Conservative Force Fields in Non-Gaussian Statistics

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Abstract

In this letter, we determine the $\kappa$-distribution function for a gas in the presence of an external field of force described by a potential $U(r)$. In the case of a dilute gas, we show that the $\kappa$-power law distribution including the potential energy factor term can rigorously be deduced in the framework of kinetic theory with basis on the Vlasov equation. Such a result is significant as a preliminary to the discussion on the role of long range interactions in the Kaniadakis thermostatistics and the underlying kinetic theory.

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I. INTRODUCTION

Over the last two decades, a great deal of attention has been paid to the so-called power-law statistics, both from theoretical and observational viewpoints. The main motivation is the lack of a comprehensive and rigorous treatment including gravitational and Coulombian fields, or more generally, any long range interaction for which the assumed additivity of the entropy present in the standard approach is not valid (see, e.g., [1, 2, 3, 4, 5, 6, 7] and references therein).

In this context, the Tsallis [1, 2, 3, 4, 5, 6, 7] and Kaniadaki [8, 9, 10, 11, 12, 13, 14, 15, 16] power-law statistics are the most investigated frameworks. In the former case, the standard Boltzmann-Gibbs formalism is extended through a new analytic form for the entropy $S_q = k_B(1 - \sum_i p_i^q)/(q - 1)$, where $k_B$ is the standard Boltzmann constant, $p_i$ is the probability of the $i$th microstate, and $q$ is a parameter quantifying the degree of nonextensivity. This expression has been introduced in order to extend the applicability of statistical mechanics to system with long range interactions and has the standard Gibbs-Jaynes-Shannon entropy as a particular limiting case ($q = 1$). Even the so-called $q$-nonextensive kinetic theory [4] has also been developed and applied for many different contexts ranging from plasmas [5] to gravitational systems [6].

On the other hand, recent studies on the kinetic foundations of the $\kappa$-statistics also leads to a power-law distribution function and a $\kappa$-entropy which emerges naturally in the framework of the kinetic interaction principle (see, for instance, Ref. [8, 9, 10]). Several physical features of the so-called $\kappa$-distribution have also been theoretically investigated, among them: i) the self-consistent relativistic statistical theory [10], ii) the framework of nonlinear kinetics [11] and iii) the H-theorem and generalization of the chaos molecular hypothesis [12]. Actually, this $\kappa$-framework leads to a class of one parameter deformed structures with interesting mathematical properties [13]. In particular, the canonical quantization of a classical system [14], and the so-called Lesche stability have also been discussed in the $\kappa$-framework [15]. Still more important, a consistent form for the entropy (linked with a two-parameter deformations of logarithm function), which generalizes the Tsallis, Abe and Kaniadakis logarithm behaviours [16] have also been found.

In the experimental front, there also exist some evidence closely related to the behavior predicted by the $\kappa$-statistics, namely, cosmic rays flux, rain events in meteorology [13], quark-
gluon plasma \[17\], kinetic models describing a gas of interacting atoms and photons \[18\], fracture propagation phenomena \[19\], and income distribution \[20\], and even the possibility of improved financial models has also been investigated \[21\].

Mathematically, the \(\kappa\)-framework is based on \(\kappa\)-exponential and \(\kappa\)-logarithm functions which are defined by

\[
\exp_{\kappa}(f) = (\sqrt{1 + \kappa^2 f^2 + \kappa f})^{1/\kappa}, \tag{1}
\]

\[
\ln_{\kappa}(f) = (f^\kappa - f^{-\kappa})/2\kappa, \tag{2}
\]

\[
\ln_{\kappa}(\exp_{\kappa}(f)) = \exp_{\kappa}(\ln_{\kappa}(f)) \equiv f. \tag{3}
\]

The \(\kappa\)-entropy associated with this \(\kappa\)-framework is given by

\[
S_{\kappa}(f) = -\int d^3 p f \left[ \frac{f^\kappa - f^{-\kappa}}{2\kappa} \right], \tag{4}
\]

which recovers standard Boltzmann-Gibbs entropy, \(S_{\kappa=0}(f) = -\int f \ln f d^3 p\), in the limit \(\kappa \rightarrow 0\) (see Ref. \[8, 9\] for details).

The so-called Kaniadakis entropy reads \[8, 25\]

\[
S_{\kappa} = -\int d^3 p f \ln_{\kappa} f = -(\ln_{\kappa}(f)), \tag{5}
\]

while the equilibrium velocity distribution can be written as \[8, 9, 10, 12\]

\[
f_0(v) = A_{\kappa} \left[ \sqrt{1 + \kappa^2 \left( -\frac{mv^2}{2k_B T} \right)^2} + \kappa \left( -\frac{mv^2}{2k_B T} \right) \right]^{\frac{1}{\kappa}}. \tag{6}
\]

In this expression \(k_B\) is the Boltzmann constant while the \(\kappa\) index denotes a continuous parameter associated to the gas entropy, and whose main effect at the level of the distribution function is to replace the standard Gaussian form by a power law. The quantity \(A_{\kappa}\) is a normalization constant fixed by the total number of particles in a given volume. As it should be expected, the above expression reduces to the standard Maxwellian distribution in the limit \(\kappa \rightarrow 0\) for which \(A_0 = (m/2\pi k_B T)^{3/2}\).

In this letter we explore how the potential energy term can rigorously be introduced in the \(\kappa\)-distribution \[22\]. More precisely, we deduce an analytical expression for the equilibrium distribution of a dilute gas under the action of a conservative force field with basis on the stationary solution of the collisionless Boltzmann equation. As we shall see, this result is significant as a preliminary to the discussion of long range interactions according to Kaniadakis thermostatistics and the underlying kinetic theory.
II. VLASOV EQUATION AND THE BOLTZMANN FACTOR

In this section we discuss briefly the standard case, i.e., the kinetic description of a classical gas under stationary conditions and immersed in a conservative force field, \( F = -\nabla U(r) \). Typical examples are a gas in the earth’s gravitational field and ions in an external magnetic field \([23, 24]\). This kind of problem is important on their own because it permits to understand how the molecular motion is modified by force-fields different from those exerted by the containing vessel or even by the other particles of the gas. As widely known, its distribution function differs from the Maxwellian velocity statistics through an extra exponential factor involving the potential energy whose general form reads

\[
f(r, v) = n_0 \left( \frac{m}{2\pi k_B T} \right)^{3/2} \exp \left( -\frac{1}{2} \frac{mv^2 + U(r)}{k_B T} \right),
\]

(7)

where \( m \) is the mass of the particles, \( T \) is the temperature and \( n_0 \) is the particle number density in the absence of the external force field. In addition, since this distribution function is normalized, it is easy to see that the number density is given by

\[
n(r) = n_0 \exp \left[ -\frac{U(r)}{k_B T} \right],
\]

(8)

where the factor, \( \exp[-U(r)/k_B T] \), which is responsible for the inhomogeneity of \( f(r, v) \), is usually called the Boltzmann factor. Expression (7) follows naturally from an integration of the collisionless Boltzmann’s equation

\[
\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial r} + \frac{F}{m} \cdot \frac{\partial f}{\partial v} = 0,
\]

(9)

when stationary conditions are adopted along with the assumption that the total distribution can be factored

\[
f(r, v) = f_0(v)\chi(r),
\]

(10)

where \( f_0(v) \) represents the Maxwell equilibrium distribution function, and \( \chi(r) \) is a scalar function of \( r \). As one may show, after a simple normalization, the resulting expression for \( \chi(r) \) is exactly the Boltzmann factor for the potential energy of the external field, namely:

\[
\chi(r) = \exp \left[ -\frac{U(r)}{k_B T} \right],
\]

(11)

and combining this result with equation (10) the Boltzmann stationary distribution (7) is readily obtained.
Let us now consider a spatially inhomogeneous dilute gas supposed in nonequilibrium stationary state at temperature $T$. The gas is immersed in a conservative external field in such a way that $f(r, v)d^3vdr$ is the number of particles with velocity lying within a volume element $d^3v$ about $v$ and positions lying within a volume element $d^3r$ around $r$. In principle, this distribution function must be determined from the $\kappa$-type Boltzmann equation:

$$\mathbf{v}.\nabla_rf - \frac{1}{m}\nabla_r U.\nabla_v f = C_\kappa(f)$$

(12)

where $C_\kappa$ denotes the $\kappa$-collisional term. The left-hand-side of the above equation is just the total time derivative of the distribution function. Hence, the effects appearing from the $\kappa$-approach can be explicitly incorporated only through the collisional term. In particular, this means that the Vlasov equation, or the stationary Boltzmann equation takes the standard form (see Eq. (7))

$$\mathbf{v}.\nabla_rf - \frac{1}{m}\nabla_r U.\nabla_v f = 0.$$  

(13)

In order to introduce the $\kappa$-statistics effects we first recall that the factorizability condition as given by (10) is modified in this extended framework. This means that the standard starting assumption must be extended. In the spirit of the $\kappa$-formalism, a consistent $\kappa$-generalization of (10) is

$$f(r, v) = A_\kappa \exp_\kappa \left[ \ln_\kappa \left( \frac{f_0}{A_k} \right) + \ln_\kappa \chi(r) \right],$$

(14)

where $f_0$ denotes the $\kappa$-velocity distribution and the constant normalization has been introduced for mathematical convenience, and the functions $\exp_\kappa(f)$, $\ln_\kappa(f)$, were previously defined by Eqs. (1) and (2).

Note that in the limit $\kappa \rightarrow 0$ the above identity reproduces the usual properties of the exponential and logarithm functions. In addition, since $\exp_\kappa(\ln_\kappa f) = \ln_\kappa(\exp_\kappa(f)) = f$, the standard factored decomposition (10) is readily recovered in the limit $\kappa = 0$. In what follows, the properties of $\kappa$-exponential and $\kappa$-log differentiation

$$\frac{d\ln_\kappa f}{dx} = \left( f^{\kappa-1} + f^{-(\kappa+1)} \right) \frac{df}{dx},$$

(15)

$$\frac{d\exp_\kappa(f)}{dx} = \frac{\exp_\kappa(f)}{\sqrt{1 + \kappa^2 f^2}} \frac{df}{dx},$$

(16)
will also be extensively used. In particular, for the total \( \kappa \)-distribution (14), we obtain

\[
\nabla_r f(r, v) = \frac{\exp_\kappa[\ln_\kappa f_0(v) + \ln_\kappa \chi(r)]}{\exp_\kappa[\ln_\kappa f_0(v) + \ln_\kappa \chi(r)]} \nabla_r \ln_\kappa \chi(r) \times \left\{ 1 + \frac{\kappa \left( \ln_\kappa \chi(r) - \frac{mv^2}{2k_B T} \right)}{[1 + \kappa^2(\ln_\kappa f_0(v) + \ln_\kappa \chi(r))^2]^{1/2}} \right\} \tag{17}
\]

\[
\nabla_v f(r, v) = \frac{\exp_\kappa[\ln_\kappa f_0(v) + \ln_\kappa \chi(r)]}{\exp_\kappa[\ln_\kappa f_0(v) + \ln_\kappa \chi(r)]} \left( -\frac{mv}{k_B T} \right) \times \left\{ 1 + \frac{\kappa \left( \ln_\kappa \chi(r) - \frac{mv^2}{2k_B T} \right)}{[1 + \kappa^2(\ln_\kappa f_0(v) + \ln_\kappa \chi(r))^2]^{1/2}} \right\} \tag{18}
\]

Now, substituting \( \nabla_r f \) and \( \nabla_v f \) given above into the stationary Boltzmann equation (12), and performing the elementary calculations one obtains

\[
\nabla_r \ln \chi \cdot dr = -\frac{1}{k_B T} \nabla U(r) \cdot dr \tag{19}
\]

the solution of which is

\[
\chi(r) = \exp_\kappa \left( -\frac{U(r)}{k_B T} + C \right), \tag{20}
\]

where \( C \) is an arbitrary constant.

Here, inserting (20) into (14), and integrating the result in the velocity space it follows that

\[
\int A_k \exp_\kappa \left[ \ln_\kappa (zf_0) - \frac{U}{k_B T} + C \right] d^3v = n(r). \tag{21}
\]

Finally, by substituting the expression of \( f_0 \) and considering a region where \( U(r) = 0 \), one finds

\[
A_k \int \exp_\kappa \left( -\frac{mv^2}{2k_B T} + C \right) d^3v = n_0, \tag{22}
\]

and from normalization condition, \( n_0 = \int f_0(v) d^3v \), it follows that the unique allowed value for the integration constant is \( C = 0 \). Consequently, (20) becomes

\[
\chi(r) = \exp_\kappa \left[ -\frac{U(r)}{k_B T} \right], \tag{23}
\]

which is the \( \kappa \)-generalized Boltzmann factor.
FIG. 1: The $\kappa$-velocity distribution function $f_\kappa(r, v)$ for typical values of the $\kappa$ parameter on the interval $[-1,1]$ ($z = A_\kappa^{-1}$). In contrast to Tsallis statistics [1, 2], this power law distributions does not exhibit a thermal cutoff, that is, a maximum value allowed for the velocity of the particles. The $\kappa$-distribution also presents a striking mathematical property, namely, $f_\kappa(\eta) = f_{-\kappa}(\eta)$.

Finally, by inserting this result into (14), we obtain the complete $\kappa$-distribution function in the presence of an external field

$$f_\kappa(r, v) = A_\kappa \left[ \sqrt{1 + \kappa^2 \left( -\frac{mv^2}{2k_BT} - \frac{U(r)}{k_BT} \right)^2 + \kappa \left( -\frac{mv^2}{2k_BT} - \frac{U(r)}{k_BT} \right)} \right]^{1/\kappa}$$

$$\equiv A_\kappa \exp_\kappa(-E/k_BT), \quad (24)$$

where $E$ is the total energy of the particles. It thus follows that a generalized $\kappa$-exp factor for Kaniadakis’ thermostatistics can exactly be deduced if the standard approach is slightly modified. In Fig. 1 we show $zf(E)$ for some selected values of the $\kappa$-parameter (where $z = A_\kappa^{-1}$). Different from Tsallis power-law functions which can become finite for some
values of the q-parameter, the $\kappa$-distributions are not finite. In other words, the thermal cutoff on the velocity space is not present in such distributions regardless of the values of $\kappa$. It is worth emphasizing, however, that this $\kappa$-distribution presents the following mathematical behavior, viz., $f_\kappa(\eta) = f_{-\kappa}(\eta)$.

IV. CONCLUDING REMARKS

In the last few years, several applications of the $\kappa$-power law velocity distribution have been done in many disparate branches of physics [8-21]. However, such investigations are usually related with the $\kappa$-velocity distribution function as given by equation (6). On the other hand, many physical systems involve naturally the presence of a conservative force field as happens for example with ions in a magnetic field. Probably, the most popular problem of a gas in a force-field is the planetary atmosphere. In the standard simplified treatment, the temperature is uniform and the three-dimensional motion occurs under the action of a constant gravitational field along the $z$-direction. To all this sort of problems, the extended $\kappa$-distribution deduced here with basis on the Vlasov equation, namely

$$f_\kappa(r, v) = A_\kappa \left[ \sqrt{1 + \kappa^2 \left( \frac{m v^2}{2 k_B T} - \frac{U(r)}{k_B T} \right)^2} + \kappa \left( -\frac{m v^2}{2 k_B T} - \frac{U(r)}{k_B T} \right) \right]^{1/\kappa},$$

can be applied, and, might prove to be of extreme wide usefulness. Note also that a giroscopic term may also be added to the above power law distribution. In a rotating frame with constant angular velocity, the whole effect is just to add a Coriolis term $-1/2 m \omega^2 R^2$ to the potential energy, where $R$ is the perpendicular distance from the axis of rotation. In the Newtonian framework, such a term simulates a change in the potential energy due to gravity. Finally, it is worth mentioning that the present consistency between Vlasov equation and Kaniadakis thermostatistics also is valid in the context of Tsallis nonextensive framework [22].

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