MORI CONIC BUNDLES WITH A REDUCED LOG TERMINAL BOUNDARY

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ABSTRACT. We study the local structure of Mori contractions $f: X \rightarrow Z$ of relative dimension one under an additional assumption that there exists a reduced divisor $S$ such that $K_X + S$ is plt and anti-ample.

INTRODUCTION

Let $f: X \rightarrow Z$ be a Mori conic bundle, i.e., $f: X \rightarrow Z$ is an equi-dimensional contraction from threefold $X$ with only terminal singularities onto a normal surface $Z$ such that $-K_X$ is $f$-ample. We study the analytic situation, so we assume that $(Z \ni o)$ is an analytic germ at a point and $X \supset C$ is a germ along a reduced curve such that $f^{-1}(o)_{\text{red}} = C$. Such a contractions were studied in [P], [P1], [P2]. They are interesting in his own sake and for applications to the three-dimensional birational geometry [Isk], [Isk1].

In this paper we consider the case when there exists an integer (irreducible) Weil divisor $S$ such that $f(S)$ is a curve $R \subset Z$, $K_X + S$ is purely log terminal and anti-ample. The reason to consider contractions with a reduced boundary is that, even we start with no reduced boundary components, these may appear after some appropriate blow-up (cf. [Sh]).

To study the three-dimensional case we have to understand the structure of surface contractions to curves. Such contractions are classified in Sect. 1. The classification can be obtained also by purely graph theoretical technique (see [KeM], §11). We use complements [Sh] because this method is more useful for applications. In Sect. 2 we apply these results to threefolds. The main result is that the log canonical divisor $K_X + S$ is either 1- or 2-complementary (Theorem 2.11). Also a partial classification of such contractions is given.

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1
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Notation. We will work over the field \( \mathbb{C} \) of complex numbers. A \textit{contraction} is a projective morphism \( f : X \to Z \) of normal varieties such that \( f_* \mathcal{O}_X = \mathcal{O}_Z \). A contraction is said to be \textit{extremal} if \( \rho(X/Z) = 1 \). Usually we assume that contraction is \( K_X \)-negative, i.e. \( -K_X \) is \( f \)-ample. By \( \text{Diff}_S(B) \) we denote the different (see [Sh]). For definition and properties of complements we refer to [Sh, §5] or [Ut, Ch. 19].

1. Two-dimensional log terminal contractions

1.1. Notation. Let \( (S \supset C) \) be a germ of normal surface \( S \) with only log terminal singularities along a reduced curve \( C \), and \( (R \ni o) \) be a smooth curve germ. Let \( f : (S, C) \to (R, o) \) be a \( K_S \)-negative contraction such that \( f^{-1}(o)_{\text{red}} = C \). Then it is easy to prove that \( p_a(C) = 0 \) and each irreducible component of \( C \) is isomorphic to \( \mathbb{P}^1 \). Everywhere if we do not specify the opposite we will assume that \( C \) is irreducible (or, equivalently, \( \rho(S/R) = 1 \), i.e. \( f \) is extremal). Let \( S_{\text{min}} \to S \) be the minimal resolution. Since the composition map \( f_{\text{min}} : S_{\text{min}} \to R \) is flat, the fiber \( f_{\text{min}}^{-1}(o) \) is a tree of rational curves. Therefore it is possible to define the dual graph of \( f_{\text{min}}^{-1}(o) \). We will draw it in the following way: \( \bullet \) denotes the proper transform of \( C \), while \( \circ \) denotes the exceptional curve. We attach the selfintersection number to the corresponding vertex. It is clear that the proper transform of \( C \) is the only \((-1)\)-curve into \( f_{\text{min}}^{-1}(o) \), so we usually omit \(-1\) over \( \bullet \).

Example 1.2. Let \( \mathbb{P}^1 \times \mathbb{C}^1 \to \mathbb{C}^1 \) be the natural projection. Consider the following action of \( \mathbb{Z}_m \) on \( \mathbb{P}^1_{x,y} \times \mathbb{C}^1_u \)

\[
(x, y; u) \to (x, \varepsilon^q y; \varepsilon^q u), \quad \varepsilon = \exp 2\pi i/m, \quad (m, q) = 1.
\]

Then the morphism \( f : S = (\mathbb{P}^1 \times \mathbb{C}^1)/\mathbb{Z}_m \to \mathbb{C}^1/\mathbb{Z}_m \) satisfies the conditions above. The surface \( S \) has exactly two singular points which are of types \( \frac{1}{m}(1, q) \) and \( \frac{1}{m}(1, -q) \). Morphism \( f \) is toric, so \( K_S \) is 1-complementary. One can check that the minimal resolution of \( S \) has the dual graph

\[
\overset{-b_1}{\circ} - \cdots - \overset{-b_s}{\circ} - \bullet - \overset{-a_r}{\circ} - \cdots - \overset{-a_1}{\circ},
\]

where \( (b_1, \ldots, b_s) \) and \( (a_r, \ldots, a_1) \) are uniquely defined by \( n/q \) and \( n/(n - q) \), respectively (see e.g. [Br]).
Proposition 1.3 (see also [KeM] (11.5.12)). Let $f: (S, C) \to (R, o)$ be a contraction as in 1.1, not necessary extremal. Assume that $S$ has only DuVal singularities. Then $S$ is analytically isomorphic to a surface into $\mathbb{P}^2_x \times \mathbb{P}^1_y \times \mathbb{C}_z^1$ which is given by one of the following equations.

(i) $x^2 + y^2 + t^n z^2 = 0$, then the central fiber is a reducible conic and $S$ has only one singular point which is of type $A_{n-1}$,

(ii) $x^2 + ty^2 + tz^2 = 0$, then the central fiber is a non-reduced conic and $S$ has exactly two singular points which are of type $A_1$,

(iii) $x^2 + ty^2 + t^2 z^2 = 0$, then the central fiber is a non-reduced conic and $S$ has only one singular point which is of type $A_3$,

(iv) $x^2 + ty^2 + t^n z^2 = 0$, $t \geq 3$ then the central fiber is a non-reduced conic and $S$ has only one singular point which is of type $D_{n+1}$.

Proof. One can show that the linear system $|-K_S|$ is very ample and determines an embedding $S \subset \mathbb{P}^2 \times R$. Then $S$ must be given by the equation $x^2 + t^ky^2 + t^n z^2 = 0$. $lacktriangle$

1.4. Construction. Notation and assumptions as in 1.1. Let $d$ be the index of $C$ on $S$, i.e. the smallest positive integer such that $dC \sim 0$. If $d = 1$, then $C$ is a Cartier divisor and $S$ must be smooth along $C$, because so is $C$. If $d > 1$, then there exists the following commutative diagram

\[
\begin{array}{ccc}
\hat{S} & \xrightarrow{g} & S \\
\downarrow f & & \downarrow f \\
\hat{R} & \xrightarrow{h} & R,
\end{array}
\]

where $\hat{S} \to S$ is a cyclic étale outside $\text{Sing}(S)$ cover of degree $d$ defined by $C$ and $\hat{S} \to \hat{R} \to R$ is the Stein factorization. Then $\hat{f}: \hat{S} \to \hat{R}$ is also a $K_S$-negative contraction but not necessary extremal. By the construction the central fiber $\hat{C} := \hat{f}^{-1}(\hat{o})$ is reducible. Therefore $\hat{S}$ is smooth outside $\text{Sing}(\hat{S})$. We distinguish two cases.

(A) $\hat{C}$ is irreducible. Then $\hat{S}$ is smooth and $\hat{S} \cong \mathbb{P}^1 \times \hat{R}$. We have the case of Example 1.2.

(B) $\hat{C}$ is reducible. Then the group $\mathbb{Z}_d$ permutes components of $\hat{C}$ transitively. Since $p_a(\hat{C}) = 0$, this gives us that all the components of $\hat{C}$ passes through one point, say $\hat{P}$ and do not intersect each other elsewhere. The surface $\hat{S}$ is smooth outside $\hat{P}$. Note that in this case $K_{\hat{S}} + C$ is not purely log terminal, because so is $K_S + C$.

Corollary 1.5. In notation of 1.4 $S$ has at most two singular points on $C$. 

3
Proof. In the case (B) these points are \(g(\bar{P})\) and, possible, the image of other \(d\) points on \(\bar{C}\) with nontrivial stabilizer.

1.6. **Additional notation.** In the case (B) we denote \(P := g(\bar{P})\) and if \(S\) has two singular points, let \(Q\) be another singular point. To distinguish exceptional divisors over \(P\) and \(Q\) in the corresponding Dynkin graph we reserve the notation \(\circ\) for exceptional divisors over \(P\) and \(\odot\) for exceptional divisors over \(Q\).

**Corollary 1.7.** In conditions above, if \(S\) has two singular points on \(C\), then \(K_S + C\) is log terminal near \(Q\).

**Lemma 1.8.** Notation as in 1.1, 1.4 and 1.6. Let \(S' \to S\) be a finite étale in codimension 1 cover. Then there exists the decomposition \(\hat{S} \to S' \to S\). In particular, \(S' \to S\) is cyclic and the preimage of \(P\) on \(S'\) consists of one point.

**Proof.** Let \(S''\) be the normalization of \(S' \times_S \hat{S}\). Consider the Stein factorization \(S'' \to R'' \to R\). Then \(S'' \to R''\) is a flat generically \(\mathbb{P}^1\)-bundle. Therefore for the central fiber \(C''\) one has \((-K_{S''} \cdot C'') = 2\), where \(C''\) is reduced and it is the preimage of \(\hat{C}\). On the other hand, \((-K_{S''} \cdot C'') = n(-K_{\hat{S}} \cdot \hat{C}) = 2n\), where \(n\) is the degree of \(S'' \to \hat{S}\). Whence \(n = 1\), \(S'' \simeq \hat{S}\). This gives us the assertion.

**Lemma 1.9.** Let \(f : S \to (R \ni o)\) be an extremal contraction as in 1.4 (with irreducible \(C\)). Assume that \(K_S + C\) is log terminal. Then

(i) \(f : S \to (R \ni o)\) is analytically isomorphic to the contraction from Example 1.2. In particular \(S\) has exactly two singular points on \(C\) which are of types \(\frac{1}{m}(1, q)\) and \(\frac{1}{m}(1, -q)\).

(ii) \(K_S + C\) is 1-complementary.

**Proof.** In the construction 1.4 we have the case (A). Therefore \(f\) is toric, so \(K_S + C\) is 1-complementary.

The following result gives us the classifications of surface log terminal contractions of relative dimension 1.

**Theorem 1.10.** Let \(f : (S \supset C) \to (R \ni o)\) be an extremal contraction as in 1.4 (with irreducible \(C\)). Then \(K_S\) is 1, 2 or 3-complementary. Moreover, there are the following cases

(i) Case \(A^*\): \(K_S + C\) is log terminal, then \(K_S + C\) is 1-complementary and \(f\) is toric (see Example 1.2).
(ii) Case $D^*$: $K_S + C$ is log canonical, but not log terminal, then $K_S + C$ is 2-complementary and $f$ is a quotient of a conic bundle of type (ii) of Proposition 1.3 by a cyclic group $\mathbb{Z}_{2m}$ which permutes components of the central fiber and acts on $S$ freely in codimension 1. The minimal resolution of $S$ is

\[ \begin{array}{cccccccc}
-a_1^* & -a_2^* & \cdots & -a_r^* & -2^* \\
-\circ & -\circ & -\circ & -\circ & -\circ
\end{array} \]

where $s, r \neq 0$ (recall that $S$ can be non-singular outside $P$, so $r = 0$ is also possible).

(iii) Case $A^{**}$: $K_S$ is 1-complementary, but $K_S + C$ is not log canonical. The minimal resolution of $S$ is

\[ \begin{array}{cccccccc}
-a_1^* & -a_2^* & \cdots & -a_r^* & -2^* & -2^* & -2^* \\
-\circ & -\circ & -\circ & -\circ & -\circ & -\circ & -\circ
\end{array} \]

where $r \geq 4, i \neq 1, r$.

(iv) Case $D^{**}$: $K_S$ is 2-complementary, but not 1-complementary and $K_S + C$ is not log canonical. The minimal resolution of $S$ is

\[ \begin{array}{cccccccc}
-a_r^* & -\cdots & -a_i^* & -\cdots & -a_1^* & -b^* & -2^* \\
-\circ & -\circ & -\circ & -\circ & -\circ & -\circ & -\circ
\end{array} \]

where $r \geq 2, i \neq r$.

(v) Case $E_6^*$: $K_S$ is 3-complementary, but not 1- or 2-complementary. The minimal resolution of $S$ is

\[ \begin{array}{cccccccc}
-3^* & -2^* & -2^* & -2^* & -2^* & -2^* & -2^* \\
-\circ & -\circ & -\circ & -\circ & -\circ & -\circ & -\circ
\end{array} \]

(it is possible that $b = 2$ and $Q \in S$ is non-singular).
Remark 1.11. (i) In the case $D^* K_S$ can be 1-complementary
a) if $P \in S$ is DuVal (see L.3 (ii)) or
b) if $s = 0$, $a_1 = \cdots = a_r = 2$, $b = r + 2$.
(ii) In cases $D^*$, $A^{**}$, and $D^{**}$ there are additional restrictions on the
graph of the minimal resolution. For example, in the case $A^{**}$ one
easily can check that $(\sum_{j=1}^{i-1} a_j) - (i - 1) = (\sum_{j=i+1}^{r} a_j) - (r - i)$
and $a_i = \text{(number } \odot \text{-vertices}) + 2$.

Proof. If $K_S + C$ is log terminal, then by Lemma 1.9 we have $A^*$. If
$K_S + C$ is log canonical but not log terminal, then in Construction
L.4 $K_{\tilde{S}} + \tilde{C}$ is also log canonical but not log terminal [Sh, 2.2], [Ut, 20.4]. Since $\tilde{C}$ is a Cartier divisor, $K_{\tilde{S}}$ is canonical. Hence $\tilde{f}$ is such
as in Proposition L.3, (i). We get the case $D^*$. To prove that $K_S + C$
is 2-complementary we take on the minimal resolution $S_{\min}$ of $S$ the
divisor $D_{\min}$ with coefficients

\[
\begin{array}{cccccccc}
1/2 & \circ & & & & & & \circ \\
| & & & & & & | & \\
1 & - & 1 & - & \cdots & 1 & - & 1 & - & 1 & \cdots & 1 & \circ \\
| & & & & & & | & \\
1/2 & \circ & & & & & & \circ 
\end{array}
\]

where $\circ$ corresponds to an additional incomplete curve. One can check
that $2(K_{S_{\min}} + D_{\min}) \sim 0$, so we have an integer Weil divisor $D = C + B$
on $S$ such that $K_S + C + B$ is log canonical and $2(K_S + C + B) \sim 0$.

Assume that $K_S$ is 1-complementary, but $K_S + C$ is not log terminal.
Then there exists a reduced divisor $D$ such that $K_S + D$ is log canonical
and linearly trivial. By our assumption and by [K, 9.6] $C \not\subseteq D$. Let
$P \in S$ is a point of index $> 1$. Then $P \in C \cap D$ and again by [K, 9.6]
there are two components $D_1, D_2 \subset D$ passing through $P$. But since
$D \cdot L = 2$, where $L$ is a generic fiber of $f$, $D = D_1 + D_2$, $P \in D_1 \cap D_2$
and $P$ is the only point of index $> 1$ on $S$.

Now we claim that $K_S$ is 1-, 2- or 3-complementary. Suppose that
$K_S$ is not 1-complementary. Then by Lemma L.9 for some $\alpha \leq 1$ the
log divisor $K_S + \alpha C$ is log canonical, but not Kawamata log terminal.
Consider the log terminal blow-up $\varphi: (\tilde{S}, \sum E_i + \alpha \tilde{C}) \to (S, \alpha C)$
[Sh], where $\sum E_i$ is the reduced exceptional divisor, $\tilde{C}$ is the proper
transform of $C$ and $K_{\tilde{S}} + \sum E_i + \alpha \tilde{C} = \varphi^*(K_S + \alpha C)$ is log terminal.
Applying the $(K_{\tilde{S}} + \sum E_i)$-MMP to $\tilde{S}$ we obtain on the last step
the blow-up $\sigma: \tilde{S} \to S$ with irreducible exceptional divisor $E$. Moreover,
$\sigma^*(K_S + \alpha C) = K_{\tilde{S}} + E + \alpha \tilde{C}$ is log canonical, where $\tilde{C}$ is the
proper transform of $C$ and $K_{\tilde{S}} + E$ is purely log terminal and negative over $S$. Since $K_{\tilde{S}} + E + (\alpha - \epsilon)\tilde{C}$ is anti-ample for $0 < \epsilon \ll 1$, the curve $\tilde{C}$ can be contracted in the appropriate MMP over $R$ and this gives us $(\tilde{S}, \tilde{C}) \to R$ with purely log terminal $K_{\tilde{S}} + \tilde{E}$, i.e. by Lemma 1.9 $(\tilde{S}, \tilde{E}) \to R$ is such as in Example 1.2. If $K_{\tilde{S}} + E$ is non-negative on $\tilde{C}$, then by [Sh1, 4.4] we can pull back 1-complements from $\tilde{S}$ on $\tilde{S}$ and then pull down them on $S$ [Sh, 5.4]. So below we assume that $-(K_{\tilde{S}} + E)$ is ample over $R$. Then by [Ut, 19.6] complements for $K_E + \text{Diff}_E(0)$ can be extended on $\tilde{S}$. According to [Sh, 3.9] or [Ut, 16.6, 19.5] $\text{Diff}_E(0) = \sum_{i=1}^3 (1 - 1/m_i)P_i$, where for $(m_1, m_2, m_3)$ there are the following possibilities: $(2, 2, m)$, $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$. Further, $\tilde{S}$ has exactly two singular points and these are of type $\frac{1}{m}(1, q)$ and $\frac{1}{m}(1, -q)$, respectively (see Lemma 1.3). Since $\tilde{C}$ intersects $E$ at only one point, this point must be singular and we have two more points with $m_i = m_j$. We get two cases

1) $(2, 2, m)$, $\tilde{C} \cap E = \{P_3\}$, there is 2-complement.
2) $(2, 3, 3)$, $\tilde{C} \cap E = \{P_1\}$, there is 3-complement.

This proves the claim.

Now assume that $K_{\tilde{S}}$ is 2-complementary, but not 1-complementary and $K_{\tilde{S}} + C$ is not log canonical. Then we are in the case 1), so $(\tilde{S} \ni P_1) \simeq (\tilde{S} \ni P_2) \simeq (\mathbb{C}^2, 0)/\mathbb{Z}_2(1, 1)$ and $(\tilde{S} \ni P_3) \simeq (\mathbb{C}^2, 0)/\mathbb{Z}_m(1, q)$, $(m, q) = 1$. Take the minimal resolution $S_{\min} \to \tilde{S}$ of $P_1, P_2, P_3 \in \tilde{S}$. Over $P_1$ and $P_2$ we have only single $(-2)$-curves and over $P_3$ we have a chain which must intersect the proper transform of $\tilde{C}$, because $\tilde{C}$ passes through $P_3$. Since the fiber of $\tilde{S}_{\min} \to R$ over $o$ is a tree of rational curves, there are no tree of them passing through one point. Whence proper transforms of $E$ and $\tilde{C}$ on $\tilde{S}_{\min}$ are disjoint. Moreover, the proper transform of $E$ cannot be a $(-1)$-curve. Indeed, otherwise contracting it we get three components of the fiber over $o \in R$ passing through one point. It gives us that $\tilde{S}_{\min}$ coincides with the minimal resolution $S_{\min}$ of $S$. Therefore configuration of curves on $S_{\min}$ looks like that in (iv). We have to show only that all the curves in the down part have the selfintersections $-2$. Indeed, contracting $(-1)$-curves over $R$ we obtain $\mathbb{P}^1$-bundle. Each time, we contract a $(-1)$-curve, we have the configuration of the same type. If there is a vertex with
selfintersection $< -2$, then on some step we get the configuration

$$\cdots \circ \longrightarrow \bullet \longrightarrow \circ \cdots$$

$$\circ$$

It is easy to see that this configuration cannot be contracted to a non-singular point over $o \in R$, because contraction of the central $(-1)$-curve gives us configuration curves which is not a tree.

The case $E_6^*$ is very similar to $D^{**}$. We omit it.

Remark 1.12. In cases $A^*$, $A^{**}$, $D^*$ and $D^{**}$ the canonical divisor $K_S$ can be 1- or 2-complementary with integer $K_S + B$. Indeed, it is sufficient to check only in the case $D^{**}$. As in the case $D^*$ we take on the minimal resolution $S_{\text{min}}$ of $S$ the divisor $B_{\text{min}}$ with coefficients

$$\begin{array}{cccccccc}
    & 1 & \cdots & 1 & \circ & \cdots & 1 & \circ \\
\circ & | & | & | & 1/2 & 1/2 & | & \circ \\
\circ & | & 0 & \bullet & 0 & | & 0 & \circ \\
\circ & | & \cdots & 0 & \circ
\end{array}$$

One can check that $2(K_{S_{\text{min}}} + B_{\text{min}}) \sim 0$, so we have an integer Weil divisor $B$ on $S$ such that $B \cap C = \{P\}$, $K_S + B$ is log canonical and $2(K_S + B) \sim 0$.

Corollary 1.13. In cases $D^*$ and $D^{**}$ there exist a double étale outside $P$ cover $g: S_1 \rightarrow S$, a contraction onto a curve $f_1: S_1 \rightarrow R_1$ and a double cover $h: R_1 \rightarrow R$ such that $hf_1 = fg$. The central fiber $C_1$ of $f_1$ has exactly two components which are intersects each other at $P_1 := g^{-1}(P)$.

Proof. Let $K_S + B$ be a 2-complement from 1.12 (with integer $B$). Then $2(K_S + B) \sim 0$ and $K_S + B \not\sim 0$. It gives us the desired cover $g: S_1 \rightarrow S$. The decomposition $S_1 \rightarrow R_1 \rightarrow R$ is the Stein factorization.

2. Three-dimensional contractions

In this section we apply the results of the previous section to study three-dimensional contractions with a reduced boundary. First we simplify the assertion [19, 19.6].
Lemma 2.1. Let \((X, S)\) be a purely log terminal pair with reduced \(S \neq 0\) and let \(f: X \to Z\) be a projective morphism such that \(-(K_X + S)\) is nef and big. Given an \(n\)-complement \(B_S^+\) of \(K_S + \text{Diff}_S(0)\), then in a neighborhood of any fiber of \(f\) meeting \(S\) there exists an \(n\)-complement \(S + B^+\) of \(K_X + S\) such that \(B_S^+ = \text{Diff}_S(B^+)\).

Proof. By [Sh, 3.9] the multiplicities \(b_i\) of \(\text{Diff}_S(0)\) are standard, i.e. \(b_i = 1 - 1/m_i\), \(m_i \in \mathbb{N}\). Consider a good resolution of singularities \(g: Y \to X\). We can pull back the complements from \(S\) on its proper transform \(S_Y\) by [Sh1, 4.4]. Then applying [Ut, 19.6] we get \(n\)-complement on \(Y\). Finally, we push down it on \(X\) by [Sh, 5.4].

As an easy consequence of Theorem 1.10 and the lemma above we have the following

Proposition 2.2. Let \((X \supset C)\) be a germ of a normal threefold along an irreducible reduced curve, let \((Z \ni o)\) be a normal surface germ and let \(f: X \to Z\) be an extremal \(K_X\)-negative contraction (so \(f^{-1}(o)\) is irreducible). Assume that there exists an irreducible surface \(S \subset X\) containing \(f^{-1}(o)\) such that \(f(S)\) is a curve on \(Z\) and \(K_X + S\) is purely log terminal and assume that the intersection of the singular locus of \(X\) with \(S\) is zero-dimensional. Then \(K_X + S\) is 1-, 2- or 3-complementary.

Remark 2.3. The same arguments as in the proof of Theorem 2.11 shows that the case \(E_6^*\) is impossible, so \(K_X + S\) is 1- or 2-complementary.

For terminal singularities we have more stronger results. In the rest of this paper we fix the following notation.

2.4. Notation. Let \(f: X \to Z\) be a Mori conic bundle, i.e. \(f: X \to Z\) is an equi-dimensional contraction from threefold \(X\) with only terminal singularities onto a normal surface \(Z\) such that \(-K_X\) is \(f\)-ample and \(\rho(X/Z) = 1\). We study the analytic situation, so we assume that \((S \ni o)\) is an analytic germ at a point and \(X \supset C\) is a germ along a reduced curve such that \(f^{-1}(o)_{\text{red}} = C\). Since \(\rho(X/Z) = 1, C \simeq \mathbb{P}^1\) [11, 1.1.1]. We assume that there exists an (irreducible) curve \(R \subset Z\) such that \(K_X + S\) is purely log terminal for \(S := f^{-1}(R)\). By the adjunction \(S\) is normal and has only log terminal singularities. The following fact gives us that \(f: S \to R\) is such as in Sect. 1.

Lemma 2.5. Let \(f: X \to Z, S\) and \(R\) be as in 2.4. Then \(K_Z + R\) is purely log terminal. In particular, \(R\) is non-singular.

Proof. Let \(H\) be a generic hyperplane section of \(X\). By Bertini theorem \(K_H + \text{Diff}_H(S)\) is purely log terminal. Consider the restriction \(f: H \to
Z. Then \( \text{Diff}_H(S) = H \cap S = f^* R \) and by the ramification formula
\[ K_H + \text{Diff}_H(S) - \Delta = f^*(K_Z + R), \]
where \( \Delta \) is the ramification divisor. By [KL] §2 \( K_Z + R \) is purely log terminal, because so is \( K_H + \text{Diff}_H(S) - \Delta \).

Assuming \( S \) is Cartier, we get a semistable contraction (cf. e. g. [K], [Sh2]).

**Theorem 2.6.** Notation as in §4. Assume additionally that \( S \) is Cartier and \( f: X \to Z \) is not a (usual) conic bundle. Then

(i) \( S \) is of type \( A^* \) or \( A^{**} \).
(ii) \( K_X + S \) is \( 1 \)-complementary. Moreover, \( K_X \) is \( 1 \)-complementary with canonical \( K_X + D \).
(iii) The base surface \( Z \) is non-singular.
(iv) \( X \) contains exactly one point of index \( > 1 \) and, possibly, one more Gorenstein terminal point.
(v) If \( S \) is of type \( A^* \), then \( f: S \to R \) is isomorphic the contraction from Example 1.2 with \( n = 4 \). In particular \( X \) is of index 2 in this case (see Example 2.10 below).

Note that Mori conic bundles of index 2 are completely classified in [P1], §3.

**Proof.** By Lemma 2.7 below \( S \) has only cyclic quotient or DuVal singularities and by our assumption \( S \) has at least one non-DuVal point. So it is of type \( A^* \), \( A^{**} \) or \( D^* \) (with \( s = 0, a_1 = \cdots = a_r = 2, b = r + 2 \)). But the last case is also impossible because \( (P \in S) \) is not of class \( T \) (see Lemma 2.9). Therefore \( S \) is of type \( A^* \) or \( A^{**} \). Then by Lemma 2.1 there exists a \( 1 \)-complement \( K_X + S + B \). Since \( S \sim 0, K_X + B \) is linearly trivial and canonical (cf. [Sh2]). Since \( K_Z + R \) is purely log terminal and \( R \) is Cartier, \( Z \) is non-singular [K]. Finally, in the case \( A^* \) \( S \) contains exactly two singular points which are of types \( 1_n(1, q) \) and \( 1_n(1, n - q) \). By Corollary 2.8 these singularities are of class \( T \) iff \( n = 4 \).

**Lemma 2.7 ([K], [KSB]).** Let \( (X \ni P) \) be a germ of a three-dimensional terminal singularity of index \( m \) and let \( S \subset X \) be an effective Cartier divisor such that \( K_X + S \) is purely log terminal. Then one of the following holds

(i) \( (S \ni P) \) is DuVal or smooth, and then \( (X \ni P) \) is cDV or smooth, respectively,
(ii) \( (S \ni P) \) is isomorphic to \( 1_m(msm' - 1) \), where \( (m, m') = 1 \) and \( m \) is the index of \( (X \ni P) \).
The singularities \( \frac{1}{m^2}(1, msm' - 1), (m, m') = 1 \) in [KSB] are called singularities of class \( T \). It is easy to see that the index of such a singularity is equal to \( m \).

**Corollary 2.8.** If the singularity \( \frac{1}{n}(1, q) \) is of class \( T \), then \( n \) divides \( (q + 1)^2 \).

Minimal resolutions of singularities of class \( T \) has very well description.

**Lemma 2.9** ([KSB]).

(i) The singularities
\[
\begin{align*}
&\frac{-4}{1} - \frac{-3}{1} - \frac{-2}{1} - \cdots - \frac{-2}{1} - \frac{-3}{1}
\end{align*}
\]
are of class \( T \).

(ii) If the singularity \( \frac{-a_1}{1} - \cdots - \frac{-a_k}{1} \) is of class \( T \), then so are
\[
\begin{align*}
&\frac{-2}{1} - \frac{-a_1}{1} - \cdots - \frac{-a_{k-1}}{1} - \frac{-a_k}{1} - \frac{-2}{1} \quad \text{and} \quad \frac{-a_1}{1} - \frac{-a_2}{1} - \cdots - \frac{-a_k}{1} - \frac{-2}{1}.
\end{align*}
\]

(iii) Every singularity of class \( T \) that is not DuVal can be obtained by starting with one of the singularities described in (i) and iterating the steps described in (ii).

Below we give an example of a semistable Mori conic bundle.

**Example 2.10** (cf. [P], 4.4). Let \( Y \subset \mathbb{P}^3_{x,y,z,t} \times \mathbb{C}^2_{u,v} \) is given by the equations
\[
\begin{align*}
&\begin{align*}
xy - z^2 &= ut^2 \\
x^2 &= uy^2 + v(z^2 + t^2).
\end{align*}
\end{align*}
\]
Define the action of \( \mathbb{Z}_2 \) on \( Y \):
\[
x \to -x, \quad y \to -y, \quad z \to -z, \quad t \to t, \quad u \to u, \quad v \to v.
\]
Then \( f: X = Y/\mathbb{Z}_2 \to \mathbb{C}^2 \) is a Mori conic bundle with a unique singular point of type \( \frac{1}{2}(1, 1, 1) \). Take a \( \mathbb{Z}_2 \)-invariant divisor \( F := Y \cap \{u = v\} \) and let \( S = F/\mathbb{Z}_4 \). Then \( S \to f(S) \) is a contraction of type \( A^* \). More precisely, \( S \) has two singular points which are DuVal of type \( A_3 \) and a cyclic quotient of type \( \frac{1}{4}(1, 1) \). By the inversion of adjunction [Sh] \( K_X + S \) is purely log terminal.

**Theorem 2.11.** Notation as in 2.4. Then

(i) \( S \) cannot be of type \( E_6^\ast \).

(ii) If \( S \) is of type \( A^* \) or \( A^{**} \), then \( K_X + S \) is \( 1 \)-complementary.

(iii) If \( S \) is of type \( A^* \), then \( f: X \to Z \) is either toric (see [P1, 2.1]) or is a quotient of another Mori conic bundle of index 2 such as in Theorem 2.7 (v) by a cyclic group which acts on \( X \) free in codimension 2.
(iv) If $S$ is of type $D^*$ or $D^{**}$, then $K_X + S$ is 2-complementary. In these cases $f: X \to Z$ is a quotient of another Mori conic bundle $f': X' \to Z'$ of index 1 or 2 and a reducible central fiber by a cyclic group of even order which acts on $X$ free in codimension 2. Moreover the preimage $S'$ of $S$ on $X'$ is normal and has the following type of the minimal resolution

\[
\text{Case } D^*. \quad \begin{cases} \cdots \times -2 \times -3 \times -2 \times -3 \times -2 \times -2 \end{cases},
\]

\[
\text{Case } D^{**}. \quad \begin{cases} \cdots \times -2 \times -3 \times -2 \times -3 \times -2 \times -3 \end{cases}
\]

**Proof.** Consider the base change \([P, 3.1], [P1, 1.9]\)

\[
X' \xrightarrow{g} X \xrightarrow{f} Z' \xrightarrow{h} Z
\]

where $Z'$ is smooth and $g, h$ are quotient morphisms by a finite group, say $G_0$. Let $S'$ be the preimage of $S$. Then $S'$ is Cartier and $K_{X'} + S'$ is purely log terminal \([11, \S 2], [13, 20.4]\). By construction the action of $G_0$ on $Z'$ is free outside $o'$ and the (non-singular) curve $R' := f'^5 S'$ is invariant under this action. Therefore $G_0$ is cyclic. If $X'$ has only points of index 1, then $f': X' \to Z'$ is a usual conic bundle (possible singular). Such quotients are described in \([P1]\). In our situation the central fiber of $f'$ cannot be multiple, so $f''$ either is smooth or has a reduced reducible central fiber. From \([P1]\, \text{Theorem 2.4}\) we get that $f: X \to Z$ must be toric \([P1, 2.1]\) ($S$ is of type $A^*$) or it is such as in example \(2.13\) ($S$ is of type $D^*$). All the assertions hold in these cases.

Below we assume that $X'$ has at least one point of index $> 1$ (and so is $S'$). By Lemma \([3.8]\) the preimage of $P$ consists of one point, say $P'$. This gives as the decomposition $(\mathbb{C}^2 \ni 0) \to (S' \ni P') \to (S \ni P)$ and the corresponding exact sequence

\[
1 \to G_1 \to G \to G_0 \to 1,
\]

where $(S \ni P) \simeq (\mathbb{C}^2 \ni 0)/G$ and $(S' \ni P') \simeq (\mathbb{C}^2 \ni 0)/G_1$. By the construction $G_0$ is cyclic. Whence $G_1 \supset [G, G]$. On the other hand, by Lemma \([2.7]\) the group $G_1$ either is cyclic or $G_1 \subset \text{SL}_2(\mathbb{C})$.

**Case $E_6^\ast$.** In this case $(P \in S) \simeq (\mathbb{C}^2 \ni 0)/G$, where the image of $G$ in $\text{PGL}_2(\mathbb{C})$ is isomorphic to the alternating group $A_4$ \([B1]\). So both $G_0$ and $G_1$ cannot be cyclic. Therefore $(S' \ni P')$ is a DuVal point and
\((X' \ni P')\) is of index 1. Now let \(Q'_1, \ldots, Q'_k\) be the preimages of \(Q\). Since \((S \ni Q) \simeq \mathbb{C}^2/\mathbb{Z}_{2b-3}(1, b-2), (S' \ni Q'_i) \simeq \mathbb{C}^2/\mathbb{Z}_c(1, b-2)\), where \(c\) divides \((2b-3)\). Thus \((1 + b - 2, c) = 1\) and by \(2.8\) \((S' \ni Q'_i)\) cannot be of class \(T\). Therefore \((S' \ni Q'_i)\) is smooth and \(X'\) is Gorenstein, a contradiction with our assumptions.

**Cases** \(A^*\) **and** \(A^{**}\). Then \(K_X + S\) is 1-complementary by \([1.10]\) and \(2.1\). In the case \(A^* \ f'\) is (usual) conic bundle or such as in Theorem \(2.6\) (v). This proves (ii) and (iii).

**Cases** \(D^*\) **and** \(D^{**}\). We claim that there is a decomposition \((X', S') \rightarrow (X_1, S_1) \rightarrow (X, S)\), where \(S_1\) is such as in Corollary \([1.13]\). Let \(K_S + B\) be an integer 2-complement from \([1.12]\). It is sufficient to prove that \(K_S + B' \sim 0\), where \(B'\) is the preimage of \(B\) on \(S'\). Assume the opposite. Note that \(K_S + B'\) is log canonical, but not purely log terminal near \(P'\), since \(S' \rightarrow S\) is étale in codimension 1. Recall also that \(K_S + B' \sim 0\) and purely log terminal outside \(P'\). If \(B'\) has locally near \(P'\) two components, then \(K_S + B' \sim 0\) also near \(P'\), a contradiction. Therefore \(B'\) is locally irreducible (and non-singular) near \(P'\). If \((P' \in S')\) is not DuVal, then by the classification of log canonical singularities with a reduced boundary \([K]\) the minimal resolution of \((P' \in S')\) has the form

\[
\begin{array}{ccc}
B' \\
\bullet \\
-2 \\
\circ \\
-2 \\
-2 \\
\cdots \\
-2 \\
\circ \\
-2 \\
\circ
\end{array}
\]

But by Lemma \([2.9]\) it can not be a singularity of type \(T\). Whence \((P' \in S')\) is DuVal with the minimal resolution of the form

\[
\begin{array}{cccccc}
-2 \\
\circ \\
-2 \\
\circ \\
-2 \\
-2 \\
\cdots \\
-2 \\
\circ \\
-2 \\
\circ
\end{array}
\]

By our assumptions \(S'\) has at least one non-DuVal point. Then \(S\) can not be of type \(D^{**}\), because in this case \(S\) is DuVal at \(Q'\). So \(S\) is of type \(D^*\), \(B' = C'\) near \(P'\) and, therefore \(C'\) is irreducible. In this case it is easy to see that \(S'\) is such as in Proposition \([1.3]\) (iii)-(iv), a contradiction with our assumptions.

Now let \((X', S') \rightarrow (X_1, S_1) \rightarrow (X, S)\) be a decomposition, where \(S_1\) is such as in Corollary \([1.13]\). By our construction \(K_{X_1} + S_1\) is 1-complementary and so is \(K_{S_1}\). Therefore \(S_1\) has only cyclic quotient
singularities. The group $G_0$ acts on the minimal resolution of $S'$ and permutes (two) components of $C'$ in the case $D^*$ and permutes (two) components of $B'_S$ in the case $D'^*$. Therefore the dual graph of $S'$ is symmetric, in particular the dual graph of $(P' \in S')$ is symmetric. It is easy to see that a singularity of class $T$ is symmetric iff it is as in (i). But if the dual graph of $(P' \in S')$ is $\ast^4$ then $G_1$ is contained in the center of $G$. Since $G/G_1$ is cyclic and $G$ is not abelian, this is impossible. Keeping in mind that in the case $D^*$ the fiber $C'$ has exactly two components and in the case $D'^*$ $K_S + C'$ is not log canonical it is easy to see that for $S'$ we have only one of two graphs in (vi). In particular either $(P' \in S')$ and $(P' \in X')$ are of index 2. This proves our theorem.

Remark 2.12. In cases $D^*$ and $D'^*$ the point $(P \in S)$ is of even index. By Brieskorn's classification \cite{Br} $(P \in S)$ is isomorphic to $\mathbb{C}^2/G$, where $G = \langle \mathbb{Z}_{4m}, \mathbb{Z}_{2m}; D_n, C_{2n} \rangle$, $(2, m) = 1$, $(n, m) = 1$.

Consider few examples of contractions such as in Theorem 2.11.

Example 2.13 (cf. \cite{P}, 4.1.2. (i)). Let $X' \subset \mathbb{P}^2 \times \mathbb{C}^2$ be a hypersurface which in some coordinate system $(x, y; u, v)$ is defined by the equation

$$x^2 + y^2 + (u^{2n} - v^{2n})z^2 = 0,$$

Let $X' \to \mathbb{C}^2$ be the natural projection and let $S'$ is the section $X' \cap \{v = 0\}$. Then the projection $S' \to R' := \{v = 0\}$ is a contraction as in (i). Define the action of $\mathbb{Z}_2$ by

$$u \to -u, \quad v \to -v, \quad x \to -x, \quad y \to y, \quad z \to z.$$ 

$X := X'/\mathbb{Z}_2 \to \mathbb{C}^2/\mathbb{Z}_2$ is a Mori conic bundle. $S \to R := R'/\mathbb{Z}_2$ is a contraction of type $A'^*$ (if $n = 1$) or $D^*$ (if $n \geq 2$). By the inversion of adjunction $K_X + S$ is purely log terminal. Note that $S$ has a unique singular point which is DuVal so it is such as in (iii) or (iv).

Example 2.14 (cf. \cite{P}, 4.4). Let $Y \subset \mathbb{P}^3_{x,y,z,t} \times \mathbb{C}^2_{u,v}$ is given by the equations

$$\begin{cases} xy = ut^2 \\ z^2 = u(x^2 + y^2) + vt^2 \end{cases}$$

Consider the action of $\mathbb{Z}_4$ on $Y$:

$$x \to y, \quad y \to -x, \quad z \to iz, \quad t \to t, \quad u \to -u, \quad v \to -v.$$ 

As above $f: Y/\mathbb{Z}_4 \to \mathbb{C}^2/\mathbb{Z}_2$ is a Mori conic bundle with a unique singular point of type $\frac{1}{4}(1, 1, 3)$. Take a $\mathbb{Z}_4$-invariant divisor $F := Y \cap \{u = v\}$ and let $S = F/\mathbb{Z}_4$. Then $S \to f(S)$ is a contraction of type.
More precisely, $S$ has two singular points which are DuVal of type $A_1$ and toric of type $\frac{1}{8}(1, 5)$.

**Example 2.15** (cf. [P], 4.3). Let $Y \subset \mathbb{P}^3_{x,y,z,t} \times \mathbb{C}^2_{u,v}$ be a smooth subvariety given by the equations

\[
\begin{align*}
xy &= ut^2 \\
(x + y + z)z &= vt^2
\end{align*}
\]

Consider the following action of $\mathbb{Z}_8$ on $Y$:

\[
x \to \varepsilon^{-3}z, \quad y \to \varepsilon(x + y + z), \quad z \to \varepsilon^{-3}y, \quad t \to t, \\
u \to \varepsilon^{-2}v, \quad v \to \varepsilon^{-2}u,
\]

where $\varepsilon := \exp(2\pi i/8)$. We obtain a Mori conic bundle $f: Y/\mathbb{Z}_8 \to \mathbb{C}^2/\mathbb{Z}_4$. Let $S := (Y \cap \{u = v\})/\mathbb{Z}_8$. Then $S \to f(S)$ is a contraction of type $A^{**}$. More precisely, $S$ has exactly one singular point which is toric of type $\frac{1}{16}(1, 5)$.

Note that $K_X$ is 1-complementary in all these examples [P].

2.16. **Final remark.** Of course it is possible to consider Mori conic bundles $f: X \to Z$ such that $K_X + S$-negative and with the condition $f(S) = Z$ instead of $\dim f(S) = 1$. But in this case $S$ is a section of $f$ over a generic point, i.e. $(S \cdot L) = 1$ for a generic fiber $L$ of $f$. By an easy construction [P, 3.1] or [P1, 1.9] $f: X \to Z$ is a quotient of another Mori conic bundle $f': X' \to Z'$ (not necessary extremal) over a smooth surface $Z'$ by a cyclic group. The preimage of $S$ is again a section over a generic point. Since we are considering a germ over $o$, $f'$ must be smooth outside $o'$. Then by using [Isk, Lemma 4] we can prove that $f': X' \to Z'$ is smooth (see proof of Lemma 1.1 in [P2]). In this situation $f: X \to (Z \ni o)$ is analytically isomorphic to $\mathbb{C}^2 \times \mathbb{P}^1/\mathbb{Z}_n(1, -1; a, 0) \to \mathbb{C}^2/\mathbb{Z}_n$ [P1]. Thus this case is not very interesting.

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