Data Dissemination Problem in Wireless Networks

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Abstract—In this work, we formulate and study a data dissemination problem, which can be viewed as a generalization of the index coding problem and of the data exchange problem to networks with an arbitrary topology. We define \( r \)-solvable networks, in which data dissemination can be achieved in \( r > 0 \) communications rounds. We show that the optimum number of transmissions for any one-round communications scheme is given by the minimum rank of a certain constrained family of matrices. For general \( r \)-solvable networks, we derive an upper bound on the minimum number of transmissions in any scheme with \( \geq r \) rounds. We experimentally compare the obtained upper bound to a simple lower bound.

Index Terms—Data dissemination, data exchange, index coding.

I. INTRODUCTION

A problem of index coding with side information considers a communications scenario with one broadcast transmitter and several receivers. All receivers possess some partial information available to the transmitter and request additional information. The goal is to design a communications scheme, which minimizes the total number of transmissions. Index coding problem was proposed first in [4]: it was suggested therein to use coding in order to minimize a number of transmissions. Later, in [3], the minimum number of transmissions in the index coding problem was shown to be equal to the minimum rank of a properly defined family of matrices. Generally, computing the minimum rank of a family of the matrices is an NP-hard problem, yet in some special cases there exist efficient algorithms to compute it [3], [8].

Index coding problem was intensively studied in the recent years, see for example [1], [5], [6], [7], [16], [19]. It was shown in [9], [10] that index coding problem is equivalent to a network coding problem [2], [17]. In index coding, however, the underlying network graph is very simple, it is a directed star graph, where the transmitter is the root of that graph.

A variation of index coding, termed data exchange problem, was studied in [11]. In the data exchange problem, unlike the index coding problem, every node can serve as both a transmitter and a receiver. The underlying network graph is a complete directed graph. Before the communications take place, each node possesses some partial information. The goal is to deliver all information to all the nodes in a minimum number of transmissions. It was shown in [11] that the minimum number of transmissions in the data exchange problem can also be described as a rank minimization problem of a certain constrained family of matrices, thus resembling some of the results for index coding.

Another related problem is a set reconciliation [12], [20], [21], [22]. The set reconciliation problem is usually defined over a network of arbitrary topology, either wired or wireless. In that problem, similarly to the data exchange problem, the goal is to deliver all information to all the nodes. However, by contrast, it is assumed that no node knows what information is possessed by the other nodes. This makes the set reconciliation problem more difficult than the data exchange problem.

In this work, we introduce a data dissemination problem, which further generalizes both the index coding and the data exchange problems, such that the underlying directed connectivity graph of the network is an arbitrary graph. This model, in particular, represents cached networks of arbitrary topology. The data dissemination problem can also be viewed as a generalization of the set reconciliation problem. In data dissemination problem, every node can serve as both a transmitter and a receiver. Moreover, each node possesses some partial information and requests some additional information.

Example I.1. Consider an example network in Figure 2. There are five nodes \( v_1, v_2, v_3, v_4 \) and \( v_5 \), which in total possess three bits of information \( x_1, x_2, x_3 \). If \( v_1 \) transmits \( x_1 + x_2 \) and \( v_2 \) transmits \( x_2 + x_3 \), then the requests of all nodes will be satisfied with only two transmissions.

![Fig. 1. Example network](image_url)

Let \( \mathcal{A} \) be a data dissemination algorithm. Important parameters in the analysis are communications cost \( \text{COMMUNICATION}(\mathcal{A}) \) (the worst case number of symbols sent between the devices), and \( \text{ROUNDS}(\mathcal{A}) \) (the number of communications rounds in the algorithm, will be defined more explicitly in the sequel).

In this work, we present the following results. First, we formulate and study a data dissemination problem. We de-
fine $r$-solvable networks, in which data dissemination can be achieved in $r > 0$ communications rounds. We show that the optimal number of transmissions for any one-round communications scheme is given by the minimum rank of a certain constrained family of matrices. For general $r$-solvable networks, by using similar techniques, we derive an upper bound on the minimum number of transmissions in $\geq r$ rounds. We experimentally compare the obtained upper bound to a simple lower bound.

II. NOTATION

Denote $[n] \triangleq \{1, 2, \cdots, n\}$ (in particular, $[0]$ denotes the empty set). We use $0$ to denote the all-zero vector, when the length of the vector is clear from the context. Similarly, we use $e_i$ to denote the unit vector which has $1$ in position $i$ and zeros everywhere else. We assume hereafter that all vectors are column vectors.

Let $\mathbb{F}$ be a finite field $\mathbb{F}_q$, where $q$ is a prime power. Take $A$ to be a matrix over $\mathbb{F}$. Denote by $A^{(i)}$ the $i$-th row of $A$ and by $(A)_{i,j}$ the entry in the $i$-th row and $j$-th column of $A$. We use the notation rowvec$(A)$ to denote the row space of the matrix $A$, and notation $A \otimes B$ for the standard tensor product of the matrices $A$ and $B$. For the row vector $v = (v_1, v_2, \cdots, v_n)$, we denote by $\text{diag}(v)$ the $n \times n$ matrix as follows:

$$
\text{diag}(v)_{i,j} = \begin{cases} 
v_i & \text{if } i = j \\
0 & \text{otherwise} \end{cases}
$$

Fix an ambient vector space $V \subseteq \mathbb{F}^n$. Let $W$ be a subspace of $V$. The orthogonal vector space of $W$ is given by

$$
W^\perp \triangleq \{v \in V \mid \forall w \in W \ : \ v \cdot w = 0 \},
$$

where $v \cdot w$ denotes the inner product of the two vectors.

Let $U, W \subseteq V$ be two vector subspaces. Define

$$
U + W = \{u + w \mid u \in U \text{ and } w \in W\} \subseteq V.
$$

If $U \cap W = \{0\}$, then we also write $U \oplus W$ instead of $U + W$.

Let $\mathcal{G}(\mathcal{V}, \mathcal{E})$ be a directed graph with the vertex set $\mathcal{V}$ and the edge set $\mathcal{E}$. For each $\ell \in \mathcal{V}$, introduce the notations

$$
\mathcal{N}_{in}(\ell) = \{v \in \mathcal{V} \ : \ (v, \ell) \in \mathcal{E}\}
$$

and

$$
\mathcal{N}_{out}(\ell) = \{v \in \mathcal{V} \ : \ (\ell, v) \in \mathcal{E}\}.
$$

Let $E$ be the all-one square matrix. The size of $E$ will be apparent from the context. Similarly, let $I$ be the identity matrix. Finally, denote by $1_n$ the all-one column vector of length $n$.

III. PROBLEM SETUP

First, we define the data dissemination problem. Consider a wireless network with a topology given by a finite directed connected graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = [k]$ is the set of nodes and $\mathcal{E}$ is the set of edges. Let $x = (x_1, \cdots, x_n) \in \mathbb{F}^n$ be an information vector. Each node $\ell \in \mathcal{V}$ possesses some side information consisting of the symbols $x_j, j \in \mathcal{P}_\ell \subseteq [n]$, and is interested in receiving all of the symbols $x_i, i \in \mathcal{T}_\ell \subseteq [n] \setminus \mathcal{P}_\ell$. In the data dissemination problem, the goal is to find a coded transmission schedule with the minimum number of transmissions, such that all nodes could recover all their respective requested symbols. However, unlike in [1], the network might not have full connectivity.

Throughout this paper we make the following assumptions.

- The graph $\mathcal{G}$ is an arbitrary directed graph.
- All transmissions are broadcast, i.e. the messages transmitted by the node $\ell$ are always received by all the nodes in $\mathcal{N}_{out}(\ell)$.
- The transmissions take place in rounds. If the node $\ell$ receives some symbol during round $\eta$, but it did not have that symbol before the beginning of round $\eta$, it cannot retransmit it within the same round.
- The coding is linear, i.e. each node $\ell \in \mathcal{V}$ transmits $n_\ell$ messages of the form $s^{(i)}_\ell \cdot x, i \in [n_\ell], 0 \leq n_\ell \leq n$, and all $s^{(i)}_\ell \in \mathbb{F}^n$.
- There is a central entity that knows $\mathcal{G}$ and all the sets $\mathcal{P}_i$ and $\mathcal{T}_i$ for all $i \in \mathcal{V}$. This entity is running an algorithm for finding an optimal communications scheme.

In our work, we will be interested in minimizing COMMUNICATION$(\Omega)$.

Definition III.1. The network based on the graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ is said to be $r$-solvable, $r \in \mathbb{N}$, if for any combination of the sets $\mathcal{P}_i$ and $\mathcal{T}_i, i \in \mathcal{V}$, $r$ communications rounds are sufficient to satisfy all the node requests, but $r-1$ rounds are not sufficient. If the network is not $r$-solvable for any $r \in \mathbb{N}$, then we say that it is not solvable.

Lemma III.1. The network is $r$-solvable for some $r \in \mathbb{N}$ if the maximum of the shortest length of the directed path from the node $i$ to the node $\ell$ in $\mathcal{G}$, for any two nodes $i, \ell \in \mathcal{V}$, is exactly $r$.

The proof of this lemma appears in the appendix.

We define the transposed $k \times k$ integer adjacency matrix $D$ of the graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ as follows:

$$
(D)_{i,j} = \begin{cases} 
1 & \text{if } (j, i) \in \mathcal{E} \\
0 & \text{otherwise} \end{cases}
$$

Corollary III.2. The network is $r$-solvable if $r$ is the smallest integer such that all the entries in the matrix $D^r$ are strictly positive.

IV. PROBLEM SETTINGS

Let the graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, the information vector $x = (x_1, \cdots, x_n) \in \mathbb{F}^n$, and the sets $\mathcal{P}_\ell$ and $\mathcal{T}_\ell$ for $\ell \in \mathcal{V}$ be defined as above. We represent a matrix family $\mathbb{A}$ over $\mathbb{F}$ as a matrix over $\mathbb{F} \cup \{\star\}$, where ‘$\star$’ is a special symbol. The entry, which can take any value from $\mathbb{F}$ in $\mathbb{A}$ is marked as ‘$\star$’.

For each node $\ell \in \mathcal{V}$, we define the family $\mathbb{A}_\ell$ of $n \times n$ matrices as follows.

$$
(\mathbb{A}_\ell)_{i,j} = \begin{cases} 
\star & \text{if } j \in \mathcal{P}_\ell \\
0 & \text{otherwise} \end{cases}.
$$

(1)
Define the family $\mathcal{A}$ of $(kn) \times n$ block matrices as:

$$
\mathcal{A} \triangleq \begin{bmatrix}
A_{i_1} \\
A_{i_2} \\
\vdots \\
A_{i_d}
\end{bmatrix}, 
$$

(2)

Given $A \in \mathcal{A}$, the $j$-th $n \times n$ sub-matrix of $A$ will be denoted as $A_j$. We will also use the notation $A_{N_i(n)}$ to denote the $dn \times n$ matrix

$$
A_{N_i(n)} = \begin{bmatrix}
A_{i_1} \\
A_{i_2} \\
\vdots \\
A_{i_d}
\end{bmatrix},
$$

where $N_i(n) = \{i_1, i_2, \ldots, i_d\}$, and $d$ is an in-degree of $i$ in $G$.

For each $\ell \in \mathcal{V}$, we define an $n \times n$ information matrix $P_\ell = (P_{\ell})_{i \in [n], j \in [n]}$, $P_{\ell}(i, j) = \begin{cases} 1 & \text{if } i = j \text{ and } i \in \mathcal{P}_{\ell} \\ 0 & \text{otherwise} \end{cases}$.

Similarly, for each $\ell \in \mathcal{V}$, we define an $n \times n$ query matrix $T_\ell = (T_{\ell})_{i \in [n], j \in [n]}$, $T_{\ell}(i, j) = \begin{cases} 1 & \text{if } i = j \text{ and } i \in \mathcal{T}_{\ell} \\ 0 & \text{otherwise} \end{cases}$.

**Theorem V.1.** Consider a wireless network defined by the graph $G(\mathcal{V}, \mathcal{E})$. Let $\mathcal{A}$ be an $nk \times n$ matrix family defined as above. For all nodes $\ell \in \mathcal{V}$, let $P_\ell$ and $T_\ell$ be the corresponding possession and query matrices. Then, the minimal number of transmissions needed to satisfy the demands of all nodes in $\mathcal{V}$ in one round of communications is

$$
\tau = \min_{\mathcal{A} \in \mathcal{A}} \left\{ \sum_{\ell \in \mathcal{V}} \text{rank} \left( A_{\ell} \right) \right\}, 
$$

(3)

where for all $\ell \in \mathcal{V}$

$$
\text{rowspace}\left( \begin{bmatrix} A_{N_i(n)} \\ P_{\ell} \end{bmatrix} \right) \supseteq \text{rowspace}(T_{\ell}).
$$

(4)

If the above matrix $A \in \mathcal{A}$ as above does not exist then there is no algorithm that satisfies all requests in one round.

The proof of this theorem appears in the appendix.

**V. DATA EXCHANGE PROTOCOL EXTENSION TO MANY ROUNDS**

In this section, we consider a more general scenario. Here, the underlying network graph $G(\mathcal{V}, \mathcal{E})$ is an arbitrary directed graph. For each node $\ell \in \mathcal{V}$, we require that $\mathcal{P}_{\ell} \cup \mathcal{T}_{\ell} = [n]$.

Our goal is to minimize the number of transmissions. For $r$-solvable network, we are aiming at an algorithm which minimizes COMMUNICATIONS$(\mathcal{A})$, while $\text{ROUNDS}(\mathcal{A}) = r$.

**Proposition V.1.** For a node $\ell \in \mathcal{V}$, and for $i \in [n]$, denote by $d_\ell(x_i)$ the length of the shortest path from a set of vertices having $x_i$ in their possession to $\ell$. Let $d_{\ell} = \sum_{i \in \mathcal{T}_{\ell}} d_\ell(x_i)$ and $d_{\text{max}} = \max_{\ell \in \mathcal{V}} d_{\ell}$.

Then, the minimum number of transmissions in any algorithm for data dissemination problem is at least $d_{\text{max}}$.

The proof of this proposition appears in the appendix.

Let the matrix families $\mathcal{A}_i$, for $i \in \mathcal{V}$, be as defined in (1), and $\mathcal{A}$ be as defined in (2).

**Definition V.1.** The maximum rank of the matrix family $\mathcal{A}$ is defined as

$$
\max \text{rank}(\mathcal{A}) = \max \text{rank}(A) 
$$

(5)

Given the matrix family $\mathcal{A}_i$, we define an operator $\Gamma(\cdot)$, which replaces the symbols `$\ast$' in the maximal number of the first rows with linearly independent canonical vectors, and replaces the symbols `$\ast$' in the remaining rows with zeros.

Similarly, operator $\Gamma_{\ell}(\cdot)$, $\ell \in \mathcal{V}$, takes as an input the possession matrix from $\mathcal{A}_i$ and returns $\Gamma_{\ell}(\mathcal{A}_i) = \Gamma(\mathcal{A}_i)$.

**Example V.1.** For a fixed $\ell \in \mathcal{V}$, let $\mathcal{A}_i = \begin{bmatrix}
\ast & 0 & \ast & \ast & 0 \\
\ast & 0 & \ast & \ast & 0 \\
\ast & 0 & \ast & \ast & 0 \\
\ast & 0 & \ast & \ast & 0 \\
\ast & 0 & \ast & \ast & 0
\end{bmatrix}.

After replacing the symbols `$\ast$’, we obtain

$$
\Gamma(\mathcal{A}_i) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}.
$$

**Algebra of matrix families:** Denote by $\mathbb{F}^*$ the alphabet $\mathbb{F} \cup \{\ast\}$. In what follows, we represent families of matrices over $\mathbb{F}$ as matrices over $\mathbb{F}^*$. In the sequel, we define operations on the matrices over $\mathbb{F}^*$, in a way which allows to describe algebraically the data dissemination in the network. In particular, we define two operations, the addition `$+$' and the multiplication `$\cdot$' of two elements $a, b \in \mathbb{F}^*$, in such way that if $a, b \in \mathbb{F}$, then these operations coincide with usual addition and multiplication in the field $\mathbb{F}$.

Addition and multiplication of two elements, where at least one of the elements is `$\ast$’, are given in the following tables. Addition table:

$$
\begin{array}{c|ccc}
+ & b & \ast \\
\hline
a & a+b & \ast \\
\ast & \ast & \ast \\
\end{array}
$$

(6)

Multiplication table:

$$
\begin{array}{c|cccc}
\cdot & 0 & b \neq 0 & \ast \\
\hline
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\neq 0 & 0 & a \cdot b & \ast \\
\ast & 0 & \ast & \ast \\
\end{array}
$$

(7)

where $a$ and $b$ are any two elements in $\mathbb{F}$. 


The addition and multiplication operations over $\mathbb{F}^*$ can be naturally extended to operations on matrices over $\mathbb{F}^*$.

**Example V.2.** Let a $3 \times 3$ matrix $B$ over the field $\mathbb{F}$ and a $3 \times 3$ matrix family $A$ over $\mathbb{F}$ be given by

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{bmatrix}.$$  

By multiplying the matrix family $A$ from the left by the matrix $B$, we obtain:

$$B \cdot A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{bmatrix} = \begin{bmatrix} * & 0 & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}.$$  

We also define the multiplication of integer matrices by the matrix families over $\mathbb{F}$.

**Remark V.1.** In what follows, we use a binary operation of matrix multiplication, where one of the arguments is an integer matrix and the second argument is a family of matrices over $\mathbb{F}$, and the result is a family of matrices over $\mathbb{F}$. In order to be able to do so, by slightly abusing the notation, we use the product of an integer matrix with a matrix over $\mathbb{F}^*$, according to the rules defined in (6) and (7). The result of this operation is a matrix over $\mathbb{F}^*$, which can be interpreted as a family of matrices over $\mathbb{F}$.

**Example V.3.** Let a $3 \times 3$ integer matrix $B$ be

$$B = \begin{bmatrix} 1 & 2 & 0 \\ 4 & 5 & 6 \\ 0 & 7 & 8 \end{bmatrix},$$

and $A$ be a $3 \times 3$ matrix family over $\mathbb{F}$ as in Example V.2.

Multiplying $B$ by $A$ yields

$$B \cdot A = \begin{bmatrix} 1 & 2 & 0 \\ 4 & 5 & 6 \\ 0 & 7 & 8 \end{bmatrix} \cdot \begin{bmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{bmatrix} = \begin{bmatrix} * & 0 & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}.$$  

A. Adjacency matrix in set recombination

**Lemma V.2.** Let $A$ be the possession matrix as defined in Equation 2. Let $D$ be the adjacency matrix of the graph $G$. Let $E$ be an $n \times n$ identity matrix. After performing one round of the protocol, the new possession matrix $A_+$ is related to $A$ as

$$A_+ = (D \otimes E) \cdot A.$$  

Proof: From the definition of $A$ in Equation 2, the matrix families $A_i, i \in V$, have $n$ identical rows. Then, we can write $A_i = A_i^{(1)} \otimes 1_n$, where $A_i^{(1)}$ is a row vector over $\mathbb{F}^*$ of length $n$ which consists of a single row of $A_i$. From the definition of the tensor product, we have $A = A_0 \otimes 1_n$, where

$$A_0 = \begin{bmatrix} A_0^{(1)} \\ A_0^{(2)} \\ \vdots \\ A_0^{(k)} \end{bmatrix}.$$  

The right hand side of the claim can be re-written as

$$(D \otimes E) \cdot A = (D \otimes E) \cdot (A_0 \otimes 1_n) = (D \cdot A_0) \otimes (n1_n) = (D \cdot A_0) \otimes 1_n.$$

Here, the transition $(\cdot)$ is due to the properties of the tensor product, and the transition $(\otimes)$ is due to Remark V.1.

Next, assume that

$$A_0 = \left( \hat{A}_{i,j} \right)_{i \in [k], j \in [n]} \quad \text{and} \quad D = \left( d_{j,i} \right)_{j \in [k], i \in [n]} \quad \text{and} \quad D \cdot \hat{A}_0 = \left( \theta_{i,j} \right)_{i \in [k], j \in [n]}.$$  

By using tables in (6) and (7), for all $i \in [k], j \in [n]$, we have

$$\theta_{i,j} = \sum_{i \in [k]} d_{e,i} \cdot \hat{a}_{i,j}.$$  

Assume that there is an edge $(i, \ell) \in E$ for some $i \in [k]$, and that the node $i$ has $x_j$. Then, $d_{e,i} \neq 0$ and $\hat{a}_{i,j} = *$. In that case, we obtain $\theta_{i,j} = *$. This correctly represents the situation that the node $i$ delivers $x_j$ to the node $\ell$.

We conclude that the matrix $(D \cdot \hat{A}_0) \otimes 1_n$ correctly represents the possession matrix of the graph $G$ after one round of the protocol execution.

**Lemma V.2** can be naturally extended to protocols with several communications rounds. In the sequel, we denote by $A^{(i)}, i \in \mathbb{N}$, the possession matrix after the $i$-th round of the protocol. For the sake of convenience, we also use the notation $A^{(0)} = A$.

**Corollary V.3.** The possession matrix after the $i$-th round of the protocol execution is given by

$$A^{(i)} = (D^i \otimes E) \cdot A^{(0)}.$$  

B. Data dissemination using rank optimization

The following theorem is the main result of this section.

**Theorem V.4.** Let $G$ be an underlying directed graph of an $r_0$-solvable network defined by the adjacency matrix $D^*$. Let $A$ be the corresponding possession matrix of the network. Then there exists an iterated data exchange protocol with $r$ rounds, for any $r \geq r_0$, and $\tau$ transmissions, where

$$\tau = \sum_{i=1}^{r} \min_{A^{(i)} \in (D^{r-i} \otimes E) \cdot A} \left\{ \sum_{j=1}^{k} \text{rank} \left( A^{(i)}_j \right) \right\}$$  

for matrices $A^{(i)}$ which are subject to

$$\forall j \in [k] : \text{rank} \left( \left( \text{diag} (D^{(j)}_j) \otimes I \right) \cdot A^{(i)}_j \right) = \text{max-rank} \left( (\text{diag}(e_j) \otimes I) \cdot (D^{(j)}_j \otimes E) \cdot A \right),$$  

where the matrices $I$ and $E$ are both $n \times n$.

The proof of this theorem appears in the appendix.
In this section, we describe experimental study of the tightness of the bound in Theorem V.4. The instance of the problem consists of two main ingredients: the adjacency matrix of the graph and the possession matrix of the network. We generate the adjacency matrix of the graph randomly, while fixing the number of vertices in the graph and the diameter. We also generate randomly the possession matrix of the network.

In general, enumeration of all the matrices in a matrix family has exponential complexity. In order to facilitate this process, we use a randomized algorithm. It picks random matrices from a given matrix family, and then checks if that matrix satisfies the conditions of the theorem. We use two different types of networks: in the first case the diameter of the graph \( G \) was two, and in the second case it was three. In both cases, the number of nodes was 4 and the number of information bits was 4.

For each randomly chosen network, we compute the number of transmissions guaranteed by Theorem V.4 and the lower bound on the number of transmissions in Proposition V.1. We compute the ratio of these two quantities. The tables in Figures 2 and 3 present the distribution of this ratio. In order to compute the maximum rank of a matrix family, we use the algorithm in [13] (see also [15]).

### VI. EXPERIMENTAL RESULTS

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| Range      | 1.1, 1.2 | 1.2, 1.4 | 1.4, 1.6 | 1.6, 1.8 | 1.8, 2.0 | 2.0, ∞ |
|------------|---------|---------|---------|---------|---------|--------|
| Occurrence | 54      | 22      | 6       | 4       | 0       | 14     |

Fig. 2. The efficiency of the algorithm for graphs of diameter 2

| Range      | 1.1, 1.2 | 1.2, 1.4 | 1.4, 1.6 | 1.6, 1.8 | 1.8, 2.0 | 2.0, ∞ |
|------------|---------|---------|---------|---------|---------|--------|
| Occurrence | 30      | 18      | 24      | 0       | 6       | 22     |

Fig. 3. The efficiency of the algorithm for graphs of diameter 3
Proof: (Lemma III.1)

1) Consider an algorithm, where in each round, each node broadcasts all the symbols \(x_j\) that it has in its possession (including the messages that it received in the previous rounds). Pick some \(\ell \in V\). Let \((v_0, v_1, v_2, \ldots, v_t = \ell)\) be a shortest path from \(v_0\) to \(\ell\) of length \(t \leq r\). Then, after \(i\) rounds the node \(v_i\) obtains all the symbols \(x_j\) that \(v_0\) has in its possession. Therefore, after \(r\) rounds, \(\ell\) has all the messages possessed by all the nodes in \(V\).

2) Next, we show that \(r - 1\) rounds are not sufficient. Let \(v_0\) and \(\ell\) be two vertices, such that the shortest path between them is of length \(r\). Denote this path \((v_0, v_1, v_2, \ldots, v_r = \ell)\). Then, the shortest path between \(v_0\) and \(v_i\) is \(i\) for all \(i \in [r]\). Assume that \(\mathcal{P}_{v_0} = [n]\) and for any \(i \in V\setminus\{v_0\}\), \(\mathcal{P}_i = \emptyset\). Assume that \(n \geq 1\) and \(\mathcal{T}_i = \{1\}\). Then, clearly, after one iteration \(v_2\) can not know \(x_1\) (because the shortest path from \(v_0\) to \(v_2\) is of length two, and the same symbol is not retransmitted within the same round.) More generally, for the same reason, for any \(i \in [r - 1]\), after \(i\) iterations \(v_{i+1}\) can not know \(x_1\).

Before we prove Theorem IV.1 we formulate and prove the following two lemmas.

Lemma VIII.1. (3) Let \(V\) be an ambient space and \(W \subseteq V\) be a linear subspace. If there is a vector \(x \in V\) such that \(x \notin W\), then there exists \(y \in W^\perp\) such that \(x \cdot y \neq 0\).

Proof: Let \(x \in V\) be such that \(x \notin W\). By contrary, assume that for all \(y \in W^\perp\) we have \(x \cdot y = 0\). Then, \(x \in (W^\perp)^\perp = W\). This is in contradiction to the conditions of the lemma.

Lemma VIII.2. Let \(V\) be an ambient space and \(W \subseteq V\) be its linear subspace. If \(x \in W^\perp\), then for every subspace \(W' \subseteq W\) and for every vector \(y \in W'\) it holds \(x \cdot y = 0\).

Proof: Let \(x \in W^\perp\). Pick any \(y \in W'\). Then \(y \in W\). We obtain that \(x \cdot y = 0\).

Proof: (Theorem IV.1)
The statement of the theorem is proven in two steps.

1) We construct an exact coding scheme, which uses \(\tau\) symbol transmissions over \(F\). We show that under this scheme, for all \(\ell \in V\), the node \(\ell\) can recover the bit \(x_\eta\) for all \(\eta \in \mathcal{T}_\ell\).

Let \(A \in \mathbb{A}\) be the matrix, which minimizes the value of \(\tau\) in (3). Assume that Equation (4) holds, and take some \(\eta \in \mathcal{T}_\ell\), \(\ell \in V\). Then,

\[
\mathbf{e}_\eta^T = \sum_{i \in \mathcal{N}_n(\ell), j \in [n]} \alpha_{i,j} A_i^{[j]} + \sum_{j \in [n]} \beta_j P_{\ell}^{[j]},
\]

where all \(\alpha_{i,j}\) and \(\beta_j\) are in \(F\). Then,

\[
x_\eta = e_\eta^T x = \sum_{i \in \mathcal{N}_n(\ell), j \in [n]} \alpha_{i,j}(A_i^{[j]} \cdot x) + \sum_{j \in [n]} \beta_j(P_{\ell}^{[j]} \cdot x).
\]

Each (sending) node \(i \in V\) will use some basis \(B_i\) of the rowspace of \(A_i\), and will transmit the messages \((b_i \cdot x) \in F\) for all \(b_i \in B_i\). It is straightforward to verify that in such way each node transmits combinations of bits that it has in its possession. The total number of messages that the node \(i\) transmits is \(\text{rank}(A_i)\) and the total number of messages transmitted in the scheme is

\[
\sum_{i \in V} \text{rank}(A_i).
\]

Each (receiving) node \(\ell \in V\) will be able to compute the values \((A_i^{[j]} \cdot x)\) for all \(i \in \mathcal{N}_n(\ell), j \in [n]\), from the messages \((b_i \cdot x)\). It will also be able to compute the values \(P_{\ell}^{[j]} \cdot x\) for all \(j \in [n]\). Therefore, the node \(\ell\) will be able to compute \(x_\eta\), as required.

2) We show that if there exists another linear code which satisfies the requests of all the nodes in \(G\), then it is possible to construct a corresponding matrix \(A\) as in Equation (4), which satisfies the conditions of the theorem.

Consider the transmission scheme with the optimal number of transmissions \(\tau_{\text{opt}}\). Assume that for each \(\ell \in V\), the node \(\ell\) transmits \(n_\ell\) messages of the form \(s^{(i)} \cdot x\), \(i \in [n]\), \(0 \leq n_\ell \leq n\). Here \(\sum_{\ell \in V} n_\ell = \tau_{\text{opt}}\) is the total number of transmissions.

Next, we show that for all \(\ell \in V\) and for all \(\eta \in [n]\), if \(\eta \in \mathcal{T}_\ell\) then the vector \(e_\eta \in F^n\) belongs to \(W_\ell \subseteq F^n\), where \(W_\ell\) is the linear span of the vector set

\[
\left( \bigcup_{j \in \mathcal{N}_n(\ell)} \{s_j^{(i)}\} \right) \cup \left( \bigcup_{j \in \mathcal{P}_\ell} \{e_j\} \right).
\]

Fix some \(\ell \in V\). By contrary, assume that \(e_\eta \notin W_\ell\). Then, by Lemma VIII.1 there exists \(x \in W_\ell^\perp\) such that \(e_\eta \cdot x \neq 0\).

From the definition of \(W_\ell\) and Lemma VIII.2 we have that \(x \cdot s_j^{(i)} = 0\) for every transmitter \(j \in \mathcal{N}_n(\ell), i \in [n]\); (ii) the side information symbols \(x_i, i \in \mathcal{P}_\ell\), available to the node \(\ell\) are all 0.

Thus, the node \(\ell\) cannot distinguish between the information vector \(x\) and the zero vector 0. Therefore, our assumption that \(e_\eta \notin W\) is false. We conclude that \(e_\eta \in W_\ell\).
Next, we construct the $n \times n$ matrices $A_\ell$ for all $\ell \in \mathcal{V}$. For that sake, we take
\[
A_\ell^{(i)} = \begin{cases} s_\ell^{(i)} & \text{if } i \in [n_\ell] \\ 0 & \text{otherwise} \end{cases}
\]
We obtain that for each $\ell \in \mathcal{V}$, $\text{rank}(A_\ell) \leq n_\ell$, and therefore
\[
\sum_{\ell \in \mathcal{V}} \text{rank}(A_\ell) \leq \tau_{\text{opt}}.
\]
By construction, the resulting $A$ belongs to the family $\hat{A}$, and therefore the corresponding code satisfies equation (3). We conclude that $\tau$ in expression (3) is indeed the minimum number of transmissions.

Proof: (Proposition VIII.3)
The proof of this proposition is straightforward: let $\ell \in \mathcal{V}$ be the node that maximizes the expression (5). Then, at least $d_\ell$ transmissions are needed in order to satisfy all the requests of $\ell$.

Proof: (Corollary VIII.4)
We have:
\[
A^{(i)} = (D \otimes E)^i \cdot \hat{A}^{(0)} \\
= (D^i \otimes E^i) \cdot \hat{A}^{(0)} \\
= (D^i \otimes n^{i-1} E) \cdot \hat{A}^{(0)} \\
= n^{i-1}(D^i \otimes E) \cdot \hat{A}^{(0)} \\
(\S) \quad (D^i \otimes E) \cdot \hat{A}^{(0)}
\]
Here, the transition (\S) holds due to Remark VIII.1. Thus, any non-zero integer entry in $(D^i \otimes E)$ is mapped to the element 1 $\in \mathbb{F}$, and, therefore, the factor $n^{i-1}$ can be omitted.

Before we turn to proving Theorem VIII.4, we formulate and prove the following lemma.

Lemma VIII.3. Let $G$ be a directed graph defined by the adjacency matrix $D^T$. Let the possession matrix family of the graph $G$ be $\hat{A}$ as defined in Equation (2). There exists a transmission matrix $A \in \hat{A}$ such that
\[
\text{rank} \left( \left( \begin{array}{c} \text{diag}(D[\beta]) \otimes I \end{array} \right) \cdot A \right) \\
= \text{max-rank} \left( \left( \begin{array}{c} \text{diag}(e_j) \otimes I \end{array} \right) \cdot (D \otimes E) \cdot \hat{A} \right)
\]
for all $j \in [k]$.

Proof: We analyze the left and the right-hand side of equation (10) separately.

1) The right-hand side of equation (10) can be written as
\[
\text{rank} \left( \left( \begin{array}{c} \text{diag}(D[\beta]) \otimes I \end{array} \right) \cdot A \right) \\
= \text{max-rank} \left( \left( \begin{array}{c} \text{diag}(e_j) \otimes I \end{array} \right) \cdot (D \otimes E) \cdot \hat{A} \right)
\]
2) Consider the upper-block part of the matrix in the left-hand side of (10). Denote
\[
A = (\hat{a}_{i,j})_{i \in [n]} \cdot j \in [n].
\]
In the sequel, we show that the values of the elements $\hat{a}_{i,j}$ in $A$ can be chosen such that the equation (10) holds.
The matrix $\text{diag}(D[\beta]) \otimes I$ is the diagonal block matrix, namely,
\[
\text{diag}(D[\beta]) \otimes I = \begin{bmatrix} D_{j,1} \ldots 0_n \\ \vdots \ldots \vdots \\ 0_n \ldots D_{j,k} \end{bmatrix},
\]
where for all $i \in [k]$, $D_{j,i}$ is a diagonal $n \times n$ matrix as follows:
\[
D_{j,i} = \begin{bmatrix} d_{j,i} \ldots 0 \\ 0 \ldots d_{j,i} \\ \vdots \ldots \vdots \\ 0 \ldots 0 \end{bmatrix},
\]
and $0_n$ is an $n \times n$ all-zero matrix.
Then, \((\text{diag}(D^{[j]})) \otimes I) \cdot A = \)

\[
\begin{bmatrix}
    d_{j,1} \cdot a_{1,1} & \cdots & d_{j,1} \cdot a_{1,n} \\
    \vdots & \ddots & \vdots \\
    d_{j,1} \cdot a_{n,1} & \cdots & d_{j,1} \cdot a_{n,n}
\end{bmatrix}
\]

(13)

The element in the \(j\)-th row and \(\ell\)-th column in the matrix in Equation (12) is \('*'\) if there exists \(i\) such that \(d_{j,i} \neq 0\) and \(a_{i,\ell} = '.*'.\) In that case, we can pick \(s \in [n]\) and set \(a_{(i-1)n+s,\ell} = 1\). We obtain that \(d_{j,i} \cdot a_{(i-1)n+s,\ell} \neq 0\).

Since \(s \in [n]\), different \(s\) can be chosen for every \(\ell \in [n]\). After setting \(a_{(i-1)n+s,\ell} \) to 1 in every column \(\ell \in [n]\), we set the values of all other elements in \(A\) to 0. Because the ones in the matrix in Equation (13) are all in the distinct rows and in the distinct columns, the rank of the matrix \((\text{diag}(D^{[j]}) \otimes I) \cdot A\) equals to the max-rank of the family \((\text{diag}(e_j) \cdot D \cdot A) \otimes I_1\).

If \(a_{i,\ell} = '.*', then \(a_{(i-1)n+s,\ell} = '.*', for \(s \in [n]\), and only these elements are set to 1 in \(A\). Therefore, \(A \in \mathcal{A}\).

From the construction of \(A \in \mathcal{A}\), we have that \((\text{diag}(D^{[j]}) \otimes I) \cdot A\) has a single one in some row, for every column where there is \('.*' in \((\text{diag}(e_j) \otimes I) \cdot (D \otimes E) \cdot A\). The transposed adjacency matrix \(D\) has ones in the main diagonal. Therefore, if there exists a column with \('.*' in \((\text{diag}(e_j) \cdot D \cdot A) \otimes 1_n\), then there also exists \('.*' in the same column of \(A\).

Thus, if there is a single one in a row in any column of \((\text{diag}(D^{[j]}) \otimes I) \cdot A\), then there is a single one in a row in the corresponding column of \(\Gamma_j(A)\). As

\[
\text{rank}\((\text{diag}(D^{[j]}) \otimes I) \cdot A\) = \text{max-rank}\((\text{diag}(e_j) \otimes I) \cdot (D \otimes E) \cdot A\),
\]

then

\[
\text{rowspace}(\Gamma_j(A)) \subseteq \text{rowspace}\((\text{diag}(D^{[j]}) \otimes I) \cdot A\),
\]

and condition (11) holds.

The proof of the last lemma showed that the transmission matrix \(A\) exists. However, it may not be optimal. We next turn to proving Theorem (4).

Proof: (Theorem (4))

From Lemma VIII.3 there exist matrices \(A^{(i)}\) satisfying (10). Take any such matrices, and write them as

\[
A^{(i)} = \left(a_{\rho,\eta}^{(i)}\right)_{\rho \in [kn]}^{\eta \in [n]}
\]

For all \(s \in [k]\), let the vectors

\[
t_s^{(i)} = \left(t_s^{(i)}, t_s^{(i)} \cdot x_m \right)_{r \in [n]}
\]

be the \(r\)-th row of \(A_s^{(i)}\). These vectors can be viewed as the linear coefficients multiplying the symbols transmitted by the node \(s\) during the \(i\)-th round of the protocol. In the \(i\)-th round of the protocol, the messages transmitted by the node \(s\) are given by the non-zero vectors in

\[
Y_s^{(i)} = \left\{ \sum_{m \in [n]} t_s^{(i)} \cdot x_m \right\}_{r \in [n]} = \left\{ t_s^{(i)} \cdot x \right\}_{r \in [n]}
\]

where \(x = (x_1, x_2, \ldots, x_n)^T\).

The number of transmissions of the node \(s\) during the \(i\)-th round of the protocol is the rank of \(A_s^{(i)}\). When summing for all \(s \in [k]\), we obtain the number of transmissions \(\tau\) as stated in the right-hand side of Theorem (4) with respect to this \(A^{(i)}\).

Observe that the node \(\ell\) receives all the messages from the node \(s\) if \(d_{\ell,s} \neq 0\). Therefore, the node \(\ell\) receives all the messages of the form

\[
d_{\ell,s} \sum_{m \in [n]} t_s^{(i)} \cdot x_m
\]

for all \(s \in [k], r \in [n]\).

Since \(n \cdot t_s^{(i)} = \sum_{m \in [n]} t_s^{(i)} \cdot x_m\), the messages received by the node \(t\) are the entries of the vector given by:

\[
\begin{bmatrix}
    d_{1,1} \cdot a_{1,1}^{(i)} & \cdots & d_{1,1} \cdot a_{1,n}^{(i)} \\
    \vdots & \ddots & \vdots \\
    d_{1,1} \cdot a_{n,1}^{(i)} & \cdots & d_{1,1} \cdot a_{n,n}^{(i)}
\end{bmatrix}
\]

(15)

From Lemma (8) the matrix family \((D^i \otimes E) \cdot A\) is the possession matrix of the network after the round \(i\). Thus, the matrix family \((\text{diag}(e_j) \otimes I) \cdot (D^i \otimes E) \cdot A\) is the possession matrix of the node \(j\) after the round \(i\). The max-rank of this matrix family is the number of symbols the node \(j\) has after the completion of the \(i\)-th round of the protocol.
To this end, the matrices $A^{(i)}$ as above satisfy the condition (10), and the number of transmission in the protocol based on it is given by the right-hand side of the equality (9). Therefore, in order to minimize the number of transmissions in the protocol, one has to choose the matrices $A^{(i)}$ that satisfy (10) and minimize the right-hand side of the equality (9).