String Approximation for Cooper Pair in High-T<sub>c</sub> Superconductivity

V. Dzhunushaliev
Universität Potsdam, Institute für Mathematik, 14469, Potsdam, Germany
and Phys. Dept., KRSU, 720000, Bishkek, Kyrgyzstan

It is assumed that in some sense the High-T<sub>c</sub> superconductivity is similar to the quantum chromodynamics (QCD). This means that the phonons in High-T<sub>c</sub> superconductor have the strong interaction between themselves like to gluons in the QCD. At the experimental level this means that in High-T<sub>c</sub> superconductor exists the nonlinear sound waves. It is possible that the existence of the strong phonon-phonon interaction leads to the confinement of phonons into a phonon tube (PT) stretched between two Cooper electrons like a hypothesized flux tube between quark and antiquark in the QCD. The flux tube in the QCD brings to a very strong interaction between quark-antiquark, the similar situation can be in the High-T<sub>c</sub> superconductor: the presence of the PT can essentially increase the binding energy for the Cooper pair. In the first rough approximation the PT can be approximated as a nonrelativistic string with Cooper electrons at the ends. The BCS theory with such potential term is considered. It is shown that Green’s function method in the superconductivity theory is a realization of discussed Heisenberg idea proposed by him for the quantization of nonlinear spinor field. A possible experimental testing for the string approximation of the Cooper pair is offered.

I. INTRODUCTION

In the quantum chromodynamics (QCD) there is a very fruitful analogy between superconductivity and QCD: according to the ‘t Hooft and Mandelstam assumption [1], [2] a confinement in the QCD is similar to the dual Meissner effect in a superconductor. Such confinement scenario assumes that the ground state of QCD is a condensate of magnetic monopoles [16] which one confines (pinches) the color electric field of color charges into flux tubes [17]. In other words, the magnetic charges, defined as the Dirac monopoles in ground state of QCD, should condense in the vacuum in just the same way as the Cooper pair in a superconductor. This analogy works in the direction: superconductivity ⇒ quantum chromodynamics.

An analogy can be offered in the opposite direction: from the QCD to the High-T<sub>c</sub> superconductivity by the following manner. An interaction between gluons exists in the QCD and it is so strong that the force lines of color gauge field are stretched into a flux tube between quark and antiquark. The most important in this picture is the strong interaction between gluons. This one distinguishes the quantum electrodynamics (QED) from the quantum chromodynamics: QED does not have the photon-photon interaction. Thus, our basic assumption is that in the High-T<sub>c</sub> superconductor there is a strong phonon-phonon interaction. We assume that the jump from the ordinary superconductivity to High-T<sub>c</sub> one is analogous to the jump from QED to QCD.

\[
\begin{array}{c|c|c}
\text{ordinary superconductivity} & \approx & \text{QED} \\
\text{High-T}_c \text{ superconductivity} & \approx & \text{QCD}
\end{array}
\]

The consequence of such conjectural analogy is that the interaction between phonons is so strong that the phonons between the Cooper pair are confined into a phonon tube (PT) [3].

The phonon Hamiltonian \( \mathcal{H}_{ph} \) in the continuum limit we can write as follows

\[
\mathcal{H}_{ph} = \int d^3 r \left[ \frac{\rho}{2} \dot{s}_i^2 (r, t) + c^{kl} \partial_k s_i \partial_l s^k + V(r, t) \right],
\]

where \( \rho \) is the mass density, \( s_i = \{ s_i \} \) \( (i = x, y, z) \) is the ion deviation from the equilibrium state, \( V \) is the potential energy, \( c^{kl} \) are some coefficients. Our assumption indicates that the potential term \( V \) in the High-T<sub>c</sub> superconductor can not be simplified as

\[
V \approx \frac{\partial^2 V}{\partial s_i \partial s_j} s_i s_j
\]
even though for the small deviations $s_i$. The same takes place in the QCD with Lagrangian

$$\mathcal{L}_{QCD} \propto F^a_{\mu\nu}F^{a\mu\nu},$$

(3)

here $\mu, \nu = t, x, y, z$; $F^a_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + gf_{abc}A^b_\mu A^c_\nu$; $A^a_\mu$ is the SU(3) gauge potential; $f_{abc}$ are the structural constants of the SU(3) gauge group; $a, b, c$ are the color indexes. It is easy to see that in this QCD Lagrangian we have $(A)^3$ and $(A)^4$ terms. The most important is that coupling constant $g$ is not small, i.e. in contrast with QED we have a very strong interaction between gluons that probably leads to the appearance of a nonlocal object - flux tube stretched between quark and antiquark.

II. BCS-THEORY + STRING APPROXIMATION.

The basic idea of BCS theory is that in the presence of even a weak interaction between electrons they cooperate in the Cooper pairs that leads to decreasing the ground state energy of superconductor in comparison with the normal state. According to the above-mentioned assumption about the possible analogy with the QCD we suppose

- There is a strong interaction between phonons (nonlinear potential term in expression) which leads to appearing of a tube filled by phonons (like the flux tube filled by chromodynamical color fields in the QCD). We can name such tube as the PT.

- In the first approximation, neglecting by cross section of the PT, such object is like a string. Analogously to the QCD in the first rough approximation the PT in the High-$T_c$ superconductor can be considered as a string.

- As in the QCD we assume that the interaction between Cooper electrons is described by the nonrelativistic string potential

$$V = kl,$$

(4)

where $V$ is the potential energy of the phonon interaction between Cooper electrons, $k$ is a coefficient and $l$ is a length of the string.

In this model the Cooper pair is modeled as a string with two Cooper electrons attached at its ends. In the QCD such construction is a string with quark and antiquark at the ends. The interaction quark-antiquark is so strong that we have confinement: if we increase the string length then in some moment the string is torn and again we will have the quark-antiquark pair, i.e. we can not obtain a single quark. By applying such model to the High-$T_c$ superconductivity we should be delicate: certainly a single electron exists but the basic idea is the same: the string interaction leads to an essential increase of the binding energy of the Cooper pair.

Now we would like to repeat the calculations in the microscopic BCS theory with the potential (4) (following, for example, to Ref. [4]). Let us assume that $|00\rangle$ is a quantum amplitude that the Cooper pair is unoccupied and $|11\rangle$ that it is occupied, then the wave function is

$$\Phi = \prod_k \psi_k,$$

(5)

$$\psi_k = u_k|00\rangle + v_k|11\rangle,$$

(6)

where $|u_k|^2$ is the probability that the Cooper pair $(k, \uparrow), (-k, \downarrow)$ is unoccupied and accordingly $|v_k|^2$ is the probability that it is occupied; $k$ is the wave vector. We have the normalization condition

$$|v_k|^2 + |u_k|^2 = 1.$$  

(7)

The normal ground state is given by

$$u_k = 0 \quad \text{and} \quad v_k = 1 \quad \text{for} \quad |k| < k_F,$$

(8)

$$u_k = 1 \quad \text{and} \quad v_k = 0 \quad \text{for} \quad |k| > k_F,$$

(9)

where $k_F$ is the Fermi wave vector.

The energy of ground state is calculated by the variational method

$$\delta \langle \Phi | \hat{H} - \mu \hat{N} | \Phi \rangle = 0$$

(10)
with the constraint $\langle \Phi | \hat{N} | \Phi \rangle = N$, where $\hat{N}$ is the number operator and $N$ is the mean number of particles in this system, $\mu$ is the Fermi energy.

The calculation for kinetic energy term $\hat{K}$ gives us

$$\langle \Phi | \hat{K} | \Phi \rangle = 2 \sum_k \varepsilon_k |v_k|^2,$$

where the single particle energies $\varepsilon_k$ are measured from the Fermi surface $\varepsilon_F$, the factor 2 appears as $|v_k|^2$ is the occupation probability for the Cooper pair. The expectation value of the potential energy part is

$$\langle \Phi | \hat{V} | \Phi \rangle = \sum_{k,k'} V_{kk'} u_k^* v_{k'} u_{k'}^* v_k,$$

where $u_k^* v_k$ is the amplitude for the initial state in which the Cooper pair $k$ is occupied and pair $k'$ is unoccupied; $u_{k'}^* v_k$ is the amplitude for the final state in which the reverse is true. Thus we have

$$\langle \Phi | \hat{H} - \mu \hat{N} | \Phi \rangle = 2 \sum_k \varepsilon_k |v_k|^2 + \sum_{k,k'} V_{kk'} u_k^* v_{k'} u_{k'}^* v_k.$$  

As usual, we minimize this functional with respect to $v_k^*$ that leads to the equation

$$\Delta_k = -\sum_{k'} V_{kk'} \frac{\Delta_{k'}}{2E_{k'}}$$

where

$$E_{k'}^2 = \varepsilon_{k'}^2 + \Delta_{k'}^2,$$

$$\Delta_k = \sum_k V_{kk'} u_k v_{k'},$$

As usual we simplify

$$V_{kk'} = \begin{cases} -V & \text{if } |\varepsilon_k|, |\varepsilon_{k'}| < \hbar \omega_D \\ 0 & \text{otherwise} \end{cases}$$

here $\hbar \omega_D$ is the Debye energy of phonons.

Now we can include our string assumption: we suppose that $V$ in the (17) expression is

$$V = kl,$$

where $k$ is some constant, $l$ is the length of Cooper pair (length of the string).

As usual the further calculations give us

$$|\Delta| = \frac{\hbar \omega_D}{\sinh \left( \frac{1}{N_0 V} \right)},$$

where $N_0$ is the density of states in energy at the Fermi surface.

III. DISCUSSION

We can hope that in the High-$T_c$ superconductor the quantity $V$ in the (19) expression will be much more than the corresponding quantity in the ordinary superconductors

$$V_{\text{high}} \gg V_{\text{ordinary}}$$

as it takes place in the QCD: the strong interaction between quarks is much more than the electromagnetic one between electrons (positrons). In the QCD the interaction between gluons is so strong that the force lines are confined into the flux tube stretched between quark and antiquark.
FIG. 1: Phonon exchange between two electrons. Here the momenta of the incoming electrons are $\hbar k$ and $\hbar k'$ and the momentum of the exchanged phonon is $\hbar q$.

The greatest difficulty here is the microscopical calculation of the quantity $V$. This difficulty is connected with the presence of the nonlinear potential term in the Hamiltonian. This means that in the presence of the strong self-interaction between phonons the expression for the exchange of one phonon between two electrons (see Fig. 1)

$$V(k,k') = \frac{g^2 \hbar \omega}{(\varepsilon_{k+q})^2 - (\hbar \omega_q)^2}$$  \hspace{1cm} (21)

now is not correct. The same problem exists in the QCD: now we cannot deduce the existence of the flux tube (string in the first rough approximation) from the SU(3) Lagrangian. This problem is connected with that we do not have appropriate mathematical tools in the nonperturbative QCD, i.e., without the Feynman diagram techniques.

One of the first attempts of nonperturbative calculations in the QCD was a Veneziano amplitude (see Fig. 2)

$$A(s,t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s) - \alpha(t))}$$  \hspace{1cm} (22)

where $s = -(p_1 + p_2)^2$ and $t = -(p_3 + p_4)^2$ are the Mandelstam variables; $\Gamma$ is the gamma function; $\alpha(s) = \alpha(0) + \alpha'$; $\alpha(0)$ and $\alpha'$ are some constants. This approach has not resulted in a great success in the QCD as this amplitude is some rough approximation for the correct 4-point Green’s function. It is possible to assume that such approximation can be made in the superconductivity theory: the scattering amplitude (21) of two electrons $V(k,k')$ is the Veneziano amplitude (22). Certainly (just as in QCD) it is a rough approximation for the correct Green’s function.

FIG. 2: Elastic scattering of colliding particles with impulses $p_1$ and $p_2$ and divergent particles with impulses $p_3$ and $p_4$. 
We can try to obtain an expression for $V$ on the basis of the dimensional reasons. The result, for example, can be

$$k \approx \frac{\hbar \omega_D}{\sqrt{S}},$$  \hspace{1cm} (23)

$$V \approx \frac{\hbar \omega_D}{\sqrt{S}} l,$$  \hspace{1cm} (24)

where $S$ is the cross section of the PT. If we suppose that the PT is located inside of a layer of an anisotropical superconductor then $\sqrt{S} \approx \delta$, where $\delta$ is the thickness of the layer. In this case these expressions will be

$$k \approx \frac{\hbar \omega_D}{\delta},$$  \hspace{1cm} (25)

$$V \approx \frac{\hbar \omega_D}{\delta} l,$$  \hspace{1cm} (26)

because of $l \gg \delta$ we have $V \gg \hbar \omega_D$.

IV. PHONON TUBE

What is the physical meaning of the PT? In the QCD the flux tube is filled with the color field which is parallel to the tube axis and equal to zero outside this tube. In our case we can suppose that there can be one from the following possibilities

$$\langle s \rangle \neq 0 \text{ and/or } \langle s^2 \rangle \neq 0 \text{ and/or } \langle s^3 \rangle \neq 0 \ldots \text{ inside PT and}$$  \hspace{1cm} (27)

$$\langle s \rangle \approx 0 \text{ and } \langle s^2 \rangle \approx 0 \text{ and } \langle s^3 \rangle \approx 0 \ldots \text{ outside PT.}$$  \hspace{1cm} (28)

The first possibility $\langle s \rangle \neq 0$ means that the nonzero static deviation of the ion lattice exists only inside PT. The second one $\langle s^2 \rangle \neq 0$ means that the nonzero mean square of the deviation exists only inside PT only and so on.

V. HEISENBERG QUANTIZATION MODEL FOR THE PT

A possible strong phonon-phonon interaction can obstruct the application of Feynman diagram techniques in our case. Some time ago W. Heisenberg had conceived of the difficulties in applying an expansion in small parameters to quantum field theories having strong interactions. He had investigated the Dirac equation with nonlinear terms (Heisenberg equation) (see, for example, Ref’s [6] - [7]). In these papers he repeatedly underscored that a nonlinear theory with a large parameter requires the introduction of another quantization rule. He worked out a quantization method for strong nonlinear field unusing the expansion in a small parameter. It is possible that in the High - $T_c$ superconductivity the interaction between phonons is strong making it necessary take into account the interaction between phonons to correctly calculate the energy of the Cooper pairs.

A. Heisenberg quantization of nonlinear spinor field

Heisenberg’s basic idea proceeds from the fact that the n-point Green functions must be found from some infinity differential equations system derived from the field equation for the field operator. For example, we present Heisenberg quantization for nonlinear spinor field.

The basic equation (Heisenberg equation) has the following form:

$$\gamma^\mu \partial_\mu \hat{\psi}(x) - i^2 \Im \left[ \hat{\psi}(\hat{\bar{\psi}} \hat{\psi}) \right] = 0,$$  \hspace{1cm} (29)

where $\gamma^\mu$ are Dirac matrices; $\hat{\psi}(x)$ is the field operator; $\hat{\bar{\psi}}$ is the Dirac adjoint spinor; $\Im[\hat{\psi}(\hat{\bar{\psi}} \hat{\psi})] = \hat{\psi}(\hat{\bar{\psi}} \hat{\psi}$ or $\hat{\psi}\gamma^5(\hat{\bar{\psi}} \gamma^5 \hat{\psi}$ or $\hat{\psi}\gamma^\mu(\hat{\bar{\psi}} \gamma^\mu \hat{\psi}$ or $\hat{\psi}\gamma^\mu \gamma^5(\hat{\bar{\psi}} \gamma^\mu \gamma^5 \hat{\psi})$. Heisenberg emphasizes that the 2-point Green function $G_2(x_2, x_1)$ in this theory differs strongly from the propagator in linear theory. This difference lies in its behaviour on the light cone. $G_2(x_2, x_1)$ oscillates strongly on the light cone in contrast to the propagator of the linear theory which has a $\delta$-like singularity. Then Heisenberg introduces the $\tau$ functions:

$$\tau(x_1 x_2 \ldots | y_1 y_2 \ldots) = \langle 0 | T \psi(x_1)\psi(x_2)\ldots \psi^*(y_1)\psi^*(y_2)\ldots | \Phi \rangle,$$  \hspace{1cm} (30)
where $T$ is the time ordering operator. $|\Phi\rangle$ is a system state characterized by the fundamental Eq. (29). Relationship (30) allows us to establish a one-to-one correspondence between the system state $|\Phi\rangle$ and the function set $\tau$. This state can be defined using the infinite function set of (30). Applying Heisenberg’s equation (29) to (31) we can obtain the following infinite equations system:

$$\gamma^\mu_{(r)} \frac{\partial}{\partial x^\mu_{(r)}} \tau(x_1 \ldots x_n | y_1 \ldots y_n) = \Im \left[ \tau(x_1 \ldots x_n | y_1 \ldots y_r) \right] +$$

$$\delta(x_r - y_1) \tau(x_1 \ldots x_{r-1} x_{r+1} \ldots x_n | y_2 \ldots y_{r-1} y_{r+1} \ldots y_n) +$$

$$\delta(x_r - y_2) \tau(x_1 \ldots x_{r-1} x_{r+1} \ldots x_n | y_1 y_2 \ldots y_{r-1} y_{r+1} \ldots y_n) + \ldots \ldots$$

(31)

Heisenberg then employs the Tamm - Dankoff method for getting approximate solutions to the infinite equations system of (31). The key to this method lies in the fact that the system of equation has an approximate solution derived after cutting off the infinite equation system (31) to a finite equation system.

It is necessary to note that a method of solution to Eq. (31) can be various. For example, we can try to determine the Green’s functions using the numerical lattice calculations. Here the important point is the following: The technique of expansion in small parameters (Feynman diagrams) can not be employed for strong nonlinear fields. It is possible that as in quantum chromodynamics, where quarks are thought to interact strongly by means of flux tubes, so too in High-$T_c$ superconductivity phonons may strongly interact among themselves.

In Ref. 11 such a mechanism is applied for the QCD. In this paper we apply a variant of Heisenberg’s quantization method to solutions of the classical SU(3) Yang-Mills field equations which have bad asymptotic behavior. After quantization it has been found that the bad features (i.e. divergent fields and energy densities) of these solutions are moderated. From these results is argued that in general the n-point Green’s functions for Yang-Mills theories can have nonperturbative pieces which can not be represented as the sum of Feynman diagrams.

In Heisenberg’s theory the matter and the interacting fields are identical: fundamental spinor field $|\psi\rangle$. From a more recent perspective this is not the case. An interaction is carried by some kind of boson field. In superconductivity this is the phonons, in quantum chromodynamics it is the nonabelian SU(3) gauge field - gluons.

In conclusion of this section we emphasize again Heisenberg’s statement that the perturbation theory is inapplicable to strong nonlinear fields.

B. Heisenberg quantization method for High-$T_c$ superconductivity

Thus, the basic assumption supposed here is the following: The energy of Cooper pair has an essential contribution coming from an interaction of phonons. This means that the corresponding sound wave is a nonlinear wave.

1. An application of Heisenberg quantization method for the Green’s function method.

In this subsection we would like to show that Heisenberg idea about quantization of strongly interacting fields in fact had been applied in the superconductivity theory for definition of Green’s function. Here we would like to show that the calculations which was made in Ref. 11 is an application of the Heisenberg quantization idea. In this section we follow to Ref. 12.

The Hamiltonian of the system electrons describing the properties of a metal in the superconductivity state is

$$H = \int \left[ -\frac{\psi^\dagger \nabla^2}{2m} \psi_\alpha + \frac{\lambda}{2} \left( \psi^\dagger_\alpha \left( \psi^\dagger_\beta \psi_\alpha \right) \psi_\beta \right) \right] dV,$$

(32)

where $\psi_\alpha$ is the operator of spinor field describing electrons, $m$ is the electron mass, $\lambda$ is some constant and $\alpha, \beta$ are the spinor indexes. As usually (it is the same as Heisenberg equation (24) for nonlinear spinor field) the operators $\hat{\psi}$ and $\hat{\psi}^\dagger$ obey the following operator equations

$$\left( i \frac{\partial}{\partial t} + \frac{\nabla^2}{2m} \right) \hat{\psi}_\alpha(x) - \lambda \left( \hat{\psi}^\dagger_\alpha(x) \hat{\psi}_\beta(x) \right) \hat{\psi}_\alpha(x) = 0,$$

(33)

$$\left( i \frac{\partial}{\partial t} - \frac{\nabla^2}{2m} \right) \hat{\psi}^\dagger_\alpha(x) + \lambda \hat{\psi}^\dagger_\alpha(x) \left( \hat{\psi}^\dagger_\beta(x) \hat{\psi}_\beta(x) \right) = 0.$$

(34)
As well as in Heisenberg method for nonlinear spinor field we have an equation for the 2-point Green’s function $G_{\alpha\beta}(x, x') = -i(T(\hat\psi_\alpha(x)\hat\psi^\dagger_\beta(x')))$

\[
\left(\frac{i}{\hbar} \frac{\partial}{\partial t} + \frac{\nabla^2}{2m}\right) G_{\alpha\beta}(x, x') + i\lambda(T(\hat\psi^+_\alpha(x)\hat\psi_\gamma(x)\hat\psi_\alpha(x)\hat\psi^+_\beta(x'))) = \delta(x - x'). \tag{35}
\]

Further we have to write an equation for term $T(\hat\psi^+_\alpha(x)\hat\psi_\gamma(x)\hat\psi_\alpha(x)\hat\psi^+_\beta(x'))$ and so on. After this we will have an infinite set for Green’s function. The main problem here is: how can we cut off this infinite equation system? Let us cite Ref. [12]: “... For interacting particles, the product of four \(\hat\psi\)-operators can be expressed in terms of the vertex part, i.e., it includes the contributions from various scattering processes. In the weak-interaction model under consideration, these scattering processes involving collisions of particles can be neglected, but at the same time, it must be borne in mind that the ground state of the system differs from the usual state with a filled Fermi sphere, because of the presence of bound pairs of electrons. ...”

This means that in the textbook [14] it was made the following approximation: the operators \(\hat\psi^+_\alpha\psi_\gamma\) and \(\hat\psi^+_\gamma\hat\psi_\alpha\) contain terms corresponding to the annihilation and creation of bound pairs (Cooper pairs). It allows to state

\[
\left\langle T\left(\hat\psi_\alpha(x_1)\hat\psi^+_\beta(x_2)\hat\psi^+_\gamma(x_3)\hat\psi^+_\delta(x_4)\right)\right\rangle \approx \left\langle T\left(\hat\psi_\alpha(x_1)\hat\psi^+_\beta(x_2)\hat\psi^+_\gamma(x_3)\hat\psi_\delta(x_4)\right)\right\rangle + \left\langle T\left(\hat\psi_\alpha(x_1)\hat\psi^+_\beta(x_2)\hat\psi_\gamma(x_3)\hat\psi^+_\delta(x_4)\right)\right\rangle + \left\langle T\left(\hat\psi_\alpha(x_1)\hat\psi^+_\beta(x_2)\hat\psi^+_\gamma(x_3)\hat\psi_\delta(x_4)\right)\right\rangle + \left\langle T\left(\hat\psi_\alpha(x_1)\hat\psi^+_\beta(x_2)\hat\psi^+_\gamma(x_3)\hat\psi^+_\delta(x_4)\right)\right\rangle. \tag{36}
\]

where \(|N\rangle\) and \(|N + 2\rangle\) are ground states of system with \(N\) and \(N + 2\) particles (Cooper pairs), respectively. We should note that this expression is approximate one and following to Heisenberg idea it allows us to cut off the above-mentioned infinite equation set for Green’s function. The key role here plays the last term in expression (36). This expression allows us to split the 4-point Green’s function with the nonperturbative way not using Feynman diagram techniques.

Eq. (36) means that we have neglected all effects of scattering particles by each other and the presence of the interaction has been taken into account only as leading to the formation of bound pairs. The third term in the right-hand side of Eq. (36) has been written in complete analogy with the case of a Bose gas in the correspondence with the fact that all bound pairs (Cooper pairs) are “condensed on the lowest level”. The quantity

\[
\left\langle N\left| T\left(\hat\psi_\alpha\hat\psi^+_\beta\right)\right| N + 2\right\rangle \left\langle N + 2\left| T\left(\hat\psi^+_\gamma\hat\psi^+_\delta\right)\right| N\right\rangle,
\]

obviously has the same order of magnitude as the density of pairs. We can introduce the following functions

\[
e^{-2i\mu t} F_{\alpha\beta}(x - x') = \left\langle N\left| T\left(\hat\psi_\alpha(x)\hat\psi^+_\beta(x')\right)\right| N + 2\right\rangle, \tag{38}
\]

\[
e^{2i\mu t} F^+_{\alpha\beta}(x - x') = \left\langle N + 2\left| T\left(\hat\psi^+_\gamma(x)\hat\psi^+_\delta(x')\right)\right| N\right\rangle, \tag{39}
\]

where \(\mu\) is a chemical potential. We now substitute (38) into the equation (33) for the Green’s function. We can everywhere omit the first two terms in the right-hand side of (38) since they lead to an additive correction to the chemical potential in the equations for the functions \(G, F, F^+\). As a result we obtain the following equation connecting \(G\) and \(F^+\):

\[
\left(\frac{\partial}{\partial t} + \frac{\nabla^2}{2m}\right) G(x - x') - i\lambda F(0+) F^+(x - x') = \delta(x - x'). \tag{40}
\]

The quantity \(F(0+)\) is defined as

\[
F_{\alpha\beta}(0+) = e^{2i\mu t} \left\langle N\left| \hat\psi_\alpha(x)\hat\psi^+_\beta(x)\right| N + 2\right\rangle = \lim_{x' \to x'} \lim_{t' \to t} F_{\alpha\beta}(x - x'). \tag{41}
\]

An equation for \(F^+(x - x')\) can be obtained in a similar way by using the equation (34)

\[
\left(\frac{i}{\hbar} \frac{\partial}{\partial t} - \frac{\nabla^2}{2m} - 2\mu\right) F^+(x - x') + i\lambda F^+(0+) G(x - x') = 0. \tag{42}
\]
Analogously to Eq. (41) we have

$$F_{\alpha\beta}^+(0+) = e^{-2i\mu t} \left( N + 2 \left[ \psi_\alpha^+(x)\psi_\beta^+(x) \right] N \right)$$

Now we have Eq.'s (40) and (42) as the finite set of equations for the 2-point Green's functions $G$ and $F$.

Thus, above-mentioned reasonings should convince us that this Green's function method in the superconductivity theory is a realization of Heisenberg idea developed by him for the quantization of non-linear spinor field.

2. Phonon-phonon interaction

In this subsection we would like to consider a possibility for the phonons to form a static configuration. Let us write again the Lagrangian (3) for phonons in continuous limit

$$L_{ph} = \int d^3r \left[ \frac{\rho}{2} s_i^2(r,t) + c^{kl} \partial_k s_k \partial_l s^l - V(r,t) \right].$$

Thus, in this model it is assumed that operators of strong nonlinear fields $s_i$ must satisfy the following equation (which is implied from the Lagrangian (44)):

$$\frac{\partial^2 \hat{s}_i}{\partial t^2} + c^{kl} \frac{\partial \hat{s}_i}{\partial x^k} \partial_{x^l} = - \frac{dV(\hat{s}_i)}{d\hat{s}_i}.$$  

The multitime formalism of Heisenberg's method (when in $\tau(t_1, t_2, \cdots), t_1 \neq t_2 \neq \cdots$) allows us to investigate the scattering processes in quantum theory. The simultaneous formalism (when $\tau(t_1, t_2, \cdots), t_1 = t_2 = \cdots = t$) allows us to calculate the mean value of the field, the energy, or any combination of field powers.

Let us consider, for example, the simplest case of the scalar field when the vector $s$ is replaced by the scalar field $\varphi$. In this case the Lagrangian is

$$\mathcal{L} = \int d^3r \left[ (\partial_\mu \varphi)^2 - V(\varphi) \right]$$

where $\mu = t, x, y, z$ and the potential term

$$V(\varphi) = \frac{\lambda}{4} (\varphi^2 - \varphi_0^2).$$

The classical field equation is

$$\Box \varphi(r) = -\lambda \varphi(r) \left( \varphi^2(r) - \varphi_0^2 \right).$$

The quantization of (46) equation gives us

$$\Box \hat{\varphi}(r) = \varphi^3(r) - \varphi(r)\varphi_0^2.$$  

It is easy to see that the mean value $\langle \varphi(r) \rangle = \langle 0|\hat{\varphi}(r)|0 \rangle$ satisfies the following equation:

$$\Box \langle \varphi(r) \rangle = \langle \varphi^3(r) \rangle - \varphi_0^2 \langle \varphi(r) \rangle.$$  

For the definition of $\langle \varphi^3(r) \rangle$ we turn to Eq. (49) and obtain $(\langle \varphi^3(r) \rangle = \tau(\varphi r))$ in Heisenberg's notation:

$$\Box \langle \varphi^3(r) \rangle = 3\lambda \left( \langle \varphi^3(r) \rangle - \varphi_0^2 \langle \varphi^3(r) \rangle \right),$$

here $\langle \varphi^3(r) \rangle = \tau(\varphi r r r)$. Analogously it can be used to derive the infinite equation system for calculating $\langle \varphi^n(r) \rangle$. In the first approximation we can solve this equation system using the following assumption:

$$\langle \varphi^3(r) \rangle \approx \langle \varphi(r) \rangle^3,$$

and then we can derive the equation

$$\Box \langle \varphi(r) \rangle = \lambda \langle \varphi(r) \rangle \left( \langle \varphi(r) \rangle^2 - \varphi_0^2 \right).$$
This equation is very interesting for us. Derrick’s Theorem \cite{8} states that Eq. (53) has a static solution with the finite energy only in two dimensional spacetime \((t,x)\). This solution is

\[
\langle \varphi(x) \rangle = \varphi_1 \tanh \left[ \frac{1}{2} m (x - x_0) \right]
\]

(54)

where \(\varphi_1, m\) and \(x_0\) are some constants. The first assumption is that such solution can be realized on the layer of High-T\(c\) superconductor as a line (wall) separating regions with different vacua \((\varphi = \pm \varphi_0)\). The existence of regions with different vacua can have very interesting physical consequences for a distribution of free electrons in the ion lattice. Following Ref.\cite{13} we present a model example of a fermion coupled to solution (54). Let we have 2-dimensional Dirac equation

\[
(\gamma^\mu \partial_\mu + g\varphi(x)) \psi(x) = 0
\]

(55)

where \(\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}\) is the two-component spinor; \(g\) is the coupling constant; \(\varphi(x)\) is the soliton \(\varphi_1\); \(\gamma^\mu\) are the Dirac matrices, which we can choose to be the Pauli matrices

\[
\gamma^1 = \sigma_1, \quad \gamma^4 = \sigma_3,
\]

(56)

Now we search a static solution of Dirac equation in the presence of the soliton

\[
(\sigma_1 \partial_x + g\varphi) \psi = E \sigma_3 \psi = 0
\]

(57)

here we consider case with energy \(E = 0\). In this case we have the following normalizable zero mode

\[
\psi = \psi_0 \begin{pmatrix} \cosh \left[ \frac{m}{2} (x - x_0) \right]^{-2m/m_0} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{pmatrix}
\]

(58)

where \(m_\psi = g\varphi_0\). We see that it is strongly localized at the position \(x_0\) of the soliton (see, Fig.3). In the context of mechanism with different vacua this toy model can mean that the free electrons on the layer of the High-T\(c\) superconductor will be located on the boundaries between regions with different vacua.

![FIG. 3: Zero energy fermion in a soliton background.](image)

Evidently, the appearance of localized solutions is connected with the presence of two vacuums \((\varphi = \pm \varphi_0)\). It allows us to assume that the mechanism offered here for the formation of Cooper pair in High-T\(c\) superconductivity depends on the existence of several vacuums for phonon field, \(i.e.\) on the presence of several equilibrium states (stable and unstable) for ions of the lattice.

Such interpretation depends on the existence of different vacua. Thus, this mechanism takes place only in the presence of different equilibrium states for ions of the lattice. It is necessary to note that can exist other mechanism of the field localization in some region. It is possible, for example, that such mechanism (without different vacua of gauge field) occurs in QCD by forming a hypothesized flux tube between quark-antiquark.

Another possibility can be connected with cut-in of an external gauge field. In the Ref.\cite{14} it was shown that the scalar field coupled with the electromagnetic field can forms the flux tube (Nielsen - Olesen flux tube). The detailed investigation of such possibility in the context of the Gingbug-Landau theory is given in Ref.\cite{15}.

It should be pointed out that the investigation of \(\tau(r_1 \cdots) = \langle \varphi^n(r) \rangle\) gives us the information about the mean value of the field \(\varphi(r)\). For the investigation of questions on the scattering or interaction of phonons it is necessary to explore the functions, \(\tau(r_1 \cdots r_n) = \langle 0|\varphi(r_1)\cdots\varphi(r_n)|0\rangle\).
VI. AN EXPERIMENTAL TESTING

How can this string model for the Cooper pair in the High-T\textsubscript{c} superconductor be proved? The best way is the direct observation of the PT between Cooper electrons. Evidently it should be connected with the detection of the ion lattice between Cooper electrons. If we can establish that $\langle s^n_i \rangle \neq 0$ inside of the tube and $\langle s^n_i \rangle \approx 0$ outside of the PT then probably it will indicate the existence of the PT. Certainly, such measurements of the state of ion lattice are very difficult.

Another indirect way is a test for the nonlinear potential $V(r)$. The presence of such nonlinear term can lead to: (a) the nonlinear effects by propagation of the sound wave in the High-T\textsubscript{c} superconductor along the layer; (b) the different equilibrium states of ions in the lattice. Such effects can be: (a) nonlinear scattering, propagation, absorption and so on for the sound waves; (b) appearance of regions with different vacua. Certainly, the presence of such kind of nonlinear effects is not the direct demonstration of the PT but it can indicate the important role of the strong phonon-phonon interaction in the High-T\textsubscript{c} superconductor.

VII. CONCLUSIONS

The basic goal of this paper is an assumption that the quantum solid theory with the strong nonlinear interaction of ions in the lattice can be similar to the QCD in which there is the strong gluon-gluon interaction. Such similarity can lead to the appearance of nonlocal objects in the quantum solid theory like the flux tube in the QCD. It is possible that such a mechanism is realized in the High-T\textsubscript{c} superconductor by such a way that the phonons are confined into a nonlocal object (tube) located between two Cooper electrons.

In this paper we have considered only one mechanism probably leading in emergence the PT. The problem on existence of other alternative such mechanisms was not discussed in this paper and remains while open.

It is very important to note that our analysis shows that the Green’s function method in the superconductivity theory is a realization of an algorithm proposed by Heisenberg for quantization of nonlinear spinor field. We have shown that Green’s function method is such realization of discussed Heisenberg idea. We would like to emphasize that the nonperturbative Heisenberg quantization method is much more powerful than a perturbative Feynman diagram techniques.

Finally, we can presuppose that in the quantum field theory nonlinearity can lead to nonlocality \[1\]: nonlinear terms $(A)^3$ and $(A)^4$ in the QCD lead to the flux tube and probably such a mechanism in the High-T\textsubscript{c} superconductivity leads to the PT.

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we can name such condensate as a dual superconductor

in ordinary superconductor this happens with magnetic field

this means that the Cooper pair can be destroyed

in the case of strong nonlinearity, of course

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