$L^2$—interpolation with error and size of spectra

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Abstract
Given a compact set $S$ and a uniformly discrete sequence $\Lambda$, we show that "approximate interpolation" of delta–functions on $\Lambda$ by a bounded sequence of $L^2$–functions with spectra in $S$ implies an estimate on measure of $S$ through the density of $\Lambda$.

1 Introduction

Suppose $S$ is a bounded set on the real line $\mathbb{R}$. By $PW_S$ we shall denote the corresponding Paley–Wiener space:

$$PW_S := \{ f \in L^2(\mathbb{R}); \hat{f} = 0 \text{ on } \mathbb{R} \setminus S \},$$

where

$$\hat{f}(t) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{itx} f(x) \, dx$$

denotes the Fourier transform. It is well–known that each function $f \in PW_S$ can be extended to the complex plane as an entire function of finite exponential type.

Given a discrete set $\Lambda = \{ \lambda_j, j \in \mathbb{Z} \} \subset \mathbb{R}$, one says that $\Lambda$ is a uniformly discrete if

$$\inf_{j \neq k} |\lambda_j - \lambda_k| > 0.$$  

This infimum is called the separation constant of $\Lambda$. The following inequality is well known (see [7], p. 82):

$$\|f\|_{L^2} \geq C \|f|_\Lambda\|_\ell^2,$$  

for every $f \in PW_S$. (1)

Here $C > 0$ is a constant which depends only on the separation constant of $\Lambda$ and $S$, $f|_\Lambda$ denotes the restriction of $f$ on $\Lambda$, and

$$\|f\|_{L^2}^2 := \int_{\mathbb{R}} |f(x)|^2 \, dx, \quad \|f|_\Lambda\|_\ell^2 := \sum_{j \in \mathbb{Z}} |f(\lambda_j)|^2.$$
One can therefore regard the restriction $f|_\Lambda$ as an element of $l^2(\mathbb{Z})$, the $j-$th coordinate of $f|_\Lambda$ being $f(\lambda_j)$.

**Definition.** $\Lambda$ is called a set of interpolation for $PW_S$, if for every data $c \in l^2(\mathbb{Z})$ there exists $f \in PW_S$ such that

$$f|_\Lambda = c.$$  \hfill (2)

A classical interpolation problem is to determine when $\Lambda$ is a set of interpolation for $PW_S$.

The upper uniform density of a uniformly discrete set $\Lambda$ is defined as

$$D^+(\Lambda) := \lim_{r \to \infty} \max_{a \in \mathbb{R}} \frac{\#(\Lambda \cap (a, a+r))}{r}.$$  

A fundamental role of this quantity in the interpolation problem, in the case when $S$ is a single interval, was found by A. Beurling and J-P. Kahane. Kahane proved in [2] that for $\Lambda$ to be an interpolation set for $PW_S$ it is necessary that

$$D^+(\Lambda) \leq \frac{1}{2\pi} \text{mes } S,$$  \hfill (3)

and it is sufficient that

$$D^+(\Lambda) < \frac{1}{2\pi} \text{mes } S.$$  

Beurling ([1]) proved that the last inequality is necessary and sufficient for interpolation in the Bernstein space of all bounded on $\mathbb{R}$ functions with spectrum on the interval $S$.

The situation becomes much more delicate for disconnected spectra, already when $S$ is a union of two intervals. For the sufficiency part, not only the size but also the arithmetics of $\Lambda$ is important. On the other hand, Landau [4] extended the necessity part to the general case:

**Theorem A** Let $S$ be a bounded set. If a uniformly discrete set $\Lambda$ is an interpolation set for $PW_S$ then condition (3) is fulfilled.

### 2 Main result

Denote by $\{e_j, j \in \mathbb{Z}\}$ the standard orthogonal basis in $l^2(\mathbb{Z})$. When $S$ is compact, it is shown in [6] that Theorem A remains true under
a weaker assumption that only $e_j, j \in \mathbb{Z}$, admit interpolation by functions from $PW_S$ whose norms are uniformly bounded.

Let us say that $\delta$—functions on $\Lambda$ can be approximated with error $d$ by functions from $PW_S$, if for every $j \in \mathbb{Z}$ there exists $f_j \in PW_S$ satisfying

$$\|f_j|_\Lambda - e_j\|_2 \leq d, \ j \in \mathbb{Z}. \quad (4)$$

The aim of this paper is to show that this ‘approximate’ interpolation of $e_j$ already gives an estimate on the measure of $S$. The result below extends both Theorem A (for compact $S$) and the mentioned result from [6].

**Theorem 1** Let $S$ be a compact set, and $\Lambda$ a uniformly discrete set. Suppose there exist functions $f_j \in PW_S$ satisfying (4) for some $0 < d < 1$ and

$$\sup_{j \in \mathbb{Z}}\|f_j\|_{L^2} < \infty. \quad (5)$$

Then

$$D^+(\Lambda) \leq \frac{1}{2\pi(1 - d^2)} \text{mes } S. \quad (6)$$

Bound (6) is sharp for every $d$.

This result was announced in [5].

Theorem 1 will be proved in sec. 4. A variant of this result holds also when the norms of $f_j$ have a moderate growth, see sec. 5.

### 3 Lemmas

**3.1. Concentration.**

**Definition:** Given a number $c, 0 < c < 1$, we say that a linear subspace $X$ of $L^2(\mathbb{R})$ is $c$-concentrated on a set $Q$ if

$$\int_Q|f(x)|^2\,dx \geq c\|f\|_{L^2}^2, \ f \in X.$$

**Lemma 1** Given sets $S, Q \subset \mathbb{R}$ of positive measure and a number $0 < c < 1$, let $X$ be a linear subspace of $PW_S$ which is $c$-concentrated on $Q$. Then

$$\dim X \leq \frac{(\text{mes } Q)(\text{mes } S)}{2\pi c}.$$
This lemma follows from H. Landau’s paper [4] (see statements (iii) and (iv) in Lemma 1, [4]).

3.2. A remark on Kolmogorov’s width estimate.

**Lemma 2** Let $0 < d < 1$, and $\{u_j\}, 1 \leq j \leq n$, be an orthonormal basis in an $n$-dimensional complex Euclidean space $U$. Suppose that $\{v_j\}, 1 \leq j \leq n$, is a family of vectors in $U$ satisfying

$$\|v_j - u_j\| \leq d, \ j = 1, \ldots, n.$$  \tag{7}

Then for any $\alpha$, $1 < \alpha < 1/d$, one can find a linear subspace $X$ in $\mathbb{C}^n$ such that

(i) $\dim X > (1 - \alpha^2 d^2) n - 1$,

(ii) The estimate

$$\|\sum_{j=1}^{n} c_j v_j\|^2 \geq (1 - \frac{1}{\alpha}) \sum_{j=1}^{n} |c_j|^2,$$

holds for every vector $(c_1, ..., c_n) \in X$.

The classical equality for Kolmogorov’s width of “octahedron” (see [3]) implies that the dimension of the linear span of $v_j$ is at least $(1 - d^2) n$. This means that there exists a linear space $X \subset \mathbb{C}^n$, $\dim X \geq (1 - d^2) n$ such that the quadratic form

$$\|\sum_{j=1}^{n} c_j v_j\|^2$$

is positive on the unite sphere of $X$. The lemma above shows that by a small relative reduction of the dimension, one can get an estimate of this form from below by a positive constant independent of $n$.

We are indebted to E. Gluskin for the following simple proof of this lemma.

**Proof.** Given an $n \times n$ matrix $T = (t_{k,l}), k, l = 1, ..., n$, denote by $s_1(T) \geq ... \geq s_n(T)$ the singular values of this matrix (=the positive square roots of the eigenvalues of $TT^*$).

The following properties are well-known:

(a) (Hilbert–Schmidt norm of $T$ via singular values)

$$\sum_{j=1}^{n} s_j^2(T) = \sum_{k,l=1}^{n} |t_{k,l}|^2.$$
(b) (Minimax–principle for singular values)
\[ s_k(T) = \max_{L_k} \min_{x \in L_k, \|x\| = 1} \|Tx\|, \]
where the maximum is taken over all linear subspaces \( L_k \subseteq \mathbb{C}^n \) of dimension \( k \).

(c) \( s_{k+j}(T_1 + T_2) \leq s_k(T_1) + s_j(T_2) \), for all \( k + j \leq n \).

Denote by \( T_1 \) the matrix, whose columns are the coordinates of \( v \) in the basis \( u_k \), and \( T_2 := I - T_1 \), where \( I \) is the identity matrix. Then property (a) and (7) imply:

\[ \sum_{j=1}^{n} s_j^2(T_2) < d^2 n, \]
and hence:
\[ s_j^2(T_2) \leq d^2 \frac{n}{j}, \quad 1 \leq j \leq n. \]

Now (c) gives:
\[ s_k(T_1) \geq 1 - s_{n-k}(T_2) \geq 1 - d \sqrt{\frac{n}{n - k}}. \]

Taking the appropriate value of \( k \), one can obtain from (b) that there exists \( X \) satisfying conclusions of the lemma.

4 Proof of Theorem 1

1. Fix a small number \( \delta > 0 \) and set \( S(\delta) := S + [-\delta, \delta] \). Set
\[ g_j(x) := f_j(x)\varphi(x - \lambda_j), \quad \varphi(x) := \left( \frac{\sin(\delta x/2)}{\delta x/2} \right)^2. \tag{8} \]
Clearly, \( \varphi \in PW_{[-\delta, \delta]} \), so that \( g_j \in PW_{S(\delta)} \). Also, since \( \varphi(0) = 0 \) and \( |\varphi(x)| \leq 1, x \in \mathbb{R} \), it follows from (4) that each \( g_j|_{\Lambda} \) approximates \( e_j \) with an \( l^2 \)-error \( \leq d \):
\[ \|g_j|_{\Lambda} - e_j\|_{l^2} \leq d, \quad j \in \mathbb{Z}. \tag{9} \]

2. Given two numbers \( a \in \mathbb{R} \) and \( r > 0 \), set
\[ \Lambda(a, r) := \Lambda \cap (a - r, a + r), \quad n(a, r) := \#\Lambda(a, r). \]
For simplicity of presentation, in what follows we assume that \( \Lambda(a, r) = \{ \lambda_1, \ldots, \lambda_{n(a, r)} \} \).

For every \( g \in PW_{S(\delta)} \), we regard the restriction \( g|_{\Lambda(a, r)} \) as a vector in \( \mathbb{C}^{n(a, r)} \). It follows from (9) that the vectors \( v_j := g_j|_{\Lambda(a, r)} \) satisfy (7), where \( \{ u_j, j = 1, \ldots, n(a, r) \} \) is the standard orthogonal basis in \( \mathbb{C}^{n(a, r)} \).

In the rest of this proof, we shall denote by \( C \) different positive constants which do not depend on \( r \) and \( a \).

Fix a number \( \alpha > 1 \). By Lemma 2, there exists a subspace \( X = X(a, r) \subset \mathbb{C}^{n(a, r)} \), \( \dim X \geq (1 - \alpha^2 d^2)n(a, r) - 1 \), such that

\[
\left\| \left( \sum_{j=1}^{n(a, r)} c_j g_j \right)|_{\Lambda(a, r)} \right\|_2^2 \geq (1 - \frac{1}{\alpha}) \sum_{j=1}^{n(a, r)} |c_j|^2, \ (c_1, \ldots, c_{n(a, r)}) \in X.
\]

By (1), this gives:

\[
\| \sum_{j=1}^{n(a, r)} c_j g_j \|_{L^2}^2 \geq C \sum_{j=1}^{n(a, r)} |c_j|^2, \ (c_1, \ldots, c_{n(a, r)}) \in X. \tag{10}
\]

3. By (5), we have

\[
|f_j(x)|^2 = \left| \frac{1}{\sqrt{2\pi}} \int_S e^{-itx} \hat{f}_j(t) \, dt \right|^2 \leq \frac{\text{mes } S}{2\pi} \| \hat{f}_j \|_{L^2}^2 \leq C, \ j \in \mathbb{Z}.
\]

Observe also that, since \( \Lambda \) is uniformly discrete, we have \( n(a, r) \leq Cr \), for every \( a \in \mathbb{R} \) and \( r > 1 \).

4. Since \( |x - \lambda_j| \geq \delta r \) whenever \( \lambda_j \in (a-r, a+r) \) and \( |x-a| \geq r+\delta r \), the inequalities in step 3 and (8) give

\[
\int_{|x-a| \geq r+\delta r} \left| \sum_{j=1}^{n(a, r)} c_j g_j(x) \right|^2 \, dx =
\]

\[
\int_{|x-a| \geq r+\delta r} \left( \sum_{j=1}^{n(a, r)} c_j f_j(x) \right)^2 \, dx \leq
\]

\[
Cr \left( \sum_{j=1}^{n(a, r)} |c_j|^2 \right) \int_{|x| > \delta r} \left( \frac{2}{\delta x} \right)^4 \, dx \leq \frac{C}{\delta^3 r^3} \left( \sum_{j=1}^{n(a, r)} |c_j|^2 \right).
\]
This and (10) show that for every \( \epsilon > 0 \) there exists \( r(\epsilon) \) such that the linear space of functions

\[
g(x) = \sum_{j=1}^{n(a,r)} c_j g_j(x), \ (c_1, \ldots, c_n(a,r)) \in X,
\]

is \((1 - \epsilon)\)–concentrated on \((a - r - \delta r, a + r + \delta r)\) for every \( r \geq r(\epsilon) \), and every \( a \in \mathbb{R} \).

5. Lemma 1 now implies

\[
\text{mes } S(\delta) \geq 2\pi (1 - \epsilon) \frac{\dim X}{\text{mes } (a - r - \delta r, a + r + \delta r)} \geq \frac{2\pi (1 - \epsilon) (1 - \alpha^2 d^2) \# (\Lambda \cap (a - r, a + r) - 1)}{1 + \delta}.
\]

Taking the limit as \( r \to \infty \), where \( a = a(r) \) is such that the relative number of points of \( \Lambda \) in \((a - r, a + r)\) tends to \( D^+(\Lambda) \), we get

\[
\text{mes } S(\delta) \geq \frac{2\pi (1 - \epsilon)}{1 + \delta} (1 - \alpha^2 d^2) D^+(\Lambda).
\]

Since this is true for every \( \epsilon > 0, \delta > 0 \) and \( \alpha > 1 \), we conclude that (6) is true.

Let us now check that estimate (6) in Theorem 1 is sharp.

**Example.** Pick up a number \( a, 0 < a < \pi \), and set \( S := [-a, a] \), \( \Lambda := \mathbb{Z} \) and

\[
f_j(x) := \frac{\sin a(x - j)}{\pi (x - j)} \in \text{PW}_S, \ j \in \mathbb{Z}.
\]

We have for every \( j \in \mathbb{Z} \) that

\[
\|f_j|_Z - e_j\|_2^2 = \|f_0|_Z - e_0\|_2^2 = \sum_{k \neq 0} \left( \frac{\sin a k}{\pi k} \right)^2 + \left( \frac{a}{\pi} - 1 \right)^2 = \frac{a}{\pi} - \frac{a^2}{\pi^2} + \left( \frac{a}{\pi} - 1 \right)^2 = 1 - \frac{a}{\pi}.
\]

Hence, the assumptions of Theorem 1 are fulfilled with \( d^2 = 1 - a/\pi \).

On the other hand, since \( D^+(\mathbb{Z}) = 1 \), we see that \( \text{mes } S = 2\pi (1 - d^2) D^+(\mathbb{Z}) \), so that estimate (6) is sharp.
5 Interpolation with moderate growth of norms

When the norms of functions satisfying (4) have a moderate growth
\[ \|f_j\|_{L^2} \leq Ce^{|j|^\gamma}, \quad j \in \mathbb{Z}, \]  
where \( C > 0 \) and \( 0 < \gamma < 1 \), the statement of Theorem 1 remains true, provided the density \( D^+(\Lambda) \) is replaced by the upper density \( D^*(\Lambda) \),
\[ D^*(\Lambda) := \limsup_{a \to \infty} \frac{\#(\Lambda \cap (-a, a))}{2a}. \]
Observe that \( D^*(\Lambda) \leq D^+(\Lambda) \).

**Theorem 2** Let \( S \) be a compact set, and \( \Lambda \) a uniformly discrete set. Suppose there exist functions \( f_j \in PW_S \) satisfying (4) for some \( 0 < d < 1 \) and (11). Then
\[ D^*(\Lambda) \leq \frac{1}{2\pi(1 - d^2)} \text{mes } S. \]  
(12)

The upper density in this theorem cannot be replaced with the upper uniform density, see Theorem 2.5 in [6]. The growth estimate (11) can be replaced with every ‘nonquasianalytic growth’. However, we do not know if it can be dropped.

**Proof of Theorem 2.** The proof is similar to the proof of Theorem 1.

1. Fix numbers \( \delta > 0 \) and \( \beta, \gamma < \beta < 1 \). There exists a function \( \psi \in PW_{(-\delta, \delta)} \) with the properties:
\[ \psi(0) = 1, \quad |\psi(x)| \leq 1, \quad |\psi(x)| \leq Ce^{-|x|^\beta}, \quad x \in \mathbb{R}, \]  
(13)
where \( C > 0 \) is some constant. Such a function can be constructed as a product of \( \sin(\delta_j x)/(\delta_j x) \) for certain sequence of \( \delta_j \) (see Lemma 2.3 in [6]).

Set
\[ h_j(x) := f_j(x)\psi(x - \lambda_j), \quad j \in \mathbb{Z}. \]
Then each \( h_j|_{\Lambda} \) belongs to \( PW_{S(\delta)} \) and approximates \( e_j \) with an \( l^2 \)-error \( \leq d \).

2. Set
\[ \Lambda_r := \Lambda \cap (-r, r), \]
and denote by $C$ different positive constants independent on $r$.

The argument in step 2 of the previous proof shows that there exists a linear space $X = X(r)$ of dimension $\geq (1 - \alpha^2 d^2)\#\Lambda_r - 1$ such that

$$\| \sum_{j \in \Lambda_r} c_j h_j(x) \|_{L^2}^2 \geq C \sum_{j \in \Lambda_r} |c_j|^2,$$

for every vector $(c_j) \in X$.

3. Since $\Lambda$ is uniformly discrete, we have $\# \Lambda_r \leq Cr$ and $\max \{|j|, j \in \Lambda_r\} \leq Cr, r > 1$. The latter estimate and (11) give:

$$|f_j(x)|^2 \leq \left( \frac{1}{\sqrt{2\pi}} \int_S |\hat{f}_j(t)| \, dt \right)^2 \leq \frac{\mes S}{2\pi} \|\hat{f}_j\|_{L^2}^2 \leq Ce^{Cr\gamma}, \quad j \in \mathbb{Z}.$$

4. Using the estimates in step 3 and (13), we obtain:

$$\int_{|x| \geq r + \delta r} \left| \sum_{j \in \Lambda_r} c_j h_j(x) \right|^2 \, dx =$$

$$\int_{|x| \geq r + \delta r} \left| \sum_{j \in \Lambda_r} c_j f_j(x)\psi(x - \lambda_j) \right|^2 \, dx \leq$$

$$\left( \sum_{j \in \Lambda_r} |c_j|^2 \right) \left( Cr e^{Cr\gamma} \int_{|x| > \delta r} e^{-2|x|^\beta} \, dx \right).$$

Since $\beta > \gamma$, the last factor tends to zero as $r \to \infty$. This and the estimate in step 2 show that for every $\epsilon > 0$ there exists $r(\epsilon)$ such that the linear space of functions

$$\sum_{j \in \Lambda_r} c_j h_j(x), \quad (c_j) \in X,$$

is $(1 - \epsilon)$-concentrated on $(-r - \delta r, r + \delta r)$, for all $r \geq r(\epsilon)$.

5. Now, by Lemma 1, we obtain:

$$\mes S(\delta) \geq \frac{2\pi (1 - \epsilon) (1 - \alpha^2 d^2) \# (\Lambda \cap (-r, r) - 1)}{1 + \delta}.$$

By taking the upper limit as $r \to \infty$, this gives

$$\mes S(\delta) \geq \frac{2\pi (1 - \epsilon)}{1 + \delta} (1 - \alpha^2 d^2) D^*(\Lambda).$$

Since this is true for every $\epsilon > 0, \delta > 0$ and $\alpha > 1$, we conclude that (12) is true.
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