NON-NEGATIVELY CURVED MANIFOLDS WITH MAXIMAL SYMMETRY RANK IN LOW DIMENSIONS

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ABSTRACT. We show that a closed, simply-connected, non-negatively curved 5-manifold admitting an (almost) effective, isometric $T^3$ action is diffeomorphic to one of $S^5$, $S^3 \times S^2$ or $S^3 \times S^1$. If we allow only $T^2$ symmetry, the Wu manifold $SU(3)/SO(3)$ may also occur and we conclude that the corresponding manifold is diffeomorphic to any of these four. As a direct consequence, we can show that for any manifold, of dimensions up to and including 9 under the same hypotheses, the maximal symmetry rank is equal to $\left[\frac{2n}{3}\right]$ and the free rank is less than or equal to one half that value, which is consistent with the maximal symmetry rank of the closed, simply-connected $n$-manifold of non-negative curvature equal to the product of $m$ copies of $S^3$ for $n = 3m$, $m - 1$ copies of $S^1$ and an $S^1$ when $n = 3m + 1$ and $m$ copies of $S^3$ and an $S^2$ when $n = 3m + 2$.

1. INTRODUCTION

As in the study of closed, simply-connected Riemannian manifolds of positive (sectional) curvature, a typical problem for manifolds with non-negative curvature is to classify those manifolds whose isometry groups are large. In the context of this paper, “large” will refer to the symmetry rank of a Riemannian manifold $M$, defined as the rank of the isometry group of $M$. In particular, we are interested in manifolds with non-negative curvature that have maximal or almost maximal symmetry rank.

In the case of positive curvature, manifolds with maximal symmetry rank were classified up to diffeomorphism by Grove and Searle [GS]. In the case for positively curved manifolds of almost maximal symmetry rank, Rong [R] found topological restrictions for all dimensions (distinguishing the cases for even and odd) and showed that a closed, simply-connected positively curved 5-manifold with almost maximal symmetry rank must be...
homeomorphic to the 5-sphere (in fact, diffeomorphic, as a consequence of the Generalized Poincaré conjecture). Later Wilking [W] was able to improve these results significantly for simply-connected manifolds considering actions of rank approximately $\frac{1}{4}$ of the dimension of the manifold.

In the case of non-negative curvature, Kleiner [K] and Searle and Yang [SY] independently classified closed, simply-connected 4-manifolds with an isometric circle action, corresponding to the almost maximal symmetry rank case.

In this paper we address the remaining cases in dimensions 3, 4 and 5 and determine the maximal symmetry rank of closed simply-connected, non-negatively curved $n$-manifolds of dimension $n \leq 9$.

The paper is divided into three parts. In the first part, Section 2, we recall some basic notions and outline the proof of the classification of simply-connected, non-negatively curved 5-manifolds with an isometric $T^3$ or $T^2$ action (see Theorem A, below). In this section we also prove several preliminary results that will be used in subsequent sections. In the second part, comprising Sections 3–6, we consider the classification of non-negatively curved 3-manifolds with an isometric $T^1$ or $T^2$ action, simply-connected non-negatively curved 4-manifolds also with an isometric $T^1$ or $T^2$ action and finally simply-connected, non-negatively curved 5-manifolds with an isometric $T^2$ or $T^3$ action, thus covering the maximal and almost maximal symmetry rank cases.

To obtain our classification results we use the geometry of the torus action $T \times M \to M$ to obtain sufficient topological information so as to be able to identify $M$. When the orbit space $M/T$ has dimension 1 or 2, corresponding to $M$ of dimension 3 or 4 and a $T^1$ or $T^2$ action, and $M$ of dimension 5 with a $T^3$ action, we use general classification results of smooth manifolds with smooth torus actions, such as Oh’s classification of simply-connected 5-manifolds with a smooth $T^3$ action [Oh]. In the case of a non-negatively curved 5-manifold with an isometric $T^2$ action, we generalize Rong’s almost maximal symmetry rank result in dimension 5 [R] and apply the Barden-Smale classification of simply-connected 5-manifolds [Ba, Sm].

In the third part, comprising Section 7, we use our work in the previous sections to determine the maximal symmetry rank of a simply-connected, non-negatively curved $n$-manifold of dimension $n \leq 9$. Our main results are the following:

**Theorem A.** Let $M^5$ be a closed, simply-connected, non-negatively curved 5-manifold. If $T^k$ acts isometrically and (almost) effectively on $M^5$, then $k \leq 3$. In particular, if $k = 3$, then $M^5$ is diffeomorphic to one of $S^5$, $S^3 \times S^2$ or $S^3 \times \tilde{S}^2$. If $k = 2$ then $M^5$ is diffeomorphic to one of the above manifolds or the Wu manifold $SU(3)/SO(3)$.
Theorem B (Maximal Symmetry Rank). Let $T^k$ be an (almost) effective isometric action on a closed, simply-connected, non-negatively curved $n$-manifold $M^n$. If $n \leq 9$, then $k \leq \left\lfloor \frac{2n^3}{3} \right\rfloor$. Further, the free rank of the action is at most $\left\lfloor \frac{n^3}{3} \right\rfloor$.

We remark that, as a consequence of our work, a closed non-negatively curved simply-connected 6-manifold with maximal symmetry rank must be diffeomorphic to $S^3 \times S^3$ (cf. Corollary 7.2).

2. Tools and preliminary results

In this section we gather several definitions and results that we will use in subsequent sections. We have also included an outline of the proof of Theorem A, and note that Theorem B will follow by work done for Theorem A.

2.1. Transformation groups. Let $G$ be a Lie group acting (on the left) on a smooth manifold $M$. We denote by $G_x = \{ g \in G : gx = x \}$ the isotropy group at $x \in M$ and by $Gx = \{ gx : g \in G \} \simeq G/G_x$ the orbit of $x$. The ineffective kernel of the action is the subgroup $K = \cap_{x \in M} G_x$. We say that $G$ acts effectively on $M$ if $K$ is trivial. The action is called almost effective if $K$ is finite. We will often denote the fixed point set $M^G = \{ x \in M : gx = x, g \in G \}$ of this action by $\text{Fix}(M, G)$ and define its dimension as $\dim \text{Fix}(M, G) = \max \{ \dim N : N$ is a component of $\text{Fix}(M, G) \}$. When convenient, we will also denote the orbit space $M/G$ by $X$. We will denote by $\pi$ the image of a point $p \in M$ under the projection map $\pi : M \to M/G$.

One measurement for the size of a transformation group $G \times M \to M$ is the dimension of its orbit space $M/G$, also called the cohomogeneity of the action. This dimension is clearly constrained by the dimension of the fixed point set $M^G$ of $G$ in $M$. In fact, $\dim(M/G) \geq \dim(M^G) + 1$ for any non-trivial action. In light of this, the fixed-point cohomogeneity of an action, denoted by $\text{cohomfix}(M, G)$, is defined by

$$\text{cohomfix}(M, G) = \dim(M/G) - \dim(M^G) - 1 \geq 0.$$ 

A manifold with fixed-point cohomogeneity 0 is also called a fixed-point homogeneous manifold. We will use the latter term throughout this article. We further remark that the fixed point set of a fixed-point homogeneous action has codimension 1 in the orbit space.

Remark. In the rest of the paper, we will always consider (almost) effective actions.
2.2. **Alexandrov geometry.** Recall that a finite dimensional length space \((X, \text{dist})\) is an **Alexandrov space** if it has curvature bounded from below (cf. \[BBI\]). When \(M\) is a complete, connected Riemannian manifold and \(G\) is a compact Lie group acting (almost) effectively on \(M\) by isometries, the orbit space \(X = M/G\) is equipped with the orbital distance metric induced from \(M\), i.e., the distance between \(p\) and \(q\) in \(X\) is the distance between the orbits \(Gp\) and \(Gq\) as subsets of \(M\). If, in addition, \(M\) has sectional curvature bounded below \(\sec M \geq k\), the orbit space \(X\) is an Alexandrov space with \(\text{curv} X \geq k\).

The **space of directions** of a general Alexandrov space at a point \(x\) is by definition the completion of the space of geodesic directions at \(x\). In the case of orbit spaces \(X = M/G\), the space of directions \(S_p^\perp X\) at a point \(p \in X\) consists of geodesic directions and is isometric to \(S_p^\perp /G_p\), where \(S_p^\perp\) is the normal sphere to the orbit \(Gp\) at \(p \in M\).

We now state Kleiner’s Isotropy Lemma (cf. \[K\]), which we will use to obtain information on the distribution of the isotropy groups along minimal geodesics joining two orbits and, in consequence, along minimal geodesics joining two points in the orbit space \(M/G\).

**Isotropy Lemma 2.1.** Let \(c : [0, d] \rightarrow M\) be a minimal geodesic between the orbits \(Gc(0)\) and \(Gc(d)\). Then, for any \(t \in (0, d)\), \(G_{c(t)} = G_c\) is a subgroup of \(G_{c(0)}\) and of \(G_{c(d)}\).

Recall that the \(q\)-extent \(\text{xt}_q(X)\), \(q \geq 2\), of a compact metric space \((X, d)\) is the maximum average distance between \(q\) points in \(X\):

\[
\text{xt}_q(X) = \left(\frac{q}{2}\right)^{-1} \max \left\{ \sum_{1 \leq i < j \leq q} d(x_i, x_j) : \{x_i\}_{i=1}^n \subset X \right\}.
\]

We note that the Extent Lemma stated below will allow us to bound the total number of isolated singular points in \(X = M/G\).

**Extent Lemma 2.2.** Let \(\bar{p}_0, \ldots, \bar{p}_q\) be \(q + 1\) distinct points in \(X = M/G\). If \(\text{curv}(X) \geq 0\), then

\[
\frac{1}{q + 1} \sum_{i=0}^q \text{xt}_q(S_{\bar{p}_i}X) \geq \pi/3.
\]

We remark that in the case of strictly positive curvature, the inequality is also strict.

We will also use the following analogue for orbit spaces of the Cheeger-Gromoll Soul Theorem to obtain information on the geometry of the orbit space \(M/G\).
Soul Theorem 2.3. If $\text{curv } M/G \geq 0$ and $\partial M/G \neq \emptyset$, then there exists a totally convex compact subset $\Sigma \subset M/G$ with $\partial \Sigma = \emptyset$, which is a strong deformation retract of $M/G$. If $\text{curv } M/G > 0$, then $\Sigma = \overline{\Sigma}$ is a point, and $\partial M/G$ is homeomorphic to $S_{\pi}M/G \simeq S^2_{\pi}/G_{\pi}$.

When $M$ is a non-negatively curved fixed-point homogeneous Riemannian $G$-manifold, the orbit space $X$ is a non-negatively curved Alexandrov space and $\partial X$ contains a component $N$ of $\text{Fix}(M, G)$. Let $C \subset X$ denote the set at maximal distance from $N \subset \partial X$ and let $B = \pi^{-1}(C)$. The Soul Theorem 2.3 then implies that $M$ can be written as the union of neighborhoods $D(N)$ and $D(B)$ along their common boundary $E$, i.e.,

$$M = D(N) \cup E \cup D(B).$$

In particular, when $G = S^1$ and $C$ is another fixed point set component with maximal dimension, one has the following result from [SY].

Double Soul Theorem 2.4. Let $M$ be a complete, non-negatively curved Riemannian manifold admitting an isometric $S^1$ action. If $\text{Fix}(M, S^1)$ contains 2 connected codimension 2 components $X$ and $Y$, with one of them being compact, then $X$ is isometric to $Y$, $\text{Fix}(M, S^1) = X \cup Y$, and $M$ is diffeomorphic to an $S^2$-bundle over $X$ with $S^1$ as its structure group. In other words, there is a principal $S^1$-bundle $P$ over $X$ such that $M$ is diffeomorphic to $P \times_{S^1} S^2$.

2.3. Outline of the proof of Theorem A. The classification of simply-connected non-negatively curved 5-manifolds with an isometric $T^3$ action is a consequence of Oh’s classification of simply-connected 5-manifolds with a smooth $T^3$ action [Oh] (cf. Section 4). In the case of an isometric $T^2$ action on a simply-connected non-negatively curved manifold $M^5$ we use the geometry of the action to obtain enough information on the structure of $M^5$ to compute $H_2(M^5)$ and then appeal to the classification of simply-connected 5-manifolds due to Barden and Smale (cf. Section 5). In this case we consider two possibilities: when there is some circle $T^1 \subset T^2$ such that $\text{Fix}(M^5, T^1)$ has a 3-dimensional component, and when no circle has a 3-dimensional fixed point set. In the latter case, the fixed point set of any circle $T^1 \subset T^2$ is the union of circles, and in subsection 2.4 we show that there can be at most four such circles.

When there is some circle subgroup with a 3-dimensional fixed point set, our basic strategy is to understand and classify closed, non-negatively curved Riemannian codimension 2 submanifolds, $N^3 \subset M^5$, that admit an isometric $T^k$ action, $k = 1, 2$ (cf. Section 3). We observe that the technique can be generalized for any dimension $n$. That is, if we have a $T^k$ action on $M^n$ that guarantees the existence of a codimension 2 fixed point set for some circle, then we need to classify closed, non-negatively curved
Riemannian manifolds of dimension $n - 2$ admitting a $T^{k-1}$ action, in order to be able to identify the manifold $M^n$. That is, the results we obtain for a given dimension $n$ play a crucial role in the classification for dimension $n + 2$.

This is of course not sufficient to prove Theorem A. We must also understand the set at maximal distance from the codimension 2 submanifold $N^3$ in the orbit space. In the general case, let $M^n$ be a fixed-point homogeneous $S^1$-manifold and let $N^{n-2}$ be a component of $\text{Fix}(M^n, S^1)$ of codimension 2. Let $C_k$ be the set at maximal distance from $N^{n-2}$ in the orbit space $X^{n-1}$ and let $\pi^{-1}(C_k) = B^{k'}$ be the inverse image of $C_k$ under the projection map $\pi : M^n \to X^{n-1}$, where $k' \in \{k, k + 1\}$. We can write $M$ as the union of neighborhoods of $N$ and $B^{k'}$. We note that in our particular situation, since $N^{n-2}$ is the fixed point set component of a circle action, $C_k$ and $B^{k'}$ will be invariant under the torus action and this allows us to limit the possibilities for both.

Analyzing the possible $C_k$ as in [GG], we can then compile a list of possible $B^{k'}$ and, using the fact that $M^n$ can be written as the union of neighborhoods $D(N^{n-2})$ and $D(B^{k'})$, we obtain homological information on $M^n$ via the Mayer-Vietoris sequence. In particular, when $n = 5$ we compute $H_2(M^5)$ and then identify $M^5$ using Barden’s and Smale’s results [Ba, Sm]. More generally, we note that under the hypothesis that $M^n$ is simply-connected and non-negatively curved, we have the following restrictions on the fundamental group of $N^{n-2}$.

**Proposition 2.5.** Let $M^n$ be a closed, simply-connected, non-negatively curved manifold of dimension $n \geq 4$ with an isometric $S^1$ action and suppose that $\text{Fix}(M^n, S^1)$ contains an $(n-2)$-dimensional component $N^{n-2}$. Let $C_k$ be the set at maximal distance from $N^{n-2}$ in the orbit space $X^{n-1}$.

1. If $C_k$ has dimension $k = n - 2$, then $C_k$ is fixed by the $S^1$ action, is isometric to $N^{n-2}$ and $N^{n-2}$ is simply-connected.

2. If $C_k$ has dimension $k \leq n - 4$, then $N^{n-2}$ is simply-connected.

**Proof.** First we prove (1). If we suppose that $C$ is not fixed, then $B$, its inverse image in $M$ under the orbit projection map, is of dimension $n - 1$. We may decompose $M$ as a union of neighborhoods of $N^{n-2}$ and $B$, which we will denote $V$ and $U$, respectively. Their common boundary will be a circle bundle over $N^{n-2}$ which we will denote by $\partial V$. Observe that both $V$ and $\partial V$ are orientable, but that $U$ is not (this follows directly from the Mayer-Vietoris sequence of the triple $(M, V, U)$). Since $M^n$ is simply-connected it follows by duality that $H_{n-1}(M) = 0$ and that the torsion subgroup of $H_{n-2}(M)$ is trivial. Further, since $\partial V$ is a compact, orientable
manifold of dimension $n - 1$, the torsion subgroup of $H_{n-2}(\partial V)$ is also trivial. Likewise, since $V$ deformation retracts onto $N^{n-2}$, $H_{n-2}(V) = \mathbb{Z}$. Since $B$ is non-orientable, the torsion subgroup of $H_{n-2}(U)$ is equal to $\mathbb{Z}_2$.

If we write down the first few elements of the Mayer-Vietoris sequence of the triple $(M, V, U)$ we have:

$$0 \rightarrow H_n(M) \rightarrow H_{n-1}(\partial V) \rightarrow H_{n-1}(U) \oplus H_{n-1}(V) \rightarrow H_{n-1}(M)$$

$$\rightarrow H_{n-2}(\partial V) \rightarrow H_{n-2}(U) \oplus H_{n-2}(V) \rightarrow H_{n-2}(M).$$

Substituting known values we obtain:

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow 0 \oplus 0 \rightarrow 0$$

$$\rightarrow \text{Fr}(H_{n-2}(\partial V)) \rightarrow \text{Fr}(H_{n-2}(U)) \oplus \mathbb{Z}_2 \oplus \mathbb{Z} \rightarrow \text{Fr}(H_{n-2}(M)).$$

The sequence is clearly not exact and thus this case cannot occur. This in turn implies that the inverse image of $C^{n-2}$ in $M$ must be exactly $C^{n-2}$ and thus $C^{n-2}$ is a component of $\text{Fix}(M; S^1)$. It then follows from the Double Soul Theorem 2.4 that $M$ is an $S^2$ bundle over $N^{n-2}$ and in particular, it follows immediately that $N^{n-2}$ must be simply-connected.

To prove (2), let $\gamma$ be a loop in $N^{n-2} \subset M^n$. Since $M^n$ is simply-connected, $\gamma$ bounds a 2-disk $D^2$. Let $B^{k'} = \pi^{-1}(C^k)$ and observe that $k' \leq n - 3$. By transversality, we can perturb $D^2$ so as to lie in the complement of $D(B^{k'})$, a neighborhood of $B^{k'}$, while keeping $\gamma = \partial D^2$ in $N^{n-2}$. The conclusion follows after observing that $D^2$ is now contained in $D(N^{n-2})$, which deformation retracts onto $N^{n-2}$. □

**Observation.** The assertions in Proposition 2.5 hold trivially in dimension 2, since in this case the fixed point set components of codimension 2 are isolated points. We note that in dimension 3, however, Proposition 2.5 fails, since a 1-dimensional fixed point set component, being compact must be a circle, and thus has infinite fundamental group.

### 2.4. Estimating the number of sets with $S^1$ isotropy

We observe first that if we are given a $T^2$ action on a non-negatively curved 5-manifold, we may not have a codimension 2 fixed point set for any circle. In this case, we will need to determine the number of isolated circles fixed by a given circle subgroup. Theorem 2.6 below shows that this number is at most four. This result follows from an argument used in Kleiner’s thesis [K] to show that an isometric circle action on a non-negatively curved 4-manifold has at most four isolated fixed points. The key observation that allows us to apply the techniques in [K] to our situation is that the normal sphere at a point to each one of the circles fixed by some $S^1 \subset T^2$ is 3-dimensional. We include the proof of the theorem here for the sake of completeness since Kleiner’s result was never published. For the remainder of this subsection, we will consider
closed, non-negatively curved Riemannian 5-manifolds with an isometric $T^2$ action.

We observe as well, that unlike in the positively curved case, where the Extent Lemma tells us that we could have no more than 3 such isolated circle orbits (cf. [R]), here the Extent Lemma only tells us that there can be at most 5 such isolated circle orbits.

**Theorem 2.6.** Let $M^5$ be a closed non-negatively curved Riemannian manifold with an isometric $T^2$ action. Then there are at most 4 circles fixed by some $S^1 \subset T^2$.

The proof of Theorem 2.6 will occupy the remainder of this subsection. We begin by fixing some notation and recasting several lemmas from [K] to meet our needs. We assume that all geodesics have unit speed unless stated otherwise.

Let $\{p_i\}_{i=1}^4$ be four distinct points in a Riemannian manifold. For $1 \leq i, j \leq 4$, let $\Gamma_{ij}$ be the set of minimizing normal geodesics from $p_i$ to $p_j$ and, for each triple $1 \leq i, j, k \leq 4$, let

$$\alpha_{ijk} = \angle_{p_i}(p_j, p_k) = \min \{ \angle(\gamma'_j(0), \gamma'_k(0)) : \gamma_j \in \Gamma_{ij}, \gamma_k \in \Gamma_{ik} \}.$$ 

For each pair of points $p_i, p_j, i \neq j$, let $\text{Dir}_{p_i}(p_j)$ be the set of initial directions of a normal minimizing geodesic from $p_i$ to $p_j$, i.e.,

$$\text{Dir}_{p_i}(p_j) = \{ \gamma'_j(0) : \gamma_j \in \Gamma_{ij} \}.$$ 

We will assume for the remainder of this subsection:

**Lemma 2.7.** If there are 4 isolated circle orbits $\{N_i\}_{i=1}^4$, then, for distinct points $p_i \in N_i$, $1 \leq i \leq 4$, and every quadruple of distinct integers $1 \leq i, j, k, l \leq 4$, we have

$$\alpha_{ijk} + \alpha_{ijl} + \alpha_{ikl} = \pi.$$ 

**Proof.** In the orbit space $X^3 = M^5/T^2$, the 4 circles $\{N_i\}_{i=1}^4$ correspond to 4 points $\{\bar{p}_i\}_{i=1}^4$. By Toponogov’s theorem for Alexandrov spaces (cf. [BGP]), we know that the sum of the angles of a geodesic triangle in $X^3$ will be greater than or equal to $\pi$. Connecting each pair of distinct points in $\{\bar{p}_i\}_{i=1}^4$ by a minimal geodesic we obtain a configuration of 4 triangles and the total sum of the angles in this configuration will be greater than or equal to $4\pi$.

For each one of the four points $\{p_i\}_{i=1}^4$, the tangent space $T_{p_i}M^5$ decomposes as $T_{p_i}N_i \oplus T_{p_i}N_i^\perp$ and the normal space $T_{p_i}N_i^\perp$ is invariant under the action of the isotropy subgroup of $p_i$. Further, the isotropy subgroup acts freely on the normal space $T_{p_i}N_i^\perp$ and the quotient of the unit normal sphere $S^3 \subset T_{p_i}N_i^\perp$ is $S^2(\frac{1}{2})$, the round sphere of radius $1/2$. Hence $S_{p_i}X^3 = S^2(\frac{1}{2})$.
Lemma 2.8. If there are 5 isolated circle orbits \( \{N_i\}_{i=1}^5 \) then, for fixed \( 1 \leq i \leq 5 \) and points \( p_j \in N_j, 1 \leq j \leq 5, j \neq i, \) the following hold.

(a) The sets \( \text{Dir}_{p_i}(p_j), j \neq i, \) consist of a single \( S^1 \) orbit and come in mutually orthogonal pairs.

(b) The isotropy representation of \( S^1 \) on the unit normal sphere \( S^3 \subset T_{p_i}N_i^\perp \) is orthogonally equivalent to the Hopf action.

Proof. We first prove part (a). For convenience, let \( i = 5 \). We know by Lemma 2.7 that for the 4 points \( p_j, p_k, p_l, p_5 \), with \( j, k, l \neq 5 \), we have

\[
\alpha_{5jk} + \alpha_{5kl} + \alpha_{5lj} = \pi.
\]

For \( m = j, k, l \), let \( D_m = \text{Dir}_{p_5}(p_m)/S^1 \), where \( S^1 \) is the isotropy subgroup of the isolated circle orbit \( N^5 \). Observe that \( D_m \subset S^3/S^1 = S^2(1/2) \). Then we know that

\[
\frac{1}{2}\angle(D_i, D_j) + \frac{1}{2}\angle(D_j, D_k) + \frac{1}{2}\angle(D_k, D_i) \geq \pi,
\]

by Toponogov. But this implies that the sets \( D_m \) lie on a great circle, consist of a single point each and must pair off as antipodal points, thus proving part (a) of the lemma.

Part (b) follows from the fact that the isotropy representation of \( S^1 \) on the normal space \( T_{p_5}N_5^\perp \) contains more than one pair of mutually orthogonal orbits in the normal sphere \( S^3 \subset T_{p_5}N_5^\perp \), that is, the action is orthogonal and reducible. \( \square \)

Lemma 2.9. Let \( S^1 \) act on \( \mathbb{C}^2 \) by scalar multiplication and suppose \( v, w \in S^3(1) \subset \mathbb{C}^2 \). Then either \( \angle(S^1(v), S^1(w)) = \frac{\pi}{2} \) or there exists a unique \( t \in S^1 \) such that \( \angle(v, tw) = \angle(S^1(v), S^1(w)) = \angle(v, S^1(w)) \).

Proof. Let \( \pi : S^3(1) \to S^2(1/2) \). The Riemannian submersion metric on \( S^3(1)/S^1 \) is isometric to \( S^2(1/2) \subset \mathbb{R}^3 \). The images of the orbits \( S^1(v), S^1(w) \) in \( S^3(1)/S^1 \) are either separated by \( \frac{\pi}{2} \) or they are joined by a unique minimizing geodesic segment \( \tilde{\gamma} \subset S^3(1)/S^1 \) with \( \text{Length}(\tilde{\gamma}) = \angle(S^1(v), S^1(w)) \).

If \( w_0 \in S^1(w) \) satisfies

\[
\angle(v, w_0) = \angle(\pi(S^1(v)), \pi(S^1(w))) = \angle(S^1(v), S^1(w)),
\]

Using the fact that \( \text{xt}_q(S^2(1/2)) = \pi/3 \), it is easily seen that for any triple of distinct points \( x_j, x_k, x_l \in S^2(1/2) \), we have

\[
\text{dist}(x_j, x_k) + \text{dist}(x_j, x_l) + \text{dist}(x_k, x_l) \leq \pi.
\]

Thus summing over all the triangles formed by the points \( \{\tilde{p}_i\}_{i=1}^4 \) we find that the sum of their angles should be less than or equal to \( 4\pi \). Thus this sum of angles must be exactly \( 4\pi \). \( \square \)
and \( \gamma \) is a minimizing geodesic segment from \( v \) to \( w_0 \) in \( S^3(1) \), then \( \gamma \) projects to the minimizing curve \( \bar{\gamma} \). This implies that \( \gamma \) is the unique horizontal lift of \( \bar{\gamma} \) starting at \( v \) and \( w_0 \) is unique. \( \square \)

**Proof of Theorem 2.6.** We will assume that there are at least 5 isolated circle orbits \( \{N_i\}_{i=1}^5 \) and will obtain a contradiction. For each \( 1 \leq i \leq 5 \), let \( p_i \in N_i \) and observe that for each pair of points \( p_i \in N_i, p_j \in N_j, i \neq j \), \( \text{Dir}_{p_i}(p_j) \) is a subset of the unit normal sphere \( S^3 \subset T_{p_i}N_i^\perp \). We will now show that when \( \alpha_{ijk} = \pi/2 \), the set \( \text{Dir}_{p_i}(p_k) \) cannot be a single \( S^1 \) orbit in the unit normal sphere \( S^3 \subset T_{p_i}N_i^\perp \), in contradiction with Lemma 2.8.

Assume after relabeling points that \( \alpha_{123} = \pi/2 \). Let \( \gamma_2, \gamma_3 \) be minimal normal geodesics from \( p_1 \) to \( p_2 \) and \( p_1 \) to \( p_3 \), respectively. By Lemma 2.8, \( \angle_{p_1}(\gamma_2(0), \gamma_3(0)) = \pi/2 \). This implies that there exists a flat, totally geodesic triangular surface \( \triangle^2 \subset M^5 \) with geodesic edges \( \gamma_2, \gamma_3 \) and \( \eta \), where \( \eta \) is a minimal geodesic from \( p_2 \) to \( p_3 \).

Now, if we replace \( \gamma_3 \) with \( t\gamma_3 \), where \( t \in S^1 \), we obtain another flat, totally geodesic triangular surface \( \triangle^2_t \subset M^5 \) with geodesic edges \( \gamma_2, t\gamma_3 \) and \( \eta_t \), where \( \eta_t \) is a minimal geodesic from \( p_2 \) to \( p_3 \). In particular, \( \angle_{p_2}(\gamma_2, \eta_t) = \angle_{p_2}(\gamma_2, \eta) = \angle_{p_2}(p_1, p_3) < \frac{\pi}{2} \). Then, by Lemma 2.9 there is a unique \( t_0 \in S^1 \) such that \( \angle_{p_2}(\gamma_2, \eta_{t_0}) = \angle_{p_2}(p_1, p_3) \). It then follows that, for this \( t_0 \), \( \eta_{t_0} = \eta \) and thus \( \triangle^2_{t_0} = \triangle^2 \). Hence \( t = e \) and we have a contradiction. \( \square \)

**Observation.** We can easily generalize this result to higher dimensions. Applying the methods here we obtain the following result:

**Theorem 2.10.** Let \( M^n \) be a non-negatively curved \( n \)-manifold, \( n \geq 4 \), with an isometric \( T^2 \) action. Then there are at most 4 codimension 4 submanifolds fixed by some circle subgroup of \( T^2 \) (the submanifolds can be fixed by different circle subgroups).

3. **Non-negatively curved 3- and 4-manifolds with maximal and almost maximal symmetry rank**

In this section we consider the classification of 3- and 4-manifolds with an isometric action of a torus \( T^2 \) or \( T^1 \), corresponding, respectively, to the maximal and almost maximal symmetry rank cases.

3.1. **Dimension 3.** We classify closed, orientable, non-negatively curved 3-manifolds admitting an isometric \( T^2 \) or \( T^1 \) action. In the case of a \( T^2 \) action, we have the following result, without appealing to any curvature assumptions.

**Theorem 3.1.** A closed, orientable, 3-manifold with an isometric \( T^2 \) action is equivariantly diffeomorphic to \( S^3, L_{p,q}, S^2 \times S^1 \) or \( T^3 \).
Proof. This is a direct consequence of Mostert’s and Neumann’s classification of closed cohomogeneity one 3-manifolds \([M, N]\). We observe that there are 3 different types of actions, depending on how many circles can act freely. If we first assume that \(T^2\) acts freely, we see that the only closed, orientable, cohomogeneity one 3-manifold admitting a free \(T^2\) action is \(T^3\). If we assume that just one circle can act freely, then the quotient space \(M^3/T^1\) is a 2-dimensional closed, orientable manifold. By the classification of 2-manifolds we see that \(M^3/T^1\) is either a \(T^2\) or \(S^2\) and the other circle must fix points in this quotient. Since the former does not admit a non-trivial action with fixed points, \(M^3/T^1 = S^2\). In particular, the other circle will fix exactly 2 points of \(S^2\) and thus \(M^3\) will have two isolated circle orbits. In particular, \(M^3\) is an \(S^1\)-bundle over \(S^2\) and thus one of \(S^3\), \(L_{p,q}\) or \(S^2 \times S^1\). If both circles fix, then we see from the orbit structures that the only possibilities are \(S^3\) and \(L_{p,q}\). □

In the case of a \(T^1\) action, we have the following result (cf. \([GG]\) for the more general case where \(M^3\) need not be orientable), which follows from the Orlik-Raymond-Seifert classification of 3-manifolds with a smooth \(T^1\) action \([OR]\).

**Theorem 3.2.** Let \(T^1\) act on \(M^3\), a closed, orientable 3-manifold of non-negative curvature. Then \(M^3\) is equivariantly diffeomorphic to \(S^3\), \(L_{p,q}\), \(S^2 \times S^1\), \(T^3\) or \(\mathbb{R}P^3 \# \mathbb{R}P^3\).

Proof. If the action is free, then \(M^3/T^1\) is a closed, orientable 2-manifold of non-negative curvature and thus \(M^2 = S^2\) or \(M^2 = T^2\). Since \(M^3\) is a principal circle bundle over \(M^2\), we know that \(M^3\) is one of \(S^3\), \(L_{p,q}\), \(S^2 \times S^1\) or \(T^3\).

If the action is not free, then there is a fixed point set of dimension 1 with at most 2 components, and the components are circles. If \(\text{Fix}(M^3, S^1)\) contains two components, then by the Double Soul Theorem we see that \(M^3\) is one of the two \(S^1\) bundles over \(S^2\) and since \(M^3\) is assumed to be orientable it is \(S^2 \times S^1\). If \(\text{Fix}(M^3, S^1)\) consists of a single component \(F^1\), then we note that \(X^2 = M^3/S^1\) is a 2-dimensional Alexandrov space of non-negative curvature with boundary \(F^1 \cong S^1\). Thus \(X^2\) is an orientable, non-negatively curved topological manifold with boundary and the only possibilities are the 2-disc \(D^2\) and the cylinder. Since we only have one circle in our fixed point set, we see that \(X^2 = D^2\).

We observe that there can be at most two points with finite isotropy in \(X^2\). The set at maximal distance from \(F^1\) can have dimension at most 1 and will contain the points of finite isotropy. If \(C^0 = \{p\}\), then the point is either of finite isotropy \(Z_p\) or not. In the first case, we see that \(M^3 = L_{p,q}\) and in the second case \(M^3\) would be \(S^3\). If \(C\), the set at maximal distance is
of dimension 1, it can be $S^1$ or an interval, $I$. In the first case, it is clear that none of the points in $C^1 = S^1$ can have finite isotropy by Kleiner’s isotropy lemma and in the second case, we see that there can be at most 2 points of finite isotropy. In the second case, we can add one more space to our list, namely $M^3 = \mathbb{R}P^3 \# \mathbb{R}P^3$.

Thus $M^3$ can only be $S^3, L_{p,q}, S^2 \times S^1, T^3$ or $\mathbb{R}P^3 \# \mathbb{R}P^3$ and all of these manifolds support isometric $T^1$ actions with non-negative curvature yielding the possible orbit structures.

□

3.2. Dimension 4. Given Perelman’s work on the Poincaré conjecture [P1, P2, P3], the classification of closed, simply-connected, non-negatively curved 4-manifolds admitting a $T^1$ isometric action can be seen as a simple consequence of earlier classification results in a curvature-free setting. The case of an isometric $T^2$ action will similarly follow from the classification of closed, simply-connected 4-manifolds with a smooth $T^2$ action. We first recall these classification results. The first theorem is due to Orlik and Raymond [OR2].

**Theorem 3.3.** A closed, simply-connected 4-manifold with a $T^2$ action is equivariantly diffeomorphic to a connected sum of $S^4, \pm \mathbb{C}P^2$ and $S^2 \times S^2$.

The second theorem is due to work of Fintushel [F1, F2], in combination with work of Pao [Pa] and Perelman’s proof of the Poincaré conjecture [P1, P2, P3].

**Theorem 3.4.** A closed, simply-connected 4-manifold with a $T^1$ action is equivariantly diffeomorphic to a connected sum of $S^4, \pm \mathbb{C}P^2$ and $S^2 \times S^2$.

Let $M^4$ be a closed, simply-connected, non-negatively curved 4-manifold and let $\chi(M^4)$ be the Euler characteristic of $M^4$. It follows from work in [K, SY] that $2 \leq \chi(M^4) \leq 4$. Combining this with Theorems 3.3 and 3.4 yields the following result in the case of non-negative curvature.

**Theorem 3.5.** A closed, simply-connected non-negatively curved manifold with an isometric $T^2$ or $T^1$ action is diffeomorphic to $S^4, \mathbb{C}P^2, S^2 \times S^2$ or $\mathbb{C}P^2 \# \pm \mathbb{C}P^2$.

4. Non-negatively curved 5-manifolds with maximal symmetry rank

In this section we classify closed, simply-connected, non-negatively curved 5-manifolds with an isometric $T^3$ action, corresponding to the maximal symmetry rank case in dimension 5. We first recall Oh’s classification of simply-connected 5-dimensional $T^3$-manifolds [Oh]. We will denote the second Stiefel-Whitney class of a manifold $M$ by $w_2(M)$ and the non-trivial $S^3$-bundle over $S^2$ by $S^3 \times S^2$. 
Theorem 4.1. Let $M^5$ be a closed, simply-connected 5-manifold with a $T^3$ action and $k$ orbits with isotropy $T^2$. Then $M^5$ is equivariantly diffeomorphic to one of the following:

- $S^5$, if $k = 3$;
- $(k - 3)(S^2 \times S^3)$, if $w_2(M^5) = 0$;
- $(S^3 \times S^2)(k - 4)(S^2 \times S^3)$, if $w_2(M^5) \neq 0$.

The classification of simply-connected non-negatively curved 5-manifolds with maximal symmetry rank follows easily from Oh’s classification and the following lemma.

Lemma 4.2. Let $M^5$ be a closed, simply-connected non-negatively curved 5-manifold. If $T^3$ acts isometrically on $M^5$, then there are at most 4 points with $T^2$ isotropy.

The proof of this lemma follows by applying the same arguments utilized in the proof of Theorem 2.10 after observing that a $T^2$ action on a normal 3-sphere yields a space of directions equal to an interval of length at most $\pi/2$.

Observe that, when $M^5$ is a simply-connected non-negatively curved 5-manifold with an isometric $T^3$ action, the orbit space is a 2-disc and is an Alexandrov space with non-negative curvature. We obtain the following result by combining Lemma 4.2 and Theorem 4.1.

Theorem 4.3. Let $M^5$ be a closed, simply-connected non-negatively curved 5-manifold with an isometric $T^3$ action. Then $M^5$ is diffeomorphic to $S^5$, $S^2 \times S^3$ or $S^3 \times S^2$.

We observe first that this is the maximal symmetry rank of $M^5$. If we assume the existence of a $T^4$ action we first note that if no circle acts freely, the action is not effective and such an action cannot have more than one circle acting freely, since more than one circle would imply the existence of a free action on $M^4 = M^5/T^1$ and this is impossible. Likewise, a $T^3$ action on $M^4$ with all circles fixing is not effective and one can then show that some circle subgroup of the $T^4$ action must fix all of $M^5$.

We remark that we can also prove this theorem directly in a more “geometric” fashion, using the same techniques as in case 1 of the proof of Theorem 5.1.
5. Non-negatively curved 5-manifolds with almost maximal symmetry rank

We now classify closed, simply-connected, non-negatively curved 5-manifolds with an isometric $T^2$ action, corresponding to the almost maximal symmetry rank case in dimension 5. In the next section we provide examples of such actions. We summarize our results in the following theorem.

**Theorem 5.1.** Let $M^5$ be a closed, simply-connected 5-manifold with non-negative curvature and an isometric $T^2$ action. Then $M^5$ is diffeomorphic to $S^5$, $SU(3)/SO(3)$, $S^3 \times S^2$ or $S^3 \tilde{\times} S^2$.

We note first that $T^2$ cannot act freely. For if this were the case, then we would have a free circle action on $N^4 = M^5/S^1$, a closed, simply-connected manifold. As mentioned previously, this is impossible. Thus at least one circle in the $T^2$ action must fix points in $M^5$.

First we consider the case where one circle in $T^2$ acts freely. Here $M^5$ fibers over one of the four manifolds $S^4$, $CP^2$, $S^2 \times S^2$ or $CP^2 \# \pm CP^2$, mentioned in Theorem 3.5. Now note that only $CP^2$, $S^2 \times S^2$ or $CP^2 \# - CP^2$ admit such a fibration, and the corresponding total space is diffeomorphic to $S^5$, $S^3 \times S^2$ or the non-trivial $S^3$ bundle over $S^2$.

We now consider the case where no circle acts freely. There are two cases to consider here: the dimension of $\text{Fix}(M^5; S^1)$ is 3 or the dimension is 1. We break the proof into two parts correspondingly. Our main strategy here will be to compute $H_2(M^5)$. The conclusions of the Theorem 5.1 will then follow from the Barden-Smale classification of simply-connected 5-manifolds [Bal, Sm].

5.1. Case $1$: $\dim(\text{Fix}(M^5; S^1)) = 3$. Suppose first that $C^k$, the set at maximal distance from $F^3$ in the orbit space $X^4$ is of dimension 0 or 1. By Proposition 2.5 $F^3$ is simply-connected and hence $F^3$ must be $S^3$. If $C^k$ has dimension 0, then it is already the soul of $X^4$ and, as in [GS], $M^5$ is diffeomorphic to $S^5$. If $C^k$ is 1-dimensional, it must be an interval or a circle. Since $F^3 \cong S^3$ is homeomorphic to the boundary of a neighborhood of $C^1$, we must have that $C$ is an interval. By Kleiner’s Isotropy Lemma, the only points in $C^1$ that may have finite non-trivial isotropy $\mathbb{Z}_k$ are the endpoints. In the presence of points with finite isotropy, $F^3$ is a lens space or the connected sum of two lens spaces, which is a contradiction. Hence all the points in $C^1$ are regular and $X^4$ is a manifold with boundary $F^3$, the soul of $X^4$ is a regular point and again we conclude that $M^5$ is $S^5$.

We now consider the case where the set at maximal distance, $C^k$ is of dimension 2 and note that while $F^3$ is still an orientable submanifold, it no longer need be simply-connected. We know that it admits a $T^1 = T^2/S^1$
action and by Theorem\[3.2\] we may conclude that \( M^3 \) is one of \( S^3, L_{p,q}, S^2 \times S^1, T^3 \) or \( \mathbb{R}P^3 \# \mathbb{R}P^3 \).

Let \( B = \pi^{-1}(C^2) \subset M^5 \) be the lift of \( C^2 \) under the orbit projection map \( \pi : M^5 \to X^4 = M^5/\mathbb{S}^1 \). We note that as in the proof of Theorem\[4.3\] we have a \( T^2 \) action on \( B \subset M^5 \) and a circle action on its image in \( X^4 \). Note that \( B^3 \) is a 3-dimensional, totally convex subspace of \( M^5 \), invariant under the \( T^2 \) action. In particular, it is also totally geodesic and thus of non-negative curvature. Since \( C^2 = B^3/\mathbb{S}^1 \) is a 2-dimensional Alexandrov space of non-negative curvature, we know that it is a topological manifold of non-negative curvature \([BBI]\). Hence \( C^2 \) can be one of \( S^2, \mathbb{R}P^2, T^2, D^2, \text{Kl} \) or the Möbius band \( \text{Mb} \). We note that the \( T^2 \) action on \( B^3 \) has only a finite number of circle orbits (as we saw before, there can be at most 4 isolated circle orbits in all of \( M^5 \)). It is thus clear that \( B^3 \) Seifert fibers over \( C^2 \), and if we analyze case by case, we see that \( B^3 \) is also a topological manifold. In particular, it is a cohomogeneity one 3-manifold, and we know by results of \([MN]\) that \( B^3 \) can be one of \( T^3, \text{Kl} \times S^1, \text{A}, S^3, L_{p,q}, S^2 \times S^1, S^2 \times S^1 \) or \( \mathbb{R}P^3 \# \mathbb{R}P^3 \), where \( \text{A} \) is \((\text{Mb} \times S^1) \cup (S^1 \times \text{Mb}) \) intersecting canonically in \( S^1 \times S^1 \), and \( S^2 \times S^1 \) denotes the non-trivial \( S^2 \) bundle over \( S^1 \).

Note that as before, \( M^5 \) will decompose as a union of disc bundles over \( F^3 \) and \( B^3 \) and thus \( D(F^3) \cap D(B^3) \) must be a circle bundle over both \( F^3 \) and \( B^3 \).

By considering all possible combinations of \( F^3 \) and \( B^3 \) we see that there is only one possibility that can occur and that we have not yet considered, namely \( F^3 = S^3 \) and \( B^3 = S^2 \tilde{\times} S^1 \), the non-trivial \( S^2 \) bundle over \( S^1 \). With this last combination \( M^5 \) has the same homology type as the Wu manifold \( SU(3)/SO(3) \).

We consider now the case where \( \dim(\text{Fix}(M^5; \mathbb{S}^1)) = 1 \).

5.2. Case 2: \( \dim(\text{Fix}(M^5; \mathbb{S}^1)) = 1 \).

**Theorem 5.2.** Let \( M^5 \) be a closed, simply-connected, non-negatively curved Riemannian manifold admitting an isometric \( T^2 \) action. If no circle has fixed point set of codimension less than 4, then \( M^5 \) is diffeomorphic to \( S^5, S^3 \times S^2 \) or \( S^3 \tilde{\times} S^2 \).

The proof of Theorem\[5.2\] will occupy the remainder of this section. We have divided the proof into two cases: Case A, where there is no finite isotropy and Case B, where there is finite isotropy.

5.2.1. Case A: \( T^2 \) acts without finite isotropy. In this case, the following proposition determines \( M^5 \).
Proposition 5.3. Let $M^5$ be a closed, simply-connected, non-negatively curved Riemannian manifold admitting an isometric $T^2$ action with no finite isotropy. If no circle has fixed point set of codimension less than 4, then $M^5$ is diffeomorphic to $S^5$, $S^3 \times S^2$ or the non-trivial $S^3$ bundle over $S^2$.

Before proving Proposition 5.3, we recall Lemma 2.4 from [R], taking into account Perelman’s proof of the Poincaré conjecture [P1, P2, P3].

Lemma 5.4. Let $M^5$ be a closed, simply-connected, non-negatively curved Riemannian manifold admitting an isometric $T^2$ action. Then the quotient space $M^* = M^5 / T^2$ is a 3-sphere.

Note that Rong states in [R] that $M^*$ is only a homotopy 3-sphere, since at the time the Poincaré conjecture was as yet not proven.

Proof of Proposition 5.3. Let $M_0$ denote the manifold with boundary obtained by removing a small tubular neighborhood around each isolated fixed circle. Let $M^*_0$ denote the quotient space $M_0 / T^2$. By the standard transversality argument we know that

$$\pi_1(M_0) \cong \pi_1(M) = \{1\}$$

and

$$\pi_2(M_0) \cong \pi_2(M).$$

By Theorem 2.6 there are at most 4 isolated circle orbits and we observe that there are also at least 3 such orbits by the following argument of [R].

Since there is no isotropy group of finite order we obtain a fibration

$$T^2 \rightarrow M_0 \rightarrow M^*_0,$$

and therefore a long exact sequence in homotopy:

$$0 \rightarrow \pi_2(M_0) \rightarrow \pi_2(M^*_0) \rightarrow \pi_2(T^2) \rightarrow \pi_1(M_0) \rightarrow \pi_1(M^*_0).$$

Since $\pi_1(M) \cong \pi_1(M_0) = \{1\}$ this implies that $\pi_1(M^*_0) = \{1\}$. Since $M^*$ is a 3-sphere, applying the Mayer-Vietoris sequence to the pair $(M^*_0, cl(M^* \setminus M^*_0))$, it follows easily that $H_2(M^*_0) \cong \mathbb{Z}^r$, where $(r + 1)$ is the number of fixed circles. By the Hurewicz isomorphism it follows that $\pi_2(M^*_0) \cong H_2(M^*_0) \cong \mathbb{Z}^r$ and the above exact sequence in homotopy becomes

$$0 \rightarrow \pi_2(M_0) \rightarrow \mathbb{Z}^r \rightarrow \mathbb{Z} \oplus \mathbb{Z} \rightarrow 0.$$ 

We conclude that $2 \leq r \leq 3$. When $r = 2$, we have 3 isolated circle orbits and we see that $\pi_2(M) = \pi_2(M_0) = 0$ and when $r = 3$ we see that $\pi_2(M) = \mathbb{Z}$. Thus by the Hurewicz isomorphism we see that $H_2(M) = 0$ or $\mathbb{Z}$, respectively, and therefore by [Sm] $M^5$ is diffeomorphic to $S^5$ or one of the $S^3$ bundles over $S^2$. 

$\square$
5.2.2. Case B: $T^2$ acts with finite isotropy. We must now consider the case where the $T^2$ action has isotropy groups of finite order. Since the fixed point sets of isotropy groups of finite order will contain circle orbits we know that there are at most 4 different isotropy subgroups of finite order.

Rong considers in [R] the case where there are 3 circle orbits and the manifold is positively curved and shows that $M^5 = S^5$. We will present another (shorter) proof of this result here as well as its extension to non-negative curvature.

We note that, as with fixed point sets of circles, fixed point sets of finite cyclic groups (of odd order) will be totally geodesic submanifolds of even codimension and of non-negative sectional curvature, as a result of being totally geodesic. If the order of the group is even, then the fixed point set components may be of odd codimension as well. In particular if the fixed point set components are of even codimension they will be orientable.

We note that if $\mathbb{Z}_2$ does fix a submanifold of odd codimension, then the same $\mathbb{Z}_2$ acts on the normal even-dimensional sphere to this fixed point component and the action must be orientation reversing. However, this implies that the $\mathbb{Z}_2$ action on $M^5$ is orientation reversing and thus the entire $T^2$ action on $M^5$ is also orientation reversing. Note that we are assuming here that some circle subgroup, $S^1$, fixes at least one 1-dimensional submanifold in $M^5$ (thus a circle). In particular, $S^1$ acts on the normal $S^3$ to any point of this fixed point set component and since the action must be orientation reversing, it follows immediately that the action must fix a point. But this implies that $S^1$ fixes a codimension 2 submanifold in $M^5$, contrary to our hypothesis. Thus the $T^2$ action must be orientation preserving and no $\mathbb{Z}_2$ subgroup will fix a submanifold of odd codimension.

We will now show that any finite isotropy group can only fix 3-dimensional components. In particular this 3-dimensional component will be orientable.

**Lemma 5.5.** Let $T^2$ act isometrically on $M^5$, a closed, simply-connected, non-negatively curved Riemannian manifold, and assume that we have only isolated circle orbits and finite isotropy. Then the fixed point set of the finite isotropy group is 3-dimensional, orientable and one of either $L_{p,q}$ (where $p = q = 1$ is also considered), $T^3$ or $S^2 \times S^1$.

**Proof.** First observe that $T^2$ will act on the fixed point set of any finite subgroup. As observed above, the fixed point set of finite isotropy will be of even codimension and thus of dimension 1 or 3 only.

If the fixed point set is 1-dimensional, we know it must be a circle and at least one circle subgroup of the $T^2$ action must fix it. So we have an $S^1 \times \mathbb{Z}_p$ action on the normal $S^3$ to this fixed point set. Analyzing the possibilities we see that either some circle fixes a codimension 2 submanifold
of $M^5$ (contradicting our hypothesis) or some finite group fixes a larger dimensional submanifold.

Therefore, the fixed point set must be of dimension 3 and we will denote it by $N^3_H$. Observe that $N^3_H$ is a closed, orientable 3-manifold of non-negative curvature admitting an isometric $T^2$ action. By work of Mostert and Neumann [M, N], we know that the closed, orientable 3-manifolds of cohomogeneity one admitting a $T^2$ action are $T^3$, $S^3$, $L_{p,q}$ and $S^2 \times S^1$. We observe that if $N^3_H$ contains at least one circle orbit then $T^3$ is excluded as a possibility. We further observe that in all the other cases, $N^3_H$ will contain exactly two isolated circle orbits. This leaves us then with only the following candidates when we assume that $N^3_H$ must contain an isolated circle orbit: $L_{p,q}$ and $S^2 \times S^1$. We note that we must include the case $p = q = 1$, that is where $L_{p,q} = S^3$.

\[
18 \text{ GALAZ-GARCÍA AND SEARLE}
\]

Observe that it suffices to consider the case where there is a unique finite isotropy group $H = \mathbb{Z}_k$ whose fixed point set $\text{Fix}(M^5; \mathbb{Z}_k)$ contains a 3-dimensional component $N^3_H$. Indeed, if there is another finite subgroup, $H'$, fixing points in $M^5$, we can simply mod out this action from the $T^2$ and $T^2/H' \cong T^2$ will now act without $H'$ isotropy. Hence from now on we will assume that there is a unique finite isotropy group $H = \mathbb{Z}_k$ with $N^3_H \subset \text{Fix}(M^5; \mathbb{Z}_k)$.

We note as well that if the fixed point set of the finite isotropy group does not contain any of the isolated circle orbits, it follows directly from the proof of Theorem 2.6 that there can be at most one 3-dimensional component $N^3_H$ when we have 3 isolated circle orbits. In the case where there are 4 isolated circle orbits $\text{Fix}(M^5; \mathbb{Z}_k)$ must contain at least one of the isolated circle orbits. We will consider each of these two cases separately, and the case where $N^3_H$ contains no isolated circle orbit last. Observe that $N^3_H$ may be simply-connected (cf. Example 6.6 in Section 6).

We now have two sub-cases to consider: Case B 1, where there are exactly 3 isolated circle orbits and Case B 2, where there are exactly 4 isolated circle orbits.

**Case B 1: There are exactly 3 isolated circle orbits.** We have the following theorem.

**Theorem 5.6.** Let $T^2$ act isometrically on $M^5$, a closed, simply-connected, non-negatively curved manifold, and assume that no circle fixes a codimension 2 submanifold. If there are exactly 3 circle orbits and there is finite isotropy, then $M^5 = S^5$.
To prove this theorem we will show that $H_2(M^5) = 0$ for all possible fixed point sets of the finite isotropy group (cf. Lemma 5.5). The conclusion will then follow from [Ba, Sm]. We first observe that proceeding as in the proof of Lemma 3.1 in [R], we obtain the following lemma:

**Lemma 5.7.** Suppose $T^2$ acts isometrically on $M^5$, a closed, simply-connected Riemannian manifold. If there are exactly 3 isolated circle orbits, then $H_2(M^5)$ is at most finite.

Using this lemma we will show that $H_2(M^5)$ is actually trivial and appealing to results of [Sm] and [Ba] it follows immediately that $M^5$ is $S^5$.

**Proof of Theorem 5.6.** Let $Z_k$ be the finite isotropy group of the $T^2$ action and let $N^3_H$ be a 3-dimensional component of $\text{Fix}(M^5; Z_k)$. Let $V$ denote a small closed $\delta$ tubular neighborhood of $N^3_H$ and let $U$ denote the complement of an open $\frac{\delta}{2}$ tubular neighborhood of $N^3_H$. Then $M^5 = U \cup V$ and $U \cap V = \partial V$ is an $S^1$-bundle over $N^3_H$. By Lemma 5.5 $N^3_H$ can be one of $L_{p,q}$, $S^2 \times S^1$ or $T^3$. We will analyze each case separately.

**CASE B 1.1: $N^3_H = L_{p,q}$, a lens space.**

Assume that $N^3_H = L_{p,q}$. We include the case where $p = q = 1$ and $N^3_H = S^3$. Thus $V$ is topologically equivalent to $L_{p,q}$ and $\partial V$ is a circle bundle over $L_{p,q}$ and using duality and the Gysin sequence of the fibration it follows easily that $\partial V$ has the homology of $L_{p,q} \times S^1$.

We will use the following theorem due to Matveev [Ma] to show that $\pi_2(M^5)$ is trivial in the case where $\pi_2(V) = \pi_2(U) = 0$ and thus by the Hurewicz isomorphism, $H_2(M^5) = 0$ when $N^3_H = L_{p,q}$.

**Theorem 5.8.** Let the topological space $X$ be split into two subspaces $X_+$ and $X_-$ such that $\pi_2(X_+) = \pi_2(X_-) = 0$. Suppose further that $X_0 = X_+ \cap X_- \neq \emptyset$ is connected and that the homomorphisms $\phi_\pm : \pi_1(X_0) \to \pi_1(X_\pm)$ are epimorphisms. Let $N_\pm = \ker(\phi_\pm)$. Then $\pi_2(X) = (N_+ \cap N_-)/[N_+, N_-]$.

We now suppose that there is no other finite isotropy in $M^5$. It follows from the Van Kampen theorem that $\pi_1(U)$ is $\mathbb{Z}$ or $\mathbb{Z}_n$. Let $U_0$ be $U$ with a tubular neighborhood around the other isolated circle orbit excised. If we consider the long exact sequence in homotopy of the fibration $T^2 \to U_0 \to U_0^*$, where $U_0^* = U_0/T^2$, (2)\[ 0 \to \pi_2(U_0) \to \pi_2(U_0^*) \to \mathbb{Z} \oplus \mathbb{Z} \to \pi_1(U_0) \to \pi_1(U_0^*). \]

We claim that $\pi_1(U_0^*) = 0$ and $\pi_2(U_0^*) = \mathbb{Z}$ which then implies, since $\pi_1(U) \cong \pi_1(U_0)$, that $\pi_1(U) \cong \mathbb{Z}$. The proof that $\pi_1(U_0^*) = 0$ and $\pi_2(U_0^*) = \mathbb{Z}$ is exactly the same as in [R] and we repeat the argument for the sake of completeness. Note that $M^* = S^3$ and since $V^* = I$, it follows that $U^*$
has the homotopy type of a point. Thus \( \pi_1(U_0^*) = \pi_1(U^*) = 0 \). Since 
\( H_2(U^*) = 0 \), it is easy to see that \( H_2(U_0^*) = \mathbb{Z} \) and using the Hurewicz isomorphism we see that \( \pi_2(U_0^*) = \mathbb{Z} \).

By hypothesis, \( \pi_2(V) = 0 \). By transversality \( \pi_2(U) = \pi_2(U_0) \) and it 
follows from the long exact homotopy sequence (2) above that \( \pi_2(U_0) = 0 \), so 
\( \pi_2(U) = 0 \). Since \( \phi_+ : \pi_1(V) \to \pi_1(U) \) and \( \phi_- : \pi_1(V) \to \pi_1(U) \) are 
onto (see [R] p. 168), it follows from Theorem 5.8 that \( \pi_2(M) = (\mathbb{N}_+ \cap \mathbb{N}_-)/[\mathbb{N}_+, \mathbb{N}_-] \), where \( \mathbb{N}_\pm = \ker \phi_\pm \). Given \( \pi_1(U) = \mathbb{Z} \), it follows that 
\( \mathbb{N}_+ = \mathbb{Z} \) and \( \mathbb{N}_- = \mathbb{Z}_p \) and thus \( \mathbb{N}_+ \cap \mathbb{N}_- = \{1\} \). We conclude that 
\( \pi_2(M) = 0 \) and, by the Hurewicz theorem, that \( H_2(M) = 0 \). Finally, it 
follows from work of Smale and Barden [Sm, Ba] that \( M = S^5 \).

We note that in this case, it follows directly from the Mayer-Vietoris sequence that \( N^3_H \) must be \( S^3 \) and that \( U \) will have the homology groups of a circle.

**CASE B 1.2: \( N^3_H = S^2 \times S^1 \).**

Here \( N^3_H \) has two circle orbits and so \( U \) will contain one. We once again 
assume that the isolated circle orbit in \( U \) is not contained in the fixed point 
set of a finite isotropy subgroup. Since \( \partial V \) is an orientable \( S^1 \) bundle over 
\( N^3_H \), we can show, using the long exact sequence in homology, duality and 
the Gysin sequence in homology, that the homology groups of \( \partial V \) are the 
same as those of \( S^3 \times S^1, L_p \times S^1 \) or \( S^2 \times T^2 \).

In the first two cases, when we consider the Mayer-Vietoris sequence of the 
triple \( (M, U, V) \) we see that the restrictions on the homology groups of 
\( M^5 \), i.e., that \( H_2(M^5) \) is at most finite, are inconsistent with \( N^3_H = S^2 \times S^1 \) 
and \( \partial V \) homotopic to one of \( S^3 \times S^1 \) or \( L_p \times S^1 \). Thus neither case occurs.

We consider then the case where \( \partial V \) is homotopic to \( S^2 \times T^2 \). We note 
that the map from \( H_2(S^2 \times T^1) \) to \( H_2(S^2 \times S^1) \) is onto with kernel equal 
to \( \mathbb{Z} \). However, when we analyze the Mayer-Vietoris sequence of the triple 
\( (M^5, U, V) \) this implies that \( H_3(M^5) \) contains a \( \mathbb{Z} \) factor, contrary to our 
hypothesis that \( H_3(M^5) \) is trivial and thus this case does not occur.

**CASE B 1.3: \( N^3_H = T^3 \) and contains no isolated circle orbit.** In this case 
we have 3 isolated circle orbits, and one 3-dimensional submanifold fixed 
by a finite subgroup that does not contain any of the isolated circle orbits. 
If this is the case, then \( N^3_H \) is \( T^3 \).

**CASE B 1.3.1: Fix(\( M; \mathbb{Z}_k \)) = T^3.** Suppose first that \( N^3_H = \text{Fix}(M^5; \mathbb{Z}_k) = T^3 \). As we did before, we decompose \( M^5 \) into two closed subsets \( V \) and \( U \),
where $V$ is a closed $\delta$ tubular neighborhood around $T^3$ and $U$ is the complement of an open $\frac{\delta}{2}$ tubular neighborhood around $T^3$. Clearly $U$ contains the three isolated circle orbits.

We can now decompose $U$ into two closed subsets $W$ and $W'$, where $W$ is the $\delta$ tubular neighborhood around one isolated circle orbit and $W'$ is the complement of an open $\frac{\delta}{2}$ tubular neighborhood around the same isolated circle orbit. We have $\partial W = S^3 \times S^1$ (in particular $\partial W$ is cobordant to $\partial V$ which is orientable and thus $\partial W$ is also orientable). The boundary of $W'$ is the disjoint union of two components, $\partial V$ and $\partial W$. We can then show that the relative sequence of $(W', \partial W')$ is not exact (if we assume that the interior of $W'$ is orientable this follows directly from Lefschetz duality with $\mathbb{Z}$ coefficients, and if the interior of $W'$ is not orientable this follows from Lefschetz duality with $\mathbb{Z}_2$ coefficients). Thus this case does not occur.

We must also consider the cases where there is finite isotropy in $U$. However if any of the isolated circle orbits is contained in a 3-dimensional submanifold fixed by a finite isotropy group, then we may once again consider the decomposition around this fixed point set, leaving the $T^3$ and the other isolated circle orbit in the complement of $U$.

We consider first the case where the two isolated circle orbits are contained in $L_{p,q}$ and then the case where these orbits are contained in $S^2 \times S^1$.

**CASE B 1.3.2:** Fix $(M; \mathbb{Z}_K) = T^3 \cup L_{p,q}$. As we mention above, we suppose first that $V$ is a closed $\delta$ tubular neighborhood around $L_{p,q}$ and that $U$ is the complement of an open $\frac{\delta}{2}$ tubular neighborhood around $L_{p,q}$. We then decompose $U$ into 2 closed subsets $W$ and $W'$, where $W$ is the $\delta$ tubular neighborhood around $T^3$ and $W'$ is the complement of an open $\frac{\delta}{2}$ tubular neighborhood around the $T^3$. In particular, this means that $W'$ has two boundary components: $T^4$ and $L_{p,q} \times S^1$. We can then show that the relative sequence of $(W', \partial W')$ is not exact (if we assume that the interior of $W'$ is orientable this follows directly from Lefschetz duality with $\mathbb{Z}$ coefficients, and if the interior of $W'$ is not orientable this follows from Lefschetz duality with $\mathbb{Z}_2$ coefficients). Thus this case does not occur.

**CASE B 1.3.3:** Fix $(M; \mathbb{Z}_K) = T^3 \cup S^2 \times S^1$. We note that using the same procedure as in Case 3.1 we may also show that this case never occurs, noting that since $V$ will be the closed $\delta$ tubular neighborhood around $S^2 \times S^1$, there are 3 possibilities for its boundary, $\partial V$, that we must consider. Calculating as above in each case we see that all fail for the same reasons as above.

□
Case B 2: There are exactly 4 isolated circle orbits. We must now consider the case where there are 4 circle orbits and \( \text{Fix}(M^5; \mathbb{Z}_k) \) is non-empty.

We have the following theorem.

**Theorem 5.9.** Let \( T^2 \) act isometrically on \( M^5 \), a closed, simply-connected, non-negatively curved manifold, and assume that no circle fixes a codimension 2 submanifold. If there are exactly 4 circle orbits and there is finite isotropy, then \( M^5 \) is one of the two \( S^3 \) bundles over \( S^2 \).

To prove Theorem 5.9 it suffices to show that \( H_2(M^5) \cong \mathbb{Z} \) and thus by \( [\text{Sm}, \text{Ba}] \) \( M^5 \) is one of the two \( S^3 \) bundles over \( S^2 \). Before beginning the proof, we first observe that, proceeding as in the proof of Lemma 3.1 in Rong \( [R] \), one obtains the following lemma.

**Lemma 5.10.** Suppose \( T^2 \) acts isometrically on \( M^5 \), a closed, simply-connected, non-negatively curved Riemannian manifold. If there are exactly 4 isolated circle orbits, then \( H_2(M^5) \) is at most a finite extension of \( \mathbb{Z} \).

**Proof of Theorem 5.9.** As before we must consider the two possibilities for \( N^3_H \). Note that since we have 4 isolated circle orbits, either we will have one 3-dimensional fixed point set of finite isotropy or two distinct 3-dimensional fixed points sets of finite isotropy.

When there are exactly two 3-dimensional fixed point sets containing all 4 isolated circle orbits, then we will show that \( N^3_H_i, i = 1, 2 \) can only be the (same) lens space, \( L_{p,q} \).

**CASE B 2.1:** \( N^3_H = L_{p,q} \), a lens space

We assume first that \( N^3_H = L_{p,q} \) and that the other two isolated circle orbits are not fixed by any finite isotropy. We note that \( N^3_H \) will contain exactly 2 of the isolated circle orbits and thus the other two must be contained in \( U \).

Then \( N^3_H \subset V \) and \( V \) is homotopy equivalent to \( L_{p,q} \) and \( \partial V \) has the homology groups of \( L_{p,q} \times S^1 \).

It follows from the Van Kampen theorem that \( \pi_1(U) \) is \( \mathbb{Z} \) or \( \mathbb{Z}_q \). We may calculate the homology groups of \( U \) using the Mayer-Vietoris sequence of the triple \( (M, U, V) \). Since we have two isolated circle orbits in \( U \), we may divide \( U \) into \( W \) and \( W' \), where \( W \) is the closed \( \delta \) tubular neighborhood of one isolated circle orbit (with boundary \( S^3 \times S^1 \) (which is cobordant to \( \partial V \) and therefore orientable) and \( W' \) the complement of an open \( \frac{1}{2} \) tubular neighborhood of the same isolated circle orbit. We may then consider the relative sequence in homology of \( (W', \partial W') \) and we find that with both \( \mathbb{Z} \) and \( \mathbb{Z}_2 \) coefficients (respectively assuming that the interior of \( W' \) is orientable or not) that the sequence is not exact by using Lefschetz duality. Thus this case cannot occur.
If the other two isolated circle orbits are contained in a 3-dimensional submanifold of finite isotropy, they are contained in an $S^2 \times S^1$ or an $L_{p,q}$. In the first case, since $\partial V = \partial U$ is a circle bundle over both $S^2 \times S^1$ and over $L_{p,q}$ it follows that $\partial V = S^3 \times S^1$. It follows directly from the Mayer-Vietoris sequence of the triple $(M, U, V)$ that $H_2(M) = \mathbb{Z}$ and thus $M^5$ is diffeomorphic to one of the two $S^3$ bundles over $S^2$. Likewise, in the second case, where both 3-dimensional fixed point set components are $L_{p,q}$, it follows immediately that they must be the same $L_{p,q}$. When we analyze the Mayer-Vietoris sequence of the triple $(M, U, V)$ it follows immediately that $H_2(M) = \mathbb{Z}$ and thus once again $M^5$ is diffeomorphic to one of the two $S^3$ bundles over $S^2$. Thus only these last two cases may occur.

**CASE B 2.2: $N^3_H = S^2 \times S^1$**

Here, once again, $N^3_H$ will contain exactly 2 of the isolated circle orbits and thus the other two must be contained in $U$.

We have that $\partial V$ can be one of $S^3 \times S^1, L_{p,q} \times S^1$ or $S^2 \times T^2$. We consider first the case where $\partial V = S^3 \times S^1$. From the Mayer-Vietoris sequence of the triple $(M, U, V)$ it follows that $H_1(U) = 0$ and using Van Kampen’s theorem we see that $\pi_1(U) = 0$. We consider the fibration $T^2 \rightarrow U_0 \rightarrow U^*_0$ we see that $\pi_1(U) = \pi_1(U_0) = \pi_1(U^*_0) = 0$ and as we have shown before it follows easily that $H_2(U^*_0) = \mathbb{Z}^2$ and thus $\pi_2(U^*_0) = \mathbb{Z}^2$. In particular this tells us that $\pi_2(U) = \pi_2(U_0) = 0$ and by the Hurewicz isomorphism we see that $H_2(U) = 0$. Analyzing the Mayer-Vietoris sequence, it follows immediately that $H_2(M^5) = \mathbb{Z}$ and thus $M^5$ is diffeomorphic to one of the two $S^3$ bundles over $S^2$.

We now consider the cases where $\partial V = L_{p,q} \times S^1$ and $\partial V = S^2 \times T^2$. We note that we may use the same procedure we used in Case 1 to show that neither of these cases occur. Namely we divide $M$ into subsets $U$ ad $V$ and we then divide $U$ into subsets $W$ and $W'$ and show that the relative sequence in homology of $(W', \partial W')$ is never exact. Thus these cases do not occur.

Finally we consider the case where the other two isolated circle orbits are contained in another 3-dimensional submanifold fixed by the same finite isotropy group. We note that we already considered the case where one 3-dimensional submanifold is $L_{p,q}$ and the other is $S^2 \times S^1$ in Case 1. Thus it remains to consider the case where both 3-dimensional submanifolds are $S^2 \times S^1$.

We note that no matter what the common boundary of $U$ and $V$ is, none of these cases is possible. This is easily seen by computing the Mayer-Vietoris sequence of the decomposition.
6. SOME EXAMPLES OF ISOMETRIC $T^2$ ACTIONS ON SIMPLY-CONNECTED, NON-NEGATIVELY CURVED 5-MANIFOLDS

6.1. Examples of actions with codimension 2 fixed point set. It is easy to find examples of such actions and we list a few here.

Example 6.1. Let $T^2$ act on $S^5$ as follows:
\[
(\theta_1, \theta_2), (z_1, z_2, z_3) \mapsto (e^{2\pi i \theta_1}z_1, e^{2\pi i \theta_2}z_2, z_3)
\]
where $(\theta_1, \theta_2) \in T^2$, $(z_1, z_2, z_3) \in S^5 \subset \mathbb{C}^3$. Here both circles $\theta_1$ and $\theta_2$ fix a 3-sphere.

Example 6.2. Let $T^2$ act on $S^3 \times S^2$ as follows. Given $(\theta_1, \theta_2) \in T^2$ and $(z_1, z_2, x_1, x_2, x_3) \in S^3 \times S^2 \subset \mathbb{C}^2 \times \mathbb{R}^3$, let
\[
((\theta_1, \theta_2), (z_1, z_2, x_1, x_2, x_3)) \mapsto (e^{2\pi i \theta_1}z_1, e^{2\pi i \theta_2}z_2, x_1, x_2, x_3).
\]
Here both circles $\theta_1$ and $\theta_2$ fix an $S^2 \times S^1$ and the action is the product of the cohomogeneity one action on $S^3$ combined with the trivial action on $S^2$.

Example 6.3. Let $T^2$ act on $S^3 \times S^2$ as follows. Given $(\theta_1, \theta_2) \in T^2$, $(z_1, z_2, x_1, x_2, x_3) \in S^3 \times S^2 \subset \mathbb{C}^2 \times \mathbb{R}^3$, send the pair
\[
((\theta_1, \theta_2), (z_1, z_2, x_1, x_2, x_3))
\]
to
\[
(z_1, e^{2\pi i \theta_1}z_2, \cos(\theta_2)x_1 + \sin(\theta_2)x_2, -\sin(\theta_2)x_1 + \cos(\theta_2)x_2, x_3).
\]
Here the circle $\theta_1$ will fix an $S^2 \times S^1$ and the circle $\theta_2$ will fix an $S^3$.

Example 6.4. Note that there is a cohomogeneity one action on the Wu manifold, $SU(3)/SO(3)$ given by the following diagram $G \subset \{K_1, K_2\} \subset H$ where $G = U(2)$, $K_1 = \{e^i, 1\} \cdot H$, $K_2 = \{(e^{j\pi}, e^{j\theta})\}, H = \{(j, i)\}$ and the orbits are given by $G/H = S^3 \times S^1$, $G/K_2 = S^3$ and $G/K_1$ is the nontrivial $S^2$ bundle over $S^1$. Clearly the maximal torus, $T^2$, of $U(2)$ also acts on $SU(3)/SO(3)$.

6.2. Examples of actions with finite isotropy. We give examples of actions with finite isotropy, with 3 and 4 isolated circle orbits on $S^5$ and on $S^3 \times S^2$. The action on $S^5$ was given by Rong [R] and we include it here for the sake of completeness.

Example 6.5. Let $T^2$ act on $S^5$ as follows.
\[
(\theta_1, \theta_2), (z_1, z_2, z_3) \mapsto (e^{2\pi i (\theta_1 + p\theta_2)}z_1, e^{2\pi i (\theta_1 + q\theta_2)}z_2, e^{2\pi i (\theta_1 + r\theta_2)}z_3)
\]
where $(\theta_1, \theta_2) \in T^2$, $(z_1, z_2, z_3) \in S^5 \subset \mathbb{C}^3$ and $p, q, r$ are pairwise relatively prime and without loss of generality $p > q > r$. Here there are 3
isolated circle orbits and the finite groups $\mathbb{Z}_{p-r}, \mathbb{Z}_{q-r}$ and $\mathbb{Z}_{p-q}$ each fix an $S^3(1) \subset S^5(1)$ containing exactly two isolated circle orbits.

**Example 6.6.** Let $T^2$ act on $S^3 \times S^2$ as follows. Given $(\theta_1, \theta_2) \in T^2$ and $v = (z_1, z_2, x_1, x_2, x_3) \in S^3 \times S^2 \subset \mathbb{C}^2 \times \mathbb{R}^3$, we let $(\theta_1, \theta_2)$ act on $v$ by

$$(\theta_1, \theta_2), v) \mapsto A(\theta_1, \theta_2)v,$$

where $A(\theta_1, \theta_2)$ is the matrix

$$
\begin{pmatrix}
    e^{2\pi i (\theta_1 + p\theta_2)} & 0 & 0 & 0 & 0 \\
    0 & e^{2\pi i (\theta_1 + q\theta_2)} & 0 & 0 & 0 \\
    0 & 0 & \cos(\theta_1 + r\theta_2) & \sin((\theta_1 + r\theta_2) & 0 \\
    0 & 0 & -\sin((\theta_1 + r\theta_2) & \cos(\theta_1 + r\theta_2) & 0 \\
    0 & 0 & 0 & 0 & 1
\end{pmatrix},
$$

$p, q, r$ are pairwise relatively prime integers and, without loss of generality, $p > q > r$. Here there are 4 isolated circle orbits and the finite groups $\mathbb{Z}_{p-r}, \mathbb{Z}_{q-r}$ each fix 2 disjoint copies of $S^2 \times S^1$'s whereas the finite group $\mathbb{Z}_{p-q}$ fixes two disjoint copies of $S^3$.

7. **Maximal Symmetry Rank of Low-Dimensional Manifolds with Non-Negative Curvature**

**Theorem 7.1.** Let $T^k$ act isometrically on a closed, simply-connected, non-negatively curved $n$-manifold $M^n$. If $n \leq 9$, then $k \leq \left[\frac{2n}{3}\right]$.

**Proof.** This theorem is an easy consequence of our work in dimensions 3, 4 and 5. As we observed earlier the maximal symmetry rank of a closed, simply-connected $n$-manifold with non-negative curvature is 2 if $n = 3, 4$, and 3 if $n = 5$. We note that for $n \geq 5$, given a $T^k$ action on $M^n$, we may always assume that at least one circle acts freely and thus by induction the maximal symmetry rank of an $(n+1)$-manifold is always at most $k+1$ when the maximal symmetry rank of $M^n$ is $k$. It follows that $M^6$ has maximal symmetry rank 4.

Note that a $T^4$ action on $M^6$ must have at least one circle acting freely, since if all circles fix, then one acts ineffectively. This can be seen using the same inductive argument found in [GS]. To show that $M^7$ also has maximal symmetry rank 4, we note first that by our work in dimension 5, if $T^5$ acts on $M^7$, then there is at most one free circle action, since none of the manifolds obtained by a $T^3$ action on $M^5$ are the base of a simply-connected bundle with fiber $T^2$. Thus we have two cases to consider: first where one circle may act freely and second, where none act freely. A $T^5$ action with one circle acting freely implies that we have a $T^4$ action on $M^6 = M^7/S^1$ where no circle acts freely and we have just seen that this means we have one circle acting ineffectively on $M^6$ and thus on $M^7$. Likewise, if no
circle in $T^5$ acts freely on $M^7$, then one circle acts ineffectively. Thus the maximal symmetry rank of $M^7$ is 4. By induction, it follows that the maximal symmetry rank of $M^8$ is 5 and the maximal symmetry rank of $M^9$ is 6. It follows easily that the free rank of these actions is bounded above by $\left\lfloor \frac{n}{3} \right\rfloor$.

\[\Box\]

We observe that using the fact that the maximal symmetry rank of an $(n + 1)$-manifold is always at most $k + 1$ when the maximal symmetry rank of $M^n$ is $k$, and the result for dimension 9, it is easy to conclude that the maximal symmetry rank of $M^n$ is less than or equal to $n - 3$.

We also obtain the following corollary.

**Corollary 7.2.** Let $T^k$ act isometrically on $M^n$, a closed, simply-connected, non-negatively curved $n$-manifold with $n \leq 6$ and $k$ equal to the maximal symmetry rank. Then $M^3 = S^3$, $M^4$ is one of the 5 manifolds $S^4, \mathbb{C}P^2, S^2 \times S^2$ or $\mathbb{C}P^2 \# \pm \mathbb{C}P^2$, $M^5$ is one of the 3 manifolds $S^5, S^3 \times S^2$ or $S^3 \times S^2$ and $M^6 = S^3 \times S^3$.

**Proof.** By our work in dimension 5 and results in dimensions 3 and 4, we only need to show the result in dimension 6. If we have a $T^4$ action on $M^6$, we know that at least one circle must act freely and at most 2 may. In either case, we obtain a $T^3$ action on $M^6/S^1$ and thus $M^5$ is one of the three manifolds $S^5, S^3 \times S^2$ or $S^3 \times S^2$. Only the last two are the base of a principal circle bundle that is closed, simply-connected and non-negatively curved, namely $S^3 \times S^3$. In particular, it is clear that $M^6$ is a homology $S^3 \times S^3$ and thus by work of Kreck [Kr] and Wall [Wa] it follows that $M^6$ is diffeomorphic to $S^3 \times S^3$.

\[\Box\]

We observe that in order to obtain the same information in dimensions 7, 8 and 9 it should suffice to understand the case when there is a $T^3$ action on $M^6$ with no circle acting freely, i.e. $M^6$ is $S^1$ fixed point cohomogeneity 1.

Finally, we point out that the maximal symmetry rank and maximal free symmetry rank results we have obtained for dimensions 9 and less are consistent with the maximal symmetry rank and maximal free symmetry rank of

- $M^{3n} = S^3 \times \cdots \times S^3$ (the product of $n$ copies of $S^3$);
- $M^{3n+1} = S^3 \times \cdots \times S^3 \times S^4$ (the product of $n - 1$ copies of $S^3$ and one $S^4$);
- $M^{3n+2} = S^3 \times \cdots \times S^3 \times S^2$ (the product of $n$ copies of $S^3$ and one $S^2$);
and conclude with the conjecture that Theorem 7.1 is true in all dimensions and that we have rigidity in dimension $3n$:

**Conjecture 7.3.** Let $T^k$ act isometrically on a closed, simply-connected non-negatively curved $n$-manifold $M^n$. Then $k \leq \left\lfloor \frac{2n}{3} \right\rfloor$. Further, if $n = 3k$, $M^{3k} = S^3 \times \cdots \times S^3$, the product of $k$ copies of $S^3$.

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