The group of causal automorphisms

Do-Hyung Kim

Department of Applied Mathematics, College of Applied Science, Kyung Hee University, Seocheon-dong, Giheung-gu, Yongin-si, Gyeonggi-do 446-701, Korea

E-mail: mathph@khu.ac.kr

Received 13 February 2010, in final form 14 February 2010
Published 17 June 2010
Online at stacks.iop.org/CQG/27/155005

Abstract

The group of causal automorphisms on Minkowski spacetime is given and its structure is analyzed.

PACS numbers: 02.40.−k, 04.20.−q, 04.20.Cv, 04.20.Gz, 02.40.Ma

1. Introduction

In 1964, Zeeman showed that any causal automorphism on Minkowski spacetime \( \mathbb{R}^{n+1}_1 \) can be represented by a composite of orthochronous transformation, translation and homothety when \( n \geq 3 \) (see [1]). Therefore, if we let \( G \) be the group of all causal automorphisms on \( \mathbb{R}^{n+1}_1 \) with \( n \geq 3 \), then \( G \) is isomorphic to the semi-direct product of \( \mathbb{R}^+ \times O \) and \( \mathbb{R}^{n+1}_1 \) where \( \mathbb{R}^+ \) is the group of positive real numbers and \( O \) is the group of orthochronous transformations on \( \mathbb{R}^{n+1}_1 \). However, as he remarked, Zeeman’s theorem does not hold when \( n = 2 \).

In connection with this, recently, the standard form of causal automorphism on \( \mathbb{R}^{2+1}_1 \) has been given in [2]. In this paper, we improve the result in [2] and by use of this, the structure of the causal automorphism group is analyzed when \( n = 2 \).

2. Causal automorphisms on \( \mathbb{R}^{2+1}_1 \)

The following is a known result.

**Theorem 2.1.** Let \( F : \mathbb{R}^{2+1}_1 \rightarrow \mathbb{R}^{2+1}_1 \) be a causal automorphism. Then, there exist a continuous function \( g : \mathbb{R} \rightarrow \mathbb{R} \) and a homeomorphism \( f : \mathbb{R} \rightarrow \mathbb{R} \) which satisfy \( \sup(g \pm f) = \infty \), \( \inf(g \pm f) = -\infty \) and \( \left| \frac{f(t+\delta t)-f(t)}{\delta t} \right| < 1 \) for all \( t \) and \( \delta t \), such that if \( f \) is increasing, then \( F \) is given by \( F(x, y) = \left( \frac{f(x+y)+f(x-y)}{2} + \frac{g(x+y)+g(x-y)}{2}, \frac{f(x+y)-f(x-y)}{2} + \frac{g(x+y)-g(x-y)}{2} \right) \), and if \( f \) is decreasing, then we have \( F(x, y) = \left( \frac{f(x+y)+f(x-y)}{2} + \frac{g(x+y)+g(x-y)}{2}, \frac{f(x+y)-f(x-y)}{2} + \frac{g(x+y)-g(x-y)}{2} \right) \).

Conversely, for any functions \( f \) and \( g \) which satisfy the above conditions, the function \( F : \mathbb{R}^{2+1}_1 \rightarrow \mathbb{R}^{2+1}_1 \) defined as above is a causal automorphism.

**Proof.** See theorem 4.4 in [2].
We remark that in the above theorem, for the given causal automorphism \( F(x, y) = (F_1(x, y), F_2(x, y)) \), the homeomorphism \( f \) and the continuous function \( g \) are uniquely determined by \( f(t) = F_1(t, 0) \) and \( g(t) = F_2(t, 0) \). By use of this, we improve the above theorem.

**Theorem 2.2.** Let \( F : \mathbb{R}_1^2 \rightarrow \mathbb{R}_1^2 \) be a causal automorphism on \( \mathbb{R}_1^2 \). Then, there exist unique homeomorphisms \( \varphi \) and \( \psi \) of \( \mathbb{R} \), which are either both increasing or both decreasing, such that if \( \varphi \) and \( \psi \) are increasing, then we have \( F(x, y) = \frac{1}{2}(\varphi(x+y) + \psi(x-y), \varphi(x+y) - \psi(x-y)) \), or if \( \varphi \) and \( \psi \) are decreasing, then we have \( F(x, y) = \frac{1}{2}(\varphi(x+y) + \psi(x+y), \varphi(x+y) - \psi(x+y)) \).

Conversely, for any given homeomorphisms \( \varphi \) and \( \psi \) of \( \mathbb{R} \), which are either both increasing or both decreasing, the function \( F \) defined as above is a causal automorphism of \( \mathbb{R}_1^2 \).

**Proof.** For any given causal automorphism \( F : \mathbb{R}_1^2 \rightarrow \mathbb{R}_1^2 \), by the previous theorem, we can get a unique homeomorphism \( f \) and a unique continuous function \( g \). Let \( \varphi = f + g \) and \( \psi = f - g \). Then, clearly, \( \varphi \) and \( \psi \) are continuous.

The conditions \( \sup(g \pm f) = \infty, \inf(g \pm f) = -\infty \) imply that \( \varphi \) and \( \psi \) are surjective. If \( \varphi(t) = \varphi(t') \) with \( t \neq t' \), then this implies that \( f(t') = f(t) \), which contradicts the condition \( |\frac{g(t+h) - g(t)}{f(t+h) - f(t)}| < 1 \) for all \( t \) and \( h \). Therefore, \( \varphi \) is injective and likewise, we can show that \( \psi \) is injective. Since \( \varphi \) and \( \psi \) are continuous bijections from \( \mathbb{R} \) to \( \mathbb{R} \), the topological domain of invariance implies that \( \varphi \) and \( \psi \) are homeomorphisms.

The equation \( \varphi(t_2) - \varphi(t_1) = (f(t_2) - f(t_1))\left[1 + \frac{g(t_2) - g(t_1)}{f(t_2) - f(t_1)}\right] \) shows that, by the condition \( \frac{g(t+h) - g(t)}{f(t+h) - f(t)} \rightarrow 1 \) as \( h \rightarrow 0 \), \( \varphi \) is increasing if and only if \( f \) is increasing. Likewise, we can show that \( \psi \) is increasing if and only if \( f \) is increasing.

By simple calculation, we can show that \( F \) has the desired form when expressed in terms of \( \varphi \) and \( \psi \). This completes the proof of the first part.

To prove the converse, let \( \varphi \) and \( \psi \) be increasing homeomorphisms on \( \mathbb{R} \). Let \( f = \frac{1}{2}(\varphi + \psi) \) and \( g = \frac{1}{2}(\varphi - \psi) \). Then \( g \) is continuous and \( f \) is an increasing homeomorphism. Since \( \varphi \) and \( \psi \) are homeomorphisms, we have \( \sup(f + g) = \sup\varphi = \infty, \sup(f - g) = \sup\psi = \infty, \inf(f + g) = \inf\varphi = -\infty \) and \( \inf(f - g) = \inf\psi = -\infty \).

We now show that \( f \) and \( g \) satisfy the inequality \( |\frac{g(t+h) - g(t)}{f(t+h) - f(t)}| < 1 \). Let \( \Delta = \frac{g(t_1) - g(t_0)}{f(t_1) - f(t_0)} = \frac{\varphi(t_1) - \varphi(t_0) - \varphi(t_1) - \varphi(t_0)}{f(t_1) - f(t_0)} \). Without loss of generality, we can assume that \( t > t_0 \) and it is easy to see that \( -1 < \Delta < 1 \), since \( \varphi \) and \( \psi \) are increasing. Therefore, by theorem 2.1, the function \( F \) as defined in the statement is a causal automorphism. By exactly the same argument, we can show that the assertion also holds when both \( \varphi \) and \( \psi \) are decreasing homeomorphisms.

\( \square \)

3. The group of causal automorphisms on \( \mathbb{R}_1^2 \)

In this section, we denote the group of all causal automorphisms on \( \mathbb{R}_1^2 \) by \( G \), and we analyze its group structure. For this we let \( H(\mathbb{R}) \) be the group of all homeomorphisms on \( \mathbb{R} \) and let \( H = H^+ \cup H^- \) where \( H^+ = \{(\varphi, \psi) \in H(\mathbb{R}) \times H(\mathbb{R}) | \varphi, \psi \) are increasing\} and \( H^- = \{(\varphi, \psi) \in H(\mathbb{R}) \times H(\mathbb{R}) | \varphi, \psi \) are decreasing\}. Then \( H \) is a subgroup of \( H(\mathbb{R}) \times H(\mathbb{R}) \) under the operation induced from \( H(\mathbb{R}) \times H(\mathbb{R}) \).

From theorem 2.2, we can see that any causal automorphism \( F \) on \( \mathbb{R}_1^2 \) corresponds to a unique element in \( H \) and, conversely, each element in \( H \) uniquely determines a causal automorphism on \( \mathbb{R}_1^2 \). Thus, there exists a one-to-one correspondence between \( G \) and \( H \) as a
set. It might seem that $G$ is isomorphic to $H$. However, we cannot obtain an isomorphism in this way and we define a new operation on $H$ as follows.

We define a $\mathbb{Z}_2$-action on $H$ by $a \cdot (\varphi, \psi) = (\varphi, \psi)$ if $a = 0$ and $a \cdot (\varphi, \psi) = (\psi, \varphi)$ if $a = 1$. If we define a map $\pi : H \to \mathbb{Z}_2$ by $\pi(x) = 0$ if $x \in H^+$ and $\pi(x) = 1$ if $x \in H^-$, then $\pi$ is a group homomorphism with $H$ equipped with the operation induced from $H(\mathbb{R}) \times H(\mathbb{R})$.

Note that $\pi(\varphi, \psi) = \pi(\varphi^{-1}, \psi^{-1}) = \pi(\psi, \varphi)$ and $\pi((a, b) \cdot (\varphi, \psi)) = \pi(\varphi, \psi)$ for any $(a, b) \in H$.

To get an isomorphism from $G$ to $H$, we define a new operation $\ast$ on $H$ by $(\alpha, \beta) \ast (\varphi, \psi) = (\alpha, \beta) \circ \pi(\alpha, \beta) \cdot (\varphi, \psi)$ where $\circ$ is the $\mathbb{Z}_2$-action defined above and $\cdot$ is the operation induced from $H(\mathbb{R}) \times H(\mathbb{R})$.

**Theorem 3.1.** The set $H$ under $\ast$ is a group and is isomorphic to $G$.

**Proof.** To show associativity, by calculation, we have,

\[
(a, b) \ast ((c, d) \ast (e, f)) = (a, b) \ast ((c, d) \circ \pi(c, d) \cdot (e, f)) = \pi(a, b) \cdot (c, d) \circ \pi(c, d) \cdot (e, f).
\]

Thus, we have the following four cases.

| $\pi(a, b)$ | $\pi(c, d)$ | $(a, b) \ast ((c, d) \ast (e, f))$ |
|------------|------------|----------------------------------|
| 0          | 0          | $(a, b) \circ (c, d) \circ (e, f) = (ace, bdf)$ |
| 0          | 1          | $(a, b) \circ (c, d) \circ (f, e) = (acf, bde)$ |
| 1          | 0          | $(a, b) \circ (df, ce) = (adf, bce)$ |
| 1          | 1          | $(a, b) \circ (de, cf) = (ade, bcf)$ |

On the other hand, we have

\[
[(a, b) \ast (c, d)] \ast (e, f) = [(a, b) \circ \pi(a, b) \cdot (c, d)] \ast (e, f)
\]

\[
= [(a, b) \circ \pi(a, b) \circ (c, d)] \circ \pi((a, b) \circ \pi(a, b) \circ (c, d)) \cdot (e, f)
\]

\[
= [(a, b) \circ \pi(a, b) \circ (c, d)] \circ [\pi(a, b) \pi(a, b) \circ (c, d)] \cdot (e, f)
\]

(\because \pi \text{ is a group homomorphism})

\[
= [(a, b) \circ \pi(a, b) \circ (c, d)] \circ [\pi(a, b) \pi(c, d)] \cdot (e, f)
\]

(\because \pi(a, b) \circ (c, d) = \pi(c, d))

Thus, we have the following four cases.

| $\pi(a, b)$ | $\pi(c, d)$ | $[(a, b) \ast (c, d)] \ast (e, f)$ |
|------------|------------|----------------------------------|
| 0          | 0          | $(a, b) \circ (c, d) \circ (e, f) = (ace, bdf)$ |
| 0          | 1          | $(a, b) \circ (c, d) \circ (f, e) = (acf, bde)$ |
| 1          | 0          | $(a, b) \circ (df, ce) = (adf, bce)$ |
| 1          | 1          | $(a, b) \circ (de, cf) = (ade, bcf)$ |

The above two tables show that the operation $\ast$ satisfies the associative law.

The following two formulae show that $(id, id)$ is the identity element in $H$ under $\ast$:

\[
(id, id) \ast (\varphi, \psi) = (id, id) \circ \pi(id, id) \cdot (\varphi, \psi)
\]

\[
= (id, id) \circ (\varphi, \psi) = (\varphi, \psi),
\]

\[
(\varphi, \psi) \ast (id, id) = (\varphi, \psi) \circ \pi(\varphi, \psi) \cdot (id, id)
\]

\[
= (\varphi, \psi) \circ (id, id) = (\varphi, \psi).
\]
The following two formulae show that, for given \( (\varphi, \psi) \in H \), \( \pi(\varphi, \psi) \cdot (\varphi^{-1}, \psi^{-1}) \) is the inverse element of \( (\varphi, \psi) \):

\[
(\varphi, \psi) \ast \{ \pi(\varphi, \psi) \cdot (\varphi^{-1}, \psi^{-1}) \} = (\varphi, \psi) \circ \pi(\varphi, \psi) \cdot \{ \pi(\varphi, \psi) \cdot (\varphi^{-1}, \psi^{-1}) \}
\]

\[
= (\varphi, \psi) \circ \{ \pi(\varphi, \psi) \pi(\varphi, \psi) \} \cdot (\varphi^{-1}, \psi^{-1})
\]

\[
= (\varphi, \psi) \circ (\varphi^{-1}, \psi^{-1}) = (id, id).
\]

\[
\{ \pi(\varphi, \psi) \cdot (\varphi^{-1}, \psi^{-1}) \} \ast (\varphi, \psi) = [\pi(\varphi, \psi) \cdot (\varphi^{-1}, \psi^{-1})]
\]

\[
\circ \{ \pi(\varphi, \psi) \cdot (\varphi^{-1}, \psi^{-1}) \} \ast (\varphi, \psi) = [\pi(\varphi, \psi) \cdot (\varphi^{-1}, \psi^{-1})] \circ [\pi(\varphi, \psi) \cdot (\varphi, \psi)] = (id, id).
\]

Therefore, \( H \) forms a group under the operation \( \ast \).

We now show that the groups \( G \) and \( H \) under \( \ast \) are isomorphic. For the given causal automorphism \( F(x, y) = (F_1(x, y), F_2(x, y)) \), theorem 2.2 shows that \( F_1 + F_2 \) and \( F_1 - F_2 \) determine a unique element in \( H \). Therefore, we can define \( \Pi : G \rightarrow H \) by

\[
\Pi(F) = (\text{homeomorphism determined by } F_1 + F_2, \text{ homeomorphism determined by } F_1 - F_2)
\]

By theorem 2.2, \( \Pi \) is a bijection and we only need to show that \( \Pi \) is a homomorphism. We have the following four cases.

(i) Let \( G = \frac{1}{2}(\alpha(x + y) + \beta(x - y), \alpha(x + y) - \beta(x - y)) \) and \( F = \frac{1}{2}(\varphi(x + y) + \psi(x - y), \varphi(x + y) - \psi(x - y)) \) where both \( (\alpha, \beta) \) and \( (\varphi, \psi) \) are in \( H^\ast \). Then, we have \( G \circ F(x, y) = \frac{1}{2}(\alpha \circ \varphi(x + y) + \beta \circ \varphi(x - y), \alpha \circ \varphi(x + y) - \beta \circ \varphi(x - y)) \) and thus \( \Pi(G \circ F) = (\alpha \circ \varphi, \beta \circ \varphi) \). Since \( \Pi(G) = (\alpha, \beta) \) and \( \Pi(F) = (\varphi, \psi) \), we have

\[
\Pi(G \circ F) = \Pi(G) \ast \Pi(F).
\]

(ii) Let \( G = \frac{1}{2}(\alpha(x + y) + \beta(x - y), \alpha(x + y) - \beta(x - y)) \) and \( F = \frac{1}{2}(\varphi(x - y) + \psi(x + y), \varphi(x - y) - \psi(x + y)) \) where \( (\alpha, \beta) \) is in \( H^\ast \) and \( (\varphi, \psi) \) is in \( H^\ast \). Then, we have \( G \circ F(x, y) = \frac{1}{2}(\alpha \circ \varphi(x - y) + \beta \circ \psi(x + y), \alpha \circ \varphi(x - y) - \beta \circ \psi(x + y)) \) and thus \( \Pi(G \circ F) = (\alpha \circ \varphi, \beta \circ \varphi) \). Since \( \Pi(G) = (\alpha, \beta) \) and \( \Pi(F) = (\varphi, \psi) \), we have

\[
\Pi(G \circ F) = \Pi(G) \ast \Pi(F).
\]

(iii) Let \( G = \frac{1}{2}(\alpha(x - y) + \beta(x + y), \alpha(x - y) - \beta(x + y)) \) and \( F = \frac{1}{2}(\varphi(x + y) + \psi(x - y), \varphi(x + y) - \psi(x - y)) \) where \( (\alpha, \beta) \) is in \( H^\ast \) and \( (\varphi, \psi) \) is in \( H^\ast \). Then, we have \( G \circ F(x, y) = \frac{1}{2}(\alpha \circ \varphi(x - y) + \beta \circ \varphi(x + y), \alpha \circ \varphi(x - y) - \beta \circ \varphi(x + y)) \) and thus \( \Pi(G \circ F) = (\alpha \circ \varphi, \beta \circ \varphi) \). Since \( \Pi(G) = (\alpha, \beta) \) and \( \Pi(F) = (\varphi, \psi) \), we have

\[
\Pi(G \circ F) = \Pi(G) \ast \Pi(F).
\]

(iv) Let \( G = \frac{1}{2}(\alpha(x - y) + \beta(x + y), \alpha(x - y) - \beta(x + y)) \) and \( F = \frac{1}{2}(\varphi(x - y) + \psi(x + y), \varphi(x - y) - \psi(x + y)) \), where both \( (\alpha, \beta) \) and \( (\varphi, \psi) \) are in \( H^\ast \). Then, we have \( G \circ F(x, y) = \frac{1}{2}(\alpha \circ \varphi(x + y) + \beta \circ \varphi(x - y), \alpha \circ \varphi(x + y) - \beta \circ \varphi(x - y)) \) and thus \( \Pi(G \circ F) = (\alpha \circ \varphi, \beta \circ \varphi) \). Since \( \Pi(G) = (\alpha, \beta) \) and \( \Pi(F) = (\varphi, \psi) \), we have

\[
\Pi(G \circ F) = (\alpha \circ \varphi, \beta \circ \varphi) = (\alpha, \beta) \circ \pi(\alpha, \beta) \cdot (\varphi, \psi) = \Pi(G) \ast \Pi(F).
\]

This shows that \( \Pi \) is an isomorphism and the proof is completed. \( \square \)

In [3], it is shown that \( H(\mathbb{R}) \) is a subgroup of \( G \) and this can also be seen in the above theorem as follows. If we define a map \( \Omega : H(\mathbb{R}) \rightarrow H \) by \( \Omega(f) = (f, f) \), then it is easy to see that \( \Omega \) is an injective homomorphism and thus, \( H(\mathbb{R}) \) is a subgroup of \( G \) through an injective homomorphism \( \Pi^{-1} \circ \Omega \). Zeeman’s result shows that the group of causal automorphisms on \( \mathbb{R}^n_+ \) is finite dimensional when \( n \geq 3 \) and our result shows that the group is infinite dimensional when \( n = 2 \).
Acknowledgments

This work was supported by a grant from the College of Applied Science, Kyung Hee University research professor fellowship.

References

[1] Zeeman E C 1964 Causality implies the Lorentz group *J. Math. Phys.* 5 490
[2] Kim D-H 2010 Causal automorphisms of two-dimensional Minkowski space-time *Class. Quantum. Grav.* 27 075006
[3] Kim D-H 2009 An imbedding of Lorentzian manifolds *Class. Quantum. Grav.* 26 075004