Majorization and Rényi Entropy Inequalities via Sperner Theory

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Abstract

A natural link between the notions of majorization and strongly Sperner posets is elucidated. It is then used to obtain a variety of consequences, including new Rényi entropy inequalities for sums of independent, integer-valued random variables.

1 Introduction

It was observed by Erdős [3] in 1945 that the lemma of Littlewood and Offord [4] on small ball probabilities of weighted sums of Bernoulli random variables actually follows from Sperner’s theorem [5] on the maximal size of antichains in the Boolean lattice. Subsequently Stanley [6] and Proctor [7] used similar ideas to attack more difficult problems; a very nice review of the key ideas can be found in [8]. The goal of this paper is to further develop the core idea of relating properties of posets to the “distributional spread” of weighted sums of independent, integer-valued random variables. We do this in two steps. First, we elucidate a natural link, which does not seem to have been explicitly observed in the literature, between the strong Sperner property of posets and its behavior for product posets on the one hand, and majorization inequalities on the other. Second, we follow a classical approach, similar to that used in our earlier papers [9, 1, 10], to demonstrate new Rényi entropy inequalities for sums of independent random variables using the majorization inequalities. The entropy inequalities are of interest in information theory and probability, and were our original motivation for this work— they are discussed at length in Section 2.

In order to state our main results, we need to develop some terminology. For a non-negative function $f : \mathbb{Z} \to \mathbb{R}_+$ over the integers, the support $\text{Supp}(f)$ is defined by $\{x \in \mathbb{Z} : f(x) > 0\}$. We identify sets with their indicator functions; thus, for example, $f = 0.4\{0, 3\} + 0.2\{2\}$ means $f(0) = f(3) = 0.4$, $f(2) = 0.2$, and $f(x) = 0$ for $x \in \mathbb{Z} \setminus \{0, 2, 3\}$.

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Definition 1.1. Suppose $f : \mathbb{Z} \to \mathbb{R}_+$ is finitely supported, with $|\text{Supp}(f)| = n + 1$. Then we may write $\text{Supp}(f) = \{x_0, \cdots, x_n\}$ with $x_0 < \cdots < x_n$, and we may represent $f$ in the form

$$f = \sum_{r=0}^{n} a_r \{x_r\},$$

where $a_i > 0$ for each $i \in \{0, \ldots, n\}$. Given the non-negative function $f$, we define $f^\#$ by

$$f^\#: = \sum_{r=0}^{n} a_r \{r\}.$$

Thus, $f^\#$ is supported on $\{0, \cdots, n\}$ and it takes the same functional values as $f$. If we consider the graph, we may think of $f^\#$ as a “squeezed rearrangement” of $f$, where we preserve the order of the function values but eliminate gaps in the support.

Definition 1.2. We say $f$ is $\#$-log-concave if $f^\#$ is log-concave, i.e., $f^\#(i)^2 \geq f^\#(i-1)f^\#(i+1)$ for any $i \in \mathbb{Z}$. We say that a random variable $X$ taking values in the integers is $\#$-log-concave if its probability mass function is $\#$-log-concave. Given a random variable $X$ with probability mass function $f$, we write $X^\#$ for a random variable with probability mass function $f^\#$.

In the terminology of Definition 1.1, since $a_r = 0$ for $r \in \mathbb{Z} \setminus \{0, \cdots, n\}$, $f$ is $\#$-log-concave if and only if $a_r^2 \geq a_{r-1}a_{r+1}$.

We also need the classical notion of majorization. We use $f^{[i]}$ to denote the $i$-th largest value of $f$, allowing for the possibility of multiple ties. For example, $f^{[i]} = f^{[i+1]}$ when $i$-th large value appears at two different arguments.

Definition 1.3. Consider two finitely supported functions $f$ and $g$ from $\mathbb{Z}$ to $\mathbb{R}_+$, and assume $|\text{Supp}(f)| = |\text{Supp}(g)| = n + 1$. We say $f$ is majorized by $g$ (and write $f < g$) if

$$\sum_{i=1}^{k} f^{[i]} \leq \sum_{i=1}^{k} g^{[i]} \quad \text{for all } k = 1, \cdots, n, \quad (1.1)$$

and

$$\sum_{i=1}^{n+1} f^{[i]} = \sum_{i=1}^{n+1} g^{[i]} \quad (1.2)$$

For random variables $X$ and $Y$ with probability mass functions $f$ and $g$ respectively, we write $X < Y$ if $f < g$.

Our first main theorem is a majorization inequality for convolutions that holds under a log-concavity condition. Recall that, given independent random variables $X, Y$ with probability mass functions $f, g$, the sum $X + Y$ has the probability mass function $f * g$, where $*$ denotes convolution, i.e., $f * g(k) = \sum_{i \in \mathbb{Z}} f(i)g(k-i)$ for each $k \in \mathbb{Z}$.

Theorem 1.4. Let $N$ be a finite number. If $X_1, \cdots, X_N$ are independent and $\#$-log-concave over $\mathbb{Z}$, then

$$X_1 + \cdots + X_N < X_1^\# + \cdots + X_N^\#. \quad (1.3)$$
The proof of Theorem 1.4 is based on the strong Sperner property of the product of weighted chain posets; Section 3 summarizes the necessary background on poset theory, and the proof of the theorem is detailed in Section 4. We mention in passing that although we focus on $\mathbb{Z}$-valued random variables with finite support in this paper, Theorem 1.4 has an extension to the case where the random variables have infinite support using a similar procedure to that in [1, 11, 10].

Let us discuss a pleasing application of Theorem 1.4 to proving a key ingredient in rearrangement inequalities on the integers proved by Gabriel [12] (generalizing a result of Hardy and Littlewood [13]) and popularized in the book by Hardy, Littlewood and Pólya [14]. For a finite set $A$ in $\mathbb{Z}$, note that $A^\# = \{0, 1, \ldots , |A| - 1\}$; here, as before, we identify the sets $A$ and $A^\#$ with their indicator functions.

**Corollary 1.5.** If $A_1, A_2, \cdots , A_N$ are finite sets (or indicator functions) in $\mathbb{Z}$, then

$$A_1 \star A_2 \cdots \star A_N < A_1^\# \star A_2^\# \cdots \star A_N^\#.$$ 

To see how Corollary 1.5 follows from Theorem 1.4, suppose random variables $X_1, X_2, \cdots , X_N$ are uniformly distributed on the sets $A_1, A_2, \cdots , A_N$ respectively. By applying Theorem 1.4 and re-normalizing the probability into the total number of points, the desired result follows.

We note that a version of Gabriel’s inequality was, in fact, extended to the prime cyclic groups $\mathbb{Z}/p\mathbb{Z}$ by Lev [15]. In our companion paper [10], we develop a further generalization of such rearrangement inequalities (see [10, Theorem 6.2]) in the prime cyclic groups, with the crucial in our proofs being the leveraging of Lev’s set majorization lemma (see [15, Theorem 1] for the full statement). The results of Gabriel [12], Lev [15] and the authors [10] for general non-negative functions rather than indicator functions of sets require additional assumptions because one has to take into account the “shape” of the convolved functions. It is a nice feature of the statement and proof above that it does not require such assumptions.

Our second theorem is related to a beautiful and well known result of Sárközy and Szemerédi [16] related to what they called the Erdős-Moser problem (although the paper of Katona [17], which they cite a pre-publication version of, does not discuss it in the published paper, and the problem posed by Erdős in 1947 in the American Mathematical Monthly with solutions given by Moser [18] as well as several others, which is cited by several later papers on the Sárközy-Szemerédi result, seems only tangentially related). In any case, the “Erdős-Moser problem” is the following: Given $N$ i.i.d. Bernoulli random variables $Y_1, \cdots , Y_N$, estimate the maximal probability of independent weighted sums over distinct weights:

$$\sup_{k \in \mathbb{Z}} \sup_{0 < a_1, \ldots , a_N} \mathbb{P}(a_1 Y_1 + a_2 Y_2 + \cdots + a_N Y_N = k).$$

Sárközy and Szemerédi [16] asserted that Erdős and Moser had shown that the maximal probability is of order $\left(\frac{\log N}{N}\right)^{3/2}$ and had conjectured that the logarithmic term could be removed; they proved this conjecture, thus showing that the maximal probability is of order $N^{-3/2}$. However, identification of an extremal set of weights remained open until Stanley [6] used tools from algebraic geometry to show that $(a_1, a_2, \ldots , a_N) = (1, 2, \ldots , N)$ is extremal. A more elementary algebraic proof was soon after given by Proctor [7]. Much more recently, Nguyen [19] not only observed that the maximal probability is in fact $\left[\sqrt{24/\pi} + o(1)\right] N^{-3/2}$, but he also showed a stability result around the extremal configuration.

With this background, we are ready to state our second main result.
Theorem 1.6. Assume that $0 < a_1 < a_2 < \cdots < a_N$. If $Y_i$’s are independent random variables following Binomial $(m_i, \frac{1}{2})$ for $1 \leq m_N \leq m_{N-1} \leq \cdots \leq m_1$, then
\[ a_1Y_1 + a_2Y_2 + \cdots + a_NY_N < Y_1 + 2Y_2 + \cdots + NY_N. \] (1.4)

The proof of Theorem 1.6 is based on the strong Sperner property of some products of posets, and is detailed in Section 5.

As a direct application of Theorem 1.6, we can go beyond the Bernoulli assumption in the prior studies of [16, 6] discussed above, and identify the extremal weights for the wider class of binomially distributed random variables.

Corollary 1.7. Let $0 < a_1 \neq \cdots \neq a_N$. Let $Y_i$ be i.i.d. random variables with the Binomial $(m, \frac{1}{2})$ distribution. Then
\[ P(a_1Y_1 + a_2Y_2 + \cdots + a_NY_N = k) \leq P \left( Y_1 + 2Y_2 + \cdots + NY_N = \frac{mN(N + 1)}{4} \right). \]

To see how Corollary 1.7 follows from Theorem 1.6, observe that since $Y_i$ are i.i.d. in the former, we may assume that the $a_i$ are ordered. Then the conclusion follows from Theorem 1.6, and the optimal case is achieved at the midpoint of the range as the distribution of $Y_1 + 2Y_2 + \cdots + NY_N$ is symmetric and unimodal (this latter fact is confirmed by Lemma 5.3, which we discuss later). Of course, Theorem 1.6 can be applied without an identically distributed assumption, but we make this assumption in Corollary 1.7 for simplicity of statement.

This paper is organized as follows. Our original motivation for pursuing this work came from a search for Rényi entropy power inequalities for integer-valued random variables, which is a problem of significant interest in information theory. We explain this motivation and describe how our main results may be applied to obtain new entropy power inequalities in Section 2. The rest of the paper focuses on the proofs of our main results– Section 3 recalls the necessary background on Sperner theory, and the proofs of the two main theorems are detailed in Sections 4 and 5.

2 Applications to Rényi Entropy Inequalities

2.1 Background on entropy power inequality

We first define the one-parameter family of Rényi entropies for a probability mass function on the integers (these can be defined on more general spaces by using a reference measure other than counting measure, but we do not need the more general notion here).

Definition 2.1. Let $X$ be an integer-valued random variable with probability mass function $f$. The Rényi entropy of order $\alpha \in (0, 1) \cup (1, +\infty)$ is defined by
\[ H_\alpha(X) = \frac{1}{1 - \alpha} \log \left( \sum_{i \in \mathbb{Z}} f(i)^\alpha \right). \]

For limiting cases of $\alpha$, define
\[ H_0(X) = \log |\text{supp}(f)|, \]
\[ H_1(X) = \sum_{i \in \mathbb{Z}} -f(i) \log f(i), \]
\[ H_\infty(X) = -\log \sup_{i \in \mathbb{Z}} f(i). \]
The three special cases are defined in a manner consistent with taking limits of $H_\alpha(X)$ for $\alpha \in (0,1) \cup (1, +\infty)$. Thus the Rényi entropy of order $\alpha \in [0,\infty]$ is well-defined. In particular, $H_1(\cdot)$ is simply the Shannon entropy $H(\cdot)$.

Entropy inequalities (even just for Shannon entropy) are powerful tools that have found use in virtually all parts of mathematics. For example, just within discrete mathematics, they have been used to obtain bounds for enumeration problems (see, e.g., [20, 21, 22]), to prove sunset inequalities in additive combinatorics (see, e.g., [23, 24, 25, 26, 27]), and to study probabilistic models of discrete phenomena (e.g., independent sets [28], card shuffles [29]), colorings [30]). Among various entropy inequalities, the so-called “entropy power inequality” in Euclidean spaces has been very successfully applied to prove coding theorems or to determine channel capacities for communication problems involving Gaussian noise in Information Theory (see, e.g., [31, 32]). The entropy power inequality also plays an important role in Probability Theory (see, e.g., [33]) and Convex Geometry (see, e.g., [34, 35, 36]).

The entropy power inequality can be formulated in two different ways. Firstly, the original formulation, which was suggested by Shannon [37] and proved by Stam [38], is the following. For independent random variables $X$ and $Y$ in $\mathbb{R}^d$,

$$e^{\frac{d}{2} h(X+Y)} \geq e^{\frac{d}{2} h(X)} + e^{\frac{d}{2} h(Y)},$$

where $h(X)$ represents the differential entropy of $X$. Formally, if $X$ has a density function $f$ in $\mathbb{R}^d$, then $h(X) = -\int_{\mathbb{R}^d} f(x) \log f(x) dx$. The inequality shows the superadditivity of the “entropy power” with respect to the sum of two independent random variables. Secondly, an equivalent sharp formulation (see, e.g., [9] for discussion) states that

$$h(X + Y) \geq h(Z_X^* + Z_Y^*),$$

where $Z_X^*$ and $Z_Y^*$ are two independent Gaussian distributions with $h(X) = h(Z_X^*)$ and $h(Y) = h(Z_Y^*)$.

The entropy power inequality stated above focuses on the continuous setting of $\mathbb{R}^d$. It has been extensively studied and many refinements exist (see, e.g., [39, 40, 41]). On the other hand, we only have a limited understanding of analogues of the entropy power inequality on discrete domains such as the integers or cyclic groups.

One of the main difficulties is that useful analytic tools in the continuous domain cannot be naturally translated into the discrete domain. For example, one can derive the entropy power inequality in $\mathbb{R}^d$ from the sharp form of Young’s inequality for convolution developed by Beckner [42], as observed by Lieb [43], or using inequalities for Fisher information, which is defined using derivatives of the probability density function, as done by Stam [38]. Unfortunately, a non-trivial sharp Young’s inequality cannot be achieved in the discrete setting. It is also not obvious what the right definition of Fisher information should be for the discrete setting because discrete derivatives do not satisfy the chain rule (see, e.g., [44, 45, 46] for possible definitions of discrete Fisher informations). Owing to the difficulty of fitting such approaches into a discrete setting, a general and sharp analogue of the entropy power inequality on the integers has not yet been established.

Nevertheless, it is a natural and interesting question to find a fully satisfactory entropy power inequality on the integers—earlier attempts in this direction include [47, 48, 1, 2]. In our companion paper [10], we established a lower bound on the entropy of sums in prime cyclic groups (including the integers) based on rearrangement inequalities and functional ordering by majorization where the rearrangement of a function $f$ is achieved by shuffling (permuting) the domain of $f$. While details may be found in [10], the goal of these rearrangement
inequalities is to identify optimal permutations that maximize or minimize a sum of pairwise products.

In this paper, we focus on the integer domain or the integer lattice domain. We continue to leverage the idea of majorization used in our companion paper [10]. However, instead of establishing rearrangement inequalities, we take a different path to establish the lower bound inequality of the entropy of sums in integers. Our approach is to establish and utilize the similarity between the strong Sperner property of posets (the origin of this notion lies, of course, in Sperner’s theorem [5], but the way we use this notion is inspired by Erdős [3] as described in Section 1) and functional ordering by majorization.

2.2 Two entropy inequalities

A key application of Theorem 1.4 (and also Theorem 1.6) lies in establishing a lower bound on the Rényi entropy of convolutions. As the main tool in translating majorization results to entropy inequalities, we use the following basic lemma.

**Lemma 2.2.** [49, Proposition 3-C.1] Assume that \( f \) and \( g \) are finitely supported non-negative functions in \( \mathbb{Z} \) and \( f \prec g \). For any convex function \( \Phi : \mathbb{R} \to \mathbb{R} \),

\[
\sum_{i \in \mathbb{Z}} \Phi (f(i)) \leq \sum_{i \in \mathbb{Z}} \Phi (g(i)).
\]

We note that by choosing a convex function \( \Phi(x) = -x^\alpha \) for \( \alpha \in (0, 1) \), \( \Phi(x) = x \log x \) for \( \alpha = 1 \), and \( \Phi(x) = x^\alpha \) for \( \alpha \in (1, +\infty) \), and by taking limits when \( \alpha \in \{0, \infty\} \), inequalities for Rényi entropies of all orders follow from Lemma 2.2 whenever we have a probability mass function majorized by another. In particular, our two main theorems combined with Lemma 2.2 yield the following propositions.

**Proposition 2.3.** If \( X_1, \cdots, X_N \) are independent and \#-log-concave over \( \mathbb{Z} \), then

\[
H_\alpha (X_1 + \cdots + X_N) \geq H_\alpha \left(X_1^\# + \cdots + X_N^\#\right),
\]

for \( \alpha \in [0, \infty) \).

Rényi entropy inequalities such as this (and others of similar form in our companion paper [10]) have already begun finding utility (see, e.g., [50, 51]).

**Proposition 2.4.** Let \( 0 < a_1 < \cdots < a_N \). If \( Y_i \)'s are independent random variables following Binomial \((m_i, 1/2)\) for \( 1 \leq m_N \leq m_{N-1} \leq \cdots \leq m_1 \), then

\[
H_\alpha (a_1Y_1 + a_2Y_2 + \cdots + a_NY_N) \geq H_\alpha (Y_1 + 2Y_2 + \cdots + NY_N),
\]

for \( \alpha \in [0, \infty) \).

Nguyen [19] observed that the optimal solution of the Erdős-Moser problem (for Bernoulli \( Y_i \)) minimizes the variance of \( a_1Y_1 + a_2Y_2 + \cdots + a_NY_N \) among all choices of distinct positive weights, i.e., for \( 0 < a_1 \neq \cdots \neq a_N \),

\[
\text{Var} (Y_1 + 2Y_2 + \cdots + NY_N) \leq \text{Var} (a_1Y_1 + a_2Y_2 + \cdots + a_NY_N).
\]

Proposition 2.4 implies that the optimal solution of the Erdős-Moser problem also minimizes the Rényi entropy for any order \( \alpha \in [0, +\infty) \) (and even for the more general binomial setting).
2.3 Inequalities for uniform distributions on subsets of \( \mathbb{Z}^d \)

In [2], we proved a discrete entropy power inequality for uniform distributions over finite subsets of the integers \( \mathbb{Z} \). In the following lemma, we extend [2, Theorem II.2] from the \( \alpha = 1 \) case to any Rényi entropy of order \( \alpha \geq 1 \).

**Lemma 2.5.** If \( X \) and \( Y \) are independent and uniformly distributed over finite sets \( A \subset \mathbb{Z} \) and \( B \subset \mathbb{Z} \) respectively,

\[
N_\alpha(X + Y) + 1 \geq N_\alpha(X) + N_\alpha(Y),
\]

(2.3)

where \( N_\alpha(X) = e^{(1+\alpha)H_\alpha(X)} \) for \( \alpha \geq 1 \).

**Proof.** Since the \( \alpha = 1 \) case is proved in [2], we assume that \( \alpha > 1 \). Since any uniform distribution over a finite set is \(-\log\)-concave, Theorem 2.3 implies that

\[
H_\alpha(X + Y) \geq H_\alpha(X^\# + Y^\#).
\]

Since \( N_\alpha(X) = N_\alpha(X^\#) \) and \( N_\alpha(Y) = N_\alpha(Y^\#) \) hold trivially, it suffices for proving the inequality (2.3) to only consider the case where \( A \) and \( B \) are sets of consecutive integers. Indeed, if we proved this special case, we would have

\[
N_\alpha(X + Y) + 1 \geq N_\alpha(X^\# + Y^\#) + 1 \geq N_\alpha(X^\#) + N_\alpha(Y^\#) = N_\alpha(X) + N_\alpha(Y),
\]

which is the desired statement.

What remains is to prove (2.3) for uniform distributions on finite sets of consecutive integers. Let \( |A| = n \) and \( |B| = m \). Since the roles of \( X \) and \( Y \) are symmetric, we may assume that \( n \geq m \). While \( H_\alpha(X) = \log n \) and \( H_\alpha(Y) = \log m \) due to uniformity, a direct calculation easily shows that

\[
H_\alpha(X + Y) = \frac{1}{1 - \alpha} \log \left[ \sum_{i=1}^{m-1} \frac{i^\alpha}{m^\alpha n^\alpha} + (n - m + 1) \frac{1}{n^\alpha} \right].
\]

(2.4)

First, observe that if \( m = 1 \), then \( H_\alpha(X + Y) = H_\alpha(X) \) and \( H_\alpha(Y) = 0 \). Then \( N_\alpha(X + Y) + 1 = N_\alpha(X) + 1 = N_\alpha(X) + N_\alpha(Y) \), and hence the inequality (2.3) is sharp. Next, consider the expression inside the logarithm in the formula (2.4):

\[
\sum_{i=1}^{m-1} \frac{i^\alpha}{m^\alpha n^\alpha} + (n - m + 1) \frac{1}{n^\alpha} = n^{-\alpha} m^{-\alpha} \sum_{i=1}^{m-1} i^\alpha + n^{1-\alpha} - mn^{-\alpha} + n^{-\alpha}
\]

\[
\leq n^{-\alpha} m^{-\alpha} \int_1^m x^\alpha dx + n^{1-\alpha} - mn^{-\alpha} + n^{-\alpha}
\]

\[
= n^{1-\alpha} \left[ 1 + \frac{1}{n} - \frac{\alpha}{1+\alpha} \frac{m}{n} - \frac{1}{1+\alpha} \frac{m^{-\alpha}}{n} \right].
\]

Plugging this bound into (2.4), setting \( k = m/n \in [0, 1] \) and writing

\[
\xi(k, n) = 1 + \frac{1}{n} - \frac{\alpha}{1+\alpha} k - \frac{1}{1+\alpha} k^{-\alpha} n^{-(1+\alpha)},
\]

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we obtain the following lower bound for the Rényi entropy of $X + Y$:

$$H_\alpha(X + Y) \geq \frac{1}{1 - \alpha} \left[ \log n^{1 - \alpha} + \log \xi(k, n) \right].$$

Then

$$N_\alpha(X + Y) = e^{(1 + \alpha) H_\alpha(X + Y)} \geq e^{(1 + \alpha) \log n e^{\frac{1}{1 + \alpha} \log \xi(k, n)}} =: \nu(k, n).$$

Since the $m = 1$ case is already proved, if the following inequality is true, we are done.

$$\nu(k, n) = e^{(1 + \alpha) \log n e^{\frac{1}{1 + \alpha} \log \xi(k, n)}} \geq e^{(1 + \alpha) \log n e^{(1 + \alpha) \log kn}} = N_\alpha(X) + N_\alpha(Y).$$

By rearranging terms, the above inequality is equivalent to

$$\xi(k, n) \leq (1 + k^{1 + \alpha}) \frac{1}{1 + \alpha}.$$

When $k = \frac{2}{n}$, we can directly show the following inequality is true by elementary calculation:

$$\xi \left( \frac{2}{n}, n \right) = 1 + \frac{1 - \alpha}{1 + \alpha} \frac{1}{n} - \frac{2^\alpha}{1 + \alpha} \frac{1}{n} \leq \left( 1 + 2^{1 + \alpha} n^{-(1 + \alpha)} \right) \frac{1}{1 + \alpha}.$$

Furthermore, it is easy to show that for $k \in [0, 1]$ and $\alpha > 1$

$$(1 + k^{1 + \alpha}) \frac{1}{1 + \alpha} \geq 1 - \left( 1 - 2^{1 + \alpha} \frac{1}{1 + \alpha} \right) k.$$

Hence it suffices to show

$$\xi(k, n) \leq 1 - \left( 1 - 2^{1 + \alpha} \frac{1}{1 + \alpha} \right) k$$

for all $n \geq 2$ and $k \geq \frac{2}{n}$. Let $\phi(k, n) := 1 - \left( 1 - 2^{1 + \alpha} \frac{1}{1 + \alpha} \right) k - \xi(k, n)$. For a fixed $n \geq 2$,

$$\frac{\partial \phi}{\partial k} = -1 + 2^{1 + \alpha} + \frac{\alpha}{1 + \alpha} \left( 1 - k^{-(1 + \alpha)} n^{-(1 + \alpha)} \right).$$

Then $\frac{\partial \phi}{\partial k} = 0$ at

$$k^* = \frac{1}{n} \alpha^{\frac{1 + \alpha}{1 + \alpha}} \left[ -1 + 2^{1 + \alpha} + 2^{1 + \alpha} \alpha \right]^{-\frac{1}{1 + \alpha}},$$

and $\frac{\partial \phi}{\partial k} > 0$ for $k > k^*$ and $\frac{\partial \phi}{\partial k} < 0$ for $k < k^*$. Finally, by elementary calculation, we can easily show that

$$\frac{1}{n} \leq k^* < \frac{2}{n}.$$

Thus, $\phi(k, n)$ is minimized at $k = \frac{3}{n}$ for $k \geq \frac{3}{n}$. By elementary calculation, we can also confirm that $\phi \left( \frac{2}{n}, n \right) < 0$ for some $n$; this is why $\phi(k, n) \geq 0$ is only proved when $k \geq \frac{3}{n}$ rather than $k \geq \frac{2}{n}$.
We remark that +1 term on the left side of inequality (2.3) is only necessary for one-point mass distributions. In other words, if $X$ and $Y$ are independent and uniformly distributed over finite sets of cardinality at least 2, then we in fact have

$$
\mathcal{N}_\alpha(X + Y) \geq \mathcal{N}_\alpha(X) + \mathcal{N}_\alpha(Y).
$$

(2.5)

However, we highlight the formulation with +1 both because of the similarity with the Cauchy-Davenport Theorem [52], and because the discrete entropy power inequality over the integers in Lemma 2.5 can be extended to the integer lattice $\mathbb{Z}^d$.

**Theorem 2.6.** If $X$ and $Y$ are uniform distributions over finite sets $A$ and $B$ in $\mathbb{Z}^d$,

$$
\mathcal{N}_\alpha(X + Y) + 1 \geq \mathcal{N}_\alpha(X) + \mathcal{N}_\alpha(Y),
$$

where $\mathcal{N}_\alpha(X) = e^{(1+\alpha)H_\alpha(X)}$ for $\alpha \geq 1$.

**Proof.** Consider a point $z = (z_1, \cdots, z_d)$ in $\mathbb{Z}^d$ where $z_i \geq 0$ for each $i$. We regard $z$ as a $q$-ary representation of an integer value, where $q$ is large and chosen later. In other words, the point $z \in \mathbb{Z}^d$ can be mapped to a unique integer value in $\mathbb{Z}$:

$$
z = (z_1, \cdots, z_d) \mapsto z_1 q^{d-1} + z_2 q^{d-2} + \cdots + z_d \in \mathbb{Z}.
$$

(2.6)

For the set $A$ in $\mathbb{Z}^d$, without loss of generality, we can shift $A$ so that each point contains only non-negative components. Let $A'$ be the set in $\mathbb{Z}$ equivalent to $A$ via the $q$-ary representation (2.6). Similarly we can find the set $B'$ in $\mathbb{Z}$ equivalent to $B$ in $\mathbb{Z}^d$. We choose $q$ large enough so that $A \ast B$ in $\mathbb{Z}^d$ maps to $A' \ast B'$ in $\mathbb{Z}$ via the $q$-ary representation.

Let $X'$ and $Y'$ be uniform distributions on $A'$ and $B'$ in $\mathbb{Z}$, respectively. This implies

$$
H_\alpha(X + Y) = H_\alpha(X' + Y') \geq H_\alpha(X' \# + Y' \#).
$$

Then the conclusion follows by applying Lemma 2.5 and from the fact that $H_\alpha(X) = H_\alpha(X')$ and $H_\alpha(Y) = H_\alpha(Y')$.

Theorem 2.6 uses the exponent $c = 1 + \alpha$; Rényi entropy power inequalities with the same exponent in $\mathbb{R}$ were recently explored by Bobkov and Marsiglietti [53] (although it was shown soon after by Li [54] that this exponent can be improved). In fact, these authors proved similar inequalities in $\mathbb{R}^d$, with the exponent $\frac{1+\alpha}{d}$, mimicking the $2/d$ exponent in the original Shannon-Stam entropy power inequality.

The inequality we derived in Theorem 2.6 is independent of the dimensional factor $d$, and one might wonder whether a entropy power inequality in the integer lattice $\mathbb{Z}^d$ that respects the dimension exists. Even for the subclass of uniform distributions and for $\alpha = 1$, however, one can easily construct counterexamples showing that an exponent of $2/d$ fails in general in $\mathbb{Z}^d$. We remark that in order to develop a discrete Brunn-Minkowski inequality in the integer lattice, Gardner and Gronchi [55] imposed a natural and appropriate dimensional assumption, the main point of which is that at least two points should be assigned to each axis direction. However, the dimensional assumption from [55] is still not sufficient to obtain an improvement of Theorem 2.6 with exponent $\frac{1+\alpha}{d}$ (as can be checked by counterexamples). Hence we leave the discovery of appropriate dimensional entropy inequalities in the integer lattice as an open question for future works.
3 Background on Sperner Theory

In this section, we summarize the basic elements of Sperner Theory as needed for our proofs. A comprehensive summary can be found in books by Stanley [56] and Engel [57].

3.1 Partially ordered set (poset)

A set $S$ with a binary relation $\preceq$ is said to be partially ordered if the relation $\preceq$ satisfies the reflexive, anti-symmetric, and transitive properties, i.e., for any $a, b, c \in S$,

- $a \preceq a$ (reflexive),
- if $a \preceq b$ and $b \preceq a$, then $a = b$ (antisymmetric),
- if $a \preceq b$ and $b \preceq c$, then $a \preceq c$ (transitive).

We emphasize that we use the symbol $<$ to represent majorization, and this has no relation to the partial order $\preceq$. If $S$ is partially ordered, we call $S$ a partially ordered set, or a poset. For $a, b \in S$, $a$ and $b$ are comparable if $a \preceq b$ or $b \preceq a$. Otherwise, $a$ and $b$ are incomparable.

A chain poset is a poset in which any two elements are comparable. A subset $C$ of $S$ is called a chain of $S$ if $C$ is a chain poset as a sub-poset of $S$. We define the length of a chain $C$ to be the number of elements in $C$.

A subset $A$ of $S$ is called an antichain if any two distinct elements of $A$ are incomparable. A subset $K$ of $S$ is called a $k$-family of $S$ if it is a union of at most $k$ antichains. We say a poset $S$ is weighted if each element has a positive weight. The weight function $w : S \to \mathbb{R}_+$ defines the weight of each element in $S$. We use a triple $(S, w, \preceq)$ to represent the weighted poset, but we sometimes omit to mention the weight function $w$ explicitly when we describe a weighted poset. If the poset has no weight function (or unweighted), we implicitly assume that each weight of an element is 1.

A chain $C$ of $S$ is called maximal if there is no larger chain $C'$ such that $C \subseteq C'$. An element $s$ in $S$ is called minimal if $t \preceq s$ implies $s = t$ in $S$. The element $s$ of $S$ is said to cover the element $t$ of $S$ if $t \preceq s$ and if $t \preceq s' \preceq s$ implies $s = s'$ when $s' \neq t$. If every maximal chain of the poset $S$ has length $n + 1$, we call $S$ a graded poset with rank $n$. In such a case, we can define a unique rank function $\rho : S \to \{0, 1, \cdots, n\}$ of $S$ such that $\rho(a) = 0$ if $a$ is a minimal element of $S$, and $\rho(b) = \rho(a) + 1$ if $b$ covers $a$. Then the rank of $a$ is assigned to be $\rho(a)$. Given a weighted and ranked poset $(S, \preceq, w)$, the sum of all weights at the same rank $r \in \{0, 1, \cdots, n\}$ is called the weighted Whitney number of the rank $r$. Similarly, if the poset is unweighted, the Whitney number of the rank $r$ is the total number of elements at the rank $i$. We say that weighted Whitney numbers are log-concave if the sequence of weighted Whitney numbers is log-concave in an increasing order of the rank. We say that weighted Whitney numbers are rank-symmetric if the sequence of weighted Whitney numbers is symmetric in an increasing order of the rank. Similarly, we say that weighted Whitney numbers are rank-unimodal if the sequence of weighted Whitney numbers is unimodal in an increasing order of the rank.

The weighted and ranked poset $(S, \preceq, w)$ has $k$-Sperner property if the maximum total weight among all $k$-families in $S$ equals the largest sum of $k$ weighted Whitney numbers. The weighted and ranked poset $(S, \preceq, w)$ is strongly Sperner (or has the strong Sperner property) if it is $k$-Sperner for all $k = 1, 2, \cdots$.

The product of the posets $S$ and $T$ is defined to be the Cartesian product $S \times T$, equipped with the partial order defined by the requirement that $(s, t) \preceq (s', t')$ in $S \times T$ if and only if...
s \preceq s' in \mathcal{S} and t \preceq t' in \mathcal{T}. If \mathcal{S} and \mathcal{T} are weighted with weight functions w_\mathcal{S} and w_\mathcal{T}, then the weight function w_{\mathcal{S} \times \mathcal{T}} of \mathcal{S} \times \mathcal{T} is defined to be w_{\mathcal{S} \times \mathcal{T}}(s, t) = w_\mathcal{S}(s)w_\mathcal{T}(t).

### 3.2 Normalized matching property

Consider a ranked and weighted poset \((\mathcal{S}, w, \preceq)\) with the rank function \(\rho\). For any subset \(A\) of \(\mathcal{S}\), we define the upper shade of \(A\), denoted \(\nabla(A)\), as the set of all elements covering \(A\). If \(a' \in \nabla(A)\), then there exists an element \(a \in A\) such that \(a \preceq a'\) and \(\rho(a') = \rho(a) + 1\).

Let \(N_r\) be the collection of all elements at rank \(r\). A ranked and weighted poset \((\mathcal{S}, w, \preceq)\) is called normal if for any antichain \(A\) subject to a subset of elements of rank \(r\), the weight sum ratio of \(A\) with respect to the weighted Whitney number of rank \(r\) is less than or equal to the weighted sum ratio of the shade of \(A\) at rank \(r + 1\) with respect to the weighted Whitney number of rank \(r + 1\), i.e.,

\[
\frac{w(A)}{w(N_r)} \leq \frac{w(\nabla(A))}{w(N_{r+1})}
\]  

(3.1)

where \(A \subseteq N_i\) is an antichain, \(w(A)\) is the sum of all weights of elements in \(A\), and \(w(N_r)\) is the weighted Whitney number of the rank \(r\). Hsieh and Kleitman [58] proved that the normalized matching property is preserved under the product of normal posets if it assumes log-concave weighted Whitney numbers.

**Proposition 3.1** (See [57, Theorem 4.6.2] or [58]). A product of two normal posets with log-concave weighted Whitney numbers is again a normal poset with log-concave weighted Whitney numbers.

**Proposition 3.2** (See [57, Corollary 4.5.3]). A normal poset is strongly Sperner.

Applying Proposition 3.1 and Proposition 3.2 to chain posets, we have the following Corollary.

**Corollary 3.3.** For each \(i \in \{1, \cdots, N\}\), assume that \(S(m_i)\) has log-concave weighted Whitney numbers. Then the product of chain posets \(S(m_1, \cdots, m_N)\) is strongly Sperner with log-concave weighted Whitney numbers.

**Proof.** Since the weight sum ratio in (3.1) of any antichain for a chain poset is always 1, any weighted chain is normal. Thus, the conclusion follows from Proposition 3.1 and Proposition 3.2.

For further discussion, we need to define an order isomorphism between two posets. We say that two posets \((Q, \preceq)\) and \((R, \preceq)\) are isomorphic if there exists a bijective map \(\phi : Q \to R\) such that \(q_1 \preceq q_2\) iff \(\phi(q_1) \preceq \phi(q_2)\) for \(q_1, q_2 \in Q\) and \(\phi(q_1), \phi(q_2) \in R\).

### 3.3 Strongly Sperner posets

Let \(N, n_1, \cdots, n_N\) be fixed positive integers. A product of chain posets \(S(n_1, \cdots, n_N)\) can be defined to be a collection of \(N\)-tuples of integers \((a_1, \cdots, a_N)\) such that \(0 \leq a_i \leq n_i\) for each \(i \in \{1, \cdots, N\}\). The relation \(a = (a_1, \cdots, a_N) \preceq b = (b_1, \cdots, b_N)\) iff \(a_i \leq b_i\) for each \(i \in \{1, \cdots, N\}\). Figure 1 shows an example of the product poset \(S(2, 3)\).

Next, we introduce the poset \(M(m)\), a collection of \(m\)-tuples \((a_1, \cdots, a_m)\) such that \(0 = a_1 = \cdots = a_i < a_{i+1} < \cdots < a_m \leq m\) with \(i \in \{0, \cdots, m\}\). As noted, we allow one
exceptional case \( i = 0 \). If \( i = 0 \), we mean \( a_1 > 0 \). So \( 0 < a_1 < a_2 < \cdots < a_m \leq m \). The relation \( a = (a_1, \ldots, a_n) \preceq b = (b_1, \ldots, b_m) \) iff \( a_i \leq b_i \) for all \( i \). The rank \( \rho(a) = \sum_{i=1}^{m} a_i \).

Stanley \cite{Stanley6} originally proved that \( M(n) \) is rank-symmetric, rank-unimodal, and strongly Sperner leveraging ideas from algebraic geometry. After that, Proctor \cite{Proctor7} gave a more accessible proof requiring the background of basic linear algebra. Following conventional terminology, we say that a ranked poset is Peck if the poset is rank-symmetric, rank-unimodal, and strongly Sperner.

**Lemma 3.4** (See \cite{Stanley6, Proctor7}). The poset \( M(m) \) is a Peck poset.

Proctor, Saks, and Sturtevant \cite{Proctor59} proved that the Peck property is invariant under the product of posets.

**Lemma 3.5** (See \cite{Proctor59, Theorem 3.2}). A product of Peck posets is again Peck, and hence strongly Sperner.

### 4 Proof of Theorem 1.4

We establish the link between a non-negative function and a weighted chain poset. Consider a \#-log-concave function \( f = \sum_{r=0}^{n} a_r \{ x_r \} \), where \( x_0 < \cdots < x_n \) and \( a_r^2 \geq a_{r-1}a_{r+1} \). By letting \( S_f := \text{Supp}(f) = \{ x_0, \cdots, x_n \} \) with a weight function \( f(x_r) = a_r \), \((S_f, f, \preceq)\) forms a ranked and weighted chain poset. Thus, we regard a non-negative function with finite support as a weighted chain poset:

\[
f \equiv (S_f, f, \preceq),
\]

where the relation \( \preceq \) is the same as the usual \( \leq \). Since \( f \) is \#-log-concave, weighted Whitney numbers of \( S_f \) are log-concave. Similarly \( f^\# = \sum_{r=0}^{n} a_r \{ r \} \) forms a weighted chain poset \( S_{f^\#} = \{ 0, \cdots, n \} \) with log-concave Whitney numbers. Based on the construction, \( S_f \) is isomorphic to \( S_{f^\#} \) by mapping \( \phi(x_r) = r \), so \( f(x_r) = f^\#(\phi(x_r)) \). i.e. the isomorphism map \( \phi \) can be chosen to be the rank function of \( S_f \).

Next, consider \( N \) non-negative functions \( f_1, \cdots, f_N \), all of which are \#-log-concave. Define \( F(x^{(1)}, \cdots, x^{(N)}) := f_1(x^{(1)}) \cdots f_N(x^{(N)}) \). Similarly define \( F^\#(x^{(1)}, \cdots, x^{(N)}) := \)
Figure 2: Poset of $M(5)$
$f_1^\#(x^{(1)}) \cdots f_N^\#(x^{(N)})$. As shown above, there exists an isomorphic map $\phi_i$ between $S_{f_i}$ and $S_{f_i^\#}$ for each $i = 1, \ldots, N$. Thus, we choose $\phi : S_{f_1} \times \cdots \times S_{f_N} \to S_{f_1^\#} \times \cdots \times S_{f_N^\#}$ by

$$\phi(x^{(1)}_1, \ldots, x^{(N)}_N) = (\phi_1(x^{(1)}_1), \ldots, \phi_N(x^{(N)}_N)). \quad (4.1)$$

Then $S_{f_1} \times \cdots \times S_{f_N}$ is isomorphic to $S_{f_1^\#} \times \cdots \times S_{f_N^\#}$ by $\phi$, i.e.

$$(S_{f_1} \times \cdots \times S_{f_N}, F, \preceq) \equiv (S_{f_1^\#} \times \cdots \times S_{f_N^\#}, F^\#, \preceq),$$

where $F(x^{(1)}_1, \ldots, x^{(N)}_N) = F^\#(\phi_1(x^{(1)}_1), \ldots, \phi_N(x^{(N)}_N)).$

**Lemma 4.1.** $S_{f_1^\#} \times \cdots \times S_{f_N^\#}$ forms a normal poset with log-concave weighted Whitney numbers.

**Proof.** Each $S_{f_i}^\#$ is a chain, thus it is a normal poset with log-concave weights. Corollary 3.3 confirms that the product of normal posets is again normal with log-concave weighted Whitney numbers. □

Next, we establish a link between a product of posets and a convolution of non-negative functions through an antichain. We define a level set

$$L[x] := \{ (x^{(1)}, \ldots, x^{(N)}) : x^{(1)} + \cdots + x^{(N)} = x, x^{(i)} \in S_{f_i} \text{ for } i = 1, \ldots, N \}.$$

**Lemma 4.2.** $\phi(L[x])$ forms an antichain in $S_{f_1^\#} \times \cdots \times S_{f_N^\#}$.

**Proof.** Note that $\phi$ in (4.1) is a bijective and order-preserving map. Thus, it suffices to consider elements in $L[x]$. Suppose that there exist two distinct comparable elements $x := (x^{(1)}, \ldots, x^{(N)})$ and $y := (y^{(1)}, \ldots, y^{(N)})$ such that $x, y \in L[x]$ and $x \preceq y$. This implies $x^{(i)} \leq y^{(i)}$ for each $i = 1, \ldots, N$. Since $x$ and $y$ are distinct, there exists some $j$ such that $x^{(j)} < y^{(j)}$. Hence

$$x^{(1)} + \cdots + x^{(N)} < y^{(1)} + \cdots + y^{(N)}.$$

This contradicts the fact that both $x$ and $y$ are in $L[x]$. □

Since $S_{f_1^\#} \times \cdots \times S_{f_N^\#}$ is strongly Sperner, majorization follows.

**Proposition 4.3.** If $f_1, \ldots, f_N$ are $\#$-log-concave probability mass functions,

$$f_1 \cdots \cdots f_N \prec f_1^\# \cdots \cdots f_N^\#.$$

**Proof.** Let $f_L := f_1 \cdots \cdots f_N$ and $f_R := f_1^\# \cdots \cdots f_N^\#$. Convolutions in $f_L$ can be fully represented by

$$f_L(x) = \sum_{(x^{(1)}, \ldots, x^{(N)}) \in L[x]} f_1(x^{(1)}) \cdots f_N(x^{(N)}).$$
The isomorphism $\rho$ in (4.1) and Lemma 4.2 imply that $f_L(x)$ can be regarded as a sum of weights of an antichain in $S_{f_1} \times \cdots \times S_{f_n}$. Since $S_{f_1} \times \cdots \times S_{f_n}$ is a normal poset by Lemma 4.1, $S_{f_1} \times \cdots \times S_{f_n}$ is strongly Sperner. Therefore

$$\sum_{i=1}^{k} f_L[i] \leq \sum_{i=1}^{k} f_R[i],$$

where the left-hand side corresponds to the sum of $k$ antichains and the right-hand side corresponds to the sum of $k$ largest Whitney numbers in $S_{f_1} \times \cdots \times S_{f_n}$. Since $f_L$ and $f_R$ are still probability mass functions,

$$\sum_{i=1}^{M} f_L[i] = \sum_{i=1}^{M} f_R[i] = 1,$$

for some sufficiently large $M > 0$. Thus majorization follows.

We finally note that Theorem 1.4 is a restatement of Proposition 4.3 using the notions of random variables.

5 Proof of Theorem 1.6

We assume that $0 < a_1 < \cdots < a_N$. Let $X_{i,j}$ for $1 \leq i \leq N$ and $1 \leq j \leq m_i \leq N$ be independent random variables following Bernoulli $\left( \frac{1}{2} \right)$. From the assumption, $1 \leq m_N \leq m_{N-1} \leq \cdots \leq m_1$. Then, we can decompose each $Y_i$ as follows:

$$Y_1 := X_{1,1} + \cdots + X_{1,m_1},$$

$$\vdots \quad \vdots$$

$$Y_N := X_{N,1} + \cdots + X_{N,m_N}.$$

By the construction, $Y_i$’s are independent random variables following Binomial $\left( m_i, \frac{1}{2} \right)$ for all $1 \leq m_N \leq m_{N-1} \leq \cdots \leq m_1$. We denote $n_j$ by the number of defined $X_{i,j}$’s for each $1 \leq j \leq N$. Let $Z_j := (X_{1,j}, \cdots, X_{m_j,j})$.

For each $j$, consider an element $i_j = (b_1, \cdots, b_{m_j})$ in $M(m_j)$. We encode $b_k = i > 0$ for some $k$ in $i_j$ iff $X_{i,j} = 1$. Otherwise, $b_k = 0$. For example, when $m_j = 5$,

$$i_j = (0, 0, 0, 2, 4) \quad \text{iff} \quad Z_j = (X_{1,j} = 0, X_{2,j} = 1, X_{3,j} = 0, X_{4,j} = 1, X_{5,j} = 0).$$

Thus as described above, there exists a bijective link between an element $i_j$ in $M(m_j)$ and each realization of the random vector $Z_j$. Furthermore, we are able to construct another bijective map between an element $(i_1, \cdots, i_N)$ in $M(m_1) \times \cdots \times M(m_N)$ and each realization of the random array $(Z_1, \cdots, Z_N)$.

Let $L_j(Z_j) := a_1X_{1,j} + \cdots + a_jX_{m_j,j}$ for each $1 \leq j \leq N$. Based on the construction, we see that

$$Y_L := a_1Y_1 + \cdots + a_NY_N = L_1(Z_1) + \cdots + L_N(Z_N).$$

As demonstrated in Section 4, we similarly define a level set $L_j(x_j)$ in $M(m_j)$ as follows:

$$L_j(x_j) := \{ i_j \in M(m_j) : \text{$i_j$ bijectively corresponds to $Z_j$ such that $L_j(Z_j) = x_j$} \}.$$
Lemma 5.1. \( L_j[x_j] \) forms an antichain in \( M(m_j) \).

**Proof.** Suppose that there exist two distinct elements \( i_j \) and \( i'_j \) in \( L_j[x_j] \) such that \( i_j \preceq i'_j \). Assume that \( i_j \) and \( i'_j \) correspond to \( Z_j \) and \( Z'_j \), respectively. Since \( i_j = (b_1, \cdots, b_{m_j}) \) and \( i'_j = (b'_1, \cdots, b'_{m_j}) \) are distinct, there exists some \( k > 0 \) such that \( b_k < b'_k \). Since \( a_i > 0 \), \( 0 < a_{b_k} < a_{b'_k} \). It implies that

\[
L_j(Z_j) < L_j(Z'_j).
\]

This contradicts that both \( i_j \) and \( i'_j \) are in \( L_j[x_j] \). \( \square \)

More generally, we define a level set \( L[x] \) in \( M(m_1) \times \cdots \times M(m_N) \) as

\[
L[x] := \{(i_1, \cdots, i_N) \in M(m_1) \times \cdots \times M(m_N) : (i_1, \cdots, i_N) \text{ bijectively corresponds to } (Z_1, \cdots, Z_N) \text{ such that } L_1(Z_1) + \cdots + L_N(Z_N) = x\}.
\]

Following the same argument in Lemma 5.1, we see that \( L[x] \) forms an antichain. We omit the proof for simplicity.

Lemma 5.2. \( L[x] \) forms an antichain in \( M(m_1) \times \cdots \times M(m_N) \).

Next, Lemma 3.5 implies that a product of posets \( M(m_1) \times \cdots \times M(m_N) \) is again Peck, so it is strongly Sperner.

Lemma 5.3. \( M(m_1) \times \cdots \times M(m_N) \) is Peck, thus strongly Sperner.

We note that \( |M(m_j)| = 2^{m_j} \), so \( |M(m_1) \times \cdots \times M(m_N)| = 2^{m_1 + \cdots + m_N} \). Then,

\[
P(Y_L = x) = \frac{|L[x]|}{2^{m_1 + \cdots + m_N}}.
\]

Before finding the link between strong Sperner property and majorization, it is necessary to identify the saturated case meaning that it achieves the maximal sum. When \( a_1 = 1, \cdots, a_N = N \), let

\[
Y_R := Y_1 + \cdots + NY_N = R_1(Z_1) + \cdots + R_N(Z_N),
\]

where \( R_j(Z_j) := X_{1,j} + \cdots + m_jX_{m_j,j} \) for each \( 1 \leq j \leq N \). Then as Stanley and Proctor explained in [6, 7], we see that the size of each level set of \( R_j(Z_j) \) has a bijective correspondence to a Whitney number of \( M(m_j) \) by matching the rank to the value \( R_j(Z_j) \). Hence each level set of \( Y_R \) has a bijective correspondence to a Whitney number of \( M(m_1) \times \cdots \times M(m_N) \) by applying the property of the product of posets. Therefore we confirm that \( Y_R \) is the saturated case.

Now it remains to establish majorization through the strong Sperner property of \( M(m_1) \times \cdots \times M(m_N) \). Let \( Z_0 \) be a collection of non-negative integers. Observe that

\[
(2^{m_1 + \cdots + m_N}) \sup_{C \subset Z_0^+ \mid |C| = k} \sum_{c \subset Z_0^+ \mid |c| = k} P(Y_L \in C) \leq (2^{m_1 + \cdots + m_N}) \sup_{C \subset Z_0^+ \mid |C| = k} \sum_{c \subset Z_0^+ \mid |c| = k} P(Y_R \in C), \tag{5.1}
\]

where the left-hand side corresponds to the sum of weights from \( k \) antichains in \( M(n_1) \times \cdots \times M(n_N) \) and the right-hand side exactly corresponds to the sum of \( k \)-largest Whitney numbers from \( M(m_1) \times \cdots \times M(m_N) \). We see that the equation (5.1) is confirming the condition (1.1). The condition (1.2) follows using the fact that the total sum equals 1 as probability mass functions. Thus, the conclusion follows by re-stating it using the notions of random variables.
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References

[1] L. Wang, J. O. Woo, M. Madiman, A lower bound on the Rényi entropy of convolutions in the integers, in: Proc. IEEE Intl. Symp. Inform. Theory, Honolulu, Hawaii, 2014, pp. 2829–2833.

[2] J. O. Woo, M. Madiman, A discrete entropy power inequality for uniform distributions, in: Proc. IEEE Intl. Symp. Inform. Theory, Hong Kong, China, 2015.

[3] P. Erdős, On a lemma of Littlewood and Offord, Bull. Amer. Math. Soc. 51 (1945) 898–902.

[4] J. E. Littlewood, A. C. Offord, On the number of real roots of a random algebraic equation. III, Rec. Math. [Mat. Sbornik] N.S. 12(54) (1943) 277–286.

[5] E. Sperner, Ein Satz über Untermengen einer endlichen Menge, Math. Z. 27 (1) (1928) 544–548. doi:10.1007/BF01171114. URL https://doi-org.udel.idm.oclc.org/10.1007/BF01171114

[6] R. P. Stanley, Weyl groups, the hard Lefschetz theorem, and the Sperner property, SIAM J. Algebraic Discrete Methods 1 (2) (1980) 168–184. doi:10.1137/0601021. URL https://doi-org.udel.idm.oclc.org/10.1137/0601021

[7] R. A. Proctor, Solution of two difficult combinatorial problems with linear algebra, Amer. Math. Monthly 89 (10) (1982) 721–734. doi:10.2307/2975833. URL https://doi-org.udel.idm.oclc.org/10.2307/2975833

[8] M. Krishnapur, Anti-concentration inequalities, Lecture notes.

[9] L. Wang, M. Madiman, Beyond the entropy power inequality, via rearrangements, IEEE Trans. Inform. Theory 60 (9) (2014) 5116–5137. URL arXiv:1307.6018

[10] M. Madiman, L. Wang, J. O. Woo, Rényi entropy inequalities for sums in prime cyclic groups, Preprint, arXiv:1710.00812.

[11] J. O. Woo, Information theoretic inequalities, limit theorems, and universal compression over unknown alphabets, Ph.D. thesis, Yale University, New Haven (May 2015).

[12] R. M. Gabriel, The rearrangement of positive Fourier coefficients, Proceedings of the London Mathematical Society 2 (1) (1932) 32–51.

[13] G. H. Hardy, J. E. Littlewood, Notes on the Theory of Series (VIII): An Inequality, J. London Math. Soc. 3 (2) (1928) 105–110. doi:10.1112/jlms/s1-3.2.105. URL https://doi-org.udel.idm.oclc.org/10.1112/jlms/s1-3.2.105
[14] G. H. Hardy, J. E. Littlewood, G. Pólya, Inequalities, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1988, reprint of the 1952 edition.

[15] V. F. Lev, Linear equations over \( \mathbb{F}_p \) and moments of exponential sums, Duke Math. J. 107 (2) (2001) 239–263. doi:10.1215/S0012-7094-01-10722-9. URL http://dx.doi.org/10.1215/S0012-7094-01-10722-9

[16] A. Sárközi, E. Szemerédi, Über ein Problem von Erdős und Moser, Acta Arith. 11 (1965) 205–208. doi:10.4064/aa-11-2-205-208. URL https://doi-org.udel.idm.oclc.org/10.4064/aa-11-2-205-208

[17] G. Katona, On a conjecture of Erdős and a stronger form of Sperner’s theorem, Studia Sci. Math. Hungar 1 (1966) 59–63.

[18] P. Erdős, L. Moser, Elementary Problems and Solutions: Solutions: E736, Amer. Math. Monthly 54 (4) (1947) 229–230. doi:10.2307/2304711. URL https://doi-org.udel.idm.oclc.org/10.2307/2304711

[19] H. H. Nguyen, A new approach to an old problem of Erdős and Moser, J. Combin. Theory Ser. A 119 (5) (2012) 977–993. doi:10.1016/j.jcta.2012.01.003. URL https://doi-org.udel.idm.oclc.org/10.1016/j.jcta.2012.01.003

[20] J. Radhakrishnan, Entropy and counting, in: IIT Kharagpur Golden Jubilee Volume, 2001. URL http://www.tcs.tifr.res.in/~jaikumar/mypage.html

[21] M. Madiman, P. Tetali, Information inequalities for joint distributions, with interpretations and applications, IEEE Trans. Inform. Theory 56 (6) (2010) 2699–2713.

[22] O. Johnson, I. Kontoyiannis, M. Madiman, Log-concavity, ultra-log-concavity, and a maximum entropy property of discrete compound Poisson measures, Discrete Appl. Math. 161 (2013) 1232–1250, doi: 10.1016/j.dam.2011.08.025.

[23] I. Z. Ruzsa, Sumsets and entropy, Random Structures Algorithms 34 (1) (2009) 1–10. doi:10.1002/rsa.20248. URL http://dx.doi.org/10.1002/rsa.20248

[24] T. Tao, Sumset and inverse sumset theory for Shannon entropy, Combin. Probab. Comput. 19 (4) (2010) 603–639. doi:10.1017/S0963548309990642. URL http://dx.doi.org/10.1017/S0963548309990642

[25] M. Madiman, A. Marcus, P. Tetali, Entropy and set cardinality inequalities for partition-determined functions, Random Struct. Alg. 40 (2012) 399–424. URL http://arxiv.org/abs/0901.0055

[26] E. Abbe, J. Li, M. Madiman, Entropies of weighted sums in cyclic groups and an application to polar codes, Entropy, Special issue on “Entropy and Information Inequalities” (edited by V. Jog and J. Melbourne) 19 (9).

[27] M. Madiman, I. Kontoyiannis, Entropy bounds on abelian groups and the Ruzsa divergence, IEEE Trans. Inform. Theory 64 (1) (2018) 77–92, available online at arXiv:1508.04089.
[28] J. Kahn, An entropy approach to the hard-core model on bipartite graphs, Combinatorics, Probability and Computing 10 (2001) 219–237.

[29] B. Morris, Improved mixing time bounds for the Thorp shuffle, Combin. Probab. Comput. 22 (1) (2013) 118–132. doi:10.1017/S0963548312000478. URL https://doi-org.udel.idm.oclc.org/10.1017/S0963548312000478

[30] R. Peled, Y. Spinka, Rigidity of proper colorings of $Z^d$, Preprint, arXiv:1808.03597.

[31] P. Bergmans, A simple converse for broadcast channels with additive white Gaussian noise, IEEE Trans. Inform. Theory 20 (2) (1974) 279–280.

[32] H. Weingarten, Y. Steinberg, S. Shamai, The capacity region of the Gaussian multiple-input multiple-output broadcast channel, IEEE Trans. Inform. Theory 52 (9) (2006) 3936–3964. doi:10.1109/TIT.2006.880064. URL http://dx.doi.org/10.1109/TIT.2006.880064

[33] O. Johnson, Information theory and the central limit theorem, Imperial College Press, London, 2004.

[34] A. Dembo, T. Cover, J. Thomas, Information-theoretic inequalities, IEEE Trans. Inform. Theory 37 (6) (1991) 1501–1518.

[35] R. J. Gardner, The Brunn-Minkowski inequality, Bull. Amer. Math. Soc. (N.S.) 39 (3) (2002) 355–405 (electronic). doi:10.1090/S0273-0979-02-00941-2. URL http://dx.doi.org/10.1090/S0273-0979-02-00941-2

[36] M. Madiman, J. Melbourne, P. Xu, Forward and reverse entropy power inequalities in convex geometry, in: E. Carlen, M. Madiman, E. M. Werner (Eds.), Convexity and Concentration, Vol. 161 of IMA Volumes in Mathematics and its Applications, Springer, 2017, pp. 427–485, available online at arXiv:1604.04225.

[37] C. Shannon, A mathematical theory of communication, Bell System Tech. J. 27 (1948) 379–423, 623–656.

[38] A. Stam, Some inequalities satisfied by the quantities of information of Fisher and Shannon, Information and Control 2 (1959) 101–112.

[39] S. Artstein, K. M. Ball, F. Barthe, A. Naor, Solution of Shannon’s problem on the monotonicity of entropy, J. Amer. Math. Soc. 17 (4) (2004) 975–982 (electronic).

[40] M. Madiman, A. Barron, Generalized entropy power inequalities and monotonicity properties of information, IEEE Trans. Inform. Theory 53 (7) (2007) 2317–2329.

[41] M. Madiman, F. Ghassemi, Combinatorial entropy power inequalities: A preliminary study of the Stam region, IEEE Trans. Inform. Theory (to appear)Available online at arXiv:1704.01177.

[42] W. Beckner, Inequalities in Fourier analysis, Ann. of Math. (2) 102 (1) (1975) 159–182.

[43] E. H. Lieb, Proof of an entropy conjecture of Wehrl, Comm. Math. Phys. 62 (1) (1978) 35–41.

[44] I. Kontoyiannis, P. Harremoës, O. Johnson, Entropy and the law of small numbers, IEEE Trans. Inform. Theory 51 (2) (2005) 466–472.
[45] M. Madiman, Topics in information theory, probability and statistics, Ph.D. thesis, Brown University, Providence RI (August 2005).

[46] A. D. Barbour, O. Johnson, I. Kontoyiannis, M. Madiman, Compound Poisson approximation via information functionals, Electron. J. Probab. 15 (42) (2010) 1344–1368.

[47] O. Johnson, Y. Yu, Monotonicity, thinning, and discrete versions of the entropy power inequality, IEEE Trans. Inform. Theory 56 (11) (2010) 5387–5395. doi:10.1109/TIT.2010.2070570.
URL http://dx.doi.org/10.1109/TIT.2010.2070570

[48] S. Haghighatshoar, E. Abbe, E. Telatar, A new entropy power inequality for integer-valued random variables, IEEE Trans. Inform. Th. 60 (7) (2014) 3787–3796.

[49] A. W. Marshall, I. Olkin, B. C. Arnold, Inequalities: theory of majorization and its applications, 2nd Edition, Springer Series in Statistics, Springer, New York, 2011. doi:10.1007/978-0-387-68276-1.
URL https://dx-doi.org/10.1007/978-0-387-68276-1

[50] M. Madiman, J. Melbourne, P. Xu, Rogozin’s convolution inequality for locally compact groups, Preprint, arXiv:1705.00642.

[51] P. Xu, J. Melbourne, M. Madiman, A new bound on the $L^p$ Lebesgue constant, Preprint, arXiv:1808.07732.

[52] I. Z. Ruzsa, Sumsets and structure, in: Combinatorial number theory and additive group theory, Adv. Courses Math. CRM Barcelona, Birkhäuser Verlag, Basel, 2009, pp. 87–210. doi:10.1007/978-3-7643-8962-8.
URL http://dx.doi.org/10.1007/978-3-7643-8962-8

[53] S. G. Bobkov, A. Marsiglietti, Variants of the entropy power inequality, IEEE Trans. Inform. Theory 63 (12) (2017) 7747–7752. doi:10.1109/TIT.2017.2764487.
URL https://doi-org.udel.idm.oclc.org/10.1109/TIT.2017.2764487

[54] J. Li, Rényi entropy power inequality and a reverse, Studia Math. 242 (2018) 303–319.

[55] R. J. Gardner, P. Gronchi, A Brunn-Minkowski inequality for the integer lattice, Trans. Amer. Math. Soc. 353 (10) (2001) 3995–4024 (electronic). doi:10.1090/S0002-9947-01-02763-5.
URL http://dx.doi.org/10.1090/S0002-9947-01-02763-5

[56] R. P. Stanley, Enumerative combinatorics. Volume 1, 2nd Edition, Vol. 49 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2012.

[57] K. Engel, Sperner theory, Vol. 65 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 1997. doi:10.1017/CBO9780511574719.
URL https://doi-org.udel.idm.oclc.org/10.1017/CBO9780511574719

[58] W. N. Hsieh, D. J. Kleitman, Normalized matching in direct products of partial orders, Studies in Appl. Math. 52 (1973) 285–289.
[59] R. A. Proctor, M. E. Saks, D. G. Sturtevant, Product partial orders with the Sperner property, Discrete Math. 30 (2) (1980) 173–180. doi:10.1016/0012-365X(80)90118-1. URL https://doi-org.udel.idm.oclc.org/10.1016/0012-365X(80)90118-1