Wigner coefficient for Lie algebras of series $B, C, D$ and a base of Gelfand-Tsetlin type.∗

Dedicated to the 100-th anniversary of I.M. Gelfand

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For the Lie algebras $g_n = \mathfrak{o}_{2n+1}, \mathfrak{sp}_{2n}, \mathfrak{o}_{2n}$ a simple construction of a base in an irreducible representation is given. The construction of this base uses the method of $Z$-invariants of Zhelobenko and the technique of Wigner coefficients, which was applied by Biedenharn and Baird to the construction of a Gelfand-Tsetlin base in the case $\mathfrak{gl}_n$. A relation between matrix elements and Wigner coefficients for $g_n$ and analogous objects for $\mathfrak{gl}_{n+1}$ is established.

1 Introduction

One can construct explicitly a representation of a simple Lie algebra of the type $A$ using the technique of Wigner coefficients. In the case $\mathfrak{sl}_2$ such a construction was obtained by Wigner and Racah, and in the case $\mathfrak{sl}_n$ it was obtained by Biedenharn, Baird, Louck and others. The aim of the present paper is a generalization of this technique to the case of simple Lie algebras $g_n$ of types $B, C, D$. Such constructions can be applied in the nuclear physics, for example in the theory of nuclear shells.

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The construction given in the present paper establishes a remarkable relation between Gelfand-Tsetlin bases for the algebras of series $A$ and of series $B, C, D$. Moreover we express matrix elements for generators in the base and Wigner coefficients for algebras of series $B, C, D$ through the analogous objects of series $A$.

The construction in the paper is inductive. For the algebras $g_n = o_{2n+1}, sp_{2n}, o_{2n}$ the constructions are different in the case $n = 1$ and also in the case $n = 2$. But the consideration for $n > 2$ are similar for all algebras and also they are similar to analogous constructions for the algebra $gl_{n+1}$. This fact plays a key role in all constructions. It is a direct corollary of the fact that for the Dynkin diagrams $A_{n+1}, B_n, C_n, D_n$ transform in a similar way when we change $2$ to $n$.

The construction uses a well-known inductive procedure of Gelfand and Tsetlin. The main step in this construction is an investigation of a branching of an irreducible representation of $g_n$ when one restricts the algebra $g_n \downarrow g_{n-1}$. For this investigation we use the method of $Z$-invariants of Zhelobenko.

To obtain formulas for the action of generators of the algebra in the base we use the technique of Wigner coefficients. More precise, the matrix elements are expressed through the simplest Wigner coefficients that correspond to a decomposition of a tensor product of an arbitrary representation with a standard representation of the algebra. Earlier analogous results were obtained by Biedenharn and Baird. They expressed matrix elements for operators $E_{i,i-1}$ in the Gelfant-Tsetlin base for the algebras of series $A$ through Wigner coefficients [1].

I.M. Gelfand in the late 80-th several times said about the desire to generalize the works of Biedenharn and Louck. We have understood this as a problem of a generalization of the construction of Biedenharn and Louck for the Lie algebras of series $A$ to the case of algebras of series $B, C, D$. This is done in the present paper.

The problem of construction of base for orthogonal algebras algebras was investigated in the works of Gelfand and Tsetlin, but it’s construction is
based on restrictions $\mathfrak{o}_N \downarrow \mathfrak{o}_{N-1}$.

In [2] using the method of $Z$-invariants of Zhelobenko there was constructed a Gelfand-Tsetlin type base for orthogonal algebras (based on and for symplectic algebras (see also [3], [4], [5]). But in these papers only an indexation is constructed, the formulas for the action of generators are not obtained.

Nevertheless there exists a construction of a Gelfand-Tsetlin type base for the algebras of series $B, C, D$ which belongs to Molev [6]. But it uses a difficult technique of Mickelsson-Zhelobenko algebras and the action of Jangians on the multiplicity spaces.

The construction given in the present paper is much simpler. It’s main advantage is that it establishes a remarkable relation between Gelfand-Tsetlin bases for series $A, B, C, D$. Also as a by-product of our construction we obtain explicit formulas for Wigner coefficients for algebras of series $B, C, D$. The indexation of base vectors is the same as Molev’s. But also our base is not orthonormal as Molev’s.

The Gelfand-Tsetlin type bases for Lie $g_n$ of types $B_n, C_n, D_n$ based on restrictions $g_n \downarrow g_{n-1}$ are important in nuclear physics. Such a base is used in problems where the algebra of five-dimensional quasi-spin (isomorphic to $\mathfrak{o}_5 = \mathfrak{sp}_4$) is involved, for example in the theory of nuclear shells, [7], in the Bohr-Mottelson model [8], in the model of interacting bosons [9], in the models of high-temperature superconductivity [10], [11], [12].

The typical application of Wigner coefficients is the following. If one identify an irreducible representation with a (quasi)particle, then Wigner coefficients that define a decomposition of a tensor product of two representation describe a spectrum of (quasi)particles that appear after their interaction (см. [13]).

1 Thus this base is not a base for a series $B$ or $D$ since the algebras of both series are involved in it’s construction

2 An indexation of base vectors in the base that is constructed in the present paper is the same as in the Gelfand-Tsetlin-Molev [6]. But we have not managed to construct an explicit isomorphism between our base and the Gelfand-Tsetlin-Molev base.
spond to the adding of one (quasi)particle. Thus in the case of a nuclear shell model the multiplication to the standard representation corresponds to the adding to the system of one nucleon that is a proton or a neutron. The explicit formulas for Wigner coefficients allow to write the selection rulers for the quantum numbers that define the states of quasi-spin and calculate the probabilities of transitions into these states.

There exist also completely different applications. Thus the high-temperature decomposition in the classical $N$-vector model is described using Wigner coefficients of the orthogonal algebra $\mathfrak{o}_N$ [14].

In all these problems mostly the Wigner coefficients that define the decomposition of two symmetric representations are used. Such coefficients were explicitly obtained in many particular cases in [15], [16], [17], [18], [19]. But mention the authors of these papers used the Gelfand-Tsetlin base whose construction is based on restrictions $\mathfrak{o}_N \downarrow \mathfrak{o}_{N-1}$ ($\mathfrak{o}_5 \downarrow \mathfrak{o}_4 \downarrow \mathfrak{o}_3$ in the case $\mathfrak{o}_5$). This base is not natural form the physical point of view. In the present paper we use the base for a representation of $\mathfrak{o}_5$, whose construction is based on restrictions within the series $B$.

### 1.1 The structure of the paper

The structure of the paper is the following. In Section 2 definitions and notations are introduced.

The technical details, the discussion of definitions is placed in Appendix 8.

In Section 3 the method of $Z$-invariants is explained, it allows to solve effectively the problem of description of a branching of a representation when one restricts an algebra.

In Section 4 we give the scheme of a solution of the problem of restriction $g_n \downarrow g_{n-1}$\footnote{When we write "the problem of restriction $g_n \downarrow g_{n-1}$" we mean the problem of an explicit description of a base in the space of $g_{n-1}$-highest vectors in a $g_n$-representation.}. We explain how to construct a Gelfand-Tselin type base, obtain coefficients and restricted Wigner coefficients and matrix elements of the generators.
The main construction is given in Sections 5, 6, 7. In these Section the Gelfand-Tsetlin type base for series $B, C, D$ and explicit formulas for the action of generators are constructed for $n = 1, n = 2, n > 2$ respectively. Note that Wigner coefficients for $g_{n-1}$ are calculated when the algebra $g_n$ is considered.

The cases $n = 1, n = 2$ are considered in Sections 5, 6 separately for each series $B, C, D$. In Section 7 further steps are discussed, they are similar for all series.

In all cases all values are expressed through the analogous values for the algebras $gl_N$, which were obtained in an explicit form by Biedenharn and Baird in [1] and also by Zhelobenko in [2].

In Appendix the facts from the representation theory that are not well-known are given. Mostly that can be found in [20]. The reader can find them here of in Appendix 8.

2 Basic definitions

In the present Section the basic definitions and notations are introduced. See also Appendix 8.

In the paper we use the Lie algebras $sp_{2n}$ and $o_N$ in split realization. These algebras act in the space with coordinates $x_{-n}, ..., x_{-1}, x_1, ..., x_n$, in the cases $sp_{2n}$ and $o_{2n}$, and in the space with coordinates $x_{-n}, ..., x_{-1}, x_0, x_1, ..., x_n$ in the case $o_{2n+1}$. The generators of the symplectic algebra are the matrices

$$F_{i,j} = E_{i,j} - \text{sign}(i)\text{sign}(j)E_{-j,-i},$$

and the generators of the orthogonal algebra are the matrices

$$F_{i,j} = E_{i,j} - E_{-j,-i}.$$

2.1 Gelfand-Tsetlin tableaux

In the present paper a Gelfand-Tsetlin base in an irreducible representation of $g_n$ is constructed. Base vectors are indexed by Gelfand-Tsetlin type tableaux, these tableaux have similar structure, let us describe it here.
Let us be given an irreducible representation $V$ of the algebra $g_n$ with the highest weight $[m_{-n,n}, ..., m_{-1,n}]$. Then the base vectors are indexed by tableaux $(m)$ of type

$$
(m) = \begin{pmatrix}
    [m]_n \\
    [m']_n \\
    [m]_{n-1} \\
    [m']_{n-1} \\
    \vdots \\
    [m]_1 \\
    [m']_1
\end{pmatrix}.
$$

(1)

The row $[m]_n$ is the highest weight of the considered representation $V$, the row $[m]_{n-1}$ is the highest weight of an irreducible $g_{n-1}$-representation that contains the vector $(m)$. The row $[m']_{n-1}$ is a base element in the space of $g_{n-1}$-highest vectors with highest weight $[m]_{n-1}$ and so on. The structure of the rows depends on the series $B_n$, $C_n$ or $D_n$.

The weight of the vector $(m)$ is denoted as $\Delta(m)$.

Denote the tableau in which all indices take the maximum possible values as $(m)_{\text{max}}$. The vector $(m)_{\text{max}}$ is a highest vector.

### 2.2 Wigner coefficients and reduced Wigner coefficients

Let us define Wigner coefficients, reduced Wigner coefficients and let us give a solution of the multiplicity problem. See also Appendix $\S$

Denote an irreducible representation with highest weight $[m]_n$ as $V^{[m]_n}$. The tableau $(m)$ from which the first row is removed is denoted as $(m)_{n-1}$.

The Wigner coefficients are matrix elements of an interwining operator $\Phi : V^{[m]_n} \to V^{[M]_n} \otimes V^{[m]_n}$. All such operators are indexed by tableaux such that

$$
\Phi((m)_{\text{max}}) = (\Gamma) \otimes (m)_{\text{max}} + \text{l.o.t.}, \quad (\Gamma) \in V^{[M]_n}
$$

where l.o.t. (lower order terms) denotes a sum of tensor products of weight vectors where the second vector has a weight lower than $[m]_n$. 

The corresponding Wigner coefficient is denoted as

$$< \left( \begin{array}{c} [\bar{m}]_n \\ (\bar{m})_{n-1} \end{array} \right) \left( \begin{array}{c} (\Gamma)_{n-1} \\ [M]_n \\ (M)_{n-1} \end{array} \right) \left( \begin{array}{c} [m]_n \\ (m)_{n-1} \end{array} \right) >. \quad (2)$$

This coefficient can be non-zero only if $[\bar{m}]_n = \Delta(\Gamma) + [m]_n$. The corresponding Wigner coefficient is denoted as

$$< \left( \begin{array}{c} [\bar{m}]_n \\ [\bar{m}']_n \\ [\bar{m}]_{n-1} \end{array} \right) \left( \begin{array}{c} (\Gamma)_{n-1} \\ [M]_n \\ (\gamma)_{n-1} \end{array} \right) \left( \begin{array}{c} [m]_n \\ [m']_n \\ [m]_{n-1} \end{array} \right) > \quad (3)$$

### 2.3 Fundamental Wigner coefficients

In the present paper only the Wigner coefficients are considered for which $[M]_n = [1, 0, ..., 0] = [\hat{1}]_n$, that is when the tensor factor $V^{[M]}_n$ is a standard representation. Such Wigner coefficients are called fundamental.

Note that weight vectors $(m)$ of the standard representations are completely defined by their weights $\Delta(m) = [0, ..., \pm 1, ..., 0]$, where $\pm 1$ occurs at the place $i$. If $i = 0$ then only 1 is allowed. Thus the Wigner coefficient is of type can be denoted as

$$\left( \begin{array}{c} i \\ [\hat{1}]_n \\ j \end{array} \right) \quad (4)$$

### 3 The method of Z-invariants

Let us explain the method of Z-invariants, that allows to describe effectively the space of $g_{n-1}$-highest vectors in a $g_n$-representation.
3.1 Realization of a representation on the space of functions on upper triangular matrices.

Let \( g \) be a classical Lie algebra, denote as \( n \) its rank, and let \( G \) be the corresponding Lie group.

Let
\[
G = Z - DZ_+\]
be the Gauss decomposition. Denote the group \( Z_+ \) of upper triangular matrices with units on the diagonal shortly as \( Z \).

Let us construct a realization of a representation with the highest weight \([m_{-n,n}, \ldots, m_{-1,n}]\) on the space of polynomial functions on \( Z \).

The elements of the matrix \( z \in Z \) are denoted as \( z_{ij} \). Thus the functions are polynomials in variables \( z_{ij} \).

Define the function \( \alpha \) on the space of diagonal matrices as follows
\[
\alpha (\delta) = \delta^{m_{-n,n}} \ldots \delta^{m_{-1,n}},
\]
where \( \delta \in D \) and \( \delta_{-n}, \ldots, \delta_{-1} \) are diagonal elements of \( \delta \).

Define the action \( T_g \) of the element \( g \in G \) on a function \( f(z), z \in Z \) as follows
\[
(T_g f)(z) = \alpha(\tilde{\delta}) f(\tilde{z}), \tag{5}
\]
where \( \tilde{\delta} \) and \( \tilde{z} \) through the Gauss decomposition of \( zg \)
\[
zg = \tilde{\zeta} \tilde{\delta} \tilde{z}, \quad \tilde{\zeta} \in Z_-, \quad \tilde{\delta} \in D, \quad \tilde{z} \in Z.
\]

In the space of function on \( Z \) the subspace of functions \( f \), that form an irreducible representation with the highest weight \( m = [m_{-n,n}, \ldots, m_{-1,n}] \) is defined as follows.

Let \( \partial_i \) be an operator on the space of functions on \( Z \), which is a left infinitesimal shift on the \( i \)-root element. Then the space of functions we are looking for is the solution space of the following system of differential equations which is called the indicator system
\[
\partial_i^{r_i+1} f = 0, \quad r_i = \frac{2(m_i, \omega_i)}{(\omega_i, \omega_i)}, \quad i = 1, \ldots, n, \tag{6}
\]

where $\omega_i$ are fundamental weights for $g_n$. We call $r_i$ the exponents of the system (6).

### 3.2 The highest vectors for the subalgebra $g_{n-1} \subset g_n$, that preseves the coordinates $x_{-n}, x_n$.

Let $G_n = Sp_{2n}, O_{2n+1}, O_{2n}$. Identify $G_{n-1} \subset G_n$ with a subgroup in $G_n$, that preserves coordinates $x_{-1}, x_1$. The subgroup $Z$ in $G_n$ is denoted as $Z_n$. Obviously $Z_{n-1} = Z_n \cap G_{n-1}$. Then $G_{n-1}$-highest vectors correspond to polynomials that are invariant under the action of the subgroup $Z_{n-1}$ and satisfy the indicator system.

Let us find polynomials invariant under the action of $Z_{n-1}$. In the case $Sp_{2n}$ it is done in [2], in the case $O_{2n+1}, O_{2n}$ this can be done in a similar way. The answer is the following.

The polynomials invariant under the action of $Z_{n-1}$ are polynomials in variables $z_{-1,1}, \ldots, z_{-1,n}, z_{1,2}, \ldots, z_{1,n}$ in the case of orthogonal groups and in variables $z_{-1,1}, z_{-1,2}, \ldots, z_{-1,2}, z_{1,2}, \ldots, z_{1,n}$ in the case of symplectic groups.

### 4 The sketch of the construction

Let us give a sketch the construction of the Gelfand-Tsetlin type base, the calculation of Wigner coefficients and reduced Wigner coefficients and the derivation of formulas for the action of generators of the algebra.

The induction in $n$ is used. The subalgebra $g_{n-1} \subset g_n$ is identified with the subalgebra that preserves the coordinates $x_{-1}, x_1$.

At first the case $n = 1$ is considered, this is the case of algebras $\mathfrak{o}_3, \mathfrak{sp}_2, \mathfrak{o}_2$, then the case $n = 2$ is considered, this is the case of algebras $\mathfrak{o}_5, \mathfrak{sp}_4, \mathfrak{o}_4$.

Finally the the passage from $g_2$ to $g_n$ is considered. This passage is done simultaneously for all algebras $B, C, D$. In this step the Gelfand-Tsetlin base for $g_n$ is constructed, Wigner coefficients for $g_{n-1}$ corresponding to tensor multiplication on the standard representation are obtained. As before the Wigner coefficients in the case $g_2$ are calculated separately for the three series of algebras and the Wigner coefficients for $g_n, n > 2$ are calculated in
a similar way. Finally the explicit formulas for the action of generator $g_n$ in
the base are derived.

Note that the Wigner coefficients for $g_{n-1}$ are calculated when the alge-
bra $g_n$ is considered. Let us stress that the passage from $g_2$ to $g_n$ is very
close to the passage from $\mathfrak{gl}_3$ to $\mathfrak{gl}_{n+1}$. This relation plays a key role in the
construction.

Is is not necessary to calculate the formulas for the action of all generators.
Indeed, it is enough to obtain the formulas for the action of $e_{\pm\alpha}$, where $\alpha$
are simple roots and also formulas for the action of Cartan elements. Also let us note that the matrices that define the action the element $e_{-\alpha}$ can
be obtained from the matrix that correspond to $e_{-\alpha}$ by conjugation, hence
it is enough to consider only one of the roots $\pm\alpha$.

It is well-know that for the Gelfand-Tsetlin type base the following state-
ment is true. The subalgebra $g_k \subset g_n$ changes only the part of the Gelfans-
Tsetlin tableau that corresponds to $g_k$, and this action does not depend on
the upper rows.

Thus we obtain that it is enough to calculate the weight of the base vector
and also the action of the operators $F_{-1,0}$ for $n = 1$, and $F_{-1,-2}, F_{-2,1}$ for
$n = 2$ and $F_{-1,-2}$ for $n > 2$.

4.1 Notations

For the rows $[m]_n = [m_{-n,n}, ..., m_{-1,n}]$ and $[m]_{n-1} = [m_{-n,n-1}, ..., m_{-2,n-1}]$
introduce as in $[20]$ the following symbols.

Let

$$
\begin{vmatrix}
  n \\
  i_1 : n - 1
\end{vmatrix}^{[m]_n, [m]_{n-1}} = \left( (m_{-2,n-1} - m_{-1,n}) \Pi_{j=-1}^{n-1} (m_{j,n} - m_{i_1,n-1} - j + i_1 + 1) \right)^{\frac{1}{2}},
$$

be the reduced matrix element for the $\mathfrak{g}l_{n-1}$-tensor operator $E_{n,i}$. Let

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$^4$In $[20]$ it is suggested that $m_{-1,n} = 0$, that is why the formula (7) is slightly different
from $[20]$. 

10
be a $\mathfrak{gl}_n$ reduced Wigner coefficient. Here $S(x) = \text{sign}(x)$, $S(0) = 1$. Let

$$i : n \begin{pmatrix} m[n], m[n-1] \\ n-1 \end{pmatrix} = S(i_2 - i_1) \left( \frac{\prod_{j=1}^{-n} (m_{j,n-1} - m_{i,n} - j + i_1)}{\prod_{j=1}^{-n,j \neq i_1} (m_{j,n} - m_{i,n} - j + i_1)} \right)^\frac{1}{2}, \quad (8)$$

be a $\mathfrak{gl}_n$ reduced Wigner coefficient. Here $S(x) = \text{sign}(x)$, $S(0) = 1$. Let

$$i : n \begin{pmatrix} m[n], m[n-1] \\ n-1 \end{pmatrix} = \left( \frac{\prod_{j=-1}^{-n,j \neq i_2} (m_{j,n} - m_{i_2,n-1} - j + i_2 + 1)}{\prod_{j=-1}^{-n,j \neq i_2} (m_{j,n-1} - m_{i_2,n-1} - j + i_2 + 1)} \right)^\frac{1}{2}, \quad (9)$$

be $\mathfrak{gl}_n$ Wigner coefficient.

5. $n = 1$: the construction for the algebras $\mathfrak{o}_3$, $\mathfrak{sp}_2$, $\mathfrak{o}_2$.

Let us construct the Gelfand-Tsetlin base and derive formulas for the action of generators of these algebras.

For the algebras $\mathfrak{sp}_2$, $\mathfrak{o}_2$ this is a trivial task, since these algebras are one-dimensional and their irreducible representations are also one-dimensional. They are defined by the highest weight which is just one number $m_{-1}$. Consider the nontrivial case $\mathfrak{o}_3$.

5.1 The case $\mathfrak{o}_3$

The Gelfand-Tsetlin base coincides with the standard weight base in a $\mathfrak{sl}_2$-representation. But an indexation that we construct differs from the standard one.

The nonstandard indexation is useful in further investigations since it allows to establish a relation for Gelfand-Tsetlin bases for the algebras of series $B$ and $A$ for $n = 2$. 

11
Let us use the realization of a representation on the space of functions on $Z$. The indicator system in the case of the highest weight $m_{-1,1}$ (which is an inter or a half-integer number) is

$$\left( \frac{\partial}{\partial z_{-1,0}} \right)^{2m_{-1,1}} f = 0. \quad (10)$$

Thus the base vectors in the representation in this realization are monomials $z_{-1,0}^k$, $k = 0, ..., 2m_{-1,1}$. In [2] it is shown that the vector corresponding to the monomial $z_{-1,0}^{m_{-1,1}-1}$, has a weight $m_{-1,1}$.

However when the Wigner coefficient for $\mathfrak{sl}_2$ are calculated the orthonomal base is used, that is the base

$$e_\mu = \frac{z_{-1,0}^{m_{-1,1}-\mu}}{\sqrt{(m_{-1,1}-\mu)!((m_{-1,1}+\mu)!},}$$

$\mu = -m_{-1,1}, ..., m_{-1,1}$ is the weight of the vector. Below we use this base.

Let us construct two numbers. Put $m'_{-1,1} = \lfloor \frac{k}{2} \rfloor$, where $\lfloor . \rfloor$ is an integer part and let $\sigma_{-1} = 0, 1$ be the residue of the division of $k$ by 2. A base vector can be encoded by a tableau

$$m_{-1,1}, \quad \sigma_{-1} m'_{-1,1}, \quad (11)$$

The following inequality holds $m_{-1,1} \geq m'_{-1,1} \geq 0$. Also if $m_{-1,1} = m'_{-1,1}$ and $m_{-1,1}$ is an integer then $\sigma_{-1} = 0$.

The weight of the vector encoded by a tableau can be calculated by the formula $m_{-1,1} - 2m'_{-1,1} - \sigma_1$.

Let us find the action of the operator $F_{-1,0}$. Put $g = exp(tF_{-1,0}) \in Z$, then $T_g f(z) = f(zg)$. Under the action of $F_{-1,0}$ the vector $z_{-1,0}^k$ is mapped to $kz_{-1,0}^{k-1}$. One can easily write how the numbers $m'_{-1,1}$ and $\sigma_{-1}$ change under the action of $F_{-1,0}$. This gives us a necessary condition for the matrix element to be non-zero. In this case the matrix element equals to the reduced matrix element of a $\mathfrak{so}_1$-tensor operator $F_{-1,0}$. Thus the following theorem takes place.

Put $[m]_1 = [m_{-1,1}, 0]$, $[m']_1 = [m'_{-1,1}]$. 12
Theorem 1. Under the action of $F_{-1,0}$ the tableau $(m)$ changes in the following way.

1. If $\sigma_{-1} = 0$, then $\sigma_{-1}$ diminishes by 1, $m'_{-1,1}$ remain unchanged, the resulting tableau is multiplied to \[
\begin{vmatrix}
2 & [m_1, [m']_1] \\
-2 & 1
\end{vmatrix}.
\]

2. If $\sigma_{-1} = 1$, then $m'_{-1,1}$ diminishes by 1, $\sigma_{-1}$ turns to 0, the resulting tableau is multiplied to \[
\begin{vmatrix}
2 & [m_1, [m']_1] \\
-2 & 1
\end{vmatrix}.
\]

6. $n = 2$: the construction for the algebras $\mathfrak{o}_5$, $\mathfrak{sp}_4$, $\mathfrak{o}_4$.

In the present Section the restrictions $\mathfrak{o}_5 \downarrow \mathfrak{o}_3$, $\mathfrak{sp}_4 \downarrow \mathfrak{sp}_2$, $\mathfrak{o}_4 \downarrow \mathfrak{o}_2$ are investigated. Then the Gelfand-Tsetlin base is constructed and the action of the generators in this base is investigated.

The construction in the three cases $\mathfrak{o}_5$, $\mathfrak{sp}_4$, $\mathfrak{o}_4$ are different. For $\mathfrak{o}_5$ and $\mathfrak{sp}_4$ the corresponding problem of restriction is solved by establishing a relation with the analogous problem of restriction $\mathfrak{gl}_3 \downarrow \mathfrak{gl}_1$.

Note that the Wigner coefficients for $\mathfrak{o}_3 = \mathfrak{sl}_2$ are known thus for $n = 2$ it is not necessary to calculate the Wigner coefficients.

6.0.1 The auxiliary problem of restriction $\mathfrak{gl}_3 \downarrow \mathfrak{gl}_1$

Identify the algebra $\mathfrak{gl}_3$ with the algebra of endomorphisms of the space with coordinates $x_{-2}, x_{-1}, x_1$, and $\mathfrak{gl}_1$ is the subalgebra that preserves the last two coordinates. The problem of restriction $\mathfrak{gl}_3 \downarrow \mathfrak{gl}_1$ is equivalent to the problem of construction of weight base in a $\mathfrak{gl}_3$-representation.

We use the realization of a representation with the highest weight $[\lambda_{-2}, \lambda_{-1}, \lambda_1]$ on the space of polynomials on the group $Z$, that is on the space of polynomials in variables $z_{-2,-1}, z_{-2,1}, z_{-1,1}$. The indicator system is of type
\[
\left( \frac{\partial}{\partial z_{-2,-1}} + z_{-1,1} \frac{\partial}{\partial z_{-2,1}} \right)^{r_{-2}+1} f = 0, \quad (12)
\]

\[
\left( \frac{\partial}{\partial z_{-1,1}} \right)^{r_{-1}+1} f = 0,
\]

where \( r_{-2} = \lambda_{-2} - \lambda_{-1}, r_{-1} = \lambda_{-1} \).

The solution space of this system is spanned by polynomials of type

\[
f = f_0(z_{-2,-1}, z_{-2,1}, z_{-1,1}) z_{-1,1}^p,
\]

where \( f_0 \) is a polynomial solution of the first equation, that is not divisible by \( z_{-1,1} \). The following inequality must hold

\[
deg_{z_{-1,1}} f_0 + p \leq r_{-1},
\]

where \( deg_{z_{-1,1}} f_0 \) is the degree of \( f_0 \) as a polynomial in \( z_{-1,1} \).

There exist a base in the representation whose vectors are encoded by Gelfand-Tsetlin tableaux

\[
\begin{array}{ccc}
\lambda_{-2} & \lambda_{-1} & \lambda_1 \\
\lambda_{-2,2} & \lambda_{-1,2} \\
\lambda_{-2,1}
\end{array}
\]

(14)

6.1 The construction in the case \( \mathfrak{o}_5 \)

Consider a representation of \( \mathfrak{o}_5 \) with the highest weight \([m_{-2,2}, m_{-1,2}]\).

6.1.1 The problem of restriction \( \mathfrak{o}_5 \downarrow \mathfrak{o}_3 \)

Using the method of \( Z \)-invariants (see Section 3.2) let us investigate how an \( \mathfrak{o}_5 \)-representation branches when we restrict the algebra \( \mathfrak{o}_5 \downarrow \mathfrak{o}_3 \). The polynomials on \( Z_{\mathfrak{o}_5} \), that are invariant under the action of \( Z_{\mathfrak{o}_3} \), are polynomials in variables \( z_{-2,-1}, z_{-2,1}, z_{-1,0} \).

The indicator system is
\[
\frac{\partial}{\partial z_{-2,-1}} + z_{-1,1} \frac{\partial}{\partial z_{-2,1}})^{r-2+1} f = 0, \\
\left(\frac{\partial}{\partial z_{-1,0}}\right)^{r-1+1} f = 0,
\]

where \( r_{-2} = m_{-2,2} - m_{-1,2}, \) \( r_{-1} = 2m_{-1,2}. \) Note that \( z_{-1,1} = \frac{-z^2}{2}. \)

The solution space of this system is spanned by polynomials of type

\[
f = f_0(z_{-2,-1}, z_{-2,1}, z_{-1,1}) z_{-1,0}^p,
\]

where \( f_0 \) is a polynomial solution of the first equation that is not divisible by \( z_{-1,1}. \) The following inequality must take place

\[
2 \text{deg}_{z_{-1,1}} f_0 + p \leq r_{-1}.
\]

Let us establish a correspondence between the problems of restriction \( \mathfrak{so}_5 \downarrow \mathfrak{so}_3 \) and \( \mathfrak{gl}_3 \downarrow \mathfrak{gl}_1. \) Consider the cases when \( p = 2p' \) is even and when \( p = 2p' + 1 \) is odd.

1. In the case \( p = 2p' \) to the solution (16) with the parameter \( p \) of the system (15) with exponents \( r_{-2}, r_{-1} \) there corresponds a solution (13) with the parameter \( p' \) of the system (12) with exponents \( r_{-2}, \frac{r_{-1}}{2} \)

\[
f_0(z_{-2,-1}, z_{-2,1}, z_{-1,1}) z_{-1,0}^{\frac{p}{2}} \mapsto f_0(z_{-2,-1}, z_{-2,1}, z_{-1,1}) z_{-1,1}^{\frac{p}{2}}.
\]

This correspondence is a bijection between the solution space of (15) with an even parameter \( p \) and the solution space of (12).

2. In the case \( p = 2p' + 1 \) to the solution (16) with the parameter \( p \) of the system (15) with exponents \( r_{-2}, r_{-1} \) there corresponds a solution (13) with the parameter \( p' \) of the system (12) with exponents \( r_{-2}, \frac{r_{-1}}{2} \)

\[
f_0(z_{-2,-1}, z_{-2,1}, z_{-1,1}) z_{-1,0}^{\frac{p}{2}} \mapsto f_0(z_{-2,-1}, z_{-2,1}, z_{-1,1}) z_{-1,1}^{\frac{p}{2}}.
\]

This correspondence is a bijection between the solution space of (15) with an odd parameter \( p \) and the solution space of (12).
6.1.2 The Gelfand-Tsetlin base for $\mathfrak{so}_5$

In the case $\mathfrak{gl}_3$ the indicator system with exponents $r_-, \frac{r_1}{2}$ corresponds to the highest weight $[m_{-2,2}, m_{-1,2}, 0]$. Base vectors of a representation of $\mathfrak{gl}_3$ are encoded by tableaux (14).

Denote as $\sigma_{-1}$ the residue of the division of $p$ by 2. Using the correspondence that was established in the previous subsection one obtains that $\mathfrak{so}_3$-highest vectors in a $\mathfrak{so}_5$-representation are encoded by $\mathfrak{gl}_3$-tableaux (from which the zero in the upper row is removed) with an additional number $\sigma_{-2}$, that is by tableaux of type

\[
\begin{array}{llll}
  m_{-2,2} & m_{-1,2} \\
  \sigma_{-2} & m'_{-2,2} & m'_{-1,2} \\
  m_{-2,1}.
\end{array}
\]  

(17)

The following inequalities must take place

\[
m_{-2,2} \geq m'_{-2,2} \geq m_{-1,2} \geq m'_{-1,2} \geq 0 
\]  

(18)

\[
m'_{-2,2} \geq m_{-2,1} \geq m'_{-1,2}.
\]  

(19)

Also if $m'_{-2,2} = 0$, then $\sigma_{-2} = 0$. Indeed, consider the inequality $2\text{deg}_{z_1,1} f_0 + p \leq r_{-1}$, and divide it by two. One obtaines $\text{deg}_{z_1,1} f_0 + p' + \sigma_{-2} \leq \frac{r_1}{2}$. From here we conclude that if $r_{-1}$ is even (that is when the highest weight $[m_{-2,2}, m_{-1,2}]$ is integer) and $p'$ takes the maximum value then $\sigma_{-2} = 0$. The fact that $p'$ takes the maximum value means that $m'_{-2,2} = 0$ (see explicit formula for the polynomial on $Z$, corresponding to a $\mathfrak{gl}_3$-tableau in [2]).

To be able to use this indexation of $\mathfrak{so}_3$-highest vectors for the construction of the Gelfand-Tsetlin type base we must prove the following Proposition.

**Proposition 1.** The number $m_{-2,1}$ in the tableau (17) is the $\mathfrak{so}_3$-weight of the vector that is encoded by the tableau.

The proof is elementary, it can be found in Appendix in Section 8.6.
Using the inductive procedure of the construction of the Gelfand-Tsetlin type base we obtain that the vectors of the representation are encoded by a tableau \( (m) \), that is obtained by adding to the tableau \( (17) \) the tableau \( (11) \), that is by a tableau \( (m) \) of type

\[
\begin{array}{cccc}
m_{-2,2} & m_{-1,2} \\
\sigma_{-2} & m'_{-2,2} & m'_{-1,2} \\
m_{-2,1} \\
\sigma_{-1} & m'_{-2,1}.
\end{array}
\]

The elements of this tableau must satisfy the inequalities corresponding to the tableaux \( (17) \), \( (11) \).

Let us give formulas for the weight \([\Delta(m)_{-2}, \Delta(m)_{-1}]\) of the vector encoded by tableau. It is enough to give a formula for \( \Delta(m)_{-1} \).

**Proposition 2.** \( \Delta(m)_{-1} = -2 \sum_i m'_{i,2} + \sum_i m_{i,2} + m_{-2,1} - \sigma_{-2} \)

The proof can be found in Appendix in Section 8.6.

### 6.1.3 Reduced matrix elements for \( \mathfrak{o}_5 \)

To calculate matrix elements of generators in the base that we have constructed we need explicit formulas for reduced matrix elements of operators \( F_{-1,-2} \) and \( F_{1,-2} \), viewed as \( \mathfrak{o}_3 \)-tensor operators acting between \( \mathfrak{o}_3 \)-representations into which an \( \mathfrak{o}_5 \)-representation splits.

Let \( (\bar{m})_{\text{red}}, (m)_{\text{red}} \) be two tableaux of type \( (17) \) that define \( \mathfrak{o}_3 \)-highest vectors in a \( \mathfrak{o}_5 \) representation. Let \( (\bar{m})^*_{\text{red}}, (m)^*_{\text{red}} \) be two \( \mathfrak{gl}_3 \)-tableaux, that are obtained from \( (\bar{m})_{\text{red}} \), \( (m)_{\text{red}} \) by removing the number \( \sigma_1 \) and adding 0 to the upper row in the right.

The reduced matrix element of a \( \mathfrak{o}_3 \)-tensor operator \( F_{\pm1,-2} \) depends on tableaux \( (\bar{m})_{\text{red}}, (m)_{\text{red}} \). Denote it \( <(\bar{m})_{\text{red}}|F_{\pm1,-2}|(m)_{\text{red}}>_{\text{red}}. \)

Denote as \( <(\bar{m})^*_{\text{red}}|_{\pm1,-2}|(m)^*_{\text{red}}>_{\text{\text{gl}_3}} \) a \( \mathfrak{gl}_3 \)-matrix element.

The following lemma takes place.
Lemma 1. The matrix elements for $\mathfrak{o}_5$ and $\mathfrak{gl}_3$ are equal

$$< (\tilde{m})_{\text{red}} | F_{\pm 1, -2} | (m)_{\text{red}} >_{\text{red}} = < (\tilde{m})^*_{\text{red}} | F_{\pm 1, -2} | (m)^*_{\text{red}} >$$  \hspace{1cm} (21)

Proof. The proof uses the following fact. Let $(m)_{\text{max}}$ be a $\mathfrak{o}_5$-tableau that is obtained from $(m)_{\text{red}}$ by adding maximal rows below.

Proposition 3. The following equality takes place

$$< (\tilde{m})_{\text{max}} | F_{\pm 1, -2} | (m)_{\text{max}} >= < (\tilde{m})_{\text{red}} | F_{\pm 1, -2} | (m)_{\text{red}} >_{\text{red}}$$ \hspace{1cm} (22)

The proof can be found in Appendix in Section 9.

Let us return to the proof of Lemma. It enough to prove that

$$< (\tilde{m})_{\text{max}} | F_{\pm 1, -2} | (m)_{\text{max}} >= < (\tilde{m})_{\text{red}} | F_{\pm 1, -2} | (m)_{\text{red}} >_{\text{red}}$$

Let $f = z_{-1, 0}^p f_0(z_{-2, -1}, z_{-2, 1}, z_{-1, 1})$ be a polynomial corresponding to $(m)_{\text{max}}$. One can easily check that under the action of $e^{tF_{-2, -1}}$ it is transformed into the polynomial $z_{-1, 0}^p f_0(z_{-2, -1} + t, z_{-2, 1}, z_{-1, 1})$.

To the vector $(m)_{\text{red}}$ there corresponds a polynomial $f^* = z_{-1, 1}^p f_0(z_{-2, -1}, z_{-2, 1}, z_{-1, 1})$. One can easily check that under the action of $e^{tE_{-2, -1}}$ it is transformed into the polynomial $z_{-1, 1}^p f_0(z_{-2, -1} + t, z_{-2, 1}, z_{-1, 1})$.

Thus the actions of $F_{-2, -1}$ and $E_{-2, -1}$ on the highest $\mathfrak{o}_3$ and $\mathfrak{gl}_1$ vectors are agreed with the correspondence that was constructed in Section 6.1.1. Thus the matrix elements of these operators are equal. Hence the matrix elements of the operators $F_{-1, -2}$ and $E_{-1, -2}$ are equal. Using Proposition 3 we prove the lemma for the operators $F_{-1, -2}$ and $E_{-1, -2}$.

Analogously under the action of the operator $e^{tF_{-2, 1}}$, the polynomial $f$ is transformed into the polynomial $z_{-1, 0}^p f_0(z_{-2, -1}, z_{-2, 1} + t, z_{-1, 1})$. The polynomial $f^* = z_{-1, 1}^p f_0(z_{-2, 1}, z_{-2, 1}, z_{-1, 1})$ corresponding to $(m)_{\text{red}}$ under the action if the operator $e^{tE_{-2, 1}}$ is transformed into the polynomial $z_{-1, 1}^p f_0(z_{-2, -1}, z_{-2, 1} + t, z_{-1, 1})$.

Form this fact we obtain the statement of Lemma for operators $F_{1, -2}$ and $E_{1, -2}$.

\[\square\]
Let us find reduced matrix elements for operators $F_{-1,2}$ and $F_{1,2}$ using the lemma.

Put $[m]_2 = [m_{-2,2}, m_{-1,2}, 0]$, $[m']_2 = [m'_{-2,2}, m'_{-1,2}]$, $[m]_1 = [m_{-2,1}]$.

Let us prove theorems

Theorem 2. The reduced matrix element $<(\tilde{m})_{\text{red}} | F_{1,-2} | (m)_{\text{red}} >_{\text{red}}$ can be nonzero only if the following condition holds. There exist a unique index $i_1 = -2$ or $-1$ such that $\tilde{m}_{i_1,2} = m'_{i_1,2} - 1$, and for the other index $i$ one has $\tilde{m}_{i,2} = m'_{i,2}$. Also $\tilde{m}_{-2,1} = m_{-2,1} - 1$.

If this condition holds one has

$$< (\tilde{m})_{\text{red}} | F_{1,-2} | (m)_{\text{red}} >_{\text{red}} = \begin{vmatrix} 3 & [m]_2, [m']_2 & i_1 : 2 \\ i_1 : 2 & -2 : 1 \end{vmatrix}.$$  \hspace{1cm} (23)

Proof. By Lemma [1] it is enough to find matrix elements for the operator $E_{1,-2}$.

In [1] in the case $\mathfrak{gl}_n$ the following formula is proved

$$< (\tilde{m}) | E_{1,-2} | (m) > = \begin{vmatrix} n & [m]_n, [m]_{n-1} & i_1 : n - 1 \\ i_1 : n - 1 & i_2 : n - 2 \\ i_2 : n - 2 & n - 3 \end{vmatrix},$$ \hspace{1cm} (24)

where $[m]_k$ is the $k$-th row in the Gelfand-Tsetlin tableau $(m)$.

It is suggested that one has $\tilde{m}_{i,j} = m_{i,j} - 1$, and for $k \neq i$ one has $\tilde{m}_{i,j} = m_{i,j}$. In all other cases $<(\tilde{m}) | E_{1,i} | (m)>$ vanishes.

If one considered $E_{1,-2}$ as a $\mathfrak{gl}_{n-2}$-tensor operator then the first two factors are a reduced matrix element and the last is a Wigner coefficient.

If one puts $n = 3$ then (since $i_2 = -2$) one gets

$$< (\tilde{m})_{\text{red}} | E_{1,-2} | (m)_{\text{red}} >_{\text{red}} = \begin{vmatrix} 3 & [m]_3, [m]_2 & i_1 : 2 \\ i_1 : 2 & i_2 : 2 & -2 : 1 \end{vmatrix}.$$ \hspace{1cm} (25)

Using now Lemma [1] we prove the Theorem. 

\hspace{1cm} \square
Let us find the formula for the reduced matrix element of the operator $F_{-1,-2}$.

**Theorem 3.** The reduced matrix element $<(\bar{m})_{\text{red}} | F_{-1,-2} | (m)_{\text{red}} >_{\text{red}}$ can be nonzero only if the following condition holds. One has $\bar{m}_{-2,1} = m_{-2,1} - 1$ and all other entries of $(\bar{m})$ and $(m)$ are equal.

If this condition holds one has

$$< (\bar{m})_{\text{red}} | F_{-1,-2} | (m)_{\text{red}} >_{\text{red}} = \frac{2}{[-2 : 1]^{[m']_2,[m]_1}},$$

(26)

**Proof.** By Lemma 1 it is enough to find matrix elements $E_{-1,-2}$. One has the formula

$$< (\bar{m})_{\text{red}} | E_{-1,-2} | (m)_{\text{red}} >_{\text{red}} = \begin{vmatrix} \frac{2}{[m]_n,[m]_{n-1}} \\ i_1 : n - 1 \\ n - 2 \\ n \end{vmatrix},$$

(27)

where $[m]_k$ is the $k$-th row of the Gelfand-Tsetlin tableau $(m)$. It is suggested that $\bar{m}_{i_1,n} = m_{i_1,n} - 1$ and $\bar{m}_{j,n} = m_{j,n}$ for $j \neq i_1$. The first factor is the reduced matrix element. Put $n = 2$, using Lemma 1 we obtain the statement of the Theorem.

\[\square\]

### 6.1.4 Matrix elements

Using calculation that were done in the previous sections let us derive formulas for matrix elements of generators of $\mathfrak{so}_5$ in the base that we have constructed.

It is enough to give formulas for the matrix elements of operators $F_{-1,-2}$ and $F_{1,-2}$. The following theorem takes place.

**Theorem 4.** The matrix element $<(\bar{m})_{\text{red}} | F_{1,-2} | (m)_{\text{red}} >_{\text{red}}$ can be nonzero only if the following condition holds. There exist a unique index
$i_1 = -2$ or $-1$ such that $\tilde{m}_{i_1,2} = m'_{i_1,2} - 1$, and for the other index $i$ one has $\tilde{m}_{i,2} = m'_i$. Also $\tilde{m}_{-2,1} = m_{-2,1} - 1$.

If this condition holds one has

$$< (\tilde{m})|F_{1,-2}|(m) > = \begin{vmatrix} 3 & [m]_{2},[m']_{2} ; i_1 : 2 & -2 : 1 \\ 2 & [m']_{1},[m]_{1} ; -2 : 1 \\ -2 : 1 \\ 1 \end{vmatrix}.$$ \hspace{1cm} (28)

The matrix element $< (\tilde{m}) | F_{-1,-2} | (m) >$ can be nonzero only if the following condition holds. One has $\tilde{m}_{-2,1} = m_{-2,1} - 1$ and all other entries of $(\tilde{m})$ and $(m)$ are equal.

If this condition holds one has

$$< (\tilde{m})|F_{-1,-2}|(m) > = \begin{vmatrix} 2 & [m']_{2},[m]_{1} ; -2 : 1 \\ -2 : 1 \\ 1 \end{vmatrix}.$$ \hspace{1cm} (29)

**Proof.** The Theorem is immediately proved using the Wigner-Eckart theorem.

\hspace{1cm} $\Box$

6.2 The construction in the case $\mathfrak{sp}_4$

Consider an irreducible representation of $\mathfrak{sp}_4$ with the highest weight $[m_{-2,2}, m_{-1,2}]$. Since the algebra $\mathfrak{sp}_2$ is one dimensional the problem of restriction $\mathfrak{sp}_4 \downarrow \mathfrak{sp}_2$ is equivalent to the problem of construction of a base in a $\mathfrak{sp}_4$-representation.

6.2.1 The problem of restriction $\mathfrak{sp}_4 \downarrow \mathfrak{sp}_2$

Let us investigate the problem of restriction $\mathfrak{sp}_4 \downarrow \mathfrak{sp}_2$ using the method of $Z$-invariants. The polynomials $Z_{\mathfrak{sp}_4}$ that are invariant under $Z_{\mathfrak{sp}_2}$ are polynomials in variables $z_{-2,-1}, z_{-2,1}, z_{-1,1}$.

The indicator system is

$$\frac{\partial}{\partial z_{-2,-1}} + z_{-1,1} \frac{\partial}{\partial z_{-2,1}})^{r-2+1}f = 0$$

$$\frac{\partial}{\partial z_{-1,1}})^{r-1+1}f = 0,$$ \hspace{1cm} (30)
where \( r_2 = m_{2,2} - m_{-1,2} \), \( r_1 = m_{-1,2} \).

The system (30) is exactly the system (12), that appears in the problem of restriction \( \mathfrak{gl}_3 \uparrow \mathfrak{gl}_1 \).

6.2.2 The Gelfand-Tsetlin type base for \( \mathfrak{sp}_4 \)

In the case \( \mathfrak{gl}_3 \) the indicator system with exponents \( r_2, r_1 \) corresponds to the highest weight \( [m_{-2,2}, m_{-1,2}, 0] \). Base vectors of a \( \mathfrak{gl}_3 \)-representation are encoded by tableaux (14). Hence the base vectors of a \( \mathfrak{sp}_4 \)-representation can be encoded by a \( \mathfrak{gl}_3 \)-tableau (form which we remove zero in the upper row), that is by tableaux of type

\[
\begin{array}{c}
m_{-2,2} & m_{-1,2} \\
m'_{-2,2} & m'_{-1,2} \\
m_{-2,1}.
\end{array}
\] (31)

The entries of this tableau must satisfy the inequalities

\[
\begin{align*}
m_{-2,2} & \geq m'_{-2,2} \geq m_{-1,2} \geq m'_{-1,2} \geq 0 \\
m'_{-2,2} & \geq m_{-2,1} \geq m'_{-1,2}.
\end{align*}
\] (32) (33)

The following Proposition takes place

**Proposition 4.** The number \( m_{-2,1} \) in the tableau (31) is the \( \mathfrak{sp}_4 \)-weight of the corresponding vector.

The proof is analogous to the proof of the Proposition 1.

Let us give the formula for the weight \( [\Delta(m)_{-2}, \Delta(m)_{-1}] \) of the vector encoded by the tableau \( (m) \). It is enough to consider \( \Delta(m)_{-1} \).

**Proposition 5.** \( \Delta(m)_{-1} = -2 \sum_i m'_{i,2} + \sum_i m_{i,2} + m_{-2,1} \)

The proof of this Proposition is analogous to the proof of Proposition 5.
6.2.3 Matrix elements for $\mathfrak{sp}_4$

Matrix elements for $F_{-1,-2}$ and $F_{1,-2}$ can be calculated if we consider these operators as $\mathfrak{sp}_2$-tensor operators acting between $\mathfrak{sp}_2$-representations into which a $\mathfrak{sp}_4$-representation splits.

Let us write the Wigner-Eckart theorem for $F_{-1,-2}$ and $F_{1,-2}$. Since the algebra $\mathfrak{sp}_2$ is one dimensional, the corresponding Wigner coefficient is zero. Hence the matrix elements of $F_{-1,-2}$ and $F_{1,-2}$ equal to the reduced matrix elements.

As in Section 6.1.3 let us find the relation of these reduced matrix elements with the analogous matrix elements for $\mathfrak{gl}_3$.

Let $(\bar{m})$, $(m)$ be two tableaux of type (31), and let $(\bar{m})^*$, $(m)^*$ be two tableaux for $\mathfrak{gl}_3$, that are obtained from $(\bar{m})$, $(m)$ by adding zero 0 to the upper row in the right. The following Lemma takes place.

**Lemma 2.** Matrix elements for $\mathfrak{sp}_4$ and $\mathfrak{gl}_3$ are equal

$$< (\bar{m}) | F_{\pm 1,2} | (m) > = < (\bar{m})^* | \pm 1,2 | (m)^* >$$

The proof is analogous to the proof of Lemma 1.

Introduce notations $[m]_2 = [m_{-2,2}, m_{-1,2}, 0]$, $[m']_2 = [m'_{-2,2}, m'_{-1,2}]$, $[m]_1 = [m_{-2,1}]$.

Using Lemma 2 let us calculate matrix elements for $F_{-1,2}$ and $F_{1,2}$. They are given by the following theorem.

**Theorem 5.** The matrix element $< (\bar{m}) | F_{-1,-2} | (m) >$ can be nonzero only if the following condition holds. For one index $i_1 = -2$ or $-1$ one has $\bar{m}'_{i_1,2} = m'_{i_1,2} - 1$, and for another one has $\bar{m}_{ij} = m_{ij}$. Also $\bar{m}_{2,1} = m_{2,1} - 1$.

If this condition holds one has

$$< (\bar{m}) | F_{-1,-2} | (m) > = \frac{3}{2} ^{i_1 : 2} | m_{1,2}, m'_{1,2}, m_{1,1} |_{1,1} \frac{1}{2} ^{-2} | m'_{2,1}, m_{2,1} >,$$  \hspace{1cm} (35)$$

The matrix element $< (\bar{m}) | F_{-1,-2} | (m) >$ can be nonzero only if the following condition holds. One has $\bar{m}_{-2,1} = m_{-2,1} - 1$ and all other entries of $(\bar{m})$ and $(m)$ are equal. If this condition holds one has
\begin{equation}
\langle (\tilde{m}) \mid F_{-1,-2} \mid (m) \rangle = \frac{2}{-2:1}^{[m']_2,[m]_1},
\end{equation}

The proof of this Theorem is based on Lemma 2 and is analogous to the proof of Theorem 2.

6.3 The construction in the case $\mathfrak{o}_4$

Let us be given a $\mathfrak{o}_4$-representation with the highest weight $[m_{-2,2}, m_{-1,2}]$. Since the algebra $\mathfrak{o}_2$ is one dimensional the problem of restriction $\mathfrak{o}_4 \downarrow \mathfrak{o}_2$ is equivalent to the problem of construction of a base in a $\mathfrak{o}_4$-representation.

6.3.1 The problem of restriction $\mathfrak{o}_4 \downarrow \mathfrak{o}_2$

Consider the problem of restriction $\mathfrak{o}_4 \downarrow \mathfrak{o}_2$. The polynomials on $Z_{\mathfrak{o}_4}$ that are invariant under $Z_{\mathfrak{o}_2}$ are the polynomials in variables $z_{-2,1}, z_{-2,1}$.

The indicator system is

\begin{equation}
\left( \frac{\partial}{\partial z_{-2,-1}} \right)^{r_{-2} + 1} f = 0,
\left( \frac{\partial}{\partial z_{-2,1}} \right)^{r_{-1} + 1} f = 0,
\end{equation}

where $r_{-2} = m_{-2,2} - m_{-1,2}, r_{-1} = m_{-2,2} + m_{-1,2}$.

All polynomial solutions are linear combination of the following polynomials (we use the standard normalization for $\mathfrak{sl}_2$)

\begin{equation}
f = \frac{z^p_{-2,-1}}{\sqrt{p!(2r_{-1} - p)!}} \frac{z^q_{-2,1}}{\sqrt{q!(2r_{-2} - q)!}},
0 \leq p \leq r_{-2},
0 \leq q \leq r_{-1}.
\end{equation}
6.3.2 The Gelfand-Tsetlin type base for \( \mathfrak{o}_4 \)

We see that there exist a base whose vectors are defined by two numbers \( p, q \). Following Molev (see [6]) let us define another indexation. Put

\[
m'_{-2,2} = m_{-2,2} - \min\{p, q\}, \quad m_{-2,1} = m_{-1,2} - p + q.
\]  

(39)

**Proposition 6.** The numbers \( p \) and \( q \) are reconstructed as follows

1. \( \text{Ecau} \) \( m'_{-2,1} - m_{-1,2} > 0 \), \( \text{mo} \) \( p = m_{-2,2} - m'_{-2,2}, \ q = m_{-2,2} - m'_{-2,2} + m'_{-2,1} - m_{-1,2} \).

2. \( \text{Ecau} \) \( m'_{-2,1} - m_{-1,2} \leq 0 \), \( \text{mo} \) \( q = m_{-2,2} - m'_{-2,2}, \ p = m_{-2,2} - m'_{-2,2} - m'_{-2,1} + m_{-1,2} \).

Let us construct the Gelfand-Tsetlin type tableau in the following way

\[
\begin{array}{cc}
  m_{-2,2} & m_{-1,2} \\
  m'_{-2,2} & \\
  m_{-2,1} & \\
\end{array}
\]

(40)

The equalities (38) are satisfied if and only if (see [6]), when the following inequalities take place

\[
m_{-2,2} \geq m'_{-2,2} \geq |m_{-1,2}| \tag{41}
\]

\[
m'_{-2,2} \geq |m_{-2,1}| \tag{42}
\]

One has

**Proposition 7.** The number \( m_{-2,1} \) in the tableau (40) is the \( o_2 \)-weight of the corresponding vector.

The proof is analogous to the proof of Proposition 1

Let us give formulas for the weight \([\Delta(m)_{-2}, \Delta(m)_{-1}]\) of the vector encoded by the tableau \((m)\). It is enough to consider the component \( \Delta(m)_{-1} \).

**Proposition 8.** \( \Delta(m)_{-1} = -2 \sum_i m'_{i,2} + \sum_i m_{i,2} + m_{-2,1} \)

The proof is analogous to the proof of Proposition 2
6.3.3 Matrix elements

Since the construction of the Gelfand-Tsetlin type base for $\mathfrak{o}_4$ has another character than the construction for the algebras $\mathfrak{o}_5$ and $\mathfrak{sp}_4$ the calculation of matrix elements does not use the Wigner-Eckart theorem. It is based on the isomorphism $\mathfrak{o}_4 \simeq \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$.

The operator $F_{-1,-2}$ increases the number $p$ by 1 and multiplies the vector on the reduced matrix element $\sqrt{p!(2r-2-p)!}$.

The operator $F_{-1,2}$ increases the number $q$ by 1 and multiplies the vector on the reduced matrix element $\sqrt{q!(2r-1-q)!}$.

When we pass to the Gelfand-Tsetlin type tableaux then using Proposition 6, we obtain the theorem

**Theorem 6.** The action of $F_{-1,-2}$ on the tableau (10) is described as follows. If $m'_{2,1} - m_{-1,2} + 1 \leq 0$, then $F_{-1,-2}$ diminishes $m_{-2,1}$ by 1. If $m'_{2,1} - m_{-1,2} + 1 > 0$ then also $m'_{-2,2}$ diminishes by 1. In both cases the vector is multiplied onto $\sqrt{p!(2r-2-p)!}$ (see Proposition 6).

The action of $F_{-1,2}$ on the tableau (10) is described as follows. If $m'_{2,1} - m_{-1,2} - 1 \geq 0$, then $F_{-1,2}$ increases $m_{-2,1}$ by 1. If $m'_{2,1} - m_{-1,2} + 1 < 0$ then also $m'_{-2,2}$ diminishes by 1. In both cases the vector is multiplied by $\sqrt{q!(2r-1-q)!}$ (see Proposition 6).

7 The construction in the case $g_n$

In the present section in the case $n > 2$ we construct a Gelfand-Tsetlin type base in a $g_n$-representation, the construction is similar for all algebras. Then we calculate the Wigner coefficients for $g_{n-1}$, the calculations for $g_2$ are different for series $B, C, D$ but the calculations for $g_n, n > 2$ are similar for all algebras. Then we derive explicit formulas for the action of generators in the Gelfand-Tsetlin type base.

Consider a $g_n$-representation with the highest weight $[m_{-n,n}, \ldots, m_{-1,n}]$. 

26
7.1 The indicator system in the problem of restriction $g_n \downarrow g_{n-1}$

The indicator system $I_n$ can be presented as a union of two systems of equation: $I'_n$ and $I_2$. Here $I_2$ is an indicator system that appears in the problem of restriction $g_2 \downarrow g_1$, it is different for series $B$, $C$, $D$ (see Sections 6.1.1, 6.2.1, 6.3.1), and the system $I'_n$ is the same for all algebras, this is the following system

\[
\begin{align*}
(z_{-n+1,-1} \frac{\partial}{\partial z_{-n,-1}} + z_{-n+1,1} \frac{\partial}{\partial z_{-n,1}})^{r_{-n+1}} f &= 0, \\
&\quad \ldots \\
(z_{-2,-1} \frac{\partial}{\partial z_{-3,-1}} + z_{-2,1} \frac{\partial}{\partial z_{-3,1}})^{r_{-3}} f &= 0.
\end{align*}
\]

The exponents $r_i$ are defined by formulas

\[r_{-n} = m_{-n,n} - m_{-n+1,n}, \ldots, r_{-3} = m_{-3,n} - m_{-2,n}.\]  

The indicator system has this type not only in the problem of restriction $g_n \downarrow g_{n-1}$ for all algebras $g_n = \mathfrak{o}_{2n+1}, \mathfrak{sp}_{2n}, \mathfrak{o}_{2n}$, but also in the problem of restriction $\mathfrak{gl}_{n+1} \downarrow \mathfrak{gl}_{n-1}$.

Using this remark let us construct a base in the solution space of $I_n$, starting from a base in the solution space of $I_2$. Firstly let us show how this base is constructed in the case $\mathfrak{gl}_{n+1}$, and then let us show that this construction is valid also in the case $g_n$.

Consider the algebra $\mathfrak{gl}_{n+1}$. Let this algebra act in the space with coordinates $x_{-n}, x_{-n+1}, \ldots, x_{-1}, x_1$. Then $\mathfrak{gl}_{n-1}$-highest vector are indexed by tableaux

\[
\begin{align*}
m_{-n,n}, & \quad m_{-n+1,n}, \quad m_{-n+2,n}, \ldots, \quad m_{-3,n}, \quad m_{-2,n}, \quad m_{-1,n}, \quad m_{1,n} \\
m_{-n,n-1}, & \quad m_{-n+1,n-1}, \ldots, \quad m_{-3,n-1}, \quad m_{-2,n-1}, \quad m_{-1,n-1} \\
m_{-n,n-2}, & \quad m_{-n+1,n-2}, \ldots, \quad m_{-3,n-2}, \quad m_{-2,n-2}
\end{align*}
\]

the entries of this tableau must satisfy the betweenness conditions.
The procedure of construction of solutions consists of two steps.

**Step 1.** Let us start with a solution \( f(z_{-2,-1}, z_{-2,1}, z_{-1,1}) \) of the system \( I_2 \), corresponding to a \( gl_3 \)-tableau

\[
\begin{align*}
&m_{-2,n}, m_{-1,n}, m_{1,n} \\
&m_{-2,n-1}, m_{-1,n-1}, \\
&m_{-2,n-2},
\end{align*}
\]

Also \( f \) is a solution of the system \( I_n \), which is encoded by a tableau

\[
\begin{align*}
&n, m_{-n,n}, m_{-n+1,n}, m_{-n+2,n}, \ldots, m_{-3,n}, m_{-2,n}, m_{-1,n}, m_{1,n} \\
&m_{-n,n}, m_{-n+1,n}, \ldots, m_{-3,n}, m_{-2,n-1}, m_{-1,n-1}, \\
&m_{-n,n}, m_{-n+1,n}, \ldots, m_{-3,n}, m_{-2,n-2},
\end{align*}
\]

**Step 2.** Let us transform the solution \( f \) of the system \( I_n \) to the solution that corresponds to the tableau (45). In [2] it is shown that the polynomial corresponding to the tableau (45) is of type

\[
\prod_{i=-n}^{-2} \nabla_{-1,i}^{m_{i,n-1}-m_{i,n-2}} \prod_{i=-n}^{-1} z_{i,1}^{m_{i,n}-m_{i,n-1}}
\]

where

\[
\nabla_{k,i} = \sum_{j_1 < \ldots < j_s} c_{j_1 \ldots j_s} E_{k,j_1} E_{j_1,j_2} \ldots E_{j_s,j},
\]

\[
c_{j_1 \ldots j_s} = \prod_{k>j>i,j \neq j_k} (E_{i,i} - E_{j,j} + j + i).
\]

One can easily check that the operators \( \nabla_{-1,i} \) and the operator of multiplication on \( z_{j,1} \) commute for \( i < j \), thus the polynomial corresponding to (45) can be written as

\[
(\prod_{i=-n}^{-3} \nabla_{-1,i}^{m_{i,n-1}-m_{i,n-2}} \prod_{i=-n}^{-3} z_{i,1}^{m_{i,n}-m_{i,n-1}})(\nabla_{-1,-2}^{m_{-2,n-1}-m_{-2,n-2}} z_{-2,1}^{m_{-2,n}-m_{-2,n-1}} z_{-1,1}^{m_{-1,n}-m_{-1,n-1}}),
\]

(50)
The second expression in (50) is the polynomial \( f \) up to multiplication on a constant. Introduce a notation for the first factor.

\[
\Omega^{\mathfrak{g}_{n+1}} = \prod_{i=-n}^{-3} \nabla_{-1,i}^{m_{i,n-1}-m_{i,n-2}} \prod_{i=-n}^{-3} z_{i,1}^{m_{i,n}-m_{i,n-1}}. \quad (51)
\]

Thus to the tableau (45) there corresponds the vector

\[
\Omega^{\mathfrak{g}_{n+1}} f. \quad (52)
\]

Let us formulate the statement that we have proved. For this let us give a definition.

**Definition 1.** Let \( f \) be a polynomial corresponding to the tableau (46). A set of operators of type (51) is called admissible, if after application of operators from this set to \( f \) one obtains all solutions of \( I'_{n} \) in the space of polynomials \( F(z_{-n,1}, z_{-n,1}, ..., z_{-3,1}, z_{-2,1}, z_{-2,1}) \) with coefficients in \( \mathbb{C} \) with the initial condition \( F(0,0,...,0,z_{-2,1}, z_{-2,1}) = f(z_{-2,1}, z_{-2,1}) \). And also \( \Omega_{1} f \neq \Omega_{2} f \), where \( \Omega_{1} \neq \Omega_{2} \) are admissible operators.

The previous discussion shows that the set of admissible operators depend on the elements \( m_{-2,n}, m_{-2,n-1}, m_{-2,n-2} \) of the tableau (46) only. An operator \( \Omega^{\mathfrak{g}_{n+1}} \) is admissible if and only if the numbers \( m_{i,n}, m_{i,n-1}, m_{i,n-2} \) satisfy

\[
m_{-n,n} \geq m_{-n,n-1} \geq m_{-n+1,n} \geq \ldots \geq m_{-3,n} \geq m_{-3,n-1} \geq m_{-2,n} \]
\[
m_{-n,n-1} \geq m_{-n,n-2} \geq m_{-n+1,n-1} \geq \ldots \geq m_{-3,n-1} \geq m_{-3,n-2} \geq m_{-2,n-1}, \quad (53)
\]

(Using explicit formula for the polynomial \( f \) (see [2]) corresponding to the tableau (46), one can see that the numbers \( m_{-2,n}, m_{-2,n-1}, m_{-2,n-2} \) are defined by the degrees of variables \( z_{-2,1}, z_{-2,1} \) in the polynomial \( f \).

Put \( z_{-1,1} = 1 \), one gets the proposition.

**Proposition 9.** In the case \( \mathfrak{g}_{n+1} \) all solution of the system \( I'_{n} \) in the space of polynomials of type \( F(z_{-n,1}, z_{-n,1}, ..., z_{-3,1}, z_{-2,1}, z_{-2,1}) \) with coefficients in \( \mathbb{C} \) with the initial condition \( F(0,0,...,0,z_{-2,1}, z_{-2,1}) = f(z_{-2,1}, z_{-2,1}) \). can
be written as $\Omega^{\mathfrak{gl}_{n+1}} f$, where the set of admissible operators $\Omega^{\mathfrak{gl}_{n+1}}$ is defined by the monomials of $f$ or by inequalities (53).

Let us give an analogous construction in the case $g_n$. Put $k = C(z_{-1,1})$ in the case $o_{5}$, put $k = C(z_{-1,0})$ in the case $sp_4$, put $k = C$ in the case $o_4$.

Note that in the case $g_n$ the operator $F_{ij}$ acts on a polynomial in variables $z_{-2,-1}, z_{-2,1}, \ldots, z_{-n,-1}, z_{-n,1}$ exactly in the same way as the operator $E_{ij}$ acts on this polynomial.

**Definition 2.** Define a operator $\Omega^{g_n}$ by formulas (51), (49), where $E_{ij}$ is replaced to $F_{ij}$.

Now let us construct the set of admissible operators $\Omega^{g_n}$. Let us be given a $g_2$-tableau $D$ and denote the corresponding polynomial as $f$. Our next purpose is to describe the set of admissible operators $\Omega^{g_n}$, such that applying them to $f$ we obtain all solutions of $I'_{n}$ with the initial condition $f$ as in Proposition 9.

In the cases $o_{2n+1}$, $sp_{2n}$ in Sections 6.1.1, 6.2.1 a correspondence between problems of restriction $\mathfrak{gl}_3 \downarrow \mathfrak{gl}_1$ and $o_5 \downarrow o_3$, $sp_4 \downarrow sp_2$ was established. This correspondence is generated by a correspondence between solution spaces of indicator systems. The later correspondence preserves the degrees of variables $z_{-2,-1}, z_{-2,1}$.

The system $I'_n$ is the same in the case $g_n$ and in the case $\mathfrak{gl}_{n+1}$. Thus one gets that the condition providing that an operator is admissible in the cases $\Omega^{o_{2n+1}}$, $\Omega^{sp_{2n}}$ and in the case $\Omega^{\mathfrak{gl}_{n+1}}$ are the same. This condition is the set of inequalities (53).

Now let us find the admissible operators in the case $o_{2n}$. In the case $\mathfrak{gl}_3$ the element $m_{-2,n-1}$ of the Gelfand-Tsetlin tableau can be characterized as follows. We act on the vector corresponding to the tableau by the raising operator $E^k_{-1,-2}$ for the maximum possible $k$. We obtain a $\mathfrak{gl}_2$-highest vector. Then $m_{-2,n-1}$ is a $(-2)$-component of its highest weight.

Consider a vector $\mathfrak{gl}_3$-representation defined by a polynomial $z^p_{-2,-1}z^q_{-2,1}$, it is a linear combination of tableaux. Let us give a formula for the biggest element $m_{-2,n-1}$ of these tableaux. Under the action $E^k_{-1,-2}$ this polynomial is
transformed into \( const z_{-2,1}^{p-k} z_{-2,1}^{q-k} \). Thus the maximum value of \( k \) is \( \min(p, q) \). Hence

\[
m_{-2,n-1} = m_{-2,n} - \min(p, q).
\]  

(54)

In the case \( \mathfrak{g}_4 \) to a Gelfand-Tsetlin type tableax there corresponds a polynomial \( const z_{-2,1}^{p-k} z_{-2,1}^{q-k} \). The formula for the element \( m'_{-2,2} \) of this tableau is just (54).

Also in the case of the problem of restriction \( \mathfrak{g}_3 \downarrow \mathfrak{g}_1 \) the element \( m_{-2,n-2} \) is the weight of the corresponding vector relatively the subalgebra \( \mathfrak{g}_1 \). In the case \( g_n = \mathfrak{g}_4 \) and the problem of the restriction \( g_n \downarrow g_{n-1} \) the element \( m_{-2,1} \) is the weight relatively \( g_{n-1} = \mathfrak{o}_2 \).

Thus in the case \( \Omega^{\mathfrak{o}_2} \) and in the case \( \Omega^{\mathfrak{o}_2} \) are the same. This condition is the set of inequalities (53).

The following analog of the Proposition 9 takes place.

**Proposition 10.** In the case \( g_n \) all solutions of the system \( I_n' \) in the space of polynomials \( F(z_{-n-1}, z_{-n}, ..., z_{-3,1}, z_{-2,-1}, z_{-2,1}) \) with coefficients in \( \mathfrak{g} \) with the initial condition \( F(0,0, ..., 0, z_{-2,-1}, z_{-2,1}) = f(z_{-2,-1}, z_{-2,1}) \) can be written as \( \Omega^{\mathfrak{o}_2} f \), where only the admissible operators \( \Omega^{\mathfrak{o}_2} \) are taken.

Note that if \( f \) is a polynomial over \( \mathbb{C} \) then \( \Omega^{\mathfrak{o}_2} f \) is a polynomial over \( \mathbb{C} \).

Using an explicit description of admissible operators one obtains a proposition

**Proposition 11.** In the case \( g_n \) all polynomial solutions \( I_n \) can be written as \( \Omega^{\mathfrak{o}_2} f \), they are encoded by tableaux of type

\[
\begin{align*}
m_{-n,n}, & \quad m_{-n+1,n}, \quad m_{-n+2,n}, \quad ..., \quad m_{-3,n}, \\
m'_{-n,n}, & \quad m'_{-n+1,n}, \quad ..., \quad m'_{-3,n}, \quad D
\end{align*}
\]  

(55)

where \( D \) is a Gelfand-Tsetlin type tableaux for a \( g_2 \)-representation with the highest weight \( [m_{-2,n}, m_{-1,n}] \). The following inequalities must take place

\[
\begin{align*}
m_{-n,n} & \geq m'_{-n,n} \geq m_{-n+1,n} \geq ... \geq m_{-3,n} \geq m'_{-3,n} \geq m_{-2,n} \geq ... \\
m'_{-n,n} & \geq m_{-n,n-1} \geq m'_{-n+1,n} \geq ... \geq m'_{-3,n} \geq m_{-3,n-1} \geq m'_{-2,n} \geq ...,
\end{align*}
\]  

(56)
supplied with the inequalities corresponding to the $g_2$-tableau $D$.

The polynomial $f$ is defined by $D$, and $\Omega g_n$ is defined by the rest part of the tableau (55).

7.2 The Gelfand-Tsetlin type base for $g_n$

To use this indexation of $g_{n-1}$-highest obtain in Proposition 11 vectors for the construction of the Gelfand-Tsetlin type base we must prove the following Proposition.

**Proposition 12.** The lower row in (55) is the $g_{n-1}$-weight of the corresponding $g_{n-1}$-highest vector.

**Proof.** The proof is based on the following fact. Let us be given a polynomial from $\mathfrak{g}[z_{-3,-1}, z_{-3,1}, ..., z_{-n,-1}, z_{-n,1}]$, corresponding to the tableau (55). The action on it of the operators $E_{i,i} i = -n, ..., -3$, in the case $\mathfrak{g}l_{n+1}$, and of the operators $F_{i,i}$, in the case $g_n$, coincide.

For the operators $E_{i,i}$ this polynomial is an eigenvector with the eigenvalue $m_{i,n-2}$. Thus for the components of the weight with indices $i = -n, ..., -3$ the statement is proved. For $i = -2$ the statement follows form the analogous statement for $g_2$.

Applying the standard procedure of construction of the Gelfand-Tsetlin base we obtain the Theorem.

**Theorem 7.** In a representation of $g_n$ there exist a base called the Gelfand-Tsetlin type base. Its vector are encoded by tableaux (1). The rows $[m], [m'], [m]_{k-1}$ of these tableaux are of type (55), where $D$ is a Gelfand-Tsetlin tableau for $g_2$.

The elements of these tableaux satisfy the inequalities (56), supplied with inequalities corresponding to the $g_2$-tableau $D$.

Let us give the formulas for the weight of the vector encoded by a tableau $(m)$. Denote it as

$$\Delta(m) = [\Delta(m)_{-n}, ..., \Delta(m)_{-1}],$$

(57)
Proposition 13.

$\Delta(m)_{-k+n-1} = -2 \sum_i m'_{i,k} + \sum_i m_{i,k} + \sum_i m_{i,k-1} \text{ in the cases } \mathfrak{sp}_{2n}, \mathfrak{o}_{2n},$

$\Delta(m)_{-k+n-1} = -2 \sum_i m'_{i,k} + \sum_i m_{i,k} + \sum_i m_{i,k-1} - \sigma_{-k} \text{ in the cases } \mathfrak{o}_{2n+1}$

The proof can be found in Appendix in Section 8.7.

7.3 Reduced matrix elements

To calculate Wigner coefficients and matrix elements of generators we need reduced matrix elements of the operator $F_{-1,-2}$, viewed as $g_{n-1}$-tensor operator that acts between $g_{n-1}$-representations into which a $g_n$-representation splits.

Let $(\bar{m}), (m)$ be two tableaux for the algebra $g_n$. In the calculation of the reduced matrix elements we can suggest that these tableaux are maximal with respect to the subalgebra $g_{n-1}$.

**Definition 3.** Denote as $(m)_{red}$ the part of the tableau $(m)$, formed by three upper rows (that is the tableau (55)) from which the $g_2$-tableau $D$ is removed. Thus the three upper rows of $(m)$ (on which the reduced matrix element depend) can be written as $(m)_{red}D$.

Note that $(m)_{red}$ is also a part of a $\mathfrak{gl}_{n+1}$-tableau. Its three upper rows can be written as $(m)_{red}D$, where $D$ is a $\mathfrak{gl}_3$-tableau. The action of $E_{-1,-2}$ on the tableau $(m)_{red}D$ is know (see [2]). The result is a linear combination of tableaux, the $i$-th tableau in this combination is obtained by subtracting 1 from the $i$-th element of the third row.

Thus one can write

$$E_{-1,-2}(m)_{red}D = (E_{-1,-2}(m)_{red})D + (m)_{red}(E_{-1,-2}D). \quad (58)$$

To obtain an analogous equality for $F_{-1,-2}$, let us use the formula for the polynomial $f$, corresponding to $(m)_{red}D$ that was obtained in [7,1]. One has
where \( c \) is a polynomial, corresponding to \( D \).

The action of \( E_{-1, -2} \) on \( f \) is described as follows. We multiply the expression (59) onto \( E_{-1, -2} \) in the left and then move \( E_{-1, -2} \) to \( f_0 \). From the commutation relations new summands appear. They correspond to the member \((E_{-1, -2} D)_{\text{red}}\) in the expression (58). The member corresponding to the action of \( E_{-1, -2} \) on \( f_0 \), corresponds to the member \((m)_{\text{red}}(E_{-1, -2} D)_{\text{red}}\)

Let us formulate the ruler for calculation of \( <(\bar{m})_{\text{red}} | E_{-1, -2} | (m)_{\text{red}} >_{\text{red}}\).

**Lemma 3.**

\[
F_{-1, -2}(m)_{\text{red}} D = (E_{-1, -2} (m)_{\text{red}}) D + (m)_{\text{red}} (E_{-1, -2} D).
\]  

**Proof.** The proof is an immediate consequence of the following fact. The correspondence: \( E_i, -2 \mapsto F_i, -2 \), \( i = -n, ..., -1 \) is agreed with commutators. 

Let us give an explicit formula for the reduced matrix elements.

Note that when we apply the operator

\[
\Omega_{\text{gl}_n +1} = \prod_{i=-n}^{-3} \nabla_{-1,i}^{-m_i,n-1} \prod_{i=-n}^{-3} z_{i,1}^{m_i,n-m_i'n},
\]

to the function, which is identically equal to one, we get a vector corresponding to the diagram

\[
m_{-n,n}, m_{-n+1,n}, m_{-n+2,n}, ..., m_{-3,n}, m_{-2,n}, m_{-1,n}, m_{1,n} \\
m'_{-n,n}, m'_{-n+1,n}, ..., m'_{-3,n}, m_{-2,n}, m_{-1,n}, \\
m_{-n,n-2}, m_{-n+1,n-1}, ..., m_{-3,n-1}, m_{-2,n},
\]

where the \( \text{gl}_3 \)-tableau on the right is maximal. Denote this \( \text{gl}_3 \)-tableau as \( D_{\text{max}} \).

Define the symbol \( <(\bar{m})_{\text{red}} | E_{-1, -2} | (m)_{\text{red}} >_{\text{red}} \) as follows. If there exists \( i_1 = -n, ..., -3 \), such that \( \bar{m}_{i_1,n-1} = m_{i_1,n-1} - 1 \) and all other elements
of rows $(\bar{m})_{\text{red}}, (m)_{\text{red}}$ coincide that it equals the $\mathfrak{gl}_{n+1}$-matrix element $< (\bar{m})_{\text{red}}D_{\text{max}} | E_{-1,-2} | (m)_{\text{red}}D_{\text{max}} >$. Otherwise it is zero.

An explicit formula for the matrix element $< (\bar{m})_{\text{red}}D_{\text{max}} | E_{-1,-2} | (m)_{\text{red}}D_{\text{max}} >$ is obtained in [2]. It equals

$$< (\bar{m})_{\text{red}}D_{\text{max}} | E_{-1,-2} | (m)_{\text{red}}D_{\text{max}} > = \prod_{i=-n}^{-1}(l'_{i,n} - l_{i,n-1}) \prod_{i=-n,i \neq i}(l_{i,n-1} - l_{i,n-1}),$$

where

$$l'_{i,n} = m'_{i,n} + i, \quad l_{i,n-1} = m_{i,n-1} + i.$$  

(63)

Theorem 8.

$$< (\bar{m})_{\text{red}}D | F_{-1,-2} | (m)_{\text{red}}D >_{\text{red}} = \delta_{D,D} < (\bar{m})_{\text{red}} | E_{-1,-2} | (m)_{\text{red}} >_{\text{red}} + \delta(\bar{m})_{\text{red}}(m)_{\text{red}} < \bar{D} | F_{-1,-2} | D >_{\text{red}}.$$  

(65)

It is suggested that $\bar{m}_{i,1,n-1} = m_{i,1,n-1} - 1$ and $\bar{m}_{j,n-1} = m_{j,n-1}$ for $j \neq i$.

7.4 Wigner coefficients

Let us obtain formulas for the Wigner coefficients for $g_{n-1}$. Following [1], let us first obtain formulas for the coefficients

$$< (\bar{m}) \begin{pmatrix} j \\ \end{pmatrix}_{[10]_{n-1}}^{j} (m) >.$$  

(66)

We use the following fact. Let us be given a diagram $(m)$ that define a vector in $s g_{n-1}$-representation with the highest weight $[m_{-n,n-1}, \ldots, m_{-2,n-1}]$.

A polynomial on the group $Z_{n-1}$ corresponds to this vector. Consider it as a polynomial on a bigger group $Z_n$.  

35
Proposition 14. The vector that corresponds to this polynomial belong to a \( g_n \)-representation with the highest weight \([m_{-n,n-1}, ..., m_{-2,n-1}, 0]\). The corresponding tableau is of type \( \left( \max \frac{m}{m} \right) \), this is a \( g_n \)-tableau that is obtained form \( (m) \) by adding two maximum row.

The proof can be found in Appendix in Section 10.

Now return to the calculation of the Wigner coefficient (66). Let us be given two \( g_{n-1} \)-tableaux \((\vec{m})\) and \((m)\). We can suggest that they are \( g_{n-2} \)-maximal. The three upper rows of these tableau are of type \((\vec{m})_{\text{red}}\vec{D}\) and \((m)_{\text{red}}D\). As in Proposition 14 add to them two maximal \( g_n \)-rows, denote the five rows that we obtain as \((\vec{m})_{\text{red}}\vec{D}\) and \((m)_{\text{red}}D\).

Let us prove the equality.

Proposition 15.

\[
\langle (\vec{m}) \begin{pmatrix} j \\ [10]_{n-1} \\ -2 \end{pmatrix} (m) \rangle = \langle (\vec{m})_{\text{red}}\vec{D} | F_{-1,-2} | (m)_{\text{red}}D \rangle . \tag{67}
\]

Proof. Let us apply the Wigner-Eckart theorem to the matrix element on the right. It equals to the product of a reduced matrix element and a Wigner coefficient that occurs on the left side of the equality. Since the upper two rows of tableaux \((\vec{m})_{\text{red}}D\), \((m)_{\text{red}}D\) are maximal the reduced matrix element equals to 1. This proves the equality.

\[\Box\]

Let us calculate the matrix element that occurs on the right in the equality 15.

Let us be given a tableau that defines a \( g_{n-2} \)-highest vector in a \( g_n \)-representation. This is a tableau of type...
\( m_{-n,n}, \ldots, m_{-4,n} \)
\( m'_{-n,n}, \ldots, m'_{-4,n} \)
\( m_{-n,n-1}, \ldots, m_{-4,n-1} \quad C \) \hspace{1cm} (68)
\( m'_{-n,n-1}, \ldots, m'_{-4,n-1} \)
\( m_{-n,n-2}, \ldots, m_{-4,n-2} \)

where \( C \) is a \( g_3 \)-tableau. Denote the tableau (68) shortly as \((k)C\). Using the technique of raising operators \( \nabla_{ij} \) (see [2]), and applying the arguments that were used in the proof of the formula (52), one obtains that the polynomial that corresponds to this tableau can be written as \( \Omega f \), where \( f \) is a polynomial that corresponds to a \( g_3 \)-tableau \( C \) and the operator \( \Omega \) is defined as follows

\[
\Omega = \prod_{i=-n}^{-3} \nabla_{-3,i}^{m_{i,n-1}-m_{i,n-2}} \prod_{i=-n}^{-3} \nabla_{-2,i}^{m_{i,n-1}-m'_{i,n-1}} \prod_{i=-n}^{-3} \nabla_{-1,i}^{m'_{i,n}-m_{i,n-1}} \prod_{i=-n}^{-3} \nabla_{z,i}^{m_{i,n}-m'_{i,n}}
\] \hspace{1cm} (69)

The considered tableau \(((\tilde{m})_{\text{red}}D)\) is of type (68). One has \((n) = (\tilde{m})_{\text{red}}\) and \( C = \tilde{D} \). By analogy with the proof of Lemma 3 one concludes that

\[
F_{-1,-2}((\tilde{m})_{\text{red}}D) = (E_{-1,-2}(m)_{\text{red}})\tilde{D} + (m)_{\text{red}}F_{-1,-2}\tilde{D}.
\] \hspace{1cm} (70)

When one passes to matrix elements, one gets

\[
<((\tilde{m})_{\text{red}}\tilde{D}) | F_{-1,-2} | (\tilde{m})_{\text{red}}D> = \delta_{\tilde{D},\tilde{D}} <(\tilde{m})_{\text{red}} | E_{-1,-2} | (\tilde{m})_{\text{red}}> + \delta_{\tilde{m}_{\text{red}},(m)_{\text{red}}} <\tilde{D} | F_{-1,-2} | \tilde{D}>
\] \hspace{1cm} (71)

Thus we have proved a Theorem
Theorem 9.

\[
< (\tilde{m}) \left( \begin{array}{c} j \\ \left[ 10 \right]_{n-1} \\ -2 \end{array} \right) (m) > = \\
= \delta_{\tilde{D},D} < (\tilde{m})_{\text{red}} \mid E_{-1,-2} \mid (m)_{\text{red}} > + \delta_{(\tilde{m})_{\text{red}},(m)_{\text{red}}} < \tilde{D} \mid F_{-1,-2} \mid D > .
\]

(72)

Let us give rulers for calculation of summands that occur on the right hand side in Theorem 9.

7.4.1 The matrix element \(< (\tilde{m})_{\text{red}} \mid E_{-1,-2} \mid (\tilde{m})_{\text{red}} >\)

As in previous section this matrix element can expressed through a matrix element of the algebra \(\mathfrak{gl}_{n+1}\).

Theorem 10. If there exists \(i_1 = -n, ..., -3\), such that \(\tilde{m}_{i_1,n-1} = m_{i_1,n-1} - 1\), and all other elements of rows \((\tilde{m})_{\text{red}}, (m)_{\text{red}}\) coincide than the considered matrix elements equals

\[
\prod_{i=-n}^{-1} (l_{i,n-1} - l_{i_1,n-2}) \\
\prod_{i=-n,i\neq i_1}^{-2} (l_{i,n-2} - l_{i_1,n-2}),
\]

(73)

where

\[
l_{i,n-1} = m_{i,n-1} + i, \quad l_{i,n-2} = m'_{i,n-1} + i.
\]

(74)

If there exist no such index than the considered matrix element equals zero.

7.4.2 The matrix element \(< \tilde{D} \mid F_{-1,-2} \mid \tilde{D} >\). The case \(\mathfrak{o}_5\)

This matrix element equals to a \(\mathfrak{o}_5\)-Wigner coefficients

\[
< \tilde{D} \mid F_{-1,-2} \mid \tilde{D} > = < \tilde{D} \left( \begin{array}{c} j \\ \left[ 10 \right]_{\mathfrak{o}_5} \\ -2 \end{array} \right) D > .
\]

(75)
Let us calculate it directly. One can suggest that \(\mathfrak{o}_5\)-tableaux \(\bar{D}\) and \(D\) are \(\mathfrak{o}_3\)-maximal.

To the tableau \(D\) there corresponds a polynomial \(f\) on \(Z_{\mathfrak{o}_5}\). Since \(D\) is \(\mathfrak{o}_3\)-maximal, then \(f = f(z_{-3,-2}, z_{-3,2})\). By Proposition 14, to the tableau \(\tilde{D}\) there corresponds the same polynomial, but considered as a polynomial on \(Z_{\mathfrak{o}_5}\).

Let find the action of the operator \(e^{tF_{-1,-2}}\) on the polynomial \(f\). The explicit calculation gives that

\[
f(z_{-3,-2}, z_{-3,2}) \mapsto (1 + tz_{-3,-1})^{m_{-2,2} - m_{-1,2}}(z_{-3,-2} + tz_{-3,-1}, z_{-3,2})
\]

When we were defining the Gelfand-Tsetlin type base for \(\mathfrak{o}_5\) in Section 6.1.1 for the polynomial \(f\) we have constructed a polynomial \(f^*\) on the group \(Z_{\mathfrak{gl}_4}\).

By explicit calculations it can be shown that on the polynomial \(f^*\) the operator \(e^{tF_{-1,-2}}\) acts by the same formula (76). Thus the correspondence conjugates the actions of \(e^{tF_{-1,-2}}\) and \(e^{tE_{-1,-2}}\).

In Section 6.1.2 using this correspondence the Gelfand-Tsetlin type base for \(\mathfrak{o}_5\) was constructed. To a \(\mathfrak{o}_5\)-tableau \(D\), that defines a \(\mathfrak{o}_3\)-highest vector (and a polynomial \(f\)), there corresponds a \(\mathfrak{gl}_3\)-tableau \(D^*\), which is obtained from \(D\) by removing the zero from the upper row and \(\sigma_{-2}\) (to the tableau \(D^*\) there corresponds the polynomial \(f^*)\).

Thus we have

\[
< \tilde{D} | F_{-1,-2} | \tilde{D} > = < \tilde{D}^* | E_{-1,-2} | \tilde{D}^* >
\]

The matrix element on the right in (77) is a \(\mathfrak{gl}_3\)-Wigner coefficient.

Thus we have proved the theorem

**Theorem 11.** The Wigner coefficient for \(\mathfrak{o}_5\) and \(\mathfrak{gl}_3\) are equal

\[
< \bar{D} \begin{pmatrix} \frac{j}{2} \\ [10]_{\mathfrak{o}_5} \end{pmatrix} D > = < \tilde{D}^* \begin{pmatrix} \frac{j}{2} \\ [100]_{\mathfrak{gl}_3} \end{pmatrix} D^* > = \left| j : 3 \right|^{[m]_2, [m']_2} \left| \frac{m}{2} \right|,
\]

(78)
where \([m]_2 = [m_{-2,2}, m_{-1,2}, 0]\) u \([m']_2 = [m'_{-2,2}, m'_{-1,2}]\).

**Следствие 1.**
\[
< \tilde{D} \mid F_{-1,-2} \mid \tilde{D} > = \left| \begin{array}{c} j : 3 \\ 2 \end{array} \right|^{[m]_2,[m']_2},
\]
(79)

where \([m]_2 = [m_{-2,2}, m_{-1,2}, 0]\) u \([m']_2 = [m'_{-2,2}, m'_{-1,2}]\).

### 7.4.3 The matrix element \(< \tilde{D} \mid F_{-1,-2} \mid \tilde{D} >\). The case \(\mathfrak{sp}_4\)

In this case the matrix element equals to a \(\mathfrak{sp}_4\)-Wigner coefficient
\[
< \tilde{D} \mid F_{-1,-2} \mid \tilde{D} > = < \tilde{D} \left( \begin{array}{c} j \\ [10]_{\mathfrak{sp}_4} \\ -2 \end{array} \right) D >.
\]
(80)

In section 6.2.2 the Gelfand-Tsetlin type base for \(\mathfrak{sp}_4\) was defined. To a \(\mathfrak{sp}_4\)-tableau \(D\) there corresponds a \(\mathfrak{gl}_3\)-tableau \(D^*\) from which the zero in the upper row is removed.

Analogously to the case \(\mathfrak{o}_5\) one can prove

**Theorem 12.** The Wigner coefficient for \(\mathfrak{sp}_4\) and \(\mathfrak{gl}_3\) are equal
\[
< \tilde{D} \left( \begin{array}{c} j \\ [10]_{\mathfrak{sp}_4} \\ -2 \end{array} \right) D > = < \tilde{D}^* \left( \begin{array}{c} j \\ [100]_{\mathfrak{gl}_3} \\ -2 \end{array} \right) D^* > = \left| \begin{array}{c} j : 3 \\ 2 \end{array} \right|^{[m]_2,[m']_2},
\]
(81)

where \([m]_2 = [m_{-2,2}, m_{-1,2}, 0]\) u \([m']_2 = [m'_{-2,2}, m'_{-1,2}]\).

**Следствие 2.**
\[
< \tilde{D} \mid F_{-1,-2} \mid \tilde{D} > = \left| \begin{array}{c} j : 3 \\ 2 \end{array} \right|^{[m]_2,[m']_2},
\]
(82)

where \([m]_2 = [m_{-2,2}, m_{-1,2}, 0]\) u \([m']_2 = [m'_{-2,2}, m'_{-1,2}]\).
7.4.4 The matrix element $\langle \tilde{D} \mid F_{-1,-2} \mid \tilde{D} \rangle$. The case $o_4$

In this case the matrix element equals to a $o_4$-Wigner coefficient.

$$\langle \tilde{D} \mid F_{-1,-2} \mid \tilde{D} \rangle = \langle \tilde{D} \begin{pmatrix} j \\ -2 \end{pmatrix} D \rangle .$$  

Let us calculate directly the $o_4$-Wigner coefficient on the right. The index $j$ can take values $-2$ and $-1$. If $j = -2$ then

$$\bar{m}_{-2} = m_{-2} + 1, \quad \bar{m}_{-1} = m_{-1},$$  

and if $j = -1$ then

$$\bar{m}_{-2} = m_{-2}, \quad \bar{m}_{-1} = m_{-1} + 1 .$$  

For $r_{-2} = m_{-2} - m_{-1}$ and $r_{-1} = m_{-2} + m_{-1}$ (these are highest weights for two $\mathfrak{sl}_2$ copies), one has in the case $j = -2$

$$\bar{r}_{-2} = r_{-2} + 1, \quad \bar{r}_{-1} = r_{-1} + 1,$$

and in the case $j = -1$

$$\bar{r}_{-2} = r_{-2} - 1, \quad \bar{r}_{-1} = r_{-1} - 1 .$$

The weights $p$ and $q$ for two copies of $\mathfrak{sl}_2$ are expressed through the elements of a $o_4$-tableau using the Proposition 6.

Thus one gets the Theorem

**Theorem 13.** Put $[m^1]_2 = \left[ \begin{array}{c} m_{-2} - m_{-1} - 1/2 \\ m_{-1} - 1/2 \\ 0 \end{array} \right]$, $[m^2]_2 = \left[ \begin{array}{c} m_{-2} + m_{-1} - 1/2 \\ m_{-1} - 1/2 \\ 0 \end{array} \right]$.

For $j = -2$ one has

$$\langle \tilde{D} \begin{pmatrix} j \\ [10]_{o_4} \end{pmatrix} D \rangle = \begin{vmatrix} \frac{1}{2} : 2 & [m^1]_{2,p} \\ 1 & 1 \end{vmatrix} \begin{vmatrix} \frac{1}{2} : 2 & [m^2]_{2,q} \\ 1 & 1 \end{vmatrix} .$$  

For $j = -1$ one has
\[ < \hat{D} \left( \begin{array}{c} j \\ [10]_{n-1} \\ -2 \end{array} \right) D > = \left| \frac{1}{2} : 2 \right|^2 \left[ m^1 \right]_{2,p} \left| \frac{-1}{2} : 2 \right|^2 \left[ m^2 \right]_{2,q}. \quad (89) \]

### 7.5 Reduced Wigner coefficients

For the Wigner coefficients

\[ < (\tilde{m}) \left( \begin{array}{c} j \\ [10]_{n-1} \\ i \end{array} \right) (m) > \quad (90) \]

the following formula takes place

\[ < (\tilde{m}) \left( \begin{array}{c} j_1 \\ [10]_{n-1} \\ i \end{array} \right) (m) > = \left( \begin{array}{c} j_{-i} \\ -i \end{array} \right) \prod_{l=1}^{\infty} \left| \begin{array}{c} \left[ 10 \right]_{n-l} \\ j_{l+1} \end{array} \right| \quad (91) \]

To calculate all Wigner coefficients we must obtain a formula for the reduced Wigner coefficients \[ \left| \begin{array}{c} j_l \\ [10]_{n-l} \\ j_{l+1} \end{array} \right|. \]

Let us obtain the formula for the reduced Wigner coefficients using the previous calculations. The following theorem takes place.

**Theorem 14.**

\[ \left| \begin{array}{c} j_1 \\ [10]_{n-1} \\ j_2 \end{array} \right| = \left( \begin{array}{c} j_1 \\ [10]_{n-1} \\ -2 \end{array} \right) < (\tilde{m}')_{red} \hat{D}' | F_{-2-3} | (m')_{red} \hat{D'} >_{red}, \quad (92) \]

where ' means that we take only the part of the tableaux that correspond to \( g_{n-1} \).

**Proof.** To prove the theorem let us calculate the matrix element \( < (\tilde{m}) | F_{-1-3} | (m) >, i < -2 \) in two ways.
Firstly apply to the matrix element the Wigner-Eckart theorem and decompose the Wigner coefficient into the product of a reduced Wigner coefficient and a Wigner coefficient. One has

\[
< (\bar{m}) | F_{-1-3} | (m) > = < (\bar{m})_{\text{red}} D | F_{-1-3} | (m)_{\text{red}} D >_{\text{red}} \begin{pmatrix} j_1 \\ [10]_{n-1} \\ j_2 \\ -3 \end{pmatrix}.
\]

(93)

Secondly using the commutation relation \[ F_{-1-3} = [F_{-1-2}, F_{-2-3}] \] we obtain

\[
< (m') | F_{-1-3} | (m) > = < (\bar{m})_{\text{red}} D | F_{-1-2} | (m)_{\text{red}} D >_{\text{red}} \begin{pmatrix} j_1 \\ [10]_{n-1} \\ j_2 \\ -2 \end{pmatrix}.
\]

\cdot < (\bar{m'})_{\text{red}} D' | F_{-2-3} | (m')_{\text{red}} D' >_{\text{red}} \begin{pmatrix} j_2 \\ [10]_{n-2} \\ -3 \end{pmatrix},
\]

(94)

where \( ' \) means that we take only the part of the tableaux that correspond to \( g_{n-1} \).

Compare two expressions, one obtains

\[
\begin{pmatrix} j_1 \\ [10]_{n-1} \\ j_2 \\ -2 \end{pmatrix} = \begin{pmatrix} j_1 \\ [10]_{n-1} \\ j_2 \end{pmatrix} < (\bar{m})_{\text{red}} D | F_{-1-3} | (m)_{\text{red}} D >_{\text{red}}
\]

(95)

The theorem is proved

\[ \square \]

The expressions in Theorem [13] are obtained in previous Sections.
7.6 Matrix elements

Using the previous results let us write the formulas for the action of generators of $g_n$ in the base that we have constructed. It is enough to give a formula for the action of $F_{-1,-2}$. The following theorem takes place.

Theorem 15.

\[ < (\bar{m}) | F_{-1,-2} | (m) > = < (\bar{m})_{\text{red}} | \bar{D} | F_{-1,-2} | (m)_{\text{red}} D >_{\text{red}} < (\bar{m})_{\text{red}} | [0]_{n-1} | (m) >, \]

(96)

where the expression for the factors are given in theorems 8 and 9.

Proof. The theorem is proved by application of the Wigner-Eckart theorem \qed

8 Appendix

8.1 Lie algebras.

8.1.1 The symplectic algebra $sp_{2n}$

Take the space $\mathbb{C}^{2n}$. Choose a base and let us index its elements by numbers $-n, ..., -1, 1, ..., n$. Fix a skew-symmetric form

\[ \omega = \sum_{i=1}^{n} x_i \wedge x_{-i}. \]

The group $Sp_{2n}$ consists of isomorphisms of $\mathbb{C}^N$, that preserve this skew-symmetric form. It’s Lie algebra is denoted as $sp_{2n}$.

Define the generators of the Lie algebra

\[ F_{ij} = E_{ij} - \text{sign}(i)\text{sign}(j)E_{-j-i}. \]

The only relations between them are

\[ F_{ij} = -\text{sign}(i)\text{sign}(j)F_{-j-i}. \]
The generators $F_{ij}$, in the case $i > j$, correspond to negative roots. The generators $F_{ij}$, in the case $i < j$, correspond to positive roots. In the case $i = j$ the generator belongs to the Cartan subalgebra.

### 8.1.2 The orthogonal algebra $\mathfrak{o}_N$

Take the space $\mathbb{C}^N$. Let $n$ be such that $N = 2n$ in the case of even $N$, and $N = 2n + 1$ in the case of odd $N$.

Choose a base in $\mathbb{C}^N$ and index its elements by $-n, ..., -1, 1, ..., n$ in the case of even $N$ and by numbers $-n, ..., -1, 0, 1, ..., n$ in the case of odd $N$.

In the case of even $N$ take a quadratic form

$$x_{-n}x_n + ... + x_{-1}x_1,$$

and in the case of odd $N$ take a quadratic form

$$x_{-n}x_n + ... + x_{-1}x_1 + x_0^2.$$

The group $O_N$ consists of isomorphisms of $\mathbb{C}^N$, that preserve this quadratic form. It’s Lie algebra is denoted as $\mathfrak{o}_N$.

Introduce generators

$$F_{ij} = E_{ij} - E_{-j-i}.$$

The only relations between them are

$$F_{ij} = -F_{-j-i}.$$

The generators $F_{ij}$, in the case $i > j$, correspond to negative roots. The generators $F_{ij}$, in the case $i < j$, correspond to positive roots. In the case $i = j$ the generator belongs to the Cartan subalgebra.

### 8.2 Tensor operators and Wigner coefficients

In this section the Wigner coefficients are defined, the solution of the multiplicity problem is given, the reduced Wigner coefficients are introduced. We follow the analogous discussion for the case of the algebra $\mathfrak{gl}_n$ in [20].

The Wigner coefficients are closely related with irreducible tensor operators.
Definition 4. An irreducible tensor operator of type $[M]_n$, where $[M]_n$ is a dominant $g_n$-weight, is an indexed by vectors $(M) \in V^{[M]}_n$, the set of linear mappings

$$f(M): V^{[\bar{m}]}_n \to V^{[m]}_n,$$

which has the following property. For $g \in g_n$ one has

$$[g, f(M)] = f_{g(M)}.$$  

For given $[\bar{m}], [m]$ and $[M]_n$ the tensor operator $V^{[\bar{m}]}_n \to V^{[m]}_n$ of type $[M]_n$ is not unique. If it is not unique one says that for this tensor operator the multiplicity problem takes place.

By Wigner-Eckart theorem the matrix elements of a tensor operator decompose into a product of a factor that depends only on highest weights $\nu$, $\nu$, $\nu$ (this factor is called the reduced matrix element) and a Wigner coefficient that defines an interwining operator $\Phi: V^{[\bar{m}]}_n \to V^{[M]}_n \otimes V^{[m]}_n$.

8.3 The solution of the multiplicity problem

As it is know the interwining operator $\Phi: V^{[\bar{m}]}_n \to V^{[M]}_n \otimes V^{[m]}_n$ is in general not unique. On the other language this means that one irreducible representation $V^{[\bar{m}]}_n$ can occur in splitting of tensor product not once. But the parametrization of all interwining operators is well-known.

All such operators are indexed by tableaux $(\Gamma) \in V^{[M]}_n$, such that

$$[\bar{m}]_n = \Delta(\Gamma) + [m]_n,$$

where $\Delta(\Gamma)$ is the weight of $\nu$. This vector $\Phi$ is defined by the operator as follows

$$\Phi((m)_{\text{max}}) = \Delta(\Gamma) \otimes (m)_{\text{max}} + \text{l.o.t.},$$

where l.o.t. (lower order terms) denotes a sum of tensor products of weight vectors where the second vector has a weight lower than $[m]_n$. For the group $U(n)$ this was proved by Biedenharn and Baird in [21].
The corresponding Wigner coefficient is denoted as

$$
< \begin{pmatrix} [\bar{m}]_n \\ (m')_{n-1} \\ [\bar{m}]_{n-2} \\ (\bar{m})_{n-2} \end{pmatrix} \begin{pmatrix} (\Gamma)_{n-1} \\ [M]_n \\ (M)_{n-1} \end{pmatrix} \begin{pmatrix} [m]_n \\ [m']_n \\ [m]_{n-1} \\ ([m]_{n-1}) \end{pmatrix} > \quad (97)
$$

This coefficient can be nonzero only if the following equality holds $[\bar{m}]_n = \Delta(\Gamma) + [m]_n$.

### 8.4 Reduced Wigner coefficients

Take a Wigner coefficient\(^{(97)}\) for the algebra $g_n$. It also defines a tensor operator for the algebra $g_{n-1}$. Decompose it into a sum of irreducible tensor operators and apply the Wigner-Eckart theorem. One gets

$$
< \begin{pmatrix} [\bar{m}]_n \\ [\bar{m}]'_n \\ [\bar{m}]_{n-2} \\ (\bar{m})_{n-2} \end{pmatrix} \begin{pmatrix} (\Gamma)_{n-1} \\ [M]_n \\ (M)_{n-1} \end{pmatrix} \begin{pmatrix} [m]_n \\ [m']_n \\ [m]_{n-1} \\ ([m]_{n-1}) \end{pmatrix} > = \sum_{(\gamma)_{n-2}} < \begin{pmatrix} [\bar{m}]_n \\ [\bar{m}]'_n \\ [\bar{m}]_{n-2} \\ (\bar{m})_{n-2} \end{pmatrix} \begin{pmatrix} (\Gamma)_{n-1} \\ [M]_n \\ [M]_{n-1} \end{pmatrix} \begin{pmatrix} [m]_n \\ [m']_n \\ [m]_{n-1} \\ ([m]_{n-1}) \end{pmatrix} >
$$

In this formula the first factor on the right is a notation for the reduced matrix element of an irreducible $g_{n-1}$-tensor operator.

This factor is called the reduced Wigner coefficient.

The tableau $(\gamma)_{n-1}$ is obtained by adding to the tableau $(\gamma)_{n-2}$ the rows $[M']_n$ and $[M]_{n-1}$. Note that the reduced matrix element does not depend on the rows $[\bar{m}]_{n-2}$, $[m]_{n-2}$ and below. Thus the reduced Wigner coefficient can be denoted as

$$
< \begin{pmatrix} [\bar{m}]_n \\ [\bar{m}]'_n \\ [\bar{m}]_{n-1} \\ (\bar{m})_{n-1} \end{pmatrix} \begin{pmatrix} (\Gamma)_{n-1} \\ [M]_n \\ [m']_n \\ (\gamma)_{n-1} \end{pmatrix} \begin{pmatrix} [m]_n \\ [m']_n \\ [m]_{n-1} \end{pmatrix} > \quad (98)
$$
This coefficient is nonzero only if the following holds: $[\tilde{m}]_n = \Delta(\Gamma) + [m]_n$ and $[\tilde{m}]_{n-1} = \Delta((\gamma)_{n-1}) + [m]_n$.

### 8.5 Fundamental operators

In the present paper only the Wigner coefficients are considered for which $[M]_n = [1, 0, ..., 0] = [\tilde{1}]_n$, that is when the tensor factor $V^{|M|}_n$ is a standard representation. Such Wigner coefficients are called fundamental.

Note that weight vectors $(m)$ of the standard representations are completely defined by their weights $\Delta(m) = [0, ..., \pm 1, ..., 0]$, where $\pm 1$ occurs at the place $i$. If $i = 0$ then only 1 is allowed. Thus the Wigner coefficient is of type can be denoted as

$$
\begin{pmatrix}
  i \\
  [\tilde{1}]_n \\
  j
\end{pmatrix}
$$

### 8.6 The proof of Propositions 1 and 2

The Proposition 1 is the following statement.

**Proposition.** The number $m_{-2,1}$ in the tableau (17) is the $\mathfrak{o}_3$-weight of the corresponding vector.

**Proof.** Take a polynomial

$$
\sum_{k,l,r,s} z_{-2}^k z_{-1}^{l} z_{-1,1}^{r} z_{-1,0}^s.
$$

Suggest it defines a $\mathfrak{o}_3$-highest vector in a $\mathfrak{o}_5$-representation with the highest weight $[m_{-2}, m_{-1}]$, then to each it’s monomial there corresponds a function

$$
\delta_{-2}^{m_{-2,2}-k-l} \delta_{-1}^{m_{-1,2}+k-2r-s-l}.
$$

Then the $\mathfrak{o}_3$-weight of the corresponding vector equals $m_{-2} - k - l$.

48
Also to (100) there corresponds a polynomial

\[ \sum_{k,l,r,s}^k z_{-2,-1}^l z_{-1,1}^r z_{-1,1}^{\frac{s}{2}}. \]  

(102)

that defines a \( \mathfrak{gl}_1 \)-highest vector in a \( \mathfrak{gl}_3 \)-representation with the highest weight \([m_{-2,2}, m_{-1,2}, 0]\), to each it’s monomial there corresponds a function

\[ \delta_{m_{-2,2} - k - l} \delta_{m_{-1,2} + k - r - s} \delta_{r + l}. \]  

(103)

The weight of the vector that corresponds to this polynomial equals \( m_{-2,2} - k - l \).

Thus the weight of the \( \mathfrak{a}_3 \)-highest vector that corresponds to (100) equals the weight of the \( \mathfrak{gl}_1 \)-highest vector that corresponds to (102). But the weight of the last vector equals \( m_{-2,1} \). The Proposition is proved.

Let us prove the Proposition 2.

**Proposition.** \( \Delta(m)_{-1} = -2 \sum_i m'_{i,2} + \sum_i m_{i,2} + m_{-2,1} - \sigma_{-2} \)

**Proof.** Take a polynomial (100) that corresponds to a Gelfand-Tsetlin tableaux. Consider the expression (101). One obtains

\[ \Delta(m)_{-1} = m_{-1,2} + k - 2r - s - l. \]  

(104)

Consider the polynomial the corresponding (102) that defines a \( \mathfrak{gl}_1 \)-highest vector in a \( \mathfrak{gl}_3 \)-representation, its components with indices \(-1\) and \(1\) are equal to \( m_{-1,2} + k - r - s \) and \( r + l \). Their difference equals

\[ m_{-1,2} + k - 2\left[\frac{s}{2}\right] - 2 - l. \]

This equals (104), if \( s \) is even and differs by one from (104), if \( s \) is odd. Thus the expression

\[ m_{-1,2} + k - 2\left[\frac{s}{2}\right] - 2 - l - \sigma_{-1} \]  

(105)

equals (104).
Thus $\Delta(m)_{-1}$ is a difference of $\mathfrak{gl}_3$-weights with numbers $-1$ and $1$ minus $\sigma_{-1}$.

Note that in the case $\mathfrak{gl}_3$ the component of the weight with the index $-1$ equals $\sum_i m_{i,2}' - \sum_i m_{i,2}$, and the component of the weight with the index $1$ equals $m_{-1,1} - \sum_i m_{i,2}'$.

\[\Box\]

### 8.7 The proof of Proposition [13]

Let us prove the following statement

**Proposition.**

$$
\Delta(m)_{-k+n-1} = -2 \sum_i m_{i,k}' + \sum_i m_{i,k} + \sum_i m_{i,k-1} \text{ in the cases } \mathfrak{sp}_{2n}, \mathfrak{o}_{2n},
$$

$$
\Delta(m)_{-k+n-1} = -2 \sum_i m_{i,k}' + \sum_i m_{i,k} + \sum_i m_{i,k-1} - \sigma_{-k} \text{ in the cases } \mathfrak{o}_{2n+1}
$$

**Proof.** It is enough to prove the formula for $\Delta(m)_{-1}$. One can suggest that the tableau $(\Gamma)$ is maximal with respect to $\mathfrak{g}_{n-1}$. To this tableau there corresponds a polynomial $f$, the procedure of it’s construction is described in Section 7.1. The polynomial $f$ is of type

$$
f = cf_0,
$$

where $c \in \mathfrak{k}$ (see Definition of the field $\mathfrak{k}$ in Section 7.1), and $f_0 \in \mathbb{C}(z_{-3,-1}, z_{-3,1}, ..., z_{-n,-1}, z_{-n,1})$.

The weight of the vector corresponding to $f$ in calculated as follows (see [2]). To each variable $z_{ij}$ (also to variables from $\mathfrak{k}$) the correspond the multiplicator $\delta_{ij}^{-1}. \delta_j$. We suggest that $\delta_0 = 1$, $\delta_1 = \delta_{-1}^{-1}$. The multiplicators are multiplied onto the function $\delta_{-n}^{m_{-n}} ... \delta_{-1}^{m_{-1}}$, corresponding to the highest weight. The degree of $\delta_{-1}$ is the weight $\Delta(m)_{-1}$.

From the structure of $f$ one sees that the change of transformation of the weight $[m_{-n,n}, ..., m_{-1,n}]$ under the action of $f$ equals to the sum of transformations under the action of $c$ and $f_0$. The transformation under the action of $c$ was investigated when we considered the case $g_2$, it equals
\[-2 \sum_{i=-2}^{T} m'_{i,k} + \sum_{i=-2}^{-1} m_{i,n} + m_{-2,n-1} \text{ where } T = -1 \text{ or } -2 \text{ in the case } \mathfrak{sp}_{2n}, \mathfrak{o}_{2n},\]

\[-2 \sum_{i=-2}^{-1} m'_{i,k} + \sum_{i=-2}^{-1} m_{i,n} + m_{-2,n-1} - \sigma_{-n} \text{ in the case } \mathfrak{o}_{2n+1}.\]

Since the transformation of the weight under the action of $f_0$ is the same for all $g_n$ and $\mathfrak{gl}_{n+1}$ then using result for $\mathfrak{gl}_{n+1}$ we obtain that under the action of $f_0$ to $\Delta(\Gamma)_{-1}$ the following value is added

\[-2 \sum_{i=-n}^{-3} m'_{i,k} + \sum_{i=-n}^{-3} m_{i} + \sum_{i=-n}^{-3} m_{-2,n-1}\]

Adding the transformations of the weight corresponding to $c$ and $f_0$ we prove the Proposition.

\[\square\]

9 The proof of Proposition \[3\]

Proposition. The following equality takes place

\[< (\bar{m})_{\text{max}} \mid F_{\pm 1, -2} \mid (m)_{\text{max}} > = < (\bar{m})_{\text{red}} \mid F_{\pm 1, -2} \mid (m)_{\text{red}} >_{\text{red}} \]  \hspace{1cm} (106)

The equality is proved using the Wigner-Eckart theorem. One has

\[< (\bar{m})_{\text{max}} \mid F_{-1, -2} \mid (m)_{\text{red}} >_{\text{max}} = < (\bar{m})_{\text{red}} \mid F_{-1, -2} \mid (m)_{\text{red}} > \cdot \]

\[< \left( \frac{[\bar{m}]_{n-1}}{\text{max}} \right) \mid \left( \begin{array}{c} j \\ [10]_{n-1} \end{array} \right) \mid \left( \frac{(m)_{\text{red}}}{\text{max}} \right) >, \]  \hspace{1cm} (107)

where the Wigner coefficient equals to 1 if there exists $j$, such that $[\bar{m}]_{n-1} = [m]_{n-1} + [0, \ldots, 1 \text{ at the place } j, \ldots, 0]$, and equals zero otherwise. Thus one has

\[< (m)_{\text{red}} \mid F_{-1, -2} \mid (m)_{\text{red}} >_{\text{red}} = < (\bar{m})_{\text{max}} \mid F_{-1, -2} \mid (m)_{\text{max}} >. \]  \hspace{1cm} (108)
10 The proof of Proposition 14

Let us be given a diagram \((m)\) that define a vector in \(s g_{n-1}\)-representation with the highest weight \([m_{-n,n-1}, \ldots, m_{-2,n-1}]\).

A polynomial on the group \(Z_{n-1}\) corresponds to this vector. Consider it as a polynomial on a bigger group \(Z_n\).

**Proposition.** \(g_n\)-representation with the highest weight \([m_{-n,n-1}, \ldots, m_{-2,n-1}, 0]\).

The corresponding tableau is of type \(\left( \frac{\text{max}}{m} \right)\), this is a \(g_n\)-tableau that is obtained form \((m)\) by adding two maximum row.

**Proof.** One has

\[
(m) = \zeta_1 \cdots \zeta_t (\text{max}),
\]

where \(\zeta_i \in \mathbb{Z}_{n-1}\) and \((\text{max})\) is the highest vector.

Let use the realization on the group \(Z\). The highest vector is the function that equals identically to one.

One has \((m) = (T_{\zeta_1} \cdots T_{\zeta_t} 1)(z) = \alpha_{n-1}(z\zeta_1 \cdots \zeta_t)\).

Also one has

\[
\left( \frac{\text{max}}{m} \right) = \zeta_1 \cdots \zeta_t (\text{max}),
\]

where \(\zeta_i \in \mathbb{Z}_{n-1} \subset \mathbb{Z}_n\).

In the space of functions on \(Z_n\), one has

\[
\left( \frac{\text{max}}{m} \right) = (T_{\zeta_1} \cdots T_{\zeta_t} 1)(z) = \alpha_n(z\zeta_1 \cdots \zeta_t) \tag{110}
\]

If \(\alpha_n\) corresponds to the highest weight \([m_{-n,n-1}, m_{-n+1,n-1}, \ldots, m_{-2,n-1}, 0]\), then \(\alpha_{n-1}(z\zeta_1 \cdots \zeta_t) = \alpha_n(z\zeta_1 \cdots \zeta_t)\).

Thus the polynomials corresponding to \((m)\) and \(\left( \frac{\text{max}}{m} \right)\) coincide. \(\square\)
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