Monoid hypersurfaces

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Abstract. A monoid hypersurface is an irreducible hypersurface of degree $d$ which has a singular point of multiplicity $d - 1$. Any monoid hypersurface admits a rational parameterization, hence is of potential interest in computer aided geometric design. We study properties of monoids in general and of monoid surfaces in particular. The main results include a description of the possible real forms of the singularities on a monoid surface other than the $(d-1)$-uple point. These results are applied to the classification of singularities on quartic monoid surfaces, complementing earlier work on the subject.

1 Introduction

A monoid hypersurface is an (affine or projective) irreducible algebraic hypersurface which has a singularity of multiplicity one less than the degree of the hypersurface. The presence of such a singular point forces the hypersurface to be rational: there is a rational parameterization given by (the inverse of) the linear projection of the hypersurface from the singular point.

The existence of an explicit rational parameterization makes such hypersurfaces potentially interesting objects in computer aided design. Moreover, since the “space” of monoids of a given degree is much smaller than the space of all hypersurfaces of that degree, one can hope to use monoids efficiently in (approximate or exact) implicitization problems. These were the reasons for considering monoids in the paper [17]. In [12] monoid curves are used to approximate other curves that are close to a monoid curve, and in [13] the same is done for monoid surfaces. In both articles the error of such approximations are analyzed – for each approximation, a bound on the distance from the monoid to the original curve or surface can be computed.

In this article we shall study properties of monoid hypersurfaces and the classification of monoid surfaces with respect to their singularities. Section 2 explores properties of monoid hypersurfaces in arbitrary dimension and over an arbitrary base field. Section 5 contains results on monoid surfaces, both over arbitrary fields and over $\mathbb{R}$. The last section deals with the classification of monoid surfaces of degree four. Real and complex quartic monoid surfaces were first studied by Rohn [15], who gave a fairly complete description of all possible cases. He also remarked [15, p. 56] that some of his results on quartic monoids hold for monoids of arbitrary degree; in particular, we believe he was aware of many of the results in Section 5.
Higuchi [19] classify complex quartic monoid surfaces, but do not refer to Rohn. (They cite Jessop [7]; Jessop, however, only treats quartic surfaces with double points and refers to Rohn for the monoid case.) Here we aim at giving a short description of the possible singularities that can occur on quartic monoids, with special emphasis on the real case.

## 2 Basic properties

Let \( k \) be a field, let \( \overline{k} \) denote its algebraic closure and \( \mathbb{P}^n := \mathbb{P}^n_k \) the projective \( n \)-space over \( \overline{k} \). Furthermore we define the set of \( k \)-rational points \( \mathbb{P}^n(k) \) as the set of points that admit representatives \((a_0 : \cdots : a_n)\) with each \( a_i \in k \).

For any homogeneous polynomial \( F \in \overline{k}[x_0, \ldots, x_n] \) of degree \( d \) and point \( p = (p_0 : p_1 : \cdots : p_n) \in \mathbb{P}^n \) we can define the multiplicity of \( Z(F) \) at \( p \). We know that \( p_r \neq 0 \) for some \( r \), so we can assume \( p_0 = 1 \) and write

\[
F = \sum_{i=0}^{d} a_i^{d-i} f_i(x_1 - p_1 x_0, x_2 - p_2 x_0, \ldots, x_n - p_n x_0)
\]

where \( f_i \) is homogeneous of degree \( i \). Then the multiplicity of \( Z(F) \) at \( p \) is defined to be the smallest \( i \) such that \( f_i \neq 0 \).

Let \( F \in \overline{k}[x_0, \ldots, x_n] \) be of degree \( d \geq 3 \). We say that the hypersurface \( X = Z(F) \subset \mathbb{P}^n \) is a monoid hypersurface if \( X \) is irreducible and has a singular point of multiplicity \( d - 1 \).

In this article we shall only consider monoids \( X = Z(F) \) where the singular point is \( k \)-rational. Modulo a projective transformation of \( \mathbb{P}^n \) over \( k \) we may – and shall – therefore assume that the singular point is the point \( O = (1 : 0 : \cdots : 0) \).

Hence, we shall from now on assume that \( X = Z(F) \), and

\[
F = x_0 f_{d-1} + f_d,
\]

where \( f_i \in \overline{k}[x_1, \ldots, x_n] \subset k[x_0, \ldots, x_n] \) is homogeneous of degree \( i \) and \( f_{d-1} \neq 0 \). Since \( F \) is irreducible, \( f_d \) is not identically 0, and \( f_{d-1} \) and \( f_d \) have no common (non-constant) factors.

The natural rational parameterization of the monoid \( X = Z(F) \) is the map

\[
\theta_F: \mathbb{P}^{n-1} \to \mathbb{P}^n
\]

given by

\[
\theta_F(a) = (f_d(a) : -f_{d-1}(a)a_1 : \ldots : -f_{d-1}(a)a_n),
\]

for \( a = (a_1 : \cdots : a_n) \) such that \( f_{d-1}(a) \neq 0 \) or \( f_d(a) \neq 0 \).

The set of lines through \( O \) form a \( \mathbb{P}^{n-1} \). For every \( a = (a_1 : \cdots : a_n) \in \mathbb{P}^{n-1} \), the line

\[
L_a := \{(s : ta_1 : \cdots : ta_n) | (s : t) \in \mathbb{P}^1\}
\]

(1)

intersects \( X = Z(F) \) with multiplicity at least \( d - 1 \) in \( O \). If \( f_{d-1}(a) \neq 0 \) or \( f_d(a) \neq 0 \), then the line \( L_a \) also intersects \( X \) in the point

\[
\theta_F(a) = (f_d(a) : -f_{d-1}(a)a_1 : \ldots : -f_{d-1}(a)a_n).
\]
Hence the natural parameterization is the “inverse” of the projection of $X$ from the point $O$. Note that $\theta_F$ maps $Z(f_{d-1}) \setminus Z(f_d)$ to $O$. The points where the parameterization map is not defined are called base points, and these points are precisely the common zeros of $f_{d-1}$ and $f_d$. Each such point $b$ corresponds to the line $L_b$ contained in the monoid hypersurface. Additionally, every line of type $L_b$ contained in the monoid hypersurface corresponds to a base point.

Note that $Z(f_{d-1}) \subset \mathbb{P}^{n-1}$ is the projective tangent cone to $X$ at $O$, and that $Z(f_d)$ is the intersection of $X$ with the hyperplane “at infinity” $Z(x_0)$.

Assume $P \in X$ is another singular point on the monoid $X$. Then the line $L$ through $P$ and $O$ has intersection multiplicity at least $d - 1 + 2 = d + 1$ with $X$. Hence, according to Bezout’s theorem, $L$ must be contained in $X$, so that this is only possible if $\dim X \geq 2$.

By taking the partial derivatives of $F$ we can characterize the singular points of $X$ in terms of $f_d$ and $f_{d-1}$:

**Lemma 1.** Let $\nabla = (\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n})$ be the gradient operator.

(i) A point $P = (p_0 : p_1 : \cdots : p_n) \in \mathbb{P}^n$ is singular on $Z(F)$ if and only if $f_{d-1}(p_0, \ldots, p_n) = 0$ and $p_0 \nabla f_{d-1}(p_1, \ldots, p_n) + \nabla f_d(p_0, \ldots, p_n) = 0$.

(ii) All singular points of $Z(F)$ are on lines $L_a$ where $a$ is a base point.

(iii) Both $Z(f_{d-1})$ and $Z(f_d)$ are singular in a point $a \in \mathbb{P}^{n-1}$ if and only if all points on $L_a$ are singular on $X$.

(iv) If not all points on $L_a$ are singular, then at most one point other than $O$ on $L_a$ is singular.

**Proof.** (i) follows directly from taking the derivatives of $F = x_0 f_{d-1} + f_d$, and (ii) follows from (i) and the fact that $F(P) = 0$ for any singular point $P$. Furthermore, a point $(s : t a_1 : \cdots : t a_n)$ on $L_a$ is, by (i), singular if and only if $s \nabla f_{d-1}(ta_1) + \nabla f_d(ta) = t^{d-1} (s \nabla f_{d-1}(a) + t \nabla f_d(a)) = 0$.

This holds for all $(s : t) \in \mathbb{P}^1$ if and only if $\nabla f_{d-1}(a) = \nabla f_d(a) = 0$. This proves (iii). If either $\nabla f_{d-1}(a)$ or $\nabla f_d(a)$ are nonzero, the equation above has at most one solution $(s_0 : t_0) \in \mathbb{P}^1$ in addition to $t = 0$, and (iv) follows.

Note that it is possible to construct monoids where $F \in k[x_0, \ldots, x_n]$, but where no points of multiplicity $d - 1$ are $k$-rational. In that case there must be (at least) two such points, and the line connecting these will be of multiplicity $d - 2$. Furthermore, the natural parameterization will typically not induce a parameterization of the $k$-rational points from $\mathbb{P}^{n-1}(k)$.

### 3 Monoid surfaces

In the case of a monoid surface, the parameterization has a finite number of base points. From Lemma 1(ii) we know that all singularities of the monoid other than $O$, are on lines $L_a$ corresponding to these points. In what follows we will develop the theory for singularities on monoid surfaces — most of these results were probably known to Rohn [15, p. 56].

We start by giving a precise definition of what we shall mean by a monoid surface.
Definition 2. For an integer \(d \geq 3\) and a field \(k\) of characteristic 0 the polynomials \(f_{d-1} \in k[x_1, x_2, x_3]_{d-1}\) and \(f_d \in k[x_1, x_2, x_3]_d\) define a normalized non-degenerate monoid surface \(Z(F) \subset \mathbb{P}^3\), where \(F = x_0 f_{d-1} + f_d \in k[x_0, x_1, x_2, x_3]\) if the following hold:

(i) \(f_{d-1}, f_d \neq 0\)
(ii) \(\gcd(f_{d-1}, f_d) = 1\)
(iii) The curves \(Z(f_{d-1}) \subset \mathbb{P}^2\) and \(Z(f_d) \subset \mathbb{P}^2\) have no common singular point.

The curves \(Z(f_{d-1}) \subset \mathbb{P}^2\) and \(Z(f_d) \subset \mathbb{P}^2\) are called respectively the tangent cone and the intersection with infinity.

Unless otherwise stated, a surface that satisfies the conditions of Definition 2 shall be referred to simply as a monoid surface.

Since we have finitely many base points \(b\) and each line \(L_b\) contains at most one singular point in addition to \(O\), monoid surfaces will have only finitely many singularities, so all singularities will be isolated. (Note that Rohn included surfaces with nonisolated singularities in his study [15].) We will show that the singularities other than \(O\) can be classified by local intersection numbers.

Definition 3. Let \(f, g \in k[x_1, x_2, x_3]\) be nonzero and homogeneous. Assume \(p = (p_1 : p_2 : p_3) \in Z(f, g) \subset \mathbb{P}^2\), and define the local intersection number

\[
I_p(f, g) = \lg \frac{k[x_1, x_2, x_3]_{m_p}}{(f, g)},
\]

where \(k\) is the algebraic closure of \(k\), \(m_p = (p_2 x_1 - p_1 x_2, p_3 x_1 - p_1 x_3, p_3 x_2 - p_2 x_3)\) is the homogeneous ideal of \(p\), and \(\lg\) denotes the length of the local ring as a module over itself.

Note that \(I_p(f, g) \geq 1\) if and only if \(f(p) = g(p) = 0\). When \(I_p(f, g) = 1\) we say that \(f\) and \(g\) intersect transversally at \(p\). The terminology is justified by the following lemma:

Lemma 4. Let \(f, g \in k[x_1, x_2, x_3]\) be nonzero and homogeneous and \(p \in Z(f, g)\). Then the following are equivalent:

(i) \(I_p(f, g) > 1\)
(ii) \(f\) is singular at \(p\), \(g\) is singular at \(p\), or \(\nabla f(p)\) and \(\nabla g(p)\) are nonzero and parallel.
(iii) \(s \nabla f(p) + t \nabla g(p) = 0\) for some \((s, t) \neq (0, 0)\)

**Proof.** (ii) is equivalent to (iii) by a simple case study: \(f\) is singular at \(p\) if and only if (iii) holds for \((s, t) = (1, 0)\), \(g\) is singular at \(p\) if and only if (iii) holds for \((s, t) = (0, 1)\), and \(\nabla f(p)\) and \(\nabla g(p)\) are nonzero and parallel if and only if (iii) holds for some \(s, t \neq 0\).

We can assume that \(p = (0 : 0 : 1)\), so \(I_p(f, g) = \lg S\) where

\[
S = \frac{k[x_1, x_2, x_3]_{(x_1, x_2)}}{(f, g)}.
\]
Furthermore, let \( d = \deg f, e = \deg g \) and write
\[
f = \sum_{i=1}^{d} f_i x_3^{d-i} \quad \text{and} \quad g = \sum_{i=1}^{e} g_i x_3^{e-i}
\]
where \( f_i, g_i \) are homogeneous of degree \( i \).

If \( f \) is singular at \( p \), then \( f_1 = 0 \). Choose \( \ell = ax_1 + bx_2 \) such that \( \ell \) is not a multiple of \( g_1 \). Then \( \ell \) will be a nonzero non-invertible element of \( S \), so the length of \( S \) is greater than 1.

We have \( \nabla f(p) = (\nabla f_1(p), 0) \) and \( \nabla g(p) = (\nabla g_1(p), 0) \). If they are parallel, choose \( \ell = ax_0 + bx_1 \) such that \( \ell \) is not a multiple of \( f_1 \) (or \( g_1 \)), and argue as above.

Finally assume that \( f \) and \( g \) intersect transversally at \( p \). We may assume that \( f_1 = x_1 \) and \( g_1 = x_2 \). Then \( (f, g) = (x_1, x_2) \) as ideals in the local ring \( \bar{k}[x_1, x_2, x_3(x_1, x_2)] \).

This means that \( S \) is isomorphic to the field \( \bar{k}(x_3) \). The length of any field is 1, so \( I_p(f, g) = \lg S = 1 \).

Now we can say which are the lines \( L_b \), with \( b \in Z(f_{d-1}, f_d) \), that contain a singularity other than \( O \):

**Lemma 5.** Let \( f_{d-1} \) and \( f_d \) be as in Definition 2. The line \( L_b \) contains a singular point other than \( O \) if and only if \( Z(f_{d-1}) \) is nonsingular at \( b \) and the intersection multiplicity \( I_b(f_{d-1}, f_d) > 1 \).

**Proof.** Let \( b = (b_1 : b_2 : b_3) \) and assume that \( (b_0 : b_1 : b_2 : b_3) \) is a singular point of \( Z(F) \). Then, by Lemma 2, \( f_{d-1}(b_1, b_2, b_3) = f_d(b_1, b_2, b_3) = 0 \) and \( b_0 \nabla f_{d-1}(b_1, b_2, b_3) + \nabla f_d(b_1, b_2, b_3) = 0 \), which implies \( I_b(f_{d-1}, f_d) > 1 \). Furthermore, if \( f_{d-1} \) is singular at \( b \), then the gradient \( \nabla f_{d-1}(b_1, b_2, b_3) = 0 \), so \( f_d \), too, is singular at \( b \), contrary to our assumptions.

Now assume that \( Z(f_{d-1}) \) is nonsingular at \( b = (b_1 : b_2 : b_3) \) and the intersection multiplicity \( I_b(f_{d-1}, f_d) \) > 1. The second assumption implies \( f_{d-1}(b_1, b_2, b_3) = 0 \) and \( s \nabla f_{d-1}(b_1, b_2, b_3) = t \nabla f_d(b_1, b_2, b_3) \) for some \( (s, t) \neq (0, 0) \).

Since \( Z(f_{d-1}) \) is nonsingular at \( b \), we know that \( \nabla f_{d-1}(b_1, b_2, b_3) \neq 0 \), so \( t \neq 0 \). Now \( (-s/t) : b_1 : b_2 : b_3) \neq (1 : 0 : 0 : 0) \) is a singular point of \( Z(F) \) on the line \( L_b \).

Recall that an \( A_n \) singularity is a singularity with normal form \( x_1^2 + x_2^3 + x_3^{n+1} \), see [3] p. 184.

**Proposition 6.** Let \( f_{d-1} \) and \( f_d \) be as in Definition 2 and assume \( P = (p_0 : p_1 : p_2 : p_3) \neq (1 : 0 : 0 : 0) \) is a singular point of \( Z(F) \) with \( I_{(p_1, p_2, p_3)}(f_{d-1}, f_d) = m \). Then \( P \) is an \( A_{m-1} \) singularity.

**Proof.** We may assume that \( P = (0 : 0 : 0 : 1) \) and write the local equation
\[
g := F(x_0, x_1, x_2, 1) = x_0 f_{d-1}(x_1, x_2, 1) + f_d(x_1, x_2, 1) = \sum_{i=2}^{d} g_i \tag{2}
\]
with \( g_i \in \bar{k}[x_0, x_1, x_2] \) homogeneous of degree \( i \). Since \( Z(f_{d-1}) \) is nonsingular at \( 0 := (0 : 0 : 1) \), we can assume that the linear term of \( f_{d-1}(x_1, x_2, 1) \) is equal to \( x_1 \).
The quadratic term of \( g \) is then \( g_2 = x_0x_1 + ax^2_1 + bx_1x_2 + cx^2_2 \) for some \( a, b, c \in k \).

The Hessian matrix of \( g \) evaluated at \( P \) is

\[
\begin{pmatrix}
0 & 1 & 0 \\
1 & 2a & b \\
b & 0 & 2c
\end{pmatrix}
\]

which has corank 0 when \( c \neq 0 \) and corank 1 when \( c = 0 \). By [2, p. 188], \( P \) is an \( A_1 \) singularity when \( c \neq 0 \) and an \( A_n \) singularity for some \( n \) when \( c = 0 \).

The index \( n \) of the singularity is equal to the Milnor number

\[
\mu = \dim_k \ker [k[x_0,x_1,x_2]_{x_0,x_1,x_2}] = \dim_k \ker \left( \frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right).
\]

We need to show that \( \mu = I_0(f_{d-1}, f_d) - 1 \). From the definition of the intersection multiplicity, it is not hard to see that

\[
I_0(f_{d-1}, f_d) = \dim_k \ker [k[x_1,x_2]_{x_1,x_2}] / (f_{d-1}(x_1,x_2,1), f_d(x_1,x_2,1)).
\]

The singularity at \( p \) is isolated, so the Milnor number is finite. Furthermore, since \( \gcd(f_{d-1}, f_d) = 1 \), the intersection multiplicity is finite. Therefore both dimensions can be calculated in the completion rings. For the rest of the proof we view \( f_{d-1} \) and \( f_d \) as elements of the power series rings \( k[[x_1,x_2]] \subset k[[x_0,x_1,x_2]] \), and all calculations are done in these rings.

Since \( Z(f_{d-1}) \) is smooth at \( O \), we can write

\[
f_{d-1}(x_1,x_2,1) = (x_1 - \phi(x_2)) u(x_1,x_2)
\]

for some power series \( \phi(x_2) \) and invertible power series \( u(x_1,x_2) \). To simplify notation we write \( u = u(x_1,x_2) \subset k[[x_1,x_2]] \).

The Jacobian ideal \( J_g \) is generated by the three partial derivatives:

\[
\frac{\partial g}{\partial x_0} = (x_1 - \phi(x_2)) u,
\]

\[
\frac{\partial g}{\partial x_1} = x_0 \left( u + (x_1 - \phi(x_2)) \frac{\partial u}{\partial x_1} \right) + \frac{\partial f_d}{\partial x_1}(x_1,x_2),
\]

\[
\frac{\partial g}{\partial x_2} = x_0 \left( -\phi'(x_2) u + (x_1 - \phi(x_2)) \frac{\partial u}{\partial x_2} \right) + \frac{\partial f_d}{\partial x_2}(x_1,x_2)
\]

By using the fact that \( x_1 - \phi(x_2) \in \left( \frac{\partial g}{\partial x_0} \right) \) we can write \( J_g \) without the symbols \( \frac{\partial u}{\partial x_1} \) and \( \frac{\partial u}{\partial x_2} \):

\[
J_g = \left( x_1 - \phi(x_2), x_0 u + \frac{\partial f_d}{\partial x_1}(x_1,x_2), -x_0 u \phi'(x_2) + \frac{\partial f_d}{\partial x_2}(x_1,x_2) \right)
\]
To make the following calculations clear, define the polynomials $h_i$ by writing

$$f_d(x_1, x_2, 1) = \sum_{i=0}^{d} x_1^i h_i(x_2).$$

Now

$$J_y = (x_1 - \phi(x_2), x_0 u + \sum_{i=1}^{d} i x_1^{i-1} h_i(x_2), -x_0 u \phi'(x_2) + \sum_{i=0}^{d} x_1^i h_i'(x_2)), $$

so

$$\frac{\tilde{k}[x_0, x_1, x_2]}{J_y} = \frac{\tilde{k}[x_2]}{(A(x_2))},$$

where

$$A(x_2) = \phi'(x_2) \left( \sum_{i=1}^{d} i \phi(x_2)^{i-1} h_i(x_2) \right) + \left( \sum_{i=0}^{d} \phi(x_2)^i h_i'(x_2) \right).$$

For the intersection multiplicity we have

$$\frac{\tilde{k}[x_1, x_2]}{(f_{d-1}(x_1, x_2, 1), f_d(x_1, x_2, 1))} = \frac{\tilde{k}[x_1, x_2]}{(x_1 - \phi(x_2), \sum_{i=0}^{d} x_1^i h_i(x_2))} = \frac{\tilde{k}[x_2]}{(B(x_2))},$$

where $B(x_2) = \sum_{i=0}^{d} \phi(x_2)^i h_i(x_2)$. Observing that $B'(x_2) = A(x_2)$ gives the result $\mu = I_0(f_{d-1}, f_d) - 1$.

**Corollary 7.** A monoid surface of degree $d$ can have at most $\frac{1}{2}d(d - 1)$ singularities in addition to $O$. If this number of singularities is obtained, then all of them will be of type $A_1$.

**Proof.** The sum of all local intersection numbers $I_a(f_{d-1}, f_d)$ is given by Bézout’s theorem:

$$\sum_{a \in \mathbb{Z}(f_{d-1}, f_d)} I_a(f_{d-1}, f_d) = d(d - 1).$$

The line $L_a$ will contain a singularity other than $O$ only if $I_a(f_{d-1}, f_d) \geq 2$, giving a maximum of $\frac{1}{2}d(d - 1)$ singularities in addition to $O$. Also, if this number is obtained, all local intersection numbers must be exactly 2, so all singularities other than $O$ will be of type $A_1$.

Both Proposition 6 and Corollary 7 were known to Rohn, who stated these results only in the case $d = 4$, but said they could be generalized to arbitrary $d$ [15, p. 60].

For the rest of the section we will assume $k = \mathbb{R}$. It turns out that we can find a real normal form for the singularities other than $O$. The complex singularities of type $A_n$ come in several real types, with normal forms $x_1^2 \pm x_2^2 \pm x_3^{n+1}$. Varying the $\pm$ gives two types for $n = 1$ and $n$ even, and three types for $n \geq 3$ odd. The real type with normal form $x_1^2 - x_2^2 + x_3^{n+1}$ is called an $A_n$ singularity, or of type $A^-$, and is what we find on real monoids:

**Proposition 8.** On a real monoid, all singularities other than $O$ are of type $A^-$. 
The power series
\[
g(x_0, x_1, x_2) = g_0(x_0, x_1, x_2 + \psi(x_2)) = x_0x_1 + f(x_1 + \phi(x_2), x_2)u^{-1}(x_1 + \phi(x_2), x_2)
\]
where \(\psi(x_1, x_2) \in \mathbb{R}[x_1, x_2]\). Write \(\psi(x_1, x_2) = x_1\psi_1(x_1, x_2) + \psi_2(x_2)\) and define
\[
g_0(x_0, x_1, x_2) = g_0(x_0, x_1, x_2 + \psi_2(x_2)) = x_0x_1 + \psi_1(x_1, x_2).
\]
The power series \(\psi_2(x_2)\) can be written on the form
\[
\psi_2(x_2) = sx_2^n(a_0 + a_1x_2 + a_2x_2^2 + \ldots)
\]
where \(s = \pm 1\) and \(a_0 > 0\). We see that \(g_0(2)\) is right equivalent to \(g_0(3) = x_0x_1 + sx_2^n\) since
\[
g_0(2)(x_0, x_1, x_2) = g_0(3)\left( x_0, x_1, x_2 \sqrt[2n]{a_0 + a_1x_2 + a_2x_2^2 + \ldots} \right).
\]
Finally we see that
\[
g_0(4)(x_0, x_1, x_2) := g_0(3)\left( sx_0 - sx_1, x_0 + x_1, x_2 \right) = s(x_0^2 - x_1^2 + x_2^n)
\]
proves that \(u^{-1}g\) is right equivalent to \(s(x_0^2 - x_1^2 + x_2^n)\) which is an equation for an \(A_{n-1}\) singularity with normal form \(x_0^2 - x_1^2 + x_2^n\).

Note that for \(d = 3\), the singularity at \(O\) can be an \(A_{1^+}\) singularity. This happens for example when \(f_2 = x_0^2 - x_1^2 + x_2^2\).

For a real monoid, Corollary 11 implies that we can have at most \(\frac{1}{d}d(d - 1)\) real singularities in addition to \(O\). We can show that the bound is sharp by a simple construction:

**Example.** To construct a monoid with the maximal number of real singularities, it is sufficient to construct two affine real curves in the \(xy\)-plane defined by equations \(f_{d-1}\) and \(f_d\) of degrees \(d - 1\) and \(d\) such that the curves intersect in \(d(d - 1)/2\) points with multiplicity 2. Let \(m \in \{d - 1, d\}\) be odd and set
\[
f_m = e - m \prod_{i=1}^{m} \left( x \sin \left( \frac{2i\pi}{m} \right) + y \cos \left( \frac{2i\pi}{m} \right) + 1 \right).
\]
For \(\varepsilon > 0\) sufficiently small there exist at least \(\frac{m+1}{2}\) radii \(r > 0\), one for each root of the univariate polynomial \(f_m|_{x=0}\), such that the circle \(x^2 + y^2 - r^2\) intersects \(f_m\) in \(m\) points.
with multiplicity 2. Let $f_{2d-1-m}$ be a product of such circles. Now the homogenizations of $f_{d-1}$ and $f_d$ define a monoid surface with $1 + \frac{1}{2}d(d - 1)$ singularities. See Figure 1.

Proposition 6 and Bezout’s theorem imply that the maximal Milnor number of a singularity other than $O$ is $d(d - 1) - 1$. The following example shows that this bound can be achieved on a real monoid:

**Example.** The surface $X \subset \mathbb{P}^3$ defined by $F = x_0(x_1x_2^{d-2} + x_3^{d-1}) + x_1^d$ has exactly two singular points. The point $(1 : 0 : 0 : 0)$ is a singularity of multiplicity $d - 1$ with Milnor number $\mu = (d^2 - 3d + 1)(d - 2)$, while the point $(0 : 0 : 1 : 0)$ is an $A_{d(d-1)-1}$ singularity. A picture of this surface for $d = 4$ is shown in Figure 2.

### 4 Quartic monoid surfaces

Every cubic surface with isolated singularities is a monoid. Both smooth and singular cubic surfaces have been studied extensively, most notably in [16], where real cubic surfaces and their singularities were classified, and more recently in [18], [4], and [8]. The site [9] contains additional pictures and references.

In this section we shall consider the case $d = 4$. The classification of real and complex quartic monoid surfaces was started by Rohn [15]. (In addition to considering the singularities, Rohn studied the existence of lines not passing through the triple point, and that of other special curves on the monoid.) In [19], Takahashi, Watanabe, and Higuchi described the singularities of such complex surfaces. The monoid singularity of a quartic monoid is minimally elliptic [21], and minimally elliptic singularities have the same complex topological type if and only if their dual graphs are isomorphic [10]. In [10] all possible dual graphs for minimally elliptic singularities are listed, along with example equations.

Using Arnold’s notation for the singularities, we use and extend the approach of Takahashi, Watanabe, and Higuchi in [19].
Consider a quartic monoid surface, $X = Z(F)$, with $F = x_0 f_3 + f_4$. The tangent cone, $Z(f_3)$, can be of one of nine (complex) types, each needing a separate analysis.

For each type we fix $f_3$, but any other tangent cone of the same type will be projectively equivalent (over the complex numbers) to this fixed $f_3$. The nine different types are:

1. Nodal irreducible curve, $f_3 = x_1 x_2 x_3 + x_2^2 + x_3^2$.
2. Cuspidal curve, $f_3 = x_1^3 - x_2^2 x_3$.
3. Conic and a chord, $f_3 = x_3 (x_1 x_2 + x_2^2)$.
4. Conic and a tangent line, $f_3 = x_3 (x_1 x_3 + x_2^2)$.
5. Three general lines, $f_3 = x_1 x_2 x_3$.
6. Three lines meeting in a point, $f_3 = x_2^3 - x_2 x_3^2$.
7. A double line and another line, $f_3 = x_2 x_3^3$.
8. A triple line $f_3 = x_3^3$.
9. A smooth curve, $f_3 = x_3^3 + x_2^3 + x_3^3 + 3 a x_0 x_1 x_3$ where $a^3 \neq -1$.

To each quartic monoid we can associate, in addition to the type, several integer invariants, all given as intersection numbers. From [19] we know that, for the types 1–3, 5, and 9, these invariants will determine the singularity type of $O$ up to right equivalence. In the other cases the singularity series, as defined by Arnol’d in [11] and [2], is determined by the type of $f_3$. We shall use, without proof, the results on the singularity type of $O$ due to [19]; however, we shall use the notations of [11] and [2].

We complete the classification begun in [19] by supplying a complete list of the possible singularities occurring on a quartic monoid. In addition, we extend the results to the case of real monoids. Our results are summarized in the following theorem.

**Theorem 9.** On a quartic monoid surface, singularities other than the monoid point can occur as given in Table [1]. Moreover, all possibilities are realizable on real quartic monoids.

*Fig. 2. The surface defined by $F = x_0 (x_1 x_2^{d-2} + x_3^{d-1}) + x_4^d$ for $d = 4$.***
monoids with a real monoid point, and with the other singularities being real and of type $A^-$. 

| Case | Triple point | Invariants and constraints | Other singularities |
|------|-------------|----------------------------|--------------------|
| 1    | $T_{3,3,4}$ | $m = 2, \ldots, 12$          | $A_{m-1}, \sum m_i = 12$ |
|      | $T_{3,3,4+m}$ |                        | $A_{m-1}, \sum m_i = 12 - m$ |
| 2    | $Q_10$       | $m = 2, 3$                | $A_{m-1}, \sum m_i = 12$ |
| 3    | $T_{3,4+r_0,4+r_1}$ | $r_0 = \max(j_0, k_0), r_1 = \max(j_1, k_1)$, $j_0 > 0 \iff k_0 > 0, \min(j_0, k_0) \leq 1$, $j_1 > 0 \iff k_1 > 0, \min(j_1, k_1) \leq 1$ | $A_{m-1}, \sum m_i = 4 - k_0 - k_1$, $A_{m'-1}, \sum m_i' = 8 - j_0 - j_1$ |
| 4    | $S$ series   | $j_0 \leq 8, k_0 \leq 4, \min(j_0, k_0) \leq 2$, $j_0 > 0 \iff k_0 > 0, j_1 > 0 \iff k_0 > 1$ | $A_{m-1}, \sum m_i = 4 - k_0$, $A_{m'-1}, \sum m_i' = 8 - j_0$ |
| 5    | $T_{4+j_k,4+j_l,4+j_m}$ | $m_1 + l_1 \leq 4, k_2 + m_2 \leq 4$, $k_3 + l_3 \leq 4, k_3 > 0 \iff k_3 > 0$, $l_1 > 0 \iff l_3 > 0, m_1 > 0 \iff m_2 > 0$, $\min(k_2, k_3) \leq 1, \min(l_1, l_3) \leq 1$, $\min(m_1, m_2) \leq 1$, $j_k = \max(k_2, k_3)$, $j_1 = \max(l_1, l_3)$, $j_m = \max(m_1, m_2)$ | $A_{m-1}, \sum m_i = 4 - m_1 - l_1$, $A_{m'-1}, \sum m_i' = 4 - k_2 - m_2$, $A_{m''-1}, \sum m_i'' = 4 - k_3 - l_3$ |
| 6    | $U$ series   | $j_1 > 0 \iff j_2 > 0 \iff j_3 > 0$, at most one of $j_1, j_2, j_3 > 1$, $j_1, j_2, j_3 \leq 4$ | $A_{m-1}, \sum m_i = 4 - j_1$, $A_{m'-1}, \sum m_i' = 4 - j_2$, $A_{m''-1}, \sum m_i'' = 4 - j_3$ |
| 7    | $V$ series   | $j_0 > 0 \iff k_0 > 0, \min(j_0, k_0) \leq 1$, $j_0 \leq 4, k_0 \leq 4$ | $A_{m-1}, \sum m_i = 4 - j_0$, $A_{m'-1}, \sum m_i' = 4 - j_0$, $A_{m''-1}, \sum m_i'' = 4 - j_0$ |
| 8    | $V'$ series  | None                       | $A_{m-1}, \sum m_i = 12$ |
| 9    | $P_5 = T_{3,3,3}$ |                            | $A_{m-1}, \sum m_i = 12$ |

Table 1. Possible configurations of singularities for each case

Proof. The invariants listed in the “Invariants and constraints” column are all nonnegative integers, and any set of integer values satisfying the equations represents one possible set of invariants, as described above. Then, for each set of invariants, (positive) intersection multiplicities, denoted $m_i$, $m_i'$ and $m_i''$, will determine the singularities other than $O$. The column “Other singularities” give these and the equations they must satisfy. Here we use the notation $A_0$ for a line $L_0$ on $Z(F)$ where $O$ is the only singular point.

The analyses of the nine cases share many similarities, and we have chosen not to go into great detail when one aspect of a case differs little from the previous one. We end the section with a discussion on the possible real forms of the tangent cone and how this affects the classification of the real quartic monoids.
In all cases, we shall write
\[ f_4 = a_1x_1^4 + a_2x_1^3x_2 + a_3x_1^2x_3 + a_4x_1x_2^3 + a_5x_1^3x_2x_3 + a_6x_1^2x_3^2 + a_7x_1x_2^2x_3 + a_8x_1x_2x_3^2 + a_9x_1x_2x_3^3 + a_{10}x_1x_3^3 + a_{11}x_2^3 + a_{12}x_2^2x_3 + a_{13}x_2x_3^2 + a_{14}x_2x_3^3 + a_{15}x_3^4 \]

and we shall investigate how the coefficients \( a_1, \ldots, a_{15} \) are related to the geometry of the monoid.

**Case 1.** The tangent cone is a nodal irreducible curve, and we can assume
\[ f_3(x_1, x_2, x_3) = x_1x_2x_3 + x_2^3 + x_3^3. \]

The nodal curve is singular at \((1 : 0 : 0)\). If \( f_4(1, 0, 0) \neq 0 \), then \( O \) is a \( T_{3,3,4} \) singularity [19]. We recall that \((1 : 0 : 0)\) cannot be a singular point on \( Z(f_4) \) as this would imply a singular line on the monoid, so we assume that either \((1 : 0 : 0) \notin Z(f_4)\) or \((1 : 0 : 0)\) is a smooth point on \( Z(f_4) \). Let \( m \) denote the intersection number \( I_{(1,0,0)}(f_3, f_4) \). Since \( Z(f_3) \) is singular at \((1 : 0 : 0)\) we have \( m \neq 1 \). From [19] we know that \( O \) is a \( T_{3,3,3+m} \) singularity for \( m = 2, \ldots, 12 \). Note that some of these complex singularities have two real forms, as illustrated in Figure 3.

**Fig. 3.** The monoids \( Z(x^3 + y^3 + 5xyz - z^3(x + y)) \) and \( Z(x^3 + y^3 + 5xyz - z^3(x - y)) \) both have a \( T_{3,3,5} \) singularity, but the singularities are not right equivalent over \( \mathbb{R} \). (The pictures are generated by the program [5].)

Bézout’s theorem and Proposition 6 limit the possible configurations of singularities on the monoid for each \( m \). Let \( \theta(s, t) = (-s^3 - t^3, s^2t, st^2) \). Then the tangent cone \( Z(f_3) \) is parameterized by \( \theta \) as a map from \( \mathbb{P}^1 \) to \( \mathbb{P}^2 \). When we need to compute the intersection numbers between the rational curve \( Z(f_3) \) and the curve \( Z(f_4) \), we can do
that by studying the roots of the polynomial \( f_4(\theta) \). Expanding the polynomial gives

\[
f_4(\theta)(s, t) = a_1s^{12} - a_2s^{11}t + (-a_3 + a_4)s^{10}t^2 + (4a_1 + a_5 - a_7)s^9t^3 \\
+ (-3a_2 + a_6 - a_8 + a_{11})s^8t^4 + (-3a_3 + 2a_4 - a_9 + a_{12})s^7t^5 \\
+ (6a_1 + 2a_5 - a_7 - a_{10} + a_{13})s^6t^6 \\
+ (-3a_2 + 2a_6 - a_8 + a_{14})s^5t^7 + (-3a_3 + a_4 - a_9 + a_{15})s^4t^8 \\
+ (4a_1 + a_5 - a_{10})s^3t^9 + (-a_2 + a_6)s^2t^{10} - a_3st^{11} + a_4t^{12}.
\]

This polynomial will have roots at \((0 : 1)\) and \((1 : 0)\) if and only if \(f_4(1, 0, 0) = a_1 = 0\). When \(a_1 = 0\) we may (by symmetry) assume \(a_3 \neq 0\), so that \((0 : 1)\) is a simple root and \((1 : 0)\) is a root of multiplicity \(m - 1\). Other roots of \(f_4(\theta)\) correspond to intersections of \(Z(f_3)\) and \(Z(f_4)\) away from \((1 : 0 : 0)\). The multiplicity \(m_i\) of each root is equal to the corresponding intersection multiplicity, giving rise to an \(A_{m_i - 1}\) singularity if \(m_i > 0\), as described by Proposition 6, or a line \(L_0 \subset Z(F)\) with \(O\) as the only singular point if \(m_i = 1\).

The polynomial \(f_4(\theta)\) defines a linear map from the coefficient space \(k^{15}\) of \(f_4\) to the space of homogeneous polynomials of degree 12 in \(s\) and \(t\). By elementary linear algebra, we see that the image of this map is the set of polynomials of the form

\[
b_0s^{12} + b_1s^{11}t + b_2s^{10}t^2 + \cdots + b_{12}t^{12}
\]

where \(b_0 = b_{12}\). The kernel of the map corresponds to the set of polynomials of the form \(\ell f_3\) where \(\ell\) is a linear form. This means that \(f_4(\theta) \equiv 0\) if and only if \(f_3\) is a factor in \(f_4\), making \(Z(F)\) reducible and not a monoid.

For every \(m = 0, 2, 3, 4, \ldots, 12\) we can select \(r\) parameter points

\[
p_1, \ldots, p_r \in \mathbb{P}^1 \setminus \{(0 : 1), (1 : 0)\}
\]

and positive multiplicities \(m_1, \ldots, m_r\) with \(m_1 + \cdots + m_r = 12 - m\) and try to describe the polynomials \(f_4\) such that \(f_4(\theta)\) has a root of multiplicity \(m_i\) at \(p_i\) for each \(i = 1, \ldots, r\).

Still assuming \(a_3 \neq 0\) whenever \(a_1 = 0\), any such choice of parameter points \(p_1, \ldots, p_r\) and multiplicities \(m_1, \ldots, m_r\) corresponds to a polynomial \(q = b_0s^{12} + b_1s^{11}t + \cdots + b_{12}t^{12}\) that is, up to a nonzero constant, uniquely determined.

Now, \(q\) is equal to \(f_3(\theta)\) for some \(f_4\) if and only if \(b_0 = b_{12}\). If \(m \geq 2\), then \(q\) contains a factor \({s^{m-1}}\), so \(b_0 = b_{12} = 0\), giving \(q = f_4(\theta)\) for some \(f_4\). In fact, when \(m \geq 2\) any choice of \(p_1, \ldots, p_r\) and \(m_1, \ldots, m_r\) with \(m_1 + \cdots + m_r = 12 - m\) corresponds to a four dimensional space of equations \(f_4\) that gives this set of roots and multiplicities in \(f_4(\theta)\). If \(f'_4\) is one such \(f_4\), then any other is of the form \(\lambda f'_4 + \ell f_3\) for some constant \(\lambda \neq 0\) and linear form \(\ell\). All of these give monoids that are projectively equivalent.

When \(m = 0\), we write \(p_i = (\alpha_i : \beta_i)\) for \(i = 1, \ldots, r\). The condition \(b_0 = b_{12}\) on the coefficients of \(q\) translates to

\[
a_1^{m_1} \cdots a_r^{m_r} = \beta_1^{m_1} \cdots \beta_r^{m_r}.
\]
This means that any choice of parameter points $(\alpha_1 : \beta_1), \ldots, (\alpha_r : \beta_r)$ and multiplicities $m_1, \ldots, m_r$ with $m_1 + \cdots + m_r = 12$ that satisfy condition (3) corresponds to a four dimensional family $\lambda f_4 + \ell f_3$, giving a unique monoid up to projective equivalence.

For example, we can have an $A_{11}$ singularity only if $f_4(\theta)$ is of the form $(\alpha s - \beta t)^{12}$. Condition (3) implies that this can only happen for 12 parameter points, all of the form $(1 : \omega)$, where $\omega^{12} = 1$. Each such parameter point $(1 : \omega)$ corresponds to a monoid uniquely determined up to projective equivalence. However, since there are six projective transformations of the plane that maps $Z(f_3)$ onto itself, this correspondence is not one to one. If $\omega^{12} = \omega_2^{12} = 1$, then $\omega_1$ and $\omega_2$ will correspond to projectively equivalent monoids if and only if $\omega_1^3 = \omega_2^3$ or $\omega_1^3 \omega_2^3 = 1$. This means that there are three different quartic monoids with one $T_{3,3,4}$ singularity and one $A_{11}$ singularity. One corresponds to those $\omega$ where $\omega^3 = 1$, one to those $\omega$ where $\omega^3 = -1$, and one to those $\omega$ where $\omega^6 = -1$. The first two of these have real representatives, $\omega = \pm 1$.

It easy to see that for any set of multiplicities $m_1 + \cdots + m_r = 12$, we can find real points $p_1, \ldots, p_r$ such that condition (3) is satisfied. This completely classifies the possible configurations of singularities when $f_3$ is an irreducible nodal curve.

**Case 2.** The tangent cone is a cuspidal curve, and we can assume $f_3(x_1, x_2, x_3) = x_1^3 - x_2^2 x_3$. The cuspidal curve is singular at $(0 : 0 : 1)$ and can be parameterized by $\theta$ as a map from $\mathbb{P}^1$ to $\mathbb{P}^2$ where $\theta(s, t) = (s^2 t, s^3, t^3)$. The intersection numbers are determined by the degree 12 polynomial $f_4(\theta)$. As in the previous case, $f_4(\theta) \equiv 0$ if and only if $f_3$ is a factor of $f_4$, and we will assume this is not the case. The multiplicity $m$ of the factor $s$ in $f_4(\theta)$ determines the type of singularity at $O$. If $m = 0$ (no factor $s$), then $O$ is a $Q_{10}$ singularity. If $m = 2$ or $m = 3$, then $O$ is of type $Q_{9+m}$. If $m > 3$, then $(0 : 0 : 1)$ is a singular point on $Z(f_4)$, so the monoid has a singular line and is not considered in this article. Also, $m = 1$ is not possible, since $f_4(\theta(s, t)) = f_4(s^2 t, s^3, t^3)$ cannot contain $st^{11}$ as a factor.

For each $m = 0, 2, 3$ we can analyze the possible configurations of other singularities on the monoid. Similarly to the previous case, any choice of parameter points $p_1, \ldots, p_r \in \mathbb{P}^1 \setminus \{(0 : 1)\}$ and positive multiplicities $m_1, \ldots, m_r$ with $\sum m_i = 12 - m$ corresponds, up to a nonzero constant, to a unique degree 12 polynomial $q$.

When $m = 2$ or $m = 3$, for any choice of parameter values and associated multiplicities, we can find a four dimensional family $f_4 = \lambda f_4 + \ell f_3$ with the prescribed roots in $f_4(\theta)$. As before, the family gives projectively equivalent monoids.

When $m = 0$, one condition must be satisfied for $q$ to be of the form $f_4(\theta)$, namely $b_{11} = 0$, where $b_{11}$ is the coefficient of $st^{11}$ in $q$.

For example, we can have an $A_{11}$ singularity only if $q$ is of the form $(\alpha s - \beta t)^{12}$. The condition $b_{11} = 0$ implies that either $q = \lambda s^{12}$ or $q = \lambda t^{12}$. The first case gives a surface with a singular line, while the other gives a monoid with an $A_{11}$ singularity (see Figure 2). The line from $O$ to the $A_{11}$ singularity corresponds to the inflection point of $Z(f_3)$.

For any set of multiplicities $m_1, \ldots, m_r$ with $m_1 + \cdots + m_r = 12$, it is not hard to see that there exist real points $p_1, \ldots, p_r$ such that the condition $b_{11} = 0$ is satisfied. It suffices to take $p_1 = (\alpha_1 : 1)$, with $\sum m_i \alpha_i = 0$ (the condition corresponding to $b_{11} = 0$). This completely classifies the possible configurations of singularities when $f_3$ is a cuspidal curve.
Case 3. The tangent cone is the product of a conic and a line that is not tangent to the conic, and we can assume $f_3 = x_3(x_1x_2 + x_3^2)$. Then $Z(f_3)$ is singular at $(1 : 0 : 0)$ and $(0 : 1 : 0)$, the intersections of the conic $Z(x_1x_2 + x_3^2)$ and the line $Z(x_3)$. For each $f_4$ we can associate four integers:

$$j_0 := I_{(1:0:0)}(x_1x_2 + x_3^2, f_4), \quad k_0 := I_{(1:0:0)}(x_3, f_4),$$

$$j_1 := I_{(0:1:0)}(x_1x_2 + x_3^2, f_4), \quad k_1 := I_{(0:1:0)}(x_3, f_4).$$

We see that $k_0 > 0 \iff f_4(1 : 0 : 0) = 0 \iff j_0 > 0$, and that $Z(f_4)$ is singular at $(1 : 0 : 0)$ if and only if $k_0$ and $j_0$ both are bigger than one. These cases imply a singular line on the monoid, and are not considered in this article. The same holds for $k_1, j_1$ and the point $(0 : 1 : 0)$.

Define $r_i = \max(j_i, k_i)$ for $i = 1, 2$. Then, by [19], $O$ will be a singularity of type $T_{3,4+r_1,4+r_2}$ if $r_0 \leq r_1$, or of type $T_{3,4+r_1,4+r_2}$ if $r_0 \geq r_1$.

We can parameterize the line $Z(x_3)$ by $\theta_1$ where $\theta_1(s, t) = (s, t, 0)$, and the conic $Z(x_1x_2 + x_3^2)$ by $\theta_2$ where $\theta_2(s, t) = (s^2, t^2, st)$. Similarly to the previous cases, roots of $f_4(\theta_1)$ correspond to intersections between $Z(f_4)$ and the line $Z(x_3)$, while roots of $f_4(\theta_2)$ correspond to intersections between $Z(f_4)$ and the conic $Z(x_1x_2 + x_3^2)$.

For any legal values of of $j_0, j_1, k_0$ and $k_1$, parameter points

$$(\alpha_1 : \beta_1), \ldots, (\alpha_m : \beta_m) \in \mathbb{P}^1 \setminus \{(0 : 1), (1 : 0)\},$$

with multiplicities $m_1, \ldots, m_r$ such that $m_1 + \cdots + m_r = 4 - k_0 - k_1$, and parameter points

$$(\alpha'_1 : \beta'_1), \ldots, (\alpha'_m : \beta'_m) \in \mathbb{P}^1 \setminus \{(0 : 1), (1 : 0)\},$$

with multiplicities $m'_1, \ldots, m'_r$ such that $m'_1 + \cdots + m'_r = 8 - j_0 - j_1$, we can fix polynomials $q_1$ and $q_2$ such that

- $q_1$ is nonzero, of degree 4, and has factors $s^{k_0}t^{\alpha_i} \cdot (\beta_is - \alpha_it)^{m_i}$ for $i = 1, \ldots, r$,
- $q_2$ is nonzero, of degree 8, and has factors $s^{l_0}t^{\alpha_i} \cdot (\beta_is - \alpha_it)^{m_i}$ for $i = 1, \ldots, r'$.

Now $q_1$ and $q_2$ are determined up to multiplication by nonzero constants. Write $q_1 = b_0s^4 + \cdots + b_4t^4$ and $q_2 = c_0s^8 + \cdots + c_8t^8$.

The classification of singularities on the monoid consists of describing the conditions on the parameter points and nonzero constants $\lambda_1$ and $\lambda_2$ for the pair $(\lambda_1q_1, \lambda_2q_2)$ to be on the form $(f_4(\theta_1), f_4(\theta_2))$ for some $f_4$.

Similarly to the previous cases, $f_4(\theta_1) \equiv 0$ if and only if $x_3$ is a factor in $f_4$ and $f_4(\theta_2) \equiv 0$ if and only if $x_1x_2 + x_3^2$ is a factor in $f_4$. Since $f_3 = x_3(x_1x_2 + x_3^2)$, both cases will make the monoid reducible, so we only consider $\lambda_1, \lambda_2 \neq 0$.

We use linear algebra to study the relationship between the coefficients $a_1 \ldots a_{15}$ of $f_4$ and the polynomials $q_1$ and $q_2$. We find $(\lambda_1q_1, \lambda_2q_2)$ to be of the form $(f_4(\theta_1), f_4(\theta_2))$ if and only if $\lambda_1b_0 = \lambda_2c_0$ and $\lambda_1b_4 = \lambda_2c_8$. Furthermore, the pair $(\lambda_1q_1, \lambda_2q_2)$ will fix $f_4$ modulo $f_3$. Since $f_4$ and $\lambda f_4$ correspond to projectively equivalent monoids for any $\lambda \neq 0$, it is the ratio $\lambda_1/\lambda_2$, and not $\lambda_1$ and $\lambda_2$, that is important.
Recall that $k_0 > 0 \Leftrightarrow j_0 > 0$ and $k_1 > 0 \Leftrightarrow j_1 > 0$. If $k_0 > 0$ and $k_1 > 0$, then $b_0 = c_0 = b_4 = c_8 = 0$, so for any $\lambda_1, \lambda_2 \neq 0$ we have $(\lambda_1 q_1, \lambda_2 q_2) = (f_4(\theta_1), f_4(\theta_2))$ for some $f_4$. Varying $\lambda_1/\lambda_2$ will give a one-parameter family of monoids for each choice of multiplicities and parameter points.

If $k_0 = 0$ and $k_1 > 0$, then $b_0 = c_0 = 0$. The condition $\lambda_1 b_4 = \lambda_2 c_8$ implies $\lambda_1/\lambda_2 = c_8/b_4$. This means that any choice of multiplicities and parameter points will give a unique monoid up to projective equivalence. The same goes for the case where $k_0 > 0$ and $k_1 = 0$.

Finally, consider the case where $k_0 = k_1 = 0$. For $(\lambda_1 q_1, \lambda_2 q_2)$ to be of the form $(f_4(\theta_1), f_4(\theta_2))$ we must have $\lambda_1/\lambda_2 = c_8/b_4 = c_0/b_0$. This translates into a condition on the parameter points, namely

$$
\frac{(\beta_1')^m \cdots (\beta_r')^m}{\beta_1^{m_1} \cdots \beta_r^{m_r}} = \frac{(\alpha_1')^n \cdots (\alpha_r')^n}{\alpha_1^{n_1} \cdots \alpha_r^{n_r}}.
$$

(4)

In other words, if condition (4) holds, we have a unique monoid up to projective equivalence.

It is easy to see that for any choice of multiplicities, it is possible to find real parameter points such that condition (4) is satisfied. This completes the classification of possible singularities when the tangent cone is a conic plus a chordal line.

Case 4. The tangent cone is the product of a conic and a line tangent to the conic, and we can assume $f_3 = x_3(x_1 x_3 + x_2^2)$. Now $Z(f_3)$ is singular at $(1 : 0 : 0)$. For each $f_4$ we can associate two integers

$$
j_0 := I_{(1:0:0)}(x_3, x_3^2, f_4) \quad \text{and} \quad k_0 := I_{(1:0:0)}(x_3, f_4).
$$

We have $j_0 > 0 \Leftrightarrow k_0 > 0, j_0 > 1 \Leftrightarrow k_0 > 1$. Furthermore, $j_0$ and $k_0$ are both greater than 2 if and only if $Z(f_4)$ is singular at $(1 : 0 : 0)$, a case we have excluded. The singularity at $O$ will be of the $S$ series, from [1], [2].

We can parameterize the conic $Z(x_1 x_3 + x_2^2)$ by $\theta_2$ and the line $Z(x_3)$ by $\theta_1$ where $\theta_2(s : t) = (s^2 : st : -t^2)$ and $\theta_1(s : t) = (s : t : 0)$. As in the previous case, the monoid is reducible if and only if $f_4(\theta_1) \equiv 0$ or $f_4(\theta_2) \equiv 0$. Consider two nonzero polynomials

$$
q_1 = b_0 s^4 + b_1 s^3 t + b_2 s^2 t^2 + b_3 s t^3 + b_4 t^4
$$

$$
q_2 = c_0 s^8 + c_1 s^7 t + \cdots + c_7 s t^7 + c_8 t^8.
$$

Now $(\lambda_1 q_1, \lambda_2 q_2) = (f_4(\theta_1), f_4(\theta_2))$ for some $f_4$ if and only if $\lambda_1 b_0 = \lambda_2 c_0$ and $\lambda_1 b_1 = \lambda_2 c_1$. As before, only the cases where $\lambda_1, \lambda_2 \neq 0$ are interesting. We see that $(\lambda_1 q_1, \lambda_2 q_2) = (f_4(\theta_1), f_4(\theta_2))$ for some $\lambda_1, \lambda_2 \neq 0$ if and only if the following hold:

- $b_0 = 0 \Leftrightarrow c_0 = 0$ and $b_1 = 0 \Leftrightarrow c_1 = 0$
- $b_0 c_1 = b_1 c_0$

The classification of other singularities (than $O$) is very similar to the previous case. Roots of $f_4(\theta_1)$ and $f_4(\theta_2)$ away from $(1 : 0)$ correspond to intersections of $Z(f_3)$ and $Z(f_4)$ away from the singular point of $Z(f_3)$, and when one such intersection is multiple, there is a corresponding singularity on the monoid.
Now assume \((\lambda_1 q_1, \lambda_2 q_2) = (f_4(\theta_1), f_4(\theta_2))\) for some \(\lambda_1, \lambda_2 \neq 0\) and some \(f_4\). If \(b_0 \neq 0\) (equivalent to \(c_0 \neq 0\)) then \(j_0 = k_0 = 0\) and \(\lambda_1/\lambda_2 = c_0/b_0\). If \(b_0 = c_0 = 0\) and \(b_1 \neq 0\) (equivalent to \(c_1 \neq 0\)), then \(j_0 = k_0 = 1\), and \(\lambda_1/\lambda_2 = c_1/b_1\). If \(b_0 = b_1 = c_0 = c_1 = 0\), then \(j_0, k_0 > 1\) and any value of \(\lambda_1/\lambda_2\) will give \((\lambda_1 q_1, \lambda_2 q_2)\) of the form \((f_4(\theta_1), f_4(\theta_2))\) for some \(f_4\). Thus we get a one-dimensional family of monoids for this choice of \(q_1\) and \(q_2\).

Now consider the possible configurations of other singularities on the monoid. Assume that \(j_0^\prime \leq 8\) and \(k_0^\prime \leq 4\) are nonnegative integers such that \(j_0 > 0 \leftrightarrow k_0 > 0\) and \(j_0 > 1 \leftrightarrow k_0 > 1\). For any set of multiplicities \(m_1, \ldots, m_r\) with \(m_1 + \cdots + m_r = 4 - k_0^\prime\) and \(m_1^\prime, \ldots, m_r^\prime\) with \(m_1^\prime + \cdots + m_r^\prime = 8 - j_0^\prime\), there exists a polynomial \(f_4\) with real coefficients such that \(f_4(\theta_1)\) has real roots away from \((1 : 0)\) with multiplicities \(m_1, \ldots, m_r\), and \(f_4(\theta_2)\) has real roots away from \((1 : 0)\) with multiplicities \(m_1^\prime, \ldots, m_r^\prime\). Furthermore, for this \(f_4\) we have \(k_0 = k_0^\prime\) and \(j_0 = j_0^\prime\). Proposition 6 will give the singularities that occur in addition to \(O\).

This completes the classification of the singularities on a quartic monoid (other than \(O\)) when the tangent cone is a conic plus a tangent.

**Case 5.** The tangent cone is three general lines, and we assume \(f_3 = x_1 x_2 x_3\).

For each \(f_4\) we associate six integers,

\[
\begin{align*}
k_2 &:= I_{(1:0:0)}(f_4, x_2), \\
l_1 &:= I_{(0:1:0)}(f_4, x_1), \\
m_1 &:= I_{(0:0:1)}(f_4, x_1), \\
k_3 &:= I_{(1:0:0)}(f_4, x_3), \\
l_3 &:= I_{(0:1:0)}(f_4, x_3), \\
m_2 &:= I_{(0:0:1)}(f_4, x_2).
\end{align*}
\]

Now \(k_2 > 0 \leftrightarrow k_3 > 0, l_1 > 0 \leftrightarrow l_3 > 0,\) and \(m_1 > 0 \leftrightarrow m_2 > 0\). If both \(k_2\) and \(k_3\) are greater than 1, then the monoid has a singular line, a case we have excluded. The same goes for the pairs \((l_1, l_3)\) and \((m_1, m_2)\).

When the monoid does not have a singular line, we define \(j_k = \max(k_2, k_3), j_l = \max(l_1, l_3)\) and \(j_m = \max(m_1, m_2)\). If \(j_k \leq j_l \leq j_m\), then [19] gives that \(O\) is a \(T_4+\cdot\cdot\cdot+4+p+4+j\) singularity.

The three lines \(Z(x_1), Z(x_2)\) and \(Z(x_3)\) are parameterized by \(\theta_1, \theta_2\) and \(\theta_3\) where \(\theta_1(s : t) = (0 : s : t), \theta_2(s : t) = (s : 0 : t)\) and \(\theta_3(s : t) = (s : t : 0)\). Roots of the polynomial \(f_4(\theta_1)\) away from \((1 : 0)\) and \((0 : 1)\) correspond to intersections between \(Z(f_4)\) and \(Z(x_i)\) away from the singular points of \(Z(f_4)\).

As before, we are only interested in the cases where none of \(f_4(\theta_1) = 0\) for \(i = 1, 2, 3\), as this would make the monoid reducible.

For the study of other singularities on the monoid we consider nonzero polynomials

\[
\begin{align*}
q_1 &= b_0 s^4 + b_1 s^3 t + b_2 s^2 t^2 + b_3 s t^3 + b_4 t^4, \\
q_2 &= c_0 s^4 + c_1 s^3 t + c_2 s^2 t^2 + c_3 s t^3 + c_4 t^4, \\
q_3 &= d_0 s^4 + d_1 s^3 t + d_2 s^2 t^2 + d_3 s t^3 + d_4 t^4.
\end{align*}
\]

Linear algebra shows that \((\lambda_1 q_1, \lambda_2 q_2, \lambda_3 q_3) = (f_4(\theta_1), f_4(\theta_2), f_4(\theta_3))\) for some \(f_4\) if and only if \(\lambda_1 b_0 = \lambda_2 d_1, \lambda_1 b_4 = \lambda_2 c_4,\) and \(\lambda_2 c_0 = \lambda_3 d_0\). A simple analysis shows the following: There exist \(\lambda_1, \lambda_2, \lambda_3 \neq 0\) such that

\[
(\lambda_1 q_1, \lambda_2 q_2, \lambda_3 q_3) = (f_4(\theta_1), f_4(\theta_2), f_4(\theta_3))
\]

for some \(f_4\), and such that \(Z(f_4)\) and \(Z(f_3)\) have no common singular point if and only if all of the following hold:
identically zero, so we assume that neither is identically zero. For each \( \theta \)

The tangent cone is singular along the line \( \lambda \). This completes the classification of the singularities (other

multiplicity of the factor \( \lambda \)) that will be greater than one.) The singularity will be of the

Case 6. The tangent cone is three lines meeting in a point, and we can assume that \( f_3 = x_2^3 - x_2x_3^2 \). We write \( f_3 = \ell_1\ell_2\ell_3 \) where \( \ell_1 = x_2, \ell_2 = x_2 - x_3 \) and \( \ell_3 = x_2 + x_3 \), representing the three lines going through the singular point \( 1 : 0 : 0 \).

For each \( f_4 \) we associate three integers \( j_1, j_2 \) and \( j_3 \) defined as the intersection numbers \( j_i = I_{(1,0,0)}(f_4, \ell_i) \). We see that \( j_1 = 0 \iff j_2 = 0 \iff j_3 = 0 \), and that \( Z(f_4) \) is singular at \( 1 : 0 : 0 \) if and only if two of the integers \( j_1, j_2, j_3 \) are greater then one. (Then all of them will be greater than one.) The singularity will be of the \( U \) series \([1], [2] \).

The three lines \( Z(\ell_1), Z(\ell_2) \) and \( Z(\ell_3) \) can be parameterized by \( \theta_1, \theta_2 \), and \( \theta_3 \) where \( \theta_1(s : t) = (s : 0 : t), \theta_2(s : t) = (s : t : t) \) and \( \theta_2(s : t) = (s : t : -t) \).

For the study of other singularities on the monoid we consider nonzero polynomials

\[
\begin{align*}
q_1 &= b_0s^4 + b_1s^3t + b_2st^2 + b_3st^3 + b_4t^4, \\
q_2 &= c_0s^4 + c_1s^3t + c_2st^2 + c_3st^3 + c_4t^4, \\
q_3 &= d_0s^4 + d_1s^3t + d_2st^2 + d_3st^3 + d_4t^4. 
\end{align*}
\]

Linear algebra shows that \( (\lambda_1q_1, \lambda_2q_2, \lambda_3q_3) = (f_4(\theta_1), f_4(\theta_2), f_4(\theta_3)) \) for some \( f_4 \) if and only if \( \lambda_1b_0 = \lambda_2c_0 = \lambda_3d_0 \), and \( 2\lambda_1b_1 = \lambda_2c_1 + \lambda_3d_1 \). There exist \( \lambda_1, \lambda_2, \lambda_3 \neq 0 \) such that \( (\lambda_1q_1, \lambda_2q_2, \lambda_3q_3) = (f_4(\theta_1), f_4(\theta_2), f_4(\theta_3)) \) for some \( f_4 \) and such that \( Z(f_4) \) and \( Z(f_3) \) have no common singular point if and only if all of the following hold:

\[
\begin{align*}
&b_0 = 0 \iff c_0 = 0 \iff d_0 = 0, \\
&\text{If } b_0 = c_0 = d_0 = 0, \text{ then at least two of } b_1, c_1, \text{ and } d_1 \text{ are different from zero,} \\
&2b_1c_0d_0 = b_0c_1d_0 + b_0c_0d_1. 
\end{align*}
\]

As in all the previous cases we can classify the possible configurations of other singularities for all possible \( j_1, j_2, j_3 \). As before, the first bullet point only affect the multiplicity of the factor \( t \) in \( q_1, q_2 \) and \( q_3 \). For any set of multiplicities for the rest of the roots, we can find \( q_1, q_2, q_3 \) with real roots of the given multiplicities such that the last bullet point is satisfied. This completes the classification of the singularities (other than \( O \)) when \( Z(f_3) \) is three lines meeting in a point.

Case 7. The tangent cone is a double line plus a line, and we can assume \( f_3 = x_2x_3^2 \). The tangent cone is singular along the line \( Z(x_3) \). The line \( Z(x_2) \) is parameterized by \( \theta_1 \) and the line \( Z(x_3) \) is parameterized by \( \theta_2 \) where \( \theta_1(s : t) = (s : 0 : t) \) and \( \theta_2(s : t) = (s : t : 0) \). The monoid is reducible if and only if \( f_4(\theta_1) \) or \( f_4(\theta_2) \) is identically zero, so we assume that neither is identically zero. For each \( f_4 \) we associate
two integers, \( j_0 := \Gamma(1:0:0)(f_4, x_2) \) and \( k_0 := \Gamma(1:0:0)(f_4, x_3) \). Furthermore, we write \( f_4(\theta_2) \) as a product of linear factors

\[
f_4(\theta_2) = \lambda s^{k_0} \prod_{i=0}^{r} (\alpha_i s - t)^{m_i}.
\]

Now the singularity at \( O \) will be of the \( V \) series and depends on \( j_0, k_0 \) and \( m_1, \ldots, m_r \).

Other singularities on the monoid correspond to intersections of \( Z(f_4) \) and the line \( Z(x_2) \) away from \((1:0:0)\). Each such intersection corresponds to a root in the polynomial \( f_4(\theta_1) \) different from \((1:0)\). Let \( j'_0 \leq 4 \) and \( k'_0 \leq 4 \) be integers such that \( j_0 > 0 \leftrightarrow k_0 > 0 \). Then, for any homogeneous polynomials \( q_1, q_2 \) in \( s, t \) of degree 4 such that \( s \) is a factor of multiplicity \( j'_0 \) in \( q_1 \) and of multiplicity \( k'_0 \) in \( q_2 \), there is a polynomial \( f_4 \) and nonzero constants \( \lambda_1 \) and \( \lambda_2 \) such that \( k_0 = k'_0, j_0 = j'_0 \) and \( (\lambda_1 q_1, \lambda_2 q_2) = (f_4(\theta_1), f_4(\theta_2)) \). Furthermore, if \( q_1 \) and \( q_2 \) have real coefficients, then \( f_4 \) can be selected with real coefficients. This follows from an analysis similar to case 5 and completes the classification of singularities when the tangent cone is a product of a line and a double line.

**Case 8.** The tangent cone is a triple line, and we assume that \( f_3 = x_3^3 \). The line \( Z(x_3) \) is parameterized by \( \theta \) where \( \theta(s,t) = (s,t,0) \). Assume that the polynomial \( f_4(\theta) \) has \( r \) distinct roots with multiplicities \( m_1, \ldots, m_r \). (As before \( f_4(\theta) \equiv 0 \) if and only if the monoid is reducible.) Then the type of the singularity at \( O \) will be of the \( V' \) series [3, p. 267]. The integers \( m_1, \ldots, m_r \) are constant under right equivalence over \( \mathbb{C} \). Note that one can construct examples of monoids that are right equivalent over \( \mathbb{C} \), but not over \( \mathbb{R} \) (see Figure 4).

![Fig. 4. The monoids \( Z(z^3 + xy^3 + x^3 y) \) and \( Z(z^3 + xy^3 - x^3 y) \) are right equivalent over \( \mathbb{C} \) but not over \( \mathbb{R} \).](image)

The tangent cone is singular everywhere, so there can be no other singularities on the monoid.
Case 9. The tangent cone is a smooth cubic curve, and we write \( f_3 = x_1^3 + x_2^3 + x_3^3 + 3a_1 x_1 x_2 x_3 \) where \( a^3 \neq -1 \). This is a one-parameter family of elliptic curves, so we cannot use the parameterization technique of the other cases. The singularity at \( O \) will be a \( P_6 \) singularity (cf. [3] p. 185)), and other singularities correspond to intersections between \( Z(f_3) \) and \( Z(f_4) \), as described by Proposition 6.

To classify the possible configurations of singularities on a monoid with a nonsingular (projective) tangent cone, we need to answer the following question: For any positive integers \( m_1, \ldots, m_r \) such that \( \sum_{i=1}^r m_i = 12 \), does there, for some \( a \in \mathbb{R} \setminus \{-1\} \), exist a polynomial \( f_4 \) with real coefficients such that \( Z(f_3, f_4) = \{p_1, \ldots, p_r\} \in \mathbb{P}^2(\mathbb{R}) \) and \( I_{p_i}(f_3, f_4) = m_i \) for \( i = 1, \ldots, r \)? Rohn [15] p. 63] says that one can always find curves \( Z(f_3), Z(f_4) \) with this property. Here we shall show that for any \( a \in \mathbb{R} \setminus \{-1\} \) we can find a suitable \( f_4 \).

In fact, in almost all cases \( f_4 \) can be constructed as a product of linear and quadratic terms in a simple way. The difficult cases are \((m_1, m_2) = (11, 1), (m_1, m_2, m_3) = (8, 3, 1)\), and \((m_1, m_2) = (5, 7)\). For example, the case where \((m_1, m_2, m_3) = (3, 4, 5)\) can be constructed as follows: Let \( f_4 = \ell_1 \ell_2 \ell_3^2 \) where \( \ell_1 \) and \( \ell_2 \) define tangent lines at inflection points \( p_1 \) and \( p_3 \) of \( Z(f_3) \). Let \( \ell_3 \) define a line that intersects \( Z(f_3) \) once at \( p_3 \) and twice at another point \( p_2 \). Note that the points \( p_1, p_2 \) and \( p_3 \) can be found for any \( a \in \mathbb{R} \setminus \{-1\} \).

The case \((m_1, m_2) = (11, 1)\) is also possible for every \( a \in \mathbb{R} \setminus \{-1\} \). For any point \( p \) on \( Z(f_3) \) there exists an \( f_4 \) such that \( I_p(f_3, f_4) \geq 11 \). For all except a finite number of points, we have equality [11], so the case \((m_1, m_2) = (11, 1)\) is possible for any \( a \in \mathbb{R} \setminus \{-1\} \). The case \((m_1, m_2, m_3) = (8, 3, 1)\) is similar, but we need to let \( f_4 \) be a product of the tangent at an inflection point with another cubic.

The case \((m_1, m_2) = (5, 7)\) is harder. Let \( a = 0 \). Then we can construct a conic \( C \) that intersects \( Z(f_3) \) with multiplicity five in one point and multiplicity one in an inflection point, and choosing \( Z(f_4) \) as the union of \( C \) and twice the tangent line through the inflection point will give the desired example. The same can be done for \( a = -4/3 \).

By using the computer algebra system SINGULAR [6] we can show that these constructions can be continuously extended to any \( a \in \mathbb{R} \setminus \{-1\} \). This completes the classification of singularities on a monoid when the tangent cone is smooth.

In the Cases 3, 5, and 6, not all real equations of a given type can be transformed to the chosen forms by a real transformation.

In Case 3 the conic may not intersect the line in two real points, but rather in two complex conjugate points. Then we can assume \( f_3 = x_3(x_1 x_3 + x_1^2 + x_2^2) \), and the singular points are \((1 : \pm i : 0)\). For any real \( f_4 \), we must have

\[ I_{(1:+0)}(x_1 x_3 + x_1^2 + x_2^2, f_4) = I_{(1:-i:0)}(x_1 x_3 + x_1^2 + x_2^2, f_4) \]

and

\[ I_{(1:+0)}(x_3, f_4) = I_{(1:-i:0)}(x_3, f_4), \]

so only the cases where \( j_0 = j_1 \) and \( k_0 = k_1 \) are possible. Apart from that, no other restrictions apply.

In Case 5, two of the lines can be complex conjugate, and we assume \( f_3 = x_3(x_1^2 + x_2^2) \). A configuration from the previous analysis is possible for real coefficients of \( f_4 \) if
and only if $m_1 = m_2$, $k_2 = l_1$, and $k_3 = l_3$. Furthermore, only the singularities that correspond to the line $Z(x_3)$ will be real.

In Case 6, two of the lines can be complex conjugate, and then we may assume $f_3 = x_2^3 + x_3^3$. Now, if $j_3$ denotes the intersection number of $Z(f_4)$ with the real line $Z(x_2 + x_3)$, precisely the cases where $j_1 = j_2$ are possible, and only intersections with the line $Z(x_2 + x_3)$ may contribute to real singularities.

This concludes the classification of real and complex singularities on real monoids of degree 4.

**Remark.** In order to describe the various monoid singularities, Rohn computes the “class reduction” due to the presence of the singularity, in (almost) all cases. (The class is the degree of the dual surface [14, p. 262].) The class reduction is equal to the local intersection multiplicity of the surface with two general polar surfaces. This intersection multiplicity is equal to the sum of the Milnor number and the Milnor number of a general plane section through the singular point [20, Cor. 1.5, p. 320]. It is not hard to see that a general plane section has either a $D_4$ (Cases 1–6, 9), $D_5$ (Case 7), or $E_6$ (Case 8) singularity. Therefore one can retrieve the Milnor number of each monoid singularity from Rohn’s work.

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