On the structure of the algebra generated by the non-commutative operator graph demonstrating superactivation for a zero-error capacity

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Abstract

Recently M.E. Shirokov [1] introduced the non-commutative operator graph depending on the complex parameter θ to construct channels with positive quantum zero-error capacity having vanishing n-shot capacity. We study the algebraic structure of this graph. The relations for the algebra generated by the graph are derived. In the limiting case θ = ±1 the graph becomes abelian and degenerates into the direct sum of four one-dimensional irreducible representations of the Klein group.

1 Introduction

The superactivation of a quantum channel capacity was discovered in [2]. It appeared that the quantum capacity for a tensor product of two quantum channels may be positive while the quantum capacity of each of these channels is zero. In [3, 4] it was shown that a value of the quantum capacity is closely related to the so-called non-commutative graph of a quantum channel. The same phenomenon for the zero-error classical capacities was established in [5]. In [6, 7] some techniques for studying superactivation by means of non-commutative graphs was introduced. It allows to give low-dimensional examples of superactivation for a quantum capacity. In our paper we shall analyse the algebra generated by the non-commutative graph introduced in [1].

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2 Preliminaries

Denote $B(H)$ the algebra of all bounded operators and $\mathcal{S}(H)$ the convex set of quantum states (positive unit trace operators) in a finite-dimensional Hilbert state $H$. Let $\Phi : B(H_A) \to B(H_B)$ be a quantum channel, i.e. a completely positive trace-preserving linear map transmitting $\mathcal{S}(H_A)$ to the subset of $\mathcal{S}(H_B)$ in the Hilbert spaces $H_A$ and $H_B$ [8]. Then the dual map $\Phi^* : B(H_B) \to B(H_A)$ is defined by the formula

$$Tr(\rho\Phi^*(x)) = Tr(\Phi(\rho)x), \, \rho \in \mathcal{S}(H_A), \, x \in B(H_B).$$

The Stinespring theorem results in the representation of a channel $\Phi$ in the form

$$\Phi(\rho) = Tr_{H_E} V \rho V^*, \, \rho \in \mathcal{S}(H_A), \tag{1}$$

where $H_E$ is a Hilbert space of the environment and $V : H_A \to H_B \otimes H_E$ is an isometry. The representation (1) allows to define a complementary quantum channel $\hat{\Phi} : \mathcal{S}(H_A) \to \mathcal{S}(H_E)$ by the formula [8, 9]

$$\hat{\Phi}(\rho) = Tr_{H_B} V \rho V^*, \, \rho \in \mathcal{S}(H_A). \tag{2}$$

Taking into account (1) it is possible to derive the Kraus representation

$$\Phi(\rho) = \sum_{k=1}^{\dim H_E} V_k \rho V_k^*, \, \rho \in \mathcal{S}(H_A), \tag{3}$$

where $\{V_k\}$ is a set of linear operators $V_k : H_A \to H_B$, $\sum_j V_k^* V_k = I_{H_A}$. Using (3) the complementary channel (2) can be represented as follows

$$\hat{\Phi}(\rho) = \sum_{j,k=1}^{\dim H_E} Tr[V_j \rho V_k]| j > < k|. \tag{4}$$

The non-commutative graph [3, 5] $\mathcal{G}(\Phi)$ of a quantum channel $\Phi$ is a closed linear span of the Kraus operators $\{V_j^* V_k\}_{j,k=1}^{\dim H_E}$. It follows from (1) that $\mathcal{G}(\Phi)$ is equal to the subspace $\hat{\Phi}^*(B(H_E))$. The operator space $S$ is a non-commutative graph for some quantum channel iff $x \in S$ implies $x^* \in S$ and the identity operator $I \in S$ [3, 6]. In [1] the following operator graph was introduced

$$\begin{pmatrix}
  a & b & c\theta & d \\
  b & a & d & c/\theta \\
  c/\theta & d & a & b \\
  d & c\theta & b & a
\end{pmatrix} \tag{5}$$

for $\theta \in \mathbb{C}$, $|\theta| = 1$. The operator graph (5) is associated with pseudo-diagonal channels [10]

$$\Phi(\rho) = \sum_{j,k} c_{j,k} < \psi_j | \rho | \psi_k > | j > < k| \tag{6}$$

where $\{c_{ij}\}$ is a Gram matrix of a collection of unit vectors, $\{|\psi_i >\}$ is a collection of vectors in $H$ such that $\sum_i |\psi_i > < \psi_i | = I$. Here we recover the structure of the algebra $\mathcal{M}_\theta$ generated by the graph (5) for any complex parameter $\theta \in \mathbb{C}$, $\theta \neq 0$. 

\[2\]
3 The structure of the algebra $\mathcal{M}_\theta$ generated by the graph.

It is straightforward to check that the graph (5) is a linear envelope of the identity $I$ and the following three matrices:

$$X = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 & \theta & 0 \\ 0 & 0 & 0 & 1/\theta \\ 1/\theta & 0 & 0 & 0 \\ 0 & \theta & 0 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$ (7)

The matrices (7) satisfy the relations

$$X^2 = Y^2 = Z^2 = I, \quad XZ = ZX, \quad YZ = ZY.$$ (8)

Let us consider the group $G$ generated by formal variables $x, y, z$ satisfying the relations: $x^2 = y^2 = z^2 = 1, xz = zx, yz = zy$. Note that adding to the last relations $xy = yx = z$ we obtain the Klein group. Consider the subalgebra $\mathcal{M}_\theta \subset \text{Mat}_4(\mathbb{C})$ generated by the matrices $X, Y, Z$. Thus, we have a well-defined morphism of algebras: $\phi_\theta : \mathbb{C}G \to \mathcal{M}_\theta$ defined by the rule: $x \mapsto X, y \mapsto Y, z \mapsto Z$. Also, the morphism $\phi_\theta$ defines a 4-dimensional representation of the group $G$.

**Theorem 1.** We have the following statements:

- If $\theta \neq \pm 1$ then the algebra $\mathcal{M}_\theta$ is a direct sum of two matrix algebras of size 2. In this case $\dim_{\mathbb{C}} \mathcal{M}_\theta = 8$.
- If $\theta = \pm 1$ then the algebra $\mathcal{M}_\theta$ is isomorphic to the group algebra of the Klein group. In this case $\dim_{\mathbb{C}} \mathcal{M}_\theta = 4$.

Proof.

Consider a normal subgroup $P \triangleleft G$ of index 2 generated by the elements $g = xy$ and $z$. Note that we have the following relation: $xgx = g^{-1}$. Using the relations of the group $G$, we obtain that the group $P$ is abelian. Hence, all irreducible representations of $P$ are 1-dimensional. Any 1-dimensional representation is said to be a character of $P$. Thus, a dimension of irreducible representations of $G$ is less or equal to 2. Let us describe characters of $P$. A character $\chi$ of $P$ is a morphism: $\chi : P \to \mathbb{C} \setminus \{0\}$, $\chi(ab) = \chi(a)\chi(b)$, $a, b \in P$. The character $\chi$ is determined by two numbers $\chi(g) \in \mathbb{C} \setminus \{0\}$ and $\chi(z)$, $\chi(z)^2 = 1$. The last condition implies $\chi(z) = \pm 1$.

One can describe the standard construction of a $G$-representation $V_\chi$ induced by the character $\chi$ of $P$ as follows. Consider a vector $v$ such that $av = \chi(a)v, a \in P$. Also, we can consider a "formal" vector $x \cdot v$. Thus, the vector space generated by $v$ and $x \cdot v$ is a space of $V_\chi$. Elements of $P$ act on $V_\chi$ by the rule: $av = \chi(a)v$. Using the relations $x^2 = 1$ and $xax = a^{-1}$, we get $a(x \cdot v) = x(axa \cdot v) = xa^{-1} \cdot v = \chi(a^{-1})x \cdot v = \chi^{-1}(a)x \cdot v$. It means that $V_\chi$ as representation of $P$ is a direct sum of 1-dimensional representations corresponding to characters $\chi$ and $\chi^{-1}$. The element $x$ acts by the formula: $xv = x \cdot v$ and $xx \cdot v = x^2v = v$. Also, one can show that any 2-dimensional irreducible representation of $G$ is induced by a character of $P$. Consider the
representation \( \phi_\theta \) of the group algebra \( \mathbb{C}G \) for \( \theta \neq \pm 1 \). In this case there are two irreducible submodules \( V_\chi \) and \( V_{-\chi} \) of representation \( \phi_\theta \), where characters \( \chi \) and \( -\chi \) are defined by rule: \( \chi(g) = \theta \) and \( -\chi(g) = -\theta \). Using the standard arguments, we get that the \( G \)-representation \( \phi_\theta \) is a direct sum \( V_\chi \oplus V_{-\chi} \). It means that if \( \theta \neq \pm 1 \) then algebra \( M_\theta \) is a direct sum of the matrix algebras of size 2. Thus, we get the first statement. In the case \( \theta = \pm 1 \), one can check that the representation \( \phi \) is a sum of four 1-dimensional representations.

We can see that there is a gap in dimension of the algebras \( M_\theta \) in the case \( \theta = \pm 1 \). We can change this situation as follows. We will construct a family of algebras \( A_\theta \) that the representation \( \phi_\theta \) is a direct sum \( V_\chi \oplus V_{-\chi} \). It means that if \( \theta \neq \pm 1 \) then algebra \( M_\theta \) is a direct sum of the matrix algebras of size 2. Thus, we get the first statement. In the case \( \theta = \pm 1 \), one can check that the representation \( \phi \) is a sum of four 1-dimensional representations.

Let us show that a dimension of the algebra \( A_\theta \) is equal to 8. Actually, one can show that if \( \theta \neq \pm i \) the algebra \( A_\theta \) has the following basis: \( 1, g, g^2, g^3, x, xg, xg^2, xg^3 \), where \( g = xy \). In the case \( \theta = \pm i \) one can show that the algebra \( A_\theta \) has the basis: \( 1, g, x, z, xg, xz, gz, xgz \). Thus, in the case \( \theta \neq \pm 1 \), \( \psi \) is bijective, hence, \( \psi \) is isomorphism.

Consider the case \( \theta = \pm 1 \). In this case we have the relation: \( g + g^{-1} = \pm 2z \). Thus, \( (g + g^{-1})^2 = 4z^2 = 4 \). And, hence, we obtain the following relation:

\[
(g^2 - 1)^2 = 0.
\] (9)

Consider the ideal \( J \) of the algebra \( A \) generated by \( g^2 - 1 \). One can see that \( J \) has the following basis \( g^2 - 1, x(g^2 - 1), g(g^2 - 1), xg(g^2 - 1) \). Also, consider \( J^2 = \langle t_1t_2, t_1 \in J \rangle \). Using the relation (9), we get \( J^2 = 0 \). The ideal satisfying this property is said to be a radical. Therefore, the algebra \( A_\theta \) for \( \theta = \pm 1 \) has a 4-dimensional radical. One can check that \( \psi(J) = 0 \).

\[\square\]

## 4 Conclusion

We have found the general features of the algebra generated by the non-commutative operator graph playing an important role in quantum information theory. We have presented only a
sketch of the theory. We are planning to continue the study in the future to discover the algebraic nature of the quantum superactivation.

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References

[1] M.E. Shirokov, Quantum Information Processing, 14:8 3057-3074 (2015)
[2] G. Smith, J. Yard, Science, 321, 1812 (2008); arXiv:0807.4935
[3] R. Duan, arXiv: 0906.2527 (2009).
[4] T.S. Cubitt, J. Chen, A.W. Harrow, arXiv:0906.2547 (2009).
[5] R. Duan, S. Severini, A. Winter, IEEE Trans. Inf. Theory, 59:2, 1164-1174 (2013); arXiv:1002.2514
[6] M.E. Shirokov, T.V. Shulman, Comm. Math. Phys., 335:3, 11591179 (2015); arXiv:1309.2610
[7] M.E. Shirokov, T.V. Shulman, Problems Inform. Transmission. 50:3, 232-246 (2014); arXiv:1312.3586
[8] A.S. Holevo, Quantum systems, channels, information. A mathematical introduction, Berlin, DeGruyter, 2012.
[9] A.S. Holevo, Probability Theory and Applications, 51:1, 134-143 (2006); arXiv:quant-ph/0509101.
[10] T.S. Cubitt, M.B. Ruskai, G. Smith, J. Math. Phys., 49:2 (2008) 102104; arXiv:0802.1360