Spectral Structure of Elastic Neumann–Poincaré Operators

Yoshihisa Miyanishi\textsuperscript{1}, Kazunori Ando\textsuperscript{2} and Hyeonbae Kang\textsuperscript{3}

\textsuperscript{1} Center for Mathematical Modeling and Data Science, Osaka University, Osaka 560-8631, Japan
\textsuperscript{2} Department of Electrical and Electronic Engineering and Computer Science, Ehime University, Ehime 790-8577, Japan
\textsuperscript{3} Department of Mathematics and Institute of Applied Mathematics, Inha University, Incheon 22212, S. Korea

E-mail: miyanishi@sigmath.es.osaka-u.ac.jp

Abstract. In [1, 2], it is proved that the elastic Neumann–Poincaré operator defined on the smooth boundary of a bounded domain, which is known to be non-compact, is in fact polynomially compact. As a consequence, it is shown that the spectrum of the elastic Neumann–Poincaré operator consists of non-empty sets of eigenvalues accumulating to certain numbers determined by Lamé parameters. The purpose of this paper is to review these results and their proofs, and to discuss about some questions related to these results.

1. Introduction: Polynomial compactness and spectral structure

The elastic Neumann–Poincaré (abbreviated by NP) operator is a boundary integral operator which appears naturally when solving classical boundary value problems for the Lamé system [4, 6]. The Lamé system is defined in terms of a pair of Lamé constants $(\lambda, \mu)$ satisfying the strong convexity condition: $\mu > 0$ and $d\lambda + 2\mu > 0$, where $d$ denotes the space dimension, i.e., $d = 2, 3$. The isotropic elasticity tensor $C = (C_{ijkl})_{i,j,k,l=1}^{d}$ and the corresponding Lamé system $L_{\lambda,\mu}$ are defined by

\begin{equation}
C_{ijkl} := \lambda \delta_{ij}\delta_{kl} + \mu (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})
\end{equation}

and

\begin{equation}
L_{\lambda,\mu}u := \nabla \cdot C\nabla u = \mu \Delta u + (\lambda + \mu) \nabla \nabla \cdot u,
\end{equation}

where $\nabla \cdot$ denotes the symmetric gradient, namely,

\begin{equation}
\nabla u := \frac{1}{2} \left( \nabla u + \nabla u^T \right) \quad (T \text{ for transpose}).
\end{equation}

Let $\Gamma(x) = (\Gamma_{ij}(x))_{i,j=1}^{d}$ be the Kelvin matrix of the fundamental solution to the Lamé operator $L_{\lambda,\mu}$, namely,

\begin{equation}
\Gamma_{ij}(x) = \begin{cases}
\frac{\alpha_1}{4\pi} \frac{\delta_{ij}}{|x|} - \frac{\alpha_2}{4\pi} \frac{x_i x_j}{|x|^3}, & \text{if } d = 3, \\
\frac{\alpha_1}{2\pi} \delta_{ij} \log |x| - \frac{\alpha_2}{2\pi} \frac{x_i x_j}{|x|^2}, & \text{if } d = 2
\end{cases}
\end{equation}
where
\[ \alpha_1 = \frac{1}{2} \left( \frac{1}{\mu + \frac{1}{2\mu + \lambda}} \right) \quad \text{and} \quad \alpha_2 = \frac{1}{2} \left( \frac{1}{\mu - \frac{1}{2\mu + \lambda}} \right). \]

The NP operator for the Lamé system is defined by
\[ K[f](x) := \text{p.v.} \int_{\partial \Omega} \partial_{\nu_x} \Gamma(x - y) f(y) d\sigma(y) \quad \text{a.e.} \ x \in \partial \Omega. \]  

Here, p.v. stands for the Cauchy principal value, and the conormal derivative \( \partial_{\nu_x} \Gamma(x - y) \) of the Kelvin matrix with respect to \( x \)-variables is defined by
\[ \partial_{\nu_x} \Gamma(x - y) b = \partial_{\nu_x} (\Gamma(x - y) b) \]

for any constant vector \( b \) (We note that our definition of \( K \) is the \( L^2 \)-adjoint of that in [1]).

The NP operator \( K \) is related to the boundary value problem in the following way. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^d \) with the smooth boundary, and consider the Neumann boundary value problem with the given traction \( g \) on \( \partial \Omega \): \( \mathcal{L}_{\lambda,\mu} u = 0 \) in \( \Omega \) and \( \partial_{\nu} u = g \) on \( \partial \Omega \). Here and throughout this paper, \( \partial_{\nu} \) denotes the conormal derivative on \( \partial \Omega \), namely,
\[ \partial_{\nu} u := (\nabla \wedge u)n = \lambda (\nabla \cdot u)n + 2\mu (\nabla u)n \quad \text{on} \ \partial \Omega, \]
where \( n \) is the outward unit normal to \( \partial \Omega \). One may seek the solution in the form \( u = S[f] \) for some vector-valued function \( f \) on \( \partial \Omega \), where \( S \) is the single layer potential, namely,
\[ S[f](x) := \int_{\partial \Omega} \Gamma(x - y) f(y) d\sigma(y) \quad x \in \Omega. \]

Then \( u \) satisfies the equation: \( \mathcal{L}_{\lambda,\mu} u = 0 \) in \( \Omega \). The conormal derivative of the single layer potential is related to the NP operator by the jump relation:
\[ \partial_{\nu} S[f](x) = \left( \frac{1}{2} I + K \right) [f](x) \quad \text{a.e.} \ x \in \partial \Omega, \]

where \( I \) is the identity operator. It means that to solve the Neumann boundary value problem, it suffices to solve the integral equation on \( \partial \Omega \):
\[ \left( \frac{1}{2} I + K \right) [f] = g. \]

Here arises the crux of this paper’s investigation. Unlike the electro-static NP operator, the elastic NP operator is a genuine singular integral operator and is not a compact operator even if \( \partial \Omega \) is smooth (see section 2), which implies that the Fredholm index theory cannot be applied to solve (10). It is proved in [4] that \( -1/2 I + K \) is invertible on \( H^{-1/2}_{0}(\partial \Omega)^d \). Here \( H^{-1/2} \) denotes the Sobolev space of order \(-1/2\), and the subscript 0 means that the members are orthogonal to rigid motions.

Since the elastic NP operator \( K \) is not compact even on smooth domains, a naturally arising question is how its spectrum looks like. We emphasize that even though \( K \) is not self-adjoint on \( L^2(\partial \Omega)^d \), it can be realized as a self-adjoint operator on \( H^{-1/2}(\partial \Omega)^d \) (see [1, 5]). Thus its spectrum on \( H^{-1/2}(\partial \Omega)^d \) consists of discrete and continuous ones. In relation to the spectrum of \( K \), it is recently proved in [1, 2] that \( K \) is in fact polynomially compact. An operator \( A \) is said to be polynomially compact if there is a polynomial \( p \) such that \( p(A) \) is compact. In fact, the following theorem is obtained. Throughout this paper we let \( \mathcal{H} := H^{-1/2}(\partial \Omega)^d \) for ease of notation.
\textbf{Theorem 1.1} ([1, 2]). Let
\begin{equation}
k_0 = \frac{\mu}{2(2\mu + \lambda)}.
\end{equation}
Then we have
\begin{enumerate}[(i)]
    \item In two dimensions \(K^2 - k_0^2 I\) is compact on \(H\) if \(\partial \Omega\) is \(C^{1,\alpha}\) for some \(\alpha > 0\),
    \item In three dimensions \(K(K^2 - k_0^2 I)\) is compact on \(H\) if \(\partial \Omega\) is \(C^{\infty}\)-smooth. Moreover, \(K(K - k_0 I), K(K + k_0 I)\) and \(K^2 - k_0^2 I\) are not compact.
\end{enumerate}

As a consequence, the following theorem regarding the spectral structure of the elastic NP operator.

\textbf{Theorem 1.2} ([1, 2]). Let \(K\) be the elastic NP operator on \(\partial \Omega\).
\begin{enumerate}[(i)]
    \item In two dimensions the spectrum of \(K\) on \(H\) consists of two non-empty sets of eigenvalues accumulating at \(k_0\) and \(-k_0\), respectively (provided that \(\partial \Omega\) is \(C^{1,\alpha}\) for some \(\alpha > 0\))
    \item In three dimensions the spectrum of \(K\) on \(H\) consists of three non-empty sets of eigenvalues accumulating at \(k_0\), \(0\) and \(-k_0\), respectively (provided that \(\partial \Omega\) is \(C^{\infty}\)).
\end{enumerate}

Some questions arise from above mentioned results. An apparent question is why the smoothness of \(\partial \Omega\), in stead of \(C^{1,\alpha}\) for two dimensions, is required in three dimensions. In fact, this assumption is forced because of the method of proofs in [2]. A less apparent but quite interesting question is whether there is a physical meaning on those eigenvalues converging to different accumulation points. Another question is about the invertibility of \(-1/2I + K\). As mentioned before, the Fredholm theory cannot be applied since \(K\) is not compact. But, what if \(K\) is polynomially compact? We discuss about these questions in this paper. For that purpose we first review the main ideas of proofs of the above mentioned results.

We end the introduction with a remark. Recently there is rapidly growing interest in the spectral properties of the NP operator (electro-static and elastic) in relation to plasmon resonance and stress concentration. For these relations we refer to [1].

2. Review of proofs
Straightforward computations using the definition (6) yield that
\begin{equation}
\partial_{\nu_y} \Gamma(x - y) = -k_0 K_1(x, y) + K_2(x, y),
\end{equation}
where
\begin{align}
K_1(x, y) &= \frac{2[n_x(x - y)^T - (x - y)n_x^T]}{\omega_d |x - y|^d}, \quad (13) \\
K_2(x, y) &= \frac{\mu}{2\mu + \lambda} \frac{(x - y) \cdot n_x}{\omega_d |x - y|^d} I + \frac{2(\mu + \lambda)}{2\mu + \lambda} \frac{(x - y) \cdot n_x}{\omega_d |x - y|^d} (x - y)(x - y)^T, \quad (14)
\end{align}
where \(\omega_d\) is \(2\pi\) if \(d = 2\) and \(4\pi\) if \(d = 3\), \(I\) is the \(d \times d\) identity matrix. The superscript \(T\) denotes transpose.

Note that if \(\partial \Omega\) is \(C^{1,\alpha}\) for some \(\alpha > 0\), then
\[|(x - y) \cdot n_x| \leq C |x - y|^{1 + \alpha}\]
for some constant \(C\), and hence
\[\frac{|(x - y) \cdot n_x|}{|x - y|^d} \leq \frac{C}{|x - y|^{d-1-\alpha}}\]
for all $x, y \in \Omega$, in other words, it is weakly singular. It means that the integral operator on $\partial \Omega$ defined by $K_2$ is compact, and the integral operator defined by $K_1(x, y)$ is responsible for non-compactness of $K$.

Let

$$T[f](x) := \text{p.v.} \int_{\partial \Omega} K_1(x, y) f(y) d\sigma(y), \quad x \in \partial \Omega.$$  \hspace{1cm} (15)

Then we have

$$K = -k_0 T.$$  \hspace{1cm} (16)

Here and throughout the expression $A \equiv B$ for operators $A$ and $B$ on $\mathcal{H}$ indicates that $A - B$ is compact on $\mathcal{H}$. We emphasize that $T$ is a singular integral operator and bounded on $\mathcal{H}$ as well as on $L^2(\partial \Omega)^d$ (see [3]).

In two dimensions $K_1$ takes the form

$$K_1(x, y) = \frac{1}{\pi |x - y|^2} \begin{bmatrix} 0 & K(x, y) \\ -K(x, y) & 0 \end{bmatrix},$$  \hspace{1cm} (17)

where

$$K(x, y) := -n_2(x)(x_1 - y_1) + n_1(x)(x_2 - y_2).$$

Let $R$ is the Hilbert transformation on $\partial \Omega$, namely,

$$R[f] = g \quad \text{on} \quad \partial \Omega,$$

where $g$ is the boundary value of $u^\perp$ which is a harmonic conjugate of $u$ such that $\Delta u = 0$ in $\Omega$ and $u = f$ on $\partial \Omega$. It is proved in [1] that

$$T = \begin{bmatrix} 0 & -R \\ R & 0 \end{bmatrix}.$$  \hspace{1cm} (18)

Since $R^2 = -I$, we have $T^2 = I \equiv 0$ and (i) of Theorem 1.1 follows.

Since existence of harmonic conjugates is used for proofs in two dimensions, the same argument may not be applied to the three dimensional case. In three dimensions $K_1$ takes the form

$$K_1(x, y) = \frac{1}{2\pi |x - y|^3} \begin{bmatrix} 0 & K_{12}(x, y) & K_{13}(x, y) \\ -K_{12}(x, y) & 0 & K_{23}(x, y) \\ -K_{13}(x, y) & -K_{23}(x, y) & 0 \end{bmatrix},$$  \hspace{1cm} (19)

where

$$K_{12}(x, y) = n_1(x)(x_2 - y_2) - n_2(x)(x_1 - y_1),$$

$$K_{13}(x, y) = n_1(x)(x_3 - y_3) - n_3(x)(x_1 - y_1),$$

$$K_{23}(x, y) = n_2(x)(x_3 - y_3) - n_3(x)(x_2 - y_2).$$

Let

$$T_{ij}[f](x) := \int_{\partial \Omega} K_{ij}(x, y) \frac{f(y)}{2\pi |x - y|^3} d\sigma(y),$$  \hspace{1cm} (20)

so that

$$T = \begin{bmatrix} 0 & T_{12} & T_{13} \\ -T_{12} & 0 & T_{23} \\ -T_{13} & -T_{23} & 0 \end{bmatrix}.$$  \hspace{1cm} (21)
It is helpful to consider the case when $\Omega$ is the upper half-space even if it is not a bounded domain. In this case, $n_z = (0, 0, -1)^T$, and hence

$$T = \begin{bmatrix} 0 & 0 & R_1 \\ 0 & 0 & R_2 \\ -R_1 & -R_2 & 0 \end{bmatrix},$$

(22)

where $R_j$ is the Riesz transform, i.e.,

$$R_j[f](x_1, x_2) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x_j - y_j}{[(x_1 - y_1)^2 + (x_2 - y_2)^2]^{3/2}} f(y_1, y_2) \, dy_1 \, dy_2, \quad j = 1, 2.$$

Since $R_1^2 + R_2^2 = -I$ (see [7]), one can see easily from (22) that $T^3 - T = 0$.

It is shown in [2] that the operator $T$ on a bounded surface $\partial \Omega$ can be well approximated by the surface Riesz transforms which is defined by local coordinates charts. Let $U$ be a coordinate chart in $\partial \Omega$ so that there is an open set $D$ in $\mathbb{R}^2$ and a parametrization $\Phi : D \to U$, namely,

$$\mathbf{x} = (x(s, t), y(s, t), z(s, t)), \quad \mathbf{x} \in U, \ (s, t) \in D.$$

(23)

Then the metric tensor of the surface, denoted by $(g_{ij}(s, t))_{i,j=1}^2$, is given by

$$dx^2 + dy^2 + dz^2 = (x_s^2 + y_s^2 + z_s^2) ds^2 + 2(x_s x_t + y_s y_t + z_s z_t) ds \otimes dt + (x_t^2 + y_t^2 + z_t^2) dt^2$$

$$= g_{11} ds^2 + g_{12} ds \otimes dt + g_{21} dt \otimes ds + g_{22} dt^2.$$

The unit normal vector is given by

$$\mathbf{n}(s, t) = (x_s(s, t), y_s(s, t), z_s(s, t)) \times (x_t(s, t), y_t(s, t), z_t(s, t))/| (x_s, y_s, z_s) \times (x_t, y_t, z_t) |$$

and the surface element is given by $d\sigma(s, t) = |(x_s, y_s, z_s) \times (x_t, y_t, z_t)| \, ds \wedge dt$.

Let $x' = (x_1, x_2) = (s_1, t_1)$ and $y' = (y_1, y_2) = (s_2, t_2)$, and let

$$L(x', x' - y') = \frac{1}{|g_{11}(x')(x_1 - y_1)^2 + 2g_{12}(x')(x_1 - y_1)(x_2 - y_2) + g_{22}(x')(x_2 - y_2)^2|^{3/2}}.$$

Then the surface Riesz transform $R_j^g$, $j = 1, 2$, is defined by

$$R_j^g[f](x') = \frac{1}{2\pi} \int_{\mathbb{R}^2} (x_j - y_j)L(x', x' - y') f(y') \, dy'.$$

(24)

It is then shown that

$$T_{12} \equiv (z_t g_{11} - z_s g_{12}) R_1^g - (z_s g_{22} - z_t g_{21}) R_2^g,$$

$$T_{13} \equiv (y_t g_{11} - y_s g_{12}) R_1^g - (y_s g_{22} - y_t g_{21}) R_2^g,$$

$$T_{23} \equiv (x_t g_{11} - x_s g_{12}) R_1^g - (x_s g_{22} - x_t g_{21}) R_2^g.$$

We emphasize that above identities hold locally on coordinate charts.

To show that

$$T^3 - T \equiv 0$$

(25)

in three dimensions, calculus of pseudo-differential operator is adapted. For this reason the smoothness assumption on $\partial \Omega$ is required. In fact, it is shown that the symbol of $R_j^g$ is given by

$$\frac{-i}{\sqrt{\det(g_{jk}(x))}} \sum_k g^{jk}(x') \xi_k \sqrt{\sum_{i,j} g^{ik}(x') \xi_i \xi_j}.$$
Using this it is shown that the following identity holds locally: Here $Op(Z)$ denotes the $\psi$DO defined by the symbol $Z$.

\[
K \equiv \begin{bmatrix}
0 & -Op(Z) & -Op(Y) \\
Op(Z) & 0 & -Op(X) \\
Op(Y) & Op(X) & 0
\end{bmatrix},
\]

where

\[
X = i\sqrt{\det(g^{jk}(x))}(x_1\xi_1 - x_2\xi_2),
\]

\[
Y = i\sqrt{\det(g^{jk}(x))}(y_1\xi_1 - y_2\xi_2),
\]

\[
Z = i\sqrt{\det(g^{jk}(x))}(z_1\xi_1 - z_2\xi_2).
\]

Since $X^2 + Y^2 + Z^2 = -1$, (25) follows, and so does (ii) of Theorem 1.1.

These are the main ideas of proofs in [1, 2], even though there are several technical difficulties which are mentioned in this review.

3. Discussions

It is now apparent why the smoothness assumption on $\partial \Omega$ is required in three dimensions: it is because calculus of $\psi$DO is adapted. It is likely that the result for the three dimensional case is valid for domains with $C^{1,\alpha}$ boundaries like the two dimensional case. To prove it, it is necessary to compute the compositions of surface Riesz potentials, which are singular integral operators. We will pursue this in future.

Theorem 1.2 shows that there are two groups of eigenvalues converging to $k_0$ and $-k_0$, respectively, in two dimensions, and three groups of eigenvalues converging to $k_0$, 0, and $-k_0$, respectively, in three dimensions. It is quite interesting to find out relevant physical meanings of the corresponding eigenspaces. In this respect the following remark is instructive. It is proved in [1] that $k_0$ and $-k_0$ are actually eigenvalues (not accumulation points of eigenvalues) of $K^*$ when $\Omega$ is a disk (there are two more eigenvalues of multiplicities one). The eigenfunctions corresponding to $k_0$ and $-k_0$ are

- $k_0$:

\[
\varphi_m^{(1)} = \begin{bmatrix}
\cos m\theta \\
\sin m\theta
\end{bmatrix}, \quad \varphi_m^{(2)} = \begin{bmatrix}
-sin m\theta \\
\cos m\theta
\end{bmatrix}, \quad m = 2, 3, \ldots,
\]

- $-k_0$:

\[
\psi_m^{(1)} = \begin{bmatrix}
\cos m\theta \\
-sin m\theta
\end{bmatrix}, \quad \psi_m^{(2)} = \begin{bmatrix}
\sin m\theta \\
\cos m\theta
\end{bmatrix}, \quad m = 1, 2, \ldots
\]

If we apply the single layer potential operator $S$ to these eigenfunctions, we obtain

\[
u_m^{(1)} := S[\varphi_m^{(1)}] = (-1/2 + k_0) \begin{bmatrix}
r^m \cos m\theta \\
r^m \sin m\theta
\end{bmatrix}, \quad u_m^{(2)} := S[\varphi_m^{(2)}] = (-1/2 + k_0) \begin{bmatrix}
r^m \sin m\theta \\
r^m \cos m\theta
\end{bmatrix},
\]

\[
u_m^{(1)} := S[\psi_m^{(1)}] = (-1/2 - k_0) \begin{bmatrix}
r^m \cos m\theta \\
-r^m \sin m\theta
\end{bmatrix}, \quad v_m^{(2)} := S[\psi_m^{(2)}] = (-1/2 - k_0) \begin{bmatrix}
r^m \sin m\theta \\
r^m \cos m\theta
\end{bmatrix}.
\]
If we identify the two-dimensional vector \((a_1, a_2)^T\) with the complex number, then \(u_m^{(j)}\) is analytic and \(v_m^{(j)}\) is anti-analytic for \(j = 1, 2\). In particular,

\[
\text{div } v_m^{(j)} = 0,
\]

in other words, \(v_m^{(j)}\) is a shear displacement. We also mention that the convergence rates of eigenvalues at \(k_0\) and \(-k_0\) are different on ellipses (see [1]).

The third question we discuss is regarding the Fredholm alternative, which says that \(I + A\) is invertible if and only if it is injective provided that \(A\) is a compact operator. What if \(A\) is polynomially compact, not compact?

Acknowledgements

This work is supported by grant A3 Foresight Program among China (NSF), Japan (JSPS), and Korea (NRF 2014K2A2A6000567). Work of H.K. is supported by NRF 2016R1A2B4011304. Work of Y.M. is partially supported by a MEXT Grant-in-Aid for Scientific Research on Innovative Areas (16H06576).

References

[1] Ando K, Ji Y, Kang H, Kim K and Yu S 2017 Spectral properties of the Neumann-Poincaré operator and cloaking by anomalous localized resonance for the elasto-static system, *Euro. J. Appl. Math.* S095672517000080 (Preprint arXiv: 1510.00989)

[2] Ando K, Kang H and Miyanishi Y 2017 Elastic Neumann–Poincaré operators on three dimensional smooth domains: Polynomial compactness and spectral structure: Elastic Neumann–Poincaré operators on three dimensional smooth domains, *Int. Math. Res. Not.* rnx258 (Preprint arXiv: 1702.03415)

[3] Coifman R R, McIntosh A and Meyer Y 1982 *Ann. Math.* 116 361

[4] Dahlberg B E J, Kenig C E and Verchota G C 1988 *Duke Math. J.* 57(3) 795

[5] Khavinson D, Putinar M and Shapiro H S 2007 *Arch. Rational Mech. Anal.* 185 143

[6] Kupradze V D 1965 *Potential methods in the theory of elasticity* (New York: Daniel Davey & Co.)

[7] Stein E M and Weiss G 1971 *Introduction to Fourier analysis on Euclidean spaces* (Princeton: Princeton University Press)