TRANSPORT AND GENERATION OF MACROSCOPICALLY
MODULATED WAVES IN DIATOMIC CHAINS

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ABSTRACT. We derive and justify analytically the dynamics of a small macro-
scopically modulated amplitude of a single plane wave in a nonlinear diatomic
chain with stabilizing on-site potentials including the case where a wave gener-
ates another wave via self-interaction. More precisely, we show that in typical
chains acoustical waves can generate optical but not acoustical waves, while
optical waves are always closed with respect to self-interaction.

1. INTRODUCTION

The present work constitutes a generalization of previous work of the author, see
[3], to a case of vector-valued displacement in nonlinear lattices. As the technically
most simple but yet generic case we consider a nonlinear diatomic chain. For the
physical derivation, interpretation and discussion of several applications of the har-
monic diatomic chain we refer to [2]. Various questions concerning diatomic lattices
have been addressed up to now, see e.g. [1, 4, 7, 8, 9]. Here we focus on the anal-
lytical justification of the dynamics of small macroscopic amplitude modulations, see
(12). More precisely, we consider the diatomic chain

\[ \begin{align*}
\ddot{x}_{2j+1} &= V'_i(x_{2j+2} - x_{2j+1}) - V'_i(x_{2j+1} - x_{2j}) - W'_i(x_{2j+1}), \\
\ddot{x}_{2j} &= V'_i(x_{2j+1} - x_{2j}) - V'_i(x_{2j-1} - x_{2j}) - W'_i(x_{2j}),
\end{align*} \]

(1) with nearest-neighbor interaction and on-site potentials \( V_i, W_i \in C^4(\mathbb{R}), i = 1, 2 \), such that

\[ \begin{align*}
V'_i(x) &= v_{i,1}x + v_{i,2}x^2 + \bar{V}'_i(x), & \bar{V}'_i(x) = O(|x|^3), \\
W'_i(x) &= w_{i,1}x + w_{i,2}x^2 + \bar{W}'_i(x), & \bar{W}'_i(x) = O(|x|^3).
\end{align*} \]

(2) Setting \( u_j = \begin{pmatrix} u_{j,1} \\ u_{j,2} \end{pmatrix} := \begin{pmatrix} x_{2j+1} \\ x_{2j} \end{pmatrix} \), \( j \in \mathbb{Z} \), and using the Taylor-expansions (2), the
diatomic chain (1) takes the form

\[ \ddot{u} = \mathcal{L} u + \mathcal{M}(u), \]

(3) \[ \begin{align*}
(\mathcal{L} u)_j &:= \begin{pmatrix}
v_{1,1}(u_{j+1,1} - 2u_{j,1} + u_{j,2}) - w_{1,1}u_{j,1} \\
v_{2,1}(u_{j,1} - 2u_{j,2} - u_{j-1,1}) - w_{2,1}u_{j,2}
\end{pmatrix}, \\
(\mathcal{M}(u))_j &:= \begin{pmatrix}
v_{1,2}((u_{j+1,2} - u_{j,2})^2 - (u_{j,1} - u_{j,2})^2) - w_{1,2}u_{j,1}^2 \\
v_{2,2}((u_{j,1} - u_{j,2})^2 - (u_{j,2} - u_{j-1,1})^2) - w_{2,2}u_{j,2}^2
\end{pmatrix}.
\end{align*} \]
we discuss whether a given plane wave solution can generate via self-interaction another plane wave. In a diatomic chain we are interested in resonance conditions that are equivalent to finding a non-trivial plane-wave solution for all $\vartheta \in (-\pi, \pi)$. This is equivalent to

\begin{equation}
\omega^2 = \omega_2^2(\vartheta) := \frac{c_1 + c_2}{2} \pm \frac{1}{2} \sqrt{(c_1 - c_2)^2 + 8v_{1,1}v_{2,1}(\cos \vartheta + 1)}.
\end{equation}

Assuming $c_1 + c_2 > 0$, $c_1c_2 > 4v_{1,1}v_{2,1} > 0$, we obtain

\begin{equation}
\omega_{\pm}(\vartheta) := \sqrt{\frac{1}{2} \left( c_1 + c_2 \pm \sqrt{(c_1 - c_2)^2 + 8v_{1,1}v_{2,1}(\cos \vartheta + 1)} \right)} > 0
\end{equation}

for all $\vartheta \in (-\pi, \pi)$ and the additional assumption $c_1 \neq c_2$ yields the strict separation of the optical and acoustical branches of the frequency,

\[2\omega_2^2(\vartheta) \geq c_1 + c_2 + |c_1 - c_2| > c_1 + c_2 - |c_1 - c_2| \geq 2\omega_2^2(\vartheta) \quad \forall \vartheta \in (-\pi, \pi).
\]

All of the above assumptions are satisfied in the case $w_{i,1} > 0$, $4v_{i,1} + w_{i,1} > 0$, $v_{1,1}v_{2,1} > 0$, $2v_{2,1} + w_{2,1} > 2v_{1,1} + w_{1,1}$, which we assume in the following.

The eigenvectors $A$ to the eigenfrequencies $\omega = \omega_{\pm}(\vartheta)$ are given by

\begin{equation}
A^{(2)} = -\rho A^{(1)}, \quad \rho := \frac{\omega^2 - c_1}{v_{1,1}(e^{i\vartheta} + 1)} = \frac{v_{2,1}(e^{-i\vartheta} + 1)}{\omega^2 - c_2} \neq 0, \quad \text{if } \vartheta \neq \pm \pi
\end{equation}

and

\begin{equation}
A = \begin{pmatrix} A^{(1)} \\ 0 \end{pmatrix} \quad \text{for } \omega = \omega_-(\pm \pi), \quad A = \begin{pmatrix} 0 \\ A^{(2)} \end{pmatrix} \quad \text{for } \omega = \omega_+(\pm \pi).
\end{equation}

The plan of the paper is as follows. In Section 2 we discuss whether a given plane wave solution $E$ can generate via self-interaction another plane wave $E^2$. Then, taking into account also this possibility, in Section 3 we derive formally the macroscopic equations for the first order amplitudes $A_{1,n}$ of two waves $n = 1, 2$, and finally, in Section 4, we justify the derived equations.

2. Resonances

Since we are interested in the self-interaction of a plane wave $E$, which means that $E^2$ is also a plane wave, in a diatomic chain we are interested in resonance conditions like the ones on the left hand side below. Making in (6) the substitutions $c := (\cos \vartheta + 1)/2 \in [0, 1]$, $d_1 := (c_1 + c_2)^2/f > 0$, $d_2 := (c_1 - c_2)^2/f > 0$ with $f := 16v_{1,1}v_{2,1} > 0$ and $d_1 - d_2 > 1$, the problem of finding a $\vartheta \in (-\pi, \pi)$ satisfying one of these resonance conditions is equivalent to finding a $c \in [0, 1]$ for given $d_1 > d_2 + 1 > 1$ satisfying the corresponding equation on the right hand side:

\begin{equation}
2\omega_{\pm}(\vartheta) = \omega_{\pm}(2\vartheta) \iff 4 \left( \sqrt{d_1} (\pm) \sqrt{d_2 + c} \right) = \sqrt{d_1} \pm \sqrt{d_2 + (2c - 1)^2} \quad \text{if } \vartheta \neq \pm \pi
\end{equation}

\begin{equation}
\quad \iff 3\sqrt{d_1} = (\mp) 4\sqrt{d_2 + c} \pm \sqrt{d_2 + (2c - 1)^2}.
\end{equation}
By the positivity of all appearing square roots we immediately see that a resonance $2\omega_+(\vartheta) = \omega_-(2\vartheta)$, i.e., an optical wave generating an acoustical one, is not possible. Moreover, since

$$3\sqrt{d_1} > \sqrt{d_1} > \sqrt{d_2 + 1} > -4\sqrt{d_2 + c + \sqrt{d_2 + (2c - 1)^2}},$$

we see that an optical wave cannot generate another optical one, i.e., $2\omega_+(\vartheta) \neq \omega_+(2\vartheta)$ \(\forall \vartheta \in (-\pi, \pi]\). Thus, an optical wave is closed under self-interaction of order 2.

However, an acoustical wave can generate an optical one by self-interaction, i.e., for appropriate choice of the harmonic parts of the interaction and on-site potentials there exist $\vartheta \in (-\pi, \pi]$ such that $2\omega_-(\vartheta) = \omega_+(2\vartheta)$. After taking squares on the left and right hand sides, the corresponding condition (9) reads

$$9d_1 = 17d_2 + 16c + (2c - 1)^2 + 8\sqrt{d_2 + c}\sqrt{d_2 + (2c - 1)^2},$$

and we want to prove the existence of a $c \in [0, 1]$ that satisfies this condition for the $d_1, d_2$ given above. We restrict ourselves to the case $v_{1,1} = a > 0, v_{2,1} = \gamma a, \gamma > 1, w_{1,1} = w_{2,1} = b > 0$. This setting satisfies all conditions posed so far on the harmonic coefficients, and we obtain

$$d_1 = \frac{(\gamma + 1)^2}{4\gamma} + \frac{1}{\gamma} \left( (\gamma + 1)\frac{b}{a} + \frac{b^2}{a^2} \right), \quad d_2 = \frac{(\gamma - 1)^2}{4\gamma} =: \delta$$

(which obviously satisfies $d_1 > d_2 + 1 > 1$). Inserting these values into (10), we get

$$\frac{9}{\gamma} \left( (\gamma + 1)\frac{b}{a} + \frac{b^2}{a^2} \right) = 8\delta - 9 + 16c + (2c - 1)^2 + 8\sqrt{\delta + c}\sqrt{\delta + (2c - 1)^2}.$$ 

Hence, for every $c \in [0, 1]$ such that $16c \geq 9 - 8\delta$ there exists a $\frac{b}{a}$ such that (10) is satisfied. Since $\delta > 0$, we can always find such a $c$.

Furthermore, the resonance condition for the generation of an acoustical wave from an acoustical one, $2\omega_-(\vartheta) = \omega_-(2\vartheta)$, is equivalent to

$$3\sqrt{d_1} = 4\sqrt{d_2 + c - \sqrt{d_2 + (2c - 1)^2}}$$

Concerning the case just considered, we observe that for $d_2 = \delta$, the r.h.s. is nonnegative only for $c \in [c_e, 1]$ with $c_e := \max\{0, \frac{5 - \sqrt{13} + 2\sqrt{2}}{2}\}$ (and hence for all $c \in [0, 1]$ when $\delta \geq 1/15$). Restricting our analysis to the set $[c_e, 1]$ (non-empty for all $\delta > 0$), we obtain by squaring and insertion of the values (11) as above

$$\frac{9}{\gamma} \left( (\gamma + 1)\frac{b}{a} + \frac{b^2}{a^2} \right) = 8\delta - 9 + 16c + (2c - 1)^2 - 8\sqrt{\delta + c}\sqrt{\delta + (2c - 1)^2},$$

although with a minus sign in front of the square root. Due to the existent on-site potential (where $b > 0$), in order to obtain resonances the r.h.s. $g$ needs to be strictly positive for some $c \in [c_e, 1]$. However, a careful analysis reveals that $g(c) \leq 0$ for $c \in [c_e, 1]$, and we obtain that in the case $v_{2,1} = \gamma a > a = v_{1,1}, w_{1,1} = w_{2,1} = b > 0$, an acoustical wave cannot generate another acoustical one by self-interaction.

Finally, we conclude by showing that $\omega_-(\vartheta) + \omega_+(2\vartheta) \neq \omega_+(3\vartheta)$ for all $\vartheta \in (-\pi, \pi]$. Indeed, after squaring the left and right hand sides we see that the equality is equivalent to

$$-\sqrt{d_2 + c + \sqrt{d_2 + (2c - 1)^2}} + 2\sqrt{(\sqrt{d_1} - \sqrt{d_2 + c})(\sqrt{d_1} + \sqrt{d_2 + (2c - 1)^2})} = \sqrt{d_2 + (4c - 3)^2 c - \sqrt{d_1}}$$

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for \( d_1 > d_2 + 1 > 1 \). Since \((4c - 3)^2 c = (\cos(3\theta) + 1)/2 \in [0, 1] \), the r.h.s. of this equation is always < 0, and it suffices to show that the l.h.s. is ≥ 0 even for \( c ≥ (2c - 1)^2 \). Hence, since \( d_1 > d_2 + 1 \), it is sufficient to show that the l.h.s. with \( \sqrt{d_1} \) replaced by \( \sqrt{d_2} + 1 \) is ≥ 0 for \( c \in [1/4, 1] \). Comparing in this modified l.h.s. the square of the first two terms with the square of the third one, and adding a suitable term, this is equivalent to showing that

\[
4(1 - c) ≥ \left( \sqrt{d_2 + c} - \sqrt{d_2 + (2c - 1)^2} \right)
\]

\[
\left( \left( \sqrt{d_2 + c} - \sqrt{d_2 + (2c - 1)^2} \right) + 4\left( \sqrt{d_2 + 1} - \sqrt{d_2 + c} \right) \right)
\]

for \( c \in [1/4, 1] \). Since the r.h.s. is positive and strictly decreasing as a function of \( d_2 > 0 \) when \( c > (2c - 1)^2 \), it suffices to show

\[
g(c) := 4(1 - c) - \left( \sqrt{c} - \sqrt{(2c - 1)^2} \right) \left( \left( \sqrt{c} - \sqrt{(2c - 1)^2} \right) + 4(1 - \sqrt{c}) \right) ≥ 0
\]

for \( c \in [1/4, 1] \), which holds true (as an elementary analysis shows), with \( g(1) = 0 \).

### 3. Formal derivation

We are interested in solutions of (3) which in first order in \( \varepsilon \) are a sum of two macroscopically modulated plane-wave solutions with small amplitudes

\[
(12) \quad u = U^{A,1}_\varepsilon + O(\varepsilon^2), \quad (U^{A,1}_\varepsilon)_{j}(t) := \varepsilon \sum_{n=1}^{2} A_{1,n}(\varepsilon t, \varepsilon j)E_n(t, j) + c.c.
\]

where \( A_{1,n} = (A^{1}_{1,n}, A^{2}_{1,n})^T : \mathbb{R} \times \mathbb{R} \to \mathbb{C}^2 \) and \( E_n(t, j) := e^{i(\omega_n t + j\vartheta_n)} \) with \( (\omega_n, \vartheta_n) \) satisfying (4).

However, due to the scaling of \( A_{1,n} \) by \( \varepsilon \) and the macroscopic nature of its time and space variables, its dynamics will include terms of second order in \( \varepsilon \). Hence, taking into account the nonlinearity of our original system (3) and the fact that we consider two different plane waves, we insert into (3) the improved approximation

\[
(13) \quad U^{A,2}_\varepsilon := U^{A,1}_\varepsilon + \varepsilon^2 \left( \sum_{n=1}^{2} \left( A_{2,n}E_n + A_{2,\{n\},n}E_n^2 \right) + A_{2,\{1,2\}}E_1E_2 \right)
\]

\[
+ A_{2,\{1,-2\}}E_{-1}E_2 + \frac{1}{2}A_{2,\{1,(-1)\} + c.c.},
\]

where \( A_{2,i} = (A^{1}_{2,i}, A^{2}_{2,i})^T : \mathbb{R} \times \mathbb{R} \to \mathbb{C}^2, i \in \{1, 2\} \cup I, I := \{(1, 1), (2, 2), (1, 2), (1, -2), (1, -1)\} \), are again functions of the macroscopic variables \( \tau = \varepsilon t, y = \varepsilon j \), and where \( E_{-n} = \overline{E_n} \). Thereby, we use the Taylor expansions

\[
A_{1,n}(\cdot, \pm \varepsilon) = A_{1,n}(\cdot, \pm \varepsilon) + \varepsilon \partial_y A_{1,n} + \varepsilon^2 \frac{1}{2} \partial_y^2 A_{1,n} \pm \xi, \quad \partial_y A_{1,n}(\cdot, \pm \xi \varepsilon) := \partial_y A_{1,n}(\tau, y \pm \xi \varepsilon),
\]

\[
A_{2,i}(\cdot, \pm \varepsilon) = A_{2,i}(\cdot, \pm \varepsilon) + \varepsilon \partial_y A_{2,i} \pm \xi_2, \quad \partial_y A_{2,i}(\cdot, \pm \xi_2) := \partial_y A_{2,i}(\tau, y \pm \xi_2) \varepsilon
\]

with \( \xi, \xi_2 \in (0, 1) \), assuming \( A_{1,n}(\tau, \cdot) \in C^2(\mathbb{R}; \mathbb{C}^2) \). \( A_{2,i}(\tau, \cdot) \in C^1(\mathbb{R}; \mathbb{C}^2) \).

Carrying out the usual (lengthy but straightforward) formal expansion in terms of \( \varepsilon \) and \( E_n \), we obtain that \( \hat{U}^{A,2}_\varepsilon = \mathcal{L}U^{A,2}_\varepsilon + \mathcal{M}(U^{A,2}_\varepsilon) \) is equivalent to

\[
\varepsilon \left( \sum_{n=1}^{2} H(\omega_n, \vartheta_n)A_{1,n}E_n + c.c. \right) +
\]
\[ + \varepsilon^2 \left\{ \sum_{n=1}^{2} \left( \frac{2\omega_n \partial F A_{1,n}^{(1)} + v_{1,1} e^{i\rho_n} \partial_y A_{1,n}^{(2)}}{-2\omega_n \partial F A_{1,n}^{(1)} - v_{2,1} e^{-i\rho_n} \partial_y A_{1,n}^{(1)}} \right) + H(\omega_n, \partial_n) A_{2,n} \right\} E_n \]

\[ + \sum_{n=1}^{2} \left( H(2\omega_n, 2\partial_n) A_{2,n} + K_{(n,n)} \right) E_n^2 \]

\[ + \left( \frac{H(\omega_1 + \omega_2, \partial_1 + \partial_2) A_{2,(1,2)} + K_{(1,2)}}{H(\omega_1 - \omega_2, \partial_1 - \partial_2) A_{2,(1,-2)} + K_{(1,-2)}} \right) E_1 E_2 \]

\[ + \left( \frac{H(0,0) A_{2,(1,1)} + K_{(1,-1)} + c.c.}{\text{res}(U^A_{(1,1)})} \right) + \text{res}(U^A_{(1,2)}) = 0 \]

with the explicit expressions for \( K_i, \ i \in I \), and \( \text{res}(U^A_{(1,2)}) = \mathcal{O}(\varepsilon^3) \) given in the Appendix. Hence, in order for our ansatz (13) to satisfy (3) up to order \( \varepsilon \), taking into account that \( E_1 \neq E_2 \), the systems \( H(\omega_n, \partial_n) A_{1,n} = 0 \) have to be satisfied. As we have already seen, since \( \det H(\omega_n, \partial_n) = 0 \), this gives the relation between first and second component of \( A_{1,n} \) (7), (8) with \( A, \rho, \omega, \vartheta \) replaced by \( A_{1,n}, \rho_n, \omega_n, \vartheta_n \).

Next, we assume that

\[ \det H(\omega, \vartheta) \neq 0 \quad \text{for} \quad (\omega, \vartheta) = (2\omega_n, 2\partial_n), (\omega_1 \pm \omega_2, \vartheta_1 \pm \vartheta_2), \]

which means in particular that \( E_1 E_1, E_1 E_2, E_1 E_2 \neq E_1, E_2 \). (Note here that \( \det H(0,0) \neq 0 \) is always satisfied due to our stability assumption \( c_1 c_2 > 4v_{1,1} v_{2,1} \)).

In this case and for \( \vartheta_n \neq \pm \pi \), we obtain from the equations for \( \varepsilon^2 E_n \)

\[ \rho_n A_{2,n}^{(1)} + A_{2,n}^{(2)} = \frac{1}{v_{1,1}(e^{i\rho_n} + 1)} \left( 2\omega_n \partial_y A_{1,n}^{(1)} - v_{1,1} e^{i\rho_n} \partial_y A_{1,n}^{(2)} \right) \]

\[ = \frac{1}{\omega^2_n - c_2} \left( 2\omega_n \partial_y A_{1,n}^{(1)} + v_{2,1} e^{-i\rho_n} \partial_y A_{1,n}^{(1)} \right). \]

Inserting \( A_{1,n}^{(2)} = -\rho_n A_{1,n}^{(1)} \), and noting that (5) gives

\[ \omega^{(\pm)}(\vartheta) = \frac{-v_{1,1} v_{2,1} \sin \vartheta}{\omega^2 \vartheta(\vartheta) - c_1 - c_2}, \]

we obtain from the equality of the right hand sides of (15)

\[ \partial_x A_{1,n}^{(1)} - \omega^{(\pm)}(\vartheta) \partial_y A_{1,n}^{(1)} = 0 \quad \text{for} \quad \omega_n = \omega^{(\pm)}(\vartheta_n). \]

Analogously, in the case \( \vartheta_n = \pm \pi \) we get from (8) (for \( A = A_{1,n} \))

\[ \partial_x A_{1,n}^{(1)} = 0, \quad A_{2,n}^{(2)} = \frac{v_{2,1}}{c_2 - c_1} \partial_y A_{1,n}^{(1)} \quad \text{for} \quad \omega^2_n = \omega^{(\pm)}(\pm \pi) = c_1, \]

\[ \partial_x A_{2,n}^{(2)} = 0, \quad A_{1,n}^{(1)} = \frac{v_{1,1}}{c_2 - c_1} \partial_y A_{1,n}^{(1)} \quad \text{for} \quad \omega^2_n = \omega^{(\pm)}(\pm \pi) = c_2. \]

Thus, we conclude that if the non-resonance conditions (14) hold, which means in particular that neither wave generates a new one via self-interaction, the dynamics of the amplitudes \( A_{1,n} \) are given by uncoupled transport equations where the velocity is the group velocity of the corresponding carrier wave. Hence, setting in particular \( A_{1,2}(0, \cdot) = 0 \) we obtain that the dynamics of \( A_{1,1} \) are given, unsurprisingly, by a homogeneous transport equation. Moreover, since the \( K_i, \ i \in I \), are known, as they depend only on the first order amplitudes \( A_{1,n} \) (see Appendix), and since (15), (17)2, (18)2 determine the relation between the components of \( A_{2,n} \), we obtain by (14) all \( A_{2} \), except for one component of \( A_{2,n} \), which can be assumed to be equivalently vanishing.
However, it is possible that \((\omega_2, \vartheta_2) = (2\omega_1, 2\vartheta_1)\), i.e. \(E_2 = E_2^2\), namely for \(\omega_1 = \omega_- (\vartheta_1)\), \(\omega_2 = \omega_+ (\vartheta_2)\), which moreover implies that \(E_1^1 = E_1 E_2\), \(E_1^2 = E_2^2\) do not characterize plane waves, as we have shown in Section 2. In this case the formal expansion gives

\[
\varepsilon \left\{ \sum_{n=1}^{2} H(\omega_n, \vartheta_n) A_{1,n} E_n + c.c. \right\}
+ \varepsilon^2 \left\{ \left( -2i\omega_1 \partial_1 A_{1,1}^{(1)} + v_{1,1} e^{i\vartheta_1} \partial_y A_{1,1}^{(2)} \right) + H(\omega_1, \vartheta_1) A_{2,1} + K(1,-2) \right\} E_1
+ \left( -2i\omega_2 \partial_1 A_{1,2}^{(1)} + v_{1,1} e^{i\vartheta_2} \partial_y A_{1,1}^{(2)} \right) + H(\omega_2, \vartheta_2) A_{2,2} + K(1,1) \right\} E_2
+ \left( H(2\omega_2, 2\vartheta_2) A_{2,2} + K(2,2) \right) E_1^3 + \left( H(2\omega_2, 2\vartheta_2) A_{2,2} + K(1,2) \right) E_2^3
+ \frac{1}{2} H(0,0) A_{2,1} + K(1,-1) + c.c. \right\} + \text{res}(U_x^A, 2) = 0
\]

The equations for \(\varepsilon E_n\) are the same as before, and hence (7) and (8) (with \(A_{1,n}, \rho_n, \omega_n, \vartheta_n\)) are still valid. Then, using \(\rho_n\), we obtain from the equations for \(\varepsilon^2 E_n\) in the case \(\vartheta_n \neq \pm \pi\)

\[
\rho_1 A_{2,1}^{(1)} + A_{2,1}^{(2)} = \frac{1}{v_{1,1}(c_1 + 1)} \left( 2i\omega_1 \partial_1 A_{1,1}^{(1)} - v_{1,1} e^{i\vartheta_1} \partial_y A_{1,1}^{(2)} + K(1,1) \right)
+ \frac{1}{\omega_1^2 - c_2} \left( 2i\omega_1 \partial_1 A_{1,2}^{(1)} + v_{1,2} e^{i\vartheta_2} \partial_y A_{1,2}^{(2)} + K(2,1) \right),
\]

\[
\rho_2 A_{2,1}^{(1)} + A_{2,2}^{(2)} = \frac{1}{v_{1,1}(c_2 + 1)} \left( 2i\omega_2 \partial_1 A_{1,1}^{(1)} - v_{1,2} e^{i\vartheta_2} \partial_y A_{1,1}^{(2)} + K(1,1) \right)
+ \frac{1}{\omega_2^2 - c_2} \left( 2i\omega_2 \partial_1 A_{1,2}^{(1)} + v_{1,2} e^{i\vartheta_2} \partial_y A_{1,2}^{(2)} + K(2,1) \right),
\]

and inserting (7) into the equalities on the right hand side we get for \(\vartheta_1 \neq \pm \frac{\pi}{2}, \pi\), \(\vartheta_2 = 2\vartheta_1\), \(\omega_1 = \omega_- (\vartheta_1)\), \(\omega_2 = \omega_+ (\vartheta_2) = 2\omega_1\)

\[
\begin{aligned}
\partial_1 A_{1,1}^{(1)} - \omega_- (\vartheta_1) \partial_y A_{1,1}^{(1)} &= \frac{d_1}{\omega_1^2 - c_2} \left( \partial_1 \omega_1 - c_2 \partial_y A_{1,1}^{(1)} \right), \\
\partial_1 A_{1,1}^{(2)} - \omega_- (\vartheta_2) \partial_y A_{1,1}^{(2)} &= \frac{d_1}{\omega_1^2 - c_2} \left( \partial_1 \omega_1 - c_2 \partial_y A_{1,1}^{(2)} \right),
\end{aligned}
\]

with

\[
\begin{aligned}
d_1 := v_{1,1}^2 (2 + 4 \cos \vartheta_1 + 2 \cos \vartheta_2),
\end{aligned}
\]

\[
\begin{aligned}
d_2 := v_{1,1} (2 + 4 \cos \vartheta_1 + 2 \cos \vartheta_2),
\end{aligned}
\]

\[
\begin{aligned}
d := \frac{v_{1,1}^2 (2 + 4 \cos \vartheta_1 + 2 \cos \vartheta_2)}{(\omega_1^2 - c_2)^2 (\omega_2^2 - c_2)}.
\end{aligned}
\]
Analogously, for $\vartheta_1 = \pm \frac{\pi}{2}, \vartheta_2 = 2\vartheta_1, \omega_1 = \omega_-(\vartheta_1), \omega_2 = \omega_+(\vartheta_2) = \sqrt{c_2} = 2\omega_1$ we obtain

$$
\begin{aligned}
\partial_\vartheta A_{1,1}^{(1)} - \omega_-(\pm \frac{\pi}{2}) \partial_\vartheta A_{1,1}^{(2)} &= \frac{d_1}{\omega_1} \omega_1^2 \omega_1^2 - \omega_1 - c_2 \hat{A}_{1,1}^{(1)} A_{1,2}^{(2)} \\
\partial_\vartheta A_{1,2}^{(2)} &= \frac{1}{2\omega_2} \left( 2\omega_2 \hat{A}_{1,1}^{(1)} (1 + \rho_1(1 \mp i)) - w_{2,2} \rho_1^2 \right) (A_{1,1}^{(1)})^2 
\end{aligned}
$$

with $d_1 = \left( \frac{v_{2,2}}{v_{2,1}} + i \frac{w_{2,2}}{\omega_1 - c_2} \right) (\omega_1^2 - c_1) + v_{1,2} \rho_1(1 \mp i) + 2$ and

$$
\rho_1 A_{2,1}^{(1)} + A_{2,1}^{(2)} = \rho_1 \left( \frac{1}{2} \mp i \partial_\vartheta A_{1,1}^{(2)} \right) - \frac{2i\omega_1}{\omega_1^2 - c_2} \partial_\vartheta A_{1,1}^{(1)} + 2\rho_1 \left( \frac{v_{2,2}}{v_{2,1}} + i \frac{w_{2,2}}{\omega_1^2 - c_2} \right) \hat{A}_{1,1}^{(1)} A_{1,2}^{(2)} \\
A_{2,2}^{(1)} = \frac{1}{c_2 - c_1} \left( v_{2,1} \partial_\vartheta A_{1,1}^{(2)} + \left( 2v_{2,2}(\rho_1^2 + \rho_1(1 \mp i)) + w_{1,2} \right) (A_{1,1}^{(1)})^2 \right)
$$

and for $\vartheta_1 = \pm \pi, \vartheta_2 = 2\vartheta_1, \omega_1 = \omega_-(\vartheta_1) = \sqrt{c_1}, \omega_2 = \omega_+(\vartheta_2) = \omega_+(0) = 2\omega_1$ we get, using (8) and (7), the equations (19) with $\omega'_-(\vartheta_1) = \omega'_+(\vartheta_2) = 0$ and $d_1 = d_2 = -w_{1,2,2}$, and

$$
A_{2,1}^{(2)} = \frac{1}{c_2 - c_1} \left( v_{2,1} \partial_\vartheta A_{1,1}^{(1)} + 4v_{2,2}(1 + \rho_2) \hat{A}_{1,1}^{(1)} A_{1,2}^{(2)} \right), \\
\rho_2 A_{2,2}^{(1)} + A_{2,2}^{(2)} = \rho_2 \left( \frac{1}{2} \partial_\vartheta A_{1,1}^{(2)} - \frac{2i\omega_2}{\omega_2^2 - c_2} \partial_\vartheta A_{1,1}^{(1)} \right).
$$

Hence, in the case $E_2 = E_2^n$ we obtain two coupled equations for $A_{1,1}^{(1,2)}$, the solutions of which (cf. about their well-posedness Lemma 4.2) determine again $U^{A,2}$ up to one component of $A_{2,n}$. We would like to stress that in order to obtain non-trivial dynamics for $A_{1,1}$ we have to consider also the dynamics of the generated wave $A_{1,2}$ while if only interested in $A_{1,2}$ we could ignore the generating wave $A_{1,1}$, see e.g. (19). Note in this context, that even for initial data $A_{1,2}(0, \cdot) = 0$ an amplitude $A_{1,2} \neq 0$ emerges, which motivates the notion of generation of waves.

### 4. Justification

The equations obtained by the formal derivation constitute only necessary conditions on the amplitudes $A_{1,n}$ of the ansatz (12). The purpose of the justification is to show that indeed solutions $u$ of such a form exist.

**Theorem 4.1.** Let $V_i, W_i \in C^4(\mathbb{R}), i = 1, 2, m (2)$ satisfy

$$
v_{1,1} = \frac{v_1}{M}, \quad v_{2,1} = \frac{v_1}{m}, \quad w_{1,1} > 0, \quad 4v_1 + \min\{MW_{1,1}, MW_{2,1}\} > 0, \quad M, m > 0,
$$

let $\omega_2 > \omega_1 > 0$ and $\vartheta_n \in (-\pi, \pi], n = 1, 2$, satisfy $\det H(\omega_n, \vartheta_n) = 0$ and

- either $\det H(2\omega_n, 2\vartheta_n) \neq 0$, $\det H(\omega_1 \pm \omega_2, \vartheta_1 \pm \vartheta_2) \neq 0$,
- or $(\omega_2, \vartheta_2) = (2\omega_1, 2\vartheta_1 \mod 2\pi)$, $\det H(k\omega_1, k\vartheta_1) \neq 0, k = 3, 4$.

with the dispersion matrix $H$ in (4), and let $A_{1,n}^{(1,2)} : [0, \tau_0] \times \mathbb{R} \to \mathbb{C}, \tau_0 > 0$, be, respectively, the unique solutions of either (16) (or (17) or (18)) or (19) (or (20)) with $A_{1,n}^{(1,2)}(0, \cdot) \in H^4(\mathbb{R}; \mathbb{C})$.

Then, for the corresponding approximation $U^{A,1}_\varepsilon$ and every $c > 0, \beta \in (1, 3/2]$ there exist $\varepsilon_0, C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and all $t \in [0, \tau_0/\varepsilon]$ any solution $u$ of (3) satisfies

$$
\left\| \left( \frac{u - U_{A,1}^{A,1}}{\hat{u} - U_{A,1}^{A,1}} \right)(0) \right\|_{(L^2)^4} \leq C\varepsilon^\beta \quad \Rightarrow \quad \left\| \left( \frac{u - U_{A,1}^{A,1}}{\hat{u} - U_{A,1}^{A,1}} \right)(t) \right\|_{(L^2)^4} \leq C\varepsilon^\beta.
$$
Proof. The idea of the proof is classical, see e.g. [5]. We write the microscopic model (3) as a first order system in $Y := \mathcal{L}^2$ with $(\mathcal{L}^2)^4 = (\mathcal{L}^2)^2 \times (\mathcal{L}^2)^2 = (\mathcal{L}^2 \times \mathcal{L}^2) \times (\mathcal{L}^2 \times \mathcal{L}^2)$ and $\mathcal{L}^2 = \mathcal{L}^2(\mathbb{Z})$.

\begin{equation}
\tilde{u} = \tilde{u} + \tilde{M}(\tilde{u}) \quad \text{with} \quad \tilde{u} := \left(\begin{array}{c}
u \\
\nu \end{array}\right), \quad \tilde{M}(\tilde{u}) := \left(\begin{array}{c}0 \\
\mathcal{M}(\nu)\end{array}\right),
\end{equation}

where $\mathcal{L} : (\mathcal{L}^2)^2 \to (\mathcal{L}^2)^2$ is the identity. Then, the flow of the linearized system $\tilde{u} = \tilde{L}\tilde{u}$ preserves the energy norm on $Y$,

$$
\|\tilde{u}\|_Y^2 := \|u\|_E^2 + \|u\|_M^2
$$

$$
= \sum_{j \in \mathbb{Z}} \left(\varepsilon_1 \left(|u_{j+1,2} - u_{j,1}|^2 + |u_{j,1} - u_{j,2}|^2\right) + M w_{1,1}|u_{j,1}|^2 + m w_{2,1}|u_{j,2}|^2\right)
$$

$$
+ \sum_{j \in \mathbb{Z}} \left(M |u_{j,1}|^2 + m |u_{j,2}|^2\right) \quad \text{for} \quad \tilde{u} = \left(\begin{array}{c}
u \\
\nu \end{array}\right),
$$

i.e., its associated semi-group $e^{t\tilde{L}}$ satisfies $\|e^{t\tilde{L}}\|_{Y \to Y} = 1$, and from (21) it follows by Fourier transformation that the norms $\|\|_{\mathcal{L}^2}$, $\|\|_{\mathcal{L}^2}$ and $\|\|_{\mathcal{L}^2}$, and hence also $\|\|_{\mathcal{L}^2}$, $\|\|_{\mathcal{L}^2}$, $\|\|_{\mathcal{L}^2}$, are equivalent: $\hat{k}_1\|\hat{u}\|_{\mathcal{L}^2} \leq \|\hat{u}\|_{\mathcal{L}^2} \leq \hat{k}_2\|\hat{u}\|_{\mathcal{L}^2}$, $\hat{k}_1\|\hat{u}\|_{\mathcal{L}^2} \leq \|\hat{u}\|_{\mathcal{L}^2}$, with $\hat{k}_1, \hat{k}_2 > 0$ and $\|\hat{u}\|_{\mathcal{L}^2}^2 = \|u\|_E^2 + \|u\|_M^2$ for $u = (u_1, u_2)^T$, $\|\hat{u}\|_{\mathcal{L}^2}^2 = \|u\|_E^2 + \|u\|_M^2$.

We consider the error $\varepsilon - \tilde{R}_\varepsilon := \tilde{u} - \tilde{U}_\varepsilon^A, \tilde{u} = \tilde{U}_\varepsilon^A, \tilde{U}_\varepsilon^A, \varepsilon \tilde{U}_\varepsilon^A, \varepsilon \tilde{U}_\varepsilon^A$ between an original solution $u$ of (3) and the improved approximation $U_\varepsilon^A$ given by (13) with the $A_1, A_2, \varepsilon$ determined by the formal derivation. Since for this $U_\varepsilon^A$ we have $L U_\varepsilon^A + \mathcal{M}(U_\varepsilon^A, \varepsilon) \tilde{U}_\varepsilon^A = \varepsilon(U_\varepsilon^A, \varepsilon)$, inserting $\tilde{R}_\varepsilon$ into (22) we obtain the differential equation

$$
\tilde{R}_\varepsilon = \tilde{L}\tilde{R}_\varepsilon + \varepsilon^{-\beta} \mathcal{M}(U_\varepsilon^A + \varepsilon^{-\beta} \tilde{R}_\varepsilon) - \mathcal{M}(U_\varepsilon^A) + \mathcal{M}(U_\varepsilon^A).
$$

Taking the energy norm of its integral formulation, assuming $\|\tilde{R}_\varepsilon(0)\|_Y \leq d$, and applying Lemma 4.2 c), we get

\begin{equation}
\|\tilde{R}_\varepsilon(t)\|_Y \leq d + \varepsilon^{3/2 - \beta} \tau_0 c_r + \varepsilon^{-\beta} \int_0^t \|\mathcal{M}(U_\varepsilon^A + \varepsilon^{-\beta} \tilde{R}_\varepsilon) - \mathcal{M}(U_\varepsilon^A)\|_M ds
\end{equation}

for $\varepsilon \in (0, \varepsilon_0], \ t \in (0, \tau_0/\varepsilon]$. From (3) and (2) we get by the mean value theorem

\begin{equation}
\|\mathcal{M}(u) - \mathcal{M}(u)\|_M \leq c_M \|u\|_{\mathcal{L}^2} \|u\|_M \quad \text{for} \quad \|u\|_{\mathcal{L}^2}, \|u\|_M \leq c_0
\end{equation}

with $\|u\|_{\mathcal{L}^2} = \max\{|u_1|_{\mathcal{L}^2}, |u_2|_{\mathcal{L}^2}\}$ for $u = (u_1, u_2)^T \in (\mathcal{L}^2)^2$, and $c_M$ depending only on $\mathcal{V}_i, \mathcal{W}_i$ and $c_0 > 0$.

We set $D := (d + \varepsilon^{3/2 - \beta} \tau_0 c_r) e^{\tau_0 c_M A_1 \varepsilon^{-1}/\tau_2}$ with $c_A$ from Lemma 4.2 a) and $\varepsilon_0 > 0$ such that $\tau_0 D/c_0 \leq \varepsilon_0 c_A \leq c_0/2$. Since $\|\tilde{R}_\varepsilon(0)\|_Y \leq d < D$ and $\|\tilde{R}_\varepsilon(t)\|_Y$ is continuous, there exists for every $\varepsilon \in (0, \varepsilon_0]$, such that $\|\tilde{R}_\varepsilon(t)\|_Y \leq D$ for $t \in [0, t_0]$. Then, for $\varepsilon \in (0, \varepsilon_0]$ and $t \in [0, \min\{\tau_0/\varepsilon, t_0\}]$ (24) gives

$$
\|\mathcal{M}(U_\varepsilon^A + \varepsilon^{-\beta} \tilde{R}_\varepsilon) - \mathcal{M}(U_\varepsilon^A)\|_M \leq \varepsilon^{\beta+1} (\tau_0 c_M A_1 \varepsilon^{-1}/\tau_2) \|\tilde{R}_\varepsilon\|_Y.
$$

Inserting this estimate into (23) and applying Gronwall’s Lemma, we get

$$
\|\tilde{R}_\varepsilon(t)\|_Y \leq \left(d + \varepsilon^{3/2 - \beta} \tau_0 c_r e^{2\tau_0 c_M A_1 \varepsilon^{-1}/\tau_2} \leq D \quad \text{for} \quad \varepsilon \in (0, \varepsilon_0], \ t \in (0, \tau_0/\varepsilon].
$$
Finally, with $d := \kappa_2 c + \varepsilon_0^{3/2-\beta} c_f$ and $C := (D + \varepsilon_0^{3/2-\beta} c_f)/\kappa_1$ we obtain from Lemma 4.2) b) and the equivalence of $\|\cdot\|_{\ell^2}$ and $\|\cdot\|_Y$ the assertion of the theorem.

Lemma 4.2. For $U^A,2$ given by (13) with $A_{1,n}, A_{2,n}$ as determined in Section 3 and initial data $A_{1,n,i}^i(0, \cdot) \in H^4(\mathbb{R}; \mathbb{C})$, there exist $\tau_0, \varepsilon_0, c_A, c_f, c_r > 0$ such that for all $\varepsilon \in [0, \varepsilon_0], \varepsilon t \in [0, \tau_0]$

\begin{align*}
&\text{a) } \|U_{\varepsilon}^{A,2}\|_{\ell^\infty} \leq \varepsilon c_A, \\
&\text{b) } \|\tilde{U}_{\varepsilon}^{A,2} - U_{\varepsilon}^{A,1}\|_Y \leq \varepsilon^{3/2} c_f, \\
&\text{c) } \|\text{res}(U_{\varepsilon}^{A,2})\|_M \leq \varepsilon^{5/2} c_r.
\end{align*}

Proof. Inserting into (12), (13) and res$(U_{\varepsilon}^{A,2})$ (see Appendix) $A_{1,2}^{(2)}(\varepsilon) = -\rho_n A_{1,1}^{(1)}(\varepsilon)$ (with $\rho_n = 0$ for $\vartheta_n = \pm \pi$, $\omega_n = \sqrt{\varepsilon}$), $A_{2,2}^{(1)} = 0$ or $A_{2,2}^{(2)} = 0$ for $\vartheta_n = \pm \pi$, $\omega_n = \sqrt{\varepsilon}$, and the $A_{2,2}^{(2)}$ (or $A_{2,2}^{(1)}$, respectively), $A_{2,2}$, $\varepsilon \in I$, specified in Section 3, recalling $|V'_\vartheta(x)|, |W'_\vartheta(x)| = O(|x|^3)_{x \to 0}$ and the equivalence of $\|\cdot\|_{\ell^2}, \|\cdot\|_M$, $\|\cdot\|_{\ell^2}$, and using the corollary of Sobolev’s embedding theorem (cf., e.g., [3, Lemma 3.1])

\[
\|\varphi(\varepsilon \cdot + \xi)\|_{\ell^2} \leq c^{-1/2} \|\varphi\|_{H^1(\mathbb{R}; \mathbb{C})}, \quad \xi : \mathbb{Z} \to [-1, 1], \ \varepsilon \in (0, \varepsilon_0),
\]

and $A, B \in C([0, \tau_0]; H^4(\mathbb{R}; \mathbb{C})) \Rightarrow AB \in C([0, \tau_0]; H^4(\mathbb{R}; \mathbb{C}))$, we obtain that the above estimates are satisfied, provided

\[
\partial_x^\vartheta \partial_y^A A_{1,2}^{(1,2,\text{resp})} \in C([0, \tau_0]; H^4(\mathbb{R}; \mathbb{C})) \quad \text{for } \|(p, q)\| \leq 2, \ (p, q) = (3, 0), (2, 1).
\]

Since the macroscopic equations for $A_{1,2}^{(1,2)}$, $n = 1, 2$, are semilinear autonomous transport systems with smooth nonlinearities, standard results of semigroup theory (cf., e.g., [6, Th. 6.1.7]) yield that for initial data $A_{1,2}^{(1,2)}(0, \cdot) \in H^m(\mathbb{R}; \mathbb{C})$, $m \geq 1$, there exist unique classical solutions with $\partial_x^\vartheta \partial_y^A A_{1,2}^{(1,2)} \in C([0, \tau_0]; H^1(\mathbb{R}; \mathbb{C}))$ for $\|(p, q)\| \leq m - 1$ up to some $\tau_0 \in (0, \infty)$. Hence, for $m = 4$ we obtain the statement of the lemma.

5. Appendix

For completeness we present here the $K_i = (K_i^{(1)}, K_i^{(2)})^T, \ i \in I$, and res$(U_{\varepsilon}^{A,2}) = \text{res}(U_{\varepsilon}^{A,2}), \text{res}(U_{\varepsilon}^{A,2})^T$ derived in Section 3, with $i = 1, 2$ (where $i + 1 = 1$ for $i = 2$) and with the upper sign to i be.

\begin{align*}
&K_{(n,n)}^{(i)} \ := \pm v_{i,2} \left( A_{i,1}^{(i+1)}(e^{\vartheta \vartheta_i + i}) - 2 A_{i,1}^{(i+1)}(e^{\vartheta \vartheta_i - i}) - w_{i,2} A_{i,1}^{(i)} \right), \\
&K_{(1,2)}^{(i)} \ := \pm w_{i,2} A_{i,1}^{(i+1)} \left( e^{\vartheta \vartheta_i + i} - 2 A_{i,1}^{(i+1)}(e^{\vartheta \vartheta_i - i}) - w_{i,2} A_{i,1}^{(i)} \right), \\
&K_{(1,-2)}^{(i)} \ := \pm v_{i,2} A_{i,1}^{(i+1)} \left( e^{\vartheta \vartheta_i - i} - 2 A_{i,1}^{(i+1)}(e^{\vartheta \vartheta_i + i}) - w_{i,2} A_{i,1}^{(i)} \right), \\
&K_{(1,-1)}^{(i)} \ := \pm v_{i,2} A_{i,1}^{(i+1)} \left( e^{\vartheta \vartheta_i - i} - 2 A_{i,1}^{(i+1)}(e^{\vartheta \vartheta_i + i}) - w_{i,2} A_{i,1}^{(i)} \right), \\
&K_{(2,1)}^{(i)} \ := \pm w_{i,2} A_{i,1}^{(i+1)} \left( e^{\vartheta \vartheta_i - i} - 2 A_{i,1}^{(i+1)}(e^{\vartheta \vartheta_i + i}) - w_{i,2} A_{i,1}^{(i)} \right), \\
&K_{(2,1)}^{(i)} \ := \pm v_{i,2} A_{i,1}^{(i+1)} \left( e^{\vartheta \vartheta_i - i} - 2 A_{i,1}^{(i+1)}(e^{\vartheta \vartheta_i + i}) - w_{i,2} A_{i,1}^{(i)} \right), \\
&K_{(2,1)}^{(i)} \ := \pm w_{i,2} A_{i,1}^{(i+1)} \left( e^{\vartheta \vartheta_i - i} - 2 A_{i,1}^{(i+1)}(e^{\vartheta \vartheta_i + i}) - w_{i,2} A_{i,1}^{(i)} \right), \\
&\text{res}(U_{\varepsilon}^{A,2}) \ := \varepsilon^3 \left( - T_i \pm v_{i,1} F_i \pm v_{i,2} (D_i E_i - (A_1 - A_2)(B_1 - B_2)) \right), \\
&\pm \varepsilon^3 \left( - T_i \pm v_{i,1} F_i \pm v_{i,2} (D_i E_i - (A_1 - A_2)(B_1 - B_2)) \right) + \varepsilon^4 \left( - S_i \pm v_{i,2} (E_i^2 + 2 D_i (F_i^2 + (B_1 - B_2))^2) \right) \pm \varepsilon^5 v_{i,2} E_i F_i \pm \varepsilon^6 v_{i,2} F_i^2
\end{align*}
\[ T_i := \sum_{n=1}^{2} \left( \partial_{2}^2 A^{(i)}_{1,n} + \partial_{r} A^{(i)}_{2,2,n} - 2i\omega_n \right) E_n + \partial_{r} A^{(i)}_{2,2(n,n)} 4i\omega_n E^2_n \]
\[ + \partial_{r} A^{(i)}_{2,2(1,2)} 2i(\omega_1+\omega_2) E_1 E_2 + \partial_{r} A^{(i)}_{2,2(1,-2)} 2i(\omega_1-\omega_2) E_1 E_{-2} + \text{c.c.}, \]
\[ S_i := \sum_{n=1}^{2} \left( \partial_{2}^2 A^{(i)}_{2,2(n,n)} E_n + \partial_{r}^2 A^{(i)}_{2,2(n,n)} E^2_n \right) + \partial_{r}^2 A^{(i)}_{2,2(1,2)} E_1 E_2 + \partial_{r}^2 A^{(i)}_{2,2(1,-2)} E_1 E_{-2} + \frac{1}{2} A^{(i)}_{2,2(1,-1)} + \text{c.c.}, \]
\[ A_i := \sum_{n=1}^{2} A^{(i)}_{1,n} E_n + \text{c.c.}, \quad D_i := \pm \sum_{n=1}^{2} \left( A^{(i+1)}_{1,n} e^{\pm i\theta_n} - A^{(i)}_{1,n} \right) E_n + \text{c.c.}, \]
\[ B_i := \sum_{n=1}^{2} \left( A^{(i)}_{2,n} E_n + A^{(i)}_{2,2(n,n)} E^2_n \right) + A^{(i)}_{2,2(1,2)} E_1 E_2 + A^{(i)}_{2,2(1,-2)} E_1 E_{-2} + \frac{1}{2} A^{(i)}_{2,2(1,-1)} + \text{c.c.}, \]
\[ E_i := \sum_{n=1}^{2} \left( \left( \partial_{2}^2 A^{(i+1)}_{2,1,n} e^{\pm i\theta_n} \mp A^{(i)}_{2,2(n,n)} \mp A^{(i)}_{2,2(n,n)} \right) E_n + \left( \left( A^{(i+1)}_{2,2(n,n)} e^{\pm i\theta_n} \mp A^{(i)}_{2,2(n,n)} \mp A^{(i)}_{2,2(n,n)} \right) E^2_n \right) \right. \]
\[ + \left( A^{(i+1)}_{2,2(1,2)} e^{\pm i(\theta_1+\theta_2)} - A^{(i)}_{2,2(1,2)} \right) E_1 E_2 \pm \left( \left( A^{(i+1)}_{2,2(1,-2)} e^{\pm i(\theta_1-\theta_2)} - A^{(i)}_{2,2(1,-2)} \right) E_1 E_{-2} \right. \]
\[ + \frac{1}{2} A^{(i)}_{2,2(1,-1)} + \text{c.c.}, \]
\[ F_i := \sum_{n=1}^{2} \left( \left( \pm \frac{1}{2} \partial_{2}^2 A^{(i+1)}_{2,2(n,n)} \pm \partial_{r} A^{(i+1)}_{2,2(n,n)} e^{\pm i\theta_n} E_n + \partial_{r} A^{(i+1)}_{2,2(n,n)} e^{\pm i\theta_n} E^2_n \right) \right. \]
\[ + \partial_{r} A^{(i+1)}_{2,2(1,2),\xi_2 \pm \epsilon} e^{\pm i(\theta_1+\theta_2)} E_1 E_2 + \partial_{r} A^{(i+1)}_{2,2(1,-2),\xi_2 \pm \epsilon} e^{\pm i(\theta_1-\theta_2)} E_1 E_{-2} \]
\[ + \frac{1}{2} \partial_{r} A^{(i+1)}_{2,2(1,-1),\xi_2 \pm \epsilon} + \text{c.c.} \]

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