Covariant hamiltonian formalism
for field theory: Hamilton-Jacobi
equation on the space $G$

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January 12, 2022

Abstract
Hamiltonian mechanics of field theory can be formulated in a generally
covariant and background independent manner over a finite
dimensional extended configuration space. The physical symplectic structure of the
theory can then be defined over a space $G$ of three-dimensional surfaces
without boundary, in the extended configuration space. These surfaces
provide a preferred over-coordinatization of phase space. I consider the
covariant form of the Hamilton-Jacobi equation on $G$, and a canonical
function $S$ on $G$ which is a preferred solution of the Hamilton-Jacobi
equation. The application of this formalism to general relativity is equiv-
alent to the ADM formalism, but fully covariant. In the quantum domain,
it yields directly the Ashtekar-Wheeler-DeWitt equation. Finally, I apply
this formalism to discuss the partial observables of a covariant field theory
and the role of the spin networks –basic objects in quantum gravity– in
the classical theory.

1 Introduction

Hamiltonian mechanics is a clean and general formalism for describing a physical
system, its states and its observables, and provides a road towards quantum
theory. In its traditional formulation, however, the hamiltonian formalism is
badly non covariant. This is a source of problems already for finite dimensional
systems. For instance, the notions of state and observable are not very clean in
the hamiltonian mechanics of the systems where evolution is given in parametric form, especially if the evolution cannot be deparametrized (as in certain
cosmological models). But the problem is far more serious in field theory. The
conventional field theoretical hamiltonian formalism breaks manifest Lorentz
invariance badly. More importantly, in a generally covariant context the conventional hamiltonian formalism is cumbersome and its physical interpretation is far from being transparent.

In my opinion, a proper understanding of the generally covariant structure of mechanics is necessary in order to make progress in quantum gravity. The old notions of observable, state, evolution, hamiltonian, and so on, and the tools that are conventionally employed to relate quantum field theory with experience—S matrix, n-point functions and so on, cease to make sense in a genuinely general covariant and background independent context. Therefore we need to understand the the generally covariant version of these notions. In particular, if we want to understand quantum field theory in a truly background independent context, we must find a proper definition of quantum field theoretical transition amplitudes, in a form with a clear operational interpretation that makes sense also when there is no background spacetime.

A covariant formulation of the hamiltonian mechanics of finite dimensional systems is possible. Several versions of this formulation can be found in the literature, with different degrees of developments. Perhaps the first to promote this point of view was Lagrange himself, who first understood that the proper covariant definition of phase space is as the space of the physical motions [1], or the space of the solutions of the equations of motion (modulo gauges). Notable contributions, among many others, are Arnold’s identification of the presymplectic space with coordinates \((t, q^i, p_i)\) (time, lagrangian variables and their momenta) as the natural home for mechanics [2], and the beautiful and well developed, but little known, formalism of Souriau [3]. Here, I use the covariant version of hamiltonian mechanics described in reference [4], which builds on previous results. This formalism is based on the physical notion of “partial observable” [5]. The partial observables of a non-relativistic finite dimensional mechanical system are the quantities \((t, q^i)\), treated on the same footing. In particular, the formalism treats the time variable \(t\) on the same footing as the lagrangian variables \(q^i\). The space of the partial observables is the extended configuration space \(\mathcal{C}\) and the hamiltonian formalism is built over this space. The phase space \(\Gamma\) is identified with a space of one-dimensional curves in \(\mathcal{C}\). The elements of \(\mathcal{C}\) and \(\Gamma\) provide the proper relativistic generalization of the notions of observable and state, consistent with the modification of the notions of space and time introduced by general relativity [4]. The formalism can be manifestly Lorentz covariant and deals completely naturally with reparametrization invariant systems such as the cosmological models.

The extension of these ideas to field theory require the identification of the partial observables of field theory. These are finite in number. They include the coordinates of the spacetime \(M\) and the coordinates of the target space \(T\) on which the fields take value. Thus the extended configuration space of a field theory is the finite dimensional space \(\mathcal{C} = M \times T\). A hamiltonian formalism for field theory built on the finite dimensional space \(\mathcal{C} = M \times T\) has been developed by many authors in several variants, developing classical works by Cartan, Weyl [6] and DeDonder [7] on the calculus of variations. See for instance [8] and especially the beautiful work [9] and the extended references therein. Here, I
refer to the version of hamiltonian mechanics for field theory described in [10], where the accent is on general covariance and relation to observability. The (infinite dimensional) phase space \( \Gamma \) is identified as a space of four-dimensional surfaces in \( C \). The physical symplectic form of \( \Gamma \), which determines the Poisson brackets, was not given in [10]. Here, I consider a definition of the symplectic form on \( \Gamma \), in this context. (On a covariant definition of the symplectic structure on the space of the solutions of the field equations, see also [11], whose relation with this work will be briefly discussed below.)

The key of the construction is to introduce the space \( G \). The space \( G \) is the space of the boundary – “initial and final” – lagrangian data (no momenta). In the finite dimensional case, a typical element \( \gamma \) of \( G \) is a pair of points in \( C \). Generically, two points in \( C \) – say \((t_0, q_0^i)\) and \((t, q^i)\) – identify a motion. In the field theoretical case, \( G \) is a space of three-dimensional surfaces \( \gamma \), without boundary, in \( C \). Again, \( \gamma \) is a set of Lagrangian data sufficient to identify a motion. Indeed \( \gamma \) defines a closed hypersurface in spacetime and the value of the fields over it. I show below that there is a canonical two-form \( \omega_G \) on \( G \), and the physical phase space and its physical symplectic form, namely its Poisson brackets, follow immediately from the pair \((G, \omega_G)\).

I then study a covariant version of the Hamilton-Jacobi theory, defined on \( G \), and I observe that there exist a preferred, canonical, solution \( S[\gamma] \) of the Hamilton-Jacobi equation on \( G \). The Hamilton-Jacobi formalism is a window open towards quantum theory. Schrödinger introduced the Schrödinger equation by interpreting the Hamilton-Jacobi equation as the optical approximation of a wave equation [12]. This means searching for an equation for a wave function \( \psi \), solved to lowest order in \( \hbar \) by \( \psi = A e^{i/\hbar S} \), if \( S \) solves the Hamilton-Jacobi equation. On the basis of this idea, Schrödinger found his celebrated equation by simply replacing each partial derivative of \( S \) in the Hamilton-Jacobi function with \((-i\hbar \times)\) a partial derivative operator [13]. This same procedure can be used in the covariant formulation of mechanics. The covariant Hamilton-Jacobi equation yields then directly the quantum dynamical equation of the theory. This is the Schrödinger equation in the case of a non-covariant system, or the appropriate “Wheeler-DeWitt” equation for covariant systems. (In fact, the Wheeler-DeWitt equation as well was first found replacing partial derivatives with partial derivative operators in the Hamilton-Jacobi equation of general relativity [14].) For a parametrized system, this procedure shortcuts Dirac’s recipe for the quantization of first class constraints, which is cumbersome and has a very cloudy interpretation when applied to such systems. Furthermore, if \( S[\gamma] \) is the canonical solution of the Hamilton-Jacobi equation mentioned above, then \( \psi[\gamma] = A[\gamma] e^{i/\hbar S[\gamma]} \) is the propagator of the Schrödinger equation, which was identified in [15] as the quantity providing a direct operational interpretation to a generally covariant quantum system.

In the field theoretical context, several authors have developed a covariant Hamilton-Jacobi formalism based the Weyl-DeDonder approach. In this formalism the Hamilton-Jacobi equation is a partial differential equation for a multiplet of Hamilton-Jacobi functionals defined on the extended configuration space. (For a review, see [16]). Here I do not utilize this formalism, because I
do not see how it could lead to the quantum theory (on this, see [17]). Instead, I discuss a Hamilton-Jacobi function over $G$, which provides a general covariant setting for a functional Hamilton-Jacobi equation.

I then apply this formalism to general relativity (following also [18] and [10]). I use self-dual variables, which much simplify the equations [19, 20]. The extended configuration space is identified with the finite-dimensional space $\tilde{C} = M \times \mathcal{C}$, where $M$ is the four-dimensional manifold of the spacetime coordinates and $\mathcal{C} = R^4 \times sl(2,C)$ where $sl(2,C)$ is the Lorentz algebra. The formalism is simple and straightforward. It shortcuts the intricacies of the conventional hamiltonian formalism of general relativity, and further simplify the one described in [10]. The Hamilton-Jacobi equation yields immediately to the Ashtekar form of the Wheeler-DeWitt equation. (See extended references in [21]. On early use of Hamilton-Jacobi theory in general relativity, see [22].) In the classical as well as in the quantum theory, the $M$ (spacetime) component of $\tilde{C}$ ends up playing only an auxiliary role and it disappears from observables and states. This reflects the diffeomorphism invariance of the theory.

Finally, I discuss the physical interpretation of the formalism. This is of interest for the interpretation of quantum gravity and, more in general, any background independent quantum field theory. The physical predictions of the theory are given in terms of correlations between partial observables. These can be obtained directly from $S[\gamma]$. In the context of general relativity, this leads to the introductions of spin networks in the classical context, opening a bridge towards the spin networks used in the quantum theory (spin networks were introduced in quantum gravity in [23] and then developed in [24]. See [25] and extended references therein.)

The paper is structured as follows. The main notions are introduced in Section 2 in the finite dimensional context. For concreteness, I exemplify all structures introduced by computing them explicitly in the simple case of a free particle. The field theoretical formalism is developed in Section 3. As an example, I describe a self-interacting scalar field in Minkowski space. General relativity is treated in Section 4. In this paper, I am not concerned with global issues: I deal only with aspects of the theory which are local in $C$.

2 Finite dimensional systems

The exercise that we perform in this Section, following and extending the results of [4], is to carefully reformulate classical hamiltonian mechanics and its interpretation in a form that does not require any of the non-relativistic notions that loose meaning in a generally covariant context. These notions are for instance: instantaneous state of the system, evolution in time, evolution of the observables in time, and so on. The alternative notion that we use (correlations, motions ...) are introduced in detail in [4]. The motivation of this exercise is to introduce the ideas that will then be used for field theory in Section 3 and for general relativity in Section 4.
2.1 Relativistic mechanics

Consider a system with \( n \) degrees of freedom governed by a Hamiltonian function \( H_0(t, q^i, p_i) \), where \( q^i \) with \( i = 1, \ldots, n \), are coordinates on the configuration space, \( p_i \) the corresponding momenta and \( t \) the time variable. Let \( \mathcal{C} \) be the \((n+1)\)-dimensional extended configuration space, that is, the product of the configuration space with the real line, with coordinates \( q^a = (t, q^i) \), with \( a = 0, \ldots, n \). From now on we work on this extended configuration space and we treat \( t \) on the same footing as the configuration space variables. In this form mechanics is general enough to describe fully relativistic systems.

Let \( \Omega = T^* \mathcal{C} \) be the cotangent space to \( \mathcal{C} \), and \( p_a = (p_0, p_i) \equiv (\pi, p_i) \) the momenta conjugate to \( q^a \). Being a cotangent space, \( \Omega \) carries the canonical one-form \( \theta_\Omega = p_a dq^a \). Dynamics is defined on \( \Omega \) by the relativistic hamiltonian (or hamiltonian constraint)

\[
H(q^a, p_a) = \pi + H_0(q^a, p_i) = 0 \tag{1}
\]

This equation defines a surface \( \Sigma \) in \( \Omega \). It is convenient to coordinatize \( \Sigma \) with the coordinates \((q^a, p_i) = (t, q^i, p_i)\). We denote \( \theta \) the restriction of \( \theta_\Omega \) to \( \Sigma \) and \( \omega \) the pull back of \( \theta \). An orbit of \( \omega \) is a curve without boundaries \( m : \tau \rightarrow (q^a(\tau), p_i(\tau)) \) in \( \Sigma \) whose tangent \( X = (\dot{q}^a \partial_a + \dot{p}_i \partial_i) \) satisfies

\[
\omega(X) = 0 \tag{2}
\]

Here the dot indicates the derivative with respect to \( \tau \) while \( \partial_a \) and \( \partial_i \) form the basis of tangent vectors in \( \Sigma \) associated with the coordinates \((q^a, p_i)\). The last equation is equivalent to the Hamilton equations. The projection \( \tilde{m} \) of \( m \) to \( \mathcal{C} \), that is the curve \( \tilde{m} : \tau \rightarrow q^a(\tau) = (t(\tau), q^i(\tau)) \), is a solution of the equations of motion, given in parametric form.

Let \( \Gamma \) be the space of the orbits. There is a natural projection

\[
\Pi : \Sigma \rightarrow \Gamma \tag{3}
\]

that sends a point to the orbits to which it belongs. There is also a unique two-form \( \omega_\Gamma \) on \( \Gamma \) such that its pull back to \( \Sigma \) is \( \omega \):

\[
\Pi_* \omega_\Gamma = \omega \tag{4}
\]

The symplectic space \((\Gamma, \omega_\Gamma)\) is the physical phase space with its physical symplectic structure.

This formalism, as well as its interpretation, can be immediately generalized to the case in which the coordinates \( q^a \) of \( \mathcal{C} \) do not split into \( t \) and \( q^i \) and the relativistic hamiltonian does not have the particular form (1). Therefore it remains valid for generally covariant or reparametrization invariant systems.

**Example: free particle**

Consider a free particle in one dimension. Then \( n = 1 \), the extended configuration space has coordinates \( t \) (the time) and \( x \) (the position of the particle).
Denote \( \pi \) and \( p \) the corresponding momenta. The constraint (4) that defines the free motion is
\[
H = \pi + \frac{1}{2m}p^2 = 0. \tag{5}
\]

The restriction of the canonical form \( \theta, \omega = \pi dt + p dx \) to the surface \( \Sigma \) defined by (5) is
\[
\theta = -\frac{p^2}{2m} dt + p dx \tag{6}
\]
where we have taken the coordinates \((t, x, p)\) for \( \Sigma \). The two form \( \omega \) on \( \Sigma \) is therefore
\[
\omega = d\theta = -\frac{p}{m} dp \wedge dt + dp \wedge dx = dp \wedge \left(dx - \frac{p}{m} dt\right). \tag{7}
\]

A curve \((t(\tau), x(\tau), p(\tau))\) on \( \Sigma \) has tangent \( X = \dot{t} \partial_t + \dot{x} \partial_x + \dot{p} \partial_p \) and inserting this and (7) in the main equation (2) we obtain
\[
\omega(X) = -\frac{p}{m} \dot{p} dp + \dot{x} dp - \dot{p} \left(dx - \frac{p}{m} dt\right) = 0. \tag{8}
\]

Equating to zero each component, we have
\[
\dot{x} = \frac{p}{m}, \quad \dot{p} = 0. \tag{9}
\]

With the solution \( x(\tau) = \frac{p}{m} t(\tau) + Q \) and \( p(\tau) = P \) where \( Q \) and \( P \) are constants and \( t(\tau) \) arbitrary. The projection of this orbit on \( C \) gives the motions \( x = \frac{P}{m} t + Q \).

The space of the orbits is thus parametrized by the two integration constants \( Q \) and \( P \). The point \((t, x, p)\) in \( \Sigma \), belongs to the orbit \( Q = x - \frac{p}{m} t \) and \( P = p \). Thus the projection (4) is given by \( \Pi(t, x, p) = (Q(t, x, p), P(t, x, p)) = (x - \frac{p}{m} t, p) \). Then \( \omega_\Gamma = dP \wedge dQ \), because
\[
\Pi_\ast \omega_\Gamma = dP(t, x, p) \wedge dQ(t, x, p) \tag{10}
= dp \wedge d\left(x - \frac{p}{m} t\right) = dp \wedge \left(dx - \frac{p}{m} dt\right) = \omega,
\]
as required by the definition (4). See [4] for examples of relativistic systems.

### 2.2 The space \( \mathcal{G} \)

We now introduce a space which is important for what follows. Let \( \mathcal{G} \) be defined as
\[
\mathcal{G} = C \times C. \tag{11}
\]

That is, an element \( \gamma \) of \( \mathcal{G} \) is an ordered pair of elements of the extended configuration space \( C \):
\[
\gamma = (q^a, q^a_0) = (t, q^i, t_0, q^i_0). \tag{12}
\]
We think at $\gamma$ as initial and final conditions for a physical motion: the motion begins at $q_0^a$ at time $t_0$ and ends at $q^a$ at time $t$. Generically, given $\gamma = (q^a, q_0^a)$ there is a unique solution of the equations of motion that goes from $q_0^a$ to $q^a$. That is, there is a curve $(q^a(\tau), p_i(\tau))$, with boundaries, in $\Sigma$, with $\tau \in [0,1]$ such that

$$q^a(0) = q_0^a, \quad q^a(1) = q^a, \quad \omega(X) = 0. \quad(13)$$

We denote $m_\gamma$ this curve, $\tilde{m}_\gamma$ its projection to $C$ (namely $q^a(\tau)$). Thus $\gamma$ is the boundary of $\tilde{m}_\gamma$, which we write as $\gamma = \partial \tilde{m}_\gamma$. We denote $s$ and $s_0$ the initial and final points of $m_\gamma$ in $\Sigma$. Notice that $s = (q^a, p_i)$ and $s_0 = (q_0^a, p_{0i})$, where in general $p_i$ and $p_{0i}$ depend both on $q^a$ as well as on $q_0^a$.

There is a natural map $i: \mathcal{G} \rightarrow \Gamma$ which sends each pair to the orbit that the pair defines. Thus we can define a two-form $\omega_\mathcal{G}$ on $\mathcal{G}$ as $\omega_\mathcal{G} = i^* \omega_\Gamma$. In other words, $\gamma = (q^a, q_0^a) = (t, q^a, t_0, q_0^a)$ can be taken as a natural (over-) coordinatization of the phase space. Instead of coordinatizing a motion with initial and final positions and momenta, we coordinatize it with initial and final positions. In these coordinates, the symplectic form is given by $\omega_\mathcal{G}$.

For what follows, it is important to notice that there is an equivalent, alternative definition of $\omega_\mathcal{G}$, which can be obtained without going first through $\Gamma$. Indeed, let $\delta \gamma = (\delta q^a, \delta q_0^a)$ be a vector (an infinitesimal displacement) at $\gamma$. Then the following is true:

$$\omega_\mathcal{G}(\gamma)(\delta_1 \gamma, \delta_2 \gamma) = \omega_\mathcal{G}(q^a, q_0^a)((\delta_1 q^a, \delta_1 q_0^a), (\delta_2 q^a, \delta_2 q_0^a))$$

$$= \omega(s)(\delta_1 s, \delta_2 s) - \omega(s_0)(\delta_1 s_0, \delta_2 s_0). \quad (14)$$

Notice that $\delta_1 s$, the variation of $s$, is determined by $\delta_1 q$ as well as by $\delta_1 q_0$, and so on. This equation expresses $\omega_\mathcal{G}$ directly in terms of $\omega$. As we shall see, this equation admits an immediate generalization in the field theoretical framework, where $\omega$ will be a five-form, but $\omega_\mathcal{G}$ is a two-form.

Now fix a pair $\gamma = (q^a, q_0^a)$ and consider a small variation of only one of its elements. Say

$$\delta \gamma = (\delta q^a, 0). \quad (15)$$

This defines a vector $\delta \gamma$ at $\gamma$ on $\mathcal{G}$, which can be pushed forward to $\Gamma$. If the variation is along the direction of the motion, then the push forward vanishes, that is $i_* \delta \gamma = 0$, because $\gamma$ and $\gamma + \delta \gamma$ define the same motion. It follows that if the variation is along the direction of the motion, $\omega_\mathcal{G}(\delta \gamma) = 0$. Thus clearly the equation

$$\omega_\mathcal{G}(X) = 0. \quad (16)$$

gives again the solutions of the equations of motion.

Thus, the pair $(\mathcal{G}, \omega_\mathcal{G})$ contains all the relevant information on the system. The null directions of $\omega_\mathcal{G}$ define the physical motions, and if we divide $\mathcal{G}$ by these null directions, the factor space is the physical phase space, equipped with the physical symplectic structure.\footnote{More precisely, we define $\mathcal{G}$ as the set of pairs for which this solution exists. If there is more than one solution, we choose the one with minimal action. See below.}
Example: free particle

The space $G$ has coordinates $\gamma = (t, x, t_0, x_0)$. Given this point in $G$, there is clearly one motion that goes from $(t_0, x_0)$ to $(t, x)$, which is

$$
t(\tau) = t_0 + (t - t_0)\tau, \quad (17)$$
$$
x(\tau) = x_0 + (x - x_0)\tau. \quad (18)$$

Along this motion,

$$
p = m \frac{x - x_0}{t - t_0}, \quad (19)$$
$$
\pi = -\frac{(x - x_0)^2}{2m(t - t_0)^2}. \quad (20)$$

The map $i : G \to \Gamma$ is thus given by

$$
P = p = m \frac{x - x_0}{t - t_0}, \quad (21)$$
$$
Q = x - \frac{p}{m} t = x - \frac{x - x_0}{t - t_0} t, \quad (22)$$

and therefore the two-form $\omega_G$ is

$$
\omega_G = i^* \omega_\Gamma = dP(t, x, t_0, x_0) \wedge dQ(t, x, t_0, x_0)
= \frac{m}{t - t_0} d\left(\frac{x - x_0}{t - t_0} t\right) \wedge \left(dx_0 - \frac{x - x_0}{t - t_0} dt_0\right) \wedge \left(dx - \frac{x - x_0}{t - t_0} dt\right). \quad (23)$$

It is immediate to see that a variation $\delta \gamma = (\delta t, \delta x, 0, 0)$ (at constant $(x_0, t_0)$) such that $\omega_G(\delta \gamma) = 0$ must satisfy

$$
\delta x = \frac{x - x_0}{t - t_0} \delta t. \quad (24)$$

This is precisely a variation of $x$ and $t$ along the physical motion (determined by $(x_0, t_0)$). Therefore $\omega_G(\delta \gamma) = 0$ gives again the equations of motion. The two null directions of $\omega_G$ are thus given by the two vector fields

$$
X = \frac{x - x_0}{t - t_0} \partial_x + \partial_t, \quad (25)$$
$$
X_0 = \frac{x - x_0}{t - t_0} \partial_{x_0} + \partial_{t_0}, \quad (26)$$

which are in involution (their Lie bracket vanishes), and therefore define a foliation of $G$ with two-dimensional surfaces. These surfaces are parametrized by $P$ and $Q$, given in (21,22), and in fact

$$
X(P) = X(Q) = X_0(P) = X_0(Q) = 0. \quad (27)$$

In fact, we have simply recovered in this way the physical phase space: the space of these surfaces is the phase space $\Gamma$ and the restriction of $\omega_G$ to it is the physical symplectic form $\omega_\Gamma$. 

8
2.3 The function $S$ on the space $\mathcal{G}$: a preferred solution of the Hamilton-Jacobi equation

Let us now introduce an important function on $\mathcal{G}$. We define $S(\gamma) = S(q^a, q_0^a)$ by

$$S(\gamma) = \int_{m_\gamma} \theta. \quad (28)$$

It is easy to see that this is the value of the action along the path $m_\gamma$. In fact

$$S(\gamma) = \int_{m_\gamma} p_a dq^a$$

$$= \int_0^1 p_a(\tau) \dot{q}^a(\tau) d\tau = \int_0^1 \left( \pi(\tau) \dot{\theta}(\tau) + p_i(\tau) \dot{q}_i(\tau) \right) d\tau$$

$$= \int_0^1 \left( -H_0(\tau) \dot{\theta}(\tau) + p_i(\tau) \dot{q}_i(\tau) \right) d\tau = \int_{t_0}^t \left( -H_0(\tau) + p_i(\tau) \frac{dq_i(t)}{dt} \right) dt$$

$$= \int_{t_0}^t L \left( q^i, \frac{dq^i(t)}{dt} \right) dt, \quad (30)$$

where $L$ is the Lagrangian. We have from the definition (28) of $S$,

$$\frac{\partial S(q^a, q_0^a)}{\partial q^a} = p_a(q^a, q_0^a) \quad (31)$$

where $p_a$ is the value of the momenta in $s$ (which depends on $q^a$ and $q_0^a$). The derivation of this equation is less obvious than what it looks at first sight: see appendix A.

It follows from (31) that $S$ satisfies the (covariant) Hamilton-Jacobi equation

$$H \left( q^a, \frac{\partial S(q^a, q_0^a)}{\partial q^a} \right) = 0. \quad (32)$$

More precisely, $S$ satisfies the Hamilton-Jacobi equation in both sets of variables, namely it satisfies also

$$H \left( q_0^a, \frac{\partial S(q^a, q_0^a)}{\partial q_0^a} \right) = 0, \quad (33)$$

where the minus sign comes from the fact that the second set of variable is in the lower integration boundary in (28). $S(\gamma)$, defined in (28), is thus a preferred solution of the Hamilton-Jacobi solution.

In view of the generalization to field theory, it is convenient to extend the definition of $\mathcal{G}$ and $S(\gamma)$ as follow. Define

$$S(q_i^a, q^a, q_0^a) = \int_{m_{q_i^a, q^a}} \theta + \int_{m_{q^a, q_0^a}} \theta. \quad (34)$$
This can be seen as an extension of the definition (28) in the following sense. We can view the path \( m = m_q^a, q^a \cup m_q^a, q^a \) as a path in \( C \) bounded by the three points \( (q_f^a, q^a, q_i^a) \). This form of \( S \) will be convenient for extracting physical information from it, and for the extension to field theory.

**Example: free particle**

The function \( S(\gamma) \) is easily computed.

\[
S(t, x, t_0, x_0) = \int_0^t (\pi t + p \dot{x}) dt + p \int_{x_0}^x dx = \frac{m(x - x_0)^2}{2(t - t_0)} + m \frac{(x - x_0)^2}{t - t_0} = \frac{m(x - x_0)^2}{2(t - t_0)}.
\]

(35)

We now verify that it satisfies the Hamilton-Jacobi equation of the relativistic hamiltonian (36), which is

\[
H \left( q^a, \frac{\partial S}{\partial q^a} \right) = \frac{\partial S}{\partial t} + \frac{1}{2m} \left( \frac{\partial S}{\partial x} \right)^2 = 0.
\]

(36)

Easily

\[
\frac{\partial S}{\partial t} = -\frac{m(x - x_0)^2}{2(t - t_0)^2}, \quad \frac{\partial S}{\partial x} = \frac{m(x - x_0)}{(t - t_0)}.
\]

(37)

So that the Hamilton-Jacobi equation (36) is satisfied.

Notice that \( S(x, t, x_0, t_0) \) is strictly related to the quantum theory. The Shrödinger equation can be obtained from the relativistic Hamiltonian-Jacobi equation as

\[
H \left( q^a, -i\hbar \frac{\partial}{\partial q^a} \right) \psi(q^a) = 0.
\]

(38)

for a wave function \( \psi(q^a) \) on the extended phase space. In fact, this gives immediately

\[
\left( -i\hbar \frac{\partial}{\partial t} - \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \right) \psi(x, t) = 0.
\]

(39)

which is the conventional time dependent Shrödinger equation. Its propagator \( W(x, t, x_0, t_0) \), which satisfies the equation itself

\[
\left( -i\hbar \frac{\partial}{\partial t} - \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \right) W(x, t, x_0, t_0) = 0
\]

(40)

is

\[
W(x, t, x_0, t_0) = \frac{1}{\sqrt{t - t_0}} e^{\pm S(x, t, x_0, t_0)}.
\]

(41)

Therefore \( S \) is the phase of the propagator.
2.4 Physical predictions

Finally, let us discuss the relation between the formalism described and the physical interpretation of the theory. If the function $S(\gamma)$ on $\mathcal{G}$ is known explicitly, the general solution of the equation of motion is

$$f^a(q^a, q_0^a, p^0_a) = \frac{\partial S(q^a, q_0^a)}{\partial q_0^a} + p^0_a = 0. \quad (42)$$

We view (42) as an equation for $q^a$; the quantities $p^0_a$ and $q_0^a$ are constants determining one solution. The solution defines a curve in $\mathcal{C}$, namely a motion. Physically, once we have determined a motion by means of $p^0_a$ and $q_0^a$, equation (42) determines whether or not the correlation $q^a$ can be observed. (See [4] for more details.) In general, there is a redundancy in the system (42): one of the equations ($a = 0$ for non relativistic systems) is a consequence of the others.

For instance, in the case of a free particle, inserting (35) into (42) gives, for $q^a = x$

$$\frac{\partial S(t, x, t_0, x_0)}{\partial x_0} + p^0 = 0. \quad (43)$$

Inserting the explicit form of $S$ given in (35), we obtain

$$x - x_0 = \frac{p^0}{m}(t - t_0). \quad (44)$$

which is the correct relation that relates $x$ and $t$, at constant values of $x_0, t_0, p^0$. The other equation, obtained for $q^a = t$, does not give anything new.

There is another way of using the solution of the Hamilton-Jacobi to obtain physical predictions, which is of interest in view of it generalization to quantum field theory. Fix two points $q^a_i$ and $q^a_f$ in $\mathcal{C}$. We can ask if a third point $q^a$ can lay on the same motion as $q^a_i$ and $q^a_f$. This means asking whether or not we could observe the correlation $q^a$, given that the correlations $q^a_i$ and $q^a_f$ are observed. A moment of reflection will convince the reader that the answer to this question is positive if and only if

$$\frac{\partial S(q^a_f, q^a_i)}{\partial q^a_f} = \frac{\partial S(q^a_f, q^a)}{\partial q^a_f}, \quad (45)$$

$$\frac{\partial S(q^a_f, q^a_i)}{\partial q^a_i} = \frac{\partial S(q^a, q^a_i)}{\partial q^a_i}. \quad (46)$$

Using the definition (34), this becomes

$$\frac{\partial S(q^a_f, q^a, q^a_i)}{\partial q^a_f} = \frac{\partial S(q^a_f, q^a_i)}{\partial q^a_f}, \quad (47)$$

with $\tau = i, f$.

Alternatively, we may notice that a first order variation of $q^a$ does not change $S(q_0^a, q^a, q_i^a)$, because the two derivatives of the first and second term in (34)
cancel out if $m$ is a motion (and therefore the two momenta at $q^a$ are equal). Thus the condition can be reformulated as

$$\frac{\partial S(q^a, q^b, q^c)}{\partial q^a} = 0.$$ \hspace{1cm} (48)

This equation is easily recognized as a corollary of the principle that the action is an extremum on the motions.

Consider now the same question—whether or not we can observe the correlation $q^a$ in the quantum theory. The quantum theory does not provide a deterministic yes/no answer to a physical question; it only provides the probability (amplitude) for a positive answer. In the classical theory, the two correlations $q^a_i$ and $q^a_f$ determine a state. In the quantum theory, a (generalized) state of a finite dimensional system is determined by a single point in $C$, because the detection of the correlation $q^a_i$ is incompatible with the detection of the correlation $q^a_f$, in the sense of Heisenberg: the detection of the second erases the information on the first. If the state is the generalized state determined by the correlation $q^a_i$, then the probability density amplitude for detecting the correlation $q^a$ is determined by the propagator $W(q^a, q^c)$, defined on $G$. See [15] for details. Since $S(q^a, q^c)$ is the phase of the propagator, equations (46) follow then from the standard optical approximation for the behavior of the wave packets. In this sense the classical interpretation of the formalism can be recovered from the quantum one.

3 Field theory

3.1 Field theoretical relativistic mechanics

Consider a field theory on Minkowski space $M$. Let $x^\mu$, where $\mu = 0, 1, 2, 3$, be Minkowski coordinates and call $\phi^A(x^\mu)$ the field, where $A = 1, \ldots, N$. The field is a function $\phi : M \to T$, where $T = \mathbb{R}^N$ is the target space, namely the space in which the field takes values. The extended configuration space of this theory is the finite dimensional space $C = M \times T$, with coordinates $q^a = (x^\mu, \phi^A)$. In fact, the coordinates of this space correspond to the $(4+N)$ partial observables whose relations are described by the theory [5, 10]. A solution of the equations of motion defines a four-dimensional surface $\tilde{m}$ in $C$. If we coordinatize this surface using the coordinates $x^\mu$, then this surface is given by $[x^\mu, \phi^A(x^\mu)]$, where $\phi^A(x^\mu)$ is a solution of the field equations. If, alternatively, we use an arbitrary parametrization with parameters $\tau^\rho, \rho = 0, 1, 2, 3$, then the surface is given by $[x^\mu(\tau^\rho), \phi^A(\tau^\rho)]$, and $\phi^A(x^\mu)$ is determined by $\phi^A(x^\mu(\tau^\rho)) = \phi^A(\tau^\rho)$.

In the case of a finite number of degrees of freedom (and no gauges), motions are given by one-dimensional curves. At each point of the curve, there is one tangent vector, and momenta coordinatize the one-forms. In field theory, motions are four-dimensional surfaces, and have four independent tangents at each point. Accordingly, momenta coordinatize the four-forms. Let $\Omega = \Lambda^4 T^*C$, be the bundle of the four-forms $p_{abcd}dq^a \wedge dq^b \wedge dq^c \wedge dq^d$ over
A point in $\Omega$ is thus a pair $(q^\mu, p_{\mu\nu\rho\sigma})$. The space $\Omega$ carries the canonical four-form $\theta_\Omega = p_{\mu\nu\rho\sigma} dq^\mu \wedge dq^\nu \wedge dq^\rho \wedge dq^\sigma$. It is convenient to use the notation $p_{\mu\nu\rho\sigma} = \pi_{\mu\nu\rho\sigma}$ and $p_{A\mu\nu\rho\sigma} = \pi_{A\mu\nu\rho\sigma}$.

The Hamiltonian theory can be defined on $\Omega$ by the relativistic Hamiltonian

$$H = \pi + H_0(x^\mu, \phi^A, p_A) = 0, \quad (49)$$

where $H_0$ is DeDonder’s covariant Hamiltonian (see below for an example). This system defines a surface $\Sigma$ in $\Omega$. It is convenient to take coordinates $(x^\mu, \phi^A, p_A)$ on $\Sigma$. As before, we denote $\theta$ the restriction of $\theta_\Omega$ to $\Sigma$ and $\omega = d\theta$.

On the surface defined by (49), $\theta_\Sigma$ becomes the canonical four-form

$$\theta = \pi d^4x + p_A^\mu d\phi^A \wedge d^3 x_\mu, \quad (51)$$

where we have introduced the notation $d^4x = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$ and $d^3x_\mu = d^4x(\partial_\mu) = \frac{1}{4!}\epsilon_{\mu\nu\rho\sigma} dx^\nu \wedge dx^\rho \wedge dx^\sigma$. On $\Sigma$, defined by (49) and (50),

$$\theta = -H_0(x^\mu, \phi^A, p_A^\mu) d^4x + p_A^\mu d\phi^A \wedge d^3 x_\mu, \quad (52)$$

and $\omega$ is the five-form

$$\omega = -dH_0(x^\mu, \phi^A, p_A^\mu) \wedge d^4x + dp_A^\mu \wedge d\phi^A \wedge d^3 x_\mu. \quad (53)$$

An orbit of $\omega$ is a four-dimensional surface $m$ immersed in $\Sigma$, such that at each of its points a quadruplet $X = (X_1, X_2, X_3, X_4)$ of independent tangents to the surface satisfies

$$\omega(X) = 0. \quad (54)$$

The projection of an orbit on $C$ gives a solution of the field equations.

More in detail, let $(\partial_\mu, \partial_A, \partial_A^\mu)$ be the basis in the tangent space of $\Sigma$ determined by the coordinates $(x^\mu, \phi^A, p_A^\mu)$. Parametrize the surface with arbitrary parameters $\tau^\rho$, so that the surface is given by $[x^\mu(\tau^\rho), \phi^A(\tau^\rho), p_A^\mu(\tau^\rho)]$. Let $\partial_\mu = \partial/\partial\tau^\mu$. Then let

$$X_\mu = \partial_\mu x^\mu(\tau^\rho) \partial_\mu + \partial_\rho \phi^A(\tau^\rho) \partial_A + \partial_\rho p_A^\mu(\tau^\rho) \partial_A^\mu. \quad (55)$$

Then $X = X_0 \otimes X_1 \otimes X_2 \otimes X_3$ is a rank four tensor on $\Sigma$. If $\omega(X) = 0$, then $\phi^A(x^\mu)$ determined by $\phi^A(x^\mu) = \phi^A(\tau^\rho)$ is a solution of the equations of motion.

The formalism as well as its interpretation can be immediately generalized to the case in which the coordinates of $C$ do not split into $x^\mu$ and $\phi^A$ and the relativistic Hamiltonian does not have the particular form (49-50) (10).

**Example: scalar field**

As an example, consider a scalar field $\phi(x^\mu)$ on Minkowski space, satisfying the field equations

$$\partial_\mu \partial^\mu \phi(x^\mu) + m^2 \phi(x^\mu) + V'(\phi(x^\mu)) = 0. \quad (56)$$
Here the Minkowski metric has signature $[+,-,-,-]$ and $V'(\phi) = dV(\phi)/d\phi$. The field is a function $\phi : M \rightarrow T$, where here $T = \mathbb{R}$. The extended configuration space of this theory is the five dimensional space with coordinates $(x^\mu, \phi)$. The space $\Omega$ has coordinates $(x^\mu, \phi, \pi, p^\mu)$ (equation (49) is trivially satisfied) and carries the canonical four-form

$$\theta_{\Omega} = \pi \, d^4x + p^\mu \, d\phi \wedge d^3x_\mu; \quad (57)$$

The dynamics is defined on this space by the DeDonder relativistic hamiltonian

$$H_0 = \frac{1}{2} \left( p^\mu p_\mu + m^2 \phi^2 + 2V(\phi) \right). \quad (58)$$

The form $\omega$ is thus the five-form

$$\omega = - \left( p^\mu dp_\mu + m^2 \phi d\phi + V'(\phi) d\phi \right) \wedge d^4x + dp^\mu \wedge d\phi \wedge d^3x_\mu. \quad (59)$$

A straightforward calculation shows that $\omega(X) = 0$ gives

$$\partial_\mu \phi(x^\mu) = p_\mu(x^\mu), \quad (60)$$
$$\partial_\mu p^\mu(x^\mu) = -m^2 \phi(x^\mu) - V'(\phi(x^\mu)). \quad (61)$$

and therefore precisely the field equations (56). Notice that the formalism is manifestly Lorentz covariant, and that no equal time initial data surface has to be chosen.

### 3.2 The space $\mathcal{G}$ and the physical symplectic structure

The phase space $\Gamma$ is defined as the space of the orbits, as in the finite dimensional case. However, notice that now there is no natural projection map $\pi$ from $\Sigma$ to $\Gamma$, because a point in $\Sigma$ may belong to many different orbits. It follows that we cannot define a symplectic two-form on the phase space $\Gamma$ by simply requiring that its pull back with $\pi$ is $\omega$. As we shall see now, however, the problem can be circumvented.

The key step is to identify the space $\mathcal{G}$. Recall that in the finite dimensional case $\mathcal{G}$ was the Cartesian product of the extended configuration space with itself. The same cannot be true in the field theoretical context, because the proper characterization of $\mathcal{G}$ is as the space of the boundary configuration data that can specify a solution. In field theory, we obviously need an infinite number of data to characterize a solution, therefore $\mathcal{G}$ must be infinite dimensional. The key observation is that in the finite dimensional case $\mathcal{G}$ is the space of the possible boundaries of a portion of a motion in $\mathcal{C}$. In the field theoretical context, a portion of a motion is a 4d surface in $\mathcal{C}$ with boundaries. Its boundary is a three-dimensional surface $\gamma$. The surface $\gamma$ bounds a four-dimensional surface $\tilde{m}$, and therefore has no boundaries itself.

Thus, we take $\mathcal{G}$ to be a space of oriented three-dimensional surfaces $\gamma$ without boundaries in $\mathcal{C}$. The 3d surface $\gamma$ does not need to be connected. In fact,
it is sometimes convenient to think at \( \gamma \) as having two connected components: the initial component and the final component.

Let us coordinatize \( \gamma \) with coordinates \( \vec{\tau} = (\tau^1, \tau^2, \tau^3) \). Then \( \gamma \) is given as

\[
\gamma = [x^\mu(\vec{\tau}), \phi^A(\vec{\tau})].
\]

Notice that \( x^\mu(\vec{\tau}) \) defines a 3d surface without boundaries in Minkowski space, which we call \( \gamma_M \), while \( \phi^A(\vec{\tau}) \) determines the value of the field on this surface. The surface in Minkowski space \( \gamma_M \) is the boundary of a connected region \( V_M \) of \( M \). A solution of the equation of motion is determined by the value of the field on the boundary. (This is the generic situation, since if two solutions agree on a closed 3d surface, generically they agree in the interior.) Thus, \( \gamma \) determines a solution \( \tilde{m} \) of the equations of motion in the interior \( V_M \).

Furthermore, let \( m \) be the lift of \( \tilde{m} \) to \( \Sigma \). That is, let \( m \) be the portion of an orbit of \( \omega \) that projects down to \( \tilde{m} \). Finally, let \( s_\gamma \) be the 3d surface in \( \Sigma \) that bounds \( m \). That is, \( s_\gamma = [x^\mu(\vec{\tau}), \phi^A(\vec{\tau}), p^\mu_A(\vec{\tau})] \), where \( p^\mu_A(\vec{\tau}) \) is determined by the solution of the field equations determined by the entire \( \gamma \).

We can now define a two-form on \( G \) as follows

\[
\omega_G[\gamma] = \int_{s_\gamma} \omega.
\]  

(62)

The form \( \omega_G \) is a two-form: it is the integral of a five-form over a 3d surface. More precisely, let \( \delta \gamma \) be a small variation of \( \gamma \). This variation can be seen as a vector field \( \delta \gamma(\vec{\tau}) \) defined on \( \gamma \). This variation determines a corresponding small variation \( \delta s_\gamma \), which, in turn, is a vector field \( \delta s_\gamma(\vec{\tau}) \) over \( s_\gamma \). Then

\[
\omega_G[\gamma](\delta_1 \gamma, \delta_2 \gamma) = \int_{s_\gamma} \omega(\delta_1 s_\gamma, \delta_2 s_\gamma).
\]  

(63)

Thus, the five-form \( \omega \) on the finite dimensional space \( \Sigma \) defines the two-form \( \omega_G \) on the infinite dimensional space \( G \).

Now, consider a small local variation \( \delta \gamma \) of \( \gamma \). This means varying the surface \( \gamma_M \) in Minkowski space, as well as varying the value of the field over it. Assume that this variation satisfies the field equations: that is, the variation of the field is the correct one, for the solution of the field equations determined by \( \gamma \). We have

\[
\omega_G[\gamma](\delta \gamma) = \int_{s_\gamma} \omega(\delta s_\gamma).
\]  

(64)

But the variation \( \delta s_\gamma \) is by construction along the orbit, namely in the null direction of \( \omega \) and therefore the right hand side of this equation vanishes. It follows that if \( \delta \gamma \) is an infinitesimal physical motion, then

\[
\omega_G(\delta \gamma) = 0.
\]  

(65)

In conclusion, the pair \((G, \omega_G)\) contains all the relevant information on the system. The null directions of \( \omega_G \) determine the variations of the three-surfaces \( \gamma \) along the physical motions. The space \( G \) divided by these null directions, namely the space of the orbits of these variations is the physical phase space \( \Gamma \), and the \( \omega_G \), restricted to this space, is the physical symplectic two-form of the system.
Example: scalar field

Let us now compute $\omega_G$ in a slightly more explicit form for the example of the scalar field. From the definition (62),

$$
\omega_G[\gamma] = \int_{s_\gamma} \omega = \int_{s_\gamma} d\pi \wedge d^4x + dp^\mu \wedge d\phi \wedge d^3x_\mu
$$

$$
= \int_{s_\gamma} (p^\nu dp_\nu + \phi d\phi + V' d\phi) \wedge d^4x + dp^\mu \wedge d\phi \wedge d^3x_\mu
$$

$$
= \int_{\gamma_M} d^3x_\nu \left( (p_\mu - \partial_\mu \phi) dp^\mu \wedge dx^\nu + (\phi + V' + \partial_\mu p^\mu) d\phi \wedge dx^\nu + dp^\nu \wedge d\phi \right)
$$

$$
= \int_{\gamma_M} d^3x_\nu \ dp^\nu \wedge d\phi.
$$

(66)

where we have used the $x^\mu$ coordinates themselves as integration variables, and therefore the integrand fields are the functions of the $x^\mu$’s. Notice that since the integral is on $s_\gamma$, the $p^\mu$ in the integrand is the one given by the solution of the field equation determined by the data on $\gamma$. Therefore it satisfies the equations of motion (60-61), which we have used above. Using (60) again, we have

$$
\omega_G[\gamma] = \int_{\gamma_M} d^3x \ n_\nu \ d(\nabla^\nu \phi) \wedge d\phi.
$$

(67)

In particular, if we consider variations $\delta \gamma$ that do not move the surface and such that the change of the field on the surface is $\delta \phi(x)$, we have

$$
\omega_G[\gamma](\delta_1 \gamma, \delta_2 \gamma) = \int_{\gamma_M} d^3x \ n_\nu \left( \delta_1 \phi \nabla^\nu \delta_2 \phi - \delta_2 \phi \nabla^\nu \delta_1 \phi \right).
$$

(68)

This formula can be directly compared with the expression of the symplectic two-form given on the space of the solutions of the field equations in [1]. The expression is the same, but with a nuance in the interpretation: $\omega_G$ is not defined on the space of the solutions of the field equations – it is defined on the space of the lagrangian data $\mathcal{G}$, and the normal derivative $n_\nu \nabla^\nu \phi$ of these data is determined by the data themselves via the field equations.

3.3 Hamilton-Jacobi

Let us now construct the function $S$ on $\mathcal{G}$. We define as in the finite dimensional case

$$
S[\gamma] = \int_{m_\gamma} \theta.
$$

(69)

Again, it is easy to see that this is in fact the value of the action of the solution $\tilde{m}_\gamma$. For the scalar field, for instance

$$
S[\gamma] = \int_{m_\gamma} \theta = \int_{m_\gamma} (\pi d^4x + p^\mu d\phi \wedge d^3x_\mu) = \int_{V_\gamma} (\pi + p^\mu \partial_\mu \phi) d^4x
$$

16
\[
\int_{V} \left( -\frac{1}{2} p^{\mu} p_{\mu} - \frac{1}{2} m^2 \phi^2 - V(\phi) + p^{\mu} \partial_{\mu} \phi \right) d^4 x \tag{70}
\]
\[
= \int_{V} \left( \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{2} m^2 \phi^2 - V(\phi) \right) d^4 x
\]
\[
= \int_{V} L(\phi, \partial_{\mu} \phi) d^4 x, \tag{71}
\]

where \( L \) is the Lagrangian density, and we have used the equation of motion \( p_{\mu} = \partial_{\mu} \phi \).

We have from the definition
\[
\frac{\delta S[\gamma]}{\delta x^\mu(\vec{\tau})} = \pi(\vec{\tau}) n_{\mu}(\vec{\tau}) + \epsilon_{\mu\nu\rho\sigma} p^{\nu}(\vec{\tau}) \partial_{\rho} \phi(\vec{\tau}) \partial_{\sigma} x^\sigma(\vec{\tau}) \epsilon^{ijk} \tag{72}
\]
where \( \pi \) depends on the full \( \gamma \), and \( n_{\mu}(\vec{\tau}) = \frac{1}{3!} \epsilon_{\mu\nu\rho\sigma} \partial_{\nu} x^\nu(\vec{\tau}) \partial_{\rho} x^\rho(\vec{\tau}) \partial_{\sigma} x^\sigma(\vec{\tau}) \) is the normal to the three-surface \( \gamma_M \). Also
\[
\frac{\delta S[\gamma]}{\delta \phi(\vec{\tau})} = p^\mu(\vec{\tau}) n_{\mu}(\vec{\tau}). \tag{73}
\]

The derivation of these two equations requires steps analogous to the one we used to derive (31). See the appendix for details.

Now, from (50) and (58) we have, for the scalar field
\[
\pi + \frac{1}{2} \left( p^\mu p_{\mu} + m^2 \phi^2 + 2V(\phi) \right) = 0. \tag{74}
\]

We split \( p_{\mu} \) in its normal \( (p = p^\mu n_{\mu}) \) and tangential \( (p^i) \) components (so that \( p^\mu = p^i \partial_i x^\mu + pn_{\mu} \)) and from (72) we have
\[
n^\mu(\vec{\tau}) \frac{\delta S[\gamma]}{\delta x^\mu(\vec{\tau})} = \pi(\vec{\tau}) - p^i(\vec{\tau}) \partial_i \phi(\vec{\tau}). \tag{75}
\]

Using this, (74), and the field equations (72), we obtain
\[
\frac{\delta S[\gamma]}{\delta x^\mu(\vec{\tau})} n_{\mu}(\vec{\tau}) + \frac{1}{2} \left[ \left( \frac{\delta S[\gamma]}{\delta \phi(\vec{\tau})} \right)^2 + \partial_j \phi(\vec{\tau}) \partial^j \phi(\vec{\tau}) + m^2 \phi^2(\vec{\tau}) + 2V(\phi(\vec{\tau})) \right] = 0. \tag{76}
\]

This is the Hamilton-Jacobi equation of the theory. Notice that the function \( S[\gamma] = S[\chi^\mu(\vec{\tau}), \phi(\vec{\tau})] \) is a function of the surface, not the way the surface is parametrized. Therefore it is invariant under a change of parametrization. It follows that
\[
\frac{\delta S[\gamma]}{\delta x^\mu(\vec{\tau})} \partial_j x^{\mu}(\vec{\tau}) + \frac{\delta S[\gamma]}{\delta \phi(\vec{\tau})} \partial_j \phi(\vec{\tau}) = 0. \tag{77}
\]

(This equation can be obtained also from the tangential component of (72).) The two equations (70) and (71) govern the Hamilton-Jacobi function \( S[\gamma] \).
The connection with the non-relativistic field theoretical Hamilton-Jacobi formalism is the following. We can restrict the formalism to a preferred choice of parameters \( \overline{\tau} \). Choosing \( \tau_j = x^j \), we obtain \( S \) in the form \( S[t(x), \phi(x)] \) and the Hamilton-Jacobi equation (76) becomes

\[
\frac{\delta S}{\delta t(x)} + \frac{1}{2} \left[ \left( \frac{\delta S}{\delta \phi(x)} \right)^2 + \partial_j \phi \partial^j \phi + m^2 \phi^2 + 2V(\phi) \right] = 0.
\]

Further restricting the surfaces to the ones of constant \( t \) gives the functional \( S[t, \phi(x)] \), satisfying the Hamilton-Jacobi equation

\[
\frac{\partial S}{\partial t} + \frac{1}{2} \int d^3 \vec{x} \left[ \left( \frac{\delta S}{\delta \phi(x)} \right)^2 + |\nabla \phi|^2 + m^2 \phi^2 + 2V(\phi) \right] = 0,
\]

which is the usual non-relativistic Hamilton-Jacobi equation

\[
\frac{\partial S}{\partial t} + H(\phi, \nabla \phi, \partial_t \phi) = 0,
\]

where \( H(\phi, \nabla \phi, \partial_t \phi) \) is the non-relativistic hamiltonian.

### 3.4 Physical predictions

As in the case of finite dimensional systems, if \( S[\gamma] = S[x^\mu(\overline{\tau}), \phi(\overline{\tau})] \) is known explicitly, the general solution of the equation of motion can be obtained by derivations. For instance, let \( \gamma \) be formed by two connected components that can be viewed as a past and a future Cauchy surfaces \( \gamma_{\text{in}} \) and \( \gamma_{\text{out}} \), parametrized by \( \overline{\tau}_{\text{in}} \) and \( \overline{\tau}_{\text{out}} \) respectively. Consider the equation

\[
\frac{\delta S[\gamma_{\text{out}} \cup \gamma_{\text{in}}]}{\delta \phi(\overline{\tau}_{\text{in}})} = \delta S[\tilde{\gamma}_{\text{out}} \cup \gamma_{\text{in}}] \quad \delta \phi(\overline{\tau}_{\text{in}})
\]

for the variable \( \tilde{\gamma}_{\text{out}} \), where \( \gamma_{\text{in}} \) and \( \gamma_{\text{out}} \) are held fix. All the solutions \( \tilde{\gamma}_{\text{out}} \) of this equation sit on the same motion. That is, this equation determines which are the \( \tilde{\gamma}_{\text{out}} \) that are compatible with a given state. The situation is completely analogous to the finite dimensional case.

However, these are not the most interesting physical predictions, because operationally well defined observables are local in spacetime. In order to deal with these, fix a surface \( \gamma \) that determines a motion (for instance, formed by two parallel Cauchy surfaces), and consider a single correlation \( (x^\mu, \phi^A) \) in \( \mathcal{C} \). A well posed question is whether or not the point \( (x^\mu, \phi^A) \) sits on the motion defined by \( \gamma \). That is, whether or not the value of the field at \( x^\mu \) is \( \phi^A \), on the solution of the field equations determined by the boundary conditions \( \gamma \). To answer this question in the Hamilton-Jacobi formalism, observe that there exist a surface \( \gamma \cup (x^\mu, \phi^A) \) in \( \mathcal{G} \) and we can consider \( S[\gamma \cup (x^\mu, \phi^A)] \). More precisely, pick a small \( \epsilon \) and let \( B_\epsilon^x \) be a 3d surface with radius \( \epsilon \) surrounding
the point \( x^\mu \) in \( M \). Let \( \gamma^\epsilon_{(x^\nu, \phi^A)} \) be the 3d surface in \( C \) defined by the constant value \( \phi^A \) and by \( x^\mu \in B^\epsilon_{x^\nu} \). Then

\[
S[\gamma \cup (x^\mu, \phi^A)] = \lim_{\epsilon \to 0} S[\gamma \cup \gamma^\epsilon_{(x^\nu, \phi^A)}]. \tag{82}
\]

Using this definition, \( (x^\mu, \phi^A) \) is on the motion determined by \( \gamma \) iff

\[
\frac{\delta S[\gamma \cup (x^\mu, \phi^A)]}{\delta \phi(\bar{\tau})} = \frac{\delta S[\gamma]}{\delta \phi(\bar{\tau})} \tag{83}
\]

where \( \bar{\tau} \) parametrizes \( \gamma \). This is the field theoretical generalization of (47). This can be generalized to an arbitrary number of correlations \((x^\mu_1, \phi^A_1), \ldots, (x^\mu_n, \phi^A_n)\).

These are compatible with the initial data \( \gamma \) iff

\[
\frac{\delta S[\gamma \cup (x^\mu_1, \phi^A_1) \cup \ldots \cup (x^\mu_n, \phi^A_n)]}{\delta \phi(\bar{\tau})} = \frac{\delta S[\gamma]}{\delta \phi(\bar{\tau})} \tag{84}
\]

In fact, it is clear that if the correlations \((x^\mu_j, \phi^A_j)\) are on \( m \), then the insertion does not change the momenta on \( \gamma \). Thus, \( \gamma \) determines a state, and equation (83) determines the correlations in the extended configuration space \( C \) that are compatible with this state. Clearly equation (83) is the field theoretical generalization of (47). Alternatively, we can write, as in (48),

\[
\frac{\partial S[\gamma \cup (x^\mu, \phi^A)]}{\partial \phi^A} = 0. \tag{85}
\]

In conclusion, there are local predictions of the theory. Given a state, the theory can predict whether or not individual correlations (points in \( C \)), or sets of correlations, can be observed.

4 General Relativity

4.1 Covariant hamiltonian formulation

General relativity can be formulated on the finite dimensional configuration space \( \tilde{C} \) with coordinates \((x^\mu, A^I_\mu)\). (See [18], [10] and [20].) Here \( i = 1, 2, 3 \) and \( A^I_\mu \) is a complex matrix. We raise and lower the \( i,j,\ldots \) indices with \( \delta_{ij} \).

Assuming immediately (19), the corresponding space \( \Omega \) has coordinates \((x^\mu, A^I_\mu, \pi, p^{\mu\nu})\) and carries the canonical four-form

\[
\theta_\Omega = \pi \, d^4x + p^{\mu\nu} \, dA^I_\nu \wedge d^3x_\mu. \tag{86}
\]

It is convenient to introduce the following notation. We define the gauge covariant differential on all quantities with internal indices as

\[
Dv^i = dv^i + \epsilon^i_{jk} A^j_\mu p^k d x^{\mu}. \tag{87}
\]
so that, in particular,

$$DA_i^\mu = dA_i^\mu + \epsilon_{jk} A_j^\nu A_\mu^k dx^\nu.$$  \hfill (88)

Using this notation, the canonical form (86) reads

$$\theta_\Omega = p d^4x + p_{\mu\nu}^i DA_i^\mu \wedge d^3x_\nu.$$  \hfill (89)

where $p = \pi - p_{\mu\nu}^i A_j^\nu A_k^\mu \epsilon_{ijk}$. We also define

$$E_i^\mu = \epsilon_{\mu\nu\rho\sigma} \delta^{ij} p_{ij}^{\rho\sigma}$$  \hfill (90)

and the forms $A^i = A_i^\mu dx^\mu, DA_i^\mu = dA_i^\mu \wedge dx^\mu + A_j^\nu A_k^\mu \epsilon_{ijk} dx^\nu \wedge dx^\mu, E^i = E_i^\mu dx^\mu \wedge dx^\nu$, and so on, on $\Omega$.

General relativity is defined by the Hamiltonian system

\begin{align*}
p &= 0, \quad \hfill (91) \\
p_{\mu\nu}^i + p_{\nu\mu}^i &= 0, \quad \hfill (92) \\
\bar{E}^i \wedge E^j &= 0 \quad \hfill (93)
\end{align*}

Let me now show that this indeed general relativity. The key point is that the constraints (93), (94) imply that there exists a real four by four matrix $e^I_\mu$, where $I = 0, 1, 2, 3$, such that $E_i^\mu$ is the self-dual part of $e^I_\mu e^J_\nu$. This means the following. Let $P^{ij}_{IJ}$ be the self-dual projector, that is

\begin{align*}
P^{ij}_{jk} &= \epsilon_{jk}, \quad \hfill (95) \\
P^{ij}_{j0} &= -P^{ij}_{0j} = i \delta^i_j. \quad \hfill (96)
\end{align*}

Then it is easy to check that (93) and (94) are solved by

$$E_i^\mu = P^{ij}_{IJ} e^J_\nu \wedge e^i_\mu.$$  \hfill (97)

and the counting of degrees of freedom indicates that this is the sole solution.

Therefore we can use the coordinates $(x^\mu, A_i^\mu, e^I_\mu)$ on the constraint surface $\Sigma$ (where $A_i^\mu$ is complex and $e^I_\mu$ is real) and the induced canonical four-form is simply

$$\theta = P_{IJ} e^I \wedge e^J \wedge DA_i^\mu.$$  \hfill (98)

Indeed, the orbits $(x^\mu, A_i^\mu(x^\mu), e^I_\mu(x^\mu))$ of $\omega = d\theta$ satisfy the Einstein equations, in the form

\begin{align*}
e^I \wedge (de_J + P_{JKi} A_i^J \wedge e^K) &= 0, \quad \hfill (99) \\
P_{IJ} e^I \wedge e^J \wedge F^i &= 0, \quad \hfill (100)
\end{align*}

where $F^i_{\mu\nu}$ is the curvature of $A_i^\mu$. From these equations it follows that $g_{\mu\nu}(x) \equiv \eta_{IJ} e^I_\mu(x)e^J_\nu(x)$ is Ricci flat. The demonstration is a straightforward calculation.

Thus, rather remarkably, the simple and natural form (83), defined on the finite dimensional space $\Sigma$ with coordinates $(x^\mu, A_i^\mu, e^I_\mu)$, defines general relativity entirely.
4.2 Hamilton-Jacobi equation and the $S$ solution

Let $\gamma$ be a three-dimensional surface in $\tilde{\mathcal{C}}$. Thus $\gamma = [x^\mu(\vec{\tau}), A^i_\mu(\vec{\tau})]$, where $\vec{\tau} = (\tau^1, \tau^2, \tau^3, \tau^0)$. Define the functional

$$S[\gamma] = \int_{m_{\gamma}} \theta,$$

as above. That is, $m$ is the four-dimensional surface in $\Sigma$ which is (part of) an orbit of $d\theta$, and therefore a solution of the field equations, and such that the projection of its boundary to $\tilde{\mathcal{C}}$ is $\gamma$. From the definition,

$$\frac{\delta S[\gamma]}{\delta A^i_\mu(\vec{\tau})} = P_{\mu JK} e^{\mu \rho \sigma} e^i_\rho(\vec{\tau}) e^K_\sigma(\vec{\tau}) n_\nu(\vec{\tau}).$$

(102)

Since from this equation we have immediately

$$n_\mu(\vec{\tau}) \frac{\delta S[\gamma]}{\delta A^i_\mu(\vec{\tau})} = 0,$$

(103)

it follows that the dependence of $S[\gamma]$ on $A^i_\mu(\vec{\tau})$ is only through the restriction of $A^i(\vec{\tau})$ to the three-surface $\gamma_M$. That is, only through the components

$$A^i_\mu(\vec{\tau}) = \partial_a X^\mu(\vec{\tau}) A^i_a(\vec{\tau}).$$

(104)

Thus

$$S = S[x^\mu(\vec{\tau}), A^i_a(\vec{\tau})]$$

and

$$\frac{\delta S[\gamma]}{\delta A^i_a(\vec{\tau})} = P_{ijkl} e^{\alpha \beta \gamma \delta} \partial_\alpha X^\rho(\vec{\tau}) \partial_\beta X^\sigma(\vec{\tau}) e^i_\rho(\vec{\tau}) e^j_\sigma(\vec{\tau}) n_\nu(\vec{\tau}) \equiv i E^a_i(\vec{\tau}).$$

(105)

The projection of the field equations (100) on $\gamma_M$, written in terms of $E^a_i$, read $D_a E^a_i = 0$, $F^i_{ab} E^a_i = 0$ and $F^i_{ab} E^a_i E^{bc} \epsilon^{ijk} = 0$, where $D_a$ and $F^i_{ab}$ are the covariant derivative and the curvature of $A^i_a$. Using (105) these give the three equations

$$D_a \frac{\delta S[\gamma]}{\delta A^i_a(\vec{\tau})} = 0,$$

(106)

$$\frac{\delta S[\gamma]}{\delta A^i_a(\vec{\tau})} F^i_{ab} = 0,$$

(107)

$$\epsilon^{ijk} F^i_{ab}(\vec{\tau}) \frac{\delta S[\gamma]}{\delta A^i_a(\vec{\tau})} \frac{\delta S[\gamma]}{\delta A^i_b(\vec{\tau})} = 0.$$

(108)

These equations have a well known interpretation. In fact, the first could have been obtained by simply observing that $S[\gamma]$ is invariant under local $SU(2)$ gauge transformations on the three-surface. Under one such transformation generated by a function $f^i(\vec{\tau})$ the variation of the connection is $\delta f^i A^i_a = D_a f^i$. Therefore $S$ satisfies

$$0 = \delta f S = \int d^3 \vec{\tau} \delta f A^i_a(\vec{\tau}) \frac{\delta S[\gamma]}{\delta A^i_a(\vec{\tau})} = \int d^3 \vec{\tau} D_a f^i(\vec{\tau}) \frac{\delta S[\gamma]}{\delta A^i_a(\vec{\tau})}$$

(109)

$$= - \int d^3 \vec{\tau} f^i(\vec{\tau}) D_a \frac{\delta S[\gamma]}{\delta A^i_a(\vec{\tau})}.$$
This gives (106). Next, the action is invariant under a change of coordinates on the three surface $\gamma_M$. Under one such transformation generated by a function $f^a(\vec{\tau})$ the variation of the connection is
\[ \delta f A^i_a = f^b \partial_b A^i_a + A^j_b \partial_a f^i. \]
Integrating by parts as in (110) this gives
\[ \partial_b A^i_a \frac{\delta S[\gamma]}{\delta A^i_a(\vec{\tau})} + (\partial_b A^i_a) \frac{\delta S[\gamma]}{\delta A^i_a(\vec{\tau})} = 0, \tag{110} \]
which, combined with (106) gives (107). Thus, (106) and (107) are simply the requirement that $S[\gamma]$ is invariant under internal gauge and changes of coordinates on the three-surface. The three equations (106), (107) and (108) govern the dependence of $S$ on $A^i_a(\vec{\tau})$.

On the other hand, it is easy to see that $S$ is independent from $x^\mu(\vec{\tau})$. A change of coordinates $x^\mu(\vec{\tau})$ tangential to the surface cannot affect the action, which is independent from the coordinates used. More formally, the invariance under change of parameters $\vec{\tau}$ implies
\[ \frac{\delta S[\gamma]}{\delta x^\mu(\vec{\tau})} \partial_j x^\mu(\vec{\tau}) = \frac{\delta S[\gamma]}{\delta A^i_a(\vec{\tau})} \delta A^i_a(\vec{\tau}), \tag{111} \]
and we have already seen that the right hand side vanishes. The variation of $S$ under a change of $x^\mu(\vec{\tau})$ normal to the surface is governed by the Hamilton-Jacobi equation proper, Equation (76). In the present case, following the same steps as for the scalar field, we obtain
\[ \frac{\delta S[\gamma]}{\delta x^\mu(\vec{\tau})} n^\mu(\vec{\tau}) + \epsilon_{ijk} F^i_{ab} \frac{\delta S[\gamma]}{\delta A^j_a(\vec{\tau})} \frac{\delta S[\gamma]}{\delta A^k_b(\vec{\tau})} = 0. \tag{112} \]
But the second term vanishes because of (108). Therefore $S[\gamma]$ is independent from tangential as well as normal parts of $x^\mu(\vec{\tau})$: $S$ depends only on $[A^i_a(\vec{\tau})]$.

We can thus drop altogether the spacetime coordinates $x^\mu$ from the extended configuration space. Define a smaller extended configuration space $\mathcal{C}$ as the 9d complex space of the variables $A^i_a$. Geometrically, this can be viewed as the space of the linear mappings $A : D \to \text{sl}(2,C)$, where $D = R^3$ is a “space of directions” and we have chosen the complex selfdual basis in the $\text{sl}(2,C)$ algebra. We then identify the space $\mathcal{G}$ as a space of parametrized three-dimensional surfaces $[A^i_a(\vec{\tau})]$ without boundaries in $\mathcal{C}$, modulo reparametrizations – where, however, two parametrized surfaces are considered equivalent if $A_a^i(\vec{\tau}) = \frac{\partial x^a}{\partial \tau^i} A^i_a(\vec{\tau})$. The dynamics of the theory is entirely contained in the equations (106), (107) and (108) for the functional $S[A^i_a(\vec{\tau})]$ on $\mathcal{G}$.

It is then immediate to obtain the dynamical equation of the quantum theory. This is obtained by replacing the functional derivative of $S$ with a functional derivative operator, and acting over a wave function that has the same argument of $S$, namely a wave functional $\Psi[A^i_a(\vec{\tau})]$ on the extended configuration space $\mathcal{C}$. (A different point of view on the use of the covariant formalism in quantum theory is developed in [17].) Equations (106) and (107) do not change and demand that $\Psi[A] = \Psi[A^i_a(\vec{\tau})]$ is invariant under gauge transformations and
changes of coordinates $\tau$, while (108) becomes

$$F_{ab}^{ij}(\tau) \frac{\delta}{\delta A_a^i(\tau)} \frac{\delta}{\delta A_b^j(\tau)} \Psi[A] = 0. \quad (113)$$

This is the Ashtekar-Wheeler-DeWitt equation, which the basic equation of canonical quantum gravity. See for instance [21].

### 4.3 Physical predictions

What are the quantities predicted by the theory in the case of general relativity? Let $\gamma$ determine a motion $m$. Notice that we cannot simply ask whether a single point $c$ of $C$ is in $m$, because of the non-trivial transformation properties of $A^i_a$ under change of parametrization. More precisely, if we want to interpret a point in $C$ as a constant field over the 3d surface of a small ball, as we did for the scalar field case, we run in the difficulty that a constant connection on a three-sphere is trivial. We thus have to look for invariant extended objects in $C$. These can be defined as follows.

Choose a closed unknotted curve $\alpha : s \mapsto (x^\mu(s), A^i_a(s))$ in $C$. Given a motion $m$, we can ask whether or not the set of correlations forming the loop $\alpha$ can be realized. More in general, let $\Gamma$ be a graph (a set of points $p_i$ joined by lines $l_{ij}$) imbedded in $C$. We can ask whether the collection of correlations forming $\Gamma$ is realizable in a given state, determined by a point $\gamma$ in $G$. The answer is positive if

$$\frac{\delta S[\gamma]}{\delta A^i_a(\tau)} = \frac{\delta S[\gamma \cup \Gamma]}{\delta A^i_a(\tau)}. \quad (114)$$

Here $\tau$ parametrizes $\gamma$ and $S[\gamma \cup \Gamma]$ is the integral of $\theta$ over a motion $m_{\gamma \cup \Gamma}$ which has $\gamma$ and $\Gamma$ as boundaries. This means $\Gamma$ is in $m_{\gamma \cup \Gamma}$. More precisely, we can thicken out the $M$ section of the graph $\Gamma$ as we did for the scalar field. Here however we do not obtain a ball of radius $\epsilon$, but rather a sort of “tubular structure”. The boundary of this tubular structure is a three dimensional surface $\Gamma_\epsilon$ in $G$ and we pose $S[\gamma \cup \Gamma] = \lim_{\epsilon \to 0} S[\gamma \cup \Gamma_\epsilon]$.

An important observation follows. Consider a simple curve $\alpha$ in $C$. Define now the “holonomy” of $\alpha$

$$T_\alpha = Tr U_\alpha = Tr Pe^{\int_\alpha ds \, dx^\mu A^i_a(s) \tau^i}, \quad (115)$$

where $\tau_i$ is a basis in the $su(2)$ algebra. Let $\alpha'$ be another closed unknotted curve in $C$, distinct from $\alpha$, but with the same holonomy – that is such that $T_{\alpha'} = T_\alpha$. A moment of reflection shows that, due to the internal gauge and diffeomorphism invariance of $S$, we have

$$S[\gamma \cup \alpha] = S[\gamma \cup \alpha']. \quad (116)$$

Therefore the predictions of the theory do not distinguish $\alpha$ from $\alpha'$, as far as $T_{\alpha'} = T_\alpha \equiv T$. The prediction depend only on $T$. More in general: to any closed
cycle $\alpha = l_{i_1} \cup l_{i_2}$ in $\Gamma$, let the holonomy $T_\alpha$ be defined as in (113). Denote $T_\Gamma = (T_{\alpha_1}, \ldots, T_{\alpha_n})$ the collection of these holonomies. Let $[\Gamma]$ be the knot-class (the equivalence class under diffeomorphisms) to which the restriction of $\Gamma$ to $M$ belongs, and call $s = ([\Gamma], T_\Gamma)$ a “spin-network”. Then to graphs $\Gamma$ and $\Gamma'$ cannot be distinguished if they belong to the same spin-network $s$. That is, we have

$$S[\gamma \cup \Gamma] = S[\gamma \cup \Gamma'] \equiv S[\gamma \cup s].$$

(117)

Therefore what the theory predicts is, for a given state, whether or not a spin network $s$ is realizable. A set of correlations determined by a spin network $s$ is realizable iff

$$\frac{\delta S[\gamma \cup s]}{\delta A^a_i(\vec{r})} = \frac{\delta S[\gamma]}{\delta A^a_i(\vec{r})}.$$ 

(118)

The detection of a given $s$ can be realized in principle as follows (see also [10]). Imagine we set up an experience in which we parallel transport a reference system along finite paths $l_{ij}$ in spacetime. This can be realized macroscopically by transporting a gyroscope and a devise keeping track of local acceleration, or, microscopically, by paralleling transport a particle with spin. For instance a left handed neutrino, whose parallel transport is directly described by the self-dual connection $A^a_i$. We can then compare the result of the parallel transport at the points $p_1$, thus effectively measuring the quantities $T_\alpha$ as angles and relative velocities. The gauge and diffeomorphism invariant information provided by such a measurement is then in the topology of the graph formed by the paths and the invariant values of these relative angles and velocities. This is what is contained in the spin network $s$. In the quantum theory, we expect then a quantum state to determine the probability amplitude for any spin network $s$ to be realizable [25].

5 Conclusions and open issues

I think that a proper understanding of the generally covariant structure of mechanics is necessary in order to make progress in quantum gravity. In this paper I have made several steps in this direction. My focus has been on searching a

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2 One may object that this setting can be physically realized only if the $l_{ij}$ are all timelike and future oriented. However, this is not a serious experimental limitation. First, we have obviously $U_{ij}^{-1} = U_{ij}^{-1}$, therefore future orientation is not a limitation in measurability. Second, the measurement of a spacelike $l_{ij}$ can be obtained in principle as a limit of timelike ones, as in fact we do in practice. That is, divide $l_{ij}$ in a sequence of $N$ (spacelike) segments. For each such spacelike segment, bounded by the points $p_1$ and $p_2$, pick a point $p$ in the common past of both $p_1$ and $p_2$, and consider the two timelike geodesics that go from $p_1$ to $p$ and from $p$ to $p_2$. Then replace the spacelike segment with the union of these two timelike geodesics. It is clear that (dividing $l_{ij}$ and picking the points $p$ appropriately) the parallel transport along the timelike curve obtained in this way converges to the parallel transport along the spacelike curve $l_{ij}$ for large $N$. That is, spacelike measurements can be seen as bookkeeping for results of measurements obtained by timelike motion.
physically viable language for background independent quantum field theory and not in mathematical completeness. From the mathematical point of view, the structures that I have introduced in this paper certainly can (and need to) be refined.

A result of this paper is the derivation of the symplectic structure and the construction of the Hamilton-Jacobi formalism, in a general covariant hamiltonian formulation of mechanics in field theory. The main ingredient for this is the introduction of the space $\mathcal{G}$. The preferred solution $S[\gamma]$ of the Hamilton-Jacobi equation, defined on $\mathcal{G}$, contains the full dynamical information on the system. In finite dimensional systems, the preferred solution $S[\gamma]$ is also the classical limit of the quantum propagator, which contains the full dynamical information on the quantum system in a form which makes sense in a generally covariant context, and has a direct operational interpretation [15]. In the field theoretical context, operationally realistic measurements are local. Outcome of these can be derived from $S[\gamma]$ as well. I think that the precise relation between $S[\gamma]$ and the Wightman amplitudes in quantum field theory deserves to be explored.

As far as the gravitational field is concerned, the covariant hamiltonian formulation described here is remarkably simple. In fact, general relativity is entirely defined on the finite dimensional space with coordinates $(x^\mu, A^a_i, e^I_\mu)$, by the simple and natural form (18) (see also [18]). This formulation leads directly to the basic equation of quantum gravity, and to a classical notion of spin network, which may be of help in clarifying the physical interpretation of the quantum spin networks.

Finally, I think that there should be a proper definition of the extended configuration space $\mathcal{C}$ in which spacetime coordinates $x^\mu$ play no role at all, and with a clean operational interpretation. Here and in [10] I have made some steps in this direction, but I think the matter could be further clarified. Ideally, I think we should get to a clean operational definition of the partial observables of the gravitational field, and of the transition amplitudes between correlations of these. These are the amplitudes that a properly covariant and background independent quantum theory of gravity should allow us to compute.

I thank Igor Kanatchikov for useful comments, corrections on the first draft of this work, and help with the literature on the subject.

Appendix

Here we derive equations (31), (72) and (73). The right hand side of (31) is given by the variation of the boundary in the integral (28). However, this boundary variation is not the only variation to be considered, because if $q^a$ changes, the entire curve $m_\gamma$ may change, becoming a curve $m_{\gamma+\delta\gamma}$. Thus

$$\frac{\partial S(q^a, q_0^a)}{\partial q^a} \delta q^a = \int_{m_{\gamma+\delta\gamma}} \theta - \int_{m_\gamma} \theta.$$ (119)
Consider now the closed line integral of $\theta$ along the path $\alpha$ formed as $\alpha = (m_\gamma, \delta s, (m_\gamma + \delta \gamma)^{-1}, \delta s_0^{-1})$

$$\oint_\alpha \theta = \int_{m_\gamma} \theta + \int_{\delta s} \theta - \int_{m_\gamma + \delta \gamma} \theta - \int_{\delta s_0} \theta.$$  \hspace{1cm} (120)

The path $\alpha$ path bounds a surface (a strip) $\sigma$ which is everywhere tangent to the orbits of $\omega$. Therefore the restriction of $\omega = d\theta$ to $\sigma$ vanishes. Therefore

$$0 = \int_\sigma d\theta = \oint_\alpha \theta = 0.$$  \hspace{1cm} (121)

From (119), (120) and (121), we have

$$\frac{\partial S(q^a, q_0^a)}{\partial q^a} \delta q^a = \int_{\delta s} \theta - \int_{\delta s_0} \theta = p_a \delta q^a - p_0 a \delta q_0^a.$$  \hspace{1cm} (122)

And since we are varying $q^a$ at fixed $q_0^a$, we have (21).

Let us now come to equation (72) and (73). Consider a surface $\gamma$ and a small variation $\delta \gamma = (\delta x^\mu(\vec{\tau}), \delta \phi(\vec{\tau}))$. Consider the five dimensional strip $\sigma$ in $\Sigma$ bounded by $m_\gamma, m_\gamma + \delta \gamma$, and $\delta s$, where $s$ is the boundary of $m$ and $\delta s$ its variation. The five-form $\omega$ vanishes when restricted to $\sigma$ because $\sigma$ is tangent to the orbits of $\omega$. Therefore

$$0 = \int_\sigma \omega = \int_{\partial \sigma} \theta = \int_{m_\gamma + \delta \gamma} \theta - \int_{m_\gamma} \theta - \int_{\delta s} \theta.$$  \hspace{1cm} (123)

By linearity,

$$\int_{m_\gamma + \delta \gamma} \theta - \int_{m_\gamma} \theta = \int d^3 \vec{\tau} \left( \frac{\delta S[\gamma]}{\delta x^\mu(\vec{\tau})} \delta x^\mu(\vec{\tau}) + \frac{\delta S[\gamma]}{\delta \phi(\vec{\tau})} \delta \phi(\vec{\tau}) \right).$$  \hspace{1cm} (124)

From the last two equations, (72) and (73) follow with a short calculation.

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