18726777/12125x^2 + 14 + 73835768/675x^2 + 11 + 45376879/3375x^9 + 156447592/675x^7 + 6257091 \\
\text{568/1125x^2} \\
\text{n5 } 5 + 1804572x^2 + 18 - 14266116/729x^2 + 16 + 433739468/279x^2 + 14 + 217405856/696x^2 + 12 + 1482487848/243x^2 + 10 - 7790 \\
\text{8008112/729x^2 + 16 - 155816870024/729x^2} \\
\text{n6 } 614400x^2 + 21 + 3057057251/1286/52x^2 + 15 + 30857572752/795x^3 + 17 + 18127478711/7125x^5 + 15 + 6025670088 \\
\text{98/2672x^5} + 13 + 38731728972/35/11x + 1 + 267266180931/560x^3 + 9 + 166106864627/636/1286x/57 \\
\text{n7 } 1801341x^2 + 24 - 1058546368/441x^2 + 22 + 61657801328/383x^5 \\
\text{n8 } 5373952x^2 + 27 - 107332145874/142x/35/1255x^5 + 25 + 25847767716/274848/26/21382875x^2 + 23 + 392434518138411x/28 \\
\text{n9 } 1447138x^2 + 30 - 13583587219/175x^2 + 12/8652x^2 + 28 + 99974195467/2879/21214476315x^2 + 12 - 5682724535269/8393x^3 + 8/6821 \\
\text{n10 } 3711204x^2 + 33 - 780006730844/19/152x^2 + 27 + 7752463742768632/296/2691765x^2 + 29 - 19385946862125 \\
\text{n11 } 1193215/656x^2 + 27 + 773568271/532687648/9367/652x^3 + 19 - 24084667642/495x^6/3946/868/26183x^2 + 17 + 26017431241312 \\
\text{n12 } 1268721x^2 + 39 - 23849377484/212/24809099/2/18504958515x^2 + 32 - 182391666667 \\
\text{n13 } 4955490x^2 + 42 - 1665494686932685621/2773437975x^4 + 48 + 5743914694231/26648366262/26/1685385670446675x^5 + 38 \\
\text{n14 } 4955490x^2 + 42 - 1665494686932685621/2773437975x^4 + 48 + 5743914694231/26648366262/26/1685385670446675x^5 + 38 \\
\text{3511879177/7702441108x/158394/2948511471/5755x/31 + 186964193928381610783585395576/744 + 61017286170698397x^3} \\
\text{3x/32 = 6572819310041306297x/53756x/30268 + 1820849636380986x/52 + 2027938314177852849280287x/31365/95 \\
\text{5x/32 = 13801514971/32/1030924688x/80767889/2126 - 1164133719148275/264384786/21673647392/2653368122454677x^x/24 + 3436 \\
\text{31383809348331325251927x/866809/23821675166x/16869/20803918457x/32 + 1546466787276132x/8 + 8641262319679221897/3071930482132867x/32 + 13862934 \\
\text{68002796785x^7 + 20 + 5984132779241532181714846662/801233287x/8 + 541563290896875x^5 + 15 - 58458911950851354359808375x^2 + 15 + 6082345221412433753335396487/626/8/8/826/18/2623495 \\
\text{2439052x/35x} \\
\end{array}
\]
### 12.3 Table 3.

| n  | \( x = x + (x^2 + 4/3) \) |
|----|---------------------------------|
| 0  | \( (24) \)                        |
| 2  | \( (26) * x^2 * (x^4 - 8/9x^2 - 16/23) \) |
| 3  | \( (2864) * (x - 2) * (x + 2) * x^3 * (x^4 - 8/27x^2 - 16/27) \) |
| 1  | \( (1120) * (x - 2) * (x + 2) * x^4 * (x^6 - 3872/5681x^4 + 11312/5681x^2 + 4664/1687) \) |
| 4  | \( (49152) * (x - 2) * (x + 2) * x^5 * (x^6 - 51968/3375x^6 + 650144/18125x^4 - 198976/18125x^2 + 95488/3375) \) |
| 6  | \( (18044) * (x - 2) * (x + 2) * x^6 * (x^10 - 1554984/621881x^10 + 373957344/16769187x^6 - 452733568/621881x^6 + 127128488/5589729x^6 + 2488086532/6761987) \) |
| 8  | \( (64488) * (x - 2) * (x + 2) * x^7 * (x^12 - 48092658/9468675x^12 + 18438205584/1929375x^4 - 1687848576/5788125x^2 - 3375569712/9684755) \) |
| 10 | \( (9) \)                        |
| 12 | \( (577952) * (x - 2) * (x + 2) * x^8 * (x^16 - 73253258992/1281498857xx^14 + 164497611264/11533147875xx^12 - 4645427448388/336085375xx^10 + 23135518702/11533147875xx^8 - 146963387581772/464716573xx^6 + 288326776220/464716573xx^4 + 2526257 \) |
| 14 | \( (8) \)                        |
| 16 | \( (1447188) * (x - 2) * (x + 2) * x^10 * (x^18 - 119828714684/46563957625xx^16 + 118536982545627738/52375597245/52375597245xx^14 + 143799693137766/13998147417xx^12 - 21818166102/5513795739/5513795739xx^10 - 61748798857449849925/9691985739xx^8 - 20468539529268794/9691985739xx^6 + 264878958587723/9691985739xx^4 + 578125 \) |
| 18 | \( (10) \)                        |
| 20 | \( (40) \)                        |

**Factored \( C_n(x) \) (Conjecture 1.)**

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106 *op. cit.*
### 12.4 Table 4.

#### n: 3

| m  | 1  | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|----|----|---|---|---|---|---|---|---|---|----|
| 31 | 1823X | 18,495X² | 17,789X³ | 45,767X⁴ | 41,658,308,875X⁵ |
| 32 | 27,648 | 2519,424 | 18,345,885,696 | 34,828,517,376 | 3,327,916,660,116,655,468 |
| 711,997X⁶ | 1,663,962,743,405X⁷ | 1,821,044,125X⁸ |
| 7,763,518,787,253,184 | 2,944,327,674,199,668,893,831,168 |
| 339,351,370,311,776,179,508,288 |

#### n: 4

| m  | 1  | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|----|----|---|---|---|---|---|---|---|---|----|
| 13 | 1893X | 47X² | 628,001X³ | 653X⁴ |
| 32 | 16,384 | 8,192 | 2,147,483,648 | 67,168,864 |
| 9,383,515X⁵ | 52,677X⁶ | 2,206,741,887X⁷ |
| 35,184,372,882,832 | 8,796,953,622,288 |
| 18,446,744,073,709,551,616 |
| 36,628,797,919,963,968 |

#### n: 5

| m  | 1  | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|----|----|---|---|---|---|---|---|---|---|----|
| 79 | 42,877X | 12,957X² | 1,335,816,657X³ | 1,493,611,263X⁴ | 1,458,495,926,643X⁵ |
| 200 | 648,009 | 2,000,000 | 3,276,868,088 | 80,000,000,000 | 2,097,152,000,000,000 |
| 64,664,568,664,389X⁶ | 3,494,046,888,864,913,731X⁷ |
| 23,644,862,224,068,813X⁸ |
| 2,867,280,000,000,000,000,000,000 |
| 3,568,709,120,000,000,000,000,000 |
| 1,376,256,888,000,000,000,000,000 |
| 77,191X⁹ |

#### n: 6

| m  | 1  | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|----|----|---|---|---|---|---|---|---|---|----|
| 39 | 42,584,624 | 19,059,972 | 1,835,855,555,787,866 |
| 4,921,648,952,328 | 118,895,751,725,676,756,992 |
| 36,826,135,396,421,541X⁴ |
| 28,845,419,590,657,658,847X⁵ |
| 53,674,329,848,187,738,667X⁶ |
| 624,462,781,816,782,892,113,920 |
| 9,354,238,358,195,389,311,446,368,256 |
| 698,893,631,231,484,559,139,312,580,736 |

#### n: 7

| m  | 1  | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|----|----|---|---|---|---|---|---|---|---|----|
| 151 | 165,229 | 107,365 | 30,493,858,865 | 2,771,867,459X⁴ |
| 192 | 2,458,634 | 15,659,972 | 18,315,055,555,787,866 |
| 32,501,648,952,328 | 118,895,751,725,676,756,992 |
| 36,826,135,396,421,541X⁴ |
| 28,845,419,590,657,658,847X⁵ |
| 53,674,329,848,187,738,667X⁶ |
| 624,462,781,816,782,892,113,920 |
| 9,354,238,358,195,389,311,446,368,256 |
| 698,893,631,231,484,559,139,312,580,736 |

#### n: 8

| m  | 1  | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|----|----|---|---|---|---|---|---|---|---|----|
| 247 | 158,671 | 13,589,191 | 69,901,012,827 |
| 648 | 2,239,488 | 1,836,668,096 | 128,367,356,051,456 |
| 366,776,621,371X⁸ |
| 3,257,500,444,698,134,635X⁹ |
| 108,551,656,689,834,559X¹⁰ |
| 10,282,945,612,677,128 | 1,768,551,357,765,866,663,198,208 |
| 1,283,597,865,037,611,254,415,366 |
| 2,222,620,238,316,981,329,883,361X¹¹ |
| 641,719,347,824,464,135,628,559X¹² |
| 633,718,259,619,258,583,956,804,550,860,648 |
| 4,737,105,877,202,748,268,298,391,042,949,120 |

#### n: 9

| m  | 1  | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|----|----|---|---|---|---|---|---|---|---|----|
| 19 | 673X | 791X² | 59,679X³ | 2,194,921X⁴ |
| 10 | 18,000 | 32,758 | 138,000,000 | 58,593,756,000 |
| 254,289,321X⁶ | 89,594,891,393X⁷ |
| 87,629,178,911X⁸ |
| 2,734,375,088,000,000 |

---

**Fourier coefficients** $a_m(n)$ (conjecture 2.)

---

**Notebook “conjecture 2.nb”**
12.5 Table 5.

| n: 1 | 1 |
|---|---|
| n: 0 | 1 3 x^2 2 8 |
| n: 1 | 3 64 128 1024 |
| n: 2 | 3 373 21 809 x^4 7961 x^6 16367 x^8 3 x^{10} |
| n: 3 | 72 800 172 800 1382 498 18432 000 65536 |
| n: 4 | 4754 693 1 562 805 647 616 81 257 768 976 186 156 872 72 131 837 393 88 10 8  |
| n: 5 | 8 241 137 8 241 137 x^4 2 556 608 000 7 225 344 000 825 765 600  |
| n: 6 | 165 768 344 647 | 165 768 344 647 x^4 | 6 257 828 776 189 x^6 | 4 912 564 839 x^8 | 38 381 472 457 973 x^{10} |
| n: 7 | 108 108 ibid. |
| n: 8 | 124 848 553 281 | 124 848 553 281 x^2 | 185 194 889 077 x^4 | 1351 158 418 331 x^6 | 8 955 889 117 293 x^8 | 41 x^{10} |

Polynomials $A_n(x)$ (conjecture 2.)
### Table 6.

| n  | \( (1, 1) \) |
|----|--------------|
| n=1| \[
\left\{ \left\{ \frac{3}{3} x^4, \frac{4}{3} x^3, 1 \right\}, \right. \] |
| n=2| \[
\left\{ \left\{ \frac{1}{128}, \left\{ -2 x, 1 \right\}, \left\{ 2 x, 1 \right\}, \left\{ 16, \frac{8}{27} x^2 + x^6, 1 \right\} \right. \right. \] |
| n=3| \[
\left\{ \left\{ \frac{5681}{5681}, \left\{ -2 x, 1 \right\}, \left\{ -2 x, 1 \right\}, \left\{ 6464, 11312 x^2 - 38732 x^6 + x^8, 1 \right\} \right. \right. \] |
| n=4| \[
\left\{ \left\{ 65536, \left\{ -2 x, 1 \right\}, \left\{ 2 x, 1 \right\}, \left\{ 95448, 190976 x^2 - 660144 x^6 + 51968 x^8, 1 \right\} \right. \right. \] |
| n=5| \[
\left\{ \left\{ \frac{23083}{16769}, \left\{ -2 x, 1 \right\}, \left\{ -2 x, 1 \right\}, \left\{ 4868895632, 11217201408 x^2 - 452733566 x^6 + 373957344 x^8 + 15545948 x^{10}, 1 \right\} \right. \right. \] |
| n=6| \[
\left\{ \left\{ \frac{3}{137438953472}, \left\{ -2 x, 1 \right\}, \left\{ -2 x, 1 \right\}, \left\{ 818207482 580 736, 818207482 580 736 x^2 - 25262578 868 992 928 x^6 + 1469613 675 817 728 x^{10}, 1 \right\} \right. \right. \] |

\[ A_n(x) \text{ FACTORED IN Mathematica (conjecture 2.)} \quad \text{(109)} \]

\[ \text{(op. cit.)} \]
12.7 Table 7.

\[
\begin{array}{c|c|c}
\hline
n & \text{factor} & \text{value} \\
\hline
1 & x^2 + 4 & 1 \\
2 & x^2 + 2 & 0 \\
3 & x^2 + 1 & 0 \\
4 & x^2 + 1 & 0 \\
5 & x^2 + 1 & 0 \\
6 & x^2 + 1 & 0 \\
\hline
\end{array}
\]

\[A_n(x) \text{ factored in } SageMath \text{ (conjecture 2.)} \]

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\footnote{Notebook “conjecture 2 clause 1b.ipynb”}

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