AN UPPER BOUND ON DISTANCE DEGENERATE HANDLE
ADDITIONS

YANQING ZOU

Abstract. We prove that for any distance at least 3 Heegaard splitting and a boundary
component $F$, there is a diameter finite ball in the curve complex $C(F)$ so that it contains
distance degenerate curves or slopes in $F$.

Keywords: Heegaard Distance, Curve Complex, Handle Addition.

AMS Classification: 57M50

1. Introduction

Let $M$ be a compact orientable 3-manifold with a boundary component $F$. Then it
admits a Heegaard splitting $V \cup_S W$, where $F \subset V$. Here $S$ is $\partial_+ V$ (resp. $\partial_+ W$) and
$\partial_- V = \partial V - S$ (resp. $\partial_- W = \partial W - S$).

Let $r$ be a slope or an essential simple closed curve in $F$. Then a Dehn filling or a handle
addition along $r$ on $M$ produces a 3-manifold $M(r)$. Since $V \cup_S W$ is a Heegaard splitting
of $M$, $M(r)$ admits a Heegaard splitting $V(r) \cup_S W$, where $V(r)$ is obtained from attaching
a 2-handle along $r$ on $V$ and capping a possible 2-sphere by a 3-ball.

There is a long story on studying handle additions or Dehn fillings on a 3-manifold.
Lickorish [13] proved that every closed orientable 3-manifold is a Dehn surgery along some
link or knot in $S^3$. For any knot $K$ in $S^3$, Gordon and Luecke [6] proved that only the trivial
Dehn surgery produces $S^3$. In general, Culler, Gordon, Luecke and Shalen introduced a
cyclic surgery theorem, see [2]. One of its corollaries is that only integer surgery on a non
torus knot can produce a cyclic fundamental group.

Given a knot $K$ in $S^3$, it is either prime or a connected sum of some prime knots. Let
$\eta(K)$ be the regular neighborhood of $K$ in $S^3$. For any knot $K$ in $S^3$, Gordon and Luecke [6] proved that only the trivial
Dehn surgery produces $S^3$. In general, Culler, Gordon, Luecke and Shalen introduced a
cyclic surgery theorem, see [2]. One of its corollaries is that only integer surgery on a non
torus knot can produce a cyclic fundamental group.

Given a knot $K$ in $S^3$, it is either prime or a connected sum of some prime knots. Let
$\eta(K)$ be the regular neighborhood of $K$ in $S^3$. If $K$ is prime, then it is either hyperbolic, i.e., $E(K) = S^3 \setminus \eta(K)$ admits a complete hyperbolic metric, or a torus knot or a satellite
knot. But if $K$ is a connected sum of some knots, then there is a properly embedded
essential annulus in $E(K)$. In this case, by Thurston’s Haken hyperbolic theorem, $E(K)$
adopts no hyperbolic structure. Thus with respect to the geometry of $E(K)$, hyperbolic
knots are mostly concerned. For a hyperbolic knot $K$, Thurston [40] proved that all but
finitely many Dehn fillings on $E(K)$ produce hyperbolic 3-manifolds. It was conjectured

Date: April 14th, 2017.
This work was partially supported by NSFC No. 11601065.
by Gordon \cite{4} that (1) there are at most 10 non hyperbolic Dehn fillings (there are 10 non hyperbolic Dehn fillings for the figure eight knot); (2) the intersection number of any two non hyperbolic slopes is at most 8. Recently, Agol \cite{1} proved that for all but finitely many one cusped hyperbolic 3-manifolds, the intersection number is 5 while there are at most 8 non hyperbolic Dehn fillings. Later Lackenby and Meyerhoff \cite{14} proved this conjecture completely. Moreover, by Thurston’s Geometrization conjecture \cite{40} proved by Perelman \cite{29,30,31}, except a small Seifert fiber space, every closed orientable non hyperbolic 3-manifold is either reducible or toroidal. So if we consider the reducible Dehn fillings on $E(K)$, i.e., the resulted 3-manifold is reducible, then the number 10 is reduced to 2, see \cite{5}. For more cases, see \cite{3}.

It is natural to extend these Dehn fillings results into a handle addition on a hyperbolic 3-manifold. Then we consider a hyperbolic 3-manifold $M$ with a totally geodesic boundary component $F$. Before stating some results about handle additions on $F$, we introduce a definition. An essential simple closed curve $r \subset F$ is called a non degenerate curve if $M(r)$ is also hyperbolic. Otherwise, it is degenerate. Then if $r$ is degenerate in $F$, by Thurston’s Haken hyperbolic theorem, $M(r)$ is reducible, or boundary reducible, or annular or toroidal. So to figure out all non degenerate curves in $F$, it is sufficient to give a classification of all degenerate curves from the topology of $M(r)$. Scharlemann and Wu \cite{37} studied all those degenerate curves on $F$ and proved that there are finitely many basic degenerate curves in $F$ so that either each degenerate curve is basic or it bounds a pair of pants with a basic degenerate curve. It means that for most of all essential simple closed curves, $M(r)$ is hyperbolic. Unfortunately there is no upper bound on their intersection numbers among all degenerate curves in $F$, for example, two complicated intersecting degenerate curves with respect to a same basic degenerate curve. However, if we consider two separating reducible handle addition curves, then their intersection number is at most 2, see \cite{32}. Meanwhile, Lackenby \cite{12} introduced a handlebody addition along $F$ and proved that there is an upper bound on all non hyperbolic handlebody additions.

It is known that every Heegaard surface of $M$ is also a Heegaard surface of $M(r)$. The properties of a Heegaard splitting of $M$ under a Dehn filling or a handle addition are concerned, such as the minimal genus, Heegaard distance. It is not hard to see that the minimal Heegaard genus of $M(r)$ is not larger than $M$’s. Then it is interesting to know that when they have the same minimal Heegaard genera. There are some results as follows: Rieck \cite{34} proved that for most of all $r$ in $F$, the minimal Heegaard genus of $M(r)$ is at most one less then $M$’s; Moriah and Sedgwick \cite{28} proved that for all but finitely many curves in $F$, $M(r)$ has the same genus as $M$; Li \cite{20} proved that if the gluing map of a handlebody addition is sufficiently complicated, then the resulted 3-manifold has the same minimal Heegaard genus as $M$.

Hempel \cite{9} introduced the Heegaard distance for studying a Heegaard splitting. More precisely, let $\{\alpha_0, ..., \alpha_n\}$ be a collection of essential simple closed curves in $S$ so that for any $1 \leq i \leq n$, $\alpha_i$ is disjoint from $\alpha_{i-1}$. Then for a Heegaard splitting $V \cup_S W$, the Heegaard distance $d(V, W)$ is the minimum of all $n$ so that $\alpha_0$ (resp. $\alpha_n$) bounds a disk in $V$ (resp. $W$). Since each essential disk in $V$ is also an essential disk of $V(r)$, $d(V(r), W) \leq d(V, W)$. So there is a question.
**Question 1.1.** Is \(d(V(r), W) = d(V, W)\)?

Unfortunately, for \(M = E(K)\), some high distance knot \(K\) (see Minsky, Moriah and Schleimer [24]), if \(r\) is the meridian, then \(M(r)\) is \(S^3\). By Waldhausen theorem [41], every genus at least 2 Heegaard splitting of \(S^3\) is stabilized and hence has distance 0. So the answer to Question 1.1 is no.

However, by those results of Hempel [9], Hartshorn [7] and Schlarlemann [35], if \(M\) admits a distance at least 3 Heegaard splitting, then it is irreducible, boundary irreducible, atoroidal and anannular. Then by Thurston’s Haken hyperbolic theorem, it is hyperbolic. Compared with Schlarlemann and Wu’s hyperbolic handle addition theorem, it was conjectured by Ma and Qiu [25] that \(d(V(r), W) = d(V, W)\) for most of all curves.

To quote such an exceptional curve as the meridian of a knot, it is proper to introduce the definition of a distance degenerate curve. We say \(r\) is a distance degenerate curve in \(F\) if \(d(V(r), W) < d(V, W)\). Furthermore, attaching 2-handle to a 3-manifold along a distance degenerate curve is called a distance degenerate handle addition. By standard techniques, there is a theorem as follows.

**Theorem 1.1.** If the Heegaard distance of \(V \cup S W\) is at least 3, then there are an essential simple closed curve \(c \subset F\) and a real number \(R > 0\) so that for any distance degenerate curve \(r\) in \(F\), \(d_{C(F)}(c, r) < R\).

**Note 1.2.** There is a precise description of \(R\) in Page 22, Section 4.

It is not hard to see that for a distance at least 3 Heegaard splitting, every degenerate curves in Schlarlemann and Wu’s hyperbolic addition theorem is also a distance degenerate curve. So Theorem 1.1 gives a bound for all those degenerate curves in the curve complex.

**Remark 1.1.** If \(M\) is \(T^2 \times I\), then it admits an unique strongly irreducible Heegaard splitting. It is known that for any slope \(r \subset \partial M\), \(M(r)\) is a solid torus. So every Heegaard splitting of \(M(r)\) is weakly reducible and hence has distance at most 1. It means that for any slope \(r \subset \partial M\), \(r\) is a distance degenerate slope with respect to this strongly irreducible Heegaard splitting.

**Remark 1.2.** Lustig and Moriah [14] proved that there is a measure defined on the curve complex \(C(F)\) so that for any \(R > 0\) and any essential simple closed curve \(c\), the measure of a \(R\)-ball of \(c\) is 0. Under this circumstance, for almost all choices of \(r\) in \(F\), \(d(V(r), W) = d(V, W)\).

**Remark 1.3.** Let \(c_1\) and \(c_2\) be two separating essential simple closed curves in \(S\). Suppose that \(d_{C(S)}(c_1, c_2) = l \geq 3\). Then attaching two 2-handles along \(c_1\) and \(c_2\) from two different sides of \(S\) produces a Heegaard splitting, denoted by \(V \cup S W\). Since \(V\) (resp. \(W\)) has only one essential disk up to isotopy, the distance of \(V \cup S W\) is equal to \(l\). Then by Theorem 1.1, we can attach 2-handles to its boundary and some 3-balls so that \(V\) (resp \(W\)) is changed into a handlebody \(H_1\) (resp. \(H_2\)) and furthermore \(d(H_1, H_2) = l\), see also in [10, 33, 32].

**Remark 1.4.** If \(V \cup S W\) is genus two Heegaard splitting, Ma, Qiu and Zou [26] proved the main theorem by a different method. Meanwhile, there is a result proved by Liang, Lei...
and Li [15], which says that if the Heegaard splitting is locally complicated, then there is a bound for all distance degenerate curves in $C(F)$.

Remark 1.5. If $d(V, W) \geq 2g(S)$, then by Scharlemann and Tomova’s result [36], $V \cup W$ is a minimal Heegaard splitting. Then by the proof of Theorem 1.1 we can attaching a handlebody $H$ along distance non degenerate slopes or curves in $F$ so that $V(H) \cup S W$ is still a minimal Heegaard splitting. So it gives a description of Li’s sufficiently complicated gluing map between a handlebody and $M$ in [20].

We call a knot $K$ in $S^3$ a high distance knot if $E(K)$ admits a distance at least 3 Heegaard splitting. It is known that for any knot $K \subset S^3$ and any distance at least 3 Heegaard splitting of $E(K)$, the meridian is a distance degenerate slope. For $M(r)$ is $S^3$ and by Waldhausen theorem [41], every genus at least 2 Heegaard splitting is stabilized and thus has distance 0. Then we choose the meridian as the center among all distance degenerate slopes of $E(K)$’s all distance at least 3 Heegaard splittings. So Theorem 1.1 is updated into the following corollary.

**Corollary 1.2.** For any high distance knot $K \subset S^3$, there is a $R_K$-ball of the meridian in $C[\partial E(K)]$ so that it contains all distance degenerate slopes of $E(K)$’s all distance at least 3 Heegaard splittings.

This paper is organized as follows. We introduce some results of a compression body in Section 2 and some lemmas of the curve complex in Section 3. Then we give proofs of Theorem 1.1 and Corollary 1.2 in Section 4.

**Acknowledgement.** We would like to thank Jiming Ma and Ruifeng Qiu for pointing out Question 1.1, thank Ruifeng Qiu for many discussions and pointing out some mistakes in our early draft.

2. **Subsurface projection of the disk complex**

Let $S$ be a closed orientable genus at least 2 surface. Harvey [8] introduced the curve complex on $S$, denoted by $C(S)$, as follows. The vertices consist of all isotopy classes of essential, i.e., incompressible and non peripheral, simple closed curves in $S$. A $k$-simplex is a collection of $k + 1$ vertices which are presented by pairwise non isotopy and disjoint essential simple closed curves.

Let $F$ be a compact orientable surface. If $F$ is an at most once punctured torus, then $C(F)$ is defined as follows. The vertices consist of all isotopy classes of essential, i.e., incompressible and non peripheral, simple closed curves in $F$. A $k$-simplex is the collection of $k + 1$ vertices which are presented by pairwise non isotopy and intersecting one point essential simple closed curves. If $F$ is a fourth punctured 2-sphere, then the definition of $C(F)$ is slightly different, which is defined as follows. The vertices consist of all isotopy classes of essential, i.e., incompressible and non peripheral, simple closed curves in $F$. A $k$-simplex is the collection of $k + 1$ vertices which are presented by pairwise non isotopy and intersecting twice essential simple closed curves. In general, if $\chi(F) \leq -2$, the definition of $C(F)$ is similar to $C(S)$. 
It is assumed that the length of an edge in $C(F)$ is 1. Then for any two vertices $\alpha$ and $\beta$, $d_{C(F)}(\alpha, \beta)$ is defined to be the minimum of lengths of paths from $\alpha$ to $\beta$ in $C(F)$. So if $\alpha$ is disjoint from but not isotopic to $\beta$, then there is an edge between them and so $d_{C(F)}(\alpha, \beta) = 1$. What if $\alpha$ intersects $\beta$?

**Lemma 2.1.** If $\alpha$ intersects $\beta$ in $N$ points up to isotopy, then $d_{C(F)}(\alpha, \beta) \leq 2 \log_2 2^N + 1$.

**Proof.** See the proof of Lemma 1.21 in [39]. □

Suppose $F \subset S$ is an essential subsurface, i.e., $\partial F$ is incompressible in $S$. Masur and Minsky [23] introduced the subsurface projection from $C(S)$ to $C(F)$ as follows. Let $\alpha$ be a vertex in $C(S)$, where $\alpha \cap F \neq \emptyset$ up to isotopy. Then either $\alpha$ is an essential simple closed curve in $F$ or $\alpha \cap \partial F \neq \emptyset$. In the former case, the subsurface projection of $\alpha$, denoted by $\pi_F(\alpha)$, is $\alpha$. In the latter case, $\alpha$ intersects $\partial F$ efficiently, i.e., there is no bigon bounded by them in $S$. Let $a$ be an arbitrary one arc of $\alpha \cap F$. Then $\pi_F(\alpha)$ is defined to be an arbitrary one essential simple closed curve of $\partial N(a \cup \partial F)$ in $F$. Under the definition of the subsurface projection, if two disjoint curves $\alpha$ and $\beta$ both cut $F$, i.e., neither $\pi_F(\alpha)$ nor $\pi_F(\beta)$ is an empty set, then $d_{C(F)}(\pi_F(\alpha), \pi_F(\beta)) \leq 2$.

If $\partial F$ is not connected, some essential simple closed curve of $F$ cutting out a planar surface while some one doesn’t. To distinguish these two kinds of essential simple closed curves in $F$, we introduce the definition of a strongly essential curve, see also in [43].

**Definition 2.1.** An essential simple closed curve $C \subset F$ is strongly essential if $C$ doesn’t cut out a planar surface in $F$.

Similarly, for a properly embedded essential arc $a \subset F$, $a$ is strongly essential if $\pi_F(a)$ is strongly essential in $F$. Otherwise, it is not strongly essential in $F$.

If $S = \partial_+ V$, then there is a disk complex defined on $S$, denoted by $D(S)$. The vertices consist of all isotopy classes of boundary curves of essential disk of $V$. A $k$-simplex is the collection of $k + 1$ vertices which are pairwise non isotopy and disjoint. It is not hard to see that $D(S)$ is a subcomplex of $C(S)$. Thus for an essential subsurface $F \subset S$, there is a subsurface projection from $D(S)$ to $C(F)$. Throughout the finer structure of $D(S)$, Li [18], Masur and Schleimer [27] proved that if $\partial F$ is disk-busting, i.e., it intersects the boundary of every essential disk nonempty, then there is a bound on the diameter of subsurface projection of the disk complex for almost all cases. More precisely, it is written as follows.

**Lemma 2.2.** Let $F$ be a connected subsurface of $S$ so that each component of $\partial F$ is disk-busting. Then

1. either $V$ is an I-bundle over a compact surface, $F$ is a component of the horizontal boundary of this I-bundle, and the vertical boundary of this I-bundle is a single annulus, or
2. $\pi_F(D(S))$ has diameter at most 12 in $C(F)$.

**Note 2.2.** In Lemma 2.2 if $V$ is a twisted I-bundle of $F$, then the vertical boundary of this I-bundle is non separating.
Let \( N = \min\{|\partial D \cap \partial F | \mid D : \text{an essential disk of } V\} \). Suppose \( D \) realizes the minimum \( N \). Then there is a more interesting result in Li’s proof of Lemma 2.2.

**Lemma 2.3.** Let \( V, S \) and \( F \) be the same ones in Lemma 2.2. If \( N > 4 \), then for any essential disk \( E \) of \( V \) with \( \partial E \subset S \), there is a component of \( \partial E \cap F \) disjoint from a component of \( \partial D \cap F \).

**Proof.** Suppose the conclusion is false. Then each component of \( \partial D \cap F \) intersects every component of \( \partial E \cap F \) nontrivially. In Li’s proof of Lemma 3.4 [18], for \( D \) and \( E \), it is assumed that there is no cycle in their intersection. Then there is an outermost disk \( \Delta \) in \( E \) bounded by an arc \( \delta \subset D \cap E \) and an arc \( \delta' \subset \partial E \). Moreover, for \( \Delta \), there are only two types: a triangle or a quadrilateral. Then Li [18] proved that for both of these two cases, there is a new disk \( D_1 \) so that \( |\partial D_1 \cap \partial F| < |\partial D \cap \partial F| \). But it contradicts the choice of \( D \). \( \Box \)

3. The SE-position of an essential disk

Let \( V \) be a nontrivial compression body with \( F \subset \partial_- V \).

If \( \partial_- V = F \), then there are finitely many disjoint and parwise non isotopy essential disks \( \cup_{i=1}^{s} B_i \subset V \) satisfying \( F \)-condition, i.e., their complement in \( V \) is \( F \times I \). So there is a subsurface \( S_F = \overline{S - \cup_{i=1}^{s} B_i} \) in \( F \times I \), see Figure 3.1.

![Figure 3.1](image)

Let \( r \) be an essential simple closed curve in \( F \). Attaching a 2-handle along \( r \) on \( V \) (capping a possible 2-sphere by a 3-ball) produces a new compression body or handlebody, denoted by \( V(r) \). It is not hard to see that there are at least one more essential disks in \( V(r) \) than \( V \), for example, an essential disk \( D \) containing \( r \). Since each \( B_i \) is also an essential disk in \( V(r) \), it is interesting to know how they intersect.

It is assumed that \( D \) and \( \cup_{i=1}^{s} B_i \) are in a general position. Then they intersect in some arcs or cycles. It is known that both a compression body and a handlebody are irreducible. Then there is no cycle in their intersection up to isotopy. So \( D \cap \cup_{i=1}^{s} B_i \) consists of finitely many arcs. If \( D \cap \cup_{i=1}^{s} B_i = \emptyset \), then \( \partial D \) is strongly essential in \( S_F \). For if not, then \( \partial D \) cuts out a planar surface in \( S_F \). So \( \partial D \) is a band sum of some components of \( \partial S_F \).

Since \( \partial S_F \) consists of some essential disks’ boundary curve in \( V \), \( D \) is a band sum of some essential disks in \( V \). Therefore \( D \) is an essential disk in \( V \). It contradicts the fact that \( D \) is not in \( V \). If \( D \cap \cup_{i=1}^{s} B_i \neq \emptyset \), then there is an outermost disk in \( D \) so that it is bounded by an arc \( \gamma \subset \partial D \) and an arc of \( D \cap B_i \) for some \( 1 \leq i \leq s \). In this case, \( \gamma \) is
either strongly essential in $S_F$ or not. If $\gamma$ is not strongly essential in $S_F$, then there is a boundary compression on $B_i$ along this outermost disk so that it produces two essential disks $B^1_i$ and $B^2_i$. By a standard argument, there is at least one of them, says $B^1_i$, so that \{ $B_1, ..., B_{i-1}, B^1_i, B_{i+1}, ..., B_s$ \} satisfies the $F$-condition. Then we consider the intersection between $D$ and \{ $B_1, ..., B_{i-1}, B^1_i, B_{i+1}, ..., B_s$ \}, which has a fewer number of intersection arcs. If neither $D$ is disjoint from all these disks nor there is a strongly essential outermost disk in $D$, then there is a boundary compression on these new disks again. Since there are only finitely many arcs between $D$ and $\bigcup^s_{i=1} B_i$, finally either $D$ is strongly essential and disjoint from all these disks or there is a strongly essential outermost disk in $D$. Then we say $D$ is in SE-position with respect to $\bigcup^s_{i=1} B_i$. In general, for an essential disk $D$ in $V(r)$ but not in $V$, there is also a SE-position for it in $V(r)$ as follows.

Let $A$ be a collection of all non separating spanning annuli in $V$, where none of their boundary curves lies in $F$. Let $B$ be the collection of all non separating essential disks of $V$. Then there is an annu-disc system of $V$ in $A \cup B$, says \{ $A_1, A_2, ..., A_l, B_1, ..., B_s$ \} so that (1) they are pairwise disjoint; (2) their complement in $V$ is connected; (3) the complement of their boundary curves in $S$, denoted by $S_{l,s}$, has genus $g(F)$, $2[\gamma(S) - g(F)]$ boundaries. In this case, $l + s = g(S) - g(F)$.

For any annu-disc system \{ $A_1, A_2, ..., A_l, B_1, ..., B_s$ \} of $V$, either $D$ is disjoint from them or they intersect nontrivially. Since $V$ is irreducible, it is assumed that there is no cycle in their intersection. In the later case, their intersection consists of finitely many arcs. Then there is an outermost disk in $D$, which is bounded by some arc $\gamma \subset \partial D$ and some arc $a_1$ in their intersection. We call an annu-disc system \{ $A_1, A_2, ..., A_l, B_1, ..., B_s$ \} tamed for $D$ if there is a component $\gamma \subset \partial D \cap S_{l,s}$ so that (1) $\gamma$ lies in an outermost disk in $D$; (2) it is strongly essential in $S_{l,s}$. Otherwise, it is untamed.

If a given annu-disc system is untamed for $D$, then there is some outermost disk and an arc $\gamma \subset \partial D \cap S_{l,s}$ so that $\pi_{S_{l,s}}(\gamma)$ bounds a disk in $V$. Then doing a boundary compression along this outermost disk on $A_i$ (resp. $B_j$) produces a new non separating spanning annulus $A^1_i$ (resp. $B^1_j$). It is known that $A^1_i$ shares the same boundary curve in $\partial V$ with $A_j$. Then there is a new annu-disc system \{ $A_1, ..., A^1_i, ..., A_l, B_1, ..., B_s$ \} (resp. \{ $A_1, ..., A_i, B_1, ..., B^1_j, ..., B_s$ \}). It is not hard to see that the intersection number between $D$ and the new annu-disc system is less than before. So we cyclically do this operation until this annu-disc system is transformed into a tamed annu-disc system.

In all, for the essential disk $D$, there is a tamed annu-disc system for it. To find a tamed annu-disc system for $D$ in $V$, there are some surgeries introduced on this tamed annu-disc system \{ $A_1, A_2, ..., A_l, B_1, ..., B_s$ \}.

For two spanning annuli $A_1$ and $A_2$, which lie in the same component of $\partial V$, there is an arc $a_{1,2}$ in $\partial V$ connecting them, whose interior is disjoint from this tamed annu-disc system. So the I-bundle $a_{1,2} \times I$ connects $A_1$ and $A_2$ disjoint from this tamed annu-disc system. If $\gamma$ is disjoint from $a_{1,2} \times I$, then cutting the complement of this annu-disc system along it produces a 3-manifold $V_{a_{1,2}}$. So the subsurface $S_{l,s}$ is cut into an essential subsurface $S'_{l,s,a_{1,2}}$, see Figure 3.2.

It is not hard to see that $\gamma \cap S'_{l,s,a_{1,2}}$ is also strongly essential in $S_{l,s,a_{1,2}}$. 

Let $D(\gamma)$ be the disk bounded by $\gamma$ and some arc in this annu-disc system in $V(r)$. If $\gamma$ intersects $a_{1,2} \times I \cap S_{l,s}$ nontrivially, then $D(\gamma)$ intersects this I-bundle some arcs up to isotopy, where all these arcs have their ends in $a_{1,2} \times I \cap S_{l,s}$. Then there is an outermost disk in $D(\gamma)$ bounded by $\gamma_1$ and some arc in $a_{1,2} \times I$. If this outermost disk is also in $V$, then doing a boundary compression on $a_{1,2} \times I$ along it produces a I-bundle and a disk. Here we still use $a_{1,2} \times I$ representing this new I-bundle. It is said that this new I-bundle has less intersection number with $D(\gamma)$. Cyclically doing this operation until either $\gamma$ is disjoint from the resulted I-bundle or there is an outermost disk bounded by $\gamma_1$ and some arc in $D(\gamma)$ is in $V(r)$ but not in $V$. In all, the strongly essential arc is denoted by $\gamma_1$. In this case $\gamma_1$ is strongly essential and bounds an essential disk in $V(r)$ not in $V$ with some arc in $\partial S_{l,s,a_{1,2}}$. Let $A_{1,2}$ be the band sum of $A_1$ and $A_2$ along $a_{1,2} \times I$. Then it is also a spanning annulus in $V$. Furthermore, there is a collection of annuli and disks $\{A_{1,2}, A_3, \ldots A_l, B_1, \ldots, B_s\}$ so that $\gamma_1$ lies in the complement of it in $V$.

Cyclically doing the above operation until there is no spanning annulus in this tamed annu-disc system. Then at last it is transformed into a tamed disc system for $D$, i.e., a collection of essential disk in $V$. It is known that one component of their complement in $V$ is $F \times I$. Let $S_F \subset S$ be the component of their boundary’s complement in $S$, which lies in $F \times I$. Then there is an arc $\gamma^* \in \partial D \cap S_F$ so that it not only lies in an outermost disk in $D$ but also is strongly essential in $S_F$.

We summarize the above argument into a lemma as follows:

**Lemma 3.1.** For any essential disk $D$ in $V(r)$ but not in $V$, there are finitely many essential disks $\{B_1, \ldots, B_s\}$ of $V$ so that (1) one component of their complement in $V$ is $F \times I$; (2) the other components are some closed surfaces I-bundles if possible; (3) $D$ is in a SE-position with respect to $\bigcup_{i=1}^s B_i$, i.e., for some component $\gamma^* \subset \partial D \cap S_F$, $\pi_{S_F}(\gamma^*)$ not only is strongly essential in $S_F$ but also bounds an essential disk in $V(r)$.

4. AN UPPER BOUND ON DISTANCE DEGENERATE HANDLE ADDITIONS

Suppose $V \cup_S W$ has distance $m \geq 3$. An essential separating disk $B \subset V$ is called a $F$-disk if one component of $V - B$ is $F \times I$. Let

$$N = \min\{|B \cap E| \mid E: \text{an essential disk in } W; B: \text{a } F \text{-disk}\}.$$
Then there are an essential disk $E \subset W$ and a $F$-disk $B \subset V$ so that $\mathcal{N} = |B \cap E|$.

Since cutting $V$ along a $F$-disk $B$ produces a closed surface I-bundle $F \times I$, there is a component $S_1 \subset S - \partial B$ in $F \times I$. It has been discussed that $\partial E$ intersects $S_1$ nontrivially. So the subsurface projection $\pi_{S_1}(\partial E)$ is an essential simple closed curve in $S_1$. Since $S_1$ lies in $F \times I$, there is an essential simple closed curve $c \subset F$ so that the union of $c$ and $\pi_{S_1}(\partial E)$ bound a spanning annulus in $V$. Moreover, $c$ is unique up to isotopy. Therefore to get an upper bound for all distance degenerate curves in $F$, it is sufficient to give an upper bound of distances between all these degenerate curves and $c$ in $\mathcal{C}(F)$. More precisely, let $r \subset F$ be a distance degenerate curve for $V \cup_s W$. It is known that $d_{\mathcal{C}(F)}(r, c) \leq d_{\mathcal{C}(F)}(r, \gamma_l) + d_{\mathcal{C}(F)}(\gamma_l, b) + d_{\mathcal{C}(F)}(b, c)$. We will give an upper bound of $d_{\mathcal{C}(F)}(r, \gamma_l)$ in Subsection 4.1, an upper bound of $d_{\mathcal{C}(F)}(b, c)$ in Subsection 4.2 and an upper bound of $d_{\mathcal{C}(F)}(\gamma_l, b)$ in Subsection 4.3. Then they together give an upper bound of $d_{\mathcal{C}(F)}(r, c)$.

4.1. $d_{\mathcal{C}(F)}(r, \gamma_l) \leq 2l \log_2 4|g(S) - g(F)| + l + 1$. Since $r \subset F$ is a distance degenerate curve for $V \cup_s W$, $d(V(r), W) = l \leq m - 1$. By the definition of a Heegaard distance, there is a collection of finitely many essential simple closed curves on $S$, say $\{\alpha_0, \ldots, \alpha_l\}$, so that (I) $\alpha_0$ (resp. $\alpha_l$) bounds an essential disk $D_0$ (resp. $E_l$) in $V(r)$ (resp. $W$); (II) for any $1 \leq i \leq l$, $\alpha_i$ is disjoint from $\alpha_{i-1}$. Here $D_0$ is an essential disk in $V(r)$ but not in $V$. For if not, then $d(V, W) \leq l < m$. Then by Lemma 3.1, there are finitely many essential disks $\{B_1, \ldots, B_s\}$ in $V$ so that either $\alpha_0$ is disjoint from all these disks or there is a strongly essential outermost arc $\gamma \subset \alpha_0$ so that $\gamma$ and one arc in $D_0 \cap B_i$, for some $1 \leq i \leq s$, bounds an essential disk. In the first case, $\alpha_0$ is strongly essential in $S_F$. In the later case, $\gamma$ is strongly essential in $S_F$ and so is $\pi_{S_E}(\gamma)$. So in both of these two cases, there is an essential disk bounded by $\pi_{S_E}(\gamma)$ or $\alpha_0$ in the component of $V(r) - \cup_{i=1}^s B_i$, says $V_F(r)$. For simplicity, they are both denoted by $\pi_{S_E}(\gamma)$.

Since $V(r)$ is obtained from attaching a 2-handle along $r$ on $V$, there is also an essential disk bounded by $r^*$ in $V_F(r)$, which bounds a spanning annulus $\mathcal{A}_r$ with $r$ in $V$. So how does $r^*$ intersect $\pi_{S_E}(\gamma)$ in $V_F(r)$?

Lemma 4.1. (1) If $r$ is separating in $F$, then for some choice of $r^*$, $\pi_{S_E}(\gamma)$ is isotopic to $r^*$ in $V_F(r)$;

(2) If $r$ is non separating in $F$, then for some choice of $r^*$, $\pi_{S_E}(\gamma)$ is disjoint from $r^*$ in $V_F(r)$.

Proof. If $r$ is separating in $F$, then $V_F(r)$ contains only one essential disk $D_r$ up to isotopy. For if not, then there is another essential disk in $V_F(r)$. In this case, either it is disjoint from the disk bounded by $r^*$ or it intersects $D_r$ nontrivially. But since $V_F(r)$ is homeomorphic to two closed surface I-bundles linked by a 1-handle, it is impossible. It is known that $\pi_{S_E}(\gamma)$ bounds an essential disk in $V_F(r)$. So $\pi_{S_E}(\gamma)$ is isotopic to $r^*$.

If $r$ is not separating in $F$, then $V_F(r)$ contains an essential non separating disk $D_r$. Since $\pi_{S_E}(\gamma)$ also bounds an essential disk $D_1$ in $V_F(r)$, either $D_1$ is disjoint from $D_r$ or they intersects nontrivially. In the former case, $\pi_{S_E}(\gamma)$ is disjoint from $r$. In the later case, it is assumed that there is no circle in their intersection. Then it consists of some arcs.
Therefore there is an outermost disk in $D_1$, which is bounded by a component $\eta \subset \partial D_1$ and some arc in $D_r \cap D_1$.

Cutting $\partial_+ V_F(r)$ along $\partial D_r$ produces a compact surface $S_{V_F(r), \partial D_r}$, whose boundary curves are two copies of $\partial D_r$. Then $\eta \subset S_{V_F(r), \partial D_r}$ is strongly essential and its two ends lie in the same copy of $\partial D_r$, says $\partial D^1_r$. For if not, then it cuts out an annulus in $S_{V_F(r), \partial D_r}$, which contains $\partial D^2_r$ as one boundary. It is not hard to see that for any arc of $\pi S_F(\gamma) \cap S_{V_F(r), \partial D_r}$, if it has one end in $\partial D^2_r$, then the other end of it is in $\partial D^1_r$, see Figure 4.1.

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{figure4.1}
\caption{Figure 4.1.}
\end{figure}

However, the existence of $\eta$ shows that there are at least two more points in $\pi S_F(\gamma) \cap \partial D^1_r$ than in $\pi S_F(\gamma) \cap \partial D^2_r$. So it contradicts the fact that $\partial D^1_r$ is isotopic to $\partial D^2_r$. In this case, $\pi S_{V_F(r), \partial D_r}(\eta)$ bounds an essential disk in $V_F(r) - D_r$. But $V_F(r) - D_r$ is a closed surface I-bundle and so contains no essential disk, a contradiction. \hfill $\square$

So $\pi S_F(\gamma)$ is disjoint from $\gamma^*$ after some isotopy. For simplicity, $\pi S_F(\gamma)$ is abbreviated by $\gamma^*_0$.

For any $1 \leq i \leq l$, by the definition of Heegaard distance, $\alpha_i$ intersects every essential disk of $V$ nontrivially.

**Lemma 4.2.** Let $F$ be a genus at least one proper subsurface of $S$, i.e., essential but not isotopic, so that each component of $\partial F$ bounds an essential disk of $V$. Then for any essential simple closed curve $\alpha \subset S$, if $\alpha$ is disk busting in $V$, then there are $t \leq 2[g(S) - g(F)]$ arcs $\{a_1, ..., a_t\}$ of $\alpha \cap F$ so that one boundary curve of $N(\partial F \cup \bigcup_{i=1}^t a_i)$ is strongly essential in $F$, says $\gamma^*$.

**Proof.** If $\partial F$ is connected, then every component of $\alpha \cap F$ is strongly essential. Let $\gamma^*$ be an arbitrary one essential curve of $N(\partial F \cup \alpha)$ in $F$. So we assume that $\partial F$ is not connected. If one component of $\alpha \cap F$ is strongly essential in $F$, then let $a_1$ be the one. So $t = 1$. Otherwise, none of $\alpha \cap F$ is strongly essential. So there is at least one arc of $\alpha \cap F$ connecting two different boundary curves of $F$. For if not, then each arc of $\alpha \cap F$ cuts out a planar surface in $F$. So there is an essential boundary curve of $N(\partial F \cup (\alpha \cap F))$, denoted by $C$, so that it cuts out a planar surface in $F$. Then $C$ bounds an essential disk in $V$. By the construction of $C$, it is disjoint from $\alpha$. Therefore $\alpha$ is not disk busting in $V$.

Since $F$ has a finite genus and finitely many boundary curves, there are finitely many disjoint but nonisotopic essential arcs in $\alpha \cap F$, says $\{a_1, ..., a_t\}$, so that each of them connects two different boundary curves of $F$. Then one boundary curve of $N(\partial F \cup \bigcup_{i=1}^t a_i)$ is
strongly essential in $F$. For if not, then for any choice of these essential arcs in $\alpha \cap F$, there is an essential but not strongly essential simple closed curve $C \subset F$ so that it is disjoint from them. For any two essential simple closed curves $C_1$ and $C_2$, there is a partial order $\prec$ defined. We say $C_1 \prec C_2$ if $C_2$ is essential in the planar surface bounded by the union of $C_2$ and $\partial F$. So for any sequence of essential simple closed curves as above, there is a maximal one, denoted by $C$. Moreover $C$ is disjoint from $\alpha$. For if not, then $C$ intersects $\alpha$ nontrivially. Since $C$ is a union of some components $\alpha \cap F$ and some boundary arcs of $F$, $\alpha$ intersects these boundary arcs nontrivially. It means that there is some arc $a$ of $\alpha \cap F$ which is not contained in the planar surface bounded by $C$ and $\partial F$. Then there is an essential but not strongly essential simple closed curve $C_*$ in $\partial \Sigma(\partial F \cup a \cup C)$ so that $C_*$ is essential in the planar surface bounded by $C$ and $\partial F$. Then it contradicts the maximality of $C$. Since $C$ is a band sum of $\partial F$, $C$ bounds an essential disk in $V$. So $\alpha$ is not disk busting.

Since $F$ is an essential subsurface of $S$, $\partial F$ has at most $2[\log(S) - g(F)]$ components. If $\gamma_i^*$ is the $\pi_F(\gamma_i)$, then $i = 1 \leq 2[\log(S) - g(F)]$. Otherwise, there are some pairwise disjoint and nontrivial arcs $\{a_1, ..., a_l\}$ in $F$ so that $\gamma_i^*$ is a boundary component of $\Sigma(\partial F \cup (\bigcup_{i=1}^{l-1} a_i))$, where $t$ is minimal. Since cutting $F$ along $a_i$ once reduces the number of $\partial F$ by one, the extreme case is that $F - \bigcup_{i=2}^{l} a_i$ is connected. Then $t \leq 2[\log(S) - g(F)]$.

Since $\partial S_F$ consists of finitely many disks’ boundary curves, for any $1 \leq i \leq l$, by Lemma 4.2 there is a strongly essential simple closed curve $\gamma_i^*$ for $a_i$ in $S_F$. Since $S_F$ lies in $F \times I$, for any $0 \leq i \leq l$, there is an essential simple closed curve $\gamma_i$ so that the union of $\gamma_i$ and $\gamma_i^*$ bound a spanning annulus $\mathcal{A}_i$ in $V$.

By Lemma 4.1, $\gamma_0^*$ is disjoint from $r^*$ in $V_F(r)$. Since $\gamma_0^* \cup \gamma_0$ (resp. $r^* \cup r$) bounds a spanning annulus $\mathcal{A}_0$ (resp. $\mathcal{A}_r$), the intersection number $\gamma_0 \cap r$ is not larger than $\gamma_0^* \cap r^*$. So $\gamma_0$ is disjoint from $r$ in $F$. For any $1 \leq i \leq l$, since $\alpha_i \cap \alpha_i \neq \emptyset$, by Lemma 4.2, $\gamma_i^*$ intersects $\gamma_i^{-1}$ in at most $2[\log(S) - g(F)]$ points in $S_F$ and therefore $\gamma_i$ intersects $\gamma_i^{-1}$ in at most $2[\log(S) - g(F)]$ points in $F$. By Lemma 2.1, $d_{\mathcal{C}(F)}(\gamma_i, \gamma_{i-1}) \leq 2 \log 2 \cdot 4[\log(S) - g(F)] + 1$. So $d_{\mathcal{C}(F)}(r, \gamma_i) \leq 2 \log 2 \cdot 4[\log(S) - g(F)] + t + 1$. Thus to get an upper bound of the distance between $r$ and $c$ in $\mathcal{C}(F)$, it is sufficient to give an upper bound of $d_{\mathcal{C}(F)}(c, \gamma_1)$.

By Lemma 4.2, there are at most $t \leq 2[\log(S) - g(F)]$ arcs $\{a_1, ..., a_l\}$ of $\partial E \cap S_F$ so that one boundary curve of $\partial \Sigma(\partial S_F \cup (\bigcup_{i=1}^{l} a_i))$, says $\beta$, is strongly essential in $S_F$. Moreover, there is an essential simple closed curve $b$ in $F$ so that $\beta \cup b$ bound a spanning annulus in $V$. Thus to get an upper bound of $d_{\mathcal{C}(F)}(c, \gamma_1)$, it is enough to give these two estimations of $d_{\mathcal{C}(F)}(c, b)$ and $d_{\mathcal{C}(F)}(b, \gamma_1)$.

4.2. $d_{\mathcal{C}(F)}(c, b) \leq 2 \log 2 \cdot 2N - 1$. Since $\partial D \cap \partial E = 2N$, $\pi_{S_1}(\partial E)$ intersects $\partial E$ in at most $2N$ points. So is $\pi_{S_1}(\partial E) \cap (\bigcup_{i=1}^{l} a_i))$. If $\pi_{S_1}(\partial E)$ is contained in $S_F$, then $\pi_{S_1}(\partial E)$ intersects $\partial E$ nontrivially, it becomes more subtler. For explaining it, there is a lemma introduced.

**Lemma 4.3.** There are two essential simple closed curve $\beta^* \subset S$, $c^* \subset S$ and $b \subset F$ so that

□
1. \( c^* \cup c \) (resp. \( \pi_{S_1}(\partial E) \cup c \)) bounds a spanning annulus in \( V \);
2. \( \beta^* \cup b \) (resp. \( \beta \cup b \)) bounds a spanning annulus in \( V \);
3. \( |\beta^* \cap c^*| \leq 2N \).

Proof. By the construction of \( \beta \), it is a union of some arcs of \( \partial S_F \) and some arcs in the interior of \( S_F \). The former arc is marked by + while the later one is marked by –. Then + arcs and – arcs appear in \( \beta \) alternatively.

It is possible that there are some more points in \( \beta \cap \pi_{S_1}(\partial E) \) than in \( \beta \cap \partial E \). In this case, all these new points belong to the intersection points between + arcs and \( \partial E \). Therefore for removing all these new points, it is necessary to make some surgeries on both \( \beta \) and \( \pi_{S_1}(\partial E) \).

Since \( \pi_{S_1}(\partial E) \cup c \) bounds a spanning annulus \( A \) in \( V \), by the standard innermost circle and outermost disk argument, there is an outermost disk in \( A \) which is bounded by the union of an arc in \( e_{i,1} \subset B_i \) and an arc \( \eta_{B_i} \subset \partial E \). Because this outermost disk is contained in \( V \), \( \eta_{B_i} \) is not strongly essential in \( S_F \). So \( \eta_{B_i} \) cuts \( S_F \) out a planar surface, denoted by \( S_{\eta_{B_i}} \).

If \( \beta \cap \eta_{B_i} = \emptyset \), then for any point \( p \in A \), there is a surgery on \( \beta \) along \( \eta_{B_i} \) so that \( p \) is removed from \( \beta \cap \pi_{S_1}(\partial E) \), denoted by \( \beta^1 \), see Figure 4.2 for example.

![Figure 4.2.](image_url)

It is not hard to see that \( \beta^1 \cup b \) also bound a spanning annulus in \( V \). Moreover, there is no new point generated in this process. If \( \beta \cap \eta_{B_i} = \emptyset \), then let \( \beta^1 = \beta \). So \( \beta^1 \) is a union of some + arcs and some – arcs. Then (1) \( \beta^1 \cup b \) bound a spanning annulus in \( V \); (2) \( \beta^1 \) is disjoint from \( S_{\eta_{B_i}} \); (3) there are at most \( 2N \) points in intersection of – arcs between \( \beta^1 \) and \( \partial A \).

Since \( e_{i,1} \) cuts out a disk in \( B_i \), there is an outermost disk \( B_{i,1} \subset B_i \) for the spanning annulus \( A \). In this case, \( \partial B_{i,1} \) consists of an arc in \( B_i \) and an arc in \( S_{\eta_{B_i}} \). Then doing a boundary compression along \( B_{i,1} \) cuts \( A \) into a spanning annulus \( A_1 \) and an essential disk in \( V \). In this process, since \( S_{\eta_{B_i}} \) is disjoint from \( \beta^1 \), there is no new point generated in \( \partial A_1 \cap \beta^1 \). So – arcs of \( \beta^1 \) intersects \( \partial A_1 \) in at most \( 2N \) points.

Let \( c^1 \) be \( A_1 \cap S \). If \( c^1 \) doesn’t intersect \( \partial S_F \) essentially, then there is a bigon bounded by the union of \( c^1 \) and \( \partial S_F \) in \( S \). We assume that \( \beta^1 \) is disjoint from this bigon. For if
there is a smaller bigon bounded by $\beta \cup c$, then we push $c$ over this smaller bigon so that it vanishes. In this process, there is no new point generated. So we push $c$ over this bigon, bounded by $c \cup \partial S_F$, so that it vanishes. Also in this process, there is no new point $\beta \cap c$ generated for $\beta \cup c$. In all, $-\text{arcs of } \beta$ intersects $c$ in at most $2N$ points.

After finitely many steps, $c$ intersects $\cup_i \partial(B_i)$ essentially and $c \cup \partial S$ bounds a spanning annulus in $V$ too. Moreover $-\text{arcs of } \beta$ intersects $c$ in at most $2N$ points.

Cyclicly doing this operation until $c^*$ is disjoint from $\partial S_F$. Under this circumstance, $-\text{arcs of } \beta^*$ intersects $c^*$ in at most $2N$ points. Since $c^*$ is disjoint from $+\text{arcs of } \beta^*$, $\beta^*$ intersects $c^*$ in at most $2N$ points. Moreover, both $c^* \cup c$ and $\beta^* \cup b$ bound spanning annuli in $V$. \hfill $\square$

By Lemma 4.3 $|c^* \cup b^*| \leq 2N$. So $|c \cup b| \leq 2N$. Therefore, by Lemma 2.1 $d_{C(F)}(c, b) \leq 2 \log_2 2N + 1$.

4.3. **An upper bound of** $d_{C(F)}(b, \gamma_l)$. Since the Heegaard distance $d(V, W)$ is at least 3, by Lemma 2.2 and 2.3 for these two disks $E$ and $E_{\alpha_l}$ bounded by $\alpha_l$, there is some connection between them. So do $\beta$ and $\gamma_l^*$. Then there is an upper bound of $d_{C(F)}(b, \gamma_l)$ obtained from it as follows.

**Lemma 4.4.**

$$g(S) - g(F) \geq 2, d_{C(F)}(b, \gamma_l) <$$

$$\frac{1}{2\sqrt{6}}[(\sqrt{6} - 2)^2 - 14 \log_2 2g(S) + 7 + (2 - \sqrt{6}) \log_2 2(50\sqrt{6} - 14 \log_2 2g(S) - 117)] +$$

$$+ (\sqrt{6} + 2)(-14 \log_2 2g(S) - 7 + (2 + \sqrt{6}) \log_2 2(50\sqrt{6} + 14 \log_2 2g(S) + 117)) \frac{1}{\sqrt{6} + 1};$$

$$g(S) - g(F) = 1, d_{C(F)}(b, \gamma_l) \leq 52.$$

**Proof.** There is a mathematical induction in this proof.

(I) $S_F$ has only one boundary component.

Since $d(V, W) \geq 3$, $W$ is neither a product I-bundle of $S_F$ nor a twisted I-bundle of $S_F$. By Lemma 2.2 $d_{C(F)}(\beta, \gamma_l^*) \leq 12$. It is assumed that $S_F$ has only one boundary component. Then every essential simple closed curve in $S_F$ is also strongly essential in it. So $d_{C(F)}(b, \gamma_l) \leq 12$.

(II) $S_F$ has exactly two boundary components.

Since $d(V, W) \geq 3$, $N > 4$. For if not, then there is an essential disk $E_0 \subset W$ so that $\partial E_0 \cap \partial S_F$ has 2 or 4 points. If $\partial E_0 \cap \partial S_F$ has 2 points, then $d(V, W) \leq 1$. If $\partial E_0 \cap \partial S_F$ has 4 points, either all these 4 points lies in a same component of $\partial S_F$ or there are 2 intersecting points for each boundary component of $S_F$ individually. But in both of these two cases, $d(V, W) \leq 2$.

By Lemma 2.3 there is an essential disk $E_1 \subset W$ so that (1) it intersects $\partial S_F$ minimally; (2) a component $e_{1,1}$ (resp. $e_{1,2}$) of $\partial E_1 \cap S_F$ is disjoint from a component $e$ (resp. $e_l$) of $\partial E \cap S_F$ (resp. $\partial E_l \cap S_F$).
Let’s firstly consider the case that all of these four arcs are strongly essential in $S_F$. Then each one of $\{\pi_{S_F}(e), \pi_{S_F}(e_{1,1}), \pi_{S_F}(e_{1,2}), \pi_{S_F}(e_l)\}$ is strongly essential in $S_F$. For $\pi_{S_F}(e)$, there is an essential simple closed curve in $F$ so that the union of them bounds a spanning annulus in $V$. In order to not introduce too many labels, this essential simple closed curve in $F$ for $\pi_{S_F}(e)$ is still denoted by itself. So do the left three curves.

It is not hard to see that
\[ |\pi_{S_F}(e) \cap \pi_{S_F}(e_{1,1})| \leq 1; \]
\[ |\pi_{S_F}(e_{1,1}) \cap \pi_{S_F}(e_{1,2})| \leq 1; \]
\[ |\pi_{S_F}(e_{1,2}) \cap \pi_{S_F}(e_l)| \leq 1. \]

Then
\[ d_C(F)(\pi_{S_F}(e), \pi_{S_F}(e_{1,1})) \leq 2; \]
\[ d_C(F)(\pi_{S_F}(e_{1,1}), \pi_{S_F}(e_{1,2})) \leq 2; \]
\[ d_C(F)(\pi_{S_F}(e_{1,2}), \pi_{S_F}(e_l)) \leq 2. \]

So
\[ d_C(F)(\pi_{S_F}(e), \pi_{S_F}(e_l)) \leq 6. \]

Since $\beta$ is a union of at most two components of $\partial E \cap S_F$ and some arcs of $\partial S_F$, $\beta$ intersects $\pi_{S_F}(e)$ in at most one point. It is known that the union of $\beta$ and $b$ bounds a spanning annulus in $V$. So $b$ intersects $\pi_{S_F}(e)$ in at most one point in $F$ up to isotopy. Then $d_C(F)(b, \pi_{S_F}(e)) \leq 2$. Similarly, $d_C(F)(\pi_{S_F}(e_l), \gamma_l) \leq 2$. Hence $d_C(F)(b, \gamma_l) \leq 10$.

The worst scenario is that none of $\{\partial E_1 \cap S_F, \partial E_1 \cap S_F, \partial E_l \cap S_F\}$ is strongly essential in $S_F$ while none of any two arcs of $\{e, e_{1,1}, e_{1,2}, e_l\}$ is isotopic. Under this circumstance, each component of $\{\partial E_1 \cap S_F, \partial E_1 \cap S_F, \partial E_l \cap S_F\}$ has its two ends in different boundary components of $S_F$. For if not, let’s consider $\partial E \cap S_F$ for example. Then there is one arc of $\partial E \cap S_F$ so that it cuts out a planar surface of $S_F$ and a subsurface $S_{F,E}$, where $\partial S_{F,E}$ is connected. It is known that $\partial E \cap \partial S_{F,E}$ is not an empty set. Then for each component of $\partial E \cap S_{F,E}$, it is a sub-arc of some component of $\partial E \cap S_{F,E}$, see Figure 4.3. It means that there is a strongly essential arc of $\partial E \cap S_F$ in $S_F$.

![Figure 4.3](image-url)

**Claim 4.5.** In the worst scenario, $d_C(F)(b, \gamma_l) \leq 52$. 

---

**Note:** The above text contains mathematical content that includes equations and diagrams. The diagrams are not rendered here, but they are referenced in the text. The image URLs for the diagrams are not provided, but they would be included in the actual document. The text is formatted to resemble a research paper, with proper citation of claims and theorems.
Proof. Let $S_{F,e}$ be the subsurface obtained from cutting $S_F$ along $e$. Then $\partial S_{F,e}$ is connected. Since $\beta$ is the union of two components of $\partial E \cap S_F$ and two sub-arcs of $\partial S_F$, $\beta$ intersects $\pi_{S_{F,e}}(\partial E)$ in at most 2 points, see Figure 4.4. By Lemma 4.3, $d_{C(F)}(b, \pi_{S_{F,e}}(\partial E_1)) \leq 2$.

![Figure 4.4.](image-url)

Since $e_{1,1}$ is not isotopic to $e$, the union of $e_{1,1}$, $e$ and two boundary sub-arcs is a strongly essential simple closed curve in $S_F$, see Figure 4.4 for example. Moreover, it is isotopic to $\pi_{S_{F,e}}(\partial E_1)$. By Lemma 2.2, since $W$ is not an I-bundle of $S_{F,e}$, $d_{C(S_{F,e})}(\pi_{S_{F,e}}(\partial E), \pi_{S_{F,e}}(\partial E_1)) \leq 12$.

Since $\partial S_{F,e}$ is connected, for any essential simple closed curve in $S_{F,e}$, there is an essential simple closed curve in $F$ so that the union of them bound a spanning annulus in $V$. In order to not introduce too many symbols, from now on, if there is no further notation, for any strongly essential simple closed curve $C \subset S_F$, the corresponding essential simple closed curve in $F$ is also represented by itself. So $d_{C(F)}(b, \pi_{S_{F,e}}(\partial E_1)) \leq 14$.

Cutting $S_F$ along $e_{1,1}$ produces a subsurface $S_{F,e_{1,1}}$, where $\partial S_{F,e_{1,1}}$ is connected. Then by Lemma 2.2 since $W$ is not an I-bundle of $S_{F,e_{1,1}}$, $d_{C(S_{F,e_{1,1}})}(\pi_{S_{F,e_{1,1}}}(\partial E), \pi_{S_{F,e_{1,1}}}(\partial E_1)) \leq 12$. Since $\partial S_{F,e_{1,1}}$ is connected, for any essential simple closed curve in $S_{F,e_{1,1}}$, there is an essential simple closed curve in $F$ so that the union of them bound a spanning annulus in $V$. So $d_{C(F)}(\pi_{S_{F,e_{1,1}}}(\partial E_1), \pi_{S_{F,e_{1,1}}}(\partial E)) \leq 12$.

It is not hard to see that the union of $e$, $e_{1,1}$ and two boundary arcs is also isotopic to $\pi_{S_{F,e_{1,1}}}(\partial E)$. Then $\pi_{S_{F,e_{1,1}}}(\partial E)$ is isotopic to $\pi_{S_{F,e}}(\partial E_1)$. By the triangle inequality, $d_{C(F)}(b, \pi_{S_{F,e_{1,1}}}(\partial E_1)) \leq 26$.

Similarly, $d_{C(F)}(\gamma_1, \pi_{S_{F,e_{1,2}}}(\partial E_1)) \leq 26$.

Moreover, the union of $e_{1,1}, e_{1,2}$ and two boundary arcs is isotopic to not only $\pi_{S_{F,e_{1,1}}}(\partial E_1)$ but also $\pi_{S_{F,e_{1,2}}}(\partial E_1)$. Then by the triangle inequality again, $d_{C(F)}(b, \gamma_1) \leq 52$.

In general, let’s firstly consider the case that $e_{1,1}$ is strongly essential in $S_F$.

**Claim 4.6.** If $e_{1,1}$ is strongly essential in $S_F$, then $d_{C(F)}(b, \gamma_1) \leq 40$. 

So $W$ is not an I-bundle of inequality, since $\gamma$ is in $S$. It is not hard to see that every essential simple closed curve in $S$ so that $\gamma$ intersects $\pi_{S_F} (e)$ in at most two points in $F$. So $d_{c(F)}(b, \pi_{S_F} (e)) \leq 4$.

If $e$ is not strongly essential in $S_F$, then there is an essential subsurface $S_{F,e}$ of $S_F$ obtained from cutting $S_F$ along $e$. By Lemma 2.2 since $W$ is not an I-bundle of $S_{F,e}$,

$$d_{c(S_{F,e})}(\pi_{S_{F,e}} (\partial E), \pi_{S_{F,e}} (\partial E_1)) \leq 12.$$

On one hand, $e_{1,1} \cap S_{F,e}$ is an essential arc of $\partial E_1 \cap S_{F,e}$. Then $\pi_{S_{F,e}}(e_{1,1})$ is isotopic to $\pi_{S_{F,e}} (\partial E_1)$. Since $e_{1,1}$ is strongly essential, $\partial e_{1,1}$ lies in the same boundary. Therefore $\pi_{S_F}(e_{1,1})$ is isotopic to both $\pi_{S_{F,e}}(e_{1,1})$ and $\pi_{S_F}(\partial E_1)$. On the other hand, $\beta$ intersects $\pi_{S_F \setminus E}$ in at most two points. So $d_{c(F)}(b, \pi_{S_F \setminus E} (\partial E)) \leq 2$. By the triangle inequality, $d_{c(F)}(b, \pi_{S_{F,e}} (\partial E_1)) \leq 14$. In all,

$$(1) \quad d_{c(F)}(b, \pi_{S_F} (e_{1,1})) \leq 14.$$

**Case 4.6.1.** $e_{1,2}$ is strongly essential in $S_F$.

Then by the above argument, $d_{c(F)}(\gamma_l, \pi_{S_F} (e_{1,2})) \leq 14$. Since $e_{1,1}$ is disjoint from $e_{1,2}$, $d_{c(F)}(\pi_{S_F} (e_{1,1}), \pi_{S_F} (e_{1,2})) \leq 2$. Then $d_{c(F)}(b, \gamma_l) \leq 30$.

**Case 4.6.2.** $e_{1,2}$ is not strongly essential in $S_F$.

Then there is an essential subsurface $S_{F,e_{1,2}} \subset S_F$ obtained from cutting $S_F$ along $e_{1,2}$ so that $\pi_{S_F} (e_{1,1})$ is isotopic to $\pi_{S_{F,e_{1,2}}} (\partial E_1)$.

**Subcase 4.6.2.1.** $e_{1}$ is strongly essential in $S_F$.

Since $e_{1,2}$ is disjoint from $e_{1}$, $\pi_{S_F} (e_{1})$ is isotopic to $\pi_{S_{F,e_{1,2}}} (\partial E_1)$. Since $W$ is not a I-bundle of $S_{F,e_{1,2}}$, by Lemma 2.2

$$d_{c(S_{F,e_{1,2}})}(\pi_{S_{F,e_{1,2}}} (\partial E_1), \pi_{S_{F,e_{1,2}}} (\partial E_1)) \leq 12.$$ So

$$d_{c(S_{F,e_{1,2}})}(\pi_{S_F} (e_{1,1}), \pi_{S_F} (e_{1})) \leq 12.$$ It is not hard to see that every essential simple closed curve in $S_{F,e_{1,2}}$ is strongly essential in $S_F$. Therefore

$$d_{c(F)}(\pi_{S_F} (e_{1,1}), \pi_{S_F} (e_{1})) \leq 12.$$ Since $\gamma_{l}^*$ intersects $\pi_{S_F} (e_{1})$ in at most one point, $d_{c(F)}(\pi_{S_F} (e_{1}), \gamma_l) \leq 2$. By the triangle inequality,

$$d_{c(F)}(\pi_{S_F} (e_{1,1}), \gamma_l) \leq 14.$$ So

$$d_{c(F)}(b, \gamma_l) \leq d_{c(F)}(b, \pi_{S_F} (e_{1,1})) + d_{c(F)}(\pi_{S_F} (e_{1,1}), \gamma_l) \leq 28.$$

**Subcase 4.6.2.2.** $e_{1}$ is not strongly essential in $S_F$.

Then there is an essential subsurface $S_{F,e_{1}}$ obtained from cutting $S_F$ along $e_{1}$. Since $e_{1,2}$ is not strongly essential, either $e_{1}$ is isotopic to $e_{1,2}$ or not. In the first case, since $W$ is not an I-bundle of $S_{F,e_{1}}$, by Lemma 2.2

$$d_{c(S_{F,e_{1}})}(\pi_{S_{F,e_{1}}} (\partial E_1), \pi_{S_{F,e_{1}}} (\partial E_1)) \leq 12.$$ So
$d_{C(F)}(\pi_{S_{F,e_1}}(\partial E_1), \pi_{S_{F,e_1}}(\partial E_1)) \leq 12$. By the construction of $\gamma_l$, $\gamma_l$ intersects $\pi_{S_{F,e_1}}(\partial E_l)$ in at most two points. Then $d_{C(F)}(\gamma_l, \pi_{S_{F,e_1}}(\partial E_l)) \leq 2$. So

$$d_{C(F)}(\pi_{S_{F,e_1}}(\partial E_1), \gamma_l) \leq 14.$$  

Since $\pi_{S_F}(e_{1,1})$ is isotopic to $\pi_{S_{F,e_1,2}}(\partial E_1)$, $\pi_{S_F}(e_{1,1})$ is isotopic to $\pi_{S_{F,e_1}}(\partial E_1)$. Then

$$d_{C(F)}(\pi_{S_F}(e_{1,1}), \gamma_l) \leq 14.$$  

By Equation 1

$$d_{C(F)}(b, \gamma_l) \leq d_{C(F)}(b, \pi_{S_F}(e_{1,1})) + d_{C(F)}(\pi_{S_F}(e_{1,1}), \gamma_l) \leq 28.$$  

The left case is that $e_l$ is not isotopic to $e_{1,2}$. Then one of them has its ends in two different boundary curves of $S_F$, says $e_{1,2}$. For if not, then at least one of $\{e_{1,2}, e_l\}$ is strongly essential. So $\pi_{S_F}(e_{1,1})$ is isotopic to $\pi_{S_{F,e_1,2}}(\partial E_1)$. If $e_l$ also has its two ends in two different boundary curves of $S_F$, then by the argument of Claim 4.5

$$d_{C(F)}(\gamma_l, \pi_{S_{F,e_1,2}}(\partial E_1)) \leq 26.$$  

Then

$$d_{C(F)}(\pi_{S_F}(e_{1,1}), \gamma_l) \leq 26.$$  

Since $d_{C(F)}(b, \pi_{S_F}(e_{1,1})) \leq 14$,

$$d_{C(F)}(b, \gamma_l) \leq 40.$$  

If $e_l$ has its two ends in a same boundary component of $S_F$, then it cuts out a subsurface $S_{F,e_l}$ containing no $e_{1,1}$, see Figure 4.5.

![Figure 4.5](image.png)

On one hand, $\pi_{S_F}(e_{1,1})$ (resp. $\gamma_l$) is isotopic to $\pi_{S_{F,e_1}}(\partial E_1)$ (resp. $\pi_{S_{F,e_1}}(\partial E_l)$). On the other hand, since $W$ is not an I-bundle of $S_{F,e_l}$, by Lemma 2.2

$$d_{C(S_{F,e_l})}(\pi_{S_{F,e_1}}(\partial E_1), \pi_{S_{F,e_1}}(\partial E_l)) \leq 12.$$  

So

$$d_{C(F)}(\pi_{S_F}(e_{1,1}), \gamma_l) \leq 12.$$  

Since $d_{C(F)}(b, \pi_{S_F}(e_{1,1})) \leq 14$,

$$d_{C(F)}(b, \gamma_l) \leq 26.$$
Similarly, for the case that $e_i$ has its ends in two different boundary components in $S_F$,

$$d_{C(F)}(b, \gamma_l) \leq 26.$$ 

\[\square\]

Similarly, if $e_{1,2}$ is strongly essential in $S_F$, then $d_{C(F)}(b, \gamma_l) \leq 40$. The left case is that neither $e_{1,1}$ nor $e_{1,2}$ is strongly essential. If $e_{1,1}$ is not isotopic to $e_{1,2}$, then either the union of $e_{1,1}$, $e_{1,2}$ and two sub-arcs of $\partial S_F$ is a closed strongly essential curve in $S_F$ or one of them, says $e_{1,1}$ for example, cuts out an essential subsurface $S_{F,e_{1,1}}$ containing no $e_{1,2}$. For the first case, the closed strongly essential curve is isotopic to both $\pi_{S_{F,e_{1,2}}}(\partial E_1)$ and $\pi_{S_{F,e_{1,1}}}(\partial E_1)$. By the proof of Claim 4.6,

$$d_{C(F)}(\pi_{S_{F,e_{1,2}}}(\partial E_1), \gamma_l) \leq 26.$$ 

Similarly,

$$d_{C(F)}(\pi_{S_{F,e_{1,1}}}(\partial E_1), b) \leq 26.$$ 

So

$$d_{C(F)}(b, \gamma_l) \leq 52.$$ 

For the later case, without loss of generality, we assume that there is an essential subsurface $S_{F,e_{1,1}} \subset S_F$ of $e_{1,1}$ so that it doesn’t contain $e_{1,2}$. Let $S_{F,e_{1,2}}$ be the surface obtained from cutting $S_F$ along $e_{1,2}$.

Then it is not hard to see that $\pi_{S_{F,e_{1,2}}}(\partial E_1)$ is isotopic to $\pi_{S_{F,e_{1,1}}}(\partial E_1)$, see Figure 4.6. By the proof of Claim 4.6,

$$d_{C(F)}(\pi_{S_{F,e_{1,2}}}(\partial E_1), \gamma_l) \leq 26.$$ 

Similarly,

$$d_{C(F)}(\pi_{S_{F,e_{1,1}}}(\partial E_1), b) \leq 26.$$ 

So

$$d_{C(F)}(b, \gamma_l) \leq 52.$$
If \( e_{1,1} \) is isotopic to \( e_{1,2} \), by the similar argument,

\[
\begin{align*}
    d_{C(F)}(b, \pi_{S,F,e_{1,1}}(\partial E)) & \leq 14; \\
    d_{C(F)}(\pi_{S,F,e_{1,1}}(\partial E), \pi_{S,F,e_{1,2}}(\partial E)) & \leq 12; \\
    d_{C(F)}(\pi_{S,F,e_{1,2}}(\partial E), \gamma_l) & \leq 14.
\end{align*}
\]

Therefore,

\[
d_{C(F)}(b, \gamma_l) \leq 40.
\]

In all,

\[
d_{C(F)}(b, \gamma_l) \leq 52.
\]

(III) \( S_F \) has \( n \geq 3 \) boundary components.

Since \( d(V,W) \geq 3 \), by a similar argument in case 2, \( N > 4 \). By Lemma 2.3, there is an essential disk \( E_1 \subset W \) so that (1) it intersects \( \partial S_F \) minimally, (2) a component \( e_{1,1} \) of \( \partial E_1 \cap S_F \) is disjoint from a component \( e \) of \( \partial E \cap S_F \). Similarly, there is a component \( e_{1,2} \) of \( \partial E_2 \cap S_F \) disjoint from a component \( e_l \) of \( \partial E_l \cap S_F \).

The most complicated case is that none of \( \{ e, e_{1,1}, e_{1,2}, e_l \} \) is strongly essential in \( S_F \) while none of any two curves is isotopic. If \( e \) has its two ends in a same boundary component of \( S_F \), then cutting \( S_F \) along it produces a nonplanar subsurface \( S_{F,e} \). If \( e \) has its two ends in different boundary components of \( S_F \), then cutting \( S_F \) along it produces a subsurface \( S_{F,e,e} \).

Since \( d(V,W) \geq 3 \), by Lemma 4.2 there is a strongly essential simple closed curve \( \beta_{1,e} \) in \( S_{F,e} \) for \( \partial E \). Since \( \beta \) is either disjoint from \( e \) or intersects \( e \) in at most two points, \( \beta \) intersects \( \beta_{1,e} \) in at most \( 2g(S) \) points. For the essential disk \( E_1 \), by Lemma 4.2 again, there is a strongly essential simple closed curve \( \theta \) (resp. \( \theta_{1,e} \)) for \( E_1 \) in \( S_F \) (resp. \( S_{F,e} \)).

Let \( S_{F,e_{1,1}} \) be the nonplanar subsurface obtained from cutting \( S_F \) along \( e_{1,1} \). By Lemma 4.2 there is a strongly essential simple closed curve \( \beta_{1,e_{1,1}} \) (resp. \( \theta_{1,e_{1,1}} \)) for \( \partial E \) (resp. \( \partial E_{1,e_{1,1}} \)) in \( S_F,e_{1,1} \). It is not hard to see that \( \beta_{1,e_{1,1}} \) (resp. \( \theta_{1,e_{1,1}} \)) intersects \( \beta_{1,e_{1,1}} \) (resp. \( \theta_{1,e_{1,1}} \)) in at most \( 2g(S) \) points.

Let \( S_{F,e_{1,2}} \) be the nonplanar subsurface obtained from cutting \( S_F \) along \( e_{1,2} \). By Lemma 4.2 there is a strongly essential simple closed curve \( \theta_{1,e_{1,2}} \) for \( \partial E_1 \). Since \( e_{1,1} \) is disjoint from \( e_{1,2} \), \( \theta_{1,e_{1,1}} \) intersects \( \theta_{1,e_{1,2}} \) in at most \( 2g(S) \) points. By the similar argument, there is a strongly essential simple closed curve \( \gamma_{l,e_{1,2}} \) for \( \partial E_l \) in \( S_{F,e_{1,2}} \).

Let \( S_{F,e_l} \) be the nonplanar subsurface obtained from cutting \( S_F \) along \( e_l \). By Lemma 4.2 there is a strongly essential simple closed curve \( \gamma_{l,e_l} \) (resp. \( \gamma_{l,e_{1,2}} \)) for \( \partial E_1 \) (resp. \( \partial E_l \)) in \( S_{F,e_{1,2}} \). It is not hard to see that \( \gamma_{l,e_l} \) intersects \( \gamma_l \) in at most \( 2g(S) \) points.

Let \( S_{F,e_{1,2},e_l} \) be the nonplanar subsurface obtained from cutting \( S_F \) along the union of \( e_{1,2} \) and \( e_l \). By Lemma 4.2 there is a strongly essential simple closed curve \( \theta_{1,e_{1,2},e_l} \) (resp. \( \gamma_{l,e_{1,2},e_l} \)) for \( \partial E_1 \) (resp. \( \partial E_l \)) in \( S_{F,e_{1,2},e_l} \). Since \( e_{1,2} \) is disjoint from \( e_l \), \( \theta_{1,e_l} \) (resp. \( \gamma_{l,e_{1,2},e_l} \)) intersects \( \theta_{1,e_{1,2},e_l} \) (resp. \( \gamma_{l,e_{1,2},e_l} \)) in at most \( 2g(S) \) points.
Therefore,
\[
\begin{align*}
  d_{C(F)}(b, \gamma_l) & \leq d_{C(F)}(b, \beta_{1,e}) + d_{C(F)}(\beta_{1,e}, \theta_{1,e}) + d_{C(F)}(\theta_{1,e}, \theta_{1,e,\theta_1,1}) \\
  & + d_{C(F)}(\theta_{1,e,\theta_1,1}, \beta_{1,e,\theta_1,1}) + d_{C(F)}(\beta_{1,e,\theta_1,1}, \beta_{1,e_1,1}) + d_{C(F)}(\beta_{1,e_1,1}, \theta_{1,e_1,1}) \\
  & + d_{C(F)}(\theta_{1,e_1,1}, \theta_{1,e_1,2}) + d_{C(F)}(\theta_{1,e_1,2}, \gamma_{e_1,2}) + d_{C(F)}(\gamma_{e_1,2}, \gamma_{e_1,2,e_1}) \\
  & + d_{C(F)}(\gamma_{e_1,2,e_1}, \theta_{1,e_1,2,e_1}) + d_{C(F)}(\theta_{1,e_1,2,e_1}, \theta_{1,e_1}) + d_{C(F)}(\theta_{1,e_1}, \gamma_{e_1}) \\
  & + d_{C(F)}(\gamma_{e_1}, \gamma_l) 
\end{align*}
\]

For any one of
\[
\{\beta \cap \beta_{1,e}, \theta_{1,e} \cap \theta_{1,e,\theta_1,1}, \beta_{1,e,\theta_1,1} \cap \beta_{1,e_1,1}, \theta_{1,e_1,1} \cap \theta_{1,e_1,2}, \gamma_{e_1,2} \cap \gamma_{e_1,2,e_1}, \theta_{1,e_1,2,e_1} \cap \theta_{1,e_1}, \gamma_{e_1} \cap \gamma_l\},
\]

it has at most \(2g(S)\) points. It means that each one of
\[
\{b \cap \beta_{1,e}, \theta_{1,e} \cap \theta_{1,e,\theta_1,1}, \beta_{1,e,\theta_1,1} \cap \beta_{1,e_1,1}, \theta_{1,e_1,1} \cap \theta_{1,e_1,2}, \gamma_{e_1,2} \cap \gamma_{e_1,2,e_1}, \theta_{1,e_1,2,e_1} \cap \theta_{1,e_1}, \gamma_{e_1} \cap \gamma_l\},
\]

has at most \(2g(S)\) points. Then by Lemma 2.41
\[
\begin{align*}
  d_{C(F)}(b, \beta_{1,e}) & \leq 2 \log_2 2g(S) + 1; \\
  d_{C(F)}(\theta_{1,e}, \theta_{1,e,\theta_1,1}) & \leq 2 \log_2 2g(S) + 1; \\
  d_{C(F)}(\beta_{1,e,\theta_1,1}, \beta_{1,e_1,1}) & \leq 2 \log_2 2g(S) + 1; \\
  d_{C(F)}(\theta_{1,e_1,1}, \theta_{1,e_1,2}) & \leq 2 \log_2 2g(S) + 1; \\
  d_{C(F)}(\gamma_{e_1,2}, \gamma_{e_1,2,e_1}) & \leq 2 \log_2 2g(S) + 1; \\
  d_{C(F)}(\theta_{1,e_1,2,e_1}, \theta_{1,e_1}) & \leq 2 \log_2 2g(S) + 1; \\
  d_{C(F)}(\gamma_{e_1}, \gamma_l) & \leq 2 \log_2 2g(S) + 1. 
\end{align*}
\]

So,
\[
\begin{align*}
  d_{C(F)}(b, \gamma_l) & \leq d_{C(F)}(\beta_{1,e}, \theta_{1,e}) + d_{C(F)}(\theta_{1,e,\theta_1,1}, \beta_{1,e_1,1}) + d_{C(F)}(\beta_{1,e_1,1}, \theta_{1,e_1,1}) \\
  & + d_{C(F)}(\theta_{1,e_1,1}, \theta_{1,e_1,2}) + d_{C(F)}(\gamma_{e_1,2}, \theta_{1,e_1,2}) + d_{C(F)}(\gamma_{e_1,2,e_1}, \theta_{1,e_1,2,e_1}) \\
  & + 14 \log_2 2g(S) + 7. 
\end{align*}
\]

For any one of \(\{S_{F,e}, S_{F,e_1,1}, S_{F,e_1,2}, S_{F,e_1}\}\), it has at most \(n - 1\) boundary curves; for any one of \(\{S_{F,e,\theta_1,1}, S_{F,e_1,2}\}\), it has at most \(n - 2\) boundary curves. Thus to get an upper bound, it is enough to consider the extreme case. Then there is a formula introduced.
\[
\begin{align*}
  f(n) & = 4f(n - 1) + 2f(n - 2) + 14 \log_2 2g(S) + 7, n \geq 3; \\
  f(2) & = 52, f(1) = 12, 
\end{align*}
\]

where \(\{f(n), n \in N^+\}\) is a Fibonacci series. So there is a transformation of it as follows.
\[
\begin{align*}
  f(n) + rf(n - 1) + t & = s(f(n - 1) + rf(n - 2) + t), n \geq 3; \\
  f(2) & = 52, f(1) = 12. 
\end{align*}
\]
So

\[ s - r = 4; \]
\[ rs = 2; \]
\[ st - t = 14 \log 2g(S) + 7. \]

Then there are two solutions, which are

\[ s_1 = \sqrt{6} + 2, \quad r_1 = \sqrt{6} - 2; \]
\[ s_2 = 2 - \sqrt{6}, \quad r_2 = -2 - \sqrt{6}. \]

Therfore

\[ (r_1 - r_2)f(n) + r_1t_2 - r_2t_1 = r_1s_2^{n-2}[f(2) + r_2f(1) + t_2] - r_2s_1^{n-2}[f(2) + r_1f(1) + t_1]; \]
\[ f(n) = \frac{1}{2\sqrt{6}}[(\sqrt{6} - 2)^{14 \log 2g(S)} + 7 + (2 - \sqrt{6})^{n-2}(50\sqrt{6} - 14 \log 2g(S) - 117) \frac{\sqrt{6}}{\sqrt{6} - 1} + \frac{(\sqrt{6} + 2)^{14 \log 2g(S)} - 7 + (2 + \sqrt{6})^{n-2}(50\sqrt{6} + 14 \log 2g(S) + 117)}{\sqrt{6} + 1}], \quad n \geq 3; \]
\[ f(2) = 52, \quad f(1) = 12. \]

Since \( S_F \) has at most \( 2[g(S) - g(F)] \) boundary components, \( d_{C_F}(b, \gamma_l) \leq f[2g(S) - 2g(F)] \). Therefore

\[ d_{C_F}(b, \gamma_l) \leq \frac{1}{2\sqrt{6}}[(\sqrt{6} - 2)^{14 \log 2g(S)} + 7 + (2 - \sqrt{6})^{2g(S)-2g(F)-2}(50\sqrt{6} - 14 \log 2g(S) - 117) \frac{\sqrt{6}}{\sqrt{6} - 1} + \frac{(\sqrt{6} + 2)^{2g(S)-2g(F)-2}(50\sqrt{6} + 14 \log 2g(S) + 117)}{\sqrt{6} + 1}], \]
\[ g(S) - g(F) \geq 2; \]
\[ d_{C_F}(b, \gamma_l) \leq 52, \quad g(S) - g(F) = 1. \]

For a general case, by the similar argument, \( d_{C_F}(b, \gamma_l) \) is not larger than the upper bound in the extreme case.
So
\[ d_{C}(r, c) \leq d_{C}(r, \gamma_{0}) + d_{C}(\gamma_{0}, \gamma_{1}) + d_{C}(\gamma_{1}, b) + d_{C}(b, c); \]
\[ d_{C}(r, \gamma_{0}) \leq 1; \]
\[ d_{C}(\gamma_{0}, \gamma_{1}) \leq 2l \log 2g(S) + l < 2m \log 2g(S) + m; \]
\[ d_{C}(b, c) \leq 2 \log 2N + 1; \]
\[ d_{C}(b, \gamma_{1}) \leq \frac{1}{2\sqrt{6}}[\sqrt{6} - 2] \left( \frac{14 \log 2g(S) + 7 + (2 - \sqrt{6})^{2g(S) - 2g(F) - 2}(5\sqrt{6} - 14 \log 2g(S) - 117)}{\sqrt{6} - 1} \right) + \frac{1}{2\sqrt{6}}[\sqrt{6} + 2] \left( \frac{-14 \log 2g(S) - 7 + (2 + \sqrt{6})^{2g(S) - 2g(F) - 2}(5\sqrt{6} + 14 \log 2g(S) + 117)}{\sqrt{6} + 1} \right), \]
\[ g(S) - g(F) \geq 2; \]
\[ d_{C}(b, \gamma_{1}) \leq 52, g(S) - g(F) = 1. \]

Then
\[ d_{C}(r, c) < 2 \log 2N + 2m \log 2g(S) + m + 2 + \frac{1}{2\sqrt{6}}[\sqrt{6} - 2] \left( \frac{14 \log 2g(S) + 7 + (2 - \sqrt{6})^{2g(S) - 2g(F) - 2}(5\sqrt{6} - 14 \log 2g(S) - 117)}{\sqrt{6} - 1} \right) + \frac{1}{2\sqrt{6}}[\sqrt{6} + 2] \left( \frac{-14 \log 2g(S) - 7 + (2 + \sqrt{6})^{2g(S) - 2g(F) - 2}(5\sqrt{6} + 14 \log 2g(S) + 117)}{\sqrt{6} + 1} \right), \]
\[ g(S) - g(F) \geq 2; \]
\[ d_{C}(r, c) < 2 \log 2N + 2m \log 2g(S) + m + 54, g(S) - g(F) = 1. \]

Let
\[ R = \max\{2 \log 2N + 2m \log 2g(S) + m + 2 + \frac{1}{2\sqrt{6}}[\sqrt{6} - 2] \left( \frac{14 \log 2g(S) + 7 + (2 - \sqrt{6})^{2g(S) - 2g(F) - 2}(5\sqrt{6} - 14 \log 2g(S) - 117)}{\sqrt{6} - 1} \right) + \frac{1}{2\sqrt{6}}[\sqrt{6} + 2] \left( \frac{-14 \log 2g(S) - 7 + (2 + \sqrt{6})^{2g(S) - 2g(F) - 2}(5\sqrt{6} + 14 \log 2g(S) + 117)}{\sqrt{6} + 1} \right), \]
\[ 2 \log 2N + 2m \log 2g(S) + m + 54 \}. \]

Then \( d_{C}(r, c) < R \). Hence the proof of Theorem 1.1 ends.

4.4. The proof of Corollary 1.2. By Scharlemann and Tomova’ result [36], every Heegaard splitting of \( M \) has distance at most \( \max\{d(V, W), 2g(S)\} \). It is a result of Kobayashi and Rieck [11] (an extended result of Schleimer [38] for compact 3-manifolds) that if \( t \) is the number of tetrahedra and truncated tetrahedra in \( M \), then every genus at least \( 76t + 26 \) Heegaard splitting has distance at most 2. Therefore for any distance at least 3 Heegaard splitting of \( M \), it has genus at most \( 76t + 25 \) and distance at most \( \max\{d(V, W), 2g(S)\} \).
By the generalized Waldhausen conjecture proved by Li [16,17], there are finitely many same genus but non isotopic Heegaard splittings for \( M \). So there are finitely many non isotopic distance at least 3 Heegaard splittings of \( M \). Therefore there are a maximum \( \mathcal{N} \) for all of these distance at least 3 Heegaard splittings and finitely many choices of \( c \). So there is a curve \( c^* \) and a universal bound \( R^* \) in \( \mathcal{C}(F) \) so that for any distance degenerate curve \( r \) among all its distance at least 3 Heegaard splittings, \( d_{\mathcal{C}(F)}(c^*, r) < R^* \).

In particular, if \( M = E(K) \) for some knot \( K \subset S^3 \), then the meridian is a distance degenerate slope for any distance at least 3 Heegaard splitting. So we write Corollary 1.2 as follows.

**Corollary 4.7.** For any high distance knot \( K \subset S^3 \), there is a \( R_K \)-ball of the meridian in \( \mathcal{C}[^OE(K)] \) so that it contains all degenerate slopes of its all distance at least 3 Heegaard splittings.

**Proof.** It is a direct result of the above argument. \( \square \)

**References**

[1] I. Agol, *Bounds on exceptional Dehn filling II*, Geom. Topol. 14 (2010), 1921-1940.

[2] M. Culler, C. Gordon, J. Luecke and P. Shalen, *Dehn surgery on knots*, Ann. of Math. (2) 125 (1987), no. 2, 237-300.

[3] C. Gordon, *Dehn surgery and 3-manifolds*, Low dimensional topology, 21-71, IAS/Park City Math. Ser., 15, Amer. Math. Soc., Providence, RI, 2009.

[4] C. Gordon, *Dehn filling a survey*, Knot theory (Warsaw, 1995), Polish Acad. Sci., Warsaw 1998, 129-144.

[5] C. Gordon and J. Luecke, *Only integral Dehn surgeries can yield reducible manifolds*, Math. Proc. Cambridge Philos. Soc. 102(1), 97-101 (1987).

[6] C. Gordon and J. Luecke, *Knots are determined by their complements*, J. Amer. Math. Soc. 2 (1989), no. 2, 371-415.

[7] K. Hartshorn, *Heegaard splittings of Haken manifolds have bounded distance*, Pacific J. Math. 204, 61-75(2002).

[8] W.Harvey, *Boundary structure of the modular group*, in: Riemann Surfaces and Related Topics, Ann. of Math. Stud., vol. 97, Princeton University Press, Princeton, NJ, 1981, pp. 245-251.

[9] J. Hempel, *3-manifolds as viewed from the curve complex*, Topology 40(2001), 631-657.

[10] A. Ido, Y. Jang, and T. Kobayashi, *Heegaard splittings of distance exactly n*, Algebr. Geom. Topol. 14:3 (2014), 1305-1411.

[11] T. Kobayashi and Y. Rieck, *Hyperbolic volume and Heegaard distance*, Comm. Anal. Geom. 22 (2014), no. 2, 247-268.

[12] M. Lackenby, *Attaching handlebodies to 3-manifolds*, Geom. Topol. 6(2002), 889-904.

[13] W.B. A. Lickorish, *A representation of orientable combinatorial 3-manifolds*, Ann. of Math. (2) 76 1962 531-540.

[14] M. Lackenby and R. Meyerhoff, *The maximal number of exceptional Dehn surgeries*, Invent. Math. 191 (2013), 241-382.

[15] L. Liang, F.C. Lei and F.L. Li, *Distance degenerating handle additions*, Proc. Amer. Math. Soc. 144 (2016), no. 1, 423-434.

[16] T. Li, *Heegaard surfaces and measured laminations, II: Non-Haken 3-manifolds*, J. Amer. Math. Soc. 19:3 (2006), 625-657.

[17] T. Li, *Heegaard surfaces and measured laminations, I: The Waldhausen conjecture*, Invent. Math. 167:1 (2007), 135-177.
2. T. Li, *Images of the disk complex*, Geom. Dedicata 158 (2012), 121-136.
3. T. Li, *Small 3-manifolds with large Heegaard distance*, Math. Proc. Cambridge Philos. Soc. 155:3 (2013), 431-441.
4. T. Li, *Heegaard surfaces and the distance of amalgamation*, Geom. Topol. 14 (2010), no. 4, 1871-1919.
5. M. Lustig and Y. Moriah, *Horizontal Dehn surgery and genericity in the curve complex*, J. Topol. 3 (2010), no. 3, 691-712.
6. H. Masur and Y. Minsky, *Geometry of the complex of curves. I. Hyperbolicity*, Invent. Math. 138(1999) 103-149.
7. H. Masur and Y. Minsky, *Geometry of the complex of curves. II. Hierarchical structure*, Geom. Funct. Anal. 10(2000) 902-974.
8. Y. Minsky, Y. Moriah and S. Schleimer, *High distance knots*, Algebr. Geom. Topol. 7 (2007), 1471-1483.
9. J. Ma and R. Qiu, *Degenerating slopes with respect to Heegaard distance*, arXiv:0907.4419
10. J. Ma, R. Qiu and Y. Zou, *Non degenerating Dehn fillings on genus two Heegaard splittings of knots’ complements*, to appear in Science China Mathematics.
11. H. Masur and S. Schleimer, *The geometry of the disk complex*, J. Amer. Math. Soc. 26(2013), no. 1, 1-62.
12. Y. Moriah and E. Sedgwick, *The Heegaard structure of Dehn filled manifolds (English summary)*, Workshop on Heegaard Splittings, 233-263, Geom. Topol. Monogr., 12, Geom. Topol. Publ., Coventry, 2007.
13. G. Perelman, *The entropy formula for the Ricci flow and its geometric applications*, arXiv:math.DG/0211159
14. G. Perelman, *Ricci flow with surgery on three-manifolds*, arXiv:math.DG/0303109
15. G. Perelman, *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds*, arXiv:math.DG/0307245
16. R. Qiu and M. Zhang, *Reducible and ∂-reducible handle additions*, Trans. Amer. Math. Soc. 361 (2009), no. 4, 1867-1884.
17. R. Qiu, Y. Zou and Q. Guo, *The Heegaard distances cover all non-negative integers*, Pacific J. Math. 275 (2015), no. 1, 231-255.
18. Y. Rieck, *Heegaard structures of manifolds in the Dehn filling space*, Topology 39 (2000), no. 3, 619-641.
19. M. Scharlemann, *Proximity in the curve complex: boundary reduction and bicompressible surfaces*, Pacific J. Math. 228, 325-348(2006).
20. M. Scharlemann and M. Tomova, *Alternate Heegaard genus bounds distance*, Geom. Topol. 10 (2006), 593-617.
21. M. Scharlemann and Y.Q. Wu, *Hyperbolic manifolds and degenerating handle additions*, J. Austral. Math. Soc. Ser. A 55 (1993), no. 1, 72-89.
22. S. Schleimer, *The disjoint curve property*, Geom. Topol. 8 (2004), 77-113.
23. S. Schleimer, *Notes on the complex of curves*, http://homepages.warwick.ac.uk/~masgar/Maths/notes.pdf
24. W.P. Thurston, *Three-dimensional manifolds, Kleinian groups and Hyperbolic geometry*, Bull. Amer. Math. Soc., 1982, 6(3):357-381.
25. F. Waldhausen, *Heegaard-Zerlegungen der 3-Sphäre. (German)*, Topology 7, 1968, 195-203.
26. F. Zhang, R.Qiu and Y. Zou, *Infinitely many hyperbolic 3-manifolds admitting distance-d and genus-g Heegaard splittings*, Geom. Dedicata 181 (2016), 213-222.
27. Y. Zou, K. Du, Q. Guo and R. Qiu, *Unstabilized self-amalgamation of a Heegaard splitting*, Topology Appl. 160 (2013), no. 2, 406-411.

Yanqing Zou

Department of Mathematics, Dalian Minzu University

yanqing@dlnu.edu.cn; yanqing_dut@163.com