THE ZILBER-PINK CONJECTURE AND THE GENERALIZED
COSMETIC SURGERY CONJECTURE

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Abstract. In this paper, we generalize the Cosmetic Surgery Conjecture to an \( n \)-cusped hyperbolic 3-manifold and prove it under the assumption of another well-known conjecture in number theory, so called the Zilber-Pink Conjecture. For \( n = 1 \) and 2, we show them without the assumption.

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1. Introduction

1.1. Main Results

Dehn filling is one of the most fundamental topological operations in the field of low dimensional topology. More than 50 years ago, W. Lickorish and A. Wallace showed that any closed connected orientable 3-manifold can be obtained by a Dehn filling on a link complement. In the late 1970’s, W. Thurston, as a part of his revolutionary work, showed Dehn filling behaves very nicely under hyperbolic structure by proving that if the original 3-manifold is hyperbolic, then almost all of its Dehn fillings are also hyperbolic. Since then, understanding hyperbolic Dehn filling has become a central topic in the study of 3-dimensional geometry and topology.

There are many questions and conjectures regarding Dehn filling. Among them, the following conjecture, which was proposed by C. Gordon in 1990 [7] (see also Kirby’s problem list [12]), is one of the well-known conjectures in the topic and has been largely studied by many people ([2], [13], [17], [18], [20], [21], etc):

**Conjecture 1** (Cosmetic Surgery Conjecture (Hyperbolic Case)). Let $M$ be a 1-cusped hyperbolic 3-manifold. Let $M(p/q)$ and $M(p'/q')$ be the $p/q$ and $p'/q'$-Dehn filled manifolds of it (respectively) which are also hyperbolic. If

$$p/q \neq p'/q',$$

then there is no orientation preserving isometry between $M(p/q)$ and $M(p'/q')$.

The goal of this paper is to suggest a new approach to solve the above conjecture and consider its generalization to an $n$-cusped manifold. First we resolve the conjecture for a 1-cusped hyperbolic 3-manifold whose cusp shape is not quadratic possibly with finitely many exceptions as follows:

**Theorem 1.1.** Let $M$ be a 1-cusped hyperbolic 3-manifold whose cusp shape is non-quadratic. Then, for sufficiently large $|p| + |q|$ and $|p'| + |q'|$,

$$M(p/q) \cong M(p'/q')$$

if and only if

$$p/q = p'/q'$$

where $\cong$ represents an orientation preserving isometry.

The above theorem is a consequence of the following theorem.

**Theorem 1.2.** Let $M$ be a 1-cusped hyperbolic 3-manifold whose cusp shape is non-quadratic. Let $t_{p/q}$ (where $|t_{p/q}| > 1$) be the holonomy of the core geodesic of its $p/q$-Dehn filling. Then

$$t_{p/q} = t_{p'/q'}$$ (1.1)

if and only if

$$p/q = p'/q'$$

for sufficiently large $|p| + |q|$ and $|p'| + |q'|$.

If the Dehn filling coefficients are sufficiently large, then the core geodesic is the shortest geodesic for each Dehn filled manifold. Thus if there exists an isometry between $M(p/q)$ and $M(p'/q')$, the holonomy of the core geodesic of $M(p/q)$, which represents the core geodesic of $M(p/q)$, maps to the holonomy of the core geodesic of $M(p'/q')$. 
A naive extension of the above theorem to a 2-cusped manifold is not true. For example, if $\mathcal{M}$ is the Whitehead link complement, there is a self-isometry sending one cusp to the other and so
\[
\mathcal{M}(p_1/q_1, p_2/q_2) \cong \mathcal{M}(p_2/q_2, p_1/q_1)
\]
for any $p_1/q_1$ and $p_2/q_2$. Thus extending Theorem 1.1 to a more cusped manifold requires a necessary condition to assure that there are no symmetries between the given cusps. Let $\tau_1$ and $\tau_2$ be two cusp shapes of a 2-cusped hyperbolic 3-manifold $\mathcal{M}$. If there is an isometry sending one cusp to the other, then
\[
\tau_1 = \frac{a\tau_2 + b}{c\tau_2 + d}
\]
for some $a, b, c, d \in \mathbb{Z}$ such that $ad - bc = \pm 1$. So if
\[
1, \tau_1, \tau_2, \tau_1 \tau_2
\]
are linearly independent over $\mathbb{Q}$, then it guarantees that there is no symmetry between two cusps of $\mathcal{M}$. Inspired from this, we extend (1.2) to an $n$-cusped manifold as follows:

**Definition 1.3.** Let $\mathcal{M}$ be an $n$-cusped manifold and $\tau_1, \ldots, \tau_n$ be its cusp shapes. We say $\mathcal{M}$ has rationally independent cusp shapes if the elements in
\[
\{\tau_{i_1} \cdots \tau_{i_t} \mid 1 \leq i_1 < \cdots < i_t \leq n\}
\]
are linearly independent over $\mathbb{Q}$.\(^2\)

Having this definition, our second main result is the following theorem, which we can view as a natural generalization of the Cosmetic Surgery Conjecture:

**Theorem 1.4.** Let $\mathcal{M}$ be an $n$-cusped ($n \geq 2$) hyperbolic 3-manifold having non-quadratic and rationally independent cusp shapes. If the Zilber-Pink conjecture is true, then
\[
\mathcal{M}(p_1/q_1, \ldots, p_n/q_n) \cong \mathcal{M}(p'_1/q'_1, \ldots, p'_n/q'_n)
\]
if and only if
\[
(p_1/q_1, \ldots, p_n/q_n) = (p'_1/q'_1, \ldots, p'_n/q'_n)
\]
for sufficiently large $|p_i| + |q_i|$ and $|p'_i| + |q'_i|$ ($1 \leq i \leq n$).

The statement of the Zilber-Pink conjecture will be given in Subsection 1.2 below. The above theorem is a consequence of the following theorem:

**Theorem 1.5.** Let $\mathcal{M}$ be the same manifold given in Theorem 1.4 with
\[
\{t_1^{(p_1/q_1, \ldots, p_n/q_n)} \cdots t_n^{(p_1/q_1, \ldots, p_n/q_n)} \mid (t^{(p_1/q_1, \ldots, p_n/q_n)} | > 1)\}
\]
the set of holonomies of its $(p_1/q_1, \ldots, p_n/q_n)$-Dehn filling of it. If the Zilber-Pink conjecture is true, then
\[
\{t_1^{(p_1/q_1, \ldots, p_n/q_n)} \cdots t_n^{(p_1/q_1, \ldots, p_n/q_n)} \} = \{t_1^{(p'_1/q'_1, \ldots, p'_n/q'_n)} \cdots t_n^{(p'_1/q'_1, \ldots, p'_n/q'_n)} \}
\]
if and only if
\[
(p_1/q_1, \ldots, p_n/q_n) = (p'_1/q'_1, \ldots, p'_n/q'_n)
\]
for sufficiently large $|p_i| + |q_i|$ and $|p'_i| + |q'_i|$ ($1 \leq i \leq n$).

For 2-cusped manifolds, as mentioned earlier, we generalize it without the Zilber-Pink conjecture under a slightly stronger condition on two cusp shapes as follows:

\(^2\)Note that this definition is also independent of the choice of basis.
Theorem 1.6. Let $M$ be a 2-cusped hyperbolic 3-manifold with two cusp shapes $\tau_1$ and $\tau_2$. Suppose that the elements in
\[
\{\tau^j_1\tau^j_2 | 0 \leq i, j \leq 2\}
\]
appears linearly independent over $\mathbb{Q}$. Then
\[
M(p_1/q_1, p_2/q_2) \cong M(p'_1/q'_1, p'_2/q'_2)
\]
if and only if
\[
(p_1/q_1, p_2/q_2) = (p'_1/q'_1, p'_2/q'_2)
\]
for sufficiently large $|p_i| + |q_i|$ and $|p'_i| + |q'_i|$ ($1 \leq i \leq 2$).

This is a consequence of the following theorem:

Theorem 1.7. Let $M$ be the same manifold given in Theorem 1.6 with
\[
\left\{t^1_{(p_1/q_1, p_2/q_2)}, t^2_{(p_1/q_1, p_2/q_2)} \right\} \quad \left(|t^i_{(p_1/q_1, p_2/q_2)}| > 1\right)
\]
the set of holonomies of its $(p_1/q_1, p_2/q_2)$-Dehn filling. Then
\[
\left\{t^1_{(p'_1/q'_1, p'_2/q'_2)}, t^2_{(p'_1/q'_1, p'_2/q'_2)} \right\} = \left\{t^1_{(p'_1/q'_1, p'_2/q'_2)}, t^2_{(p'_1/q'_1, p'_2/q'_2)} \right\}
\]
if and only if
\[
(p_1/q_1, p_2/q_2) = (p'_1/q'_1, p'_2/q'_2)
\]
for sufficiently large $|p_i| + |q_i|$ and $|p'_i| + |q'_i|$ ($1 \leq i \leq 2$).

We also prove the following theorem which is of independent interest and will be used in the proof of a main theorem:

Theorem 1.8. Let $X$ be a holonomy of an $n$-cusped hyperbolic 3-manifold $M$ with
\[
\left\{t^1_{(p_1/q_1, \ldots, p_n/q_n)}, \ldots, t^n_{(p_1/q_1, \ldots, p_n/q_n)} \right\}
\]
the set of holonomies of the Dehn filling coefficient $(p_1/q_1, \ldots, p_n/q_n)$ with $|p_i| + |q_i|$ ($1 \leq i \leq n$) sufficiently large. If the Zilber-Pink conjecture is true, then the elements in (1.3) are multiplicatively independent for sufficiently large $|p_i| + |q_i|$ ($1 \leq i \leq n$).

See Definition 3.13 for the definition of being multiplicatively independent.

1.2. Unlikely Intersections and the Zilber-Pink Conjecture

The study of unlikely intersections concerns about intersections between an algebraic variety and multiplicative algebraic subgroups. In particular, it is interested in how the points on intersection parts behave as one varies algebraic subgroups. The study was first initiated by E. Bombieri, D. Masser, and U. Zannier in the 1990’s, has grown and become an active research area in number theory nowadays 22. One of the main conjectures in the field is the one so called the Zilber-Pink conjecture, which was independently stated by B. Zilber and R. Pink in 2002 and 2005 respectively. This is a generalization of many classical conjectures such as the Mumford-Manin, Andre-Oort and Mordell-Lang conjectures. There are various versions of this, but the one that we need is the following, which was stated in 5:
Conjecture 2 (Zilber-Pink Conjecture). Let \( G_m^{r+s} \) be either \((\mathbb{Q}_s^*)^{r+s}\) or \((\mathbb{C}^*)^{r+s}\). For every variety \( X \) of dimension \( r \) in \( G_m^{r+s} \) defined over \( \mathbb{Q} \) and irreducible over \( \overline{\mathbb{Q}} \), there is a finite union \( U = \bigcup(X) \) of proper algebraic subgroups of \( G_m^{r+s} \) such that \( X \cap H_{s-1} \) is contained in \( U \) where \( H_{s-1} \) be the set of algebraic subgroups of dimension \( s-1 \) in \( G_m^{r+s} \). In particular, possibly except for finitely many points, \( X \cap H_{s-1} \) is contained in a finite number of anomalous subvarieties of \( X \).

See Definition 3.4 for the definition of an anomalous subvariety. In general, if \( H \) is an algebraic subgroup of dimension \( s-1 \), then the intersection between \( H \) and \( X \) is empty. So when it has non-empty intersection, it means they are not in general position (this is a heuristic meaning of “unlikely intersections”). What the conjecture says is that if there are infinitely many points on \( X \) arising from unlikely intersections, these points are not randomly distributed but rather lying in a finite number of algebraic subgroups. In some sense, this conjecture follows the spirit of Falting’s theorem or, more generally, the Bombieri-Lang conjecture, implying the set of rational points on a given variety (of general type) is not Zariski dense but contained in its finitely many subvarieties.

The Zilber-Pink conjecture was proved for the curve case by G. Maurin [14], but is widely open for other cases.

1.3. Key Idea

The reason that theories in the study of unlikely intersections are applicable to various problems of hyperbolic Dehn filling is one can interpret this geometric and topological phenomenon as an algebro-geometric one. More precisely, we can view an \( n \)-cusped hyperbolic 3-manifold \( \mathcal{M} \) as an \( n \)-dimensional algebraic variety \( \mathcal{X} \), and Dehn filling on \( \mathcal{M} \) as the intersection between \( \mathcal{X} \) and an algebraic subgroup whose index is given by the Dehn filling coefficient. For example, if \( \mathcal{M} \) is a 1-cusped hyperbolic 3-manifold, then, by the work of Thurson, its representation variety

\[
\mathcal{X} := \text{Hom} \left( \pi_1(\mathcal{M}), \text{SL}_2(\mathbb{C}) \right) / \sim
\]

(1.4)

(where \( \sim \) is an equivalence under conjugation) is known to be a 1-dimensional algebraic variety. (See Section 2.2 for the precise definition of this variety.) In particular, this variety is parameterized by the holonomies of the longitude and meridian of the cusp. Let \( M, L \) be the parameters of those holonomies and \( f(M, L) = 0 \) be a polynomial representing (1.4). If \( \pi_1(\mathcal{M}) \) is of the following form

\[
\{G \mid R\},
\]

then, by the Seifert-Van Kampen theorem, the fundamental group of \( \mathcal{M}(p/q) \) is

\[
\{G \mid R, m^p l^q = 1\}
\]

where \( m \) and \( l \) are the meridian and longitude of the cusp. So we obtain the (discrete faithful) representation of \( \pi_1(\mathcal{M}(p/q)) \) (into \( \text{SL}_2(\mathbb{C}) \)) by finding an intersection point between \( f(M, L) = 0 \) and \( M^p L^q = 1 \).

Now suppose

\[
\mathcal{M}(p/q) \cong \mathcal{M}(p'/q')
\]

(1.5)

with sufficiently large \( |p| + |q| \) and \( |p'| + |q'| \), and consider \( \mathcal{X} \times \mathcal{X} \) defined by

\[
f(M, L) = 0, \quad f(M', L') = 0.
\]

(1.6)
We associate $\mathcal{M}(p/q)$ and $\mathcal{M}(p'/q')$ with the first and second coordinates of $\mathcal{X} \times \mathcal{X}$ respectively. As mentioned earlier, a part of Thurston’s Dehn filling theorem shows $m'r^s$ (where $-qr + ps = 1$) represents the shortest geodesic of $\mathcal{M}(p/q)$ for $|p| + |q|$ sufficiently large. Since the shortest geodesic of $\mathcal{M}(p/q)$ maps to the shortest geodesic of $\mathcal{M}(p'/q')$ under the isometry given in (1.5), the holonomies of two shortest geodesics are the same and so

$$M' L^s = (M')^{r'} (L')^{s'}$$

where $-q'r' + p's' = 1$. In other words, the existence of an isometry between $\mathcal{M}(p/q)$ and $\mathcal{M}(p'/q')$ guarantees the existence of an intersection point between

$$f(M, L) = 0, \quad f(M', L') = 0$$

and a 1-dimensional algebraic subgroup defined by

$$M p L^s = 1,$$

$$(M')^{r'} (L')^{s'} = 1,$$

$$M' L^s = (M')^{r'} (L')^{s'}.$$  \hspace{1cm} (1.8)

So if there are infinitely many different pairs

$$\{(p_i/q_i, p'_i/q'_i) \mid i \in \mathcal{I}\}$$

such that

$$\mathcal{M}(p_i/q_i) \cong \mathcal{M}(p'_i/q'_i) \quad (i \in \mathcal{I}),$$

then it implies there are infinitely many 1-dimensional algebraic subgroups \{H_i \mid i \in \mathcal{I}\} having nonempty intersection points with (1.7). Thus the problem naturally fits into the framework of the Zilber-Pink conjecture. In view of the Zilber-Pink conjecture, all those intersection points are not irregularly distributed, but instead contained in a finite number of algebraic subgroups. When $f(M, L) = 0$ arises as the representation variety of a 1-cusped hyperbolic 3-manifold, we show the only possible algebraic subgroup containing all those points is

$$M = M', \quad L = L'.$$

This implies

$$p_i/q_i = p'_i/q'_i$$

and resolves the Cosmetic Surgery Conjecture.

1.4. Outline of the paper

The paper consists of four parts after Introduction, the background part and other three parts that will cover the proofs of three main theorems. In Sections 2 and 3, we study some necessary background on both hyperbolic geometry and number theory. In Sections 4, 5, and 6, we establish our three major theorems, Theorem 1.2, Theorem 1.7 and Theorem 1.5 respectively. For each section, we provide preliminary lemmas and theorems in its subsection that play key roles in proving a main theorem. But we encourage audience to go over the

\[3\] That is, we identify $\mathcal{M}(p/q)$ as

$$f(M, L) = 0, \quad M p L^s = 1,$$

and $\mathcal{M}(p'/q')$ as

$$f(M', L') = 0, \quad (M')^{r'} (L')^{s'} = 1.$$
statements only first without seeing any of the proofs at first reading. Also the proof of the 1-cusped case is the simplest one, but provide the essential ideas. So we suggest the reader go over it thoroughly before reading other proofs. In particular, the proof of Theorem 1.7 follows the same strategy and scheme presented in the proof of Theorem 1.2. The proof of Theorem 1.5 is somewhat different and independent from the proofs of other theorems, so it would be reasonable to skip Sections 4 and 5 and directly go to Section 6 at first reading.

1.5. **Acknowledgement**

This paper is an extension of the author’s previous work [11]. They partially overlap each other. In particular, Theorem 1.2 and a slightly different version of Theorem 1.6 were already announced in the earlier paper. Note that the statement of Theorem 1.6 is different from the earlier version (Theorem 1.8 [11]) as follows. First, in Theorem 1.6 we remove the assumption on the coefficient of the Neumann-Zagier potential function, which originally appeared in Theorem 1.8 [11]. Also the required condition on two cusp shapes of a given hyperbolic 3-manifold in the statement of Theorem 1.6 is a bit stronger than the one presented in the statement of Theorem 1.8 [11].
2. Background I (Hyperbolic Geometry)

2.1. Gluing variety

In this section, we follow the same scheme in [16]. Suppose that $\mathcal{M}$ is an $n$-cusped hyperbolic 3-manifold whose hyperbolic structure is realized as a union of $k$ geometric tetrahedra having modulus $z_v$ ($1 \leq v \leq k$). Then the gluing variety of $\mathcal{M}$ is defined by the following form of $k$ equations where each represents the gluing condition at each edge of a tetrahedron:

$$\prod_{v=1}^{k} z_v^{\theta_1(r,v)} \cdot (1 - z_v)^{\theta_2(r,v)} = \pm 1$$

(2.1)

for $1 \leq r \leq k$, $\theta_1(r,v), \theta_2(r,v) \in \mathbb{Z}$. It is known that there is redundancy in the above equations so that exactly $k-n$ of them are independent [16]. We the gluing variety by $\text{Hol}(\mathcal{M})$ and the point corresponding to the complete structure on $\text{Hol}(\mathcal{M})$ by $z^0 = (z^0_1, \ldots, z^0_k)$.

Let $T_i$ be a torus cross-section of the $i^{\text{th}}$-cusp and $l_i, m_i$ be the chosen longitude-meridian pair of $T_i$ ($1 \leq i \leq n$). For each $z \in \text{Hol}(\mathcal{M})$, by giving similarity structures on the tori $T_i$, the dilation components of the holonomies (of the similarity structures) of $l_i$ and $m_i$ are represented in the following forms:

$$\delta(l_i)(z) = \pm \prod_{v=1}^{k} z_v^{\lambda_1(i,v)} \cdot (1 - z_v)^{\lambda_2(i,v)}$$

$$\delta(m_i)(z) = \pm \prod_{v=1}^{k} z_v^{\mu_1(i,v)} \cdot (1 - z_v)^{\mu_2(i,v)}.$$  

(2.2)

Then $\delta(l_i)(z)$ and $\delta(m_i)(z)$ behave very nicely near $z^0$ [16]:

**Theorem 2.1.** For each $i$ ($1 \leq i \leq n$), $\delta(l_i)(z) = 1$ and $\delta(m_i)(z) = 1$ are equivalent in a small neighborhood of $z^0$.

**Theorem 2.2.** $z^0$ is a smooth point of $\text{Hol}(\mathcal{M})$ and the unique point near $z^0$ with all $\delta(l_i)(z) = 1$ ($1 \leq i \leq n$).

By taking logarithms locally near the point $z^0$, we rewrite (2.1) as

$$\sum_{v=1}^{k} \left( \theta_1(r,v) \cdot \log(z_v) + \theta_2(r,v) \cdot \log(1 - z_v) \right) = c(r) \quad (1 \leq r \leq k - n)$$

(2.3)

where $c(r)$ are some suitable constants. If we let

$$u_i(z) = \log \left( \delta(l_i)(z) \right) \quad (1 \leq i \leq n)$$

$$v_i(z) = \log \left( \delta(m_i)(z) \right) \quad (1 \leq i \leq n),$$

(2.4) (2.5)

then $v_1, \ldots, v_n$ are parameterized holomorphically in terms of $u_1, \ldots, u_n$ as below [16]:

**Theorem 2.3.** In a neighborhood of the origin in $\mathbb{C}^n$ with $u_1, \ldots, u_n$ as coordinates, the following statements hold for each $i$ ($1 \leq i \leq n$):

1. $v_i = u_i \cdot \tau_i(u_1, \ldots, u_n)$ where $\tau_i(u_1, \ldots, u_n)$ is an even holomorphic function of its arguments with non-real algebraic number $\tau_i(0, \ldots, 0) = \tau_i$ ($1 \leq i \leq n$).

2. There is a holomorphic function $\Phi(u_1, \ldots, u_n)$ such that $v_i = \frac{1}{2} \frac{\partial \Phi}{\partial u_i}$ ($1 \leq i \leq n$) and
\( \Phi(0, \ldots, 0) = 0 \).

(3) \( \Phi(u_1, \ldots, u_n) \) is even in each argument and it has Taylor expansion of the following form:

\[
\Phi(u_1, \ldots, u_n) = (\tau_1 u_1^2 + \cdots + \tau_n u_n^2) + \text{(higher order)}.
\]

We call \( \tau_i \) the cusp shape of \( T_i \) with respect to \( l_i, m_i \) and \( \Phi(u_1, \ldots, u_n) \) the Neumann-Zagier potential function of \( \mathcal{M} \) with respect to \( \{l_i, m_i\}_{1 \leq i \leq n} \). Considering \( u_i, v_i \) \( (1 \leq i \leq n) \) as coordinates, we denote by \( \text{Def}(\mathcal{M}) \) the complex manifold, locally near \((0, \ldots, 0) \in \mathbb{C}^{2n}\) defined by the following holomorphic functions

\[
v_i = u_i \cdot \tau_i(u_1, \ldots, u_n) \quad (1 \leq i \leq n).
\]

Note that \( \text{Def}(\mathcal{M}) \) is biholomorphic to a small neighborhood of \( z^0 \) in \( \text{Hom}(\mathcal{M}) \).

### 2.2. Holonomy variety

Thinking of the holonomies of the meridian-longitude pairs \( \delta(l_i)(z) \) and \( \delta(m_i)(z) \) in (2.2) as new variables, we consider the algebraic variety defined by the following equations in \( \mathbb{C}^{k + 2n} \):

\[
\prod_{v=1}^{k} z_v^{\theta_1(v,r,v)} \cdot (1 - z_v)^{\theta_2(v,r,v)} = \pm 1
\]

\[
L_i = \pm \prod_{v=1}^{k} z_v^{\lambda_1(v,i,v)} \cdot (1 - z_v)^{\lambda_2(v,i,v)}
\]

\[
M_i = \pm \prod_{v=1}^{k} z_v^{\mu_1(v,i,v)} \cdot (1 - z_v)^{\mu_2(v,i,v)}
\]

where \( 1 \leq r \leq k - n \) and \( 1 \leq i \leq n \). We call this the holonomy variety of \( \mathcal{M} \) and denote it by \( \mathcal{X} \). Then the point corresponding to the complete structure is

\[
(z_0^1, \ldots, z_0^n, 1, \ldots, 1),
\]

and, by abusing the notation, we still denote (2.8) by \( z^0 \).

**Remark 1.** Throughout the paper, we are only interested in a small neighborhood of \( z^0 \) (see Theorem 2.4 and Theorem 2.5), and, in the proofs of the main theorems, usually work with \( \text{Def}(\mathcal{M}) \) instead of \( \mathcal{X} \) since \( \text{Def}(\mathcal{M}) \) is easier to deal with. For instance, if \( H \) is an algebraic variety defined by

\[
M_1^{a_{11}} L_1^{b_{11}} \cdots M_n^{a_{1n}} M_n^{b_{1n}} = 1,
\]

\[
\cdots
\]

\[
M_1^{a_{m1}} L_1^{b_{m1}} \cdots M_n^{a_{mn}} M_n^{b_{mn}} = 1,
\]

then, by taking logarithms to each coordinates, it is equivalent to

\[
a_{11} u_1 + b_{11} v_1 + \cdots + a_{1n} u_n + b_{1n} v_n = 0,
\]

\[
\cdots
\]

\[
a_{m1} u_1 + b_{m1} v_1 + \cdots + a_{mn} u_n + b_{mn} v_n = 0,
\]
and so, by Theorem 2.3, \( \mathcal{X} \cap H \) is locally biholomorphic (near \( z^0 \)) to the complex manifold defined by
\[
a_{11}u_1 + b_{11}u_1\tau_1(u_1, \ldots, u_n) + \cdots + a_{1n}u_n + b_{1n}u_n\tau_n(u_1, \ldots, u_n) = 0,
\]
\[
\vdots
\]
\[
a_{mn}u_1 + b_{m1}u_1\tau_1(u_1, \ldots, u_n) + \cdots + a_{mn}u_n + b_{mn}u_n\tau_n(u_1, \ldots, u_n) = 0.
\]
We find the dimension of \( \mathcal{X} \times \mathcal{X} \) holonomy variety, \( M \) we use the following coordinates for \( \text{Def}(M) \) and denote by \((z_1, \ldots, z_n, L_1, \ldots, L_n, z_1', \ldots, z_n', M_1', \ldots, M_n')\), and denote by \((z_1^0, z_n^0)\) the point in \( \mathcal{X} \times \mathcal{X} \) corresponding to the complete structure. Similarly, we use the following coordinates for \( \text{Def}(M) \times \text{Def}(M) \) in \( \mathbb{C}^{2n} \times \mathbb{C}^{2n} \):
\[
(u_1, v_1, \ldots, u_n, v_n, u_1', v_1', \ldots, u_n', v_n').
\]

**Remark 2.** As explained in the first section, we often need to consider two copies of a holonomy variety, \( \mathcal{X} \times \mathcal{X} \) in \( \mathbb{C}^{k+2n} \times \mathbb{C}^{k+2n} \). In this case, we use the following coordinates for its ambient space \( \mathbb{C}^{k+2n} \times \mathbb{C}^{k+2n} \):
\[
(z_1, \ldots, z_n, M_1, \ldots, L_n, z_1', \ldots, z_n', M_1', \ldots, M_n'),
\]
and denote by \((z_1^0, z_n^0)\) the point in \( \mathcal{X} \times \mathcal{X} \) corresponding to the complete structure. Similarly, we use the following coordinates for \( \text{Def}(M) \times \text{Def}(M) \) in \( \mathbb{C}^{2n} \times \mathbb{C}^{2n} \):
\[
(u_1, v_1, \ldots, u_n, v_n, u_1', v_1', \ldots, u_n', v_n').
\]

**Remark 3.** For a given cusped hyperbolic 3-manifold, its gluing variety (holonomy variety and potential function as well) depends on a choice of meridian-longitude pair, but they are all isomorphic. Sometimes changing basis from one to another and working with a different (isomorphic) variety are quite useful, and we use this technique several times in the proofs of the main theorems.

### 2.3. Dehn Filling

Hyperbolic Dehn filling can be defined in a few slightly different ways. In this paper, we adopt the definition that, after attaching a new torus, the core of the torus is always isotopic to a geodesic of the Dehn filled manifold.

Let
\[
\mathcal{M}(p_1/q_1, \ldots, p_n/q_n)
\]
be the \((p_1/q_1, \ldots, p_n/q_n)\)-Dehn filled manifold of \( \mathcal{M} \). By the Seifert-Van Kampen theorem, the fundamental group of \( \mathcal{M}(p_1/q_1, \ldots, p_n/q_n) \) is obtained by adding the relations
\[
m_1^{p_1} t_1^{q_1} = 1, \ldots, m_n^{p_n} t_n^{q_n} = 1
\]
to the fundamental group of \( \mathcal{M} \). Hence, on the holonomy variety of \( \mathcal{M} \), the hyperbolic structure of \( \mathcal{M}(p_1/q_1, \ldots, p_n/q_n) \) is identified with a point satisfying the additional equations corresponding to the above relations. More precisely, if the holonomy variety \( \mathcal{X} \) of \( \mathcal{M} \) is defined by
\[
f_i(z_1, \ldots, z_n, M_1, \ldots, M_n, L_1, \ldots, L_n) = 0 \quad (1 \leq i \leq s),
\]
then a holonomy representation of \( \mathcal{M} \) inducing \( \mathcal{M}(p_1/q_1, \ldots, p_n/q_n) \) is a point on \( \mathcal{X} \) satisfying the following equations:
\[
M_1^{p_1} L_1^{q_1} = 1, \ldots, M_n^{p_n} L_n^{q_n} = 1.
\]
We call \( \text{Def}(2.11) \) the **Dehn filling equations** with coefficient \((p_1/q_1, \ldots, p_n/q_n)\) and the points inducing the hyperbolic structure on \( \mathcal{M}(p_1/q_1, \ldots, p_n/q_n) \) the **Dehn filling points** corresponding to \( \mathcal{M}(p_1/q_1, \ldots, p_n/q_n) \). Let
\[
(M_1, L_1, \ldots, M_n, L_n) = (t_1^{-q_1}, t_1^{p_1}, \ldots, t_n^{-q_n}, t_n^{p_n})
\]
(2.13)
be the last \((2n)\)-coordinates of a Dehn filling point corresponding to \(\mathcal{M}(p_1/q_1, \ldots, p_n/q_n)\) such that \(|t_i| > 1\) \((1 \leq i \leq n)\). Then the holonomy of each core geodesic \(m^i_{t_i} t^i_{s_i}\) where 

\[ -q_i r_i + p_i s_i = 1 \]

is the set of holonomies of the Dehn filling coefficient \((p_1/q_1, \ldots, p_n/q_n)\). We define

\[ \{t_1, \ldots, t_n\} \]

as the set of holonomies of the Dehn filling coefficient \((p_1/q_1, \ldots, p_n/q_n)\). If there is no confusion, we simply identify \((2.13)\) with the corresponding Dehn filling point on \(X\) and call it the Dehn filling point associated with \(\mathcal{M}(p_1/q_1, \ldots, p_n/q_n)\).

The following theorems are parts of Thurston’s hyperbolic Dehn filling theory [16][19].

**Theorem 2.4.** Using the same notation as above,

\[
(t_1^{-q_1}, t_1^{p_1}, \ldots, t_n^{-q_n}, t_n^{p_n})
\]

converges to \((1, \ldots, 1)\) as \(|p_i| + |q_i|\) goes to \(\infty\) for \(1 \leq i \leq n\).

**Theorem 2.5.** Using the same notation as above,

\[ \{t_1, \ldots, t_n\} \]

converges to \((1, \ldots, 1)\) as \(|p_i| + |q_i|\) goes to \(\infty\) for \(1 \leq i \leq n\).

### 2.4. Cosmetic Surgery Point

For an \(n\)-cusped hyperbolic 3-manifold \(\mathcal{M}\), suppose

\[ \mathcal{M}(p_1/q_1, \ldots, p_n/q_n) \equiv \mathcal{M}(p'_1/q'_1, \ldots, p'_n/q'_n) \]

with sufficiently large \(|p_i| + |q_i|\) and \(|p'_i| + |q'_i|\) \((i = 1, 2)\). If

\[
(t_1, \ldots, t_n) \quad \text{(resp. } (t'_1, \ldots, t'_n)\text{)}
\]

is the set of holonomies of the Dehn filling coefficient \((p_1/q_1, \ldots, p_n/q_n)\) (resp. \((p'_1/q'_1, \ldots, p'_n/q'_n)\)), then, by Thurston’s Dehn filling theory, we get

\[ \{t_1, \ldots, t_n\} = \{t'_1, \ldots, t'_n\} \]

and so

\[
(t_1, \ldots, t_n) = (t'_\sigma(1), \ldots, t'_\sigma(n))
\]

for some \(\sigma \in S_n\). We consider \(X \times X\), and associate \(\mathcal{M}(p_1/q_1, \ldots, p_n/q_n)\) and \(\mathcal{M}(p'_1/q'_1, \ldots, p'_n/q'_n)\) to the first and second coordinate of it respectively. Then the following point

\[
(t_1^{-q_1}, \ldots, t_n^{p_n}, (t'_1)^{-q_1'}, \ldots, (t'_n)^{p_n'})
\]

is the image of an intersection point between \(X \times X\) and

\[
M^i t_i^i L^i_{s_i} = 1, \quad (M'_i)^{p'_i} (L'_i)^{q'_i} = 1, \quad M^i t_i^i L^i_{s_i} = (M'_i)^{p'_i} (L'_i)^{q'_i} (1 \leq i \leq n),
\]

under the following projection map:

\[
(z_1, \ldots, z_k, M_1, \ldots, L_n, z'_1, \ldots, z'_k, M'_1, \ldots, L'_n) \longrightarrow (M_1, \ldots, L_n, M'_1, \ldots, L'_n).
\]

We call \((2.14)\) the cosmetic surgery point associated with \(\mathcal{M}(p_1/q_1, \ldots, p_n/q_n)\) and \(\mathcal{M}(p'_1/q'_1, \ldots, p'_n/q'_n)\), and, by a slight abuse of terminology, identify it with the original corresponding intersection point between \(X \times X\) and \((2.15)\).
3. Background II (Number Theory)

3.1. Height

The height \( h(\alpha) \) of an algebraic number \( \alpha \) is defined as follows:

**Definition 3.1.** Let \( K \) be a number field containing \( \alpha \), \( V_K \) be the set of places of \( K \), and \( K_v, \mathbb{Q}_v \) be the completions at \( v \in V_K \). Then
\[
h(\alpha) = \sum_{v \in M_K} \log \left( \max \{1, |\alpha|_v\} \right) \left[ K_v : \mathbb{Q}_v \right] / \left[ K : \mathbb{Q} \right].
\]

Note that the above definition does not depend on the choice of \( K \). That is, for any number field \( K \) containing \( \alpha \), it gives us the same value.

The first property in the following theorem is a classical result due to D. Northcott and the rest can be easily deduced from the definition [6].

**Theorem 3.2.** (1) There are only finitely many algebraic numbers of bounded height and degree.
(2) \( h(\alpha^n) = |n|h(1/\alpha) \) for \( \alpha \in \overline{\mathbb{Q}} \).
(3) \( h(\alpha_1 + \cdots + \alpha_r) \leq \log r + h(\alpha_1) + \cdots + h(\alpha_r) \) for \( \alpha_1, \ldots, \alpha_r \in \overline{\mathbb{Q}} \).
(4) \( h(\alpha_1 \cdots \alpha_r) \leq h(\alpha_1) + \cdots + h(\alpha_r) \) for \( \alpha_1, \ldots, \alpha_r \in \overline{\mathbb{Q}} \).

If \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \overline{\mathbb{Q}}^n \) is a \( n \)-tuple of algebraic numbers, the definition can be generalized as follows:

**Definition 3.3.** Let \( K \) be an any number field containing \( \alpha_1, \ldots, \alpha_n \), \( V_K \) be the set of places of \( K \), and \( K_v, \mathbb{Q}_v \) be the completions at \( v \). Then
\[
h(\alpha) = \sum_{v \in V_K} \log \left( \max \{1, |\alpha_1|_v, \ldots, |\alpha_n|_v\} \right) \left[ K_v : \mathbb{Q}_v \right] / \left[ K : \mathbb{Q} \right].
\]

Similar to Theorem 3.2, the following inequalities hold:
\[
\max \{h(\alpha_1), \ldots, h(\alpha_n)\} \leq h(\alpha) \leq h(\alpha_1) + \cdots + h(\alpha_n).
\] (3.1)

3.2. Anomalous Subvarieties

In this section, we identify \( G^n_m \) with the non-vanishing of the coordinates \( x_1, \ldots, x_n \) in the affine \( n \)-space \( \overline{\mathbb{Q}}^n \) or \( \mathbb{C}^n \) (i.e. \( G^n_m = (\overline{\mathbb{Q}}^*)^n \) or \( (\mathbb{C}^*)^n \)). An algebraic subgroup \( H_\Lambda \) of \( G^n_m \) is defined as the set of solutions satisfying equations \( x_1^{a_1} \cdots x_n^{a_n} = 1 \) where the vector \( (a_1, \ldots, a_n) \) runs through a lattice \( \Lambda \subset \mathbb{Z}^n \). If \( \Lambda \) is primitive, then we call \( H_\Lambda \) an irreducible algebraic subgroup or algebraic torus. By a coset \( K \), we mean a translate \( gH \) of some algebraic subgroup \( H \) by some \( g \in G^n_m \). For more properties of algebraic subgroups and \( G^n_m \), see [6].

**Definition 3.4.** An irreducible subvariety \( \mathcal{V} \) of \( \mathcal{X} \) is anomalous (or better, \( \mathcal{X} \)-anomalous) if it has positive dimension and lies in a coset \( K \) in \( G^n_m \) satisfying
\[
\dim K \leq n - \dim \mathcal{X} + \dim \mathcal{Y} - 1.
\]

The quantity \( \dim \mathcal{X} + \dim K - n \) is what one would expect for the dimension of \( \mathcal{X} \cap K \) when \( \mathcal{X} \) and \( K \) were in general position. Thus we can understand anomalous subvarieties of
\( \mathcal{X} \) as the ones that are unnaturally large intersections of \( \mathcal{X} \) with cosets of algebraic subgroups of \( G^n_m \).

**Definition 3.5.** The deprived set \( \mathcal{X}^{oa} \) is what remains of \( \mathcal{X} \) after removing all anomalous subvarieties.

**Definition 3.6.** An anomalous subvariety of \( \mathcal{X} \) is maximal if it is not contained in a strictly larger anomalous subvariety of \( \mathcal{X} \).

The following theorem tells us the structure of anomalous subvarieties (Theorem 1 of [4]).

**Theorem 3.7.** Let \( \mathcal{X} \) be an irreducible variety in \( G^n_m \) of positive dimension defined over \( \mathbb{Q} \).

(a) For any torus \( H \) with
\[
1 \leq n - \dim H \leq \dim \mathcal{X},
\]
the union \( Z_H \) of all subvarieties \( \mathcal{Y} \) of \( \mathcal{X} \) contained in any coset \( K \) of \( H \) with
\[
\dim H = n - (1 + \dim \mathcal{X}) + \dim \mathcal{Y}
\]
is a closed subset of \( \mathcal{X} \), and the product \( HZ_H \) is not Zariski dense in \( G^n_m \).

(b) There is a finite collection \( \Psi = \Psi_{\mathcal{X}} \) of such tori \( H \) such that every maximal anomalous subvariety \( \mathcal{Y} \) of \( \mathcal{X} \) is a component of \( \mathcal{X} \cap gH \) for some \( H \) in \( \Psi \) satisfying (3.2) and (3.3) and some \( g \) in \( Z_H \). Moreover \( \mathcal{X}^{oa} \) is obtained from \( \mathcal{X} \) by removing the \( Z_H \) of all \( H \) in \( \Psi \), and thus it is open in \( \mathcal{X} \) with respect to the Zariski topology.

Following the definition given in [5], we have a refined version of Definition 3.4 as follows:

**Definition 3.8.** We say that an irreducible subvariety \( \mathcal{Y} \) of \( \mathcal{X} \) is \( b \)-anomalous \((b \geq 1)\) if it has positive dimension and lies in some coset of dimension
\[
n - (b + \dim \mathcal{X}) + \dim \mathcal{Y}.
\]

The following strengthening version of Theorem 3.7 is also true [5]:

**Theorem 3.9.** Suppose that \( \mathcal{X} \) is an irreducible variety in \( G^n_m \) defined over \( \mathbb{Q} \).

(a) The union of all subvarieties \( \mathcal{Y} \) of \( \mathcal{X} \) contained in any coset \( K \) of \( H \) with
\[
\dim H = n - (b + \dim \mathcal{X}) + \dim \mathcal{Y}
\]
is a closed subset of \( \mathcal{X} \).

(b) There is a finite collection \( \Psi = \Psi_{\mathcal{X}} \) of algebraic tori such that every maximal \( b \)-anomalous subvariety \( \mathcal{Y} \) of \( \mathcal{X} \) lies in a coset \( K \) of some \( H \) in \( \Psi \) satisfying (3.4).

Now we state the bounded height theorem due to P. Habegger.

**Theorem 3.10.** Let \( \mathcal{X} \subset G^n_m \) be an irreducible variety over \( \mathbb{Q} \). The height is bounded in the intersection of \( \mathcal{X}^{oa} \) with the union of algebraic subgroups of dimension \( \leq n - \dim \mathcal{X} \).

Using the proof of the above theorem, the author proved the following theorem [9][10]:

**Theorem 3.11.** Let \( \mathcal{M} \) be an \( n \)-cusped hyperbolic 3-manifold. Then the height of any Dehn filling point of \( \mathcal{M} \) is uniformly bounded.

A corollary of the above theorem is the following:

**Corollary 3.12.** There are only a finite number of hyperbolic 3-manifolds of bounded volume and trace field degree.

---

4By Theorem 2.1, for each \( i \) \((1 \leq i \leq k)\), the component of \( \mathcal{X} \cap (M_i = L_i = 1) \) containing \( z^0 \) is an anomalous subvariety of \( \mathcal{X} \).

5In [5], only the second statement was mentioned, but it is not difficult to show the first statement following the same idea given in the proof of Theorem 3.7.
3.3. Miscellaneous

In this section, we mention a couple of lemmas and a theorem which will be used crucially in the proofs of the main theorems.

**Definition 3.13.** We say \( \eta_1, \ldots, \eta_r \in \mathbb{Q} \) are multiplicatively dependent if there exist \( a_1, \ldots, a_r \in \mathbb{Z} \) such that
\[
\eta_1^{a_1} \cdots \eta_r^{a_r} = 1. \tag{3.5}
\]
Otherwise, we say they are multiplicatively independent.

The following is Lemma 7.1 in [3].

**Lemma 3.14.** [3] Given \( r \geq 1 \) there are positive constants \( c(r) \) and \( k(r) \) with the following property. Let \( K \) be a cyclotomic extension of degree \( d \) over \( \mathbb{Q} \) and let \( \eta_1, \ldots, \eta_r \) be multiplicatively independent non-zero algebraic numbers with \( [K(\eta_1, \ldots, \eta_r) : K] = \tilde{d} \). Then
\[
h(\eta_1) \cdots h(\eta_r) \geq \frac{c(r)}{d(\log(3))^{k(r)}}.
\]

**Lemma 3.15** (Siegel). Consider the following linear equations:
\[
\sum_{i=1}^{n} a_{ij} X_i \quad (1 \leq j \leq r). \tag{3.6}
\]
Let \( v_j = (a_{1j}, \ldots, a_{nj}) \) \( (1 \leq j \leq r) \) and \( \prod = |v_1| \cdots |v_r| \) where \( |v_j| \) represents its Euclidean length. Then there exist an universal constant \( c > 0 \) and \( (n-r) \)-independent vectors \( b_i \in \mathbb{Z}^n \) \( (1 \leq i \leq n-r) \) which vanish at the forms in (3.6) and satisfy
\[
|b_1| \leq \cdots \leq |b_{n-r}|, \quad |b_1| \cdots |b_{n-r}| \leq c \prod.
\]

See [6] for a proof of the above lemma. The following theorem was proved in [5].

**Theorem 3.16.** [5] Let \( C \) be a complex algebraic curve and \( H_2 \) be the set of all the algebraic subgroups of co-dimension 2. If \( H_2 \cap C \) is not finite, then \( C \) is contained in an algebraic subgroup.

3.4. Strong Geometric Isolation and Anomalous Subvarieties

Strong geometric isolation was first introduced by W. Neumann and A. Reid in [15]. Geometrically, it simply means one subset of cusps moves independently without affecting the rest. Using Theorem 4.3 in [13], we give one of the equivalent forms of the definition as follows:

**Definition 3.17.** Let \( M \) be a 2-cusped hyperbolic 3-manifold. Following the same notation given in Theorem 2.3 we say two cusps of \( M \) are strongly geometrically isolated (SGI) if \( v_1 \) only depends on \( u_1 \) and \( v_2 \) only depends on \( u_2 \).

The following theorems were proved in [9].

**Theorem 3.18.** Let \( M \) be a 2-cusped manifold with rationally independent cusp shapes and \( \mathcal{X} \) be its holonomy variety. Then a maximal anomalous subvariety of \( \mathcal{X} \) containing \( z^0 \) is either contained in \( M_1 = L_1 = 1 \) or \( M_2 = L_2 = 1 \).

**Theorem 3.19.** Let \( M \) and \( \mathcal{X} \) be same as Theorem 3.18. Then \( \mathcal{X}^{\text{out}} = \emptyset \) if and only if two cusps of \( M \) are SGI each other.
4. 1-cusped case

4.1. Preliminaries

In this section, we prove several lemmas will be used in the proofs of the main theorems. The proofs of the lemmas are purely computational and elementary, and so a trusting reader can skip ahead at first reading.

**Lemma 4.1.** Let

\[
\begin{pmatrix}
  a_1 & b_1 & c_1 & d_1 \\
  a_2 & b_2 & c_2 & d_2 \\
\end{pmatrix}
\]  

be an integer matrix of rank 2, and \( \tau \) be a non-quadratic number. If the rank of the following \((2 \times 2)\)-matrix

\[
\begin{pmatrix}
  a_1 + b_1 \tau & c_1 + d_1 \tau \\
  a_2 + b_2 \tau & c_2 + d_2 \tau \\
\end{pmatrix}
\]  

is equal to 1, then (4.1) is either of the following forms:

\[
\begin{pmatrix}
  a_1 & b_1 & ma_1 & mb_1 \\
  a_2 & b_2 & ma_2 & mb_2 \\
\end{pmatrix}
\]

(4.3)  

for some nonzero \( m \in \mathbb{Q} \) or

\[
\begin{pmatrix}
  a_1 & b_1 & 0 & 0 \\
  a_2 & b_2 & 0 & 0 \\
\end{pmatrix}
\]

(4.4)  

or

\[
\begin{pmatrix}
  0 & 0 & c_1 & d_1 \\
  0 & 0 & c_2 & d_2 \\
\end{pmatrix}
\]

(4.5)

**Proof.** Since the rank of (4.2) is 1, we have

\[(a_1 + b_1 \tau)(c_2 + d_2 \tau) = (a_2 + b_2 \tau)(c_1 + d_1 \tau).\]

By the assumption, \( \tau \) is not quadratic, so we get

\[a_1c_2 = a_2c_1,\]

(4.6)

\[b_1d_2 = b_2d_1,\]

(4.7)

\[b_1c_2 + a_1d_2 = a_2d_1 + b_2c_1.\]

(4.8)

(1) If none of \( a_i, b_i, c_i, d_i \) \((i = 1, 2)\) are zero, then there exist \( m, n \in \mathbb{Q} \) such that

\[(a_1, a_2) = m(c_1, c_2), \quad (b_1, b_2) = n(d_1, d_2).\]

(4.9)

By (4.8), we get

\[nd_1c_2 + mc_1d_2 = mc_2d_1 + nd_2c_1,\]

which is equivalent to

\[(n - m)(d_1c_2 - d_2c_1) = 0.\]

(a) If \( m = n \), then (4.1) is of the form given in (4.3).

(b) If \( d_1c_2 - d_2c_1 = 0 \), then there exists \( l \in \mathbb{Q} \) such that \( l(d_1, d_2) = (c_1, c_2) \). Combining with (4.9), we get

\[(a_1, a_2) = m(c_1, c_2) = ml(d_1, d_2) = mln(b_1, b_2).\]

This implies that \((a_1, b_1, c_1, d_1) = t(a_2, b_2, c_2, d_2)\) for some \( t \in \mathbb{Q} \), and it contradicts the fact that (4.1) is a matrix of rank 2.
One of \(a_i, b_i, c_i, d_i\) is equal to 0. By symmetry, it is enough to consider the case \(a_1 = 0\). If \(a_1 = 0\), then \(a_2 = 0\) or \(c_1 = 0\) by (4.6). We consider each case step by step below.

**(1)** If \(a_1 = a_2 = 0\), then, by (4.7) and (4.8), we get
\[
\begin{align*}
    b_1d_2 &= b_2d_1, \\
    b_1c_2 &= b_2c_1.
\end{align*}
\]

(a) If none of \(b_i, c_i, d_i\) are zero, then \((b_1, b_2) = m(c_1, c_2)\) and \((b_1, b_2) = n(d_1, d_2)\) for some \(m, n \in \mathbb{Q}\). But this contradicts the fact that (4.1) is a matrix of rank 2.

(b) Suppose \(b_1 = 0\) or \(b_2 = 0\). Without loss of generality, we assume \(b_1 = 0\). By (4.10), if \(b_2 \neq 0\), then \(c_1 = d_1 = 0\) and so \(a_1 = b_1 = c_1 = d_1 = 0\). But this contradicts the fact that (4.1) is a matrix of rank 2. Thus \(b_2 = 0\) and the matrix (4.1) is of the following form:
\[
\begin{pmatrix}
    0 & 0 & c_1 & d_1 \\
    0 & 0 & c_2 & d_2
\end{pmatrix},
\]
which is the case given in (4.5).

(c) Suppose \(b_1, b_2 \neq 0\) and one of \(c_i, d_i\) is zero. Without loss of generality, we consider \(c_1 = 0\). Then, by (4.10), \(c_2 = 0\). Since \(b_1d_2 = d_1b_2\), we have \(\ell(b_1, b_2) = (d_1, d_2)\) for some \(\ell \in \mathbb{Q}\). But this contradicts the fact that the rank of (4.1) is 2.

**(2)** If \(a_1 = c_1 = 0\), then
\[
\begin{align*}
    b_1d_2 &= b_2d_1, \\
    b_1c_2 &= a_2d_1.
\end{align*}
\]

(a) If none of \(b_i, d_i, c_2, d_2\) are zero, then \(m(b_1, d_1) = (b_2, d_2)\) and \(n(b_1, d_1) = (a_2, c_2)\) for some \(m, n \in \mathbb{Q}\). So the matrix (4.1) is of the following form
\[
\begin{pmatrix}
    0 & b_1 & 0 & d_1 \\
    nb_1 & mb_1 & nd_1 & md_1
\end{pmatrix},
\]
which is of the form given in (4.3).

(b) If \(b_1 = 0\), then \(a_1 = b_1 = c_1 = 0\). Since (4.1) is a matrix of rank 2, we assume \(d_1 \neq 0\). Then, by (4.11), we have \(a_2 = b_2 = 0\) and so (4.1) is of the form given in (4.5).

(c) If \(d_1 = 0\), then \(a_1 = c_1 = d_1 = 0\). Since (4.1) is a matrix of rank 2, we assume \(b_1 \neq 0\). Then, by (4.11), we have \(c_2 = d_2 = 0\) and so (4.1) is of the form given in (4.4).

(d) If \(b_2 = 0\) (with \(b_1 \neq 0\)), then \(d_2 = 0\) by (4.11). Since we already dealt with the cases \(a_2 = 0\) or \(d_1 = 0\) above, we assume \(a_2 \neq 0\) and \(d_1 \neq 0\). By (4.11), we have \(m(b_1, d_1) = (a_2, c_2)\) for some \(m \in \mathbb{Q}\), and so (4.1) is of the following form
\[
\begin{pmatrix}
    0 & b_1 & 0 & d_1 \\
    mb_1 & 0 & md_1 & 0
\end{pmatrix},
\]
which is the case given in (4.3).

(e) If \(d_2 = 0\), then \(b_2 = 0\) or \(d_1 = 0\) by (4.11). But we already considered these two cases above.

(f) If \(c_2 = 0\), then \(a_2 = 0\) or \(d_1 = 0\) by (4.11). We also covered these cases above.
\(\square\)
Lemma 4.2. Let
\[
\begin{pmatrix}
a_1 & b_1 & c_1 & d_1 \\
a_2 & b_2 & c_2 & d_2
\end{pmatrix}
\]
be an integer matrix of rank 2 and \((p, q), (p', q')\) be two co-prime pairs such that
\[
\begin{align*}
-q a_1 + p b_1 - q' c_1 + p' d_1 &= 0, \\
-q a_2 + p b_2 - q' c_2 + p' d_2 &= 0.
\end{align*}
\] (4.13)
For a non-quadratic number \(\tau\), if the rank of the following \((2 \times 2)\)-matrix
\[
\begin{pmatrix}
a_1 + b_1 \tau & c_1 + d_1 \tau \\
a_2 + b_2 \tau & c_2 + d_2 \tau
\end{pmatrix}
\] (4.14)
is equal to 1, then we have either \((p, q) = (p', q')\) or \((p, q) = (-p', -q')\). In particular, if \((p, q) = (p', q')\), then (4.12) is of the form
\[
\begin{pmatrix}
a_1 & b_1 & c_1 & d_1 \\
a_2 & b_2 & c_2 & d_2
\end{pmatrix}
\]
and if \((p, q) = (-p', -q')\), then (4.12) is of the form
\[
\begin{pmatrix}
a_1 & b_1 & -a_1 & -b_1 \\
a_2 & b_2 & -a_2 & -b_2
\end{pmatrix}
\].

Proof. By Lemma 4.1, the matrix (4.12) is one of the forms given in (4.3) or (4.4) or (4.5). If it is of the form in (4.4) or (4.5), then we have either
\[
\begin{align*}
-q a_1 + p b_1 &= 0, \\
-q a_2 + p b_2 &= 0
\end{align*}
\] (4.15)
or
\[
\begin{align*}
-q' c_1 + p' d_1 &= 0, \\
-q' c_2 + p' d_2 &= 0
\end{align*}
\] (4.16)
which induces either
\[(p, q) = m_1(a_1, b_1) = m_2(a_2, b_2)\]
or
\[(p', q') = n_1(c_1, d_1) = n_2(c_2, d_2)\]
for some nonzero \(m_i, n_i \in \mathbb{Q}\) \((i = 1, 2)\) respectively. But either case contradicts the fact that (4.12) is a matrix of rank 2.

Now we assume that (4.12) is of the form given in (4.3). Since the rank of the matrix is 2, we further assume \(a_1 b_2 - a_2 b_1 \neq 0\). By (4.13), we have
\[
\begin{pmatrix}
a_1 & b_1 \\
a_2 & b_2
\end{pmatrix}
\begin{pmatrix}
-q \\
-p
\end{pmatrix}
= m
\begin{pmatrix}
a_1 & b_1 \\
a_2 & b_2
\end{pmatrix}
\begin{pmatrix}
q' \\
-p'
\end{pmatrix}
\]
Since \(a_1 b_2 - a_2 b_1 \neq 0\), this induces that
\[
(-q, p) = m(q', -p')
\]
for some \(m \in \mathbb{Q}\). Since \((p, q)\) and \((p', q')\) are relatively co-prime pairs, we get \(m = 1\) or \(m = -1\). This completes the proof. \(\square\)
Lemma 4.3. Let

\[
\begin{pmatrix}
  a_1 & b_1 & c_1 & d_1 \\
  a_2 & b_2 & c_2 & d_2
\end{pmatrix}
\]  \tag{4.17}

be an integer matrix of rank 2, and \(\tau_1, \tau_2\) be algebraic numbers such that \(1, \tau_1, \tau_2, \tau_1 \tau_2\) are linearly independent over \(\mathbb{Q}\). If the rank of the following \((2 \times 2)\)-matrix

\[
\begin{pmatrix}
  a_1 + b_1 \tau_1 & c_1 + d_1 \tau_2 \\
  a_2 + b_2 \tau_1 & c_2 + d_2 \tau_2
\end{pmatrix}
\]  \tag{4.18}

is equal to 1, then (4.17) is either

\[
\begin{pmatrix}
  a_1 & b_1 & 0 & 0 \\
  a_2 & b_2 & 0 & 0
\end{pmatrix}
\]  \tag{4.19}

or

\[
\begin{pmatrix}
  0 & 0 & c_1 & d_1 \\
  0 & 0 & c_2 & d_2
\end{pmatrix}
\]  \tag{4.20}

Proof. Since the rank of (4.18) is 1, and as \(1, \tau_1, \tau_2, \tau_1 \tau_2\) are linearly independent over \(\mathbb{Q}\), we have

\[
a_1 c_2 - c_1 a_2 = 0,  \tag{4.21}
\]
\[
b_1 c_2 - c_1 b_2 = 0,  \tag{4.22}
\]
\[
a_1 d_2 - d_1 a_2 = 0,  \tag{4.23}
\]
\[
b_1 d_2 - d_1 b_2 = 0.  \tag{4.24}
\]

If none of \(a_i, b_i, c_i, d_i\) \((i = 1, 2)\) are zero, then (4.21)-(4.24) imply the two nonzero vectors \((a_1, b_1, c_1, d_1)\) and \((a_2, b_2, c_2, d_2)\) are linearly dependent over \(\mathbb{Q}\). But this is impossible because (4.17) is a matrix of rank 2. Without loss of generality, let us assume \(a_1 = 0\). Then, by (4.21) and (4.23), we have the following two cases:

(1) \(a_2 = 0\).

In this case, the problem is reduced to the following:

\[
b_1 c_2 - c_1 b_2 = 0,  \tag{4.25}
\]
\[
b_1 d_2 - d_1 b_2 = 0.  \tag{4.26}
\]

Similar to above, if none of \(b_i, c_i, d_i\) \((i = 1, 2)\) are zero, then (4.25) and (4.26) contradict the fact that (4.17) is a matrix of rank 2. So at least one of \(b_i, c_i, d_i\) \((i = 1, 2)\) is zero and the situation is divided into the following two subcases.

(a) \(b_1 = 0\) or \(b_2 = 0\).

By symmetry, it is enough to consider the case \(b_1 = 0\). If \(b_1 = 0\), then \(b_2 = 0\) or \(c_1 = 0\) by (4.25), and \(d_2 = 0\) or \(d_1 = 0\) by (4.26). If \(b_2 = 0\), then we get the desired result (i.e. \(a_1 = a_2 = b_1 = b_2 = 0\)). Otherwise, if \(c_1 = d_1 = 0\), it contradicts the fact that \((a_1, b_1, c_1, d_1)\) is a nonzero vector.

(b) \(c_1 = 0\) or \(c_2 = 0\) or \(d_1 = 0\) or \(d_2 = 0\) (with \(b_1, b_2 \neq 0\)).

Here, also by symmetry, it is enough to consider the first case \(c_1 = 0\). If \(b_1, b_2 \neq 0\) and \(c_1 = 0\), then \(c_2 = 0\) by (4.25), and \((d_1, d_2) = m(b_1, b_2)\) for some \(m \in \mathbb{Q}\) by (4.26). But this contradicts the fact that (4.17) is a matrix of rank 2.
(2) \( a_2 \neq 0 \) and so \( c_1 = d_1 = 0 \).

Since \((a_1, b_1, c_1, d_1)\) is a nonzero vector, \( b_1 \) is nonzero and \( c_2 = d_2 = 0 \) by (4.22) and (4.24). As a result, we get \( c_1 = c_2 = d_1 = d_2 = 0 \), which is the second desired result of the statement.

So the lemma holds. \( \square \)

**Lemma 4.4.** Let

\[
\begin{pmatrix}
  a_1 & b_1 & c_1 & d_1 \\
  a_2 & b_2 & c_2 & d_2 \\
\end{pmatrix}
\]

be an integer matrix of rank 2, and \((p, q), (p', q')\) be two nonzero pairs (i.e. \( p, q, p', q' \neq 0 \)) such that

\[
\begin{align*}
-qa_1 + pb_1 - q'c_1 + p'd_1 &= 0, \\
-qa_2 + pb_2 - q'c_2 + p'd_2 &= 0.
\end{align*}
\]

(4.28)

If \( \tau_1 \) and \( \tau_2 \) are two non-real, non-quadratic algebraic numbers which are rationally independent, then the rank of the following matrix

\[
\begin{pmatrix}
  a_1 + b_1 \tau_1 & c_1 + d_1 \tau_2 \\
  a_2 + b_2 \tau_1 & c_2 + d_2 \tau_2 \\
\end{pmatrix}
\]

is always equal to 2.

**Proof.** By Lemma 4.3, (4.27) is of the form given in (4.19) or (4.20). And, combining with (4.28), we have either

\[
-qa_1 + pb_1 = 0, \quad -qa_2 + pb_2 = 0
\]

or

\[
-q'c_1 + p'd_1 = 0, \quad -q'c_2 + p'd_2 = 0.
\]

which induces

\[
(p, q) = m_1(a_1, b_1) = m_2(a_2, b_2)
\]

or

\[
(p', q') = n_1(c_1, d_1) = n_2(c_2, d_2)
\]

for some nonzero \( m_i, n_i \in \mathbb{Q} \) (\( i = 1, 2 \)) respectively. But either case contradicts the fact that (4.27) is a matrix of rank 2. \( \square \)

**Lemma 4.5.** Let \( \mathcal{M} \) be a 1-cusped hyperbolic 3-manifold whose cusp shape is non-quadratic and \( \mathcal{X} \) be its holonomy variety. Consider \( \mathcal{X} \times \mathcal{X} \) in \( \mathbb{C}^{k+2} \times \mathbb{C}^{k+2} \) with the following coordinates:

\[
(z_1, \ldots, z_k, M, L, z_1', \ldots, z_k', M', L').
\]

Let \( H \) is an algebraic subgroup of codimension 2, defined by

\[
\begin{align*}
M^{a_1} L^{b_1} (M')^{c_1} (L')^{d_1} &= 1, \\
M^{a_2} L^{b_2} (M')^{c_2} (L')^{d_2} &= 1.
\end{align*}
\]

Similarly, we use the following coordinates for \( \text{Def}(\mathcal{M}) \times \text{Def}(\mathcal{M}) \) in \( \mathbb{C}^4 \):

\[
(u, v, u', v').
\]

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Suppose that \((p,q)\) and \((p',q')\) are co-prime pairs such that
\[
-q a_1 + p b_1 - q' c_1 + p' d_1 = 0, \\
-q a_2 + p b_2 - q' c_2 + p' d_2 = 0.
\] (4.29)

If the component of \((X \times X) \cap H\) containing \((z^0, z^0)\) is a 1-dimensional anomalous subvariety of \(X \times X\), then \((p,q) = (p',q')\) or \((p,q) = -(p',q')\).

**Proof.** As we explained in Section 2.2, taking logarithm to each coordinate, \((X \times X) \cap H\) is locally biholomorphic to the complex manifold defined by
\[
a_1 u + b_1 v + c_1 u' + d_1 v' = a_1 u + b_1 (\tau u + \cdots) + c_1 u' + d_1 (\tau u' + \cdots) = 0, \\
a_2 u + b_2 v + c_2 u' + d_2 v' = a_2 u + b_2 (\tau u + \cdots) + c_2 u' + d_2 (\tau u' + \cdots) = 0.
\] (4.30)

If \((X \times X) \cap H\) contains a 1-dimensional anomalous subvariety containing \((z^0, z^0)\), then, equivalently, two equations in (4.30) define a 1-dimensional complex manifold containing \(u = v = u' = v' = 0\). Thus the Jacobian of (4.30) at \(u = u' = 0\), which is equal to
\[
\begin{pmatrix}
a_1 + b_1 \tau & c_1 + d_1 \tau \\
a_2 + b_2 \tau & c_2 + d_2 \tau
\end{pmatrix},
\]
has rank 1. By Lemma 4.2, the result follows. \(\square\)

### 4.2. Proof of Theorem 1.2

Using the above lemma, we now prove Theorem 1.2. The strategy of the proof goes as follows. For a given pair of two isometric Dehn filled manifolds, we find the cosmetic surgery point \(P\) associated with these two manifolds. Recall that we identify \(P\) as an intersection point between \(X \times X\) and an algebraic subgroup of codimension 3 (see Section 2.3). Using Siegel’s lemma, we find an algebraic subgroup \(H\) of codimension 2, containing \(P\). If \(P\) is an isolated point of \(H \cap (X \times X)\), then we first prove the degree of \(P\) is uniformly bounded. Since the height of \(P\) is uniformly bounded as well, by Northcott’s theorem, we get the desired. Otherwise if they intersect unlikely, that is, if the component of \(H \cap (X \times X)\) containing \(P\) is an anomalous subvariety of \(X \times X\), then we show \(P\) lies in \(M = M', L = L'\), implying two initially given isometric manifolds must have the same Dehn filling coefficient.

Note that for the first half of the proof, we follow along the same lines given in the proof of Lemma 8.1 in [3].

**Theorem 4.6.** Let \(M\) be a 1-cusped hyperbolic 3-manifold whose cusp shape is non-quadratic. Let \(t_{p/q}\) (where \(|t_{p/q}| > 1\)) be the holonomy of the core geodesic of \((p,q)\)-Dehn filling. For \(p/q\) and \(p'/q'\) such that \(|p| + |q|\) and \(|p'| + |q'|\) are sufficiently large, if
\[
t_{p/q} = t_{p'/q'}
\] (4.31)
then
\[
p/q = p'/q'.
\] (4.32)

**Proof.** Let
\[
P = \left( t_{p/q}, t_{p/q}, t_{p'/q}, t_{p'/q} \right)_{20}
\]
be the cosmetic surgery point associated with \( \mathcal{M}(p/q) \) and \( \mathcal{M}(p'/q') \). By Theorem 3.11, there exists an universal constant \( B \) such that
\[
h(P) \leq B.
\]
Let
\[
v = (-q, p, -q', p'),
\]
and then, by the properties of height (i.e. Theorem 3.2 and (3.1)), we can find \( c_1 \) such that
\[
|v|h(t_{p/q}) \leq c_1 B.
\] (4.33)
By Siegel’s lemma, there exists \( b_1, b_2, b_3 \in \mathbb{Z}^4 \) which vanishes at
\[
-qx_1 + px_2 - q'x_3 + p'x_4 = 0
\] (4.34)
and satisfies
\[
|b_1||b_2||b_3| \leq |v|,
\] (4.35)
\[
|b_1| \leq |b_2| \leq |b_3|.
\]
Let \( b_i = (a_i, b_i, c_i, d_i) \) \((i = 1, 2)\) and \( H \) be defined by
\[
M^{a_1}L^{b_1}(M')^{c_1}(L')^{d_1} = 1,
\]
\[
M^{a_2}L^{b_2}(M')^{c_2}(L')^{d_2} = 1.
\]
We first consider the case that \( P \) is an isolated point of \( (\mathcal{X} \times \mathcal{X}) \cap H \).

**Claim 4.7.** If \( P \) is an isolated point of \( (\mathcal{X} \times \mathcal{X}) \cap H \), then the degree of \( P \) is bounded by some number which depends only on \( \mathcal{X} \).

**Proof.** By standard degree theory in arithmetic geometry, it is well-known that the degree of \( H \) is bound by \( c_2|b_1||b_2| \) for some constant \( c_2 \), and thus by \( c_2|v|^{2/3} \) by (4.35). By Bézout’s theorem, the degree \( D \) of \( P \) is bounded by the product of the degrees of \( \mathcal{X} \times \mathcal{X} \) and \( H \). Thus we have
\[
D \leq c_3|v|^{2/3}
\] (4.36)
for some constant \( c_3 \) depending on \( \mathcal{X} \). By Lemma 3.14
\[
h(t_{p/q}) \geq \frac{1}{c_4D(\log 3D)^\kappa}
\]
for some \( \kappa \) and \( c_4 \). Combining with (4.33), we deduce \( |v| \leq c_5D(\log 3D)^\kappa B \) for some constant \( c_5 \) and, together with (4.36), we get \( D \leq c_6(D(\log 3D)^\kappa B)^{2/3} \) for some constant \( c_6 \) depending only on \( \mathcal{X} \). This completes the proof. \( \square \)

Next we consider the case that the component of \( (\mathcal{X} \times \mathcal{X}) \cap H \) containing \( P \) is an anomalous subvariety of \( \mathcal{X} \times \mathcal{X} \). Denote this anomalous subvariety by \( \mathcal{Y} \). If \( \mathcal{Y} \) contains \((z^0, z^0)\), then, by Lemma 4.5, we get \( p/q = p'/q' \) as desired. If \( \mathcal{Y} \) does not contain \((z^0, z^0)\) and \( \mathcal{X} \times \mathcal{X} \) has only a finite number of anomalous subvarieties near \((z^0, z^0)\), then, by shrinking the size of a neighborhood if necessary, we exclude those cosmetic surgery points contained in \( \mathcal{Y} \).

Now we assume \( \mathcal{X} \times \mathcal{X} \) contains infinitely many anomalous subvarieties near \((z^0, z^0)\) and each contains a cosmetic surgery point arising from two isometric Dehn filled manifolds of different filling coefficients. More precisely, we consider the following situation. Let \( (p_i/q_i)_{i \in I} \) and \( (p'_i/q'_i)_{i \in I} \) be two infinite sequences of co-prime pairs such that, for each \( i \),
\[
(p_i/q_i) \neq (p'_i/q'_i),
\] (4.37)
but

\[ t_i = t'_i, \]

where \( t_i \) (resp. \( t'_i \)) is the holonomy of the core geodesic of \( \mathcal{M}(p_i/q_i) \) (resp. \( \mathcal{M}(p'_i/q'_i) \)). Let

\[ P_i = \left( t_i^{-q_i}, t_i^{p_i}, (t'_i)^{-q'_i}, (t'_i)^{p'_i} \right) \]

be the cosmetic surgery point in \( \mathcal{X} \times \mathcal{X} \) (associated with \( \mathcal{M}(p_i/q_i) \) and \( \mathcal{M}(p'_i/q'_i) \)), and \( H_i \) be an algebraic subgroup containing \( P_i \) and obtained by the same procedure given above (i.e. using Siegel’s lemma). Let \( H_i \) be defined by the following equations

\[ M^{a_{i1}} L^{h_{i1}} (M')^{a'_{i1}} (L')^{h'_{i1}} = 1, \]

\[ M^{a_{2i}} L^{h_{2i}} (M')^{a'_{2i}} (L')^{h'_{2i}} = 1 \quad (4.38) \]

for each \( i \in \mathcal{I} \). We further assume that the component of \( (\mathcal{X} \times \mathcal{X}) \cap H_i \) (say \( \mathcal{Y}_i \)) containing the cosmetic surgery point \( P_i \) is an anomalous subvariety of \( \mathcal{X} \times \mathcal{X} \), and any neighborhood of \((z^0, z^0)\) contains infinitely many \( \mathcal{Y}_i \). Then we have the following claim:

**Claim 4.8.** For each \( i \), the component of \( (\mathcal{X} \times \mathcal{X}) \cap H_i \) containing \((z^0, z^0)\) is an anomalous subvariety of \( \mathcal{X} \times \mathcal{X} \). (Thus each \((\mathcal{X} \times \mathcal{X}) \cap H_i \) contains at least two anomalous subvarieties.)

**Proof.** Let \( b \) be the largest number such that there are infinitely many \( \mathcal{Y}_i \) such that each \( \mathcal{Y}_i \) is a \( b \)-anomalous subvariety of \( \mathcal{X} \times \mathcal{X} \) but not a \((b + 1)\)-anomalous subvariety. By Lemma 3.9, we find an algebraic subgroup \( H^{(0)} \) such that, for infinitely many \( i \), \( \mathcal{Y}_i \) is contained in \((\mathcal{X} \times \mathcal{X}) \cap g_i H^{(0)} \) for some \( g_i \).

We claim \( g_i H^{(0)} \subset H_i \) for each \( i \). Suppose \( g_i H^{(0)} \not\subset H_i \). Since \( \mathcal{Y}_i \subset (\mathcal{X} \times \mathcal{X}) \cap H_i \) and \( \mathcal{Y}_i \subset (\mathcal{X} \times \mathcal{X}) \cap g_i H^{(0)} \), we have \( \mathcal{Y}_i \subset (\mathcal{X} \times \mathcal{X}) \cap (H_i \cap g_i H^{(0)}) \). If \( g_i H^{(0)} \not\subset H_i \), then \( H_i \cap g_i H^{(0)} \) is an algebraic coset whose dimension is less than \( g_i H^{(0)} \). But this contradicts to the maximality of \( b \). Thus \( g_i H^{(0)} \subset H_i \) and this further implies \( H^{(0)} \subset H_i \) for each \( i \). Since the component of \((\mathcal{X} \times \mathcal{X}) \cap H^{(0)} \) containing \((z^0, z^0)\) is an anomalous subvariety of \( \mathcal{X} \times \mathcal{X} \), we get the desired result. \( \square \)

Thus, again by Lemma 4.5, we have \( p_i/q_i = p'_i/q'_i \) for each \( i \), which contradicts to the assumption \((4.37)\).

In conclusion, if \((4.31)\) holds, then either \((4.32)\) holds or the degree of \( t_{p/q} \) is bounded. By Theorem 3.11, the height of \( t_{p/q} \) is uniformly bounded and, by Northcott’s theorem, there are only a finite number of choices for \( t_{p/q} \). Combining with Theorem 2.3, we conclude there are only a finite number of Dehn filling coefficients having the same holonomy. Thus, except for those finitely many choices, the only case that makes \((4.31)\) possible is \((4.32)\). This completes the proof. \( \square \)

**Remark 4.** The above proof is the prototype of the other proofs below. Basically we will follow the same strategies given here to prove Theorem 6.57 as well as Theorem 1.7. In particular, Claims 4.7 and 4.8 will be used repeatedly there.
5. 2-cusped case

5.1. Preliminaries

In this subsection, we collect several lemmas and Theorems needed to prove our second main theorem. Similar to Subsection 4.1, the proofs of the lemmas and theorems in this subsection are very technical (but elementary), so we suggest the reader skip them at first reading.

**Lemma 5.1.** Let $M$ be a 2-cusped hyperbolic 3-manifold having rationally independent cusp shapes and $X$ be its holonomy variety. Let $H$ be a 2-dimensional algebraic subgroup defined by

\[
M_t^{21} L_1^{b_1} M_2^{c_1} L_2^{d_1} = 1,
\]

\[
M_t^{22} L_1^{b_2} M_2^{c_2} L_2^{d_2} = 1.
\]

Suppose that $(p_1, q_1)$ and $(p_2, q_2)$ are two nonzero pairs (i.e. $p_i, q_i \neq 0$ where $i = 1, 2$) such that

\[
-q_1 a_1 + p_1 b_1 - q_2 c_1 + p_2 d_1 = 0,
\]

\[
-q_1 a_2 + p_1 b_2 - q_2 c_2 + p_2 d_2 = 0.
\]

Then $z^0$ is an isolated component of $X \cap H$.

**Proof.** As explained earlier, $X \cap H$ is locally biholomorphic (near $z^0$) to the complex manifold defined by

\[
a_1 u_1 + b_1 (\tau_1 u_1 + \cdots) + c_1 u_2 + d_1 (\tau_2 u_2 + \cdots) = 0,
\]

\[
a_2 u_1 + b_2 (\tau_1 u_1 + \cdots) + c_2 u_2 + d_2 (\tau_2 u_2 + \cdots) = 0.
\]

The Jacobian matrix of (5.2) at $u_1 = u_2 = 0$ is equal to

\[
\begin{pmatrix}
  a_1 + b_1 \tau_1 & c_1 + d_1 \tau_2 \\
  a_2 + b_2 \tau_1 & c_2 + d_2 \tau_2
\end{pmatrix}.
\]

If the component of $X \cap H$ containing $z^0$ is an anomalous subvariety, then the rank of (5.3) is strictly less than 2. However, by Lemma 4.4 it is impossible. □

For the 2-cusped case, we prove Theorem 1.8 unconditionally, following a idea similar to the one presented in the proof of Theorem 1.2.

**Theorem 5.2.** Let $M$ a hyperbolic 2-cusped manifold and $X$ be its holonomy variety. Let $t_1^{(p_1/q_1, p_2/q_2)}$ and $t_2^{(p_1/q_1, p_2/q_2)}$ be two holonomies of $(p_1/q_1, p_2/q_2)$-Dehn filling. If $|p_i|+|q_i|$ ($i = 1, 2$) are sufficiently large, then $t_1^{(p_1/q_1, p_2/q_2)}$ and $t_2^{(p_1/q_1, p_2/q_2)}$ are multiplicatively independent.

**Proof.** To simplify the notation we denote $t_1^{(p_1/q_1, p_2/q_2)}$ and $t_2^{(p_1/q_1, p_2/q_2)}$ by $t_1$ and $t_2$ respectively. If they are multiplicatively dependent, then

\[
t_1 = \xi^{s_1} \eta^{e_1}, \quad t_2 = \xi^{s_2} \eta^{e_2}
\]

for some $\eta$ such that $|\eta| \neq 1$, a $N$-th primitive root of unity $\xi$ and $s_i, e_i \in \mathbb{Z}$ ($1 \leq i \leq 2$). Let

\[
P = \left(t_1^{-q_1}, t_1^{p_1}, t_2^{-q_2}, t_2^{p_2}\right) = \left((\xi^{s_1} \eta^{e_1})^{-q_1}, (\xi^{s_1} \eta^{e_1})^{p_1}, (\xi^{s_2} \eta^{e_2})^{-q_2}, (\xi^{s_2} \eta^{e_2})^{p_2}\right)
\]

\[
= (\xi^{l_1} \eta^{-e_1 q_1}, \xi^{l_2} \eta^{e_1 p_1}, \xi^{l_3} \eta^{-e_2 q_2}, \xi^{l_4} \eta^{e_2 p_2}) \quad (0 \leq l_j < N, 1 \leq j \leq 4)
\]

be the Dehn filling point associated with $\mathcal{M}(p_1/q_1, p_2/q_2)$, and consider the following forms:

\[
NX_0 + l_1 X_1 + l_2 X_2 + l_2 X_3 + l_2 X_4, \quad -e_1 q_1 X_1 + e_1 p_1 X_2 - e_2 q_2 X_3 + e_2 p_2 X_4
\]

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In conclusion, if \( X \) are independent cusp shapes and \( \Lambda \) is multiplicatively independent. \( \square \)

of Claim 4.7, we get that the degree of \( P \) lies in an algebraic torus \( H \) defined by

\[
M_1^{b_{11}}L_1^{b_{21}}M_2^{b_{31}}L_2^{b_{41}} = 1,
M_1^{b_{12}}L_1^{b_{22}}M_2^{b_{32}}L_2^{b_{42}} = 1.
\]

If \( P \) is an isolated point of \( \mathcal{X} \cap H \), then following the same procedure given in the proof of Claim 4.7 we get that the degree of \( P \) is bounded by some constant depending only on \( \mathcal{X} \). Combining with Theorem 3.11 and Northcott’s property, we have the desired result.

If the component of \( \mathcal{X} \cap H \) containing \( P \) (denote this anomalous subvariety by \( Y \)) is an anomalous subvariety of \( \mathcal{X} \), then it falls into the following two cases.

1. First note that \( Y \) does not contain \( z^0 \) by Lemma 5.1. So if \( \mathcal{X} \) has only a finite number of anomalous subvarieties near \( z^0 \) and \( Y \) is one of them, by shrinking the size of a neighborhood of \( z^0 \) if necessary, we discard all the Dehn filling points contained in \( Y \).

2. Now suppose \( \mathcal{X} \) contains infinitely many anomalous subvarieties and consider the following situation. Let

\[
\mathcal{M}(p_{1i}/q_{1i}, p_{2i}/q_{2i})_{i \in \mathcal{I}}
\]

be an infinite sequence of Dehn fillings of \( \mathcal{M} \) such that two holonomies of each filling are multiplicatively dependent. Let \( P_i \) be the Dehn filling point associated with \( \mathcal{M}(p_{1i}/q_{1i}, p_{2i}/q_{2i}) \), and \( H_i \) be an algebraic subgroup of codimension 2 containing \( P_i \) constructed via the same procedure above (i.e. using Siegel’s lemma). Denote the component of \( \mathcal{X} \cap H_i \) containing \( P_i \) by \( Y_i \). If \( \{ Y_i \}_{i \in \mathcal{I}} \) is a family of infinitely many anomalous subvarieties of \( \mathcal{X} \), then, by the same idea given in the proof of Claim 4.8 it can be shown the component of \( \mathcal{X} \cap H_i \) containing \( z^0 \) is also an anomalous subvariety of \( \mathcal{X} \) for each \( i \in \mathcal{I} \). However this is impossible by Lemma 5.1.

In conclusion, if \( |p_i|+|q_i| (i = 1, 2) \) are sufficiently large, then two holonomies of \( \mathcal{M}(p_1/q_1, p_2/q_2) \) are multiplicatively independent. \( \square \)

**Lemma 5.3.** Let \( \mathcal{M} \) be a 2-cusped hyperbolic 3-manifold having non-quadratic, rationally independent cusp shapes and \( \mathcal{X} \) be its holonomy variety. Let \( H \) be an algebraic subgroup
defined by the following forms of equations
\[ M_1^{a_1} L_1^{b_1} (M_1')^{a_1'} (L_1')^{b_1'} = 1, \]
\[ M_1^{a_2} L_1^{b_2} (M_1')^{a_2'} (L_1')^{b_2'} = 1, \]
\[ M_2^{d_1} L_2^{d_1'} (M_2')^{d_1'} (L_2')^{d_1'} = 1, \]
\[ M_2^{d_2} L_2^{d_2'} (M_2')^{d_2'} (L_2')^{d_2'} = 1, \]
and \((p_1, q_1), (p_2, q_2), (p'_1, q'_1), (p'_2, q'_2)\) be four co-prime pairs satisfying
\[ -q_1 a_1 + p_1 b_1 - q'_1 a'_1 + p'_1 b'_1 = 0, \]
\[ -q_1 a_2 + p_1 b_2 - q'_1 a'_2 + p'_1 b'_2 = 0, \]
\[ -q_2 c_1 + p_2 d_1 - q'_2 c'_1 + p'_2 d'_1 = 0, \]
\[ -q_2 c_2 + p_2 d_2 - q'_2 c'_2 + p'_2 d'_2 = 0. \]

Let \(Y\) be the component of \((X \times X) \cap H\) containing \((z^0, z^0)\). If \(Y\) is a 1-dimensional anomalous subvariety of \(X \times X\), then \(Y\) is contained in either
\[ M_1 = M_1', \quad L_1 = L_1', \quad M_2 = M_2' = L_2 = L_2' = 1 \]
or
\[ M_1 = (M_1')^{-1}, \quad L_1 = (L_1')^{-1}, \quad M_2 = M_2' = L_2 = L_2' = 1 \]
or
\[ M_2 = M_2', \quad L_2 = L_2', \quad M_1 = M_1' = L_1 = L_1' = 1 \]
or
\[ M_2 = (M_2')^{-1}, \quad L_2 = (L_2')^{-1}, \quad M_1 = M_1' = L_1 = L_1' = 1. \]

Moreover, in each case, we have
\[ (p_1, q_1) = (p'_1, q'_1), \quad a_i = -a'_i, \quad b_i = -b'_i \quad (i = 1, 2) \]
or
\[ (p_1, q_1) = -(p'_1, q'_1), \quad a_i = a'_i, \quad b_i = b'_i \quad (i = 1, 2) \]
or
\[ (p_2, q_2) = (p'_2, q'_2), \quad c_i = -c'_i, \quad d_i = -d'_i \quad (i = 1, 2) \]
or
\[ (p_2, q_2) = -(p'_2, q'_2), \quad c_i = c'_i, \quad d_i = d'_i \quad (i = 1, 2) \]
respectively. If \(Y\) is a 2-dimensional anomalous subvariety of \(X \times X\), then we have either
\[ (p_1, q_1) = (p'_1, q'_1), \quad (p_2, q_2) = (p'_2, q'_2) \]
or
\[ (p_1, q_1) = (p'_1, q'_1), \quad (p_2, q_2) = -(p'_2, q'_2) \]
or
\[ (p_1, q_1) = -(p'_1, q'_1), \quad (p_2, q_2) = (p'_2, q'_2) \]
or
\[ (p_1, q_1) = -(p'_1, q'_1), \quad (p_2/q_2) = -(p'_2/q'_2). \]
Proof. By taking logarithm to each coordinate, \( \mathcal{Y} \) is locally biholomorphic (near \((z^0, z^0)\)) to the complex manifold defined by

\[
\begin{align*}
& a_1 u + b_1 (\tau_1 u + \cdots) + a'_1 u' + b'_1 (\tau_1 u' + \cdots) = 0, \\
& a_2 u + b_2 (\tau_1 u + \cdots) + a'_2 u' + b'_2 (\tau_1 u' + \cdots) = 0, \\
& c_1 u + d_1 (\tau_2 u + \cdots) + c'_1 u' + d'_1 (\tau_2 u' + \cdots) = 0, \\
& c_2 u + d_2 (\tau_2 u + \cdots) + c'_2 u' + d'_2 (\tau_2 u' + \cdots) = 0.
\end{align*}
\tag{5.5}
\]

The rank of the Jacobian of \((5.5)\) at \((u, u') = 3\). Likewise, if \(Y\) is a 1-dimensional anomalous subvariety, then the equations in \((5.5)\) describe a 1-dimensional complex manifold. So, by the implicit function theorem, the rank of \((5.6)\) is equal to 3. Likewise, if \(Y\) is a 2-dimensional anomalous subvariety, the rank of \((5.6)\) is equal to 2.

1. First if the rank of \((5.6)\) is 3, then the rank of one of the following two matrices is 1 and the other is 2:

\[
\begin{pmatrix}
  a_1 + b_1 \tau_1 & a'_1 + b'_1 \tau_1 & 0 & 0 \\
  a_2 + b_2 \tau_1 & a'_2 + b'_2 \tau_1 & 0 & 0 \\
  0 & 0 & c_1 + d_1 \tau_2 & c'_1 + d'_1 \tau_2 \\
  0 & 0 & c_2 + d_2 \tau_2 & c'_2 + d'_2 \tau_2
\end{pmatrix}.
\tag{5.6}
\]

Without loss of generosity, we assume the rank of the first one is 1 and the rank of the second one is 2. By Lemma 4.2, we have either

\[
(p_1, q_1) = (p'_1, q'_1),
\]

\[
a_1 = -a'_1, b_1 = -b'_1, a_2 = -a'_2, b_2 = -b'_2,
\tag{5.7}
\]

or

\[
(p_1, q_1) = -(p'_1, q'_1),
\]

\[
a_1 = a'_1, b_1 = b'_1, a_2 = a'_2, b_2 = b'_2.
\tag{5.8}
\]

Since the rank of the following matrix

\[
\begin{pmatrix}
  c_1 + d_1 \tau_2 & c'_1 + d'_1 \tau_2 \\
  c_2 + d_2 \tau_2 & c'_2 + d'_2 \tau_2
\end{pmatrix}
\]

is equal to 2, by the implicit function theorem, we get \(u_2 = 0, u'_2 = 0\) (i.e. \(M_2 = M'_2 = 1\)). In other words, the complex manifold defined by \((5.5)\) is contained in either

\[
u_1 = u'_1, \quad v_2 = u'_2 = 0
\]

or

\[
u_1 = -u'_1, \quad v_2 = u'_2 = 0.
\]

Equivalently, \(\mathcal{Y}\) is contained in either

\[
M_1 = M'_1, \quad M_2 = M'_2 = 1 \quad (\text{and so} \quad L_1 = L'_1, \quad L_2 = L'_2 = 1)
\]

or

\[
M_1 = (M'_1)^{-1}, \quad M_2 = M'_2 = 1 \quad (\text{and so} \quad L_1 = (L'_1)^{-1}, \quad L_2 = L'_2 = 1).
\]
(2) Second, if the rank of (5.6) is equal to 2, then the rank of the following two matrices is equal to 1:

\[
\begin{pmatrix}
  a_1 + b_1 \tau_1 & a_1' + b_1' \tau_1 \\
  a_2 + b_2 \tau_1 & a_2' + b_2' \tau_1 \\
  c_1 + d_1 \tau_2 & c_1' + d_1' \tau_2 \\
  c_2 + d_2 \tau_2 & c_2' + d_2' \tau_2
\end{pmatrix},
\]

Again, by Lemma 4.2 the result follows.

\[\Box\]

**Lemma 5.4.** Let \( \mathcal{X} \) be the same as the one given in Lemma 5.3. Let \( H \) be an algebraic subgroup defined by

\[
M_1^{a_1} L_1^{b_1} (M_2)^{a_1'} (L_2)^{b_1'} = 1,
\]

\[
M_1^{a_2} L_1^{b_2} (M_2)^{a_2'} (L_2)^{b_2'} = 1,
\]

\[
M_2^{d_1} L_2^{d_1'} (M_1)^{c_1'} (L_1)^{d_1'} = 1,
\]

\[
M_2^{d_2} L_2^{d_2'} (M_1)^{c_2'} (L_1)^{d_2'} = 1
\]

and \((p_1, q_1), (p_2, q_2), (p_1', q_1'), (p_2', q_2')\) be four co-prime pairs satisfying

\[-q_1 a_1 + p_1 b_1 - q_1' a_1' + p_1' b_1' = 0,\]

\[-q_1 a_2 + p_1 b_2 - q_1' a_2' + p_1' b_2' = 0,\]

\[-q_2 c_1 + p_2 d_1 - q_2' c_1' + p_2' d_1' = 0,\]

\[-q_2 c_2 + p_2 d_2 - q_2' c_2' + p_2' d_2' = 0.\]

Then \((z^0, z^0)\) is an isolated component of \((\mathcal{X} \times \mathcal{X}) \cap H\).

**Proof.** Let \( \mathcal{Y} \) be the component of \((\mathcal{X} \times \mathcal{X}) \cap H\) containing \((z^0, z^0)\). Then \( \mathcal{Y} \) is locally biholomorphic (near \((z^0, z^0)\)) to the complex manifold defined by

\[
\begin{align*}
  a_1 u_1 + b_1 (\tau_1 u_1 + \cdots) + a_1' u_2 + b_1' (\tau_2 u_2 + \cdots) &= 0, \\
  a_2 u_1 + b_2 (\tau_1 u_1 + \cdots) + a_2' u_2 + b_2' (\tau_2 u_2 + \cdots) &= 0, \\
  c_1 u_2 + d_1 (\tau_2 u_2 + \cdots) + c_1' u_1 + d_1' (\tau_1 u_1 + \cdots) &= 0, \\
  c_2 u_2 + d_2 (\tau_2 u_2 + \cdots) + c_2' u_1 + d_2' (\tau_1 u_1 + \cdots) &= 0,
\end{align*}
\]

and the Jacobian of (5.9) at \((u_1, u_1', u_2, u_2') = (0, 0, 0, 0)\) is

\[
\begin{pmatrix}
  a_1 + b_1 \tau_1 & a_1' + b_1' \tau_1 & 0 & 0 \\
  a_2 + b_2 \tau_1 & a_2' + b_2' \tau_1 & 0 & 0 \\
  0 & 0 & c_1 + d_1 \tau_2 & c_1' + d_1' \tau_1 \\
  0 & 0 & c_2 + d_2 \tau_2 & c_2' + d_2' \tau_1
\end{pmatrix}.
\]

If \( \mathcal{Y} \) is an anomalous subvariety containing \((z^0, z^0)\) the rank of (5.10) is strictly less than 4. However, this is impossible by Lemma 4.4. \(\Box\)

**Lemma 5.5.** Let \( \mathcal{M} \) and \( \mathcal{X} \) be the same as Lemma 5.3. We further assume that two cusps of \( \mathcal{M} \) are not SGI. Let \( H \) be an algebraic subgroup defined by

\[
M_1^{a_1} L_1^{b_1} (M_2)^{a_1'} (L_2)^{b_1'} = 1,
\]

\[
M_2^{d_1} L_2^{d_1'} (M_1)^{c_1'} (L_1)^{d_1'} = 1,
\]

(5.11)
and $Z$ be the component of $H \cap (X \times X)$ containing $(z^0, z^0)$. If $Z$ is a maximal 3-dimensional anomalous subvariety of $X \times X$, then $Z$ is either contained in

$$M_1 = L_1 = 1 \quad \text{or} \quad M_2 = L_2 = 1$$

or

$$M'_1 = L'_1 = 1 \quad \text{or} \quad M'_2 = L'_2 = 1.$$  

Proof. $Z$ is locally biholomorphic (near $(z^0, z^0)$) to the complex manifold defined by

$$a_1u_1 + b_1(\tau_1u_1 + \cdots) + c_1u_2 + d_1(\tau_2u_2 + \cdots) + a'_1u'_1 + b'_1(\tau_1u'_1 + \cdots) + c'_1u'_2 + d'_1(\tau_2u'_2 + \cdots) = 0,$$

$$a_2u_1 + b_2(\tau_1u_1 + \cdots) + c_2u_2 + d_2(\tau_2u_2 + \cdots) + a'_2u'_1 + b'_2(\tau_1u'_1 + \cdots) + c'_2u'_2 + d'_2(\tau_2u'_2 + \cdots) = 0,$$

Since $Z$ is a 3-dimensional anomalous subvariety, the rank of the Jacobian of (5.14) at $(0, \ldots, 0)$,

$$\begin{pmatrix}
    a_1 + b_1\tau_1 & c_1 + d_1\tau_2 & a'_1 + b'_1\tau_1 & c'_1 + d'_1\tau_2 \\
    a_2 + b_2\tau_1 & c_2 + d_2\tau_2 & a'_2 + b'_2\tau_1 & c'_2 + d'_2\tau_2
\end{pmatrix},$$

is equal to 1.

(1) First if

$$c_1 = d_1 = c_2 = d_2 = c'_1 = d'_1 = c'_2 = d'_2 = 0,$$

then (5.15) is reduced to

$$\begin{pmatrix}
    a_1 + b_1\tau_1 & 0 & a'_1 + b'_1\tau_1 & 0 \\
    a_2 + b_2\tau_1 & 0 & a'_2 + b'_2\tau_1 & 0
\end{pmatrix}.$$

Since the rank of the above matrix is 1, by Lemma 1.1 we have either

$$a_1 = b_1 = a_2 = b_2 = 0, \quad a'_1b'_2 - a'_2b'_1 \neq 0$$

or

$$a'_1 = b'_1 = a'_2 = b'_2 = 0, \quad a_1b_2 - a_2b_1 \neq 0$$

or

$$\begin{pmatrix}
    a'_1 & b'_1 \\
    a'_2 & b'_2
\end{pmatrix} = \begin{pmatrix}
    ma_1 & mb_1 \\
    ma_2 & mb_2
\end{pmatrix} \quad (\text{where } a_1b_2 - a_2b_1 \neq 0)$$

for some nonzero $m \in \mathbb{Q}$. If (5.16) or (5.17) is true, then (5.14) is equivalent to either

$$u'_1 = v'_1 = 0$$

or

$$u_1 = v_1 = 0,$$

which implies $Z$ is contained in either

$$M'_1 = L'_1 = 1$$

or

$$M_1 = L_1 = 1$$

respectively. If (5.18) holds, then (5.14) is of the following form

$$a_1u_1 + b_1v_1 + ma_1u'_1 + mb_1v'_1 = 0,$$

$$a_2u_1 + b_2v_1 + ma_2u'_1 + mb_2v'_1 = 0,$$

and, since $a_1b_2 - a_2b_1 \neq 0$, (5.19) is further simplifies as

$$u_1 + mu'_1 = 0, \quad v_1 + mv'_1 = 0.$$
Since two cusps of $\mathcal{M}$ are not SGI each other, $v_1$ (resp. $v'_1$) contains a nonzero term of the form $u'_1 u'_2$ (resp. $(u'_1)^i (u'_2)^j$). So $u_2$ depends on $u'_1, u'_2$ from the second equation in (5.20). Since $u_1$ depends on $u'_1$, (5.20) defines a 2-dimensional complex manifold in general.

(2) Now we assume either $(c_1, d_1, c_2, d_2) \neq (0, 0, 0, 0)$ or $(c'_1, d'_1, c'_2, d'_2) \neq (0, 0, 0, 0)$.

Without loss of generality, suppose $(c_1, d_1, c_2, d_2) \neq (0, 0, 0, 0)$. Since the rank of the following matrix

$$
\begin{pmatrix}
    a_1 + b_1 \tau_1 & c_1 + d_1 \tau_2 \\
    a_2 + b_2 \tau_1 & c_2 + d_2 \tau_2 \\
\end{pmatrix}
$$

is 1, by Lemma 4.3, either the rank

$$
\begin{pmatrix}
    a_1 & b_1 & c_1 & d_1 \\
    a_2 & b_2 & c_2 & d_2 \\
\end{pmatrix}
$$

is 1 or

$$a_1 = b_1 = a_2 = b_2 = 0. \quad (5.22)
$$

(a) Suppose the rank of (5.21) is 1, and, applying Gauss Elimination, we assume $a_1 = b_1 = c_1 = d_1 = 0$ and (5.15) is of the following form:

$$
\begin{pmatrix}
    a_1 + b_1 \tau_1 & c_1 + d_1 \tau_2 & a'_1 + b'_1 \tau_1 & c'_1 + d'_1 \tau_2 \\
    a'_2 + b'_2 \tau_1 & c'_2 + d'_2 \tau_2 \\
\end{pmatrix}
$$

(5.23)

Since the rank of

$$
\begin{pmatrix}
    c_1 + d_1 \tau_2 & a'_1 + b'_1 \tau_1 \\
    0 & a'_2 + b'_2 \tau_1 \\
\end{pmatrix}
$$

is 1, again, by Lemma 4.3, either the rank of

$$
\begin{pmatrix}
    c_1 & d_1 & a'_1 & b'_1 \\
    0 & 0 & a'_2 & b'_2 \\
\end{pmatrix}
$$

is 1 or

$$a'_1 = b'_1 = a'_2 = b'_2 = 0. \quad (5.25)
$$

(i) If the rank of (5.24) is 1, then, applying Gauss elimination, we further assume $a'_2 = b'_2 = 0$ and (5.22) is of the following form:

$$
\begin{pmatrix}
    a_1 + b_1 \tau_1 & c_1 + d_1 \tau_2 & a'_1 + b'_1 \tau_1 & c'_1 + d'_1 \tau_2 \\
    0 & 0 & c'_2 + d'_2 \tau_2 \\
\end{pmatrix}
$$

(5.26)

Since the rank of the following matrix

$$
\begin{pmatrix}
    c_1 + d_1 \tau_2 & c'_1 + d'_1 \tau_2 \\
    0 & c'_2 + d'_2 \tau_2 \\
\end{pmatrix}
$$

is 1, by Lemma 4.1, either

$$
\begin{pmatrix}
    c'_1 & d'_1 \\
    c'_2 & d'_2 \\
\end{pmatrix} = \begin{pmatrix}
    l c_1 & l d_1 \\
    0 & 0 \\
\end{pmatrix}
$$

(5.27)

for some nonzero $l \in \mathbb{Q}$ or

$$c'_1 = c'_2 = d'_1 = d'_2 = 0.$$

---

Two equations in (5.20) define a 3-dimensional complex manifold if and only if $v_1$ (resp. $v'_1$) is independent of $u_2$ (resp. $u'_2$), that is, two cusps of $\mathcal{M}$ are SGI each other.
But, in either case, it contradicts the fact that (5.11) is an algebraic subgroup of codimension 2.

(ii) Suppose (5.25) is true. Then (5.23) is of the following form:

$$\begin{pmatrix} a_1 + b_1 \tau & c_1 + d_1 \tau_2 & 0 & c_1' + d_1' \tau_2 \\ 0 & 0 & 0 & c_2' + d_2' \tau_2 \end{pmatrix}.$$  \hspace{1cm} (5.28)

Since the rank of the following matrix

$$\begin{pmatrix} c_1 + d_1 \tau_2 & c_1' + d_1' \tau_2 \\ 0 & c_2' + d_2' \tau_2 \end{pmatrix}$$

is 1, by Lemma 4.1, either

$$\begin{pmatrix} c_1' & d_1' \\ c_2' & d_2' \end{pmatrix} = \begin{pmatrix} lc_1 & ld_1 \\ 0 & 0 \end{pmatrix}$$ \hspace{1cm} (5.29)

for some nonzero $l \in \mathbb{Q}$ or

$$c_1' = c_2' = d_1' = d_2' = 0.$$

Similar to the previous case, both contradict the fact that (5.11) is an algebraic subgroup of codimension 2.

(b) Suppose (5.22) holds, and let us assume (5.15) is of the following form:

$$\begin{pmatrix} 0 & c_1 + d_1 \tau_2 & a_1' + b_1' \tau_1 & c_1' + d_1' \tau_2 \\ 0 & c_2 + d_2 \tau_2 & a_2' + b_2' \tau_1 & c_2' + d_2' \tau_2 \end{pmatrix}.$$ \hspace{1cm} (5.30)

Since the rank of

$$\begin{pmatrix} c_1 + d_1 \tau_2 & a_1' + b_1' \tau_1 \\ e_2 + d_2 \tau_2 & a_2' + b_2' \tau_1 \end{pmatrix}$$

is 1, by Lemma 4.3 either

$$\begin{pmatrix} c_1 & d_1 & a_1' & b_1' \\ c_2 & d_2 & a_2' & b_2' \end{pmatrix}$$ \hspace{1cm} (5.31)

is a matrix of rank 1 or

$$a_1' = b_1' = a_2' = b_2' = 0.$$ \hspace{1cm} (5.32)

(i) If (5.31) is a matrix of rank 1, then, applying Gauss elimination, we assume (5.30) is of the following form:

$$\begin{pmatrix} 0 & c_1 + d_1 \tau_2 & a_1' + b_1' \tau_1 & c_1' + d_1' \tau_2 \\ 0 & 0 & 0 & c_2' + d_2' \tau_2 \end{pmatrix}.$$ \hspace{1cm} (5.33)

Since the rank of

$$\begin{pmatrix} c_1 + d_1 \tau_2 & c_1' + d_1' \tau_2 \\ 0 & c_2' + d_2' \tau_2 \end{pmatrix}$$

is 1, by Lemma 4.1, we have either

$$\begin{pmatrix} c_1' & d_1' \\ c_2' & d_2' \end{pmatrix} = \begin{pmatrix} lc_1 & ld_1 \\ 0 & 0 \end{pmatrix}$$ \hspace{1cm} (5.34)

for some nonzero $l \in \mathbb{Q}$ or

$$c_1' = c_2' = d_1' = d_2' = 0.$$ But either contradicts the fact that (5.11) an algebraic subgroup of codimension 2.
(ii) If \((5.32)\) is true, then \((5.30)\) is of the following form:

\[
\begin{pmatrix}
0 & c_1 + d_1 \tau_2 & 0 & c'_1 + d'_1 \tau_2 \\
0 & c_2 + d_2 \tau_2 & 0 & c'_2 + d'_2 \tau_2 \\
\end{pmatrix}.
\]

(5.35)

Since the rank of

\[
\begin{pmatrix}
c_1 + d_1 \tau_2 & c'_1 + d'_1 \tau_2 \\
c_2 + d_2 \tau_2 & c'_2 + d'_2 \tau_2 \\
\end{pmatrix}
\]

is 1, by Lemma 4.1, either

\[
\begin{pmatrix}
c'_1 & d'_1 \\
c'_2 & d'_2 \\
\end{pmatrix} = \begin{pmatrix}
lc_1 & ld_1 \\
lc_2 & ld_2 \\
\end{pmatrix} \quad \text{(where } c_1 d_2 - c_2 d_1 \neq 0) \tag{5.36}
\]

for some nonzero \(l \in \mathbb{Q}\) or

\[
c'_1 = c'_2 = d'_1 = d'_2 = 0. \tag{5.37}
\]

(A) In the first case, since \(c_1 d_2 - c_2 d_1 \neq 0\), we assume \((5.35)\) is of the following form

\[
\begin{pmatrix}
0 & u_2 & 0 & lu'_2 \\
0 & v_2 & 0 & lv'_2 \\
\end{pmatrix}.
\]

(5.38)

In other words, the following equations

\[u_2 + lu'_2 = 0, \quad v_2 + lv'_2 = 0\]

define a 3-dimensional complex manifold. But, as observed above, this is not true unless two cusps of \(\mathcal{M}\) are SGI each other.

(B) If \((5.37)\) is true, then \((5.35)\) is of the following form:

\[
\begin{pmatrix}
0 & c_1 + d_1 \tau_2 & 0 & 0 \\
0 & c_2 + d_2 \tau_2 & 0 & 0 \\
\end{pmatrix}.
\]

(5.39)

In other words, \((5.14)\) is equal to

\[c_1 u_2 + d_1 v_2 = 0, \quad c_2 u_2 + d_2 v_2 = 0.\]

(5.40)

Since \(c_1 d_2 - c_2 d_1 \neq 0\), \((5.40)\) is equivalent to

\[u_2 = v_2 = 0,\]

(5.41)

meaning that \(H\) is equivalent to

\[M_2 = L_2 = 1.\]

\[\Box\]

**Lemma 5.6.** Let \(\mathcal{M}\) and \(\mathcal{X}\) be the same as Lemma 5.5. Then \(\mathcal{X} \times \mathcal{X}\) does not have infinitely many maximal 3-dimensional anomalous subvarieties.

**Proof.** Suppose that \(\mathcal{X} \times \mathcal{X}\) has infinitely many 3-dimensional anomalous subvarieties. By Theorem 3.7, there exists an algebraic subgroup \(H\) of codimension 2 such that any anomalous subvariety is contained in a translation of \(H\). By the previous lemma, \(H\) is equivalent to either

\[M_1 = L_1 = 1 \quad \text{or} \quad M_2 = L_2 = 1 \quad \text{or} \quad M'_1 = L'_1 = 1 \quad \text{or} \quad M'_2 = L'_2 = 1,\]

and, without loss of generality, we assume \(H\) is

\[M_1 = L_1 = 1.\]

(5.42)
By moving to $\text{Def}(\mathcal{M}) \times \text{Def}(\mathcal{M})$, $(\mathcal{X} \times \mathcal{X}) \cap H$ is locally biholomorphic (near $(z^0, z^0)$) to
$$u_1 = v_1 = 0.$$Since a translation of $H$ contains a 3-dimensional anomalous subvariety of $\mathcal{X} \times \mathcal{X}$, we get that
$$u_1 = \xi_1, \quad v_1 = \xi_2$$defines a 3-dimensional complex manifold for $\xi_1, \xi_2 \in \mathbb{C}$ sufficiently close to 0. However this is possible if and only if $v_1$ depends only on $u_1$. That is, two cusps of $\mathcal{M}$ are SGI.

**Lemma 5.7.** Let $\mathcal{M}$ be a 2-cusped hyperbolic 3-manifold with $\tau_1, \tau_2$ its two cusp shapes. Suppose two cusps of $\mathcal{M}$ are not SGI, and the elements in
$$\{\tau_i | 0 \leq i, j \leq 2\}$$are linearly independent over $\mathbb{Q}$. If $\{\mathcal{Y}_i\}_{i \in I}$ is a family of infinitely many 2-dimensional anomalous subvarieties of $\mathcal{X} \times \mathcal{X}$ near $(z^0, z^0)$, then $\mathcal{Y}_i$ is contained in either
$$M_i = L_i = 1 \quad (i = 1 \text{ or } 2),$$or
$$M'_i = L'_i = 1 \quad (i = 1 \text{ or } 2),$$or
$$(M_1(M'_1)^{-1})^a = (M_2M'_2)^b \quad \text{for some } a, b \in \mathbb{Z},$$or
$$(M_1(M'_1)^{-1})^a = (M_2(M'_2)^{-1})^b \quad \text{for some } a, b \in \mathbb{Z},$$or
$$(M_1M'_1)^a = (M_2M'_2)^b \quad \text{for some } a, b \in \mathbb{Z},$$or
$$(M_1M'_1)^a = (M_2(M'_2)^{-1})^b \quad \text{for some } a, b \in \mathbb{Z},$$or a translation of
$$M_1 = L_1 = M_2 = L_2 = 1,$$or a translation of
$$M'_1 = L'_1 = M'_2 = L'_2 = 1.$$

**Proof.** By Theorem 3.7 there exists a finite collection $\Theta$ of algebraic subgroups of codimension 3 such that $\mathcal{Y}_i$ is a component of
$$K \cap (\mathcal{X} \times \mathcal{X})$$where $K$ is a coset of some $H \in \Theta$. Let $H$ be defined by
$$M_1^{a_1}L_1^{d_1}M_2^{a_2}L_2^{d_2}(M_1')^{a_1}(L_1')^{b_1}(M_2')^{c_1}(L_2')^{d_1} = 1,$$or
$$M_1^{a_2}L_1^{d_2}M_2^{a_2}L_2^{d_2}(M_1')^{a_2}(L_1')^{b_2}(M_2')^{c_2}(L_2')^{d_2} = 1,$$or
$$M_1^{a_3}L_1^{b_3}M_2^{a_3}L_2^{b_3}(M_1')^{a_3}(L_1')^{b_3}(M_2')^{c_3}(L_2')^{d_3} = 1,$$and suppose translations of $H$ contain infinitely many $\mathcal{Y}_i$. By Gauss elimination and changing the basis if necessary, we further assume $H$ is of the following form:
$$L_1^{a_1}L_2^{d_1}(M_1')^{a_1}(L_1')^{b_1}(M_2')^{c_1}(L_2')^{d_1} = 1,$$or
$$M_1^{a_2}(M_1')^{a_2}(L_1')^{b_2}(M_2')^{c_2}(L_2')^{d_2} = 1,$$or
$$M_2^{a_3}(M_1')^{a_3}(L_1')^{b_3}(M_2')^{c_3}(L_2')^{d_3} = 1.$$
By taking logarithm to each coordinate, $H \cap (X \times X)$ is locally biholomorphic (near $(z^0, z^0)$) to the complex manifold defined by

$$b_1 v_1 + d_1 v_2 + a_1' u_1' + b_1' v_1' + c_1' u_2' + d_1' v_2' = 0,$$

$$a_2 u_1 + a_2' u_1' + b_2' v_1' + c_2' u_2' + d_2' v_2' = 0,$$

$$c_3 u_2 + a_3' u_1' + b_3' v_1' + c_3' u_2' + d_3' v_2' = 0,$$

and the Jacobian of (5.49) at $(0, 0, 0, 0)$ is

$$
\begin{pmatrix}
b_1 \tau_1 & d_1 \tau_2 & a_1' + b_1' \tau_1 & c_1' + d_1' \tau_2 \\
a_2 & 0 & a_2' + b_2' \tau_1 & c_2' + d_2' \tau_2 \\
0 & c_3 & a_3' + b_3' \tau_1 & c_3' + d_3' \tau_2
\end{pmatrix}.
$$

(5.50)

Since the component of $H \cap (X \times X)$ containing $(z^0, z^0)$ is a 2-dimensional anomalous subvariety of $X \times X$, the equations in (5.49) define a 2-dimensional complex manifold and (5.50) is a matrix of rank 2. To prove the theorem, equivalently, we show if a translation of (5.49) are 2-dimensional complex manifolds, it is either contained in

$$u_i = v_i = 0 \quad (i = 1 \ or \ 2),$$

or

$$u_i' = v_i' = 0 \quad (i = 1 \ or \ 2),$$

or

$$u_1 - u_1' = m(u_2 + u_2') \quad \text{for some } m \in \mathbb{Q},$$

or

$$u_1 - u_1' = m(u_2 - u_2') \quad \text{for some } m \in \mathbb{Q},$$

or

$$u_1 + u_1' = m(u_2 + u_2') \quad \text{for some } m \in \mathbb{Q},$$

or

$$u_1 + u_1' = m(u_2 - u_2') \quad \text{for some } m \in \mathbb{Q},$$

or a translation of

$$u_1 = v_1 = u_2 = v_2 = 0,$$

or a translation of

$$u_1' = v_1' = u_2' = v_2' = 0.$$

We prove the statement by considering each possible case step by step.

1. $a_2 = c_3 = 0$.

   (a) $b_1 = d_1 = 0$.

   By applying Gauss elimination again, we assume (5.49) is of the following form

$$a_1' u_1' + b_1' v_1' + c_1' u_2' + d_1' v_2' = 0,$$

$$b_2' v_1' + c_2' u_2' + d_2' v_2' = 0,$$

$$c_3' u_2' + d_3' v_2' = 0,$$

and the Jacobian of (5.51) at $(u_1', u_2') = (0, 0)$ is

$$
\begin{pmatrix}
a_1' + b_1' \tau_1 & c_1' + d_1' \tau_2 \\
b_2' \tau_1 & c_2' + d_2' \tau_2 \\
0 & c_3' + d_3' \tau_2
\end{pmatrix}.
$$

(5.52)
whose rank is equal to 2. By the implicit function theorem, any translation of
\( (5.51) \) is equivalent to
\[ u_1' = \xi_1, \quad u_2' = \xi_2, \]  
for some \( \xi_1, \xi_2 \in \mathbb{C} \) (and so \( v_1', v_2' \) are constants as well). Thus it falls into the
last case in the statement.

(b) \( b_1 \neq 0 \) or \( d_1 \neq 0 \).
Since the rank of \( (5.50) \) is 2, the rank of
\[
\begin{pmatrix}
0 & 0 & a'_2 + b'_2 \tau_1 & c'_2 + d'_2 \tau_2 \\
0 & 0 & a'_3 + b'_3 \tau_1 & c'_3 + d'_3 \tau_2
\end{pmatrix}
\]  
is 1. Thus, by Lemma 4.3, we get either
\[ a'_2 = a'_3 = b'_2 = b'_3 = 0 \]  
(5.55)
or
\[ c'_2 = c'_3 = d'_2 = d'_3 = 0 \]  
(5.56)
or
\[ (a'_2, b'_2, c'_2, d'_2) = m(a'_3, b'_3, c'_3, d'_3) \]  
(5.57)
for some \( m \in \mathbb{Q} \setminus \{0\} \). But we ignore the last case since \( H \) is an algebraic
subgroup of codimension 3. By Gauss elimination if necessary, we get (5.49) is
either
\[ b_1 v_1 + d_1 v_2 + a'_1 u_1' + b'_1 v_1' + c'_1 u_2' + d'_1 v_2 = 0, \]  
u_1' = v_1' = 0,  
(5.58)
or
\[ b_1 v_1 + d_1 v_2 + a'_1 u_1' + b'_1 v_1' + c'_1 u_2' + d'_1 v_2 = 0, \]  
u_2' = v_2' = 0.  
(5.59)

Claim 5.8. If a translation of \( (5.58) \) or \( (5.59) \) is a 2-dimensional complex
manifold, then it is contained in either
\[ u_1' = v_1' = 0 \]  
(5.60)
or
\[ u_2' = v_2' = 0 \]  
(5.61)
respectively.

Proof. Suppose that
\[ b_1 v_1 + d_1 v_2 + a'_1 u_1' + b'_1 v_1' + c'_1 u_2' + d'_1 v_2 = \xi_1, \]  
u_1' = \xi_2, \quad v_1' = \xi_3  
(5.62)
define a 2-dimensional complex manifold for some \( \xi_i \) (\( 1 \leq i \leq 3 \)) sufficiently
close to 0. If \( \xi_2, \xi_3 \neq 0 \), then, since \( v_1' \) contains a nonzero term of the form
\( (u_1')^j(u_2')^j \), \( u_2' \) is a constant determined by \( \xi_2 \) and \( \xi_3 \). Since \( b_1 \neq 0 \) or \( d_1 \neq 0 \) in
the first equation, \( u_1 \) and \( u_2 \) are depending each other. So (5.62) defines a
1-dimensional complex manifold. \( \square \)

By the claim, in this case, it falls into the first or the second case in the statement.

(2) \( (a_2 = 0, c_3 \neq 0) \) or \( (c_3 = 0, a_2 \neq 0) \).
Without loss of generality, we only consider the first case, and further split the
problem into the following four cases.
(a) \( b_1 = d_1 = 0. \)

This is reduced to the case similar to the one that we considered above.

(b) \( b_1 \neq 0 \) and \( d_1 = 0. \)

This contradicts the fact that the rank of \( (5.50) \) is equal to 2.

(c) \( b_1 \neq 0 \) and \( d_1 \neq 0. \)

This also contradicts the fact that the rank of \( (5.50) \) is equal to 2.

(d) \( b_1 = 0 \) and \( d_1 \neq 0. \)

In this case, \( (5.50) \) is of the following form
\[
\begin{pmatrix}
0 & d_1 \tau_2 & a_1' + b_1' \tau_1 & c_1' + d_1' \tau_2 \\
0 & 0 & a_2' + b_2' \tau_1 & c_2' + d_2' \tau_2 \\
0 & c_3 & a_3' + b_3' \tau_1 & c_3' + d_3' \tau_2
\end{pmatrix}.
\]
\( (5.63) \)

Since \( (5.63) \) is a matrix of rank 2, we get
\[
y(0, c_3, a_3' + b_3' \tau_1, c_3' + d_3' \tau_2) + x(0, 0, a_2' + b_2' \tau_1, c_2' + d_2' \tau_2)
\]
\( (5.64) \)

for some \( x, y \in \mathbb{C} \). By computing the second coordinates, we get \( y = \frac{d_1}{c_3} \tau_2 \) and so \( (5.64) \) is equivalent to
\[
a_1' + b_1' \tau_1 = \frac{d_1}{c_3} \tau_2(a_3' + b_3' \tau_1) + x(a_2' + b_2' \tau_1),
\]
\( (5.65) \)

\[
c_1' + d_1' \tau_2 = \frac{d_1}{c_3} \tau_2(c_3' + d_3' \tau_2) + x(c_2' + d_2' \tau_2).
\]
\( (5.66) \)

Since \( (a_3', b_3', c_3', d_3') \neq (0, 0, 0, 0) \), we have either \( a_2' + b_2' \tau_1 \neq 0 \) or \( c_2' + d_2' \tau_2 \neq 0. \)

(i) \( a_2' + b_2' \tau_1 \neq 0. \)

In this case, \( (5.65) \) is equivalent to
\[
x = \frac{a_1' + b_1' \tau_1 - \frac{d_1}{c_3} \tau_2(a_3' + b_3' \tau_1)}{a_2' + b_2' \tau_1}
\]
\( (5.67) \)

and
\[
c_1' + d_1' \tau_2 = \frac{d_1}{c_3} \tau_2(c_3' + d_3' \tau_2) + \frac{a_1' + b_1' \tau_1 - \frac{d_1}{c_3} \tau_2(a_3' + b_3' \tau_1)}{a_2' + b_2' \tau_1}(c_2' + d_2' \tau_2).
\]
\( (5.68) \)

Now \( (5.67) \) is equivalent to
\[
(a_2' + b_2' \tau_1)(c_1' + d_1' \tau_2) = (a_2' + b_2' \tau_1)\frac{d_1}{c_3} \tau_2(c_3' + d_3' \tau_2) + (a_1' + b_1' \tau_1 - \frac{d_1}{c_3} \tau_2(a_3' + b_3' \tau_1))(c_2' + d_2' \tau_2),
\]
\( (5.69) \)

which is equivalent to
\[
c_3(a_2' + b_2' \tau_1)(c_1' + d_1' \tau_2) = d_1 \tau_2(a_2' + b_2' \tau_1)(c_3' + d_3' \tau_2) + (c_3(a_1' + b_1' \tau_1) - d_1 \tau_2(a_3' + b_3' \tau_1))(c_2' + d_2' \tau_2),
\]
\( (5.70) \)

which is equivalent to
\[
c_3(a_2' + b_2' \tau_1)(c_1' + d_1' \tau_2) + d_1 \tau_2(a_2' + b_2' \tau_1)(c_3' + d_3' \tau_2) = d_1 \tau_2(a_2' + b_2' \tau_1)(c_3' + d_3' \tau_2) + c_3(a_1' + b_1' \tau_1)(c_2' + d_2' \tau_2).
\]
\( (5.71) \)

Since the elements in
\[
\{ \tau_1 \tau_2^j \mid 0 \leq i, j \leq 2 \}
\]
are linearly independent over \( \mathbb{Q} \), we get
\[
c_2(\tau_2) + b_2(\tau_2)(c' + d_1') = c_3(\tau_2) + b_3(\tau_2)(c' + d_2'),
\]
and, since \( c_3 \neq 0 \) and \( d_1 \neq 0 \), we have
\[
(a' + b_2(\tau_2))(c' + d_1') = (a' + b_1(\tau_1))(c' + d_2'),
\]
(5.72)
\[
(a' + b_3(\tau_1))(c' + d_2') = (a' + b_2(\tau_2))(c' + d_3'),
\]
(5.73)
By Lemma 4.3, (5.72) implies either
\[
a_1' + b_1' = b_2 = 0 \tag{5.74}
\]
or
\[
c_1' = c_2' = d_1' = d_2' = 0 \tag{5.75}
\]
or
\[
(a_1', b_1', c_1', d_1') = m(a_2', b_2', c_2', d_2') \tag{5.76}
\]
for some \( m \in \mathbb{Q} \setminus \{0\} \). But the first case (5.74) contradicts the fact that \( a_2' + b_2(\tau_2) \neq 0 \), and for the third case (5.76), by applying Gauss elimination, we can assume
\[
a_1' = b_1' = c_1' = d_1' = 0. \tag{5.77}
\]
In conclusion, (5.72) implies either
\[
c_1' = c_2' = d_1' = d_2' = 0 \tag{5.78}
\]
or
\[
a_1' = b_1' = c_1' = d_1' = 0. \tag{5.79}
\]
Similarly (5.73) implies either
\[
c_2' = c_3' = d_2' = d_3' = 0 \tag{5.78}
\]
or
\[
a_3' = b_3' = c_3' = d_3' = 0. \tag{5.79}
\]
Now we consider each possible case separately.
(A) \( c_2' = d_1' = d_2' = c_3' = d_3' = 0. \)

In this case, (5.49) is of the following form:
\[
d_1v_2 + a_1'u_1 + b_1'v_1 = 0,
\]
\[
a_2'u_1 + b_2'v_1 = 0,
\]
\[
c_3u_2 + a_3'u_1 + b_3'v_1 = 0. \tag{5.80}
\]

Claim 5.9. A nontrivial translation of (5.80) is not a 2-dimensional complex manifold.

Proof. First, by Gauss elimination, we further assume (5.80) is of the following form:
\[
d_1v_2 + a_1'u_1 = 0,
\]
\[
a_2'u_1 + b_2'v_1 = 0, \tag{5.81}
\]
\[
c_3u_2 + a_3'u_1 = 0.
\]
Suppose
\[ d_1 v_2 + a'_1 u'_1 = \xi_1, \]
\[ a'_2 u'_1 + b'_2 v'_1 = \xi_2, \]
\[ c_3 u_2 + a'_3 u'_1 = \xi_3, \]
\[ (5.82) \]
is a 2-dimensional complex manifold for some \( \xi_i \) (\( 1 \leq i \leq 3 \)) sufficiently close to 0. The Jacobian of
\[ a'_2 u'_1 + b'_2 v'_1 = \xi_2, \]
\[ c_3 u_2 + a'_3 u'_1 = \xi_3, \]
\[ (5.83) \]
with respect to \( u_2 \) and \( u'_1 \) at some \( (u_2, u'_1, u'_2) \) sufficiently close to \( (0, 0, 0) \) is approximately
\[ \begin{pmatrix} 0 & a'_2 + b'_2 \gamma_1 (\neq 0) \\ c_3 (\neq 0) & a'_3 \end{pmatrix}. \]
\[ (5.84) \]
Thus, by applying the implicit function theorem to
\[ a'_2 u'_1 + b'_2 v'_1 = \xi_2, \]
\[ c_3 u_2 + a'_3 u'_1 = \xi_3 \]
\[ (5.85) \]
we represent \( u_2 \) and \( u'_1 \) as holomorphic functions of \( u'_2 \). Let
\[ u_2 = g_1(u'_2), \quad u'_1 = g_2(u'_2) \]
\[ (5.86) \]
for some holomorphic functions \( g_1 \) and \( g_2 \). Since \( v_2 \) contains a nonzero term of the form \( u'_1 u'_2 \), by plugging \((5.86)\) to the first equation in \((5.82)\), we get that \( u_1 \) also depends on \( u'_2 \). Thus \((5.82)\) defines a 1-dimensional complex manifold. \( \square \)

(B) \( c'_1 = c'_2 = d'_1 = d'_3 = 0 \) and \( a'_3 = b'_3 = c'_3 = d'_3 = 0 \).

In this case, \((5.49)\) is of the following form:
\[ d_1 v_2 + a'_1 u'_1 = 0, \]
\[ a'_2 u'_1 + b'_2 v'_1 = 0, \]
\[ u_2 = 0. \]
\[ (5.87) \]

First if \( a'_1 b'_2 - b'_1 a'_2 = 0 \), then by Gauss elimination, we further assume
\[ (5.87) \]
is of the following form:
\[ a'_2 u'_1 + b'_2 v'_1 = 0, \]
\[ u_2 = v_2 = 0. \]
\[ (5.88) \]

By an argument similar to the one in the proof of Claim 5.8, if a translation of \((5.88)\) is a 2 dimensional complex manifold, then it is contained in \( u_2 = v_2 = 0 \).

Now suppose \( a'_1 b'_2 - b'_1 a'_2 \neq 0 \). Then, by Gauss elimination, \((5.87)\) is further reduced to
\[ d_1 v_2 + a'_1 u'_1 = 0, \]
\[ a'_2 u'_1 + b'_2 v'_1 = 0, \]
\[ u_2 = 0. \]
\[ (5.89) \]

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Using the same argument in the proof of Claim 5.9, it can be shown that a nontrivial translation of (5.89) is a 1-dimensional complex manifold.

(C) \( a'_1 = b'_1 = c'_1 = d'_1 = 0 \) and \( c'_2 = c'_3 = d'_2 = d'_3 = 0 \).

In this case, (5.49) is of the following form:

\[
\begin{align*}
v_2 &= 0, \\
    a'_2 u'_1 + b'_2 v'_1 &= 0, \\
    c'_3 u_2 + a'_3 u'_1 + b'_3 v'_1 &= 0.
\end{align*}
\]

(5.90)

By an argument similar to the one in the proof of Claim 5.9, we can show that a nontrivial translation of (5.90) is a 1-dimensional complex manifold.

(D) \( a'_1 = b'_1 = c'_1 = d'_1 = 0 \) and \( a'_3 = b'_3 = c'_3 = d'_3 = 0 \).

In this case, (5.49) is of the following form:

\[
\begin{align*}
    a'_2 u'_1 + b'_2 v'_1 + c'_2 u'_2 + d'_2 v'_2 &= 0, \\
    v_2 &= u_2 = 0.
\end{align*}
\]

(5.91)

By an argument similar to the one given in the proof of Claim 5.8, if a translation of (5.91) is a 2-dimensional complex manifold, then it is contained in \( u_2 = v_2 = 0 \).

(ii) \( c'_2 + \tau_2 d'_2 \neq 0 \).

Recall (5.65) which is given as follows:

\[
\begin{align*}
    a' + b'_1 \tau_1 &= \frac{d'_1}{c_3} \tau_2 (a'_3 + b'_3 \tau_1) + x (a'_2 + b'_2 \tau_1), \\
    c' + d'_1 \tau_2 &= \frac{d'_1}{c_3} \tau_2 (c'_3 + d'_3 \tau_2) + x (c'_2 + d'_2 \tau_2).
\end{align*}
\]

(5.92)

If \( c'_2 + \tau_2 d'_2 \neq 0 \), then

\[
x = \frac{c'_1 + d'_1 \tau_2 - \frac{d'_1}{c_3} \tau_2 (c'_3 + d'_3 \tau_2)}{c'_2 + d'_2 \tau_2}
\]

(5.93)

and so

\[
a' + b'_1 \tau_1 = \frac{d'_1}{c_3} \tau_2 (a'_3 + b'_3 \tau_1) + \frac{c'_1 + d'_1 \tau_2 - \frac{d'_1}{c_3} \tau_2 (c'_3 + d'_3 \tau_2)}{c'_2 + d'_2 \tau_2} (a'_2 + b'_2 \tau_1).
\]

(5.94)

Now (5.94) is equivalent to

\[
(c'_2 + d'_2 \tau_2)(a'_1 + b'_1 \tau_1) = \frac{d'_1}{c_3} \tau_2 (c'_2 + d'_2 \tau_2) (a'_3 + b'_3 \tau_1) + (c'_1 + d'_1 \tau_2 - \frac{d'_1}{c_3} \tau_2 (c'_3 + d'_3 \tau_2)) (a'_2 + b'_2 \tau_1),
\]

(5.95)

which is equivalent to

\[
c_3 (c'_2 + d'_2 \tau_2) (a'_1 + b'_1 \tau_1) = d_1 \tau_2 (c'_2 + d'_2 \tau_2) (a'_3 + b'_3 \tau_1) + (c_3 (c'_1 + d'_1 \tau_2) - d_1 \tau_2 (c'_3 + d'_3 \tau_2)) (a'_2 + b'_2 \tau_1),
\]

(5.96)

which is equal to

\[
c_3 (c'_2 + d'_2 \tau_2) (a'_1 + b'_1 \tau_1) + d_1 \tau_2 (c'_3 + d'_3 \tau_2) (a'_2 + b'_2 \tau_1) = d_1 \tau_2 (c'_2 + d'_2 \tau_2) (a'_3 + b'_3 \tau_1) + c_3 (c'_1 + d'_1 \tau_2) (a'_2 + b'_2 \tau_1).
\]

(5.97)
Since the elements in 
\[ \{ \tau_i \tau_j \mid 0 \leq i, j \leq 2 \} \]
are linearly independent over \( \mathbb{Q} \), we get
\[ c_3 (c'_2 + d'_2 \tau_2)(a'_1 + b'_1 \tau_1) = c_3 (c'_1 + d'_1 \tau_2)(a'_2 + b'_2 \tau_1) \] (5.98)
and
\[ d_1 \tau_2 (c'_3 + d'_3 \tau_2)(a'_2 + b'_2 \tau_1) = d_1 \tau_2 (c'_2 + d'_2 \tau_2)(a'_3 + b'_3 \tau_1) \] (5.99)
which imply (since \( c_3 \neq 0 \) and \( d_1 \neq 0 \))
\[ (c'_2 + d'_2 \tau_2)(a'_1 + b'_1 \tau_1) = (c'_1 + d'_1 \tau_2)(a'_2 + b'_2 \tau_1) \] (5.100)
and
\[ (c'_3 + d'_3 \tau_2)(a'_2 + b'_2 \tau_1) = (c'_2 + d'_2 \tau_2)(a'_3 + b'_3 \tau_1) \] (5.101)
respectively. By Lemma 4.3, (5.100) implies either
\[ a'_1 = b'_1 = c'_1 = d'_1 = 0 \]
or
\[ a'_1 = b'_1 = a'_2 = b'_2 = 0, \]
and (5.101) implies either
\[ a'_3 = b'_3 = a'_2 = b'_2 = 0 \]
or
\[ a'_3 = b'_3 = c'_3 = d'_3 = 0. \]
We consider each possible case separately.
(A) \( a'_1 = b'_1 = c'_1 = d'_1 = a'_3 = b'_3 = a'_2 = b'_2 = 0. \)
In this case, (5.49) is of the following form:
\[ v_2 = 0, \]
\[ c'_2 u_2' + d'_2 v_2' = 0, \]
\[ c_3 u_2 + c'_3 u_2' + d'_3 v_2' = 0. \] (5.102)
If \( c'_2 d'_3 - c'_3 d'_2 = 0 \), then, by Gauss elimination, (5.102) is reduced to
\[ u_2 = v_2 = 0, \]
\[ c'_2 u_2' + d'_2 v_2' = 0, \] (5.103)
Thus it falls into the case that we already discussed in Claim 5.8. If
\( c'_2 d'_3 - c'_3 d'_2 \neq 0 \), then, by Gauss elimination, we assume (5.102) is of
the following form:
\[ v_2 = 0, \]
\[ c'_2 u_2' + d'_2 v_2' = 0, \]
\[ c_3 u_2 + c'_3 u_2' = 0. \] (5.104)
Using an argument similar to the one given in the proof of Claim 5.9, one can show that a nontrivial translation of (5.104) is a 1-dimensional complex manifold.
Recall the equations and matrix given in (5.49) and (5.50), which are respectively. Since the rank of (5.110) is 2, we have

\[ \begin{pmatrix} a_1 \tau_1 & d_1 \tau_2 & a_1' \tau_1 + b_1' \tau_1 & c_1' + d_1' \tau_2 \\ a_2 & 0 & a_2' \tau_1 + b_2' \tau_1 & c_2' + d_2' \tau_2 \\ 0 & c_3 & a_3' \tau_1 + b_3' \tau_1 & c_3' + d_3' \tau_2 \end{pmatrix} \]

respectively. Since the rank of (5.110) is 2, we have

\[ \begin{pmatrix} a_1 \tau_1 & d_1 \tau_2 & a_1' \tau_1 + b_1' \tau_1 & c_1' + d_1' \tau_2 \\ a_2 & 0 & a_2' \tau_1 + b_2' \tau_1 & c_2' + d_2' \tau_2 \\ 0 & c_3 & a_3' \tau_1 + b_3' \tau_1 & c_3' + d_3' \tau_2 \end{pmatrix} \]
which implies
\[
\begin{align*}
a_1' + b_1' \tau_1 &= \frac{b_1}{a_2} \tau_1 (a_2' + b_2' \tau_1) + \frac{d_1}{c_3} \tau_2 (a_3' + b_3' \tau_1), \\
c_1' + d_1' \tau_2 &= \frac{b_1}{a_2} \tau_1 (c_2' + d_2' \tau_2) + \frac{d_1}{c_3} \tau_2 (c_3' + d_3' \tau_2).
\end{align*}
\] (5.112)

Since the elements in
\[
\{\tau_i^j | 0 \leq i, j \leq 2\}
\]
are linearly independent over \(\mathbb{Q}\), we get
\[
\begin{align*}
a_1' &= 0, \quad b_1' = 0, \quad d_1a_3' = 0, \quad d_1b_3' = 0, \quad b_1' - \frac{b_1a_2'}{a_2} = 0, \\
c_1' &= 0, \quad b_1c_2' = 0, \quad d_1d_3' = 0, \quad b_1d_2' = 0, \quad d_1' - \frac{d_1c_3'}{c_3} = 0.
\end{align*}
\] (5.113)

Now we consider each possible case (depending on whether \(b_1, d_1\) are zero or not) separately.

(a) \(b_1 = d_1 = 0\).

This contradicts the fact that the rank of (5.110) is 2.

(b) \(b_1 \neq 0\) and \(d_1 = 0\) (and so \(b_2' = c_2' = d_2' = d_1' = 0, \quad b_1' - \frac{b_1a_2'}{a_2} = 0\)).

In this case, (5.110) is of the following form
\[
\begin{align*}
b_1v_1 + b_1'v_1' &= 0, \\
a_2u_1 + a_2'u_1' &= 0, \\
c_3u_2 + a_3'u_1' + b_3'v_1' + c_3'u_2' + d_3'v_2' &= 0,
\end{align*}
\] (5.114)

and its Jacobian at \((u_1, u_2, u_1', u_2') = (0, 0, 0, 0)\) is
\[
\begin{pmatrix}
b_1' \tau_1 (\neq 0) & 0 & b_1' \tau_1 & 0 \\
an_2' (\neq 0) & 0 & a_2' & 0 \\
0 & c_3(\neq 0) & a_3' + b_3' \tau_1 & c_3' + d_3' \tau_2
\end{pmatrix}.
\]

Since \(a_2b_1' - a_2'b_1 = 0\), we have either \(b_1' = a_2' = 0\) or \(b_1' \neq 0, a_2' \neq 0\).

(i) \(b_1' = a_2' = 0\).

First if \(a_3' + b_3' \tau_1 \neq 0\) or \(c_3' + d_3' \tau_2 \neq 0\), then it is reduced to a case similar to the one considered in Claim 5.8. If \(a_3' + b_3' \tau_1 = c_3' + d_3' \tau_2 = 0\), (5.114) is of the following form
\[
\begin{align*}
v_1 &= u_1 = u_2 = 0,
\end{align*}
\] (5.115)

and it falls into a case in the statement of the theorem.

(ii) \(b_1' \neq 0, a_2' \neq 0\).

Since \(a_2b_1' - a_2'b_1 = 0\), without loss of generality, we assume (5.114) is of the following form:
\[
\begin{align*}
v_1 + b_1'v_1' &= 0, \\
u_1 + b_1'u_1' &= 0, \\
c_3u_2 + a_3'u_1' + b_3'v_1' + c_3'u_2' + d_3'v_2' &= 0.
\end{align*}
\] (5.116)
If translations of (5.116) are 2-dimensional complex manifolds, then there exist holomorphic functions $h(t), g(t)$ such that the following equations

$$
u_1 + b'_1 u'_1 = t,$$
$$v_1 + b'_1 v'_1 = h(t),$$
$$c_3 u_2 + a'_3 u'_1 + b'_3 v'_1 + c'_3 u'_2 + d'_3 v'_2 = g(t),$$

(5.117)

define a 2-dimensional complex manifold for each $t \in \mathbb{C}$ sufficiently close to 0. In other words, the following two equations

$$v_1 + b'_1 u'_1 = h(u_1 + b'_1 u'_1),$$
$$c_3 u_2 + a'_3 u'_1 + b'_3 v'_1 + c'_3 u'_2 + d'_3 v'_2 = g(u_1 + b'_1 u'_1)$$

(5.118)
define a 3-dimensional complex manifold (thus two equations in (5.118) are equivalent to each other). We show that it is impossible.

**Claim 5.10.** The equations in (5.118) do not define a 3-dimensional complex manifold.

**Proof.** Suppose (5.118) defines a 3-dimensional complex manifold. We first claim $c'_2 = d'_3 = 0$. Let $u'_1 = 0$, then (5.118) becomes

$$v_1 = h(u_1),$$
$$c_3 u_2 + c'_3 u'_1 + d'_3 v'_2 = g(u_1),$$

(5.119)

which describes a 2-dimensional complex manifold. Since $v_1$ contains a nonzero term of the form $u'_1 u'_2$, $u_2$ depends on $u_1$ in the first equation. In the second equation, if $c'_3 \neq 0$ or $d'_3 \neq 0$, then $u'_2$ depends on $u_1$ and $u_2$, and so (5.119) defines a 1-dimensional complex manifold. But this contradict the assumption that (5.119) describes a 2-dimensional complex manifold. Thus $c'_3 = d'_3 = 0$ and (5.118) is simplified as

$$v_1 + b'_1 u'_1 = h(u_1 + b'_1 u'_1),$$
$$c_3 u_2 + a'_3 u'_1 + b'_3 v'_1 = g(u_1 + b'_1 u'_1).$$

(5.120)

Now let $u_1 = 0$. Then (5.120) is

$$b'_1 v'_1 = h(b'_1 v'_1),$$
$$c_3 u_2 + a'_3 u'_1 + b'_3 v'_1 = g(b'_1 u'_1),$$

(5.121)

which defines a 2-dimensional complex manifold. In the first equation (of (5.121)), $u'_2$ depends on $u'_1$, and in the second equation, $u_2$ depends on $u'_1$ and $u'_2$ (recall $c_3 \neq 0$). So both $u'_2$ and $u_2$ depend on $u'_1$, and thus the equations in (5.121) describe a 1-dimensional complex manifold. But, again, this is a contradiction. \[\square\]

\begin{enumerate}[\text{(c)}]
\item $b_1 = 0$ and $d_1 \neq 0$ (and so $a'_3 = b'_3 = d'_3 = b'_1 = 0, d'_1 - \frac{d_1 c'_3}{c_3} = 0$).
\end{enumerate}

In this case, (5.109) is of the following form

$$d_1 v_2 + d'_1 v'_2 = 0,$$
$$a_2 u_1 + a'_2 u'_1 + b'_2 v'_1 + c'_2 u'_2 + d'_2 v'_2 = 0,$$
$$c_3 u_2 + c'_3 u'_2 = 0,$$

(5.122)
and its Jacobian at \((u_1, u_2, u'_1, u'_2) = (0, 0, 0, 0)\) is
\[
\begin{pmatrix}
0 & d_1\tau_2 & 0 & d'_1\tau_2 \\
0 & 0 & a' + b_2\tau_1 & c_2 + d'_2\tau_2 \\
0 & c_3 & 0 & c'_3
\end{pmatrix}.
\]

This can be handled similarly to the previous case.

(d) \(b_1 \neq 0, d_1 \neq 0\) (and so \(c'_2 = b'_2 = d'_2 = a'_3 = b'_3 = d'_3 = 0\) and \(d'_1 - \frac{d_1c'_3}{c_3} = b'_1 - \frac{b_1a'_2}{a_2} = 0\)).

In this case, (5.109) is of the following form
\[
b_1v_1 + d_1v_2 + b'_1v'_1 + d'_1v'_2 = 0,
\]
\[
a_2u_1 + a'_2u'_1 = 0, \quad (5.123)
\]
\[
c_3u_2 + c'_3u'_2 = 0
\]
and the Jacobian of the above matrix at \((u_1, u_2, u'_1, u'_2) = (0, 0, 0, 0)\) is
\[
\begin{pmatrix}
b_1\tau_1(\neq 0) & d_1\tau_2(\neq 0) & b'_1\tau_1 & d'_1\tau_2 \\
0 & c_3(\neq 0) & 0 & c'_3
\end{pmatrix}.
\]

Now we split the problem into the following four cases depending on whether \(a'_2, c'_3\) are zero or not.

(i) \(a'_2 = c'_3 = 0\).

In this case, (5.124) is of the following form:
\[
\begin{pmatrix}
b_1\tau_1(\neq 0) & d_1\tau_2(\neq 0) & b'_1\tau_1 & d'_1\tau_2 \\
0 & c_3(\neq 0) & 0 & c'_3
\end{pmatrix}.
\]

Since (5.125) is a matrix of rank 2, we have \(b'_1 = d'_1 = 0\), and thus (5.123) is of the following form
\[
b_1v_1 + d_1v_2 = 0,
\]
\[
u_1 = u_2 = 0. \quad (5.126)
\]
Any translation of (5.126) is a 2-dimensional complex manifold, and it falls into a case in the statement of the theorem.

(ii) \(a'_2 = 0\) and \(c'_3 \neq 0\).

In this case, (5.124) is of the following form:
\[
\begin{pmatrix}
b_1\tau_1(\neq 0) & d_1\tau_2(\neq 0) & b'_1\tau_1 & d'_1\tau_2 \\
0 & c_3(\neq 0) & 0 & c'_3
\end{pmatrix}.
\]

Since (5.127) is a matrix of rank 2, we get \(b'_1 = 0\). Also since \(d'_1c_3 - d_1c'_3 = 0\), without loss of generality, we assume (5.123) is of the following form:
\[
b_1v_1 + c_3v_2 + c'_3v'_2 = 0,
\]
\[
u_1 = 0,
\]
\[
c_3u_2 + c'_3u'_2 = 0. \quad (5.128)
\]

**Claim 5.11.** A translation of (5.128) is not a 2-dimensional complex manifold.
Proof. Consider
\[ b_1 v_1 + c_3 v_2 + c_3' v_2' = \xi_1, \]
\[ u_1 = \xi_2, \]
\[ c_3 u_2 + c_3' u_2' = \xi_3 \]
with \((\xi_1, \xi_2, \xi_3) \in C^3\) sufficiently close to the origin. From the third equation in (5.129), we know that \(u_2\) depends on \(u_2'\). Since \(v_2'\) contains a nonzero term of the form \(u_1' \cdot u_2'\), we get \(u_2'\) is determined by \(u_2\) from the first equation. In conclusion, (5.129) defines a 1-dimensional complex manifold. \(\square\)

(iii) \(a_2' \neq 0\) and \(c_3' = 0\).
In this case, (5.124) is of the following form:
\[
\begin{pmatrix}
    b_1 \tau_1 (\neq 0) & d_1 \tau_2 (\neq 0) & b_1' \tau_1 & d_1' \tau_2 \\
    a_2 (\neq 0) & 0 & a_2' (\neq 0) & 0 \\
    0 & c_3 (\neq 0) & 0 & 0
\end{pmatrix}.
\] (5.130)
Since (5.130) is a matrix of rank 2, we have \(d_1' = 0\). Also since \(b_1' a_2 - b_1 a_2' = 0\), without loss of generality, we assume (5.123) is of the following form:
\[ a_2 v_1 + d_1 v_2 + a_2' v_1' = 0, \]
\[ a_2 u_1 + a_2' u_1' = 0, \]
\[ u_2 = 0. \] (5.131)
Using an argument similar to the one given in the proof of Claim 5.11, it can be shown that a translation of (5.131) is a 1-dimensional complex manifold.

(iv) \(a_2' \neq 0\) and \(c_3' \neq 0\).
Since \(c_3 d_1' - d_1 c_3' = a_2 b_1' - b_1 a_2' = 0\), we have
\[ n_1(a_2, a_2') = (b_1, b_1'), \quad n_2(c_3, c_3') = (d_1, d_1') \]
for some \(n_1, n_2 \in \mathbb{Q} \setminus \{0\}\). Without loss of generality, we assume (5.123) is of the following form:
\[ a_2 v_1 + m c_3 v_2 + a_2' v_1' + m c_3' v_2' = 0, \]
\[ a_2 u_1 + a_2' u_1' = 0, \]
\[ c_3 u_2 + c_3' u_2' = 0 \] (5.132)
for some \(m \in \mathbb{Q} \setminus \{0\}\). We first claim the following:

Claim 5.12. If equations in (5.132) describe a 2-dimensional complex manifolds, then
\[ \frac{a_2}{a_2'} = \pm 1, \quad \frac{c_3}{c_3'} = \pm 1. \] (5.133)

Proof. Let
\[ \Phi(u_1, u_2) = \sum_{i,j \geq 0; \text{even}} m_{ij} u_1'^i u_2'^j \] (5.134)
be a Neumann-Zagier potential function\textsuperscript{8} of \( \mathcal{M} \). If (5.132) defines a 2-dimensional complex manifold, by plugging the second and third equations to the first one, we get

\[
a_2 v_1(u_1, u_2) + mc_3 v_2(u_1, u_2) + a_2' v_1 \left( -\frac{a_2}{a_2'} u_1, -\frac{c_3}{c_3'} u_2 \right) + mc_3' v_2 \left( -\frac{a_2}{a_2'} u_1, -\frac{c_3}{c_3'} u_2 \right),
\]

(5.135)

and this must be equal to zero for any \( u_1 \) and \( u_2 \). In other words, (5.135) is simply the zero function. Now we rewrite (5.135) as follows:

\[
a_2 \frac{\partial \Phi}{\partial u_1}(u_1, u_2) + mc_3 \frac{\partial \Phi}{\partial u_2}(u_1, u_2) + a_2' \frac{\partial \Phi}{\partial u_1}(u_1, u_2) + mc_3' \frac{\partial \Phi}{\partial u_2}(u_1, u_2),
\]

(5.136)

which is equivalent to

\[
a_2 \left( \frac{1}{2} \sum_{i,j \geq 0; even} im_{ij} u_1^{i-1} u_2^j \right) + mc_3 \left( \frac{1}{2} \sum_{i,j \geq 0; even} jm_{ij} u_1^i u_2^{j-1} \right)
\]

\[
+ a_2' \left( \frac{1}{2} \sum_{i,j \geq 0; even} im_{ij} \left( -\frac{a_2}{a_2'} u_1 \right)^{i-1} \left( -\frac{c_3}{c_3'} u_2 \right)^j \right) + mc_3' \left( \frac{1}{2} \sum_{i,j \geq 0; even} jm_{ij} \left( -\frac{a_2}{a_2'} u_1 \right)^i \left( -\frac{c_3}{c_3'} u_2 \right)^{j-1} \right).
\]

(5.137)

Since two cusps of \( \mathcal{M} \) are not SGI each other, there exists \( u_1^i u_2^j \) such that \( i, j \geq 2 \) and \( m_{ij} \neq 0 \). The coefficients of \( u_1^{i-1} u_2^j \) and \( u_1^i u_2^{j-1} \) in (5.137) are

\[
a_2 m_{ij} + a_2' m_{ij} \left( -\frac{a_2}{a_2'} \right)^{i-1} \left( -\frac{c_3}{c_3'} \right)^j
\]

(5.138)

and

\[
mc_3 m_{ij} + c_3' m_{ij} \left( -\frac{a_2}{a_2'} \right)^i \left( -\frac{c_3}{c_3'} \right)^{j-1}
\]

(5.139)

respectively. Since (5.137) is the zero function, (5.138) and (5.139) are all equal to zero. So

\[
a_2 m_{ij} = -a_2' m_{ij} \left( -\frac{a_2}{a_2'} \right)^{i-1} \left( -\frac{c_3}{c_3'} \right)^j,
\]

(5.140)

\[
mc_3 m_{ij} = -c_3' m_{ij} \left( -\frac{a_2}{a_2'} \right)^i \left( -\frac{c_3}{c_3'} \right)^{j-1},
\]

(5.141)

which imply

\[
\left( \frac{a_2}{a_2'} \right)^{i-2} \left( \frac{c_3}{c_3'} \right)^j = \left( \frac{a_2}{a_2'} \right)^i \left( \frac{c_3}{c_3'} \right)^{j-2} = 1
\]

and

\[
\left( \frac{a_2}{a_2'} \right)^2 = \left( \frac{c_3}{c_3'} \right)^2.
\]

(5.142)

Finally we get

\[
\frac{a_2}{a_2'} = \pm \frac{c_3}{c_3'}
\]

\textsuperscript{8} We define \( m_{00} \) as 0.

\textsuperscript{9} Otherwise, \( u_1 \) and \( u_2 \) depend each other and so (5.132) defines a 1-dimensional complex manifold.
from (5.142) and
\[ \frac{\alpha_2}{a_2} = \pm 1, \quad \frac{c_3}{c_3} = \pm 1 \]
from (5.141). □

Without loss of generality, we suppose \( \frac{\alpha_2}{a_2} = -1, \quad \frac{c_3}{c_3} = -1 \) and rewrite (5.132) as follows:
\[ v_1' + mv_2' - v_1 - mv_2 = 0, \]
\[ u_2' - u_2 = 0, \]
\[ u_1' - u_1 = 0. \]

**Claim 5.13.** If translations of (5.143) are 2-dimensional complex manifolds, then
\[ u_2 - u_2' = -m(u_1 - u_1'). \]  

**Proof.** If translations of (5.143) are 2-dimensional complex manifolds, then it means there exist holomorphic functions \( h(t), g(t) \) such that the following equations
\[ v_1' + mv_2' - v_1 - mv_2 = h(t), \]
\[ u_2' - u_2 = g(t), \]
\[ u_1' - u_1 = t \]
define a 2-dimensional complex manifold for \( t \in \mathbb{C} \) sufficiently close to 0. To simplify the proof, we assume \( m = -1 \) and show \( g(t) = t \). (The proof for the general case would be similar.) By substituting
\[ u_2' = u_2 + g(t), \]
\[ u_1' = u_1 + t \]
into the first equation in (5.145), we get
\[ v_1(u_1 + t, u_2 + g(t)) - v_2(u_1 + t, u_2 + g(t)) - v_1(u_1, u_2) + v_2(u_1, u_2) - h(t), \]
and this must be equal to zero for any \( u_1, u_2 \) and \( t \). Recall the Neumann-Zagier potential function of \( \mathcal{M} \) given in (5.134) and rewrite (5.147) as
\[ \frac{\partial \Phi}{\partial u_1}(u_1 + t, u_2 + g(t)) - \frac{\partial \Phi}{\partial u_2}(u_1 + t, u_2 + g(t)) - \frac{\partial \Phi}{\partial u_1}(u_1, u_2) + \frac{\partial \Phi}{\partial u_2}(u_1, u_2) - h(t), \]
which is equivalent to
\[ \left( \frac{1}{2} \sum_{i,j:(\geq 0);{\text{even}}} i m_{ij}(u_1 + t)^{i-1}(u_2 + g(t))^j \right) - \left( \frac{1}{2} \sum_{i,j:(\geq 0);{\text{even}}} j m_{ij}(u_1 + t)^i(u_2 + g(t))^{j-1} \right) \]
\[ - \left( \frac{1}{2} \sum_{i,j:(\geq 0);{\text{even}}} i m_{ij} u_1^{i-1} u_2^j \right) + \left( \frac{1}{2} \sum_{i,j:(\geq 0);{\text{even}}} j m_{ij} u_1^i u_2^{j-1} \right) - h(t). \]  

(5.149)
We denote (5.149) by $\Psi(u_1, u_2, t)$. Note that the coefficient of $u_1^k u_2^j$-term in $\Psi(u_1, u_2, t)$ is equal to
\[
\frac{\partial^{i+j} \Psi(u_1, u_2, t)}{\partial u_1^i \partial u_2^j}(0, 0, t), 
\] (5.150)
and (5.150) is equal to 0 for any $i, j$ since $\Psi(u_1, u_2, t)$ is the zero function.

We now prove $g(t) = t$ via the following three subclaims.

**Subclaim 1.** $g(t)$ is of the following form:
\[ g(t) = t + \cdots. \]

**Proof.** Let $g(t)$ be of the following form:
\[ g(t) = g_1 t + g_2 t^2 + \cdots. \]
Since two cusps of $M$ are not SGI each other, there exist $i, j \geq 2$ such that $m_{i,j} \neq 0$. By the chain rule,
\[
\frac{\partial^{i+j-2} \Psi}{\partial u_1^{i-1} \partial u_2^{j-1}}(0, 0, t)
\] is
\[ \frac{i}{2}(i-1)!j!m_{i,j}(u_2 + g(t)) - \frac{j}{2}i!(j-1)!m_{i,j}(u_1 + t) - \frac{i}{2}(i-1)!j!m_{i,j}u_1 + (\text{higher order}), \]
and so
\[
\frac{\partial^{i+j-2} \Psi}{\partial u_1^{i-1} \partial u_2^{j-1}}(0, 0, t)
\] is
\[ \frac{i}{2}(i-1)!j!m_{i,j}g(t) - \frac{j}{2}i!(j-1)!m_{i,j}t + (\text{higher order}). \] (5.151)
The coefficient of $t$-term in (5.151) is
\[ \frac{i}{2}(i-1)!j!m_{i,j}g_1 - \frac{j}{2}i!(j-1)!m_{i,j}, \]
which is equal to 0, and so
\[ i!j!m_{i,j}g_1 = i!j!m_{i,j}, \]
implying $g_1 = 1$. \hfill $\square$

**Subclaim 2.** For $k \geq 2$ and any $i$ satisfying $0 \leq i \leq k$, we have
\[ (i + 1)(i + 1)m_{i+2, k-i-2} = (k-i)(k-i-1)m_{i, k-i}. \]

**Proof.** By the chain rule
\[
\frac{\partial^{k-2} \Psi}{\partial u_1^{i-1} \partial u_2^{k-i-2}}(u_1, u_2, t)
\]

To get the linear terms, it is enough consider the following terms of $v_1$ and $v_2$:
\[
v_1 = \cdots + \frac{i}{2}m_{i+2, k-i-2}u_1^{i+1}u_2^{k-i-2} + \frac{i}{2}m_{i, k-i-1}u_1^{i+1}u_2^{k-i-2} + \cdots, \\
v_2 = \cdots + \frac{k-i}{2}m_{i, k-i}u_1^{i-1}u_2^{k-i} + \frac{k-i-2}{2}m_{i+2, k-i-2}u_1^{i-1}u_2^{k-i-3} + \cdots.
\]
By Subclaim 1 the coefficient of $t$ in (5.152) is
\[
\frac{i + 2}{2}(i + 1)!\!\!(k - i - 2)!m_{i+2,k-i-2} - \frac{k - i}{2}!\!!(k - i - 1)!m_{i,k,i},
\]
and, since this is equal to 0, we get
\[
(i + 2)(i + 1)m_{i+2,k-i-2} = (k - i)(k - i - 1)m_{i,k,i}.
\]
□

Now we expand
\[
\frac{1}{2} \frac{\partial \Phi}{\partial u_1} (u_1 + t, u_2 + g(t)) - \frac{1}{2} \frac{\partial \Phi}{\partial u_2} (u_1 + t, u_2 + g(t))
\]
as follows:
\[
\frac{1}{2} \left( \sum_{i,j,(i \geq 0), even} \!\!\!\! m_{ij}(u_1 + t)^i(u_2 + g(t))^j \right) - \frac{1}{2} \left( \sum_{i,j,(i \geq 0), even} \!\!\!\! j m_{ij}(u_1 + t)^i(u_2 + g(t))^{j-1} \right)
\]
\[
= \frac{1}{2} \sum_{k=2, even}^{\infty} \left( \sum_{i=0, even}^{k-2} (i + 2)m_{i+2,k-i-2}(u_1 + t)^i(u_2 + g(t))^{k-i-2} - (k - i)m_{i,k-i}(u_1 + t)^i(u_2 + g(t))^{k-i-1} \right).
\]  
(5.153)

By Subclaim 2 we have
\[
\frac{(k - i)m_{i,k-i}}{(i + 2)m_{i+2,k-i-2}} = \frac{(k - i)(i + 1)}{(i + 2)(k - i)(k - i - 1)} = \frac{i + 1}{k - i - 1} = \frac{\binom{k-1}{i}}{\binom{k-1}{i+1}},
\]  
(5.154)

and, combining with this, rewrite (5.153) as follows:
\[
\frac{1}{2} \sum_{k=2, even}^{\infty} \left( \sum_{i=0, even}^{k-2} (i + 2)m_{i+2,k-i-2}(u_1 + t)^i(u_2 + g(t))^{k-i-2} - (k - i)m_{i,k-i}(u_1 + t)^i(u_2 + g(t))^{k-i-1} \right)
\]
\[
= \frac{1}{2} \sum_{k=2, even}^{\infty} \!\!\!\! km_{k,0} \left( \binom{k-1}{1}(u_1 + t)^{k-1} - \binom{k-1}{0}(u_1 + t)^{k-2}(u_2 + g(t)) + \cdots - \binom{k-1}{k-1}(u_2 + g(t))^{k-1} \right)
\]
\[
= \sum_{k=2, even}^{\infty} \!\!\!\! \frac{km_{k,0}}{2} ((u_1 + t) - (u_2 + g(t)))^{k-1}.
\]

Subclaim 3. $g(t) = t$.  
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Proof. By the above argument,

\[
\Psi(u_1, u_2, t) = \frac{1}{2} \frac{\partial \Phi}{\partial u_1}(u_1 + t, u_2 + g(t)) - \frac{1}{2} \frac{\partial \Phi}{\partial u_2}(u_1 + t, u_2 + g(t)) - \frac{1}{2} \frac{\partial \Phi}{\partial u_1}(u_1, u_2) + \frac{1}{2} \frac{\partial \Phi}{\partial u_2}(u_1, u_2) - h(t)
\]

\[
= \sum_{k=2, even}^{\infty} \frac{km_{k,0}}{2} \left( (u_1 + t) - (u_2 + g(t)) \right)^{k-1} - \sum_{k=2, even}^{\infty} \frac{km_{k,0}}{2} (u_1 - u_2)^{k-1} - h(t).
\]

Suppose \(g(t) \neq t\) and \(g(t)\) is of the following form:

\[g(t) = t + g_1 t^i + \cdots\]

where \(t^i\) is the second smallest degree term of \(g(t)\) having nonzero coefficient \(g_i\). If \(u_2 = 0\), then

\[
\Psi(u_1, 0, t) = \tau_1 t + \tau_2 g(t) + \sum_{k=4, even}^{\infty} \frac{km_{k,0}}{2} (u_1 - g_1 t^i + \cdots)^{k-1} - \sum_{k=4, even}^{\infty} \frac{km_{k,0}}{2} u_1^{k-1} - h(t).
\]

Let \(l\) is the smallest number such that \(l \geq 4\) and \(m_{l,0} \neq 0\). Then the coefficient of \(u_1\)-term in \(\Psi(u_1, 0, t)\) is

\[
\frac{lm_{l,0}}{2} \left( \frac{l-1}{l-2} \right) (-g_1 t^i + \cdots)^{l-2} + \text{(higher order)}
\]

(5.155)

and the coefficient of \(t^i\)-term in (5.155) is

\[
\frac{lm_{l,0}}{2} \left( \frac{l-1}{l-2} \right) (-g_1)^{l-2}.
\]

(5.156)

Since (5.155) is 0 for any \(t\), (5.156) is 0 and so \(g_1 = 0\). This contradicts our assumption \(g_i \neq 0\). So \(g(t) = t\).

\[
\square
\]

\[
\square
\]

5.2. Proof of Theorem 1.7

This section is devoted to prove Theorem 1.7. We divide the problem into two cases and first prove it for manifolds whose two cusps are SGI each other. If two cusps of a given hyperbolic 3-manifold are SGI each other, it is well-known that its holonomy variety is represented as the product of two curves, where each curve corresponds to a holonomy variety of a 1-cusped manifold. So, in this case, we prove the theorem by using the result of the 1-cusped case.

**Theorem 5.14.** Let \(\mathcal{M}\) be a 2-cusped hyperbolic 3-manifold having non-quadratic, rationally independent cusp shapes and whose two cusps are SGI. Let

\[
\left\{ t_{(p_1/q_1, p_2/q_2)}^1, t_{(p_1/q_1, p_2/q_2)}^2 \right\}
\]

be the set of holonomies of its \((p_1/q_1, p_2/q_2)\)-Dehn filling. Then

\[
\left\{ t_{(p_1/q_1, p_2/q_2)}^1, t_{(p_1/q_1, p_2/q_2)}^2 \right\} = \left\{ t_{(p_1'/q_1', p_2'/q_2')}^1, t_{(p_1'/q_1', p_2'/q_2')}^2 \right\}
\]

if and only if

\[(p_1/q_1, p_2/q_2) = (p_1'/q_1', p_2'/q_2')\]

for sufficiently large \(|p_i| + |q_i|\) and \(|p_i'| + |q_i'|\) \((1 \leq i \leq 2)\).
Proof. First, if two cusps of $\mathcal{M}$ are SGI each other, then a holonomy variety $\mathcal{X}$ of $\mathcal{M}$ is of the form (see Theorem 4.2 in [9]):

$$\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2$$

where each $\mathcal{X}_i$ (resp. $\mathcal{X}_2$) represents the variety obtained by having 2nd (resp. 1st) cusp complete.\(^{11}\)

To simplify the notation, we denote $(t_{(p_1/q_1,p_2/q_2)}, t_{(p_1/q_1,p_2/q_2)})$ and $(t_{(p'_1/q'_1,p'_2/q'_2)}, t_{(p'_1/q'_1,p'_2/q'_2)})$ by $(t_1, t_2)$ and $(t'_1, t'_2)$ respectively. By Theorem \([1.2]\) if

$$(p_1/q_1,p_2/q_2) \neq (p'_1/q'_1,p'_2/q'_2)$$

then

$$t_1 \neq t'_1, \quad t_2 \neq t'_2$$

for sufficiently large $|p_i| + |q_i|$ and $|p'_i| + |q'_i|$ ($i = 1, 2$). Now suppose

$$t_1 = t'_2, \quad t'_1 = t_2,$$

Since two cusps of $\mathcal{M}$ are SGI each other, in this case, $t_1$ and $t'_2$ (resp. $t_2$ and $t'_1$) are two holonomies of $(p_1/q_1,p'_2/q'_2)$-Dehn filling of $\mathcal{M}$ (resp. $(p'_1/q'_1,p_2/q_2)$-Dehn filling). But this contradicts Theorem \([5.2]\). Thus

$$t_1 \neq t'_2, \quad t_2 \neq t'_1,$$

and we conclude

$$\{t_1, t_2\} = \{t'_1, t'_2\}$$

if and only if

$$(p_1/q_1,p_2/q_2) = (p'_1/q'_1,p'_2/q'_2)$$

for sufficiently large $|p_i| + |q_i|$ and $|p'_i| + |q'_i|$ ($i = 1, 2$). \(\square\)

Now we prove the theorem for manifolds whose two cusps are not SGI. The general strategy of the proof is very similar to the one given in the proof of Theorem \([1.2]\). But as it was shown in Lemma \([5.7]\), since $\mathcal{X} \times \mathcal{X}$ has several different types of infinitely many anomalous subvarieties, it requires more work than the proof of Theorem \([1.2]\).

**Theorem 5.15.** Let $\mathcal{M}$ be a 2-cusped hyperbolic 3-manifold whose two cusps are not SGI. Let $\tau_1, \tau_2$ be its two cusp shapes and

$$\left\{ t_{(p_1/q_1,p_2/q_2)}, t_{(p_1/q_1,p_2/q_2)} \right\}$$

be the set of holonomies of its $(p_1/q_1,p_2/q_2)$-Dehn filling. Suppose the elements in

$$\{\tau_i^j \mid 0 \leq i, j \leq 2\}$$

are linearly independent over $\mathbb{Q}$. Then

$$\left\{ t_{(p_1/q_1,p_2/q_2)}, t_{(p_1/q_1,p_2/q_2)} \right\} = \left\{ t_{(p'_1/q'_1,p'_2/q'_2)}, t_{(p'_1/q'_1,p'_2/q'_2)} \right\},$$

(5.157)

if and only if

$$(p_1/q_1,p_2/q_2) = (p'_1/q'_1,p'_2/q'_2)$$

(5.158)

for sufficiently large $|p_i| + |q_i|$ and $|p'_i| + |q'_i|$ ($i = 1, 2$).

\(^{11}\)So each $\mathcal{X}_i$ can be considered as the holonomy variety of a 1-cusped manifold.
Proof. To simplify the notation, we denote \((t_1^{p_1/q_1, p_2/q_2}, t_2^{p_1/q_1, p_2/q_2})\) by \((t_1, t_2)\) and \((t_1', t_2')\) respectively. The condition \(5.157\) implies either
\[ t_1 = t_1', \quad t_2 = t_2' \]  
(5.159)
or
\[ t_1 = t_2', \quad t_2 = t_1' \]  
(5.160)
We show that, in both cases, either it satisfies \(5.158\) or the degree of \((t_1, t_2)\) is uniformly bounded. Since the whole proof is fairly long, we divide it into two parts. We consider \(5.159\) in Part I first and \(5.160\) in Part II later.

**Part I** Note that, by Lemma 5.2, \(t_1\) and \(t_2\) are multiplicatively independent. Let
\[ P_1 = (M_1, L_1, M_1', L_1') = (t_1^{-q_1}, t_1^{-q_1'}, t_1^{p_1'}, t_1^{p_1'}), \]
\[ P_2 = (M_2, L_2, M_2', L_2') = (t_2^{-q_2}, t_2^{-q_2'}, t_2^{p_2'}, t_2^{p_2'}). \]
Using Siegel’s lemma, we get two algebraic subgroups \(H_i\) \((i = 1, 2)\) of the forms
\[ M_i^{a_i}L_i^{b_i}(M_i')^{c_i}(L_i')^{d_i} = 1, \]
\[ M_i^{a_2}L_i^{b_2}(M_i')^{c_2}(L_i')^{d_2} = 1, \]  
(5.161)
satisfying \(P_i \in H_i\) and
\[ |b_{1i}| |b_{2i}| \leq |v_i|^{2/3} \]
where
\[ b_{ij} = (a_{ij}, b_{ij}, c_{ij}, d_{ij}), \]
\[ v_i = (-q_i, p_i, -q_i', p_i') \]
\((1 \leq i, j \leq 2)\). Let
\[ H = H_1 \cap H_2, \]
and
\[ P = (M_1, L_1, M_2, L_2, M_1', L_1', M_2', L_2') = (t_1^{-q_1}, t_1^{-q_1'}, t_2^{-q_2}, t_2^{-q_2'}, t_1^{p_1'}, t_1^{p_1'}, t_2^{p_2'}, t_2^{p_2'}) \]
be the cosmetic surgery point associated with \(M(p_1/q_1, p_2/q_2)\) and \(M(p_1'/q_1', p_2'/q_2')\).

First if \(P\) is an isolated point of \((\mathcal{X} \times \mathcal{X}) \cap H\), then, similar to Claim 4.7, we get that the degree of \(P\) is uniformly bounded:

**Claim 5.16.** If \(P\) is an isolated point of \((\mathcal{X} \times \mathcal{X}) \cap H\), then the degree of \(P\) is bounded by \(D\) depending only on \(\mathcal{X}\).

**Proof.** By Theorem 3.11, there exists an universal constant \(B\) such that \(h(P) \leq B\) and, by the properties of height, we can find \(c_1\) such that
\[ |v_1| h(t_1) \leq c_1 B, \]
\[ |v_2| h(t_2) \leq c_1 B. \]  
(5.162)
(5.163)
By standard degree theory, the degree of \(H\) is bounded by \(c_2 \prod_{i=1}^{2} |b_{1i}| |b_{2i}|\), and so by \(c_2(|v_1| |v_2|)^{2/3}\) for some constant \(c_2\). By Bézout’s theorem, the degree \(D\) of the cosmetic surgery point \(P\) is bounded by the product of the degrees of \(\mathcal{X} \times \mathcal{X}\) and \(H\): that is
\[ D \leq c_3 (|v_1| |v_2|)^{2/3} \]  
(5.164)
where $c_3$ is a constant depending on $\mathcal{X}$. By Lemma 3.14
\[ h(t_1)h(t_2) \geq \frac{1}{c_4D(\log 3D)^\kappa} \]
for some $\kappa$ and $c_4$. On the other hand, by (5.162), we deduce $|v_1||v_2| \leq c_5D(\log 3D)^\kappa B^2$
for some constant $c_5$, and thus, combining with (5.164), we get $D \leq c_6(D(\log 3D)^\kappa B^2)^{2/3}$
where $c_6$ is a constant depending only on $\mathcal{X}$. This completes the proof.

Next we consider the case that the component of $(\mathcal{X} \times \mathcal{X}) \cap H$ containing $P$ is an anomalous
subvariety of $\mathcal{X} \times \mathcal{X}$. Denote this component by $\mathcal{Y}$. If $\mathcal{Y}$ contains $(z^0, z^0)$, by Lemma 5.3
we get either
\[ (p_1/q_1, p_2/q_2) = (p'_1/q'_1, p'_2/q'_2) \]
or $\mathcal{Y}$ is contained in
\[ M_1 = (M'_1)^{\pm 1}, \quad L_1 = (L'_1)^{\pm 1}, \]
\[ M_2 = M'_2 = L_2 = L'_2 = 1, \]
or
\[ M_1 = M'_1 = 1 = L_1 = L'_1 = 1, \]
\[ M_2 = (M'_2)^{\pm 1}, \quad L_2 = (L'_2)^{\pm 1}. \]
For the first case, we get the desired result, and the last two cases contradict the fact
that $t_1, t_2 \neq 1$. If $\mathcal{Y}$ does not contain $(z^0, z^0)$ and $\mathcal{X} \times \mathcal{X}$ contains only a finite
number of anomalous subvarieties near $(z^0, z^0)$, then, by shrinking the size of a neighborhood of
$(z^0, z^0)$, we exclude the cosmetic surgery points contained in those anomalous subvarieties.

Now assume $\mathcal{X} \times \mathcal{X}$ contains infinitely many anomalous subvarieties near $(z^0, z^0)$, and,
similar to the proof of Theorem 4.2, consider the following situation. Suppose $(p_{1i}/q_{1i}, p_{2i}/q_{2i})_{i \in I}$
and $(p'_{1i}/q'_{1i}, p'_{2i}/q'_{2i})_{i \in I}$ are two infinite sequences such that, for each $i \in I$,
\[ (p_{1i}/q_{1i}, p_{2i}/q_{2i}) \neq (p'_{1i}/q'_{1i}, p'_{2i}/q'_{2i}), \]  (5.165)
and
\[ t_{1i} = t'_{1i}, \quad t_{2i} = t'_{2i} \]
where $\{t_{1i}, t_{2i}\}$ and $\{t'_{1i}, t'_{2i}\}$ are the sets of holonomies of $\mathcal{M}(p_{1i}/q_{1i}, p_{2i}/q_{2i})$ and $\mathcal{M}(p'_{1i}/q'_{1i}, p'_{2i}/q'_{2i})$
respectively. Let
\[ P_i = \left( t_{1i}^{-q_{1i}}, t_{1i}^{p_{1i}}, t_{2i}^{-q_{2i}}, t_{2i}^{p_{2i}}, (t'_{1i})^{-q'_{1i}}, (t'_{1i})^{p'_{1i}}, (t'_{2i})^{-q'_{2i}}, (t'_{2i})^{p'_{2i}} \right) \]  (5.166)
be the cosmetic surgery point associated with $\mathcal{M}(p_{1i}/q_{1i}, p_{2i}/q_{2i})$ and $\mathcal{M}(p'_{1i}/q'_{1i}, p'_{2i}/q'_{2i})$,
and $H_i$ be an algebraic subgroup containing $P_i$, obtained by the same procedure shown
earlier (i.e. using Siegel’s lemma). So $H_i$ is defined by equations of the following forms:
\[ M_1^{a_{1i}}L_1^{b_{1i}}(M'_1)^{\alpha_{1i}}(L'_1)^{\beta_{1i}} = 1, \]
\[ M_1^{a_{2i}}L_1^{b_{2i}}(M'_1)^{\alpha_{2i}}(L'_1)^{\beta_{2i}} = 1, \]
\[ M_2^{c_{1i}}L_2^{d_{1i}}(M'_2)^{\gamma_{1i}}(L'_2)^{\delta_{1i}} = 1, \]
\[ M_2^{c_{2i}}L_2^{d_{2i}}(M'_2)^{\gamma_{2i}}(L'_2)^{\delta_{2i}} = 1. \]  (5.167)
We further assume that, for each $i \in I$, the component of $(\mathcal{X} \times \mathcal{X}) \cap H_i$ containing $P_i$, denoted
by $\mathcal{Y}_i$, is an anomalous subvariety of $\mathcal{X} \times \mathcal{X}$ near $(z^0, z^0)$, and $\{\mathcal{Y}_i\}_{i \in I}$ are all different. That
is, for a given small neighborhood of $(z^0, z^0)$, we have a family of infinitely anomalous

subvarieties \( \{Y_i\}_{i \in I} \) all intersecting the neighborhood. Now we consider the following two cases.

(1) All \( \{Y_i\}_{i \in I} \) are contained in a finite list \( \Theta \) of maximal anomalous subvarieties of \( X \times X \) containing \( (z^0, z^0) \). Let \( Z \in \Theta \) and, without loss of generality, suppose \( Z \) contains all \( Y_i \). If \( \dim Z = 3 \), then, by Lemma 5.5, \( Z \) is contained in either

\[
M_1 = L_1 = 1 \quad \text{or} \quad M_2 = L_2 = 1 \quad \text{or} \quad M_1' = L_1' = 1 \quad \text{or} \quad M_2' = L_2' = 1.
\]

But this contradicts the fact that \( t_{1i}, t_{2i} \neq 1 \) for \( i \in I \). Thus \( \dim Z = 2 \). Let \( H \) be an algebraic subgroup defined by

\[
M_1^{a_1} L_1^{b_1} M_2^{a_2} L_2^{b_2} (M_1')^{d_1} (L_1')^{e_1} (M_2')^{d_2} (L_2')^{e_2} = 1,
\]

\[
M_1^{a_2} L_1^{b_2} M_2^{a_3} L_2^{b_3} (M_1')^{d_2} (L_1')^{e_2} (M_2')^{d_3} (L_2')^{e_3} = 1,
\]

such that \( Z \) is the component of \( H \cap (X \times X) \) containing \( (z^0, z^0) \). Since \( t_{1i} \) and \( t_{2i} \) are multiplicatively independent for each \( i \), all \( P_i \) are contained in

\[
M_1^{a_1} L_1^{b_1} (M_1')^{d_1} (L_1')^{e_1} = 1,
\]

\[
M_1^{a_2} L_1^{b_2} (M_1')^{d_2} (L_1')^{e_2} = 1,
\]

\[
M_1^{a_3} L_1^{b_3} (M_1')^{d_3} (L_1')^{e_3} = 1,
\]

\[
M_2^{a_1} L_2^{b_1} (M_2')^{d_1} (L_2')^{e_1} = 1,
\]

\[
M_2^{a_2} L_2^{b_2} (M_2')^{d_2} (L_2')^{e_2} = 1,
\]

\[
M_2^{a_3} L_2^{b_3} (M_2')^{d_3} (L_2')^{e_3} = 1.
\]

**Claim 5.17.** The ranks of

\[
\{(a_j, b_j, d_j', b_j') \mid j = 1, 2, 3\}
\]

and

\[
\{(c_j, d_j, c_j', d_j') \mid j = 1, 2, 3\}
\]

are at most 2.

**Proof.** Suppose that the rank of either \( 5.169 \) or \( 5.170 \) is 3, then, by applying Gauss elimination, we assume \( P_i \) are contained in either

\[
M_1^{a_1} L_1^{b_1} (M_1')^{d_1} (L_1')^{e_1} = 1,
\]

\[
L_1^{b_2} (M_1')^{d_2} (L_1')^{e_2} = 1,
\]

\[
(M_1')^{d_3} (L_1')^{e_3} = 1
\]

or

\[
M_2^{a_1} L_2^{b_1} (M_2')^{d_1} (L_2')^{e_1} = 1,
\]

\[
L_2^{b_2} (M_2')^{d_2} (L_2')^{e_2} = 1,
\]

\[
(M_2')^{d_3} (L_2')^{e_3} = 1.
\]

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But this implies \((p_{1i}', q_{1i}') = (a_i', b_i')\) or \((p_{2i}', q_{2i}') = (c_i', d_i')\) for all \(i \in I\), which contradicts the fact \((p_{1i}', q_{1i}')\) and \((p_{2i}', q_{2i}')\) (and so \(t_{1i}'\) and \(t_{2i}'\) as well) are not stationary.\(^{12}\) Thus the ranks of \((5.169)\) and \((5.170)\) are at most 2.

(a) The ranks of \((5.169)\) and \((5.170)\) are all equal to 2.

By applying Gauss elimination if necessary, we assume \(H\) defined by

\[
\begin{align*}
M_1^{a_1} L_1^{b_1} M_2^{a_2} L_2^{d_1} (M'_1)^{\epsilon_1} (L'_1)^{\delta_1} (M'_2)^{\epsilon_2} (L'_2)^{\delta_2} &= 1, \\
M_1^{a_2} L_1^{b_2} (M'_1)^{\epsilon_2} (L'_1)^{\delta_1} (L'_2)^{\delta_2} &= 1, \\
M_2^{a_3} L_2^{d_3} (M'_2)^{\epsilon_3} (L'_2)^{\delta_2} &= 1,
\end{align*}
\]

\[(5.171)\]

and \(Z\) is the component of \((X \times X) \cap H\) containing \((z^0, z^0)\). By moving to \(\text{Def}(M) \times \text{Def}(M)\), \(Z\) is locally biholomorphic (near \((z^0, z^0)\)) to the complex manifold defined by

\[
a_1 u_1 + b_1 (\tau_1 u_1 + \cdots) + c_1 u_2 + d_1 (\tau_2 u_2 + \cdots) + a_1' u_1' + b_1' (\tau_1 u_1' + \cdots) + c_1' u_2' + d_1' (\tau_2 u_2' + \cdots) = 0, \\
a_2 u_1 + b_2 (\tau_1 u_1 + \cdots) + a_2' u_1' + b_2' (\tau_1 u_1' + \cdots) = 0, \\
c_3 u_2 + d_3 (\tau_2 u_2 + \cdots) + c_3' u_2' + d_3' (\tau_2 u_2' + \cdots) = 0.
\]

(5.172)

The Jacobian of \((5.172)\) at \((u_1, u_1', u_2, u_2') = (0, 0, 0, 0)\) is

\[
\begin{pmatrix}
a_1 + \tau_1 b_1 & a_1' + \tau_1 b_1' & c_1 + \tau_2 d_1 & c_1 + \tau_2 d_1 \\
a_2 + \tau_1 b_2 & a_2' + \tau_1 b_2' & 0 & 0 \\
0 & 0 & c_3 + \tau_2 d_3 & c_3' + \tau_2 d_3
\end{pmatrix}
\]

(5.173)

Since the rank of \((5.173)\) is 2, we get the ranks of the following two matrices,

\[
\begin{pmatrix}
a_1 + \tau_1 b_1 & a_1' + \tau_1 b_1' \\
a_2 + \tau_1 b_2 & a_2' + \tau_1 b_2'
\end{pmatrix}
\]

(5.174)

and

\[
\begin{pmatrix}
c_1 + \tau_2 d_1 & c_1 + \tau_2 d_1 \\
c_3 + \tau_2 d_3 & c_3' + \tau_2 d_3'
\end{pmatrix}
\]

(5.175)

are all equal to 1. By Lemma \[4.2\],

\[
\begin{pmatrix}
a_1 & b_1 & a_1' & b_1' \\
a_2 & b_2 & a_2' & b_2'
\end{pmatrix}
\]

is of the form either

\[
\begin{pmatrix}
a_1 & b_1 & -a_1 & -b_1 \\
a_2 & b_2 & -a_2 & -b_2
\end{pmatrix}
\]

or

\[
\begin{pmatrix}
a_1 & b_1 & a_1 & b_1 \\
a_2 & b_2 & a_2 & b_2
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
c_1 & d_1 & c_1' & d_1' \\
c_3 & d_3 & c_3' & d_3'
\end{pmatrix}
\]

is of the form either

\[
\begin{pmatrix}
c_1 & d_1 & -c_1 & -d_1 \\
c_3 & d_3 & -c_3 & -d_3
\end{pmatrix}
\]

or

\[
\begin{pmatrix}
c_1 & d_1 & c_1 & d_1 \\
c_3 & d_3 & c_3 & d_3
\end{pmatrix}
\]

\[12\] If \((p_{1i}', q_{1i}') = (a_i', b_i')\) or \((p_{2i}', q_{2i}') = (c_i', d_i')\) for \(i \in I\), then, by Thurston’s Dehn filling theory (see Theorem \[2.4\], \(P_i\) do not converge to \((z^0, z^0)\). But this contradicts to our initial assumption.
Without loss of generality, we assume the first cases. Thus \(5.168\) is of the following form:

\[
M_1^{a_1}L_1^{b_1}M_2^{a_2}L_2^{b_2}(M'_1)^{-a_1}(L'_1)^{-b_1}(M'_2)^{-c_1}(L'_2)^{-d_1} = 1,
\]
\[
M_1^{a_2}L_2^{b_2}(M'_1)^{-a_2}(L'_1)^{-b_2} = 1,
\]
\[
M_2^{c_2}L_2^{d_2}(M'_2)^{-c_2}(L'_2)^{-d_2} = 1.
\]

Again since \(t_{11}\) and \(t_{21}\) are multiplicatively independent each other, \(P_i\) are contained in

\[
M_1^{a_1}L_1^{b_1}(M'_1)^{-a_1}(L'_1)^{-b_1} = 1,
\]
\[
M_1^{a_2}L_2^{b_2}(M'_1)^{-a_2}(L'_1)^{-b_2} = 1,
\]
\[
M_2^{c_1}L_2^{d_1}(M'_2)^{-c_1}(L'_2)^{-d_1} = 1,
\]
\[
M_2^{c_2}L_2^{d_2}(M'_2)^{-c_2}(L'_2)^{-d_2} = 1.
\]

Since \(a_1b_2 - a_2b_1 \neq 0\) and \(c_1d_3 - c_3d_1 \neq 0\), we conclude \(P_i\) are contained in

\[
M_1 = M'_1, \quad M_2 = M'_2, \quad L_1 = L'_1, \quad L_2 = L'_2,
\]

implying

\[
(p_{1i}/q_{1i}, p_{2i}/q_{2i}) = (p'_{1i}/q'_{1i}, p'_{2i}/q'_{2i}).
\]

But this contradicts our initial assumption \(5.165\).

(b) Either the rank of \(5.169\) or \(5.170\) is 1.

Without loss of generality, assume that the rank of \(5.169\) is 2 and the rank of \(5.170\) is 1. Applying Gauss elimination if necessary, let \(H\) be defined by

\[
M_1^{a_1}L_1^{b_1}(M'_1)^{a_1}(L'_1)^{b_1} = 1,
\]
\[
M_1^{a_2}L_2^{b_2}(M'_1)^{a_2}(L'_1)^{b_2} = 1,
\]
\[
M_2^{c_1}L_2^{d_1}(M'_2)^{c_1}(L'_2)^{d_1} = 1.
\]

Since \(Z\) is the component of \((X \times X) \cap H\) containing \((z^0, z^0)\), by moving to Def \((M) \times\) Def \((M)\), it is locally biholomorphic (near \((z^0, z^0)\)) to

\[
a_1u_1 + b_1(\tau \tau u_1 + \cdots) + a'_1u'_1 + b'_1(\tau \tau u'_1 + \cdots) = 0,
\]
\[
a_2u_1 + b_2(\tau \tau u_1 + \cdots) + a'_2u'_1 + b'_2(\tau \tau u'_1 + \cdots) = 0,
\]
\[
c_1u_2 + d_1(\tau \tau u_2 + \cdots) + c'_1u'_2 + d'_1(\tau \tau u'_2 + \cdots) = 0,
\]

and the Jacobian of \(5.180\) at \((u_1, u'_1, u_2, u'_2) = (0, 0, 0, 0)\) is

\[
\begin{pmatrix}
  a_1 + \tau_1b_1 & a'_1 + \tau_1b'_1 & 0 & 0 \\
  a_2 + \tau_1b_2 & a'_2 + \tau_1b'_2 & 0 & 0 \\
  0 & 0 & c_1 + \tau_2d_1 & c'_1 + \tau_2d'_1 \\
\end{pmatrix}.
\]

Since the rank of \(5.181\) is 2, the rank of

\[
\begin{pmatrix}
  a_1 + \tau_1b_1 & a'_1 + \tau_1b'_1 \\
  a_2 + \tau_1b_2 & a'_2 + \tau_1b'_2 \\
\end{pmatrix}
\]

is 1. Thus, by Lemma \(4.2\) we have either

\[
\begin{pmatrix}
  a_1 & b_1 & a'_1 & b'_1 \\
  a_2 & b_2 & a'_2 & b'_2 \\
\end{pmatrix}_{55} = \begin{pmatrix}
  a_1 & b_1 & -a_1 & -b_1 \\
  a_2 & b_2 & -a_2 & -b_2 \\
\end{pmatrix}.
\]
or
\[
\begin{pmatrix}
  a_1 & b_1 & a_1' & b_1' \\
  a_2 & b_2 & a_2' & b_2'
\end{pmatrix} = \begin{pmatrix}
  a_1 & b_1 & a_1 & b_1 \\
  a_2 & b_2 & a_2 & b_2
\end{pmatrix}.
\] (5.184)

Without loss of generality, we assume the first case. So \( H \) is of the following form
\[
\begin{align*}
M_1^{a_1} L_1^{b_1} (M_1')^{-a_1} (L_1')^{-b_1} &= 1, \\
M_1^{a_2} L_1^{b_2} (M_1')^{-a_2} (L_1')^{-b_2} &= 1, \\
M_2^{c_1} L_2^{d_1} (M_2')^{-c_1} (L_2')^{-d_1} &= 1.
\end{align*}
\] (5.185)

Since \( a_1 b_2 - a_2 b_1 \neq 0 \), we further simplify \( H \) as
\[
\begin{align*}
M_1 &= M_1', \\
L_1 &= L_1', \\
M_2^{c_1} L_2^{d_1} (M_2')^{-c_1} (L_2')^{-d_1} &= 1.
\end{align*}
\] (5.186)

Now consider \( H(\equiv (\mathbb{C}^\ast)^{2k+5}) \) as an ambient space and \( Z \) as a 2-dimensional algebraic variety in \( (\mathbb{C}^\ast)^{2k+5} \). More precisely, let
\[
a_1 = (c_1, d_1, c_1', d_1'),
\] (5.187)

and find
\[
a_1 = (c_i, d_i, c_i', d_i') \in \mathbb{Z}^4 \quad (i = 2, 3, 4)
\]
such that \( \{a_1, a_2, a_3, a_4\} \) is a primitive basis of \( \mathbb{Z}^4 \). If
\[
\begin{align*}
\tilde{L}_2 &= M_2^{c_1} L_2^{d_1} (M_2')^{c_1} (L_2')^{d_1}, \\
\tilde{M}_2 &= M_2^{c_2} L_2^{d_2} (M_2')^{c_2} (L_2')^{d_2}, \\
\tilde{L}_2' &= M_2^{c_4} L_2^{d_4} (M_2')^{c_4} (L_2')^{d_4}, \\
\tilde{M}_2' &= M_2^{c_5} L_2^{d_5} (M_2')^{c_5} (L_2')^{d_5},
\end{align*}
\] (5.188)

then \( H \) is equivalent to
\[
\begin{align*}
M_1 &= M_1', \\
L_1 &= L_1', \\
\tilde{L}_2 &= 1
\end{align*}
\] (5.189)

under (5.188), and
\[
\text{Proj} : (z_1, \ldots, z_k, M_1, \ldots, L_2, z_1', \ldots, z_k', M_1', \ldots, L_2') \rightarrow (z_1, \ldots, z_k, M_1, L_1, z_1', \ldots, z_k', \tilde{M}_2, \tilde{L}_2, \tilde{M}_2')
\] (5.190)

is a projection map onto \( H \). Recall \( P_i \) is an intersection point between \( X \times X \) and an algebraic subgroup defined by
\[
\begin{align*}
M_1^{p_{1i}} L_1^{q_{1i}} &= 1, \quad (M_1')^{p_{1i}} (L_1')^{q_{1i}} = 1, \\
M_1^{r_{1i}} L_1^{s_{1i}} &= (M_1')^{r_{1i}} (L_1')^{s_{1i}}, \\
M_2^{p_{2i}} L_2^{q_{2i}} &= 1, \quad (M_2')^{p_{2i}} (L_2')^{q_{2i}} = 1, \\
M_2^{r_{2i}} L_2^{s_{2i}} &= (M_2')^{r_{2i}} (L_2')^{s_{2i}}.
\end{align*}
\] (5.191)

\[^{13}\text{So } p_{1i}/q_{1i} = p_{1i}'/q_{1i}', \text{ for each } i \in \mathcal{I}.\]
Then the equations in (5.191) project onto the following forms of equations under (5.190):

\[
M_1^{p_{1i}} L_1^{q_{1i}} = 1, \\
\tilde{M}_2^{e_{2i}} L_2^{f_{2i}} \tilde{M}_2^{g_{2i}} = 1, \\
M_2^{e_{2i}} L_2^{f_{2i}} \tilde{M}_2^{g_{2i}} = 1
\]  
(5.192)

for some \((e_{ji}, f_{ji}, g_{ji}) \in \mathbb{Z}^3 (j = 1, 2)\). As the image of \(Z\) under (5.190) (denote it by \(\text{Proj } Z\)) is still a 2-dimensional algebraic variety in \(H\) and it intersects with (5.192), an algebraic subgroup of codimension 3, nontrivially at \(\text{Proj } P_i\), we again fall into a problem of unlikely intersections and so repeat the methods used earlier. First, using Siegel’s lemma, we construct a 3-dimensional algebraic subgroup \(K_i\) containing (5.192) (as well as \(\text{Proj } P_i\)). Since \(t_{1i}\) and \(t_{2i}\) are multiplicatively independent each other, \(K_i\) is defined by either one of the following forms:

\[
M_1^{p_{1i}} L_1^{q_{1i}} = 1, \\
\tilde{M}_2^{m_{1i}} L_2^{n_{1i}} \tilde{M}_2^{l_{1i}} = 1, \\
\tilde{M}_2^{m_{2i}} L_2^{n_{2i}} \tilde{M}_2^{l_{2i}} = 1.
\]  
(5.193)

or

\[
\tilde{M}_2^{m_{1i}} L_2^{n_{1i}} \tilde{M}_2^{l_{1i}} = 1, \\
M_2^{m_{2i}} L_2^{n_{2i}} \tilde{M}_2^{l_{2i}} = 1.
\]  
(5.194)

(i) \(\text{Proj } P_i\) is an isolated point of \(K_i \cap \text{Proj } Z\).

By a similar argument given in the proof of Claim 5.16, it can be shown the degree of \(\text{Proj } P_i\) is uniformly bounded. Since the height of \(\text{Proj } P_i\) is also uniformly bounded as well, we obtain the desired result by Northcott’s theorem.

(ii) The component of \(K_i \cap \text{Proj } Z\) containing \(\text{Proj } P_i\) is a 1-dimensional anomalous subvariety of \(\text{Proj } Z\) (denote this component by \(W_i\)). In this case, the problem is further divided into the following two subcases:

(A) \(\text{Proj } Z\) has only finitely many maximal anomalous subvarieties containing \((1, 1, 1, 1, 1)\) and \(W_i\) is one of them.

Claim 5.18. If \(W_i\) is a 1-dimensional anomalous subvariety containing \((1, 1, 1, 1, 1)\), then \(W_i\) is contained in

\[
\tilde{M}_2 = L_2 = M_2' = 1.
\]  
(5.195)

Proof. Since \(W_i\) is a component of \(K_i \cap \text{Proj } Z\), by putting it back to the original ambient space, we consider \(W_i\) as a 1-dimensional anomalous subvariety of \(\mathcal{X} \times \mathcal{X}\) (or \(Z\)) contained in

\[(\mathcal{X} \times \mathcal{X}) \cap (H \cap \text{Proj}^{-1}(K_i)) \quad \text{(or } Z \cap \text{Proj}^{-1}(K_i)).\]  
(5.196)

But, in this case, as already discussed earlier (see Lemma 5.3 and the discussion after the proof of Claim 5.16), \(W_i\) is contained in

\[
M_2 = L_2 = M_2' = L_2' = 1.
\]  
(5.197)

\footnote{Note that each algebraic subgroup defined by \((5.191)\) is contained in \(H\). (Otherwise, it contradicts the fact that none of the coordinates of \(P_i\) is cyclotomic.)}
The image of (5.197) under (5.190) is clearly (5.195), completing the proof. □

However if \( W_i \) is contained in (5.195), it contradicts the fact that none of the coordinates in \( P_i \) is cyclotomic.

(B) Proj \( Z \) has **infinitely** many different anomalous subvarieties and \( \{ W_i \mid i \in I \} \) is a subset of them.

- We first suppose \( K_i \) is of the form given in (5.193). By the same idea given in the proof of Claim 4.8, there exists \( K^{(0)} \) such that
  - \( K^{(0)} \cap \text{Proj} \ Z \) contains a 1-dimensional anomalous subvariety of Proj \( Z \);
  - for each \( i \), \( W_i \subset g_i K^{(0)} \cap \text{Proj} \ Z \) for some \( g_i \);
  - for each \( i \), \( K^{(0)} \subset K_i \).

Since \( (p_i, q_i) \) are different for each \( i \) (in (5.193)), \( K^{(0)} \) is contained in

\[
M_1 = 1, \quad L_1 = 1
\]

and thus (for each \( i \)) there exist \( \xi_{1i}, \xi_{2i} \) such that

\[
M_1 = \xi_{1i}, \quad L_1 = \xi_{2i}
\]

contains an anomalous subvariety of Proj \( Z \). But this is impossible unless two cusps of \( M \) are SGI each other.

- Now suppose \( K_i \) is of the form given in (5.194). Similar to the above case, there exists \( K^{(0)} \) such that each \( W_i \) is contained in a translation of \( K^{(0)} \) and \( K^{(0)} \) is contained in \( K_i \) for every \( i \in I \).

Thus \( K^{(0)} \) lies in an algebraic subgroup defined by

\[
\tilde{M}_2 = \tilde{L}_2 = \tilde{M}'_2 = 1.
\]

In other words, for every \( i \), there exist \( \tilde{\xi}_{1i}, \tilde{\xi}_{2i} \) and \( \tilde{\xi}_{3i} \) such that each \( W_i \) is contained in

\[
\tilde{M}_2 = \tilde{\xi}_{1i}, \quad \tilde{L}_2 = \tilde{\xi}_{2i}, \quad \tilde{M}'_2 = \tilde{\xi}_{3i}.
\]

However, as discussed in the previous case, there is no anomalous subvariety contained in (5.201) unless two cusps of \( M \) are SGI each other.

(2) There exist **infinitely** many **maximal** anomalous subvarieties \( \{ Z_i \}_{i \in I} \) of \( X \times Y \) such that \( Y_i \subset Z_i \) for each \( i \in I \). By Lemma 5.6, \( X \times Y \) does not have infinitely many anomalous subvarieties of dimension 3, so we have either \( \dim Z_i = 2 \) or \( \dim Z_i = 1 \).

(a) \( \dim Z_i = 2 \).

Since \( Z_i \) contains \( P_i \) and each coordinate of \( P_i \) is non-cyclotomic, \( Z_i \) is not contained in

\[
M_j = L_j = 1 \quad (j = 1 \text{ or } 2)
\]

or

\[
M'_j = L'_j = 1 \quad (j = 1 \text{ or } 2).
\]

\[\text{Since Proj} \ Z \text{ lies in (5.186), (5.199) implies}
\]

\[
M_1 = M'_1 = \xi_{1i}, \quad L_1 = L'_1 = \xi_{2i}.
\]

If two cusps of \( M \) are not SGI, then \( M_2 \) and \( M'_2 \) are determined by \( L_1 \) and \( L'_1 \) and so (5.200) defines a point on \( X \times X \).
Thus, by Lemma 5.7, we fall into the following two cases.

(i) Each \( Z_i \) is contained in a translation of either

\[
M_1 = L_1 = M_2 = L_2 = 1 
\]

or

\[
M'_1 = L'_1 = M'_2 = L'_2 = 1. 
\]

Without loss of generality, we only consider the first case, and so \( Z_i \) is a component of the intersection between \( X \times X \) and

\[
M_1 = \xi_{1i}, \quad L_1 = \xi'_{1i}, \quad M_2 = \xi_{2i}, \quad L_2 = \xi'_{2i} 
\]

(5.204) for some \( \xi_{1i}, \xi_{2i}, \xi'_{1i} \) and \( \xi'_{2i} \) in \( C \). By projecting onto the second coordinate of \( X \times X \), we simply view \( Y_i \) as an anomalous subvariety of \( X \). By Theorems 3.18 and 3.19, \( X \) has only finitely many varieties and a maximal subvariety of \( X \) is either contained in \( M_1 = L_1 = 1 \) or \( M_2 = L_2 = 1 \). But this contradicts the fact that any coordinate of \( P_i \) is non-cyclotomic.

(ii) Suppose \( Z_i \) is contained in either

\[
(M_1(M'_1)^{-1})^a = (M_2M'_2)^b \quad \text{for some } a, b \in \mathbb{Z},
\]

or

\[
(M_1(M'_1)^{-1})^a = (M_2(M'_2)^{-1})^b \quad \text{for some } a, b \in \mathbb{Z},
\]

or

\[
(M_1M'_1)^a = (M_2M'_2)^b \quad \text{for some } a, b \in \mathbb{Z},
\]

or

\[
(M_1M'_1)^a = (M_2(M'_2)^{-1})^b \quad \text{for some } a, b \in \mathbb{Z}.
\]

Recall that \( t_{1i} \) and \( t_{2i} \) are multiplicatively independent each other in (5.166), and so \( P_i \) is contained in either

\[
M_1 = M'_1, \quad M_2 = M'_2 \quad \text{(and so } L_1 = L'_1, \quad L_2 = L'_2),
\]

or

\[
M_1 = M'_1, \quad M_2 = (M'_2)^{-1} \quad \text{(and so } L_1 = L'_1, \quad L_2 = (L'_2)^{-1}),
\]

or

\[
M_1 = (M'_1)^{-1}, \quad M_2 = M'_2 \quad \text{(and so } L_1 = (L'_1)^{-1}, \quad L_2 = L'_2),
\]

or

\[
M_1 = (M'_1)^{-1}, \quad M_2 = M'_2 \quad \text{(and so } L_1 = (L'_1)^{-1}, \quad L_2 = L'_2).
\]

Either case implies

\[
(p_{1i}/q_{1i}, p_{2i}/q_{2i}) = (p'_{1i}/q'_{1i}, p'_{2i}/q'_{2i}),
\]

but it contradicts our initial assumption (5.165).

(b) \( \dim Z_i = 1 \) (and so \( Y_i = Z_i \)).

In this case, each \( Y_i \) is a maximal anomalous subvariety of \( X \times X \), and so using a idea similar to the one given in the proof of Claim 4.8, we find \( H^{(0)} \) such that, for each \( i \),

- \( H^{(0)} \subset H_i \);
- \( Y_i \subset (X \times X) \cap g_iH^{(0)} \) with some \( g_i \);
• \((X \times X) \cap H^{(0)}\) contains an anomalous subvariety \(Y^{(0)}\) of \(X \times X\) containing \((z^0, z^0)\).

First, if \(Y^{(0)}\) is a 2-dimensional anomalous subvariety, then, since

\[
Y^{(0)} \subset (X \times X) \cap H_i
\]

for every \(i\), we have

\[
p_{1i}/q_{1i} = p'_{1i}/q'_{1i}, \quad p_{2i}/q_{2i} = p'_{2i}/q'_{2i}
\]

for each \(i \in I\) by Lemma 5.3. But this contradicts to the initial assumption (5.165).

Second, if \(Y^{(0)}\) is a 1-dimensional anomalous subvariety, then, again by Lemma 5.3, \(Y^{(0)}\) is contained in either

\[
M_1 = M'_1, \quad L_1 = L'_1, \quad M_2 = M'_2 = L_2 = L'_2 = 1,
\]

(5.205)

or

\[
M_1 = (M'_1)^{-1}, \quad L_1 = (L'_1)^{-1}, \quad M_2 = M'_2 = L_2 = L'_2 = 1,
\]

or

\[
M_2 = M'_2, \quad L_2 = L'_2, \quad M_1 = M'_1 = L_1 = L'_1 = 1,
\]

or

\[
M_2 = (M'_2)^{-1}, \quad L_2 = (L'_2)^{-1}, \quad M_1 = M'_1 = L_1 = L'_1 = 1.
\]

Without loss of generality, we assume \(Y^{(0)}\) is contained in the algebraic subgroup defined by (5.205). So, for each \(i \in I\), we have

\[
(p_{1i}, q_{1i}) = (p'_{1i}, q'_{1i}),
\]

and \(H_i\) is defined by the following simpler forms of equations:

\[
M_1 = M'_1, \quad L_1 = L'_1,
\]

\[
M_2^{c_1i} L_2^{d_1i} (M'_2)^{c'_1i} (L'_2)^{d'_1i} = 1,
\]

(5.206)

\[
M_2^{c_2i} L_2^{d_2i} (M'_2)^{c'_2i} (L'_2)^{d'_2i} = 1,
\]

for each \(i \in I\). Recall \(H^{(0)}\) is contained in \(H_i\) for every \(i \in I\) and so

\[
H^{(0)} \subset \bigcap_{i \in I} H_i.
\]

We consider the following three cases which can happen depending on the dimension of \(\bigcap_{i \in I} H_i\). In each case, it is shown either it falls into a contradiction or a case that considered earlier.

(i) \(\bigcap_{i \in I} H_i\) is an algebraic torus of codimension 4.

As each \(H_i\) is an algebraic subgroup of codimension 4, it means all \(H_i\) are equal. But this contradicts the assumption that all \(Y_i\) are different.

(ii) \(\bigcap_{i \in I} H_i\) is an algebraic torus of codimension 6.

In this case, \(\bigcap_{i \in I} H_i\) is equal to

\[
M_1 = M'_1, \quad L_1 = L'_1,
\]

\[
M_2 = M'_2 = L_2 = L'_2 = 1.
\]
Let $H^{(1)}$ and $H^{(2)}$ be algebraic tori defined by

$$M_1 = M_1', \quad L_1 = L_1',$$

and

$$M_2 = M_2' = L_2 = L_2' = 1$$

respectively. Since $Y_i \subset H_i$ and $H_i$ is of the form given in (5.206), we have $Y_i \subset H^{(1)}$ for all $i \in I$. Also each $Y_i$ is contained in a translation of $H^{(1)} \cap H^{(2)}$, so we conclude

$$Y_i \subset \left( N \cap X \right) \cap (H^{(1)} \cap g_i H^{(2)}) \quad (5.207)$$

for some $g_i$. By moving to $\text{Def}(M) \times \text{Def}(M)$, equivalently, there are infinitely many $(\xi_1, \xi_2, \xi_3, \xi_4) \in (\mathbb{C})^4$

such that the intersection between $\text{Def}(M) \times \text{Def}(M)$ and the manifold defined by

$$u_1 = u_1', \quad v_1 = v_1', \quad u_2 = \xi_1, \quad u_2' = \xi_2, \quad v_2 = \xi_3, \quad v_2' = \xi_4 \quad (5.208)$$

is a 1-dimensional complex manifold. In other words, if we let

$$v_1 = \frac{1}{2} \frac{\partial \Phi}{\partial u_1}(u_1, u_2) = \tau_1 u_1 + 2m_{40}u_1^3 + m_{22}u_1u_2^2 + \cdots, \quad (5.209)$$

then (5.208) is equivalent to

$$\frac{1}{2} \frac{\partial \Phi}{\partial u_1}(u_1, \xi_1) = \frac{1}{2} \frac{\partial \Phi}{\partial u_1}(u_1, \xi_2), \quad (5.210)$$

$$\xi_3 = \frac{1}{2} \frac{\partial \Phi}{\partial u_2}(u_1, \xi_1) = \tau_2 u_2 + 2m_{04}u_2^3 + m_{22}u_2u_1^2 + \cdots, \quad (5.211)$$

$$\xi_4 = \frac{1}{2} \frac{\partial \Phi}{\partial u_2}(u_1, \xi_2) = \tau_2 \xi_2 + 2m_{04}\xi_2^3 + m_{22}\xi_2u_1^2 + \cdots, \quad (5.212)$$

and it defines a 1-dimensional complex manifold parameterized by $u_1$. However this is impossible because we have only finitely many possibilities for $u_1$ in (5.211) (or (5.212)). In other words, (5.210) - (5.212) define a 1-dimensional complex manifold if and only if $v_1$ is independent of $u_2$ and $v_2$ is independent of $u_1$. But this contradicts to our assumption that two cusps of $M$ are not SGI.

(iii) $\bigcap_{i \in I} H_i$ is an algebraic torus of codimension 5. In this case, $H^{(0)}$ is contained in an algebraic subgroup defined by

$$M_1 = M_1, \quad L_1 = L_1, \quad M_2^c L_2^d (M_2')^c (L_2')^d = 1, \quad M_2^c L_2^d (M_2')^c (L_2')^d = 1, \quad M_2^c L_2^d (M_2')^c (L_2')^d = 1. \quad (5.213)$$
Applying Gauss elimination if necessary, we assume (5.213) is of the following forms of equations:

\[
M_1 = M_1, \quad L_1 = L_1, \\
M_2^{d_1} L_2^{d_1} (M_2')^{d_1} (L_2')^{d_1} = 1, \\
M_2^{d_2} L_2^{d_2} (M_2')^{d_2} = 1, \\
M_2^{d_3} L_2^{d_3} = 1.
\]  

(5.214)

Changing the basis if necessary\(^{16}\) we further simplify (5.214) as

\[
M_1 = M_1, \quad L_1 = L_1, \\
M_2^{d_1} L_2^{d_1} (M_2')^{d_1} (L_2')^{d_1} = 1, \\
L_2^{d_2} (M_2')^{d_2} = 1, \\
M_2 = 1.
\]  

(5.215)

Taking logarithm to each coordinate, \((\mathcal{X} \times \mathcal{X}) \cap H^{(0)}\) is locally biholomorphic (near \((z^0, z^0)\)) to the complex manifold defined by

\[
\begin{align*}
&c_1 u_2 + d_1 v_2 (u_1, u_2) + c_1' u_2' + d_1' v_2 (u_1, u_2') = 0, \\
d_2 v_2 (u_1, u_2) + c_2' u_2' = 0, \\
u_2 = 0.
\end{align*}
\]  

(5.216)

If translations of (5.216) are 1-dimensional complex manifolds, using \(u_2\) as a parameter, we get holomorphic functions \(\phi(u_2)\) and \(\phi'(u_2)\) such that the following equations

\[
\begin{align*}
&\frac{1}{2} \frac{\partial \Phi}{\partial u_1} (u_1, u_2) = \frac{1}{2} \frac{\partial \Phi}{\partial u_1} (u_1, u_2') \\
&c_1 u_2 + \frac{d_1}{2} \frac{\partial \Phi}{\partial u_2} (u_1, u_2) + c_1' u_2' + \frac{d_1'}{2} \frac{\partial \Phi}{\partial u_2} (u_1, u_2') = \phi(u_2), \\
&\frac{d_2}{2} \frac{\partial \Phi}{\partial u_2} (u_1, u_2) + c_2' u_2' = \phi(u_2)
\end{align*}
\]  

(5.217, 5.218, 5.219)

\(^{16}\) Keep the basis of the first cusp the same and change the basis of the second cusp by letting

\[
m^*_2 = m_2^{c_3} r_2^{d_3}, \quad l^*_2 = m_2^d r_2^d
\]

where \(c_3 s - d_3 r = 1\). We apply this basis change to both factors in \(\mathcal{X} \times \mathcal{X}\), and get a new holonomy variety \(\mathcal{X}^* \times \mathcal{X}^*\). Note that the Neumann-Zagier potential function obtained from this new basis also satisfies the given condition on one of its coefficients (i.e. \(m_{22} \neq 0\).

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describe a 1-dimensional complex manifold for each $u_2$ sufficiently close to 0. We rewrite (5.217)-(5.219) as follows:\(^1\)
\[
\frac{1}{2} \frac{\partial \Phi}{\partial u_1}(u_1, u_2) = \frac{1}{2} \frac{\partial \Phi}{\partial u_1}(u_1, u_2'), \\
c_1 u_2 + \frac{d_1}{2} \frac{\partial \Phi}{\partial u_2}(u_1, u_2) + c_2' u_2' + \frac{d_2'}{2} \frac{\partial \Phi}{\partial u_2}(u_1, u_2) = \phi'(u_2), \\
u_2' = \frac{1}{c_2} \phi(u_2) - \frac{d_2}{2c_2} \frac{\partial \Phi}{\partial u_2}(u_1, u_2).
\]
Denote the complex manifold defined by (5.220) by $\Psi(u_1, u_2)$. So if $Z$ is the algebraic surface containing all $\mathcal{Y}_i$, then $Z$ is locally biholomorphic to $\Psi(u_1, u_2)$. Now we prove the following claim:

**Claim 5.19.** $\phi'(u_2) = 0$ in (5.218).

**Proof.** Since $\mathcal{Y}_i$ is contained in
\[(X \times X) \cap (H(1) \cap g_i H(2))\]
for some $g_i$, equivalently, there exists $\xi_i$ such that $\mathcal{Y}_i$ is locally biholomorphic to $\Psi(u_1, \xi_i)$. On the other hand, if we fix $u_1$ and consider $u_2$ as a variable, then, for $\zeta$ sufficiently close to 0, $\Psi(\zeta, u_2)$ intersects with holomorphic images of infinitely many $\mathcal{Y}_i$. Thus if
\[Z_\xi := Z \cap (M_1 = \xi)\]
for $\xi$ sufficiently close to 1, $Z_\xi$ meets with infinitely many $\mathcal{Y}_i$. Recall that $\mathcal{Y}_i \subset H_i$ and $H_i$ is defined by equations given in (5.206). By projecting $Z_\xi$ under the following

\[
\text{Proj} : (z_1, \ldots, z_k, M_1, \ldots, L_2) \times (z_1', \ldots, z_k', M_1', \ldots, L_2') \rightarrow (M_2, L_2, M_2', L_2'),
\]
we get $\text{Proj} Z_\xi$ intersects with infinitely many algebraic subgroups defined by the following types of equations
\[
M_2^{c_1} L_2^{d_1} (M_2')^{c_1'} (L_2')^{d_1'} = 1, \\
M_2^{c_2} L_2^{d_2} (M_2')^{c_2'} (L_2')^{d_2'} = 1.
\]
So, by Maurin’s theorem (i.e. Theorem 3.16), $\text{Proj} Z_\xi$ is contained in some algebraic subgroup $H_\xi$ for each complex number $\xi$. Since there are uncountably many complex numbers but only countably many algebraic subgroups, we have infinitely many (indeed uncountably many) $\xi$ such that $\text{Proj} Z_\xi$ is contained in the same algebraic subgroup. Let $H$ be an algebraic group containing infinitely many $Z_\xi$. As $H$ contains a Zariski-dense subset of $Z$, it contains $Z$ as well. Without loss of generality, we suppose $H$ is defined by
\[
M_2^{c_1} L_2^{d_1} (M_2')^{c_1'} (L_2')^{d_1'} = 1, \\
M_2^{c_2} L_2^{d_2} (M_2')^{c_2'} (L_2')^{d_2'} = 1. 
\]
\(^1\)If $c_2' = 0$, then, for each fixed $u_2$, there are only a finite number of choices for $u_1$ in (5.219). Also for fixed $u_1$ and $u_2$, there are only a finite number of choices for $u_2'$ in (5.217) and (5.218), which contradicts to the fact that (5.217) - (5.219) define a 2-dimensional complex manifold. Thus we assume $c_2' \neq 0$.

\(^2\)So we consider (5.220) as a 2-dimensional complex manifold parameterized by $u_1$ and $u_2$. 

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Part II

Now we consider the second case

Let \( H \) be an algebraic subgroup defined by

\[
M_1 = M_1', \quad L_1 = L_1', \quad M_2^{d_2} L_2^{d_3} (M_2')^{c_1} (L_2')^{d_1} = 1
\]

and \( Z \) contains infinitely many 1-dimensional anomalous subvarieties \( \mathcal{Y}_i \).

But we already dealt with this case earlier.

In conclusion, \( Z \) is a 2-dimensional anomalous subvariety of \( \mathcal{X} \times \mathcal{X} \) contained in an algebraic subgroup of defined by the following forms of equations

\[
\mathcal{X} \times \mathcal{X}
\]

following a similar argument given in the proof of Claim 5.16, if \( P \) is a isolated point of \((\mathcal{X} \times \mathcal{X}) \cap H\), we get the degree of \( P \) is uniformly bounded by some constant depending only on \( \mathcal{X} \). So suppose the component of \((\mathcal{X} \times \mathcal{X}) \cap H\) containing \( P \), denoted by \( \mathcal{Y} \), is an anomalous subvariety of \( \mathcal{X} \times \mathcal{X} \). First note that, by Lemma 5.4, \( \mathcal{Y} \) does not contain \((z^0, z^0)\).

Thus if there are only finitely many such anomalous subvarieties, by shrinking the size of a neighborhood of \((z^0, z^0)\), we exclude those cosmetic surgery points contained in finitely many such \( \mathcal{Y} \) and get the desired result.

Now we assume \( \mathcal{X} \times \mathcal{X} \) contains infinitely many anomalous subvarieties near \( z^0 \) and each of them contains a cosmetic surgery point arising from two isometric Dehn filled manifolds.
with different filling coefficients. More precisely, we consider two infinite sequences of two co-prime pairs \((p_{1i}/q_{1i}, p_{2i}/q_{2i})\) \(i \in \mathcal{I}\) and \((p'_{1i}/q'_{1i}, p'_{2i}/q'_{2i})\) \(i \in \mathcal{I}\) such that, for each \(i \in \mathcal{I}\),
\[
(p_{1i}/q_{1i}, p_{2i}/q_{2i}) \neq (p'_{1i}/q'_{1i}, p'_{2i}/q'_{2i}),
\]
and
\[
t_{1i} = t'_{2i}, \quad t_{2i} = t'_{1i}
\]
where \(\{t_{1i}, t_{2i}\}\) and \(\{t'_{1i}, t'_{2i}\}\) are the sets of holonomies of \(\mathcal{M}(p_{1i}/q_{1i}, p_{2i}/q_{2i})\) and \(\mathcal{M}(p'_{1i}/q'_{1i}, p'_{2i}/q'_{2i})\) respectively. Let
\[
P_i = \left( t_{1i}^{-q_{1i}}, t_{1i}^{-p_{1i}}, t_{2i}^{-q_{2i}}, t_{2i}^{-p_{2i}}, (t'_{1i})^{-q_{1i}}, (t'_{1i})^{-p_{1i}}, (t'_{2i})^{-q_{2i}}, (t'_{2i})^{-p_{2i}} \right)
\]
be the cosmetic surgery point associated with \(\mathcal{M}(p_{1i}/q_{1i}, p_{2i}/q_{2i})\) and \(\mathcal{M}(p'_{1i}/q'_{1i}, p'_{2i}/q'_{2i})\) in \(\mathcal{X} \times \mathcal{X}\), and \(H_i\) be an algebraic subgroup containing \(P_i\) and obtained by the same procedure shown earlier (i.e. using Siegel’s lemma). Thus \(H_i\) is defined by following forms of equations:
\[
\begin{align*}
M_1^{a_{11}} L_1^{b_{11}} (M_2')^{c_{11}} (L_2)_{d_{11}} & = 1, \\
M_1^{a_{21}} L_1^{b_{21}} (M_2')^{c_{21}} (L_2)_{d_{21}} & = 1, \\
M_2^{a_{13}} L_2^{b_{13}} (M_1')^{c_{13}} (L_1)_{d_{13}} & = 1, \\
M_2^{a_{23}} L_2^{b_{23}} (M_1')^{c_{23}} (L_1)_{d_{23}} & = 1.
\end{align*}
\]
(5.226)

Denote the component of \(H_i \cap (\mathcal{X} \times \mathcal{X})\) containing \(P_i\) by \(\mathcal{Y}_i\), and assume \(\{\mathcal{Y}_i\}_{i \in \mathcal{I}}\) is a family of infinitely many anomalous subvarieties near \((z^0, z^0)\). Then the following two cases occur.

1. All \(\{\mathcal{Y}_i\}_{i \in \mathcal{I}}\) are contained in a finite list \(\Theta\) of maximal anomalous subvarieties of \(\mathcal{X} \times \mathcal{X}\) containing \((z^0, z^0)\). For \(Z \in \Theta\), as we checked in **Part I**, it is enough to assume \(\text{dim } Z = 2\). Let \(H\) be an algebraic subgroup defined by
\[
\begin{align*}
M_1^{a_{11}} L_1^{b_{11}} M_2^{d_{11}} (M_2')^{c_{11}} (L_1)_{d_{11}} & = 1, \\
M_1^{a_{21}} L_1^{b_{21}} M_2^{d_{21}} (M_2')^{c_{21}} (L_1)_{d_{21}} & = 1, \\
M_2^{a_{13}} L_2^{b_{13}} M_1^{d_{13}} (M_1')^{c_{13}} (L_2)_{d_{13}} & = 1, \\
M_2^{a_{23}} L_2^{b_{23}} M_1^{d_{23}} (M_1')^{c_{23}} (L_2)_{d_{23}} & = 1.
\end{align*}
\]
(5.227)
such that \(Z \subset H \cap (\mathcal{X} \times \mathcal{X})\). Since \(t_{1i}\) and \(t_{2i}\) are multiplicatively independent for each \(i\), \(P_i\) are contained in
\[
\begin{align*}
M_1^{a_{11}} L_1^{b_{11}} (M_2')^{c_{11}} (L_2)_{d_{11}} & = 1, \\
M_1^{a_{21}} L_1^{b_{21}} (M_2')^{c_{21}} (L_2)_{d_{21}} & = 1, \\
M_2^{a_{13}} L_2^{b_{13}} (M_1')^{c_{13}} (L_1)_{b_{13}} & = 1, \\
M_2^{a_{23}} L_2^{b_{23}} (M_1')^{c_{23}} (L_1)_{b_{23}} & = 1, \\
M_2^{a_{23}} L_2^{b_{23}} (M_1')^{c_{23}} (L_1)_{b_{23}} & = 1.
\end{align*}
\]
(5.228)

By Claim 5.17, the ranks of
\[
\{(a_j, b_j, c_j', d_j') \mid j = 1, 2, 3\}
\]
and
\[
\{(c_j, d_j, a_j', b_j') \mid j = 1, 2, 3\}
\]
are at most 2. So we have the following two subcases.
(a) The ranks of \( (5.228) \) and \( (5.229) \) are all equal to 2. By applying Gauss elimination if necessary, we assume that \( (5.227) \) is of the following form:

\[
\begin{align*}
M_1^{a_1} L_1^{b_1} M_2^{c_1} L_2^{d_1} (M_1')^{c_1'} (L_1')^{b_1'} (M_2')^{c_1'} (L_2')^{d_1'} &= 1, \\
M_1^{a_2} L_1^{b_2} (M_2')^{c_2'} (L_2')^{d_2'} &= 1, \\
M_2^{c_2} L_2^{d_2} (M_1')^{c_1'} (L_1')^{b_1'} &= 1,
\end{align*}
\]

and, by moving to Def (\( \mathcal{M} \)) \( \times \) Def (\( \mathcal{M} \)) is locally biholomorphic (near \( (z^0, z^0) \)) to

\[
\begin{align*}
&\quad a_1 u_1 + b_1 v_1 + c_1 u_2 + d_1 v_2 + a_1' u_1' + b_1' v_1' + c_1' u_2' + d_1' v_2' = 0, \\
&\quad a_2 u_1 + b_2 v_1 + c_2 u_2 + d_2 v_2 = 0, \\
&\quad c_3 u_2 + d_3 v_2 + a_3' u_1' + b_3' v_1' = 0.
\end{align*}
\]

The Jacobian of \( (5.231) \) at \((u_1, u_1', u_2, u_2') = (0, 0, 0, 0)\) is

\[
\begin{pmatrix}
a_1 + \tau_1 b_1 & a_1' + \tau_1 b_1' & c_1 + \tau_2 d_1 & c_1' + \tau_2 d_1' \\
a_2 + \tau_1 b_2 & 0 & 0 & c_2' + \tau_2 d_2'
\end{pmatrix}
\]

(5.232)

Since the rank of \( (5.232) \) is 2, we get the ranks of the following two matrices,

\[
\begin{pmatrix}
a_1 + \tau_1 b_1 & c_1 + \tau_2 d_1 \\
a_2 + \tau_1 b_2 & c_2 + \tau_2 d_2
\end{pmatrix}
\]

(5.233)

and

\[
\begin{pmatrix}
a_1' + \tau_1 b_1' & c_1 + \tau_2 d_1 \\
a_3' + \tau_1 b_3' & c_3 + \tau_2 d_3
\end{pmatrix}
\]

(5.234)

are all equal to 2. But this is impossible by Lemma 4.4.

(b) Either the rank of \( (5.228) \) or \( (5.229) \) is 1. Without loss of generality, assume the rank of \( (5.228) \) is 2 and the rank of \( (5.229) \) is 1. By applying Gauss elimination if necessary, we further assume \( (5.227) \) is of the following form:

\[
\begin{align*}
M_1^{a_1} L_1^{b_1} (M_2')^{c_1'} (L_2')^{d_1'} &= 1, \\
M_1^{a_2} L_1^{b_2} (M_2')^{c_2'} (L_2')^{d_2'} &= 1, \\
M_2^{c_2} L_2^{d_2} (M_1')^{c_1'} (L_1')^{b_1'} &= 1.
\end{align*}
\]

Then, near \((z^0, z^0)\), \( (5.235) \) is locally biholomorphic to

\[
\begin{align*}
&\quad a_1 u_1 + b_1 v_1 + c_1 u_2 + d_1 v_2 = 0, \\
&\quad a_2 u_1 + b_2 v_1 + c_2' u_2' + d_2' v_2' = 0, \\
&\quad c_3 u_2 + d_3 v_2 + a_3' u_1' + b_3' v_1' = 0,
\end{align*}
\]

and the Jacobian of \( (5.236) \) at \((u_1, u_1', u_2, u_2') = (0, 0, 0, 0)\) is

\[
\begin{pmatrix}
a_1 + \tau_1 b_1 & 0 & 0 & c_1' + \tau_2 d_1' \\
a_2 + \tau_1 b_2 & 0 & 0 & c_2' + \tau_2 d_2'
\end{pmatrix}
\]

(5.237)

\[
\begin{pmatrix}
a_3' + \tau_1 b_3' & c_3 + \tau_2 d_3 & 0
\end{pmatrix}
\]
Since the rank of \((5.237)\) is 2, the rank of
\[
\begin{pmatrix}
a_1 + \tau_1 b_1 & c'_1 + \tau_2 d'_1 \\
a_2 + \tau_1 b_2 & c'_2 + \tau_2 d'_2
\end{pmatrix}
\] (5.238)
is 1, but again this is impossible by Lemma 4.4.

(2) There exist infinitely many maximal anomalous subvarieties of \(\{Z_i\}_{i \in I} \times \mathcal{X} \times \mathcal{X}\) such that \(Y_i \subset Z_i\) for each \(i \in I\). As we checked in Part I, it is enough to consider the following case:
\[
Z_i = Y_i \quad \text{and} \quad \dim Z_i = \dim Y_i = 1
\]
for each \(i \in I\). Then, following a similar argument given in the proof of Claim 4.8, the component of \(H_i \cap (\mathcal{X} \times \mathcal{X})\) containing \(z^0\) is also an anomalous subvariety of \(\mathcal{X} \times \mathcal{X}\) for each \(i\). But this is impossible by Lemma 5.4.

\(\square\)
6. *n*-cusped case

6.1. Preliminaries

The goal of this section is proving the following theorem, which is a generalization of Theorem 3.18:

**Theorem 6.1.** Let $\mathcal{X}$ be the holonomy variety of an $n$-cusped hyperbolic 3-manifold having rationally independent cusp shapes. Let $H$ be an algebraic subgroup such that a component of $\mathcal{X} \cap H$ containing $(1, \ldots, 1)$ is an anomalous subvariety of $\mathcal{X}$. Then this variety is contained

$$M_i = L_i = 1$$

for some $1 \leq i \leq n$.

Before proving the theorem, we collect some preliminary definitions and lemmas.

**Definition 6.2.** Let $\tilde{V}$ be a vector space and $V = \{v_1, \ldots, v_n\}$ be a basis of $\tilde{V}$. We say $v \in \tilde{V}$ is interchangeable with $v_i$ in $\tilde{V}$ (or $v_i$ is interchangeable with $v$ in $\tilde{V}$) if

$$\{v_1, \ldots, v_{i-1}, v, v_{i+1}, \ldots, v_n\}$$

is a basis of $\tilde{V}$. Similarly we say that we say $A \subset \tilde{V}$ is interchangeable with $B(\subset V)$ if

$$(V \setminus B) \cup A$$

is a basis of $\tilde{V}$.

For example, if $\tilde{V}$ is a vector space generated by $v_1, v_2, v_3$, then $v_1 + v_2$ is interchangeable with $v_1$ or $v_2$ in $\tilde{V}$.

The following lemma can be proved easily.

**Lemma 6.3.** Let $\tilde{V}_1 \subset \cdots \subset \tilde{V}_m$ be a sequence of vector spaces. Let

$$V_1 = \{v_1, \ldots, v_{h_1}\}$$

be a basis of $\tilde{V}_1$ and

$$V_1 \cup \{v_{h_1+1}, \ldots, v_{h_2}\}$$

be a basis of $\tilde{V}_2$. Inductively, let

$$V_{i+1} = \{v_{h_{i+1}}, \ldots, v_{h_{i+1}}\}$$

(6.1)

and

$$V_1 \cup \cdots \cup V_{i+1}$$

be a basis of $\tilde{V}_{i+1}$. Suppose that, for each $1 \leq i \leq m$, there exist $v_{n_i} \in V_i$ and $v'_{n_i} \in \tilde{V}_i$ such that $v_{n_i}$ is interchangeable with $v'_{n_i}$ in $\tilde{V}_i$. Then

$$\{v'_{n_1}, \ldots, v'_{n_m}\}$$

is interchangeable with

$$\{v_{n_1}, \ldots, v_{n_m}\}$$

in $\tilde{V}_m$. 68
Proof. Rearranging if necessary, we assume
\[ v_{n_1} = v_{h_1} \]
(for each \( 1 \leq i \leq m \)) and represent each \( v'_{n_i} \) as\(^{19}\)
\[ v'_{n_i} = \sum_{j=1}^{h_i} a_{ij} v_j. \]

The matrix representation of the linear transformation from \( \bigcup_{i=1}^{m} V_i \) to
\[ \left( V_1 \cup \cdots \cup V_m \cup \{ v'_{n_1}, \ldots, v'_{n_m} \} \right) \setminus \{ v_{n_1}, \ldots, v_{n_m} \} \]
(6.2)
with respect to the following basis
\[ V_1 \cup \cdots \cup V_m \]
is a triangular form with determinant \( \prod_{i=1}^{m} a_{ih_i} \neq 0 \), which concludes (6.2) is a basis of \( \tilde{V}_m \). \( \square \)

The following lemma will play the central role in the proof of Theorem 6.1.

Lemma 6.4. Let
\[ \{ v_1, w_1, \ldots, v_n, w_n \} \]
be a set of vectors in \( \mathbb{Q}^n \) satisfying the following property: if
\[ \{ u_1, \ldots, u_n \} \quad (6.4) \]
is a subset of (6.3) such that \( u_i = v_i \) or \( w_i \) for each \( 1 \leq i \leq n \), then the vectors in (6.4) are linearly dependent. Then there exists \( \{ i_1, \ldots, i_m \} \subset \{ 1, \ldots, n \} \) such that the dimension of the vector space spanned by
\[ \{ v_{i_1}, w_{i_1}, \ldots, v_{i_m}, w_{i_m} \} \]
(6.5)
is at most \( m \).

Note that we allow the zero vector for \( v_i \) or \( w_i \). So if
\[ v_i = w_i = 0, \]
for some \( i \), the lemma is clearly true by taking (6.5) to be \( \{ v_i, w_i \} \).

Proof. Let
\[ U = \{ u_{i_1}, \ldots, u_{i_h} \} \]
(6.6)
be a subset of (6.3) satisfying
- \( u_i = v_i \) or \( w_i \) for each \( i \in \{ i_1, \ldots, i_h \} \);
- the vectors in (6.6) are linearly independent over \( \mathbb{Q} \);
- the cardinality of \( U \) is the biggest among all the subsets of (6.3) satisfying (1) and (2).

Let
\[ U' = \{ u'_j, \ldots, u'_j \} \]
(6.7)
be another subset of (6.3) associated with \( U \) satisfying:
- \( \{ j_1, \ldots, j_k \} \subset \{ i_1, \ldots, i_h \} \);
- \( u'_j = w_j \) if \( u_j = v_j \) and \( u'_j = v_j \) if \( u_j = w_j \) for each \( j \in \{ j_1, \ldots, j_k \} \);
- \( u'_{i_1}, \ldots, u'_{i_h}, u'_{j_1}, \ldots, u'_{j_k} \) are linearly independent over \( \mathbb{Q} \);

\(^{19}\)So \( a_{ih_i} \neq 0 \) for each \( i \) by the assumption.
• the cardinality of $U'$ is the biggest among all the sets satisfying (1), (2), and (3).

There are many different choices for $U$ and $U'$, but we choose one of them. Rearranging if necessary, we assume

$$U = \{v_1, \ldots, v_h\}$$

(6.8)

and

$$U' = \{w_1, \ldots, w_k\}$$

(6.9)

where $k \leq h < n$. To simplify the notation, we denote (6.8) and (6.9) by $V_H$ and $W_K$ respectively. Also, for each $i$ ($1 \leq i \leq n$), we call $v_i$ (resp. $w_i$) the counter vector of $w_i$ (resp. $v_i$).

Claim 6.5. If $h = k$, then $v_i = w_i = 0$ for $h + 1 \leq i \leq n$.

Proof. Suppose $v_i \neq 0$ for some $i$ ($h + 1 \leq i \leq n$). If $v_i \notin V_H$, then

$$V_H \cup \{v_i\}$$

is the set of $(h + 1)$-linearly independent vectors, which contradict the assumption on $h$. So $v_i \in \tilde{V}_H$. Since $w_1, \ldots, w_h$ are linearly independent vectors not contained in $\tilde{V}_H$, $w_1, \ldots, w_h, v_i$

are $(h + 1)$-linearly independent vectors. Again this also contradicts the assumption on $h$.

Thus if $h = k$, by letting (6.5) be

$$\{v_{h+1}, w_{h+1}, \ldots, v_n, w_n\},$$

we get the desired result.

Now suppose $h > k$, and let $V_{H\setminus K}$ be

$$\{v_{k+1}, \ldots, v_h\}$$

and $\tilde{V}_{H\setminus K}$ be the vector space spanned by $V_{H\setminus K}$. We also denote

$$\{v_{h+1}, \ldots, v_n\}$$

and

$$\{w_{h+1}, \ldots, w_n\}$$

by $V_{N\setminus H}$ and $W_{N\setminus H}$ respectively. (See Table 1.)

Claim 6.6. $V_{N\setminus H}, W_{N\setminus H} \subset \tilde{V}_{H\setminus K}$.

Proof. Suppose $v_i \in V_{N\setminus H}$ but $v_i \notin \tilde{V}_{H\setminus K}$ ($h + 1 \leq i \leq n$).

(1) If $v_i \in \tilde{V}_H$, then, since $v_i \notin \tilde{V}_{H\setminus K}$,

$$v_{k+1}, \ldots, v_h, v_i$$

are linearly independent vectors (in $\tilde{V}_H$). Since $\tilde{V}_H \cap \tilde{W}_K = \{0\}$,

$$w_1, \ldots, w_k, v_{k+1}, \ldots, v_h, v_i$$

are linearly independent as well. But this contradicts the assumption on $h$.

(2) If $v_i \notin \tilde{V}_H$, then

$$v_1, \ldots, v_k, v_{k+1}, \ldots, v_h, v_i$$

are linearly independent, and again it contradicts the assumption on $h$. 

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Similarly, we can show \( w_i \in \tilde{V}_{H \setminus K} \) for all \( h + 1 \leq i \leq n \).

Define \( V_1 \) to be a subset of
\[
V_{H \setminus K}
\]
such that it is the set of all the interchangeable vectors either with \( v \in V_{N \setminus H} \) or \( w \in W_{N \setminus H} \) in \( \tilde{V}_{H \setminus K} \). If \( V_1 = \emptyset \), it means, by the definition, none of the vectors in
\[
V_{N \setminus H} \cup W_{N \setminus H}
\]
is interchangeable with any \( v \in V_{H \setminus K} \). By Claim 6.6 since
\[
\tilde{V}_{N \setminus H} \cup \tilde{W}_{N \setminus H} \subset \tilde{V}_{H \setminus K}
\]
we conclude
\[
\tilde{V}_{N \setminus H} \cup \tilde{W}_{N \setminus H} = \{0\},
\]
meaning that \( V_{H \setminus K} \) and \( W_{H \setminus K} \) are sets of zero vectors. So, by letting (6.5) be
\[
V_{H \setminus K} \cup W_{H \setminus K},
\]
we get the desired result.

Now we suppose \( V_1 \neq \emptyset \), and, rearranging if necessary, let \( V_1 \) be given by
\[
\{v_{h_1+1}, \ldots, v_h\} \quad (k + 1 \leq h_1 + 1 \leq h).
\]
Let \( \tilde{V}_1 \) be the vector space spanned by \( V_1 \) and \( W_1 \) be the set of all the counter vectors of \( V_1 \).
(See Table 1.)

Claim 6.7. \( \tilde{V}_{N \setminus H}, \tilde{W}_{N \setminus H} \subset \tilde{V}_1 \).

Proof. By the above claim, for any \( v \in V_{N \setminus H} \) (or \( w \in W_{N \setminus H} \)), we have
\[
v = a_{k+1}v_{k+1} + \cdots + a_hv_h
\]
for some \( a_{k+1}, \ldots, a_h \in \mathbb{Q} \). For \( k + 1 \leq j \leq h \), \( a_j \neq 0 \) if and only if \( v_j \) is interchangeable with \( v \) in \( \tilde{V}_{H \setminus K} \). By the definition of \( V_1 \), if \( v_j \) is interchangeable with \( v \in V_{N \setminus H} \), then \( v_j \in V_1 \). Thus we have
\[
a_j = 0
\]
for all \( j \) such that \( v_j \notin V_1 \). In other words, \( v \in \tilde{V}_1 \).

Claim 6.8. \( W_1 \subset \tilde{V}_{H \setminus K} \).

Proof. Suppose there exists \( w_i \in W_1 \) such that \( w_i \notin \tilde{V}_{H \setminus K} \) \((h + 1 \leq i \leq h)\). Since \( v_i \in V_1 \) by the definition of \( V_1 \), \( v_i \) is interchangeable either with some \( v_j \in V_{N \setminus H} \) or \( w_j \in W_{N \setminus H} \) \((h + 1 \leq j \leq n)\) in \( \tilde{V}_{H \setminus K} \). Without loss of generality, we assume \( v_j \) is the one.

(1) \( w_i \in \tilde{V}_H \).

Since \( v_j \) is interchangeable with \( v_i \) in \( \tilde{V}_{H \setminus K} \) and \( w_i \notin \tilde{V}_{H \setminus K} \), the following
\[
(V_{H \setminus K} - \{v_i\}) \cup \{v_j\} \cup \{w_i\}
\]
are linearly independent vectors (in \( \tilde{V}_H \)). Since \( \tilde{V}_H \cap \tilde{W}_K = \{0\} \), the following \((h + 1)\)-vectors
\[
W_K \cup (V_{H \setminus K} - \{v_i\}) \cup \{v_j\} \cup \{w_i\}
\]
are also linearly independent. But this contradicts the assumption on \( h \). \(\Box\)

\(\Box\) \(v_i\) is the counter vector of \( w_i \).
(2) \( w_i \notin \tilde{V}_H \).

In this case, the following \((h + 1)\)-vectors
\[
V_K \cup (V_H \setminus K - \{v_i\}) \cup \{v_j\} \cup \{w_i\}
\]
are linearly independent, and this also contradicts the assumption on \( h \).

\[\square\]

We define \( V_2 \) as a subset of \( V_{H \setminus K} - V_1 \) such that the set of all the interchangeable vectors with \( w \in W_1 \) in \( \tilde{V}_{H \setminus K} \). First if \( V_2 = \emptyset \), it means none of the vectors in \( W_1 \) is interchangeable with a vector contained in \( V_{H \setminus K} - V_1 \).

Since \( W_1 \subset \tilde{V}_{H \setminus K} \) by Claim 6.8, we conclude \( W_1 \subset \tilde{V}_1 \). Thus the rank of
\[
V_1 \cup W_1 \quad \text{(6.13)}
\]
is at most \( |V_1| = |W_1| \), and we get the desired result.

Now we assume \( V_2 \neq \emptyset \) and, rearranging if necessary, let \( V_2 \) is given by
\[
\{v_{h+1}, \ldots, v_{h+1}\} \quad (k + 1 \leq h + 1 \leq h).
\]

Let \( \tilde{V}_2 \) be the vector space spanned by \( V_1 \cup V_2 \) and \( W_2 \) be the set of all the counter vectors of \( V_2 \). (See Table 1.)

Claim 6.9. \( W_1 \subset \tilde{V}_2 \).

Proof. For any \( w \in W_1 \), by the previous claim, we have \( w \in \tilde{V}_{H \setminus K} \) and so
\[
w = a_{k+1}v_{k+1} + \cdots + a_hv_h \quad \text{(6.15)}
\]
for some \( a_{k+1}, \ldots, a_h \in \mathbb{Q} \). For \( k + 1 \leq j \leq h \), \( a_j \neq 0 \) if and only if \( v_j \) is interchangeable with \( w \) in \( \tilde{V}_{H \setminus K} \). By the definition, \( V_2 \) is a subset of \( V_{H \setminus K} - V_1 \), and, if an element of \( V_{H \setminus K} \) is interchangeable with \( w \) in \( \tilde{V}_{H \setminus K} \), then it is contained in either \( V_1 \) or \( V_2 \). Thus \( a_j = 0 \) for all \( k + 1 \leq j \leq h_1 \) in (6.15). In other words, \( w \) belongs to the vector space spanned by \( V_1 \) and \( V_2 \), \( \tilde{V}_2 \). \( \square \)

Before proceeding to the next step, for reader’s convenience, we summarize what we have collected so far.

- \( \tilde{V}_{N \setminus H} \subset \tilde{V}_1 \);
- \( \tilde{W}_{N \setminus H} \subset \tilde{V}_1 \);
- \( W_1 \subset \tilde{V}_2 \);
- \( \tilde{V}_1 \subset \tilde{V}_2 \);
- for any \( v_i \in V_1 \), there exists \( v_j \in \tilde{V}_{N \setminus H} \) or \( w_j \in \tilde{W}_{N \setminus H} \) such that \( v_i \) is interchangeable with \( v_j \) or \( w_j \) in \( \tilde{V}_{H \setminus K} \) (or \( \tilde{V}_1 \));
- for any \( v_i \in V_2 \), there exists \( w_j \in W_1 \) such that \( v_i \) is interchangeable with \( w_j \) in \( \tilde{V}_{H \setminus K} \) (or \( \tilde{V}_2 \)).

Claim 6.10. \( W_2 \subset \tilde{V}_{H \setminus K} \).
Now we split the problem into two cases.

### Case 1

For $h_1 < h$, we have $V_{h_1} \supset V_{h_1 + 1}$.

**Lemma 6.3**

If $v_j \in V_{h_1}$, then $v_j \in V_{h_1 + 1}$.

### Case 2

For $h_1 = h$, we have $V_{h_1} = V_{h_1 + 1}$.

**Lemma 6.4**

If $v_j \in V_{h_1}$, then $v_j \in V_{h_1 + 1}$.

---

**Proof.** Suppose $w_{j_2} \in W_2$ but $w_{j_2} \notin \tilde{V}_{H \setminus K}$. By the definition of $V_2$, there exists $w_{j_1} \in W_1 (h + 1 \leq j_1 \leq h)$ such that $w_{j_1}$ is interchangeable with $v_{j_2}$ in $V_2$.

Also, by the definition of $V_1$, there exists $v_{j_0} \in V_{N \setminus H} (h_1 \leq j_0 \leq n)$ such that it is interchangeable with $v_{j_1}$ in $V_1$. Without loss of generality, we assume $w_{j_0}$ is the one. By Lemma 6.3,

$$\{w_{j_1}, w_{j_0}\}$$

is interchangeable with

$$\{v_{j_2}, v_{j_1}\}$$

in $\tilde{V}_{H \setminus K}$ and so the following set is a basis of $\tilde{V}_{H \setminus K}$:

$$\left( V_{H \setminus K} - \{v_{j_2}, v_{j_1}\} \right) \cup \{w_{j_1}, w_{j_0}\},$$

(6.16)

Since $w_{j_2} \notin \tilde{V}_{H \setminus K}$, by adding $w_{j_2}$ to (6.16), we get the following set of $(h + 1)$-independent vectors:

$$\left( V_{H \setminus K} - \{v_{j_2}, v_{j_1}\} \right) \cup \{w_{j_2}, w_{j_1}, w_{j_0}\}.$$  

(6.17)

Now we split the problem into two cases.

1. If $w_{j_2} \in \tilde{V}_H$, then the elements in the following set

$$\{w_1, \ldots, w_k\} \cup \left( V_{H \setminus K} - \{v_{j_2}, v_{j_1}\} \right) \cup \{w_{j_2}, w_{j_1}, w_{j_0}\},$$

(6.18)

are $(h + 1)$-linearly independent vectors (since $\tilde{W}_K \cap \tilde{V}_H = \{0\}$). But this contradicts the assumption on $h$. (Note that the indexes of the vectors appeared in (6.25) are all different.)

2. If $w_{j_2} \notin \tilde{V}_H$, then the elements in the following set

$$\{v_1, \ldots, v_k\} \cup \left( V_{H \setminus K} - \{v_{j_2}, v_{j_1}\} \right) \cup \{w_{j_2}, w_{j_1}, w_{j_0}\},$$

(6.19)

are $(h + 1)$-linearly independent vectors. Again this contradicts the assumption on $h$.

$\square$

---

More generally, for $m \geq 3$, we suppose

- $V_i \neq \emptyset$ for $1 \leq i \leq m - 1$;
- $W_{i-1} \subset \tilde{V}_i$ (1 \leq i \leq m - 1);
- $W_{m-1} \subset \tilde{V}_{H \setminus K}$;

$\overline{v_{j_2}}$ is the counter vector of $w_{j_2}$.
and construct $V_m$ as follows:
- $V_m$ is a subset of $V_{H\setminus K} - (V_1 \cup \cdots \cup V_{m-1})$;
- $V_m$ is the union of all the interchangeable vectors with some $w_i \in W_{m-1}$ in $\tilde{V}_{H\setminus K}$.

Similarly, for each $1 \leq i \leq m$, define $\tilde{V}_i$ as the vector space spanned by $V_1 \cup \cdots \cup V_i$.

Extending the argument presented in the proof of Claim 6.9, we prove Claim 6.11.

**Claim 6.11.** $W_{m-1} \subset \tilde{V}_m$.

**Proof.** For $w \in W_{m-1}$, by the assumption, we have $w \in \tilde{V}_{H\setminus K}$ and so
\[ w = a_{k+1}v_{k+1} + \cdots + a_hv_h \] (6.20)
for some $a_{k+1}, \ldots, a_h \in \mathbb{Q}$. For $k + 1 \leq j \leq h$, $a_j \neq 0$ if and only if $v_j$ is interchangeable with $w$ in $\tilde{V}_{H\setminus K}$. By the definition, $V_m$ is a subset of
\[ V_{H\setminus K} - (V_1 \cup \cdots \cup V_{m-1}) \]
and, if an element of $V_{H\setminus K}$ is interchangeable with $w$ in $\tilde{V}_{H\setminus K}$, it is contained in
\[ V_1 \cup \cdots \cup V_m. \] (6.21)
Thus $a_j = 0$ for all $k + 1 \leq j \leq h$ such that $v_j$ is not in (6.21). In other words, $w$ belongs to the vector space spanned by (6.21), $\tilde{V}_m$. \[ \square \]

If $V_m = \emptyset$, by Claim 6.11 it means
\[ W_{m-1} \subset \tilde{V}_{m-1}, \]
and thus
\[ (W_1 \cup \cdots \cup W_{m-1}) \subset \tilde{V}_{m-1}. \]
Since $\tilde{V}_{m-1}$ is the vector space spanned by $V_1 \cup \cdots \cup V_{m-1}$, we conclude the rank of
\[ V_1 \cup \cdots \cup V_{m-1} \cup W_1 \cup \cdots \cup W_{m-1} \]
(6.22) is equal to the rank of $V_1 \cup \cdots \cup V_{m-1}$.

So we get the desired result.

Now we suppose $V_m \neq \emptyset$ and prove the following claim which is a generalization of Claim 6.10.

**Claim 6.12.** $W_m \subset \tilde{V}_{H\setminus K}$.

**Proof.** Suppose there exists $w_{jm} \in W_m$ such that $w_{jm} \notin \tilde{V}_{H\setminus K}$. By the definition of $V_m$, there exists $w_{jm-1} \in W_{m-1}$ such that $w_{jm-1}$ is interchangeable with $v_{jm}$ in $\tilde{V}_m$.

Similarly, for each $i$ ($1 \leq i \leq m - 1$), we find $w_{ji} \in W_i$, which is interchangeable with $v_{ji+1}$ in $\tilde{V}_{i+1}$.

\[ v_{jm} \] is the counter vector of $w_{jm}$.
Finally there exists $v_{j_0} \in \tilde{V}_{N \setminus H}$ or $w_{j_0} \in \tilde{W}_{N \setminus H} \ (h + 1 \leq j_0 \leq n)$, which is interchangeable with $v_{j_1}$ in $\tilde{V}_1$. Without loss of generality, we assume $w_{j_0}$ is the one. By Lemma 6.3, 

$$\{w_{j_m-1}, \ldots, w_{j_0}\}$$

is interchangeable with 

$$\{v_{j_m}, \ldots, v_{j_1}\}$$

in $\tilde{V}_{H \setminus K}$, and so

$$(V_{H \setminus K} - \{v_{j_m}, \ldots, v_{j_1}\}) \cup \{w_{j_m-1}, \ldots, w_{j_0}\}$$

(6.23)

is a basis of $\tilde{V}_{H \setminus K}$. Since $w_{j_m} \notin \tilde{V}_{H \setminus K}$ (by the assumption), by adding $w_{j_m}$ to (6.23), we get the following set of $(h - k + 1)$-independent vectors:

$$(V_{H \setminus K} - \{v_{j_m}, \ldots, v_{j_1}\}) \cup \{w_{j_m}, w_{j_m-1}, \ldots, w_{j_0}\}.$$

(6.24)

Now we split the problem into two cases:

1. If $w_{j_m} \in \tilde{V}_{H}$, then the elements in the following set

$$\{w_1, \ldots, w_k\} \cup (V_{H \setminus K} - \{v_{j_m}, \ldots, v_{j_1}\}) \cup \{w_{j_m}, w_{j_m-1}, \ldots, w_{j_0}\}.$$

(6.25)

are $(h+1)$-linearly independent vectors (since $\tilde{V}_{K} \cap \tilde{V}_{H} = \{0\}$). But this contradicts the assumption on $h$. (Note that the indexes of the vectors appeared in (6.25) are all different.)

2. If $w_{j_m} \notin \tilde{V}_{H}$, then the elements in the following set

$$\{v_1, \ldots, v_k\} \cup (V_{H \setminus K} - \{v_{j_m}, \ldots, v_{j_1}\}) \cup \{w_{j_m}, w_{j_m-1}, \ldots, w_{j_0}\}.$$

(6.26)

are $(h+1)$-linearly independent vectors. Again this also contradicts the assumption on $h$.

By Claims 6.11 and 6.12, it satisfies the required assumptions to define $V_{m+1}$ and we repeat the same process as above. Since $V_{m+l} = \emptyset$ for some $l$, we eventually get the desired result. \[\square\]

Using the above lemma, we now prove Theorem 6.1 using induction.

**Theorem 6.13.** Let $\mathcal{X}$ be the holonomy variety of an $n$-cusped hyperbolic 3-manifold having rationally independent cusp shapes. Suppose $H$ be an algebraic subgroup such that the component of $\mathcal{X} \cap H$ containing $z^0$ is a maximal anomalous subvariety of $\mathcal{X}$. Then this component is contained in

$$M_i = L_i = 1$$

for some $1 \leq i \leq n$.

**Proof.** Let $H$ be defined by

$$M_1^{a_{11}} L_1^{b_{11}} \cdots M_n^{a_{1n}} L_n^{b_{1n}} = 1,$$

$$\cdots$$

$$M_1^{a_{ln}} L_1^{b_{ln}} \cdots M_n^{a_{ln}} L_n^{b_{ln}} = 1.$$  

(6.27)

For $l = 2$ and any $n$, the theorem is easily deduced from Lemma 4.3 (for instance, see Lemma 4.3 in [9]). We skip the proof here.
(1) We consider the case \( n = l \) first. As remarked above, the claim is true for \( n = l = 2 \). So we let \( n = l \geq 3 \) and suppose the statement is true for \( 1, \ldots, n - 1 \). Moving to Def(\( \mathcal{M} \)), \( \mathcal{X} \cap H \) is locally biholomorphic (near \( z^0 \)) to

\[
a_{11}u_1 + b_{11}(\tau_1 u_1 + \cdots) + \cdots + a_{1n}u_n + b_{1n}(\tau_n u_n + \cdots) = 0, \\
\cdots \\
a_{n1}u_1 + b_{n1}(\tau_1 u_1 + \cdots) + \cdots + a_{nn}u_n + b_{nn}(\tau_n u_n + \cdots) = 0.
\]

and the Jacobian of it at \((u_1, \ldots, u_n) = (0, \ldots, 0)\) is

\[
\begin{pmatrix}
a_{11} + \tau_1 b_{11} & \cdots & a_{1n} + \tau_1 b_{1n} \\
a_{21} + \tau_1 b_{21} & \cdots & a_{2n} + \tau_1 b_{2n} \\
\vdots & \ddots & \vdots \\
a_{n1} + \tau_1 b_{n1} & \cdots & a_{nn} + \tau_1 b_{nn}
\end{pmatrix}.
\]

(a) If the determinant of (6.29) is nonzero, then, by the inverse function theorem, (6.28) is equivalent to

\[
u_1 = \cdots = u_n = 0,
\]

and so \( z^0 \) is an isolated point of \( \mathcal{X} \cap H \) and clearly it is contained in

\[M_1 = \cdots = M_n = L_1 = \cdots = L_n = 1.\]

(b) Suppose the determinant of (6.29) is zero, and consider the following matrix:

\[
\begin{pmatrix}
a_1 & b_1 & \cdots & a_n & b_n \\
\begin{pmatrix} a_{11} & b_{11} & \cdots & a_{1n} & b_{1n} \\
a_{21} & b_{21} & \cdots & a_{2n} & b_{2n} \\
\vdots & \ddots & \vdots & \vdots \\
a_{n1} & b_{n1} & \cdots & a_{nn} & b_{nn}
\end{pmatrix}
\end{pmatrix}.
\]

Since the cusp shapes are rationally independent and the determinant of (6.29) is equal to zero, the determinant of any matrix of the following form

\[
\begin{pmatrix}
c_1 & \cdots & c_n \\
\end{pmatrix}
\]

where \( c_i = a_i \) or \( b_i \) (\( 1 \leq i \leq n \)) is equal to 0. Thus, by Lemma 6.4 there exists

\[
\{a_{i_1}, b_{i_1}, \ldots, a_{i_m}, b_{i_m}\}
\]

where \( \{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\} \) such that the dimension of the vector space spanned by (6.31) is at most \( m \). Without loss of generality, we suppose

\[
\{i_1, \ldots, i_m\} = \{1, \ldots, m\}.
\]
Since the rank of (6.31) is at most \( m \), applying Gauss elimination, we assume (6.30) and \( H \) are of the following forms

\[
\begin{pmatrix}
a_{11} & \ldots & b_{1m} & a_{1(m+1)} & \ldots & b_{1n} \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
a_{m1} & \ldots & b_{mm} & a_{m(m+1)} & \ldots & b_{mn} \\
0 & \ldots & 0 & a_{(m+1)(m+1)} & \ldots & b_{(m+1)n} \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & 0 & a_{n(m+1)} & \ldots & b_{nn}
\end{pmatrix},
\]

and

\[
M_1^{a_{11}} L_1^{b_{11}} \cdots M_m^{a_{m1}} L_m^{b_{m1}} \cdots M_m^{a_{m1}} L_m^{b_{m1}} \cdots M_m^{a_{m1}} L_m^{b_{m1}} = 1,
\]

\[
M_m^{a_{m1}} L_m^{b_{m1}} \cdots M_m^{a_{m1}} L_m^{b_{m1}} = 1,
\]

\[
M_m^{a_{m1}} L_m^{b_{m1}} \cdots M_m^{a_{m1}} L_m^{b_{m1}} = 1,
\]

\[
M_m^{a_{m1}} L_m^{b_{m1}} \cdots M_m^{a_{m1}} L_m^{b_{m1}} = 1.
\]

respectively. We also rewrite (6.29) as follows:

\[
\begin{pmatrix}
a_{11} + \tau_1 b_{11} & \ldots & a_{1m} + \tau_m b_{1m} & a_{1(m+1)} + \tau_{m+1} b_{1(m+1)} & \ldots & a_{1n} + \tau_n b_{1n} \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
a_{m1} + \tau_1 b_{m1} & \ldots & a_{mm} + \tau_m b_{mm} & a_{m(m+1)} + \tau_{m+1} b_{m(m+1)} & \ldots & a_{mn} + \tau_n b_{mn} \\
0 & \ldots & 0 & a_{(m+1)(m+1)} + \tau_{m+1} b_{(m+1)(m+1)} & \ldots & a_{(m+1)n} + \tau_n b_{(m+1)n} \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \ldots & 0 & a_{n(m+1)} + \tau_{m+1} b_{n(m+1)} & \ldots & a_{nn} + \tau_n b_{nn}
\end{pmatrix}.
\]

Consider the following submatrix of (6.32)

\[
\begin{pmatrix}
a_{(m+1)(m+1)} + \tau_{m+1} b_{(m+1)(m+1)} & \ldots & a_{(m+1)n} + \tau_n b_{(m+1)n} \\
\vdots & \ddots & \ddots \\
a_{n(m+1)} + \tau_{m+1} b_{n(m+1)} & \ldots & a_{nn} + \tau_n b_{nn}
\end{pmatrix},
\]

(6.33)

and let \( H' \) be an algebraic subgroup defined by

\[
M_m^{a_{m1}} L_m^{b_{m1}} \cdots M_n^{a_{m1}} L_n^{b_{m1}} = 1,
\]

\[
M_m^{a_{m1}} L_m^{b_{m1}} \cdots M_n^{a_{m1}} L_n^{b_{m1}} = 1.
\]

Then \( \mathcal{X} \cap H' \) is locally biholomorphic (near \( z^0 \)) to the complex manifold defined by

\[
a_{(m+1)(m+1)} u_{m+1} + b_{(m+1)(m+1)} (\tau_{m+1} u_{m+1} + \cdots) + \cdots + a_{(m+1)n} u_n + b_{(m+1)n} (\tau_n u_n + \cdots) = 0,
\]

\[
\cdots
\]

\[
a_{n(m+1)} u_{m+1} + b_{n(m+1)} (\tau_{m+1} u_{m+1} + \cdots) + \cdots + a_{(m+1)n} u_n + b_{(m+1)n} (\tau_n u_n + \cdots) = 0,
\]

(6.34)
and \((6.33)\) is the Jacobian of \((6.34)\) at \((u_1, \ldots, u_n) = (0, \ldots, 0)\).

(i) If the rank of \((6.33)\) is \(n - m\), then, by the implicit function theorem, \((6.34)\) is equivalent to
\[
    u_{m+1} = \cdots = u_n = 0,
\]
and so the component of \(X \cap H'\) containing \(z^0\) lies in
\[
    M_{m+1} = \cdots = M_n = L_{m+1} = \cdots = L_n = 1.
\]
(ii) If the rank of \((6.33)\) is strictly less than \(n - m\), then the component of \(X \cap H'\) containing \(z^0\) is an anomalous subvariety of \(X\). Since \(H'\) is an algebraic subgroup of codimension \(n - m\), by the induction hypothesis, the component is contained in
\[
    M_i = L_i = 1
\]
for some \(1 \leq i \leq n\).

(2) Next we assume \(l > n\). Let \(H'\) be an algebraic subgroup defined by
\[
    M_1^{a_{11}} L_1^{b_{11}} \cdots M_n^{a_{1n}} L_n^{b_{1n}} = 1, \\
    \cdots \\
    M_1^{a_{n1}} L_1^{b_{n1}} \cdots M_n^{a_{nn}} L_n^{b_{nn}} = 1,
\]
which is the first \(n\)-equations of \((6.27)\). Then \(X \cap H'\) is locally biholomorphic (near \(z^0\)) to the complex manifold defined by
\[
    a_{11} u_1 + b_{11} (\tau_1 u_1 + \cdots) + \cdots + a_{1n} u_n + b_{1n} (\tau_n u_n + \cdots) = 0, \\
    \cdots \\
    a_{n1} u_1 + b_{n1} (\tau_1 u_1 + \cdots) + \cdots + a_{nn} u_n + b_{nn} (\tau_n u_n + \cdots) = 0,
\]
and the Jacobian of \((6.35)\) at \((u_1, \ldots, u_n) = (0, \ldots, 0)\) is
\[
    \begin{pmatrix}
        a_{11} + \tau_1 b_{11} & \cdots & a_{1n} + \tau_1 b_{1n} \\
        a_{21} + \tau_1 b_{21} & \cdots & a_{2n} + \tau_1 b_{2n} \\
        \vdots & \ddots & \vdots \\
        a_{n1} + \tau_1 b_{n1} & \cdots & a_{nn} + \tau_1 b_{nn}
    \end{pmatrix}.
\]
If the determinant of \((6.36)\) is nonzero, then, by the inverse function theorem, \((6.35)\) is equivalent to
\[
    u_1 = \cdots = u_n = 0.
\]
Thus \(z^0\) is an isolated point of \(X \cap H'\) (or \(X \cap H\)) and it is clearly contained in
\[
    M_1 = \cdots = M_n = 1.
\]
If the determinant of \((6.36)\) is zero, then the component of \(X \cap H'\) containing \(z^0\) is an anomalous subvariety of \(X\) and so it falls into the previous case (i.e. the case when \(n = l\)).

(3) Now we consider the remaining case \(n > l \geq 3\). We first assume the theorem is true for any \((m, k)\) satisfying
\[
    1 \leq m \leq n, \quad 1 \leq k \leq l - 1
\]
or
\[
    1 \leq m \leq n - 1, \quad 1 \leq k \leq l.
\]
We show that the result is true for \((n, l)\) as well. Moving to \(\text{Def}(\mathcal{M})\), \(X \cap H\) is locally (near \(z^0\)) biholomorphic to
\[
 a_{11}u_1 + b_{11}(\tau_1 u_1 + \cdots) + \cdots + a_{1n}u_n + b_{1n}(\tau_n u_n + \cdots) = 0, \\
\cdots \\
 a_{l1}u_1 + b_{l1}(\tau_1 u_1 + \cdots) + \cdots + a_{ln}u_n + b_{ln}(\tau_n u_n + \cdots) = 0,
\]
and its Jacobian at \((u_1, \ldots, u_n) = (0, \ldots, 0)\) is
\[
\begin{pmatrix}
 a_{11} + \tau_1b_{11} & \cdots & a_{1n} + \tau_1b_{1n} \\
 a_{21} + \tau_1b_{21} & \cdots & a_{2n} + \tau_1b_{2n} \\
 \vdots & \ddots & \vdots \\
 a_{l1} + \tau_1b_{l1} & \cdots & a_{ln} + \tau_1b_{ln}
\end{pmatrix}.
\]
(6.37)

Since the component of \(X \cap H\) containing \(z^0\) is an anomalous subvariety of \(X\), the rank of (6.37) is strictly less than \(l\). Let \(M\) be
\[
\begin{pmatrix}
 a_1 & b_1 & \cdots & a_n & b_n \\
 a_{11} & b_{11} & \cdots & a_{1n} & b_{1n} \\
 a_{21} & b_{21} & \cdots & a_{2n} & b_{2n} \\
 \vdots & \vdots & \ddots & \vdots & \vdots \\
 a_{l1} & b_{l1} & \cdots & a_{ln} & b_{ln}
\end{pmatrix},
\]
and pick the first \(2l\) columns of \(M\) as below:
\[
\begin{pmatrix}
 a_1 & b_1 & \cdots & a_l & b_l \\
 a_{11} & b_{11} & \cdots & a_{l1} & b_{l1} \\
 a_{21} & b_{21} & \cdots & a_{l2} & b_{l2} \\
 \vdots & \vdots & \ddots & \vdots & \vdots \\
 a_{l1} & b_{l1} & \cdots & a_{ll} & b_{ll}
\end{pmatrix}.
\]
Since the rank of the following matrix
\[
\begin{pmatrix}
 a_{11} + \tau_1b_{11} & \cdots & a_{1l} + \tau_1b_{1l} \\
 \vdots & \ddots & \vdots \\
 a_{l1} + \tau_1b_{l1} & \cdots & a_{ll} + \tau_1b_{ll}
\end{pmatrix}
\]
(6.38)
is strictly less than \(l\) and the cusp shapes are rationally independent, the determinant of any matrix of the following form
\[
\begin{pmatrix}
 c_1 & \cdots & c_i \\
 \vdots & \vdots & \vdots \\
 c_1 & \cdots & c_i
\end{pmatrix}
\]
where \(c_i = a_i\) or \(b_i\) \((1 \leq i \leq l)\) is always equal to 0. By Lemma 6.4 we get
\[
\{a_{i_1}, b_{i_1}, \ldots, a_{i_m}, b_{i_m}\}
\]
(6.39)
where \(\{i_1, \ldots, i_m\} \subset \{1, \ldots, l\}\) such that the dimension of the vector space spanned by (6.39) is strictly less than \(m\). Without loss of generality, we suppose \(\{i_1, \ldots, i_m\} = \)
{1, \ldots, m}. The rank of the vector space spanned by (6.39) is at most \( m \) and so, by applying Gauss elimination, we assume \( M \) and \( H \) are of the following forms

\[
\begin{pmatrix}
  a_{11} & \ldots & a_{1m} & \ldots & a_{1(m+1)} & \ldots & a_{1n} \\
  \vdots & \ddots & \ddots & \ddots & \vdots & \ldots & \vdots \\
  a_{m1} & \ldots & a_{mm} & \ldots & a_{m(m+1)} & \ldots & a_{mn} \\
  0 & \ldots & 0 & a_{(m+1)(m+1)} & \ldots & a_{(m+1)n} & \ldots & a_{ln} \\
  \vdots & \ddots & \ddots & \ddots & \vdots & \ldots & \vdots \\
  0 & \ldots & 0 & a_{l(m+1)} & \ldots & a_{ln} & \ldots & b_{ln} \\
\end{pmatrix},
\]

and

\[
M_1^{a_{11}} \ldots M_1^{a_{1m}} L_1^{b_{1m}} M_1^{a_{1(m+1)}} L_1^{b_{1(m+1)}} \ldots M_1^{a_{1n}} L_1^{b_{1n}} = 1,
\]

\[
\vdots
\]

\[
M_1^{a_{m1}} \ldots M_1^{a_{mm}} L_1^{b_{mm}} M_1^{a_{m(m+1)}} L_1^{b_{m(m+1)}} \ldots M_1^{a_{mn}} L_1^{b_{mn}} = 1,
\]

\[
M_1^{a_{(m+1)(m+1)}} L_1^{b_{(m+1)(m+1)}} \ldots M_1^{a_{(m+1)n}} L_1^{b_{(m+1)n}} = 1,
\]

\[
\vdots
\]

\[
M_1^{a_{l(m+1)}} L_1^{b_{l(m+1)}} \ldots M_1^{a_{ln}} L_1^{b_{ln}} = 1.
\]

respectively. Also (6.37) is simplified as

\[
\begin{pmatrix}
  a_{11} + \tau_1 b_{11} & \ldots & a_{1m} + \tau_m b_{1m} & a_{1(m+1)} + \tau_{m+1} b_{1(m+1)} & \ldots & a_{1n} + \tau_n b_{1n} \\
  \vdots & \ddots & \ddots & \ddots & \vdots & \ldots & \vdots \\
  a_{m1} + \tau_1 b_{m1} & \ldots & a_{mm} + \tau_m b_{mm} & a_{m(m+1)} + \tau_{m+1} b_{m(m+1)} & \ldots & a_{mn} + \tau_n b_{mm} \\
  0 & \ldots & 0 & a_{(m+1)(m+1)} + \tau_{m+1} b_{(m+1)(m+1)} & \ldots & a_{(m+1)n} + \tau_n b_{(m+1)n} \\
  \vdots & \ddots & \ddots & \ddots & \vdots & \ldots & \vdots \\
  0 & \ldots & 0 & a_{l(m+1)} + \tau_{m+1} b_{l(m+1)} & \ldots & a_{ln} + \tau_n b_{ln} \\
\end{pmatrix}.
\]

(a) First if the rank of the following submatrix

\[
\begin{pmatrix}
  a_{11} + \tau_1 b_{11} & \ldots & a_{1m} + \tau_m b_{1m} \\
  \vdots & \ddots & \ddots \\
  a_{m1} + \tau_1 b_{m1} & \ldots & a_{mm} + \tau_m b_{mm} \\
\end{pmatrix}
\]

of (6.40) is equal to \( m \), then the rank of

\[
\begin{pmatrix}
  a_{(m+1)(m+1)} + \tau_{m+1} b_{(m+1)(m+1)} & \ldots & a_{(m+1)n} + \tau_n b_{(m+1)n} \\
  \vdots & \ddots & \ddots \\
  a_{l(m+1)} + \tau_{m+1} b_{l(m+1)} & \ldots & a_{ln} + \tau_n b_{ln} \\
\end{pmatrix}
\]

of (6.40) is equal to \( m \), then the rank of

\[
\begin{pmatrix}
  a_{(m+1)(m+1)} + \tau_{m+1} b_{(m+1)(m+1)} & \ldots & a_{(m+1)n} + \tau_n b_{(m+1)n} \\
  \vdots & \ddots & \ddots \\
  a_{l(m+1)} + \tau_{m+1} b_{l(m+1)} & \ldots & a_{ln} + \tau_n b_{ln} \\
\end{pmatrix}
\]

is also equal to \( m \).
is strictly less than $m$ (otherwise, it contradicts the fact that the rank of $\mathbf{6.40}$ is strictly less than $l$). So if $H'$ is an algebraic subgroup defined by

$$M_{m+1}^{a(m+1)(m+1)} L_{m+1}^{b(m+1)(m+1)} \cdots M_n^{a(m+1)n} L_n^{b(m+1)n} = 1,$$

$$M_{m+1}^{a(m+1)} L_{m+1}^{b(m+1)} \cdots M_n^{a(n)} L_n^{b(n)} = 1,$$

the component of $\mathcal{X} \cap H'$ containing $z^0$ is an anomalous subvariety of $\mathcal{X}$. Since $H'$ is an algebraic subgroup of codimension $l - m$, by the induction hypothesis, this anomalous subvariety is contained in

$$M_i = L_i = 1 \quad (6.43)$$

for some $1 \leq i \leq n$.

(b) Now suppose the rank of $\mathbf{6.41}$ is strictly less than $m$, and consider the following submatrix of $M$:

$$\begin{pmatrix}
  a_{11} & b_{11} & \ldots & b_{1m} \\
  a_{21} & b_{21} & \ldots & b_{2m} \\
  \vdots & \ddots & \ddots & \vdots \\
  a_{m1} & b_{m1} & \ldots & b_{mm}
\end{pmatrix} \quad (6.44)$$

By Lemma $6.3$, there exists

$$\{a_{i_1}, b_{i_1}, \ldots, a_{i_{m'}}, b_{i_{m'}}\} \quad (6.45)$$

where $\{i_1, \ldots, i_{m'}\} \subset \{1, \ldots, m\}$ such that the dimension of the vector space spanned by $\mathbf{6.45}$ is strictly less than $m'$. Without loss of generality, we assume

$$\{i_1, \ldots, i_{m'}\} = \{1, \ldots, m'\},$$

and, applying Gauss elimination if necessary, $\mathbf{6.49}$ is a matrix of the following form:

$$\begin{pmatrix}
  a_{11} & \ldots & b_{1m'} & a_{1(m'+1)} & \ldots & b_{1m} \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
  a_{m'1} & \ldots & b_{m'm'} & a_{m'(m'+1)} & \ldots & b_{m'm} \\
  0 & \ldots & 0 & a_{(m'+1)(m'+1)} & \ldots & b_{(m'+1)m} \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
  0 & \ldots & 0 & a_{m(m'+1)} & \ldots & b_{mm}
\end{pmatrix} \quad (6.46)$$
Note that, in this case, (6.40) is of the following form:

\[
\begin{pmatrix}
    a_{11} & \cdots & b_{1m'} & a_{1(m'+1)} & \cdots & b_{1m} & a_{1(m+1)} & \cdots & b_{1n} \\
    \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
    a_{m'1} & \cdots & b_{m'm'} & a_{m'(m'+1)} & \cdots & b_{m'm} & a_{m'(m+1)} & \cdots & b_{m'n} \\
    0 & \cdots & 0 & a_{(m'+1)(m'+1)} & \cdots & b_{(m'+1)m} & a_{(m'+1)(m+1)} & \cdots & b_{(m'+1)n} \\
    \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
    0 & \cdots & 0 & a_{m(m+1)} & \cdots & b_{mm} & a_{m(m+1)} & \cdots & b_{mn} \\
    0 & \cdots & \cdots & \cdots & \cdots & 0 & a_{(m+1)(m+1)} & \cdots & b_{(m+1)n} \\
    \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
    0 & \cdots & \cdots & \cdots & \cdots & 0 & a_{l(m+1)} & \cdots & b_{ln} \\
\end{pmatrix}
\]

(6.47)

Now we repeat a similar procedure. First if the rank of

\[
\begin{pmatrix}
    a_{11} + b_{11} \tau_1 & \cdots & a_{1m'} + b_{1m'} \tau_{m'} \\
    \vdots & \ddots & \ddots \\
    a_{m'1} + b_{m'1} \tau_1 & \cdots & a_{m'm'} + b_{m'm'} \tau_{m'} \\
\end{pmatrix}
\]

is equal to \(m'\), then the rank of

\[
\begin{pmatrix}
    a_{m'+1} + \tau_m b_{m(m'+1)} & \cdots & a_{m(m'+1)+m} + \tau_m b_{m(m'+1)m} & a_{m(m'+1)(m+1)} + \tau_{m+1} b_{m(m'+1)(m+1)} & \cdots & a_{m(m'+1)n} + \tau_{m+1} b_{m(m'+1)n} \\
    \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
    a_{m(m'+1)} + \tau_m b_{m(m'+1)} & \cdots & a_{m(m+1)} + \tau_{m+1} b_{m(m+1)} & a_{m(m'+1)(m+1)} + \tau_{m+1} b_{m(m'+1)(m+1)} & \cdots & a_{m(m'+1)n} + \tau_{m+1} b_{m(m'+1)n} \\
    \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
    0 & \cdots & 0 & a_{l(m+1)} + \tau_m b_{l(m+1)} & \cdots & a_{l(m+1)n} + \tau_{l} b_{l(m+1)n} \\
\end{pmatrix}
\]

is strictly less than \(l - m'\). (Otherwise it contradicts the fact that the rank of (6.40) (or (6.47)) is strictly less than \(l\)) So if we let \(H'\) be an algebraic subgroup defined by

\[
M_{m'+1}^{a_{m'+1}(m'+1)} L_{m'+1}^{b_{m'(m'+1)(m'+1)}} \cdots M_{m+1}^{a_{m+1}(m+1)} L_{m+1}^{b_{m'(m+1)(m+1)}} \cdots M_n^{a_{m+1}(m+1)n} L_n^{b_{m'(m+1)n}} = 1,
\]

\[
\cdots
\]

\[
M_{m'+1}^{a_{m}(m'+1)} L_{m'+1}^{b_{m'(m+1)}} \cdots M_{m+1}^{a_{m+1}(m+1)} L_{m+1}^{b_{m(m+1)}} \cdots M_n^{a_{m+1}(m+1)n} L_n^{b_{m(m+1)n}} = 1,
\]

\[
\cdots
\]

\[
M_{m+1}^{a_{m}(m+1)} L_{m+1}^{b_{m(m+1)}} \cdots M_n^{a_{m+1}(m+1)n} L_n^{b_{m(m+1)n}} = 1,
\]

the component of \(X \cap H'\) containing \(z^0\) is an anomalous subvariety of \(X\). Since \(H'\) is an algebraic subgroup of codimension \(l - m'\), by the induction hypothesis, we get the desired result. If the rank of (6.48) is strictly less than \(m'\), then we
apply Lemma [6.4] to
\[
\begin{pmatrix}
  a_{11} & b_{11} & \cdots & b_{1m'} \\
  a_{21} & b_{21} & \cdots & b_{2m'} \\
  \vdots & \ddots & \ddots & \vdots \\
  a_{m1} & b_{m1} & \cdots & b_{mm'}
\end{pmatrix}
\] (6.49)

and repeat the same procedure again. Clearly these procedures terminate after a certain finite number of times and we get the desired conclusion eventually.

Combining the above theorem with the Zilber-Pink conjecture, we prove the following theorem which is a generalization of Theorem [5.2].

**Theorem 6.14.** Let $M$ be an $n$-cusped ($n \geq 3$) hyperbolic 3-manifold having rationally independent cusp shapes. Let
\[
\left\{ t_{(p_1/q_1, \ldots, p_n/q_n)}^1, \ldots, t_{(p_1/q_1, \ldots, p_n/q_n)}^n \right\}
\] (6.50)
be the set of holonomies of the Dehn filling coefficient $(p_1/q_1, \ldots, p_n/q_n)$ with $|p_i| + |q_i| (1 \leq i \leq n)$ sufficiently large. If the Zilber-Pink conjecture is true, then the elements in (6.50) are multiplicatively independent.

**Proof.** To simplify the notation we denote
\[
t_{(p_1/q_1, \ldots, p_n/q_n)}^1, \ldots, t_{(p_1/q_1, \ldots, p_n/q_n)}^n
\]
by $t_1, \ldots, t_n$ respectively. Let
\[
P = (t_1^{-q_1}, t_1^{p_1}, \ldots, t_n^{-q_n}, t_n^{p_n})
\]
be the Dehn filling point associated with the given Dehn filling coefficient $(p_1/q_1, \ldots, p_n/q_n)$. Since $t_1, \ldots, t_n$ are multiplicatively dependent, there exist $m_1, \ldots, m_n \in \mathbb{Z}$ such that
\[
t_1^{m_1} \cdots t_n^{m_n} = 1,
\] (6.51)
and thus $P$ is contained in
\[
(M_1^{-r_1} L_1^{r_1})^{m_1} \cdots (M_n^{-r_n} L_n^{r_n})^{m_n} = 1
\] (6.52)
where $-q_ir_i + p_is_i = 1$ for each $1 \leq i \leq n$. In other words, $P$ lies in the intersection between an $n$-dimensional algebraic variety $\mathcal{X}$ and an algebraic subgroup of codimension $n + 1$. By the Zilber-Pink conjecture, it is contained in one of a finite list of anomalous subvarieties of $\mathcal{X}$. Since $|p_i| + |q_i| (1 \leq i \leq n)$ are sufficiently large, without loss of generality, we assume $P$ is contained in an anomalous subvariety of $\mathcal{X}$ containing $z^0$. By Theorem 6.13 every anomalous subvariety of $\mathcal{X}$ containing $z^0$ is contained in
\[
M_i = L_i = 1
\] for some $i$, but this contradicts the fact that $t_i \neq 1$ for any $1 \leq i \leq n$. \[\square\]
6.2. Proof of Theorem 1.5

Now we prove Theorem 1.5 which we restate below:

**Theorem 6.15.** Let $\mathcal{M}$ be an $n$-cusped ($n \geq 2$) hyperbolic 3-manifold having non-quadratic and pairwise rationally independent cusp shapes. Let

$$\left\{ t_{(p_1/q_1,\ldots,p_n/q_n)}^1,\ldots,t_{(p_1/q_1,\ldots,p_n/q_n)}^n \right\}$$

where $|t_{(p_1/q_1,\ldots,p_n/q_n)}^i| > 1$ be the set of holonomies corresponding to the core geodesics of $(p_1/q_1,\ldots,p_n/q_n)$-Dehn filling. If the Zilber-Pink conjecture is true, then

$$\left\{ t_{(p_1/q_1,\ldots,p_n/q_n)}^1,\ldots,t_{(p_1/q_1,\ldots,p_n/q_n)}^n \right\} \subset \mathcal{M}(p_1/q_1,\ldots,p_n/q_n)$$

if and only if

$$(p_1/q_1,\ldots,p_n/q_n) = (p_1'/q_1',\ldots,p_n'/q_n')$$

for sufficiently large $|p_1| + |q_1|$ and $|p_i'| + |q_i'|$ ($1 \leq i \leq n$).

**Proof.** Similar to the previous theorems, we consider the situation as follows. Suppose $(p_{1i}/q_{1i},\ldots,p_{ni}/q_{ni}) \in \mathcal{I}$ and $(p_{1i}'/q_{1i}',\ldots,p_{ni}'/q_{ni}') \in \mathcal{I}$ are two infinite sequences such that

$$\mathcal{M}(p_{1i}/q_{1i},\ldots,p_{ni}/q_{ni}) \cong \mathcal{M}(p_{1i}'/q_{1i}',\ldots,p_{ni}'/q_{ni}')$$

but

$$(p_{1i}/q_{1i},\ldots,p_{ni}/q_{ni}) \neq (p_{1i}'/q_{1i}',\ldots,p_{ni}'/q_{ni}).$$

(6.53)

Denote the sets of holonomies of $\mathcal{M}(p_{1i}/q_{1i},\ldots,p_{ni}/q_{ni})$ and $\mathcal{M}(p_{1i}'/q_{1i}',\ldots,p_{ni}'/q_{ni}')$ by

$$\{t_{1i},\ldots,t_{ni}\} \quad \text{and} \quad \{t_{1i}',\ldots,t_{ni}'\}$$

respectively. Since

$$\{t_{1i},\ldots,t_{ni}\} = \{t_{1i}',\ldots,t_{ni}'\}$$

for each $i \in \mathcal{I}$, there exists $\sigma \in S_n$ such that

$$t_{1i} = t_{1i}' \sigma(1)i; \ldots; t_{ni} = t_{ni}' \sigma(n)i.$$

Let

$$P_i = \left( t_{1i}^{-q_{1i}}, t_{1i}'^{p_{1i}}, \ldots, t_{ni}^{-q_{ni}}, t_{ni}'^{p_{ni}}, (t_{1i}')^{-q_{1i}'}, (t_{1i}')^{p_{1i}'}, \ldots, (t_{ni}')^{-q_{ni}'}, (t_{ni}')^{p_{ni}'} \right)$$

be the cosmetic surgery point associated with $\mathcal{M}(p_{1i}/q_{1i},\ldots,p_{ni}/q_{ni})$ and $\mathcal{M}(p_{1i}'/q_{1i}',\ldots,p_{ni}'/q_{ni}')$. Since $P_i$ is an intersection point between a $(2n)$-dimensional algebraic variety $\mathcal{X} \times \mathcal{X}$ and an algebraic subgroup $H_i$ of codimension $3n$ defined by

$$M_j^{P_{1i}j} L_j^{q_{1i}j} = 1 \quad (1 \leq j \leq n),$$

$$(M_{\sigma(1)i})^{p_{1i}'(1)i}(L_{\sigma(1)i})^{q_{1i}'(1)i} = 1,$$

$$(M_j^{P_{1i}j} L_j^{q_{1i}j} = (M_{\sigma(1)i})^{p_{1i}'(1)i}(L_{\sigma(1)i})^{q_{1i}'(1)i}\right)$$

where $-q_{1i}r_{1i} + p_{1i}s_{1i} = -q_{1i}'r_{1i}' + p_{1i}'s_{1i}'$ and $\sigma(1)i = 1$, by the Zilber-Pink conjecture, there exists an algebraic subgroup $H$ containing infinitely many $P_i$. We assume $H$ is an algebraic subgroup of the smallest dimension containing infinitely many $P_i$ and it is defined by

$$M_1^{a_{1i}1} L_1^{b_{1i}1} (M_1')^{a_{1i}'1}(L_1')^{b_{1i}'1} \ldots M_n^{a_{ni}1} L_n^{b_{ni}1} (M_n')^{a_{ni}'1}(L_n')^{b_{ni}'1} = 1,$$

$$\ldots$$

(6.55)
By Lemma 5.2, \( t_{i_1}, \ldots, t_{n_i} \) are multiplicatively independent each other, so each \( P_i \) is contained in

\[
M_{j_1}^{a_{j_1}} L_{j_1}^{b_{j_1}} (M'_{\sigma(j_1)})^{\alpha_{\sigma(j_1)}(j_1)} (L'_{\sigma(j_1)})^{\beta_{\sigma(j_1)}(j_1)} = 1
\]

where \( 1 \leq l \leq m \) and \( 1 \leq j \leq n \). By Claim 5.17, the dimension of the following set is at most 2 for each \( j \):

\[
\{(a_{jk}, b_{jk}, a'_{jk}, b'_{jk}) \mid 1 \leq k \leq n \}. \tag{6.57}
\]

Now we split

\[
\{1, \ldots, n\}
\]

into two parts. First let \( J_1 \) be a subset of \( \{1, \ldots, n\} \) such that the dimension of \( (6.57) \) is 2 for each \( j \in J_1 \). Similarly we define \( J_2 \) as a subset of \( \{1, \ldots, n\} \) such that the dimension of \( (6.57) \) is 1 for each \( j \in J_2 \). As \( H \) is an algebraic subgroup of the smallest dimension containing infinitely many \( P_i \), it is defined by

\[
M_{j_1}^{a_{j_1}} L_{j_1}^{b_{j_1}} (M'_{\sigma(j_1)})^{\alpha_{\sigma(j_1)}(j_1)} (L'_{\sigma(j_1)})^{\beta_{\sigma(j_1)}(j_1)} = 1,
\]

\[
M_{j_2}^{a_{j_2}} L_{j_2}^{b_{j_2}} (M'_{\sigma(j_2)})^{\alpha_{\sigma(j_2)}(j_2)} (L'_{\sigma(j_2)})^{\beta_{\sigma(j_2)}(j_2)} = 1.
\]

where \( j_1 \in J_1, j_2 \in J_2 \) and \( J_1 \cup J_2 \subset \{1, \ldots, n\} \). Recall the definition of \( H_i \) given in \( (6.58) \) and note that

\[
H_i \subset H
\]

for infinitely many \( i \in \mathcal{I} \).

Claim 6.16. If \( j_1 \in J_1 \), then \( j_1 = \sigma(j_1) \).

Proof. Let \( K_{j_1} \) be an algebraic subgroup defined by

\[
M_{j_1}^{a_{j_1}} L_{j_1}^{b_{j_1}} (M'_{\sigma(j_1)})^{\alpha_{\sigma(j_1)}(j_1)} (L'_{\sigma(j_1)})^{\beta_{\sigma(j_1)}(j_1)} = 1,
\]

\[
M_{j_2}^{a_{j_2}} L_{j_2}^{b_{j_2}} (M'_{\sigma(j_2)})^{\alpha_{\sigma(j_2)}(j_2)} (L'_{\sigma(j_2)})^{\beta_{\sigma(j_2)}(j_2)} = 1.
\]

Then \( (\mathcal{X} \times \mathcal{X}) \cap K_{j_1} \) is locally biholomorphic (near \( z_0 \)) to the complex manifold defined by

\[
a_{j_1} u_{j_1} + b_{j_1} \tau_{j_1} u_{j_1} + \cdots + a'_{\sigma(j_1)} u'_{\sigma(j_1)} + b'_{\sigma(j_1)} \tau_{\sigma(j_1)} u'_{\sigma(j_1)} + \cdots = 0,
\]

\[
a_{j_2} u_{j_2} + b_{j_2} \tau_{j_2} u_{j_2} + \cdots + a'_{\sigma(j_2)} u'_{\sigma(j_2)} + b'_{\sigma(j_2)} \tau_{\sigma(j_2)} u'_{\sigma(j_2)} + \cdots = 0. \tag{6.60}
\]

The Jacobian of \( (6.60) \) at \( (0, \ldots, 0) \) is

\[
\begin{pmatrix}
-a_{j_1} + b_{j_1} \tau_{j_1} & a'_{\sigma(j_1)} + b'_{\sigma(j_1)} \\
-a_{j_2} + b_{j_2} \tau_{j_2} & a'_{\sigma(j_2)} + b'_{\sigma(j_2)}
\end{pmatrix}. \tag{6.61}
\]

If \( j_1 \neq \sigma(j_1) \), then the rank of \( (6.61) \) is 2 by Lemma 4.4. By the implicit function theorem and \( (6.60) \), we get

\[
u_{j_1} = u'_{\sigma(j_1)} = 0.
\]

In other words, the component of \( (\mathcal{X} \times \mathcal{X}) \cap K_{j_1} \) containing \( (z^0, z^0) \) is contained in

\[
M_{j_1} = M'_{\sigma(j_1)} = 1.
\]

But this contradicts the fact that \( t_{j_1} \neq 1 \) and \( t_{\sigma(j_1)} \neq 1 \). \( \square \)

\[23\text{Otherwise, it contradicts the fact that none of the coordinates of } P_i \text{ is cyclotomic.} \]
Now we rewrite (6.59) as follows:
\[ M_{j_1}^{a_{j_1}^1} L_j^{b_j^{11}} (M'_{j_1})^{a_{j_1}^2} (L'_{j_1})^{b_{j_1}^2} = 1, \]  
(6.62)
\[ M_{j_1}^{a_{j_1}^1} L_j^{b_j^{21}} (M'_{j_1})^{a_{j_1}^2} (L'_{j_1})^{b_{j_1}^2} = 1, \]  
(6.63)
\[ M_{j_2}^{a_{j_2}'} L_j^{b_j^{'21}} (M'_{j_2})^{a_{j_2}^2} (L'_{j_2})^{b_{j_2}^2} = 1, \]
where \( j_1 \in J_1 \) and \( j_2 \in J_2 \).

**Claim 6.17.** We further simplify (6.62)-(6.63) as
\[ M_{j_1} = M'_{j_1}, \quad L_{j_1} = L'_{j_1}. \]
or
\[ M_{j_1} = (M'_{j_1})^{-1}, \quad L_{j_1} = (L'_{j_1})^{-1}. \]

**Proof.** Let \( K_{j_1} \) be an algebraic group defined by (6.62) and (6.63). Then \((X \cap X) \cap K_{j_1}\) is locally biholomorphic (near \( z^0 \)) to
\[
\begin{align*}
  a_{j_1} u_{j_1} + b_{j_1} (\tau_{j_1} u_{j_1} + \cdots) + a'_{j_1} u_{j_1} + b'_{j_1} (\tau_{j_1} u_{j_1} + \cdots) &= 0, \\
  a_{j_2} u_{j_1} + b_{j_2} (\tau_{j_2} u_{j_1} + \cdots) + a'_{j_2} u_{j_1} + b'_{j_2} (\tau_{j_2} u_{j_1} + \cdots) &= 0,
\end{align*}
\]
(6.64)
and the Jacobian of (6.64) at \((0, \ldots, 0)\) is
\[
\begin{pmatrix}
  a_{j_1} + b_{j_1} \tau_{j_1} & a'_{j_1} + b'_{j_1} \tau_{j_1} \\
  a_{j_2} + b_{j_2} \tau_{j_1} & a'_{j_2} + b'_{j_2} \tau_{j_1}
\end{pmatrix}.
\]
(6.65)
If the rank of (6.65) is 2, then, by the implicit function theorem, (6.64) is equivalent to
\[ u_{j_1} = u'_{j_1} = 0, \]
which implies that the component of \((X \cap X) \cap K_{j_1}\) containing \((z^0, z^0)\) is contained in
\[ M_{j_1} = M'_{j_1} = 1. \]
But this contradicts the fact that \( t_{j_1} \neq 1 \) and \( t'_{j_1} \neq 1 \) for each \( j_1 \). Thus the rank of (6.65) is 1, and, by Lemma 4.2, we get either
\[
\begin{pmatrix}
  a_{j_1} & b_{j_1} \\
  a_{j_2} & b_{j_2}
\end{pmatrix} = \begin{pmatrix}
  a'_{j_1} & b'_{j_1} \\
  a'_{j_2} & b'_{j_2}
\end{pmatrix}
\]
or
\[
\begin{pmatrix}
  a_{j_1} & b_{j_1} \\
  a_{j_2} & b_{j_2}
\end{pmatrix} = \begin{pmatrix}
  -a'_{j_1} & -b'_{j_1} \\
  -a'_{j_2} & -b'_{j_2}
\end{pmatrix}
\]
for each \( j_1 \in J_1 \). This implies (6.62)-(6.63) are equivalent to either
\[ M_j^{a_{j_1}^1} L_j^{b_j^{11}} = M_j^{a_{j_1}^1} L_j^{b_j^{11}}, \]
or
\[ M_j^{a_{j_1}^1} L_j^{b_j^{11}} = M_j^{a_{j_1}^1} L_j^{b_j^{11}}, \]
\[ M_j^{a_{j_2}^1} L_j^{b_j^{12}} = M_j^{a_{j_2}^1} L_j^{b_j^{12}}, \]
\[ M_j^{a_{j_2}^1} L_j^{b_j^{12}} = M_j^{a_{j_2}^1} L_j^{b_j^{12}}. \]
Since \( a_{j_1} b_{j_2} - a_{j_2} b_{j_1} \neq 0 \), (6.62)-(6.63) are equivalent to either
\[ M_{j_1} = (M'_{j_1})^{-1}, \quad L_{j_1} = (L'_{j_1})^{-1}. \]
or

\[ M_{j_1} = M'_{j_1}, \quad L_{j_1} = L'_{j_1}. \]

Without loss of generality, we suppose \([6.59]\) is simplified as

\[ M_{j_1} = M'_{j_1}, \quad L_{j_1} = L'_{j_1}, \]

\[ M^{a_{j_2}L^{b_{j_2}}_{j_2}}(M'_{\sigma(j_2)})^{\nu_{\sigma(j_2)}}(L'_{\sigma(j_2)})^{\nu_{\sigma(j_2)}} = 1 \]

(6.66)

where \(j_1 \in J_1\) and \(j_2 \in J_2\). Now \((\mathcal{X} \times \mathcal{X}) \cap H\) is locally biholomorphic to

\[
\begin{align*}
    u_{j_1} &= u'_{j_1}, \quad v_{j_1} = v'_{j_1}, \\
    (a_{j_2} + b_{j_2} \tau_{j_2})(u_{j_2} + \cdots) + (a'_{\sigma(j_2)} + b'_{\sigma(j_2)} \tau_{\sigma(j_2)})(u'_{\sigma(j_2)} + \cdots),
\end{align*}
\]

which is equivalent to

\[ u_{j_2} = -\frac{a_{\sigma(j_2)} + b_{\sigma(j_2)} \tau_{\sigma(j_2)}}{a_{j_2} + b_{j_2} \tau_{j_2}}u'_{\sigma(j_2)} + \cdots \]

(6.67)

where \(j_1 \in J_1\) and \(j_2 \in J_2\). Clearly \([6.67]\) defines a complex manifold of dimension at most

\[ 2n - |J_1| - |J_2|, \]

and so \(H \cap (\mathcal{X} \times \mathcal{X})\) is an algebraic subvariety of dimension at most \(2n - |J_1| - |J_2|\).

Now consider

\[ H(\cong (\mathbb{C}^*)^{2k+4n-(2|J_1|+|J_2|)}) \]

as an ambient space and \(H \cap (\mathcal{X} \times \mathcal{X})\) as an algebraic variety in \((\mathbb{C}^*)^{2k+4n-(2|J_1|+|J_2|)}\). Since \(H \cap H_i\) (which is equal to \(H_i\)) is an algebraic group of dimension \(2k+n\) in \((\mathbb{C}^*)^{2k+4n-(2|J_1|+|J_2|)}\), if

\[ |J_1| < n, \]

then

\[ \dim (H \cap (\mathcal{X} \times \mathcal{X})) + \dim (H \cap H_i) \leq (2n - |J_1| - |J_2|) + (2k+n) = 2k + 3n - |J_1| - |J_2| \]

\[ < 2k + 4n - (2|J_1| + |J_2|). \]

(6.68)

In other words, the sum of the dimensions of \(H \cap (\mathcal{X} \times \mathcal{X})\) and \(H \cap H_i\) is strictly less than the dimension of the ambient space. Since \(H \cap (\mathcal{X} \times \mathcal{X})\) intersects with \(H \cap H_i\) at \(P_i\) for each \(i\), by the Zilber-Pink conjecture again, we find an algebraic subgroup \(H'(\subseteq \overline{H})\) such that \(H' \cap \mathcal{X} \times \mathcal{X}\) is an anomalous subvariety of \(H \cap (\mathcal{X} \times \mathcal{X})\) and \(H'\) contains infinitely many \(P_i\). However, this contradicts the fact that \(H\) is an algebraic subgroup of the smallest dimension containing infinitely many \(P_i\). So \(|J_1| = n\) and \(|J_2| = 0\). In other words, \(H\) is defined by

\[ M_j = M'_{j}, \quad L_j = L'_{j} \]

where \(1 \leq j \leq n\). Since each \(P_i\) is contained in \(H\), we conclude

\[ (p_{1i}/q_{1i}, \ldots, p_{ni}/q_{ni}) = (p'_{1i}/q'_{1i}, \ldots, p'_{ni}/q'_{ni}) \]

for infinitely many \(i\), which contradicts our initial assumption \([6.53]\). \(\square\)

\(^{24}\)Recall \(H\) is an algebraic subgroup of codimension \(|J_1| + |J_2|\) in \((\mathbb{C}^*)^{k+2n} \times (\mathbb{C}^*)^{k+2n} \).
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