Feynman-Kac formula for the heat equation driven by time-homogeneous white noise potential.

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Abstract

We present a Feynman-Kac formula for the 1-dimensional stochastic heat equation (SHE) driven by a time-homogeneous Gaussian white noise potential, where the noise is interpreted in the Wick-Itô-Skorokhod sense. Our approach consists in constructing a Wong-Zakai-type approximation for the SHE from which we are able to obtain an “approximating Feynman-Kac” representation via the reduction of the approximated SHE to a deterministic partial differential equation (PDE). Then we will show that those “approximating Feynman-Kac” converge to a well defined object we will call “formal Feynman-Kac” representation which happens to coincide with the unique solution of SHE.

Key words and phrases: Stochastic heat equation, Feynman-Kac, Parabolic Anderson Model, Wick product, Wiener chaos expansion.

AMS 2000 classification: 60H10; 60H30; 60H05.

1 Introduction.

In this work we will deal with the 1-dimensional stochastic heat equation

\[
\begin{aligned}
\partial_t u(t, x) &= \frac{1}{2} \partial_{xx}^2 u(t, x) + u(t, x) \circ \frac{d}{dx} W_x, \quad (t, x) \in [0, T] \times \mathbb{R} \\
\partial_0 u(x) &= u_0(x),
\end{aligned}
\]

(1.1)

driven by the (distributional) derivative of the Brownian motion \( \{W_x\}_{x \in \mathbb{R}} \). From now on, in a slight abuse of notation we will denote \( \tilde{W}_x = \frac{d}{dx} W_x \).

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The initial condition \( u_0 : \mathbb{R} \to \mathbb{R} \) is assumed to be a bounded, deterministic Borel-measurable function. In (1.1) the symbol “\( \diamond \)” denotes the Wick product (e.g. Holden et al., 2009 or Gjessing et al., 2020) and implies that the noise is interpreted in the Itô-Skorohod sense.

In the case of space-time white noise potential equation (1.1) has been extensively studied (e.g. Potthoff, Våge, and Watanabe, 1998, Hu and Nualart, 2009, Hu, Nualart, and Song, 2011, Bertini and Cancrini, 1995 and references therein). On the other hand if the noise is assumed to be space-homogeneous white noise the equation (1.1) is just a particular case of the well known Zakai equation from non-linear filtering theory (see for instance the original paper Zakai, 1969 or the review Heunis, 1990 and references therein).

Nevertheless the case of space-only white noise hasn’t received the same attention. In Uemura, 1996 the author has showed that in this one-dimensional setting, equation (1.1) admits a unique weak-solution which is square integrable for any \((t, x) \in [0, T] \times \mathbb{R}\); such a solution is constructed employing the Wiener Chaos expansion (see also Theorem 3.9 of Hu, Huang, et al., 2015).

In Hu, 2001 Hu, 2002 the author treated the SHE with time-homogeneous noise in the \(d\)-dimensional case, showing existence, and providing numerous estimations of the Lyapunov exponents of the solutions. The former treats the case of fractional Brownian motion, while in the latter the solution is showed to exist in a flat Hilbert space similar to those introduced by Kondratiev (see for instance Holden et al., 2009).

In Hu, Huang, et al., 2015 the authors studied, among other things, the existence and regularity of the multidimensional version of (1.1) when the covariance structure of \( \{W_x\}_{x \in \mathbb{R}^d} \) satisfies certain conditions. They also propose some formal Feynman-Kac representations for the solutions of the SHE with space-time, and time-homogeneous Gaussian noise both in the Skorohod and Stratonovich sense. Nevertheless no representation was proposed for the solution of (1.1).

In Kim and S. V. Lototsky, 2017 and Hyun-Jung Kim, 2019 the authors study the SHE with time-homogeneous white noise potential in a bounded interval using the concept of Wiener chaos solution and propagator introduced in Mikulevicius and B. Rozovskii, 1993 (see also S. Lototsky, Remigijus Mikulevicius, and Boris L Rozovskii, 1997). They obtain estimation of the regularity of the solutions and well as existence and uniqueness results.

The aim of this article is to construct a Feynman-Kac representation for the unique solution of (1.1), this hasn’t been achieved in Hu, Huang, et al., 2015 due to the great generality under which the authors analyze the problem. In this simpler framework we were able to use some of the techniques proposed by them to construct such a representation.

Our analysis relies on the use of a “Wong-Zakai-type” approximation of equation (1.1) where we replace the singular white noise \( \{W_x\} \) with a truncated Karuhnen-Loève-like series.
As in Lanconelli and Scorolli, 2021b this “approximating equation” is reduced to a deterministic partial differential equation for which we are able to derive a Feynman-Kac representation. Then we show that as the number of terms in the truncated series goes to infinity this representation converges to a well-defined random variable (for fixed $t$ and $x$) and that this limit-object is the unique solution of (1.1) present in the literature. It’s worth noticing that due to the structure of the approximated noise the sequence of “approximated solutions” converges not only in the $L^p$ norm for any $p \in [1, \infty)$, but also almost surely.

2 Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space which carries a one dimensional Brownian motion $\{W_x\}_{x \in \mathbb{R}}$ indexed by the real line. Then consider the following Gaussian Hilbert space

$$\mathcal{H}(W) := \left\{ \int_{\mathbb{R}} h(x)dW_x; h \in L^2(\mathbb{R}) \right\},$$

where the stochastic integral over the real line is defined in Janson, 1997 (Chapter 7, section 2). From now on we will let $\mathcal{F} = \sigma(\mathcal{H}(W))$, i.e. the sigma algebra generated by the Gaussian Hilbert space $\mathcal{H}(W)$ then we have that the family of “stochastic exponentials” (also known as Wick exponentials)

$$\{ \mathcal{E}^f := \exp \left( \int_{\mathbb{R}} f(x)dW_x - \frac{1}{2} \int_{\mathbb{R}} \left| f(x) \right|^2dx \right), f \in L^2(\mathbb{R}) \},$$

is total in $L^2(\Omega, \mathcal{B}, \mathbb{P})$ ($L^2(\mathbb{P})$ for short). According to the Wiener chaos decomposition, any random variable $X \in L^2(\mathbb{P})$ can be represented as:

$$X = \sum_{n=0}^{\infty} I_n(h_n), \text{ convergence in } L^2(\mathbb{P})$$

where $I_n(\bullet)$ denotes the $n$-th multiple stochastic integral (e.g. Janson, 1997 Theorem 7.26), and the kernels $h_n \in L^2(\mathbb{R}^n)$ are symmetric deterministic functions.

If $A : L^2(\mathbb{P}) \to L^2(\mathbb{P})$ is a bounded linear operator and we assume that $A$ is a contraction, then its “second quantization operator” $\Gamma(A) : L^2(\mathbb{P}) \to L^2(\mathbb{P})$ is given by the action:

$$\Gamma(A)X = \sum_{n=0}^{\infty} I_n(A^{\otimes n}h_n),$$

notice that $A$ being a contraction is a sufficient condition for $\Gamma(A)$ to map $L^2(\mathbb{P})$ into itself.
In the following we will also need a complete orthonormal system (CONS for short) of the Hilbert space $L^2_p \mathbb{R}^q$. In particular we will use the family of Hermite functions, $\{e_j\}_{j \in \mathbb{N}}$ that are defined by:

$$e_j(x) = (-1)^{j-1} \left(\sqrt{\pi} 2^{j-1} (j-1)!\right)^{-1/2} e^{-x^2/2} \left(\frac{d}{dx}\right)^{j-1} e^{-x^2}, j \in \mathbb{N}$$

(2.1)

where we used a different indexing in order to avoid having a 0-order element.

It's straightforward to see that taking tensor products we obtain a CONS for the space $L^2_p \mathbb{R}^n$, i.e. $\{e_{i_1} \otimes \cdots \otimes e_{i_n}\}_{(i_1,\ldots,i_n) \in \mathbb{N}^n}$.

One could show that the family $\{e_j\}_{j \in \mathbb{N}}$ belongs to the Schwartz space of rapidly decreasing functions $S(\mathbb{R})$, and this is particularly useful since this implies that we can grab any element of its dual space $S'(\mathbb{R})$ (space of tempered distributions) and expand it into series of Hermite functions (e.g. Picard, 1991). In particular we will work with the Dirac's delta function $\delta_x \in S'(\mathbb{R}), x \in \mathbb{R}$, that can be written as:

$$\delta_x(\bullet) = \sum_{n=0}^{\infty} e_j(x)e_j(\bullet),$$

(2.2)

where the series clearly diverges in the classical sense.

We will now briefly discuss a particular type of product between random variables; let $X, Y \in L^2_p \mathbb{P}^W$ with Wiener chaos expansions given by

$$X = \sum_{n=0}^{\infty} I_n(h_n), \quad Y = \sum_{n=0}^{\infty} I_n(g_n)$$

the new element

$$X \circ Y = \sum_{n=0}^{\infty} I_n(k_n), \quad k_n := \sum_{j=1}^{n} h_j \odot g_{n-j},$$

where " $\odot$ " denotes the symmetric tensor product is called the Wick product of $X$ and $Y$. It’s worth noticing that $L^2_p \mathbb{P}^W$ is not closed under Wick multiplication. This product has great relevance in several areas of stochastic analysis and quantum physics, the interested reader is referred to Holden et al., 2009 and Kuo, 2018 from a complete account on the Wick product.

Finally all throughout this article we will denote with $p_{t-s}(x-y)$ the heat kernel

$$p_{t-s}(x-y) := \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(x-y)^2}{2(t-s)}},$$

and $(P_t f)(x)$ will denote the action of the heat semigroup on the function $f$, i.e.

$$(P_t f)(x) = \int_{\mathbb{R}} p_t(x-y)f(y)dy.$$
3 Construction of the approximating equation.

From now on we will work on the probability space \((\Omega, \mathcal{B}, \mathbb{P}^W)\) introduced in the previous section. In this section we will propose an approximation of equation (1.1), and thus the first thing to do is to construct an opportune smooth approximation of the singular White noise process \(\{\dot{W}_x\}_{x \in \mathbb{R}}\). It’s known that the singular white noise at \(x \in \mathbb{R}\) could be formally seen as

\[
\dot{W}_x = \int_{\mathbb{R}} \delta_x(y)dW_y,
\]

i.e. the stochastic integral of a Dirac delta function with mass at \(x \in \mathbb{R}\) (see Kuo, 2018).

One possible approximation can be obtained by truncating the series in (2.2) up to a certain finite value \(K\) yielding:

\[
\dot{W}_x^K := \sum_{j=1}^{K} e_j(x) \int_{\mathbb{R}} e_j(y)dW_y,
\]

(3.1)

(notice that the latter is nothing more than the derivative of a truncated Karhunen-Loève expansion of the Brownian motion \(W\)) clearly \(\dot{W}_x^K\) converges to \(\dot{W}\) in some space of generalized random variables (see formula 2.3.33 of Holden et al., 2009).

If we substitute the singular white noise in (1.1) with (3.1) we obtain the following “approximating equation”:

\[
\begin{cases}
\hat{c}_t u^K_{t,x} = \frac{1}{2} \hat{c}^2_{x,x} u^K_{t,x} + u^K_{t,x} \circ \dot{W}_x^K, \ (t, x) \in [0, T] \times \mathbb{R} \\
u(0, x) = u_0(x).
\end{cases}
\]

(3.2)

Since the equation above involves non-trivial operations, such as taking the Wick product between the solution and a random potential we should state what a “solution” of the latter actually is. Following Hu, Nualart, and Song, 2011 we have:

**Definition 3.1.** Let \(K\) be any arbitrary positive integer, then random field \(u^K : [0, T] \times \mathbb{R} \times \Omega \to \mathbb{R}\) is said to be a weak solution of (3.2) if for any fixed \((t, x) \in [0, T] \times \mathbb{R}\) we have that \(u^K_{t,x} \in L^2(\mathbb{P}_W)\) and for any random variable \(F \in \mathbb{D}^{1,2}\) it holds that:

\[
\mathbb{E} \left[ u^K_{t,x} F \right] = (P_t u_0)(x) \mathbb{E}[F] \\
+ \mathbb{E} \left[ \int_0^t \int_{\mathbb{R}} p_{t-s}(x-y) \left( \sum_{j=1}^{K} e_j(y) e_j(\bullet) \right) u^K_{s,y} dy ds, D(\bullet) F \right]_{L^2(\mathbb{R})},
\]

(3.3)

where \(D\) denotes the Malliavin derivative and \(\mathbb{D}^{1,2}\) the Sobolev-Malliavin space Nualart, 2006.
If in (3.2) we consider a space-independent potential $\hat{W}_t^K$ the change of variables $v^K(t, x) = u^K(t, x) \cdot \exp^{\int_0^t \hat{W}_s^K ds}$ (see section 3.6 of Holden et al., 2009 for a detailed explanation) reduces (3.2) to a standard homogeneous heat equation. Unfortunately, since the potential in our case is time-homogeneous (space-dependent) this approach is not applicable. For this reason in order to deal with equation (3.2) we will employ the approach proposed Lanconelli and Scorolli, 2021b (see also Lanconelli and Scorolli, 2021a ). The following remark offers a brief explanation of the latter.

**Remark 3.2.** From now on we will let $Z_j^K(\omega) := \int_\mathbb{R} e_j(y) dW_y(\omega), j \in \{1, \ldots, K\}, \omega \in \Omega$, then we can rewrite equation (3.2) as

$$
\begin{align*}
\hat{\partial}_t u_{t,x}^K &= \frac{1}{2} \partial^2_{xx} u_{t,x}^K + \sum_{j=1}^K e_j(x) u_{t,x}^K \cdot Z_j, (t, x) \in [0, T] \times \mathbb{R} \\
u(0, x) &= u_0(x).
\end{align*}

(3.4)
$$

It’s known (e.g. formula (2.5) of Hu and Øksendal, 1996 or Theorem 9.20 of Kuo, 2018 for an alternative but equivalent formulation) that the Wick product between a random variable $X \in \mathbb{D}^{1,2}$ and a random variable $I(g), g \in \mathbb{L}^2(\mathbb{R})$ in first Wiener Chaos is given by:

$$
X \circ I(g) = X \cdot I(g) - \langle DX, g \rangle_{L^2(\mathbb{R})}.
$$

Then proceeding formally we can write:

$$
u^K_{t,x} \circ Z_j = \nu^K_{t,x} \cdot Z_j - \langle Du^K_{t,x}, e_j \rangle_{L^2(\mathbb{R})},
$$

and thus we are left to consider the equation

$$
\begin{align*}
\hat{\partial}_t \nu^K_{t,x} &= \frac{1}{2} \partial^2_{xx} \nu^K_{t,x} + \nu^K_{t,x} \cdot Z_j - \langle Du^K_{t,x}, e_j \rangle_{L^2(\mathbb{R})}, (t, x) \in [0, T] \times \mathbb{R} \\
u(0, x) &= u_0(x).
\end{align*}
$$

If we assume that the solution of the equation above is of the form

$$
u^K_{t,x}(\omega) = u^K(t, x, Z_1(\omega), \ldots, Z_K(\omega))
$$

for some continuous function $u : [0, T] \times \mathbb{R} \times \mathbb{R}^K \to \mathbb{R}$ that must be determined we can use the chain rule for Malliavin derivatives and obtain the following partial differential equation (PDE for short):

$$
\begin{align*}
\hat{\partial}_t u^K_{t,x,z} &= \frac{1}{2} \partial^2_{xx} u^K_{t,x,z} \nu^K_{t,x,z} \left( \sum_{j=1}^K e_j(x) z_j \right) - \sum_{j=1}^K e_j(x) \partial_{z_j} u^K_{t,x,z}, (t, x, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^K \\
u^K(0, x) &= u_0(x).
\end{align*}
$$

Upon multiplying both sides of the equation by $\exp\left(-\frac{1}{2} \sum_{j=1}^K z_j^2\right)$ and defining $v^K_{t,x,z} := u^K_{t,x,z} \exp\left(-\frac{1}{2} \sum_{j=1}^K z_j^2\right)$, we are able to get rid of the zero-order term above and obtain...
the following:

\[
\begin{aligned}
\hat{c}_t v^K = \frac{1}{2} \hat{c}_{xx} v^K - \sum_{j=1}^K e_j(x) \hat{c}_{x_j} v^K, \\
v^K(0, x) = u_0(x) \times \exp \left( \frac{1}{2} \sum_{j=1}^K z_j^2 \right).
\end{aligned}
\]  

(3.5)

Formally applying the classical Feynman-Kac formula we obtain:

\[
v^K(t, x, z) = \mathbb{E}^B \left[ u_0(B_t^x) \exp \left\{ -\frac{1}{2} \sum_{j=1}^K \left( z_j - \int_0^t e_j(B_s^x) ds \right)^2 \right\} \right],
\]

(3.6)

where \( \{B_t\}_{t \in [0, T]} \) is a 1-dimensional Brownian motion defined on the auxiliary filtered probability space \((\tilde{\Omega}, \mathcal{G}, \{\mathcal{G}_t^B\}_{t \in [0, T]}, \mathbb{P}^B)\) and \( \mathbb{E}^B \) denotes the expectation in this space.

The expression given by (3.6) is sometimes referred to as a “generalized solution” of (3.5). It is worth mentioning that the latter becomes a classical solution if suitable regularity assumptions on the coefficients of (3.5) are in force. For more details see Freidlin, 2016 (page 122). By definition the latter implies that

\[
u^K(t, x, z) = \mathbb{E}^B \left[ u_0(B_t^x) \times \exp \left\{ \sum_{j=1}^K z_j \left( \int_0^t e_j(B_s^x) ds \right) - \frac{1}{2} \sum_{j=1}^K \left( \int_0^t e_j(B_s^x) ds \right)^2 \right\} \right].
\]

Letting \( u^K_{t,x}(\omega) := u^K(t, x, Z_1(\omega), ..., Z_K(\omega)) \) we obtain the formula in theorem 4.1.

4 Statements of the theorems.

**Theorem 4.1.** The random field \( u^K : [0, T] \times \mathbb{R} \times \Omega \to \mathbb{R} \) defined by:

\[
u^K_{t,x} = \mathbb{E}^B \left[ u_0(B_t^x) \times \exp \left\{ \int_\mathbb{R} \sum_{j=1}^K \left( \int_0^t e_j(B_s^x) ds \right) e_j(y) dW(y) - \frac{1}{2} \sum_{j=1}^K \left( \int_0^t e_j(B_s^x) ds \right)^2 \right\} \right],
\]

(4.1)

is a weak solution of (3.2) (in the sense of definition 3.1).

**Theorem 4.2.** The family of random variables \( \{\Psi^K_{t,x} ; K \in \mathbb{N}\} \) given by:

\[
\Psi^K_{t,x} := \int_\mathbb{R} \sum_{j=1}^K \left( \int_0^t e_j(B_s^x) ds \right) e_j(y) dW(y) - \frac{1}{2} \sum_{j=1}^K \left( \int_0^t e_j(B_s^x) ds \right)^2,
\]

converges in \( L^2(\mathbb{P}^W \otimes \mathbb{P}^B) \) to a well defined random variable denoted by

\[
\Psi_{t,x} = \int_0^t \int_\mathbb{R} \delta(B_s^x - y) dW(y) ds - \frac{1}{2} \int_\mathbb{R} \int_\mathbb{R} L_a(t)^2 da.
\]

Furthermore conditional on \( \mathcal{G}_t^B \) it holds that \( \Psi_{t,x} \sim N \left( -\frac{1}{2} \int_\mathbb{R} |L_a(t)|^2 da, \int_\mathbb{R} |L_a(t)|^2 da \right) \), where \( \{L_a(t) ; (t, a) \in [0, T] \times \mathbb{R}\} \) is the local time of \( \{B_t\}_{t \in [0, T]} \).

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Theorem 4.3. For fixed \((x, t) \in [0, T] \times \mathbb{R}, p \in [1, \infty]\), let \(u^K_{t,x} := \mathbb{E}^B [u_0(B_t^x) \exp\{\Psi^K_{t,x}\}]\) and denote \(u_{t,x} := \mathbb{E}^B [u_0(B_t^x) \exp\{\Psi_{t,x}\}]\), then it holds that:
\[
\lim_{K \to \infty} \|u^K_{t,x} - u_{t,x}\|_{L^p} = 0, \quad (4.2)
\]
and
\[
\lim_{K \to \infty} u^K_{t,x} = u_{t,x}, \mathbb{P}^W - \text{a.s.} \quad (4.3)
\]
Furthermore we have that
\[
[0, T] \times \mathbb{R} \times \Omega \ni (t, x, \omega) \mapsto u_{t,x}(\omega)
\]
is the unique solution for \((1.1)\) (in the sense of theorem 3.1 of Uemura, 1996).

Using our Feynman-Kac representation we are able to derive the following formulae for the moments of the solution (this formula has also been obtained in Hu, Huang, et al., 2015 but no proof is provided).

Theorem 4.4. Let \(q \geq 2\) then the \(q\)-th moment of the unique solution of \((1.1)\) is given by
\[
\mathbb{E}^W [(u_{t,x})^q] = \mathbb{E}^B \left[ \left( \prod_{i=1}^q u_0(B_t^{(i)} + x) \right) \exp \left\{ \sum_{i<j}^q \int_0^t \int_0^t \delta_0(B_s^{(i)} - B_r^{(j)})dsdr \right\} \right], \quad (4.4)
\]
\((B^{(1)}, ..., B^{(q)})\) are \(q\) independent 1-dimensional Brownian motions and \(\int_0^t \int_0^t \delta_0(B_s - B_r)dsdr\) denotes the intersection local time of the Brownian motions \(B\) and \(B'\) (e.g. Le Gall, 1994)

5 Proofs of theorem 4.1
Letting \(F = \mathcal{E}^\xi, \xi \in \mathbb{L}^2(\mathbb{R})\) we have that \(\mathbb{E}[u^K_{t,x} F]\) coincides with the \(S\)-transform (e.g. Kuo, 2018) of \(u^K_{t,x}\); i.e.
\[
\mathbb{E}^W [u^K_{t,x} F] = \mathcal{S}(u^K_{t,x}) (\xi) = \mathbb{E}^W \left[ \mathbb{E}^B \left[ u_0(B_t^x) \times \exp\{\Psi^K_{t,x}\} \right] \times \mathcal{E}^\xi \right]
\]
\[
= \mathbb{E}^B \left[ u_0(B_t^x) \mathbb{E}^W \left[ \exp\{\Psi^K_{t,x}\} \times \mathcal{E}^\xi \right] \right]
\]
\[
= \mathbb{E}^B \left[ u_0(B_t^x) \exp \left\{ \int_0^t \sum_{j=1}^K e_j(B_s^x) \int_{\mathbb{R}} e_j(y)\xi(y)dy \right\} \right],
\]
hence by the classical Feynman-Kac formula we can see that the latter is the solution of
\[
\begin{align*}
\partial_t S_{t,x}(\xi) &= \frac{1}{2} \mathcal{E}^2_{xx} S_{t,x}(\xi) + S_{t,x}(\xi) \cdot \left( \sum_{j=1}^K e_j(x) \int_{\mathbb{R}} e_j(y)\xi(y)dy \right), \quad (t, x) \in [0, T] \times \mathbb{R} \\
S_{0,x}(\xi) &= \mathcal{S}_{0,x}(\xi) = u_0(x),
\end{align*}
\]
for any $\xi \in L^2(\mathbb{R})$.

The solution of this equation can be written in mild form as

$$S_{t,x}(\xi) = (P_t u_0)(x) + \int_0^t \int_{\mathbb{R}} p_{t-s} (x-y) S_{s,y}(\xi) \sum_{j=1}^K e_j(y) \int_{\mathbb{R}} e_j(z) \xi(z) dz \, dy,$$

$$= (P_t u_0)(x) + \int_0^t \int_{\mathbb{R}} p_{t-s} (x-y) \mathbb{E}^W [u_{t,s}^{Kx}] \sum_{j=1}^K e_j(y) \int_{\mathbb{R}} e_j(z) \xi(z) dz \, dy,$$

or which is equivalent,

$$\mathbb{E}^W [u_{t,s}^{Kx}] = (P_t u_0)(x) + \mathbb{E}^W \left[ \left< \int_0^t \int_{\mathbb{R}} p_{t-s} (x-y) \left( \sum_{j=1}^K e_j(y) e_j(\bullet) \right) u_{s,y}^K dy ds, \xi(\bullet) \mathcal{E}^x \right>_{L^2(\mathbb{R})} \right],$$

which implies (3.3) since $\mathbb{E}^W [\mathcal{E}^x] = 1$ and $D_\bullet \mathcal{E}^x = \xi(\bullet) \mathcal{E}^x$ and the fact that the stochastic exponentials are a dense family in $D^{1,2}$.

## 6 Proof of theorem 4.2

We start by showing that $\{\Psi_{t,x}^{Kx}\}_{K \in \mathbb{N}}$ is a Cauchy sequence in $L^2(\mathbb{P}^W \otimes \mathbb{P}^B)$, and we let $\| \bullet \|_p$ be the norm on the Hilbert space $L^p(\mathbb{P}^W \otimes \mathbb{P}^B)$.

Without lost of generality assume that $N \geq M$

$$\| \Psi_{t,x}^N - \Psi_{t,x}^M \|^2 = \left\| \sum_{j=M+1}^N z_j \times \left( \int_0^t e_j(B^x_s) ds \right) - \frac{1}{2} \sum_{j=1}^K \left( \int_0^t e_j(B^x_s) ds \right)^2 \right\|^2$$

$$= \mathbb{E}^W \left[ \left\| \sum_{j=M+1}^N z_j \times \left( \int_0^t e_j(B^x_s) ds \right) - \frac{1}{2} \sum_{j=M+1}^N \left( \int_0^t e_j(B^x_s) ds \right)^2 \right\|^2 \right]$$

$$= \mathbb{E}^W \left[ \left( \sum_{j=M+1}^N z_j \times \left( \int_0^t e_j(B^x_s) ds \right) \right)^2 \right]$$

$$- \left( \sum_{j=M+1}^N z_j \times \left( \int_0^t e_j(B^x_s) ds \right) \right) \left( \sum_{j=M+1}^N \left( \int_0^t e_j(B^x_s) ds \right)^2 \right)$$

$$+ \frac{1}{4} \left( \sum_{j=M+1}^N \left( \int_0^t e_j(B^x_s) ds \right)^2 \right)^2 \right]$$

$$= \mathbb{E}^W \left[ \sum_{j=M+1}^N \left( \int_0^t e_j(B^x_s) ds \right)^2 + \frac{1}{4} \left( \sum_{j=M+1}^N \left( \int_0^t e_j(B^x_s) ds \right)^2 \right)^2 \right].$$

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Then if would suffices to show that
\[
\lim_{N,M \to \infty} \mathbb{E}^B \left[ \sum_{j=M+1}^{N} \left( \int_0^t e_j(B_s^x)ds \right)^2 + \frac{1}{4} \left( \sum_{j=M+1}^{N} \left( \int_0^t e_j(B_s^x)ds \right)^2 \right)^2 \right] = 0.
\]

Using (9.1) we have that
\[
\sum_{j=M+1}^{N} \left( \int_0^t e_j(B_s^x)ds \right)^2 \leq \int_{\mathbb{R}} |L_a(t)|^2 da = \alpha_t,
\]
for any positive integers \( N \geq M \). This together with the fact that the random variable \( \alpha_t \) is exponentially integrable (e.g. Le Gall, 1994 p. 178) allows us to use the Dominated Convergence theorem to bring the limit inside the expectation.

Finally from (9.1) we know that the sequence
\[
S_n = \sum_{j=1}^{n} \left( \int_0^t e_j(B_s^x)ds \right)^2,
\]
is \( \mathbb{P}^B \)-a.s. convergent and thus we have that
\[
|S_N - S_M| = \sum_{j=M+1}^{N} \left( \int_0^t e_j(B_s^x)ds \right)^2 \to 0, \quad \mathbb{P}^B - a.s.
\]
when \( N, M \to \infty \).

This implies that
\[
\lim_{N,M \to \infty} \| \Psi_{t,x}^N - \Psi_{t,x}^M \|_2^2 = 0.
\]

Furthermore notice that conditional on \( G_T^B \) the random variable \( \Psi_{t,x}^N \) is a Gaussian random variable with mean \(-\frac{1}{2} \sum_{j=1}^{K} \left( \int_0^t e_j(B_s^x)ds \right)^2 \) and variance \( \sum_{j=1}^{K} \left( \int_0^t e_j(B_s^x)ds \right)^2 \) and since the \( L^2(P^W) \) limit preserves the Gaussianity we have that conditional on \( G_T^B \) the random variable \( \Psi_{t,x} \sim N \left( -\frac{1}{2} \int_{\mathbb{R}} |L_a(t)|^2 da, \int_{\mathbb{R}} |L_a(t)|^2 da \right) \).

7 Proof of theorem 4.3

The proof of our main theorem will be done in several steps:

1. Show that the “approximated Feynman-Kac” formula converges in \( L^2(\mathbb{P}^W) \) to the “formal Feynman-Kac”.

2. Obtain the Wiener Chaos expansion of the “approximated Feynman-Kac”.
3. Show that the latter converges in $L^2(P^W)$ (as $K \to \infty$) to the solution of (1.1) represented by the formal series given in Uemura, 1996 and Hu, 2002.

Then since the limit in $L^2(P^W)$ is $P^W$-a.s. unique, we conclude that the solution given by the formal Wiener chaos series in Uemura, 1996 and Hu, 2002 coincides with the “formal Feynman-Kac” formula.

The previous can be summarized by the following diagram,

\[
\begin{array}{c}
\mathbb{E}^B \left[ u_0(B^x_t) \exp \{ \Psi^K_{t,x} \} \right] \\
\xrightarrow[K \to \infty]{} \sum_{n=0}^{\infty} I_n(f^K_n(t,x)) \\
\frac{\sum_{n=0}^{\infty} I_n(f^K_n(t,x))}{K \to \infty} = \mathbb{E}^B \left[ u_0(B^x_t) \exp \{ \Psi_{t,x} \} \right]
\end{array}
\]

\text{Step 1:}

We start by showing the convergence of the “approximated Feynman-Kac” formula:

\[
\mathbb{E}^W \left[ |u^K_{t,x} - u_{t,x}|^p \right] = \mathbb{E}^W \mathbb{E}^B \left[ u_0(B^x_t) \left( \exp \{ \Psi^K_{t,x} \} - \exp \{ \Psi_{t,x} \} \right) \right] \\
\leq \| u_0 \|^p \mathbb{E}^W \mathbb{E}^B \left[ \| \exp \{ \Psi^K_{t,x} \} - \exp \{ \Psi_{t,x} \} \|^p \right]
\]

Since $\Psi^K_{t,x} \to \Psi_{t,x}$ in $L^2(P^W \otimes P^B)$, then $\exp \{ \Psi^K_{t,x} \} \to \exp \{ \Psi_{t,x} \}$ in probability, in order to show the desired result we just need to prove that $\| \exp \{ \Psi^K_{t,x} \} \|^p \to \| \exp \{ \Psi_{t,x} \} \|^p$.

Using the Tower rule and the fact that conditional on $G_T^B$ the random variables $\Psi_{t,x}$ and $\Psi^K_{t,x}$ are Gaussian we have that

\[
\left\| \exp \{ \Psi_{t,x} \} \right\|^p = \mathbb{E}^W \mathbb{E}^B \left| \exp \{ p \Psi_{t,x} \} \right| \\
= \mathbb{E}^B \left[ \mathbb{E}^W \left[ \exp \{ p \Psi_{t,x} \} \left| G_T^B \right. \right] \right] \\
= \mathbb{E}^B \left[ \exp \left\{ \frac{p(p-1)}{2} \int_R |L_0(t)|^2 da \right\} \right] < \infty,
\]

and

\[
\left\| \exp \{ \Psi^K_{t,x} \} \right\|^p = \mathbb{E}^W \mathbb{E}^B \left| \exp \{ p \Psi^K_{t,x} \} \right| \\
= \mathbb{E}^B \left[ \mathbb{E}^W \left[ \exp \{ p \Psi_{t,x} \} \left| F_T^B \right. \right] \right] \\
= \mathbb{E}^B \left[ \exp \left\{ \frac{p(p-1)}{2} \sum_{j=1}^{K} \left( \int_0^t e_j(B_s^x) ds \right)^2 \right\} \right] .
\]

At this point we can use Monotone convergence theorem to bring the limit inside the expectation, the continuity of the exponential function and (9.1) implies the desired result.
Step 2:

Now we need to obtain the Wiener chaos decomposition of (4.1) and start by noticing that conditional on $G^B_T$ we can write

$$\exp\{\Psi^K_{t,x}\} = \sum_{n=0}^{\infty} \frac{1}{n!} I_n\left(g^K_n(t,x,\bullet)\right)$$

where the $n$-th kernel is given by:

$$g^K_n(t,x,\bullet) = \sum_{i_1=1}^{K} \cdots \sum_{i_n=1}^{K} \frac{1}{n!} \left( \int_{[0,t]^n} e_{i_1}(B_{s_1}^x) \cdots e_{i_n}(B_{s_n}^x) ds_1 \cdots ds_n \right) (e_{i_1} \otimes \cdots \otimes e_{i_n})(\bullet).$$

From the latter it follows that

$$u^K_{t,x} = \mathbb{E}^B\left[ u_0(B^x_t) \times \sum_{n=0}^{\infty} I_n \left( \sum_{i_1=1}^{K} \cdots \sum_{i_n=1}^{K} \frac{1}{n!} \left( \int_{[0,t]^n} e_{i_1}(B_{s_1}^x) \cdots e_{i_n}(B_{s_n}^x) ds_1 \cdots ds_n \right) e_{i_1} \otimes \cdots \otimes e_{i_n} \right) \right].$$

Since the series is convergent in $L^2(\mathbb{P}^W)$ we can apply Jensen inequality and monotone convergence to interchange the series with the expectation yielding

$$\sum_{n=0}^{\infty} I_n \left( \sum_{i_1=1}^{K} \cdots \sum_{i_n=1}^{K} \mathbb{E}^B \left[ u_0(B^x_t) \frac{1}{n!} \int_{[0,t]^n} e_{i_1}(B_{s_1}^x) \times \cdots \times e_{i_n}(B_{s_n}^x) ds_1 \cdots ds_n \right] e_{i_1} \otimes \cdots \otimes e_{i_n} \right).$$

Since we are summing over all possible combinations of the indexes $(i_1,\ldots,i_n)$ the expression above equals

$$\sum_{n=0}^{\infty} I_n \left( \sum_{i_1=1}^{K} \cdots \sum_{i_n=1}^{K} \mathbb{E}^B \left[ \int_{T_{t,n}} u_0(B^x_t) e_{i_1}(B_{s_1}^x) \times \cdots \times e_{i_n}(B_{s_n}^x) ds_1 \cdots ds_n \right] e_{i_1} \otimes \cdots \otimes e_{i_n} \right)$$

where the time integrals are taken over the simplex

$$T_{t,n} := \{(s_1, s_2, \ldots, s_n); 0 \leq s_1 \leq s_2 \leq \cdots \leq s_n \leq t\}.$$

An application of Fubini-Tonelli lemma shows that the kernel of the $n$-fold multiple stochastic integral is given by:

$$\int_{T_{t,n}} \sum_{i_1=1}^{K} \cdots \sum_{i_n=1}^{K} e_{i_1} \otimes \cdots \otimes e_{i_n}(\bullet) \mathbb{E}^B \left[ u_0(B^x_t) e_{i_1}(B_{s_1}^x) \times \cdots \times e_{i_n}(B_{s_n}^x) \right] ds_1 \cdots ds_n,$$

the conclusion is stated in the following proposition.
Proposition 7.1. Let $u_{t,x}^K$ be given by (4.1) then it holds that:

$$u_{t,x}^K = \sum_{n=0}^{\infty} I_n(f_n^K(t,x)),$$

(7.1)

where

$$\begin{cases}
  f_0^K(t,x) = (P_t u_0)(x), \\
  f_n^K(t,x) = \int_{\mathbb{R}^{n+1}} \sum_{i_1=1}^{K} \cdots \sum_{i_n=1}^{K} e_{i_1} \otimes \cdots \otimes e_{i_n}(\bullet) \\
  \times \int_{\mathbb{R}^{n+1}} p_{t-s_n}(x_n - x_{n-1}) \cdots p_{s_1}(x_0 - x) u_0(x_n) e_{i_1}(x_{n-1}) \times \cdots \times e_{i_n}(x_0) dx \, ds.
\end{cases}$$

(7.2)

where $dx := dx_0 \cdots dx_n$, $ds := ds_1 \cdots ds_n$.

In Uemura, 1996 the author has shown that if the initial condition is deterministic and square integrable, then equation (1.1) has a unique weak solution given by the Wiener Chaos expansion:

$$u(t,x) = \sum_{n=0}^{\infty} I_n(f_n(t,x)),$$

(7.3)

where

$$\begin{cases}
  f_0(t,x) = (P_t u_0)(x), \\
  f_n(t,x;x_1, \ldots, x_n) = \int_{\mathbb{R}^{n+1}} \int_{\mathbb{R}} p_{t-s_n}(x - x_n) \cdots p_{s_1}(x_1 - x_0) u_0(x_0) dx_0 ds,
\end{cases}$$

(7.4)

see also equations (4.2) and (4.3) of Hu, 2002.

Now the idea is to show that (7.1) converges in $L^2(\mathbb{P}^W)$ to (7.3) as $K \to \infty$ and for that we will need the following:

Definition 7.2. Let $K$ be some fixed positive integer then $A_K : S'(\mathbb{R}) \to L^2(\mathbb{R})$ is a self-adjoint projection operator defined by the action

$$A_K f = A_K \left( \sum_{j=1}^{\infty} \langle f, e_j \rangle e_j \right) = \sum_{j=1}^{K} \langle f, e_j \rangle e_j,$$

(7.5)

for any $f \in S'(\mathbb{R})$, i.e. the orthogonal projection on the linear span of the first $K$ elements of the CONS $\{e_j\}_{j \in \mathbb{N}}$.

Proposition 7.3. Let $u(t,x), (t,x) \in [0,T] \times \mathbb{R}$ denote the weak solution of (1.1) given in Uemura, 1996 (eq. (3.5)). Then for any $(t,x) \in [0,T] \times \mathbb{R}$ it holds that:

$$u_{t,x}^K = \Gamma(A_K) u(t,x),$$

(7.6)

where $\Gamma(A_K)$ stands for the second quantization of the projection operator $A_K$. 
Proof. In order to prove the latter we need to show that

$$f_n^K(t, x, \bullet) = (A_K^n f_n(t, x))(\bullet) = \sum_{i_1=1}^K \cdots \sum_{i_n=1}^K \left\langle f_n, e_{i_1} \otimes \cdots \otimes e_{i_n} \right\rangle_{L^2(\mathbb{R}^n)} e_{i_1} \otimes \cdots \otimes e_{i_n}(\bullet),$$

where $f_n^K$ and $f_n$ are defined by (7.2) and (7.4) respectively.

For the sake of simplicity we consider the case with $n = 2$, the general case does not present particular difficulties besides the more complex notation. In this case (7.2) takes the form:

$$f_n^K(t, x, \bullet) = \int_0^t \int_0^{s_2} \sum_{i_1=1}^K \sum_{i_2=1}^K e_{i_1} \otimes e_{i_2}(\bullet)$$

$$\times \int_{\mathbb{R}^3} p_{t-s_2}(x_2-x_1)p_{s_2-s_1}(x_1-x_0)p_{s_1}(x_0-x)u_0(x_2)e_{i_1}(x_1)e_{i_2}(x_0)dx \, ds.$$

We will define a new set of variables according to the prescription:

$$r_1 := t - s_2,$$

$$r_2 := t - s_1,$$

$$y_0 := x_2,$$

$$y_1 := x_1,$$

$$y_2 := x_0,$$

then we can rewrite the expression above as

$$\int_0^t \int_0^{r_2} \sum_{i_1=1}^K \sum_{i_2=1}^K e_{i_1} \otimes e_{i_2}(\bullet) \int_{\mathbb{R}^3} p_{r_1}(y_1-y_0)p_{r_2-r_1}(y_2-y_1)p_{r_2-r_1}(x-y_2)u_0(y_0)e_{i_1}(y_1)e_{i_2}(y_2)dy \, dr_2 \, dr_1$$

$$= \int_0^t \int_0^{r_2} \sum_{i_1=1}^K \sum_{i_2=1}^K e_{i_1} \otimes e_{i_2}(\bullet) \int_{\mathbb{R}^3} p_{t-r_2}(x-y_2)p_{r_2-r_1}(y_2-y_1)p_{r_1}(y_1-y_0)u_0(y_0)e_{i_1}(y_1)e_{i_2}(y_2)dy \, dr,$$

which is equal to $(A_K^2 f_2(t, x))(\bullet)$. From here it’s easy to see that

$$u_{t,x}^n = \sum_{n=0}^{\infty} I_n(f_n^K(t, x)) = \sum_{n=0}^{\infty} I_n(A_K^n f_n(t, x)) = \Gamma(A_K)u(t, x),$$

which proves the result.

Step 3:

It’s straightforward to see that

$$\left\| \sum_{n=0}^{\infty} I_n(f_n^K(t, x)) - \sum_{n=0}^{\infty} I_n(f_n(t, x)) \right\|_{L^2(\mathbb{R}^n)}^2 = \left\| \sum_{n=0}^{\infty} I_n(f_n(t, x) - A_K^n f_n(t, x)) \right\|_{L^2(\mathbb{R}^n)}^2$$

$$= \sum_{n=0}^{\infty} n! \|f_n(t, x) - (A_K^n f_n(t, x))\|_{L^2(\mathbb{R}^n)}^2 \to 0$$

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as \( K \to \infty \). This together with the results obtained in Step 1 and the unicity of the \( L^2(\mathbb{P}^W) \) limit we conclude that

\[
\mathbb{E}^B \left[ u_0(B_t^+) \exp \left\{ \Psi_{t,x} \right\} \right] = \sum_{n=0}^{\infty} I_n(f_n(t,x)).
\]

On the other hand from the propositions 7.3 and 10.1 we see that

\[
u^K_{t,x} = \mathbb{E}^W \left[ u(t,x) | \sigma(Z_1, ..., Z_K) \right],
\]

and we also notice that \( \sigma(Z_1, ..., Z_K) \uparrow \mathfrak{B} := \sigma(\mathfrak{F}(W)) \). Then the martingale convergence theorem (e.g. theorem 35.6 of Billingsley, 2008) gives us the \( \mathbb{P}^W \)-a.s. convergence

\section{Proof of theorem 4.4}

The convergence in \( L^p(\mathbb{P}^W) \), \( p \in [1, \infty) \) of \( u^K_{t,x} \) to the solution of \( u_{t,x} \) implies that for any \( q \in \mathbb{N} \) the \( q \)-th moment of \( u^K_{t,x} \) converges to that of the solution.

\[
\mathbb{E}^W \left[ (u^K_{t,x})^q \right] = \mathbb{E}^W \left[ \prod_{i=1}^{q} \mathbb{E}^B \left[ u_0(B_t^{(i)} + x) \exp \left\{ \Psi^K_{t,x} \right\} \right] \right]
= \mathbb{E}^B \left[ \left( \prod_{i=1}^{q} u_0(B_t^{(i)} + x) \right) \mathbb{E}^W \left[ \exp \left\{ \sum_{i=1}^{q} \Psi^K_{t,x} \right\} | \mathcal{F}_T^B \right] \right]
= \mathbb{E}^B \left[ \left( \prod_{i=1}^{q} u_0(B_t^{(i)} + x) \right) \mathbb{E}^W \left[ \exp \left\{ \int_{\mathbb{R}} \sum_{i=1}^{q} \sum_{k=1}^{K} \left( \int_0^t e_k(B_s^{(i)} + x)ds \right) e_k(y)dW_y \right\} | \mathcal{F}_T^B \right] \right]
\times \exp \left\{ -\frac{1}{2} \sum_{i=1}^{q} \left( \int_0^t e_k(B_s^{(i)} + x)ds \right)^2 \right\},
\]

using the fact that conditional on \( \mathcal{G}_T^B \) the stochastic integral appearing in the exponential is a centered Gaussian random variable we can see that the latter equals

\[
= \mathbb{E}^B \left[ \left( \prod_{i=1}^{q} u_0(B_t^{(i)} + x) \right) \exp \left\{ \left\| \sum_{i=1}^{q} \sum_{k=1}^{K} \left( \int_0^t e_k(B_s^{(i)} + x)ds \right) e_k(\bullet) \right\|_{L^2(\mathbb{R})}^2 \right\} \right]
\times \exp \left\{ -\frac{1}{2} \sum_{i=1}^{q} \left( \int_0^t e_k(B_s^{(i)} + x)ds \right)^2 \right\}
= \mathbb{E}^B \left[ \left( \prod_{i=1}^{q} u_0(B_t^{(i)} + x) \right) \exp \left\{ \sum_{i<j}^{q} \sum_{k=1}^{K} \left( \int_0^t e_k(B_s^{(i)} + x)ds \right) \left( \int_0^t e_k(B_r^{(j)} + x)dr \right) \right\} \right].
\]
Now we must take the limit for $K \to \infty$ (we can see that the exponential function is dominated by $\exp\{q \max_{1 \leq i \leq q} \int_{\mathbb{R}} |L_i^{(i)}(t)|^2 da\}$ which is integrable) yielding

$$
\mathbb{E}^W [(u_{t,x})^q] = \lim_{K \to \infty} \mathbb{E}^W [(u_{t,x}^K)^q] = \mathbb{E}^B \left[ \left( \prod_{i=1}^q u_0(B_t^{(i)} + x) \right) \exp \left\{ \sum_{i<j}^q \int_0^t \int_0^t \delta_0(B_s^{(i)} - B_r^{(j)}) dr ds \right\} \right].
$$

(8.1)

9 Appendix A: Local time

Consider the Brownian local time of a one-dimensional Brownian motion $\{B_t\}_{t \in [0,T]}$ starting at $x \in \mathbb{R}$, at level $a \in \mathbb{R}$ and time $t \in [0,T]$

$$
L_a^x(t) = \int_0^t \delta_a(B_s^x) ds,
$$

and notice that the latter can be seen as the usual Brownian local time $L_{a-x}(t)$.

It’s known (e.g. the proof of proposition XIII-2.1. of Revuz and Yor, 2013) that for a fixed $t$ the map $\mathbb{R} \ni a \mapsto L_a(t)$ is a.s. continuous and has compact support, hence it follows that

$$
\alpha_t = \int_{\mathbb{R}} |L_a(t)|^2 da < \infty, \text{a.s.,}
$$

this together with the invariance of Lebesgue measure implies that $a \mapsto L_a^x(t)$ belongs to $L^2(\mathbb{R})$ almost surely.

Then the following Fourier-like series expansion holds a.s.

$$
L_a^x(t) = \sum_{j=1}^\infty \left( \int_{\mathbb{R}} L_a^x(t) e_j(y) dy \right) e_j(a) = \sum_{j=1}^\infty \left( \int_0^t e_j(B_s^x) ds \right) e_j(a),
$$

where in the last equality we’ve used the occupation time formula.

By the Parseval’s identity we have:

$$
\sum_{j=1}^\infty \left( \int_0^t e_j(B_s^x) ds \right)^2 = \int_{-\infty}^\infty |L_a^x(t)|^2 da = \int_{-\infty}^\infty |L_a(t)|^2 da < \infty \text{ a.s.} \quad (9.1)
$$
Appendix B: Second quantization and Conditional expectation

Let \((\Omega, \mathcal{A}, \mathbb{P}^W)\) be a probability space then it’s well know that if \(X \in L^2(\Omega, \mathcal{A}, \mathbb{P}^W)\) and \(\mathcal{G} \subset \mathcal{A}\) is a sub-sigma-algebra, the conditional expectation \(\mathbb{E}[X|\mathcal{G}]\) can be seen as the orthogonal projection of \(X\) on \(L^2(\Omega, \mathcal{G}, \mathbb{P}^W)\). In this appendix we will show an analogous property of the second quantization operator.

**Proposition 10.1.** Let \(A_K\) be the projection operator of definition 7.2 then the second quantization operator \(\Gamma_{A_K}\) coincides with the conditional expectation \(\mathbb{E}[\cdot|\sigma(Z_1, ..., Z_K)]\) where \(\sigma(Z_1, ..., Z_K)\) is the sigma algebra generated by the family of i.i.d Gaussian random variables \((Z_1, ..., Z_K)\).

**Proof.** We consider again the complete probability space \((\Omega, \mathcal{B}, \mathbb{P}^W)\) treated in the introduction. Let \(X \in L^2(\mathbb{P}^W)\) then a result by Cameron and Martin \(1947\) tells us that \(X\) has a series expansion of the form

\[
X = \sum_{\alpha \in J} x_\alpha \mathcal{H}_\alpha, \text{ convergence in } L^2(\mathbb{P}^W)
\]

where \(J\) is the space of of all sequences \(\alpha = (\alpha_1, \alpha_2, ...)\) with elements \(\alpha_i \in \mathbb{N}_0\) and with compact support and

\[
\mathcal{H}_\alpha := \prod_{j=1}^{\infty} H_{\alpha_j}(Z_j),
\]

where \(H_n(\bullet)\) is the \(n\)-th Hermite polynomial and \(Z_j := \int_{\mathbb{R}} e_j(x) dW_x\), are known as “generalized Hermite polynomials” or “Wick polynomials” \(\text{e.g. Holden et al., 2009}\) and

\[
x_\alpha = \left( \prod_{j=1}^{\infty} \alpha_j! \right)^{-1} \mathbb{E}[X\mathcal{H}_\alpha].
\]

Now lets take the conditional expectation of \(X\) given the sigma algebra \(\sigma_K := \sigma(Z_1, ..., Z_K)\). It’s well known that we are allowed to interchange conditional expectation with an \(L^2\) convergent series, yielding

\[
\mathbb{E}[X|\sigma_K] = \sum_{\alpha \in J} x_\alpha E[\mathcal{H}_\alpha|\sigma_K]
\]

\[
= \sum_{\alpha \in J} x_\alpha \mathbb{E}\left[\prod_{j=1}^{\infty} H_{\alpha_j}(Z_j) \big| \sigma_K\right],
\]

and at this point we notice that the terms of the product involving \(Z_j\) for \(j \in \{1, 2, ..., N\}\) are \(\sigma_K\)-measurable and hence can be pulled outside the conditional expectation,

\[
\mathbb{E}[X|\sigma_K] = \sum_{\alpha \in J} x_\alpha \prod_{i=1}^{K} H_{\alpha_i}(Z_i) \mathbb{E}\left[\prod_{j=K+1}^{\infty} H_{\alpha_j}(Z_j) \big| \sigma_K\right],
\]

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all the remaining terms are independent from $\sigma_K$, and mutually independent which implies that

$$
E[X|\sigma_K] = \sum_{\alpha \in J^K} x_\alpha \prod_{i=1}^K H_{\alpha_i}(Z_i) \prod_{j=K+1}^\infty \mathbb{E} \left[ H_{\alpha_j}(Z_j) \bigg| \sigma_K \right].
$$

Furthermore since the Hermite polynomials of a centered Gaussian random variables can be seen as its Wick power i.e. $H_n(Z_j) = Z_j^{\otimes n}$ (Janson, 1997 Theorem 3.19), and since $\mathbb{E}(Z_j^{\otimes n}) = \mathbb{E}(Z_j)^n = 0$ we see that the only non-vanishing terms are those corresponding to the $\alpha$’s containing only positive values in the first $K$ entries (remember that $H_0(\cdot) \equiv 1$). This allows us to conclude that

$$
E[X|\sigma_K] = \sum_{\alpha \in J^K} x_\alpha H_\alpha
$$

(10.1)

where $J^K := \{ \alpha \in J : \alpha_i = 0, \forall i > K \}$.

On the other hand we could write the Chaos decomposition in terms of multiple Wiener integrals, i.e.

$$
X = \sum_{n=0}^\infty I_n(f_n),
$$

where the kernel $f_n$ is a symmetric function in $L^2(\mathbb{R})$. Then by definition of the second quantization operator we have

$$
\Gamma(A_K)X = \sum_{n=0}^\infty I_n(\Gamma(A_K)^{\otimes n} f_n)
$$

$$
= \sum_{n=0}^\infty \sum_{\alpha \in J^K_n} \left\langle f_n, \otimes_{j=1}^K e_j^{\otimes \alpha_j} \right\rangle_{L^2(\mathbb{R})^{\otimes n}} I_n \left( \otimes_{j=1}^K e_j^{\otimes \alpha_j} \right)
$$

$$
= \sum_{n=0}^\infty \sum_{\alpha \in J^K_n} x_\alpha H_\alpha
$$

$$
= \sum_{\alpha \in J^K} x_\alpha H_\alpha
$$

$$
= \sum_{\alpha \in J^K} x_\alpha H_\alpha
$$

$$
= \mathbb{E}[X|\sigma_K],
$$

where we have used the following identity proved by Itô Itô, 1951.

$$
H_\alpha = I_n \left( \otimes_{j=1}^K e_j^{\otimes \alpha_j} \right),
$$
and for $\alpha \in J_n^K := \{ \alpha \in J : |\alpha| = n, \alpha_i = 0, \forall i > K \}$ we let

$$x_{\alpha} = \left< f_n, \bigotimes_{j=1}^K e_j^{\alpha_j} \right>_{L^2(\mathbb{R})^\otimes n} = n! \left< f_n, \bigotimes_{j=1}^K e_j^{\alpha_j} \right>_{L^2(\mathbb{R}^n)}.$$

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