On the homology theory of fiber spaces

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In this paper the homology theory of fibre spaces is studied by introducing additional algebraic structure in homology and cohomology.

All modules are assumed to be over an arbitrary associative ring \(\Lambda\) with unit; by a differential algebra, coalgebra, module, or comodule we mean these objects graded by non-negative integers; \(\hat{a}\) denotes \((-1)^{deg a}\).

The category \(A(\infty)\). An \(A(\infty)\)-algebra in the sense of Stasheff is defined to be a graded \(\Lambda\)-module \(M\), endowed with a set of operations \(\{m_i : \otimes^i M \to M, i = 1, 2, \ldots\}\) satisfying the conditions

\[
\sum_{k=0}^{i-1} \sum_{j=1}^{i-k} (-1)^k m_{i-j+1}(\hat{a}_1 \otimes \ldots \otimes \hat{a}_k \otimes m_j(a_{k+1} \otimes \ldots \otimes a_{k+j}) \otimes a_{k+j+1} \otimes \ldots \otimes a_i) = 0
\]

for any \(a_i \in M\) and \(i \geq 1\). A morphism of \(A(\infty)\)-algebras \((M, \{m_i\}) \to (M', \{m'_i\})\) is a set of homomorphisms \(\{f_i : \otimes^i M \to M', i = 1, 2, \ldots\}\) satisfying the conditions

\[
\sum_{k=0}^{i-1} \sum_{j=1}^{i-k} (-1)^k f_{i-j+1}(\hat{a}_1 \otimes \ldots \otimes \hat{a}_k \otimes m_j(a_{k+1} \otimes \ldots \otimes a_{k+j}) \otimes \ldots \otimes a_i) = \sum_{t=1}^i \sum_{S(t, i)} m'_t(f_{k_1}(a_1 \otimes \ldots \otimes a_{k_1}) \otimes \ldots \otimes f_{k_t}(a_{i-k_1+1} \otimes \ldots \otimes a_i))
\]

where \(S(t, i) = \{k_1, \ldots, k_t \in N, \sum k_p = i\}\). The \(A(\infty)\)-algebras together with these morphisms form a category, which we denote by \(A(\infty)\).

The specification on \(M\) of an \(A(\infty)\)-algebra structure \((M, \{m_i\})\) is equivalent to the specification on the tensor coalgebra \(T_c(M) = \Lambda + M + M \otimes M + \ldots\) with the grading \(dim(a_1 \otimes \ldots \otimes a_n) = \sum dim a_i + n\) and comultiplication

\[
\Delta(a_1 \otimes \ldots \otimes a_n) = \sum_{i=0}^n (a_1 \otimes \ldots \otimes a_i) \otimes (a_{i+1} \otimes \ldots \otimes a_n)
\]
of a differential \( d_m : T^c(M) \to T^c(M) \) that turns \( T^c(M) \) into a differential coalgebra; this set \( \{ m_i \} \) determines the differential \( d_m \) by

\[
d_m(a_1 \otimes \ldots \otimes a_n) = \sum_{k=0}^{n-1} \sum_{j=1}^{n-k} (-1)^k \hat{a}_1 \otimes \ldots \hat{a}_k \otimes m_j(a_{k+1} \otimes \ldots \otimes a_{k+j}) \otimes \ldots \otimes a_n,
\]

and the differential coalgebra \( (T^c(M), d_m) \) is called the \( \tilde{B} \)-construction of the \( A(\infty) \)-algebra \( (M, \{ m_i \}) \) (Stasheff [4]) and is denoted by \( \tilde{B}(M, \{ m_i \}) \). The specification of an \( A(\infty) \)-algebra morphism \( \{ f_i \} : (M, \{ m_i \}) \to (M', \{ m'_i \}) \) is equivalent to that of a differential coalgebra mapping \( f : \tilde{B}(M, \{ m_i \}) \to \tilde{B}(M', \{ m'_i \}) \); the morphism \( \{ f_i \} \) determines the mapping \( f \) by

\[
f(a_1 \otimes \ldots \otimes a_n) = \sum_{i=1}^{n} \sum_{S(t,i,n)} f_{k_1}(a_1 \otimes \ldots \otimes a_{k_1}) \otimes \ldots \otimes f_{k_t}(a_{n-k_1+1} \otimes \ldots \otimes a_i).
\]

Thus the category \( A(\infty) \) can be identified with a full subcategory of the category of differential coalgebras.

An arbitrary object in \( A(\infty) \) of the form \( (M, \{ m_1, m_2, 0, 0, \ldots \}) \) is identified with the differential algebra \( (M, \partial, \cdot) \) where \( \partial = m_1 \) and \( a_1 \cdot a_2 = -m_2(\hat{a}_1 \otimes a_2) \). For such an object the \( \tilde{B} \)-construction coincides with the usual \( B \)-construction, any morphism of such objects of the form \( \{ f_1, 0, 0, \ldots \} \) is identified with the differential algebra mapping \( f_1 : (M, \partial, \cdot) \to (M, \partial, \cdot) \). Thus the category of differential algebras is a subcategory of \( A(\infty) \), while the category \( DASH \) (see [2]) is the full subcategory of \( A(\infty) \) generated by differential algebras, and the functor \( \tilde{B} \) is an extension of \( B \) from this subcategory to \( A(\infty) \).

**Theorem 1** For any differential algebra \( C \) with free \( H_i(C), i \geq 0 \) it is possible to introduce on \( H(C) \) an \( A(\infty) \)-algebra structure

\[
(H(C), \{ X_i \}), \quad X_i : \otimes^i H(C) \to H(C), \quad i = 1, 2, 3, \ldots
\]

such that \( X_1 = 0 \), \( X_2(a_1 \otimes a_2) = -\hat{a}_1 \cdot a_2 \) and there exists an \( A(\infty) \)-morphism

\[
\{ f_i \} : (H(C), \{ X_i \}) \to (C, \{ m_1, m_2, 0, 0, \ldots \})
\]

for which \( f_1 : H(C) \to C \) induces an identical isomorphism in homology.

**Proof.** We need to construct two sets of homomorphisms

\[
\{ X_i : \otimes^i H(C) \to H(C), \quad i = 1, 2, 3, \ldots \}, \quad \{ f_i : \otimes^i H(C) \to C, \quad i = 1, 2, 3, \ldots \},
\]

satisfying the conditions in the definition of the category \( A(\infty) \):

\[
\sum_{j=1}^{i-k} \sum_{k=0}^{j-1} (-1)^k X_{i-j+1} (\hat{a}_1 \otimes \ldots \hat{a}_k \otimes X_j(a_{k+1} \otimes \ldots \otimes a_{k+j}) \otimes a_{k+j+1} \otimes \ldots \otimes a_i) = 0,
\]

(1)
Corollary 1. The mapping of differential coalgebras

\[ f : B(H(C), \{X_i\}) \to B(C) \]

induces an isomorphism in homology.

\textbf{The category} \(M(\infty)\). An \(A(\infty)\)-module over an \(A(\infty)\)-algebra \((M, \{m_i\})\) we define to be a graded \(\Lambda\)-module \(P\), endowed with a set of operations \(\{p_i :
$$\left(\otimes^{i-1} M\right) \otimes P \to P, \ i = 1, 2, 3, \ldots$$
satisfying the conditions $p_i((\otimes^{i-1} M) \otimes P)_q \subset M_{q+i-2}$ and

$$\sum_{k=0}^{i-2} \sum_{j=1}^{i-k-1} (-1)^k p_{i-j+1} (\hat{a}_1 \otimes \ldots \otimes \hat{a}_k \otimes m_j (a_{k+1} \otimes \ldots \otimes a_{k+j}) \otimes \ldots \otimes a_{i-1} \otimes b) + \sum_{k=0}^{i-1} (-1)^k p_{i-1} (\hat{a}_1 \otimes \ldots \otimes \hat{a}_k \otimes p_{i-k} (a_{k+1} \otimes \ldots \otimes a_{i-1} \otimes b)) = 0.$$

The specification on $P$ of an $A(\infty)$-module structure over $(M, \{m_i\})$ is equivalent to the specification on $\hat{B}(M, \{m_i\}) \otimes P$ of a differential that turns it into a differential comodule over $\hat{B}(M, \{m_i\})$. The objects of the category $M(\infty)$ are defined to be the pairs $((M, \{m_i\}), (P, \{p_i\}))$, where $(M, \{m_i\})$ is an $A(\infty)$-algebra, and $(P, \{p_i\})$ is an $A(\infty)$-module over it. A morphism is defined to be a pair of sets of homomorphisms $\{f_i\}, \{g_i\}$ where $f_i : (M, \{m_i\}) \to (M', \{m'_i\})$ is a morphism of $A(\infty)$-algebras and $g_i : (\otimes^{i-1} M) \otimes P \to P'$, $i = 1, 2, 3, \ldots$

is a set satisfying the conditions $g_i((\otimes^{i-1} M) \otimes P)_q \subset P'_{q+i-1}$ and

$$\sum_{k=0}^{i-2} \sum_{j=1}^{i-k-1} (-1)^k g_{i-j+1} (\hat{a}_1 \otimes \ldots \otimes \hat{a}_k \otimes m_j (a_{k+1} \otimes \ldots \otimes a_{k+j}) \otimes \ldots \otimes a_{i-1} \otimes b) = \sum_{k=0}^{i-1} \sum_{j=1}^{k} f_{k,j} (a_{k+1} \otimes \ldots \otimes a_k) \otimes f_{k+1,j} (a_{k+1} \otimes \ldots \otimes a_{k+j}) \otimes \ldots \otimes g_k (a_{i-k+1} \otimes \ldots \otimes a_{i-1} \otimes b);$$

These conditions ensure that the mapping

$$g : \hat{B}(M, \{m_i\}) \otimes P \to \hat{B}(M', \{m'_i\}) \otimes P'$$
given by

$$g(a_1 \otimes \ldots \otimes a_{i-1} \otimes b) = \sum_{k=1}^{i} \sum_{S(t,i)} f_{k,1} (a_1 \otimes \ldots \otimes a_k) \otimes \ldots \otimes f_{k-1} (a_{k+t-1} \otimes \ldots \otimes a_{k+1+t} \otimes \ldots \otimes a_{i-1} \otimes b)$$

is a differential comodule mapping compatible with $f$. With the obvious morphisms the category of pairs $(C, D)$, where $C$ is a differential algebra and $D$ is a differential module over it, forms a subcategory of $M(\infty)$.

**Theorem 2** If $C$ is a differential algebra and $D$ is a differential module over it such that $H_i(C)$ and $H_i(D)$ are free, then on $H(D)$ it is possible to introduce the structure of an $A(\infty)$-module $(H(D), \{Y_i\})$, $Y_i : (\otimes^{i-1} H(C)) \otimes H(D) \to H(D)$, $p = 1, 2, 3, \ldots$ over the homology $A(\infty)$-algebra $(H(C), \{X_i\})$ such that $Y_1 = 0$, $Y_2(a \otimes b) = -\hat{a} \cdot b$ and there exists a morphism $\{(f_i), \{g_i\} : ((H(C), \{X_i\}), (H(D), \{Y_i\})) \to (C, D) \}$ of $M(\infty)$ for which $f_1 : H(C) \to C$ and $g_1 : H(D) \to D$ induce identical isomorphisms in homology.

**Proof.** The sets $\{g_i\}$ and $\{Y_i\}$ are constructed by induction on $i$ just as in the proof of Theorem 1. Using the fact that $H_i(D)$ is free, we define $g_1$ to be a
cycle-choosing homomorphism, while $Y_1 = 0$, and the conditions of the category $M(\infty)$ are satisfied for $i = 1$. Let

\[
V_n(a_1 \otimes ... \otimes a_{n-1} \otimes b) = \\
\sum_{i=1}^{n-1} \left( \begin{array}{c} n-1 \\ i \end{array} \right) p_x(s_1 \otimes \ldots \otimes s_i \otimes g_{n-s}(a_{s+1} \otimes \ldots \otimes a_{n-1} \otimes b)) + \\
\sum_{k=0}^{n-2} \sum_{j=2}^{n-1} (-1)^{k+1} g_{j-1}(a_{k+1} \otimes \ldots \otimes a_{k+j}) \otimes \ldots \otimes a_{n-1} \otimes b) + \sum_{k=1}^{n-2} (-1)^k g_{k+1}((a_1 \otimes \ldots \otimes a_k) \otimes Y_{n-k}(a_{k+1} \otimes \ldots \otimes a_{n-1} \otimes b)),
\]

then $\partial V_n = 0$, therefore, we define $Y_n = \{V_n\}$. Since $g_1$ is a cycle-choosing homomorphism, $g_1 Y_n - V_n$ is a cycle in $D$ homologous to zero. Using the fact that $H_i(D)$ is free, we define $g_1 : (\otimes^{n-1}) \otimes H(D) \to H(D)$ to be a homomorphism for which $\partial g_n = g_1 V_n - V_n$. The conditions of the category $M(\infty)$ are satisfied for the $Y_n$ and $g_n$ thus defined.

**Twisted tensor products.** The twisted tensor products of Brown \[8\] can be generalized from the case of differential algebras and modules to the case of $A(\infty)$-algebras and $A(\infty)$-modules: for an arbitrary differential coalgebra $(K, d)$ and an $A(\infty)$-algebra $(M, \{m_i\})$ a $\sim$-twisting cochain we define to be a homomorphism $\phi : K \to M$ of degree $-1$ that satisfies the condition

\[
\phi d = \sum_{i=1}^{\infty} m_i(\phi \otimes \phi \otimes \ldots \otimes \phi)\Delta^i,
\]

where $\Delta^i : K \to K \otimes K$ is the homomorphism defined by $\Delta^1 = id_K$, $\Delta^2 = \Delta : K \to K \otimes K$, $\Delta^i = (id_K \otimes \Delta^{i-1})\Delta$. The specification of a $\sim$-twisting cochain $\phi : K \to M$ is equivalent to that of a mapping of differential coalgebras $f_\phi : (K, d) \to B(M, \{m_i\})$. For any $(K, d)$ and $((M, \{m_i\}), (P, \{p_i\})) \in M(\infty)$ any $\sim$-twisting cochain $\phi : K \to M$ on the tensor product $K \otimes P$ determines by

\[
\partial_\phi = d \otimes id_P + \sum_{i=1}^{\infty} (id_\otimes p_i)(id_K \otimes \phi \otimes \ldots \otimes \phi \otimes id_P)(\Delta^i \otimes id_P)
\]

a differential, turning $(K \otimes P, \partial_\phi)$ into a differential comodule over $(K, d)$; this differential comodule is called the $\sim$-twisted tensor product $K \otimes_\phi P$. If $M$ is an $A(\infty)$-algebra of the form $(M, \{m_1, m_2, 0, 0, \ldots\})$, and $P$ is an $A(\infty)$-module of the form $(P, \{p_1, p_2, 0, 0, \ldots\})$, then $\phi$ is the usual twisting cochain, and $K \otimes_\phi P$ coincides with the usual twisted tensor product $K \otimes P$.

We need the concept of equivalence of twisting cochains (see \[4\], \[5\], \[6\]). We say that $\phi, \psi : K \to C$ are *equivalent* if there is a homomorphism $c : K \to C$ of degree $0$ for which $c_0 = c|C_0 = 0$ and $\psi = (1 + c) \ast \phi$ where

\[
(1 + c) \ast \phi = (1 + c) \cdot \phi \cdot (1 + c)^{-1} - (cd + dc) \cdot (1 + c)^{-1};
\]

$\phi \sim \psi$ if and only if $f_\phi, f_\psi : K \to B$ are homotopic in the sense of \[2\] (coderivation homotopy): $f_\phi - f_\psi = \partial D + D\partial$ with $(D \otimes f_\phi + f_\psi \otimes D)\Delta = \Delta D$.

**Theorem 3** If $(K, d)$ is a differential coalgebra with free $K_1$, and $\phi : K \to C$ is an arbitrary twisting cochain, then there exists a $\sim$-twisting cochain $\phi^* : K \to H(C)$ such that $\phi$ and $f^* \phi^* = \sum_{i=1}^{\infty} f_\phi(\phi \ast \ldots \otimes \phi^*)\Delta^i$ are equivalent.
Proof. To construct $\phi^*$ we prove the following inductive assertion: for any $i > 0$ there exists a twisting cochain $\phi^{(i)} : K \to C$ and a homomorphisms $\phi^*_i : K_i \to H_{i-1}(C)$ and $c_i : K_i \to C_i$ such that

(a) $\phi^*_i d = \sum_{t=2}^i \sum_{S(t,i)} X_i(\phi^*_{k_1} \otimes \cdots \otimes \phi^*_{k_t}) \Delta^i$;

(b) $\phi^{(i)} = (1 + c_i) * \phi^{(i-1)}$;

(c) $\phi^*_i = \sum_{t=1}^i \sum_{S(t,i)} f_i(\phi^*_{k_1} \otimes \cdots \otimes \phi^*_{k_t}) \Delta^i$.

For $i = 1$ we take $\phi^*_1 = \{\phi_1\}$. Since the difference $(\phi_1 - f_1 \phi^*_1)(k)$ is homologous to zero for each $k \in K_1$ and $K_1$ is free, we obtain a homomorphism $c_1 : K_1 \to C_1$ for which $-\partial c_1 = \phi_1 - f_1 \phi^*_1$. We define $\phi^{(1)} = (1 + c_1) * \phi$, so $\phi^{(1)} = \phi_1 + \partial c_1 = f_1 \phi^*_1$. Suppose now that $\phi^{(i)}$, $\phi^*_i$, and $c_i$ have already been constructed in such a way that (a), (b), and (c) hold for $i < n$. Let

$W_n = \phi_n^{(n-1)} - \sum_{t=2}^n \sum_{S(t,n)} f_i(\phi^*_{k_1} \otimes \cdots \otimes \phi^*_{k_t}) \Delta^i$;

A direct check shows that $\partial W_n = 0$; we define $\phi^*_n = \{W_n\}$. Since the difference $W_n - f_1 \phi^*_n$ is homological to zero and $K_n$ is free, we can construct a $c_n : K_n \to C_n$ such that $-\partial c_n = W_n - f_1 \phi^*_n$; let $\phi^{(n)} = (1 + c_n) * \phi^{(n-1)}$. Then

$\phi_n^{(n)} = \phi_n^{(n-1)} + \partial c_n = f_1 \phi^*_n - W_n = \sum_{t=1}^n \sum_{S(t,n)} f_i(\phi^*_{k_1} \otimes \cdots \otimes \phi^*_{k_t}) \Delta^i$,

consequently, (b) and (c) hold for $\phi^{(n)}$, $\phi^*_n$, and $c_n$, and the validity of (a) can be checked directly. From (a) we see that $\phi^* = \sum_{i} \phi^*_i$ is a ~-twisting cochain, and from (b) and (c) we deduce that $f^* \phi^* = \phi^\infty$, where $\phi^\infty = \Pi_i (1 + c_i) * \phi \sim \phi$.

It follows from Theorem 2 that for any differential coalgebra mapping $g : K \to B$ there exists a $G^* : K \to \hat{B}(H(C), \{X_i\})$ for which $g$ and $fg^*$ are homotopic in the sense of 2. This assertion implies uniqueness mentioned above for the structure of homology $A(\infty)$-algebra: if $(H(C), \{X_i\})$ and $(H(C), \{X_i\})$ are two structures of homology $A(\infty)$-algebra on $H(C)$, then by taking $K = \hat{B}(H(C), \{X_i\})$ and $g = f : \hat{B}(H(C), \{X_i\}) \to B(C)$, we obtain a

$g^* : \hat{B}(H(C), \{X_i\}) \to \hat{B}(H(C), \{X_i\})$

for which $fg^* \sim g$. Then the first component of the $A(\infty)$-algebra morphism $\{g^*_i : (H(C), \{X_i\}) \to (H(C), \{X_i\})$ induced by $g^*$ is $g^*_1 = id_{H(C)}$, and this implies that $\{g^*_i\}$ is an isomorphism in $A(\infty)$.

The next result follows from Theorems 1 and 2, and 8.

Corollary 2 $K \otimes_\phi D$ and $K \otimes_{\phi^*} H(D)$ have isomorphic homology under the conditions of Theorems 1, 2, and 8.

The results obtained have the following applications.

The first proposition is obtained from Corollary 1 by taking $C = \hat{C}_s(G)$, where $G$ is a connected topological group such that the $H_i(G)$ are free, bearing in mind that $H(B(C)) = H_s(BG)$. 

Proposition 1 The homology of the $\tilde{\mathcal{B}}$-construction $\tilde{\mathcal{B}}(\bar{H}^*(G),\{X_i\})$ are isomorphic to that of classifying space $B_G$.

The next proposition is obtained from Corollary 1 by taking $C = C^*(B,b_0)$, where $B$ is a simply connected space with free groups $H^i(B,b_0)$, and bearing in mind that $H(B(C)) = H^*(\Omega C)$.

Proposition 2 The homology of the $\tilde{\mathcal{B}}$-construction $\tilde{\mathcal{B}}(\bar{H}^*(B,b_0),\{X_i\})$ are isomorphic to the cohomology of the loop space $\Omega B$.

Let $\xi = (X,p,B,G)$ be a principal $G$-fibration with paracompact base and connected $G$, let $F$ be a $G$-space, and $\xi[F] = (E,p,B,F,G)$ the associated fiber bundle, with the $H_i(G)$ and $H(i)(F)$ free. The final proposition is obtained from Corollary 2 by taking $C = C_*(G)$, $D = C_*(F)$, and $\phi$ a twisting cochain of the fibration $\xi$ (3).

Proposition 3 The homology of the $\sim$-twisted tensor product $C_*(B) \otimes_{\phi^*} H_*(F)$ is isomorphic to that of $E$.

This proposition generalizes a result of Shih [7]: if $G$ is $(n-1)$-connected, then the components $\phi^*_i \in C^i(B,H_{i-1}(G))$ vanish for $0 < i < n+1$, therefore, the differentials $d^i$ of the spectral sequence of $\xi[F]$ are trivial for $1 < i < n+1$, and the components are cocycles for $n < i < 2n+1$, consequently, $d^i$ can be expressed for $n < i < 2n+1$ in terms of certain characteristic classes of $\xi$ and the operation $Y_2 : H_*(G) \otimes H_*(F) \rightarrow H_*(F)$; we remark that higher operations $Y_j$ are needed for computing $d^i$, $j > 2n$, in terms of $\phi^*$.

Theorems 1, 2 and Proposition 3 were announced in [8] and Proposition 2 in [9].

References

[1] J.D. Stasheff, Homotopy associativity of H-spaces, I, II, Trans. Amer. Math. Soc. 108 (1963), 27-313. MR 28 1623.
[2] H. Munkholm, The Eilenberg-Moore spectral sequence and strongly homotopy multiplicative maps, J. Pure and Appl. Algebra 5 (1974), 1-50. MR 50 3227.
[3] E. H. Brown, Twisted tensor products, I, Ann. Of Math. (2) 69 (1959), 223-246. MR 21 4423.
[4] N. A. Berikashvili, The differentials of a spectral sequence, Bull. Of Georg. Acad. Sci., 51 (1968), 9-14. MR 41 9258.
[5] N. A. Berikashvili, Homology theory of spaces, Bull. Of Georg. Acad. Sci., 86 (1977), 529-532. MR 57 13949.
[6] V. A. Smirnov, The functor $D$ for twisted tensor products, Mat. Zametki 20 (1976), 465-472. MR 55 4172.
[7] W. Shih, Homologie des espaces fibres, Inst. Hautes Etudes Sci. Publ. Math. 1962, no. 13, 88. MR 26 1893.

[8] T. Kadeishvili, On the differentials of spectral sequence of a fiber bundle, Bull. Of Georg. Acad. Sci., 82 (1976), 285-288. MR 55 6430.

[9] T. Kadeishvili, On the homology of classifying spaces, Proc. 7th All Union Topology Conference, Minsk 1977.