ON COMPLETE SYSTEM OF COVARIANTS FOR THE BINARY FORM OF DEGREE 8

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Abstract. A minimal system of homogeneous generating elements of the algebra of covariants for the binary form of degree 8 is calculated.

1. Introduction

Let $V_n$ be a vector $\mathbb{C}$-space of the binary forms of degree $d$ considered with natural action of the group $G = SL(2, \mathbb{C})$. Let us extend the action of the group $G$ to the polynomial functions algebra $\mathbb{C}[V_d \oplus \mathbb{C}^2]$. Denote by $C_d = \mathbb{C}[V_d \oplus \mathbb{C}^2]^G$ the corresponding subalgebra of $G$-invariant functions. In the vocabulary of classical invariant theory the algebra $C_d$ is called the algebra of covariants of the binary form of $d$-th degree. Let $C_d^+$ be an ideal of $C_d$ generated by all homogeneous elements of positive power. Denote by $\overline{C}_d$ a set of homogeneous elements of $C_d^+$ such that their images in $C_d^+/(C_d^+)^2$ form a basis of the vector space. The set $\overline{C}_d$ is called complete system of covariants of the $d$-th degree binary form. Elements of $\overline{C}_d$ form a minimal system of homogeneous generating elements of the invariants algebra $C_d$. Denote by $c_d$ a number of elements of the set $\overline{C}_d$.

The complete systems of covariants was a topic of major research interest in classical invariant theory of the 19th century. It is easy to show that $c_1 = 0$, $c_2 = 2$, $c_3 = 4$. A complete system of covariants in the case $d = 4$ was calculated by Bool, Cayley, Eisenstein, see survey [1]. The complete systems of invariants and covariants in the cases $d = 5, 6$ were calculated by Gordan, see [2]. In particular, $c_4 = 5$, $c_5 = 23$, $c_6 = 26$.

Gall’s attempt [3] to discover the complete system for the case $d = 7$ was unsuccessful. He did offered a system of 151 covariants but the system was not a minimal system, see [11], [20]. Also, Sylvester’s attempts [6], [7] to find the cardinality $c_7$ and covariant’s degree-order distribution of the complete system were mistaken, see [11]. Therefore, the problem of finding a minimal system of homogeneous generating elements (or even a cardinality of the system) of the algebra of covariants for the binary form of degree 7 is still open.

The case $d = 8$ was studied by Sylvester and Gall but they have obtained completely different results. By using Sylvester-Cayley technique, Sylvester in [6] got that $c_8 = 69$. Gall in [9], evolving the Gordan’s constructive method, offered 68 covariants as a complete system of covariants for the case $d = 8$. Also they have got a different degree-order distribution of the covariants, see in [4], a part of their long discussion.

For the cases $d = 9, 10$ Sylvester in [7] calculated the cardinalities $c_9$ and $c_{10}$ but the present author, using a computer, found numerous mistakes in those computations.

Therefore, the complete systems of covariants for the binary form are so far known only up to degree 6, see [5], [1].
For the case \( d = 8 \) Gall have found the 68 invariants in implicit way by the symbolic method. A covariant has very simple symbolic representation but it is too hard to check out and be sure if the covariant is an irreducible one. The verification could be done for an covariants in their explicit representation but some of offered by Gall invariants is impossyble even for computer calculation due to hight tranvectant’s order.

To solve the computation problem we offer a form of represenation of the covariants, which is an intermediate form between the highly unwieldy explicit represenation and too "compressed" symbolic represenation. Also we have found new symbolic representation of the covariants, which is different from Gall’s representation. The representation use transvectants of low orders. A first step in the simplification of a calculation is calculation a semi-invariants instead calculation of covariants. Let us consider a covariant as a polynomial of generating functions of the polynomial functions algebra \( \mathbb{C}[V_d \oplus \mathbb{C}^2] \). Then a semi-invariant is just a leading coefficient of the polynomial with respect to usual lexicographical ordering, see [15], [5]. A semi-invariant is an invariant of upper unipotent matrix subalgebra of Lie algebra \( \mathfrak{sl}_2 \).

Let us indentify the algebra \( \mathbb{C}[V_d] \) with the algebra \( \mathbb{C}[X_d] := \mathbb{C}[t, x_1, x_2, \ldots, x_d] \), and the algebra \( \mathbb{C}[V_d \oplus \mathbb{C}^2] \) identify with the polynomial algebra \( \mathbb{C}[t, x_1, x_2, \ldots, x_n, Y_1, Y_2] \).

The generating elements \( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \) of the tangent Lie algebra \( \mathfrak{sl}_2 \) act on \( \mathbb{C}[V_d] \) by derivations

\[
D_1 := t \frac{\partial}{\partial x_1} + 2 x_1 \frac{\partial}{\partial x_2} + \cdots + d x_{d-1} \frac{\partial}{\partial x_d},
\]

\[
D_2 := d x_1 \frac{\partial}{\partial t} + (d - 1) x_2 \frac{\partial}{\partial x_1} + \cdots + x_d \frac{\partial}{\partial x_{d-1}}.
\]

It follows that the semi-invariant algebra coincides with an algebra \( \mathbb{C}[X_d]^{D_1} \), of polynomial solutions of the following first order PDE, see [14], [15]:

\[
\frac{t}{x_1} \frac{\partial u}{\partial x_1} + 2 x_1 \frac{\partial u}{\partial x_2} + \cdots + d x_{d-1} \frac{\partial u}{\partial x_d} = 0,
\]

where \( u \in \mathbb{C}[X_d] \), and

\[
\mathbb{C}[X_d]^{D_1} := \{ f \in \mathbb{C}[X_d] | D_1(f) = 0 \}.
\]

It is easy to get an explicit form of the algebra \( \mathbb{C}[X_d]^{D_1} \), see, for example, [13]. Namely –

\[
\mathbb{C}[X_d]^{D_1} = \mathbb{C}[t, z_2, \ldots, z_d][\frac{1}{t}] \cap \mathbb{C}[X],
\]

here \( z_i \) are some functional independent semi-invariants of power \( i \). The polynomials \( z_i \) arised in the first time in Cayley, see [15]. Therefore, any semi-invariant we may write as rational fraction of \( \mathbb{C}[Z_d][\frac{1}{t}] := \mathbb{C}[t, z_2, \ldots, z_d][\frac{1}{t}] \). This form of semi-invariants is more compact than their standard form as a polynomial of \( \mathbb{C}[X_d] \). In this case, a semi-invariant has terms number in tens times less than terms number of the corresponding covariant that make a computation crucial easy. By using Robert’s theorem, see [12], knowing a semi-invariant one may restore a corresponding covariant.

In the paper on explicit form have found a complete system of covariants for the binary form of degree 8. The system consists of 69 covariants, i.e. \( c_8 = 69 \). Also, the covariants degree-order distribution coincides completely with Sylvester’s offered distribution.
All calculation were done with Maple.

2. Preliminaries.

Before any calculation we try make a simplification of a covariant represenation and their computation. Let $\kappa : C_d \rightarrow \mathbb{C}[X_d]^{D_1}$ be the $\mathbb{C}$-linear map takes each homogeneous covariant of order $k$ to his leading coefficient, i.e., a coefficient of $Y_1^k$. Follow by classical tradition an element of the algebra $\mathbb{C}[X_d]^{D_1}$ is called semi-invariants, a degree of a homogeneous covariant with respect to the variables set $X_d$ is called degree of the covariant and its degree with respect to the variables set $Y_1, Y_2$ is called order.

Suppose $F = \sum_{i=0}^{m} f_i \binom{m}{i} Y_1^{m-i} Y_2^i$ be a covariant of order $m$, $\kappa(F) = f_0 \in \mathbb{C}[X_d]^{D_1}$. The classical Robert’s theorem, \cite{12}, states that the covariant $F$ is completelly and uniquely determined by its leading coefficient $f_0$, namely

$$F = \sum_{i=0}^{m} D_2^i(f_0) Y_1^{m-i} Y_2^i.$$ 

On the other hand, every semi-invariant is a leading coefficient of some covariant, see \cite{15}, \cite{5}. This give us well defined explicit form of the inverse map $\kappa^{-1} : \mathbb{C}[X_d]^{d_1} \rightarrow C_d$, namely

$$\kappa^{-1}(a) = \sum_{i=0}^{\text{ord}(a)} \frac{D_2^i(a)}{i!} Y_1^{\text{ord}(a) - i} Y_2^i,$$

here $a \in \mathbb{C}[X_d]^{d_1}$ and $\text{ord}(a)$ is an order of the element $a$ with respect to the locally nilpotent derivation $D_2$, i.e. $\text{ord}(a) := \max\{s, D_2^s(a) \neq 0\}$. For example, since $\text{ord}(t) = d$, we have

$$\kappa^{-1}(t) = \sum_{i=0}^{\text{ord}(t)} \frac{D_2^i(t)}{i!} Y_1^{\text{ord}(t) - i} Y_2^i = t Y_1^{d} + \sum_{i=1}^{d} \binom{d}{i} x_i Y_1^{d-i} Y_2^i.$$

As we see the $\kappa^{-1}(t)$ is just the basic binary form. From polynomial functions point of view the covariant $\kappa^{-1}(t)$ is the evaluation map.

Thus, the problem of finding of complete system of the algebra $\mathcal{C}_d$ is equivalent to the problem of finding of complete system of semi-covariants’s algebra $\mathbb{C}[X]^{D_1}$. It is well known classical results.

A structure of constants algebras for such locally nilpotent derivations can be easy determined, see for example \cite{13}. In particular, for the derivation $D_1$ we get

$$\mathbb{C}[X_d]^{D_1} = \mathbb{C}[t, \sigma(x_2), \ldots, \sigma(x_d)][\frac{1}{t}] \cap \mathbb{C}[X_d],$$

where $\sigma : \mathbb{C}[X_d] \rightarrow \mathbb{C}(X_d)^{D_1}$ is a ring homomorphism defined by

$$\sigma(a) = \sum_{i=0}^{\infty} d_i^1(a) \frac{\lambda^i}{i!}, \lambda = -\frac{x_1}{t}.$$
Hence a generating elements of the semi-invariant algebra $\mathbb{C}_\sigma$ is

$$\sigma(x_i) = \frac{z_{i+1}}{t^i},$$

where $z_i \in \mathbb{C}(X_d)^{D_1}$ and

$$z_i := \sum_{k=0}^{i-2} (-1)^k \binom{i}{k} x_{i-k} x_1^k t^{i-k-1} + (i-1)(-1)^{i+1} x_1, i = 2, \ldots, d.$$

Especially

$$z_2 = x_2 t - x_1$$
$$z_3 = x_3 t^2 + 2 x_1^3 - 3 x_1 x_2 t$$
$$z_4 = x_4 t^3 - 3 x_1^4 + 6 x_1^2 x_2 t - 4 x_1 x_3 t^2$$
$$z_5 = x_5 t^4 + 4 x_1^5 - 10 x_1^3 x_2 t + 10 x_1^2 x_3 t^2 - 5 x_1 x_4 t^3$$
$$z_6 = x_6 t^5 - 5 x_1^6 + 15 x_1^4 x_2 t - 20 x_1^3 x_3 t^2 + 15 x_1^2 x_4 t^3 - 6 x_1 x_5 t^4$$
$$z_7 = x_7 t^6 + 6 x_1^7 - 21 x_1^5 x_2 t + 35 x_1^4 x_3 t^2 - 35 x_1^3 x_4 t^3 + 21 x_1^2 x_5 t^4 - 7 x_1 x_6 t^5$$
$$z_8 = 28 x_1^6 x_2 t - 56 x_1^5 x_3 t^2 - 56 x_1^4 x_4 t^3 + 28 x_1^3 x_5 t^4 - 8 x_1 x_7 t^6 - 7 x_1^8 + 70 x_1^4 x_4 t^3 + x_8 t^7$$

Thus we obtain

$$\mathbb{C}[X_d]^{D_1} = \mathbb{C}[t, z_2, \ldots, z_d][\frac{1}{t}] \cap \mathbb{C}[X_d].$$

Hence, a generating elements of the semi-invariant algebra $\mathbb{C}(X_d)^{D_1}$ we may looking as a rational fraction $\frac{f(z_2, \ldots, z_d)}{t^s}$, $f \in \mathbb{C}[Z_d] := \mathbb{C}[t, z_2, \ldots, z_d]$, $s \in \mathbb{Z}_+$. To make a calculation with an invariants in such representation we need know an action of the operator $D_2$ in new coordinates $t, z_2, \ldots, z_d$. Denote by $D$ extention of the derivation $D_2$ to the algebra $\mathbb{C}[Z_d][\frac{1}{t}]$:

$$D := D_2(t) \frac{\partial}{\partial t} + D_2(z_2) \frac{\partial}{\partial z_2} + \ldots + D_2(z_d) \frac{\partial}{\partial z_d}.$$
To calculate the semi-invariants we need have an analogue of the transvectants. Suppose
\[ F = \sum_{i=0}^{m} f_i \binom{m}{i} Y_1^{m-i} Y_2^i, \quad G = \sum_{i=0}^{k} g_i \binom{k}{i} Y_1^{k-i} Y_2^i, \quad f_i, g_i \in \mathbb{C}[Z_d][\frac{1}{t}], \]
are two covariants of the degrees \( m \) and \( k \) respectively. Let
\[ (F, G)^r = \sum_{i=0}^{r} (-1)^i \binom{r}{i} \frac{\partial^i F}{\partial Y_1^{r-i} \partial Y_2^i} \frac{\partial^r G}{\partial Y_1^r \partial Y_2^{r-i}}, \]
be their \( r \)-th transvectant. The following lemma give us rule how to find the semi-invariant \( \kappa((F, G)^r) \) without of direct computing of the covariant \( (F, G)^r \).

**Lemma 1.** The leading coefficient \( \kappa((F, G)^r) \) of the covariant \( (F, G)^r \), \( 0 \leq r \leq \min(m, k) \) is calculating by the formula
\[
\kappa((F, G)^r) = \sum_{i=0}^{r} (-1)^i \binom{r}{i} \frac{D^i(\kappa(F))}{[m]_i} \bigg|_{x_1=0, \ldots, x_r=0} \frac{D^{r-i}(\kappa(G))}{[k]_{r-i}} \bigg|_{x_1=0, \ldots, x_r=0},
\]
here \( [a]_i := a(a-1) \ldots (a-(i-1)), a \in \mathbb{Z} \).

The proof is in [20].

Let \( f, g \) be two semi-invariants. Their numer are a polynomials of \( z_2, \ldots, z_d \) with rational coefficients. Then the semi-invariant \( \kappa((\kappa^{-1}(f), \kappa^{-1}(g))^r) \) be a fraction and their numer be a polynomial of \( z_2, \ldots, z_d \) with rational coefficients too. Therefore we may multiply \( \kappa((\kappa^{-1}(f), \kappa^{-1}(g))^r) \) by some rational number \( q_r(f, g) \in \mathbb{Q} \) such that the numer of the expression \( q_r(f, g)\kappa((\kappa^{-1}(f), \kappa^{-1}(g))^r) \) be now a polynomial with an integer coprime coefficients. Put
\[
[f, g]^r := q_r(f, g)\kappa((\kappa^{-1}(f), \kappa^{-1}(g))^r), \quad 0 \leq r \leq \min(\text{ord}(f), \text{ord}(g)).
\]
The expression \([f, g]^r\) is said to be the \( r \)-th semitransvectant of the semi-invariants \( f \) and \( g \).

The following statements is direct consequences of corresponding transvectant properties, see [15]:

**Lemma 2.** Let \( f, g \) be two semi-invariants. Then the following conditions hold
\begin{enumerate}
  \item[(i)] the semitransvectant \([t, f g]^i\) is reducible for \( 0 \leq i \leq \min(d, \max(\text{ord}(f), \text{ord}(g)))\);
  \item[(ii)] if \( \text{ord}(f) = 0 \), then \([t, f g]^i = f[t, g]^i]\);
  \item[(iii)] \( \text{ord}([f, g]^i) = \text{ord}(f) + \text{ord}(g) - 2i \);
  \item[(iv)] \( \text{ord}(z_2^{i_1} z_3^{i_2} \cdots z_d^{i_d}) = d(i_2 + i_3 + \cdots + i_d) - 2(2i_2 + 3i_3 + \cdots + d i_d) \).
\end{enumerate}

3. Calculation

Let \( \overline{C}_{8,i} := (C_8)_i \) be a subset of \( \overline{C}_8 \) whose elements has degree \( i \). Let \( C_+ \) be the ideal \( \sum_{i>0} C_{8,i} \) of \( C_8 \), and \( \overline{C}_{8,i} := \overline{C}_8 \cap C_{8,i} \). The number \( \delta_i \) of linearly independent irreducible invariants of degree \( i \) is calculated by the formula \( \delta_i = \dim C_{8,i} - \dim(C_+^2) \). A dimension of the vector space \( C_{d,i} \) is calculated by Sylvester-Cayley formula, see for example [14], [16]:
\[
\dim C_{d,i} = \left( \frac{1 - T^{d+1})(1 - T^{d+2}) \cdots (1 - T^{d+i})}{(1 - T^2)(1 - T^3) \cdots (1 - T^d)} \right)_{T=1}.
\]
A dimension of the vector space \((C^2_x)_i\) is calculating by the formula \(\dim(C^2_x)_i = \sigma_i - \dim S_i\). Here \(\sigma_i\) is a coefficient of \(T^i\) in Poincaré series \(\prod_{k<i}(1 - T^k)^{\delta_k}\) of a graded algebra generated by the system of homogeneous elements \(\cup_{k<i}C_{8,k}\), and \(S_i\) is a vector subspace of \((C^2_x)_i\) generated by syzygies. The dimension \(\dim S_i\) one may find by direct Maple calculation.

If we already have calculated the set \(\overline{C}_{8,i}\), then the elements of the set \(\overline{C}_{8,i+1}\) we are seeking as an irreducible elements of a basis of a vector space generated by semitransvectants of the form \([t, u v]^r, u \in C_{8,i}, v \in C_{8,k}, l + k = i, \max(\text{ord}(u), \text{ord}(v)) \leq r \leq 8\). It is a standard linear algebra problem.

The unique semi-invariant of the degree one obviously is \(t, \text{ord}(t) = 8\).

For \(i = 2\) have \(\dim C_{8,2} = 5, \sigma_2 = 1\), thus \(\delta_{8,2} = 4\). The semi-transvectants \([t, t]^k\) are equal to zero for odd \(i\). Put

\[
\begin{align*}
dv_1 &:= [t, t]^2 = z_2 = x_2 t - x_1^2, \\
dv_2 &:= [t, t]^4 = \frac{z_4 + 3 z_2^2}{t^2} = x_4 t - 4 x_1 x_3 + 3 x_2^2, \\
dv_3 &:= [t, t]^6 = \frac{z_6 + 15 z_2 z_4 - 10 z_3^2}{t^4} = x_6 t - 6 x_1 x_5 + 15 x_2 x_4 - 10 x_3^2, \\
dv_4 &:= [t, t]^8 = \frac{z_8 + 28 z_2 z_6 - 56 z_3 z_5 + 35 z_4^2}{t^6} = -8 x_1 x_7 + x_8 t + 28 x_2 x_6 - 56 x_3 x_5 + 35 x_4^2, \\
\text{ord}(dv_1) &= 12, \text{ord}(dv_2) = 8, \text{ord}(dv_3) = 4, \text{ord}(dv_4) = 0.
\end{align*}
\]

The polynomials \(t^2, dv_1, dv_2, dv_3, dv_4\) are linear independent. Therefore, the set \(\overline{C}_{8,2}\), consists of the irreducible semi-invariants \(dv_1, dv_2, dv_3, dv_4\).

For \(i = 3\) we have \(\dim C_{8,3} = 13, \sigma_3 = 5, \dim S_3 = 0\), thus \(\delta_{8,3} = 8\).

The set \(\overline{C}_{8,3}\) consists of the following 8 irreducible semi-invariants:

\[
\begin{align*}
tr_1 &= [t, dv_1]^3, \text{ord}(tr_1) = 14, \quad tr_2 = [t, dv_1]^4, \text{ord}(tr_2) = 12, \\
tr_3 &= [t, dv_1]^5, \text{ord}(tr_3) = 10, \quad tr_4 = [t, dv_1]^7, \text{ord}(tr_4) = 6, \\
tr_5 &= [t, dv_2]^4, \text{ord}(tr_5) = 8, \quad tr_6 = [t, dv_2]^6, \text{ord}(tr_6) = 4, \\
tr_7 &= [t, dv_2]^8, \text{ord}(tr_5) = 0, \quad tr_8 = [t, dv_1], \text{ord}(tr_6) = 18.
\end{align*}
\]

For \(i = 4\) we have \(\dim C_{8,4} = 33, \sigma_4 = 23, \dim S_4 = 0\), thus \(\delta_{8,4} = 10\). The set \(\overline{C}_{8,4}\) consists of the following 10 irreducible semi-invariants:

\[
\begin{align*}
ch_1 &= [t, tr_4]^2, \text{ord}(ch_1) = 10, \quad ch_2 = [t, tr_4]^5, \text{ord}(ch_2) = 4, \\
ch_3 &= [t, tr_5]^8, \text{ord}(ch_3) = 0, \quad ch_4 = [t, tr_6]^4, \text{ord}(ch_4) = 4, \\
ch_5 &= [t, tr_1]^4, \text{ord}(ch_5) = 14, \quad ch_6 = [t, tr_1]^5, \text{ord}(tr_6) = 12, \\
ch_7 &= [t, tr_1]^6, \text{ord}(ch_7) = 10, \quad ch_8 = [t, tr_1]^7, \text{ord}(ch_8) = 8, \\
ch_9 &= [t, tr_2], \text{ord}(ch_9) = 18, \quad ch_{10} = [t, tr_2]^7, \text{ord}(ch_{10}) = 6.
\end{align*}
\]

For \(i = 5\) we have \(\dim C_{8,5} = 73, \sigma_5 = 65\). Below is typical instance how \(\dim S_i\) is calculated.

The vector space \((C^2_x)_5 = t C_{8,4} + C_{8,2} C_{8,3}\) is generated by the following 65 elements:

\[
\begin{align*}
t ch_1, \ldots, t ch_{10}, \\
t^2 tr_1, \ldots, t^2 tr_8, \\
dv_1 tr_1, \ldots, dv_4 tr_8, \\
t^3 dv_1, \ldots, t^3 dv_4, \\
t^5.
\end{align*}
\]
To find a basis of the vector space $S_5$ of syzygies let us equate the system:

$$\alpha_1 t \ ch_1 + \alpha_2 t \ ch_2 + \cdots + \alpha_{65} t^5 = 0.$$  

By solving it we get

$$\begin{align*}
(55 \ tr_1 \ dv_3 - 55 \ tr_3 \ dv_2 + ch_7 \ t - 12 \ ch_1 \ t) \ \alpha_{19} + \\
+ (\frac{383}{5} \ tr_4 \ t^2 + ch_5 \ t - \frac{176}{5} \ tr_8 \ dv_3 + \frac{176}{5} \ tr_3 \ dv_1) \ \alpha_{24} + \\
+ (126 \ tr_1 \ dv_1 - ch_9 \ t + tr_3 \ t^2 - 126 \ tr_8 \ dv_2) \ \alpha_{11} = 0.
\end{align*}$$

Therefore, the vector space $S_5$ is generated by the 3 syzygies:

$$\begin{align*}
-12 \ ch_1 \ t + 55 \ tr_1 \ dv_3 - 55 \ tr_3 \ dv_2 + ch_7 \ t = 0, \\
5 \ ch_5 \ t + 383 \ tr_4 \ t^2 - 176 \ tr_8 \ dv_3 + 176 \ tr_3 \ dv_1 = 0, \\
- \ ch_9 \ t - 126 \ tr_8 \ dv_2 + 126 \ tr_1 \ dv_1 + tr_3 \ t^2 = 0.
\end{align*}$$

Thus $\dim S_5 = 3$ and $\delta_{8,5} = 11$ and the set $\overline{\sigma}_{8,5}$ consists of 11 irreducible semi-invariants. We search the semi-invariants as semitransvectants of the form $[t, u]^i$, $u \in (C_+^2)^4$. By using Lemma 2 we have that irreducibles covariants be only for the following values of $u$:

$$ch_1, \ ch_2, \ ch_4, \ ch_5, \ ch_6, \ ch_7, \ ch_8, \ ch_9, \ ch_{10}, \ dv_3^2.$$  

We calculate the 65 semitransvectants of the forms $[t, ch_i]^k$, $k = 1, \ldots, \min(8, \ord(ch_i))$, $i = 1, 2, 4, \ldots, 10$, $[t, dv_3^2]^k$, $k = 5, \ldots, 8$, and select 11 linearly independent such that ones dont belong to the vector space $(C_+^2)^5$ :

$$\begin{align*}
pt_1 &= [t, dv_3^2]^6, \ord(pt_1) = 4, \\
pt_2 &= [t, dv_3^2]^7, \ord(pt_2) = 2, \\
pt_3 &= [t, dv_3^2]^8, \ord(pt_3) = 0, \\
pt_4 &= [t, ch_1]^2, \ord(pt_4) = 14, \\
pt_5 &= [t, ch_1]^4, \ord(pt_5) = 10, \\
pt_6 &= [t, ch_1]^5, \ord(pt_6) = 8, \\
pt_7 &= [t, ch_1]^7, \ord(pt_7) = 4, \\
pt_8 &= [t, ch_2], \ord(pt_8) = 10, \\
pt_9 &= [t, ch_4], \ord(pt_9) = 10, \\
pt_{10} &= [t, ch_4]^3, \ord(pt_{10}) = 6, \\
pt_{11} &= [t, dv_3^2]^5, \ord(pt_{11}) = 6.
\end{align*}$$

Thus, $\overline{\sigma}_{8,5} = \{pt_1, pt_2, \ldots, pt_{11}\}$.

For $i = 6$ we have $\dim C_{8,6} = 151$, $\sigma_5 = 172$, $\dim S_6 = 30$. Thus $\delta_6 = 151 - (172 - 30) = 9$, the set $\overline{\sigma}_{8,6}$ consists of the 9 irreducible semi-invariants:

$$\begin{align*}
sh_1 &= [t, tr_6 \ dv_3]^6, \ord(sh_1) = 4, \\
sh_2 &= [t, tr_6 \ dv_3]^7, \ord(sh_2) = 2, \\
sh_3 &= [t, tr_6 \ dv_3]^8, \ord(sh_3) = 0, \\
sh_4 &= [t, pt_5]^5, \ord(sh_4) = 8, \\
sh_5 &= [t, pt_6]^5, \ord(sh_5) = 6, \\
sh_6 &= [t, pt_8]^6, \ord(sh_6) = 6, \\
sh_7 &= [t, pt_9]^4, \ord(sh_7) = 6, \\
sh_8 &= [t, pt_9]^7, \ord(sh_8) = 4, \\
sh_9 &= [t, pt_9]^2, \ord(sh_9) = 10.
\end{align*}$$

For $i = 7$ we have $\dim C_{8,7} = 289$, $\sigma_7 = 385$, $\dim S_7 = 104$. Thus $\delta_7 = 8$, the set $\overline{\sigma}_{8,7}$ consists of the 8 irreducible semi-invariants:

$$\begin{align*}
si_1 &= [t, ch_{10} \ dv_3]^7, \ord(si_1) = 4, \\
si_2 &= [t, tr_6^2]^5, \ord(si_2) = 6, \\
si_3 &= [t, tr_6^2]^7, \ord(si_3) = 2, \\
si_4 &= [t, tr_6^2]^8, \ord(si_4) = 0, \\
si_5 &= [t, sh_9]^6, \ord(si_5) = 6, \\
si_6 &= [t, sh_9]^7, \ord(si_6) = 4, \\
si_7 &= [t, ch_4 \ dv_3]^5, \ord(si_7) = 6, \\
si_8 &= [t, ch_{10} \ dv_3]^8, \ord(si_8) = 2.
\end{align*}$$
For \( i = 8 \) we have \( \dim C_{8,8} = 289, \sigma_8 = 385, \dim S_8 = 104 \). Thus \( \delta_8 = 8 \). The set \( \mathcal{C}_{8,8} \) consists of the 7 irreducible semi-invariants:

\[
\begin{align*}
vi_1 &= [t, ch_4 tr_6]^7, \text{ord}(vi_1) = 2, & vi_2 &= [t, ch_4 tr_6]^8, \text{ord}(vi_2) = 0, \\
vi_3 &= [t, ch_2 tr_6]^5, \text{ord}(vi_3) = 6, & vi_4 &= [t, pt_10 dv_3]^7, \text{ord}(vi_4) = 4, \\
vi_5 &= [t, pt_10 dv_3]^8, \text{ord}(vi_5) = 2, & vi_6 &= [t, ch_4 tr_6]^5, \text{ord}(vi_6) = 6, \\
vi_7 &= [t, ch_4 tr_6]^6, \text{ord}(vi_7) = 4.
\end{align*}
\]

For \( i = 9 \) we have \( \dim C_{8,9} = 910, \sigma_8 = 1782, \dim S_8 = 877 \). Thus \( \delta_8 = 5 \). The set \( \mathcal{C}_{8,9} \) consists of the 5 irreducible semi-invariants:

\[
\begin{align*}
de_1 &= [t, vi_2]^6, \text{ord}(de_1) = 4, & de_2 &= [t, vi_2]^7, \text{ord}(de_2) = 2, \\
de_3 &= [t, sh_2 dv_3]^6, \text{ord}(de_3) = 2, & de_4 &= [t, pt_1 tr_6]^8, \text{ord}(de_4) = 0, \\
de_5 &= [t, ch_4]^7, \text{ord}(de_5) = 2.
\end{align*}
\]

For \( i = 10 \) we have \( \dim C_{8,10} = 1514, \sigma_{10} = 3673, \dim S_{10} = 2162 \). Thus \( \delta_{10} = 3 \). The set \( \mathcal{C}_{8,10} \) consists of the 3 irreducible semi-invariants:

\[
\begin{align*}
des_1 &= [t, pt_1 ch_4]^8, \text{ord}(des_1) = 0, & des_2 &= [t, si_2 dv_3]^8, \text{ord}(des_2) = 2, \\
des_3 &= [t, pt_1 ch_4]^8, \text{ord}(des_3) = 2.
\end{align*}
\]

For \( i = 11 \) we have \( \dim C_{8,11} = 2430, \sigma_{11} = 7355, \dim S_{11} = 4927 \). Thus \( \delta_{11} = 2 \). The set \( \mathcal{C}_{8,11} \) consists of the 2 irreducible semi-invariants:

\[
\begin{align*}
odn_1 &= [t, si_3 tr_6]^6, \text{ord}(odn_1) = 2, & odn_2 &= [t, vi_7 dv_3]^7, \text{ord}(odn_2) = 2.
\end{align*}
\]

For \( i = 12 \) we have \( \dim C_{8,12} = 3788, \sigma_{12} = 14520, \dim S_{12} = 10733 \). Thus \( \delta_{12} = 1 \). The set \( \mathcal{C}_{8,12} \) consists of the unique irreducible semi-invariant:

\[
dvan = [t, vi_5 tr_6]^6, \text{ord}(dvan) = 2.
\]

It is follows from \cite{3} that \( \delta_i = 0 \) for \( i > 12 \).

The cardinalities of the sets \( \mathcal{C}_{8,i} \) and the covariant’s degree-order distributions of Sylvester’s results, see \cite{6}.

Summarizing the above results we get

**Theorem.** The system of the 69 covariants:

\[
\begin{align*}
t, & \quad dv_1, dv_2, dv_3, dv_4, \\
tr_1, tr_2, tr_3, tr_4, tr_5, tr_6, tr_7, tr_8, & \quad ch_1, ch_2, ch_3, ch_4, ch_5, ch_6, ch_7, ch_8, ch_9, ch_{10}, \\
pt_1, pt_2, pt_3, pt_4, pt_5, pt_6, pt_7, pt_8, pt_9, pt_{10}, pt_{11}, & \quad sh_1, sh_2, sh_3, sh_4, sh_5, sh_6, sh_7, sh_8, sh_9, \\
si_1, si_2, si_3, sh_4, sh_5, sh_6, sh_7, si_8, & \quad vi_1, vi_2, vi_3, vi_4, vi_5, vi_6, vi_7, \\
de_1, de_2, de_3, de_4, de_5, & \quad des_1, des_2, des_3, \\
odn_1, odn_2, & \quad dvan.
\end{align*}
\]

is a complete system of the covariants for the binary form of degree 8.
ON COMPLETE SYSTEM OF COVARIANTS FOR THE BINARY FORM OF DEGREE 8

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4. APPENDIX

The degree-order distribution for $C_8$. 
### Order

|   | 0  | 2  | 4  | 6  | 8  | 10 | 12 | 14 | 16 | 18 |
|---|----|----|----|----|----|----|----|----|----|----|
| 1 |    |    |    |    |    |    |    |    |    |    |
| 2 | $dv_4$ | $dv_3$ | $dv_5$ | $dv_2$ | $dv_1$ |    |    |    |    |    |
| 3 | $tr_7$ | $tr_6$ | $tr_4$ | $tr_5$ | $tr_3$ | $tr_2$ | $tr_1$ | $tr_8$ |    |    |
| 4 | $ch_3$ | $ch_2, ch_4$ | $ch_{10}$ | $ch_8$ | $ch_1, ch_7$ | $ch_6$ | $ch_5$ | $ch_9$ |    |    |
| 5 | $pt_3$ | $pt_2$ | $pt_1, pt_7$ | $pt_{10}, pt_{11}$ | $pt_6$ | $pt_5, pt_8, pt_9$ | $pt_4$ |    |    |    |
| 6 | $sh_3$ | $sh_2$ | $sh_1, sh_8$ | $sh_5, sh_6, sh_7$ | $sh_4$ | $sh_9$ |    |    |    |    |
| 7 | $si_4$ | $si_3, si_8$ | $si_1, si_6$ | $si_2, si_5, si_7$ |    |    |    |    |    |    |
| 8 | $vi_2$ | $vi_1, vi_5$ | $vi_4, vi_7$ | $vi_3, vi_6$ |    |    |    |    |    |    |
| 9 | $de_4$ | $de_2, de_3, de_5$ | $de_1$ |    |    |    |    |    |    |    |
| 10 | $des_1$ | $des_2, des_3$ |    |    |    |    |    |    |    |    |
| 11 | $odn_1, odn_2$ |    |    |    |    |    |    |    |    |    |
| 12 | $dvan$ |    |    |    |    |    |    |    |    |    |
| 13 |    |    |    |    |    |    |    |    |    |    |

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