Canonical Forms and Automorphisms in the Projective Space

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Abstract

Let $\mathcal{C}$ be a sequence of multisets of subspaces of a vector space $\mathbb{F}_q^k$. We describe a practical algorithm which computes a canonical form and the stabilizer of $\mathcal{C}$ under the group action of the general semilinear group. It allows us to solve canonical form problems in coding theory, i.e. we are able to compute canonical forms of linear codes, $\mathbb{F}_q$-linear block codes over the alphabet $\mathbb{F}_q^s$ and random network codes under their natural notion of equivalence. The algorithm that we are going to develop is based on the partition refinement method and generalizes a previous work by the author on the computation of canonical forms of linear codes.

Key words: automorphism group, additive code, canonical form, canonization, linear code, random network code

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1 Introduction

In this paper we consider the canonization problem defined on group actions in the following sense: Let $G$ be a group acting on a set $X$ from the left. Furthermore, let $\mathcal{L}(G)$ be the set of subgroups of $G$.

Problem 1 (Canonization). Determine a function

$$\text{Can}_G : X \to X \times G \times \mathcal{L}(G)$$
$$x \mapsto (\text{CF}_G(x), \text{TR}_G(x), \text{Stab}_G(x))$$

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∀x ∈ X, ∀g ∈ G : CF_G(x) = CF_G(gx) \quad (1)
∀x ∈ X : CF_G(x) = TR_G(x) \quad (2)
∀x ∈ X : Stab_G(x) = \{ g ∈ G \mid gx = x \} \quad (3)

The element CF_G(x) is called the canonical form of x and the element TR_G(x) ∈ G a transporter element. The element TR_G(x) is well-defined up to the multiplication with the stabilizer Stab_G(x) from the right.

In the case of a finite group G the orbit Gx is finite as well. Hence, applying an orbit-stabilizer algorithm and defining CF(x) := \min Gx already solves this problem. Our goal is to define Can_G in such a way that there is an algorithm with good practical performance to compute a canonical form. Indeed, Can_G is implicitly defined via the result of the algorithm. We included the stabilizer computation to the canonization process since the Homomorphism Principle (Theorem 5) [8], which we will apply as a key tool, must have this data available.

In this work, we will provide a practical canonization algorithm for sequences of (multi-) sets of subspaces under the action of the semilinear group. It will be a natural generalization of the algorithm [4], where the author solves the same problem in the special case that all occurring subspaces are one-dimensional. Since this is a question arising from coding theory, the algorithm was formulated using generator matrices of linear codes. In fact, it canonizes generator matrices of linear codes. Similarly, the present problem also has applications in coding theory as well, see Section 4.

[11] investigate the computational complexity of the code equivalence problem for linear codes over finite fields. They show that this problem is not NP-complete. On the other hand, it is at least as hard as the graph isomorphism problem. The latter problem has been studied for decades, but until now there is no polynomial time algorithm solving it. Therefore, we can not expect to give a polynomial time algorithm solving our present problem. We therefore measure efficiency in terms of running times on selected non-trivial examples.

The paper is structured as follows: The next section will describe general methods for providing practical canonization algorithms like the partition refinement approach. The program nauty [10] is a prominent example using this idea: it canonizes a given graph under the action of the symmetric group, i.e. the relabeling of vertices. In Section 3 we give a reformulation of the original problem such that we are able to use the partition refinement idea, too. The subsequent section deals with the origins of this problem from coding theory. Subsection 5.1 summarizes the necessary modifications of Section 2 in order
to canonize linear codes. In the following, we give the details of the canon-
ization algorithm for sequences of subspaces and finish this work with some
applications of the algorithm in Section 6 and a conclusion.

2 General canonization algorithms

This section surveys four principle attacks for the canonization of an object
$x \in X$ under the action of $G$. It is a summary of [6]. Therefore, we will omit
the proofs.

2.1 The direct approach

The first method is the most desirable. It is directly attacking the problem,
which means that we understand the group action in such a way that we are
able to define $\text{CF}_G(x)$ and the group element $\text{TR}_G(x)$ and to give a polynomial-
time algorithm for its computation without making (explicitly) use of the
group structure.

Example 2. Let $X$ be a totally ordered set. The symmetric group $S_n$ acts on
$X^n$ via

$$\pi(x_1, \ldots, x_n) := (x_{\pi^{-1}(1)}, \ldots, x_{\pi^{-1}(n)}), \text{ for all } \pi \in S_n, (x_1, \ldots, x_n) \in X^n.$$  

We may compute a canonical form of a given sequence $(x_1, \ldots, x_n)$ by lexico-
graphically sorting the elements of the vector. It is easy to define a transporter
element. The stabilizer of $(x_1, \ldots, x_n)$ is the subgroup of elements in $S_n$ which
interchanges equal entries of the vector.

Example 3. Let $F_q$ be the finite field with $q$ elements, where $q = p^r$ for some
prime $p$. The general linear group $\text{GL}_k(q)$ is the set of all invertible $k \times k$-
matrices with entries in $F_q$. It acts on the set $F_q^{k \times r}$ of all $k \times r$-matrices using
the usual matrix multiplication from the left. We may define a canonical form
for $M \in F_q^{k \times r}$ using the reduced row echelon form RREF($M$) of $M$. Gaussian
elimination is a polynomial-time algorithm to compute the canonical form
and a transporter element under this action. Furthermore, the stabilizer of $M$
could be easily given using the stabilizer of the canonical form RREF($M$).

Similarly, a canonical form of $M \in F_q^{k \times r}$ under the action of $\text{GL}_r(q)$ – given by
$(A, M) \mapsto MA^T$ – could be defined to be the one in reduced column echelon
form RCEF($M$).
2.2 Homomorphism Principle

**Definition 4.** Let $G$ be a group acting on a set $X$ and $H$ another group acting on $Y$. A pair of mappings $(\theta : X \to Y, \varphi : G \to H)$ where $\varphi$ is a group homomorphism is called a **homomorphism of group actions** if the mappings commute with the actions, i.e. $\theta(gx) = \varphi(g)\theta(x)$, $\forall g \in G, x \in X$.

In the case that $\varphi$ is the identity on $G$, i.e. $G = H$, we call the function $\theta$ a **$G$-homomorphism**. If the action on the right is trivial, i.e. $hy = y$ for all $y \in Y$ and $h \in H$, we call the function $\theta$ **$G$-invariant**. In this case we could always suppose that $|H| = 1$.

**Theorem 5** (Homomorphism Principle, [8]). Let $(\theta : X \to Y, \varphi : G \to H)$ be a homomorphism of group actions, with surjective mappings $\theta$ and $\varphi$. Then

- the stabilizer subgroup $\text{Stab}_G(x)$ is a subgroup of $\text{Stab}_G(\theta(x)) := \varphi^{-1}(\text{Stab}_H(\theta(x)))$, and
- we can define a canonization map $\text{Can}_G(x)$ in the following way:
  1. Compute $\text{Can}_H(\theta(x)) = (\text{CF}_H(\theta(x)), \text{TR}_H(\theta(x)), \text{Stab}_H(\theta(x)))$ for some fixed canonization map $\text{Can}_H$.
  2. Compute $g \in \varphi^{-1}(\text{TR}_H(\theta(x)))$ and $G' := \text{Stab}_G(g\theta(x)) = g\text{Stab}_G(\theta(x))g^{-1}$.
  3. Define $\text{Can}_G(x) := (\text{CF}_{G'}(gx), \text{TR}_{G'}(gx)g, g^{-1}\text{Stab}_{G'}(gx)g)$ for some fixed canonization map $\text{Can}_{G'}$.

**Example 6.** Let $G = (V, E)$ be a graph with finite vertex set $V := \{1, \ldots, n\}$ and edges $E \subseteq \{\{x, y\} \mid x, y \in V : x \neq y\}$. There is a natural action of $\pi \in S_n$ on $G$ defined in the following way:

$$\pi G := (V, \pi E) := (V, \{\{\pi(x), \pi(y)\} \mid \{x, y\} \in E\}).$$

If $\{x, y\} \in E$ we say that $y$ is a neighbor of $x$. Now, let $N(x)$ count the number of neighbors of $x$ and define the $S_n$-homomorphism $N(G) := (N(1), \ldots, N(n))$.

The Homomorphism Principle tells us

1. to canonize this sequence under the action of the symmetric group $S_n$, for instance by sorting the sequence lexicographically.
2. If $\pi \in S_n$ is the corresponding permutation, we have to relabel the graph via the application of $\pi$ and
3. canonize the relabeled graph under the stabilizer $\text{Stab}_{S_n}(\pi N(G))$.

We may interpret the result of $N(\pi G)$ as some coloring on the vertices. In the following this coloring has to be preserved by the group action. This allows us to apply the Homomorphism Principle recursively since in the following we can count neighbors of a single color class as well.
2.3 The lifting approach

Let \( H \) be a subgroup of \( G \), short: \( H \leq G \). A subset \( T \subseteq G \) is called a right (left) transversal of \( H \) in \( G \) if it is a minimal but complete set of right (left) coset representatives, i.e. \( Ht \neq Ht' \) for all \( t, t' \in T \) and \( G = \bigcup_{t \in T} Ht \).

**Proposition 7.** Let \( G \) be a group acting on a totally ordered set \( X \). Suppose that there is already some canonization \( \text{Can}_H \) available for \( H < G \) and let \( T \) be a right transversal of \( H \) in \( G \). Then, we can define the canonization map \( \text{Can}_G \) for \( x \in X \) in the following way:

- \( \text{CF}_G(x) := \min_{t \in T} \text{CF}_H(tx) \).
- Let \( t_1 \in T \) be a transversal element with \( \text{CF}_G(x) = \text{CF}_H(t_1x) \). Define \( \text{TR}_G(x) := \text{TR}_H(t_1x)t_1 \).
- Let \( t_1, \ldots, t_m \in T \) be those elements of \( T \) which define a canonical form \( \text{CF}_H(t_ix) = \text{CF}_G(x) \). The stabilizer \( \text{Stab}_G(x) \) is generated by \( \{t_it_i^{-1} \mid i = 2, \ldots, m\} \) and \( \text{Stab}_H(x) \).

**Example 8.** Like in the example above, let \( G = (V,E) \) be a graph with \( n \) vertices and let \( H := \text{Stab}_{S_n}(1) \) be the stabilizer of \( 1 \in V \). Then, we may define the canonization of \( G \) under the action of \( S_n \) by comparing the canonical forms under the action of \( H \) for the \( n \) graphs derived by interchanging the vertices \( 1 \) and \( i, i = 1, \ldots, n \).

For example, we may apply this approach if the number of neighbors is constant on \( G \). Then, the separation of \( 1 \in V \) allows us to color the vertex \( 1 \) differently from all others and to count neighbors by colors again. This may result in different values and would allow us to define the canonization under \( H \) with the help of the Homomorphism Principle.

2.4 Partitions and Refinements

As we have seen in Example 8 it makes sense to combine the methods of Subsections 2.2 and 2.3. The basic idea is to alternate between both methods and is known as the partition refinement method: \( \text{Can}_G(x) \) is recursively computed via

1. the application of the Homomorphism Principle for a well-defined sequence of homomorphisms of group actions which may lead to a smaller stabilizer \( G' \) and the element \( x' = gx \).

2. If the group \( G' \) is not trivial, we apply the lifting approach for a well-defined subgroup \( H \leq G' \) and recursively continue the computation of \( \text{Can}_H(tx') \) for \( t \in T \) in a similar way. Otherwise, we just return \( (x', \text{id}_{G'}, \{\text{id}_{G'}\}) \).
In this formulation, the different canonization processes \( \text{Can}_H(tx') \) for the right transversal elements \( t \in T \) in the lifting approach are carried out independently. Of course, making use of some global information in this processes could further reduce the computational complexity. The partition refinement method also considers this problem as we will see later. For this reason, we will replace the above formulation by a backtracking approach.

Partition refinement methods are widely used in the canonization of combinatorial objects, for equivalence tests and automorphism group computations, for instance \[4,5,9,10\]. In most cases the authors restrict themselves to the action of the symmetric group or some special subgroups. The formulation above shows that the ideas presented there are also applicable for arbitrary groups.

Nevertheless, we will similarly formulate our algorithm only for the action of the symmetric group. The main reason for this restriction is an easier description of the algorithm and some observations which we can only give in this special case. We will later see, that there is an action of the symmetric group in our problem, too.

A partition of \([n] := \{1, \ldots, n\}\) is a set \( p = \{P_1, \ldots, P_l\} \) of disjoint nonempty subsets of \([n]\) whose union is equal to \([n]\). We call the subsets \( P \in p \) cells of the partition. Cells of cardinality 1 are singletons and the partition \( p \) is discrete if all its cells are singletons. If all cells of a partition \( p \) are intervals we call \( p \) a standard partition. In the following we will always use upper-case letters for standard partitions. The stabilizer

\[
S_p := \text{Stab}_{S_n}(p) := \bigcap_{P \in p} \text{Stab}_{S_n}(P)
\]

of the (standard) partition \( p \) is a (standard) Young subgroup of \( S_n \). With \( \text{Fixed}(p) := \{i \in [n] \mid \{i\} \in p\} \) we refer to those indices which define singletons of \( p \), i.e. fixed points under the group action of \( S_p \).

The partition \( p \) is finer than the partition \( p' \) if each cell \( P \in p \) is a subset of some cell of \( p' \). We also call \( p \) a refinement of \( p' \) and say that \( p' \) is coarser than \( p \).

Differently to \[9,10\] our approach only uses standard partitions where the ordering of the cells is naturally defined by the elements they contain. This difference is due to the fact that we maintain a coset \( S_p \pi \), of a standard Young subgroup, which is the key data structure in all algorithms, by the pair \((\Phi, \pi)\) instead.
Suppose there is the group action of a standard Young subgroup $S_{P_0}$ on a set $X$. For an element $x \in X$ we can compute its unique canonical form and its stabilizer using a backtrack procedure on the following search tree, see also Figure 1:

- The root node of the search tree is $(P_0, \text{id})$ and we will apply a refinement on it as described below.
- The nodes $(P, \pi)$ where $P$ is discrete define leaves of the tree.
- Otherwise, i.e. in the case that $P$ is not discrete, we perform an individualization-refinement step:
  - Choose a well-defined cell $P \in \mathfrak{P}$ which is not a singleton, called the target cell and use the lifting approach for $S_{P'} := \text{Stab}_{S_{P'}}(m) \leq S_{\mathfrak{P}}$ where $m = \min(P)$: The refinement $P'$ of $P$ is derived by separating the minimal element $m \in P$, i.e. replace $P$ by $\{m\}$ and $P \setminus \{m\}$. If $T$ is a right transversal of $S_{P'}$ in $S_{\mathfrak{P}}$, the $|T| = |P|$ different children of the actual node are constructed by applying the permutations $t \in T$.
  - A refinement of $P'$ for the node $(P', t\pi)$ could be computed via the application of the Homomorphism Principle using a fixed $S_{P'}$-homomorphism $f_{P'} : X \to Y$. We choose the action and the canonical forms in $Y$ in such a way that their stabilizers are again standard Young subgroups.

Let $\sigma := \text{TR}_{S_{P'}}(f_{P'}(t\pi x))$ and $S_{\mathfrak{R}} := \text{Stab}_{S_{P'}}(f_{P'}(\sigma t\pi x))$ be the result of the canonization of $f_{P'}(t\pi x)$. The principle tells us that we have to canonize the element $\sigma t\pi x$ under the action of the group $S_{\mathfrak{R}}$ which is based again on further individualization-refinement steps.

Traversing this tree in a depth-first search manner corresponds to the aforementioned alternating application of the Homomorphism Principle and the lifting approach. We are able to define the canonical representative $\text{Can}_{S_{\mathfrak{P}}}(\pi x)$

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1 The target cell selection is an $S_{\mathfrak{P}}$-invariant.
if we have visited all children of the node \((\mathcal{P}, \pi)\). So far, the canonization under the action of \(S_{\mathcal{P}}\) are still independent processes for different \(t \in T\). The following definition of a total ordering \(\leq_{\mathcal{P}}\) on \(X\) allows us to change this:

\[
x \leq_{\mathcal{P}} y \iff f_{\mathcal{P}}(x) \prec_Y f_{\mathcal{P}}(y) \lor (f_{\mathcal{P}}(x) = f_{\mathcal{P}}(y) \land x \leq_X y)
\]

where \(\leq_X\) and \(\leq_Y\) are still some arbitrary total orderings on \(X\) and \(Y\) respectively.

This ordering will be used in the lifting approach for the definition of the minimum and it allows us to prune the search tree, i.e. skip the canonization \(\text{Can}_{S_{\mathcal{P}}}(t_2 \pi x)\) in the following situation: If the canonical form \(\sigma_1 f_{\mathcal{P}}(t_1 \pi x)\) is smaller than the canonical form \(\sigma_2 f_{\mathcal{P}}(t_2 \pi x)\) in the Homomorphism Principle for two nodes arising in an individualization step, we prune the subtree rooted in \((\mathcal{P}', t_2 \pi)\).

**Remark 9.** For the sake of simplicity, we did not use homomorphisms of group actions in the formulation of the refinement step and we restricted the formulations to \(S_{\mathcal{P}}\)-homomorphisms \(f_{\mathcal{P}}\). The function \(f_{\mathcal{P}}\) itself might be a concatenation of several functions which allow a successive application of the Homomorphism Principle. In this case, we adapt the ordering \(\leq_{\mathcal{P}}\) such that we may prune the tree in some intermediate step as well.

In the case that \(f_{\mathcal{P}}(x) = f_{\mathcal{P}}(y)\), we may recursively use \(x \leq_{\mathcal{P}} y \iff x \geq_{X} y\), where \(S_{\mathcal{P}} := \text{Stab}_{S_{\mathcal{P}}}(x)\), to compare \(x\) and \(y\). Only in the case that \(\mathcal{P}\) is discrete, i.e. the corresponding node is a leaf, we use some fixed ordering on \(X\). This also shows that we are not only allowed to compare the children of a fixed node among each other. In fact, we can prune a node \((\mathcal{P}', \pi)\) of the search tree if there is another node \((\mathcal{P}', \sigma)\) on the same level having the same values for all \(S_{\mathcal{P}}\)-homomorphism \(f_{\mathcal{P}}\) applied from the root down to these nodes and whose actual image \(f_{\mathcal{P}}(\pi x)\) is larger than \(f_{\mathcal{P}}(\sigma x)\).

**Theorem 10.** Let \(\mathcal{D}\) denote the discrete partition of \([n]\). Suppose that \((\mathcal{D}, \pi)\) is the last visited leaf of this pruned search tree. The mapping

\[
\text{Can}_{\mathcal{G}}(x) := (\pi x, \pi, \{\pi^{-i} \pi_i \mid i \in [a]\})
\]

defines a canonization, where \(\pi_1, \ldots, \pi_a\) are those permutations leading to all other leaf nodes \((\mathcal{D}, \pi_i)\) with \(\pi_i x\) equal to \(\pi x\).

**Proof.** Let \(F := (f_{\mathcal{P}_0}, \ldots, f_{\mathcal{P}_r})\) be the sequence of \(S_{\mathcal{P}}\)-homomorphism, \(i = 0, \ldots, r\) applied in the generation process to the leaf node \((\mathcal{D}, \pi)\). The sequence \(F\) defines a total ordering on \(X\):

\[
x \leq_{F} y \iff (f_{\mathcal{P}_0}(x), \ldots, f_{\mathcal{P}_r}(x), x) \leq (f_{\mathcal{P}_0}(y), \ldots, f_{\mathcal{P}_r}(y), y)
\]
The pruning ensures that all other leaves \((\mathcal{O}, \sigma)\) of this search tree will lead to orbit elements \(\pi x \leq_F \sigma x\). This shows that all leaves which were not visited correspond to orbit elements that compare strictly larger than \(\pi x\). Therefore, \(\pi x\) is the minimal orbit representative of \(S_{\mathcal{P}_0}x\) under \(\leq_F\). It is not difficult to prove that starting this backtracking algorithm for some other element \(x' \in S_{\mathcal{P}_0}x\) will lead to the same orbit representative \(\pi' x' = \pi x\).

Obviously, the depth-first-search strategy only allows the pruning of a subtree based on some partial information. In particular, we have to explore subtrees which will later be discarded. A breadth-first-search strategy would avoid this behavior. Nevertheless, there are more advantages of a depth-first search approach: First of all, there are no storage limitations since we only have to maintain the path from the root node to the actual node. Furthermore, it is possible to discover automorphisms of the object \(x\), since those can only be computed by the comparison of leaf nodes. The group of known automorphisms of \(x\) allows us to perform a further pruning of the search tree. For this goal, we use the methods described in [4, Section 5.2]: We store the subgroup of already known automorphisms \(A \leq S_n\) by a complete labeled branching. [4, Lemma 5.9] now gives a simple criterion if the coset \(S_{\mathcal{P}} \pi\), i.e. the subtree below the node \((\mathcal{P}, \pi)\), has to be traversed or not.

Finally, we would like to mention that [12] discusses a mixture of both strategies in the computation of a canonical form of a graph. We think that this approach might be applicable in our case as well, but we are not yet sure about all the consequences because we have to incorporate a second group action on \(x\) at the same time, as we will see in the following section.

3 A reformulation of the problem

The projective space \(P_q(k)\) is the set of all subspaces of \(\mathbb{F}^k_q\). As usual, we call the one-dimensional subspaces points and the \((k-1)\)-dimensional subspaces hyperplanes. Let \(\text{Aut}(\mathbb{F}_q)\) denote the automorphism group of \(\mathbb{F}_q\). Recall that any automorphism \(\alpha \in \text{Aut}(\mathbb{F}_q)\) is a power \(\tau^a\) of the Frobenius automorphism \(\tau: x \mapsto x^p\). It applies to vectors and matrices element-wise. The set of all semilinear mappings, i.e. the general semilinear group \(\Gamma L_k(q) := \text{GL}_k(q) \rtimes \text{Aut}(\mathbb{F}_q)\), decomposes as a semidirect product with multiplication \((A, \tau^a)(B, \tau^b) := (A \tau^a(B), \tau^{a+b})\).

**Remark 11.** Let \(N, H\) be arbitrary groups and \(\varphi: H \rightarrow \text{Aut}(N)\) be a homomorphism between the group \(H\) and the automorphism group of \(N\). Although the multiplication of elements \((n_1, h_1), (n_2, h_2) \in N \rtimes_{\varphi} H\) depends on the choice of \(\varphi\), i.e. \((n_1, h_1)(n_2, h_2) := (n_1 \varphi(h_1)(n_2), h_1h_2)\), we will not give the exact definition of the homomorphism \(\varphi\) when introducing semidirect products.
of groups in the following. We believe that the right choice of \( \varphi \) can always be observed from the context.

There is a natural action of \( \Gamma L_k(q) \) on the projective space \( P_q(k) \) from the left, i.e.

\[
\Gamma L_k(q) \times P_q(k) \rightarrow P_q(k) \\
((A, \tau^a), U) \mapsto A\tau^a(U).
\]

Since this action is not faithful, one may also factor out the kernel resulting in the action of \( P\Gamma L_k(q) := \Gamma L_k(q)/\mathbb{F}_q^\ast \) on \( P_q(k) \), where \( \mathbb{F}_q^\ast \) denotes the multiplicative group of \( \mathbb{F}_q \). We use both groups and both actions interchangeably.

The goal of this paper is the description of a practical canonization algorithm for a given sequence of (multi-) sets \( C = (C_1, \ldots, C_m) \), with \( C_i \subseteq P_q(k) \) under the action of \( \Gamma L_k(q) \). The stabilizer subgroup

\[
\Aut(C) := \text{Stab}_{\Gamma L_k(q)}(C) := \bigcap_{i=1}^m \{(A, \tau^a) \in \Gamma L_k(q) \mid \forall U \in C_i : (A, \tau^a)U \in C_i\}
\]

is computed by the algorithm at the same time without any additional effort. We apply this algorithm to solve canonization problems in coding theory, see Section 4.

The remaining part of this section deals with further modifications of the given sequence \( C \) we could make:

- We may assume that the multisets \( C_i \) are in fact disjoint subsets. Otherwise, we could distinguish the occurring subspaces by their sequence of multiplicities, which leads to a sequence of disjoint subsets \( C' \). This defines an \( \Gamma L_k \)-homomorphism and the stabilizer of \( C' \) acts trivially on \( C \).
- The action of \( \Gamma L_k(q) \) preserves the dimension of any \( U \in P_q(k) \). Hence, asking for a canonization algorithm for a set \( C_i \) is equivalent to ask for a canonization of the sequence \( \{U \in C_i \mid \dim(U) = s\} \). Therefore, we can assume that all elements of a subset \( C_i \) have fixed dimension \( 0 \leq s_i \leq k \).
- If some subset \( C_i \) is empty or equal to \( \{U \in P_q(k) \mid \dim(U) = s\} \), i.e. the subset of all \( s \)-dimensional subspaces of \( \mathbb{F}_q^k \) for some \( s = 0, \ldots, k \), we can remove \( C_i \) from the sequence since the action of \( \Gamma L_k(q) \) on this subset is trivial. Therefore we could suppose that \( 1 \leq s_i \leq k - 1 \).
- The union \( \bigcup_{i=1}^m C_i \) spans the whole space, otherwise we would be able to solve the problem in a smaller ambient space \( \mathbb{F}_q^{k'}, k' < k \).

In the following, we suppose that the sequence \( C = (C_1, \ldots, C_m) \) and therefore also the parameters \( q, k, m, n_i, s_i, n = \sum_{i=1}^m n_i \) will be fixed.
Proposition 12. For a given subspace \( U \in \mathbb{P}_q(k) \) let \( U^\perp := \{ v \in \mathbb{F}_q^k \mid v^Tu = 0 \} \) be its dual subspace. The dual subspace of \((A, \tau^a)\)\( U \) for \((A, \tau^a) \in \Gamma \mathbb{L}_k(q)\) is equal to \((A^{-1}T, \alpha)\)\( U^\perp\).

Proof. Let \( U \in \mathbb{P}_q(k)\), \((A, \tau^a) \in \Gamma \mathbb{L}_k(q)\) and \( u \in U, v \in U^\perp \) be arbitrary. The equation

\[
((A^{-1}T, \tau^a)v)^T(A, \tau^a)u = \tau^a(v)^T A^{-1} A \tau^a(u) = \tau^a(v^Tu) = \tau^a(0) = 0
\]

shows that \((A^{-1}T, \tau^a)U^\perp \subseteq ((A, \tau^a)U)^\perp\). But both subspaces have dimension \( n - \dim(\mathcal{U}) \) and hence must be equal. \(\square\)

Remark 13. If we define \( \mathcal{C}_i^+ := \{ U \mid U \in \mathcal{C}_i \} \) and \( \mathcal{C}^\perp := (\mathcal{C}_1^+, \ldots, \mathcal{C}_n^+) \) then we may also canonize \( \mathcal{C}^\perp \), i.e. compute \( \text{Can}_{\Gamma \mathbb{L}_k(q)}(\mathcal{C}^\perp) \), and define the canonical form \( \text{CF}_{\Gamma \mathbb{L}_k(q)}(\mathcal{C}) := \left( \text{CF}_{\Gamma \mathbb{L}_k(q)}(\mathcal{C}^\perp) \right)^\perp \). The automorphism group of \( \mathcal{C} \) is equal to

\[
\left\{ (A^{-1}T, \tau^a) \mid (A, \tau^a) \in \text{Aut}(\mathcal{C}^\perp) \right\}.
\]

This transformation will always be applied if we suppose that the computational effort of computing \( \text{Can}_{\Gamma \mathbb{L}_k(q)}(\mathcal{C}^\perp) \) is less expensive than the computation of \( \text{Can}_{\Gamma \mathbb{L}_k(q)}(\mathcal{C}) \).

Let \( \mathbb{F}_q^{k \times n,s} \) denote the set of \( k \times n \)-matrices of rank \( s \). The algorithm we are going to develop is a generalization of the canonization algorithm for linear codes, see [4][5] and Section 5.1 for a short summary. Instead of working on linear codes directly, the problem is transferred to generator matrices of linear codes, i.e. matrices whose rows form an \( \mathbb{F}_q \)-basis of the linear code. Two matrices \( \Gamma, \Gamma' \in \mathbb{F}_q^{k \times n,k} \) generate equivalent codes, if their orbits under the group action of \((\Gamma \mathbb{L}_k(q) \times \mathbb{F}_q^{n,k}) \rtimes (S_n \times \text{Aut}(\mathbb{F}_q))\) are the same. It is a well-known fact [2][9.1.2] that there is a one-to-one correspondence of these orbits and the orbits of \( \Gamma \mathbb{L}_k(q) \) on multisets of at most \( n \) points in the projective space, which span a vector space of dimension \( k \). Therefore, the canonization algorithm for linear codes already solves the canonization problem for any multiset of points. A closer look reveals that the algorithm similarly transfers the multiset to a sequence of disjoint sets of points.

Now, represent the element \( U \in \mathcal{C}_i \) by some matrix \( U \in \mathbb{F}_q^{k \times n,s} \) whose columns generate \( \mathcal{U} \). The set of all matrices generating \( \mathcal{U} \) in this regard is equal to the orbit \( \Gamma \mathbb{L}_s(q)^{U} \) \( \{ U A^T \mid A \in \Gamma \mathbb{L}_s(q) \} \). Analogously, we can identify the set \( \mathcal{C}_i := \{ U_1, \ldots, U_n \} \) with the orbit of \( S_n \) on \( (U_1, \ldots, U_n) \). In summary, there is a natural one-to-one correspondence between the set \( \mathcal{C}_i \) and the orbit of \((U_1, \ldots, U_n) \) under the action of \( \Gamma \mathbb{L}_s(q)^{n_1} \rtimes S_n \). This semidirect product is equal to the wreath product \( \Gamma \mathbb{L}_s(q) \wr S_n \).
In the case that $s_i = 1$ we know that $GL_1(q) = F_q^*$ and the group $GL_1(q)^{n_i} \rtimes S_{n_i} = F_q^*$ isomorphic to the group of $F_q^*$-monomial matrices, i.e. the set of permutation matrices whose nonzero entries got replaced by elements from $F_q^*$. In this regard, we can view the wreath product $GL_{s_i}(q) \rtimes S_{n_i}$ as the group of $GL_{s_i}(q)$-monomial matrices by replacing the nonzero entries of a permutation matrix by arbitrary elements from $GL_{s_i}(q)$ and the zero entries by $(s_i \times s_i)$-zero matrices.

Altogether, we can identify the sequence $C$ with the orbit of $(U^{(i)}_1, \ldots, U^{(i)}_{n_i})_{i \in [m]}$ under the action of $\prod_{i=1}^m (F_{k \times s_i \times s_i})^{n_i}$

under the action of $\prod_{i=1}^m (GL_{s_i}(q)^{n_i} \rtimes S_{n_i})$ which could be interpreted as the group of block diagonal matrices whose $m$ nonzero blocks are equal to $GL_{s_i}(q)$-monomial matrices.

Finally, taking the action of $\Gamma L_k(q)$ into account we have to canonize the sequence $(U^{(i)}_1, \ldots, U^{(i)}_{n_i})_{i \in [n]}$ under the action of

$$\left( GL_k(q) \times \prod_{i=1}^m (GL_{s_i}(q)^{n_i} \rtimes S_{n_i}) \right) \rtimes Aut(F_q),$$

where the action is defined as follows:

Let $\left( A, (B^{(i)}_1, \ldots, B^{(i)}_{n_i}, \pi^{(i)})_{i \in [m]}, \tau^a \right)$ be an element of the acting group then

$$\left( A \tau^a \left( U^{(i)}_{\pi^{(i)}-1(1)} B^{(i)}_1, \ldots, A \tau^a \left( U^{(i)}_{\pi^{(i)}-1(n_i)} B^{(i)}_{n_i} \right) \right) \right)_{i \in [m]}.$$

In order to apply the methods developed in the Subsection 2.4, we change the order in which we compose the group:

$$\left( GL_k(q) \times \prod_{i=1}^m GL_{s_i}(q)^{n_i} \rtimes Aut(F_q) \right) \rtimes \prod_{i=1}^m S_{n_i} \simeq \left( GL_k(q) \times \prod_{i=1}^m GL_{s_i}(q)^{n_i} \rtimes Aut(F_q) \right) \rtimes S_{\mathfrak{P}_0}$$

and replace the permutational part of this group using the standard Young subgroup $S_{\mathfrak{P}_0}$ to the partition $\mathfrak{P}_0 := \{1, \ldots, n_1\}, \ldots, \{n-n_m+1, \ldots, n\}$ of $[n]$. 
A final reformulation of our problem will be given in Section 5 since we would like to motivate the algorithm with the observations given in [4]. It will show that we could observe a homomorphic group action of the symmetric group $S_{q}$. This allows us to apply the ideas developed in Subsection 2.4.

As we have seen above, the comparison of the objects we are working with plays a central role in the canonization process. In our case, if nothing else is stated, we will suppose that $\mathbb{F}_q$ is totally ordered such that $0 < 1 \leq \mu$ for all $\mu \in \mathbb{F}_q^*$. Then we can totally order the set of $k \times n$-matrices by interpreting them as lexicographically ordered sequences of colexicographically ordered column vectors.

Furthermore, we will access submatrices of a matrix $U \in \mathbb{F}_q^{k \times n}$ in the following way:

- $U_{*,i}$ denotes the $i$-th column of $U$. Similarly, we write $U_{i,*}$ for the $i$-th row.
- For a sequence $I := (i_1, \ldots, i_m)$ of indices $i_j \in [n]$ we write $U_{*,I} := (U_{*,i_1}, \ldots, U_{*,i_m})$ for the projection of the matrix onto the columns given by $I$. We also use this notation for the set $I := \{i_1, \ldots, i_m\}$ which should be interpreted as the lexicographically ordered sequence of its elements.
- Finally, if $J$ is a sequence of indices in $[k]$, then $U_{J,*}$ denotes a similar access to the rows of $U$ and $U_{J,I} := (U_{J,*})_{*,I}$.

4 Coding theory

Let $(M_1, d_1)$ and $(M_2, d_2)$ be two metric spaces. A map $\iota : M_1 \rightarrow M_2$ is an isometry if it respects distances, i.e. $d_2(\iota(x), \iota(y)) = d_1(x, y)$ for all $x, y \in M_1$.

4.1 Random network codes

The subspace distance is a metric on the projective space $\mathbb{P}_q(k)$ given by

$$d_S(\mathcal{U}, \mathcal{V}) := \dim(\mathcal{U} + \mathcal{V}) - \dim(\mathcal{U} \cap \mathcal{V}) = \dim(\mathcal{U}) + \dim(\mathcal{V}) - 2 \dim(\mathcal{U} \cap \mathcal{V})$$

for any $\mathcal{U}, \mathcal{V} \in \mathbb{P}_q(k)$. It is a suitable distance for coding over the operator channel using so-called random network codes $\mathcal{C} \subset \mathbb{P}_q(k)$, see [7].

Obviously, the action of an element of the general semilinear group preserves the subspace distance. On the other hand, [13] showed that $\operatorname{PGL}_k(q)$ is isomorphic to the group of isometries on $\mathbb{P}_q(k)$ which preserve the dimension of each element in $\mathbb{P}_q(k)$. The dimension is another basic property of a codeword.
which should be preserved, too. Therefore, it makes sense to define equivalence of random network codes by means of this group action. It shows that the canonization of random network codes is a special case of our algorithm for sequences of length one.

4.2 Additive codes

An $\mathbb{F}_q$-linear block code over the alphabet $\mathbb{F}_q$, $s \geq 1$ is an $\mathbb{F}_q$-linear subset of $\mathbb{F}_q^n$ equipped with the usual Hamming distance $d_{\text{Ham}}$. Additive codes with $s = 1$ are classical linear codes. For $s > 1$, those codes are sometimes also called additive codes. They gained more and more interest in the past years since for example self-orthogonal additive codes over $\mathbb{F}_q^2$ could be used for quantum error-correction, see [1].

With an $\mathbb{F}_q$-linear representation of the elements of $\mathbb{F}_q^s$ in $\mathbb{F}_q^n$, the $\mathbb{F}_q$-linear code can be represented by a generator matrix with entries in $\mathbb{F}_q$. Let $T : \mathbb{F}_q^s \rightarrow \mathbb{F}_q^n$ denote the corresponding $\mathbb{F}_q$-linear mapping. Defining the distance

$$d_{\text{Ham}}(x, y) := d_{\text{Ham}}(T^{-1}(x), T^{-1}(y)) = \begin{cases} 0 & x = y \text{ for } x, y \in \mathbb{F}_q^s \\ 1, & \text{else} \end{cases}$$

and extending this definition as usual to $\mathbb{F}_q^n$ we are able to find all isometries on $\mathbb{F}_q^n$, mapping $\mathbb{F}_q$-linear codes onto $\mathbb{F}_q$-linear codes:

- The multiplication of $\mathbb{F}_q^s$ by an invertible matrix $A \in \text{GL}_s(q)$ defines an $\mathbb{F}_q$-linear isometry.
- The same holds for the permutation of the $n$ components of $\mathbb{F}_q^n$.
- The element-wise application of an automorphism of $\mathbb{F}_q$ defines an isometry on $\mathbb{F}_q^n$, which maps $\mathbb{F}_q$-linear codes onto $\mathbb{F}_q$-linear codes.

Altogether, this defines a group action of $\text{GL}_s(q)^n \rtimes (S_n \times \text{Aut}(\mathbb{F}_q))$ on the set of $\mathbb{F}_q$-linear subsets of $\mathbb{F}_q^n$. Since isometries are injective, we could also restrict this action to act on subsets $C$ with $\dim_{\mathbb{F}_q}(C) = k$. Each such subset $C$ could be represented by a generator matrix $\Gamma \in \mathbb{F}_q^{k \times s \times n}$. The set of all generator matrices of $C$ is equal to the orbit $\text{GL}_k(q)\Gamma$. Hence, we are interested in the canonization of a generator matrix under the group action of

$$(\text{GL}_k(q) \times \text{GL}_s(q)^n) \rtimes (S_n \times \text{Aut}(\mathbb{F}_q)).$$

Since every $s$ consecutive columns may define the same subspace, the code could be identified with a multiset of subspaces of $\mathbb{F}_q^k$. 

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5 The Algorithm

In this section we develop a practical algorithm which computes the automorphism group and a canonical form of a given sequence $C = (C_1, \ldots, C_m)$. In Section 2 we have seen why it is useful to combine the searches for both tasks.

5.1 The algorithm for linear codes revisited

First of all, we want to motivate our procedure by a reformulation of the canonization algorithm in [4] for linear codes using the language developed in Section 2. Some of the ideas we are going to introduce are based on [5] which gives a more detailed description of the backtracking approach. The algorithm that we are going to develop in the subsequent subsections can be seen as a natural generalization of the one for linear codes.

We first observe that we could compute a canonical form of a $k$-dimensional linear code $C$ with generator matrix $\Gamma \in \mathbb{F}_q^{k \times n}$ using the ideas presented in Section 2.4: For simplicity, let $G^{(sl)} := \left(\text{GL}_k(q) \times \mathbb{F}^{\times n}_q\right) \rtimes \text{Aut}(\mathbb{F}_q)$. The group $S_{\mathbb{F}_q}$ acts on the set of orbits $G^{(sl)} \backslash \mathbb{F}_q^{k \times n} := \{G^{(sl)} \Gamma' \mid \Gamma' \in \mathbb{F}_q^{k \times n}\}$ and we can define a homomorphism of group actions

$$\begin{pmatrix}
\theta : \mathbb{F}_q^{k \times n} & \rightarrow & G^{(sl)} \backslash \mathbb{F}_q^{k \times n},
\varphi : G^{(sl)} \ltimes S_{\mathbb{F}_q} & \rightarrow & S_{\mathbb{F}_q},
\Gamma & \mapsto & G^{(sl)} \Gamma
\end{pmatrix}.$$

Before we provide the details of the canonization $\text{Can}_{S_{\mathbb{F}_q}}(G^{(sl)} \Gamma)$ we explain how to define $\text{Can}_{G^{(sl)} \rtimes S_{\mathbb{F}_q}}(G^{(sl)} \Gamma)$ using the Homomorphism Principle: First of all, in [4] it is observed that there is a direct and efficient canonization algorithm $\text{Can}_{G^{(sl)}}$ for the action of $G^{(sl)}$ on $\mathbb{F}_q^{k \times n}$. Let $\pi = \text{TR}_{S_{\mathbb{F}_q}}(G^{(sl)} \Gamma)$ and $G^{(sl)} \rtimes H = \text{Stab}_{G^{(sl)} \rtimes S_{\mathbb{F}_q}}(\pi G^{(sl)} \Gamma)$ be the result of the canonization. Since we know that $(G^{(sl)} \rtimes H) \pi \Gamma = G^{(sl)} \pi \Gamma$ it remains to define $\text{CF}_{G^{(sl)} \rtimes H}(\pi \Gamma) := \text{CF}_{G^{(sl)}}(\pi \Gamma)$ and the transporter $\text{TR}_{G^{(sl)} \rtimes H}(\pi \Gamma) := (\text{TR}_{G^{(sl)}}(\pi \Gamma), \text{id})$. There is also a simple way to compute the automorphism group $\text{Stab}_{G^{(sl)} \rtimes H}(\Gamma)$ using the canonization under the action of $G^{(sl)}$. The details are left to the reader. We will later see that all necessary data is already computed in the computation of $\text{Can}_{S_{\mathbb{F}_q}}(G^{(sl)} \Gamma)$. 

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5.1.1 Backtrack search

This shows that we are able to give a practical canonization algorithm for linear codes if we are able to give a practical algorithm for the computation of $\text{Can}_{\mathcal{S}_0}(G^{(sl)}\Gamma)$. This algorithm will be based on the partition refinement idea. In this algorithm, it is necessary to compare the leaves of the backtrack search tree. Therefore, we have to define a total ordering on $G^{(sl)}(\mathbb{F}_q^{k \times n,k})$. Let $\text{Can}_{G^{(sl)}}$ be a canonization algorithm for the action of $G^{(sl)}$ on the set $\mathbb{F}_q^{k \times n,k}$, then we may order the orbits via the ordering on their canonical forms.

It remains to give the $S_{\mathbb{F}}$-homomorphisms which will be applied to a node $(\mathfrak{P}, \pi)$. The first is closely related to $\text{Can}_{G^{(sl)}}$. Let $\overrightarrow{\text{Fixed}}(\mathfrak{P}, \pi)$ be the sequence of elements of $\text{Fixed}(\mathfrak{P})$ in the order they appeared as singletons in the refinement process $\mathfrak{P}_0 \geq \ldots \geq \mathfrak{P}$ leading to this node $(\mathfrak{P}, \pi)$. We say that a matrix $\Gamma(\mathfrak{P}, \pi)$ is a semicanonical representative of the node $(\mathfrak{P}, \pi)$ if $\Gamma(\mathfrak{P}, \pi) \in G^{(sl)}(\pi)\Gamma$ and

$$\left(\Gamma(\mathfrak{P}, \pi)\right)_{*, \overrightarrow{\text{Fixed}}(\mathfrak{P}, \pi)} \leq \left(\Gamma'\right)_{*, \overrightarrow{\text{Fixed}}(\mathfrak{P}, \pi)}$$

for all $\Gamma' \in \pi G^{(sl)}\Gamma = G^{(sl)}(\pi)\Gamma$.

**Proposition 14.** The projection $\pi G^{(sl)}\Gamma \mapsto \left(\Gamma(\mathfrak{P}, \pi)\right)_{*, \overrightarrow{\text{Fixed}}(\mathfrak{P}, \pi)}$ is $S_{\mathbb{F}}$-invariant.

This invariant is applied immediately after each individualization step and after each refinement which leads to a new singleton in the partition $\mathfrak{P}$. Since it is an invariant, it will not refine the partition $\mathfrak{P}$. But, it will give us the possibility to prune the search tree.

For a child $(\mathfrak{R}, \sigma \pi)$ of $(\mathfrak{P}, \pi)$ the semicanonical representative could be easily computed from the semicanonical representative of $(\mathfrak{P}, \pi)$. For this computation, we only need to know the stabilizer $\text{Inn}(\mathfrak{P}, \pi) \leq G^{(sl)}$ of $\left(\Gamma(\mathfrak{P}, \pi)\right)_{*, \overrightarrow{\text{Fixed}}(\mathfrak{P}, \pi)}$, where the action is defined by

$$(A, b, \tau^a)\Gamma' := (A, (b_j)_{j \in \overrightarrow{\text{Fixed}}(\mathfrak{P}, \pi)}, \tau^a)\Gamma' \text{ for all } \Gamma' \in \mathbb{F}_q^{k \times |\overrightarrow{\text{Fixed}}(\mathfrak{P})|}.$$

For more details see [4].

Therefore, it makes sense to add this data to the nodes of the backtrack tree. We modify the nodes $(\mathfrak{P}, \pi)$ of Section 2.4 such that the algorithm additionally maintains the orbit $G^{(sl)}(\pi)\Gamma$ by the pair $(\Gamma(\mathfrak{P}, \pi), \text{Inn}(\mathfrak{P}, \pi))$. We will call the action by $\text{Inn}(\mathfrak{P}, \pi)$ the *inner group action* and $\text{Inn}(\mathfrak{P}, \pi)$ the *inner stabilizer*. The computation of the semicanonical representative of $(\mathfrak{P}, \pi)$ will also be called the *inner minimization process*.

**Remark 15.** The computation of a semicanonical representative itself could be seen as an application of the Homomorphism Principle applied to the $G^{(sl)}$-
homomorphism $\Gamma \mapsto \Gamma_{*,\text{Fixed}(\mathcal{P},\pi)}$. If $\mathcal{P}$ is discrete the semicanonical representative defines a canonical form of the orbit $G^{(d)}\pi \Gamma$.

**Remark 16.** There is also a second interpretation of the subtree below some node $(\mathcal{P}, \pi, \Gamma^{(\mathcal{P},\pi)}, \text{Inn}^{(\mathcal{P},\pi)})$. It could be identified as the canonization of $\Gamma^{(\mathcal{P},\pi)}$ under the action of $\text{Inn}^{(\mathcal{P},\pi)} \rtimes S_{\mathcal{P}}$. For the root node, the group $\text{Inn}^{(\mathcal{P}_0,\text{id})} \rtimes S_{\mathcal{P}_0}$ is equal to $G^{(d)} \rtimes S_{\mathcal{P}_0}$, i.e. the action we are actually interested in. We have motivated this backtracking with the canonization under $S_{\mathcal{P}_0}$, since

- we already proved its correctness in Section [2.4],
- the test on the group of known automorphisms is restricted to standard Young subgroups, and
- the homomorphism of group actions we are going to apply should have this special structure, i.e. they will be either $S_{\mathcal{P}}$-homomorphisms or equal to the inner minimization process. This will avoid the occurrence of complex subgroups of $G^{(d)} \rtimes S_{\mathcal{P}_0}$ for which it would be difficult to define appropriate homomorphisms of group actions in the refinement steps. Furthermore, the computation of the transversal $T$ in a individualization step would become more complicated, too.

Apart from some further $S_{\mathcal{P}}$-homomorphisms which make use of $(\Gamma^{(\mathcal{P},\pi)}, \text{Inn}^{(\mathcal{P},\pi)})$, there is another very important $S_{\mathcal{P}}$-homomorphism used to derive further refinements. In particular, this $S_{\mathcal{P}}$-homomorphism works already very well on nodes on the first levels of the backtrack tree. We are going to generalize it in the following. Again, we will give a more general description of this function than given in [16]. The basic idea is a modification of Leon’s algorithm, [9], for the computation of the automorphism group of a linear code:

### 5.1.2 Leon’s invariant set of codewords

Suppose that $W := \{c^{(1)}, \ldots, c^{((q-1)h)}\} \subseteq C$ is a set of codewords, which is invariant under the automorphism group of $C$. In fact, if one wants to apply Leon’s algorithm, [9], to test two linear codes for equivalence the mapping $C \mapsto W$ has to be an $(\mathbb{F}_q^n \rtimes (S_n \times \text{Aut}(\mathbb{F}_q)))$-homomorphism. For simplicity suppose that $W$ is formed by all words of minimal nonzero weight $d$. This set is once computed for the root node and fixed for the whole backtracking algorithm.

For each codeword $c^{(j)}$ there is a well-defined information word $v^{(j)} \in \mathbb{F}_q^n$ such that $v^{(j)} \Gamma = c^{(j)}$. Since $W$ is closed under scalar multiplication, we may restrict ourselves to the set $\overline{W}$ of projective representatives and define $H := \{v^{(j)} | j = 1, \ldots, h\} := \{v \in \mathbb{F}_q^n | v \Gamma \in \overline{W}\}$ the corresponding set of information words.

The standard inner product of $\langle v^{(j)} \Gamma, \Gamma_{*,i} \rangle = v^{(j)^T} \Gamma_{*,i}$ is equal to $c_i^{(j)}$, the $i$-th
coordinate of the vector \( e^{(j)} \). Therefore, the set \( \mathcal{H} \) is the well defined subset of all normal vectors of those hyperplanes containing exactly \( n - d \) points \((\Gamma_{s,i})\).

This gives a bipartite, vertex-colored subgraph of the subspace lattice of the projective space, whose vertices are labeled by \([n]\) and \(\{n+1, \ldots, n+h\}\) respectively. Initially, the colors only distinguish vertices by dimension and in the case of points additionally by the cell they are contained in. Since the action by \(\Gamma_{k}(q)\) obviously preserves the graph structure, this graph is independent from the actual representative of \(G^{(sl)}\). Furthermore, it is well-defined up to the action of \(S_{h}\) on the vertices \(\{n+1, \ldots, n+h\}\). The permutation of the columns of the generator matrix results in a relabeling of the vertex set \([n]\).

Cell-wise counting of neighbors for each vertex allows us to define an \(S_{[n]} \times S_{h}\)-homomorphism and hence to apply the Homomorphism Principle to refine the partition (coloring) of \([n+h]\). Since the projection on the first \(n\) components also obviously defines an \(S_{[n]}\)-homomorphism, this could be also seen as a refinement of the root node of the backtrack search tree. The resulting permutation in the application of the Homomorphism Principle gives a simultaneous relabeling of the graph and a permutation of the columns of the generator matrix.

This homomorphism on the incidence graph is also used in the computation of a canonical form of an arbitrary graph [10]. We furthermore observe that the finer partitions \(\mathfrak{R}_0\) of \([n]\) and \(\mathfrak{Q}_0\) of \([h]\) allow us to call this invariant iteratively. Hence, the nodes of the backtrack tree additionally maintain a partition \(\mathfrak{Q}\) of the set \([h]\). Furthermore, instead of storing the relabeled graph, we just maintain a second permutation \(\sigma \in S_{h}\) which stores the relabeling on the vertices \(\{n+1, \ldots, n+h\}\). Any refinement on the partition \(\mathfrak{P}\) during the backtrack search gives us the possibility to restart this refinement process on the relabeled, newly colored incidence graph again.

5.2 Preprocessing and the backtrack tree

In the same manner, we start the algorithm for the sequence \(\mathcal{C}\) by some preprocessing routine, which is in fact an \(\Gamma_{k}(q)\)-homomorphism computing a set \(\mathcal{H}\) of hyperplanes: For each hyperplane \(\langle v \rangle^\perp \in P_{q}(k)\) we may compute the number of elements it contains from each cell of \(\mathfrak{P}_0\). This results in a unique partition of the set of all hyperplanes. We choose a well-defined subset \(\mathcal{H} := \bigcup_{i=1}^{m} \mathcal{H}_i\), which is a union of the blocks of this partition in such a way that it contains \(k\) linearly independent normal vectors. This set should also be reasonably small.

**Example 17.** Let \(\mathcal{C} := \{\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3\} \subset P_{3}(4)\) with \(\mathcal{U}_i\) generated by \(U_i\):
The hyperplanes $H_1 := \{ (0,1,0,0)^T, (0,0,0,1)^T \}$ contain exactly 2 elements of $C$, whereas the elements $\langle (0,1,0,\nu)^T \rangle^\perp$, $\langle (\mu,1,0,0)^T \rangle^\perp$ and $\langle (0,0,\mu,1)^T \rangle^\perp$ for $\nu \in F_3^*, \mu \in F_3$ contain a single element of $C$. They form the set $H_2$. The remaining hyperplanes contain none of the elements from $C$. We choose the set $H = H_1 \cup H_2$ since it is the smallest set spanning the whole vector space.

In difference to the algorithm for linear codes described above, we do not only use this set as a tool for the refinement of a node. In fact, we append the sequence $(H_1, \ldots, H_{m'})$ to $C$ and it plays an active role in the construction of the backtrack search tree, as we will see later. We are allowed to do so because of the following proposition:

**Proposition 18.** Let $G$ be a group which acts on $X$ and $f : X \to Y$ a $G$-homomorphism. Suppose there is a canonization algorithm $\text{Can}_G((x,f(x)))$ for the action of $G$ on $\{(x,y) \mid x \in X, y \in Y\}$. Then, this defines a canonization algorithm on $X$ via

$$\text{Can}_G(x) := (\text{CF}_G((x,f(x)))_1, \text{TR}_G((x,f(x))), \text{Stab}_G((x,f(x)))).$$ 

One reason for this decision is the fact that the partition $Q$ of $[h], h := |\mathcal{H}|$ allows us to perform the individualization step on a cell with smaller cardinality and hence results in a smaller branching factor, which we see as an advantage. On the other hand, we realized that it is much more difficult to give an efficient canonization algorithm for the action of

$$\left(\text{GL}_k(q) \times \prod_{i=1}^m \text{GL}_{s_i}(q)^{n_i}\right) \rtimes \text{Aut}(F_q)$$

on the sequence of subspaces whose coordinates are fixed by $S_\mathcal{P}$.

Let $\mathcal{P}$ be a refinement of $\{\{1,\ldots,n\}, \{n+1,\ldots,n+h\}\}$. In the following we use $\mathcal{P}_C$ to refer to the corresponding partition of $[n]$ and $\mathcal{P}_H$ to refer to the partition of the set $[h]$ arising from the partition of the last $h$ coordinates given by $\mathcal{P}$.

Instead of representing the hyperplane $H \in \mathcal{H}$ using a matrix $U \in F_q^{k \times (k-1) \times k-1}$ we use its dual space which could be represented by a single vector $v \in F_q^k$ - a normal vector of $H$. Since $(A,\alpha)H^\perp = \langle (A^{-1}^T, \alpha)v \rangle^\perp$, see Proposition 12, we have to keep in mind that the action is differently defined on them.
Altogether, we are going to develop a canonization algorithm for the action of 
\(G^{(sl)} \rtimes S_0\) on \(\prod_{i=1}^m (\mathbb{F}_q^k \times s_i)^{n_i} \times (\mathbb{F}_q^h)^h\) where

\[G^{(sl)} := \left( \text{GL}_k(q) \times \prod_{i=1}^m \text{GL}_{s_i}(q)^{n_i} \times \mathbb{F}_q^h \right) \rtimes \text{Aut}(\mathbb{F}_q)\]

and \(S_0\) is the partition of \([n+h]\) given by the different subsets. The action of \(G^{(sl)}\) on a vector \((U, V)\) is defined in the following way:

\[
\begin{align*}
\left( A, (B_i)_{i \in [n]}, (b_j)_{j \in [h]}, \tau^a \right) & \cdot \left( (U_i)_{i \in [n]}, (v_j)_{j \in [h]} \right) \\
& = \left( (A \tau^a(U_i) B_i^T)_{i \in [n]}, (A^{-1} \tau^a(v_j)b_j)_{j \in [h]} \right)
\end{align*}
\]

Before we are going to describe the rules for building up the backtrack tree, in particular how to define an analogue inner minimization procedure and suitable \(S_0\)-homomorphisms, we shortly summarize which information should be contained in each node:

- A permutation \(\pi \in S_{n+h}\) and a standard partition \(P\) of \([n+h]\) describing the state of the backtrack tree analogously to Section 2.4.
- A vector \((U(P_0, \pi), V(P_0, \pi)) \in \left( \prod_{i=1}^m \mathbb{F}_q^k \times s(i), (\mathbb{F}_q^h)^h \right)\), storing the semicanonical representative of this node.
- A subgroup \(\text{Inn}(P, \pi) \leq \left( \text{GL}_n(q) \times \prod_{i=1}^m \text{GL}_{s(i)}(q) \times (\mathbb{F}_q^h)^h \right) \rtimes \text{Aut}(\mathbb{F}_q)\) which stores the stabilizer under the inner group action.

Similar to Remark 16, we can interpret this backtracking as an algorithm which computes \(\text{Can}_{S_0}(G^{(sl)}(U, V))\) or as the canonization \(\text{Can}_{G^{(sl)} \rtimes S_0}(U, V)\). Altogether, this solves our initial canonization problem for sequences of subsets in the projective space.

**Example 19** (Example 17 continued). We can choose

\[
(U^{(\mathbb{P}_0, \text{id})}, V^{(\mathbb{P}_0, \text{id})}) := \begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 & 1 & 2 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 2 & 1 & 1 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

as a semicanonical representative of the root node \((\mathbb{P}_0, \text{id})\) with the initial partition \(\mathbb{P}_0 := \{\{1, 2, 3\}, \{4, \ldots, 11\}, \{12, 13\}\}\).

In the above representation of \((U^{(\mathbb{P}_0, \text{id})}, V^{(\mathbb{P}_0, \text{id})})\) we already included the partition \(\mathbb{P}_0\); dashed lines mark the end of the matrices \(U_i^{(\mathbb{P}_0, \text{id})}\), whereas solid
Before starting the backtracking, we refine \( \mathfrak{P}_0 \) based on the incidence graph like in the case of linear codes, see also Subsection 5.4.2 for a detailed description in our case:

- We observe that \( \mathcal{U}_3 \) is different from \( \mathcal{U}_1, \mathcal{U}_2 \) since it is contained in both hyperplanes in \( \mathcal{H}_1 \) whereas the other two elements are only contained in a single hyperplane.
- The hyperplanes which contain \( \mathcal{U}_3 \) can be distinguished from all others.

This leads to the following refinement \( \mathfrak{R}_0 \) of the root node:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 1 & 1 & 0 & 1
\end{pmatrix}
\]

where we also applied the permutation \((2, 3)(7, 10)(8, 11) \in S_{\mathfrak{P}_0}\). 

In the following subsections we discuss the generalization of the inner minimization process and the definition of suitable \( S_{\mathfrak{P}} \)-homomorphisms. In Subsection 5.5 at the end of this section the whole backtracking procedure will be summarized.

### 5.3 Inner Minimization

One main observation of [4] is that (in the computation of a canonical form of a linear code) the \( GL_k(q) \)-component of the stabilizer \( Inn(\mathfrak{P}, \pi) \) could be easily stored by a pair \((t, \mathfrak{p})\) where \( 0 \leq t \leq k \) and \( \mathfrak{p} \) is a partition of \([t]\), i.e. it is equal to

\[
GL_k^{(t, \mathfrak{p})}(q) := \left\{ \begin{pmatrix} D & B_1 \\ 0 & B_2 \end{pmatrix} \middle| D \in \mathbb{F}_q^{t \times t} \text{ diagonal matrix and constant on all } P \in \mathfrak{p}, B_1 \in \mathbb{F}_q^{t \times (k-t)}, B_2 \in GL_{k-t}(q) \right\}
\]

In the case that \( \mathfrak{p} \) is the discrete partition we simply denote this group by \( GL_k^{(t)}(q) := GL_k^{(t, \{\{1\}, \ldots, \{t\}\})}(q) \). We will make a similar observation in this work, but since the result is achieved by the action on the normal vectors of the hyperplanes we have to use the transposed group.
\[ \text{GL}_k^{(t,p)}(q)^T := \{ A^T \mid A \in \text{GL}_k^{(t,p)}(q) \} \]

instead. This group has furthermore the following nice property:

**Proposition 20.** The multiplication of some matrix \( A \in \text{GL}_k^{(t,p)}(q)^T \) from the left stabilizes the first \( t \) rows of a matrix \( U \in \mathbb{F}_q^{k \times s} \) up to scalars.

In the following, let \( s(j) \) denote the number of columns of the matrix \( U_j, j \in [n] \). We describe the inner minimization process which is always applied after the partition \( \mathfrak{P} \) has been refined to \( \mathfrak{R} \) combined with the application of some permutation \( \sigma \in S_{\mathfrak{P}} \), i.e. after each individualization step and any successful refinement. The semicanonical representative of \((\mathfrak{R}, \sigma \pi)\) is the one which is derived from the following sequence of applications of the Homomorphism Principle:

- Let \( \Pi_{\text{Fixed}(\mathfrak{R})} \) and \( \varphi_{\text{Fixed}(\mathfrak{R})} \) define a homomorphism of group actions.

- We will later prove that the \( \text{GL}_k(q) \)-component of the stabilizer in the previous step is a subgroup of \( \text{GL}_k^{(t,p)}(q)^T \). In particular, we can use the parameter \( t \) in the following definition:

Let \( \Pi_{\text{Fixed}(\mathfrak{R}_C)} \) and \( \varphi_{\text{Fixed}(\mathfrak{R}_C)} \) define a homomorphism of group actions.

- We also apply a third homomorphism of group actions which restricts the components \( \text{GL}_{s(i)}(q) \) for \( i \in [n] \setminus \text{Fixed}(\mathfrak{R}_C) \). It is only called in special cases and therefore we will state this at some more appropriate place, see Subsection 5.3.2.3.

### 5.3.1 The structure of the inner stabilizer and semicanonical representatives

We start with a description of the inner stabilizer \( \text{Inn}^{(\mathfrak{P}, \pi)} \) which is computed in each step: For simplicity we just write \( \text{Inn} \) and \( (U, V) \) in the following. The description of the computation of a semicanonical representative is given...
afterward. It is based on a recursive method and the full details are stated in Subsection 5.3.2.

After the inner minimization process, the group Inn can be expressed by the parameters

- \((t, p)\) – describing the multiplication from the left
- \((t_i)_{i\in[n]}\) – describing the multiplication from the right for each sequence element
- \(e\) – describing the subgroup of field automorphisms

in the following way: The group Inn is the subgroup of

\[
\left(\text{GL}_k^{(t,p)}(q)^T \times \prod_{i=1}^n \text{GL}_{s(i)}^{(t_i)}(q) \times (\mathbb{F}_q^*)^h\right) \rtimes \langle \tau^e \rangle
\]

containing all elements \(\left(\begin{array}{cc} D & 0 \\ A_1 & A_2 \end{array}\right), \left(\begin{array}{cc} E_i & F_i \\ 0 & G_i \end{array}\right)\) with the following properties:

\[
\begin{align*}
\left(\begin{array}{cc} D & 0 \\ A_1 & A_2 \end{array}\right)^{T-1} v_j b_j &= v_j, \forall j \in \text{Fixed} (\mathfrak{P}_H) \\
D(U_i)_{[t],[t_i]} E^T_{i} &= (U_i)_{[t],[t_i]}, \forall i \in \text{Fixed} (\mathfrak{P}_C)
\end{align*}
\]

The integer \(t\) is well defined by the rank of \(V_{*,\text{Fixed}(\mathfrak{P}_H)}\). Furthermore, the inner minimization ensures that this matrix is in reduced row echelon form. Similarly, the integer \(t_i\) is well-defined by the rank of the submatrix \((U_i)_{[t],[t_i]}\) consisting of the first \(t\) rows of \(U_i\). The inner minimization produces a special structure of these matrices, i.e.: \((U_i)_{[t],[t_i]} = \left(\begin{array}{cc} (U_i)_{[t],[t_i]} & 0 \end{array}\right)\) where \((U_i)_{[t],[t_i]} \in \mathbb{F}_{q^{t_i}}^{t \times t_i}\) is in reduced column echelon form up to scalars.

The partition \(p\) is equal to the finest partition whose cells contain the supports of all vectors \(v_j, j \in \text{Fixed} (\mathfrak{P}_H)\) and the supports of the columns of \((U_i)_{[t],[t_i]}\), \(i \in \text{Fixed} (\mathfrak{P}_C)\). The exponent \(e\) is equal to the least positive power of the Frobenius automorphism which fixes all entries of \(v_j, j \in \text{Fixed} (\mathfrak{P}_H)\) and \((U_i)_{[t],[t_i]}\), \(i \in \text{Fixed} (\mathfrak{P}_C)\).

**Corollary 21.** For \(A \in \text{GL}_k^{(t,p)}(q)^T\) there exists a group element

\[
(A, B(A), b(A), \tau^0) \in \text{Inn}.
\]

Furthermore, \((I_k, I_{s(1)}, \ldots, I_{s(n)}, 1^h, \tau^e)\) is an element of Inn.
5.3.2 Inner Minimization Process

In the following, we are going to describe the inner minimization process in detail. Let \((\mathcal{P}, \pi)\) be the partition of the predecessor node and \((\sigma\pi, \mathcal{R})\) the actual node. For simplicity we suppose that the permutation \(\sigma\) was already applied to the sequence \((U^{(\mathcal{P}, \pi)}, V^{(\mathcal{P}, \pi)})\) and that we have to compute \((U^{(\mathcal{R}, \sigma\pi)}, V^{(\mathcal{R}, \sigma\pi)})\).

In the case that \(\text{Fixed}(\mathcal{P}) = \text{Fixed}(\mathcal{R})\) there is nothing to do. Otherwise, we successively modify the elements of the sequence corresponding to the indices \(i \in \text{Fixed}(\mathcal{R})\) starting from those components corresponding to the hyperplanes. Additionally, we have to give rules how to change \(\text{Inn}\) in each step in order to guarantee that the \(i\)-th entry of the semicanonical representative does fulfill Equations (4) and (5).

The procedures of the next two paragraphs are converted from the algorithm for linear codes, see [4, Algorithm 1].

5.3.2.1 Increased Rank  Suppose that the normal vector \(v_j, j \in \text{Fixed}(\mathcal{R}_H) \setminus \text{Fixed}(\mathcal{P}_H)\) contains some nonzero entry in the set \(\{t+1, \ldots, k\}\). In this case we can perform some elementary row operations in order to map \(v_j\) to the unit vector \(e_{t+1}\), i.e. there is some matrix \(A = \begin{pmatrix} I_t & A_1 \\ 0 & A_2 \end{pmatrix} \in GL_{k+1}^{(\mathbb{F})}\) such that \(Av_j = e_{t+1}\).

In particular, applying the group element \((A^{-1T}, I_{s(1)}, \ldots, I_{s(n)}, 1^h, \tau^e) \in \text{Inn}\) leads to this result.

The new stabilizer can be described by \(t+1\) and the partition \(\mathcal{P} \cup \{t+1\}\). Condition [4] ensures that this vector can not be changed anymore.

5.3.2.2 Same Rank  In the second case, the support of the newly fixed normal vector \(v_j, j \in \text{Fixed}(\mathcal{R}_H) \setminus \text{Fixed}(\mathcal{P}_H)\) is contained in the set \([t]\). In this case we can use each block \(P \in \mathcal{P} : P \cap \text{supp}(v_j) \neq \emptyset\) in order to map the nonzero entry \((v_j)_i\) with \(i := \max(P \cap \text{supp}(v_j))\) onto \(1_{F_q}\). This is done using the simultaneous multiplication of all rows indexed by \(i' \in P\) with \((v_j)_{i'}^{-1}\) which corresponds to the multiplication by the matrix \(A := \begin{pmatrix} D_{i,i'} & 0 \\ 0 & I_{k-t} \end{pmatrix} \in GL_k^{(\mathbb{F})}(q)\) with \(D_{i,i'} = (v_j)_{i'}^{-1}\) for \(i' \in P\). Corollary [21] ensures that we find a group element having the necessary \(\text{GL}_k(q)\)-component \(A_{T-1}\). Furthermore, we can choose such a group element such that \(b(A_{T-1})_j = 1\).

Finally, we may use the remaining field automorphisms \(\langle (I_k, I_{s(1)}, \ldots, I_{s(n)}, 1^h, \tau^e) \rangle \leq \text{Inn}\) to minimize the remaining nonzero, non-identity entries of the vector \(v_j\) starting from the highest index.

In the following, we are not allowed to multiply two different nonzero elements of this vector by different units since we are only able to revert these
multiplications by the multiplication of the whole column with the same element \( b_j \). Hence, the new stabilizer can be described by \( t \) and 
\[ p' := \left\{ P \in p \mid P \cap \text{supp}(v_j) = \emptyset \right\} \cup \left\{ \bigcup_{P \in p : P \cap \text{supp}(v_j) \neq \emptyset} P \right\}. \]

The stabilizer under the field automorphisms can be expressed by the smallest multiple \( e' \) of \( e \) such that 
\[ \tau^{e'}(v_j) = v_j. \]

**Example 22** (Example 19 continued). Suppose that the target cell selection told us to split the last cell \( \{12, 13\} \in \mathcal{R}_{0} \) in the individualization step. Suppose we are in the branch of the backtracking where we applied the identity element. The refined partition \( \mathcal{P}' \) contains two singletons \( \{12\} \) and \( \{13\} \). Minimizing \( v_9 \) yields the representative

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 2 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

and the inner stabilizer with parameters \( t = 1, p = \{1\} \). The horizontal line shows the parameter \( t \), which decomposes the matrices \( U_i \) into two submatrices. Similarly, we perform some elementary row operations on the second fixed coordinate \( v_{10} \), leading to

\[
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 2 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

and the inner stabilizer parameter \( t = 2, p = \{1, 2\} \).

Suppose that we would also have to minimize the entry \( v_7 = (2, 1, 0, 0)^T \) in the next step under this stabilizer. Then we would be able to minimize \( v_7 \) by the multiplication with the matrix 
\[ A := \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_2 \end{pmatrix}, \]

i.e. we would apply a group element

\[ (A^{-1}^T, B(A^{-1}^T), b(A^{-1}^T), \tau^0). \]

The stabilizer would be modified such that the \( \text{GL}_4(3) \)-component would be equal to \( \text{GL}_4^{(2,1,2)}(3) \).

For the minimization of the sequence elements \( U_i \) of \( U \) we distinguish two cases. The first is, that we fixed at least one normal vector of a hyperplane
whose minimization led to an increased parameter \( t \), see Subsection 5.3.2.1. In this case we have to update all matrices \( U_i \) for \( i \in [n] \), i.e. we have to perform the following procedure:

5.3.2.3 Increasing rank for the sequence of newly fixed normal vectors In this case, we know that the \( GL_k \)-component \( \Pi_{GL_k}(\text{Inn}) \) of \( \text{Inn} \) is a subgroup of \( GL_k^{(t)}T \) after finishing 5.3.2.1. Proposition 20 ensures that these matrices stabilize \( (U_i)_{[t],*} \) up to scalar multiplications of the rows. Hence, we can use the action from the right in order to map \( (U_i)_{[t],*} \) onto its well-defined reduced column echelon form \( \text{RCEF} \left( (U_i)_{[t],*} \right) \). In fact, it is sufficient to produce the reduced column echelon form up to multiplications of the columns by elements in \( \mathbb{F}_q^* \). This is due to the fact that at this point there is no decision on the ordering of the elements of those cells of \( \mathfrak{R}_C \) which are not singletons.

This canonization corresponds to the missing third homomorphism of group actions:

\[
(U, V) := \left( \left( GL_t^{(l)}(q) \times GL_{s(i)}(q) \right) \text{RCEF}(\text{Rows}_t(U_i)) \right)_{i \in [n]}
\]

\[
\varphi_{[n]}^{(l)} : \text{Inn} \rightarrow \left( GL_t(q)^T \times \prod_{i=1}^{n} GL_{s(i)}(q) \right) \rtimes \text{Aut}(\mathbb{F}_q)
\]

\[
(A, B, b, \tau^a) \mapsto (A_{[t],[t]_1}, B, \tau^a)
\]

**Remark 23.** The mapping \( G^{(sl)}(U, V) \mapsto \text{Can}_{(GL_t(q)^T \times \prod_{i=1}^{n} GL_{s(i)}(q)) \rtimes \text{Aut}(\mathbb{F}_q)}(\theta(U, V)) \) is an \( S_{\mathfrak{R}_C} \)-homomorphism, but it is not \( S_{\mathfrak{R}_C} \)-invariant. Therefore, we also could derive a refinement of \( \mathfrak{R}_C \) in the application of this minimization. Nevertheless, we included this refinement in the inner minimization process, since it allows us to reduce the inner stabilizer, too.

**Example 24** (Example 22 continued). Since the minimization of the normal vectors changed the parameter \( t \) of the inner stabilizer, we have to perform this step for each \( i \in [3] \), resulting in \( U \) equal to

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1
\end{bmatrix}
\]

Since we want to preserve the \((2 \times 1)\) upper left submatrices of \( U_1 \) and \( U_2 \) up to row and column multiplications, we choose the parameters \((t_1, t_2, t_3) = (1, 1, 0)\). This also leads to a refinement of the cell \( \{1, 2\} \).
Finally, in a last step we guarantee the minimality of \((U_i)_{[t],*}\) for those indices \(i \in \text{Fixed}(\mathcal{R}_c)\). We use the methods described in the following in order to guarantee that \((U_i)_{[t],[t_i]}\) is minimal and unchanged.

5.3.2.4 **Singletons** This procedure is applied to all \(i \in \text{Fixed}(\mathcal{R}_c)\). We can prove that the set of matrices we are allowed to apply on this component via a multiplication with the transpose from the right is equal to the subgroup \(GL_{s(i)}^{(t_i,p_i)}\) for some partition \(p_i\) of \([t_i]\). The minimization in Subsection 5.3.2.3 guarantees that the submatrix \((U_i)_{[t],*}\) decomposes as \((U_i)_{[t],*} = ((U_i)_{[t],[t_i]},0)\) where \((U_i)_{[t],[t_i]}\) is up to scalars a \((t \times t_i)\)-matrix in reduced column echelon form. The action of \(GL_k^{(t,p)}\) from the left and \(GL_{s(i)}^{(t_i,p_i)}\) from the right only multiplies rows or columns of \((U_i)_{[t],[t_i]}\) by nonzero entries of the field \(\mathbb{F}_q\). We can easily compute the smallest possible image of \((U_i)_{[t],[t_i]}\) under this simultaneous action by examining the operation column by column (the same algorithm as for the normal vectors, cf. 5.3.2.2). Similarly, in a subsequent step, we use the remaining automorphisms of the field, i.e. the group \(\langle \tau^e \rangle\) for the minimization of the nonzero entries.

**Example 25.** We give a bigger example over \(\mathbb{F}_{16}\). Let \(\xi \in \mathbb{F}_{16}\) be a primitive element and suppose we have to minimize

\[
U_i := \begin{pmatrix}
\xi^8 & 0 & 0 \\
0 & \xi^{10} & 0 \\
0 & 0 & \xi^8 \\
\xi^{12} & \xi^4 & 0 \\
\xi & \xi^7 & \xi^2
\end{pmatrix}
\]

under the action of the inner stabilizer, whose \(\Gamma L_{5}(16)\)-component is defined by \(t = 4, p = \{\{1,2\}, \{3\}, \{4\}\}\) and \(e = 1\). The index \(i \in [n]\) should be a newly produced singleton of the actual partition \(\mathfrak{P}\) and we already produced a reduced column echelon form (up to scalars) by the methods in the previous paragraph.

First of all we must minimize the first column leading to \(\overline{U}_i\) using the diagonal matrix with diagonal entries \((\xi^7, \xi^7, 1, \xi^3)\).
\[ U_i := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \xi^2 & 0 \\ 0 & \xi^8 & 0 \\ \xi & \xi^7 & \xi^2 \end{pmatrix} \quad \tilde{U}_i := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \xi^{10} & 0 \\ 0 & 1 & 0 \\ \xi & \xi^7 & \xi^2 \end{pmatrix} \quad U'_i := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \xi^5 & 0 \\ 0 & 1 & 0 \\ \xi^2 & \xi^{14} & \xi^4 \end{pmatrix} \]

The Frobenius automorphism stabilizes all entries of the newly produced column. In the following, we are only allowed to apply diagonal matrices which are constant on the cells \{1, 2, 4\} and \{3\}. In the next step, we minimize the second column using those row multiplications. In order to map the entry \(\xi^7\) to 1 we use the simultaneous multiplication by \((\xi^8, \xi^8, 1, \xi^8)\). Note that we can revert this multiplication on the first column via the multiplication of the whole column with \(\xi^7\). Furthermore, we multiply the third row by \(\xi^7\), leading to \(\tilde{U}_i\). Finally, we use the Frobenius automorphism to minimize the only nonzero and non-identity entry \(\xi^{10}\) to \(\xi^5\), see \(U'_i\). The \(\Gamma\mathcal{L}_{6}(16)\)-component of the inner stabilizer is defined by \(t' = 4, p' = \{1, 2, 3, 4\}\) and \(e' = 2\).

### 5.4 Refinements

One of the most crucial tasks in the algorithm is the pruning of subtrees. We have already mentioned how the group of known automorphisms could be used for this task. Furthermore, Subsection 5.3 allows us to prune subtrees based on the semicanonical representative of the node. Since those mappings heavily depend on Inn\((\mathcal{P}, \pi)\), they usually will have poor performance on the first levels of the search tree. The same holds for the homomorphism we are going to introduce in the next subsection.

#### 5.4.1 Inner Minimization Refine

Similar to Remark 23 the mapping

\[
\theta_{\text{Subset}} : \mathcal{G}^{(\text{str})}(U, V) \mapsto \left(\text{Subset}_{[t]}(\text{supp}(v^{(\mathcal{P}, \pi)}_j)) \right)_{j \in [h]},
\]

where \(\text{Subset}_{[t]}(X) := \begin{cases} 1, & \text{if } X \subseteq [t] \\ 0, & \text{else} \end{cases}\) for \(X \subseteq [k]\), defines an \(S_{\mathcal{P}_{\pi}}\)-homomorphism.
Furthermore, we can predict the result of the inner minimization for the unfixed normal vectors and use this as an $S_{\Psi,H}$-homomorphism as well:

$$\theta_{\text{min},H} : G^{(sl)}(U,V) \mapsto \min_{(A,\tau^a) \in GL_k^{(t,p)}(q) \times \langle \tau^e \rangle} (A \tau^a(v_j^{(\Psi,\pi)}))$$

This function is easily computable. In the case that $\theta_{\text{Subset}}((U,V))_j = 0$ we know that $\theta_{\text{min},H}(U,V)_j = e_{i+1}$. Otherwise, we can use the methods described in Subsection 5.3.2.2 in order to compute the smallest possible representative.

We can make similar computations on the positions $i \in [n]$, i.e. we can define the $S_{\Psi,C}$-homomorphism $\theta_{\text{min},C}$ with

$$\left(\theta_{\text{min},C}(G^{(sl)}(U,V))\right)_i := \min_{(D,E,\tau^a) \in (GL_{t_i}^{(t,p)}(q) \times GL_{t_i}^{(t_i)}(q)) \times \langle \tau^e \rangle} D \tau^a \left(\left(U_i^{(\Psi,\pi)}\right)_{[t],\tau_{[t]}^a}\right)^T E^T$$

Again, this function is easily computable using the methods described in Subsection 5.3.2.4.

### 5.4.2 Colored Incidence Graph

Associated with the semicanonical representative $(U^{(\Psi,\pi)}, V^{(\Psi,\pi)})$ of a node is the bipartite Graph $G$ with vertex set $[n+h]$ and edges

$$\left\{ \{i,j\} \mid i \in [n] ; j \in ([n+h] \setminus [n]) : v_j^{(\Psi,\pi)} U_i^{(\Psi,\pi)} = 0 \right\}.$$

Using the partition $\Psi$ we may cell-wise count the neighbors of a vertex $u \in [n+h]$ which defines an $S_{\Psi}$-homomorphism.

Finally, we have the possibility to color the edges of this graph as well. Therefore, we investigate the result of $v_j^{(\Psi,\pi)} U_i^{(\Psi,\pi)} \neq 0$ under the action of the inner stabilizer. For some arbitrary $A \in GL_k^{(t,p)}(q)$, $B := \begin{pmatrix} E & B_1 \\ 0 & B_2 \end{pmatrix} \in GL_{t_i}^{(t_i)}(q), b_j \in \mathbb{F}_q^*$ and $\tau^a \in \langle \tau^e \rangle$ we have

$$\left(\left(A^T \tau^a(v_j^{(\Psi,\pi)}b_j)^T \right) \left(\left(A \tau^a(U_i^{(\Psi,\pi)}) \begin{pmatrix} E & B_1 \\ 0 & B_2 \end{pmatrix}^T \right) = b_j \tau^a \left(v_j^{(\Psi,\pi)} U_i^{(\Psi,\pi)} \right) \begin{pmatrix} E & B_1 \\ 0 & B_2 \end{pmatrix}^T \right.$$
Now, substitute $v_j^{(P, \pi)^T} U_i^{(P, \pi)} = (w_1, w_2)$ with $w_1 \in \mathbb{F}_q^{t_i}$ and $w_2 \in \mathbb{F}_q^{s(i) - t_i}$. The action of $\text{Inn}^{(P, \pi)}$ changes the result of this product as given in the equation above. Hence, we can distinguish the edges (introduce colors) based on the orbits of $(w_1, w_2)$ under the group action of $\left( \text{GL}_{s(i)} \times \mathbb{F}_q^* \right) \rtimes \langle \tau^e \rangle$.

Canonical representatives of these orbits could be easily computed using the following observation: In the case that $w_2 \neq 0$, we are able to find some matrix $B$ to map the vector $(w_1, w_2)$ onto the $(t_i + 1)$-th unit vector. In the case that $w_2$ is equal to 0, we observe that $E$ is a diagonal matrix, hence the support of the vector $w_1$ is fixed by the application of this matrix.

Again, cell-wise counting of neighbors distinguished by the coloring of the edges defines an $S_{P^*}$-homomorphism. The condition (5) for the positions $i \in \text{Fixed}(\mathcal{P}_c)$ furthermore restricts the diagonal matrices $E$ even more. In this case, it is even possible to give a refined coloring on the edges which allows the definition of a stronger $S_{P^*}$-homomorphism.

**Example 26.** Suppose that the partition $\mathcal{P}$ of the actual node contains the cells $\{1, 2\}$ and $\{1 + n, 2 + n\}$. Furthermore, the inner stabilizer is defined by $t = 3, \mathcal{P} = \{\{1\}, \{2, 3\}\}$ and $e = 2$. The example should be over $\mathbb{F}_4$ with $k = 6$ and

$$
(U_1, U_2, v_1, v_2) = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \xi \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & \xi & \xi \\
0 & \xi & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & \xi & \xi & 1 & 1
\end{pmatrix}
$$

We first give the results of the multiplication $v_j^T U_i$:

| $(w_1^{(i,j)}, w_2^{(i,j)})$ | $i=1$ | $i=2$ |
|-----------------------------|-------|-------|
| $j=1$                       | $(1, \xi^2, 0, 0)$ | $(1, 1, 1, \xi^2)$ |
| $j=2$                       | $(1, \xi, 0, 1)$   | $(1, \xi, 0, 0)$   |

There is exactly one entry in each column and each row whose $w_2^{(i,j)}$ component is equal to zero. The support of the corresponding $w_1^{(i,j)}$ part is in both cases equal to $\{1, 2\}$. Hence, the observation of the vertex- and edge-colored graph would not lead to refinement in this case.
In contrary, if the partition \( p \) would be \( \{1, 2, 3\} \) instead, we would observe that the matrices which are multiplied from the right to the elements \( U_1 \) and \( U_2 \) under the action of the inner stabilizer must be elements in \( GL_4^{2\{1,2\}}(4) \). Hence, the orbits under this restricted action would be \( \{(1, \xi^2, 0, 0), (\xi, 1, 0, 0), (\xi^2, \xi, 0, 0)\} \) and \( \{(1, \xi, 0, 0), (\xi, \xi^2, 0, 0), (\xi^2, 1, 0, 0)\} \). Therefore, we would be able to color the edges \( (1, 1 + n) \) and \( (2, 2 + n) \) differently. This leads to a refinement of both cells.

### 5.4.3 Iterative Refinements

In the case that one of these refinements leads to a new singleton in the partition \( \Psi \), we use the inner minimization procedure of Section 5.3 in order to get some smaller group \( \text{Inn}(\Psi, \pi) \). The result of this minimization is compared with the candidate for the canonical form to prune the tree at an early stage.

In the case that the inner minimization leads to a smaller group \( \text{Inn}(\Psi, \pi) \), we use the refinements described in Subsection 5.4.1 immediately after the inner minimization procedure.

Since the refinement based on Subsection 5.4.2 is the most expensive, we try to avoid its application as long as possible. In the case that all other refinements fail and that we have updated \( \text{Inn}(\Psi, \pi) \) by some smaller group or that we have replaced the partition \( \Psi \) by a finer partition since the last call of this function, the incidence graph may provide a refinement.

Rather than computing the \( S_{\Psi} \)-homomorphism for all indices \( i \in [n + h] \) at once, we compute the result iteratively for each pair \( P \in \Psi_H, Q \in \Psi_C \) of cells and call the refinement after each step. The ordering of the cells in this regard should be determined based on the information of all previous steps and we are not yet sure about the optimal strategy.

### 5.5 The algorithm

Algorithm 1 gives a recursive description of the backtrack tree generation and Figure 2 visualizes this process. As we already mentioned, this tree is traversed in a depth-first search approach. Therefore we store some candidate for the canonical form in a global variable \( (U_{\text{Can}}, V_{\text{Can}}) \). This is the element which compares less than all other leaf nodes already visited including all comparisons performed on the paths to these nodes. At the end, this candidate is defined to be the unique representative of the orbit. If this variable is uninitialized (\( = \text{NIL} \)), the leaf node which will be visited next becomes the candidate for the canonical form. Two further global variables \( A, T \) maintain the group of known automorphisms, which could be used for further pruning
the tree. The subroutine TARGETCELL chooses the target cell in this step. Similarly, REFINEMENTFUNC defines the next $S_p$-homomorphism which has to be applied. This function also may return NIL which indicates that the refinement process should finish.

The function INNERMINIMIZATION implements the inner minimization as described in Section 5.3. In the case that the result of the minimization ($U'$, $V'$) is smaller than ($U^\text{Can}$, $V^\text{Can}$) \( \neq \) NIL, this function sets the global variable ($U^\text{Can}$, $V^\text{Can}$) to NIL. On the other hand, if there is a smaller candidate for the canonical form, the function returns is_leq = false. Otherwise, this flag is set to true. Similarly, REFINEMENT implements the Homomorphism Principle for some given $S_p$-homomorphism $f$. It updates the variables in the same way. Furthermore, it also calls the INNERMINIMIZATION function in the case that a further singleton appeared in $\mathfrak{P}'$.

During the backtracking it is not necessary to maintain a group element $(A, B, b, \tau^a) \in G^{(sl)}$, which maps the root node to the semicanonical representative of the actual node. The permutation $\pi \in S_{n+h}$ and the corresponding path to the leaf $(\mathfrak{D}, \pi)$ define an element $(A^{(\pi)}, B^{(\pi)}, b^{(\pi)}, \tau^{a^{(\pi)}}) \in G^{(sl)}$ which maps the initial sequence $(U, V)$ of the root node to the semicanonical representative of this leaf. The element $(A^{(\pi)}, B^{(\pi)}, b^{(\pi)}, \tau^{a^{(\pi)}})$ is well defined up to the multiplication by $\text{Inn}(\mathfrak{D}, \pi)$ from the left and it is only computed for some few leaves. Furthermore, since we are only interested in a canonization map for $\mathcal{C}$, we may restrict this computation to its $\Gamma_{L_k}(q)$-component $(A^{(\pi)}, \tau^{a^{(\pi)}})$.

Let $\pi^\text{Can} \in S_{n+h}$ be the permutation leading to the canonical form and let $\sigma_1,\ldots,\sigma_z \in S_{n+h}$ define generators of the automorphism group $\text{Aut}$ used for pruning the search tree. The canonical form of $\text{Can}_{\Gamma_{L_k}(q)}(\mathcal{C})$ is defined to be the sequence of subsets of subspaces given by the column spaces of $U_i^\text{Can}$. A transporter element $\text{TR}_{\Gamma_{L_k}(q)}(\mathcal{C})$ is given by $(A^{(\pi^\text{Can})}, \tau^{a^{(\pi^\text{Can})}})$.

**Proposition 27.** With the help of the elements $(A^{(\pi^\text{Can}\sigma_i)}, \tau^{a^{(\pi^\text{Can}\sigma_i)}})$, \( i \in [z] \), the group $\text{Inn}(\mathfrak{D}, \pi^\text{Can})$ and $(A^{(\pi^\text{Can})}, \tau^{a^{(\pi^\text{Can})}})$ we are able to compute generators of the automorphism group $\text{Aut}(\mathcal{C}) \leq \Gamma_{L_k}(q)$: $\text{Aut}(\mathcal{C})$ is generated by

$$
\Pi_{\Gamma_{L_k}} \left( \text{Inn}(\mathfrak{D}, \text{id}) \right) = \left( A^{(\pi^\text{Can})}, \tau^{a^{(\pi^\text{Can})}} \right)^{-1} \Pi_{\Gamma_{L_k}(q)} \left( \text{Inn}(\mathfrak{D}, \pi^\text{Can}) \right) \left( A^{(\pi^\text{Can})}, \tau^{a^{(\pi^\text{Can})}} \right)
$$

and

$$
\left\{ \left( A^{(\pi^\text{Can}\sigma_i)}, \tau^{a^{(\pi^\text{Can}\sigma_i)}} \right)^{-1} \left( A^{(\pi^\text{Can}\sigma_i)}, \tau^{a^{(\pi^\text{Can}\sigma_i)}} \right) \mid i \in [z] \right\}
$$

Note that $(\mathfrak{D}, \pi^\text{Can}\sigma_i)$ may not appear as a node of the pruned search tree because of the pruning based on the group of known automorphisms. In this case, we still know in which order we have to apply the methods from Sec-
Algorithm 1 BACKTRACK

Require: global variable $(U^{\text{Can}}, V^{\text{Can}})$ – candidate for the canonical form
Require: global variable $\pi^{\text{Can}}$ – the permutation leading to the candidate
Require: global variable $\text{Aut} \leq S_{n+h}$ – the group of known automorphisms
Require: global variable $T \subseteq S_{n+h}$ – a left transversal of $\text{Aut}$ in $S_{n+h}$
Require: $(\Psi, \pi, U, V)$ a node of the backtrack tree

Ensure: individualization-refinement step on $(\Psi, \pi, U, V)$

1: procedure BACKTRACK($\Psi, \pi, U, V, \text{Inn}$)  
2: if $\Psi$ is discrete then  
3: if $(U^{\text{Can}}, V^{\text{Can}}) = \text{NIL}$ then  
4: $(U^{\text{Can}}, V^{\text{Can}}, \pi^{\text{Can}}) \leftarrow (U, V, \pi)$ \Comment{a new candidate}  
5: else  
6: $\text{Aut} \leftarrow \langle \text{Aut}, \pi^{-1}\pi^{\text{Can}} \rangle$ \Comment{a new automorphism}  
7: $T \leftarrow$ left transversal of $\text{Aut}$ in $S_{n+h}$  
8: end if  
9: return  
10: end if  
11: $P \leftarrow \text{TARGETCELL}(\Psi, \pi, \text{Inn}), m \leftarrow \min(P)$ \Comment{target cell selection}  
12: $\Psi' \leftarrow (\Psi \setminus P) \cup \{m\}, P \setminus \{m\}$  
13: for $j \in P$ do \Comment{individualization}  
14: $t \leftarrow (m, j), \pi' \leftarrow t\pi, (U', V') \leftarrow t(U, V)$  
15: $(\text{is}\text{.eq.}(\Psi', \pi', U', V', \text{Inn}')) \leftarrow \text{INNERMINIMIZATION}(\Psi', \pi', U', V', \text{Inn})$  
16: $f \leftarrow \text{REFINEMENTFUNC}(\Psi', \text{Inn}')$ \Comment{choose an $S_{\Psi'}$-homomorphism}  
17: while $\text{is}\text{.eq.}$ and $f \neq \text{NIL}$ do  
18: if $T \cap S_{\Psi'}\pi = \emptyset$ then \Comment{see \cite[Lemma 5.9]{4}}  
19: return  
20: end if  
21: $\text{REFINEMENT}(f, \Psi', \pi', U', V', \text{Inn}')$  
22: $f \leftarrow \text{REFINEMENTFUNC}(\Psi', \text{Inn}')$  
23: end while  
24: if not $\text{is}\text{.eq.}$ then  
25: return \Comment{the actual candidate $(U^{\text{Can}}, V^{\text{Can}})$ is smaller}  
26: end if  
27: BACKTRACK($\Psi', \pi', U', V', \text{Inn}')  
28: end for  
29: end procedure  

Theorem 28. The mapping $C \mapsto (\text{Can}_{\Gamma L_{k}(q)}(C), \text{TR}_{\Gamma L_{k}(q)}(C), \text{Aut}(C))$, where $\text{Can}_{\Gamma L_{k}(q)}$ and $\text{TR}_{\Gamma L_{k}(q)}$ are defined as above, solves the canonization problem and Algorithm 1 is a practical algorithm to compute the data.
Fig. 2. Backtrack tree generation

6 Applications

In [4,5] we gave running times for the computation of the automorphism groups of a family of almost perfect nonlinear (APN-) function $f^{(d)} : \mathbb{F}_2^d \rightarrow \mathbb{F}_2^d, x \mapsto x^3$. These computations were done using a reformulation as a linear code $C_f^{(d)} \subseteq \mathbb{F}_2^{2d}$ of dimension $2d + 1$.

Definition 29. A set $\mathcal{C}$ of $d$-dimensional subspaces in $\mathbb{P}_q(k + 1)$ with $|\mathcal{C}| = 1 + (q^{d+1} - 1)/(q - 1)$ is called a $(d - 1)$-dimensional dual hyperoval if the
intersection of any two distinct elements of $C$ is a point and any three have an empty intersection.

[14] gives a construction of $(d - 1)$-dimensional dual hyperovals $C_f^{(d)}$ in $P_2(2d)$ using quadratic APN-functions $f^{(d)}: \mathbb{F}_2^d \to \mathbb{F}_2^d$. Again we use the quadratic function $x \mapsto x^3$ to produce a family of subsets $C_f^{(d)}$ to test our algorithm. By [3] we know that the automorphism group of $C_f^{(d)}$ and $C_f^{(d)}$ are identical for $d \geq 4$. This allows us on the one hand to test the algorithm for correctness and on the other to compare its performance with the algorithm for linear codes. Table 6 shows the running times for different $d$ on a single core of a 2.4 GHz Intel Quad 2 processor.

### 7 Conclusion

This work presents a practical algorithm which solves the canonization problem for sequences of subsets of $P_q(k)$. From the reduction to the graph isomorphism problem, we know that we could not expect to give an algorithm that runs in polynomial time.

The algorithm itself relies on many heuristics, for instance the choice of the target cell, the choice of the homomorphism of group actions which has to be applied next or when to stop the refinements since they are more expensive than performing an individualization step. These problems are well-known from the canonization of graphs where possible modifications of the basic algorithm are discussed in several papers.

\[
\begin{array}{cccccc}
 k = 2d & s = d & n = 2^d & h & |\text{Aut}(C_f^{(d)})| & \text{time } C_f^{(d)} & \text{time } C_f^{(d)} \\
 6 & 3 & 8 & 28 & 1344 & 0.1 \text{ s} & 0.1 \text{ s} \\
 8 & 4 & 16 & 20 & 5760 & 0.1 \text{ s} & 0.1 \text{ s} \\
 10 & 5 & 32 & 496 & 4960 & 0.5 \text{ s} & 0.1 \text{ s} \\
 12 & 6 & 64 & 336 & 24192 & 0.2 \text{ s} & 0.1 \text{ s} \\
 14 & 7 & 128 & 8128 & 113792 & 3 \text{ s} & 0.3 \text{ s} \\
 16 & 8 & 256 & 5440 & 522240 & 2.5 \text{ s} & 0.3 \text{ s} \\
 18 & 9 & 512 & 130816 & 2354688 & 4 \text{ min} & 45 \text{ s} \\
 20 & 10 & 1024 & 87296 & 10475520 & 2 \text{ min} & 6 \text{ s} \\
 22 & 11 & 2048 & 2096128 & 46114816 & 8 \text{ h} & 4 \text{ h} \\
 24 & 12 & 4096 & 1397760 & 201277440 & 8 \text{ h} & 6 \text{ min} \\
\end{array}
\]

Table 1
Running times for $C_f^{(d)}$ compared to $C_f^{(d)}$ for $f(x) = x^3$
Similarly, the improvement of these heuristics is still part of our current research and we are not yet sure about an optimal strategy. This is also the reason of not giving the full implementation details in this regard.

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References

[1] A. Ashikhmin and E. Knill. Nonbinary quantum stabilizer codes. *IEEE Trans. Inform. Theory*, 47(7):3065–3072, 2001.

[2] A. Betten, M. Braun, H. Fripertinger, A. Kerber, A. Kohnert, and A. Wassermann. *Error-correcting linear codes. Classification by isometry and applications. With CD-ROM*. Algorithms and Computation in Mathematics 18. Berlin: Springer. xxix, 798 p., 2006. [http://linearcodes.uni-bayreuth.de](http://linearcodes.uni-bayreuth.de).

[3] Y. Edel and U. Dempwolff. Dimensional dual hyperovals and APN functions with translation groups (preprint). [http://www.mathi.uni-heidelberg.de/~yves/Papers/eqdho.html](http://www.mathi.uni-heidelberg.de/~yves/Papers/eqdho.html), 2012.

[4] T. Feulner. The automorphism groups of linear codes and canonical representatives of their semilinear isometry classes. *Adv. Math. Commun.*, 3(4):363–383, 2009.

[5] T. Feulner. Canonization of linear codes over $\mathbb{Z}_4$. *Adv. Math. Commun.*, 5(2):245–266, 2011.

[6] R. Gugisch. *Construction of isomorphy classes of oriented matroids. (Konstruktion von Isomorphieklassen orientierter Matroide.).* PhD thesis, Bayreuther Mathematische Schriften 72. Bayreuth: Univ. Bayreuth, Mathematisches Institut; Bayreuth: Univ. Bayreuth, Fakultät für Mathematik und Physik (Dissertation). x, 130 p., 2005. [http://opus.ub.uni-bayreuth.de/volltexte/2006/229/](http://opus.ub.uni-bayreuth.de/volltexte/2006/229/).

[7] R. Kötter and F. R. Kschischang. Coding for errors and erasures in random network coding. *Information Theory, IEEE Transactions on*, 54(8):3579–3591, August 2008.

[8] R. Laue. Constructing objects up to isomorphism, simple 9-designs with small parameters. In *Algebraic combinatorics and applications* (Gößweinstein, 1999), pages 232–260. Springer, Berlin, 2001.
[9] J. S. Leon. Computing automorphism groups of error-correcting codes. *IEEE Trans. Inf. Theory*, 28:496–511, 1982.

[10] B. D. McKay. Practical graph isomorphism. Numerical mathematics and computing, Proc. 10th Manitoba Conf., Winnipeg/Manitoba 1980, Congr. Numerantium 30, 45-87 (1981)., 1981.

[11] Erez Petrank and Ron M. Roth. Is code equivalence easy to decide? *IEEE Transactions on Information Theory*, 43:1602–1604, 1997.

[12] A. Piperno. Search space contraction in canonical labeling of graphs (preliminary version). [http://arxiv.org/abs/0804.4881v2](http://arxiv.org/abs/0804.4881v2), 2011.

[13] A.-L. Trautmann, F. Manganiello, M. Braun, and J. Rosenthal. Cyclic orbit codes. [http://arxiv.org/abs/1112.1238v1](http://arxiv.org/abs/1112.1238v1), December 2011.

[14] Satoshi Yoshiara. Dimensional dual hyperovals associated with quadratic APN functions. *Innov. Incidence Geom.*, 8:147–169, 2008.