On the Fourth Power Moment of Fourier Coefficients of Cusp Form

Jinjiang Li∗ & Panwang Wang† & Min Zhang‡
Department of Mathematics, China University of Mining and Technology∗†‡
Beijing 100083, P. R. China

Abstract: Let \( a(n) \) be the Fourier coefficients of a holomorphic cusp form of weight \( \kappa = 2n \geq 12 \) for the full modular group and \( A(x) = \sum_{n \leq x} a(n) \). In this paper, we establish an asymptotic formula of the fourth power moment of \( A(x) \) and prove that
\[
\int_{1}^{T} A^4(x)dx = \frac{3}{64\kappa \pi^4} s_{4;2}(\tilde{a}) T^{2\kappa} + O(T^{2\kappa - \delta_4 + \varepsilon})
\]
with \( \delta_4 = 1/8 \), which improves the previous result.

Keywords: Cusp form; Fourier coefficient; mean value; asymptotic formula

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1 Introduction and main result

Let \( a(n) \) be the Fourier coefficients of a holomorphic cusp form of weight \( \kappa = 2n \geq 12 \) for the full modular group. In 1974, Deligne [2] proved the following profound result
\[
a(n) \ll n^{(\kappa-1)/2} d(n),
\]
where \( d(n) \) denotes the Dirichlet divisor function and the implied constant in \( \ll \) is absolute. Suppose \( x \geq 2 \) and define
\[
A(x) := \sum_{n \leq x} a(n).
\]
It is well known that \( A(x) \) has no main term and \( A(x) \ll x^{\kappa/2-1/6} + \varepsilon \). In 1973, Joris [5] proved that
\[
A(x) = \Omega_{\pm} \left( x^{\kappa/2-1/4} \log \log \log x \right).
\]

†Corresponding author.

E-mail addresses: jinjiang.li.math@gmail.com (J. Li), panwangw@gmail.com (P. Wang), min.zhang.math@gmail.com (M. Zhang).
In 1990, Ivić [3] showed that there exist two points $t_1$ and $t_2$ in the interval $[T, T + CT^{1/2}]$ such that

$$A(t_1) > Bt_1^{\kappa/2-1/4}, \quad A(t_2) < -Bt_2^{\kappa/2-1/4},$$

where $B > 0$, $C > 0$ are constants. It is conjectured that

$$A(x) \ll x^{(\kappa-1)/2+1/4+\varepsilon}$$

is true for every $\varepsilon$. The evidence in support of this conjecture has been given by Ivić [3], who proved the following square mean value formula of $A(x)$, i.e.

$$\int_1^T A^2(x)dx = C_2 T^{\kappa+1/2} + B(T),$$

where

$$C_2 = \frac{1}{(4\kappa + 2)\pi^2} \sum_{n=1}^{\infty} a^2(n) n^{-\kappa-1/2},$$

$$B(T) \ll T^\kappa \log^5 T, \quad B(T) = \Omega\left(T^{\kappa-1/4} \frac{(\log \log \log T)^3}{\log T}\right).$$

In [3], Ivić also proved the upper bound of eighth power moment of $A(x)$, that is

$$\int_1^T A^8(x)dx \ll T^{4\kappa-1+\varepsilon}.$$  

Cai [1] studied the third and fourth power moments of $A(x)$. He proved that

$$\int_1^T A^3(x)dx = C_3 T^{(6\kappa+1)/4} + O(T^{(6\kappa+1)/4-\delta_3+\varepsilon}),$$  

$$\int_1^T A^4(x)dx = C_4 T^{2\kappa} + O(T^{2\kappa-\delta_4+\varepsilon}),$$  

where $\delta_3 = 1/14$, $\delta_4 = 1/23$ and

$$C_3 := \frac{3}{4(6\kappa + 1)\pi^3} \sum_{n,m,k \in \mathbb{N}} \frac{(nmk)^{-\kappa/2-1/4}a(n)a(m)a(k)},$$

$$C_4 := \frac{3}{64\kappa\pi^4} \sum_{n,m,k,\ell \in \mathbb{N}} \frac{(nmk\ell)^{-\kappa/2-1/4}a(n)a(m)a(k)a(\ell)}{\sqrt{n+m+\sqrt{m+n}+\sqrt{k+\ell}}}.$$

In [10], Zhai proved that (1.3) holds for $\delta_3 = 1/4$. Following the approach of Tsang [9], Zhai [10] proved that the equation (1.4) holds for $\delta_4 = 2/41$. This approach used the method of exponential sums. In particular, if the exponent pair conjecture is true, namely, if $(\varepsilon, 1/2 + \varepsilon)$ is an exponent pair, then the equation (1.4) holds for $\delta_4 = 1/14$. 

2
Later, combining the method of [4] and a deep result of Robert and Sargos [8], Zhai [12] proved that the equation (1.4) holds for $\delta_4 = 3/28$. By a unified approach, Zhai [11] proved that the asymptotic formula

$$\int_1^T A^k(x) dx = C_k T^{1+k(2\kappa-1)/4} + O(T^{1+k(2\kappa-1)/4 - \delta_k + \varepsilon})$$

holds for $3 \leq k \leq 7$, where $C_k$ and $0 < \delta_k < 1$ are explicit constants.

The aim of this paper is to improve the value of $\delta_4 = 3/28$, which is achieved by Zhai [12]. The main result is the following

**Theorem 1.1** We have

$$\int_1^T A^4(x) dx = \frac{3}{64\pi^4} s_{4,2}(\tilde{a}) T^{2\kappa} + O(T^{2\kappa - \delta_4 + \varepsilon})$$

with $\delta_4 = 1/8$, where

$$s_{4,2}(\tilde{a}) = \sum_{n, m, k, \ell \in \mathbb{N}^*} \frac{a(n)a(m)a(k)a(\ell)}{(nmk\ell)^{\kappa/2 + 1/4}}.$$

**Notation.** Throughout this paper, $a(n)$ be the Fourier coefficients of a holomorphic cusp form of weight $\kappa = 2n \geq 12$ for the full modular group; $d(n)$ denote the Dirichlet divisor function; $\tilde{a}(n) := a(n)n^{-\kappa/2 + 1/2}$; $\|x\|$ denotes the distance from $x$ to the nearest integer, i.e., $\|x\| = \min_{n \in \mathbb{Z}} |x - n|$. $[x]$ denotes the integer part of $x$; $n \sim N$ means $N < n \leq 2N$; $n \asymp N$ means $C_1 N \leq n \leq C_2 N$ with positive constants $C_1, C_2$ satisfying $C_1 < C_2$. $\varepsilon$ always denotes an arbitrary small positive constant which may not be the same at different occurrences. We shall use the estimates $d(n) \ll n^\varepsilon$. Suppose $f : \mathbb{N} \to \mathbb{R}$ is any function satisfying $f(n) \ll n^\varepsilon$, $k \geq 2$ is a fixed integer. Define

$$s_{k,\ell}(f) := \sum_{n_1, \ldots, n_k, n_{k+1}, \ldots, n_{k+\ell} \in \mathbb{N}^*} \frac{f(n_1)f(n_2) \cdots f(n_k)}{(n_1n_2 \cdots n_k)^{3/4}}, \quad 1 \leq \ell < k. \quad (1.5)$$

We shall use $s_{k,\ell}(f)$ to denote both of the series (1.5) and its value. Suppose $y > 1$ is a large parameter, and we define

$$s_{k,\ell}(f; y) := \sum_{n_1, \ldots, n_k, n_{k+1}, \ldots, n_{k+\ell} \leq y} \frac{f(n_1)f(n_2) \cdots f(n_k)}{(n_1n_2 \cdots n_k)^{3/4}}, \quad 1 \leq \ell < k.$$

**2 Preliminary Lemmas**

**Lemma 2.1** If $g(x)$ and $h(x)$ are continuous real-valued functions of $x$ and $g(x)$ is monotonic, then

$$\int_a^b g(x)h(x)dx \ll \left( \max_{a \leq x \leq b} |g(x)| \right) \left( \max_{a \leq u < v \leq b} \left| \int_u^v h(x)dx \right| \right).$$
Proof. See Tsang [9], Lemma 1. ■

Lemma 2.2 Suppose $A, B \in \mathbb{R}, A \neq 0$. Then we have
$$\int_{T}^{2T} t^{\alpha} \cos(A\sqrt{t} + B) dt \ll T^{1/2 + \alpha}|A|^{-1}.$$  
Proof. It follows from Lemma 2.1 easily. ■

Lemma 2.3 If $n, m, k, \ell \in \mathbb{N}$ such that $\sqrt{n} + \sqrt{m} \pm \sqrt{k} - \sqrt{\ell} \neq 0$, then there hold
$$|\sqrt{n} + \sqrt{m} \pm \sqrt{k} - \sqrt{\ell}| \gg (nmk\ell)^{-1/2} \max(n, m, k, \ell)^{-3/2},$$  
respectively. Proof. See Kong [7], Lemma 3.2.1. ■

Lemma 2.4 Let $f : \mathbb{N} \to \mathbb{R}$ be any function satisfying $f(n) \ll n^{\varepsilon}$. Then we have
$$|s_{k, \ell}(f) - s_{k, \ell}(f; y)| \ll y^{-1/2 + \varepsilon}, \quad 1 \leq \ell < k,$$
where $k \geq 2$ is a fixed integer. Proof. See Zhai [11], Lemma 3.1. ■

Lemma 2.5 Suppose $1 \leq N \leq M$, $1 \leq K \leq L$, $N \leq K$, $M \approx L$, $0 < \Delta \ll L^{1/2}$. Let $\mathcal{A}(N, M, K, L; \Delta)$ denote the number of solutions of the following inequality
$$0 < |\sqrt{n} + \sqrt{m} \pm \sqrt{k} - \sqrt{\ell}| < \Delta$$  
with $n \sim N, m \sim M, k \sim K, \ell \sim L$. Then we have
$$\mathcal{A}(N, M, K, L; \Delta) \ll \Delta L^{1/2} N M K + NKL^{1/2 + \varepsilon}.$$  
Especially, if $\Delta L^{1/2} \gg 1$, then
$$\mathcal{A}(N, M, K, L; \Delta) \ll \Delta L^{1/2} N M K.$$
Proof. See Zhai [12], Lemma 5. ■

Lemma 2.6 Suppose $N_j \geq 2$ ($j = 1, 2, 3, 4$), $\Delta > 0$. Let $\mathcal{A}_{\pm}(N_1, N_2, N_3, N_4; \Delta)$ denote the number of solutions of the following inequality
$$0 < |\sqrt{n_1} + \sqrt{n_2} \pm \sqrt{n_3} - \sqrt{\ell_4}| < \Delta$$  
with $n_j \sim N_j$ ($j = 1, 2, 3, 4$), $n_j \in \mathbb{N}^*$. Then we have
$$\mathcal{A}_{\pm}(N_1, N_2, N_3, N_4; \Delta) \ll \prod_{j=1}^{4} \left(\Delta^{1/4} N_j^{7/8} + N_j^{1/2}\right) N_j^{\varepsilon}.$$  
Proof. See Zhai [12], Lemma 3. ■
3 Proof of Theorem 1.1

In this section, we shall prove the theorem. We begin with the following truncated formula, which is proved by Jutila [6], i.e.,

\[
A(x) = \frac{1}{\sqrt{2\pi}} \sum_{n \leq N} \frac{a(n)}{n^{\kappa/2+1/4}} x^{\kappa/2-1/4} \cos(4\pi \sqrt{n} x - \pi/4) + O(x^{\kappa/2+\varepsilon} N^{-1/2}),
\]

where \(1 \leq N \ll x\).

Suppose \(T \geq 10\). By a splitting argument, it is sufficient to prove the result in the interval \([T, 2T]\). Take \(y = T^{3/4}\). For any \(T \leq x \leq 2T\), by the truncated formula (3.1), we get

\[
A(x) = \frac{1}{\sqrt{2\pi}} \mathcal{R}(x) + O(x^{\kappa/2+\varepsilon} y^{-1/2}),
\]

where

\[
\mathcal{R}(x) := x^{\kappa/2-1/4} \sum_{n \leq y} \frac{a(n)}{n^{\kappa/2+1/4}} \cos(4\pi \sqrt{n} x - \pi/4).
\]

We have

\[
\int_T^{2T} A^4(x) \, dx = \frac{1}{4\pi^4} \int_T^{2T} \mathcal{R}^4(x) \, dx + O(T^{2\kappa+1/4+\varepsilon} y^{-1/2} + T^{2\kappa+1+\varepsilon} y^{-2})
\]

\[
= \frac{1}{4\pi^4} \int_T^{2T} \mathcal{R}^4(x) \, dx + O(T^{2\kappa-1/8+\varepsilon}).
\]

(3.3)

Let

\[
g = g(n, m, k, \ell) := \begin{cases} a(n)a(m)a(k)a(\ell) \frac{1}{(nmk\ell)^{\kappa/2+1/4}}, & \text{if } n, m, k, \ell \leq y, \\ 0, & \text{otherwise.} \end{cases}
\]

According to the elementary formula

\[
\cos a_1 \cos a_2 \cdots \cos a_h = \frac{1}{2^{h-1}} \sum_{(i_1, i_2, \ldots, i_h-1) \in \{0,1\}^{h-1}} \cos (a_1 + (-1)^{i_1} a_2 + \cdots + (-1)^{i_{h-1}} a_h),
\]

we can write

\[
\mathcal{R}^4(x) = S_1(x) + S_2(x) + S_3(x) + S_4(x),
\]

(3.4)

where

\[
S_1(x) := \frac{3}{8} \sum_{n,m,k,\ell \leq y} g x^{2\kappa-1},
\]

\[
S_2(x) := \frac{3}{8} \sum_{n,m,k,\ell \leq y} g x^{2\kappa-1} \cos (4\pi (\sqrt{n} + \sqrt{m} - \sqrt{k} - \sqrt{\ell}) \sqrt{x}),
\]

\[
S_3(x) := \frac{3}{8} \sum_{n,m,k,\ell \leq y} g x^{2\kappa-1} \cos (4\pi (\sqrt{n} + \sqrt{m} + \sqrt{k} + \sqrt{\ell}) \sqrt{x}),
\]

\[
S_4(x) := \frac{3}{8} \sum_{n,m,k,\ell \leq y} g x^{2\kappa-1} \cos (4\pi (\sqrt{n} - \sqrt{m} + \sqrt{k} - \sqrt{\ell}) \sqrt{x}),
\]

\[
S_5(x) := \frac{3}{8} \sum_{n,m,k,\ell \leq y} g x^{2\kappa-1} \cos (4\pi (\sqrt{n} - \sqrt{m} - \sqrt{k} + \sqrt{\ell}) \sqrt{x}),
\]

\[
S_6(x) := \frac{3}{8} \sum_{n,m,k,\ell \leq y} g x^{2\kappa-1} \cos (4\pi (\sqrt{n} + \sqrt{m} + \sqrt{k} - \sqrt{\ell}) \sqrt{x}),
\]

\[
S_7(x) := \frac{3}{8} \sum_{n,m,k,\ell \leq y} g x^{2\kappa-1} \cos (4\pi (\sqrt{n} - \sqrt{m} - \sqrt{k} - \sqrt{\ell}) \sqrt{x}),
\]

\[
S_8(x) := \frac{3}{8} \sum_{n,m,k,\ell \leq y} g x^{2\kappa-1} \cos (4\pi (\sqrt{n} - \sqrt{m} + \sqrt{k} + \sqrt{\ell}) \sqrt{x}).
\]
By (1.1) and Lemma 2.4, we get
\[
\int_T^{2T} S_1(x)dx = \frac{3}{8}s_{4.2}(a(n)n^{-1/2}; y)\int_T^{2T} x^{2\kappa-1}dx
\]
\[= \frac{3}{8}s_{4.2}(\hat{a}; y)\int_T^{2T} x^{2\kappa-1}dx
\]
\[= \frac{3}{8}s_{4.2}(\hat{a})\int_T^{2T} x^{2\kappa-1}dx + O(T^{2\kappa-1/2+\epsilon})
\]
\[= \frac{3}{8}s_{4.2}(\hat{a})\int_T^{2T} x^{2\kappa-1}dx + O(T^{2\kappa-3/8+\epsilon}). \tag{3.5}
\]
We now proceed to consider the contribution of \(S_4(x)\). Applying Lemma 2.2 and (1.1), we get
\[
\int_T^{2T} S_4(x)dx = \frac{1}{8}\sum_{n,m,k,\ell \leq y} g\int_T^{2T} x^{2\kappa-1} \cos \left(4\pi(\sqrt{n} + \sqrt{m} + \sqrt{k} + \sqrt{\ell})\frac{x}{2}\right) dx
\]
\[\ll \sum_{n,m,k,\ell \leq y} T^{2\kappa-1/2} a(n)a(m)a(k)a(\ell)
\]
\[= T^{2\kappa-1/2} \sum_{n,m,k,\ell \leq y} \frac{a(n)a(m)a(k)a(\ell)}{(nmk\ell)^{(n-1)/2}(nmk\ell)^{3/4}} \cdot \frac{1}{\sqrt{n} + \sqrt{m} + \sqrt{k} + \sqrt{\ell}}
\]
\[\ll T^{2\kappa-1/2} \sum_{n,m,k,\ell \leq y} \frac{d(n)d(m)d(k)d(\ell)}{(nmk\ell)^{3/4}\ell^{1/2}}
\]
\[\ll T^{2\kappa-1/2+\epsilon} \sum_{n,m,k,\ell \leq y} \frac{1}{(nmk\ell)^{3/4}\ell^{5/4}}
\]
\[\ll T^{2\kappa-1/2+\epsilon} y^{1/2} \ll T^{2\kappa-1/8+\epsilon}. \tag{3.6}
\]
Now let us consider the contribution of \(S_2(x)\). By the first derivative test, we have
\[
\int_T^{2T} S_2(x)dx \ll \sum_{n,m,k,\ell \leq y} g \min \left(T^{2\kappa}, \frac{T^{2\kappa-1/2}}{\sqrt{n} + \sqrt{m} + \sqrt{k} + \sqrt{\ell}}\right)
\]
\[\ll x^\epsilon \mathcal{G}(N, M, K, L), \tag{3.7}
\]
where
\[
\mathcal{G}(N, M, K, L) = \sum_{n \sim N, m \sim M, k \sim K, \ell \sim L} g \cdot \min \left(T^{2\kappa}, \frac{T^{2\kappa-1/2}}{\sqrt{n} + \sqrt{m} + \sqrt{k} + \sqrt{\ell}}\right).
\]
If $M \geq 200L$, then $|\sqrt{n} + \sqrt{m} - \sqrt{k} - \sqrt{t}| \gg M^{1/2}$, so the trivial estimate yields

$$
G(N, M, K, L) \ll \frac{T^{2\kappa-1/2+\varepsilon} \ell}{(NML)^{3/4}M^{1/2}} \ll T^{2\kappa-1/2+\varepsilon} y^{1/2} \ll T^{2\kappa-1/8+\varepsilon}.
$$

If $L \geq 200M$, we can get the same estimate. So later we always suppose that $M \asymp L$.

Let $\eta = \sqrt{n} + \sqrt{m} - \sqrt{k} - \sqrt{t}$. Write

$$
G(N, M, K, L) = G_1 + G_2 + G_3,
$$

where

$$
G_1 := T^{2\kappa} \sum_{0 < \eta < T^{-1/2}} g, \\
G_2 := T^{2\kappa-1/2} \sum_{T^{-1/2} < \eta \leq 1} g|\eta|^{-1}, \\
G_3 := T^{2\kappa-1/2} \sum_{|\eta| > 1} g|\eta|^{-1}.
$$

We estimate $G_1$ first. By Lemma 2.5, we get

$$
G_1 \ll \frac{T^{2\kappa+\varepsilon}}{(NML)^{3/4}} A_1(N, M, K, L; T^{-1/2}) \\
\ll \frac{T^{2\kappa+\varepsilon}}{(NML)^{3/4}} (T^{-1/2} L^{1/2} N M K + N K L^{1/2}) \\
\ll T^{2\kappa-1/2+\varepsilon} (N K)^{1/4} + T^{2\kappa+\varepsilon} (N K)^{1/4} L^{-1} \\
\ll T^{2\kappa-1/2+\varepsilon} y^{1/2} + T^{2\kappa+\varepsilon} (N K)^{1/4} L^{-1} \\
\ll T^{2\kappa-1/8+\varepsilon} + T^{2\kappa+\varepsilon} (N K)^{1/4} L^{-1}. \quad (3.9)
$$

On the other hand, by Lemma 2.6, without loss of generality, we assume that $N \leq K \leq L$ and obtain

$$
G_1 \ll \frac{T^{2\kappa+\varepsilon}}{(NML)^{3/4}} A_2(N, M, K, L; T^{-1/2}) \\
\ll \frac{T^{2\kappa+\varepsilon}}{(NML)^{3/4}} (T^{-1/8} N^{7/8} + N^{1/2}) (T^{-1/8} K^{7/8} + K^{1/2}) (T^{-1/4} L^{7/4} + L) \\
\ll T^{2\kappa+\varepsilon} (N K)^{-1/4} L^{-1/2} (T^{-1/8} N^{7/8} + 1) (T^{-1/8} K^{7/8} + 1) (T^{-1/4} L^{7/4} + 1) \\
\ll T^{2\kappa+\varepsilon} (N K)^{-1/4} L^{-1/2} (T^{-1/4} (N K)^{3/8} + T^{-1/8} K^{3/8} + 1) (T^{-1/4} L^{3/4} + 1) \\
\ll T^{2\kappa-1/4+\varepsilon} (N K)^{1/8} L^{-1/2} + T^{2\kappa+\varepsilon} (N K)^{-1/4} L^{-1/2} (T^{-1/8} K^{3/8} + 1) (T^{-1/4} L^{3/4} + 1) \\
\ll T^{2\kappa-1/4+\varepsilon} L^{-1/4} + T^{2\kappa+\varepsilon} (N K)^{-1/4} L^{-1/2} (T^{-3/8} L^{9/8} + 1) \\
\ll T^{2\kappa-1/4+\varepsilon} + T^{2\kappa+\varepsilon} (N K)^{-1/4} L^{-1/2} (T^{-3/8} L^{9/8} + 1). \quad (3.10)
$$
From (3.9) and (3.10), we get

$$G_1 \ll T^{2\kappa - 1/8 + \epsilon} + T^{2\kappa + \epsilon} \cdot \min\left(\frac{(NK)^{1/4}}{L}, \frac{T^{-3/8} L^{9/8} + 1}{(NK)^{1/4} L^{1/2}}\right).$$

**Case 1** If $L \gg T^{1/3}$, then $T^{-3/8} L^{9/8} \gg 1$, we get

$$G_1 \ll T^{2\kappa - 1/8 + \epsilon} + T^{2\kappa + \epsilon} \cdot \min\left(\frac{(NK)^{1/4}}{L}, \frac{T^{-3/8} L^{9/8}}{(NK)^{1/4} L^{1/2}}\right)$$

$$\ll T^{2\kappa - 1/8 + \epsilon} + T^{2\kappa + \epsilon} \left(\frac{L}{(NK)^{1/4}}\right)^{1/2} \left(\frac{T^{-3/8} L^{9/8}}{(NK)^{1/4} L^{1/2}}\right)^{1/2}$$

$$\ll T^{2\kappa - 1/8 + \epsilon} + T^{2\kappa - 3/16 + \epsilon} L^{-3/16} \ll T^{2\kappa - 1/8 + \epsilon}. \quad (3.11)$$

**Case 2** If $L \ll T^{1/3}$, then $T^{-3/8} L^{9/8} \ll 1$. By noting that $M \asymp L \asymp \max(N, M, K, L)$ and Lemma 2.3, we have

$$T^{-1/2} \gg |\eta| \gg (nmk\ell)^{-1/2} \max(n, m, k, \ell)^{-3/2} \asymp (NK)^{-1/2} L^{-5/2}.$$

Hence, we obtain

$$G_1 \ll T^{2\kappa - 1/8 + \epsilon} + T^{2\kappa + \epsilon} \min\left(\frac{(NK)^{1/4}}{L}, \frac{1}{(NK)^{1/4} L^{1/2}}\right)$$

$$\ll T^{2\kappa - 1/8 + \epsilon} + T^{2\kappa + \epsilon} \left(\frac{L}{(NK)^{1/4}}\right)^{1/4} \left(\frac{1}{(NK)^{1/4} L^{1/2}}\right)^{3/4}$$

$$= T^{2\kappa - 1/8 + \epsilon} + T^{2\kappa + \epsilon} \frac{(NK)^{-1/4}}{L^{-5/8}}$$

$$\ll T^{2\kappa - 1/8 + \epsilon} + T^{2\kappa + \epsilon} (T^{-1/2})^{1/4} \ll T^{2\kappa - 1/8 + \epsilon}. \quad (3.12)$$

Combining (3.11) and (3.12), we get

$$G_1 \ll T^{2\kappa - 1/8 + \epsilon}. \quad (3.13)$$

Now, we estimate $G_2$. By a splitting argument, we get that there exists some $\delta$ satisfying $T^{-1/2} \ll \delta \ll 1$ such that

$$G_2 \ll \frac{T^{2\kappa - 1/2 + \epsilon}}{(NK)^{3/4} \delta} \sum_{\delta < |\eta| \leq 2\delta, \eta \neq 0} 1.$$

By Lemma 2.5, we get

$$G_2 \ll \frac{T^{2\kappa - 1/2 + \epsilon}}{(NK)^{3/4} \delta} \sum_{\delta < |\eta| \leq 2\delta} \Phi_1(N, M, K, L; 2\delta)$$

$$\ll \frac{T^{2\kappa - 1/2 + \epsilon}}{(NK)^{3/4} \delta} \left(\delta L^{1/2} N M K + N K L^{1/2}\right)$$

$$= T^{2\kappa - 1/2 + \epsilon} (NK)^{1/4} + T^{2\kappa - 1/2 + \epsilon} \delta^{-1} (NK)^{1/4} L^{-1}$$

$$\ll T^{2\kappa - 1/2 + \epsilon} \frac{y^{1/2}}{2} + T^{2\kappa - 1/2 + \epsilon} \delta^{-1} (NK)^{1/4} L^{-1}$$

$$\ll T^{2\kappa - 1/8 + \epsilon} + T^{2\kappa - 1/2 + \epsilon} \delta^{-1} (NK)^{1/4} L^{-1}. \quad (3.14)$$
On the other hand, by Lemma 2.6, without loss of generality, we assume that $N \leq K \leq L$ and obtain

$$
\mathcal{G}_2 \ll \frac{T^{2\kappa-1/2+\varepsilon}}{(N M K L)^{3/4} \delta} \times \sigma_\alpha(N, M, K, L; 2\delta)
$$

$$
\ll \frac{T^{2\kappa-1/2+\varepsilon}}{(N M K L)^{3/4} \delta} \left( \delta^{1/4} N^{7/8} + N^{1/2} \right) \left( \delta^{1/4} K^{7/8} + K^{1/2} \right) \left( \delta^{1/2} L^{7/4} + L \right)
$$

$$
\ll T^{2\kappa-1/2+\varepsilon} (NK)^{-1/4} L^{-1/2} \delta^{-1} \left( \delta^{1/4} N^{3/8} + 1 \right) \left( \delta^{1/4} K^{3/8} + 1 \right) \left( \delta^{1/2} L^{3/4} + 1 \right)
$$

$$
\ll T^{2\kappa-1/2+\varepsilon} (NK)^{-1/4} L^{-1/2} \delta^{-1} \left( \delta^{1/4} N^{3/8} + 1 \right) \left( \delta^{1/4} K^{3/8} + 1 \right) \left( \delta^{1/2} L^{3/4} + 1 \right)
$$

$$
+ T^{2\kappa-1/2+\varepsilon} (NK)^{-1/4} L^{-1/2} \delta^{-1} \left( \delta^{1/4} K^{3/8} + 1 \right) \left( \delta^{1/2} L^{3/4} + 1 \right)
$$

$$
\ll T^{2\kappa-1/4+\varepsilon} L^{-1/4} + T^{2\kappa-1/2+\varepsilon} (NK)^{-1/4} L^{-1/2} \delta^{-1} \left( \delta^{3/4} L^{9/8} + 1 \right).
$$

(3.15)

From (3.14) and (3.15), we get

$$
\mathcal{G}_2 \ll T^{2\kappa-1/8+\varepsilon} + T^{2\kappa-1/2+\varepsilon} \delta^{-1} \cdot \min \left( \frac{(NK)^{1/4}}{L}, \frac{\delta^{3/4} L^{9/8} + 1}{(NK)^{1/4} L^{1/2}} \right).
$$

Case 1 If $\delta \gg L^{-3/2}$, then $\delta^{3/4} L^{9/8} \gg 1$, we get (recall $\delta \gg T^{-1/2}$)

$$
\mathcal{G}_2 \ll T^{2\kappa-1/8+\varepsilon} + T^{2\kappa-1/2+\varepsilon} \delta^{-1} \cdot \min \left( \frac{(NK)^{1/4}}{L}, \frac{\delta^{3/4} L^{9/8}}{(NK)^{1/4} L^{1/2}} \right)
$$

$$
\ll T^{2\kappa-1/8+\varepsilon} + T^{2\kappa-1/2+\varepsilon} \delta^{-1} \left( \frac{(NK)^{1/4}}{L} \right)^{1/2} \left( \frac{\delta^{3/4} L^{9/8}}{(NK)^{1/4} L^{1/2}} \right)^{1/2}
$$

$$
\ll T^{2\kappa-1/8+\varepsilon} + T^{2\kappa-1/2+\varepsilon} \delta^{-5/8} L^{-3/16}
$$

$$
\ll T^{2\kappa-1/8+\varepsilon} + T^{2\kappa-1/2+\varepsilon} T^{5/16} L^{-3/16} \ll T^{2\kappa-1/8+\varepsilon}.
$$

(3.16)

Case 2 If $\delta \ll L^{-3/2}$, then $\delta^{3/4} L^{9/8} \ll 1$. By Lemma 2.3, we have

$$
\delta \gg |\eta| \gg (nmk\ell)^{-1/2} \max(n, m, k, \ell)^{3/2} \gg (NK)^{-1/2} L^{-5/2}.
$$

Therefore, we obtain (recall $\delta \gg T^{-1/2}$)

$$
\mathcal{G}_2 \ll T^{2\kappa-1/8+\varepsilon} + T^{2\kappa-1/2+\varepsilon} \delta^{-1} \cdot \min \left( \frac{(NK)^{1/4}}{L}, \frac{1}{(NK)^{1/4} L^{1/2}} \right)
$$

$$
\ll T^{2\kappa-1/8+\varepsilon} + T^{2\kappa-1/2+\varepsilon} \delta^{-1} \left( \frac{(NK)^{1/4}}{L} \right)^{1/4} \left( \frac{1}{(NK)^{1/4} L^{1/2}} \right)^{3/4}
$$

$$
\ll T^{2\kappa-1/8+\varepsilon} + T^{2\kappa-1/2+\varepsilon} \delta^{-1} (NK)^{-1/8} L^{-5/8}
$$

$$
\ll T^{2\kappa-1/8+\varepsilon} + T^{2\kappa-1/2+\varepsilon} \delta^{-1} \delta^{1/4}
$$

$$
\ll T^{2\kappa-1/8+\varepsilon} + T^{2\kappa-1/2+\varepsilon} T^{3/8} \ll T^{2\kappa-1/8+\varepsilon}.
$$

(3.17)

Combining (3.16) and (3.17), we get

$$
\mathcal{G}_2 \ll T^{2\kappa-1/8+\varepsilon}.
$$

(3.18)
For $G_3$, by a splitting argument and Lemma 2.5 again, we get

$$G_3 \ll \frac{T^{2\kappa-1/2+\varepsilon}}{(NMKL)^{3/4\delta}} \times \sum_{\delta<|\eta|\leq 2\delta} 1$$

$$\ll \frac{T^{2\kappa-1/2+\varepsilon}}{(NMKL)^{3/4\delta}} \cdot \delta L^{1/2} NMK \ll T^{2\kappa-1/2+\varepsilon} (NK)^{1/4}$$

$$\ll T^{2\kappa-1/2+\varepsilon} y^{1/2} \ll T^{2\kappa-1/8+\varepsilon}.$$  (3.19)

Combining (3.7), (3.8), (3.13), (3.18) and (3.19), we get

$$\int_T^{2T} S_2(x)dx \ll T^{2\kappa-1/8+\varepsilon}.  \quad (3.20)$$

In the same way, we can prove that

$$\int_T^{2T} S_3(x)dx \ll T^{2\kappa-1/8+\varepsilon}.  \quad (3.21)$$

From (3.3)-(3.6), (3.20) and (3.21), we get

$$\int_T^{2T} A_4(x)dx = \frac{3}{32\pi^2} s_{4;2}(\tilde{a}) \int_T^{2T} x^{2\kappa-1}dx + O(T^{2\kappa-1/8+\varepsilon}),  \quad (3.22)$$

which implies Theorem 1.1 immediately.

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