COBORDISM GROUP OF MORSE FUNCTIONS ON UNORIENTED SURFACES

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Abstract. Ikegami and Saeki have proved that the cobordism group of Morse functions on oriented surfaces is an infinite cyclic group. Their method is applicable with a little modification to the computation of the cobordism group of Morse functions on unoriented surfaces. We prove that this group is isomorphic to the direct sum of the infinite cyclic group and the finite group of order two.

1. Introduction

Ikegami and Saeki [2] initiated the study of the oriented cobordism group of Morse functions on oriented surfaces. Summarizing in an informal language, two Morse functions $f_0: M_0 \to \mathbb{R}$ and $f_1: M_1 \to \mathbb{R}$ on closed surfaces $M_0$ and $M_1$ are cobordant, if there exists a smooth map $F: X \to \mathbb{R} \times [0,1]$ with only fold singularities (i.e., a fold map; fold singularities are the higher dimensional analogues of the non-degenerate critical points of Morse functions; for a precise definition of a fold map, see §2) from a compact 3-manifold $X$ with $\partial X = M_0 \sqcup M_1$, and $F$ restricted to $\partial X$ is identified with $f_0 \sqcup f_1$, where $\sqcup$ denotes the disjoint union (while the normal direction is mapped onto the normal direction isomorphically). Such an $F$ is a cobordism between $f_0$ and $f_1$. One can define the cobordism group of Morse functions on surfaces, which describes the structure of Morse functions on surfaces from the viewpoint of the cobordism relation. Ikegami and Saeki [2] have transformed the problem of Morse functions and their cobordisms into a problem of maps from graphs to $\mathbb{R}$ and maps from 2-dimensional polyhedra to $\mathbb{R} \times [0,1]$. This transformation has been realized by using the Stein factorization: the Stein factorization of a cobordism between two Morse functions provides a cobordism in an abstract sense between the Stein factorizations of the Morse functions (called also the Reeb functions).

The computation of the oriented cobordism group of Morse functions on oriented surfaces [2] involves two steps:

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(1) computing the abstract cobordism group of abstract Reeb functions, which are maps of graphs to $\mathbb{R}$ like Reeb functions, where a cobordism between two such maps is a map from a compact 2-dimensional polyhedron to $\mathbb{R} \times [0,1]$ like the Stein factorization of a cobordism between two Morse functions,

(2) showing that the abstract cobordism group is isomorphic to the cobordism group of Morse functions.

This method enables us to solve the problem in a purely combinatorial context.

More precisely step (1) consists of the following. A cobordism between two Morse functions is a generic map of a compact 3-dimensional manifold into $\mathbb{R} \times [0,1]$. The local structure of the Stein factorizations of such maps is completely described in [4, 5]. Using this description, Ikegami and Saeki [2] gave an alternative description of an abstract cobordism of Reeb functions, which is more algorithmic: if we change a function on a graph by a finite iteration of simple local moves together with a homotopy, we get a function cobordant to the original function in the abstract sense. Using this observation, Ikegami and Saeki [2] gave representatives of the elements of the abstract cobordism group, and proved that it is isomorphic to $\mathbb{Z}$. In step (2) [2] realizes an abstract cobordism between Reeb functions of two Morse functions as the Stein factorization of a fold map of an oriented 3-manifold.

In this paper, we follow the same line of arguments in order to compute the cobordism group of Morse functions\footnote{The author was informed, when this paper had already been written, that Ikegami [1] had computed the analogous group in each dimension. His method is quite different and more sophisticated than our approach.} on unoriented surfaces. However, we have to modify the method of [2]: in step (1), when we want to compute the abstract cobordism group, we have to consider functions and maps coming from the Stein factorizations of Morse functions on nonorientable surfaces and those of fold maps of nonorientable 3-manifolds, respectively. Levine [5] describes these maps completely. With these new cases, we get an equivalent algorithmic description of an abstract cobordism, which differs from [2] only in some new moves on a few new types of Reeb graphs. It turns out that the method of the computation of the abstract cobordism group given in [2] is applicable in our case as well: we give a set of representatives of the elements of this group by using the algorithm of [2], and we show that the group is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_2$. Step (2) also needs some modifications. On the one hand this step becomes simpler since we are allowed to use nonorientable 3-manifolds for the realization of abstract cobordisms between Reeb functions. On the other hand this step becomes more complicated, since the method used in [2] cannot be applied here. An abstract cobordism between two given Reeb functions is a map from a compact 2-dimensional polyhedron to $\mathbb{R} \times [0,1]$. In order to realize the neighbourhood of
an appropriate 1-skeleton of this polyhedron as a Stein factorization of a fold map of a 3-
manifold $X$ with boundary, one can apply the method described in [2], but this realization
is not a cobordism between the two given Reeb functions. To obtain a cobordism we attach
2-disk bundles and their fold maps to the appropriate part of the boundary of $X$, as will
be described in §4.

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2. Preliminaries

**Definition 2.1.** Let $Q^q$ and $N^n$ be smooth manifolds of dimensions $q$ and $n$ respectively
($q \geq n$). Let $p \in Q^q$ be a singular point of $f$. A smooth map $f: Q^q \to N^n$ has a fold
singularity at the singular point $p$, if we can write $f$ in some local coordinates at $p$ and
$f(p)$ in the form

$$f(x_1, \ldots, x_q) = (x_1, \ldots, x_{n-1}, \pm x_n^2 \pm \cdots \pm x_q^2).$$

A smooth map $f: Q^q \to N^n$ is called a fold map, if $f$ has only fold singularities. A smooth
function on a manifold which has only fold singularities is called a Morse function.

Let $\sqcup$ denote the disjoint union.

**Definition 2.2.** Two Morse functions $f_0: M_0 \to \mathbb{R}$ and $f_1: M_1 \to \mathbb{R}$ on closed surfaces
$M_0$ and $M_1$ are cobordant, if there exists a smooth map $F: X \to \mathbb{R} \times [0, 1]$ with only fold
singularities from a compact 3-manifold $X$ with $\partial X = M_0 \sqcup M_1$ such that $F|_{M_0 \times [0, \epsilon]} = f_0 \times \text{id}_{[0, \epsilon]}$ and $F|_{M_1 \times (1-\epsilon, 1]} = f_1 \times \text{id}_{[1-\epsilon, 1]}$, where $M_0 \times [0, \epsilon)$ and $M_1 \times (1-\epsilon, 1]$ are small
collar neighbourhoods of $\partial X$ with the identifications $M_0 = M_0 \times \{0\}$, $M_1 = M_1 \times \{1\}$,
and $\text{id}_A(a) = a$ ($a \in A$) for an arbitrary set $A$.

We call the map $F$ a cobordism between $f_0$ and $f_1$.

This clearly defines an equivalence relation.

**Definition 2.3.** We can define a group operation on the set of the cobordism classes of
Morse functions as follows. If $[f_0: M_0 \to \mathbb{R}]$ and $[f_1: M_1 \to \mathbb{R}]$ are cobordism classes of
Morse functions, then the sum of $[f_0]$ and $[f_1]$ is the cobordism class represented by the
Morse function $f_0 \sqcup f_1: M_0 \sqcup M_1 \to \mathbb{R}$.

It is easy to show that the cobordism class of the Morse function $f_0 \sqcup f_1$ does not depend on the choice of the representatives $f_0$ and $f_1$.

This group operation is clearly commutative, so we have an abelian group, which we
denote by $\text{Cob}_{\text{f}}(2, -1)$, where the lower index “$f$” comes from “fold map”.
Remark 2.4. One can define the analogous cobordism groups $\text{Cob}_f(q,k)$ ($q \in \mathbb{N}$, $k \in \mathbb{Z}$, $-q \leq k \leq 0$) of fold maps $f: Q^q \to \mathbb{R}^{q+k}$ from closed $q$-dimensional manifolds $Q^q$ to $\mathbb{R}^{q+k}$ as well. The integer $k$ is called the codimension of the map $f$.

For $k \geq 0$ there are many results concerning the cobordism groups $\text{Cob}_\tau(q,k)$, where $\tau$ is a set of singularity types, the elements of the group $\text{Cob}_\tau(q,k)$ are cobordism classes of smooth maps with only singularities in $\tau$, and a cobordism between two such maps has only singularities in $\tau$. See for example [3, 7, 9, 10].

The aim of this paper is to prove

Theorem 2.5. The cobordism group $\text{Cob}_f(2,-1)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_2$.

We use the notion of the Stein factorization of a smooth map $f: Q^q \to N^n$, where $Q^q$ and $N^n$ are smooth manifolds of dimensions $q$ and $n$ respectively ($q \geq n$). Two points $p_1, p_2 \in Q^q$ are equivalent if and only if $p_1$ and $p_2$ lie on the same component of an $f$-fiber. Let $W_f$ denote the quotient space of $Q^q$ with respect to this equivalence relation and $q_f: W_f \to N^n$ the quotient map. Then there exists a unique continuous map $\tilde{f}: W_f \to N^n$ such that $f = \tilde{f} \circ q_f$. The space $W_f$ or the factorization of the map $f$ into the composition of $q_f$ and $\tilde{f}$ is called the Stein factorization of the map $f$. We call the map $\tilde{f}$ the Stein factorization of the map $f$ as well.

In the case of a Morse function $f: Q^q \to \mathbb{R}$ on a closed manifold $Q^q$, $W_f$ is a graph (since every regular fiber is a 1-codimensional submanifold in $Q^q$ and the set of critical points of $f$ is finite and discrete), which is called the Reeb graph, and the function $\tilde{f}: W_f \to \mathbb{R}$ is called the Reeb function of the Morse function $f$.

We may assume that the Morse function $f$ is in general position, that is, every fiber contains at most one critical point, since a small perturbation of the function $f$ is a Morse function cobordant to $f$.

3. Abstract cobordism

Let $f: M \to \mathbb{R}$ be a Morse function on a closed surface $M$. One can see (for example, see [5, 8]) that the Reeb function of $f$ in a neighbourhood of the $q_f$-image of a critical point is equivalent to one of the functions as depicted in Fig. 1. In case (a) the corresponding critical point of $f$ is a local minimum or a local maximum, in case (b) the corresponding critical point of $f$ has index 1 with sign $+1$ or $-1$, and a sign is associated to each vertex of degree three as in Fig. 1. Case (c) can occur only on a nonorientable surface. The corresponding critical point of $f$ is also of index 1. Note that the Reeb function restricted to an edge of the Reeb graph is always an embedding.

Definition 3.1. We say that a smooth map $F: X \to \mathbb{R}^2$ from a closed 3-manifold $X$ has a definite fold singularity at a singular point $p \in X$ if we can write $F$ in some local
coordinates at $p$ and $F(p)$ in the form
\[ F(x_1, x_2, x_3) = (x_1, x_2^2 + x_3^2), \]
and we say that $F$ has an indefinite fold singularity at a singular point $p \in X$ if we can write $F$ in some local coordinates at $p$ and $F(p)$ in the form
\[ F(x_1, x_2, x_3) = (x_1, x_2^2 - x_3^2). \]
We say that $F$ has a simple indefinite fold singularity at a singular point $p \in X$ if the map $F$ has an indefinite fold singularity at $p$ and each component of $F^{-1}(F(p))$ contains at most one singular point. Furthermore, $F$ has a nonsimple indefinite fold singularity at $p$ if $F$ has an indefinite fold singularity at $p$ and there is a component of $F^{-1}(F(p))$ which contains at least two singular points.

Let $F : X \to \mathbb{R}^2$ be a generic smooth map of a closed 3-manifold $X$ with only fold singularities. By \[5\] the Stein factorization $\tilde{F} : W_F \to \mathbb{R}^2$ of $F$ in a neighbourhood of the $q_F$-image of a singular point is equivalent to one of the maps as depicted in Fig. 2, where the maps in question are projections to the plane of the paper. The pictures (f)–(i) can
Figure 2. The Stein factorization of a fold map $F$ in a neighbourhood of the $q_F$-image of a singular point

occur only if $X$ is nonorientable. The lines in bold indicate the $q_F$-image of the set of singular points of $F$, which is a 1-dimensional submanifold of $X$ and is immersed into $\mathbb{R}^2$ by $F$. The dot in bold indicates the $q_F$-image of the singular point under study. Case (a) corresponds to a regular value, case (b) corresponds to the image of a definite fold singularity, case (c) corresponds to the image of a simple indefinite fold singularity, cases (d) and (e) correspond to a nonsimple indefinite fold singularity, and these exhaust all the possibilities when $X$ is orientable. Case (f) corresponds to the image of a simple indefinite fold singularity, and cases (g), (h), and (i) correspond to a nonsimple indefinite fold singularity. Details can be found in [2], [4] and [5].

Based on these observations, we give a definition of abstract Reeb functions and their cobordism group analogous to [2] as follows.
Definition 3.2. Let $G$ be a finite graph such that each of its vertices is of degree 1, 2 or 3. A continuous function $r: G \to \mathbb{R}$ is said to be an abstract Reeb function if $r$ is an embedding on each edge, and $r$ is equivalent in a neighbourhood of every vertex to one of the functions as depicted in Fig. 1.

Definition 3.3. Let $r_i: G_i \to \mathbb{R}$, $i = 0, 1$, be abstract Reeb functions in the sense of Definition 3.2. We say that $r_0$ and $r_1$ are cobordant if there exists a continuous map $R: P \to \mathbb{R} \times [0, 1]$ of a compact 2-dimensional polyhedron $P$ such that

(i) $G_i = R^{-1}(\mathbb{R} \times \{i\})$, $i = 0, 1$, are subcomplexes of $P$ with regular neighbourhoods of the forms $G_0 \times [0, \varepsilon)$ and $G_1 \times (1 - \varepsilon, 1]$, where $G_i$ corresponds to $G_i \times \{i\}$, $i = 0, 1$,

(ii) $R|_{G_0 \times [0, \varepsilon)} = r_0 \times \text{id}_{[0, \varepsilon)}$ and $R|_{G_1 \times (1 - \varepsilon, 1]} = r_1 \times \text{id}_{[1 - \varepsilon, 1]}$,

(iii) in a neighbourhood of each point of $P \setminus (G_0 \cup G_1)$, the polyhedron $P$ and the map $R$ are equivalent to one of the pictures as depicted in Fig. 2.

Furthermore, we call the map $R: P \to \mathbb{R} \times [0, 1]$ a cobordism between $r_0$ and $r_1$.

This clearly defines an equivalence relation.

Definition 3.4. We can define a group operation on the set of the cobordism classes of abstract Reeb functions as follows. If $[r_0]: G_0 \to \mathbb{R}$ and $[r_1]: G_1 \to \mathbb{R}$ are cobordism classes of abstract Reeb functions, then the sum of $[r_0]$ and $[r_1]$ is the cobordism class represented by the abstract Reeb function $r_0 \amalg r_1: G_0 \amalg G_1 \to \mathbb{R}$.

It is easy to show that the cobordism class of the abstract Reeb function $r_0 \amalg r_1$ does not depend on the choice of the representatives $r_0$ and $r_1$.

Let us denote the cobordism group of abstract Reeb functions by $\mathcal{A}$.

Proposition 3.5. The cobordism group $\mathcal{A}$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_2$.

Proof. By the facts explained in [2], we see that two abstract Reeb functions $r_0: G_0 \to \mathbb{R}$ and $r_1: G_1 \to \mathbb{R}$ are cobordant if and only if we can deform $r_0$ to $r_1$ by a finite iteration of the eleven local moves as depicted in Fig. 3 up to a homotopy in the space of abstract Reeb functions.

By [2] we can transform every abstract Reeb function whose domain (source) graph has no vertices of degree 2, using the first seven local moves of Fig. 3 together with a homotopy in the space of abstract Reeb functions, into one of the abstract Reeb functions as depicted in Fig. 4 where $n$ stands for the sum of the signs over all vertices of degree 3.

Lemma 3.6. Every abstract Reeb function is cobordant to one of the functions as depicted in Figs. 4 and 5.
Proof. With the help of the local moves (j) and (k) of Fig. 3 we can apply the algorithm of [2] even if a graph has vertices of degree 2, since we can lower any vertex of degree 2 through a vertex of degree 3. Furthermore by the local move (i) of Fig. 3 we can arrange so that the graph has at most one vertex of degree 2. □
Now let us define a map \( \sigma : \mathcal{A} \to \mathbb{Z} \oplus \mathbb{Z}_2 \) from the cobordism group of abstract Reeb functions as follows. For a given abstract Reeb function \( r \) put

\[
\sigma(r) = (t(r), d(r)) \in \mathbb{Z} \oplus \mathbb{Z}_2,
\]

where \( t(r) \in \mathbb{Z} \) is the sum of the signs over all vertices of degree 3, and \( d(r) \in \mathbb{Z}_2 \) is the number of vertices of degree 2 modulo two. Then \( \sigma(r) \) depends only on the cobordism
class of an abstract Reeb function \( r \), since the local moves of Fig. 3 or a homotopy in the space of abstract Reeb functions do not change the values of \( t \) nor \( d \). So \( \sigma \) defines a well-defined map of the set of cobordism classes, which is clearly a homomorphism, and by Lemma 3.6 it is an isomorphism. This completes the proof of Proposition 3.5. \( \square \)

4. Relation between cobordism and abstract cobordism

If we have a cobordism \( F: X \to \mathbb{R} \times [0,1] \) between two Morse functions \( f_0: M_0 \to \mathbb{R} \) and \( f_1: M_1 \to \mathbb{R} \) on closed surfaces, then the Stein factorization \( \overline{F}: W_F \to \mathbb{R} \times [0,1] \) of the possibly perturbed \( F \) provides a cobordism between the abstract Reeb functions \( r_0: G_0 \to \mathbb{R} \) and \( r_1: G_1 \to \mathbb{R} \) associated with the Morse functions \( f_0 \) and \( f_1 \) respectively, where \( r_j = \overline{f}_j \) and \( G_j = W_{f_j}, \ j = 0,1 \) (for details, see [2]). So the following definition makes sense.

**Definition 4.1.** Let \( \varphi: \text{Cob}_f(2,-1) \to \mathcal{A} \) be the map associating to the cobordism class of a Morse function \( f \) the cobordism class of its Reeb function \( \overline{f}: W_f \to \mathbb{R} \). This is clearly a well-defined homomorphism between the two cobordism groups \( \text{Cob}_f(2,-1) \) and \( \mathcal{A} \).

**Proposition 4.2.** The homomorphism \( \varphi \) is an isomorphism.

**Proof.** The map \( \varphi \) is clearly surjective. For the injectivity let \( R: P \to \mathbb{R} \times [0,1] \) be a cobordism between two abstract Reeb functions \( r_0: G_0 \to \mathbb{R} \) and \( r_1: G_1 \to \mathbb{R} \), and suppose that \( r_0 \) and \( r_1 \) correspond to Morse functions \( f_0 \) and \( f_1 \) on closed surfaces \( M_0 \) and \( M_1 \) respectively.

It is sufficient to show that there exists a cobordism \( F \) between the given Morse functions \( f_0 \) and \( f_1 \). For this, it suffices to show that

1. there exists a cobordism \( \overline{R}: \tilde{P} \to \mathbb{R} \times [0,1] \) between the two Reeb functions \( r_0: G_0 \to \mathbb{R} \) and \( r_1: G_1 \to \mathbb{R} \), and

2. there exists a fold map \( \tilde{F}: \tilde{X} \to \mathbb{R} \times [0,1] \) from a compact 3-manifold \( \tilde{X} \) with boundary such that \( \tilde{F} \) is a composition of a map \( \tilde{X} \to \tilde{P} \) and the map \( \overline{R} \), and \( \tilde{R} \) is identified with the Stein factorization of \( \tilde{F} \). In this case we say that the map \( \tilde{F} \) is over \( \overline{R} \).

Starting in the same way as in [2], we consider the decomposition

\[ P = N(Q) \cup N(V) \cup N(\Sigma) \cup T \]

as follows. We put \( Q = G_0 \cup G_1 \), and let \( N(Q) \) (\( \cong (G_0 \times [0, \varepsilon]) \sqcup (G_1 \times [1-\varepsilon, 1]) \)) be the regular neighbourhood of \( Q \) in \( P \). Let \( V \) be the set of the points of \( P \) which have regular neighbourhoods as in (d), (e), (g), (h), and (i) of Fig. 2 and let \( N(V) \) be
its small regular neighbourhood in \( P \). Let \( \Sigma \) be the set of the points of \( P \) which have regular neighbourhoods as in (b)–(i) of Fig. 2, and let \( \tilde{N}(\Sigma) \) be its regular neighbourhood in \( P \). Then \( N(\Sigma) \) is the closure of \( \tilde{N}(\Sigma) \setminus (N(Q) \cup N(V)) \). Let \( T \) be the closure of \( P \setminus (N(Q) \cup N(V) \cup N(\Sigma)) \). Note that \( T \) is a compact surface with boundary.

By the construction of Mata-Lorenzo \([6]\) we can construct a 3-dimensional compact manifold \( X \) with boundary and a map \( F: X \rightarrow \mathbb{R} \times [0, 1] \) over \( R |_{N(Q) \cup N(V) \cup N(\Sigma)} \) such that \( \partial X = M_0 \amalg M_1 \amalg S \), \( F |_{M_0 \times [0, \varepsilon]} = f_0 \circ \text{id} |_{[0, \varepsilon]} \), \( F |_{M_1 \times (1-\varepsilon, 1]} = f_1 \circ \text{id} |_{(1-\varepsilon, 1]} \), and \( S \) is a 2-dimensional manifold fibered over \( \partial T \): in fact, we consider \((M_0 \times [0, \varepsilon]) \amalg (M_1 \times [1-\varepsilon, 1])\) over \( N(Q) \), and over \( N(V) \cup N(\Sigma) \) we consider a manifold similar to that constructed in \([2]\).

It is not clear whether we can extend \( X \) and \( F \) like in \([2]\) over \( T \) so that the extension is a cobordism between the given Morse functions \( f_0 \) and \( f_1 \), since now \( S \) can be nonorientable. From \([5]\) \( T \) is an orientable surface with boundary and \( R |_{T}: T \rightarrow \mathbb{R} \times [0, 1] \) is an immersion. Over \( \partial T \) we have an \( S^1 \)-bundle (= \( S \)). Summarizing in an informal language, we have constructed a 3-manifold \( X \) with boundary \( M_0 \amalg M_1 \amalg S \) and its fold map into \( \mathbb{R} \times [0, 1] \) whose Stein factorization gives a neighbourhood of the 1-skeleton of the 2-dimensional polyhedron \( P \), where 1-skeleton means \( Q \cup \Sigma \). It remained to extend this construction over the 2-cells of \( P \), that is over \( T \). This was easy in the oriented case considered in \([2]\), since \( S \) was a trivial circle bundle over the boundary of \( T \) in \( P \). In the present case \( S \) may contain a nonorientable circle bundle. Such a nonorientable circle bundle is a fibration of the Klein bottle over \( S^1 \). This fibration can be extended to a fold map of the (nonorientable) solid Klein bottle \( S^1 \times_{\mathbb{Z}_2} D^2 \) to the annulus \( S^1 \times [0, 1] \) with only definite fold singularities, where the boundary Klein bottle \( S^1 \times_{\mathbb{Z}_2} S^1 \) is mapped onto \( S^1 \times \{1\} \) by the above fibration, and the zero section \( S^1 \times_{\mathbb{Z}_2} \{0\} \) coincides with the set of the singular points and is mapped onto \( S^1 \times \{0\} \) diffeomorphically. (For this, it suffices to consider the function \((x, y) \mapsto x^2 + y^2 \) on each 2-disk fiber.) Attaching such a solid Klein bottle or its orientable version to \( X \) along each component of \( S \), we obtain a compact 3-dimensional manifold and its fold map into \( \mathbb{R} \times [0, 1] \), which gives a cobordism between the Morse functions \( f_0 \) and \( f_1 \). This completes the proof of Proposition 4.2 and Theorem 2.5. \( \square \)

**Remark 4.3.** We see that the two functions as depicted in Fig. 6 represent generators of the cobordism group of abstract Reeb functions. The function (a) has infinite order, while the function (b) has order 2. We can realize the function (a) as the Reeb function associated with a height function of a 2-sphere appropriately embedded in \( \mathbb{R}^3 \) (see \([2]\)), and the function (b) can be realized as the Reeb function associated with a Morse function on \( \mathbb{R}P^2 \) with three critical points whose indices are 0, 1 and 2.
Figure 6. Two generators of the abstract cobordism group

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