CONNECTED COMPONENTS OF THE SPACE OF
CIRCLE-VALUED MORSE FUNCTIONS ON SURFACES

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Abstract. We classify the path-components of the space of circle-valued Morse functions on compact surfaces: two Morse functions $f, g : M \to S^1$ belong to same path-component of this space if and only if they are homotopic and have equal numbers of critical points at each index.

1. Introduction

Let $M$ be a smooth ($C^\infty$) connected compact surface, orientable or not, with boundary $\partial M$ or without it, and $P$ a one-dimensional manifold either the real line $\mathbb{R}^1$ or the circle $S^1$. Let $\mathcal{M}(M, P)$ denotes the subspace of $C^\infty(M, P)$ consisting of Morse mappings $M \to P$. It is well-known (e.g. Milnor [3]) that for the case $\partial M = \emptyset$ the set $\mathcal{M}(M, P)$ is an everywhere dense open in $C^\infty(M, P)$ with the $C^\infty$ Whitney topology of $C^\infty(M, P)$.

Recently, S. V. Matveev (his proof is included and generalized in E. Kudryavtseva [1]) and V. V. Sharko [4] have obtained a full description of the set $\pi_0\mathcal{M}(M, \mathbb{R}^1)$ of connected path-components of $\mathcal{M}(M, \mathbb{R}^1)$. Their methods are independent and based on different ideas.

For orientable closed surfaces the classification of $\pi_0\mathcal{M}(M, S^1)$ was initially given in author’s Ph.D, see [2]. The problem was proposed to the author by V. V. Sharko. In this note, we extend the results of [2] to all compact surfaces (Theorem 1.0.2) and simplify their proof.

To begin with, let us fix, once and for all, some Riemannian metric on $M$ and some orientation of $P$.

A $C^\infty$-mapping $f : M \to P$ is Morse if the following conditions hold true:

1. all critical points of $f$ are non-degenerated and belong to the interior of $M$;
2. $f$ is constant at every connected component of $\partial M$ though its values on different components may differ each from other.

Let $f : M \to P$ be a Morse function and $z$ a non-degenerated critical point $f$. Define the index of $z$ to be the usual one with respect to an
arbitrary local representation $M \supset \mathbb{R}^2 \xrightarrow{f} \mathbb{R}^1 \subset P$ of $f$ in which $\phi$ preserves orientations. Denote by $c_i(f)$ ($i = 0, 1, 2$) the number of critical points of $f$ of index $i$.

Further let $V$ be a connected component of $\partial M$. Since $f$ has no critical points on $V$, it follows that the gradient vector field $\nabla f$ is transversal to $V$ in every point $z \in V$.

Consider the function $\varepsilon_f : \pi_0 \partial M \to \{-1, 1\}$ such that for every connected component $V$ of $\partial M$, regarded as an element of $\pi_0 \partial M$, we have $\varepsilon_f(V) = +1$ provided $\nabla f$ is directed outward on all of $V$ and $\varepsilon_f(V) = -1$ otherwise. We will call $V$ either $f$-positive or $f$-negative in accordance with $\varepsilon_f(V)$. Then the following quadruple:

(1.1) $K(f) = (c_0(f), c_1(f), c_2(f), \varepsilon_f)$

will be called the critical type of a Morse mapping $f$. Notice that the reversion of the orientation of $P$ interchanges $c_0(f)$ and $c_2(f)$ and replaces $\varepsilon_V$ by $-\varepsilon_V$.

Finally we will say that two Morse mappings $f, g : M \to P$ are $\Sigma$-homotopic (belong to same connected path-component of $\mathcal{M}(M, P)$), and write $f \cong g$, if there is a continuous mapping $F : M \times I \to P$ such that for every $t \in I$ the function $f_t(x) = F(x, t) : M \to P$ is Morse.

1.0.1. **Theorem** (S. Matveev [1], V. Sharko [4]). Two Morse functions $f, g : M \to \mathbb{R}$ are $\Sigma$-homotopic iff $K(f) = K(g)$. Moreover, suppose that $f = g$ is a neighborhood of some open-closed subset $V$ of $\partial M$. Then $f \cong g$ with respect to some neighborhood of $V$.

The main result of this note is the following theorem:

1.0.2. **Theorem.** Two Morse functions $f, g : M \to S^1$ are $\Sigma$-homotopic if and only if they are homotopic and $K(f) = K(g)$.

The proof is heavily based on Theorem 1.0.1 and the structure of minimal Morse functions on $M$.

2. **Preliminaries**

We will regard $S^1$ as $\mathbb{R}/\mathbb{Z}$. Let $f : M \to S^1$ be a Morse mapping. Then a point $z \in S^1$ will be called a regular value of $f$ if $f^{-1}(z)$ contains no critical points of $f$ and no connected components of $\partial M$.

The following construction will often be used. Let $x = 0 \in S^1$ be a regular value of $f$. Let us cut $M$ along $f^{-1}(0)$ and denote the obtained surface by $\tilde{M}$. Let also $p : \tilde{M} \to M$ be the factor-map and $q(t) = e^{2\pi it} : \mathbb{R} \to S^1$ a universal covering. Then there is a
Morse function \( \tilde{f} : \tilde{M} \to [0, 1] \), such that the following diagram is commutative:

\[
\begin{array}{ccc}
\tilde{M} & \xrightarrow{\tilde{f}} & [0, 1] \\
p & & q \\
M & \xrightarrow{f} & S^1.
\end{array}
\]

2.1. Orientation of level-sets. Let \( f : M \to S^1 \) be a Morse mapping. Since \( f \) is constant on components of \( \partial M \) we have the following homomorphism \( f^* : H^1(S^1) \to H^1(M, \partial M) \). Let \( \xi \in H^1(S^1) \approx \mathbb{Z} \) be the generator that yields chosen positive orientation of \( S^1 \). Then for every oriented closed curve \( \omega : S^1 \to M \) we have

\[
\text{deg}(f \circ \omega) = f^*(\xi)(\omega).
\]

Suppose that \( M \) is oriented. Then there is an orientation of level-sets of \( f \) such that for every regular point \( x \in M \) of \( f \) and a tangent vector \( v \) to \( f^{-1}(x) \) at \( x \) the pair \( (\nabla f(x), v) \) gives the positive orientation of \( T_xM \). Thus the level-sets of \( f \) can be regarded as elements of \( H_1(M, \partial M) \).

Recall that there is an intersection form on \( M \)

\[
\langle \cdot, \cdot \rangle : H_1(M, \partial M) \times H_1(M, \partial M) \to \mathbb{Z}
\]

such that the correspondence \( Z \mapsto \langle Z, \cdot \rangle, Z \in H_1(M, \partial M) \), yields an isomorphism \( H_1(M, \partial M) \approx H^1(M, \partial M) \). Then for \( z \in S^1 \) we have

\[
\text{deg}(f \circ \omega) = f^*(\xi)(\omega) = \langle f^{-1}(z), \omega \rangle.
\]

2.1.1. Lemma. Let \( f, g : M \to S^1 \) be two smooth functions which take constant values on connected components of \( M \). Then the following conditions are equivalent:

1. \( f \) and \( g \) are homotopic;
2. \( f^* = g^* \);
3. for every \( x, y \in S^1 \) the 1-cycles \( f^{-1}(x) \) and \( g^{-1}(y) \) are homological in \( H_1(M, \partial M) \).

Proof. Equivalence (1)\( \Leftrightarrow \) (2) is well-known.

(2)\( \Leftrightarrow \) (3). Let \( x, y \in S^1 \) and \( X = f^{-1}(x) \) and \( Y = g^{-1}(y) \). Then \( X = Y \) in \( H_1(M, \partial M) \) iff \( \langle X, \omega \rangle = \langle Y, \omega \rangle \) for every oriented closed curve in \( M \). In view of (2.4) this is equivalent to the statement that \( f^*(\xi) = g^*(\xi) \). \( \square \)
2.2. **Minimal Morse functions.** Let \( V_0 \) and \( V_1 \) be two disjoint open-closed subsets of \( \partial M \) (we do not require that \( V_0 \cup V_1 = \partial M \)). Then \( V_0 \) and \( V_1 \) consist of connected components of \( \partial M \).

Recall that a Morse function \( f : M \to \mathbb{R} \) is **minimal**, provided \( f \) has minimal number of critical points at each index among all Morse function on \( M \).

The following statement is well-known, see e.g. [5]

2.2.1. **Lemma.** Let \( \varepsilon : \pi_0 \partial M \to \{-1, 1\} \) be an arbitrary function such that \( \varepsilon(V_0) = 0 \) and \( \varepsilon(V_1) = 1 \). Then there exists a minimal Morse function \( f : M \to [0, 1] \) such that \( f^{-1}(0) = V_0 \), \( f^{-1}(1) = V_1 \), and \( \varepsilon_f = \varepsilon \). Moreover, for every such a function we have

1) \( c_f(0) = 0 \) provided \( \varepsilon^{-1}(0) \neq \emptyset \); otherwise \( c_f(0) = 1 \).
2) Similarly, if \( \varepsilon^{-1}(1) \neq \emptyset \), then \( c_f(2) = 0 \), otherwise \( c_f(2) = 1 \).

Finally, every Morse function can be obtained from some minimal one by adding proper number of pairs of critical points of indexes 0 and 1 or 1 and 2. \( \square \)

2.3. **Unessential components.** Let \( f : M \to S^1 \) be a Morse mapping, \( x \) a regular value of \( f \) and \( X = f^{-1}(x) \).

A connected component \( C \) of \( M \setminus X \) will be called **essential** if either \( X = \emptyset \) or \( f(C) = S^1 \), otherwise \( C \) is **unessential**.

Let \( C \) be an unessential component of \( M \setminus X \). Then \( C \) lower if \( f(C) \subset [x, x + d) \), for some \( d \in (0, 1) \). Otherwise, \( f(C) \subset (x - d, x] \) for some \( d \in (0, 1) \) and \( C \) will be called **upper**.

Finally, we will say that \( f \) is \( x \)-**reduced**, provided all connected components of \( M \setminus X \) are essential.

2.3.1. **Lemma.** In the above notations, \( f \) is \( \Sigma \)-homotopic to an \( x \)-reduced Morse mapping.

**Proof.** Let \( C \) be an unessential component of \( M \setminus X \). We can assume that \( C \) is lower so that \( f(C) = [x, d] \), where \( 0 < x < d < 1 \) and the interval \( [0, x] \) consists of regular values of \( f \) only. Denote by \( D \) the connected component of \( f^{-1}[0, d] \) including \( C \).

Let also \( \mu : [0, 1] \to [0, 1] \) be a \( C^\infty \)-function such that \( \mu(0) = 1 \) and \( \mu(1) = x \). Then it is easy to verify that the following mapping \( F : M \times I \to S^1 \) defined by

\[
F(s, t) = \begin{cases} 
\mu(t)f(s), & s \in D \\
    f(s), & s \in M \setminus C.
\end{cases}
\]

is a \( \Sigma \)-homotopy between \( f = F_0 \) and the mapping \( g = F_1 \) such that \( g(C) \subset [x^2, xd] \subset [0, x] \), whence \( g^{-1}(x) = X \setminus C \).

Then our lemma follows by the induction on the number of connected components of \( X \). \( \square \)
2.4. Construction of Morse functions with given regular level-set. Let \( \gamma = \{ \gamma_1, \ldots, \gamma_n \} \subset \text{Int}M \) be a family of mutually disjoint two-sided simple closed curves, \( f : M \to S^1 \) a Morse function, \( x \in S^1 \) a regular value of \( f \), and \( X = f^{-1}(x) \).

2.4.1. Definition. We will say that \( \gamma \) is \((f, x)\)-regular if \( X \cap \gamma = \emptyset \) and for every connected component \( C \) of \( M \setminus (X \cup \gamma) \) we have

1. \( \overline{C} \cap X \neq \emptyset \) and \( \overline{C} \cap \gamma \neq \emptyset \);
2. \( f(\overline{C}) \neq S^1 \).

It follows from (2) that either \( f(\overline{C}) = [x, x + d] \) or \( f(\overline{C}) \subset [x - d, x] \) for some \( d \in (0, 1) \). We will call \( C \) lower in the first case and upper in the second.

2.4.2. Lemma. Suppose that \( \gamma \) is \((f, x)\)-regular. Then there exists a Morse function \( h : M \to S^1 \) such that

1. \( h^{-1}(x) = X = f^{-1}(x) \) and \( h = f \) near \( X \);
2. \( h^{-1}(y) = \gamma \) for some regular value \( y \) of \( h \);
3. \( K(h) = K(f) \).

Then it follows from Lemma 3.0.3 that \( h \overset{\Sigma}{\sim} f \) with respect to a neighborhood of \( X \).

Proof. We can assume that \( x = 0 \) and \( y = \frac{1}{2} \). Let \( C \) be a connected component of \( M \setminus (X \cup \gamma) \). If \( C \) is lower, then it follows from Definition 2.4.1 and Lemma 2.2.1 that there exists a minimal Morse function \( h_C : \overline{C} \to [0, \frac{1}{2}] \) such that \( h_C^{-1}(\frac{1}{2}) = \overline{C} \cap \gamma \), \( h_C^{-1}(0) = \overline{C} \cap X \), and \( \varepsilon_h = \varepsilon_f \).

Similarly, if \( C \) is upper, then we can construct a minimal Morse function \( h_C : \overline{C} \to [\frac{1}{2}, 1] \) such that \( h_C^{-1}(\frac{1}{2}) = \overline{C} \cap \gamma \), \( h_C^{-1}(1) = \overline{C} \cap X \), and \( \varepsilon_h = \varepsilon_f \).

Then the union of all functions \( h_C \), where \( C \) runs all connected components of \( M \setminus (X \cup \gamma) \), gives a function \( \hat{h} : M \to S^1 \) without critical points of indexes 0 and 2 and such that \( \hat{h}^{-1}(0) = X \), \( \hat{h}^{-1}(\frac{1}{2}) = \gamma \).

Moreover, we can choose these functions so that \( \hat{h} \) is smooth near \( X \cup \gamma \). Then adding to \( \hat{h} \) a necessary number of pairs of critical points of indexes 0 and 1 or 1 and 2 we can obtain a Morse function \( h : M \to S^1 \) satisfying the conditions (1)-(3) of our lemma.

Evidently, the condition (1) implies that \( h \) is homotopic to \( f \), whence by Lemma 3.0.3 we get \( f \overset{\Sigma}{\sim} h \) with respect to a neighborhood of \( X \). \( \square \)

3. Proof of Theorem 1.0.2

The necessity is obvious, therefore we will consider only sufficiency. Let \( f, g : M \to S^1 \) be two Morse mappings that are homotopic and
\( K(f) = K(g) \). We have to show that \( f \overset{\Sigma}{\sim} g \). First consider one particular case.

3.0.3. **Lemma.** Suppose that there exists a common regular value \( x \) of \( f \) and \( g \) such that \( f^{-1}(x) = g^{-1}(x) \), and \( f = g \) in a neighborhood of \( f^{-1}(x) \). Then \( f \overset{\Sigma}{\sim} g \).

**Proof.** Denote \( X = f^{-1}(x) \). If \( X = \emptyset \), then \( f \) and \( g \) are mappings \( M \to S^1 \setminus \{ x \} \approx \mathbb{R} \), whence by Theorem 1.0.1, \( f \overset{\Sigma}{\sim} g \).

Thus suppose that \( X \neq \emptyset \) and let \( x = 0 \). Then \( X \) is a disjoint union of two-sided simple closed curves. Using the notations of (2.2), we cut \( M \) along \( X \) and obtain liftings \( \tilde{f}, \tilde{g} : \tilde{M} \to [0,1] \) of \( f \) and \( g \) respectively. Then \( \tilde{f} = \tilde{g} \) near \( \tilde{X} = p^{-1}(X) \).

3.0.4. **Claim.** \( f \) is \( \Sigma \)-homotopic to a Morse function \( h : M \to S^1 \) such that \( K(h|\overline{D}) = K(g|\overline{D}) \) for every connected component \( C \) of \( M \setminus X \).

It follows from this claim that \( h \) yields a Morse map \( \tilde{h} : \tilde{M} \to [0,1] \) such that \( K(\tilde{h}|\overline{D}) = K(\tilde{g}|\overline{D}) \) for every connected component \( D \) of \( \tilde{M} \). Then from Theorem 1.0.1 we obtain that \( \tilde{h}|\overline{D} \overset{\Sigma}{\sim} \tilde{g}|\overline{D} \) with respect to a neighborhood of \( \tilde{X} \cap \overline{D} \). Hence \( \tilde{h} \overset{\Sigma}{\sim} \tilde{g} \) with respect to a neighborhood of \( \tilde{X} \) and therefore \( h \overset{\Sigma}{\sim} g \) with respect to a neighborhood of \( X \). Thus \( f \overset{\Sigma}{\sim} h \overset{\Sigma}{\sim} g \). This will prove Lemma 3.0.3.

**Proof of Claim.** It follows from Lemma 2.2.1 that such a function \( h \) can be obtained from \( f \) by moving pairs of critical points of indexes 0 and 1 and indexes 1 and 2 from some connected components of \( M \setminus X \) to another ones.

Consider the partition of \( M \) by the connected components of level-sets of \( f \). Recall that the factor-space of \( M \) by this partition admits a natural structure of a graph called Reeb graph of \( f \).

Evidently, moves of pairs of critical points yield transformations of Reeb graph of \( f \) shifting edges with vertexes of degree 1, see Figure 3.1, where bold points are the vertices of degree 2.

Every such a transformation can be realized by some \( \Sigma \)-homotopy \( f_t, (t \in [0,1]) \).

Moreover, let \( x \) be a value of \( f \) corresponding to the level-set denoted in Figure 3.1 by long horizontal line. Then \( f_t \) can be chosen so that \( f_0^{-1}(x) = f_1^{-1}(x) \) and \( f_0 = f_1 \) near \( f_0^{-1}(x) \), while it is possible that \( f_0^{-1}(x) \neq f_t^{-1}(x) \) for some \( t \in (0,1) \).

Thus properly moving edges with vertexes of degree 1 we can obtain from \( f \) a Morse function \( h \) satisfying the statement of this lemma. □
Now Theorem 1.0.2 is implied by the following two propositions and previous Lemma 3.0.3.

3.0.5. Proposition. Let $x$ and $y$ be regular values of $f$ and $g$ respectively. Suppose that $f$ and $g$ are reduced with respect to $x$ and $y$, and $f^{-1}(x) \cap g^{-1}(y) = \emptyset$. Then $f \Sigma \sim g$.

3.0.6. Proposition. The functions $f$ and $g$ are $\Sigma$-homotopic to Morse mappings $f_1$ and $g_1$ respectively such that $x$ and $y$ are regular values of $f_1$ and $g_1$ respectively, and $f_1^{-1}(x) \cap g_1^{-1}(y) = \emptyset$. Then by Proposition 3.0.5, $f_1 \Sigma \sim g_1$, whence $f \Sigma \sim g$.

4. Proof of Proposition 3.0.5

Denote $X = f^{-1}(x)$ and $Y = g^{-1}(y)$, so we have $X \cap Y = \emptyset$. It suffices to prove the following statement.

4.0.7. Claim. Let $C$ be a connected component of $M \setminus (X \cup Y)$. Then $f(C) \neq S^1$ and $C \cap Y \neq \emptyset$. Similarly, $g(C) \neq S^1$ and $C \cap X \neq \emptyset$. Thus $Y$ is $(f, x)$-regular and $X$ is $(g, y)$-regular.

It will follow from Lemma 2.4.2 that there exists a Morse function $h$ such that $h^{-1}(x) = X$, $h^{-1}(y) = Y$, $h = f$ near $X$, and $h = g$ near $Y$. Then by Lemma 3.0.3 we will get $f \Sigma \sim h \Sigma g$.

Proof of Claim 4.0.7. (1) First suppose that $M$ is orientable. Let us assume that $x = 0$. Cutting $M$ along $X$ we obtain a connected surface $\tilde{M}$, the projection $p : \tilde{M} \to M$, and a Morse function $\tilde{f} : \tilde{M} \to [0,1]$ such that the commutative diagram (2.2) holds true.

Denote $\tilde{X}_0 = \tilde{f}^{-1}(0)$, $\tilde{X}_1 = \tilde{f}^{-1}(1)$, $\tilde{Y} = p^{-1}(Y)$, and $D = p^{-1}(C)$.

Since $f$ and $g$ are homotopic and $M$ is orientable, it follows from Lemma 2.4.1 that 1-cycles $[X]$ and $[Y]$ are homological modulo $\partial M$. Whence $\tilde{Y}$ separates $\tilde{X}_0$ and $\tilde{X}_1$ in $\tilde{M}$, i.e. for every connected subset $U \subset \tilde{M}$ such that $U \cap \tilde{X}_i \neq \emptyset$ ($i = 0, 1$) we have that $U \setminus \tilde{Y}$ can be
represented as a union of two disjoint open-closed subsets $U_i$ such that $U \cap \tilde{X}_i \subset U_i$.

Suppose that $f(C) = S^1$. Then $\tilde{f}(\tilde{D}) = [0, 1]$. Therefore $\tilde{D} \cap \tilde{X}_i \neq \emptyset$ for $i = 0, 1$. Hence $\tilde{D} \setminus \tilde{Y}, \tilde{C} \setminus Y$, and therefore $C \setminus Y$ are not connected which contradicts to the assumption.

If $\overline{C} \cap Y = \emptyset$, then $C$ is a connected component of $M \setminus X$. But $f(C) \neq S^1$, whence $C$ is unessential for $(f, x)$, i.e. $f$ is not $x$-reduced.

(2) Suppose now that $M$ is non-orientable. Let $\tau : \hat{M} \to M$ be the oriented double covering, $\hat{f} = f \circ \tau, \hat{g} = g \circ \tau : \hat{M} \to S^1$, $\hat{X} = \tau^{-1}(X) = \hat{f}^{-1}(x), \hat{Y} = \tau^{-1}(Y) = \hat{g}^{-1}(y)$, and $\hat{C} = \tau^{-1}(C)$.

If $C$ is orientable, then $\hat{C}$ consists of two components each homeomorphic to $C$. Otherwise, $C$ is non-orientable, and $\hat{C}$ is an oriented double covering of $C$.

Notice that $\hat{f}$ is $x$-reduced and $\hat{g}$ is $y$-reduced. Indeed, for every connected component $D$ of $\hat{M} \setminus \hat{X}$ the set $\tau(D)$ is a connected component of $M \setminus X$, whence $\hat{f}(D) = f(\tau(D)) \neq S^1$. The proof for $\hat{g}$ is similar.

Let $\hat{D}$ be a connected component of $\hat{C}$. Since $\hat{X} \cap \hat{Y} = \emptyset$, we get from orientable case of this claim that

$$f(C) = \hat{f}(\hat{D}) \neq S^1 \quad \text{and} \quad \overline{C} \cap Y \supset \tau(\overline{D} \cap \hat{Y}) \neq \emptyset.$$  

This completes Claim 4.0.7 and Proposition 3.0.6  

5. Proof of Proposition 3.0.6

We can assume that the intersection $X$ and $Y$ is transversal. Let $n = \#(X \cap Y)$. If $n = 0$, then our statement is just Proposition 3.0.5.

Suppose that $n > 0$. We will show how to reduce the number of intersection points $X \cap Y$ by $\Sigma$-homotopy.

For simplicity let $x = 0$. We can also assume that $f$ is $x$-reduced.

Cutting $M$ along $X$ we obtain a connected surface $\hat{M}$, the projection $p : \hat{M} \to M$, and a Morse function $\hat{f} : \hat{M} \to [0, 1]$ such that the commutative diagram (2.2) holds true. Denote $\hat{X}_0 = \hat{f}^{-1}(0), \hat{X}_1 = \hat{f}^{-1}(1)$, and $\hat{Y} = p^{-1}(Y)$.

Notice that $\hat{Y}$ consists of simple closed curves and arcs with ends at $\hat{X}$. Let us divide $\hat{Y}$ by the following four disjoint subsets:

$$L_0, L_1, L_0^1, L_c,$$

where $L_0$ ($L_1$) consists of arcs whose both ends belong to $\hat{X}_0$ ($\hat{X}_1$), $L_0^1$ consists of arcs of $\hat{Y}$ connecting $\hat{X}_0$ with $\hat{X}_1$, and $L_c$ consists of simple closed curves of $\hat{Y}$. 
Since \( f \) and \( g \) are homotopic, it follows from (2.3), that the restriction \( f|_Y \) is null-homotopic. Hence \( L_0 \neq \emptyset \) and \( L_1 \neq \emptyset \).

Let \( U \) be a regular neighborhood of \( \tilde{X}_0 \cup L_0 \) such that \( \partial U \) does not intersect \( L^1 \cup L_c \) and transversely intersects every arc of \( L_0 \) at a unique point.

Let also \( V \) be the union of closures of those connected components \( D \) of \( \tilde{M} \setminus (\tilde{X} \cup \gamma) \) for which \( \overline{D} \cap \tilde{X} = \emptyset \).

Denote \( U' = U \cup V, \tilde{\gamma} = \partial U', \) and \( \gamma = p(\tilde{\gamma}). \) Then

\[
\#(\gamma \cap Y) < \#(X \cap Y) = n.
\]

5.0.8. **Claim.** \( \gamma \) is \((f, x)\)-regular.

Then by Lemma 2.4.2 we can construct a Morse function \( h : M \to S^1 \) such that \( h^{-1}(0) = X, h^{-1}(1/2) = \gamma, \) \( h = f \) near \( X, \) and \( h \sim f. \) It will follow from (5.5) by the induction on \( n \) that \( h \sim g. \)

**Proof of Claim 5.0.8.** We have to show that for every connected component \( C \) of \( M \setminus (X \cup \gamma) \) the following conditions hold true:

\[
C \cap \gamma \neq \emptyset, \quad C \cap X \neq \emptyset.
\]

Denote \( D = p^{-1}(C). \) Then (5.6) are equivalent to the following ones:

\[
\tilde{f}(\overline{D}) \notin [0, 1], \quad \overline{D} \cap \tilde{\gamma} \neq \emptyset, \quad \overline{D} \cap \tilde{X} \neq \emptyset.
\]

Since \( \partial U \) separates \( \tilde{X}_0 \) and \( \tilde{X}_1 \) in \( \tilde{M} \), we get \( \tilde{f}(\overline{D}) \notin [0, 1]. \)

Moreover, if \( \overline{D} \cap \tilde{\gamma} = \emptyset \), then \( D \) is in fact a connected component of \( \tilde{M} \). But \( \tilde{f}(\overline{D}) \notin [0, 1] \) implies \( f(\overline{C}) \notin S^1 \), whence \( C \) is unessential with respect to \((f, x)\) which contradicts to the assumption.

Suppose that \( \overline{D} \cap \tilde{X} = \emptyset \). Then \( D \) is a connected component of \( \tilde{M} \setminus \gamma \).

Therefore, \( \overline{D} \cap U \subset \partial U \), whence \( D \subset V. \) But every connected component of \( V \) evidently intersect \( \tilde{X}_0 \), thus \( D \cap \tilde{X}_0 \neq \emptyset \) which contradicts to the assumption. \( \square \)

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