Some Algebraic Properties of a subclass of Finite Normal Form Games

Ratnik Gandhi and Samaresh Chatterji

Dhirubhai Ambani Institute of Information and Communication Technology,
Post Bag No. 4, Gandhinagar 382007, India
{ratnik_gandhi,samaresh_chatterji}@daiict.ac.in

Abstract. We study the problem of computing all Nash equilibria of a subclass of finite normal form games. With algebraic characterization of the games, we present a method for computing all its Nash equilibria. Further, we present a method for deciding membership to the class of games with its related results. An appendix, containing an example to show working of each of the presented methods, concludes the work.

Key words: Galois group, Multivariate Newton-Raphson Method, Nash equilibria, Polynomial Algebra, Finite normal form games, Game Theory.

1 Introduction

Game theory has become an important technique for analyzing interactions amongst players (rational decision makers) in a competitive scenario, where each player tries to maximize his profit. A fundamental concept in game theory is that of a Nash equilibrium in which every player is satisfied with his move. The concept was introduced by John Nash in his celebrated 1951 paper [1]. Nash also proved that every mixed game has a Nash equilibrium. Nash’s proof was a pure existence proof, and did not indicate any methods for computing Nash equilibria.

In recent years, however, the problem of computing Nash equilibria has gained prominence, and has generated substantial research literature. In this work we consider methods for computing Nash equilibria of finite normal form games – games known to have finitely many solutions[1] – that emphasize use of polynomial algebra.

Nash equilibria of a game can be viewed as solutions to a system of equations and inequalities defined over payoffs and strategies. More specifically, they are the states of the game in which no player can obtain a more favourable outcome by a unilateral change of strategy. This system of inequalities can be converted into a system of polynomial equations that we call the game system (GS). We adopt this characterization of Nash equilibria and apply polynomial algebra as a computational framework. Note that the conversion of inequalities to equalities causes GS to have more solutions than just the Nash equilibria. Our main objective is to propose a method for computing all the Nash equilibria for a suitable subclass of finite normal form games.

[1] For the class of games that we consider, Harsanyi[2] shows that all the equilibria of the game are isolated and are odd in numbers.
games, that serves as an alternate to existing numerical or algebraic methods.

Tight complexity bounds have recently been presented for the problem of computing an equilibrium for finite normal form games. In the light of this and related results, it is of interest to focus on restricted classes of games and develop methods for computing their Nash equilibria. This is also of value in terms of applications of game theory in particular domains.

Another aspect of our work relates to is that of finding all Nash equilibria rather than a single one. From an application perspective, this helps in better strategic decision making.

Algorithms for computing all Nash equilibria, characterized as solutions of $GS$, typically iterate the procedure for a single solution (sample solution). In their investigation of these algorithms, McKelvey and McLennan [5] raised an important question: Whether a method can be found for computing all the equilibria of the input game, given a single equilibrium (referred to hereafter as a sample equilibrium), without repeating the solution procedure for the sample equilibrium.

Motivated by the question raised by McKelvey and McLennan, and the tight complexity bound for computing an equilibrium, we consider the problem of computing all Nash equilibria of subclass of finite normal form games. The subclass of finite normal form games that we consider have all integral payoffs and all irrational equilibria. The class of games are called integer payoff irrational equilibria (IPIE) games. We develop algorithms for computing all its Nash equilibria using a sample solution, thus answering McKelvey and McLennan’s question in the affirmative for games in this class. We further present an algorithm for deciding membership to the class of IPIE games.

Our overall philosophy is to exploit the Galois group of univariate polynomial in $I$ of $GS$ along with a single sample solution to extend our knowledge about the remaining solutions of the $GS$, which include all the Nash equilibria. It is known that knowledge of the Galois group does not presuppose knowledge of all the roots. In the remaining treatment, it is therefore usually assumed that the Galois group of irreducible univariate polynomials in the the ideal $I$ of $GS$ is known.

The primary setting of our work remains the Galois theory over commutative rings [7]. Accordingly, several subsidiary results of an essentially algebraic nature are derived in the course of our development. We also briefly consider the possibility of games over finite fields.

### 1.1 Related Work

To our knowledge, a method for computing Nash equilibria of IPIE games with Galois groups has not been considered earlier. An algorithm for fast decomposition of univariate polynomials, over the field of rational numbers, with known Galois group is presented by Enge and Morain [8]. The algorithm decomposes univariate polynomials with Galois or non-Galois field extensions. Segal and Ward [9] also considers the use of known Galois group in the problem of computing weight distributions in irreducible cyclic codes. The use of

---

2 For $n \geq 2$-player games [3] and [4] show that problem of computing an equilibrium is PPAD-complete.

3 A sample solution is a solution tuple of $GS$ with all irrational coordinates.

4 A method for computing Galois group using Tschirnhaus Transformations is presented in [6].
Galois group for computing roots of a polynomial is also mentioned in [10,11].

In our method, we consider systems of multivariate polynomials and their solutions. All the algebraic extensions we consider are extensions of the ring of integers $\mathbb{Z}$. Further, we are able to obtain results related to games with payoff values belonging to a discrete set, a finite set or a finite field.

Existing methods for computing Nash equilibrium, such as the approach based on the Gröbner bases are computationally inefficient. Homotopy continuation methods have the added drawback that they provide solutions via approximation. The method presented in [12] is highly dependent on the probability distributions chosen. Our method offers a reduction of computational time (compared to existing methods). Further, it computes exact equilibria for a subclass of IPIE games.

Section 2 outlines the underlying model for the class of games that we consider. Section 3 presents the algorithm for computing all equilibria of IPIE games. Results related to the games are presented in Section 4. Issue relating to the computational complexity of the algorithm is discussed in Section 5. Section 6 outlines a method for deciding membership to the class of integer payoff games. Section 7 presents our conclusions and suggests directions for future work. An appendix, presenting detailed examples to show working of the methods, concludes the work.

## 2 Underlying Model

In this section we give the underlying model for the class of games we are going to consider. The required definitions have also been presented.

**Definition 1.** A strategic finite normal form game $T_S$ is a 3-tuple $\langle N, S_i, c_i : i \in N \rangle$, where, $N$ is a non-empty finite set of players, $S_i$ is a non-empty finite set of strategies employed by player $i$ and for each player $i$ its payoff $c_i$ is of the form $c_i : \times_{k \in N} S_k \to \mathbb{R}$.

Each player $i$’s mixed strategy $\Delta(S_i)$ is a probability distribution on its set of pure strategies $S_i$, i.e. from $S_i$ player $i$ chooses strategy $j$ with probability $x_i^j$, where $x_i^j \in \Delta(S_i)$. A finite normal form game with mixed strategies and expected payoff $\alpha_i : \times_{k \in N} \Delta(S_k) \to \mathbb{R}$ is called mixed extension $T_M$ of strategic game $T_S$.

Let $T_M$ be a finite normal form game with $n = |N|$ players. Each player $i$ has $k_i \geq 2$ strategies, $|S_i| = k_i$. We write $K^+ = \sum_{i=1}^n k_i$ and $K^* = \prod_{i=1}^n k_i$. $A^i_{j_1 j_2 \ldots j_n}$ denotes the payoff received by player $i$ when each player adopts strategy $j_m$ for $1 \leq j_m \leq k_m$ and $m = 1, \ldots, n$. The probability that player $i$ chooses strategy $j_i \in \{1, 2, \ldots, k_i\}$ is denoted by $x^i_{j_i}$,

$$0 \leq x^i_{j_i} \leq 1. \quad (1)$$

Moreover, for each player $i$,

$$\sum_{j_i=1}^{k_i} x^i_{j_i} = 1. \quad (2)$$

\[5\] cf. Proposition 7.
Expected payoff for player $i$, 

$$\alpha_i = \sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \cdots \sum_{j_n=1}^{k_n} A_{j_1j_2\ldots j_n}^i x_{j_1}^1 x_{j_2}^2 \cdots x_{j_n}^n.$$  \hspace{1cm} (3)

Definition 2. Given mixed extension of strategic game, mixed Nash equilibrium is an action profile $\{x_{j_i}^i\} \in \Delta(S_i) \forall i, j_i$ such that each player’s mixed strategy maximizes his payoff if the strategies of the other players are held fixed.

In a Nash equilibrium, the following holds:

$$\alpha_i \geq \sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \cdots \sum_{j_{i-1}=1}^{k_{i-1}} \sum_{j_{i+1}=1}^{k_{i+1}} \cdots \sum_{j_n=1}^{k_n} A_{j_1j_2\ldots j_i-1j_i+1\ldots j_n}^i x_{j_1}^1 x_{j_2}^2 \cdots x_{j_{i-1}}^{i-1} x_{j_{i+1}}^{i+1} \cdots x_{j_n}^n,$$

for every $j_i \in S_i$ and for every $i \in \{1, \ldots, n\}$.  \hspace{1cm} (4)

The class of games that we consider can be defined as follows.

Definition 3. Finite normal form games with all integer payoffs and all irrational equilibria are called Integer Payoff Irrational Equilibria (IPIE) game $T$.

It is clear that $T \subset T_M$. Applying (1) and (2) to (4), we obtain a system $\mathcal{GS}$ of polynomial equations, called game system,

$$x_{j_i}^i (\alpha_i - \sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \cdots \sum_{j_{i-1}=1}^{k_{i-1}} \sum_{j_{i+1}=1}^{k_{i+1}} \cdots \sum_{j_n=1}^{k_n} A_{j_1j_2\ldots j_i-1j_i+1\ldots j_n}^i x_{j_1}^1 x_{j_2}^2 \cdots x_{j_{i-1}}^{i-1} x_{j_{i+1}}^{i+1} \cdots x_{j_n}^n) = 0,$$

for every $j_i \in S_i$ and for every $i \in \{1, \ldots, n\}$.  \hspace{1cm} (5)

The class of games that we consider has integer payoffs and so coefficient $A_{j_1j_2\ldots j_n}^i \in \mathbb{Z}$. All equilibria of these games are irrational and so (1) changes to

$$0 < x_{j_i}^i < 1.$$  \hspace{1cm} (6)

This changes (5) to

$$\alpha_i - \sum_{j_1=1}^{k_1} \sum_{j_2=1}^{k_2} \cdots \sum_{j_{i-1}=1}^{k_{i-1}} \sum_{j_{i+1}=1}^{k_{i+1}} \cdots \sum_{j_n=1}^{k_n} A_{j_1j_2\ldots j_i-1j_i+1\ldots j_n}^i x_{j_1}^1 x_{j_2}^2 \cdots x_{j_{i-1}}^{i-1} x_{j_{i+1}}^{i+1} \cdots x_{j_n}^n = 0,$$

for every $j_i \in S_i$ and for every $i \in \{1, \ldots, n\}$.  \hspace{1cm} (7)
For the problem of computing Nash equilibria of IPIE games, we characterize games with a $GS$ of the form \((7)\). For deciding membership to the class of IPIE games, we characterize games with a $GS$ of the form \((5)\).

Our convention is to write totally mixed real-irrational Nash equilibria as irrational Nash equilibria. Note that all Nash equilibria of the game correspond to solutions of $GS$, but the converse is not necessarily true.

Next, we define some algebraic concepts required for our method.

**Definition 4.** Let $G$ be a group and $X$ be a set. Then an action of $G$ on $X$ is a function of form $G \times X \rightarrow X$.

We shall be specifically interested in the following situation.

**Definition 5.** [\(\text{[13]}\)] Let $S$ be an extension of commutative ring with $R$, i.e. $R$ is a subring of $S$. Let $G$ be a finite group acting as $R$-algebra automorphisms on $S$. Then we define $S^G$ as the subring

$$S^G = \{ s \in S \mid \forall \sigma \in G, \sigma s = s \},$$

and say that $S$ is a Galois extension with group $G$, if

- $S^G = R$, and
- for any maximal ideal $m$ in $S$ and any $\sigma \in G \setminus \{1\}$, there is an $s \in S$ such that $\sigma s - s \notin m$.

We now consider the particular situation when $S$ is an extension of $R$ of the form $R(\alpha)$ where $\alpha$ is a root of a polynomial $p(x) \in R[x]$. It is known that the Galois group $G$ of the extension $S$ acts as a permutation group on the roots of the polynomial $p(x)$.

When Galois group acts on a subset of roots, due to group actions, we get the other elements of the set of roots. A transitive group action by a Galois group on an element of the root set produces all other elements of the root set. Transitivity of a group can be formally defined as follows:

**Definition 6.** A subgroup $H \subset S_n$ is transitive if for every pair of elements $i, j \in \{1, 2, \ldots, n\}$, there is $\tau \in H$ such that $\tau(i) = j$.

A conjugate element of an element of a set is an element generated by a group action on the element. Set of all conjugate elements of an element is called an orbit of the element under the action of a particular group. Formally,

**Definition 7.** For every $x \in X$ we put $Gx = \{ gx : \forall g \in G \}$, and call it the orbit of $x$ under $G$, or simply $G$-orbit of $x$.

Orbit of an element under Galois group is called Galois-orbit of the element. If $G$ acts transitively on $X$ then there is only one $G$-orbit, $X$ itself.

For more game theoretic concepts refer \([14]\), and for Galois theory related concepts refer Cox \([15]\).

3 Method

In this section we present outline of an algorithm for computing all Nash equilibria of IPIE games. The algorithm has two stages: a sample solution computation and the group action. Various methods for computing a sample solution are presented in \([3]\). In this work we use a version of Multivariate Newton Raphson.
method (MVNRM).

Over all approach of the algorithm is as follows: In the first stage compute a sample solution of the \( GS \). Once the sample solution is available, apply group action by Galois group to produce conjugate solutions of the sample solution. Rejecting all non-equilibria solutions from the set of solutions gives all the equilibria of an IPIE game.

Note that the input to the algorithm is an IPIE game, and Nash [16] gives existence of at least one mixed strategy equilibrium (irrational equilibrium in our case). This implies that for each indeterminate variable, we are guaranteed to get a polynomial irreducible over the base ring \( \mathbb{Z} \).

In the first stage, MVNRM starts with a guessed solution of the \( GS \). For the class of games all the mixed strategy Nash equilibria (probability tuples) form a subset of the set of solutions of \( GS \). This allows choosing an initial guess of a solution to be either all 0’s or 1’s or some value between \((0, 1)\). The choice of a solution tuple speeds up the convergence rate of MVNRM. Next we convert the approximate solution of \( GS \) into algebraic form.

For constructing minimal polynomials of approximate roots, output of MVNRM is fed as input to the KLL algorithm [17]. The KLL algorithm requires at least \( O(d^2 + d \cdot \log H) \) bits of an approximate root for computing its minimal polynomial, where \( d \) is the degree bound and \( H \) is magnitude bound of the coefficients of the minimal polynomial. If we let \((x_{k1}, x_{k2}, \ldots, x_{kn})\) denote the approximate solution generated by MVNRM in \( k^{th} \) iteration, and \( x_k = \sqrt{x_{k1}^2 + x_{k2}^2 + \ldots + x_{kn}^2} \), then the bound on number of bits required by the KLL algorithm sets the stopping criteria for MVNRM.

**Proposition 1.** MVNRM must compute approximate solution of \( GS \) till the number of zero bits in the binary representation of \( |x_{k+1} - x_k| \) is bounded above by \( O(d^2 + d \cdot \log H) \).

**Proof.** MVNRM computes approximate roots of \( GS \), which form the input to the KLL algorithm. Required precision, \( (d^2 + d \cdot \log H) \), for the input to KLL algorithm then immediately sets the criteria for the amount of precision required by MVNRM. \( \square \)

Bounds on the value of \( d \) and \( H \) can be obtained as follows. The variety of polynomials in a Gröbner basis and the polynomial system are same (Hilbert’s Nullstellensatz). By change in a monomial order we can obtain a univariate polynomial in the desired indeterminate variable. It is easy to see that the bound on number of solutions of the system of polynomial equations also bounds the degree of each these univariate polynomials.

Each polynomial in a Gröbner basis of a polynomial system is essentially an S-polynomial. An S-polynomial is constructed by taking modulus of all the polynomials with a pair of distinct polynomials, for every pair of polynomials in the basis. Maximum value of coefficients in the \( GS \) then bounds the coefficient values of univariate polynomials in a Gröbner basis of the \( GS \). A univariate polynomial in each indeterminate variable has a minimal polynomial as its one of the factors. And so the maximum magnitude of the coefficient in the minimal polynomial is bounded above by the maximum coefficient value of the game system \( GS \).

At the end of the first stage, the algorithm generates a sample solution of the \( GS \).
With the samples solution, available either in algebraic form or in numeric form, in the next stage of the algorithm, we apply group actions by Galois groups $G$. These groups are associated with the irreducible univariate polynomials in the ideal $I$ of $G\mathcal{S}$. Note that, we assume Galois groups are known. For IPIE games the Galois groups are associated with ring extensions over $\mathbb{Z}$, and they generate conjugate solutions of the sample solution of the $G\mathcal{S}$. The group action is transitive and produces a single orbit for each indeterminant variable. Using all the orbits we can determine all the irrational solutions of the $G\mathcal{S}$.

Recall that all the solutions of the $G\mathcal{S}$ need not be Nash equilibria. For rejecting unwanted non-equilibrium solutions, we apply the Nash equilibrium verification algorithm.

**Algorithm 3.1** Phase 1: Computing a sample solution of $G\mathcal{S}$ with MVNRM and KLL Algorithm

1: $X' = \emptyset$. // Initialize an empty set to store a sample solution of the $G\mathcal{S}$.
2: Characterize all the Nash Equilibria of the input game as solutions to the system of form $G\mathcal{S}$.
3: while one sample solution of polynomial system $G\mathcal{S}$ is not constructed do
4: Apply MVNRM with a starting solution tuple $x_0$ consisting entirely of zeros.
5: while inequality in Proposition 1 holds true. do
6: Compute approximate solution of $G\mathcal{S}$.
7: end while
8: Apply KLL Algorithm and compute minimal polynomial.
9: if minimal polynomials of some of the indeterminate variables in the approximate solution tuple is reducible over $\mathbb{Q}$ then
10: Divide $G\mathcal{S}$ with those factors and go to step Step 4 with the updated game system.
11: else
12: Save the solution tuple in $X'$.
13: end if
14: end while

The following algorithm computes transitive Galois group action of every indeterminate on its corresponding root in $X'$. It then generates Galois-conjugates of roots and saves them in solution tuple at appropriate position.

For the division by linear factor in Step 10, of Algorithm 3.1 we use multivariate polynomial division algorithm from [18]. It is important to note that due to the existence of a mixed Nash equilibrium and the fact that all equilibria are irrational for the input game, we are guaranteed to get one solution of $G\mathcal{S}$ in $X'$ and so Algorithm 3.1 reaches Step 14 every time. Moreover, the finiteness of the group and the variety on which it acts guarantees that the Algorithm 3.2 reaches Step 10. At the end of Step 10, Algorithm 3.2 generates solutions of the system $G\mathcal{S}$ in $X$, all of which may not be Nash equilibria. At last, for rejecting non-equilibrium solutions, the polynomial time Nash equilibrium verification algorithm [19] is used.

Note that the method above computes solutions to a system of polynomial equation using its sample solution and its Galois group. After the computation of a sample solution, all other solutions computed are without factorization of the system of polynomial equations.
Algorithm 3.2 Phase 2: Computing polynomial time Galois group action on the sample solution in $X'$ to generate solutions of $\mathcal{GS}$.

1: Initialize processed-element list $X$ and unprocessed-elements list $U$ as $X = U = X'$.
2: while $U$ is not empty do
3: Let $u$ be the first element of $U$. Delete $u$ from $U$.
4: for each element $g$ in Galois group $G$ do
5: Compute transitive Galois group action $u^g$.
6: if $u^g \notin X$ then
7: $X = X \cup \{u^g\}$ and $U = U \cup \{u^g\}$.
8: end if
9: end for
10: end while

4 Results

Observe that Step 9 in the Algorithm 3.1 rejects not only integer roots but rational roots. This forces the polynomial system not to consider rational extensions over the ring of integers. Such rejections can be justified with the following result.

Proposition 2. Algorithm 3.1 and Algorithm 3.2 cannot be used for computing all Nash equilibria of games with integral payoffs and rational equilibria.

Proof. Let $T$ be a game with integer payoffs and one or more rational equilibria of form $a/b$, where $a, b \neq 0 \in \mathbb{Z}$. This forces an extension $S = \mathbb{Z}(a/b)$ over $\mathbb{Z}$. The group $G$ of automorphisms of $S$ which fix $\mathbb{Z}$ can be computed as follows.

Let $c, d \in \mathbb{Z}$, for any $c + (a/b)d \in S$ and for any $\sigma \neq id \in G$,

$$\sigma(c + (a/b)d) = \sigma(c) + \sigma(a/b)\sigma(d) = c + \sigma(a/b)d,$$

and $\sigma(\alpha \cdot b) = a \Rightarrow \sigma(\alpha)\sigma(b) = a \Rightarrow \sigma(\alpha) = a/b \Rightarrow \sigma = \text{identity}$.

This means, the group of automorphisms of rational extensions of the ring of integers turns out to be a trivial identity group. And so, the group doesn’t provide necessary information for producing conjugate solutions of the $\mathcal{GS}$.

Now, in order to prove the validity of the proposed method for IPIE games, we establish 3 more results, as follows:

Proposition 3. For any IPIE game its Galois group is non-trivial.

Proof. Any IPIE game, by definition, produce irrational ring extensions over the ring of integers $\mathbb{Z}$. We first show that each such extension produces a Galois extension. Let $\alpha = \sqrt[n]{m}$, for $m, n \geq 2$, be an irrational number. Let $p \neq n^m$ be a positive prime integer\(^7\). Then it can be verified that $p\mathbb{Z}(\alpha)$ forms a prime and thus

\(^7\)It is easy to see that if $p = n^m$ then any element $\beta \in p\mathbb{Z}(\alpha)$ is not a prime element.
a maximal ideal of \( \mathbb{Z} \). We now choose a \( s \in \mathbb{Z}(\alpha) \) such that its co-prime to \( p \). Following Definition 5 if we let \( S = \mathbb{Z}(\alpha) \) and \( R = \mathbb{Z} \), then for any \( \rho \in G(S/R) \backslash \{1\} \) and for chosen \( s, ps - s \notin p\mathbb{Z}(\alpha) \). This shows that the extension is indeed a Galois extension.

Next, suppose the Galois group for the irrational extension \( \mathbb{Z}(\alpha) \) is trivial. Then the minimal polynomial of \( \sqrt[n]{n} \) has all factors linear over \( \mathbb{Z} \), and hence \( \alpha \in \mathbb{Z} \). This is impossible for IPIE games. And so the result follows.

The next result sets the criteria for the MVNRM to converge to a solution of a \( \mathcal{GS} \).

**Proposition 4.** Let \( x = (x_1, x_2, \ldots, x_n) \) be the vector strategy tuple with each \( x_i \) denoting a pure strategy for the players and let \( f(x) = (f_1(x), f_2(x), \ldots, f_n(x)) \), for all polynomials \( f_i \in \mathcal{GS} \). Then MVNRM converges to a sample solution of \( \mathcal{GS} \) if the following condition holds: \( |f(x) J^2(f(x))| < |J(f(x))^2| \).

**Proof.** In MVNRM, an approximation of the \( n^{th} \) strategy tuple \( x_n \) is computed using

\[
x_n = x_{n-1} - \frac{f(x_{n-1})}{Jf(x_{n-1})}.
\]

If we let

\[
\phi(x) = x - \frac{f(x)}{Jf(x)} \tag{8}
\]

then for overall convergence of MVNRM we need \( |\frac{d}{dx} \phi(x)| < 1 \). Taking the derivative of (8) and simplifying it, we get \( |f(x) J^2(f(x))| < |J(f(x))^2| \). With this condition, MVNRM converges to a sample solution of the \( \mathcal{GS} \). \( \square \)

With the required tools in hand, we can now show the correctness of the method for computing all Nash equilibria of IPIE games.

**Proposition 5.** Algorithms 3.1 and 3.2 for computing all equilibria of IPIE games works, i.e. these algorithms generate all irrational equilibria of IPIE game at termination - and they do not generate any spurious solutions.

**Proof.** The input to the Algorithm 3.1 is an IPIE game \( T \) with \( n \) players. All the Nash equilibria of this game are characterized by a polynomial system \( \mathcal{GS} \) of the form (7). The polynomial system comes from the inequalities on expected payoffs and payoffs at pure strategies. These inequalities cause the system to have more solutions then just equilibria.

Algorithm 3.1 computes a sample solution of the \( \mathcal{GS} \) using MVNRM and saves it in \( X' \). MVNRM computes an approximate solution of the game system \( \mathcal{GS} \), starting with an approximate solution value either 0,1 or \( \frac{1}{2} \). Nash [1] guarantees that a solution of \( \mathcal{GS} \) exists in (0,1). With the convergence criteria from Proposition 4, MVNRM converges to a solution tuple of \( \mathcal{GS} \). This may not be a sample solution. It is known that the input game has only irrational equilibria. If there are integer or rational solutions, they are rejected with the condition in Step 9 of the Algorithm 3.1. This guarantees a sample solution of \( \mathcal{GS} \). The polynomial bound given in Proposition 1 on the amount of precision required for KLL algorithm, forces MVNRM to stop. KLL algorithm computes the minimal polynomial for each root in the sample solution and terminates.
Roots in the sample solution extend the ring of integers $\mathbb{Z}$ to some Galois extension $S$ of it. This is due to the Galois correspondence established by Chase et al. [7]. More so, irreducible polynomials of univariate polynomials in ideal $I$ of $\mathbb{G}S$ forces transitivity of Galois groups. This gives us a meaningful non-trivial transitive Galois group $G = Gal(S/\mathbb{Z})$.

The transitive group action has only one orbit. Polynomials that we consider are irreducible after elimination of integer and rational roots. Due to our assumption that the Galois group of irreducible polynomials are known, we get group action of a transitive group $G = Gal(S/\mathbb{Z})$ on elements of set $X'$. System (7) has finitely many real solutions and so the group action terminates. This enables Algorithm 3.2 to reach Step 10. And so the algorithm generates all conjugate solutions of the sample solution containing all Nash equilibria of game $T$.

The solutions of $\mathbb{G}S$ may be more than just the Nash equilibria. Unwanted solutions are further rejected by calling a polynomial time algorithm to verify Nash equilibrium. What remains are all irrational equilibria of the game $T$. And so the result follows.

With the algebra and algorithms discussed above, we further ask, whether we can consider finite normal form games with all irrational equilibria and all payoff values from either finite fields or some finite set. This question can be partially answered as follows:

**Proposition 6.** The algebra and algorithms for IPIE games cannot be extended to work over finite fields and their extensions.

**Proof.** If we define a finite normal form game over some finite number field, then the only polynomial algebra that we can consider is congruent-modulo algebra. i.e. polynomial system of form (7) will be modulo some prime or prime power. This forces the expected cost function codomain values to be restricted to the finite number field. The payoff functions in games must provide every player a choice over his strategies by suggesting an order between elements in the codomain, where the function maps strategies. It is known that, finite number fields are not ordered fields and so they fail to provide a total order amongst player strategies. Moreover, the available order over finite fields conflict with field operations and we cannot perform polynomial algebra. So, we cannot meaningfully define games, and consider polynomial algebra such as suggested in the Algorithms in Section 3 for computing Nash equilibria of such games. □

However, we can consider some discrete codomain of the payoff function such as the ring of integers with total order or some finite set with arbitrarily defined total order and consider games over it.

Now, we consider a result that talks about computation of Nash equilibria in closed form. It is known that if a polynomial defined over fields has a solvable Galois group, then all its roots can be computed with radicals. If the result generalizes over rings then we can generalize the solvability by radical result. i.e. for some ring $S$ and a subring $R$ the following holds:

$$R = \mathbb{Z} = L_0 \subset L_1 \subset \ldots \subset L_n = S,$$

and $\exists \alpha_i \in L_{i+1}$, a natural number $n_i$, such that $L_{i+1} = L_i(a_i)$ and $\alpha_i^{n_i} \in L_i$, then solvability by radicals can be extended for a subclass of IPIE games. All finite ring extensions need not be radical. With this restriction on the extension of the ring and the definition of Galois theory over rings, we have following result.
Proposition 7. If the ring extensions associated with the IPIE games are radical then all the equilibria of IPIE games can be computed in closed form.

Proof. Follows immediately from the discussion above. □

5 Computational Complexity

The characterization of equilibria as solutions to a system of polynomial equation is a polynomial time operation in the size of input payoff matrix, where the size of the matrix is $K^+$. The while loop in side the Algorithm 3.1 of Steps (3-14) runs until a sample solution of the $\mathcal{G}S$ is computed. For $i \in \{1, \ldots, K^+\}$ and for each indeterminate variable $x_i$, let $d_i$ denote the degree of its univariate polynomial in $I$ of the $\mathcal{G}S$. The while loop of steps (3-14) runs for at most $d = \max_i d_i$ times. Average case running time analysis of the Newton’s method – for computing approximation of all the roots of a univariate polynomial – is studied by Smale [20,21]. A sufficient number of the steps for the Newton’s method to obtain an approximate zero of a polynomial $f$, are polynomially bounded by the degree $d_i$ of the polynomial and $1/\rho$, where $\rho \in (0, 1)$ is the probability that the method fails. Kuhn’s algorithm improves efficiency by a polynomial factor and provides global convergence. On the other hand Renegar [22] studies the problem of computing approximate solutions of multivariate system of equations using homotopy method and presents an efficient algorithm. Note that these results on the complexity analysis has underlying assumption that the numerical method converges.

For the Algorithm 3.1 number of operations, for constructing a minimal polynomial and checking its irreducibility over $\mathbb{Q}$, are bounded by a polynomial in the size of degree $d$ and maximum norm $H$ of the minimal polynomial [18]. The operation of multivariate polynomial division of Step 10 runs in $O(M_1 \cdot M_2)$ time [23], where each $M_i$ is the maximum number of terms in the polynomials considered for division. Without loss of generality we let $M = M1 > M2$.

With these details, we present the following complexity bound for computing a sample solution with the Algorithm 3.1.

Proposition 8. Algorithm 3.1 runs polynomial in $O(K^+ d(1/\rho + H + dH + cM))$.

Proof. The while loop of (3-14) runs for at most $d$ times. Considering the complexity of computing an approximate root of each univariate polynomial with Newton-Raphson’s method, the MVNRM with Proposition 4 runs polynomial in $O(K^+ d \cdot 1/\rho)$. The KLL Algorithm runs in $O(dH)$, requiring at most $K^+$ repetition in worst case. The operation of checking irreducibility of a minimal polynomial, in worst case, is required for each indeterminate variable and for every factor of the univariate polynomials. The irreducibility check runs polynomial in $O(dH)$. Finally, the multivariate division is called with only a single root of the univariate polynomial, making $M_2$ a constant $c$. The division algorithm is also required for $K^+ d$ times. Summing up all and rearranging terms we get the result. □

We are not considering the issue of computing the Galois groups in this work, i.e. we consider that the Galois groups are known. But to make the discussion complete, we give below, the complexity of computing a Galois group of the given polynomial. Computation of a Galois group requires polynomial time in the size of the input polynomial and the order of its Galois group. If $f(x)$ has degree $d$ then its Galois group can have at most $d!$ elements and so in worst case the computation takes exponential time. This is best know upper bound due to Landau [24]. Lanstra [25] surveys computational complexity result of computing Galois
groups and other related problems.

Once a Galois group $G$ is completely known, we must find the Galois-orbit $Gx$ of every known root $x$ of indeterminate variable in the $\mathcal{GS}$. An orbit construction takes polynomial time with algorithm suggested by Luks [26]. In the worst case, the algorithm requires action of each of the Galois group generator $g' \in G' \subseteq G$ to each element of the set of roots. This gives worst case time $O(|G'| \cdot |X|)$. Finally, the operation of verification of a Nash equilibrium solution, runs polynomial in the size of total number of strategies $K^+$.

6 Membership

With the characterization of games as game system $\mathcal{GS}$, in this section we outline a method for deciding membership to the class of IPIE games.

6.1 Method

The games that we consider are known to have integer payoff values and all irrational equilibria. The irrational equilibrium solutions induce irrational ring extensions. With this property, an intuitive and naive approach to answer the problem of deciding membership is as follows. Given an input game, we characterize all its equilibria as solutions to the system of polynomial equations of form $\mathcal{GS}$ in (5). After the characterization we ask, whether for each indeterminate variable its univariate polynomial has linear factors over the field of rational numbers or not. If the polynomial has no linear factors over $\mathbb{Q}$, then we must verify whether the solutions – consisting of linear factors of the polynomial over $\mathbb{Q}$ – are Nash equilibria of the input game or not. If any of the solutions is an equilibrium then the game is a non-member to IPIE games. Otherwise it is. For the polynomial irreducibility test over $\mathbb{Q}$, we make use of a polynomial time univariate factorization algorithm over $\mathbb{Q}$ from [18]. For constructing a univariate polynomial from multivariate system of polynomial equation $\mathcal{GS}$ we use a Gröbner basis of the $\mathcal{GS}$.

We start the membership decision with checking payoff values of the input game. If the payoff values are non-integer then we declare the input game a non-member to the class of IPIE games. In the case otherwise we do the following.

Note that the condition of checking irreducibility of each univariate polynomial over $\mathbb{Q}$ rather than $\mathbb{Z}$ lets us use field algebra, providing a richer set of tools to work with.

Next, we analyze various possibilities of a root generation during the process of factorization of the univariate polynomials in $\mathcal{GS}$.

There are mainly five possibilities: (1). The first univariate polynomial in the Gröbner basis of $\mathcal{GS}$ has complete linear factorization over $\mathbb{Q}$. In this situation the game is a non-member. (2). The first univariate polynomial has some linear factors over $\mathbb{Q}$. Further, substitution of all these linear factors in the triangular

---

8 It is important to note here that the membership to the class of IPIE games can be decided by considering irreducibility of the univariate polynomial over $\mathbb{Q}$. Justification of this fact comes from the following: if a polynomial has all its factor linear over $\mathbb{Q}$, then its roots are either integers or rationals. If the polynomial has no linear factors over $\mathbb{Q}$ then the roots are either irrational or complex. By [11] every game has a mixed strategy Nash equilibrium and so strictly a real number. This means that there is at least one irrational root of the polynomial.
Algorithm 6.1 Algorithm for deciding membership to the class of IPIE games.

1: **for** each of the indeterminate variable \((i = 1 \text{ to } K^+)** **do**
2: Apply the Buchberger’s Algorithm with the lexicographical order \((x_i \prec x_j), \forall j \neq i\) and compute a univariate polynomial in the Gröbner bases of the \(\mathcal{GS}\).
3: For the univariate polynomial produced in Step 2, check its irreducibility over \(\mathbb{Q}\).
4: **if** the univariate polynomial has linear factors over \(\mathbb{Q}\) **then**
5: Substitute each root in the triangular form of the Gröbner basis and compute a complete solution tuple corresponding to the root.
6: **if** the solution tuple verifies to be a Nash equilibrium of the input game **then**
7: Declare the input game a non-member to the class of IPIE games and stop.
8: **end if**
9: **end if**
10: **end for**
11: Declare the input game a member to the class of IPIE games.

form of the Gröbner basis produce univariate polynomials (in other indeterminate variables) with some linear factors. The substitution process generates solutions tuples with rational and irrational coordinates. There are two possibilities for each of these solution tuples. Either the solution is a Nash equilibrium or it is not. In former case the game is a non-member while in later it may be a member. (3). The first univariate polynomial has some linear factors but the subsequent univariate polynomials have all the irreducible factors over \(\mathbb{Q}\). This case is subsumed in case (2) mentioned above. (4). The first univariate polynomial has all irreducible factors over \(\mathbb{Q}\), and subsequent substitutions produce polynomials with rational or integer roots. This case is also subsumed in case (2). Finally, (5). The first univariate polynomial has all irreducible factors over \(\mathbb{Q}\), and subsequent substitutions produce all the roots that extend \(\mathbb{Q}\). In this situation, due to existence of at least one mixed Nash equilibrium \([16]\), we are guaranteed to get at least one irrational solution of the \(\mathcal{GS}\). And so the game is a member.

The analysis above, of roots generation during the factorization in Algorithm 6.1, suggests that a solution tuple with rational coordinates must be verified to be a Nash equilibrium. In this situation if it is guaranteed that substitution of an irrational root produce all irrational roots in subsequent substitution, then the repeated factorization and verification of the roots can reduce. We call this condition the membership condition. The condition will also improve efficiency of the membership algorithm.

In the following section, we present a result that improves running time of the membership Algorithm 6.1. The result primarily focuses on an algebraic property of the ideal \(I\) of \(\mathcal{GS}\).

6.2 Result

In this section we present a result that allows us to decide membership to the classes of games with relatively few root computation, factorization and verifications.

**Proposition 9.** If the polynomial ideal \(I\) of the game polynomial \(\mathcal{GS}\) is zero-dimensional, radical and in general position. Then for deciding membership to the class of IPIE games exactly one irreducible univariate polynomial in the Gröbner basis of the \(\mathcal{GS}\) is sufficient.
Proof. If the GS over a field \( \mathbb{Z} \) has following form

\[
\begin{align*}
    x_1 - h_1(x_p) &= 0 \\
    x_2 - h_2(x_p) &= 0 \\
    \quad & \vdots \\
    x_{p-1} - h_{p-1}(x_p) &= 0 \\
    h_p(x_p) &= 0.
\end{align*}
\]

(10)

And if \( h_p \) is irreducible over \( \mathbb{Z} \) with \( 1 < \deg h_i < \deg h_p, \ 1 \leq i < p \). Then, for all \( i \), root value of \( x_i \) will extend \( \mathbb{Z} \). This is a sufficient condition for the membership condition.

One of the ways to obtain the condition above for GS defined over some field \( \mathbb{F} \subset \mathbb{Z} \) is the Shape Lemma \[27\].

The shape lemma states: Let the ideal \( \mathcal{I} \) of a polynomial system be a zero-dimensional ideal in \( \mathbb{F}[x_1, \ldots, x_n] \) which is in general position with respect to \( x_1 \), i.e. the projection of \( \mathcal{V}_K(\mathcal{I}) \) onto the 1-st coordinate is injective. Then \( \sqrt{\mathcal{I}} \) has a lexicographical reduced Gröbner basis with respect to \( x_n \prec \cdots \prec x_1 \) of the form:

\[
\sqrt{\mathcal{I}} = \langle g_n(x_n), x_{n-1} - g_{n-1}(x_n), \ldots, x_2 - g_2(x_n), x_1 - g_1(x_n) \rangle
\]

where \( g_n \) is a square-free polynomial and the degree of every \( g_i \) doesn’t exceed degree \( d \) of \( g_n \). \( \mathbb{K} \) is algebraic closure of \( \mathbb{F} \).

The requirement in the shape lemma gives us the required condition, and so the result. \( \Box \)

The result above throws more light on the structure of a game system GS for the class of games that we consider. Moreover, it improves the computational complexity of Algorithm 6.1.

Next we present the correctness of the membership Algorithm 6.1.

**Proposition 10.** The Algorithm 6.1 for deciding membership to the class of IPIE games works.

Proof. Input to the Algorithm 6.1 is a finite normal form game characterized as GS of the form (5). All the coefficients of the GS must be integral, otherwise the game is a non-member to the class of IPIE games.

In case all the payoff values are integer, then the for loop of Steps 1-10 executes. The other property, of the input game, that we have to check is, whether all its solutions are irrational or not. For checking the irrationality of each indeterminate variable in the GS, we must check whether its univariate polynomial has all its factors irreducible over \( \mathbb{Q} \) or not. To obtain a univariate polynomial in each indeterminate variable, with different lexicographical orders, the Buchberger’s algorithm is called. Due to the finiteness of number of equilibria solutions of the input game and Buchberger [28] we are guaranteed to get a univariate polynomial every time.

For checking irreducibility of each of the univariate polynomials over \( \mathbb{Q} \), algorithm from [18] is used. The polynomial time irreducibility check algorithm takes Algorithm 6.1 to Step 4 every time.

\[9\] We recall that the type of games that we consider, in this work, are known to have finitely many equilibria solutions.
For each of the linear factors of a univariate polynomial a solution tuple is constructed. These solutions are then verified to be Nash equilibrium of the input game. If one of these solutions is a Nash equilibrium, then the method stops with declaring the input game a non-member to the class of IPIE games.

With the finiteness of the following: degree bound of the degrees of the $\mathcal{GS}$, the degrees of each univariate polynomial and number of strategies, we are guaranteed to reach Step 10 of the Algorithm 6.1.

At the end of the for loop, univariate polynomials of all the indeterminate variables have all non-rational factors and so the input game is declared a member to the class of IPIE games.

Note that in the light of Proposition 9, the for loop in the Algorithm 6.1 must be run exactly once. And so Algorithm 6.1 works with the condition in Proposition 9.

Running time of the Algorithm 6.1 is primarily dominated by the preprocessing task of constructing univariate polynomials. The Buchberger’s algorithm for constructing univariate polynomials takes doubly exponential time in the number of indeterminate variables $K^+$. Our game system is of finite size in terms of degree, number of indeterminate variables and its norm. And so, keeping aside running time of the Buchberger’s algorithm, the membership Algorithm 6.1 runs in polynomial time. Further, Proposition 9 improves running time efficiency of the naive algorithm.

It is important to note that, the method for deciding membership does not assume that the Galois groups are known. Also, for deciding the membership, the algorithm 6.1 does not compute all the solutions of the $\mathcal{GS}$.

7 Conclusion

In this article, we considered the problem of computing all the equilibria of a subclass of finite normal form. Our intention was to use knowledge of a sample equilibrium for computing all the other equilibria. By defining the class of IPIE games, and presenting an algorithm for computing all its equilibria, we have addressed the problem partially. For computing a sample solution we used MVNRM with the KLL algorithm. Though MVNRM is not globally convergent but offers significant speed. For convergence guarantee we derived a condition. With the use of KLL algorithm we computed equilibria in algebraic form contrary to traditional approach of keeping solutions in approximation form. We also factored out unwanted roots during the sample solutions generation, keeping the MVNRM away from local minima. The sample solution computation algorithm, with the convergence condition, is a definitive method for computing exact equilibria rapidly.

Further, for computing all the other equilibria we used knowledge of Galois groups. The assumption of known Galois groups can easily be relaxed by adjoining an algorithm for computing Galois groups. Construction of minimal polynomials using the KLL Algorithm and the use of Tschruhaus transformation over minimal polynomials help relax the criteria.

Algorithm that we suggest for solving system of equation may not be time efficient for large problems in practice, but it is time efficient compared to similar method based only on Gröbner bases. Algebraic approach
that we consider in this paper throws more light on structure of solutions of class of games.

Based on the algebraic model in the Section 2, we presented an outline of a method for deciding membership to the class of IPIE games. We also presented a result for improving efficiency of the naive method, and presented the correctness of the membership algorithm.

The complimentary work to the presented work would be to prove a result concerning the existence of IPIE games in general.

For the class of IPIE games, the algorithms we suggest are new and should be considered as precursor to efficient algorithms in future.

8 Appendix: Example

In this section we present an example to show working of the algorithms presented. The input 3 players 2 strategy game is given by the following payoff matrix. The game is defined in [29].

\[
\begin{array}{ccc}
\text{A} & \text{B} \\
1 & 3, 0, 2 & 0, 2, 0 \\
2 & 0, 1, 0 & 1, 0, 0 \\
\end{array}
\]

\[
\begin{array}{ccc}
\text{a} & 1, 0, 0 & 0, 1, 0 \\
\text{b} & 0, 3, 0 & 2, 0, 3 \\
\end{array}
\]

Table 1. Payoff matrix of a 3-player 2-strategy IPIE game. Player 1 and 2’s strategies are indicated by a, b and A, B respectively. Player 3’s strategies are 1 and 2. Each entry in the matrix indicates player 1, 2 and 3’s payoff for their respective strategies.

For \(i \in \{1, 2\}\), we denote \(x_i, y_i\) and \(z_i\) as player 1, 2 and 3’s \(i^{th}\) strategy, respectively. First, we characterize all the Nash equilibria of the game in Table 1 as solutions of the \(G\mathcal{S}\).

First we decide membership of the input game to class of IPIE games. After the characterization of all the Nash equilibria of the game above, we apply the Buchberger’s algorithm to a generate univariate polynomial in a Gröbner basis with lexicographical order \(z_1 \prec x_1 \prec y_1\) and get,

\[y_1^4 + 7y_1^3 - 11y_1^2 + 3y_1 = 0.\]  

(11)

The polynomial has \(y_1(y_1 - 1)(y_1^2 + 8y_1 - 3)\) as its factorization over \(\mathbb{Q}\). Substituting rational roots in the triangular form of the Gröbner basis we get \((0, 0, 0)\) and \((1, 1, 1)\) as its two solutions with rational coordinates. Verification of these solutions as Nash equilibrium of the input game reveals that neither constitute an equilibrium of the game. We can repeat the procedure above for \(x_1\) and \(z_1\) with different lexicographical order\(^{10}\). We see that the ideal of the game polynomial for this game follows the shape lemma and so the

\(^{10}\) We do not require to repeat the procedure for \(x_2, y_2\) and \(z_2\) because its a two strategy game.
irreducible factor of indeterminate $y_1$ guarantees that the game is a member to the class of IPIE game.

Next we apply equilibria computation algorithm for the game above and compute its equilibria using group action. With the initial guess of the solution tuple consisting of all the 0’s, $d = 2$ and $H = 3$ we apply MVNRM and compute an approximate sample solution tuple as follow.

$$x_1 := 0.7282202113; y_1 := 0.3588989435; z_1 := 0.4717797888$$  \hspace{1cm} (12)

Applying the KLL algorithm on the solution above generates univariate polynomials as follows.

$$5x_1^2 - 16x_1 + 9 = 0; \quad y_1^2 + 8y_1 - 3 = 0; \quad 5z_1^2 + 4z_1 - 3 = 0$$  \hspace{1cm} (13)

These polynomials are irreducible over $\mathbb{Z}$ and has a Galois group \{id,conjugate\}, isomorphic to $\mathbb{Z}_2$. With minimal polynomials, we compute a solution of the GS in closed form. Let one such solution be,

$$x = \frac{1}{5}(8 + \sqrt{19}); y = -4 - \sqrt{19}; z = \frac{1}{5}(-2 - \sqrt{19}).$$  \hspace{1cm} (14)

This is a sample solution of the game system. Next we perform Galois group action on the sample solution. Once all the solutions are computed, we reject non-equilibria solution of the game with the polynomial time verification algorithm \[19\]. This gives us the unique irrational equilibrium of the game depicted in Table 1,

$$x = \frac{1}{5}(8 - \sqrt{19}); y = -4 + \sqrt{19}; z = \frac{1}{5}(-2 + \sqrt{19}).$$  \hspace{1cm} (15)

References

1. Nash, J.: Non-cooperative games. The Annals of Mathematics, Second Series, Issue 2 54 (1951) 286–295
2. Harsanyi, J.C.: Oddness of the number of equilibrium points: A new proof. International Journal of Game Theory 2 (1973) 235–250
3. Chen, X., Deng, X.: Settling the complexity of two-player nash equilibrium. In: FOCS ’06: Proceedings of the 47th Annual IEEE Symposium on Foundations of Computer Science, Washington, DC, USA, IEEE Computer Society (2006) 261–272
4. Daskalakis, C., Goldberg, P.W., Papadimitriou, C.H.: The complexity of computing a nash equilibrium. In: STOC ’06: Proceedings of the thirty-eighth annual ACM symposium on Theory of computing, New York, NY, USA, ACM (2006) 71–78
5. McKelvey, R.D., McLennan, A.: Computation of Equilibria in Finite Games. In: Handbook of Computational Economics. Elsevier (1996) 87–142
6. Geissler, K., Kluners, J.: Galois group computation for rational polynomials. Journal of Symbolic Computation 11 (2000) 1–23
7. Chase, S.U., Harrison, D.K., Rosenberg, A.: Galois theory and galois cohomology of commutative rings. Memoirs of the American Mathematical Society 52 (1965) 15–33
8. Enge, A., Morain, F.: Fast decomposition of polynomials with known galois group. In: Applied Algebra, Algebraic Algorithms and Error-Correcting Codes. Volume 2643. (2003) 254–264
9. Segal, R., Ward, R.L.: Weight distributions of some irreducible cyclic codes. Mathematics of Computation 46(173) (1986) 341–354
10. Kiernan, B.M.: The development of galois theory from lagrange to artin. Archive for History of Exact Sciences 8(1-2) (1971) 40–154
11. Brodzik, A.: On the fourier transform of finite chirps. IEEE Signal Processing Letters 13(9) (2006) 541–544
12. Bárány, I., Vempala, S., Vetta, A.: Nash equilibria in random games. Random Struct. Algorithms 31(4) (2007) 391–405
13. Jensen, C.U., Ledet, A., Yui, N.: Generic Polynomials: Constructive Aspects of the Inverse Galois Problem. Number 45 in Mathematical Sciences Research Institute Publications. Cambridge University Press (2002)
14. Osborne, M.J., Rubinstein, A.: A Course in Game Theory. MIT Press (1999)
15. Cox, D.: Galois Theory. John Wiley & Sons (2004)
16. Nash, J.F.: Equilibrium points in n-person games. In: Proceedings of the National Academy of Sciences of the United States of America. Volume 36 of (Jan. 15, 1950). (1950) 48–49
17. Kannan, R., Lenstra, A.K., Lovasz, L.: Polynomial factorization and nonrandomness of bits of algebraic and some transcendental numbers. In: STOC ’84: Proceedings of the annual ACM symposium on Theory of computing, ACM (1984) 191–200
18. Gathen, J.V.Z., Gerhard, J.: Modern Computer Algebra. Cambridge University Press, New York, NY, USA (2003)
19. Gandhi, R.: Selfish routing and network creation games. Master’s thesis, Dhirubhai Ambani Institute of Information and Communication Technology, Gandhinagar, Gujarat, India (2005)
20. Smale, S.: The fundamental theorem of algebra and complexity theory. Bulletines of American Mathematical Society 4 (1981) 1–36
21. Smale, S. In: Complexity theory and numerical analysis. Cambridge University Press (1997) 523–551
22. Renegar, J.: On the efficiency of newton’s method in approximating all zeros of a system of complex polynomials. Mathematics of Operations Research 12(1) (1987) 121–148
23. Monagan, M., Pearce, R.: Polynomial Division Using Dynamic Arrays, Heaps, and Packed Exponent Vectors. Lecture Notes in Computer Science. In: Computer Algebra in Scientific Computing, Springer (2007) 295–315
24. Landau, S.: Polynomial time algorithms for galois groups. In: EUROSAM ’84: Proceedings of the International Symposium on Symbolic and Algebraic Computation, London, UK, Springer-Verlag (1984) 225–236
25. Lenstra, J.H.W.: Algorithms in algebraic number theory. Bulletin of the American Mathematical Society 26(2) (1992) 211–244
26. Luks, E.M.: Permutation groups and polynomial-time computation. In: Groups and Computation II, DIMACS series in Discrete Mathematics and Theoretical Computer Science. Volume 11. (1993) 139–175
27. Gonzalez-Lopez, M.J., Gonzalez-Vega, L. Number 251 in London Mathematical Society Lectures Notes Series. In: Newton Identities in the multivariate case: Pham Systems. Cambridge University Press (1998) 351 – 366
28. Buchberger, B.: Groebner bases: A short introduction for systems theorists. In Moreno-Diaz, R., Buchberger, B., Freire, J., eds.: EUROCAST 2001 - 8th International Conference on Computer Aided Systems Theory - Formal Methods and Tools for Computer Science. Volume 2178 of Lecture Notes in Computer Science., Berlin, Springer-Verlag (2001) 1 – 19
29. Nau, R., Canovas, S.G., Hansen, P.: On the geometry of nash equilibria and correlated equilibria. International Journal of Game Theory 32 (2003) 443–453