Crystal Structure on the Category of Modules over Colored Planar Rook Algebra

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Abstract

Colored planar rook algebra is a semigroup algebra in which the basis element has a diagrammatic description. The category of finite dimensional modules over this algebra is completely reducible and suitable functors are defined on this category so that it admits a crystal structure in the sense of Kashiwara. We show that the category and functors categorify the crystal bases for the polynomial representations of quantized enveloping algebra $U_q(gl_{n+1})$.

1 Introduction

Let $gl_{n+1}(\mathbb{C})$ be the general Linear Lie algebra and let $\mathfrak{h}$ be the Cartan subalgebra of $gl_{n+1}(\mathbb{C})$ consisting of all diagonal $(n + 1) \times (n + 1)$ matrices over $\mathbb{C}$. Take $e_i \in \mathfrak{h}^*$ such that $e_i(E_{jj}) = \delta_{ij}$ for $1 \leqslant i \leqslant n + 1$. We denote by $P$ and $Q$ the weight lattice and the root lattice respectively, i.e.

\[ P = \{ \mu = \sum_{i=1}^{n+1} \mu_i e_i \mid \mu_i \in \mathbb{Z} \}, \]

\[ Q = \{ \sum_{i=1}^{n} \mu_i (e_i - e_{i+1}) \mid \mu_i \in \mathbb{Z} \}. \]

A set $B$ with maps $(\text{wt}, \tilde{e}_i, \tilde{f}_i, e_i, \varphi_i)$

\[ \text{wt} : B \rightarrow P, \]

\[ e_i, \varphi_i : B \rightarrow \mathbb{Z} \cup \{-\infty\}, \]

\[ \tilde{e}_i, \tilde{f}_i : B \rightarrow B \cup \{0\} \]

for $1 \leqslant i \leqslant n$, is called a $gl_{n+1}(\mathbb{C})$-crystal if the following axioms are satisfied:
(a) For $b \in B$, $\varphi_i(b) = e_i(b) + \text{wt}(b)(E_{ii} - E_{i+1,i+1})$.

(b) For $b \in B$ with $\tilde{e}_ib \in B$, $\text{wt}(\tilde{e}_ib) = \text{wt}(b) + e_i - e_{i+1}$;
   For $b \in B$ with $\tilde{f}_ib \in B$, $\text{wt}(\tilde{f}_ib) = \text{wt}(b) - e_i + e_{i+1}$.

(c) For $b_1, b_2 \in B$, $\tilde{f}_ib_2 = b_1$ if and only if $\tilde{e}_ib_1 = b_2$.

(d) If $e_i(b) = -\infty$, then $\tilde{e}_ib = \tilde{f}_ib = 0$.

The crystal graph associated to $B$ is a graph with the set of vertices $B$ and there is an
arrow colored by $i$ from $b_1 \in B$ to $b_2 \in B$ if $\tilde{f}_i(b_1) = b_2$. In a series of works\cite{4,5,6,7,8}, Kashiwara developed the crystal base theory to study the representation and
the structure of the quantized enveloping algebra $U_q(g)$ associated to a Kac-Moody
algebra $g$. For any irreducible highest weight integrable $U_q(g)$-module $V$, Kashiwara
found a basis for $V$ which admits a $g$-crystal structure when $q = 0$. Take $g = gl_{n+1}(\mathbb{C})$
as an example. If $V_1$ and $V_2$ are polynomial $U_q(gl_{n+1})$-modules with crystal bases $B_1$
and $B_2$, it is known that $V_1 \otimes V_2$ has a crystal basis $B_1 \otimes B_2 = \{b_1 \otimes b_2 \mid b_i \in B_i\}$ with

\[
\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} 
\tilde{e}_i(b_1) \otimes b_2 & \text{if } \varphi_i(b_1) \geq e_i(b_2), \\
b_1 \otimes \tilde{e}_i(b_2) & \text{if } \varphi_i(b_1) < e_i(b_2), 
\end{cases}
\]

\[
\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} 
\tilde{f}_i(b_1) \otimes b_2 & \text{if } \varphi_i(b_1) > e_i(b_2), \\
b_1 \otimes \tilde{f}_i(b_2) & \text{if } \varphi_i(b_1) \leq e_i(b_2), 
\end{cases}
\]

\[
\text{wt}(b_1 \otimes b_2) = \text{wt}(b_1) + \text{wt}(b_2),
\]

\[
e_i(b_1 \otimes b_2) = \max\{e_i(b_1), e_i(b_2) - \text{wt}(b_1)(E_{ii} - E_{i+1,i+1})\},
\]

\[
\varphi_i(b_1 \otimes b_2) = \max\{\varphi_i(b_1), \varphi_i(b_1) + \text{wt}(b_2)(E_{ii} - E_{i+1,i+1})\}.
\]

Moreover, the decomposition of $V_1 \otimes V_2$ into a direct sum of irreducible submodules
is compatible with the decomposition of $B_1 \otimes B_2$ into connected components. Hence
crystal basis theory works as an important combinatorial tool to study the representation
theory of the quantized enveloping algebra.

Various realizations of the crystal bases are also considered by mathematicians
such as Young tableaux realization, Littelmann’s path model or geometric models
\cite{8,9,10,11,12}. In this paper, by studying the colored planar rook algebra and its
representations, we realize the highest weight crystal basis $B(\lambda)$ for irreducible
polynomial $U_q(g)$-module $V(\lambda)$. The colored planar rook algebra $\mathbb{C}P_m$ is first defined
and studied by Mousley, Schley and Shoemaker in \cite{12} to give an alternative proof
of the recursive formula for multinomial coefficients

\[
\binom{m}{m_0, m_1, \ldots, m_n} = \sum_{i=0}^{n} \binom{m}{m_0, \ldots, m_i - 1, \ldots, m_n}.
\]
They describe all irreducible finite dimensional modules over the colored planar rook algebra $CP_m$ in [12] which generalizes the work of Flath, Halverson and Herbig[1]. The above equality is proved then by taking the dimension of certain $CP_m$-module which restricts to a $CP_{m-1}$-module through an embedding

$$CP_{m-1} \rightarrow CP_m.$$ 

Their work inspires the author to consider other sorts of embeddings from $CP_{m-1}$ to $CP_m$ which help to define suitable functors on the category of $CP_m$-modules and finally realize the crystal.

We organize the paper as follows. In section 2, we recall the definition of the the colored planar rook algebra and summarize the main results in [12]. In section 3, restriction and induction functors are defined through various embeddings $CP_{m-1} \rightarrow CP_{m-1}$ and their properties are investigated. In the last section, we show that the category of finite dimensional $CP_m$-modules categorifies the crystal $B(m\epsilon_1)$, and furthermore, by studying the representations of the tensor product $CP_{\lambda_1} \otimes \cdots \otimes CP_{\lambda_k}$, we realize the crystal $B(\lambda)$.

2 Colored Planar Rook Algebra

We fix a positive integer $n$ and denote by $\mathbb{Z}_2^n$ the direct product of $n$ copies of the ring $\mathbb{Z}_2$. Take $u_i \in \mathbb{Z}_2^n$ whose $i^{th}$ component is 1 and the others are 0. Given a positive integer $m$, let $P_n^m$, or $P_m$ for simplicity, denote the set consisting of all $m \times m$ matrices over $\mathbb{Z}_2^n$ which satisfies the following conditions,

(a) there is at most one non-zero element from the set $\{u_i \mid 1 \leq i \leq n\}$ in each row and each column.

(b) there is no $2 \times 2$ submatrix of the following form for any $1 \leq i \leq n$,

$$\begin{pmatrix} 0 & u_i \\ u_i & 0 \end{pmatrix}.$$ 

For example, the matrix $B$ below is an element in $P_5$,

$$B = \begin{pmatrix} 0 & u_1 & 0 & 0 & 0 \\ u_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & u_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & u_2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$
It can be easily checked that $P_m$ is a semigroup under matrix multiplication. To each element $A = (a_{ij})_{m \times m}$ in $P_m$, one can associate a diagram consisting of two rows of $m$ vertices, where the $i$th vertex in the top row and the $j$th vertex in the bottom row are connected by an edge colored with $1 \leq k \leq n$ if $a_{ij} = u_k$. Thus the diagram corresponding to $B$ above is as follows, where red and blue are used for the color 1 and 2 respectively.

![Figure 1: the planar diagram associated to $B$](image)

It is required by (b) that the edges with the same color cannot cross in such so-called planar diagrams. We also mention that given 2 planar diagrams $d_1$ and $d_2$, the product $d_1d_2$ is the diagram obtained by stacking $d_1$ on top of $d_2$, and then removing the concatenation of 2 edges with different colors or the edges connecting to isolated vertices. See that $d_1d_2$ is still planar and the associated matrix is exactly the product $D_1D_2$ of two matrices where $D_i \in P_m$ corresponds to the diagram $d_i$ for $i = 1, 2$. Thus we do not distinguish the matrices in $P_m$ and the planar diagrams hereafter.

The $n$ colored planar rook algebra $\mathbb{C}P^m_n$, or $\mathbb{C}P^m$, is then defined to be the semigroup algebra associated to $P_m$, i.e. it consists of all formal expressions

$$\sum_{d \in P_m} a_d d$$

where $a_d \in \mathbb{C}$ and the multiplication is given by linearly extending the product in $P_m$. Note that there is an embedding of algebras from $\mathbb{C}P^m_1 \otimes \mathbb{C}P^m_2$ to $\mathbb{C}P^m_{m_1+m_2}$ through which, for $d_i \in P_m$, we regard $d_1 \otimes d_2$ as the planar diagram in $P_{m_1+m_2}$ by putting $d_1$ and $d_2$ next to each other. For $1 \leq i \leq n$, let $I_i$ denote the diagram in $P_1$ consisting of two vertices connected by an edge colored $i$ and let $I_0 \in P_1$ be the single pair of isolated vertices. Then the identity element in $\mathbb{C}P^m$ is

$$e_m = e_1 \otimes \cdots \otimes e_1,$$

where $e_1 = \sum_{i=1}^n I_i - (n - 1)I_0 \in \mathbb{C}P_1$.

In this paper, we consider only (left) $\mathbb{C}P^m$-modules, or called representations of $\mathbb{C}P^m$ which are unital, i.e. given a $\mathbb{C}P^m$-module $M$, $e_m x = x$ for all $x \in M$. To decompose the regular representation of $\mathbb{C}P^m$, a new basis for the algebra is given as
the following

\[ x_d = \sum_{d' \subseteq d} (-1)^{\text{size}(d) - \text{size}(d')} d' \in \mathbb{C}P_m \mid d \in P_m, \]

where \( d' \subseteq d \) means \( d' \) is a sub-diagram of \( d \) obtained by deleting some of the edges in \( d \) and \( \text{size}(d) \) denotes the number of edges in \( d \).

For \( d \in P_m \) and \( 1 \leq i \leq n \), let \( \tau_i(d) \) (resp. \( \beta_i(d) \)) be the set of all vertices in the top (resp. bottom) row of \( d \) which is connected by an edge colored \( i \). Also we denote by \( \tau_0(d) \) (resp. \( \beta_0(d) \)) the set of all isolated top (resp. bottom) vertices in \( d \). Set \( \tau(d) = (\tau_0(d), \tau_1(d), \cdots, \tau_n(d)) \) and \( \beta(d) = (\beta_0(d), \beta_1(d), \cdots, \beta_n(d)) \). Taking \( B \in \mathbb{P}_2 \) in Figure 1 for example, one has

\[ \tau(B) = ([5], [1, 3], [2, 4]), \quad \beta(B) = ([4], [2, 3], [1, 5]). \]

Let \( X_m = \{ \tau(d) \mid d \in P_m \} \). For \( T = (T_0, \cdots, T_n), S = (S_0, \cdots, S_n) \in X_m \), we say that \( T \subseteq S \) if \( T_i \subseteq S_i \) for all \( 1 \leq i \leq n \).

**Proposition 2.1.** (\([12] \)) For \( d, d' \in P_m \),

\[
\begin{align*}
    d' \cdot x_d &= \begin{cases}
        x_{d'd} & \text{if } \tau(d) \subseteq \beta(d'), \\
        0 & \text{otherwise},
    \end{cases}\\
    x_{d'd} \cdot d &= \begin{cases}
        x_{d'd} & \text{if } \beta(d') \subseteq \tau(d), \\
        0 & \text{otherwise},
    \end{cases}\\
    x_{d'd} \cdot x_d &= \begin{cases}
        x_{d'd} & \text{if } \beta(d') = \tau(d), \\
        0 & \text{otherwise}.
    \end{cases}
\end{align*}
\]

For \( T \in X_m \), let \( W^m_T \) denote the subspace of \( \mathbb{C}P_m \) spanned by all \( x_d \) with \( \beta(d) = T \). The following theorem follows easily from Proposition 2.1.

**Theorem 2.2.** (\([12] \))

1. For \( T \in X_m \), \( W^m_T \) is an irreducible \( \mathbb{C}P_m \)-module. Moreover, \( \mathbb{C}P_m \) is semisimple, i.e. the regular module \( \mathbb{C}P_m = \bigoplus_{T \in X_m} W^m_T \).

2. For \( T, T' \in X_m \), \( W^m_T \cong W^m_{T'} \), as \( \mathbb{C}P_m \)-modules if and only if \( |T_i| = |T'_i| \) for all \( 0 \leq i \leq n \).

For \( d \in P_m \), we define \( \omega(d) \) to be the planar diagram in \( P_m \) by flipping \( d \) about the horizontal axis, i.e. \( \omega(d) \) is the transpose of \( d \) when viewed as a matrix. We denote by \( \omega \) the linear transformation of \( \mathbb{C}P_m \) by linearly extending the map \( d \mapsto \omega(d) \). One can check that \( \omega \) is an anti-involution and

\[
\omega(d_1 \otimes d_2) = \omega(d_1) \otimes \omega(d_2), \quad \omega(x_d) = x_{\omega(d)}. \tag{1}
\]
3 Restriction and Induction Functors

Let $\mathcal{C}_m$ denote the category of finite dimensional $\mathbb{C}P_m$-modules. By the well-known Artin-Wedderburn theorem and Theorem 2.2, $\mathcal{C}_m$ is completely reducible and each irreducible $\mathbb{C}P_m$-module is isomorphic to some $W^m_T$. For $T \in X_m$, let

$$T = ([1, \ldots, m_0], [m_0 + 1, \ldots, m_0 + m_1], \ldots, [\sum_{i=0}^{n-1} m_i, \ldots, m]) \in X_m$$

where $m_i = |T_i|$ for $0 \leq i \leq n$. Then $W^m_T$, or denoted by $W^m_{m_0, \ldots, m_n}$, can be naturally chosen as a representative of the isomorphism class $[W^m_T]$.

Apart from the embedding given in [12], we consider, for each $1 \leq i \leq n$, the embedding of algebras

$$\psi_i : \mathbb{C}P_{m-1} \hookrightarrow \mathbb{C}P_m$$

which takes $x \in \mathbb{C}P_{m-1}$ to $x \otimes (I_i - I_0) \in \mathbb{C}P_m$. Since $I_i - I_0$ is an idempotent in $\mathbb{C}P_1$, the map $\psi_i$ indeed preserves the product and we note also that $\psi_i$ is not unital, i.e. the image of the identity element $e_{m-1}$ is not $e_m$ but an idempotent $e_{m-1} \otimes (I_i - I_0)$ instead.

Given a $\mathbb{C}P_m$-module $M$ in $\mathcal{C}_m$, it is equivalent to say that there is a homomorphism of algebras

$$\phi : \mathbb{C}P_m \rightarrow \text{End}_\mathbb{C}(M).$$

The composition of $\phi$ and $\psi_{i,m}$ is then again a homomorphism of algebras

$$\phi \circ \psi_{i,m} : \mathbb{C}P_{m-1} \rightarrow \text{End}_\mathbb{C}(M).$$

Mention that we cannot regard $M$ as a $\mathbb{C}P_{m-1}$-module through this map, since it is not unital. To obtain a $\mathbb{C}P_{m-1}$-module, it needs a little modification. We denote the idempotent $e_{m-1} \otimes (I_i - I_0)$ by $e$ for simplicity. Then $\mathbb{C}P_{m-1}$ is actually embedded into $e\mathbb{C}P_m e$ by $\psi_{i,m}$, i.e.

$$\overline{\psi}_{i,m} : \mathbb{C}P_{m-1} \hookrightarrow e\mathbb{C}P_m e, \quad x \mapsto x \otimes (I_i - I_0).$$

Here $e$ becomes the identity element of the subalgebra $e\mathbb{C}P_m e$ and hence $\overline{\psi}_{i,m}$ is a unital homomorphism of algebras. See that $eM$ is an $e\mathbb{C}P_m e$-module, i.e. there is a homomorphism of algebras

$$\overline{\phi} : e\mathbb{C}P_m e \rightarrow \text{End}_\mathbb{C}(eM).$$

Composing $\overline{\psi}_{i,m}$ and $\overline{\phi}$, one has

$$\overline{\phi} \circ \overline{\psi}_{i,m} : \mathbb{C}P_{m-1} \rightarrow \text{End}_\mathbb{C}(eM),$$

where $m_i = |T_i|$ for $0 \leq i \leq n$. Then $W^m_T$, or denoted by $W^m_{m_0, \ldots, m_n}$, can be naturally chosen as a representative of the isomorphism class $[W^m_T]$. 

Apart from the embedding given in [12], we consider, for each $1 \leq i \leq n$, the embedding of algebras

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Here $e$ becomes the identity element of the subalgebra $e\mathbb{C}P_m e$ and hence $\overline{\psi}_{i,m}$ is a unital homomorphism of algebras. See that $eM$ is an $e\mathbb{C}P_m e$-module, i.e. there is a homomorphism of algebras

$$\overline{\phi} : e\mathbb{C}P_m e \rightarrow \text{End}_\mathbb{C}(eM).$$

Composing $\overline{\psi}_{i,m}$ and $\overline{\phi}$, one has

$$\overline{\phi} \circ \overline{\psi}_{i,m} : \mathbb{C}P_{m-1} \rightarrow \text{End}_\mathbb{C}(eM).$$
through which $eM$ can be seen as a $\mathbb{C}P_{m-1}$-module. The restriction functor $Res_{i,m}$ is then defined as follows,

$$Res_{i,m} : \mathcal{C}_m \rightarrow \mathcal{C}_{m-1}$$

which takes each $M \in ob(\mathcal{C}_m)$ to $Res_{i,m}(M) \triangleq eM = e_{m-1} \otimes (I_i - I_0)M$. Restricting each homomorphism $f : M \rightarrow N$ of $\mathbb{C}P_m$-modules to $eM$, one obtains a homomorphism of $\mathbb{C}P_{m-1}$-modules

$$Res_{i,m}(f) : eM \rightarrow eN.$$

**Lemma 3.1.** For $d \in P_m$ and $1 \leq i \leq n$,

$$\left(e_{m-1} \otimes (I_i - I_0)\right) \cdot x_d = \begin{cases} x_d & \text{if } m \in \tau_i(d), \\ 0 & \text{otherwise}. \end{cases} \quad (2)$$

*Proof.* Suppose $m \in \tau_i(d)$. See that $e_{m-1} \otimes I_0$ is a linear combination of planar diagrams of the form

$$I_{i_1} \otimes \cdots \otimes I_{i_{m-1}} \otimes I_0,$$

where $0 \leq i_j \leq n$ for $1 \leq j \leq m - 1$. Let $d' = I_{i_1} \otimes \cdots \otimes I_{i_{m-1}} \otimes I_0 \in P_m$ Clearly $\tau(d') = \beta(d')$ and $m \in \beta_0(d')$. Hence $m \notin \beta_i(d') \neq \tau_i(d)$, which implies by Proposition 2.1 that

$$d' x_d = 0.$$

Then we have $(e_{m-1} \otimes I_0) \cdot x_d = 0$. It can be similarly proved that $(e_{m-1} \otimes I_j) \cdot x_d = 0$ for $1 \leq j \leq n$ with $j \neq i$. Since $e_m = e_{m-1} \otimes (\sum_{k=1}^n I_k - (n - 1)I_0)$, then we have

$$e_m \cdot x_d = (e_{m-1} \otimes I_i) \cdot x_d = x_d.$$

Hence $(e_{m-1} \otimes (I_i - I_0)) \cdot x_d = x_d$.

If $m \notin \tau_i(d)$, then $m \in \tau_j(d)$ for some $0 \leq j \leq n$ with $j \neq i$. First assume $j \neq 0$. One can prove similarly as above that

$$(e_{m-1} \otimes I_i) \cdot x_d = (e_{m-1} \otimes I_0) \cdot x_d = 0,$$

and the conclusion follows. Second, assume that $j = 0$. We write $e_{m-1} \otimes (I_i - I_0)$ as a linear combination of vectors of the form

$$I_{i_1} \otimes \cdots \otimes I_{i_{m-1}} \otimes (I_i - I_0).$$

It is easy to check that $\beta(I_{i_1} \otimes \cdots \otimes I_{i_{m-1}} \otimes I_i) \supseteq \tau(d)$ if and only if $\beta(I_{i_1} \otimes \cdots \otimes I_{i_{m-1}} \otimes I_0) \supseteq \tau(d)$. By Proposition 2.1 and the fact $(I_{i_1} \otimes \cdots \otimes I_{i_{m-1}} \otimes I_i) \cdot d = (I_{i_1} \otimes \cdots \otimes I_{i_{m-1}} \otimes I_0) \cdot d$, one has

$$(I_{i_1} \otimes \cdots \otimes I_{i_{m-1}} \otimes I_i) \cdot x_d = (I_{i_1} \otimes \cdots \otimes I_{i_{m-1}} \otimes I_0) \cdot x_d,$$

which completes the proof. \qed
It follows from Lemma 3.1 that for $T \in X_m$ and $1 \leq i \leq n$, 

$$
\text{Res}_{i,m}(W^m_T) = \mathbb{C}\text{-span}\{e_{x,\beta} | d \in P_m, \beta(d) = T\}
= \mathbb{C}\text{-span}\{x_d | d \in P_m, \beta(d) = T, m \in \tau_i(d)\}.
$$

**Theorem 3.2.** For $T \in X_m$ and $1 \leq i \leq n$, $\text{Res}_{i,m}(W^m_T) = 0$ if $T_i = \emptyset$, otherwise, there is an isomorphism of $\mathbb{C}P_{m-1}$-modules

$$\text{Res}_{i,m}(W^m_T) \cong W^{m-1}_{m_0, m_1, \ldots, m_{i-1}, m_i, \ldots, m_n},$$

where $m_i = |T_i|$.

**Proof.** It is obvious for the case $T_i = \emptyset$. Hence we assume that $T_i \neq \emptyset$. Note that $T$ corresponds to a row of $m$ vertices colored by the set $\{0, 1, \ldots, n\}$. Let $\hat{T} \in X_{m-1}$ correspond to the row of $m - 1$ vertices obtained by deleting the rightmost vertex colored by $i$. For $d \in P_m$ with $\beta(d) = T$ and $m \in \tau_i(d)$, we denote by $\hat{d}$ the planar diagram in $P_{m-1}$ by deleting the rightmost edge colored by $i$ and the two vertices connected. Observe that $\beta(\hat{d}) = \hat{T}$ and

$$\tau(\hat{d}) = (\tau_0(d), \ldots, \tau_i(d) \setminus \{m\}, \ldots, \tau_n(d)).$$

Linearly extending the bijection $\gamma : x_d \mapsto x_{\hat{d}}$, we have an isomorphism of $\mathbb{C}$-vector spaces

$$
\gamma : \text{Res}_{i,m}(W^m_T) \longrightarrow W^{m-1}_{\hat{T}}.
$$

Since $W^{m-1}_{\hat{T}} \cong W^{m-1}_{m_0, m_1, \ldots, m_{i-1}, m_i, \ldots, m_n}$ by Theorem 2.2, it remains to prove that $\gamma$ commutes with the action of $\mathbb{C}P_{m-1}$. For $d' \in P_{m-1}$ and $d \in P_m$ with $\beta(d) = T, m \in \tau_i(d)$,

$$
\gamma(d'x_d) = \gamma(\psi_{i,m}(d') \cdot x_d)
= \gamma((d' \otimes (I_i - I_0)) \cdot x_d)
= \gamma((d' \otimes I_i) \cdot x_d),
$$

where the third equality follows from the fact $(d' \otimes I_0) \cdot x_d = 0$. See that $\beta(d' \otimes I_i) \not\supseteq \tau(d)$ if and only if $\beta(d') \not\supseteq \tau(\hat{d})$. Hence if $\beta(d' \otimes I_i) \not\supseteq \tau(\hat{d})$, then

$$
\gamma(d'x_d) = \gamma(0) = 0 = d' \cdot x_{\hat{d}} = d' \cdot \gamma(x_d).
$$

If $\beta(d' \otimes I_i) \supseteq \tau(d)$, one has

$$
\gamma(d'x_d) = \gamma(x_d \otimes I_i) \cdot x_{\hat{d}} = x_{d' \otimes I_i} \cdot x_{\hat{d}} = d' \gamma(x_d).
$$

\[ \square \]
Lemma 3.4. where the fourth equality is implied by Lemma 3.3.

Next, for any embedding $\psi_{l,m} : \mathbb{C}P_{m-1} \rightarrow \mathbb{C}P_m$, we define an induction functor

$$\text{Ind}_{l,m} : \mathcal{C}_{m-1} \rightarrow \mathcal{C}_m$$

by $\text{Ind}_{l,m}(M) = \mathbb{C}P_m \otimes_{\mathbb{C}P_{m-1}} M$ for $M \in \text{ob}(\mathcal{C}_{m-1})$. Here

$$\text{Ind}_{l,m}(M) = \mathbb{C}P_m(e + (e_m - e)) \otimes_{\mathbb{C}P_{m-1}} M$$

$$= \mathbb{C}P_m e \otimes_{\mathbb{C}P_{m-1}} M + \mathbb{C}P_m(e - e_m) \otimes_{\mathbb{C}P_{m-1}} M$$

$$= \mathbb{C}P_m e \otimes_{\mathbb{C}P_{m-1}} M + \mathbb{C}P_m(e - e_m) e \otimes_{\mathbb{C}P_{m-1}} e_m M$$

$$= \mathbb{C}P_m e \otimes_{\mathbb{C}P_{m-1}} M + \mathbb{C}P_m(e - e_m) e \otimes_{\mathbb{C}P_{m-1}} M$$

$$= \mathbb{C}P_m e \otimes_{\mathbb{C}P_{m-1}} M,$$

where $e = e_m \otimes (I_l - I_0)$.

Lemma 3.3. For $d \in P_m$ and $1 \leq i \leq n$,

$$x_d \cdot (e_{m-1} \otimes (I_l - I_0)) = \begin{cases} x_d & \text{if } m \in \beta_i(d), \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

Proof. The result is straightforward by applying $\omega$ to [2].

Suppose $M = W_T^{m-1}$ for some $T \in X_{m-1}$. Let $d_T$ be the unique planar diagram with $\tau(d_T) = \beta(d_T) = T$. We have $x_{d_T} \in W_T^{m-1}$ and

$$W_T^{m-1} = \mathbb{C}P_{m-1} x_{d_T}$$

since $W_T^{m-1}$ is irreducible. Hence

$$\text{Ind}_{l,m}(W_T^{m-1}) = \mathbb{C}P_m e \otimes_{\mathbb{C}P_{m-1}} W_T^{m-1}$$

$$= \mathbb{C}P_m e \otimes_{\mathbb{C}P_{m-1}} \mathbb{C}P_{m-1} x_{d_T}$$

$$= \mathbb{C}P_m e \otimes_{\mathbb{C}P_{m-1}} x_{d_T}$$

$$= \mathbb{C}\text{-span}\{x_d \otimes_{\mathbb{C}P_{m-1}} x_{d_T} | d \in P_m, m \in \beta_i(d)\},$$

where the fourth equality is implied by Lemma 3.3.

Lemma 3.4. For $T \in P_m$ and $1 \leq i \leq n$,

$$x_{d_T} \otimes (I_l - I_0) = x_{d_T} \otimes I_l.$$
Proof. For \( d' \subseteq d_T \otimes I_i \), \( d' = d_1 \otimes I_i \) or \( d' = d_2 \otimes I_0 \) where \( d_1, d_2 \subseteq d_T \). Thus

\[
x_{d_T} \otimes I_i = \sum_{d' \subseteq d_T \otimes I_i} (-1)^{\text{size}(d_T \otimes I_i, d')} d'
\]

\[
= \sum_{d_1 \subseteq d_T} (-1)^{\text{size}(d_T, d_1)} d_1 \otimes I_i + \sum_{d_2 \subseteq d_T} (-1)^{\text{size}(d_T, d_2)+1} d_2 \otimes I_0
\]

\[
= \sum_{d_1 \subseteq d_T} (-1)^{\text{size}(d_T, d_1)} d_1 \otimes I_i - \sum_{d_1 \subseteq d_T} (-1)^{\text{size}(d_T, d_1)} d_1 \otimes I_0
\]

\[
= \sum_{d_1 \subseteq d_T} (-1)^{\text{size}(d_T, d_1)} d_1 \otimes (I_i - I_0)
\]

\[
x_{d_T} \otimes (I_i - I_0),
\]

where we write \( \text{size}(d_1, d_2) \) for \( \text{size}(d_1) - \text{size}(d_2) \). \( \square \)

By Proposition 2.1, \( x_{d_T} \cdot x_{d_T} = x_{d_T} \) for \( T \in X_{m-1} \). Then we have, for \( d \in P_m \) with \( m \in \beta_i(d) \),

\[
x_d \otimes \mathbb{C}P_{m-1} x_{d_T} = x_d \otimes \mathbb{C}P_{m-1} x_{d_T} x_{d_T}
\]

\[
= x_d \psi_{i,m}(x_{d_T}) \otimes \mathbb{C}P_{m-1} x_{d_T}
\]

\[
= x_d x_{d_T} \otimes \mathbb{C}P_{m-1} x_{d_T}
\]

\[
= \delta_{\beta(d), \tau(d_T \otimes I_i)} x_d \otimes \mathbb{C}P_{m-1} x_{d_T}
\]

\[
= \delta_{\beta(d), \tau(d_T \otimes I_i)} x_d \otimes \mathbb{C}P_{m-1} x_{d_T},
\]

which implies that

\[
\text{Ind}_{i,m}(W^{m-1}_T) = \mathbb{C}\text{-span}\{x_d \otimes \mathbb{C}P_{m-1} x_{d_T} \mid d \in P_m, \beta(d) = \overbar{T}^i\},
\]

where \( \overbar{T}^i = (T_0, \cdots, T_i \cup \{m\}, \cdots, T_n) \in X_m \). The following theorem is then an immediate consequence.

**Theorem 3.5.** For \( T \in X_{m-1} \) and \( 1 \leq i \leq n \), \( \text{Ind}_{i,m}(W^{m-1}_T) \cong W^m_{\overbar{T}^i} \) as \( \mathbb{C}P_m \)-modules.

For \( M \in \text{ob}(\mathcal{C}_m) \), we define \( \text{Res}_{0,m}(M) = (e_{m-1} \otimes I_0)M \) and restrict the action of \( \mathcal{C}_m \) to \( \mathcal{C}_{m-1} \) on \( \text{Res}_{0,m}(M) \), through the embedding of algebras

\[
\psi_{0,m} : \mathbb{C}P_{m-1} \longrightarrow \mathbb{C}P_m
\]

which takes \( x \) to \( x \otimes I_0 \). This gives a restriction functor

\[
\text{Res}_{0,m} : \mathcal{C}_m \longrightarrow \mathcal{C}_{m-1}.
\]

Via the embedding \( \psi_{0,m} \), an induction functor \( \text{Ind}_{0,m} : \mathcal{C}_{m-1} \longrightarrow \mathcal{C}_m \) is also given by \( \text{Ind}_{0,m}(N) = \mathbb{C}P_m \otimes_{\mathbb{C}P_{m-1}} N \). Similarly one can show the following theorem.
Theorem 3.6.

(1) For $T \in X_m$, $Res_{0,m}(W^m_T) = 0$ if $T_0 = \emptyset$; otherwise $Res_{0,m}(W^m_T) \cong W_{m_{0-1},m_1,\ldots,m_n}$ as $\mathbb{C}P_{m-1}$-modules.

(2) For $S \in X_{m-1}$, $Ind_{0,m}(W_{S_{m-1}}) \cong W_{m_{0+1},m_1,\ldots,m_n}$ as $\mathbb{C}P_m$-modules, where $m_i = |S_i|$.

It is easy to show that for $0 \leq i \leq n$, $Ind_{i,m}$ is left adjoint to $Res_{i,m}$, i.e.

$$\text{Hom}_{\mathbb{C}P_m}(Ind_{i,m}(M), N) \cong \text{Hom}_{\mathbb{C}P_{m-1}}(M, Res_{i,m}(N)),$$

which is natural in $M \in ob(\mathcal{C}_m)$ and $N \in ob(\mathcal{C}_m)$.

4 Crystal Structure

4.1 Crystal Bases for $U_q(gl_n)$-modules

Fix an indeterminant $q$. The quantized enveloping algebra associated to $gl_{n+1} \mathbb{C}$, denoted by $U_q(gl_{n+1})$, is a $\mathbb{C}(q)$-algebra generated by $E_i$, $F_i$ and $q^h$ for $1 \leq i \leq n$ and $h \in \mathfrak{h}$. We omit the generating relations of $U_q(gl_{n+1})$ and refer the readers to [3] for more details.

The representation theory of $U_q(gl_{n+1})$ is paralleled to that of $gl_{n+1} \mathbb{C}$. It is known that finite dimensional irreducible polynomial representation of $U_q(gl_{n+1})$, or $U_q(gl_{n+1})$-modules, are indexed by the set of partitions of the form

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{n+1}),$$

where $\lambda_i \in \mathbb{N}$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n+1} \geq 0$. Indeed, every finite dimensional irreducible $U_q(gl_{n+1})$-module $V$ has a decompositions of weight spaces

$$V = \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu,$$

where $V_\mu = \{ v \in V \mid q^h v = \lambda(h)v \}$. Furthermore, there is a highest weight $\lambda$ among all $\mu$’s such that $V_\mu \neq 0$ in the following sense,

$$\lambda - \mu \in \sum_{i=1}^n \mathbb{N}(\epsilon_i - \epsilon_{i+1}).$$

The polynomial representations we considered are those whose weights can be written as a combination of $\epsilon_i$ with non-negative integer coefficients. The highest weight $\lambda$ of an irreducible polynomial representation is known to be dominant, i.e.

$$\lambda = \lambda_1 \epsilon_1 + \cdots + \lambda_{n+1} \epsilon_{n+1} \in \mathfrak{h}^*,$$
where $\lambda_i \in \mathbb{N}$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n+1} \geq 0$. Hence each dominant weight corresponds to a partition, or, a Young diagram and we do not distinguish them. We use $V(\lambda)$ to denote the finite dimensional irreducible polynomial $U_q(gl_{n+1})$-module of highest weight $\lambda$. It is well known that the basis vectors of $V(\lambda)$ can be parameterized by semistandard Young tableaux of shape $\lambda$. The crystal basis theory developed by Kashiwara not only reinterprets this classical result, but also give a combinatorial explanation to the Littlewood-Richardson rule. Indeed, one can realize the crystal graph of $B(\lambda)$ with semistandard Young tableaux of shape $\lambda$. For example, when $\lambda = e_1$, $V(e_1)$ is the natural $U_q(gl_{n+1})$-module of dimension $n+1$ with the following crystal graphs

\[
\begin{array}{cccccc}
0 & \overset{\tilde{f}_1}{\rightarrow} & 1 & \overset{\tilde{f}_2}{\rightarrow} & 2 & \cdots \overset{\tilde{f}_n}{\rightarrow} n,
\end{array}
\]

where $\text{wt}(T) = e_{j+1}$, $e_i(T) = \delta_{i,j}$ and $\varphi_i(T) = \delta_{i,j+1}$. Mention that for convenience, the Young tableaux in this paper are Young diagrams filled with numbers in $\{0, 1, \cdots, n\}$ instead of those in $\{1, \cdots, n+1\}$ though the latter is more often used. Since every irreducible polynomial $U_q(gl_{n+1})$-module $V(\lambda)$ is a direct summand of $V(e_1) \otimes \lambda$, there is an embedding of crystals from $B(\lambda)$ to $B(e_1) \otimes \lambda$. Note that usually the ways of embedding are not unique, among which we choose the following one, called the Middle-Eastern reading,

$$\Psi : B(\lambda) \longrightarrow B(e_1) \otimes \lambda,$$

which maps a semistandard Young tableau $T \in B(\lambda)$ to the tensor product of boxes in $T$ from right to left in each row and from top to bottom. For instance,

$$\Psi(\begin{array}{ccc}
0 & 1 & 1 \\
2 & 3 \\
3
\end{array}) = 3 \otimes 1 \otimes 1 \otimes 0 \otimes 3 \otimes 2 \otimes 3.$$

Lifting the actions of Kashiwara operators $\tilde{e}_i$ and $\tilde{f}_i$ on $B(e_1) \otimes \lambda$ to those on $B(\lambda)$, one obtains a crystal structure on $B(\lambda)$.

### 4.2 $B(m e_1)$

In this subsection, we categorify the crystal basis $B(\lambda)$ for $\lambda = m e_1$ via representation theory of colored planar rook algebras. In this case, $\lambda$ can be viewed as a Young diagram which has only one row of $m$ boxes. As discussed in last subsection, we can realize $B(m e_1)$ with semistandard Young tableaux of shape $(m, 0, \cdots, 0)$.

We remind the readers that in Section 3, for $0 \leq i \leq n$, two functors

$$\text{Res}_{i,m} : \mathcal{C}_m \longrightarrow \mathcal{C}_{m-1},$$
are introduced. We denote by \([\mathcal{C}_m]\) the set of all isomorphism classes of irreducible objects in \(\mathcal{C}_m\), i.e.

\[
[\mathcal{C}_m] = \{[W^m_{n_0,m_1,\ldots,m_n}] \mid m \geq 0, \sum_{i=0}^n m_i = m\}.
\]

Then there are two maps induced by the functors \(\text{Res}_{i,m}\) and \(\text{Ind}_{i,m}\),

\[
[\text{Res}_{i,m}] : [\mathcal{C}_m] \longrightarrow [\mathcal{C}_{m-1}] \cup \{0\},
\]

\[
[\text{Ind}_{i,m}] : [\mathcal{C}_{m-1}] \longrightarrow [\mathcal{C}_m].
\]

For \(1 \leq i \leq n\), we define two functors \(\tilde{\text{e}}_{i,m}, \tilde{\text{f}}_{i,m} : \mathcal{C}_m \longrightarrow \mathcal{C}_m\)

\[
\tilde{\text{e}}_{i,m} = \text{Res}_{i,m+1} \circ \text{Ind}_{i-1,m+1},
\]

\[
\tilde{\text{f}}_{i,m} = \text{Res}_{i-1,m+1} \circ \text{Ind}_{i,m+1}.
\]

Let \(\tilde{e}_{i,m}\) and \(\tilde{f}_{i,m}\) be the maps induced by \(\tilde{e}_{i,m}\) and \(\tilde{f}_{i,m}\) respectively, i.e.

\[
\tilde{e}_{i,m} = [\tilde{e}_{i,m}] : [\mathcal{C}_m] \longrightarrow [\mathcal{C}_m] \cup \{0\},
\]

\[
\tilde{f}_{i,m} = [\tilde{f}_{i,m}] : [\mathcal{C}_m] \longrightarrow [\mathcal{C}_m] \cup \{0\}.
\]

By Theorem 3.2, 3.5 and 3.6 we have the following proposition.

**Proposition 4.1.** For \(1 \leq i \leq n\) and \(m_0, \ldots, m_n \in \mathbb{N}\) with \(m = \sum_{i=0}^n m_i\),

\[
\tilde{e}_{i,m}([W^m_{n_0,m_1,\ldots,m_n}]) = \begin{cases} [W^m_{n_0,\ldots,m_i-1,m_{i-1},\ldots,m_n}] & \text{if } m_i > 0 \\ 0 & \text{if } m_i = 0. \end{cases}
\]

\[
\tilde{f}_{i,m}([W^m_{n_0,m_1,\ldots,m_n}]) = \begin{cases} [W^m_{n_0,\ldots,m_{i-1},m_i-1,\ldots,m_n}] & \text{if } m_{i-1} > 0 \\ 0 & \text{if } m_{i-1} = 0. \end{cases}
\]

We define the weight of \([W^m_{n_0,m_1,\ldots,m_n}]\) to be

\[
m_0 \epsilon_1 + m_1 \epsilon_2 + \cdots + m_n \epsilon_{n+1},
\]

denoted by \(\text{wt}(W^m_{n_0,m_1,\ldots,m_n})\). For \(1 \leq i \leq n\), two maps \(\varepsilon_{i,m}, \varphi_{i,m} : [\mathcal{C}_m] \longrightarrow \mathbb{N}\) are defined by

\[
\varepsilon_{i,m}([W^m_{n_0,m_1,\ldots,m_n}]) = m_i,
\]

\[
\varphi_{i,m}([W^m_{n_0,m_1,\ldots,m_n}]) = m_{i-1}.
\]
Proposition 4.2. $[C_m]$ with $(\text{wt}, \tilde{e}_i, m, \tilde{f}_i, \epsilon_i, \varphi_i, \psi_i)$ is a crystal.

The proposition can be checked directly by Proposition 3.1. It remains to show that $[C_m]$ is isomorphic to $B(me_1)$ as crystals.

We consider the map

$$\rho : [C_m] \rightarrow B(me_1)$$

where the number of $i$'s in the tableau is $m_i$ for $0 \leq i \leq n$. Observe that $\rho$ is a bijection and it preserves the weight. We only show that $\rho$ commutes with the Kashiwara operator $\tilde{e}_i$, i.e.

$$\rho \tilde{e}_i = \tilde{e}_i \rho,$$

(4)

and the case for $\tilde{f}_i$ is similar. The proof proceeds by induction on $m$. It is trivial when $m = 1$ and we assume that (4) holds if $m < k$. Consider the case when $m = k$. If $m_i = 0$, we have

$$\tilde{e}_i([W_{m_0, \ldots, m_n}]) = 0$$

and it only needs to show that

$$\tilde{e}_i(pq \cdots r) = 0$$

(5)

where $\rho^{-1}(r \cdots q \cdot p)$ denotes the image of $[W_{m_0, \ldots, m_n}]$ under $\rho$ with $0 \leq p \leq q \leq \cdots \leq r \leq n$. Through the Middle-Eastern reading, (5) is equivalent to the equality

$$\tilde{e}_i(r \cdot q \cdots p) = 0,$$

(6)

Since there is no $i$ in the tableau $\rho^{-1}(r \cdots q \cdot p)$, we have $p \neq i$ and hence

$$\varphi_i(q \cdots r) \geq \epsilon_i(p) = 0.$$

It follows that

$$\varphi_i(r \cdot q \cdots p) = \varphi_i([W_{m_0, \ldots, m_n}]) = 0 < \epsilon_i(p) = 1.$$

Next, suppose $m_i \geq 0$. In order to prove (4), we only need to show $\tilde{e}_i$ acts on $pq \cdots r$ by changing the leftmost $i$ to $i - 1$. Similarly, we consider the image of $q \cdot r \cdot p$ under $\tilde{e}_i$. If $p = i$, one has $q \geq p = i$ and by assumption,

$$\varphi_i(q \cdot r \cdot p) = \varphi_i([W_{m_0, \ldots, m_n - 1, \ldots, m_n}]) = 0 < \epsilon_i(p) = 1.$$

Hence

$$\tilde{e}_i(q \cdot r \cdot p) = q \cdot r \cdot \tilde{e}_i(p).$$

If $p \neq i$, then $p < i$. It follows that

$$\varphi_i(q \cdot r \cdot p) \geq 0 = \epsilon_i(p),$$

and then

$$\tilde{e}_i(q \cdot r \cdot p) = \tilde{e}_i(q \cdot r \cdot p) \cdot \tilde{e}_i(p)$$

which completes the proof by the assumption.

Theorem 4.3. $[C_m] \simeq B(me_1)$ as crystals.
4.3 Tensor Product of Crystals

Let $\lambda = (\lambda_1, \cdots, \lambda_k)$ be a composition of an integer $l$, i.e.
\[
\sum_{i=1}^{k} \lambda_i = l,
\]
where $\lambda_i$ is a positive integer for $1 \leq i \leq k$. We denote by $\mathbb{CP}_{\lambda}$ the algebra
\[
\mathbb{CP}_{\lambda_1} \otimes \cdots \otimes \mathbb{CP}_{\lambda_k},
\]
which can be viewed as a subalgebra of $\mathbb{CP}_l$. The following result is straightforward.

**Proposition 4.4.** $\mathbb{CP}_{\lambda}$ is semisimple and $\mathbb{CP}_{\lambda}$-modules of the form
\[
W^{\lambda_1}_{\lambda_{10}, \cdots, \lambda_{1n}} \otimes \cdots \otimes W^{\lambda_k}_{\lambda_{k0}, \cdots, \lambda_{kn}}
\]
with $\lambda_i = \sum_{j=0}^{n} \lambda_{ij}$ exhaust all finite dimensional irreducible $\mathbb{CP}_{\lambda}$-modules up to isomorphism.

Let $\mathcal{C}_\lambda$ be the category of finite dimensional $\mathbb{CP}_{\lambda}$-modules and let $[\mathcal{C}_\lambda]$ be the set of all isomorphism classes of irreducible modules in $\mathcal{C}_\lambda$. By Proposition 4.4
\[
[\mathcal{C}_\lambda] = \{ [W^{\lambda_1}_{\lambda_{10}, \cdots, \lambda_{1n}} \otimes \cdots \otimes W^{\lambda_k}_{\lambda_{k0}, \cdots, \lambda_{kn}}] \mid \lambda_i = \sum_{j=0}^{n} \lambda_{ij}, \lambda_{ij} \in \mathbb{N} \}.
\]

We define suitable operators $\tilde{e}_{l,\lambda}$ and $\tilde{f}_{l,\lambda}$ on $[\mathcal{C}_\lambda]$ so that it admits a crystal structure. For $1 \leq i \leq n$, we assign a sequence of $-\prime$s and $+\prime$s to each $[M] = [W^{\lambda_1}_{\lambda_{10}, \cdots, \lambda_{1n}} \otimes \cdots \otimes W^{\lambda_k}_{\lambda_{k0}, \cdots, \lambda_{kn}}]$ as the following
\[
\begin{align*}
\text{the } 1^\text{st} \text{ component} &\quad \begin{array}{c}
\lambda_i \\
\vdots \quad \vdots \quad \vdots \\
\lambda_{i-1} \end{array}, \\
\text{the } k^\text{th} \text{ component} &\quad \begin{array}{c}
\lambda_i \\
\vdots \quad \vdots \quad \vdots \\
\lambda_{i-1} \end{array}.
\end{align*}
\]

(7)

Deleting all $(+, -)$-pairs in the above sequence, we obtain a new sequence of the form
\[
(-, \cdots, -, +, \cdots, +).
\]

(8)

Take $1 \leq s, t \leq k$ such that the rightmost $-$ and the leftmost $+$ in (8) belong to the $s^{th}$ and the $t^{th}$ component in the original sequence (7) respectively. Then define
\[
\tilde{e}_{l,\lambda}([M]) = [W^{\lambda_1}_{\lambda_{10}, \cdots, \lambda_{1n}} \otimes \cdots \otimes \tilde{e}_{l,\lambda_1}(W^{\lambda_2}_{\lambda_{20}, \cdots, \lambda_{2n}}) \otimes \cdots \otimes W^{\lambda_k}_{\lambda_{k0}, \cdots, \lambda_{kn}}],
\]
\[
\tilde{f}_{l,\lambda}([M]) = [W^{\lambda_1}_{\lambda_{10}, \cdots, \lambda_{1n}} \otimes \cdots \otimes \tilde{f}_{l,\lambda_1}(W^{\lambda_2}_{\lambda_{20}, \cdots, \lambda_{2n}}) \otimes \cdots \otimes W^{\lambda_k}_{\lambda_{k0}, \cdots, \lambda_{kn}}].
\]

(9)
Also, we define

\[ \text{wt}(M) = \sum_{j=1}^{k} \text{wt}(W_{\lambda_{j},\lambda_{j-1}}), \] (11)

\[ \varepsilon_{i,\lambda}(M) = \max\{j \geq 0 \mid \tilde{e}^j_{i,\lambda}(M) \neq 0\} , \] (12)

\[ \varphi_{i,\lambda}(M) = \max\{j \geq 0 \mid \tilde{f}^j_{i,\lambda}(M) \neq 0\}. \] (13)

See that there is a natural bijection between \([\mathcal{C}_{\lambda}]\) and the Cartesian product \([\mathcal{C}_{\lambda_1}] \times \cdots \times [\mathcal{C}_{\lambda_k}]\). Since \([\mathcal{C}_{\lambda_i}] \cong B(\lambda_i \varepsilon_1)\) as crystals and the actions of \(\tilde{e}_{i,\lambda}\) and \(\tilde{f}_{i,\lambda}\) on \([\mathcal{C}_{\lambda}]\) coincide with those of \(\tilde{e}_i\) and \(\tilde{f}_i\) on the tensor product \([\mathcal{C}_{\lambda_1}] \otimes \cdots \otimes [\mathcal{C}_{\lambda_k}]\), we have the following theorem.

**Theorem 4.5.** For a composition \(\lambda = (\lambda_1, \cdots, \lambda_k)\) of \(l\), there is an isomorphism of crystals \([\mathcal{C}_{\lambda}] \cong B(\lambda_1 \varepsilon_1) \otimes \cdots \otimes B(\lambda_k \varepsilon_1)\).

Furthermore, if \(\lambda = (\lambda_1, \cdots, \lambda_k)\) is a partition of \(l\) with length \(k \leq n + 1\), with the help of the Middle-Eastern reading, it is not hard to deduce that the connected component of the crystal graph \([\mathcal{C}_{\lambda}]\) which contains \([W_{\lambda_1,0,0,\cdots,0} \otimes W_{\lambda_2,0,0,\cdots,0} \otimes \cdots \otimes W_{\lambda_k,0,0,\cdots,0}]\) is isomorphic to \(B(\lambda)\) as crystals.

**References**

1. D. Flath, T. Halverson, and K. Herbig, *The planar rook algebra and Pascals triangle*, Enseign. Math. (2), 55(1-2):77C92, (2009).

2. W. Fulton, *Young Tableaux*, volume 35 of London Mathematical Society Student Texts. Cambridge University Press, 1997.

3. J. Hong and S.-J. Kang, *Introduction to quantum groups and crystal bases*, Graduate Studies in Mathematics, Vol. 42, Amer. Math. Soc. Providence, RI, (2002).

4. M. Kashiwara, *Crystalizing the q-analogue of universal enveloping algebras*, Comm. Math. Phys. 133 (1990), 249-260.

5. M. Kashiwara, *On crystal bases of the q-analogue of universal enveloping algebras*, Duke. Math. J. 63 (1991), 465-516.

6. M. Kashiwara, *The crystal bases and Littelmann’s refined Demazure character formula*, Duke. Math. J. Vol. 71. No.3 (1993), 839-858.

7. M. Kashiwara, *Global crystal bases of quantum groups*, Duke. Math. J. Vol. 69. No. 2 (1993), 455-485.
8 M. Kashiwara, *Crystal bases of modified enveloping algebra*, Duke. Math. J. Vol. 73, No. 2 (1994), 383-413.

9 M. Kashiwara, *Realizations of crystals*, Combinatorial and geometric representation theory (Seoul, 2001), Contemp. Math. Vol. 325, Amer. Math. Soc., Providence, RI, 2003, 133-139.

10 P. Littelmann, *A Littlewood-Richardson rule for symmetrizable Kac-Moody algebras*, Invent. Math. 116 (1994), 329-346.

11 P. Littelmann, *Path and root operators in representation theory*, Ann. of Math. 142 (1995), No.3, 499-525.

12 S. Mousley, N. Schley and A. Shoemaker, *Planar rook algebra with colors and Pascal’s simplex* [arXiv:1211.0663] 2012.

13 A. Savage, *Geometric and combinatorial realizations of crystal graphs*, Algebr. Represent. Theory 9 (2006), No. 2, 161C199.