The Gaps of Consecutive Eigenvalues of Laplacian on Riemannian Manifolds

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Abstract
In this paper, we investigate the Dirichlet problem of Laplacian on complete Riemannian manifolds. By constructing new trial functions, we obtain a sharp upper bound of the gap of the consecutive eigenvalues in the sense of the order, which affirmatively answers to a conjecture proposed by Chen-Zheng-Yang. In addition, we also exploit the closed eigenvalue problem of Laplacian and obtain a similar optimal upper bound. As some important examples, we investigate the eigenvalues of the eigenvalue problem of the Laplacian on the unit sphere and cylinder, compact homogeneous Riemannian manifolds without boundary, connected bounded domain and compact complex hypersurface without boundary in the standard complex projective space $CP^n(4)$ with holomorphic sectional curvature 4, and some intrinsic estimates for the eigenvalue gap is obtained. As the author know, for the Dirichlet problem, the gap $\lambda_{k+1} - \lambda_k$ is bounded by the first $k$-th eigenvalues in the previous literatures. However, by a large number of numerical calculations, the author surprisingly find that the gap of the consecutive eigenvalues of the Dirichlet problem on the $n$-dimensional Euclidean space $R^n$ can be bounded only by the first two eigenvalues. Therefore, we venture to conjecture that all of the eigenvalues satisfy: $\lambda_{k+1} - \lambda_k \leq \Xi_i(\Omega)(\lambda_2 - \lambda_1)k^{1/n}$, where $\Xi_i(\Omega), i = 1, 2$ denote the first shape coefficient and the second shape coefficient. In particular, if we consider the second shape coefficient, then there is a close connection between this universal inequality and the famous Panye-Pólya-Weinberger conjecture in general form. By calculating some important examples, we adduce some good evidence on the correctness of this conjecture.

Keywords: sharp gap; Laplacian; consecutive eigenvalue; Riemannian manifold.

2000 MSC 35P15, 53C40.

1 Introduction

Let $\Omega$ be a bounded domain with piecewise smooth boundary $\partial \Omega$ in an $n$-dimensional complete Riemannian manifold $M^n$ and $\Delta$ be the Laplacian on $M^n$. We consider the following Dirichlet problem:
\begin{equation}
\begin{aligned}
\Delta u = -\lambda u, \quad \text{in} \quad \Omega, \\
u = 0, \quad \text{on} \quad \partial\Omega,
\end{aligned}
\end{equation}

which is also called a membrane problem (1.1). This eigenvalue problem has nontrivial solutions \( u \) only for a discrete set of eigenvalues \( \{\lambda_k\}_{k \in \mathbb{Z}^+} \). The corresponding eigenfunctions \( \{u_k\}_{k \in \mathbb{Z}^+} \) provide an orthonormal basis of \( L^2(\Omega) \). We may enumerate the eigenvalues in increasing order as follows:

\[ 0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_k \leq \cdots \to +\infty, \]

where each eigenvalue is repeated according to its multiplicity. When \( M^n \) is an \( n \)-dimensional Euclidean space \( \mathbb{R}^n \), Payne, Pólya and Weinberger [27] and [28] exploited the eigenvalues of the Dirichlet problem (1.1) and obtained the following universal inequality:

\[ \lambda_{k+1} - \lambda_k \leq \frac{4}{nk} \sum_{i=1}^{k} \lambda_i. \]  

(1.2)

In fact, the universal inequality of Payne, Pólya and Weinberger is extended by many mathematicians in some differential backgrounds. However, there are two main contributions due to Hile and Protter [22] and Yang [36]. In 1980, Hile and Protter [22] obtained the following universal inequality:

\[ \frac{1}{n} \sum_{i=1}^{k} \frac{\lambda_i}{\lambda_{k+1} - \lambda_i} \geq \frac{n k}{4}, \]  

(1.3)

which is sharper than (1.2). Furthermore, Yang [36] (cf. [15]) obtained a very sharp universal inequality of eigenvalues:

\[ \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \lambda_i. \]  

(1.4)

From the inequality (1.4), one can yield

\[ \lambda_{k+1} \leq \frac{1}{k} \left( 1 + \frac{4}{n} \right) \sum_{i=1}^{k} \lambda_i. \]  

(1.5)

The inequalities (1.4) and (1.5) are called by Ashbaugh Yang’s first inequality and second inequality, respectively (cf. [8], [9]). Also we note that Ashbaugh and Benguria gave an optimal estimate for \( k = 1 \), see [3–5]. By utilizing the Chebyshev’s inequality, it is not difficult to prove the following inequalities (cf. [9]):

\[ (1.4) \Rightarrow (1.5) \Rightarrow (1.3) \Rightarrow (1.2). \]

In 2007, Cheng and Yang established a celebrated recursion formula [17]. By making use of this recursion formula, they obtained a sharp upper bound of the \((k+1)\)-th eigenvalue, this is, they proved the following:
\[ \lambda_{k+1} \leq C_0(n,k)k^{\frac{2}{n}}\lambda_1, \quad \text{(1.6)} \]

where \( C_0(n,k) \leq 1 + \frac{4}{n} \) is a constant depending on \( n \) and \( k \) (see Cheng and Yang’s paper [17]). From the Weyl’s asymptotic formula (cf. [11, 34, 35]):

\[ \lambda_k \sim \frac{4\pi^2}{(\omega_n V(\Omega))^\frac{2}{n}}k^\frac{2}{n} \quad \text{as} \quad k \to +\infty, \quad \text{(1.7)} \]

where \( \omega_n \) and \( V(\Omega) \) denote the volumes of the \( n \)-dimensional unit ball \( \mathbb{B}^n(1) \subset \mathbb{R}^n \) and the bounded domain \( \Omega \), we know that the upper bound (1.6) of Cheng and Yang is best possible in the meaning of the order on \( k \). If \( M^n \) is a complete Riemannian manifold isometrically immersed in a Euclidean space \( \mathbb{R}^{n+p} \), Chen-Cheng [13] derived an extrinsic estimates as follows:

\[
\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4}{n} \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \left( \lambda_i + \frac{n^2||H||^2}{4} \right). \quad \text{(1.8)}
\]

\( H \) is the mean curvature vector field of \( M^n \) with \( ||H||^2 = \sup_{\Omega} |H|^2 \). Further, by using the Cheng-Yang’s recursive formula in [15], they also obtained an upper bound estimates, this is,

\[
\left( \lambda_{k+1} + \frac{n^2||H||^2}{4} \right) \leq C_0(n,k) \left( \lambda_1 + \frac{n^2||H||^2}{4} \right). \quad \text{(1.9)}
\]

Suppose that \( \Omega \) is a bounded connected domain in a unit sphere \( \mathbb{S}^n(1) \), Cheng and Yang [15] obtained an upper bound estimate for the gap of the consecutive eigenvalues of the eigenvalue problem (1.1):

\[
\lambda_{k+1} - \lambda_k \leq 2 \left[ \left( \frac{2}{n} \sum_{i=1}^{k} \lambda_i + \frac{n}{2} \right)^2 - \left( 1 + \frac{4}{n} \right) \frac{1}{k} \sum_{j=1}^{k} \left( \lambda_j - \frac{1}{k} \sum_{i=1}^{k} \lambda_i \right) \right]^{\frac{1}{2}}. \quad \text{(1.10)}
\]

In [16], Cheng and Yang investigated the Dirichlet problem (1.1) of the Laplacian on a connected bounded domain of the standard complex projective space \( \mathbb{C}P^n(4) \) with holomorphic sectional curvature 4. They proved the following universal inequality

\[
\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \leq 2 \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) (\lambda_i + 2n(n+1)). \quad \text{(1.10)}
\]

By (1.10), Cheng and Yang gave an explicit estimate of the \( k + 1 \)-th eigenvalue of Laplacian on such objects by its first \( k \) eigenvalues and proved the following inequality (cf. [16]):

\[
\lambda_{k+1} - \lambda_k \leq 2 \left[ \left( \frac{1}{n} \sum_{i=1}^{k} \lambda_i + 2n(n+1) \right)^2 - \left( 1 + \frac{2}{n} \right) \frac{1}{k} \sum_{j=1}^{k} \left( \lambda_j - \frac{1}{k} \sum_{i=1}^{k} \lambda_i \right)^2 \right]^{\frac{1}{2}}. \quad \text{(1.11)}
\]
Let $\Omega$ be a bounded domain on an $n$-dimensional Euclidean space $\mathbb{R}^n$, Chen, Zheng and Yang \cite{14} recently established a gap of consecutive eigenvalues of the eigenvalue problem (1.1),

$$\lambda_{k+1} - \lambda_k \leq C_{n,\Omega} k^\frac{1}{n},$$

(1.12)

where

$$C_{n,\Omega} = 4\lambda_1 \sqrt{\frac{C_0(n)}{n}},$$

and the constant $C_0(n)$ is the same as the one in (1.6). By a direct calculation and using the Weyl’s asymptotic formula, we know the order of the upper bound of the gap of the consecutive eigenvalues of $\mathbb{S}^n$ with standard metric is $k^\frac{1}{n}$. Therefore, for general Dirichlet problem of the Laplacian on Riemannian manifolds, Chen, Zheng and Yang presented a conjecture as follows:

**Conjecture 1.1.** (cf. \cite{14}) Let $(M^n, g)$ be an $n$-dimensional Riemannian manifold, and $\lambda_i$ be the $i$-th ($i = 1, 2, \cdots, k$) eigenvalue of the eigenvalue problem (1.1). Then we have

$$\lambda_{k+1} - \lambda_k \leq C_{n,\Omega} k^\frac{1}{n},$$

(1.13)

where $C_{n,\Omega}$ is a constant dependent on $\Omega$ itself and the dimension $n$.

Furthermore, as an excellent example to support conjecture 1.1 Chen, Zheng and Yang also investigated the eigenvalues of Laplacian on hyperbolic space. They proved the eigenvalue inequality (1.13) also holds for some hyperbolic space with some curvature conditions.

In addition, the famous fundamental gap conjecture states that, for the Dirichlet eigenvalue problem of the Schrödinger operator,

$$\begin{cases}
\Delta u - V(x)u = -\lambda u, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega,
\end{cases}$$

(1.14)

where $V(x)$ is a convex potential, Then the eigenvalues of (1.14) satisfy $\lambda_2 - \lambda_1 \geq \frac{3\pi^2}{D^2}$. Many mathematicians have contributed much to this conjecture (cf. \cite{12,32,37,38} and the references therein), and it was finally solved by B. Andrews and J. Clutterbuck in \cite{1}.

In this paper, we exploit the Dirichlet problem (1.1) of the Laplacian on the complete Riemannian manifolds. Suppose that $M^n$ is an $n$-dimensional complete Riemannian manifolds isometrically immersed in a Euclidean space. For this case, we answer to the conjecture 1.1. This is to say, we prove the following:

**Theorem 1.2.** Let $(M^n, g)$ be an $n$-dimensional complete Riemannian manifolds isometrically immersed in a Euclidean space $\mathbb{R}^{n+p}$, and $\lambda_i$ be the $i$-th ($i = 1, 2, \cdots, k$) eigenvalue of the Dirichlet problem (1.1). Then we have

$$\lambda_{k+1} - \lambda_k \leq C_{n,\Omega} k^\frac{1}{n},$$

(1.15)

where $C_{n,\Omega}$ is a constant dependent on $\Omega$ itself and the dimension $n$. 
Next, assume that $M^n$ is an $n$-dimensional closed Riemannian manifold. We also consider the closed eigenvalue problem of Laplacian:

$$\Delta u = -\lambda u, \quad \text{in} \quad M^n. \quad (1.16)$$

It is well known that the spectrum of the closed eigenvalue problem (1.16) is discrete and satisfies the following:

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots \to +\infty,$$

where $\lambda_k$ is the $k$-th eigenvalue of the closed eigenvalue problem (1.16) and each eigenvalue is repeated according to its multiplicity.

When $M^n$ is an $n$-dimensional compact homogeneous Riemannian manifold, for the closed eigenvalue problem (1.16), Li [24] proved

$$\lambda_k^{k+1} - \lambda_k - \frac{2}{k+1} \left( \sum_{i=1}^k \lambda_i \right)^2 + (k+1) \sum_{i=1}^k \lambda_i \lambda_1 + \sum_{i=1}^k \lambda_i \lambda_1 \right), \quad (1.17)$$

When $M^n$ is an $n$-dimensional compact minimal submanifold in a unit sphere $S^N(1)$, P. C. Yang and Yau [37] proved the eigenvalues of the closed eigenvalue problem satisfy the following eigenvalue inequality:

$$\lambda_k^{k+1} - \lambda_k \leq n + \frac{2}{n(k+1)} \left( \sum_{i=1}^k \lambda_i \right)^2 + n^2(k+1) \sum_{i=1}^k \lambda_i \lambda_1 + \sum_{i=1}^k \lambda_i \lambda_1 \right), \quad (1.17)$$

Furthermore, Harrel II and Michel (see [19] and [20]) obtained an abstract inequality of algebraic version. By applying the algebraic inequality, they proved that, if $M^n$ is an $n$-dimensional compact minimal submanifold in a unit sphere $S^N(1)$, then

$$\lambda_k^{k+1} - \lambda_k \leq n + \frac{4}{n(k+1)} \sum_{i=1}^k \lambda_i, \quad (1.17)$$

and if $M^n$ is an $n$-dimensional compact homogeneous Riemannian manifold, then

$$\lambda_k^{k+1} - \lambda_k \leq \frac{4}{k+1} \sum_{i=1}^k \lambda_i + \lambda_1, \quad (1.18)$$

Therefore, the above inequalities (1.17) and (1.18) made significant improvement to earlier estimates of differences of consecutive eigenvalues of Laplacian introduced by P. C. Yang and Yau [37], Leung [23], Li [24] and Harrel II [19]. Cheng and Yang [15] also considered the same eigenvalue problem and proved that, when $M^n$ is an $n$-dimensional compact homogeneous Riemannian manifold without boundary, then the eigenvalues of the close eigenvalue problem (1.16) satisfy
\[ \bar{\lambda}_{k+1} - \bar{\lambda}_k \leq \left[ \frac{4}{k+1} \sum_{i=1}^{k} \bar{\lambda}_i + \frac{n}{2} \right] - \frac{20}{k+1} \sum_{i=0}^{k} \left( \bar{\lambda}_i - \frac{1}{k+1} \sum_{j=1}^{k} \bar{\lambda}_j \right)^2; \]

and when \( M^n \) is an \( n \)-dimensional compact minimal submanifold without boundary in a unit sphere \( S^n(1) \), then the eigenvalues of the close eigenvalue problem (1.16) satisfy

\[ \bar{\lambda}_{k+1} - \bar{\lambda}_k \leq 2 \left[ \frac{2}{nk} \sum_{i=0}^{k} \bar{\lambda}_i + \frac{n}{2} \right]^2 - \frac{1}{k+1} \sum_{i=0}^{k} \left( \bar{\lambda}_i - \frac{1}{k+1} \sum_{j=0}^{k} \bar{\lambda}_j \right)^2. \]

In [16], Cheng and Yang investigated the closed eigenvalue problem (1.16) of the Laplacian on a compact complex hypersurface without boundary in the standard complex projective space \( \mathbb{C}P^n(4) \) with holomorphic sectional curvature 4. They proved the following universal inequality

\[ \sum_{i=0}^{k} \left( \bar{\lambda}_{k+1} - \bar{\lambda}_i \right)^2 \leq \frac{2}{nk} \sum_{i=0}^{k} \left( \bar{\lambda}_{k+1} - \bar{\lambda}_i \right) \left( \bar{\lambda}_i + 2n(n+1) \right). \] (1.19)

By (1.19), Cheng and Yang gave an explicit estimate of the \( k + 1 \)-th eigenvalue of Laplacian on such objects by its first \( k \) eigenvalues and proved the following inequality (cf. [16]):

\[ \bar{\lambda}_{k+1} - \bar{\lambda}_k \leq 2 \left[ \frac{1}{nk+1} \sum_{i=1}^{k} \bar{\lambda}_i + 2(n+1) \right]^2 - \frac{1}{k+1} \sum_{i=0}^{k} \left( \bar{\lambda}_i - \frac{1}{k+1} \sum_{j=0}^{k} \bar{\lambda}_j \right)^2. \] (1.20)

In this paper, we investigate the eigenvalues of the closed eigenvalue problem (1.16) of the Laplacian on the closed Riemannian manifolds and prove the following:

**Theorem 1.3.** Let \((M^n, g)\) be an \( n \)-dimensional closed Riemannian manifold, which is isometrically immersed into \((n+p)\)-dimensional Euclidean space \( \mathbb{R}^{n+p} \), and \( \bar{\lambda}_i \) be the \( i \)-th \((i = 0, 1, 2, \cdots)\) eigenvalue of the closed eigenvalue problem (1.16). Then, for any \( k \geq 1 \), we have

\[ \bar{\lambda}_{k+1} - \bar{\lambda}_k \leq \bar{C}_{n,M^n} k^\epsilon, \] (1.21)

where \( \bar{C}_{n,M^n} \) is a constant dependent on \( M^n \) itself and the dimension \( n \).

This paper is organized as follows. In section 2 we prove several key lemmas and establish several general formulas of the eigenvalues of the Dirichlet problem. In addition, by the same method, we also yield the corresponding general formulas with respect to the closed eigenvalue problem. By utilizing those general formulas, we give the proofs of theorems 1.2 and theorem 1.3 in section 3. We exploit the eigenvalue of the Dirichlet problem on the unit sphere and cylinder in section 4. In section 5 we investigate the eigenvalues of the eigenvalue problem of the Laplacian on a connected bounded domain and on a compact complex hypersurface without boundary in the standard complex projective
space $\mathbb{C}P^n(4)$ with holomorphic sectional curvature 4. In addition, we consider the eigenvalues of the closed eigenvalue problem of Laplacian on the compact Riemannian manifolds without boundary in section 6. In the last section, we give some gap conjectures of consecutive eigenvalues of the Dirichlet problem on complete Riemannian manifolds. As a further interest, we provide some important examples to support those conjectures proposed in this section.

2 Some Technical Lemmas

In order to give the proofs of theorem 1.2 and theorem 1.3 we would like to prove some key lemmas in this section. At first, we recall the following algebraic inequality which is proved by Chen, Zheng and Yang in [14]. By applying this algebraic inequality, Chen, Zheng and Yang established the following general formula (see lemma 2.1 in [14]).

**Lemma 2.1.** Let $(M^n, g)$ be an $n$-dimensional complete Riemannian manifold and $\Omega$ a bounded domain with piecewise smooth boundary $\partial \Omega$ on $M^n$. Assume that $\lambda_i$ is the $i$th eigenvalue of the Dirichlet problem (1.1) and $u_i$ is an orthonormal eigenfunction corresponding to $\lambda_i$, $i = 1, 2, \ldots$, such that

$$\begin{cases} \Delta u_i = -\lambda_i u_i, & \text{in } \Omega, \\ u_i = 0, & \text{on } \partial \Omega, \\ \int_{\Omega} u_i u_j dv = \delta_{ij}, & \text{for any } i, j = 1, 2, \ldots. \end{cases}$$

Then, for any function $h(x) \in C^3(\Omega) \cap C^2(\overline{\Omega})$ and any integer $k, i \in \mathbb{Z}^+$, $(k > i \geq 1)$, eigenvalues of the Dirichlet problem (1.1) satisfy

$$((\lambda_{k+2} - \lambda_i) + (\lambda_{k+1} - \lambda_i))\|\nabla h u_i\|^2 \leq 2 \sqrt{(\lambda_{k+2} - \lambda_i)(\lambda_{k+1} - \lambda_i)}\|\nabla h\|^2 \|u_i\|^2 + \|2\langle \nabla h, \nabla u_i \rangle + u_i \Delta h\|^2, \quad (2.1)$$

where

$$\|h(x)\|^2 = \int_{\Omega} h^2(x) dv.$$

For the closed eigenvalue problem, we can also prove the following by the same method given by:

**Lemma 2.2.** Let $(M^n, g)$ be an $n$-dimensional closed Riemannian manifold. Assume that $\lambda_i$ is the $i$th eigenvalue of the eigenvalue problem (1.16) and $u_i$ is an orthonormal eigenfunction corresponding to $\lambda_i$, $i = 0, 1, 2, \ldots$, such that

$$\begin{cases} \Delta u_i = -\lambda_i u_i, & \text{in } \Omega, \\ \int_{\Omega} u_i u_j dv = \delta_{ij}, & \text{for any } i, j = 0, 1, 2, \ldots. \end{cases}$$
Then, for any function $h(x) \in C^3(\Omega) \cap C^2(\overline{\Omega})$ and any integer $k, i \in \mathbb{Z}$, $(k > i \geq 0)$, eigenvalues of the closed eigenvalue problem (1.16) satisfy

$$((\lambda_{k+2} - \lambda_i) + (\lambda_{k+1} - \lambda_i))\|\nabla h u_i\|^2 \leq 2 \sqrt{(\lambda_{k+2} - \lambda_i)(\lambda_{k+1} - \lambda_i)}\|\nabla h\|^2 u_i + \|2\nabla h, \nabla u_i\| + u_i \Delta h|^2,$$

where

$$\|h(x)\|^2 = \int_\Omega h^2(x)dv.$$

**Proof.** Recall that the proof of lemma 2.1 given by Chen-Zheng-Yang in [14] is very fascinating and the key strategy is to apply the Rayleigh-Ritz inequality and Lagrange method of multipliers in real Banach spaces. By the same strategy as the one in [14], it is not difficult to give the proof of this lemma if one notices to count the number of eigenvalues from 0. Here, we omit it. □

By applying lemma 2.1, we have

**Lemma 2.3.** Let $\rho$ be a constant such that, for any $i = 1, 2, \ldots, k, \lambda_i + \rho > 0$. Under the assumption of the lemma 2.1 for any $j = 1, 2, \ldots, l$, and any real value function $h_j \in C^3(\Omega) \cap C^2(\overline{\Omega})$, we have

$$\sum_{j=1}^l a_j^2 + b_j (\lambda_{k+2} - \lambda_{k+1})^2 \leq 4(\lambda_{k+2} + \rho) \sum_{j=1}^l \|2\nabla h_j, \nabla u_i\| + u_i \Delta h_j|^2,$$

where

$$a_j = \sqrt{\|\nabla h_j u_i\|^2},$$

$$b_j = \sqrt{\|\nabla h_j u_i\|^2},$$

$$a_j^2 \geq b_j,$$

and

$$\|h(x)\|^2 = \int_\Omega h^2(x)dv.$$

**Proof.** By the assumption in this lemma, we have

$$\frac{a_j^2 - b_j}{2} (\sqrt{\lambda_{k+2} - \lambda_i} + \sqrt{\lambda_{k+1} - \lambda_i})^2 \geq 0,$$

which is equivalent to the following:

$$a_j^2 (\lambda_{k+2} - \lambda_i) + (\lambda_{k+1} - \lambda_i) - 2b_j \sqrt{(\lambda_{k+2} - \lambda_i)(\lambda_{k+1} - \lambda_i)}$$

$$\geq a_j^2 + b_j (\sqrt{\lambda_{k+2} - \lambda_i} - \sqrt{\lambda_{k+1} - \lambda_i})^2.$$
By (2.5) and (2.2), we have
\[
a_j^2 + b_j \frac{\lambda_{k+2} - \lambda_i - \sqrt{\lambda_{k+1} - \lambda_i}}{2} \leq 2\langle \nabla h_j, \nabla u_i \rangle + u_i \Delta h_j \| \|^2.
\]
Taking sum over \( j \) from 1 to \( l \), we yield
\[
\sum_{j=1}^{l} a_j^2 + b_j \frac{\lambda_{k+2} - \lambda_i - \sqrt{\lambda_{k+1} - \lambda_i}}{2} \leq \sum_{j=1}^{l} 2\langle \nabla h_j, \nabla u_i \rangle + u_i \Delta h_j \|^2.
\]
(2.6)

Multiplying (2.6) by \( \left( \sqrt{\lambda_{k+2} - \lambda_i - \sqrt{\lambda_{k+1} - \lambda_i}} \right)^2 \) on both sides, one can infer that
\[
\sum_{j=1}^{l} a_j^2 + b_j \frac{\lambda_{k+2} - \lambda_i - \sqrt{\lambda_{k+1} - \lambda_i}}{2} \leq \sum_{j=1}^{l} 2\langle \nabla h_j, \nabla u_i \rangle + u_i \Delta h_j \|^2\left( \sqrt{\lambda_{k+2} - \lambda_i + \sqrt{\lambda_{k+1} - \lambda_i}} \right)^2
\]
\[
= \sum_{j=1}^{l} 2\langle \nabla h_j, \nabla u_i \rangle + u_i \Delta h_j \|^2
\]
\[
\times \left( \sqrt{\lambda_{k+2} + \rho} - \lambda_i + \sqrt{\lambda_{k+1} + \rho} - \lambda_i + \rho \right)^2
\]
\[
\leq 4(\lambda_{k+2} + \rho) \sum_{j=1}^{l} 2\langle \nabla h_j, \nabla u_i \rangle + u_i \Delta h_j \|^2.
\]
which is the inequality (2.3). Therefore, we finish the proof of this lemma.

□

**Remark 2.4.** Recall that, under the assumption that \( \| \nabla h \| = 1 \), by utilizing (2.2), Chen, Zheng and Yang [14] obtained
\[
(\lambda_{k+2} - \lambda_{k+1})^2 \leq 4\lambda_{k+2} \sum_{j=1}^{l} 2\langle \nabla h_j, \nabla u_i \rangle + u_i \Delta h_j \|^2,
\]
(2.7)
which plays a significant role in estimating the gap of \( \lambda_{k+1} - \lambda_k \). If there is no this assumption, it will encounter great difficulties of computing or estimating the detailed value of the term \( \| \nabla h \|^2 u_i \) and thus to obtain (2.7) even if \( h \) is a standard coordinate function on Euclidean space. However, we notice that the assumption that \( | \nabla h | = 1 \) can be replaced by the assumption that the trial function \( h \) satisfies the following condition:
\[
\| \nabla h u_i \|^2 \geq \sqrt{\| \nabla h \|^2 u_i \|^2}.
\]
Under the assumption, we can obtain the inequality (2.3), which plays a significant role in estimating the gap of eigenvalues of Laplacian on general Riemannian manifolds.

By the same method as the proof of lemma 2.3, we can prove the following lemma if one notices to count the number of eigenvalues from 0.
Lemma 2.5. Let $\rho$ be a constant such that, for any $i = 0, 1, 2, \cdots, k$, $\overline{\lambda}_i + \rho > 0$. Under the assumption of the lemma 2.2 for any $j = 0, 1, 2, \cdots, l$, and any real value function $h_j \in C^2(M^n)$, we have

$$
\sum_{j=1}^{l} \frac{a_j^2 + b_j}{2} \left(\overline{\lambda}_{k+2} - \overline{\lambda}_{k+1}\right)^2 \leq 4(\overline{\lambda}_{k+2} + \rho) \sum_{j=1}^{l} \|2\langle \nabla h_j, \nabla u_i \rangle + u_i \Delta h_j\|^2,
$$

(2.8)

where

$$
a_j = \sqrt{\|\nabla h_j u_i\|^2},
$$

$$
b_j = \sqrt{\|\nabla h_j^2 u_i\|^2},
$$

$$
a_j^2 \geq b_j,
$$

(2.9)

and

$$
\|h(x)\| = \int_{\Omega} h(x) dv.
$$

3 Proofs of theorem 1.2 and theorem 1.3

In this section, we would like to give the proofs of theorem 1.2 and theorem 1.3. Firstly, we need the following lemma which can be found in [13].

Lemma 3.1. For an $n$-dimensional submanifold $M^n$ in Euclidean space $\mathbb{R}^{n+p}$, let $y = (y^1, y^2, \cdots, y^{n+p})$ is the position vector of a point $p \in M^n$ with $y^\alpha = y^\alpha(x_1, \cdots, x_n)$, $1 \leq \alpha \leq n + p$, where $(x_1, \cdots, x_n)$ denotes a local coordinate system of $M^n$. Then, we have

$$
\sum_{\alpha=1}^{n+p} g(\nabla y^\alpha, \nabla y^\alpha) = n,
$$

$$
\sum_{\alpha=1}^{n+p} g(\nabla y^\alpha, \nabla u)g(\nabla y^\alpha, \nabla w) = g(\nabla u, \nabla w),
$$

for any functions $u, w \in C^1(M^n)$,

$$
\sum_{\alpha=1}^{n+p} (\Delta y^\alpha)^2 = n^2 H^2,
$$

$$
\sum_{\alpha=1}^{n+p} \Delta y^\alpha \nabla y^\alpha = 0,
$$

where $H$ is the mean curvature of $M^n$.
Proof of theorem 1.2 Let $a_1, a_2, \cdots, a_{n+p}$ are $(n+p)$ positive number. We define $(n+p)$ scarling coordinate functions $h_j(x) = \alpha_j x^j$, such that

$$a_j^2 = \|\nabla h_j u_i\|^2 \geq \sqrt{\|\nabla h_j u_i\|^2} = b_j \geq 0,$$

and

$$\sum_{j=1}^{n+p} \int_{\Omega} 2u_i \langle \nabla h_j, \nabla u_i \rangle \Delta h_j dv = 0,$$

where $j = 1, 2, \cdots, n+p$, and $x^j$ denotes the $j$-th standard coordinate function of the Euclidean space $\mathbb{R}^{n+p}$. Let

$$\alpha = \min_{1 \leq j \leq n+p} \{\alpha_j\},$$

$$\overline{\alpha} = \max_{1 \leq j \leq n+p} \{\alpha_j\},$$

$$\beta = \min_{1 \leq j \leq n+p} \{b_j\},$$

and $l = n + p$, then, by lemma 2.1, we have

$$\sum_{j=1}^{l} \frac{a_j^2 + b_j}{2} = \sum_{j=1}^{n+p} \frac{a_j^2 + b_j}{2} \geq \frac{1}{2} \left( n\alpha^2 + \sum_{j=1}^{n+p} b_j \right) \geq \frac{1}{2} \left( n\alpha^2 + (n + p)\beta \right),$$

and

$$\sum_{j=1}^{n+p} (\Delta h_j)^2 \leq \overline{\alpha}^2 n^2 H^2,$$

and

$$\sum_{j=1}^{n+p} \int_{\Omega} \langle \nabla h_j, \nabla u_i \rangle^2 dv \leq \overline{\alpha}^2 \sum_{j=1}^{n+p} \int_{\Omega} \langle \nabla x^j, \nabla u_i \rangle^2 dv = \overline{\alpha}^2 \lambda_i.$$

Since eigenvalues are invariant under isometries, defining

$$c = \frac{1}{4} \inf_{\psi \in \Psi} \max_{\Omega} n^2 H^2 > 0,$$

where $\Psi$ denotes the set of all isometric immersions from $M^n$ into a Euclidean space, by lemma 3.1, (3.2), (3.4), and (3.5), we have
\[ 4(\lambda_{k+2} + c) \sum_{j=1}^{n+p} \langle \nabla h_j, \nabla u_i \rangle + u_i \Delta h_j \|^{2} \leq 4(\lambda_{k+2} + c)\alpha^2 \left( 4\lambda_i + \int_{\Omega} u_i^2 n^2 H^2 dv \right) \leq 16\lambda_{k+2}^2 (\lambda_i + c). \] (3.6)

Let \( i = 1, \rho = c \), then, substituting (3.3) and (3.6) into (2.3), we have

\[ (\lambda_{k+2} - \lambda_{k+1})^2 \leq \frac{32\alpha^2 (\lambda_{k+2} + c)}{n\alpha^2 + (n + p)\beta} (\lambda_1 + c), \] (3.7)

Therefore, we deduce from (3.7) that,

\[ \lambda_{k+2} - \lambda_{k+1} \leq \sqrt{\frac{32\alpha^2 C_0(n)}{n\alpha^2 + (n + p)\beta}} (k + 1)^{\frac{1}{2}} \]

\[ = C_{n,\Omega}(k + 1)^{\frac{1}{2}}, \]

where

\[ C_{n,\Omega} = (\lambda_1 + c) \sqrt{\frac{32\alpha^2 C_0(n)}{n\alpha^2 + (n + p)\beta}}. \]

Therefore, we complete the proof of theorem 1.2.

\(\Box\)

**Remark 3.2.** In the theorem 1.2 one can obtain an even stronger result. Indeed, in the proof of this theorem, there exist a positive integer \( 1 \leq j_0 \leq n + p \) such that we can choose \( n + p \) positive numbers \( \alpha_1, \alpha_2, \cdots, \alpha_{n+p} \) satisfy the following:

\[ a_j^2 = \| \nabla h_j u_i \|^2 = \sqrt{\| \nabla h_j \|^2 u_i} = b_j \geq 0, \text{ where } j = 1, 2, \cdots, j_0 - 1, j_0 + 1, \cdots, n + p, \]

\[ a_{j_0}^2 = \| \nabla h_{j_0} u_i \|^2 \leq \sqrt{\| \nabla h_{j_0} \|^2 u_i} = b_{j_0} \geq 0, \]

and

\[ \sum_{j=1}^{n+p} \int_{\Omega} 2u_i \langle \nabla h_j, \nabla u_i \rangle \Delta h_j dv = 0. \]

**Corollary 3.3.** Assume that \((M^n, g)\) is an \( n \)-dimensional complete Riemannian manifolds, which is isometrically immersed into \((n + \overline{p})\)-dimensional Euclidean space \( \mathbb{R}^{n+\overline{p}} \). Let \( \lambda_i \) be the \( i \)-th \((i = 1, 2, \cdots, k)\) eigenvalue of the Dirichlet problem (1.1). Then we have
\[ \lambda_{k+1} - \lambda_k \leq C_{n,\Omega} k^\frac{1}{2}, \quad (3.8) \]

where

\[ C_{n,\Omega} = (\lambda_1 + c) \sqrt{\frac{32\pi^2 C_0(n)}{n\alpha^2 + (n + p)\beta^2}}. \]

\[ C_0(n) \text{ is the same as the one in (1.6), and} \]

\[ c = \frac{1}{4} \inf_{\psi \in \Psi} \max_{\Omega} n^2 H^2 > 0, \]

where \( \Psi \) denotes the set of all isometric immersions from \( M^n \) into a Euclidean space \( \mathbb{R}^{n+p} \). Furthermore, assume that \( (M^n, g) \) is an \( n \)-dimensional complete minimal submanifold which is isometrically immersed into \( (n + p) \)-dimensional Euclidean space \( \mathbb{R}^{n+p} \), and then the constant \( c \) is given by \( c = 0 \).

**Remark 3.4.** We shall note that when \( p \) tends to infinity, it dose not mean that the constant \( C_{n,\Omega} \) will be asymptotic to zero. This is because, for any \( j = 1, 2, \cdots, n + p \), we have

\[ a_j^2 = \|\nabla h_j u_i\|^2 \geq \sqrt{\|\nabla^2 h_j u_i\|^2} = b_j \geq 0, \]

which implies that \( (n + 1)\rho \leq n\alpha^2 \).

**Remark 3.5.** Usually, we choose the standard coordinate functions to construct the trial functions to obtain the universal inequalities or the estimate for the bounds of eigenvalues. However, we do not choose standard coordinate functions but the scarling coordinate functions which satisfy some conditions to construct the trial functions in the proof of theorem \( 1.2 \).

From the proof of theorem \( 1.2 \) we have

**Remark 3.6.** If \( M^n \) is an \( n \)-dimensional Euclidean space, then we have \( H = 0 \), and thus \( c = 0 \). Let \( \alpha_j = 1 \), where \( j = 1, 2, \cdots, n + p \), then \( h_j = x^j \). Thus, we have

\[ \alpha = 1, \]

and

\[ \sum_{j=1}^{n+p} b_j = n, \]

which implies that

\[ C_{n,\Omega} = (\lambda_1 + c) \sqrt{\frac{32\pi^2 C_0(n)}{n\alpha^2 + \sum_{j=1}^{n+p} b_j}} = 4\lambda_1 \sqrt{\frac{C_0(n)}{n}}. \]
Therefore, the eigenvalue inequality (3.3) in theorem 1.2 generalize the eigenvalue inequality (1.12) given by Chen-Zheng-Yang in [14].

Proof of theorem 1.3. By lemma 2.2, lemma 2.5 and lemma 3.1, we can give the proof by using the same method as the proof of theorem 1.2.

□

Similarly, we have the following:

Corollary 3.7. Let \((M^n, g)\) be an n-dimensional closed Riemannian manifold, which is isometrically immersed into \((n + \overline{p})\)-dimensional Euclidean space \(\mathbb{R}^{n+\overline{p}}\), and \(\lambda_i\) be the i-th \((i = 0, 1, 2, \cdots, k)\) eigenvalue of the closed eigenvalue problem \((1.16)\). Then, for any \(k = 1, 2, \cdots\), there exist some constants \(\alpha'\) and \(b'_j\), \(j = 1, 2, \cdots, n + p\), such that

\[
\lambda_{k+1} - \lambda_k \leq C_{n,\Omega}k^{\frac{1}{4}},
\]

where

\[
C_{n,\Omega} = (\lambda_1 + c) \sqrt{\frac{32\alpha'^2C_0(n)}{n\alpha'^2 + \sum_{j=1}^{n+p}b'_j}},
\]

and \(C_0(n)\) is the same as the one in \((1.6)\), and

\[
\overline{c} = \frac{1}{4} \inf_{\Psi} \max_{M^n} n^2 H^2 > 0,
\]

where \(\Psi\) denotes the set of all isometric immersions from \(M^n\) into a Euclidean space \(\mathbb{R}^{n+\overline{p}}\). Furthermore, assume that \((M^n, g)\) is an n-dimensional closed minimal submanifold which is isometrically immersed into \((n + \overline{p})\)-dimensional Euclidean space \(\mathbb{R}^{n+\overline{p}}\), and then,

\[
\overline{c} = 0. \tag{3.9}
\]

4 Estimates for the Eigenvalues on the Unit Sphere and Cylinder

In this section, we investigate the eigenvalues on the n-dimensional unit sphere \(\mathbb{S}^n(1)\) and cylinder \(\mathbb{R}^{n-m} \times \mathbb{S}^m(1)\) with \(m < n\). However, when \(n = m\), we assume that \(\mathbb{R}^{n-m} \times \mathbb{S}^m(1)\) is exactly an n-dimensional unit sphere. Under those assumptions, we have

Theorem 4.1. Let \(M^n\) be an n-dimensional unit sphere \(\mathbb{S}^n(1)\) or cylinder \(\mathbb{R}^{n-m} \times \mathbb{S}^m(1)\), and \(\lambda_i\) be the i-th \((i = 1, 2, \cdots, k)\) eigenvalue of the eigenvalue problem \((1.1)\). Then, we have

\[
\lambda_{k+1} - \lambda_k \leq C_{n,\Omega}k^{\frac{1}{4}}, \tag{4.1}
\]
where

\[ C_{n,\Omega} = 4 \left( \frac{\lambda_1 + \frac{m^2 + mn}{8}}{n^2 + (n + 1)\gamma} \right) \sqrt{\frac{2C_0(n)}{n^\delta + (n + 1)\gamma}}. \]

**Proof.** We denote the position vector of the \(n\)-dimensional unit round cylinder \(\mathbb{R}^{n-m} \times S^m(1)\) in \((n + 1)\)-dimensional Euclidean space \(\mathbb{R}^{n+1}\) by

\[ x = (v, w) = (x^1, x^2, \ldots, x^{n-m}, x^{n-m+1}, x^{n-m+2}, \ldots, x^n, x^{n+1}), \]

where \(v = (x^1, x^2, \ldots, x^{n-m}), w = (x^{n-m+1}, x^{n-m+2}, \ldots, x^n, x^{n+1}).\) In particular, when \(n = m, x = w.\) Then, we obtain

\[ \sum_{j=n-m+1}^{n+1} (x^j)^2 = 1, \quad \sum_{j=1}^{n+1} |\nabla x^j|^2 = n. \] (4.2)

It is not difficult to see that, when \(n > m,\)

\[ \Delta x^j = \begin{cases} 0, & \text{if } j = 1, \ldots, n-m, \\ -mx^j, & \text{if } j = n-m+1, \ldots, n+1; \end{cases} \] (4.3)

and when \(n = m,\)

\[ \Delta x^j = -nx^j, \text{if } j = 1, \ldots, n+1. \] (4.4)

For any \(j (j = 1, 2, \ldots, n+1),\) let \(l = n + 1\) and \(h_j(x) = \delta_j x^j\) and \(\delta_j > 0,\) such that

\[ \sum_{j=1}^{n+1} \int 2u_i \langle \nabla (\delta_j x^j), \nabla u_i \rangle \Delta (\delta_j x^j) dv = 0, \] (4.5)

and

\[ a_j^2 = ||\nabla h_j u||^2 \geq \sqrt{||\nabla h_j^2 u_i||^2} = b_j \geq 0, \]

Let

\[ \delta = \min_{1 \leq j \leq n+p} \{\delta_j\}, \]

\[ \overline{\delta} = \max_{1 \leq j \leq n+p} \{\delta_j\}, \]

\[ \gamma = \min_{1 \leq j \leq n+p} \sup_{\Omega} \{b_j\}. \]

Then, we have
\[
\sum_{j=1}^{l} \frac{a_j^2 + b_j}{2} = \sum_{j=1}^{n+1} \frac{\sqrt{\|\nabla (\delta_j x') u_i\|^2} + \sqrt{\|\nabla (\delta_j x')^2 u_i\|^2}}{2} \\
\geq \frac{1}{2} \left( n\delta + \sum_{j=1}^{n+1} b_j \right) \\
\geq \frac{1}{2} (n\delta + (n+1)\gamma). 
\] (4.6)

For any fixed point \(x_0 \in \Omega\), we can find a coordinate system \((\bar{x}^1, \bar{x}^2, \cdots, \bar{x}^{n+1})\) of the \(n\)-dimensional unit round cylinder \(\mathbb{R}^{n-m} \times S^m(1)\) such that at the point \(x_0\)
\[
\bar{x}^1 = \cdots = \bar{x}^n = 0, \quad \bar{x}^{n+1} = 1, \\
\nabla \bar{x}^{n+1} = 0; \quad \nabla_p x^q = \delta^q_p (p, q = 1, 2, \cdots, n+1). 
\] (4.7)

In fact, we can choose a constant \((n+1) \times (n+1)\) type orthonormal matrix \(a_i^j\) satisfying
\[
\sum_{\alpha=1}^{n+1} a_p^\alpha a_q^\alpha = \delta_{pq},
\]
such that
\[
x^p = \sum_{\alpha=1}^{n+1} a_p^\alpha \bar{x}^\alpha
\]
and (4.7) is satisfied at the point \(x_0\). By a direct computation, at the point \(x_0\), we yield
\[
\sum_{p=1}^{n+1} \langle \nabla x^p, \nabla u_i \rangle^2 = |\nabla u_i|^2.
\]
Since \(x_0\) is an arbitrary point, we know that for any point \(x \in \Omega\),
\[
\sum_{p=1}^{n+1} \langle \nabla x^p, \nabla u_i \rangle^2 = |\nabla u_i|^2.
\]
On the other hand, by using (4.2), we have
\[
\sum_{p=n-m+1}^{n+1} \nabla (x^p)^2 = 0, 
\] (4.8)
and
\[
\sum_{p=n-m+1}^{n+1} |\nabla x^p|^2 = - \sum_{p=1}^{n+1} x^p \Delta x^p = m. 
\] (4.9)
Let

$$\Psi = \sum_{j=1}^{l} ||2\langle \nabla h_j, \nabla u_i \rangle + u_i \Delta h_j ||^2 = \sum_{j=1}^{n+1} ||2\langle \nabla (\delta_j x^j), \nabla u_i \rangle + u_i \Delta (\delta_j x^j) ||^2$$  \hspace{1cm} (4.10)$$

Then, by making use of (4.3), (4.5), (4.8), (4.9) and (4.10), we deduce

$$\Psi = \sum_{j=1}^{n+1} ||2\langle \nabla (\delta_j x^j), \nabla u_i \rangle + u_i \Delta (\delta_j x^j) ||^2$$

$$= 4 \sum_{j=1}^{n+1} \int_{\Omega} \langle \nabla (\delta_j x^j), \nabla u_i \rangle^2 d\mu + m^2 \sum_{j=n-m+1}^{n+1} \int_{\Omega} u_i^2 (\delta_j x^j)^2 d\mu,$$  \hspace{1cm} (4.11)

Furthermore, by (4.11), we have

$$\Psi \leq 4\delta^2 \sum_{j=1}^{n+1} \int_{\Omega} \langle x^j, \nabla u_i \rangle^2 d\mu + m^2 \delta^2 \sum_{j=n-m+1}^{n+1} \int_{\Omega} u_i^2 (x^j)^2 d\mu = 4\delta^2 \lambda_i + m^2 \delta^2.$$  \hspace{1cm} (4.12)

Let

$$c = \rho = \frac{m^2}{4}.$$  \hspace{1cm} Then, we deduce from (2.3), (4.6) and (4.12) that,

$$\frac{1}{2} (n\delta + (n+1)\gamma) (\lambda_{k+2} - \lambda_{k+1})^2 \leq 4(\lambda_{k+2} + c) \left(4\delta \lambda_i + m^2 \delta^2\right).$$  \hspace{1cm} (4.13)

Therefore, by utilizing (1.9) and (4.13), we yield

$$\lambda_{k+2} - \lambda_{k+1} \leq 4\delta \sqrt{\frac{2}{n\delta + (n+1)\gamma}} \sqrt{\lambda_1 + c} \sqrt{\lambda_{k+2} + c}$$

$$\leq 4\delta (\lambda_1 + c) \sqrt{\frac{2C_0(n)}{n\delta + (n+1)\gamma}} (k+1)^{\frac{\gamma}{2}}$$

$$= C_{n,\Omega} (k+1)^{\frac{\gamma}{2}},$$

where

$$C_{n,\Omega} = 4\delta (\lambda_1 + c) \sqrt{\frac{2C_0(n)}{n\delta + (n+1)\gamma}}.$$  \hspace{1cm} (4.13)

Therefore, we complete the proof of this theorem. \hspace{1cm} □

**Remark 4.2.** It is easy to see that inequality (4.1) is also an intrinsic inequality. In particular, when \(n = m\), inequality (4.1) gives an intrinsic estimates for the gap of the consecutive eigenvalues on the sphere space form \(\mathbb{S}^n(1)\) with unit radius.
5 Eigenvalues on Complex Projective Spaces

In this section, we investigate the eigenvalues of the eigenvalue problem of the Laplacian on a connected bounded domain and on a compact complex hypersurface without boundary in the standard complex projective space $\mathbb{C}P^n(4)$ with holomorphic sectional curvature $4$. We shall give an explicit gap estimate of the consecutive eigenvalues $\lambda_{k+1} - \lambda_k$. Firstly, we prove

**Theorem 5.1.** Let $\Omega$ a connected bounded domain in the standard complex projective space $\mathbb{C}P^n(4)$ with holomorphic sectional curvature $4$, and $\lambda_i$ be the $i$-th ($i = 1, 2, \cdots, k$) eigenvalue of the eigenvalue problem (1.1). Then, we have

$$\lambda_{k+2} - \lambda_{k+1} \leq C(n, \Omega)k^{\frac{1}{2n}},$$

where

$$C(n, \Omega) = 4 \sqrt{\frac{4 \left( \sum_{s=1}^{n} \frac{\theta_s^2}{\theta_{n+1}^2} \right)^2 + \sum_{s=1}^{n} \frac{\theta_s^4}{\theta_{n+1}^2} + \frac{2\theta_1^2}{\theta_{n+1}^2}}{\sum_{s=1}^{n} \frac{\theta_s^2}{\theta_{n+1}^2} + \frac{1}{2}(n+1)^2 \beta}},$$

and $C_0(n)$ is the same as the one in (1.6).

**Proof.** Let $Z = (Z^1, Z^2, \cdots, Z^{n+1})$ be a homogeneous coordinate system of $\mathbb{C}P^n(4), (Z^s \in \mathbb{C})$. Defining $\Psi_{st}$ for $s, t = 1, 2, \cdots, n+1$, by

$$\Psi_{st}(\theta_1, \cdots, \theta_{n+1}, Z^1, \cdots, Z^{n+1}) = \frac{(\theta_s Z^s)(\overline{\theta_t Z^t})}{\sum_{r=1}^{n+1} (\theta_r Z^r)(\overline{\theta_r Z^r})},$$

where $\theta_s, s = 1, 2, \cdots, n+1$, are $(n+1)$ coefficients of scarling coordinate system which are determined later, we have

$$\Psi_{st} = \overline{\Psi_{st}}, \quad \sum_{s,t=1}^{n+1} \Psi_{st} \overline{\Psi_{st}} = 1.$$

For any fixed point $P \in M^n$, we can choose a new homogeneous coordinate system of $\mathbb{C}P^n(4)$, which satisfies that, at $P$

$$\overline{Z}^1 = \overline{Z}^2 = \cdots = \overline{Z}^n = 0, \quad \overline{Z}^{n+1} \neq 0 \quad (5.1)$$

and
\[ Z^s = \sum_{r=1}^{n+1} A_{sr} \overline{Z}^r, \]

where

\[ A = (A_{sr}) \in U(n + 1) \]

is an \((n + 1) \times (n + 1)\)-unitary matrix, that is, \(A_{st}\) satisfies

\[ \sum_{r=1}^{n+1} A_{sr} \overline{A}_{rt} = \sum_{r=1}^{n+1} A_{sr} \overline{A}_{tr} = \delta_{st}. \]

Then, we know that

\[ z = (z^1, \ldots, z^n), \quad \overline{z}^s = \frac{\theta_i \overline{Z}^s}{\theta_{n+1} \overline{Z}^{n+1}}, \]

is a local holomorphic coordinate system of \(\mathbb{C}P^n(4)\) in a neighborhood \(U\) of the point \(P \in M^n\) and (5.1) implies that, at \(P\),

\[ z^1 = \cdots = z^n = 0. \]

Hence, we infer, for \(s, t = 1, 2, \ldots, n + 1\),

\[ \Psi_{st}(\theta_1, \cdots, \theta_{n+1}, Z^1, \cdots, Z^{n+1}) = \frac{(\theta_i \overline{Z}^s)(\theta_j \overline{Z}^t)}{\sum_{r=1}^{n+1} (\theta_r \overline{Z}^r)(\theta_r \overline{Z}^r)} = \frac{z^s \overline{z}^t}{1 + \sum_{r=1}^{n} z^r \overline{z}^r} \]

(5.2)

Putting

\[ \mathcal{G}_{st}(\theta_1, \cdots, \theta_{n+1}, Z^1, \cdots, Z^{n+1}) = \text{Re}(\Psi_{st}(\theta_1, \cdots, \theta_{n+1}, Z^1, \cdots, Z^{n+1})) \]

and

\[ \mathcal{F}_{st}(\theta_1, \cdots, \theta_{n+1}, Z^1, \cdots, Z^{n+1}) = \text{Im}(\Psi_{st}(\theta_1, \cdots, \theta_{n+1}, Z^1, \cdots, Z^{n+1})) \]

for \(s, t = 1, 2, \cdots, n + 1\), then, we have

\[ \sum_{s,t=1}^{n+1} (\mathcal{G}_{st}^2 + \mathcal{F}_{st}^2) = \sum_{s,t=1}^{n+1} \Psi_{st} \overline{\Psi}_{st} = \sum_{s,t=1}^{n+1} \overline{\Psi}_{pq} \Psi_{pq} = 1 \]

(5.3)

\[ \sum_{s,t=1}^{n+1} (\nabla \mathcal{G}_{st} \mathcal{G}_{st} + \nabla \mathcal{F}_{st} \mathcal{F}_{st}) = 0. \]

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Next, according to special proportion, we define the corresponding weighted transformation (or we say that they are some scarling transformations) to the variables $z^s$, where $s = 1, 2, \ldots, n + 1$, as follows:

$$y^s = \frac{\theta_{n+1}}{\theta_s} z^s.$$  

We note that those weighted transformations play a significant role in the calculation. Then, it follows from (5.2) that

$$\tilde{\Psi}_{st}(\theta_1, \ldots, \theta_{n+1}, Z^1, \ldots, Z^{n+1}) = \frac{\theta_1 \theta_t y^s y^t}{\theta_{n+1}^2 + \sum_{r=1}^n \theta_r^2 y^r y^r},$$  

(5.4)

and

$$\tilde{\Psi}_{(n+1)(n+1)}(\theta_1, \ldots, \theta_{n+1}, Z^1, \ldots, Z^{n+1}) = 1$$  

(5.5)

Let

$$g = \sum_{s,t=1}^n g^s_t dy^s dy^t$$

be the Fubini-Study metric of $\mathbb{C}P^n(4)$. Then,

$$g^s_t = \frac{\delta^s_t}{1 + \sum_{r=1}^n |y^r|^2 \left[ 1 + \sum_{r=1}^n |y^r|^2 \right]^2} - \frac{y^s y^t}{1 + \sum_{r=1}^n |y^r|^2 \left[ 1 + \sum_{r=1}^n |y^r|^2 \right]^2}$$  

(5.6)

Under the local coordinate system, for any smooth function $\Psi$, it follows from (5.6) that

$$\Delta \Psi = 4 \sum_{s,t=1}^n \frac{\partial^2}{\partial y^s \partial y^t} \Psi,$$

And, by the definition of $\tilde{\Psi}_{st}$, we know that, at $P$,
\[
\Delta = 4 \sum_{r=1}^{n} \frac{\partial^2}{\partial y^r \partial y^r},
\]
\[
\nabla \tilde{\Psi}_{st} = 0, \text{ if } s \neq 0 \text{ and } t \neq 0
\]
\[
\nabla \tilde{\Psi}_{st} = 0,
\]
\[
\Delta \tilde{\Psi}_{st} = 0, \text{ if } s \neq t
\]
\[
\Delta \tilde{\Psi}_{(n+1)(n+1)} = -4 \sum_{s=1}^{n} \frac{\theta_s^2}{\theta_{n+1}^2}; \Delta \tilde{\Psi}_{r\sigma} = \frac{4\theta_r^2}{\theta_{n+1}^2}, \quad r = 1, \ldots, n.
\]

Thus, we obtain from (5.3), (5.4) and (5.5), at \( P \),
\[
\sum_{s,t=1}^{n+1} (\nabla G_{st} \cdot \nabla G_{st} + \nabla \mathcal{F}_{st} \cdot \nabla \mathcal{F}_{st}) = -\sum_{s,t=1}^{n+1} (G_{st} \Delta G_{st} + \mathcal{F}_{st} \Delta \mathcal{F}_{st})
\]
\[
= -\text{Re} \sum_{s,t=1}^{n+1} \tilde{\Psi}_{st} \Delta \tilde{\Psi}_{st}
\]
\[
= -\text{Re} \sum_{s,t=1}^{n+1} \sum_{r,w=1}^{n+1} A_{sr} A_{tw} \tilde{\Psi}_{rw} \sum_{u,v=1}^{n+1} A_{su} A_{tv} \Delta \tilde{\Psi}_{uv}
\]
\[
= -\sum_{s,t=1}^{n+1} \text{Re} \tilde{\Psi}_{st} \Delta \tilde{\Psi}_{st}
\]
\[
= -\tilde{\Psi}_{(n+1)(n+1)} \Delta \tilde{\Psi}_{(n+1)(n+1)}
\]
\[
= 4 \sum_{s=1}^{n} \frac{\theta_s^2}{\theta_{n+1}^2}.
\]

By a similar calculation, we have, at \( P \),
\[
\sum_{s,t=1}^{n+1} (\nabla G_{st} \Delta G_{st} + \nabla \mathcal{F}_{st} \Delta \mathcal{F}_{st}) = \text{Re} \sum_{s,t=1}^{n+1} \nabla \tilde{\Psi}_{st} \Delta \tilde{\Psi}_{st} = 0. \quad (5.8)
\]
where and

\[ \sum_{s,t=1}^{n+1} (\Delta G_{st} \Delta G_{st} + \Delta F_{st} \Delta F_{st}) = \text{Re} \sum_{s,t=1}^{n+1} \overline{\Delta \Psi_{st}} \Delta \Psi_{st} \]

\[ = \text{Re} \sum_{s,t=1}^{n+1} \overline{\Delta \Psi_{st}} \Delta \Psi_{st} \]

\[ = \left( -4 \sum_{s=1}^{n} \frac{\theta_s^2}{\theta_{n+1}^2} \right) \times \left( -4 \sum_{s=1}^{n} \frac{\theta_s^2}{\theta_{n+1}^2} \right) + 4 \times 4 \times \sum_{s=1}^{n} \frac{\theta_s^4}{\theta_{n+1}^4} \]

\[ = 16 \left( \sum_{s=1}^{n} \frac{\theta_s^2}{\theta_{n+1}^2} \right)^2 + 16 \sum_{s=1}^{n} \frac{\theta_s^4}{\theta_{n+1}^4}. \]

and

\[ \sum_{s,t=1}^{n+1} \left( \langle \nabla G_{st}, \nabla u_t \rangle^2 + \langle \nabla F_{st}, \nabla u_t \rangle^2 \right) = \text{Re} \sum_{s,t=1}^{n+1} \langle \overline{\nabla \Psi_{st}}, \nabla u_t \rangle \langle \overline{\nabla \Psi_{st}}, \nabla u_t \rangle \]

\[ = \text{Re} \sum_{s,t=1}^{n+1} \langle \overline{\nabla \Psi_{st}}, \nabla u_t \rangle \langle \overline{\nabla \Psi_{st}}, \nabla u_t \rangle \]

\[ \leq \frac{2 \tilde{\theta}^2}{\theta_{n+1}^2} |\nabla u_t|^2, \]

where

\[ \tilde{\theta} = \max_{1 \leq s \leq n+1} \{ \theta_s \}. \]

Since \( P \) is arbitrary, we have at any point \( x \in M^n \),

\[ \sum_{s,t=1}^{n+1} (\nabla G_{st} \cdot \nabla G_{st} + \nabla F_{st} \cdot \nabla F_{st}) = 4 \sum_{s=1}^{n} \frac{\theta_s^2}{\theta_{n+1}^2}. \]

\[ \sum_{s,t=1}^{n+1} (\nabla G_{st} \Delta G_{st} + \nabla F_{st} \Delta F_{st}) = 0. \]

\[ \sum_{s,t=1}^{n+1} (\Delta G_{st} \Delta G_{st} + \Delta F_{st} \Delta F_{st}) = 16 \left[ \left( \sum_{s=1}^{n} \frac{\theta_s^2}{\theta_{n+1}^2} \right)^2 + \sum_{s=1}^{n} \frac{\theta_s^4}{\theta_{n+1}^4} \right]. \]

\[ \sum_{s,t=1}^{n+1} \left( \langle \nabla G_{st}, \nabla u_t \rangle^2 + \langle \nabla F_{st}, \nabla u_t \rangle^2 \right) \leq \frac{2 \tilde{\theta}^2}{\theta_{n+1}^2} |\nabla u_t|^2. \]

By applying the Lemma 2.3 to the functions \( G_{st} \) and \( F_{st} \) and taking sum on \( s \) and \( t \) from 1 to \( n+1 \), we infer from (5.7), (5.8), (5.9), (5.10) and (5.11) that
\[
\sum_{s,t=1}^{n+1} \left( \|u_t \nabla G_{st}\|^2 + \|u_t \nabla F_{st}\|^2 \right) = 4 \int_{M^n} \sum_{s,t=1}^{n+1} \left( \langle \nabla G_{st} \Delta G_{st}, u_t \nabla u_t \rangle + \langle \nabla F_{st} \Delta F_{st}, u_t \nabla u_t \rangle \right) dv \\
+ \int_{M^n} \sum_{s,t=1}^{n+1} \left( \Delta G_{st} \Delta G_{st} + \Delta F_{st} \Delta F_{st} \right) u_t^2 dv \\
+ 4 \int_{M^n} \sum_{s,t=1}^{n+1} \left( \langle \nabla G_{st}, \nabla u_t \rangle^2 + \langle \nabla F_{st}, \nabla u_t \rangle^2 \right) dv \\
\leq 16 \left[ \left( \sum_{s=1}^{n} \frac{\theta^2_s}{\theta^2_{n+1}} \right)^2 + \sum_{s=1}^{n} \frac{\theta^4_s}{\theta^4_{n+1}} \right] + \frac{8\bar{\theta}^2}{\theta^2_{n+1}} \lambda_i.
\]

We choose \((n + 1)\) positive real numbers \(\theta_s\), such that, for all \(s = 1, 2, \cdots, n + 1\),

\[
\sum_{s,t=1}^{n+1} \left[ \int 2u_t \langle \nabla G_{st}, \nabla u_t \rangle \Delta G_{st} dv + \int 2u_t \langle \nabla F_{st}, \nabla u_t \rangle \Delta F_{st} dv \right] = 0, \tag{5.13}
\]

and

\[
a^2_{st} = \|\nabla G_{st} u_t\|^2 + \|\nabla F_{st} u_t\|^2 \geq \sqrt{\|\nabla G_{st}\|^2 u_t\|^2} + \sqrt{\|\nabla F_{st}\|^2 u_t\|^2} = b_{st} \geq 0.
\]

Let

\[
\beta = \min_{1 \leq s,t \leq n+1} \{b_{st}\},
\]

and \(l = n + 1\), then, by lemma 2.1, we have

\[
\sum_{s,t=1}^{n+1} \frac{a^2_{st}}{2} + \frac{b_{st}}{2} = 2 \sum_{s=1}^{n} \frac{\theta^2_s}{\theta^2_{n+1}} + \sum_{s,t=1}^{n+1} \frac{b_{st}}{2} \geq 2 \sum_{s=1}^{n} \frac{\theta^2_s}{\theta^2_{n+1}} + \frac{1}{2} (n + 1)^2 \beta. \tag{5.14}
\]

From (2.3), (5.12), (5.13) and (5.14), we obtain

\[
\left[ 2 \sum_{s=1}^{n} \frac{\theta^2_s}{\theta^2_{n+1}} + \frac{1}{2} (n + 1)^2 \beta \right] (\lambda_{k+2} - \lambda_{k+1})^2 \leq 4(\lambda_{k+2} + \rho) \left[ 16 \sum_{s=1}^{n} \frac{\theta^2_s}{\theta^2_{n+1}} \left( \sum_{s=1}^{n} \frac{\theta^4_s}{\theta^4_{n+1}} + 1 \right) + \frac{8\bar{\theta}^2}{\theta^2_{n+1}} \lambda_1 \right],
\]

for any \(i, i = 1, 2, \cdots, k\), which implies

\[
\lambda_{k+2} - \lambda_{k+1} \leq \sqrt{\frac{4(\lambda_{k+2} + \rho) \left[ 16 \left( \sum_{s=1}^{n} \frac{\theta^2_s}{\theta^2_{n+1}} \right)^2 + \sum_{s=1}^{n} \frac{\theta^4_s}{\theta^4_{n+1}} \right] + \frac{8\bar{\theta}^2}{\theta^2_{n+1}} \lambda_1}{2 \sum_{s=1}^{n} \frac{\theta^2_s}{\theta^2_{n+1}} + \frac{1}{2} (n + 1)^2 \beta}}. \tag{5.15}
\]
In order to complete the proof, we need a recursion formula given by Cheng and Yang in [17] as follows: Let \( \mu_1 \leq \mu_2 \leq \ldots \leq \mu_{k+1} \) be any positive real numbers satisfying

\[
\sum_{i=1}^{k} (\mu_{k+1} - \mu_i)^2 \leq \frac{4}{n} \sum_{i=1}^{k} \mu_i (\mu_{k+1} - \mu_i).
\]

Define

\[
\Gamma_k = \frac{1}{k} \sum_{i=1}^{k} \mu_i, \quad \mathcal{E}_k = \frac{1}{k} \sum_{i=1}^{k} \mu_i^2, \quad \mathcal{H}_k = \left(1 + \frac{2}{n}\right) \Gamma_k^2 - \mathcal{E}_k.
\]

Then, we have

\[
\mathcal{H}_{k+1} \leq C(n, k) \left(\frac{k+1}{k}\right)^{\frac{4}{2}} \mathcal{H}_k,
\]

where

\[
C(n, k) = 1 - \frac{1}{3n} \left(\frac{k}{k+1}\right)^{\frac{4}{2}} \left(1 + \frac{2}{n}\right) \left(1 + \frac{4}{n}\right) < 1.
\]

Let \( \mu_i = \lambda_i + 2n(n+1) \). By (1.10) and (5.16), we yield

\[
\sum_{i=1}^{k} (\mu_{k+1} - \mu_i)^2 \leq \frac{4}{2n} \sum_{i=1}^{k} (\mu_{k+1} - \mu_i) \mu_i,
\]

which implies

\[
\lambda_{k+1} + 2n(n+1) \leq C_0(n)(\lambda_1 + 2n(n+1)) k^{\frac{4}{2}}, \quad (5.17)
\]

\( C_0(n) \) is the same as the one in (1.6). Therefore, putting \( \rho = 2n(n+1) \) and synthesizing (5.15) and (5.17), we obtain

\[
\lambda_{k+2} - \lambda_{k+1} \leq \sqrt{4(\lambda_{k+2} + 2n(n+1) \left(16 \left[\sum_{s=1}^{n} \frac{\theta_s^2}{\theta_{s+1}^2} + \sum_{s=1}^{n} \frac{\theta_s^4}{\theta_{s+1}^4} \right] + 8\theta^2 \lambda_1 \right) + 2 \sum_{s=1}^{n} \frac{\theta_s^2}{\theta_{s+1}^2} + \frac{1}{2}(n+1)^2 \beta}
\]

\[
\leq 4 \sqrt{\frac{\sum_{s=1}^{n} \theta_s^2}{\theta_{s+1}^2} + \sum_{s=1}^{n} \theta_s^4} + \frac{2\theta^2}{\theta_{s+1}^2} \lambda_1}
\]

\[
\leq C(n, \Omega) k^{\frac{4}{2}},
\]

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where

\[ C(n, \Omega) = 4 \left( \frac{\left( \sum_{s=1}^{n} \frac{\theta_s^2}{\theta_{n+1}^2} \right)^2 + \sum_{s=1}^{n} \frac{\theta_s^4}{\theta_{n+1}^4}}{\frac{2\theta_1^2}{\theta_{n+1}^2} \lambda_1} \right) \cdot \sqrt{C_0(n)(\lambda_1 + 2(n + 1))}. \]

Therefore, we finish the proof of this theorem. \( \square \)

Next, we shall consider the eigenvalue problem of the Laplacian on a compact complex hypersurface \( M^n \) without boundary in \( \mathbb{C}P^{n+1}(4) \):

**Theorem 5.2.** Let \( M^n \) a compact complex hypersurface with empty boundary in the standard complex projective space \( \mathbb{C}P^n(4) \) with holomorphic sectional curvature 4, and \( \lambda_i \) be the \( i \)-th \( (i = 1, 2, \cdots, k) \) eigenvalue of the eigenvalue problem (1.16). Then, we have

\[ \lambda_{k+2} - \lambda_k \leq C(n, M^n)(k + 1)^{\frac{1}{2}}, \quad (5.18) \]

where

\[ C(n, M^n) = \sqrt{4 \left( \frac{\left( \sum_{s=1}^{n} \frac{\theta_s^2}{\theta_{n+2}^2} \right)^2 + \sum_{s=1}^{n} \frac{\theta_s^4}{\theta_{n+2}^4}}{\frac{2\theta_1^2}{\theta_{n+2}^2} \lambda_1} \right) \cdot \sqrt{C_0(n)(\lambda_1 + 2n(n + 1))}, \]

and \( C_0(n) \) is the same as the one in (1.16).

**Proof.** Since the method of proof is the same as in the proof of Theorem 1, we shall only give its outline. Let \( Z = (Z^1, Z^2, \cdots, Z^{n+2}) \) be a homogeneous coordinate system of \( \mathbb{C}P^{n+1}(4) \), \( (Z_s \in \mathbb{C}) \). Defining \( \Psi_s \), for \( s, t = 1, 2, \cdots, n + 2 \), by

\[ \Psi_s(\theta_1, \cdots, \theta_{n+2}, Z^1, \cdots, Z^{n+2}) = \frac{(\theta_s Z^s)(\bar{\theta}_s \bar{Z}^s)}{\sum_{r=1}^{n+2} (\theta_r Z^r)(\bar{\theta}_r \bar{Z}^r)}, \]

where \( \theta_s, s = 1, 2, \cdots, n + 2 \), are \( (n + 2) \) coefficients of scarling coordinate system, which will be determined later, we have

\[ \Psi_s = \Psi_s, \quad \sum_{s,t=1}^{n+2} \Psi_s \Psi_s = 1. \]
For any fixed point $P \in M^n$, we can choose a new homogeneous coordinate system of $\mathbb{C}P^n(4)$, which satisfies, at $P$

$$\overline{Z}^1 = \overline{Z}^2 = \cdots = \overline{Z}^{n+1} = 0, \quad \overline{Z}^{n+2} \neq 0$$

and

$$Z^s = \sum_{r=1}^{n+2} C_{sr} \overline{Z}^r,$$

where $A = (A_{st}) \in U(n + 2)$ is an $(n + 2) \times (n + 2)$-unitary matrix, that is, $A_{st}$ satisfies

$$\sum_{r=1}^{n+2} A_{rs} \overline{A}_{rt} = \sum_{r=1}^{n+2} A_{sr} \overline{A}_{tr} = \delta_{st}.$$ 

Let

$$z^s = \frac{\theta_s \overline{Z}^s}{\theta_{n+2} \overline{Z}^{n+2}}, \text{ for } s = 1, 2, \cdots, n + 2,$$

Then, we know that $z = (z^1, \cdots, z^n)$ is a local holomorphic coordinate system of $M$ in a neighborhood $U$ of the point $P \in M$ and $z^{n+1} = h(z^1, z^2, \cdots, z^n)$ is a holomorphic function of $z^1, z^2, \cdots, z^n$ and satisfying,

$$\frac{\partial h}{\partial z^s} \bigg|_P = 0, \text{ for } s = 1, 2, \cdots, n,$$

At the point $P$, one has

$$z^1 = \cdots = z^{n+1} = 0.$$

Hence, for any $s, t = 1, 2, \cdots, n + 2$, we have,

$$\Psi_{st} (\theta_1, \cdots, \theta_{n+2}, Z^1, \cdots, Z^{n+2}) = \frac{\overline{(\theta_s \overline{Z}^s)} (\overline{\theta_t Z^t})}{\sum_{r=1}^{n+2} (\overline{\theta_r Z^r}) (\overline{\theta_r Z^r})} = \frac{z^s z^t}{1 + \sum_{r=1}^{n+2} z^r \overline{z}^r}$$

(5.19)

Putting

$$\mathcal{G}_{st} (\theta_1, \cdots, \theta_{n+2}, Z^1, \cdots, Z^{n+2}) = \text{Re}(\Psi_{st} (\theta_1, \cdots, \theta_{n+2}, Z^1, \cdots, Z^{n+2}))$$

and

$$\mathcal{F}_{st} (\theta_1, \cdots, \theta_{n+2}, Z^1, \cdots, Z^{n+2}) = \text{Im}(\Psi_{st} (\theta_1, \cdots, \theta_{n+2}, Z^1, \cdots, Z^{n+2})),$$

for $s, t = 1, 2, \cdots, n + 2$, then, we infer
\[
\sum_{s,t=1}^{n+2} (G_{st}^2 + F_{st}^2) = \sum_{s,t=1}^{n+2} \Psi_{st} \Psi_{st} = \sum_{s,t=1}^{n+2} \Psi_{pq} \Psi_{pq} = 1
\]
\[
\sum_{s,t=1}^{n+2} (G_{st} \nabla G_{st} + F_{st} \nabla F_{st}) = 0.
\]

Similarly, we define the corresponding weighted transformations of the variables \(z^s\), where \(s = 1, 2, \ldots, n + 2\), as follows:

\[
y^s = \frac{\theta_{n+2}}{\theta_s} z^s.
\]

Then, it follows from (5.19) that

\[
\Psi_{st} = \frac{\theta_s \theta_t y^s y^t}{\theta_{n+2} + \sum_{r=1}^{n+1} \theta_r^2 y^r y^r}.
\]

(5.20)

It is easy to see that, under the local coordinate system, for \(z \in U\), the metric can be written as the following:

\[
g_M = \sum_{s,t=1}^{n} \left(1 + O\left(|z|^{2}\right)\right) dz^s d\zeta^t,
\]

Thus, for any smooth function \(\Psi\), we have

\[
\Delta \Psi = 4 \sum_{s,t=1}^{n} g^{st} \frac{\partial^2}{\partial y^s \partial y^t} \Psi,
\]

By a direct calculation, we obtain, at \(P\),

\[
\Delta = 4 \sum_{r=1}^{n} \frac{\partial^2}{\partial y^r \partial y^r},
\]

\(\nabla \Psi_{st} = 0\), if \(s \neq n + 2\) and \(t \neq n + 2\)

\(\nabla \Psi_{ss} = 0\),

\(\nabla \Psi_{(n+2)(n+2)} = \nabla \Psi_{(n+2)(n+2)} = \nabla \Psi_{(n+2)(n+2)} = 0\), for \(s = 1, 2, \ldots, n\),

\(\Delta \Psi_{st} = 0\), if \(s \neq t\), \(\Delta \Psi_{(n+1)(n+1)} = 0\),

\(\Delta \Psi_{(n+2)(n+2)} = -4 \sum_{s=1}^{n} \frac{\theta^2}{\theta_{n+2}}; \quad \Delta \Psi_{r,r} = \frac{4\theta^2}{\theta_{n+2}}\), \(r = 1, \ldots, n\).
Similarly, one can check the following:

$$\sum_{s,t=1}^{n+2} (\nabla G_{st} \cdot \nabla G_{st} + \nabla F_{st} \cdot \nabla F_{st}) = 4 \sum_{s=1}^{n+2} \frac{\theta_s^2}{\theta_{s+1}^2};$$

$$\sum_{s,t=1}^{n+2} (\nabla G_{st} \Delta G_{st} + \nabla F_{st} \Delta F_{st}) = 0;$$

$$\sum_{s,t=1}^{n+2} (\Delta G_{st} \Delta G_{st} + \Delta F_{st} \Delta F_{st}) = 16 \left[ \left( \sum_{s=1}^{n} \frac{\theta_s^2}{\theta_{s+1}^2} \right)^2 + \sum_{s=1}^{n} \frac{\theta_s^4}{\theta_{s+1}^4} \right];$$

$$\sum_{s,t=1}^{n+2} (\nabla G_{st}, \nabla u_t)^2 + (\nabla F_{st}, \nabla u_t)^2 \leq \frac{2\bar{\theta}}{\theta_{n+2}^2} |\nabla u_t|^2,$$

where

$$\bar{\theta} = \max_{1 \leq s \leq n+2} \{ \theta_s \}.$$

Hence, by (5.21), if \( \theta_1 = \theta_2 = \cdots = \theta_{n+2} = 1 \), we have

$$\left\| \sum_{s,t=1}^{n+2} (\nabla G_{st} \cdot \nabla G_{st} + \nabla F_{st} \cdot \nabla F_{st}) \right\|_{1,1,\cdots,1} = 4n;$$

$$\left\| \sum_{s,t=1}^{n+2} (\nabla G_{st} \Delta G_{st} + \nabla F_{st} \Delta F_{st}) \right\|_{1,1,\cdots,1} = 0;$$

$$\left\| \sum_{s,t=1}^{n+2} (\Delta G_{st} \Delta G_{st} + \Delta F_{st} \Delta F_{st}) \right\|_{1,1,\cdots,1} = 16n (n + 1);$$

$$\left\| \sum_{s,t=1}^{n+2} (\nabla G_{st}, \nabla u_t)^2 + (\nabla F_{st}, \nabla u_t)^2 \right\|_{1,1,\cdots,1} = 2|\nabla u_t|^2.$$

Therefore, it follows from (5.22) that

$$\sum_{s,t=1}^{n+2} \left( ||u_t \nabla G_{st}||^2 + ||u_t \nabla F_{st}||^2 \right) = 4n,$$

and

$$\sum_{s,t=1}^{n+2} \left( ||\nabla G_{st}, \nabla u_t||^2 + ||\nabla F_{st}, \nabla u_t||^2 \right) \leq 16n(n + 1) + \frac{8\bar{\theta}}{\theta_{n+2}^2} \lambda_i.$$

Recall that Cheng and Yang established the following general formula in [16]:

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Applying (5.25) to the functions $G_{\overline{s}}(1, \cdots, 1, Z', \cdots, Z^{n+2})$ and $F_{\overline{s}}(1, \cdots, 1, Z', \cdots, Z^{n+2})$, we yield

$$\sum_{i=0}^{k} (\overline{\lambda}_{k+1} - \overline{\lambda}_i)^2 \|h \nabla u_i\|^2 \leq \sum_{i=0}^{k} (\overline{\lambda}_{k+1} - \overline{\lambda}_i)(\|\nabla h, \nabla u_i\| + u_i \Delta h\|^2).$$

(5.25)

Taking sum on $s$ and $t$ from 1 to $n + 1$, one infers that

$$\sum_{i=1}^{k} (\overline{\lambda}_{k+1} - \overline{\lambda}_i)^2 \sum_{s, t=1}^{n+2} (\|u_i \nabla G_{\overline{s}}\|^2 + \|u_i \nabla F_{\overline{s}}\|^2) \leq \sum_{s, t=1}^{n+2} (\overline{\lambda}_{k+1} - \overline{\lambda}_i)(\|\nabla G_{\overline{s}}, \nabla u_i\| + u_i \Delta G_{\overline{s}}\|^2 + \|\nabla F_{\overline{s}}, \nabla u_i\| + u_i \Delta F_{\overline{s}}\|^2).$$

(5.26)

Substituting (5.23) and (5.24) into (5.26), we obtain

$$\sum_{i=1}^{k} (\overline{\lambda}_{k+1} - \overline{\lambda}_i)^2 \sum_{s, t=1}^{n+2} (\|u_i \nabla G_{\overline{s}}\|^2 + \|u_i \nabla F_{\overline{s}}\|^2) \leq \sum_{s, t=1}^{n+2} (\overline{\lambda}_{k+1} - \overline{\lambda}_i)(\|\nabla G_{\overline{s}}, \nabla u_i\| + u_i \Delta G_{\overline{s}}\|^2 + \|\nabla F_{\overline{s}}, \nabla u_i\| + u_i \Delta F_{\overline{s}}\|^2).$$

(5.27)

Let $\mu_i = \overline{\lambda}_i + 2n(n + 1)$. By (5.27) and (5.16), we yield

$$\overline{\lambda}_{k+1} + 2n(n + 1) \leq C_0(n)(\overline{\lambda}_1 + 2n(n + 1))(k + 1)\frac{1}{k},$$

(5.28)

where $C_0(n)$ is the same as the one in (1.6). Therefore, we have

$$\int_{M^o} \sum_{s, t=1}^{n+2} (\|u_i \nabla G_{\overline{s}}\|^2 + \|u_i \nabla F_{\overline{s}}\|^2) dv = 16 \sum_{s=1}^{n} \frac{\theta^2_s}{\theta^4_{s+2}} \left( \sum_{s=1}^{n} \frac{\theta^2_s}{\theta^4_{s+2}} + 1 \right) + \frac{8\overline{\theta}^2}{\theta^2_{n+2}} \overline{\lambda}_i.$$  

(5.29)

We choose $n + 2$ positive real numbers $\theta_s$, such that, for all $s = 1, 2, \cdots, n + 2$,

$$\sum_{s, t=1}^{n+1} \left[ \int 2u_i \langle \nabla G_{\overline{s}}, \nabla u_i \rangle \Delta G_{\overline{s}} dv + \int 2u_i \langle \nabla F_{\overline{s}}, \nabla u_i \rangle \Delta F_{\overline{s}} dv \right] = 0,$$

(5.30)

and

$$a^2_{\overline{s}} = \|\nabla G_{\overline{s}} u_i\|^2 + \|\nabla F_{\overline{s}} u_i\|^2 \geq \sqrt{\|\nabla G_{\overline{s}}^2 u_i\|^2 + \|\nabla F_{\overline{s}}^2 u_i\|^2} = b_{\overline{s}} \geq 0.$$

Let

$$\beta = \min_{1 \leq s, t \leq n+2} \{b_{\overline{s}}\},$$

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and \( l = n + 2 \). Then, according to lemma 2.1, it follows from (2.3), (5.29) and (5.30) that
\[
2 \sum_{s=1}^{n} \frac{\theta_s^2}{\theta_{n+1}^2} + \frac{1}{2} (n+1)^2 \beta \left( \overline{\lambda}_{k+2} - \overline{\lambda}_{k+1} \right)^2 \leq 4(\lambda_{k+2} + \rho) \left\{ 16 \left[ \left( \sum_{s=1}^{n} \frac{\theta_s^2}{\theta_{n+2}^2} \right)^2 + \sum_{s=1}^{n} \frac{\theta_s^4}{\theta_{n+2}^4} \right] + \frac{8s^2}{\theta_{n+2}^2} \right\},
\]
for any \( i, \ i = 1, 2, \ldots, k \). Therefore, by the above inequality, we obtain
\[
\overline{\lambda}_{k+2} - \overline{\lambda}_{k+1} \leq \sqrt{4(\lambda_{k+2} + \rho) \left\{ 16 \left[ \left( \sum_{s=1}^{n} \frac{\theta_s^2}{\theta_{n+2}^2} \right)^2 + \sum_{s=1}^{n} \frac{\theta_s^4}{\theta_{n+2}^4} \right] + \frac{8s^2}{\theta_{n+2}^2} \right\}}.
\] (5.31)
Furthermore, we put \( \rho = 2n(n+1) \). Then, synthesizing (5.31) and (5.28), we obtain
\[
\overline{\lambda}_{k+2} - \overline{\lambda}_{k+1} \leq \sqrt{4(\lambda_{k+2} + 2n(n+1)) \left\{ 16 \left[ \left( \sum_{s=1}^{n} \frac{\theta_s^2}{\theta_{n+2}^2} \right)^2 + \sum_{s=1}^{n} \frac{\theta_s^4}{\theta_{n+2}^4} \right] + \frac{8s^2}{\theta_{n+2}^2} \right\}} \cdot \sqrt{C_0(n)(\overline{\lambda}_1 + 2n(n+1)) \cdot (k + 1) \frac{1}{2}}
\]
\[
= C(n, M^n)(k + 1) \frac{1}{2},
\]
where
\[
C(n, M^n) = \sqrt{4 \left[ \left( \sum_{s=1}^{n} \frac{\theta_s^2}{\theta_{n+2}^2} \right)^2 + \sum_{s=1}^{n} \frac{\theta_s^4}{\theta_{n+2}^4} \right] + \frac{2s^2}{\theta_{n+2}^2} \overline{\lambda}_1} \cdot \sqrt{C_0(n)(\overline{\lambda}_1 + 2n(n+1))}. \]

Therefore, we finish the proof of this theorem.

\[\square\]

**Remark 5.3.** In the proofs of theorem 5.1 and theorem 5.2, the calculations of inequality (5.11) and inequality (5.21) is the same as in [10].
6 Eigenvalues on Compact Homogeneous Riemannian Manifolds

In this section, we investigate the eigenvalue of the Laplacian on the compact homogeneous Riemannian manifolds. More specifically, we prove the following theorem.

**Theorem 6.1.** Let $M^n$ be an $n$-dimensional compact homogeneous Riemannian manifold without boundary. If $\lambda_i, = 0, 1, 2, \cdots$, is the $i$-th eigenvalue of the closed eigenvalue problem (1.16), then

\[ \bar{\lambda}_{k+1} - \bar{\lambda}_k \leq C_{n,M^n}(k + 1), \]

where

\[ C(n, M^n) = 4 \sqrt{\frac{\alpha^2 [4\lambda_1^2\sigma^2 + \lambda_1^2\sigma^2]}{\alpha\sigma^2\bar{\lambda}_1 + d\beta}} \frac{\sqrt{5C_0(n)\bar{\lambda}_1}}{2}. \]

**Proof.** Recall that Cheng and Yang [16] proved the following

\[ \sum_{i=0}^{k} (\bar{\lambda}_{k+1} - \bar{\lambda}_i)^2 \leq 4 \sum_{i=0}^{k} (\bar{\lambda}_{k+1} - \bar{\lambda}_i)(\bar{\lambda}_i + \frac{1}{4}\bar{\lambda}_1). \]  

(6.1)

By the (5.16) and (6.1), we obtain

\[ \bar{\lambda}_{k+1} \leq C_0(n)(k + 1)^2\bar{\lambda}_1. \]  

(6.2)

We assume that $\{g_p\}_{p=1}^l$ is an orthonormal basis corresponding to the first eigenspace $E_{\lambda_1}$ of the eigenvalue problem

\[ \Delta f = -\bar{\lambda}_1 f, \text{ on } M. \]

It is equivalent to say that, the orthonormal basis $\{g_p\}_{p=1}^l$ satisfies the following

\[ \Delta g_p = -\bar{\lambda}_1 g_p, \text{ on } M. \]  

(6.3)

Since $M$ is an $n$-dimensional compact homogeneous Riemannian manifold without boundary, we know that

\[ \sum_{p=1}^{l} g_p^2 = \sigma^2 \]

is constant (cf. Proposition 1 of Li [24]). Since the sum

\[ \sum_{p=1}^{l} g_p^2 = \sigma^2 \]

is a constant, we infer
\[
\sum_{p=1}^{l} g_p \nabla g_p = 0, \quad \text{(6.4)}
\]

\[
\sum_{p=1}^{l} \nabla g_p \cdot \nabla g_p = - \sum_{p=1}^{l} g_p \Delta g_p = \lambda_1 \sigma^2.
\]

Hence, we infer from (6.4)

\[
\sum_{p=1}^{l} ||u_i \nabla g_p||^2 = \lambda_1 \sigma^2. \quad \text{(6.5)}
\]

Let \( \alpha_1, \alpha_2, \cdots, \alpha_d \) are \( d \) positive numbers. We define \( d \) scarling eigenfunctions \( h_p(x) = \alpha_p g_p \), such that

\[
\sum_{p=1}^{l} \int u_i \langle \nabla h_p, \nabla u_i \rangle \Delta h_p dv = 0, \quad \text{(6.6)}
\]

and

\[
a_p^2 = ||\nabla h_p u_i||^2 \geq \sqrt{||\nabla h_p \|^2 u_i ||^2} = b_p \geq 0,
\]

where \( p = 1, 2, \cdots, d \). Let

\[
\alpha = \min \{ \alpha_j \}_{1 \leq j \leq d},
\]

\[
\overline{\alpha} = \max \{ \alpha_j \}_{1 \leq j \leq d},
\]

\[
\beta = \min \{ b_j \}_{1 \leq j \leq d},
\]

and \( l = n + p \), then, by lemma 2.3 and (6.5), we have

\[
\sum_{j=1}^{l} \frac{a_j^2}{2} + b_j = \sum_{p=1}^{l} \frac{||u_i \nabla (\alpha_p g_p)||^2}{2} + \frac{\sqrt{\|\nabla (\alpha_p g_p)^2 u_i \|^2}}{2} \geq \frac{\alpha \sigma^2 \overline{\lambda_1} + d \beta}{2}. \quad \text{(6.7)}
\]

Furthermore, by (6.3), one can deduce that
\[
\sum_{p=1}^{d} ||2\langle \nabla h_p, \nabla u_i \rangle + u_i \Delta h_p \|^2 = \sum_{p=1}^{d} ||2\langle \nabla (\alpha_p g_p), \nabla u_i \rangle + u_i \Delta (\alpha_p g_p) \|^2 \\
= \sum_{p=1}^{d} \int_{M} \left\{ 4\alpha_p^2 (\nabla g_p \cdot \nabla u_i)^2 - 4\lambda_1 \alpha_p g_p u_i \nabla g_p \cdot \nabla u_i + \dot{\lambda}_1^2 \alpha_p^2 g_p^2 u_i^2 \right\} dv \\
\leq 2 \sum_{p=1}^{d} \int_{M} \left\{ 4\alpha_p^2 (\nabla g_p \cdot \nabla u_i)^2 + \dot{\lambda}_1^2 \alpha_p^2 g_p^2 u_i^2 \right\} dv \\
\leq 2\alpha^2 \left[ \int_{M} \sum_{p=1}^{d} 4(\nabla g_p \cdot \nabla u_i)^2 dv + \dot{\lambda}_1^2 \sigma^2 \right].
\]

Since
\[
\langle \nabla g_p, \nabla u_i \rangle^2 \leq |\nabla g_p|^2 |\nabla u_i|^2
\]
and
\[
\sum_{p=1}^{d} |\nabla g_p|^2 = \lambda_1 \sigma^2,
\]
we infer from (6.6), (6.8) and (6.9) that
\[
\sum_{p=1}^{d} ||2\langle \nabla h_p, \nabla u_i \rangle + u_i \Delta h_p \|^2 \leq 2\alpha^2 \left[ \int_{M} 4\lambda_1 \sigma^2 |\nabla u_i|^2 dv + \dot{\lambda}_1^2 \sigma^2 \right]
\]
(6.10)
Since $\lambda_1 \alpha^2$ is positive, i.e., $\lambda_1 \alpha^2 > 0$, by making use of (2.8), (6.7), (6.9) and (6.10), we have
\[
\frac{\alpha \sigma^2 - \lambda_1}{2} \left( \lambda_{k+2} - \lambda_{k+1} \right)^2 \leq 8(\lambda_{k+2} + \rho) \sigma^2 \left[ 4\lambda_1 \sigma^2 \lambda_i + \lambda_1^2 \sigma^2 \right],
\]
(6.11)
for any $i = 1, 2, \cdots, k$. Let
\[
\rho = \frac{1}{4} \lambda_1.
\]
Then, by using (6.2) and (6.11), one can infer that
\[
\lambda_{k+2} - \lambda_{k+1} \leq 4 \sqrt{\frac{\left(\lambda_{k+2} + \frac{1}{4} \lambda_1\right)^2}{\alpha \sigma^2 \lambda_1 + d\beta}} \cdot \frac{4 \lambda_1 \sigma^2 \lambda_1 + \lambda_1^2}{\alpha \sigma^2 \lambda_1 + d\beta}
\]

\[
\leq 4 \sqrt{\frac{\alpha^2}{\alpha \sigma^2 \lambda_1 + d\beta}} \cdot \frac{4 \lambda_1 \sigma^2 \lambda_1 + \lambda_1^2}{\alpha \sigma^2 \lambda_1 + d\beta} \cdot \sqrt{\frac{5C_0(n) \lambda_1}{2}} \cdot (k+2)
\]

\[
= C(n, M^n)(k+2),
\]

where

\[
C(n, M^n) = 4 \sqrt{\frac{\alpha^2}{\alpha \sigma^2 \lambda_1 + d\beta}} \cdot \sqrt{\frac{5C_0(n) \lambda_1}{2}}.
\]

Thus, we finish the proof of this theorem. \(\Box\)

### 7 Gap Coefficients

In theorem 1.2 the best constant \(C_{n,\Omega}\) is called the gap coefficient. In this section, we pay attention to investigating the gap coefficient \(C_{n,\Omega}\). It is worth noting that it is very difficult for us to give the explicit expression of the optimal gap coefficient, even if \(\Omega\) are some special domains in the Euclidean space with dimension \(n\). However, we find that the eigenvalues depend on the shape of the bounded domain \(\Omega \subset \mathbb{R}^n\). Therefore, we introduce two new notations which will play significant roles in the estimating for the eigenvalues.

**Definition 7.1.** Assume that \(\Sigma_1\) and \(\Sigma_2\) are two cubes in \(\mathbb{R}^n\), where \(n \geq 2\), such that \(\Sigma_1 \subset \Omega \subset \Sigma_2\). We define

\[
\mathcal{G}_1(\Omega) = \begin{cases} 
\sup_{\Sigma_1 \subset \mathbb{R}^n, \Sigma_2 \subset \mathbb{R}^n} \frac{d_1^2}{d_2^2}, & n \geq 2, \\
1, & n = 1,
\end{cases}
\]

and call it the first shape coefficient, where \(d_1\) denotes the length of the side of the cube \(\Sigma_1\) and \(d_2\) denotes the length of the side of the cube \(\Sigma_2\), respectively. Assume that \(B_1\) and \(B_2\) are two balls in \(\mathbb{R}^n\) such that \(B_1 \subset \Omega \subset B_2\). We define

\[
\mathcal{G}_2(\Omega) = \begin{cases} 
\sup_{B_1 \subset \mathbb{R}^n, B_2 \subset \mathbb{R}^n} \frac{r_1^2}{r_2^2}, & n \geq 2, \\
1, & n = 1,
\end{cases}
\]

and call it the second shape coefficient, where \(r_1\) denotes diameter of the ball \(B_1\) and \(r_2\) denotes diameter of the ball \(B_2\), respectively.
Remark 7.2. When $\Omega$ is a cube then, $\Sigma_1 = \Sigma_2$, therefore, we have $\mathcal{E}_1(\Omega) = 1$. Similarly, when $\Omega$ is a ball, then $B_1 = B_2$, therefore, we have $\mathcal{E}_2(\Omega) = 1$.

According to a great amount of numeric calculation for some special examples, we venture to propose the following:

**Conjecture 7.3.** Let $\Omega$ be a bounded domain with piecewise smooth boundary $\partial \Omega$ on an n-dimensional Euclidean space $\mathbb{R}^n$. If $\lambda_i$ is the $i$-th eigenvalue of Dirichlet problem (1.1), then, for any positive integer $k$,

$$
\lambda_{k+1} - \lambda_k \leq \mathcal{E}_1(\Omega)(\lambda_2 - \lambda_1)k^{\frac{1}{2}}. \quad (7.1)
$$

**Conjecture 7.4.** Let $\Omega$ be a bounded domain with piecewise smooth boundary $\partial \Omega$ on an n-dimensional Euclidean space $\mathbb{R}^n$. If $\lambda_i$ is the $i$-th eigenvalue of Dirichlet problem (1.1), then, for any positive integer $k$,

$$
\lambda_{k+1} - \lambda_k \leq \mathcal{E}_2(\Omega)(\lambda_2 - \lambda_1)k^{\frac{1}{2}}. \quad (7.2)
$$

**Remark 7.5.** As we know, for the Dirichlet problem (1.1) on the n-dimensional Euclidean space $\mathbb{R}^n$, the gap of the consecutive eigenvalues $\lambda_{k+1} - \lambda_k$ is bounded by the first $k$-th eigenvalues in the previous literatures. However, from the above conjecture, we know that the gap of the consecutive eigenvalues is bounded only by the first two eigenvalues.

To exploit the gap coefficients, we discuss some important examples in the Euclidean space $\mathbb{R}^n$. We note that there maybe exist more examples in the complete Riemannian manifolds to be found to suppose the conjecture.

**Example 7.1.** the interval $(0, L)$

When the dimension is one, Dirichlet problem (1.1) reads:

$$
\begin{align*}
\Delta u &= -\lambda u, & \text{in} & \quad [0, L], \\
\quad u &= 0, & \text{on} & \quad \{0, L\}. 
\end{align*} \quad (7.3)
$$

It is not difficult to infer that

$$
\lambda_{k+1} - \lambda_k = \frac{((k + 1)\pi)^2}{L^2} - \frac{(k\pi)^2}{L^2} = (2k + 1)\frac{\pi^2}{L^2} \leq \frac{3\pi^2}{L^2}k = (\lambda_1 - \lambda_2)k.
$$

which implies that the conjecture (7.1) is true when the dimension $n = 1$.

**Example 7.2.** The cuboid in $\mathbb{R}^n$
Assume that \( n \geq 2 \), and \( \Sigma_0(\subset \Omega \subset \mathbb{R}^n) \) is a cuboid satisfying
\[
V(\Sigma_0) = \sup_{\Sigma \in \Omega} V(\Sigma).
\]
We define the gap coefficient as follows:
\[
S_1(\Omega) = \lambda_2(\Sigma_0) - \lambda_1(\Sigma_0),
\]
where \( \lambda_2(\Sigma_0) \) and \( \lambda_1(\Sigma_0) \) are the first eigenvalue and the second eigenvalue of the Dirichlet problem (1.1) of Laplacian on the cube \( \Sigma_0 \subset \mathbb{R}^2 \), respectively. Under the above assumptions, we present the following:

**Conjecture 7.6.** Let \( \Omega \) be a bounded domain with piecewise smooth boundary \( \partial \Omega \) on an \( n \)-dimensional Euclidean space \( \mathbb{R}^n \). If \( \lambda_i \) is the \( i \)-th eigenvalue of Dirichlet problem (1.1), then
\[
\lambda_{k+1} - \lambda_k \leq S_1(\Omega)k^\frac{1}{n}.
\]

**Remark 7.7.** Suppose that \( \Omega \) is an arbitrary cuboid, and \( \Omega_* \) is cube with the same volume as \( \Omega \), i.e., \( V(\Omega) = V(\Omega_*) \). Let \( R_* \) be the inscribe radii of the cube \( \Omega_* \). Assume that \( \lambda_1(\Omega_*) \) and \( \lambda_2(\Omega_*) \) are the first eigenvalue and the second eigenvalue of the Dirichlet problem (1.1) of Laplacian on the cuboid \( \Omega_* \subset \mathbb{R}^2 \), respectively. Then, it is easy to check that
\[
\lambda_2(\Omega_*) - \lambda_1(\Omega_*) = \frac{3n\pi^2}{D^2(\Omega_*)}.
\]
Then, we have the following eigenvalue inequality (see [31]):
\[
(\lambda_2 - \lambda_1)R_0^2 \leq (\lambda_2(\Omega_*) - \lambda_1(\Omega_*))R_*^2,
\]
where \( R_0 \) denotes the inradius of \( \Omega \). Assume that the eigenvalues of the Dirichlet problem (1.1) of Laplacian on the cuboid \( \Omega \subset \mathbb{R}^2 \) satisfying (7.1). Then, from (7.5) and (7.6), we have
\[
\lambda_{k+1} - \lambda_k \leq (\lambda_2(\Omega_*) - \lambda_1(\Omega_*))\frac{R_*^2}{R_0^2}k^\frac{1}{n}
\]
\[
= \frac{3n\pi^2}{D^2(\Omega_*)}\frac{R_*^2}{R_0^2}k^\frac{1}{n}.
\]

We assume that \( \Omega \) is the open \( n \)-dimensional rectangle \( \Omega = (0, a_1) \times \cdots \times (0, a_n) \subset \mathbb{R}^n \), then, for the Dirichlet eigenvalue problem on \( \Omega \), the eigenvalues are given by the collection \( \{\lambda_{k_1 \cdots k_n}\} \), where
\[
\lambda_{k_1 \cdots k_n} = \left( \frac{k_1^2}{a_1^2} + \cdots + \frac{k_n^2}{a_n^2} \right)\pi^2
\]
and each \( k_j, j = 1, \cdots, n \), ranges over the positive integers. For any fixed value of \( a_i, i = 1, 2, \cdots, n \), we can arrange all of the eigenvalues in order of size such that
\[ \{ \lambda_i \}_{i=1}^{+\infty} = \{ \lambda_{k_1 \cdots k_n} | k_j \in \mathbb{N}^+, j = 1, 2, \ldots, n \} . \]

Assume that \( a_1 = a_2 = \cdots = a_n, k \leq 100. \) By a direct calculation, one can obtain the fundamental gap:

\[ \lambda_2 - \lambda_1 = \frac{3n\pi^2}{D^2(\Omega)}. \]  (7.8)

Furthermore, by the numerical calculation, one can easily check that conjecture (7.6) is true. This is, by (7.8), we can prove the following

**Proposition 7.8.** Let \( \Omega \) be a cube on the n-dimensional Euclidean space \( \mathbb{R}^n. \) If \( \lambda_i \) is the i-th eigenvalue of Dirichlet problem (1.1), then, for any \( k \leq 100, \)

\[ \lambda_{k+1} - \lambda_k \leq \frac{3\pi^2}{[d(\Omega)]^2} k^{\frac{1}{n}}, \]  (7.9)

where \( d(\Omega) \) denotes the length of side of the cube \( \Omega. \)

**Example 7.3.** The triangle in \( \mathbb{R}^2 \)

Assume that \( \Omega \) is a triangle on the plane \( \mathbb{R}^2, \) many mathematicians investigated the bounds for the eigenvalues of the Dirichlet problem (1.1) of Laplacian on \( \Omega, \) for example, in \([18, 21, 25, 26, 29-31]\).

In particular, for any triangle \( \Omega \subset \mathbb{R}^2 \) with diameter \( D(\Omega), \) Lu and Rowlett \([25]\) obtained a sharp lower bound of the fundamental gap as follows:

\[ \lambda_2 - \lambda_1 \geq \frac{64\pi^2}{9D^2(\Omega)}, \]

where equality holds if and only if \( \Omega \) is equilateral, which affirmatively answers to a conjecture proposed by Antunes-Freitas in \([2]\). Furthermore, we assume that \( \Pi_0 \subset \Omega \subset \mathbb{R}^2 \) is an equilateral triangle satisfying

\[ V(\Pi_0) = \sup_{\Pi \subset \Omega} V(\Pi). \]

Define the gap coefficient as follows:

\[ S_2(\Omega) = \lambda_2(\Pi_0) - \lambda_1(\Pi_0) = \frac{64\pi^2}{9D^2(\Pi_0)}, \]

where \( D(\Pi_0) \) denotes the diameter of the domain \( \Pi_0. \) Under those assumptions, we similarly present the following:

**Conjecture 7.9.** Let \( \Omega \) be a bounded domain with piecewise smooth boundary \( \partial \Omega \) on an n-dimensional Euclidean space \( \mathbb{R}^n. \) If \( \lambda_i \) is the i-th eigenvalue of Dirichlet problem (1.1), then

\[ \lambda_{k+1} - \lambda_k \leq S_2(\Omega) \sqrt{k}. \]  (7.10)
Remark 7.10. Assume that $\Omega$ is an arbitrary triangle, and $\Omega_*$ is an equilateral triangle with the same volume as $\Omega$, i.e., $V(\Omega) = V(\Omega_*)$. Let $R_*$ be the inscribe radii of the equilateral triangle $\Omega_*$. Suppose that $\lambda_1(\Omega_*)$ and $\lambda_2(\Omega_*)$ are the first eigenvalue and the second eigenvalue of the Dirichlet problem (1.1) of Laplacian on the equilateral triangle $\Omega_* \subset \mathbb{R}^2$, respectively. Then, we have

$$\lambda_2(\Omega_*) - \lambda_1(\Omega_*) = \frac{64\pi^2}{9D^2(\Omega_*)}.$$  \hfill (7.11)

In [31], B. Siudeja proved the following eigenvalue inequality:

$$\lambda_2 - \lambda_1 \leq \frac{\lambda_2(\Omega_*) - \lambda_1(\Omega_*)}{R_*^2},$$  \hfill (7.12)

where $R_0$ denotes the inradius of $\Omega$. Assume that the eigenvalues of the Dirichlet problem (1.1) of Laplacian on a triangle $\Omega \subset \mathbb{R}^2$ satisfying (7.10). Then, from (7.11) and (7.12), we have

$$\lambda_{k+1} - \lambda_k \leq \frac{\lambda_2(\Omega_*) - \lambda_1(\Omega_*)}{R_*^2} \sqrt{k} = \frac{64\pi^2}{9D^2(\Omega_*)} \frac{R_*^2}{R_0^2} \sqrt{k}.$$  \hfill (7.13)

Next, we suppose that $\Omega$ is the open equilateral triangle in the 2-dimensional Euclidean space $\mathbb{R}^2$, then, for the Dirichlet eigenvalue problem on $\Omega$, the eigenvalues are given by the collection $\{\lambda_{mn} | m, n \in \mathbb{N}^+\}$, where

$$\lambda_{m,n} = \frac{16\pi^2(m^2 + mn + n^2)}{9D^2(\Omega)},$$  \hfill (7.14)

and the positive integers $m$ and $n$ range over the set of positive integer $\mathbb{Z}^+$. According to the size of the eigenvalues, these elements of the set $\{\lambda_{mn}\}$ can be put in increasing order such that

$$\{\lambda_i\}_{i=1}^{+\infty} = \{\lambda_{mn} | m, n \in \mathbb{N}^+\}.$$  

It is easy to see that the spectral structure of the equilateral triangle hinges upon the number theoretic properties of the binary quadratic form $m^2 + mn + n^2$. Therefore, from the point of view of number theory, it is very difficult to obtain the estimates for the gap of the eigenvalues. However, according to the numerical calculation, it is not difficult to check that conjecture 7.9 is true for any $k \leq 100$, i.e., noticing (7.11), one can prove the following:

**Proposition 7.11.** Assume that $\Omega$ is an equilateral triangle on the plane $\mathbb{R}^2$, then the eigenvalues of the Dirichlet problem (1.1) of Laplacian satisfy the inequality:

$$\lambda_{k+1} - \lambda_k \leq \frac{64\pi^2}{9D^2(\Omega)} \sqrt{k},$$  \hfill (7.15)

for any $k \leq 100$. 38
Remark 7.12. In fact, the order $k$ can be less than any finite positive integer in proposition\[7.8\] and proposition\[7.11\]. However, if the condition $k \leq 100$ is removed, then, from the point of view of number theory, we shall encounter an essential difficulty in the proofs of proposition\[7.8\] and proposition\[7.11\].

**Example 7.4.** the $n$-dimensional Ball $\mathbb{B}^n$ in $\mathbb{R}^n$

Suppose that $\Omega_*$ is an $n$-dimensional ball with the same volume as $\Omega$, i.e., $Vol(\Omega) = Vol(\Omega_*)$. Let $\lambda_1(\Omega_*)$ and $\lambda_2(\Omega_*)$ denote the first eigenvalue and the second eigenvalue of the Dirichlet problem of Laplace operator on $\Omega_*$, respectively. Recall that the famous Panye-Pólya-Weinberger conjecture (cf. [6,7,27,28,33]) is to say that, the ratios of the consecutive eigenvalues of Dirichlet problem (1.1) satisfy the following

$$\frac{\lambda_{k+1}}{\lambda_k} \leq \left( \frac{j_n/2,1}{j_n/2-1,1} \right)^2,$$

where $j_{p,k}$ is the $k$-th positive zero of the Bessel function $J_p(t)$. In particular, when $k = 1$, (7.16) becomes

$$\frac{\lambda_2}{\lambda_1} \leq \left( \frac{j_n/2,1}{j_n/2-1,1} \right)^2,$$

which is solved by Ashbaugh and Benguria([3–5]).

**Remark 7.13.** Let $\Omega$ be a bounded domain with piecewise smooth boundary $\partial \Omega$ on an $n$-dimensional Euclidean space $\mathbb{R}^n$. Suppose that $\lambda_i$ is the $i$-th eigenvalue of Dirichlet problem (1.1) and satisfies

$$\lambda_{k+1} - \lambda_k \leq (\lambda_2 - \lambda_1)k^{\frac{1}{2}},$$

then, by (7.17), we can obtain the following estimate for the gap:

$$\lambda_{k+1} - \lambda_k \leq \lambda_1 \left( \frac{j_n/2,1}{j_n/2-1,1} - 1 \right)k^{\frac{1}{2}}.$$

Therefore, eigenvalue inequality (7.19) can be viewed as an algebraic inequality of the Panye-Pólya-Weinberger type in the sense of the version of the gap of the consecutive eigenvalues. Obviously, it is a universal inequality.

Assume that $R_0$ is the supremum of the radii among all of the disks contained in $\Omega$ and $R_*$ is the radii of the ball $\Omega_*$, then we have (see [4,31])

$$\lambda_{k+1} - \lambda_k \leq (\lambda_2(\Omega_*) - \lambda_1(\Omega_*)\frac{R_*^2}{R_0^2})k^{\frac{1}{2}},$$

which implies

$$\lambda_{k+1} - \lambda_k \leq \left( \frac{j_n^2/2,1 - j_n^2/2-1,1}{R_0^2} \right)k^{\frac{1}{2}}.$$

(7.20)
In [32], Singer-Wong-Yau-Yau obtained the following:

\[
\lambda_2 - \lambda_1 \leq \frac{n\pi^2}{R_0^2}. \tag{7.21}
\]

Therefore, by utilizing (7.21), we yield

\[
\lambda_{k+1} - \lambda_k \leq \frac{n\pi^2}{R_0^2} k^{\frac{1}{n}}.
\]

Acknowledgment. The author would like to thank Professor Zuoqin Wang for his interest and helpful discussions. In particular, the author would like to express his gratitude to professor Chiu-Yen Kao and Braxton Osting for their useful comments and presenting an counterexample on the conjecture in the previous version of this paper. The author is supported by the National Nature Science Foundation of China (Grant No. 11401268).

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