SHALLOW WATER MODELS FOR STRATIFIED EQUATORIAL FLOWS

ANNA GEYER\(^1\) AND RONALD QUIRCHMAYR\(^2\)

Abstract. Our aim is to study the effect of a continuous prescribed density variation on the propagation of ocean waves. More precisely, we derive KdV-type shallow water model equations for unidirectional flows along the Equator from the full governing equations by taking into account a prescribed but arbitrary depth-dependent density distribution. In contrast to the case of constant density, we obtain for each fixed water depth a different model equation for the horizontal component of the velocity field. We derive explicit formulas for traveling wave solutions of these model equations and perform a detailed analysis of the effect of a given density distribution on the depth-structure of the corresponding traveling waves.

1. Introduction

The dynamics of inviscid fluids are governed by Euler’s equations of motion. In the context of water wave models one usually assumes the density function to be constant throughout the entire fluid body, which is a reasonable simplification since the compressibility of water is negligible: even very high amounts of hydrostatic or hydrodynamic pressure have almost no effect on the water density. In fact, the pressure acts like a Lagrange multiplier, enforcing the incompressibility condition (see the discussion in \([3]\)). However, other factors like temperature and salinity have a considerably higher effect on the density and give rise to stratified flows which become relevant e.g. in the context of ocean dynamics. A significant increase of density due to changes in temperature or salinity gives rise to a pycnocline—a layer that separates surface water of lower density from water of higher density in greater depths. The propagation of disturbances of a pycnocline gives rise to internal waves, which are much larger than surface waves but hard to detect since there is only weak interaction between internal waves and surface waves, cf. \([19, 11]\) and the references therein.

In recent studies of stratified equatorial ocean flows, cf. \([4, 11, 12]\), the pycnocline has been considered as an infinitely thin interface between two layers of different but constant densities. This approach opened up rich developments in the theoretical investigations of equatorial flows, including the derivation of exact solutions \([5, 6, 7]\), a generalization to three-dimensional fluid flows \([14]\), discovery of the Hamiltonian structure of the model \([8, 9, 22, 23]\) and qualitative analysis of the thermocline see e.g. \([32]\); see also the overview articles \([26, 27]\) and the reference therein. A comprehensive account of phenomena pertaining to physical oceanography in general and to equatorial ocean dynamics in particular are presented in \([13]\).

\(^1\)Delft University of Technology, Delft Institute of Applied Mathematics, Faculty of EEMCS, Mekelweg 4, 2628 CD Delft, The Netherlands.

\(^2\)KTH Royal Institute of Technology, Department of Mathematics, Lindstedtsvägen 25 100 44 Stockholm, Sweden.

E-mail addresses: a.geyer@tudelft.nl, ronaldq@kth.se

Key words and phrases. equatorial shallow water equations; continuous density stratification; explicit solutions.

MSC2010. 86A05; 76B70.
While the above approach assumes a predefined discontinuous piecewise constant density function with a jump at the pycnocline, the aim of the present paper is to study the effect of a continuous change of density on the propagation of water waves. To this end we derive the weakly nonlinear KdV-type shallow water equation

$$2u_\tau - 2\Omega_0 u_\xi + \nu uu_\xi + \frac{1}{3}u_{\xi\xi\xi} = 0$$

for prescribed but arbitrary continuously differentiable density distributions. Here the unknown $u$, depending on time $\tau$ and space $\xi$, describes the horizontal velocity component of the flow field beneath a unidirectional free surface shallow water wave of small amplitude at a specific water depth; the coefficient $\nu$ of the nonlinear term $uu_\xi$, to be specified in the sequel, depends on the prescribed density distribution and the water depth. The major novelty compared to the case of constant density is the depth dependence of the horizontal velocity component of the flow at leading order. In particular, the vorticity of these flows does not vanish, cf. Remark 3.3. This is expected in view of the general principle that stratified flows can never be irrotational, see the discussion in [3] and [33]. The constant $\Omega_0$ incorporates the Coriolis effect along the Equator. We obtain a similar equation for the elevation of the free surface. By a thorough study of the explicit traveling wave solutions of these KdV type equations we acquire a solid theoretical understanding of the effect of the density distribution on the size and shape of these traveling waves at various depths. This analysis shall serve as a basis for a better understanding on how a continuous change of density affects the dynamics of more general types of waves.

2. Governing equations in dimensionless form

In this section we introduce the governing equations for unidirectional gravity water waves; we provide the equations in physical variables and bring them into dimensionless form in the sequel. The main difference compared to the classical water waves problem is that we allow for a non-constant “background” density distribution depending smoothly on the vertical direction caused by e.g. increasing salinity or decreasing temperature with depth. Such stratification phenomena occur in particular in equatorial regions, where waves typically propagate in purely azimuthal direction, so that a unidirectional setting is a reasonable simplification, cf. [31] and the discussion in [11]. Motivated by this geophysical point of view, in which the effects of the earth’s rotation on large waves become relevant, we consider the $f$-plane approximation for the inviscid Euler equations for two-dimensional unidirectional flows near the Equator. The motion of the water between the flat bed and the free surface is therefore governed by the following set of equations:

$$\begin{align*}
\bar{u}_\bar{t} + \bar{u}\bar{u}_\bar{x} + \bar{w}\bar{u}_\bar{z} + 2\bar{\Omega}\bar{w} &= -\bar{\rho}^{-1}\bar{P}_x \\
\bar{w}_\bar{t} + \bar{u}\bar{w}_\bar{x} + \bar{w}\bar{w}_\bar{z} - 2\bar{\Omega}\bar{u} &= -\bar{\rho}^{-1}\bar{P}_z - \bar{g} \\
\bar{\rho}_\bar{t} + (\bar{\rho}\bar{u})_\bar{x} + (\bar{\rho}\bar{w})_\bar{z} &= 0,
\end{align*}$$

(1)

where the bars distinguish these physical variables from dimensionless variables appearing later. The independent variables $\bar{x}$ and $\bar{z}$ denote the directions of increasing azimuth and vertical elevation respectively; the horizontal and vertical fluid velocity component in the direction of increasing azimuth and elevation are given by $\bar{u} = \bar{u}(\bar{x}, \bar{z}, \bar{t})$ and $\bar{w} = \bar{w}(\bar{x}, \bar{z}, \bar{t})$. The pressure is denoted by $\bar{P} = \bar{P}(\bar{x}, \bar{z}, \bar{t})$. We consider a prescribed but arbitrary continuously differentiable density distribution $\bar{\rho}$ of the water which may vary with depth only, that is

$$\bar{\rho} = \bar{\rho}(\bar{z}) > 0.$$  

(2)

The gravitational acceleration is denoted by $\bar{g} \approx 9.81 m s^{-2}$ and the constant $\bar{\Omega} \approx 7.29 \cdot 10^{-5} \text{ rad s}^{-1}$
denotes the rotational speed of the Earth around the polar axis. Thus, the two $\bar{\Omega}$-terms in (1) capture the effects of the so-called Coriolis force. Let us emphasize at this point that all considerations we make in this paper equally apply to the corresponding non-geophysical settings, i.e. to unidirectional inviscid gravity water waves with a prescribed smooth depth-dependent density distribution. In this case we simply set $\bar{\Omega} = 0$.

As already mentioned, we assume that the fluid body is bounded from below by a flat bed at $\bar{z} = 0$; it is bounded from above by the free water surface which is assumed to be described by the graph of the function $\bar{h}_0 + \bar{\eta}(\cdot, \bar{t})$ at any instant of time $\bar{t}$. The average water level $\bar{h}_0$ is indicated by a dashed line, $\bar{\lambda}$ shows the distance between two consecutive crests and $\bar{a}$ is the vertical deviation of a typical crest from $\bar{h}_0$. Fig. 1b shows a prescribed depth dependent density distribution $\bar{\rho}(\bar{z})$ with a significant density increase in the region between the two dotted horizontal lines close to the surface giving rise to a pycnocline.

To bring the set of equations (1) and boundary conditions (3) into dimensionless form, we use the following standard reference length scales: a typical amplitude of the surface wave $\bar{a}$ and the average undisturbed water depth $\bar{h}_0$ as the vertical scales, and a typical wave length $\bar{\lambda}$ as the horizontal scale,
SHALLOW WATER MODELS FOR STRATIFIED EQUATORIAL FLOWS

cf. [10], [24] and [25] for more details. We introduce the following set of non-dimensional variables:

\[ \bar{x} = \lambda x, \quad \bar{\varepsilon} = \bar{h}_0 z, \quad \bar{t} = \frac{\lambda}{\sqrt{g \bar{h}_0}} t, \quad \bar{u} = \sqrt{g \bar{h}_0} \frac{\lambda}{h_0} u, \quad \bar{w} = \sqrt{g \bar{h}_0} \frac{\lambda}{h_0} w, \quad \bar{\eta} = \bar{a} \eta, \]

\[ \bar{P} = \bar{P}_{\text{atm}} - \bar{g} \int_{h_0}^{\bar{\varepsilon}} \bar{\rho}(s) \, ds + \bar{g} \bar{h}_0 \bar{\rho} P, \quad \bar{\Omega} = \frac{\sqrt{g \bar{h}_0}}{h_0} \Omega, \]

where the dimensionless pressure \( \bar{P} \) acts as a measure for the deviation of \( \bar{P} \) from the hydrostatic pressure distribution. The density \( \bar{\rho} \) is scaled according to

\[ \bar{\rho}(\bar{z}) = \bar{\rho}_0 \rho(z), \]

where \( \bar{\rho}_0 = \bar{\rho}(\bar{h}_0) \).

By employing (4)–(5) and introducing the two fundamental dimensionless parameters

\[ \varepsilon := \frac{\bar{a}}{\bar{h}_0}, \quad \delta := \frac{\bar{h}_0}{\lambda}, \]

referred to as amplitude and shallowness parameter, the set of equations (1) and boundary conditions (3) transform into the following dimensionless form:

\[ u_t + u u_x + w u_x + 2\Omega w = -P_x \]

\[ \delta^2 (w_t + w w_x + w w_z) - 2\Omega u = -\frac{(\rho P)_z}{\rho} \]

\[ u_x + w_x + \frac{\rho_z}{\rho} w = 0 \]

\[ P = \frac{1}{\rho(1)} \int_1^{\varepsilon} \rho(s) \, ds \quad \text{on} \quad z = 1 + \varepsilon \eta(x, t) \]

\[ w = \varepsilon (\eta_t + u \eta_x) \quad \text{on} \quad z = 1 + \varepsilon \eta(x, t) \]

\[ w = 0 \quad \text{on} \quad z = 0. \]

In order to transfer the free surface to a fixed boundary, we rewrite the boundary conditions at the free surface by means of Taylor expansions of the involved variables \( u, w \) and \( P \) about \( z = 1 \). This yields the expressions

\[ P + \varepsilon \eta P_z + \varepsilon^2 \eta^2 / 2 \rho_z + \mathcal{O}(\varepsilon^3) = \varepsilon \eta - \varepsilon^2 \eta^3 \rho_z / 2 + \mathcal{O}(\varepsilon^3) \quad \text{on} \quad z = 1 \]

\[ w = \varepsilon (\eta_t + \eta \eta_x - \eta w_z) + \mathcal{O}(\varepsilon^2) \quad \text{on} \quad z = 1 \]

where we have used the fact that \( \rho(1) = 1 \) in view of the nondimensionalization (5). Finally, following [3], we perform the usual scaling

\[ u \mapsto \varepsilon u, \quad w \mapsto \varepsilon w, \quad P \mapsto \varepsilon P \]

on the system (7) and set

\[ \Omega = \varepsilon \Omega_0, \]

where \( \Omega_0 \) is an appropriate constant. The latter scaling is a valid manoeuvre since the dimensionless parameters \( \Omega \) and \( \varepsilon \) are of the same order of magnitude when assuming typical values of ocean depth and wave amplitude for offshore waves near the Equator, see the discussion in [21]. Additionally
we exploit that the shallowness parameter \( \delta \) in system (7) can be scaled out in favor of \( \varepsilon \) via the transformation

\[
x \mapsto \frac{\delta}{\sqrt[3]{\varepsilon}} x, \quad z \mapsto z, \quad t \mapsto \frac{\delta}{\sqrt[3]{\varepsilon}} t, \\
P \mapsto P, \quad \eta \mapsto \eta, \quad u \mapsto u, \quad w \mapsto \frac{\sqrt[3]{\varepsilon}}{\delta} w, \tag{11}
\]

where the scaling of \( w \) is required to ensure conservation of mass in the resulting system. We note that this transformation remains non-singular for \( \varepsilon \) and \( \delta \) tending to zero only if these parameters satisfy a certain asymptotic relation. That is, by imposing the transformation (11) one considers the restriction to shallow water waves of small amplitude:

\[
\delta \ll 1, \quad \varepsilon = O(\delta^2). \tag{12}
\]

The application of (9), (10) and (11) on the system (7) yields the following system of governing equations for unidirectional equatorial waves in scaled and dimensionless form:

\[
\begin{align*}
  u_t + \varepsilon(ww_x + wu_z) + 2\varepsilon\Omega_0 w &= -P_x \\
  \varepsilon(w_t + \varepsilon(ww_x + wu_z)) - 2\varepsilon\Omega_0 u &= -\frac{(\rho P)_z}{\rho} \\
  u_x + w_z + \frac{\rho z}{\rho} w &= 0
\end{align*}
\]

\[
\begin{align*}
P &= \eta - \varepsilon\eta\left(\frac{\eta}{2}\rho z - P_z\right) \quad \text{on} \quad z = 1 \\
w &= \eta_t + \varepsilon(u\eta_t - \eta w_z) \quad \text{on} \quad z = 1 \\
  w &= 0 \quad \text{on} \quad z = 0.
\end{align*}
\]

3. Derivation of a shallow water model

The system (13) of governing equations for equatorial waves serves as a starting point for our derivations. We follow the techniques presented in [24, 25] to obtain model equations for the surface elevation \( \eta \) and the horizontal velocity \( u \). By retaining only the leading order terms of (13) (i.e. by setting \( \varepsilon = 0 \)) we obtain the linear system

\[
\begin{align*}
u_t &= -P_x \\
0 &= (\rho P)_z \\
u_x + w_z + \frac{\rho z}{\rho} w &= 0
\end{align*}
\]

\[
P = \eta \quad \text{on} \quad z = 1 \\
w = \eta_t \quad \text{on} \quad z = 1 \\
w = 0 \quad \text{on} \quad z = 0,
\]

from which we infer that the leading order approximation of \( \eta \) (again denoted by \( \eta \)) satisfies the linear wave equation:

\[
\eta_{tt} - \eta_{xx} = 0. \tag{15}
\]
To see this, we first note that the second equation in (14) and the dynamic boundary condition imply that \( \eta = \rho P \) for all \( z \in [0,1] \). In view of the first equation this yields that \( \rho u_t = -\eta_x \). Next we invoke the equation of mass conservation which is equivalent to \( (\rho w)_z = -\rho u_x \), hence
\[
\eta = \rho P \quad \text{for all } z \in [0,1].
\]

Differentiating both sides with respect to \( t \), using the boundary condition at the top and the fact that \( \rho(1) = 1 \), we obtain that \( \eta_{tt} = -\int_0^1 \rho u_x(t) \, ds \) on \( z = 1 \). Hence, we infer that \( \eta \) satisfies (15).

The general solution of (15) is of the form
\[
\eta(x,t) = F(x-t) + F(x+t)
\]
for arbitrary functions \( F \) and \( F \).

This motivates the introduction of the far-field variables
\[
\xi = x - t, \quad \tau = \varepsilon t
\]
(16) to follow right-propagating waves governed by (13). Rewriting (13) in terms of the far-field variables (16) yields the system
\[
-u\xi + \varepsilon(u\tau + uu\xi + uwz + 2\Omega_0 w) = -P\xi
\]
\[
\varepsilon(-w\xi + \varepsilon(w\tau + uw\xi + wwz) - 2\Omega_0 u) = \frac{(\rho P)\xi}{\rho}
\]
\[
(\rho w)_z = -\rho u\xi
\]
(17)

To obtain an asymptotic solution of system (17), we formally expand the respective variables \( \eta, u, w \) and \( p \) in the form \( q \sim \sum_{n=0}^{\infty} q_n \varepsilon^n \). At leading order we obtain the linear system
\[
u_{0\xi} = P_{0\xi}
\]
\[
0 = (\rho P_0)_z
\]
\[
(\rho w_0)_z = -\rho u_{0\xi}
\]
(18)

Integrating the second equation and using the boundary condition on the pressure we obtain that \( \eta_0 = \rho P_0 \) for all \( z \in [0,1] \). In view of the first equation we may therefore deduce that
\[
u_0 = \rho^{-1} \eta_0,
\]
(19)
where we have invoked the assumption that any change in \( u \) can occur only due to a perturbation of \( \eta \), see [24]. From the equation of mass conservation, the bottom boundary condition and (19) we deduce
that \( w_0 = -z \rho^{-1} \eta_0 \xi \). Next we consider the first order system

\[
-u_1 \xi + \rho^{-1} \eta_0 \tau + \rho^{-2} (1 + z \rho^{-1} \rho_z) \eta_0 \eta_0 \xi - 2 \Omega_0 z \rho^{-1} \eta_0 \xi = -P_1 \xi
\]

\[
z \eta_0 \xi - 2 \Omega_0 \eta_0 = - (\rho P_1)_z
\]

\[
(\rho w_1)_z = - \rho u_1 \xi
\]

(20)

where we have written all leading order expressions in terms of \( \eta_0 \). Viewing the second equation as a linear ODE in \( P_1 \) with \( z \) being the independent variable, we can solve it using the pressure boundary condition and obtain that

\[
P_1(z) = \rho^{-1} \left[ P_1(1) + \int_1^z 2 \Omega_0 \eta_0 - s \eta_0 \xi \xi \, ds \right].
\]

This results in the following representation of \( P_1 \) in terms of \( \eta_0 \) and \( \eta_1 \):

\[
P_1(z) = \rho^{-1} \left\{ \eta_1 + \frac{\rho_z}{2} \eta_0^2 + 2 \Omega_0 \eta_0 (z - 1) - \frac{z^2 - 1}{2} \eta_0 \xi \right\}
\]

Next we solve the equation of mass conservation as a linear ODE in \( w_1 \). By expressing the inhomogeneity using the first equation, and incorporating the boundary condition on \( z = 0 \) and \( P_1 \xi \) obtained in the previous step, we find that

\[
w_1(z) = - \frac{1}{\rho} \int_0^z \rho w_1 \xi \, ds
\]

\[
= - \frac{1}{\rho} \int_0^z \left[ \eta_1 + \eta_0 \tau + (2 - \rho_z) \eta_0 \eta_0 \xi \right. - \frac{z^2 - 1}{2} \eta_0 \xi \xi \xi + \left. \left( \rho + \frac{s \rho_z}{z} \right) \eta_0 \eta_0 \xi \right] \, ds.
\]

(21)

Finally, we calculate the integral in (21), evaluate the resulting expression for \( w_1 \) at \( z = 1 \) and compare the outcome with the corresponding boundary condition in (20). This yields the weakly nonlinear third order partial differential equation

\[
2 \eta_1 - 2 \Omega_0 \eta_1 \xi + \nu(\rho) \eta_1 \xi + \frac{1}{3} \eta_1 \xi \xi \xi = 0,
\]

(22)

where

\[
\nu(\rho) = 2 - \rho_z (1 + \int_0^1 \rho_z + \frac{(z \rho_z)}{\rho^2} \, dz) \in \mathbb{R},
\]

(23)

which governs the leading order approximation of the free surface \( \eta \), where \( \eta \sim \eta_0 + O(\varepsilon) \). The equation for the leading-order approximation of the horizontal velocity \( u \) at any depth \( z \in [0, 1] \) has a depth-dependent coefficient of the nonlinear term:

\[
2 u_1 - 2 \Omega_0 u_1 \xi + \rho(z) \nu(\rho) u_1 \xi + \frac{1}{3} u_1 \xi \xi \xi = 0.
\]

(24)

Remark 3.1. For constant density \( \rho \equiv 1 \) and vanishing Coriolis force, both equations (22) and (24) reduce to the standard KdV [29]

\[
2 u_1 + 3 u_1 \xi + \frac{1}{3} u_1 \xi \xi \xi = 0,
\]

(25)
which approximates right-propagating shallow water waves of small amplitude at leading order. We observe that for fixed \( z \in [0, 1] \) and for any given density function \( \rho \) the coefficient of the nonlinear term in both equations is a fixed real number. Therefore, these equations can always be brought to the form (25) via the transformation

\[
U = (\alpha/3)^{3/5}u, \quad T = (\alpha/3)^{3/5}\tau, \quad X = (\alpha/3)^{1/5}(\xi + \Omega_0\tau),
\]

(26)

where \( \alpha = \nu(\rho) \) and \( u = \eta \) for (22) and \( \alpha = \rho(z)\nu(\rho) \) for (24). This transformation is valid only as long as \( \alpha > 0 \). The sign of \( \alpha \) can be determined for any arbitrary given density distribution \( \rho \); we note that \( \alpha \) is strictly positive for realistic density distributions in the context of (geophysical) water waves and provide examples for different densities at the end of Section 4.

**Remark 3.2.** In view of Remark 3.1 several remarkable properties of the standard KdV (25), like soliton interaction, bi-Hamiltonian structure, the existence of a Lax pair, integrability, global Birkhoff coordinates, see e.g. [20, 30, 18, 28], and stability of traveling wave solutions (cf. Section 4.3 for some references regarding stability) remain valid for equations (22) and (24) by means of the transformation (26).

**Remark 3.3.** The depth dependent coefficient in the velocity equation (24) due to non-constant density \( \rho \) represents a major difference compared to the standard KdV, which models both the free surface \( \eta \) and the horizontal velocity \( u \) at any depth. In the next section we will consider explicit traveling wave solutions of (22) and (24) for certain prescribed density distributions to illustrate the change of the corresponding wave profiles with depth. Moreover, a non-constant density distribution \( \rho \) entails a non-vanishing vorticity\(^1\) of the flow field at leading order, cf. (18) and (19). This is to be expected in view of the rotational nature of stratified flows, see [3, 33].

4. **Explicit traveling wave solutions**

In this section we present explicit traveling wave solutions of equations (22) and (24). We study the effects of the non-constant density on the \( z \)-structure of solutions in general and for two concrete examples of density distributions. Traveling wave solutions of (22) of the form \( \eta(\xi, \tau) = \varphi(\xi - c\tau) \) with wave speed \( c > 0 \) satisfy the ordinary differential equation

\[
-2(c + \Omega_0)\varphi' + \nu(\rho)\varphi\varphi' + \frac{1}{3}\varphi''' = 0,
\]

(27)

with \( \nu(\rho) \) defined in (23).

**4.1. Solitary traveling waves.** Assuming decay at infinity we may integrate this equation twice using \( \varphi' \) as an integrating factor in the second integration. The solitary traveling wave equation then reads

\[
(\varphi')^2 = (6(c + \Omega_0) - \nu(\rho))\varphi^2.
\]

(28)

We see that real solutions can exist only if \( \varphi \leq \frac{6(c + \Omega_0)}{\nu(\rho)} \). Following [17] we integrate the equation by writing

\[
\int \frac{d\varphi}{\varphi\sqrt{6(c + \Omega_0) - \nu(\rho)}} = \pm \int ds,
\]

\(^1\)The vorticity of the dimensionless flow field is given by \( \omega = u_z - \varepsilon w_x \).
where \( s = \xi - ct \), and using the substitution \( \varphi = \frac{6(c + \Omega_0)}{\nu(\rho)} \sech^2(\theta) \). We obtain that the solitary wave solutions of (22) are given by
\[
\varphi(\xi - ct) = \frac{6(c + \Omega_0)}{\nu(\rho)} \sech^2 \left( \frac{\sqrt{6(c + \Omega_0)}}{2} (\xi - ct) \right).
\] (29)

Notice that for constant \( \rho = 1 \) and \( \Omega_0 = 0 \) this reduces to the familiar \( \sech^2 \)-solutions of the standard KdV (25).

**Remark 4.1.** We observe that \( \rho \) appears only as a multiplicative factor in front of the solution and hence any changes in \( \rho \) affect only the amplitude, cf. Fig. 2a. This is different from the dependence on the Coriolis parameter and the wave speed: as \( c \) or \( \Omega_0 \) increase, the profiles become taller as well as narrower, see [21]. Moreover, observe that we have made no assumption on the sign of \( \nu(\rho) \). Therefore, in principle our approach allows us to obtain solitary waves of elevation when this number is positive and waves of depression when it is negative, see the remark at the end of Example 2.

For the horizontal velocity component \( u \) traveling wave solutions of the equation (24) of the form
\[
u(\xi, z, \tau) = \phi(\xi - ct, z)
\] can be obtained directly from the explicit traveling wave solutions of (22) in view of the relation between the surface and the velocity component given in (19). This relation provides an explicit measure for the attenuation of the amplitude with depth which is inversely proportional to \( \rho(z) \), see also Fig. 2b. Therefore, we obtain a traveling wave solution of (24) for each depth \( z \in [0, 1] \) given by
\[
\phi(\xi - ct, z) = \rho(z)^{-1} \varphi(\xi - ct),
\] (30)
where \( \varphi \) is a solution of (27). We observe that the amplitude of the wave does not vanish at the bottom.

**Example 1.** Consider the linear density function
\[
\rho(z) = 1 + A(1 - z), \quad z \in [0, 1]
\] (31)
where \( A > 0 \), i.e. \( \rho(z) \) is increasing with depth. In this case the coefficient of the nonlinear term in the equation (22) is \( \nu(\rho) = 2 \ln(1 + A)/A + 1 \in (1, 3) \), which is a strictly monotonically decreasing function of \( A \). Therefore, for every fixed \( z \in [0, 1] \), the profiles of the solitary traveling wave solutions become taller and also wider as \( A \) increases, cf. (29) and Fig. 2a. The amplitude of the velocity component \( \phi(\xi - ct, z) \) decreases with depth like \( (Az)^{-1} \) in view of (19) and (31), see Fig. 2b.

**Example 2.** Next we consider the density function
\[
\rho(z) = a_0 - a_1 \arctan (a_2(z - a_3)), \quad z \in [0, 1],
\] (32)
for suitable parameters \( a_i \in \mathbb{R} \), to obtain a more realistic density distribution as described in the introduction and depicted in Fig. 1b. We choose the parameters to match the actual physical situation in the mid-equatorial Pacific, with an average water depth of about 4000 m and a pycnocline located at a depth of roughly 200 m. In this oceanic region, the density of the water at the surface is about 1027 kg/m\(^3\) with a density increase of 1% from the surface to the seabed, cf. [11]. Nondimensionalizing and scaling these quantities to depths \( z \in [0, 1] \) and \( \rho(1) = 1 \) leads us to the following choice of parameter values: \( a_3 = 0.95 \) determines where the region of biggest density gradient is located, i.e. at the center of the pycnocline; \( a_2 = 300 \) regulates the thickness of the pycnocline; with the last two parameters \( a_0 = 1.0047 \) and \( a_1 = 0.0034 \) we ensure the correct density increase. This choice results in the density distribution depicted in Fig. 3a.
It is interesting to see how this density profile is reflected in the decay of the amplitude of the solitary wave profile \( \phi(\xi - c\tau, z) \) with depth. This is because the solitary waves \( \phi(\xi - c\tau, z) \) of the velocity equation (24) decay with depth inversely proportional to \( \rho(z) \) according to relation (30), see Fig. 3b.

Finally, we observe that for the choice of parameters described above the coefficient of the nonlinear term \( \nu(\rho) = 2.97 \) defined in (23) is positive. We note that in this setting negative values of \( \nu(\rho) \), which give rise to solitary waves of depression, would arise only for a density increase of almost 300% which is not a physically relevant scenario.

4.2. Periodic traveling waves. Periodic traveling waves of (22) can be obtained in a similar fashion, cf. [17] or [10]. To this end we integrate (27) twice to obtain

\[
(\phi')^2 = F(\phi) \quad \text{where} \quad F(\phi) := -\nu(\rho)\phi^3 + 6(c + \Omega_0)\phi^2 + A\phi + B,
\]

where \( A, B \) are constants of integration. One can show that this equation has real and bounded solutions only if \( F(\phi) \) has three simple roots satisfying \( \varphi_3 < \varphi_2 < \varphi_1 \) when \( \nu(\rho) > 0 \) (the other situation being similar), in which case we can write

\[
F(\phi) = -\nu(\rho)(\phi - \varphi_1)(\phi - \varphi_2)(\phi - \varphi_3).
\]

The solution oscillates between \( \varphi_2 \leq \phi \leq \varphi_1 \) with period \( 2 \int_{\varphi_2}^{\varphi_1} \frac{d\phi}{\sqrt{F(\phi)}} \). Therefore we can express the solution implicitly as

\[
s = s_1 \pm \int_{\rho}^{\varphi_1} \frac{dg}{\sqrt{F(g)}},
\]

\[\text{Note that, since the density change of 1% in } \rho \text{ and the resulting impact on the amplitude decay in depth is quite small compared to the amplitude of the solitary wave, we have used a larger value for } a_1 \text{ to produce Fig. 3b to make this effect more visible.}\]
In Fig. 3a we see a plot of the density profile $\rho(z) = a_0 - a_1 \arctan \left( a_2 (z - a_3) \right)$ defined in (32) for suitable choices of parameter values $a_i \in \mathbb{R}$ to model a density increase of 1% from surface to bed. Fig. 3b shows a schematic representation of the fact that the amplitude of solitary wave solutions $\phi(\xi - c_T, z)$ decays with depth inversely proportional to $\rho(z)$, cf. (30).

where $\varphi(s_1) = \varphi_1$. Using the substitution $g = \varphi_1 - (\varphi_1 - \varphi_2) \sin^2(\theta)$ this can be transformed into the standard elliptic integral

$$ (s - s_1) \frac{\sqrt{\nu(\rho)(\varphi_1 - \varphi_2)}}{2} = \pm \int_0^\varphi \frac{d\theta}{\sqrt{1 - m \sin^2(\theta)}} =: \pm v, $$

with elliptic modulus $m = \frac{\varphi_1 - \varphi_2}{\varphi_1 - \varphi_2}$ such that $m \in (0,1)$ and $\varphi = \varphi_2 + (\varphi_1 - \varphi_2) \cos^2(\phi)$. Finally, using Jacobi elliptic functions and the fact that $\text{cn}(v|m) = \cos(\phi)$ is an even function, we obtain the periodic cnoidal solution

$$ \varphi(s) = \varphi_2 + (\varphi_1 - \varphi_2) \text{cn}^2 \left( (s - s_0) \frac{\sqrt{\nu(\rho)(\varphi_1 - \varphi_3)}}{2} \right) \frac{\varphi_1 - \varphi_2}{\varphi_1 - \varphi_3} \quad (34) $$

in terms of the three distinct roots of $F(\varphi)$. These roots are related to the parameters of the traveling wave equation (33) via the nonlinear relations

$$ \varphi_1 + \varphi_2 + \varphi_3 = \frac{6(c + \Omega_0)}{\nu(\rho)}, \quad \varphi_1 \varphi_2 + \varphi_1 \varphi_3 + \varphi_2 \varphi_3 = \frac{-A}{\nu(\rho)}, \quad \varphi_1 \varphi_2 \varphi_3 = \frac{B}{\nu(\rho)}, $$

which one can solve as needed for concrete choices of density distributions $\rho$. We see from the explicit expression (34) for periodic cnoidal wave solutions that both the amplitude $\frac{1}{2}(\varphi_1 - \varphi_2)$ and the wave length $4K(m)(\nu(\rho)(\varphi_1 - \varphi_3))^{-1/2}$, where $K(m) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - m \sin^2(\theta)}}$, depend on the density distribution $\rho$ in an intricate way. As in the case of solitary waves, explicit periodic traveling wave solutions of the velocity equation (24) can be obtained directly from (34) in view of relation (19) for each depth $z \in [0,1]$. We refrain here from any further qualitative discussion on periodic waves, which can be carried out along the lines of our analysis for solitary waves as illustrated in Section 4.1.
4.3. Stability. Having established existence and some qualitative properties of traveling waves, we may ask about the stability of these solutions. The orbital stability of periodic and solitary traveling waves of the standard KdV have been studied by many authors, see for instance [1] [2] [15]. Recall the observation presented in Remark 3.1 that for every fixed depth $z \in [0,1]$ the KdV type equations (22) and (24) derived in Section 3 can be transformed into the standard KdV. For this reason, we infer that the explicit solitary and periodic traveling wave solutions (29) and (30) of equations (22) and (24), respectively, are orbitally stable as well.

Acknowledgments. Both authors acknowledge the support of the Erwin Schrödinger International Institute for Mathematics and Physics (ESI) during the program “Mathematical Aspects of Physical Oceanography”. R. Quirchmayr acknowledges the support of the European Research Council, Consolidator Grant No. 682537. Finally, we thank the reviewers for useful suggestions.

References

[1] T.B. Benjamin, The Stability of Solitary Waves, Proc. Roy. Soc. London A 328 153–183 (1972).
[2] J.L. Bona, P.E. Souganidis and W. Strauss, Stability and Instability of Solitary Waves of Korteweg-de Vries Type, Proc. Roy. Soc. London A 411, 395–412 (1987).
[3] A. Constantin, Nonlinear Water Waves with Applications to Wave-Current Interactions and Tsunamis, CBMS-NSF Regional Conference Series in Applied Mathematics, SIAM, Philadelphia, 2011.
[4] A. Constantin, On the modelling of equatorial waves, Geophys. Res. Lett. 39(5), Art. No. L05602 (2012).
[5] A. Constantin, An exact solution for equatorially trapped waves, J. Geophys. Res. Oceans. 117, Art. No. C05029 (2012).
[6] A. Constantin, Some Three-Dimensional Nonlinear Equatorial Flows, J. Phys. Oceanogr. 43, 165–175 (2013).
[7] A. Constantin, Some Nonlinear, Equatorially Trapped, Nonhydrostatic Internal Geophysical Waves, J. Phys. Oceanogr. 44, 781–789 (2014).
[8] A. Constantin and R.I. Ivanov, A hamiltonian approach to wave-current interactions in two-layer fluids, Phys. Fluids. 27(8), Art. No. 086603 (2015).
[9] A. Constantin, R.I. Ivanov and C.I. Martin, Hamiltonian formulation for wave-current interactions in stratified rotational flows, Arch. Ration. Mech. Anal. 221, 1417–1447 (2016).
[10] A. Constantin and R. S. Johnson, On the Non-Dimensionalisation, Scaling and Resulting Interpretation of the Classical Governing Equations for Water Waves, J. Nonlinear Math. Phys. 15(2), 58–73 (2008).
[11] A. Constantin and R. S. Johnson, The dynamics of waves interacting with the Equatorial Undercurrent, Geophys. Astrophys. Fluid Dyn. 109(4), 311–358 (2015).
[12] A. Constantin and R. S. Johnson, An exact, steady, purely azimuthal equatorial flow with a free surface, J. Phys. Oceanogr. 46, 1935–1945 (2016).
[13] A. Constantin and R. S. Johnson, Current and future prospects for the application of systematic theoretical methods to the study of problems in physical oceanography, Phys. Lett. A 380, 3007–3012 (2016).
[14] A. Constantin and R. S. Johnson, A nonlinear, three-dimensional model for ocean flows, motivated by some observations of the Pacific Equatorial Undercurrent and thermocline, Phys. Fluids. 29(5), 056604 (2017).
[15] B. Deconinck and T. Kapitula, The orbital stability of the cnoidal waves of the Korteweg-de Vries equation, Phys. Lett. A. 374, 4018–4022 (2010).
[16] M.W. Dingemans, Water wave propagation over uneven bottoms, World Scientific, 1997.
[17] P. G. Drazin and R. S. Johnson, Solitons: An Introduction, Cambridge University Press, Cambridge, UK, 1989.
[18] L. D. Faddeev and V. E. Zakharov, Korteweg-de Vries equation: a completely integrable Hamiltonian system, Funct. Anal. Appl. 5, 288–287 (1971).
[19] A. V. Fedorov and J. N. Brown, Equatorial waves, Encyclopedia of Ocean Sciences, edited by J. Steele, pp. 3679-3695, Academic Press: New York (2009).
[20] C. S. Gardner, J. M. Greene, M. D. Kruskal and R. M. Miura, Method for solving the Korteweg-de Vries equation, Phys. Rev. Lett. 19, 1095–1097 (1967).
[21] A. Geyer and R. Quirchmayr, Shallow water equations for equatorial tsunami waves, Phil. Trans. R. Soc. A 376, 20170100 (2018).
[22] D. Ionescu-Kruse and C.I. Martin, Periodic equatorial water flows from a Hamiltonian perspective, J. Differ. Equ. 262, 4451–4474 (2017).
[23] R.I. Ivanov, Hamiltonian model for coupled surface and internal waves in the presence of currents, Nonlinear Anal. Real World Appl. 34, 316–334 (2017).
[24] R. S. Johnson, A Modern Introduction to the Mathematical Theory of Water Waves, Cambridge University Press, Cambridge, UK, 1997.
[25] R. S. Johnson, Camassa-Holm, Korteweg-de Vries and related models for water waves, J. Fluid Mech. 455, 63–82 (2002).
[26] R. S. Johnson, An ocean undercurrent, a thermocline, a free surface, with waves: a problem in classical fluid mechanics, J. Nonlinear Math. Phys. 22, 475–493 (2015).
[27] R. S. Johnson, Application of the ideas and techniques of classical fluid mechanics to some problems in physical oceanography, Phil. Trans. R. Soc. A 376, 20170092 (2018).
[28] T. Kappeler and J. Pöschel, KdV & KAM, Ergeb. der Math. und ihrer Grenzgeb., Springer, Berlin-Heidelberg-New York, 2003.
[29] D. J. Korteweg and G. de Vries, On the change of form of long waves advancing in a rectangular canal and on a new type of long stationary waves, Phil. Mag. 39, 422–443 (1895).
[30] P. Lax, Integrals of nonlinear equations of evolution and solitary waves, Comm. Pure Appl. Math. 21, 467–490 (1968).
[31] P. H. LeBlond and L. A. Mysak, Waves in the Ocean, Elsevier, Amsterdam, 1978.
[32] C. I. Martin, Dynamics of the thermocline in the equatorial region of the Pacific ocean, J. Nonlinear Math. Phys. 22, 516–522 (2015).
[33] S. Walsh, Stratified steady periodic water waves, SIAM J. Math. Anal. 41, 1054–1105 (2009).