Duality and Couplings of 3-Form-Multiplets in $N = 1$ Supersymmetry

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ABSTRACT

In this paper we study 3-form gauge fields in four-dimensional $N = 1$ supersymmetric theories. We give the sigma model action together with its Poincaré dual action for massless and massive 3-forms. The resulting target space geometries are Kähler where the respective field variables and superspace couplings are related by a Legendre transformation.

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1 Introduction

Supersymmetric multiplets which contain $p$-form gauge fields are of particular interest in string theory as they generically appear in the low energy effective action. In standard Calabi-Yau compactifications these $p$-forms are massless while in generalized compactifications and/or string backgrounds with non-trivial background fluxes they can be massive gaining their mass via a Stückelberg mechanism [1–3].

In this paper we study massless and massive 3-form gauge fields $C_{npq}$ in four-dimensional ($D = 4$) $N = 1$ globally supersymmetric theories. When $C_{npq}$ is massless its equation of motion forces the field strength to be a constant and thus no propagating degree of freedom is left [4]. For this reason massless 3-forms have also been discussed in connection with the problem of the cosmological constant [5–9]. The 3-form can gain a mass via the Stückelberg mechanism by “eating” a 2-form and as a consequence a massive 3-form has one physical scalar degree of freedom.

In $N = 1$ supersymmetric theories the massless 3-form resides in a supermultiplet together with two scalar and two fermionic physical degrees of freedom [10–12] and Poincaré duality relates this multiplet to a non-minimal scalar multiplet discussed in [13, 14]. A massive 3-form multiplet gains its mass by “eating” a linear multiplet which contains a 2-form as one of its components. The linear multiplet also has two scalar and two fermionic physical degrees of freedom so that a massive 3-form multiplet altogether has four scalar and four fermionic degrees of freedom. Its Poincare dual multiplet is closely related to the complex linear multiplet studied in [18, 19]. The massive 3-form multiplet appears in the effective description of gaugino condensation and was used to describe a (massive) pseudo-scalar glueball [20–22].

In this paper we systematically study the massless and massive 3-form multiplet, their renormalizable and non-renormalizable actions and their Poincaré dual descriptions. We pay particular attention to the issue of boundary terms which in the non-supersymmetric case were discussed in [6–9] while, as far as we know, the supersymmetric version does not exist in the literature so far.

For the case of a non-renormalizable sigma model we find that the scalar fields of the 3-form multiplets span a Kähler manifold. The dual scalars have the same geometry but with the Kähler metric expressed in terms of the Legendre transform of the original Kähler potential. A massive 3-form multiplet features an additional real scalar whose sigma model metric is related to the mass matrix of the 3-forms. In the dual action the massive 3-form is replaced by another scalar and the resulting geometry is the product of two Kähler manifolds.

This work is organized as follows. In section 2.1 we introduce the 3-form multiplet and give its field strength as well as its gauge and supersymmetry transformations. In section 2.2 we give the renormalizable kinetic action and discuss the necessary supersymmetric boundary terms. In section 2.3 we introduce mass and Fayet-Iliopoulos terms and compute the component action, while the modification of the mass spectrum in the presence of a superpotential is discussed in 2.4. The dual actions in the massless and massive case are derived in sections 2.5 and 2.6 respectively. Section 3 generalizes the

\footnote{Since 3-forms can also play the role of the Yang-Mills Chern-Simons term, the 3-form multiplet has been used in the description of gauge anomalies in supersymmetric theories [15–17].}
2 The 3-form multiplet

2.1 Components, field strength and gauge transformation

The superfield \( U \) which contains a 3-form \( C_{npq} \) can be constructed from a vector multiplet

\[ U = U^a = B + i\chi^a + \theta^a \chi + \theta^a M^a - \frac{i}{4} D_{\alpha} U, \]

where \( B \) and \( D \) are real scalars, \( M \) is a complex scalar, \( \chi \) and \( \lambda \) are Weyl spinors.

The 3-form multiplet has a 2-form field strength, with another 2-form as its dual. The field strength of the 3-form on the other hand is a 4-form

\[ H_{mnpq} = \frac{1}{4!} \varepsilon_{mnpq} C_{mnpq}, \]

with a 0-form \( H \) as its dual.

\[ S = -\frac{1}{4} D^2 U. \]

For this reason a field strength for the 3-form multiplet cannot be constructed like the vector multiplet's \( W_\alpha = -\frac{1}{4} D^2 V \) defined in (B.6). Instead it is defined by [10–12, 23]

\[ H_{mnpq} = 0, \]

\[ H = \frac{\varepsilon_{mnpq}}{4} C_{mnpq}, \]

\[ H_{mnpq} = \frac{\varepsilon_{mnpq}}{4} H. \]

2 Our conventions are summarized in Appendix A while basic facts about the various supermultiplets used in this paper are recalled in Appendix B.

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3. Compared to the usual normalization we rescaled \( U \) in (2.2) and (2.5) by a factor of 16 for later convenience.
which implies that $S$ is chiral
\[ \bar{D}_{\dot{\alpha}} S = 0 . \] (2.6)

Its expansion in component fields reads
\[ S = M + i \theta \sigma^m \bar{\theta} \partial_m M + \frac{1}{4} i \theta \bar{\theta} \theta \bar{\theta} \Box M + \sqrt{2} \theta \lambda + \frac{i}{\sqrt{2}} \theta \bar{\theta} \bar{\sigma}^m \partial_m \lambda + \theta (D + iH) . \] (2.7)

Since $U$ is real $S$ is not a generic chiral superfield. Indeed, the imaginary part of the $\theta \theta$-component of $S$ is the dual field strength $H$, which, being a total divergence, is not an unconstrained scalar field.

From its definition (2.5) we see that $S$ is invariant under the gauge transformation
\[ U \rightarrow U - L , \] (2.8)
where $L$ is a linear multiplet obeying $D^2 L = \bar{D}^2 L = 0$. In particular, $L$ contains a real scalar $E$, a Weyl fermion $\eta$ and the field strength of a two-form $\partial_{[n} B_{pq]}$ as component fields (see Appendix B.3 for further details). Using (B.8) one infers the gauge transformation of the component fields to be
\begin{align*}
B &\rightarrow B - E , \quad \chi \rightarrow \chi - \eta , \quad M \rightarrow M , \\
C_{npq} &\rightarrow C_{npq} - \partial_{[n} B_{pq]} , \quad \lambda \rightarrow \lambda , \quad D \rightarrow D ,
\end{align*} (2.9)
which renders $B$ and $\chi$ gauge degrees of freedom. Thus the massless 3-form multiplet $U$ has four scalar and four fermionic off-shell degrees of freedom, which occur in the field strength multiplet (2.7).

The supersymmetry transformations of the 3-form multiplet are given by [11, 12]
\begin{align*}
\delta_\xi M &= \sqrt{2} \xi \lambda , \\
\delta_\xi \lambda &= \sqrt{2} i \sigma^m \bar{\xi} \partial_m M + \sqrt{2} \xi (D + iH) , \\
\delta_\xi C_{npq} &= \varepsilon_{mnpq} \xi \left( \frac{1}{\sqrt{2}} \sigma^m \bar{\lambda} + \sigma^m \partial_l \chi \right) + \text{h.c.} , \\
\delta_\xi B &= i \xi \chi - i \bar{\xi} \bar{\chi} , \\
\delta_\xi \chi &= -2 i \xi M^* + \sigma^m \bar{\xi} \left( -\frac{i}{3} \varepsilon_{mnpq} C^{npq} + \partial_m B \right) .
\end{align*} (2.10)

Note that the gauge invariant components of $S$ transform among themselves as in an ordinary chiral multiplet. Also note that the second term in the supersymmetry variations of the 3-form (which does not appear in the references [11, 12]) constitutes only a gauge transformation since it can be written as
\[ \varepsilon_{mnpq} \xi \sigma^m \partial_l \chi = \frac{3}{2} \partial_{[n} \varepsilon_{pq]} m \xi \sigma^m \chi . \] (2.11)

\[ ^4\text{Recall that every chiral superfield can be expressed as } \Phi = \bar{D}^2 F \text{ with } F \text{ being complex.} \]

\[ ^5\text{Note that since } S \text{ is a chiral superfield it naturally carries mass dimension } 1. \text{ From (2.5) and the fact that } D_\alpha \text{ has dimension } 1/2 \text{ it follows that } U \text{ has mass dimension } 0. \text{ } \theta_\alpha \text{ has dimension } -1/2 \text{ and thus (2.2) implies dimension } 0 \text{ for } B \text{ and } 1/2 \text{ for } \chi \text{, i.e. both fields have non-canonical mass dimensions. Therefore, when they enter the massive 3-form action as will be described in sec. 2.3, they have to be rescaled in order to get kinetic terms of the standard form.} \]
2.2 Renormalizable action of the massless 3-form multiplet

We are now prepared to construct the gauge invariant kinetic action for $N_3$ massless
3-form multiplets $U^a$, $a = 1, \ldots, N_3$ with field strengths $S^a$. Since the $S^a$ are chiral, the
kinetic action is given by the standard expression \[14\]

$$S_3 = \int d^8 z \delta_{ab} S^a \bar{S}^b = \int d^4 x \delta_{ab} \left( - \partial_m M^a \partial^m M^b - i \lambda^a \sigma^m \partial_m \lambda^b + D^a D^b + H^a H^b \right), \tag{2.12}$$

where $D^b = D^b$ and $H^b = H^b$ since both are real.\(^6\) The $D^a$ are auxiliary fields which
vanish by their equations of motion $D^a = 0$. The action (2.12) contains the correct
kinetic term for the 3-form

$$H^a H^a = - \frac{1}{24} H^a_{mnqp} H^{mnqp}. \tag{2.13}$$

However, as we discuss in more detail in Appendix C, one has to add a boundary term to
the action (2.12) in order to impose the gauge invariant variational boundary condition
$\delta H^a |_{\partial M} = 0$ instead of the gauge variant $\delta C^a_{npq} |_{\partial M} = 0$ required by (2.12) \cite{7,8}. (Here $M$
is the integration volume.) In a supersymmetric theory the boundary condition should
in addition be supersymmetry invariant and thus we demand

$$\delta S^a |_{\partial M} = 0, \quad D_\alpha (\delta S^a) |_{\partial M} = 0 \tag{2.14}$$

rather than $\delta U^a |_{\partial M} = 0, \; D_\alpha (\delta U^a) |_{\partial M} = 0$. This can be achieved by adding to (2.12) the
boundary term

$$B = \frac{1}{4} \int d^8 z \; D^a \left( S^a D_\alpha U^a - (D_\alpha S^a) U^a \right) + \text{h.c.} \tag{2.15}$$

$$= \text{Re} \int d^8 z \; D^a \left( S^a D_\alpha U^a \right) - \frac{1}{2} \text{Re} \int d^8 z \; D^2 \left( S^a U^a \right),$$

where the first term in the second line of (2.15) contains the boundary term for the
3-forms

$$- \frac{1}{3} \int d^4 x \; \partial_m (H^a \varepsilon^{mnqp} C^a_{npq}) \tag{2.16}.$$

One can easily check that the variation of the action

$$S'_3 = S_3 + B \tag{2.17}$$

with respect to $U$ is given by

$$\delta S'_3 = \frac{1}{4} \int d^8 z \left[ - (D^2 S^a) \delta U^a + D^a ( (\delta S^a) D_\alpha U^a - D_\alpha (\delta S^a) U^a ) + \text{h.c.} \right]. \tag{2.18}$$

Applying the constraint (2.14) to the variation (2.18) leads to the superfield equations
of motion

$$D^2 S^a + \tilde{D}^2 \tilde{S}^a = 0. \tag{2.19}$$

\(^6\)To obtain the component action we used partial integration for the fields $M^a$ and $\lambda^a$ which might
seem questionable since we cannot assume that boundary terms involving the 3-form field strengths $H^a$
vanish and the latter transform into these fields under supersymmetry. The issue of the boundary terms
is discussed in more detail in the following and in Appendix C. However, for now we can note that
boundary terms do not affect the equations of motion and therefore it is legitimate to drop them here.
They include the equations of motion for $C_{npq}^{a}$

$$\varepsilon^{mnpq} \partial_{m} H^{a} = 0,$$  \hspace{1cm} (2.20)

which imply that the 3-form field strengths are constants, i.e. $H^{a} = c^{a}$ with $c^{a} \in \mathbb{R}$. Thus the action (2.12) describes $2N_{3}$ bosonic and $2N_{3}$ fermionic degrees of freedom on-shell. Due to the presence of the boundary term (2.16) one can check that the 3-forms contribute a positive constant potential [5–9]

$$V = c^{a}c^{a},$$  \hspace{1cm} (2.21)

corresponding to a positive correction of the bare cosmological constant $\Lambda$.

$$\Lambda = \Lambda_{0} + 8\pi G c^{a}c^{a}.$$  \hspace{1cm} (2.22)

The supersymmetry variation of $\lambda$ given in (2.10) shows that it transforms inhomogeneously for $H^{a} = c^{a} \neq 0$ and thus supersymmetry is spontaneously broken with $\lambda$ being the Goldstone fermion.

### 2.3 Renormalizable action of the massive 3-form multiplet

Let us now consider a massive 3-form multiplet by adding a gauge invariant mass term to the action (2.12) with the help of the St"uckelberg mechanism. One introduces $N_{3}$ additional linear multiplets $L^{a}$ with the transformation law

$$L^{a} \rightarrow L^{a} - L^{a},$$  \hspace{1cm} (2.23)

where the transformation parameters $L^{a}$ are also linear superfields. Due to (2.8) the combination $U^{a} - L^{a}$ is gauge invariant so that one can add to the action the mass term [22]

$$S_{\text{mass}} = \int d^{8}z \left( -\frac{1}{2} m^{2}_{ab}(U^{a} - L^{a})(U^{b} - L^{b}) + \xi_{a}(U^{a} - L^{a}) \right),$$  \hspace{1cm} (2.24)

where $m_{ab} = m_{ba}$ is a symmetric mass matrix and the $\xi_{a}$ parametrize possible Fayet-Iliopoulos terms.\(^{8}\) The additional degrees of freedom introduced by the $L^{a}$ can be absorbed into the $U^{a}$ by fixing a gauge. In the following we work in the “unitary” gauge $L^{a} = 0$. Furthermore, for simplicity we assume that there are no massless modes and thus $m_{ab}$ is invertible.\(^{9}\) In the gauge $L^{a} = 0$ we have the action\(^{10}\)

$$S_{3} = \int d^{8}z \left( \delta_{ab} S^{a} S^{b} - \frac{1}{2} m^{2}_{ab} U^{a} U^{b} + \xi_{a} U^{a} \right)$$

$$= \int d^{4}x \left( -\partial_{m} M^{a} \partial^{m} M^{a} - i \lambda^{a} \sigma^{m} \partial_{m} \lambda^{a} + D^{a} D^{a} + H^{a} H^{a} \right. \hspace{1cm} (2.25)\left. \right.$$  

$$-\frac{1}{2} m^{2}_{ab} (i \chi^{a} m^{b} \partial_{m} \chi^{b} - \sqrt{2} i \chi^{a} \lambda^{b} + \sqrt{2} i \bar{\lambda}^{a} \bar{\lambda}^{b} + 2 M^{a} M^{b} + 2 B^{a} D^{b} - \frac{1}{2} B^{a} \Box B^{b} + \frac{1}{2} C_{npq}^{a} C_{lmpq}^{b} + \xi_{a} (D^{a} - \frac{1}{4} \Box B^{a}) \right).$$

\(^{7}\)Without the boundary term the contribution would be negative (for a detailed discussion see Appendix C).

\(^{8}\)Note that cubic terms in $U$ are non-renormalizable.

\(^{9}\)The action for massless modes was already given in section 2.2.

\(^{10}\)The boundary terms (2.15) should be added to the massive action as well. Since we are not going to eliminate the massive 3-forms here, this is however not relevant for our purposes.
The auxiliary fields $D^a$ can be eliminated by their equations of motion

\[ 2\delta_{ab}D^b - m_{ab}^2B^b + \xi_a = 0 \].

This is done most conveniently by “completing the square” as described in Appendix D, leading to the on-shell action

\[
S_3 = \int dx \left( -\partial_m M^a \partial^m M^{a*} - i\lambda^a \sigma^m \partial_m \bar{\lambda}^a + H^a H^a - m_{ab}^2B^b - \frac{\xi_a}{\sqrt{2}} \bar{\chi}^a - \frac{i}{\sqrt{2}} \bar{\chi}^a \bar{\chi}^b + m^a M^{ba} \right.
\]

\[ + \frac{1}{4} \partial^m B^a \partial_m B^b + \frac{1}{6} C^a_{npq} C^{bnpq} - \frac{1}{4} (m_{ab}^2 B^b - \xi_a)(m_{ac}^2 B^c - \xi_a) \] \tag{2.27}

As we already mentioned in footnote 5, $B^a$ and $\chi^a$ do not have standard mass dimensions. In the previous section this was irrelevant as both fields dropped out as gauge degrees of freedom. For a massive 3-form multiplet however, they become physical and in order for their kinetic terms to have the canonical form we need the following field redefinitions

\[
B^a := \frac{1}{2} (\delta^ab m_{bc} B^c - m^{-1ab} \xi_b), \quad \chi^a := -\frac{i}{\sqrt{2}} \delta^{ab} m_{bc} \chi^c \] \tag{2.28}

Note that we also shifted $B$ by a constant proportional to the FI-parameter $\xi$. From (2.27) we see that as long as $m_{ac}^2$ has maximal rank $\xi$ merely induces a vacuum expectation value for $B$ but does not break supersymmetry. However, whenever $m_{ac}^2$ has a zero eigenvalue and the corresponding $\xi$ is non-zero, supersymmetry is spontaneously broken. In the following we assume $m_{ac}^2$ to have maximal rank and perform the field redefinition given in (2.28). Then the on-shell action is independent of $\xi$ and reads

\[
S_3 = \int dx \left( -\partial_m M^a \partial^m M^{a*} - m_{ab}^2 M^a M^{ba} - \partial^m B^a \partial_m B^a - m_{ab}^2 B^a B^b \right.
\]

\[ - i\lambda^a \sigma^m \partial_m \bar{\lambda}^a - i\chi^a \sigma^m \partial_m \bar{\chi}^a - m_{ab} \chi^a \lambda^b - m_{ab} \bar{\chi}^a \bar{\chi}^b + H^a H^a - \frac{1}{6} m_{ab}^2 C^a_{npq} C^{bnpq} \] \tag{2.29}

We see that the fermions $\lambda^a$ and $\chi^a$ form $N_3$ massive Dirac-spinors corresponding to $4N_3$ fermionic degrees of freedom. The $N_3$ massive 3-forms now contribute one bosonic on-shell degree of freedom each, because their equations of motion

\[
-\delta_{ab} \varepsilon^{mnpq} \partial_m H^b = m_{ab}^2 C^{bnpq}, \tag{2.30}
\]

which imply

\[
\partial^n C^b_{npq} = 0, \quad \delta_{ab} \Box C^b_{npq} = m_{ab}^2 C^b_{npq} \] \tag{2.31}

remove $3N_3$ of the $4N_3$ off-shell degrees of freedom. Together with $N_3$ complex scalars $M^a$ and the $N_3$ real scalars $B^a$ we thus also have $4N_3$ massive bosonic on-shell degrees of freedom.
2.4 Including a superpotential

Since the $S^a$ are chiral superfields, one may also add a superpotential to the action. This is an alternative way to introduce masses for the components of the 3-form multiplet, although the 3-form itself cannot gain a mass in this way as we will see below. A superpotential can also lead to spontaneous supersymmetry breaking as in the case of ordinary chiral multiplets. For simplicity we drop the mass term of the previous section and start with the action

\[ S_3 = \int d^4x \left[ \int d^2\theta d^2\bar{\theta} S^a \bar{S}^a + \int d^2\bar{\theta} W(S) + \int d^2\theta W^*(\bar{S}) \right] \]

\[ = \int d^4x \left( - \partial_m M^a \partial^m M^{a*} - i\lambda^a \sigma^m \partial_m \bar{\lambda}^a + D^a D^a + H^a H^a \right. \]

\[ + W_a(D^a + iH^a) + W^*_a(D^a - iH^a) - \frac{1}{2} W_{ab} \bar{\lambda}^a \lambda^b - \frac{1}{2} W^*_{ab} \bar{\lambda}^a \lambda^b \) \]

(2.32)

where

\[ W_a(M) := \frac{\partial W}{\partial S^a} \bigg|_{\theta=\bar{\theta}=0}, \quad W_{ab}(M) := \frac{\partial^2 W}{\partial S^a \partial S^b} \bigg|_{\theta=\bar{\theta}=0}. \]

(2.33)

The auxiliary fields $D^a$ can be easily eliminated from (2.32), creating a contribution to the scalar potential

\[ V_D = \left( \text{Re}(W_a) \right)^2. \]

(2.34)

However, also the massless 3-forms have to be eliminated in order to find the effective scalar potential. Their equations of motion read

\[ \varepsilon^{mnpq} \partial_m (H_a - \text{Im}(W_a)) = 0, \]

(2.35)

where $H_a = \delta_{ab} H^b$, and they have the solution $H_a = \text{Im}(W_a) + c_a$ with $c_a \in \mathbb{R}$. In order to impose gauge invariant boundary conditions one again has to add appropriate boundary terms to the action (2.32). We will not do this in a supersymmetric way here, but only give the correct boundary term for the 3-forms which reads (cf. (2.16))

\[ B_3 = -\frac{1}{3} \int d^4x \partial_m \left( (H_a - \text{Im}(W_a)) \varepsilon^{mnpq} C_{npq}^a \right). \]

(2.36)

When this term is added to the action, the contribution of the 3-forms to the scalar potential is found to be

\[ V_3 = H_a H_a \bigg|_{H_a = \text{Im}(W_a) + c_a}. \]

(2.37)

Thus the effective scalar potential is given by

\[ V = W_a W^*_a + 2c_a \text{Im}(W_a) + c_a c_a = (W_a + i c_a)(W^*_a - i c_a). \]

(2.38)

This coincides with the result for ordinary chiral multiplets with the modified superpotential [11, 22]

\[ \tilde{W}(S) = W(S) + ic_a S^a. \]

(2.39)

Thus the analysis of spontaneous supersymmetry breaking and the mass spectrum can be performed in exactly the same way as for the well known O'Raifeartaigh models [24, 25].
We find that a superpotential can create masses for the scalars $M^a$ and Weyl spinors $\lambda^a$ but the 3-forms remain massless.

One may also consider an action

$$S_3 = \int d^4x \left[ \int d^2\theta d\bar{\theta} \left( S^a S^a - \frac{1}{2} m_{ab} U^a U^b + \xi_a U^a \right) + \int d^2\theta W(S) + \int d^2\bar{\theta} W^*(S) \right]$$

(2.40)

that contains both a superpotential and mass and FI terms for the $U^a$ as done in [20–22] for the low energy effective description of $N = 1$ SYM theory. The 3-forms are then dynamical field variables that cannot be eliminated from the action. The only auxiliary fields of the theory are the $D^a$ that couple both to the $B^a$ as in (2.25) and to the real part of $W_a$ as in (2.32). After their elimination and rescaling of $B$ and $\chi$ as in (2.28) one obtains the action (dropping the prime on $B^a$ and $\lambda^a$)

$$S_3 = \int d^4x \left( - \partial_m M^a \partial^m M^{a*} - \lambda^a \sigma^m \partial_m \bar{\lambda}^a + \partial^m B^a \partial_m B^a - \partial^m M^a \partial_m \bar{\lambda}^a \right.$$

$$- m_{ab} \left( \lambda^a \lambda^b + \bar{\lambda}^a \bar{\lambda}^b \right) - \frac{1}{2} \partial^a W_a \lambda^b - \frac{1}{2} \partial^b W_b \bar{\lambda}^a - m_{ab}^2 M^a M^b -$$

$$- \frac{1}{6} m_{ab}^2 C^a_{npq} C^{bnpq} - 2 \text{Im}(W_a) H^a - \left( m_{ab} B^b - \text{Re}(W_a) \right)^2 \right).$$

(2.41)

The equation of motion for $C^a_{npq}$ reads

$$-\varepsilon^{mnpq} \partial_m (H_a - \text{Im}(W_a)) = m_{ab}^2 C^{bnpq}.$$ 

(2.42)

Since the massive 3-forms carry one physical degree of freedom each, they can be represented by the real scalars

$$\phi^a := m^{-1ab}(H_b - \text{Im}(W_b)).$$

(2.43)

It follows from (2.42) that this scalar satisfies the equation of motion

$$\delta_{ab} \Box \phi^b = m_{ab}^2 \phi^b + m_{ab} \text{Im}(W_b),$$

(2.44)

while the equations of motion for $B^a$ and $M^a$ are

$$\delta_{ab} \Box B^b = m_{ab}^2 B^b - m_{ab} \text{Re}(W_b),$$

$$\delta_{ab} \Box M^b = m_{ab}^2 M^b + W_{ab}^* W^b + W_{ab}^* m_{bc} \left( - B^c + i \phi^c \right).$$

(2.45)

These equations call for the definition of the $N_3$ complex scalar fields

$$N^a := - B^a + i \phi^a.$$ 

(2.46)

In the theory dual to (2.41) where the 3-forms are replaced by the scalars $\phi^a$, the scalar potential is given by

$$\mathcal{V} = m_{ab}^2 M^a M^{b*} + (m_{ab} N^b + W_a)(m_{ac} N^{c*} + W_a^*),$$

(2.47)

as can be easily seen from the equations of motion for $M^a$ and $N^a$. As $\mathcal{V}$ vanishes for $M^a = 0$, $N^a = -m^{-1ab} W_b |_{M=0}$, supersymmetry again remains unbroken by virtue of the non-singular mass matrix $m_{ab}$. Moreover we find that $\langle S^a \rangle = 0$. 

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As linear terms in \( W \) only induce a shift in the VEVs of the \( N^a \), one can absorb them into the \( N^a \) by redefining \( N^a \rightarrow \langle N^a \rangle + N^a \). Thus we can assume without loss of generality that the superpotential is of the form

\[
W(S) = \frac{1}{2} \mu_{ab} S^a S^b + \mathcal{O}(S^3)
\]

(2.48)

with a symmetric matrix \( \mu \). Then the mass terms of the scalar potential (2.47) can be identified as

\[
V = (M^\dagger N^\dagger) \left( \begin{array}{cc} \mu^* & m \\ m & m^2 \end{array} \right) \left( \begin{array}{c} M \\ N \end{array} \right) + \text{higher order terms}.
\]

(2.49)

The quadratic mass matrix for the fermions \( \lambda^a \) and \( \chi^a \) can be directly read off from the action (2.40) and is given by

\[
m_{\lambda,\chi}^2 = \left( \begin{array}{cc} \mu^* & m \\ m & m^2 \end{array} \right) = \left( \begin{array}{cc} \mu^* & m^2 \\ m & m^2 \end{array} \right).
\]

(2.50)

It is identical with the mass matrix for the scalars \( M^a \) and \( N^a \) found in (2.49). In the one dimensional case \( (N_3 = 1) \) the mass eigenvalues are given by

\[
m_{\pm} = \left| \frac{1}{2} |\mu| \pm \sqrt{\frac{1}{4} |\mu|^2 + m^2} \right|.
\]

(2.51)

Note that this coincides (for real \( \mu \)) with the result given in [22] with the correspondence \( \mu = m_{11}, \ m = m_{12} \). Obviously, the four bosonic and fermionic mass eigenstates organize into two chiral supermultiplets with masses \( m_+ \) and \( m_- \) respectively (as this is the only \( N = 1 \) supermultiplet which contains only particles of spin 0 and 1/2).

### 2.5 Dualization of the massless action

It is possible to reproduce a physical action from a ‘first order action’ by introducing additional fields with algebraic equations of motion [26]. Alternatively the original fields can be eliminated from the first order action, giving rise to a dual action. There is a one-to-one map between the on-shell degrees of freedom of action and dual action which is defined by the Euler-Lagrange equations of the first order action.

For the case at hand the massless action (2.17) can be reproduced from the first order action [14]

\[
S_{\text{first}} = \int d^8 z \left( -\delta^{ab} F_a \bar{F}_b + F_a S^a + \bar{F}_a \bar{S}^a \right) + B_{\text{first}},
\]

(2.52)

including the boundary term

\[
B_{\text{first}} = \frac{1}{4} \int d^8 z \left( \bar{D}_a \left( F_a \bar{D}^\alpha U^a - \bar{D}^\alpha F_a U^a \right) + \text{h.c.} \right).
\]

(2.53)

Here the \( F_a \) are unconstrained superfields with a component field expansion

\[
F_a = f_a + \theta \psi_a + \sqrt{2} \bar{\theta} \bar{\varphi}_a + \theta \theta h_a + \bar{\theta} \bar{\theta} n_a + \theta \sigma^m \bar{\theta} w_{am} \\
+ \theta \bar{\theta} \bar{\theta} a + \bar{\theta} \bar{\theta} \bar{\theta} (\zeta_a - \frac{i}{\sqrt{2}} \sigma^m \partial_m \bar{\varphi}_a) + \theta \bar{\theta} \theta (d_a - \frac{1}{4} f_a - \frac{i}{2} \partial_m w^m_a),
\]

(2.54)
where $f_a$, $h_a$, $n_a$ and $d_a$ are complex scalars, $w_{am}$ is a complex vector and $\psi_a$, $\varphi_a$, $\vartheta_a$ and $\zeta_a$ are Weyl spinors. The $F_a$ can be eliminated from (2.52) by inserting their equations of motion
\[
\delta^{ab} F_b = S^a , \tag{2.55}
\]
reproducing the action (2.17).

To find the dual action one inserts (2.5) to obtain
\[
S_{\text{first}} = \int d^8 z \left( - \frac{1}{2} \delta^{ab} F_a F_b - \frac{1}{4} F_a \bar{D}^2 U^a + \frac{1}{4} \bar{D}_a \left( F_a \bar{D}^\alpha U^\alpha - \bar{D}^\alpha F_a U^\alpha \right) + \text{h.c.} \right) 
= \int d^8 z \left( - \delta^{ab} F_a F_b - \frac{1}{4} \left( \bar{D}^2 F_a + D^2 \bar{F}_a \right) U^a \right) , \tag{2.56}
\]
where the Leibniz rule for the covariant superspace derivatives was applied. Note that due to the boundary term the first order action has a simple form whose variation with respect to $U^a$ yields immediately (and without dropping any boundary term) a constraint for $F_a$
\[
0 = - \frac{1}{4} \left( \bar{D}^2 F_a + D^2 \bar{F}_a \right) 
= n_a + n^*_a + \theta \zeta_a + \bar{\theta} \bar{\zeta}_a + \theta \partial_a + \bar{\theta} \bar{\partial}_a + i \theta \sigma^m \bar{\partial}_m (n_a - n^*_a) 
+ \frac{i}{2} \theta \bar{\theta} \sigma^m \partial_m \zeta_a + \frac{i}{2} \bar{\theta} \theta \sigma^m \bar{\partial}_m \bar{\zeta}_a + \frac{i}{2} \theta \bar{\theta} \bar{\theta} \bar{\partial} (n_a + n^*_a) . \tag{2.57}
\]
As usual, Poincaré duality has exchanged the equation of motion with the constraint (cf. (2.19)). We will see below that the condition (2.57) is special for the massless case in that it reduces the number of degrees of freedom in $F_a$ while in the massive case the $F_a$ remain unconstrained superfields. As implied by (2.57) $\zeta_a$ and $d_a$ vanish while the $n_a$ become purely imaginary constants,
\[
\zeta_a = 0 , \quad d_a = 0 , \quad n_a = i \hat{c}_a , \quad \text{with} \quad \hat{c}_a \in \mathbb{R} . \tag{2.58}
\]
Therefore $F_a$ takes the form
\[
F_a = f_a + \theta \psi_a + \sqrt{2} \theta \bar{\varphi}_a + \theta \theta h_a + i \theta \bar{\theta} \hat{c}_a + \theta \sigma^m \bar{\partial} w_{am} + \theta \bar{\theta} \bar{\partial} \bar{\varphi}_a - \frac{i}{\sqrt{2}} \theta \bar{\theta} \sigma^m \partial_m \bar{\varphi}_a 
+ \theta \bar{\theta} \bar{\theta} (\frac{i}{2} \partial_m w^m_a - \frac{1}{4} \Box f_a) , \tag{2.59}
\]
containing 12 bosonic and 12 fermionic off-shell degrees of freedom. Using (2.57) and (2.59) we obtain as the dual component action
\[
S_{\text{dual}} = \int d^8 z \left( - \delta^{ab} F_a F_b \right) 
= \int d^4 x \left( f^*_a \left( \frac{i}{2} \partial_m w^m_a + \frac{1}{4} \Box f_a \right) + \frac{1}{2} \psi_a \vartheta_a - \frac{i}{2} \varphi_a \sigma^m \partial_m \bar{\varphi}_a 
- \frac{1}{2} h_a h^*_a - \frac{1}{2} \hat{c}_a \hat{c}_a + \frac{1}{4} w^*_a w^m_a + \text{h.c.} \right) . \tag{2.60}
\]
After eliminating the auxiliary fields $\psi_a$, $\vartheta_a$, $h_a$ and $w_{am}$ we obtain
\[
S_{\text{dual}} = \int d^4 x \left( - \partial_m f_a \bar{\partial}^m f_a^* - i \varphi_a \sigma^m \partial_m \bar{\varphi}_a - \hat{c}_a \hat{c}_a \right) . \tag{2.61}
\]
Just like the original action (2.12), the dual action describes $N_3$ complex scalars and $N_3$ Weyl spinors. The field strengths of the 3-forms are replaced by the constants $\hat{c}_a$, that also create a constant positive potential. In fact, the superfield equation of motion (2.55) includes the duality relation

$$H^a = \delta^{ab} \text{Im} \eta_b^* = -\delta^{ab} \hat{c}_b,$$  

(2.62)

so that the cosmological constants of action and dual action coincide.

Before we proceed let us note that in the dualization of the massless action a new multiplet $F_a$ appeared. It differs from the complex linear multiplet described in Appendix B.4 only by the free constant $\hat{c}$ (for $\hat{c} = 0$ they coincide). This difference arises from the fact that $S$ is not a general chiral superfield but constructed from a real superfield $U$ via (2.5). If $U$ was complex then $\bar{D}^2 F_a$ and $D^2 \bar{F}_a$ had to vanish separately in (2.57) as in the known duality between the chiral and the complex linear multiplet [19].

### 2.6 Dualization of the massive action

Let us now determine the dual action of $N_3$ massive 3-form multiplets. In this case the first order action is given by

$$S_{\text{first}} = \int d^8 z \left( - \delta^{ab} F_a \bar{F}_b + F_a S^a + \bar{F}_a \bar{S}^a - \frac{1}{2} m_{ab}^2 U^a U^b + \xi_a U^a \right),$$  

(2.63)

where we simply added the mass and Fayet-Iliopoulos terms to (2.52). Since the equations of motion for $F_a$ are the same as in the massless case, the massive action (2.25) is correctly reproduced when the $F_a$ are eliminated from (2.63).

In order to find the dual action we first rewrite (2.63) as in (2.56)

$$S_{\text{first}} = \int d^8 z \left( - \delta^{ab} F_a \bar{F}_b - \frac{1}{4} U^a (\bar{D}^2 F_a + D^2 \bar{F}_a) - \frac{1}{2} m_{ab}^2 U^a U^b + \xi_a U^a \right),$$  

(2.64)

and then again eliminate the 3-form multiplets $U^a$ by their equations of motion

$$-\frac{1}{4} (\bar{D}^2 F_a + D^2 \bar{F}_a) - m_{ab}^2 U^b + \xi_a = 0.$$  

(2.65)

In contrast to the massless case (2.57) the superfields $F_a$ now remain unconstrained and the complex scalars $d_a, n_a$ and the Weyl spinors $\zeta_a$ no longer drop out of the dual action. Substituting (2.65) into (2.64) and using the abbreviation

$$\Omega_a := \frac{1}{4} (\bar{D}^2 F_a + D^2 \bar{F}_a),$$  

(2.66)

we obtain the dual action

$$S_{\text{dual}} = \int d^8 z \left( - \delta^{ab} F_a \bar{F}_b + \frac{1}{2} m^{-2ab} (\Omega_a - \xi_a)(\Omega_b - \xi_b) \right).$$  

(2.67)

Using the $\theta$-expansions of $F_a$ and $\Omega_a$ as given in (2.54) and (2.57) respectively, we find the component action

$$S_{\text{dual}} = \int d^4 x \left( - f_a d_a^* - f_a^* d_a - \frac{1}{2} \partial_m f_a \partial^m f_a^* + \frac{i}{2} \partial_m f_a w_a^m - \frac{i}{2} w_a^m \partial_m f_a^* + \frac{1}{2} w_a^m w_a^m \right. $$

$$+ \frac{1}{2} \psi_a \psi_a^* + \frac{1}{4} \bar{\varphi}_a \bar{\varphi}_a + \frac{1}{\sqrt{2}} \bar{\varphi}_a \zeta_a + \frac{1}{\sqrt{2}} \bar{\varphi}_a \zeta_a - i \varphi_a \bar{\sigma}^m \partial_m \bar{\varphi}_a - h_a h_a^* - n_a n_a^* $$

$$+ \frac{1}{2} \bar{\varphi}_a \bar{\varphi}_a + \frac{1}{2} \zeta_a \bar{\sigma}^m \partial_m \bar{\varphi}_a + d_a d_a^*) \right).$$  

(2.68)
Note that the FI-parameters $\xi_a$ have dropped out of the action due to the fact that the highest component of $\Omega_a$ is a total space-time divergence (remember that they also dropped out of the original action (2.25) by a field redefinition). The action (2.68) still contains the auxiliary fields $h_a, \psi_a, \vartheta_a, d_a$ and $w^m_a$. Eliminating them by their equations of motion and performing the field redefinitions

$$n'_a := \delta_{ab} n^{-1bc} n_c, \quad \zeta'_a := -\frac{1}{\sqrt{2}} \delta_{ab} n^{-1bc} \zeta_c,$$

yields the on-shell action

$$S_{\text{dual}} = \int d^4 x \left( -\partial_m f_a \partial^m f^*_a - m_{ab} f_a f^*_b - \partial_m n'_a \partial^m n'^*_a - m_{ab} n'_a n'^*_b - i \varphi_a \sigma^m \partial_m \bar{\varphi}_a - i \zeta'_a \sigma^m \partial_m \bar{\zeta}'_a - m_{ab} (\varphi_a \zeta'_b + \bar{\varphi}_a \bar{\zeta}'_b) \right).$$

The action (2.70) is dual to the renormalizable massive action of the 3-form multiplet given in (2.29) and describes the dynamics of $2N_3$ massive complex scalars $f_a, n'_a$ and $N_3$ massive Dirac spinors $\varphi_a, \zeta'_a$. The massive 3-forms $C_{anpq}^a$ and the real scalars $B^a$ that appear in (2.29) are represented in the dual action by the complex scalars $n'_a$, so that action and dual action again contain an equal number of on-shell degrees of freedom.

### 3 Non-renormalizable action

#### 3.1 From the superfield Lagrangian to the on-shell action

In this section we drop the requirement of renormalizability and consider an action with arbitrary real functions $K(S, \bar{S})$ and $G(U - L')$ given by

$$S_3 = \int d^8 z \left( K(S, \bar{S}) - G(U - L') \right).$$

$S_3$ is invariant under the combined gauge transformations (2.8), (2.23) and as before we choose the gauge $L' = 0$. For simplicity we restrict our analysis to the bosonic part of the action from now on by setting all fermionic components to zero. Using (2.2) and (2.7) we obtain the component form of (3.1)

$$S_3 = \int d^4 x \left( K_{ab} \left( -\partial^m M^a \partial_m M^{b\ast} + D^a D^b + H^a H^b \right) - 2(\text{Im}K_{ab})H^a D^b - G_a \left( D^a - \frac{1}{4} \square B^a \right) - \frac{1}{2} G_{ab} \left( 2M^a M^{b\ast} + \frac{1}{3} C_{nmpq} C^{nmpq} \right) \right),$$

where we defined

$$K_{a_1...a_n b_1...b_m}(M, M^*) := \left. \frac{\partial K}{\partial S^{a_1} \ldots \partial S^{a_n} \partial \bar{S}^{b_1} \ldots \partial \bar{S}^{b_m}} \right|_{\bar{\theta} = \theta = 0},$$

$$G_{a_1...a_n}(B) := \left. \frac{\partial G}{\partial U^{a_1} \ldots \partial U^{a_n}} \right|_{\theta = \bar{\theta} = 0}.$$

\[\text{We choose a minus sign for the G-term in order to have canonical kinetic terms for the scalars } B^a \text{ for a positive definite } G_{ab}.\]
We see that the complex scalar fields $M^a$ can be viewed as coordinates of a Kähler manifold with metric $K_{ab}$ derived from the Kähler potential $K$. $K_{ab}$ is taken to be positive definite and hence also its symmetric part, which for a Kähler metric coincides with its real part is positive definite and in particular invertible. The equations of motion for the auxiliary fields $D^a$ imply

$$D^a = (\text{Re}K)^{-1ab} \left( \frac{1}{2} G_b - (\text{Im}K)_{bc} H^c \right),$$

(3.4)

where $(\text{Im}K)_{bc}$ denotes the imaginary part of the Kähler metric while $(\text{Re}K)^{-1ab}$ denotes the inverse of its real part. Inserting (3.4) into (3.2) we obtain

$$S_3 = \int d^4x \left[ K_{ab} \left( - \partial^a M^a \partial_m M^b + H^a H^b \right) - G_{ab} \left( M^a M^{b*} + \frac{1}{6} C_{anpq} C^{b}_{npq} \right) \right. $$

$$+ \frac{1}{4} G_{ab} \delta^{ab} - \frac{1}{4} \left( G_{ab} + 2 H^c (\text{Im}K)_{ca} \right) (\text{Re}K)^{-1ab} (G_b - 2 (\text{Im}K)_{bd} H^d) \right]$$

$$= \int d^4x \left[ - K_{ab} \partial^a M^a \partial_m M^b - G_{ab} \left( \frac{1}{4} \partial^a B^a \partial_m B^b + M^a M^{b*} + \frac{1}{6} C_{anpq} C^{b}_{npq} \right) \right. $$

$$+ g_{ab} H^a H^b + G_{a} \left( \text{Re}K \right)^{-1ab} (\text{Im}K)_{bc} H^c - \frac{1}{4} G_{a} (\text{Re}K)^{-1ab} G_b \bigg],$$

(3.5)

where in the second step integration by parts was used. Note that $G_{ab}$ as the sigma model metric of $B^a$ coincides with the mass matrix of $M^a$ and $C^a$. In (3.5) we also defined the real metric

$$g_{ab} := (\text{Re}K)_{ab} + (\text{Im}K)_{ac} (\text{Re}K)^{-1ad} (\text{Im}K)_{db} = K_{ac} (\text{Re}K)^{-1ad} K_{db},$$

(3.6)

where the second expression for $g_{ab}$ shows explicitly that it is positive definite and that its inverse is given by

$$g^{-1ab} = \text{Re}(K^{-1\bar{a}b}).$$

(3.7)

The last term in (3.5) plays the role of a scalar potential. Depending on the choice of the functions $K$ and $G$ it can lead to non-vanishing vacuum expectation values of the fields $B^a$ and $M^a$ as in the renormalizable action (2.27).

The non-renormalizable action for massless 3-form multiplets can be obtained from (3.1), or (3.5) respectively, by setting $G = 0$. As we learned in section 2.2 in this case it is important to add appropriate boundary terms to the action (3.1). These terms should be such that they cancel all variational boundary terms containing $\delta U^a$ in favor of boundary terms containing $\delta S^a$, which we assume to vanish.

Let us pause and outline the general prescription for finding the correct boundary terms for an arbitrary gauge invariant action $S_3$ involving the 3-form multiplets

$$S_3 = \int d^8 z K(S, \bar{S}, F, \bar{F}),$$

(3.8)

where $F$ denotes possible other superfields whose variations are assumed to vanish at the boundary. Each term in the variation of this action with respect to $U^a$

$$\delta S_3 = -\frac{1}{4} \int d^8 z \left( \frac{\partial K}{\partial S^a} \bar{D}^2 (\delta U^a) + \bar{\partial} K / \partial S^a \bar{D}^2 (\delta U^a) \right),$$

(3.9)
contains exactly one \( \delta U^a \) with two superspace derivatives acting on it. Thus one has to apply integration by parts twice for each term in \( \delta S_3 \)

\[
-\frac{1}{4} \int d^{8} \bar{D}_{\dot{a}} \left( \frac{\partial K}{\partial S^a} \bar{D}^{\dot{a}} (\delta U^a) - \left( \bar{D}^{\dot{a}} \frac{\partial K}{\partial S^a} \right) \delta U^a \right) + \text{h.c.} \quad (3.10)
\]

In order to exchange these for terms depending only on \( \delta S^a, \bar{D}^{\dot{a}} (\delta S^a) \) and their complex conjugates, one adds to the action (3.8) the boundary terms

\[
B = \frac{1}{4} \int d^{8} \bar{D}_{\dot{a}} \left( \frac{\partial K}{\partial S^a} \bar{D}^{\dot{a}} U^a - \left( \bar{D}^{\dot{a}} \frac{\partial K}{\partial S^a} \right) U^a \right) + \text{h.c.} \quad (3.11)
\]

Indeed, the boundary terms in the variation \( \delta (S_3 + B) \) are then given by

\[
\frac{1}{4} \int d^{8} \bar{D}_{\dot{a}} \left( \left( \delta \frac{\partial K}{\partial S^a} \right) \bar{D}^{\dot{a}} U^a - \left( \bar{D}^{\dot{a}} \delta \frac{\partial K}{\partial S^a} \right) U^a \right) + \text{h.c.} \quad (3.12)
\]

which vanish by the variational constraints.

Following this prescription we add to the action (3.1) the boundary term

\[
B = -\frac{1}{4} \int d^{8} \bar{D}_{\dot{a}} (\bar{D}^{\dot{a}} K_a(S, \bar{S}) U^a - K_a(S, \bar{S}) \bar{D}^{\dot{a}} U^a) + \text{h.c.}
\]

\[
= \text{Re} \int d^{8} \bar{D}_{\dot{a}} (K_a(S, \bar{S}) \bar{D}^{\dot{a}} U^a) - \frac{1}{2} \text{Re} \int d^{8} \bar{D}^2 (K_a(S, \bar{S}) U^a) .
\quad (3.13)
\]

The first term in the second line of (3.13) contains all the boundary terms involving the 3-forms \( C_{npq}^a \) without derivatives, which are

\[
B_3 = -\frac{1}{3} \int d^{4} x \partial_m \left( (\text{Re}K)_{ab} H^b - (\text{Im}K)_{ab} D^b \right) \varepsilon^{mnpq} C_{npq}^a
\]

\[
= -\frac{1}{3} \int d^{4} x \partial_m \left( g_{ab} H^a \varepsilon^{mnpq} C_{npq}^b \right)
\quad (3.14)
\]

where we used (3.4) in the second line. Now we are ready to eliminate the 3-forms from the massless sigma model action given by

\[
S_3 = \int d^{4} x \left( -K_{ab} \partial^m M^a \partial_m M^b + g_{ab} H^a H^b \right) + B_3 \quad (3.15)
\]

The equations of motion for \( H^a \) enforce

\[
g_{ab} H^b = c_a , \quad \text{with} \quad c_a \in \mathbb{R} .
\quad (3.16)
\]

Inserted back into (3.15) one obtains

\[
S_3 = \int d^{4} x \left( -K_{ab} \partial^m M^a \partial_m M^b - \text{Re}(K^{-1})_{ab} c_a c_b \right) .
\quad (3.17)
\]

We see that the massless 3-forms generate a (positive) potential for the scalars \( M^a \). Since \( K_{ab} \) is positive definite, a positive cosmological constant is induced and supersymmetry is spontaneously broken whenever one \( c_a \neq 0 \). As we will see below, the same phenomenon occurs in the dual action.
3.2 Dual action in the massless case

We now want to find a dual action for the massless action (3.1), i.e. with $G = 0$. For the first order action we make the ansatz

$$S_{\text{first}} = \int d^8z \left( -\hat{K}(F, \bar{F}) + F_a S^a + \bar{F}_b \bar{S}^b \right),$$

where $\hat{K}$ is real. The equations of motion for $F_a$ then read

$$\frac{\partial \hat{K}}{\partial F_a} = S^a. \quad (3.19)$$

In order to reproduce (3.1) (with $G = 0$) $\hat{K}$ has to fulfill the equation

$$K \left( \frac{\partial \hat{K}}{\partial F}, \frac{\partial \hat{K}}{\partial \bar{F}} \right) = F_a \frac{\partial \hat{K}}{\partial F_a} + \bar{F}_b \frac{\partial \hat{K}}{\partial \bar{F}_b} - \hat{K}(F, \bar{F}). \quad (3.20)$$

This relation is satisfied when $K$ is the Legendre transform of $\hat{K}$ and vice versa (the Legendre transformation is its own inverse, see App. F). Therefore we assume here that $K$ is strictly convex so that a Legendre transform as defined in Appendix F exists. Then the relation (3.19) is invertible and equivalent to

$$F_a = \frac{\partial K}{\partial S^a} \equiv K_a(S, \bar{S}). \quad (3.21)$$

The dual action is obtained by eliminating $U^a$ from the first order action (3.18) together with the boundary terms (2.53). Note that the latter exactly reproduce the terms given in (3.13) for $F_a = K_a(S, \bar{S})$. Just like in (2.56), we can then write the action in the form

$$S_{\text{first}} + B_{\text{first}} = \int d^8z \left( -\hat{K}(F, \bar{F}) - \frac{1}{4} U^a (\bar{D}^2 F_a + D^2 \bar{F}_a) \right). \quad (3.22)$$

As before, variation with respect to $U^a$ yields the constraint (2.57) and thus $F_a$ again takes the form (2.59). Inserted into (3.22) one obtains the dual action

$$S_{\text{dual}} = -\int d^8z \hat{K}(F, \bar{F})$$

$$= -\int d^4x \left[ \hat{K}^a \left( -\frac{1}{4} \Box f_a - \frac{i}{2} \partial_m w^m a \right) + \hat{K}^\bar{a} \left( -\frac{1}{4} \Box f_a^* + \frac{i}{2} \partial_m w^m a^* \right) \right.$$

$$+ \hat{K}^{ab} \left( -\frac{1}{4} w_{am} w_{b}^{m} + i \hat{c}_a h_b \right) + \hat{K}^{\bar{a} \bar{b}} \left( -\frac{1}{4} w_{am}^{*} w_{b}^{m*} - i \hat{c}_a h_b^* \right)$$

$$+ \hat{K}^{ab} \left( -\frac{1}{2} w_{am} w_{b}^{m*} + h_a h_b + \hat{c}_a \hat{c}_b \right) \right]. \quad (3.23)$$

The fields $h_a$ and $w_{am}$ have purely algebraic equations of motion and can thus be eliminated. This is most conveniently done for the $h_a$ by completing the square as described

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12In Appendix E we discuss the special situation of a Kähler potential with an additional shift symmetry.
in Appendix D with the result\(^\text{13}\)

\[
S_{\text{dual}} = \int d^4x \left[ \frac{1}{4} \hat{K}^{ab} w_a^m w_{bm} + \frac{1}{4} \hat{K}^{ab} w_a^{m*} w_{bm}^* + \frac{1}{2} \hat{K}^{ab} w_a^m w_{bm}^* \\
- \partial_m \hat{K}^a \left( \frac{1}{4} \partial^m f_a + i w_a^m \right) - \partial_m \hat{K}^\alpha \left( \frac{1}{4} \partial^m f_a^* - i w_a^{m*} \right) \right. \\
+ \left( \hat{K}^{ac} \hat{K}_{cd}^{-1} \hat{K}^{db} - \hat{K}^{ab} \right) \hat{c}_a \hat{c}_b \right].
\]  
\[(3.24)\]

Following the prescription given in Appendix D, we can also complete the square with respect to the fields \(w_a^m\) by writing the action in the form

\[
S_{\text{dual}} = \int d^4x \left[ \frac{1}{4} \left( (w_a^m + u_a^m) (w_a^{m*} + u_a^{m*}) \right) \left( \hat{K}^{ab} \hat{K}^{\alpha b} \hat{K}_{ab} \hat{K}_{\alpha b} \right) \right. \\
- \frac{1}{4} \left( u_a^m u_a^{m*} \right) \left( \hat{K}^{ab} \hat{K}^{\alpha b} \hat{K}_{ab} \hat{K}_{\alpha b} \right) \left( u_{bm}^* + u_{bm} \right) - \frac{1}{4} \partial^m f_a \partial_m \hat{K}^a - \frac{1}{4} \partial^m f_a^* \partial_m \hat{K}^a \\
+ \left( \hat{K}^{ac} \hat{K}_{cd}^{-1} \hat{K}^{db} - \hat{K}^{ab} \right) \hat{c}_a \hat{c}_b \right],
\]

where the \(u_a\) solve the equations

\[
\left( \begin{array}{cc}
\hat{K}^{ab} & \hat{K}^{\alpha b} \\
\hat{K}^{\alpha b} & \hat{K}_{ab} \end{array} \right) \left( \begin{array}{c}
u_{bm}^* + u_{bm} \\
u_{bm} \end{array} \right) = -i \left( \begin{array}{c}
\partial_m \hat{K}^a \\
-\partial_m \hat{K}^\alpha \end{array} \right).
\]
\[(3.26)\]

The action \((3.25)\) depends on the Hesse matrix of the Legendre transformed Kähler potential \(\hat{K}(f, f^*)\) which, as derived in Appendix F, is the inverse of the Hesse matrix of \(K(M, M^*)\). However, let us ignore this fact for the moment and only notice that Hess \(\hat{K}\) is invertible with its inverse given by (cf. (D.12))

\[
(Hess \hat{K})^{-1} = \begin{pmatrix} C & D \\ D^* & C^* \end{pmatrix}
\text{ where } D_{ab} = \left( \hat{K}^{ba} - \hat{K}^{bc} \hat{K}_{cd}^{-1} \hat{K}^{da} \right)^{-1},
\]
\[(3.27)\]

The equations of motion for the \(w_{am}\) and \(w_{\bar{a}m}^*\) imply that the term in the first line of \((3.25)\) (the “square”) vanishes and thus we obtain the on-shell action

\[
S_{\text{dual}} = \int d^4x \left[ -\frac{1}{2} \left( \partial^m \hat{K}^a D_{ab} \partial_m \hat{K}^b \right) + \frac{1}{2} \left( \partial^m \hat{K}^a C_{ab} \partial_m \hat{K}^b + \partial^m \hat{K}^\alpha C_{\alpha b} \partial_m \hat{K}^b \right) \\
- \frac{1}{4} \partial^m f_a \partial_m \hat{K}^a - \frac{1}{4} \partial^m f_a^* \partial_m \hat{K}^a + \left( \hat{K}^{ac} \hat{K}_{cd}^{-1} \hat{K}^{db} - \hat{K}^{ab} \right) \hat{c}_a \hat{c}_b \right].
\]
\[(3.28)\]

To simplify this expression, we substitute \(C_{ab} = -D_{bc} \hat{K}_{cd}^{-1} \hat{K}^{da}\), use the chain rule for \(\partial^m \hat{K}^a\) and the relation

\[
\hat{K}^{cd} \hat{K}_{da}^{-1} \hat{K}^{\alpha e} = \hat{K}^{\alpha e} - D^{\alpha e}.
\]
\[(3.29)\]

This leads to

\[
S_{\text{dual}} = \int d^4x \left( -D_{ab} \partial^m \hat{K}^a \partial_m \hat{K}^b - \text{Re} (D^{-1 \bar{ba}}) \hat{c}_a \hat{c}_b \right).
\]
\[(3.30)\]

\(^{13}\) As \(K\) is the Legendre transform of \(\hat{K}\), it is implicit in formula \((F.10)\) (with \(K\) and \(\hat{K}\) exchanged) that \(\hat{K}^{ab}\) is invertible.
Using \((\text{Hess } \hat{K})^{-1} = \text{Hess } K\) we obtain from (3.27) (cf. (F.10))
\[
D_{ab}(f, f^*) = K_{ab}(M, M^*),
\]  
(3.31)
where \(K_{ab}\) has to be evaluated at
\[
M^a = \hat{K}^a(f, f^*), \quad M^{a\bar{a}} = \hat{K}^{\bar{a}}(f, f^*).
\]  
(3.32)
Equation (3.32) is just the lowest component of the Legendre relation (3.19) that appeared as the equation of motion for \(F_a\) in the first order action (3.18). Using (3.31) and (3.32) we see that the kinetic term of the dual on-shell action (3.30) is equal to that of the original action (3.2). In particular the “new” metric
\[
D_{ab} = \left(\hat{K}^{-1} - \hat{K}^{-1}_{ab} \hat{K}^{-1}_{cd} \hat{K}^{cd}_{ab}\right)^{-1}
\]  
(3.33)
appearing in the dual action is again Kähler with respect to its natural variables \(\hat{K}^a\) and \(\hat{K}^{\bar{b}}\).

Let us now show that also the potential coincides upon using (3.32) and the duality relation of the constants \(c_a\) and \(\hat{c}_a\) which is contained in (3.19). The \(\theta\theta\) and \(\bar{\theta}\bar{\theta}\)-components of (3.19) with constrained \(F_a\) (i.e., \(n_a = i\hat{c}_a\)) read
\[
\hat{K}^{ab} h_b - i \hat{K}^{ab} \hat{c}_b = D^a + i H^a, \quad \hat{K}^{ab} h^*_b + i \hat{K}^{ab} \hat{c}_b = 0. \]  
(3.34)
The second equation in (3.34) is just the equation of motion for the auxiliary field \(h_a\) which we already used to compute the on-shell action (3.30). Inserting the solution for \(h^*_b\) into the complex conjugate of the first equation in (3.34) and using (3.33), one finds the on-shell duality relation

\[
H^a = \text{Re} \left( \hat{K}^{ab} \hat{K}^{-1} \hat{K}^{cd} \hat{K}^{-1}_{cd} \hat{K}^{ad} \right) \hat{c}_d = -\text{Re} \left( D^{-1a\dot{d}} \right) \hat{c}_d. \]  
(3.35)
From this last equation, (3.7) and (3.31) one can derive the relation between the constants \(c_a\) and \(\hat{c}_a\) appearing in the on-shell action (3.17) and dual action (3.30) to be
\[
c_a = g_{ab} H^b = -\hat{c}_a. \]  
(3.36)
Now we see that the sigma model action of the massless 3-form multiplet (3.17), is indeed equal to its dual action (3.30) by use of the two duality relations (3.32) and (3.36).

In conclusion, we discuss the relation of the action (3.17) with its dual (3.25). The two superfield equations of motion of the first order action (3.19) and (2.57) can be used to eliminate the corresponding superfields from the action so that it becomes the original action \(S_3\) of the 3-form multiplet or the dual action \(S_{\text{dual}}\) respectively. Together they contain all equations of motion of the first order action, in particular also those of the auxiliary fields which were used to eliminate them from action and dual action to obtain the final on-shell actions (3.17) and (3.30). Thus it is clear that one can translate these on-shell actions into each other by making use of all the information contained in

\[\text{For the case of the complex linear multiplet, i.e. for } \hat{c}_a = 0, \text{ ref. [18] derives a different result that apparently does not lead to a Kähler geometry in the dual action. We thank J. Gates for discussing this issue.}\]
However, the situation is not that simple because these two superfield equations also contain the equations of motion of the physical fields $M^a$ and $f_a$ appearing in the on-shell action and dual action. Therefore one might not expect that these actions can be translated into each other only by use of the duality relations (3.32) and (3.36). It has to be considered as coincidence that this is nevertheless possible for the massless case, so that the transition from action to dual action can be described as a simple field redefinition. Clearly, it would not be possible if the number of off-shell degrees of freedom of action and dual action did not coincide. (In the massive case they do not coincide and one has to make use of the equations of motion of the physical fields to translate the dual action back into the 3-form action.) On the other hand, using the duality relations (3.32) and (3.36) as a field redefinition to re-express the dual action in terms of the fields $M^a$ and constants $c_a$ one will find an action whose equations of motion are equivalent to those of the dual action by these relations. As the massless 3-form action (3.17) has the same property, it is not very surprising that they coincide.

### 3.3 Dual action in the massive case

Let us now turn to the massive case where the potential $G(U)$ is nontrivial. We simply add this term to the first order action (3.18) and consider

$$S_{\text{first}} = \int d^8z \left( - \hat{K}(F, \bar{F}) + F_a S^a + \bar{F}_b S^b - G(U) \right) ,$$

where $\hat{K}$ is again the Legendre transform of $K$. Since the Euler-Lagrange equations (3.19) do not change, the original action (3.1) is reproduced correctly. We then rewrite the action as

$$S_{\text{first}} = \int d^8z \left( - \hat{K}(F, \bar{F}) - \frac{1}{4} U^a (\tilde{D}^2 F_a + D^2 \bar{F}_a) - G(U) \right) ,$$

and determine the Euler-Lagrange equation for $U^a$ to be

$$\frac{\partial G}{\partial U^a} = -\frac{1}{4} (\tilde{D}^2 F_a + D^2 \bar{F}_a) = \Omega_a .$$

To eliminate the $U^a$ from the action we have to assume that $G$ has a Legendre transform $\hat{G}$. When this is the case, we find as a dual action

$$S_{\text{dual}} = \int d^8z \left( - \hat{K}(F, \bar{F}) + \hat{G}(\Omega) \right) .$$

Using (2.54) and (2.57) we obtain the component action

$$S_{\text{dual}} = - \int d^4x \left[ \hat{K}^a \left( d_a - \frac{1}{4} \Box f_a - \frac{i}{2} \partial_m w^m_a \right) + \hat{K}^{ab} \left( -\frac{1}{4} w^m_a w^m_b + h_a n_b \right) + \text{h.c.} \right.$$

$$+ \hat{K}^{ab} \left( -\frac{1}{2} w^m_a w^*_{bm} + h_a h^*_b + n_a n^*_b \right) + \hat{G}^{ab} \left( d_a d^*_b - \partial_m n_a \partial^m n^*_b \right) \right] .$$

\(^{15}\)Again, in the massive case it is legitimate to drop boundary terms.
Note that compared to (3.23), the complex scalars $d_a$ and $n_a$ also appear in the massive dual action since now the $F_a$ remain unconstrained. The equations of motion for the auxiliary fields $d_a$ and $h_a$ implied by (3.41) are

$$-\dot{\dot{K}}^a + \dot{\dot{G}}^{ab}d_b^* = 0 \quad , \quad \dot{\dot{K}}^{ab}n_b + \dot{\dot{K}}^{ab}h_b^* = 0 \quad .$$

(3.42)

Inserted into (3.41) yields\(^{16}\)

$$S_{\text{dual}} = \int d^4x \left[ \frac{1}{4} \dot{\dot{K}}^{ab}w^a_mw^b_m + \frac{1}{4} \dot{\dot{K}}^{ab}w^a_{m*}w^b_{m*} + \frac{1}{2} \dot{\dot{K}}^{ab}w^a_mw^b_{m*} 
- \partial_m \dot{\dot{K}}^a \left( \frac{i}{4} \partial^m f_\bar{a} + \frac{i}{2} w^m_a \right) - \partial_m \dot{\dot{K}}^\bar{a} \left( \frac{i}{4} \partial^m f_a - \frac{i}{2} w^m_a \right) 
- D^{-1\bar{a}b}n_a n_b^* - \dot{\dot{K}}^\bar{b} \dot{\dot{G}}^{-1}_{\bar{a}b} \dot{\dot{K}}^\bar{a} - \dot{\dot{G}}^{ab} \partial^m n_a \partial_m n_b^* \right] ,$$

(3.43)

where $D^{-1\bar{a}b} = \dot{\dot{K}}^{\bar{a}b} - \dot{\dot{K}}^{ac} \dot{\dot{K}}_{\bar{d}c} \dot{\dot{K}}^{\bar{d}\bar{b}}$. Note that the action (3.43) differs from the massless action (3.24) only by the three terms given in the last line of (3.43). Therefore the auxiliary fields $w^m_a$ can be eliminated using the same steps as in the massless case and we obtain the massive dual action

$$S_{\text{dual}} = \int d^4x \left[ - D_{\bar{a}b} \partial^m \dot{\dot{K}}^\bar{a} \partial_m \dot{\dot{K}}^\bar{b} - \dot{\dot{G}}^{ab} \partial^m n_a \partial_m n_b^* - D^{-1\bar{a}b}n_a n_b^* - \dot{\dot{K}}^\bar{a} \dot{\dot{G}}^{-1}_{\bar{a}b} \dot{\dot{K}}^\bar{b} \right] .$$

(3.44)

Just as for the renormalizable action discussed in section 2.6, the massive 3-forms are no longer dual to constants but are represented, together with the real scalars $B_a$, by the complex scalars $n_a$. Note that $\dot{\dot{G}}^{ab}$ can be viewed as a Kähler metric derived from the Kähler potential $\dot{\dot{G}}(n, n^*) = \dot{\dot{G}}(n + n^*)$ which has a shift symmetry. The last two terms in (3.44) form the potential of the scalars $f_a$ and $n_a$.

4 Coupling to chiral fields

4.1 Renormalizable couplings

In this section we study the coupling of $N_3$ 3-form multiplets $U^a$ to $N_c$ chiral multiplets $\Phi^i$. We start with the renormalizable massive action (2.25) and add kinetic and interaction terms for $\Phi^i$. The action is then of the form\(^{17}\)

$$S = \int d^4x \left[ \int d^2\theta d\bar{\theta} \left( S^a \bar{\xi}^a + \Phi^i \bar{\Phi}^i - \frac{1}{2} m^2_{ab} U^a U^b + \xi_a U^a \right) + \left( \int d^2\theta W(S, \Phi) + \text{h.c.} \right) \right] \, ,$$

(4.1)

where we split $W$ as

$$W(S, \Phi) = W^\text{int.}(S, \Phi) + W^S(S) + W^\Phi(\Phi) \quad .$$

(4.2)

\(^{16}\)Note that $\dot{\dot{G}}^{ab}$ is invertible with $\dot{\dot{G}}^{-1}_{\bar{a}b} = G_{ab}$.

\(^{17}\)Note that this is a gauge fixed action, with the gauge specified in section 2.3. Furthermore, for renormalizable theories $m^2_{ab}$ has to be constant and we do not consider a $\Phi U$ coupling because it can be rewritten as a $\Phi S$ coupling in the superpotential using $\int d^2\bar{\theta} \Phi U = -\frac{1}{4} \bar{D}^2(\Phi U) + \text{tot. divergence.}$
In order for the action to be renormalizable \( W \) can be at most cubic in the superfields. \( \Phi \) has components \( A, \psi \) and \( F \) defined in Appendix B.1 while for \( S \) we use (2.7). This yields (cf. (2.25))

\[
S = \int d^4x \left[ - \partial^m M^a \partial_m M^{a*} - i \lambda^a \sigma^m \partial_m \tilde{\lambda}^a - \partial_m A^i \partial^m A^{i*} - i \psi^i \sigma^m \partial_m \tilde{\psi}^i \\
+ D^a D^a + H^a H^a + F^i F^i - m_{ab} \left( i 2 \lambda^a \sigma^m \partial_m \tilde{\chi}^b - i 2 \chi^a \tilde{\lambda}^b + M^a M^{b*} \right) \\
+ B^a D^b - \frac{1}{4} B^a \Box B^b + \frac{1}{6} C_{npq} C^{npq} \right] + \xi_a (D^a - \frac{1}{4} \Box B^a) \\
+ \left( W_a (D^a + i H^a) + W_i F^i - \frac{1}{2} W_{ij} \psi^i \psi^j - \frac{1}{2} W_{ab} \lambda^a \lambda^b - W_{a1} \lambda^a \psi^i + \text{h.c.} \right) .
\]

(4.3)

Compared with the action (2.40), here we have the additional auxiliary fields \( F^i \) that create a contribution \( W_i W_i^* \) to the scalar potential. Thus, after eliminating the auxiliary fields, redefining \( B \) and \( \chi \) as in (2.28) and dualizing the 3-forms to scalars

\[
\phi^a := m^{-1ab} (H_b - \text{Im}(W_b)) ,
\]

we obtain the action

\[
S = \int d^4x \left[ - \partial_m A^i \partial^m A^{i*} - i \psi^i \sigma^m \partial_m \tilde{\psi}^i - \partial^m M^a \partial_m M^{a*} - i \lambda^a \sigma^m \partial_m \tilde{\lambda}^a - i \chi^a \sigma^m \partial_m \tilde{\chi}^a \\
- (m_{ab} \tilde{\lambda}^a \lambda^b + W_{a1} \lambda^a \psi^i + \frac{1}{2} W_{ab} \lambda^a \lambda^b + \frac{1}{2} W_{ij} \psi^i \psi^j + \text{h.c.}) - \partial_m N^a \partial^m N^{a*} - \mathcal{V} \right] .
\]

(4.5)

As in section 2.4 we defined \( N^a := -B^a + i \phi^a \) and the scalar potential is given by

\[
\mathcal{V} = m_{ab}^2 M^a M^{b*} + W_i W_i^* + (m_{ab} N^b + W_a) \left( m_{ac} N^{c*} + W_a^* \right) .
\]

(4.6)

The form of this potential implies that supersymmetry is unbroken if and only if there is a field configuration for which the equations

\[
m_{ab} N^b + W_a = 0 , \quad W_i = 0 , \quad M^a = 0 \quad (4.7)
\]

are fulfilled. If \( m_{ab} \) is invertible the first equation in (4.7) always has a solution which determines the \( N^b \). For renormalizable interactions we can assume that \( W_{\text{int.}} \) in (4.2) is of the form

\[
W_{\text{int.}}(S, \Phi) = \mu_{a,i} S^a \Phi^i + \rho_{a,ij} S^a \Phi^i \Phi^j + \gamma_{a,ib} S^a S^b \Phi^i ,
\]

(4.8)

and as consequence the second and third equation in (4.7) can be summarized as

\[
W_i \Phi^i (A) = W_i |_{M=0} = 0 .
\]

(4.9)

This implies that in the class of models considered here supersymmetry is broken for exactly the same O’Raifeartaigh superpotentials \( W^\Phi \) as in the well known chiral theories [24]. It is particularly interesting that supersymmetry cannot be broken by \( W_{\text{int.}} \) and \( W^S \) for a non-singular mass matrix \( m_{ab} \) as the first equation in (4.7) always has a solution. This also prevented the Fayet-Iliopoulos term from breaking supersymmetry in (2.25).

Let us now analyze the mass spectrum of the theory for the special case \( W^S(S) = 0 = W^\Phi(\Phi) \). Note that in this case supersymmetry is unbroken since the potential (4.6) vanishes for \( M^a = N^a = A^i = 0 \). Since all fields have vanishing vacuum expectation
values, contributions to the mass matrices only come from terms that are quadratic in the fields.\textsuperscript{18} For the scalars $M^a$ these can be found in
\begin{equation}
    m_{ab}^2 M^a M^b + W_i^* W_i = \left( m_{ab}^2 + \mu_{a,i}^* \mu_{b,i} \right) M^a M^b + \ldots,
\end{equation}
where dots denote terms that are at least cubic. Next we collect the mass terms for the scalars $N^a$ and $A^i$ which are entirely contained in the term
\begin{equation}
    \left| m_{ab} N^b + W_a \right|^2 = \left| m_{ab} N^b + \mu_{a,i} A^i \right|^2 + \ldots = \left( N^b a^* A^j \right) \left( m_{ba} \mu_{a,i} \right) \left( N^b A^i \right) + \ldots.
\end{equation}
Thus we have determined the quadratic mass matrix for the $A^i$ and $N^a$ which can be written in matrix notation as
\begin{equation}
    m_{N,A}^2 = \begin{pmatrix} m_{\mu} \end{pmatrix} \begin{pmatrix} m_{\mu} \end{pmatrix} = \begin{pmatrix} m^2 & \mu \mu \end{pmatrix} \begin{pmatrix} \mu^\dagger m & \mu^\dagger \mu \end{pmatrix}.
\end{equation}
Obviously, $m_{N,A}^2$ has (at least) $N_c$ zero eigenvalues corresponding to the $N_c$ linearly independent vectors in the kernel of the $N_3 \times (N_3 + N_c)$ matrix $(m_{\mu})$. The remaining $N_3$ eigenvalues of $m_{N,A}^2$ coincide with the eigenvalues of the hermitian $N_3 \times N_3$ matrix
\begin{equation}
    Q := \begin{pmatrix} m_{\mu} \end{pmatrix} \begin{pmatrix} m_{\mu}^\dagger \end{pmatrix} = m^2 + \mu \mu^\dagger.
\end{equation}
The corresponding eigenvectors of $m_{N,A}^2$ are given by
\begin{equation}
    \begin{pmatrix} m_{\mu} \end{pmatrix} v_a = \begin{pmatrix} \mu^\dagger v_a \end{pmatrix}
\end{equation}
when $v_a, a = 1, \ldots, N_3$, are the eigenvectors of $Q$. Note that the quadratic mass matrix for the $M^a$ given in (4.10) coincides with $Q^*$ so that for each eigenvalue of $Q$ there are two complex scalar fields (i.e., four on-shell degrees of freedom) with this mass.\textsuperscript{19} Let us also analyze the fermionic mass spectrum in order to determine the physical supermultiplets into which the mass eigenstates are organized. The quadratic mass matrix for the fermions of the theory, $\lambda^a, \chi^a$ and $\psi^i$, is found to be
\begin{equation}
    m_{N,A,\lambda,\chi,\psi}^2 = \begin{pmatrix} 0 & m & \mu \end{pmatrix} \begin{pmatrix} 0 & m & \mu \end{pmatrix} = \begin{pmatrix} m^2 + \mu^* \mu^T & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & m^2 & m \mu \end{pmatrix}.
\end{equation}
The $\lambda^a$ mix among themselves to form mass eigenstates and their mass matrix coincides with that of the $M^a$. As the corresponding linear combinations of the $\lambda^a$ and $M^a$ are contained in the same chiral superfield (which is a linear combination of the $S^a$), it is clear that also the quanta associated to these fields reside in one chiral supermultiplet. Furthermore, the mass matrix for the $\chi^a$ and $\psi^i$ is identical with $m_{N,A}^2$. Thus there are also $2N_c$ massless fermionic states, which join the $2N_c$ massless bosonic states to form $N_c$ massless chiral multiplets. Furthermore, there are another $2N_3$ fermionic mass eigenstates for the eigenvalues of $Q$ which reside in $N_3$ chiral multiplets with the corresponding linear combinations of the $N^a$ and $A^i$ (or, more precisely, the physical states associated to these fields).

\textsuperscript{18} We could also include terms that are linear in $S$ as they would only shift the VEVs of the $N^a$. By redefining $N^a \rightarrow (N^a) + N^a$, we would reduce the problem to the case where $W^S(S) = 0$.

\textsuperscript{19} This is special for the case of vanishing superpotentials. It is generally not true when $W^S$ or $W^\Phi$ is nontrivial, e.g. for $W^\Phi = \frac{i}{2} \mu_{ij}^2 \Phi^i \Phi^j$. 

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4.2 Non-renormalizable coupling

In the non-renormalizable case we allow for arbitrary couplings between the $U^a$ and $\Phi^i$ as well as non-renormalizable kinetic terms that mix $\Phi^i$ and $S^a$. Thus we start with the expression

$$S = \int d^4x \left[ \int d^2\theta d^2\bar{\theta} \left( K(S, \bar{S}, \Phi, \bar{\Phi}) - G(U, \Phi, \bar{\Phi}) \right) + \int d^2\theta W(S, \Phi) + \int d^2\bar{\theta} W^*(\bar{S}, \bar{\Phi}) \right].$$

(4.16)

We again consider only the bosonic part of the action, which has the component form \(^{20}\)

$$S = \int d^4x \left[ K_{ab} \left( - \partial_m M_a \partial^n M^*b + D^a D^b + H^a H^b \right) + 2(\text{Im}K_{ab})H^a D^b \right.$$

$$+ P_{ij} \left( - \partial_m A^i \partial^n A^j + F^i F^j \right) - \left[ K_{ai} \left( - \partial_m M^*a \partial^n A^i + (D^a - iH^a)F^i \right) + \text{h.c.} \right]$$

$$- \frac{1}{2} M B^a \partial^m B^b + M^a M^b + \frac{i}{6} C_{aNPQ} C^{anpq}$$

$$- G_{ab} \left( G_{ai} M^a F^i - G_{aj} M^a F^j + \frac{1}{6} i (G_{ai} \partial^m A^i - G_{aj} \partial^m A^j) \varepsilon_{mNPQ} C^{anpq} \right)$$

$$+ W_i F^i + W_i^* F^i + W_a \left( D^a + iH^a \right) + W_{a*} \left( D^a - iH^a \right) \right].$$

(4.17)

where we defined $P_{ij} := K_{ij} - G_{ij}$. The action (4.17) still contains the auxiliary fields $F^i$ and $D^a$. To eliminate them, we once more follow the prescription given in Appendix D and first consider only the part of the Lagrangian containing the $F^i$. With the definitions

$$Z_i := -iH^a K_{ai} - G_{ai} M^a + W_i , \quad J_i := D^a K_{ai} + Z_i ,$$

(4.18)

this is \(^{21}\)

$$\mathcal{L}_F = F^i P_{ij} F^j + J_i F^i + J_j^* F^j$$

$$= \left( F^i + J_i^* P^{-1ki} \right) P_{ij} \left( F^j + P^{-1jk} J_k \right) - J_j^* P^{-1ji} J_i .$$

(4.19)

After elimination of the $F^i$ only the second term in the second line of (4.19) survives. Then the part of the Lagrangian containing the fields $D^a$ becomes (note that $J_i$ also depends on the $D^a$)

$$\mathcal{L}_D = D^a (K_{ab} - K_{aj} P^{-1ji} K_{ib}) D^b + Q_a D^a$$

$$= (D^a + \frac{1}{2} Q_a R^{-1ca}) R_{ab} (D^b + \frac{1}{2} R^{-1bd} Q_d) - \frac{1}{4} Q_a R^{-1ab} Q_b$$

(4.20)

where

$$Q_a := -2(\text{Im}K_{ab})H^b - G_a + 2\text{Re}(W_a - K_{aj} P^{-1ji} Z_i) ,$$

$$R_{ab} := \text{Re}(K_{ab} - K_{aj} P^{-1ji} K_{ib}) .$$

\(^{20}\)Here integration by parts was applied to rewrite the terms with d’Alambert operators. Note that the ‘mixed kinetic’ terms proportional to $\partial_m A^i \partial^n B^a$ (and complex conjugate), that arise in this step, cancel due to the different signs of the $\Box$-terms in $U$ and $\Phi$. One can argue that such terms cannot appear in a supersymmetric theory as their supersymmetry variation cannot be canceled by the variation of a kinetic term for their superpartners $\psi$ and $\chi$ because $B$ is real while $A$ is complex.

\(^{21}\)Since $P_{ij}$ is the sigma model metric for the scalars $A^i$ we assume it to be invertible.
Now the $D^a$ can also be eliminated from the action: again the first term in the second line of (4.20) vanishes and one finds that all the terms in the action (4.17) containing the auxiliary fields $F^i$ and $D^a$ are replaced by

$$\int d^4x \left( - Z^*_j P^{-1j i} Z_i - \frac{i}{4} Q_a R^{-1ab} Q_b \right).$$

This expression contains potential terms for the scalars $M^a, B^a$ and $A^i$ as well as terms involving the field strengths $H^a$. To separate them, let us define

$$\tilde{Z}_i := -G_{ai} M^a + W_i,$$

$$\tilde{Q}_a := -G_a + 2 \Re (W_a - K_{aj} P^{-1ji} \tilde{Z}_i).$$

Then the on-shell action can be written as

$$S = \int d^4x \left[ - K_{ab} \partial_\mu M^a \partial^\mu M^b - P_{ij} \partial_\mu A^i \partial^\mu A^j - K_{ai} \partial_\mu M^a \partial^\mu A^a - K_{ai} \partial_\mu M^s \partial^\mu A^i \\
- G_{ab} \left( \frac{1}{4} \partial_\mu B^a \partial^\mu B^b + \frac{1}{6} C_{npq} C^{bpq} \right) + \frac{i}{6} \left( G_{ai} \partial_\mu A^i - G_{aj} \partial_\mu A^j \right) \varepsilon^{mpq} C_{npq} \\
+ \hat{g}_{ab} H^a H^b + \Im \left( \tilde{Q}_a R^{-1ab} K_{bc} - 2 \tilde{Z}^*_j P^{-1ji} K_{ie} - 2 W_c \right) H^c - \mathcal{V} \right],$$

where the 3-form field strengths come with the metric

$$\hat{g}_{ab} = \Re (K_{ab} - K_{aj} P^{-1ji} K_{ib}) + (\Im K_{ac}) R^{-1cd} (\Im K_{db})$$

and the scalar potential is given by

$$\mathcal{V} = G_{ab} M^a M^b + \tilde{Z}^*_j P^{-1ji} \tilde{Z}_i + \frac{1}{4} \tilde{Q}_a R^{-1ab} \tilde{Q}_b.$$

### 5 Conclusion

In this paper we determined the couplings and dualities of 3-forms in $N = 1, D = 4$ globally supersymmetric theories. We gave the actions for massless and massive 3-form multiplets including supersymmetric boundary terms. We first discussed renormalizable interactions where we also allowed for the presence of a superpotential. When dualizing these actions we found that a cosmological constant arises in the massless case while the dual action of massive 3-forms contains an additional scalar field. In analogy to the known duality between the chiral and complex linear multiplet, there appears a new multiplet that, in the massless case, differs from the complex linear multiplet only by a constant.

In the non-renormalizable case we focused on the scalar geometry of the non-linear sigma model for 3-form multiplets. Elimination of the massless 3-forms from the action was demonstrated in order to find the on-shell action with the scalar potential. We derived the Poincaré dual action and showed that in the massless case the transition from action to dual action can be described by a field redefinition so that the dual action comes with the same Kähler geometry as the original one. In the massive case an
additional real scalar appears and its sigma model metric is related to the mass matrix of the 3-forms. In the dual action this scalar, together with the massive 3-form, is replaced by a complex scalar and the resulting geometry is the product of two Kähler manifolds.

Finally we coupled the massive 3-form multiplets to chiral multiplets, studied the condition for supersymmetric backgrounds and determined a typical mass spectrum.

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Appendix

A Conventions

In this paper we use the conventions of [25]. The Minkowski metric is $\eta = \text{diag}(-1,1,1,1)$ and we fix the totally antisymmetric tensor $\varepsilon^{mnpq}$ in four space-time dimensions by

$$\varepsilon^{0123} = 1, \quad \varepsilon_{0123} = \det g = -1 \quad \text{for} \quad g = \eta . \tag{A.1}$$

Spinor indices are raised and lowered with the antisymmetric tensor $\varepsilon^{\alpha\beta}$ as follows

$$\psi_\alpha = \varepsilon_{\alpha\beta} \psi^\beta, \quad \bar{\psi}_\dot{\alpha} = \varepsilon^{\dot{\alpha}\dot{\beta}} \bar{\psi}_{\dot{\beta}} . \tag{A.2}$$

Two spinors can form Lorentz invariant products by contraction of their indices:

$$\psi \chi = \psi^\alpha \chi_\alpha = \chi \psi , \quad \bar{\psi} \bar{\chi} = \bar{\psi}_{\dot{\alpha}} \bar{\chi}^{\dot{\alpha}} = \bar{\chi} \bar{\psi} . \tag{A.3}$$

The Pauli matrices $\sigma^m$ and $\bar{\sigma}^m$ are defined by

$$\sigma^m_{\alpha\dot{\alpha}} := (-1, \sigma^1, \sigma^2, \sigma^3)_{\alpha\dot{\alpha}} , \quad \bar{\sigma}^{m\dot{\alpha}\alpha} := \varepsilon^{\dot{\alpha}\dot{\beta}} \varepsilon_{\alpha\beta} \sigma^m_{\beta\beta} . \tag{A.4}$$

and the conventions for the superspace integration are

$$\int \! d^8 z := \int \! d^4 x \int \! d^2 \theta d^2 \bar{\theta} , \quad d^2 \theta := \frac{1}{4} \varepsilon^{\alpha\beta} d\theta_\alpha d\theta_\beta , \quad d^2 \bar{\theta} := -\frac{1}{4} \varepsilon^{\dot{\alpha}\dot{\beta}} d\bar{\theta}_{\dot{\alpha}} d\bar{\theta}_{\dot{\beta}} . \tag{A.5}$$
The generators of supersymmetry \( Q_\alpha, \bar{Q}_\dot{\alpha} \) were chosen to be represented by

\[
Q_\alpha = \frac{\partial}{\partial \theta^\alpha} - i\sigma^m_{\alpha\dot{\alpha}} \bar{\theta}^\dot{\alpha} \partial_m, \quad \bar{Q}_{\dot{\alpha}} = -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} + i\theta^\alpha \sigma^m_{\alpha\dot{\alpha}} \partial_m, \tag{A.6}
\]

while the covariant superspace derivatives \( D_\alpha, \bar{D}_{\dot{\alpha}} \) are defined as

\[
D_\alpha = \frac{\partial}{\partial \theta^\alpha} + i\sigma^m_{\alpha\dot{\alpha}} \bar{\theta}^\dot{\alpha} \partial_m, \quad \bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha \sigma^m_{\alpha\dot{\alpha}} \partial_m. \tag{A.7}
\]

## B N=1 supersymmetry multiplets

For completeness we recall various \( N = 1 \) supermultiplets in this appendix.

### B.1 Chiral multiplet

The chiral superfield is defined by

\[
\bar{D} \dot{\alpha} \Phi = 0, \tag{B.1}
\]

and can always be expressed via an unconstrained complex superfield \( F \) as \( \Phi = \bar{D}^2 F \). In terms of component fields it has the generic form

\[
\Phi = A + i\theta \sigma^m \bar{\theta} \partial_m A + \frac{1}{4} \theta \theta \bar{\theta} \bar{\theta} \partial^2 A + \sqrt{2} \theta \psi - \frac{i}{\sqrt{2}} \theta \theta \bar{\theta} \partial_m \psi \sigma^m \bar{\theta} + \theta \theta F, \tag{B.2}
\]

containing a complex scalar \( A \), a Weyl fermion \( \psi \) and an auxiliary field \( F \). Its renormalizable kinetic action is given by

\[
S = \int d^8 z \Phi \bar{\Phi} = \int d^4 x \left( - \partial_m A \partial^m A^* - i \psi \sigma^m \partial_m \bar{\psi} + FF^* \right). \tag{B.3}
\]

### B.2 Vector multiplet

The vector multiplet is represented by a real superfield \( V = \bar{V} \). Its \( \theta \)-expansion can be written as

\[
V = B + i\theta \chi - i\bar{\theta} \bar{\chi} + \theta \theta M^* + \bar{\theta} \bar{\theta} M + 2\theta \sigma^m \bar{\theta} v_m \\
+ \theta \bar{\theta} \left( \sqrt{2} \lambda + \frac{1}{2} \sigma^m \partial_m \chi \right) + \bar{\theta} \theta \left( \sqrt{2} \lambda - \frac{1}{2} \sigma^m \partial_m \bar{\chi} \right) + \theta \bar{\theta} \bar{\theta} \left( D - \frac{1}{4} \Box B \right), \tag{B.4}
\]

with real scalars \( B \) and \( D \), a complex scalar \( M \), a real vector \( v_m \) and Weyl spinors \( \chi, \lambda \). The vector multiplet is used for the description of supersymmetric gauge theories with \( v_m \) being the gauge boson. A gauge transformation is implemented as

\[
V \to V + \Phi + \bar{\Phi}, \quad v_m \to v_m + \frac{i}{2} \partial_m (A - A^*), \tag{B.5}
\]

with a chiral superfield \( \Phi \). The (Abelian) field strength multiplet of \( V \), invariant under (B.5), is defined by

\[
W_\alpha = -\frac{1}{4} \bar{D}^2 D_\alpha V, \tag{B.6}
\]

and it contains the field strength \( v_{mn} = \partial_m v_n - \partial_n v_m \). With the help of (B.5) one can go to the Wess-Zumino gauge where the components \( B, \chi \) and \( M \) in (B.4) are set to zero.
B.3 Linear multiplet

A real multiplet \( L = \bar{L} \) that satisfies the additional constraint [27]

\[
D^2 L = 0 , \quad \bar{D}^2 L = 0
\]

is called linear multiplet. Its component form is given by [28]

\[
L = E + i\theta \eta - i\bar{\theta} \bar{\eta} + \frac{1}{3} \theta \sigma^m \bar{\epsilon}_{mpq} \bar{\partial}^{[m} B^{pq]} + \frac{1}{2} \theta \partial \bar{\partial} \sigma^m \partial_m \eta - \frac{1}{2} \bar{\theta} \bar{\partial} \theta \sigma^m \partial_m \bar{\eta} - \frac{1}{4} \theta \bar{\theta} \bar{\theta} \bar{\partial} \square E ,
\]

containing a real scalar \( E \), the field strength of a 2-form \( B^{pq} \) and a Weyl spinor \( \eta \). The action reads

\[
S = -\int d^8 z L^2 = \int d^4 x \left( -\frac{1}{2} \partial_m E \partial^m E - i\eta \sigma^m \partial_m \bar{\eta} - \frac{1}{3} \partial_{[m} B_{pq]} \bar{\partial}^{[m} B^{pq]} \right).
\]

The linear multiplet carries, like the chiral multiplet, (4 + 4) (4 bosonic and 4 fermionic) degrees of freedom off-shell. On-shell it has (2 + 2) degrees of freedom and contains no auxiliary fields. There is a duality between the chiral and linear multiplet that corresponds to the on-shell equivalence of a scalar field and a 2-form.

B.4 Complex linear multiplet

The complex linear multiplet \( \Sigma \) is defined by a condition similar to (B.7) but with \( \Sigma \) being complex [18, 19]:

\[
D^2 \Sigma = 0 , \quad \bar{D}^2 \Sigma = 0.
\]

This implies the component expansion

\[
\Sigma = f + \theta \psi + \sqrt{2} \bar{\theta} \bar{\varphi} + \theta \sigma^m \bar{\theta} w_m + \theta \bar{\theta} \bar{\varphi} - \frac{i}{\sqrt{2}} \theta \bar{\theta} \theta \sigma^m \partial_m \bar{\varphi}
\]

\[
+ \bar{\theta} \theta \bar{\varphi} \left( -\frac{i}{2} \partial_m w^m - \frac{1}{4} \square f \right) ,
\]

where \( f, h \) are complex scalars, \( w_m \) is a complex vector and \( \psi, \varphi, \bar{\psi}, \bar{\varphi} \) are Weyl spinors. Altogether these are (12 + 12) off-shell degrees of freedom. The action for \( \Sigma \) reads

\[
S = -\int d^8 z \Sigma \bar{\Sigma} = \int d^4 x \left( \frac{1}{2} f^* \partial_m w^m - \frac{1}{2} f \partial_m w^m + \frac{1}{2} f \bar{\square} f^* + \frac{1}{2} \psi \bar{\theta} + \frac{1}{2} \bar{\psi} \theta 
\]

\[
- i \varphi \sigma^m \partial_m \bar{\varphi} - hh^* + \frac{1}{2} w_m^* w^m \right) .
\]

After elimination of the auxiliary fields \( w_m, h, \vartheta \) and \( \psi \) one obtains the on-shell action

\[
S = \int d^4 x \left( -\partial_m f \partial^m f^* - i \varphi \sigma^m \partial_m \bar{\varphi} \right).
\]

Like the action of the chiral multiplet, it describes a complex scalar and a Weyl spinor. Therefore the chiral multiplet can alternatively be dualized to a complex linear multiplet [19].
C The massless 3-form action: boundary terms and duality

In this appendix we discuss the action of a massless 3-form, its dualization and its connection to the cosmological constant. We deal with the issue of boundary terms and show that the appropriate variational constraint says that the variation of the scalar field strength has to vanish at the boundary of the integration volume.

The canonical action of a massless 3-form is

\[ S_3 = -\frac{1}{24} \int d^4x H_{mnpq} H^{mnpq} = \int d^4x H^2, \]  

where \( H_{mnpq} = 4 \partial_m C_{npq} = -\varepsilon_{mnpq} H \). The equation of motion for the 3-form

\[ \varepsilon^{mnpq} \partial_m H = 0 \]  

forces the field strength to be a constant, \( H = c \) with \( c \in \mathbb{R} \), or

\[ H_{mnpq} = -c \varepsilon_{mnpq}. \]  

For this reason the massless 3-form has been studied in the context of the problem of the cosmological constant [5–9].

However, the action (C.1) is not the full story since its variation includes a boundary term of the form

\[ \delta S_3 = \frac{1}{3} \int d^4x \partial_m \left( H \varepsilon^{mnpq} \delta C_{npq} \right) - \frac{1}{3} \int d^4x \left( \partial_m H \right) \varepsilon^{mnpq} \delta C_{npq}. \]  

Thus, for the action (C.1) one has to impose

\[ \delta C_{npq} \bigg|_{\partial \mathcal{M}} = 0, \]  

in order to make the boundary term vanish (\( \partial \mathcal{M} \) denotes the boundary of the integration volume \( \mathcal{M} \)).\(^{23}\) One might already doubt that (C.5) is a valid boundary condition as it is not gauge invariant. Moreover, it has been pointed out in ref. [6] that substituting the solution (C.3) back into (C.1) yields the wrong sign for the correction of the bare cosmological constant \( \Lambda_0 \). The correct value of the effective cosmological constant can be found by coupling the 3-form to gravity via the action

\[ S_{3, \text{EH}} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} (R - 2\Lambda_0) + \int d^4x \sqrt{-g} H^2, \]  

where

\[ H = \frac{1}{24\sqrt{-g}} \varepsilon^{mnpq} H_{mnpq}, \quad H_{mnpq} = -\frac{1}{\sqrt{-g}} \varepsilon_{mnpq} H. \]  

(Here we used (A.1) which implies for example \( \varepsilon^{mnpq} \varepsilon_{mnpq} = 24g \) and the definition of \( H \) was chosen such that it is a Lorentz scalar.)

\(^{22}\)The usual normalization includes another factor of 1/2 which we omit for convenience.

\(^{23}\)Alternatively one could demand \( \delta C^{npq} (x) \to 0 \) for \( x \to \infty \) sufficiently fast when one integrates over the whole Minkowski space.
The equation of motion of the 3-form again reads

\[ \varepsilon^{mnpq} \partial_m H = 0 \]

which is solved by

\[ H = c, \quad \text{or} \quad \sqrt{-g} = \frac{1}{24c} \varepsilon^{mnpq} H_{mnpq}. \]  

Inserting this solution into the stress energy tensor

\[ T^{mn} = -g^{mn} H^2 \]  

which appears in the Einstein equations

\[ R^{mn} - \frac{1}{2} g^{mn} R = -\Lambda_0 g^{mn} + 8\pi G T^{mn}, \]

one computes an effective cosmological constant

\[ \Lambda = \Lambda_0 - \frac{8\pi G}{3} c^2. \]  

On the other hand, substituting (C.8) into the action (C.6) yields

\[ \Lambda = \Lambda_0 + \frac{8\pi G}{3} c^2. \]

This discrepancy is clearly a result of the incompatibility of the variational constraint (C.5) with the solution (C.8). More precisely, in order to implement (C.5) in the on-shell action, (C.8) constrains the variations of the metric by

\[ \int_M d^4x \sqrt{-g} \delta C_{npq} = 0, \]  

which is not a reasonable constraint. In fact, there is no way at all to implement the constraint \( \delta C_{npq} \big|_{\partial M} = 0 \) in the on-shell action since for given \( \delta g_{mn} \) the solution (C.8) fixes \( \delta C_{npq} \) only up to a gauge transformation. Thus it is not possible to derive a consistent on-shell action from the action (C.6). To cure this, one imposes a different variational constraint on the 3-form by demanding

\[ \delta H \big|_{\partial M} = 0. \]

This condition is automatically fulfilled by (C.8). In order to apply (C.11) one modifies the action (C.1) by adding the following boundary term

\[ S'_3 = \int_M d^4x \left( -\phi^2 + 2\phi H \right) - \frac{1}{3} \int_M d^4x \partial_m (H \varepsilon^{mnpq} C_{npq}), \]  

which does not alter the equations of motion. \(^ {24} \) Indeed, the variation of (C.12) is given by

\[ \delta S'_3 = -\frac{1}{3} \int d^4x (\partial_m H) \varepsilon^{mnpq} \delta C_{npq} - \frac{1}{3} \int d^4x \partial_m (\delta H \varepsilon^{mnpq} C_{npq}). \]

Substituting the solution (C.3) into \( S'_3 \), we find that the boundary term has the opposite sign than the kinetic term and is twice as big resulting in

\[ \langle S'_3 \rangle = c^2 - 2c^2 = -c^2, \]

which is indeed the correct positive contribution to the cosmological constant. \(^ {25} \)

Finally, let us discuss the dual action of (C.12) including the boundary term. One couples the scalar field strength \( H \) to another real scalar \( \phi \) via the first-order action

\[ S_{\text{first}} = \int d^4x \left( -\phi^2 + 2\phi H \right) - \frac{1}{3} \int d^4x \partial_m (\phi \varepsilon^{mnpq} C_{npq}) \]

\[ = \int d^4x \left( -\phi^2 - \frac{1}{3}(\partial_m \phi) \varepsilon^{mnpq} C_{npq} \right). \]

\(^ {24} \)Interestingly, this boundary term formally breaks the gauge invariance of the 3-form action.

\(^ {25} \)Ref. [8] also argued in the context of the Baum-Hawking-Coleman mechanism that adding the boundary term is necessary in order to get the right behavior of the quantity \( \exp(-\langle S_3, \varepsilon H \rangle) \) which is maximized for \( \Lambda \to 0_+ \) and thus could provide a statistical explanation for the smallness of the cosmological constant.
The equation of motion for $\phi$ is $\phi = H$ which, when inserted into (C.15), reproduces (C.12) correctly, including the boundary term. On the other hand, the equation of motion for $C_{npq}$ constrains $\phi$ to be a constant, $\phi = c$ with $c \in \mathbb{R}$. Like in the original action (C.12), the boundary term ensures these equations without imposing that $\delta C_{npq}$ should vanish on the boundary. Here in the first order action the necessity of adding a boundary term becomes even more obvious, as it allows for the elimination of the 3-form from the action, leading to the dual action\footnote{Variations of the dual action with respect to the constant field $c$ are not allowed since we impose the constraint $\delta c|_{\partial M} = 0$, i.e., $\delta c = 0$.}

$$S_{\text{dual}} = \int d^4x \ (-c^2) \ . \quad (C.16)$$

\section{Elimination of auxiliary fields}

Most of the known supermultiplets feature auxiliary fields that do not correspond to on-shell degrees of freedom. These can be eliminated from the action by using their purely algebraic equations of motion. The elimination of multiple auxiliary fields can become computationally involved, especially for complex fields. However, when they occur quadratically in the action, a generalization of the well known technique of “completing the square” simplifies this task a lot. Since we use this technique numerous times in the main text, we outline the general procedure in this appendix.

Suppose we have $N$ real auxiliary fields $D^a$ that occur in the Lagrangian as

$$\mathcal{L} = D^a M_{ab} D^b + J_a D^a + C \ , \quad (D.1)$$

where $M_{ab}$, $J_a$ and $C$ are arbitrary functions of all other fields contained in the action. In order for the Lagrangian to be real, $J_a$ and $C$ have to be real and $M$ has to be a hermitian matrix, even though only its symmetric, i.e. real part contributes in (D.1), which we take to be invertible here.\footnote{If $\text{Re}M$ was not invertible, each of its zero eigenvalues would account for a constraint on the fields that couple to the $D^a$ which reads $v^b J^b = 0$, where $v$ is a corresponding zero eigenvector.} To eliminate the $D^a$, we could simply insert their equations of motion

$$2 \text{Re}(M_{ab}) D^b + J_a = 0 \quad \Rightarrow \quad D^b = -\frac{1}{2} (\text{Re} M)^{-1ba} J_a \quad (D.2)$$

into the Lagrangian (D.1). However the same can be achieved in a more elegant way by shifting the $D^a$,

$$\tilde{D}^a := D^a + \frac{1}{2} (\text{Re} M)^{-1ac} J_c \ , \quad (D.3)$$

to have the Lagrangian (D.1) assume the form

$$\mathcal{L} = \tilde{D}^a (\text{Re} M)_{ab} \tilde{D}^b - \frac{1}{4} J_a (\text{Re} M)^{-1ab} J_b + C \ . \quad (D.4)$$

Now we immediately see that the first term in (D.4) (the “square”) vanishes by the equations of motion for the $D^a$ (or $\tilde{D}^a$ respectively) and we can easily read off the final Lagrangian.
Now suppose that there are complex auxiliary fields $F^a$ that occur in the Lagrangian as
\[ \mathcal{L} = F^a K^{ab} F^{*b} + J_a F^a + J_b^* F^{*b} + C , \quad (D.5) \]
where $K$ is an invertible hermitian matrix, $J_a$ is a complex and $C$ a real function of the other fields. Here the square is completed by shifting
\[ \tilde{F}^a := F^a + J_a^* K^{-1} , \quad (D.6) \]
so that the Lagrangian becomes
\[ \mathcal{L} = \tilde{F}^a K^{ab} \tilde{F}^{*b} - J_a^* K^{-1ab} J_b + C . \quad (D.7) \]
Again the “square” vanishes by the equations of motion for the $F^a$.

Note that (D.5) does not give the most general form of quadratic terms for $N$ complex auxiliary fields. One could also have terms proportional to $FF$ and $F^*F^*$ multiplied by a symmetric matrix $M$ and its complex conjugate, i.e.
\[ \mathcal{L} = F^a M_{ab} F^b + F^{*a} M^{*b} F^{*b} + 2F^a K_{ab} F^{*b} + J_a F^a + J_b^* F^{*b} + C . \quad (D.8) \]
Now the task of completing the square is more complicated than for (D.5). However, by making the ansatz
\[ \mathcal{L} = ((F + T)^T (F + T)^* \left( \begin{array}{cc} M & K \\ K^* & M^* \end{array} \right) \left( \begin{array}{c} F + T \\ F^* + T^* \end{array} \right) - (T^T T^*) \left( \begin{array}{cc} M & K \\ K^* & M^* \end{array} \right) \left( \begin{array}{c} T \\ T^* \end{array} \right) + C , \quad (D.9) \]
one finds that the $T^a$ have to satisfy
\[ 2 \left( \begin{array}{cc} M & K \\ K^* & M^* \end{array} \right) \left( \begin{array}{c} T \\ T^* \end{array} \right) = \left( \begin{array}{c} J \\ J^* \end{array} \right) . \quad (D.10) \]
Provided that $K$ and the matrix
\[ H := \left( \begin{array}{cc} M & K \\ K^* & M^* \end{array} \right) \quad (D.11) \]
are invertible, the inverse is of the form
\[ H^{-1} = \left( \begin{array}{cc} N & G \\ G^* & N^* \end{array} \right) , \quad \text{where} \quad G = \left( K^* - M^* K^{-1} M \right)^{-1} , \quad N = - \left( K^{-1} M G \right)^* . \quad (D.12) \]
Then the on-shell Lagrangian becomes (note that $G$ is hermitian)
\[ \mathcal{L}_{\text{on-shell}} = - \frac{1}{4} \left( J^T J^\dagger \right) H^{-1} \left( \begin{array}{c} J \\ J^* \end{array} \right) + C \]
\[ = - \frac{1}{2} J_a N^{a\bar{b}} J_b + J_a^* N^{*a\bar{b}} J_b^* - \frac{1}{2} J_a G^{a\bar{b}} J_b^* + C . \quad (D.13) \]
E Dual sigma model actions with a shift symmetry

Not every sigma model action with 3-form multiplets can be dualized in the way described in sections 3.2 and 3.3. As an important application for string theory let us consider the specific class of Kähler potentials with a shift symmetry where $K$ only depend on the real parts of the $S^a$, i.e.

$$K(S, \bar{S}) = K(S + \bar{S}) \Rightarrow \frac{\partial K}{\partial S^a} = \frac{\partial K}{\partial \bar{S}^a}.$$  \hspace{1cm} (E.1)

Then the arguments $F_a$ of the Legendre transform $\hat{K}$ of $K$ also have to be real and a first order action is, in the massless case, given by

$$S_{\text{first}} = \int d^8z \left( - \hat{K}(F) + F_a (S^a + \bar{S}^a) \right) + B_{\text{first}},$$  \hspace{1cm} (E.2)

with boundary terms as before (cf. (2.53))

$$B_{\text{first}} = \frac{1}{4} \int d^8z \left[ D_a \left( F_a D^a \bar{U}^a - D^a F_a U^a \right) + \text{h.c.} \right].$$  \hspace{1cm} (E.3)

Since the $F_a$ are real, their component expansion can be written as

$$F_a = f_a + \theta \theta n_a + \bar{\theta} \bar{\theta} n^*_a + \theta \sigma^m \bar{\sigma} w_{am} + \theta \theta \bar{\theta} \left( d_a - \frac{1}{4} \Box f_a \right),$$  \hspace{1cm} (E.4)

where $f_a, d_a$ and $w_{am}$ are real. The superfield equations of motion for the action (E.2) are

$$S^a + \bar{S}^a = \frac{\partial \hat{K}}{\partial F_a}, \quad (D^2 + \bar{D}^2) F_a = 0.$$  \hspace{1cm} (E.5)

The second equation (which is used to eliminate the 3-form multiplets from the action and find a dual action) imposes the constraints

$$d_a = 0, \quad \partial_m w^m_a = 0, \quad n_a = i \hat{c}_a, \quad \hat{c}_a \in \mathbb{R}$$  \hspace{1cm} (E.6)

on the components of $F_a$. The second condition is solved by $w_{am} = \varepsilon_{mnpq} \bar{\sigma}^{[n} B^p_m q]$ with a 2-form $B^{pq}$. Therefore we find as a dual action

$$S_{\text{dual}} = - \int d^8z \hat{K}(F) = \int d^4x \hat{K}^{ab} \left( - \frac{1}{4} \partial^m f_a \partial_m f_b - \frac{2}{3} \bar{\sigma}^{[n} B^p_m q] \partial_n B_{bpq} - \hat{c}_a \hat{c}_b \right).$$  \hspace{1cm} (E.7)

The $2N_3$ bosonic on-shell degrees of freedom contained in the $M^a$ are distributed in the dual action among the real scalars $f_a$ and the 2-forms $B^{pq}_a$, each with $N_3$ degrees of freedom.

As an example consider the Kähler potential

$$K(S, \bar{S}) = - \log(S + \bar{S}).$$  \hspace{1cm} (E.8)

The massless action including the boundary term for the 3-form is then given by

$$S = \int d^4x \frac{1}{(2\text{Re} M)^2} \left( - \partial_m M \partial^m M^* + H^2 \right) - \frac{1}{3} \int d^4x \partial_m \left( \frac{1}{(2\text{Re} M)^2} H \varepsilon^{mpq} C_{npq} \right).$$  \hspace{1cm} (E.9)

To find $\hat{K}(F)$, use the Legendre relation $F = \partial K/\partial S = -(S + \bar{S})^{-1}$ and

$$\hat{K}(F) = -K(S, \bar{S}) + F(S + \bar{S}) = \log(-F) - 1.$$  \hspace{1cm} (E.10)

Thus the dual acion is given by

$$S_{\text{dual}} = - \int d^8z \hat{K}(F) = \int d^4x \frac{1}{f^2} \left( - \frac{1}{4} \partial^m f \partial_m f - \frac{3}{2} \bar{\sigma}^{[n} B^p_m q] \partial_n B_{bpq} - \hat{c}^2 \right).$$  \hspace{1cm} (E.11)
F Legendre transformation

In this appendix we assemble a few facts about the Legendre transformation. For a smooth, strictly convex function $K : \mathbb{R}^n \to \mathbb{R}$ the gradient function

$$s : \mathbb{R}^n \to \mathbb{R}^n, \quad s_i(x) := \frac{\partial K}{\partial x^i}(x)$$

is invertible [29] and the inverse is denoted by $x(s)$. Then the Legendre transform of $K$ is defined by

$$\hat{K}(s) := s_i x^i(s) - K(x(s)) .$$

(F.2)

In particular, $\hat{K}$ satisfies

$$\hat{K} \left( \frac{\partial K}{\partial x} \right) = \frac{\partial K}{\partial x^i} x^i - K(x) \quad \text{for all } x \in \mathbb{R}^n .$$

(F.3)

For the derivative of $\hat{K}$ one finds

$$\frac{\partial \hat{K}}{\partial s_i}(s) = s_j \frac{\partial x^j}{\partial s_i} + x^i - \frac{\partial K}{\partial x^j}(x(s)) \frac{\partial x^j}{\partial s_i} = x^i(s) .$$

(F.4)

If we denote the variable of the double Legendre transform $\hat{\hat{K}}$ as $\tilde{x}$, the function $s(\tilde{x})$ is defined by

$$\frac{\partial \hat{\hat{K}}}{\partial s_i}(s(\tilde{x})) = \tilde{x}^i .$$

(F.5)

Thus equation (F.4) shows that $x(s(\tilde{x})) = \tilde{x}$ (i.e., $\tilde{x}$ is really the original variable $x$) which implies that the Legendre transformation is its own inverse:

$$\hat{\hat{K}}(\tilde{x}) = \tilde{x}^i s_i(\tilde{x}) - \hat{K}(s(\tilde{x}))$$

$$= \tilde{x}^i s_i(\tilde{x}) - \left( s_i(\tilde{x}) x^i(s(\tilde{x})) - K(x(s(\tilde{x}))) \right) = K(\tilde{x}) .$$

(F.6)

From the relations (F.1) and (F.4) it follows that

$$\frac{\partial^2 K}{\partial x^i \partial x^j} \frac{\partial^2 \hat{K}}{\partial s_j \partial s_k} = \frac{\partial s_i}{\partial x^j} \frac{\partial x^j}{\partial s_k} = \delta^k_i , \quad \text{or} \quad \text{Hess } K = \left( \text{Hess } \hat{K} \right)^{-1} ,$$

(F.7)

where the derivatives of $\hat{K}$ have to be evaluated at $s(x) = \partial K/\partial x$ when those of $K$ are evaluated at $x$.

In the case of a Kähler potential $K(z, \bar{z})$ with Kähler metric

$$K_{ij} = \frac{\partial^2 K}{\partial z^i \partial \bar{z}^j}$$

one has

$$\text{Hess } K = \begin{pmatrix} K_{ij} & K_{i\bar{j}} \\ K_{j\bar{i}} & K_{\bar{i}\bar{j}} \end{pmatrix} , \quad \text{Hess } \hat{K} = \begin{pmatrix} \hat{K}_{ij} & \hat{K}_{i\bar{j}} \\ \hat{K}_{j\bar{i}} & \hat{K}_{\bar{i}\bar{j}} \end{pmatrix} .$$

(F.9)

The inverse of a block matrix of the form of Hess $\hat{K}$ is given in equation (D.12). Thus we obtain the formula

$$K_{ij} = \left( \hat{K}_{\bar{j}i} - \hat{K}_{\bar{j}k} \hat{K}_{k\bar{i}}^{-1} \hat{K}_{i\bar{j}} \right)^{-1} .$$

(F.10)
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