Unified Field Theory from Enlarged Transformation Group: Spherically Symmetric Solutions

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Abstract. Previously a theory has been presented which extends the geometrical structure of a real four-dimensional space-time via a field of orthonormal tetrads with an enlarged transformation group. This new transformation group, called the conservation group, contains the group of diffeomorphisms as a proper subgroup. Field equations were obtained from a variational principle which is invariant under the conservation group. In this paper, this theory is further extended by development of a suitable Lagrangian for a field with sources. Spherically symmetric solutions for both the free field and the field with sources are given. A stellar model and an external, free-field model are developed. The theory implies that the external stress-energy tensor has non-compact support and hence may give the geometrical foundation for dark matter. The resulting models are compared to the internal and external Schwarzschild models. The theory may explain the Pioneer anomaly and the corona heating problem.

PACS numbers: 04.50.-h, 12.10.-g, 04.40.-b

Submitted to:
1. Introduction

Let $X^4$ be a 4-dimensional space with orthonormal tetrad $h^i_{\mu}$. Then a metric $g_{\mu\nu}$ may be defined on $X^4$ by $g_{\mu\nu} = \eta_{ij} h^i_{\mu} h^j_{\nu}$ where $\eta_{ij} = \text{diag}\{-1,1,1,1\}$. Whereas Einstein extended special relativity to general relativity by extending the group of transformations from the Lorentz group to the group of diffeomorphisms, Einstein later suggested that a unified field theory may be obtained by extending the diffeomorphisms to a larger group [1]. Einstein was also led by the principle that the speed of light was constant. Consistent with Einstein’s approach, we look for the largest group of transformations for which the wave equation, $\Psi^{\alpha \beta} = 0$, is covariant. This is the guiding principle for our theory.

Let $\tilde{V}^\alpha$ be a vector density of weight +1. Then a conservation law of the form $\tilde{V}^\alpha = 0$ is invariant under all transformations satisfying

$$ x^{\nu} \tilde{\alpha} (x^{\beta} \nu, \mu - x^{\beta} \mu, \nu) = 0. $$

This property defines the group of conservative transformations, of which, the group of diffeomorphisms is a proper subgroup [2]. Since the wave equation may be written as $\tilde{V}^\alpha = 0$ with $\tilde{V}^\alpha = \sqrt{-g} \Psi^\alpha$, we see that the conservation group is "the largest group of coordinate transformations under which the equation for the propagation of light is covariant" [3].

Although we may view the space $X^4$ as a Riemannian manifold, the space is more general than a manifold [2-6]. Suppose that $x^\mu$ are coordinates on a Riemannian manifold $M$. Then for functions $f$ with continuous second derivatives, the commutator of partial derivatives is zero, i.e., $[\partial_{\mu}, \partial_{\nu}] f = 0$. If $x^\alpha_{\mu}$ is non-diffeomorphic but satisfies (1) we see that $[\partial_{\mu}, \partial_{\nu}] f$ is nonzero. Thus the geometry associated with the group of conservative transformations is more general than a manifold. It is not simply an abandonment of the diffeomorphisms condition that $[\partial_{\mu}, \partial_{\nu}] = 0$, however, because we require that the transformation be conservative.

The argument that accelerated observers should be on equal footing led Einstein to general relativity. We have argued that requiring that quantum observers be on equal footing leads to the conservation group [3]. If we, as observers consider ourselves to be classical (non-quantum) observers, we will have a preference for the manifold view for what we observe. In truth, we are are quantum observers and hence some "fuzziness" in our observations as well as the observations of other observers is present. Suppose $x^\alpha$ are used as coordinates on a neighborhood of our manifold. If $x^\mu_{\beta}$ is non-diffeomorphic but satisfies (1), then $x^\beta$ is similar to anholonomic coordinates [7]. However, the conservative group property (1) ensures that observers, one using $x^\alpha$ and the other using $x^\beta$ as their coordinates, are on equal footing. When we take the manifold view, we interpret the change of coordinates from $x^\alpha$ to $x^\beta$ as only making sense at the differential (i.e. infinitesimal) level. Thus, we see the neighborhoods upon which the coordinate systems of the second observer make sense to us as the first observer have shrunk from global (special relativity) to local (general relativity) to infinitesimal (conservation group theory). We stipulate that we may begin the setup of our theory as if it were a manifold,
defining \( h^i_\alpha \) as a function of \( x^\mu \). On equal footing there are many coordinate systems related to \( x^\mu \) via conservative transformations, however we, as observers, have a natural preference for the manifold view and hence prefer \( x^\mu \).

In an effort to model dark matter cosmic acceleration, many theorists have simply modified general relativity in some fashion. We claim that our modification which is based on an enlargement of the transformation group is perhaps the only one with a solid guiding principle. Einstein himself felt it was a mistake to simply add a cosmological constant \( \Lambda \). Recently, theoretical developments of \( f(R) \) gravity [8], quintessence [9] and other modifications of general relativity [10] have a similar *ad hoc* flavor.

The geometrical content of the theory based on the conservation group is determined by \( C_\alpha \equiv h^\nu_\iota (h^i_\alpha,\nu - h^i_\nu,\alpha) = \gamma^{\mu}_{\alpha\mu} \), where the Ricci rotation coefficient is given by \( \gamma^{i}_{\mu\nu} = h^{i}_{\mu\nu} \) [2-6]. Pandres calls this the curvature vector. He shows that \( C_\alpha \) is invariant under transformations from \( x^\mu \) to \( x^{\tilde{\mu}} \) if and only if the transformation is conservative and thus satisfies (1). A suitable scalar Lagrangian for the free field is given by

\[
\mathcal{L}_f = \frac{1}{16\pi} \int C^\alpha C_\alpha \, h \, d^4x
\]

where \( h = \sqrt{-g} \) is the determinant of the tetrad.

Using \( h^I_\mu = h^{J\mu}_\iota \Lambda^I_j \), we have extended the field variables [5] to include the tetrad \( h^I_\mu \) and 4 internal vectors \( \Lambda^I_j \), with internal space variable \( x^I \). With this extension, the covariant derivative has been extended to be invariant under a larger group of transformations on \( x^I \) as well as \( x^\mu \) [5]. We assume that the metric on the \( x^I \) space is also \( \eta_{IJ} = diag(-1,1,1,1) \). The definition of the Ricci rotation coefficient is also extended using the \( \Lambda^I_j \) to

\[
\Upsilon^{\alpha}_{\mu\nu} \equiv h^\alpha_I h^{\mu}_{\nu,I} + h^\alpha_I h^I_{\mu\nu} \Lambda^I_j
\]

and the definition of \( C_\alpha \) is also extended to \( C_\alpha \equiv \Upsilon^\alpha_{\alpha\mu} \). Using these extended Ricci rotation coefficients, one finds that

\[
C^\alpha C_\alpha = R + \Upsilon^{\alpha\beta\nu} \Upsilon_{\alpha\nu\beta} - 2C^\alpha_{;\alpha} - \eta^{ij} h^\alpha_j h^\alpha_i (\Lambda^I_j \alpha,\nu - \Lambda^I_j \nu,\alpha)
\]

where \( R \) is the usual Ricci scalar curvature. Thus, when the physical space is interpreted as a manifold, the Lagrangian density of the free field contains terms corresponding to non-gravitational interactions [4,6].

The motion of a free particle or photon in the inertial coordinate system is given by

\[
\frac{d^2x^i}{ds^2} = 0,
\]

where \(-ds^2 = \eta_{ij} dx^i dx^j \). This equation when transformed to internal coordinates, \( x^I \) is

\[
\frac{d^2x^I}{ds^2} = -\Lambda^I_j \Lambda^i_j \frac{dx^j}{ds} \frac{dx^K}{ds},
\]

where the right hand side of this equation is zero when there are no internal forces. Since \( \eta_{IJ} \) corresponds to the flat metric, we naturally interpret the right hand side of (6) as a
force. The $\Lambda^i_I$ are thus internal fields that via $\Lambda^i_{I,J}$ correspond to electroweak and strong interactions. In the manifold view, with coordinates $x^\alpha$ equation (6) becomes

$$\frac{d^2 x^\alpha}{ds^2} + \Gamma^\alpha_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = -\frac{\gamma^\alpha_{\mu\nu}}{ds} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds}. \quad (7)$$

The right hand is partly generated from the internal forces since from (3) one sees that this equation of motion depends on $\Lambda^i_{I,\nu}$.

Setting the variations of $\mathcal{L}_f$ with respect to $h^I_{\mu}$ and $\Lambda^i_I$ equal to zero along with the assumption that we may always choose $\Lambda^i_I$ to correspond to a complex Lorentz transformation (since $h^I_{\mu} = h^I_{\mu} \Lambda^i_I$), yields the field equations

$$C_{\mu} = 0. \quad (8)$$

One feature of the extended theory with field variables $h^I_{\mu}$ and $\Lambda^i_I$ is that the internal fields associated with $\Lambda^i_I$ may be specified after finding a tetrad $h^I_{\alpha}$ which satisfies the condition $h^I_{\alpha}(h^I_{\mu,\nu} - h^I_{\nu,\mu}) = 0$. If this tetrad is a function of position, then $h^I_{\alpha}$ yields a Riemannian manifold (manifold view) with corresponding metric, $g_{\mu\nu} = \eta_{IJ} h^I_{\mu} h^J_{\nu}$. Changes in $\Lambda^i_I$ have no effect on this manifold [5]. If $\Lambda^i_I$ is a constant field such that $\eta_{ij} = \eta_{IJ} \Lambda^i_I \Lambda^j_J = \text{diag}(-1,1,1,1)$, then the field equations are satisfied, however $\Lambda^i_I$ may be non-constant. There exist nonconstant (non-diffeomorphic) values of $\Lambda^i_I$ that satisfy the conservative condition, $\Lambda^i_I(\Delta^i_{I,J} - \Delta^i_{J,I}) = 0$. With $\Lambda^i_I$ that satisfy this condition, the field equations, $C_{\mu} = 0$ remain satisfied. Since this paper is concerned with gravitational implications of the the theory we will assume for the remainder of this paper that we are working with a solution of the field equations for which $\Lambda^i_I = \delta^i_I$. In this case, an identity for the Einstein tensor is

$$G_{\mu\nu} = C_{\mu\nu} - C_{\alpha} \gamma^\alpha_{\mu\nu} - g_{\mu\nu} C^\alpha_{;\alpha} - \frac{1}{2} g_{\mu\nu} C^\alpha C^\alpha$$

$$+ \gamma^\alpha_{\mu,\nu} + \gamma^\alpha_{\sigma\nu} \gamma^\sigma_{\mu\alpha} + \frac{1}{2} g_{\mu\nu} \gamma^{\alpha\beta\sigma} \gamma_{\alpha\beta\sigma}$$

This expression is not manifestly symmetric in $\mu$ and $\nu$, but the left-hand side is symmetric in its lower indices and hence the right-hand side must be as well. Thus we use a symmetrized expression to ensure this. Define for general $K_{\mu\nu}$, the symmetrized tensor by $K_{(\mu\nu)} = \frac{1}{2}(K_{\mu\nu} + K_{\nu\mu})$. Using (8) we see that the field equations may be also expressed in the form

$$G_{\mu\nu} = \gamma^\alpha_{(\mu;\nu)} + \gamma^\alpha_{\sigma(\nu} \gamma^\sigma_{\mu)\alpha} + \frac{1}{2} g_{\mu\nu} \gamma^{\alpha\beta\sigma} \gamma_{\alpha\beta\sigma} \equiv 8\pi \langle T_\mu T_\nu \rangle \quad (9)$$

with free field stress energy tensor $T_f$. The terms of $T_f$ suggest that, when interpreted in Riemannian geometry, this new geometry produces a stress energy tensor with additional terms that could be the stress energy tensor for dark matter or dark energy [6].

In the presence of sources the Lagrangian is of the form

$$\mathcal{L} = \mathcal{L}_f + \mathcal{L}_s = \int \left( \frac{1}{16\pi} C^\alpha C_{\alpha} + L_s \right) h^{4} x \quad (10)$$

where $L_s = L_s(x^\alpha)$ is the appropriate Lagrangian density function for the source. In this case $C_{\alpha}$ is nonzero and variation of (10) with respect to the tetrad results in

$$\int \left[ \frac{1}{16\pi} \left( C_{(\mu;\nu)} - C_{\alpha} \gamma^\alpha_{(\mu;\nu)} - \frac{1}{2} g_{\mu\nu} C^\alpha_{\alpha} - g_{\mu\nu} C^\alpha_{;\alpha} \right) - \frac{1}{2} \langle T_s \rangle_{\mu\nu} \right] h^{4} h^{I}_{\nu} \delta_{I}^{\mu} d^4 x = 0$$
Here, \((T_s)_{\mu\nu}\) is the usual stress-energy tensor of the source for the standard theory \([11]\). Thus
\[
C_{(\mu;\nu)} - C_\alpha \nabla^\alpha \gamma_{(\mu;\nu)} - \frac{1}{2} g_{\mu\nu} C^\alpha C_\alpha - g_{\mu\nu} C_\alpha^\alpha = 8\pi(T_s)_{\mu\nu}
\]
and also we have the following identity for the Einstein tensor,
\[
G_{\mu\nu} = \left( \nabla_\alpha \nabla^\alpha \gamma_{(\mu;\nu)} + \frac{1}{2} g_{\mu\nu} \nabla^\alpha \nabla_\alpha \gamma - \frac{1}{2} g_{\mu\nu} C^\alpha C_\alpha \right) - 8\pi(T_s)_{\mu\nu}
\]
or
\[
G_{\mu\nu} = 8\pi(T_f)_{\mu\nu} + 8\pi(T_s)_{\mu\nu}
\]
We call \(T_f\) the free field stress energy and \(T_s\) the stress energy for the source.

2. Spherically symmetric solutions.

2.1. Free Fields.

We now exhibit spherically symmetric solutions of the field equations for a free field \((5)\). Let \(r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}\). If \(f(r)\) is a positive differentiable function of \(r\), then the tetrad field given by
\[
h^i_\mu = \delta_0^i \delta_0^\mu \sqrt{f(r)} + \frac{1}{\sqrt{f(r)}} \left( \delta_1^i \delta_1^\mu + \delta_2^i \delta_2^\mu + \delta_3^i \delta_3^\mu \right)
\]
yields \(C_\mu = 0\) and hence is a solution of the field equations \((5)\). The line element (metric) in spherical coordinates is given by
\[
ds^2 = -f(r) dt^2 + \frac{1}{\sqrt{f(r)}} dr^2 + \frac{r^2}{\sqrt{f(r)}} d\theta^2 + \frac{r^2 \sin^2 \theta}{\sqrt{f(r)}} d\phi^2
\]
This is the line element (metric) is isotropic spherical coordinates. Now change the radial coordinate \(r \rightarrow \bar{r}\) so that \(\bar{r}^2 = \frac{r^2}{f(r)}\) and \(f(r) = e^{2\Phi(r)}\). Since these are differentiable functions, this change of coordinates \((t, r, \theta, \phi) \rightarrow (t, \bar{r}, \theta, \phi)\) is a diffeomorphism and hence the field equations remain satisfied. The mapping \(r \rightarrow \bar{r}\) is the simply the inverse of the function \(r = r(\bar{r}) = \bar{r} e^{\frac{1}{2} \Phi(r)}\). After this change in the radial coordinate, we will now rename \(\bar{r}\) as simply \(r\). The tetrad in spherical coordinates may be expressed by
\[
h^i_\mu = \begin{bmatrix}
  e^\Phi & 0 & 0 & 0 \\
  0 & (1 + \frac{1}{2} r \Phi') \sin \theta \cos \phi & r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\
  0 & (1 + \frac{1}{2} r \Phi') \sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos \phi \\
  0 & (1 + \frac{1}{2} r \Phi') \cos \theta & -r \sin \theta & 0
\end{bmatrix}
\]
where the upper index refers to the row and the prime indicates differentiation with respect to \(r\). One finds that \(C_\mu = 0\) for this tetrad. The new metric is
\[
ds^2 = -e^{2\Phi(r)} dt^2 + (1 + \frac{1}{2} r \Phi'(r))^2 dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2
\]
After a long, but straightforward calculation, one finds that the Einstein tensor equals a diagonal tensor which is in general nonzero: \( G_{\mu\nu} = 8\pi(T)_{\mu\nu} \). The non-zero components are (with \( \Phi \) representing \( \Phi(r) \))

\[
G_{tt} = 8\pi(T)_{tt} = \frac{e^{2\Phi} \left( \frac{1}{3}(r\Phi')^3 + \frac{3}{4}(r\Phi')^2 + 2r\Phi' + r^2\Phi'' \right)}{r^2 \left( 1 + \frac{1}{2}r\Phi' \right)^3}, \quad (18)
\]

\[
G_{rr} = 8\pi(T)_{rr} = \frac{r^2\Phi' - \frac{1}{4}(r\Phi')^2}{r^2}, \quad (19)
\]

and

\[
G_{\theta\theta} = \frac{8\pi T_{\theta\theta}}{r^2} = \frac{\frac{1}{2}(r\Phi')^3 + (r\Phi')^2 + \frac{1}{4}r\Phi' + \frac{1}{2}r^2\Phi''}{r^2 \left( 1 + \frac{1}{2}r\Phi' \right)^3}, \quad (20)
\]

\[
G_{\phi\phi} = \frac{8\pi T_{\phi\phi}}{r^2 \sin^2 \theta} = \frac{\frac{1}{2}(r\Phi')^3 + (r\Phi')^2 + \frac{1}{4}r\Phi' + \frac{1}{2}r^2\Phi''}{r^2 \left( 1 + \frac{1}{2}r\Phi' \right)^3}. \quad (21)
\]

One difference between this and the Schwarzschild metric [12] is that there is only one unknown function (\( \Phi(r) \)) instead of two (the standard \( \Lambda(r) \) and \( \Phi(r) \) functions).

We will first work on the \( G_{tt} \) term. One finds that

\[
e^{-2\Phi(r)}G_{tt} = \frac{2}{r^2} \cdot \frac{d}{dr} \left( \frac{r}{2} - \frac{r}{2(1 + \frac{1}{2}r\Phi')} \right) = \frac{2}{r^2} w'(r) \equiv 8\pi \rho_f, \quad (21)
\]

where \( w(r) \equiv \frac{r}{2} - \frac{r}{2(1 + \frac{1}{2}r\Phi')} \). Hence

\[
\Phi'(r) = \frac{2}{r} \left[ \left( 1 - \frac{2w(r)}{r} \right)^{-\frac{1}{2}} - 1 \right]. \quad (22)
\]

Thus

\[
g_{rr} = \left( 1 + \frac{1}{2}r\Phi' \right)^2 = \left( 1 - \frac{2w(r)}{r} \right)^{-1}, \quad (23)
\]

and

\[
g_{tt} = -e^{2\Phi(r)}, \text{ where } \Phi(r) = \int \frac{2}{r} \left[ \left( 1 - \frac{2w(r)}{r} \right)^{-\frac{1}{2}} - 1 \right] dr \quad (24)
\]

(this defines \( \Phi(r) \) up to a constant). The function \( w(r) \) (as shown below) is related to the mass inside a ball of radius \( r \) for the free field and \( \rho_f \) represents the density of the free field in the manifold interpretation.

Let \( p_R \) represent the radial pressure of the free field. Then one finds [12] that the radial pressure of the free field is given by

\[
8\pi p_R = \frac{G_{rr}}{(1 + \frac{1}{2}r\Phi')^2} = \frac{r\Phi' - \frac{1}{4}(r\Phi')^2}{r^2 \left( 1 + \frac{1}{2}r\Phi' \right)^3}, \quad (25)
\]

and from (22) one finds that

\[
8\pi p_R = \frac{4r\sqrt{1 - \frac{2w(r)}{r} - 4r + 6w(r)}}{r^3}. \quad (26)
\]
Let the tangential pressure of the free field be denoted by \( p_T \). We also find that
\[
8\pi p_T = \frac{Gmn}{r^{2\sin\theta}} = \frac{G_{\theta\theta}}{r^{2\sin\theta}} + \frac{G_{\phi\phi}}{r^{2\sin\theta}}.
\]
and thus,
\[
8\pi p_T = \frac{\frac{1}{2}(r\Phi')^3 + (r\Phi')^2 + \frac{1}{2}r\Phi' + \frac{1}{2}r^2\Phi''}{r^2(1 + \frac{1}{2}r\Phi')^3}.
\] (27)
Using (22), the tangential pressure may be expressed in terms of \( w(r) \) and \( r \) by
\[
8\pi p_T = \frac{8r - 9w(r) - 8r\sqrt{1 - \frac{2w(r)}{r} + rw'(r)}}{r^3}.
\] (28)
Since \( p_R \neq p_T \) there are shear stresses and we see that \( (T^r_f)_{\mu\nu} \) does not model a perfect fluid. We note that \( (T^r_f)_{\mu\nu} = \text{diag}[-\rho, p_R, p_T, p_T] \). The conservation of energy condition, \( T_{\nu;\mu} = 0 \) is vacuous for \( \nu = 0, 2 \) and 3. The only nontrivial condition is when \( \nu = 1 \) representing the radial coordinate and in this case yields
\[
(\rho + p_R)\Phi' + p_R' - \frac{2}{r}(p_T - p_R) = 0,
\] (29)
which indicates that the resultant force on a fluid element is zero.

2.2. Field with Sources.

In spherical coordinates, a spherically symmetric tetrad with \( r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} \) may be expressed by
\[
h^i_{\mu} = \begin{pmatrix}
e^{\Phi(r)} & 0 & 0 & 0 \\
e^{\Lambda(r)} \sin \theta \cos \phi & r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\
e^{\Lambda(r)} \sin \theta \sin \phi & r \cos \theta \sin \phi & r \sin \theta \cos \phi \\
e^{\Lambda(r)} \cos \theta & -r \sin \theta & 0
\end{pmatrix}
\] (30)
where the upper index refers to the row. The curvature vector for this tetrad field is given by
\[
C_{\mu} = \frac{e^{\Lambda}}{r} \begin{pmatrix}0, \ 2 - e^{-\Lambda}(r\Phi' + 2), \ 0, \ 0 \end{pmatrix}
\] (31)
where components are in the order \([t, r, \theta, \phi]\) and the prime denotes the derivative with respect to \( r \). The tetrad (30) leads to the metric
\[
ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2.
\] (32)
Comparison of metrics (17) and (32) implies that for the metric of (17), \((r\Phi' + 2) = 2e^\Lambda\) which then implies that \( C_{\mu} \) in equation (31) would be identically zero. From (31) we see that the general spherically symmetric tetrad field does not generally yield \( C_{\mu} = 0 \), hence we consider whether there exists a spherically symmetric solution of the field equations which flow from (10). The metric (32) leads to a diagonal Einstein tensor with nonzero elements:
\[
G^t_t = \frac{1}{r^2}(-2re^{-2\Lambda}\Lambda' + e^{-2\Lambda} - 1) = -\frac{2}{r^2} \frac{d}{dr} \left[ \frac{1}{2} r (1 - e^{-2\Lambda}) \right],
\] (33)
\[
G^r_r = \frac{1}{r^2}(2re^{-2\Lambda}\Phi' + e^{-2\Lambda} - 1)
\] (34)
and
\[ G^\theta_\theta = G^\phi_\phi = \frac{e^{-2\Lambda}}{r}(r\Phi'' + r(\Phi')^2 - r\Phi'\Lambda' + \Phi' - \Lambda') \quad (35) \]

Using \( G_{\mu\nu} = 8\pi T_{\mu\nu} \), we now decompose the stress-energy tensor using (13). From
\[ 8\pi( T_{\mu\nu}) = \Upsilon_{\mu\nu} + \Upsilon_{\sigma\nu}\Upsilon^\sigma_{\mu\alpha} + \frac{1}{2}g_{\mu\nu}\Upsilon_{\alpha\beta\gamma}\Upsilon_{\alpha\sigma\beta}, \]
one finds that \( T_{\mu\nu} \) is diagonal with elements
\[ 8\pi( T_{\mu\nu})_{tt} = e^{2\Phi - 2\Lambda} \frac{r^2(\Phi'')^2 - r^2(\Phi')^2 + 2r\Phi' + 2e^\Lambda - e^{2\Lambda} - 1}{r^2}, \quad (36) \]
\[ 8\pi( T_{\mu\nu})_{rr} = \frac{1}{r^2} \left( -\frac{1}{2}(\Phi')^2 + e^{2\Lambda} - 1 \right) \quad \text{and} \quad (37) \]
\[ 8\pi( T_{\mu\nu})_{\theta\theta} = \frac{8\pi( T_{\mu\nu})_{\phi\phi}}{r} = \frac{e^{-2\Lambda}(\frac{1}{2}r(\Phi')^2 - \Phi' + \Lambda' + e^\Lambda\Phi')}{r}. \quad (38) \]

As indicated by (12) and (13), \( T_{\mu\nu} \) is determined by variation of the \( L_s \) term in the Lagrangian (10).

3. Models for the Interior of a Star.

We will use the general spherical tetrad and the field equations which are derived from the Lagrangian (10) with \( L_s = \rho(r) \), where \( \rho(r) \) is the density as a function of \( r \). It is well known that this Lagrangian with appropriate thermodynamic conditions lead to the usual perfect fluid stress-energy tensor [13,14]. With a tetrad that corresponds to a stationary basis (velocity of the observer is zero if \( h_{0\mu} = 0 \) for \( \mu = 1, 2 \text{ and } 3 \) ), one finds [12]
\[ (T_s)^\mu_\nu = \begin{bmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix}. \quad (39) \]

Using the tetrad field of (30), we require that the radial and tangential pressures of the corresponding source stress-energy tensor (11) be equal, leading to the following differential equation with primes denoting derivatives with respect to \( r \):
\[ r^2\Phi'' - (r^2\Lambda' + re^\Lambda)\Phi' = 2 - 2e^{2\Lambda} + 2r\Lambda' \quad (40) \]
After multiplying by an integrating factor and integrating, (40) implies that
\[ (r\Phi' + 2)e^{-\Lambda} = 2 - \kappa re^{(r^2e^\Lambda)} \quad (41) \]
where \( \kappa \) is arbitrary.

For convenience of interpretation we define \( \rho_s(r) \equiv \rho(r) - \frac{1}{8\pi}C^\alpha_{\alpha\beta} \) and thus the new source Lagrangian term is \( L_s = \rho_s(r) = \rho(r) - \frac{1}{8\pi}C^\alpha_{\alpha\beta} \). Since addition of a pure covariant divergence leaves the field equations unchanged, this does not affect any of our conclusions thus far. In order to leave the energy unchanged this induces the definition
\( p_s \equiv p + \frac{1}{8\pi} C_\alpha^\alpha. \) (The enthalpy \([14]\) given by \( \frac{\rho + p}{n} \), where \( n \) is the baryon number density, is unchanged.) We also note that we assume that \( C_\alpha \) has compact support and is a smooth function and hence integration of the \( C_\alpha^\alpha \) term over the region of support results in a value of zero and hence does not affect the overall mass as well.

With these definitions from (33-37), (40) and (41) we find that

\[
8\pi p_s = \frac{1}{2} \left( ke \int (r^{-1} e^\Lambda) \right)^2
\]  

(42)

and

\[
8\pi p_s = \frac{ke \int (r^{-1} e^\Lambda)}{r} - \frac{1}{2} \left( ke \int (r^{-1} e^\Lambda) \right)^2.
\]  

(43)

We also note that for this internal solution that the curvature vector in the order \( t, r, \theta, \phi \) is given by

\[
 C_\mu = \left[ 0, \frac{ke e \int (r^{-1} e^\Lambda)}{r}, 0, 0 \right].
\]  

(44)

and this gives \( C^\mu C_\mu = \kappa^2 e^2 \int (r^{-1} e^\Lambda) \).

For the total stress-energy tensor \( T^\mu_\nu \) with nonzero components given by (33-35), one finds indeed that \( T^\mu_\nu\mid_{\nu=\mu} = 0 \). From (33) with \( G_{tt} = -\frac{8\pi \rho}{R} \), we also interpret the mass as a function of \( r \) to be given by

\[
 m(r) = \frac{1}{2} r^2 \left( \frac{1}{2} r (1 - e^{-2\Lambda}) \right)'
\]  

(45)

and hence the mass within a sphere of radius \( r \) is given by the function

\[
 m(r) = \frac{1}{2} r (1 - e^{-2\Lambda})
\]  

(46)

This implies that \( e^{2\Lambda} = (1 - 2m/r)^{-1} \) which matches with the standard \( g_{rr} \) value of the interior Schwarzschild metric. From (34) and (46) with \( G_{rr} = 8\pi p_R \), we get

\[
 8\pi p_R = \frac{2}{r^2} \left[ 2 \sqrt{1 - 2m/r} + \frac{3m}{r} - 2 - \kappa r \sqrt{1 - 2m/r} e^{\int \frac{1}{r \sqrt{1 - 2m/r}}} \right]
\]  

(47)

and from (35) and (46) with \( G^\theta_\theta = G^\phi_\phi = 8\pi p_T \), we get

\[
 8\pi p_T = \frac{1}{r^2} \left[ \kappa r \left( 3 \sqrt{1 - 2m/r} - 5 \right) e^{\int \frac{1}{r \sqrt{1 - 2m/r}}} + \kappa^2 r^2 e^{\int \frac{2}{r \sqrt{1 - 2m/r}}} \right. + \left. 4 \left( 1 - \sqrt{1 - 2m/r} \right)^2 + m' - \frac{m}{r} \right]
\]  

(48)

There are 2 constants that may be chosen for convenience of interpretation. The value of \( \kappa \) may be determined by conditions on the pressure. A second constant is the constant of integration in solving for \( \Phi(r) \) from (41), which may be determined by appropriate continuity conditions.

**Constant Density Model.** As a reasonable model, suppose that \( \rho(r) = \frac{3\alpha^2}{8\pi} \), where \( \alpha \) is an arbitrary constant and the factors of 3 and \( 8\pi \) are chosen for convenience. This constant value of \( \rho \) from (45-46) implies that \( m(r) = \frac{1}{2} \alpha^2 r^3 \) and \( e^{-2\Lambda} = 1 - \alpha^2 r^2 \) and this model only makes sense for \( 0 \leq r \leq 1/\alpha \). We note that the integral that appears in (42-44) and (47-48) may be easily integrated. Let \( \hat{\kappa} = \frac{\kappa}{\alpha} \), then
\[ \kappa r \int_{\sqrt{1-2m/r}} = \hat{\kappa} \left( 1 - \sqrt{1 - \alpha^2 r^2} \right) \] and hence \( C^\mu C_\mu = \hat{\kappa}^2 \left( \frac{1}{r} - \sqrt{1 - \alpha^2 r^2} \right)^2 \). We note that this also implies that \( r \Phi' = (2 - \hat{\kappa}) \left( \left( 1 - \alpha^2 r^2 \right)^{-\frac{1}{2}} - 1 \right) \).

In this constant density model, the resulting radial pressure is given by \( 8\pi p_R = \frac{\hat{\kappa}}{r^2} \left[ 2 \left( 2 - \hat{\kappa} \right) \left( 1 - \alpha^2 r^2 \right)^{\frac{1}{2}} - 1 \right] + (3 - 2\hat{\kappa}) \alpha^2 r^2 \). We note that \( \lim_{r \to 0} (8\pi p_R) = (1 - \hat{\kappa}) \alpha^2 \) which suggests that a reasonable value of \( \hat{\kappa} \) is less than 1. The radial pressure approaches a value: \( 8\pi p_R (r = 1/\alpha) = -2(2 - \hat{\kappa}) \alpha^2 \) which is less than zero. At some intermediate value, it will match with the corresponding external radial pressure. If we use a result that is given in the next section, we may estimate the radial pressure at the surface to be approximately \( \frac{1}{2} \alpha^2 \). Using this approximate value, we find that the radial pressure matches the external radial pressure at \( r = \frac{1}{\alpha} \sqrt{1 - \frac{9}{(5-4\hat{\kappa})^2}} \) and we also find that this implies that \( \hat{\kappa} < 1/2 \).

In order to work out the value of the tangential pressure we use \( (48) \) which yields \( 8\pi p_T = \frac{\hat{\kappa}}{r^2} \left[ 2 \left( 2 - \hat{\kappa} \right) \left( 1 - \sqrt{1 - \alpha^2 r^2} \right) - (3 - 3\hat{\kappa} + \hat{\kappa}^2) \alpha^2 r^2 \right] \). As \( r \to 0 \), \( 8\pi p_T \to (1 - \hat{\kappa}) \alpha^2 \) which is the same as the radial pressure. For \( r > 0 \), however, we see that \( p_R \neq p_T \). As \( r \to \frac{1}{\alpha} \), \( 8\pi p_T \to \left( \hat{\kappa}^2 - 5\hat{\kappa} + 5 \right) \alpha^2 \) which is positive when \( \hat{\kappa} < \frac{1}{2} \). Graphs of \( 8\pi p_R \) and \( 8\pi p_T \) for \( \hat{\kappa} = \frac{1}{10} \) and \( \alpha = \frac{1}{100} \) are shown in Figure 1.
4. External Solutions.

In order for the external solution to agree with the weak-field solution as \( r \to \infty \), we will identify 
\[ w(r) = \frac{1}{2} M \] for \( r \geq R \), the radius of the star. Thus from (22) we have
\[ \frac{1}{2} M = \frac{r}{2} - \frac{r}{2(1 + r \Phi')}^{2} \] and hence
\[ \Phi(r) = \int \left[ \frac{2}{r \sqrt{1 - \frac{M}{r}}} - \frac{2}{r} \right] \, dr \] (49)
which can be easily integrated to find
\[ \Phi(r) = 4 \ln \left( 1 + \sqrt{1 - \frac{M}{r}} \right) + \frac{1}{2} \ln C \] for some arbitrary \( C > 0 \). Thus
\[ g_{tt} = -e^{2\Phi(r)} = -C \left( 1 + \sqrt{1 - \frac{M}{r}} \right)^{8} \] . (50)
The arbitrary constant \( C \) is determined by the usual weak field approximation [12] which is 
\[ g_{tt} \approx -1 + \frac{2M}{r} \]. This implies that
\[ C = \frac{1}{256} \]. Hence
\[ g_{tt} = -\frac{1}{256} \left( 1 + \sqrt{1 - \frac{M}{r}} \right)^{8} \] . (51)
We thus obtain the following line element:
\[ ds^{2} = -\frac{1}{256} \left( 1 + \sqrt{1 - \frac{M}{r}} \right)^{8} dt^{2} + \left( 1 + \frac{M}{r} + \frac{M^{2}}{r^{2}} \right) dr^{2} + r^{2} d\theta^{2} + r^{2} \sin^{2} \theta d\phi^{2} . \] (52)

Expanding \( g_{tt} \) and \( g_{rr} \) in powers of \( \frac{M}{r} \), we find that asymptotically (for \( r \gg M \)), to second order,
\[ ds^{2} \approx -\left( 1 - \frac{2M}{r} + \frac{5M^{2}}{4r^{2}} \right) dt^{2} + \left( 1 + \frac{M}{r} + \frac{M^{2}}{r^{2}} \right) dr^{2} + r^{2} d\theta^{2} + r^{2} \sin^{2} \theta d\phi^{2} . \] (53)

Using (18-20), the Einstein field equations for the external solution are
\[ G_{tt} = 8\pi T_{tt} = 0 \]
\[ G_{rr} = 8\pi T_{rr} = \frac{M\left( 3\sqrt{1 - \frac{M}{r}} - 1 \right)}{r^{3}\left( 1 + \sqrt{1 - \frac{M}{r}} \right)} \] (54)
\[ \frac{G_{\theta\theta}}{r^{2}} = \frac{G_{\phi\phi}}{r^{2}\sin^{2} \theta} = \frac{8\pi T_{\theta\theta}}{r^{2}} = \frac{8\pi T_{\phi\phi}}{r^{2}\sin^{2} \phi} = \frac{-M\left( 9\sqrt{1 - \frac{M}{r}} - 7 \right)}{2r^{2}\left( 1 + \sqrt{1 - \frac{M}{r}} \right)} \] .

Or
\[ 8\pi \rho = 0 \]
\[ 8\pi p_{R} = \frac{M\left( 3\sqrt{1 - \frac{M}{r}} - 1 \right)}{r^{3}\left( 1 + \sqrt{1 - \frac{M}{r}} \right)} \] (55)
\[ 8\pi p_{T} = \frac{-M\left( 9\sqrt{1 - \frac{M}{r}} - 7 \right)}{2r^{2}\left( 1 + \sqrt{1 - \frac{M}{r}} \right)} . \]
Asymptotically for \( r \gg M \), we have
\[
8\pi p_R \approx \frac{M}{r^3} \left( 1 - \frac{M}{2r} \right),
\]
\[
8\pi p_T \approx -\frac{M}{2r^3} \left( 1 - \frac{2M}{r} \right).
\] (56)

How do we interpret these equations? The field equations for the space surrounding a mass of \( M \) has noncompact stress-energy which is zero if and only if \( M = 0 \), and the metric is the Lorentz metric if and only if \( M = 0 \) as well. The energy in this halo is a direct consequence of the mass \( M \). This likely corresponds to dark matter. It is purely geometric in origin and has the appearance and properties associated with an actual mass. Its gravitational field and effects are equivalent to that of regular matter, but it is dark in the sense that its non-gravitational effects are feeble. Its electro-weak interactions are not as dominant as the effects it has on other massive objects. Independent of the nature of the mass \( M \) (whether it is a sun, a planet or a sphere of ice) this halo has the same form which depends only on the value of \( M \).

Although the stress-energy tensor does not correspond to a perfect fluid, the pressure gradients prevent fluid elements from moving inward or outward. Using \( T^\mu_{\nu;\mu} = 0 \), with \( T^\mu_\nu = \text{diag}(-\rho, p_R(r), p_T(r), p_T(r)) \) one easily finds from (29) that
\[
-\frac{dp_R}{dr} + 2\frac{1}{r} \left( p_T - p_R \right) = (\rho + p_R) \Phi'.
\] (57)
The left-hand side of this equation corresponds to the outward force due to the pressure of the ”fluid.” We see from (57) that the outward force due to pressures equals the inward force due to the gravitational force. From (49), we see that asymptotically \( \Phi' \approx \frac{M}{r^2} \) and thus asymptotically
\[
-\frac{dp_R}{dr} + 2\frac{1}{r} \left( p_T(r) - p_R(r) \right) \approx \frac{M^2}{8\pi r^5}.
\] (58)

4.1. A Noncompact External Solution.

If the density outside (for \( r \geq R \)) is small but nonzero, a realistic function that agrees with the weak field solution must have the property, \( \lim_{r \to \infty} w(r) = \frac{1}{2} M \). Thus noncompact solutions may be constructed that realistically model the exterior of a star or the distribution of matter in a star cluster. One particularly simple model is given by
\[
w(r) = \frac{M}{2} - \frac{M^2}{8r}.
\] (59)

With this choice, \( 1 - 2w(r) = 1 - \frac{M}{r} + \frac{M^2}{2r^2} = (1 - \frac{M}{2r})^2 \). Thus, using (17), (21), (26) and (28) we have (the approximation assumes \( r \gg M \))
\[
ds^2 = -\left( 1 - \frac{M}{2r} \right)^4 dt^2 + \left( 1 - \frac{M}{r} \right)^2 dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2
\approx -\left( 1 - \frac{2M}{r} + \frac{3M^2}{r^2} \right) dt^2 + \left( 1 + \frac{M}{r} + \frac{3M^2}{4r^2} \right) dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2
\] (60)
and hence
\[ 8\pi\rho = \frac{M^2}{4r^4} \]
\[ 8\pi\rho_r = \frac{M}{r^3} \left( 1 - \frac{3M}{4r} \right) \tag{61} \]
\[ 8\pi\rho_T = \frac{M}{2r^3} \left( 1 - \frac{5M}{2r} \right) \]

These equations are exact. Equation (58) is also correct for this noncompact solution. The comments about dark matter which immediately follow equation (56) apply here as well. It may be that the size and properties of the dark matter halo will determine the function \( w(r) \).

5. Motion of a Test Particle and Comparison with Predictions of General Relativity.

We now investigate the motion of a test particle in the external field solution. An efficient procedure for doing this is one that extremalizes an appropriate Lagrangian. We follow de Felice and Clarke [14] with a Lagrangian for a particle in a field with non-zero stress energy tensor. Specifically, we use the Lagrangian (10) with the source term given by
\[ L_s = \rho(x) = \frac{\mu}{\sqrt{-g}} \int \delta^4(\epsilon(x - \gamma(s))(-u^\mu u_\mu)^{\frac{3}{2}} ds \tag{62} \]
where \( \delta^4(\epsilon) \) approximates the Dirac delta function with a space-like volume of \( \epsilon \) which yields the usual Dirac delta function in the limit as \( \epsilon \rightarrow 0 \). The path of the particle is given by \( \gamma(s) \) and its velocity is \( u^\mu = \frac{d\gamma^\mu}{ds} \). For convenience, we will use the "dot" notation for the components of \( u^\alpha \), i.e. \( u^\alpha = (\dot{t}, \dot{r}, \dot{\theta}, \dot{\phi}) \). Let \( \mu \) denote the mass of the particle. As noted in de Felice and Clarke ([14] page 222), the condition \( T^\beta_{\gamma\beta} = 0 \) leads to
\[ \epsilon (T_f)^{\beta\alpha}_{\;\beta\;\beta} + \frac{\mu}{\sqrt{-g}} u^\alpha u^\beta_{\;\beta} = 0 \tag{63} \]
The only nonzero component of \( (T_f)^{\beta\alpha}_{\;\beta\;\beta} \) is the radial component \( (\alpha = 1) \) as we saw in (29). The \( u^\beta u^\alpha_{\;\beta} \) term corresponds to the geodesic equation. When \( \alpha \neq 1 \), (63) is equivalent to the geodesic equation for \( u^\alpha \).

First we note that the \( \theta \) component (when \( \alpha = 2 \)) of (63) after multiplying by \( \frac{\sqrt{-g} u^\alpha}{\mu} \) yields
\[ \frac{d^2\theta}{dT^2} + \frac{2}{r} \frac{d\theta}{dT} - (\sin \theta \cos \theta) \left( \frac{d\phi}{dT} \right)^2 = 0 \tag{64} \]
We note that \( \theta \equiv \frac{\pi}{2} \) is a solution of this equation and symmetry considerations imply that we may safely assign this value of \( \theta \) since particle motion takes place in a plane through the origin \( (r = 0) \).
We next look at the $t$ component (when $\alpha = 0$) of (63) which yields:

$$\frac{d^2 t}{d\tau^2} + \frac{4M}{r^2 \sqrt{1 - M/r} \left(1 + \sqrt{1 - M/r}\right)} \frac{dr}{d\tau} \frac{dt}{d\tau} = 0 .$$

(65)

After multiplying by an integrating factor we find that this equation may be written as

$$\frac{d}{d\tau} \left[ \frac{1}{256} (1 + \sqrt{1 - M/r})^8 \frac{dt}{d\tau} \right] = 0$$

and hence

$$\dot{t} = 256 E \left(1 + \sqrt{1 - M/r}\right)^{-8}$$

(66)

where $E$ is a constant representing the energy of the particle.

The $\phi$ component (when $\alpha = 3$) of (63) yields:

$$\frac{d^2 \phi}{d\tau^2} + \frac{2d\phi}{r} + \frac{1}{d\tau} \frac{d\phi}{d\tau} = 0$$

(67)

and after multiplying by $r^2$ this equation may be written as

$$\frac{d}{d\tau} \left[ r^2 \frac{d\phi}{d\tau} \right] = 0 .$$

Thus

$$\dot{\phi} = \frac{L}{r^2}$$

(68)

where the constant $L$ represents the angular momentum which is conserved.

The $r$ component requires some careful interpretation. We will assume the the particle is small in the sense that the curvature of space does not change appreciably over the space-like regions associated with its motion. We will also assume that the particle has spherical symmetry. Thus there is an external field associated with the particle that is carried along with it (halo). It seems reasonable that the pressures in this particle halo, similar to those of equations (55) or (61), will have very little effect on the motion of the particle. The density of the particle’s halo will be incorporated into the calculation of the mass $\mu$ of the particle. If the density of the particle is identical to the density determined by $(T_i)^{\mu}_{\nu}$, i.e. equal to $(T_i)^{0}_{0}$, (i.e. identical to the density of the fluid elements of the halo associated with the mass $M$) then the net force on the particle would be zero. However, we find that when the density differs from the fluid element density, then the $(T_i)^{\beta\alpha}_{;\beta}$ term has a nonzero contribution. As is usual for the perfect fluid type stress-energy tensor, the components of $(T_i)^{\beta\alpha}_{;\beta}$ are in units of force per unit volume. We find that the gravitational action on the particle is accounted for in the $\sqrt{-u^\mu u_\nu} u^\alpha u^\beta$ term of (63). Thus, the corresponding term of $(T_i)^{\beta\alpha}_{;\beta}$ should be omitted. (Recall that $\epsilon$ represents the volume of the particle.) This implies that there is an additional outward force, $F_p(r)$, from the radial and tangential pressures, since

$$\epsilon (T_i)^{\beta\alpha}_{;\beta} + \frac{\mu}{\sqrt{-u^\mu u_\nu}} u^\beta u^\alpha_{;\beta} = 0$$

$$\epsilon \left( \left( p'_R - \frac{2}{r} (p_T - p_R) \right) + \frac{\mu}{\sqrt{-u^\mu u_\nu}} u^\beta u^\alpha_{;\beta} \right) = 0$$

(69)

i.e.

$$\frac{\mu}{\sqrt{-u^\mu u_\nu}} u^\beta u^\alpha_{;\beta} = \epsilon \left( -p'_R + \frac{2}{r} (p_T - p_R) \right) \equiv F_p .$$

(70)
The $r$ component of (63) (when $\alpha = 1$) after multiplying by $\frac{\sqrt{-u^\mu u_\mu}}{\mu}$ is given by

$$\ddot{r} + \frac{M(1 + \sqrt{1 - M/r})^7 \sqrt{1 - M/r}}{128r^2} \dot{t}^2 - \frac{M}{2r^2(1 - M/r)} \dot{r}^2 - r(1 - M/r)\dot{\phi}^2 = \frac{\sqrt{-u^\mu u_\mu}}{\mu} F_p$$

Using (66) and (68) we find that

$$\ddot{r} + \frac{512ME^2(1 - M/r)^3}{r^2 (1 + \sqrt{1 - M/r})^3} - \frac{M}{2r^2(1 - M/r)} \dot{r}^2$$

$$- \frac{L^2(1 - M/r)}{r^3} = \frac{\sqrt{-u^\mu u_\mu}}{\mu} F_p$$

We now impose a normalization on the velocity $u^\mu$: $-u^\nu u_\nu \approx 1$. (It is actually should be $-u^\nu u_\nu - g_{tt} \int \frac{F_p}{\mu} dt = 1$ with the constant of integration chosen so that the integral term vanishes as $r \to \infty$. This correction to $-u^\nu u_\nu$ is much less than $\frac{M}{r}$. Furthermore it is multiplied by $\frac{F_p}{\mu}$ which is also small.) Therefore $\frac{1}{256} \left(1 + \sqrt{1 - M/r}\right)^8 \dot{t}^2 - (1 - \frac{M}{r})^{-1} \dot{r}^2 - r^2 \dot{\phi}^2 = 1$. Using (66) and (68) we may eliminate the $\dot{t}$ and $\dot{\phi}$ terms. This leads to $256E^2(1 + \sqrt{1 - M/r})^{-8} = (1 - \frac{M}{r})^{-1} \dot{r}^2 + \frac{L^2}{r^3} + 1$. Substituting this into (72), we arrive at

$$\ddot{r} + \frac{2M \sqrt{1 - \frac{M}{r}}}{r^2 \left(1 + \sqrt{1 - \frac{M}{r}}\right)} = \frac{L^2 \sqrt{1 - \frac{M}{r}} \left(3 \sqrt{1 - \frac{M}{r}} - 2\right)}{r^3}$$

$$+ \frac{M \left(3 \sqrt{1 - \frac{M}{r}} - 1\right)}{2r^2 \left(1 - \frac{M}{r}\right) \left(1 + \sqrt{1 - \frac{M}{r}}\right)} \dot{r}^2 = \frac{1}{\mu} F_p$$

A similar computation with the metric given by (60) results in

$$\ddot{r} + \frac{M(1 - \frac{M}{2r})}{r^2} + \frac{M}{2r^2(1 - \frac{M}{2r})} \dot{r}^2 - \frac{L^2(1 - \frac{M}{2r}) \left(1 - \frac{3M}{2r}\right)}{r^3} = \frac{1}{\mu} F_p$$

From (58) we see that $\frac{1}{\mu} F_p \approx \frac{M^2}{8\pi \rho r^5}$. Let the average density of the particle be given by $\bar{\rho}$, then $\bar{\rho} = \frac{\mu}{r}$ and so we see that

$$\frac{1}{\mu} F_p \approx \frac{M^2}{8\pi \rho r^5}$$

The mean radius of the earth is $6.3675 \times 10^8$ cm with a mass in geometrized units of 0.4438 cm. This yield a value of $\bar{\rho}$ of approximately $4.0971 \times 10^{-28}$ cm$^{-2}$. Typical values of $\bar{\rho}$ for planets range between $3 \times 10^{-29}$ cm$^{-2}$ and $5 \times 10^{-28}$ cm$^{-2}$. Consider the ratio of $F_p$ to the magnitude of the (Newtonian) gravitational force of the sun, $F_{\text{grav}} \equiv \frac{uM}{r^3}$. This ratio is $\frac{F_p}{F_{\text{grav}}} \approx \frac{M}{8\pi \rho r^3}$. For the planet Mercury this ratio is approximately $7.53 \times 10^{-8}$.

Table 1 gives values of $\frac{1}{\mu} F_p$, $\frac{F_p}{F_{\text{grav}}}$, $\frac{M}{r}$, and $\frac{M^2}{r^3}$.

**Kepler’s Law.** The angular velocity is given by $\omega = \frac{\dot{\phi}}{t}$, and so when the orbit is circular ($\ddot{r} = \dot{r} = 0$) we see generally that (70) and the normalization $-u^\alpha u_\alpha = 1$ imply

$$\Gamma^r_{tt} \dot{t}^2 + \Gamma^r_{\phi\phi} \dot{\phi}^2 = -\frac{1}{\mu} F_p \left( g_{tt} \dot{t}^2 + g_{\phi\phi} \dot{\phi}^2 \right)$$

(76)
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
Planet & $\frac{1}{\mu} F_p$ & $\frac{F_p}{F_{grav}}$ & $\frac{M}{r}$ & $\frac{M^2}{r^2}$ \\
\hline
Mercury & $3.31 \times 10^{-28}\text{cm}^{-1}$ & $7.53 \times 10^{-8}$ & $2.55 \times 10^{-5}$ & $1.12 \times 10^{-28}\text{cm}^{-1}$ \\
Earth & $2.82 \times 10^{-30}\text{cm}^{-1}$ & $4.28 \times 10^{-9}$ & $9.86 \times 10^{-9}$ & $6.51 \times 10^{-30}\text{cm}^{-1}$ \\
Jupiter & $3.08 \times 10^{-33}\text{cm}^{-1}$ & $1.26 \times 10^{-10}$ & $1.90 \times 10^{-9}$ & $4.62 \times 10^{-32}\text{cm}^{-1}$ \\
Neptune & $3.60 \times 10^{-37}\text{cm}^{-1}$ & $4.94 \times 10^{-13}$ & $3.28 \times 10^{-10}$ & $2.39 \times 10^{-34}\text{cm}^{-1}$ \\
\hline
\end{tabular}
\caption{Values of $F_p$, $\frac{M}{r}$ and $\frac{M^2}{r^2}$ for various planets}
\end{table}

Solving for $\omega^2$ and multiplying by $r^3$ yields

$$r^3 \omega^2 = \frac{-r^3 \left( \Gamma_{lt}^r + \frac{1}{\mu} F_p g_{tt} \right)}{\Gamma_{\phi\phi}^r + \frac{1}{\mu} F_p g_{\phi\phi}}$$

We will assume that $\frac{M}{r} << 1$ and that $\frac{F_p}{F_{grav}} \approx \frac{M}{8\pi r^2}$ is small and is approximately the same size as $\frac{M}{r}$. These assumptions are supported by the values in Table 1. For the metric of (52) we find that $\Gamma_{lt}^r = \frac{M}{128\pi^2} \left( 1 + \sqrt{1 - M/r} \right) \sqrt{1 - M/r}$ and $\Gamma_{\phi\phi}^r = -r \left( 1 - \frac{M}{r} \right)$. Thus using this and (52) we find

$$\omega^2 r^3 \approx M \left( 1 - \frac{5M}{4} - \frac{M}{8\pi \rho r^3} \right)$$

For the motion under the metric (60) one gets $\Gamma_{lt}^r = \frac{M}{r^2} \left( 1 - \frac{M}{2r} \right)^5$ and $\Gamma_{\phi\phi}^r = -r \left( 1 - \frac{M}{2r} \right)^2$ and hence under these assumptions we find

$$\omega^2 r^3 \approx M \left( 1 - \frac{3M}{2r} - \frac{M}{8\pi \rho r^3} \right)$$

Thus we see that when $\frac{M}{r}$ and $\frac{F_p}{F_{grav}}$ are very small that we have excellent agreement with Kepler’s Law.

**Radial Motion.** For pure radial motion ($L = 0$), (73) with $r >> M$ asymptotically yields

$$i \approx -\frac{M}{r^2} \left( 1 - \frac{M}{4r} \right) - \frac{M}{2r^2} \left( 1 + \frac{M}{2r} \right) r^2 + \frac{M^2}{8\pi \rho r^5}$$

From the metric (60), one finds from (74) with $r >> M$, that the pure radial motion to be approximately given by

$$i \approx -\frac{M}{r^2} \left( 1 - \frac{M}{2r} \right) - \frac{M}{2r^2} \left( 1 + \frac{M}{2r} \right) r^2 + \frac{M^2}{8\pi \rho r^5}$$

The magnitude of the $i^2$ terms in (80-82) do not appear to be large enough to explain the Pioneer anomaly. The Pioneer spacecraft is traveling out of the solar system. A small acceleration toward the sun which cannot be explained by general relativity has been observed over a period of years [15]. For Pioneer, the magnitude of these terms in (80 - 81) at planet Pluto is approximately $10^{-15}$ m s$^{-2}$ which is much less that the anomalous value of about $8.74 \times 10^{-10}$ m s$^{-2}$.

However, we do see that there is an explanation of the Pioneer anomaly. These radial equations (80-81) have additional outward accelerations that are not part of the
standard external Schwarzschild solution equations. From (80) we see an additional outward acceleration given by
\[
a_{\text{out}} = \frac{M^2}{4r^3} + \frac{M^2}{8\pi\tilde{\rho}r^5}.\tag{82}
\]
However the $\tilde{\rho}$ value is for the Pioneer spacecraft instead of the planet’s mean density. A rough estimate of the volume of the Pioneer spacecraft is $540,000 \text{ cm}^3$ for the main compartment and approximately $200,000 \text{ cm}^3$ for the remaining components (note: this is a rough estimate). Thus $\tilde{\rho} \approx \frac{1}{35} \text{ g/cm}^3$. This yields $\frac{M^2}{8\pi\tilde{\rho}r^5} \approx 4.67 \times 10^{-29} \text{ cm}^{-1}$ at a 1 A.U. from the sun. At Earth, we see using the value of $\frac{M^2}{r^7}$ from Table 1, that
\[
a_{\text{out}} \approx 0.25 \times 6.51 \times 10^{-30} + 4.67 \times 10^{-29} \text{ cm}^{-1}.
\]
Thus for the Pioneer spacecraft at Earth’s distance from the sun has an outward acceleration of
\[
a_{\text{out}} \approx 4.83 \times 10^{-29} \text{ cm}^{-1} \quad \text{(at Earth).}\tag{83}
\]
This is an extra outward acceleration due to the fact that our theory differs from general relativity and also includes a nonzero stress-energy tensor. At a distance of Jupiter from the Sun, with the same value of $\tilde{\rho}$ yields $\frac{M^2}{8\pi\tilde{\rho}r^5} \approx 1.23 \times 10^{-32} \text{ cm}^{-1}$. Thus, using the value of $\frac{M^2}{r^7}$ from Table 1, we see that
\[
a_{\text{out}} \approx 1.39 \times 10^{-32} \text{ cm}^{-1} \quad \text{(at Jupiter).}\tag{84}
\]
For distances that are greater than the distance from the Sun to Jupiter we see that $\Delta a_{\text{out}} \approx 4.83 \times 10^{-29} \text{ cm}^{-1}$ and under the general relativity model, this would be interpreted as an additional Sun-ward acceleration. Converting this value to standard units yields
\[
\Delta a_{\text{out}} \approx 4.34 \times 10^{-10} \text{ m s}^{-2} \quad \text{(85)}
\]
which is about 50% of the anomalous acceleration. For the metric (60) at earth we have
\[
a_{\text{out}} \approx 4.99 \times 10^{-29} \text{ cm}^{-1} \quad \text{which yields a value of}
\]
\[
\Delta a_{\text{out}} \approx 4.49 \times 10^{-10} \text{ m s}^{-2}
\]
which is 51% of the anomalous acceleration. The difference may be explained by thermal forces [16].

**Redshift.** The difference between the values of $g_{tt}$ in this model and the standard Schwarzschild solution would produce small differences in the predicted redshift. The redshift $z = \frac{\Delta \lambda}{\lambda} = |g_{tt}|^{-\frac{1}{2}} - 1$ for stationary objects. From (50) we find that
\[
z = 16 \left(1 + \sqrt{1 - \frac{M}{r}}\right)^{-4} - 1 \approx \frac{M}{r} + \frac{7M^2}{8r^2}, \quad r >> M, \tag{87}
\]
and from (60) we find
\[
z = \left(1 - \frac{M}{2r}\right)^{-2} - 1 \approx \frac{M}{r} + \frac{3M^2}{4r^2}, \quad r >> M. \tag{88}
\]
Asymptotically, these results agree with the value found in the Schwarzschild geometry, i.e. $z \approx \frac{M}{r}$. At the distance of the earth from the sun, one finds the value given by (87) differs from the standard value by $8.5 \times 10^{-17}$, with a relative difference of $8.6 \times 10^{-9}$. 

From (88) we find the value of $z$ differs from the standard value by $7.3 \times 10^{-17}$ with a relative difference of $7.4 \times 10^{-9}$.

**Precession of Perihelion.** We now consider the precession of perihelion problem. Assuming spherical symmetry and using the $-u^\alpha u_\alpha = 1$ normalization, we have $g^{tt}t^2 + g_{rr}r^2 + r^2 \hat{\theta}^2 = -1$ with motion restricted (without loss of generality) to the $\theta = \frac{\pi}{2}$ plane. Now $\dot{t} = -g^{tt}E$ and $\dot{\phi} = \frac{L}{r^2}$. Hence $g^{tt}E^2 + g_{rr}r^2 + \frac{L^2}{r^2} = -1$. After differentiating this equation we see that equations (73) and (74) are not recovered unless a term is added, specifically, we get $g^{tt}E^2 + g_{rr}r^2 + \frac{L^2}{r^2} - g_{rr} \int \frac{E}{\mu} = -1$. Using the approximation $\frac{E}{\mu}$ in (75) we find

$$i^2 = -\left(\frac{g^{tt}}{g_{rr}}\right)E^2 - \frac{1}{g_{rr}} \left(1 + \frac{L^2}{r^2}\right) - \frac{M^2}{16\pi \rho r^4}$$ \quad (89)

Using $\frac{dv}{d\phi} = \frac{v^2}{L} \frac{dv}{dr}$, with $u \equiv M/r$ and $L \equiv \frac{L}{M}$, one finds that

$$\left(\frac{L}{M} \frac{du}{d\phi}\right)^2 = \left(-\frac{g^{tt}}{g_{rr}}\right)E^2 - \frac{1}{g_{rr}} \left(1 + (L^4)u^2\right) - \frac{1}{16\pi \rho M^2} u^4 \equiv f(u) \quad .$$ \quad (90)

When $E$ is large $f(u) > 0$ and the value of $u$ oscillates. When the orbit is circular at $u = u_0 = \frac{M}{r_0}$, the function $f(u)$ has a maximum with both $f(u_0) = 0$ and $f'(u_0) = 0$. Hence $f(u) \approx \frac{1}{2} f''(u_0)(u - u_0)^2$. Via the chain rule, one has $2(L^4)u \frac{du}{d\phi} \frac{du}{d\phi} = f'(u) \frac{du}{d\phi}$. Thus, one finds that,

$$\frac{d^2}{d\phi^2} \left(u - u_0\right) - \frac{f''(u_0)}{2(L^4)} \left(u - u_0\right) = 0 \quad .$$ \quad (91)

When $f''(u_0) < 0$, the solution is periodic with

$$\text{Period} = \frac{2\pi}{\sqrt{-\frac{f''(u_0)}{2(L^4)}}} \quad .$$ \quad (92)

From the metric given in (52), we find that

$$-\frac{f''(u_0)}{2(L^4)} \approx 1 - \frac{15}{4} u_0 + \frac{12u_0^2}{16\pi \rho L^2} \quad .$$ \quad (93)

For Mercury, the $u_0^2$ term (last term) of this expression is approximately $10^{-4}$ times the value of the preceding term and thus the perihelion is shifted by

$$\Delta \phi \approx \left(\frac{15\pi}{4}\right) \frac{M}{r_0} \quad .$$ \quad (94)

where $r_0$ is the radius of the near-circular orbit. If the metric (60) is used one finds

$$-\frac{f''(u_0)}{2(L^4)} \approx 1 - \frac{7}{2} u_0 + \frac{12u_0^2}{16\pi \rho L^2} \quad .$$ \quad (95)

and hence

$$\Delta \phi \approx \left(\frac{7\pi}{2}\right) \frac{M}{r_0} \quad .$$ \quad (96)

Both of these results are less than the standard result of $\frac{6\pi M}{r_0^2}$, with (94) being $\frac{5}{8}$ of the standard result and (96) being $\frac{7}{12}$ of the standard result. In the Newcomb's calculation
of the precession of Mercury in 1882, [17], it was stated that "a planet or a group of planets between Mercury and the Sun" could explain the additional 43.03" per century. It seems reasonable to define the average pressure by $\bar{p} = \frac{1}{3}(p_r + 2p_T)$ and thus the inertial mass per unit volume of the halo is given by $\rho + \bar{p}$. The inertial mass of the halo between the Sun and Mercury for the metric given by (52) is given by

$$\Delta m \approx \int_{r_{\text{sun}}} \leq r \leq r_{\text{merc}} \frac{M^2}{8\pi \cdot 2r^4} \, d^3x \approx 0.0784 \text{ cm}.$$  \hspace{1cm} (97)

This is about 17.7% of the mass of Earth. For the metric of (60), the inertial mass is $\frac{5}{3}$ times larger, giving 0.131 cm which represents 29.4% of the Earth’s mass. It is possible that these values of $\Delta m$ may explain the remaining fraction of the anomalous precession that is not explained by (94) and (96). Since it is spherically symmetric, other effects on the orbit of Mercury should be minimal.

**Isotropic Form and Temperature of the Corona.** For most problems in astrophysics, the isotropic form of the metric is preferred. From the metric of (15), which is generated by tetrad of (14), the field equations $C_\mu = 0$ are satisfied. For the weak field approximation to hold, $f(r) \approx 1 - \frac{2M}{r}$ for large $r$. The tetrad that produces the isotropic form for the metric given in (60) above, is generated by the transformation, $r \rightarrow r + \frac{M}{2}$. The resulting metric is

$$ds^2 = -\left(1 + \frac{M}{2r}\right)^{-2} dt^2 + \left(1 + \frac{M}{2r}\right)^2 \left[dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2\right]$$ \hspace{1cm} (98)

with

$$8\pi \rho = \frac{M^2}{4r^4(1 + \frac{M}{2r})^4},$$

$$8\pi p_r = \frac{M}{r^3(1 + \frac{M}{2r})^2}\left(1 - \frac{M}{4r}\right),$$

$$8\pi p_T = \frac{-M}{2r^3(1 + \frac{M}{2r})^4}\left(1 - \frac{2M}{r}\right)$$ \hspace{1cm} (99)

To first order, the isotropic metric and its corresponding stress energy tensor are equivalent to the metric of (60).

We again note that, unlike most other alternatives to general relativity, the theory based on the conservation group when interpreted as a manifold has a non-vanishing stress energy tensor. Suppose we apply the ideal gas law to the halo with a pressure of $\bar{p} = \frac{1}{3}(p_r + 2p_T)$ and assume that the halo is comprised of particles with mass $\hat{m}$ ev. The resulting temperature is $T = \frac{7}{3}\hat{m}$ cm = $2.70 \times 10^4$ K. If the masses of the constituent matter in the halo are approximately 36 ev, the resulting temperature would be approximately $10^6$ K and hence would explain the high temperature of the corona. These considerations also suggest that the dark matter may be a mixture of the neutrinos $\nu_e$, $\nu_\mu$ and $\nu_\tau$.

**Deflection of Light and Time Delay.** For null rays which model photon motion, $ds^2 = 0$, and we see from (15) that for position vector $\mathbf{r}$,

$$\left|\frac{d\mathbf{r}}{dt}\right| = [f(r)]^{\frac{3}{2}} \approx \left(1 - \frac{2M}{r}\right)^{\frac{3}{2}} \approx 1 - \frac{3M}{2r} \approx \frac{1}{1 + \frac{4M}{2r}}.$$ \hspace{1cm} (100)
The denominator of the last expression in (100) represents a refraction index of 
\( n_0(r) \equiv 1 + \frac{3M}{2r} \). This is a general result and, as a check, one easily sees that the
isotropic metric given in (98) satisfies this condition. We see that \( n_0(r) - 1 = \frac{3M}{2r} \) is thus
precisely 75% of the value in general relativity (where \( n(r) - 1 = \frac{\beta M}{\sqrt{r^2 - M^2}} \)). In our theory,
however, the resulting density and pressures \( \bar{\rho} \) indicate a stressed medium through
which the electromagnetic radiation passes. Let \( \bar{p} = \frac{1}{3}(p_r + p_T + p_T) \) be the average
pressure. We propose that additional refraction occurs due to the medium and the value of
\( n_1(r) - 1 \) is proportional to \( \rho + \bar{p} \), viz.

\[
n_1(r) - 1 = 8\pi \alpha (\rho + \bar{p})
\]

where \( \alpha \) is a positive constant and the factor of \( 8\pi \) is included for convenience. This
formula may be justified by the Lorentz-Lorenz relation \[18\].

For the deflection of light problem we will follow the analysis of de Felice and
Clarke \[14, p 354\]. We see from the stress energy tensor \( (56) \) that \( 8\pi (\rho + \bar{p}) \approx \frac{M^2}{4a^2} \), and
for the stress energy tensor of \( (61) \), \( 8\pi (\rho + \bar{p}) \approx \frac{5M^2}{4a^2} \). The values computed from the
isotropic form of the metric are the same to the order of approximation used. Hence
\( n_1(r) \approx 1 + \frac{\beta M^2}{r^2} \), with \( \beta = \frac{1}{2} \alpha \) for \( (56) \) and \( \beta = \frac{5}{6} \alpha \) for the stress energy tensor \( (61) \). We
multiply this by the corresponding refractive index which is calculated from the metric, hence

\[
n(r) = n_0(r) \cdot n_1(r) \approx 1 + \frac{3M}{2r} + \frac{\beta M^2}{r^4}
\]

From \[14\] we find that the angle of deflection of light passing near the surface of the
sun (i.e. a minimum radius of \( r_0 \)) is given by

\[
\Delta \phi = \int_{1 + \frac{3M}{2r_0} + \frac{\beta M^2}{r_0}}^{\infty} \frac{2dr}{r \sqrt{r^2 - 1}} - \int_{1}^{\infty} \frac{2dr}{r \sqrt{r^2(1 + \frac{3M}{2r} + \frac{\beta M^2}{r^4})^2 - 1}}
\]

Defining \( A \equiv \frac{3M}{2r_0} \), \( B \equiv \frac{\beta M^2}{r_0} \) and changing variables to \( w \equiv \frac{r}{r_0} \) we find that

\[
\Delta \phi = \int_{1 + A + B}^{\infty} \frac{2dw}{w \sqrt{w^2 - 1}} - \int_{1}^{\infty} \frac{2dw}{w \sqrt{w^2(1 + \frac{A}{w} + \frac{B}{w^2})^2 - 1}}
\]

Noting that \( A \) and \( B \) are much less than 1, we find that \( 104 \) yields

\[
\Delta \phi \approx - 2A - \frac{3\pi}{2} B + (2A + 8B) \sqrt{2(A + B)}
\]

In the calculation of the time delay we use the approach of Misner, Thorne and
Wheeler \[12\]. Suppose the photon is moving along a path which is approximated in
Cartesian coordinates by \( y = b, z = 0 \) for \( -a_T \leq x \leq a_R \). We also assume that
\( a_T >> b > 0 \) and \( a_R >> b > 0 \). Using \( ds^2 = 0 \), one finds that \( dt = \left( 1 + \frac{3M}{2\sqrt{x^2 + b^2}} \right) dx \).

We modify this by replacing it with the corresponding index of refraction \( 102 \). Thus
the total time of transit from transmitter to reflector and back is

\[
t_{TRT} = 2 \int_{-a_T}^{a_T} \left( 1 + \frac{3M}{2\sqrt{x^2 + b^2}} + \frac{\beta M}{(x^2 + b^2)^2} \right) dx
\]

\[106\]
The second and third terms of this integral correspond to the delay effect. We see that the second term (which when integrated will be called $\Delta \tau$) is 75% of the general relativity value. The value of $\Delta \tau$ is

$$\Delta \tau = 3M \ln \left| \frac{\sqrt{a_R^2 + b^2} + a_R}{\sqrt{a_T^2 + b^2} + a_T} \right|.$$  \hspace{1cm} (107)

The third term which will be called $\Delta(\Delta \tau)$ when integrated has a value

$$\Delta(\Delta \tau) \equiv \frac{\beta M^2 \pi}{2b^3} \left[ \arctan \left( \frac{x}{b} \right) + \frac{bx}{x^2 + b^2} \right] \bigg|_{-a_T}^{a_R} \approx \frac{\beta M^2 \pi}{2b^3}.$$  \hspace{1cm} (108)

when $a_T$ and $a_R$ are large compared to $b$. We assume that $\frac{da_T}{d\tau} \approx \frac{da_R}{d\tau} \approx 0$ and hence the rate of change of the total time delay is

$$\frac{d}{d\tau} \left( \Delta \tau + \Delta(\Delta \tau) \right) \approx \frac{-6M}{b} \left( 1 + \frac{\beta M \pi}{2b^3} \right) \frac{db}{d\tau}.$$  \hspace{1cm} (109)

Thus, agreement with the general relativity value would occur if $\frac{\beta M \pi}{2b^3} = \frac{1}{3}$ and hence if $\beta = \frac{2b^3}{3\pi M}$. If $b = r_0 \approx 6.960 \times 10^{10}$ cm, then we find that $\beta \approx 4.844 \times 10^{26}$ cm$^2$. For the stress energy tensor of (56), we find $\alpha \approx 9.688 \times 10^{26}$ cm$^2$ and for the stress energy tensor of (61), we find $\alpha \approx 5.813 \times 10^{26}$ cm$^2$. We note that these values of $\alpha$ are fairly typical. For example, the corresponding value of $\alpha$ for hydrogen ($H_2$) gas is approximately $7.9 \times 10^{26}$ cm$^2$. However, the Lorentz-Lorenz relation in its most basic form [18] relates the number density to the refraction. If the consideration of the corona temperature is correct, the number density of the halo near the sun is approximately $5 \times 10^7$ times that of hydrogen gas. Thus, the dark matter is seen to interact weakly.

As already noted, with $b = r_0$ and $\beta = \frac{2b^3}{3\pi M}$, (109) leads to the general relativity result of $\frac{-8M}{b} \frac{db}{d\tau}$ for the total delay. For the deflection of light problem, we find that $B = \frac{2b^3}{3\pi r_0}$ and hence (105) yields $\Delta \phi \approx \frac{-4M}{r_0} + \left( \frac{2M}{r_0} + \frac{16M}{3\pi r_0} \right) \sqrt{\frac{3M}{r_0} + \frac{4M}{3\pi r_0}}$. For $M \approx 1.477 \times 10^5$ cm and $r_0 \approx 6.960 \times 10^{10}$ cm, we find that $\Delta \phi \approx -8.462 \times 10^{-6}$. We note that this value of $\Delta \phi$ is approximately 1.75".

6. Conclusion.

The theory based on the conservative transformation group may provide a theoretical basis for a unified field theory and may also provide a theoretical basis for dark matter and the correct modification of general relativity. The Lagrangian for the field with sources may be used in a variety of applications, including quantization. The internal solution and its corresponding stellar model needs additional work to produce more realistic models. The external solutions, being non-compact, show promise for explaining dark matter. Excellent agreement is found with Kepler’s Law and redshift. The theory also gives a realistic explanation for the Pioneer anomaly and the high temperature of the corona. While there are differences in the precession of perihelia, light deflection and time delay predictions, these may be explained by the fact that the stress-energy tensor is non-zero, yielding densities and pressures that affect the motion of planets and photons.
Acknowledgments

The author would like to thank Dave Pandres for many helpful suggestions. Also the author would like to thank Peter Musgrave, Denis Pollney and Kayll Lake for the GRTensorII software package which was very helpful.

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