A new class of non-Hermitian Hamiltonians with real spectra

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Abstract

We construct a new class of non-Hermitian Hamiltonians with real spectra. The Hamiltonians possess one explicitly known eigenfunction.

1 Introduction

By now, non-Hermitian Hamiltonians attract a lot of attention. Such Hamiltonians are used in optics [1, 2], in field theory [3] and other branches of theoretical physics.

Among the non-Hermitian Hamiltonians much attention was devoted to investigation of properties of the so-called PT symmetric Hamiltonians [4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. A Hamiltonian is called to be PT symmetric if $P^{-1}HP = HP$, where $P$ is the parity operator, i.e. $Pf(x) = f(-x)$, and $T$ is the complex conjugation operator. The main reason for this interest was an assumption that their spectra were entirely real as long as the PT symmetry was not spontaneously broken.

There are several ways to build a non-Hermitian Hamiltonian with real spectrum. For this purpose it was proposed to use supersymmetric quantum mechanics [14].
Recently, Mostafazadeh generalized $PT$ symmetry by pseudo-Hermiticity [13]. The idea of pseudo-Hermiticity was introduced by Dirac [16]. A Hamiltonian $H$ is said to be $\eta$-pseudo-Hermitian if

$$H^+ = \eta H \eta^{-1},$$

where $^+$ denotes the operation of adjoint. In [15] it was proposed a new class of non-Hermitian Hamiltonians with real spectra which are obtained using pseudo-supersymmetry.

Mostafazadeh also showed [17] that the operator $H$ with complete set of biorthonormal eigenvectors has a real spectrum if and only if there exists a linear invertible operator $O$ such that $H$ is $\eta$-pseudo-Hermitian, where $\eta = O^+ O$.

In the paper we construct a new class of pseudo-Hermitian operators with real spectra using $O$ as a first order differential operator.

## 2 Pseudo-Hermiticity

Suppose that non-Hermitian Hamiltonian $H$ is $\eta$-pseudo-Hermitian:

$$\eta H = H^+ \eta.$$

Here, we choose another form of pseudo-Hermiticity to avoid a necessity of $\eta$ invertibility (the form (2) is mentioned in [15]).

Choose an operator $\eta$ to be an Hermitian operator. Then $\eta H$ is an Hermitian operator, too: $(\eta H)^+ = H^+ \eta^+ = H^+ \eta = \eta H$. Consider an eigenfunction $\psi$ and the corresponding eigenvalue $E$ of $H$. Then, because of Hermiticity of $\eta H$ as well as of $\eta$,

$$\int \psi^* \eta H \psi dx = E \int \psi^* \eta \psi dx,$$

both integrals are real and except for the case

$$\int \psi^* \eta \psi dx = 0$$

the eigenvalue $E$ is also real. On the contrary, if $\int \psi^* \eta \psi dx = 0$ then the left integral of (4) has to be zero, too. In this case $E$ can be either a real or a complex number.
For a general form of $\eta$ it is difficult to find if there exist such eigenfunctions which satisfy (4). To simplify the study of the case of $\int \psi^*\eta\psi dx = 0$ we concretize the form of $\eta$ to be

$$\eta = O^+ O.$$  \hfill (5)

For this case the integral $\int \psi^*O^+O\psi dx = \int |O\psi|^2 dx$ is greater than zero except for the case of $\psi$ belonging to the kernel of $O$. So we have to solve

$$O\phi = 0$$  \hfill (6)

and verify if solutions of this equation are the eigenfunctions of $H$.

In the following section we build such a pair of Shrödinger Hamiltonian

$$H = -\frac{d^2}{dx^2} + V(x)$$  \hfill (7)

and $O^+O$ that satisfies condition (2).

3 \hspace{1em} \textit{O as the first order differential operator}

Choose $O$ in the following form

$$O = \frac{d}{dx} + f(x) + ig(x),$$  \hfill (8)

where $f$, $g$ are regular, real-valued functions. Then

$$O^+ = -\frac{d}{dx} + f(x) - ig(x).$$  \hfill (9)

Substituting (7), (8) and (9) into (2) and collecting terms with $\frac{d^2}{dx^2}$ operator we obtain

$$\text{Im} V = -2g'.$$  \hfill (10)

The terms without differential operators lead to $4g'(f'+f^2) + 2g(f'+f^2)' = g'''$. Multiplying this equation by $g$ and integrating it we obtain

$$f^2 - f' = \frac{2gg'' - g'^2 + \alpha}{4g^2}.$$  \hfill (11)
where $\alpha$ is a real constant of integration.

The terms with $\frac{d^2}{dx^2}$ give $2\text{Re}V' = 2(f^2 - f' - g^2)'. $ Integrating it and substituting (11) we can rewrite the real part of potential as

$$\text{Re}V = f^2 - f' - g^2 + \beta = \frac{2gg'' - g^2 + \alpha}{4g^2} - g^2 + \beta, \tag{12}$$

where $\beta$ is a real constant of integration. In equations (10-12) $g$ plays a role of generating function. In order to obtain a $PT$ symmetric Hamiltonian the generating function $g$ must be an even function, i.e. $g(x) = g(-x)$.

It should be noted that the choice (8) leads to

$$\eta = -\frac{d^2}{dx^2} - 2ig\frac{d}{dx} + f^2 - f' + g^2 - ig' \text{ and } \eta \text{ plays a role of a second order Darboux operator. It intertwines } H \text{ and } H^+ \text{ which are superpartners of the second order supersymmetry [18].}

Formulae (10-12) are similar to the corresponding results of [19].

The next step is to check whether the solution of (6) is an eigenfunction of $H$. In terms of $f$ and $g$ we can express this solution as:

$$\phi = e^{-\int (f + ig)dx}. \tag{13}$$

Considering $\phi$ as an eigenfunction of (7) and using (10), (12) we obtain

$$-i(g' + 2fg) + \beta = E, \tag{14}$$

where $E = E_r + iE_i$ is the complex eigenvalue of $H$ ($H\phi = E\phi$). We see that $\beta = E_r$. Then

$$f = -\frac{E_i + g'}{2g}. \tag{14}$$

Now, from (14) and (11) we have two different relations between $f$ and $g$. To compare them we substitute $f$ from (14) into (11) and after some simplification we obtain $E_i^2 = \alpha$. So we can state that $\phi$ can be an eigenfunction of (11) only if $\alpha \geq 0$. Note that (14) for the case $E_i^2 = \alpha$ is the solution of (11).

So choosing any $g$ and $\alpha < 0$ we can be sure that the spectrum of the corresponding Hamiltonian is entirely real, but we are not sure that it is not empty. By choosing for $\alpha = 0$ a suitable $g$ one can construct the Hamiltonian with real spectrum and that also possesses one explicitly known eigenfunction. Choosing $g$ and $\alpha > 0$ we have to check if the corresponding $\phi$ does not belong to $L_2$ space to obtain an Hamiltonian with real spectrum.

In the following section we illustrate these results.
4 Examples

For constructing Hamiltonians we use formulae (10) and (12) to represent imaginary and real part of the potential as well as (14) to express $f$. There are two ways to obtain regular expression for $f$. The first is to choose $g$ without sign changing and any value of $E_i$ or $\alpha$. It is illustrated with example 1. The second way is to choose $g$ as a function with a simple zero. In this case we have to fix value of the $E_i$ to avoid singularity. This way is illustrated with examples 2 and 3.

**Example 1**

Choosing the generating function $g$ as the even function

$$g = e^{-x^2}$$

we obtain $PT$ symmetric Hamiltonian

$$H = -\frac{d^2}{dx^2} + x^2 + \frac{\alpha}{4} e^{2x^2} - e^{-2x^2} - 4ixe^{-x^2} + \beta - 1$$  \hspace{1cm} (15)

which possesses real spectrum for $\alpha < 0$, for $\alpha = 0$ we know one eigenfunction $\psi_{E=0} = \exp(-\frac{x^2}{2} - i \int e^{-x^2} dx)$ and for $\alpha = E_i^2 > 0$ the eigenfunction $\psi_{E=\beta+\delta E_i} = \exp(-\frac{x^2}{2} + \frac{E_i}{2} \int e^{x^2} dx - i \int e^{-x^2} dx)$ does not belong to $L_2$ space. So we can state that spectrum of (15) is entirely real for any value of $\alpha$.

**Example 2**

Choose the generating function $g$ in the form

$$g = \sinh(x),$$

then, to obtain regular $f = -\frac{E_i + \cosh(x)}{2\sinh(x)}$ one must set $E_i = -1$ and then $f = -\frac{1}{2} \tanh \frac{1}{2} x$. Then $\phi = \cosh(\frac{1}{2} x) e^{-i \cosh(x)}$ does not belong to $L_2$. So spectrum of

$$H = -\frac{d^2}{dx^2} - 2i\cosh(x) - \sinh^2(x)$$

is real.

**Example 3**

Choose the generating function $g$ in the form

$$g = \tanh(x),$$
then, avoiding singularity, we set $E_i = -1$ and obtain

$$H = -\frac{d^2}{dx^2} - \frac{2i - \frac{1}{4}}{\cosh^2(x)} + \beta - \frac{3}{4}$$

(16)

with eigenfunction $\psi_{E=-i} = \frac{1}{\sqrt{\cosh(x)}} e^{-i \ln(\cosh(x))}$. The spectrum of (16) can be found using supersymmetric methods and it easy to show that this eigenvalue is unique.

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