On symplectic caps

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Abstract An important class of contact 3–manifolds are those that arise as links of rational surface singularities with reduced fundamental cycle. We explicitly describe symplectic caps (concave fillings) of such contact 3–manifolds. As an application, we present a new obstruction for such singularities to admit rational homology disk smoothings.

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We dedicate this paper to Oleg Viro on the occasion of his 60th birthday.

1 Introduction

Our understanding of topological properties of (weak) symplectic fillings of certain contact 3–manifolds showed a dramatic improvement in the recent past. These developments rested on recent results in symplectic topology, most notably on McDuff’s characterization of (closed) rational symplectic 4–manifolds [14]. In order to apply results of McDuff, however, symplectic caps were needed to close up the fillings at hand. The general results of Eliashberg and Etnyre [5, 6] showed that such caps do exist in general, but these results can be used powerfully only in case a detailed description of the cap is also available. This was the case, for example, for lens spaces with their standard contact structures [12], or for certain 3–manifolds which can be given as links of isolated surface singularities [2, 3, 16].

In the following we will show an explicit construction of symplectic caps for contact 3–manifolds which can be given as links (with their Milnor fillable
structures) of rational singularities with reduced fundamental cycle. In topological terms it means that the 3–manifold can be given as a plumbing of spheres along a negative definite tree, with the additional assumption that the absolute value of the framing at each vertex is at least the valency of the vertex. The
construction of the cap in this case relies on a symplectic handle attachment along the binding component of a compatible open book decomposition. In the terminology of open book decompositions, our construction coincides with the cap-off procedure initiated and further studied by Baldwin [1].

The success of the rational blow–down procedure (initiated by Fintushel and Stern [8] and then extended by J. Park [17]) led to the search for isolated surface singularities which admit rational homology disk smoothings. Strong restrictions on the combinatorics of the resolution graph of such a singularity were found in [20], and by identifying Neumann’s \( \mu \)–invariant with a Heegaard Floer theoretic invariant of the underlying 3–manifold, further obstructions for the existence of such a smoothing were given in [19]. More recently the question has been answered for all singularities with starshaped resolution graphs (in particular, for weighted homogeneous singularities) in [3], but the question remained open in general. Motivated by our construction of a symplectic cap for special types of Milnor fillable contact 3–manifolds, we show examples of surface singularities which pass all tests provided by [19] [20] but still do not admit rational homology disk smoothings.

The paper is organized as follows. In Section 2 we describe the symplectic handle attachment which caps off a boundary component of a compatible open book decomposition. Section 3 is devoted to the detailed description of the topology of the symplectic cap, and also an example is worked out. In Section 4 we show that certain singularities do not admit rational homology disk smoothings.

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2 Symplectic handle attachments

Throughout this section suppose that \((Y, \xi)\) is the strongly convex boundary of a symplectic 4–manifold \((X, \omega)\), that \(\xi\) is supported by an open book decomposition with page \(\Sigma\), binding \(B = \partial \Sigma\) and monodromy \(h\), and that \(K\)
is a component of $B$. Let $\text{pf}(K)$ denote the page–framing of $K$, the framing induced by the page $\Sigma$. Note that if $Y_K$ is the result of performing surgery on $Y$ along $K$ with framing $\text{pf}(K)$, and if $K \neq B$, then the open book on $Y$ induces a natural open book on $Y_K$ with page $\Sigma_K$ equal to $\Sigma \cup_K D^2$, the result of capping off $K$ with a disk, and with monodromy equal to $h$ extended by the identity on the cap $D^2$. (In [1] this construction has been examined from the Heegaard Floer theoretic point of view.)

If instead $Y_K$ is obtained by surgery along $K$ with framing $\text{pf}(K) \pm 1$, then the open book on $Y$ induces a natural open book on $Y_K$ with page $\Sigma_K = \Sigma$ and with monodromy $h_K = h \circ \tau_K^{\pm 1}$, where $\tau_K$ is a right-handed Dehn twist along a circle in the interior of $\Sigma$ parallel to $K$. In fact, if $K \neq B$ then surgery with framing $\text{pf}(K) - 1$ coincides with Legendrian surgery along a Legendrian realization of $K$ on the page, hence the 4–dimensional cobordism resulting from the construction supports a symplectic structure. In the following we extend the existence of such a symplectic structure to the cases where the surgery coefficient is $\text{pf}(K)$ and $\text{pf}(K) + 1$.

**Theorem 2.1** Suppose that $X_K \supset X$ is the result of attaching a 2–handle $H$ to $X$ along $K$ with framing $\text{pf}(K)$. Suppose furthermore that $[K] \neq 0$ in $H_1(Y;\mathbb{Z})$ or that $[K] = 0$ but that the Seifert framing of $K$ is not equal to $\text{pf}(K)$. (In particular, $K \neq B$.) Then $\omega$ extends to a symplectic form $\omega_K$ on $X_K$ and the new boundary $Y_K$ is $\omega_K$–convex. The new contact structure $\xi_K$ is supported by the natural open book on $Y_K$ described above.

**Proof** Let $\pi: Y \setminus B \rightarrow S^1$ be the fibration associated to our given open book on $Y$, and let $\pi_K: Y_K \setminus (B \setminus K) \rightarrow S^1$ be the fibration for the induced open book on $Y_K$. Let $Z$ be $[-1,0] \times Y$ together with the 2–handle $H$ attached along $\{0\} \times K \subset \{0\} \times Y$, and identify $Y$ with $\{0\} \times Y$. Thus $Z$ is a cobordism from $[-1] \times Y$ to $Y_K$ and $Y \cap Y_K$ is nonempty and is in fact the complement of a neighborhood of $K$ in $Y$. We will show that there is a symplectic structure $\eta$ on $Z$ which, on $[-1,0] \times Y$, is equal to the symplectization of a certain contact form $\alpha$ on $Y$ supported by $(B,\pi)$ and such that $Y_K$ is $\eta$–convex, with induced contact structure $\xi_K$ supported by the natural open book $(B \setminus K, \pi_K)$ on $Y_K$ described above. This proves the theorem.

Let $(r,\mu,\lambda)$ be coordinates on a neighborhood $\nu \cong D^2 \times S^1$ of $K$, with $(r,\mu)$ being polar coordinates on the $D^2$–factor and $\lambda$ being the $S^1$–coordinate, in such a way that $\mu = \pi|_{\nu}$. Thus the pages are the level sets for $\mu$. Let $\nu_K$ be the corresponding neighborhood in $Y_K$ of the belt–sphere for $H$, with corresponding coordinates $(r_K, \mu_K, \lambda_K)$, with the natural diffeomorphism from
\( \nu \setminus \{ r = 0 \} \rightarrow \nu_K \setminus \{ r_K = 0 \} \) given by \( r_K = r, \mu_K = -\lambda \) and \( \lambda_K = \mu \).

Note that \( \pi_K|_{\nu_K} = \lambda_K \), which is defined on all of \( \nu_K \). We begin with three observations:

1. First we claim that, for any positive constants \( c_1, c_2, c_3 \) and \( c_4 \) with \( c_3 > c_2 \) and \( c_4 > \frac{1}{2} c_2^2 \), there exists a contact form \( \alpha \) on \( Y \) which is supported by our open book such that on \( \nu \) we have \( \alpha = \frac{1}{2} g(r)^2 d\mu + (-\frac{1}{2} f(r)^2 + c_4) d\lambda \), where \( f \) and \( g \) have the following properties (illustrated in Figure 1):

   a. The function \( f \) is monotone increasing with \( f'(0) = 0 \) and \( f'(r) > 0 \) for \( r > 0 \).
   b. \( f(0) = c_2 \) and \( f(r) = r \) for \( r \geq c_3 \).
   c. \( g(0) = 0 \).
   d. The function \( g \) is monotone increasing, with \( g'(r) > 0 \) on \([0, c_3)\).  
   e. \( g(r) = c_1 \) for \( r \geq c_3 \).

Note that this result depends critically on the fact that \( K \neq B \). In terms of the contact structure \( \xi = \ker \alpha \), we are simply claiming that we can find a contact structure \( \xi \) supported by \( (B, \pi) \) which makes slightly more than a 1/4 turn inside the neighborhood \( \nu \) of \( K \), where a 1/4 turn is measured in terms of the coordinates \( \mu \) and \( \lambda \) as \( r \) increases from 0 to the radius of \( \nu \). Equivalently, we are stating that we can choose \( \xi \) so that there is a circle parallel to \( K \) in each page where \( \xi \) is tangent to the page. This is not possible if \( K \) is the only component of \( B \). If there are more than one components of \( B \), however, we can choose a 1–form \( \delta \) on \( \Sigma \) such that \( d\delta > 0 \) and on a collar neighborhood \((-\epsilon, 0] \times S^1\) of \( K \subset \partial \Sigma \), with corresponding coordinates \((t, \eta)\), \( \delta = (t - \epsilon/2)d\eta \). When \( \delta \) is used in the standard way to construct a contact structure supported by our open book, the circles \( \{ t = \epsilon/2 \} \) on each page will be the circles parallel to \( K \) where \( \xi \) is tangent to the pages.

2. Secondly we claim that there exists a closed 1–form \( \alpha_0 \) on \( Y \setminus K \) such that \( \alpha_0|_\nu = m_0 d\mu + l_0 d\lambda \) for some \( m_0, l_0 \in \mathbb{R} \) with \( l_0 > 0 \). This follows directly from the condition that either \( |K| \neq 0 \) or \( |K| = 0 \) and the Seifert framing of \( K \) is different from \( \text{pf}(K) \).

3. Given the above, we can choose the constants \( c_1, c_2, c_3, c_4 \) and an additional positive constant \( c_5 \) so that \( (c_5 - \frac{1}{2} c_2^2)/c_4 = m_0/l_0 \).

Now embed \( \nu \) and \( \nu_K \) in \( \mathbb{R}^4 \) as follows, using polar coordinates \((r_1, \theta_1, r_2, \theta_2)\) on \( \mathbb{R}^4 \): The embedding of \( \nu \) is given by \( (r_1 = f(r), \theta_1 = -\lambda, r_2 = g(r), \theta_2 = \mu) \).

The embedding of \( \nu_K \) is given by \( (r_1 = r_K, \theta_1 = \mu_K, r_2 = c_1, \theta_2 = \lambda_K) \). This is illustrated in Figure 2 which also shows that the region between \( \nu \) and \( \nu_K \) is
Figure 1: Graphs of the functions $f$ and $g$

precisely our 2–handle $H$ attached along $K$ with framing $p_\text{f}(K)$. The overlap $\nu \cap \nu_K$ is the set $\{ r_1 \geq c_3, r_2 = c_1 \}$, which in $\nu$–coordinates is $\{ r \geq c_3 \}$ and in $\nu_K$–coordinates is $\{ r_K \geq c_3 \}$.

Figure 2: Embeddings of $\nu$, $\nu_K$ and $H$ into $\mathbb{R}^4$

Consider the standard symplectic form $\omega_0 = r_1 dr_1 d\theta_1 + r_2 dr_2 d\theta_2$ on $\mathbb{R}^4$. Note that $\omega_0|_\nu = gg' drd\mu - ff' drd\lambda = d\alpha$, so that $H$ equipped with this symplectic form can be glued symplectically to $[-1,0] \times Y$ with the symplectization of $\alpha$.

Next note that $\omega_0|_{\nu_K} = r_K dr_K d\mu_K = d\alpha_K$, where $\alpha_K = \frac{1}{2} r_K^2 d\mu_K + c_5 d\lambda$. (We need $c_5 > 0$ for $\alpha_K$ to be a positive contact form.) On the overlap $\nu \cap \nu_K \subset \mathbb{R}^4$, 5
using the coordinates \((r, \mu, \lambda)\) from \(\nu\), we see that 
\[
\alpha_K = c_5 d\mu - \frac{1}{2} r^2 d\lambda = \alpha + (c_5 - \frac{1}{2} c_1^2) d\mu + c_4 d\lambda = \alpha + k \alpha_0 \quad \text{for some } k > 0.
\]
Thus we see that \(\alpha_K\) extends to all of \(Y_K\). It remains to verify that \(\alpha_K = \alpha + k \alpha_0\) is contact and is supported by the open book \((B \setminus K, \pi_K)\). This will then complete the proof.

We know that \(\alpha_K\) is contact and supported by the open book \((B \setminus K, \pi_K)\) on \(\nu_K\). To see that \(\alpha_K\) is contact on the rest of \(Y_K\), which means checking it on \(Y \cap Y_K = Y \setminus \nu\), we compute \(\alpha_K \wedge d\alpha_K = \alpha \wedge d\alpha + k \alpha_0 \wedge d\alpha\). By multiplying \(c_i\) \((i = 1, \ldots, 5)\) by a suitable small positive constant we can make \(k\) as small as we like, and then we will have \(\alpha_K \wedge d\alpha_K > 0\). To see that \(\alpha_K\) is supported by the open book \((B \setminus K, \pi_K)\) on \(Y \setminus \nu\), we need to check that \(d\pi_K \wedge d\alpha_K > 0\) and that \(R_{\alpha_K}\) is positively tangent to \(B \setminus K\). These are conditions only on \(d\pi_K\) which equals \(d\alpha\), so the conclusion obviously follows.

In fact, 2–handles can be attached with framing \(\text{pf}(K) + 1\) to boundary components of a compatible open book, and the symplectic structure will still extend. In this case, however, the convex boundary will become concave. More precisely:

**Theorem 2.2** Suppose that \(K = B\) and that \(X_K \supset X\) is the result of attaching a 2-handle \(H\) to \(X\) along \(K\) with framing \(\text{pf}(K) + 1\). Then \(\omega\) extends to a symplectic form \(\omega_K\) on \(X_K\) and the new boundary \(Y_K\) is \(\omega_K\)–concave. The new (negative) contact structure \(\xi_K\) is supported by the natural open book on \(Y_K\) described above.

**Proof** This is Theorem 1.2 in [9]. However in that paper, which predates Giroux’s work on open book decompositions, the terminology is slightly different. Definition 2.4 of [9] defines what it means for a transverse link \(L\) in a contact 3–manifold \((M, \xi)\) to be “nicely fibered”. It is easy to see that if \(L\) is the binding of an open book supporting \(\xi\) then \(L\) is nicely fibered. (The notion of “nicely fibered” is more general because, in open book language, it allows for “pages” whose boundaries multiply cover the binding.) Theorem 1.2 in [9] then says that if we attach 2–handles to all the components of a nicely fibered link in the strongly convex boundary of a symplectic 4–manifold, with framings which are more positive than the framings coming from the fibration, then the symplectic form extends across the 2–handles to make the new boundary strongly concave. In our case we have a single component and we are attaching with framing exactly one more than the framing coming from the fibration. Finally, Addendum 5.1 of [9] characterizes the negative contact structure induced on the new boundary as follows: There exists a constant \(k\) such that \(\alpha_K = kd\pi - \alpha\) on
the complement of the surgery knots. (Here we are identifying $Y \setminus K$ with the complement in $Y_K$ of the belt sphere for $H$ in the obvious way.) The constant $k$ is simply the appropriate constant so that $\alpha_K$ extends to all of $Y_K$. Then $d\pi \wedge d\alpha_K = -d\pi \wedge d\alpha$ which is positive on $-Y_K$. Since $d\alpha_K = -d\alpha$, and the Reeb vector field for $\alpha$ is tangent to the level sets for the radial function $r$ on a neighborhood of $K$ (see Definition 2.4 in [9]), the Reeb vector field for $\alpha_K$ is necessarily tangent to the new binding of $Y_K$ and it is not hard to check that it points in the correct direction, so that $\alpha_K$ is supported by the natural open book on $Y_K$.

**Corollary 2.3** If the open book on $Y$ is planar (i.e. $\text{genus}(\Sigma) = 0$) then $(X, \omega)$ embeds in a closed symplectic 4–manifold $(Z, \eta)$ which contains a symplectic $(+1)$–sphere disjoint from $X$.

**Proof** Let the components of $B$ be $K_1, \ldots, K_n$. Attach 2–handles to $K_1, \ldots, K_{n-1}$ with framings $\text{pf}(K_i)$, as in Theorem 2.1. This gives $(X', \omega') \supset (X, \omega)$ with $\omega'$–convex boundary $(Y', \xi')$. Now attach a 2–handle to $K_n$ with framing $\text{pf}(K_n) + 1$ as in Theorem 2.2; the resulting concave end is $S^3$ with its negative contact structure supported by the standard disk open book, i.e. the contact structure is the standard negative tight contact structure. Thus we can fill in the concave end with the standard symplectic structure on $B^4$. Alternatively, we can note that, on $Y'$, the positive contact structure $\xi'$ is supported by an open book with page diffeomorphic to a disk. In other words, $Y'$ is diffeomorphic to $S^3$ and $\xi'$ is the standard positive tight contact structure on $S^3$. Thus we can remove a standard $(B^4, \omega_0)$ from $\mathbb{CP}^2$ with its standard Kähler form, and replace $(B^4, \omega_0)$ with $(X', \omega')$ to get $(Z, \eta)$. Since there is a symplectic $(+1)$–sphere in $\mathbb{CP}^2$ disjoint from $B^4$, we end up with a symplectic $(+1)$–sphere in $(Z, \eta)$ disjoint from $X'$, and hence disjoint from $X$.

By [14] the symplectic 4–manifold $Z$ found in the proof of Corollary 2.3 is diffeomorphic to a blowup of $\mathbb{CP}^2$. Let $Z'$ be the result of anti-blowing down the symplectic $(+1)$–sphere in $Z$ (i.e. $Z'$ is the union of the 4–manifold $X'$ in the proof of the corollary above with $B^4$). Then $Z'$ (still containing $X$) is diffeomorphic to the connected sum of a number of copies of $\overline{\mathbb{CP}^2}$. Let $D$ be the closure of $Z' \setminus X$ in $Z'$; we will call this the *dual configuration* (or *compactification*) for $X$. Thus we get embeddings of the intersection forms $H_2(X; \mathbb{Z})$ and $H_2(D; \mathbb{Z})$ into a negative definite diagonal lattice, and therefore both $H_2(X; \mathbb{Z})$ and $H_2(D; \mathbb{Z})$ are negative definite.

**Remark 2.4** A very similar compactification has been found by Némethi and Popescu-Pampu in [13], using rather different methods.

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3 Examples: rational surface singularities with reduced fundamental cycle

Suppose that $\Gamma$ is a plumbing tree of spheres which is negative definite, and at each vertex the absolute value of the framing is at least the number of edges emanating from the vertex. Every negative definite plumbing graph $\Gamma$ gives rise to a (not necessarily unique) surface singularity, and the further assumptions on $\Gamma$ ensure that the singularity has reduced fundamental cycle. According to Laufer’s algorithm, for example, this property implies that the singularity is rational, cf. [19, Section 3]. The Milnor fillable contact structure on such a 3–manifold is known to be compatible with a planar open book decomposition [7, 18]. A fairly explicit description of such an open book decomposition can be given by a construction resting on results of [10]: By [10] Proposition 5.3 the Milnor fillable contact structure is compatible with an open book decomposition resting on a toric construction (cf. [10, Section 4]), and therefore by [10] Proposition 4.2 a compatible planar open book can be explicitly given as follows.

View the tree $\Gamma$ as a planar graph in $\mathbb{R}^2$ and consider the boundary sphere of an $\epsilon$ neighborhood of it in $\mathbb{R}^3$. Suppose that $v$ is a vertex of $\Gamma$ with framing $e_v$ and valency $d_v$. Then near $v$ drill $-e_v - d_v \geq 0$ holes on the sphere. The resulting planar surface will be the page of the open book decomposition. Consider a parallel circle to each boundary component, and further curves near each edge, as shown by the example of Figure 3. The monodromy of the open book decomposition is simply the product of the right handed Dehn twists defined by all these curves on the planar surface.

![Figure 3: Light circles on the punctured sphere define the monodromy of the open book](image)

Consider now the Kirby diagram for $Y$ based on the open book decomposition as follows: regard the planar page as a multipunctured disk. (This step involves a choice of an 'outer circle'.) Every hole on the disk defines a 0–framed unknot.
linking the boundary of the hole, while the light circles defining the monodromy through right handed Dehn twists give rise to a \((-1)\)-framed unknots. In fact, the 0–framed unknots can be turned into dotted circles, and then viewed as 4–dimensional 1–handles. These will build up a Lefschetz fibration with fiber diffeomorphic to the page of the open book, and the addition of the \((-1)\)–framed circles correspond to the vanishing cycles of the Lefschetz fibration, giving the right monodromy.

Having this Kirby diagram for \(Y\), a relative handlebody diagram for the dual configuration \(D\) (built on \(-Y\)) can be easily deduced by performing 0–surgery along all the boundary circles except the outer one. This operation corresponds to capping off all but the last boundary component of the open book defining the Milnor fillable structure on \(Y\). Since after all the capping off we get an open book with a disk as a page, the 4–manifold \(D\) is a cobordism from \(-Y\) to \(S^3\).

It is usually more convenient to have an absolute handlebody than a relative one, and since the other boundary component of \(D\) is \(S^3\), by turning \(D\) upside down we can easily derive a handlebody description first for \(-D\) and then, after the reversal of the orientation, for \(D\). After appropriate handleslides, in fact, the diagram for \(D\) can be given by a simple algorithm. Since we only dualize 2–handles, \(D\) can be given by attaching 2–handles to \(D^4\). The framed link can be given by a braid, which is derived from the plumbing tree by the following inductive procedure. To start, we choose a vertex \(v\) where the strict inequality \(-e_v - d_v > 0\) holds. (Such a vertex always exists, for example, we can take a leaf.) We will choose the outer circle to be the boundary of one of the holes near \(v\). Now associate to every inner boundary component a string and to every light circle a box symbolizing a full negative twist of the strings passing through the box, which in our case comprise of those strings which correspond to the boundary components encircled by the light circle. The framing on a string is given by the negative of the ‘distance’ of the boundary component from the outer circle: this distance is simply the number of light circles we have to cross when traversing from the boundary component to the outer circle. Another (obviously equivalent) way of describing the same braid purely in terms of the graph \(\Gamma\) goes as follows: choose again a vertex \(v\) with \(-e_v - d_v > 0\), and consider \(-e_u - d_u\) strings for each vertex \(u\), except for \(v\) for which we take only \(-e_v - d_v - 1\) strings. Introduce a full negative twist on the resulting trivial braid (corresponding to the light circle parallel to the outer circle), and then introduce a further full negative twists for every edge \(e\) in the graph, where the strings affected by the negative twist can be characterized by the property that they correspond to vertices which are in a component of
\[ \Gamma - \{e\} \] not containing the distinguished vertex \( v \). Finally, equip every string corresponding to a vertex \( u \) by \( r_{uv} - 2 \) where \( r_{uv} \) is the negative of the minimal number of edges we traverse when passing from \( u \) to \( v \).

We will demonstrate this procedure through an explicit family of examples. To this end, suppose that the graph \( \Gamma_n \) is given by Figure 4. It is easy to see that

\[
\begin{align*}
-4 & -2 & -n - 1 & -3 & -3 \\
-4 & & -3 \\
-2 & & \\
\vdots & & n - 1 \\
-2 & & \\
\end{align*}
\]

Figure 4: An interesting family of plumbing graphs.

the graphs in the family for \( n \geq 1 \) are all negative definite, and for \( n \geq 2 \) define a rational singularity with reduced fundamental cycle. Assume that \( n \geq 3 \) and choose a boundary circle near the \((n - 1)\)-framed vertex to be the outer circle. The page of the planar open book, together with the light circles (giving rise to the monodromy through right handed Dehn twists) are pictured by Figure 5 (with the circle \( C_1 \) disregarded for a moment). The 0–framed unknots originating from the 1–handles of the Lefschetz fibration become unknots which each link one of the interior boundary components of the punctured disk once and the exterior boundary once. In the diagram, the unknot labelled \( C_1 \) is one of these unknots; we have not drawn the rest because they would only complicate the picture needlessly, but it is important to remember that there is one such unknot for each interior boundary. Putting \((n - 1)\)-framings to all light circles we get a convenient description of \( Y \). Now add framing 0 to all boundary components except the outer one. The result is a cobordism \( D \) from \(-Y\) to \( S^3 \). Mark all these circles (for example, use the convention of [11] by replacing all framing \( a \) with \( \langle a \rangle \)) and turn \( D \) upside down: add 0–framed meridians to the circles corresponding to the boundary components of the open book (these are the curves along which we ’capped off’ the open book). Now sliding and blowing down marked curves only, we end up with the diagram of \(-D\), and by reversing all crossings and multiplying all framings by \((n - 1)\) eventually we get a Kirby diagram for \( D \) as it is shown by Figure 6. (Every box in the diagram means a full negative twist.)
There are $n$ concentric light circles around the boundary component labelled by $K$ and there are $n - 3$ boundary circles on the right hand side of the disk. For each of the interior boundary components there should be a corresponding unknot $C_i$ linking it and the exterior boundary component; here we have only drawn $C_1$.

4 The nonexistence of rational homology disk smoothings

Next we will demonstrate how the explicit topological description of the dual $D$ can be applied to study smoothings of surface singularities. We start with a simple observation providing an obstruction for a 3–manifold to bound a rational homology disk, i.e. a 4–manifold $V$ with $H_*(V; \mathbb{Q}) = H_*(D^4; \mathbb{Q})$. 

Figure 5: The light circles on the disk define the monodromy of the open book. There are $n$ concentric light circles around the boundary component labelled by $K$ and there are $n - 3$ boundary circles on the right hand side of the disk. For each of the interior boundary components there should be a corresponding unknot $C_i$ linking it and the exterior boundary component; here we have only drawn $C_1$. 

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Theorem 4.1  Suppose that the 3–manifold $-Y$ is the boundary of a compact 4–manifold $D$ with the property that $\text{rk} H_2(D;\mathbb{Z}) = n$ and that the intersection form $(H_2(D;\mathbb{Z}), Q_D)$ does not embed into the negative definite diagonal lattice $n\langle -1 \rangle$ of the same rank. Then $Y$ cannot bound a rational homology disk.

Proof  Suppose that such a rational homology disk $V$ exists; then $Z = V \cup D$ is a closed, negative definite 4–manifold. By Donaldson’s Theorem [4] the intersection form of $Z$ is diagonalizable over $\mathbb{Z}$, and by our assumption on $V$ we get that $\text{rk} H_2(Z;\mathbb{Z}) = \text{rk} H_2(D;\mathbb{Z}) = n$. Since $H_2(D;\mathbb{Z}) \subset H_2(Z;\mathbb{Z})$ does not embed into $n\langle -1 \rangle$, we get a contradiction, implying the result.

Consider now the plumbing graph $\Gamma_n$ of Figure 4 and denote the corresponding 3–manifold by $Y_n$.

Proposition 4.2  The 3–manifold $Y_n$ does not bound a rational homology disk 4–manifold once $n \geq 7$.

Remark 4.3  Notice that elements of this family pass all the tests provided by [20] since these graphs are elements of the family $\mathcal{A}$ of [20]: change the framing.
of the single \((-4)\)-framed vertex with valency two to \((-1)\) and blow down the graph until it becomes the defining graph of the family \(A\). Also, using the algorithm described, e.g. in [19] it is easy to see that \(\det \Gamma_n \equiv n \pmod{2}\), hence for odd \(n\) the 3–manifold \(Y_n\) admits a unique spin structure. The corresponding Wu class can be given by the \((-3)\)-framed vertex of valency three, the unique \((-4)\)-framed vertex on the long chain and then every second \((-2)\)-framed vertex. A simple count then shows that for \(n\) odd we have that \(\pi(Y_n) = 0\), hence the result of [19] provides no obstruction for a rational homology disk smoothing. (For the terminology used in the above argument, see [19].)

**Proposition 4.4** The lattice determined by the intersection form of the dual \(D_n\) given by Figure 6 for \(n \geq 7\), does not embed into the same rank negative definite diagonal lattice.

**Proof** The labels on the components of the braid in Figure 6 will be used to represent the corresponding basis elements for the lattice determined by the intersection form of \(D_n\). The rank is \(n + 7\). Let \(E = \{e_1, \ldots, e_{n+7}\}\) be the standard basis for the negative definite diagonal lattice of rank \(n + 7\), so \(e_i \cdot e_j = -\delta_{ij}\). Suppose that the lattice for \(D_n\) does embed into the definite diagonal lattice. Then without loss of generality, since \(K_i \cdot K_i = -2\) and \(K_i \cdot K_j = -1\) otherwise, we may assume that \(K_i = e_1 + e_{10+i}\). Furthermore, without loss of generality we may assume that every other one of the basis elements \(A, B, \ldots, J\) is of the form \(e_1 + x\) where \(x\) is an expression in \(e_2, \ldots, e_{10}\). Thus each basis element whose square is \(-3\) (i.e. \(F\) and \(G\)) must be of the form \(e_1 \pm u \pm v\) where \(u\) and \(v\) are distinct elements of the set \(\{e_2, \ldots, e_{10}\}\).

Each element whose square is \(-4\) (i.e. \(A, B, C, D, H, I\) and \(J\)) must be of the form \(e_1 \pm q \pm r \pm s\) where \(q, r\) and \(s\) are distinct elements of the set \(\{e_2, \ldots, e_{10}\}\).

Now we can assume that \(F = e_1 + e_2 + e_3\) and \(G = e_1 + e_2 + e_4\) (noting that \(F \cdot G = -2\)). Then we note that none of the expressions for \(A, B, C, D, H, I\) or \(J\) can contain \(e_2, e_3\) or \(e_4\) for the following reason: For each of \(X = A, B, C, D, H, I, J\) there is another basis element \(Y\) from this set such that \(X \cdot Y = -3\) while \(X \cdot X = Y \cdot Y = -4\). Thus if we write \(X = e_1 + \alpha a + \beta b + \gamma c\) with \(a, b, c \in E\) and \(\alpha, \beta, \gamma \in \{-1, 1\}\), then \(Y\) must be \(Y = e_1 + \alpha a + \beta b + \delta d\), with \(d \in E\) and \(\delta \in \{-1, 1\}\), where \(a, b, c\) and \(d\) are distinct elements from the set \(\{e_2, \ldots, e_{10}\}\). Now noting that \(X \cdot F = X \cdot G = Y \cdot F = Y \cdot G = -1\), we see that if \(a = e_2\) then \(b, c, d\) must be in \(\{e_3, e_4\}\) which cannot happen because \(b, c\) and \(d\) must be distinct. Similarly \(b\) cannot be \(e_2\). If \(a = e_3\) then \(b\) or \(c\) must be \(e_2\), but we have just seen that it cannot be \(b\), so \(c = e_2\). But the same
argument also shows that \( d = e_2 \), but \( c \neq d \). Similarly we can rule out \( a = e_4 \) and also \( b = e_3 \) and \( b = e_4 \). But if one of \( c \) or \( d \) is in the set \( \{e_2, e_3, e_4\} \) then one of \( a \) or \( b \) must also be, so finally we see that none of them can be.

Thus we can now take \( H = e_1 + e_5 + e_6 + e_7 \). There are then two possibilities for \( I \) and \( J \) (up to relabelling the members of the sets \( \{e_8, e_9, e_{10}\} \) and \( \{e_5, e_6, e_7\} \)).

**Case I:** \( I = e_1 + e_5 + e_6 + e_8 \) and \( J = e_1 + e_5 + e_6 + e_9 \). In this case we can see that \( A, B, C \) and \( D \) cannot contain \( e_7, e_8 \) or \( e_9 \). So then the only remaining possibilities are all equivalent (after changing signs of basis elements in \( E \)) to \( A = e_1 + e_5 - e_6 + e_{10} \), but then we can not find any candidates for \( B \) which give \( A \cdot B = -3 \). This rules out Case I.

**Case II:** \( I = e_1 + e_5 + e_7 + e_8 \) and \( J = e_1 + e_5 + e_6 + e_8 \). To rule out this case, write \( A = e_1 + a a + \beta b + \gamma c \), \( a, b, c \in \{e_5, e_6, e_7, e_8, e_9, e_{10}\} \) and \( \alpha, \beta, \gamma \in \{-1, 1\} \). In order to have \( A \cdot H = -1 \), either 0 or 2 of \( a, b, c \) must be in the set \( \{e_5, e_6, e_7\} \), but not 1 or 3 of them. Similarly, using \( A \cdot I = -1 \), either 0 or 2 must be in \( \{e_5, e_7, e_8\} \), and using \( A \cdot J = -1 \), either 0 or 2 must be in \( \{e_5, e_6, e_8\} \). If it is 0 in one of these cases it must be 0 for all three, but that leaves only \( e_9 \) and \( e_{10} \) for \( a, b \) and \( c \), an impossibility. Thus it is 2 in each case. We cannot have one of them to be \( e_5 \), because then we could not have exactly 2 from all three sets. So we must have \( a = e_6 \), \( b = e_7 \), \( c = e_8 \). But exactly the same argument holds for \( B \), and we can never get \( A \cdot B = -3 \).

Thus Case II is ruled out, concluding the proof of the proposition. \( \square \)

**Proof of Proposition 4.2** Combine Theorem 4.1 and Proposition 4.4.

**Corollary 4.5** Suppose that \((S_\Gamma, 0)\) is an isolated surface singularity with resolution graph given by Figure 4. If \( n \geq 7 \), then \((S_\Gamma, 0)\) admits no rational homology disk smoothing, i.e., it has no smoothing \( V \) with \( H_*(V; \mathbb{Z}) = H_*(D^4; \mathbb{Z}) \).

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