BURST STATISTICS AS A CRITERION FOR IMMINENT FAILURE

Srutarshi Pradhan\textsuperscript{1}, Alex Hansen\textsuperscript{2} and Per C. Hemmer\textsuperscript{3}

Department of Physics, Norwegian University of Science and Technology, N–7491 Trondheim, Norway, \textsuperscript{1}pradhan.srutarshi@ntnu.no, \textsuperscript{2}alex.hansen@ntnu.no, \textsuperscript{3}per.hemmer@ntnu.no

Abstract: The distribution of the magnitudes of damage avalanches during a failure process typically follows a power law. When these avalanches are recorded close to the point at which the system fails catastrophically, we find that the power law has an exponent which differs from the one characterizing the size distribution of all avalanches. We demonstrate this analytically for bundles of many fibers with statistically distributed breakdown thresholds for the individual fibers. In this case the magnitude distribution $D(\Delta)$ for the avalanche size $\Delta$ follows a power law $\Delta^{-\xi}$ with $\xi = 3/2$ near complete failure, and $\xi = 5/2$ elsewhere. We also study a network of electric fuses, and find numerically an exponent 2.0 near breakdown, and 3.0 elsewhere. We propose that this crossover in the size distribution may be used as a signal for imminent system failure.

Key words: Failure, Fiber bundle model, Fuse model, Burst statistics, Crossover

1 INTRODUCTION

Catastrophic failures\textsuperscript{[1,2,3,4,5]} are abundant in nature: earthquakes, landslides, mine-collapses, snow-avalanches etc. are well-known examples. A sudden catastrophic failure is a curse to human society due to the devastation it causes in terms of properties and human lives. Therefore a fundamental challenge is to detect reliable precursors of such catastrophic events. This is also an important issue in strength considerations of materials as well as in construction engineering.

During the failure process of composite materials under external stress, avalanches of different size are produced where an avalanche consists of simultaneous rupture of several elements. Such avalanches closely correspond to the bursts of acoustic emissions\textsuperscript{[6,7]} which are observed experimentally during the failure process of several materials. If one counts all avalanches till the complete failure, their size distribution is typically a power law. However we observed recently that if one records avalanches not for the entire failure process, but within a finite interval, a clear crossover behavior\textsuperscript{[8,9]} is seen between two power laws, with a large exponent when the system is far away from the failure point and a much smaller exponent for avalanches in the vicinity of complete failure. Therefore such crossover behavior in the burst statistics can be taken as a criterion for imminent failure. In this report we discuss the crossover behavior in two different fracture-breakdown models- fiber bundle models\textsuperscript{[10-18]} which describe the failure of composite...
material under external load and fuse models [11][19][20] which demonstrate the breakdown of electrical networks. An analytic derivation of the crossover behavior is given for fiber bundle models and numerical study confirms similar behavior in a fuse model. Thus we claim that such crossover behavior can be used as a signal of imminent failure. A recent observation [21] of the existence of a crossover behavior in the magnitude distribution of earthquakes within Japan has strengthened this claim.

2 FIBER BUNDLE MODEL

Our fiber bundle model consists of $N$ elastic and parallel fibers, clamped at both ends, with statistically distributed thresholds for breakdown of individual fibers (Figure 1). The individual thresholds $x_i$ are assumed to be independent random variables drawn from the same cumulative distribution function $P(x)$ and a corresponding density function $p(x)$:

$$\text{Prob}(x_i < x) = P(x) = \int_0^x p(u) \, du. \quad (1)$$

Figure 1. A fiber bundle of $N$ parallel fibers clamped at both ends. The externally applied force $F$ corresponds to a stretching by an amount $\delta$.

Whenever a fiber experiences a force equal to or greater than strength threshold $x_i$, it breaks immediately and does not contribute to the strength of the bundle thereafter. The maximal load the bundle can resist before complete breakdown is called the critical load and its value depends upon the probability distribution of the thresholds. Two popular examples of threshold distributions are the uniform distribution

$$P(x) = \begin{cases} \frac{x}{x_r} & \text{for } 0 \leq x \leq x_r \\ 1 & \text{for } x > x_r, \end{cases} \quad (2)$$

and the Weibull distribution

$$P(x) = 1 - \exp\left(-\left(\frac{x}{x_r}\right)^\kappa\right). \quad (3)$$

Here $x_r$ is a reference threshold, and the dimensionless number $\kappa$ is the Weibull index (Figure 2.)
Fiber bundle models differ in the mechanism for how the extra stress caused by a fiber failure is redistributed among the unbroken fibers. The simplest models are the equal-load-sharing models, in which the load previously carried by a failed fiber is shared equally by all the remaining intact fibers in the system. In the present article we study the statistics of burst avalanches for equal-load-sharing models.

2.1 Burst statistics

The burst distribution $D(\Delta)$ is defined as the expected number of bursts in which $\Delta$ fibers break simultaneously when the bundle is stretched steadily until complete breakdown. Hemmer and Hansen performed a detailed statistical analysis [14] to find the burst distribution for this quasi-static situation. For a bundle of many fibers they calculated the probability of a burst of size $\Delta$ starting at fiber $k$ with threshold value $x_k$ as

$$
\frac{\Delta^{\Delta-1}}{\Delta!} \left[ 1 - \frac{x_k p(x_k)}{Q(x_k)} \right] \left[ \frac{x_k p(x_k)}{Q(x_k)} \right]^{\Delta-1} \times \exp \left[ -\Delta \frac{x_k p(x_k)}{Q(x_k)} \right],
$$

where $Q(x) = 1 - P(x)$. Since a burst of size $\Delta$ can occur at any point before complete breakdown, the above expression has to be integrated over all possible values of $x_k$, i.e., from 0 to $x_c$ where $x_c$ is the maximum amount of stretching beyond which the bundle collapses completely. Therefore the burst distribution is given by [14]

$$
\frac{D(\Delta)}{N} = \frac{\Delta^{\Delta-1} e^{-\Delta}}{\Delta!} \int_0^{x_c} p(x) r(x) [1 - r(x)]^{\Delta-1} \exp [\Delta r(x)] \, dx,
$$

where

$$
r(x) = 1 - \frac{x p(x)}{Q(x)} = \frac{1}{Q(x)} \frac{d}{dx} [x Q(x)].
$$
The integration yields the asymptotic behavior

\[ D(\Delta)/N \propto \Delta^{-\frac{5}{2}}, \]  

(7)

which is universal under mild restrictions on the threshold distributions \[15\]. As a check, we have done simulation experiments for different threshold distributions. Figure 3 show results for the uniform threshold distribution (A) and the Weibull distribution (B) with index \(\kappa = 5\).

![Figure 3](image)

**Figure 3.** The burst distribution \(D(\Delta)/N\) for the uniform distribution (A) and the Weibull distribution with index 5 (B). The dotted lines represent the power law with exponent \(-5/2\). Both figures are based on 20000 samples of bundles each with \(N = 10^6\) fibers.

### 2.2 Crossover behavior near failure point

![Figure 4](image)

**Figure 4.** The distribution of bursts for thresholds uniformly distributed in an interval \((x_0, x_c)\), with \(x_0 = 0\) and with \(x_0 = 0.9x_c\). The figure is based on 50 000 samples, each with \(N = 10^6\) fibers.

When all the bursts are recorded for the entire failure process, we have seen that the burst distribution \(D(\Delta)\) follows the asymptotic power law \(D \propto \Delta^{-5/2}\). If we just sample bursts that occur near criticality, a different behavior is seen. As an illustration we consider the uniform
threshold distribution, and compare the complete burst distribution with what one gets when one samples merely burst from breaking fibers in the threshold interval \((0.9 x_c, x_c)\). Figure 4 shows clearly that in the latter case a different power law is seen.

From Eq. \(6\) we see that \(r(x)\) vanishes at the point \(x_c\). If we have a situation in which the weakest fiber has its threshold \(x_0\) just a little below the critical value \(x_c\), the contribution to the integral in the expression \(5\) for the burst distribution will come from a small neighborhood of \(x_c\). Since \(r(x)\) vanishes at \(x_c\), it is small here, and we may in this narrow interval approximate the \(\Delta\)-dependent factors in \(5\) as follows

\[
(1 - r)^\Delta e^{\Delta r} = \exp[\Delta \ln(1 - r) + r] = \exp[-\Delta r^2/2 + O(r^3)] \approx \exp[-\Delta r(x)^2/2] \tag{8}
\]

We also have

\[
r(x) \approx r'(x_c)(x - x_c). \tag{9}
\]

Inserting everything into Eq. \(5\), we obtain to dominating order

\[
\frac{D(\Delta)}{N} = \frac{\Delta e^{-\Delta} \int_{x_0}^{x_c} p(x_c) r'(x_c)(x - x_c)e^{-\Delta r'(x_c)^2(x - x_c)^2/2} dx}{\Delta!} = \frac{\Delta e^{-\Delta} p(x_c)}{|r'(x_c)| \Delta!} \left[ 1 - e^{-\Delta r'(x_c)^2(x - x_c)^2/2} \right]_{x_0} = \frac{\Delta e^{-\Delta} p(x_c)}{|r'(x_c)| \Delta!} \left[ 1 - e^{-\Delta / \Delta_c} \right], \tag{10}
\]

with

\[
\Delta_c = \frac{2}{r'(x_c)^2(x_c - x_0)^2}. \tag{11}
\]

By use of the Stirling approximation \(\Delta! \approx \Delta e^{-\Delta} \sqrt{2\pi\Delta}\), the burst distribution \(10\) may be written as

\[
\frac{D(\Delta)}{N} = C \Delta^{-3/2} \left( 1 - e^{-\Delta / \Delta_c} \right), \tag{12}
\]

with a nonzero constant

\[
C = (2\pi)^{-1/2} p(x_c) / |r'(x_c)|. \tag{13}
\]

We can see from \(12\) that there is a crossover at a burst length around \(\Delta_c\):

\[
\frac{D(\Delta)}{N} \propto \begin{cases} 
\Delta^{-3/2} & \text{for } \Delta \ll \Delta_c \\
\Delta^{-5/2} & \text{for } \Delta \gg \Delta_c 
\end{cases} \tag{14}
\]

We have thus shown the existence of a crossover from the generic asymptotic behavior \(D \propto \Delta^{-5/2}\) to the power law \(D \propto \Delta^{-3/2}\) near criticality, i.e., near global breakdown. The crossover is a universal phenomenon, independent of the threshold distribution \(p(x)\).
Figure 5. The distribution of bursts for the uniform threshold distribution (left) with $x_0 = 0.80x_c$ and for a Weibull distribution (right) with $x_0 = 1$ (square) and $x_0 = 1.7$ (circle). Both the figures are based on 50000 samples with $N = 10^6$ fibers each. The straight lines represent two different power laws, and the arrows locate the crossover points $\Delta_c \simeq 12.5$ and $\Delta_c \simeq 14.6$, respectively.

For the uniform distribution $\Delta_c = (1 - x_0/x_c)^{-2}/2$, so for $x_0 = 0.8x_c$, we have $\Delta_c = 12.5$. For the Weibull distribution $P(x) = 1 - \exp(-(x - 1)^{10})$, where $1 \leq x \leq \infty$, we get $x_c = 1.72858$ and for $x_0 = 1.7$, the crossover point will be at $\Delta_c \simeq 14.6$. Such crossover is clearly observed (Figure 5) near the expected values $\Delta = \Delta_c = 12.5$ and $\Delta = \Delta_c = 14.6$, respectively, for the above distributions.

Figure 6. The distribution of bursts for the uniform threshold distribution for a single fiber bundle with $N = 10^7$ fibers. Results with $x_0 = 0$, i.e., when all avalanches are recorded, are shown as squares and data for avalanches near the critical point ($x_0 = 0.9x_c$) are shown by circles.

The simulation results we have shown so far are based on averaging over a large number of samples. For applications it is important that crossover signal can be seen also in a single sample. We show in Figure 6 that equally clear crossover behavior is seen in a single fiber bundle when $N$ is large enough. Also, as a practical tool one must sample finite intervals $(x_i, x_f)$ during the fracture process. The crossover will be observed when the interval is close to the failure point [9].
3 BURST STATISTICS IN THE FUSE MODEL

![Diagram of a fuse model and burst distribution](image)

*Figure 7. A fuse model and the burst distribution: System size is $100 \times 100$ and averages are taken for 300 samples.*

On the average, catastrophic failure sets in after 2097 fuses have blown. The circles denote the burst distribution measured throughout the entire breakdown process. The squares denote the burst distribution based on bursts appearing after the first 1000 fuses have blown. The triangles denote the burst distribution after 2090 fuses have blown. The two straight lines indicate power laws with exponents $\xi = 3$ and $\xi = 2$, respectively.

To test the crossover phenomenon in a more complex situation than for fiber bundles, we have studied burst distributions in the fuse model [1]. It consists of a lattice in which each bond is a fuse, i.e., an ohmic resistor as long as the electric current it carries is below a threshold value. If the threshold is exceeded, the fuse burns out irreversibly. The threshold $t$ of each bond is drawn from an uncorrelated distribution $p(t)$. The lattice is placed between electrical bus bars and an increasing current is passed through it. The lattice is a two-dimensional square one placed at $45^\circ$ with regards to the bus bars, and the Kirchhoff equations are solved numerically at each node assuming that all fuses have the same resistance.

When one records all the bursts, the distribution follows a power law $D(\Delta) \propto \Delta^{-\xi}$ with $\xi = 3$, which is consistent with the value reported in recent studies [19, 20]. We show the histogram in Figure 7. With a system size of $100 \times 100$, 2097 fuses blow on the average before catastrophic failure sets in. When measuring the burst distribution only after the first 2090 fuses have blown, a different power law is found, this time with $\xi = 2$. After 1000 blown fuses, on the other hand, $\xi$ remains the same as for the histogram recording the entire failure process (Figure 7).

In Figure 8, we show the power dissipation $E$ in the network as a function of the number of blown fuses and as a function of the total current. The dissipation is given as the product of the voltage drop across the network $V$ times the total current that flows through it. The breakdown process starts by following the lower curve, and follows the upper curve returning to the origin. It is interesting to note the linearity of the unstable branch of this curve. In Figure 9, we record the avalanche distribution for power dissipation, $D_d(\Delta)$. Recording, as before, the avalanche distribution throughout the entire process as well as recording only close to the point at which the system catastrophically fails, result in two power laws, with exponents $\xi = 2.7$ and $\xi = 1.9$, respectively. It is interesting to note that in this case there is not a difference of unity between the
two exponents. The power dissipation in the fuse model corresponds to the stored elastic energy in a network of elastic elements. Hence, the power dissipation avalanche histogram would in the mechanical system correspond to the released energy. Such a mechanical system could serve as a simple model for earthquakes.

The Gutenberg-Richter law [1, 2] relating the frequency of earthquakes with their magnitude is essentially a measure of the elastic energy released in the earth’s crust, as the magnitude of an earthquake is the logarithm of the elastic energy released. Hence, the power dissipation avalanche histogram $D_d(\Delta)$ in the fuse model corresponds to the quantity that the Gutenberg-Richter law addresses in seismology. Furthermore, the power law character of $D_d(\Delta)$ is consistent with the form of the Gutenberg-Richter law. It is then intriguing that there is a change in exponent $\xi$ also for this quantity when failure is imminent.

![Figure 8. Power dissipation $E$ as a function of the number of broken bonds (left) and as a function of the total current $I$ flowing in the fuse model (right).](image)

![Figure 9. The power dissipation avalanche histogram $D_d(\Delta)$ for the fuse model. The slopes of the two straight lines are $-2.7$ and $-1.9$, respectively. The circles show the histogram of avalanches recorded through the entire process, whereas the squares show the histogram recorded after 2090 fuses have blown.](image)
4 CONCLUDING REMARKS

Establishing a signature of imminent failure is the principal objective in our study of different breakdown phenomena. The same goal is of course central to earthquake prediction scheme. As in different failure situations, bursts can be recorded from outside -without disturbing the ongoing failure process, burst statistics are much easily available and also contain reliable information of the failure process. Therefore, any signature in burst statistics that can warn of imminent system failure would be very useful in the sense of wide scope of applicability. The crossover behavior in burst distributions, we found in the fiber bundle models, is such a signature which signals that catastrophic failure is imminent. Similar crossover behavior is also seen in the burst distribution and energy distributions of the fuse model. Most important is that this crossover signal does not hinge on observing rare events and is seen also in a single system. Therefore, such signature has a strong potential to be used as useful detection tool. It should be mentioned that most recently, Kawamura [21] has observed a change in exponent values of the local magnitude distributions of earthquakes in Japan, before the onset of a mainshock (Figure 10). This observation has definitely strengthened our claim of using crossover signals in burst statistics as a criterion for imminent failure.

Figure 10. Crossover signature in the local magnitude distributions of earthquakes in Japan. The exponent of the distribution during 100 days before a mainshock is about 0.60, much smaller than the average value 0.88 [21].

ACKNOWLEDGMENT

S. P. thanks the Research Council of Norway (NFR) for financial support through Grant No. 166720/V30.
References

[1] H. J. Herrmann and S. Roux, eds. *Statistical Models for the Fracture of Disordered Media* (Elsevier, Amsterdam, 1990).

[2] B. K. Chakrabarti and L. G. Benguigui *Statistical Physics and Breakdown in Disordered Systems* (Oxford University Press, Oxford, 1997).

[3] D. Sornette, *Critical Phenomena in Natural Sciences*, Springer-Verlag, Berlin (2000).

[4] M. Sahimi, *Heterogeneous Materials II: Nonlinear and Breakdown Properties*, Springer-Verlag, Berlin (2003).

[5] P. Bhattacharyya and B. K. Chakrabarti (Eds), *Modelling Critical and Catastrophic Phenomena in Geoscience*, Springer, Berlin (2006).

[6] A. Petri, G. Paparo, A. Vespignani, A. Alippi, and M. Costantini, Phys. Rev. Lett **73** 3423 (1994).

[7] A. Garcimartin, A. Guarino, L. Bellon, and S. Ciliberto, Phys. Rev. Lett. **79** 3202 (1997).

[8] S. Pradhan, A. Hansen, and P. C. Hemmer, Phys. Rev. Lett. **95**, 125501 (2005).

[9] S. Pradhan, A. Hansen, and P. C. Hemmer, Phys. Rev. E **74**, 026106 (2006).

[10] F. T. Peirce, J. Text. Ind. **17**, 355 (1926).

[11] H. E. Daniels, Proc. Roy. Soc. London A**183**, 405 (1945).

[12] R. L. Smith, Ann. Prob. **10**, 137 (1982).

[13] S. L. Phoenix and R. L. Smith, Int. J. Sol. Struct. **19**, 479 (1983).

[14] P. C. Hemmer and A. Hansen, ASME J. Appl. Mech. **59**, 909 (1992).

[15] M. Kloster, A. Hansen, and P. C. Hemmer, Phys. Rev. E **56**, 2615 (1997).

[16] S. Pradhan, P. Bhattacharyya, and B. K. Chakrabarti, Phys. Rev. E **66** 016116 (2002).

[17] P. Bhattacharyya, S. Pradhan, and B. K. Chakrabarti, Phys. Rev. E **67**, 046122 (2003).

[18] R. C. Hidalgo, Y. Moreno, F. Kun, and H. J. Herrmann, Phys. Rev. E **65**, 046148 (2002).

[19] A. Hansen and P. C. Hemmer, Trends in Statistical Physics **1**, 213 (1994).

[20] S. Zapperi, P. Ray, H. E. Stanley, and A. Vespignati, Phys. Rev. Lett. **85**, 2865 (2000).

[21] H. Kawamura, arXiv: *cond-mat/0603335* (2006).