The quantum 2-sphere as a complex quantum manifold

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Abstract

We describe the quantum sphere of Podleš for $c = 0$ by means of a stereographic projection which is analogous to that which exhibits the classical sphere as a complex manifold. We show that the algebra of functions and the differential calculus on the sphere are covariant under the coaction of fractional transformations with $SU_q(2)$ coefficients as well as under the action of $SU_q(2)$ vector fields. Going to the classical limit we obtain the Poisson sphere. Finally, we study the
invariant integration of functions on the sphere and find its relation with the translationally invariant integration on the complex quantum plane.
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1 INTRODUCTION

Quantum spheres can be defined in any number of dimensions by normalizing a vector of quantum Euclidean space. The differential calculus on quantum Euclidean space induces a calculus on the quantum sphere. The case of two-spheres in three space is special in that there are many more possibilities than the one obtained from the general construction. These have been studied by P. Podleś who has also shown how to define a noncommutative differential calculus on them. In this paper we study in detail a particular case of Podleś spheres which is one of those special to three space dimensions. In this case the algebra of functions on the sphere is a subalgebra of the algebra of functions on $SU_q(2)$ and the differential calculus on the sphere can be inferred from a differential calculus on $SU_q(2)$. We can also define a stereographic projection and describe the coaction of $SU_q(2)$ on the sphere by fractional transformations on the complex variable in the plane analogous to the classical ones. The quantum sphere appears then as the quantum deformation of the classical two-sphere described as a complex manifold.

Our quantization of the sphere is not symmetric between the north and the south pole. This asymmetry is also apparent when we go to the classical limit of the Poisson sphere and it seems to be unavoidable in our approach. A description of Podleś spheres was given in an interesting paper by Šťovíček. He shows that the sphere can be understood as the patching of two complex quantum planes. His choice of variable is symmetric between the two planes, but the coaction of $SU_q(2)$ is very complicated in terms of his variable. Also, Šťovíček does not consider the noncommutative calculus on the sphere.
2 $S^2_q$ AS A COMPLEX MANIFOLD

In Ref.\[3\], a family of quantum 2-spheres was introduced. There, the algebra of functions over the sphere is generated by 3 coordinates, subjected to a condition that reduces the number of independent generators to 2. The case of $c = 0$ is of special interest\[9\]. In this case, the algebra is generated by $b_+ = \gamma \delta, b_- = \alpha \beta, b_3 = \alpha \delta$, (where $\alpha, \beta, \gamma, \delta \in SU_q(2)$) with commutations

\[
\begin{align*}
  b_+ b_- & = (1 - q^{-2}) b_- + q^{-2} b_3, \\
  b_3 b_+ & = b_+ (1 - q^2) + q^2 b_+ b_3, \\
  q^{-2} b_- b_+ & = q^2 b_+ b_- + (q^{-1} - q)(b_3 - 1),
\end{align*}
\]

and constraint

\[
  b_3^2 = b_3 + q^{-1} b_- b_+.
\]

The $*$-algebra structure is $b_+^* = -q^{-1} b_+, b_3^* = b_3$, and $q^* = q$.

One can construct a stereographic projection to go from the 3 coordinates $b_\pm, b_3$ to the complex plane $z, \bar{z}$. Define

\[
\begin{align*}
  z & = -q b_- (1 - b_3)^{-1} = \alpha \gamma^{-1}, \\
  \bar{z} & = b_+ (1 - b_3)^{-1} = -\delta \beta^{-1},
\end{align*}
\]

which is the projection from the north pole of the sphere to the plane with coordinates $z, \bar{z}$. It is easy to derive the commutation relation

\[
  z \bar{z} = q^{-2} \bar{z} z + q^{-2} - 1
\]

and the $*$-structure $z^* = \bar{z}$. This differs from the usual quantum plane by an additional inhomogeneous constant term. One can check directly that Eq.\[7\] is covariant under the fractional transformation, with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU_q(2)$,

\[
  z \to (a z + b)(c z + d)^{-1}, \quad \bar{z} \to -(c - d \bar{z})(a - b \bar{z})^{-1},
\]

which is induced from the $SU_q(2)$ coproduct, interpreted as a left transformation. Here $a, b, c$ and $d$ commute with $z$ and $\bar{z}$. 

\[2\]
3 DIFFERENTIAL CALCULUS

In Refs. [4, 5, 6], differential structures on $S^2_q$ are studied and classified. In this section we give a differential calculus on $S^2_q$ in terms of the complex coordinates $z$ and $\bar{z}$. Just as the algebras of functions and vector fields on $S^2_q$ can be inferred from those of $SU_q(2)$, so can the differential calculus.

For $SU_q(2)$ there are several well-known calculi [11, 12]: the 3D left- and right-covariant differential calculi, and the $4D_+, 4D_-$ bi-covariant calculi. The 4D bi-covariant calculi have one extra dimension in their space of one-forms compared with the classical case. The right-covariant calculus will not give a projection on $S^2_q$ in a closed form in terms of $z$, $\bar{z}$, which are defined to transform from the left. Therefore we shall choose the left-covariant differential calculus.

It is straightforward to obtain the following relations from those for $SU_q(2)$:

$$zd\bar{z} = q^{-2}dzz, \quad \bar{z}d\bar{z} = q^2d\bar{z}$$  \hspace{1cm} (9)

$$zd\bar{z} = q^{-2}d\bar{z}z, \quad \bar{z}d\bar{z} = q^2d\bar{z}\bar{z},$$  \hspace{1cm} (10)

$$(dz)^2 = (d\bar{z})^2 = 0,$$  \hspace{1cm} (11)

and

$$dzd\bar{z} = -q^{-2}d\bar{z}dz.$$  \hspace{1cm} (12)

We can also define derivatives $\partial, \bar{\partial}$ such that on functions,

$$d = dz\partial + d\bar{z}\bar{\partial}.$$  \hspace{1cm} (13)

From the requirement $d^2 = 0$ and the undeformed Leibniz rule for $d$ together with Eqs. (9) to (11), it follows that:

$$\partial z = 1 + q^{-2}z\partial, \quad \partial \bar{z} = q^2\bar{z}\partial, \quad (14)$$

$$\bar{\partial} z = q^{-2}z\bar{\partial}, \quad \bar{\partial} \bar{z} = 1 + q^2\bar{z}\bar{\partial}, \quad (15)$$
and

\[ \partial \bar{\partial} = q^{-2} \bar{\partial} \partial. \quad (16) \]

It can be checked explicitly that these commutation relations are covariant under the transformation (8) and

\[ dz \to dz(q^{-1}cz + d)^{-1}(cz + d)^{-1}, \quad (17) \]
\[ \partial \to (cz + d)(q^{-1}cz + d)\bar{\partial}, \quad (18) \]

which follow from (8) and the fact that \( d \) is invariant.

The \( \ast \)-structure also follows from that of \( SU_q(2) \):

\[ (dz)^\ast = d\bar{z}, \quad (19) \]
\[ \partial^\ast = -q^{-2} \bar{\partial} + (1 + q^{-2})z\rho^{-1}, \quad (20) \]
\[ \bar{\partial}^\ast = -q^2 \partial + (1 + q^2)\rho^{-1} \bar{z}. \quad (21) \]

where we have introduced

\[ \rho = 1 + \bar{z}z \quad (22) \]

(remember that the \( \ast \)-involution inverts the order of factors in a product).

The inhomogeneous pieces on the RHS of the Eqs. (20) and (21) reflect the fact that the sphere has curvature. Incidentally all the commutation relations in this section admit another possible involution:

\[ (dz)^\ast = d\bar{z}, \quad (23) \]
\[ \partial^\ast = -q^2 \bar{\partial}, \quad (24) \]
\[ \bar{\partial}^\ast = -q^{-2} \partial. \quad (25) \]

This involution is not covariant under the fractional transformations and cannot be used for the sphere. However, it can be used when we have a quantum plane defined by the same algebra of functions and calculus.

We shall take Eqs. (9) to (22) as the definition of the differential calculus on \( S^2_q \).
It is interesting to note that there exist two different types of symmetries in the calculus. The first symmetry is that if we put a bar on all unbarred variables \((z, \, dz, \, \partial)\), take away the bar from any barred ones and at the same time replace \(q\) by \(1/q\) in any statement about the calculus, the statement is still true.

The second symmetry is the consecutive operation of the two \(*\)-involutions above, so that

\[
\partial \rightarrow -q^2 \bar{\partial}^* = q^4 \partial - q^2 (1 + q^2) \rho^{-1} \bar{z},
\]

\[
\bar{\partial} \rightarrow -q^{-2} \partial^* = q^{-4} \bar{\partial} - q^{-2} (1 + q^{-2}) z \rho^{-1},
\]

with \(z, \bar{z}, dz, d\bar{z}\) unchanged. This replacement can be iterated \(n\) times and gives a symmetry which resembles that of a gauge transformation on a line bundle:

\[
\partial \rightarrow \partial^{(n)} \equiv q^{4n} \partial - q^2 [2n]_{q} \rho^{-1} \bar{z}
\]

\[
= q^{4n} \rho^{2n} \partial \rho^{-2n},
\]

\[
\bar{\partial} \rightarrow \bar{\partial}^{(n)} \equiv q^{-4n} \bar{\partial} - q^{-2} [2n]_{1/q} \rho^{-1}
\]

\[
= q^{-4n} \rho^{2n} \bar{\partial} \rho^{-2n},
\]

where \([n]_q = \frac{q^{2n}-1}{q^2-1}\). For example, we have

\[
\partial^{(n)} z = 1 + q^{-2} z \partial^{(n)}.
\]

Making a particular choice of \(\partial, \bar{\partial}\) is like fixing a gauge.

Many of the features of a calculus on a classical complex manifold are preserved. Define \(\delta = dz\partial\) and \(\bar{\delta} = d\bar{z}\bar{\partial}\) as the exterior derivatives on the holomorphic and antiholomorphic functions on \(S^2_q\) respectively. We have:

\[
[\delta, \bar{z}] = dz, \quad [\delta, \bar{z}] = 0,
\]

\[
[\bar{\delta}, z] = 0, \quad [\bar{\delta}, \bar{z}] = d\bar{z},
\]

\[
d = \delta + \bar{\delta}.
\]
The action of $\delta$ and $\bar{\delta}$ can be extended consistently on forms as follows

$$\delta dz = dz\delta = 0, \quad \bar{\delta} \bar{z} = d\bar{z}\bar{\delta} = 0,$$

$$\{\delta, d\bar{z}\} = 0, \quad \{\bar{\delta}, dz\} = 0,$$

$$\delta^2 = \bar{\delta}^2 = 0,$$

$$\{\delta, \bar{\delta}\} = 0,$$

where $\{\cdot, \cdot\}, [\cdot, \cdot]$ are the anticommutator and commutator respectively.

4 THE RIGHT INVARIANT VECTOR FIELDS
ON \(S^2_q\)

In this section we want to define vector fields on \(S^2_q\) which generate the fractional transformation mentioned above. We will see that these vector fields can be inferred from those on \(SU_q(2)\).

First let us recall some well-known facts about the vector fields on \(SU_q(2)\) (see for example Ref.[13]). The enveloping algebra \(\mathcal{U}\) of \(SU_q(2)\) is usually said to be generated by the left-invariant vector fields \(H_L, X_{L\pm}\) which are arranged in two matrices \(L^+\) and \(L^-\). The action of these vector fields corresponds to infinitesimal right transformation: \(T \rightarrow TT'\). What we want now is the infinitesimal version of the left transformation given by Eqs.(8), hence we shall use the right-invariant vector fields \(H_R, X_{R\pm}\). Since only the right-invariant ones will be used, we will drop the subscript \(R\) hereafter.

The properties of the right-invariant vector fields are similar to those of the left-invariant ones. Note that if an \(SU_q(2)\) matrix \(T\) is transformed from the right by another \(SU_q(2)\) matrix \(T'\), then it is equivalent to say that the \(SU_{1/q}(2)\) matrix \(T^{-1}\) is transformed from the left by another \(SU_{1/q}(2)\) matrix \(T'^{-1}\). Therefore one can simply write down all properties of the left-invariant vector fields and then make the replacements: \(q \rightarrow 1/q, \ T \rightarrow T^{-1}\) and left-invariant fields $\rightarrow$ right-invariant fields.
Using the matrices:

\[
M^+ = \begin{pmatrix}
q^{-H/2} & q^{-1/2}\lambda X_+ \\
0 & q^{H/2}
\end{pmatrix}, \quad M^- = \begin{pmatrix}
q^{H/2} & 0 \\
-q^{1/2}\lambda X_- & q^{-H/2}
\end{pmatrix},
\]  

(40)

the commutation relations between the vector fields are given by,

\[
R_{12}M^+_2 M^+_1 = M^+_1 M^+_2 R_{12},
\]

(41)

\[
R_{12}M^-_2 M^-_1 = M^-_1 M^-_2 R_{12},
\]

(42)

\[
R_{12}M^+_2 M^-_1 = M^-_1 M^+_2 R_{12},
\]

(43)

while the commutation relations between the vector fields and the elements of the quantum matrix in the smash product of \( \mathcal{U} \) and \( SU_q(2) \) are,

\[
T_1 M^+_2 = M^+_2 \mathcal{R}_{12} T_1,
\]

(44)

\[
T_1 M^-_2 = M^-_2 \mathcal{R}_{21}^{-1} T_1,
\]

(45)

where \( T \) is a \( SU_q(2) \) matrix, \( \mathcal{R} = q^{-1/2}R \) and \( R \) is the \( GL_q(2) \) R-matrix.

Clearly \( M^+ \), and \( M^- \) are the right-invariant counterparts of \( L^+ \) and \( L^- \). The commutation relations between the \( M \)'s and the \( T \)'s tell us how the functions on \( SU_q(2) \) are transformed by the vector fields \( H, X_+, X_- \). It is convenient to define a different basis for the vector fields,

\[
Z_+ = X_+ q^{H/2},
\]

(46)

\[
Z_- = q^{H/2} X_-
\]

(47)

and

\[
\mathcal{H} = [H]_q = \frac{q^{2H-1}}{q^2-1}.
\]

(48)

They satisfy the commutation relations

\[
\mathcal{H}Z_+ - q^4 Z_+ \mathcal{H} = (1 + q^2) Z_+,
\]

(49)

\[
Z_- \mathcal{H} - q^4 \mathcal{H}Z_- = (1 + q^2) Z_-
\]

(50)
Using the expressions of \( z, \bar{z} \) in terms of \( \alpha, \beta, \gamma, \delta \), one can easily find the action of the vector fields on the variables \( z, \bar{z} \) on the sphere,

\[
\begin{align*}
Z_+ z &= q^2 z Z_+ + q^{1/2} z^2, \\
\bar{Z}_+ \bar{z} &= q^{-2} \bar{z} \bar{Z}_+ + q^{-3/2}, \\
\mathcal{H}z &= q^4 z \mathcal{H} + (1 + q^2)z, \\
\mathcal{H}\bar{z} &= q^{-4} \mathcal{H} - q^{-4}(1 + q^2)\bar{z}, \\
\bar{Z}_- z &= q^2 z \bar{Z}_- - q^{1/2}
\end{align*}
\]

and

\[
\begin{align*}
\bar{Z}_- \bar{z} &= q^{-2} \bar{z} \bar{Z}_- - q^{-3/2} \bar{z}^2.
\end{align*}
\]

It is clear that a \( * \)-involution can be given:

\[
\begin{align*}
Z^*_+ &= Z_-, & \mathcal{H}^* &= \mathcal{H}.
\end{align*}
\]

Since all the relations listed above are closed in the vector fields and \( z, \bar{z} \) (this would not be the case if we had used the left-invariant fields), we can now take these equations as the definition of the vector fields that generate the fractional transformation on \( S^2_\nu \). We shall take our vector fields to commute with the exterior differentiation \( d \). This is consistent for right-invariant vector fields in a left-covariant calculus and allows us to obtain the action of our vector fields on the differentials \( dz \) and \( d\bar{z} \), as well as on the derivatives \( \partial \) and \( \bar{\partial} \). For instance (52) gives

\[
\begin{align*}
\mathcal{Z}_+ dz &= q^2 dz Z_+ + q^{1/2}(dz \bar{z} + \bar{z} dz)
\end{align*}
\]

and

\[
\begin{align*}
\partial \mathcal{Z}_+ &= q^2 \mathcal{Z}_+ \partial + q^{-3/2}(1 + q^2)z \partial.
\end{align*}
\]
5 MORE ABOUT THE CALCULUS

The calculus described in the previous section has a very interesting property. There exists a one-form $\Xi$ having the property that

$$\Xi f \mp f\Xi = \lambda df,$$

where, as usual, the minus sign applies for functions or even forms and the plus sign for odd forms. Indeed, it is very easy to check that

$$\Xi = \xi - \xi^*$$

and

$$\xi = qdz\rho^{-1}\bar{z}$$

satisfies Eq. (61) and

$$\Xi^* = -\Xi.$$

It is also easy to check that

$$d\Xi = 2qd\bar{z}\rho^{-2}dz$$

and

$$\Xi^2 = q\lambda d\bar{z}\rho^{-2}dz.$$  

Suitably normalized, $d\Xi$ is the natural area element on the quantum sphere. Notice that $\Xi^2$ commutes with all functions and forms, as required for consistency with the relation

$$d^2 = 0.$$  

The existence of the form $\Xi$ within the algebra of $z, \bar{z}, dz, d\bar{z}$ is especially interesting because no such form exists for the 3-D calculus on $SU_q(2)$, from which we have derived the calculus on the quantum sphere (a one-form analogous to $\Xi$ does exist for the two bicovariant calculi on $SU_q(2)$, but we have explained before why we didn't choose either of them). It is also interesting that $d\Xi$ and $\Xi^2$ do not vanish (as the corresponding expressions do in the
bicovariant calculi on the quantum groups or in the calculus on quantum Euclidean space). We see here an example of Connes’ calculus\cite{14} of the type $F^2 = 1$ rather than $F^2 = 0$.

The one-form $\Xi$ is regular everywhere on the sphere, except at the point $z = \bar{z} = \infty$, which classically corresponds to the north pole. We shall discuss this question in Sec.\footnote{7} where we argue that the pole singularity at that point can be included by allowing forms with distribution valued coefficients. The area element $d\Xi$ is regular everywhere on the sphere.

It is interesting to see how $\Xi$ and $d\Xi$ transform under the action of the right invariant vector fields or under the coaction of the fractional transformations (8). Using (52) to (57) one finds

$$Z_+\Xi = \Xi Z_+ + q^{-1/2}dz$$

and

$$H\Xi = \Xi H. \quad (69)$$

These equations are consistent with (61). For instance,

$$Z_+(\lambda dz - \Xi z + z\Xi) = \frac{q^2}{z}(\lambda dz - \Xi z + z\Xi)Z_+ + q^{1/2}(\lambda dz - \Xi z^2 + z^2\Xi) - q^{-1/2}(dzz - q^2zdz). \quad (70)$$

Eqs. (68) and (69) imply that $d\Xi$ commutes with $Z_\pm$ and $H$, as expected for the invariant area element.

For the fractional transformation (8) one finds $\xi \rightarrow \xi'$ where

$$\xi' - \xi = -q(dz)cd^{-1}(1 + cd^{-1}z)^{-1} \quad (71)$$

and a similar formula for $\xi^*$. The right hand side of (71) is a closed one-form, since $(dz)^2 = 0$, so one could write

$$\xi' - \xi = -qd[\log_q(1 + cd^{-1}z)] \quad (72)$$
with a suitably defined quantum function $\log_q$. At any rate

$$d\xi' = d\xi$$

(73)

so that the area element two-form is invariant under finite transformations as well.

6 PATCHING TWO QUANTUM PLANES

The variables $z$ and $\bar{z}$ cover the sphere with the exception of the north pole. In analogy with the classical case, we can introduce new variables $w = z^{-1}$ and $\bar{w} = \bar{z}^{-1}$ which describe the sphere without the south pole. These variables satisfy the commutation relation

$$w\bar{w} = q^{-2}\bar{w}w + (q^{-2} - 1)w\bar{w}^2w$$

(74)

which is covariant under the transformation

$$w \rightarrow (dw + c)(bw + a)^{-1}, \quad \bar{w} \rightarrow -(a\bar{w} - b)(c\bar{w} - d)^{-1}.$$  

(75)

Notice that the commutation relation (74) is different from that satisfied by $z$ and $\bar{z}$; our way of quantizing the sphere is inherently asymmetric between the north and the south pole.

The calculus in $z$ and $\bar{z}$ induces a calculus in $w$ and $\bar{w}$. It is not hard to derive the commutation relations for this $w, \bar{w}$ calculus as well as the mixed commutation relations. For example, we have

$$wdw = q^2dww,$$

(76)

$$\partial_w w = 1 + q^2w\partial_w$$

(77)

and

$$dzw = q^{-2}wdz.$$  

(78)
Since $w$ and $\bar{w}$ are functions of $z$ and $\bar{z}$, Eq. (61) is valid for functions and forms in $w$ and $\bar{w}$, with the same $\Xi$. In terms of $w$ and $\bar{w}$ the one-forms $\xi$ and $\xi^*$ are given by

$$
\xi = -w^{-1}dw(1 + \bar{w}w)^{-1}, \quad \xi^* = -(1 + \bar{w}w)^{-1}d\bar{w}\bar{w}^{-1}.
$$

Clearly they are singular at the north pole $w = \bar{w} = 0$. This polar singularity is an intrinsic feature of our asymmetric quantization and of our calculus. We believe that it can be controlled by allowing distributions, rather than just functions as the elements of our algebra and as coefficients of differential forms. In order to avoid the need to develop the concept of distribution in the framework of noncommutative algebra, we explain our point of view in the next section for the limit of the Poisson sphere.

## 7 THE POISSON SPHERE

The commutation relations of the previous sections give us, in the limit $q \to 1$, a Poisson structure on the sphere. The Poisson Brackets (P.B.s) are obtained as usual as a limit

$$
(f, g) = \lim_{h \to 0} \left( \frac{fg - gh}{h} \right), \quad q^2 = e^h = 1 + h + [h^2].
$$

For instance, the commutation relation (7) gives

$$
zz = (1 - h)\bar{z}z - h + [h^2]
$$

and therefore

$$
(\bar{z}, z) = \rho.
$$

Similarly one finds

$$
(dz, z) = zdz, \quad (d\bar{z}, z) = zd\bar{z},
$$

$$
(dz, \bar{z}) = -\bar{z}dz, \quad (d\bar{z}, \bar{z}) = -\bar{z}d\bar{z}
$$

12
and

\[(d\bar{z}, dz) = d\bar{z}dz.\]  (85)

In this classical limit functions and forms commute or anticommute according to their even or odd parity, as usual. The P.B. of any quantity with itself vanishes. The P.B. of two even quantities or of an even and an odd quantity is antisymmetric, that of two odd quantities is symmetric. It is

\[d(f, g) = (df, g) \pm (f, dg)\]  (86)

where the plus (minus) sign applies for even (odd) \(f\). Notice that we have enlarged the concept of Poisson bracket to include differential forms. This is very natural when considering the classical limit of our commutation relations.

In the classical limit, Eq. (61) becomes

\[(\Xi, f) = df\]  (87)

where

\[\Xi = \xi - \xi^*\]  (88)

and

\[\xi = d\bar{z}\bar{z}\rho^{-1}, \quad \xi^* = d\bar{z}z\rho^{-1}\]  (89)

are ordinary classical differential forms. Now

\[d\Xi = 2d\bar{z}dz\rho^{-2}\]  (90)

and

\[\Xi^2 = 0.\]  (91)

As before, the variables \(z\) and \(\bar{z}\) cover the sphere except for the north pole, while \(w\) and \(\bar{w}\) miss the south pole. It is

\[(\bar{w}, w) = \bar{w}w(1 + \bar{w}w).\]  (92)
The Poisson structure is not symmetric between the north and south pole. All P.B.s of regular functions and forms vanish at the north pole \( w = \bar{w} = 0 \). Therefore, for Eq.(87) to be valid, the one-form \( \Xi \) must be singular at the north pole. Indeed one finds

\[
\xi = \frac{dw\bar{w}}{1 + \bar{w}w} - \frac{d\bar{w}}{\bar{w}}, \quad \xi^* = \frac{d\bar{w}w}{1 + \bar{w}w} - \frac{dw}{\bar{w}},
\]

(93)

and

\[
\Xi = \frac{wd\bar{w} - \bar{w}dw}{\bar{w}(1 + \bar{w}w)}.
\]

(94)

On the other hand the area two-form

\[
d\Xi = 2\frac{d\bar{w}dw}{1 + \bar{w}w} \equiv \Omega
\]

(95)

is regular everywhere on the sphere.

The singularity of \( \Xi \) at the north pole is not a real problem if we treat it in the sense of the theory of distributions. Consider a circle \( C \) of radius \( r \) encircling the origin of the \( w \) plane in a counter-clockwise direction and set

\[
w = re^{i\theta}, \quad \bar{w} = re^{-i\theta}.
\]

(96)

Using (93), we have

\[
\int_C \Xi = \int_C \frac{\bar{w}dw - \bar{w}d\bar{w}}{1 + \bar{w}w} - 4\pi i.
\]

(97)

As \( r \to 0 \) the integral in the right hand side tends to zero because the integrand is regular at the origin. Stokes theorem can be satisfied even at the origin if we modify Eq.(95) to read

\[
d\Xi = \Omega - 4\pi i\delta(w)\delta(\bar{w})d\bar{w}dw.
\]

(98)

It is

\[
\int_{S^2} \Omega = 4\pi i
\]

(99)
so that
\[ \int_{S^2} d\bar{z} = 0 \] (100)
as it should be for a compact manifold without boundary. Notice that the additional delta function term in (98) also has zero P.B.s with all functions and forms as required by consistency.

8 INTEGRATION

We now return to the quantum case. For the integral of a function \( f \) over the sphere we shall use the notation \( \langle f \rangle \). A left-invariant integral can be defined, up to a normalization constant, by requiring invariance under the action of the right-invariant vector fields
\[
\langle \mathcal{O} f(z, \bar{z}) \rangle = 0, \quad \mathcal{O} = \mathcal{Z}_+, \mathcal{Z}_-, \mathcal{H}. \tag{101}
\]

Using \( \mathcal{H} \) and Eqs.(54) and (55) one finds that
\[
\langle z^k \bar{z}^l g(\bar{z}z) \rangle = 0, \text{ unless } k = l. \tag{102}
\]
(Here \( g \) is a convergence function.) Therefore we can restrict ourselves to integrals of the form \( \langle f(\bar{z}z) \rangle \).

Eqs.(52) and (53) imply
\[
\mathcal{Z}_+ \rho = \rho \mathcal{Z}_+ + q^{1/2} z \rho \tag{103}
\]
and
\[
\mathcal{Z}_+ \rho^{-l} = \rho^{-l} \mathcal{Z}_+ - q^{-3/2[l]} z \rho^{-l}. \tag{104}
\]
From \( \langle \mathcal{Z}_+ (\bar{z} \rho^{-l}) \rangle = 0, \ l \geq 1, \) one finds easily the recursion formula
\[
[l + 1]_q < \rho^{-l} > = [l]_q < \rho^{-l+1} >, \ l \geq 1, \tag{105}
\]
which gives
\[
\langle \rho^{-l} \rangle = \frac{1}{[l + 1]_q} < 1 >, \ l \geq 0. \tag{106}
\]
Similarly
\[
< \frac{\bar{z}z}{(1 + \bar{z}z)^l} > = \left( \frac{1}{[l]_q} - \frac{1}{[l + 1]_q} \right) < 1 >, \quad l \geq 1. \tag{107}
\]

We leave it to the reader to find the expression for
\[
< \frac{(\bar{z}z)^k}{(1 + \bar{z}z)^l} >, \quad l \geq k. \tag{108}
\]

The above results can also be obtained by using the relations (3) and (4) for \( z \) and \( \bar{z} \) in terms of the \( SU_q(2) \) parameters and known results \cite{10,13} for the Haar measure of \( SU_q(2) \). However we wanted to show that one can formulate the integration directly for the sphere.

As an application of the stereographic projection, we can define an integration on the complex quantum plane \( C_q \) by inserting an appropriate measure factor \( \rho^2 \). \( C_q \) has the same algebra (5) and differential calculus (6) to (10), but a different \( * \)-structure (23) to (25). Classically, it holds
\[
\int_{C_q} dzd\bar{z}/2\pi if(z, \bar{z}) = \int_{S^2} \rho^2 f(z, \bar{z}) \]
Motivated by this, we define an integration over the quantum plane as,
\[
\int_{C_q} f(z, \bar{z}) \equiv < \rho^2 f(z, \bar{z}) > . \tag{109}
\]

We need to check that this integration is translationally invariant, namely, \( \int \partial f = \int \bar{\partial} f = 0 \). To show this, we must find relations between the infinitesimal generators \( \partial, \bar{\partial} \) on the plane and \( \mathcal{Z}_+, \mathcal{Z}_-, \mathcal{H} \) on the sphere. Introduce the differential operators,
\[
C = 1 - \lambda q^{-1}z\partial, \tag{110}
\]
\[
D = 1 + \lambda q\bar{z}\bar{\partial} \tag{111}
\]
and
\[
B = 1 - \lambda q^{-1}z\partial + \lambda q\bar{z}\bar{\partial} - \lambda^2 q^{-2}\rho \bar{\partial} \partial. \tag{112}
\]

One finds the following realizations of \( \mathcal{Z}_+, \mathcal{Z}_-, \mathcal{H} \) as pseudo-differential operators, which satisfy Eqs. (49) to (58):
\[
q^{3/2} \mathcal{Z}_+ = (z^2 \partial + q^2 \bar{\partial}B^{-1})C^{-1}, \tag{113}
\]
\[
-q^{3/2} \mathcal{Z}_- = (q^2 \bar{z}^2 \bar{\partial} + \partial B^{-1})D^{-1}. \tag{114}
\]
and

\[ \mathcal{H} = \frac{1 - B^{-2}}{1 - q^2}. \]  \hspace{1cm} (115)

One also has

\[ q^{-1} \rho^2 \partial = (Z_- z Z_+ - q^4 Z_+ z Z_- + q^{1/2}(1 + q^2)Z_+)B \]  \hspace{1cm} (116)

and

\[ q^{-1} \rho^2 \partial = (q^4 Z_+ \bar{z} Z_- - \bar{z} Z_- Z_+ - q^{1/2}(1 + q^2)Z_-)B. \]  \hspace{1cm} (117)

Together with the definition \((109)\), we have

\[ \int_{C_q} \partial f = \langle \rho^2 \partial f \rangle = \langle \partial Z_- \cdots \rangle - \langle \partial Z_+ \cdots \rangle \]  \hspace{1cm} (118)

and

\[ \int_{C_q} \bar{\partial} f = \langle \rho^2 \bar{\partial} f \rangle = \langle \bar{\partial} Z_- \cdots \rangle - \langle \bar{\partial} Z_+ \cdots \rangle, \]  \hspace{1cm} (119)

which are both zero since the integral on the sphere is defined by \( \langle O f \rangle = 0 \) for \( O = \mathcal{Z}_\pm, \mathcal{H} \). So the integral defined by \((109)\) is translational invariant.

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