Cosmological framework for renormalization
group extended gravity at the action level

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Abstract. Since renormalization group (RG) effects may lead to corrections to general
relativity at cosmological distances, we develop further the approach in which RG effects
are fully encoded in a classical effective action (including the scale settings, which enter as
constraints) and apply it to cosmology. The framework is presented up to the first order
cosmological perturbations and it is such that the RG effects cannot change the background
cosmology, only the dynamics of the perturbations. An arbitrary number of RG scales is con-
sidered, but for applying to cosmology we find that two scales are sufficient. The evaluations
are done without fixing specific RG functions (β-functions and RG-scale settings), only their
dependencies with assumptions on analyticity. The emphasis here is on analytical results
and qualitative understanding of the implied cosmology. In particular, we parametrize the
equations for the cosmological scalar potentials by the gravitational slip and other commonly
employed functions to parametrize modified or extended gravity. Observational constraints
are derived and possible impacts on the interpretation of dark matter and dark energy are
discussed.

Keywords: General relativity extensions; cosmology; renormalization group.
1 Introduction

Gravity is peculiar among the fundamental interactions. Although a complete and consistent theory for quantum gravity is still unknown, quantum field theory in curved spacetime (QFTCS) has lead to some important results, like Hawking radiation and the spectrum of primordial cosmological perturbations. Within QFTCS, the Einstein-Hilbert action needs to be supplemented by higher derivative terms, in order for properly quantizing the matter sector. These high derivative terms are dynamically relevant at small distance scales, but their importance decreases as one moves towards larges scales. It is remarkable that the couplings of the high derivative terms can be shown to have trivial renormalization group (RG) flows in the infrared, in the sense that their $\beta$-functions become zero, and hence at large distances they simply behave as constants $[1, 2]$. This behaviour is similar to the quantum electrodynamics (QED) case, where in the infrared limit the coupling can be shown to become a constant (see e.g., ref. [3] for a derivation from fundamental principles and ref. [4] for an example on the measurement of the effective running of the coupling). However, the two other couplings, $G$ and $\Lambda$ can run in the far infrared. Such case is a possibility both in the framework of QFTCS and the quantum gravity context $[5-9]$. If indeed $G$ and $\Lambda$ run at large distance scales, this could be an important window towards understanding gravity beyond GR. Here we elaborate further on the framework and cosmological consequences of this possibility.

Renormalization group effects in gravity at large distances and their consequences are being pursued from different approaches, including QFTCS and asymptotic safety quantum
gravity [9–16]. The approach that we continue to develop here uses RG principles as the basis for providing GR corrections at large distances. The relevant fields and their interactions are suggested from the RG context, while the complete picture is either impossible or very unnatural to be achieved from other well known GR extensions (like scalar-tensor, $f(R)$, scalar-vector-tensor or bigravity). A central hypothesis is that, at large distances, there must exist an effective description that is fully in the classical framework. In particular, there must be a complete classical action capable of effectively describing the full large scale RG effects.\footnote{It is an extension of the improved action approach described in ref. [10]. All the information is put in a classical action, including the meaning of the RG scale. See also ref. [17].} Another well studied possibility is implementing the RG effects at the level of the GR field equations (see e.g., [8, 10, 18, 19]). For the latter case, a full classical action is not considered, and it may even fail to exist. A second important consideration of ours is that we explore the RG application with respect to certain spacetime background which is fixed from the physical context. Thus, the perturbations of a given spacetime are subjected to RG effects, but there are no consequences to the background. For cosmology the most natural background to consider is the cosmological one (given by the Friedmann-Lemaître-Robertson-Walker spacetime).\footnote{Perhaps it is worth recalling that one of the greatest success of QFTCS relies on the origin of the cosmological perturbations, which come from quantum effects on the perturbations, not on the cosmological background.} Therefore, the cosmological background in this work is the same one of the standard cosmology, the differences rely in the perturbations. We do not know of other works in this context that use these two considerations above.

Here, considering the application to cosmology, we extend the approach of ref. [14] (which in turn extends the approaches of refs. [10, 11, 20–22]). As detailed in this work, the application of [14] to cosmology is not immediate. The application is straightforward for systems in which either the Ricci scalar is constant at background level, or if a single coordinate is sufficient to describe the full spacetime metric (e.g., a static and spherically symmetric spacetime). These conditions are not satisfied in perturbative cosmology, which will lead to the need for a second RG scale.

This work is organized as follows: in the next section we review and extend the action presented in [14]. The extension comes from considering an arbitrary number of RG scales, instead of a single one. In section 3 the field equations are studied and two RG scales are specified. Section 4 considers cosmological constraints and discuss the consequences. Since the present work has the main purpose of establishing a consistent framework, detailed and recent observational data will be considered in a future work. At last, in section 5, we present our conclusions and perspectives.

2 Infrared renormalization group effects in gravity at the action level

Here we briefly review the approach developed in ref. [14], which is specially based on refs. [10, 11, 20, 21]. Many approaches to nontrivial renormalization group flows for gravity on the largest scales use information appended to a given action, in ref. [14] we have presented a way to fully implement such effects directly in the action (see also [17] for a related approach). To put all the relevant information in the action is important for understanding the system dynamics and symmetries (which are effectively classical at large scales). By appending additional equations to those that come from an action, one is giving up on using the action as a fundamental principle for the theory, opening the way for possible hidden issues that
are straightforwardly absent in theories that are fully developed from an action principle. For instance, the additional equations may break symmetries present in the action, and they may introduce constraints that are not compatible with the dynamics from the action.\footnote{We recall that the proper way to deal with constraints is with Lagrange multipliers, to simply append constraints at the level of the field equations is in general not consistent.}

In refs. [10, 11, 20, 21, 23, 24], it is argued in favour of the following action capable of enclosing the large scale Renormalization Group effects for gravity,

\[
S[g] = \frac{1}{16\pi} \int \frac{R - 2\Lambda}{G} \sqrt{-g} d^4x. \tag{2.1}
\]

In the above, \(G\) and \(\Lambda\) are not constants, they are external scalar fields (that is, no variation with respect to either \(G\) or \(\Lambda\) should be considered in this action), whose running is determined from \(\beta\)-functions. Clearly, although this simple action has some interesting properties (e.g., [10, 11, 21]), not all relevant physical information is included in it. The dependences of \(G\) and \(\Lambda\) on the RG scale are not explicit, also the physical meaning of the RG scale (the scale setting) is not in this action, these informations are appended at the level of the field equations.

To achieve scale setting at the action level, and without recourse to external scalar fields, we use [14],

\[
S[g, \mu, \lambda, \Psi] = \int \left[ \frac{R - 2\Lambda(\mu)}{16\pi G(\mu)} + \lambda [\mu - f(g, \Psi)] \right] \sqrt{-g} d^4x + S_{\text{matter}}[g, \Psi]. \tag{2.2}
\]

In the above, \(G\) and \(\Lambda\) are not external fields, both depend on the RG scale \(\mu\), and the latter is seen as a fundamental field (i.e., one of those that the action should be varied with respect to it). It should be noted that \(\mu\) only enters in the action (2.2) as an auxiliary field: it can be completely removed by solving \(\mu - f(g, \Psi) = 0\), as detailed in Appendix A of ref. [14]. We remark that this is also in agreement with the RG framework, in the sense that the RG scale must not be a new independent field with its own dynamics. See also ref. [17] for similar arguments.

Fixing the dependence of \(\Lambda\) and \(G\) on \(\mu\) corresponds to fixing their \(\beta\)-functions. In general the action above imposes a relation between these two \(\beta\)-functions [14] (see also [10, 11]), this relation is indirectly related to diffeomorphism invariance and energy-momentum conservation [14]. If for one of them the \(\beta\)-function is settled considering natural arguments from the RG group, for the other it is found from the field equations. To make explicit this distinction, we introduced \(F\{\phi\}\) for a dependence of \(F\) on a field \(\phi\) that is not fixed at the action level, but that it must be derived from the field equations. Hence, the specific dependence form of the latter case may depend on constants that characterize the matter content, and thus it is system dependent. This is in contrast to the standard use of potentials in scalar-tensor gravity, in which all of them are assumed to be system independent.

In general, RG effects need not to depend on a single scale and mutiscale RG methods can be found [25–27]. The application of the above action to some systems may demand additional scales, which are implemented as follows

\[
S[g, \mu_p, \lambda, \Psi] = \frac{1}{16\pi} \int \left[ \frac{R - 2\Lambda(\mu_p)}{G(\mu_p)} + \sum_p \lambda_p [\mu_p - f_p(g, \Psi)] \right] \sqrt{-g} d^4x + S_{\text{matter}}[g, \Psi]. \tag{2.3}
\]
The field equations are

\[ G_{\alpha\beta} + \Lambda g_{\alpha\beta} + f_{\alpha\beta} = 8\pi G T_{\alpha\beta}, \]  

(2.4)

\[ \frac{1}{16\pi} \sum_p \int \lambda'_p \frac{\delta f'_p}{\delta \Psi} \sqrt{-g'} d^4x' = \frac{\delta S_{\text{matter}}}{\delta \Psi}, \]  

(2.5)

\[ \mu_p - f_p = 0, \]  

(2.6)

\[ 2 \frac{\partial}{\partial \mu_p} \Lambda G - R \frac{\partial}{\partial \mu_p} G - \frac{1}{g} = \lambda_p, \]  

(2.7)

where a prime indicates dependence on \( x' \), instead of \( x \), and

\[ G_{\alpha\beta} \equiv G_{\alpha\beta} + G \Box^{-1} G_{\alpha\beta} - G \nabla_\alpha \nabla_\beta G^{-1}, \]  

(2.8)

\[ f_{\alpha\beta} \equiv -G \sqrt{-g} \sum_p \int \lambda'_p \frac{\delta f'_p}{\delta g^\alpha\beta} \sqrt{-g'} d^4x', \]  

(2.9)

\[ T_{\alpha\beta} \equiv -\frac{2}{\sqrt{-g}} \delta S_{\text{matter}} \delta g_{\alpha\beta}. \]  

(2.10)

From the diffeomorphism invariance of \( S_{\text{matter}} \) (e.g., [28]), \( T_{\alpha\beta} \) is not conserved in general since \( \delta S_{\text{matter}} / \delta \Psi \neq 0 \), but

\[ 0 = \delta \xi S_{\text{matter}}[g, \Psi] = \int \left( -\frac{1}{2} T_{\alpha\beta} \sqrt{-g} \nabla^\alpha \nabla^\beta + \frac{\delta S_{\text{matter}}}{\delta \Psi} \delta \xi \right) d^4x \]  

(2.11)

\[ = \int \left( \frac{1}{2} \nabla^\alpha T_{\alpha\beta} \sqrt{-g} \xi^\beta + \frac{1}{16\pi} \sum_p \int \lambda'_p \frac{\delta f'_p}{\delta \Psi} \sqrt{-g'} d^4x' \delta \xi \right) d^4x, \]  

(2.12)

where \( \delta \xi \) represents an infinitesimal change of coordinates, given by a Lie derivative along the vector \( \xi^\alpha \). Hence, any violation of the energy-momentum tensor conservation is proportional to \( \lambda_p \).

3 Cosmology

3.1 Cosmological backgrounds and perturbations

Similarly to standard cosmology, it is assumed that spacetime can be foliated and that the universe at large scales can be described by a spatially homogeneous and isotropic metric, added by non-homogenous perturbations. Only scalar perturbations are considered here, since they are the most relevant for the large scale structure. For clarity, considering that the main purpose of the present work is to establish the cosmological framework, the spatial slices are taken to be flat. Hence, the metric in the Newtonian gauge can be written as

\[ g_{00} = -a^2(\eta) \left( 1 + 2\Phi \right), \]  

(3.1)

\[ g_{ij} = a^2(\eta) \left( 1 - 2\Psi \right) \delta_{ij}, \]  

(3.2)

where \( \eta \) is the conformal time, \( a \) is the scale factor, \( \Phi \) and \( \Psi \) are the first order metric perturbations.
There are two relevant background concepts: one is the zeroth order expansion of the metric about the average cosmological homogeneous spacetime, it is denoted by \( g_{\alpha\beta}^{(0)} \) and satisfies
\[
(0)\, g_{\alpha\beta} = a^2(\eta) \eta_{\alpha\beta},
\]  
where \( \eta_{\alpha\beta} \) is the Minkowski metric. The other background concept is the RG background, denoted by \( \gamma_{\alpha\beta} \). It is this background that selects which perturbations are sensitive to the RG effects.

From the definition of the RG background, if the complete spacetime metric \( g_{\alpha\beta} \) coincides with the RG background \( \gamma_{\alpha\beta} \), there should be no RG effects. On the other hand, if
\[
\gamma_{\alpha\beta} = g_{\alpha\beta}^{(0)},
\]  
then at the background level there will be no RG induced corrections, but there will be in general RG contributions to the cosmological perturbations.

Considering eq. (3.4), for a perfect fluid with background energy density \( \epsilon_0 \) and background pressure \( p_0 \), the background equations are exactly the Friedmann equations. For convenience and latter reference, these are
\[
3H^2 - \Lambda_0 a^2 = 8\pi G_0 a^2 \epsilon_0, \tag{3.5}
\]
\[
2H' + H^2 - \Lambda_0 a^2 = -8\pi G_0 a^2 p_0, \tag{3.6}
\]
where \( H \equiv a'(\eta)/a(\eta) \). The constant \( \Lambda_0 \) is the background value of \( \Lambda \).

### 3.2 The first RG scale as a measure of the spacetime perturbations

Let the difference between the spacetime metric and the background one be given by
\[
h_{\alpha\beta} \equiv g_{\alpha\beta} - \gamma_{\alpha\beta}. \tag{3.7}
\]
The RG effects should depend on \( h_{\alpha\beta} \) trough an RG scale since. As previously considered, when \( h_{\alpha\beta} = 0 \) there should be no RG effects.

In the context of a fluid with 4-velocity \( U^\alpha \), a simple scalar choice for the first RG scale is [14]
\[
\mu_1 = f_1(U^\alpha U^\beta h_{\alpha\beta}), \tag{3.8}
\]
where \( f_1 \) is some function. The scalar \( W \equiv U^\alpha U^\beta h_{\alpha\beta} \) constitutes a measurement of the spacetime perturbations and we will consider henceforth that in a spacetime region in which \( W = 0 \) there will be no RG effects. This consideration will have impact on the boundary conditions for \( \Lambda \) (as explained in the next section), and this stricter version (instead of only stating “\( h_{\alpha\beta} = 0 \) implies no RG effects”) will be sufficient to eliminate ambiguities on the integration constants.

The coupling of the action (2.2) to the complete action of a relativistic fluid was addressed in ref. [14], where a fluid formulation in which \( U^\alpha \) is one of the fundamental fields was used [29]. Since the action dependence on \( \gamma_{\alpha\beta} \) is only through \( f_1 \), this implies, from the field equations, that \( \lambda_1 = 0 \) [14]. As commented by the end of section 3.1, if there were no other RG scales, \( \lambda_1 = 0 \) would be sufficient to guarantee energy-momentum conservation.
3.3 Vacuum solution

For vacuum, which can be pictured a fluid in the limit of zero density, a single scale (3.8) can be shown to be sufficient to find a consistent Λ expression. The demonstration below derives the same final result, but extends that from [14] in the sense that considers possible additional scales. From eq. (2.7), up to the first perturbation order and using that \( R = 4\Lambda_0 \),

\[
\frac{\partial}{\partial \mu_1} \Lambda = \Lambda_0 G_0 \frac{\partial}{\partial \mu_1} G^{-1},
\]

(3.9)

\[
\frac{\partial}{\partial \mu_{\hat{p}}} \Lambda = \Lambda_0 G_0 \frac{\partial}{\partial \mu_{\hat{p}}} G^{-1} + \frac{1}{2} G_0 \lambda_{\hat{p}},
\]

(3.10)

where \( \hat{p} = 2, 3, 4, ... \), and the constants \( G_0 \) and \( \Lambda_0 \) are the values of \( G \) and \( \Lambda \) respectively at background level.

The general solution of eq. (3.9) reads

\[
\Lambda = \Lambda_0 G_0 G^{-1},
\]

(3.11)

where we used

\[
G|_{W=0} = G_0 \quad \text{and} \quad \Lambda|_{W=0} = \Lambda_0
\]

(3.12)

Equation (3.12) is a boundary condition for \( \Lambda \), as commented in section 3.2.4.

From eqs. (3.10, 3.11), one sees that \( \lambda_{\hat{p}} = 0 \), for all \( \hat{p} \), independently on whether \( G \) depends on a second RG scale or not. Consequently, the vacuum solution becomes a Brans-Dicke theory whose scalar field is given by \( \phi_{\text{BD}} = G^{-1} \), with Brans-Dicke \( \omega_{\text{BD}} \) parameter null and with a potential given by \( V_{\text{BD}} \propto \phi_{\text{BD}}^2 \), which is the same vacuum solution described in [14]. The solution (3.11) is the same that was found in ref. [14], but here we extended its derivation considering that there may be more than one RG scale.

We stress that we are considering requirements for achieving a consistent classical picture, even though the underlying fundamental reason for the running of \( G \) and \( \Lambda \) is not classical. For instance, in the present approach one needs not to consider whether there is or there is not an infrared fixed point in the RG flow, as assumed in refs. [30, 31]. Nonetheless, we note that in those references the relation between the \( \Lambda \) and \( G \) is also given by eq. (3.11).

3.4 Field equations and the framework for perturbative cosmology

From eqs. (2.7, 3.8) up to the first perturbation order, one can write an extension of eqs. (3.9, 3.10),

\[
\partial_{\mu_1} \Lambda = \lambda_{\hat{p}} G_0 \partial_{\mu_1} G^{-1},
\]

(3.13)

\[
\partial_{\mu_{\hat{p}}} \Lambda = \lambda_0 G_0 \partial_{\mu_{\hat{p}}} G^{-1} + \frac{1}{2} G_0 \lambda_{\hat{p}},
\]

(3.14)

where we introduced the (background) scalar

\[
\lambda_0 = \frac{R}{2} - \Lambda_0 = \Lambda_0 - 4\pi G_0 \frac{(0)}{T}.
\]

(3.15)

4One can verify that the less strict boundary condition \( \Lambda|_{h_{\alpha\beta}=0} = \Lambda_0 \) leaves an undetermined \( C(\mu_{\hat{p}}) \) function.
To find a solution for $\Lambda(\mu_1)$ from the above equations, it must be possible to express $\lambda_0$ as a function of the RG scales. Since $\lambda_0$ is a function of $\eta$ alone, while $\mu_1$ depends in general on all the spacetime coordinates, from eq. (3.13) there is no general solution of the form $\Lambda(\mu_1)$, and a new scale (at least one more) becomes necessary for finding a consistent $\Lambda$ solution. There is not a unique possibility, but we will proceed with the simplest one capable of leading to nontrivial results. We will consider that $\lambda_0$ is independent from $\mu_1$ and that $G$ only depends on the first RG scale. Then, from eq. (3.13) one finds

$$\Lambda = \Lambda_0 + \lambda_0(G_0G^{-1} - 1) = \Lambda_0 + \lambda_0\delta_G,$$  

(3.16)

where the boundary condition (3.12) was used and

$$\delta_G \equiv G_0G^{-1} - 1.$$  

(3.17)

Inserting this result into (3.14),

$$\lambda_\hat{p} = 2G^{-1}\frac{\partial\lambda_0}{\partial\mu_\hat{p}}.$$  

(3.18)

We remark that the equations above are second order field equations, unless one introduces some RG scale that depends on higher order derivatives. We will not consider such introduction.

If $\mu_\hat{p}$ (or $f_\hat{p}$) does not depend on $\Psi$, the left hand side of eq. (3.20) is zero, thus leading to the standard field equations for the matter fields. And if $\mu_\hat{p}$ also does not introduce higher order derivative terms, then the single possibility for $\mu_\hat{p}$ is $\mu_\hat{p} = f_\hat{p}(R)$. As detailed in Appendix A, in this case eq. (3.19) becomes identical to Einstein field equation, and hence this is a class of scale settings in which there is no difference from GR. On the other hand, for the case in which $\mu_\hat{p}$ does not depend on the metric, the integral term in eq. (3.19) becomes zero, and the field equations are not the same of GR.

Considering the previous results, and looking for the simplest cases that extend GR, we consider henceforth the following framework in the presence of a fluid with 4-velocity $U^\alpha$:

1. One of the RG scales is named $\mu_1$ and it is a function of the scalar

$$W \equiv U^\alpha U^\beta h_{\alpha\beta}.$$  

(3.21)

In a comoving frame ($U^i = 0$) and up to first order, $W \equiv -2U^0U^0a^2\Phi \approx 2\Phi$.

2. Neither the $\beta$-function of $G$ nor the relation between $W$ and $\mu_1$ need to be explicitly specified, it is only considered that

$$G = G(\mu_1(W)) = G_0 + G_0\nu W + O(W^2) \approx G_0 + 2G_0\nu \Phi + O(\Phi^2).$$  

(3.22)
3. Other RG scales \((\mu_p)\) do not depend explicitly on the spacetime metric.\(^5\)

4. Perturbative orders are counted with respect to the \(h_{\alpha\beta}\) order. It is not assumed that the value of \(\nu\) is small before evaluating the physical bounds.

5. The explicit expression of \(\Lambda\) as a function of all the RG scales \((\mu_p)\) should be uncovered from the field equations.\(^6\)

6. Matter is assumed to be described by a fluid-like \(T_{\alpha\beta}\), namely

\[
T_{\alpha\beta} = U^\alpha U^\beta \epsilon + (U^\alpha U^\beta + g^{\alpha\beta}) p.
\]

(3.23)

The full conservation of \(T_{\alpha\beta}\) is not assumed, but the conservation of \(T_{\alpha\beta}\) is guaranteed to hold at background level.\(^7\)

3.5 Equations and solutions for the scalar perturbations

From section 3.4 considerations, the field equations for the scalar perturbations are

\[
3\mathcal{H}(\Psi' + \nu \Phi') - \nabla^2(\Psi + \nu \Phi) + 3\mathcal{H}^2 \Phi + \frac{(1) a^2}{2} = 4\pi G_0 a^2 (T_0^0 + 2\nu \Phi T_0^0),
\]

(3.24)

\[
\partial_i \left[ \Psi' + \nu \Phi' + \mathcal{H} (1 - \nu) \right] = -4\pi G_0 a^2 T_0^i,
\]

(3.25)

\[
\left( \Psi'' + \nu \Psi'' + \mathcal{H} (\Phi' + 2\Psi' + \nu \Phi') + \frac{1}{2} \nabla^2 (\Phi - \Psi - 2\nu \Phi) + \Phi \left( 2\mathcal{H}' + \mathcal{H}^2 \right) + \frac{(1) a^2}{2} \right) \delta^i_j -
\]

\[- \frac{1}{2} \partial_j \partial^i (\Phi - \Psi - 2\nu \Phi) = 4\pi G_0 a^2 (T_j^i + 2\nu \Phi T_j^i).
\]

(3.26)

In the above and henceforth, a prime (‘) denotes derivative with respect to the conformal time \(\eta\).

From the non-diagonal part of eq. (3.26), one infers the gravitational slip parameter (in a comoving frame) as \([32, 33]\)

\[
\frac{\Psi}{\Phi} = 1 - 2\nu.
\]

(3.27)

Using this result, the field equations can be written in the Fourier space as

\[
3\mathcal{H} (1 - \nu) \Phi' + 3\mathcal{H}^2 \Phi + 8\pi G_0 \epsilon_0 a^2 \nu \Phi + k^2 (1 - \nu) \Phi + \frac{(1) a^2}{2} = -4\pi G_0 a^2 \delta \epsilon,
\]

(3.28)

\[
(\Phi' + \mathcal{H} \Phi) (1 - \nu) = 4\pi G_0 a^2 (\epsilon_0 + p_0) \frac{\theta}{k^2},
\]

(3.29)

\[
(1 - \nu) \Phi'' + 3\mathcal{H} (1 - \nu) \Phi' + (2\mathcal{H}' + \mathcal{H}^2) \Phi - 8\pi G_0 a^2 p_0 \nu \Phi + \frac{(1) a^2}{2} = 4\pi G_0 a^2 \delta p.
\]

(3.30)

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\(^5\)This avoids the hidden GR limit, see Appendix A, and the integral term in eq. (3.19) becomes zero.

\(^6\)For first order perturbative cosmology, eq. (3.16) can be used.

\(^7\)This since the integral term in eq. (3.20) is not zero in general.
In the above, eq. (3.29) is the divergence of eq. (3.25) and it was introduced $\theta \equiv \partial^iv_i$. These equations also show that $\nu = 1$ is a very special case, as it will be further detailed in the next sections.

The $\Lambda$ expression comes from eq. (3.16), and it reads

$$\Lambda = \left( \Lambda_0 - \frac{R}{2} \right) 2\nu \Phi.$$

(3.31)

The perturbative solution for a universe with $\Lambda_0 = 0$ and with either dust or radiation can be presented in analytical form. From eqs. (3.28, 3.30) with $p = we$ and constant $w$, one finds

$$\left[ \Phi'' + 3\mathcal{H}(1+w)\Phi' \right] (1-\nu) + \left[ w k^2 (1-\nu) + (1+3w)\mathcal{H}'^2 + 2\mathcal{H}' + \right. \left. (1+w)\nu \Lambda_0 a^2 - 3\nu (1+w)(\mathcal{H}' + \mathcal{H}^2) \right] \Phi = 0.$$

(3.32)

One can directly check that for $\omega = 1/3$ and negligible $\Lambda_0$ there will be no RG effects on $\Phi$, that is, for radiation fluid the solution for $\Phi$ is the same of GR. The $\Psi$ solution will also be equal to the GR solution, apart from a constant factor, which comes from the slip parameter. Hence one should expect that the RG effects for a radiation fluid will be mild ones, in particular it will be shown that the lens potential $\Sigma$ satisfies $\Sigma = 1$ in subhorizon scales.

For the case of a universe with dust only, where $w = 0$ and $\Lambda_0 = 0$, eq. (3.32) becomes

$$\left( \Phi'' + 3\mathcal{H}\Phi' \right) (1-\nu) - \frac{3}{2} \nu \mathcal{H}'^2 \Phi = 0,$$

(3.33)

where it was used that $\mathcal{H}' = -\frac{1}{2} \mathcal{H}^2$. The solution reads

$$\Phi(\eta) = C_1 \eta^{-\frac{5}{2}} \left( 1 + \sqrt{1 + \frac{24}{25} \frac{\nu}{1-\nu}} \right) + C_2 \eta^{-\frac{5}{2}} \left( 1 + \sqrt{1 + \frac{24}{25} \frac{\nu}{1-\nu}} \right),$$

(3.34)

where $C_1$ and $C_2$ are integration constants with respect to $\eta$, they depend on the wavenumber $k$. The GR solution (i.e., $\Phi = C_1 + C_2 \eta^{-5}$) is reproduced in the limit $\nu \rightarrow 0$.

Considering an expansion on $\nu$ up to its first order, and neglecting the decreasing terms, eq. (3.34) becomes specially simple

$$\Phi(\eta) \approx C_1 \left( 1 + \frac{6}{5} \nu \ln \eta \right).$$

(3.35)

Hence, for a universe that is dust dominated, the first nontrivial RG correction is the introduction of a logarithm time dependence in the Newtonian potential, contrasting to the GR case of constant Newtonian potential. To summarize, for negligible $\Lambda_0$, in a radiation dominated universe ($T = 0$) there are no RG corrections on $\Phi$, while for a dust dominated universe the RG effects add a logarithm time evolution on $\Phi$. These results do not depend on the RG scales beyond the first one.

Before proceeding to observational results, we address the second RG scale and the matter field equations.
3.6 The second RG scale and the matter field equations

From diffeomorphism invariance of the complete action, we already know some necessary properties of the matter field equations (2.12, 3.20). Actually, there are many possible choices of the second RG scale that lead to the same dynamics.

Considering the relativistic perfect fluid formulation of ref. [29], and using a notation closely related to that of ref. [14], its action reads

\[
S_{\text{fluid}} = \int \left[ -\epsilon(n, s) + \eta_1 (1 + U^\alpha U_\alpha) + \eta_2 \nabla_\alpha (n U^\alpha) + \eta_3 U^\alpha \nabla_\alpha X + \eta_4 U^\alpha \nabla_\alpha s \right] \sqrt{-g} \, d^4x. \tag{3.36}
\]

In the above, \( S_{\text{fluid}} = S_{\text{fluid}}[g, U, n, s, \eta_1, X] \), \( n \) is the fluid mass density, \( s \) the rest specific entropy, \( \eta_m \) stands for the four Lagrange multipliers and \( \epsilon(n, s) \) is the energy density. The quantity \( X \) is relevant for the description of fluids with rotational flow [29].

From the above fluid formulation, the most simple and natural choice is a function of the fluid mass density \( n \),

\[
\mu_2 = f_2(n). \tag{3.37}
\]

There is no need to specify a particular \( f_2 \) function, since, from eq. (3.20), one sees that the precise form of \( f_2 \) is not relevant, as long as it is a differentiable function that depends only on \( n \).

As shown explicitly in ref. [14], \( \mu_1 \) introduces an a priori change on the fluid equations since \( \mu_1 \) depends on the quadrivelocity \( U^\alpha \), but the complete field equations are such that \( \lambda_1 = 0 \), leading in the end to no change on the fluid dynamics due to \( \mu_1 \). On the other hand, \( \mu_2 \) will have an impact on the field equations, as developed below. From eq. (3.20), one finds that

\[
2\eta_1 = -nU^\alpha \partial_\alpha \eta_2 = n\partial_n \epsilon - \frac{1}{2} \delta_G n \partial_n (T), \tag{3.38}
\]

\[
T_{\alpha\beta} = 2\eta_1 U_\alpha U_\beta + g_{\alpha\beta} (-nU^\alpha \partial_\alpha \eta_2 - \epsilon)
\]

\[
= (n\partial_n \epsilon - \frac{1}{2} \delta_G n \partial_n (T)) U_\alpha U_\beta + g_{\alpha\beta} (n\partial_n \epsilon - \frac{1}{2} \delta_G n \partial_n (T) - \epsilon) \tag{3.39}
\]

\[
= (\epsilon_{\text{eff}} + p_{\text{eff}}) U_\alpha U_\beta + g_{\alpha\beta} p_{\text{eff}}, \tag{3.40}
\]

where

\[
p_{\text{eff}} = p + \frac{1}{2} \delta_G n \partial_n (\epsilon_0 - 3p_0), \tag{3.41}
\]

\[
\epsilon_{\text{eff}} = \epsilon, \tag{3.42}
\]

with \( p = n\partial_n \epsilon - \epsilon \). Both \( p \) and \( \epsilon \) are identical to the pressure and energy density used within GR at background level, while at first order the correspondence is not valid in general.

From the diffeomorphism invariance of the complete action (2.12),

\[
\nabla^\alpha T_{\alpha\beta} = \frac{1}{4\pi G_0} \delta_G \frac{\partial \lambda_0}{\partial \beta n} \partial_n \beta n
\]

\[
= \delta_G \frac{\partial T}{\partial n} \partial_\beta n, \tag{3.43}
\]

\[ -10 - \]
where we used $\delta \xi n = \xi^\alpha \partial_\alpha n$ and eq. (3.18). As an example, for a dust fluid at the background level (which implies $\epsilon_0 \propto n^\gamma$), the above expression becomes, up to first order,

$$\nabla^\alpha T_{\alpha \beta} = - \delta G \partial_\beta \epsilon_0. \quad (3.44)$$

In general, and up to the first order, one can always write

$$\nabla_\alpha T^{\alpha \beta} = Q^\beta, \quad (3.45)$$

where $Q^\beta$ is a first order quantity whose single non-null component is the zeroth one.

In a frame that is comoving with the fluid at background level, the fluid equations can be written in a form that is independent from $Q^\beta$, up to the first order, as we show below.

The previous equation, for an effective perfect fluid, can be written as

$$\nabla_\alpha \left( (\epsilon_{\text{eff}} + p_{\text{eff}}) U^\alpha + g^{\alpha \beta} p_{\text{eff}} \right) = Q^\beta. \quad (3.46)$$

Multiplying by $U_\beta$,

$$- \nabla_\alpha [(\epsilon_{\text{eff}} + p_{\text{eff}}) U^\alpha] + \frac{Dp_{\text{eff}}}{D\tau} = U_\beta Q^\beta = U_0 Q^0, \quad (3.47)$$

where, for any quantity $X$, $DX/D\tau \equiv U^\alpha \nabla_\alpha X$. Inserting this result into eq. (3.46),

$$U_\beta \frac{Dp_{\text{eff}}}{D\tau} + (\epsilon_{\text{eff}} + p_{\text{eff}}) \frac{DU^\beta}{D\tau} + \nabla^\beta p_{\text{eff}} = Q^\beta + U^\beta U_0 Q^0. \quad (3.48)$$

The above equation is the same one that can be found from GR for a fluid with energy density $\epsilon_{\text{eff}}$ and pressure $p_{\text{eff}}$, apart from the limit $Q^\beta \to 0$. However, the previous limit is not necessary, since the right hand side is already zero up to the first order, as it is evident once the cases $\beta = 0$ and $\beta = i$ are considered. Therefore, up to the first order,

$$(\epsilon_{\text{eff}} + p_{\text{eff}}) \frac{DU^\beta}{D\tau} + \nabla^\beta p_{\text{eff}} + U_\beta \frac{Dp_{\text{eff}}}{D\tau} = 0, \quad (3.49)$$

just like a standard relativistic fluid. In particular, for $p_{\text{eff}} = 0$, one finds the geodesic equation $DU^\beta/D\tau = 0$. It is important to stress that these are first order results which hold in any frame that is comoving with the fluid at the background level.

In the following, to simplify the notation, we will no longer use the “eff” with $\epsilon$ and $p$.

4 Consequences and bounds

4.1 Density contrast evolution and the Jeans length

This subsection aims to explore qualitatively the RG effects for structure formation, and such we proceed in a context that allows for analytical expressions. The case of interest here is that of a universe dominated by matter and with negligible influence of $\Lambda_0$. Here we will use results from section 3.6, but two of the main results found in this section, the density contrast equation and the Jeans length, can be derived without specifying $\mu_2$ or other possible scales, assuming that they exist is sufficient.

Recalling that $\nabla_\alpha T^{\alpha \beta} = Q^\beta$, $p = w \epsilon$ (with constant $w$), $\delta p = c_s^2 \delta \epsilon$, and since $Q^0$ is a first order quantity while $Q^i$ is zero, one finds, up to first order,\(^8\)

\(^8\)Apart from the $Q^0$ term, see for instance ref. [34].
\[ \delta' + 3\mathcal{H}(c^2_s - w)\delta = (1 + w)(3\Phi' - \theta) - \frac{Q_0}{\epsilon_0}, \quad (4.1) \]

\[ \theta' + \mathcal{H}(1 - 3w)\theta = k^2 \left( \Phi + \frac{c^2_s}{1 + w}\delta \right). \quad (4.2) \]

Using eq. (3.44) and that, at background level, \( c'_0 + 3\mathcal{H}\epsilon_0 = 0 \),

\[ Q_0 = -\delta G\epsilon'_0 = 6\nu \Phi \mathcal{H}\epsilon_0. \quad (4.3) \]

By deriving eq. (4.1) and combining it with eqs. (3.6, 3.27, 4.2, 4.3), it is possible to find a second order equation that governs density contrast dynamics. Such equation, in a matter dominated universe, takes the form

\[ \delta'' + 6\mathcal{H}\delta' + \left( \frac{3}{2}\mathcal{H}^2 + k^2 \right) c^2_s \delta = 3(1 - 2\nu)\Phi'' + 3(1 - 4\nu)\mathcal{H}\Phi' - (k^2 + 3\nu \mathcal{H}^2)\Phi. \quad (4.4) \]

In the above, we used \( w = 0 \) and \( c^2_s \ll 1 \).

For a universe with only dust, it is straightforward to solve the above equation analytically for \( c_s = 0 \),

\[ \delta = C_1 \eta^{\frac{1}{2}(\xi - 5)} \left(-2 - \xi + \nu(\xi - 1) - \frac{1}{6}(1 - \nu)\eta^2 k^2 \right) + C_2, \quad (4.5) \]

with \( \xi \equiv \frac{\sqrt{1 + 24\eta}}{1 - \nu} \), and where \( C_1 \) and \( C_2 \) depend on \( \nu \). This solution is exact on \( \nu \) and puts an upper bound on it, namely

\[ \nu < 1. \quad (4.6) \]

The violation of the energy-momentum tensor has a quantitative impact on the solution (4.5), and this impact is only present at large scales (small \( k \)) or early times (small \( \eta \)). That is, by neglecting \( Q_0 \) from eq. (4.1) there would be a change on the constant part that appears inside the parenthesis which multiplies \( C_1 \), but no change on the \( \eta^2 k^2 \) dependent part.

For computing the Jeans length, we are interested in the subhorizon limit and without neglecting \( c_s \). Recalling that, in the subhorizon limit, eq. (3.29) implies \( \Phi' = -\mathcal{H}\Phi \), then eq. (4.4) can be written as

\[ \delta'' + 6\mathcal{H}\delta' + \left( k^2 c^2_s - 4\pi G_0 \frac{\delta}{1 - \nu}\epsilon_0 \right) \delta = 0. \quad (4.7) \]

Therefore, the Jeans length is

\[ \lambda_J \equiv \frac{2\pi a}{k} = c_s \sqrt{\frac{(1 - \nu)\pi}{G_0 \epsilon_0}}. \quad (4.8) \]

Hence, for small scales and for \( 0 < \nu < 1 \), the RG effects reduce \( \lambda_J \) and enhance the collapse of structures, while \( \nu < 0 \) decreases structure formation. This is contrary to what could be naively expected from the change of \( G \) considering eq. (3.22). Since \( \Phi < 0 \), one sees that a positive \( \nu \) decreases the numerical value of \( G \). This is possible since the main dynamical effect comes from the derivative terms that act on \( G \), not from the \( G \) factor that multiplies \( T_{\alpha\beta} \); thus the “force” that acts on test particles is enhanced for \( \nu > 0 \), similarly to [21, 35].

\(^9\)Instead of using the divergence of the the energy momentum tensor, this equation can also be found from the field equations (3.19), hence actually it does not depend on the choice for \( \mu_2 \) (only that \( \mu_2 \) exists and its variation with respect to the metric is zero).
4.2 Modified gravity parametrizations and clusters of galaxies

Besides the slip parameter, another relevant parameter for describing cosmological models comes from the extended Poisson equation (3.28), and it is sometimes designated by $Q(a, k)$, where $Q$ is such that [32, 36, 37]

$$-k^2 \Psi = 4\pi G_0 Q(a, k) \, a^2 \epsilon \Delta_\epsilon,$$

with

$$\Delta_\epsilon \equiv \delta_\epsilon + 3(1 + w)H\theta/k^2.$$  \hspace{1cm} (4.9)

If more than one fluid is being considered, then there should be a sum on $\delta_\epsilon$, $w$ and $\theta$.

From eq. (3.28) and using eqs. (3.5, 3.22, 3.27, 3.29, 3.31), we find

$$Q(a, k) = 1 - 2\nu \frac{1 - 2\nu}{1 - \nu + 12\pi G_0 \nu(\epsilon_0 + p_0)a^2/k^2}.$$  \hspace{1cm} (4.11)

We remark that the found expressions for the gravitational slip $\Psi/\Phi$ and $Q(a, k)$ are not usual ones, in particular they differ from Brans-Dicke and $f(R)$ gravity expressions. Here we develop further on these potentials and use ref. [33] to constraint $\nu$, since their constraints do not depend on a specific time evolution for the potentials (in particular, the slip here derived is exactly a constant, while other potentials are not constant). Many references consider specific time evolutions on these potentials that are not compatible with the effects here studied (e.g., [38, 39]).

Besides $\Psi/\Phi$ and $Q(a, k)$, there are other potentials that are also useful and commonly employed to understand cosmological evolution of modified gravity. Since the derived gravitational slip is constant, it is trivial to convert the $Q$ result into an expression for $Y$, that is, the analogous quantity with $\Psi$ replaced by $\Phi$ in the left hand side in eq. (4.9). It reads

$$Y = \frac{1}{1 - \nu + 12\pi G_0 \nu(\epsilon_0 + p_0)a^2/k^2} = \frac{Q}{\Psi/\Phi}.$$  \hspace{1cm} (4.12)

And the lensing potential (i.e., the potential for $\Psi + \Phi$) reads

$$\Sigma = \frac{1 - \nu}{1 - \nu + 12\pi G_0 \nu(\epsilon_0 + p_0)a^2/k^2}.$$  \hspace{1cm} (4.13)

For sufficiently small scales, one sees that $\Sigma = 1$ (for any $\nu$), while $Y$ can be used to infer the Jeans length. Indeed, in subhorizon limit $Y$ is constant and one can re-interpret it as a redefinition of $G_0$ that absorbs the constant $1 - \nu$, which is the same one that appears in eq. (4.8).

The expressions for $Y$ and $\Sigma$ can be simplified if it is possible to neglect $p_0$ (which is the pressure from the matter part), and can be written as a function of $H^2/k^2$ in the case that $\Lambda_0$ can also be neglected, namely, by using that

$$12\pi G_0 \nu(\epsilon_0 + p_0) \frac{a^2}{k^2} \frac{p_0}{\epsilon_0} \frac{3}{2} \nu \frac{1}{k^2} (3H^2 - \Lambda_0 a^2) \frac{\Lambda_0 \leq H^2}{\frac{9}{2} \nu \frac{H^2}{k^2}}.$$  \hspace{1cm} (4.14)

Care should be taken when comparing the results above with other theories, or using the potentials above to infer cosmological constraints. Most of the constraints on these modified gravity potentials are presented assuming some given variation on time and assuming energy-momentum tensor conservation. On the first issue, we will employ a bound that is valid
specifically for a constant gravitational slip. On the energy-momentum conservation, we note the following particularities of this specific case of non-conservation: i) at background level, energy-momentum tensor is conserved; ii) any energy-momentum tensor whose trace is constant at background level is also conserved at first order level; and iii) if \( \nabla_\alpha T^{\alpha\beta} \) is not null, then, in any frame that is comoving with matter at background level, one finds \( \nabla_\alpha T^{\alpha\beta} = 0 \). Consequently, this is a particularly mild case of non-conservation, since the trajectory of light for a given metric is the same of GR. Also, test particles follow geodesics up to the first perturbative order [see eq. (3.49)].

The bounds on the gravitational slip proposed in ref. [33] are based on a comparison between the potential \( \Phi \) inferred from the internal dynamics of clusters of galaxies with lensing effects from the same clusters. The bound for this case is found to be \( |1 - \Psi/\Phi| \leq 0.09 \), at 2\( \sigma \) level. Consequently, from (3.27),

\[
|\nu| \leq 0.04,
\]

(4.15)
at 2\( \sigma \) level. This is a significant constraint, but still the RG effects are orders of magnitude larger than the second order effects. Also, we stress that this bound depends on an assumption for the dark matter halo of clusters of galaxies. The test above works in the following way: assuming that current observations are in agreement with \( \Lambda CDM \), it states what could be the largest gravitational slip deviation from the fiducial value of 1 that would be still in agreement with observations.

In the following we consider the possible impact for the dark sector and afterwards issues in galactic and solar system scales.

### 4.3 On the cosmological dark matter and dark energy interpretation

Can the dynamical change provided by these RG corrections have a direct impact on dark matter at cosmological scales? Considering changes of the 10% order on large local structures (say a dark matter filament or a large cluster of galaxies), our answer is no. But there could be a relevant impact for phenomena usually attributed to dark energy, as explained below.

From the expressions of \( Y \) and \( \Sigma \) in subhorizon scales, one can define effective (and constant) values for \( G \) given by \( G_Y = G_0/(1 - \nu) \) and \( G_{\Sigma} = G_0 \), respectively. Hence the bound (4.15) implies that the differences between \( G_Y \), \( G_{\Sigma} \) and \( G_0 \) are at most of 4\%. This means that, for a given local matter distribution (compatible with linear cosmological perturbation theory), the internal dynamics from either pure GR or GR with RG corrections would be very similar, hence the dark matter distribution inferred from pure GR would be essentially the same of the case with RG effects.

For larger scales, the role of \( \nu \) is inverted, that is, a positive value of \( \nu \) decreases the gravitational attraction. To explore the gravitational attraction among massive particles at large scales, we only have to study the relative change of \( Y \) as a function of \( k \). That is, for a universe of dust without cosmological constant, we have to evaluate the variation of the function \( Y_e \equiv (1 - \frac{9}{2} \frac{\nu}{1 - \nu} \frac{H^2}{k^2})^{-1} \). For \( \nu = 0.04 \), a 10\% or larger decrease on the value of \( Y_e \) needs \( k \leq 1.3 \, H \); while for \( \nu = -0.04 \), a 10\% or larger decrease on the value of \( Y_e \) needs \( k \leq 1.4 \, H \). These are very low values for \( k \), implying that the 10\% effect on the gravitational force would only be present at scales about or larger than the cosmological horizon; which are too large distances for the type of dynamical influence commonly expected from dark matter, but it could be perhaps more naturally interpreted as dark energy effects. Such effects are not identical to a cosmological constant effect, but may have an impact on our dark energy understanding: it will not eliminate the cosmological constant \( \Lambda_0 \) (or change its
value, which is a background quantity), but may have an impact on certain $\Lambda$CDM tensions [40, 41]. Further and more detailed tests, using in particular the CMB data, are necessary and constitute a work in progress.

### 4.4 Galaxies and the solar system

In Refs. [21, 42, 43] some of us considered data on rotation curves and dispersion curves of disk and elliptical galaxies in order to infer whether RG effects could mimic dark matter at least in part, thus continuing Refs. [11, 20] proposals. It was found that, for the considered data and the assumed non-universality of the function $G(W)$ (details below), no dark matter was necessary; that is, all the dark matter-like effects could be in principle attributed to RG effects in gravity. These galactic results, in the light of the present notation, were based in the following considerations: i) Spacetime and RG background are given by the Minkowski metric (i.e., $\gamma_{\alpha\beta} = \delta_{\alpha\beta}$). ii) There is a single RG scale, which is a function of $W$.\(^{10}\) iii) It was used a particular expression for $G(W)$, which can be written as: $G \approx G_{0}^{\text{gal}} [1 + \nu_{\text{gal}} \ln (1 + W/W_{0})]$, where $W_{0}$ is a small constant such that, for rotation curve analyses, $1 + W/W_{0} \approx W/W_{0}$; while for sufficiently large distances, $1 + W/W_{0} \approx 1$.\(^{11}\) The constant $\nu_{\text{gal}}$ would not be common among all the galaxies and its typical values in galaxies, in order to remove dark matter and agree with observational data, would be about $10^{-4}$ (with a correlation with galactic baryonic mass). If dark matter is present as well, $\nu_{\text{gal}}$ would be smaller.

In ref. [35] the Will-Nordtvedt PPN analysis is considered, together with the considerations i and ii above, while for iii the $G(W)$ is taken to be $G \approx G_{0}^{\text{SS}} [1 + \nu_{1,\text{SS}} W + \nu_{2,\text{SS}} W^{2}]$. Among other results, it is found that, for the solar system, $|\nu_{1,\text{SS}}| \lesssim 10^{-9}$ (due to the PPN $\alpha_{2}$ parameter). Details on the correspondence and implications between the two $G(W)$ expressions can be found in Refs. [35, 44].\(^{12}\) For other solar system tests related to this RG approach see Refs. [45, 46].

A key difference between the two cases above and the cosmological results presented in this work is the background. While for cosmological studies the most natural background is the FRW one, for sufficiently small and isolated structures, like field galaxies, Minkowski should be the background. This difference is the one that leads to the necessity of a second RG scale cosmologically and the lack of conservation of the energy momentum-tensor at distances of the order of the cosmological (Hubble) horizon. The cosmological results point out that the matter dynamics, in the subhorizon limit, behave as if the energy-momentum tensor is conserved (see section 4.1); thus in agreement of the expected transition between cosmological and Minkowski backgrounds.

For many extended or modified gravity models, it is not straightforward to employ solar system test results to cosmology (e.g., due to screening effects). For the present case, a detailed understanding of the transition between the cosmological and Minkowski backgrounds is still in need, in particular in order to unveil the cosmological contribution to the galactic environment (the external potential effect [35, 44]). There is however a specific case presented in ref. [35] which is not sensitive to the external effect (and hence possibly not sensitive to the cosmological contribution), whose $G$ non-covariant expression is given by

\(^{10}\)Since the Ricci scalar is zero at background level, a single scale is sufficient for deriving the $\Lambda$ solution.

\(^{11}\)More precisely, the works on galaxies used $W/W_{0}$ inside the logarithm (or equivalently $\Phi/\Phi_{0}$, and in [35, 44] the use of $1 + W/W_{0}$ was introduced. As detailed in those references, this introduction does not change the results on galaxies for a large range of possible $W_{0}$ values.

\(^{12}\)To understand the relation between the two different expressions for $G(W)$ one should take care to consider the effect of external structures (the external potential effect) to a give system, since $W$ (likewise the perturbation $\Phi$) is sensitive not only to the system under consideration, but to its environment as well.
\(G^{-1} = G_0^{-1}(1 + \nu^{ss}_1 \Phi + O(\Phi^3)),\) where \(\Phi\) is the Newtonian potential, and here it cannot be approximated by \(W\), since that would include corrections smaller than the \(\Phi^3\) order. For this case, if \(\nu^{ss}\) is taken to be a universal constant, it is possible\(^ {13}\) that it would be the same constant that appears in cosmology (\(\sim \nu\)) and hence that from the solar system one could derive a strong bound on \(\nu\) (i.e., \(|\nu| < 10^{-9}\)). This particular case would imply that the RG effects would be irrelevant for the first order cosmological perturbations.

The most relevant relation between cosmological and galactic system tests seems to come from the opposite direction, that is, from the cosmological constraint to galactic physics. The main impact of the constraint (4.15) is that in this context cosmological dark matter cannot be appreciably mimicked by RG effects. Hence, since there should be cosmological dark matter with abundance close to the standard cosmological model, in galaxies there should also be dark some relevant amount of matter; and hence replacing large amounts of galactic dark matter in favor of RG effects is not plausible from this cosmological picture.

5 Conclusions

Here we have extended the implementation of Renormalization Group (RG) effects to General Relativity (GR) towards cosmology, this within the approach in which all the relevant information is in the action \([14]\). In this context, the background to be considered is the Friedmann-Robertson-Walker (FRW) one, and the RG corrections of the gravitational complying \(G\) depend on perturbations on such background (specifically, \(G\) depends on the scalar \(W\), which is the basis of the first RG scale \(\mu_1\)). Therefore, the background is the same of \(\Lambda\)CDM and any corrections appear at the perturbative level. It was found that a second RG scale is necessary for consistence, and that \(\Lambda\) corrections will depend on this second scale. Actually, apart from the case \(\mu_2 = f_2(R)\) which leads to GR, as explained in the Appendix A, there is not much freedom left for the second scale, and many results can be derived without explicitly specifying the scale (as in section 4.1). For concreteness, and demonstration that there are consistent choices for \(\mu_2\), the second scale was taken as a function of the relativistic fluid mass density.

A cosmological development based on ref. \([17]\) can be found in ref. \([47]\), which has some similarities with this work in the sense that the RG improvement is made at the action level, and not at the field equations level. It should be pointed out that there is a number of important differences, in particular the RG scale of \([47]\) depends on time only and it enters in the action as an external field. There are more examples of cosmological works that consider RG improvements at the field equations level, not at the action level (e.g., \([8, 10, 18, 19]\)), and we stress that their results cannot be directly compared to ours, since the cosmological framework is significantly different.

This work is centered on analytical solutions and qualitative understanding of the cosmology with RG effects. As such, detailed numerical evaluations with CMB and LSS data are not our aim here, but constitute a work in progress. Further developments on the theoretical side, as a Hamiltonian formulation (e.g., \([48]\)), have also to be addressed. A useful and commonly employed method to understand qualitatively the dynamics of some cosmological model is to parametrize the scalar perturbations (\(\Phi\) and \(\Psi\)) field equations by some functions that are commonly denoted by \(Q, Y, \Psi/\Phi\) and \(\Sigma\). This we did in sections 3.5 and 4.2, where it was found that the gravitational slip is exactly a constant and proportional.

\(^{13}\)Apart from the issue on how to make that \(G\) expression covariant without introducing second order corrections.
to the parameter that regulates the magnitude of the RG corrections ($\nu$). This behaviour lead us to use ref. [33] results, implying the bound (4.15). In section 4.3 we consider the possible implications of the RG cosmological implications for dark matter and dark energy, and find that cosmologically there is not much room for the RG effects to mimic dark matter effects, but there seem to be a relation with dark energy effects which would not change the background value of $\Lambda$, but changes the gravitational dynamics at scales comparable to the Hubble horizon and could have an impact in certain $\Lambda$CDM tensions [40, 41].

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A A hidden GR limit from a class of RG scales

Certain choices for the RG scales lead to GR for arbitrary $\nu$ values. Namely, if $\mu_2 = f_2(R)$ and there are no RG scales beyond the second one, then one find that the field equations become GR field equations, as detailed in this appendix.

In this case, $\lambda_0$ is a function of $\mu_2$ alone and $\delta \lambda_0 / \delta \Psi = 0$, thus eq. (3.20) leads to the usual relation $\delta S_{\text{matter}} / \delta \Psi = 0$. The term with an integral in eq. (3.19), which is $f_{\alpha\beta}$, does not vanish, namely

$$f_{\alpha\beta} = -\frac{2}{\sqrt{-g}} \int \delta G \frac{\partial \lambda_0}{\partial \mu_2} \frac{\delta f_2}{\delta g^\alpha{}_{\beta}} \sqrt{-g} \, d^4x'$$

$$= -\frac{1}{\sqrt{-g}} \int \delta G \frac{\delta R}{\delta g^\alpha{}_{\beta}} \sqrt{-g} \, d^4x'$$

$$= - (\delta G R_{\alpha\beta} + g_{\alpha\beta} G_0 \Box G^{-1} - G_0 \nabla\alpha \nabla\beta G^{-1}) . \quad (A.1)$$

Inserting the above equation together with eq. (3.16) into the field equation (2.4), one finds, up to the first order,

$$G_{\alpha\beta} + g_{\alpha\beta} \left[ \Lambda_0 + \left( \frac{R}{2} - \Lambda_0 \right) \delta G \right] - \delta G R_{\alpha\beta} = 8\pi G T_{\alpha\beta} , \quad (A.2)$$

$$(G_{\alpha\beta} + g_{\alpha\beta} \Lambda_0)(2 - G_0 G^{-1}) = 8\pi G T_{\alpha\beta} . \quad (A.3)$$

Hence, since $2 - G_0 G^{-1} \approx 1 + GG_0^{-1}(1 - G_0 G^{-1}) = GG_0^{-1}$,

$$G_{\alpha\beta} + g_{\alpha\beta} \Lambda_0 = 8\pi G_0 T_{\alpha\beta} , \quad (A.4)$$

which is the Einstein field equation. The demonstration above is only for first order perturbations, but actually the result is an exact one. To show the exact version one needs to start from the field equations, instead of (3.20), and proceed analogously to the above. If higher order derivatives are considered, say $\mu_2 = f_2(R_{\alpha\beta} R_{\alpha\beta})$, then one can show that there will be differences with respect to GR.
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