The method of approximate inverse for the normal Radon transform operator

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Abstract. We propose an approach for reconstructing a three-dimensional function from the known values of the Radon transform. The approach is based on the method of approximate inverse. The result obtained is the basis of two approaches for reconstructing a potential part of the vector and symmetric 2-tensor fields from the known values of the normal Radon transform. The first approach allows the recovery of components of the potential part of fields, and the second one reconstructs a potential of the potential part of fields.

1. Introduction
Suppose a symmetric \( m \)-tensor field \( (m = 0, 1, 2) \) is distributed in a limited domain of the space \( \mathbb{R}^3 \). It is required to reconstruct the field by its known values of the Radon transform (for \( m = 0 \)) or the normal Radon transform (for \( m = 1, 2 \)).

We may mention the published work [1], in which the inversion of the normal Radon transform is investigated. The singular value decomposition of the Radon transform operator is well known [2]. While the singular value decompositions of the normal Radon transform operators acting on the vector [3]–[5] and the symmetric 2-tensor fields [6] have been discussed relatively recently.

We propose to use the method of approximate inverse [7]–[9] to solve the \( m \)-tensor tomography problem \( (m = 0, 1, 2) \). This method was successfully applied, in particular, to solve the scalar [10]–[12], the vector [13]–[17] and the tensor tomography problems [18], [19].

2. Definitions and preliminary information
We use the following notations: \( B = \{x \in \mathbb{R}^3 \mid |x| = \sqrt{x_1^2 + x_2^2 + x_3^2} < 1 \} \) is the unit ball, \( \partial B = \{x \in \mathbb{R}^3 \mid |x| = 1 \} \) is the unit sphere, \( Z = \{(s, \xi) \mid \xi \in \mathbb{R}^3, |\xi| = 1, s \in (-1, 1)\} \) is the cylinder.

In this paper, we give some definitions for an arbitrary rank \( m \) of the tensor fields, but only \( m = 0 \) (functions), \( m = 1 \) (vector fields) and \( m = 2 \) (symmetric 2-tensor fields) are used. A set of the symmetric \( m \)-tensor fields \( \mathbf{w}(x) = (w_{i_1...i_m}(x)), \mathbf{u}(x) = (u_{i_1...i_m}(x)), \mathbf{v}(x) = (v_{i_1...i_m}(x)) \), \( i_1, \ldots, i_m = 1, 2, 3 \), in \( B \) is denoted by \( S^m(B) \). We use the spaces of the square-integrable symmetric \( m \)-tensor fields \( L_2(S^m(B)) \). The inner product in the space \( L_2(S^m(B)) \) is defined by

\[
\langle \mathbf{u}, \mathbf{v} \rangle_{L_2(S^m(B))} = \int_B \langle \mathbf{u}(x), \mathbf{v}(x) \rangle dx = \int_B \sum_{i_1, \ldots, i_m=1}^3 u_{i_1...i_m}(x)v_{i_1...i_m}(x)dx.
\]
The Sobolev spaces of the symmetric $m$-tensor fields are denoted by $H^k(S^m(B))$, $H^k_0(S^m(B))$. Also we use the space $L_2(Z)$.

We use the following differential operators:

1) The inner derivation operator $d : H^k(S^m(B)) \rightarrow H^{k-1}(S^{m+1}(B))$ acts on the potential $\psi$ and the vector field $v$ by the rules

$$(dv)_i = \frac{\partial v_i}{\partial x_i}, \quad (d\psi)_i = \frac{1}{2} \left( \frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right).$$

2) The curl operator $\text{curl} : H^k(S^1(B)) \rightarrow H^{k-1}(S^1(B))$ acts on the vector field $w$ by the formula

$$\text{curl} w = \left( \frac{\partial w_3}{\partial x_2} - \frac{\partial w_2}{\partial x_3}, \frac{\partial w_2}{\partial x_1} - \frac{\partial w_1}{\partial x_2}, \frac{\partial w_1}{\partial x_3} - \frac{\partial w_3}{\partial x_1} \right).$$

3) The divergence operator $\text{div} : H^k(S^{m+1}(B)) \rightarrow H^{k-1}(S^m(B))$ acts on the symmetric $m$-tensor field $w$ as follows:

$$(\text{div} w)_{i_1...i_m} = \sum_{j=1}^{3} \frac{\partial w_{i_1...i_mj}}{\partial x_j}.$$ 

Let us recall that a symmetric $m$-tensor field $w \in H^k(S^m(B))$ is solenoidal if $\text{div} w = 0 \in H^{k-1}(S^{m-1}(B))$. It is clear that the vector field $w = \text{curl} u$ is solenoidal. Analogously, the symmetric 2-tensor field $w$ is solenoidal if $(w_{i1},w_{i2},w_{i3}) = \text{curl} v^i$, $i = 1,2,3$ for some vector fields $v^i$. A field $u \in H^k(S^m(B))$ is potential if there exists a symmetric $(m-1)$-tensor field $v \in H^{k+1}(S^{m-1}(B))$ such that $u = d\text{curl} v$.

It is well known [20] that every vector field $v \in L^2(S^1(B))$ can be uniquely represented as the sum

$$v = \text{curl} u + d\phi, \quad u \in H^1(S^1(B)), \quad \phi \in H^1_0(B).$$

In [6], it is shown that every symmetric 2-tensor field can be uniquely decomposed as follows

$$v = w + d(\text{curl} u) + d^2\phi,$$

where

$$w \in H^1(S^2(B)), \quad \text{div} w = 0, \quad u \in H^2(S^1(B)), \quad \text{curl} u \in H^1_0(S^1(B)), \quad \phi \in H^2_0(B).$$

The plane $P_{\xi,s}$ in the space $\mathbb{R}^3$ is defined by the normal equation $(\xi,x) - s = 0$. Here $x$ is a point, $\xi$ is the normal vector of the plane, $|\xi| = 1$, $|s|$ is a distance from the plane $P_{\xi,s}$ to the origin.

The Radon transform $Rf : L^2(B) \rightarrow L^2(Z)$ of the function $f(x)$ is defined by the rule

$$[Rf](s,\xi) = \int_{\mathbb{R}^3} f(x) \delta(\langle \xi, x \rangle - s) \, dx = \int_{P_{\xi,s}} f(x) \, dx,$$

where $\delta$ denotes the delta distribution.

The normal Radon transform $R^*_m : L^2(S^2(B)) \rightarrow L^2(Z)$ acts on the symmetric $m$-tensor field $u(x)$ by the formula

$$[R^*_m u](s,\xi) = \int_{\mathbb{R}^3} \langle u(x), \xi^m \rangle \delta(\langle \xi, x \rangle - s) \, dx = \int_{P_{\xi,s}} \langle u(x), \xi^m \rangle \, dx.$$
Obviously, the Radon transform is the normal Radon transform for \( m = 0 \). In contrast to the Radon transform, the normal Radon transform of the vector \( (m = 1) \) and the symmetric 2-tensor fields \( (m = 2) \) have nonzero kernels. We formulate as two lemmas the description of the kernels of the normal Radon transforms, as well as the relations between the normal Radon transforms of their potentials (for more details, see [4]–[6]).

**Lemma 1.** Let a vector field \( \mathbf{v} \) have the form

\[
\mathbf{v} = \nabla \times \mathbf{u} + d\phi,
\]

\( \mathbf{u} \in H^1_0(S^1(B)), \quad \phi \in H^1_0(B). \)

Then the following statements are true.

1) The solenoidal part \( \nabla \times \mathbf{u} \) of the vector field \( \mathbf{v} \) belongs to the kernel of the normal Radon transform \( R^\perp_1 \), i.e. we have

\[
[R^\perp_1 \nabla \times \mathbf{u}] (s, \xi) = 0.
\]

2) The normal Radon transform of the vector field \( \mathbf{v} \) is related to the Radon transform of the potential \( \phi \) by the relation

\[
[R^\perp_1 \mathbf{v}] (s, \xi) = [R^\perp_1 d\phi] (s, \xi) = \partial_s [R \phi] (s, \xi).
\]

Thus, by the known values of the normal Radon transform \( [R^\perp_1 \mathbf{v}] \) only the potential part \( d\phi, \phi \in H^1_0(B) \) of the field \( \mathbf{v} \) can be restored.

**Lemma 2.** Let a symmetric 2-tensor field \( \mathbf{v} \) have the form

\[
\mathbf{v} = \mathbf{w} + d(\nabla \times \mathbf{u}) + d^2\phi,
\]

where \( \mathbf{u} \in H^2_0(S^1(B)), \phi \in H^2_0(B) \) and for the components of the solenoidal part \( \mathbf{w} \) of the field \( \mathbf{v} \) the following rules hold

\[
(w_{i1}, w_{i2}, w_{i3}) = \nabla \times \mathbf{u}^i, \quad \mathbf{u}^i \in H^1_0(S^1(B)), \quad i = 1, 2, 3.
\]

Then the following statements are true.

1) The solenoidal part \( \mathbf{w} \) and the potential part \( d(\nabla \times \mathbf{u}) \) of the symmetric 2-tensor field \( \mathbf{v} \) belong to the kernel of the normal Radon transform \( R^\perp_2 \), i.e. we have

\[
[R^\perp_2 \mathbf{w}] (s, \xi) = 0, \quad [R^\perp_2 d(\nabla \times \mathbf{u})] (s, \xi) = 0.
\]

2) The normal Radon transform of the symmetric 2-tensor field \( \mathbf{v} \) is related to the Radon transform of the potential \( \phi \) by the relation

\[
[R^\perp_2 \mathbf{v}] (s, \xi) = [R^\perp_2 d^2\phi] (s, \xi) = \partial^2_{ss} [R \phi] (s, \xi).
\]

Thus, by the known values of the normal Radon transform \( [R^\perp_2 \mathbf{v}] \) only the potential part \( d^2\phi, \phi \in H^2_0(B) \) of the field \( \mathbf{v} \) can be restored.

Let us prove two corollaries of the above lemmas.

**Corollary 1.** The Radon transform of the components of the potential vector field \( d\phi, \phi \in H^1_0(B) \) is related to the normal Radon transform by the formula

\[
[R (d\phi)]_i (s, \xi) = \xi_i [R^\perp_1 d\phi] (s, \xi), \quad i = 1, 2, 3.
\]
\textbf{Proof.} We use the property of the Radon transform (see, for example, [21])

\[ R \frac{\partial \phi}{\partial x_i}(s, \xi) = \xi_i \frac{\partial}{\partial s}[R\phi](s, \xi), \]

the second item of Lemma 1 and obtain

\[ [R(d\phi)]_i(s, \xi) = \xi_i \frac{\partial}{\partial s}[R\phi](s, \xi) = \xi_i[R^\perp 1d\phi](s, \xi). \]

\textbf{Corollary 2.} The Radon transform of the components of the potential symmetric 2-tensor field \( d^2\phi, \phi \in H^2_0(B) \) is related to the normal Radon transform by the rule

\[ [R(d^2\phi)_{ij}](s, \xi) = \xi_i \xi_j[R^\perp 2d\phi](s, \xi), \quad i, j = 1, 2, 3. \]

\textbf{Proof.} Similar to the proof of Corollary 1, we have

\[ [R(d^2\phi)_{ij}](s, \xi) = \xi_i \xi_j[R^\perp 2d^2\phi](s, \xi). \]

\textbf{Statement of the problem.} In this paper, we consider the following three tasks:

1) Let a function \( f \) be distributed in the unit ball. It is required to reconstruct this function by its known values of the Radon transform.

2) Let a vector field \( v \) be distributed in the unit ball. It is required to recover the potential part \( d\phi \) of this vector field by its known values \([R^\perp 1v]\) of the normal Radon transform.

3) Let a symmetric 2-tensor field \( u \) be distributed in the unit ball. It is required to recover the potential part \( d^2\phi \) of this field by its known values \([R^\perp 2u]\) of the normal Radon transform.

To solve the problems, we propose the approaches based on the method of approximate inverse.

3. The method of approximate inverse for the Radon transform operator

This section gives the results for the Radon transform in \( \mathbb{R}^3 \). In the space \( \mathbb{R}^2 \), similar results were obtained in [11].

The adjoint operator for the Radon transform operator \( R \) is the back projection operator \( R^* \) defined for the function \( g(s, \xi), \ g \in L_2(Z) \) by the equality

\[ [R^*g](x) = \int_0^\pi \int_0^{2\pi} g(x, \xi) d\alpha d\beta. \]

Here \( (\alpha, \beta) = (\cos \alpha \ sin \ beta, \sin \alpha \ sin \beta, \cos \beta) \). The Radon transform is the injective operator. One of the inversion formulas is the following (see, for example, [22])

\[ f = \frac{1}{8\pi^2} R^* I^{-2} R f, \quad (1) \]

where \( I^{-2} \) is the Riesz potential defined by the rule \( F_1[I^{-2}g](\tilde{s}, \xi) = \tilde{s}^2 F_1[g](\tilde{s}, \xi) \). Here \( F_1[g](\tilde{s}, \xi) = (2\pi)^{-1/2} \int_{\mathbb{R}} g(s, \xi) e^{-i\tilde{s}s} ds \)

is the one-dimensional Fourier transform acting by \( s \).
Let $e \in L_2(\mathbb{R}^3)$ be a function with the property $\|e\|_{L_1(\mathbb{R}^3)} = 1$. Using the operator of translating and dilating $T_{1,\gamma}^y : L_2(\mathbb{R}^3) \rightarrow L_2(\mathbb{R}^3)$ we form the mollifier $e_\gamma^y$ from the function $e$ by the formula

$$e_\gamma^y(x) = T_{1,\gamma}^y e(x) = \gamma^{-3} e\left(\frac{x - y}{\gamma}\right), \quad x, y \in \mathbb{R}^3, \gamma > 0.$$ 

Namely, for the function $e_\gamma^y(x)$ and for arbitrary function $f \in L_2(\mathbb{R}^3)$ we have the equality

$$\lim_{\gamma \to 0} \langle f, e_\gamma^y \rangle_{L_2(\mathbb{R}^3)} = f(y).$$

For the fixed point $y \in \mathbb{R}^3$ the operator of translating and dilating $T_{2,\gamma}^y : L_2(Z) \rightarrow L_2(Z)$ acting on a function $g \in L_2(Z)$ is defined by the formula

$$T_{2,\gamma}^y g(s, \xi) = \gamma^{-3} g\left(\frac{s - \langle y, \xi \rangle}{\gamma}, \xi\right).$$

**Theorem 1.** The operator $T_{2,\gamma}^y$ is related to the operator $T_{1,\gamma}^y$ by the equality $\mathcal{R}^* T_{2,\gamma}^y = T_{1,\gamma}^y \mathcal{R}^*$.

**Proof.** We have

$$[\mathcal{R}^*(T_{2,\gamma}^y g)](x) = \gamma^{-3} \int_0^{\pi} \int_0^{2\pi} g\left(\frac{x}{\gamma} + \frac{(s - \langle y, \xi \rangle)}{\gamma}, \xi\right) d\alpha d\beta = \gamma^{-3} \int_0^{\pi} \int_0^{2\pi} g(\langle x, \xi \rangle - \langle y, \xi \rangle, \xi) d\alpha d\beta$$

$$= \gamma^{-3} (\mathcal{R}^* g)\left(\frac{x - y}{\gamma}\right) = T_{1,\gamma}^y \mathcal{R}^* g(x).$$

**Corollary 3.** Let the function $e \in L_2(\mathbb{R}^3)$, $\|e\|_{L_1(\mathbb{R}^3)} = 1$ belong to the range of the operator $\mathcal{R}^*$ and $\psi$ is the solution of the equation

$$\mathcal{R}^* \psi = e,$$  

then for the fixed parameter $\gamma$ and the point $y$ the function

$$\psi_\gamma^y = T_{2,\gamma}^y \psi$$

is the solution of the equation $\mathcal{R}^* \psi_\gamma^y = e_\gamma^y$, where $e_\gamma^y$ is the mollifier.

This statement follows directly from Theorem 1 and the injectivity of the Radon transform. The function $\psi_\gamma^y$ is called the reconstruction kernel associated with the mollifier $e_\gamma^y$. We have

$$\langle f, e_\gamma^y \rangle_{L_2(\mathbb{R}^3)} = \langle f, \mathcal{R}^* \psi_\gamma^y \rangle_{L_2(\mathbb{R}^3)} = \langle \mathcal{R} f, \psi_\gamma^y \rangle_{L_2(Z)}.$$ 

Thus, we obtain the formula for the approximate inversion of the Radon transform (for small $\gamma$)

$$f(y) \approx \langle \mathcal{R} f, \psi_\gamma^y \rangle_{L_2(Z)}.  \quad (4)$$

Finding the reconstruction kernels $\psi_\gamma^y$ associated with the mollifier $e_\gamma^y$ is the separate important task. From formula (3) it follows that in order to construct the reconstruction kernel associated with the mollifier $e_\gamma^y$, it is necessary to apply the operator $T_{2,\gamma}^y$ to the solution of the equation (2). The solution of this equation can be obtained using the inversion formula (1)

$$\psi = \frac{1}{8\pi^2} \mathcal{I}^{-2} \mathcal{R} e.  \quad (5)$$

To calculate $\psi$ by formula (5) we use the projection theorem for the Radon transform (see, for example, [22]). Namely, for the function $f \in L_2(\mathbb{R}^3)$ we have the equality

$$F_1[\mathcal{R} f](\tilde{s}, \xi) = 2\pi F_3[f](\tilde{s} \xi), \quad \tilde{s} \in \mathbb{R}.  \quad (6)$$
Here \( F_3[\cdot] \) is the three-dimensional Fourier transform. We do not consider the functions \( e(x) \) of a general form, since for constructing mollifiers we can use the functions that are invariant with respect to rotation, i.e. \( e(x) = e^{i\Omega(|x|)} \).

**Theorem 2.** Let a function \( e(x) \in L_2(\mathbb{R}^3) \) be invariant with respect to rotation and \( \|e\|_{L_1(\mathbb{R}^3)} = 1 \). Then the function

\[
\psi(s) = (2\pi)^{-3/2} \int_{-\infty}^{\infty} \tilde{s}^2 F_3[e](\tilde{s}\xi_0) \cos(s\tilde{s})d\tilde{s},
\]

(7)

where \( \xi_0 = (1, 0, 0) \), is the solution of equation (2).

**Proof.** Using (5) and (6), we get

\[
F_1[\psi](\tilde{s}, \xi) = (8\pi^2)^{-1} F_1[L_2^{-1} Re](\tilde{s}, \xi) = (8\pi^2)^{-1} \tilde{s}^2 F_1[Re](\tilde{s}, \xi) = (4\pi)^{-1} \tilde{s}^2 F_3[e](\tilde{s}\xi).
\]

Since the function \( e \) is invariant with respect to rotation, the Fourier transform \( F_3[e] \) is also independent of \( \xi \), i.e. \( F_3[e](\tilde{s}\xi) = F_3[e](\tilde{s}\xi_0) \). Applying the one-dimensional Fourier transform, we obtain

\[
\psi(s) = \frac{1}{2(2\pi)^{3/2}} \int_{-\infty}^{\infty} \tilde{s}^2 F_3[e](\tilde{s}\xi_0) e^{is\tilde{s}}d\tilde{s} = (2\pi)^{-3/2} \int_{0}^{\infty} \tilde{s}^2 F_3[e](\tilde{s}\xi_0) \cos(s\tilde{s})d\tilde{s}.
\]

**An example.** The Gauss function

\[
e_G(x) = (2\pi)^{-3/2} \exp(-|x|^2/2)
\]

is invariant with respect to rotation and \( \|e_G\|_{L_1(\mathbb{R}^3)} = 1 \). Using formula (7) and taking into account the equality

\[
F_3[e_G](\tilde{s}\xi_0) = (2\pi)^{-3/2} \exp(-\tilde{s}^2/2),
\]

we calculate a reconstruction kernel

\[
\psi_G(s) = (2\pi)^{-3} \int_{0}^{\infty} \tilde{s}^2 \exp(-\tilde{s}^2/2) \cos(s\tilde{s})d\tilde{s} = -(2\pi)^{-3} \int_{0}^{\infty} \frac{\partial}{\partial \tilde{s}} (\exp(-\tilde{s}^2/2)) \tilde{s} \cos(s\tilde{s})d\tilde{s}
\]

\[
= (2\pi)^{-3} \int_{0}^{\infty} \exp(-\tilde{s}^2/2) \frac{\partial}{\partial \tilde{s}} (\tilde{s} \cos(s\tilde{s}))d\tilde{s}
\]

\[
= (2\pi)^{-3} \left[ \int_{0}^{\infty} \exp(-\tilde{s}^2/2) \cos(s\tilde{s})d\tilde{s} - s \int_{0}^{\infty} \exp(-\tilde{s}^2/2) \tilde{s} \sin(s\tilde{s})d\tilde{s} \right]
\]

\[
= (2\pi)^{-3} \left[ \int_{0}^{\infty} \exp(-\tilde{s}^2/2) \cos(s\tilde{s})d\tilde{s} + s \int_{0}^{\infty} \frac{\partial}{\partial \tilde{s}} (\exp(-\tilde{s}^2/2)) \sin(s\tilde{s})d\tilde{s} \right]
\]

\[
= (2\pi)^{-3} (1 - s^2) \int_{0}^{\infty} \exp(-\tilde{s}^2/2) \cos(s\tilde{s})d\tilde{s} = \frac{1}{2(2\pi)^{3/2}} (1 - s^2) \exp(-s^2/2).
\]

At the last step, we use formula (7.4.6) from [23].

Note that if it is impossible to analytically calculate the values of the reconstruction kernel associated with the mollifier, then to calculate the values of the reconstruction kernel, we can use the singular value decomposition of the operators of the Radon transform [2] and the normal Radon transform [3–6].
4. The method of approximate inverse for the normal Radon transform operator acting on the vector and the 2-tensor fields

This section substantiates two approaches for reconstructing the vector and the symmetric 2-tensor fields based on the method of approximate inverse. The first approach allows one to reconstruct the potential part of the field componentwise, while when using the second approach, the potential of the potential part is recovered.

**Theorem 3.** Let a vector field \( \mathbf{v} \) have the form \( \mathbf{v} = \text{curl} \mathbf{u} + d \phi \), where \( \mathbf{u} \in H^1_0(S^1(B)) \), \( \phi \in H^2_0(B) \). Then for an approximate reconstruction of components of the potential part \( d \phi \) of the vector field \( \mathbf{v} \) by known values of the normal Radon transform \( [R^+_1 \mathbf{v}](s, \xi) \), we can use the formula (for small \( \gamma \))

\[
(d \phi)_i(y) \approx \langle R^+_1 \mathbf{v}, \xi_i \psi^y_i \rangle_{L_2(Z)},
\]

where \( \psi^y_i \) is the reconstruction kernel for the Radon transform.

**Proof.** Using formula (4), Corollary 1 and the second item of Lemma 1, for small \( \gamma \) we get

\[
(d \phi)_i(y) \approx \langle R(d \phi), \psi^y_i \rangle_{L_2(Z)} = \langle \xi_i [R^+_1 d \phi], \psi^y_i \rangle_{L_2(Z)} = \langle R^+_1 d \phi, \xi_i \psi^y_i \rangle_{L_2(Z)} = \langle R^+_1 \mathbf{v}, \xi_i \psi^y_i \rangle_{L_2(Z)}.
\]

**Theorem 4.** Let a symmetric 2-tensor field \( \mathbf{v} \) have the form \( \mathbf{v} = \mathbf{w} + d(\text{curl} \mathbf{u}) + d^2 \phi \), where \( \mathbf{u} \in H^1_0(S^1(B)) \), \( \phi \in H^2_0(B) \) and for the components of the solenoidal part \( \mathbf{w} \) of the field \( \mathbf{v} \) the equalities \( (w_{i1}, w_{i2}, w_{i3}) = \text{curl} \mathbf{u}^i \), \( \mathbf{u}^i \in H^1_0(S^1(B)) \), \( i = 1, 2, 3 \), hold. Then for an approximate reconstruction of components of the potential part \( d^2 \phi \) of the field \( \mathbf{v} \) by known values of the normal Radon transform \( [R^+_2 \mathbf{v}](s, \xi) \) we have the formula (for small \( \gamma \))

\[
(d^2 \phi)_{ij}(y) \approx \langle R^+_2 \mathbf{v}, \xi_i \xi_j \psi^y_{ij} \rangle_{L_2(Z)},
\]

where \( \psi^y_{ij} \) is the reconstruction kernel for the Radon transform.

**Proof.** Using formula (4), Corollary 2 and the second item of Lemma 2, for small \( \gamma \) we obtain

\[
(d^2 \phi)_{ij}(y) \approx \langle R(d \phi)_{ij}, \psi^y_{ij} \rangle_{L_2(Z)} = \langle \xi_i \xi_j [R^+_2 d^2 \phi], \psi^y_{ij} \rangle_{L_2(Z)} = \langle R^+_2 d^2 \phi, \xi_i \xi_j \psi^y_{ij} \rangle_{L_2(Z)}
= \langle R^+_2 \mathbf{v}, \xi_i \xi_j \psi^y_{ij} \rangle_{L_2(Z)}.
\]

To obtain formulas for an approximate inversion of the operators of the normal Radon transform, to restore the potentials of potential parts of the vector and the symmetric 2-tensor fields, we use the Fourier transform. The Parseval equality

\[
\langle g, h \rangle_{L_2(\mathbb{R})} = \langle F_1[g], F_1[h] \rangle_{L_2(\mathbb{R})}
\]

holds (see, for example, [24]). By \( Z_0 = \{(s, \xi) : s \in \mathbb{R}, \xi \in \mathbb{R}^3, |\xi| = 1\} \) we denote an infinite cylinder. Suppose \( g, h \in L_2(Z_0) \) and a support at least of one of the functions belongs to \( Z \subset Z_0 \). Then it follows from (8) that the equality

\[
\langle g, h \rangle_{L_2(Z)} = \langle g, h \rangle_{L_2(Z_0)} = \langle F_1[g], F_1[h] \rangle_{L_2(Z_0)}
\]

holds. Here and further the Fourier transform acts with respect to \( s \). Below we use the well known properties of the Fourier transform

\[
F_1 \left[ \frac{\partial^k f}{\partial z^k} \right](\bar{z}) = (i\bar{z})^k F_1[f](\bar{z})
\]

\[
F_1[f * g](\bar{z}) = \sqrt{2\pi} F_1[f](\bar{z}) \cdot F_1[g](\bar{z}).
\]
**Theorem 5.** Let a vector field $\mathbf{v}$ have the form $\mathbf{v} = \text{curl} \mathbf{u} + \partial \phi$, where $\mathbf{u} \in H^1_0(S^1(B))$, $\phi \in H^1_0(B)$. Then for an approximate reconstruction of the potential $\phi$ of the potential part $\partial \phi$ of the vector field $\mathbf{v}$ by known values of the normal Radon transform $[\mathcal{R}^+_1 \mathbf{v}](s, \xi)$, the formula

$$\phi(y) \approx \langle \mathcal{R}^+_1 \mathbf{v}, \Psi^y_{1,1} \rangle_{L_2(Z)}$$

holds (for small $\gamma$). Here

$$\Psi^y_{1,1}(s, \xi) = \frac{1}{2} \text{sgn}(s) *_s \psi^y_2(s, \xi),$$

where $\psi^y_2$ is the reconstruction kernel for the Radon transform and $*_s$ means the convolution with respect to $s$.

**Proof.** Taking into account the second item of Lemma 1 and formula (10), we get

$$F_1[\mathcal{R}^+_1 \mathbf{v}](\tilde{s}, \xi) = F_1[\mathcal{R}^+_1 \partial \phi](\tilde{s}, \xi) = F_1 \left[ \frac{\partial [\mathcal{R} \phi]}{\partial s} \right](\tilde{s}, \xi) = i\tilde{s} F_1[\mathcal{R} \phi](\tilde{s}, \xi).$$

Using (4), for small $\gamma$ we obtain

$$\phi(y) \approx (\mathcal{R} \phi, \psi^y_2)_{L_2(Z)} = \langle F_1[\mathcal{R} \phi], F_1[\psi^y_2] \rangle_{L_2(Z_0)} = \langle i\tilde{s} F_1[\mathcal{R} \phi], (i\tilde{s})^{-1} F_1[\psi^y_2] \rangle_{L_2(Z_0)}
\hspace{1cm} (\approx) \langle F_1[\mathcal{R}^+_1 \mathbf{v}], (i\tilde{s})^{-1} F_1[\psi^y_2] \rangle_{L_2(Z_0)} = \langle \mathcal{R}^+_1 \mathbf{v}, F_1^{-1} [(i\tilde{s})^{-1} F_1[\psi^y_2]] \rangle_{L_2(Z)},$$

where $F_1^{-1}[\cdot]$ means the one-dimensional inverse Fourier transform.

Using $(i\tilde{s})^{-1} = \sqrt{\pi/2} F_1[\text{sgn}(\tilde{s})]$ (see, for example, [25]) and (11), we get

$$F_1^{-1} [(i\tilde{s})^{-1} F_1[\psi^y_2]](s, \xi) = \frac{1}{2} \text{sgn}(s) *_s \psi^y_2(s, \xi).$$

By $\Psi^y_{1,1}(s, \xi)$ we denote the right-hand side part of the last equality and get the statement of the theorem.

**Theorem 6.** Let a symmetric 2-tensor field $\mathbf{v}$ have the form $\mathbf{v} = \mathbf{w} + d(\text{curl} \mathbf{u}) + \partial^2 \phi$, where $\mathbf{u} \in H^1_0(S^1(B))$, $\phi \in H^2_0(B)$ and for the components of the solenoidal part $\mathbf{w}$ of the field $\mathbf{v}$ the equalities $(w_{11}, w_{12}, w_{13}) = \text{curl} \mathbf{u}^i$, $\mathbf{u}^i \in H^1_0(S^1(B))$, $i = 1, 2, 3$, hold. Then for an approximate reconstruction of the potential $\phi$ of the potential part $\partial^2 \phi$ of the field $\mathbf{v}$ by known values of the normal Radon transform $[\mathcal{R}^+_2 \mathbf{v}](s, \xi)$ the formula

$$\phi(y) \approx \langle \mathcal{R}^+_2 \mathbf{v}, \Psi^y_{2,2} \rangle_{L_2(Z)}$$

holds (for small $\gamma$). Here

$$\Psi^y_{2,2}(s, \xi) = \frac{1}{2} |s| *_s \psi^y_2(s, \xi),$$

where $\psi^y_2$ is the reconstruction kernel for the Radon transform.

The proof of Theorem 6 is similar to the proof of Theorem 5.

For fixed $y = y_0 = (0, 0, 0)$ and $\gamma = 1$ we introduce the function

$$\Psi_m(s, \xi) = \Psi^y_{1,m}(s, \xi) = \frac{1}{2} \left( s^{m-1} \text{sgn}(s) \right) *_s \psi(s, \xi), \hspace{1cm} m = 1, 2.$$

The following theorem establishes a relation between functions $\Psi^y_{1,m}(s, \xi)$ and $\Psi_m(s, \xi), m = 1, 2$. 
Theorem 7. For the functions $\Psi_{\gamma,m}^y(s,\xi) \Psi_m(s,\xi)$, $m = 1, 2$, defined by formulas (12), (14) and (15) the following equality

$$\Psi_{\gamma,m}^y(s,\xi) = \gamma^{m-3} \Psi_m\left(\frac{s-\langle y,\xi \rangle}{\gamma}\right),$$

holds.

Proof. Using the relation between the functions $\psi^y_x(s,\xi)$ and $\psi(s,\xi)$, we have

$$\Psi_{\gamma,m}^y(s,\xi) = \frac{1}{2} \left(s^{m-1} \operatorname{sgn}(s)\right) *_s \psi^y_x(s,\xi) = \frac{1}{2} \int_{-\infty}^{\infty} \left(s-t\right)^{m-1} \operatorname{sgn}(s-t) \psi^y_x(t,\xi) \, dt$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \left(s-t\right)^{m-1} \operatorname{sgn}(s-t) \gamma^{-3} \psi\left(\frac{t-\langle y,\xi \rangle}{\gamma}\right) \, dt.$$ 

A change of the variable $p = (t-\langle y,\xi \rangle)/\gamma$ leads to the equality

$$\Psi_{\gamma,m}^y(s,\xi) = \frac{1}{2} \int_{-\infty}^{\infty} \left(s-p\gamma\langle y,\xi \rangle\right)^{m-1} \operatorname{sgn}(s-p\gamma\langle y,\xi \rangle) \gamma^{-2} \psi(p,\xi) \, dp$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \gamma^{m-3} \left(s-\frac{\langle y,\xi \rangle}{\gamma}-p\right)^{m-1} \operatorname{sgn}\left(\frac{s-\langle y,\xi \rangle}{\gamma}-p\right) \psi(p,\xi) \, dp$$

$$= \gamma^{m-3} \Psi_m\left(\frac{s-\langle y,\xi \rangle}{\gamma}\right).$$

Acknowledgements

This research was partially supported by RFBR and DFG according to the research project 19-51-12008.

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