We present a formulation for the internal motion of equilibrium configurations with a rotational Killing vector in general relativity. As an approximation, this formulation is applicable to investigation of the internal motion of quasi-equilibrium configurations such as binary neutron stars. Based on this simple formulation, a condition for the general relativistic counter rotation has been obtained, though in the recent work by Bonazzola, Gourgoulhon and Marck, their condition for the counter rotation is not enough to specify the internal velocity field. Under the condition given in this paper, the internal velocity field can be determined completely. Indeed, in the counter-rotating case, we have also derived Poisson equations for the internal velocity, which take tractable forms in numerical implementation.

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I. INTRODUCTION

Kilometer-size interferometric gravitational wave detectors, such as LIGO and VIRGO and TAMA are now under construction, which are increasing our expectation of direct detection of gravitational waves. Coalescing binary neutron stars are among the most promising sources for such detectors. The main reasons are that (1) we can expect to detect the signal of coalescence of binary neutron stars about several times per year, and (2) the waveform from coalescing binaries can be predicted with a high accuracy compared with other sources. Informations carried by gravitational waves tell us not only various physical parameters of neutron stars, but also the cosmological parameters if and only if we can make a detailed comparison between the observed signal with theoretical prediction during the epoch of the so-called inspiraling phase where the orbital separation is much larger than the radius of component stars.

To obtain such a large amount of informations of the gravitational waves, we must understand the theoretical mechanism of merging and construct templates for signals a priori. When the orbital separation of binary neutron stars is \( \leq 10GM/c^2 \), where \( M \) is the total mass of binary neutron stars and \( c \) is the light velocity, they move approximately in circular orbits because the timescale of the energy loss due to gravitational radiation \( t_{GW} \) is much longer than the orbital period \( P \) as

\[
\frac{t_{GW}}{P} \approx 15 \left( \frac{dc^2}{10GM} \right)^{5/2} \left( \frac{M}{4\mu} \right)^{3/2}.
\]

where \( \mu \) and \( d \) are the reduced mass and the separation of binary neutron stars. (As for higher order corrections, see for instance.) Thus, binary neutron stars evolve adiabatically radiating gravitational waves. As the orbital separation approaches \( 6GM/c^2 \), however, they cannot be described by the circular orbital motion because of instabilities due to the general relativistic gravity and/or the tidal field. As a consequence of such instabilities, binary neutron stars will plunge to merge. Thus, it is expected that the property of the signal of gravitational waves changes drastically around this transition epoch. This means that gravitational waves emitted at this transition region may bring us an important information about the internal structure of neutron stars, since the location where the instability occurs, the so-called innermost stable circular orbit (ISCO) significantly depends on the equation of state of neutron stars.

A strategy to search the ISCO accurately is as follows: Close binary neutron stars have two dominant time scales, the orbital period and the time scale of the energy loss. The former is much shorter than the latter according to Eq. (1.1). From this physical point of view, we may consider that binary neutron stars evolve in the quasi-stationary manner, so that we can take the following procedure: first, we neglect the effect due to gravitational radiation and construct equilibrium configuration, and then the radiation reaction is taken into account as a correction to the equilibrium
configuration. However, how to separate the stationary part from the non-stationary part is one of the most important but difficult problems in general relativity. For instance, Detweiler proposed that a stationary solution of the Einstein equation with standing gravitational waves will be constructed by adding incoming waves from infinity, and may be a valuable approximation to physically realistic solutions [1]. However, these solutions are not asymptotically flat because the total energy of gravitational waves inside a radius \( r \) grows linearly with \( r \). Thus, the boundary condition for them is significantly different from that of physical solutions.

In the numerical approach, Wilson and Mathews [17] proposed a semi-relativistic approximate method in order to obtain the equilibrium configurations of binary neutron stars just before merging. In their method, they assumed the spatially conformal flat metric for binary neutron stars. Practically, following their approach, numerical calculations have been done by some people [18–20]. However binary neutron stars are genuinely anisotropic, so that the assumption of the spatially conformal flatness should break down at some level. As a result, a lot of debates are continuing [21–24].

In the post-Newtonian approximation, the metric and the material quantities are expanded with respect to \( c^{-1} \) assuming the slow motion and weak gravity [25–29]. In this approximation, we can identify the radiation reaction terms which begin at the 2.5PN order [29]. Thus, it is possible to construct the equilibrium configuration of binary neutron stars in the 2PN approximation [30,31]. Asada and Shibata presented a formalism to obtain equilibrium configurations of uniformly rotating fluid at the 2PN order [31]. In this formalism, the equations to be solved are reduced to Poisson equations. This formulation is applicable to construction of non-axisymmetric uniformly rotating equilibrium configurations which include synchronized binary neutron stars and the Jacobi ellipsoid.

Nevertheless, owing to Kochanek’s investigation, the viscosity inside close binary neutron stars may be significantly small, so that synchronization may not be a good approximation [32]. Rather, irrotation (counter rotation) may be preferred for close binary neutron stars. Recently, Bonazzola, Gourgoulhon and Marck presented a formulation of the general relativistic Euler equation for perfect fluid hydrodynamics under the maximal slice [33]. Especially, in the case of the counter rotation, they gave Poisson equations to determine internal velocity fields, where they put only one condition on some scalar so that the counter rotation could be chosen from various kinds of internal motions allowed by the equation of motion. However, in order to specify the internal spatial velocity field which can be expressed as an essentially three vector, we must adopt three conditions. Indeed, although one of their equations (Eq.(47) in [33]) is the Poisson equation for the internal velocity, it turns out to be the equation projected perpendicularly to the internal velocity. This will be described in detail in section 7. Therefore, only their equation is insufficient to determine the velocity field. The main purpose of this paper is to investigate carefully the equations to determine the internal velocity field with respect to the orbital motion, and present the equations to determine completely the internal velocity field in the case of the general relativistic counter rotation. In section 2, we assume the rotational Killing vector and introduce basic quantities. We consider the conservation law with the above point in mind in section 3. In section 4, the counter-rotating case is considered. In section 5, some basic equations are derived, and based on this formulation, the condition for the counter rotation is presented. Next, equations to determine completely the internal velocity field are derived in section 6. In Section 7, we discuss the relation with this formulation with the previous work. Section 8 summarizes the conclusions. Appendix shows the relation between the counter rotation and the irrotation.

We use the units of \( c = G = 1 \) in this paper. Greek and Latin indices take 0, 1, 2, 3 and 1, 2, 3, respectively.

II. KILLING VECTOR AND TWO FRAMES

A. Killing vector

Motivated by the idea discussed in the Introduction, let us consider the equilibrium configurations by assuming the Killing vector such as

\[ l^\alpha = t^\alpha + \Omega m^\alpha, \tag{2.1} \]

where \( t^\alpha = \left( \frac{\partial}{\partial t} \right)^\alpha \) and \( m^\alpha = \left( \frac{\partial}{\partial \phi} \right)^\alpha \). If one uses the Cartesian coordinate, \( m^\alpha \) may be expressed as

\[ m^\alpha = -y \left( \frac{\partial}{\partial x} \right)^\alpha + x \left( \frac{\partial}{\partial y} \right)^\alpha. \tag{2.2} \]

The Killing equation is

\[ \mathcal{L}_l g_{\alpha\beta} = \nabla_\alpha l_\beta + \nabla_\beta l_\alpha = 0, \tag{2.3} \]

where \( \mathcal{L} \) denotes the Lie derivative.
We consider a foliation of the spacetime described by the hypersurface \( (\Sigma_t) \) whose unit normal vector is

\[
\begin{align*}
  n_\mu &= (-N, 0), \\
  n^\mu &= \left( \frac{1}{N}, -\frac{N^i}{N} \right),
\end{align*}
\]

where \( N \) and \( N^i \) are the lapse function and the shift vector respectively. The line element is expressed as

\[
ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt),
\]

where our sign convention for the shift vector is opposite to that of Bonazzola et al. We consider the internal motion which may deviate from the orbital motion (co-rotation). Therefore, it is convenient to introduce two frames for the purpose of the detailed investigation: one is the non-rotating frame and the other is the co-rotating frame.

(1) non-rotating frame

At first, we consider the non-rotating frame as follows: The temporal vector \( t^\alpha \) is related with the unit normal vector \( n^\alpha \) as

\[
t^\alpha = N n^\alpha + N^\alpha.
\]

Then, in this frame, we rewrite the Killing vector \( l^\alpha \) as

\[
l^\alpha = N n^\alpha + B^\alpha,
\]

where we defined \( B^\alpha \) as

\[
B^\alpha = N^\alpha + \Omega m^\alpha.
\]

We define the extrinsic curvature as

\[
K_{\alpha\beta} = -(P_n)^\mu_\alpha (P_n)^\nu_\beta \nabla_\mu n_\nu,
\]

where the projection tensor \( (P_n)^\beta_\alpha \) was defined as

\[
(P_n)^\beta_\alpha = \delta^\beta_\alpha + n_\alpha n^\beta.
\]

Then, we obtain

\[
\nabla_\alpha n_\beta = -K_{\alpha\beta} - n_\alpha D_\beta \ln N,
\]

where \( D_\alpha \) denotes the spatial covariant derivative defined on \( \Sigma_t \) and we used \( n_\mu dx^\mu = -N dt \).

(2) co-rotating frame

The decomposition with respect to \( l^\alpha \) is useful to describe the internal motion relative to the co-rotation along \( l^\alpha \). Let us introduce the unit tangent vector along \( l^\alpha \) as

\[
v^\alpha = e^{-\Phi} l^\alpha,
\]

and the projection tensor \( (P_v)^\beta_\alpha \) as

\[
(P_v)^\beta_\alpha = \delta^\beta_\alpha + v_\alpha v^\beta.
\]

Thus, in the co-rotating frame, we use the projection tensor \( (P_v)^\beta_\alpha \), in place of \( (P_n)^\beta_\alpha \) in the non-rotating frame. From Eq.(2.13), \( v_\alpha v^\alpha = -1 \) becomes

\[
e^{2\Phi} = N^2 - B_\alpha B^\alpha.
\]

From Eq.(2.13), we obtain

\[
\nabla_\beta v_\alpha = \omega_{\alpha\beta} - v_\beta \nabla_\alpha \Phi,
\]

where using the anti-symmetrization \( \{ \} \) we defined

\[
\omega_{\alpha\beta} = -(P_v)^\mu_\alpha (P_v)^\nu_\beta \nabla_{[\mu} v_{\nu]}.
\]
Here, introducing
\[ \varepsilon^\alpha\beta\gamma = \nu_\mu \epsilon^{\alpha\mu\beta\gamma}, \]  
(2.18)
we define the vorticity of \( v^\alpha \) as
\[ \omega^\alpha = -\frac{1}{2} \varepsilon^{\alpha\mu\nu} \omega_{\mu\nu}, \]  
(2.19)
where we used
\[ \varepsilon^\alpha\beta\gamma \varepsilon^{\mu\nu\gamma} = (P_v)_{\alpha}^\mu (P_v)_{\beta}^\nu - (P_v)_{\nu}^\nu (P_v)_{\alpha}^\mu. \]  
(2.20)

### III. Conservation Law

#### A. Adiabatic Flow

Since we consider the perfect fluid in this paper, the stress energy tensor is written as
\[ T^{\mu\nu} = (e + P)u^\mu u^\nu + Pg^{\mu\nu}. \]  
(3.1)
We decompose the four velocity in terms of \( v^\alpha \) as
\[ u^\mu = \Gamma (v^\mu + V^\mu), \]  
(3.2)
where we defined \( \Gamma \) and \( V^\mu \) as
\[ \Gamma = -v_\mu u^\mu, \]  
(3.3)
\[ V^\alpha = \Gamma^{-1} (P_v)^\mu_\alpha u^\mu. \]  
(3.4)
Then, from \( u_\mu u^\mu = -1 \), we obtain
\[ \Gamma = (1 - V_\mu V^\mu)^{-1/2}. \]  
(3.5)
Next, for the adiabatic flow, the first law of the Thermodynamics leads to
\[ \nabla_\alpha e \frac{e + P}{e + P} = \nabla_\alpha n \frac{n}{n}, \]  
(3.6)
where we assumed the barotropic equation of state, \( e = e(n) \) and \( P = P(n) \). Instead of the specific enthalpy
\[ H = \frac{e + P}{m_B n}, \]  
(3.7)
where \( m_B \) is the baryon rest mass, we use the logarithmic enthalpy as
\[ h = \ln H. \]  
(3.8)
Eq. (3.6) is rewritten as
\[ \nabla_\alpha h = \frac{\nabla_\alpha P}{e + P}. \]  
(3.9)
B. conservation law

Here, we consider the conservation law \[33\]. From its decomposition with respect to \(v_\nu\), we obtain

\[
\nabla_\mu V^\mu + V^\nu \nabla_\nu \left( \ln \left( \Gamma^2 n \right) + \Phi + h \right),
\]

(3.10)

and

\[
\nabla_\mu (V^\nu V^\alpha) + 2 \omega^\alpha_\mu V_\mu + \nabla^\alpha \Phi + \Gamma^{-2} \nabla^\alpha h
+ V^\alpha V^\mu \nabla_\mu \left( \ln \left( \Gamma n \right) - \Phi \right) = 0.
\]

(3.11)

Moreover, by decomposing Eq.(3.11) with respect to \(V_\alpha\), we obtain

\[
V^\mu \nabla_\mu (\ln \Gamma + \Phi + h) = 0,
\]

(3.12)

which can be taken as the expression for a general relativistic version of the Bernoulli theorem. Namely, \(\ln \Gamma + \Phi + h\) is constant along the internal flow \(V_\alpha\). Then, Eq.(3.10) becomes

\[
\nabla_\mu V^\mu + V^\mu \nabla_\mu \ln \left( n \Gamma \right) = 0,
\]

(3.13)

which is exactly the equation for the baryon number conservation. Furthermore, we obtain the general relativistic analogue of the Euler’s equation as

\[
V^\mu \nabla_\mu V^\alpha + 2 \varepsilon^{\alpha \mu \nu} \omega_\mu V_\nu + \nabla^\alpha \Phi + \Gamma^{-2} \nabla^\alpha h - V^\alpha V^\mu \nabla_\mu \Phi = 0,
\]

(3.14)

where we used Eq.(3.12).

In order to obtain simple expressions later, it will be useful to introduce the vorticity of the internal motion as

\[
\Omega^\alpha = \frac{1}{2} \varepsilon^{\alpha \mu \nu} \nabla_\mu V_\nu.
\]

(3.15)

Using the internal vorticity, we obtain the relation

\[
V^\mu \nabla_\mu V^\alpha = \Gamma^{-2} \nabla^\alpha \ln \Gamma + 2 \varepsilon^{\alpha \mu \nu} \Omega_\mu V_\nu,
\]

(3.16)

where we used Eq.(3.3). Using this relation, Eq.(3.14) becomes

\[
\nabla^\alpha \left( \ln \Gamma + \Phi + h \right) + \Gamma^2 \left( 2 \varepsilon^{\alpha \mu \nu} (\Omega_\mu + \omega_\mu) V_\nu + V_\mu V^\mu (P_V)_\nu^\alpha \nabla_\nu \Phi \right) = 0,
\]

(3.17)

where we defined the projection operator with respect to the internal velocity as

\[
(P_V)_\beta^\alpha = \delta_\beta^\alpha - \frac{V_\alpha V^\beta}{V_\mu V^\mu}.
\]

(3.18)

IV. COUNTER-ROTATION

In order to simplify the expression, we introduce the scalar \(\mathcal{G}\) as

\[
\mathcal{G} = \ln \Gamma + \Phi + h.
\]

(4.1)

This can be re-expressed as

\[
\mathcal{G} = \ln \left( -u_\nu l^\nu H \right).
\]

(4.2)

Then, Eqs. (3.12) and (3.17) become respectively

\[
V^\mu \nabla_\mu \mathcal{G} = 0,
\]

(4.3)
and

$$2\varepsilon_{\alpha\mu\nu}(\Omega^{\mu} + \omega^{\mu})V^{\nu} + V_{\mu}V^{\mu}(P_{V})_{\alpha}^{\nu} \nabla_{\nu} \Phi + \Gamma^{-2} \nabla_{\alpha} G = 0. \quad (4.4)$$

It should be noted that we do not need to solve Eq. (4.3), since it is obtained by contracting Eq. (4.4) with $V^\alpha$. In the Newtonian limit, the counter rotation means

$$\Omega_{\mu} + \omega_{\mu} \rightarrow 0. \quad (4.5)$$

Therefore, a simple condition for the counter rotation would be

$$\Omega_{\mu} + \omega_{\mu} = 0. \quad (4.6)$$

In this case, however, Eq. (4.4) becomes

$$\Gamma^{2}V_{\mu}V^{\mu}(P_{V})_{\alpha}^{\nu}\nabla_{\nu} \Phi + \nabla_{\alpha} G = 0, \quad (4.7)$$

for which such a $G$ does not always exist.

It seems difficult to find out conditions for the internal motion in the counter-rotating case from Eq. (4.4). Therefore, it is desirable to change Eq. (4.4) into the form easier to investigate. Indeed, we can reformulate the equation as follows: First, we should note the relation

$$\Omega^{\alpha} + \frac{1}{2}\varepsilon^{\alpha\mu\nu}V_{\mu} \nabla_{\nu} \Phi = \tilde{\Omega}^{\alpha}, \quad (4.8)$$

where we defined the *renormalized* vorticity as

$$\tilde{\Omega}^{\alpha} = \frac{1}{2}e^{\Phi}\varepsilon^{\alpha\mu\nu}\nabla_{\mu}(e^{-\Phi}V_{\nu}). \quad (4.9)$$

Then, we obtain

$$\tilde{\varepsilon}_{\alpha\mu\nu}\Omega^{\mu}V^{\nu} + \frac{1}{2}V_{\nu}V^{\nu}(P_{V})_{\alpha}^{\beta}\nabla_{\beta} \Phi = \tilde{\varepsilon}_{\alpha\mu\nu}\tilde{\Omega}^{\mu}V^{\nu}. \quad (4.10)$$

Using this relation, we rewrite Eq. (4.4) in the simple form as

$$\tilde{\varepsilon}_{\alpha\mu\nu} (\tilde{\Omega}^{\mu} + \omega^{\mu})V^{\nu} + \frac{1}{2}\Gamma^{-2} \nabla_{\alpha} G = 0. \quad (4.11)$$

Owing to the simple form of Eq. (4.11), we can define the counter rotation by requiring $\tilde{\Omega}^{\nu}$ such as

$$\tilde{\Omega}^{\nu} + \omega^{\nu} = 0. \quad (4.12)$$

In the Newtonian limit, this coincides with the condition for the Newtonian counter rotation. Under Eq. (4.12), Eq. (4.11) means simply

$$G = \text{constant}, \quad (4.13)$$

which satisfies a general relativistic version of the Bernoulli theorem [3.12]. It is noteworthy that the condition (4.12) satisfies the irrotation condition. (For instance, see [3.17] for the definition of the vorticity). This is shown in the appendix.

**V. BASIC EQUATIONS FOR INTERNAL VELOCITY FIELD**

**A. decomposition of $V^{\alpha}$**

In the previous section, we have presented Eq. (4.12), the condition for the general relativistic version of the counter rotation, by the use of Eq. (4.4) which is an algebraic equation for the vorticity $\Omega^{\alpha}$. For the numerical implementation, it is desirable to derive the equations for the internal velocity field in the case of the counter rotation. First, with respect to $\Sigma_t$, we decompose the internal velocity $V^{\alpha}$ into the following form
\[ V^\alpha = V_\perp^\alpha + V_\parallel^\alpha, \quad (5.1) \]

where we defined
\[ V_\perp^\alpha = (P_n)^\alpha_\beta V^\beta, \quad \text{ (5.2)} \]
\[ V_\parallel^\alpha = V_\parallel n^\alpha, \quad \text{ (5.3)} \]
\[ V_\parallel = -n_\beta V^\beta. \quad \text{ (5.4)} \]

Since the spatial vector \( v^\alpha \) expresses only three degrees of freedom in the four velocity, \( V_\parallel \) is related with \( V_\perp^\alpha \).

Indeed, from \( v_\alpha V^\alpha = 0 \), we obtain
\[ V_\parallel = \frac{1}{N}B_i V_i. \quad \text{ (5.5)} \]

For the sake of the further decomposition, we introduce the spatial alternating tensor as
\[ \epsilon^{\alpha\beta\gamma} = n_\mu \epsilon^{\mu\alpha\beta\gamma}. \quad \text{ (5.6)} \]

This new alternating tensor is related with \( \epsilon^{\alpha\beta\gamma} \) as
\[ \epsilon^{\alpha\beta\gamma} = e^{-\Phi} \left\{ n_\mu \epsilon^{\mu\alpha\beta\gamma} + B_\mu \left( n_\alpha \epsilon^{\mu\beta\gamma} + n_\beta \epsilon^{\mu\gamma\alpha} + n_\gamma \epsilon^{\mu\alpha\beta} \right) \right\}. \quad \text{ (5.7)} \]

In turn, with respect to \( n^\alpha \), we decompose the covariant derivative of the internal velocity as
\[ \nabla_\alpha v_\beta = D_\alpha V_\perp, \beta + \frac{n_\beta}{N} [B, V_\perp]_\beta + (n_\alpha K_\beta - n_\beta K_\alpha) V_\perp^n_\gamma - n_\alpha n_\beta V_\perp^n_\gamma D_\gamma \ln N + n_\beta \nabla_\alpha V_\parallel - V_\parallel K_\alpha - n_\beta D_\ln N, \quad \text{ (5.8)} \]

where we defined the Lie bracket as
\[ [B, V_\perp]^\alpha = B^\beta \nabla_\beta V_\perp^\alpha - V_\perp^\beta \nabla_\beta B^\alpha. \quad \text{ (5.9)} \]

Here, we used the following relation for the vector \( X^\alpha \) on \( \Sigma_t \)
\[ n^\beta \nabla_\beta X^\alpha = n^\alpha X^\beta D_\beta \ln N - X_\beta K^\alpha - \frac{1}{N} \ln [B, X]^\alpha, \quad \text{ (5.10)} \]

which is derived from
\[ \mathcal{L}_t X^\alpha = 0. \quad \text{ (5.11)} \]

Therefore, by the frequent use of Eq. (5.10) in Eq. (4.4), we obtain
\[ \check{\Omega}^\alpha = \frac{1}{2} e^{-\Phi} \epsilon^{\alpha\mu\nu} \left\{ N D_\mu V_\perp^\nu + B_\mu \left\{ -\frac{1}{N} [B, V_\perp]_\perp - 2 K_\nu \sigma V_\perp^\sigma - V_\parallel D_\nu \ln N + D_\nu V_\parallel \right\} + \left( N V_\perp - V_\parallel B_\mu \right) - \frac{1}{N} V_\perp B_\mu B_\nu B_\sigma D_\Phi \right\} + \frac{1}{2} e^{-\Phi} n_\alpha \epsilon^{\lambda\mu\nu} B_\lambda \left( D_\mu V_\perp^\nu + V_\perp D_\nu \Phi \right), \quad \text{ (5.12)} \]

where we defined the Lie bracket on \( \Sigma_t \) as
\[ [B, V_\perp]_\perp^\nu = B^\mu D_\mu V_\perp^\nu - V_\perp^\mu D_\mu B_\nu. \quad \text{ (5.13)} \]

From Eqs. (2.17), (2.18) and (5.7), we obtain
\[ \omega^\alpha = \frac{1}{2} e^{-2\Phi} \epsilon^{\alpha\mu\nu} \left( N D_\mu B_\nu + 2 B_\mu D_\nu N + 2 K_\nu B_\mu B_\nu \right) + \frac{1}{2} e^{-2\Phi} n_\alpha B_\sigma \epsilon^{\sigma\mu\nu} D_\mu B_\nu. \quad \text{ (5.14)} \]

From Eqs. (5.12) and (5.14), Eq. (4.12) becomes
\[ \epsilon^{\alpha\mu\nu} \left\{ N D_\mu V_\perp^\nu + B_\mu \left\{ -\frac{1}{N} [B, V_\perp]_\perp - 2 K_\nu \sigma V_\perp^\sigma - V_\parallel D_\nu \ln N + D_\nu V_\parallel \right\} + \left( N V_\perp - V_\parallel B_\mu \right) D_\nu \Phi - \frac{1}{N} V_\perp B_\mu B_\nu B_\sigma D_\Phi \right. \]
\[ + \left. e^{-\Phi} \left( N D_\mu B_\nu + 2 B_\mu D_\nu N + 2 K_\nu B_\mu B_\nu \right) \right\} + n_\alpha \epsilon^{\lambda\mu\nu} B_\lambda \left( D_\mu V_\perp^\nu + V_\perp D_\nu \Phi + e^{-\Phi} D_\mu B_\nu \right) = 0. \quad \text{ (5.15)} \]
This equation has been already decomposed manifestly with respect to $n^\alpha$. Therefore, using the spatial indices, equations to be solved can be rewritten as

$$
\hat{\epsilon}^{ijk}
\left[N D_j V_{\perp k} + B_j \left( -\frac{1}{N} [B, V_{\perp}]_{\perp k} - 2K_{kl}V_{\perp}^l - V_{\parallel}D_k \ln N + D_kV_{\parallel} \right)
+ \left(N V_{\perp j} - V_{\parallel}B_j \right)D_k \Phi - \frac{1}{N} V_{\perp j} B_k B^l D_l \Phi
+ e^{-\Phi} \left( N D_j B_k + 2B_jD_k N + 2K^l_jB_lB_k \right) \right] = 0,
\tag{5.16}
$$

and

$$
\hat{\epsilon}^{ijk} B_i \left( D_j V_{\perp k} + V_{\perp j} D_k \Phi + e^{-\Phi} D_j B_k \right) = 0.
\tag{5.17}
$$

It should be noted that we need not solve the latter equation, since it can be derived from the projection of the former onto $B_i$.

By taking the contraction of Eq.(5.16), we obtain

$$
\nabla_\alpha V^\alpha = D_\alpha V^\alpha_{\perp} + V^\alpha_{\parallel} D_\alpha \ln N - \frac{B^\alpha}{N} D_\alpha V_{\parallel} - V_{\parallel} K,
\tag{5.18}
$$

where we used the following identity for arbitrary vectors $E^\alpha$ and $F^\alpha$ on $\Sigma_t$

$$
n^\alpha [E, F]_\alpha = 0.
\tag{5.19}
$$

Thus, with the spatial indices, Eq.(3.13) becomes

$$
D_iV_{\perp}^i + V_{\perp}^i D_i \ln (\mathcal{N}n\Gamma) - \frac{V_{\parallel}^i}{N}B^i D_i \ln (\mathcal{V}_{\parallel} n\Gamma) - V_{\parallel} K = 0.
\tag{5.20}
$$

Since we have derived the basic equations (5.16) and (5.20) to be solved, all we must do is to investigate the boundary condition for these equations. Here, we concentrate our attention on conditions on the surface of the matter. By multiplying Eq.(5.20) with $n$ and evaluating it on the surface of the matter, we obtain

$$
\left( V_{\perp}^i - \frac{B^i B_{\perp}}{N^2} \right) (D_i n)|_{\text{surface}} = 0,
\tag{5.21}
$$

where we used the fact that $n$ must vanish on the surface. This is the boundary condition for Eqs.(5.16) and (5.20).

VI. POISSON EQUATIONS FOR THE INTERNAL VELOCITY FIELD

In order to reduce the problem of solving Eqs.(5.16) and (5.20) to that of solving the Poisson equations, we decompose $V_{\perp}^i$ as follows;

$$
V_{\perp}^i = \hat{\epsilon}^{ijk} D_j A_k + D^i \psi,
\tag{6.1}
$$

where in order to fix gauge freedoms in this decomposition, we adopt the Coulomb gauge

$$
D_i A^i = 0.
\tag{6.2}
$$

Applying this decomposition to Eq.(5.16), we obtain

$$
D_j D^j A^i = 3 R^i_k A^k
+ \hat{\epsilon}^{ijk} \left[ -B_j \left( \frac{1}{N^2} B_{\perp} V_{\perp}^i + \frac{V_{\parallel}}{N^2} D_k N - \frac{1}{N} D_k V_{\parallel} \right)
+ V_{\perp j} - \frac{V_{\parallel}}{N} B_j \right) D_k \Phi - \frac{1}{N^2} V_{\perp j} B_k B^l D_l \Phi
+ e^{-\Phi} \left( D_j B_k + \frac{2}{N} B_j D_k N + \frac{2}{N} K^l_j B_l B_k \right) \right],
\tag{6.3}
$$

8
where $3R^i_k$ denotes the Ricci tensor defined on $\Sigma_t$ and we used

$$\hat{\epsilon}^{ijk}D_jV_{\perp k} = -D_jD^iA^j + 3R^i_kA^k. \quad (6.4)$$

Similarly, from Eq.(5.20), we obtain

$$D_jD^j\psi = -V^i_\perp D_i\ln (N n^\Gamma) + \frac{V^i_N}{N}B^i D_i\ln (V_{\parallel n^\Gamma}) + V_{\parallel K}. \quad (6.5)$$

VII. DISCUSSION

Bonazzola et al. [33] have recently developed the formulation for the internal motion of quasi-equilibrium configurations. However, they put only one condition for a scalar, so that the counter rotation could be chosen from various kinds of internal motions. Namely, in our notation, their condition is expressed as

$$G = \text{constant}, \quad (7.1)$$

which is not enough for full specifying the internal vorticity. We can understand the reason by counting the degrees of freedom: Although the spatial velocity has three degrees of freedom, Eq.(7.1) fixes only one degree of freedom.

We can also realize the situation in the following concrete form: The equation (Eq.(50) in [33]), which was claimed to determine the internal velocity field, is rewritten as

$$(P_V)_{\beta}^\alpha (\Omega^\beta + \omega^\alpha) = -\frac{1}{2}\epsilon^{\alpha\mu\nu}V_\mu \nabla_\nu \Phi, \quad (7.2)$$

which is obtained by contracting Eq.(4.4) for $G = \text{constant}$ with $\epsilon^{\mu \nu \alpha}V_\nu$. Since this manipulation is the projection perpendicular to $V^\alpha$, the resultant equation (7.2) shows clearly the projection, though in the original form (Eq.(50) in [33]) the projection of $\Omega^\alpha$ is separated into two parts, one in the left hand side and the other in the right hand side. Therefore, this equation is not enough to determine fully the internal velocity field. On the other hand, in this paper, we have defined the counter rotation by requiring Eq.(4.12), so that the full determination of the internal motion can be achieved by Eqs.(6.3) and (6.5).

VIII. CONCLUSION

In this paper, we have presented a formulation for the internal motion of equilibrium configurations with a rotational Killing vector in general relativity. As an approximation, this formulation is applicable to investigation of the internal motion of quasi-stationary configurations such as binary neutron stars. In particular, by careful treatment, we have derived the simple equation (4.11) to determine the internal velocity relative to the orbital motion. Thus, using this simple form of the equation, we could find easily a condition (4.12) for the general relativistic counter rotation. This condition determines completely the internal velocity field. In this counter-rotating case, we have also presented Poisson equations for the internal velocity, which take tractable forms in numerical implementation. For instance, an iterative procedure has been proposed [33]. Thus, the formalism presented here will be useful to investigate quasi-equilibrium configurations for counter-rotating binary neutron stars and/or the Riemann ellipsoid. In addition, it is noteworthy that slice conditions are not specified in the present paper.

We focused our attention on the conservation law, which gives us the equation of motion. In order to construct quasi-equilibrium configurations, we must solve the Einstein equation. The post-Newtonian approximation is useful to find out the stationary parts in the Einstein equation. In the counter-rotating case, the equations to determine the metric will be derived in the second post-Newtonian approximation, in the similar manner in the synchronized case [31]. This will be done in the future.

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Here, we shall show that the counter rotation (4.12) means the irrotation (vanishing vorticity), in the spacetime with the rotational Killing vector. The vorticity is defined as (for instance, [37])

\[
\Omega_{\alpha\beta} = -P^\mu_\alpha P^\nu_\beta \nabla_{[\mu} u_{\nu]}.
\] (A1)

For the fluid with the barotropic equation of state, the vorticity is rewritten as

\[
\Omega_{\alpha\beta} = -\frac{1}{H} \nabla_{\alpha}(Hu_{\beta}),
\] (A2)

where we used the projection of the conservation law with respect to \(u^\alpha\) [33]

\[
u^\mu \nabla_\mu u^\alpha + \nabla^\alpha h + u^\alpha u^\mu \nabla_\mu h = 0.
\] (A3)

From Eqs. (2.19) and (4.9), we obtain

\[
\tilde{\Omega}_{\mu} + \omega_{\mu} = \frac{1}{2} e^\Phi \epsilon_{\mu\rho\sigma} \nabla^\rho \left(e^{-\Phi} (V^\sigma + v^\sigma)\right).
\] (A4)

From Eqs. (3.8), (4.1) and (4.13), we can take \(e^{-\Phi}\) as

\[
e^{-\Phi} = CH,
\] (A5)

where \(C\) is some constant. Then, Eq. (A4) becomes

\[
\tilde{\Omega}_{\mu} + \omega_{\mu} = \frac{1}{2} C e^\Phi \epsilon_{\mu\rho\sigma} \nabla^\rho (Hu^\sigma).
\] (A6)

Inserting Eq. (A2) into Eq. (A6), we obtain

\[
\tilde{\Omega}_{\mu} + \omega_{\mu} = -\frac{1}{2} C H e^\Phi \epsilon_{\mu\rho\sigma} \Omega^{\rho\sigma}.
\] (A7)

Thus, we find

\[
\epsilon_{\mu\alpha\beta\gamma}(\tilde{\Omega}^\mu + \omega^\mu) = -2CH e^\Phi (v_\alpha \Omega_{\beta\gamma} + v_\beta \Omega_{\gamma\alpha} + v_\gamma \Omega_{\alpha\beta}).
\] (A8)

Since the left hand side vanishes in the counter rotation, we obtain

\[
v_\alpha \Omega_{\beta\gamma} + v_\beta \Omega_{\gamma\alpha} + v_\gamma \Omega_{\alpha\beta} = 0.
\] (A9)

Contracting \(u^\beta v^\gamma\) with this, we find

\[
\Omega_{\alpha\gamma} v^\gamma = 0.
\] (A10)

Hence, contracting \(v^\gamma\) with Eq. (A8), we obtain

\[
\Omega_{\alpha\beta} = 0.
\] (A11)

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