ABSTRACT. Bitangential interpolation problems in the class of matrix valued functions in the generalized Schur class are considered in both the open unit disc and the open right half plane, including problems in which the solutions is not assumed to be holomorphic at the interpolation points. Linear fractional representations of the set of solutions to these problems are presented for invertible and singular Hermitian Pick matrices. These representations make use of a description of the ranges of linear fractional transformations with suitably chosen domains that was developed in [23].

1. Introduction

The main objective of this paper is to study bitangential interpolation problems in the generalized Schur class $S_{p \times q}(\Omega_+)$ of $p \times q$ matrix valued functions that are meromorphic in $\Omega_+$ and for which the kernel

\begin{equation}
\Lambda_s^\kappa(\lambda) = \frac{I_p - s(\lambda)s(\omega)^*}{\rho_\omega(\lambda)}
\end{equation}

has $\kappa$ negative squares in $\mathfrak{h}_s^+ \times \mathfrak{h}_s^+$ (see [38]), where $\mathfrak{h}_s^+$ denotes the domain of holomorphy of $s$ in $\Omega_+$, $\Omega_+$ is either equal to $\mathbb{D} = \{ \lambda \in \mathbb{C} : |\lambda| < 1 \}$ or $\Pi_+ = \{ \lambda \in \mathbb{C} : \lambda + \overline{\lambda} > 0 \}$, and

$\rho_\omega(\lambda) = \begin{cases} 1 - \lambda \omega, & \text{if } \Omega_+ = \mathbb{D}; \\ \lambda + \overline{\omega}, & \text{if } \Omega_+ = \Pi_. \end{cases}$

Thus, in both cases

$\Omega_+ = \{ \omega \in \mathbb{C} : \rho_\omega(\omega) > 0 \}$

and $\Omega_0 = \{ \omega \in \mathbb{C} : \rho_\omega(\omega) = 0 \}$ is the boundary of $\Omega_+$. Correspondingly we set

\begin{equation}
\Omega_- = \mathbb{C} \setminus (\Omega_+ \cup \Omega_0) = \{ \omega \in \mathbb{C} : \rho_\omega(\omega) < 0 \}.
\end{equation}

The normalized standard inner product $\langle f, g \rangle_{nst}$ is defined as

$\langle f, g \rangle_{nst} = \begin{cases} \frac{1}{2\pi} \int_0^{2\pi} g(e^{i\theta})^* f(e^{i\theta}) d\theta & \text{if } \Omega_0 = \mathbb{D} \\ \frac{1}{2\pi} \int_{-\infty}^{\infty} g(i\mu)^* f(i\mu) d\mu & \text{if } \Omega_0 = i\mathbb{R}. \end{cases}$

The symbol $H_{p \times q}^2$ (resp., $H_{p \times q}^\infty$) stands for the class of $p \times q$ mvf’s with entries in the Hardy space $H_2$ (resp., $H_\infty$); $H_2^p$ is short for $H_{2 \times 1}^p$ and $(H_2^p)^\perp$ is the orthogonal complement of $H_2^p$ in $L_2^p$ with respect to the standard inner product on $\Omega_0$.

Most of the other notation that we use will be fairly standard:
mvf for matrix valued function, vvf for vector valued function, and \( R \) for rational mvf's; \( \ker A \) and \( \rng A \) for the kernel and range of a matrix \( A \), and, if \( A \) is square, \( \sigma(A) \) for its spectrum and \( \nu_-(A) \) (resp., \( \nu_+(A) \)) for the number of its negative (resp., positive) eigenvalues (counting multiplicities). If \( f(\lambda) \) is a mvf, then

\[
f^#(\lambda) = f(\lambda^*)^*, \quad \text{where} \quad \lambda^* = \begin{cases} \frac{1}{\lambda} & \text{if } \Omega_+ = \mathbb{D}, \lambda \neq 0; \\ -\lambda & \text{if } \Omega_+ = \Pi_+ \end{cases}
\]

\( \mathfrak{h}_f = \{ \lambda \in \mathbb{C} \text{ at which } f(\lambda) \text{ is holomorphic} \} \) and \( \mathfrak{h}_f^\pm = \mathfrak{h}_f \cap \Omega_\pm \).

By a fundamental result of Krein and Langer [38], every generalized Schur function \( s \in S_{\kappa}^{p \times q} := S_0^{p \times q}(\Omega_+) \) admits a pair of coprime factorizations

\[
(1.3) \quad s(\lambda) = b^\ell(\lambda)^{-1}s^\ell(\lambda) = s^r(\lambda)b^r(\lambda)^{-1} \quad (\lambda \in \mathfrak{h}_s^+),
\]

where \( b^\ell \) and \( b^r \) are Blaschke–Potapov products of sizes \( p \times p \) and \( q \times q \), respectively, of degree \( \kappa \), and the mvf's \( s^\ell \) and \( s^r \) both belong to the Schur class \( S_{\kappa}^{p \times q} := S_0^{p \times q}(\Omega_+) \).

Various interpolation problems in the class of generalized Schur mvf's were considered in [44], [41], [33], [12], [13], [14], [2], [22], [16].

The data for the interpolation problem we consider in this paper for \( \Omega_+ = \mathbb{D} \) is coded into a set of four matrices, \( M, N, P \in \mathbb{C}^{n \times n} \) and \( C \in \mathbb{C}^{m \times n} \), that are subject to the following constraints:

(A1) The pencil \( M - \lambda N \) is invertible on \( \Omega_0 = \mathbb{T} \), i.e., the resolvent set of this pencil,

\[
\rho(M, N) \overset{\text{def}}{=} \{ \lambda \in \mathbb{C} : \det(M - \lambda N) \neq 0 \}
\]

contains \( \Omega_0 \).

(A2) \( P \) is a Hermitian solution of the Lyapunov-Stein equation

\[
(1.4) \quad M^* PM - N^* PN = C^* j_{pq} C,
\]

where

\[
(1.5) \quad j_{pq} = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix}, \quad p > 0, \quad q > 0 \quad \text{and} \quad m = p + q.
\]

(A3) The matrices \( C_1, C_2 \) determined by the decomposition

\[
(1.6) \quad C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}, \quad C_1 \in \mathbb{C}^{p \times n}, \quad C_2 \in \mathbb{C}^{q \times n},
\]

satisfy the rank conditions

\[
(1.7) \quad \text{rank } \begin{bmatrix} M - \lambda N \\ C_2 \end{bmatrix} = n \quad \text{and} \quad \text{rank } \begin{bmatrix} \lambda M - N \\ C_1 \end{bmatrix} = n \quad \text{for every } \lambda \in \Omega_+.
\]

(A4) There exists an \( n \times n \) Hermitian matrix \( X \) that meets the conditions specified in (B4) below.

In particular, it follows from (A3) that the triple \( (C, M, N) \) is observable:

\[
(1.8) \quad \bigcap_{\lambda \in \rho(M, N)} \ker C(M - \lambda N)^{-1} = \{0\},
\]

see Proposition 3.5, and for additional discussion of observability and controllability of a fairly general class of pencils, Theorem 3.5 of [5].
The basic **bitangential interpolation** problem under consideration for $\Omega_+ = \mathbb{D}$ corresponds to the decompositions:

\[(1.9) \quad M = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix}, \quad N = \begin{bmatrix} I_{n_1} & 0 \\ 0 & A_2 \end{bmatrix}\]

and

\[C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} : \mathbb{C}^{n_1} \rightarrow \mathbb{C}^{p}, \]  

where $n = n_1 + n_2$, $n_1 > 0$, $n_2 > 0$, $A_1 \in \mathbb{C}^{n_1 \times n_1}$ and $A_2 \in \mathbb{C}^{n_2 \times n_2}$. These matrices are subject to the following constraints:

(B1) $\sigma(A_1) \cup \sigma(A_2) \subset \mathbb{D}$.

(B2) $P$ is a Hermitian solution of the Lyapunov-Stein equation $\text{(1.4)}$ with $M$ and $N$ as in $\text{(1.9)}$.

(B3) The pairs $(C_{12}, A_2)$ and $(C_{21}, A_1)$ are observable.

(B4) There exists an $n \times n$ Hermitian matrix $X$ such that:

(i) $XPX = X$.

(ii) $PXP = P$.

(iii) $\text{rang } X$ is invariant for $M$ and $N$, i.e., $Mx \in \text{rang } X$ and $Nx \in \text{rang } X$ if $x \in \text{rang } X$. If $P$ is invertible, then (B4) is superfluous, since it is automatically satisfied by $X = P^{-1}$.

The **one sided tangential interpolation** problem corresponds to the case when either $n_1 = 0$ or $n_2 = 0$ (but not both). In these two cases the formulations in (B1) and (B3) must be interpreted properly. For ease of future reference we summarize the changes in the following remark:

**Remark 1.1.** If $n_2 = 0$, then $n_1 = n > 0$, $M = A_1$, $N = I_n$, (B1) reduces to $\sigma(A_1) \subset \mathbb{D}$ and (B3) to $(C_2, A_1)$ is observable.

If $n_1 = 0$, then $n_2 = n > 0$, $M = I_n$, $N = A_2$, (B1) reduces to $\sigma(A_2) \subset \mathbb{D}$ and (B3) to $(C_1, A_2)$ is observable.

In both of these cases $C_1$ and $C_2$ are determined by the decomposition $(1.6)$.

The **bitangential interpolation problem** corresponding to the data set $M, N, C, P$ is formulated in terms of the Krein-Langer factorizations $(1.3)$ of the mvf $s \in S^p_{\kappa \times q}$, the mvf $F(\lambda) = C(M - \lambda N)^{-1}$ and the Hermitian matrix $P_\kappa \in \mathbb{C}^{n \times n}$ that is defined by formula $(3.1)$ as follows:

Describe the set $\hat{S}_\kappa(M, N, C, P)$ of mvf’s $s \in S^p_{\kappa \times q}$ which satisfy the three conditions:

(C1) $[b_t \quad -s_t^*] Fu \in H^q_2$ for every $u \in \mathbb{C}^n$;

(C2) $[-s_t^* \quad b_t^*] Fu \in (H^q_2)\perp$ for every $u \in \mathbb{C}^n$;

(C3) $P_\kappa \leq P$.

The set of mvf’s $s \in \hat{S}_\kappa(M, N, C, P)$ for which the equality $P_\kappa = P$ prevails in (C3) will be denoted by $S_\kappa(M, N, C, P)$. We will show in Theorem 3.3 that $P_\kappa$ is a solution of the Lyapunov-Stein equation $(1.4)$ and $P_\kappa = P$ for every $s \in \hat{S}_\kappa(M, N, C, P)$ and hence $\hat{S}_\kappa(M, N, C, P) = S_\kappa(M, N, C, P)$.

We shall also write

\[S_\kappa(M, N, C, P) = \begin{cases} 
S_\kappa(A_1, A_2, C, P) & \text{if } n_1 > 0 \text{ and } n_2 > 0, \\
S_\kappa(A_1, C, P) & \text{if } n_1 = n \text{ and } n_2 = 0, \\
S_\kappa(A_2, C, P) & \text{if } n_1 = 0 \text{ and } n_2 = n.
\]
Therefore, they admit inner-outer and outer-inner factorizations (1.17) shown in [23], the mvf's where

\[ T_W[\mathcal{S}_{\kappa_2}^{p \times q}] = \{T_W[\varepsilon] : \varepsilon \in \mathcal{S}_{\kappa_2}^{p \times q}\} \]

is the range of the linear fractional transformation (1.13) based on the \( m \times m \) mvf (1.14), the class \( \mathcal{U}_{\kappa_1}(j_{pq}) \) of \( m \times m \) mvf's \( W \) meromorphic in \( \Omega_+ \) such that the kernel (1.15)

\[ K_{\omega}^W(\lambda) := \frac{j_{pq} - W(\lambda)j_{pq}W(\omega)^*}{\rho_\omega(\lambda)} \]

has \( \kappa_1 \) negative squares on \( h_{1\omega}^+ \times h_{2\omega}^+ \) and \( W(\mu)j_{pq}W(\mu)^* = j_{pq} \) a.e. on \( \Omega_0 \). The class of mvf's \( W \in \mathcal{U}_{\kappa}(j_{pq}) \) which satisfy

\[ s_{21} := -w_{22}^{-1}w_{21} \in \mathcal{S}_{\kappa}^{q \times p} \]

will be denoted \( \mathcal{U}_{\kappa}(j_{pq}) \). Here \( s_{21} \) is the lower left hand corner block in the Potapov-Ginzburg transform

\[ S = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} \]

of \( W \) that is defined in (2.17).

The characterization of the set \( T_W[\mathcal{S}_{\kappa_2}^{p \times q}] \) given in Theorem 2.8 was obtained in [23]; it is a generalization to an indefinite setting of a result from [30]. The formulation of the result requires some facts about the reproducing kernel de Branges-Krein space \( \mathcal{K}(W) \) associated with the kernel \( K_{\omega}^W(\lambda) \) and an indefinite analog of the de Branges-Rovnyak space \( \mathcal{D}(s) \), developed in [4, 3] and [23]. The needed facts are reviewed in the next section for the convenience of the reader.

Then, as follows from Theorem 2.8, the set of solutions of the problem \((C1)-(C3)\) when \( P \) is not invertible can be described via a formula that is similar to (1.11) (see Theorem 3.23).

Finer analysis of the set of solutions is connected with the factorization of the resolvent matrix \( W(\lambda) \) that is presented in Theorem 4.10. In Lemma 4.8 we show that if \((B1)-(B4)\) are in force, then \( s_{21} \in \mathcal{S}_{\kappa}^{q \times p} \) and hence \( W \in \mathcal{U}_{\kappa}(j_{pq}) \). Let \( s_{21} \) have Krein-Langer factorizations (1.17)

\[ s_{21}(\lambda) := b_\ell(\lambda)^{-1}s_\ell(\lambda) = s_r(\lambda)b_r(\lambda)^{-1}, \quad \lambda \in h_{s}^+ \]

where \( b_\ell, b_r \) are Blaschke-Potapov products of degree \( \kappa \) and \( s_\ell, s_r \in \mathcal{S}_{\kappa}^{q \times p} \). Then, as was shown in [23], the mvf's \( s_{11}b_r \) and \( b_\ell s_{22} \) belong to the classes \( \mathcal{S}_{\kappa}^{p \times q} \) and \( \mathcal{S}_{\kappa}^{q \times q} \), respectively. Therefore, they admit inner-outer and outer-inner factorizations (1.18)

\[ s_{11}b_r = b_1\varphi_1, \quad b_\ell s_{22} = \varphi_2b_2, \]
where \( b_1 \in S_{in}^{p \times p}, \) \( b_2 \in S_{out}^{p \times q}, \) \( \varphi_1 \in S_{in}^{p \times p}, \) \( \varphi_2 \in S_{out}^{q \times q}. \) In keeping with the terminology used in [10], the pair \( \{b_1, b_2\} \) is called an associated pair of the mvf \( W \in U_{\kappa}^c(J) \) and denoted as \( \{b_1, b_2\} \in \text{ap}(W). \) In the definite case the formulas in (1.18) simplify to the inner-outer factorization of \( w_{11}^{-1} \) and the outer-inner factorization of \( w_{22}^{-1} \) (see [3], [9]).

Using the results of [23] we show that the matrices \([w_{11} \; w_{12}]\) and \([w_{21} \; w_{22}]\) admit coprime factorization over \( \Omega_- \) and \( \Omega_+, \) respectively:

\[
(1.19) \quad [w_{11} \; w_{12}] = b_1 \begin{bmatrix} \varphi_{11} & \varphi_{12} \end{bmatrix} = (b_1^{-1})^{-1} \begin{bmatrix} \varphi_{11} & \varphi_{12} \end{bmatrix},
\]

\[
(1.20) \quad [w_{21} \; w_{22}] = b_2^{-1} \begin{bmatrix} \varphi_{21} & \varphi_{22} \end{bmatrix},
\]

where \( \{b_1, b_2\} \in \text{ap}(W), \) \( \varphi_{11} \in \mathcal{R} \cap H_{\infty}^{p \times p}(\Omega_-), \) \( \varphi_{12} \in \mathcal{R} \cap H_{\infty}^{p \times q}(\Omega_-), \) \( \varphi_{21} \in \mathcal{R} \cap H_{\infty}^{q \times p} \) and \( \varphi_{22} \in \mathcal{R} \cap H_{\infty}^{q \times q}. \)

Then applying the Krein-Langer generalization of Rouche’s theorem we show that for every mvf \( \varepsilon \in S_{k-\kappa 1}^{p \times q}, \) with the Krein-Langer factorizations

\[
(1.21) \quad \varepsilon = \theta_\ell^{-1} \varepsilon_\ell = \varepsilon_r \theta_r^{-1}
\]

the mvf’s \( \theta_\ell \varphi_{11}^\# + \varepsilon_\ell \varphi_{12}^\# \) and \( \varphi_{21} \varepsilon_r + \varphi_{22} \theta_r \) have exactly \( \kappa \) zeros in \( \Omega_+. \)

The main results of this paper is the following description of \( S_{k}(M, N, C, P) \) when \( \Omega_+ = \mathbb{D} \) and the analogous description for \( \Omega_+ = \Pi_+ \) that is presented in Section 6.

**Theorem 1.2.** Let \((B1)-(B4)\) be in force, let \( \nu_-(P) = \kappa \) and let

\[
(1.22) \quad \nu = \text{rank } (M^*P^2M + N^*P^2N + C^*C) - \text{rank } P.
\]

Then there are unitary matrices \( U \in \mathbb{C}^{p \times p}, V \in \mathbb{C}^{q \times q}, \) such that \( s \in S_{k}(M, N, C, P) \) if and only if \( s \) belongs to \( S_{k}^{p \times q} \) and is of the form \( s = T_W[\varepsilon], \) where

\[
(1.23) \quad \varepsilon = U \begin{bmatrix} \bar{\varepsilon} & 0 \\ 0 & I_\nu \end{bmatrix} V^*, \quad \text{and} \quad \varepsilon \in S_{\nu \times \nu}^{(p - \nu) \times (q - \nu)}.
\]

If \( \varepsilon \in S_{k}^{p \times q}, \) then \( T_W[\varepsilon] \in S_{k}^{p \times q} \) if and only if

(a) the factorization \( w_{11}^\# + \varepsilon w_{12}^\# = (\bar{\varphi}_{11}^\# + \varepsilon_\ell \varphi_{12}^\#) b_1^{-1} \) is coprime over \( \Omega_- \) and

(b) the factorization \( w_{21} \varepsilon + w_{22} = b_2^{-1}(\varphi_{21} \varepsilon + \varphi_{22}) \) is coprime over \( \Omega_+. \)

**Proof.** See subsection 5.2.

If \( P \) is invertible, the statement of this theorem is simpler, since \( \nu = 0 \) (see Corollary 5.6), and we can also treat the case when \( \nu_-(P) < \kappa. \)

**Theorem 1.3.** Let the data set \((M, N, C, P)\) satisfy the assumptions \((B1)-(B3), \) let \( P \) be invertible, \( \kappa_1 = \nu_-(P) \leq \kappa, \) and let the mvf’s \( W, b_1, b_2 \) be defined by (1.14), (1.20), (1.19).

Then:

(I) \( s \in S_{k}(M, N, C, P) \) if and only if \( s = T_W[\varepsilon], \) where \( \varepsilon \in S_{k-\kappa_1}^{p \times q} \) and \( T_W[\varepsilon] \in S_{k}^{p \times q}. \)

(II) If \( \varepsilon \in S_{k-\kappa_1}^{p \times q}, \) \( \theta_\ell, \theta_r, \varepsilon_\ell, \varepsilon_r \) are a choice of its Krein-Langer factorizations as in (1.21), then \( T_W[\varepsilon] \in S_{k}^{p \times q} \) if and only if the factorizations

\[
(1.24) \quad \theta_\ell w_{11}^\# + \varepsilon_\ell w_{12}^\# = (\theta_\ell \bar{\varphi}_{11}^\# + \varepsilon_\ell \varphi_{12}^\#) b_1^{-1},
\]

\[
(1.25) \quad w_{21} \varepsilon_r + w_{22} \theta_r = b_2^{-1}(\varphi_{21} \varepsilon_r + \varphi_{22} \theta_r)
\]

are coprime over \( \Omega_+. \)

**Proof.** See subsection 5.3.

The set of mvf’s \( s \in S_{k}^{p \times q} \) which satisfy (C1)-(C3) and the supplementary condition
(C4) $s$ is holomorphic in $\sigma(A_1) \cup \overline{\sigma(A_2)}$, is denoted by $\mathcal{T}\mathcal{N}_\kappa(M, N, C, P)$, and $s \in \mathcal{T}\mathcal{N}_\kappa(M, N, C, P)$ is said to be a solution of the Takagi-Nudelman problem.

For the Takagi-Nudelman problem the conditions (1.24) and (1.25) of Theorem 1.3 are replaced by the single condition:

$$ (\varphi_2\varepsilon_r + \varphi_2\theta_t)^{-1} \text{ is holomorphic in } \sigma(A_1) \cup \overline{\sigma(A_2)}. $$

which reduces to a condition in [11] when $\varepsilon \in S^{p \times q}$ and reduces to (19.2.6) in [14] when $\varepsilon$ is a rational mvf in $S^{p \times q}$.

In the scalar case ($p = q = 1$, $n_2 = 0$, $\nu_-(P) = \kappa$) the two conditions (1.24) and (1.25) are in force if and only if (1.26) holds. Consequently, every solution $s$ from the set $\mathcal{S}_\kappa(A_1, C, P)$ is holomorphic on $\sigma(A_1)$. A new effect which is revealed in the matrix case is that there are mvf’s $s$ which belong to $\mathcal{S}_\kappa(A_1, A_2, C, P)$ but are not holomorphic on $\sigma(A_1) \cup \overline{\sigma(A_2)}$ (see Example 5). Thus, there are mvf’s that satisfy the conditions (1.24) and (1.25) but do not satisfy (1.26). Therefore, the inclusion

$$ \mathcal{T}\mathcal{N}_\kappa(M, N, C, P) \subseteq \mathcal{S}_\kappa(M, N, C, P) $$

can be proper.

The parameter $\varepsilon \in S^{p \times q}$ in (1.13) is said to be excluded for the problem (C1)-(C4) if $s = T_W[\varepsilon] \notin \mathcal{T}\mathcal{N}_\kappa(M, N, C, P)$. If $\sigma(A_1) \cap \overline{\sigma(A_2)} = \emptyset$, the condition

$$ K^W_\omega(\omega) \geq 0 \text{ for all } \omega \in \sigma(A_1)^0 \cup \overline{\sigma(A_2)^0}, $$

which is formulated in terms of the kernel (1.15), suffices to guarantee that the problem (C1)-(C4) has no excluded parameters. In the scalar case sufficient conditions for the Nevanlinna-Pick problem in the generalized Nevanlinna class to have no excluded parameters were found in [24] (see also [1] and [6] for the matrix case).

In subsection 5.6, given a data set $(M, N, C, P)$ satisfying the assumptions (B1)-(B3), we will consider an associated pair $(b_1, b_2)$ for the mvf $W$ and a rational mvf $K$ holomorphic in $\Omega_+$ such that

$$ (1.27) \quad \text{the mvf } b_1^{-1}(s - K)b_2^{-1} \text{ has exactly } \kappa \text{ poles (counting multiplicities) in } \Omega_+ $$

for every $s \in \mathcal{S}_\kappa(M, N, C, P)$. Every mvf $s \in S^{p \times q}_{\kappa'} (\kappa' \leq \kappa)$ which satisfies (1.27) is called a solution of the Takagi-Sarason problem with data set $(b_1, b_2, K)$, and the symbol $T\mathcal{S}_\kappa(b_1, b_2, K)$ is used to denote the set of such solutions $s$. We shall show that for $(b_1, b_2, K)$ corresponding to the problem (C1)-(C3) $T\mathcal{S}_\kappa(b_1, b_2, K) = T_W[S^{p \times q}_{\kappa - \nu_-(P)}]$, and, hence, that

$$ \mathcal{T}\mathcal{N}_\kappa(M, N, C, P) \subseteq \mathcal{S}_\kappa(M, N, C, P) \subseteq T\mathcal{S}_\kappa(b_1, b_2, K). $$

Rational solutions of the Takagi-Nudelman and Takagi-Sarason problems in the case of invertible $P$ have been described earlier in [14].

The paper is organized as follows. In Section 2 the basic notions are introduced. Section 3 focuses on the bitangential interpolation problem in the open unit disc $\mathbb{D}$. Pole and zero multiplicities and a factorization formula for the resolvent matrix for the interpolation problem considered in Section 3 are developed in Section 4. Theorems 1.2 and 1.3 which provide parametrizations of the set of all solutions to this problem both when $P$ is invertible and $P$ is not invertible are completed in the first part of Section 5. The latter part discusses the Tagaki-Nudelman problem and excluded parameters. The overall strategy for analyzing the bitangential interpolation problem in the open right half plane $\Pi_+$ (and the open upper half plane $\mathbb{C}_+$) is much the same as for $\mathbb{D}$. The changes in the formulas and the main conclusions for $\Omega_+ = \mathbb{D}$ are discussed briefly in Section 6, without proof.
2. Preliminaries

2.1. The generalized Schur class. Recall that a Hermitian kernel $K_\omega(\lambda) : \Omega \times \Omega \to \mathbb{C}^{m \times m}$ is said to have $\kappa$ negative squares and is written as

$$sq_- K = \kappa,$$

if for every positive integer $n$ and every choice of $\omega_j \in \Omega$ and $u_j \in \mathbb{C}^m$ ($j = 1, \ldots, n$) the matrix

$$\left( \langle K_\omega(\omega_k)u_j, u_k \rangle \right)_{j,k=1}^n$$

has at most $\kappa$ negative eigenvalues and for some choice of $\omega_j \in \Omega$ and $u_j \in \mathbb{C}^m$ exactly $\kappa$ negative eigenvalues.

The class $S_{\kappa}^{p \times q} := S_0^{p \times q}(\Omega_+)$ is the usual Schur class. Recall that a mvf $s \in S^{p \times q}$ is called inner (resp., $*$-inner), if $s(\mu)$ is an isometry (resp., co-isometry) for a.e. $\mu \in \Omega_0$, that is

$$I_q - s(\mu)^*s(\mu) = 0 \quad (\text{resp., } I_p - s(\mu)s(\mu)^* = 0)$$

for almost all points $\mu \in \Omega_0$.

Let $S_{in}^{p \times q}$ ($S_{\kappa}^{p \times q}$) denote the set of all inner (resp., $*$-inner) mvf’s $s \in S^{p \times q}$. An example of an inner square mvf is provided by the Blaschke–Potapov product, that in the case of the unit disc ($\Omega_+ = \mathbb{D}$) is given by

$$s(\lambda) = b_\ell(\lambda)^{-1}s_\ell(\lambda) \quad \text{for } \lambda \in \mathbb{D}^+,$$

where $b_\ell$ is a Blaschke–Potapov product of degree $\kappa$, $s_\ell$ is in the Schur class $S_{\kappa}^{p \times q}(\Omega_+)$ and

$$\ker s_\ell(\lambda)^* \cap \ker b_\ell(\lambda)^* = \{0\} \quad \text{for } \lambda \in \Omega_+.$$

The representation (2.2) is called a left Krein–Langer factorization. The assumption (2.3) can be rewritten in the equivalent form

$$\text{rank } \left[ \begin{array}{c} b_\ell(\lambda) \\ s_\ell(\lambda) \end{array} \right] = p \quad \text{for } \lambda \in \Omega_+.$$

If $\alpha_j \in \mathbb{D}$ ($j = 1, \ldots, n$) are all the zeros of $b_\ell$ in $\Omega_+$, then the noncancellation condition (2.3) ensures that $S_{in}^+ = \Omega_+ \setminus \{\alpha_1, \ldots, \alpha_n\}$. The left Krein–Langer factorization (2.2) is essentially unique in a sense that $b_\ell$ is defined uniquely up to a left unitary factor $V \in \mathbb{C}^{p \times p}$.

Similarly, every generalized Schur function $s \in S_{\kappa}^{p \times q}(\Omega_+)$ admits a right Krein–Langer factorization

$$s(\lambda) = s_\tau(\lambda)b_\tau(\lambda)^{-1} \quad \text{for } \lambda \in \mathbb{D}^+,$$

where $b_\tau$ is a Blaschke–Potapov product of degree $\kappa$ and $s_\tau \in S_{\kappa}^{p \times q}(\Omega_+)$ satisfies the condition

$$\ker s_\tau(\lambda) \cap \ker b_\tau(\lambda) = \{0\}, \quad \text{for } \lambda \in \Omega_+.$$

This condition can be rewritten in the equivalent form

$$\text{rank } \left[ \begin{array}{c} b_\tau(\lambda)^* \\ s_\tau(\lambda)^* \end{array} \right] = q \quad \text{for } \lambda \in \Omega_+.$$

Under assumption (2.6) the mvf $b_\tau$ is uniquely defined up to a right unitary factor $V' \in \mathbb{C}^{q \times q}$. The generalized Schur class.

2.2. The generalized Schur class.
Lemma 2.1. A mvf $s_\ell \in S^{p\times q}$ and a finite Blaschke-Potapov product $b_\ell \in S_{in}^{p\times p}$ meet the rank condition (2.7), if and only if there exists a pair of mvf’s $c \in H^{q\times p}_\infty$ and $d \in H^{q\times p}_\infty$ such that

\begin{equation}
\label{2.10}
b_\ell(\lambda)c(\lambda) + s_\ell(\lambda)d(\lambda) = I_p \quad \text{for} \ \lambda \in \Omega_+.
\end{equation}

Lemma 2.1 is a matrix version of the Carleson Corona Theorem. A proof, which is adapted from Fuhrmann [34] who treated the case $p = q$, is furnished in [23].

A dual statement for Lemma 2.1 is obtained by applying Lemma 2.1 to transposed vvf’s.

Lemma 2.2. A mvf $s_r \in S^{p\times q}$ and a finite Blaschke-Potapov product $b_r \in S_{in}^{q\times q}$ meet the rank condition (2.7), if and only if there exists a pair of mvf’s $c \in H^{q\times q}_\infty$ and $d \in H^{q\times q}_\infty$ such that

\begin{equation}
\label{2.9}
c(\lambda)b_r(\lambda) + d(\lambda)s_r(\lambda) = I_q \quad \text{for} \ \lambda \in \Omega_+.
\end{equation}

The factorization (2.2) is called a left coprime factorization of $s$ if $s_\ell$ and $b_\ell$ satisfy (2.8). Similarly, the factorization (2.5) is called a right coprime factorization of $s$ if $s_r$ and $b_r$ satisfy (2.9).

Every vvf $h(\lambda)$ from $H^p_\infty (H^2_0\perp)$ has nontangential limits $h(\mu)$ ($\mu \in \Omega_0$) a.e. on the boundary $\Omega_0$. These nontangential limits identify the vvf $h$ uniquely. In what follows we often identify a vvf $h \in H^p_\infty (H^2_0\perp)$ with its boundary values $h(\mu)$.

Let $P_+$ and $P_-$ denote the orthogonal projections from $L^2_k$ onto $H^k_\infty$ and $H^2_\perp$, respectively, where $k$ is a positive integer that will be understood from the context. The Hilbert spaces

\begin{equation}
\label{2.10}
\mathcal{H}(b_r) = H^q_2 \oplus b_r H^q_2, \quad \mathcal{H}_+(b_\ell) := (H^p_2)^\perp \oplus b_\ell^*(H^p_2)^\perp
\end{equation}

and the operators

\begin{equation}
\label{2.11}
X_r : h \in \mathcal{H}(b_r) \mapsto P_- sh, \quad X_\ell : h \in \mathcal{H}_+(b_\ell) \mapsto P_+ s^* h
\end{equation}

based on $s \in S^{p\times q}_\infty$ will play an important role.

Lemma 2.3. (cf. [21]) If $s \in S^{p\times q}_\infty$, then:

(i) The operator $X_\ell$ maps $\mathcal{H}_+(b_\ell)$ injectively onto $\mathcal{H}(b_r)$.

(ii) The operator $X_r$ maps $\mathcal{H}(b_r)$ injectively onto $\mathcal{H}_+(b_\ell)$.

(iii) $X_\ell = X^*_r$.

Proof. (i) For every $h \in \mathcal{H}_+(b_\ell)$ and $f = b_r h_+ \in b_r H^q_2$, it is readily checked that

\[
\langle P_+ s^* h, f \rangle_{nst} = \langle s^* h, b_r h_+ \rangle_{nst} = \langle h, s h_+ \rangle_{nst} = 0 \quad \forall h_+ \in H^q_2,
\]

i.e., $X_\ell$ maps $\mathcal{H}_+(b_\ell)$ into $\mathcal{H}(b_r)$. Therefore, since $\mathcal{H}_+(b_\ell)$ and $\mathcal{H}(b_r)$ are finite dimensional spaces of the same dimension, and

\[
\dim \mathcal{H}_+(b_\ell) = \dim \ker X_\ell + \dim \text{range } X_\ell,
\]

it suffices to show that $\ker X_\ell = \{0\}$. But, if $h \in \mathcal{H}_+(b_\ell)$ and $P_+ s^* h = 0$, then $b_r h \in H^p_2$ and, in view of Lemma 2.1, $X_\ell h = 0$. Since $b_\ell(\omega) \not\equiv 0$, this implies that $h(\omega) \equiv 0$.

Statement (ii) can be obtained by similar calculations, and (iii) is easy.
Definition 2.4. Let
\[ (2.12) \quad \Gamma_{\ell} : f \in L^2_{\ell} \rightarrow X_{\ell}^{-1}P_{H(b_{\ell})}f \in \mathcal{H}_{+}(b_{\ell}) \quad \text{and} \quad \Gamma_{r} : g \in L^2_{r} \rightarrow X_{r}^{-1}P_{H_{+}(b_{r})}g \in \mathcal{H}(b_{r}), \]
where \( X_{\ell} \) and \( X_{r} \) are defined in formula (2.11) and \( P_{X} \) denotes the orthogonal projection of \( L^2_{k} \) onto a closed subspace \( \mathcal{X} \) of \( L^2_{k} \) (where \( k \) will always be understood from the context).

It is readily checked that
\[ (2.13) \quad P_{+}s^{*}\Gamma_{\ell}f = P_{H(b_{\ell})}f \quad \text{and} \quad P_{-}\Gamma_{r}g = P_{H_{+}(b_{r})}g \quad \text{for} \ f \in L^2_{\ell} \ \text{and} \ g \in L^2_{r}. \]

Lemma 2.5. The operators \( \Gamma_{\ell} \) and \( \Gamma_{r} \) satisfy the equalities:
\begin{enumerate}
  \item \( \Gamma_{\ell}^{*} = \Gamma_{r} \);
  \item \( \langle (\overline{\psi}\Gamma_{\ell} - \Gamma_{\ell}\overline{\psi})f, g \rangle_{\text{nst}} = 0 \) for \( f \in \mathcal{H}(b_{r}), \ g \in \mathcal{H}_{+}(b_{\ell}) \) and \( \psi \) a scalar inner function.
\end{enumerate}

Proof. (i) Let \( f \in L^2_{\ell} \) and \( g \in L^2_{r} \). Then, since \( X_{\ell} \) maps \( \mathcal{H}_{+}(b_{\ell}) \) onto \( \mathcal{H}(b_{r}) \) and \( X_{r}^{*} = X_{r} \),
\[ \langle \Gamma_{\ell}f, g \rangle_{\text{nst}} = \langle X_{\ell}^{-1}P_{H(b_{\ell})}f, P_{H_{+}(b_{r})}g \rangle_{\text{nst}} = \langle P_{H(b_{\ell})}f, X_{r}^{-1}P_{H(b_{r})}g \rangle_{\text{nst}} = \langle f, \Gamma_{r}g \rangle_{\text{nst}}. \]
(ii) If \( f \in \mathcal{H}(b_{r}) \) and \( g \in \mathcal{H}_{+}(b_{\ell}) \), then
\[ \langle (\overline{\psi}\Gamma_{\ell} - \Gamma_{\ell}\overline{\psi})f, g \rangle_{\text{nst}} = \langle \overline{\psi}\Gamma_{\ell}f, P_{-}\Gamma_{r}g \rangle_{\text{nst}} - \langle P_{+}s^{*}\Gamma_{\ell}f, \overline{\psi}\Gamma_{r}g \rangle_{\text{nst}} = 0. \]
\[ \square \]

2.2. Reproducing kernel Pontryagin spaces. In this subsection we review some facts and notation from [11,15] on the theory of indefinite inner product spaces for the convenience of the reader. A linear space \( \mathcal{K} \) equipped with a sesquilinear form \( \langle \cdot, \cdot \rangle_{\mathcal{K}} \) on \( \mathcal{K} \times \mathcal{K} \) is called an indefinite inner product space. A subspace \( \mathfrak{F} \) of \( \mathcal{K} \) is called positive (negative) if \( \langle f, f \rangle_{\mathcal{K}} > 0 \) \((< 0)\) for all \( f \in \mathfrak{F}, f \neq 0 \). If the full space \( \mathcal{K} \) is positive and complete with respect to the norm \( \| f \| = \langle f, f \rangle_{\mathcal{K}}^{1/2} \), then it is a Hilbert space.

An indefinite inner product space \((\mathcal{K}, \langle \cdot, \cdot \rangle_{\mathcal{K}})\) is called a Pontryagin space, if it can be decomposed as the orthogonal sum
\[ (2.14) \quad \mathcal{K} = \mathcal{K}_{+} \oplus \mathcal{K}_{-} \]
of a positive subspace \( \mathcal{K}_{+} \) which is a Hilbert space and a negative subspace \( \mathcal{K}_{-} \) of finite dimension. The number \( \text{ind}_{\mathcal{K}} := \dim \mathcal{K}_{-} \) is referred to as the negative index of \( \mathcal{K} \). The convergence in a Pontryagin space \((\mathcal{K}, \langle \cdot, \cdot \rangle_{\mathcal{K}})\) is meant with respect to the Hilbert space norm
\[ (2.15) \quad \| h \|^{2} = \langle h_{+}, h_{+} \rangle_{\mathcal{K}} - \langle h_{-}, h_{-} \rangle_{\mathcal{K}} \quad \text{when} \ h = h_{+} + h_{-} \quad \text{with} \ h_{\pm} \in \mathcal{K}_{\pm}. \]

It is easily seen that the convergence does not depend on a choice of the decomposition (2.14).

A Pontryagin space \((\mathcal{K}, \langle \cdot, \cdot \rangle_{\mathcal{K}})\) of \( \mathbb{C}^{m} \)-valued functions defined on a subset \( \Omega \) of \( \mathbb{C} \) is called a reproducing kernel Pontryagin space associated with the Hermitian kernel \( K_{\omega}(\lambda) : \Omega \times \Omega \rightarrow \mathbb{C}^{m \times m} \) if:
\begin{enumerate}
  \item for every \( \omega \in \Omega \) and \( u \in \mathbb{C}^{m} \) the vvf \( K_{\omega}(\lambda)u \) belongs to \( \mathcal{K} \);
  \item for every \( h \in \mathcal{K}, \ \omega \in \Omega \) and \( u \in \mathbb{C}^{m} \) the following identity holds
  \[ (2.16) \quad \langle h, K_{\omega}u \rangle_{\mathcal{K}} = u^{*}h(\omega). \]
\end{enumerate}

It is known (see [43]) that for every Hermitian kernel \( K_{\omega}(\lambda) : \Omega \times \Omega \rightarrow \mathbb{C}^{m \times m} \) with a finite number of negative squares on \( \Omega \times \Omega \) there is a unique Pontryagin space \( \mathcal{K} \) with reproducing kernel \( K_{\omega}(\lambda) \), and that \( \text{ind}_{\mathcal{K}} = \text{sq}_{-}K = \kappa \). In the case \( \kappa = 0 \) this fact is due to Aronszajn [7].
2.3. The class $\mathcal{U}_w(j_{pq})$ and the space $\mathcal{K}(W)$. Recall (see [1]) that the lower right hand $q \times q$ corner $w_{22}(\lambda)$ of every $m \times m$ mvf $W \in \mathcal{U}_w(j_{pq})$ is invertible for all $\lambda \in \mathcal{H}_W^+$ except for at most $\kappa$ points. The Potapov-Ginzburg transform of $W(\lambda)$, which is defined on $\mathcal{H}_W^+$ by the formulas
\[
S(\lambda) = PG(W) := \left[ \begin{array}{cc} w_{11}(\lambda) & w_{12}(\lambda) \\ 0 & I_q \end{array} \right] \left[ \begin{array}{cc} I_p & 0 \\ w_{21}(\lambda) & w_{22}(\lambda) \end{array} \right]^{-1} = \left[ \begin{array}{cc} I_p & -w_{12}(\lambda) \\ 0 & -w_{22}(\lambda) \end{array} \right]^{-1} \left[ \begin{array}{cc} w_{11}(\lambda) & 0 \\ w_{21}(\lambda) & -I_q \end{array} \right],
\]
belongs to the class $S_{\kappa}^{m \times m}$. Since
\[
W = PG(S) \Longrightarrow S = PG(W),
\]
the mvf $W$ is of bounded type. Thus, the nontangential limits $W(\mu)$ exist a.e. on $\Omega_0$ and assumption (2) makes sense. It guarantees that $W(\lambda)$ is invertible in $\Omega_+$ except for an isolated set of points. Define $W$ in $\Omega_-$ by the formula
\[
W(\lambda) = j_{pq}W^#(\lambda)^{-1}j_{pq} = j_{pq}W(\lambda)^{-*}j_{pq} \quad \text{for } \lambda \in \Omega_-.
\]
Since the nontangential limits
\[
W_{\pm}(t) = \angle \lim_{\lambda \rightarrow t} \{W(\lambda) : \lambda \in \Omega_{\pm}\}
\]
coincide a.e. in $\Omega_0$, $W$ in $\Omega_-$ is a pseudo-meromorphic extension of $W$ in $\Omega_+$. If $W(\lambda)$ is rational this extension is meromorphic on $\mathbb{C}$. Formula (2.19) implies that $W(\lambda)$ is holomorphic and invertible in $\Omega_W := \mathcal{H}_W \cap \mathcal{H}_W^*$. Let $W \in \mathcal{U}_w(j_{pq})$ and let $\mathcal{K}(W)$ be the reproducing kernel Pontryagin space associated with the kernel $K^W_\omega(\lambda)$. The kernel $K^W_\omega(\lambda)$ extended to $\Omega_+ \cup \Omega_-$ by the equality (2.19) has the same number $\kappa$ of negative squares. This fact is due to a generalization of the Ginzburg inequality (see [3] Theorem 2.5.2).

The following example of a $\mathcal{K}(W)$-space will play an important role in Section 4.

**Example 1.** Let $C \in \mathbb{C}^{m \times n}$, let $M, N \in \mathbb{C}^{n \times n}$ and assume that $P \in \mathbb{C}^{n \times n}$ is an invertible Hermitian matrix, that $\rho(M, N) \neq \emptyset$ and that the observability condition (1.3) is in force. Then the linear space of vvf's
\[
\mathcal{M} = \{F(\lambda)u : u \in \mathbb{C}^n\}
\]
based on the mvf
\[
F(\lambda) = C(M - \lambda N)^{-1} \quad \text{for } \lambda \in \rho(M, N)
\]
and endowed with the inner product
\[
\langle Fu, Fv \rangle_{\mathcal{M}} = v^*Pu
\]
is an RKPS (reproducing kernel Pontryagin space) with RK (reproducing kernel)
\[
K_\omega(\lambda) = F(\lambda)P^{-1}F(\omega)^*
\]
and negative index $\text{ind}_-\mathcal{M} = \nu_-(P)$. The assumption (1.8) insures that the inner product (2.22) is well defined.

We will need the following criterion for the space $\mathcal{M}$ to be a $\mathcal{K}(W)$ space:
Theorem 2.6. \[ \text{Let } M, N, P \in \mathbb{C}^{n \times n}, C \in \mathbb{C}^{m \times n} \text{ and assume that } P \text{ is invertible, } \rho(M, N) \neq \emptyset \text{ and that } 11 \text{ holds. Then the RKPS } \mathcal{M} \text{ considered in Example 7 with kernel } K_\omega^W(\lambda) \text{ of the form } (2.23) \text{ is a } \mathcal{K}(W) \text{ space if and only if } P \text{ is a solution of the equation} \]

\[ (2.24) \quad M^*PM - N^*PN = C^*JC \quad \text{when } \Omega_+ = \mathbb{D}, \]

\[ (2.25) \quad M^*PN + N^*PM + C^*JC = 0 \quad \text{when } \Omega_+ = \Pi_+, \]

where \( J \) is a signature matrix \( (J = J^* = J^{-1}) \). The mvf \( W \) is uniquely defined by the formula

\[ (2.26) \quad W(\lambda) = I_m - \rho_\mu(\lambda)F(\lambda)P^{-1}F(\mu)^*J \quad \text{where } \mu \in \Omega_0 \cap \rho(M, N), \]

up to a constant \( J \)-unitary factor on the right (that depends upon \( \mu \)).

2.4. Linear fractional transformations. Let

\[ (2.27) \quad T_W[\varepsilon] := (w_{11}(\lambda)\varepsilon(\lambda) + w_{12}(\lambda))(w_{21}(\lambda)\varepsilon(\lambda) + w_{22}(\lambda))^{-1} \]

denote the linear fractional transformation of a mvf \( \varepsilon \in S_{\kappa_2}^{p \times q} \) \((\kappa_2 \in \mathbb{Z}_+)\) based on the block decomposition

\[ (2.28) \quad W(\lambda) = \begin{bmatrix} w_{11}(\lambda) & w_{12}(\lambda) \\ w_{21}(\lambda) & w_{22}(\lambda) \end{bmatrix} \]

of a mvf \( W \in U_{\kappa_1}(j_{pq}) \) with blocks \( w_{11}(\lambda) \) and \( w_{22}(\lambda) \) of sizes \( p \times p \) and \( q \times q \), respectively. The transformation \( T_W[\varepsilon] \) is well defined for those points \( \lambda \in h_W \cap h^+_\varepsilon \) for which

\[ \det(w_{21}(\lambda)\varepsilon(\lambda) + w_{22}(\lambda)) \neq 0. \]

Lemma 2.7. If \( W \in U_{\kappa_1}(j_{pq}) \) and \( \varepsilon \in S_{\kappa_2}^{p \times q} \), then:

1. \( T_W[\varepsilon] \) admits the supplementary representation

\[ (2.29) \quad T_W[\varepsilon] = (w_{11}^\#(\lambda) + \varepsilon(\lambda)w_{12}^\#(\lambda))^{-1}(w_{21}^\#(\lambda) + \varepsilon(\lambda)w_{22}^\#(\lambda)) \quad \text{for } \lambda \in \Omega_W \cap h^+_\varepsilon \cap h^*_\varepsilon. \]

2. \( s := T_W[\varepsilon] \in S_{\kappa'}^{p \times q} \) with \( \kappa' \leq \kappa_2 + \kappa_1. \)

The set \( T_W[S_{\kappa_1}^{p \times q}] \cap S_{\kappa}^{p \times q} \) is characterized by three conditions in Theorem 1.1 of of \([30]\), which is repeated immediately below for the convenience of the reader; it is a generalization of Theorem 3.1 of \([30]\), which treated the case \( \kappa_1 = \kappa_2 = 0; \) see also \([35]\) for more general formulations in a Hilbert space setting. The interpolation conditions \((C1)\)–\((C3)\) coincide with the three conditions in the next theorem.

The Krein-Langer representations for mvf’s \( s \in S_{\kappa}^{p \times q} \) insure that the nontangential limits \( s(t) \) exist and are contractions for almost all points \( t \in \Omega_0 \). Therefore,

\[ (2.30) \quad \Delta_s(t) := \begin{bmatrix} I_p & -s(t) \\ -s(t)^* & I_q \end{bmatrix} \geq 0 \quad \text{for almost all points } \ t \in \Omega_0. \]

Theorem 2.8. Let \( \kappa, \kappa_1 \in \mathbb{N} \cup \{0\}, \ (\kappa_1 \leq \kappa), \ \text{let } W \in U_{\kappa_1}(j_{pq}) \ \text{and let } s \in S_{\kappa}^{p \times q} \text{ admit the Krein-Langer factorizations } s = b_1^{-1}s_1 = s_2b_2^{-1} \text{ and let } \Delta_s(\mu) \text{ and } \Gamma_\ell \text{ be defined by } (2.30) \text{ and } (2.12). \text{ Then } s \in T_W[S_{\kappa_1}^{p \times q}] \text{ if and only if the following conditions hold:} \]

1. \[ [b_\ell \ -s_\ell] \ f \in H^p_2 \text{ for every } f \in \mathcal{K}(W); \]
2. \[ [-s_\ell^* \ b_\ell^*] \ f \in (H^p_2)^\perp \text{ for every } f \in \mathcal{K}(W); \]
3. \[ \langle \Delta_s + \Delta_s \begin{bmatrix} 0 & \Gamma_\ell \\ \Gamma_\ell^* & 0 \end{bmatrix} \Delta_s \rangle f, f \rangle_{\mathcal{K}(W)} \leq \langle f, f \rangle_{\mathcal{K}(W)} \text{ for every } f \in \mathcal{K}(W). \]
3. Bitangential interpolation in the unit disc.

3.1. Main assumptions. In this section the basic theorem (Theorem 2.8) will be used to obtain a linear fractional parametrization of the set of solutions \( \mathcal{S}_s(A_1, A_2, C, P, X) \) of the interpolation problem that is discussed in the Introduction. Note that if \( M \) and \( N \) are as in (1.9), then

\[
\rho(M, N) = \{ \lambda \in \mathbb{C} : \det(\lambda I_n - A_1) \neq 0 \text{ and } \det(I_n - \lambda A_2) \neq 0 \}.
\]

Let \( P_s \in \mathbb{C}^{n \times n} \) be defined by the formula

\[
v^* P_s u = \left\{ \Delta_s + \Delta_s \begin{bmatrix} 0 & \Gamma^*_\ell \\ \Gamma^*_\ell & 0 \end{bmatrix} \Delta_s \right\} F_s, F_v \right\}_{n}\), \quad \text{for } u, v \in \mathbb{C}^n,
\]

where the mvf \( \Delta_s \) is given by (2.30), \( \Gamma^*_\ell \) is the operator from \( L^q_2 \) onto \( \mathcal{H}_s(b_\ell) \) defined by (2.12) and \( \Gamma^*_\ell \) is the adjoint of \( \Gamma^*_\ell \) with respect to the standard inner product.

Remark 3.1. The mvf \( s \in \mathcal{S}_s(A_1, A_2, C, P) \) if and only if \( \Delta_s Fu \) belongs to the de Branges-Rovnyak space \( D(s) \) and

\[
\langle \Delta_s Fu, \Delta_s Fu \rangle_D(s) \leq u^* Pu, \quad \text{for every } u \in \mathbb{C}^n.
\]

Therefore, since the left hand side of (3.2) is equal to \( u^* P_s u \) and (C3) implies that \( P_s \leq P \), the inequality

\[
\nu_-(P) \leq \kappa
\]

is necessary for the problem (C1)–(C3) to be solvable in the class \( \mathcal{S}_\kappa^{p \times q} \). The space \( D(s) \) is considered in detail in [23]; see also [3] and the references cited therein.

Remark 3.2. Although the condition (B3) is not necessary for the problem (C1)-(C3) to be solvable, it can be shown that for every data set \( (A_1, A_2, C, P) \) satisfying (B1), (B2) there exists a data set \( (\tilde{A}_1, \tilde{A}_2, \tilde{C}, \tilde{P}) \) satisfying (B1)–(B3), such that

\[
\mathcal{S}_\kappa(A_1, A_2, C, P) = \mathcal{S}_\kappa(\tilde{A}_1, \tilde{A}_2, \tilde{C}, \tilde{P}).
\]

The same statement for the sets of rational mvf’s in \( \mathcal{S}_\kappa(A_1, A_2, C, P) \) and \( \mathcal{S}_\kappa(\tilde{A}_1, \tilde{A}_2, \tilde{C}, \tilde{P}) \) was proved in [14]. It will be shown below that the assumptions (B1)-(B4) combined with (3.3) are sufficient for the problem (C1)-(C3) to be solvable in the class \( \mathcal{S}_\kappa^{p \times q} \). The condition (B4) will be discussed in Subsection 3.4.

Theorem 3.3. If (B1) is in force and \( s \in \mathcal{S}_\kappa^{p \times q} \) satisfies (C1) and (C2), then \( P_s \) is a solution of the Lyapunov-Stein equation (1.4).

Proof. For every \( u \in \mathbb{C}^n \) let us set

\[
h_1(\lambda) = C(M - \lambda N)^{-1} Mu, \quad h_2(\lambda) = C(M - \lambda N)^{-1} Nu
\]

and

\[
g_1 = P_+ \begin{bmatrix} -s^* & I_q \end{bmatrix} h_1, \quad g_2 = P_- \begin{bmatrix} I_p & -s \end{bmatrix} h_2.
\]

Since \( s \) satisfies the assumptions (C1), (C2), one has \( g_1 \in \mathcal{H}(b_r), \quad g_2 \in \mathcal{H}_s(b_\ell) \). It follows from the identity

\[
(M - \lambda N)^{-1} M = I_n + \lambda(M - \lambda N)^{-1} N
\]

that

\[
h_1(\lambda) = Cu + \lambda h_2(\lambda).
\]
Using the formulas (3.1), (3.5) one obtains
\[
u^* M^* P_s M u - u^* N^* P_s N u = \langle \Delta_s h_1, h_1 \rangle_{nst} - \langle \Delta_s h_2, h_2 \rangle_{nst} \\
+ \Re \left[ \left[ \begin{array}{c} 0 \Gamma_t \\
\Gamma_t * 0 \end{array} \right] \Delta_s h_1, \Delta_s h_1 \right]_{nst} - \Re \left[ \left[ \begin{array}{c} 0 \Gamma_t \\
\Gamma_t * 0 \end{array} \right] \Delta_s h_2, \Delta_s h_2 \right]_{nst} \\
= \langle \Delta_s h_1, h_1 \rangle_{nst} - \langle \Delta_s (h_1 - Cu), (h_1 - Cu) \rangle_{nst} \\
+ 2\Re \left[ \left[ \begin{array}{c} 0 \Gamma_t \\
\Gamma_t * 0 \end{array} \right] \Delta_s h_1, \Delta_s h_1 \right]_{nst} - 2\Re \left[ \left[ \begin{array}{c} 0 \Gamma_t \\
\Gamma_t * 0 \end{array} \right] \Delta_s h_2, \Delta_s h_2 \right]_{nst} \\
= 2\Re \langle \Delta_s h_1, Cu \rangle_{nst} + 2\Re X_1 - 2\Re X_2 - \langle \Delta_s Cu, Cu \rangle_{nst},
\]
where \(X_1\) and \(X_2\) are given by
\[
X_1 = \langle \Gamma_t [-s^* I_q h_1, [I_p - s] h_1 \rangle_{nst} \\
= \langle \Gamma_t [-s^* I_q h_1, [I_p - s](Cu + \lambda h_2) \rangle_{nst},
\]
\[
X_2 = \langle \Gamma_r [I_p - s] h_2, [-s^* I_q h_2 \rangle_{nst} \\
= \langle \Gamma_r [I_p - s] h_2, \lambda [-s^* I_q h_1 - Cu \rangle_{nst}.
\]
Decomposing the matrix \(C\) as in (1.6) and using the definitions (3.4) of \(g_1, g_2\) one obtains
\[
X_1 = \langle \Gamma_t g_1, \lambda g_2 \rangle_{nst} - \langle s^* \Gamma_t g_1, C_2 u \rangle_{nst},
\]
\[
X_2 = \langle \Gamma_r g_2, \lambda g_1 \rangle_{nst} + \langle s \Gamma_r g_2, \lambda C_1 u \rangle_{nst}.
\]
The relations (3.4), (3.9) and (3.10) imply
\[
\langle \Delta_s h_1, Cu \rangle_{nst} + X_1 - X_2 = \langle [I_p - s] h_1, C_1 u \rangle_{nst} + \langle [-s^* I_q h_1, C_2 u \rangle_{nst} \\
+ \langle \lambda \Gamma_t g_1, g_2 \rangle_{nst} - \langle s^* \Gamma_t g_1, C_2 u \rangle_{nst} - \langle g_2, \lambda g_1 \rangle_{nst} - \langle s \Gamma_r g_2, \lambda C_1 u \rangle_{nst} \\
= \langle [I_p - s] (Cu + \lambda h_2), C_1 u \rangle_{nst} + \langle g_1, C_2 u \rangle_{nst} \\
+ \langle \lambda \Gamma_t g_1, g_2 \rangle_{nst} - \langle s^* \Gamma_t g_1, C_2 u \rangle_{nst} - \langle g_2, \lambda g_1 \rangle_{nst} - \langle s \Gamma_r g_2, \lambda C_1 u \rangle_{nst} \\
= \langle g_1 - s^* \Gamma_t g_1, C_2 u \rangle_{nst} + \langle g_2 - s \Gamma_r g_2, \lambda C_1 u \rangle_{nst} \\
+ \langle \lambda \Gamma_t g_1, g_2 \rangle_{nst} - \langle g_2, \lambda \Gamma_t g_1 \rangle_{nst} + \langle (C_1 - s C_2) u, C_1 u \rangle_{nst},
\]
Since \(g_1 \in \mathcal{H}(b_r), g_2 \in \mathcal{H}_*(b_r)\) the equations (2.13) characterizing the vectors \(\Gamma_t g_1 \in \mathcal{H}_*(b_r)\) and \(\Gamma_r g_2 \in \mathcal{H}(b_r)\) can be rewritten as
\[
P_+(g_1 - s^* \Gamma_t g_1) = 0, \quad P_-(g_2 - s \Gamma_r g_2) = 0,
\]
and hence (3.11) takes the form
\[
\langle \Delta_s h_1, Cu \rangle_{nst} + X_1 - X_2 = \langle \lambda \Gamma_t g_1, g_2 \rangle_{nst} - \langle g_2, \lambda g_1 \rangle_{nst} + \langle (C_1 - s C_2) u, C_1 u \rangle_{nst}.
\]
Substituting (3.13) in (3.6) one obtains
\[
u^* M^* P_s M u - u^* N^* P_s N u = 2\Re \langle (C_1 - s C_2) u, C_1 u \rangle_{nst} - \langle \Delta_s Cu, Cu \rangle_{nst} \\
+ \langle \lambda \Gamma_t - \Gamma_t \lambda \rangle g_1, g_2 \rangle_{nst} + \langle g_2, (\lambda \Gamma_t - \Gamma_t \lambda) g_1 \rangle_{nst} \\
= \langle C_1 u, C_1 u \rangle_{nst} - \langle C_2 u, C_2 u \rangle_{nst} + 2\Re \langle \lambda \Gamma_t - \Gamma_t \lambda \rangle g_1, g_2 \rangle_{nst}.
\]
In view of Lemma 2.5 the last term in (3.14) is equal 0 and, hence, (3.14) can be rewritten as
\[
M^* P_s M - N^* P_s N = C_1^* C_1 - C_2^* C_2.
\]
□
Theorem 3.3 implies that the problem (C1)-(C3) possesses the Parseval identity property in the sense of [37].

**Corollary 3.4.** If (B1) and (B2) are in force and \( s \in \hat{S}_\kappa(A_1, A_2, C, P) \), then

\[
P_s = P,
\]

and hence

\[
\hat{S}_\kappa(A_1, A_2, C, P) = S_\kappa(A_1, A_2, C, P).
\]

**Proof.** Let \( P \) be decomposed conformally with \( M \) and \( N \):

\[
P = \begin{bmatrix}
P_{11} & P_{12} \\
M & P_{21} \\
N & P_{22}
\end{bmatrix}, \quad P_{ij} \in \mathbb{C}^{n_i \times n_j}, \quad (i, j = 1, 2).
\]

Then, as \( P \) is Hermitian, equation (1.4) is equivalent to the system of equations

\[
A_1^* P_{11} A_1 - P_{11} = C_{11}^* C_{11} - C_{21}^* C_{21},
\]

(3.18)

\[
P_{22} - A_2^* P_{22} A_2 = C_{12}^* C_{12} - C_{22}^* C_{22},
\]

(3.19)

\[
P_{21} A_1 - A_2^* P_{21} = C_{12}^* C_{11} - C_{22}^* C_{21}.
\]

(3.20)

Equations (3.18) and (3.19) have unique solutions \( P_{11} \) and \( P_{22} \), respectively, since \( \sigma(A_1) \cup \sigma(A_2^*) \subset \mathbb{D} \). Therefore, since \( P_s \) is also a solution of the Lyapunov-Stein equation (1.4), it must be of the form

\[
P_s = \begin{bmatrix}
P_{11} & \tilde{P}_{12} \\
\tilde{P}_{21} & P_{22}
\end{bmatrix}, \quad \tilde{P}_{ij} \in \mathbb{C}^{n_i \times n_j}, \quad (i, j = 1, 2).
\]

Thus, the inequality \( P_s \preceq P \) implies (3.15) and so too (3.16).

Let \( C \) be decomposed as \( C = [C_1, C_2] \) (\( C_j \in \mathbb{C}^{m \times n_j}, \quad j = 1, 2 \)) and define the mvf’s

\[
F_1(\lambda) = C_1(A_1 - \lambda I_{n_1})^{-1} \quad \text{and} \quad F_2(\lambda) = C_2(I_{n_2} - \lambda A_2)^{-1}
\]

which belong to \( R \cap (H^m_{2^*})^\perp \) and \( R \cap H^m_{2^*} \), respectively. Our first objective is to check that the condition (1.8) is satisfied.

**Proposition 3.5.** If (B1) and (B3) are in force, then (1.8) holds and

\[
\ker \begin{bmatrix} M - \lambda N \\ C \end{bmatrix} = \{0\} \quad \text{and} \quad \ker \begin{bmatrix} \lambda M - N \\ C \end{bmatrix} = \{0\} \quad \text{for every } \lambda \in \mathbb{C}.
\]

**Proof.** If \( \text{col}(u_1, u_2) \in \ker(C(M - \lambda N)^{-1} \quad \text{for all } \lambda \in \rho(M, N) \) and some \( u_1 \in \mathbb{C}^{n_1}, \) \( u_2 \in \mathbb{C}^{n_2} \), then

\[
F_1(\lambda) u_1 + F_2(\lambda) u_2 \equiv 0, \quad \text{for } \lambda \in \rho(M, N).
\]

Since \( F_1 u_1 \in (H^m_2)^\perp \) and \( F_2 u_2 \in H^m_2 \), it follows that

\[
F_1(\lambda) u_1 = C_1(A_1 - \lambda I_{n_1})^{-1} u_1 \equiv 0
\]

and

\[
F_2(\lambda) u_2 = C_2(I_{n_2} - \lambda A_2)^{-1} u_2 \equiv 0 \quad \text{for } \lambda \in \rho(M, N),
\]

and hence that \( u_1 = 0, \) \( u_2 = 0 \) in view of (B3).

To complete the proof it suffices to verify the rank condition implicit in (3.23) for all points \( \lambda \in \mathbb{D} \). But, if \( \lambda \in \mathbb{D} \), then, in view of (B3),

\[
n \geq \text{rank} \begin{bmatrix} M - \lambda N \\ C \end{bmatrix} \geq \text{rank} \begin{bmatrix} M - \lambda N \\ C_2 \end{bmatrix} = n.
\]
This proves the first equality in (3.23). The proof of the remaining assertion in (3.23) is similar.

3.2. Regular case. We now parametrize the set of solutions \( S_\kappa(A_1, A_2, C, P) \) of the problem (C1)–(C3) assuming that the matrices \( A_1, A_2, C, P \) satisfy the constraints (B1)–(B3) and \( P \) is invertible.

**Theorem 3.6.** Let (B1)–(B3) be in force, let \( P \) be invertible and let \( \kappa_1 := \nu_-(P) \leq \kappa \). Then

\[
S_\kappa(A_1, A_2, C, P) = S_\kappa^{p \times q} \cap T_W[S_\kappa^{p \times q}],
\]

where the mvf \( W(\lambda) \) is given by formula (2.26).

**Proof.** In view of Example 1, the linear space \( W \) where the mvf \( S \) (3.25) and \( \alpha \) \( W \) where the mvf \( s \) (3.25) of that theorem. □

3.3. Examples. We next present a few examples to illustrate the interpolation problem (C1)–(C3).

**Example 2.** (Interpolation with multiplicities one). Let

\[
C = \begin{bmatrix}
\xi_1 & \cdots & \xi_n \\
\eta_1 & \cdots & \eta_n
\end{bmatrix}, \quad (\xi_j \in \mathbb{C}^p, \; \eta_j \in \mathbb{C}^q, \; 1 \leq j \leq n)
\]

and let the matrices \( M \) and \( N \) be as in (1.9) with

\[
A_1 = \text{diag}(\alpha_1, \cdots, \alpha_{n_1}), \quad A_2 = \text{diag}(\overline{\alpha}_{n_1+1}, \cdots, \overline{\alpha}_n)
\]

and \( \alpha_j \in \mathbb{D}, \; (j = 1, \ldots, n) \). Then the interpolation condition (C1) is met if and only if

\[
\frac{b_t(\xi_j) - s_t(\lambda)\eta_j}{\alpha_j - \lambda} \in H_2^p \quad \text{for} \quad 1 \leq j \leq n_1,
\]

and

\[
\frac{b_t(\xi_j) - s_t(\lambda)\eta_j}{1 - \lambda \overline{\alpha}_j} \in H_2^p \quad \text{for} \quad n_1 + 1 \leq j \leq n.
\]

The second set of conditions is automatically fulfilled, and the first set can be rewritten as

\[
b_t(\alpha_j)\xi_j = s_t(\alpha_j)\eta_j \quad \text{for} \quad 1 \leq j \leq n_1.
\]

If the \( b_t(\alpha_j) \) are invertible, then the constraints in (3.27) take the form

\[
s(\alpha_j)\eta_j = \xi_j \quad \text{for} \quad 1 \leq j \leq n_1.
\]

Similarly, condition (C2) holds if and only if

\[
\frac{s(\xi_j) - b^\#(\lambda)\eta_j}{1 - \lambda \overline{\alpha}_j} \in (H_2^p)^\perp \quad \text{for} \quad n_1 + 1 \leq j \leq n,
\]

or, equivalently, if and only if

\[
\xi_j^* s(\alpha_j) = \eta_j^* b^r(\alpha_j) \quad \text{for} \quad n_1 + 1 \leq j \leq n.
\]
If the $b_r(\alpha_j)$ are invertible, then the constraints in (3.29) take the form
\[(3.30)\]
$$\xi_j^* s(\alpha_j) = \eta_j^* \quad (n_1 + 1 \leq j \leq n).$$

If $\sigma(A_1) \cap \sigma(A_2^*) = \emptyset$, then the Lyapunov-Stein equation (1.4) has exactly one solution. Therefore, $P_s = P$, since both of these matrices are solutions of (1.4) and hence the condition (C3) is automatically met.

If $\sigma(A_1) \cap \sigma(A_2^*) \neq \emptyset$, then, since $P_s$ and $P$ are both Hermitian, (1.4) is equivalent to the system (3.18)-(3.20) for the blocks $P_{11}$, $P_{22}$ and $P_{21}$ of $P$, respectively. Since the first two equations are uniquely solvable,
$$P - P_s = \begin{bmatrix} 0 & X^* \\ X & 0 \end{bmatrix}$$
with $X = P_{21} - (P_s)_{21}$.

Therefore, (C3) holds if and only if $s \in S_{p\times q}^{\kappa}$ is such that $P_{21} = (P_s)_{21}$. This imposes an extra interpolation condition on $s$ (see e.g., p.368 of [29] for an explicit example when $\kappa = 0$). Because of this, (C1)–(C2) was called the basic interpolation problem and (C1)–(C3) was called the augmented interpolation problem in [27].

Example 3. (Nevanlinna-Pick matrix interpolation problem). Let $n_2 = 0$, $n_1 = tq$, and let the matrices $A_1$ and $C$ have the following block form
\[(3.31)\]
$$A_1 = \text{diag} (\alpha_1 I_q, \ldots, \alpha_t I_q), \quad C = \begin{bmatrix} s_1 & \cdots & s_t \\ I_q & \cdots & I_q \end{bmatrix}$$
where $\alpha_j$ are distinct points in $\mathbb{D}$ and $s_j \in \mathbb{C}^{p\times q}$ for $j = 1, \ldots, t$. Then the interpolation condition (C1) reduces to
\[(3.32)\]
$$s_\ell(\alpha_j) = b_\ell(\alpha_j) s_j \quad \text{for } j = 1, \ldots, t.$$
If $s(\lambda)$ is holomorphic at $\alpha_j$ for $j = 1, \ldots, t$, then these conditions take the form
\[(3.33)\]
$$s(\alpha_j) = s_j \quad \text{for } j = 1, \ldots, t.$$
The data $\{A_1, C\}$ specified in (3.31) satisfies the conditions (B1)–(B3) and the corresponding Lyapunov-Stein equation (1.4) has a unique solution that may be written in block form as
\[(3.34)\]
$$P = \begin{bmatrix} I_q - s_j^* s_k \\ \alpha j_{,k=1} \end{bmatrix}^t.$$  
Let $\kappa_1 = \nu_-(P)$. If $P$ is invertible and $\kappa_1 \leq \kappa$, then, by I of Theorem 1.3, the set of solutions to the problem (3.32) is described by the formula
$$S_\kappa(A_1, C, P) = T_W[S_{\kappa - \kappa_1}] \cap S_{p\times q}^{\kappa},$$
where $W$ is given by (2.26).

The Nevanlinna-Pick problem in the class of mvf’s $s \in S_{p\times q}^{\kappa}$ that are holomorphic at the interpolation points was investigated by Golinskii in [33].

Example 4. Suppose now that
\[A_1 = \begin{bmatrix} \alpha & 1 \\ \vdots & \ddots \\ 1 & \alpha \end{bmatrix} = \alpha I_{n_1} + N_1 \quad \text{and} \quad A_2 = \begin{bmatrix} \beta & 1 \\ \vdots & \ddots \\ 1 & \beta \end{bmatrix} = \beta I_{n_2} + N_2\]
are Jordan cells of size $n_1 \times n_1$ and $n_2 \times n_2$, respectively, and that $\alpha, \beta \in \mathbb{D}$. Then (C1) holds if and only if

$$[b_\ell(\lambda) - s_\ell(\lambda)] \begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix} (\lambda I_{n_1} - A_1)^{-1}$$

is holomorphic in $\mathbb{D}$. A necessary condition for this is that the contour integral around the unit circle $\mathbb{T}$

$$\frac{1}{2\pi i} \int_{\mathbb{T}} [b_\ell(\zeta) - s_\ell(\zeta)] \begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix} (\zeta I_{n_1} - A_1)^{-1} d\zeta = 0.$$ 

But upon substituting the formula

$$(\zeta I_{n_1} - A_1)^{-1} = \sum_{j=0}^{n_1-1} \frac{N^j_1}{(\zeta - \alpha)^{j+1}}$$

into the last integral and invoking Cauchy’s formula this is readily seen to reduce to the constraint

$$\sum_{j=0}^{n_1-1} \frac{b^{(j)}_\ell(\alpha)}{j!} C_{11} N^j_1 = \sum_{j=0}^{n_1-1} \frac{s^{(j)}_\ell(\alpha)}{j!} C_{21} N^j_1,$$

or, equivalently, in terms of the columns $\xi_1, \ldots, \xi_{n_1}$ of $C_{11}$ and the columns $\eta_1, \ldots, \eta_{n_1}$ of $C_{21}$,

$$\begin{bmatrix} a_0 & 0 & \cdots & 0 \\ a_1 & a_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n_1-1} & a_{n_1-2} & \cdots & a_0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_{n_1} \end{bmatrix} = \begin{bmatrix} b_0 & 0 & \cdots & 0 \\ b_1 & b_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ b_{n_1-1} & b_{n_1-2} & \cdots & b_0 \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_{n_1} \end{bmatrix},$$

where

$$a_j = \frac{b^{(j)}_\ell(\alpha)}{j!} \text{ and } b_j = \frac{s^{(j)}_\ell(\alpha)}{j!}.$$ 

It is readily checked that this condition is sufficient as well as necessary, and that it is equivalent to the asymptotic condition

$$(3.35) \quad b_\ell(\lambda)(\xi_1 + (\lambda - \alpha)\xi_2 + \cdots + (\lambda - \alpha)^{n_1-1}\xi_{n_1})$$

$$= s_\ell(\lambda)(\eta_1 + (\lambda - \alpha)\eta_2 + \cdots + (\lambda - \alpha)^{n_1-1}\eta_{n_1}) + O(\lambda - \alpha)^{n_1} \quad \text{as } \lambda \to \alpha.$$ 

Similarly, the condition (C2) is met if and only if the columns of the mvf

$$[-s^\#, b^\#] \begin{bmatrix} C_{21} \\ C_{22} \end{bmatrix} (I_{n_2} - \lambda A_2)^{-1}$$

belong to $(H^3_2)^\perp$, or, equivalently, if and only if

$$(\lambda I - A^*_2)^{-1} \begin{bmatrix} C_{21}^* \\ C_{22}^* \end{bmatrix} \begin{bmatrix} -s_r \\ b_r \end{bmatrix}$$

is holomorphic in $\mathbb{D}$, i.e., if and only if

$$\sum_{j=0}^{n_2-1} (N^*_2)^j C_{21}^* \frac{s^{(j)}_r(\beta)}{j!} = \sum_{j=0}^{n_2-1} (N^*_2)^j C_{22}^* \frac{b^{(j)}_r(\beta)}{j!}.$$
The last constraint can be rewritten in terms of the columns $\xi_{n_1+1}, \ldots, \xi_n$ of $C_{21}$ and the columns $\eta_{n_1+1}, \ldots, \eta_n$ of $C_{22}$ as

$$
\begin{bmatrix}
\xi^*_{n_1+1} & \cdots & \xi_n
\end{bmatrix}
= 
\begin{bmatrix}
c_0 & c_1 & \cdots & c_{n_2-1} \\
0 & c_0 & \cdots & c_{n_2-2} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & c_0
\end{bmatrix} 
\begin{bmatrix}
\eta^*_{n_1+1} & \cdots & \eta_n
\end{bmatrix} ,
$$

where

$$
c_j = \frac{s^{(j)}(\beta)}{j!} \quad \text{and} \quad d_j = \frac{b^{(j)}(\beta)}{j!} \quad \text{for} \quad j = 0, \ldots, n_2 - 1 .
$$

This condition is equivalent to

$$
(\xi^*_{n_1+1} + (\lambda - \beta)\xi^*_{n_1+2} + \cdots + (\lambda - \beta)^{n_2-1}\xi^*_n) s_r(\lambda) = (\eta^*_{n_1+1} + (\lambda - \beta)\eta^*_{n_1+2} + \cdots + (\lambda - \beta)^{n_2-1}\eta^*_n) b_r(\lambda) + O((\lambda - \beta)^{n_2}) \quad \text{as} \lambda \rightarrow \beta .
$$

Interpolation problems in which $A_1$ and/or $A_2$ are made up of several Jordan blocks lead to similar sets of formulas.

If $s$ is holomorphic on $\sigma(A_1) \cup \sigma(A_2)$ then the interpolation conditions (3.35) and (3.36) take the form

$$
\begin{align*}
(\xi^*_{n_1+1} + (\lambda - \beta)\xi^*_{n_1+2} + \cdots + (\lambda - \beta)^{n_2-1}\xi^*_n) & = (\eta^*_{n_1+1} + (\lambda - \beta)\eta^*_{n_1+2} + \cdots + (\lambda - \beta)^{n_2-1}\eta^*_n) + O((\lambda - \beta)^{n_2}) \quad \text{as} \lambda \rightarrow \beta .
\end{align*}
$$

One sided versions of the problem considered in Example 4 ($\ell_2 = 0$) of the problem have been considered by T. Takagi [11] and A. A. Nudelman [11], respectively. Rational solutions of the two-sided Takagi-Nudelman problem (3.37)–(3.38) were described in [14].

Remark 3.7. If (C4) is in force, i.e., if $b_\ell(\lambda)$ and $b_r(\lambda)$ are invertible at every point $\lambda \in \sigma(A_1) \cup \sigma(A_2)$, then the conditions (C1)–(C3) are equivalent to the residue conditions

$$
\sum_{j=1}^{n_1} \text{res}\{s(\lambda)C_{21}(\lambda I_{n_1} - A_1)^{-1}\} = C_{11} ,
$$

$$
\sum_{j=n_1+1}^{n_1+n_2} \text{res}\{(\lambda I_{n_2} - A_2^*)^{-1}C_{12}^*s(\lambda)\} = C_{22}^* ,
$$

and

$$
\sum_{j=1}^{n_1+n_2} \text{res}\{(\lambda I_{n_2} - A_2^*)^{-1}C_{12}^*s(\lambda)C_{21}(\lambda I_{n_1} - A_1)^{-1}\} = P_{21} ,
$$

in [14], respectively.

3.4. Resolvent matrix in the singular case. If $P$ is not invertible, then the construction of the model space $\mathcal{M}$ depends upon assumption (B4); cf. [30], [17]. Since $\text{rng} X$ is presumed to be invariant under $M$ and $N$, there exist matrices $M_0 \in \mathbb{C}^{n \times n}$ and $N_0 \in \mathbb{C}^{n \times n}$ such that

$$
MX = XM_0 \quad \text{and} \quad NX = XN_0 .
$$

Moreover, the matrices $M_0$ and $N_0$ may be chosen so that

$$
\sigma(M_0) \subseteq \sigma(M) \quad \text{and} \quad \sigma(N_0) \subseteq \sigma(N) ;
$$

see e.g., [31].
Lemma 3.8. If (B4) is in force and $M$ and $N$ are as in (1.9), then

\begin{equation}
\nu_-(X) = \nu_-(P), \quad \text{rank } X = \text{rank } P
\end{equation}

and

\begin{equation}
\text{rng } X = \left( \text{rng } X \cap \left[ \mathbb{C}^n_1 \right] \right) + \left( \text{rng } X \cap \left[ \mathbb{C}^n_2 \right] \right).
\end{equation}

Proof. The equality $XPX = X$ implies that $\nu_-(X) \leq \nu_-(P)$ and rank $X \leq$ rank $P$; whereas the equality $PXP = P$ implies the opposite inequalities. Therefore, (3.41) holds.

Next, since $\mathbb{C}^n = \left[ \mathbb{C}^n_1 \right] + \left[ \mathbb{C}^n_2 \right]$, to verify (3.42), it suffices to show that

if $u \in \text{rng } X$, then \[ \begin{bmatrix} 0 & 0 \\ 0 & I_{n_2} \end{bmatrix} u \in \text{rng } X. \]

The invariance assumption implies that $MX = XM_0$ for some $M_0 \in \mathbb{C}^{n \times n}$ with $\sigma(M_0) \subseteq \sigma(M)$. Therefore, $M^kX = XM_0^k$, and since $\sigma(A_1) \subseteq \mathbb{D}$, the entries in $M_0^k$ are bounded and so a subsequence $M_0^{k_j}$ tends to a limit $H \in \mathbb{C}^{n \times n}$ as $k_j$ tends to infinity. Thus, if $u = Xv$, then

\[ \begin{bmatrix} 0 & 0 \\ 0 & I_{n_2} \end{bmatrix} u = \lim_{j \to \infty} M_0^{k_j} v = XHv. \]

□

Lemma 3.9. If (B1), (B2) and (B4) are in force, then $X$ is a solution of the Riccati equation

\begin{equation}
XM^*PMX - XN^*PNX = XC^*j_{pq}CX,
\end{equation}

or, equivalently (in view of (3.39)),

\begin{equation}
M_0^*XM_0 - N_0^*XN_0 = XC^*j_{pq}CX.
\end{equation}

Proof. This follows from (1.4) upon multiplying it on the left and on the right by $X$ and using (3.39). □

Let $\mathcal{M}$ be the space of rational vvf's

\begin{equation}
\mathcal{M} = \{ F(\lambda)Xu : u \in \mathbb{C}^n \}
\end{equation}

endowed with the inner product

\begin{equation}
\langle FXu, FXv \rangle_{\mathcal{M}} = v^*Xu.
\end{equation}

If the condition (B3) holds, then the inner product in $\mathcal{M}$ is well defined, since by Proposition 3.5 the identity $F(\lambda)Xu \equiv 0$ implies $Xu = 0$ and hence $\langle FXu, FXv \rangle_{\mathcal{M}} = 0$ for all $v \in \mathbb{C}^n$. $\mathcal{M}$ is a reproducing kernel space with kernel

\begin{equation}
K_\omega(\lambda) = F(\lambda)XF(\omega)^* \quad \text{for } \lambda, \omega \in \rho(M, N).
\end{equation}

Lemma 3.10. Let (B1)–(B4) be in force, let the reproducing kernel space $\mathcal{M}$ be given by (3.45), (3.46) and let the mvf $W(\lambda)$ be given by (1.14) Then $\mathcal{M}$ is a finite dimensional de Branges-Krein space $\mathcal{K}(W)$ with reproducing kernel

\begin{equation}
K_\omega(\lambda) = F(\lambda)XF(\omega)^* = K_\omega^{W}(\lambda) := \frac{j_{pq} - W(\lambda)j_{pq}W(\omega)^*}{\rho_\omega(\lambda)}
\end{equation}

and negative index $\nu_-(P)$.
Proof. Upon substituting (1.14) into (1.15), direct calculations show that
\begin{equation}
K_\\omega^W(\lambda) = F(\lambda)X(M^* - \overline{\mu}N^*)^{-1} \frac{Y}{\rho_\omega(\lambda)}(M - \mu N)^{-1}XF(\omega)^*,
\end{equation}
where, with the help of the Lyapunov-Stein equation (1.4) and the formulas
\begin{equation}
X(M^* - \overline{\mu}N^*)^{-1} = X(M^* - \overline{\mu}N^*)^{-1}PX \quad \text{and} \quad (M - \mu N)^{-1}X = XP(M - \mu N)^{-1}X
\end{equation}
based on the invariance of the rng $X$ under $M$ and $N$, the central term
\begin{equation}
Y = \rho_\mu(\lambda)(M^* - \overline{\mu}N^*)P(M - \mu N)
\end{equation}
\begin{equation}
+ \rho_\omega(\mu)(M^* - \overline{\mu}N^*)P(M - \lambda N) - \rho_\mu(\lambda)\rho_\omega(\mu)C^*_j\mu C
\end{equation}
\begin{equation}
= \rho_\omega(\lambda)(M^* - \overline{\mu}N^*)P(M - \mu N).
\end{equation}
The verification of the formula for $Y$ rests on the evaluations
\begin{equation}
F(\lambda)XFP(\mu) = F(\lambda)X(M^* - \overline{\mu}N^*)^{-1}(M^* - \overline{\mu}N^*)F(\omega)^*
\end{equation}
\begin{equation}
= F(\lambda)X(M^* - \overline{\mu}N^*)^{-1}(M^* - \overline{\mu}N^*)PXF(\omega)^*
\end{equation}
\begin{equation}
= F(\lambda)X(M^* - \overline{\mu}N^*)^{-1}(M^* - \overline{\mu}N^*)P(M - \mu N)(M - \mu N)^{-1}XF(\omega)^*.
\end{equation}
Now (3.49) and (3.50) yield (3.48). \qed

Lemma 3.11. If the Hermitian matrices $X, P \in \mathbb{C}^{n \times n}$ satisfy $XPX = X$ and $\text{rank } X = \text{rank } P$, then
\begin{equation}
\mathbb{C}^n = \ker P + \text{rng } X
\end{equation}
Proof. (Cf. [30].) If $u = Xv$ and $Pu = 0$, then $0 = XPXv = Xv = u$. Thus, the indicated sum is direct. The rest follows from the fact that
\begin{equation}
n = \dim \text{rng } P + \dim \ker P,
\end{equation}
and, by assumption, $\dim \text{rng } P = \dim \text{rng } X$. \qed

The assumptions of Lemma 3.11 are fulfilled if $X$ and $P$ satisfy (i) and (ii) in (B4).

Lemma 3.12. If $\lambda \in \rho(M, N) \cap \rho(N^*, M^*)$, $W$ is as in (1.14) and
\begin{equation}
\tilde{C} = F(\mu)X(\mu M^* - N^*),
\end{equation}
then
\begin{equation}
j_{pq}W^\#(\lambda)j_{pq}F(\lambda)X = \tilde{C}(\lambda M^* - N^*)^{-1}
\end{equation}
and
\begin{equation}
j_{pq}W^\#(\lambda)j_{pq}F(\lambda)v = F(\mu)(I_n - XP)(M - \mu N)(M - \lambda N)^{-1}v \quad \text{for } v \in \ker P.
\end{equation}
Proof. Formula (1.14) implies that
\begin{equation}
j_{pq}W^\#(\lambda)j_{pq}F(\lambda) = C[I_n - (\lambda - \mu)(M - \mu N)^{-1}X(\lambda M^* - N^*)^{-1}C^*_j\mu C](M - \lambda N)^{-1}
\end{equation}
\begin{equation}
= C(M - \mu N)^{-1}[M - \mu N - (\lambda - \mu)X(\lambda M^* - N^*)^{-1}C^*_j\mu C](M - \lambda N)^{-1}.
\end{equation}
Therefore, since
\begin{equation}
C^*_j\mu C = M^*PM - N^*PN = M^*P(M - \lambda N) + (\lambda M^* - N^*)PN,
\end{equation}
it follows that
\[
  j_{pq} W^\#(\lambda) j_{pq} F(\lambda) = F(\mu)[-(\lambda - \mu)X(\lambda M^* - N^*)^{-1}M^*P]
  + F(\mu)[M - \mu N - (\lambda - \mu)XPN](M - \lambda N)^{-1}
\]
\[
= F(\mu)[I_n - (\lambda - \mu)X(\lambda M^* - N^*)^{-1}M^*P]
+ (\lambda - \mu)F(\mu)(I_n - XP)N(M - \lambda N)^{-1}.
\]
(3.55)

Thus, as
\[
(I_n - XP)N(M - \lambda N)^{-1}X = 0 \quad \text{and} \quad X(\lambda M^* - N^*)^{-1}M^*(I_n - PX) = 0,
\]
it is readily seen that
\[
X(\lambda M^* - N^*)^{-1}M^*PX = X(\lambda M^* - N^*)^{-1}M^*,
\]
and hence that formula (3.55) implies that
\[
j_{pq} W^\#(\lambda) j_{pq} F(\lambda)X = F(\mu)X[I_n - (\lambda - \mu)(\lambda M^* - N^*)^{-1}M^*]
= F(\mu)X[\mu M^* - N^*](\lambda M^* - N^*)^{-1}.
\]

This proves (3.53).

Next, if \(v \in \ker P\), then formula (3.55) implies that
(3.56)
\[
j_{pq} W^\#(\lambda) j_{pq} F(\lambda)v = F(\mu)[I_n + (\lambda - \mu)(I_n - XP)N(M - \lambda N)^{-1}]v
= F(\mu)[I_n + (I_n - XP)\{(\lambda N - M) + (M - \mu N)\}(M - \lambda N)^{-1}]v
= F(\mu)XPv + F(\mu)(I_n - XP)(M - \mu N)(M - \lambda N)^{-1}v.
\]
This proves (3.54) since \(F(\mu)XPv = 0\).

\[\Box\]

Remark 3.13. Lemma 3.12 is similar to [28, Theorem 5.1] (see also [16, Lemma 2.1]). If \(P\) is invertible, then \(P^{-1}\) satisfies the Lyapunov-Stein equation
\[
MP^{-1}M^* - NP^{-1}N^* = \tilde{C}^* j_{pq} \tilde{C}.
\]

Lemma 3.14. Let \(W\) and \(\tilde{C}\) be given by (1.14) and (3.52), respectively. Then
(3.57)
\[
W(\lambda) = D + C \begin{bmatrix} (A_1 - \lambda I_{n_1})^{-1} & 0 \\ 0 & \lambda(I_{n_2} - \lambda A_2)^{-1} \end{bmatrix} \tilde{C}^* j_{pq}, \quad \lambda \in \rho(M, N)
\]
and
(3.58)
\[
W^{-1}(\lambda) = j_{pq} D^* j_{pq} - \tilde{C} \begin{bmatrix} \lambda(I_{n_1} - \lambda A_1^*)^{-1} & 0 \\ 0 & (A_2^* - \lambda I_{n_2})^{-1} \end{bmatrix} C^* j_{pq}, \quad \lambda \in \rho(N^*, M^*),
\]
where
\[
D = I_m - C \begin{bmatrix} \bar{\mu}I_{n_1} & 0 \\ 0 & I_{n_2} \end{bmatrix} XF(\mu)^* j_{pq}.
\]

Proof. This is a tedious but straightforward computation based on the identities
\[
(1 - \bar{\mu}\lambda)(A_1 - \lambda I_{n_1})^{-1} = \bar{\mu}I_{n_1} + (A_1 - \lambda I_{n_1})^{-1}(I_{n_1} - \bar{\mu}A_1),
\]
\[
(1 - \bar{\mu}\lambda)(I_{n_2} - \lambda A_2)^{-1} = I_{n_2} + \lambda(I_{n_2} - \lambda A_2)^{-1}(A_2 - \bar{\mu}I_{n_2})
\]
and
\[
W^{-1}(\lambda) = j_{pq} W^\#(\lambda) j_{pq}.
\]
\[\Box\]
**Lemma 3.15.** Let \((A_1, A_2, C, P)\) satisfy \((B1) - (B4)\), let \(X\) be defined in \((B4)\), and let

\[
\tilde{C}_1 = \tilde{C} \left[ \begin{array}{c} I_{n_1} \\ 0 \end{array} \right] \quad \text{and} \quad \tilde{C}_2 = \tilde{C} \left[ \begin{array}{c} 0 \\ I_{n_2} \end{array} \right].
\]

Then

\[
\mathcal{L}_1 := \ker \left[ \begin{array}{c} \tilde{C}_1 \\ \tilde{C}_1 A_1^* \\ \vdots \\ \tilde{C}_1 A_1^{n_1-1} \end{array} \right] = \ker \left[ \begin{array}{c} X_{11} \\ X_{21} \end{array} \right] \quad \text{and} \quad \mathcal{L}_2 := \ker \left[ \begin{array}{c} \tilde{C}_2 \\ \tilde{C}_2 A_2^* \\ \vdots \\ \tilde{C}_2 A_2^{n_2-1} \end{array} \right] = \ker \left[ \begin{array}{c} X_{12} \\ X_{22} \end{array} \right].
\]

**Proof.** Since the subspace \(\mathcal{L}_1\) is invariant under \(A_1^*\), it can be decomposed into the sum of algebraic subspaces of \(A_1^*|_{\mathcal{L}_1}\). Let \(u_1, u_2, \ldots, u_k\) be a maximal chain of generalized eigenvectors of \(A_1^*|_{\mathcal{L}}\) corresponding to an eigenvalue \(\alpha\) and let \(U = [u_1 \cdots u_k]\). Then

\[
\tilde{C}_1 (A_1^*)^j U = 0 \quad \text{for } j = 0, 1, \ldots \quad \text{and} \quad A_1^* U = U (\alpha I_k + Z),
\]

where \(Z\) is an upper triangular \(k \times k\) Jordan cell with 0 on the diagonal. Therefore, since \(\sigma(A_1) \subset \mathbb{D}\),

\[
-C(M - \mu N)^{-1} \left[ \begin{array}{c} X_{11} \\ X_{21} \end{array} \right] U = \tilde{C}_1 (I_{n_1} - \mu A_1^*)^{-1} U = \sum_{j=0}^\infty \tilde{C}_1 \mu^j (A_1^*)^j U = 0.
\]

Thus, if

\[
\tilde{u}_j = \left[ \begin{array}{c} u_j \\ 0 \end{array} \right] \quad \text{for } j = 1, \ldots, k \quad \text{and correspondingly} \quad \tilde{U} = \left[ \begin{array}{c} U \\ 0 \end{array} \right],
\]

then

\[
C(M - \mu N)^{-1} X \tilde{U} = 0.
\]

The Lyapunov-Stein equation implies that

\[
M^* P (M - \mu N) + (\mu M^* - N^*)PN = C^* j_{pq} C
\]

and hence that

\[
X (\mu M^* - N^*)^{-1} M^* PX + X PN (M - \mu N)^{-1} X = X (\mu M^* - N^*)^{-1} C^* j_{pq} C (M - \mu N)^{-1} X.
\]

In view of the presumed invariance of the range of \(X\) under multiplication by \(M\) and \(N\),

\[
X (\mu M^* - N^*)^{-1} M^* PX = X (\mu M^* - N^*)^{-1} M^*
\]

and

\[
X PN (M - \mu N)^{-1} X = N (M - \mu N)^{-1} X.
\]

Therefore, equations \((3.64)\) and \((3.62)\) imply that

\[
X (\mu M^* - N^*)^{-1} M^* \tilde{U} + N (M - \mu N)^{-1} X \tilde{U} = 0.
\]

However, since

\[
X (\mu M^* - N^*)^{-1} M^* \tilde{U} = \left[ \begin{array}{c} X_{11} \\ X_{21} \end{array} \right] (\mu A_1^* - I_{n_1})^{-1} A_1^* U
\]

\[
= \left[ \begin{array}{c} X_{11} \\ X_{21} \end{array} \right] U (\beta I_k + \mu Z)^{-1} (\alpha I_k + \mu Z),
\]

\[
\text{and} \quad X PN (M - \mu N)^{-1} X = N (M - \mu N)^{-1} X,
\]

then

\[
X (\mu M^* - N^*)^{-1} M^* \tilde{U} = \left[ \begin{array}{c} X_{11} \\ X_{21} \end{array} \right] U (\beta I_k + \mu Z)^{-1} (\alpha I_k + \mu Z) = 0.
\]
where \( \beta = \mu \alpha - 1 \neq 0 \), the matrix
\[
E = \begin{bmatrix} X_{11} \\ X_{12} \end{bmatrix} U
\]
is a solution of the equation
\[
E(\beta I_k + \mu Z)^{-1}(\alpha I_k + Z) + N(M - \mu N)^{-1}E = 0.
\]
Moreover, since
\[
(\beta I_k + \mu Z)^{-1}(\alpha I_k + Z) = \gamma I_k + (1 - \mu \gamma)(\beta I_k + \mu Z)^{-1}Z \quad \text{with} \quad \gamma = \alpha / \beta
\]
and
\[
\gamma I_n + N(M - \mu N)^{-1} \beta^{-1}(\alpha M - N)(M - \mu N)^{-1},
\]
it is readily seen that
\[
(\alpha M - N)(M - \mu N)^{-1}E = (\mu \gamma - 1)\beta E(\beta I_k + \mu Z)^{-1}Z.
\]
Let \( e_j, j = 1, \ldots, k \), denote the \( j \)’th column of \( I_k \). Then, since
\[
C(M - \mu N)^{-1}E = 0, \quad Ze_1 = 0 \quad \text{and} \quad Ze_{j+1} = e_j \quad \text{for} \quad j = 1, \ldots, k - 1,
\]
it follows that
\[
\begin{bmatrix} \alpha M - N \\ C \end{bmatrix} (M - \mu N)^{-1}Ee_1 = 0
\]
and hence that \( Ee_1 = 0 \). Next, proceeding inductively, suppose that \( Ee_1 = \cdots = Ee_j = 0 \). Then
\[
(\alpha M - N)(M - \mu N)^{-1}Ee_{j+1} = (\mu \gamma - 1)\beta E(\beta I_k + \mu Z)^{-1}Ze_{j+1}
\]
\[
= (\mu \gamma - 1)\beta E(\beta I_k + \mu Z)^{-1}e_j
\]
\[
= (\mu \gamma - 1)\{ Ee_j - \mu E(\beta I_k + \mu Z)^{-1}Ze_j \}
\]
\[
= (\mu / \beta)(\alpha M - N)(M - \mu N)^{-1}Ee_j = 0.
\]
Therefore,
\[
\begin{bmatrix} \alpha M - N \\ C \end{bmatrix} (M - \mu N)^{-1}Ee_{j+1} = 0
\]
too. This proves that the columns of \( U \) are in the kernel of \( \text{col}(X_{11}, X_{21}) \). The opposite inclusion follows from the presumed invariance:
\[
\tilde{C}_1(A_1^*)^j F(\mu)X(\mu M^* - N^* )^{-1}(M^*)^j[ \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix} ] = F(\mu)X(\mu M^* - N^* )^{-1}(M^*)^j PX [ \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix} ].
\]
This completes the proof of the first assertion. The proof of the second is similar. \( \square \)

**Corollary 3.16.** Let the assumptions of Lemma 3.15 hold. Then
\[
\text{rng} \left[ \tilde{C}_1^* A_1 \tilde{C}_1^* \ldots A_1^{n_1-1} \tilde{C}_1^* \right] = \text{rng} \left[ X_{11} \quad X_{12} \right],
\]
\[
\text{rng} \left[ \tilde{C}_2^* A_2 \tilde{C}_2^* \ldots A_2^{n_2-1} \tilde{C}_2^* \right] = \text{rng} \left[ X_{21} \quad X_{22} \right].
\]
3.5. Interpolation with singular $P$. Given a set of matrices $(A_1, A_2, C, P, X)$ that satisfy (B1)–(B4), let $\tilde{S}_\kappa(A_1, A_2, C, P, X)$ denote the set of mvf’s $s \in S^{p \times q}_\kappa$ which satisfy the following conditions:

(X1) $\begin{bmatrix} b_\ell & -s_\ell \end{bmatrix} FXu \in H^1_\kappa$ for every $u \in \mathbb{C}^n$.

(X2) $\begin{bmatrix} -s^\#_\ell & b^\#_\ell \end{bmatrix} FXu \in (H^1_\kappa)^\perp$ for every $u \in \mathbb{C}^n$.

(X3) $XP_s X \leq XPX = X$.

**Theorem 3.17.** If (B1)–(B4) are in force and $\kappa \geq \kappa_1 = \nu_-(P)$, then

$$\tilde{S}_\kappa(A_1, A_2, C, P, X) = T_W[S^{p \times q}_{\kappa, -\kappa_1}] \cap S^{p \times q}_\kappa,$$

where the mvf $W$ is given by (1.14).

**Proof.** 1) Consider the linear space $M$ defined by (3.45) with inner product (3.46). By Lemma 3.10 $M$ is a finite dimensional de Branges-Krein space $K(W)$ with $W$ given by (1.14).

Let $s \in \tilde{S}_\kappa(A_1, A_2, C, P, X)$. Then it follows from (X3) that $XP_s X \leq X$ and for every $u \in \mathbb{C}^n$

$$(3.68) \quad \left\langle \left\{ \Delta_s + \Delta_s \begin{bmatrix} 0 & \Gamma_\ell \\ \Gamma_\ell & 0 \end{bmatrix} \Delta_s \right\} FXu, FXu \right\rangle_{n_{st}} \leq \langle FXu, FXu \rangle_{K(W)}.$$  

Due to Theorem 2.8 this implies that $s \in T_W[S^{p \times q}_{\kappa, -\kappa_1}]$. This proves the inclusion

$$(3.69) \quad \tilde{S}_\kappa(A_1, A_2, C, P, X) \subseteq S^{p \times q}_\kappa \cap T_W[S^{p \times q}_{\kappa, -\kappa_1}].$$

2) Conversely, let $s \in S^{p \times q}_\kappa \cap T_W[S^{p \times q}_{\kappa, -\kappa_1}]$. Then, by Theorem 2.8 the conditions (X1), (X2) and the inequality (3.68) hold. In view of (3.1) and (3.46) this implies $XP_s X \leq X$. Therefore $s \in \tilde{S}_\kappa(A_1, A_2, C, P, X)$ which serves to prove the opposite inclusion in (3.69). □

Clearly,

$$S_\kappa(A_1, A_2, C, P) \subseteq \tilde{S}_\kappa(A_1, A_2, C, P, X),$$

since

$$s \in S_\kappa(A_1, A_2, C, P) \implies P_s = P \implies XP_s X = XPX = X.$$  

In fact $XP_s X = X$ for every solution $s \in \tilde{S}_\kappa(A_1, A_2, C, P, X)$, but we will not verify that in this paper. Every mvf $s \in S_\kappa(A_1, A_2, C, P)$ satisfies also an extra condition that will be presented below in Lemma 3.19. It is convenient, however, to first establish a preliminary fact from linear algebra.

**Lemma 3.18.** If $A \in \mathbb{C}^{n \times n}$ is positive semidefinite, $B \in \mathbb{C}^{n \times n}$ is Hermitian and $\nu_-(A+B) = \nu_-(B)$, then $\ker(A+B) \subseteq \ker A$.

**Proof.** Let $k = \nu_-(A+B) = \nu_-(B)$. Since the conclusion of the lemma is clear if $k = 0$, suppose that $k > 0$ and let $\lambda_1 \leq \cdots \leq \lambda_n$ denote the eigenvalues of $A+B$, let $x_1, \ldots, x_n$ be a corresponding set of eigenvectors and suppose further that $\ker(A+B) \neq \emptyset$. Then, $\lambda_1 \leq \cdots \leq \lambda_k < 0$, $\lambda_{k+1} = 0$, and $\langle Ax_{k+1}, x_{k+1} \rangle \geq 0$. However, if $\langle Ax_{k+1}, x_{k+1} \rangle > 0$, then

$$\langle Bx_{k+1}, x_{k+1} \rangle = -\langle Ax_{k+1}, x_{k+1} \rangle < 0.$$  

Therefore, if $X \in \mathbb{C}^{n \times k}$ denotes the matrix the matrix with columns $x_1, \ldots, x_{k+1}$, then

$$k + 1 = \nu_-(X^*BX) \leq \nu_-(B) = k,$$

which is impossible. Therefore $Ax_{k+1} = 0$, as claimed. □
Lemma 3.19. Let $(B1)$–$(B4)$ be in force, let $\nu_-(P) = \kappa$ and let $s \in S_\kappa(A_1, A_2, C, P)$. Then
\[
\Delta_s Fv = 0 \quad \text{a.e. on } \Omega_0 \text{ for every } v \in \ker P.
\]

Proof. Let $s \in S_\kappa(A_1, A_2, C, P)$ and let $P_s$ be defined by $(\ref{3.1})$. Then, since $P_s = P$ by Corollary $3.4$, the result follows from Lemma $3.18$ upon letting $u_1, \ldots, u_n$ be any orthonormal basis of $C^n$ and setting $A$ and $B$ be the matrices with entries
\[
a_{ij} = \langle \Delta_s F X u_j, F X u_i \rangle_{nst} \quad \text{and} \quad b_{ij} = \left\langle \left\{ \Lambda_s \left[ 0 \quad \Gamma_{\ell} \right] \right\} F X u_j, F X u_i \right\rangle_{nst},
\]
respectively. \hfill \Box

Lemma 3.20. Let $(B1)$–$(B4)$ be in force, and let $\nu_-(P) = \kappa$. Then $s \in S_\kappa(A_1, A_2, C, P)$ if and only if $s \in \tilde{S}_\kappa(A_1, A_2, C, P, X)$ and \((\ref{3.70})\) holds.

Proof. If $s \in S_\kappa(A_1, A_2, C, P)$, then clearly, $s \in \tilde{S}_\kappa(A_1, A_2, C, P, X)$ and \((\ref{3.70})\) is implied by Lemma $3.19$.

Conversely, if $s \in \tilde{S}_\kappa(A_1, A_2, C, P, X)$ and \((\ref{3.70})\) holds, then, in view of Lemma $3.11$, (C1) and (C2) are satisfied. Moreover, it follows from \((\ref{3.70})\) and formula \((\ref{3.1})\) that
\[
u^* P_s v = 0 \quad \text{for every } v \in \ker P \text{ and } u \in C^n,
\]
i.e., $\ker P \subseteq \ker P_s$. Thus, if $u = u_1 + X u_2$ with $u_1 \in \ker P$ and $u_2 \in C^n$, then
\[
u^* P_s u = (u_1 + X u_2)^* P_s (u_1 + X u_2) = u_2^* X P_s X u_2 \
\leq u_2^* X P X u_2 = (u_1 + X u_2)^* P (u_1 + X u_2) = u^* P u.
\]
This proves (C3) and hence, in view of Corollary $3.4$ that $P_s = P$. Therefore, $s \in S_\kappa(A_1, A_2, C, P)$. \hfill \Box

Lemma 3.21. If $(B1)$–$(B4)$ are in force, $W$ is defined by $(\ref{1.14})$, $\varepsilon \in S_{p \times q}$ and $s = T_W[\varepsilon]$, then condition \((\ref{3.70})\) holds if and only if
\[
(I_p - \varepsilon(\lambda)) F(\mu)v = 0
\]
for every $\lambda \in \mathcal{S}^+_\kappa$ and every $v \in \ker P$, where $\mu \in \Omega_0$ is the point chosen to normalize $W$ in $(\ref{1.14})$ (i.e., $W(\mu) = I_m$).

Proof. Let $s = T_W[\varepsilon]$ for $\varepsilon \in S_{p \times q}$ and
\[
R(\lambda) = (w_{11}^\#(\lambda) + \varepsilon(\lambda) w_{12}^\#(\lambda))^{-1} \quad \text{for } \lambda \in \Omega_W \cap \mathcal{H}_s^+.
\]
Then it follows from \((\ref{2.29})\) and \((\ref{3.72})\) that
\[
[I_p - s(\lambda)] = (w_{11}^\#(\lambda) + \varepsilon(\lambda) w_{12}^\#(\lambda))^{-1} [I_p - \varepsilon(\lambda)] j_{pq} W^\#(\lambda) j_{pq}
\]
for $\lambda \in \Omega_W \cap \mathcal{S}_{\kappa}^+ \cap \mathcal{S}_{\kappa}^+ \subset \Omega_+$. Using the formula \((\ref{3.54})\) one obtains
\[
[I_p - s(\lambda)] F(\lambda)v = R(\lambda) [I_p - \varepsilon(\lambda)] j_{pq} W^\#(\lambda) j_{pq} F(\lambda)v = R(\lambda) [I_p - \varepsilon(\lambda)] F(\mu)(I_n - XP)(M - \mu N)(M - \lambda N)^{-1} v.
\]
Since
\[
(I_n - XP)(M - \mu N)(M - \lambda N)^{-1} X = (I_n - XP) X (M_0 - \mu N_0)(M_0 - \lambda N_0)^{-1} = 0,
\]
it follows that $[I_p - s(\lambda)] F(\lambda)v \equiv 0$ for all $v \in \ker P$ if and only
\[
[I_p - \varepsilon(\lambda)] F(\mu)(I_n - XP) \equiv 0,
\]
which is equivalent to \((\ref{3.71})\). \hfill \Box
Lemma 3.22. If (B1) and (B2) are in force and \( \mu \in \Omega_0 \), then the subspace \( F(\mu)\ker P \) is a \( \mu \)-neutral subspace of \( \mathbb{C}^n \).

Proof. If \( v, w \in \ker P \), then the Lyapunov-Stein equation \((1.4)\) implies that
\[
w^*F(\mu)^*j_{pq}F(\mu)v = w^*(M^* - \overline{\mu}N^*)^{-1}\{M^*P(M - \mu N) + (\mu M^* - N^*)PN\}(M - \mu N)^{-1}v = w^*\{(M^* - \overline{\mu}N^*)^{-1}M^*P + \mu PN(M - \mu N)^{-1}\}v = 0.
\]
This proves the statement. \( \square \)

Theorem 3.23. If (B1)--(B4) are in force and \( \nu_-(P) = \kappa \), then there are unitary matrices \( U \in \mathbb{C}^{p \times p}, V \in \mathbb{C}^{q \times q} \) such that
\[
S_\nu(A_1, A_2, C, P) = \left\{ T_W \left[ \begin{array}{cc} \tilde{\varepsilon} & 0 \\ 0 & I_\nu \end{array} \right] V^* : \tilde{\varepsilon} \in S^{(p-\nu) \times (q-\nu)} \} \cap S_\nu^{p \times q},
\]
where \( W \) and \( \nu \) are given by \((1.14)\) and \((1.22)\), respectively.

Proof. Let \( s \in S_\nu(A_1, A_2, C, P) \). Then, in view of Lemma 3.20, \( s \) is a solution of (X1)–(X3) such that \( (3.70) \) holds. By Theorem 3.17 \( s \) admits the representation \( s = T_W[\varepsilon] \), where \( W \) is given by \((1.14)\), \( \varepsilon \in S_\nu^{p \times q} \) and, by Lemma 3.21
\[
(3.76) \quad [I_\nu - \varepsilon(\lambda)] F(\mu)u = 0 \quad \text{for} \quad u \in \ker P \text{ and } \mu \in \Omega_0.
\]
Thus, if \( \text{col}(x,y) \) is a nonzero vector in the subspace \( \mathcal{K}_\mu = F(\mu)\ker P \) for some choice of \( x \in \mathbb{C}^p, y \in \mathbb{C}^q \) and \( \mu \in \Omega_0 \), then, since \( \mathcal{K}_\mu \) is \( \mu \)-neutral,
\[
x = \varepsilon(\lambda)y \quad \text{and} \quad x^*x = y^*y.
\]
Therefore, (see [26, Lemma 0.13]) \( \varepsilon(\lambda) \) admits the representation
\[
\varepsilon(\lambda) = U \left[ \begin{array}{cc} \tilde{\varepsilon}(\lambda) & 0 \\ 0 & I_\nu \end{array} \right] V^*,
\]
where \( U \in \mathbb{C}^{p \times p} \) and \( V \in \mathbb{C}^{q \times q} \) are unitary matrices, \( \nu = \dim \mathcal{K}_\mu \) and \( \tilde{\varepsilon} \in S^{(p-\nu) \times (q-\nu)} \).

Conversely, if \( \varepsilon \) is of the form \((1.23)\), then \((3.76)\) holds; and \( s \in S_\nu(A_1, A_2, C, P) \) by Lemma 3.22

Next, to verify \((1.22)\), we first observe that if \( \mu \in \Omega_0 \), then \( M - \mu N \) defines an invertible map of
\[
(3.77) \quad \mathcal{L} = \ker PM \cap \ker PN \cap \ker C \quad \text{onto} \quad \mathcal{L}_\mu = \ker F(\mu) \cap \ker P.
\]

Let \( u \in \mathbb{C}^n \) and \( v = (M - \mu N)^{-1}u \). Then it is readily checked that \( v \in \mathcal{L} \implies u \in \mathcal{L}_\mu \), i.e., \( (M - \mu N)\mathcal{L} \subseteq \mathcal{L}_\mu \). Conversely, if \( u \in \mathcal{L}_\mu \), then clearly \( P(M - \mu N)v = 0 \) and \( CV = 0 \). Therefore, the Lyapunov-Stein equation \((1.4)\) implies that
\[
0 = C^*j_{pq}Cv = M^*PMv - N^*PNv = M^*P(M - \mu N)v + (\mu M^* - N^*)PNv
\]
and hence that \( PNv = 0 \). Thus, \( (M - \mu N)\mathcal{L} = \mathcal{L}_\mu \) and
\[
(3.78) \quad \dim \mathcal{L}_\mu = \dim \mathcal{L} = \dim \ker (M^*P^2M + N^*P^2N + C^*C).
\]

Therefore,
\[
\nu = \dim \mathcal{K}_\mu = \dim \ker P - \dim \mathcal{L}_\mu = \dim \ker P - \dim \ker (M^*P^2M + N^*P^2N + C^*C),
\]
which is equivalent to formula \((1.22)\). \( \square \)
4. Resolvent matrices

4.1. Pole and zero multiplicities. Let $G(\lambda)$ be a $p \times q$ mvf that is meromorphic on $\Omega_+$ with a Laurent expansion

$$G(\lambda) = (\lambda - \lambda_0)^{-k}G_{-k} + \cdots + (\lambda - \lambda_0)^{-1}G_{-1} + G_0 + \cdots$$

in a neighborhood of a pole $\lambda_0 \in \Omega_+$. The pole multiplicity $M_\pi(G, \lambda_0)$ is defined by (see [38])

$$M_\pi(G, \lambda_0) = \text{rank } T(G, \lambda_0), \quad \text{where } T(G, \lambda_0) = \begin{bmatrix} G_{-k} & 0 \\ \vdots & \ddots \\ G_{-1} & \cdots & G_{-k} \end{bmatrix}.$$

The pole multiplicity of $G$ over $\Omega_+$ is given by

$$M_\pi(G, \Omega_+) = \sum_{\lambda \in \Omega_+} M_\pi(G, \lambda).$$

The zero multiplicity of a square mvf $G$ over $\Omega_+$ is defined by

$$M_\zeta(G, \Omega_+) = M_\pi(G^{-1}, \Omega_+) \quad \text{if } \det G(\lambda) \neq 0.$$

Note that definitions (4.3), (4.4) of pole and zero multiplicities coincide with those based on Smith-McMillan representations of $G$ (see [14]); and the degree of a Blaschke-Potapov product $b$ of the form (2.1) coincides with $M_\zeta(b, \Omega_+)$.

Let $H_{\kappa, \infty}^{p \times q}(\Omega_+)$ denote the class of $p \times q$ mvf’s $G$ of the form $G = F + B$, where $B$ is a rational $p \times q$ mvf of pole multiplicity $M_\pi(B, \Omega_+) \leq \kappa$ and $F \in H_{\kappa, \infty}^{p \times q}(\Omega_+)$.

It follows from the results of [38], [22] (see also [36, Theorem 5.2]) that every mvf $G \in H_{\kappa, \infty}^{p \times q}(\Omega_+)$ admits coprime factorizations

$$G(\lambda) = b_\ell(\lambda)^{-1}\varphi_\ell(\lambda) = \varphi_r(\lambda)b_r(\lambda)^{-1},$$

where $b_\ell \in S_m^{p \times p}$, $b_r \in S_m^{q \times q}$ are Blaschke-Potapov factors of degree $M_\pi(G, \Omega_+)$ and $\varphi_\ell, \varphi_r \in H_{\kappa, \infty}^{p \times q}$.

Left and right coprime factorizations may be characterized in terms of the pole and zero multiplicities of its factors:

**Proposition 4.1.** ([23, Proposition 3.4] and [14] in the rational case) Let $H_\ell, H_r \in H_{\infty}^{p \times q}$, $G_\ell, G_r \in H_{\infty}^{p \times q}$ be a pair of mvf’s such that $G_\ell^{-1}, G_r^{-1} \in H_{\kappa, \infty}^{p \times p}$ for some $\kappa \in \mathbb{N} \cup \{0\}$. Then:

(i) The pair $G_\ell$, $H_\ell$ is left coprime $\iff M_\pi(G_\ell^{-1}H_\ell, \Omega_+) = M_\pi(G_\ell^{-1}, \Omega_+)$.  
(ii) The pair $G_r$, $H_r$ is right coprime $\iff M_\pi(H_rG_r^{-1}, \Omega_+) = M_\pi(G_r^{-1}, \Omega_+)$.

4.2. Associated pairs. Let $b_1 \in S_m^{p \times p}$ and $b_2 \in S_m^{q \times q}$ be the Blaschke-Potapov factors that are uniquely defined (up to right and left constant unitary factors, respectively) by the coprime factorizations

$$C_{12}(I_{n_2} - \lambda A_2)^{-1} \left[ \begin{array}{cc} X_{21} & X_{22} \end{array} \right] = b_1(\lambda)\varphi_1(\lambda), \quad \lambda \in \Omega_- \setminus \sigma(A_2)^{-1}, \quad \varphi_1 \in \mathcal{R} \cap H_{\infty}^{p \times n}(\Omega_-),$$

and

$$C_{21}(A_1 - \lambda I_{n_1})^{-1} \left[ \begin{array}{cc} X_{11} & X_{12} \end{array} \right] = b_2(\lambda)^{-1}\varphi_2(\lambda), \quad \lambda \in \Omega_+ \setminus \sigma(A_1), \quad \varphi_2 \in \mathcal{R} \cap H_{\infty}^{q \times n}(\Omega_+).$$

**Remark 4.2.** Since $(C_{21}, A_1)$ and $(C_{12}, A_2)$ are observable pairs and $\sigma(A_1) \subset \mathbb{D}$ and $\sigma(A_2) \subset \mathbb{D}$, Theorem 2.6 can be exploited to obtain Blaschke-Potapov products $\tilde{b}_1(\lambda)$ and $\tilde{b}_2(\lambda)$ such that

$$\tilde{b}_1(\lambda)^{-1}C_{12}(I_{n_2} - \lambda A_2)^{-1} \in H_{\infty}^{p \times n_2}(\Omega_-) \quad \text{and} \quad \tilde{b}_2(\lambda)C_{21}(A_1 - \lambda I_{n_1})^{-1} \in H_{\infty}^{q \times n_1}(\Omega_+).$$
In particular, the Stein equations \( Q_1 - A^*Q_1A_1 = C^*_{21}C_{21} \) and \( Q_2 - A^*Q_2A_2 = -C^*_{12}C_{12} \) have unique solutions

\[
Q_1 = \sum_{j=0}^{\infty} (A_1^*)^j C^*_{21}C_{21}A_1^j = \frac{1}{2\pi} \int_0^{2\pi} (A_1^* - e^{-i\theta}I_{n_1})^{-1}C^*_{21}C_{21}(A_1 - e^{i\theta}I_{n_1})^{-1}d\theta
\]

and

\[
Q_2 = -\sum_{j=0}^{\infty} (A_2^*)^j C^*_{12}C_{12}A_2^j = -\frac{1}{2\pi} \int_0^{2\pi} (I_{n_2} - e^{-i\theta}A_2^*)^{-1}C^*_{12}C_{12}(I_{n_2} - e^{i\theta}A_2)^{-1}d\theta,
\]

respectively. Moreover, \( Q_1 \) is positive definite and \( Q_2 \) is negative definite. Therefore, Theorem 2.6 (with \( C = C_{21}, M = A_1, N = I_{n_1} \) and \( J = -I_q \)) implies that

\[
\tilde{b}_2(\lambda) = I_q + (\lambda - \mu)C_{21}(A_1 - \mu I_{n_1})^{-1}Q_1^{-1}(\lambda A_1^* - I_{n_1})^{-1}C^*_{21};
\]

see also Theorem 5.4 in [28] for additional discussion, if need be. Similarly,

\[
\tilde{b}_1(\lambda) = I_p + (1 - \pi\lambda)C_{12}(I_{n_2} - \lambda A_2)^{-1}Q_2^{-1}(I_{n_2} - \pi A_2)^{-1}C^*_{12}.
\]

Furthermore, if \( F_{12}(\lambda) = C_{12}(I_{n_2} - \lambda A_2)^{-1} \),

\[
\tilde{\mathcal{M}}_2 = \{ F_{12}u : u \in \mathbb{C}^{n_2} \} \quad \text{and} \quad \mathcal{M}_2 = \{ F_{12}[X_{21} X_{22}]u : u \in \mathbb{C}^n \},
\]

then \( \mathcal{M}_2 \) is a subspace of \( \tilde{\mathcal{M}}_2 \) that is invariant under the backwards shift operator

\[
R_\alpha : f \to \frac{f(\lambda) - f(\alpha)}{\lambda - \alpha} \quad \text{for} \quad \alpha \in \mathbb{D}.
\]

In particular,

\[
R_0F_{12}(\lambda)[X_{21} X_{22}] = F_{12}(\lambda)A_2[X_{21} X_{22}] = F_{12}(\lambda)[0 I_{n_2}]NX
\]

\[
= F_{12}(\lambda)[0 I_{n_2}]XPNX = F_{12}(\lambda)[X_{21} X_{22}]PNX.
\]

Thus, upon endowing these two spaces with the normalized standard inner product, it follows from the either the arguments cited earlier in this remark or the Beurling-Lax theorem that

\[
\mathcal{M}_2 = \mathcal{H}(b_1) \subseteq \mathcal{H}(\tilde{b}_1) = \tilde{\mathcal{M}}_2.
\]

Similar considerations imply that if \( F_{21}(\lambda) = C_{21}(A_1 - \lambda I_{n_1})^{-1} \),

\[
\tilde{\mathcal{M}}_1 = \{ F_{21}u : u \in \mathbb{C}^{n_1} \} \quad \text{and} \quad \mathcal{M}_1 = \{ F_{21}[X_{11} X_{12}]u : u \in \mathbb{C}^n \},
\]

then

\[
\mathcal{M}_1 = \mathcal{H}_s(b_2) \subseteq \mathcal{H}_s(\tilde{b}_2) = \tilde{\mathcal{M}}_1.
\]

These conclusions also follow from Lemmas 4.3 and 4.7 below, which are based on state space calculations. Moreover,

\[
b_1(\lambda) = \tilde{b}_1(\lambda) \quad \text{if} \quad \text{rank} \ [X_{21} X_{22}] = n_2 \quad \text{and} \quad b_2(\lambda) = \tilde{b}_2(\lambda) \quad \text{if} \quad \text{rank} \ [X_{11} X_{12}] = n_1.
\]

**Lemma 4.3.** Let the data set \((A_1, A_2, C, P)\) satisfy the assumptions (B1), (B2) and (B4). Then:

\[
b_2^{-1}H_2^p = \{ C_{21}(A_1 - \lambda I_{n_1})^{-1} [X_{11} X_{12}] u + h_+ : u \in \mathbb{C}^n, h_+ \in H_2^p \},
\]

(4.10)

\[
b_1(H_2^p)\perp = \{ C_{12}(I_{n_2} - \lambda A_2)^{-1} [X_{21} X_{22}] u + h_- : u \in \mathbb{C}^n, h_- \in (H_2^p)^\perp \}.
\]

(4.11)
Proof. The first step in the verification of (4.10) is to observe that if \( g \in H_2^{n_1} \) and \( v \in \mathbb{C}^{n_1} \), then
\[
(A_1 - \lambda I_{n_1})^{-1}(g(\lambda) - v) \in H_2^{n_1} \iff v = \frac{1}{2\pi i} \int_T (\zeta I_{n_1} - A_1)^{-1}g(\zeta) d\zeta.
\]

Next, recall that since \( b_2 \) and \( \varphi_2 \) are left coprime, there exist a pair of mvf's \( c \in H^{q \times q}_\infty \) and \( d \in H^{n \times q}_\infty \) such that \( b_2 c + \varphi_2 d = I_q \) in \( \Omega_+ \). Let \( f \in H_2^n \) and let
\[
g = [X_{11} \ X_{12}] df \quad \text{and} \quad v = \frac{1}{2\pi i} \int_T (\zeta I_{n_1} - A_1)^{-1}g(\zeta) d\zeta.
\]
Then
\[
b_2^{-1} f = b_2^{-1} (b_2 c + \varphi_2 d) f = cf + C_{21} (A_1 - \lambda I_{n_1})^{-1}(g - v) + C_{21} (A_1 - \lambda I_{n_1})^{-1}v,
\]
and since \( g \in H_2^{n_1} \), the first two terms on the far right belong to \( H_2^n \). Moreover, since
\[
(A_1 - \zeta I_{n_1})^{-1}[X_{11} \ X_{12}] = [I_{n_1} \ 0](M - \zeta N)^{-1}X = [I_{n_1} \ 0]XP(M - \zeta N)^{-1}X
\]
\[
= [X_{11} \ X_{12}]P(M - \zeta N)^{-1}X,
\]
it follows that \( v = [X_{11} \ X_{12}]w \) for some \( w \in \mathbb{C}^n \) and hence that
\[
C_{21} (A_1 - \lambda I_{n_1})^{-1}v = b_2^{-1} \varphi_2 w = C_{21} (A_1 - \lambda I_{n_1})^{-1}[X_{11} \ X_{12}]w.
\]
Thus, the left-hand side of (4.10) is a subset of the right-hand side. The opposite inclusion is easy to check and is left to the reader, as is the verification of (4.11). \( \square \)

Lemma 4.4. Let the assumptions of Lemma 4.3 be in force and let \( \tilde{b}_1 \) and \( \tilde{b}_2 \) be defined by (4.9) and (4.8), respectively.

(1) If the pair \( (C_{12}, A_2) \) is observable, then
\[
\{P_-(\lambda I_{n_2} - A_2^*)^{-1}h : h \in H_2^{n_2}\} = \{P_- (\lambda I_{n_2} - A_2^*)^{-1}C_{12}^* g : g \in \mathcal{H}(\tilde{b}_1)\}.
\]

(2) If the pair \( (C_{21}, A_1) \) is observable, then
\[
\{P_+(A_1^* - \lambda I_{n_1})^{-1}h : h \in (H_2^{n_1})^+\} = \{P_+(A_1^* - \lambda I_{n_1})^{-1}C_{21}^* g : g \in \mathcal{H}_+(\tilde{b}_2)\}.
\]

Proof. If \( h \in H_2^{n_2} \) and
\[
x = \frac{1}{2\pi i} \int_T (\zeta I_{n_2} - A_2^*)^{-1}h(\zeta) d\zeta, \quad \text{then} \quad P_- (\lambda I_{n_2} - A_2^*)^{-1}\{h(\lambda) - x\} = 0.
\]
Therefore,
\[
\{P_-(\lambda I_{n_2} - A_2^*)^{-1}h : h \in H_2^{n_2}\} \subseteq \{(\lambda I_{n_2} - A_2^*)^{-1}x : x \in \mathbb{C}^{n_2}\}.
\]
On the other hand, if \( g = C_{12}(I_{n_2} - \lambda A_2)^{-1}u \) for some \( u \in \mathbb{C}^{n_2} \), then
\[
P_- (\lambda I_{n_2} - A_2^*)^{-1}C_{12}^* g = (\lambda I_{n_2} - A_2^*)^{-1}v,
\]
where
\[
v = \frac{1}{2\pi i} \int_T (\zeta I_{n_2} - A_2^*)^{-1}C_{12}^* C_{12}(I_{n_2} - \zeta A_2)^{-1} d\zeta u = \frac{1}{2\pi} \int_0^{2\pi} (I_{n_2} - e^{-i\theta} A_2)^{-1}C_{12}^* C_{12}(I_{n_2} - e^{i\theta} A_2)^{-1} d\theta u = Q_2 u,
\]
where \( Q_2 \) is the negative definite matrix introduced in Remark 4.2. Thus, as \( Q_2 \) is invertible, and \( g \in \mathcal{H}(\tilde{b}_1) \)
\[
\{(\lambda I_{n_2} - A_2^*)^{-1}x : x \in \mathbb{C}^{n_2}\} \subseteq \{P_- (\lambda I_{n_2} - A_2^*)^{-1}C_{12}^* g : g \in \mathcal{H}(\tilde{b}_1)\}.
\]
This completes the proof of (1). The proof of (2) is similar. \( \square \)
Lemma 4.5. Let the data set \((A_1, A_2, C, P)\) satisfy the assumptions (B1)–(B4) and let \(W\) and \(F_1\) be given by \(1.14\) and \(3.22\), respectively. Then:

(i) \(P_W H^m_2 = \{F_1 [X_{11}, X_{12}] u : u \in \mathbb{C}^n\}\).

(ii) \(W H^m_2 = \{F_1 [X_{11}, X_{12}] u + h : u \in \mathbb{C}^n, h \in H^m_2 \& P_W^{-1} (F_1 [X_{11}, X_{12}] u + h) = 0\}\).

(iii) \(f \in H^m_2, \quad 0 \quad I_q W f \in H^m_2 \implies W f \in H^m_2\).

(iv) \(0 \quad I_q W (\mathcal{R} \cap H^m_2) = b_2^{-1} (\mathcal{R} \cap H^m_2)\).

Proof. (i) Since

\[
(4.17) \quad \tilde{C}^* = (\pi M - N) X F(\mu)^* = X P(\pi M - N) X F(\mu)^*,
\]

it is readily checked with the help of formulas \(3.57\) and \(4.12\) that

\[
P_W H^m_2 \subseteq P_{-1} [X_{11}, X_{12}] H^m_2 \subseteq \{C_1 (A_1 - \lambda I_{n_1})^{-1} [X_{11}, X_{12}] u : u \in \mathbb{C}^n\}.
\]

More precisely, if \(h \in H^m_2\), then, in view of \(3.57\) and \(4.12\),

\[
P_W h = P_{-1} [X_{11}, X_{12}] \tilde{C}^*_j p h = C_1 (A_1 - \lambda I_{n_1})^{-1} x,
\]

where

\[
x = \frac{1}{2\pi i} \int_T (\zeta I_{n_1} - A_1)^{-1} \tilde{C}^*_j p h(\zeta) d\zeta.
\]

Thus, if \(h = p h \tilde{C}_1 (I_{n_1} - \zeta A_1^*)^{-1} u\), then

\[
x = Q u, \quad \text{where} \quad Q = -\frac{1}{2\pi i} \int_T (\zeta I_{n_1} - A_1)^{-1} \tilde{C}^*_j \tilde{C}_1 (I_{n_1} - \zeta A_1^*)^{-1} d\zeta.
\]

But this serves to complete the proof of (i), since \(\text{rng} Q = \text{rng} [X_{11}, X_{22}]\) by Corollary \(3.16\).

(ii) For every \(f \in H^m_2\) there is a vvf \(h \in H^m_2\) such that

\[
W f = P_{-1} (W f) + h.
\]

Thus, in view of (i), there is a vector \(u \in \mathbb{C}^n\) such that

\[
W f = C_1 (A_1 - \lambda I_{n_1})^{-1} [X_{11}, X_{12}] u + h = g \quad \text{and} \quad P_W^{-1} g = 0.
\]

(iii) Since the pair \((C_2, A_1)\) is observable the third statement is immediate from (i) and (ii).

(iv) The inclusion \(0 \quad I_q W H^m_2 \subseteq b_2^{-1} H^m_2\) follows readily from (ii) and formula \(4.17\). To verify the opposite inclusion, let \(f \in H^m_2\). Then, by Lemma \(4.3\)

\[
(4.18) \quad b_2^{-1} f = C_{21} (A_1 - \lambda I_{n_1})^{-1} [X_{11}, X_{12}] u + h_2
\]

for some choice of \(u \in \mathbb{C}^n\) and \(h_2 \in H^m_2\). In view of Lemma \(3.8\), we may assume that

\[
[X_{21}, X_{22}] u = 0 \quad \text{and hence, upon writing} \quad u = \text{col}(u_1, u_2) \quad \text{with} \quad u_j \in \mathbb{C}^{n_j} \quad \text{for} \quad j = 1, 2,
\]

\[
(4.19) \quad P_W^{-1} \left( F_1 [X_{11}, X_{12}] u + \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \right) = P_{-1} \tilde{C}_2 (\lambda I_{n_2} - A_2^*)^{-1} \{u_2 + (C_{12}^* h_1 - C_{22}^* h_2)\}
\]

for every choice of \(h_1 \in H^m_2\), since

\[
P_W^{-1} C_1 (A_1 - \lambda I_{n_1})^{-1} [X_{11}, X_{12}] u = P_{-1} \tilde{C}_2 (\lambda I_{n_2} - A_2^*)^{-1} u_2,
\]

by formula \(3.53\) and

\[
P_W^{-1} \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = P_{-1} \tilde{C}_2 (A_2^* - \lambda I_{n_2})^{-1} (C_{12}^* h_1 - C_{22}^* h_2).
\]
by formula (4.21). Now, upon choosing \( h_1 \in \mathcal{H}(\tilde{b}_1) \) so that the right hand side of (4.19) is equal to zero, it follows from (ii) that

\[
F_1[X_{11} \ X_{12}]u + \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \in WH_2^m
\]

and hence that

\[
b_2^{-1} f \in [0 \ I_q]WH_2^m,
\]
as needed. \( \square \)

**Corollary 4.6.** If \( W \) is defined by (1.14) and \( b_2 \) is defined by (1.7), then there exist a pair of rational mvf’s \( g_1 \in H_\infty^{p,q} \) and \( g_2 \in H_\infty^{q,p} \) such that

\[
w_{21}g_1 + w_{22}g_2 = b_2^{-1}.
\]

Moreover, if \((g_1,g_2)\) is any pair of functions in \( H_\infty^{p,q} \times H_\infty^{q,p} \) that satisfies equation (4.21), then the mvf

\[
K = (w_{11}g_1 + w_{12}g_2)(w_{21}g_1 + w_{22}g_2)^{-1} = (w_{11}g_1 + w_{12}g_2)b_2
\]
is holomorphic in \( \Omega_+ \).

**Proof.** By Lemma 4.5 (v) for each \( f \in \mathcal{R} \cap H_2^q \) there exists \( f_1 \in \mathcal{R} \cap H_2^p \) and \( f_2 \in \mathcal{R} \cap H_2^q \) such that

\[
w_{21}f_1 + w_{22}f_2 = b_2^{-1}f
\]
which leads easily to (4.21) by successively choosing columns of \( I_q \).

The second assertion is implied by (iii) of Lemma 4.5. \( \square \)

There is an analogue of Lemma 4.5 that focuses on \((H_2^m)^\perp\) that we state without proof.

**Lemma 4.7.** Let the data set \((A_1, A_2, C, P)\) satisfy the assumptions (B1)–(B4) and let \( W \) and \( F_2 \) be given by (1.14) and (3.22), respectively. Then:

(i) \( P_+W(H_2^p)^\perp = \{F_2(\lambda) \begin{bmatrix} X_{21} & X_{22} \end{bmatrix} u : u \in \mathbb{C}^n \} \).

(ii) \( W(H_2^m)^\perp = \{F_2[X_{21} \ X_{22}]u + h : u \in \mathbb{C}^n, h \in (H_2^m)^\perp, P_+W^{-1}(F_2[X_{21} \ X_{22}]u + h) = 0 \} \).

(iii) If \( f \in (H_2^m)^\perp \) and \([I_p \ 0]Wf \in (H_2^p)^\perp \), then \( Wf \in (H_2^m)^\perp \).

(iv) \([I_p \ 0]W(H_2^m)^\perp = b_1(H_2^p)^\perp \).

**Lemma 4.8.** Let \( W \in \mathcal{U}_\kappa(j_{pq}) \) and let the conditions (iii) and (iv) of Lemma 4.5 be in force. Then \( s_{21} \in S_{\kappa}^{p,q} \) and hence \( W \in \mathcal{U}_\kappa(j_{pq}) \).

**Proof.** Our first objective is to show that

\[
\dim P_- \begin{bmatrix} s_{11} \\ s_{21} \end{bmatrix} H_2^p = \dim P_- s_{21}H_2^p.
\]

To prove the inequality \( \leq \) assume that \( P_- s_{21}g_1 = 0 \) for some \( g_1 \in H_2^p \). Then \( g_2 = s_{21}g_1 \in H_2^q \) and, hence,

\[
\begin{bmatrix} s_{11}g_1 \\ 0 \end{bmatrix} \begin{bmatrix} w_{11}g_1 + w_{12}g_2 \\ w_{21}g_1 + w_{22}g_2 \end{bmatrix} = W \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \in WH_2^m.
\]

In view of Lemma 4.5 (iii), \( W \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \in H_2^m \). Therefore \( s_{11}g_1 \in H_2^p \) and hence the asserted inequality is justified. This completes the proof of (4.23), since the opposite inequality is self-evident.
Next, to prove that

\[(4.25) \quad P_- \begin{bmatrix} s_{11} \\ s_{21} \end{bmatrix} H_2^p \subseteq P_- \begin{bmatrix} s_{11} \\ s_{21} \end{bmatrix} H_2^p, \]

let \( h \) be an arbitrary vvf from \( H_2^q \). Then, by Lemma \[4.5\] (iv), there exists a pair of vvf’s \( g_1 \in H_2^q \) and \( g_2 \in H_2^q \) such that

\[(4.26) \quad h = w_{21}g_1 + w_{22}g_2. \]

Then

\[
\begin{bmatrix} w_{11}g_1 + w_{12}g_2 \\ w_{21}g_1 + w_{22}g_2 \end{bmatrix} = \begin{bmatrix} w_{11}g_1 + w_{12}g_2 \\ h \end{bmatrix} \in H_2^m
\]
due to Lemma \[4.5\] (iii). Therefore, \( g_2 = -w_{22}^{-1}w_{21} + w_{22}^{-1}h \), and hence

\[
\begin{bmatrix} s_{11}g_1 \\ s_{21}g_1 \end{bmatrix} + \begin{bmatrix} s_{12}h \\ s_{22}h \end{bmatrix} = \begin{bmatrix} w_{11}g_1 + w_{12}g_2 \\ g_2 \end{bmatrix} \in H_2^m.
\]

Thus, for any \( h \in H_2^q \) there exists a vvf \( g_1 \in H_2^p \) such that

\[
P_- \begin{bmatrix} s_{11}g_1 \\ s_{21}g_1 \end{bmatrix} = -P_- \begin{bmatrix} s_{12}h \\ s_{22}h \end{bmatrix},
\]

which justifies \(4.25\).

Formulas \(4.23\) and \(4.25\) imply

\[
\dim P_- s_{21} H_2^p = \dim P_- S H_2^m = \kappa.
\]

and, hence, since \( s_{21} \) is contractive on \( \Omega_0 \) than \( s_{21} \in S_\kappa^{x\mu} \).

The last statement now follows from the definition of the class \( \mathcal{U}_\kappa(j_{pq}) \) (which is given in the Introduction).

\[ \square \]

**Remark 4.9.** If the data set \((A_1, A_2, C, P)\) satisfies assumptions (B1)–(B4) and the mvf \( W \) given by \(4.14\) belongs to \( \mathcal{U}_{\kappa_1}(j_{pq}) \), then conditions (iii) and (iv) of Lemma \[4.5\] are in force and, hence, \( s_{21} = -w_{22}^{-1}w_{21} \in S_\kappa^{x\mu} \). This fact was proved in \[14\] for invertible \( P \) when (B1)-(B3) hold.

**Theorem 4.10.** Let the data set \((A_1, A_2, C, P)\) satisfy assumptions (B1)–(B4), let \( W \) be given by \(4.14\), and let the pair \( \{b_1, b_2\} \) and the mvf \( K \in H_\infty^{p\times q} \) be defined by \(4.16\), \(4.17\) and \(4.22\). Then:

1. The mvf \( W \) admits the factorization

\[(4.27) \quad W = \begin{bmatrix} b_1 & 0 \\ 0 & b_2^{-1} \end{bmatrix} \begin{bmatrix} \varphi_{11} & \varphi_{12} \\ \varphi_{21} & \varphi_{22} \end{bmatrix} \text{ a.e. in } \Omega_0 \]

where the pair \((\varphi_{11}, \varphi_{12}) \in \mathcal{R} \cap H_\infty^{p\times q}(\Omega_-) \times \mathcal{R} \cap H_\infty^{p\times q}(\Omega_-)\) is left coprime over \( \Omega_- \) and the pair \((\varphi_{21}, \varphi_{22}) \in \mathcal{R} \cap H_\infty^{p\times q}(\Omega_+) \times \mathcal{R} \cap H_\infty^{p\times q}(\Omega_+)\) is left coprime over \( \Omega_+ \).

2. The pair \( \{b_1, b_2\} \) is an associated pair for \( W \).

3. \( W \) admits the factorizations

\[(4.28) \quad W = \Theta \Phi \quad \text{and} \quad W = \widetilde{\Theta} \widetilde{\Phi}, \]

over \( \Omega_+ \) and \( \Omega_- \), respectively, with

\[
\Theta = \begin{bmatrix} b_1 & K^{-1}b_2 \\ 0 & b_2^{-1} \end{bmatrix}, \quad \widetilde{\Theta} = \begin{bmatrix} b_1 & 0 \\ K^{-1}b_1 & b_2^{-1} \end{bmatrix}
\]
Moreover, $\Phi$ is outer in $H_{\infty}^{m \times m}(\Omega_+)$, $\tilde{\Phi}$ is outer in $H_{\infty}^{m \times m}(\Omega_-)$ and
\begin{equation}
(4.31) \quad \tilde{\Theta} \# j_{pq} \Theta = j_{pq}.
\end{equation}

Proof. Formulas (3.37) and (4.17) yield the representation
\[
[w_{21}(\lambda) \quad w_{22}(\lambda)] = C_{21}(A_1 - \lambda I_{n_1})^{-1}[X_{11} \quad X_{12}]U + v(\lambda),
\]
for some choice of $U \in \mathbb{C}^{n \times m}$ and $v \in \mathcal{R} \cap H_{\infty}^{q \times m}(\Omega_+)$. Lemma 4.3 then guarantees the existence of a factorization of the form
\begin{equation}
(4.32) \quad [w_{21} \quad w_{22}] = b_2^{-1} [\varphi_{21} \quad \varphi_{22}]
\end{equation}
with $\varphi_{21} \in \mathcal{R} \cap H_{\infty}^{q \times p}(\Omega_+)$ and $\varphi_{22} \in \mathcal{R} \cap H_{\infty}^{q \times q}(\Omega_+)$. By Lemma 4.5 there are mvf’s $g_1 \in \mathcal{R} \cap H_{\infty}^{q \times q}$ and $g_2 \in \mathcal{R} \cap H_{\infty}^{q \times q}$ such that
\[
\varphi_{21}g_1 + \varphi_{22}g_2 = b_2(w_{21}g_1 + w_{22}g_2) = I_q.
\]
Thus, Lemma 2.1 implies the rank condition
\[
\text{rank } [\varphi_{21}(\lambda) \quad \varphi_{22}(\lambda)] = q \quad \text{for } \lambda \in \Omega_+.
\]
Therefore, the factorization $\varphi^{-1}_{22}\varphi_{21}$ is left coprime over $\Omega_+$ and, as
\[
\text{rank } [b_2(\lambda) \quad \varphi_{21}(\lambda) \quad \varphi_{22}(\lambda)] = q \quad \text{for } \lambda \in \Omega_+,
\]
the factorization (4.32) is also left coprime over $\Omega_+$.

Similarly, the coprimeness of the factorizations
\begin{equation}
(4.33) \quad \tilde{\varphi}_{11}^{-1} \tilde{\varphi}_{12} \quad \text{and} \quad [w_{11} \quad w_{12}] = b_1 [\tilde{\varphi}_{11} \quad \tilde{\varphi}_{12}] = (b_1^\#)^{-1} [\tilde{\varphi}_{11} \quad \tilde{\varphi}_{12}] \quad \text{over } \Omega_-
\end{equation}
follows from (3.37), (4.11) and Lemma 4.7. This completes the proof of (1).

Next, (2) follows by comparing the coprime factorizations (4.32) and (4.33) with those in [23 Corollary 4.15].

Finally, (3) follows from [23 Theorem 4.12].

\textbf{Remark 4.11.} If $W \in \mathcal{U}_{k_1}(j_{pq})$, $S = PG(W) = [s_{ij}]_{i,j=1}^2$ and the mvf’s $c_\ell \in H_{\infty}^{q \times q}$, $d_\ell \in H_{\infty}^{p \times q}$ are related to the factors in the Krein-Langer factorizations (1.17) of $s_{21}$ by the condition
\[
b_\ell c_\ell + s_\ell d_\ell = I_q,
\]
then (see e.g., [23]) $W$ admits the factorizations (4.28), (4.30), where the mvf’s $b_1$, $b_2$ are defined by (1.19) and (1.20), and
\begin{equation}
(4.34) \quad K = (-w_{11}d_\ell + w_{12}c_\ell)(-w_{21}d_\ell + w_{22}c_\ell)^{-1}.
\end{equation}
Moreover, if $\kappa_1 = 0$, then the second matrix on the right hand side of (4.27) is a (right) $\gamma$-generating matrix (in the sense of Arov); see Chapter 7 of [10] for the definition and references.

\section{5. Parametrization of solutions}

In this section we will give a parametrization of the set of solutions of the problem (C1)–(C3) in terms of the linear fractional transformation $T_W$. The main result of this section is based on Theorem 3.23 augmented by the factorization result of Theorem 4.10 and a special case of the Krein-Langer generalization of Rouche’s Theorem, which is formulated below.
5.1. Counting zeros. The analysis in this subsection is valid for \( \Omega_+ = \mathbb{D} \) and \( \Omega_+ = \Pi_+ \).

We will need the notation
\[
\tilde{L}^{p \times q}_r = \begin{cases} 
L^{p \times q}_r & \text{if } \Omega_+ = \mathbb{D}; \\
\{ f : (1 + |\mu|^2)^{-1/2} f \in L^{p \times q}_1 \} & \text{if } \Omega_+ = \Pi_+.
\end{cases}
\]

**Theorem 5.1.** [40] Let \( \varphi, \psi \in H^q_{\kappa^*} \), \( \det(\varphi + \psi) \neq 0 \) in \( \Omega_+ \), \( M_{\zeta}(\varphi, \Omega_+) < \infty \), and
\[
\|\varphi(t)^{-1}\psi(t)\| \leq 1 \quad \text{a.e. on } \Omega_+.
\]

Then \( M_{\zeta}(\varphi + \psi, \Omega_+) \leq M_{\zeta}(\varphi, \Omega_+) \) with equality if
\[
(\varphi + \psi)^{-1}\varphi|_{\Omega_0} \in \tilde{L}^{q \times q}_1.
\]

This theorem is used to estimate the zero multiplicity of the denominator in linear-fractional transformation \( T_W \) associated with the data set \((A_1, A_2, C, P)\).

**Lemma 5.2.** Let (B1)-(B4) be in force, let \( \kappa_1 = \nu_- (P) \), let the mvf’s \( W, b_1, b_2 \) be defined by (4.11), (4.4), (4.6), let \( \varphi_{21}, \varphi_{22}, \varphi_{11}, \varphi_{12} \) be defined by (4.32), (4.33) and let \( \varepsilon \in S_{\kappa_2}^{p \times q} \) admit the following Krein-Langer factorizations
\[
\varepsilon = \theta_{\varepsilon}^{-1}\varepsilon_{\theta} = \varepsilon_{\theta}^{-1} \varepsilon_{\varepsilon}.
\]

Then
\[
M_{\zeta}(\varphi_{21}\varepsilon_r + \varphi_{22}\theta_r, \Omega_+) = M_{\zeta}(\theta_{\varepsilon_{\theta}^{-1}}\varphi_{11} + \varepsilon_{\varepsilon_{\theta}^{-1}}\varphi_{12}, \Omega_+) = \kappa_1 + \kappa_2.
\]

**Proof.** 1) By (1) of Theorem 4.10 and (3) of Lemma 4.8, the factorization \( \varphi_{22}^{-1}\varphi_{21} \) is left coprime and \( \varphi_{22}^{-1}\varphi_{21} \in S_{\kappa_2}^{p \times q} \). Thus, Proposition 4.11 guarantees that \( M_{\zeta}(\varphi_{22}, \Omega_+) = \kappa_1 \). Moreover, the identities
\[
\begin{align*}
\varphi_{22}(\zeta)\varphi_{22}(\zeta)^* - \varphi_{21}(\zeta)\varphi_{21}(\zeta)^* & \equiv I_q \quad (\zeta \in \Omega_0), \\
\varphi_{22}^{-1}(\zeta)\varphi_{21}(\zeta) & = \varphi_{22}^{-1}(\zeta)\varphi_{21}(\zeta) \quad (\zeta \in \Omega_0)
\end{align*}
\]

imply that
\[
\|\varphi_{22}(\zeta)^{-1}\varphi_{21}(\zeta)\| \leq \rho < 1 \quad (\zeta \in \Omega_0)
\]

for some \( \rho < 1 \). Therefore,
\[
\|\varphi_{21}(\zeta)\varepsilon(\zeta) + \varphi_{22}(\zeta)^{-1}\varphi_{22}(\zeta)^{-1}\| \leq \|\varphi_{22}(\zeta)^{-1}\varphi_{22}(\zeta)^{-1}\|(I_q + \varphi_{22}^{-1}(\zeta)\varphi_{22}(\zeta)\varepsilon(\zeta))^{-1}\|
\]
\[
\leq \frac{\|\varphi_{22}(\zeta)^{-1}\|}{1 - \rho} \leq L \quad \text{for some } L < \infty.
\]

2) Let
\[
\psi = \varphi_{21}\varepsilon_r \quad \text{and} \quad \varphi = \varphi_{22}\theta_r.
\]

Then \( M_{\zeta}(\varphi, \Omega_+) = \kappa_1 + \kappa_2 \), since \( M_{\zeta}(\varphi_{22}, \Omega_+) = \kappa_1 \) and \( M_{\zeta}(\theta_r, \Omega_+) = \kappa_2 \). Moreover,
\[
\|\varphi(t)^{-1}\psi(t)\| \leq 1 \quad (t \in \Omega_0)
\]

and, since \( \theta_r \) is inner, it follows from (5.6) that
\[
\|\theta_r(t)^{-1}(\varphi(t) + \psi(t))\| = \|(\varphi_{21}(t)\varepsilon(t) + \varphi_{22}(t))^{-1}\| \leq L \quad \text{for } t \in \Omega_0.
\]

Therefore, by Theorem 5.1
\[
M_{\zeta}(\varphi_{21}\varepsilon_r + \varphi_{22}\theta_r, \Omega_+) = M_{\zeta}(\varphi_{22}\theta_r, \Omega_+) = \kappa_1 + \kappa_2.
\]

**Remark 5.3.** The equalities (5.4) were proved in [23] under the less restrictive assumption that \( W \in U^0_{\kappa_1} (j_{pq}) \cap \tilde{L}^{m \times m}_2 \).
Lemma 5.4. Let \( W \in \mathcal{U}_{ \kappa_1}(j_{pq}) \cap \tilde{L}_2^{m \times m} \), let the mvf \( \Phi \) be defined by (4.28), (4.29), let \( \varepsilon \in S_{\kappa_2}^{p \times q} \) and let
\[
(5.7) \quad G = T_\Phi[\varepsilon] = (\varphi_{11}\varepsilon + \varphi_{12})(\varphi_{21}\varepsilon + \varphi_{22})^{-1}.
\]
Then
\[
(5.8) \quad M_\pi(G, \Omega_+) = \kappa_1 + \kappa_2.
\]

Proof. Let \( \varepsilon \in S_{\kappa_2}^{p \times q} \) admit the Kr"{e}in-Langer factorizations (5.3). Then the factorization (5.7) of \( G \) can be rewritten as
\[
(5.9) \quad G = T_\Phi[\varepsilon] = (\varphi_{11}\varepsilon_r + \varphi_{12}\theta_r)(\varphi_{21}\varepsilon_r + \varphi_{22}\theta_r)^{-1}.
\]
Since \( \Phi \) is outer,
\[
\ker \left[ \varphi_{11}(\lambda)\varepsilon_r(\lambda) + \varphi_{12}(\lambda)\theta_r(\lambda) \right] = \ker \Phi(\lambda) \left[ \varepsilon_r(\lambda) \right] = \{0\}
\]
for every \( \lambda \in \Omega_+ \). By Lemma 2.2 (see also [23, Lemma 3.3]) the factorization (5.9) of \( G \) is right coprime over \( \Omega_+ \). Consequently, Proposition 4.1 and (5.4) imply that
\[
(5.10) \quad M_\pi(G, \Omega_+) = M_\pi((\varphi_{21}\varepsilon_r + \varphi_{22}\theta_r)^{-1}, \Omega_+) = \kappa_1 + \kappa_2.
\]

We will also need the following general noncancellation lemma from [23].

Lemma 5.5. Let \( G \in H_{\kappa,\Omega_+}^{p \times q} \), \( H_1 \in H_{\infty,\Omega_+}^{p \times q} \) and \( H_2 \in H_{\infty}^{p \times q} \). Then
\[
(5.11) \quad M_\pi(H_1G, \Omega_+) = M_\pi(G, \Omega_+) \implies M_\pi(H_1GH_2, \Omega_+) = M_\pi(GH_2, \Omega_+),
\]
whereas
\[
(5.12) \quad M_\pi(GH_2, \Omega_+) = M_\pi(G, \Omega_+) \implies M_\pi(H_1GH_2, \Omega_+) = M_\pi(H_1G, \Omega_+).
\]

5.2. Proof of Theorem 1.2. Since the first assertion (i.e., the verification of (1.23)) is covered by Theorem 3.23, it remains only to justify the second assertion. Towards this end, it is convenient to first note that if \( \varepsilon \in S_{\kappa_2}^{p \times q} \), then, by Lemma 5.2
\[
(5.13) \quad M_\pi(\varphi_{21}\varepsilon + \varphi_{22})^{-1}, \Omega_+) = M_\pi((\tilde{\varphi}_{11}^# + \varepsilon\tilde{\varphi}_{12}^#)^{-1}, \Omega_+) = \nu_-(P) = \kappa.
\]
Thus, in view of the factorization \( W = \Theta\Phi \) supplied in Theorem 4.10 and the properties of these factors, it follows that the mvf \( s = T_W[\varepsilon] \) is equal to
\[
(5.14) \quad s = T_W[\varepsilon] = T_{\Theta}[\varphi_{21}^#\varepsilon_r + \varepsilon\varphi_{12}^#] = b_1G b_2 + K,
\]
where \( G \) is defined by (5.7). Therefore, since \( K \) is holomorphic in \( \Omega_+ \),
\[
(5.15) \quad M_\pi(b_1G b_2, \Omega_+) = M_\pi(s, \Omega_+),
\]
and hence, if \( s \in S_{\kappa}^{p \times q} \), then, by (5.13),
\[
(5.16) \quad \kappa = M_\pi(b_1G b_2, \Omega_+) \leq M_\pi((\varphi_{21}\varepsilon + \varphi_{22})^{-1}b_2, \Omega_+) \leq M_\pi((\varphi_{21}\varepsilon + \varphi_{22})^{-1}, \Omega_+) = \kappa.
\]
Thus, in view of Proposition 4.1, the factorization \( b_2^{-1}(\varphi_{21}\varepsilon + \varphi_{22}) \) is coprime over \( \Omega_+ \).

Similarly, since
\[
\tilde{j}_{pq} = W^# j_{pq} W = \tilde{\Phi}^# \tilde{\Theta}^# j_{pq} \Theta \Phi = \tilde{\Phi}^# j_{pq} \Phi,
\]
the mvf \( G \) can be written as
\[
(5.17) \quad G = (\tilde{\varphi}_{11}^# + \varepsilon\tilde{\varphi}_{12}^#)^{-1}(\tilde{\varphi}_{21}^# + \varepsilon\tilde{\varphi}_{22}^#),
\]
and consequently the assumption \( s \in S_{\kappa}^{p \times q} \) and (5.13) imply that
\[
M_\pi(b_1(\tilde{\varphi}_{11}^# + \varepsilon\tilde{\varphi}_{12}^#)^{-1}, \Omega_+) = M_\pi((\tilde{\varphi}_{11}^# + \varepsilon\tilde{\varphi}_{12}^#)^{-1}, \Omega_+) = \kappa.
\]
Therefore, Proposition 4.1 implies that the factorization \((\varphi_{11}^\# + \varepsilon \varphi_{12}^\#) b_1^{-1}\) is coprime over \(\Omega_+\). This completes the proof of the implication

\[
s \in S_k^{p \times q} \implies \text{the two factorizations in (a) and (b) are coprime.}
\]

Suppose now that the two conditions (a) and (b) are met. Then by assumption (b) and Proposition (4.1)

\[
M = \pi((\varphi_{21} \varepsilon + \varphi_{22}^{-1}) b_2, \Omega_+) = M = \pi((\varphi_{21} \varepsilon + \varphi_{22}^{-1}) \Omega_+) = \kappa.
\]

Since by Lemma 5.4 \(M = \pi(G, \Omega_+) = \kappa\) it follows from Lemma 5.5 that

\[
M = \pi(G b_2, \Omega_+) = M = \pi((\varphi_{21} \varepsilon + \varphi_{22}^{-1}) b_2, \Omega_+) = \kappa.
\]

Similarly, (5.13), Lemma 5.4, Lemma 5.5 and assumption (a) imply that

\[
M = \pi(b_1 G, \Omega_+ \Omega_+) = \kappa.
\]

Therefore,

\[
M = \pi(b_1 G, \Omega_+) = \kappa,
\]

which, with the help of Lemma 5.5 implies that

\[
M = \pi(b_1 G b_2, \Omega_+) = \kappa.
\]

Consequently, \(s \in S_k^{p \times q}\), by (5.14). \(\square\)

If \(P\) is invertible, then \(\nu = 0\) and Theorem 1.2 takes the form

**Corollary 5.6.** Let the data set \((A_1, A_2, C, P)\) satisfy the assumptions (B1)-(B3), let \(P\) be invertible, let \(\nu = \nu(P)\), and let the mvf’s \(W, b_1, b_2\) be defined by (2.25), (4.7), (4.6). Then:

(I) \(s \in S_k(A_1, A_2, C, P)\) if and only if \(s\) belongs to \(S_k^{p \times q}\) and is of the form \(s = T_W[\varepsilon]\) for some \(\varepsilon \in S_k^{p \times q}\).

(II) If \(\varepsilon \in S_k^{p \times q}\), then \(T_W[\varepsilon] \in S_k^{p \times q}\) if and only if

(a) the factorization \(w_1^{\#} + \varepsilon w_1^{\#} = (\varphi_{11}^\# + \varepsilon \varphi_{12}^\#) b_1^{-1}\) is coprime over \(\Omega_+\) and

(b) the factorization \(w_2 \varepsilon + w_2 = b_2^{-1}(\varphi_{21} \varepsilon + \varphi_{22})\) is coprime over \(\Omega_+\).

**Remark 5.7.** The statement of Corollary 5.6 is a partial case of a general statement in [23, Theorem 1.3] for arbitrary mvf \(W \in \mathcal{U}_k(J_{pq})\). However the factorizations in (i), (ii) of Corollary 5.6 are simplified with respect to those in [23, Theorem 1.3] since the outer mvf’s \(\varphi_1 \in S_{out}^{p \times p}\) and \(\varphi_2 \in S_{out}^{q \times q}\) in the problem (C1)-(C3) satisfy the conditions \(\varphi_1^{-1} \in H_{\infty}^{p \times p}\), \(\varphi_2^{-1} \in H_{\infty}^{q \times q}\).

5.3. **Proof of Theorem 1.3.** Since assertion I is covered by Theorem 3.6 it remains only to justify II.

**Necessity.** Let \(s \in S_k(A_1, A_2, C, P)\). Then by Theorem 3.6 \(s = T_W[\varepsilon] \in S_k^{p \times q}\) where \(\varepsilon \in S_k^{p \times q}\). It follows from (5.14) that

\[
5.18 \quad s - K = b_1 (\varphi_{11} \varepsilon + \varphi_{12} \varepsilon + \varphi_{21} \varepsilon + \varphi_{22} \varepsilon) (\varphi_{21} \varepsilon + \varphi_{22} \varepsilon) b_1^{-1} b_2 \in S_k^{p \times q}.
\]

Due to Lemma 5.2 one has

\[
5.19 \quad M = \pi((\varphi_{21} \varepsilon + \varphi_{22} \varepsilon) b_2^{-1} b_2, \Omega_+) = \kappa,
\]

and, hence,

\[
5.20 \quad M = \pi((\varphi_{21} \varepsilon + \varphi_{22} \varepsilon) b_2^{-1} b_2, \Omega_+) = \kappa.
\]

It follows from (5.20), (5.19) and Proposition 4.1 that the factorization (1.25) is coprime over \(\Omega_+\).
Similarly, it follows from (5.14) and (5.17) that
\[(5.21) \quad s - K = b_1(\theta_1 \varphi_{11}^# + \varepsilon \bar{\varphi}_{12}^#)^{-1}(\theta_1 \varphi_{21}^# + \varepsilon \bar{\varphi}_{22}^#)b_2.\]
Since
\[(5.22) \quad M_\pi((\theta_1 \varphi_{11}^# + \varepsilon \bar{\varphi}_{12}^#)^{-1}, \Omega_+) = \kappa,\]
and \(s \in S^{p \times q}_\kappa\) one obtains
\[M_\pi(b_1(\theta_1 \varphi_{11}^# + \varepsilon \bar{\varphi}_{12}^#)^{-1}, \Omega_+) = \kappa.\]
This implies that the factorization (1.23) is coprime over \(\Omega_+\).

**Sufficiency.** The proof of sufficiency is similar to that in Theorem 1.2 except that now one should replace the factorizations (5.7) and (5.17) by (5.18), (5.21). \(\square\)

5.4. **Takagi-Nudelman problem.** Let the data set \((A_1, A_2, C, P)\) satisfy the assumptions (B1)-(B4) and let \(\kappa \in \mathbb{N} \cup \{0\}\). A mvf \(s \in S^{p \times q}_\kappa\) is said to be a solution of the Takagi-Nudelman problem if \(s\) satisfies the conditions \((C1)-(C3)\) and:

(C4) \(s\) is holomorphic on \(\sigma(A_1) \cup \overline{\sigma(A_2)}\).

The set of all mvf’s \(s \in S^{p \times q}_\kappa\) satisfying \((C1)-(C4)\) will be denoted \(TN_\kappa(A_1, A_2, C, P)\).

The following result gives a description of Takagi-Nudelman interpolation problem. It contains, as a partial case, the Nudelman description of the tangential indefinite interpolation problem considered in [41].

**Theorem 5.8.** Let the data set \((A_1, A_2, C, P)\) satisfy the assumptions (B1)-(B4), let \(\kappa = \nu_-(P)\), let the mvf \(W(\lambda)\) be defined by (1.14) and let \(\{b_1, b_2\}\) be an associated pair of the mvf \(W(\lambda)\). Then \(s \in TN_\kappa(A_1, A_2, C, P)\) if and only if \(s = W[\varepsilon],\) where \(\varepsilon \in S^{p \times q}_\kappa\) has the form (1.23) and \(\varepsilon\) satisfies one of the equivalent conditions:

(i) the mvf \((w_{11}^# + \varepsilon w_{12}^#)b_1\) has no zeros on \(\sigma(A_1) \cup \overline{\sigma(A_2)}\);

(ii) the mvf \((w_{21}^# + \varepsilon w_{22}^#)b_2\) has no zeros on \(\sigma(A_1) \cup \overline{\sigma(A_2)}\).

**Proof. Necessity.** Let \(s \in TN_\kappa(A_1, A_2, C, P)\). Then by Theorem 3.23 \(s = W[\varepsilon] \in S^{p \times q}_\kappa\), where the mvf \(\varepsilon \in S^{p \times q}_\kappa\) has the form (1.23). Since
\[M_\pi(s, \Omega_+) = M_\pi((\varphi_{21}^# + \varphi_{22}^#)^{-1}, \Omega_+) = \kappa,\]
it follows from (5.18) that poles of \((\varphi_{21}^# + \varphi_{22}^#)^{-1}\) and hence poles of \((w_{21}^# + \varepsilon w_{22}^#)^{-1}b_2^{-1}\) coincide with poles of \(s\). Then (ii) is implied by (C4). Similarly the equalities (5.21) and
\[M_\pi(s, \Omega_+) = M_\pi((\varphi_{11}^# + \varepsilon \bar{\varphi}_{12}^#)^{-1}, \Omega_+) = \kappa,\]
imply that poles of \((\varphi_{11}^# + \varepsilon \bar{\varphi}_{12}^#)^{-1}\) coincide with poles of \(s\), which serves to prove (i).

**Sufficiency.** Lemma 5.4 guarantees that
\[M_\pi(G, \Omega_+) = \kappa_1 + \kappa_2 = \kappa\]
in the present setting for the mvf \(G(\lambda)\) defined by (5.7). Moreover, in view of (1.32) and assumption (ii), \(G\) has no poles in \(\sigma(A_1) \cup \overline{\sigma(A_2)}\). Therefore,
\[M_\pi(G, \Omega_+ \setminus (\sigma(A_1) \cup \overline{\sigma(A_2)})) = M_\pi(G, \Omega_+) = \kappa.\]
Thus, as \(b_1(\lambda)\) and \(b_2(\lambda)\) are invertible in \(\sigma(A_1) \cup \overline{\sigma(A_2)}\),
\[M_\pi(b_1 G b_2, \Omega_+ \setminus (\sigma(A_1) \cup \overline{\sigma(A_2)})) = M_\pi(b_1 G b_2, \Omega_+) = \kappa\]
and hence \(s = K + b_1 G b_2 \in S^{p \times q}_\kappa\).

Similarly if (i) is in force, then (5.10), the second equality in (5.13) and the dual representation (2.29) of \(s\) lead to the desired conclusions. \(\square\)
Remark 5.9. Descriptions of the set \( T\mathcal{N}_\kappa(A_1, A_2, C, P) \) are available for \( \kappa \geq \nu_- (P) \) when \( P \) is invertible; see [14, Theorem 19.2.1] for a description of rational solutions, and, for the general case, [11, Theorem 2] for the one sided problem and [6, Proposition 4.7] for the two-sided problem with \( \sigma(A_1) \cap \sigma(A_2) = \emptyset \).

Corollary 5.10. Let \((B1)-(B3)\) be in force, let \( P \) be invertible, \( \kappa_1 := \nu_- (P) \leq \kappa \), let the mvf \( W(\lambda) \) be defined by \( (2.26) \), let \( \{b_1, b_2\} \) be an associated pair of the mvf \( W(\lambda) \), and let a mvf \( \varepsilon \in S_{\kappa-\kappa}^{p \times q} \) admits the Krein-Langer factorizations \((5.3)\). Then \( T_W [\varepsilon] \in T\mathcal{N}_\kappa(A_1, A_2, C, P) \) if and only if one of the equivalent conditions holds:

(i) the mvf \((\theta \varepsilon w_{11}^\# + \varepsilon \theta w_{12}^\#) b_1 \) has no zeros on \( \sigma(A_1) \cup \overline{\sigma(A_2)} \);
(ii) the mvf \( b_2 (w_{21} \varepsilon r + w_{22} \theta r) \) has no zeros on \( \sigma(A_1) \cup \sigma(A_2) \).

The next example shows that the inclusion
\( T\mathcal{N}_\kappa(A_1, A_2, C, P) \subseteq \mathcal{S}_\kappa(A_1, A_2, C, P) \)
may be strict.

Example 5. Let \( n_1 = n = 1 \), \( M = A_1 = 0 \), \( N = 1 \), \( C = \text{col}(2, 0, 0, 1) \) and \( \kappa = 1 \). Then the unique solution of the Lyapunov-Stein equation \((1.4)\) is \( P = -3 \), and, hence, \( \kappa_1 = 1 \). By formula \((2.26)\) with \( \mu = 1 \), the resolvent matrix \( W(\lambda) \) can be written as
\[
W(\lambda) = I_4 + \frac{1-\lambda}{3\lambda} C \bigg| C^* j_{22}
\]
Direct calculations show that
\[
W(\lambda) = \begin{bmatrix}
\frac{4-\lambda}{3\lambda} & 0 & 0 & \frac{2(1-\lambda)}{3\lambda} \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\frac{2(1-\lambda)}{3\lambda} & 0 & 0 & \frac{4\lambda-1}{3\lambda}
\end{bmatrix}, \quad W^\#(\lambda) = \begin{bmatrix}
\frac{2\lambda-1}{3} & 0 & 0 & \frac{2(\lambda-1)}{3} \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\frac{2(1-\lambda)}{3} & 0 & 0 & \frac{4-\lambda}{3}
\end{bmatrix}
\]
and hence, as follows from \((1.27)\),
\[
b_1(\lambda) = I_2 \quad \text{and} \quad b_2(\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix}.
\]
The mvf
\[
\varepsilon(\lambda) = \begin{bmatrix}
\frac{3}{4-\lambda} & \frac{2(1-\lambda)}{4-\lambda} \\
\frac{2(1-\lambda)}{4-\lambda} & \frac{3\lambda}{4-\lambda}
\end{bmatrix} \in \mathcal{S}_0^{2 \times 2},
\]
since \( \varepsilon(\lambda) \) is holomorphic in \( \mathbb{D} \) and
\[
I_2 - \varepsilon(\lambda)^* \varepsilon(\lambda) = \begin{bmatrix}
1 - |\lambda|^2 & 3 \\ |4 - \lambda|^2 & 6
\end{bmatrix} \begin{bmatrix}
3 & 6 \\ 6 & 12
\end{bmatrix} \geq 0 \quad \text{for} \ \lambda \in \mathbb{D}.
\]
However, \( \varepsilon(\lambda) \) does not satisfy condition (i) of Theorem 5.8 since the mvf
\[
(w_{11}^\# + \varepsilon w_{12}^\#)^{-1} = \begin{bmatrix}
\frac{4-\lambda}{3\lambda} & 0 \\
\frac{2(\lambda-1)}{3} & 1
\end{bmatrix}
\]
has a pole at 0. The corresponding linear fractional transform

\[ s(\lambda) = T_W[\varepsilon] = \begin{bmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{bmatrix} = b_\ell^{-1}s_\ell \quad \left( b_\ell = \begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix}, \ s_\ell = \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix} \right) \]

also has a pole at 0, and is not a solution of the Takagi-Nudelman problem (C1)-(C4). However, \( s \in S_1(A_1, C, P) \) by I of Theorem [3.1]. It is reassuring to check that

\[ [b_\ell - s_\ell]C(A_1 - \lambda)^{-1} = \frac{b_\ell\xi - s_\eta}{-\lambda - 1} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \in H_2^0 \quad \text{with} \ p = 2. \]

**Theorem 5.11.** Let (B1)-(B3) be in force, let \( P \) be invertible, and let \( \kappa_1 = \nu_-(P) \leq \kappa. \) Then the problem (C1)-(C4) is solvable.

**Proof.** Suppose first that \( q \leq p \) for the sake of definiteness and let \( \alpha_1, \ldots, \alpha_l \) be all the points in \( \sigma(A_1) \cup \sigma(A_2). \) In view of Corollary [5.10] one has to show that there is a \( \varepsilon \in S_{\kappa - \kappa_1}\) with the Krein-Langer factorizations [5.3] such that

\[ (5.23) \quad \det(\phi_{21}(\alpha_j)\varepsilon_r(\alpha_j) + \phi_{22}(\alpha_j)\theta_r(\alpha_j)) \neq 0 \quad \text{for} \quad j = 1, \ldots, l. \]

Let us choose the Blaschke-Potapov factor \( \theta_r \) of degree \( \kappa - \kappa_1 \) in such a way that \( \theta_r(\alpha_j) \) is invertible. If \( \phi_{21}(\alpha_j) = 0, \) then \( \phi_{22}(\alpha_j) \) is invertible and, hence, (5.23) is satisfied for every \( \varepsilon_r \in S_{\kappa \times \kappa}. \)

Assume now that \( r := \text{rank } \phi_{21}(\alpha_j) > 0 \) and let us show that the algebraic manifold

\[ (5.24) \quad \mathcal{M}_j = \{ Y \in \mathbb{C}^{p \times q} : \det(\phi_{21}(\alpha_j)Y + \phi_{22}(\alpha_j)\theta_r(\alpha_j)) = 0 \} \]

does not coincide with \( \mathbb{C}^{p \times q}. \) Choosing invertible matrices \( V_1 \in \mathbb{C}^{q \times q} \) and \( V_2 \in \mathbb{C}^{p \times p} \) such that

\[ V_1\phi_{21}(\alpha_j)V_2 = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \]

and setting

\[ \hat{Y} = V_2^{-1}Y, \quad Z = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = V_1\phi_{22}(\alpha_j)\theta_r(\alpha_j), \quad Z_1 \in \mathbb{C}^{r \times q}, \quad Z_2 \in \mathbb{C}^{(q-r) \times q}, \]

one can rewrite the equality in (5.24) in the form

\[ (5.25) \quad \det \left( \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \hat{Y} + \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \right) = 0. \]

Since \( \text{rank } [\phi_{21}(\alpha_j) \quad \phi_{22}(\alpha_j)] = q \) one obtains

\[ \text{rank } V_1 [\phi_{21}(\alpha_j) \quad \phi_{22}(\alpha_j)] \begin{bmatrix} V_2 & 0 \\ 0 & \theta_r(\alpha_j) \end{bmatrix} = \text{rank } \begin{bmatrix} I_r & 0 & Z_1 \\ 0 & 0 & Z_2 \end{bmatrix} = q \]

and hence \( \text{rank } Z_2 = q - r. \) Therefore, there is an invertible matrix \( V_3 \in \mathbb{C}^{q \times q} \) such that \( Z_2V_3 = [0 \ I_{q-r}]. \) Setting

\[ \begin{bmatrix} \hat{Y}_{11} & \hat{Y}_{12} \\ \hat{Y}_{21} & \hat{Y}_{22} \end{bmatrix} = \hat{Y}V_3, \quad \hat{Y}_{11} \in \mathbb{C}^{r \times r}, \quad [Z_{11} \ Z_{12}] = Z_1V_3, \quad Z_{11} \in \mathbb{C}^{r \times r}, \quad Z_{12} \in \mathbb{C}^{r \times (q-r)}, \]

one can rewrite the equality (5.25) in the form

\[ (5.26) \quad \det \left( \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{Y}_{11} & \hat{Y}_{12} \\ \hat{Y}_{21} & \hat{Y}_{22} \end{bmatrix} + \begin{bmatrix} Z_{11} & Z_{12} \\ 0 & I_{q-r} \end{bmatrix} \right) = 0 \]

or, equivalently,

\[ (5.27) \quad \det(\hat{Y}_{11} + Z_{11}) = 0. \]
It follows from (5.27) that the manifold \( M_j \) has the complex dimension at most \( pq - 1 \) in \( \mathbb{C}^{p \times q} \). Therefore, the manifold \( \cup_{j=1}^n M_j \) is nowhere dense in \( \mathbb{C}^{p \times q} \) and hence there is a constant contractive matrix \( \varepsilon \in \mathbb{C}^{p \times q} \) which satisfies (5.23).

In the case \( p < q \) one has to use the first condition (i) in Corollary 5.10

The conclusions of Theorem 5.11 may fail to hold if either (B3) is not in force or \( P \) is not invertible.

**Example 6.** Let \( n_2 = 0, M = A_1 = O_{2 \times 2}, N = I_2, C = I_2, P = -J_{11} \) and \( \kappa = 1 \). Then, clearly, assumptions (B1) and (B2) are in force (see Remark 1.1), \( P \) is invertible and \( \nu_-(P) = 1 = \kappa \), while (B3) fails to hold, since \( C_2 = [0 \ 1] \) and

\[
\bigcap_{j=0}^1 \ker C_2 A_1^j = \ker C_2 = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}.
\]

The interpolation condition (C1) can be rewritten as

\[
b_\ell(0) \xi_j = s_\ell(0) \eta_j \quad (j = 1, 2).
\]

This implies that

\[
b_\ell(0) = 0, \quad s_\ell(0) = 0,
\]

which contradicts the noncancellation condition (2.3). Therefore the problem (C1)-(C4) has no solution in \( S \), which contradicts the noncancellation condition (2.3). Therefore the problem (C1)-(C3) and

Then, clearly, assumptions (B1) and (B2) are in force (see Remark 1.1), we can give a criterion for the problem (C1)-(C4) to have no excluded parameters.

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Then, clearly, assumptions (B1) and (B2) are in force (see Remark 1.1), we can give a criterion for the problem (C1)-(C4) to have no excluded parameters.

5.5. **Excluded parameters.**

**Definition 5.12.** Let the data \( (A_1, A_2, C, P, \kappa) \) of the problem (C1)-(C4) satisfies the assumptions (B1)-(B4) and let \( \kappa_1 := \nu_-(P) (\leq \kappa) \). Then the parameter \( \varepsilon \in S_{\kappa_1}^{p \times q} \) of the form \( \text{(1.23)} \) is said to be excluded for the problem (C1)-(C4), if \( T_W[\varepsilon] \notin T(\mathbb{N}) \).

According to this definition, all the parameters \( \varepsilon \in S_0 \) are excluded for the problem (C1)-(C4) in Example 6. Theorem 5.10 leads to the following descriptions of excluded parameters for the problem (C1)-(C4).

**Proposition 5.13.** Under the assumptions of Theorem 5.8 the parameter \( \varepsilon \in S_{\kappa_1}^{p \times q} \) of the form \( \text{(1.23)} \) is excluded for the problems (C1)-(C4), if and only if

\[
\det(\varphi_{21}(\alpha_j) \varepsilon_r(\alpha_j) + \varphi_{22}(\alpha_j) \theta_r(\alpha_j)) = 0.
\]

for at least one of the point \( \alpha_j \in \sigma(A_1) \cup \sigma(A_2) \) \( (j = 1, \ldots, l) \).

In accordance with the above statement the parameter \( \varepsilon \in S_{\kappa_1}^{p \times q} \) will be said to be excluded at the point \( \alpha_j \in \sigma(A_1) \cup \sigma(A_2) \) if (5.28) holds. In a special case when \( \nu_-(P) = \kappa \) we can give a criterion for the problem (C1)-(C4) to have no excluded parameters.
Proposition 5.14. Let (B1)-(B4) be in force, and let $\nu_-(P) = \kappa$. Then the problem (C1)-(C4) has no excluded parameters at $\alpha_j \in \sigma(A_1) \cup \overline{\sigma(A_2)}$ $(j = 1, \ldots, l)$ if and only if

\begin{equation}
\varphi_{21}(\alpha_j)\varphi_{21}(\alpha_j)^* - \varphi_{22}(\alpha_j)\varphi_{22}(\alpha_j)^* < 0.
\end{equation}

Proof. A matrix $\varepsilon \in S^{p \times q}$ is an excluded parameter at $\alpha_j$ if and only if there is a vector $v \in \mathbb{C}^q$, $(v \neq 0)$ such that

\begin{equation}
v^*\varphi_{21}(\alpha_j)\varepsilon(\alpha_j) + v^*\varphi_{22}(\alpha_j) = 0.
\end{equation}

The tangential interpolation problem (5.30) has a solution $\varepsilon \in S^{p \times q}$ if and only if the corresponding Pick matrix

\begin{equation}
(1 - |\alpha_j|^2)^{-1}(v^*\varphi_{21}(\alpha_j)\varphi_{21}(\alpha_j)^*v - v^*\varphi_{22}(\alpha_j)\varphi_{22}(\alpha_j)^*v)
\end{equation}

is nonnegative. Therefore the problem (5.30) has no solutions for every $v \in \mathbb{C}^q$ if and only if the matrix in (5.29) is negative. \hfill \Box

Corollary 5.15. Let (B1) - (B4) be in force, let $\nu_-(P) = \kappa$ and let the conditions (5.29) are satisfied for all $\alpha_j \in \sigma(A_1) \cup \overline{\sigma(A_2)}$ $(j = 1, \ldots, l)$. Then the problem (C1)-(C4) has a solution $s \in S^{p \times q}$. This solution is unique if and only if

\begin{equation}
\text{rank } (M^*P^2M + N^*P^2N + C^*C) - \text{rank } P = \min\{p, q\}.
\end{equation}

The proof is immediate from Theorem 5.8 and Proposition 5.13.

5.6. The Schur-Takagi interpolation problem. In this subsection $\Omega_+$ is either $D_+$ or $\Pi_+$. Let $b_1 \in S_{1\times p}^{\kappa \times q}$, $b_2 \in S_{m \times q}^{\kappa \times q}$ be inner mvf’s, let $K \in H_{\kappa \times \kappa}^{p \times q}$ and let $\kappa \in \mathbb{N} \cup \{0\}$. Consider the following problem.

Definition 5.16. A $p \times q$ mvf $s$ is called a solution of the Schur-Takagi interpolation problem in the class $S^{p \times q}_\kappa$, if $\kappa \geq 1$, $s \in S^{p \times q}_\kappa$ and

\begin{equation}
b_1^{-1}(s - K)b_2^{-1} \in H_{\kappa, \infty}^{p \times q} \cap H_{\kappa-1, \infty}^{p \times q}.
\end{equation}

The set of solutions of this problem is denoted $S_\kappa(b_1, b_2, K)$.

The analogue of this problem for $\kappa = 0$ (in which the right hand side of (5.32) is replaced by $H_{\infty}^{p \times q}$) has been extensively studied by D. Z. Arov; see e.g., [8] and, for a more accessible discussion and additional references, Chapter 7 of [9].

A $p \times q$ mvf $s$ is called a solution of the Takagi-Sarason problem with the data set $(b_1, b_2, K)$ if $s$ belongs to $S^{p \times q}_\kappa$ for some $\kappa' \leq \kappa$ and satisfies (5.32) (see [14]). The set of solutions of the Takagi-Sarason problem is designated as $TS_\kappa(b_1, b_2, K)$.

Theorem 5.17. Let $W \in U_{\iota_{pq}}(j_{pq}) \cap L_2^{m \times m}$, and let the mvf’s $b_1$, $b_2$, $K$ be defined by (1.19), (1.20) and (4.34), respectively. Then:

(i) $s \in TS_\kappa(b_1, b_2, K)$ if and only if $s \in T_W[S^{p \times q}_{\kappa - 1}]$;

(ii) $s \in S_\kappa(b_1, b_2, K)$ if and only if $s \in T_W[S^{p \times q}_{\kappa - 1}] \cap S^{p \times q}_\kappa$.

Proof. (i) Let $s = T_W[s]$, where $e \in S^{p \times q}_{\kappa_2}$ $(\kappa_2 = \kappa - \kappa_1)$ and let the mvf $e$ admit the Krein-Langer factorizations (5.3). Then it follows from (5.18) that

\begin{equation}
b_1^{-1}(s - K)b_2^{-1} = (\varphi_{11}e_r + \varphi_{12}\theta_r)(\varphi_{21}e_r + \varphi_{22}\theta_r)^{-1}
\end{equation}

and hence coincides with $G$ in (5.9). Then by Lemma 5.3

\begin{equation}
M_n(b_1^{-1}(s - K)b_2^{-1}, \Omega_+) = \kappa_1 + \kappa_2 = \kappa.
\end{equation}

Since $s \in S^{p \times q}_{\kappa'}$ for some $\kappa' \leq \kappa$ by Lemma 2.7, this proves that $s \in TS_\kappa(b_1, b_2, K)$. 


Conversely, let $s \in TS_\kappa(b_1, b_2, K)$. Then $s \in S^{p \times q}_\kappa$ for some $\kappa' \leq \kappa$. Consequently, the mvf 
$\varepsilon = T_{W^{-1}}[s].$

belongs to $S^{p \times q}_\kappa$, where $\kappa' - \kappa_1 \leq \kappa_2$ by Lemma 2.7 and, by an analogous argument, $\kappa_2 \leq \kappa' + \nu_+(P)$. In view of Lemma \ref{lem:5.3},

$$M_\kappa(b_1^{-1}(s - K)b_2^{-1}, \Omega_+) = \kappa_1 + \kappa_2.$$ 

Now it follows from (5.32) that $\kappa_2 = \kappa - \kappa_1$, and hence that $s \in T_{W^{-1}}[S^{p \times q}_{\kappa-\kappa_1}]$.

(ii) The second statement is immediate from (i) and Definition \ref{def:5.16}.

Corollary 5.18. Let the data set $(A_1, A_2, C, P)$ satisfy the assumptions (B1)-(B3), let $P$ be invertible, let the mvf’s $W$, $b_1$, $b_2$, $K$ be defined by (1.14), (4.6), (4.7) and (4.22), respectively, and assume that $\kappa_1 = \nu_-(P) \leq \kappa$. Then $S_\kappa(A_1, A_2, C, P) = S_\kappa(b_1, b_2, K)$.

Proof. The proof follows from the descriptions of the sets $S_\kappa(A_1, A_2, C, P)$ and $S_\kappa(b_1, b_2, K)$ in Theorem \ref{thm:6.2} and Theorem \ref{thm:5.17}, respectively.

It follows from Theorem \ref{thm:5.17} and Corollary \ref{cor:5.18} that for every data set $(A_1, A_2, C, P)$ satisfying the assumptions (B1)-(B3) with invertible $P$ there exists a data set $(b_1, b_2, K)$ ($b_1 \in S^{p \times p}_m$, $b_2 \in S^{q \times q}_m$, $K \in H^{p \times q}_\infty$), such that

$$S_\kappa(A_1, A_2, C, P) = S_\kappa(b_1, b_2, K) \subseteq TS_\kappa(b_1, b_2, K).$$

6. **Bitangential interpolation in the right half plane**

In this section we shall summarize the main formulas that come into play when $\Omega_+$ (resp., $\Omega_-$) is the open right (resp., left) half plane $\Pi_+$ (resp., $\Pi_-$) and $\Omega = i\mathbb{R}$. In this setting, the bitangential interpolation problem under consideration is formulated in terms of the matrices

$$(6.1) \quad M = A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad N = I_n \quad \text{and}, \quad J = j_{pq}, \quad C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} : \mathbb{C}^{n_1} \rightarrow \mathbb{C}^{p},$$

where $n_1 + n_2 = n$, $n_1 > 0$, $n_2 > 0$, $A_1 \in \mathbb{C}^{n_1 \times n_1}$ and $A_2 \in \mathbb{C}^{n_2 \times n_2}$. These matrices are subject to the following constraints:

(B1a) $\sigma(A_1) \subset \Pi_+$ and $\sigma(A_2) \subset \Pi_-.

(B2a) $P$ is a Hermitian solution of the Lyapunov-Stein equation

$$(6.2) \quad A^* P + P A + C^* JC = 0$$

(or, equivalently, in terms of $M$ and $N$, $M^* PN + N^* PM + C^* JC = 0$).

(B3a) The pairs $(C_{12}, A_2)$ and $(C_{21}, A_1)$ are observable.

(B4a) $X \in \mathbb{C}^{n \times n}$ is a Hermitian solution of the Riccati equation

$$(6.3) \quad X A^* + A X + XC^* JCX = 0$$

such that:

(i) $XPX = X$.

(ii) $PXP = P$.

Notice that because of the special form of (6.3), $\text{rng} X$ is automatically invariant under $M$ and $N$. If $P$ is invertible, then (B4a) is superfluous, since it is automatically satisfied by $X = P^{-1}$.

Let

$$(6.4) \quad W(\lambda) = I_m + C(A - \lambda I_n)^{-1} XC^* J.$$
Then
\[ W^\#(\lambda) = W(-\bar{\lambda})^* \quad \text{and} \quad W^{-1}(\lambda) = JW^\#(\lambda)J = I_m + JCX(A^* + \lambda I_n)^{-1}C^*. \]

**Example 7.** If assumptions (B1a)-(B4a) are in force, then the linear space of vvf's
\[ \mathcal{M} = \{ F(\lambda)Xu : u \in \mathbb{C}^n \} \]

based on the mvf \( F(\lambda) = C(A - \lambda I_n)^{-1} \) for \( \lambda \not\in \sigma(A) \) and endowed with the inner product
\[ \langle FXu, FXv \rangle_\mathcal{M} = \langle u, v \rangle \]

is an RKPS with RK
\[ W(\lambda) = F(\lambda)XF(\lambda)^* = \frac{J - W(\lambda)JW(\lambda)^*}{\lambda + \bar{\omega}} \]

and \( \nu_-(X) \) negative squares.

**Lemma 6.1.** If assumptions (B1a)-(B4a) are in force, and if \( F(\lambda) = C(A - \lambda I_n)^{-1} \) and \( W(\lambda) \) is given by (6.4), then
\[ W(\lambda)^{-1}F(\lambda)X = -CX(A^* + \lambda I_n)^{-1} \]

and
\[ W(\lambda)^{-1}F(\lambda)v = C(I_n - XP)(A - \lambda I_n)^{-1}v \quad \text{for} \quad v \in \ker P. \]

**Lemma 6.2.** If assumptions (B1a)-(B4a) are in force and if
\[ X_1 = X \begin{bmatrix} I_{n_1} \\ 0 \end{bmatrix} \quad \text{and} \quad X_2 = X \begin{bmatrix} 0 \\ I_{n_2} \end{bmatrix}, \]

then
\[ \ker \begin{bmatrix} CX_1 \\ CX_1 A_1^* \\ \vdots \\ CX_1 (A_1)^{n_1-1} \end{bmatrix} = \ker X_1, \quad \ker \begin{bmatrix} CX_2 \\ CX_2 A_2^* \\ \vdots \\ CX_2 (A_2)^{n_2-1} \end{bmatrix} = \ker X_2 \]

and
\[ \ker [X_j^* C^* A_j X_j^* C^* \cdots A_j^{n_j-1} X_j^* C^*] = \ker X_j^* \quad \text{for} \quad j = 1, 2. \]

**Lemma 6.3.** If \( g \in H_2^{n_1} \) and \( v \in \mathbb{C}^{n_1} \), then (since \( \sigma(A_1) \subset \Pi_+ \),
\[ (A_1 - \lambda I_{n_1})^{-1}(g - v) \in H_2^{n_1} \iff v = \frac{1}{2\pi} \int_{-\infty}^\infty (A_1 - itI_{n_1})^{-1}g(it)dt. \]

If \( g \in (H_2^{n_2})^\perp \) and \( v \in \mathbb{C}^{n_2} \), then (since \( \sigma(A_2) \subset \Pi_- \),
\[ (A_2 - \lambda I_{n_2})^{-1}(g - v) \in (H_2^{n_2})^\perp \iff v = -\frac{1}{2\pi} \int_{-\infty}^\infty (A_2 - itI_{n_1})^{-1}g(it)dt. \]

**Remark 6.4.** Since \( (C_{21}, A_1) \) is observable and \( \sigma(A_1) \subset \Pi_+ \), the matrix
\[ Q_1 = -\int_0^\infty e^{-tA_1}C_{21}^* C_{21} e^{-tA_1}dt = -\frac{1}{2\pi} \int_{-\infty}^\infty (A_1^* + i\mu I_{n_1})^{-1}C_{21}^* C_{21}(A_1 - i\mu I_{n_1})^{-1}d\mu \]

is a negative definite solution of the Lyapunov equation
\[ A_1^* Q_1 + Q_1 A_1 + C_{21}^* C_{21} = 0 \]

and the mvf
\[ \tilde{b}_2(\lambda) = I_q + C_{21} Q_1^{-1} (A_1^* + \lambda I_{n_1})^{-1} C_{21}^* \]
is inner with respect to \( \Pi_+ \). Similarly, since \((C_{12}, A_2)\) is observable and \( \sigma(A_2) \subset \Pi_- \), the matrix
\[
Q_2 = \int_0^\infty e^{tA_2}C_{12}^*C_{12}e^{tA_2}dt = \frac{1}{2\pi i} \int_{-\infty}^{\infty} (A_2^* + i\mu I_{n_2})^{-1}C_{12}^*C_{12}(A_2 - i\mu I_{n_2})^{-1}d\mu
\]
is a positive definite solution of the Lyapunov equation
\[
(6.14) \quad A_2^*Q_2 + Q_2A_2 + C_{12}^*C_{12} = 0
\]
and the mvf
\[
(6.15) \quad \tilde{b}_1(\lambda) = I_p + C_{12}(A_2 - \lambda I_{n_2})^{-1}Q_2^{-1}C_{12}^*
\]
is inner with respect to \( \Pi_+ \).

Let \( F_{12}(\lambda) = C_{12}(A_2 - \lambda I_{n_2})^{-1}, \)
\[
\tilde{M}_2 = \{ F_{12}u : u \in \mathbb{C}^{n_2}\} \quad \text{and} \quad M_2 = \{ F_{12}[X_{21} \quad X_{22}]u : u \in \mathbb{C}^n\}
\]
and let \( M_2 \) and \( \tilde{M}_2 \) be equipped with the (normalized standard) inner product
\[
\langle F_{12}u, F_{12}v \rangle_{nst} = u^*Q_2u = v^* \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} F_{12}(i\mu)^*F_{12}(i\mu)d\mu \right\} u.
\]
Then \( \tilde{M}_2 \) is a RKHS with RK
\[
F_{12}(\lambda)Q_2^{-1}F_{12}(\omega)^* = \frac{I_p - \tilde{b}_1(\lambda)\tilde{b}_1(\omega)^*}{\lambda + \omega},
\]
i.e.,
\[
\tilde{M}_2 = \mathcal{H}(\tilde{b}_1).
\]
Moreover, \( M_2 \) is a subspace of \( \tilde{M}_2 \) that is invariant under the backwards shift operator \( R_0 \).

Therefore, by the Beurling-Lax theorem,
\[
M_2 = \mathcal{H}(b_1) \subset \mathcal{H}(\tilde{b}_1) = \tilde{M}_2.
\]

Similar considerations based on \( F_{21}(\lambda) = C_{21}(A_1 - \lambda I_{n_1})^{-1}, \)
\[
\tilde{M}_1 = \{ F_{21}u : u \in \mathbb{C}^{n_1}\} \quad \text{and} \quad M_1 = \{ F_{21}[X_{11} \quad X_{12}]u : u \in \mathbb{C}^n\}
\]
imply that
\[
M_1 = \mathcal{H}_*(b_2) \subset \mathcal{H}_*(\tilde{b}_2) = \tilde{M}_1.
\]

The subsequent analysis will make use of the left coprime factorizations
\[
(6.16) \quad C_{21}(A_1 - \lambda I_{n_1})^{-1}[X_{11} \quad X_{12}] = b_2^{-1}\varphi_2
\]
and
\[
(6.17) \quad C_{12}(A_2 - \lambda I_{n_2})^{-1}[X_{21} \quad X_{22}] = (b_1^\#)^{-1}\varphi_1
\]
where \( b_2 \) and \( b_1 \) are inner with respect to \( \Pi_+ \), \( \varphi_2 \in \mathcal{R} \cap H_2^{q\times n} \) and \( \varphi_1 \in (H_2^{1})^{p\times n} \). If \( \text{rank}X_j = n_j \) for \( j = 1, 2 \), then \( \tilde{b}_j(\lambda) = b_j(\lambda) \).

**Lemma 6.5.** If assumptions (B1a)-(B4a) are in force, then (since \( C_{21}, A_1 \) is observable and \( \sigma(A_1) \subset \Pi_+ \))
\[
(6.18) \quad \{ P_+(A_1^* + \lambda I_{n_1})^{-1}h : h \in (H_2^{n_1})^\perp \} = \{ P_+(A_1^* + \lambda I_{n_1})^{-1}C_{12}^*g : g \in \mathcal{H}_*(\tilde{b}_2) \}
\]
\[
= \{ (A_1^* + \lambda I_{n_1})^{-1}x : x \in \mathbb{C}^{n_1} \}. 
\]
Similarly, (since \((C_{12}, A_2)\) is observable and \(\sigma(A_2) \subset \Pi_-\),
\[
\{P_-(A_2^* + \lambda I_{n_2})^{-1} h : h \in H_2^{n_2}\} = \{(A_2^* + \lambda I_{n_2})^{-1} C_{12}^* g : g \in \mathcal{H}(\tilde{b}_1)\} \\
= \{(A_2^* + \lambda I_{n_2})^{-1} x : x \in \mathbb{C}^{n_2}\}.
\]

The rest of the development for \(\Omega_+ = \Pi_+\) is pretty much the same as for \(\Omega_+ = \mathbb{D}\), with the appropriate changes of \(M\) and \(N\) and with \((B1a)-(B4a)\) in place of \((B1)-(B4)\) and leads to the following conclusions:

**Theorem 6.6.** Let \((B1a)-(B4a)\) be in force, let \(\nu_-(P) = \kappa\) and let
\[
\nu = \text{rank}(P^2 + C^*C) - \text{rank} P.
\]
Then there are unitary matrices \(U \in \mathbb{C}^{p \times p}, V \in \mathbb{C}^{q \times q}\), such that \(S_\kappa(M, N, C, P)\) if and only if \(s\) belongs to \(S_\kappa^{p \times q}\) and is of the form \(s = T_W[\varepsilon]\), where
\[
\varepsilon = U \begin{bmatrix} \tilde{\varepsilon} & 0 \\ 0 & I_\nu \end{bmatrix} V^*, \quad \text{and} \quad \tilde{\varepsilon} \in S^{(p-\nu) \times (q-\nu)}.
\]

If \(\varepsilon \in S^{p \times q}_\kappa\), then \(T_W[\varepsilon] \in S^{p \times q}_\kappa\) if and only if
(a) the factorization \(w_{11}^# + \varepsilon w_{12}^# = (\tilde{\varepsilon}_{11}^# + \varepsilon \tilde{\varepsilon}_{12}^#) b_1^{-1}\) is coprime over \(\Omega_+\) and
(b) the factorization \(w_{21} \varepsilon + w_{22} = b_2^{-1}(\varphi_{21} \varepsilon + \varphi_{22})\) is coprime over \(\Omega_+\).

**Theorem 6.7.** Let the data set \((M, N, C, P)\) satisfy the assumptions \((B1a)-(B3a)\), let \(P\) be invertible, \(\kappa_1 = \nu_-(P) \leq \kappa\), and let the muf’s \(W, b_1, b_2\) be defined as in this section. Then:
(I) \(s \in S_\kappa(M, N, C, P)\) if and only if \(s \in S_\kappa^{p \times q}\) and is of the form \(s = T_W[\varepsilon]\) for some \(\varepsilon \in S_\kappa^{p \times q}\).
(II) If \(\varepsilon \in S_\kappa^{p \times q}\) and \(\theta_\ell, \theta_r, \varepsilon_\ell, \varepsilon_r\) are a choice of its Kreĭn-Langer factorizations as in (21), then \(T_W[\varepsilon] \in S_\kappa^{p \times q}\) if and only if the factorizations
\[
\theta_\ell w_{11}^# + \varepsilon_\ell w_{12}^# = (\theta_\ell \tilde{\varepsilon}_{11}^# + \varepsilon_\ell \tilde{\varepsilon}_{12}^#) b_1^{-1},
\]
\[
w_{21} \varepsilon_r + w_{22} \theta_r = b_2^{-1}(\varphi_{21} \varepsilon_r + \varphi_{22} \theta_r)
\]
are coprime over \(\Omega_+\).

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