Matrix Boussinesq solitons and their tropical limit

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Abstract
We study soliton solutions of matrix ‘good’ Boussinesq equations, generated via a binary Darboux transformation. Essential features of these solutions are revealed via their ‘tropical limit’, as also exploited in previous work about the KP equation. This limit associates a point particle interaction picture with a soliton (wave) solution.

Keywords: Boussinesq equation, solitons, tropical limit, Darboux transformation

(Some figures may appear in colour only in the online journal)

1. Introduction

The (scalar) Boussinesq equation originated from the study of water waves propagating in a canal [1]. In this work we study the ‘good’ Boussinesq equation, in which the highest derivative term has the opposite sign, as compared to the original Boussinesq equation. It also appears as a continuum limit of certain nonlinear atomic chains, see [2], for example. From a mathematical point of view, it belongs to the main examples of completely integrable PDEs. It is the second Gelfand–Dickey reduction [3] of the KP-II equation, the famous Korteweg–deVries (KdV) equation being the first. Compared with the latter, the behavior of its soliton solutions is considerably more diverse, in particular they can move in both directions, experience head-on collision, and solitons can split or merge (see [4, 5], also see the references cited in these papers). The scalar good Boussinesq equation is also known as ‘nonlinear string equation’, see e.g. [6, 7].

In the KdV and Boussinesq case, a contour plot of a soliton solution displays a localization along a piecewise linear graph in two-dimensional space–time. The definition of the ‘tropical limit’ makes this precise and yields a method to compute it. Restricting the dependent variable to this graph, completes the tropical limit of the soliton solution, which displays the essentials of soliton interactions in a clear way. It is a very convenient tool to describe and to classify soliton solutions. The tropical limit associates with a soliton solution a point particle picture, also in the interaction region of solitons, revealing ‘virtual solitons’ (borrowing a familiar notion from perturbative quantum field theory).

In this work we address more generally the following $m \times n$ matrix potential Boussinesq equation,

$$
\phi_{tt} - 4 \beta \phi_{xx} + \frac{1}{3} \phi_{xxxx} + 2(\phi_x K \phi_x)_x \\
- 2(\phi_y K \phi_y - \phi_y K \phi_y) = 0,
$$

(1.1)

where $K$ is a constant $n \times m$ matrix, $\beta > 0$ (chosen as $\beta = 1/4$ in all plots in this work), and a subscript indicates a partial derivative of $\phi$ with respect to the respective variable $x$ or $t$. We will refer to this equation as potential BS$^qK$. It is the second Gelfand–Dickey reduction of the following KP-II equation, in a moving frame ($x \leftrightarrow x - 3 \beta t_1, t_2 = t$),

$$
- \frac{4}{3} (\phi_x + 3 \beta \phi_x)_x + \phi_x + \frac{1}{3} \phi_{xxxx} + 2(\phi_x K \phi_x)_x \\
- 2(\phi_y K \phi_y - \phi_y K \phi_y) = 0.
$$
In terms of \( u = 2\phi_a \), we obtain the \( m \times n \) matrix Boussinesq equation, or rather \( \text{Bsq}_K \)

\[
u_t - 4\beta u_{xxx} + \frac{1}{3u_{xxxx}} + (uKu)_{xx} - (K(\partial^{-1}u) - (\partial^{-1}u)Ku)_x = 0.
\]

For a solution \( u \) of the vector Boussinesq equation, where \( n = 1 \) or \( n = 1 \), \( Ku \), respectively \( uK \), is a solution of the scalar Boussinesq equation. If \( m, n > 1 \), we define the tropical limit graph via that of the scalar \( t(Ku) \), which is not in general a solution of the scalar Boussinesq equation. Our explorations and results fully substantiate this approach.

Our analysis largely parallels that in [8], where we explored line soliton solutions of the matrix KP-II equation in the tropical limit. There we concentrated on a class of solutions which we called ‘pure solitons’. Another class has been treated in [9]. Whereas pure solitons exhaust the solitons of the KdV reduction, this is not so for the Boussinesq reduction.

Matrix versions of scalar integrable equations are natural extensions and of interest as models of coupled systems. A further motivation to explore them originated from the fact that in two-soliton scattering, in particular in case of the matrix KdV [10, 11] and the vector nonlinear Schrödinger equation [12, 13], in- and outgoing polarizations (values of the dependent variable attached to in- and outgoing solitons) are related by a Yang–Baxter map, a solution of the ‘functional’ or ‘set-theoretic’ version of the famous Yang–Baxter equation. For a matrix KP equation, a Yang–Baxter map is not sufficiently to fully describe the situation [9]. It seems that we also have to go beyond Yang–Baxter in case of the second member in the family of Gel’fand–Dickey reductions of KP, which is the matrix Boussinesq equation.

The (quantum) Yang–Baxter equation is well-known to express integrability in two-dimensional quantum field theory and exactly solvable models of statistical mechanics. In the present work, as also e.g. in [8, 10–13], we meet it in a classical context. Formally, however, soliton waves may be regarded as a sort of quantization of the point particles constituting the tropical limit.

In section 2 we present a binary Darboux transformation for the potential Bsq equation (1.1). Appendix explains its origin from a general result in bidifferential calculus [14, 15]. In this work we concentrate on the case of vanishing seed solution. This leads to two cases, treated in sections 3 and 4. The soliton solutions, obtained via the binary Darboux transformation, depend on parameters that have to be roots of a cubic equation. We introduce a convenient parametrization of these roots (see section 3.1) that greatly facilitates the further analysis. Section 5 contains some concluding remarks.

5 For the scalar Boussinesq equation, \( x \mapsto -x \) and \( t \mapsto -t \) are symmetries. In the matrix case, the first is still a symmetry, but \( t \mapsto -t \) has to be accompanied by \( u \mapsto u^T \).

2. A binary Darboux transformation for the matrix Boussinesq equation

The following binary Darboux transformation is a special case of a general result in bidifferential calculus, see the appendix. Let \( N \in \mathbb{N} \). The integrability condition of the linear system

\[
\begin{align*}
\theta_t &= \theta_{xx} + 2\phi_0, \quad K\theta, \\
\theta_{xx} &= 3\beta \theta + \theta C - 3\phi_{0,x} K\theta - \frac{1}{2}C(\phi_{0,t} + \phi_{0,xx})K\theta,
\end{align*}
\]

where \( \theta \) is an \( m \times N \) and \( C \) a constant \( N \times N \) matrix, is the potential Bsq equation for \( \phi_0 \). The same holds for the adjoint linear system

\[
\begin{align*}
\chi_t &= -\chi_{xx} - 2\chi K\phi_{0,x}, \\
\chi_{xx} &= 3\beta \chi_x + C'\chi - 3\chi_x K\phi_{0,x} + \frac{3}{2}C(\phi_{0,t} + \phi_{0,xx}).
\end{align*}
\]

At space–time points where \( \Omega \) is invertible

\[
\phi = \phi_0 - \theta \Omega^{-1} \chi
\]

is then a new solution of the potential Bsq equation.

Remark 2.1. Taking the transpose of the above equations, and applying the substitution \( t \mapsto -t \), we see that, besides \( K \mapsto K^T \), we also obtain \( C' \mapsto C^T \) and \( \theta \mapsto \chi^T \). We also note that \( x \mapsto -x \), \( \phi_0 \mapsto -\phi_0 \), \( C \mapsto -C \), \( C' \mapsto -C' \) is a symmetry of the linear systems.

Remark 2.2. The equations (2.1)–(2.4) are invariant under a transformation

\[
\begin{align*}
\theta &\mapsto \theta A, \\
\chi &\mapsto B \chi, \\
C &\mapsto A^{-1} C A, \\
C' &\mapsto B C' B^{-1}, \\
\Omega &\mapsto B \Omega A,
\end{align*}
\]

with any invertible constant \( N \times N \) matrices \( A \) and \( B \).

Using (2.4) and the first of (2.3), we find

\[
\begin{align*}
\text{tr}(K\phi) &= \text{tr}(K\phi_0) - \text{tr}(K\theta \Omega^{-1} \chi) = \text{tr}(K\phi_0) - \text{tr}(\chi \theta \Omega^{-1}) = \text{tr}(K\phi_0) + \text{tr}(\Omega_x \Omega^{-1}) = \text{tr}(K\phi_0) + (\log \det \Omega)_x.
\end{align*}
\]

Hence

\[
\text{tr}(Ku) - \text{tr}(Ku_0) = 2(\log \det \Omega)_x.
\]

Such a formula is familiar in the scalar case, where \( \det \Omega \) is the Hirota \( \tau \)-function. But we will see that, also in the matrix case, \( \det \Omega \) plays a crucial role. In the following we will still
call it \( \tau \), after multiplication by a convenient factor, which preserves the relation (2.5).

### 2.1. Solutions for vanishing seed

The linear system with \( \phi_0 = 0 \) reads

\[
\theta_i = \theta_{xi}, \quad \theta_{xx} = 3 \beta \theta_i + \theta C.
\]

It possesses solutions of the form

\[
\theta = \sum_a \theta_a e^{\theta(P)},
\]

where

\[
\theta(P) = \nu x + \nu^2 t,
\]

and each \( P_a \) is a solution of the cubic equation

\[
P_a^3 = 3 \beta P_a + C.
\]

The index \( a \) runs over any number of distinct roots.

Correspondingly, the adjoint linear system takes the form

\[
\Omega_i = -\chi K \theta_i, \quad \Omega_i = -\chi K \theta_i + \chi K \theta_i, \quad C' \Omega + \Omega C = -\chi K \theta_i + \chi K \theta_i - \chi \theta_i K \theta_i + 3 \beta \chi K \theta_i.
\]

Writing

\[
\Omega = \Omega_0 + \sum_{a,b} e^{-\theta(Q_b)} W_{ba} e^{\theta(P)}
\]

these equations are solved if \( W_{ba} \) satisfies the Sylvester equation

\[
Q_b W_{ba} = W_{ba} P_a = \chi_b K \theta_b,
\]

and if the constant matrix \( \Omega_0 \) is subject to

\[
\Omega_0 C' + C \Omega_0 = 0.
\]

As a consequence of the last condition, there are two major cases. In section 3 we will address the case where \( C' = -C \). Section 4 then deals with the complement.

We note that (2.5) reduces to

\[
\text{tr}(K \Omega) = 2 \log(\text{det}(\Omega_{xx})
\]

### 3. The case \( C' = -C \)

If \( \Omega_0 \) is invertible, remark 2.2 shows that without restriction of generality we can choose \( \Omega_0 = I_{nx} \), the \( N \times N \) identity matrix. Equation (2.13) then implies \( C' = -C \). The remaining freedom of transformations, according to remark 2.2, is then given by transformations with \( B = A^{-1} \). The similarity transformation \( C \mapsto A^{-1} CA \) now allows us to assume that \( C \) has Jordan normal form.

\( P_a \) and \( Q_b \) are now solutions of the same cubic equation, so we can set

\[
Q_b = P_a,
\]

and the Sylvester equation takes the form

\[
P_a W_{ab} - W_{ab} P_a = \chi_a K \theta_b.
\]

If \( a \neq b \) and \( P_a \) and \( P_b \) have disjoint spectrum, it is well-known that there is a solution and it is unique. In this case, the sum in the expression for \( \theta \) or \( \chi \) is over a disjoint set of solutions of the cubic equation.

We will restrict our considerations to diagonal matrices\(^6\)

\[
P_a = \text{diag}(p_{1,a}, ..., p_{N,a}), \quad C = \text{diag}(c_1, ..., c_N).
\]

Equation (2.8) then requires

\[
p_a^3 = 3 \beta p_a + c_i \quad i = 1, ..., N.
\]

We will only consider real roots. Writing

\[
\chi_a = \left(\begin{array}{c}
\eta_{a,1} \\
\eta_{a,2} \\
\eta_{a,3}
\end{array}\right), \quad \theta_a = \left(\begin{array}{c}
\xi_{1,a} \\
\xi_{2,a} \\
\xi_{3,a}
\end{array}\right)
\]

and \( W_{ab} = (W_{ab,ij}) \), we find

\[
W_{ab,ij} = \frac{\eta_{b,K^i \xi_j}}{p_{a} - p_{b}}, \quad a \neq b, \quad p_{ai} \neq p_{bj}, \quad i, j = 1, ..., N,
\]

and thus

\[
\Omega_{ij} = \delta_{ij} + \sum_{a=1}^{N} \frac{\eta_{b,K^i \xi_j}}{p_{a} - p_{b}} e^{\theta(P_a) - \theta(P_b)} \quad i, j = 1, ..., N.
\]

#### 3.1. A parametrization of the roots of the cubic equation

We are only interested in real soliton solutions, hence we restrict our analysis to real roots of the cubic equation and demand that there are at least two different ones. This requires \( |c_l| \leq 2 \beta^{3/2} \). We can then express the constants \( c_j \) as follows

\[
c_j = 2 \beta^{3/2} \left(1 - 45 \lambda_1^2 + 135 \lambda_1^4 - 27 \lambda_1^6 \right) \frac{1 + 3 \lambda_1^2}{(1 + 3 \lambda_1^2)^3},
\]

where \( \lambda_i \) are real parameters. The roots of the cubic equation (3.1) are then given by

\[
p_{a,1} = -\sqrt{\beta \left(1 + 6 \lambda_1 - 3 \lambda_1^2 \right) \frac{1}{1 + 3 \lambda_1^2}},
\]

\[
p_{a,2} = -\sqrt{\beta \left(1 - 6 \lambda_1 - 3 \lambda_1^2 \right) \frac{1}{1 + 3 \lambda_1^2}},
\]

\[
p_{a,3} = 2 \sqrt{\beta \left(1 - 3 \lambda_1^2 \right) \frac{1}{1 + 3 \lambda_1^2}}.
\]

All the roots satisfy \( p_i^2 \leq 4 \beta \). We have \( \lim_{|\lambda_i| \to \infty} p_{ai} = \sqrt{\beta} \) for \( a = 1, 2 \), and \( \lim_{|\lambda_i| \to \infty} p_{a,3} = -2 \sqrt{\beta} \). Also see figure 1.

\(^6\) A treatment of the case where \( C \) contains larger than size 1 Jordan blocks is left aside in this work.
The two involutive transformations
\[
\lambda \mapsto -\lambda, \quad \lambda \mapsto \frac{1 - \lambda}{1 + 3\lambda}
\]
generate the permutation group of the three roots.

### 3.2. Pure solitons

In this subsection we select a subclass of soliton solutions, which we call 'pure solitons'. The main characterization is to restrict the expressions for the solutions of the linear systems in (2.6) and (2.9) to only involve a single root of each of the cubic equations (2.8) and (2.10). The latter equations coincide, since we assume \( C' = -C \) in this section, so we have to choose two different roots, \( P_1 = P \) and \( P_2 = Q \), of the same cubic equation. We will further assume that these matrices are diagonal, hence

\[
P = \text{diag}(p_1, \ldots, p_N) = \text{diag}(p_{1,1}, \ldots, p_{N,1}),
\]
\[
Q = \text{diag}(q_1, \ldots, q_N) = \text{diag}(p_{1,2}, \ldots, p_{N,2}).
\]

It will be convenient to allow both notations for the (diagonal) entries. The constants \( p_i, q_i \) have to solve the cubic equation
\[
z^3 = 3z^2 + c_i. \]

We will require \( q_i = p_j, \quad i = 1, \ldots, N \). Moreover, we will mostly also assume \( p_i \neq p_j \) and \( q_i \neq q_j \) for \( i \neq j \).

Writing

\[
\chi_i = \left( \begin{array}{c} \eta_{i_1} \\ \vdots \\ \eta_{i_N} \end{array} \right), \quad \theta_i = \left( \xi_1, \ldots, \xi_N \right)(Q - P),
\]

we then have

\[
\Omega_{ij} = \delta_{ij} + w_{ij} e^{\theta_i(p_i - q_j)}; \quad w_{ij} = \frac{q_j - p_i}{q_i - p_j} \eta_i K \xi_j.
\]

This is exactly the expression for \( \Omega \) that we found in [8] for the \( K_P \) equation. The only difference is that, for \( i = 1, \ldots, N \), \( q_i = p_i \) now have to satisfy the cubic equations, so that \( p_i^3 - 3p_iq_i = q_i^3 - 3q_i^2 \) holds, which is

\[
p_i^2 + p_iq_i + q_i^2 = 3\beta.
\]

But, apart from this, formulae derived in [8] for the potential \( K_P \) equation also apply to the case under consideration. Next we summarize those that are needed in this work. Introducing

\[
\tau = e^{\theta(q_i) + \cdots + \theta(q_N)} \det \Omega, \quad F = -e^{\theta(q_i) + \cdots + \theta(q_N)} e^{\theta(Q)} \text{adj} \Omega e^{-\theta(Q)} \Omega, \quad (3.6)
\]

where \( \text{adj} \Omega \) is the adjugate of the matrix \( \Omega \), we find that they have expansions

\[
\tau = \sum_{I \in \{1,2\}^N} \tau_I, \quad \tau_I = \mu_I e^{\theta_I}, \quad F = \sum_{I \in \{1,2\}^N} M_I e^{\theta_I},
\]

where \( \mu_I \) are constants, \( M_I \) constant matrices, and

\[
\theta_I = \sum_{k=1}^N \partial_i (p_k a_k) \quad I = (a_1, \ldots, a_N), \quad a_k \in \{1, 2\}.
\]

Recall that \( p_{k,1} = p_k \) and \( p_{k,2} = q_k, \quad k = 1, \ldots, N \). We have

\[
\phi = F, \quad \tau
\]

Since \( \tau_I, \quad \tau_I = p_I \tau_I \), where \( p_I = p_{1,a_1} + \cdots + p_{N,a_N} \) if \( I = (a_1, \ldots, a_N) \), we obtain

\[
u = \frac{1}{\tau^2} \sum_{I,J \in \{1,2\}^N} (p_I - p_J) (\mu_I M_J - \mu_J M_I) e^{\theta_I + \theta_J}.
\]

Note that (see [8])

\[
\text{tr}(KM_I) = \left( p_I - \sum_{i=1}^N q_i \right) \mu_I,
\]

so that

\[
\text{tr}(Ku) = 2(\log \tau)_{\text{ext}},
\]

in accordance with (2.14).

If \( \tau \neq 0 \) and if \( \mu_I > 0 \) for all \( I \) with \( \mu_I \neq 0 \) in the expression for \( \tau \), regularity of the solution is guaranteed. Let

\[
U_I = \{(x, t) \in \mathbb{R}^2 \mid \log \tau_I \geq \log \tau_J, \quad J \in \{1, 2\}^N\}.
\]

We call this the region where \( \psi_I \) dominates. As intersection of half-spaces, it is convex.

If \( \mu_I > 0 \), the tropical limit of \( \phi \) in \( U_I \) is \( ^7 \)

\[
\phi_I = \frac{M_I}{\mu_I}.
\]

For \( I = J \), the intersection \( U_I \cap U_J \) is a segment of the straight line determined by \( \log \tau_I = \log \tau_J \). On such a (visible)

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7 Appendix D in [16] explains the relation with the tropical limit defined via the Maslov dequantization formula.
segment the value of \( u \) is given by

\[
u_{ij} = \frac{1}{2} (p_i - p_j) (\phi_j - \phi_i).\]

We find \( \text{tr}(K u_{ij}) = \frac{1}{2} (p_i - p_j)^2 \) and introduce normalized values

\[
\hat{u}_{ij} = \frac{\phi_j - \phi_i}{p_j - p_i},
\]

which satisfy \( \text{tr}(K \hat{u}_{ij}) = 1 \). Using the notation

\[
I_k(a) = (a_1, \ldots, a_{k-1}, a, a_{k+1}, \ldots, a_N),
\]
the \( k \)th soliton appears in space–time on segments of the straight lines determined by \( \log \tau_k(1) = \log \tau_k(2) \), i.e.

\[
x + (p_k + q_k) t + \frac{1}{p_k - q_k} \log \frac{\tau_k(1)}{\tau_k(2)} = 0.
\]

(3.7)

This also determines the asymptotic structure of a tropical limit graph of a pure \( N \)-soliton solution. Without restriction of generality, we can order the parameters such that \( p_1 + q_1 < p_2 + q_2 < \cdots < p_\Theta + q_\Theta \). If we represent \( \rho_i \) and \( q_i \) by the first two roots in (3.4), then \( \rho_i + q_i \) is a strictly increasing function of \( |\lambda| \), hence this order is obtained by choosing \( 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_N \). Now it follows from (3.7) that, for \( t \ll 0 \), the solitons appear along the \( x \)-axis according to their numbering, and for \( t \gg 0 \) they appear in reverse order. To the left of the center is the dominating phase region \( U_{11,\ldots,1} \). Then follows counterclockwise \( U_{21,\ldots,1}, U_{22,1,\ldots,1}, \ldots \), unless we get to the region \( U_{22,\ldots,2} \) on the right-hand side. Correspondingly, starting again from \( U_{11,\ldots,1} \), we get clockwise to the regions \( U_{11,\ldots,2}, U_{12,\ldots,2}, \ldots \), until we arrive at \( U_{22,\ldots,2} \). These regions always appear in a tropical limit graph of a pure \( N \)-soliton solution. The remaining \( U_{ij} \) can only appear as bounded regions. But some may appear empty. This depends on the values of higher Boussinesq hierarchy variables, also see figure 5 below.

**Remark 3.1.** It can happen that \( \mu_l = 0 \) for some multi-index \( I \), so that \( e^{\psi} \) is absent in the expression for \( \tau \), but that this exponential appears in the numerator of the expression for \( u \), i.e. \( M_l = 0 \). Then some components of \( u \) will exhibit exponential growth in the region where this phase dominates, and a tropical limit does not exist.

### 3.2.1. Single soliton solution.

For the one-soliton solution we find

\[
\tau = e^{p_1 q_1 + q_1 r_1 + \phi_1} + e^{q_1 p_1 q_2 - \phi_2}, \quad \varphi_0 = \frac{1}{2} \log (\eta K \xi),
\]

assuming \( \eta K \xi > 0 \), and

\[
\phi = (p - q) \frac{e^{p_1 q_1 + q_1 r_1 + \phi_1} + e^{q_1 p_1 q_2 - \phi_2}}{\eta K \xi} \xi \otimes \eta.
\]

With the chosen parametrization, \( p_i + q_i = p_j + q_j \), for some \( i, j \), implies either \( p_i = p_j \) and \( q_i = q_j \), or \( p_i = q_j \) and \( p_j = q_i \). We exclude these cases.

This yields

\[
u = \frac{1}{2} (p - q)^2 sech^2 \left( \frac{1}{2} (p - q)(x + (p + q)t) + \varphi_0 \right) \xi \otimes \eta.
\]

Setting

\[
p = -\sqrt{3} \frac{1 + 6 \lambda - 3 \lambda^2}{1 + 3 \lambda^2}, \quad q = -\sqrt{3} \frac{1 - 6 \lambda - 3 \lambda^2}{1 + 3 \lambda^2},
\]

it takes the form

\[
u = 4 \beta \frac{18 \lambda^2}{(1 + 3 \lambda^2)^2} sech^2 \left( 2 \sqrt{3} \frac{3 \lambda}{1 + 3 \lambda^2} \right) \left( x - 2 \sqrt{3} \frac{1 - 3 \lambda^2}{1 + 3 \lambda^2} t \right) + \varphi_0 \frac{\xi \otimes \eta}{\eta K \xi}.
\]

Via the symmetries (3.5), one-soliton solutions with other choices of the roots are obtained from the above solution. If \( \lambda^2 < 1/3 \), the soliton moves from left to right. If \( \lambda^2 > 1/3 \), it moves from right to left. For \( \lambda^2 = 1/3 \), it is stationary. In all cases, the absolute value of the velocity is less than \( 2 \sqrt{3} \). We also note that \( 0 \leq \text{tr}(K \lambda) \leq 6 \beta \).

The tropical limit graph of the one-soliton solution is the boundary between the two dominating phase regions \( U_1 \) and \( U_2 \). It is the straight line in space–time (\( xt \)-plane), determined by

\[
x + (p + q)t + \frac{1}{p - q} \log (\eta K \xi) = 0,
\]

with slope \(-1/(p + q)\). We have

\[
u_{1,1} = \frac{1}{2} (p - q)^2 \frac{\xi \otimes \eta}{\eta K \xi}, \quad \nu_{1,2} = \frac{\xi \otimes \eta}{\eta K \xi}.
\]

#### 3.2.2. Two-soliton solution.

In this case \((N = 2)\), we find

\[
\tau = \alpha e^{A_1(p_1 + q_1) + \beta_1 (q_1 - p_1)} + \kappa_{1,2} e^{A_1 (q_1 - p_1) + \beta_2 (q_1 - p_1)} + \kappa_{2,1} e^{A_2 (p_1 - q_1) + \beta_1 (p_1 - q_1)} + \kappa_{2,2} e^{A_2 (p_1 - q_1) + \beta_2 (p_1 - q_1)},
\]

where

\[
\kappa_{ij} = \eta_i K_j \xi_j, \quad \alpha = \kappa_{1,1} \kappa_{2,2} - \frac{(q_1 - p_1)(q_2 - p_2)}{(q_1 - p_1)(q_2 - p_1)} \kappa_{1,2} \kappa_{2,1},
\]

and

\[
F = (q_1 - p_1)(q_2 - p_2) \left( \kappa_{2,2} \frac{\xi_2 \otimes \eta_1}{p_2 - q_2} + \frac{\kappa_{1,1}}{p_1 - q_1} \frac{\xi_1 \otimes \eta_2}{q_1 - p_1} + \frac{\kappa_{2,1}}{q_2 - p_1} \frac{\xi_2 \otimes \eta_1}{q_1 - p_1} + \frac{\kappa_{1,2}}{p_1 - q_1} \frac{\xi_1 \otimes \eta_2}{q_1 - p_1} \right)
\]

\[
+ (q_1 - p_1) \frac{\xi_1 \otimes \eta_1}{p_1 - q_1} e^{A_1(p_1 + q_1)} + (q_2 - p_2) \frac{\xi_2 \otimes \eta_2}{q_2 - p_2} e^{A_2(p_1 + q_1)}.
\]
Hence, if $\alpha, \kappa_{1,1}, \kappa_{2,2} \neq 0$,

$$
\phi_{1,1} = \frac{(q_1 - p_1)(q_2 - p_2)}{\alpha} \frac{\kappa_{2,2}}{p_2 - q_2} \xi_1 \otimes \eta_1 \\
+ \frac{\kappa_{1,2}}{q_1 - p_2} \xi_1 \otimes \eta_2 \\
+ \frac{\kappa_{1,1}}{p_1 - q_1} \xi_2 \otimes \eta_1 \\
$$

$$
\phi_{1,2} = \frac{p_1 - q_1}{\kappa_{1,1}} \xi_1 \otimes \eta_1, \\
\phi_{2,1} = \frac{p_2 - q_2}{\kappa_{2,2}} \xi_2 \otimes \eta_2, \\
\phi_{2,2} = 0.
$$

Choosing

$$p_1 = -\sqrt{3} \frac{1 + 6 \lambda_1 - 3 \lambda_1^2}{1 + 3 \lambda_1^2},$$

$$q_1 = -\sqrt{3} \frac{1 + 6 \lambda_2 - 3 \lambda_2^2}{1 + 3 \lambda_2^2},$$

we obtain via $\phi = F/\tau$ a two-soliton solution of the potential BsqK equation\footnote{If $\alpha, \kappa_{1,1}$ or $\kappa_{2,2}$ vanishes, the tropical limit may still be defined. But since the corresponding phase is absent in $\tau$, there is then no value $\phi_{1,1}$, $\phi_{1,2}$, respectively $\phi_{2,2}$.}. Figure 2 shows examples of corresponding tropical limit graphs. Applying the symmetries (3.5), two-soliton solutions with other choices of the roots are obtained from the above solution.

Furthermore, we find

$$\tilde{u}_{11,12} = \frac{1}{\alpha} \left( \frac{\kappa_{2,2} \kappa_{2,1}}{\kappa_{1,1}} \frac{(p_1 - q_1)^2}{(p_1 - q_2)(p_2 - q_1)(p_2 - q_2)} \xi_1 \otimes \eta_1 \\
- \frac{\kappa_{2,2}}{p_2 - q_1} \xi_1 \otimes \eta_2 \\
- \frac{\kappa_{2,1}}{(p_1 - q_1)(p_2 - q_2)} \xi_2 \otimes \eta_1 + \frac{\kappa_{1,1}}{\kappa_{2,2}} \xi_2 \otimes \eta_2 \right).$$

$$\tilde{u}_{11,21} = \frac{1}{\alpha} \left( \frac{\kappa_{1,2}}{\kappa_{1,1}} \frac{p_2 - q_2}{p_2 - q_1} \xi_1 \otimes \eta_1 \\
- \frac{\kappa_{2,2}}{p_2 - q_1} \xi_1 \otimes \eta_2 \\
- \frac{\kappa_{2,1}}{(p_1 - q_1)(p_2 - q_2)} \xi_2 \otimes \eta_1 \\
+ \frac{\kappa_{1,2}}{\kappa_{2,2}} \frac{(p_2 - q_2)^2}{(p_2 - q_1)^2} \xi_2 \otimes \eta_2 \right).$$

$$\tilde{u}_{12,22} = \frac{1}{\kappa_{1,1}} \xi_1 \otimes \eta_1, \\
\tilde{u}_{21,22} = \frac{1}{\kappa_{2,2}} \xi_2 \otimes \eta_2.$$
where

\[
R(\lambda_i, \lambda_j) = \begin{pmatrix}
\lambda_i - \lambda_j & 1 - \lambda_i - \lambda_j - 3\lambda_i\lambda_j \\
\lambda_i + \lambda_j & 1 - \lambda_i + \lambda_j + 3\lambda_i\lambda_j \\
2\lambda_i & 1 + 3\lambda_i^2 \\
\lambda_i + \lambda_j & 1 - \lambda_i + \lambda_j + 3\lambda_i\lambda_j \\
\lambda_i - \lambda_j & 1 + \lambda_i + \lambda_j - 3\lambda_i\lambda_j \\
\lambda_i + \lambda_j & 1 - \lambda_i + \lambda_j + 3\lambda_i\lambda_j
\end{pmatrix}.
\]

**Remark 3.2.** Dropping the exponential factor in (3.6), which has only been introduced to achieve a convenient numbering of phases, in the \(N = 2\) case we obtain

\[
\tau = 1 + e^{\zeta_1} + e^{\zeta_2} + \left(\frac{\kappa_{1,1}\kappa_{2,2} - \kappa_{1,2}\kappa_{2,1}}{\kappa_{1,1}\kappa_{2,2}}\right) (p_1 - p_2)(q_1 - q_2) e^{\zeta_1 + \zeta_2},
\]

where \(\zeta = \vartheta(p) - \vartheta(q) + \log \kappa_{ij}\), assuming \(\kappa_{ij} > 0\). Comparison with a known expression for the \(\tau\)-function of the two-soliton solution of the scalar Boussinesq (or KP) equation shows that this determines a solution of the scalar Boussinesq equation if \(\kappa_{1,2}\kappa_{2,1} = \kappa_{1,2}\kappa_{2,1}\). We also note that, if \(\kappa_{1,2}\kappa_{2,1} = 0\), the above expression factorizes to

\[
\tau = (1 + e^{\zeta_1})(1 + e^{\zeta_2}).
\]

In this case, the tropical limit graph is simply the superimposition of those of the factors

\footnote{This is evident from the Maslov dequantization formula, see appendix D in \[16\], for example.}, hence there is no phase shift.

### 3.2.3. Degenerations of the pure two-soliton solution of the vector Boussinesq equation.

We consider the special cases where \(p_1 = p_2\) or \(q_1 = q_2\). Then we have \(c_1 = c_2\), so that all parameters are roots of a single cubic equation. Since we represent \(p_i\) and \(q_i\) by the first two roots in (3.4), this means that \(\lambda_2 = (1 - \lambda_1)/(3\lambda_1 + 1)\), respectively \(\lambda_2 = (1 + \lambda_1)/(3\lambda_1 - 1)\). In both cases we have \(\alpha = \kappa_{1,1}\kappa_{2,2} - \kappa_{1,2}\kappa_{2,1}\), which vanishes when we address the vector Boussinesq equation. If \(q_2 = q_1\), we obtain

\[
\tau = (e^{\theta(q_1)} + \kappa_{1,1} e^{\theta(p_1)} + \kappa_{2,2} e^{\theta(p_2)}) e^{\theta(q_1)}.
\]

The factor \(e^{\theta(q_1)}\) does not influence the tropical limit graph, which is shown in figure 3. If \(p_2 = p_1\), we find

\[
\tau = (e^{-\theta(q_1)} + \kappa_{1,1} e^{-\theta(q_1)} + \kappa_{2,2} e^{-\theta(q_1)}) e^{\theta(p_1) + \theta(q_1)} e^{\theta(q_1)}.
\]

The corresponding tropical limit graphs are Y-shaped (a soliton splits into two), respectively reverse Y-shaped (two solitons merge), see figure 3. Approaching such a solution by letting \(q_2 \to q_1\), respectively \(p_2 \to p_1\), in the two-soliton solution in section 3.2.2, we see that the edge representing the virtual soliton (see the second and third graph in figure 2) gets longer and longer, in such a way that the dominating phase region \(U_{1,1}\) finally disappears at infinity.

---

**Figure 3.** Tropical limit graphs of degenerate two-soliton solutions of the \(m = 2\) vector (i.e. \(n = 1\)) Boussinesq equation. Here we chose \(K = (1, 1), v_1 = v_2 = 1, \xi_1 = (1, 0)^T, \xi_2 = (0, 1)^T\), and the parameter values \((\lambda_1, \lambda_2) = (7/10, 17/11)\) (so that \(q_2 = q_1\), respectively \((\lambda_1, \lambda_2) = (7/10, 3/31)\) (so that \(p_2 = p_1\)).

**Figure 4.** A two-soliton part of an \(N\)-soliton tropical limit graph.

In these cases the \(R\)-matrix reduces to

\[
R(\lambda_i, \lambda_j) = \begin{pmatrix}
1 - \lambda & 4\lambda \\
1 - \lambda(3\lambda + 1) & 1 - \lambda(3\lambda - 1)
\end{pmatrix},
\]

respectively

\[
R(\lambda_i, \lambda_j) = \begin{pmatrix}
1 & 4\lambda \\
1 - \lambda(3\lambda + 1) & 1 - \lambda(3\lambda - 1)
\end{pmatrix}.
\]

### 3.2.4. \(N\)-soliton solutions and phase shifts.

If \(I = (a_1, \ldots, a_N)\) and \(i < j\), let

\[
I_{ij}(a, b) = (a_i, \ldots, a_{i-1}, a, a_{i+1}, \ldots, a_{j-1}, b, a_{j+1}, \ldots, a_N).
\]

The tropical limit graph of an \(N\)-soliton solution generically contains subgraphs describing two-soliton interactions, see figure 4. In the vicinity of such a local two-soliton interaction, the solution is well approximated by only keeping the four relevant phases (since exponentials of the others are then negligible) in the function \(\tau\). Hence

\[
\tau \approx \tau_{ij} = \sum_{a,b=1,2} T_{ij}(a,b)\tau,
\]

\[
\phi \approx \phi_{ij} = \frac{1}{\tau_{ij}} \sum_{a,b=1,2} \phi_{ij}(a,b) T_{ij}(a,b)\tau.
\]
Also see, e.g. [18] (section 5 therein) for such approximations in the case of KP solitons.

The two parallel line segments corresponding to the path of the \(i\)th soliton are determined by

\[
x_{i,\text{in}} = -(p_i + q_i) t - (p_i - q_i)^{-1} \log \left( \frac{\mu_{i,1}(1)}{\mu_{i,2}(1)} \right),
\]

\[
x_{i,\text{out}} = -(p_i + q_i) t - (p_i - q_i)^{-1} \log \left( \frac{\mu_{i,1}(2)}{\mu_{i,2}(2)} \right).
\]

Their shift along the \(x\)-axis, caused by the interaction with the \(j\)th soliton, is

\[
\delta_j^i x := x_{j,\text{out}} - x_{i,\text{in}} = -(p_j - q_j)^{-1} \log(A_{k_j}).
\]

If \(N = 2\), we have \(A_{k_2} = \alpha/(k_{1,1}k_{2,2})\) (also see [17] for the scalar case).

### 3.2.5. Three-soliton solutions

If \(N = 3\), we find

\[
\tau = \gamma e^{\theta_{1,1,1}} + \alpha_{1,2} e^{\theta_{1,1,2}} + \alpha_{1,3} e^{\theta_{1,1,3}} + \alpha_{2,3} e^{\theta_{2,1,1}}
+ k_{1,1} e^{\theta_{1,2,2}} + k_{2,2} e^{\theta_{2,2,2}} + k_{3,3} e^{\theta_{3,3,3}} + e^{\theta_{3,3,3}},
\]

where \(k_{ij}, i, j = 1, 2, 3\), are defined as in (3.8), and

\[
\alpha_{ij} = k_{ij}k_{ji} - \frac{(p_i - q_j)(p_j - q_i)}{(p_i - q_i)(p_j - q_j)}
\]

\[
\gamma = k_{1,1}k_{2,2}k_{3,3} + \frac{(p_1 - q_i)(p_3 - q_3)(p_2 - q_2)}{(p_2 - q_2)(p_3 - q_3)(p_1 - q_1)} k_{1,2,2}k_{2,3}k_{3,1}
+ \frac{(p_1 - q_j)(p_3 - q_3)(p_2 - q_2)}{(p_2 - q_2)(p_3 - q_3)(p_1 - q_1)} k_{1,3,2}k_{2,1}k_{3,3}
+ \frac{(p_1 - q_k)(p_3 - q_3)(p_2 - q_2)}{(p_2 - q_2)(p_3 - q_3)(p_1 - q_1)} k_{1,3,2}k_{2,1}k_{3,3},
\]

Furthermore, if all coefficients of exponentials in \(\tau\) are positive, we have the following tropical values of \(\phi\),

\[
\phi_{1,1,1} = \frac{k_{1,1}k_{2,2}k_{3,3}}{\gamma} \left( \frac{\xi_1 \otimes \eta_1}{k_{1,1}} + (1 - \delta_{1,2}) \frac{\xi_2 \otimes \eta_2}{k_{2,2}} + (1 - \delta_{1,3}) \frac{\xi_3 \otimes \eta_3}{k_{3,3}} \right)
+ \frac{(p_1 - q_j)(p_3 - q_3)(p_2 - q_2)}{(p_2 - q_2)(p_3 - q_3)(p_1 - q_1)} k_{1,2,2}k_{2,3}k_{3,1}
+ \frac{(p_1 - q_j)(p_3 - q_3)(p_2 - q_2)}{(p_2 - q_2)(p_3 - q_3)(p_1 - q_1)} k_{1,3,2}k_{2,1}k_{3,3},
\]

\[
\phi_{1,2,1} = \frac{1}{\delta_{1,2}} \left( (p_1 - q_j) \frac{\xi_1 \otimes \eta_1}{k_{1,1}} + (p_2 - q_2) \frac{\xi_2 \otimes \eta_2}{k_{2,2}} + (q_3 - p_1)(1 - \delta_{1,2}) \frac{\xi_3 \otimes \eta_3}{k_{3,3}} \right)
+ (q_3 - p_1)(1 - \delta_{1,2}) \frac{\xi_3 \otimes \eta_3}{k_{3,3}},
\]

\[
\phi_{2,1,1} = \frac{1}{\delta_{2,3}} \left( (p_2 - q_2) \frac{\xi_2 \otimes \eta_2}{k_{2,2}} + (p_3 - q_3) \frac{\xi_3 \otimes \eta_3}{k_{3,3}} + (q_3 - p_2)(1 - \delta_{2,3}) \frac{\xi_3 \otimes \eta_3}{k_{3,3}} \right)
+ (q_3 - p_2)(1 - \delta_{2,3}) \frac{\xi_3 \otimes \eta_3}{k_{3,3}},
\]

\[
\phi_{1,2,2} = \frac{1}{\delta_{1,3}} (p_1 - q_j) \frac{\xi_1 \otimes \eta_1}{k_{1,1}},
\]

\[
\phi_{2,1,2} = \frac{1}{\delta_{2,3}} (p_2 - q_2) \frac{\xi_2 \otimes \eta_2}{k_{2,2}},
\]

\[
\phi_{2,2,1} = (p_3 - q_3) \frac{\xi_3 \otimes \eta_3}{k_{3,3}}, \quad \phi_{2,2,2} = 0,
\]

where

\[
\alpha_{ij} = 1 - \frac{(p_i - q_j)(p_j - q_i)}{(p_i - q_i)(p_j - q_j)} k_{ij}k_{ji},
\]

\[
\alpha_{kk} = 1 - \frac{(p_k - q_k)(p_k - q_k)}{(p_k - q_k)(p_k - q_k)} k_{kk}k_{kk},
\]

\[
\delta_{kk} = 1 - \frac{(p_k - q_k)(p_k - q_k)}{(p_k - q_k)(p_k - q_k)} k_{kk}k_{kk}.
Examples of corresponding tropical limit graphs are shown in Figure 5. Here we extended the phase expression (2.7) by including the next hierarchy variable:

$$\vartheta(P) = P_x + P^2 t + p^4 s.$$  

The first and the third graph correspond to large negative, respectively large positive value of \( s \). The sequences of two-soliton interactions are according to the left, respectively right-hand side of the Yang–Baxter equation. Since the polarizations along edges of a tropical limit graph do not depend on the variables \((x, t, s)\), we conclude that the Yang–Baxter equation holds.

### 3.3. Other soliton configurations

Now we consider solutions involving three roots of the cubic equation. There are then two cases, either

$$\theta = \theta_1 e^{\delta(p_1)} + \theta_2 e^{\delta(p_2)}, \quad \chi = e^{-\delta(p_1)} \xi_3,$$

or

$$\theta = \theta_3 e^{\delta(p_3)}, \quad \chi = e^{-\delta(p_1)} \xi_1 + e^{-\delta(p_2)} \xi_2.$$  

The second, ‘dual’ choice can be obtained from the first by applying the symmetry \( \phi \mapsto \phi^T, t \mapsto -t \), and using \( \chi_a \mapsto \theta_a^{T} \), \( p_a \mapsto -p_a^T \). In the following we use again the decomposition (3.2), but with the rescaling \( \xi_{ia} \mapsto (p_{i3} - p_{ia}) \xi_{ia}, \ a = 1, 2 \).

**Example 3.3.** We consider the simplest case, \( N = 1 \). Hence \( i = 1 \) in \( p_{i\alpha} \) and \( \xi_{ia} \). The corresponding index will now be suppressed. Then we have

$$\theta = (p_3 - p_1)\xi_1 e^{\delta(p_1)} + (p_3 - p_2)\xi_2 e^{\delta(p_2)}, \quad \chi = e^{-\delta(p_3)} \eta_3.$$  

Here \( \xi_{ia}, a = 1, 2 \), are \( m \)-component column vectors, \( \eta_3 \) is an \( n \)-component row vector. Then

$$\tau = e^{\delta(p_3)} \Omega = e^{\delta(p_1)} + \eta_3 K \xi_1 e^{\delta(p_1)} + \eta_3 K \xi_2 e^{\delta(p_2)},$$

and

$$\phi = \frac{1}{\tau}((p_3 - p_1)\xi_1 \otimes \eta_3 e^{\delta(p_1)} + (p_3 - p_2)\xi_2 \otimes \eta_3 e^{\delta(p_2)}).$$

We set

$$p_1 = -\frac{\sqrt{3}(1 + 6\lambda - 3\lambda^2)}{1 + 3\lambda^2},$$

$$p_2 = -\frac{\sqrt{3}(1 - 6\lambda - 3\lambda^2)}{1 + 3\lambda^2},$$

$$p_3 = \frac{2\sqrt{3}(1 - 3\lambda^2)}{1 + 3\lambda^2}.$$  

The solution describes the merging of two solitons into a single one, also see Figure 6. If \( \lambda \in \{0, \pm 1/3, \pm 1\} \), two of the \( p_\alpha \) are equal and the solution reduces to a one-soliton solution. The dual case describes the splitting of a single soliton into two.

**Remark 3.4.** In contrast to the KdV reduction of KP, where a quadratic equation rules the game, \( B_{sq} \) thus admits solutions with a tropical limit graph in space–time having the most elementary rooted binary tree shape: three edges meeting at a vertex. We may speculate that in higher Gelfand–Dickey reductions there are corresponding limits for the number of edges forming a rooted binary tree.

**Example 3.5.** For \( N = 2 \) and \( P_{2} = \text{diag}(p_{1,a}, p_{2,a}), a = 1, 2, 3 \), we obtain

$$\phi = \frac{F}{\tau},$$

with

$$\tau = e^{\delta(p_{1,a}) + \delta(p_{2,a})} \det(\Omega)$$

$$= \sum_{a,b=1}^{2} \alpha_{ab} e^{\delta_{ab}} + \kappa_{1,1,1} e^{\delta_{1,3}} + \kappa_{1,1,2} e^{\delta_{2,3}} + \kappa_{2,2,1} e^{\delta_{3,1}} + \kappa_{2,2,2} e^{\delta_{3,2}} + \delta_{3,3}.$$
and
\[
F = (p_{1,3} - p_{1,1})(p_{2,3} - p_{2,1})\left(\frac{\kappa_{1,1} \xi_{2,1} \otimes \eta_2}{p_{1,1} - p_{1,3}} + \frac{\kappa_{2,1} \xi_{2,1} \otimes \eta_1}{p_{2,3} - p_{2,1}} + \frac{\kappa_{2,1} \xi_{1,1} \otimes \eta_1}{p_{2,1} - p_{2,3}}\right) e^{\theta_{1,1}}
\]
\[
+ (p_{1,3} - p_{1,2})(p_{2,3} - p_{2,2})\left(\frac{\kappa_{1,1} \xi_{2,2} \otimes \eta_2}{p_{1,1} - p_{1,3}} + \frac{\kappa_{2,1} \xi_{2,1} \otimes \eta_1}{p_{2,3} - p_{2,1}} + \frac{\kappa_{2,2} \xi_{1,1} \otimes \eta_1}{p_{2,1} - p_{2,3}}\right) e^{\theta_{2,1}}
\]
\[
+ (p_{1,3} - p_{1,2})(p_{2,3} - p_{2,2})\left(\frac{\kappa_{1,2} \xi_{2,2} \otimes \eta_2}{p_{1,1} - p_{1,3}} + \frac{\kappa_{2,1} \xi_{2,1} \otimes \eta_1}{p_{2,3} - p_{2,1}} + \frac{\kappa_{2,2} \xi_{1,1} \otimes \eta_1}{p_{2,1} - p_{2,3}}\right) e^{\theta_{2,2}}
\]
\[
+ (p_{1,3} - p_{1,2})(p_{2,3} - p_{2,2})\left(\frac{\kappa_{1,2} \xi_{2,2} \otimes \eta_2}{p_{1,1} - p_{1,3}} + \frac{\kappa_{2,1} \xi_{2,1} \otimes \eta_1}{p_{2,3} - p_{2,1}} + \frac{\kappa_{2,2} \xi_{1,1} \otimes \eta_1}{p_{2,1} - p_{2,3}}\right) e^{\theta_{2,3}}
\]

Here we set
\[
\tilde{\vartheta}_{ab} = \tilde{\vartheta}_{(p_{1,a})} + \tilde{\vartheta}_{(p_{2,b})} \quad a, b = 1, 2, 3,
\]
\[
\kappa_{ij,a} = \eta_a \varpi \xi_{j,a} \quad i, j = 1, 2, a = 1, 2,
\]
\[
\alpha_{ab} = \kappa_{1,a} \kappa_{2,2,b} - \frac{\eta_1}{(p_{1,a} - p_{2,b})} \quad a, b = 1, 2,
\]

using (3.2) with
\[
\lambda_3 = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}
\]

Example 3.6. According to proposition 3.7 below, in the scalar and vector case there is no regular solution in the class given by example 3.5, with all possible phases present in the expression for \(\tau\). But corresponding regular solutions exist, for example, in the 2 \(\times\) 2 matrix case. Figure 7 shows two examples of tropical limit graphs. Here we chose \(K = \text{diag}(1, 1)\) and \(\lambda_1 = 7/10, \lambda_2 = 4\).

3.3.1. Solutions of the vector Boussinesq equation. In this subsection we consider the case of the m-component vector \((n = 1)\) Boussinesq equation. This includes the scalar Boussinesq case \((m = 1)\).

Proposition 3.7. For the vector Boussinesq equation, and with real parameters, it is not possible that all coefficients in the expression for \(\tau\) in the \(N = 2\) case (example 3.5) are positive.

Proof. In the vector case, where \(\eta_i\) are scalars, we have \(\kappa_{2,1,a} \kappa_{1,2,b} = \kappa_{1,1,a} \kappa_{2,2,b}\), hence
\[
\alpha_{ab} = \kappa_{1,1,a} \kappa_{2,2,b} \cdot \alpha_{ab},
\]
\[
\tilde{\alpha}_{ab} = \frac{(p_{2,3} - p_{1,3})(p_{1,a} - p_{2,b})}{(p_{1,a} - p_{2,b})(p_{2,3} - p_{1,3})},
\]

Let us assume \(\kappa_{i,a} > 0\), \(i = 1, 2, a = 1, 2, 3\). This implies that, for \(a = 1, 2, b = 1, 2\), either
\[
(p_{2,3} - p_{1,3})(p_{1,a} - p_{2,b}) > 0 \quad \text{and} \quad (p_{1,a} - p_{2,b})(p_{2,3} - p_{1,3}) > 0,
\]
or
\[
(p_{2,3} - p_{1,3})(p_{1,a} - p_{2,b}) < 0 \quad \text{and} \quad (p_{1,a} - p_{2,b})(p_{2,3} - p_{1,3}) < 0
\]
holds. Next we observe that, for \(i = 1, 2\), the solutions (3.4) of the cubic equation satisfy one of the following sets of inequalities (also see figure 1)

1. \(p_{1,3} \leq -\sqrt{3} \leq p_{1,2} \leq \sqrt{3} < p_{2,1}\)
2. \(p_{2,2} \leq -\sqrt{3} \leq p_{2,3} \leq \sqrt{3} \leq p_{1,1}\)
3. $p_{i,2} \leq -\sqrt{3} \leq p_{i,1} \leq \sqrt{3} \leq p_{i,3}$.

This leaves us with nine cases. In all of them, an analysis of the inequalities leads to a contradiction.

As a consequence of the above proposition, if all possible terms are present in the $N = 2$ function $\tau$ (example 3.5), in the vector Boussinesq case the solution has a singularity. Regular solutions are then only possible if at least one of the coefficients of exponentials in $\tau$ vanishes. In order to achieve this, we have to choose special values of some parameters, or special relations between parameters.

To see what happens if one of the coefficients of exponentials in $\tau$ vanishes, let us consider the case where $K^{\delta}_{i,2} = 0$ (other cases lead to the same conclusions). Then the $N = 2$ function $\tau$ reduces to

$$
\tau = \alpha_{i,1} e^{\beta_{i,1}} + \alpha_{i,2} e^{\beta_{i,2}} + \alpha_{i,1,1} e^{\beta_{i,1,1}} + \kappa_{2,2,1} e^{\beta_{2,2,1}} + \kappa_{2,2,2} e^{\beta_{2,2,2}} + e^{\beta_{i,3}}
$$

where we set $\eta = 1$, which is no restriction of generality in the case under consideration, and $\delta_{i,1}, \delta_{i,2}$ are defined in (3.11). In the last expression in (3.12), the part in the first brackets corresponds to an $N = 1$ soliton configuration, see example 3.3. If the part in the other brackets had the same coefficients, $\tau$ would factorize and we would obtain a tropical limit graph which is a superimposition of that of the $N = 1$ Y-shaped solution and a single soliton. Different coefficients lead to a deformation, introducing phase shifts. Figure 8 shows an example.

The condition $K^{\delta}_{i,2} = 0$ eliminates some phases from the function $\tau$, i.e. we have $\mu^{ab}_{i} = 0$ for some $a, b$. If we choose $\xi_{i,2} = (1, -1)^T$, keeping otherwise the data specified in figure 8, then we still have $K^{\delta}_{i,2} = 0$, but some $M^{ab}_{i}$ is different from zero, although $\mu^{ab}_{i} = 0$.

The last proposition tells us that, however small (with respect to a suitable norm) we choose a neighborhood of such a regular solution, in the set of solutions, it contains singular solutions.

An $N > 2$ solution locally consists approximately of $N = 2$ solutions. We should thus expect the above proposition to extend to $N > 2$. But we will not attempt to provide a rigorous proof here. In the following we show that solutions with $N = 3$ exist, where the singularities only appear in a compact region of space-time.

N = 3. We set $P = \text{diag}(p_{i}, q_{i}, r_{i})$, $i = 1, 2, 3$, and obtain

$$
\tau = \sum_{a,b,c=1}^{3} \mu^{abc}_{i} e^{\delta^{abc}_{i}}
$$

with $\delta^{abc}_{i} = \delta(p_{i}) + \delta(q_{i}) + \delta(r_{i})$ and

$$
\mu^{abc}_{i} = \frac{(p_{i} - q_{i})(p_{i} - r_{i})(q_{i} - r_{i})(p_{i} - q_{i})(p_{i} - r_{i})(q_{i} - r_{i})}{\eta^{1}_{1,1} \eta^{1}_{1,2} \kappa_{1,1} \kappa_{2,2} \kappa_{3,3}}
$$

where $\delta^{abc}_{i}$ is the Kronecker symbol. Furthermore

$$
\Phi^{abc}_{i} = \frac{(p_{i} - q_{i})(q_{i} - r_{i})(p_{i} - r_{i})}{\kappa_{1,1}}
$$

If $\tau$ has also negative summands, strictly its tropical limit is not defined. Notwithstanding this, we can determine

---

**Figure 7.** Tropical limit graphs of $2 \times 2$ Boussinesq solutions according to example 3.6. In the first example, we set $\eta = (10^2, 0)$, $\eta = (0, 10^{-2})$, $\delta = (1, 1)^T$, $\delta = (1, 0)^T$, $\delta = (0, 1)^T$, $\delta = (1, 2)^T$. In the second we chose $\eta = (10^{-4}, 0)$, $\eta = (0, 10^4)$, $\delta = (1, 0)^T$, $\delta = (1, -2)^T$, $\delta = (2, 1)^T$, $\delta = (1, 1)^T$. Only seven of the nine phases appearing in the expression for $\tau$ are visible.
and plot) the regions where the logarithm of the absolute value of a summand of $\tau$ dominates. On a boundary segment between such (generalized) dominating phase regions, where the corresponding summand in $\tau$ is positive for one and negative for the other, the soliton solution is singular. This is so because close to such a boundary segment contributions from all other summands are exponentially suppressed, hence negligible. Furthermore, singularities can only appear at such a boundary segment, since everywhere else either a single summand of $\tau$ dominates and all others are negligible, or we have a boundary segment along which two summands with the same sign have equal values and all other summands are negligible. The latter two cases exclude a singularity.

Figure 9 shows an example of such a modified tropical limit graph. The white regions are dominated by phases having negative contributions to the $\tau$-function. In these regions the solution is still regular, but on the interface between a ‘positive’ and a ‘negative’ phase region (plotted in red), and only there, the solution becomes singular. A similar solution of the scalar Boussinesq equation appeared in [5] (see figure 8 therein).
4. The case $C' \neq -C$

We can use the transformations in remark 2.2 to achieve that both, $C$ and $C'$, have Jordan normal form. Then (2.13) generically implies $\Omega \rho = 0$. This will be assumed in the following. Choosing diagonal matrices

$$
P_a = \text{diag}(p_{1,a}, \ldots, p_{N,a}), \quad Q_a = \text{diag}(q_{1,a}, \ldots, q_{N,a}), \quad C = \text{diag}(c_1, \ldots, c_N), \quad C' = \text{diag}(c'_1, \ldots, c'_N),
$$

(2.8) and (2.10) require

$$
p_a^3 = 3/p_{a} + c_i, \quad q_a^3 = 3/q_{a} - c'_i.
$$

For each $i = 1, \ldots, N$, we represent the roots $p_{i,a}$, $a = 1, 2, 3$, of the first cubic equation as in (3.4), using a parameter $\lambda_i$. In the same way we represent the roots $q_{i,a}$, $a = 1, 2, 3$, of the second cubic equation using a parameter $\nu_i$. We assume that $p_{i,a} \neq p_{j,b}$ and $q_{i,a} \neq q_{j,b}$ for $i \neq j$ or $a \neq b$. Now we have

$$
\theta = \sum_{a=1}^{3} \theta_a e^{\lambda(p_a)}, \quad \chi = \sum_{a=1}^{3} e^{\nu(q_a)} \chi_a.
$$

Assuming $q_{b} \neq p_{a}$ for all combinations of indices, from (2.11) and (2.12) we obtain

$$
\Omega_{ij} = \sum_{a,b=1}^{3} \eta_{ab}\kappa_{ab}\ e^{\lambda(p_a) - \lambda(q_b)}, \quad i, j = 1, \ldots, N.
$$

The Bsqk solution is given by

$$
\phi = \frac{F}{r},
$$

where now

$$
\tau = \det(\Omega) = \sum_{I, J \in \{1,2,3\}^n} \tau_{IJ}, \quad \tau_{IJ} = \mu_{IJ} e^{\hat{\vartheta}_{IJ}},
$$

$$
\hat{\vartheta}_{IJ} = \sum_{i=1}^{N} (\vartheta(p_{i,a}) - \vartheta(q_{i,b})),
$$

$$
F = -\theta \text{ adj}(\Omega) \chi = \sum_{I, J \in \{1,2,3\}^n} \phi_{IJ} \tau_{IJ},
$$

with constants $\mu_{IJ}$ and constant matrices $\phi_{IJ}$, of a certain structure. Here $\text{adj}(\Omega)$ again denotes the adjugate of the matrix $\Omega$, and we wrote $I = (a_1, \ldots, a_N)$, $J = (b_1, \ldots, b_N)$ in the expression for $\phi_{IJ}$. If a phase $\vartheta_{IJ}$ is present in $\tau$, and if the tropical limit exists, then the tropical value of $\phi$ in the corresponding dominating phase region is $\phi_{IJ}$. We decompose $\theta_a$ and $\chi_a$ as in (3.2).

In the following we restrict our considerations to the case $N = 1$. Then $\Omega$ is a scalar, consisting of at most nine sum-mands. We obtain $\phi = F/\tau$ with

$$
\tau = \Omega = \sum_{a,b=1}^{3} \kappa_{ab} e^{\nu(q_b) - \nu(q_a)}, \quad \kappa_{ab} = \chi_a K \theta_{ab},
$$

$$
F = \sum_{a,b=1}^{3} \theta_a \otimes X_b e^{\nu(q_b) - \nu(q_a)}.
$$

Hence

$$
\phi_{ab} = \frac{p_a - q_b}{\kappa_{ba}} \theta_b \otimes \chi_b.
$$

In the scalar and vector Boussinesq case, one can show that there is no regular solution of the above form, with real parameters and with all coefficients of exponentials in $\tau$ different from zero. This means that, however small (using a suitable norm) we choose a neighborhood of such a regular solution, in the set of solutions, it contains singular solutions. The first statement is not true in the matrix case.

**Example 4.1.** Let $m = n = 3$, $K = \text{diag}(1, 1, 1)$,

$$
\lambda_1 = (1 \ 0 \ 0), \quad \lambda_2 = (0 \ 1 \ 0), \quad \lambda_3 = (0 \ 0 \ 1),
$$

$$
\theta_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \theta_2 = \begin{pmatrix} -2 \\ -1 \end{pmatrix}, \quad \theta_3 = \begin{pmatrix} -1 \\ -1 \end{pmatrix},
$$

and $\lambda_i = 1/5$, $\nu_i = 3/2$. The function $\tau$, given above, is then a positive linear combination of nine independent exponentials, which is the maximal number. All nine phases, appearing in the corresponding function $\tau$, are visible in the tropical limit graph, see figure 10. The three interior phase regions are hardly expected from the plot on the left-hand side and reveal a complicated interaction pattern. Figure 11 shows another, though simpler example, where the relation is evident.

---

**Figure 10.** Plot of $\text{tr}(Ku) = 2(\log \tau)_{\nu}$ and tropical limit graph the $3 \times 3$ matrix solution as specified in example 4.1. This shows a three-soliton solution solution where none of the outgoing solitons has a velocity equal to that of one of the incoming solitons. The tropical limit graph shows that this is actually a superposition of a splitting soliton (red Y-shaped) and two merging solitons (blue reversed Y-shaped graph).
5. Conclusions

The scalar Boussinesq equation exhibits richer soliton interactions than the KdV equation. This concerns head-on collisions and inelastic scattering. All this is nicely revealed in the tropical limit. Here the function $\tau$ determines the solution via $u = 2(\log \tau)_{\alpha}$.

In the vector case ($m = 1$ or $n = 1$), we have $\text{tr}(Ku) = 2(\log \tau)_{\alpha}$, and this is a solution of the scalar Boussinesq equation. Specifying initial polarizations for sufficiently large negative time $t$, i.e., those of the incoming solitons, the distribution of polarizations over the tropical limit graph is obtained, for the class of solutions treated in section 3, by use of a Yang–Baxter $R$-matrix. For a more general class of solutions, also a tetragon map is at work, see section 8 in [9].

For the matrix ($m, n > 1$) Boussinesq equation, $\text{tr}(Ku) = 2(\log \tau)_{\alpha}$ is not in general a solution of the scalar equation. A nonlinear Yang–Baxter map is at work, which is a reduction of the corresponding matrix KP Yang–Baxter map, recently obtained in [8].

We should note that a tropical limit of a matrix soliton solution can only be expected if we arrange the parameters such that exponentials absent in $\tau$ are also absent in the numerator of $u$. Otherwise the solution will exhibit exponential growth (in some space–time direction), in which case the solution should no longer be called a soliton solution.

In the scalar and vector case, inelastic collisions are subject to severe restrictions. As expected, there is more freedom in the full matrix case.

We concentrated on regular solutions, for which all summands of $\tau$ are positive, so that the tropical limit, defined via Maslov dequantization, of $\tau$ makes sense. We did not have to compute this limit directly, however, since instead it is easier to determine the dominating phase regions and then the respective boundaries, which constitute the tropical limit graph. The latter point of view allows a generalization of the tropical limit, which also applies to singular solutions, as explained at the end of section 3.3.1. This allows us to determine, in a simple way, the locus of singularities of a solution.

Nonlinear atomic or molecular chains, modeled by the scalar Boussinesq equation in a continuum limit, disregard a possible polarization of the particles. Taking polarizations or spins into account, such a chain might be described in the continuum limit by a vector or matrix version of the Boussinesq equation.

Similar explorations, as done for matrix Boussinesq equations in this work, should be possible for other integrable equations too, in particular for discrete versions of the Boussinesq equation (see [19–21], for example).

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Appendix. Derivation of the binary Darboux transformation for the matrix Boussinesq equation

We recall a binary Darboux transformation result of bidifferential calculus [14, 15].

**Theorem A.1.** Let $(\Omega, d, \bar{d})$ be a bidifferential calculus and $\Delta, \Gamma, \lambda, \kappa$ solutions of

\[
\bar{d}\Gamma = \Gamma d\Gamma + [\kappa, \Gamma], \quad \bar{d}\kappa = \Gamma d\kappa + \kappa^2, \\
\bar{d}\Delta = (d\Delta) \Delta - [\lambda, \Delta], \quad \bar{d}\lambda = (d\lambda) \Delta - \lambda^2,
\]

and $\phi_0$ a solution of

\[
\bar{d}\bar{d}\phi + d\phi K d\phi = 0, \quad (A.1)
\]

where $dK = 0 = \bar{d}K$. Let $\theta$ and $\chi$ be solutions of the linear system

\[
\bar{d}\theta = (d\phi_0) K \theta + (d\theta) \Delta + \theta \lambda, \quad (A.2)
\]

respectively the adjoint linear system

\[
\bar{d}\chi = -\chi K d\phi_0 + \Gamma d\chi + \kappa \chi. \quad (A.3)
\]
Let $\Omega$ solve the compatible linear system

$$\Gamma \Omega - \Omega \Delta = -\eta \, K \, \theta,$$

(A.4)

$$\delta \Omega = (d\Omega) \Delta - (d\Gamma) \Omega + (d\eta) \, K \, \theta + \kappa \, \Omega + \Omega \lambda,$$

Where $\Omega$ is invertible

$$\phi = \phi_0 - \theta \, \Omega^{-1} \chi$$

(A.5)

is a new solution of (A.1).

In the above theorem, we have to assume that all objects are such that the corresponding products are defined and that $d$ and $\delta$ can be applied. Next we define a bidifferential calculus via

$$df = \{\partial_w, f\} \zeta_1 + \frac{1}{2} \{\partial_w + \partial_x^2, f\} \zeta_2,$$

$$d\bar{f} = \frac{1}{2} \{\partial_t - \partial_x^2, f\} \zeta_1 + \{\beta \, \partial_t - \frac{1}{3} \partial_x^3, f\} \zeta_2,$$

on the algebra $\mathcal{A} = \mathcal{A}_0[\partial_x, \partial_t]$, where $\mathcal{A}_0$ is the algebra of smooth functions of two variables, $x$ and $t$, and $\partial_i$ is the operator of partial differentiation with respect to $x_i$. $\zeta_1$, $\zeta_2$ are a basis of a two-dimensional vector space $\mathcal{V}$, from which we form the Grassmann algebra $\Lambda(\mathcal{V})$. $d$ and $\delta$ extend to $\Omega = \mathcal{A} \otimes \Lambda(\mathcal{V})$ in a canonical way, and to matrices with entries in $\Omega$. The equation (A.1) is then equivalent to the matrix potential $Bsq_\mathcal{V}$ equation (1.1). Choosing a solution $\phi_0$ and setting

$$\Delta = \Gamma = -\partial_x, \quad \lambda = -\frac{1}{3} \zeta_1 \zeta_2, \quad \kappa = -\frac{1}{3} \zeta_1 \zeta_2,$$

the linear system (A.2) and the adjoint linear system (A.3) lead to (2.1) and (2.2), respectively. Furthermore, (A.4) implies (2.3). According to the theorem, (A.5) yields a new solution of the matrix potential $Bsq_\mathcal{V}$ equation (1.1).

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