QUANTUM SEMIGROUPS GENERATED BY LOCALLY
COMPACT SEMIGROUPS

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Abstract. Let $S$ be a subsemigroup of a locally compact group $G$, such that $S^{-1}S = G$. We consider the $C^*$-algebra $C^*_\delta(S)$ generated by the operators of translation by all elements of $S$ in $L^2(S)$. We show that this algebra admits a comultiplication which turns it into a compact quantum semigroup.

1. Introduction

In this article we construct a class of compact quantum semigroups associated to sub-semigroups of locally compact groups. The interest of these objects is in fact that they provide natural examples of bialgebras which are co-commutative and are not however duals to functions algebras. Recall that classic examples of quantum groups belong to one of the two following types: they are either function algebras, such as the algebra $C_0(G)$ of continuous functions vanishing at infinity on a locally compact group $G$, or their duals, such as the reduced group $C^*$-algebras $C^*_r(G)$. In the semigroup situation one can go beyond this dichotomy, but at the price of no more having a Hopf algebra structure.

If $S$ is a discrete semigroup, then the algebra $C^*_\delta(S)$ which we consider coincides with the reduced semigroup $C^*$-algebra $C^*_r(S)$ which has been known since long ago [6, 7, 2, 17, 12]. If $S = G$ is a locally compact group, then $C^*_\delta(S) = C^*_\delta(G)$ is the $C^*$-algebra generated by all left translation operators in $B(L^2(G))$ [10, 4]. If $G$ is moreover abelian, then $C^*_\delta(G)$ equals to the algebra $C(G_d)$ of continuous functions on the dual of the discrete group $G_d$ [10].

The new case considered in this paper concerns non-discrete non-trivial subsemigroups of locally compact groups, and our objective is to show that they admit a natural coalgebra structure. Let $G$ be a locally compact group and $S$ its sub-semigroup such that $S^{-1}S = G$.

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Set $H_S = \{ f \in L^2(G) : \text{supp} f \subset S \}$; let $E_S$ be the orthogonal projection of $L^2(G)$ onto $H_S$ and let $J_S$ be the right inverse of $E_S$, so that $E_SJ_S = \text{Id}_{H_S}$. After a preparatory Section 2, in Section 3 we define $C^*_S(S)$ as the $C^*$-algebra generated in $B(H_S)$ by the operators $T_a = E_SL_aJ_S$ over all $a \in S$, where $L_a$ is the operator of the left translation by $a$ on $L^2(G)$. In Section 4 we show that $C^*_S(S)$ admits a comultiplication $\Delta$ such that $\Delta(T_a) = T_a \otimes T_a$. The same facts are derived in parallel for the universal $C^*$-algebra $C^*(S)$ defined axiomatically via defining relations.

In the discrete case, the construction was carried out by X. Li [11] (see also [8]). The discrete abelian case was studied in detail in [1].

The known examples of nontrivial quantum semigroups are not too numerous; among them, one should mention the families of maps on finite quantum spaces [14], quantum semigroups of quantum partial permutations [3], quantum weakly almost periodic functionals [9]. Most close to ours is the class of quantum Bohr compactifications [13, 15]); both classes include the algebras $C(\hat{G}_d)$ where $G$ is an abelian locally compact group.

2. SEMIGROUP IDEALS

Let $G$ be a locally compact group, $S$ a closed subsemigroup of $G$ containing unit $e$ and such that $G = S^{-1}S$. Denote by $\mu$ the left Haar measure on $G$.

For any subset $X \subset S$ and any $p \in S$, define

\begin{equation}
(2.1) \quad pX = \{pq : q \in X\}, \quad p^{-1}X = \{q \in S : pq \in X\}.
\end{equation}

Obviously, $pS$ is a right ideal in $S$, $eS = e^{-1}S = S$ and $p^{-1}S = S$ for any $p \in S$. It is also easy to see that $p(qX) = (pq)X$ and $p^{-1}q^{-1}X = (qp)^{-1}X$ for any $X \subset S$ and all $p, q \in S$. Moreover, $p^{-1}pX = X$ due to the fact that $X$ is a subset of a group: $pq = px$ for $p, q \in S$, $x \in X$ implies $q = x$. But in general, the products $pq^{-1}$ or $p^{-1}q$ should be viewed purely formally, and $pp^{-1}X$ might differ from $X$ (see, for example, Lemma 2.2).

More precisely, denote by $F = F(S)$ the free monoid generated by $S$ and $S^{-1} \setminus S$. Any element in $F$ is a finite word with alternating symbols in $S$ and $S^{-1}$. For every $w = p_1^{\pm 1} \cdots p_n^{\pm 1} \in F$, define by induction $wX = p_1^{\pm 1}(\ldots(p_n^{\pm 1}X)\ldots)$ for $X \subset S$. If $X = S$, then $wS$ is a right ideal in $S$. Define the family of all constructible right ideals in $S$ [11]:

\[ J = \bigcap_{i=1}^n w_iS : w_i \in F \} \cup \{\emptyset\}. \]
Suppose that \( w \in F \) has the form \( w = p_1^{-1} q_1 p_2^{-1} q_2 \ldots p_n^{-1} q_n \) with \( p_j, q_j \in S \), maybe with \( p_1 = e \) or \( q_n = e \). Then it follows from the definition that \( wS \) is the set of elements \( x \) satisfying

\[
x = p_1^{-1} q_1 \ldots p_n^{-1} q_n r_{n+1},
\]

where \( r_{n+1} \in S \) and

\[
r_k = p_k^{-1} q_k \ldots p_n^{-1} q_n r_{n+1} \in S \quad \text{for all} \quad k = 1, \ldots, n.
\]

Define a homomorphism \( F \to G \): \( w \mapsto (w)_G \), by \((p^\pm 1)_G = p^\pm 1 \) for \( p \in S \). The operation of taking inverse in \( G \) induces the operation \( w \mapsto w^{-1} \) on the monoid \( F \), by \((p_1^\pm 1 \ldots p_n^\pm 1)^{-1} = p_n^\mp 1 \ldots p_1^\mp 1 \).

There is a natural injection \( G \hookrightarrow F \), i.e. for any element \( g \in G \) we fix one of its representations \( g = p^{-1} q \) and associate it with a word \( p^{-1} q \in F \).

**Lemma 2.1.** For any \( w_1, w_2 \in F \), we have \( w_1 w_2 S \subseteq w_1 S \).

*Proof.* Follows immediately from the facts that \( pS \subseteq S, p^{-1} S = S \) for any \( p \in S \). \( \Box \)

**Lemma 2.2.** For any \( w \in F \), \( wS = w w^{-1} S \).

*Proof.* We can assume that \( w \) has the form \( w = p_1^{-1} q_1 p_2^{-1} q_2 \ldots p_n^{-1} q_n \) with \( p_j, q_j \in S \), maybe with \( p_1 = e \) or \( q_n = e \). Then every \( x \in wS \) has the form (2.2), where \( r_{n+1} \in S \) and \( r_k = p_k^{-1} q_k \ldots p_n^{-1} q_n r_{n+1} \in S \) for all \( k = 1, \ldots, n \).

Now write \( x = p_1^{-1} q_1 \ldots p_n^{-1} q_n x \). In this product, \( x \in S \) and \( r_{k+1} = q_k^{-1} p_k \ldots q_1^{-1} p_1 x \in S \) for \( k = 1, \ldots, n \), as well as \( r_k = p_k^{-1} q_k \ldots p_n^{-1} q_n r_{n+1} \in S \) for \( k = 1, \ldots, n \). It follows that \( x \in w w^{-1} S \), so \( wS \subseteq w w^{-1} S \). The inverse inclusion follows from Lemma 2.1. \( \Box \)

**Lemma 2.3.** Let a word \( w \in F \) have the form \( w = w_1 w_2 \), where \( w_1, w_2 \in F \). Then \( wS = w_1 S \cap (w_1)_G w_2 S \).

*Proof.* Suppose \( w_1 = p_1^{-1} q_1 p_2^{-1} q_2 \ldots p_i^{-1} q_i, w_2 = p_{i+1}^{-1} q_{i+1} p_{i+2}^{-1} q_{i+2} \ldots p_n^{-1} q_n \) with \( p_j, q_j \in S \), maybe with \( p_1 = e \) or \( q_n = e \). Then every \( x \in wS \) satisfies (2.2). This is equivalent to the following:

\[
x = p_1^{-1} q_1 p_2^{-1} q_2 \ldots p_i^{-1} q_i r_{i+1}',
\]

\[
r_{i+1}' \in S, \quad r_k' = p_k^{-1} q_k \ldots p_i^{-1} q_i r_{i+1}' \in S \quad \text{for} \quad k = 1, \ldots, i.
\]

and

\[
x = p^{-1} q p_{i+1}^{-1} q_{i+1} p_{i+2}^{-1} q_{i+2} \ldots p_n^{-1} q_n r_{n+1},
\]

\[
x \in S, \quad r_{n+1} \in S, \quad r_k = p_k^{-1} q_k \ldots p_n^{-1} q_n r_{n+1} \in S \quad \text{for} \quad k = i+1, \ldots, n.
\]
where $p^{-1}q = (w_1)_G$. The first part of the conditions above is the same as condition (2.2) for $x \in w_1S$, and the second is (2.2) for $x \in (w_1)_Gw_2S$. □

**Corollary 2.4.** For any $v, w \in F$, we have $vS \cap wS = ww^{-1}vS$.

**Proof.** Since $(ww^{-1})_G = e$, by Lemmas 2.2 and 2.3 we have

$$wS \cap vS = ww^{-1}S \cap vS = ww^{-1}S \cap (ww^{-1})_GvS = ww^{-1}vS.$$ □

It follows that

$$\mathcal{J} = \{wS| w \in F\} \cup \{\emptyset\}.$$

**3. The semigroup C*-algebras**

In what follows we assume the following property for $S$. If $X = \bigcup_{j=1}^n X_j$ for $X, X_1, \ldots, X_n \in \mathcal{J}$, then $X = X_j$ for some $1 \leq j \leq n$. This property is called the independence of constructible right ideals in $S$ [11].

Recall the definition of the full semigroup C*-algebra. Consider a family of isometries $\{v_p| p \in S\}$ and a family of projections $\{e_X| X \in \mathcal{J}\}$ satisfying the following relations for any $p, q \in S$, and $X, Y \in \mathcal{J}$.

(3.1) \hspace{1cm} v_{pq} = v_pv_q, \hspace{0.5cm} v_pe_Xv_p^* = e_X,$$$(3.2) \hspace{1cm} e_S = 1, \hspace{0.5cm} e_{\emptyset} = 0, \hspace{0.5cm} e_{X \cap Y} = e_Xe_Y.$$

The universal C*-algebra $C^*(S)$ of the semigroup $S$ is generated by $\{v_p| p \in S\} \cup \{e_X| X \in \mathcal{J}\}$.

The $C^*$-algebra $C^*(S)$ contains commutative $C^*$-algebra $D(S)$ generated by the family of projections $\{e_X| X \in \mathcal{J}\}$.

Consider the Hilbert space $L^2(G)$ with respect to $\mu$, and its subspace $H_S = \{f \in L^2(G) : \text{supp}f \subset S\}$ which is isomorphic to $L^2(S)$. Denote by $E_S$ the orthogonal projection of $L^2(G)$ onto $H_S$. And for any closed subset $X \subset G$ let $I_X \in L^2(G)$ be the characteristic function of $X$. Let $L: G \rightarrow B(L^2(G))$ be the left regular representation of $G$, i.e. for any $a, b \in G$, $f \in L^2(G)$

(3.3) \hspace{1cm} (La f)(b) = f(a^{-1}b).$$We define the left regular representation $T: S \rightarrow B(H_S)$ of the semigroup $S$ analogously to $L$. For any $a, b \in S$, $f \in H_S$ we have

(3.4) \hspace{1cm} (Ta f)(b) = f(a^{-1}b),
(3.5) \((T_a f)(b) = I_S(b)f(ab)\).

One can easily see that \(T_a\) is an isometry, \(T_a^*T_a = I\). Calculating the value of projection \(T_aT_a^*\) on any \(f \in H_S\) for \(a, b \in S\) we get
\[(T_aT_a^*)f(b) = I_S(a^{-1}b)f(b).\]

Clearly, \(a^{-1}b \in S\) if and only if \(b \in aS\), where \(aS\) is a constructible right ideal defined in the previous section. Hence the projection \(T_aT_a^*\) is an operator of multiplication by \(I_{aS}\). The map \(T\colon S \to B(H_S)\) is obviously a representation of \(S\). Let \(C^*_{\delta}(S)\) be the \(C^*\)-subalgebra in \(B(H_S)\), generated by operators \(T_a, T_b^*\) for all \(a, b \in S\). A finite product of the generators \(T_a, T_b^*\) for any \(a, b \in S\) is called a monomial.

If \(S = G\), then \(C^*_{\delta}(S) = C^*_{\delta}(G)\) is the \(C^*\)-algebra generated by all left translation operators in \(B(L^2(G))\) \([10, 4]\). If \(S\) is discrete, then \(C^*_{\delta}(S) = C^*_\nu(S)\) is the reduced semigroup \(C^*\)-algebra \([12]\).

To any monomial in \(C^*_{\delta}(S)\) (in its reduced form, i.e. not containing products of the form \(T_p^*T_p\)) we can associate a word in \(\mathcal{F}\), putting \(T_p \mapsto p, T_p^* \mapsto p^{-1}\). More generally, \[T_{p_1} T_{q_1}^* T_{p_2} T_{q_2}^* \cdots T_{p_n} T_{q_n}^* \mapsto p_1 q_1^{-1} \cdots p_n q_n^{-1} \].

And for any word \(w = p_1 q_1^{-1} \cdots p_n q_n^{-1} \in \mathcal{F}\), by \(T_w\) we denote the corresponding monomial (which may contain products of the form \(T_p^*T_p\)). Clearly, the word corresponding to \(T_w^*\) would be \(w^{-1}\).

**Lemma 3.1.** For any monomial \(T_w\), function \(f \in H_S\) and \(x \in G\) we have
\[(3.6) \quad (T_w f)(x) = I_{wS}(x) \cdot f((w^{-1})_G x)\]

**Proof.** Let \(k\) be the length of the word \(w\). For \(k = 1\), the word \(w\) is either \(a \in S\) or \(a^{-1}\). For \(f \in H_S\) we can multiply \(f(a^{-1}b)\) by \(I_S(a^{-1}b)\) in the formula \((3.4)\) and get the formula \((3.6)\) for \(w = a\). Due to the fact that \(a^{-1}S = S\), the formula \((3.5)\) implies \((3.6)\) for \(w = a^{-1}\).

Suppose \((3.6)\) is proved for \(k \leq n\) and \(w = vw'\) is a word in \(\mathcal{F}\) with the length equal to \(k + 1\), where the length of \(v\) equals 1. Then clearly, the length of \(w'\) is equal to \(k\). First assume that \(v = a \in S\) and denote \(g = T_w f\). Then for any \(x \in G\) we have
\[(T_w f)(x) = (T_a T_w f)(x) = (T_a g)(x) =
\]
\[= g(a^{-1}x) = (T_w f)(a^{-1}x) =
\]
\[= I_{wS}(a^{-1}x)f((w')^{-1}G a^{-1}x) =
\]
\[= I_{aw'}S(x)f(((aw')^{-1})G x).
\]
Now assume that \(v = a^{-1} \in S^{-1}\). Then for any \(x \in G\) we have
\[(T_w f)(x) = (T_w^* T_w f)(x) = (T_w^* g)(x) =
= I_S(x) g(ax) = I_S(x) (T_w f)(ax) = I_S(x) I_{wS} (ax) f((w^{-1} a) x) =
\]

Note that \(x \in S\) and \(ax \in w'S\) if and only if \(x \in a^{-1} w'S\).

\[= I_{a^{-1} w'S}(x) f(((a^{-1} w')^{-1}) g x) = I_{wS}(x) \cdot f((w^{-1}) g x).\]

And the formula (3.6) follows. \(\Box\)

For a monomial \(T_w\) define its index by \((w)_G \in G\). We have \(\text{ind} T_w^* = (w)_G^{-1}\) and \(\text{ind}(T_v T_w) = (v)_G (w)_G\). For any \(X \subset S\) let \(E_X\) denote the orthogonal projection of \(HS\) onto the subspace \(L^2(X)\). Clearly, \(E_X\) is a multiplier by \(I_X\).

**Corollary 3.2.** A monomial \(T_w\) in \(C^*_δ(S)\) is an orthogonal projection if and only if \(\text{ind} T_w = e\). And in this case \(T_w = E_{wS}\).

**Proof.** Let \(T_w\) be an orthogonal projection. Then \((ww)_G = (w)_G^2 = (w)_G\) and \(w^{-1} = w_G\). Hence, \(\text{ind} T_w = e\).

Suppose that \(\text{ind} T_w = e\). Then due to Lemma 3.1, \(T_w = E_{wS}\) which is an orthogonal projection. \(\Box\)

**Lemma 3.3.** Every projection \(E_X\) for \(X \in J\) is contained in \(C^*_δ(S)\) and equals \(T_{ww^{-1}}\) for some \(w \in F\).

**Proof.** By Corollary 2.4, family \(J\) consists of ideals \(wS\) for all \(w \in F\). So, \(X = wS\) for some \(w \in F\). Due to Corollary 3.2, if \((w)_G = e\) then \(E_{wS} \in C^*_δ(S)\).

Suppose \(w\) is an arbitrary element in \(F\). By Lemma 2.2, \(wS = ww^{-1}S\) and \(E_{wS} = E_{ww^{-1}S}\). Since \((ww^{-1})_G = e\), by Corollary 3.2, we have that \(E_{wS} = T_{ww^{-1}} \in C^*_δ(S)\). \(\Box\)

**Lemma 3.4.** There exists a surjective *-homomorphism \(\lambda: C^*(S) \to C^*_δ(S)\) called the left regular representation of the algebra \(C^*(S)\).

**Proof.** One can easily verify that operators \(T_p, T_q^*\) and \(E_X\) satisfy equations (3.1) and (3.2) for all \(p, q \in S, X \in J\). Universality of \(C^*(S)\) implies the existence of a homomorphism \(\lambda:\)

\[v_p \to T_p, \ e_X \to E_X.\]

\(\Box\)

Consider the Hilbert space \(L^2(G)\), and the left regular representation \(L: G \to B(L^2(G))\) of \(G\). Recall that \(E_S\) is an orthogonal projection of \(L^2(S)\) on \(L^2(S)\).
Lemma 3.5. The $C^*$-algebra $C_δ^*(S)$ is isomorphic to the $C^*$-subalgebra in $B(L^2(G))$ generated as a linear space by

$$E_w L_{(w)G} E_S$$

for all $w \in \mathcal{F}$.

Proof. By definition, $C_δ^*(S)$ is generated as a $C^*$-algebra by operators $T_a, T_b^*$ for all $a, b \in S$ in $B(H_S)$. This algebra can be isometrically embedded into $B(L^2(G))$:

$$T_a \mapsto E_S L_a E_S.$$ 

Any product of generators equals $T_w$ for some $w \in F$, and due to Lemma 3.1 its value on a function in $H_S$ can be represented as a composition of projection onto $wS$ and the shift $L_{(w)G}$. Hence, we have

$$T_w = E_w S L_{(w)G} E_S.$$ 

□

Applying the fact that $G$ is generated by $S$, we immediately obtain the following statement which connects $C_δ^*(S)$ with generators of the algebra $C_r^*(G)$.

Corollary 3.6. The $C^*$-algebra $C_δ^*(S)$ is generated by operators

$$\{E_S L_p E_S, p \in S\}$$

Denote by $D_δ(S)$ the $C^*$-subalgebra in $C_δ^*(S)$ generated by monomials with index equal to $e$. By Corollary 3.2 and Lemma 3.3 $D_δ(S)$ is generated by projections $\{E_X | X \in \mathcal{J}\}$, and is obviously commutative.

Lemma 3.7. The algebras $D(S)$ and $D_δ(S)$ are isomorphic.

Proof. The left regular representation restricted to $D(S)$ is surjective. Applying Lemma 2.20 in [11] and using the independence of constructible right ideals in $S$ we obtain injectivity of $\lambda|_{D(S)}$. □

There exists a natural action of the semigroup $S$ on the $C^*$-algebra $D_δ(S)$.

(3.7) $\tau_p(A) = T_p A T_p^*, \ p \in S, \ A \in D_δ(S).$

Using the formula (3.6), we obtain for $A = E_X, X \in \mathcal{J}$:

(3.8) $\tau_p(E_X) = E_{pX}$. 

4. The universal and reduced compact quantum semigroups

Consider the \( C^* \)-subalgebra \( \mathcal{A} \) in \( C^*(S) \otimes_{\text{max}} C^*(S) \) generated by the elements

\[
\{ v_p \otimes v_p, \; e_X \otimes e_X : p \in S, \; X \in \mathcal{J} \}.
\]

Clearly, these elements satisfy relations (3.1), (3.2). The universal property of \( C^*(S) \) implies the existence of an isomorphism \( \Delta_u : C^*(S) \to \mathcal{A} \), such that

\[
\Delta_u(v_p) = v_p \otimes v_p, \; \Delta_u(e_X) = e_X \otimes e_X.
\]

The map \( \Delta_u : C^*(S) \to C^*(S) \otimes_{\text{max}} C^*(S) \) is a unital *-homomorphism and admits a restriction \( D(S) \to D(S) \otimes_{\text{max}} D(S) \).

The pair \( \mathbb{Q}(S) = (C^*(S), \Delta_u) \) is a compact quantum semigroup [1].

We call the algebra \( C^*(S) \) with this structure the universal algebra of functions on the compact quantum semigroup \( \mathbb{Q}(S) \) associated with the semigroup \( S \).

We recall that a semigroup is called right reversible if every pair of non-empty left ideals has a non-empty intersection. The following theorem by Ore can be found in [5].

**Theorem 4.1.** A cancellative semigroup \( S \) can be embedded into a group \( G \) such that \( G = S^{-1}S \) if and only if it is right reversible.

Define a partial order on \( S \): \( p \leq q \) if there exists \( r \in S \) such that \( rp = q \). In this case we denote \( qp^{-1} = r \). Assuming that \( S \) generates the group \( G = S^{-1}S \) due to the theorem [4.1] we obtain that for any \( p, q \in S \) the left ideals \( \{ xp : x \in S \} \), \( \{ yq : y \in S \} \) have a non-empty intersection. Hence \( S \) is upwards directed with respect to this partial order.

Consider the directed system of \( C^* \)-algebras \( \mathcal{A}_p \) indexed by \( p \in S \), where every \( \mathcal{A}_p = D_\delta(S) \). For \( p, q \in S \) such that \( p \leq q \) we have \( qp^{-1} \in S \) and the action (3.7) generates the map \( \tau_{qp^{-1}} : \mathcal{A}_p \to \mathcal{A}_q \):

\[
\tau_{qp^{-1}}(A) = T_{qp^{-1}}AT^*_{qp^{-1}}.
\]

Clearly, \( \tau_{qp^{-1}} \) is a *-homomorphism \( D_\delta(S) \to D_\delta(S) \) and \( \tau_{qp^{-1}} = \tau_{qp^{-1}} \tau_{rp^{-1}} \) for \( p \leq r \leq q \). Let \( D_\delta^{(\infty)}(S) \) denote the \( C^* \)-inductive limit of the directed system \( \{ \mathcal{A}_p, \tau_{qp^{-1}} \} \).

For \( q \in S \) and \( X \in \mathcal{J} \) denote \( q^{-1} \cdot_G X = \{ q^{-1}x : x \in X \} \subset G \). Note that \( q^{-1}X = (q^{-1} \cdot_G X) \cap S \). We denote by \( E_{q^{-1} \cdot_G X} \in B(L^2(G)) \) the projection onto \( L^2(q^{-1} \cdot_G X) \subset G \).

**Lemma 4.2.** The \( C^* \)-algebra \( D_\delta^{(\infty)}(S) \) is isomorphic to

\[
D = C^*(\{ E_{q^{-1} \cdot_G X} : q \in S, \; X \in \mathcal{J} \}).
\]
Recall that $D_\delta(S) \subset B(H_\delta)$ by definition. Denote by $\pi: D_\delta(S) \rightarrow B(L^2(G))$ the restriction of the canonical embedding $B(H_\delta) \rightarrow B(L^2(G))$.

For any $p \in S$, the map

$$E_X \rightarrow E_{p^{-1}G X} \in D$$

for all $X \in \mathcal{J}$ generates a homomorphism $\phi_p: D_\delta(S) \rightarrow D$, namely

$$\phi_p(A) = L_p^* \pi(A) L_p,$$

for all $A \in D_\delta(S)$,

where $L_p$ is the operator defined by (3.3).

Then for $p \leq q$ and $X \in \mathcal{J}$ we have:

$$\phi_q \tau_{qp^{-1}}(E_X) = L_q^* (\pi(T_{qp^{-1}} E_X T_{qp^{-1}}^*)) L_q =$$

$$= L_q^* E_{qp^{-1}X} L_q = E_{q^{-1}E_{qp^{-1}X}} = E_{p^{-1}X} = \phi_p(E_X).$$

So the maps $\phi_p$ agree with $\tau_{qp^{-1}}$. The homomorphisms $\phi_p$ are injective since $\pi$ is obviously injective and $L_p$ is a unitary operator. It follows that the limit map $\Phi = \lim_{p \in S} \phi_p: D_\delta^{(\infty)}(S) \rightarrow D$ is injective.

To prove surjectivity of $\Phi$ it suffices to show that for any $q_1, \ldots, q_n \in S$, $X_1, \ldots, X_n \in \mathcal{J}$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ we have

$$\sum_i \lambda_i E_{q_i^{-1}G X_i} \in \Phi(D_\delta^{(\infty)}(S)).$$

Since the system $\{A_p, \tau_{qp^{-1}}\}$ is upwards directed, there exists $s \in S$ such that $q_i \leq s$, $i = 1, 2, \ldots, n$, and it implies $q_i^{-1} \cdot X_i = s^{-1} \cdot (s q_i^{-1} X_i)$. Hence

$$\sum_i \lambda_i E_{q_i^{-1}G X_i} \in \phi_s(D_\delta(S))$$

and we obtain

$$D = \bigcup_{p \in S} \phi_p(D_\delta(S))$$

Therefore $\Phi$ is surjective and we get the isomorphism $D_\delta^{(\infty)}(S) \cong D$. \hfill $\Box$

Recall that the left regular representation of $G$ on $L^2(G)$ is denoted by $L$, see (3.3). Let us show that $D$ is invariant under the adjoint action of $G$. For $g \in G$, we have:

$$L_g^* E_{g^{-1}G X} L_g = E_{(g^{-1}q^{-1})G X}.$$

Since $G = S^{-1} S$, we can write $g^{-1}q^{-1} = t^{-1}s$ with some $s, t \in S$. Then $s \cdot G X = s \cdot X \in \mathcal{J}$, and $E_{(g^{-1}q^{-1})G X} = E_{(t^{-1}s)G X} \in D$.

The isomorphism $\Phi$ defined above allows then to define an action of $G$ on $D_\delta^{(\infty)}(S)$: for $u \in D_\delta^{(\infty)}(S)$, $g \in G$ we set

$$(4.1) \quad \tau(g) u = \Phi^{-1}(L_g^* \Phi(u) L_g).$$
Lemma 4.3. The reduced crossed product $D^{(\infty)}_\delta(S) \rtimes_{r,\tau} G$ of the commutative C*-algebra $D^{(\infty)}_\delta(S)$ and the group $G$ by the action $\tau$, generated by the covariant representation $\Phi$, is isomorphic to the C*-algebra $C^*(\{E_X, L_g: X \in S^{-1} \cdot J, g \in G\})$.

Proof. The covariant representation $\Phi$ coincides with the regular representation $\text{Ind}\Phi$ generated by $\Phi$ (see [10] page 51). The integrated form $\Phi \rtimes L$ of $\text{Ind}\Phi$ is a faithful representation of $C_c(G, D^{(\infty)}_\delta(S))$ by Lemma 2.26 in [16]. The reduced crossed product $D^{(\infty)}_\delta(S) \rtimes_{r,\tau} G$ is the completion of the image of $C_c(G, D^{(\infty)}_\delta(S))$ by $\Phi \rtimes L$. Due to the fact that $\Phi(D^{(\infty)}_\delta(S))$ is generated by projections $E_X$ for $X \in S^{-1} \cdot J$ and $G$ is represented by unitaries $L_g, g \in G$, we get the required isomorphism.

Theorem 4.4. The algebra $C^*_\delta(S)$ is isomorphic to $E_S(\pi(D^{(\infty)}_\delta(S) \rtimes_{r,\tau} G))E_S$, where $\pi$ is the isomorphism in Lemma 4.3.

Proof. By Lemma 4.3, $E_S(\pi(D^{(\infty)}_\delta(S) \rtimes_{r,\tau} G))E_S$ is generated as a space by operators $E_S E_{q^{-1},c} L_a L_b E_S$ for $q, a, b \in S$, $X \in J$. Using the action $\tau$ of $G$ on $\Phi(D^{(\infty)}_\delta(S))$, for all $c \in S$, $X \in J$ we have

$$L_c E_X = E_{cX} L_c, \quad L_{c^{-1}} E_X = E_{c^{-1}X} L_{c^{-1}}.$$

Note that $r^{-1} \cdot G \cap S = r^{-1} X$ for $r \in S, X \in J$. Using these facts we obtain

$$E_S E_{q^{-1},c} L_a L_b E_S = E_{q^{-1}X} E_{a^{-1}b} E_S = E_{q^{-1}X} E_{a^{-1}b} E_S = E_{q^{-1}X} E_{a^{-1}b} E_S = E_{q^{-1}X} E_{a^{-1}b} E_S.$$

Consider the isomorphism in Lemma 3.5 and denote it by $\Psi$. Then by Lemma 3.5 we get

$$\Psi^{-1}(E_{a^{-1}b} E_S) = T_{a^{-1}b}.$$

The operator $E_{q^{-1}X} \in B(L^2(G))$ is the image of $E_{q^{-1}X} \in B(L^2(S))$, since $q^{-1} X \in J$. By Lemma 3.3 $E_{q^{-1}X}$ equals $T_{ww^{-1}}$ for some $w \in F$ which depends on $qX$. Thus $\Psi$ is the required isomorphism.

Theorem 4.5. There exists a comultiplication $\Delta: C^*_\delta(S) \to C^*_\delta(S) \otimes C^*_\delta(S)$, with which $Q(S) = (C^*_\delta(S), \Delta)$ is a compact quantum semigroup.
Proof. The map $\Delta_a$ defined at the beginning of the Section 4 is a comultiplication on $C^*(S)$ and admits a restriction $D(S) \to D(S) \otimes_{\max} D(S)$. By Lemma [3.7] $D_\delta(S)$ is isomorphic to $D(S)$. Hence $D_\delta(S)$ is endowed with a comultiplication, denote it by $\Delta$. To see that $\Delta$ extends to $D_\delta^\infty(S)$ let us note that $\Delta$ respects the maps $\tau_{qp^{-1}}$. Indeed, for a monomial $V \in \mathcal{A}_p$, $p \in S$ and $q \geq p$ we have

$$
\Delta \tau_{qp^{-1}}(V) = \Delta(T_{qp^{-1}}VT_{qp^{-1} \ast}) = T_{qp^{-1}}VT_{qp^{-1} \ast} \otimes T_{qp^{-1}VT_{qp^{-1} \ast}} = \tau_{qp^{-1}} \otimes \tau_{qp^{-1}}(\Delta(V))
$$

Using the Lemma [4.2] we get the formula for the comultiplication on the generators in $\pi(D_\delta^\infty(S))$, for $q \in S, X \in \mathcal{J}$:

$$
\Delta(E_{q^{-1},G,X}) = E_{q^{-1},G,X} \otimes E_{q^{-1},G,X}
$$

It follows that the comultiplication $\Delta$ commutes with the action of $G$ on $D_\delta^\infty(S)$ defined in [4.1]. Consequently, $\Delta$ gives rise to a comultiplication on $\pi(D_\delta^\infty(S) \rtimes_{r,\tau} G)$, which we also denote by $\Delta$. Due to the fact that $E_S \in \pi(D_\delta^\infty(S))$, using Lemma [4.4] we obtain the required comultiplication $\Delta$ on $C_\delta^*(S)$. \qed

Remark 4.1. The bialgebras $C^*(S)$ and $C_\delta^*(S)$ are co-commutative, as for example the group $C^*$-algebra $C^*(G)$ of $G$. But their dual algebras, unlike the Fourier-Stieltjes algebra $B(G)$, cannot be viewed as function algebras on $S$ or even on $G$. It is possible that $\phi, \psi \in (C_\delta^*(S))^\ast$ are nonequal but have the same values on $T_a$ and $T_a^\ast$ for all $a \in S$.

More specifically, consider $G = \mathbb{Z}$, $S = \mathbb{Z}_+$ and $\phi_k(T) = \langle T \delta_k, \delta_k \rangle$, $k \in \mathbb{Z}$. Then $\phi_k(T_a) = \phi_k(T_a^\ast) = \delta_0(a)$ for all $k \in \mathbb{Z}$, $a \in \mathbb{Z}_+$, but $\phi_k(T_aT_a^\ast) = I_{\mathbb{Z}_+}(k-a)$ while $\delta_0(T_aT_a^\ast) = \delta_0(a)$.

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