HECKE ALGEBRAS AND INVOLUTIONS IN COXETER GROUPS

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INTRODUCTION

0.1. Let $W$ be a finitely generated Coxeter group with a fixed involutive automorphism $w \mapsto w^*$ which leaves stable the set of simple reflections. An element $w \in W$ is said to be a $*$-twisted involution if $w^{-1} = w^*$. Let $I = \{w \in W; w^{-1} = w^*\}$ be the set of $*$-twisted involutions of $W$. Let $A' = \mathbb{Z}[v, v^{-1}]$ where $v$ is an indeterminate. In [LV] we have defined (geometrically) an action of the Hecke algebra of $W$ (with parameter $v^2$) on the free $A'$-module $M$ with basis $\{a_w; w \in I\}$, assuming that $W$ is a Weyl group. In [L3] a definition of the Hecke algebra action on $M$ was given in a purely algebraic way, without assumption on $W$. The purpose of this paper is to give a more conceptual approach to the definition of the Hecke algebra action on $M$, based on the theory of Soergel bimodules [S] and on the recent results of Elias and Williamson [EW] in that theory.

In this paper we interpret $M$ as a (modified) Grothendieck group associated to the category of Soergel bimodules corresponding to $W$ and to a 2-periodic functor of this category to itself, defined using $*$ and by switching left and right multiplication in a bimodule. The action of the Hecke algebra appears quite naturally in this interpretation; however, we must find a way to compute explicitly the action of a generator $T_s + 1$ of the Hecke algebra ($s$ is a simple reflection) on a basis element $a_w$ of $M$ so that we recover the formulas of [LV], [L3]. The formula has four cases depending on whether $sw$ is equal to $ws^*$ or not and on whether the length of $sw$ is smaller or larger than that of $w$. In each case, $(T_s + 1)a_w$ is a linear combination $c'a_w + c''a_{w''}$ of two basis elements $a_{w'}, a_{w''}$ where one of $w', w''$ is equal to $w$, the other is $sw$ or $sws^*$ and the length of $w'$ is smaller than that of $w''$. We cannot prove the formulas directly. Instead we compute directly the coefficient $c'$ and then observe that if $c'$ is known, then $c''$ is automatically known from the fact that we have a Hecke algebra action. The computation of $c'$ occupies Sections 4 and 5 (see Theorem 5.2). It has two cases (depending on whether $sw'$ is equal to $w's^*$ or not). The two cases require quite different proofs.
As an application of Theorem 6.2 (which is essentially a corollary of Theorem 5.2) we outline a proof (6.3) of a positivity conjecture (9.12 in [L3]) stating that, if \( y, w \in I \) and \( \delta \in \{1, -1\} \), then the polynomial \( P_{y,w}^\sigma \) introduced in [L3] (and earlier in [LV] in the case of Weyl groups) satisfies \( (P_{y,w}(u) + \delta P_{y,w}(u))/2 \in \mathbb{N}[u] \) where \( P_{y,w} \) is the polynomial introduced in [KL]. This is a refinement of the statement [EW] that \( P_{y,w}(u) \in \mathbb{N}[u] \) which holds for any \( y, w \in W \). In \S 7 we show as another application of our results that \( \mathcal{M} \) admits a filtration by Hecke algebra submodules whose subquotients are indexed by the two-sided cells of \( W \). Under a boundedness assumption we show that the Hecke algebra acts on such a subquotient by something resembling a \( W \)-graph.

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\section{1. 2-PERIODIC FUNCTORS}

\subsection*{1.1. In this section we review some results from [L1, \S 11].}

Let \( k \) be a field of characteristic zero. Let \( \mathcal{C} \) be a \( k \)-linear category, that is a category in which the space of morphisms between any two objects has a given \( k \)-vector space structure such that composition of morphisms is bilinear and such that finite direct sums exist. A functor from one \( k \)-linear category to another is said to be \( k \)-linear if it respects the \( k \)-vector space structures.

Let \( \mathcal{K}(\mathcal{C}) \) be the Grothendieck group of \( \mathcal{C} \) that is, the free abelian group generated by symbols \([A]\) for each \( A \in \mathcal{C} \) (up to isomorphism) with relations \([A \oplus B] = [A] + [B]\) for any \( A, B \in \mathcal{C} \). A \( k \)-linear functor \( M \mapsto M^\sharp, \mathcal{C} \mapsto \mathcal{C} \) is said to be 2-periodic if \( M \mapsto (M^\sharp)^\sharp \) is the identity functor \( \mathcal{C} \mapsto \mathcal{C} \). Assuming that such a functor is given we define a new \( k \)-linear category \( \mathcal{C}_2 \) as follows. The objects of \( \mathcal{C}_2 \) are pairs \((A, \phi)\) where \( A \in \mathcal{C} \) and \( \phi : A^\sharp \to A \) is an isomorphism in \( \mathcal{C} \) such that the composition \( (A^\sharp)^\sharp \xrightarrow{\phi^\sharp} A^\sharp \xrightarrow{\phi} A \) is the identity map of \( A \). Let \((A, \phi), (A', \phi')\) be two objects of \( \mathcal{C}_2 \). We define a \( k \)-linear map \( \text{Hom}_\mathcal{C}(A, A') \to \text{Hom}_\mathcal{C}(A, A') \) by \( f \mapsto f^\sharp := \phi^\sharp f^\sharp \phi^{-1} \). Note that \((f^\sharp)^\sharp = f\). By definition, \( \text{Hom}_\mathcal{C}_2((A, \phi), (A', \phi')) = \{ f \in \text{Hom}_\mathcal{C}(A, A'); f = f^\sharp \} \), a \( k \)-vector space. The direct sum of two objects \((A, \phi), (A', \phi')\) is \((A \oplus A', \phi \oplus \phi')\). Clearly, if \((A, \phi) \in \mathcal{C}_2\), then \((A, -\phi) \in \mathcal{C}_2\).

An object \((A, \phi)\) of \( \mathcal{C}_2 \) is said to be traceless if there exists an object \( B \) of \( \mathcal{C} \) and an isomorphism \( A \cong B \oplus B^\sharp \) under which \( \phi \) corresponds to an isomorphism \( B^\sharp \oplus B \xrightarrow{\cong} B \oplus B^\sharp \) which carries the first (resp. second) summand of \( B^\sharp \oplus B \) onto the second (resp. first) summand of \( B \oplus B^\sharp \).
Let $K_z(C)$ be the quotient of $K(C_z)$ by the subgroup $K^0(C_z)$ generated by the elements $[B, \phi]$ where $(B, \phi)$ is any traceless object of $C_z$. We show that:

(a) \[ [A, -\phi] = -[A, \phi] \text{ for any } (A, \phi) \in C_z. \]

Indeed, if we define $\phi' : A^2 \oplus A \to A \oplus A^2$ by $(x, y) \mapsto (y, x)$ and $\tau : A \oplus A \to A \oplus A^2$ by $(x, y) \mapsto (x + y, \phi^{-1}(x) - \phi^{-1}(y))$, then $\tau$ defines an isomorphism of $(A, \phi) \oplus (A, -\phi)$ with the traceless object $(A \oplus A^2, \phi')$.

2. A review of Soergel modules

2.1. In this section we review some results of Soergel [S] and of Elias-Williamson [EW].

Recall that $W$ is a Coxeter group. The canonical set of generators (assumed to be finite) is denoted by $S$. Let $x \mapsto l(x)$ be the length function on $W$ and let $\leq$ be the Bruhat order on $W$. Let $\mathfrak{h}$ be a reflection representation of $W$ over the real numbers $\mathbb{R}$, as in [EW]; for any $s \in S$ we fix a linear form $\alpha_s : \mathfrak{h} \to \mathbb{R}$ whose kernel is equal to the fixed point set of $s : \mathfrak{h} \to \mathfrak{h}$. Let $R$ be the algebra of polynomial functions $\mathfrak{h} \to \mathbb{R}$ with the $\mathbb{Z}$-grading in which linear functions $\mathfrak{h} \to \mathbb{R}$ have degree 2. Note that $W$ acts naturally on $R$; we write this action as $w : r \mapsto w r$ and for $s \in S$ we set $R^s = \{ r \in R; \ ^sr = r \}$, a subalgebra of $R$. Let $R^{>0} = \{ r \in R; r(0) = 0 \}$. Let $\hat{R}$ be the completion of $R$ with respect to the maximal ideal $R^{>0}$.

Let $\mathcal{R}$ be the category whose objects are $\mathbb{Z}$-graded $(R, R)$-bimodules in which for $M, M' \in \mathcal{R}$, $\text{Hom}_\mathcal{R}(M, M')$ is the space of homomorphisms of $(R, R)$-bimodules $M \to M'$ compatible with the $\mathbb{Z}$-gradings. For $M \in \mathcal{R}$ and $n \in \mathbb{Z}$, the shift $M[n]$ is the object of $\mathcal{R}$ equal in degree $i$ to $M$ in degree $i + n$. For $M, M'$ in $\mathcal{R}$ we set $MM' = M \otimes_R M'$; this is naturally an object of $\mathcal{R}$. For $M, M'$ in $\mathcal{R}$ we set

$$M'^M = \oplus_{n \in \mathbb{Z}} \text{Hom}_\mathcal{R}(M, M'[n]),$$

viewed as an object of $\mathcal{R}$ with $(rf)(m) = f(rm), (fr)(m) = f(mr)$ for $m \in M, f \in M'^M, r \in R$. For any $M \in \mathcal{R}$ we set $\hat{M} = M/\mathbb{M}R^{>0} = M \otimes_R \hat{R}$, viewed as a $\hat{R}$-module.

For $s \in S$ let $B_s = R \otimes_{R^s} R[1] \in \mathcal{R}$. More generally, for any $x \in W$, Soergel [S, 6.16] shows that there is an object $B_x$ of $\mathcal{R}$ (unique up to isomorphism) such that $B_x$ is an indecomposable direct summand of $B_{s_1}B_{s_2} \ldots B_{s_q}$ for some/an any reduced expression $w = s_1s_2 \ldots s_q$ ($s_i \in S$) and such that $B_x$ is not a direct summand of $B_{s'_1}B_{s'_2} \ldots B_{s'_p}$ whenever $s'_1, \ldots, s'_p \in S, p < q$. Let $\hat{C}$ be the full subcategory of $\mathcal{R}$ whose objects are isomorphic to finite direct sums of shifts of objects of the form $B_x$ for various $x \in W$. Let $\hat{C}$ be the full subcategory of $\mathcal{R}$ whose objects are isomorphic to finite direct sums of objects of the form $B_x$ for various $x \in W$. From [S] it follows that for $M, M' \in \hat{C}$ we have $MM' \in \hat{C}$. 
Let \( B \) be the object of \( \mathcal{C} \) whose objects are complexes of sheaves on \( B^2 \) which are (non-canonically) direct sums of objects of semisimple \( G \)-equivariant perverse sheaves with shifts. Then \( M, M' \mapsto MM' \) corresponds to convolution of complexes of sheaves.

For any \( x \in W \) let \( R_x \) be the object of \( \mathcal{R} \) such that \( R_x = R \) as a left \( R \)-module and such that for \( m \in R_x, r \in R \) we have \( mr = (xr)m \). The following result appears in [S, 6.15]:

(a) For any \( M \in \tilde{\mathcal{C}} \), \( R^M_x \) is a finitely generated graded free right \( R \)-module; hence \( \dim_R R^M_x < \infty \).

Note that \( R^M_x[n] = R^M_{x^{-1}n} \) for any \( i, n \in \mathbb{Z} \).

(In the case where \( W \) is a Weyl group of a reductive group \( G \) then \( R^M_x \) can be thought of as the dual of a stalk of a cohomology sheaf of a complex of sheaves on \( B^2 \) at a point in the \( G \)-orbit on \( B^2 \) corresponding to \( x \).)

Let \( t \in \text{Hom}_R(B_s[-1], R_s) = (R^B_s)_1 \) be the unique element such that \( t(1 \otimes \alpha_s + 1 \otimes \alpha_s) = 0, t(1 \otimes 1) = 1 \). The image of \( t \) in \( R^B_s \) is an \( R \)-basis of this one-dimensional \( R \)-vector space. Hence we have canonically \( R^B_s = R \).

2.2. Let \( x \in W \). From [EW] it follows that \( \text{Hom}_R(B_x, B_x) = R \) and from [S, 6.16] it follows that \( \dim R^B_x = 1 \). Thus \( R^B_x \) is an object of \( \mathcal{C} \) isomorphic to \( B_x \) and defined up to unique isomorphism (even though \( B_x \) was defined only up to non-unique isomorphism). From now on we will use the notation \( B_x \) for this new object.

It satisfies

\[
R^B_x \downarrow_{l(x)} = R.
\]

When \( x = s \in S \), this agrees with the earlier description of \( B_s \).

2.3. Let \( x, x' \in W, x \neq x' \). From [EW] it follows that \( \text{Hom}_R(B_{x'}, B_x) = 0 \). This, together with the equality \( \text{Hom}_R(B_x, B_x) = R \) implies that the objects \( B_x \) are simple in \( \mathcal{C} \). Conversely, it is clear that any simple object of \( \mathcal{C} \) is isomorphic to some \( B_x \).

2.4. Let \( A = \mathbb{Z}[u, u^{-1}] \) where \( u \) is an indeterminate. Let \( H \) be the free \( A \)-module with basis \( T_w, w \in W \). It is known that there is a unique associative \( A \)-algebra structure on \( H \) such that \( T_wT_{w'} = T_{ww'} \) whenever \( l(ww') = l(w) + l(w') \) and \( T_s^2 = u^2T_1 + (u^2 - 1)T_s \) for \( s \in S \). Note that \( T_1 \) is a unit element. Let \( \{c_w; w \in W\} \) be the \( A \)-basis of \( H \) which in [KL] was denoted by \( \{C'_w; w \in W\} \). Recall that

\[
(a) \quad c_w = u^{-l(w)} \sum_{y \leq w} P_{y, w}(u^2)T_y
\]

where \( P_{y, w} = 1 \) if \( y = w \) and \( P_{y, w} \) is a polynomial of degree \( \leq (l(w) - l(y) - 1)/2 \) if \( y < w \).

We regard \( \mathcal{K}(\tilde{\mathcal{C}}) \) as an \( A \)-module by \( u^n[M] = [M[-n]] \) for \( M \in \tilde{\mathcal{C}}, n \in \mathbb{Z} \).
Note that $\mathcal{K}(\tilde{C})$ is an associative $\mathcal{A}$-algebra with product defined by $[M][M'] = [MM']$ for $M \in \tilde{C}, M' \in \tilde{C}$. From [S, 1.10, 5.3] we see that

(b) the assignment $M \mapsto \sum_{y \in W, i \in \mathbb{Z}} \dim R_{yi}^M u^{-i} T_y$ defines an $\mathcal{A}$-algebra isomorphism $\chi : \mathcal{K}(\tilde{C}) \xrightarrow{\sim} \mathcal{H}$. From [EW, Theorem 1.1] it follows that

(c) $\chi(B_x) = c_x$.

3. The $\mathcal{H}$-module $\mathcal{M}$

3.1. In this section we preserve the setup of Section 2. Recall that $w \mapsto w^*$ is an involutive automorphism $W \xrightarrow{\sim} W$ leaving $S$ stable. We can assume that there exists an involutive $\mathbb{R}$-linear map $\mathfrak{h} \to \mathfrak{h}$ (denoted again by $x \mapsto x^*$) which satisfies $(wx)^* = w^* x^*$ for $w \in W, x \in \mathfrak{h}$ and satisfies $\alpha_s^* = (\alpha_s)^*$ for $s \in S$. We fix such a linear map. It induces a ring involution $R \to R$ denoted again by $r \mapsto r^*$. For $M \in \mathcal{R}$ let $M^\sharp$ be the object of $\mathcal{R}$ which is equal to $M$ as a graded $\mathbb{R}$-vector space, but left (resp. right) multiplication by $r \in R$ on $M^\sharp$ equals right (resp. left) multiplication by $r^*$ on $M$. Clearly, $(M^\sharp)^\sharp = M$. If $f : M_1 \to M_2$ is a morphism in $\mathcal{R}$ then $f$ can be also viewed as a morphism $M_1^\sharp \to M_2^\sharp$ in $\mathcal{R}$. Note that $M \mapsto M^\sharp$ is an $\mathbb{R}$-linear, 2-periodic functor $\mathcal{R} \to \mathcal{R}$. Hence $\mathcal{R}^\sharp$ is well defined, see 1.1.

If $M_1, M_2 \in \mathcal{R}$ then we have an obvious identification $M_1^\sharp M_2^\sharp = (M_2 M_1)^\sharp$ as objects in $\mathcal{R}$ (it is given by $x_1 \otimes x_2 \mapsto x_2 \otimes x_1$).

Let $s \in S$. The $\mathbb{R}$-linear isomorphism $\omega_s : B_{s^*}[-1] \xrightarrow{\sim} B_s[-1]$ given by $x \otimes_{R^s} y \mapsto y^* \otimes_{R^s} x^*$ for $x, y \in R$ can be viewed as an isomorphism $B_{s^*}^\sharp[-1] \xrightarrow{\sim} B_s[-1]$ in $\mathcal{R}$ or as an isomorphism $B_{s^*}^\sharp \xrightarrow{\sim} B_s$ in $\mathcal{R}$.

Now let $x \in W$ and let $s_1 s_2 \ldots s_k$ be a reduced expression for $x$. Since $B_x$ is an indecomposable direct summand of $B_{s_1} B_{s_2} \ldots B_{s_k}$ (and $k$ is minimal with this property) we see that $B_x^\sharp$ is an indecomposable direct summand of

$$(B_{s_1} B_{s_2} \ldots B_{s_k})^\sharp = B_{s_k}^\sharp \ldots B_{s_2}^\sharp B_{s_1}^\sharp \cong B_{s_k} \ldots B_{s_2} B_{s_1}$$

(and $k$ is minimal with this property) hence by [S, 6.16] we have

(a) $B_x^\sharp \cong B_{(x^*)^{-1}}$.

We use that $s_1^\sharp \ldots s_k^\sharp s_1^\sharp$ is a reduced expression for $(x^*)^{-1}$. In particular we have $B_x^\sharp \in C$. It follows that $M \in C \implies M^\sharp \in C$ and $M \in \tilde{C} \implies M^\sharp \in \tilde{C}$. Note that $M \mapsto M^\sharp$ are $\mathbb{R}$-linear, 2-periodic functors $C \to C$ and $\tilde{C} \to \tilde{C}$. Hence $C^\sharp, \tilde{C}^\sharp$ are defined as in 1.1 and $\mathcal{K}_\sharp(C), \mathcal{K}_\sharp(\tilde{C})$ are well defined abelian groups.

3.2. Recall that $I = \{y \in W ; y^{-1} = y^*\}$. Let $x \in W$. We define $f_x : R_x^\sharp \to R_{(x^*)^{-1}}$ by $r \mapsto f_x(r) = (x^{-1} r)^*$. This is an isomorphism in $\mathcal{R}$.

Now assume that $x \in I$; then $f_x : R_x^\sharp \to R_x$ is given by $r \mapsto f_x(r) = x (r^*)$ and $(R_x, f_x) \in \mathcal{R}_d^\sharp$; thus $(R_x[i], f_x[i]) \in \mathcal{R}_d^\sharp$ for any $i \in \mathbb{Z}$. Hence, if $(M, \phi) \in \tilde{C}^\sharp_\sharp$
and $i \in \mathbb{Z}$, then $f \mapsto f^i$, $\text{Hom}_R(M, R_x[i]) \to \text{Hom}_R(M, R_x[i])$ is defined as in 1.1. Taking direct sum over $i \in \mathbb{Z}$ we obtain a map $f \mapsto f^i$, $R^M_x \to R^M_x$ such that $(f^i)^{-1} = f$. (We always write $R^M_x$ instead of $(R_x)^M$.) From the definitions, for $f \in R^M_x$, $r \in R$ we have $(fr)^i = r^i f^i$, $(rf)^i = f^i r^i$. Since for $r \in R, b \in R_x$ we have $rb = b x^{-1} r$ we see that $R^0_x R_x = R_x R^0_x$ so that $R^0_x (R^M_x) = (R^M_x) R^0_x$; we see that $f \mapsto f^i$ induces an $R$-linear (involutive) map $R^M_x \to R^M_x$ and (for any $i$) an $R$-linear involutive map $R^M_x \to R^M_x$ denoted by $Y^M_{x, \phi, i}$. Let

$$e^i_x(M, \phi) = \text{tr}_R(Y^M_{x, \phi, i}, R^M_x) \in \mathbb{Z}.$$  

We now take $M = B_x$ (still assuming $x \in I$ so that $(B_x, \phi) \in \tilde{C}_i$ for some $\phi$). Then $R^B_{x, i}(x) = R$ hence $e^i_x(B_x, \phi) = \pm 1$. We can normalize $\phi : B^i_x \to B_x$ uniquely so that $e^i_{i}(B_x, \phi) = 1$. We shall denote this normalized $\phi$ by $\phi_x$.

Due to 2.3, we can apply [L1, 11.1.8] to $C_i^\#_i$; we see that

(a) $K^*_i(C)$ is a free abelian group with basis $\{[B_x, \phi_x]; x \in I\}$.

3.3. Let $A' = \mathbb{Z}[u, v^{-1}]$ where $v$ is an indeterminate. We view $A = \mathbb{Z}[u, u^{-1}]$ as a subring of $A'$ by setting $u = v^2$. Note that $K^*_i(\tilde{C})$ can be viewed as an $A'$-module with $v^n[M, \phi] = [M[-n], \phi]$ for $(M, \phi) \in \tilde{C}_i$, $n \in \mathbb{Z}$. We show:

(a) The map $q : A' \otimes K^*_i(C) \to K^*_i(\tilde{C})$, $v^n \otimes [M, \phi] \mapsto [M[-n], \phi]$ is an isomorphism of $A'$-modules.

The map $q$ is clearly well defined. To prove that it is surjective we shall use the functors $M \mapsto \tau_{\leq i} M$ from $\tilde{C}$ to $\tilde{C}$ (resp. $M \mapsto \mathcal{H}^i M$ from $\tilde{C}$ to $C$) defined in [EW, 6.2]. (Here $i \in \mathbb{Z}$.) These define in an obvious way functors $\tilde{C}_i \to \tilde{C}_i$ (resp. $\tilde{C}_i \to C_i$) denoted again by $\tau_{\leq i}$ (resp. $\mathcal{H}^i$). Let $(M, \phi) \in \tilde{C}_\phi$. From the definition we have an exact sequence in $\tilde{C}$ (with morphisms in $\tilde{C}_i$)

$$0 \to \tau_{\leq i-1} M \overset{e^i}{\to} \tau_{\leq i} M \overset{e^{i'}}{\longrightarrow} \mathcal{H}^i M[-i] \to 0$$

which is split but the splitting is not necessarily given by morphisms in $\tilde{C}_i$. Thus there exist morphisms

$$\tau_{\leq i-1} M \overset{f}{\leftarrow} \tau_{\leq i} M \overset{f^i}{\longrightarrow} \mathcal{H}^i M[-i]$$

in $\tilde{C}$ such that $ef = 1$, $fe = 1$, $f' e' + e f = 1$. Now $f^i, f'^i$ are defined as in 1.1 and, since $e^i = e, e'^i = e$! (notation of 1.1), we have $e^i f'^i = 1$, $f^i e = 1$, $f'^i e' + e f^i = 1$ hence setting $\tilde{f} = (f + f')/2$, $\tilde{f}' = (f' + f'^i)/2$, we have $e^i \tilde{f}' = 1$, $f^i e = 1$, $\tilde{f}' e' + e f^i = 1$ and $\tilde{f}^i = \tilde{f}$, $\tilde{f}'^i = \tilde{f}'$. Thus we obtain a new splitting of the exact sequence above which is given by morphisms in $\tilde{C}_i$. It follows that

$$(\tau_{\leq i} M, \phi) \cong (\tau_{\leq i-1} M, \phi) \oplus (\mathcal{H}^i M[-i], \phi)$$
in $\tilde{C}_2$ (the maps $\phi$ are induced by $M^2 \mapsto M$). Hence $[\tau_{\leq i} M, \phi] = [\tau_{\leq i-1} M, \phi] + [H^i M[-i], \phi]$ in $K_{\tilde{C}}(\tilde{C})$. Since $[M, \phi] = [\tau_{\leq i} M, \phi]$ for $i \gg 0$ and $0 = [\tau_{\leq i} M, \phi]$ for $-i \gg 0$ we deduce that $[M, \phi] = \sum_i [H^i M[-i], \phi]$. This proves the surjectivity of $q$.

We define $K(\tilde{C}_2) \to A' \otimes K(C_2)$ by $[M, \phi] \mapsto \sum_{n \in \mathbb{Z}} v^{-n} [H^n M, \phi_n]$ where $\phi_n$ is induced by $\phi$. This clearly induces a homomorphism $q' : K_{\tilde{C}}(\tilde{C}) \to A' \otimes K_2(C)$ which satisfies $q'q = 1$. It follows that $q$ is injective, completing the proof of (a).

3.4. Using 3.2(a), 3.3(a), we see that:

(a) $K_{\tilde{C}}(\tilde{C})$ is a free $A'$-module with basis $\{[B_x, \phi_x]; x \in I\}$, (notation of 3.2).

3.5. Let $M$ be the free $A'$-module with basis $\{a_x; x \in I\}$. For any $(M, \phi) \in \tilde{C}_2$ and any $y \in I$ we set

$$e^y(M, \phi) = \sum_{i \in \mathbb{Z}} e^y_i(M, \phi)v^{-i} \in A'.$$

The homomorphism $K(\tilde{C}_2) \to M$,

$$[M, \phi] \mapsto \sum_{y \in I} e^y(M, \phi)a_y$$

clearly factors through an $A'$-module homomorphism

(a) $\chi' : K_{\tilde{C}}(\tilde{C}) \to M$.

We show:

(b) $\chi'$ is an isomorphism.

For $x \in I$ let $\tilde{A}_x = \chi'([B_x, \phi_x])$. We can write $\tilde{A}_x = \sum_{y \in I} f_{y,x}a_y$ where $f_{y,x} \in A'$ are zero for all but finitely many $y$. In view of 3.4(a), to prove (b) it is enough to show:

(c) Let $y \in I$. If $y \not\geq x$ then $f_{y,x} = 0$. If $y \leq x$ then $f_{y,x} = v^{-l(x)}\tilde{P}_{y,x}(u)$ where $\tilde{P}_{y,x} = 1$ if $y = x$ and $\tilde{P}_{y,x}$ is a polynomial with integer coefficients of degree $\leq (l(x) - l(y) - 1)/2$ if $y < x$.

Assume that $f_{y,x} \neq 0$. Then for some $i$ we have $e^y_i(B_x, \phi_x) \neq 0$ hence $R_{y,x}^{B_x} \neq 0$. Using 2.4(b),(c) we deduce that the coefficient of $T_y$ in $c_x$ is nonzero; thus we have $y \leq x$, as required. Next we assume that $y \leq x$. We have $v^{l(x)}f_{y,x} = \sum_i e^y_i(B_x, \phi_x)v^{-i+l(x)}$ hence it is enough to show that $e^y_i(B_x, \phi_x) \neq 0$ implies $-i + l(x) \in 2\mathbb{N}$ and $-i + l(x) \leq l(x) - l(y)$ with strict inequality unless $x = y$.

Now $e^y_i(B_x, \phi_x) \neq 0$ implies $R_{y,x}^{B_x} \neq 0$. Hence it is enough to show that
\[ R^B_y \neq 0 \text{ implies } -i + l(x) \in 2\mathbb{N} \text{ and } -i + l(x) \leq l(x) - l(y) \text{ with strict inequality unless } x = y. \]

By 2.4(a),(b),(c) we have
\[ \sum_{j \in \mathbb{Z}} \dim R^B_y u^{-j + l(x)} = P_{y,x}(u^2) \]
and the desired result follows from the properties of \( P_{y,x} \) (see 2.4(a)). This proves (c) hence also (b).

Next we note that for \( y \in I, y \leq x \) and \( \delta \in \{1, -1\} \) the following holds:
\[ (P_{y,x}(u) + \delta \bar{P}_{y,x}(u))/2 \in \mathbb{N}[u]. \]
We have
\[ P_{y,x}(u) + \delta \bar{P}_{y,x}(u) = \sum_{j \in \mathbb{Z}} \dim R^B_y v^{-j + l(x)} + \delta \sum_{j \in \mathbb{Z}} c_j^y(B_x, \phi_x)v^{-j + l(x)} \]

hence it is enough to show that
\[ \dim R^B_y + \delta c_j^y(B_x, \phi_x) \in 2\mathbb{N} \]
This follows from the fact that for an involutive automorphism \( \tau \) of a real vector space \( V \) we have \( \dim(V) + \delta \text{tr}(\tau, V) \in 2\mathbb{N}. \)

### 3.6.
For any \( M \in \tilde{C} \) we define a functor \( F_M: \tilde{C}_2 \to \tilde{C}_2 \) by
\[ (M', \phi) \mapsto (MM'M^\sharp, \phi') \]
where \( \phi': (MM'M^\sharp)^\sharp = MM'^\sharp M^\sharp \to MM'M^\sharp \) is given by
\[ m_1 \otimes m' \otimes m_2 \mapsto m_2 \otimes \phi(m') \otimes m_1. \]

Note that \( F_M \) induces an \( \mathcal{A}' \)-linear map \( K(\tilde{C}_2) \to K(\tilde{C}_2) \) which clearly maps \( K^0(\tilde{C}_2) \) into itself hence it induces an \( \mathcal{A}' \)-linear map \( \bar{F}_M: K(\tilde{C}) \to K(\tilde{C}) \). If \( M_1, M_2 \in \tilde{C} \) we have \( F_{M_1 \oplus M_2} = F_{M_1}F_{M_2} \) hence \( F_{M_1 \oplus M_2} = \bar{F}_{M_1} \bar{F}_{M_2} \); moreover for any \( (M, \phi) \in \tilde{C}_2 \) we have
\[ F_{M_1 \oplus M_2}(M, \phi) = ((M_1 \oplus M_2)M(M_1^\sharp \oplus M_2^\sharp), \phi') = F_{M_1}(M, \phi) \oplus F_{M_2}(M, \phi) \oplus (\tilde{M}, \tilde{\phi}) \]
(for a suitable \( \phi' \)) where \( \tilde{M} = M_2MM_1^\sharp \oplus M_1MM_2^\sharp \) and \( \tilde{\phi}: \tilde{M}^\sharp \to \tilde{M} \) are such that \( (\tilde{M}, \tilde{\phi}) \) is a traceless object of \( \tilde{C}_2 \). It follows that \( \bar{F}_{M_1 \oplus M_2} = \bar{F}_{M_1} + \bar{F}_{M_2} \). We see that \([M] \mapsto \bar{F}_M\) makes \( K(\tilde{C}) \) into a (left) \( K(\tilde{C}) \)-module. From the definitions, for any \( M \in \tilde{C}, (M', \phi) \in \tilde{C}_2, n \in \mathbb{Z} \) we have \( F_{M[n]}(M', \phi) = F_M(M'[2n], \phi) \). Hence for \( h \in K(\tilde{C}), h' \in K(\tilde{C}), n \in \mathbb{Z} \) we have \((u^n h)h' = u^{2n}(hh') = u^n(hh') \). Via the isomorphism \( \chi : K(\tilde{C}) \sim \mathbf{H} \) in 2.4(b) and the isomorphism \( \chi': K(\tilde{C}) \sim \mathcal{M} \) in 3.5(a),(b), \( \mathcal{M} \) becomes a (left) \( \mathbf{H} \)-module (with \( u \in \mathbf{H} \) acting on \( \mathcal{M} \) as multiplication by \( u = v^2 \)).
4. Some exact sequences

4.1. In this section we fix $s \in S$ and we write $\alpha$ instead of $\alpha_s$ so that $\alpha^* = \alpha_s^*$. Let $R^{s^2,>0} = R^{s_s^*} \cap R^{>0}$. Let $R = R/R^{s^2,>0}R$, a $\mathbb{Z}$-graded $R$-algebra which can be naturally identified with $\mathbb{R}[\alpha^*]/(\alpha^*^2)$ (it is zero except in degree 0 and 2). Let $\mathcal{R}$ be the category whose objects are $\mathbb{Z}$-graded right $R$-modules. For any $M' \in \mathcal{R}$ we write $M' = M'/R^{s^2,>0} = M' \otimes_{R^{s^2,>0}} R$ where $R = R^{s^2}/R^{s^2,>0}$ is viewed as a $R^{s^2}$-algebra in the obvious way. Note that $M'$ is naturally an object of $\mathcal{R}$.

4.2. For any $M \in \mathcal{R}$ we write $R.M$ (resp. $M.R$) instead of $R \otimes_{R^{s^2}} M \in \mathcal{R}$ (resp. $M \otimes_{R^{s^2}} R \in \mathcal{R}$); for $r \in R, m \in M$ we write $r.m$ (resp. $m.r$) instead of $r \otimes m \in R.M$ (resp. $m \otimes r \in M.R$). Note that any element of $R.M$ (resp. $M.R$) can be written uniquely in the form $\sum_{i \in \{0,1\}} \alpha^i.m_i$ (resp. $\sum_{i \in \{0,1\}} m_i \alpha^i$) where $m_i \in M$.

For $M, N \in \mathcal{R}$ let $\text{hom}(M, N)$ (resp. $\text{hom}'(M, N)$) be the set of maps $M \to N$ which are homomorphisms of $(R^{s^2}, R)$-bimodules (resp. $(R, R^{s^2})$-bimodules) and are compatible with the $\mathbb{Z}$-gradings; let

$$\text{hom}^\bullet(M, N) = \oplus_{i \in \mathbb{Z}} \text{hom}(M, N[i]), \quad \text{hom}'^\bullet(M, N) = \oplus_{i \in \mathbb{Z}} \text{hom}'(M, N[i]).$$

The statements (i)-(ii) below are easily verified.

(i) There is a unique group isomorphism

$$\text{hom}^\bullet(M, N) \xrightarrow{\sim} N^{R.M} \quad \text{resp.} \quad \text{hom}'^\bullet(M, N) \xrightarrow{\sim} N^{M.R},$$

$f \mapsto F$, such that for $m \in M$ we have

$$F(1.m) = f(m), F(\alpha.m) = \alpha f(m) \quad \text{resp.} \quad F(1.m) = f(m), F(m.\alpha^*) = f(m)\alpha^*;$$

this is in fact an isomorphism in $\mathcal{R}$, provided that $\text{hom}^\bullet(M, N)$ (resp. $\text{hom}'^\bullet(M, N)$) is viewed as an object of $\mathcal{R}$ with $(rf)(m) = r(f(m)), (fr)(m) = (f(m))r$ for $m \in M, r \in R$ and $f \in \text{hom}^\bullet(M, N)$ (resp. $f \in \text{hom}'^\bullet(M, N)$).

(ii) The map

$$f \mapsto G, \quad G(m) = \alpha f(m) + 1.f(\alpha m) \quad \text{resp.} \quad G(m) = f(m).\alpha^* + f(\alpha m).1$$

is an isomorphism

$$\text{hom}^\bullet(M, N[-2]) \xrightarrow{\sim} (R.N)^M \quad \text{resp.} \quad \text{hom}'^\bullet(M, N[-2]) \xrightarrow{\sim} (N.R)^M$$

in $\mathcal{R}$, provided that $\text{hom}^\bullet(M, N)$ (resp. $\text{hom}'^\bullet(M, N)$) is viewed as an object of $\mathcal{R}$ with $(rf)(m) = f(rm), (fr)(m) = f(mr)$ for $m \in M, r \in R$ and $f \in \text{hom}^\bullet(M, N)$ (resp. $f \in \text{hom}'^\bullet(M, N)$).

Combining (i),(ii) we see that:

(iii) The map $F \mapsto G,$

$$G(m) = \alpha F(1.m) + 1.F(\alpha m) \quad \text{resp.} \quad G(m) = F(m.1).\alpha^* + F(\alpha m.1).1$$

is an isomorphism

$$(N[-2])^{R.M} \xrightarrow{\sim} (R.N)^M \quad \text{resp.} \quad (N[-2])^{M.R} \xrightarrow{\sim} (N.R)^M$$

of $(R^{s^2}, R)$-bimodules (resp. $(R, R^{s^2})$-bimodules).

(We use that the two $(R, R)$-bimodule structures on $\text{hom}^\bullet(M, N)$ described in
(i),(ii) restrict to the same \((R^s, R)\)-bimodule structure and that the two \((R, R)\)-bimodule structures on \(\text{hom}^\bullet(M, N)\) described in (i),(ii) restrict to the same \((R, R^s)\)-bimodule structure.)

4.3. For any \(M' \in \mathcal{R}\) we write \(R.M'.R\) instead of \(R \otimes_{R^s} M' \otimes_{R^s} R \in \mathcal{R}\); for \(r, r'\) in \(R\) and \(m' \in M'\) we write \(r.m', r'\) instead of \(r \otimes m' \otimes r' \in R.M'.R\). Note that any element \(\xi \in R.M'.R\) can be written uniquely in the form \(\sum_{i,j \in \{0, 1\}} \alpha^i \xi_{ij} a^* j\) where \(\xi_{ij} \in M'\).

For \(M, N \in \mathcal{R}\) let \(\text{hom}(M, N)\) be the set of maps \(M \rightarrow N\) which are homomorphisms of \((R^s, R^s)\)-bimodules and which are compatible with the \(\mathbb{Z}\)-gradings; let \(\text{hom}^\bullet(M, N) = \bigoplus_{i \in \mathbb{Z}} \text{hom}(M, N[i])\).

We view \(\text{hom}^\bullet(M, N)\) as an object of \(\mathcal{R}\) in two ways: for \(f \in \text{hom}^\bullet(M, N), r \in R, m \in M\) we set either

(a) \((rf)(m) = r(f(m)), (fr)(m) = (f(m))r\);

(b) or \((rf)(m) = f(rm), (fr)(m) = f(mr)\).

The statements (i),(ii) below are easily verified.

(i) There is a unique group isomorphism \(\text{hom}^\bullet(M, N) \xrightarrow{\sim} N^{R,M,R}\) in \(\mathcal{R}\), \(f \mapsto F\) such that for any \(m \in M\) we have \(F(1.m.1) = f(m)\), \(F(\alpha.m.1) = \alpha f(m)\), \(F(1.m.\alpha^*) = f(m)\alpha^*, F(\alpha.m.\alpha^*) = \alpha f(m)\alpha^*\); this is in fact an isomorphism in \(\mathcal{R}\) provided that \(\text{hom}^\bullet(M, N)\) is viewed as an object of \(\mathcal{R}\) as in (a).

(ii) The map \(f \mapsto G\),

\[ G(m) = 1.f(\alpha m\alpha^*).1 + \alpha . f(m\alpha^*).1 + 1.f(\alpha m).\alpha^* + \alpha . f(m).\alpha^* \]

is an isomorphism \(\text{hom}^\bullet(M, N[-4]) \xrightarrow{\sim} (R.N.R)^M\) in \(\mathcal{R}\) provided that \(\text{hom}^\bullet(M, N)\) is viewed as an object of \(\mathcal{R}\) as in (b).

Combining (i),(ii) we see that:

(iii) The map \(F \mapsto G\),

\[ G(m) = \alpha . F(1.m.1).\alpha^* + \alpha . F(1.m.\alpha^*).1 + 1.F(\alpha . m.1).\alpha^* + 1.F(1.m.\alpha^*).1 \]

is an isomorphism \((N[-4])^{R,M,R} \xrightarrow{\sim} (R.N.R)^M\) of \((R^s, R^s)\)-bimodules.

We use that the two \((R, R)\)-bimodule structures on \(\text{hom}^\bullet(M, N)\) described in (a),(b) restrict to the same \((R^s, R^s)\)-bimodule structure.)

4.4. Let \(M \in \tilde{C}\) and let \(\omega \in W\). We define an exact sequence

\[ 0 \rightarrow R_{M}^{R,M} \varphi \rightarrow R_{\omega}^{R,M} \xrightarrow{d} R_{\omega}^{M}[2] \]

as follows. We identify \(R_{\omega}^{R,M} = \text{hom}^\bullet(M, R_{\omega})\) as objects of \(\mathcal{R}\) as in 4.2(i); then \(c\) is the obvious inclusion \(R_{\omega}^{M} \subset \text{hom}^\bullet(M, R_{\omega})\) and \(d : \text{hom}^\bullet(M, R_{\omega}) \rightarrow R_{\omega}^{M}[2]\) is given by \(f \mapsto f'\), where \(f'(m) = s(f(\alpha m) - \alpha f(m))\). (The kernel of \(d\) is clearly \(R_{\omega}^{M}\).) Now (a) induces sequences

(b) \[ 0 \rightarrow R_{\omega}^{M} \rightarrow R_{\omega}^{R,M} \rightarrow R_{\omega}^{M}[2] \rightarrow 0 \]

(c) \[ 0 \rightarrow R_{\omega}^{M} \rightarrow R_{\omega}^{R,M} \rightarrow R_{\omega}^{M}[2] \rightarrow 0 \]
We state the following result.

(d) If \( l(\omega) < l(s\omega) \), then the sequences (b), (c) are exact.

For (b) this is implicit in the proof in [S, Proposition 5.7, Corollary 5.16] of the fact that, under the assumption of (d), the alternating sum of dimensions of the terms of (b) is zero (in each degree). The statement for (c) can be reduced to that for (b) as follows. The \( R \)-modules in (c) are free of finite rank (we use 2.1(a)) and the kernel and cokernel of right multiplication by \( \alpha^* \) in these \( R \)-modules form sequences which can be identified with the sequence (b) which are already known to be exact; it follows that the sequence (c) is exact.

Next we define an exact sequence

\[(e) \quad 0 \to R^M_\omega \xrightarrow{c'} R^{M,R}_\omega \xrightarrow{d'} R^{M,M^*_\omega}[2]\]

as follows. We identify \( R^{M,R}_\omega = \text{hom}^\bullet(M, R_\omega) \) as objects of \( R \) as in 4.2(i); then \( c' \) is the obvious inclusion \( R^M_\omega \subset \text{hom}^\bullet(M, R_\omega) \) and \( d' : \text{hom}^\bullet(M, R_\omega) \to R^{M,M^*_\omega}[2] \) is given by \( f \mapsto f' \), where \( f'(m) = f(m) - f(m)\alpha^* \) (the product \( f(m)\alpha^* \) is computed in the right \( R \)-module structure of \( R_\omega \)). (The kernel of \( d' \) is clearly \( R^M_\omega \).) Now (e) induces sequences

\[(f) \quad 0 \to R^M_\omega \to R^{M,R}_\omega \to R^{M,M^*_\omega}[2] \to 0,\]

\[(g) \quad 0 \to R^M_\omega \to R^{M,R}_\omega \to R^{M,M^*_\omega}[2] \to 0.\]

We now state the following result.

(h) If \( l(\omega) < l(\omega s^*) \), then the sequences (f), (g) are exact.

For any \( x \in W \) we have an isomorphism

\[(i) \quad R^M_x \sim R^{M^*_x}_{x^{-1}}\]

as \( R \)-vector spaces (not in \( R \)) given by \( f \mapsto \tilde{f} \) where \( (\tilde{f})(m) = x^{-1}(f(m)) \) for any \( m \in M \) (we identify \( M, M^*_x \) as sets). It carries \( R^M_x R^{>0} \) onto \( R^{M^*_x}_{x^{-1}} R^{>0} \) hence it induces an isomorphism \( R^M_x \sim R^{M^*_x}_{x^{-1}} \) of graded \( R \)-vector spaces. Applying an isomorphism like (i) to each term of the sequence (f) we get a sequence

\[0 \to R^{M^*_x}_{x^{-1}} \to R^{\otimes M^*_x}_{x^{-1}} \to R^{M^*_x}_{x^{-1}}[2] \to 0;\]

(we use that \( (M,R)^d = R \otimes_{R_*^d} M^d \)). This sequence is a special case of the sequence (b) (with \( M, \omega, s \) replaced by \( M^d, \omega^{-1}, s^* \)); hence, by (d), it is exact (we use that \( l(\omega) < l(s^*\omega^{-1}) \)). It follows that the sequence (f) is exact. From this we deduce the exactness of (g) in the same way as we have deduced the exactness of (c) from that of (b).
4.5. Let \( w \in W \). We set \( N = R_w \). For \( r \in R \) and \( b \in N \) we write \( b \circ r \) for the element of \( N \) given by the right \( R \)-module structure on \( N \). We define some subsets of \( R.N.R \) as follows:

\[
Y = \{1.\alpha.b.1 + \alpha.b.1 + 1.\alpha.b'.\alpha * + \alpha.b'.\alpha *; b, b' \in N\},
\]

\[
Y' = \{1.b' \circ \alpha *.1 + \alpha.b \circ \alpha *.1 + 1.b'.\alpha * + \alpha.b.\alpha *; b, b' \in N\},
\]

\[
V = \{1.\alpha.b \circ \alpha *.1 + \alpha.b \circ \alpha *.1 + 1.\alpha.b.\alpha * + \alpha.b.\alpha *; b \in N\} = Y \cap Y',
\]

\[
Z = \{(ab + b' \circ \alpha * - ab'' \circ \alpha *).1 + \alpha.b.1 + 1.b'.\alpha * + \alpha.b''.\alpha *; b, b', b'' \in N\} = Y + Y'.
\]

It is easy to verify that \( Y, Y' \) are subobjects of \( R.N.R \) in \( R \). Hence \( V, Z \) are subobjects of \( R.N.R \) in \( R \).

By a straightforward computation we see that (a)–(d) below hold:

(a) the map \( \tau_1 : V \rightarrow N[-4], 1.\alpha.b \circ \alpha *.1 + \alpha.b \circ \alpha *.1 + 1.\alpha.b.\alpha * + \alpha.b.\alpha * \mapsto b \) is an isomorphism in \( R \);

(b) the map \( Y \rightarrow R_{ws*}[-2], 1.\alpha.b.1 + \alpha.b.1 + 1.b'.\alpha * + \alpha.b'.\alpha * \mapsto b - b' \circ \alpha *, \)

induces an isomorphism \( \tau_2 : Y/V \simto R_{ws*}[-2] \) in \( R \);

(c) the map \( Y' \rightarrow R_{sw}[-2], 1.b' \circ \alpha *.1 + \alpha.b \circ \alpha *.1 + 1.b'.\alpha * + \alpha.b.\alpha * \mapsto (b' - ab), \)

induces an isomorphism \( \tau_3 : Y'/V' \simto R_{sw}[-2] \) in \( R \);

(d) the map \( R.N.R \rightarrow R_{sw*}, \)

\[
1.b_0.1 + \alpha.b_1.1 + 1.b_2.\alpha * + \alpha.b_3.\alpha * \mapsto (-b_0 + ab_1 + b_2 \circ \alpha * - ab_3 \circ \alpha *),
\]

induces an isomorphism \( \tau_4 : R.N.R/Z \simto R_{sw*} \).

In the remainder of this section we fix \( M \in \mathcal{C} \).

Lemma 4.6. Assume that \( l(w) < l(ws^*) \). The obvious sequence

\[
0 \rightarrow \underline{V}M \rightarrow \underline{Y}M \rightarrow (Y/V)^M \rightarrow 0
\]

is exact.

We can identify \( N.R[-2] = Y \) (as objects of \( R \)) by \( b.r \mapsto \alpha.b.r + 1.ab.r \) (for \( b \in N, r \in R \)). We can identify \( N[-4] = V \) via \( \tau_1 \) in 4.5(a) and \( R_{ws*}[-2] = Y/V \) via \( \tau_2 \) in 4.5(b). Then (a) becomes a sequence

\[
0 \rightarrow \underline{(N[-4])^M} \rightarrow \underline{(N.R[-2])^M} \rightarrow \underline{(R_{ws*}[-2])^M} \rightarrow 0.
\]

By 4.2(iii) we can identify \( \underline{(N.R[-2])^M} = \underline{(N[-4])^M.R} \) (as \( R \)-vector spaces). The previous sequence becomes a sequence

\[
0 \rightarrow \underline{N^M}[4] \rightarrow \underline{N^M.R}[4] \rightarrow \underline{(R_{ws*})^M[4]}[2] \rightarrow 0.
\]

This is of the type appearing in 4.4(c) with \( M \) replaced by \( M[4] \) hence is exact by 4.4(d). The lemma is proved.
Lemma 4.7. Assume that \( l(sw) < l(sws^*) \). The obvious sequence

(a) \[ 0 \to (Z/Y)^{M} \to (R.N.R/Y)^{M} \to (R.N.R/Z)^{M} \to 0 \]

is exact.

Consider the exact sequence \( 0 \to N[-2] \xrightarrow{c} R.N \xrightarrow{c^{'}} R_{sw} \to 0 \) in which \( c \) is \( b \mapsto \alpha.b + ab.1 \) and \( c^{'}, \) maps \( r'.b \) to \( r'^{s}b \) (where \( r' \in R, b \in N \)). Applying \( \otimes_{R^{*}} R \) we obtain an exact sequence \( 0 \to N.R[-2] \to R.N.R \to R_{sw}.R \to 0 \). Here we identify \( N.R[-2] = Y \) as in the proof of 4.6 and we obtain an exact sequence \( 0 \to Y \to R.N.R \to R_{sw}.R \to 0 \) in \( R \). Hence we obtain an identification \( R.N.R/Y = R_{sw}.R \) under which \( r'^{s}b.r \in R_{sw}.R \). We identify \( Z/Y = (Y + Y')/Y = Y'/V = R_{sws}[-2] \) via the isomorphism \( \tau_{3} \) in 4.5(c) and \( R.N.R/Z = R_{sws^*} \) via the isomorphism \( \tau_{4} \) in 4.5(d). Then (a) becomes

\[ 0 \to (R_{sw}[-2])^{M} \to (R_{sw}.R)^{M} \to R_{sws^*}^{M} \to 0 \]

By 4.2(iii) we can identify \( (R_{sw}.R)^{M} = (R_{sw}[-2])^{M}.R \). The previous sequence becomes

\[ 0 \to (R_{sw}[-2])^{M} \to (R_{sw}[-2])^{M.R} \to R_{sws^*}^{M} \to 0. \]

This sequence is (up to shift) of the type appearing in 4.4(g) (with \( \omega \) replaced by \( sw \)) hence is exact by 4.4(h). The lemma is proved.

Lemma 4.8. Assume that \( l(w) < l(sw) \). The obvious sequence

(a) \[ 0 \to Y^{M} \to (R.N.R)^{M} \to (R.N.R/Y)^{M} \to 0 \]

is exact.

We identify \( Y = N.R[-2] \) as in the proof of 4.6 and \( R.N.R/Y = R_{sw}.R \) as in the proof of 4.7. Then (a) becomes the sequence

\[ 0 \to (N.R[-2])^{M} \to (R.N.R)^{M} \to (R_{sw}.R)^{M} \to 0. \]

By 4.2(iii), 4.3(iii) we can identify

\[ (N.R[-2])^{M} = (N[-4])^{M.R}, \quad (R.N.R)^{M} = (N[-4])^{R.M.R}, \]

\[ (R_{sw}.R)^{M} = (R_{sw}[-2])^{M.R} \]

and the previous sequence becomes

\[ 0 \to (N[-4])^{M.R} \to (N[-4])^{R.M.R} \to (R_{sw}[-2])^{M.R} \to 0. \]

This sequence is of the type appearing in 4.4(c) with \( M \) replaced by \( M.R[4] \), hence is exact by 4.4(d). The lemma is proved.
4.9. We set $P = \text{hom}^*(M, N)$ regarded as an object of $\mathcal{R}$ as in 4.3(a). We define subsets $\mathcal{V}, \mathcal{Y}, \mathcal{Y}', \mathcal{Z}$ of $P$ as follows:

\[ \mathcal{V} = \{ f \in P; f(\alpha m) = \alpha f(m), f(m\alpha^*) = f(m) \circ \alpha^* \text{ for all } m \in M \}; \]

\[ \mathcal{Y} = \{ f \in P; f(\alpha m) = \alpha f(m) \text{ for all } m \in M \}; \]

\[ \mathcal{Y}' = \{ f \in P; f(m\alpha^*) = f(m) \circ \alpha^* \text{ for all } m \in M \}; \]

\[ \mathcal{Z} = \{ f \in P; f(\alpha m) = f(m) \circ \alpha^* - \alpha f(m) \circ \alpha^* = 0 \text{ for all } m \in M \}. \]

Note that $\mathcal{V}, \mathcal{Y}, \mathcal{Y}', \mathcal{Z}$ are subobjects of $P$ in $\mathcal{R}$. Under the bijection $P \leftrightarrow (R.N.R)^M_{[4]}$ in 4.3(ii), $\mathcal{V}, \mathcal{Y}, \mathcal{Y}', \mathcal{Z}$ correspond respectively to the subsets $V^M, Y^M, Y'^M, Z^M$ of $(R.N.R)^M$. Thus we have natural bijections $\mathcal{V} \leftrightarrow V^M$, $\mathcal{Y} \leftrightarrow Y^M$, $\mathcal{Y}' \leftrightarrow Y'^M$, $\mathcal{Z} \leftrightarrow Z^M$ as $(R^s, R^s^*)$-bimodules. From the definitions it is clear that

(a) \[ \mathcal{V} = N^M \]

as objects of $\mathcal{R}$. Since $P \cong N_{R.M.R}^R$ as objects of $\mathcal{R}$, we see from 2.1(a) that $P$ is a finitely generated right $R$-module. Since $R$ is a Noetherian ring, it follows that $\mathcal{V}, \mathcal{Y}, \mathcal{Y}', \mathcal{Z}$ (which are subobjects of $P$) are also finitely generated right $R$-modules.

**Lemma 4.10.** Assume that $l(w) < l(ws^*)$. The map (in $\mathcal{R}$)

(a) $\mathcal{Y} \to (R_{ws^*}[2])^M$, $f \mapsto f'$, where $f'(m) = f(m\alpha^*) - f(m) \circ \alpha^*$, induces an isomorphism $\mathcal{Y}/\mathcal{V} \sim (R_{ws^*}[2])^M$ and an isomorphism $(\mathcal{Y}/\mathcal{V}) \sim (R_{ws^*}[2])^M$.

The map (a) is clearly a well defined morphism in $\mathcal{R}$ and its kernel is clearly equal to $\mathcal{V}$. Thus we have an exact sequence $0 \to \mathcal{V} \to \mathcal{Y} \to (R_{ws^*}[2])^M$ (in $\mathcal{R}$). Using 4.9(a) and the identification $\mathcal{Y} = N_{M,R}^M$ (see 4.2(i)) this exact sequence becomes an exact sequence $0 \to N^M \to N_{M,R}^M \to (R_{ws^*}[2])^M$ (in $\mathcal{R}$) which induces the exact sequence $0 \to N^M \to N_{M,R}^M \to (R_{ws^*}[2])^M \to 0$ (a special case of 4.4(c),(d)) that is an exact sequence $0 \to \mathcal{Y} \to \mathcal{Y}/\mathcal{V} \to (R_{ws^*}[2])^M \to 0$. Applying $\otimes_{R} \mathcal{R}$ to the exact sequence $0 \to \mathcal{V} \to \mathcal{Y} \to \mathcal{Y}/\mathcal{V} \to 0$ we deduce an exact sequence $\mathcal{Y} \to \mathcal{Y}/\mathcal{V} \to (R_{ws^*}[2])^M$. Now the injective homomorphism $\mathcal{Y}/\mathcal{V} \to (R_{ws^*}[2])^M$ induces an injective homomorphism $(\mathcal{Y}/\mathcal{V}) \to (R_{ws^*}[2])^M$ which becomes surjective after applying $\otimes_{R} \mathcal{R}$; hence, by the Nakayama lemma, it is surjective before applying $\otimes_{R} \mathcal{R}$. The lemma is proved.

**Lemma 4.11.** Assume that $l(w) < l(sw)$. The map (in $\mathcal{R}$)

(a) $\mathcal{Y}' \to (R_{sw}[2])^M$, $f \mapsto f'$, where $f'(m) = s(f(\alpha m) - \alpha f(m))$,

induces an isomorphism $\mathcal{Y}'/\mathcal{V} \sim (R_{sw}[2])^M$ and an isomorphism $(\mathcal{Y}'/\mathcal{V}) \sim ((R_{sw}[2])^M)$.

The proof is almost a repetition of that of Lemma 4.10. The map (a) is clearly a well defined morphism in $\mathcal{R}$ and its kernel is clearly equal to $\mathcal{V}$. Thus we have
an exact sequence $0 \to \mathcal{V} \to \mathcal{V}' \to (R_{sw}[2])^M$ (in $\mathcal{R}$). Using 4.9(a) and the identification $\mathcal{Y}' = N^{R,M}$, this exact sequence becomes an exact sequence $0 \to N^M \to N^{R,M} \to (R_{sw}[2])^M$ (in $\mathcal{R}$) which induces the exact sequence $0 \to N^M \to N^{R,M} \to (R_{sw}[2])^M \to 0$ (a special case of 4.4(b),(d)) that is an exact sequence $0 \to \mathcal{V} \to \mathcal{V}' \to (R_{sw}[2])^M \to 0$. Applying $\otimes_R \mathcal{R}$ to the exact sequence $0 \to \mathcal{V} \to \mathcal{V}' \to \mathcal{V}'/\mathcal{V} \to 0$ we deduce an exact sequence $\mathcal{V} \to \mathcal{V}' \to \mathcal{V}'/\mathcal{V} \to 0$. It follows that both $(R_{sw}[2])^M$ and $\mathcal{V}'/\mathcal{V}$ can be identified with the cokernel of the map $\mathcal{V} \to \mathcal{V}'$. Thus, $\mathcal{V}'/\mathcal{V} \xrightarrow{\sim} (R_{sw}[2])^M$. Now the injective homomorphism $\mathcal{V}'/\mathcal{V} \to (R_{sw}[2])^M$ induces an injective homomorphism $(\mathcal{V}'/\mathcal{V}) \to ((R_{sw}[2])^M)$ which becomes surjective after applying $\otimes_R \mathcal{R}$; hence, by the Nakayama lemma, it is surjective before applying $\otimes_R \mathcal{R}$. The lemma is proved.

**Lemma 4.12.** Assume that $l(w) < l(sw)$. Let $P' = \hom'(M, R_{sw})$; we view $P'$ as an object of $\mathcal{R}$ as in 4.2(i). The map (in $\mathcal{R}$)

(a) $P \to P'[2], f \mapsto f', f'(m) = \ast(f(\alpha m) - \alpha f(m))$,

induces an isomorphism $P/\mathcal{V} \xrightarrow{\sim} P'[2]$ (hence, using $P' = R_{sw}^{M,R}$, see 4.2(i)) an isomorphism $P/\mathcal{V} \xrightarrow{\sim} (R_{sw}[2])^{M,R}$; it also induces an isomorphism $(P/\mathcal{V}) \xrightarrow{\sim} ((R_{sw}[2])^{M,R})$.

The map (a) is clearly a well defined morphism in $\mathcal{R}$ and its kernel is clearly equal to $\mathcal{V}$. Thus we have an exact sequence $0 \to \mathcal{V} \to P \to P'[2]$ in $\mathcal{R}$. By 4.2(i) we can identify $\mathcal{Y} = N^{M,R}$ and our exact sequence becomes the exact sequence $0 \to N^M \to N^{R,M} \to (R_{sw}[2])^{M,R}$ in $\mathcal{R}$ which induces an exact sequence $0 \to N^M \to N^{R,M} \to (R_{sw}[2])^{M,R} \to 0$ (a special case of 4.4(b),(d) with $M$ replaced by $M,R$). Thus we have an exact sequence $0 \to \mathcal{V} \to P \to P'[2] \to 0$.

Applying $\otimes_R \hat{\mathcal{R}}$ to the exact sequences

$0 \to \mathcal{V} \to P \to P/\mathcal{V} \to 0, 0 \to \mathcal{V} \to P \to P'[2]$,

we obtain exact sequences

$0 \to \hat{\mathcal{V}} \to \hat{P} \to \hat{P}/\mathcal{V} \to 0, 0 \to \hat{\mathcal{V}} \to \hat{P} \to \hat{P}'[2]$.

From the surjectivity of $\hat{P} \to \hat{P}'[2]$ and the Nakayama lemma it follows that $\hat{\mathcal{V}} \to \hat{P}'[2]$ in the last exact sequence is surjective. Hence the obvious map $\hat{P}/\mathcal{V} \to \hat{P}'[2]$ is an isomorphism (both sides can be identified with $\text{coker}(\mathcal{V} \to \hat{P})$).

Applying $\otimes_R \hat{\mathcal{R}}$ we deduce that the obvious map $\hat{P}/\mathcal{V} \to \hat{P}'[2]$ is an isomorphism. Thus, $\hat{P}/\mathcal{V} \xrightarrow{\sim} (R_{sw}[2])^{M,R}$. Now the injective homomorphism $\hat{P}/\mathcal{V} \to (R_{sw}[2])^M$ induces an injective homomorphism $(P/\mathcal{V}) \to ((R_{sw}[2])^M)$ which becomes surjective after applying $\otimes_R \mathcal{R}$; hence, by the Nakayama lemma, it is surjective before applying $\otimes_R \mathcal{R}$. The lemma is proved.

**Lemma 4.13.** Assume that $l(w) < l(sw) < l(sw^*)$. The map (in $\mathcal{R}$) $P \to (R_{sw^*}[4])^M$,

(a) $f \mapsto f', f'(m) = \ast(f(\alpha^m m^* - \alpha f(m^*) - f(\alpha m) \circ \alpha^* + \alpha f(m) \circ \alpha^*)$,

induces an isomorphism $P/Z \xrightarrow{\sim} (R_{sw^*}[4])^M$ and an isomorphism $(P/Z) \xrightarrow{\sim} $(MATH)
The map (in \( R \)) \( Z \to (R_{sw}[2])^M \),

(b) \( f \mapsto f', f'(m) = s(f(\alpha m) - \alpha f(m)) \)

induces an isomorphism \( \overline{Z/Y} \cong (R_{sw}[2])^M \) and an isomorphism \( (Z/Y) \cong ((R_{sw}[2])^M) \).

The map (a) is clearly a well defined morphism in \( R \) and its kernel is clearly equal to \( Z \). Thus we have an exact sequence \( 0 \to Z/Y \to P/Y \to (R_{sws}^*[4])^M \).

Applying \( \otimes_R \hat{R} \) gives again an exact sequence

(c) \( 0 \to \overline{Z/Y} \to \overline{P/Y} \to ((R_{sws}^*[4])^M) \to 0 \).

From 4.4(f),(h) we have an exact sequence

(d) \( 0 \to (R_{sw}[2])^M \to (R_{sw}[2])^{M,R} \to (R_{sws}^*[4])^M \to 0 \).

Hence \( (R_{sw}[2])^{M,R} \to (R_{sws}^*[4])^M \) is surjective, that is (using 4.12) \( P/Y \to R_{sws}^M \) is surjective. Using this and Nakayama lemma we see that \( \overline{P/Y} \to (R_{sws}^M) \) is surjective. This is just the last map in (c); thus, (c) becomes an exact sequence

\[ 0 \to \overline{Z/Y} \to \overline{P/Y} \to ((R_{sws}^*[4])^M) \to 0. \]

This exact sequence of \( \hat{R} \)-modules splits since, by 2.1(a), the \( \hat{R} \)-module \( ((R_{sws}^*[4])^M) \) is free. Hence, applying \( \otimes_{\hat{R}} \hat{R} \) gives an exact sequence

(e) \( 0 \to Z/Y \to P/Y \to (R_{sws}^*[4])^M \to 0 \).

From the obvious exact sequence \( 0 \to Z/Y \to P/Y \to P/Z \to 0 \) we deduce an exact sequence \( Z/Y \to P/Y \to P/Z \to 0 \). Using this and (d) we see that both \( P/Z \) and \( (R_{sws}^*[4])^M \) can be identified with the cokernel of the map \( Z/Y \to P/Y \). Using (d) and (e), where we identify \( (R_{sw}[2])^{M,R} = P/Y \) (see 4.12), we see that both \( Z/Y \) and \( (R_{sw}[2])^M \) can be identified with the kernel of the map \( P/Y \to P/Z \).

Thus, we have \( P/Z \to (R_{sws}^*[4])^M \) and \( Z/Y \to (R_{sw}[2])^M \). Now, the injective homomorphism \( P/Z \to (R_{sws}^*[4])^M \) (resp. \( Z/Y \to (R_{sw}[2])^M \)) induces an injective homomorphism \( (P/Z) \to ((R_{sws}^*[4])^M) \) (resp. \( (Z/Y) \to ((R_{sw}[2])^M) \)) which becomes surjective after applying \( \hat{R} \); hence, by the Nakayama lemma, it is surjective before applying \( \hat{R} \). The lemma is proved.

**Lemma 4.14.** Assume that \( l(w) < l(sw) = l(ws^*) \). The obvious sequence

\[ 0 \to V \to P \to P/Y \to 0 \]

is exact.
From the exact sequence \(0 \to Y/V \to P/V \to P/Y \to 0\) we deduce an exact sequence \(0 \to (Y/V) \to (P/V) \to (P/Y) \to 0\) in which \((Y/V)\) is a free \(\hat{R}\)-module (by 4.10 and 2.1(a)) and \((P/Y)\) is a free \(\hat{R}\)-module (by 4.12 and 2.1(a)). It follows that

(a) \((P/V)\) is a free \(\hat{R}\)-module.

From the obvious exact sequence \(0 \to V \to P \to P/V \to 0\) we deduce an exact sequence \(0 \to (\hat{V}) \to (\hat{P}) \to (\hat{P}/V) \to 0\) which is split, due to (a). It follows that it remains exact after applying \(\otimes_{\hat{R}} R\). The lemma is proved.

**Lemma 4.15.** Assume that \(l(w) < l(sw) < l(sws^*)\). The obvious sequence

\[
0 \to Z/V \to P/V \to P/Z \to 0
\]

is exact.

From the obvious exact sequence \(0 \to Z/V \to P/V \to P/Z \to 0\) we deduce an exact sequence \(0 \to (Z/V) \to (P/V) \to (P/Z) \to 0\) which is split, since the \(\hat{R}\)-module \((P/Z)\) is free, by 4.13 and 2.1(a). It follows that it remains exact after applying \(\otimes_{\hat{R}} R\). The lemma is proved.

**Lemma 4.16.** Assume that \(l(w) < l(sw) < l(sws^*)\). The sum of the obvious homomorphisms \(Y/V \to Z/V\) and \(Y'/V \to Z/V\) is an isomorphism \(Y/V \oplus Y'/V \to Z/V\).

From the obvious exact sequence \(0 \to Y/V \to Z/V \to Z/Y \to 0\) we deduce an exact sequence \(0 \to (Y/V) \to (Z/V) \to (Z/Y) \to 0\) which is split, since the \(\hat{R}\)-module \((Z/Y)\) is free, by 4.13 and 2.1(a). It follows that after applying \(\otimes_{\hat{R}} R\) we get an exact sequence

\[
0 \to Y/V \to Z/V \to Z/Y \to 0.
\]

We consider the composition \(dc' : Y'/V \to Z/Y\). By 4.11 we can identify \(Y'/V = (R_{sw}(2))^M\) and by 4.13 we can identify \(Z/Y = (R_{sw}(2))^M\). Under these identifications the map \(dc'\) becomes the identity map of \((R_{sw}(2))^M\). In particular, \(dc'\) is an isomorphism. This implies immediately the statement of the lemma.

5. Trace Computations

**5.1.** To simplify notation, for \(x \in W, r \in R\) we shall write \(x^* r^*\) instead of \(x(r^*)\). Recall that if \(x \in I\), then \(r \mapsto x^* r^*\) is an involution \(R \to R\) denoted by \(f_x\) in 3.2.

In this section we fix \((M, \phi) \in \hat{C}_s^+, s \in I\) and \(w \in I\) such that \(l(w) < l(sw)\); we have automatically \(l(w) < l(sw^*)\). As in 4.5 we set \(N = R_w\). The notation \(b \circ r\) for \(b \in N, r \in R\) is as in 4.5. In the case where \(sw = ws^*\) we set \(N' = R_{sw}\). In the case where \(sw \neq ws^*\) we set \(N'' = R_{sws^*}\).
For $b \in N, r \in R$ we have
\[ f_w(b \circ r) = r^*f_w(b), \quad f_w(rb) = f_w(b) \circ r^*. \]

The involution $f \mapsto f^1, N^M \to N^M$, given by $f^1(m) = f_w(f(\phi(m)))$ induces an involution $\Theta : N^M \to N^M$. In the case where $sw = ws^*$, we have $sw \in I$ and the involution $f \mapsto f^1, N'^M \to N'^M$, given by $f^1(m) = f_w(f(\phi(m)))$ induces an involution $\Theta' : N'^M \to N'^M$. In the case where $sw \neq ws^*$, we have $sws^* \in I$ and the involution $f \mapsto f^1, N''^M \to N''^M$, given by $f^1(m) = f_w(f(\phi(m)))$ induces an involution $\Theta'' : N''^M \to N''^M$. Now $\Theta$ (or $\Theta'$ or $\Theta''$, if defined) induces a degree preserving involution of $N^M$ (or $N'^M$, or $N''^M$) denoted again by $\Theta$ (or $\Theta'$ or $\Theta''$).

By 3.6 we have $(R.M.R, \phi') \in \tilde{C}_2$ where $\phi' : R.M.R \to R.M.R$ is the $R$-linear map such that $r_1.m.r_2 \mapsto r_2^*\phi(m').r_1^*$ for $r_1, r_2 \in R, m \in M$. (Recall that $R.M.R \in \tilde{C}$ is defined in 4.3.) We have $\phi'^2 = 1$. Let $\Psi : N^{R.M.R} \to N^{R.M.R}$ be the $R$-linear involution such that for any $F \in N^{R.M.R}$ and any $r_1, r_2 \in R, m \in M$, we have
\[ \Psi(F)(r_1.m.r_2) = f_w(F(\phi'(r_1.m.r_2))) = f_w(F(r_2^*\phi(m).r_1^*)). \]
(This is a special case of the definition of $f \mapsto f^1$ in 1.1.) It induces a degree preserving involution of $N^{R.M.R}$ denoted again by $\Psi$.

We now state the main result of this section. (In this section all traces are taken over $R$.)

**Theorem 5.2.** Recall that $w \in I$, $l(w) < l(sw)$. Let $i \in Z$. If $sw \neq ws^*$ then

(a) \[ \text{tr}(\Psi, N^{R.M.R}) = \text{tr}(\Theta, N^M) + \text{tr}(\Theta'', N''^M_{i+4}). \]

If $sw = ws^*$ then

\[ \text{tr}(\Psi, N^{R.M.R}) = \text{tr}(\Theta, N^M) + \text{tr}(\Theta, N^M_{i+2}) - \text{tr}(\Theta', N'^M_{i+2}) + \text{tr}(\Theta', N'^M_{i+4}). \]

Note that the following identities (with $\phi'$ as in 5.1) are equivalent to the theorem.

(c) \[ \epsilon^w(R.M.R, \phi') = \epsilon^w(M, \phi) + \epsilon^{sws^*}(M, \phi)v^4 \text{ if } sw \neq ws^*, \]

(d) \[ \epsilon^w(R.M.R, \phi') = \epsilon^w(M, \phi)(v^2 + 1) + \epsilon^{sw}(M, \phi)(v^4 - v^2) \text{ if } sw = ws^*. \]

The proof will occupy the remainder of this section.
5.3. We identify \( N^{R,M,R} \) with \( P = \text{hom}^*(M,N) \) (as objects of \( \mathcal{R} \)) as in 4.3(i). Then \( \Psi \) becomes an involution of \( P \) denoted again by \( \Psi \). It is given by \( f \mapsto f^! \) where \( f^!(m) = \hat{f}_w(f(\phi(m))) \). This induces a degree preserving involution of \( P \) denoted again by \( \Psi \). For any \( i \) we have clearly

\[
\text{(a)} \quad \text{tr}(\Psi, N^{R,M,R}) = \text{tr}(\Psi, P_i).
\]

5.4. In this subsection we assume that \( sw \neq ws^* \) so that \( l(w) < l(sw) < l(ws^*) \) and \( ws^* \in \mathbf{I} \). Let \( \mathcal{V}, \mathcal{Y}, \mathcal{Y}', \mathcal{Z} \) be the subobjects of \( P \) defined in 4.9. From the definition we see that \( \Psi : P \to P \) preserves \( \mathcal{V} \) and \( \mathcal{Z} \); it interchanges \( \mathcal{Y} \) and \( \mathcal{Y}' \). Now for any \( \xi \in P \) we have \( \Psi(\xi R^{>0}) = R^{>0}\Psi(\xi) = \Psi(\xi) R^{>0} \). (We use that \( R^{>0} b = b \circ R^{>0} \) for \( b \in N \).) It follows that \( \mathcal{Y} R^{>0}, \mathcal{Z} R^{>0} \) are preserved by \( \Psi \) and \( \mathcal{Y}' R^{>0}, \mathcal{Y}'' R^{>0} \) are interchanged by \( \Psi \). Hence \( \Psi \) induces involutions of \( \mathcal{Y}_i, P/\mathcal{Z}_i, \mathcal{Z}/\mathcal{Y}_i \) (denoted again by \( \Psi \)) and the two summands \( \mathcal{Y}/\mathcal{Y}_i, \mathcal{Y}'/\mathcal{Y}_i \) of \( \mathcal{Z}/\mathcal{Y}_i \) (see 4.16) are interchanged by \( \Psi : \mathcal{Z}/\mathcal{Y}_i \to \mathcal{Z}/\mathcal{Y}_i \). Hence we have \( \text{tr}(\Psi, \mathcal{Z}/\mathcal{Y}_i) = 0 \) and (using 4.14, 4.15) we have

\[
\text{(a)} \quad \text{tr}(\Psi, P_i) = \text{tr}(\Psi, \mathcal{Y}_i) + \text{tr}(\Psi, P/\mathcal{V}_i) = \text{tr}(\Psi, \mathcal{Y}_i) + \text{tr}(\Psi, P/\mathcal{Z}_i).
\]

We now show that the map (say \( \tau \), \( P \to R_{sws^*}[4] \)) in 4.13(a) satisfies

\[
\text{(b)} \quad \tau(\Psi(f)) = \Theta''(\tau(f))
\]

for any \( f \in P \). For \( m \in M \) we have

\[
\tau(\Psi(f))(m) = s(\hat{f}_w(f(\phi(\alpha m^*)))) - \alpha \hat{f}_w(f(\phi(\alpha m^*))) - \hat{f}_w(f(\phi(\alpha m))) \circ \alpha + \alpha \hat{f}_w(f(\phi(\alpha m))) \circ \alpha^*,
\]

\[
\Theta''(\tau(f))(m) = f_{sws^*}(\tau(f)(\phi(m)))
\]

\[
= f_{sws^*}(s(f(\alpha^* (\phi(m)^*)) - \alpha f(\phi(m)^*) - f(\alpha^* (\phi(m)) \circ \alpha + \alpha f(\phi(m)) \circ \alpha^*).
\]

It is enough to show that for any \( m' \in M \) we have

\[
s(\hat{f}_w(f(m'))) = f_{sws^*}(s(f(m')))
\]

or that

\[
s(w(f(m')^*)) = s(ws^*)(s(f(m')^*)).
\]

This is clear; (b) is proved. Using (b) and 4.13 we deduce that

\[
\text{(c)} \quad \text{tr}(\Psi, P/\mathcal{Z}_i) = \text{tr}(\Theta'', N^{R}M_{i+4}).
\]

We have clearly \( \mathcal{V} = N^M \) and \( \text{tr}(\Psi, \mathcal{Y}_i) = \text{tr}(\Theta, N^M\mathcal{Y}_i) \). Introducing this and (c) into (a) and using 5.3(a) we obtain 5.2(a) and (equivalently) 5.2(c).
5.5. In the remainder of this section we assume that $sw = ws^*$ so that $sw \in I$. Note that we have $w\alpha^s = \alpha$, hence $b \circ \alpha^s = ab$ for $b \in N$.

In this case the involution $\Psi : P \to P$ preserves $PR^s,>0$ (more precisely, $\Psi(fR^s,>0) = \Psi(f)R^s,>0$ for any $f \in P$) hence it induces an involution of $P$ denoted again by $\Psi$. (We use that $w(R^s) = R^s$ hence $w(R^s \cap R^{>0}) = R^s \cap R^{>0}$). Moreover the involution $\Psi$ of $P$ is $R$-linear. (We use that $w\alpha^s = \alpha$.)

Let $\Phi : (R.N.R)^M \to (R.N.R)^M$ be the $R$-linear involution which corresponds to $\Psi : P \to P$ under the bijection $P[-4] \xrightarrow{\sim} (R.N.R)^M$ in 4.3(ii). Since this bijection is compatible with the $(R^s, R^{s^*})$-bimodule structures, it follows that $\Phi$ preserves the subset $(R.N.R)^M R^{s^*,>0}$, more precisely we have

(a) $\Phi(\xi R^{s^*,>0}) = \Phi(\xi) R^{s^*,>0}$ for any $\xi \in (R.N.R)^M$,

hence $\Phi$ induces an $R$-linear involution of $(R.N.R)^M$ (which is not necessarily $R$-linear). For any $i$ we have from the definition:

\[(b) \quad \text{tr}(\Phi, (R.N.R)^M) = \text{tr}(\Psi, P_{i-4}).\]

Note that $P$ is a free right $R[\alpha^s]/(\alpha^s)^2$-modules. Hence we have exact an sequence of $R$-vector spaces

\[0 \to P_{i-6} \to P_{i-4} \xrightarrow{d} P_{i-4} \to 0,\]

where $c$ is induced by right multiplication by $\alpha^s$ and we have $d\Theta = \Theta d, c\Theta = \Theta c$. It follows that we have

\[\text{tr}(\Psi, P_{i-4}) = \text{tr}(\Psi, P_{i-4}) + \text{tr}(\Theta, P_{i-6}).\]

Introducing this in (b) we obtain

\[(c) \quad \text{tr}(\Phi, (R.N.R)^M) = \text{tr}(\Psi, P_{i-4}) + \text{tr}(\Psi, P_{i-6}).\]

5.6. We define a map $\xi \mapsto \xi^s$, $R.N.R \to R.N.R$ by

\[(a) \quad 1.b_0.1 + \alpha.b_1.1 + 1.b_2.\alpha^s + \alpha.b_3.\alpha^s \mapsto 1.f_w(b_0).1 + \alpha.f_w(b_1).1 + 1.wf_w(b_1).\alpha^s + \alpha.f_w(b_3).\alpha^s\]

where $b_i \in N$. Then $\xi \mapsto \xi^s$ is an involution of the $R$-vector space $R.N.R$ such that $(r_1.b.r_2)^s = r^s_2.f_w(b).r^s_1$ for $r_1, r_2 \in R, b \in N$. Hence in the $(R, R)$-bimodule structure of $R.N.R$ we have $(r\xi)^s = \xi^s r^s$, $(\xi r)^s = r^s \xi$ for $r \in R, \xi \in R.N.R$. Thus $(R.N.R, \xi \mapsto \xi^s) \in \mathcal{R}_+^s$.

From the definitions we see that $\Phi : (R.N.R)^M \to (R.N.R)^M$ is given explicitly by $G \mapsto G^t$ where for any $G \in (R.N.R)^M$ and any $m \in M$ we have

\[(b) \quad G^t(m) = (G(\phi(m)))^t;\]

(This is a special case of the definition of $f \mapsto f^t$ in 1.1.)
5.7. Let $V,Y$ be the subsets of $R.N.R$ defined as in 4.5. (They are subobjects in $\mathcal{R}$.) In addition to the subsets $V,Y$ we shall need the following subsets of $R.N.R$:

$$X = \{1.ab' \circ \alpha^* .1 + \alpha.b.1 + 1.b.\alpha^* + \alpha.b'.\alpha^* ; b,b' \in N\},$$

$$U = \{1.(ab-b' \circ \alpha^* + ab'' \circ \alpha^* ).1 + \alpha.b.1 + 1.b'.\alpha^* + \alpha.b''.\alpha^* ; b,b',b'' \in N\} = X+Y.$$

Note that $X \cap Y = V$. Using our assumptions on $w$, it is easy to verify that $X$ is a subobject of $R.N.R$ in $\mathcal{R}$ hence $U = X + Y$ is a subobject of $R.N.R$ in $\mathcal{R}$.

By a straightforward computation we see that (a),(b) below hold:

(a) the map $X \to N[-2], 1.ab' \circ \alpha^* .1 + \alpha.b.1 + 1.b.\alpha^* + \alpha.b'.\alpha^* \mapsto *(b-\alpha b')$

induces an isomorphism $X/V \cong N[-2]$ in $\mathcal{R}$;

(b) the map $R.N.R \to N'$,

$$1.b_0.1 + \alpha.b_1.1 + 1.b_2.\alpha^* + \alpha.b_3.\alpha^* \mapsto *(-b_0 + \alpha b_1 - \alpha b_2 + \alpha^2 b_3),$$

induces an isomorphism $R.N.R/U \cong N'$ in $\mathcal{R}$.

**Lemma 5.8.** The obvious sequence

$$(a) \quad 0 \to (U/Y)^M \to (R.N.R/Y)^M \to (R.N.R/U)^M \to 0$$

is exact.

Note that $N'.R = N.R$. Indeed, it is enough to show that $N = N'$ as $(R,R^*)$-bimodules. It is also enough to show that if $r \in R^*$, then $w_r = ws^*r$; this follows from $s^*r = r$. We identify $R.N.R/Y = R_{sw}.R = N'.R$ (hence $R.N.R/Y = N.R$) as in the proof of 4.7, $U/Y = (X+Y)/Y = X/V = N[-2]$ as in 5.1(a) and $R.N.R/U = N'$ as in 5.1(b). Then (a) becomes a sequence

$$0 \to (N[-2])^M \to (N.R)^M \to N'^M \to 0$$

or equivalently a sequence

$$0 \to (N[-2])^M \to (N[-2])^{M,R} \to N'^M \to 0$$

which is exact by 4.4(g),(h). The lemma is proved.

5.9. We write $N^0 = R^s$, $N^1 = \alpha R^s = R^s \circ \alpha^*$ so that $N = N^0 \oplus N^1$ as an $(R^s,R^s)$-bimodule. For $b \in N$ we can write uniquely $b = b^0 + b^1$ where $b^i \in N^i$. For $i \in \{0,1\}$ let

$$(R.N.R)^i = \{1.b_0.1 + \alpha.b_1.1 + 1.b_2.\alpha^* + \alpha.b_3.\alpha^* \in R.N.R; b_i \in N^i \text{ for } i = 0,1,2,3\}.$$

Using the fact that $\alpha^2 \in R^s$ we see that $(R.N.R)^i$ is a subobject of $R.N.R$ in $\mathcal{R}$. Thus, we have $R.N.R = (R.N.R)^0 \oplus (R.N.R)^1$ as objects of $\mathcal{R}$. For $i \in \{0,1\}$ we set

$$X^i = X \cap (R.N.R)^i = \{1.ab' \circ \alpha^*.1 + \alpha.b.1 + 1.b.\alpha^* + \alpha.b'.\alpha^* ; b,b' \in N^i\},$$
We prove (b). We must show that for any \( X \) is induced by the identity map, the second one is the restriction to \( X \). Setting \( \beta \) for (i) we must show that given \( b \) so that \( b \), \( b \) for (ii) we must show that given \( b, b', b'' \) in \( N \) there exist unique \( \beta \in N^0, \beta' \in N^0, b'' \in N \) such that

\[
1.ab' \circ \alpha^* \cdot 1 + 1.b.ab' \circ \alpha^* + 1.b'.ab' \circ \alpha^* = 1.\alpha b' \circ \alpha^* \cdot 1 + 1.\alpha.\beta \cdot \alpha + 1.\beta'.\alpha^* + 1.\alpha b'' \circ \alpha^* + 1.\alpha b' \circ \alpha^* + 1.\alpha b'' \circ \alpha^*
\]

or equivalently \( b = \beta + \alpha b'' \circ \alpha^* \cdot 1 + 1.b.ab' \circ \alpha^* + 1.b'.ab' \circ \alpha^* \). Setting \( b'' = b - b' \) we see that we must show that there are unique \( \beta \in N^0, \beta' \in N^0 \) such that \( b - \alpha b' = \beta - \alpha b' \). This is obvious.

We prove (c). It is enough to show that

(i) \( R.N.R = U + (R.N.R)^0 \),

(ii) \( U \cap ((R.N.R)^0 + X^1) = X \).

For (i) we must show that given \( b_1, b_2, b_3, b_4 \in N \) there exist \( b, b', b'' \in N \) and \( \beta_1, \beta_2, \beta_3, \beta_4 \in N^0 \) such that

\[
b_1 = b + \beta_1, b_2 = b' + \beta_2, b_3 = b'' + \beta_3, b_4 = \alpha b - b' \circ \alpha + \alpha b'' \circ \alpha + \alpha \beta_4.
\]

Setting \( \beta_2 = \beta_1, b = b_1 - \beta_1, b' = b_2, b'' = b_3 \) we see that it is enough to show that there exist \( \beta_1, \beta_4 \in N^0 \) such that

\[
b_4 - \alpha b_1 + \alpha b_2 - \alpha^2 b_3 = \beta_4 - \alpha \beta_1.
\]

This is obvious.

For (ii) we must show that given \( b, b', b'' \in N \) and \( \beta, \beta' \in N^1 \) such that

\[
1.(\alpha b - b' \circ \alpha + \alpha b'' \circ \alpha^*).1 + 1.b.ab' \circ \alpha + 1.b'.ab' \circ \alpha^* - (1.\alpha b' \circ \alpha^* \cdot 1 + 1.\alpha \beta_1 + 1.\beta.\alpha^* + 1.\alpha b'' \circ \alpha^*) \in (R.N.R)^0,
\]

we have \( b = b' \). Our assumption implies \( b^1 = \beta, b'^1 = \beta, b''^1 = \beta' \), \( (\alpha b - \alpha b' + \alpha^2 b'' \circ \alpha^*)^1 = \alpha^2 \beta' \) (that is \( b - b' + \alpha b'' \circ \alpha^* = \alpha \beta' \)). Thus, \( (b - b')^0 = 0 \) and \( (b - b')^0 = 0 \), so that \( b - b' = 0 \). This proves (c).

Now (a),(b) yield isomorphisms (in \( \mathcal{R} \)) \( X^0 \to X/V \), \( X^1 \to X/V \); the first one is induced by the identity map, the second one is the restriction to \( X^1 \) of the first projection \( X = V \circ X^0 \to V \). Moreover, (a),(c) yield isomorphisms (in \( \mathcal{R} \)) \( (R.N.R/X)^0 \to R.N.R/U \), \( (R.N.R/X)^1 \to U/X \); the first one is induced by the identity map, the second one is the restriction to \( (R.N.R)^1 \) of the first projection \( R.N.R/X = U/X \circ (R.N.R/X)^0 \to U/X \).
Lemma 5.10. The obvious sequence

(a) \[ 0 \rightarrow X^M \rightarrow (R.N.R)^M \rightarrow (R.N.R/X)^M \rightarrow 0 \]

is exact.

Consider the obvious commutative diagram with exact horizontal and vertical lines

\[
\begin{array}{ccccccccc}
0 & 0 & 0 \\
\downarrow & & & & & & & & \downarrow \\
0 & \rightarrow & V & \rightarrow & X & \rightarrow & U/Y & \rightarrow & 0 \\
\downarrow & & & & & & & & \downarrow \\
0 & \rightarrow & Y & \rightarrow & R.N.R & \rightarrow & R.N.R/Y & \rightarrow & 0 \\
\downarrow & & & & & & & & \downarrow \\
0 & \rightarrow & Y/V & \rightarrow & R.N.R/X & \rightarrow & R.N.R/U & \rightarrow & 0 \\
\downarrow & & & & & & & & \downarrow \\
0 & 0 & 0 
\end{array}
\]

Here the non-middle horizontal maps are split as exact sequences in \( \mathcal{R} \). Indeed by the results in 5.9 they can be identified with the obvious split exact sequences

\[ 0 \rightarrow X^1 \rightarrow X^0 \oplus X^1 \rightarrow X^0 \rightarrow 0, \]

\[ 0 \rightarrow (R.N.R)^1 \rightarrow (R.N.R)^0 \oplus (R.N.R)^1 \rightarrow (R.N.R)^0 \rightarrow 0. \]

From this we deduce the commutative diagram

\[
\begin{array}{ccccccccc}
0 & 0 & 0 \\
\downarrow & & & & & & & & \downarrow \\
0 & \rightarrow & V^M & \rightarrow & X^M & \rightarrow & (U/Y)^M & \rightarrow & 0 \\
\downarrow & & & & & & & & \downarrow \\
0 & \rightarrow & Y^M & \rightarrow & (R.N.R)^M & \rightarrow & (R.N.R/Y)^M & \rightarrow & 0 \\
\downarrow & & & & & & & & \downarrow \\
0 & \rightarrow & (Y/V)^M & \rightarrow & (R.N.R/X)^M & \rightarrow & (R.N.R/U)^M & \rightarrow & 0 \\
\downarrow & & & & & & & & \downarrow \\
0 & 0 & 0 
\end{array}
\]
in which the middle horizontal line and the non-middle vertical lines are exact sequences (see 4.8, 4.6, 5.8) and in which the non-middle horizontal lines are (split) exact sequences. This implies, by diagram chasing, that the middle vertical line is an exact sequence. The lemma is proved.

5.11. From 5.6(a) we see that $X, (R.N.R)^0, (R.N.R)^1$ are stable under the involution $\xi \mapsto \tilde{\xi}$ of $R.N.R$. Hence that involution induces involutions on $X$, on $(R.N.R)/X$, on $X^i$ and on $(R.N.R/X)^i$ (for $i = 0, 1$) which in turn induce (by formulas like 5.6(b)) involutions on $X^M, ((R.N.R)/X)^M$, on $(X^i)^M$ and on $((R.N.R/X)^i)^M$ (for $i = 0, 1$) which are denoted again by $\Phi$. Using 5.5(a) we see that each of these involutions preserve the image of right multiplication by $R^s, > 0$ hence we have induced involutions on $X^M, ((R.N.R)/X)^M$, on $(X^i)^M$ and on $((R.N.R/X)^i)^M$ (for $i = 0, 1$) which are denoted again by $\Phi$.

Using the definitions we see that the exact sequence 5.10(a) is compatible with the involutions $\Phi$ on each of its terms. Using the definitions we also see that the obvious direct sum decompositions

$$X^M = (X^0)^M \oplus (X^1)^M,$$

$$(R.N.R/X)^M = ((R.N.R/X)^0)^M \oplus ((R.N.R/X)^1)^M$$

are compatible with the involutions $\Phi$ on each of their terms. It follows that for $i \in \mathbb{Z}$ we have

$$\text{tr}(\Phi, (R.N.R)^M) = \text{tr}(\Phi, X^M) + \text{tr}(\Phi, (R.N.R/X)^M),$$

(a) $$= \text{tr}(\Phi, (X^0)^M) + \text{tr}(\Phi, (X^1)^M) + \text{tr}(\Phi, ((R.N.R/X)^0)^M) + \text{tr}(\Phi, ((R.N.R/X)^1)^M).$$

5.12. By a straightforward computation we see that the maps $t_i$ in (a)–(d) below are isomorphisms in $\mathcal{R}$:

(a) $t_1 : X^1 \to N[-4], 1.\alpha \beta' \circ \alpha^* \cdot 1 + \alpha.\beta.1 + 1.\beta.\alpha^* + \alpha.\beta'.\alpha^* \mapsto \alpha^{-1}\beta + \beta'$ with $\beta, \beta' \in N^1$;

(b) $t_1 : X^0 \to N[-2], 1.\alpha \beta' \circ \alpha^* \cdot 1 + \alpha.\beta.1 + 1.\beta.\alpha^* + \alpha.\beta'.\alpha^* \mapsto \beta + \alpha \beta'$ with $\beta, \beta' \in N^0$;

(c) $t_3 : (R.N.R/X)^1 \to N'[\bar{2}]$ induced by

$$1.\beta_0.1 + \alpha.\beta_1.1 + 1.\beta_2.\alpha^* + \alpha.\beta_3.\alpha^* \mapsto 1.\beta_1 - \beta_2 + \alpha^{-1} \beta_0 - \alpha \beta_3$$

with $\beta_0, \beta_1, \beta_2, \beta_3 \in N^1$;

(d) $t_4 : (R.N.R/X)^0 \to N'$ induced by

$$1.\beta_0.1 + \alpha.\beta_1.1 + 1.\beta_2.\alpha^* + \alpha.\beta_3.\alpha^* \mapsto 1.\beta_1 - \alpha \beta_2 + \beta_0 - \alpha^2 \beta_3$$

with $\beta_0, \beta_1, \beta_2, \beta_3 \in N^0$. 
5.13. The identities (a)–(d) below express a connection between \( \xi \mapsto \tilde{\xi} \) and the isomorphisms \( t_j \) in 5.12:

(a) if \( \xi \in X^1 \) then \( t_1(\tilde{\xi}) = w(t_1(\xi))^* \);
(b) if \( \xi \in X^0 \) then \( t_2(\tilde{\xi}) = w(t_2(\xi))^* \);
(c) if \( \xi \in (R.N.R/X)^1 \) then \( t_3(\tilde{\xi}) = sw(t_3(\xi))^* \);
(d) if \( \xi \in (R.N.R/X)^0 \) then \( t_4(\tilde{\xi}) = sw(t_4(\xi))^* \).

Here \( t_i(\xi) \) is viewed as an element of \( R \) and the shift is ignored.

We prove (a). Let \( \xi = 1.\alpha\beta' \circ \alpha^*.1 + \alpha.\beta.1 + 1.\beta.\alpha^* + \alpha.\beta'.\alpha^* \in X^1 \) be as in 5.3(a). Then

\[
\tilde{\xi} = 1.\alpha w \beta'^* \circ \alpha^*.1 + \alpha. w \beta^*.1 + 1. w \beta^*. \alpha^* + \alpha. w \beta'^*. \alpha^*
\]

and we must show that \( \alpha^{-1} w \beta^* + w b^* = w (\alpha^{-1} \beta + \beta')^* \). This follows from the equality \( w \alpha^* = \alpha \).

We prove (b). In this case we must show that \( w \beta^* + \alpha w b'^* = w (\beta + \alpha b')^* \) for \( \beta, \beta' \in N^0 \). This again follows from the equality \( w \alpha^* = \alpha \).

We prove (c). Let \( \xi = 1.\beta_0.1 + \alpha.\beta_1.1 + 1.\beta_2.\alpha^* + \alpha.\beta_3 \alpha^* \) be as in 5.3(c). Then

\[
\tilde{\xi} = 1. w \beta_0^*.1 + \alpha. w \beta_2^*.1 + 1. w \beta_1^*. \alpha^* + \alpha. w \beta_3^*. \alpha^*
\]

and we must show that

\[
w \beta_2^* - w \beta_1^* + \alpha^{-1} w \beta_0^* - \alpha w \beta_3^* = sw (\beta_1 - \beta_2 + \alpha^{-1} \beta_0 - \alpha \beta_3)^*.
\]

Since \( \beta_i \in N^1 \) we have \( s^* \beta_i^* = -b_i^* \); we have also \( s^* \alpha^* = -\alpha^* \) and \( sw = ws^* \). Thus

\[
sw (\beta_1 - \beta_2 + \alpha^{-1} \beta_0 - \alpha \beta_3)^* = w (s^* \beta_1^* - s^* \beta_2^* - (\alpha^*)^{-1} s^* \beta_0^* + \alpha^* s^* \beta_3^*)
\]

\[
= w (-\beta_1^* + \beta_2^* + (\alpha^*)^{-1} \beta_0^* - \alpha^* \beta_3^*)
\]

as desired.

We prove (d). In this case we must show that

\[
\alpha w \beta_2^* - \alpha w \beta_1^* - w \beta_0^* + \alpha^2 w \beta_3^* = sw (\alpha \beta_1 - \alpha \beta_2 - \beta_0 + \alpha^2 \beta_3)^*
\]

for \( \beta_i \in N_0 \). We have \( s^* \beta_i^* = b_i^* \); we have also \( s^* \alpha^* = -\alpha^* \) and \( sw = ws^* \). Thus

\[
sw (\alpha \beta_1 - \alpha \beta_2 - \beta_0 + \alpha^2 \beta_3)^* = w (-\alpha^* s^* \beta_1^* + \alpha^* s^* \beta_2^* - s^* \beta_0^* + \alpha^2 s^* \beta_3^*)
\]

as desired.

5.14. The involution \( f \mapsto f^!, N^M \to N^M \), given by \( f^!(m) = f_w(f(\phi(m))) \) induces an involution \( \Theta : N^M \to N^M \) (see 5.1) and also an involution \( N^M \to N^M \) denoted again by \( \Theta \). The involution \( f \mapsto f^!, N'^M \to N'^M \), given by \( f^!(m) = f_{sw}(f(\phi(m))) \) induces an involution \( \Theta' : N'^M \to N'^M \) (see 5.1) and also an involution \( N'^M \to N'^M \) denoted again by \( \Theta' \). Using that \( w^{-1} \alpha = \alpha^* \) and \( (sw)^{-1} \alpha = -\alpha^* \), we see that:
(a) \( \Theta : N^M \to N^M \) is \( R[\alpha^*]/(\alpha^*2) \)-linear; \( \Theta' : N'^M \to N'^M \) is only \( R \)-linear
and satisfies \( \Theta'(f \alpha^*) = -\Theta'(f) \alpha^* \) for \( f \in N'^M \).

Note that \( N^M, N'^M \) are free right \( R[\alpha^*]/(\alpha^*2) \)-modules. Hence for any \( i \) we have exact sequences of \( R \)-vector spaces

\[
0 \to N^M_{i-2} \xrightarrow{c} N^M_i \xrightarrow{d} N^M_i \to 0,
\]

\[
0 \to N'^M_{i-2} \xrightarrow{c'} N'^M_i \xrightarrow{d'} N'^M_i \to 0,
\]

where \( c, c' \) are induced by right multiplication by \( \alpha^* \) and we have \( d \Theta = \Theta d, d' \Theta' = \Theta' d, c \Theta = \Theta c, c' \Theta' = -\Theta' c'. \) (We use (a).) It follows that for any \( i \in \mathbb{Z} \) we have

(b) \( \text{tr}(\Theta, N^M_i) = \text{tr}(\Theta, N^M_{i-2}) \),

(c) \( \text{tr}(\Theta', N'^M_i) = \text{tr}(\Theta, N'^M_{i-2}) - \text{tr}(\Theta, N'^M_{i-2}) \).

Using the isomorphisms in 5.12 and the identities in 5.13 we see that for any \( i \in \mathbb{Z} \) we have

\[
\text{tr}(\Phi, (X^0)^M) = \text{tr}(\Theta, N^M_{i-4}), \quad \text{tr}(\Phi, (X^1)^M) = \text{tr}(\Theta, N^M_{i-2}),
\]

\[
\text{tr}(\Phi, ((R.N.R/X)^0)^M) = \text{tr}(\Theta', N'^M_{i-2}),
\]

\[
\text{tr}(\Phi, ((R.N.R/X)^1)^M) = \text{tr}(\Theta', N'^M_i).
\]

Introducing this into 5.11(a) we deduce

\[
\text{tr}(\Phi, (R.N.R)^M) = \text{tr}(\Theta, N^M_{i-4}) + \text{tr}(\Theta, N^M_{i-2}) + \text{tr}(\Theta', N'^M_{i-2}) + \text{tr}(\Theta', N'^M_i),
\]

from which (taking into account (b),(c) and 5.5.(c)) we deduce

\[
\text{tr}(\Psi, P_{i-4}) + \text{tr}(\Psi, P_{i-6})
\]

\[
= \text{tr}(\Theta, N^M_{i-4}) + \text{tr}(\Theta, N^M_{i-6}) + \text{tr}(\Theta, N^M_{i-2}) + \text{tr}(\Theta, N^M_{i-4})
\]

\[
+ \text{tr}(\Theta', N'^M_{i-2}) - \text{tr}(\Theta', N'^M_{i-4}) + \text{tr}(\Theta', N'^M_i) - \text{tr}(\Theta', N'^M_{i-2}).
\]

We multiply this equality by \( v^{-i} \) and sum over all \( i \). We get

\[
\sum_i \text{tr}(\Psi, P_{i-4}) v^{-i} + \sum_i \text{tr}(\Psi, P_{i-6}) v^{-i}
\]

\[
= \sum_i \text{tr}(\Theta, N^M_{i-4}) v^{-i} + \sum_i \text{tr}(\Theta, N^M_{i-6}) v^{-i} + \sum_i \text{tr}(\Theta, N^M_{i-2}) v^{-i}
\]

\[
+ \sum_i \text{tr}(\Theta', N'^M_{i-4}) v^{-i} + \sum_i \text{tr}(\Theta', N'^M_{i-2}) v^{-i} - \sum_i \text{tr}(\Theta', N'^M_{i-4}) v^{-i}
\]

\[
+ \sum_i \text{tr}(\Theta', N'^M_i) v^{-i} - \sum_i \text{tr}(\Theta', N'^M_{i-2}) v^{-i},
\]

\[
= \sum_i \text{tr}(\Theta, N^M_{i-4}) v^{-i} + \sum_i \text{tr}(\Theta, N^M_{i-6}) v^{-i} + \sum_i \text{tr}(\Theta, N^M_{i-2}) v^{-i}
\]

\[
+ \sum_i \text{tr}(\Theta', N'^M_{i-4}) v^{-i} + \sum_i \text{tr}(\Theta', N'^M_{i-2}) v^{-i} - \sum_i \text{tr}(\Theta', N'^M_{i-4}) v^{-i}
\]

\[
+ \sum_i \text{tr}(\Theta', N'^M_i) v^{-i} - \sum_i \text{tr}(\Theta', N'^M_{i-2}) v^{-i}.
\]
that is (using also 5.3(a)):
\[ e^w(R.M.R, \phi')v^{-4} + e^w(R.M.R, \phi')v^{-6} \]
\[ = e^w(M, \phi)v^{-4} + e^w(M, \phi)v^{-6} + e^w(M, \phi)v^{-2} + e^w(M, \phi)v^{-4} \]
\[ + e^{sw}(M, \phi)v^{-2} - e^{sw}(M, \phi)v^{-4} + e^{sw}(M, \phi) - e^{sw}(M, \phi)v^{-2} , \]
where \( \phi' \) is as in 5.1. We divide both sides by \( v^{-4} + v^{-6} \); we obtain
\[ e^w(R.M.R, \phi') = e^w(M, \phi) + e^w(M, \phi)v^2 - e^{sw}(M, \phi)v^2 + e^{sw}(M, \phi)v^4 . \]
This proves 5.2(d) and (equivalently) 5.2(b). Theorem 5.2 is proved.

6. Applications

6.1. Theorem 6.2 below describes the action of \( T_s + 1 \in H \) in the \( H \)-module \( M \) (see 3.5, 3.6) for a fixed \( s \in S \). We set
\[ I' = \{ z \in I; l(z) < l(sz) \}, \quad I'' = \{ z \in I; l(z) > l(sz) \} \]
\[ I_e = \{ z \in I; sz = zs^* \}, \quad I_n = \{ z \in I; sz \neq zs^* \} \]
\[ I'_e = I' \cap I_e, \quad I'_n = I' \cap I_n, \quad I''_e = I'' \cap I_e, \quad I''_n = I'' \cap I_n . \]
We denote by \( w \mapsto \tilde{w} \) the involution of \( I \) given by \( w \mapsto sw \) if \( w \in I_e \) and \( w \mapsto sws^* \) if \( w \in I_n \).

**Theorem 6.2.** In the \( H \)-module \( M \) the following identities hold for any \( z \in I \):
\[ (T_s + 1)a_z = (u + 1)(a_z + a_{\tilde{z}}) \text{ if } z \in I'_e, \]
\[ (T_s + 1)a_z = (u^2 - u)(a_z + a_{\tilde{z}}) \text{ if } z \in I''_e, \]
\[ (T_s + 1)a_z = a_z + a_{\tilde{z}} \text{ if } z \in I'_n, \]
\[ (T_s + 1)a_z = u^2(a_z + a_{\tilde{z}}) \text{ if } z \in I''_n . \]
(Recall that \( u = v^2 \).) We define a map \( I \rightarrow I' \), \( z \mapsto \tilde{z} \) by \( z \mapsto z \) if \( z \in I' \) and \( z \mapsto \tilde{z} \) if \( z \in I'' \). For any \( z \in I \) we set
\[ (T_s + 1)a_z = \sum_{y \in I} c_{y,z} a_y \]
where \( c_{y,z} \in A' \). The following equality (for any \( (M, \phi) \in \tilde{C}_2 \)) is a reformulation of Theorem 5.2:
\[ \sum_{y \in I'} \sum_{z \in I} \epsilon^z(M, \phi)c_{y,z}a_y = \sum_{y \in I'_n} (\epsilon^y(M, \phi)v^4 + \epsilon^y(M, \phi))a_y \]
\[ + \sum_{y \in I'_e} (\epsilon^y(M, \phi)v^4 - \epsilon^y(M, \phi)v^2 + \epsilon^y(M, \phi)v^2 + \epsilon^y(M, \phi))a_y . \]
Taking \((M, \phi) = (B_x, \phi_x)\) (see 3.2) we see that for any \(x \in I\) we have
\[
\sum_{z \in I} \epsilon^z(B_x, \phi_x) c_{y,z} = \epsilon^y(B_x, \phi_x) v^4 + \epsilon^y(B_x, \phi_x) \text{ if } y \in I_n
\]
\[
\sum_{z \in I} \epsilon^z(B_x, \phi_x) c_{y,z} = \epsilon^y(B_x, \phi_x)(v^4 - v^2) + \epsilon^y(B_x, \phi_x)(v^2 + 1) \text{ if } y \in I_e.
\]
Since the functions \(z \mapsto [x \mapsto \epsilon^z(B_x, \phi_x)]\) from \(I\) to the set of maps \(I \to \mathcal{A}'\) are linearly independent (see 3.4, 3.5) we deduce that for \(y \in I_n, z \in I\) we have
\[
c_{y,z} = v^4 \text{ if } y = \bar{z}; c_{y,z} = 1 \text{ if } y = z; c_{y,z} = 0 \text{ if } z \notin \{y, \bar{y}\};
\]
and for \(y \in I_e, z \in I\) we have
\[
c_{y,z} = v^4 - v^2 \text{ if } y = \bar{z}; c_{y,z} = v^2 + 1 \text{ if } y = z; c_{y,z} = 0 \text{ if } z \notin \{y, \bar{y}\}.
\]
Thus for any \(z \in I\) we have
\[
(T_s + 1)a_z = r_z a_{\bar{z}} + \sum_{y \in I} c_{y,z}a_y
\]
where \(r_z = v^4\) if \(z \in I_n, r_z = 1\) if \(z \in I_e, r_z = v^4 - v^2\) if \(z \in I'_n, r_z = v^2 + 1\) if \(z \in I'_e\).

We apply \((T_s + 1)\) to both sides of (a) and we use that \((T_s + 1)^2 = (u^2 + 1)(T_s + 1)\) in \(H\). We obtain
\[
(u^2 + 1)r_z a_{\bar{z}} + \sum_{y \in I} (u^2 + 1)c_{y,z}a_y
\]
\[
= r_z r_{\bar{z}} a_{\bar{z}} + r_z \sum_{y \in I} c_{y,\bar{z}}a_y + \sum_{y \in I'} r_y c_{y,\bar{z}}a_{\bar{y}} + \sum_{y \in I', y' \in I'} c_{y,z}c_{y',\bar{y}}a_{y'}
\]
for any \(z \in I\). Taking the coefficients of \(a_y\) with \(y \in I'\) in the two sides of the last equality we obtain
\[
(u^2 + 1)r_z \delta_{y,\bar{y}} = r_z r_{\bar{z}} \delta_{\bar{y},\bar{z}} + r_{\bar{y}} c_{\bar{y},z}.
\]
We see that if \(y \in I'\) then \(c_{\bar{y},z} = 0\) unless \(y = \bar{z}\) in which case we have \(c_{\bar{y},z} = r_{\bar{y}}^{-1}r_{\bar{z}}((u^2 + 1) - r\bar{z})\). The theorem follows.

6.3. By Theorem 6.2, the \(H\)-module \(M\) is identified with the \(H\)-module denoted in [L3, 0.3] by \(\bar{M}\) in such a way that to \(a_y \in M\) corresponds to \(a_y \in \bar{M}\) in [L3]. The duality functor \(M \mapsto D(M)\) [S, 5.9] can be used to define a \(Z\)-linear map
\[
-: \mathcal{K}_2(\bar{C}) \to \mathcal{K}_2(\bar{C})
\]
which satisfies \(u^n \xi = v^{-n} \xi\) for \(\xi \in \mathcal{K}_2(\bar{C}), n \in Z\), satisfies
\[
[B_x, \phi_x] = [B_x, \phi_x]
\]
for any \(x \in I\) and satisfies \(u^{-1}(T_s + 1) \xi = u^{-1}(T_s + 1) \xi\) for any \(s \in S\) and any \(\xi \in \mathcal{K}_2(\bar{C})\). It follows that the operator \(-: \mathcal{K}_2(\bar{C}) \to \mathcal{K}_2(\bar{C})\)
corresponds under the bijection \( \chi' \) in 3.5(a),(b) to the operator \( \sim : \mathcal{M} \to \mathcal{M} \) given by [L3, 0.2]. It follows that for \( x \in I \), \( \tilde{A}_x = v^{-l(x)} \sum_{y \in I : y \leq x} \bar{P}_{y,x}(u)a_y \in \mathcal{M} \) (see 3.5) is fixed by the operator \( \sim : \mathcal{M} \to \mathcal{M} \) in [L3, 0.2] where \( \bar{P}_{y,x} \) are as in 3.5(c). Using [L3, 0.4], it follows that for \( x \in I \) we have \( \tilde{A}_x = A_x \) (notation of [L3, 0.4]) and that for \( y \in I, y \leq x, \bar{P}_{y,x} \) coincides with the polynomial \( P_{x,y}^\sigma \) introduced in [L3, 0.4]. Using now 3.5(d), we see that for \( y \in I, y \leq x \) and \( \delta \in \{1,-1\} \), the following holds:

\[
(P_{y,x}(u) + \delta P_{x,y}^\sigma(u))/2 \in \mathbb{N}[u].
\]

This proves Conjecture 9.12 in [L3]. (In the case where \( W \) is a Weyl group this was already known from [LV].)

6.4. For \( x, y \in W \) we have \( c_x c_y = \sum_{z \in W} h_{x,y,z}(u) c_z \) where \( h_{x,y,z}(u) \in \mathbb{N}[u,u^{-1}] \).

Hence for \( z, w \in W \) we have \( c_z c_w c_{z+1} = \sum_{w' \in W} \tilde{h}_{z,w,w'}(u) c_w \) where

\[
\tilde{h}_{z,w,w'}(u) = \sum_{z' \in W} h_{z,w,z'}(u) h_{z',z+1,w'}(u) = \sum_{z' \in W} h_{w,z',z+1,w'}(u). 
\]

For \( z \in W, w \in I \) we write \( c_z A_w = \sum_{w' \in I} b_{z,w,w'}(v) A_{w'} \) where \( b_{z,w,w'}(v) \in \mathcal{A}' \).

For \( z \in W, w, w' \in I \) and \( \delta \in \{1,-1\} \) the following holds:

\[
(\tilde{h}_{z,w,w'}(u) + \delta b_{z,w,w'}(u))/2 \in \mathbb{N}[u,u^{-1}].
\]

The proof is analogous to that of 6.3(a). (In the case where \( W \) is a Weyl group and \( * = 1 \) this was stated in [LV, 5.1]). In particular,

7. The \( \mathbb{H} \)-module \( \mathcal{M}_c \)

7.1. Let \( \leq_L, \leq_{LR} \) be the preorders on \( W \) defined as in [KL]; let \( \sim_L, \sim_{LR} \) be the associated equivalence relations on \( W \). In this section we fix an equivalence class \( c \) for \( \sim_{LR} \) that is, a two-sided cell of \( W \). For \( w \in W \) we write \( w \leq_{LR} c \) if \( w \leq_{LR} w' \) for some \( w' \in c \); we write \( w <_{LR} c \) if \( w \leq_{LR} c \) and \( w \notin c \). Let \( \mathcal{M}_{\leq c} \) (resp. \( \mathcal{M}_{< c} \)) be the \( \mathcal{A}' \)-submodule of \( \mathcal{M} \) generated by the elements \( A_x \) with \( x \in I \) such that \( x \leq_{LR} c \) (resp. \( x <_{LR} c \)). We show:

(a) \( \mathcal{M}_{\leq c} \) is an \( \mathbb{H} \)-submodule of \( \mathcal{M} \).

With the notation in 6.4 it is enough to show that, if \( z \in W \) and \( w, w' \in I \) satisfy \( b_{z,w,w'}(v) \neq 0 \) and \( w \leq_{LR} c \) then \( w' \leq_{LR} c \). Using 6.4(a) we have \( \tilde{h}_{z,w,w'}(u) \neq 0 \) hence \( \sum_{z' \in W} h_{z,w,z'}(u) h_{z',z+1,w'}(u) \neq 0 \). It follows that for some \( z' \in W \) we have \( h_{z,w,z'}(u) \neq 0 \) and \( h_{z',z+1,w'}(u) \neq 0 \) hence \( w' \leq_{LR} z' \leq_{LR} w \) and \( w' \leq_{LR} w \) so that \( w' \leq_{LR} c \), as required.

A similar proof shows:

(b) \( \mathcal{M}_{< c} \) is an \( \mathbb{H} \)-submodule of \( \mathcal{M} \).

We now define \( \mathcal{M}_c = \mathcal{M}_{\leq c}/\mathcal{M}_{< c} \). From (a),(b) we see that \( \mathcal{M}_c \) inherits an \( \mathbb{H} \)-module structure from \( \mathcal{M}_{\leq c} \). For \( x \in I \cap c \) we denote the image of \( A_x \in \mathcal{M}_{\leq c} \) in \( \mathcal{M}_c \) again by \( A_x \). Note that \( \{A_x; x \in I \cap c\} \) is an \( \mathcal{A}' \)-basis of \( \mathcal{M}_c \).
7.2. In the remainder of this paper we assume that \((W, l)\) satisfies the boundedness property in [L2, 13.2]. (This holds automatically when \(W\) is finite or an affine Weyl group, and it probably holds in general.) Then the function \(a : W \rightarrow N\) is defined as in [L2, 13.6]. We recall the following properties:

(i) if \(z, z'\) in \(W\) satisfy \(z \sim_{LR} z'\) then \(a(z) = a(z')\);
(ii) if \(z, z'\) in \(W\) satisfy \(z \leq_L z'\) and \(a(z) = a(z')\) then \(z \sim_L z'\).
(See [L2, Ch.14, P4, P9] and [L2, Ch.15]; the assumptions 15.1(a),(b) are satisfied by [EW].

In this subsection we fix \(s \in S\). For \(w \in W\) we set \(\epsilon_w = (-1)^{l(w)}\). Let \(y, w \in I\).

As in [L2, 13.6], we define

\[ M_{y,w} = \mu''_{y,w} - \sum_{x \in I; y < x < w; s < x} \mu'_{y,x} \mu''_{x,w} - \delta_{sw,w} \mu'_{y,w} + \mu'_{s,y,w} \delta_{sy,sw}. \]

We have the following result.

Let \(w \in I \cap c\). In the \(H\)-module \(M_c\) we have the following identities:

(b) if \(sw < w\), then \(c_s A_w = (u + u^{-1}) A_w\);
(c) If \(sw > w\), then \(c_s A_w = \Xi + \sum_{z \in I; c; z < sw, \epsilon_z = \epsilon_w} M_{z,w} A_z\);

where \(\Xi\) is given by

\[ \Xi = A_{sw^*} \text{ if } sw \neq ws^* > w \text{ and } sw^* \in c, \]
\[ \Xi = 0, \text{ otherwise.} \]

To prove (b),(c) we make use of the formula for \(c_s A_w\) given in [L2, 4.3] (for Weyl groups) and [L3, 6.3] in the general case and show that all terms of that formula which involve \((v + v^{-1})\) belong to \(M_{ccc}\) and can therefore be neglected. It is enough to prove the following statements:

(d) If \(sw = ws^* > w\) then \(sw <_{LR} c\).
(e) If \(sw > w\) and \(z \in I, \epsilon_z = -\epsilon_w, s < z < sw, \mu'_{z,w} \neq 0\), then \(z <_{LR} c\).

We prove (d). Since \(sw > w\) we have \(sw \leq_L w\). If \(sw \sim_L w\) then by [KL, 2.4], for any \(t \in S\) such that \((sw)t < sw\) we have \(wt < w\); in particular, since \(sw^* = w < sw\) we have \(ws^* < w\), a contradiction. Thus, we have \(sw \not\sim_L w\).

From \(sw \leq_L w\), \(sw \not\sim_L w\), we deduce that \(sw \not\sim_{LR} w\). (If \(sw \sim_{LR} w\) then \(a(sw) = a(w)\), see (i); from \(sw \leq_L w, a(sw) = a(w)\) we deduce \(sw \sim_L w\) by (ii), a contradiction.) Now (d) follows.

We prove (e). Since \(\mu'_{z,w} \neq 0\), the coefficient of \(v^{l(w) - l(z) - 1}\) in \(P_{z,w}^\sigma(v)\) is \(\neq 0\). Using 6.3(a) we deduce that the coefficient of \(v^{l(w) - l(z) - 1}\) in \(P_{z,w}^\sigma(v)\) is \(\neq 0\). Since \(sz < z < sw > w\), the last coefficient is known to be equal to \(h_{s,w,z} (\text{an integer})\), see [KL]. Thus we have \(h_{s,w,z} \neq 0\) so that \(z \leq_L w\). If \(z \sim_L w\) then by [KL, 2.4],...
for any \( t \in S \) such that \( zt < z \) we have \( wt < w \); but from \( sz < z \), \( z \in I \), we deduce \( zs^* < z \) hence \( ws^* < w \). From \( ws^* < w \) and \( w \in I \) we deduce \( sw < w \), a contradiction. Thus we have \( z \not\sim_L w \). From \( z \leq_L w \), \( z \not\sim_L w \), we deduce that \( z \not\sim_{LR} w \). (If \( z \sim_{LR} w \) then \( a(z) = a(w) \) by (i); from \( z \leq_L w \), \( a(z) = a(w) \) we deduce \( z \sim_L w \) by (ii), a contradiction.) Now (e) follows.

This completes the proof of (b) and (c).

**7.3.** For \( \delta \in \{1, -1\} \) let \( M^\delta_c \) be the \( A' \)-submodule of \( M_c \) generated by \( \{A_x; x \in I \cap c, \epsilon_x = \delta\} \). From 7.2(b),(c) we see that \( M^\delta_c \) is an \( H \)-submodule of \( M \). Clearly, we have \( M_c = M^1_c \oplus M^{-1}_c \) as \( H \)-modules.

**7.4.** The formulas 7.2(b),(c) for the action of \( c_s \) in the basis \( \{A_x; x \in I \cap c\} \) of \( M_c \) are similar to those in a \( W \)-graph (see [KL]) since the coefficients in the right hand side of 7.2(c) are integer constants (but unlike the case of \( W \)-graph these integer constants can in principle depend on \( s \)). Note that the action of left multiplication by \( c_s \) in the basis \( \{A_x; x \in I\} \) of \( M \) is not given by a \( W \)-graph, due to the appearance of terms involving \( v + v^{-1} \).

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