Normal Bandits of Unknown Means and Variances: 
Asymptotic Optimality, Finite Horizon Regret Bounds, and a Solution to an Open Problem

Wesley Cowan
Department of Mathematics
Rutgers University
110 Frelinghuysen Rd., Piscataway, NJ 08854, USA
CWCOWAN@MATH.RUTGERS.EDU

Michael N. Katehakis
Department of Management Science and Information Systems
Rutgers University
100 Rockafeller Rd., Piscataway, NJ 08854, USA
MNK@RUTGERS.EDU

Abstract

Consider the problem of sampling sequentially from a finite number of \(N \geq 2\) populations, specified by random variables \(X_i^k, i = 1, \ldots, N,\) and \(k = 1, 2, \ldots;\) where \(X_i^k\) denotes the outcome from population \(i\) the \(k^{th}\) time it is sampled. It is assumed that for each fixed \(i,\) \(\{X_i^k\}_{k \geq 1}\) is a sequence of i.i.d. normal random variables, with unknown mean \(\mu_i\) and unknown variance \(\sigma_i^2\). The objective is to have a policy \(\pi\) for deciding from which of the \(N\) populations to sample form at any time \(n = 1, 2, \ldots\) so as to maximize the expected sum of outcomes of \(n\) samples or equivalently to minimize the regret due to lack on information of the parameters \(\mu_i\) and \(\sigma_i^2\).

In this paper, we present a simple inflated sample mean (ISM) index policy that is asymptotically optimal in the sense of Theorem 4 below. This resolves a standing open problem from Burnetas and Katehakis (1996b).

Additionally, finite horizon regret bounds are given.

Keywords: Inflated Sample Means, Multi-armed Bandits, Sequential Allocation

1. Introduction and Summary

Consider the problem of sampling sequentially from a finite number of \(N \geq 2\) populations or ‘bandits’, where the measurements from population \(i\) are specified by a sequence of i.i.d. random variables \(\{X_i^k\}_{k \geq 1}\), taken to be normal with finite mean \(\mu_i\) and finite variance \(\sigma_i^2\). The means \(\{\mu_i\}\) and variances \(\{\sigma_i^2\}\) are taken to be unknown to the controller. It is convenient to define the maximum mean, \(\mu^* = \max_i \{\mu_i\}\), and the bandit discrepancies \(\{\Delta_i\}\) where \(\Delta_i = \mu^* - \mu_i \geq 0\).

In this paper, given \(k\) samples from population \(i\) we will take the estimators: \(\bar{X}_i^k = \frac{1}{k} \sum_{t=1}^k X_i^t\) and \(S_i^2(k) = \frac{1}{k} \sum_{t=k+1}^n (X_i^t - \bar{X}_i^k)^2\) for \(\mu_i\) and \(\sigma_i^2\) respectively. Note that the use of the biased estimator for the variance, with the \(1/k\) factor in place of \(1/(k-1)\), is largely for aesthetic purposes - the results presented here adapt to the use of the unbiased estimator as well.

For any adaptive, non-anticipatory policy \(\pi, \pi(t) = i\) indicates that the controller samples bandit \(i\) at time \(t\). Define \(T_i^k(n) = \sum_{t=1}^n 1\{\pi(t) = i\}\), denoting the number of times bandit \(i\) has been sampled during the periods \(t = 1, \ldots, n\) under policy \(\pi\); we take, as a convenience, \(T_i^0(0) = 0\) for all \(i, \pi\). The value of a policy \(\pi\) is the expected sum of the first \(n\) outcomes under \(\pi\), which we define to be the function \(V_\pi(n)\):

\[
V_\pi(n) = \mathbb{E} \left[ \sum_{i=1}^N \sum_{k=1}^n X_i^k \right] = \sum_{i=1}^N \mu_i \mathbb{E} \left[ T_i^k(n) \right],
\tag{1}
\]

©2015 Wesley Cowan and Michael N. Katehakis.
where for simplicity the dependence of $V_{\pi}(n)$ on the true, unknown, values of the parameters $\mu = (\mu_1, \ldots, \mu_N)$ and $\sigma^2 = (\sigma^2_1, \ldots, \sigma^2_N)$, is suppressed. The regret of a policy is taken to be the expected loss due to ignorance of the parameters $\mu$ and $\sigma^2$ by the controller. Had the controller complete information, she would at every round activate some bandit $i^*$ such that $\mu_{i^*} = \mu^* = \max_i \{ \mu_i \}$. For a given policy $\pi$, we define the expected regret of that policy at time $n$ as

$$R_{\pi}(n) = n\mu^* - V_{\pi}(n) = \sum_{i=1}^n \Delta_i \mathbb{E} \left[ T^i_{\pi}(n) \right].$$

(2)

It follows from Eqs. (1) and (2) that maximization of $V_{\pi}(n)$ with respect to $\pi$ is equivalent to minimization of $R_{\pi}(n)$. This type of loss due to ignorance of the means (regret) was first introduced in the context of an $N = 2$ problem by Robbins (1952) as the ‘loss per trial’ $L_{\pi}(n)/n = \mu^* - \sum_{i=1}^N \frac{T^i_{\pi}(n)}{n} \sigma_i$ (for which $R_{\pi}(n) = \mathbb{E}[L_{\pi}(n)]$) who constructed a a modified (along two sparse sequences) ‘play the winner’ policy, $\pi_R$, such that $L_{\pi_R}(n) = o(n)$ (a.s.) and $R_{\pi_R}(n) = o(n)$, using for his derivation only the assumption of the Strong Law of Large Numbers. Following Burnetas and Katehakis (1996b) when $n \to \infty$, if $\pi$ is such that $R_{\pi}(n) = o(n)$ we say policy $\pi$ is uniformly convergent (UC) (since then $\lim_{n \to \infty} V_{\pi}(n)/n = \mu^*$). However, if under a policy $\pi$, $R_{\pi}(n)$ grew at a slower pace, such as $R_{\pi}(n) = o(n^{1/2})$, or better $R_{\pi}(n) = o(n^{1/100})$ etc., then the controller would be assured that $\pi$ is making a effective trade-off between exploration and exploitation. It turns out that it is possible to construct ‘uniformly fast convergent’ (UFC) policies, defined as the policies $\pi$ for which:

$$R_{\pi}(n) = o(n^\alpha),$$

for all $\alpha > 0$ for all $(\mu, \sigma^2)$.

The existence of UFC policies in the case considered here is well established, e.g., Auer et al. (2002) (fig. 4. therein) presented the following UFC policy $\pi_{ACF}$:

**Policy $\pi_{ACF}$ (UCB1-NORMAL).** At each $n = 1, 2, \ldots$:

i) Sample from any bandit $i$ for which $T^i_{\pi_{ACF}}(n) < [8 \ln n]$

ii) if $T^i_{\pi_{ACF}}(n) > [8 \ln n]$, for all $i = 1, \ldots, N$, sample from bandit $\pi_{ACF}(n+1)$ with

$$\pi_{ACF}(n+1) = \arg \max_i \left\{ \overline{X}^i_{T^i_{\pi}(n)} + 4 \cdot S_i(T^i_{\pi}(n)) \sqrt{\frac{\ln n}{T^i_{\pi}(n)}} \right\}.$$

(3)

(Taking, in this case, $S^2_i(k)$ as the unbiased estimator.)

Additionally, Auer et al. (2002) (in Theorem 4. therein) gave the following bound:

$$R_{\pi_{ACF}}(n) \leq M_{ACF}(\mu, \sigma^2) \ln n + C_{ACF}(\mu),$$

for all $n$ and all $(\mu, \sigma^2)$,

(4)

with

$$M_{ACF}(\mu, \sigma^2) = 256 \sum_{i: \mu_i \neq \mu^*} \frac{\sigma_i^2}{\Delta_i} + 8 \sum_{i=1}^N \Delta_i,$$

(5)

$$C_{ACF}(\mu) = (1 + \frac{\pi^2}{2}) \sum_{i=1}^N \Delta_i.$$

(6)

Ineq. (4) readily implies that $R_{\pi_{ACF}}(n) \leq M_{ACF}(\mu, \sigma^2) \ln n + o(\ln n)$. Thus, since $\ln n = o(n^\alpha)$ for all $\alpha > 0$ and $R_{\pi_{ACF}}(n) \geq 0$, it follows that $\pi_{ACF}$ is uniformly fast convergent.
Given that UFC policies exist, the question immediately follows: just how fast can they be? The primary motivation of this paper is the following general result, from [Burnetas and Katehakis (1996b)], where they showed that for any UFC policy \( \pi \), the following holds:

\[
\liminf_{n} \frac{R(n)}{\ln n} \geq M_{BK}(\mu, \sigma^2), \quad \text{for all } (\mu, \sigma^2), \tag{7}
\]

where the bound itself \( M_{BK}(\mu, \sigma^2) \) is determined by the specific distributions of the populations, in this case:

\[
M_{BK}(\mu, \sigma^2) = \sum_{i: \mu_i \neq \mu^*} \frac{2\Delta_i}{\ln\left(1 + \frac{\Delta_i^2}{\sigma^2_i}\right)}.
\tag{8}
\]

For comparison, depending on the specifics of the bandit distributions, there is a considerable distance between the logarithmic term of the upper bound of Eq. (4) and the lower bound implied by Eq. (8).

The derivation of Ineq. (7) implies that in order to guarantee that a policy is uniformly fast convergent, sub-optimal populations have to be sampled at least a logarithmic number of times. The above bound is a special case of the theoretical bounds derived in Burnetas and Katehakis (1996b) for distributions with multi-parameters being unknown (such as in the current problem of Normal populations with both the mean and the variance being unknown). Previously, Lai and Robbins (1985) had obtained such lower bounds for distributions with one-parameter (such as in the current problem of Normal populations with unknown mean but known variance). Allocation policies that achieved the lower bounds were called asymptotically efficient or optimal in Lai and Robbins (1985).

As in Burnetas and Katehakis (1996b), Ineq. (7) motivates the definition of a uniformly fast convergent policy \( \pi \) as having a uniformly maximal convergence rate (UM) or simply being asymptotically optimal, within the class of uniformly fast convergent policies, if \( \lim_{n} R_{\pi}(n)/\ln n = M_{BK}(\mu, \sigma^2) \).

Burnetas and Katehakis (1996b) proposed the following index policy \( \pi_{BK} \) as one that could achieve this lower bound:

Policy \( \pi_{BK} \) (ISM-NORMAL). At each \( n = 1, 2, \ldots \):

i) For \( n = 1, 2, \ldots, 2N \) sample each bandit twice, and

ii) for \( n \geq 2N \), sample from bandit \( \pi_{BK}(n+1) \) with

\[
\pi_{BK}(n+1) = \arg \max \left\{ \hat{X}_i^j + S_i(T^i_{n+1}) \frac{\Delta_i}{\sqrt{nT^i_{n+1}}} - 1 \right\}.
\tag{9}
\]

Burnetas and Katehakis (1996b) were not able to establish the asymptotic optimality of the \( \pi_{BK} \) policy because they were not able to establish a sufficient condition (Condition A3 therein), which we express here as the following equivalent conjecture.

Conjecture 1 For each \( i \), for every \( \epsilon > 0 \), and for \( k \to \infty \), the following is true:

\[
P(\hat{X}_i^j + S_i(n) \sqrt{\frac{k^2}{j^2}} - 1 < \mu_i - \epsilon \text{ for some } 2 \leq j \leq k) = o(1/k).
\tag{10}
\]

1. In Burnetas and Katehakis (1996b), \( M_{BK}(\mu, \sigma^2) \) is given as a special case of a more general result (part 1 of Theorem 1 therein) that in the case of normal populations with unknown means and variances specializes to: \( M_{BK}(\mu, \sigma^2) = \sum_{\mu \neq \mu^*} \Delta_i \frac{1}{K_i(\mu, \sigma^2)} \) with \( K_i(\mu, \sigma^2) = \inf\{I(\mu_i, \sigma^2_i; \mu_i' > \mu^*, \sigma^2_i > 0) \} = (1/2) \ln(1 + \frac{\Delta_i}{\sigma_i^2}) \), cf. subsection 4.4 therein.
We show that the above conjecture is false (cf. Proposition 6 in the Appendix). While $\pi_{BK}$ may in fact be UM (i.e., asymptotically optimal), this failure means that the techniques established in [Burnetas and Katehakis (1996b)] are insufficient to verify its optimality. All is not lost, however. One of the central results of this paper is to establish that with an asymptotically negligible change, the policy $\pi_{BK}$ may be modified to one that is provably optimal. We introduce in this paper the policy $\pi_{CK}$ defined in the following way:

**Policy $\pi_{CK}$ (ISM-NORMAL$^2$).** At each $n = 1, 2, \ldots$

i) For $n = 1, 2, \ldots, 3N$ sample each bandit three times, and

ii) For $n \geq 3N$, sample from bandit $\pi_{CK}(n + 1)$ with

\[
\pi_{CK}(n + 1) = \arg \max_i \left\{ \frac{X_i^n}{T_i^n(n)} + \sigma_i \sqrt{\frac{2 \ln n}{T_i^n(n)}} \right\}
\]

The central result of this paper (Theorem 3) establishes a finite horizon bound on the regret of $\pi_{CK}$. From this bound, it follows that $\pi_{CK}$ is asymptotically optimal (Theorem 4), and we additionally give a bound on the remainder term (Theorem 5).

**Remark 1**

1) Note that policy $\pi_{CK}$ is only a slight modification of policy $\pi_{BK}$, the only difference between their indices is the $-2$ in the power on $n$ under the radical, i.e., $2/(T^n_i(n) - 2)$ in $\pi_{CK}(n + 1)$ replacing $2/T^n_i(n)$ in $\pi_{BK}(n + 1)$. This change, while asymptotically negligible, has a profound effect on what is provable about $\pi_{CK}$. 

2) We note that the indices of policy $\pi_{CK}$ are a significant modification of those of the optimal allocation policy $\pi_{CK}$ for the case of normal bandits with known variances, cf. [Burnetas and Katehakis (1996b)] and [Katehakis and Robbins (1995)], which are:

\[
\pi_{CK}^*(n + 1) = \arg \max_i \left\{ \frac{X_i^n}{T_i^n(n)} + \sigma_i \sqrt{\frac{2 \ln n}{T_i^n(n)}} \right\}
\]

the difference being the replacing the term $\sigma_i \sqrt{\frac{2 \ln n}{T_i^n(n)}}$ in $\pi_{CK}^*$ by $S_i(T^n_i(n)) \sqrt{\frac{2}{n}}$ in $\pi_{CK}$. However, the indices of policy $\pi_{ACF}$ are a minor modification of the optimal policy $\pi_{CK}^*$, the difference being replacing the term $\sigma_i \sqrt{\frac{2 \ln n}{T_i^n(n)}}$ in $\pi_{CK}^*$ by $S_i(T^n_i(n)) \sqrt{\frac{16 \ln n}{T_i^n(n)}}$ in $\pi_{ACF}$. 

3) The $\pi_{BK}$ and $\pi_{CK}$ policies can be seen as connected in the following way, however, observing that $2 \ln n / T^n_i(n)$ is a first-order approximation of $n^2 / T^n_i(n) - 1 = e^{2 \ln n / T^n_i(n)} - 1$.

Following [Robbins (1952)], and additionally [Gittins (1979), Lai and Robbins (1985) and Weber (1992)] there is a large literature on versions of this problem, cf. [Burnetas and Katehakis (2003), Burnetas and Katehakis (1997b)] and references therein. For recent work in this area we refer to [Audibert et al. (2009), Auer and Ortner (2010), Gittins et al. (2011), Bubeck and Slivkins (2012), Cappé et al. (2013), Kaufmann (2015), Li et al. (2014), Cowan and Katehakis (2015a), Cowan and Katehakis (2015b), and references therein. For more general dynamic programming extensions we refer to [Burnetas and Katehakis (1997a), Butenko et al. (2003), Tewari and Bartlett (2008), Audibert et al. (2009), Littman (2012), Feinberg et al. (2014) and references therein.]

To our knowledge, outside the work in [Lai and Robbins (1985), Burnetas and Katehakis (1996b) and Burnetas and Katehakis (1997a), asymptotically optimal policies have only been developed for the different problem of finite known support in [Honda and Takemura (2011)] and [Honda and Takemura (2010)] constructed optimal policies, cyclic and randomized, that are simpler to implement than those consider in [Burnetas and Katehakis (1996b)].
The main results of this paper, that Conjecture 1 is false (cf. Proposition 6 in the Appendix) and the bounds on the behavior of \( \pi_{CK} \), all depend on the following general probability bound:

**Proposition 2** Let \( Z, U \) be independent random variables, \( Z \sim N(0,1) \) a standard normal, and \( U \sim \chi^2_d \) a chi-squared distribution with \( d \) degrees of freedom, where \( d \geq 2 \).

For \( \delta > 0 \), \( p > 0 \), the following holds for all \( k \geq 1 \):

\[
\frac{1}{2} \mathbb{P} \left( \sum_{i=1}^k \frac{Z_i^2}{k^{d/p}} \geq \delta^2 \right) \leq \mathbb{P} \left( \delta + \sqrt{U} \sqrt{k^{2/p} - 1} < Z \right) \leq \frac{e^{-(1+\delta^2)/2} p k^{(1-d)/p}}{2\delta^2 \sqrt{d} \ln k}. \tag{12}
\]

**Proof** [of Proposition 2] The proof is given in the Appendix. Tighter bounds are possible, but these are sufficient for this paper.

**Theorem 3** For a policy \( \pi_{CK} \) as defined above, the following bounds hold for all \( n \geq 3N \) and all \( \epsilon \in (0,1) \):

\[
R_{\pi_{CK}}(n) \leq \sum_{i: \mu_i \neq \mu^*} \left( \frac{2\ln n}{\ln (1 + \frac{\Delta_i^2}{\Delta^2 
(1+\epsilon)} \sqrt{\frac{8\sigma_i^2}{2\epsilon \Delta^3 \epsilon^3}} \ln \ln n + \frac{8\sigma_i^2}{\Delta^2 \epsilon^2} + 4) \Delta_i. \right) \tag{13}
\]

Before giving the proof of this bound, we present two results, the first demonstrating the asymptotic optimality of \( \pi_{CK} \), the second giving an \( \epsilon \)-free version of the above bound, which gives a bound on the sub-logarithmic remainder term. It is worth noting the following: The bounds of Theorem 3 can actually be improved, through the use of a modified version of Proposition 2, to eliminate the \( \ln \ln n \) dependence, so the only dependence on \( n \) is through the initial \( \ln n \) term. The cost of this, however, is a dependence on a larger power of \( 1/\epsilon \). The particular form of the bound given in Eq. (13) was chosen to simplify the following two results, cf. Remark 4 in the proof of Proposition 2.

**Theorem 4** For a policy \( \pi_{CK} \) as defined above, \( \pi_{CK} \) is asymptotically optimal in the sense that

\[
\lim_{n \to \infty} \frac{R_{\pi_{CK}}(n)}{\ln n} = M_{BK}(\mu, \sigma^2). \tag{14}
\]

**Proof** [of Theorem 4] For any \( \epsilon \) such that \( 0 < \epsilon < 1 \), we have from Theorem 3 that the followings holds:

\[
\limsup_n \frac{R_{\pi_{CK}}(n)}{\ln n} \leq \sum_{i: \mu_i \neq \mu^*} \frac{2\Delta_i}{\ln (1 + \frac{\Delta_i^2}{\sigma^2 (1+\epsilon)})}. \tag{15}
\]

Taking the infimum over all such \( \epsilon \),

\[
\limsup_n \frac{R_{\pi_{CK}}(n)}{\ln n} \leq \sum_{i: \mu_i \neq \mu^*} \frac{2\Delta_i}{\ln (1 + \frac{\Delta_i^2}{\sigma^2})} = M_{BK}(\mu, \sigma^2), \tag{16}
\]
and observing the lower bound of Eq. (7) completes the result.

\[ R_{\pi_{CK}}(n) \leq M_{BK}(\mu, \sigma^2) \ln n + O((\ln n)^{3/4} \ln \ln n), \]

**Theorem 5** For a policy \( \pi_{CK} \) as defined above, \( R_{\pi_{CK}}(n) \leq M_{BK}(\mu, \sigma^2) \ln n + M_{CK}(\mu, \sigma^2)(\ln n)^3/4 \ln \ln n \)

\[
M_{CK}(\mu, \sigma^2) = M_{BK}(\mu, \sigma^2) \\
M_{CK}^2(\mu, \sigma^2) = 8 \sum_{i : \mu_i \neq \mu^*} \left( \sigma_i^2 \right) \\
M_{CK}^3(\mu, \sigma^2) = 6 \sum_{i : \mu_i \neq \mu^*} \left( \frac{\Delta_i^3}{\sigma_i^2 + \Delta_i^2} \right) \\
M_{CK}^4(\mu, \sigma^2) = 8 \sum_{i : \mu_i \neq \mu^*} \left( \Delta_i + \frac{\sigma_i^2}{\Delta_i} \right) \\
M_{CK}^4(\mu, \sigma^2) = 4 \sum_{i : \mu_i \neq \mu^*} \Delta_i.
\]

It is worth noting that the \((1 - (\ln n)^{-1/4})^2/(1 - (\ln n)^{-1/4})^2 \) factor converges to 1 as \( n \to \infty \). While the above bound admittedly has a more complex form than such a bound as in Eq. (4), it demonstrates the asymptotic optimality of the dominating term, and bounds the sub-linear remainder term.

**Proof** [of Theorem 5] The bound follows directly from Theorem 3 taking \( \epsilon = (\ln n)^{-1/4} \) for \( n \geq 3 \), and observing the following bound, that for \( \epsilon < 1, \)

\[
\frac{1}{\ln \left(1 + \frac{\Delta_i^2}{\sigma_i^2} \frac{(1-\epsilon)^2}{1+\epsilon}\right)} \leq \frac{1}{\ln \left(1 + \frac{\Delta_i^2}{\sigma_i^2}\right)} + \frac{3\Delta_i^2}{\sigma_i^2 + \Delta_i^2} \left(1 - \frac{\epsilon}{\Delta_i^2}(1-\epsilon)^2\right) \left(1 - \frac{\epsilon}{3}\right). \tag{19}
\]

This inequality is proven separately as Proposition 7 in the Appendix.

We make no claim that the results of Theorems 2, 3, 5 are the best achievable for this policy \( \pi_{CK} \). At several points in the proofs, choices of convenience were made in the bounding of terms, and different techniques may yield tighter bounds still. But they are sufficient to demonstrate the asymptotic optimality of \( \pi_{CK} \), and give useful bounds on the growth of \( R_{\pi_{CK}}(n) \).

**Proof** [of Theorem 1] In this proof, we take \( \pi = \pi_{CK} \) as defined above. For notational convenience, we define the index function

\[
u_i(k, j) = \bar{x}_i^j + S_i(j) \sqrt{k^{\frac{2}{\gamma^2}}}, \tag{20}
\]
The structure of this proof will be to bound the expected value of $T_i^j(n)$ for all sub-optimal bandits $i$, and use this to bound the regret $R_n(n)$. The basic techniques follow those in \cite{KatehakisRobbins1995} for the known variance case, modified accordingly here for the unknown variance case and assisted by the probability bound of Proposition 2. For any $i$ such that $\mu_i \neq \mu^*$, we define the following quantities: Let $1 > \varepsilon > 0$ and define $\bar{\varepsilon} = \Delta_i \varepsilon / 2$. For $n \geq 3N$,

\[
\begin{align*}
    n_1^i(n, \varepsilon) &= \sum_{t=3N}^{n} \mathbb{I} \{ \pi(t + 1) = \pi_i, u_i(t, T_{\bar{\pi}}^i(t)) \geq \mu^* - \bar{\varepsilon}, S_{T_{\bar{\pi}}^i(t)}^2 \leq \mu_i + \bar{\varepsilon}, S_{T_{\bar{\pi}}^i(t)}^2 \leq \sigma_i^2 (1 + \varepsilon) \} \\
    n_2^i(n, \varepsilon) &= \sum_{t=3N}^{n} \mathbb{I} \{ \pi(t + 1) = \pi_i, u_i(t, T_{\bar{\pi}}^i(t)) \geq \mu^* - \bar{\varepsilon}, S_{T_{\bar{\pi}}^i(t)}^2 \leq \mu_i + \bar{\varepsilon}, S_{T_{\bar{\pi}}^i(t)}^2 > \sigma_i^2 (1 + \varepsilon) \} \\
    n_3^i(n, \varepsilon) &= \sum_{t=3N}^{n} \mathbb{I} \{ \pi(t + 1) = \pi_i, u_i(t, T_{\bar{\pi}}^i(t)) > \mu^* - \bar{\varepsilon} \} \\
    n_4^i(n, \varepsilon) &= \sum_{t=3N}^{n} \mathbb{I} \{ \pi(t + 1) = \pi_i, u_i(t, T_{\bar{\pi}}^i(t)) \leq \mu^* - \bar{\varepsilon} \}.
\end{align*}
\]

Hence, we have the following relationship for $n \geq 3N$, that

\[
T_i^j(n + 1) = 3 + \sum_{t=3N}^{n} \mathbb{I} \{ \pi(t + 1) = \pi_i \} = 3 + n_1^i(n, \varepsilon) + n_2^i(n, \varepsilon) + n_3^i(n, \varepsilon) + n_4^i(n, \varepsilon). \tag{22}
\]

The proof proceeds by bounding, in expectation, each of the four terms. Observe that, by the structure of the index function $u_i$,

\[
\begin{align*}
    n_1^i(n, \varepsilon) &\leq \sum_{t=3N}^{n} \mathbb{I} \left\{ \pi(t + 1) = \pi_i, (\mu_i + \bar{\varepsilon}) + \sigma_i \sqrt{1 / 2 - 1 / (1 + \varepsilon)} \geq \mu^* - \bar{\varepsilon} \right\} \\
    &= \sum_{t=3N}^{n} \mathbb{I} \left\{ \pi(t + 1) = \pi_i, T_{\bar{\pi}}^i(t) \leq \frac{2 \ln n}{\ln \left( 1 + \frac{\Delta_i (1-\varepsilon)^2}{\sigma_i^2 (1+\varepsilon)} \right) + 2} \right\} \\
    &\leq \sum_{t=3N}^{n} \mathbb{I} \left\{ \pi(t + 1) = \pi_i, T_{\bar{\pi}}^i(t) \leq \frac{2 \ln n}{\ln \left( 1 + \frac{\Delta_i (1-\varepsilon)^2}{\sigma_i^2 (1+\varepsilon)} \right) + 2} \right\} \\
    &\leq \sum_{t=1}^{n} \mathbb{I} \left\{ \pi(t + 1) = \pi_i, T_{\bar{\pi}}^i(t) \leq \frac{2 \ln n}{\ln \left( 1 + \frac{\Delta_i (1-\varepsilon)^2}{\sigma_i^2 (1+\varepsilon)} \right) + 2} \right\} \\
    &\leq \frac{2 \ln n}{\ln \left( 1 + \frac{\Delta_i (1-\varepsilon)^2}{\sigma_i^2 (1+\varepsilon)} \right) + 2 + 2}.
\end{align*}
\]

The last inequality follows, observing that $T_{\bar{\pi}}^i(t)$ may be expressed as the sum of $\pi(1) = i$ indicators, and seeing that the additional condition bounds the number of non-zero terms in the above sum. The additional $+2$ simply accounts for the $\pi(1) = i$ term and the $\pi(n + 1) = i$ term.

Note, this bound is sample-path-wise.
For the second term,

\[
n_2(n, \varepsilon) \leq \sum_{t=3N}^{n} \mathbb{1}\{\pi(t + 1) = i, S_i^2(T_i^u(t)) > \sigma_i^2(1 + \varepsilon)\}
\]

\[
= \sum_{t=3N}^{n} \sum_{k=2}^{i} \mathbb{1}\{\pi(t + 1) = i, S_i^2(k) > \sigma_i^2(1 + \varepsilon), T_i^u(t) = k\}
\]

\[
= \sum_{t=3N}^{n} \sum_{k=2}^{i} \mathbb{1}\{\pi(t + 1) = i, T_i^u(t) = k\} \mathbb{1}\{S_i^2(k) > \sigma_i^2(1 + \varepsilon)\}
\]

\[
\leq \sum_{k=2}^{n} \mathbb{1}\{S_i^2(k) > \sigma_i^2(1 + \varepsilon)\} \sum_{t=k}^{n} \mathbb{1}\{\pi(t + 1) = i, T_i^u(t) = k\}
\]

\[
\leq \sum_{k=2}^{n} \mathbb{1}\{S_i^2(k) > \sigma_i^2(1 + \varepsilon)\}.
\]

The last inequality follows as, for fixed \(k\), \(\{\pi(t + 1) = i, T_i^u(t) = k\}\) may be true for at most one value of \(t\). Recall that \(kS_i^2(k)/\sigma_i^2\) has the distribution of a \(\chi^2_{k-1}\) random variable. Letting \(U_k \sim \chi^2_k\), from the above we have

\[
\mathbb{E}[n_2^2(n, \varepsilon)] \leq \sum_{k=2}^{n} \mathbb{P}(S_i^2(k) > \sigma_i^2(1 + \varepsilon))
\]

\[
\leq \sum_{k=2}^{n} \mathbb{P}(U_{k-1}/k > (1 + \varepsilon))
\]

\[
\leq \sum_{k=2}^{n} \mathbb{P}(U_{k-1}/(k-1) > (1 + \varepsilon))
\]

\[
= \sum_{k=1}^{n} \mathbb{P}(U_k > k(1 + \varepsilon))
\]

\[
\leq \frac{1}{\sqrt{1 + \varepsilon}} \frac{1}{\varepsilon^2} < \infty.
\]

The penultimate step is a Chernoff bound on the terms, \(\mathbb{P}(U_k > k(1 + \varepsilon)) \leq (e^{-\varepsilon}(1 + \varepsilon))^{k/2}\).

To bound the third term, a similar rearrangement to Eq. (24) (using the sample mean instead of the sample variance) yields:

\[
n_3(n, \varepsilon) \leq \sum_{i=3N}^{n} \mathbb{1}\{\pi(t + 1) = i, \tilde{X}_i^j > \mu_i + \tilde{\varepsilon}\} \leq \sum_{k=2}^{n} \mathbb{1}\{\tilde{X}_i^j > \mu_i + \tilde{\varepsilon}\}.
\]

Recalling that \(\tilde{X}_i^j - \mu_i \sim Z\sigma_i/\sqrt{k}\) for \(Z\) a standard normal,

\[
\mathbb{E}[n_3(n, \varepsilon)] \leq \sum_{k=2}^{n} \mathbb{P}(\tilde{X}_i^j > \mu_i + \tilde{\varepsilon}) \leq \sum_{k=1}^{n} \mathbb{P}(Z\sigma_i/\sqrt{k} > \tilde{\varepsilon}) \leq \frac{1}{\frac{\sigma_i^2}{\varepsilon^2} - 1} \leq \frac{2\sigma_i^2}{\varepsilon^2} < \infty.
\]

The penultimate step is a Chernoff bound on the terms, \(\mathbb{P}(Z > \delta/\sqrt{k}) \leq e^{-k\delta^2/2}\).

To bound the \(n_4^2\) term, observe that in the event \(\pi(t + 1) = i\), from the structure of the policy it must be true that \(u_i(T_i^u(t)) = \max_j u_j(T_i^u(t))\). Thus, if \(i^*\) is some bandit such that \(\mu_{i^*} = \mu^*, u_{i^*}(T_i^u(t)) \leq u_i(t, T_i^u(t))\).
Hence the following bound,

\[
n_t^*(n, \varepsilon) \leq \sum_{t=1}^{n} \mathbb{1}\{\pi(t+1) = i, u_{i^*}(t, T_t^i(t)) < \mu^* - \varepsilon\}
\]

\[
\leq \sum_{t=1}^{n} \mathbb{1}\{u_{i^*}(t, T_t^i(t)) < \mu^* - \varepsilon\}
\]

\[
\leq \sum_{t=1}^{n} \mathbb{1}\{u_{i^*}(t, s) < \mu^* - \varepsilon\ \text{for some} \ 3 \leq s \leq t\}.
\]

The last step follows as for \(t\) in this range, \(3 \leq T_t^i(t) \leq t\). Hence

\[
\mathbb{E}[n_t^*(n, \varepsilon)] \leq \sum_{t=1}^{n} \mathbb{P}(u_{i^*}(t, s) < \mu^* - \varepsilon\ \text{for some} \ 3 \leq s \leq t).
\]

As an aside, this is essentially the point at which the conjectured Eq. \((10)\) would have come into play for the proof of the optimality of \(\pi_{\text{NK}}\), bounding the growth of the corresponding term for that policy. We will essentially prove a successful version of that conjecture here. Define the events \(A_{i,t,\varepsilon}^* = \{u_{i^*}(t, s) < \mu^* - \varepsilon\}\). Observing the distributions of the sample mean and sample variance, we have (similar to Eq. \((40)\)) for \(Z\) a standard normal and \(U_{i-1} \sim \chi^2_{i-1}\), with \(U\), \(Z\) independent,

\[
\mathbb{P}(A_{i,t,\varepsilon}^*) = \mathbb{P}\left(\frac{\bar{\varepsilon}}{\sigma_i^*} \sqrt{3 + \sqrt{U_{i-1}}} \sqrt{\frac{2}{t+3}} - 1 < Z\right)
\]

\[
\leq \frac{e^{-(\varepsilon/\sigma_i^*)^2/2}(s-2)}{2(\varepsilon/\sigma_i^*)^2 s/(s-1)} \frac{t^{-1}}{\ln t}
\]

\[
\leq \frac{e^{-(\varepsilon/\sigma_i^*)^2/2}}{2(\varepsilon/\sigma_i^*)^2} \frac{1}{\sqrt{s}} \frac{t^{-1}}{\ln t}
\]

\[
\leq \left(\frac{1}{2(\varepsilon/\sigma_i^*)^2} \frac{e^{-(\varepsilon/\sigma_i^*)^2/2}}{\sqrt{s}}\right) \frac{t^{-1}}{\ln t}.
\]

where the first inequality follows as an application of Proposition \(2\) and the second since \(s \geq 3\). Applying a union bound to Eq. \((29)\),

\[
\mathbb{E}[n_t^*(n, \varepsilon)] \leq \sum_{t=1}^{n} \sum_{s=3}^{t} \mathbb{P}(A_{i,t,\varepsilon}^*)
\]

\[
\leq \sum_{t=1}^{n} \sum_{s=3}^{t} \left(\frac{1}{2(\varepsilon/\sigma_i^*)^2} \frac{e^{-(\varepsilon/\sigma_i^*)^2/2}}{\sqrt{s}}\right) \frac{t^{-1}}{\ln t}
\]

\[
\leq \left(\frac{1}{2(\varepsilon/\sigma_i^*)^2} \frac{e^{-(\varepsilon/\sigma_i^*)^2/2}}{\sqrt{s}}\right) \int_{s=0}^{s=\infty} ds \int_{t=0}^{t=\infty} \frac{t^{-1}}{\ln t} dt
\]

\[
= \left(\frac{1}{2(\varepsilon/\sigma_i^*)^2} \frac{\sqrt{2\pi}}{\varepsilon/\sigma_i^*} \ln \ln n\right)
\]

\[
= \frac{\pi \sigma_i^3}{2e \varepsilon^3} \ln \ln n.
\]

The bounds follow, removing the dependence of the \(s\)-sum on \(t\) by extending it to \(\infty\), and bounding the sums by integrals of the (decreasing) summands by slightly extending the range of each. From the above results,
and observing that $T^i_n \leq T^i_n(n + 1)$, it follows from Eq. (22) that for any $\varepsilon$ such that $0 < \varepsilon < 1$,

$$
E[T^i_n(n)] \leq \frac{2\ln n}{\ln \left(1 + \frac{\Delta_i^2 (1-\varepsilon)^2}{\sigma^2}ight)} + 4 + \frac{8\sigma_i^2}{\varepsilon^2} + \frac{8\Delta_i^3}{\sigma_i^2} \ln n
$$

The result then follows from the definition of regret in Eq. (2).

\textbf{Remark 2} Numerical Regret Comparison: Figure 1 shows the results of a small simulation study, implementing policies $\pi_{CK}$, $\pi_{ACF}$, and $\pi_G$ a ‘greedy’ policy that always activates the bandit with the current highest average. The simulation was done on a set of six populations, with means and variances given in the table below. Each policy was implemented over a horizon of 100,000 activations, each replicated 10,000 times to produce a good estimate of the average regret $R_{\pi}(n)$ over the times indicated.

\textbf{Remark 3} Bounds and Limits: Figure 2 shows first (left) a comparison of the theoretical bounds on the regret, $B_{\pi_{ACF}}(n)$ and $B_{\pi_{CK}}(n)$ representing the theoretical regret bounds of the RHS of Eq. (4) and Eq. (17) respectively, for the means and variances indicated in the table below. Additionally, Figure 2 (right) shows the convergence of $R_{\pi_{CK}}(n)/\ln n$ to the theoretical lower bound $M_{BK}(\mu, \sigma^2)$.

| $\mu_i$ | 8 | 8 | 7.9 | 7 | -1 | 0 |
| $\sigma_i^2$ | 1 | 1.4 | 0.5 | 3 | 1 | 4 |

Figure 1: Numerical Regret Comparison

References

Jean-Yves Audibert, Rémi Munos, and Csaba Szepesvári. Exploration–exploitation tradeoff using variance estimates in multi-armed bandits. Theoretical Computer Science, 410(19):1876–1902, 2009.

Peter Auer and Ronald Ortner. Ucb revisited: Improved regret bounds for the stochastic multi-armed bandit problem. Periodica Mathematica Hungarica, 61(1-2):55–65, 2010.
Figure 2: Left: Plots of $B_{\pi ACF}(n)$ and $B_{\pi CK}(n)$. Right: Convergence of $R_{\pi CK}(n)/\ln(n)$ to $M_{BK}(\mu, \sigma^2)$.

Peter Auer, Nicolo Cesa-Bianchi, and Paul Fischer. Finite-time analysis of the multiarmed bandit problem. *Machine learning*, 47(2-3):235–256, 2002.

Peter L Bartlett and Ambuj Tewari. Regal: A regularization based algorithm for reinforcement learning in weakly communicating mdps. In *Proceedings of the Twenty-Fifth Conference on Uncertainty in Artificial Intelligence*, pages 35–42. AUAI Press, 2009.

Sébastien Bubeck and Aleksandrs Slivkins. The best of both worlds: Stochastic and adversarial bandits. arXiv preprint arXiv:1202.4473, 2012.

Apostolos N Burnetas and Michael N Katehakis. On sequencing two types of tasks on a single processor under incomplete information. *Probability in the Engineering and Informational Sciences*, 7(1):85–119, 1993.

Apostolos N Burnetas and Michael N Katehakis. On large deviations properties of sequential allocation problems. *Stochastic Analysis and Applications*, 14(1):23–31, 1996a.

Apostolos N Burnetas and Michael N Katehakis. Optimal adaptive policies for sequential allocation problems. *Advances in Applied Mathematics*, 17(2):122–142, 1996b.

Apostolos N Burnetas and Michael N Katehakis. Optimal adaptive policies for markov decision processes. *Mathematics of Operations Research*, 22(1):222–255, 1997a.

Apostolos N Burnetas and Michael N Katehakis. On the finite horizon one-armed bandit problem. *Stochastic Analysis and Applications*, 16(1):845–859, 1997b.

Sergiy Butenko, Panos M Pardalos, and Robert Murphey. *Cooperative Control: Models, Applications, and Algorithms*. Kluwer Academic Publishers, 2003.

Olivier Cappé, Aurélien Garivier, Odalric-Ambrym Maillard, Rémi Munos, and Gilles Stoltz. Kullback–leibler upper confidence bounds for optimal sequential allocation. *The Annals of Statistics*, 41(3):1516–1541, 2013.

Wesley Cowan and Michael N Katehakis. Simple policies with (a.s.) arbitrarily slow growing regret for sequential allocation problems. *Technical Report* Rutgers Univ. NJ, Jul. 31 2015a.
Wesley Cowan and Michael N Katehakis. Multi-armed bandits under general depreciation and commitment. *Probability in the Engineering and Informational Sciences*, 29(01):51–76, 2015b.

Savas Dayanik, Warren B Powell, and Kazutoshi Yamazaki. Asymptotically optimal bayesian sequential change detection and identification rules. *Annals of Operations Research*, 208(1):337–370, 2013.

Eugene A Feinberg, Pavlo O Kasyanov, and Michael Z Zgurovsky. Convergence of value iterations for total-cost mdps and pomdps with general state and action sets. In *Adaptive Dynamic Programming and Reinforcement Learning (ADPRL), 2014 IEEE Symposium on*, pages 1–8. IEEE, 2014.

Sarah Filippi, Olivier Cappé, and Aurélien Garivier. Optimism in reinforcement learning based on kullback-leibler divergence. In *48th Annual Allerton Conference on Communication, Control, and Computing*, 2010.

John C. Gittins. Bandit processes and dynamic allocation indices (with discussion). *J. Roy. Stat. Soc. Ser. B*, 41:335–340, 1979.

John C. Gittins, Kevin Glazebrook, and Richard R. Weber. *Multi-armed Bandit Allocation Indices*. John Wiley & Sons, West Sussex, U.K., 2011.

J. Honda and A. Takemura. An asymptotically optimal policy for finite support models in the multiarmed bandit problem. *Machine Learning*, 85(3):361–391, 2011.

Junya Honda and Akimichi Takemura. An asymptotically optimal bandit algorithm for bounded support models. In *COLT*, pages 67–79. Citeseer, 2010.

Junya Honda and Akimichi Takemura. Optimality of Thompson sampling for Gaussian bandits depends on priors. arXiv preprint arXiv:1311.1894, 2013.

Wassim Jouini, Damien Ernst, Christophe Moy, and Jacques Palicot. Multi-armed bandit based policies for cognitive radio’s decision making issues. In *3rd international conference on Signals, Circuits and Systems (SCS)*, 2009.

Michael N Katehakis and Cyrus Derman. Computing optimal sequential allocation rules. In *Clinical Trials*, volume 8 of *Lecture Note Series: Adaptive Statistical Procedures and Related Topics*, pages 29–39. Institute of Math. Stats., 1986.

Michael N Katehakis and Herbert Robbins. Sequential choice from several populations. *Proceedings of the National Academy of Sciences of the United States of America*, 92(19):8584, 1995.

Michael N Katehakis and Arthur F Veinott Jr. The multi-armed bandit problem: decomposition and computation. *Math. Oper. Res.*, 12:262–68, 1987.

Emilie Kaufmann. Analyse de stratégies bayésiennes et fréquentistes pour l’allocation séquentielle de ressources. *Doctorat*, ParisTech., Jul. 31 2015.

Michail G Lagoudakis and Ronald Parr. Least-squares policy iteration. *The Journal of Machine Learning Research*, 4:1107–1149, 2003.

Tze Leung Lai and Herbert Robbins. Asymptotically efficient adaptive allocation rules. *Advances in Applied Mathematics*, 6(1):4–22, 1985.

Lihong Li, Remi Munos, and Csaba Szepesvari. On minimax optimal offline policy evaluation. arXiv preprint arXiv:1409.3653, 2014.

Michael L Littman. Inducing partially observable markov decision processes. In *ICGI*, pages 145–148, 2012.
Further, observing that

\[ P_{\gamma} \geq \mathbb{P}(\delta + \sqrt{U}k^{1/p} < Z) \]

where \( P_{\gamma} \) is taken to be the density of a \( X_2^2 \)-random variable. Letting \( U = k^{2/p}u \),

\[
P \geq \frac{1}{k^{2/p}} \int_0^\infty \int_0^\infty e^{-u^2/2} \left( \frac{\tilde{u}}{k^{2/p}} \right) \, du \, d\tilde{u}
\]

\[ = \frac{1}{k^{2/p}} \int_0^\infty \int_0^\infty e^{-u^2/2} \left( \frac{\tilde{u}}{k^{2/p}} \right) e^{-\tilde{u}^2/2} \, du \, d\tilde{u}
\]

\[ = \left( \frac{1}{k^{2/p}} \right)^{d/2} \int_0^\infty \int_0^\infty e^{-u^2/2} \left( \frac{\tilde{u}}{k^{2/p}} \right) e^{-\tilde{u}^2/2} \, du \, d\tilde{u}
\]

Observing that \( k^{2/p} \geq 1 \),

\[
P \geq \left( \frac{1}{k^{2/p}} \right)^{d/2} \int_0^\infty \int_0^\infty e^{-u^2/2} \left( \frac{\tilde{u}}{k^{2/p}} \right) e^{-\tilde{u}^2/2} \, du \, d\tilde{u}
\]

\[ = k^{-d/p} \mathbb{P}(2\sqrt{U} \leq Z \text{ and } U \geq \delta^2)
\]

\[ = \frac{1}{2} k^{-d/p} \mathbb{P}(4U \leq Z^2 \text{ and } U \geq \delta^2) = \frac{1}{2} k^{-d/p} \mathbb{P}(\frac{1}{4}Z^2 \geq U \geq \delta^2)
\]
The exchange from integral to probability is simply the interpretation of the integrand as the joint pdf of $U$ and $Z$.

For the upper bound, we utilize the classic normal tail bound, $P(x < Z) \leq e^{-x^2/(2\pi)}$.

$$P \leq \mathbb{E} \left[ \frac{e^{-\left(\delta + \sqrt{U}\sqrt{k^{2/p} - 1}\right)^2/2}}{(\delta + \sqrt{U}\sqrt{k^{2/p} - 1})\sqrt{2\pi}} \right] \leq \frac{e^{-\delta^2/2}}{\delta \sqrt{2\pi}} \mathbb{E} \left[ e^{-\delta \sqrt{U}\sqrt{k^{2/p} - 1} - \frac{1}{2} U(k^{2/p} - 1)} \right]. \quad (36)$$

Observing the bound that for positive $x$, $e^{-x} \leq 1/x$, and recalling that $d \geq 2$,

$$P \leq \frac{e^{-\delta^2/2}}{\delta \sqrt{2\pi}} \mathbb{E} \left[ \frac{e^{-\frac{1}{2} U(k^{2/p} - 1)}}{\sqrt{U}\sqrt{k^{2/p} - 1}} \right]$$

$$= \frac{e^{-\delta^2/2}}{\delta^2 \sqrt{2\pi} \sqrt{k^{2/p} - 1}} \mathbb{E} \left[ \frac{U^{-1/2} e^{-\frac{1}{2} U(k^{2/p} - 1)}}{\Gamma(d/2)\Gamma(\frac{d}{2})} \right]. \quad (37)$$

Here we utilize the following bounds: $e^t - 1 \geq (e/2)x^2$, which is easy to prove, and $\Gamma(d/2 - 1/2)/\Gamma(d/2) \leq \sqrt{2\pi/d}$, which may be proven on integer $d \geq 2$ by induction. This yields:

$$P \leq \frac{e^{-(1+\delta^2)/2} p k^{(1-d)/2}}{2\delta^2 \ln k}. \quad (38)$$

This completes the proof.

**Remark 4** Room for Improvement: The choice of the $e^t - 1 \geq (e/2)x^2$ bound above was in fact arbitrary - other bounds, such as involving alternative powers of $x$, could be used. This would influence how the resulting bound on $P$ is utilized, for instance in the proof of Theorem 3. The use of $e^{-x} \leq 1/x$ in Eq. (37) should be considered similarly.

**Proposition 6** Conjecture 1 is false and for each $i$, for $\varepsilon > 0$,

$$\mathbb{P} \left( \frac{\hat{X}_j + S_i(j) \sqrt{k^{2/j} - 1}}{1/k} < \mu_i - \varepsilon \text{ for some } 2 \leq j \leq k \right) \to \infty \text{ as } k \to \infty. \quad (39)$$

**Proof** [of Proposition 6] Define the events $A_{j,k,\varepsilon} = \{ \hat{X}_j + S_i(j) \sqrt{k^{2/j} - 1} < \mu_i - \varepsilon \}$. As the samples are taken to be normally distributed with mean $\mu_i$ and variance $\sigma_i^2$, we have that $\hat{X}_j - \mu_i \sim Z\sigma_i/\sqrt{j}$ and $S_i(j) \sim \sigma_i^2 U/j$, where $Z$ is a standard normal, $U \sim \chi^2_{j-1}$, and $Z, U$ independent. Hence,

$$\mathbb{P}(A_{j,k,\varepsilon}) = \mathbb{P} \left( Z \frac{\sigma_i}{\sqrt{j}} + \left( \frac{U}{j} \right) \sqrt{k^{2/j} - 1} < -\varepsilon \right) = \mathbb{P} \left( \frac{\varepsilon}{\sigma_i} \sqrt{j} + \sqrt{U} \sqrt{k^{2/j} - 1} < Z \right). \quad (40)$$

The last step is simply a re-arrangement, and an observation on the symmetry of the distribution of $Z$. For $j \geq 3$, we may apply Proposition 2 here for $d = j - 1$, $p = j$, to yield

$$\mathbb{P}(A_{j,k,\varepsilon}) \geq \frac{1}{2} \frac{k^{1/j}}{k} \mathbb{P} \left( \frac{1}{4} Z^2 \geq U \geq \frac{\varepsilon^2}{\sigma_i^2} \right). \quad (41)$$
For a fixed $j_0 \geq 3$, for $k \geq j_0$ we have

$$\mathbb{P} \left( A_{j,k,\epsilon}^i \text{ for some } 2 \leq j \leq k \right) \geq \mathbb{P}(A_{j_0,k,\epsilon}^i) \geq O(1/k) k^{1/j_0}. \quad (42)$$

The proposition follows immediately.

**Proposition 7** For $G > 0$, $0 \leq \epsilon < 1$, the following holds:

$$\frac{1}{\ln \left( 1 + G \frac{(1-\epsilon)^2}{1+\epsilon} \right)} \leq \frac{1}{\ln(1+G)} + \frac{3G}{(1+G) \ln(1+G)^2} \frac{\epsilon}{(1-\epsilon)^2} \left( 1 - \frac{\epsilon}{3} \right). \quad (43)$$

**Proof** We adopt the convention that for a function $F$, $F_x$ refers to the partial derivative of $F$ with respect to $x$. Let $A(G, \epsilon), B(G, \epsilon)$ be the left and right sides of the above inequality, respectively. Observing that $A(G, 0) \leq B(G, 0)$, it suffices to show that $A_{\epsilon}(G, \epsilon) \leq B_{\epsilon}(G, \epsilon)$, or equivalently (since both derivatives are positive),

$$\frac{A_{\epsilon}(G, \epsilon)}{B_{\epsilon}(G, \epsilon)} = \frac{(1+G)(1-\epsilon)^4 \ln(1+G)^2}{(1+\epsilon)(1+G(1-\epsilon)^2 + \epsilon) \ln \left( 1 + G \frac{(1-\epsilon)^2}{1+\epsilon} \right)^2} \leq 1. \quad (44)$$

Defining $C(G, \epsilon)$ as the RHS version of the above ratio $A_{\epsilon}/B_{\epsilon}$, observe that $C(G, 0) = 1$. Hence it suffices to show that $C_{\epsilon}(G, \epsilon) \leq 0$. Eliminating the positive factors in the derivative and simplifying the inequality, it is equivalent to show

$$(G(1-\epsilon)^2 + 2(1+\epsilon)) \ln \left( 1 + G \frac{(1-\epsilon)^2}{1+\epsilon} \right) \geq 2G(1-\epsilon)^2, \quad (45)$$

or in a more convenient form,

$$\left( G \frac{(1-\epsilon)^2}{1+\epsilon} + 2 \right) \ln \left( 1 + G \frac{(1-\epsilon)^2}{1+\epsilon} \right) \geq 2G \frac{(1-\epsilon)^2}{1+\epsilon}. \quad (46)$$

Taking $x = G(1-\epsilon)^2/(1+\epsilon) > 0$, the above reduces to

$$(x+2) \ln (1+x) \geq 2x. \quad (47)$$

This is a version of a common and easily verified bound on the logarithm function.