A Cavity QED Implementation of Deutsch-Jozsa Algorithm

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Abstract

The Deutsch-Jozsa algorithm is a generalization of the Deutsch algorithm which was the first algorithm written. We present schemes to implement the Deutsch algorithm and the Deutsch-Jozsa algorithm via cavity QED.

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1 INTRODUCTION

We can say that quantum computation is a young and promising subject and can revolutionize our computational way of tackling problems, especially the simulation of quantum problems numerically. Nowadays quantum computation is a branch of the wider subject quantum information which embraces besides quantum computation, for instance, quantum communication and quantum cryptography. P. Benioff and R. Feynmann were the pioneers of quantum computation suggesting the build up of computers based on the principles of quantum mechanics. Following the lead of Benioff and Feynmann, D. Deutsch in 1985 made concrete proposals exploring some properties of quantum mechanics to obtain unprecedented parallelism in computation which represented a really breakthrough to the subject which since then has developed quickly although we can still say that perhaps we are far from building a working practical and economically viable hardware based on quantum mechanics, that is, unfortunately we are still far from having, for instance, a quantum PC for helping us in working out problems and, why not, for our enjoyment. However, we should point out that although it seems that a quantum computer would speed up the solution of some problems and even turn out possible the solution of some problems intractable by classical computation, nobody still knows what is the real power of a quantum computer when compared with classical computers. Despite this fact we should go on in the enterprise of developing research on this field which can at least give us new insights on quantum theory and quantum information science.

Important quantum algorithms are the Deutsch algorithm, the Deutsch-Jozsa algorithm, the Simon algorithm, the Shor algorithm and the Grover algorithm.
In this work we present an implementation of the Deutsch \[7, 4\] and the Deutsch-Jozsa algorithms \[8, 4\], via cavity QED. The Deutsch algorithm was the first concrete proposal of computation making use of the special features of quantum mechanics. A recent alternative proposal of realization of the Deutsch algorithm is presented in [12].

2 CA VITY QED REALIZATION

Let us start revising the Deutsch problem. Consider an arbitrary Boolean function \( F : \{0, 1\} \rightarrow \{0, 1\} \). There are four such functions corresponding to two possible arguments and two possible values. For two of them \( F(0) = F(1) \) and in this case we say that \( F \) is constant. For the cases in which \( F(0) \neq F(1) \) we say that \( F \) is balanced. Suppose we do not know the function and we are given an Oracle which can evaluate it and gives us the result. Notice that in order to decide if \( F \) is constant or balanced we will have to use the oracle twice to know its value for 0 and 1. The Deutsch algorithm can solve this problem with just one call of the oracle. Let us see how the Deutsch algorithm works. First let us suppose we have a gate \((F-gate)\). The action of the \(F\)-gate is

\[
|x, y\rangle \rightarrow |x, y \oplus F(x)\rangle. \tag{2.1}
\]

The Deutsch algorithm employs a \(F\)-gate with \( F \) being our function which we want to decide if is constant or balanced. Considering that the input two qubit state is

\[
|\psi_{in}\rangle = |x\rangle \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = \frac{1}{\sqrt{2}}(|x, 0\rangle - |x, 1\rangle), \tag{2.2}
\]

the output is

\[
|\psi_{out}\rangle = \frac{1}{\sqrt{2}}(|x, F(x)\rangle - |x, 1 \oplus F(x)\rangle). \tag{2.3}
\]

Since \( f(x) = 0 \) or 1 this can be written as

\[
|\psi_{out}\rangle = (-1)^{F(x)} \frac{1}{\sqrt{2}}(|x, 0\rangle - |x, 1\rangle) = (-1)^{F(x)}|x\rangle \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle), \tag{2.4}
\]

and as we see the effect is to change the state of the \(|x\rangle\) qubit to \((-1)^{F(x)}|x\rangle\). Therefore the value of the function is in the phase of the state \(|x\rangle\).

The actual circuit of the Deutsch algorithm is shown in Fig. 1 where \( H \) is a Hadamard gate

\[
H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{2.5}
\]

and

\[
|0\rangle \rightarrow \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle),
\]

\[
|1\rangle \rightarrow \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).
\]

Then, as we see, after the action of the \(F\)-gate we have

\[
|0, 1\rangle \rightarrow \frac{1}{\sqrt{2}}[(-1)^{F(0)}|0\rangle + (-1)^{F(1)}|1\rangle] \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \tag{2.6}
\]
and, for \( F(0) = F(1) \), the upper qubit is therefore in the state \( \pm \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \) and after the application of the last Hadamard gate it will be in the state \( \pm |0\rangle \). For \( F(0) \neq F(1) \), the upper qubit is therefore in the state \( \pm \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \) and after the application of the last Hadamard gate it will be in the state \( \pm |1\rangle \). Thus a single call to the quantum oracle followed by a measure of the upper qubit in the computational basis gives us the answer of the problem. Notice that the state in which the lower qubit is left is not important.

Let us see now how we can implement experimentally the Deutsch algorithm. We start assuming that we have a cavity \( C \) prepared in the state

\[
|\rangle_C = \frac{(|0\rangle_C - |1\rangle_C)}{\sqrt{2}}. \tag{2.7}
\]

In order to prepare this state, we send a two-level atom \( A_0 \), with \(|g_0\rangle \) and \(|e_0\rangle \) being the lower and upper level respectively, in the state

\[
|\psi\rangle_{A_0} = \frac{1}{\sqrt{2}}(-i |e_0\rangle + |f_0\rangle), \tag{2.8}
\]

through \( C \), for \( A_0 \) resonant with the cavity. If \( g \) is the coupling constant and \( \tau \) the atom-field interaction time, under the Jaynes-Cummings dynamics \cite{13}, for \( g\tau = \pi/2 \), we know that the state \(|f_0\rangle|0\rangle_c \) does not evolve, however, the state \(|e_0\rangle|0\rangle_c \) evolves to \(-i|f_0\rangle|1\rangle_c \). Then, for the cavity initially in the vacuum state \( |0\rangle_c \), we have

\[
\frac{(|f_0\rangle - i|e_0\rangle)}{\sqrt{2}} \rightarrow \frac{|f_0\rangle (|0\rangle_c - |1\rangle_c)}{\sqrt{2}} = |f_0\rangle |\rangle_C \tag{2.9}
\]

Then let us assume that we have a cavity \( C \) prepared initially in a state \( |\rangle_C \) and an atom \( A_1 \) prepared initially the state

\[
|\psi\rangle_{A_1} = \frac{1}{\sqrt{2}}(|e_1\rangle + |f_1\rangle), \tag{2.10}
\]

Consider now that for an atom \( A_k \), with \(|f_k\rangle \) and \(|e_k\rangle \) being the lower and upper level respectively, the transition \(|f_k\rangle \leftarrow |e_k\rangle \) is far from resonance with the cavity central frequency as shown in Fig. 2. In this case the effective Hamiltonian is given by \cite{14}

\[
H = \hbar \frac{g^2}{\Delta} (a^\dagger a + 1) |e_k\rangle \langle e_k| - \hbar \frac{g^2}{\Delta} a^\dagger a |f_k\rangle \langle f_k|, \tag{2.11}
\]

and the time evolution operator is given by \cite{13}

\[
U(t) = e^{-i\varphi(a^\dagger a + 1)} |e_k\rangle \langle e_k| + e^{i\varphi a^\dagger a} |f_k\rangle \langle f_k|, \tag{2.12}
\]

where \( \varphi = g^2\tau / \Delta, g \) is the coupling constant, \( \Delta = \omega_e - \omega_f - \omega \) is the detuning where \( \omega_e \) and \( \omega_f \) are the frequencies of the upper and lower levels respectively and \( \omega \) is the cavity field frequency and \( \tau \) is the atom-field interaction time. Now, we are going to send \( A_1 \) through cavity \( C \) where the atom interacts dispersively with \( C \) according to \( (2.12) \). Let us take \( \varphi = \pi \). Then, after \( A_1 \) flies through \( C \) prepared in state \( (2.11) \), we have

\[
|\psi\rangle_{A_1-C} = \frac{1}{2}(-|e_1\rangle + |f_1\rangle)(|0\rangle_c + |1\rangle_c). \tag{2.13}
\]
If $\varphi = 2\pi$ we have
\[
|\psi\rangle_{A1-C} = \frac{1}{2} (|e_1\rangle + |f_1\rangle)(|0\rangle_C - |1\rangle_C).
\]
(2.14)

Now, if we use the notation
\[
|e_k\rangle = |0\rangle_{Ak},
|f_k\rangle = |1\rangle_{Ak},
\]
(2.15)
we can rewrite (2.13) and (2.14) as
\[
|\psi\rangle_{A1-C} = \pm \frac{1}{\sqrt{2}} \left[ (-1)^{F(0)} |0\rangle_{A1} + (-1)^{F(1)} |1\rangle_{A1} \right] \frac{1}{\sqrt{2}} ([0\rangle_C - (-1)^{F(0)\oplus F(1)} |1\rangle_C]
\]
(2.16)

Now we make use of the Hadamard gate $H$ (2.5). Then, in the case of $F(0) = F(1)$ the atom will be in the state $\pm \frac{1}{\sqrt{2}} (|0\rangle_{A1} + |1\rangle_{A1})$ or $\pm \frac{1}{\sqrt{2}} (|e_1\rangle + |f_1\rangle)$ and after we apply the $H$ gate we get $\pm |0\rangle_{A1}$ or $|e_1\rangle$. In the case of $f(0) \neq f(1)$ the atom will be in the state $\pm \frac{1}{\sqrt{2}} (|0\rangle_{A1} - |1\rangle_{A1})$ or $\pm \frac{1}{\sqrt{2}} (|e_1\rangle - |f_1\rangle)$ and after we apply the $H$ gate we get $\pm |1\rangle_{A1}$ or $|f_1\rangle$. Notice that the state $\frac{1}{\sqrt{2}} ([0\rangle_C - (-1)^{F(0)\oplus F(1)} |1\rangle_C]$ in which the cavity is left is not important.

Another possible implementation of the Deutsch algorithm is possible considering a three-level cascade atom $Ak$ with $|e_k\rangle$, $|f_k\rangle$ and $|g_k\rangle$ being the upper, intermediate and lower atomic states. As above, we assume that the transition $|f_k\rangle \rightleftharpoons |e_k\rangle$ is far enough from resonance with the cavity central frequency such that only virtual transitions occur between these states (only these states interact with field in cavity $C$). In addition we assume that the transition $|e_k\rangle \rightleftharpoons |g_k\rangle$ is highly detuned from the cavity frequency so that there will be no coupling with the cavity field (see Fig. 3). Here we are going to consider the effect of the atom-field interaction taking into account only levels $|f_k\rangle$ and $|g_k\rangle$. We do not consider level $|e_k\rangle$ since it will not play any role in our scheme. Therefore, we have effectively a two-level system involving states $|f_k\rangle$ and $|g_k\rangle$. Considering levels $|f_k\rangle$ and $|g_k\rangle$, we can write an effective time evolution operator (see (2.12)),
\[
U_k(t) = e^{i\varphi a^\dagger a} |f_k\rangle \langle f_k| + |g_k\rangle \langle g_k| .
\]
(2.17)

A coherent state $|\alpha\rangle$ is obtained applying the displacement operator $D(\alpha) = e^{(\alpha a^\dagger - \alpha^* a)}$ to the vacuum, that is, $|\alpha\rangle = D(\alpha)|0\rangle$, and is given by
\[
|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{(\alpha)^n}{\sqrt{n!}} |n\rangle.
\]
(2.18)

We define the even and odd coherent states as
\[
|\pm\rangle = \frac{1}{\sqrt{N^\pm}} ([\alpha] + | - \alpha\rangle) \equiv \frac{1}{\sqrt{2}} ([0\rangle_C + |1\rangle_C),
\]
(2.19)
\[
|\pm\rangle = \frac{1}{\sqrt{N^\pm}} ([\alpha] - | - \alpha\rangle) \equiv \frac{1}{\sqrt{2}} ([0\rangle_C - |1\rangle_C),
\]
with $N^\pm = 2 \left(1 \pm e^{-2|\alpha|^2}\right) \cong 2$ and $\langle \pm | - \rangle = 0$. Now, let us see how we can prepare a state $|\pm\rangle$.

Suppose we prepare cavity $C$ initially in a coherent state $| - \alpha\rangle$. Then we prepare a two-level atom $B$, with $|f\rangle$ and $|g\rangle$ being the upper and lower state respectively, in a coherent superposition, sending
$B$ in the lower state $|g\rangle$ through a first Ramsey cavity $R_1$ where the atomic states are rotated according to

$$R_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} c_g & c_f \\ -c_f & c_g \end{bmatrix},$$

(2.20)

and we get

$$|\psi\rangle_B = c_f |f\rangle + c_g |g\rangle.$$

(2.21)

After that, $B$ flies through cavity $C$ and, taking into account the time evolution operator $R_1$, after $B$ pass through $C$ the state of the system $B - C$, for $\phi = \pi$, is given by

$$|\psi\rangle_{B - C} = c_f |f\rangle |\alpha\rangle + c_g |g\rangle |-\alpha\rangle.$$  

Then, we send $B$ through a second Ramsey zone $R_2$ where the atomic states are rotated according to

$$R_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix},$$

(2.22)

that is,

$$|f\rangle \rightarrow \frac{1}{\sqrt{2}}(|f\rangle - i |g\rangle),$$

$$|g\rangle \rightarrow \frac{1}{\sqrt{2}}(-i |f\rangle + |g\rangle),$$

(2.23)

and therefore, the state of the system $B - C$ will be

$$|\psi\rangle_{B - C} = \frac{1}{\sqrt{2}}[(c_f - ic_g) |+\rangle + (c_f + ic_g) |-\rangle] |f\rangle$$

$$+ \frac{1}{\sqrt{2}}[(-ic_f + c_g) |+\rangle - (ic_f + c_g) |-\rangle] |g\rangle.$$  

Now, in order to obtain a state $|\rangle$ in cavity $C$, we detect atom $B$ in $|f\rangle$ or in $|g\rangle$. If we detect $|f\rangle$ we must have have $c_f = ic_g$ and if we detect $|g\rangle$ we we must have have $c_f = -ic_g$.

Then, we start assuming that the cavity is prepared in a state $|\rangle$ and atom $A_1$ is prepared in a state

$$|\psi\rangle_{A1} = \frac{1}{\sqrt{2}}(|f_1\rangle + |g_1\rangle).$$

(2.24)

After the atom fly through the cavity, for $\phi = \pi$, we have

$$|\psi\rangle_{A_1 - C} = \frac{1}{2}(|-f_1\rangle + |g_1\rangle)(|0\rangle_C - |1\rangle_C).$$

(2.25)

If $\phi = 2\pi$ we have

$$|\psi\rangle_{A_1 - C} = \frac{1}{2}(|f_1\rangle + |g_1\rangle)(|0\rangle_C - |1\rangle_C).$$

(2.26)

Now, if we use the notation

$$|f_k\rangle = |0\rangle_{Ak},$$

$$|g_k\rangle = |1\rangle_{Ak},$$

(2.27)
we can rewrite (2.25) and (2.26) as
\[ |ψ⟩_{A1−C} = ± \frac{1}{\sqrt{2}} [(-1)^{F(0)}|0⟩_{A1} + (-1)^{F(1)}|1⟩_{A1}] \frac{1}{\sqrt{2}} [|0⟩_{C} − |1⟩_{C}] \] (2.28)

Then, in the case of \( F(0) = F(1) \) the atom will be in the state \( ± \frac{1}{\sqrt{2}}(|0⟩_{A1} + |1⟩_{A1}) \) and after we apply the \( H \) gate we get \( ± |0⟩_{A1} \) or \( |f⟩ \). In the case of \( F(0) ≠ F(1) \) the atom will be in the state \( ± \frac{1}{\sqrt{2}}(|0⟩_{A1} − |1⟩_{A1}) \) and after we apply the \( H \) gate we get \( ± |1⟩_{A1} \) or \( |g⟩ \).

The speed up achieved in the Deutsch algorithm over the classical algorithm is just a factor of two. However, in a generalized problem, we are going to discuss now, the speedup is much more than in the classical case showing the power of the quantum computation. Let us examine now the Deutsch-Jozsa algorithm to solve the generalized Deutsch problem. Let \( F : \{0, 1\}^n \to \{0, 1\} \) be a Boolean function of a \( n \)-bit integer argument and we assume allow only those \( f \) that are either constant or yield 0 for exactly half of the arguments and 1 for the rest, that is, it is balanced. Given an oracle that evaluates the function for a given argument, the problem is to decide if \( F \) is constant or balanced. There are \( 2^n \) possible arguments and to solve the problem classically we will have to calculate the function for \( 2^n + 1 \) arguments in the worst case. Then, we see that the computational resources required to solve the problem grows exponentially with the bit size \( n \) of the input. The Deutsch-Jozsa algorithm can however solve this problem very easily. This algorithm makes use of a quantum \( f−gate \) that is a generalization of the one we used above in the Deutsch algorithm. The action in the computational basis is similar to that of the ordinary \( F−gate \), except that \( |X⟩ \) here is a computational basis state of a \( n \)-bit register. If the bottom qubit is in the state \( \frac{1}{\sqrt{2}}(|0⟩ − |1⟩) \), then the state of the upper register is transformed according to

\[ |X⟩ → (-1)^{F(X)}|X⟩. \] (2.29)

The quantum circuit of the Deutsch-Jozsa algorithm is shown in Fig. 4. The upper input is \( |0⟩|0⟩...|0⟩ \) (with \( n \) factors) and \( H^\otimes n = H \otimes H \otimes ... \otimes H \). Then, the effect of the first Hadamard gates on the top input state is

\[ H^\otimes n |0⟩|0⟩...|0⟩ = \frac{1}{\sqrt{2^n}} \sum_{X=0}^{2^n-1} |X⟩. \] (2.30)

The Hadamard gate acting on the bottom input state \( |1⟩ \) produces \( \frac{1}{\sqrt{2}}(|0⟩ − |1⟩) \). Thus the \( f−gate \) changes the state of the register to

\[ \frac{1}{\sqrt{2^n}} \sum_{X=0}^{2^n-1} (-1)^{F(X)}|X⟩. \] (2.31)

Now notice that the action of the Hadamard gates on a computational basis state is given by

\[ H^\otimes n |X⟩ = \frac{1}{\sqrt{2^n}} \sum_{Y=0}^{2^n-1} (-1)^{X.Y} |Y⟩. \] (2.32)

where \( X.Y \) is the bitwise scalar product where for \( X = x_{n−1}...x_1x_0 \) and \( Y = y_{n−1}...y_1y_0 \) we have \( X.Y = \oplus_{i=0}^{n−1} x_iy_i \) and \( \oplus \) is addition mod 2. Then, the state of the register will be

\[ \frac{1}{\sqrt{2^n}} \sum_{X=0}^{2^n-1} (-1)^{f(X)} H^\otimes n |X⟩ = \frac{1}{2^n} \sum_{X=0}^{2^n-1} \sum_{Y=0}^{2^n-1} (-1)^{F(X)+X.Y} |Y⟩. \] (2.33)

Now, the amplitude of \( |Y = 0⟩ = |0⟩|0⟩...|0⟩ \) is \( \frac{1}{2^n} \sum_{X=0}^{2^n-1} (-1)^{f(X)} \) and if \( F \) is constant this gives us \( ±1 \). On the other hand, if \( f \) is balanced one half of the terms in the sum cancels exactly the other
half and the result is 0. Then, the probability of observing $|Y = 0 \rangle = |0\rangle |0\rangle ... |0\rangle$ is 1 if $F$ is constant and is 0 if it is balanced.

Let us see now how we can implement the Deutsch-Jozsa algorithm experimentally. Considering atoms $A_1, A_2, ..., A_n$ prepared in a state like $|2.10\rangle$. After they fly through $C$ prepared in state $|2.11\rangle$, taking into account (2.12) for $\varphi = \pi$ we have

$$|\psi\rangle_{A1-C} = \frac{1}{\sqrt{2}} (-|e_1\rangle + |f_1\rangle) \frac{1}{\sqrt{2}} (-|e_2\rangle + |f_2\rangle) ... \frac{1}{\sqrt{2}} (-|e_n\rangle + |f_n\rangle) \frac{1}{\sqrt{2}} (|0\rangle_C - (-1)^n|1\rangle_C). \quad (2.34)$$

If $\varphi = 2\pi$ we have

$$|\psi\rangle_{A1-C} = \frac{1}{\sqrt{2}} (|e_1\rangle + |f_1\rangle) \frac{1}{\sqrt{2}} (|e_2\rangle + |f_2\rangle) ... \frac{1}{\sqrt{2}} (|e_n\rangle + |f_n\rangle) \frac{1}{\sqrt{2}} (|0\rangle_C - |1\rangle_C). \quad (2.35)$$

If we assume atoms $A_1, A_2, ..., A_n$ prepared in a state like $|2.24\rangle$ and cavity $C$ prepared in state $|\rangle$, taking into account (2.17) for $\varphi = \pi$ we have

$$|\psi\rangle_{A1-C} = \frac{1}{\sqrt{2}} (-|f_1\rangle + |g_1\rangle) \frac{1}{\sqrt{2}} (-|f_2\rangle + |g_2\rangle) ... \frac{1}{\sqrt{2}} (-|f_n\rangle + |g_n\rangle) \frac{1}{\sqrt{2}} (|0\rangle_C - |1\rangle_C). \quad (2.36)$$

If $\varphi = 2\pi$ we have

$$|\psi\rangle_{A1-C} = \frac{1}{\sqrt{2}} (|f_1\rangle + |g_1\rangle) \frac{1}{\sqrt{2}} (|f_2\rangle + |g_2\rangle) ... \frac{1}{\sqrt{2}} (|f_n\rangle + |g_n\rangle) \frac{1}{\sqrt{2}} (|0\rangle_C - |1\rangle_C). \quad (2.37)$$

Now, if we use the notation (2.15) and (2.27) we can rewrite (2.34), (2.35), (2.36) and (2.37) as

$$|\psi\rangle_{A1-C} = \pm \frac{1}{\sqrt{2^n}} \sum_{X=0}^{2^n-1} (-1)^{F(X)} |X\rangle_A |\psi\rangle_C \quad (2.38)$$

where, in the case of (2.34) and (2.35), if $F$ is balanced,

$$|\psi\rangle_C = \frac{1}{\sqrt{2}} (|0\rangle_C - (-1)^n|1\rangle_C), \quad (2.39)$$

and if $f$ is constant,

$$|\psi\rangle_C = \frac{1}{\sqrt{2}} (|0\rangle_C - |1\rangle_C). \quad (2.40)$$

In the case of (2.36) and (2.37),

$$|\psi\rangle_C = \frac{1}{\sqrt{2}} (|\alpha\rangle - | - \alpha\rangle) \equiv \frac{1}{\sqrt{2}} (|0\rangle_C - |1\rangle_C). \quad (2.41)$$

Now notice that the action of the $H$ gates on a computational basis state is given by

$$H^{\otimes n} |X\rangle_A = \frac{1}{\sqrt{2^n}} \sum_{Y=0}^{2^n-1} (-1)^{X.Y} |Y\rangle_A. \quad (2.42)$$

Then, the state of the register will be

$$\frac{1}{\sqrt{2^n}} \sum_{X=0}^{2^n-1} (-1)^{F(X)} H^{\otimes n} |X\rangle_A = \frac{1}{2^n} \sum_{X=0}^{2^n-1} \sum_{Y=0}^{2^n-1} (-1)^{F(X)+X.Y} |Y\rangle_A. \quad (2.43)$$
Now, the amplitude of $|0\rangle_A = |0\rangle_{A1}|0\rangle_{A2}...|0\rangle_{An}$ is $\frac{1}{2^n} \sum_{X=0}^{2^n-1} (-1)^{F(X)}$ and if $F$ is constant this gives us $\pm 1$. On the other hand, if $f$ is balanced one half of the terms in the sum cancels exactly the other half and the result is $0$. Then, the probability of observing $|0\rangle_A = |0\rangle_{A1}|0\rangle_{A2}...|0\rangle_{An}$ is $1$ if $F$ is constant and is $0$ if it is balanced. Therefore, if we detect $|0\rangle_{A1}|0\rangle_{A2}...|0\rangle_{An}$ then $F$ is constant. Concluding, a single call to the quantum oracle followed by a measure of the register and checking if the result is $|0\rangle_{A1}|0\rangle_{A2}...|0\rangle_{An}$ allows us to decide if the function is constant or balanced and we have achieved an exponential speedup over the classical computation. Finally, let us analyze the feasibility of the experimental implementation of the above algorithms. Considering Rydberg atoms of principal quantum numbers 50 or 51, the radiative time is of the order of $10^{-2}$ s and the coupling constant $g$ is of the order of $2\pi \times 25$ kHz [15, 16, 17] and the detuning. $\Delta$ is of the order of $2\pi \times 100$ kHz. Taking into account that $\varphi = g^2 \tau / \Delta$, for $\varphi = \pi$ we have an interaction time $\tau = 8 \times 10^{-5}$ s and we could, in principle, assume a time of the order of $10^{-4}$ s to realize the algorithm which is much shorter than the radiative time. We have to consider also the cavity decay time which in recent experiments, with niobium superconducting cavities at very low temperature and quality factors in the $10^9 - 10^{10}$ range, have a cavity energy damping time of the order of 10 to 100 ms, and which could be larger than the required time to perform the algorithm.

**Figure Captions**

**Fig. 1** - Quantum circuit of the Deutsch algorithm.

**Fig. 2** - Energy states scheme of a two-level atom where $|e\rangle$ is the upper state with atomic frequency $\omega_e$, $|f\rangle$ is the intermediate state with atomic frequency $\omega_f$, and $\omega$ is the cavity field frequency and $\Delta = (\omega_e - \omega_f) - \omega$ is the detuning. The transition $|f\rangle \xrightarrow{\varphi} |e\rangle$ is far enough of resonance with the cavity central frequency such that only virtual transitions occur between these levels.

**Fig. 3** - Energy states scheme of a three-level atom where $|e\rangle$ is the upper state with atomic frequency $\omega_e$, $|f\rangle$ is the intermediate state with atomic frequency $\omega_f$, $|g\rangle$ is the lower state with atomic frequency $\omega_g$ and $\omega$ is the cavity field frequency and $\Delta = (\omega_e - \omega_f) - \omega$ is the detuning. The transition $|f\rangle \xrightarrow{\varphi} |e\rangle$ is far enough of resonance with the cavity central frequency such that only virtual transitions occur between these levels (only these states interact with field in cavity $C$). In addition we assume that the transition $|e\rangle \xrightarrow{\varphi} |g\rangle$ is highly detuned from the cavity frequency so that there will be no coupling with the cavity field in $C$.

**Fig. 4** - Quantum circuit of the Deutsch-Jozsa algorithm.

**References**

[1] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information*, Cambridge Univ. Press, Cambridge, 2000.

[2] G. Chen and R. K. Brylinski Eds., *Mathematics of Quantum Computation*, Chapman & Hall/CRC, London, 2002.

[3] D. Bouwmeester, A. Ekert and A. Zeilinger (Eds.), *The Physics of Quantum Information: Quantum Cryptography, Quantum Teleportation, Quantum Computation*, Springer, New York, 2000.
[4] A. O. Pittenger, *An Introduction to Quantum Computing Algorithms*, Birkhäuser, Boston, 1999.

[5] P. Benioff, J. Stat. Physist 22, 563 (1980); Phys. Rev. Lett. 48, 1581 (1982).

[6] R. Feynmann, Int. F. Theor. Phys. 21, 467 (1982); Found. of Phys. 16, 507 (1986).

[7] D. Deutsch, Proc. R. Soc. London Scr. A 400, 97 (1985); D. Deutsch, Proc. R. Soc. London Scr. A 425, 73 (1989).

[8] D. Deutsch and R. Jozsa, Proc. R. Soc. London Scr. A 439, 553 (1992).

[9] D. Simon, Proc. 35th Annual Symposium on Foundations of Computer Science, IEEE Computer Society Press, Los Alamitos, CA, 116 (1994).

[10] P. Shor, SIAM J. Comput. 26, 1484 (1997).

[11] L. Grover, Proc. 28th Annual ACM Symposium on Theory of Computing, ACM Press, New York, 212 (1996).

[12] S-B Zheng, Phys. Rev. A 70, 034301 (2004).

[13] M. Orszag, *Quantum Optics*, Springer-Verlag, Berlin, 2000.

[14] M. J. Holland, D. F. Walls and P. Zoller, Phys. Rev. Lett. 67, 1716 (1991); G. J. Milburn and M. J. Gagen, Phys. Rev. A 46, 1578 (1992).

[15] A. Rauschenbeutel, G. Nogues, S. Osnaghi, P. Bertet, M. Brune, J. M. Raimond, and S. Haroche, Phys. Rev. Lett. 83, 5166 (1999).

[16] A. Rauschenbeutel, G. Nogues, S. Osnaghi, P. Bertet, M. Brune, J. M. Raimond, and S. Haroche, Science 288, 2024 (2000).

[17] P. Bertet, A. Auffeves, P. Maioli, S. Osnaghi, T. Meunier, M. Brune, J. M. Raimond, and S. Haroche, Phys. Rev. Lett. 89, 200402 (2002).
Fig. 1 - E. S. Guerra

\[
\begin{align*}
|X\rangle & \quad |X\rangle \\
|Y\rangle & \quad |Y\oplus F(X)\rangle
\end{align*}
\]
Fig. 2 - E. S. Guerra
Fig. 3 - E. S. Guerra

Δ

|e⟩

Δ

|f⟩

ω

|g⟩
Fig. 4 - E. S. Guerra

\[ |x\rangle \quad H^\otimes n \quad H^\otimes n \]

\[ |1\rangle \quad H \quad F \]