Boundary quantum group generators of type A

Rafael I. Nepomechie

Physics Department, P.O. Box 248046, University of Miami
Coral Gables, FL 33124 USA

Abstract

We construct boundary quantum group generators which, through linear intertwining relations, determine nondiagonal solutions of the boundary Yang-Baxter equation for the cases $A_{n-1}^{(1)}$ and $A_2^{(2)}$. 
1 Introduction

An effective means of finding solutions $R(u)$ of the Yang-Baxter equation \[1\]-\[4\]
\[
R_{12}(u - v) \ R_{13}(u) \ R_{23}(v) = R_{23}(v) \ R_{13}(u) \ R_{12}(u - v)
\] (1.1)
is the so-called quantum group approach \[5, 6\], which reduces the problem to a linear one. Indeed, $R$ matrices corresponding to vector representations of all non-exceptional affine Lie algebras were determined in this way in \[6\].

A similar approach is clearly desirable for finding solutions $K(u)$ of the boundary Yang-Baxter equation \[7, 8, 9\]
\[
R_{12}(u - v) \ K_1(u) \ R_{21}(u + v) \ K_2(v) = K_2(v) \ R_{12}(u + v) \ K_1(u) \ R_{21}(u - v).
\] (1.2)

With this goal in mind, the study of boundary quantum groups was initiated in \[10\]. In particular, for the case that the $R$ matrix corresponds to the spin $\frac{1}{2}$ representation of $A_1^{(1)}$, two matrices $Q_0(u), Q_1(u)$ were constructed which determine (up to an overall unitarization factor which does not concern us here) the $K$ matrix \[9, 11\] through the linear “intertwining” relations
\[
K(u) \ Q_j(u) = Q_j(-u) \ K(u),
\] (1.3)
where here $j = 0, 1$. This approach has recently been generalized in \[12\] to the $A_{n-1}^{(1)}$ case where vector solitons are reflected into solitons in the conjugate vector representation \[13\]. Moreover, this boundary quantum group approach has been used in \[14\] to determine $K$ matrices for higher representations of $A_1^{(1)}$.

Since the boundary quantum group “generators” $Q_j(u)$ determine (through the intertwining relations (1.3)) solutions $K(u)$ of the boundary Yang-Baxter equation, they seem to be important objects. However, very little is yet known about them.

In this Letter, we construct such boundary quantum group generators for the $A_{n-1}^{(1)}$ case where vector solitons are reflected into vector solitons (i.e., not into their conjugates), as well as for the $A_2^{(2)}$ case. The corresponding nondiagonal $K$ matrices \[13\] and \[16, 17\] are generalizations of diagonal $K$ matrices which were found earlier in \[11\] and \[18\], respectively. In Section 2 we treat the $A_{n-1}^{(1)}$ case, and in Section 3 we discuss the $A_2^{(2)}$ case. We end with a brief discussion of our results in Section 4.
2 The $A_{n-1}^{(1)}$ case

Let us consider the case that the $R$ matrix corresponds to the vector representation of $A_{n-1}^{(1)}$, $n \geq 3$. It is given by [3, 19]

$$R(u) = \sinh\left(\frac{nu}{2} + \eta\right) \sum_{k=1}^{n} E_{k,k} \otimes E_{k,k} + \sinh\left(\frac{nu}{2}\right) \sum_{k \neq l} E_{k,k} \otimes E_{l,l}$$

$$+ \sinh(\eta) \left(e^{\frac{nu}{2}} \sum_{k<l} + e^{-\frac{nu}{2}} \sum_{k>l}\right) E_{k,l} \otimes E_{l,k},$$

(2.1)

where $\eta$ is the anisotropy parameter, and $E_{l,k}$ denotes the elementary $n \times n$ matrix with matrix elements $(E_{l,k})_{\alpha\beta} = \delta_{l\alpha} \delta_{k\beta}$. We remark that we work with the $R$ matrix in the so-called homogeneous gradation.

Consider the set of generators

$$Q_0(u) = e^{nu} E_{n,1} + e^{-nu} E_{1,n} + e^{2\epsilon} e^{-\sigma(E_{1,1} + E_{n,n})},$$

(2.2)

$$Q_{k-1} = E_{k,1} + e^{-\sigma} E_{1,k} + E_{n,k} + e^{-\sigma} E_{k,n} + e^{\frac{2\pi i (k-1)}{n-1}} E_{k,k}, \quad k = 2, \ldots, n-1,$$

(2.3)

where $\epsilon$ and $\sigma$ are arbitrary boundary parameters. The intertwining relations (1.3) determine the following $K$ matrix:

$$K(u) = I - \frac{e^{-2\epsilon}}{\sinh \sigma} \left[ \sinh(nu)(E_{n,1} + E_{1,n}) + \sinh(\sigma) (e^{nu} E_{1,1} + e^{-nu} E_{n,n}) \right]$$

$$+ \sinh(nu + \sigma) \sum_{k=2}^{n-1} E_{k,k},$$

(2.4)

where $I$ is the identity matrix. This is essentially the same solution of the boundary Yang-Baxter equation (1.2) which was found by Abad and Rios [13]. Indeed, the latter solution appears to have more boundary parameters: $\rho_a, \rho_b, \rho_c, \rho_d, \epsilon$, with one constraint

$$\rho_c \rho_d = \rho_b (\rho_b + \rho_a e^{-\epsilon}).$$

(2.5)

However, by rescaling the $K$ matrix, one can set $\rho_b = 1$. By a “gauge” transformation which leaves the $R$ matrix unchanged,

$$R_{12}(u) \mapsto M_1 M_2 R_{12}(u) M_1^{-1} M_2^{-1} = R_{12}(u), \quad K(u) \mapsto MK(u)M^{-1},$$

(2.6)

with $M = diag(1, 1, \ldots, 1, \sqrt{|\rho_d|}/\sqrt{|\rho_c|})$ one can bring $\rho_c$ and $\rho_d$ to be equal, $\rho_c = \rho_d = e^{-\sigma}$. The constraint (2.3) then fixes $\rho_a = e^\epsilon (e^{-2\sigma} - 1)$. That is, there are only two independent...
boundary parameters, $\epsilon$ and $\sigma$. Finally, it should be noted that Abad and Rios work with the $R$ matrix in the so-called principal gradation, which is related to the $R$ matrix in the homogeneous gradation by the gauge transformation

$$R^{\text{prin}}_{12}(u) = M_1(u) R^{\text{hom}}_{12}(u) M_1(-u), \quad (2.7)$$

where $M(u) = \text{diag}(1, e^u, e^{2u}, \ldots)$. Hence, the $K$ matrices are also related by a corresponding transformation

$$K^{\text{prin}}(u) = M(u) K^{\text{hom}}(u) M(u). \quad (2.8)$$

We remark that the particular set of diagonal terms $e^{2\pi i(k-1)/(n-1)} E_{k,k}$ in (2.3) is merely one convenient choice. Indeed, generic diagonal terms will again lead to the same $K$ matrix (2.4).

We also emphasize that the solution (2.2) - (2.4) has two continuous boundary parameters. In contrast, for the case that vector solitons reflect into conjugate vector solitons considered in [12, 13], there are no continuous boundary parameters.

### 3 The $A_2^{(2)}$ case

We now consider the case of the Izergin-Korepin $R$ matrix [21], which corresponds to the vector representation of $A_2^{(2)}$. It can be written in the following form [3, 22]

$$R(u) = \begin{pmatrix} c & b & e & d & g & f \\ \bar{e} & \bar{d} & \bar{g} & \bar{b} & \bar{a} & \bar{f} \\ \bar{f} & \bar{g} & \bar{e} & \bar{d} & \bar{b} & \bar{c} \end{pmatrix} \quad (3.1)$$

where

$$a = \sinh(u - 3\eta) - \sinh 5\eta + \sinh 3\eta + \sinh \eta, \quad b = \sinh(u - 3\eta) + \sinh 3\eta,$$

$$c = \sinh(u - 5\eta) + \sinh \eta, \quad d = \sinh(u - \eta) + \sinh \eta,$$

$$e = -2e^{-\frac{u}{2}} \sinh 2\eta \cosh \left(\frac{u}{2} - 3\eta\right), \quad \bar{e} = -2e^{\frac{u}{2}} \sinh 2\eta \cosh \left(\frac{u}{2} - 3\eta\right),$$
Consider now the set of generators

\[ f = -2e^{-u+2\eta} \sinh \eta \sinh 2\eta - e^{-\eta} \sinh 4\eta, \quad \bar{f} = 2e^{u-2\eta} \sinh \eta \sinh 2\eta - e^{\eta} \sinh 4\eta, \]

\[ g = 2e^{-\frac{4}{3}+2\eta} \sinh \frac{u}{2} \sinh 2\eta, \quad \bar{g} = -2e^{\frac{4}{3}-2\eta} \sinh \frac{u}{2} \sinh 2\eta, \]

and \( \eta \) is again the anisotropy parameter.

For this \( R \) matrix, we find two sets of boundary quantum group generators, to which we refer as ‘type I’ and ‘type II’, following the classification scheme introduced by Lima-Santos \[15\] for the corresponding \( K \) matrices.

### 3.1 Type I

Consider the set of generators

\[ Q_0(u) = \begin{pmatrix} -ie^{-2\eta} & 0 & e^{u+\sigma} \\ 0 & 0 & 0 \\ e^{-u-\sigma} & 0 & ie^{2\eta} \end{pmatrix}, \quad Q_1 = \begin{pmatrix} e^{\eta+\epsilon} & e^\eta & 0 \\ e^{\eta-\sigma} & 0 & 1 \\ 0 & e^{-\sigma} & -e^{-\eta+\epsilon} \end{pmatrix}, \quad (3.2) \]

where \( \epsilon \) and \( \sigma \) are arbitrary boundary parameters. The intertwining relations \[13\] determine a matrix \( K(u) \) with the following matrix elements:

\[ K_{11} = 2ie^{2+\sigma+3\eta}(e^u - ie^{3\eta}) \cosh \eta + 2e^{3\eta}(e^u + ie^\eta)(e^u \cosh 2\eta - i \sinh \eta), \]

\[ K_{12} = -4e^{u+\epsilon+4\eta} \cosh \eta \sinh u, \quad K_{13} = 2ie^{u+\sigma+3\eta}(e^u + ie^\eta) \sinh u, \]

\[ K_{21} = -4e^{u+\epsilon+4\eta} \cosh \eta \sinh u, \quad K_{23} = -4ie^{2u+\epsilon+\sigma+2\eta} \cosh \eta \sinh u, \]

\[ K_{22} = 2e^{u+2\epsilon+\sigma+4\eta}(e^u + ie^{-3\eta}) \cosh \eta + ie^{2\eta}(e^u + ie^\eta)(e^u - ie^{-3\eta})(e^u - ie^{-\eta}), \]

\[ K_{31} = 2ie^{-u-\sigma+3\eta}(e^u + ie^\eta) \sinh u, \quad K_{32} = -4ie^{2u+\epsilon+2\eta} \cosh \eta \sinh u, \]

\[ K_{33} = 2ie^{2u+2\epsilon+\sigma+\eta}(e^u - ie^{3\eta}) \cosh \eta + 2e^{u+3\eta}(e^u + ie^\eta)(\cosh 2\eta - ie^u \sinh \eta). \quad (3.3) \]

Although this solution of the boundary Yang-Baxter equation may appear complicated, it is considerably simpler than the one given in \[15\], to which it can be shown to be equivalent. A shift of \( K(u) \) by \( u \mapsto u + i\pi \) is also a solution, by virtue of the periodicity \( R(u + 2i\pi) = R(u) \).

### 3.2 Type II

Consider now the set of generators

\[ Q_0(u) = \begin{pmatrix} e^\epsilon & 0 & e^{u+\sigma} \\ 0 & 0 & 0 \\ e^{-u-\sigma} & 0 & 0 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 0 & -e^\sigma & 0 \\ e^\eta & 0 & -e^\sigma \\ 0 & e^{-\eta} & 0 \end{pmatrix}, \quad (3.4) \]
where again $\epsilon$ and $\sigma$ are arbitrary boundary parameters. The intertwining relations (1.3) determine the following solution of the boundary Yang-Baxter equation:

$$K(u) = I + 2e^{-\epsilon} \begin{pmatrix}
    e^{-u} \sinh \eta & 0 & e^\sigma \sinh u \\
    0 & -\sinh(u - \eta) & 0 \\
    e^{-\sigma} \sinh u & 0 & e^u \sinh \eta
\end{pmatrix}.$$ (3.5)

This solution is equivalent to the one found by Kim [17], which is classified as type II in [16].

4 Discussion

The main results of this Letter are the expressions (2.2), (2.3) and (3.2), (3.4) for the boundary quantum group generators for the cases $A_{n-1}^{(1)}$ and $A_2^{(2)}$, respectively; and also the simplified expressions (2.4), (3.3), (3.5) for the corresponding $K$ matrices.

It remains an open question whether, for the $A_{n-1}^{(1)}$ case, the solution discussed here is the most general. Indeed, for the case of the critical $Z_n$-symmetric $R$ matrix [3, 23] with $n = 3$, which is very similar to the $A_{n-1}^{(1)}$ $R$ matrix (2.1) with $n = 3$, Yamada has recently found [24] a solution of the boundary Yang-Baxter equation with one more independent boundary parameter.

Although a principal motivation for studying boundary quantum groups is to find solutions of the boundary Yang-Baxter equation, the work so far (with the exception of [14]) has not yielded new solutions. The main difficulty is that an independent systematic method of constructing the boundary quantum group generators is not yet available. In contrast to the bulk case [5], one cannot exploit (boundary) affine Toda field theory, since appropriate classical integrable boundary conditions are not yet known [25]. We hope that by studying the known examples of boundary quantum group generators, it may become possible to uncover their basic algebraic structure, and to find generalizations to all (non-exceptional) affine Lie algebras.

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