Metric derived from Lie Groups

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Abstract

We give the expression of the metric derived from Lie groups. For the metric derived from classical Lie groups such as the unitary group, the orthogonal group and the symplectic group, we conjecture that the metric becomes the Einstein metric.

1 Introduction

In the theory of general relativity, the symmetry of the metric is quite important to classify the space-time. If we start from the metric itself, it is not easy to to find the symmetry of the metric. Then we take another approach, where we can find the symmetry of the metric in a trivial way. We start from a certain Lie group \([1, 2, 3, 4, 5]\) and derived the metric from this Lie group. Then the symmetry of the metric is that of the Lie group.

The metric derived from Lie group has the quite simple structure. The metric on \(n\)-sphere \(S^n\) is known to have the Einstein metric \(R_{\mu\nu} = 2\Lambda g_{\mu\nu}\) [6]. The symmetry of \(S^n\) is the subset of the symmetry of \(SO(n)\), then we expect that the metric derived from \(SO(n)\) becomes the Einstein metric. In this paper, we derived the metric from \(SU(2)\) Lie group in two different ways. We then confirm that the metric derived from \(SU(2)\) becomes the Einstein metric. For classical groups such as \(SO(n)\), \(SU(n)\), \(Sp(n)\), the metric derived from the Lie group is connected with the metric on the manifold which is the quadratic invariant form to define Lie groups. From this observation, we conjecture that the metric derived from the classical groups becomes the Einstein metric.

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2 Metric derived from Lie groups

In this section, we first give the general method to derive the metric from Lie groups. Next we explicitly give the expression of the metric derived from the $SU(2)$ Lie group in two different ways.

2.1 General method to derive the metric from Lie groups

We denote $U(\theta_1, \theta_2, \cdots)$ as the representation of any Lie group, which is parametrized by $\{\theta_1, \theta_2, \cdots\}$. The global Lie group transformation is given by

$$U' = VU, \quad (1)$$

where $V$ is the global Lie group element. In order to construct the metric, we define the following line element

$$(ds)^2 = k \text{Tr} (U^{-1}dUU^{-1}dU). \quad (2)$$

This line element $(ds)^2$ is the desired line element for the Lie groups. The right-hand side of Eq.(2) is invariant under the transformation Eq.(1), because we have

$$U^{-1}dU \rightarrow U^{-1}V^{-1}VdU = U^{-1}dU. \quad (3)$$

From the expression of Eq.(2), the above line element $(ds)^2$ becomes real number. Then the above line element satisfy the physically meaningful conditions, that is, $i$) it is the quadratic form of the infinitesimal quantity, $ii$) it is real, $iii$) it is invariant under the global transformation of Lie groups.

2.2 Metric derived from $SU(2)$ (I)

Here we derive the metric from $SU(2)$ Lie group. For that purpose, we first give the expression of the $SU(2)$ group element in the form

$$U = \exp \left( \frac{i}{2} \vec{\sigma} \cdot \vec{\theta} \right) = \cos \frac{|\theta|}{2} + i \vec{\sigma} \cdot \vec{\theta} \sin \frac{|\theta|}{2}. \quad (4)$$

The explicit form of the group element of $SU(3)$ is given in [7, 8]. The infinitesimal form of the group element is given by

$$dU = \left( -\frac{1}{2} \sin \frac{|\theta|}{2} + \frac{1}{2} \vec{\sigma} \cdot \vec{\theta} \cos \frac{|\theta|}{2} \right) d|\theta|.$$
\[ +i \frac{\sin(|\theta|/2)}{|\theta|} \tilde{\sigma} \cdot \theta \, d\theta - i \frac{\tilde{\sigma} \cdot \theta}{|\theta|^2} \sin \frac{|\theta|}{2} \, d\theta. \]  

(5)

The expression of \( U^{-1} \) is given by

\[ U^{-1} = \cos \frac{|\theta|}{2} - i \frac{\tilde{\sigma} \cdot \theta}{|\theta|} \sin \frac{|\theta|}{2}. \]  

(6)

and we multiply this inverse factor from the left in Eq.(5). Then we obtain

\[ U^{-1}dU = \frac{\tilde{\sigma} \cdot \theta}{|\theta|} \left( \frac{1}{2|\theta|} - \frac{\sin(|\theta|/2) \cos(|\theta|/2)}{|\theta|^2} \right) \, d|\theta| \]

\[ +i \frac{\sin(|\theta|/2) \cos(|\theta|/2)}{|\theta|} \sum_{a=1}^3 \sigma_a \, d\theta_a + i \frac{\sin^2(|\theta|/2)}{|\theta|^2} \sum_{a,b,c=1}^3 \epsilon_{abc} \sigma_a \, d\theta_a \sigma_c. \]  

(7)

After a straightforward calculation, we have

\[ (ds)^2 = k \text{Tr} \left( U^{-1}dUU^{-1}dU \right) \]

\[ = \frac{1}{|\theta|^2} \left( \sum_a \theta_a d\theta_a \right)^2 + \frac{4 \sin^2(|\theta|/2) \cos^2(|\theta|/2)}{|\theta|^4} \left( \sum_a \theta_a \sum_b (d\theta_b)^2 - \left( \sum_a \theta_a d\theta_a \right)^2 \right) \]

\[ + \frac{4 \sin^4(|\theta|/2)}{|\theta|^4} \left\{ (\theta_1 d\theta_2 - \theta_2 d\theta_1)^2 + (\theta_2 d\theta_3 - \theta_3 d\theta_2)^2 + (\theta_3 d\theta_1 - \theta_1 d\theta_3)^2 \right\}. \]  

(8)

We take \( k = 2 \) and we have the line element in the form

\[ (ds)^2 = 2 \text{Tr} \left( U^{-1}dUU^{-1}dU \right) = \sum_{a,b} g_{ab} d\theta^a d\theta^b. \]  

(9)

Then we have the expression of the metric in the form

\[ g_{ab} = \frac{\theta_a \theta_b}{|\theta|^2} + \frac{4 \sin^2(|\theta|/2)}{|\theta|^4} \left( |\theta|^2 \delta_{ab} - \theta_a \theta_b \right). \]  

(10)

We put \( \theta^a = \theta_a \), which is consistent with the above metric \( g_{ab} \). The inverse metric \( g^{ab} \), which satisfy \( \sum_b g_{ab} g^{bc} = \delta^c_a \), is given by

\[ g^{ab} = \frac{\theta^a \theta^b}{|\theta|^2} + \frac{1}{4 \sin^2(|\theta|/2)} \left( |\theta|^2 \delta^{ab} - \theta^a \theta^b \right). \]  

(11)
This metric $g_{ab}$ satisfy the Einstein equation with the cosmological constant in the form

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 0,$$

with $\Lambda = 1/4$ in this case. This is the Einstein metric, because it satisfies the equation

$$R_{ab} = 2\Lambda g_{ab}.$$  \hfill (13)

### 2.3 Metric derived from $SU(2)(II)$: Euler angle parametrization

We parametrize the $SU(2)$ group element with the Euler angle in the form

$$U = U_z(\phi)U_x(\theta)U_z(\psi)$$

$$= \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix} \begin{pmatrix} \cos(\theta/2) & i \sin(\theta/2) \\ i \sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \begin{pmatrix} e^{i\psi/2} & 0 \\ 0 & e^{-i\psi/2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{i(\phi+\psi)/2} \cos(\theta/2) & ie^{i(\phi-\psi)/2} \sin(\theta/2) \\ ie^{-i(\phi-\psi)/2} \sin(\theta/2) & e^{-i(\phi+\psi)/2} \cos(\theta/2) \end{pmatrix}.  \hfill (14)$$

From this expression, we have

$$U^{-1}dU = \begin{pmatrix} \frac{i}{2} \cos(\theta)d\phi + \frac{i}{2}d\psi \\ \frac{i}{2}e^{i\psi}d\theta + \frac{i}{2}e^{i\psi} \sin(\theta)d\phi \\ \frac{1}{2}e^{-i\psi}d\theta - \frac{1}{2}e^{-i\psi} \sin(\theta)d\phi \end{pmatrix}.  \hfill (15)$$

Then we define the line element in the form

$$(ds)^2 = 2\text{Tr} \left(U^{-1}dU U^{-1}dU\right)$$

$$= (\cos(\theta)d\phi + d\psi)^2 + (d\theta)^2 + \sin^2(\theta)(d\phi)^2$$

$$= (d\theta)^2 + (d\psi)^2 + (d\phi)^2 + 2\cos(\theta)d\psi d\phi.  \hfill (16)$$

This is the standard expression of the line element for $SU(2)$ \cite{9}. If we take $\theta^1 = \theta$, $\theta^2 = \phi$, $\theta^3 = \psi$, non-zero elements of the metric are given by

$$g_{11} = 1, \quad g_{22} = 1, \quad g_{33} = 1, \quad g_{23} = g_{32} = \cos(\theta^1),$$

$$g^{11} = 1, \quad g^{22} = \frac{1}{\sin^2(\theta^1)}, \quad g^{33} = \frac{1}{\sin^2(\theta^1)}, \quad g^{23} = g^{32} = -\frac{\cos(\theta^1)}{\sin^2(\theta^1)}.  \hfill (19)$$

In this metric, we can easily see that this line element $(ds)^2$ is invariant under

$$i) \quad \phi \to \phi' = \phi + \xi, \quad \theta \to \theta' = \theta, \quad \psi \to \psi' = \psi,$$

$$ii) \quad \psi \to \psi' = \psi + \xi, \quad \theta \to \theta' = \theta, \quad \phi \to \phi' = \phi.$$  \hfill (21)
with global parameter $\xi$. The third transformation, which makes $(ds)^2$ invariant, is quite complicated.

The above expression of the metric $g_{ab}$ also becomes the Einstein metric and it satisfies Eq.(13).

### 3 Metric derived from the quadratic invariant manifold for the classical Lie groups

Classical Lie groups such as $SU(N)$, $SO(N)$, $Sp(N)$, are defined in such a way as the set of transformations to make the special quadratic form invariant. For example, $SO(N)$ Lie group is defined in such a way as

$$\sum_{i=1}^{N} x_i^2 = \text{(invariant)},$$

under the transformation

$$\begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_N \end{pmatrix} = U \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}.$$ 

Then the metric derived on this quadratic invariant manifold have the relation with the metric derived from the classical Lie groups. The symmetry of the metric derived from the quadratic invariant manifold is the subset of the symmetry of the metric derived from the associated classical Lie groups. For $SO(N)$ case, we parametrize the above quadratic invariant form with $N - 1$ angle variables in the form $x_i = x_i(\theta^1, \theta^2, \ldots, \theta^{N-1})$. We define the vector $B_{i,a}$ in the form

$$B_{i,a} = \frac{\partial x_i}{\partial \theta^a}.$$ 

Then the metric on the quadratic invariant manifold is given by

$$g_{ab} = \sum_{i=1}^{N} B_{i,a} B_{i,b}.$$ 

For the $SO(3)$ case, we parametrize the invariant quadratic manifold

$$x_1^2 + x_2^2 + x_3^2 = 1,$$
in the form

\[ x_1 = \sin \theta \cos \phi, \quad x_2 = \sin \theta \sin \phi, \quad x_3 = \cos \theta. \] (27)

Then the vector \( \vec{B}_a \) given by

\[ \vec{B}_1 = \frac{\partial x_i}{\partial \theta} = (\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta), \] (28)

\[ \vec{B}_2 = \frac{\partial x_i}{\partial \phi} = (-\sin \theta \sin \phi, \sin \theta \cos \phi, 0). \] (29)

Then the metric is given by

\[ g_{11} = \langle \vec{B}_1, \vec{B}_1 \rangle = 1, \quad g_{22} = \langle \vec{B}_2, \vec{B}_2 \rangle = \sin^2 \theta, \quad g_{12} = g_{21} = \langle \vec{B}_1, \vec{B}_2 \rangle = 0. \] (30)

The invariant length is given by

\[ (ds)^2 = (d\theta)^2 + \sin^2 \theta (d\phi)^2. \] (31)

If we compare Eq.(31) with Eq.(16), we can see that the symmetry of the line element Eq.(31) is the subset of the symmetry of the line element of Eq.(16).

4 Summary

We give general form of the expression of the metric derived from Lie groups. Then we give the explicit expression of the metric derived from \( SU(2) \) Lie group in two different ways. We compare the metric on the manifold \( S^2 \) with the metric derived from \( SU(2) \) Lie group. From that observation, we conjecture that the metric becomes the Einstein metric \( R_{\mu\nu} = 2\Lambda g_{\mu\nu} \) if the metric is derived from classical Lie groups \( SO(n), SU(n), Sp(n) \).

References

[1] N. Jacobson, *Lie Algebras* (Dover, New York, 1979).
[2] M. Hamermesh, *Group Theory and its Application to Physical Problems* (Addison-Wesley, Reading, 1962).

[3] H. Georgi, *Lie Algebra in Particle Physics* (Benjamin/Cummings, Reading, Mass., 1982).

[4] R. Gilmore, *Lie Groups, Lie Algebras, and Some of their Applications* (John Wiley & Sons, New York, 1974).

[5] R.N. Cahn, *Semi-simple Lie Algebras and their Representations* (Benjamin/Cummings, Reading, Mass., 1982).

[6] A.L. Besse, *Einstein manifolds* (Springe-Verlag, Berlin, 1987).

[7] M. Byrd, arXiv: physics/9708015.

[8] M. Byrd and E.C.G. Sudarshan, J.Phys. A 31 (1998), 9255.

[9] S.S. Gubser, Phys. Rev. D 59 (1999), 025006.