\(N\)-fold Supersymmetry
in Quantum Mechanics
- General Formalism -

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Abstract

We report general properties of \(N\)-fold supersymmetry in one-dimensional quantum mechanics. \(N\)-fold supersymmetry is characterized by supercharges which are \(N\)-th polynomials of momentum. Relations between the anti-commutator of the supercharges and the Hamiltonian, the spectra, the Witten index, the non-renormalization theorems and the quasi-solvability are examined. We also present further investigation about a particular class of \(N\)-fold supersymmetric models which we dubbed type A. Algebraic equations which determine a part of spectra of type A models are presented, and the non-renormalization theorem are generalized. Finally, we present a possible generalization of \(N\)-fold supersymmetry in multi-dimensional quantum mechanics.

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1 Introduction

One of unique aspects of supersymmetry is its usefulness as a tool of non-perturbative analyses of quantum theories. Various non-renormalization theorems enable us to reveal non-perturbative properties of quantum theories without annoyance of perturbative corrections. An approach to quark confinement problem via $\mathcal{N} = 2$ supersymmetric QCD [1] is a good example which represents this aspect. In Refs. [2, 3], a method of calculation of non-perturbative part of the energy spectrum was developed and tested with aid of supersymmetry. It is based on the valley method [4]–[11], and together with an understanding of the Bogomolny technique [12], it correctly led to an explanation of the disappearance of the leading Borel singularity of the perturbative corrections for the ground energy when the theory becomes supersymmetric: Since the ground state of the supersymmetric theories does not receive any perturbative corrections [13, 14], the Borel singularity must vanish in this case.

The method also predicted the disappearance of the leading Borel singularity of the perturbative corrections at other values of a parameter in the theory, which do not correspond to the case when the theory becomes supersymmetric. This disappearance of the leading Borel singularity was understood by an extension of supersymmetry, which was named “$\mathcal{N}$-fold supersymmetry” [3], which supercharges are $\mathcal{N}$-th polynomials of momentum. When $\mathcal{N} = 1$, they reduce to ordinary supersymmetry. Similar higher derivative generalizations of supercharges were investigated in various different contexts [15]–[30].

In this paper, we investigate general properties of $\mathcal{N}$-fold supersymmetry. First in section 2, we define $\mathcal{N}$-fold supersymmetry in one-dimensional quantum mechanics and fix notations used throughout in this paper.

In section 3.1, we introduce the “Mother Hamiltonian” as the anti-commutator of the supercharges. In contrast to ordinary supersymmetry, it does not coincide with the Hamiltonian in general. Relations between the ordinary Hamiltonian and the Mother Hamiltonian are shown. Spectra of the $\mathcal{N}$-fold supersymmetric systems are examined in section 3.2. We investigate a relation between $\mathcal{N}$-fold supersymmetry and polynomial supersymmetry [17] in section 3.3. The Witten index is generalized to $\mathcal{N}$-fold supersymmetry in section 3.4. In section 3.5, non-renormalization theorems for $\mathcal{N}$-fold supersymmetry are briefly discussed. For $\mathcal{N}$-fold supersymmetric systems, non-renormalization theorems hold as well as ordinary supersymmetric ones. In section 3.6, we show a close relation between quasi-solvability and $\mathcal{N}$-fold supersymmetry. For $\mathcal{N}$-fold supersymmetric systems, a part of spectra (not complete spectra) can be solvable. We show that quasi-solvability is equivalent to $\mathcal{N}$-fold supersymmetry.

Two examples of $\mathcal{N}$-fold supersymmetric systems are illustrated in section 4. As the simplest but non-trivial example of $\mathcal{N}$-fold supersymmetry, 2-fold supersymmetry are examined in section 4.1. In section 4.2, a class of $\mathcal{N}$-fold supersymmetric systems which we dubbed “type A” [23] is investigated. The type A models include $\mathcal{N}$-fold supersymmetric systems found in Refs. [3, 24]. For the type A models, a part of the spectra are determined by algebraic equations. Using this equations, the non-renormalization theorem found in Refs. [3, 24] are generalized to most of type A models.

In section 5, we suggest a possible generalization of $\mathcal{N}$-fold supersymmetry in multi-
2. Definition of $\mathcal{N}$-fold supersymmetry

Let us first define $\mathcal{N}$-fold supersymmetry in one-dimensional quantum mechanics. To define the $\mathcal{N}$-fold supersymmetry, we introduce the following Hamiltonian $H_{\mathcal{N}}$,

$$H_{\mathcal{N}} = H_N^{-}\psi\psi^\dagger + H_N^{+}\psi^\dagger\psi,$$

where $\psi$ and $\psi^\dagger$ are fermionic coordinates which satisfy

$$\{\psi, \psi\} = \{\psi^\dagger, \psi^\dagger\} = 0, \quad \{\psi, \psi^\dagger\} = 1,$$

and $H_{\mathcal{N}}^{\pm}$ are ordinary Hamiltonians,

$$H_{\mathcal{N}}^{-} = \frac{1}{2}p^2 + V_{\mathcal{N}}^{-}(q), \quad H_{\mathcal{N}}^{+} = \frac{1}{2}p^2 + V_{\mathcal{N}}^{+}(q),$$

where $p = -id/dq$. The $\mathcal{N}$-fold supercharges are generically defined as

$$Q_{\mathcal{N}} = P_{\mathcal{N}}^\dagger\psi, \quad Q_{\mathcal{N}}^\dagger = P_{\mathcal{N}}\psi^\dagger,$$

where $P_{\mathcal{N}}$ is an $\mathcal{N}$-th order polynomial of $p$,

$$P_{\mathcal{N}} = w_\mathcal{N}(q)p^\mathcal{N} + w_{\mathcal{N}-1}(q)p^{\mathcal{N}-1} + \cdots + w_1(q)p + w_0(q).$$

A system is defined to be $\mathcal{N}$-fold supersymmetric if the following $\mathcal{N}$-fold supersymmetric algebra is satisfied,

$$\{Q_{\mathcal{N}}, Q_{\mathcal{N}}\} = \{Q_{\mathcal{N}}^\dagger, Q_{\mathcal{N}}^\dagger\} = 0,$$

$$[Q_{\mathcal{N}}, H_{\mathcal{N}}] = [Q_{\mathcal{N}}^\dagger, H_{\mathcal{N}}] = 0.$$

The former relation is trivially satisfied, but the latter gives the following conditions,

$$P_{\mathcal{N}}H_{\mathcal{N}}^{-} - H_{\mathcal{N}}^{+}P_{\mathcal{N}} = 0, \quad P_{\mathcal{N}}^\dagger H_{\mathcal{N}}^{-} - H_{\mathcal{N}}^{+}\dagger P_{\mathcal{N}}^\dagger = 0.$$

These conditions generally give $\mathcal{N} + 2$ differential equations for $\mathcal{N} + 3$ functions $V_{\mathcal{N}}^{-}(q)$, $V_{\mathcal{N}}^{+}(q)$ and $w_n(q)$ ($n = 0, \cdots, \mathcal{N}$), thus one function remains arbitrary. We obtain the equation $w_\mathcal{N}(q) = 0$ by comparison of the coefficient of the $\partial^{\mathcal{N}+1}$ terms in Eq. (2.8). Thus we can set $w_\mathcal{N}(q) = 1$ without losing generality.

The above definition of $\mathcal{N}$-fold supersymmetry includes ordinary supersymmetry [3, 4], which is realized when $\mathcal{N} = 1$, $w_0(q) = -iW(q)$ and

$$V_{\mathcal{N}}^{-}(q) = \frac{1}{2}(W(q)^2 - W'(q)), \quad V_{\mathcal{N}}^{+}(q) = \frac{1}{2}(W(q)^2 + W'(q)).$$
Conveniently, $\psi$ and $\psi^\dagger$ are often represented as the following $2 \times 2$ matrix form,

$$
\psi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \psi^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
$$

(2.10)

In this notation, the $\mathcal{N}$-fold supercharges are given by

$$
Q_\mathcal{N} = \begin{pmatrix} 0 & 0 \\ P_\mathcal{N}^\dagger & 0 \end{pmatrix}, \quad Q_\mathcal{N}^\dagger = \begin{pmatrix} 0 & P_\mathcal{N} \\ 0 & 0 \end{pmatrix},
$$

(2.11)

and the Hamiltonian is given by

$$
H_\mathcal{N} = \begin{pmatrix} H_\mathcal{N}^+ & 0 \\ 0 & H_\mathcal{N}^- \end{pmatrix}.
$$

(2.12)

We define the fermion number operator $F$ as follows,

$$
F = \psi^\dagger \psi = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
$$

(2.13)

Thus the form of fermionic states is

$$
\begin{pmatrix} \Phi^+ \\ 0 \end{pmatrix},
$$

(2.14)

and that of bosonic ones is

$$
\begin{pmatrix} 0 \\ \Phi^- \end{pmatrix}.
$$

(2.15)

$H_\mathcal{N}^-$ and $H_\mathcal{N}^+$ are therefore the Hamiltonians of bosonic states and fermionic ones respectively. For physical states, $\Phi^\pm$ are normalizable (square integrable) functions on $\mathbb{R}$ or on a subset of $\mathbb{R}$.

3 General properties

3.1 Mother Hamiltonian

In systems with ordinary supersymmetry, the Hamiltonian is given by the anti-commutator of the supercharges. In systems with $\mathcal{N}$-fold supersymmetry, however, this relation does not hold in general. This is evident from the fact that $Q_\mathcal{N}$ contains $\mathcal{N}$-derivatives with respect to the coordinate $q$ and therefore $\frac{1}{2}\{Q_\mathcal{N}^\dagger, Q_\mathcal{N}\}$ contains $2\mathcal{N}$-derivatives. In $\mathcal{N}$-fold supersymmetric systems, the anti-commutator has a “family resemblance” to the Hamiltonian, and is thus called “Mother Hamiltonian”:

$$
\mathcal{H}_\mathcal{N} = \frac{1}{2}\{Q_\mathcal{N}^\dagger, Q_\mathcal{N}\} = \frac{1}{2} \begin{pmatrix} P_\mathcal{N}^\dagger P_\mathcal{N} & 0 \\ 0 & P_\mathcal{N}^\dagger P_\mathcal{N} \end{pmatrix}.
$$

(3.1)
3.1 Mother Hamiltonian

The Mother Hamiltonian commutes with the $\mathcal{N}$-fold supercharges,

$$[\mathcal{H}_N, Q_N] = [\mathcal{H}_N, Q_N^\dagger] = 0. \quad (3.2)$$

To examine relations between the Mother Hamiltonian and the original one, let us introduce $\mathcal{N}$ linearly independent functions $\phi_n^{-}(q)$ ($n = 1, \cdots, \mathcal{N}$) which satisfy the following relation,

$$P_N\phi_n^{-} = 0. \quad (3.3)$$

From Eq.(2.8), the following relation holds,

$$P_N H_N^{-} \phi_n^{-} = 0, \quad (3.4)$$

thus $H_N^{-} \phi_n^{-}$ is given by a linear combination of $\phi_n^{-}$. We can therefore define the matrix $S^{-}$ as follows,

$$H_N^{-} \phi_n^{-} = \sum_m S_{n,m}^{-} \phi_m^{-}. \quad (3.5)$$

In a similar manner, for $\mathcal{N}$ independent functions which satisfy

$$P_N^\dagger \phi_n^{+} = 0, \quad (3.6)$$

the next equation holds,

$$P_N^\dagger H_N^{+} \phi_n^{+} = 0. \quad (3.7)$$

Thus we define $S^{+}$ as follows,

$$H_N^{+} \phi_n^{+} = \sum_m S_{n,m}^{+} \phi_m^{+}. \quad (3.8)$$

From these matrices $S^{\pm}$, the Mother Hamiltonian $\mathcal{H}_N$ is given as follows,

$$\mathcal{H}_N = \frac{1}{2} \text{det} M_N^{-}(H_N^{-}) \psi \psi^\dagger + \frac{1}{2} \text{det} M_N^{+}(H_N^{+}) \psi^\dagger \psi + \frac{1}{2} Q_N^+ Q_N + \frac{1}{2} q^{-Q_N^+} Q_N^+, \quad (3.9)$$

where

$$\frac{1}{2} M_N^{-}(E) \equiv E I - S^{-}, \quad \frac{1}{2} M_N^{+}(E) \equiv E I - S^{+}, \quad (3.10)$$

and $q^{\pm}$ are supercharges for at most $(\mathcal{N} - 1)$-fold supersymmetry. In $2 \times 2$ matrix notation as in Eqs.(2.10)-(2.13), this becomes

$$\mathcal{H}_N = \frac{1}{2} \begin{pmatrix} \text{det} M_N^{-}(H_N^{-}) + p^+ P_N^\dagger & 0 \\ 0 & \text{det} M_N^{-}(H_N^{-}) + p^- P_N \end{pmatrix}, \quad (3.11)$$

where $p^{\pm}$ are defined by

$$q^{\pm} = \begin{pmatrix} 0 & 0 \\ p^{\pm \dagger} & 0 \end{pmatrix}. \quad (3.12)$$
And if the $\mathcal{N}$-fold supercharges are uniquely determined for given $H_\mathcal{N}$, the Mother Hamiltonian $\mathcal{H}_\mathcal{N}$ has the following more simple form,

$$
\mathcal{H}_\mathcal{N} = \frac{1}{2} \det M_\mathcal{N}(H_\mathcal{N}) \psi \psi^\dagger + \frac{1}{2} \det M_\mathcal{N}(H_\mathcal{N}) \psi^\dagger \psi
= \frac{1}{2} \begin{pmatrix}
\det M_\mathcal{N}(H_\mathcal{N}) & 0 \\
0 & \det M_\mathcal{N}(H_\mathcal{N})
\end{pmatrix}.
$$

(3.13)

**Proof:**

First of all, note that the operator $\det M_\mathcal{N}(H_\mathcal{N})$ annihilates $\phi^m$,

$$
\det M_\mathcal{N}(H_\mathcal{N}) \phi^m = \sum_m \{ \det M_\mathcal{N}(S^-) \} \phi^m = 0.
$$

(3.14)

This is because that $\det M_\mathcal{N}(S^-)$ is identically zero by the Cayley-Hamilton theorem. From this, the form of $\det M_\mathcal{N}(H_\mathcal{N})$ is determined as

$$
\det M_\mathcal{N}(H_\mathcal{N}) = F(p,q) P_\mathcal{N},
$$

(3.15)

where $F(p,q)$ has the following form,

$$
F(p,q) = p^\mathcal{N} + f_{\mathcal{N}-1}(q)p^{\mathcal{N}-1} + \cdots + f_1(q)p + f_0(q).
$$

(3.16)

When we apply $H_\mathcal{N}$ to the above equation (3.15) from the right, the right hand side becomes

$$
F(p,q) P_\mathcal{N} H_\mathcal{N} = F(p,q) H_\mathcal{N} P_\mathcal{N},
$$

(3.17)

and the left hand side becomes

$$
\det M_\mathcal{N}(H_\mathcal{N}) H_\mathcal{N} = H_\mathcal{N} \det M_\mathcal{N}(H_\mathcal{N}) = H_\mathcal{N} F(p,q) P_\mathcal{N}.
$$

(3.18)

Thus we obtain

$$
F(p,q) H_\mathcal{N} P_\mathcal{N} = H_\mathcal{N} F(p,q) P_\mathcal{N}.
$$

(3.19)

Since any function can be written as $P_\mathcal{N} f(q)$, this equation means

$$
F(p,q) H_\mathcal{N}^+ = H_\mathcal{N} F(p,q).
$$

(3.20)

Thus if we define $p^\dagger$ as

$$
p^\dagger = P_\mathcal{N}^\dagger - F(p,q),
$$

(3.21)

it contains $\mathcal{N} - 1$ derivatives with respect to $q$ at most and satisfies

$$
p^\dagger H_\mathcal{N}^+ = H_\mathcal{N} p^\dagger.
$$

(3.22)
Using $p^-$, Eq. (3.15) is rewritten as
\[
\det M_N(H_N^-) = P_N^\dagger P_N - p^\dagger P_N. \tag{3.23}
\]
In a similar manner, the next equation can be shown,
\[
\det M_N^+(H_N^+) = P_N P_N^\dagger - p^+ P_N^\dagger, \tag{3.24}
\]
where $p_K$ is an operator which contains $N - 1$ derivatives with respect to $q$ at most and satisfies
\[
H_N^+ p^+ = p^+ H_N^- . \tag{3.25}
\]
In terms of $P_N$ and $P_N^\dagger$, the Mother Hamiltonian $H_N$ is given by
\[
H_N = \frac{1}{2} \begin{pmatrix}
P_N P_N^\dagger & 0 \\
0 & P_N^\dagger P_N
\end{pmatrix}, \tag{3.26}
\]
thus if we define $q^\pm$ as
\[
q^- = p^-\dagger \psi, \quad q^+\dagger = p^+ \psi^\dagger, \tag{3.27}
\]
we obtain Eq. (3.11) from Eqs. (3.23) and (3.24).

If the $\mathcal{N}$-fold supercharges are uniquely determined for given $H_N^\pm$, $p^\pm$ must be zero since $P_N + p^\pm$ gives new $\mathcal{N}$-fold supercharges. Therefore, in this case we obtain Eq. (3.13).

Q.E.D.

It is worth noting that Eq. (3.13) can be more simplified as
\[
H_N = \frac{1}{2} \det M_N^+(H_N) = \frac{1}{2} \det M_N^-(H_N). \tag{3.28}
\]
This will be proven in section 3.3.

### 3.2 Spectrum

Just as ordinary supersymmetry, bosonic states of $\mathcal{N}$-fold supersymmetric systems and fermionic ones have one to one correspondence unless the states are eigenstates of the Mother Hamiltonian with zero eigenvalue. To see this, let us consider a normalized bosonic state $\Phi_n^-$ which satisfy
\[
H_N^- \Phi_n^- = E_n^- \Phi_n^- . \tag{3.29}
\]
Since $H_N$ commutes with $H_N$, $\Phi_n^-$ can be simultaneously an eigenstate of the Mother Hamiltonian:
\[
H_N \begin{pmatrix} 0 \\ \Phi_n^- \end{pmatrix} = \mathcal{E}_n \begin{pmatrix} 0 \\ \Phi_n^- \end{pmatrix} . \tag{3.30}
\]
If $\mathcal{E}_n$ is not zero, the following normalized state $\Phi_n^+$ exists,

$$\Phi_n^+ \equiv \frac{P_N}{\sqrt{\mathcal{E}_n}} \Phi_n^-.$$  \hfill (3.31)

From Eq. (2.8), we can easily see that this state is an eigenstate of the fermionic Hamiltonian $H_N^+$ with the same energy $E_n^-$;

$$H_N^+ \Phi_n^+ = E_n^- \Phi_n^+.$$  \hfill (3.32)

Furthermore, this state is also the eigenstate of the Mother Hamiltonian with the same $\mathcal{E}_n$;

$$\mathcal{H}_N \left( \begin{array}{c} \Phi_n^+ \\ 0 \end{array} \right) = \mathcal{E}_n \left( \begin{array}{c} \Phi_n^+ \\ 0 \end{array} \right),$$  \hfill (3.33)

since $\mathcal{H}_N$ commutes with $Q_N^\dagger$ and

$$\left( \begin{array}{c} \Phi_n^+ \\ 0 \end{array} \right) = \frac{Q_N^\dagger}{\sqrt{\mathcal{E}_n}} \left( \begin{array}{c} 0 \\ \Phi_n^- \end{array} \right).$$  \hfill (3.34)

In a similar manner, bosonic states can be constructed from fermionic ones at each energy levels unless $\mathcal{E}_n = 0$.

For states with $\mathcal{E}_n = 0$, the eigenvalues of $H_N^+$ are determined algebraically. Bosonic states $\Phi_n^-$ with $\mathcal{E}_n = 0$ satisfy $P_N \Phi_n^- = 0$. Thus using Eq. (3.11) (or more directly Eq. (3.15)), we obtain

$$\det M_N^-(E_n^-) = 0,$$  \hfill (3.35)

where $E_n^-$ is the eigenvalue of $H_N^-$. For fermionic states with $\mathcal{E}_n = 0$, we obtain the following algebraic equation in the same way,

$$\det M_N^+(E_n^+) = 0,$$  \hfill (3.36)

where $E_n^+$ is the eigenvalue of $H_N^+$.}

### 3.3 Polynomial supersymmetry

A system is defined to have $\mathcal{N}$-th order polynomial supersymmetry [17] if the system is $\mathcal{N}$-fold supersymmetric and its Mother Hamiltonian is given by an $\mathcal{N}$-th order polynomial of Hamiltonian $H_N$. Here we show that if $H_N^- \neq H_N^+$, any $\mathcal{N}$-fold supersymmetric system have $\mathcal{M}$-th order polynomial supersymmetry with $\mathcal{M} \leq \mathcal{N}$.

First consider the case that the $\mathcal{N}$-fold supersymmetric system has a unique $\mathcal{N}$-fold supercharge for $H_N^+$. In this case, the Mother Hamiltonian is given by Eq. (3.13). Now we consider the following state

$$\left( \begin{array}{c} 0 \\ \Phi_n^- \end{array} \right),$$  \hfill (3.37)
where $\Phi^-$ satisfies
\[
H_{\bar{N}} \Phi^- = E \Phi^-.
\] (3.38)

Here we do not require normalizability of $\Phi^-$ so $E$ may be an arbitrary constant. This state is also an eigenstate of the Mother Hamiltonian and the eigenvalue $E$ becomes
\[
E = \frac{1}{2} \det M_{\bar{N}}(E).
\] (3.39)

Now we construct the following fermionic state
\[
\left( \begin{array}{c}
P_N \Phi^- \\ 0 \end{array} \right),
\] (3.40)
and apply the Mother Hamiltonian to this. Using
\[
H_{\bar{N}} P_N \Phi^- = P_N H_{\bar{N}} \Phi^- = E P_N \Phi^-,
\] (3.41)
we obtain
\[
\frac{1}{2} \det M_{\bar{N}}^+(E) = E.
\] (3.42)

Eliminating $E$ in the above equations, we obtain
\[
\det M_{\bar{N}}^-(E) = \det M_{\bar{N}}^+(E)
\] (3.43)
for any $E$. Therefore, Eq.(3.13) becomes
\[
\mathcal{H}_{\bar{N}} = \frac{1}{2} \det M_{\bar{N}}(H_{\bar{N}}) = \frac{1}{2} \det M_{\bar{N}}(H_{\bar{N}}).
\] (3.44)

Thus the system is $\bar{N}$-th order polynomial supersymmetric.

Next consider the case that the $\bar{N}$-fold supercharge is not uniquely determined for given $H_{\bar{N}}$. As is shown in section 3.1, the system has $\bar{N}_1$-fold supersymmetry with $\bar{N}_1 < \bar{N}$ in this case. If this $\bar{N}_1$-fold supercharge is uniquely determined for given $H_{\bar{N}}$, we can show in a manner similar to the above that the system has $\bar{N}_1$-th polynomial supersymmetry. If this $\bar{N}_1$-fold supercharge is not uniquely determined, we again obtain an $\bar{N}_2$-fold supercharge with $\bar{N}_2 < \bar{N}_1 < \bar{N}$. If this $\bar{N}_2$-fold supercharge is uniquely determined, the system has $\bar{N}_2$-th order polynomial supersymmetry. We continue this procedure until the obtained supercharge is uniquely determined or it becomes 0-th fold one. If the former is realized, the system is proved to have $\bar{N}_i$-th order polynomial supersymmetry with $\bar{N}_i < \bar{N}$. If the latter is realized, there exist a function $\tilde{w}_0(q)$ which satisfies
\[
\tilde{w}_0(q) H_{\bar{N}}^- = H_{\bar{N}}^+ \tilde{w}_0(q)
\] (3.45)
Comparing the first derivative terms in this equation, we find that $\tilde{w}_0$ does not depend on $q$. Thus Eq.(3.45) indicates that $H_{\bar{N}}^- = H_{\bar{N}}^+$. This contradicts the assumption and shows that the latter case is not realized.
3.4 Generalized Witten index

The Witten index of ordinary supersymmetry can be generalized to $\mathcal{N}$-fold supersymmetric systems. For polynomial supersymmetry, the generalization was first discussed in Ref. [15]. When the energy of the Mother Hamiltonian is not zero (namely $E_n \neq 0$), the bosonic and fermionic states form pairs. Thus only states with $E_n = 0$ contribute to the Witten index $\text{tr}(-1)^F$:

$$\text{tr}(-1)^F = \dim \text{Kernel } Q_{\mathcal{N}} - \dim \text{Cokernel } Q_{\mathcal{N}}. \quad (3.46)$$

The index takes integer values since the number of states with zero energy of the Mother Hamiltonian is finite ($2^N$ at most). The expression (3.46) shows that if this index is not zero, at least one $\mathcal{N}$-fold supersymmetric state exists.

3.5 Non-renormalization theorems

Non-renormalization theorems are characteristic features of supersymmetric systems. The corresponding non-renormalization theorems also hold in $\mathcal{N}$-fold supersymmetric systems. For example, non-renormalization theorems hold for the generalized Witten index. Because this index takes integer values, it is also an adiabatic invariant as well as the ordinary one and does not suffer from quantum corrections. Furthermore, by an argument analogous to ordinary supersymmetry [14], we can show that perturbation theory does not break $\mathcal{N}$-fold supersymmetry spontaneously.

There exist other kinds of non-renormalization theorems in the $\mathcal{N}$-fold supersymmetric systems. For states with $E_n \neq 0$, the bosonic spectra and the fermionic ones are the same, thus the perturbative corrections for them are also the same. This property enables us to prove the non-renormalization of the energy splittings for the $\mathcal{N}$-th and higher excited states of an asymmetric double-well potential [3]. Furthermore, it was shown that in asymmetric double well potentials [3] and periodic potentials [24], the perturbation series of the energies for states with $E_n = 0$ are convergent. This is because that $\mathcal{N}$-fold supersymmetry of these models cannot be broken by any perturbative corrections. The latter example of the non-renormalization theorem can be generalized to a class of $\mathcal{N}$-fold supersymmetric systems, which will be explained in section 4.2.

3.6 Quasi-solvability and $\mathcal{N}$-fold supersymmetry

In closing this section, we note a close relationship between quasi-solvability and $\mathcal{N}$-fold supersymmetry. For a finite order differential operator $P$, let us consider a function $\phi$ which satisfies $P\phi = 0$. A system with a Hamiltonian $H$ is defined to be “quasi-solvable” if $PH\phi = 0$ holds for any such $\phi$s. Namely, if a system is quasi-solvable, the space $\mathcal{V}$ defined by $\mathcal{V} = \{\phi | P\phi = 0\}$ is closed by the action of $H$. If we introduce the basis $\phi_n$ of $\mathcal{V}$, we obtain

$$H\phi_n = \sum_{n=1}^{\dim \mathcal{V}} S_{n,m} \phi_m \quad (3.47)$$
This means that a part of the spectra of the quasi-solvable system can be solved by the characteristic equations for $S$ which is finite dimensional.

For example, $N$-fold supersymmetric systems are quasi-solvable. The projective operators for $H_N^-$ and $H_N^+$ are $P_N$ and $P_N^\dagger$ respectively. A part of the spectra of the systems is solved by the following algebraic equations,

\[
\det M_N^-(E_n^-) = 0, \quad \det M_N^+(E_n^+) = 0,
\]

where $E_n^-$ and $E_n^+$ are eigenvalues of $H_N^-$ and $H_N^+$ respectively.

This quasi-solvability for $N$-fold supersymmetric systems comes from the $N$-fold supersymmetric algebra, but the converse is also true: If a system is quasi-solvable and $P$ is an $N$-th order differential operator, it also becomes $N$-fold supersymmetric.

Proof:

We assume that the projective operator $P$ and the Hamiltonian $H$ have the following form,

\[
P = p^N + c_{N-1}(q)p^{N-1} + \cdots + c_1(q)p + c_0(q),
\]

\[
H = \frac{1}{2}p^2 + V(q).
\]

For this $P$ and $H$, we introduce another Hamiltonian $K$ as follows,

\[
K = \frac{1}{2}p^2 + U(q), \quad U(q) = V(q) + ic'_{N-1}(q).
\]

If we introduce the operator $G(p,q) \equiv PH - KP$, it contains $N - 1$ derivatives with respect to $q$ at most and $G(p,q)\phi = 0$ for any $\phi$ which satisfy $P\phi = 0$. But as operators which contain $N - 1$ derivatives at most cannot annihilate $N$ independent functions non-trivially, this means that $G(p,q) \equiv 0$. Therefore, if we identify $P_N = P$, $H_N^- = H$ and $H_N^+ = K$, we obtain an $N$-fold supersymmetric system.

Q.E.D.

Note that all the eigenvalues of $S$ in Eq.(3.47) are not necessarily physical ones. This is because the quasi-solvability does not require that $\mathcal{V}$ is a quantum physical space, that is, $L^2$. When $\mathcal{V}$ is $L^2$, the system is often called “quasi-exactly solvable” [31, 32, 33]. In this case, all the eigenvalues are physical. Even when the elements of $\mathcal{V}$ are not normalizable, if they become normalizable in the all order of the perturbation theory, the eigenvalues of $S$ are exact in the perturbation theory. We dub this “quasi-perturbatively solvable”.

Among the known $N$-fold symmetric models, the quartic model found in [3] is quasi-perturbatively solvable, while the periodic one in [24] and the sextic one in [23] are quasi-exactly solvable, the exponential one in [23, 34] can be either of those, depending on a parameter. In the perturbation theory, all the models have normalizable eigenstates of Eq.(3.48) which are $N$-fold supersymmetric. In the quasi-exactly solvable models, they remain normalizable even if non-perturbative effects are taken into account. Thus the physical states in this type of models contain $N$-fold supersymmetric ones. But, in the quartic model, these states are no longer normalizable if non-perturbative effects are taken into account. Thus the physical states in the latter model do not contain $N$-fold supersymmetric ones.
Special cases of the correspondence between quasi-solvability and $N$-fold supersymmetry were previously reported; for the quartic potential in Ref. [3], for the periodic potential in Ref. [24], for the exponential potentials in Ref. [34], for the sextic potential in Ref. [35]. In this subsection, we have proved that the correspondence is general and does not rely on any specific models.

4 Examples

4.1 2-fold supersymmetry

The first example of $N$-fold supersymmetric models is the 2-fold supersymmetric one. Under the assumption that the Mother Hamiltonian $H_2$ becomes a polynomial of $H_2^\pm$, the 2-fold supersymmetric model was first constructed in Ref. [16]. Here we do not assume this.

In general, the 2-fold supercharges are given by

$$P_2 = p^2 + w_1(q)p + w_0(q). \quad (4.1)$$

To be 2-fold supersymmetric, the following relation must hold,

$$P_2 H_2^- - H_2^+ P_2 = 0. \quad (4.2)$$

Since the left hand side of the above is given by

$$2(P_2 H_2^- - H_2^+ P_2) \quad (4.3)$$

the following three equations have to be satisfied,

$$V_2^+ - V_2^- = iw_1', \quad (4.4)$$
$$iw_1'' - 2iw_1(V_2^+ - V_2^-) - 2w_0' + 4V_2^{-'} = 0, \quad (4.5)$$
$$w_0'' - 2w_0(V_2^+ - V_2^-) - 2iw_1 V_2^{-'} - 2V_2^{-''} = 0. \quad (4.6)$$

Eliminating $V_2^-$ and $V_2^+$ from these equations, we obtain

$$- iw_1 w_0' - 2iw_1 w_0 + \frac{1}{2} \left( iw_1''' w_1 + w_1'' w_1^2 + 2w_1' w_1^2 \right) = 0. \quad (4.7)$$

This equation is easily solved if we introduce the following function $\Omega(q)$,

$$\Omega = -iw_0 w_1^2. \quad (4.8)$$

From Eq. (4.7), the function $\Omega(q)$ satisfies

$$\Omega' = -\frac{1}{2} \left( iw_1''' w_1 + w_1'' w_1^2 + 2w_1' w_1^2 + 2iw_1' w_1^3 \right), \quad (4.9)$$
4.1 2-fold supersymmetry

thus

$$\Omega = \frac{1}{2} \left( i w_1'' w_1 - \frac{1}{2} i w_1' w_1' + w_1^2 + \frac{1}{2} i w_1^4 + C \right),$$  \hspace{1cm} (4.10)

where $C$ is an arbitrary constant. So if $w_1(q) \neq 0$, $w_0(q)$ is given as follows,

$$w_0(q) = \frac{1}{4} w_1(q)^2 + \frac{1}{2} \left( \frac{w_1''(q)}{w_1(q)} - \frac{w_1'(q)^2}{2w_1(q)^2} - \frac{i C}{w_1(q)^2} \right) - \frac{1}{2} i w_1'(q).$$  \hspace{1cm} (4.11)

The potentials $V_2^\pm$ can be also written in terms of $w_1$. Eliminating $V_2^+$ from Eqs.(4.4) and (4.5), we obtain

$$\left( i w_1' + w_1^2 - 2w_0 + 4V_2^- \right)' = 0.$$  \hspace{1cm} (4.12)

So if we omit irrelevant integral constants, $V_2^-(q)$ is given by

$$V_2^- = -\frac{1}{4} \left( i w_1' + w_1^2 - 2w_0 \right) \equiv -\frac{1}{8} w_1(q)^2 + \frac{1}{4} \left( \frac{w_1''(q)}{w_1(q)} - \frac{w_1'(q)^2}{2w_1(q)^2} - \frac{i C}{w_1(q)^2} \right) - \frac{1}{2} i w_1'(q).$$  \hspace{1cm} (4.13)

The remaining potential $V_2^+$ is obtained by Eq.(4.4).

When $w_1(q) \equiv 0$, the above solution is not valid. In this case, however, Eqs.(4.4)–(4.6) reduce to the following simple equations,

$$V_2^+ - V_2^- = 0, \quad -2w_0' + 4V_2^- = 0, \quad w_0'' - 2V_2^- = 0,$$  \hspace{1cm} (4.14)

thus the solution is easily obtained as

$$V_2^+ = V_2^- = \frac{1}{2} w_0.$$  \hspace{1cm} (4.15)

This solution is trivial and useless, since the supercharge $P_2$ coincides with the Hamiltonians $H_2^\pm$.

In summary, the non-trivial 2-fold supersymmetric system is generally given as follows:

$$P_2 = -\partial^2 - i w_1(q)\partial + w_0(q),$$  \hspace{1cm} (4.16)

$$w_0(q) = \frac{1}{4} w_1(q)^2 + \frac{1}{2} \left( \frac{w_1''(q)}{w_1(q)} - \frac{w_1'(q)^2}{2w_1(q)^2} - \frac{i C}{w_1(q)^2} \right) - \frac{1}{2} i w_1'(q),$$  \hspace{1cm} (4.17)

$$V_2^\pm = -\frac{1}{8} w_1(q)^2 + \frac{1}{4} \left( \frac{w_1''(q)}{w_1(q)} - \frac{w_1'(q)^2}{2w_1(q)^2} - \frac{i C}{w_1(q)^2} \right) \pm \frac{1}{2} i w_1'(q).$$  \hspace{1cm} (4.18)

For given $V_2^-$ and $V_2^+$, the above 2-fold supercharges are determined uniquely unless

$$w_1'' - 2i w_1 w_1' - 2i V_2^- \propto w_1'$$  \hspace{1cm} (4.19)
Aoyama, Sato and Tanaka, \(\mathcal{N}\)-fold Supersymmetry

holds. To see this, we introduce another 2-fold supercharge which are given by substitution of the following \(\hat{P}_2\) for \(P_2\):

\[
\hat{P}_2 = -\partial^2 - i\hat{w}_1(q)\partial + \hat{w}_0(q).
\]

(4.20)

\(\hat{P}_2\) also satisfies \(\hat{P}_2 H^-_2 - H_2^+ \hat{P}_2 = 0\). If we define \(\Delta w_i = w_i - \hat{w}_i (i = 0, 1)\), they satisfy

\[
i\Delta w_1' = 0,
\]

(4.21)

\[
i\Delta w_1'' - 2i\Delta w_1(V_2^+ - V_2^-) - 2\Delta w_0' = 0,
\]

(4.22)

\[
\Delta w_0'' - 2\Delta w_0(V_2^+ - V_2^-) - 2i\Delta w_1 V_2'^{-} = 0.
\]

(4.23)

From the first equation (4.21), \(\Delta w_1\) is determined as

\[
\Delta w_1 = C_1,
\]

(4.24)

where \(C_1\) is a constant. Substituting this for Eq.(4.22), we obtain

\[
2C_1 w_1' - 2\Delta w_0' = 0,
\]

(4.25)

so \(\Delta w_0\) becomes

\[
\Delta w_0 = C_1 w_1 + C_2,
\]

(4.26)

where \(C_2\) is a constant. Thus Eq.(4.23) becomes

\[
C_1(w_1'' - 2iw_1 w_1' - 2iV_2^{-}') = 2iC_2 w_1'.
\]

(4.27)

Unless

\[
w_1'' - 2iw_1 w_1' - 2iV_2^{-}' \propto w_1',
\]

(4.28)

only solution of this equation is \(C_1 = C_2 = 0\), and this means that \(\hat{w}_0 = w_0\) and \(\hat{w}_1 = w_1\).

4.2 Type A \(\mathcal{N}\)-fold supersymmetry

For the second example of \(\mathcal{N}\)-fold supersymmetry, we consider a particular class of \(\mathcal{N}\)-fold supercharges which we call type A. The form of the type A \(\mathcal{N}\)-fold supercharges \(P_N\) is defined as follows:

\[
P_N= (D + i(N - 1)E(q)) (D + i(N - 2)E(q)) \cdots (D + iE(q)) D
\]

\[
\equiv \prod_{k=0}^{N-1} (D + i k E(q)),
\]

(4.29)

where \(D = p - iW(q)\). The \(\mathcal{N}\)-fold supersymmetric models considered in Refs.[3, 24] are in this class. A type A model was also considered in Refs.[35, 38].
For this class of $\mathcal{N}$-fold supercharges, a system is $\mathcal{N}$-fold supersymmetric when the following conditions are satisfied:

\[ V_\mathcal{N}^\pm = \frac{1}{2}(W^2 + v_\mathcal{N}^\pm), \]

\[ v_\mathcal{N}^\pm = - (\mathcal{N} - 1)E(q)W(q) + \frac{(\mathcal{N} - 1)(2\mathcal{N} - 1)}{6}E(q)^2 \]

\[ - \frac{\mathcal{N}^2 - 1}{6}E'(q) \pm \mathcal{N}\left(W'(q) - \frac{\mathcal{N} - 1}{2}E'(q)\right). \]  

\[ W(q) = \frac{1}{2}E(q) + Ce^{-\int^{q} dq_1E(q_1)} \int^{q} dq_2 \left( e^{\int^{q_2} dq_3E(q_3)} \int^{q_2} dq_3 e^{\int^{q_4} dq_5E(q_5)} \right) (\mathcal{N} \geq 2), \]  

\[ E'''(q) + E(q)E'''(q) + 2E'(q)^2 - 2E(q)^2E'(q) = 0 \quad (\mathcal{N} \geq 3), \]

where $C$ is an arbitrary constant.

**Proof:**

We prove the above conditions (4.30)–(4.32) inductively. For $\mathcal{N} = 1$, Eqs. (4.30)–(4.32) reduce to

\[ V_1^\pm = \frac{1}{2}(W^2 \pm W'), \]

which is the ordinary supersymmetric case. Thus the system is $\mathcal{N}$-fold supersymmetric in this case. Next, we suppose that the conditions (4.30)–(4.32) hold for an integer $\mathcal{N}$. This assumption implies that the $\mathcal{N}$-fold superalgebra $P_\mathcal{N}H_\mathcal{N}^- = H_\mathcal{N}^+P_\mathcal{N}$ holds in this case. Then, if we put

\[ H_{\mathcal{N}+1}^+ = H_{\mathcal{N}}^+ + h_{\mathcal{N}}^+, \quad H_{\mathcal{N}+1}^- = H_{\mathcal{N}}^- + h_{\mathcal{N}}^-, \]

and use the relation $P_\mathcal{N}H_\mathcal{N}^- = H_\mathcal{N}^+P_\mathcal{N}$, we obtain

\[ P_{\mathcal{N}+1}H_{\mathcal{N}+1}^+ - H_{\mathcal{N}+1}^+P_{\mathcal{N}+1} = [D + i\mathcal{N}E, H_{\mathcal{N}}^+]P_{\mathcal{N}} - h_{\mathcal{N}}^+P_{\mathcal{N}+1} + P_{\mathcal{N}+1}h_{\mathcal{N}}^-. \]  

To facilitate the following calculation, we introduce $U$ as follows

\[ U(q) = e^{\int^{q} dq'W(q')} \]

Then the Hamiltonian $H_{\mathcal{N}}^+$ and the supercharge $P_\mathcal{N}$ are rewritten as

\[ UH_{\mathcal{N}}^+U^{-1} = \frac{1}{2}(-\partial^2 + 2W\partial + W' + v_{\mathcal{N}}^+), \]

\[ UP_\mathcal{N}U^{-1} = (-i)^{\mathcal{N}}(\partial - (\mathcal{N} - 1)E(q))(\partial - (\mathcal{N} - 2)E(q)) \cdots (\partial - E(q)) \partial \]

\[ \equiv (-i)^{\mathcal{N}} \prod_{k=0}^{\mathcal{N}-1} (\partial - kE(q)) \equiv (-i)^{\mathcal{N}} \tilde{P}_{\mathcal{N}}. \]
Now Eq. (4.35) is calculated as
\[
I_{N+1} = 2\hat{v}_{N+1}^\pm U \left( P_{N+1} H_{N+1} - H_{N+1}^+ P_{N+1} \right) U^{-1}
\]
\[
= [\partial - N E, -\partial^2 + 2W \partial + W', \hat{P}_N - 2h_N^+ \hat{P}_N + 2\hat{P}_{N+1} h_N^-]
\]
\[
= 2 \left( W' - N E' - h_N^+ + h_N^- \right) \partial \hat{P}_N
\]
\[
+ \left( \hat{v}_N^+ + W'' - N E'' + 2N E'W + 2\hat{P}_{N+1} h_N^- \right) \hat{N} \hat{P}_N + 2[\hat{P}_{N+1}, h_N^-].
\]

(4.38)

From Eq. (4.38), we see that \( I_{N+1} \) contains up to \( (N + 1) \)-th derivatives. Therefore, \( I_{N+1} = 0 \) if and only if all the coefficients of \( \partial^k \) \( (k = 0, 1, \cdots, N + 1) \) vanish. The \( \partial^{N+1} \) term comes only from the first term of the right hand side of Eq. (4.38) and thus
\[
h_N^+ - h_N^- = W' - N E'.
\]

(4.39)

When this condition (4.39) is satisfied, \( I_{N+1} \) now reads
\[
I_{N+1} = \left( \hat{v}_N^+ + W'' - N E'' + 2N E'W + 2N E^2W - 2N^2 E E' \right) \hat{P}_N + 2[\hat{P}_{N+1}, h_N^-]
\]
\[
= \left( \hat{v}_N^+ + W'' - N E'' + 2N E'W + 2N E^2W - 2N^2 E E' \right) \hat{P}_N
\]
\[
+ 2h_N^- \hat{P}_N + 2 (\partial - N E) [\hat{P}_N, h_N^-]
\]
\[
= \left( \hat{v}_N^+ + W'' - N E'' + 2N E'W + 2N E^2W - 2N^2 E E' \right) \partial \hat{N}
\]
\[
+ 2h_N^- \partial^N + 2N h_N^- \partial^N + O(\partial^{N-1}).
\]

(4.40)

To eliminate the \( \partial^N \) term, the following condition have to be satisfied,
\[
2(N + 1) h_N^- = - \left( \hat{v}_N^+ + W'' - N E'' + 2N E'W + 2N E^2W - 2N^2 E E' \right).
\]

(4.41)

From this equation, we obtain
\[
h_N^- = -\frac{1}{2(N + 1)} \left( \hat{v}_N^+ + W' - N E' + 2N E W - N^2 E^2 \right)
\]
\[
= \frac{1}{2} \left[ -E W + \frac{4N - 1}{6} E^2 - \frac{2N + 1}{6} E' - (W' - N E') \right]
\]

(4.42)

Here we omit an irrelevant constant which only affects the origin of the energy. Combining Eqs. (4.30), (4.33) and this, we finally find
\[
\hat{v}_{N+1}^\pm = \hat{v}_N^\pm + 2h_N^\pm
\]
\[
= -N E W + \frac{N(2N + 1)}{6} E^2 - \frac{N(N + 2)}{6} E' + (N + 1) \left( W' - \frac{N}{2} E' \right),
\]

(4.43)

which are nothing but the assumed form of \( \hat{v}_N^\pm \) and \( \hat{v}_N^- \) with \( N \) replaced with \( N + 1 \). When we use the condition (4.42), from the second line of the right hand side of Eq. (4.40), \( I_{N+1} \) becomes
\[
I_{N+1} = -2N h_N^- \hat{P}_N + 2 (\partial - N E) [\hat{P}_N, h_N^-]
\]
\[
= N(N + 1) \left( h_N^- W - E h_N^- \right) \partial^{N-1} + O(\partial^{N-2}).
\]

(4.44)
Thus we obtain
\[ h_{\mathcal{N}}'' - Eh_{\mathcal{N}}' = 0. \] (4.45)

From Eq. (4.42), this equation becomes
\[
\left[ (W - \frac{1}{2}E)' + E \left( W - \frac{1}{2}E \right) \right]' - E \left[ (W - \frac{1}{2}E)' + E \left( W - \frac{1}{2}E \right) \right]' - \frac{2(N' - 1)}{3} \left[ E'(E^2)'' - E(E'^2)'' \right] = 0. \] (4.46)

This equation leads to Eqs. (4.31) and (4.32). Once Eq. (4.45) holds, we can prove \( I_{\mathcal{N}+1} = 0 \), by using the following relation,
\[
\left[ \tilde{P}_{\mathcal{N}}, h_{\mathcal{N}} \right] = Mh_{\mathcal{N}}' \tilde{P}_{\mathcal{N} - 1} + \left[ \prod_{k=\mathcal{M}}^{N'-1} \left( \partial - kE \right), h_{\mathcal{N}} \right] \tilde{P}_{\mathcal{M}} \quad (0 \leq \mathcal{M} \leq \mathcal{N}), \] (4.47)

where \( \tilde{P}_0 \) and \( \prod_{k=\mathcal{N}}^{N'-1} \left( \partial - kE \right) \) should be understood as \( \tilde{P}_0 = 1 \) and \( \prod_{k=\mathcal{N}}^{N'-1} \left( \partial - kE \right) = 0 \). This relation is easily obtained by the next relation,
\[ (\partial - kE) h_{\mathcal{N}}' = h_{\mathcal{N}}' (\partial - (k - 1)E). \] (4.48)

Applying the relation (4.47) with \( \mathcal{M} = \mathcal{N} \) to the first line of the right hand side of Eq. (4.44), we immediately find that \( I_{\mathcal{N}+1} = 0 \). Q.E.D.

For illustration, we give here two examples of the solutions of Eqs. (4.31) and (4.32). The first one is
\[
W(q) = \sin(q) + \frac{N - 1}{2}i, \quad (4.49)
\]
\[ E(q) = i. \quad (4.50) \]

The potentials of this system are \[ 24 \]
\[ V_{\mathcal{N}}^\pm = \frac{1}{2} \sin^2(q) \pm \frac{N}{2} \cos(q), \] (4.51)
where we have omitted an irrelevant constant. This system is periodic and may be defined on a finite region, \( q \in (0, 2\pi] \). The second one is
\[
W(q) = C_1 q^3 + C_2 q + \frac{2C_3 - N + 1}{2q}, \quad (4.52)
\]
\[ E(q) = \frac{1}{q}, \quad (4.53) \]
\[ V_{\mathcal{N}}^\pm = \frac{1}{2} w(q)^2 + \frac{(2C_3 + N - 1)(2C_3 + N + 1)}{8q^2} \pm \left( \frac{N}{2} \pm \frac{C_3}{3} \right) w'(q) + \frac{2}{3} C_2 C_3, \] (4.54)
where \( C_i \ (i = 1, 2, 3) \) are arbitrary constants and \( w(q) = C_1 q^3 + C_2 q \). This is a sextic anharmonic oscillator with centrifugal like potential and thus may be naturally defined on the positive axis with proper boundary conditions at \( q = 0 \) and \( \infty \).

For type A \( \mathcal{N} \)-fold supersymmetric models, the matrices \( S^\pm \) defined in section 3.1 can be given explicitly. To see this, we define the following functions \( \phi_n^- \),

\[
\phi_n^- = (h)^{n-1} U^{-1} \quad (n = 1, \ldots, \mathcal{N}),
\]

where \( h \) is a function which satisfy

\[
h^{''} - E h' = 0. \tag{4.56}
\]

Integrating Eq.(4.56), we obtain

\[
h(q) = c_1 \int_0^q dq_1 e^{\int_0^{q_1} dq_2 E(q_2)} + c_2, \tag{4.57}
\]

where \( c_1 \) and \( c_2 \) are arbitrary constants. Because of Eq.(4.56), the functions \( \phi_n^- \) satisfy

\[
P_M \phi_n^- = h P_M \phi_{n-1}^- + \mathcal{M}(-i) h' P_{M-1} \phi_{n-1}^- \tag{4.58}
\]

where \( \phi_0^- \) and \( P_0 \) should be understood as \( \phi_0^- = 0 \) and \( P_0 = 1 \). From this, we obtain

\[
P_M \phi_n^- = 0 \quad (\mathcal{M} \geq n), \tag{4.59}
\]

thus all the \( \phi_n^- \)'s satisfy \( P_\mathcal{N} \phi_n^- = 0 \). And if \( h' \neq 0 \), the functions \( \phi_n^- \) are linearly independent from each other since the next equation holds,

\[
P_M \phi_{M+1}^- = \mathcal{M}!(-i)^\mathcal{M} h^{\mathcal{M}} U^{-1}. \tag{4.60}
\]

When \( h' \equiv 0 \), the independence is broken, thus we choose \( h \) as it satisfies \( h' \neq 0 \) in the following.

Using \( \phi_n^- \), we define \( S^- \) by

\[
H_n^- \phi_n^- = \sum_m S^-_{n,m} \phi_m^- \tag{4.61}
\]

Applying \( P_{\mathcal{N}-1} \) to the both sides of the above equation and using Eq.(4.59), we obtain the following equation,

\[
P_{\mathcal{N}-1} H_n^- \phi_n^- = S^-_{n,N} P_{\mathcal{N}-1} \phi_n^-, \tag{4.62}
\]

thus \( S^-_{n,N} \) is determined as

\[
S^-_{n,N} = \frac{P_{\mathcal{N}-1} H_n^- \phi_n^-}{P_{\mathcal{N}-1} \phi_n^-}. \tag{4.63}
\]

The other elements of \( S^- \) are determined inductively. They are given as follows,

\[
S^-_{n,N-m} = \frac{P_{\mathcal{N}-m-1} \left( H_n^- \phi_n^- - \sum_{k=N-m+1}^{N} S^{-}_{n,k} \phi_k^- \right)}{P_{\mathcal{N}-m-1} \phi_n^-}. \tag{4.64}
\]
where \( n = 1, \cdots, \mathcal{N} \) and \( m = 1, \cdots, \mathcal{N} - 1 \).

The matrix \( S^+_{n,m} \) are given in a similar manner as \( S^-_{n,m} \). First note that \( P_N \) and its hermitian conjugate \( P_N^\dagger \) are related by

\[
P_N^\dagger = U^2 V P_N V^{-1} U^{-2},
\]

where \( V \) is defined by

\[
V(q) = e^{-\int_0^q dq' (N-1) E(q')},
\]

(4.66)

Thus instead of \( \phi^-_n \), we introduce the following functions \( \phi^+_n \),

\[
\phi^+_n = U^2 V \phi^-_n = (h)^{-1} V U, \quad (n = 1, \cdots, \mathcal{N}),
\]

(4.67)

and define the matrices \( S^+_{n,m} \) as follows,

\[
H^+_{N} \phi^+_n = \sum_m S^+_{n,m} \phi^+_m.
\]

(4.68)

\( S^+_{n,m} \) is determined inductively as follows,

\[
S^+_{n,N} = \frac{P^\dagger_{N-1} H^+_{N} \phi^+_n}{P^\dagger_{N-1} \phi^+_n},
\]

(4.69)

\[
S^+_{n,N-m} = \frac{P^\dagger_{N-m-1} \left( H^+_{N} \phi^+_n - \sum_{k=N-m+1}^N S^+_{n,k} \phi^+_k \right)}{P^\dagger_{N-m-1} \phi^+_n},
\]

(4.70)

where \( n = 1, \cdots, \mathcal{N} \) and \( m = 1, \cdots, \mathcal{N} - 1 \).

A kind of non-renormalization theorem found in Refs.\[3, 24\] can be generalized to all the type A models which have \( q_0 \) such as \( W(q_0) = 0 \). By redefinition of the origin of the coordinate \( q \), we first set \( q_0 = 0 \). Then we introduce a coupling constant \( g \) as follows,

\[
W(q) = \frac{1}{g} w(gq), \quad E(q) = g e(gq).
\]

(4.71)

In the leading order of \( g \), the potentials \( V^\pm_N \) become harmonic ones with frequency \( |w'(0)| \),

\[
V^\pm_N = \frac{1}{2} w'(0)^2 q^2 + O(g).
\]

(4.72)

The following non-renormalization theorem holds for the first \( \mathcal{N} \) excited states of either of these harmonic potentials \( V^\pm_N \): If \( w'(0) > 0 \), perturbative corrections for the first \( \mathcal{N} \) excited states of \( V^-_N \) have a finite convergence radius in \( g^2 \), and if \( w'(0) < 0 \), those of \( V^+_N \) have a finite convergence radius in \( g^2 \). It is well-known that perturbative expansions of usual quantum mechanics become divergent series \[36\], thus this behavior means that all the possible divergent parts of the perturbative corrections vanish in type A models.
To prove the non-renormalization theorem, we adjust $c_1 = 1$ and $c_2 = 0$ in Eq.(4.57) and introduce $\eta(gq)$ as follows,

$$h(q) = \frac{1}{g} \eta(gq),$$

where

$$\eta(gq) = \int_0^{gq} dx_1 e^{\int_0^{x_1} dx_2 e(x_2)}.$$  \hspace{1cm} (4.73)

Then we consider the characteristic equations of $S^\pm$. (See, Eqs.(3.35) and (3.36)). Since $\phi^-_n$ and $\phi^+_n$ behave as

$$\phi^-_n(q) = U^{-1}(0)(q^{n-1} + O(g))e^{-w'(0)q^2/2},$$

$$\phi^+_n(q) = U(0)V(0)(q^{n-1} + O(g))e^{w'(0)q^2/2},$$

either of the eigenstates of $S^-$ or $S^+$ are normalizable, at least in the perturbation theory. Thus if $w'(0) > 0$, the eigenvalues of $S^-$ give exact spectra of $H_N^-$ in the perturbation theory, and if $w'(0) < 0$, the eigenvalues of $S^+$ give those of $H_N^+$ in the perturbation theory.\footnote{If $\phi^+_n$ or $\phi^-_n$ are normalizable without expanding by $g$, the spectra are really exact.}

Equation (4.75) shows also that either linear combinations of $\phi^+_n$ or $\phi^-_n$ give the first $N$ eigenstates of the harmonic potentials (4.72). We especially notice here that if $w'(0) > 0$, the first $N$ eigenstates of $V_N^-$ can be given by suitable linear combinations of $\phi^-_n$, and if $w'(0) < 0$, the first $N$ eigenstates of $V_N^+$ can be given by suitable linear combinations of $\phi^+_n$. Thus if $w'(0) > 0$, all order of the perturbative series for the first $N$ excited energies of $V_N^-$ are given by the eigenvalues of $S^-$, and if $w'(0) < 0$, those of $V_N^+$ are given by $S^+$. In appendix A, we will show that the characteristic equations of $S^\pm$ are polynomials of $g^2$. Therefore, the eigenvalues of $S^\pm$ have a finite convergence radius in $g^2$. Thus the theorem is proved.

As far as we know, the Mother Hamiltonians of all known type A models are polynomials of the original Hamiltonian, and the following relation holds,

$$\mathcal{H}_\mathcal{N} = \frac{1}{2} \det M^-_\mathcal{N}(H_\mathcal{N}) = \frac{1}{2} \det M^+_\mathcal{N}(H_\mathcal{N})$$  \hspace{1cm} (4.76)

where $M^\pm_\mathcal{N}$ are given by Eq.(3.10). When $\mathcal{N} = 2$, we can prove this generally. We conjecture that this holds for arbitrary $\mathcal{N}$ in the type A models.

5 $\mathcal{N}$-fold supersymmetry in multi-dimensional quantum mechanics

Finally, we will give a possible extension of $\mathcal{N}$-fold supersymmetry in multi-dimensional quantum mechanics. We denote the bosonic coordinates as $q_i$ ($i = 1, \ldots, n_b$) and the fermionic ones as $\psi_i$ ($i = 1, \ldots, n_f$). The fermionic coordinates satisfy

$$\{\psi_i, \psi_j\} = \{\psi^+_i, \psi^+_j\} = 0, \quad \{\psi_i, \psi^+_j\} = \delta_{i,j}. \hspace{1cm} (5.1)$$
The Hamiltonian of the $\mathcal{N}$-fold supersymmetric system is defined by
\[
\mathbf{H}_N = \sum_{i,j} H_N^{-(i,j)} \psi_i^\dagger \psi_j^\dagger + \sum_{i,j} H_N^{+(i,j)} \psi_i^\dagger \psi_j, \tag{5.2}
\]
where
\[
H_N^{-(i,j)} = \frac{1}{2} \sum_{k,l} G^{-(i,j)}_{N,k,l} p_k p_l + V_N^{-(i,j)},
\]
\[
H_N^{+(i,j)} = \frac{1}{2} \sum_{k,l} G^{+(i,j)}_{N,k,l} p_k p_l + V_N^{+(i,j)}. \tag{5.3}
\]
Here $G^{\pm (i,j)}_{N,k,l}$ and $V^{\pm (i,j)}_N$ are functions of $q_i$ ($i = 1, \cdots, n_b$). The $\mathcal{N}$-fold supercharges are generalized as
\[
Q_N = \sum_i P_i^\dagger \psi_i, \quad Q_N^\dagger = \sum_i P_i \psi_i^\dagger, \tag{5.4}
\]
where $P_i$ is an $\mathcal{N}$-th order polynomial of the momenta $p_m \equiv -i \partial_m$ ($m = 1, \cdots, n_b$). To satisfy the following $\mathcal{N}$-fold superalgebra,
\[
\{ Q_N, Q_N \} = \{ Q_N^\dagger, Q_N^\dagger \} = 0, \quad [ Q_N, \mathbf{H}_N ] = [ Q_N^\dagger, \mathbf{H}_N ] = 0, \tag{5.5}
\]
we put the following conditions,
\[
[P_N^\dagger, P_N^\dagger] = 0, \tag{5.6}
\]
\[
H_N^{+(i,j)} P_N^k = P_N^i H_N^{-(j,k)}, \quad H_N^{-(i,j)} P_N^k = H_N^{-(i,k)} P_N^j, \quad P_N^i H_N^{+(j,k)} = P_N^j H_N^{+(i,k)}. \tag{5.7}
\]
Equation (5.6) comes from the former equation in (5.3) and Eq. (5.7) comes from the latter. When $\mathcal{N} = 1$ and $n_b = n_f$, Eqs. (5.6) and (5.7) have the following solution,
\[
P_1^i = p_i - i \partial_i h, \quad H_1^{(i,j)} = P_1^i P_1^j, \quad H_1^{-(i,j)} = P_1^i P_1^j, \tag{5.8}
\]
where $h$ is a function of $q_i$ ($i = 1, \cdots, n_b$). This reproduces ordinary supersymmetry in multi-dimensional quantum mechanics in Ref.[14].

Extensions of supersymmetry in multi-dimensional quantum mechanics attempted in Refs.[17, 37, 38] correspond to ours with $n_f = 1$ and $n_b = 2$. When $n_f = 1$, Eq. (5.7) is simplified since the latter two equations become trivial.

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Appendix

A \ g\text{-dependence of } S^\pm \text{ in type A models}

Here we will prove that if we introduce the coupling constant \( g \) by Eqs. (4.71) and (4.73), the matrices \( S^\pm \) in section 4.2 have the following forms,

\[
S^\pm_{n,m} = g^{-n} P^\pm_{n,m}(g^2) g^n,
\]

where \( P^\pm_{n,m}(g^2) \) are polynomials of \( g^2 \). If Eq. (A.1) holds, the characteristic equations becomes the following polynomials of \( g^2 \):

\[
\det M^\pm_n(E_n) = \det (P^\pm(g^2) - E_n I) = 0.
\]

Eq. (A.1) may be proved by induction, which we will prove explicitly carry out for \( S^- \).

In a similar manner, Eq. (A.1) for \( S^+ \) can be proved. First, we calculate \( S^-_{n,N} \) by Eq. (4.63).

Since \( S^-_{n,N} \) does not depend on \( q \), it is evaluated by

\[
S^-_{n,N} = \left. \frac{P_{N-1} H^-_{\phi_n}}{P_{N-1} \phi_{\phi_{N_n}}} \right|_{q=0}.
\]

A straightforward calculation shows that

\[
H^-_{\phi_n} = g^{1-n} U^{-1}(q) \left( F^0_{N,n}(gq) + g^2 F^1_{N,n}(gq) \right),
\]

where

\[
F^0_{N,n}(gq) = (n-1)w(gq)\eta'(gq)\eta(gq)^{n-2} + \frac{1}{2} w'(gq)\eta(gq)^{n-1}
\]

\[
- \frac{N-1}{2} \epsilon(gq) w(gq) \eta(gq)^{n-1} - \frac{N}{2} \epsilon'(gq) \eta(gq)^{n-1},
\]

\[
F^1_{N,n}(gq) = - \frac{n-1}{2} \eta''(gq) \eta(gq)^{n-2} - \frac{(n-1)(n-2)}{2} \eta'(gq)^2 \eta(gq)^{n-3}
\]

\[
+ \frac{(N-1)(2N-1)}{12} \epsilon(gq)^2 \eta(gq)^{n-1} - \frac{N^2 - 1}{12} \epsilon'(gq) \eta(gq)^{n-1}
\]

\[
+ \frac{N(N-1)}{4} \epsilon'(gq) \eta(gq)^{n-1}.
\]

We also obtain

\[
P_M = (-i)^M U^{-1}(q) \tilde{P}_M U(q)
\]
Using them, we obtain

\begin{align*}
&= (-ig)^\mathcal{M}U^{-1}(q) \left( \frac{d}{d(gq)} - (\mathcal{M} - 1)e(gq) \right) \left( \frac{d}{d(gq)} - (\mathcal{M} - 2)e(gq) \right) \\
&\times \cdots \times \left( \frac{d}{d(gq)} - e(gq) \right) \frac{d}{d(gq)} U(q) \\
&\equiv (-ig)^\mathcal{M}U^{-1}(q) \prod_{k=0}^{\mathcal{M}-1} \left( \frac{d}{d(gq)} - ke(gq) \right) U(q).
\end{align*}

(A.6)

Using them, we obtain

\begin{align*}
P_{\mathcal{N}-1}H_N^N\phi_n^- \bigg|_{q=0} &= (-i)^{\mathcal{N}-1}g^{\mathcal{N}U^{-1}(0)} \prod_{k=0}^{\mathcal{N}-2} \left( \frac{d}{d(gq)} - ke(gq) \right) \\
&\times \left( \mathcal{F}_N^0(gq) + g^2\mathcal{F}_N^1(gq) \right) \bigg|_{q=0} g^{-\mathcal{N}}n,
\end{align*}

(A.7)

and

\begin{align*}
P_{\mathcal{N}-1}\phi_N^- \bigg|_{q=0} &= (-i)^{\mathcal{N}-1}(\mathcal{N} - 1)!U^{-1}(0),
\end{align*}

(A.8)

where we have used \( h'(0) = \eta'(0) = 1 \). Thus \( \mathcal{P}_{\mathcal{N}}^{-}(g^2) \) becomes

\begin{align*}
\mathcal{P}_{\mathcal{N}}^{-}(g^2) &= \frac{1}{(\mathcal{N} - 1)!} \prod_{k=0}^{\mathcal{N}-2} \left( \frac{d}{d(gq)} - ke(gq) \right) \left( \mathcal{F}_N^0(gq) + g^2\mathcal{F}_N^1(gq) \right) \bigg|_{q=0} \mathcal{P}_{\mathcal{N}}^{-}(g^2).
\end{align*}

(A.9)

Next we assume that the matrices \( \mathcal{S}^-_{\mathcal{N}-k} \) for \( k = 0, \ldots, m \) have the forms \( \mathcal{S}^-_{\mathcal{N}-k} = g^{-n}\mathcal{P}_{\mathcal{N}-k}^{-}(g^2)g^{\mathcal{N}-k} \) and \( \mathcal{P}_{\mathcal{N}-k}^{-}(g^2) \) is a polynomial of \( g^2 \). Then \( \mathcal{S}^-_{\mathcal{N}-m-1} \) is calculated by

\begin{align*}
\mathcal{S}^-_{\mathcal{N}-m-1} &= \frac{P_{\mathcal{N}-m-2}(H_N^N\phi_n^- - \sum_{k=0}^{m} \mathcal{S}^-_{\mathcal{N}-k}\phi_{\mathcal{N}-k})}{P_{\mathcal{N}-m-2}\mathcal{P}_{\mathcal{N}-m-1}^-} \bigg|_{q=0} \\
&= \frac{P_{\mathcal{N}-m-2}H_N^N\phi_n^-}{P_{\mathcal{N}-m-2}\mathcal{P}_{\mathcal{N}-m-1}^-} \bigg|_{q=0} - \sum_{k=0}^{m} \mathcal{P}_{\mathcal{N}-k}(g^2)g^{\mathcal{N}-k-n} \frac{P_{\mathcal{N}-m-2}\phi_{\mathcal{N}-k}}{P_{\mathcal{N}-m-2}\mathcal{P}_{\mathcal{N}-m-1}^-} \bigg|_{q=0},
\end{align*}

(A.10)

From this, we find that \( \mathcal{S}^-_{\mathcal{N}-m-1} \) also has the form \( \mathcal{S}^-_{\mathcal{N}-m-1} = g^{-n}\mathcal{P}_{\mathcal{N}-m-1}(g^2)g^{\mathcal{N}-m-1} \) and \( \mathcal{P}_{\mathcal{N}-m-1}(g^2) \) becomes

\begin{align*}
\mathcal{P}_{\mathcal{N}-m-1}(g^2) &= \frac{1}{(\mathcal{N} - m - 2)!} \prod_{k=0}^{\mathcal{N}-m-3} \left( \frac{d}{d(gq)} - ke(gq) \right) \left( \mathcal{F}_N^0(gq) + g^2\mathcal{F}_N^1(gq) \right) \bigg|_{q=0} \\
&\quad - \sum_{k=0}^{m} \mathcal{P}_{\mathcal{N}-k}(g^2)g^{\mathcal{N}-m-3} \prod_{k=0}^{\mathcal{N}-m-3} \left( \frac{d}{d(gq)} - ke(gq) \right) \eta^{\mathcal{N}-k-1}(gq) \bigg|_{q=0}.
\end{align*}

(A.11)

This is a polynomial of \( g^2 \).
References

[1] N. Seiberg and E. Witten, *Nucl. Phys.* B431 (1994) 484.

[2] H. Aoyama, H. Kikuchi, I. Okouchi, M. Sato and S. Wada, *Phys. Lett.* B424 (1998) 93.

[3] H. Aoyama, H. Kikuchi, I. Okouchi, M. Sato and S. Wada, *Nucl. Phys.* B553 (1999) 644.

[4] D. J. Rowe and A. Ryman, *J. Math. Phys.* 23 (1982) 732.

[5] I. I. Balitsky and A. V. Yung, *Phys. Lett.* B168 (1986) 13.

[6] P. G. Silvetrov, *Sov. J. Nucl. Phys.* 51 (1990) 1121.

[7] H. Aoyama and H. Kikuchi, *Nucl. Phys.* B369 (1992) 219.

[8] H. Aoyama and S. Wada, *Phys. Lett.* B349 (1995) 279.

[9] T. Harano and M. Sato, *hep-ph/9703457*.

[10] H. Aoyama, H. Kikuchi, T. Harano, M. Sato and S. Wada, *Phys. Rev. Lett.* 79 (1997) 4052.

[11] H. Aoyama, H. Kikuchi, T. Harano, I. Okouchi, M. Sato and S. Wada, *Prog. Theor. Phys. Supplement* 127 (1997) 1.

[12] E. B. Bogomolny, *Phys. Lett.* B91 (1980) 431.

[13] E. Witten, *Nucl. Phys.* B188 (1981) 513.

[14] E. Witten, *Nucl. Phys.* B202 (1982) 253.

[15] A. A. Andrianov, M. V. Ioffe and V. P. Spiridonov, *Phys. Lett.* A174 (1993) 273.

[16] A. A. Andrianov, M. V. Ioffe, F. Cannata and J.-P. Dedonder, *Int. J. Mod. Phys.* A10 (1995) 2683.

[17] A. A. Andrianov, M. V. Ioffe and D. N. Nishnianidze, *Phys. Lett.* A201 (1995) 103.

[18] A. A. Andrianov, M. V. Ioffe and D. N. Nishnianidze, *Theor. Math. Phys.* 104 (1995) 1129.

[19] V. G. Bagrov and B. F. Samsonov, *Theor. Math. Phys.* 104 (1995) 1051.

[20] B. F. Samsonov, *Mod. Phys. Lett.* A11 (1996) 1563.

[21] V. G. Bagrov and B. F. Samsonov, *Phys. Part. Nucl.* 28 (1997) 374.

[22] B. F. Samsonov, *Phys. Lett.* A263 (1999) 274.
REFERENCES

[23] H. Aoyama, M. Sato and T. Tanaka, Phys. Lett. B**503** (2001) 423.

[24] H. Aoyama, M. Sato, T. Tanaka and M. Yamamoto, Phys. Lett. B**498** (2001) 117.

[25] M. Plyushchay, Int. J. Mod. Phys. A**15** (2000) 3679.

[26] S. Klishevich and M. Plyushchay, Mod. Phys. Lett. A**14** (1999) 2739.

[27] J. O. Rosas-Ortiz, J. Phys. A**31** (1998) 10163.

[28] A. Khare and U. Sukhatme, J. Math. Phys. **40** (1999) 5473.

[29] D. J. Fernández C. and V. Hussin, J. Phys. A**32** (1999) 3630.

[30] D. J. Fernández C., J. Negro and L. M. Nieto, Phys. Lett. A**275** (2000) 338.

[31] A. V. Turbiner, Commun. Math. Phys. **118** (1988) 467

[32] M. A. Shifman, Int. J. Mod. Phys. A**12** (1989) 2897.

[33] A. G. Ushveridze, *Quasi-Exactly Solvable Models in Quantum Mechanics*, (IOP Publishing, Bristol, 1994), and references cited therein.

[34] S. M. Klishevich and M. S. Plyushchay, hep-th/0012023.

[35] P. Dorey, C. Dunning and R. Tateo, hep-th/0103051.

[36] For a review, see *Large-Order Behavior of Perturbation Theory*, ed. J.C. Le Guillou and J. Zinn-Justin (North-Holland, Amsterdam, 1990).

[37] A. A. Andrianov, M. V. Ioffe and D. N. Nishnianidze, solv-int/9605007

[38] S. M. Klishevich and M. S. Plyushchay, hep-th/0105135.