Hořava gravity at a Lifshitz point is a theory intended to quantize gravity by using techniques of traditional quantum field theories. To avoid Ostrogradsky’s ghosts, a problem that has been plaguing quantization of general relativity since the middle of 1970’s, Hořava chose to break the Lorentz invariance by a Lifshitz-type of anisotropic scaling between space and time at the ultra-high energy, while recovering (approximately) the invariance at low energies. With the stringent observational constraints and self-consistency, it turns out that this is not an easy task, and various modifications have been proposed, since the first incarnation of the theory in 2009. In this review, we shall provide a progress report on the recent developments of Hořava gravity. In particular, we first present four most-studied versions of Hořava gravity, by focusing first on their self-consistency and then their consistency with experiments, including the solar system tests and cosmological observations. Then, we provide a general review on the recent developments of the theory in three different but also related areas: (i) universal horizons, black holes and their thermodynamics; (ii) non-relativistic gauge/gravity duality; and (iii) quantization of the theory. The studies in these areas can be generalized to other gravitational theories with broken Lorentz invariance.

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I. INTRODUCTION

In the beginning of the last century, physics started with two triumphs, quantum mechanics and general relativity. On one hand, based on quantum mechanics (QM), the Standard Model (SM) of Particle Physics was developed, which describes three of the four interactions: electromagnetism, and the weak and strong nuclear forces. The last particle predicted by SM, the Higgs boson, was finally observed by Large Hadron Collider in 2012 [1, 2], after 40 years search. On the other hand, general relativity (GR) describes the fourth force, gravity, and predicts the existence of cosmic microwave background radiation (CMB), black holes, and gravitational waves (GWs), among other things. CMB was first observed accidentally in 1964 [3], and since then various experiments have remeasured it each time with unprecedented precisions [4–8]. Black holes have attracted a great deal of attention both theoretically and experimentally [9], and various evidences of their existence were found [10]. In particular, on Sept. 14, 2015, the LIGO gravitational wave observatory made the first-ever successful observation of GWs [11]. The signal was consistent with theoretical predictions for the GWs produced by the merger of two binary black holes, which marks the beginning of a new era: gravitational wave astronomy.

Despite these spectacular successes, we have also been facing serious challenges. First, observations found that our universe consists of about 25% dark matter (DM) [8]. It is generally believed that such matter should not be made of particles from SM with a very simple argument: otherwise we should have already observed them directly. Second, spacetime singularities exist generically [12], including those of black hole and the big bang cosmology. At the singularities, GR as well as any of other physics laws are all broken down, and it has been a cherished hope that quantum gravitational effects will step in and resolve the singularity problem.

However, when applying the well-understood quantum field theories (QFTs) to GR to obtain a theory of quantum gravity (QG), we have been facing a tremendous resistance [13–15]: GR is not (perturbatively) renormalizable. Power-counting analysis shows that this happens because in four-dimensional spacetimes the gravitational coupling constant $G_N$ has the dimension of $(\text{mass})^{-2}$ (in
units where the Planck constant \( \hbar \) and the speed of light \( c \) are one), whereas it should be larger than or equal to zero in order for the theory to be renormalizable perturbatively [16]. In fact, the expansion of a given physical quantity \( F \) in terms of the small gravitational coupling constant \( G_N \) must be in the form

\[
F = \sum_{n=0}^{\infty} a_n (G_N E^2)^n, \tag{1.1}
\]

where \( E \) denotes the energy of the system involved, so that the combination \( (G_N E^2) \) is dimensionless. Clearly, when \( E^2 \gtrsim G_N^{-1} \), such expansions diverge. Therefore, it is expected that perturbative effective QFT is broken down at such energies. It is in this sense that GR is often said not perturbatively renormalizable.

An improved ultraviolet (UV) behavior can be obtained by including high-order derivative corrections to the Einstein-Hilbert action,

\[
S_{EH} = \int d^4x \sqrt{-g} R, \tag{1.2}
\]

such as a quadratic term, \( R_{\mu\nu} R^{\mu\nu} \) [17]. Then, the gravitational propagator will be changed from \( 1/k^2 \) to

\[
\frac{1}{k^2} + \frac{1}{k^2} G_N k^4 \frac{1}{k^2} + \frac{1}{k^2} G_N k^4 \frac{1}{k^2} G_N k^4 \frac{1}{k^2} + \ldots
= \frac{1}{k^2 - G_N k^4}. \tag{1.3}
\]

Thus, at high energy the propagator is dominated by the term \( 1/k^4 \), and as a result, the UV divergence can be cured. Unfortunately, this simultaneously makes the modified theory not unitary, as now we have two poles,

\[
\frac{1}{k^2 - G_N k^4} = \frac{1}{k^2} - \frac{1}{k^2 - G_N k^4}, \tag{1.4}
\]

and the first one \((1/k^2)\) describes a massless spin-2 graviton, while the second one describes a massive one but with a wrong sign in front of it, which implies that the massive graviton is actually a ghost (with a negative kinetic energy). It is the existence of this ghost that makes the theory not unitary, and has been there since its discovery [17].

The existence of the ghost is closely related to the fact that the modified theory has orders of time-derivatives higher than two. In the quadratic case, for example, the field equations are fourth-orders. As a matter of fact, there exists a powerful theorem due to Mikhail Vasilevich Ostrogradsky, who established it in 1850 [18]. The theorem basically states that a system is not (kinematically) stable if it is described by a non-degenerate higher time-derivative Lagrangian. To be more specific, let us consider a system whose Lagrangian depends on \( \ddot{x} \), i.e., \( \mathcal{L} = \mathcal{L}(x, \dot{x}, \ddot{x}) \), where \( \ddot{x} = dx(t)/dt \), etc. Then, the Euler-Lagrange equation reads,

\[
\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \ddot{x}} + \frac{d^2}{dt^2} \frac{\partial \mathcal{L}}{\partial \dddot{x}} = 0. \tag{1.5}
\]

Now the non-degeneracy means that \( \partial^2 \mathcal{L}/\partial \dddot{x}^2 \neq 0 \), which implies that Eq.(1.5) can be cast in the form [19],

\[
\frac{d^2 x(t)}{dt^4} = \mathcal{F}(x, \dot{x}, \ddot{x}, \dddot{x}; t). \tag{1.6}
\]

Clearly, in order to determine a solution uniquely four initial conditions are needed. This in turn implies that there must be four canonical coordinates, which can be chosen as,

\[
X_1 = x, \quad P_1 = \frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \ddot{x}}, \quad \text{and} \quad X_2 = \dot{x}, \quad P_2 = \frac{\partial \mathcal{L}}{\partial \ddot{x}}. \tag{1.7}
\]

The assumption of non-degeneracy guarantees that Eq.(1.7) has the inverse solution \( \dddot{x} = A(X_1, X_2, P_2) \), so that

\[
\frac{\partial \mathcal{L}}{\partial \dddot{x}} \bigg|_{x=X_1, \dddot{x}=X_2, \dddot{x}=A} = P_2. \tag{1.8}
\]

Then, the corresponding Hamiltonian is given by

\[
H = \sum_{i=1}^{2} P_i \frac{dx}{dt} - \mathcal{L} = P_1 X_2 + P_2 A - \mathcal{L}, \tag{1.9}
\]

which is linear in the canonical momentum \( P_1 \) and implies that there are no barriers to prevent the system from decay, so the system is not stable generically.

It is remarkable to note how powerful and general that the theorem is: It applies to any Lagrangian of the form \( \mathcal{L}(x, \dot{x}, \ddot{x}) \). The only assumption is the non-degeneracy of the system,

\[
\frac{\partial^2 \mathcal{L}(x, \dot{x}, \ddot{x})}{\partial \dddot{x}^2} \neq 0, \tag{1.10}
\]

so the inverse \( \dddot{x} = A(X_1, X_2, P_2) \) of Eq.(1.7) exists. The above considerations can be easily generalized to systems with even higher order time derivatives [19].

Clearly, with the above theorem one can see that any higher derivative theory of gravity with the Lorentz invariance (LI) and the non-degeneracy condition is not stable. Taking the above point of view into account, recently extensions of scalar-tensor theories were investigated by evading the Ostrogradsky instability [20–22].

Another way to evade Ostrogradsky’s theorem is to break LI in the UV and include only high-order spatial derivative terms in the Lagrangian, while still keep the time derivative terms to the second order. This is exactly what Hořava did recently [23].

It must be emphasized that this has to be done with great care. First, LI is one of the fundamental principles of modern physics and strongly supported by observations. In fact, all the experiments carried out so far are consistent with it [24], and no evidence to show that such a symmetry must be broken at certain energy scales, although the constraints in the gravitational sector are much weaker than those in the matter sector.
At the end of this section (Sec. II.E), we consider the co-

II.A - D, which have been most intensively studied so far.

In the next section (Sec. II), we first give a brief in-

roduction to the gauge symmetry that Hořava gravity

exists a couple of excellent reviews on Hořava gravity

A supersymmetric version of Hořava gravity has not been suc-

cessfully constructed, yet [28–30].

1 A supersymmetric version of Hořava gravity has not been suc-

cessfully constructed, yet [28–30].

2 Up to the moment of writing this review, Hořava’s seminal paper

[23] has already been cited about 1400 times, see, for example,
https://inspirehep.net/search?p=find+eprint+0901.3775.

3 A more complete list of articles concerning Hořava gravity can

be found from the citation list of Hořava’s seminal paper [23]:
https://inspirehep.net/search?p=find+eprint+0901.3775.

4 It is interesting to note that big bang singularities have been

intensively studied in Loop Quantum Cosmology (LQC) [49],

and a large number of cosmological models have been considered

[50]. In all of these models big bang singularity is resolved by

quantum gravitational effects in the deep Planck regime. Similar

conclusions are also obtained for black holes [51].

5 One may never be able to prove truly that a theory is correct,

but rather disprove or more accurately constrain a hypothesis

[55]. The history of science tells us that this has been the case

so far.
exist quantum mechanically, and instead emerges at the low energy physics. Along this line of arguing, it is not unreasonable to assume that LI is broken in the UV but recovered later in the IR. Once LI is broken, one can include only high-order spatial derivative operators into the Lagrangian, so the UV behavior can be improved, while the time derivative operators are still kept to the second-order, in order to evade Ostrogradsky’s ghosts. This was precisely what Horava did [23].

Of course, there are many ways to break LI. But, Horava chose to break it by considering anisotropic scaling between time and space,

\[ t \to b^{-z} t, \quad x^i \to b^{-1} x^i, \quad (i = 1, 2, \ldots, d) \]

(2.1)

where \( z \) denotes the dynamical critical exponent, and LI requires \( z = 1 \), while power-counting renormalizability requires \( z \geq d \), where \( d \) denotes the spatial dimension of the spacetime [23, 65–69]. In this review we mainly consider spacetimes with \( d = 3 \) and take the minimal value \( z = d \), except for particular considerations. Whenever this happens, we shall make specific notice. Eq.(2.1) is a reminiscence of Lifshitz’s scalar fields in condensed matter physics [70, 71], hence in the literature Hořava gravity is also called the Hořava-Lifshitz (HL) theory. With the scaling of Eq.(2.1), the time and space have, respectively, the dimensions 6,

\[ [t] = -z, \quad [x^i] = -1. \]

(2.2)

Clearly, such a scaling breaks explicitly the LI and hence 4-dimensional diffeomorphism invariance. Hořava assumed that it is broken only down to the level

\[ t \to \xi_0(t), \quad x^i \to \xi^i(t, x^k), \]

(2.3)

so that the spatial diffeomorphism still remains. The above symmetry is often referred as to the foliation-preserving diffeomorphism, denoted by Diff(M, \( \mathcal{F} \)). To see how gravitational fields transform under the above diffeomorphism, let us first introduce the Arnowitt-Deser-Misner (ADM) variables [72],

\[ (N, N^i, g_{ij}), \]

(2.4)

where \( N, N^i \) and \( g_{ij} \), denote, respectively, the lapse function, shift vector, and 3-dimensional metric of the leaves \( t = \text{constant} \) 7. Under the rescaling (2.1) \( N, N^i \) and \( g_{ij} \) are assumed to scale, respectively, as [23],

\[ N \to N, \quad N^i \to b^2 N^i, \quad g_{ij} \to g_{ij}, \]

(2.6)

so that their dimensions are

\[ N = 0, \quad [N^i] = 2, \quad [g_{ij}] = 0. \]

(2.7)

Under the Diff(M, \( \mathcal{F} \)), on the other hand, they transform as,

\[ \delta N = \xi^k \nabla_k N + \dot{N} \xi_0 + N \xi_0, \]

\[ \delta N_i = N_k \nabla_i \xi^k + \xi^l \nabla_k N_l + g_{ik} \dot{\xi}^k + \dot{N} \xi_0 + N \xi_0, \]

\[ \delta g_{ij} = \nabla_i \xi_j + \nabla_j \xi_i + \delta_0 \delta_{ij}, \]

(2.8)

where \( N_i \equiv g_{ij} N_j \), and in writing the above we had assumed that \( \xi_0(t) \) and \( \xi^k(t, x^i) \) are small, so that only their linear terms appear. Once we know the transforms (2.8), we can construct the basic operators of the fundamental variables (2.4) and their derivatives, which turn out to be 8,

\[ K_{ij}, \quad a_i, \quad \nabla_i, \]

(2.9)

\[ \frac{\partial}{\partial t}, \quad \nabla_i a_j, \quad [R_{ij}] = 2, \quad [K_{ij}] = 3, \quad [a_i] = 1, \quad [\nabla_i] = 1. \]

(2.11)

With the basic blocks of Eq.(2.9) and their dimensions, we can build scalar operators order by order, so the total Lagrangian will finally take the form,

\[ \mathcal{L}_g = \sum_{n=0}^{2} \mathcal{L}_g^{(n)} (N, N^i, g_{ij}), \]

(2.12)

where \( \mathcal{L}_g^{(n)} \) denotes the part of the Lagrangian that contains operators of the \( n \)-th-order only. In particular, to each order of \([k] \), we have the following independent terms that are all scalars under the transformations of the foliation-preserving diffeomorphisms (2.3) [73],

\[ [k]^6 : K_{ij} K^{ij}, \quad K^2, \quad R^2, \quad R_{ij} R^{ij}, \quad R^i_j R^k_l R^i_k, \quad (\nabla R)^2, \]

(2.13)

But, in Hořava gravity the line element \( ds \) is not necessarily given by this relation [For example, see Eqs.(2.41) and (2.42)]. Instead, one can simply consider \((N, N^i, g_{ij})\) as the fundamental quantities that describe the quantum gravitational field of Hořava gravity, and their relations to the macroscopic quantities, such as \( ds \), will emerge in the IR limit. 8 Note that with the general diffeomorphism, \( x^\mu \to \xi^\mu(t, x^k) \), the fundamental quantity is the Riemann tensor \( R^\mu_{\nu\alpha\beta} \).
\[ (\nabla_i R_{jk}) (\nabla^i R_{jk}), \quad (a_i a^i)^2 R, \quad (a_i a^i) (a_j R^{ij}), \quad (a_i a^i)^3, \quad a_i \Delta^2 a_i, \quad (a_i^2) \Delta R, \ldots, \]

\[ [k]^5 : \quad K_{ij} R^{ij}, \quad \epsilon^{ijk} R_{il} \nabla_j R_{lk}, \quad \epsilon^{ijk} a_i a_l \nabla_j R_{lk}, \quad a_m a_j R^{ij}, \quad K^{ij} a_{ij}, \quad (a^i_i) K, \]

\[ [k]^4 : \quad R^2, \quad R_{ij} R^{ij}, \quad (a_i a^i)^2, \quad (a_i a^i)^2 a^j, \quad a^j a_{ij}, \quad (a_i a^i) R, \quad a_{ij} R^{ij}, \quad Ra^i, \]

\[ [k]^3 : \quad \omega_3(\Gamma), \quad \omega_4(\Gamma), \quad \omega_5(\Gamma), \quad \omega_6(\Gamma), \quad \omega_7(\Gamma), \quad \omega_8(\Gamma), \quad \omega_9(\Gamma), \quad \omega_{10}(\Gamma), \]

\[ \text{where } \omega_3(\Gamma) \text{ denotes the gravitational Chern-Simons term, } \gamma_0 \text{ is a dimensionless constant, } \Delta \equiv g^{ij} \nabla_i \nabla_j, \quad \text{and} \]

\[ a_{i_1 i_2 \ldots i_n} \equiv \nabla_{i_1} \nabla_{i_2} \ldots \nabla_{i_n} \ln(N), \quad \omega_3(\Gamma) \equiv \text{Tr} \left( \Gamma \wedge d\Gamma + \frac{2}{3} \Gamma \wedge \Gamma \wedge \Gamma \right), \]

\[ = \epsilon^{ijk} \sqrt{g} \left( \Gamma_i^{m} \partial_j \Gamma_k^{m} + \frac{2}{3} \Gamma_i^{m} \Gamma_j^{m} \Gamma_k^{m} \right). \quad (2.14) \]

Here \( g = \det(g_{ij}) \) and \( \epsilon^{ijk} \equiv \epsilon^{ijk} / \sqrt{g} \) with \( \epsilon^{123} = 1 \), etc. Note that in writing Eq.(2.13), we had not written down all the sixth order terms, as they are numerous and a complete set of it has not been given explicitly [74, 75]. Then, the general action of the gravitational part (2.6) will be the summary of all these terms. Since time derivative terms only contain in \( K_{ij} \), we can see that the kinetic part \( \mathcal{L}_K \) is the linear combination of the sixth order derivative terms,

\[ \mathcal{L}_K = \frac{1}{\zeta^2} \left( K_{ij} R^{ij} - \lambda K^2 \right), \quad (2.15) \]

where \( \zeta^2 \) is the gravitational coupling constant with the dimension

\[ [\zeta^2] = [t] \cdot [x^3] + [K^2] = -z - 3 + 2z = z - 3. \quad (2.16) \]

Therefore, for \( z = 3 \), it is dimensionless, and the power-counting analysis given between Eqs.(1.1) and (1.2) shows that the theory now becomes power-counting renormalizable. The parameter \( \lambda \) is another dimensionless coupling constant, and LI guarantees it to be one even after radiative corrections are taken into account. But, in Ho\'rava gravity it becomes a running constant due to the breaking of LI.

The rest of the Lagrangian (also called the potential) will be the linear combination of all the rest terms of Eq.(2.13), from which we can see that, without protection of further symmetries, the total Lagrangian of the gravitational sector is about 100 terms, which is normally considered very large and could potentially diminish the prediction power of the theory. Note that the odd terms given in Eq.(2.13) violate the parity. So, to eliminate them, we can simply require that parity be conserved. However, since there are only six such terms, this will not reduce the total number of coupling constants significantly.

To further reduce the number of independent coupling constants, Ho\'rava introduced two additional conditions, the projectability and detailed balance [23]. The former requires that the lapse function \( N \) be a function of \( t \) only,

\[ N = N(t), \quad (2.17) \]

so that all the terms proportional to \( a_i \) and its derivatives will be dropped out. This will reduce considerably the total number of the independent terms in Eq.(2.12), considering the fact that \( a_i \) has dimension of one only. Thus, to build an operator out of \( a_i \) to the sixth-order, there will be many independent combinations of \( a_i \) and its derivatives. However, once the condition (2.17) is imposed, all such terms vanish identically, and the total number of the sixth-order terms immediately reduces to seven, given exactly by the first seven terms in Eq.(2.13).

So, the totally number of the independently coupling constants of the theory now reduce to \( N = 14 \), even with the three parity-violated terms,

\[ K_{ij} R^{ij}, \quad \epsilon^{ijk} R_{il} \nabla_j R_{lk}, \quad \omega_3(\Gamma). \quad (2.18) \]

It is this version of the HL theory that Ho\'rava referred to as the minimal theory [41]. Note that the projectable condition (2.17) is mathematically elegant and appealing. It is preserved by the Diff(\( M, \mathcal{F} \)) (2.3), and forms an independent branch of differential geometry [76].

Inspired by condensed matter systems [77], in addition to the projectable condition, Ho\'rava also assumed that the potential part, \( \mathcal{L}_V \), can be obtained from a superpotential \( W_g \) via the relations [23],

\[ \mathcal{L}_V = w^2 E_{ij} g^{ijkl} E_{kl}, \quad E^{ij} = \frac{1}{\sqrt{g}} \frac{\delta W_g}{\delta g_{ij}}, \quad (2.19) \]

where \( w \) is a coupling constant, and \( g^{ijkl} \) denotes the generalized DeWitt metric, defined as

\[ g^{ijkl} = \frac{1}{2} \left( g^{ik} g^{jl} + g^{il} g^{jk} \right) - \lambda g^{ij} g^{kl}, \quad (2.20) \]

where \( \lambda \) is the same coupling constant, as introduced in Eq.(2.15), and the superpotential \( W_g \) is given by

\[ W_g = \int_{\Sigma} \omega_3(\Gamma) + \frac{1}{\kappa_W} \int_{\Sigma} d^3x \sqrt{g}(R - 2\Lambda), \quad (2.21) \]

where \( \Sigma \) denotes the leaves of \( t = \text{constant} \), \( \Lambda \) the cosmological constant, and \( \kappa_W \) is another coupling constant of the theory. Then, the total Lagrangian \( \mathcal{L}_g = \mathcal{L}_K - \mathcal{L}_V \), contains only five coupling constants, \( \zeta, \lambda, w, \kappa_W \) and \( \Lambda \).

Note that the above detailed balance condition has a couple of remarkable features [41]: First, it is in the same spirit of the AdS/CFT correspondence [78–82], where the superpotential is defined on the 3-dimensional leaves, \( \Sigma \), while the gravity is (3+1)-dimensional. Second, in the non-equilibrium thermodynamics, the counterpart of the
superpotential $W_g$ plays the role of entropy, while the term $E^ij$ the entropic forces [83, 84]. This might shed light on the nature of the gravitational forces [85].

However, despite of these desired features, this condition leads to several problems, including that the Newtonian limit does not exist [86], and the six-order derivative operators are eliminated, so the theory is still not power-counting renormalizable [23]. In addition, it is not clear if this symmetry is still respected by radiative corrections.

Even more fundamentally, the foliation-preserving diffeomorphism (2.3) allows one more degree of freedom in the gravitational sector, in comparing with that of general diffeomorphism. As a result, a spin-0 mode of gravitons appears. This mode is potentially dangerous and may cause ghosts and instability problems, which lead the constraint algebra dynamically inconsistent [87–90].

To solve these problems, various modifications have been proposed [34–41]. In the following we shall briefly introduce only four of them, as they have been most extensively studied in the literature so far. These are the ones: (i) with the projectability condition - the minimal theory [23, 91, 92]; (ii) with the projectability condition and an extra U(1) symmetry [93, 94]; (iii) without the projectability condition but including all the possible terms - the healthy extension [95, 96]; and (iv) with an extra U(1) symmetry but without the projectability condition [73, 97, 98].

Before considering each of these models in detail, some comments on singularities in Hořava gravity are in order, as they will appear in all of these models and shall be faced when we consider applications of Hořava gravity to black hole physics and cosmology [99]. First, the nature of singularities of a given spacetime in Hořava theory could be quite different from that in GR, which has the general diffeomorphism,

$$\delta x^\mu = \xi^\mu (t, x^k), \ (\mu = 0, 1, 2, 3). \quad (2.22)$$

In GR, singularities are divided into two different kinds: *spacetime and coordinate singularities* [100]. The former is real and cannot be removed by any coordinate transformations of the type given by Eq. (2.22). The latter is coordinate-dependent, and can be removed by proper coordinate transformations of the kind (2.22). Since the laws of coordinate transformations in GR and Hořava theory are different, it is clear that the nature of singularities are also different. In GR it may be a coordinate singularity but in Hořava gravity it becomes a spacetime singularity. Second, two different metrics may represent the same spacetime in GR but in general it is no longer true in Hořava theory. A concrete example is the Schwarzschild solution given in the Schwarzschild coordinates,

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1}dr^2 + r^2d\Omega^2. \quad (2.23)$$

Making the coordinate transformation,

$$dt_{PG} = dt + \frac{\sqrt{2m}}{r - 2m} dr, \quad (2.24)$$

the above metric takes the Painleve-Gullstrand (PG) form [101],

$$ds^2 = -dt_{PG}^2 + \left(dr + \frac{2m}{r} dt_{PG}\right)^2 + r^2d\Omega^2. \quad (2.25)$$

In GR we consider metrics (2.23) and (2.25) as describing the same spacetime (at least in the region $r > 2m$), as they are connected by the coordinate transformation (2.24), which is allowed by the symmetry (2.22) of GR. But this is no longer the case when we consider them in Hořava gravity. The coordinate transformation (2.24) is not allowed by the foliation-preserving diffeomorphism (2.3), and as a result, they describe two different spacetimes in Hořava theory. In particular, metric (2.25) satisfies the projectability condition, while metric (2.23) does not. So, in Hořava theory they belong to the two completely different branches, with or without the projectability condition. Moreover, in GR the metric

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dv^2 + 2dvdr + r^2d\Omega^2, \quad (2.26)$$

describes the same spacetime as those of metrics (2.23) and (2.25), but it does not belong to any of the two branches of Hořava theory, because $v = $ constant hypersurfaces do not define a (3 + 1)-dimensional foliation, a fundamental requirement of Hořava gravity. Third, because of the difference between the two kinds of coordinate transformations, the global structure of a given spacetime is also different in GR and Hořava gravity [102]. For example, the maximal extension of the Schwarzschild solution was achieved when it is written in the Kruskal coordinates [12],

$$ds^2 = -e^{-r/2m} dUdV + r^2(U,V)d\Omega^2. \quad (2.27)$$

But the coordinate transformations that bring metric (2.23) or (2.25) into this form are not allowed by the foliation-preserving diffeomorphism (2.3). For more details, we refer readers to [99, 102].

### A. The Minimal Theory

If we only impose the parity and projectability condition (2.17), the total action for the gravitational sector can be cast in the form [91, 92],

$$S_g = \zeta^2 \int dt d^3x N \sqrt{g} (\mathcal{L}_K - \mathcal{L}_V), \quad (2.28)$$

where $\mathcal{L}_K$ is given by Eq. (2.15), while the potential part takes the form,

$$\mathcal{L}_V = 2\Lambda - R + \frac{1}{\zeta^2} (g_2 R^2 + g_3 R_{ij} R^{ij})$$
\[ g_R R + g_s (\nabla_i R_{jk}) (\nabla^i R^{jk}) \]. (2.29)\]

Here \( \zeta^2 = 1/16\pi G \), and \( g_n (n = 2, \ldots, 8) \) are all dimensionless coupling constants. Note that, without loss of the generality, in writing Eq. (2.29) the coefficient in the front of \( R \) was set to \(-1\), which can be realized by rescaling the time and space coordinates [92]. As mentioned above, Hořava referred to this model as the \textit{minimal theory} [41].

In the IR, all the high-order curvature terms (with coefficients \( g_n \)'s) drop out, and the total action reduces to the Einstein-Hilbert action, provided that the coupling constant \( \lambda \) flows to its relativistic limit \( \lambda_{GR} = 1 \) in the IR. This has not been shown in the general case. But, with only the three coupling constants (\( \zeta, \Lambda, \lambda \)), it was found that the Einstein-Hilbert action with \( \Lambda = 0 \) is an attractor in the phase space of RG flow [103]. In addition, RG trajectories with a tiny positive cosmological constant also come with a value of \( \lambda \) that is compatible with experimental constraints.

To study the stability of the theory, let us consider the linear perturbations of the Minkowski background (with \( \Lambda = 0 \)),

\[ N = 1, \quad N_i = \partial_i B, \quad g_{ij} = e^{-2\varphi} \delta_{ij}. \] (2.30)

After integrating out the \( B \) field, the action up to the quadratic terms of \( \psi \) takes the form [104],

\[ s_\psi^{(2)} = -c^2 \int dtdx^3 \left( \frac{\dot{\psi}^2}{c^2} - \psi \left( 1 + \alpha_1 \partial^2 + \alpha_2 \partial^4 \right) \partial^2 \psi \right), \] (2.31)

where \( \partial^2 \equiv \delta^{ij} \partial_i \partial_j \), \( c^2_\psi \equiv -(\lambda - 1)/(3 \lambda - 1) \), \( \alpha_1 \equiv (8g_2 + 3g_3)/\zeta^2 \) and \( \alpha_2 \equiv -(8g_7 - 3g_8)/\zeta^2 \). Clearly, to avoid ghosts we must assume that \( c^2_\psi < 0 \), that is,

\[ (i) \ \lambda > 1 \quad \text{or} \quad (ii) \ \lambda < \frac{1}{3}. \] (2.32)

However, in these intervals the scalar mode is not stable in the IR [92, 104] \(^9\). This can be seen easily from the equation of motion of \( \psi \) in the momentum space,

\[ \ddot{\psi}_k + \omega_k^2 \psi_k = 0, \] (2.33)

where \( \omega_k^2 \equiv c^2_\psi (1 - \alpha_1 k^2 + \alpha_2 k^4) k^2 \). In the intervals of Eq. (2.32), we have \( c^2_\psi < 0 \) in the IR, so \( \omega_k^2 \) also becomes negative, that is, the theory suffers tachyonic instability. In the UV and intermediate regimes, it can be stable by properly choosing the coupling constants of the high-order operators \( g_n (n = 2, 3, 7, 8) \). Note that each of these terms is subjected to radiative corrections. It would be very interesting to show that the scalar mode is still stable, even after such corrections are taken into account.

On the other hand, from Eq. (2.31) it can be seen that there are two particular values of \( \lambda \) that make the above analysis invalid, one is \( \lambda = 1 \) and the other is \( \lambda = 1/3 \). A more careful analysis of these two cases shows that the equations for \( \psi \) and \( B \) degenerate into elliptic differential equations, so the scalar mode is no longer dynamical. As a result, the Minkowski spacetime in these two cases are stable.

It is also interesting to note that the de Sitter spacetime in this minimal theory is stable [106, 107]. See also a recent study of the issue in a closed FLRW universe [108].

In addition, this minimal theory still suffers the strong coupling problem [106, 109], so the power-counting analysis presented above becomes invalid. It must be noted that this does not necessarily imply the loss of predictability: if the theory is renormalizable, all coefficients of infinite number of nonlinear terms can be written in terms of finite parameters in the action, as several well-known theories with strong coupling (e.g., [110]) indicate. However, because of the breakdown of the (naive) perturbative expansion, we need to employ nonperturbative methods to analyze the fate of the scalar graviton in the limit. Such an analysis was performed in [34] for spherically symmetric, static, vacuum configurations and was shown that the limit is continuously connected to GR. A similar consideration for cosmology was given in [111, 112], where a fully nonlinear analysis of superhorizon cosmological perturbations was carried out, and was shown that the limit \( \lambda \rightarrow 1 \) is continuous and that GR is recovered. This may be considered as an analogue of the Vainshtein effect first found in massive gravity [113].

With the projectability condition, the Hamiltonian constraint becomes global, from which it was shown that a component which behaves like dark matter emerges as an “integration constant” of dynamical equations and momentum constraint equations [114].

Cosmological perturbations in this version of theory has been extensively studied [104, 108, 115–120], and are found consistent with current observations.

In addition, spherically symmetric spacetimes without/with the presence of matter were also investigated [121–123], and was found that the solar system tests can be satisfied by properly choosing the coupling constants of the theory.

### B. With Projectability & U(1) Symmetry

As mentioned above, the problems plaguing in Hořava gravity are closely related to the existence of the spin-0 graviton. Therefore, if it is eliminated, all the problems should be cured. This can be done, for example,
by imposing extra symmetries, which was precisely what Hořava and Melby-Thompson (HMT) did in [93]. HMT introduced an extra local U(1) symmetry, so that the total symmetry of the theory now is enlarged to,

$$U(1) \ltimes \text{Diff}(M, \mathcal{F}).$$

The extra U(1) symmetry is realized by introducing two auxiliary fields, the U(1) gauge field $A$ and the Newtonian pre-potential $\varphi$. Under this extended symmetry, the special status of time remains, so that the anisotropic scaling (2.1) is still valid, whereby the UV behavior of the theory can be considerably improved. On the other hand, under the local U(1) symmetry, the fields transform as

$$\delta \alpha A = \dot{\alpha} - N^i \nabla_i \alpha, \quad \delta \alpha \varphi = -\alpha,$$

$$\delta \alpha N_i = N \nabla_i \alpha, \quad \delta \alpha g_{ij} = 0 = \delta \alpha N,$$

(2.35)

where $\alpha (t, x^k)$ is the generator of the local U(1) gauge symmetry. Under the Diff$(M, \mathcal{F})$, the auxiliary fields $A$ and $\varphi$ transform as,

$$\delta A = \zeta' \nabla_i A + \dot{f} A + f \dot{A},$$

$$\delta \varphi = f' \varphi + \zeta \nabla_i \varphi,$$

(2.36)

while $N, N^i$ and $g_{ij}$ still transform as those given by Eq.(2.8). With this enlarged symmetry, the spin-0 graviton is indeed eliminated [93, 124].

At the initial, it was believed that the U(1) symmetry can be realized only when the coupling constant $\lambda$ takes its relativistic value $\lambda = 1$. This was very encouraging, because it is the deviation of $\lambda$ from one that causes all the problems, including ghost, instability and strong coupling \(11\).

However, this claim was soon challenged, and shown that the introduction of the Newtonian pre-potential is so strong that action with $\lambda \neq 1$ also has the local U(1) symmetry [94]. It is remarkable that the spin-0 graviton is still eliminated even with an arbitrary value of $\lambda$ first by considering linear perturbations in Minkowski and de Sitter spacetimes [94, 125], and then by analyzing the Hamiltonian structure of the theory [126].

The general action for the gravitational sector now takes the form [125],

$$S_g = \xi^2 \int dt dx^3 N \sqrt{g} \left( \mathcal{L}_K - \mathcal{L}_V + \mathcal{L}_\varphi + \mathcal{L}_A + \mathcal{L}_\lambda \right),$$

(2.37)

where $\mathcal{L}_K$ and $\mathcal{L}_V$ are given by Eqs.(2.15) and (2.29), and

$$\mathcal{L}_\varphi \equiv \varphi G^{ij} \left( 2K_{ij} + \nabla_i \nabla_j \varphi \right),$$

$$\mathcal{L}_A \equiv \frac{A}{N} \left( 2\Lambda_g - R \right),$$

$$\mathcal{L}_\lambda \equiv (1 - \lambda) \left( (\nabla^2 \varphi)^2 + 2K \nabla^2 \varphi \right),$$

$$G_{ij} \equiv R_{ij} - \frac{1}{2} g_{ij} R + \Lambda_g g_{ij}.$$

Here $\Lambda_g$ is another coupling constant, and has the same dimension of $R$. Note that the potential $\mathcal{L}_V$ takes the same form as that given in the case without the extra U(1) symmetry. This is because $g_{ij}$ does not change under the local U(1) symmetry, as it can be seen from Eq.(2.35). So, the most general form of $\mathcal{L}_V$ is still given by Eq.(2.29).

Note that the strong coupling problem no longer exists in the gravitational sector, as the spin-0 graviton now is eliminated. However, when coupled with matter, it will appear again for processes with energy higher than [125, 127],

$$\Lambda_\omega \equiv M_{pl} \left( \frac{M_{pl}}{C} \right)^{3/2} |\lambda - 1|^{5/4},$$

(2.39)

where $M_{pl}$ denotes the Planck mass, and generically $C \ll M_{pl}$. To solve this problem, one way is to introduce a new energy scale $M_*$ so that $M_* < \Lambda_\omega$, as Blas, Pujolas and Sibiryakov first introduced in the nonprojectable case [128]. This is reminiscent of the case in string theory where the string scale is introduced just below the Planck scale, in order to avoid strong coupling [42-44]. In the rest of this review, it will be referred to as the BPS mechanism. The main ideas are the following: before the strong coupling energy $\Lambda_\omega$ is reached [cf. Fig. 1], the sixth order derivative operators become dominant, so the scaling law of a physical quantity for process with $E > M_*$ will follow Eq.(2.1) instead of the relativistic one ($z = 1$). Then, with such anisotropic scalings, it can be shown that all the nonrenormalizable terms (with $z = 1$) now become either strictly renormalizable or superrenormalizable [110], whereby the strong coupling problem is resolved. For more details, we refer readers to [127, 128].

It should be noted that, in order for the mechanism to work, the price to pay is that now $\lambda$ cannot be exactly equal to one, as one can see from Eq.(2.39). In other words, the theory cannot reduce precisely to GR in the IR. However, since GR has achieved great success in low energies, $\lambda$ cannot be significantly different from one in the IR, in order for the theory to be consistent with observations.

In addition, the BPS mechanism cannot be applied to the minimal theory presented in the last subsection, because the condition $M_* < \Lambda_\omega$, together with the one that instability cannot occur within the age of the universe, requires fine-tuning $|\lambda - 1| < 10^{-24}$, as shown explicitly in [106]. However, in the current setup (with any $\lambda$), the Minkowski spacetime is stable, so such a fine-tuning does not exist.

Static and spherically symmetric spacetimes were considered in [93, 98, 102, 129-133], and solar system tests
were in turn investigated. In particular, HMT found that GR can be recovered in the IR if the lapse function $N$ of GR is related to the one $N$ of Hořava gravity and the gauge field $A$ via the relation, $N = N - A$ [93]. This can be further justified by the considerations of geometric interpretations of the gauge field $A$, in which the local U(1) symmetry was found in the first place. By requiring that the line element is invariant not only under the Diff(M, $\mathcal{F}$) but also under the local U(1) symmetry, the authors in [129] found that the lapse function $N$ and the shift vector $\mathcal{N}$ of GR should be given by $N = N - v(A - A)/c^2$ and $\mathcal{N} = N \nabla^i \varphi$, where $v$ is a dimensionless coupling constant, and

$$A \equiv -\varphi + N^i \nabla_i \varphi + \frac{1}{2} N (\nabla^i \varphi)^2.$$  (2.40)

Then, it was found that the theory is consistent with the solar system tests for both $\lambda = 1$ and $\lambda \neq 1$, provided that $|v - 1| < 10^{-5}$. To couple with matter fields, in [98] the authors considered a universal coupling between the matter and the HMT theory via the effective metric $\gamma_{\mu\nu}$,

$$ds^2 \equiv \gamma_{\mu\nu}dx^\mu dx^\nu = -c^2 N^2 dt^2 + \gamma_{ij} (dx^i + N^i dt) (dx^j + N^j dt),$$  (2.41)

where

$$\mathcal{N} \equiv (1 - a_1 \sigma) N, \quad \mathcal{N}_i = N_i + N \nabla_i \varphi,$$

$$\gamma_{ij} \equiv (1 - a_2 \sigma^2) g_{ij}, \quad \sigma \equiv \frac{A - A}{N}.$$  (2.42)

Here $a_1(\equiv v)$ and $a_2$ are two coupling constants. It can be shown that the effective metric (2.41) is invariant under the enlarged symmetry (2.34). The matter is minimally coupled with respect to the effective metric $\gamma_{\mu\nu}$ via the relation,

$$S_M = \int dt d^3 x \sqrt{\gamma} L_M (N, N^i, \gamma_{jk}; \psi_n),$$  (2.43)

where $\psi_n$ denotes collectively matter fields. With such a coupling, in [98] the authors calculated explicitly all the parameterized post-Newtonian (PPN) parameters in terms of the coupling constants of the theory, and showed that the theory satisfies the constraints [134] 12,

$$\gamma = 1 + (2.1 \pm 2.3) \times 10^{-5},$$

$$\beta = 1 + (-4.1 \pm 7.8) \times 10^{-5},$$

$$\alpha_1 < 10^{-4}, \quad \alpha_2 < 4 \times 10^{-7}, \quad \alpha_3 < 4 \times 10^{-20},$$

$$\xi < 10^{-3}, \quad \Gamma < 1.5 \times 10^{-3}, \quad \zeta_1 < 2 \times 10^{-2},$$

$$\zeta_2 < 4 \times 10^{-5}, \quad \zeta_3 < 10^{-8}, \quad \zeta_4 < 6 \times 10^{-3}.$$  (2.44)

obtained by all the current solar system tests, where

$$\Gamma \equiv 4 \beta - \gamma - 3 - \frac{10}{3} \xi - \alpha_1 + 2 \frac{1}{3} \alpha_2 - \frac{2}{3} \zeta_1 - 1 \frac{1}{3} \zeta_2.$$  (2.45)

In particular, one can obtain the same results as those given in GR [134],

$$\gamma = \beta = 1, \quad \alpha_1 = \alpha_2 = \alpha_3 = \xi = 0,$$

$$\zeta_1 = \zeta_2 = \zeta_3 = \zeta_4 = \zeta_B = 0.$$  (2.46)

for

$$(a_1, a_2) = (1, 0).$$  (2.47)

It is interesting to note that this is exactly the case first considered in HMT [93].

A remarkable feature is that the solar system tests impose no constraint on the parameter $\lambda$. As a result, when combined with the condition for the avoidance of the strong coupling problem, these conditions do not lead to an upper bound on the energy scale $M_*$ that suppresses higher dimensional operators in the theory. This is in sharp contrast to other versions of Hořava gravity without the U(1) symmetry. It should be noted that the

12 Note that in general covariant theory including GR, the term $\mathcal{B} \equiv \int \frac{\sqrt{-g}}{16\pi G} (x - x') \cdot \frac{\partial^2}{\partial t\partial t} d^3x'$ appearing in the component $h_{tt}$ can be always eliminated by the coordinate transformation $dt = \xi^0(t, x^i)$. However, in Hořava gravity, this symmetry is missing, and the term $\zeta_B \mathcal{B}$ must be included into $h_{tt}$. So, instead of the ten PPN parameters introduced in [134], here we have an additional one $\zeta_B$. For more details, we refer readers to [98].
physical meaning of the gauge field $A$ and the Newtonian prepotential $\varphi$ were also studied in [135] but with a different coupling with matter fields.

Inflationary cosmology was studied in detail in [136], and found that, among other things, the FLRW universe is necessarily flat. In the sub-horizon regions, the metric and inflaton are tightly coupled and have the same oscillating frequencies. In the super-horizon regions, the perturbations become adiabatic, and the comoving curvature perturbation is constant. Both scalar and tensor perturbations are almost scale-invariant, and the spectrum indices are the same as those given in GR, but the ratio of the scalar and tensor power spectra depends on the high-order spatial derivative terms, and can be different from that of GR significantly. Primordial non-Gaussianities of scalar and tensor modes were also studied in [137, 138] \(^{13}\).

Note that gravitational collapse of a spherically symmetric object was studied systematically in [140], by using distribution theory. The junction conditions across the surface of a collapsing star were derived under the (minimal) assumption that the junctions must be mathematically meaningful in terms of distribution theory. Lately, gravitational collapse in this setup was investigated and various solutions were constructed [141–143].

We also note that with the U(1) symmetry, the detailed balance condition can be imposed [144]. However, in order to have a healthy IR limit, it is necessary to break it softly. This will allow the existence of the Newtonian limit in the IR and meanwhile be power-counting renormalizable in the UV. Moreover, with the detailed balance condition softly breaking, the number of independent coupling constants can be still significantly reduced. This is particularly the case when we consider Hořava gravity without the projectability condition but with the U(1) symmetry. Note that, even in the latter, the U(1) symmetry is crucial in order not to have the problem of power-counting renormalizability in the UV, as shown explicitly in [73]. In particular, in the healthy extension to be discussed in the next subsection the detailed balance condition cannot be imposed even allowing it to be broken softly. Otherwise, it can be shown that the sixth-order operators are eliminated by this condition, and the resulting theory is not power-counting renormalizable.

It is also interesting to note that, using the non-relativistic gravity/gauge correspondence, it was found that this version of Hořava gravity has one-to-one correspondence to dynamical Newton-Carton geometry without torsion, and a precise dictionary was built [145].

\(^{13}\) It should be noted that such studies were carried out when matter is minimally coupled to $g_{\mu
u}$ [136], and for the universal coupling (2.42) such studies have not been worked out, yet. A preliminary study indicates that a more general coupling might be needed [139].

\section{The Healthy Extension}

Instead of eliminating the spin-0 graviton, BPS chose to live with it and work with the non-projectability condition [95, 96],

$$N = N \left(t, x^k\right).$$

(2.48)

Although again we are facing the problem of a large number of coupling constants, BPS showed that the spin-0 graviton can be stabilized even in the IR. This is realized by including the quadratic term $(a_i a^i)$ into the Lagrangian,

$$\mathcal{L}_V = 2\Lambda - R - \beta_0 a_i a^i + \sum_{n=3}^{6} \mathcal{L}_V^{(n)},$$

(2.49)

where $\beta_0$ is a dimensionless coupling constant, and $\mathcal{L}_V^{(n)}$ denotes the Lagrangian built by the $n$th-order operators only. Then, in the IR it can be shown that the scalar perturbations can be still described by Eq. (2.33), but now with [95, 96]

$$\omega_k^2 = \frac{\lambda - 1}{3\lambda - 1} \left(\frac{2}{\beta_0} - 1\right) k^2 + O \left(\frac{k^4}{M_*^2}\right),$$

(2.50)

where $M_*$ is the energy scale of the theory. While the ghost-free condition still leads to the condition (2.32), because of the presence of the $(a_i a^i)$ term, the scalar mode becomes stable for

$$0 < \beta_0 < 2.$$  

(2.51)

It is remarkable to note that the stability requires $\beta_0 \neq 0$ strictly \(^{14}\). This will lead to significantly different from the case $\beta_0 = 0$. The stability of the spin-2 mode can be shown by realizing the fact that only the following high-order terms have contributions in the quadratic level of linear perturbations of the Minkowski spacetime [74, 96],

$$\mathcal{L}^{(n \geq 3)} = \zeta^{-2} \left(\gamma_1 R^2 + \gamma_2 R_{ij} R_{ij} + \gamma_3 R \nabla a^i + \gamma_4 a_i \Delta a^i\right)$$

$$\quad + \zeta^{-4} \left[\gamma_5 \left(\nabla_i R\right)^2 + \gamma_6 \left(\nabla_i R_{jk}\right)^2 + \gamma_7 \left(\Delta R\right) \nabla_i a^i$$

$$\quad + \gamma_8 a^i \Delta^2 a_i\right],$$

(2.52)

\(^{14}\) It is interesting to note that in the case $\lambda = 1/3$ two extra second-class constraints appear [146, 147]. As a result, in this case the spin-0 graviton is eliminated even when $\beta_0 = 0$. However, since $\beta_0$ in general is subjected to radiative corrections, it is not clear which symmetry preserves this particular value. In [29], Hořava showed that at this fixed point the theory has a conformal symmetry, provided that the detailed balance condition is satisfied. But, there are two issues related to the detailed balance condition as mentioned before: (i) the resulted theory is no longer power-countering, as the sixth-order operators are eliminated by this condition; and (ii) in the IR the Newtonian limit does not exist, so the theory is not consistent with observations. For further discussions of the running of the coupling constants in terms of RG flow, see [169].
where \(\gamma_n\)'s are all dimensionless coupling constants.

As first noticed by BPS, the most stringent constraints come from the preferred frame effects due to Lorentz violation, which require \([96, 134]\),

\[
|\lambda - 1| \lesssim 4 \times 10^{-7}, \quad M_* \lesssim 10^{15} \text{ GeV.} \quad (2.53)
\]

In addition, the timing of active galactic nuclei \([148]\) and gamma ray bursts \([149]\) require

\[
M_* \gtrsim 10^{10} \sim 10^{11} \text{ GeV.} \quad (2.54)
\]

To obtain the constraint \((2.54)\), BPS used the results from the Einstein-aether theory, as these two theories coincide in the IR \([150, 151]\).

Limits from binary pulsars were also studied recently, and the most stringent constraints of the theory were obtained \([152, 153]\). However, when the allowed range of the preferred frame effects given by the solar system tests is saturated, the limit for \(\lambda\) is still given by Eq.\((2.53)\), so the upper bound \(M_*\) remains the same.

It is also interesting to note that observations of synchrotron radiation from the Crab Nebula seems to require that the scale of Lorentz violation in the matter sector must be \(M_{pl}\), not \(M_*\) \([154]\). In addition, the consistency of the theory with the current observations of gravitational waves from the events GW150914 and GW151226 were studied recently, and moderate constraints were obtained \([155]\).

Since the spin-0 graviton generically appears in this version of Hořava gravity \([156, 157]\), strong coupling problem is inevitable \([158, 159]\). However, as mentioned in the last subsection, this can be solved by introducing an energy scale \(M_\ast\) that suppresses the high-order operators \([128]\). When the energy of the system is higher than \(M_\ast\), these high-order operators become dominant, and the scaling law will be changed from \(z = 1\) to with \(z = d\) given by Eq.(2.1). With such anisotropic scalings, all the nonrenormalizable terms (in the case with \(z = 1\)) now become either strictly renormalizable or superrenormalizable, whereby the strong coupling problem is resolved.

For more details, we refer readers to \([127, 128]\)\(^\dagger\). Note that in this case the strong coupling energy is given by \([96, 128]\),

\[
\Lambda_{\omega} = |\lambda - 1|^{1/2} M_{pl}, \quad (2.55)
\]

instead of that given by Eq.(2.39) for the HMT extension with a local U(1) symmetry.

With the non-projectability condition, static space-times with \(\beta_0 = 0\) have been extensively studied, see for example, \([86, 160–166]\). Since all of these works were carried out before the realization of the importance of the term \(q_{\omega}a^4\) that could play in the IR stability \([95, 96]\), so they all unfortunately belong to the branch of the non-projectable Hořava gravity that is plagued with the instability problem \(^{16}\). Lately, the case with \(\beta_0 \neq 0\) were studied in \([167–170]\). In particular, it was claimed that slowly rotating black holes do not exist in this extension \([169]\). It was shown that this is incorrect \([171, 172]\), and slowly rotating black holes and stars indeed exist in all the four versions of Hořava gravity introduced in this review \([173]\)\(^\dagger\).

Cosmological perturbations were studied in \([174–176]\) and found that they are consistent with observations. Cosmological constraints of Lorentz violation from dark matter and dark energy were also investigated in \([177–180]\).

In addition, the odd terms given in Eq.(2.13) violate the parity, which can polarize primordial gravitational waves and lead to direct observations \([181, 182]\). In particular, it was shown that, because of both parity violation and the nonadiabatic evolution of the modes due to a modified dispersion relationship, a large polarization of primordial gravitational waves becomes possible, and could be within the range of detection of the BB, TB and EB power spectra of the forthcoming CMB observations \([182]\). Of course, this conclusion is not restricted to this version of Hořava gravity, and in principle it is true in all of these four versions presented in this review.

### D. With Non-projectability & U(1) Symmetry

A non-trivial generalization of the enlarged symmetry \((2.34)\) to the nonprojectable case \(N = N(t, x)\) was realized in \([73, 97, 98]\), and has been recently embedded into string theory by using the non-relativistic AdS/CFT correspondence \([183, 184]\). In addition, it was also found that it corresponds to the dynamical Newton-Cartan geometry with twistless torsion (hypersurface orthogonal foliation) \([145]\). A precise dictionary was then constructed, which connect all fields, their transformations and other properties of the two corresponding theories. Further, it was shown that the U(1) symmetry comes from the Bargmann extension of the local Galilean algebra that acts on the tangent space to the torsional Newton-Cartan geometries.

The realization of the enlarged symmetry \((2.34)\) in the nonprojectable case can be carried out by first noting

\(^{15}\) The mechanism requires the coupling constants of the six-order derivative terms, represented by \(\gamma_n\) \((n = 5, 6, 7, 8)\) in the action \((2.52)\), must be very large \(\gamma_n \simeq \sqrt{\lambda - 1} \geq 10^7\). This may introduce a new hierarchy \([74]\), although it was argued that this is technically natural \([96, 128]\).

\(^{16}\) In \([39]\) attention has been called on the self-consistency of the theory. In particular, one would like first resolve the instability problem before considering any application of the theory.

\(^{17}\) It should be noted that “black holes” here are defined by the existence of Killing/sound horizons. But, particles can have velocities higher than that of light, once the Lorentz symmetry is broken \([25]\). Then, Killing/sound horizons are no longer barriers to these particles, so such defined “black holes” are not really black to such particles.
the fact that $\mathcal{N}_i$ and $\sigma$ defined in Eq. (2.42) are gauge-invariant under the local U(1) transformations and are a vector and scalar under Diff(M,F), respectively. In addition, they have dimensions two and four, respectively,

$$\delta\sigma_\mathcal{N}_i = 0 = \delta\sigma_\sigma, \quad [\mathcal{N}_i] = 2, \quad [\sigma] = 4. \quad (2.56)$$

Then, the quantity $\hat{K}_{ij}$ defined by

$$\hat{K}_{ij} \equiv \frac{1}{2N} (-g_{ij} + \nabla_i N_j + \nabla_j N_i) = K_{ij} + \nabla_i \nabla_j \varphi + a_i (\nabla_j \varphi), \quad (2.57)$$

is also gauge-invariant under the U(1) transformations. In addition, $\hat{K}_{ij}$ has the same dimension as $K_{ij}$, i.e., $[\hat{K}_{ij}] = 3$, from which one can see that the quantity $\tilde{L}_K$,

$$\tilde{L}_K \equiv \hat{K}^{ij} \hat{K}_{ij} - \lambda \hat{K}^2, \quad (2.58)$$

has dimension 6. In addition, the quantity

$$\tilde{L}_S \equiv \sigma (Z_A + 2\Lambda_g - R), \quad (2.59)$$

has also dimension 6 and is gauge-invariant under the U(1) transformations, where $\tilde{Z}_A \equiv \sigma_1 a^\alpha a_i + \sigma_2 a^i$, with $\sigma_1$ and $\sigma_2$ being two dimensionless coupling constants. Then, the action

$$S_g = \zeta^2 \int \text{d}^3x \sqrt{g} \left[ \mathcal{L}_K - \gamma_1 R - 2\Lambda + \beta a^\alpha a_i + \sigma (Z_A + 2\Lambda_g - R) - \mathcal{L}_{z>1} \right], \quad (2.60)$$

gives the potential part that contains spatial derivative operators higher than second-orders. Inserting Eqs. (2.57)-(2.59) into the above action, and then after integrating it partially, the total action takes the form [98],

$$S_g = \zeta^2 \int \text{d}^3x \sqrt{g} \left[ \mathcal{L}_K - \mathcal{L}_V + \mathcal{L}_A + \mathcal{L}_\varphi + \mathcal{L}_S \right], \quad (2.61)$$

where $\mathcal{L}_K$ and $\mathcal{L}_A$ are given, respectively, by Eqs. (2.15) and (2.38), but now with

$$\mathcal{L}_\varphi \equiv \varphi \mathcal{G}^{ij} (2K_{ij} + \nabla_i \nabla_j \varphi + a_i \nabla_j \varphi) + (1 - \lambda) \left[ (\Delta \varphi + a_i \nabla_i \varphi)^2 + 2(\Delta \varphi + a_i \nabla_i \varphi) K \right] + \frac{\alpha}{3} \hat{G}^{ijk} \left[ 4(\nabla_i \nabla_j \varphi) a_k (\nabla_k \varphi) + 5 (a_i (\nabla_j \varphi) a_k (\nabla_k \varphi) + 2(\nabla_i \varphi) a_j (\nabla_k \varphi) + 6K_{ij} a_k (\nabla_k \varphi) \right],$$

$$\mathcal{L}_S \equiv \sigma (\sigma_1 a^\alpha a_i + \sigma_2 a^i), \quad (2.62)$$

where $\hat{G}^{ijk} \equiv \hat{g}^{ik} g^{jl} - \hat{g}^{ij} g^{kl}$, and $\mathcal{L}_V$ denotes the potential part of the theory made of $a_i$, $\nabla_i$ and $R_{ij}$ only. However, as mentioned previously, once the projectability condition is abandoned, it gives rise to a proliferation of a large number of independent coupling constants 

[74, 97]. To reduce the number, one way is to generalize the Horava detailed balance condition (2.19) to [97],

$$\mathcal{L}_{(V,D)} = \left( E_{ij} A_i \right) \left( \begin{array}{cc} G^{ijkl} & 0 \\ 0 & g^{ij} \end{array} \right) \left( \begin{array}{c} E_{kl} \\ A_j \end{array} \right), \quad (2.63)$$

where $E_{ij}$ and $A_i$ are given by

$$E^{ij} \equiv \frac{1}{\sqrt{g}} \frac{\delta W_g}{\delta g_{ij}}, \quad A^i \equiv \frac{1}{\sqrt{g}} \frac{\delta W_a}{\delta a_i}. \quad (2.64)$$

The super-potentials $W_g$ and $W_a$ are constructed as [18],

$$W_g = \frac{1}{w^2} \int \omega_3 (\Gamma) + \mu \int d^3x \sqrt{g} (R - 2\Lambda),$$

$$W_a = \int d^3x \sqrt{g} \sum_{n=0}^1 B_n a^i \Delta^n a_i, \quad (2.65)$$

where $B_n$ are coupling constants. However, to have a healthy infrared limit, as mentioned previously, the detailed balance condition needs to be broken softly, by adding all the low dimensional operators, so that the potential finally takes the form [97],

$$\mathcal{L}_V = 2\Lambda - \left( \beta_0 a^i a_i - \gamma_1 R \right) + \frac{1}{\zeta^2} \left( \gamma_2 R^2 + \gamma_3 R_{ij} R^{ij} \right) + \frac{1}{\zeta^2} \left( \beta_1 (a^i a_i)^2 + \beta_2 (a^i a_i)^2 + \beta_3 (a^i a_i) a_j a^j \right) + \beta_4 a^i a_j a_j + \gamma_5 C_{ij} C^{ij} + \beta_8 \left( \Delta a^i \right)^2, \quad (2.66)$$

where all the coefficients, $\beta_n$ and $\gamma_n$, are dimensionless and arbitrary, except for the ones of the sixth-order derivative terms, $\gamma_5$ and $\beta_8$, which are positive,

$$\gamma_5 > 0, \quad \beta_8 > 0, \quad (2.67)$$

as can be seen from Eqs. (2.63)-(2.65). The Cotton tensor $C_{ij}$ is defined as

$$C^{ij} = \epsilon^{ijk} \nabla_k \left( R^i_{ij} - \frac{1}{4} R \delta^i_j \right). \quad (2.68)$$

---

18. Note that in [185] a different generalization was proposed, in which $\mathcal{L}_{(V,D)}$ takes the same form in terms of $E_{ij}$ and $\mathcal{G}^{ijkl}$, as that given by Eq. (2.19), but with the superpotential including a term $a_i a^i$, i.e., $W_g = \frac{1}{w^2} \int \omega_3 (\Gamma) + \int d^3x \sqrt{g} \left[ \mu (R - 2\Lambda) + \beta a_i a^i \right]$. However, in this generalization, the six-order derivative term $(\Delta a^i)^2$ does not exist in the potential $\mathcal{L}_V$, and is a particular case of Eq. (2.66) with $\beta_8 = 0$. Without this term, the sixth-order derivative terms are absent for scalar perturbations. As a result, the corresponding theory is not power-counting renormalizable [97].
In terms of $R_{ij}$ and $R$, we have [97],
\[
C_{ij}G^{ij} = \frac{1}{2} R^3 - \frac{5}{2} R R_{ij} R^{ij} + 3 R^i_j R^j_k R^k_l + \frac{3}{8} R \Delta R \\
+ (\nabla_i R_{jk}) (\nabla^i R^{jk}) + \nabla_k G^k,
\]
where
\[
G^k = \frac{1}{2} R^{ik} \nabla_j R - R_{ij} \nabla^j R^{ik} - \frac{3}{8} R \nabla^k R. \tag{2.69}
\]

When $\sigma_1 = \sigma_2 = 0$, it was shown that the spin-0 graviton is eliminated [73, 97, 98]. This was further confirmed by Hamiltonian analysis [186]. For the universal coupling with matter given by Eqs. (2.41) - (2.43), all the PPN parameters were calculated and given explicitly in terms of the coupling constants [98, 187]. In particular, it was found that with the choice of the parameters $a_1$ and $a_2$ given by Eq. (2.47), the relativistic results (2.46) are also obtained.

When $\sigma_1 \sigma_2 \neq 0$ the spin-0 graviton appears [98, 186]. It was shown that the theory is ghost-free for the choice of $\lambda$ given by Eq. (2.32), and the spin-0 graviton is stable in both of the IR and UV, provided that the following holds [98],
\[
\beta_0 > 0, \quad \sigma_2 - \sigma_2 < \sigma^+_2, \tag{2.71}
\]
where $\sigma^+_2 \equiv 4 \left( -\gamma_1 \pm \sqrt{\gamma_1^2 - \beta_0/2} \right)$ and $\gamma_2^2 - \beta_0/2 \geq 0$.

In this case, the analysis of the PPN parameters shows that the corresponding theory is consistent with all the solar system tests in a large region of the phase space [98, 187]. In particular, when the two coupling constants $\sigma_2$ and $\beta_0$ satisfy the relations [98]
\[
\sigma_2 = 4(1 - a_1), \quad \beta_0 = -2(\gamma_1 + 1), \tag{2.72}
\]
the relativistic values of the PPN parameters given by Eq. (2.46) can be still achieved.

The strong coupling problem appearing in the gravitational and/or matter sectors can be resolved [73], by the BPS mechanism introduced previously [128].

The consistency of the theory with cosmology was studied in [188] when matter is minimally coupled to the theory. For the universal coupling of Eqs. (2.41) - (2.43), such studies have not been worked out, yet [139].

The effects of parity violation on non-Gaussianities of primordial gravitational waves were also studied recently [189]. By calculating the three point function, it was found that the leading-order contributions to the non-Gaussianities come from the usual second-order derivative terms, which produce the same bispectrum as that found in GR. The contributions from high-order spatial n-th derivative terms are always suppressed by a factor $(H/M_*)^{n-2}$ ($n \geq 3$), where $H$ denotes the inflationary energy and $M_*$ the suppression mass scale of the high-order spatial derivative operators of the theory. Thus, the next leading-order contributions come from the 3-dimensional gravitational Chern-Simons term. With some reasonable arguments, it is shown that this 3-dimensional operator is the only one that violates the parity and in the meantime has non-vanishing contributions to non-Gaussianities.

Finally we note that, instead of introducing the U(1) gauge field $A$ and the pre-Newtonian potential $\phi$, the authors in [190] introduced two Lagrange multipliers $A_{C KO}$ and $A_{C KO}$ (Don’t confuse with the U(1) gauge field $A$ and the cosmological constant $\Lambda$ introduced above), and found that in the non-projectable case the spin-0 graviton can be eliminated for certain choices of the coupling constants of the theory, while for other choices the spin-0 graviton is still present. However, it is not clear which symmetry, if there is one, will preserve these particular choices, so that the spin-0 graviton is always absent. Otherwise, it will appear generically once radiative corrections are taken into account. It is interesting to note that, in contrast to the non-projectable case, in the projectable case the spin-0 graviton is always eliminated [190].

In addition, Hořava gravity with mix spatial and time derivatives has been also studied recently in order to avoid unacceptable violations of Lorentz invariance in the matter sector [191, 192]. Unfortunately, it was found that the theory contains four propagating degrees of freedom, as opposed to three in the standard Hořava gravity, and the new degree of freedom is another scalar graviton, which is unstable at low energies [193].

E. Covariantization of Hořava Gravity

As GR is general covariant and consistent with all the observations, both cosmological and astrophysical, it is desired to write Hořava Gravity also in a general covariant form $^{20}$, in order to see more clearly the difference between the two theories. One may take one step further and consider such a process as a mechanism to obtain the IR limit of the UV complete theory of Hořava gravity with the symmetry of general covariance.

To restore the full general covariance, one way is to make the foliation dynamical by using the Stückelberg trick [194], in which the space-like hypersurfaces of $t$ constant in the (3+1) ADM decompositions are replaced by a scalar field $\phi = \text{constant}$, which is always time-like [88, 195],
\[
u_{\mu}u_{\mu} = -1, \quad u_{\mu} \equiv \frac{\phi_{,\mu}}{\sqrt{X}}, \quad X \equiv -\gamma^{\mu\nu}\phi_{,\mu}\phi_{,\nu} > 0. \tag{2.73}
\]

$^{20}$ This process in general introduces new physics, for example, the appearance of the instantaneous mode in the khrnonometric theory, which is absent in both Hořava and Einstein-aether theories, as shown below.
Clearly, choosing $\phi = t$, the original foliations are obtained. This gauge choice is often referred to as the unitary gauge [88]. Then, the gauge symmetry $t \rightarrow \xi_0(t)$ translates into the symmetry of the St"{u}ckelberg field,

$$\phi \rightarrow \tilde{\phi} = f(\phi), \quad (2.74)$$

where $f(\phi)$ is an arbitrary monotonic function of $\phi$ only, so that

$$\tilde{u}_\mu \equiv \frac{\tilde{\phi}_\mu}{\sqrt{X}} = \frac{\epsilon f \phi_\mu}{\sqrt{X}} = \epsilon f u_\mu, \quad (2.75)$$

where $\epsilon_f \equiv \text{Sign}[df(\phi)/d\phi]$. Once $u_\mu$ is introduced, all other geometric quantities can be constructed from $\gamma_{\mu\nu}$ and $u_\mu$ by following Israel's method for space-like hypersurfaces [196, 197]. For example, the extrinsic curvature $K_{\mu\nu}$ is given by,

$$K_{\mu\nu} = h_{\lambda\mu} D^\lambda u_\nu = \frac{1}{\sqrt{X}} h_{\mu}^\alpha h_{\nu}^\beta D_\alpha D_\beta \phi, \quad (2.76)$$

where $D^\lambda$ denotes the covariant derivative with respect to $\gamma_{\mu\nu}$, and $h_{\mu\nu}$ is the projection operator, defined as

$$h_{\mu\nu} \equiv \gamma_{\mu\nu} + u_\mu u_\nu. \quad (2.77)$$

Defining the compatible covariant derivative $\nabla_\mu$ on the hypersurfaces $\phi = \text{constant}$ as,

$$\nabla_\mu T_{\beta_1...\beta_l}^{\alpha_1...\alpha_k} \equiv \delta_{\alpha_1}^{\delta_1}...\delta_{\alpha_k}^{\delta_k} h_{\delta_1}^\alpha ... h_{\delta_k}^\alpha h_{\beta_1}^\beta ... h_{\beta_l}^\beta h_{\mu}^\beta D_\mu T_{\sigma_1...\sigma_l}^{\delta_1...\delta_k}, \quad (2.78)$$

we find that

$$\nabla_\mu R_{\rho\alpha\beta\gamma} = h_{\mu}^\alpha h_{\nu}^\beta h_{\rho}^\gamma D_\alpha R_{\rho\nu\beta\gamma}, \quad (2.79)$$

where $(4)R_{\alpha\beta\gamma\delta}$ denotes the 4D Riemann tensor made out of the metric $\gamma_{\mu\nu}$, and $R_{\mu\nu\rho\lambda}$ the intrinsic Riemann tensor with $R_{\mu\nu} \equiv R_{\lambda\mu\nu\lambda}$. Thus, we find that

$$R_{\mu\nu} = h_{\mu}^\alpha h_{\nu}^\beta (4)R_{\alpha\beta} + u_\alpha u_\beta (4)R_{\alpha\beta\mu\nu} + K_{\mu\nu} K_{\alpha\beta} - K_{\alpha\beta} K_{\mu\nu}, \quad (2.80)$$

$$\nabla_\mu \nabla_\nu R_{\rho\mu\nu\rho} = (4) R + K^2 - K_{\mu\nu} K_{\nu\mu} + 2D_\alpha G^\alpha, \quad (2.81)$$

Note that in writing the above expressions, we had used the relation [196, 197],

$$(4)R_{\alpha\beta} u^\alpha u^\beta = K^2 - K_{\alpha\beta} K^{\alpha\beta} + D_\alpha G^\alpha, \quad (2.82)$$

where $K \equiv K_{\lambda\gamma}$. Then, to obtain the relativistic version of Ho\v{r}ava gravity, one can simply identify the quantities appearing in Ho\v{r}ava gravity in the ADM decompositions with those obtained in the unitary gauge, which leads to the following replacements in the action,

$$N \rightarrow \frac{1}{\sqrt{X}}, \quad \nabla_i \rightarrow \nabla_\lambda, \quad (2.83)$$

where

$$\frac{\partial}{\partial \psi} \rightarrow u^\nu h_\nu^\mu D_\mu \psi, \quad \nabla_\nu \rightarrow R_{\nu}, \quad K_\mu \rightarrow K_{\mu\nu}, \quad C^{ij} \rightarrow C^{\mu\nu}, \quad (2.84)$$

and

$$\frac{\partial}{\partial \phi} \rightarrow u^\nu h_\nu^\mu D_\mu \phi + \frac{1}{2} (\nabla_\lambda \phi)^2, \quad (2.85)$$

where $\eta^{\mu\nu\alpha\beta}$ is the three-dimensional volume element defined as $\eta^{\mu\nu\alpha\beta} \equiv \eta^{\mu\nu} \delta_3^\alpha \delta_3^\beta$, and $\eta^{\mu\nu\alpha\beta}$ denotes the four-dimensional volume element with $\nabla_\alpha \eta^{\mu\nu\alpha\beta} = 0$ [195].

In the case with an extra U(1) symmetry, we also replace the gauge field $A$ and the pre-Newton potential $\varphi$, respectively, by [198],

$$A \rightarrow \frac{1}{\sqrt{X}} \left( \sigma + u^\nu h_\nu^\mu D_\mu \varphi + \frac{1}{2} (\nabla_\lambda \varphi)^2 \right), \quad (2.86)$$

where $\sigma$ defined by Eq. (2.42) and $\varphi$ are scalars under both the U(1) and the Diff$(M, F)$ transformations. Therefore, the resulting theories from the ones with the local U(1) symmetry will contain two Lagrangian multipliers $\sigma$ and $\varphi$, each of them acts like a scalar field, but none of them is dynamical, in contrast to the St"{u}ckelberg field $\phi$, introduced above. This is similar to what was done in [190], and it would be very interesting to find out if they are related one to the other.

Applying the above St"{u}ckelberg trick to the healthy extension [95, 96], it can be shown that the resulting action in the IR is identical to the hypersurface-orthogonal Einstein-aether theory [199, 200], as shown explicitly in [150, 151], provided that the aether four-vector $u_\mu$ satisfies the hypersurface-orthogonal conditions,

$$u_\mu D_\lambda u_\lambda = 0, \quad (2.87)$$

which are necessary and sufficient conditions for $u_\mu$ to be given by Eq. (2.73) [201]. From Eq. (2.85) it can be shown that

$$\omega^2 \equiv a^\mu a_\mu + (D_\alpha u_\beta)(D_\beta u^\alpha) - (D_\alpha u_\beta)(D_\beta u^\alpha), \quad (2.88)$$

vanishes identically, where $a_\mu \equiv u^\lambda D_\lambda u_\mu$. Then, one can add the term,

$$\Delta S_w \equiv c_0 \int dx^4 \sqrt{-\gamma} \omega^2, \quad (2.89)$$

where $c_0$ is an arbitrary constant and $\gamma \equiv \det(\gamma_{\mu\nu})$, into the action of the Einstein-aether theory [199, 200],

$$S_w = \int d^4 x \sqrt{-\gamma} \left[ c_1 (D_\mu u_\nu)^2 + c_2 (D_\mu u^\mu)^2 \right] + \Delta S_w.$$
to eliminate one of the four independent coupling constants \( c_i \) (\( i = 1, 2, 3, 4 \)) of the Einstein-aether theory, so that there are only three independent coupling constants in terms of the St"uckelberg field \( \phi \), which will be referred to as the \textit{khronon field}, and the corresponding theory the \textit{the khronometric theory} [96, 202], which was also referred to as the “T-theory” in [150, 151].

Some comments are in order. First, the action (2.88) is the most general action of the Einstein-aether theory in which the field equations for \( \gamma_{\mu\nu} \) and \( u_\mu \) are the second-order differential equations [199, 200]. In this theory, in addition to the spin-2 graviton encountered in GR, there are two extra modes, the spin-1 and spin-0 gravitons, each of them moves with a different velocity \( v_\mu \) [199, 200]. To avoid the gravitational Cerenkov effects [203], each of them must be not less than the speed of light, \( v_\mu \geq c \). Second, the khronometric theory is fundamentally different from the Einstein-aether theory even in the domain where the hypersurface-orthogonal conditions (2.85) hold. This can be seen clearly by the fact that there is an additional mode, the instantaneous propagation of signals, in the khronometric theory [202], which is absent in the Einstein-aether gravity. This is closely related to the fact that in terms of the khronon field \( \phi \), the action (2.88) are in general fourth-order differential equations. It is the existence of this mode that may provide a mechanism to restore the second law of thermodynamics [202]. It is also interesting to note that this instantaneous mode does not exist in the healthy extension either [95, 96]. It is introduced by the covariantization process with the use of the St"uckelberg trick.

We also note that in [88, 96], the covariantization of the projectable Ho\v{r}ava gravity was also studied, and showed that this minimal theory is always fine-tuning due to the instability of the spin-0 graviton in this version of Ho\v{r}ava gravity.

III. BLACK HOLES & THERMODYNAMICS

Black holes have been objects of intense study both theoretically and observationally over half a century now [9, 10], and so far there are many solid observational evidences for their existence in our universe, especially after the first detection of gravitational waves from two binary black holes on September 14, 2015 by aLIGO [11]. Theoretically, such investigations have been playing a fundamental role in the understanding of the nature of gravity in general, and QG in particular. They started with the discovery of the laws of black hole mechanics [204] and Hawking radiation [205], which led to the profound recognition of the thermodynamic interpretation of the four laws [206] and the reconstruction of GR as the thermodynamic limit of a more fundamental theory of gravity [207]. More recently, they play the essential role in the understanding of the AdS/CFT correspondence [78–82]

\[
+ c_3 (D^\alpha u^\alpha) (D_\nu u_\mu) - c_4 a_\mu a_\mu, \quad (2.88)
\]

and firewalls [208, 209].

Lately, such studies have gained further momenta in the framework of gravitational theories with broken LI, including Ho\v{r}ava gravity. In such theories, due to the breaking of LI, the dispersion relation of a massive particle contains generally high-order momentum terms [34–41],

\[
E^2 = m^2 + c_p p^2 \left( 1 + \sum_{n=1}^{2(z-1)} a_n \left( \frac{p}{M_*} \right)^n \right), \quad (3.1)
\]

from which we can see that both of the group and phase velocities, \( v_g = E/p \) and \( v_p = dE/dp \), become unbounded as \( p \to \infty \), where \( E \) and \( p \) are the energy and momentum of the particle considered, and \( c_p \) and \( a_n \)'s are coefficients, depending on the species of the particle, while \( M_* \) is the suppression energy scale of the higher-dimensional operators. As an immediate result, the causal structure of the spacetimes in such theories is quite different from that given in GR, where the light cone at a given point \( p \) plays a fundamental role in determining the causal relationship of \( p \) to other events [cf. Fig. 2]. However, once LI is broken, the causal structure will be dramatically changed. For example, in the Newtonian theory, time is absolute and the speeds of signals are not limited. Then, the causal structure of a given point \( p \) is uniquely determined by the time difference, \( \Delta t = t_q - t_p \), between the two events. In particular, if \( \Delta t > 0 \), the event \( q \) is to the past of \( p \); if \( \Delta t < 0 \), it is to the future; and if \( \Delta t = 0 \), the two events are simultaneous. In Ho\v{r}ava gravity, a similar situation occurs. This immediately makes the definitions of black holes given in GR [12] invalid.

To provide a proper definition of black holes, anisotropic conformal boundaries [210] and kinematics of particles [102, 167, 211–214] have been studied within the framework of Ho\v{r}ava gravity. Lately, a potential breakthrough was the discovery that there still exist absolute causal boundaries, the so-called \textit{universal horizons}, in theories with broken LI [202, 215]. Particles even with infinitely large velocities would just move around on these boundaries and cannot escape to infinity. The main idea is as follows. In a given spacetime, a globally timelike

\[
\begin{align*}
\text{FIG. 2.} \quad (a) \text{The light cone of the event } p \text{ in special relativity.} \\
\text{(b) The causal structure of the point } p \text{ in theories with broken LI. This figure is adopted from [102].}
\end{align*}
\]
scalar field \(\phi\) might exist \([216]\). Then, similar to the Newtonian theory, this field defines a global absolute time, and all particles are assumed to move along the increasing direction of the timelike scalar field, so the causality is well defined, similar to the Newtonian case [Cf. Fig. 2]. In such a spacetime, there may exist a surface as shown in Fig. 3, denoted by the vertical solid line, located at \(r = r_{UH}\). Given that all particles move along the increasing direction of the timelike scalar field, from Fig. 3 it is clear that a particle must cross this surface and move inward, once it arrives at it, no matter how large of its velocity is. This is an one-way membrane, and particles even with infinitely large speed cannot escape from it, once they are inside it. So, it acts as an absolute horizon to all particles (with any speed), which is often called the universal horizon \([202, 215, 216]\). At the universal horizon, we have \(dt\cdot d\phi = 0\), or equivalently,

\[
\zeta \cdot u = 0,
\]

where \(\zeta (= \partial_t)\) denotes the asymptotically timelike Killing vector, and \(u (= u_A dx^A)\) is defined in terms of the globally timelike scalar field \(\phi\) via Eq. (2.73). It is because of this that the globally timelike scalar field \(\phi\) is also called the khronon field. But, there is a fundamental difference between the scalar field introduced here and the one introduced in Section II.E. In particular, the one introduced in Section II.E is part of the gravitational field and represents the extra degrees of the gravitational sector, while the one introduced here is a test field, which plays the same role as a Killing vector in a given spacetime background satisfies the Killing equations, \(D_{(\alpha} \zeta_{\beta)} = 0\). To define the globally timelike scalar field \(\phi\), one way is to adopt the equations for the khronon field given by the general action (2.88), which can be written in the form \([151]\),

\[
S_{\phi} = \int d^4x \sqrt{-\gamma} \left[ \frac{1}{3} c_0 \theta^2 + c_\sigma \sigma^2 - c_\omega \omega^2 \right],
\]

where

\[
c_\theta \equiv c_{13} + 3c_2, \quad c_\sigma \equiv c_{13}, \quad c_\omega \equiv c_1 - c_3, \quad c_a \equiv c_{14},
\]

\(21\) In \([202, 215]\) the globally timelike scalar field is identified to the hypersurface-orthogonal aether field \(u_A\), in which the aether is part of the gravitational field, the existence of which violates LI. However, to apply such a concept to other theories (without an aether field), a generalization is needed. In \([216]\), the hypersurface-orthogonal aether field was promoted to a field that plays the same role as a Killing vector does in GR.

\(22\) The quantity \(\sigma\) introduced here must not be confused with one introduced in Eq. (2.42).

\(23\) Notice the difference between the signatures of the metric chosen in this paper and the ones in \([171]\).

FIG. 3. Illustration of the bending of the \(\phi = \text{constant}\) surfaces, and the existence of the universal horizon in the Schwarzschild spacetime \([216]\), where \(\phi\) denotes the globally timelike scalar field, and \(t\) and \(r\) are the Painlevé-Gullstrand coordinates. Particles move always along the increasing direction of \(\phi\). The Killing vector \(\zeta^\mu = \delta^\mu_0\) always points upward at each point of the plane. The vertical dashed line is the location of the Killing horizon, \(r = r_{KH}\). The universal horizon, denoted by the vertical solid line, is located at \(r = r_{UH}\), which is always inside the Killing horizon. This figure is adopted from \([217]\).

\[
D_\beta u_\alpha = \frac{1}{3} \theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta} - a_\alpha u_\beta,
\]

with \(c_{13} \equiv c_1 + c_3, c_{14} \equiv c_1 + c_4\), and

\[
\theta \equiv D_\lambda u^\lambda, \quad h_{\alpha\beta} = \gamma_{\alpha\beta} + u_\alpha u_\beta,
\]

\[
\sigma_{\alpha\beta} \equiv D_{(\beta} u_{\alpha)} + a_\alpha u_\beta - \frac{1}{3} \theta h_{\alpha\beta},
\]

\[
\omega_{\alpha\beta} \equiv D_{[\beta} u_{\alpha]} + a_\alpha u_\beta,
\]

where \((A, B) \equiv (AB + BA)/2\). When \(u_\mu\) is hypersurface-orthogonal, we have \(\omega^\alpha = 0\), so the last term in the above action vanishes identically, as mentioned in Section II.E. Even so, the equation for \(\phi\) still contains three free parameters, \((c_\theta, c_\sigma, c_a)\), and their physical interpretations are not clear.

Hence, the variation of \(S_{\phi}\) with respect to \(\phi\) yields the khronon equation,

\[
D_\mu A^\mu = 0,
\]

where \([171]\),

\[
A^\mu = \frac{(\delta^\mu_0 + u^\nu u_\nu)}{\sqrt{X}} E^\nu,
\]

\[
E^\nu \equiv D_\gamma J^\nu + c_4 a_\nu D^\nu u_\gamma,
\]
\[ J^\alpha_{\mu} = (c_1 g^{\alpha\beta} g_{\mu\nu} + c_2 \delta^\alpha_{\mu} \delta^3_{\nu} + c_3 \delta^\alpha_{\mu} \delta^3_{\nu} - c_4 u^\alpha u^3 g_{\mu\nu}) D_\beta u^\nu. \]  

Eq. (3.7)

To solve the above equations for \( \phi \), one can divide it into two steps: First solve Eq. (3.6) in terms of \( u_\mu \). Once \( u_\mu \) is known, one can solve \( \phi \) from the definition,

\[ \phi,_{\mu} = \sqrt{-\gamma^{\alpha\beta} \phi,_{\alpha} \phi,_{\beta}} u_\mu. \]  

Eq. (3.8)

Eq. (3.6) is a second-order differential equation for \( u_\mu \), and to uniquely determine it, two boundary conditions are needed. These two conditions can be chosen as follows [202]: (i) It will be aligned asymptotically with the timelike Killing vector,

\[ u^\mu \propto \zeta^\mu. \]  

(ii) The second condition can be that the khronon has a regular future sound horizon, which is a null surface of the effective metric,

\[ \gamma_{\mu\nu}^{(\phi)} = \gamma_{\mu\nu} - (c^2 - 1) u_\mu u_\nu, \]  

where \( c_\phi \equiv (c_{13} + c_2)/(c_{14}) \) denotes the speed of the khronon [24].

It is interesting to note that when \( c_\alpha = c_{14} = 0 \), the khronon has an infinitely large velocity, and its sound horizon now coincides with the universal horizon. Then, the second condition becomes to require that the universal horizon be regular. This is an interesting choice, and the second condition becomes to require that the universal horizon be regular. This is an interesting choice, and

\[ \text{the second condition becomes to require that the universal horizon be regular. This is an interesting choice, and} \]

the second condition becomes to require that the universal horizon be regular. This is an interesting choice, and

\[ \text{the second condition becomes to require that the universal horizon be regular. This is an interesting choice, and} \]

requires that large black holes emit a thermal flux with a temperature fixed by the surface gravity of the Killing horizon. This, in turn, suggests that the universal horizon should play no role in the thermodynamic properties of these black holes, although it must be noted that in such a setting, the khronon field is not continuous across the collapsing null shell. As mentioned above, a globally-defined khronon plays an essential role in the causality of the theory, so it is not clear how the results presented in [244] will be changed once the continuity of the khronon field is imposed.

On the other hand, using the Hamilton-Jacobi method, quantum tunneling of both relativistic and non-relativistic particles at Killing as well as universal horizons of Einstein-Maxwell-aether black holes were studied [245], after higher-order curvature corrections are taken into account. It was found that only relativistic particles are created at the Killing horizon, and the corresponding radiation is thermal with a temperature exactly the same as that found in GR. In contrary, only non-relativistic particles are created at the universal horizon and are radiated out to infinity with a thermal spectrum. However,
different species of particles, in general, experience different temperatures,
\[ T_{z=2}^{\geq 2} = \frac{2(z-1)}{z} \left( \frac{\kappa_{UH}}{2\pi} \right), \]  
(3.12)
where \( \kappa_{UH} \) is the surface gravity calculated from Eq.(3.11) and \( z \) is the exponent of the dominant term in the UV [cf. Eq.(3.1)]. When \( z = 2 \) we have the standard result,
\[ T_{z=2}^{\geq 2} = \frac{\kappa_{UH}}{2\pi}, \]  
(3.13)
which was first obtained in [241, 243].

Recently, more careful studies of ray trajectories showed that the surface gravity for particles with a non-relativistic dispersion relation (3.1) is given by [246],
\[ \kappa_{UH}^{z \geq 2} = \left( \frac{2(z-1)}{z} \right) \kappa_{UH}, \]  
(3.14)
so that Eq.(3.11) is true only for particles with \( z = 2 \). The same results were also obtained in [247]. It is remarkable to note that in terms of \( \kappa_{UH}^{z \geq 2} \) and \( T_{z=2}^{\geq 2} \), the standard relationship between the temperature and surface gravity of a black hole still holds here,
\[ T_{z=2}^{\geq 2} = \frac{\kappa_{UH}^{z \geq 2}}{2\pi}. \]  
(3.15)

Is this a coincidence? Without a deeper understanding of thermodynamics of universal horizons, it is difficult to say. But, whenever case like this raises, it is worthwhile of paying some special attention on it. In particular, does entropy of a universal horizon also depend on the dispersion relations of particles?

To understand the problem better, another important issue is: Can universal horizons be formed from gravitational collapse? so they can naturally exist in our universe [140]. To answer this question, the collapse of a spherically symmetric thin-shell in a flat background that finally forms a Schwarzschild black hole was studied [248], in which the globally timelike scalar field is taken as the Cuscuton field [249], a scalar field with infinitely large sound speed. It was shown that an observer inside the universal horizon of the Schwarzschild radius cannot send a signal outside, after a stage in collapse, even using signals that propagate infinitely fast in the preferred frame. Then, it was argued that this universal horizon should be considered as the future boundary of the classical space-time. Lately, such studies were generalized to the Reissner-Nordstrom and Kerr black holes [250] 26.

In addition, gravitational collapse of a scalar field in the Einstein-aether theory was studied in [252], prior to the discovery of universal horizons. Lately, it was revisited [253]. However, due to the specially slicing of the spacetime carried out in [252], the numerical simulations cannot penetrate inside universal horizons, so it is not clear whether or not universal horizons have been formed in such simulations.

In any case, the universal horizons defined by Eq.(3.2) are in terms of the timelike Killing vector, which exists only in stationary spacetimes. To study the dynamical formation of universal horizons from gravitational collapse, a generalization of Eq.(3.2) to non-stationary spacetimes is needed. One way, as first proposed in [254], is to replace the timelike Killing vector by a Kodama-like vector [255], which reduces to the timelike Killing vector in the stationary limit. To be more specific, let us consider spacetimes described by the metric,
\[ ds^2 = g_{ab}dx^a dx^b + R^2 (x^0, x^1) d\Sigma_k^2, \]  
(3.16)
in the coordinates, \( x^\mu = (x^0, x^1, \theta, \varpi), \) \( (\mu = 0, 1, 2, 3), \) where \( k = 0, \pm 1, \) and
\[ d\Sigma_k^2 = \begin{cases} d\theta^2 + \sin^2 \theta d\varpi^2, & k = 1, \\ d\theta^2 + d\varpi^2, & k = 0, \\ d\theta^2 + \sin^2 \theta d\varpi^2, & k = -1. \end{cases} \]  
(3.17)
The normal vector \( n_\mu \) to the hypersurface \( R = C_0 \) is given by,
\[ n_\mu \equiv \frac{\partial (R - C_0)}{\partial x^\mu} = \delta_\mu^0 R_0 + \delta_\mu^i R_i, \]  
(3.18)
where \( C_0 \) is a constant and \( R_\mu \equiv \partial R/\partial x^\mu. \) Setting
\[ \zeta^\mu \equiv \delta_\mu^i R_i - \delta_\mu^0 R_0, \]  
(3.19)
we can see that \( \zeta^\mu \) is always orthogonal to \( n_\mu, \) \( \zeta \cdot n = 0. \) The vector \( \zeta^\mu \) is often called Kodama vector and plays important roles in black hole thermodynamics [256, 257]. For spacetimes that are asymptotically flat there always exists a region, say, \( R > R_\infty, \) in which \( n_\mu \) and \( \zeta^\mu \) are, respectively, space- and time-like, that is,
\[ (n \cdot n)|_{R > R_\infty} > 0, \quad (\zeta \cdot \zeta)|_{R > R_\infty} < 0. \]  
(3.20)
An apparent horizon may form at \( R_{AH}, \) at which \( n_\mu \) becomes null,
\[ (n \cdot n)|_{R = R_{AH}} = 0. \]  
(3.21)
where \( R_{AH} < R_\infty. \) Then, in the internal region \( R < R_{AH}, \) the normal vector \( n_\mu \) becomes timelike. Therefore, we have
\[ (n \cdot n) = \begin{cases} > 0, & R > R_{AH}, \\ 0, & R = R_{AH}, \\ < 0, & R < R_{AH}. \end{cases} \]  
(3.22)

\[ \text{It should be noted that in these studies the Schwarzschild-like coordinates were used, in which the metric outside of the collapsing thin shell is valid only for radius of the shell greater than the radii of the Killing horizons. But, universal horizons occur always inside Killing horizons. In addition, the Cuscuton theory in general does not have the symmetry (2.74) that reflects the gauge symmetry \( t \to \xi_0(t) \) of Ho\'rava gravity. For other related aspects of Cuscuton theory to Ho\'rava gravity and Einstein-aether theory, see [251].} \]
Since Eq.\((3.20)\) always holds, we must have
\[
(\zeta \cdot \zeta) = \begin{cases} 
< 0, & R > R_{AH}, \\
= 0, & R = R_{AH}, \\
> 0, & R < R_{AH},
\end{cases}
\] (3.23)
that is, \(\zeta^\mu\) becomes null on the apparent horizon, and
spacelike inside it.

We define an apparent universal horizon as the hypersurface at which
\[
(u \cdot \zeta)|_{R = R_{UH}} = 0.\] (3.24)

Since \(u_\mu\) is globally timelike, Eq.\((2.19)\) is possible only when \(\zeta_\mu\) is spacelike. Clearly, this is possible only inside the apparent horizon, that is, \(R_{UH} < R_{AH}\).

In the static case, the apparent horizons defined above reduce to the Killing horizons, and the apparent universal horizons defined by Eq.\((3.24)\) are identical to those given by Eq.\((3.2)\).

The above definition is only for spacetimes in which the metric can be cast in the form \((3.16)\). However, the generalization to other cases is straightforward. In particular, it was generalized in terms of an optical scalar built with the preferred flow defined by the preferred spacetime foliation \([258]\).

### IV. NON-RELATIVISTIC GAUGE/GRAVITY DUALITY

Anisotropic scaling plays a fundamental role in quantum phase transitions in condensed matter and ultracold atomic gases \([259, 260]\). Recently, such studies have gained considerable momenta from the community of string theory in the content of gauge/gravity duality \([261, 263, 316]\). This is a duality between a QFT in D-dimensions and a QG, such as string theory, in \((D+1)\)-dimensions.

An initial example was found between the supersymmetric Yang-Mills gauge theory with maximal supersymmetry in four-dimensions and a string theory on a five-dimensional anti-de Sitter space-time in the low energy limit \([80]\). Soon, it was discovered that such a duality is not restricted to the above systems, and can be valid among various theories and in different spacetime backgrounds \([261, 263, 316]\).

One of the remarkable features of the duality is that it relates a strong coupling QFT to a weak coupling gravitational theory, or vice versa. This is particularly attractive to condensed matter physicists, as it may provide hopes to understand strong coupling systems encountered in quantum phase transitions, by simply studying the dual weakly coupling gravitational theory \([264–268]\). Otherwise, it has been found extremely difficult to study those systems. Such studies were initiated in \([269]\), in which it was shown that nonrelativistic QFTs that describe multicritical points in certain magnetic materials and liquid crystals may be dual to certain nonrelativistic gravitational theories in the Lifshitz space-time background,
\[
ds^2 = -\left(\frac{\ell}{t}\right)^{2z} dt^2 + \left(\frac{\ell}{t}\right)^2 dx^i dx^i + \left(\frac{\ell}{r}\right)^2 dr^2,\] (4.1)
where \(z\) is a dynamical critical exponent, and \(\ell\) a dimensional constant. Clearly, the above metric is invariant under the anisotropic scaling,
\[
t \to b^z t, \quad x^i \to b x^i, \quad r \to b^{-1} r.\] (4.2)
Thus, for \(z \neq 1\) the relativistic scaling is broken in the sector \((t, x^i)\), and to have the above Lifshitz space-time as a solution of GR, it is necessary to introduce gauge fields to create a preferred direction, so that the anisotropic scaling \((4.2)\) becomes possible. In \([269]\), this was realized by two p-form gauge fields with \(p = 1, 2\), and was soon generalized to different cases \([270]\).

It should be noted that the Lifshitz space-time is singular at \(r = 0\) \([269]\), and this singularity is generic in the sense that it cannot be eliminated by simply embedding it to high-dimensional spacetimes, and that test particles/strings become infinitely excited when passing through the singularity \([271, 272]\). To resolve this issue, various scenarios have been proposed. There have been also attempts to cover the singularity by horizons, and replace it by Lifshitz solitons \([270]\).

On the other hand, since the anisotropic scaling is built in by construction in Hořava gravity, it is natural to expect that the Hořava theory should provide a minimal holographic dual for non-relativistic Lifshitz-type field theories with the anisotropic scaling and dynamical exponent \(z\). Indeed, recently it was showed that the Lifshitz spacetime \((4.2)\) is a vacuum solution of Hořava gravity in \((2+1)\) dimensions, and that the full structure of the \(z = 2\) anisotropic Weyl anomaly can be reproduced in dual field theories \([273]\), while its minimal relativistic gravity counterpart yields only one of two independent central charges in the anomaly.

Note that in \([273]\), only the IR limit of the \((2+1)\)-dimensional Hořava theory was considered. Later, the effects of high-order operators on the non-relativistic Lifshitz holography were studied \([274]\), and found that the Lifshitz space-time is still a solution of the full theory of the Hořava gravity. The effects of the high-order operators on the space-time itself is simply to shift the Lifshitz dynamical exponent \(z\). However, the asymptotic behavior of a (probe) scalar field near the boundary gets dramatically modified in the UV limit, because of the presence of the high-order operators in this regime. Then, according to the gauge/gravity duality, this in turn affects the two-point correlation functions.

The above studies in the framework of Hořava gravity were soon generalized to various cases, in which various Lifshitz soliton, Lifshitz spacetimes with hyperscaling violations, and Lifshitz charged black hole solutions in terms of universal horizons \([216, 221–226, 230, 232, 234]\) as well as in terms of Killing horizons \([275–277]\) were obtained and studied.
Recently, another important discovery of the non-relativistic gauge/gravity duality is the one-to-one correspondence between the Hořava gravity with the enlarged symmetry \(2.34\) and the Newton-Cartan geometry (NCG) \([145]\). In particular, the projectable Hořava gravity discussed in Section II.B corresponds to the dynamical NCG without torsion, while the non-projectable Hořava gravity discussed in Section II.D corresponds to the dynamical NCG with twistless torsion. A precise dictionary between these two theories was established. Restricted to \((2+1)\) dimensions, the effective action for dynamical twistless torsional NCG with \(1 < z \leq 2\) was constructed by using the NCG invariance, and demonstrated that this exactly agrees with the most general forms of the Hořava actions constructed in \([73, 97, 98]\).

Further, the origin of the local \(U(1)\) symmetry was identified as coming from the Bargmann extension of the local Galilean algebra that acts on the tangent space to the torsional NCG. Such studies have already attracted lots of attention and generalized to other cases \([270]\).

In addition, it was shown that the non-projectable Hořava gravity with the enlarged symmetry \((2.34)\) \([73, 97, 98]\) can be also deduced from non-relativistic QFTs with conserved particle number, by using the symmetries: time dependent spatial diffeomorphisms acting on the background metric and \(U(1)\) invariance acting on the background fields which couple to particle number \([183, 184]\). As Hořava gravity is presumed to be UV complete, in principle this duality allows holography to move beyond the large \(N\) limit \([79–82]\).

Finally, we would like to note that one may flip the logics around and study QG by using the gauge/gravity duality from well-known QFTs \([278, 279]\).

### V. QUANTIZATION OF HOŘAVA GRAVITY

Despite numerous efforts and the vast literature on Hořava gravity, so far its quantization has not been worked out in its general form, yet. Nevertheless, particular situations have been investigated, and some (promising) results have been obtained. In particular, in \((3+1)\)-dimensional spacetimes with the projectability and detailed balance conditions \((2.17)\)-(\(2.21\)), the renormalizability of Hořava gravity was shown to reduce to the one of the corresponding \((2+1)\)-dimensional topologically massive gravity \([280]\), by using stochastic quantization \([281–283]\). Even though the renormalizability of the latter has not been rigorously proven, it is often thought that it should be the case \([284]\). As already pointed out in \([280]\), the equivalence of the renormalizability between the two theories is closely related by the detailed balance condition \((2.19)-(2.21)\), and it is not clear how to generalize it to the case without this condition.

Lately, the above obstacle is circulated by properly choosing a gauge that ensures the correct anisotropic scaling of the propagators and their uniform falloff at large frequencies and momenta \([285]\). It is this choice that guarantees the counterterms required to absorb the loop divergences to be local and marginal or relevant with respect to the anisotropic scaling. Then, gauge invariance of the counterterms is achieved by making use of the background-covariant formalism.

Along a similar line, Li et al. studied the quantization of Hořava gravity both with and without the projectability condition \((2.17)\) in \((1+1)\)-dimensional \((2D)\) spacetimes \([286, 287]\). In such spacetimes, Einstein’s theory is trivial \([288–290]\). But, this is not the case of Hořava gravity, due to the foliation-preserving diffeomorphism \((2.3)\) \([291, 292]\), although the total degree of freedom of the theories is zero \([286, 287]\). In particular, in the projectable case, when only gravity is present, the system can be quantized by following the canonical Dirac quantization \([293]\), and the corresponding wavefunction is normalizable \([286]\). It is remarkable to note that in this case the corresponding Hamilton can be written in terms of a simple harmonic oscillator, whereby the quantization can be carried out quantum mechanically in the standard way. When minimally coupled to a scalar field, the momentum constraint can be solved explicitly in the case where the fundamental variables are functions of time only. In this case, the coupled system can also be quantized by following the Dirac process, and the corresponding wavefunction is also normalizable. A remarkable feature is that orderings of the operators from a classical Hamilton to a quantum mechanical one play a fundamental role in order for the Wheeler-DeWitt equation to have nontrivial solutions. In addition, the space-time is well quantized, even when it is classically singular.

In the non-projectable case, the analysis of the \(2D\) Hamiltonian structure shows that there are two first-class and two second-class constraints \([287]\). Then, following Dirac one can quantize the theory by first requiring that the two second-class constraints be strongly equal to zero, which can be carried out by replacing the Poisson bracket by the Dirac bracket \([293]\). The two first-class constraints give rise to the Wheeler-DeWitt equations. It was found that in this case the characteristics of classical spacetimes are encoded solely in the phase of the solutions of these equations.

On the other hand, in CDTs \([52]\) a preferred frame is also part of the process of quantization, which is quite similar to the foliation specified in Hořava gravity. It was precisely this similarity that led the authors of \([294]\) to show the exact equivalence between the \(2D\) CDT and the \(2D\) projectable Hořava gravity. Such studies were further generalized to the case coupled with a scalar field \([295]\) and \((2+1)\)-dimensional spacetimes \([296, 297]\). In particular, in \([296]\) it was shown the existence of known and novel macroscopic phases of spacetime geometry, and found evidence for the consistency of these phases with solutions to the equations of motion of classical Hořava gravity. In particular, the phase diagram seemingly contains a phase transition between a time-dependent de Sitter-like phase and a time-independent phase. In \([297]\), spacetime condensation phenomena were considered and shown that a
successful condensation in (2+1)-dimensions can be obtained from a minisuperspace model of Hořava gravity, but not from GR.

The close relations between CDTs and Hořava gravity have been further explored from the point of view of spectral dimensions. In particular, after extending the definition of spectral dimension in fractal and lattice geometries to theories on smooth spacetimes with anisotropic scaling, Hořava showed that a (d+1)-dimensional spacetime with a dynamical critical exponent \( z \) has the spectral dimension \( d_s = 1 + \frac{d}{z} \).

Thus, in the IR \( (z = 1) \) the spectral dimension is identical to the macroscopic dimension \( N(= d + 1) \) of spacetimes, \( N = d_s \). But, in the UV \( (z = d) \) the spectral dimension is always two, no matter what dimensions of the spacetimes are in the IR. Therefore, (spectral) dimension is emergent and it flows from two at short distances to \( (d+1) \) at large ones. This was further confirmed in \([299, 300]\). Remarkably, this is also the qualitative behavior found in CDTs \([301]\). Even more surprising, the same results were obtained in other theories of gravity \([28]\), such as asymptotic safety \([304, 305]\), LQG \([306, 307]\), spin foams \([308]\), non-commutative theory \([309]\), Wheeler-DeWitt equation of GR \([310, 311]\), and causal set theory \([312]\), to name only a few of them. For more details, we refer readers to \([313]\).

In addition, it was shown recently that holographic renormalization of relativistic gravity in asymptotically Lifshitz spacetimes naturally reproduces the structure of Hořava gravity \([314]\): The holographic counterterms induced near anisotropic infinity take the form of the action with the anisotropic scaling \((2.1)\). In the particular case of \((3+1)\) bulk dimensions and \( z = 2 \) asymptotic scaling near infinity, a logarithmic counterterm was found, related to anisotropic Weyl anomaly of the dual conformal field theory. It is this counterterm that reproduces precisely the action of conformal gravity at a \( z = 2 \) Lifshitz point in \((2+1)\) dimensions, which is of anisotropic local Weyl invariance and satisfies the detailed balance condition. It was also shown how the detailed balance is a consequence of relations among holographic counterterms. A similar relation also holds in the relativistic case of holography in AdS \(_5\). Upon analytic continuation, similar to the relativistic case \([315, 316]\), the action of anisotropic conformal gravity produces the square of the wavefunction of the dual system.

It is also very interesting to note that recently one-loop effective action and beta functions of the projectable Hořava gravity were derived by using the heat-kernel coefficients for Laplacian operators obeying anisotropic dispersion relations in curved spacetimes \([317, 318]\), and found that the Gaussian fixed point is an infrared attractor for the renormalization group flow of Newton’s constant, and the high-energy phase of the theory is screened by a Landau pole. When coupled to \( n \) Lifshitz scalars, the Gaussian fixed point ensures that the theory is asymptotically free in the large-\( N \) expansion, indicating that the theory is perturbatively renormalizable. Earlier studies along the same line were carried out in \([74, 319–325]\), and various results were obtained.

VI. CONCLUDING REMARKS

In this review, we have summarized the recent developments in Hořava gravity. In particular, after giving a brief introduction of the general ideas of Hořava at the beginning of Section II, we have pointed out some potential issues presented in the original incarnation of the theory, and then introduced four most-studied modifications, depending on the facts being with or without the projectability condition and a local U(1) symmetry. In the presentation for each of these four versions, we have stated clearly their current status in terms of self-consistency of the theory, consistency with experiments, mainly with solar system tests and cosmological observations.

In the projectable case without the local U(1) symmetry presented in Section II.A, the minimal theory, the main issues are instability \([92, 104, 105]\), and strong coupling of the spin-0 graviton \([106, 109]\). The perturbation analysis shows that the Minkowski spacetime is not stable due to the presence of the spin-0 graviton \([92, 104, 105]\), which questions the viability of the theory (although the de Sitter spacetime is \([106, 107]\)). To solve the strong coupling problem, one way is to borrow the well-known Vainshtein mechanism \([113]\). This has been done in the static and cosmological settings \([34, 111, 112]\), but for more general cases it has not been worked out, yet.

In the projectable case with the local U(1) symmetry presented in Section II.B, the spin-0 graviton is eliminated, so the two issues, instability and strong coupling mentioned above, are absent in the gravitational sector, but the strong coupling problem re-appears when coupled with matter \([125, 127]\). To solve the problem, one way is to introduce a new energy scale \( M_s \), as shown in Fig.1, first proposed for the non-projectable case \([128]\). Therefore, this version is self-consistent. It was shown that it is also consistent with all the solar system tests \([98]\), provided the universal coupling with matter is adopted \([29]\).

\(^{27}\) Scale dependence of spacetime dimension was first considered in anisotropic cosmology \([302]\) and later in string theory where a thermodynamic dimension was also found to be two at high temperature \([303]\).

\(^{28}\) It should be noted that different definitions of dimension have been adopted in these theories, and in principle they do not necessarily agree.

\(^{29}\) It should be noted that the consistency of this version of the theory with solar system tests excludes the possibility of minimal coupling with matter \([98]\).
However, for such a coupling, the consistency with cosmological observations has not been worked out, yet, although a preliminary study indicates that a more general coupling may be needed [139]. An interesting problem in this version of the theory is the physical interpretations of the pre-Newtonian potential \( \varphi \) and the U(1) gauge field \( A \). Such understanding may also be able to shed lights on the coupling of the gravitational sector with matter. Along this direction, the non-relativistic gauge/gravity correspondence found recently in [145] may provide some hints.

In the non-projectable case without the local U(1) symmetry presented in Section II.C, the healthy extension, it is shown that the theory is self-consistent and also consistent not only with solar system tests and cosmological observations, but also with the binary pulsar and gravitational wave observations [152, 153, 155]. The strong coupling problem [158] can be solved by introducing a new energy scale \( M_\ast \), as shown explicitly in [128], by properly choosing the coupling constants. One of the concerning of this version is the possibility to introduce a hierarchy [74], although it was argued that this is technically quite nature [96, 128]. Another question is the large number of the coupling constants (which is about 100). With such a large number, the question of the prediction power of the theory raises again, although in the IR only five of them are relevant [95, 96].

In the non-projectable case with the local U(1) symmetry presented in Section II.D, the spin-0 graviton is absent only in particular cases [73, 98, 186]. But, it is always stable in a large region of the parameters space, whenever it is present. The strong coupling problem in general also exists, but can be solved by also introducing a new energy scale \( M_\ast \), as one did in the last two versions. By softly breaking the detailed balance condition, the number of the coupling constants can be significantly reduced (to 15) [73, 97], while the power-counting renormalizability is still valid. It is interesting to note that this is not the case without the local U(1) symmetry, the healthy extension, as shown explicitly in [73]. It was shown that it is also consistent with all the solar system tests, provided the universal coupling with matter is adopted [98]. As in the projectable case, for such a coupling, the consistency of the theory with cosmological observations has not been worked out, yet [139]. It is interesting to note that this version has been embedded into string theory recently by using the non-relativistic gauge/gravity correspondence [183, 184]. I has been also shown that it has one-to-one correspondence to the dynamical Newton-Cartan geometry [145].

With all the above in mind, we have considered the recent developments of Hořava gravity in three different but also related areas: (a) black holes and their thermodynamics in gravitational theories with broken Lorentz invariance, including Hořava theory; (b) non-relativistic gauge/gravity correspondence in the framework of Hořava gravity; and (c) quantization of Hořava theory. It was somehow very surprising when it was first discovered that black holes can exist (theoretically) in gravitational theories with broken Lorentz invariance [202, 215], as one would expect that one can explore the physics arbitrarily near the singularity, once particles with arbitrarily large speeds are allowed. Then, a natural question is whether such a black hole has entropy or not? Using the same arguments as we did in GR, it is not difficult to be convinced that it should have. Otherwise, the second law of thermodynamics will be violated [243]. This in turn raises several other interesting questions, and one of them is: how the four laws of thermodynamics look like now? The first law in the neutral (without charges) case can be generalized, after a new definition of surface gravity given by Eq.(3.11) is adopted [241, 243]. But, for charged black holes, such a generalization is still absent [242]. Recent investigations [245–247] with a more general dispersion relation showed that both of the temperature and surface gravity obtained by considering peering behavior of rays near the universal horizons depend on the parameter \( z \), which characterizes the leading order \( k^{2z} \) of the dispersion relation [cf. Eqs.(3.12) and (3.14)]. Then, even in the neutral case, it seems that the entropy \( S \) of the black hole should be also \( z \)-dependent, if the first law, \( dE = TdS \), still holds.

For the zeroth law, it holds automatically in all the examples considered so far, as they are either in static spacetimes or in rotating (2+1)-dimensions, in which the universal horizons always occur on a hypersurface on which we have \( r = r_{UH} = \text{constant} \). Kinematic considerations prove that this is also the case in general [299], although concrete examples in more general spacetimes have not been found so far.

For the second law of thermodynamics, Blas and Sibiryakov considered two possibilities where the missing entropy can be found [202]: (i) It is accumulated somewhere inside the black hole (BH). BS studied the stabilities of the universal horizons and found that they are linearly stable. But, they argued that after nonlinear effects are taken into account, these horizons will be turned into singularities with finite areas. One hopes that in the full Hořava gravity this singularity is resolved into a high-curvature region of finite width accessible to the instantaneous and fast high-energy modes. In this way the BH thermodynamics can be saved. (ii) A BH has a large amount of static long hairs, which have tails that can be measured outside the horizon [326]. After measuring them, an outer observer could decode the entropy that had fallen into the BH. However, BS found that spherically symmetric hairs do not exist, and to have this scenario to work, one has to use non-spherically symmetric hairs. Clearly, to have a better understanding of the second law, much work in this direction needs to be done. The same is true for the investigations of the third law.

The investigations of the non-relativistic gauge/gravity duality in the framework of Hořava gravity is still in its infancy, and various open issues remain, including the
corresponding QFTs for the versions of Hořava gravity without the U(1) symmetry, their dictionaries, and so on. By flipping the logic, another important question is: can we get deeper insights about the quantization of Hořava gravity from some well-known systems of QFTs through this correspondence? The answer seems very positive [278, 279], but systematic investigations along this direction are still absent.

Quantizing gravity is the main motivation for Hořava to propose his theory in 2009 [23], but a rigorous proof of its renormalizability is still absent, although it is power-counting renormalizable by construction. So far, it has been shown that it is renormalizable only in a few particular cases. These include (1+1)-dimensional spacetimes both with and without the projectability condition [286, 287, 294], and spacetimes with the projectable condition [285]. It is still an open question how to generalize such studies to other cases, such as the ones without the projectability conditions and the ones with the local U(1) symmetry. In addition, what is the RG flow? So far, we have assumed that the theory can arbitrarily approach GR in the IR. But, without working out the RG flow in detail, we really do not know if this is indeed the case or not. Currently, only very limited situations were considered [103, 280].

Other important issues include how the theory couples with matter [74, 98], what are the effects of the Lorentz violation [27, 31–33]. To the latter, we need at least to stay below current experimental constraints [24, 25]. As mentioned in the Introduction, this is not an easy task at all [26, 27, 31–33]. To have a proper understanding of these effects, the coupling of Hořava gravity with matter will play a crucial role.

Therefore, although Hořava gravity seems to be a very interesting and promising alternative in quantization of gravity, for it to be really a viable theory, there is still a long list of questions that need to be addressed properly.

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**APPENDIX A: LIFSHITZ SCALAR THEORY**

To introduce Lifshitz scalar theory, let us begin with a free relativistic scalar field in a (3+1)-dimensional spacetime. For the sake of simplicity and without loss of the generality, we also assume that the spacetime is flat, so that the action takes the form,

$$S^{(\text{free})} = \frac{1}{2} \int dt \int d^3x \left( \partial^2 + \phi \partial \phi \right),$$  \hspace{1cm} (A.1)

where $\Delta(=\partial^2_t)$ denotes the Laplacian operator. Clearly, it is invariant under the general transformations (2.22). Under the isotropic rescaling,

$$t \rightarrow b^{-1} t, \quad x^i \rightarrow b^{-1} x^i, \quad \phi \rightarrow b \phi,$$  \hspace{1cm} (A.2)

$S_\phi$ is also invariant $S_\phi \rightarrow S'_\phi$. Denoting the units of length, time and mass, respectively, by $L, T$ and $M$, we find that the speed $c$, energy $E$, and Planck constant $\hbar$ have the dimensions,

$$[c] = \left[ \frac{\Delta x}{\Delta t} \right] = \frac{L}{T}, \quad [E] = \left[ mc^2 \right] = \frac{ML^2}{T^2},$$  \hspace{1cm} (A.3)

and

$$[\hbar] = \left[ \frac{E}{\nu} \right] = \frac{ML^2 T^{-2}}{T^1} = \frac{ML^2}{T^2}.$$  \hspace{1cm} (A.3)

Such, choosing the natural units $c = \hbar = 1$ implies

$$L = T, \quad E = M, \quad L = M^{-1} = E^{-1}.$$  \hspace{1cm} (A.4)

In the rest of this section, we shall use the natural units. Then, for a process with $\Delta \phi \sim E$, we have

$$\Delta t \sim E^{-1}, \quad \Delta t \equiv \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2} \sim E^{-1},$$

$$\frac{\Delta \phi}{\Delta t} \sim \frac{\Delta \phi}{\Delta x^i} \sim \frac{\Delta \phi}{\Delta t} \sim E^2.$$  \hspace{1cm} (A.5)

Hence, we find that

$$\int_0^E \int d^3x \partial^2 \phi \sim \Delta t \left( \Delta t \right)^3 \left( \frac{\Delta \phi}{\Delta t} \right)^2 \sim E^{-1} \cdot E^{-3} \cdot (E^2)^2 \sim O(1),$$

$$S^{(\text{free})}_\phi = \frac{1}{2} \int dt \int d^3x \left( \partial^2 + \phi \partial \phi \right) \sim O(1).$$  \hspace{1cm} (A.6)

Thus, the integral $S^{(\text{free})}_\phi$ is always finite no matter how large $E$ will be, which is called power-counting renormalizable. What will happen when self-interaction is taken into account? For example, let us consider the term,

$$S^{(\text{s.i.})}_\phi = \int dt \int d^3x \left( \phi^2 \partial \phi \right).$$  \hspace{1cm} (A.7)
which is scaling as
\[ S^{(s.i.)}_{\phi} \rightarrow \int dt \, dx^i \left[ \phi^2 \right] [\Delta] = b^{-1-3+2+2+1} \]
\[ = b, \quad \text{(A.8)} \]
under the rescaling (A.2). Then, we find that
\[ S^{(s.i.)}_{\phi} = \int_0^E dt \, dx^3 \left( \phi^2 \Delta \phi \right) \approx E, \quad \text{(A.9)} \]
which becomes unbounded as \( E \to \infty, \) and is said (power-counting) non-renormalizable. On the other hand, as \( E \to 0, \) we have \( S^{(s.i.)}_{\phi} \to 0, \) which is said irrelevant in low energy limit \([110].\) In general, for any given operator \( O_{\phi}, \) if \( \int_0^E dt \, dx \, O_{\phi} \) scales as
\[ \int_0^E dt \, dx \, O_{\phi} \approx b^\delta, \quad \text{(A.10)} \]
under the rescaling (A.2), we have \([110]\]
\[ \int_0^E dtd^3x \, O_{\phi} \approx E^\delta \]
\[ \begin{cases} \infty, & \delta > 0, \text{ Non-renormalizable}, \\ \text{finite}, & \delta = 0, \text{ Strictly renormalizable}, \\ 0, & \delta < 0, \text{ Super-renormalizable}, \end{cases} \quad \text{(A.11)} \]
as \( E \to \infty. \) On the other hand, as \( E \to 0, \) we find that
\[ \int_0^E dtd^3x \, O_{\phi} \approx E^\delta \]
\[ \begin{cases} 0, & \delta > 0, \text{ irrelevant}, \\ \text{finite}, & \delta = 0, \text{ marginal}, \\ \infty, & \delta < 0, \text{ relevant}. \end{cases} \quad \text{(A.12)} \]
In the rest of this section, we shall use frequently the terminology given above without any further explanations. For more details, see \([110].\) In general, we have
\[ S_{\phi}^{(\text{total})} = S_{\phi}^{(\text{free})} + S_{\phi}^{(s.i.)} + \ldots, \]
\[ = O(1) + E + E^2 + \ldots \to \infty, \quad \text{(A.13)} \]
as \( E \to \infty, \) that is, the theory is not power-counting renormalizable.

To make the theory renormalizable, Lifshitz \([70, 71]\) added the following non-relativistic term into the free action (A.1),
\[ \Delta S_{\phi} \equiv \frac{1}{2} \int dtd^3x \left( \frac{\phi^2 \Delta \phi}{M_{s}^2} \right), \quad \text{(A.14)} \]
where \( z(\geq 2) \) is an integer, and \( M_s \) is a constant, which represents the energy suppressing the operator \( \phi \Delta^z \phi. \) Thus, for an integration with \( \Delta \phi \approx E, \) now we have
\[ \frac{1}{M_{s}^{2z-2}} \left( \frac{\phi \Delta^z \phi}{\phi \Delta \phi} \right) \approx \left( \frac{E}{M_s} \right)^{2(z-1)} \]
\[ = \begin{cases} < 1, & E < M_s, \\ > 1, & E > M_s. \end{cases} \quad \text{(A.15)} \]
Therefore, when \( E \ll M_s, \) we have
\[ S_{\phi} = \frac{1}{2} \int_0^E dt \, dx \left( \phi^2 + \phi \Delta \phi + \frac{\phi \Delta^2 \phi}{M_{s}^{2z-2}} \right) \]
\[ \approx \frac{1}{2} \int_0^E dt \, dx \left( \phi^2 + \phi \Delta \phi \right), \quad \text{(A.16)} \]
which is finite and invariant under the relativistic scaling (A.2), as shown above. However, when \( E \gg M_s, \) we have
\[ S_{\phi} \approx \frac{1}{2} \int_0^E dt \, dx \left( \phi^2 + \frac{\phi \Delta^2 \phi}{M_{s}^{2z-2}} \right), \quad \text{(A.17)} \]
which is invariant only under the anisotropic rescaling between time and space,
\[ t \to b^{-z} t, \quad x^i \to b^{-1} x^i, \quad \phi \to b^{\alpha \phi}. \quad \text{(A.18)} \]
Under this new rescaling, we find
\[ \int_0^E dt \, dx \phi^2 \approx b^{-z-3+2z+2\alpha \phi} = b^{-3+2\alpha \phi}, \quad \text{(A.19)} \]
where \( \alpha_{\phi} \equiv [\phi]. \) Thus, the scaling-invariance of this term requires
\[ \alpha_{\phi} = - \frac{1}{2} (z-3). \quad \text{(A.20)} \]
Thus, for \( z = d = 3 \) the scalar field becomes dimensionless, and the interacting term (A.7) now scales as
\[ S_{\phi}^{(s.i.)} \approx b^{-z-3+2\alpha_{\phi}+2+\alpha_{\phi}} = b^{-4}, \quad \text{(A.21)} \]
that is, under the new scaling, the interaction term \( S_{\phi}^{(s.i.)} \) is scaling with an negative power of \( b. \) As a result, we have
\[ S_{\phi}^{(s.i.)} \approx E^{-4/3} \to 0, \quad \text{(A.22)} \]
as \( E \to \infty, \) where we assumed that \( \Delta t \approx E^{-1} \approx p^{-1/3}, \) where \( \Delta t \approx p^{-1}. \) Thus, under the new scaling, the non-renormalizable term \( S_{\phi}^{(s.i.)} \) now becomes renormalizable! Adding all the relevant terms into (A.14), we find that the quadratic action
\[ S_{\phi}^{(\text{bare})} = \frac{1}{2} \int dtd^3x \left( \dot{\phi}^2 + \phi \ddot{\phi} \right), \quad \text{(A.23)} \]
is power-counting renormalizable, where
\[ \ddot{\phi} \equiv g_3 \frac{\Delta^3}{M_s^3} - g_2 \frac{\Delta^2}{M_s^2} + c^2 \Delta - m^2, \quad \text{(A.24)} \]
where \( g_3, \ g_2, \ c \) and \( m \) are coupling constants. Similarly, we can add the potential term, \( V(\phi), \) into the above action, so finally we have
\[ S_{\phi}^{(\text{total})} = \frac{1}{2} \int dtd^3x \left( \dot{\phi}^2 + \phi \ddot{\phi} - V(\phi) \right), \quad \text{(A.25)} \]
which is clearly also power-counting renormalizable [23, 65–67]. Note that in the most general case all the coefficients in Eq. (A.24) should be functions of $\phi$ [67–69]. In addition, all $n$-point interaction of the forms [68, 69],

$$S_\phi^{(z,n)} \equiv \lambda(z,n) \int dt d^3 x (\Delta z_i \phi^n), \quad (A.26)$$

should be also included into the action where $z_i \leq z$, which contains $2z_i$ spatial derivatives acting on $n$ $\phi$'s. The dimension of $\lambda(z,n)$ is

$$[\lambda(z,n)] = \frac{2(z - z_i)}{z}, \quad (A.27)$$

which is always non-negative for $z_i \leq z$. As a result, this term is power-counting renormalizable. Once all such terms are included into the total action, it was shown that there are an infinite number of UV divergent terms in one-loop calculations [68, 69], and to eliminate these divergences, one needs to introduce an infinite number of counterterms, which make the predictability of the theory questionable. To overcome this problem, one way is to introduce the shift symmetry [68, 69],

$$\phi \rightarrow \phi + \phi_0, \quad (A.28)$$

where $\phi_0$ is a constant. Once this symmetry is imposed, it can be shown that only finite terms of the forms (A.24) satisfy this condition. Hence, in this case only a finite number of counterterms are needed, and the theory becomes predictable.

Recently, the theory of Lifshitz scalar fields has been intensively studied and made applications to various cases [327–333].

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