Momentum recoil in the relativistic two-body problem: higher-order tails

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In the description of the relativistic two-body interaction, together with the effects of energy and angular momentum losses due to the emission of gravitational radiation, one has to take into account also the loss of linear momentum, which is responsible for the recoil of the center-of-mass of the system. We compute higher-order tail (i.e., tail-of-tail and tail-squared) contributions to the linear momentum flux for a nonspinning binary system either along hyperboliclike or ellipticlike orbits. The corresponding orbital averages are evaluated at their leading post-Newtonian approximation, using harmonic coordinates and working in the Fourier domain. The final expressions are given in a large-eccentricity (or large-angular momentum) expansion along hyperboliclike orbits and in a small-eccentricity expansion along ellipticlike orbits. We thus complete a previous analysis focusing on both energy and angular momentum losses [Phys. Rev. D 104, no.10, 104020 (2021)], providing brick-type results which will be useful, e.g., in the high-accurate determination of the radiated impulses of the two bodies undergoing a scattering process.

I. INTRODUCTION

During the coalescence process of a binary system a certain amount of the total linear momentum is carried away via gravitational wave emission and, consequently, the center-of-mass of the binary recoils, i.e., moves in order to balance the linear momentum loss. This effect can be ascribed to the mass asymmetry between the two bodies (and also to either unequal or misaligned spins in the case of spinning binaries), and has important implications in many astrophysical scenarios, especially for systems whose host structure has escape speeds comparable to the recoil velocity (see, e.g., Ref. [1] and references therein). The recoil, or kick, is also responsible for the eventual ejection of black holes from the host galaxy as galaxies merge.

The first explicit calculation of the linear momentum loss and associated recoil velocity within a Post-Newtonian (PN) framework is due to Fitchett [2] for an inspiralling binary system of two point masses in Keplerian orbit, based on earlier works by Peres [3] and Bekenstein [4]. The 2PN corrections to this result were successively computed by Wiseman [5] and by Junker and Schafer [6], who considered also the case of hyperboliclike orbits. The 2PN level of accuracy was then achieved by Blanchet, Qusailah and Will [7], who limited, however, their analysis to the quasi-circular case. Spin-orbit and spin-spin corrections were later included by Kidder [8] and by Racine, Buonanno and Kidder [9], respectively, who obtained 2PN accurate expressions for the linear momentum flux valid for any kind of orbits, but then explicit results were still specialized to the quasi-circular case. Recently, Cho, Porto and Yang [10] have reobtained and extended these results in the spin-spin sector at higher PN level following an effective field theory (EFT) approach [11] (see also Refs. [12, 13] for the computation of both spin-orbit and spin-spin radiation reaction effects).

Analytical or semi-analytical methods, including perturbation theory and effective-one-body approach (see, e.g., Refs. [14 and 15]), are in general unable to provide a sufficiently accurate estimate of the recoil velocity, since for coalescing binaries most of the linear momentum flux is emitted during the merger and ringdown phases, whereas radiation from hyperbolic encounters is plunge-dominated. Therefore, one must rely on numerical relativity simulations to make predictions for the kick velocity of the post-merger black hole, as well as for the final distribution of the radiated momentum due to the recoil over the two individual black hole at the end of a scattering process (see, e.g., Refs. [16, 17]). Nevertheless, pushing PN-based computations to higher orders will allow for a more and more accurate description of the linear momentum loss as well as a reliable estimate of the recoil velocity accumulated during the inspiral phase. In addition, the knowledge of radiative losses of energy, angular momentum and linear momentum with high-PN accuracy is necessary to evaluate the radiation-reaction contributions to physical observables, like the scattering angle in hyperbolic encounters.

After many years of works on ellipticlike orbits, a new interest in hyperboliclike motion has been recently raised, since the evaluation of the scattering angle has proven to contain all the necessary structural information to characterize the local 5PN and 6PN (conservative) Hamiltonian of the system [18, 22]. On the other side, EFT succeeded in computing the Post-Minkowskian (PM) Hamiltonian of the system at the 3PM (complete) and 4PM (partial) levels by using various complementary methods, including the classical worldline approach and a novel technique termed double-copy, which -roughly speaking- looks at gravity as the square of a Yang-Mills theory, establishing a connection between scattering amplitudes and classical dynamics (see, e.g., Refs. [23, 25]). It has
been known for a long time that the radiation-reaction linear momentum loss contributes to the total change of 4-momentum of each body undergoing the scattering process starting at 3PM order \[34, 35\]. However, the computation of the radiated impulse has been completed only very recently by using different approaches \[36, 41\]. Going to higher PM orders requires more accurate expressions for the radiative losses, which are known at their lowest PM expansion only \[52\]. Fractionally 2PN-accurate expansions of the higher PM energy, angular momentum and linear momentum losses up to 7PM have been computed in Ref. \[41\], leading to the determination of the 4PM and 5PM contributions to the radiated impulses with absolute 4.5PN accuracy.

The linear momentum flux can be written as a superposition of couplings between different radiative multipole moments \[42\], which can be in turn expressed in terms of the source multipole moments through the multipolar-post-Minkowskian (MPM) formalism \[43–47\]. The radiative moments contain both “instantaneous” terms, which depend on the source moments evaluated at the retarded time, and “hereditary” terms, which are instead given by “tail” integrals over the full past history of the source \[48, 49\]. The latter can be further decomposed as tail (quadratic in \(G\)), tail-of-tail and tail-squared (cubic in \(G\)), and higher nonlinear interaction terms \[50, 51\]. As a result, the flux will consist of an instantaneous (local in time) part and a (nonlocal) hereditary part. Tail terms up to the 2.5PN level have been computed in Ref. \[52\] for quasi-circular motion, extending previous results valid at the leading 1.5PN order \[4\]. For hyperbolic-like orbits leading order tails have been instead recently computed in Ref. \[41\].

In the present paper we evaluate higher-order tail contributions to the linear momentum flux averaged along the hyperbolic-like motion at their leading PN level, completing a previous study on the analogous effects associated with both energy and angular momentum losses \[41, 53\]. We also compute both quadratic and cubic linear momentum tails for elliptic-like orbits, extending existing results out of the circular orbit limit.

Finally, let us note in passing that we will work conveniently in the frequency domain, by using harmonic coordinates and following the notation of Ref. \[54\].

II. LINEAR MOMENTUM TAIL INTEGRALS ALONG HYPERBOLIC-LIKE ORBITS

The linear momentum flux at the leading PN order in terms of the mass-type \((U_L)\) and current-type \((V_L)\) radiative multipole moments (with \(L = i_1 i_2 \cdots i_l\) being a multi-index consisting of \(l\) spatial indices) is given by \[42\]

\[
\mathcal{F}_P^i(U) = \left(\frac{dP_i}{dU}\right)^{GW} = \frac{G}{c^5}\left[\frac{2}{63} U^{(1)}_{ijk} U^{(1)}_{jkl} + \frac{16}{45} \epsilon_{ijk} U^{(1)}_{ja} V^{(1)}_{ka}\right] + \mathcal{O}\left(\frac{1}{c^6}\right), \tag{2.1}\]

where a superscript in parenthesis denotes repeated retarded time derivatives. The flux is a function of the retarded time \(U = T - R/c\) in radiative coordinates, and it is related to the corresponding retarded time \(u = t - r/c\) in harmonic coordinates by

\[
U = t - \frac{r}{c} - \frac{2GM}{c^3} \ln \left(\frac{r}{r_0}\right) + \mathcal{O}\left(\frac{1}{c^5}\right), \tag{2.2}\]

where \(M\) denotes the total Arnowitt-Deser-Misner (ADM) mass of the system, and \(r_0\) is a constant length scale.

Expressing the radiative multipole moments in terms of the source moments \((I_L, J_L)\) \[51, 55\] allows for decomposing the flux as the sum of instantaneous and hereditary terms,

\[
\mathcal{F}_P^i(U) = \mathcal{F}_P^i_{\text{inst}}(U) + \mathcal{F}_P^i_{\text{hered}}(U), \tag{2.3}\]

where the instantaneous part depends on the dynamics of the system at the retarded instant \(U\) only, while the hereditary part is nonlocal in time depending on the full past history \[48, 49\]. The hereditary part can be further decomposed into a first-order tail part (quadratic in \(G\)) and a higher-order tail part (which is higher order in \(G\)). We have already computed in Ref. \[41\] the first-order tail contributions to the linear momentum flux at the leading order in their PN expansion. In the present work we will focus on cubic tails, which are referred to in the literature as tail-of tail and tail-squared terms \[51\], so that the hereditary part of the flux reads

\[
\mathcal{F}_P^i_{\text{hered}}(t) = \mathcal{F}_P^i_{\text{tail}(t)} + \mathcal{F}_P^i_{\text{tail}^2(t)} + \mathcal{F}_P^i_{\text{tail}^3(t)}, \tag{2.4}\]

at that level of approximation, with
\[
\mathcal{F}_{\text{P tail}}^i(t) = \frac{G^2 M}{c^6} \left\{ \frac{4}{63} I_{ijk}^{(4)}(t) \int_0^\infty d\tau \ln \left( \frac{\tau}{C_{I_2}} \right) I_{ij}^{(5)}(t - \tau) + I_{ij}^{(3)}(t) \int_0^\infty d\tau \ln \left( \frac{\tau}{C_{I_2}} \right) I_{ij}^{(6)}(t - \tau) \right\} + \frac{32}{45} \epsilon_{ijk} \left[ I_{ja}^{(3)}(t) \int_0^\infty d\tau \ln \left( \frac{\tau}{C_{J_2}} \right) J_{ja}^{(5)}(t - \tau) + J_{ja}^{(3)}(t) \int_0^\infty d\tau \ln \left( \frac{\tau}{C_{I_2}} \right) I_{ja}^{(5)}(t - \tau) \right], \tag{2.5}
\]

and

\[
\mathcal{F}_{\text{P tail}}^i(t) = \frac{G^4 M^2}{c^{13}} \left\{ \frac{32}{45} \epsilon_{ijk} \left[ J_{ja}^{(3)}(t) \int_0^\infty d\tau \ln \left( \frac{\tau}{C_{J_2}} \right) - 107 \ln \left( \frac{\tau}{C_{J_2}} \right) \right] + \right\} + \frac{4}{63} \left[ I_{ijk}^{(4)}(t) \int_0^\infty d\tau \ln \left( \frac{\tau}{C_{I_3}} \right) - \frac{107}{105} \ln \left( \frac{\tau}{C_{I_3}} \right) \right] \right\} + \frac{4}{63} \left[ I_{ijk}^{(4)}(t) \int_0^\infty d\tau \ln \left( \frac{\tau}{C_{I_3}} \right) - \frac{107}{105} \ln \left( \frac{\tau}{C_{I_3}} \right) \right] \right\} + \frac{4}{63} \left[ I_{ijk}^{(4)}(t) \int_0^\infty d\tau \ln \left( \frac{\tau}{C_{I_3}} \right) - \frac{107}{105} \ln \left( \frac{\tau}{C_{I_3}} \right) \right] \right\}, \tag{2.6}
\]

by substituting the relations (4.16), (4.17a) and (4.17b) of Ref. [51] in the general expression (2.1), PN expanding and selecting the nonlocal part only. Here we have introduced the following set of multipolar constants \((\tau_0 = c\tau_0)\)

\[
\begin{align*}
C_{I_2} &= 2\tau_0 e^{-11/12}, \\
C_{I_3} &= 2\tau_0 e^{-97/60}, \\
C_{J_2} &= 2\tau_0 e^{-7/6}, \\
\bar{C}_{I_2} &= C_{I_2} e^{-11/12}, \\
\bar{C}_{I_3} &= C_{I_3} e^{-97/60}, \\
\bar{C}_{J_2} &= C_{J_2} e^{-7/6}. \tag{2.7}
\end{align*}
\]

The quadratic-in-G term \((2.5)\) is the dominant tail at (fractional) 1.5PN order, while the two cubic-in-G order tails \((2.6)\) are both at (fractional) 3PN order. The multipolar moment expressions agree with standard literature, see e.g., Ref. [50], which we also basically follow for notation and conventions.

The previous relations \((2.5)\) and \((2.6)\) can be cast in the more compact form
\[
\mathcal{F}_{\text{P} \text{tail}}(t) = \frac{G^2 M}{c^4} \left\{ \frac{4}{63} \left[ I_{ijk}^{(4)}(t) T_{ln}^{[l]} \mathcal{C} (\mathcal{I}_n)(t) + I_{ijk}^{(3)}(t) \mathcal{C} (\mathcal{I}_n)(t) \right] + \frac{32}{45} I_{j}^{(3)}(t) T_{ln}^{[j]} \mathcal{C} (\mathcal{I}_n)(t) \right\} ,
\]
\[
\mathcal{F}_{\text{P} \text{tail(tail)}}(t) = \frac{G^2 M^2}{c^4} \left\{ \frac{32}{45} I_{ijk}^{(3)}(t) \left( T_{ln}^{[j]} \mathcal{C} (\mathcal{I}_n)(t) - \frac{107}{105} T_{kn}^{[j]} \mathcal{C} (\mathcal{I}_n)(t) \right) \right\} ,
\]
\[
\mathcal{F}_{\text{P} \text{tail}2}(t) = \frac{G^2 M^2}{c^4} \left\{ \frac{8}{63} T_{ln}^{[j]} \mathcal{C} (\mathcal{I}_n)(t) T_{kn}^{[j]} \mathcal{C} (\mathcal{I}_n)(t) + \frac{64}{45} I_{ijk} T_{ln}^{[j]} \mathcal{C} (\mathcal{I}_n)(t) \right\} ,
\]
by using the notation
\[
T_{ln}^{[n]}[X_L^{(n)}; C X_L](t) = \int_0^\infty d\tau X_L^{(n)}(t-\tau) \ln^m \left( \frac{\tau}{C X_L} \right) ,
\]
introduced in Ref. [54], with \(X_L\) denoting a generic multipolar moment, and \(m = 1, 2\) are the only powers of the log terms needed here.

These tails have been termed in Refs. [41, 54] “past tails,” since they refer to the interaction between the bodies occurring in the past, so that they are asymmetric under time-reversal. However, one can decompose them into a time-symmetric (ts) and a time-antisymmetric (tas) part. When dealing with ts tails it is enough to replace \(T_{ln}^{[n]} \rightarrow T_{ln}^{\text{ts}}\) in Eq. (2.8), with

\[
T_{ln}^{\text{ts}}[X_L^{(n)}; C X_L](t) = \int_0^\infty d\tau X_L^{(n)}(t, \tau) \ln^m \left( \frac{\tau}{C X_L} \right) ,
\]
where

\[
X_L^{(n)}(t, \tau) = \frac{1}{2} \left[ X_L^{(n)}(t-\tau) + X_L^{(n)}(t+\tau) \right] .
\]

### A. Time-integrated loss along hyperbolic-like orbits

We will evaluate below the leading PN order contribution to the orbital averages

\[
(\Delta P)_X = \int_{-\infty}^{\infty} dt \mathcal{F}_X(t) = (\Delta P)_X^{\text{ts}} + (\Delta P)_X^{\text{tas}} ,
\]
along hyperbolic-like orbits, where the label X is either tail, tail(tail), or (tail)^2, and we also distinguish among time-symmetric and time-antisymmetric contributions.

Therefore, the Newtonian description of the binary dynamics suffices. The Keplerian parametrization of the hyperbolic motion in harmonic coordinates is

\[
\begin{align*}
    r &= \bar{a}_r (e_r \cosh v - 1) , \\
    nt &= e_r \sinh v - v , \\
    \phi &= 2 \arctan \left[ \sqrt{\frac{e_r + 1}{e_r - 1}} \tanh \frac{v}{2} \right] .
\end{align*}
\]

where we have used dimensionless variables, i.e., \(r = c^2 r_{\text{phys}} / (GM)\), \(t = c^4 t_{\text{phys}} / (GM)\), and the orbital parameters \(\bar{n}, \bar{a}_r, e_r\) are given by

\[
\bar{n} = (2E)^{3/2} , \quad \bar{a}_r = \frac{1}{2E} , \quad e_r = \sqrt{1 + 2Ej^2} ,
\]
in terms of the dimensionless energy and angular momentum parameters \(E\) and \(j\). The latter are defined by

\[
E \equiv \frac{E_{\text{tot}} - M c^2}{\mu c^2} , \quad j \equiv \frac{c J}{G M \mu} ,
\]
where \(E_{\text{tot}}\) and \(J\) are the total center-of-mass energy and angular momentum of the binary system, respectively, with total mass \(M = m_1 + m_2\), reduced mass \(\mu \equiv m_1 m_2 / M\), and symmetric mass ratio \(\nu = \mu / M\). At the leading order we are interested in here we can set the total ADM mass of the system \(M = \bar{M}\). In addition we will set \(G = \bar{M} = c = 1\) for simplicity.

### III. FREQUENCY-DOMAIN COMPUTATION OF TAIL INTEGRALS

The computation of tail integrals is more conveniently performed in the Fourier domain. In fact, the time-domain convolutions entering Eq. (2.12) become multiplications by the Fourier transforms of their kernels [11].
A. Past tails

Let us consider past tails first. The $m$-type past-tail associated with the history of $X_L^{(n)}$ becomes

$$T_m[X_L^{(n)}; C_{X_L}](t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} (-i\omega)^n \hat{X}_L(\omega) A_m(\omega, C_{X_L}),$$

(3.1)

upon substituting in it the Fourier expansion

$$X_L(\tau) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega \tau} \hat{X}_L(\omega),$$

(3.2)

of the generic multipolar moment $X_L(\tau)$. Here

$$A_m(\omega, C_{X_L}) = \int_{0}^{\infty} d\xi e^{i\omega \xi} \ln^m \left( \frac{\xi}{C_{X_L}} \right),$$

(3.3)

which for $m = 1, 2$ read

$$A_1(\omega, C_{X_L}) = -\frac{\pi}{2|\omega|} - \frac{i}{|\omega|} \text{sgn}(\omega) \ln(C_{X_L} |\omega| e^\gamma),$$

$$A_2(\omega, C_{X_L}) = \frac{\pi}{|\omega|} \ln(C_{X_L} |\omega| e^\gamma) + \frac{i}{|\omega|} \text{sgn}(\omega) \left[ \ln^2(C_{X_L} |\omega| e^\gamma) - \frac{\pi^2}{12} \right].$$

(3.4)

Taking the orbital averages \(2.12\) leads to integrals of the type

$$F_m[Y_M^{(p)}, X_L^{(n)}; C_{X_L}] = \int_{-\infty}^{\infty} dt Y_M^{(p)}(t) T_m[X_L^{(n)}; C_{X_L}](t),$$

(3.5)

where in the last line, as standard, we have restricted the integration domain to positive frequencies. Using this result the linear momentum past tails \(2.12\) become

$$\Delta P_{\text{tail}} = \frac{32}{45} \int_{0}^{\infty} d\omega \frac{\omega^7 R^+_1(\omega)}{2\pi} - \frac{8}{45} \int_{0}^{\infty} d\omega \frac{\omega^7 R^-_1(\omega)}{2\pi} - \frac{2}{45} \int_{0}^{\infty} d\omega \frac{\omega^8 S^+_1(\omega)}{2\pi} + \frac{4}{63} i\pi \int_{0}^{\infty} d\omega \frac{\omega^8 S^-_1(\omega)}{2\pi},$$

$$\Delta P_{\text{tail}} = -\frac{64}{45} \int_{0}^{\infty} d\omega \frac{\omega^8 R^+_1(\omega)}{2\pi} \ln(C_{L_1} \omega e^\gamma) \ln(C_{L_2} \omega e^\gamma) - \frac{2}{45} \int_{0}^{\infty} d\omega \frac{\omega^8 S^+_1(\omega)}{2\pi} + \frac{8}{63} i\pi \int_{0}^{\infty} d\omega \frac{\omega^8 S^-_1(\omega)}{2\pi},$$

(3.6)

where we have introduced the notation

$$S^\pm_1(\omega) = I_{ijk}(-\omega) \hat{I}_{ijk}(\omega) \pm I_{ijk}(\omega) \hat{I}_{ijk}(-\omega),$$

$$R^\pm_1(\omega) = \epsilon_{ijk} \hat{I}_{jkl}(\omega) \pm \hat{I}_{jkl}(\omega),$$

(3.7)

and

$$\ln(C_{L_2} \omega e^\gamma) \ln(C_{L_1} \omega e^\gamma) = \ln^2(C_{L_2} \omega e^\gamma) - \frac{1}{4} \ln(C_{L_1} \omega e^\gamma),$$

$$\ln(C_{L_1} \omega e^\gamma) \ln(C_{L_2} \omega e^\gamma) = \ln^2(C_{L_2} \omega e^\gamma) - \frac{7}{10} \ln(C_{L_1} \omega e^\gamma).$$

(3.8)
We list below the results of the computation in a large-\(j\) expansion limit, referring to previous works for all technical details \[\text{[11, 52, 54]}\]. In all cases, \((\Delta P_x)_X = 0\). The general structure is as follows:

\[
(\Delta P_{\text{tail}}) = \nu^2 \sqrt{1 - 4\nu} \sum_{n=4}^{\infty} \frac{P_{n \text{tail}}(p_\infty)}{j^n},
\]

(3.9)

for dominant tails, and

\[
(\Delta P_{\text{Xtail}}) = \nu^2 \sqrt{1 - 4\nu} \sum_{n=5}^{\infty} \frac{1}{j^n} [P_{n \text{tail}}(p_\infty)\nu \left( \frac{P_{n \text{tail}}(p_\infty)}{j^n} \right) + P_{n \text{tail}}X(\nu)\nu \left( \frac{P_{n \text{tail}}X(\nu)}{j^n} \right) + P_{n \text{tail}}X^2(\nu)\nu \left( \frac{P_{n \text{tail}}X^2(\nu)}{j^n} \right)],
\]

(3.10)

for higher-order tails \(X = [\text{tail(tail)}, (\text{tail})^2]\). The dependence on the scale will disappear as soon as one adds to these hereditary contributions their instantaneous counterparts (when available) at the same approximation level (see Eq. (2.2) and the general discussion around Eq. (4.14) in Ref. \[\text{[51]}\]), giving so far a consistency check of both kinds of results.

Tails:

\[
(\Delta P_x)_{\text{tail}} = \nu^2 \sqrt{1 - 4\nu} \left[ \frac{1491 p_\infty^7}{400 \nu^4} + \frac{20608 p_\infty^6}{225 \nu^5} + \pi \frac{267583 p_\infty^5}{2400 \nu^6} + \frac{64576 p_\infty^4}{75 \nu^7} + O \left( \frac{1}{\nu^8} \right) \right],
\]

(3.12)

Tail of tails:

\[
(\Delta P_x)_{\text{tail(tail)}} = \nu^2 \sqrt{1 - 4\nu} \left[ \frac{547712 p_\infty^9}{4725 \nu^4} + \pi \frac{11581 p_\infty^8}{320 \nu^5} + \pi \left( \frac{5158208}{4725} + \frac{23902208}{55125} \right) \frac{p_\infty^7}{\nu^6} + O \left( \frac{1}{\nu^7} \right) \right],
\]

(3.13)

Tail squared:
\[(\Delta P_x)_{\text{tail}} = \nu^2 \sqrt{1 - 4\nu} \left[ \frac{200608}{225} \frac{p_\infty^2}{j^5} + \frac{4569}{160} \frac{\pi^2 p_\infty^2}{j^6} + \left( \frac{299008}{875} \frac{\pi^2}{2} + \frac{64576}{75} \right) \frac{p_\infty^2}{j^7} + O \left( \frac{1}{j^8} \right) \right], \]

\[(\Delta P_y)_{\text{tail}} = \nu^2 \sqrt{1 - 4\nu} \left[ \left( \frac{-503}{40} \pi^2 - \frac{778912}{1575} \ln(2) + \frac{557056}{525} \ln(2) - \frac{104792}{10125} \right) + \frac{8339456}{1575} \ln(2) \right], \]

\[(\Delta P_y)_{\text{tail}} + (\Delta P_y)_{\text{tail}}^2 \]

The coefficients of \(L^2\) in the sum \((\Delta P_y)_{\text{tail}} + (\Delta P_y)_{\text{tail}}^2\) cancel, as expected. We find

\[(\Delta P_x)_{\text{tail}} + (\Delta P_x)_{\text{tail}}^2 = \nu^2 \sqrt{1 - 4\nu} \left[ \frac{196096 \frac{p_\infty^2}{j^5} + 20719 \frac{\pi^2 p_\infty^2}{j^6} + \left( \frac{9226496}{4725} + \frac{42739712}{55125} \pi^2 \right) \frac{p_\infty^2}{j^7} + O \left( \frac{1}{j^8} \right) \right], \]

\[(\Delta P_y)_{\text{tail}} + (\Delta P_y)_{\text{tail}}^2 = \nu^2 \sqrt{1 - 4\nu} \left[ \left( \frac{41053}{2450} \ln(2) + \frac{83249503}{1029000} \pi^2 - \frac{503}{70} \pi^2 \right) \frac{p_\infty^2}{j^5} \right], \]

\[(3.14) \]

\[B. \ Time-symmetric\ tails \]

In the case of time-symmetric tails one should use instead

\[T_{\text{sym}}^{\text{in}}[X_L^{(n)}; C_{X_L}](t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} (-i\omega)^{n} X_L(\omega) \times e^{-i\omega t} A_{\text{sym}}(\omega, C_{X_L}) \] \hspace{1cm} \(3.16)\]

where

\[A_{\text{sym}}(\omega, C_{X_L}) = \frac{1}{2} \left[ A_m(\omega, C_{X_L}) + A_m(-\omega, C_{X_L}) \right] \]

so that the typical integral is

\[F_{\text{sym}}^{(p)}[Y_M^{(p)}; X_L^{(n)}; C_{X_L}] = \int_{-\infty}^{\infty} dt Y_M(t) T_{\text{sym}}^{\text{in}}[X_L^{(n)}; C_{X_L}](t) \]

\[= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} (-i\omega)^{n+p} \tilde{Y}_M(-\omega) \tilde{X}_L(\omega) A_{\text{sym}}(\omega) \]

\[= \int_{0}^{\infty} \frac{d\omega}{2\pi} (i\omega)^{n+p} [(-1)^n \tilde{Y}_M(-\omega) \tilde{X}_L(\omega) + (-1)^p \tilde{Y}_M(\omega) \tilde{X}_L(-\omega)] A_{\text{sym}}(\omega, C_{X_L}). \] \hspace{1cm} \(3.19)\]
We find

\[
\begin{align*}
(\Delta P_t)_{\text{tail}, ts} & = \frac{32}{45} \pi \int_0^\infty \frac{d\omega}{2\pi} \omega^7 R_i^+ (\omega) + \frac{4}{63} \pi \int_0^\infty \frac{d\omega}{2\pi} \omega^8 S_i^- (\omega), \\
(\Delta P_t)_{\text{tail(tail)}, ts} & = - \frac{8}{45} \pi \int_0^\infty \frac{d\omega}{2\pi} \omega^8 R_i^- (\omega) - \frac{2}{35} \pi \int_0^\infty \frac{d\omega}{2\pi} \omega^9 S_i^+ (\omega), \\
(\Delta P_t)_{\text{tail}^2, ts} & = \frac{64}{45} \pi^2 \int_0^\infty \frac{d\omega}{2\pi} \omega^9 S_i^- (\omega),
\end{align*}
\] 

(3.20)

so that

\[
\begin{align*}
(\Delta P_t)_{\text{tail}, ts} & = 0, & (\Delta P_y)_{\text{tail}, ts} & = (\Delta P_y)_{\text{tail}}, \\
(\Delta P_y)_{\text{tail(tail)}, ts} & = (\Delta P_x)_{\text{tail(tail)}}, & (\Delta P_y)_{\text{tail}^2, ts} & = 0, \\
(\Delta P_x)_{\text{tail}^2, ts} & = 0,
\end{align*}
\]

where “nolog” stands for the nonlogarithmic term of the corresponding (past tail) quantity. In the large-$j$ expansion limit we find

\[
(\Delta P_y)_{\text{tail}^2, ts} = -\nu^2 \sqrt{1 - 4\nu^2} \left[ \frac{1509}{280} \pi^2 \frac{p_\infty^6}{j^5} + \frac{260608}{1575} \pi^2 \frac{p_\infty^6}{j^5} + \frac{142391}{560} \pi^2 \frac{p_\infty^7}{j^7} + O \left( \frac{1}{j^8} \right) \right].
\]

Finally, as a consequence of the above relations (2.12) and (3.20), one immediately identifies the corresponding time-antisymmetric contributions. For example,

\[
(\Delta P_t)_{\text{tail}, tas} = -\frac{8}{45} \pi \int_0^\infty \frac{d\omega}{2\pi} \omega^8 R_i^- (\omega) - \frac{2}{45} \int_0^\infty \frac{d\omega}{2\pi} \omega^8 S_i^+ (\omega).
\]

(3.22)

Let us conclude this section by summarizing the structure of the complete expression for the energy, angular momentum and linear momentum losses, including both instantaneous and hereditary contributions. These radiative losses admit a double PM and PN expansion

\[
\begin{align*}
\frac{E_{\text{rad}}}{M} & = \nu^2 \left[ E_3 (p_\infty) \frac{j^3}{j^3} + E_4 (p_\infty) \frac{j^4}{j^4} + \cdots \right], \\
\frac{J_{\text{rad}}}{J} & = \nu \left[ J_{1,2} (p_\infty) \frac{j^2}{j^2} + J_{3,4} (p_\infty) \frac{j^3}{j^3} + \cdots \right], \\
\frac{P_{\text{rad}}}{M} & = \nu^2 \sqrt{1 - 4\nu} \left[ P_{3,5} (p_\infty) \frac{j^3}{j^3} + P_{4,6} (p_\infty) \frac{j^4}{j^4} + \cdots \right],
\end{align*}
\]

(3.23)

where the subscripts $n$ (e.g., in $E_n$) label the $n$PM order, i.e., $O(G^n)$. The subsequent expansion of the various PM coefficients in powers of $p_\infty$ then corresponds to the usual PN expansion. Quadratic tails start at the 1.5PN fractional order, whereas cubic tails at 3PN order. In the case of angular momentum there is an additional quadratic-in-$G$ contribution from a memory integral which is 2.5PN order. For instance, the first few expansion coefficients of the complete (instantaneous plus hereditary) energy loss have the following structure

\[
\begin{align*}
E_3 (p_\infty) & \sim \frac{p_\infty^4}{N \text{PN}} + \frac{p_\infty^5}{2 \text{PN}} + \frac{p_\infty^6}{3 \text{PN}} + \cdots, \\
E_4 (p_\infty) & \sim \frac{p_\infty^5}{N \text{PN}} + \frac{p_\infty^6}{2 \text{PN}} + \frac{p_\infty^7}{3 \text{PN}} + \cdots, \\
E_5 (p_\infty) & \sim \frac{p_\infty^6}{N \text{PN}} + \frac{p_\infty^7}{2 \text{PN}} + \frac{p_\infty^8}{3 \text{PN}} + \cdots + \frac{p_\infty^8}{1 \text{PN} \times \text{PN} + 1 \text{PN} \times \text{PN} + \text{PN} \times \text{PN} + \cdots}.
\end{align*}
\]

(3.24)

A final comment concerns the dependence of the various quantities by the scale $r_0$ entering the logarithmic term $L$, Eq. (3.11). This dependence should disappear as soon as one computes the full (gauge-invariant) linear momentum recoil tail effects for both spinless and spinning bodies have been investigated in Ref. [9] at

IV. THE ELLIPTICLIKE COUNTERPART OF LINEAR MOMENTUM RECOIL HEREDITARY EFFECTS

Linear momentum recoil tail effects for both spinless and spinning bodies have been investigated in Ref. [8] at
their leading PN in the simplest case of circular motion. We compute below the corresponding past tail integrals for nonrotating bodies by including the effect of the eccentricity as series expansions in a small eccentricity parameter up to the tenth order. We also evaluate higher-order past tail integrals (tail-of-tails and tail squared) through the same approximation level.

The Keplerian parametrization of the ellipticlike motion in harmonic coordinates is

\[ \ell = nt = u - \epsilon_r \sin u, \]
\[ r = a_r(1 - \epsilon_r \cos u), \]
\[ \phi = 2 \arctan \left( \frac{1 + \epsilon_r}{1 - \epsilon_r} \tan \frac{u}{2} \right), \]

with

\[ \cos \phi = \frac{\cos u - \epsilon_r}{1 - \epsilon_r \cos u}, \quad \sin \phi = \sqrt{1 - \epsilon_r^2} \sin u \left( \frac{1 - \epsilon_r \cos u}{1 - \epsilon_r} \right), \]

where we have used dimensionless variables \( t \) and \( r \) (with \( G = M = c = 1 \) as in the hyperboliclike case), and the orbital parameters \( n = \frac{2 \pi}{T_r} (T_r \text{ denoting the radial period}), a_r, \epsilon_r \) are given by

\[ n = (-2E)^{3/2}, \quad a_r = \frac{1}{(-2E)}, \quad \epsilon_r = \sqrt{1 + 2E}^2 , \]

in terms of the dimensionless energy and angular momentum parameters \( E < 0 \) and \( j \). At the Newtonian level one can invert the relation defining the mean anomaly \( \ell \) as a function of \( u \) in terms of Bessel functions of the first kind as follows

\[ u = l + \sum_{n=1}^{\infty} \frac{2}{n} J_n(ne_r) \sin nl , \]

with the property that when \( l = 0, 2\pi, u = 0, 2\pi \) too. Let us mention in passing that raising the PN approximation the parametrization of the orbit includes in general three different eccentricities, \( \epsilon_0, \epsilon_r, \) and \( \epsilon_\phi \), which however coincide at the Newtonian level considered here.

When dealing with the ellipticlike case instead of the integral Fourier transform one uses expansion in Fourier series, namely each multipolar moment \( X_L \) reads

\[ X_L(t) = \sum_{p=-\infty}^{\infty} \hat{X}_L(p) e^{ip\ell}, \]

with

\[ \hat{X}_L(p) = \int_0^{2\pi} \frac{dl}{2\pi} X_L(t) e^{-ip\ell}. \]

In view of the following use of Cartesian coordinates \( x = r \cos \phi, y = r \sin \phi \) for motion on the \( z = 0 \) plane, we also recall the useful relations

\[ x = a_r(\cos u - \epsilon_r), \quad y = a_r \sqrt{1 - \epsilon_r^2} \sin u, \]

together with their Fourier transform counterparts,

\[ \hat{x}(p) = \frac{a_r}{p} J'_p(p e_r), \]
\[ \hat{y}(p) = \frac{ia_r \sqrt{1 - \epsilon_r^2}}{p e_r} J_p(p e_r), \]

since

\[ J_n(x) = \int_0^{2\pi} \frac{du}{2\pi} e^{i(nu - x \sin u)}, \]

and

\[ J_{n-1}(x) - J_{n+1}(x) = 2 J'_n(x), \]
\[ J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x), \]

where a prime means derivative with respect to the argument, \( d/dx(J_n(x))|_{x=n\epsilon_r} \).

Let us denote the average over a period of radial motion as follows

\[ \langle \Delta P_i \rangle_x = \frac{1}{T_r} \int_0^{T_r} dt \langle F^i_{PX}(t) \rangle \]
\[ = \frac{1}{2\pi} \int_0^{2\pi} dt \langle F^i_{PX}(\ell) \rangle , \]

with \( X = \text{tail}, \text{tail(tail)}, (\text{tail})^2 \). Eqs. (2.5) and (2.6) are then simply modified.

When working in the elliptic case, instead of referring to the \( x \) and \( y \) components associated with nonrotating axes \( \partial_x \) and \( \partial_y \), it is customary to consider components with respect to the rotating axes \( e_\ell \) and \( e_\phi \), which are defined as

\[ e_\ell = \cos \phi(t) \partial_x + \sin \phi(t) \partial_y, \]
\[ e_\phi = - \sin \phi(t) \partial_x + \cos \phi(t) \partial_y , \]

along the radial and azimuthal direction, respectively, also denoted as \( n = e_\ell \) and \( \lambda = e_\phi \) in the literature. One then usually takes the following orbital averages

\[ \langle \Delta P_i \rangle_x \equiv \langle F^i_{PX}(t) n_i(t) \rangle , \]
\[ \langle \Delta P_\phi \rangle_x \equiv \langle F^i_{PX}(t) \lambda_i(t) \rangle . \]

We find...
\[ \langle (\Delta P_r)_{\text{tail}} \rangle = -\frac{\nu^2}{a_r^3} \sqrt{1 - 4\nu} \left[ \frac{928}{105} \frac{47368}{1575} + \frac{216}{7} \ln(3) - \frac{512}{35} \ln(2) \right. \]
\[ + \left. \left( \frac{416}{3} L - \frac{20792}{45} - \frac{30402}{35} \frac{\ln(3)}{21} + \frac{42932}{21} \ln(2) \right) e_r^2 \right] \]
\[ + \left( \frac{80056}{105} L - \frac{792698}{315} + \frac{390625}{42} \ln(5) + \frac{299079}{70} \ln(3) - \frac{515008}{21} \ln(2) \right) e_r^4 \]
\[ + \left( \frac{91792}{35} L - \frac{904184}{105} - \frac{660203125}{6048} \ln(5) + \frac{62965377}{1120} \ln(3) + \frac{168904676}{945} \ln(2) \right) e_r^6 \]
\[ + \left( \frac{193523}{28} L - \frac{7660219}{336} + \frac{19536910589}{55296} \log(7) + \frac{18150589625}{387072} \ln(5) \right) e_r^8 \]
\[ - \left( \frac{49766794011}{71680} \ln(3) - \frac{296036371}{315} \ln(2) \right) e_r^{10} \]
\[ + \left( \frac{2147431}{140} L - \frac{420739033}{8400} - \frac{102526991549801}{27648000} \ln(7) - \frac{5541051967875}{516096} \ln(5) \right) \]
\[ + \left( \frac{21770237937411}{7168000} \ln(3) + \frac{1548644094059}{189000} \ln(2) \right) e_r^{10} + O(e_r^{12}) \right], \quad (4.14) \]

and
\[ \langle (\Delta P_\phi)_{\text{tail}} \rangle = -\frac{\nu^2}{a_r^3} \sqrt{1 - 4\nu} \frac{\omega^{14/3}}{\tau_0} \ln \left( \frac{\omega_{\text{NS}}}{\omega} \right) \]
\[ + \frac{1822017899719}{48384000} e_r^{10} + O(e_r^{12}) \right], \quad (4.15) \]

with
\[ L = -\frac{3}{2} \ln(a_r) + \gamma + \ln(\tau_0). \quad (4.16) \]

In the quasi-circular case \((e_r = 0)\) the previous expressions reduce to
\[ \langle (\Delta P_r)_{\text{tail}} \rangle_{\text{circ}} = -\frac{928}{105} \nu^2 \sqrt{1 - 4\nu} \omega_{\text{NS}}^{14/3} \ln \left( \frac{\omega}{\omega_{\text{NS}}} \right), \]
\[ \langle (\Delta P_\phi)_{\text{tail}} \rangle_{\text{circ}} = \frac{824}{35} \nu^2 \sqrt{1 - 4\nu} \omega_{\text{NS}}^{14/3} \pi, \quad (4.17) \]

in agreement with Ref. [9], where \(\omega = a_r^{-3/2}\), and
\[ \ln \omega_{\text{NS}} = -\ln(\tau_0) + 5921 \frac{1740}{48} \ln(2) - \frac{405}{116} \ln(3) - \frac{5}{2} \ln(\tau_0). \quad (4.18) \]
Higher-order tails turn out to be

\[
\langle (\Delta P_r)_{\text{tail}}(\text{tail}) \rangle = \frac{\nu^2}{a_r^{17/2}} \sqrt{1 - 4\nu} \pi \left[ -\frac{648}{7} \ln(3) - \frac{16}{105} \ln(2) - \frac{1648}{35} L + \frac{1134068}{11025} \right. \\
+ \left( -\frac{5588}{5} \frac{L}{L} + \frac{1302389}{525} + \frac{91206}{35} \ln(3) - \frac{866948}{105} \ln(2) \right) e_r^2 \\
+ \left( -\frac{918751}{105} \frac{L}{L} + \frac{863978657}{44100} - \frac{1953125}{42} \ln(5) - \frac{897237}{70} \ln(3) + \frac{642015}{7} \ln(2) \right) e_r^4 \\
+ \left( -\frac{29096101}{720} \frac{L}{L} + \frac{8251215761}{907200} + \frac{3301015625}{6048} \ln(5) - \frac{40841261}{1120} \ln(3) - \frac{4409133443}{5040} \ln(2) \right) e_r^6 \\
+ \left( -\frac{66132493883}{483840} \frac{L}{L} + \frac{20909266973897}{67737600} - \frac{907529453125}{387072} \ln(5) - \frac{136758374123}{55296} \ln(7) \right) e_r^8 \\
+ \left( -\frac{300867436881}{71680} \ln(3) + \frac{27195624413}{53760} \ln(2) \right) e_r^{10} \\
+ \left( -\frac{404550776017}{1075200} \frac{L}{L} + \frac{173094705777757}{20321280000} + \frac{717688940848607}{27648000} \ln(7) - \frac{130779522305289}{7168000} \ln(3) \right) e_r^{12} \\
- \left( -\frac{941760801310207}{16128000} \ln(2) + \frac{2770525984375}{516096} \ln(5) \right) e_r^{14} + O(e_r^{16}),
\]  

(4.19)
\[
\langle (\Delta P_{\phi})_{\text{tail(tail)}} \rangle = \frac{\nu^2}{a_{\nu}^2} \sqrt{1 - 4\nu \left( \frac{2656}{21} + \frac{2964176}{11025} + \frac{11328}{35} \ln(2) + \frac{1296}{7} \ln(3) \right) L} \\
+ \frac{3488}{15} \ln(2)^2 - \frac{3281552}{11025} \ln(2) - \frac{59292}{245} \ln(3) + \frac{648}{7} \ln(3)^2 + \frac{1296}{7} (\ln(2) \ln(3) - \frac{664}{63} \pi^2 + \frac{278958328}{1157625}) \L \\
+ \frac{9104}{3} \bar{L}^2 \left( \frac{2047712}{105} \ln(2) - \frac{18792}{35} \ln(3) - \frac{10200152}{1575} \right) L \\
+ \frac{3151088}{105} \ln(2)^2 - \frac{2276}{9} \bar{s}^2 + \frac{2972376}{1225} \ln(3) - \frac{9396}{35} \ln(3)^2 - \frac{18792}{35} (\ln(2) \ln(3) + \frac{321000172}{55125}) \\
- \frac{267930248}{11025} \bar{L}^2 \ln(2) \dot{e}_r^2 
\]

\[
+ \frac{502100}{21} \bar{L}^2 + \left( \frac{-563662934}{11025} + \frac{195125}{21} \ln(5) - \frac{353928}{7} \ln(2) + \frac{139158}{35} \ln(3) \right) L \\
+ \frac{1953125}{21} \ln(2) \ln(5) - \frac{23828125}{196} \ln(5) + \frac{69579}{35} \ln(3)^2 + \frac{17768218079}{385875} - \frac{125525}{63} \pi^2 - \frac{5958117}{490} \ln(3) \\
+ \frac{1254559802}{11025} \ln(2) - \frac{6050836}{35} \ln(2)^2 + \frac{139158}{42} \ln(2) \ln(3) + \frac{1953125}{35} \ln(5)^2 \dot{e}_r^4 
\]

\[
+ \frac{77574}{7} \bar{L}^2 + \left( \frac{-2616344179}{11025} + \frac{215372277}{280} \ln(3) + \frac{1318656788}{945} \ln(2) - \frac{1122578125}{1512} \ln(5) \right) L \\
+ \frac{495336993249}{19600} \ln(3) + \frac{6126978125}{7056} \ln(5) + \frac{215372277}{560} \ln(3)^2 + \frac{434887577}{280} \ln(2) \ln(3) \\
- \frac{129329}{14} \pi^2 + \frac{390500482}{189} \ln(2)^2 \dot{e}_r^6 
\]

\[
+ \frac{15788093}{42} \bar{L}^2 + \left( \frac{550718046875}{193336} \ln(5) + \frac{136758374123}{27648} \ln(7) - \frac{35514677911}{44100} \ln(2) - \frac{224271483137}{21} \right) L + \frac{1315286367923}{132300} \ln(2) - \frac{1191751545929}{184320} \ln(7) + \frac{224271483137}{308700} \\
+ \frac{136758374123}{55296} \ln(7)^2 + \frac{46241895505311}{5017600} \ln(3) - \frac{235903522689}{71680} \ln(3)^2 + \frac{136758374123}{27648} \ln(2) \ln(7) \\
- \frac{1890}{1024} \ln(2) \ln(3) - \frac{5190088}{15788093} \ln(3)^2 \dot{e}_r^8 
\]

\[
+ \frac{6220217}{6} \bar{L}^2 + \left( \frac{251191187271}{23625} \ln(2) - \frac{30185397214361}{6912000} \ln(7) - \frac{358931046875}{55296} \ln(5) \right) L - \frac{5599742111}{252} \ln(3) \dot{e}_r^2 \\
- \frac{176889436093}{256000} \ln(3) - \frac{1409954253581573}{110592} \ln(2) L + \frac{704817753125}{1792000} \ln(5) \\
+ \frac{88200}{448000} \ln(7) + \frac{2221469576659}{9450} \ln(2)^2 - \frac{30185397214361}{13824000} \ln(7)^2 - \frac{6220217}{72} \pi^2 \\
+ \frac{6632998943769}{512000} \ln(3)^2 - \frac{358931046875}{55296} \ln(2) \ln(5) - \frac{30185397214361}{6912000} \ln(2) \ln(7) \dot{e}_r^{10} + O(e_r^{12}) \right) ,
\]

(4.20)
\[
\langle (\Delta P_r)_{\text{tail}} \rangle^2 = \frac{\nu^2}{a_{\nu}^{17/2}} \sqrt{1 - 4\nu} \left[ \frac{3296}{75} + \frac{6784}{105} \ln(2) - \frac{432}{7} \ln(3) \right.
+ \left( \frac{6}{35} + \frac{18727}{175} + \frac{94446}{35} \ln(3) - \frac{106220}{21} \ln(2) \right) e_r^2
+ \left( \frac{43}{24} + \frac{30585967}{3000} - \frac{5546875}{168} \ln(5) - \frac{3493611}{280} \ln(3) + \frac{5421613}{60} \ln(2) \right) e_r^4
+ \left( \frac{9211}{1120} + \frac{11966181029}{302400} + \frac{2773890625}{6048} \ln(5) - \frac{287958243}{1120} \ln(3) - \frac{2072662519}{3024} \ln(2) \right) e_r^6
+ \left( \frac{1753799}{69120} + \frac{11163364495}{82944} - \frac{25013471539}{13824} \ln(7) - \frac{98475390625}{48384} \ln(5) + \frac{12206657995}{3584} \ln(3) \right.
+ \frac{347801728481}{80640} \ln(2) \right) e_r^8
+ \left( \frac{1986762119}{32256000} + \frac{359948061095993}{96768000} + \frac{572571167150731}{27648000} \ln(7) + \frac{2428186986875}{516096} \ln(5) - \frac{15736391465067}{1024000} \ln(3) - \frac{435117633509879}{9676800} \ln(2) \right) e_r^{10} + O(e_r^{12}) \left],
\]

(4.21)
\[ \langle (\Delta P_e)_{\text{tail}}^2 \rangle = \frac{\nu^2}{a_{17/2}^2 \sqrt{1 - 4 \nu}} \left\{ \frac{3088}{105} L^2 + L \left( \frac{236704}{1575} - \frac{6176}{35} \ln(2) - \frac{432}{7} \ln(3) \right) - \frac{3088}{105} \pi^2 - \frac{834812}{4725} \right. \\
+ \frac{24704}{105} \ln(2)^2 - \frac{1728}{7} \ln(2)^3(3) + \frac{712}{7} \ln(3) + \frac{258992}{525} \ln(2) \left. \right\} + \frac{73928}{105} L^2 + L \left( \frac{5684552}{1575} - \frac{774608}{105} \ln(2) - \frac{3132}{5} \ln(3) \right) - \frac{14782}{21} \pi^2 - \frac{20083426}{4725} \\
+ \frac{4374}{7} \ln(3)^2 + \frac{47644}{35} \ln(2)^2 - \frac{27512}{35} \ln(2) \ln(3) + \frac{17082}{5} \ln(3) + \frac{4842632}{315} \ln(2) \left\} + \frac{83114}{15} L^2 + L \left( \frac{6401426}{225} - \frac{50620}{21} \ln(2) + \frac{8298}{8} \ln(3) - \frac{5548785}{168} \ln(5) \right) - \frac{6610697}{120} \pi^2 \\
+ \frac{4527301}{1350} \ln(2)^3 + \frac{212139}{35} \ln(3)^2 + \frac{173136}{21} \ln(2) \ln(3) - \frac{421875}{28} \ln(3) \ln(5) - \frac{4609375}{28} \ln(2) \ln(5) \\
+ \frac{61015625}{1008} \ln(5) + \frac{9326097}{400} \ln(2) \ln(3) - \frac{3866337}{315} \ln(3) + \frac{21541046}{315} \ln(2) \left\} \right. e_1^2 \\
+ \frac{128363}{5} L^2 + L \left( \frac{9896177}{75} + \frac{44746854}{945} \ln(2) - \frac{22512789}{80} \ln(3) + \frac{672921875}{3024} \ln(5) \right) \\
+ \frac{776091149}{900} \pi^2 = \frac{140056279}{2016} \ln(2) - \frac{35370351}{1120} \ln(2) \ln(5) - \frac{214229284}{135} \ln(2)^2 \\
+ \frac{25969675}{112} \ln(3) \ln(5) + \frac{1392240625}{1512} \ln(2) \ln(5) - \frac{4514246875}{184144} \ln(5) - \frac{45937347}{40} \ln(2) \ln(3) \\
+ \frac{423618219}{800} \ln(3) + \frac{10629299627}{14175} \ln(2) \left\} e_1^6 \\
+ \frac{5221111}{60} L^2 + L \left( \frac{805562213}{1800} + \frac{31533601}{15} \ln(2) - \frac{19666908027}{8960} \ln(3) - \frac{20763578125}{24192} \ln(5) \right) \\
+ \frac{25013471539}{13824} \ln(7) - \frac{21045404243}{2700} \ln(2)^2 - \frac{1456503047}{8064} \ln(5) - \frac{5476831929}{4480} \ln(2)^2 \\
+ \frac{2901800557}{315} \ln(2)^2 - \frac{1286759375}{13824} \ln(5) \ln(7) - \frac{71648241}{32} \ln(3) \ln(7) - \frac{3360878983}{512} \ln(2) \ln(7) \\
+ \frac{275148186929}{256} \ln(7) + \frac{1392240625}{1512} \ln(2) \ln(5) - \frac{96768}{300} \ln(2) \ln(3) + \frac{145152}{14720} \left\} e_1^8 \\
+ \frac{14394757}{60} L^2 + L \left( \frac{2221960379}{1800} + \frac{181638989123}{4725} \ln(2) - \frac{6172985809161}{716800} \ln(3) \right) \\
+ \frac{1606516634375}{1382400} \ln(5) + \frac{21115130358083}{57520000} \ln(7)^2 - \frac{257888161251}{10752000} \ln(2) - \frac{1966592023}{1350} \ln(3) \\
+ \frac{4747565109943}{1548288} \ln(2)^2 + \frac{4448974609375}{135804303533} \ln(2)^2 - \frac{2253537635733}{23625} \ln(2)^2 \\
+ \frac{36864}{400} \ln(5) \ln(7) + \frac{152480010625}{14336} \ln(3) \ln(7) - \frac{691000}{387072} \ln(2) \ln(7) \\
+ \frac{10106581528431983}{4147200000} \ln(7) - \frac{10281327292619}{1792000} \ln(2) \ln(5) \\
+ \frac{4644864}{3584000} \ln(5) + \frac{9417157620549}{1417500} \ln(2) \ln(3) + O(e_1^{12}) \right\}.
\]

\[ (4.22) \]

V. CONCLUDING REMARKS

We have computed higher-order tail (i.e., tail-of-tail and tail-squared) contributions to the linear momentum loss averaged along hyperbolic-like orbits at their leading...
PN approximation. Our computation uses harmonic coordinates and is conveniently performed in the frequency domain by applying techniques already developed in previous works \cite{41, 53, 54}. We have distinguished among past tails, time-symmetric and time-antisymmetric tails, determined by the full past interaction among the bodies, according to the proper behavior under time reversal. All results have been expressed as an expansion in the large angular momentum. This work completes a previous analysis of leading-order hereditary contributions to the loss of energy, angular momentum and linear momentum along hyperbolic-like orbits \cite{54}. Due to the increasing level of accuracy of PM-based results on gravitational radiation we expect that these results will be extremely important for their low-velocity limit check.

We have also computed quadratic and cubic past tails for elliptic-like orbits as series expansions in a small eccentricity parameter through the same level of approximation. In this case quadratic tails were known in the quasi-circular case only, whereas cubic tails were never been explicitly computed before. Completing these results by the addition of all the instantaneous terms (at the level of accuracy we have computed hereditary terms here) is still an open issue.

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