Dilute monopole gas, magnetic screening and k-tensions in hot gluodynamics

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A dilute monopole gas explains, in quarkless gluodynamics, the small ratio $\delta$ between the square of magnetic screening mass $m_M$ and spatial Wilson loop tension. This ratio is 0.0895 for $T = 0$ to 0.0594 at $T = \infty$ for any number of colours with order $N^{-2}$ corrections and equals up to a numerical factor of O(1) the diluteness. The monopoles have a size $l_M = m_M^{-1}$. The GNO classification tells us they are in a representation of the magnetic SU(N) group. Choosing the adjoint for the dilute gas predicts the k-tensions to scale as $k(N-k)$, within a percent for high T, and a few percent for low T for the seven ratios determined by lattice simulation. The transition is determined by the transition of the dilute Bose gas at $T_c = 0.174m_M$, and the transition is that of a non- or nearly-relativistic Bose gas.

1 Introduction

An ancient idea in QCD is that monopoles are responsible for flux tube formation through a dual superconductor mechanism. Another mechanism, that of the “Copenhagen vacuum” of the early eighties, proposes macroscopic Z(N) Dirac strings or Z(N) vortices.

In this talk I will discuss a specific model of the first type, that works surprisingly well in its most direct applications. A straightforward way to see its workings is to start from the plasma phase. That sounds at first sight self-defeating as there is no confinement in this phase. But it will turn out that precisely the absence of confinement renders the detection of these monopoles, or perhaps more appropriately, magnetic quasi-particles, so straightforward. We know that electric quasi-particles, the gluons, are approximately free at very high temperature. This follows from the Stephan-Boltzmann law, which is verified up to 15% on the lattice, at $T \sim 4T_c$. Note, though, that the interactions at this temperature are still quite strong.

We may guess that the same happens to the magnetic monopoles that were condensed in the cold phase.

What kind of monopoles can we expect? In the absence of any spontaneous breaking the GNO classification applies. For a theory with only Z(N) neutral fields, like the adjoint, the magnetic group is SU(N). So we can have monopoles in the fundamental, adjoint or any representation we like.

One of the consequences of the idea of a monopole condensate is the screening of the magnetic Coulomb force between two static magnetic sources. This is a straightforward generalization of the screening of the Coulomb force between two heavy quarks. There is now ample affirmation of screening from lattice simulations. The magnetic screening mass $m_M$ starts out at $T = 0$ and equals there the lowest $0^{++}$ glueball mass. It stays constant till about $2T_c$, and then starts to scale with the temperature like $g^2T$.

It is reasonable to think of the monopole as having a size of about the screening length $l_M = m_M^{-1}$. We will assume that its size is much smaller than the inter-monopole distance. That, together with the choice of multiplet, will fix the ratios of spatial string tensions. The reader who is only interested in how one computes these ratio’s should read sections 2,5 and 6.

aIn the GNO analysis the range of the monopoles is infinite. That is why quarks imply a smaller magnetic group, SU(N)/Z(N), because of the Dirac consistency relation between electric and magnetic charges. Also the existence of the monopoles was not proven. See e.g.
The remaining sections give a qualitative idea of how flux loop averages depend on the representation. The main conclusion is that the loops depend only on the N-allity. Moreover, in our model only the fully anti-symmetric representations are a practical tool to determine numerically the tension.

2 Electric flux loops

In this section we discuss the behaviour of electric flux loops in the plasma, as it will explain the basic features that are permeating the talk.

2.1 QED

Consider a plasma of ions and electrons. We will take the ions to have the same but opposite charge of the electron.

Suppose we want to compute the electric flux going through some large (with respect to the atomic size) closed loop $L$ with area $A(L)$. Normalize the flux $\Phi = \int_S d\vec{S} \cdot \vec{E}$ by the electron charge $e$ and define:

$$V(L) = \exp i2\pi \Phi/e.$$  \hfill (1)

Of course, at $T$ below the ionization temperature no flux would be detected by the loop, because there are only neutral atoms moving through the loop.

Let us now raise the temperature above $T_{\text{ionisation}}$. What will happen? Both electrons and ions are screened. For simplicity we will take the ions to have the opposite of one electron charge.

We are going to make the following simplification. The charged particles are supposed to shine their flux through the loop if they are within distance $l_D$ from the minimal area of the loop. This defines a slab of thickness $2l_D$.

Of course if we plot $|\Phi/e|$ as function of the distance of the particle to the loop you find an exponential curve with the maximum 1/2 at zero distance. For the sake of the argument we will replace that curve by a theta function of height 1/2 and width $2l_D$. If one wants to do better one has to deal with infinitesimally thin slabs, and integrate over the thickness. This yields an effective thickness of 1.6428...\(^1\). Here we will keep the factor 2. The result is parametrically the same as the one we will derive keeping the simple minded method.

Then one electron (ion) on the down side of the loop will contribute $+1/(−1/2)$ to the flux, and with opposite sign if on the up side of the loop. That is: $V(L)_{\text{one charge}} = −1$. This result is independent of the sign of the charge! The plasma is overall neutral, the loop is sensitive only to charge fluctuations. For $l$ charges inside the slab the flux adds linearly and we find:

$$V(L)_{l} = (−1)^l.$$  \hfill (2)

Assuming that all charges move independently, the average of the flux loop $V(L)$ is determined by the probability $P(l)$ that $l$ electrons (ions) are present in the slab of thickness $2l_D$ around the area spanned by the loop. Taking for $P(l)$ the Poisson distribution $\frac{1}{l!} \exp -\bar{l}$, where $\bar{l}$ is the average number of electrons (ions) in the slab— we find for the thermal average of the loop due to both ions and electrons:

$$\langle V(L) \rangle_T = \sum_l P(l)V(L)_l = \sum_l P(l)(−)l = \exp −4\bar{l}.$$  \hfill (3)
Now \( \bar{l} = A(L)2l_Dn(T) \), so the electric flux loop obeys an area law \( \exp(-\rho(T)A(L)) \), with a tension

\[
\rho(T) = 8l_Dn(T).
\] (4)

This area law distinguishes the behaviour of the loop in the plasma from that in the normal unionized state.

The flux loop has a very useful alternative formulation. Introduce the charge density \( j_0 \). Then you can write flux loop as a gauge transformation \( \exp\left(-\int d\vec{x}(\vec{E} \cdot \nabla + j_0)\omega_1\right) \). \( \omega_1 \) a gauge function that falls off fast at spatial infinity. But it has a discontinuity \( 2\pi \), when crossing the surface. So the border of the surface is of course just a closed Dirac string.

If \( \omega_1 \) has no discontinuity, integration by parts will give us the Gauss operator \( -\vec{\nabla} \cdot \vec{E} + j_0 \) in the exponent. But the discontinuity generates the extra surface term \( \frac{2\pi}{N} \Phi \). Because the flux \( \Phi \) is gauge invariant the two terms commute. So the gauge transform factorizes in a factor containing the flux and a factor containing only the Gauss operator. The latter becomes unity on the physical subspace and only the flux operator stays.

2.2 Electric flux in gluodynamics

Is there an SU(\( N \)) generalization of the QED case? In fact yes. The closed Dirac string in QED is replaced by a \( Z(N) \) vortex of strength \( k \), the ’t Hooft loop\(^3\). We introduce in the Lie algebra of traceless hermitean \( N \times N \) matrices a basis for the Cartan subalgebra, the \((N-1) \times (N-1)\) dimensional subspace of diagonal matrices. This basis of \( N-1 \) diagonal matrices \( Y_k \) is chosen such that in exponentiated form \( \exp\left(-i2\pi Y_k\right) \) it gives the \( N-1 \) centergroup elements \( \exp\left(ik\frac{2\pi}{N}\right) \).

A simple choice is

\[
Y_k = \frac{1}{N} diag(N-k, \ldots, N-k, -k, \ldots, -k).
\] (5)

The entry \( N-k \) comes in \( k \) times, and \( -k \) comes in \( N-k \) times, so the trace is 0.

The flux operator becomes in this notation:

\[
V_k(L) = \exp\left(\frac{4\pi}{g} \int d\vec{S} Tr \vec{E}Y_k\right).
\] (6)

It does correspond to the vortex operator of ’t Hooft with strength \( k \)\(^b\).

The next question is: does the gas of deconfined gluons induce an area law in this operator? The answer is yes, and the reasoning is as before. The charge of a gluon with respect to the charge \( Y_k \) is found from the form \( Y_k \) takes in the adjoint representation. This is easy: we have charge \( \pm 1 \) like in QED, but unlike in QED we have \( 2k(N-k) \) gluon species in the gluon multiplet with such a charge. All other gluons have vanishing charge so do not contribute to the flux. Since we take the gluons to be statistically independent, the charged ones all contribute a factor to \( \langle V_k \rangle \), and this factor is the same by SU(\( N \)) symmetry: \( \exp\left(-2l_Dn\right) \).

Conclusion: the expectation value of the loop is

\[
\langle V_k \rangle = \exp\left(-\rho_k(T)A(L)\right)
\] (7)

with the tension

\[
\rho_k = 4l_Dnk(N-k).
\] (8)

\(^b\)Normalization is such that \( \left[E_a^\mu(\vec{0}), A_b^\nu(\vec{x})\right] = \delta^{\mu\nu}\delta_{j,k}\delta(\vec{x}), \vec{E} = \vec{E}^a\lambda_a, [\lambda_a, \lambda_b] = if_{a,b,c}\lambda_c, f_{a,b,c}f_{d,b,c} = N\delta_{a,d}. \)
So this is the k-scaling law for the electric flux loop. It does obey large $N$ factorization:

$$\rho_k = k \rho_1. \quad (9)$$

But it has $1/N$ corrections and not, as perhaps expected in gluodynamics, $1/N^2$.

This computation is corroborated by low order perturbation theory for the loop. The expectation value of the loop can be computed from a tunneling process between adjacent vacua of an effective potential with a $Z(N)$ symmetry $^{22,24}$. This potential has, because of the $Z(N)$ symmetry, degenerate minima in the centergroup elements. Only at cubic order in the coupling $g$ the k-scaling starts to deviate slightly $^4$. Simulations in $SU(4)$ show k-scaling up to $2T_c^{19}$.

The effective potential is computed in a background that serves as profile orthogonal to the area. The minimal effective action of this profile gives the tension $\rho_k$, thereby tunneling from say 1 to $\exp i2\pi Y_k$. The profile is present in the propagators and renders the usual double line representation of the gluon propagators invalid, hence the $1/N$ corrections. The typical width of the profile around the area is the Debye screening length $l_D$.

2.3 Junctions and highest weights

It may have struck the reader that we used $Y_k$, instead of $kY_1$ in the flux formula eq.(6). The reason is that the tunneling process from 0 to $Y_k$ passes only one mountain. On the other hand tunneling from 0 to $kY_1$ passes through $k$ mountains, each of which interposes between two subsequent $Z(N)$ vacua and contributes $\rho_1$. So at first sight we find $k\rho_1$ from this operator. However, there must be a lower state $\rho_k$, as the following argument shows. Formally the operator

$$V_k(L) = \exp \frac{4\pi i}{g} \int d\vec{S} Tr \vec{E} k Y_1$$

is obtained by coalescing the product of $k$ operators $V_1(L)$. There are two relevant stages here: one where all the loops are still more than the screening distance away from each other. Then every loop will form its individual profile and wall, and give a factor $\exp -\rho_1 A(L)$. The second stage is when all loops are within screening distance. Then the individual walls emanating from each loop will form a junction at a typical distance $l_D$. At this junction $k$ unit walls merge with the $k$-wall, and the $k$-wall views the ensemble of the $k$ unit discontinuities as one discontinuity of strength $k$, on the scale of the screening length. The junction coupling involved here is non-perturbative. It may be small, certainly in our model of approximately free gluons. Hence the true $\rho_k$ may be difficult to disentangle numerically.

In ref.$^{26}$ the case $k=N$ in eq.(10) is discussed. The operator $V_{k=N}$ creates a state with energy $N\rho_1$ and involves the special junction where $k=N$. This junction has only N incoming $k=1$ walls, and no outgoing N wall. That explains the periodicity $N$ of the tension $\rho_k$.

In addition we can imagine junctions where two of the incoming unit walls are replaced by one $k=2$ wall. That junction corresponds to the matrix $(k-2)Y_1 + Y_2$.

A general junction in $SU(N)$ will be specified by a set of non-negative numbers $w_1, w_2, \ldots, w_{N-1}$. So the constraint on those numbers is:

$$\sum_{l=1}^{N-1} lw_l = k. \quad (11)$$

In our two previous examples we had $w_1 = k$ all others zero, and $w_1 = k-1, w_2 = 1$, all others zero. The corresponding matrix is then $\sum_{l=1}^{N-1} w_l Y_l$ and the reader will recognize
this matrix as being the highest weight $H_R$ of the Young tableau characterized (see fig.1) by the numbers $w_1, ..., w_{N-1}$. These Young tableaux refer to the representations of the magnetic GNO group.

Also this general junction $J(k; \{w\})$ is expected to be small for the same reasons. Especially, because of factorization, eq. (9), it vanishes in the large $N$ limit. The conclusion is that only for the partitioning $w_k = 1$, all other $w$’s zero, we obtain a direct coupling to the k-tension, and no junction is needed. All other charges will give states of non-interacting k-tensions.

Something very similar happens for the various representations involved in the Wilson loops and we will come back to this in detail in section 3.3.

3 Some basic facts in quarkless Yang-Mills

In this section we gather a few facts, partly empirical, from lattice simulations, and partly from simple physics arguments.

3.1 Magnetic screening

A feature that sets gluodynamics apart from QED is the appearance of magnetic screening. When we put two heavy monopoles far apart at a distance $r$ we find a Yukawa type potential:

$$V(r) = \frac{c}{r} \exp(-mMr).$$

A simple argument shows$^{10,8}$ that the magnetic mass is the mass of a scalar excitation of a Hamiltonian in a world with two large space dimensions and one periodic mod $1/T$. The space symmetries in such a world are $SO(2) \times P \times C \times R$. We have the rotation group in the two large dimensions, $C$ is charge conjugation, $P$ is parity in the large 2d space, and $R$ is the parity in the periodic direction and a state is written as $J^R_P C$. The state corresponding to the magnetic mass is $0^{++}$. As $T \to 0$ the rotation group becomes $SO(3)$, and $R$ and $P$ become related through a rotation, so at $T = 0$ the $0^{++}$ glueball mass results. The Debye mass, screening the electric Coulomb force between two heavy quarks, is related to the $0^{+-}$ state, and disappears below $T_c$. Below $T_c$, $Z(N)$ symmetry is restored and the string tension appears, as a state transforming non-trivially under $Z(N)$.

In contrast, the magnetic mass does not feel the $Z(N)$ transition. At temperatures where $T$ dominates all scales, the magnetic mass starts to scale like $g^2T$, as schematically shown in fig. 2.
3.2 Spatial tension

The spatial tension is defined through a spatial loop \( L \), on which an ordered product \( \mathcal{P} \exp i \oint ds \vec{A}_R \) lives. \( R \) labels the representation for the vector potential we have chosen. The spatial Wilson loop is then defined as the trace over colour:

\[
W_R(L) = Tr \mathcal{P} \exp i \oint_L ds \vec{A}_R. \tag{13}
\]

The path integral average gives an area law:

\[
\langle W_R(L) \rangle = C \exp \left( -\sigma_R(T)A(L) \right) \tag{14}
\]

for all temperatures. At zero temperature the spatial tension equals the string tension\(^4\) from a time-space loop because of Euclidean rotation invariance. But as function of temperature they behave differently. The string tension suffers a correction at low temperature of \(-\frac{\pi}{3} T^2\) due to the excitations contributing the Luescher term. Above \( T_c \) it vanishes. The spatial tension stays flat, see fig. (2). Like the magnetic mass it does not feel the \( Z(N) \) transition and starts to scale like \((g^2 T)^2\) well above the critical temperature. The dimensionless ratio of the spatial tension in the fundamental representation and the square of the magnetic mass is small, on the order of a few percent. In two extreme cases the ratio \( \delta(T) \) is accurately known from simulations: at \( T = \infty \) \(^{11,8}\) in the 3d theory, and at \( T = 0 \) \(^{12}\). In between there are simulations\(^{15,25}\) that are consistent with a slowly varying \( \delta(T) \). It is very important to have this verified with more precision.

3.3 Dependence of the tension on the representation, and \( N \)-allity

There is widespread agreement on the dependence of the tension and the string tension on the representation \( R \). It is only its \( N \)-allity \( k \) that counts. If we think of \( R \) being built up by \( f \) fundamental and \( \bar{f} \) anti-fundamental representations, the \( N \)-allity is just the difference:

\[
k = f - \bar{f}. \tag{15}\]

To get a more precise idea, let us imagine the Wilson loop is formulated in a periodic box, with the 3 space dimensions being of macroscopic length \( L \). The fourth dimension is of length \( 1/T \). The loop is now replaced by two Polyakov lines of opposite orientation in the x-direction. They are a distance \( r \) apart- and carry the representation \( R \). The expectation value of the loop is then parametrized as:

\[
\langle W_R(L) \rangle = C \exp \left( -L V_T(r) \right). \tag{16}\]

If the distance \( r \) is very small, asymptotic freedom tells us:

\[
V(r) = \frac{g^2(r)C_2(R)}{4\pi T} \tag{17}\]

where \( C_2(R) \) is the quadratic Casimir operator.

In three dimensions the Coulomb force is logarithmic.

The Casimir operator can be related to our \( N \times N \) matrices \( Y_k \) if \( R \) corresponds to a Young tableau, with the first row having \( w_1 \) more boxes than the second. The second row has \( w_2 \) more boxes than the third, and so on. By definition, the numbers \( w_k \) are never negative.

\(^4\)In what follows we reserve the name string tension for the time-space loop.
There is a one-to-one correspondence between Young diagrams and irreducible representations of SU(N).

Define the highest weight $H_R$ of $R$ as

$$H_R = \sum_k w_k Y_k$$

(18)

as we already alluded to at the end of section 2.2. An example is shown in fig.1.

Then, if $Y = \sum_k Y_k$ :

$$C_2(R) = \frac{1}{2}(Tr H_R^2 + 2Tr Y H_R).$$

(19)

If the distance is long enough (a somewhat ambiguous criterion!) the string regime sets in and we expect:

$$V(r) = \sigma_k(T)r.$$ 

(20)

The tension $\sigma_k$ only depends on the N-allity $k$. String formation between the two sources renders this dependence very plausible. The tension $\sigma_1$ of the string formed in between two fundamental sources can form a bound state in the string with tension $\sigma_2$. We can go on like this and the question is, what is the dependence on $k$? It should obey certain a priori criteria.
For large number of colours we expect factorization
\[ \sigma_k = k \sigma_1. \] (21)

From SUSY experience one also expects in that same limit:
\[ \sigma_k = k \sigma_1 + O(1/N^2). \] (22)

This is discussed in depth by Misha Shifman in this volume.

3.4 Flux representation of the Wilson loop

If we want to proceed with the Wilson loops, as we did with the 't Hooft loops, we need clearly a representation similar to eq.(6).

Remember we discussed in section 2.3 a general formula for the electric flux loop with \( Y_k \) in eq.(6) replaced by \( H_R \), the highest weight but still with N-allity \( k:V_R \exp(\frac{4\pi}{g} \int d\vec{S} \text{Tr} \vec{E} H_R). \)

Of course a natural guess is to take this electric flux formula, replace \( \vec{E} \) by \( \vec{B} \), \( \alpha = g^2/4\pi \) by \( 1/\alpha \) to get:
\[ W_{R''}'' = \exp(ig \int d\vec{S} \text{Tr} \vec{B} H_R). \] (23)

This formula cannot be true\(^{24}\). But it is true in a dynamical sense: consider the limit of a path integral average in the 3d theory with an adjoint Higgs, the electrostatic QCD Lagrangian.

For a given representation \( R \) with highest weight \( H_R \) with stability group \( S \) (i.e. \( SH_R S^{-1} = H_R \)) one finds:
\[ \langle W_R \rangle = < \int D\Omega \exp(ig \int d\vec{S} (\vec{B}_a - \frac{1}{g} f^{abc} \vec{D} n_b \wedge \vec{D} n_c) \text{Tr} \frac{\lambda_a}{2} \Omega H_R \Omega^\dagger) >. \] (24)

with \( n_a = Tr \frac{\lambda_a}{2} \Omega H_R \Omega^\dagger \) the Higgs field’s angular part that parametrizes the coset \( SU(N)/S(H_R) \). The second term in the exponent is the source term for the monopoles in the sense that it carries no long range effects in the original Higgs phase. The average of the r.h.s. is the 3d Yang-Mills average. The integration is over all gauge transforms with an arbitrary number of hedge hog configurations. The r.h.s. is obtained \(^{23}\) by taking the average of the magnetic charge operator in the 3d Higgs phase characterized by \( H_R \). Then the the VEV of the Higgs field is let to zero, which introduces the fluctuations over \( n_a \), the angular components of the Higgs field. Finally one lets the mass of the Higgs become infinite, which suppresses the radial integrations.

On the other hand \(^{17}\) the integrand of the r.h.s., without the brackets, was shown by Diakonov and Petrov to equal \( W_R \), by one dimensional quantum mechanics methods. It involves a limiting procedure which is procured in the path integral by starting out from the Higgs phase and moving to the symmetric phase as described above. With this specification we will use eq. (23) without quotation marks in what follows.

So we need a stable Higgs phase for the limiting procedure. The question is then: what are the stable Higgs phases? The answer is from simulations: only those that have a stability group \( S = SU(l) \times SU(N-l) \times U(1) \).
Summarizing: it is possible to have a magnetic flux representation in a path integral average for Wilson loops $W_R$ with special highest weight $H_R$. The highest weight should commute with a subgroup of the form $SU(l) \times SU(N-l) \times U(1)$. These invariance groups define the only stable Higgs phases in 3d. The monopoles in these phases are partly screened in the strongly coupled sectors (the non-abelian factors in the stability group). The $SU(5)$ case is nicely described by Coleman in ref\textsuperscript{5}. After the limiting procedure they are the fully screened monopoles in the symmetric phase.

To get a feeling, look at the $k=2$ representations of $SU(N)$ in fig. 3). The highest weight of the totally anti-symmetric representation is computed from eq.(5) and eq.(18). It is $Y_2$, and $SU(2) \times SU(N-2) \times U(1)$ is its invariance group. So it has a flux representation $W_R = \exp ig \int d\tilde{S}.Tr\tilde{B}Y_2$.

The symmetric $k=2$ representation has highest weight $2Y_1$. It has invariance group $SU(N-1) \times U(1)$. So its flux representation is $W_R = \exp ig \int d\tilde{S}.Tr\tilde{B}2Y_1$.

4 Monopoles and the magnetic group

In this section we briefly touch on a medley of problems related to magnetic screening.

The generators of the magnetic group as advocated by Goddard et al.\textsuperscript{5} are not known in terms of the Yang-Mills potentials. In our opinion the monopoles are collective excitations of the magnetic gluons and can be much better called magnetic quasi-particles. They are bound states of magnetic gluons. Their size is supposed to be $l_M$, the magnetic screening length.

The question is then: what representation of the magnetic group is realized by Nature for these dynamical quasi-particles? We have tried in the next section the simplest ones, the fundamental and the adjoint. The adjoint has the advantage that it is compatible with quarks. Compatible means the Dirac consistency condition is fulfilled. As already mentioned in the introduction, the consistency condition is strictly enforced in the absence of screening. If screening is present a famous example due to ’t Hooft\textsuperscript{6} tells us that screening renders the Dirac condition less restrictive than naively thought. At any rate the adjoint is clearly favoured by simulations of the Wilson loops, see section 6.

Amusing, though perhaps academic, is the observation that the flux representation of the ’t Hooft loops introduced highest weights $H_R$. These should correspond to representations of the magnetic group. If we could implement the loop with a given weight $R$ by means of a magnetic gauge potential we would have an alternative expression for the ’t Hooft loop. Once the magnetic group becomes a gauge group with gauge potentials one has an alternative for the Wilson loops as well, to wit, as $Z(N)$ discontinuous gauge transformations like the ’t Hooft loops. The magnetic gauge group provides long range excitations in the low $T$ phase in contradiction with the full magnetic screening. They should be obliterated by a Higgs mechanism that breaks fully the symmetry. And for that our adjoint multiplet is not a good candidate. It will always leave some some subgroup unbroken as we discussed in the previous section. A fundamental multiplet would fully screen, but is excluded by the lattice data.

5 Predictions for the Wilson loops

Once we are given a dilute gas of screened monopoles, we only need to specify the representation of the magnetic group for our monopoles. Then the calculation of the tension is done with our flux representation for the average of the loop, eq.(24). But in practice we can use the simple formula eq.(23). The reason is that we put the contribution of the monopole gas in by hand.
That takes care of the singular gauge transforms in eq.(24). The remaining regular ones can be dropped because we compute something gauge invariant.

So the average of a k-loop in the totally anti-symmetric representation is computed from

\[ \langle W_k \rangle = \langle \exp ig \int d\vec{S} \text{Tr} \vec{B}Y_k \rangle. \]  

(25)

For the dilute gas in the adjoint representation the computation is not different from the one with gluons in section 2.2. The adjoint monopoles have a magnetic charge equal to \( \pm \frac{2\pi}{g} \), or 0. Only the former contribute and their individual contribution to the loop in eq. (25) is \(-1\). The diluteness, and classical statistics (as the thermal wave length is \( T^{-1} \) and the typical interparticle distance is \( (g^2T)^{-1} \) this is justified, see also fig. 2.) then give for every charged species the same contribution \( \exp \left(-\frac{2l_Mn_M}{N} \right) \), \( n_M \) being the common density of a given species.

As there are \( 2k(N-k) \) charged species in the adjoint, the total results in the k-tension being:

\[ \sigma_k = 4k(N-k)l_Mn_M. \]  

(26)

One can do a similar calculation for monopoles in the fundamental representation. The counting goes now as follows. Recall eq.(5), \( Y_k = \frac{1}{N} \text{diag}(N-k,...,N-k,-k,...,-k) \) with trace zero. The \( Y_k \) charge of the highest \( k \) components of the \( N \) components of the column spinor is given by \( \left( \frac{N-k}{N} \right) \). The remaining \( N-k \) components have charge \(-\frac{k}{N}\).

It then follows from the use of the Poisson distribution that the flux of a given component is contributing \( \cos\left(\pi \frac{N-k}{N}\right) \) or \( \cos\left(\pi \frac{k}{N}\right) \). Taking into account the degeneracies the final result for the tension becomes:

\[ \sigma_k = 4l_Mn_M(N-k\cos\left(\pi \frac{N-k}{N}\right) - (N-k)\cos\left(\pi \frac{k}{N}\right)). \]  

(27)

Both tensions give ratios \( \sigma_k/\sigma_1 \) that behave like \( k \) for large \( N \) and finite \( k \). This factorization is expected for a k-loop tension caused by screened particles\(^d\). But for large \( N \) and \( k = N/2 \) the ratio for the fundamental multiplet is a factor 2 larger.

6 Comparison to lattice simulations

We have been discussing a model at very high temperature. Hence it is tested in 3d lattice simulations. The ratios found\(^14\) for the totally antisymmetric irreps are close — within a percent for the central value — as far as the adjoint multiplet of magnetic quasi-particles is concerned:

\[
\begin{align*}
\text{SU(4)} & : \frac{\sigma_2}{\sigma_1} = 1.3548 \pm 0.0064 & \text{adjoint} : 1.3333 & \text{fundamental} : 1.8182 \\
\text{SU(6)} & : \frac{\sigma_2}{\sigma_1} = 1.6160 \pm 0.0086 & \text{adjoint} : 1.6000 & \text{fundamental} : 1.9686 \\
\text{adjoint} & : 1.808 \pm 0.025 & \text{adjoint} : 1.8000 & \text{fundamental} : 2.3635
\end{align*}
\]

The results are that precise, that you see a two standard deviation from the adjoint, except for the second ratio of \( SU(6) \). This deviation is natural, since the diluteness of the magnetic quasi-particles is small, on the order of a couple of percent, as we will explain at the end of this subsection. So we expect corrections on that order to our ratios.

\(^d\)Factorization is not obvious if uncorrelated \( Z(N) \) vortices\(^2\) cause the area law: \( \sigma_k \sim (1 - \cos(2k\pi/N)) \). This is due to the macroscopic size of the vortex perimeter causing long range correlations between loops.
There is a less precise determination of the ratio $\sigma_2/\sigma_1 = 1.52 \pm 0.15$ in $SU(5)^{20}$. But the central value is within 1 to 2% of the predicted value $3/2$ from the adjoint. The fundamental gives a ratio 1.8231.

The $SU(8)$ ratios are known on a rather course lattice$^{20}$ and using a different algorithm:

\[
\begin{align*}
\sigma_2/\sigma_1 &= 1.692(29) \quad \text{adjoint : 1.714} \quad \text{fundamental : 2.106} \\
\sigma_3/\sigma_1 &= 2.160(64) \quad \text{adjoint : 2.143} \quad \text{fundamental : 2.958} \\
\sigma_4/\sigma_1 &= 2.26(12) \quad \text{adjoint : 2.286} \quad \text{fundamental : 3.256}
\end{align*}
\]

In conclusion: the seven measured ratios are consistent with the quasi-particles being independent, as in a dilute gas and in the adjoint representation. The number of quasi-particle species contributing to the k-tension is $2(k(N-k))$. This number happens to coincide with the quadratic Casimir operator of the anti-symmetric representation.

The fundamental monopoles are clearly disfavoured by the data.

For other representations we have already said what to expect. E.g. the fully symmetric representation with k boxes will show the same tension, but through a very suppressed junction, especially for large $N$. So lattice data will show predominantly the $k(\sigma_1)$ tension.

A caveat is in order. As we are in three dimensions, the Coulomb law is logarithmic. It is multiplied by the quadratic Casimir $C_2(R)$. So logarithmic confinement follows Casimir scaling!

So if the fully symmetric representation with k boxes shows ratios compatible with the $C_2(R_{sym}) \sim k(N+k)$ the strings are probably behaving somewhere in between logarithmic and linear confinement.

An invertebrate pessimist could say the same of the antisymmetric reps where the Casimir ratio happens to be the same as for our adjoint monopole model. According to him, in the fitting of the tension the plateau has been mistaken for a linear law, whereas in reality it is logarithmic. I do not know whether the data exclude this.

7 Conclusions

There is remarkable agreement with an accuracy of about a percent between numerical simulation at high $T$ and the dilute adjoint monopole gas. This diluteness appears as a small parameter for every $SU(N)$ theory. Some parameters, like the mass of the magnetic quasi-particle, are still within a large range, although its size is $l_M$. Is it heavy, on the order of the lowest glueball mass (i.e. $m_M$), then the transition is that of a non-relativistic Bose gas. Is it light (with respect to its size) then the transition is that of a near-relativistic BE transition.

Perhaps the best way to summarize our approach is to return to fig. 2. It is clear that the relation:

\[ \sigma \sim l_M n_M \]

implies that $l_M^2 \sigma$ is the diluteness $\delta$, and must be small for consistency. And lattice data have borne that out! Also flux tube models like that of Isgur-Paton$^{30}$ predict this small ratio$^{31}$ as a result of the balance of the string force and the phonon excitations of the closed string forming the lowest glueball.

Now, once we accept the idea, that on the high temperature side of the transition a dilute gas describes things so well, the constancy of that diluteness down till $T=0$ suggests that all

\footnote{For $SU(3)$ the k=2 symmetric representation still has an appreciable junction value, see ref.$^{29}$}
what changes is the thermal wavelength $\lambda_B(T)$. At temperatures on the order of the glueball mass or higher it is clear that the interparticle distance $O(1/g^2 T)$ is larger than the thermal wave length $1/T$, because the coupling is so small. But at a temperature $T_q$ the thermal wave length takes over with $T_q \sim m_{\text{monopole}}$. It is very unlikely that $T_q$ will be below $T_c$, since that would imply the BE transition as a second one below $T_c$. So the monopole mass should not be lower than $T_c$, so the transition will then be a non- or near-relativistic BE transition.

For the convenience of the reader we give the lattice numbers:

$$T_c = (0.596(4) + 0.453(30)/N^2)\sigma_1$$

and $m_{0^{++}/\sigma_1} = 3.341(76) + 1.75/N^2$ or $T_c = 0.1745(1 + 0.27(3)/N^2)m_{0^{++}}$. This gives the non-relativistic limit if the monopole is as heavy as $m_{\text{monopole}} = m_{0^{++}} = l_M$. In the unlikely case that $m_{\text{monopole}}$ is as light as $T_c$ we have the near-relativistic case.

Below $T_q$ the dilute gas gives a contribution to our ratios, but now determined by Bose statistics. For an analysis of the cold phase ratios see ref. 21.

Of course for most observables in the low $T$ phase the Bose statistics is all-important. Thus the string tension will become non-zero below $T_c$, the character of the transition, i.e. a jump or continuous behaviour in the occupation fraction of the $\vec{p} = 0$ states will be crucial to know and and is calculable in this model. Vortices in the condensed superfluid give then the flux tubes in the Isgur-Paton model of QCD.

Realistic QCD involves quarks, and there is all reason to believe they couple strongly to our monopoles. After all, the latter are bound states of magnetic gluons. In the CFL phase colour is fully broken, and it is amusing to speculate on how the adjoint monopoles behave there. A fundamental multiplet of monopoles would get confined.

It would be interesting to check this model in SUSY gluodynamics, where so many features are known analytically.

The unbearable heavity of the ground state (the energy density of our dilute gas is on the order of $1/(\text{GeV})^4$) can be taken into account by string theory calculations of the same ratios. Indeed, taking into account gravitational effects in an AdS/CFT context, does not affect the result gotten by the simple monopole gas picture: $k(N - k)$ scaling does result in a large $N$ limit, with $k/N$ fixed. Note however that in practice these calculations are done in weak gravitational bulk fields.

Note that a real time picture of our quasi-particles is lacking. According to our Euclidean picture they become, at very high $T$, a 3d gas of particles with small size $l_M$ and small inter-particle distance. The real time picture is a challenge.

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