Note on the Cardoso-Pani-Rico parametrization to test the Kerr black hole hypothesis

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The construction of a generic parametrization to describe the spacetime geometry around astrophysical black hole candidates is an important step to test the Kerr black hole hypothesis. In the last few years, the Johannsen-Psaltis metric has been the most common framework to study possible deviations from the Kerr solution with present and near future observations. Recently, Cardoso, Pani and Rico have proposed a more general parametrization. The aim of the present paper is to study this new metric in a specific context, namely the thermal spectrum of geometrically thin and optically thick accretion disks. The most relevant finding is that the spacetime geometry around objects that look like very fast-rotating Kerr black holes may still have large deviations from the Kerr solution. This is not the case with the Johannsen-Psaltis metric, which means the latter is missing an important class of non-Kerr spacetimes.

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Today, general relativity is quite well tested in the Earth gravitational field, in the Solar System, and by studying the orbital motion of pulsars in binary systems [1]. The validity of the theory in more extreme situations has still to be verified. For instance, the observed accelerated expansion of the Universe may indicate a breakdown of general relativity at large scales [2]. The strong field regime is another unexplored area and today there is an increasing interest in the possibility of testing the actual nature of astrophysical black hole (BH) candidates [3]. In classical general relativity, astrophysical BHs should be well described by the Kerr solution; for instance, the presence of accretion disks is usually completely negligible [4]. However, for the time being astrophysical BH candidates are just objects too heavy and compact to be neutron stars or clusters of neutron stars [5]. There is no evidence that the spacetime geometry around them is described by the Kerr metric. We rely on classical general relativity and we believe in the Kerr BH hypothesis. However, there are some novel theoretical arguments suggesting the possibility of macroscopic deviations from classical predictions around BHs [6].

The Kerr nature of astrophysical BH candidates can be potentially tested by studying the properties of the radiation emitted by the gas in the accretion disk. With this approach, one can check if the metric around the compact object is described by the Kerr solution, which determines the motion of the gas in the accretion disk and the propagation of the radiation from the disk to the distant observer, while it is not possible to distinguish Kerr BHs of general relativity from Kerr BHs in alternative theories of gravity [7]. The strategy is then to use a framework similar to the PPN (Parametrized Post-Newtonian) formalism [1], which has been very successful for tests of general relativity in the Solar System. The idea is to have a very general metric with a number of “deformation parameters” that measure possible deviations from the Kerr geometry. Like the traditional β and γ of the PPN formalism, the value of the deformation parameters has to be determined from observations, and a posteriori one can verify if the measurements are consistent with the predictions of general relativity. If we had a very general formalism, such a test-metric should be able to reduce to any metric describing the gravitational field around a compact object in any alternative theory of gravity for a suitable choice of the value of the deformation parameters. Unfortunately, such an approach in the strong field limit is much more complicated than its counterpart in the weak field regime, and today there is not yet a completely satisfactory formalism to test the Kerr BH hypothesis.

New parametrization — In the last few years, the most popular parametrization to test the Kerr nature of astrophysical BH candidates has been the Johannsen-Psaltis (JP) metric [8]. Such a metric has an infinite number of deformation parameters and reduces to the Kerr solution when all the deformation parameters are set to zero. Recently, Cardoso, Pani and Rico suggested a generalization of the JP parametrization. In Boyer-Lindquist coordinates, the Cardoso-Pani-Rico (CPR) parametrization reads [9]

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\begin{align*}
\text{ds}^2 &= -\left(1 - \frac{2Mr}{\Sigma} \right) (1 + h^t) \, dt^2 - 2a \sin^2 \theta \left[ \sqrt{(1 + h^t)(1 + h^r)} - \left(1 - \frac{2Mr}{\Sigma} \right) (1 + h^t) \right] \, dtd\phi \\
&\quad + \frac{\Sigma (1 + h^r)}{\Delta + h^t a^2 \sin^2 \theta} \, dr^2 + \Sigma d\theta^2 + \sin^2 \theta \left[ \Sigma + a^2 \sin^2 \theta \left[ 2\sqrt{(1 + h^t)(1 + h^r)} - \left(1 - \frac{2Mr}{\Sigma} \right) (1 + h^t) \right] \right] \, d\phi^2.
\end{align*}
\]

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where $M$ is the BH mass, $a = J/M$ the BH spin parameter, $J$ the BH spin angular momentum, $\Sigma = r^2 + a^2 \cos^2 \theta$, $\Delta = r^2 - 2Mr + a^2$, and

\[ h^t = \sum_{k=0}^{+\infty} \left( \epsilon_{2k} + \epsilon_{2k+1} \frac{Mr}{\Sigma} \right) \left( \frac{M^2}{\Sigma} \right)^k, \quad h^r = \sum_{k=0}^{+\infty} \left( \epsilon_{2k} + \epsilon_{2k+1} \frac{Mr}{\Sigma} \right) \left( \frac{M^2}{\Sigma} \right)^k. \] (2)

Here we have two infinite sets of deformation parameters, namely $\{\epsilon_k^t\}$ and $\{\epsilon_k^r\}$. The CPR parametrization reduces to the JP one for $h^t = h^r$ and to the Kerr solution when $h^t = h^r = 0$. The aim of the present paper is to figure out the possible advantages, if any, of the new metric to test the Kerr BH hypothesis.

**Constraints** — For the time being, the most trustworthy technique to probe the spacetime geometry around astrophysical BH candidates is the so-called continuum-fitting method; that is, the study of the thermal spectrum of a geometrically thin and optically thick accretion disk [10]. While it has been developed to measure the spin parameter of BH candidates under the assumption of the Kerr background, this technique can be easily extended to test the Kerr BH hypothesis [11]. With already available X-ray data, the nature of astrophysical BH candidates can be investigated even with the analysis of the iron Kα line [12], but the fact that the model has several parameters to be measured during the fitting procedure makes this technique more tricky, and at present it can just be used to rule out some very exotic spacetimes with qualitatively different iron lines [13]. Other approaches, like the study of QPOs [14] or the estimate of the jet power [15], are model-dependent and not yet mature to test fundamental physics, while high resolution observations of the accretion flow around the supermassive BH candidate in the Milky Way are not yet available [16].

At present, there are 10 stellar-mass BH candidates for which the current data allow us to get reliable constraints from the continuum-fitting method. In case we assume that the spacetime around these objects is described by the Kerr solution, it is possible to estimate the spin parameter. If we relax the Kerr BH hypothesis, it is usually possible to constrain some combination of the spin and of possible deviations from the Kerr solution, because of the strong correlation between them. Recently, the measurements of these 10 stellar-mass BH candidates have been reconsidered in the JP framework and the constraints on the dimensionless spin parameter $a_\ast = a/M$ and JP deformation parameter $\epsilon_3$ have been obtained [17]. Here we want to use the same procedure to the CPR parametrization to figure out possible differences and advantages.

Instead of reconsidering all of the 10 stellar-mass BH candidates, for the purpose of the present paper it is enough to study two extreme cases, namely an object that looks like a slow-rotating Kerr BH and one that seems to be a very fast-rotating Kerr BH. For the former case, a suitable candidate is A0620-00, whose spin has been estimated to be $a_\ast = 0.12 \pm 0.19$ (at 1σ) under the assumption of the Kerr background [19]. Proceeding as described in [17], one can translate the best value into a line on the spin–deformation parameter plane, and the uncertainty into an allowed region. The results of this analysis are reported in Fig. 1 where the left panel assumes the CPR deformation parameter $\epsilon_3^t$ and all the others are set to zero, while in the right panel the only non-vanishing deformation parameter is $\epsilon_3^r$. As we can see, $a_\ast$ and $\epsilon_3^t$ are strongly correlated, while there is almost no correlation between $a_\ast$ and $\epsilon_3^r$. Like in the JP parametrization, it is not possible to constrain the deformation parameters from an object that looks like a slow rotating Kerr BH, because observations cannot rule out large deviations from the Kerr solution.

The case of an object that looks like a very fast-rotating Kerr BH is more interesting. Within the JP framework, such an object can be used to constrain the deformation parameter; that is, it is possible to put an upper and a lower bound on the deformation parameter independently of the BH spin [17]. This is not a peculiar feature of the JP background, but it is met even in specific BH solutions [20]. In this case, we consider the BH candidate in Cygnus X-1, which has been studied in Ref. [21] and found that $a_\ast > 0.98$ (at 3σ and assuming the Kerr metric). Within the JP parametrization with $\epsilon_3$ as the only non-vanishing deformation parameter, one finds the bound $0 \lesssim \epsilon_3 \lesssim 4$ [17]. That is interesting because the bound is independent of the BH spin, and it is thus possible to say that deviations from the Kerr geometry, if any, cannot be too large. If it could be possible to

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1 In the JP metric, $\epsilon_0 = 0$ is required by asymptotic flatness, while $\epsilon_1$ and $\epsilon_2$ must be small to satisfy the Solar System tests [8]. $\epsilon_3$ is the first deformation parameter without constraints and there are no qualitative differences between $\epsilon_3$ and higher order terms [18].

2 Actually, the leading order term of $h^r$, which is not bounded by Solar System tests, is proportional to $\epsilon_3^r$. Here we consider $\epsilon_3^r$ in order to have the same corrections in $h^t$ and $h^r$, as well as a simple link to the JP metric, whose unbounded leading order correction is the term with $\epsilon_3$ and here it is recovered for $\epsilon_3^r = \epsilon_3^t$. Moreover, there are no qualitatively different constraints on $\epsilon_3^t$, $\epsilon_3^r$, or higher order deformation parameters from thin disks. In the case of the JP parametrization, Figs. 2 and 4 of Ref. [18] show that the effect of any deformation parameter is qualitatively the same. That is true even with the CPR parametrization, namely the same qualitative conclusions hold for any $\epsilon_n^t$, or $\epsilon_n^r$, if we study the constraint for a specific $n$.  

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FIG. 1. Disk thermal spectrum constraints on possible deviations from the Kerr geometry in the spacetime around the BH candidate in A0620-00. Left panel: constraints on $a_*$ and $\epsilon_3'$ assuming that all the other deformation parameters vanish. Right panel: constraints on $a_*$ and $\epsilon_3'$ assuming that all the other deformation parameters vanish. The red solid line indicates the spacetimes that, for a fixed deformation parameter, have the disk’s spectrum more similar to the one of a Kerr BH with spin $a_* = 0.12$. The blue dashed lines are the $1\sigma$ boundary.

FIG. 2. Disk thermal spectrum constraints on possible deviations from the Kerr geometry in the spacetime around the BH candidate in Cygnus X-1. Left panel: constraints on $a_*$ and $\epsilon_3'$ assuming that all the other deformation parameters vanish. Right panel: constraints on $a_*$ and $\epsilon_3'$ assuming that all the other deformation parameters vanish. The blue dashed lines are the boundaries separating objects that look more like Kerr BHs with $a_* > 0.98$ (internal area) and the ones that look more like Kerr BHs with $a_* < 0.98$ (external area).

get a stronger bound, say $a_* > 0.99$, the constraint on $\epsilon_3$ would become more stringent as well. Fig. 2 shows the constraints in the case of the CPR parametrization. In the left panel, it is assumed that the only non-vanishing deformation parameter is $\epsilon_3'$. In the right panel, the only non-vanishing deformation parameter is $\epsilon_3''$.

Comments — For Cygnus X-1, it seems that $\epsilon_3'$ can be weakly constrained, while $\epsilon_3''$ cannot be constrained at all, in the sense that observations cannot exclude large deviations from the Kerr solution. It is also understandable the constraint area of the JP parametrization found in [17], which is roughly the overlapping region from the constraint on $\epsilon_3'$ and on $\epsilon_3''$ (the JP metric with $\epsilon_3$ as the only non-vanishing deformation parameter corresponds to the case in which $\epsilon_3' = \epsilon_3''$ and all the other CPR deformation parameters vanish). However, Fig. 2 has been simply obtained by the comparison of the disk’s thermal spectrum in the corresponding backgrounds. Some of them may not be physical. For instance, for some values of the spin and the deformation parameters these spacetimes may have no event horizon, may have naked singularities, pathological regions with closed time-like curves, etc. However, these are test-metrics and we may argue that they hold up to some radius $r_*$, like in the case of exterior solutions of compact objects. In the end, we can only probe the spacetime at radii larger than the inner edge of the accretion disk, which is supposed to be at the innermost stable circular orbit (ISCO) radius. So, if we want to remain as general as possible, we have just to check that the accretion disk does not enter any
and it is plausible that objects on the right side of the red solid line are spun up. Since $E_{\text{ISCO}}$ and $L_{\text{ISCO}}$ are, respectively, the specific energy and the specific angular momentum at the ISCO.

The central object is spun up/down if the right hand side of Eq. (3) is positive/negative and the equilibrium energy and the specific angular momentum at the ISCO.

The aim of the present paper was to figure out some basic features of the new metric and see if it is indeed more useful than the JP parametrization.

The main feature of the CPR parametrization is the presence of two kinds of deformations, $h^r$ and $h^t$, which alter respectively the metric coefficients $g_{tt}$ and $g_{rr}$ in the non-rotating limit. It turns out that $h^t$ and $h^r$ have quite a different impact on possible observables. Here we have just considered the thermal spectrum of thin disks, which is today the most reliable technique to probe the spacetime geometry around BH candidates. For an object that looks like a non-rotating or a slow-rotating Kerr BH, there is a strong correlation between the spin and $h^t$, while there is almost no correlation between the spin and $h^r$. Moreover, it seems that $h^r$ cannot be constrained at all. In the case of an object that looks like a very-fast rotating Kerr BH, $h^t$ can be strongly constrained, independently of the value of the spin parameter. This was the case of the $h$ in the JP parametrization as well. On the contrary, $h^t$ remains difficult to constrain. In other words, even if we observe a BH candidate that seems to be a very fast-rotating Kerr BH, deviations from the Kerr geometry may be very large. Since this was not the case in the JP parametrization, the latter was missing an important class of non-Kerr objects and the new parametrization may be more convenient for future studies.

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FIG. 3. Spin-up/spin-down regions and ISCO stability. Objects on the left side of the red solid lines are spun up by a thin accretion disk on the equatorial plane, while objects on the right side are spun down. The blue dashed lines separate spacetimes in which the ISCO is marginally stable along the radial and the vertical direction. Left panel: plane $a_*$ and $\epsilon'_3$ assuming that all the other deformation parameters vanish; the ISCO radius is marginally radially stable except in the small area inside the blue dashed closed curve. Right panel: plane $a_*$ and $\epsilon'_r$ assuming that all the other deformation parameters vanish; for $\epsilon'_r > 0$, the ISCO radius is marginally radially stable on the left side of the blue dashed line, it is marginally vertically stable on the right side; for $\epsilon'_r < 0$, the ISCO radius is marginally radially stable.

[14] Z. Stuchlik and A. Kotrlova, Gen. Rel. Grav. 41, 1305 (2009) [arXiv:0912.5066 [astro-ph]]; T. Johannsen and D. Psaltis, Astrophys. J. 726, 11 (2011) [arXiv:1010.1009 [astro-ph.HE]]; C. Bambi, JCAP 1209, 014 (2012) [arXiv:1205.6348 [gr-qc]].

[15] C. Bambi, Phys. Rev. D 85, 043002 (2012) [arXiv:1201.1638 [gr-qc]]; C. Bambi, Phys. Rev. D 86, 123013 (2012) [arXiv:1204.6395 [gr-qc]].

[16] C. Bambi and K. Freese, Phys. Rev. D 79, 043002 (2009) [arXiv:0812.1328 [astro-ph]]; C. Bambi and N. Yoshida, Class. Quant. Grav. 27, 205006 (2010) [arXiv:1004.3149 [gr-qc]]; Z. Li, L. Kong and C. Bambi, Astrophys. J. 787, 152 (2014) [arXiv:1401.1252 [gr-qc]]; N. Tsukamoto, Z. Li and C. Bambi, JCAP 1406, 043 (2014) [arXiv:1403.0371 [gr-qc]] Z. Li and C. Bambi, Phys. Rev. D 90, 024071 (2014) [arXiv:1405.1883 [gr-qc]].

[17] L. Kong, Z. Li and C. Bambi, [arXiv:1405.1508 [gr-qc]].

[18] C. Bambi, Phys. Rev. D 85, 043001 (2012) [arXiv:1112.4663 [gr-qc]].

[19] L. Gou, J. E. McClintock, J. F. Steiner et al., Astrophys. J. 718, L122 (2010) [arXiv:1002.2211 [astro-ph.HE]].

[20] C. Bambi, Phys. Lett. B 730, 59 (2014) [arXiv:1401.4640 [gr-qc]].

[21] L. Gou, J. E. McClintock, M. J. Reid et al., Astrophys. J. 742, 85 (2011) [arXiv:1106.3690 [astro-ph.HE]]; L. Gou, J. E. McClintock, R. A. Remillard et al., Astrophys. J. 790, 29 (2014) [arXiv:1308.4760 [astro-ph.HE]].

[22] E. Barausse, V. Cardoso and G. Khanna, Phys. Rev. Lett. 105, 261102 (2010) [arXiv:1008.5159 [gr-qc]].

[23] C. Bambi, JCAP 1105, 009 (2011) [arXiv:1103.5135 [gr-qc]]; C. Bambi, Phys. Lett. B 705, 5 (2011) [arXiv:1110.0687 [gr-qc]].