Over thirty years ago, S. Coleman and E. Weinberg \cite{1} demonstrated how spontaneous symmetry breaking may occur through radiative corrections to a conformally-invariant Lagrangian in which no quadratic mass term appears. Such symmetry breaking, in which the scalar-field vacuum expectation value $\langle \phi \rangle$ is the only source of scale, is of particular relevance for the spontaneous breakdown of $SU(2) \times U(1)$ electroweak symmetry, which necessarily requires a mechanism within any embedding theory to keep any such quadratic mass term minimally contaminated by the unification mass scale — the absence of such a mass term implies that this mechanism is exact \cite{2}. We emphasize that such a mechanism, whether exact or nearly so, is a necessary component of the Standard Model, though the nature of this mechanism (possibly conformal invariance) remains unknown. In the absence of an explicit scalar-field mass term (i.e. the “exact mechanism”), the one-loop (1L) effective potential for $SU(2) \times U(1)$ gauge theory is given by \cite{1}

$$V^{(1L)}_{\text{eff}} = \frac{\lambda \phi^4}{4} + \phi^4 \left[ \frac{12\lambda^2 - 3g_\pi^2}{64\pi^2} + \frac{3(3g_2^2 + 2g_\pi^2 + g_\tau^4)}{1024\pi^2} \right] \left( \log \frac{\phi^2}{\mu^2} - \frac{25}{6} \right). \quad (1)$$

There are four distinct coupling constants appearing in Eq. (1), the $SU(2)$ coupling constant $g_2$, the $U(1)$ coupling constant $g_\pi$, the $t$-quark Yukawa coupling constant $g_\tau$, and the quartic scalar-field self-interaction coupling constant $\lambda$. The radiative symmetry breaking scenario of Ref. \cite{1}, which preceded the discovery of the massive top quark, led to a value of $\lambda$ proportional to $g_\pi^2$ and a scalar field mass of order $10\text{ GeV}$. The presence of a large Yukawa coupling [$g_\tau^2 \approx 1.0 >> g_2^2, g_\pi^2$] spoils this scenario; the $O(g_\pi^2)$ value of $\lambda$ required for radiative symmetry breaking would be so large that subsequent leading logarithm terms [e.g. $\lambda^3 \phi^4 \log^2(\phi^2/\mu^2)$] would be too large to neglect.

In the present work, we explicitly sum all such leading logarithm terms within the full perturbative series for the effective potential $\cite{3}$ to examine the viability of radiative electroweak symmetry breaking. We find the potential is minimized for a Higgs mass of $216\text{ GeV}$, and observe some evidence that this value may be stable after including contributions from subsequent-to-leading logarithms.

If we denote the dominant Standard Model couplings as $x \equiv g_\pi^2/4\pi^2$, $y \equiv \lambda/\lambda specialty, and $z \equiv g_\tau^2/4\pi^2$ [QCD contributes to leading logarithms past one-loop order], which are much larger than corresponding couplings for $g_2, g_\pi$ and non-t-quark Yukawa couplings, this series is of the form $V_{\text{eff}} = \pi^2 \phi^4 \sum C_{n,k,\ell} x^n y^k z^\ell L^\ell$, where $L(\mu) = \log(\phi^2/\mu^2)$. The leading logarithms in this series are those terms one degree lower in the power of the logarithm $L$ than in the aggregate power of the couplings $\{x, y, z\}$.

$$V_{LL} = \pi^2 \phi^4 S_{LL} = \pi^2 \phi^4 \left\{ \sum_{n=0}^{\infty} x^n \sum_{k=0}^{\infty} y^k \sum_{\ell=0}^{k-1} z^\ell C_{n,k,\ell} L^{n+k+\ell-1} \right\}, \quad (2)$$

The series $S_{LL}$ is determined entirely by one-loop contributions to the renormalization-group (RG) equation, i.e., by those contributions that either lower the power of $L$ by one or raise the aggregate power of the couplings by one $\cite{2, 3}$:

$$\left[ -2 \frac{\partial}{\partial L} + \left( \frac{9}{4} x^2 - 4xz \right) \frac{\partial}{\partial x} + \left( 6y^2 + 3y^x + \frac{3}{2} x^2 \right) \frac{\partial}{\partial y} - \frac{7}{2} z^2 \frac{\partial}{\partial z} - 3x \right] S_{LL}(x, y, z, L) = 0. \quad (3)$$

In eq. (3), the coefficients of $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ are respectively the one-loop beta-functions for $x, y, z$ (where $\beta_x = \mu \frac{dx}{d\mu}$); the final term in eq. (3) is four times the one-loop scalar field anomalous dimension. For example, the leading coefficients...
$C_{0,1,0} = 1$, $C_{1,0,0} = C_{0,0,1} = 0$, follow from the $\lambda\phi^4/4$ tree-order potential. Upon substitution of Eq. (2) into Eq. (3), one easily sees that $C_{0,2,0} = 3$, $C_{2,0,0} = -3/4$, and that the remaining four degree-2 coefficients $C_{i,j,z-i-j}$ equal zero, leading to a recovery of the $\{\lambda^2, \phi^2\}$-contributions to the potential (1).

We find it convenient to express the series (2) in the form

$$S_{LL} = yF_0(w, \zeta) + \sum_{n=1}^{\infty} x^n L^{n-1} F_n(w, \zeta)$$

(4)

where $w \equiv 1 - 3yL$ and $\zeta \equiv zL$, and where

$$F_n(w, \zeta) = \sum_{\ell=0}^{\infty} C_{n,\ell,k} \left( \frac{1 - w}{3} \right)^\ell \zeta^k = \sum_{k=0}^{n+1} f_{n,k}(\zeta) \left[ \frac{w - 1}{w} \right]^k.$$  

(5)

By using Eq. (9) to obtain sequential partial differential equations relating $F_0 = 1/w$ to $F_1(w, \zeta)$ and $F_2(w, \zeta)$, we are able to solve explicitly for these quantities. For $p \geq 3$ one can show from Eq. (9) that

$$0 = \left[ \left( 7\zeta^2/2 \right) \frac{d}{d\zeta} + 4p\zeta \right] + \left[ 2k \frac{d}{d\zeta} + 2(p - 1) + 2k \right] f_{p,k}(\zeta)$$

$$- \left[ (9p - 21)/4 + 3k \right] f_{p-1,k}(\zeta) + 3( k - 1) f_{p-1, k-1}(\zeta)$$

$$- [9(k - 1)/2] f_{p-2,k-1}(\zeta) + 9k f_{p-2,k}(\zeta) - [9(k + 1)/2] f_{p-2,k+1}(\zeta),$$

(6)

where $f_{p,k} \equiv 0$ when $k < 0$ or $k > p + 1$, and where $f_{p,k}(0)$ is finite.

We now examine possible radiative spontaneous symmetry breaking for the RG-improved effective potential $V_{eff}(\phi) = \pi^2\phi^4(S_{LL} + K)$, where $K$ is a finite $\phi^4$ counterterm coefficient. As in Ref. [1], we choose $\mu = \langle \phi \rangle = v [L = \log (\phi^2(v)/v^2)]$, in which case $V'_{eff}(v) = 0$. The counterterm $K$ facilitates the fourth-derivative renormalization condition $V''_{eff}(v) = V''_{tree}(v) = 24\pi^2y = (6\lambda)$. For any one-loop effective potential of the form $V_{eff}^{(LL)}(\phi) = \pi^2\phi^4[y + \sigma L + K]$, this fourth-derivative condition ensures that $K = -25\sigma/6$, as in Eq. (10). For our RG-improved potential, we find it convenient to expand $S_{LL}$ in powers of the logarithm $L$:

$$S_{LL} = y + B \log \left( \frac{\phi^2}{v^2} \right) + C \log^2 \left( \frac{\phi^2}{v^2} \right) + D \log^3 \left( \frac{\phi^2}{v^2} \right) + E \log^4 \left( \frac{\phi^2}{v^2} \right) + \ldots,$$

(7)

We then obtain from Eqs. (9) and (10) an exact determination of terms up to $O(L^4)$ within Eq. (10):

$$B = 3y^2 - \frac{3}{4} x^2,$$

(8)

$$C = 9y^3 + \frac{9}{4} xy^2 - \frac{9}{4} x^2y + \frac{3}{2} x^2z - \frac{9}{32} x^3,$$

(9)

$$D = 27y^4 + \frac{27}{2} xy^3 - \frac{3}{2} x^2y^2 + 3x^2yz - \frac{225}{32} x^2y^2 - \frac{23}{8} x^2z^2 + \frac{15}{16} x^3z - \frac{45}{16} x^3y + \frac{99}{256} x^4,$$

(10)

$$E = 81y^5 + \frac{243}{4} xy^4 - 9xy^3z + \frac{45}{16} x^2y^2 - \frac{69}{16} x^2y^2 - \frac{135}{8} x^2y^3 + \frac{531}{64} x^2y^2z + \frac{345}{64} x^3z^2 + \frac{207}{32} x^3yz - \frac{8343}{512} x^3y^2 - \frac{459}{512} x^4z + \frac{135}{512} x^4y + \frac{837}{1024} x^5.$$  

(11)

The procedure for obtaining the Higgs mass, as described below, is insensitive to any terms in the leading-logarithm series (11) past $O(L^4)$. Consequently, Eqs. (8)–(11) are sufficient to determine the entire leading logarithm contribution to the scalar field mass to all (contributing) orders in $\{x, y, z\}$. These equations are also obtainable via the method-of-characteristics methodology of Bando et al. [3], which has been implemented in conventional (non-radiative) Standard Model symmetry breaking by Quiros and collaborators [4]. The conditions $V'_{eff}(v) = 0$ and $V''_{eff}(v) = 24\pi^2y$ respectively imply that $y = -B/2 - K$ and $K = -1 \frac{35}{32} B + \frac{35}{32} C + 20D + 16E$. Given the phenomenological Standard Model values for the vacuum expectation value $v = 246 GeV$, the t-quark Yukawa coupling $\lambda(v) = 1/4\pi^2$, the QCD coupling $\alpha_s(M_z) = 0.120$, and the Higgs width $\Gamma_{V} \sim \alpha_s(M_z)^3 \Gamma_{V} = 0.0329$, we find these constraints taken together constitute a degree-5 equation for the scalar-field self-interaction coupling $y$. The only real positive-$y$ solution that yields a positive second derivative (hence, a local minimum) is $y = 0.0538$. Once $y$ is determined, then $B$, $C$, $D$, and $E$ are also numerically determined. To present order, we can approximate the Higgs field propagator pole with the second derivative of the effective potential at $\phi = v$. One then finds $m^2_{\phi} \equiv V''_{eff}(v) = 8\pi^2v^2(2B + C) = (216 GeV)^2.$
In assessing the viability of this result, it is of interest to consider what one would similarly obtain from the one-loop effective potential augmented by a $\pi^2\phi^4K$ counterterm. Such a potential is seen to correspond to Eq. (8) with $B$ as given by Eq. (5), but with $C = D = E = 0$. The conditions $V'(v) = 0$, $V^{(4)}(v) = 24\pi^2y$ are then seen to lead to a solution $y = 0.093$, $m_\phi = 350$ GeV. Such a mass is well outside the $O(200 \text{GeV})$ bound on $m_\phi$ from corrections to electroweak theory \cite{2}. Moreover, it is easy to demonstrate that this value for $y$ is too large to be meaningful. The contributions of $y$ alone to the $\beta$-function for its own evolution correspond to the $\beta$-function of an $O(4)$ symmetric scalar field theory, which is known to five-loop order \cite{1}:

$$\lim_{z = 0} \frac{dy}{d\mu} = 6y^2 - \frac{39}{2}y^3 + 187.85y^4 - 2698.3y^5 + 47975y^6 + \ldots$$

If $y = 0.093$, terms of this series increase after the second term, indicative of a failure to converge. By contrast, terms of the series \cite{12} decrease if $y = 0.0538$. Similar results characterize this same scalar field theory’s anomalous dimension, whose terms decrease monotonically when $y = 0.0538$, but fail to do so when $y = 0.093$. Of course it is of greater interest to estimate possible corrections to our result to 2-loop order. For our parameter values $x = 0.0253$, $y = 0.0538$, $z = 0.0329$, the two loop contributions to the Standard Model beta functions and anomalous scalar-field dimension provide corrections no larger than 17% of their one-loop counterparts. This provides us with further confidence that the 216 GeV Higgs mass prediction will be stable upon summation of subsequent next-to-leading logarithms.

In Figure 4 we compare the residual scale- ($\mu$-) dependence of $V_{eff}(\phi) = \pi^2\phi^4(S_{LL} + K)$ obtained via Eqs. (8)–(11) to that of the one-loop effective potential discussed in the preceding paragraph. Such dependence in both potentials occurs explicitly through $L(\mu)$ and implicitly through couplants whose one-loop evolution in $\mu$ is anchored to the $\mu = v$ initial values given above (e.g. $x(v) = 1/4\pi^2$). The $K\phi^4$ counterterms in both potentials are each assumed to be RG-invariant ($\beta(\phi)\phi^3(v)$), since the subleading logarithm contributions ultimately devolving from such terms are uncontrolled by Eq. (6). For $\mu = \{v/2, v, 2v\}$, the curves exhibit the dependence of the potentials on the RG-invariant initial value $\phi(v)$ for the evolution of $\phi(\mu)$ \[ (\mu/\phi)d\phi/d\mu = -3x(\mu)/4. \] Figure 4 shows that summation of leading logarithms substantially reduces the residual scale dependence of the effective potential. Moreover, if we assume such scale dependence to be indicative of next-order corrections, we can expect only modest departures from the $m_\phi = 216$ GeV prediction at $\mu = v$: $m_\phi$ varies from 208 GeV at $\mu = v/2$ to 217 GeV at $\mu = 2v$. We find such uncertainties in $m_\phi$ to dominate over much smaller ones deriving from (Standard-Model) uncertainties in the couplant values $x(v)$ and $z(v)$. We have also verified by numerical calculation of the RG equations that the method-of-characteristics methodology of Bando et al \cite{3,4} yields the same results for the effective potential and the Higgs mass to within a value of 0.2%.

Note also that the scalar field mass of order 216 GeV we obtain from the aggregate contribution of leading logarithms to the purely-radiative breakdown of $SU(2) \times U(1)$ electroweak symmetry is accompanied by a scalar-field interaction couplant $y = 0.0538$ substantially larger than that anticipated from conventional spontaneous symmetry breaking (deriving from a potential with an initially-negative quadratic term), in which a 216 GeV Higgs particle would necessarily correspond to a value $y = \lambda/4\pi^2 = m_\phi^2/(8\pi^2\phi^2) = 0.0097$. If electroweak symmetry-breaking is purely radiative, then $y$-sensitive processes such as the $W^+W^- \rightarrow ZZ$ scattering cross-section \cite{5} will necessarily be larger than anticipated from conventional spontaneous symmetry breaking. Consequently, if an $O(200 \text{GeV})$ Higgs were discovered, a clear signal of radiative symmetry breaking would be a corresponding order-of-magnitude-or-more enhancement of $\sigma(W^+W^- \rightarrow ZZ)$ over the value expected from such a Higgs mass.

One of the motivations for summing leading logarithms is to ascertain the negative large-logarithm behaviour of the effective potential, behaviour corresponding to the zero-field limit of the potential. We do not consider positive large-logarithm behaviour because of the intervening Landau singularity at $w = 0$ [Eqs. (4) and (5)], corresponding to $\phi(v) \approx 22w$, \cite{10}. When $|L|$ is very large, we find that $yF_0 \rightarrow -1/3L$, $xF_1 \rightarrow 2(x/z)/L$, and $x^2LF_2 \rightarrow -3(x^2/z^2)/2L$. The large-$|L|$ behaviour of subsequent terms in the series \cite{10} can be extracted by noting that the first term on the RHS of Eq. (5) dominates the second term when the magnitude of $\zeta (=zL)$ is large, and that $F_p(w, \zeta) \sim \sum_{k=0}^{p+1} f_{p,k}(\zeta)$ in this large-$|L|$ limit \[ [(w^{-1}) \sim 1] \]. One then finds after a little algebra that when $|\zeta|$ is large,

$$\left[(7\zeta^2/2) \frac{d}{d\zeta} + 4p\zeta\right] F_p = \frac{9p - 21}{4} F_{p-1}, \quad p \geq 3.$$  

In the large $|\zeta|$ limit, we find that $F_2 \sim \sum_{k=0}^{3} f_{2,k}(\zeta) \sim (-3/2)^{-2}$. Eq. \cite{13} implies that $F_p \sim f_p \zeta^{-p}$, where the numerical factors $f_p$ follow from $f_2 = -3/2$ via the recursion relation $f_p = (9p - 21)f_{p-1}/2p$. Note that $x^pL^{p-1}F_p(w, \zeta) \sim \left(\frac{\zeta}{w}\right)^p f_p/L$ in the large-$|L|$ limit; each term in the series \cite{11} is inversely proportional to $L$ when
FIG. 1: Residual scale dependence of the standard model effective potential with (upper three curves) and without (lower three curves) summation of leading logarithms, as discussed in the text. For the resummed curves, the solid line represents $\mu = v$, the dashed line represents $\mu = v/2$, and the dotted line represents $\mu = 2v$. For the unsummed curves, the dashed-double-dotted curve represents $\mu = v$, the dashed-single-dotted curve represents $\mu = v/2$, and the dashed curve is for $\mu = 2v$.

Let $|L|$ be large. Moreover, if $(x/z) < 2/9$, the above recursion relation for $f_p$ can be utilized to obtain the closed-form series summation $S_{LL} \sim -(\frac{1}{2}) (1 - \frac{9x}{2})^{4/3}$. For sufficiently strong QCD, this result implies that a Standard Model effective potential based upon a massless tree potential exhibits a local minimum, rather than a maximum, at $\phi = 0$ (i.e. $L \to -\infty$). Such a conclusion, however, does not follow if $x/z$ is outside its radius of convergence (i.e. if $x/z > 2/9$), as is the case for the empirical standard model [$x(v) = 1/4\pi^2$, $z(v) = \alpha_s(v)/\pi \simeq 0.033$].

Recent work [11] based upon Padé approximants constructed from the QCD $\beta$-function series suggests for up to five light flavours that the QCD couplant may exhibit the same double-valued behaviour known to characterize $N = 1$ supersymmetric Yang-Mills (SYM) theory, in which coexisting strong-couplant and (asymptotically-free) weak-couplant phases evolve toward a common infrared attractive point [12]. If the strong phase is sufficiently strong $(x/z < 2/9)$, one can envision a scenario in which the $\phi = 0$ local minimum of preserved $SU(2) \times U(1)$ symmetry is upheld by the strong phase of QCD, but is transformed into a symmetry breaking minimum at $\phi = v$ if QCD is in its weak phase. Since for the latter case the minimum at $\phi = v$ ($V(v) < 0$) is deeper than the $\phi = 0$ ($V(0) = 0$) minimum occurring when QCD is in its strong phase, the weak phase of QCD is seen to be the preferred one. Thus, if QCD is characterized by two coexisting phases, as is the case for SYM [12], the asymptotic freedom of QCD may be linked to the radiative breakdown of electroweak symmetry.

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