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Cooperative colorings of trees and of bipartite graphs

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Abstract

Given a system \((G_1, \ldots, G_m)\) of graphs on the same vertex set \(V\), a cooperative coloring is a choice of vertex sets \(I_1, \ldots, I_m\), such that \(I_j\) is independent in \(G_j\) and \(\bigcup_{j=1}^m I_j = V\). For a class \(\mathcal{G}\) of graphs, let \(m_{\mathcal{G}}(d)\) be the minimal \(m\) such that every \(m\) graphs from \(\mathcal{G}\) with maximum degree \(d\) have a cooperative coloring. We prove that \(\Omega(\log \log d) \leq m_{T}(d) \leq O(\log d)\) and \(\Omega(\log d) \leq m_{B}(d) \leq O(d/ \log d)\), where \(T\) is the class of trees and \(B\) is the class of bipartite graphs.

Mathematics Subject Classifications: 05C15, 05C69

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1 Introduction

A set of vertices in a graph is called independent if no two vertices in it form an edge. A coloring of a graph $G$ is a covering of $V(G)$ by independent sets. Given a system $(G_1, \ldots, G_m)$ of graphs on the same vertex set $V$, a cooperative coloring is a choice of vertex sets \( \{I_j \subseteq V : j \in [m]\} \) such that $I_j$ is independent in $G_j$ and $\bigcup_{j=1}^m I_j = V$. If all $G_j$'s are the same graph $G$, then a cooperative coloring is just a proper vertex coloring of $G$ by $m$ independent sets.

A basic fact about vertex coloring is that every graph $G$ of maximum degree $d$ is $(d + 1)$-colorable. It is therefore natural to ask whether $d + 1$ graphs, each of maximum degree $d$, always have a cooperative coloring. This was shown to be false:

**Theorem 1** (Theorem 5.1 of Aharoni, Holzman, Howard and Sprüssel [AHHS15]). For every $d \geq 2$, there exist $d + 1$ graphs of maximum degree $d$ that do not have a cooperative coloring.

Using the fundamental result on independent transversals of Haxell [Hax01, Theorem 2], it can be shown that $2d$ graphs of maximum degree $d$ always have a cooperative coloring. Let $m(d)$ be the minimal $m$ such that every $m$ graphs of maximum degree $d$ have a cooperative coloring. By the above, $m(1) = 2$ and

\[
d + 2 \leq m(d) \leq 2d, \text{ for every } d \geq 2. \tag{1}
\]

The theorem of Loh and Sudakov [LS07, Theorem 4.1] on independent transversals in locally sparse graphs implies that $m(d) = d + o(d)$. Neither the lower bound nor the upper bound in (1) has been improved for general $d$; even $m(3)$ is not known. However, restricting the graphs to specific classes, better upper bounds can be obtained.

**Definition 2.** For a class $\mathcal{G}$ of graphs, denote by $m_\mathcal{G}(d)$ the minimal $m$ such that every $m$ graphs belonging to $\mathcal{G}$, each of maximum degree at most $d$, have a cooperative coloring.

For example, the following was proved:

**Theorem 3** (Corollary 3.3 of Aharoni et al. [ABZ07] and Theorem 6.6 of Aharoni et al. [AHHS15]). Let $\mathcal{C}$ be the class of chordal graphs and let $\mathcal{P}$ be the class of paths. Then $m_\mathcal{C}(d) = d + 1$ for all $d$, and $m_\mathcal{P}(2) = 3$.

In this paper, we prove some bounds on $m_\mathcal{G}(d)$ for two more classes:

**Theorem 4.** Let $\mathcal{T}$ be the class of trees, and let $\mathcal{B}$ be the class of bipartite graphs. Then for $d \geq 2$,

\[
\log_2 \log_2 d \leq m_\mathcal{T}(d) \leq (1 + o(1)) \log_{4/3} d, \\
\log_2 d \leq m_\mathcal{B}(d) \leq (1 + o(1)) \frac{2d}{\ln d}.
\]
Remark 5. Let $\mathcal{F}$ be the class of forests. It is evident that $m_\mathcal{F}(d) \geq m_\mathcal{T}(d)$ as $\mathcal{F} \supset \mathcal{T}$. Conversely, when $d \geq 2$, given $m = m_\mathcal{T}(d)$ forests $F_1, \ldots, F_m$ of maximum degree $d$, we can add edges to $F_i$ to obtain a tree $F'_i$ of maximum degree $d$, and the cooperative coloring for $F'_1, \ldots, F'_m$ is also a cooperative coloring for $F_1, \ldots, F_m$. Therefore $m_\mathcal{F}(d) = m_\mathcal{T}(d)$ for $d \geq 2$.

The notions of cooperative coloring and of list coloring have a common generalization: given a system $(G_1, \ldots, G_m)$ of graphs with vertex sets $V_1, \ldots, V_m$ (which are not necessarily the same vertex set), a cooperative list coloring is then a choice of independent sets in $G_i$ whose union equals $V := V_1 \cup \cdots \cup V_m$. The notion of cooperative coloring is obtained by taking $V_i = V$, and list colorings are formed when $G_i$ is an induced subgraph of the same graph $G$ for all $i$. The upper bounds in Theorem 4 generalize to cooperative list colorings. For example, our proof of Theorem 4 for bipartite graphs readily gives the following result.

**Theorem 6.** For every system $(G_1, \ldots, G_m)$ of bipartite graphs with maximum degree $d$ with vertex sets $V_1, \ldots, V_m$, there is a cooperative list coloring if for every $v \in V_1 \cup \cdots \cup V_m$, the number of its occurrences in $V_1, \ldots, V_m$, that is $|\{i \in [m] : v \in V_i\}|$, is at least $(1 + o(1))\frac{2d}{\ln d}$.

A conjecture of Alon and Krivelevich [AK98, Conjecture 5.1] states that the choice number of any bipartite graph with maximum degree $d$ is at most $O(\log d)$ (see [AR08] for a result in this direction). This conjecture would follow if the term $(1 + o(1))\frac{2d}{\ln d}$ in Theorem 6 was strengthened to $\Theta(\log d)$.

The rest of the paper is organized as follows. In Section 2 and Section 3, we prove Theorem 4 for trees and bipartite graphs respectively. In Section 4 we discuss a further generalization of cooperative colorings.

## 2 Trees

**Proof of the lower bound on $m_\mathcal{T}(d)$**. Note that the system $\mathcal{T}_2$, consisting of two paths in Figure 1 (one in thin red, the other in bold blue), does not have a cooperative coloring.

Suppose now that $\mathcal{S} = (F_1, F_2, \ldots, F_m)$ is a system of forests on a vertex set $V$, not having a cooperative coloring. We shall construct a system $Q(\mathcal{S})$ of $m + 1$ new forests $F'_1, F'_2, \ldots, F'_m, F'_{m+1}$, again not having a cooperative coloring.

The vertex set common to the new forests is $V' = (V \cup \{z\}) \times V$, namely the vertex set consists of $|V| + 1$ copies of $V$. For every $u \in V \cup \{z\}$ and every $i \in [m]$, take a copy

![Figure 1: Construction of two paths without a cooperative coloring.](image-url)
To these we add the \((m + 1)\)st forest \(F'_{m+1}\) obtained by joining \((z, u)\) to \((u, v)\) for all \(u, v \in V\). So \(F'_{m+1}\) is a disjoint union of stars, each with \(|V|\) leaves.

Assume that there is a cooperative coloring \((I_1, I_2, \ldots, I_m, I_{m+1})\) for the system \(Q(S)\). Since the forests \(F'_1, F'_2, \ldots, F'_m\) do not have a cooperative coloring, \(I_{m+1}\) must contain a vertex from \(\{u\} \times V\) for all \(u \in V \cup \{z\}\). In particular, \(I_{m+1}\) contains a vertex \((z, u) \in I_{k+1}\) for some \(u \in V\) and a vertex \((u, v)\) for some \(v \in V\). Since \((z, u)\) is connected in \(F'_{m+1}\) to \((u, v)\), this is contrary to our assumption that \(I_{m+1}\) is independent.

Note that \(|V'| = |V|^2 + |V| \leq 2|V|^2\). Note also that the maximum degree of \(Q(S)\) is attained in \(F'_{m+1}\), and it is equal to \(|V|\). Recursively define the system \(T_m := Q(T_{m-1})\) consisting of \(m\) forests for \(m \geq 3\). Because the base \(T_2\) has 4 vertices, one can check inductively that \(|V(T_m)|\) is at most \(2^{3/2^m - 2}\) using \(|V(T_m)| \leq 2|V(T_{m-1})|^2\). Thus the maximum degree of \(T_m\) is at most \(2^{3/2^m - 3} - 1 \leq 2^{2m-1}\).

Given the maximum degree \(d \geq 2\), choose \(m := \lceil \log_2 \log_2 d \rceil\). By the choice of \(m\), the maximum degree of \(T_m\) is at most \(2^{2m-1} \leq d\). By adding a few edges between the leaves in each forest of \(T_m\), we can obtain a system of \(m\) trees of maximum degree \(d\) that does not have a cooperative coloring. This means \(m_T(d) > m > \log_2 \log_2 d\).

\(\square\)

**Proof of the upper bound on \(m_T(d)\).** Let \((T_1, T_2, \ldots, T_m)\) be a system of trees of maximum degree \(d\). We shall find a cooperative coloring by a random construction if \(m \geq (1 + o(1)) \log_{4/3} d\).

Choose arbitrarily for each tree \(T_i\) a root so that we can specify the parent or a sibling of a vertex that is not the root of \(T_i\). For each \(T_i\), choose independently a random vertex set \(S_i\), in which each vertex is included in \(S_i\) independently with probability 1/2. Set

\[ R_i := \{ v \in S_i : \text{the parent of } v \text{ is not in } S_i, \text{ or } v \text{ is a root} \}. \]

Since among any two adjacent vertices in \(T_i\) one is the parent of the other, \(R_i\) is independent in \(T_i\).
We shall show that with positive probability the sets $R_i$ form a cooperative coloring. For each vertex $v$, let $B_v$ be the event that $v \notin \bigcup_{i=1}^m R_i$. If $v$ is the root of $T_i$, then $\Pr(v \notin R_i) = 1/2$; otherwise $\Pr(v \notin R_i) = 1/4$. In any case, $\Pr(v \notin R_i) \leq 3/4$, and so $\Pr(B_v) \leq (3/4)^m$. Notice that $B_v$ is only dependent on the events $B_u$ for $u$ that is the parent, a sibling or a child of $v$ in some $T_i$. Since the degree of $v$ is at most $d$, it follows that $B_v$ is dependent on less than $2md$ other events. By the symmetric version of the Lovász Local Lemma (see for example [AS16, Chapter 5]), if

$$e \times \left(\frac{3}{4}\right)^m \times 2md \leq 1, \quad (2)$$

then with positive probability no $B_v$ occurs, meaning that the sets $R_i$ form a cooperative coloring. The inequality (2) indeed holds under the assumption that $m \geq (1+o(1)) \log_{4/3} d$. \hfill $\Box$

### 3 Bipartite graphs

**Proof of the lower bound on $m_B(d)$.** Given $d$, take $m = \lceil \log_2 d \rceil$. Let the vertex set be $\{0, 1\}^m$, and for $j \in [m]$ let $G_j$ be the complete bipartite graph between $V_j^0$ and $V_j^1$ where

$$V_j^k = \{v \in \{0, 1\}^m : v_j = k\}, \quad \text{for } k \in \{0, 1\}.$$  

Note that the degree of $G_j$ is $2^{m-1} \leq d$.

Suppose that $I_1, \ldots, I_m$ are independent sets in $G_1, \ldots, G_m$ respectively. As each $G_j$ is a complete bipartite graph, $I_j \subseteq V_j^{k_j}$ for some $k_j \in \{0, 1\}$. Thus $(1 - k_1, \ldots, 1 - k_m)$ is not in any $I_j$, and so $I_1, \ldots, I_m$ do not form a cooperative coloring. This means $m_B(d) > m \geq \log_2 d$. \hfill $\Box$

**Proof of the upper bound on $m_B(d)$.** Let $G = (G_1, \ldots, G_m)$ be a system of bipartite graphs on the same vertex set $V$ with maximum degree $d$. By a semi-random construction, we shall find a cooperative coloring if $m \geq (1 + \varepsilon) \frac{dn}{m^d}$ for fixed $\varepsilon > 0$ and $d$ sufficiently large. We may assume that $m = O(d)$ because of (1).

For each $j \in [m]$, let $(L_j, R_j)$ be a bipartition of $G_j$. Define $J_L(v) := \{j \in [m] : v \in L_j\}$ and $J_R(v) := \{j \in [m] : v \in R_j\}$ for each vertex $v \in V$, and let $A := \{v \in V : |J_L(v)| \geq m/2\}$. Set $B := V \setminus A$. Clearly, we have

$$|J_L(a)| \geq m/2, \quad \text{for all } a \in A; \quad (3a)$$

$$|J_R(b)| \geq m/2, \quad \text{for all } b \in B. \quad (3b)$$

Consider the following random process.

1. For each $a \in A$, choose $j = j(a) \in J_L(a)$ uniformly at random, and put $a$ in the set $I_j$.
2. For each $b \in B$, choose arbitrarily $j = j_R(b) \setminus \{j(a) : a \in A, (a, b) \in E(G_j)\} =: J_R(b)$ as long as it is possible, and put $b$ in the set $I_j$. 

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For any \( a, a' \in A \cap I_j, a, a' \in L_j \) and so \( (a, a') \notin G_j \). This means \( A \cap I_j \) is independent, and similarly \( B \cap I_j \) is independent. For any \( b \in B \cap I_j \) and \( (a, b) \in E(G_j) \), by the definition of \( J_R(b) \), \( j(a) \neq j \) and so \( a \notin I_j \). Therefore \( I_j \) is independent for all \( j \in [m] \).

To prove the existence of a cooperative coloring it suffices to show that \( J_R(b) \) is nonempty for all \( b \in B \) with positive probability. For a vertex \( b \in B \), let \( E_b \) be the contrary event, that is, the event that \( J_R(b) \) is empty.

For a fixed \( b \in B \), let us estimate from above the probability of \( E_b \). For every \( j \in J_R(b) \), let \( E^j \) be the event that \( j \notin J_R(b) \), that is the event that \( j(a) = j \) for some \( a \in A \) that is a neighbor of \( b \) in \( G_j \). For each \( a \in A \) that is a neighbor of \( b \) in \( G_j \), we have

\[
\Pr (j(a) = j) = \frac{1}{|J_L(a)|} \leq \frac{2}{m} \leq \frac{\ln d}{(1 + \varepsilon)d}.
\]

As there are at most \( d \) neighbors of \( b \) in \( G_j \), we have for sufficiently large \( d \) that

\[
1 - \Pr (E^j) \geq \left( 1 - \frac{\ln d}{(1 + \varepsilon)d} \right)^d \geq \exp (- (1 - \varepsilon) \ln d) = d^{\varepsilon - 1} \geq \frac{8 \ln d}{m}. \tag{4}
\]

We claim that the events \( E^j, j \in J_R(b) \), are negatively correlated. This is easier to see with the complementary events \( \bar{E}^j, j \in J_R(b) \). We have to show that for any choice of indices \( j_1, \ldots, j_t \in J_R(b) \) there holds

\[
\Pr \left( E^{j_1} \cap \bar{E}^{j_2} \cap \ldots \cap \bar{E}^{j_t} \right) \geq \Pr \left( E^{j_1} \right) .
\]

The event \( \bar{E}^{j_1} \cap \bar{E}^{j_2} \cap \ldots \cap \bar{E}^{j_t} \) means that for all \( a \in A \) if \( a \) is a neighbor of \( b \) in \( G_j \) then \( j(a) \neq j_i \). Then, for any \( j \notin \{j_1, \ldots, j_t\} \), for those vertices \( a \in A \) that are neighbors of \( b \) in \( G_j \), knowing that \( j(a) \neq j_i \) for certain \( i \in [t] \) increases the probability that \( j(a) = j \), and therefore increases the probability of \( E^j \).

By the claim, the inequality (4) and the fact that \( E_b = \bigcap_{j \in J_R(b)} E^j \), we have

\[
\Pr (E_b) \leq \prod_{j \in J_R(b)} \Pr (E^j) \leq \left( 1 - \frac{8 \ln d}{m} \right)^{\frac{m}{2}} \leq \exp \left( - \frac{8 \ln d}{m} \cdot \frac{m}{2} \right) = \frac{1}{d^t}.
\]

The event \( E_b \) is dependent on less than \( md^2 \) other events \( E_{b'} \), since for such dependence to exist it is necessary that \( b' \in B \) is at distance at most 2 from \( b \) in some graph \( G_j \). Thus, by the Lovász Local Lemma, for the positive probability that none of \( E_b \) occurs it suffices that

\[
e \times \frac{1}{d^t} \times md^2 \leq 1,
\]

which indeed holds for \( d \) sufficiently large as \( m = O(d) \). \( \square \)

### 4 Cooperative covers

Cooperative coloring of graphs is a special case of a more general concept.
Definition 7. Given a system \((C_1, \ldots, C_n)\) of (abstract) simplicial complexes, all sharing the same vertex set \(V\), a cooperative cover is a choice of faces \(f_i \in C_i\) such that \(\bigcup_{i=1}^n f_i = V\).

A cooperative coloring for \((G_1, \ldots, G_n)\) is the special case in which \(C_i\) is the independence complex \(I(G_i)\) of \(G_i\), that is, the collection of all independent sets in \(G_i\).

Definition 8. Given a hypergraph \(C\) with vertex set \(V\), the edge covering number \(\rho(C)\) is the minimal number of hyperedges from \(C\) whose union is \(V\). For a class \(C\) of simplicial complexes, let \(n_C(b)\) denote the minimal number \(n\), such that every system \((C_1, \ldots, C_n)\) of simplicial complexes belonging to \(C\) on the same vertex set \(V\) satisfying \(\rho(C_i) \leq b\) for all \(i \leq n\), has a cooperative cover. Let \(n_C(b) = \infty\) if no such \(n\) exists.

For example, consider the class \(\mathcal{I}\) of all the independence complexes of graphs. If \(G\) is bipartite, then \(\rho(I(G)) \leq 2\). Hence the fact that \(m_{\mathcal{I}}(d) \geq \log_2(d)\) for all \(d \geq 2\) (see Theorem 4) implies \(n_{\mathcal{I}}(2) = \infty\).

There are natural classes \(C\) of hypergraphs for which \(n_C\) is finite. One of these is the class of simplicial complexes associated to polymatroids, as introduced in [Edm70]. A polymatroid \((V, r)\) is defined via a rank function \(r : 2^V \to \mathbb{R}\), that is submodular (decreasing), monotone increasing and is 0 on the empty set. A k-polymatroid is a polymatroid in which every singleton set has rank at most \(k\). For example, a k-uniform hypergraph \(H\) endowed with the function \(r(E) = |\cup E|\), for every subset of hyperedges \(E\) in \(H\), is a k-polymatroid.

Following the notation in [LP86, Section 11], given a k-polymatroid \((V, r)\), a set \(M \subseteq V\) is called a matching if \(r(M) = k|M|\). By the submodularity of the rank function \(r\), the matchings in a k-polymatroid form a simplicial complex on \(V\), which we call the matching complex of a k-polymatroid.

Theorem 9. Let \(\mathcal{M}_k\) be the class of all the matching complexes of k-polymatroids. Then \(n_{\mathcal{M}_k}(b) \leq kb\) for every \(b\).

The proof uses the (homotopic) connectivity \(\eta(C)\) of a complex \(C\). We refer to [AB06, Section 2] for background. We shall use the following two topological tools. Given a complex \(C\) on \(V\) and \(U \subseteq V\), we denote by \(C[U]\) the simplicial subcomplex induced on \(U\).

Theorem 10 (Topological Hall’s theorem). Let \(C\) be a simplicial complex on the vertex set \(V\) and let \(\bigcup_{i=1}^m W_i\) be a partition of \(V\). If for all \(I \subseteq [m]\)

\[
\eta \left( C \left[ \bigcup_{i \in I} W_i \right] \right) \geq |I|,
\]

then \(C\) contains a face \(\sigma\) such that \(|\sigma \cap W_i| = 1\) for all \(i \in [m]\).

Theorem 11. If \(C\) is a matching complex on \(V\) of a k-polymatroid, then the connectivity \(\eta(C)\) of \(C\) is at least \(\nu(C)/k\), where \(\nu(C)\) is the maximal size of faces in \(C\).

The above formulation of Theorem 10 first appeared in [Mes01], attributed to the first author of the present paper (see the remark after Theorem 1.3 in [Mes01]). Theorem 11 is an unpublished result of the first two authors. The special case, where the k-polymatroid is the sum of \(k\) matroids on the same vertex set, is proved in [AB06, Theorem 6.5].
Proof of Theorem 9. Let \( n = kb \), and let \( C_1, \ldots, C_n \) be simplicial complexes associated to \( k \)-polymatroids \( (V, r_1), \ldots, (V, r_n) \) on the same vertex set \( V \) such that the edge covering number of each \( C_i \) is at most \( b \). Let \( C \) be the join of \( C_1, \ldots, C_n \) on \( V \times [n] \), that is,
\[
C := \left\{ \bigcup_{i=1}^{n} \sigma_i \times \{i\} : \sigma_i \in C_i \text{ for all } i \in [n] \right\}.
\]
A cooperative cover can be viewed as a face \( \sigma \in C \) such that \( |\sigma \cap (\{v\} \times [n])| = 1 \) for all \( v \in V \). By the topological Hall’s theorem, it suffices to prove that
\[
\eta(C[U \times [n]]) \geq |U| \text{ for all } U \subseteq V.
\]

Let \( U \) be a subset of \( V \). Note that \( C_i[U] \) is the matching complex of the \( k \)-polymatroid \( (U, r_i|_U) \). By Theorem 11, \( \eta(C_i[U]) \geq \nu(C_i[U])/k \). Since \( \nu(C_i[U]) \) is the maximal size of faces in \( C_i[U] \) and the edge covering number of \( C_i[U] \) is at most \( b \), we obtain \( \nu(C_i[U])b \geq |U| \), and so \( \eta(C_i[U]) \geq |U|/(kb) \). Notice that \( C[U \times [n]] \) is the join of \( C_1[U], \ldots, C_n[U] \). Using the superadditivity of \( \eta \) with respect to the join operator and Theorem 11, we obtain the required condition for the topological Hall’s theorem
\[
\eta(C[U \times [n]]) \geq \sum_{i=1}^{n} \eta(C_i[U]) \geq \sum_{i=1}^{n} |U|/(kb) = |U|. \]

Remark 12. It is of interest to explore the sharpness of this result.

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