The First Isomorphism Theorem on QI-algebras

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ABSTRACT—The aim of this paper is to construct the first isomorphism theorem of QI-homomorphism of QI-algebras. The concepts of normal QI-subalgebras and quotient QI-algebras are also investigated.

Keywords—QI-algebra, homomorphism, isomorphism, normal, quotient

1. INTRODUCTION

In 1966, the concept of BCK-algebras was introduced by Y. Imai and K. Iseki [4]. Moreover, K. Iseki [5] gave the definition of BCI-algebras in 1980. Both of them play an important role in the study of logical algebras. Afterwards, several structures of algebras such as BH-algebras [6], TM-algebras [7] and KU-algebras [12] were introduced and investigated. The fundamental concepts of abstract algebra such as ideals, congruences and homomorphisms were also studied on those algebraic structures (see [1], [11], [13]). Furthermore, many generalizations of BCK-algebras were introduced by several researchers. Some examples of such algebras are BH-algebras [8], B-algebras [9] and Q-algebras [10]. It turns out that many properties of these kind of algebras were extensively investigated. In 2017, A. B. Saeid, H. S. Kim and A. Razaei proposed a new algebra which is a generalization of implicative BCK-algebras, called a BI-algebra [14]. They provided the basic properties of BI-algebras and discussed about ideals and congruence relations. The properties of ideals of BI-algebras were continuously investigated in [2]. Lately, the notion of QI-algebras, which is a generalization of BI-algebras, was introduced by R. K. Bandaru [3]. The concept of ideals and some basic properties were also considered. One can see more examples of research papers in this area in [15-18].

In this paper, we gave the concept of QI-homomorphisms of QI-algebras and investigated some relate properties. The relations between QI-isomorphisms and quotient QI-algebras are also provided.

2. PRELIMINARIES

In this section, we begin with the definition of a QI-algebra which is an algebra \((X, *, 0)\) of type \((2,0)\), i.e., a nonempty set \(X\) equipped with a binary operation \(*\) and a constant \(0\). We also recall some notions and properties of QI-algebras.

**Definition 2.1.** [3] An algebra \((X, *, 0)\) of type \((2,0)\) is called a QI-algebra if

(Q1) \(x * x = 0\),

(Q2) \(x * 0 = x\),

(Q3) \(x * (y * (x * y)) = x * y\),
for all \( x, y \in X \).

The relation “\( \leq \)" on a QI-algebra \((X, *, 0)\) is defined by \( x \leq y \) if and only if \( x * y = 0 \). From (QI1), we can immediately conclude that \( \leq \) is reflexive, however \( \leq \) is not a partially ordered relation.

**Example 2.2.** Let \( X = \{0, 1, 2\} \) be a set with the following Cayley table.

\[
\begin{array}{c|ccc}
* & 0 & 1 & 2 \\
\hline
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 \\
2 & 2 & 2 & 0 \\
\end{array}
\]

Then, by using computer programming, it is easy to check that \((X, *, 0)\) is a QI-algebra.

**Definition 2.3.** \([3]\) A QI-algebra \((X, *, 0)\) is said to be *right distributive* or *left distributive*, respectively if

\[
(x * y) * z = (x * z) * (y * z)
\]

or \(z * (x * y) = (z * x) * (z * y)\), respectively, for all \( x, y, z \in X \).

**Example 2.4.** Notice that a QI-algebra \((X, *, 0)\) in Example 2.2 is not a right distributive since

\[
(1 * 1) * 1 = 0 * 1 = 2 = 0 = 0 * 0 = (1 * 1) * (1 * 1)
\]

and \((X, *, 0)\) is not left distributive QI-algebra since

\[
2 * (1 * 0) = 2 * 1 = 2 \\ 0 = 2 * 2 = (2 * 1) * (2 * 0)
\]

**Example 2.5.** \([3]\) Let \( Y = \{0, 1, 2, 3\} \) be a set with the following Cayley table.

\[
\begin{array}{c|cccc}
* & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 \\
2 & 2 & 2 & 0 & 2 \\
3 & 3 & 0 & 1 & 0 \\
\end{array}
\]

Then it is easy to check that \((Y, *, ', 0)\) is a right distributive QI-algebra.

**Proposition 2.6.** \([3]\) Let \((X, *, 0)\) be a QI-algebra.

(i) If \( X \) is a left distributive QI-algebra, then \( X = \{0\} \).

(ii) If \( X \) is a right distributive QI-algebra, then \( 0 * x = 0 \) for all \( x \in X \).

**Definition 2.7.** \([3]\) Let \((X, *, 0)\) be a QI-algebra and \( I \) be a subset of \( X \). Then \( I \) is called an *(QI)-ideal* of \( X \) if it satisfies the following:

(I1) \( 0 \in X \).

(I2) for each \( x, y \in X \), if \( x * y \in I \) and \( y \in I \) then \( x \in I \).

**Example 2.8.** \([3]\) Let \( X = \{0, 1, 2\} \) be a set with the following Cayley table.

\[
\begin{array}{c|cccc}
* & 0 & 1 & 2 & 3 \\
\hline
0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 \\
2 & 2 & 2 & 0 & 2 \\
3 & 3 & 3 & 2 & 1 \\
\end{array}
\]

Then it is easy to check that \((X, *, 0)\) is a QI-algebra. Note that \( I_1 = \{0, 1\} \) and \( I_2 = \{0, 1, 3\} \) are ideals but \( I_3 = \{0, 1, 2\} \) is not an ideal of \( X \).
3. MAIN RESULTS

In this section, we give the definition of normal QI-subalgebra, congruence relation and QI-homomorphism of QI-algebra. Note that such definitions were provided analogue to the definitions on BI-algebras given in [2]. The first isomorphism theorem on QI-algebras is proven at the end of this section.

Definition 3.1. Let $(X, \ast, 0)$ be a QI-algebra. A nonempty subset $S$ of $X$ is called a QI-subalgebra of $X$ if it is closed under the operation $\ast$, i.e., $x \ast y \in S$ for any $x, y \in S$.

Note that every QI-subalgebra contains 0 since it is nonempty and the axiom (QI1).

Definition 3.2. Let $(X, \ast, 0)$ be a QI-algebra. A nonempty subset $N$ of $X$ is called a normal subset of $X$ if for each $x, y, a, b \in X$, $x \ast y, a \ast b \in N$ implies $(x \ast a) \ast (y \ast b) \in N$.

Proposition 3.3. Let $N$ be a normal subset of a QI-algebra $(X, \ast, 0)$. Then $N$ is a QI-subalgebra of $X$.

Proof. Assume that $N$ is a normal subset of $X$. Let $x, y \in N$. Then $x \ast 0 = x \in N$ and $y \ast 0 = y \in N$. Since $N$ is normal subset of $X$, it follows that $x \ast y = (x \ast y) \ast (0 \ast 0) \in N$. Hence $N$ is closed under $\ast$. Thus $N$ is a QI-subalgebra of $X$.

From the above proposition, we will call a normal subset of a QI-algebra $(X, \ast, 0)$ a normal QI-subalgebra $X$.

In general, the converse of Proposition 3.3 does not hold as it was shown in the following examples.

Example 3.4. Let $X = \{0, 1, 2, 3\}$ be a set with the following Cayley table.

|   | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 |
| 2 | 2 | 0 | 0 | 2 |
| 3 | 3 | 0 | 3 | 0 |

Then, by using computer programming, it is easy to check that $(X, \ast, 0)$ is a QI-algebra. Notice that $A = \{0, 1, 2\}$ is a QI-subalgebra of $X$ but it is not normal since $3 \ast 3 = 0 \not\in A$, $2 \ast 3 = 2 \not\in A$ but $(3 \ast 2) \ast (3 \ast 3) = 3 \ast 0 = 3 \not\in A$.

Example 3.5. Let $X = \{0, 1, 2, 3\}$ be a set with the following Cayley table.

|   | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 1 | 0 | 1 | 1 |
| 2 | 2 | 2 | 0 | 2 |
| 3 | 3 | 3 | 3 | 0 |

Then, by using computer programming, it is easy to check that $(X, \ast, 0)$ is a QI-algebra and $N = \{0, 1\}$ is normal. Moreover, we have that $M = \{0, 1, 2\}$ is a QI-subalgebra and QI-ideal of $X$. Since $3 \ast 3 = 0 \in M$, $2 \ast 3 = 2 \in M$ and $(3 \ast 2) \ast (3 \ast 3) = 3 \ast 0 = 3 \not\in M$, we have that $M$ is not normal.
Lemma 3.6.  Let \( N \) be a normal QI-subalgebra of a QI-algebra \( (X, *, 0) \) and \( x, y \in N \). If \( x*y \in N \), then \( y*x \in N \).

**Proof.** Assume that \( x*y \in N \). Since \( N \) is QI-subalgebra of \( X \), it follows that \( y*y = 0 \in N \). The fact that \( y*y, x*y \in N \) and \( N \) is normal implies that \( y*x = (y*x)*0 = (y*x)*(y*y) \in N \). \( \square \)

Definition 3.7.  Let \( N \) be a normal QI-subalgebra of a QI-algebra \( (X, *, 0) \). A relation \( \sim_N \) is defined by for each \( x, y \in X \),

\[
x \sim_N y \quad \text{if and only if} \quad x*y \in N.
\]

Proposition 3.8.  Let \( N \) be a normal QI-subalgebra of a QI-algebra \( (X, *, 0) \). Then \( \sim_N \) is a congruence relation on \( X \).

**Proof.** Let \( x, y, z, w \in X \). Since \( x*y = 0 \in N \), we have that \( x \sim_N x \). This means that \( \sim_N \) is reflexive. From Lemma 3.6, it follows that \( x \sim_N y \) implies \( y \sim_N x \). Thus \( \sim_N \) is symmetric. To show that \( \sim_N \) is transitive, assume that \( x \sim_N y \) and \( y \sim_N z \). Then \( x*y, y*z \in N \). Since \( \sim_N \) is symmetric, \( z*y \in N \). This implies that

\[
x*z = (x*z)*0 = (x*z)*(y*y) \in N
\]

because \( x*y, z*y \in N \) and \( N \) is normal. Therefore, \( \sim_N \) is an equivalence relation on \( X \).

Next, we will show that \( \sim_N \) is a congruence relation on \( X \). Assume that \( x \sim_N y \) and \( z \sim_N w \). Then

\( x*y, y*z \in N \). Since \( N \) is normal, \( (x*z)*(y*w) \in N \). That is \( x*z \sim_N y*w \), as required. \( \square \)

Definition 3.9.  Let \( N \) be a normal QI-subalgebra of a QI-algebra \( (X, *, 0) \) and \( x \in X \). A congruence class \( [x]_N \) of \( X \) is denoted to be the set \( \{ y \in X : x \sim_N y \} \). Define \( X/\sim_N \) to be the set of all congruence class of \( X \). That is

\[
X/\sim_N = \{ [x]_N : x \in X \}.
\]

The proof of the following lemma is straightforward, we omit the proof.

Lemma 3.10.  Let \( N \) be a normal QI-subalgebra of a QI-algebra \( (X, *, 0) \) and \( x, y \in X \). Then

\[
[x]_N = [y]_N \quad \text{if and only if} \quad x \sim_N y
\]

Theorem 3.11.  Let \( N \) be a normal QI-subalgebra of a QI-algebra \( (X, *, 0) \). Then the binary operation \( *' \) on \( X/\sim_N \) defined by

\[
[x]_N *' [y]_N = [x*y]_N,
\]

for all \( x, y \in X \), makes \( \left( X/\sim_N, *', [0]_N \right) \) into a QI-algebra. Moreover, \( [0]_N = N \).

**Proof.** First, we will show that \( *' \) is well-defined. Let \( x_1, y_1, x_2, y_2 \in X \) such that \( [x_1]_N = [x_2]_N \) and \( [y_1]_N = [y_2]_N \).
Then $x_i \sim_N x_2$ and $y_i \sim_N y_2$. Since $\sim_N$ is a congruence relation, $x_i \ast y_i \sim_N x_2 \ast y_2$. From Lemma 3.10, it can be concluded that $[x_i \ast y_i]_N = [x_i \ast [y_i]_N]$, i.e., $[x_i]_N \ast [y_i]_N = [x_i]_N \ast [y_i]_N$, as required.

Next, we will show that the axioms of QI-algebra are satisfied. Let $x, y \in X$.

(Q11) $[x]_N \ast [x]_N = [x \ast x]_N = [0]_N$,

(Q12) $[x]_N \ast [0]_N = [x \ast 0]_N = [x]_N$,

(Q13) $[x]_N \ast ([y]_N \ast [x]_N) = [x \ast (y \ast x)]_N = [x \ast y]_N$.

Moreover, $[0]_N = \{x \in X : x \sim_N 0\} = \{x \in X : x \ast 0 \in N\} = \{x \in X \ast x \in N\} = N$.

The QI-algebra $X / N$ discussed in the above theorem is called the quotient QI-algebra of $X$ by $N$. Note that the normality of $N$ is required in order to show that $\sim_N$ is a congruence relation which implies that $X / N$ is a QI-algebra.

In order to state the isomorphism theorem, the definition of homomorphism in QI-algebra was provided as follows.

**Definition 3.12.** Let $(X, \ast, 0_X)$ and $(Y, [\cdot, \cdot], 0_Y)$ be QI-algebras. A QI-homomorphism is a mapping $f : X \to Y$ satisfying

$$f(x \ast y) = f(x) \ast f(y),$$

for all $x, y \in X$. An injective QI-homomorphism is called QI-monomorphism, a surjective QI-homomorphism is called QI-epimorphism. A QI-isomorphism is a QI-homomorphism which is bijective. We write $X \cong Y$ if there exists a QI-isomorphism $f : X \to Y$.

The kernel of the QI-homomorphism $f$, denoted by $\ker f$, is the set of elements of $X$ that map to $0_Y$.

**Proposition 3.13.** Let $N$ be a normal QI-subalgebra of a QI-algebra $(X, \ast, 0)$. Then the mapping $\pi : X \to X / N$ given by

$$\pi(x) = [x]_N,$$

for all $x \in X$, is a QI-epimorphism and $\ker \pi = N$.

**Proof.** Let $x, y \in X$. Then

$$\pi(x \ast y) = [x \ast y]_N = [x]_N \ast [y]_N = \pi(x) \ast \pi(y).$$

Hence $\pi$ is a QI-homomorphism. Since

$$\pi(X) = \{\pi(x) : x \in X\} = \{[x]_N : x \in N\} = X / N,$$

$\pi$ is a QI-epimorphism. \qed

The mapping $\pi$ in the above proposition is called the canonical homomorphism of $X$ onto $X / N$.

**Proposition 3.14.** Let $(X, \ast, 0_X)$, $(Y, [\cdot, \cdot], 0_Y)$ be QI-algebras and $f : X \to Y$ be a QI-homomorphism and $A \subseteq X$.

Then
(i) \( f(0_x) = 0_y. \)

(ii) If \( f \) is a QI-monomorphism, then \( \ker f = \{0_x\} \).

(iii) \( \ker f \) is a QI-subalgebra of \( X \).

(iv) If \( A \) is a QI-subalgebra of \( X \), then \( f(A) \) is a QI-subalgebra of \( Y \).

**Proof.**

(i) \( f(0_x) = f(0_x \ast 0_x) = f(0_x) \cap f(0_x) = 0_y. \)

(ii) Assume that \( f \) is a QI-monomorphism. It follows from (i) that \( 0_x \in \ker f \). To show the converse inclusion, let \( x \in \ker f \). Then \( f(x) = 0_y = f(0_x) \). Since \( f \) is injective, \( x = 0_x \). Hence \( \ker f = \{0_x\} \).

(iii) Let \( x, y \in \ker f \). Then \( f(x) = 0_y = f(y) \). Thus \( f(x \ast y) = f(x) \cap f(y) = 0_y \cap 0_y = 0_y \). Hence \( x \ast y \in \ker f \).

(iv) Suppose that \( A \) is a QI-subalgebra of \( X \). Let \( x, y \in f(A) \). Then \( x = f(a) \) and \( y = f(b) \) for some \( a, b \in A \).

Since \( A \) is a QI-subalgebra, \( x \ast y = f(a) \cap f(b) = f(a \ast b) \in f(A) \). Hence \( f(A) \) is a QI-subalgebra of \( Y \).

The following example shows that \( \ker f \) is not normal, in general.

**Example 3.15.** Consider a QI-algebra in Example 3.4. Define a mapping \( f : X \to X \) by \( f(x) = x \) for all \( x \in X \).

Then \( f \) is a QI-homomorphism and \( \ker f = \{0\} \), which is a QI-subalgebra of \( X \) but not normal since \( 2 \ast 1 = 0.3 \ast 1 = 0 \) and \( (2 \ast 3) \ast (1 \ast 1) = 2 \ast 0 = 2 \notin \ker f \).

**Definition 3.16.** A QI-algebra \((X, \ast, 0)\) is said to be a *QI*-algebra if for each \( x, y \in X \),

\[ x \ast y = 0 = y \ast x \text{ implies } x = y. \]

**Example 3.17.** Let \( X = \{0, 1, 2, 3\} \) be a set with the following Cayley table.

| *    | 0  | 1  | 2  | 3  |
|------|----|----|----|----|
| 0    | 0  | 0  | 0  | 0  |
| 1    | 1  | 0  | 1  | 2  |
| 2    | 2  | 2  | 0  | 2  |
| 3    | 3  | 3  | 3  | 0  |

Then, by using computer programming, it is easy to check that \((X, \ast, 0)\) is a QI-algebra.

**Proposition 3.18.** Let \((X, \ast, 0_x)\) be a QI-algebra, \((Y, \sqcup, 0_y)\) a QI-algebra and \( \phi : X \to Y \) a QI-homomorphism. Then \( \phi \) is QI-monomorphism if and only if \( \ker f = \{0_x\} \).

**Proof.** The necessity part is Proposition 3.14 (ii). To prove the sufficiency part, assume that \( \ker f = \{0_x\} \). Let \( x, y \in X \) such that \( \phi(x) = \phi(y) \). Then \( \phi(x \ast y) = \phi(x) \sqcup \phi(y) = \phi(x) \sqcup \phi(x) = 0_y \). That is \( x \ast y \in \ker f \). Similarly, we
can show that \( y \cdot x \in \ker f \). Since \( X \) is a QI-algebra, \( x = y \). Hence \( \phi \) is injective.

**Proposition 3.19.** Let \( M \) and \( N \) be normal QI-subalgebras of a QI-algebra \((X,\cdot,0)\) such that \( N \subseteq M \). Then \( M/N \) is a normal QI-subalgebra of \( X/N \).

**Proof.** Let \([x_1]_N \cdot [x_2]_N, [y_1]_N \cdot [y_2]_N \in M/N\). Then \([x_1 \cdot x_2]_N, [y_1 \cdot y_2]_N \in M/N\). That is \( x_1 \cdot x_2, y_1 \cdot y_2 \in M \). Since \( M \) is normal, \((x_1 \cdot x_2) \cdot (y_1 \cdot y_2), (x_1 \cdot y_1) \cdot (x_2 \cdot y_2) \in M\). Thus \([[(x_1 \cdot x_2) \cdot (y_1 \cdot y_2)]_N, [(x_1 \cdot y_1) \cdot (x_2 \cdot y_2)]_N \in M/N\).

Hence \(\left([x_1]_N \cdot [x_2]_N\right) * \left([y_1]_N \cdot [y_2]_N\right) * \left([x_1]_N \cdot [y_1]_N\right) * \left([x_2]_N \cdot [y_2]_N\right) \in M/N\). Therefore, \( M/N \) is a normal QI-subalgebra of \( X/N \).

In Example 3.5, we have shown that a QI-ideal need not be normal.

**Definition 3.20.** Let \( I \) be a QI-ideal of a QI-algebra \((X,\cdot,0)\). Then \( X \) is called a normal QI-ideal of \( X \) if it is normal.

**Example 3.21.** Let \( X = \{0,1,2,3\} \) be a set with the following Cayley table.

\[
\begin{array}{cccc}
* & 0 & 1 & 2 & 3 \\
0 & 0 & 2 & 1 & 0 \\
1 & 1 & 0 & 1 & 1 \\
2 & 2 & 0 & 2 & 0 \\
3 & 3 & 2 & 3 & 0 \\
\end{array}
\]

Then, by using computer programming, it is easy to check that \((X,\cdot,0)\) is a QI-algebra and \( I = \{0,3\} \) is a normal QI-ideal.

**Proposition 3.22.** Let \((X,\cdot,0)\) be a QI-algebra and \( I \subseteq X \). Then \( I \) is a normal QI-subalgebra of \( X \) if and only if \( I \) is a normal QI-ideal of \( X \).

**Proof.** The sufficiency part follows from Proposition 3.3. To prove the necessity part, let \( x, y \in X \) such that \( x \cdot y \in I \) and \( y \in I \). Since \( I \) is a QI-subalgebra, \( 0 \in I \). Since \( 0, y \in I \) and \( I \) is a QI-subalgebra, we have that \( 0 \cdot y \in I \). Since \( I \) is normal, \( x = x \cdot 0 = (x \cdot 0) \cdot 0 = (x \cdot 0) \cdot (y \cdot y) \in I \). Therefore, \( I \) is a QI-ideal of \( X \).

**Proposition 3.23.** Let \((X,\cdot,0_x),(Y,\cdot,0_y)\) be QI-algebras and \( f : X \rightarrow Y \) be a QI-homomorphism. Then \( \ker f \) is a QI-ideal of \( X \).

**Proof.** Since \( f(0_x) = 0_y \), we have that \( 0_x \in \ker f \). Let \( x, y \in X \) such that \( x \cdot y \in \ker f \) and \( y \in \ker f \). Then

\[
f(x) = f(x) \cdot 0_y = f(x) \cdot f(y) = f(x \cdot y) = 0_y.
\]

Thus \( x \in \ker f \). Hence \( \ker f \) is a QI-ideal of \( X \).
In Example 3.15, we have shown that a kernel of a QI-homomorphism need not be normal.

**Definition 3.24.** Let $(X, \ast, 0_x)$, $(Y, \sqcup, 0_y)$ be QI-algebras and $f : X \rightarrow Y$ be a QI-homomorphism. We say that $f$ is a normal QI-homomorphism if $\ker f$ is a normal QI-ideal of $X$.

**Theorem 3.25. (The first isomorphism theorem on QI-algebras)** Let $(X, \ast, 0_x)$ and $(Y, \sqcup, 0_y)$ be QI$_1$-algebras. If $\varphi : X \rightarrow Y$ be a normal QI-homomorphism, then

$$
\frac{X}{\ker \varphi} \cong \varphi(X)
$$

**Proof.** Since $\varphi$ is a normal QI-homomorphism, $\ker \varphi$ is normal. Then $\frac{X}{\ker \varphi}$ is a quotient QI-algebra of $X$ by $\ker \varphi$. Let $K = \ker \varphi$. Define a mapping $\phi : \frac{X}{K} \rightarrow Y$ by

$$
\phi([x]_K) = \varphi(x)
$$

for all $x \in X$. We will show that $\phi$ is well-defined. Let $[x]_K = [y]_K \in \frac{X}{K}$. Then $x \sim_K y$. It follows that $x \ast y, y \ast x \in K$. Thus $\varphi(x) \sqcup \varphi(y) = \varphi(x \ast y) = 0 = \varphi(y \ast x) = \varphi(y) \sqcup \varphi(x)$. Since $Y$ is QI$_1$-algebra, $\varphi(x) = \varphi(y)$. That is $\phi([x]_K) = \phi([y]_K)$. Since $\phi([x]_K \ast [y]_K) = \phi([x \ast y]_K) = \varphi(x \ast y) = \varphi(x) \sqcup \varphi(y) = \phi([x]_K) \sqcup \phi([y]_K)$, we have that $\phi$ is QI-homomorphism. Next, we will prove that $\phi$ is injective. Clearly, $[0]_K \in \ker \varphi$. Let $[x]_K \in \ker \varphi$. Then $\varphi(x) = \phi([x]_K) = 0_y$. Thus $x \ast 0 = x \in K$. That is $x \sim_K 0_x$. It follows that $[x]_K = [0_x]_K$. Hence $\ker \phi = \{[0]_K\}$. It implies by Proposition 3.18 that $\phi$ is QI-monomorphism. Therefore, $\frac{X}{K} \cong \varphi(X)$.

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