Long-time asymptotic analysis for defocusing Ablowitz-Ladik system with initial value in lower regularity

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Abstract

Recently, we have given the \( l^2 \) bijectivity for defocusing Ablowitz-Ladik systems in the discrete Sobolev space \( l^{2,1} \) by inverse spectral method. Based on these results, the goal of this article is to investigate the long-time asymptotic property for the initial-valued problem of the defocusing Ablowitz-Ladik system with initial potential in lower regularity. The main idea is to perform proper deformations and analysis to the correspondent Riemann-Hilbert problem with the unit circle as the jump contour \( \Sigma \). As a result, we show that when \( |\frac{n}{2t}| \leq 1 < 1 \), the solution admits Zakharov-Manakov type formula, and when \( |\frac{n}{2t}| \geq 1 > 1 \), the solution decays fast to zero.

Key Words: defocusing Ablowitz-Ladik system, inverse spectral method, long-time asymptotic property, Riemann-Hilbert problem

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1 Introduction

It’s well known that inverse scattering transform is an effective method when solving the integrable system. In 1967, the inverse scattering transform is firstly introduced when solving the KdV equation by Gardne, Greene, Kruskal and Miura [1]. This method is also applied to ZS-AKNS systems [2]. More literatures for solving the continuous and discrete integrable systems refers to [3–19]. Except for solving initial-valued problem, this method also have a great number of important results in mathematics and physics. Particularly, Deift and Zhou apply this method to the nonlinear Schrödinger equations [20, 21] and modified KdV equations [22] to obtain solutions, and further develop a nonlinear steepest descent method to study the long-time asymptotic analysis with potentials in Schwartz space. This method also has been applied to numerous integrable systems for the long-time asymptotic analysis [23–33]. Recent years, people become interested in extending the long-time asymptotic analysis for the integrable system with lower regularity. In particular, for nonlinear Schrödinger equation as one of the most important integrable systems, based on the $L^2$-Sobolev bijectivity for the inverse scattering transform [34], with a dbar steepest descent method, people investigate the long-time asymptotic property when the initial potential belongs to a weighted Sobolev space:

$$H^{1,1} = \{ f \in L^2(\mathbb{R}) : xf, x^2f' \in L^2(\mathbb{R}) \}.$$
In this paper, we focus on defocusing Ablowitz-Ladik systems
\[ i\partial_t q_n(t) = q_{n+1}(t) - 2q_n(t) + q_{n-1}(t) - |q_n(t)|^2(q_{n+1}(t) + q_{n-1}(t)), \] (1)
that is integrable systems introduced by Ablowitz and Ladik [35, 36] in 1975-1976, where
\( n \in \mathbb{Z} \) is the discrete spatial variable and \( t \in \mathbb{R}^+ \) is the continuous variable. It’s shown as
the spatial integrable discretization of the defocusing nonlinear Schrödinger equation:
\[ iu_t + u_{xx} - 2|u|^2u = 0, \]
and there are many important researches for it in aspects of mathematics and physics. In
2005, Nenciu [37] constructs the Lax pair for defocusing Ablowitz-Ladik systems by the
connection between defocusing Ablowitz-Ladik systems and the orthogonal polynomials on
the unit circle. In 2006, Chow et al [38] show the analytic doubly periodic waves pattern for
Ablowitz-Ladik systems. Using methods of algebraic geometry, Miller et al [39] obtain the
finite genus solutions for Ablowtiz-Ladik systems. Recently, Yamane [40] studies the long-
time asymptotic behavior by Deift-Zhou method with initial data bounded by \( \sup_{n \in \mathbb{Z}} |q_n| < 1 \)
and in the discrete Schwartz space:
\[ \{q_n\}_{n=-\infty}^{\infty} \in \left\{ \{a_n\}_{n=-\infty}^{\infty} : \sum_{n=-\infty}^{\infty} (1 + n^2)^k a_n^2 < \infty, \ k \in \mathbb{N}^+ \right\}. \]
Here, we present the long-time asymptotic analysis for initial-valued problem of (1) with
initial potential in lower regularity, and it satisfies
\[ \{q_n\}_{n=-\infty}^{\infty} \in l^{2,1} = \left\{ \{a_n\}_{n=-\infty}^{\infty} : \sum_{n=-\infty}^{\infty} (1 + n^2)a_n^2 < \infty \right\}, \sup_{n \in \mathbb{Z}} |q_n| < 1, \] (2)
where \( l^{2,1} \) is a discrete weighted Sobolev space.

For the initial potential satisfies (2), recently, we prove in [41] that the direct scattering
mapping maps these potentials to reflection coefficients \( r(\lambda) \) which belongs to a Sobolev
space \( H^1_\theta(\Sigma) \) and is bounded by \( \| r \|_{L^\infty(\Sigma)} < 1 \), where \( \Sigma \) is the jump contour shown in
Figure 1 and \( \theta \in [0, 2\pi] \) is the parameter of it. Reversely, by the inverse scattering mapping,
if reflection coefficients belong to $H^1_\theta(\Sigma)$ and is bounded by $\| r \|_{L^\infty(\Sigma)} < 1$, potentials also belongs to the discrete weighted Sobolev space $l^{2,1}$. In fact, by the argument in [41], these two mappings is of Lipschitz continuity. Moreover, for Ablowitz-Ladik systems, if we denote $r(\lambda, t)$ the reflection coefficient for $q_n(t)$, the time flow

$$ r(\lambda) \equiv r(\lambda, 0) \mapsto r(\lambda, t) = r(\lambda)e^{2i(\cos \theta - 1)t}, \quad \lambda = e^{i\theta} \in \Sigma, $$

persists the reflection coefficient in $H^1_\theta(\Sigma)$ and bounded by $\| r(\cdot, t) \|_{L^\infty(\Sigma)} < 1$. In (2), the weighted Sobolev space $l^{2,1}$ is the minimal condition for the solution solved by the inverse scattering transform.

Since $r \in H^1_\theta(\Sigma)$, we obtain Riemann-Hilbert (RH) problem 2.1 and it couldn’t applied proper rational approximation to $r(\lambda)$; therefore, it’s hard to extend the jump contour and make the RH transform like that in [40]. For $r \in H^1_\theta(\Sigma)$, by Sobolev embedding theorem, it’s also $\frac{1}{2}$-Hölder continuous on $\Sigma$; as a result, we deform RH problem 2.1 properly into a $\bar{\partial}$-RH problem; moreover, solution for this $\bar{\partial}$-RH problem can be factorized into a product of solutions for an RH problem and a $\bar{\partial}$ problem; and, we further analyze this two problems separately and finally obtain the long-time asymptotic property. This idea is generalized from the dbar steepest descent method that people applied to orthogonal polynomials and nonlinear Schrödinger equations [42, 43]. This method make it eligible to analyze the long-time asymptotic property under the condition $r \in H^1_\theta(\Sigma)$. In our method, the jump contour
is a unit circle and there are two separate first-order stationary phase points, which make the deformation of RH problem more tricky.

In this paper, for Hilbert spaces $P_1$ and $P_2$, we denote $B(P_1, P_2)$ as the normed linear space consisting of all linear bounded operators from $P_1$ to $P_2$, and simply write $B(P_1, P_1)$ as $B(P_1)$.

The article is organized as follows. In Section 2, according to Lax pair, we give the direct scattering including Jost solutions, modified Jost solution and the reflection coefficient, and then perform inverse scattering transform to obtain the reconstructed formula by constructing the correspondent RH problem. In Section 3, we analyze the long-time asymptotic behavior on the region $-1 < -V_0 \leq \xi \leq V_0 < 1$. In Section 4 and 5, we analyze the long-time asymptotics for $|\xi| \geq V_0 > 1$.

### 1.1 Main results

In this paper, we study the long-time asymptotic analysis on three regions as shown in Figure 2.

When $-1 < -V_0 \leq \xi \leq V_0 < 1$, we see that first-order stationary phase points $S_j = i\xi + \sqrt{1 - \xi^2}$ are on the jump contour $\Sigma$. Since these stationary phase points appear on
Σ, the reflection coefficient at $S_j$ has impact to the solution; so, we come to investigate the local RH problem at $S_j$ on Section 3.4, and make scaling transform for them to obtain the model RH problem that is related to parabolic cylinder functions, which refers to [44]. By the model RH problem, we obtain the oscillatory leading term that is $O(t^{-\frac{1}{2}})$ as shown in (3), and we obtain that as $t \to +\infty$, the solution in Zakharov-Manakov type formulas (3).

It’s notable that under the condition $\{ q_n \}_{n=-\infty}^{\infty} \in L^{2,1}$, the decaying rate is $O(t^{-1})$.

When $|\xi| \geq V_0 > 1$, stationary phase points are pure imaginary number and off the jump contour $\Sigma$; in this case, the leading term in (3) vanishes, and $q_n(t)$ decays fast when $t \to +\infty$, and we call these regions as fast decaying region.

**Theorem 1.1.** For the initial-valued problem (1)-(2), the solution $q_n(t)$ admits the following long-time asymptotic formula in the sectors obtained by dividing the half plane by rays $\frac{\pi}{2\pi} = \pm 1$, which is shown in Figure 2.

a) The Zakharov-Manakov region $-1 < -V_0 \leq \xi \leq V_0 < 1$ with some positive constant $V_0$: as $t \to +\infty$, the solution admits Zakharov-Manakov type formulas:

$$q_n(t) = i2t^{-\frac{1}{4}}(1 - \xi^2)^{-\frac{1}{4}}\delta^{-1}(0)\sum_{j=1}^{2} \delta_{j0}[M_{1}\lambda_{j}^{L,j}]_{1,2} + O(t^{-\frac{1}{2}}), \quad (3)$$

where $r(S_j)$ is the value of reflection coefficients at stationary phase points $S_j$, $j = 1, 2$, $\int_{S_2}^{S_1}$ denotes the integral on $\Sigma$ from $S_2$ to $S_1$, $\Gamma(\lambda)$ is the Euler’s gamma function,

$$\begin{align*}
S_j &= -i\xi + (-1)^j \sqrt{1 - \xi^2}, \quad \beta_j = \frac{it^{-\frac{1}{2}}}{2}(1 - \xi^2)^{-\frac{1}{2}}S_j, \quad \nu_j = -\frac{1}{2\pi} \ln(1 - |r(S_j)|^2) \\
\delta^{-1}(0) &= e^{\frac{1}{2\pi} \int_{S_2}^{S_1} s^{-1} \ln(1 - |r(s)|^2) ds}, \quad \phi(S_j) = 2((-1)^j \sqrt{1 - \xi^2} - \xi \arg S_j - 1), \\
\alpha_j(S_j) &= \frac{1}{2\pi i} \int_{S_2}^{S_1} \frac{\ln(1 - |r(s)|^2) - \ln(1 - |r(S_j)|^2)}{s - S_j} ds, \\
\delta_{j0} &= e^{\alpha_j(S_j) - \frac{\nu_j}{2}\phi(S_j)} \left( \frac{(-1)^{j-1} \beta_j}{S_1 - S_2} \right)^{1 - \nu_j}. 
\end{align*}$$
b) The fast decaying region $|\xi| \geq V_0 > 1$: as $t \to +\infty$, the solution decays to zero

$$q_n(t) \sim O(t^{-1}).$$

2 Riemann-Hilbert problem and the solution for defocusing Ablowitz-Ladik systems

In this section, we investigate Jost solutions: $X^\pm$ for the Lax pair correspondent to the initial-valued problem, make a transform for Jost solutions to obtain a modified Jost solution: $X^\pm \to Y^\pm$, and then construct the RH problem. The Lax pair for (1) is:

$$X(z, n+1, t) = \begin{bmatrix} z & q_n(t) \\ \frac{q_n(t)}{z} & z^{-1} \end{bmatrix} X(z, n, t), \quad \text{(4a)}$$

$$\partial_t X(z, n, t) = i \left[ \frac{1}{2} (z - z^{-1})^2 + \frac{q_n(t)q_{n-1}(t)}{q_n(t)z - q_{n-1}(t)z} \right] \frac{1}{2} (z - z^{-1})^2 - \frac{q_n(t)}{q_{n-1}(t)} \frac{1}{2} (z - z^{-1})^2 - \frac{q_{n-1}(t)}{q_n(t)} X(z, n, t). \quad \text{(4b)}$$

where $z$ is the spectral parameter and the Lax pair admits the Jost solution,

$$X^\pm \equiv X^\pm(z, n, t) \sim z^{n\sigma_3} e^{-\frac{i}{2}(z-z^{-1})^2\sigma_3}, \quad n \to \pm \infty. \quad \text{(5)}$$

Seeing from the spatial problem (4a), we derive that

$$\det X^\pm = \prod_{k=-\infty}^{n-1} (1 - |q_k(t)|^2), \quad \det X^+ = \prod_{k=n}^{\infty} (1 - |q_k(t)|^2)^{-1}. \quad \text{(6)}$$

Introducing

$$c_n(t) = \prod_{k=n}^{\infty} (1 - |q_k(t)|^2), \quad n \in \mathbb{Z} \cup \{-\infty\},$$

because of assumption (2), from the result of [41], we learn that $c_{-\infty} = \lim_{t \to -\infty} c_n(t)$ is nonzero and independent on $t$; therefore, for any fixed $t \leq 0$,

$$1 - |q_n(t)|^2 \neq 0, \quad n \in \mathbb{Z},$$
which combined with (4a) deduces that $X^\pm$ are invertible and fundamental solutions for the Lax pair; moreover, by the uniqueness of solution, there is a unique $2 \times 2$ matrix-valued function depending only on $z$ such that

$$X^-(z, n, t) = X^+(z, n, t)S(z), \quad S(z) = \begin{bmatrix} a(z) & b(z) \\ b(z) & \bar{a}(z) \end{bmatrix}. \tag{7}$$

$S(z)$ is also known as the scattering matrix. Since $\left[\begin{array}{cc} 0 & q_n(t) \\ q_n(t) & 0 \end{array}\right]$ is Hermitian, we learn from the spatial problem (4a) that $X^\pm$ admit the following symmetric property

$$X^\pm(z, n, t) = \sigma_1 X^\pm(\bar{z}, n, t)\sigma_1, \quad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

which combined with (7) deduces symmetry properties of $S(z)$:

$$\bar{a}(z) = a(\bar{z}^{-1}), \quad \bar{b}(z) = b(\bar{z}^{-1}), \quad z \in \mathbb{C}. \tag{8}$$

Making the transformation

$$Y^\pm(z, n, t) = \begin{bmatrix} 1 & 0 \\ 0 & z \end{bmatrix} X^\pm(z, n, t)z^{-n}\sigma_3 e^{\frac{2}{z-\bar{z}^{-1}}^2\sigma_3} \begin{bmatrix} 1 & 0 \\ 0 & z^{-1} \end{bmatrix}, \tag{9}$$

from property (5), we learn that $Y^\pm(z, n, t)$ satisfy the normalization property

$$Y^\pm(z, n, t) \sim I, \quad n \to \pm\infty. \tag{10}$$

Making the spectral parameter transform: $\lambda = z^2$, seeing (2) and (8), we deduce that $Y^\pm(z, n, t) = Y^\pm(-z, n, t)$; thus, we simply write $Y^\pm(\lambda, n, t) = Y^\pm(z, n, t)$. Seeing from [41], we learn that $Y_1^+$ and $Y_2^-$ are both holomorphic on $D^- = \{\|\lambda\| < 1\}$ and continuously extended to $D^- \cup \Sigma$, $Y_1^-$ and $Y_2^+$ are both holomorphic on $D^+ = \{\|\lambda\| > 1\}$ and continuously extended to $D^+ \cup \Sigma$. By (7) and (9), we derive that

$$a(z) = c_n(t) \det[Y_1^-, Y_2^+](\lambda, n, t), \quad zb(z) = c_n(t)e^{-it\phi(\lambda, n, t)} \det[Y_1^+, Y_1^-](\lambda, n, t), \tag{11}$$

where

$$\phi(\lambda, n, t) = \lambda + \lambda^{-1} + 2i\xi \log \lambda - 2;$$
since then, we write \( a(\lambda = z^2) \) instead of \( a(z) \). Because \( [Y_1^+, Y_2^-] \) is holomorphic on \( D^- \) and continuously extended to \( D^- \cup \Sigma \), by (11), \( a(\lambda) \) is holomorphic on \( D^- \) and continuously extended to \( D^+ \cup \Sigma \), too; moreover, because of the continuity of \( [Y_1^+, Y_1^-] \) on \( \Sigma \), \( zb(z) \) is also continuous on \( \Sigma \); therefore, we define the reflection coefficients on the circle \( \lambda = e^{i\theta} \in \Sigma \):

\[
r(\lambda) = \frac{zb(z)}{a(z)},
\]

which is well-defined and belongs to \( H_0^1(\Sigma) \) according to [41]. Sometimes, we denote \( r(\theta) = r(e^{i\theta}) \) without confusion of notation on the jump contour \( \Sigma \). Since \( r \in H_0^1(\Sigma) \), by Sobolev embedding theory, we learn that \( r(\theta) \) is \( \frac{1}{2} \)-Hölder continuous and bounded on \( \Sigma \). By (6), (7) and (8), we see that

\[
1 - |r(\theta)|^2 = \frac{c^{-\infty}}{|a(e^{i\theta})|} > 0, \quad \theta \in [0, 2\pi].
\]

Seeing from [41] and setting \( Y^\pm = [Y_1^\pm, Y_2^\pm] \), we learn that as \( \lambda \to \infty \) and \( \lambda \to 0 \), \( Y^\pm(z, n, t) \) admit the following asymptotic property:

\[
[Y_1^+, Y_2^-](\lambda, n, t) \sim \begin{cases} 
\begin{bmatrix} c_n^{-1}(t) + O(\lambda) & q_n^{-1}(t) + O(\lambda) \\
-q_n^{-1}(t)q_n(t)\lambda + O(\lambda^2) & 1 + O(\lambda) \end{bmatrix}, & \lambda \to 0, \\
\begin{bmatrix} 1 + O(\lambda^{-1}) & -c_n^{-1}(t)q_n(t)\lambda^{-1} + O(\lambda^{-2}) \\
q_n^{-1}(t) + O(\lambda^{-1}) & c_n^{-1}(t) + O(\lambda^{-1}) \end{bmatrix}, & \lambda \to \infty.
\end{cases}
\]

(13a) (13b)

From (11) and (13b), we learn that

\[
\lim_{\lambda \to \infty} a(\lambda) = 1.
\]

(14)

We come to the construction of correspondent RH problems. Set a \( 2 \times 2 \) matrix-valued function

\[
M \equiv M(\lambda, n, t) = \begin{cases} 
\begin{bmatrix} 1 & 0 \\
-q_n^{-1}(t)c_n(t) & c_n(t) \end{bmatrix} \begin{bmatrix} Y_1^+ \ Y_2^- \end{bmatrix}, & |\lambda| < 1, \\
\begin{bmatrix} 1 & 0 \\
-q_n^{-1}(t)c_n(t) & c_n(t) \end{bmatrix} \begin{bmatrix} Y_1^- \ Y_2^+ \end{bmatrix}, & |\lambda| > 1;
\end{cases}
\]

(15)
then, we claim that $M$ admits the following RH problem. The first item comes from that of $Y^\pm(\lambda, n, t)$ and $a(\lambda)$. The second item naturally follows after the definition of $M(\lambda, n, t)$, (9) and (15) deduce the third item. Here, the orientation of jump contour is clockwise, and as general notation, we call the left/right side of the jump contour as $+/-$ side. As follows, we obtain the reconstructed formula from (8), (13a), (14) and (15) that the reconstructed formula is

$$q_n(t) = M_{1,2}(0, n + 1, t).$$  \hspace{1cm} (16)

**RH problem 2.1.** Find a $2 \times 2$ matrix-valued function $M$ such that

- $M$ is analytic on $\Sigma$.
- As $\lambda \to \infty$, $M \sim I + O(\lambda^{-1})$.
- On $\lambda \in \Sigma$,

$$M_+ = M_- V, \quad V \equiv V(\lambda, n, t) = \begin{bmatrix} 1 - |r(\lambda)|^2 & -r(\lambda)e^{-it\phi(\lambda, n, t)} \\ r(\lambda)e^{it\phi(\lambda, n, t)} & 1 \end{bmatrix}. $$

## 3 Long-time asymptotic analysis on $-1 < -V_0 \leq \xi \leq V_0 < 1$

In this section, we study the case for the region $-1 < -V_0 \leq \xi \leq V_0 < 1$. In this case, we derive that there are two first-order stationary phase points $S_1$ and $S_2$ on the unit circle $\Sigma$: therefore, we obtain that

$$\phi(\lambda, n, t) - \phi(S_j, n, t) \sim \frac{\phi''(S_j, n, t)}{2}(\lambda - S_j)^2 + O(\lambda - S_j)^3, $$

$$\phi''(S_j, n, t) = (-1)^j S_j^{-2} \sqrt{1 - \xi^2}. $$
3.1 Deformation on the jump contour

In this part, we study an RH problem transform: \( M \leadsto M^{(1)} = M\delta^{-\sigma}; \) and the new jump matrix admits a proper factorization, seeing in RH problem 3.3.

Introducing the scalar function

\[
\delta \equiv \delta(\lambda) = e^{\frac{2\pi i}{\Omega}} \int_{\hat{S}_2}^{S_1} \frac{\ln(1 - |r(\lambda)|)}{s - \lambda} ds,
\]

where the integral is along the lower-half arc on \( \Sigma: \hat{S}_2S_1 \) and this arc’s orientation is from \( S_2 \) to \( S_1 \) along \( \Sigma \), we derive that \( \delta \) admits properties shown in Proposition 3.1

**Proposition 3.1.** Since \( r \in H^1_{\Omega}(\Sigma) \), the scalar function \( \delta \) satisfies:

(a) \( \delta \) is analytic on \( \mathbb{C} \setminus \hat{S}_2S_1 \).

(b) As \( \lambda \to \infty \), \( \delta(\lambda) \sim 1 + O(\lambda^{-1}) \).

(c) On the arc \( \lambda \in \hat{S}_2S_1 \), \( \delta_+(\lambda) = \delta_-(\lambda)(1 - |r(\lambda)|^2) \).

(d) On \( \lambda \in \mathbb{C} \setminus \hat{S}_2S_1 \), \( \delta \) admits the symmetry:

\[
\delta(\lambda) = \frac{\delta(0)}{\delta(\lambda^{-1})}.
\]
(e) On the neighborhood of $S_1$ and $S_2$, $\delta$ admits following asymptotic properties:

$$\delta(\lambda) \sim \left(\frac{\lambda - S_1}{\lambda - S_2}\right)^{i\nu_j} e^{\alpha_j(S_j)} + O(|\lambda - S_j|^{\frac{1}{2}}), \quad \lambda \to S_j,$$

$$\alpha_j(\lambda) = \frac{1}{2\pi i} \int_{S_2}^{S_1} \frac{\ln(1 - |r(s)|^2) - \ln(1 - |r(S_j)|^2)}{s - \lambda} \, ds,$$

$$\nu_j = -\frac{\ln(1 - |r(S_j)|^2)}{2\pi}, \quad j = 1, 2.$$

Proof. (a) comes from the analyticity of $Y^\pm(\lambda, n, t)$, $a(\lambda)$ and (15). (b) comes from (13b) and (15). (c) comes from (7) and (9). Since $\delta(\lambda)$ satisfies (a-c) which determine an RH problem with the jump contour $S_2S_1$, we also verify by computation that $\delta(0)/\delta(\lambda^*)$ is also the solution of this RH problem; then, by uniqueness of solution, we derive (d). Changing the variable on $\in \Sigma$: $\lambda \to \lambda'$, $s \to s'$,

$$\lambda = \sqrt{S_1S_2} \sqrt{\frac{S_1 + \sqrt{S_2} \lambda'}{\sqrt{S_2} + \sqrt{S_1} \lambda'}}, \quad s = \sqrt{S_1S_2} \sqrt{\frac{S_1 + \sqrt{S_2} s'}{\sqrt{S_2} + \sqrt{S_1} s'}},$$

we obtain that

$$\int_{S_2}^{S_1} \left(\frac{f(s) - f(S_1)}{2\pi i}(s - \lambda) \, ds = \frac{\sqrt{S_2} + \sqrt{S_1} \lambda'}{2\pi i} \int_{-\infty}^{0} \frac{f(s) - f(S_1)}{\sqrt{S_2} + \sqrt{S_1} s'}(s' - \lambda') \, ds' \right). \quad (19)$$

where $f(s) = \ln(1 - |r(s)|^2)$. Since $r(s) \in H^1_0(\Sigma) \subset L^\infty(\Sigma)$, it naturally follows that

$$F(s') = \begin{cases} \frac{f(s) - f(S_1)}{\sqrt{S_2} + \sqrt{S_1} s'}, & s' \leq 0, \\ 0, & s' > 0, \end{cases} \quad (20)$$

belongs to $H^1(\mathbb{R})$ and

$$F(s = 0) = 0;$$

therefore, we apply Lemma 23.3 in [45] and get that for any $\lambda \in \mathbb{C} \setminus \mathbb{R}$

$$\left| \int_{-\infty}^{+\infty} \frac{F(s')}{(2\pi i)(s' - \lambda')} \, ds' - \int_{-\infty}^{+\infty} \frac{F(s')}{2\pi i s'} \, ds' \right| \lesssim \| F \|_{H^1} |\lambda'|^{\frac{1}{2}} \quad (21)$$

Considering (19), (20) and (21), we obtain (e) for $j = 1$, and the proof for $j = 2$ is parallel.

We confirm the result. \qed
Remark 3.2. Seeing property (c) and (d) in Proposition 3.1, we learn that for $\lambda \in \bar{S}_2S_1$,

$$|\delta_+(\lambda)|^2 = (1 - |r(s)|^2)\delta(0), \quad |\delta_-(\lambda)|^2 = \delta(0)(1 - |r(\lambda)|^2)^{-1}.$$  

which are nonzero and finite by (12) and (19); moreover, $\delta(\lambda) \to 1$ as $\lambda \to \infty$; so, $\delta$ and $\delta^{-1}$ are both bounded function by (a) in Proposition 3.1 and the maximal modular theorem.

Here, we confirm that a new $2 \times 2$ matrix-valued function

$$M^{(1)} = M\delta^{-\sigma_3}, \quad (22)$$

satisfies RH problem 3.3, which is the natural consequence of RH problem 2.1, (a-c) in Proposition 3.1 and (22).

**RH problem 3.3.** Find a $2 \times 2$ matrix-valued function such that

- $M^{(1)}$ is analytic on $\mathbb{C} \setminus \Sigma$.
- As $\lambda \to \infty$, $M^{(1)}(\lambda, n, t) \sim I + O(\lambda^{-1})$.
- On $\lambda \in \Sigma$,

$$M^{(1)}_+ = M^{(1)}_V,$$

where $V^{(1)}(\lambda, n, t)$ is the jump matrix

$$V^{(1)} = \begin{cases}
\left[ \begin{array}{cc} 1 & \delta^2 \frac{r(\lambda)}{1 - |r(\lambda)|^2} e^{-i\phi(\lambda, n, t)} \\ r(\lambda) \delta^2(\lambda) e^{-i\phi(\lambda, n, t)} & 1 \end{array} \right] & \lambda \in \bar{S}_2S_1, \\
\left[ \begin{array}{cc} \delta^2 & 0 \\ 0 & 1 \end{array} \right] & \lambda \in \Sigma \setminus \bar{S}_2S_1.
\end{cases}$$
3.2 Split the circle

In this part, we introduce a new RH problem transform: $M^{(1)} \sim M^{(2)}$. As a result, the jump contour $\Sigma$ is deformed into $\Sigma^{(2)}$, where the jump matrix is decaying.

Introducing the jump contour $\Sigma^{(2)} = L \cup \tilde{L}$ as shown in Figure 4, we see that $L$ and $\tilde{L}$ are both closed curves consisting of straight line segments and arcs centering at the origin. As for orientation of the jump contour, we denote it in Figure 4. Again seeing Figure 4, the complex plane is divided into six parts by $\Sigma^{(2)}$ and $\Sigma$: $\Omega_j$, $j = 1, \ldots, 6$. Since we have constructed the jump contour, introduce a $2 \times 2$ matrix-valued function

$$M^{(2)} = M^{(1)}P,$$  (23)

Figure 4: Jump contour $\Sigma^{(2)} = L \cup \tilde{L}$. 

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where the $2 \times 2$ matrix-valued function $\mathcal{P}$ is invertible and written as

$$
\mathcal{P} \equiv \mathcal{P}(\lambda, n, t) = \begin{cases}
\begin{bmatrix} 1 & 0 \\ -R_1(\lambda)\delta^{-2}(\lambda)e^{it\phi(\lambda, n, t)} & 1 \end{bmatrix} & \lambda \in \Omega_1, \\
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \lambda \in \Omega_3, \\
\begin{bmatrix} R_4(\lambda)\delta^{-2}(\lambda)e^{it\phi(\lambda, n, t)} & 0 \\ 1 & -R_6(\lambda)\delta^2(\lambda)e^{-it\phi(\lambda, n, t)} \end{bmatrix} & \lambda \in \Omega_4, \\
\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} & \lambda \in \Omega_6, \\
I & \lambda \in \Omega_2 \cup \Omega_5
\end{cases}
$$

and we denote $\lambda = \rho e^{i\theta}$, $\rho \geq 0$, $\theta \in [0, 2\pi]$, and

$$R_1(\lambda) = r(\theta), \quad R_3(\lambda) = \frac{r(\theta)}{1 - |r(\theta)|^2}, \quad R_4(\lambda) = \frac{r(\theta)}{1 - |r(\theta)|^2}, \quad R_6(\lambda) = \frac{r(\theta)}{1 - |r(\theta)|^2}.$$  

Recalling (12) and that the reflection coefficients $r$ is bounded on $\Sigma$, we see by (25) that $R_1(\lambda)$, $R_3(\lambda)$, $R_4(\lambda)$ and $R_6(\lambda)$ are bounded on $\mathbb{C}$. Then, as a consequence of RH problem 3.3 and (23), we deduce that $M^{(2)}$ satisfies $\bar{\partial}$-RH problem 3.4, and the jump matrix admits the lower/upper triangular factorization:

$$V^{(2)} = (1 - w_-)^{-1}(1 + w_+),$$
where we denote
\[ w = w_+ + w_-, \] (26)

\[
w_+ = \begin{cases} \begin{bmatrix} 0 & -\delta^2(\lambda)R_3(\lambda)e^{-it\phi(\lambda, n, t)} \\ 0 & 0 \\ 0 & -\delta^2(\lambda)R_6(\lambda)e^{-it\phi(\lambda, n, t)} \end{bmatrix} & \text{if } \tilde{L} \cap D_+, \\ 0 & \text{if } \tilde{L} \cap D_, \end{cases} \] (27)

\[
w_- = \begin{cases} \begin{bmatrix} 0 & 0 \\ \delta^2(\lambda)R_1(\lambda)e^{it\phi(\lambda, n, t)} & 0 \\ \delta^2(\lambda)R_4(\lambda)e^{it\phi(\lambda, n, t)} & 0 \end{bmatrix} & \text{if } L \cap D_+, \\ 0 & \text{if } L \cap D_, \end{cases} \] (28)

and it is easy to check that \( w_\pm \) are \( 2 \times 2 \) nilpotent matrices.

\textbf{\( \bar{\partial} \)-RH problem 3.4.} Find a \( 2 \times 2 \) matrix-valued function \( M^{(2)} \) such that

- \( M^{(2)} \) belongs to \( C^0(\mathbb{C} \setminus (L \cup \mathring{L})) \) and its first-order partial derivatives are continuous on \( \mathbb{C} \setminus (\Sigma \cup L \cup \mathring{L}) \).
- As \( \lambda \to \infty \), \( M^{(2)} \sim I + O(\lambda^{-1}) \).
- On \( \lambda \in \Sigma \),

\[ M^{(2)}_+ = M^{(2)}_- V^{(2)}, \]
where

\[
V^{(2)}(\lambda, n, t) = \begin{cases}
1 & \lambda \in L \cap D_+, \\
R_1(\lambda)\delta^{-2}(\lambda)e^{it\phi(\lambda, n, t)} & \lambda \in \bar{L} \cap D_+, \\
1 & \lambda \in \bar{L} \cap D_-, \\
0 & \lambda \in \tilde{L} \cap D_-.
\end{cases}
\]

On \(\mathbb{C} \setminus (\Sigma \cup L \cup \tilde{L})\),

\[
\bar{\partial}M^{(2)} = M^{(2)}\bar{\partial}\mathcal{P},
\]

where \(\bar{\partial}\mathcal{P}\) is a nilpotent matrix and

\[
\bar{\partial}\mathcal{P}(\lambda, n, t) = \begin{cases}
0 & \lambda \in \Omega_1, \\
-\bar{\partial}R_1(\lambda)\delta^{-2}(\lambda)e^{it\phi(\lambda, n, t)} & \lambda \in \Omega_3, \\
0 & \lambda \in \Omega_4, \\
0 & \lambda \in \Omega_6, \\
0 & \lambda \in \Omega_2 \cup \Omega_5.
\end{cases}
\]

### 3.3 Deformation of the \(\bar{\partial}\)-RH problem

In this part, we deform the solution for \(\bar{\partial}\)-RH problem 3.4 into the product of solutions for RH problem 3.5 and \(\bar{\partial}\)-problem 3.6:

\[
M^{(2)} = M^{(2, D)}M^{(2, R)},
\]

(29)
where \( M^{(2,R)} \) admits the same jump condition as \( M^{(2)} \)'s, and \( M^{(2,D)} \) is continuous over complex plane \( \mathbb{C} \). Seeing from RH problem 3.5, \( M^{(2,R)} \) has no pole and on \( \lambda \in L \cup \tilde{L} \),

\[
\det M^{(2,R)}_+(\lambda, n, t) = \det M^{(2,R)}_-(\lambda, n, t);
\]

so, it is analytic in the whole complex plane; moreover, we see that as \( \lambda \to \infty \),

\[
\det M^{(2,R)}(\lambda, n, t) \sim 1 + \mathcal{O}(\lambda^{-1}),
\]

which deduce by Liouville’s Theorem that

\[
\det M^{(2,R)} \equiv 1,
\]

and \( M^{(2,R)} \) is invertible. Since \( M^{(2,R)} \) is invertible, we obtain that (29) is well-defined.

**RH problem 3.5.** Find a \( 2 \times 2 \) matrix-valued function \( M^{(2,R)} \) such that:

- \( M^{(2,R)} \) is analytic on \( \mathbb{C} \setminus L \cap \tilde{L} \).
- As \( \lambda \to \infty \), \( M^{(2,R)} \sim I + \mathcal{O}(\lambda^{-1}) \).
- On \( \lambda \in L \cap \tilde{L} \), \( M^{(2,R)}_+ = M^{(2,R)}_- P^{(2)} \).

**\( \bar{\partial} \)-problem 3.6.** Find a \( 2 \times 2 \) matrix-valued function \( M^{(2,D)} \) such that

- \( M^{(2,D)} \) belongs to \( C^0(\mathbb{C}) \) and its first-order partial derivatives are continuous on \( \mathbb{C} \setminus (\Sigma \cap L \cap \tilde{L}) \).
- As \( \lambda \to \infty \), \( M^{(2,D)} \sim I + \mathcal{O}(\lambda^{-1}) \).
- On \( \lambda \in L \cap \tilde{L} \), \( \bar{\partial} M^{(2,D)} = M^{(2,D)} \tilde{P} \), where \( \tilde{P} = M^{(2,R)} \bar{\partial} P (M^{(2,R)})^{-1} \).

Introducing a Cauchy-like operator \( C_w \), for a \( 2 \times 2 \) matrix-valued function \( f \):

\[
C_w f = C_+(fw_-) + C_-(fw_+), \quad w = w_+ + w_-,
\]

\[
C_\pm f(\lambda) = \lim_{\lambda' \to \lambda, \lambda' \text{ on } \pm \text{ side of } \Sigma^{(2)}} \int_{\Sigma^{(2)}(s)} \frac{f(s)ds}{s - \lambda'},
\]
for RH problem 3.3, since \((1 - C_w)^{-1} \in \mathcal{B}(L^2(\Sigma^{(2)}))\), which is shown in Section 3.8, we obtain Beals-Coifman solutions:

\[
M^{(2,R)}(\lambda, n, t) = I + \frac{1}{2\pi i} \int_{\Sigma} \frac{(1 - C_w)^{-1}Iw(s, n, t)ds}{s - \lambda}.
\] (30)

### 3.4 RH problems at stationary phase points

In this part, we introduce two crosses contained in \(\Sigma^{(2)}\) associated to the two stationary phase points. And then, by these crosses, we analyze the solution of RH problem 3.5 at \(\lambda = 0\). We find that as \(t \to +\infty\), the leading terms is related to RH problems on each of the crosses, and the remaining part decays not faster than \(O(t^{-1})\), i.e. the result in (61).

Set \(\epsilon_0 > 0\) fixed and introduce contours

\[
\Sigma_1 = \bigcup_{l=1}^{4} \Sigma_{1l}, \quad \Sigma_{1l} = \left\{ S_l(1 + xe^{2i\pi j}) : x \in [0, \epsilon_0) \right\},
\]

\[
\Sigma_2 = \bigcup_{l=1}^{4} \Sigma_{2l}, \quad \Sigma_{2l} = \left\{ S_l(1 + xe^{2i\pi j}) : x \in [0, \epsilon_0) \right\}.
\]

We choose \(\epsilon_0\) sufficiently small such that \(\Sigma' \triangleq \Sigma_1 \cup \Sigma_2 \subset \Sigma^{(2)}\), and on \(\lambda \in \Sigma', (17)\) guarantees that

\[
\text{Im} \phi(\lambda, n, t) \geq x^2 \sqrt{1 - \xi^2}, \quad \lambda = S_j(1 + xe^{-\frac{1}{2}j\pi}), \quad x \in (-\epsilon_0, \epsilon_0),
\] (31a)

\[
\text{Im} \phi(\lambda, n, t) \leq -x^2 \sqrt{1 - \xi^2}, \quad \lambda = S_j(1 + xe^{-\frac{1}{2}j\pi}), \quad x \in (-\epsilon_0, \epsilon_0).
\] (31b)
We also reasonably restrict $\epsilon_0$ such that for $\lambda \in \Sigma'$,

$$|\text{Re}\lambda| \geq \frac{1}{2} |\text{Re} S_1| = \frac{\sqrt{1 - \xi^2} - 1}{2} \geq \frac{\sqrt{1 - V_0^2}}{2}.$$  \hspace{1cm} (32)

Set $2 \times 2$ matrix-valued functions supported on $\Sigma'$:

$$w' = w'_- + w'_+, \quad w'_\pm = \begin{cases} w_\pm & \lambda \in \Sigma', \\ 0 & \lambda \in \Sigma^{(2)} \setminus \Sigma'. \end{cases}$$ \hspace{1cm} (33)

Similar to $C_w$, we also define a Cauchy-like operator $C_w'$, and in Section 3.8, we show that $(1 - C_w')^{-1} \in B(L^2(\Sigma^{(2)}))$ for sufficiently large $t$. Taking $\lambda = 0$ in (30), since $1 - C_w$ and $1 - C_w'$ is invertible on $L^2(\Sigma^{(2)})$, we have by second resolvent identity that

$$M^{(2, R)}(0, n, t) = I + \frac{1}{2\pi i} \int_{\Sigma^{(2)}} \frac{((1 - C_w)^{-1} Iw)(s, n, t)ds}{s} + I_1 + I_2,$$ \hspace{1cm} (34)

where

$$I_1 = \frac{1}{2\pi i} \int_{\Sigma^{(2)}} s^{-1} [(1 - C_w)^{-1} I(w - w')](s, n, t)ds,$$

$$I_2 = \frac{1}{2\pi i} \int_{\Sigma^{(2)}} s^{-1} [(1 - C_w)^{-1} C_{w-w'}(1 - C_w')^{-1} Iw'](s, n, t)ds.$$

**Lemma 3.7.** Since $r \in H_0^1(\Sigma)$, $w_\pm$ and $w$ are bounded functions on $\Sigma^{(2)}$. Moreover, setting $w' = w - w'$ and $w'_\pm = w_\pm - w'_\pm$ on $\Sigma^{(2)}$, we obtain that there is a positive constant $C > 0$ such that for $\lambda \in \Sigma^{(2)}$:

$$|w'_\pm(\lambda)| \lesssim e^{-Ct}, \quad |w'(\lambda)| \lesssim e^{-Ct}.$$  \hspace{1cm} (35)

**Proof.** Recalling that $\delta$, $\delta^{-1}$, $R_j$ are bounded functions on $\Sigma^{(2)}$, by (27) and (28), we only have to check the boundedness of $\text{Im}\phi$; moreover, seeing Figure 3, since $\text{Im}\phi > 0$ on $L$ and $\text{Im}\phi < 0$ on $\tilde{L}$, we obtain that $w_\pm$ are bounded on $\Sigma^{(2)}$. As for the estimates of $w'_\pm$, seeing Figure 3, $L \setminus \Sigma'$ is a compact on $\{\text{Im}\phi > 0\}$, then there is a constant $C > 0$ such that

$$\text{Im}\phi |_{L \setminus \Sigma'} \geq C;$$  \hspace{1cm} (35)
similarly, choosing proper $C$, we also have
\[ \text{Im}\phi|_{L\setminus\Sigma'} \leq -C; \quad (36) \]
therefore, we confirm results for $w^\pm_\pm$ and $w^\tau$.

The next task is to prove that as $t \to +\infty$
\[ I_1, I_2 \sim \mathcal{O}(t^{-1}), \quad (37) \]
By Schwartz’s inequality, the boundedness of $(1 - C_w)^{-1}$ and $(1 - C_w')^{-1}$ shown in Section 3.8, the fact that $\Sigma^{(2)}$ is a compact set contained in $\mathbb{C} \setminus \{0\}$ and Lemma 3.7, we obtain that as $t \to +\infty$
\[ |I_1| \lesssim \|w^\tau\|_{L^2(\Sigma^{(2)})} \lesssim \|w^\tau\|_{L^\infty(\Sigma^{(2)})} \sim \mathcal{O}(t^{-1}). \quad (38) \]
Similarly for $|I_2|$, also by Schwartz’s inequality, the result in Section 3.8 and Lemma 3.7, we have that as $t \to +\infty$,
\[ |I_2| \lesssim \|C_w\|_{L^2 \to L^2} \|I\|_{L^2(\Sigma^{(2)})} \|w'\|_{L^2(\Sigma^{(2)})} \lesssim \|w'\|_{L^\infty(\Sigma^{(2)})} \sim \mathcal{O}(t^{-1}). \quad (39) \]
In (39), the second inequality is confirmed by the fact that $I$ and $w'$ are bounded on $\Sigma^{(2)}$, where the boundedness of $w'$ is supported by Lemma 3.7 and (33); the third inequality is also correct because for a $2 \times 2$ matrix function $f$ on $\Sigma^{(2)}$,
\[ \|C_w f\|_{L^2(\Sigma^{(2)})} = \|C_+(f w^\tau_+) + C_-(f w^\tau_-)\|_{L^2(\Sigma^{(2)})} \lesssim \|f w^\tau_+\|_{L^2(\Sigma^{(2)})} + \|f w^\tau_-\|_{L^2(\Sigma^{(2)})} \lesssim \|f\|_{L^2(\Sigma^{(2)})} \|w^\tau\|_{L^\infty(\Sigma^{(2)})}, \quad (40) \]
by the fact that Cauchy integral operators $C_{\pm}$ are bounded on the $L^2$ space. Seeing (34) and (37), we derive that as $t \to +\infty$,
\[ M^{(2,R)}(0, n, t) = I + \frac{1}{2\pi i} \int_{\Sigma'} \frac{((1 - C_w)^{-1} I w')(s, n, t)ds}{s} + \mathcal{O}(t^{-1}). \quad (41) \]
The remaining work is to separate the contribution of each cross in $\Sigma'$. Introducing

$$w^j = w^j_+ + w^j_- \quad j = 1, 2,$$

$$w^j_\pm = (\lambda, n, t) \begin{cases} w_\pm (\lambda, n, t) & \lambda \in \Sigma_j, \\
0 & \lambda \in \Sigma^{(2)} \setminus \Sigma_j, \end{cases}$$

we deduce that

$$w'_\pm = w^1_\pm + w^2_\pm.$$

and also define correspondent Cauchy-like operators $C_{w^j}, j = 1, 2$. By (3.17) in [22], we obtain

$$(1 - C_{w'}) (1 + C_{w^1} (1 - C_{w^1})^{-1} + C_{w^2} (1 - C_{w^2})^{-1}) = (1 - C_{w^2} C_{w^1} (1 - C_{w^1})^{-1} - C_{w^1} C_{w^2} (1 - C_{w^2})^{-1}).$$

(42)

Considering (41) and (42), we obtain that

$$M^{(2,R)}(0, n, t) = I + \sum_{j=1,2} \frac{1}{2\pi i} \int_{\Sigma_j} s^{-1} [(1 - C_{w^j})^{-1} I w^j](s, n, t) ds$$

$$+ I_3 + I_4 + I_5 + O(t^{-1}).$$

(43)

where

$$I_3 = \frac{1}{2\pi i} \int_{\Sigma'} s^{-1} [(1 + C_{w^1} (1 - C_{w^1})^{-1} + C_{w^2} (1 - C_{w^2})^{-1})$$

$$- (1 - C_{w^2} C_{w^1} (1 - C_{w^1})^{-1} - C_{w^1} C_{w^2} (1 - C_{w^2})^{-1})^{-1}$$

$$(C_{w^2} C_{w^1} (1 - C_{w^1})^{-1} + C_{w^1} C_{w^2} (1 - C_{w^2})^{-1}) I w^j](s, n, t) ds,$$

(44a)

$$I_4 = \frac{1}{2\pi i} \int_{\Sigma'} s^{-1} [(1 - C_{w^2})^{-1} C_{w^2} I w^1](s, n, t) ds,$$

(44b)

$$I_5 = \frac{1}{2\pi i} \int_{\Sigma'} s^{-1} [(1 - C_{w^1})^{-1} C_{w^1} I w^2](s, n, t) ds.$$

(44c)

For $I_3$, considering $(1 - C_{w^j})^{-1} \in \mathcal{B}(L^2(\Sigma^{(2)}))$ for $j = 1, 2$ in Section 3.8, by Lemma 3.9,

$$(1 - C_{w^2} C_{w^1} (1 - C_{w^1})^{-1} - C_{w^1} C_{w^2} (1 - C_{w^2})^{-1})^{-1} \in \mathcal{B}(L^2(\Sigma^{(2)}));$$

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also, applying the technique used in (40) on \( C_{w,j} \), we claim that \( C_{w,j} \) belongs to \( B(L^2(\Sigma^{(2)})) \), \( j = 1, 2; \) thus, we obtain by (44a) and Schwartz inequality that
\[
|I_3| \lesssim \| C_{w}C_{w,1}(1 - C_{w,1})^{-1} + C_{w,1}C_{w,2}(1 - C_{w,2})^{-1} \|_{L^2(\Sigma^{(2)})} \| w' \|_{L^2(\Sigma^{(2)})},
\]
\[
\leq \| C_{w}C_{w,1}(1 - C_{w,1})^{-1} \|_{L^2(\Sigma^{(2)})} + \| C_{w,1}C_{w,2}(1 - C_{w,2})^{-1} \|_{L^2(\Sigma^{(2)})},
\]
\[
\times \| w' \|_{L^2(\Sigma^{(2)})}.
\] (45)

By computation, applying (56a) in Lemma 3.9, Lemma 3.8, \( (1 - C_{w,1})^{-1} \in B(L^2(\Sigma^{(2)})) \) shown in Section 3.8 and the fact that
\[
\| C_{w,1}I \|_{L^2(\Sigma^{(2)})} \lesssim \| C_{+}(w_1^1) \|_{L^2(\Sigma^{(2)})} + \| C_{-}(w_1^1) \|_{L^2(\Sigma^{(2)})},
\]
\[
\leq \| w_1^1 \|_{L^2(\Sigma^{(2)})} + \| w_1^1 \|_{L^2(\Sigma^{(2)})} \lesssim t^{-\frac{1}{4}},
\] (46)

we obtain estimates:
\[
\| C_{w}C_{w,1}(1 - C_{w,1})^{-1}I \|_{L^2(\Sigma^{(2)})} \lesssim \| C_{w}C_{w,1}(1 - C_{w,1})^{-1} \|_{L^2(\Sigma^{(2)})},
\]
\[
+ \| C_{w,1}C_{w,2}I \|_{L^2(\Sigma^{(2)})} \lesssim \| C_{w,1}C_{w,2}I \|_{L^2(\Sigma^{(2)})} \lesssim (t^{-\frac{3}{2}}).
\] (47)

Similarly, we obtain that
\[
\| C_{w}C_{w,1}(1 - C_{w,1})^{-1}I \|_{L^2(\Sigma^{(2)})} \sim O(t^{-\frac{3}{2}}).
\] (48)

By (45), (47), (48) and Lemma 3.8, we confirm that
\[
I_3 \sim O(t^{-1}).
\] (49)

For \( I_4 \), we split the integral into two parts
\[
I_4 = \int_{\Sigma'} \frac{C_{w1}Iw_1^1(s, n, t)}{2\pi is} ds + \int_{\Sigma'} \frac{C_{w2}(1 - C_{w,2})^{-1}C_{w1}Iw_1^1(s, n, t)}{2\pi is} ds;
\] (50)
estimate the first part by Lemma 3.8,
\[
\left| \int_{\Sigma} \left[ C_{w^2} Iw^1 \right](s, n, t) \frac{ds}{2\pi is} \right| \leq \frac{1}{2\pi} \int_{\Sigma_1} \int_{\Sigma_2} \left| w^2(s', n, t)w^1(s, n, t) \right| ds'ds \\
\leq \frac{1}{2\pi d \inf_{s \in \Sigma_1} |s|} \left\| w_1 \right\|_{L^1(\Sigma_1)} \left\| w_2 \right\|_{L^1(\Sigma_2)} \lesssim t^{-1}; 
\]  
(51)
estimate the second part by Schwartz inequality, the boundedness of \((1 - C_{w^2})^{-1}\), Lemma 3.8 and (46)
\[
\left| \int_{\Sigma} \left[ C_{w^2}(1 - C_{w^2})^{-1} C_{w^2} Iw^1 \right](s, n, t) \frac{ds}{2\pi is} \right| \\
\leq \frac{1}{2\pi} \int_{\Sigma_1} \int_{\Sigma_2} \left| (1 - C_{w^2})^{-1} C_{w^2} Iw^2 \right| ds'ds \\
\leq \frac{1}{2\pi d \inf_{s \in \Sigma_1} |s|} \left\| (1 - C_{w^2})^{-1} C_{w^2} Iw^2 \right\|_{L^1(\Sigma_2)} \left\| w^1 \right\|_{L^1(\Sigma_1)} \\
\lesssim \left\| C_{w^2} I \right\|_{L^2(\Sigma_2)} \left\| w^1 \right\|_{L^1(\Sigma_1)} \lesssim t^{-1}; 
\]  
(52)
thus, by (50), (51) and (52), we have
\[
I_4 \sim O(t^{-1}). 
\]  
(53)
For \(I_5\), we claim that
\[
I_5 \sim O(t^{-1}), 
\]  
(54)
and the technique is parallel to that for \(I_4\). By (43), (49), (53) and (54), we get
\[
M^{(2,R)}(0, n, t) = I + \sum_{j=1}^{2} \frac{1}{2\pi i} \int_{\Sigma_j} s^{-1}[(1 - C_{w^j})^{-1} Iw^j](s, n, t)ds + O(t^{-1}). 
\]  
(55)
**Lemma 3.8.** For \(r \in H^1_1(\Sigma)\), we obtain these estimates for \(w^j\):
\[
\left\| w^j \right\|_{L^1(\Sigma^{(2)})} \lesssim t^{-\frac{1}{2}}, \quad \left\| w^j \right\|_{L^2(\Sigma^{(2)})} \lesssim t^{-\frac{1}{4}}. 
\]
**Proof.** Recalling that \(\delta, \delta^{-1}, R_j\) are bounded functions, by (27), (28) and (31), we obtain that on \(\lambda \in \Sigma_j\),
\[
|w^j(\lambda, n, t)| \lesssim e^{-t\sqrt{1-\xi^2}x^2}, \quad x = |\lambda - S_j|, 
\]  
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by which we confirm the result since \( w^j \) is supported only on \( \Sigma_j \).

\[ \square \]

**Lemma 3.9.** For \( r \in H^1_\theta(\theta) \), we claim that \( C_{w^j}C_{w^j} \), \( C_{w^i}C_{w^j} \) belong to \( B(L^2(\Sigma(2))) \) and \( B(L^\infty(\Sigma(2)), L^2(\Sigma(2))) \), and we have the following estimate

\[
\| C_{w^1}C_{w^2} \|_{L^2 \to L^2} \lesssim t^{-\frac{1}{2}}, \quad \| C_{w^1}C_{w^2} \|_{L^\infty \to L^2} \lesssim t^{-\frac{1}{2}},
\]

\( (56a) \)

\[
\| C_{w^1}C_{w^2} \|_{L^2 \to L^2} \lesssim t^{-\frac{1}{2}}, \quad \| C_{w^1}C_{w^2} \|_{L^\infty \to L^2} \lesssim t^{-\frac{1}{2}}.
\]

\( (56b) \)

**Proof.** For any \( 2 \times 2 \) matrix-valued function \( f \in L^2(\Sigma(2)) \), we obtain that

\[
C_{w^1}C_{w^2}f = C_+((f w^2_+)w^1_1) + C_-((f w^2_-)w^1_+).
\]

(57)

By the formula

\[
(C_-((f w^2_+)w^1_1))(s_1) = \frac{1}{2\pi i} \int_{\Sigma_2} \frac{f(s_2)w^2_2(s_2)w^1_+(s_1)}{s_2 - s_1} ds_2
\]

(56)

and Lemma 3.8, the \( L^2 \)-norm of the first term in (57) satisfies that

\[
\| C_+((f w^2_+)w^1_1) \|_{L^2(\Sigma(2))} \leq \frac{1}{2\pi} \left( \int_{\Sigma_1} \left\| \int_{\Sigma_2} \frac{f(s_2)w^2_2(s_2)w^1_+(s_1)}{s_2 - s_1} ds_2 \right\|^2 ds_1 \right)^{\frac{1}{2}}
\]

\[
\leq \frac{1}{2\pi d} \| f \|_{L^2(\Sigma(2))} \| w^2_+ \|_{L^2(\Sigma(2))} \| w^1_- \|_{L^2(\Sigma(2))}
\]

\[
\lesssim \begin{cases} \| f \|_{L^\infty(\Sigma(2))} \| w^2_+ \|_{L^2(\Sigma(2))} \| w^1_- \|_{L^2(\Sigma(2))} \lesssim t^{-\frac{1}{2}} \| f \|_{L^\infty(\Sigma(2))}, & \\
\| f \|_{L^2(\Sigma(2))} \| w^2_+ \|_{L^2(\Sigma(2))} \| w^1_- \|_{L^2(\Sigma(2))} \lesssim t^{-\frac{1}{2}} \| f \|_{L^2(\Sigma(2))}.
\end{cases}
\]

(58)

where \( d \) denotes the distance of \( \Sigma_1 \) and \( \Sigma_2 \). For the second term on the right of (57), we obtain the similar estimate,

\[
\| C_+((f w^2_-)w^1_+) \|_{L^2(\Sigma(2))} \lesssim \begin{cases} t^{-\frac{1}{2}} \| f \|_{L^\infty(\Sigma(2))}, & \\
t^{-\frac{1}{2}} \| f \|_{L^2(\Sigma(2))}.
\end{cases}
\]

(59)

By (57), (58) and (59), we confirm the result of \( C_{w^1}C_{w^2} \), and that of \( C_{w^2}C_{w^1} \) is similarly checked.
Define infinite crosses

\[ \Sigma'_j = S_j((1 + e^{\pm i\pi/4})R \cup (1 + e^{-\pm i\pi/4})R) \quad j = 1, 2. \]

Here, we give the infinite crosses the orientation consisting with that of \( \Sigma_j \), example as shown in Figure 6. Set \( \tilde{w}'_j \) the zero extension of \( w'_j |_{\Sigma_j} \) on \( \Sigma'_j \) and \( \tilde{w}^j = \tilde{w}'_+ + \tilde{w}'_- \); then, define Cauchy-like integral operators \( A_j: L^2(\Sigma'_j) \to L^2(\Sigma'_j), f \mapsto A_j f, \)

\[
A_j f = C^\Sigma'_j f(\tilde{w}_+) + C^\Sigma'_j f(\tilde{w}_-), \tag{60}
\]

\[
C^\Sigma'_j f(\lambda) = \lim_{\lambda' \to \lambda, \lambda' \text{ on } \pm \text{ sides of } \Sigma'_j} \int_{\Sigma'_j} \frac{f(s)ds}{s - \lambda'};
\]

combining (55) and (60), we claim that

\[
M^{(2,R)}(0, n, t) = I + \sum_{j=1}^{2} \frac{1}{2\pi i} \int_{\Sigma'_j} s^{-1}[(1 - A_j)^{-1}I\tilde{w}_j](s, n, t)ds + O(t^{-1}). \tag{61}
\]

### 3.5 Scaling and rotation

Here, we firstly introduce the scaling operator \( N_j \) according to each cross, \( j = 1, 2; \)
secondly, we apply these scaling operators to some functions and obtain the oscillatory part $\delta_{j0}$; thirdly, by these scaling operators, we obtain the Cauchy-like integral operators $\tilde{A}_j$ and their limits as $t \to +\infty$: $\tilde{A}_j^\infty$; finally, by proper evaluations and estimates, we rewrite formula (61) by $\tilde{A}_j^\infty$, which results in a long-time asymptotic formula (81) related to $M^{(2,R)}$.

From (81), we see that the leading term in the long-time asymptotic formula is determined by the model RH problem at the stationary phase point. The model RH problem is shown in the later part.

Define infinite crosses
\[
\tilde{\Sigma}_j = e^{\frac{\pi i}{4}} \mathbb{R} \cup e^{-\frac{\pi i}{4}} \mathbb{R}, \quad j = 1, 2,
\]
and scaling mapping
\[
N_j : f(\lambda) \mapsto N_j f(\zeta) = f(\beta_j \zeta + S_j).
\]
It follows from direct computation that $N_j$ belongs to $B(L^2(\Sigma'_j), L^2(\tilde{\Sigma}_j))$ and invertible; moreover, its norm satisfies that
\[
\| N_j \|_{L^2 \to L^2} = |\beta_j|^{-\frac{1}{2}} = t^{\frac{1}{4}} (1 - \xi^2)^{\frac{1}{4}}.
\]
Seeing the definition of $\Sigma'_j$ and $\tilde{\Sigma}_j$, we have the relationship
\[
\Sigma'_j = \beta_j \tilde{\Sigma}_j + S_j,
\]
and also set the orientation of $\tilde{\Sigma}_j$ consisting with that of $\Sigma'_j$ by mapping: $\tilde{\Sigma}_j \to \Sigma'_j$, $\zeta \mapsto \beta_j \zeta + S_j$.

By direct computation, we obtain that
\[
N_j(\delta^2 e^{-it\phi})(\zeta) = \zeta^{(-1)^j-12i\nu_j} e^{(-1)^j \frac{i}{2} \xi^2} \delta_{j0}^{2} \delta_{j1}(\zeta), \quad (63a)
\]
\[
\delta_{j0} = e^{\alpha_j(S_j) - \frac{i}{2} \phi(S_j)} \left( \frac{(-1)^j - 1}{S_j - S_2} \right)^{(-1)^j - 1} i^{\nu_j}, \quad (63b)
\]
\[
\delta_{j1}(\zeta) = \left( \frac{S_1 - S_2}{(1)j-1 \beta_1 \zeta + S_1 - S_2} \right)^{2(-1)^j - 1} i^{\nu_j}
\times e^{2(\alpha_j(\beta_j \zeta + S_j) - \alpha_j(S_j)) - it(\phi(\beta_j \zeta + S_j, n, t) - \phi(S_j, n, t)) + (1)^j - 1} \xi^2, \quad (63c)
\]
where $\delta_{j0}$ is the oscillatory part and $\delta_{j1}(\zeta)$ admits limits at $\zeta \to 0$ as shown in Lemma 3.10.

**Lemma 3.10.** For $r \in H_0^1(\Sigma)$, we have $\delta_{11}(\zeta) \to 1$ as $\zeta \to 0$, and on $\zeta \in \beta_j^{-1}(\Sigma_j - S_j)$,

$$|\delta_{j1}(\zeta) - 1| \lesssim t^{-\frac{1}{2}}|\zeta|^\frac{1}{2}, \quad |\delta_{j1}^{-1}(\zeta) - 1| \lesssim t^{-\frac{1}{2}}|\zeta|^\frac{1}{2}. \quad (64)$$

**Proof.** We first detail the proof for the first formula in (64) with $j = 1$ and $\zeta \in \beta_{11}^{-1}(\Sigma_{11} - S_1)$. Since $\left(\frac{S_1 - S_2}{\lambda - S_2}\right)^{2i\nu_1}$ and $it(\phi(\lambda, n, t) - \phi(S_1, n, t)) - \frac{\phi''(S_1, n, t)}{2}(\lambda - S_j)^2$ are both holomorphic on $\lambda \in \Sigma_{11}$, we deduce that

$$\left|\left(\frac{S_1 - S_2}{\lambda - S_2}\right)^{2i\nu_1} - 1\right| \lesssim |\lambda - S_1|, \quad (65a)$$

$$|it(\phi(\lambda, n, t) - \phi(S_1, n, t)) - 2\phi''(S_1, n, t)(\lambda - S_1)^2| \lesssim t|\lambda - S_1|^3; \quad (65b)$$

moreover, seeing (19) and (21), we derive that on $\lambda \in \Sigma_{11}$,

$$|\alpha_{1}(\lambda) - \alpha_{1}(S_1)| \lesssim |\lambda - S_1|^\frac{1}{2}; \quad (65c)$$

thus, by (62), (63c) and (65), we obtain that

$$|\delta_{11}(\zeta) - 1| \lesssim t^{-\frac{1}{2}}|\zeta|^\frac{1}{2},$$

Since we have proven the first formula in (64) with $j = 1$ and $\zeta \in \beta_{11}^{-1}(\Sigma_{11} - S_1)$, we generalize similarly this proof to the case for $\delta_{j1}(\zeta)$ on $\zeta \in \beta_j^{-1}(\Sigma_j - S_j)$, $j = 1, 2$. The proof for the second formula is similarly obtained. Thus, we confirm the result in this lemma. \(\square\)

Define operator $\Delta_{j0} : f \mapsto f\delta_{j0}^{\sigma_j}$, where $f$ is a $2 \times 2$ matrix-valued function; by the definition of $\delta_{j0}$, it and its inverse operator $\Delta_{j0}^{-1}$ both belong to $B(L^2(\Sigma_j))$. Define a Cauchy-like integral operator $\tilde{A}_{j} : L^2(\Sigma_j) \to L^2(\Sigma_j)$, $f \mapsto \tilde{A}_{j}f$,

$$\tilde{A}_{j}f = C_+^{\Sigma_j}(f\delta_{j0}^{-\sigma_j}N_j\tilde{w}_j^{\dagger} \delta_{j0}^{\sigma_j}) + C_-^{\Sigma_j}(f\delta_{j0}^{-\sigma_j}N_j\tilde{w}_j^{\dagger} \delta_{j0}^{\sigma_j}), \quad (66)$$

$$C_{\pm}^{\Sigma_j} f(\lambda) = \lim_{\lambda' \to \lambda, \lambda' \text{ on } \pm \text{ sides of } \Sigma_j} \int_{\Sigma_j} \frac{f(s)ds}{s - \lambda'}. \quad 28$$
By the relations (60) and (66), we have these formulas

\[ \tilde{A}_j = \Delta_{j0} N_j A_j N^{-1}_j \Delta^{-1}_{j0}, \quad j = 1, 2. \]  

(67)

Write \( \tilde{w}^j = \tilde{w}^j_+ + \tilde{w}^j_- \) on \( \tilde{\Sigma}_j \),

\[ \tilde{w}^j_\pm = \tilde{w}^j_\pm (\zeta, n, t) = \delta_{j0}^{\pm\alpha} \tilde{N}_j \tilde{w}^j_\pm (\zeta, n, t) \delta_{j0}^{\pm\beta}. \]  

(68)

By the definition of \( w^j_\pm \) and (63), we deduce that

\[
\begin{align*}
\tilde{w}^1_+ (\zeta, n, t) &= \begin{cases} 
0 & \text{otherwise} \\
-N_1 R_3 (\zeta) \zeta^{2i\nu} e^{-\frac{1}{2} \zeta^2} \delta_{11} (\zeta, n, t) & \zeta \in e^{\frac{2\pi i}{3}} (0, |\beta^{-1}_j| \epsilon_0), \\
0 & \zeta \in e^{\frac{2\pi i}{3}} (-|\beta^{-1}_j| \epsilon_0, 0), \\
-N_1 R_0 (\zeta) \zeta^{2i\nu} e^{-\frac{1}{2} \zeta^2} \delta_{11} (\zeta, n, t) & \text{otherwise}, \\
0 & 
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\tilde{w}^1_- (\zeta, n, t) &= \begin{cases} 
0 & \text{otherwise} \\
N_1 R_1 (\zeta) \zeta^{2i\nu} e^{-\frac{1}{2} \zeta^2} \delta_{11}^{\pm 1} (\zeta, n, t) & \zeta \in e^{\frac{2\pi i}{3}} (0, |\beta^{-1}_j| \epsilon_0), \\
0 & \zeta \in e^{\frac{2\pi i}{3}} (-|\beta^{-1}_j| \epsilon_0, 0), \\
0 & \text{otherwise}, \\
0 & 
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\tilde{w}^2_+ (\zeta, n, t) &= \begin{cases} 
0 & \text{otherwise} \\
-N_2 R_3 (\zeta) \zeta^{2i\nu} e^{-\frac{1}{2} \zeta^2} \delta_{21} (\zeta, n, t) & \zeta \in e^{\frac{2\pi i}{3}} (0, |\beta^{-1}_j| \epsilon_0), \\
0 & \zeta \in e^{\frac{2\pi i}{3}} (-|\beta^{-1}_j| \epsilon_0, 0), \\
0 & \text{otherwise}, \\
0 & 
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\tilde{w}^2_- (\zeta, n, t) &= \begin{cases} 
0 & \text{otherwise} \\
N_2 R_1 (\zeta) \zeta^{2i\nu} e^{-\frac{1}{2} \zeta^2} \delta_{21}^{\pm 1} (\zeta, n, t) & \zeta \in e^{\frac{2\pi i}{3}} (0, |\beta^{-1}_j| \epsilon_0), \\
0 & \zeta \in e^{\frac{2\pi i}{3}} (-|\beta^{-1}_j| \epsilon_0, 0), \\
0 & \text{otherwise}, \\
0 & 
\end{cases}
\end{align*}
\]

Taking limits of \( \tilde{w}^j_\pm \) and denoting them as \( \tilde{w}^j_\pm (\zeta) \) when \( t \to +\infty \), by direct
computation, we obtain that

\[
\tilde{w}_j^+ (\zeta) = \begin{cases} 
0 & \zeta \in e^{(2-(-1)^j-1/2)} \frac{\pi}{4} R, \\
0 - R_3 (S_j) \zeta^{2(-1)^j-1} e^{-i \nu_j} e^{(-1)^j - 1/2} \zeta^2 & \zeta \in e^{(2-(-1)^j-1/2)} \frac{\pi}{4} R^+, \\
0 - R_6 (S_j) \zeta^{2(-1)^j-1} e^{-i \nu_j} e^{(-1)^j - 1/2} \zeta^2 & \zeta \in e^{(2-(-1)^j-1/2)} \frac{\pi}{4} R^-. 
\end{cases}
\] (69a)

\[
\tilde{w}_j^- (\zeta) = \begin{cases} 
0 & \zeta \in e^{(2+(-1)^j-1/2)} \frac{\pi}{4} R, \\
0 - R_1 (S_j) \zeta^{2(-1)^j-1} e^{-i \nu_j} e^{(-1)^j - 1/2} \zeta^2 & \zeta \in e^{(2+(-1)^j-1/2)} \frac{\pi}{4} R^+, \\
0 - R_4 (S_j) \zeta^{2(-1)^j-1} e^{-i \nu_j} e^{(-1)^j - 1/2} \zeta^2 & \zeta \in e^{(2+(-1)^j-1/2)} \frac{\pi}{4} R^-.
\end{cases}
\] (69b)

**Proposition 3.11.** If \( r \in H^1_\delta (\Sigma) \), we obtain that as \( t \to +\infty \), for \( j = 1, 2 \),

\[
\| \tilde{u}^j - \tilde{w}_j^+ \|_{L^\infty (\tilde{\Sigma}_j) \cap L^2 (\tilde{\Sigma}_j) \cap L^1 (\tilde{\Sigma}_j)} \lesssim t^{-\frac{3}{2}}.
\]
Figure 8: the relation between parameters $\theta$ and $\zeta$ for $\beta_1 \zeta + S_1 \in \Sigma_{11}$.

Proof. In this proof, we claim that for $\zeta \in \tilde{\Sigma}_j$, $\tilde{w}^j$ and $\tilde{w}^{j, \infty}$ satisfy

$$|\tilde{w}^j - \tilde{w}^{j, \infty}|(\zeta, n, t) \lesssim t^{-\frac{1}{4}} |\zeta| \frac{1}{4} e^{-\frac{|\zeta|^2}{2}},$$

and then, this proposition’s results naturally follow. In the following, we will look into the case of $\zeta \in e^\frac{\pi i}{4} \mathbb{R}^+$ and $j = 1$; then, proofs for other cases are parallel to omit. For $\zeta = xe^\frac{\pi i}{4} \in \beta_1^{-1}(\Sigma_{11} - S_1) \subset e^\frac{\pi i}{4} \mathbb{R}$, we obtain the estimate

$$|N_1 R_1(\zeta) - R_1(S_1)| = |r(\theta) - r(\theta_1)| = \left| \int_{\theta_1}^{\theta} r'(\tilde{\theta}) d\tilde{\theta} \right| \leq |\theta - \theta_1| \frac{1}{2} \| r \|_{H^1},$$

with

$$\theta = \arg(\beta_1 \zeta + S_1), \quad \theta_1 = \arg S_1 \in [0, 2\pi];$$

seeing the relationship between $\zeta$ and $\theta$ in Figure 8, since $\zeta \in \beta_1^{-1}(\Sigma_{11} - S_1)$, we get the following estimate by triangular knowledge

$$|\theta_1 - \theta| \lesssim |\beta_1 \zeta| = t^{-\frac{1}{2}} (1 - \xi^2)^{-\frac{1}{4}} x;$$

therefore, by (71) and (72), recalling $|\xi| \leq V_0$, we obtain that

$$|N_1 R_1(\zeta) - R_1(S_1)| \lesssim t^{-\frac{1}{4}} x^{-\frac{1}{2}}.$$  

(73)

By definition of $\tilde{w}^j$, (69), Lemma 3.10 and (73), we estimate that

$$|\tilde{w}^j - \tilde{w}^{j, \infty}|(\zeta, n, t) \leq \left| \zeta^{-2i\nu_1} e^{\frac{\zeta^2}{2}} \right| |N_1 R_1(\zeta) \delta^{-1}_{11}(\zeta) - R_1(S_1)|$$

$$\lesssim e^{-\frac{|\zeta|^2}{2}} (|R_1(\beta_1 \zeta + S_1)| |\delta^{-1}_{11}(\zeta) - 1| + |R_1(\beta_1 \zeta + S_1) - R_1(S_1)|)$$

$$\lesssim e^{-\frac{|\zeta|^2}{2}} (\| r \|_{\infty} t^{-\frac{1}{4}} x^{\frac{1}{2}} + t^{-\frac{1}{4}} x^{\frac{1}{2}}) \lesssim t^{-\frac{1}{4}} x^{\frac{1}{2}} e^{-\frac{|\zeta|^2}{2}}.$$
For \( \zeta = xe^{\frac{i\pi}{4}} \in e^{\frac{i\pi}{4}} \mathbb{R} \setminus \beta_j^{-1}(\Sigma_{11} - S_1) \), we obtain by (69) that

\[
|\hat{w}^j - \hat{w}^{j,\infty}(\zeta, n, t)| = |\hat{w}^{j,\infty}(\zeta, n, t)| = |R_1(S_j)||e^{\frac{i\pi}{4}u_1}|e^{-x^2/2} \\
\lesssim e^{-x^2/2} \leq \left| \frac{\beta_j \zeta}{\epsilon_0} \right|^d e^{-x^2/2} \lesssim t^{-\frac{1}{4}} x^\frac{1}{2} e^{-x^2/2}.
\]

We complete the proof. \( \square \)

Here, we come to estimates of integral parts in (61). By direct computation, (67), (68) and (69), we obtain that

\[
\frac{1}{2\pi i} \int_{\Sigma_j} s^{-1}[(1 - A_j)^{-1}I\hat{w}^j](s, n, t)ds
= \frac{1}{2\pi i} \int_{\Sigma_j} s^{-1}[N_j^{-1}\Delta_j^{-1}(1 - \hat{A})^{-1}\Delta_j N_j I\delta_j^{-\sigma_3} N_j^{-1}\hat{w}^{j,\sigma_3}](s, n, t)ds
= \frac{\beta_j}{2\pi i} \int_{\Sigma_j} (\beta j s + S_j)^{-1}[(1 - \hat{A})^{-1}\delta_j^{-\sigma_3} \hat{w}^j](s, n, t)ds
= \frac{\beta_j}{2\pi i} \delta_j^{-\sigma_3} \int_{\Sigma_j} (\beta j s + S_j)^{-1}[(1 - \hat{A})^{-1}I\hat{w}^j](s, n, t)ds \delta_j^{-\sigma_3}
= \frac{\beta_j}{2\pi i} S_j^{-1} \delta_j^{-\sigma_3} \int_{\Sigma_j} [(1 - \hat{A})^{-1}I\hat{w}^{j,\infty}](s)ds \delta_j^{-\sigma_3} + \frac{\beta_j}{2\pi i} \delta_j^{-\sigma_3} (I_6 + I_7) \delta_j^{-\sigma_3},
\]

where

\[
I_6 = \int_{\Sigma_j} (\beta j s + S_j)^{-1}[(1 - \hat{A})^{-1} - (1 - \hat{A})^{-1}I\hat{w}^j](s, n, t)ds,
\]

\[
I_7 = \int_{\Sigma_j} [(1 - \hat{A})^{-1}I[(\beta j s + S_j)^{-1}\hat{w}^j - S_j^{-1}\hat{w}^{j,\infty}])(s, n, t)ds.
\]
By Schwarz inequality,

\[ |I_6| = \left| \int_{\Sigma_j} \left[ (\beta_j s + S_j)^{-1} (1 - \tilde{A}_j)^{-1} (\tilde{A}_j - \tilde{A}_j^\infty)^{-1} (1 - \tilde{A}_j^\infty)^{-1} I \tilde{w}^j \right] (s, n, t) ds \right| \]
\[ \leq \| (1 - \tilde{A}_j)^{-1} (\tilde{A}_j - \tilde{A}_j^\infty) (1 - \tilde{A}_j^\infty)^{-1} I \|_{L^2(\Sigma_j)} \| (\beta_j s + S_j)^{-1} \tilde{w}^j \|_{L^2(\Sigma_j)}, \]  
(76a)

\[ |I_7| = \int_{\Sigma_j} \left[ ((\beta_j s + S_j)^{-1} \tilde{w}^j - S_j^{-1} \tilde{w}^j) \right] (s, n, t) ds \]
\[ = \int_{\Sigma_j} \left[ (1 - \tilde{A}_j^\infty)^{-1} \tilde{A}_j I ((\beta_j s + S_j)^{-1} \tilde{w}^j - S_j^{-1} \tilde{w}^j) \right] (s, n, t) ds \]
\[ + \int_{\Sigma_j} ((\beta_j s + S_j)^{-1} \tilde{w}^j - S_j^{-1} \tilde{w}^j) (s, n, t) ds \]
\[ \leq \| (1 - \tilde{A}_j^\infty)^{-1} I \|_{L^2 \to L^2} \| \tilde{A}_j I \|_{L^2(\Sigma_j)} \| (\beta_j s + S_j)^{-1} \tilde{w}^j - S_j^{-1} \tilde{w}^j \|_{L^2(\Sigma_j)} \]
\[ + \| (\beta_j s + S_j)^{-1} \tilde{w}^j - S_j^{-1} \tilde{w}^j \|_{L^2(\Sigma_j)} \]  
(76b)

For \( I_6 \), estimating by Lemma 3.10 and \( L^2 \) boundedness of \( \tilde{w}^j \) on \( L^2(\Sigma_j) \), we get that

\[ \| (1 - \tilde{A}_j)^{-1} (\tilde{A}_j - \tilde{A}_j^\infty) (1 - \tilde{A}_j^\infty)^{-1} I \|_{L^2(\Sigma_j)} \]
\[ = \| (1 - \tilde{A}_j)^{-1} (\tilde{A}_j - \tilde{A}_j^\infty) I + (1 - \tilde{A}_j)^{-1} (\tilde{A}_j - \tilde{A}_j^\infty) (1 - \tilde{A}_j^\infty)^{-1} \tilde{A}_j^\infty I \|_{L^2(\Sigma_j)} \]
\[ \leq \| (\tilde{A}_j - \tilde{A}_j^\infty) I \|_{L^2(\Sigma_j)} + \| \tilde{A}_j - \tilde{A}_j^\infty \|_{L^2 \to L^2} \| \tilde{A}_j^\infty I \|_{L^2(\Sigma_j)} \]
\[ \leq \| \tilde{w}^j - \tilde{w}^j \|_{L^2(\Sigma_j)} + \| \tilde{w}^j - \tilde{w}^j \|_{L^2(\Sigma_j)} \| \tilde{w}^j \|_{L^2(\Sigma_j)} \]
\[ \leq t^{-\frac{1}{2}} (1 + \| \tilde{w}^j \|_{L^2(\Sigma_j)}) \]
\[ \lesssim t^{-\frac{1}{2}}; \]  
(77)

estimate the \( L^2 \)-norm by direct computation, the boundedness of \( (\beta_j \zeta + S_j)^{-1} \) on \( \zeta \in \Sigma_j \), (62), (68), Lemma 3.8 and definition of \( \beta_j \)

\[ \| (\beta_j s + S_j)^{-1} \tilde{w}^j \|_{L^2(\Sigma_j)} \lesssim \| \tilde{w}^j \|_{L^2(\Sigma_j)} = \| \delta_0^{-\sigma_3} N_j \tilde{w}^j \delta_0^{\sigma_3} \|_{L^2(\Sigma_j)} \]
\[ \lesssim \| N_j \tilde{w}^j \|_{L^2(\Sigma_j)} = \| \tilde{w}^j \|_{L^2(\Sigma_j)} |\beta_j|^{-\frac{1}{2}} \lesssim 1; \]  
(78)

therefore, we obtain by (76a), (77) and (78) that

\[ |I_6| \lesssim t^{-\frac{1}{2}}. \]  
(79)
For $I_7$, similar to Lemma 3.10, we obtain that for $j = 1, 2$,

$$\| (\beta_j + S_j)^{-1} \tilde{w}^j - S_j^{-1} \tilde{w}^j, \infty \|_{L^1(\Sigma_j) \cap L^2(\Sigma_j)} \lesssim t^{-\frac{1}{4}};$$

which deduces that

$$|I_7| \lesssim \| (1 - \tilde{A}_j^\infty)^{-1} \|_{L^2 \rightarrow L^2} \| \tilde{A}_j I \|_{L^2(\Sigma_j)} + 1 \| t^{-\frac{1}{2}} \| \lesssim t^{-\frac{1}{2}}, \quad (80)$$

by (76b), the boundedness of $(1 - \tilde{A}_j^\infty)^{-1}$ shown in Section 3.8 and

$$\| \tilde{A}_j I \|_{L^2(\Sigma_j)} \lesssim \| \tilde{w}_j \|_2 \lesssim |\beta_j|^{\frac{1}{2}} \| w_j \|_{L^2(\Sigma_j)} \lesssim 1.$$

Considering (61), (74), (79) and (80), we obtain the asymptotic formula

$$M^{(2,R)}(0, n, t) = I + \sum_{j=1}^2 \frac{\beta_j}{2\pi i} \int_{\bar{\Sigma}_j} S_j^{-1} [(1 - \tilde{A}_j^\infty)^{-1} I \tilde{w}_j, \infty](s) ds \lesssim O(t^{-\frac{3}{4}}). \quad (81)$$

**RH problem 3.12.** Find a $2 \times 2$ matrix-valued function $M^{L,j} \equiv M^{L,j}(\zeta, n, t)$ such that

- $M^{L,j}(\zeta, n, t)$ is analytic on $\mathbb{C} \setminus \tilde{\Sigma}_j$.
- As $\zeta \to \infty$, $M^{L,j}(\zeta, n, t) \sim I$.
- On $\zeta \in \tilde{\Sigma}_j$, $M^{L,j}_+(\zeta) = M^{L,j}_-(\zeta) V^{L,j}(\zeta)$, where $V^{L,j} = (I - \tilde{w}_j, \infty)^{-1}(I + \tilde{w}_j, \infty)$.

We see that RH problem 3.12 is a model RH problem that related to parabolic cylinder functions, seeing [44]. And we see that the solution of RH problem 3.12 is written in the Beals-Coifman solution:

$$M^{L,j}(\zeta) = I + \frac{1}{2\pi i} \int_{\Sigma_j} \frac{[(1 - \tilde{A}_j^\infty)^{-1} I \tilde{w}_j, \infty](s)}{s - \zeta} ds \quad (82)$$

If write $M^{L,j}(\zeta, n, t)$ by its asymptotic expansion at $\zeta \to \infty$,

$$M^{L,j}(\zeta) = I + \zeta^{-1} M^{L,j}_1 + \ldots, \quad (83)$$

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then, we derive that
\[ M_1^{L,j} = \frac{1}{2\pi i} \int_{\Sigma_j} [(1 - \tilde{A}_j^\infty)^{-1} I \tilde{a}^{j,\infty}] (s) ds; \]  
(84)
recalling (81), we obtain that
\[ M^{(2,R)}(0,n,t) = I + 2 \sum_{j=1}^{2} \beta_j S_j^{-1} \delta_j \sigma_3 M_1^{L,j} \delta_j \sigma_3 + \mathcal{O}(t^{-\frac{3}{4}}), \]  
(85)
which means that the solution of RH problem 3.12 describes the leading terms of solutions of \( M^{(2,R)}(0,n,t) \). Seeing [20, 44] and solving RH problem 3.12, we learn that \( M_1^{L,j} \) is explicitly obtain that
\[ [M_1^{L,j}]_{1,2} = -i (2\pi)^{\frac{1}{2}} e^{\pi \nu_j} e^{-\pi \nu_j} \frac{r(S_j)}{r(S_j) \Gamma(-i
u_j)}. \]  
(86)
where \( \Gamma(\lambda) \) is Euler’s Gamma functions.

### 3.6 Analysis on \( \bar{\partial} \)-problem

Since we have obtain the long-time asymptotic formula (85), in this part, we study the solution of \( \bar{\partial} \)-problem 3.6 as \( t \to +\infty \).

The solution of \( \bar{\partial} \)-problem 3.6 can be written as
\[ M^{(2,D)}(\lambda,n,t) = I - \frac{1}{\pi} \int_{\mathbb{C}} \left[ \frac{M^{(2,D)}(s,n,t)}{s - \lambda} \right] ds. \]  
(87)
(87) can also be written by operator \( S: f \mapsto Sf \)
\[ (1 - S)M^{(2,D)} = I, \quad Sf(\lambda) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{f(s) \tilde{P}(s)}{s - \lambda} dL(s), \]  
(88)
where \( L(s) \) is the Lebesgue distribution on complex plane \( \mathbb{C} \). We prove in the following that \( S \in B(L^\infty(\mathbb{C})) \) and as \( t \) is sufficiently large, \( \| S \|_{L^\infty \to L^\infty} \) decays.

**Proposition 3.13.** If \( r \in H_0^4(\Sigma) \), the operator \( S \in B(L^\infty(\mathbb{C})) \), and as \( t \to +\infty \), the norm of \( S \) satisfies
\[ \| S \|_{L^\infty \to L^\infty} \lesssim t^{-\frac{1}{4}}. \]  
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Proof. As the definition of $\tilde{\mathcal{P}}$, it is supported on $\cup_{k=1,3,4,6} \Omega_k$; therefore, the integral in (88) is only supported on $\cup_{k=1,3,4,6} \Omega_k$. To make the proof concise, we detail the case for functions supported on $\Omega_1^* \equiv \Omega_1 \cap \{\Re \lambda < 0\}$; and the other cases’ proofs are parallel.

For $2 \times 2$ matrix-valued functions $f \equiv f(\lambda)$ supported on $\Omega_1^*$, by (24), (25) and classical computation, we obtain a series of estimates:

\[ |Sf(\lambda, n, t)| = \frac{1}{\pi} \int_{\Omega_1^*} \frac{f(s)[M(2,R) \tilde{\mathcal{P}}(M(2,R))^{-1}]_{s=\lambda}}{|s-\lambda|} dL(s) \]

\[ \leq \|f\|_{L^\infty(\Omega_1^*)} \|M(2,R)\|_{L^\infty(\Omega_1^*)} \| (M(2,R))^{-1} \|_{L^\infty(\Omega_1^*)} \| \delta^{-2} \|_{L^\infty(\Omega_1^*)} \int_{\Omega_1^*} |\tilde{\partial}_R^1(s)| e^{-\text{Im} \phi(s, n, t)} dL(s) \]

\[ \approx \|f\|_{L^\infty(\Omega_1^*)} \int_{\Omega_1^*} |\tilde{\partial}_R^1(s)| e^{-\text{Im} \phi(s, n, t)} dL(s) \]

\[ \leq \|f\|_{L^\infty(\Omega_1^*)} \int_{\Omega_1^*} e^{-\text{Im} \phi(\rho, n, t)} \rho_0 d\theta d\rho |\rho e^{i\theta} - \lambda| \]

\[ = \|f\|_{L^\infty(\Omega_1^*)} \int_{\Omega_1^*} e^{-\text{Im} \phi(\rho, n, t)} \rho_0 d\theta d\rho |\rho e^{i\theta} - \lambda| \]

\[ \approx \|f\|_{L^\infty(\Omega_1^*)} \int_{\Omega_1^*} e^{-\text{Im} \phi(\rho, n, t)} \rho_0 d\theta d\rho \]

\[ \leq \|f\|_{L^\infty(\Omega_1^*)} \int_{\Omega_1^*} e^{-\frac{|\rho - \lambda|^2}{4}} d\rho \leq \|f\|_{L^\infty(\Omega_1^*)} \int_{\Omega_1^*} e^{-\frac{\nu + \varepsilon (\rho - \lambda)^2}{2}} d\rho \]

\[ \approx \|f\|_{L^\infty(\Omega_1^*)} \cdot \]

\[ (89) \]

In (89), recalling the boundedness of $\|M(2,R)\|_{L^\infty(\Omega_1^*)}$, $\| (M(2,R))^{-1} \|_{L^\infty(\Omega_1^*)}$, $\| \delta^{-1} \|_{L^\infty(\Omega_1^*)}$, we obtain estimate $\mathbb{1}$. For equality $\mathbb{2}$, $\theta_\rho$ is the unique argument such that $\rho e^{i\theta_\rho} \in \Sigma_{11}$; $\rho_0$ denotes the modular of $s$ belonging to the arc contained in $L \cap D_+$; moreover, for $s = \rho e^{i\theta}$, we obtain that

\[ dL(s) = \rho d\theta d\rho, \quad \tilde{\partial} = e^{i\theta}(\partial_\rho + i \rho^{-1} \partial_\theta)/2. \]
For the triangular function, it follows that
\[ \sin \theta \geq \sin \theta, \quad \theta \in \left[ \frac{\pi}{2}, \theta \right], \]
and then, as a consequence, we obtain (3). (4) is the consequence of Schwartz inequalities, the boundedness of \( \| r \|_{H^1} \) and the elementary fact that
\[
\int_{\frac{\pi}{2}}^{\theta} \frac{1}{|pe^{i\theta} - \lambda|^2} d\theta \leq \int_{-\pi}^{\pi} \frac{1}{|pe^{i\theta} - \lambda|^2} d\theta = \int_{-\pi}^{\pi} \left( \frac{1}{(\rho - |\lambda|)^2 + 4\rho|\lambda| \sin^2 \frac{\pi}{2}} \right) d\theta.
\]
\( \int_{-\pi}^{\pi} \frac{d\theta}{(\rho - |\lambda|)^2 + 4\rho|\lambda| \sin^2 \frac{\pi}{2}} \lesssim |\rho - |\lambda||^{-1}. \)
(5) is obtained by (31) and the fact that \( pe^{i\theta} \in \Sigma_{11} \). We derive (6) by the proof found in [42]. We complete the proof.

Since \( \| S \|_{L^\infty \rightarrow L^\infty} \) decays as \( t \rightarrow +\infty \), for sufficiently large \( t \), \( (1 - S)^{-1} \) belongs to \( \mathcal{B}(L^\infty(\mathbb{C})) \); therefore, by (88), we learn that the solution \( M^{(2,D)} \) exists and (87) is written as
\[
M^{(2,D)}(\lambda, n, t) = I - \frac{1}{\pi} \int_{\mathbb{C}} \frac{(1 - S)^{-1} IP(s, n, t)}{s - \lambda} d(s).
\]
Estimate \( M^{(2,D)}(\lambda, n, t) \) at \( \lambda = 0 \),
\[
M^{(2,D)}(0, n, t) = I - \frac{1}{\pi} \int_{\mathbb{C}} \frac{(1 - S)^{-1} IP(s, n, t)}{s} d(s). \tag{90}
\]
For sufficiently large $t$, using the technique applied on (89), we derive a series of estimates

\[
\left| \frac{1}{\pi} \int_{\Omega} \frac{[(1-S)^{-1}I\tilde{P}](s,n,t)}{s} dL(s) \right| \leq \| (1-S)^{-1}I \|_{L^\infty(\Omega^-)} \int_{\Omega^-} \frac{|	ilde{P}(s,n,t)|}{|s|} dL(s) \lesssim \int_{\Omega^-} \left| \frac{\tilde{\partial}P}{s} \right| dL(s)
\]

\[
= \int_{\rho_0}^{\rho_1} \int_{\frac{\theta_0}{2}}^{\theta_1} \rho |\rho'(\theta)| e^{-t \text{Im} \phi(\rho e^{i\theta},n,t)} \left\| \frac{1}{\delta^2(\rho e^{i\theta})} \right\| d\theta d\rho \lesssim \int_{\rho_0}^{\rho_1} \int_{\frac{\theta_0}{2}}^{\theta_1} |\rho'(\theta)| e^{-t \text{Im} \phi(\rho e^{i\theta},n,t)} \frac{\cos \theta}{\rho} d\theta d\rho
\]

\[
\lesssim t^{-\frac{1}{2}} \int_{\rho_0}^{\rho_1} e^{-\frac{\sqrt{1-V_0^2}}{2}(\rho-1)^2} d\rho \lesssim t^{-\frac{1}{2}} \int_{\rho_0}^{\rho_1} e^{-\frac{\sqrt{1-V_0^2}}{2}(\rho-1)^2} d\rho \lesssim t^{-\frac{3}{4}}. \quad (91)
\]

In addition to proof for (91), the sufficiently large $t$ and Proposition 3.13 guarantee the boundedness of $\| (1-S)^{-1}I \|_{L^\infty(\Omega^-)}$, which combined with the boundedness of $\| M^{(2,R)} \|_{L^\infty(\Omega^-)}$, $\| (M^{(2,R)})^{-1} \|_{L^\infty(\Omega^-)}$, $\| \delta^{-1} \|_{L^\infty(\Omega^-)}$ deduces $\mathbb{O}$; by the elementary fact that for $\theta \in \left[ \frac{\pi}{2}, \theta_0 \right]$

\[
\sin \theta \geq \sin \theta_0 + \frac{\cos \theta_0}{2} (\theta - \theta_0),
\]

we obtain $\mathbb{O}$; recalling $\rho e^{i\theta} \in \Sigma_{11}$ and (32), we obtain that

\[
\rho \cos \theta_0 = \text{Re}(\rho e^{i\theta}) > \frac{\sqrt{1-V_0^2}}{2} > 0, \quad (92)
\]

which deduces $\mathbb{O}$. We extend the result in (91) to that for the case when integral regions is $\mathbb{C}$

\[
\left| \frac{1}{\pi} \int_{\mathbb{C}} \frac{[(1-S)^{-1}I\tilde{P}](s,n,t)}{s} dL(s) \right| \lesssim t^{-\frac{3}{4}}; \quad (93)
\]

therefore, combining (90) and (93), we obtain that as $t \to +\infty$,

\[
M^{(2,D)} \sim I + \mathcal{O}(t^{-\frac{3}{4}}). \quad (94)
\]
3.7 the long-time asymptotics

Considering (16), (22), (23), (29), (85), (86) and (94), we obtain that
\[ q_n(t) = \delta^{-1}(0) \sum_{j=1}^{2} \beta_j S_j^{-1} \delta_j^2 [M_1^{L,j}]_{1,2} + O(t^{-\frac{3}{4}}), \]
\[ [M_1^{L,j}]_{1,2} = -i(2\pi)^\frac{3}{2} e^{\frac{2\pi}{4}} e^{-\frac{\pi}{2}} \frac{1}{r(S_j)\Gamma(-a)}, \quad j = 1, 2. \]

3.8 The invertibility of operators

In this part, we come to accomplish this section by proving the boundedness of some operators. We check the boundedness of these operators one-by-one in this order:

\[
(1 - \tilde{A}_j^{-\infty})^{-1} \rightarrow (1 - \tilde{A}_j)^{-1} \rightarrow (1 - A_j)^{-1} \\
\rightarrow (1 - C_{w^j})^{-1} \rightarrow (1 - C_w)^{-1} \rightarrow (1 - C_w)^{-1}, \quad j = 1, 2.
\]

By taking \( \tilde{A}_j^{-\infty} \) in place of \( C_{w^\infty} \) in Step 7 of [20], we obtain that \( (1 - \tilde{A}_j^{-\infty})^{-1} \in B(L^2(\tilde{\Sigma}_j)) \).

By Proposition 3.11, we learn that
\[
\| \tilde{A}_j - \tilde{A}_j^{-\infty} \|_{L^2 \rightarrow L^2} \lesssim \| \hat{w}^j - \tilde{w}^{j,\infty} \|_{L^\infty(\Sigma_j)} \lesssim t^{-\frac{3}{4}},
\]
which combined with the boundedness of \( (1 - \tilde{A}_j^{-\infty})^{-1} \) deduces that for sufficiently large \( t \),
\[
(1 - \tilde{A}_j)^{-1} = (1 - \tilde{A}_j^{-\infty})^{-1}(1 - (\tilde{A}_j - \tilde{A}_j^{-\infty})(1 - \tilde{A}_j^{-\infty})^{-1})^{-1} \in B(L^2(\tilde{\Sigma}_j)).
\]

For \( (1 - A_j)^{-1} \), by (67), we obtain that
\[
(1 - A_j)^{-1} = N_j^{-1} \Delta_j^{-1} (1 - \tilde{A}_j)^{-1} \Delta_j N_j \in B(L^2(\Sigma_j)), \quad j = 1, 2,
\]
because \( (1 - A_j)^{-1} \in B(L^2(\tilde{\Sigma}_j)) \). By Lemma 2.56 in [22], we learn that since \( w^j \) is supported on \( \Sigma_j \subset \Sigma^{(2)} \cap \Sigma'_j \), that \( (1 - C_{w^j})^{-1} \in B(L^2(\Sigma^{(2)})) \) are equivalent to that
\[(1 - A_j)^{-1} \in \mathcal{B}(L^2(\Sigma'_j)).\] By (42), boundedness of \((1 - C_{w,j})^{-1}\) and Lemma 3.9, we derive that for sufficiently large \(t\),

\[
(1 - C_{w'})^{-1} = (1 + C_{w'}(1 - C_{w'})^{-1} + C_{w'}(1 - C_{w'})^{-1})
\times (1 - C_{w'}C_{w'}(1 - C_{w'})^{-1} - C_{w'}C_{w'}(1 - C_{w'})^{-1})^{-1} \in \mathcal{B}(L^2(\Sigma'(2))).
\]

By direct computation and Lemma 3.7, we derive that

\[
\| C_w - C_{w'} \|_{L^2 \to L^2} = \| C_{w'} \|_{L^2 \to L^2} \lesssim \| w' \|_{L^\infty(\Sigma'(2))} \lesssim t^{-1}.
\] (95)

For sufficiently large \(t\), by boundedness of \((1 - C_{w'})^{-1}\) and (95), it follows that

\[
(1 - C_w)^{-1} = (1 - C_{w'})^{-1}(1 - (C_w - C_{w'})(1 - C_{w'})^{-1})^{-1} \in \mathcal{B}(L^2(\Sigma'(2))).
\]

We complete the proof of this part.

### 4 Long-time asymptotic analysis on \(\xi \geq V_0 > 1\)

In this section, we analyzing asymptotic properties for the case when \(\frac{n}{\tau} = \xi \geq V_0 > 1\). In this case, we set \(\epsilon_0 = \sqrt{V_0^2 - 1 + V_0^{-1}}\). We see that the jump matrix \(V\) admits a upper/lower triangular factorization shown in Jump condition of RH problem. The exponential part in the lower/upper triangular matrix decays fast as \(t \to +\infty\) on the \(-/+\) side of the jump contour, respectively, seeing (a) in Figure 9. According to this factorization, firstly, we deform the RH problem: \(M \rightsquigarrow M^{(3)}\), such that it admits \(\bar{\partial}\)-RH problem 4.1 and the new jump matrix decay to \(I\) as \(t \to +\infty\); secondly, we deform the solution for \(\bar{\partial}\)-RH problem into the product of solutions for RH problem 4.2 and \(\bar{\partial}\)-problem 4.3, and analyze long-time asymptotic properties of these solutions separately. Finally, we obtain the asymptotic property (110) by the reconstructed formula.
Introducing a $2 \times 2$ matrix-valued function $M^{(3)} \equiv M^{(3)}(\lambda, n, t)$

$$M^{(3)} = M \mathcal{P}^1, \quad \mathcal{P}^1(\lambda, n, t) = \begin{cases}
I & \lambda \in \Omega_7 \cup \Omega_{10}, \\
\begin{bmatrix}
1 & 0 \\
R_8(\lambda)e^{i\phi(\lambda,n,t)} & 1
\end{bmatrix} & \lambda \in \Omega_8, \\
\begin{bmatrix}
1 & R_9(\lambda)e^{-i\phi(\lambda,n,t)} \\
0 & 1
\end{bmatrix} & \lambda \in \Omega_9,
\end{cases} \quad (96a)$$

$$R_8(\lambda) = -r(\theta), \quad R_9(\lambda) = -r(\theta), \quad \lambda = |\lambda|e^{i\theta}, \quad (96b)$$

and

$$w_+^{(3)} \equiv w_+^{(3)}(\lambda, n, t) = \begin{cases}
0 & \lambda \in \Sigma_{-\epsilon_0}, \\
\begin{bmatrix}
0 & 0 \\
-R_8(\lambda)e^{i\phi(\lambda,n,t)} & 0
\end{bmatrix} & \lambda \in \Sigma_{\epsilon_0},
\end{cases} \quad (97a)$$

$$w_-^{(3)} \equiv w_-^{(3)}(\lambda, n, t) = \begin{cases}
\begin{bmatrix}
0 & R_9(\lambda)e^{-i\phi(\lambda,n,t)} \\
0 & 0
\end{bmatrix} & \lambda \in \Sigma_{-\epsilon_0}, \\
0 & \lambda \in \Sigma_{\epsilon_0},
\end{cases} \quad (97b)$$

we obtain a $\bar{\partial}$-RH problem:

**$\bar{\partial}$-RH Problem 4.1.** Find a $2 \times 2$ matrix-valued function such that

- $M^{(3)}$ belongs to $C^0(\mathbb{C} \setminus (\Sigma_{-\epsilon_0} \cup \Sigma_{\epsilon_0}))$ and its first-order partial derivatives are continuous on $\mathbb{C} \setminus (\Sigma \setminus \Sigma_{-\epsilon_0} \cup \Sigma_{\epsilon_0})$. 

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Figure 10: The jump contour: $\Sigma = \Sigma_{-\epsilon_0} \cup \Sigma_{\epsilon_0}$.

- As $\lambda \to \infty$, $M^{(3)}(\lambda, n, t) \sim I + \mathcal{O}(\lambda^{-1})$.
- On $\lambda \in (\Sigma_{\epsilon_0} \cup \Sigma_{-\epsilon_0})$, $M^{(3)}_+ = M^{(3)}_- V^{(3)}$, $V^{(3)} = (1 - w_-^{(3)})^{-1} (1 + w_+^{(3)})$.
- On $\lambda \in \mathbb{C} \setminus (\Sigma \cup \Sigma_{-\epsilon_0} \cup \Sigma_{\epsilon_0})$,

  \[
  \bar{\partial}M^{(3)} = M^{(3)} \bar{\partial}P^1,
  \]

  \[
  \bar{\partial}P^1(\lambda, n, t) = \begin{cases}
  0 & \lambda \in \Omega_7 \cup \Omega_{10}, \\
  \bar{\partial}R_8(\lambda) e^{it\phi(\lambda, n, t)} & \lambda \in \Omega_8, \\
  0 & \lambda \in \Omega_9.
  \end{cases}
  \]

For sufficiently large $t > 0$, we deform the solution for $\bar{\partial}$-RH problem 4.1 into the product of solutions for RH problem 4.2 and $\bar{\partial}$ problem 4.3:

\[
M^{(3)} = M^{(3, D)} M^{(3, R)},
\]

where $M^{(3, R)} \equiv M^{(3, R)}(\lambda, n, t)$ admits the same jump condition as $M^{(3)}$’s and $M^{(3, D)} \equiv M^{(3, D)}(\lambda, n, t)$ is continuous over complex plane $\mathbb{C}$; then, in the remaining part of this
section, by analyzing long-time asymptotic properties of these solutions, we claim that

\[ M^{(3)}(0, n, t) \sim I + \mathcal{O}(t^{-1}). \]  

Below, we study the following problems.

**RH problem 4.2.** Find a $2 \times 2$ matrix-valued function $M^{(3,R)}$ such that

- $M^{(3,R)}$ is holomorphic on $\mathbb{C} \setminus (\Sigma_{-\epsilon_0} \cup \Sigma_{\epsilon_0})$.
- As $\lambda \to \infty$, $M^{(3,R)}(\lambda, n, t) \sim I + \mathcal{O}(\lambda^{-1})$.
- On $\lambda \in \Sigma_{-\epsilon_0} \cup \Sigma_{\epsilon_0}$, $M^{(3,R)}(\lambda, n, t) = M^{(3,R)}V^{(3)}$.

**$\bar{\partial}$-problem 4.3.** Find a $2 \times 2$ matrix-valued function $M^{(3,D)}$ such that

- $M^{(3,D)}$ is continuous on $\mathbb{C}$, and its first-order partial derivatives are continuous on $\mathbb{C} \setminus (\Sigma \cup \Sigma_{-\epsilon_0} \cup \Sigma_{\epsilon_0})$.
- As $\lambda \to \infty$, $M^{(3,D)} \sim I + \mathcal{O}(\lambda^{-1})$.
- On $\lambda \in \mathbb{C} \setminus (\Sigma \cup \Sigma_{-\epsilon_0} \cup \Sigma_{\epsilon_0})$,

\[ \bar{\partial}M^{(3,D)} = M^{(3,D)}\tilde{P}^1, \quad \tilde{P}^1 = M^{(3,R)}\bar{\partial}P^1(M^{(3,R)})^{-1}. \]

For RH problem 4.2, like what we have done to RH problem 3.5, we obtain the Beals-Coifman solution for RH problem 4.2,

\[ M^{(3,R)}(\lambda, n, t) = I + \frac{1}{2\pi i} \int_{\Sigma^{(3)}} \frac{[(1 - C_{w^{(3)}_{(\lambda)-}}^{-1})Iw^{(3)}](s, n, t)}{s - \lambda} ds, \]

\[ C_{w^{(3)}_{(\lambda)-}}^{\Sigma^{(3)}} = C_{w^{(3)}_{(\lambda)-}}^{\Sigma^{(3)}}(\cdot w^{(3)}_{(3)}) + C_{w^{(3)}_{(\lambda)-}}^{\Sigma^{(3)}}(\cdot w^{(3)}_{(3)}), \quad w^{(3)} = w^{(3)}_{(3)} + w^{(3)}_{(3)}, \]

\[ C_{\pm}^{\Sigma^{(3)}} f(\lambda) = \lim_{\lambda' \to \lambda, \lambda \text{ on the } \pm \text{ side of } \Sigma^{(3)} \int_{\Sigma^{(3)}} \frac{f(s)}{2\pi i(s - \lambda')} ds. \]
Since $\Sigma_{\epsilon_0}$, $\Sigma_{-\epsilon_0}$ are compact curves and contained in the region $\{\text{Im}\phi > 0\}$, $\{\text{Im}\phi < 0\}$, respectively, there is a positive constant $C$ such that

$$\text{Im}\phi|_{\Sigma_{\epsilon_0}} \geq C, \quad \text{Im}\phi|_{\Sigma_{-\epsilon_0}} \leq -C;$$

(101)

therefore, by the fact that $r \in H^1_{\theta}(\Sigma) \subset L^\infty(\Sigma)$, (96b) and (97), we obtain that as $t \to +\infty$,

$$\|w^{(3)}_{\pm}\|_{L^\infty(\Sigma^{(3)})} \lesssim e^{-Ct};$$

(102)

as a result, using the technique applied in (40), we deduce that

$$\|C_{w^{(3)}}^{(\Sigma^{(3)})}\|_{L^2 \to L^2} \lesssim \|w^{(3)}\|_{L^\infty(\Sigma^{(3)})} \lesssim e^{-Ct}. $$

(103)

Considering (100) and (103), we obtain that as $t \to +\infty$,

$$M^{(3,R)}(0,n,t) \sim I + O(e^{-Ct}).$$

(104)

For the boundedness of $M^{(3,R)}$, by RH problem 4.2, we find that det $M^{(3,R)}$ admits no jump on $\mathbb{C}$, and as $\lambda \to \infty$,

$$\lim_{\lambda \to \infty} \det M^{(3,R)} \sim 1 + O(\lambda^{-1});$$

therefore, by Liouville’s Theorem, we derive that

$$\det M^{(3,R)} \equiv 1;$$

moreover, since $M^{(3,R)}$ admits no pole, both $M^{(3,R)}$ and $(M^{(3,R)})^{-1}$ are bounded on $\mathbb{C}$.

For $\bar{\partial}$-problem 4.3, Like that in Section 3.6, we also have the solution for $\bar{\partial}$-problem 4.3:

$$M^{(3,D)}(\lambda,n,t) = I + [S^1 M^{(3,D)}](\lambda,n,t),$$

(105)

where $S^1$: $f \mapsto S^1 f$ is a linear operator and belongs to $B(L^\infty(\mathbb{C}))$,

$$S^1 f(\lambda,n,t) = -\frac{1}{\pi} \int_C f(s) \tilde{P}^1(s,n,t) \frac{1}{s-\lambda} dL(s).$$

(106)

For this operator $S^1$, we find it decays as $t \to +\infty$ on $L^\infty(\mathbb{C})$ as shown in Proposition 4.4.
Proposition 4.4. For $r \in H^1_0(\Sigma)$, the integral operator $S^1$ belongs to $\mathcal{B}(L^\infty(\mathbb{C}))$, and it satisfies that

$$\| S^1 \|_{L^\infty \rightarrow L^\infty} \lesssim t^{-\frac{1}{4}}.$$ 

Proof. We detail the case when functions $f(s)$ are only supported on $\Omega_8$. Like the proof in Proposition 3.13, we obtain a series of estimates:

$$|S^1 f(\lambda)| \leq \frac{\int_{\Omega_8} |\partial R_8(s)| e^{-t \text{Im}(s, n, t)} d\lambda}{|s - \lambda|} \rho dL(s)$$

\[10\]

\[11\]

In (107), by the fact that for $s = \rho e^{i\theta}$,

$$\text{Im}(s, n, t) = (\rho - \rho^{-1}) \sin \theta + 2\xi \ln \rho \geq 2\xi \ln \rho - (\rho - \rho^{-1}),$$

which with the boundedness of $\| M(3, R) \|_{L^\infty(\Omega_8)}$ and $\| (M(3, R))^{-1} \|_{L^\infty(\Omega_8)}$ deduces (10); (11) is by Schwartz Inequalities and the fact that on the region $\Omega_8$, $\rho > 1$ such that

$$2\xi \ln \rho - \rho + \rho^{-1} = \left(\frac{2\xi \ln \rho}{\rho - 1} - \left(1 + \frac{1}{\rho}\right)(\rho - 1)\right) \geq 2(\rho - 1) \left(\frac{V_0 \ln(1 + \epsilon_0)}{\epsilon_0} - 1\right)$$

where $\frac{V_0 \ln(1 + \epsilon_0)}{\epsilon_0} - 1$ is a positive number because of the choosing of $\epsilon_0$ and $V_0 > 1$. We complete the proof for $f(s)$ only supported on $\Omega_8$, and the proof for general $f(s) \in L^\infty(\mathbb{C})$ is similarly obtained. \qed
From Proposition 4.4, we see that for sufficiently large $t$, $(1 - S^1)^{-1}I$ exists and is bounded on $L^\infty(\mathbb{C})$; therefore, we obtain that

$$\left| \frac{1}{\pi} \iint_{\Omega_8} \frac{[(1 - S)^{-1}I\tilde{\mathcal{P}}^1](s, n, t)}{s} dL(s) \right|$$

$$\leq \frac{1}{\pi} \| (1 - S)^{-1}I \|_{L^\infty(\mathbb{C})} \| M^{(3,R)} \|_{L^\infty(\Omega_8)} \| \tilde{\mathcal{P}}(s,n,t) \|_{L^\infty(\Omega_8)}$$

$$\leq \frac{1}{\pi} \int \int_{\Omega_8} |\partial R(s)| e^{-t s} dL(s)$$

$$\leq \frac{1}{\pi} \int \int_{\Omega_8} \left| \tilde{\partial} R(s) e^{-t s} \right| dL(s) = \int_1^{1+\epsilon_0} e^{t \rho} \int_0^{2\pi} e^{t (\rho - \frac{1}{2}) \sin \theta - 2\xi \ln \rho} d\theta d\rho$$

$$\leq \| r' \|_{L^1(\Sigma)} \int_1^{1+\epsilon_0} e^{-t(2\xi \ln \rho - \rho + \rho^{-1})} d\rho$$

$$\leq \| r \|_{L^1(\Sigma)} \int_1^{1+\epsilon_0} e^{-2t(V_0 \ln(1+\epsilon_0) - 1)(\rho - 1)} d\rho \lesssim t^{-1}; \quad (108)$$

similar to (108), we obtain the result for integral region $\mathbb{C}$ that

$$\left| \frac{1}{\pi} \iint_{\mathbb{C}} \frac{[(1 - S)^{-1}I\tilde{\mathcal{P}}^1](s, n, t)}{s} dL(s) \right| \lesssim t^{-1}. \quad (109)$$

Considering (98), (104), (105), (106) and (109), we prove (99); therefore, by (16), (96a) and (99), we obtain that

$$q_n(t) \sim O(t^{-1}).$$

5  Long-time asymptotic analysis on $\xi \leq -V_0 < -1$

In the last section, we analyze the case of $\xi \leq -V_0 < -1$. Introducing a $2 \times 2$ matrix-valued function

$$M^{(4)} = M\tilde{\delta}^{-\sigma_3}, \quad \tilde{\delta} \equiv \tilde{\delta}(\lambda) = e^{\frac{i}{\pi} \int_{K \setminus (-1,-1)} \frac{\ln(1+\phi) - 1}{x} d\phi}$$

we claim that $\tilde{\delta}$ satisfies Proposition 5.1; moreover, by Proposition 5.1 and RH problem 2.1, we obtain that $M^{(4)}$ admits RH problem 5.2. We see in RH problem 5.2 that on $\lambda \in \Sigma$, the
jump matrix admits a lower/upper triangular factorization

\[ V^{(4)} = (I - w_{-}^{(4)})^{-1}(I - w_{+}^{(4)}), \]

\[ w_{-}^{(4)}(\lambda, n, t) = \begin{bmatrix} 0 \\
\frac{\delta^2(\lambda)r(\lambda)}{1 - |r(\lambda)|^2} e^{it\phi(\lambda,n,t)} \end{bmatrix}, \]

\[ w_{+}^{(4)}(\lambda, n, t) = \begin{bmatrix} 0 \\
\frac{\delta^2(\lambda)r(\lambda)}{1 - |r(\lambda)|^2} e^{-it\phi(\lambda,n,t)} \end{bmatrix}, \]

and the exponential part of \( w_{+}^{(4)}, w_{-}^{(4)} \) decay exponentially on regions to the +, − side of \( \Sigma \) as \( t \to +\infty \), respectively, seeing (b) in Figure 9; therefore, similar to the case of \( \xi \geq V_{0} > 1 \) we obtain the long-time asymptotics when \( \xi \leq -V_{0} < -1 \), and that

\[ q_{n}(t) \sim O(t^{-1}). \]

**Proposition 5.1.** Since \( r \in H^{1}_{0}(\Sigma) \), the scalar function \( \tilde{\delta} \) satisfies:

(a) \( \tilde{\delta} \) is analytic on \( \mathbb{C} \setminus \Sigma \).

(b) As \( \lambda \to \infty \),

\[ \tilde{\delta}(\lambda) \sim 1 + O(\lambda^{-1}). \]

(c) On the unit circle \( \lambda \in \Sigma \),

\[ \tilde{\delta}_{+}(\lambda) = \tilde{\delta}_{-}(\lambda)(1 - |r(\lambda)|^2). \]

(d) \( \tilde{\delta} \) admits the symmetry:

\[ \tilde{\delta}(\lambda) = \frac{\tilde{\delta}(0)}{\tilde{\delta}(\bar{\lambda}^{-1})}. \]

**Proof.** This proof is parallel to that for Proposition 3.1.

**RH problem 5.2.** Find a \( 2 \times 2 \) matrix-valued function \( M^{(4)} \) such that
• **Analyticity:** $M^{(1)}$ is analytic on $\mathbb{C} \setminus \Sigma$.

• **Normalization:** As $\lambda \to \infty$,

\[ M^{(4)}(\lambda, n, t) \sim I + O(\lambda^{-1}). \]

• **Jump condition:** On $\lambda \in \Sigma$,

\[
M_+^{(4)} = M_-^{(4)} V^{(4)},
\]

\[
V^{(4)} \equiv V^{(3)}(\lambda, n, t) = \begin{bmatrix}
1 & -\frac{\delta^2 r^*(\lambda)}{1 - |r(\lambda)|^2} e^{-it\phi(\lambda, n, t)} \\
\frac{\delta^2 r(\lambda)r^*(\lambda)}{1 - |r(\lambda)|^2} e^{it\phi(\lambda, n, t)} & 1 - |r(\lambda)|^2
\end{bmatrix}.
\]

**References**

[1] GS Gardner, JM Greene, MD Kruskal, and RM Miura. Method for solving Korteweg-de Vries equation. *Phys. Rev. Lett.*, 19(19):1095–1097, 1967.

[2] MJ Ablowitz, DJ Kaup, AC Newell, and H Segur. The inverse scattering transform-Fourier analysis for nonlinear problems. *Stud. Appl. Math.*, 53(4):249–315, 1974.

[3] DS Wang, DJ Zhang, and JK Yang. Integrable properties of the general coupled nonlinear Schrödinger equations. *J. Math. Phys.*, 51(2):023510, 2010.

[4] G Biondini and G Kovačič. Inverse scattering transform for the focusing nonlinear Schrödinger equation with nonzero boundary conditions. *J. Math. Phys.*, 55(3):031506, 2014.

[5] G Biondini and D Kraus. Inverse scattering transform for the defocusing manakov system with nonzero boundary conditions. *SIAM J. Math. Anal.*, 47(1):706–757, 2015.

[6] D Kraus, G Biondini, and G Kovačič. The focusing Manakov system with nonzero boundary conditions. *Nonlinearity*, 28(9):3101, 2015.
[7] M Pichler and G Biondini. On the focusing non-linear Schrödinger equation with non-zero boundary conditions and double poles. *IMA J. Appl. Math.*, 82(1):131–151, 2017.

[8] J Xu and EG Fan. A riemann-hilbert approach to the initial-boundary problem for derivative nonlinear Schrödinger equation. *Acta Math. Sci.*, 34(4):973–994, 2014.

[9] Y Xiao and EG Fan. A Riemann-Hilbert approach to the Harry-Dym equation on the line. *Chin. Ann. Math. Ser. B*, 37(3):373–384, 2016.

[10] ZZ Kang, TC Xia, and X Ma. Multi-soliton solutions for the coupled modified nonlinear Schrödinger equations via Riemann–Hilbert approach. *Chin. Phys. B*, 27(7):070201, 2018.

[11] B Yang and Y Chen. High-order soliton matrices for Sasa–Satsuma equation via local Riemann–Hilbert problem. *Nonlinear Anal.-Real World Appl.*, 45:918–941, 2019.

[12] JK Yang. *Nonlinear waves in integrable and nonintegrable systems*. SIAM, 2010.

[13] Y Yang and EG Fan. Riemann–Hilbert approach to the modified nonlinear Schrödinger equation with non-vanishing asymptotic boundary conditions. *Physica D*, 417:132811, 2021.

[14] G Teschl. Inverse scattering transform for the Toda hierarchy. *Math. Nachr.*, 202(1):163–171, 1999.

[15] MJ Ablowitz, G Biondini, and B Prinari. Inverse scattering transform for the integrable discrete nonlinear Schrödinger equation with nonvanishing boundary conditions. *Inverse Probl.*, 23(4):1711–1758, 2007.

[16] MJ Ablowitz, XD Luo, and ZH Musslimani. Discrete nonlocal nonlinear Schrödinger systems: Integrability, inverse scattering and solitons. *Nonlinearity*, 33(7):3653, 2020.

[17] B Prinari. Discrete solitons of the focusing Ablowitz-Ladik equation with nonzero boundary conditions via inverse scattering. *J. Math. Phys.*, 57(8):083510, 2016.
[18] AK Ortiz and B Prinari. Inverse scattering transform for the defocusing Ablowitz-Ladik system with arbitrarily large nonzero background. *Stud. Appl. Math.*, 143(4):373–403, 2019.

[19] MS Chen and EG Fan. Riemann-Hilbert approach for discrete sine-Gordon equation with simple and double poles. *Stud. Appl. Math.*, 148(3):1180–1207, 2022.

[20] P Deift and X Zhou. Long-time behavior of the non-focusing linear Schrödinger equation-a case study. In *New series: lectures in mathematical sciences*. University of Tokyo, 1994.

[21] P Deift and X Zhou. Long-time asymptotics for integrable systems. higher order theory. *Commun. Math. Phys.*, 165(1):175–191, 1994.

[22] P Deift and X Zhou. A steepest descent method for oscillatory Riemann–Hilbert problems. Asymptotics for the MKdV equation. *Annals of Mathematics*, 137(2):295–368, 1993.

[23] K Grunert and G Teschl. Long-time asymptotics for the Korteweg–de Vries equation via nonlinear steepest descent. *Math. Phys. Anal. Geom.*, 12(3):287–324, 2009.

[24] Anne Boutet De Monvel, Aleksey Kostenko, D Shepelsky, and G Teschl. Long-time asymptotics for the Camassa–Holm equation. *SIAM J. Math. Anal.*, 41(4):1559–1588, 2009.

[25] J Xu and EG Fan. Long-time asymptotics for the Fokas–Lenells equation with decaying initial value problem: without solitons. *J. Differ. Equ.*, 259(3):1098–1148, 2015.

[26] J Xu. Long-time asymptotics for the short pulse equation. *J. Differ. Equ.*, 265(8):3494–3532, 2018.

[27] L Huang, J Xu, and EG Fan. Long-time asymptotic for the Hirota equation via non-linear steepest descent method. *Nonlinear Anal.-Real World Appl.*, 26:229–262, 2015.
[28] QZ Zhu, J Xu, and EG Fan. The Riemann–Hilbert problem and long-time asymptotics for the Kundu–Eckhaus equation with decaying initial value. *Appl. Math. Lett.*, 76:81–89, 2018.

[29] Helge Krüger and Gerald Teschl. Long-time asymptotics of the Toda lattice for decaying initial data revisited. *Rev. Math. Phys.*, 21(01):61–109, 2009.

[30] Helge Krüger and Gerald Teschl. Long-time asymptotics for the Toda lattice in the soliton region. *Math. Z.*, 262(3):585–602, 2009.

[31] Hideshi Yamane. Long-time asymptotics for the defocusing integrable discrete nonlinear Schrödinger equation II. *Symmetry Integr. Geom.*, 11:020, 2015.

[32] Hideshi Yamane. Long-time asymptotics for the integrable discrete nonlinear Schrödinger equation: the focusing case. *Funkc. Ekvacioj-Ser. Int.*, 62(2):227–253, 2019.

[33] Meisen Chen and Engui Fan. Long-time asymptotic behavior for the discrete defocusing mKdV equation. *J. Nonlinear Sci.*, 30(3):953–990, 2020.

[34] X Zhou. $l^2$-sobolev space bijectivity of the scattering and inverse scattering transforms. *Commun. Pure Appl. Math.*, 51(7):697–731, 1998.

[35] MJ Ablowitz and JF Ladik. Nonlinear differential-difference equations. *J. Math. Phys.*, 16(3):598–603, 1975.

[36] MJ Ablowitz and JF Ladik. Nonlinear differential–difference equations and Fourier analysis. *J. Math. Phys.*, 17(6):1011–1018, 1976.

[37] I Nenciu. Lax pairs for the Ablowitz-Ladik system via orthogonal polynomials on the unit circle. *International Mathematics Research Notices*, 2005(11):647–686, 2005.
[38] KW Chow, R Conte, and N Xu. Analytic doubly periodic wave patterns for the integrable discrete nonlinear Schrödinger (Ablowitz-Ladik) model. *Physics Letters A*, 349(6):422–429, 2006.

[39] PD Miller, NM Ercolani, IM Krichever, and CD Levermore. Finite genus solutions to the Ablowitz-Ladik equations. *Commun. Pure Appl. Math.*, 48(12):1369–1440, 1995.

[40] H Yamane. Long-time asymptotics for the defocusing integrable discrete nonlinear schrödinger equation. *J. Math. Soc. Jpn.*, 66(3):765–803, 2014.

[41] MS Chen, EG Fan, and JS He. $l^2$ Sobolev space bijectivity of the scattering-inverse scattering transforms related to defocusing Ablowitz-Ladik systems. *arXiv preprint arXiv:2204.06897*, 2022.

[42] M Dieng and K McLaughlin. Long-time Asymptotics for the NLS equation via dbar methods. *arXiv preprint arXiv:0805.2807*, 2008.

[43] M Borghese, R Jenkins, and K McLaughlin. Long time asymptotic behavior of the focusing nonlinear Schrödinger equation. *Ann. Inst. Henri Poincare-Anal. Non Lineaire*, 35(4):887–920, 2018.

[44] PA Deift, AR Its, and X Zhou. Long-time asymptotics for integrable nonlinear wave equations. In *Important developments in soliton theory*, pages 181–204. Springer, 1993.

[45] R Beals, P Deift, and C Tomei. *Direct and inverse scattering on the line*. Number 28. American Mathematical Society, 1988.