Conserved topological defects in non-embedded graphs in quantum gravity

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Abstract
We follow up on previous work which found that commonly used graph evolution moves lead to conserved quantities that can be expressed in terms of the braiding of the graph in its embedding space. We study non-embedded graphs under three distinct sets of dynamical rules and find non-trivial conserved quantities that can be expressed in terms of topological defects in the dual geometry. For graphs dual to two-dimensional simplicial complexes we identify all the conserved quantities of the evolution. We also indicate expected results for graphs dual to three-dimensional simplicial complexes.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Background independence in a quantum theory of gravity is the requirement that the physical content of the theory does not depend on a given geometry. This is postulated to be a desirable feature, mainly justified by analogy with the classical theory of gravity, general relativity, which is a theory of dynamical spacetime geometry. Background-independent approaches to quantum gravity include loop quantum gravity \cite{1–6}, spin foams \cite{7, 8} and causal sets \cite{9, 10}. For a review of background independence in quantum gravity, see \cite{3, 11–13}.

Probably the biggest open problem facing background-independent approaches to quantum gravity is what we shall call the low-energy problem, namely, showing that a given candidate microscopic theory at low energy reproduces the known low-energy physics, general relativity coupled to quantum fields. The problem is not whether the microscopic theory has the correct limit or not, it is in taking the limit. While background independence has many appealing features, it makes difficult the use of standard tools for taking such a limit (see, for example, \cite{14}).

Some progress has been made in this direction in loop quantum gravity using coherent states \cite{15–19} and searching for the graviton propagator \cite{20}. It is also possible to obtain
substantial results in (2+1) LQG and spin foams (see, e.g., [21–23]), however, this is largely because of the special topological nature of the theory in (2+1) dimensions and has not yet been extended to higher dimensions.

An additional direction was recently proposed in [14, 24] who suggested that a first step toward the low-energy limit of a background-independent quantum theory of gravity is to search for conserved quantities in the microscopic theory. The basic idea is that the properties and symmetries of such conserved quantities will be present in the low-energy theory and can be used to characterize it.

In [14, 24, 25], we proposed using methods from quantum information theory, such as noiseless subsystems [26, 28–30] to identify such conserved quantities. The method is promising because it can readily be applied to a large class of background-independent theories. Using this method, in [25, 31] we showed that such quantities do exist under commonly considered dynamics (evolution of networks/spin networks under local network moves). In [31], the conserved topological quantities were speculated to be quantum numbers of matter degrees of freedom, by correspondence to the preon model of [32].

The idea that matter can be encoded in topological defects of the geometry is an old one (see, for example, [33–36]). Later work [37–41] has studied in more detail the form of the conserved braiding and the propagation of braids in the graph states. Another possibility is that the conserved quantities, exhibiting different local topologies can, in conjunction with some labeling of the graph, correspond to topological geons, which themselves have the freedom of constituting matter degrees of freedom since they can exhibit both fermionic and bosonic spin statistics [42].

At the kinematical level, there are two broad classes of network states that have been considered in the quantum gravity literature: networks embedded in a manifold, and non-embedded, or abstract, networks. In the latter case, only the combinatorial information of the connectivity of the network is relevant for the physics. While the former is relevant in loop quantum gravity, the latter also regularly appear in the related literature. Abstract network states are attractive for a number of reasons. Conceptually, since the network states are more fundamental than the embedding space, the latter seems like an unnatural remnant of the classical theory that has been quantized. In fact, when spatial diffeomorphisms are taken into account in the construction of spin network states, much of the information about the embedding becomes irrelevant (see [3]). Further reasons to consider abstract graphs in loop quantum gravity are given in the new approach of algebraic loop quantum gravity [43, 44]. Abstract networks have also been used in both early works on spin foam models (e.g., [51, 54]) and the most recent developments in spin foam models [45–47]. Finally, group field theory (see [48, 49]), one of the major approaches to quantum gravity, uses exclusively abstract graphs dual to a simplicial complex.

In this paper, we return to the question of the choice between abstract and embedded networks from the new viewpoint of the search for conserved quantities under the graph evolution moves. The recent work that has motivated the present paper used a state space of embedded graphs. They found that what is conserved is the braiding of the edges of the graphs. Since there is an infinite number of braids for a finite set of graph edges, there is an infinite set of conserved quantities. It is not clear at this stage what the physical information encoded by these braids is, however, if we believe the scenario of [31, 32], this implies an infinite number of particle generations. As this may be an artifact of the embedding space and, as we discussed above, the embedding information may not be relevant at the fundamental level, we here wish to study the conserved quantities in the case of non-embedded graphs.

One may worry that since braiding of an abstract graph can be undone, the result will be trivial. In fact, we find not only that there are non-trivial topological defects that are conserved
but also that they can be nicely characterized, at least in the two-dimensional case, and there are advantages over the embedded case: there is a finite number of conserved quantities for a finite graph and there are no ambiguities in the conserved quantities resulting from the embedding topology. We study three classes of dynamics, ordered by an increasing number of dynamical rules. We find a corresponding hierarchy of conserved quantities, ordered by the inclusion relationship, with more conserved topological defects when there are more dynamical constraints. We note that the rule with the most conserved quantities has an infinite number of these but they can all be constructed by the same finite building block. This rule also has some conserved quantities that are environment dependent.

In short, we find that commonly used graph evolution moves lead to conserved quantities that can be expressed in terms of topological defects in the dual geometry and in two dimensions we give a characterization of these for three sets of evolution rules. These have not been seen before and, since they contain a substantial part of the graph information, we expect that they are of physical significance. We hope that their physical meaning can be understood in future work.

The outline of this paper is as follows. In section 2, we give the basic definitions of the graphs we consider and the construction of the dual geometry. Section 3 gives the dynamics of the three different models we study and makes explicit the hierarchy between them and the ordering, with respect to the inclusion relation, of their conserved quantities. In section 4, we find the conserved quantities for each of the models in the non-embedded case and we comment on the embedded analog. In section 5, we make some observations relevant for the interpretation of the conserved quantities as matter particles. We summarize our conclusions in section 6. Some relevant technical details are presented in appendices A–C.

2. Line graphs

We will be considering three types of mathematical objects which, in the non-embedded case are all equivalent: line graphs, framed (or ribbon) graphs and simplicial complexes.

An $n$-line graph is composed of nodes and legs (edges) that connect nodes. A leg of an $n$-line graph contains $n$ lines. At each node, $n + 1$ legs meet such that each line in one leg is connected to a line in a different leg (figure 1).

A simplicial complex is a topological space constructed by gluing together simplices. A $k$-simplex is the $k$-dimensional analog of a triangle, so a 0-simplex is a point, a 1-simplex is a segment, a 2-simplex is a triangle, a 3-simplex is a tetrahedron, etc. More generally, a $k$-simplex can be visualized as the convex hull of $(n + 1)$-independent points. A face of a $k$-simplex $K$ is an $l$-simplex at the boundary of $K$ that is formed by taking the convex hull of a subset of order $l + 1$ of the $k + 1$ points whose convex hull is $K$ and where $l$ can take any...
value between 0 and $n$. A simplicial complex, $C$, satisfies two axioms: (1) if a $k$-simplex $K$ is in $C$, then all the faces of $K$ are also in $C$ and (2) the intersection of two simplices $K_1$ and $K_2$ is a face of both $K_1$ and $K_2$. An $n$-dimensional simplicial complex is a topological space constructed by gluing together $k$-simplices where $k$ is less than or equal to $n$. A simplicial manifold is a simplicial complex which is also a piecewise linear manifold.

For $n = 2$, there is an exact bijection between $n$-line graphs and $n$-dimensional simplicial manifolds. For $n > 2$, there is an injection of $n$-dimensional simplicial manifolds into the set of $n$-line graphs but a generic $n$-line graph corresponds only to a pseudo-manifold as they contain topological defects of a different type as those which we will examine in this paper: they contain a subset of points (of measure zero) that have locally non-trivial topology (i.e. given a neighborhood of the point, one cannot always find an $n$-ball containing the point but also contained in the neighborhood) [55]. It has been suggested that this other type of topological defect (called ‘conical defect’ but not to be confused with geometrical ‘conical defects’) may also correspond to matter degrees of freedom [56]. This correspondence between $n$-line graphs and pseudo-simplicial manifolds (or simplicial manifolds when $n = 2$) defines a duality between these two mathematical structures.

The duality between the line graphs and the (pseudo)-simplicial manifolds is as follows. Suppose we have an $n$-line graph, then each node can be mapped to an $n$-dimensional simplex. Each leg emerging from the node can be mapped to an $n-1$ dimensional face of the $n$-dimensional volume simplex and every line in a leg represents an $(n-2)$-dimensional simplex bordering the $n-1$ simplex represented by the leg. For example, in figure 1, which illustrates the $n = 3$ case, a tetrahedron and its dual graph are drawn, and to make the duality more explicit, the lines and their respective dual edges are identically colored. We draw the dual graph by putting each node at the center of its dual $n$-dimensional simplex and by making each leg cross the $(n-1)$-dimensional simplex dual to it. This will result in having the lines encircle the $(n-2)$-dimensional simplex it is dual to. For $n \in \{2, 3\}$ the graph so constructed provides all the necessary information of how the $n$-dimensional simplices are glued together in the dual picture.

Finally, a framed graph is an $(n+1)$-valent graph such that all the edges are ‘thickened’ so as to take the shape of an $(n-1)$-dimensional simplex tensor product $\mathbb{R}$ (a triangular prism for $n = 3$, as shown in figure 3, and a ribbon for $n = 2$). The vertices are ‘thickened’ into $n$-simplices such that each of their $(n-1)$-simplices join with a ‘thickened edge’.

### 2.1. 2D case

If we restrict ourselves to the case $n = 2$, then the nodes are dual to triangles, the legs dual to the edges and the lines dual to points and the line graphs are exactly dual to simplicial manifolds. An example of a triangulation and its dual graph can be seen in figure 2 where a
Figure 3. Instead of being represented by three lines, each leg can be represented by a tube.

‘flat’ layout is presented. Each node sits at the center of its dual triangle, each leg crosses its dual edge and each line circles the point it is dual to.

An interesting thing to note is that, if we consider the lines of the graph to be edges of a ribbon, each triangle is dual to a 3-valent ribbon-vertex and the whole simplicial manifold corresponds to a trivalent framed graph (such a graph is what is used by [54] though in their case it is embedded in a 3-manifold and has a different interpretation). If each line is shrunk to a point, then each trivalent ribbon vertex becomes homeomorphic to the triangle it is dual to and, furthermore, they are glued together identically like their dual triangles are. Hence, the ribbon graph is a 2D surface full of holes whose boundaries, the lines, correspond to the vertices of the dual triangles. If these holes are removed by contracting their borders (the lines) to a point, then the 2-surface becomes homeomorphic to the dual triangulated surface.

2.2. 3D case

In the case \( n = 3 \), the nodes are dual to tetrahedra, the legs dual to their faces and the lines dual to their edges. As usual, the duality can be obtained by placing the nodes at the center of their dual tetrahedra and the legs crossing their dual faces. Each line loops around its dual edge.

The framed graph in the \( n = 3 \) case is obtained by considering the three lines of a leg to form a tube as shown in figure 3. One then obtains 4-valent tube graphs. This formalism is used by [37]. We have an analogous result for the tube graph as we did for the ribbon graph, namely that, if the boundaries of the tubes (2D-surfaces) are shrunk down in the direction of the length of the tube to a single line or edge, then it becomes homeomorphic to the dual triangulation.

3. Dynamics on line graphs

The dynamics for the \( n \)-simplicial complex can be given by the Pachner moves, for example as suggested in [50–52, 54]. The dynamics for simplicial complexes induces dynamics for line graphs via the duality between the two. The Pachner moves for an \( n \)-simplicial complex consist of identifying a finite subset, \( P \), of \( n \)-simplicies of the \( n \)-simplicial complex, \( C \), with a part \( B_p \) of the boundary, \( B \), of an \( (n+1) \)-simplex, \( S \), and then replacing \( P \) in \( C \) by \( (B - B_p) \). So for example, in the 2D case, the Pachner moves consist of taking \( k \in \{1, 2, 3\} \) triangles which are in the same configuration as \( k \) triangles of the boundary of a tetrahedron and replacing them by the remaining \( (4 - k) \) triangles of the tetrahedron. The four possible moves in 2D are shown in figure 4 and the 3D moves in figure 5.

Intuitively, the reason for using Pachner moves is that, by using them, the time evolution should build up \( (n + 1) \)-simplices which are connected together to form an \( (n + 1) \)-simplicial complex. One would thus expect these moves to create a \( (n + 1) \)-dimensional manifold from \( n \)-dimensional slices, though this not necessarily the case in practice, a counter-example is shown in [59], where fractal structures occur.
A possible inconvenience of such dynamics is that the Pachner moves leave the topology of the simplicial complex invariant. In fact, Pachner proved in 1987 that two closed $n$-simplicial manifolds are homomorphic if and only if one can be transformed to the other by a sequence of Pachner moves [58]. The use of Pachner moves for the dynamics therefore restricts us to evolutions that do not change the topology. But general relativity, which we expect to be reproduced in an appropriate limit of the theory, does not have this restriction: a 3-geometry can evolve into a topologically distinct manifold.

It is worth mentioning that, viewed in the graph representation, Pachner moves have non-trivial restrictions. At least in two and three dimensions, all the expanding moves (i.e., the Pachner moves that increase the total number of triangles or tetrahedra) can be performed without restrictions. The contracting moves are less trivial though. For example, in the 2D
class, the diagram in figure 6 cannot be contracted with a $3 \to 1$ move because it lacks a closed line. Less trivially, the diagram in figure 7 cannot be contracted by a $3 \to 2$ even though it looks like it could be because it does not correspond to the required configuration: it is not homeomorphic to three tetrahedra forming part of the boundary of a 4-simplex.

The above observations suggest that extra restrictions can be imposed on when one is permitted to do a Pachner move. The restrictions one imposes depend on what characteristics one is looking for in the theory. We will consider four ‘natural’ rules: the basic rule, the closed rule, a modified version of the closed rule (first introduced by Smolin and Wan in [37]) and what we shall call the embedded rule.

### 3.1. Basic rule

The basic rule consists simply of applying the Pachner moves without any restrictions. More precisely, what we will call the basic rule is the set of rules for which all Pachner moves are permitted by the dynamics. We will later consider dynamics where this is not the case and Pachner moves are forbidden in certain situations. As such, the basic rule is the least restrictive dynamical rule. This rule, having the least restrictions, will have the least number of quantities conserved by the evolution.

### 3.2. Closed rule

The closed rule is motivated by the idea that the microscopic evolution of space can only happen around areas which are not convoluted and twisted but rather have trivial topology. In classical general relativity it is always the case that, locally, space has trivial topology. As such, under closed rule, the Pachner moves, which act on only a few simplices of space at a
time, can only act if these simplices form a structure of trivial topology. That is, the evolution is generated by Pachner moves but there is an extra restriction as to when a Pachner move may be applied: the Pachner move may only be applied when the simplices supporting the move are homeomorphic to a closed (each simplex is closed topologically) \( n \)-ball (a disk in 2D and a ball in 3D).

### 3.3. Smolin–Wan (open) rule

In [37], Smolin and Wan suggested a modification of the closed rule. The same principle is applied as in the closed rule except that the boundary of the union of the tetrahedra which are acted upon by the Pachner move is disregarded. That is, one takes the topological interior of the union of the simplices supporting the Pachner move and if the resulting set is homeomorphic to an open \( n \)-ball, the Pachner move is allowed, if not, the Pachner move is forbidden. For example, in the configuration of the two triangles on the top of figure 8, according to the closed rule, we cannot apply the \( 2 \rightarrow 2 \) Pachner move because the configuration is not homeomorphic to a closed disk, but according to Smolin and Wan’s rule, the \( 2 \rightarrow 2 \) move can be applied because the interior is homeomorphic to an open disk.

An alternative way (slightly less elegant but practical in actual evaluations) to enforce Smolin and Wan’s rule is simply to work with the truncated graph. The truncated graph is the subgraph containing only the nodes which take part in the Pachner move and the legs joined to these nodes. By truncating the graph thus, we disregard any relations between the exterior lines (the lines which have been cut by the truncation because they are not part of a leg linking nodes which participate in the Pachner move) that are dual to the boundary of the set of simplices on which the Pachner move is applied. By doing that, we lose track of possible identifications on the boundary which result from the relations between the exterior lines. This contrasts with the closed rule where we must track how the exterior lines link up.

### 3.4. Embedded rule

The three previous rules apply when the line graph or framed graph is abstract graph, not a graph embedded in a 3-manifold. We can ask whether these rules carry over to embedded line graphs or framed graphs. (Note that we get different models for embedded framed graphs or
an embedded line graphs: an embedded line graph has many more degrees of freedom and is much more complex.)

The answer is that, in the case of embedded graphs, a further restriction must be added to the pre-existing restrictions. The extra restriction is as follows: a Pachner move may be carried out if and only if it can be implemented by continuously deforming the framed graph. We must impose this further restriction or else the Pachner moves would be ambiguous. This ambiguity arises from the fact that the topology of the graph (not that of the dual simplicial complex) can change during a Pachner move and so it may be impossible to embed the resulting graph in the same way as the initial graph was embedded. This is illustrated in figure 9.

To resolve the ambiguity, we impose that the initial state must continuously deform into the final state. This uniquely defines the embedding of the final state.

An abstract graph can always be ‘continuously deformed’ in the sense that, for example, we can see the graph as being embedded in $\mathbb{R}^4$ where all knots and braids can be undone. Without knots and braids, all Pachner moves can be carried out by continuously deforming the graph.

3.5. Rule hierarchy

It is important to note that there is a hierarchy of rules. The basic rule has the least constraints. Smolin and Wan’s rule has all the constraints of the basic rules plus some extra ones. Closed rule has all the constraints of Smolin and Wan’s rule plus some extra ones. These three rules can therefore be ordered from the most constraining (closed rule) to the least constraining (basic rule). These relations imply similar relations in terms of the conserved quantities of each dynamical model. The fact that Smolin and Wan’s rule respects all the constraints of the basic rule means that any quantity that is conserved for the basic rules is also conserved for Smolin and Wan’s rule. And similarly, the fact that the closed rule respects all the constraints of Smolin and Wan’s rule implies that any quantity that is conserved for Smolin and Wan’s rules will also be conserved for the closed rule.

The embedded rule is independent from the previous three rules in the sense that it can be applied in conjunction with any of the three aforementioned rules and simply adds extra constraints and therefore extra conserved quantities (infinitely many extra conserved quantities in fact, due to all the possible knots which are conserved) when the graph is an embedded one.

We now turn to the focus of this paper. It would be very nice to see whether matter degrees of freedom can be realized as topological defects in the 3D case, but to help us see what we should be looking for and what to expect we will first thoroughly analyze the 2D case.
4. Conserved quantities in the 2D case

We shall now turn to the conserved quantities under each of the above rules. As mentioned previously, the basic rule has the least conserved quantities and each successive rule adds more conserved quantities. We will, therefore, first consider the basic rule and then, for each other rule, find what conserved quantities are added.

4.1. The basic rule: classifying topology in 2D

The topology of connected 2D manifolds can be classified by whether the manifold is orientable or not and a natural number. A good way to see this is through prime decomposition which decomposes manifolds into the connected sum of prime manifolds. The connected sum, $M \# N$, of two $n$-manifolds, $M$ and $N$, is obtained by cutting out an $n$-ball from each manifold and gluing the resulting boundaries together (with the proper orientation if the manifolds are orientable). The result is unique up to homeomorphism. The connected sum of two tori is illustrated in figure 10.

The connected sum is obviously commutative and it is also associative, the unit element is the $n$-sphere. A prime manifold is a manifold which is not homeomorphic to any non-trivial connected sum, i.e., it cannot be decomposed as the connected sum of two other manifolds other than the connected sum of itself with an $n$-ball. In 2D, there are only two prime manifolds: the torus and the real projective plane (the projective plane is the 2-sphere with each point identified with its antipodal point, it is also the Möbius strip with its edge contracted to a single point). We can therefore characterize the global topology of the manifold with two numbers: the number of tori and the number of projective planes in its prime decomposition [60]. The decomposition is not unique. As we will see later, if there is at least one projective plane, then each torus can be replaced by two projective planes without changing the topology. If the prime decomposition does not contain any projective plane, then the manifold is orientable and is characterized by the number of tori in its prime decomposition, also called its genus. If the prime decomposition contains a projective plane, then the manifold is not orientable and it is homeomorphic to a manifold with a prime decomposition containing only projective planes. Therefore, a non-orientable manifold $M$ is characterized by the number of projective planes it takes in a connected sum without tori to build a manifold homeomorphic to $M$.

To recap, the topology of connected 2D manifolds can be categorized by whether the manifold is orientable or not and a natural number. If the manifold is orientable, the natural number represents its genus (the number of holes or ‘handles’ the manifold contains) and, in the case of a non-orientable surface, the natural number represents the number of Möbius strips (also called cross-caps) glued to the boundary of a disk. To add a cross-cap to a surface, one has only to make a slight cut on the surface and then reglue the two sides together with opposite orientation as shown in figure 11.

Adding two cross-caps we get a ‘Klein bottle handle’ which can also be represented by cutting out two circles and regluing them with the same orientation. As shown in figure 12,
Figure 11. Operation of adding a cross-cap to a surface. A piece of the oriented surface is cut out, flipped orientation, and glued back in. As a result, the surface has acquired a cross-cap.

Figure 12. (a) Gluing a normal handle on a surface. (b) Gluing a Klein bottle handle on a surface. For the normal handle, the two circles have opposite orientation whereas for the Klein bottle handle, the circles have the same orientation.

This is very similar to a normal handle except that, for a normal handle, the two circles which are glued back together have opposite orientation.

Since, as mentioned in section 3, the Pachner moves allow the transition from any initial state to any final state as long as the initial state and the final state have the same topology, we must conclude that the only conserved quantity for the basic rule is the global topology. In other words, the conserved quantity is $Q_B \equiv 2n_T + n_{PP}$ and the orientation of the manifold, where $n_T$ is the number of tori in a prime decomposition of the manifold and $n_{PP}$ is the number of projective planes in the same decomposition.

In a theory based on such graph evolution, it may be interesting to consider the handles and cross-caps to be pseudo-particles. We say pseudo-particles because even though they can propagate, diffuse and interact as we will see, it is not clear how they relate to real particles in some semiclassical background. In any case, the handles and the cross-caps are not always precisely located and can spread across arbitrarily large distances. This could be a problem if we expect that a handle spread over macroscopic distances and make space topologically non-trivial on observable scales.

Understanding the significance of these pseudo-particles requires being able to recognize them in our graph. This is not so obvious. In figure 32 of appendix C we can see three configurations which corresponded to (a) a disk, (b) and (c) a cross-cap and (d) a handle. The situation is much more complicated in general as, in order to figure out, for example, what topology the joining of subgraph of figure 32(b) with the subgraph of figure 32(d) corresponds to, we need to manipulate the graph until we get a familiar shape.

One way to start reading the topology off the graphs is to recognize that, in addition to what we already know about the subgraphs in figure 32, the subgraphs presented in figure 13 correspond to (if embedded in a larger graph such that the three lines of each subgraph form three different loops, i.e., that the subgraph represents a triangle with three distinct vertices): (a) adding a disk to the larger graph, i.e. not changing the topology of the larger graph, (b) adding a cross-cap to the larger graph, (c) adding two cross-caps to the larger graph and (d) adding a handle to the larger graph. Adding a disk means taking the connected sum with
For each subgraph in this figure, if it is embedded in a larger graph such that the three lines of the subgraph close into three different loops then, (a) adds a disk to the larger graph (i.e. does not change the topology of the larger graph), (b) adds a cross-cap to the larger graph, (c) adds two cross-caps to the larger graph and (d) adds a handle to the larger graph.

Inserting the subgraph of figure 13(b) in a larger graph corresponds, topologically, to adding a cross-cap to it.

A 2-sphere (this leaves the original manifold invariant), adding a handle refers to taking the connected sum with a torus, as shown in figure 10 and adding a cross-cap refers to taking the connected sum with a projective sphere. Adding a cross-cap is equivalent to cutting out a disk from the original manifold and gluing the edge of a Möbius strip on the edge of the hole, or equivalently, to performing the cut of figure 11 on the manifold. Inserting figure 13 in a larger graph such that each line of figure 13 closes into a different loop adds a cross-cap to the larger graph as shown in figure 14.

Therefore, to figure out the dual topology, we gradually remove all subgraphs of the type shown in figure 13, counting how many cross-caps and handles have been removed, until we are left with a graph with at most two loops, at which point we use the Pachner moves to get to a configuration of where we know the topology.

4.2. Smolin and Wan’s rule

As mentioned previously, because Smolin and Wan’s rule is the basic rule with added restrictions, we already know that the conserved quantities of the basic rule are still conserved with Smolin and Wan’s rule. Therefore, the number of handles and cross-caps is also fixed in Smolin and Wan’s rule. The new restriction brought about in the evolution by Smolin and Wan’s rule is that a $j \rightarrow k$ Pachner move can now only be carried out if the subgraph
containing only the \( j \) nodes on which the Pachner move is to be applied is homeomorphic to an open \( n \)-ball. In 2D we have three possible Pachner moves: \( 3 \to 1 \), \( 1 \to 3 \) and \( 2 \to 2 \). We will check these moves one by one to see what restrictions Smolin and Wan’s rule adds.

\( 3 \to 1 \): To perform the \( 3 \to 1 \) Pachner move under the basic rule, one must start with the subgraph on the left in figure 15(a). When the free lines of the subgraph on the left in figure 15(a) are connected to nodes other than the three nodes in the subgraph, Smolin and Wan’s rule allows for the \( 3 \to 1 \) Pachner move to be performed. This is because the subgraph containing the three nodes is then dual to a triangle homeomorphic to a open disk. The only time the \( 3 \to 1 \) move could possibly be forbidden is when the free lines are directly connected to each other. There are only two ways to link them to each other, of which only one, presented on the left in figure 15(b), makes the dual not homeomorphic to a open disk (it is homeomorphic to a Möbius strip). Therefore, with respect to the \( 3 \to 1 \) move, Smolin and Wan’s rule gives a single further restriction with respect to the basic move: the move presented in figure 15(b) is forbidden with Smolin and Wan’s rule whereas it was not with the basic rule.

\( 1 \to 3 \): To perform the \( 1 \to 3 \) Pachner move, one needs to simply start with a node. If the three free lines of the node are all linked to other nodes, then Smolin and Wan’s move will not restrict the application of the \( 1 \to 3 \) move as the subgraph containing only the node will be dual to a triangle homeomorphic to a open disk. For possible restrictions coming from Smolin and Wan’s rule one must therefore have the links between the three lines. There are only two possible ways to link the lines, those presented in figure 16. The subgraph in figure 16(a) is dual to a triangle with two of its edges glued together to form a cone, which is homeomorphic to an open disk and therefore Smolin and Wan’s rule does not restrict such a node from being expanded via the \( 1 \to 3 \) move. The subgraph presented in figure 16(b), on the other hand, is dual to a triangle with two edges identified so as to form a Möbius strip (as shown in figure 17) and so the \( 1 \to 3 \) move which expands such a subgraph is forbidden. This restricted Pachner move is the reverse of the move in figure 15(b).
2 → 2: Following the same procedure, one finds that the 2 → 2 Pachner moves which are forbidden by Smolin and Wan’s rule are those presented in figure 18. Note that 2-node-subgraphs with leg crossings are equivalent to 2-node-subgraphs without leg crossings but with extra line crossings (see appendix C). This is why leg crossings are not considered below. Of these, the move in figure 18(c) does nothing, and so, even though forbidding it with Smolin and Wan’s rule can potentially change the dynamics by modifying transition amplitudes, forbidding this move has strictly no effect on conserved quantities. A similar situation occurs for the move in figure 18(b). In this case the move does in fact change the graph, but a sequence of legitimate moves (under Smolin and Wan’s rule) can achieve the same result as shown in figure 27 of appendix B. Hence, the fact that this move is forbidden under Smolin and Wan’s rule has no influence on the conserved quantities.

Therefore, the only restrictions that will add new conserved quantities to those of the basic rule are the Pachner move shown in figure 15(b) and its reverse and the Pachner move shown in figure 18(a) and its reverse. These moves are precisely all the moves which create or destroy the subgraphs of figure 16(b) which correspond to a single simplex Möbius strip (cross-cap).

We have then found that the quantities conserved by Smolin and Wan’s rule are the global topology (the number of handles and cross-caps as in the basic rule) and the number of simple simplex Möbius strips dual to figure 16(b).

4.3. Closed rule

The closed rule, which forbids Pachner moves when the closure of their support is not homeomorphic to a closed disk, is more restrictive than Smolin and Wan’s rule. Passing from
Figure 19. This configuration of $n$ nodes (where $n = 2k$ and the node in brackets is not included if $n$ is even and $n = 2k + 1$ and the node in brackets is included if $n$ is odd) is stable under the closed rule because none of the simplices are homeomorphic to a closed disk. Note that this configuration is not stable under Smolin and Wan’s rule because the nodes marked by the dotted circles can be expanded under Smolin and Wan’s rule as the interior of their dual is homeomorphic to an open disk.

Smolin and Wan’s rule to the closed rule, one adds a countably infinite number of conserved quantities. For each $n \in \mathbb{N}$ there is a finite number of configurations involving $n$ simplices which are conserved by the evolution. This is quite different from Smolin and Wan’s rule in which only configurations involving a single simplex are conserved. To see that for every $n$ there is at least one configuration that is conserved by the evolution, we look at figure 19 whose dual consists of $n = 2k$ simplices (or $n = 2k + 1$ simplices if we include the node in brackets), none of which are homeomorphic to a closed disk. All these simplices are glued together and cannot be separated or modified by the evolution because any choice of a subset of those simplices is not homeomorphic to a closed disk. One can equally see that, for every $n$, there is only a finite number of such configurations because there is only a finite number of ways to glue $n$ simplices together. For this reason, we will only investigate such conserved quantities made up of only up to two simplices.

4.3.1. Conserved quantities up to two simplices. A stable (with respect to the closed rule) configuration of two simplices can either have the two simplices linked by one leg or two legs. They cannot be linked by all three legs, otherwise there will not be any free legs to link with the rest of the graph. Also, an interesting phenomenon happens with the closed rule: a subgraph can look locally (i.e., with respect to the graph connectivity) as if it is dual to simplices which are homeomorphic to a closed disk while, in fact, looking at the whole graph, this is not that case (see figure 20).

For a specific configuration to be stable under the closed rule, no single simplex that composes the stable substructure can be homeomorphic to a closed disk. This is because, if one of the simplices is homeomorphic to a closed disk, we can expand it to three simplices via the $1 \to 3$ Pachner move. As shown in figure 21 there are only three ways for a simplex to fail to be homeomorphic to a disk: two vertices of the triangle are identified, three vertices of
Figure 20. Taken out of context, the subgraph inside the shaded area seems to be dual to two triangles joined along one edge forming a parallelogram because the two parallel line segments seem to be part of different lines. If that were the case, Pachner moves could be applied to these nodes in accordance with the closed rule. We need to look at the whole graph to see that these two line segments are part of the same line and that therefore no Pachner moves can be applied to the nodes inside the shaded areas.

Figure 21. The one simplex conserved configurations under the closed rule. On the left is the line graph depiction of the conserved configurations, with similarly colored line segments belonging to the same line. On the right is the simplicial complex dual to the line graph with the points having the same color as their dual lines. Colored bars across legs of the line graph indicate the color of the dual edge in the simplicial depiction. In category A are the conserved configurations which are conserved independently of their environment. In category B are configurations which are conserved only in certain environments. Each of the above configurations give rise to a conserved quantity: the number of times they appear in a simplicial complex.

the triangle are identified or two of the edges are identified, so as to form a Möbius strip (the only other way they could be identified forms a cone which is homeomorphic to a disk).

The only possible stable structures under the closed rule made of a single triangle are therefore those shown in figure 21, but the fact that they cannot be expanded via a $1 \rightarrow 3$ move does not guarantee their stability. To guarantee their stability, we must make sure that no matter how it is connected to other triangles, no $2 \rightarrow 2$ or $3 \rightarrow 1$ Pachner moves can be applied to a set of triangles containing one of the structures of figure 21.
It is easy to see that this is in fact the case for figure 21 A(a) because the simplex forms a non-orientable surface and so, no matter how it is joined to other simplices, any set of simplices containing this non-orientable simplex will itself be non-orientable and thus not homeomorphic to a closed disk. Thus, no Pachner move can be applied to a set of simplices containing the simplex of figure 21 A(a). Therefore, figure 21 A(a) is a conserved quantity under the closed rule. Of course, we could have deduced this from the fact that figure 21 A(a) is the same as figure 16(b) which we already saw is conserved under Smolin and Wan’s rule and therefore must be conserved under closed rule because of the rule hierarchy.

Figure 21 A(b) is also conserved under the closed rule. This is not as easy to see as in the case of figure 21 A(a) but it follows from the fact that, to do a $2 \to 2$ move on figure 21 A(b), it must be connected to a triangle that ‘fills in’ two of its holes. The only way to do that is to have a simplicial structure of the type shown in the second graph of figure 22 which is not orientable and as such not homeomorphic to a closed disk. Furthermore, to perform a $3 \to 1$ on a subset of triangles containing figure 21 A(b), at least one of the line segments of the line graph of figure 21 A(b) must form a closed loop going through exactly three nodes. But this is impossible because all the line segments are part of the same line, and so the minimum number of nodes that line must go through before forming a closed loop is four.

It is not true in general, however, that the simplex configurations shown in category B of figure 21 will be invariant under the closed rule. In fact one can see that, if one of the holes of figure 21 B(a) is ‘filled in’ with a simplicial complex homeomorphic to a closed disk, then the resulting simplicial complex will itself be homeomorphic to a closed disk. For example, suppose the two edges of the rim of the 2-triangle cone of figure 33(a) of appendix C were glued to the two edges bordering one of the hole of figure 21 B(a). The resulting simplicial complex is homeomorphic to a closed disk and, under the closed rule, can be acted upon by the $3 \to 1$ Pachner move to get the simplicial dual to figure 16(a).

On the other hand, if the free edges of the simplicial configuration of category B are glued to a simplicial complex which either (i) has non-trivial topology (i.e. is not homeomorphic to a closed disk) or (ii) contains at least one configuration of category A, then the simplicial configuration of category B will necessarily be conserved by the closed rule because its ‘holes’ will not be ‘filled in’ with simplices which can take part in a Pachner move with it.

Therefore, a single simplex configuration of category A of figure 21 is exactly conserved under the closed rule independently of the greater simplicial complex it is in; but for a simplicial complex of category B, whether it is conserved or not depends on its environment. We therefore define two categories of locally conserved simplicial configurations under the closed rule: category A simplicial configurations which are conserved independently of their environment and category B simplicial configurations which are conserved only in certain environments. The overall number of each conserved simplicial configuration in a given simplicial complex defines a conserved quantity. A category A quantity (the number of a specific category A simplicial configuration in a simplicial complex) is always a conserved quantity. A category B quantity is conserved only if the simplicial complex has non-trivial topology or contains at least one category A conserved configuration.

Thus, we have seen that, up to 1-simplex, the conserved quantities of the closed rule are those shown in figure 21. Similarly, the conserved quantities of the closed rule made up of two simplices are those shown in figures 22 and 23. Note that, apart from the third graph of figure 22, the 2-simplex conserved configurations which are part of category A are
Figure 22. 2-simplex configurations conserved under the closed rule. On the left is the line graph depiction of the conserved configurations, with line segments of the same color belonging to the same line. On the right is the simplicial complex dual to the line graph with the points having the same color as their dual lines. Colored bars across legs of the line graph indicate the color of the dual edge in the simplicial depiction. These are the conserved configurations which are conserved independently of their environment, i.e., category A configurations. Each of the above configurations give rise to a conserved quantity: the number of times they appear in a simplicial complex.

easily identifiable because they are either non-orientable or made up entirely from 1-simplex category A conserved configurations.

4.4. Embedded rule

There is, in fact, not one embedded rule but many. All the previously mentioned rules can be implemented in the embedded case, where the line graph or the framed graph is seen as embedded in a 3-manifold. The rules that were applied in the non-embedded case are still valid in the embedded case, but as explained in section 3.4, a new rule is necessary for the graph to be continuously deformed from the initial state to the final state inside the 3-manifold. This means that, in the embedded case, a specific model will have all the same conserved quantities as in the non-embedded case but will also gain an infinite number of new conserved quantities due to the braiding and knotting which are now possible and can restrict move evolution as shown in figure 24.

More specifically, in the case of embedded framed graphs, the 1 → 3 Pachner move can always be carried out in the embedded case without any further restrictions (i.e., if and only if it can be performed in the non-embedded case). The 2 → 2 Pachner moves can be performed in the embedded case if it can be performed in the non-embedded case and if the leg linking the two nodes is not knotted. The 3 → 1 Pachner move can be performed in the embedded
case if it can be performed in the non-embedded case and if the legs connecting the three nodes together are not knotted and there is no leg running through the hole formed by these three legs.

We therefore conclude that the knots such as the one shown in figure 24 can never spread or propagate throughout the graph and always stays localized on the graph.

5. Particles and their propagation and interaction in the 2D case

Having found the conserved quantities in 2D, in this section we make some brief comments on the possible interpretation of these as particles or pseudo-particles.

The first problem we run into in trying to identify particles is actually defining what a particle is. Furthermore something that may look like a particle or quasi-particle at the microscopic level may not have the properties of a particle in an effective low-energy limit of the theory or model and vice versa. This is why we chose to concentrate our efforts on conserved quantities in this paper as they will still be conserved in some effective limit.

At least with the basic rule, we only have a limited number of non-trivial structures and so we can formulate some sort of pseudo-particles with interactions and propagation. The
only non-trivial structures with the basic rule are handles and cross-caps. We recall that the conserved quantity in the basic rule is the number of cross-caps plus two times the number of handles.

The very interesting properties of these pseudo-particles is that, as absolutely any subgraph, they can propagate, simply expanding the 'space' behind the particle by a $1 \to 3$ Pachner move and contracting the space in front of it by a $3 \to 1$ Pachner move. This is very nice since simply evolving the space through all possible Pachner moves will make the particle sample all possible paths closely simulating a path integral.

The other very nice property is that our two types of pseudo-particle, the handles and the cross-caps can interact: if one end of a handle crosses a cross-cap the handle splits into two cross-caps (this does not change the global topology). We thus have some sort of decay process of handles into two cross-caps and, obviously, also the reverse process is possible. This process is shown in figure 25.

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{figure24.png}
\caption{If the framed graph is not embedded, there is a problem in performing the $2 \to 2$ Pachner move because whether the crossing is over or under is irrelevant and so we can pass from (a) to (b) which ultimately permits the crucial (c) to (d) step of contracting the joining leg. On the other hand, if the framed graph is embedded, we cannot continuously deform the graph from (a) to (e) because of the knot on the leg which cannot be undone.}
\end{figure}
6. Expectations for the 3D case

The theorem which applies in the basic rule for the 2D case equally applies to the 3D case. Therefore, in the basic rule of the 3D case, the only conserved quantity that will be conserved is the global topology. In three dimensions, the prime decomposition theorem tells us that what will be replacing the handles and cross-caps of the 2D case will be the trivial bundle $S^2 \times S^1$, the non-orientable $S^2$ bundle over $S^1$ and irreducible manifolds. This same theorem also suggests that the number of each irreducible prime manifold will be an exactly conserved quantity while the two different types of $S^2$ bundles over $S^1$ can transform into one another like the handles and cross-caps of two dimensions and only a certain combination of them is conserved analogously to the handles and cross-caps.

In addition, in 3D, the ‘topological conical defects’ mentioned in section 2 will start to play a role and likely provide other possibilities for matter degrees of freedom. What is interesting in this case is that the points exhibiting ‘topological conical defects’ will form string-like structures [55] hinting at matter being fundamentally string-like.

The results in two dimensions suggest that the conserved quantities for Smolin and Wan’s rule in three dimensions (other than the global topology) are single tetrahedra with some of their faces and edges identified so as to have non-trivial topology.

Furthermore, it is worth noting that all of what has been done in [37] can be exactly transposed to the non-embedded Smolin and Wan’s rule context since these papers did not, in actual fact, make use of the restrictions on Pachner moves that arise from an embedded setting.

As for the 3D case of the closed rule, we can already tell that something similar to the 2D case will occur. So that none of the tetrahedra composing the conserved structure can be expanded via the 3D 1 → 4 Pachner move, all these tetrahedra must have some identifications to make the topologically non-trivial. Hence we can expect that the conserved quantities of the 3D closed rule are a subset of the finite set of single tetrahedra with identifications and all possible multi-tetrahedral structures that can be built from the tetrahedra of the subset. We also expect the environmental dependence of some conserved quantities to persist in three dimensions.
7. Conclusions

We studied three different models of evolving graphs and simplicial complexes based on the loop quantum gravity and spin foam framework: the basic rule, Smolin and Wan’s rule and the closed rule. Each of the three models can be considered either in the abstract case or the embedded case. In this paper we mostly focused on the abstract case where we found the conserved quantities for each model. The conserved quantities are subject to the hierarchy of the rules. That is to say that the conserved quantities of the basic rule will also be conserved quantities of Smolin and Wan’s rule and the conserved quantities of Smolin and Wan’s rule will also be conserved quantities of the closed rule.

The quantity which is conserved for all three rules is the global topology which can be characterized by a single number, $T$:

$$T = 2 \times \text{number of handles} + \text{number of cross-caps}. \quad (1)$$

In general we have the following conserved quantities:

- Basic rule: $T$.
- Smolin and Wan’s rule: $T + \text{number of single-simplex cross-caps}$.
- Closed rule: $T + \text{the number of single-simplex cross-caps} + \text{number of each structure whose building blocks are triangles with at least two vertices identified}$.

It is interesting that the closed rule has the special feature that some of the conserved quantities are only conserved in a given type of environment. This seems very unusual but is not necessarily unheard of: in electromagnetism the existence of a magnetic monopole implies the quantization of electric charge. It is such a dependence on the environment that is exhibited in the closed rule.

We also saw that, in the basic rule, handles and cross-caps can be seen as pseudo-particles that interact via the process of a handle and two cross-caps decaying into three cross-caps or vice versa. In Smolin and Wan’s rule, there is only one extra locally conserved quantity, so we may obtain a limited and finite particle spectrum. The closed rule, in contrast has an infinite number of conserved quantities and so, naively, one would expect an infinite number of particles in such a theory. In a more realistic theory, the legs or lines of the graph will be labeled, and, depending on how the labeling is done and how it changes with evolution, the labeling can be the source of more conserved quantities, in particular these may lead to conical defects.

Finally, as mentioned in the introduction, geometrical conical defects, combined with the topological conserved quantities we found may correspond to topological geons which can exhibit a variety of different spin statistics, including the standard bosonic and fermionic statistics [42]. We have not attempted to relate the conserved quantities we found with other notions of 2D particles that can be found in the literature (see, e.g., [53] and references therein).

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Appendix A. Equivalence between adding a cross-cap and performing a surgery

The following surgery is equivalent to adding a cross-cap to the manifold:

1. Consider a trivial portion of a manifold with a disk removed from it (the dark grey area in figure 26(a)).
2. Cut the portion of the manifold in half while remembering where we will have to glue it back later. (Figure 26(b): the simple grey arrows on the two top edges mean that the edges must be identified so that the arrows point in the same direction. The same applies for the double arrows on the bottom edges. The two points marked with a grey circle must be identified and similarly for the two points marked with a grey star.)
3. Place the cross-cap or Möbius strip in the void between the two parts of the manifold without gluing it yet and we continuously deform the border of the disk (figure 26(c)).
4. Cut the cross-cap in the middle while marking the points and edges which must be identified. (Figure 26(d): note that the points marked by a white circle and the points marked by a square are in fact the same points because they are the point lying at the end of the single black arrow.)
Figure 27. The forbidden $2 \rightarrow 2$ move. The two edges identified with the double arrows are identified such that the arrows match.

Figure 28. The most general line graph representation of how two simplices can be joined along two edges.

(5) Glue half of the edge of the cross-cap with the right half of the edge of the removed disk. (Figure 26(e): this identifies the points marked by a black star with the points marked by a grey star and the points marked by a black circle with the points marked by a grey circle.)
To undo the leg crossing, we can think of the graph as embedded in $\mathbb{R}^3$ (even though it is not) and then rotate the subgraph inside the dotted lines by $\pi$ along the leg coming out of the dotted box on the left.

(6) Glue the other half of the edge. (Figure 26(f): the identification of the circle with the star and the square with the triangle requires us to flip the cross-cap around before making the gluing.)

(7) Continuously deform the previously obtained manifold (figure 26(g)).

(8) Finally, glue back together all the identified edges with the exception of the edge identified with the triple black arrow (figure 26(h)).

We have thus achieved the prescription of figure 11 for adding a cross-cap to a 2-manifold.

Appendix B. Forbidden $2 \rightarrow 2$ Pachner move under Smolin and Wan’s rule

The $2 \rightarrow 2$ Pachner move shown in figure 18(b) is forbidden under Smolin and Wan’s rule because the dual triangles form a Möbius strip whose interior is still a Möbius strip and therefore not homeomorphic to a disk. We can nevertheless achieve the same result through a sequence of Pachner moves which are allowed by Smolin and Wan’s rule. Such a sequence is shown in figure 27 in the triangulation picture.

Appendix C. The different ways to link two simplices along two edges

Leg crossings in 2-node-subgraphs are equivalent to the same subgraph without leg crossings but with extra line crossings. The possible crossing of the legs can be removed by ‘rotating’ one of the two central nodes along the horizontal axis. This is shown in figure 29. The two-line permutations at the extreme left and right of figure 28 can also similarly be removed by
Figure 30. To undo the line crossing, we can think of the graph as embedded in $\mathbb{R}^3$ (even though it is not) and then rotate the subgraph inside the dotted lines by $\pi$ along the leg coming out of the dotted box on the right. The same can be done to undo $\rho$ if necessary.

Figure 31. The local degrees of freedom.

Figure 32. The four possible configurations of the local degrees of freedom of the pseudo-particles. Note that (b) and (c) are locally the same, they only have a different orientation with respect to the bigger graph.
Figure 33. (a) The triangulation dual to the embedding of the subgraph of figure 32(a) in the graph dual to $\mathbb{R}^2$ shown in figure 2. The subgraph of figure 32(a) is dual to the two triangles forming the cone. (b) Drawing of the dual triangulation to the embedding of the subgraph of figure 32(d) in the graph dual to $\mathbb{R}^2$ shown in figure 2. The subgraph of figure 32(d) is dual to the two white and orange triangles at the center of the cylinder.

Figure 34. Drawing of the dual triangulation to the embedding of the subgraph of figure 32(b). The subgraph of figure 32(b) is dual to the Möbius strip.

‘rotating’ the node which connects to our subgraph through these permutations. This is shown in figure 30. This means that, for the local degrees of freedom, we can restrict ourselves to consider only subgraphs of type shown in figure 31. This leaves us with only four possibilities as shown in figure 32, but, in fact, two of them are the same, (b) and (c). They are simply different orientations, two different ways to ‘embed’ the subgraph in the larger graph.

One can easily see that the same procedure also works in the embedded case to remove any braidings of the legs at the cost of extra line braidings. The fact that this works both in the embedded and unembedded cases is a particularity of the 2D case ($n = 2$, three-legged nodes). In the 3D case ($n = 3$, four-legged-nodes), the leg crossings of the equivalent configuration can be removed at the expense of extra line crossings in the unembedded case. However, in the embedded case, a generic braiding of the legs is not equivalent to an unbraided legs with extra line braidings.

The simplicial dual to the configuration of figure 32(a) is topologically only a 2-ball and therefore nothing special as it does not change the topology of the graph within which it is put. A rendition of the dual triangulation is given in figure 33(a) where the subgraph in figure 32(a) is dual to a cone.

The configuration given in figure 32(d) is more interesting, its dual consists of two triangles joined together so as to form a cylinder. This makes it into a micro-wormhole linking together two regions of the whole graph. This is shown in figure 33(b).

Finally, the configuration given in figure 32(d) (and analogously for figure 32(c)) has for dual the Möbius strip as is shown in figure 34. This is also what we will call a cross-cap: cutting a disk out of a 2-manifold and gluing the border of the surface in figure 34 to the
border of the hole formed by removing the disk adds a cross-cap. The last procedure is exactly equivalent, topologically, as the procedure described in figure 11. That this is the case is shown in figure 26 of appendix A.

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