Linearizability of Non-expansive Semigroup Actions on Metric Spaces

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Abstract
We show that a non-expansive action of a topological semigroup $S$ on a
metric space $X$ is linearizable iff its orbits are bounded. The crucial point
here is to prove that $X$ can be extended by adding a fixed point of $S$, thus
allowing application of a semigroup version of the Arens-Eells linearization,
iff the orbits of $S$ in $X$ are bounded.

Introduction
By a well-known construction due to Arens and Eells [1], every metric space
can be isometrically embedded as a closed metric subspace of a normed (linear) space.
Using this construction (or other linear extensions like the free Banach space), one
can show [2, 3] that a non-expansive action $\pi$ of a topological semigroup $S$ on
a metric space is linearizable, i.e. arises by restricting an action of $S$ by linear
contractions on a normed space $V$ to a metric subspace of $V$, if $\pi$ has a fixed point
$z$ (which then serves as the 0 of $V$). The question of when an action $\pi$ is linearizable
in general thus reduces to the question of when $\pi$ can be extended by adding a fixed
point.

It is trivial to observe that if $X$ is bounded, then $\pi$ may be extended by adding a
fixed point: introduce a new point $z$, make $z$ a fixed point of $S$, and put $d(z, x) = c$
for all $x \in X$, where $c > \text{diam}(X)$. It is then easy to check that the distance
function $d$ thus defined on $X \cup \{z\}$ is a metric, and that the action of $S$ on the
extended space is non-expansive. Here, we improve on this construction by giving
a necessary and sufficient criterion: $\pi$ may be extended by adding a fixed point iff
its orbits are bounded sets. We thus obtain an exact linearizability criterion: $\pi$ is
linearizable iff its orbits are bounded.

1 Preliminaries
Throughout the exposition, fix a topological semigroup $S$. We shall generally be
concerned with non-expansive actions $\pi : S \times X \to X$, with $\pi(s, x)$ denoted as $s \cdot x$,
of $S$ on metric spaces $(X, d)$, i.e. $d(s \cdot x, s \cdot y) \leq d(x, y)$ for all $s \in S$ and all $x, y \in X$.
In the special case that $(X, d)$ is a real normed space $V$, we say that $\pi$ is linear if the
translation maps $s : x \mapsto s \cdot x$ are linear maps on $V$. In this case, non-expansivity
of $\pi$ means that the $s$ are contracting, i.e. $\|s \cdot x\| \leq \|x\|$ for all $x$. We say that a
map $f : X \to Y$ is equivariant w.r.t. an action of $S$ on $Y$ if $f(s \cdot x) = s \cdot f(x)$ for all
$x \in X$.

We note an observation from [3], omitting the (straightforward) proof:
Lemma 1. For a non-expansive action \( \pi : S \times X \to X \) on a metric space \((X, d)\), the following are equivalent.

1. The action \( \pi \) is jointly continuous.
2. The action \( \pi \) is separately continuous.
3. The restriction \( \pi : S \times Y \to X \) to some dense subspace \( Y \) of \( X \) is separately continuous.

We shall henceforth implicitly include the requirement that \( S \times X \to X \) is continuous in the term 'non-expansive action' (thus avoiding the term 'non-expansive continuous action', which is a bit of a mouthful). As an immediate consequence of the preceding lemma, we obtain the following extension result [3]:

Lemma 2. A linear non-expansive action of \( S \) on a normed space \( V \) extends (uniquely) to a linear non-expansive action of \( S \) on the completion of \( V \).

We denote the orbit \( \{ s \cdot x \mid s \in S \} \) of \( x \in X \) under \( S \) by \( S \cdot x \). Note that orbits need not be disjoint, elements of an orbit need not have the same orbit, and \( x \) need not be contained in its orbit \( S \cdot x \). In case \( S \) is a monoid, however, \( x \in S \cdot x \) for all \( x \in X \).

2 Fixed Points and Linearizations

We now give the announced criterion for extendibility by a fixed point:

Theorem 3. Let \((X, d)\) be a metric space equipped with a non-expansive action of \( S \). Then the following are equivalent:

1. The space \( X \) can be extended by adding a fixed point of \( S \), i.e. there exists a metric space \((Y, d)\) equipped with a non-expansive action of \( S \) that has a fixed point, and an isometric and equivariant embedding of \( X \) into \( Y \).
2. The orbits \( S \cdot x \) of \( S \) in \( X \) are bounded sets.

The following definition will be useful in the proof:

Definition 4. Let \((X, d)\) be a metric space. For \( A \subseteq X \) and \( x \in X \), we put

\[
\supdist(x, A) = \sup_{y \in A} d(x, y) \in [0, \infty].
\]

Proof. (Theorem 3) (1) \(\iff\) (2): We can assume that \( X \) is a subspace of \( Y \). Let \( z \in Y \) be a fixed point of \( S \). Then we have, for \( x \in X \) and \( s, t \in S \),

\[
d(s \cdot x, t \cdot x) \leq d(s \cdot x, z) + d(z, t \cdot x) = d(s \cdot x, s \cdot z) + d(t \cdot z, t \cdot x) \leq 2d(x, z),
\]

i.e. \( \text{diam}(S \cdot x) \leq 2d(x, z) \).

(2) \(\iff\) (1): To begin, we reduce to the case that \( S \) is a monoid, as follows. For a semigroup \( S \), we have the free monoid \( S_\epsilon \) over \( S \), constructed by taking \( S_\epsilon = S \cup \{ e \} \), where \( e \notin S \), and putting \( eS = se = s \) for all \( s \in S_\epsilon \). The action of \( S \) on \( X \) is extended to a (non-expansive) action of \( S_\epsilon \) by \( e \cdot x = x \) for all \( x \in X \). The orbits \( S_\epsilon \cdot x = \{ x \} \cup S \cdot x \) are bounded (by \( d(x, s \cdot x) + \text{diam}(S \cdot x) \) for any \( s \in S \)). By the monoid case of the theorem, we obtain an extended space \((Y, d)\) in which \( S_\epsilon \)
has a fixed point $z$; the action of $S_z$ restricts to an action of $S$ on $Y$, and $z$ trivially remains a fixed point of $S$.

When $S$ is a monoid, then $x \in S \cdot x$ for all $x \in X$. We can assume w.l.o.g. that there exists a point $x_0 \in X$ which is not fixed under $S$. We put $Y = X \cup \{z\}$, where $z \notin X$,

$$d(z, x) = d(x, z) = \supdist(x_0, S \cdot x)$$

for $x \in X$, and $d(z, z) = 0$. We have to check that this makes $(Y, d)$ a metric space. To begin, $d(x, z) > 0$ for $x \in X$: we have $\supdist(x_0, S \cdot x_0) > 0$ because $x_0$ is not fixed under $S$, and for $x \neq x_0$, $\supdist(x_0, S \cdot x) \geq d(x_0, x) > 0$ (using $x \in S \cdot x$). Symmetry holds by construction. Moreover, for $x \in X$, $d(x_0, s \cdot x) \leq d(x_0, x) + d(x, s \cdot x) \leq d(x_0, x) + \text{diam}(S \cdot x)$ for all $s \in S$ (again using $x \in S \cdot x$) and hence $d(x, z) \leq d(x_0, x) + \text{diam}(S \cdot x) < \infty$ by (2). It remains to prove the triangle inequality. There are only two non-trivial cases to prove:

(a) $d(x, z) \leq d(x, y) + d(y, z)$ for $x, y \in X$, and
(b) $d(x, y) \leq d(x, z) + d(y, z)$ for $x, y \in X$.

*Ad (a)*: Let $s \in S$. Then $d(x_0, s \cdot x) \leq d(x_0, s \cdot y) + d(s \cdot y, s \cdot x) \leq d(x_0, s \cdot y) + d(y, x)$. Thus, $\supdist(x_0, S \cdot x) \leq d(x, y) + \supdist(x_0, S \cdot y)$.

*Ad (b)*: We have

$$d(x, y) \leq d(x, x_0) + d(y, x_0) \leq \supdist(S \cdot x, x_0) + \supdist(S \cdot y, x_0) = d(x, z) + d(y, z),$$

where the second inequality uses $x \in S \cdot x$.

We then extend the action of $S$ to $Y$ by letting $z$ be fixed under $S$. It is clear that this really defines an action of $S$; we have to check that this action is non-expansive. For $x \in X$ and $s \in S$, we have

$$d(s \cdot x, s \cdot z) = d(s \cdot x, z) = \supdist(x_0, S \cdot (s \cdot x)) \leq \supdist(x_0, S \cdot x) = d(x, z),$$

where the inequality uses $S \cdot (s \cdot x) \subseteq S \cdot x$.

It remains to prove that $S \times Y \to Y$ is continuous, i.e. by Lemma 4 that the orbit maps $S \to Y, s \mapsto s \cdot y$, are continuous. For $y \in X$, this follows from continuity of the action on $X$, and for $y = z$, the orbit map is constant.

**Remark 5.** In case $S$ is a group, one can identify the space $Y$ constructed in the above proof with the subspace $\{\{x\} \mid x \in X\} \cup \{S \cdot x_0\}$ of the space of bounded subsets of $X$, equipped with the Hausdorff pseudometric

$$d(A, B) := \max\{\sup\{d(a, B) : a \in A\}, \sup\{d(A, b) : b \in B\}\}$$

and the natural action taking $A$ to $s \cdot A$ for $s \in S$. For arbitrary semigroups $S$, however, orbits will in general fail to be fixed points under the natural action.

**Remark 6.** Analogously as in the proof of (1) $\implies$ (2) in the above theorem, one shows that for $x, y \in X$, $\text{diam}(S \cdot y) \leq 2d(x, y) + \text{diam}(S \cdot x)$. Thus, for boundedness of all orbits it suffices to require that there exists a bounded orbit.
We now briefly recall the Arens-Eells extension of a pointed metric space \((X, d, z)\) (i.e. \(z \in X\)). One constructs a real normed space \((A(X), \|\cdot\|)\) by taking as the elements of \(A(X)\) the formal linear combinations

\[
\sum_{i=1}^{n} c_i (x_i - y_i),
\]

with \(x_i, y_i \in X\) and \(c_i \in \mathbb{R}\) and putting for \(u \in A(X)\)

\[
\|u\| = \inf \left\{ \sum_{i=1}^{n} |c_i| d(x_i, y_i) \mid u = \sum_{i=1}^{n} c_i (x_i - y_i) \right\}.
\]

The space \((X, d)\) is isometrically embedded into \(A(X)\) (as a closed subspace) by taking \(x \in X\) to \(x - z\). It is shown in [3] (Proposition 2.10) that a non-expansive action of \(S\) on \(X\) can be extended to a linear non-expansive action of \(S\) on \(A(X)\) by putting

\[
s \cdot \sum_{i=1}^{n} c_i (x_i - y_i) = \sum_{i=1}^{n} c_i (s \cdot x_i - s \cdot y_i).
\]

(A similar construction can be found already in [4]; moreover, the Arens-Eells extension may be replaced by other linear extensions [2], e.g. the free Banach space over \(X\) as in [4].)

We then immediately obtain the announced exact linearizability criterion.

**Theorem 7.** For a non-expansive action of \(S\) on a metric space \((X, d)\), the following are equivalent:

1. There exists a Banach space \(V\), equipped with a linear non-expansive action of \(S\), and an equivariant isometric embedding of \((X, d)\) into \(V\).

2. The orbits \(S \cdot x\) of \(S\) in \(X\) are bounded sets.

**Proof.** \((1) \implies (2):\) By the corresponding direction of Theorem 3 as 0 is a fixed point of \(S\) in \(V\).

\((2) \implies (1):\) By Theorem 3 we may assume that \(S\) has a fixed point \(z\) in \(X\). By Lemma 2 it suffices to construct \(V\) as a normed space. We thus may take \(V\) as the Arens-Eells extension of \((X, d, z)\), equipped with the \(S\)-action described above.

**References**

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