Many-body filling-factor dependent renormalization of Fermi velocity in graphene in strong magnetic field

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We present the theory of many-body corrections to cyclotron transition energies in graphene in strong magnetic field due to Coulomb interaction, considered in terms of the renormalized Fermi velocity. A particular emphasis is made on the recent experiments where detailed dependencies of this velocity on the Landau level filling factor for individual transitions were measured. Taking into account the many-body exchange, excitonic corrections and interaction screening in the static random-phase approximation, we successfully explained the main features of the experimental data, in particular that the Fermi velocities have plateaus when the 0th Landau level is partially filled and rapidly decrease at higher carrier densities due to enhancement of the screening. We also explained the features of the nonmonotonous filling-factor dependence of the Fermi velocity observed in the earlier cyclotron resonance experiment with disordered graphene by taking into account the disorder-induced Landau level broadening.

I. INTRODUCTION

Massless Dirac electrons in single-layer graphene offer an opportunity to study condensed-matter counterparts of relativistic effects and to achieve new regimes in quantum many-body systems \cite{1,2}. Low-energy electronic excitations in this material obey the Dirac equation and static, many-body systems \cite{1–3}. Low-energy electronic excitations in this material obey the Dirac equation and relativistic effects and to achieve new regimes in quantum many-body systems \cite{1,2}. Unlike graphene Landau levels was first confirmed by the half-integer quantum Hall effect \cite{2}, and direct observations of these levels using the scanning tunneling spectroscopy had followed (see the review of experiments in \cite{5}).

Another way to study Landau levels in graphene is to induce electron interlevel transitions by an electromagnetic radiation, typically in the infrared range. The selection rules for photon absorption \cite{6} require $|\Delta n|=\pm 1$, implying the intraband $-n-1\rightarrow -n, n\rightarrow n+1$ and interband $-n-1\rightarrow n$ (which will be referred to as $T_{n+1}^{-}$) and $-n\rightarrow n+1$ (referred to as $T_{n+1}^{+}$) transitions. The interband transitions $T_{n+1}^{\pm}$, which are more widely studied, have the energies

$$E_{n+1}-E_{n}=v_{F}\sqrt{2}\hbar c/\sqrt{n+1}$$

in the ideal picture of massless Dirac electrons \cite{11} in the absence of interaction and disorder.

In a series of cyclotron resonance measurements, mainly on epitaxial graphene, transition energies in very good agreement with Eq. (2) were reported (see \cite{5,6} and references therein). However, the other experiments \cite{9–12} demonstrated deviations from Eq. (2) due to many-body effects and, possibly, disorder. Similar deviations were discovered in magneto-Raman scattering for both cyclotron $T_{n+1}^{-}$ and symmetric interband $-n\rightarrow n$ transitions. Indeed, the Kohn’s theorem \cite{17}, which protects cyclotron resonance energies of usual massive electrons against many-body renormalizations, is not applicable to graphene \cite{18,29}. The observed energies of $T_{n+1}$ can be described by the counterpart of Eq. (2)

$$\Omega_{n+1}^{\pm}=v_{F}\sqrt{2}\hbar c/\sqrt{n+\sqrt{n+1}}$$

with the bare Fermi velocity $v_{F}$ replaced by the renormalized velocity $v_{F}^{\pm}$. While the former one, $v_{F}$, should be close to $0.85 \times 10^{6}$ m/s, as indicated by theoretical fits to experimental data or by first-principle calculations (see, e.g., \cite{29,61}), the latter one, $v_{F}^{\pm}$, range from $10^{6}$ m/s to $1.4 \times 10^{6}$ m/s depending on carrier density, magnetic field and substrate material \cite{3,10}. The existing theory describes renormalization of Fermi velocity in magnetic field with reasonable accuracy in the Hartree-Fock \cite{15,22,24} and static random-phase \cite{26,29,52} approximations.

In two very recent experiments \cite{12,13}, the energies of the $T_{n+1}^{\pm}$ transitions were measured with high accuracy as functions of the Landau level filling factor $\nu$, that may provide an especially deep insight into the many-body physics of graphene in magnetic field. Unlike graphene without magnetic field, where $v_{F}^{\pm}$ diverges logarithmically upon approach to the charge neutrality point \cite{3,33}, here it saturates to a finite value at $\nu \rightarrow 0$, and, in the most cases, has even a broad plateau in the range $-2<\nu<2$.

In this article, we calculate the energies of the $T_{n+1}^{\pm}$ transitions as functions of the filling factor $\nu$ with taking into account many-body effects. Our approach, which is
described in Sec. II and Appendices A, B and C takes into account the screening of the Coulomb interaction as one of the key points. It was applied earlier to reproduce experimental data on both Landau levels and interlevel transition [29] energies. In Sec. II we analyze the electron-hole asymmetry of transition energies and the presence of plateaus at $-2 < \nu < 2$, following from the properties of interaction matrix elements.

In Sec. IV we present the results of numerical calculations, which reproduce the main features of the experimental $v_F(\nu)$ dependencies from Refs. 12, 13: a) the plateaus in $v_F^2$ at $-2 < \nu < 2$ when the 0th Landau level is partially filled, b) the rapid decrease of $v_F^2$ at $|\nu| > 2$ with increasing carrier density, c) the decrease of $v_F^2$ at $\nu = \text{const}$ at increasing magnetic field. We have found good agreement between the experiments and the theory using the bare Fermi velocity $v_F = 0.85 \times 10^6 \text{m/s}$ and realistic values of the dielectric constant $\varepsilon$.

Additionally, we have considered the nonmonotonous dependence $v_F^2(\nu)$ for the $T^\pm$ transition observed in [11] with the maximum at $\nu = 0$ and minima at $\nu = \pm 2$. Taking into account a disorder-induced broadening of Landau levels, we have explained this dependence with good accuracy in Sec. V. Our conclusions are presented in Sec. VI.

II. THEORETICAL APPROACH

Dynamical conductivity of graphene can be calculated using the Kubo formula [32]:

$$\sigma_{\alpha\beta}(q, \omega) = \frac{1}{\hbar \omega S} \int_0^\infty dt e^{i(\omega+i\delta)t} \langle [j_\alpha(q, t), j_\beta(-q, 0)] \rangle, \quad (4)$$

where $j_\beta(q, t)$ is the $\alpha$-axis projection of the Fourier component of the current density operator $j_\beta(q) = e^{i\mathbf{q}\cdot\mathbf{r}} \int d\mathbf{r} \Psi^+(\mathbf{r}) \sigma_\beta \Psi(\mathbf{r}) e^{-i\mathbf{q}\cdot\mathbf{r}}$ evolving in time in the Heisenberg representation, $\Psi(\mathbf{r})$ is the two-component field operator for Dirac electrons, $S$ is the system area, and $\delta \to +0$.

Diagrammatic representation of the conductivity, shown in Fig. 1(a), allows its calculation in terms of the current vertex matrix $\Gamma_\beta$, which would be equal to $\sigma_\beta$ in the absence of interaction and disorder. To find it, we use the mean field approximation, where the excitonic ladder [Fig. 1(b)] for the vertex $\Gamma_\beta$ and the one-loop self-energy corrections [Fig. 1(c)] for the single-particle Green functions $G$ are taken into account. Using the interaction, which is statically screened in the random-phase approximation [Fig. 1(d)], greatly simplify the calculations. If we additionally neglect the mixing of different pairs of electron and hole Landau levels, appearing in the excitonic ladder (which was shown to be weak under typical conditions with using the screened interaction [29]), the optical conductivity $\sigma_{\alpha\beta}(\omega) \equiv \sigma_{\alpha\beta}(0, \omega)$ is (see the details of calculations in Appendix C):

$$\sigma_{\alpha\beta}(\omega) = \frac{ie^2v_F^2}{\omega} \sum_{n_1n_2} \frac{f_{n_2} - f_{n_1}}{\hbar \omega - \Omega_{n_1n_2} + i\delta} \times \text{Tr} \left[ \Phi_{n_1n_2}(0) \sigma_\alpha \right] \text{Tr} \left[ \Phi^{+}_{n_2n_2}(0) \sigma_\beta \right]. \quad (5)$$

Here $f_n$ is the occupation number ($0 \leq f_n \leq 1$) of the $n$th Landau level, and the matrix $\Phi_{n_1n_2}(0)$, which is defined by (33) and (34), determines the selection rules $|n_1| = |n_2| \pm 1$ for each $n_2 \to n_1$ transition. The resonant transition energy $\Omega_{n_1n_2}$, where $\sigma_{\alpha\beta}$ has a pole, consists of the difference between the bare Landau level energies $E_{n_1} - E_{n_2}$, the difference between electron self-energies $\Sigma_{n_1} - \Sigma_{n_2}$, and the excitonic correction $\Delta E_{n_1n_2}$ (see the similar formula in [29]):

$$\Omega_{n_1n_2} = E_{n_1} - E_{n_2} + \Sigma_{n_1} - \Sigma_{n_2} + \Delta E^{(exc)}_{n_1n_2}. \quad (6)$$

In the mean field approximation, the self-energy

$$\Sigma_{n} = -\sum_{n'} f_{n'} \langle m'n'|V|n'n \rangle, \quad (7)$$

as shown in Appendix D is a sum of the exchange matrix elements

$$\langle m'n'|V|n'n \rangle = 2^{\delta_{m'0}+\delta_{n'0}} \frac{\hbar^2 \pi}{2} \int dq V(q) \times \left| s_{n's'} \phi_{|n'-1,n'|} - 1 \langle a_q \rangle + \phi_{|n'|n'} \langle a_{-q} \rangle \right|^2$$

of the screened Coulomb interaction $V(q)$ between the $n$th and all filled $n'$th Landau levels [15, 18, 19], where the functions $\phi_{nk}$ are defined in (34), and $a_q \equiv \mp i^j |e_2 \times \mathbf{q}|$. The excitonic correction

$$\Delta E^{(exc)}_{n_1n_2} = -(f_{n_2} - f_{n_1}) \langle n_1n_2|V|n_1n_2 \rangle \quad (9)$$

FIG. 1: (a) Diagrammatic relationship between the current Green function and the vertex. (b), (c) Equations for, respectively, the vertex function and the electron Green function in the mean-field approximation. (d) Coulomb interaction screening in the random-phase approximation.
is the direct interaction matrix element
\[
\langle n_1 n_2 | V | n_1 n_2 \rangle = 2 \delta_{n_1,n_2} \delta_{\pm n_2} - \frac{2 q^2}{2 \pi} \int dq \, V(q)
\]
\[
\times \left\{ \phi_{n_1 n_2 \pm 1, n_1 n_2}(a_q) + \phi_{n_1 n_2}(a_q) \right\} \tag{10}
\]
with the minus sign, weighted with the difference of occupation numbers of the final and initial levels.

The dynamically screened interaction in the random-phase approximation is [see Fig. 1(d)]
\[
V(q, i\omega) = \frac{v_g}{1 - v_g \Pi(q, i\omega)}, \tag{11}
\]
where \(v_g = 2\pi e^2 / e q\) is the bare Coulomb interaction weakened by the surrounding medium with the dielectric constant \(\varepsilon\), and
\[
\Pi(q, i\omega) = g \sum_{n n'} F_{n n'}(q) \frac{f_n - f_{n'}}{i\omega + E_n - E_{n'}} \tag{12}
\]
is the polarizability (or density response function) of non-interacting Dirac electrons \([3, 35, 36]\). Here
\[
F_{n n'}(q) = 2 \delta_{n n'} + \delta_{n n'}^{-2}
\]
\[
\times \left\{ s_n s_{n'} \delta_{|n| - 1, |n'| - 1}(a_q) + \phi_{|n| |n'|}(a_q) \right\}^2 \tag{13}
\]
is the form-factor of Landau level wave functions and \(g = 4\) is the degeneracy of electron states by valleys and spin. The statically screened interaction \(V(q)\) is obtained from (11), (12) by taking \(i\omega = 0\).

In our model, there are three mechanisms leading to dependence of \(\Omega_{n_1 n_2}\) on the filling factor \(\nu\) via the occupation numbers
\[
f_n = \begin{cases} 0, & \text{if } \nu \leq 4n - 2, \\ (\nu - 4n + 2)/4, & \text{if } 4n - 2 < \nu < 4n + 2, \\ 1, & \text{if } \nu \geq 4n + 2, \end{cases} \tag{14}
\]
i.e.; through exchange energies \([4]\), excitonic corrections \([2]\), and polarizability \([12]\).

Note that the sum \([3]\) over the filled Landau levels \(n'\) in the valence band diverges at \(n' \to -\infty\), so we impose the cutoff \(n' > -n_\text{c}\) to obtain finite results. The physical reason of this cutoff is a finite actual number of Landau levels in the valence band, which can be found from the total electron density: \(n_\text{c} = 2\pi \hbar c / \sqrt{3} a^2 e H \approx 39600 / B [T]\), where \(a \approx 2.46 \, \text{Å} \,[29, 34]\).

III. ELECTRON-HOLE ASYMMETRY AND PLATEAUS AT \(-2 < \nu < 2\)

The selection rule \(|n_1| = |n_2| \pm 1\) for the interband \(n_2 \to n_1\) transitions implies \(n_1, n_2 = n + 1, -n\) (the \(T_{n+1}^+\) transition) or \(n_1, n_2 = n, -n - 1\) (the \(T_{n+1}^-\) transition). In the idealized Dirac model without interactions, the energies of these transitions \([2]\) are equal. However this is no longer the case when exchange self-energies are taken into account. Any nonzero doping \(\nu \neq 0\) introduces an asymmetry between \(\Omega_{n+1}^+\) and \(\Omega_{n+1}^-\), at least, in the mean-field approximation. Looking at (14) and taking into account that \((n + 1, -n) | V | n + 1, -n\) = \((-n - 1, n) | V | -n - 1, n\), we have:
\[
\Omega_{n+1}^- - \Omega_{n+1}^+ = \Sigma_{n+1} - \Sigma_n = \Sigma_{-n} - \Sigma_{-n} + (f_n + f_{n+1} - f_{-n} - f_{-n+1})/(n + 1, -n) | V | n + 1, -n \tag{15}
\]
The electron-hole asymmetry in graphene, which is induced by the exchange interaction in the absence of magnetic field and is similar in scale to our case, was found in \([41]\).

The first line of (15) is a contribution of exchange self-energies to the asymmetry. Let us separate the occupation numbers \(f_n = f_n^{(0)} + \Delta f_n\) on those of undoped graphene \(f_n^{(0)}\) and the doping-induced part \(\Delta f_n\), and define \(\Sigma_n = - \sum_n f_n^{(0)} (n' n'') | V | n' n''\). Using (8) and (16), and neglecting a difference of small matrix elements at \(n' \approx -n_\text{c}\), we get \(\Sigma_{n+1} + \Sigma_{-n+1} = \Sigma_n - \Sigma_{-n}\). Thus the exchange energy contribution to (15) arises only at nonzero doping \(\nu \neq 0\).

The second line of (15) corresponding to excitonic effects is nonzero only when either \(\pm n\) or \(\pm (n + 1)\) level is partially filled, i.e. at \(4n - 2 < |\nu| < 4n + 6\). Since the polarization \([12]\) and hence the screened interaction \(V(q)\) are even functions of \(\nu\), both parts of (15) change sign at \(\nu \to -\nu\), so
\[
\Omega_{n+1}^- (\nu) = \Omega_{n+1}^+ (-\nu). \tag{16}
\]
This is illustrated in Fig. 2, where the typical calculated \(v_\nu^p\) are shown as functions of \(\nu\).

The case of \(n = 0\) is the special one. The explicit structure of the wave functions \([82]\) imply the following relationships connecting the matrix elements of direct and exchange interaction (valid even for non-Coulomb potentials):
\[
(\pm 1, 0) | V | 1, 0 + (\pm 1, 0) | V | 0, \pm 1 = (00) | V | 00. \tag{17}
\]

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**FIG. 2:** Renormalized Fermi velocities \(v_\nu^p\) for (a) the \(T_{n+1}^+\) and (b) the \(T_{n+1}^-\) transitions calculated with the screened interaction at \(v_F = 0.85 \times 10^6 \, \text{m/s}, \varepsilon = 3.27, B = 8 \, \text{T}\). Solid lines show the velocities found from the weighted transition energies \([20]\).
In result, the doping-induced changes of exchange and excitonic parts of (6) due to $f_0$ cancel each other at $-2 < \nu < 2$, when the 0th level is partially filled. Additionally, the polarizability (12) and hence $V(q)$ are also unchanged in this range of $\nu$, thus we expect plateaus in both $\Omega_1^\pm$:

$$\Omega_1^+(\nu) = \Omega_1^-(\nu) = \text{const. \ at \ } -2 < \nu < 2,$$

as seen in Fig. 2(a). For $n \neq 0$ this is no longer the case, although variations of $\Omega_1^\pm$ at $-2 < \nu < 2$ are typically very small [see Fig. 2(b)].

In experiments, $\Omega_2^+\pm_1$ and $\Omega_2^-\pm_1$ can be separated by observing cyclotron resonant absorption of light with opposite circular polarizations. Using linear polarization, one can observe a mixture of these transitions with relative intensities $I_2^+\pm_1 = f_{-n}^\pm f_{n+1}^\pm$ and $I_2^-\pm_1 = f_{n}^\pm f_{-n-1}^\pm$, equal to occupation number differences in final and initial states. Assuming that experimental apparatus does not resolve the individual lines $\Omega_2^\pm_1$ and $\Omega_2^-\pm_1$, we will calculate the weighted transition energy

$$\langle \Omega_{n+1} \rangle = \frac{\Omega_{n+1}^+ I_{n+1}^+ + \Omega_{n+1}^- I_{n+1}^-}{I_{n+1}^+ + I_{n+1}^-}$$

and compare it with the experiments in the next section. From the particle-hole symmetry relationship $f_{\nu}(-\nu) = 1 - f_{\nu}(\nu)$ we see that $\langle \Omega_{n+1} \rangle$ is even function of $\nu$. At $-2 < \nu < 2$, $\langle \Omega_{n+1} \rangle$ is linear (via $f_0$) and at the same time even function of $\nu$, so

$$\langle \Omega_{n+1} \rangle = \text{const. \ at \ } -2 < \nu < 2,$$

as seen in Figs. 2(a,b). Thus our model predicts plateaus in all weighted transition energies $\langle \Omega_{n+1} \rangle$ at $-2 < \nu < 2$. Similar conclusions about existence of the electron-hole asymmetry [15] and the conjugation property [16] were made in the recent theoretical work [25], which considers transition energies in the Hartree-Fock approximation.

IV. CALCULATION RESULTS

First we compare our calculations of the renormalized Fermi velocities

$$v_F^\pm = \frac{\langle \Omega_{n+1} \rangle}{\sqrt{2B\hbar/c}}$$

with the data of Ref. [12] where $\Omega_1 \ldots \Omega_6$ as functions of $\nu$ were measured at three magnetic fields $B = 5, 8, 11$ T. We fit the experimental points in three approximations:

1) Hartree-Fock approximation, where the unscreened Coulomb potential $v_F$ is used in all calculations.

2) Static random-phase approximation, where the potential $V(q)$ is screened [11] with using the polarizability of noninteracting electron gas in magnetic field.

3) Self-consistent screening approximation, where the polarizability is multiplied by $v_F^\pm/v_F^0$ to take into account weakening of the screening caused by many-body

increase of the energy differences $E_\nu - E_n$ in denominators of (12). Similarly to our previous works [29, 54], this semi-phenomenological model is aimed to achieve a self-consistency between many-body renormalizations of transition energies and screening. Using the iterative procedure, we take $v_F^\pm$, obtained on each step, to renormalize the screening when calculating new $v_F^\pm$ on the next step. About 5-6 iterations are usually sufficient to achieve a convergence.

In all calculations, we assume the bare Fermi velocity $0.85 \times 10^6$ m/s, which seems to be the most suitable based on theoretical fits of experimental data or first-principle calculations (see, e.g., [23, 30, 12]). The optimal dielectric constant of the surrounding medium $\varepsilon$ is the only adjustable parameter in our model, and we find it by performing the least square fitting of the experimental points for all $n$ and $B$ simultaneously.

The first line of Table I shows the optimal $\varepsilon$ used to fit the cyclotron resonance data of Ref. [12] where high-mobility graphene samples were encapsulated from both sides in hexagonal boron nitride monolayers and placed on an oxidized silicon. Fig. 3 shows the experimental points together with our calculations at these $\varepsilon$ in the three approximations described above. The calculation with the unscreened Coulomb interaction (Hartree-Fock approximation) demonstrates two significant drawbacks. First, the dielectric constant $\varepsilon = 7.72$ is unrealistically high, because in this approximation it should imitate the interaction screening by Dirac electrons in graphene in addition to the screening by an external medium. Second, the falloff of $v_F^\pm$ at $|\nu| > 2$ turns out to be insufficient, because the increase of the screening strength (and, consequently, suppression of the upward renormalization of the Fermi velocity) following the carrier density, is absent here. For the $T_1$ transitions, the calculated $v_F^\pm$ even increases at $|\nu| > 2$, in contradiction with the experiment, because the excitonic correction [9], which normally decrease $v_F^\pm$, become suppressed due to partial filling of 1th or $-1$th Landau level.

The similar drawbacks of the Hartree-Fock approximations were mentioned in our previous works [29, 54].

The screening allows us to achieve much better agreement with the experimental points at more realistic $\varepsilon = 3.27$, and the falloff of $v_F^\pm$ at $|\nu| > 2$ is reproduced very well. The iterative calculations with the self-consistent screening provide almost the same curves, but at somewhat higher $\varepsilon \approx 4.36$. This distinction arises because the

| Experiment | Interaction | Interaction | Screening |
|------------|-------------|-------------|-----------|
| Russell et al. [12] | 7.72 | 3.27 | 4.36 |
| Sonntag et al. [13] | 5.50 | 1.05 | 2.55 |
higher $\varepsilon$ is needed to compensate the screening weakening caused by an upward renormalization of energy denominators in [12].

Our calculations with taking into account the screening are thus able to fit the data of Ref. [12] at three different $B$ and for six resonances $T_{n+1}$ simultaneously with the single adjustable parameter $\varepsilon$. We can explain both the decrease of $v_F^\nu$ at $\nu = 0$ as $B$ gets higher, the plateaus at $|\nu| < 2$, and the rapid falloff of $v_F^\nu$ at $|\nu| > 2$, $n \geq 1$ due to increase of the screening strength. The exceptions are some inconsistencies of $v_F^\nu$ at specific resonances ($T_2$ and $T_6$ at $B = 5$ T, $T_3$ and $T_6$ at $B = 8$ T) and the local maxima at $\nu = 0$ for the $T_1$ transitions. Moreover, the local minima of $v_F^\nu$ for $T_2$ at $\nu = \pm 4$, when the 1th or −1th Landau level is half-filled, which occur only at $B = 8$ T and are absent in other fields, are not predicted by our approach.

Another experiment we analyze is Ref. [13] where graphene is suspended 160 nm above oxidized silicon, and the filling-factor dependence of the $T_2$ transition energy was measured at $B = 3$ T by observing its avoided crossing with the phonon energy in Raman spectrum. In Fig. [4] we plot the results of our calculations for this transition in the three approximations at optimal $\varepsilon$ listed in the second line of Table [I]. We observe the same regularities as in the previous case. The Hartree-Fock approximations requires overestimated $\varepsilon$ and cannot explain the rapid falloff of $v_F^\nu$ at $|\nu| > 2$. At $|\nu| > 6$ we see even slight increase of $v_F^\nu$ due to suppression of the excitonic correction when the 2nd or −2nd Landau level start to be partially filled. In contrast, with taking into account the screening we obtain the realistic $\varepsilon$ for graphene suspended above the

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**FIG. 3: Renormalized Fermi velocities $v_F^\nu$ at (a) $B = 5$, (b) 8, and (c) 11 T for the set of the $T_{n+1}$ transitions ($-n \to n+1/ -n-1 \to n$), taken from the experiment [12] (crosses), and calculated theoretically in the Hartree-Fock approximation (dashed lines), with taking into account the interaction screening (solid lines) and with the self-consistent screening (dotted lines). The dielectric constants $\varepsilon$, used in each calculation, are listed in the first line of Table [I].**

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**FIG. 4: Renormalized Fermi velocity $v_F^\nu$ for the $T_2$ transition ($-1 \to 2/ -2 \to 1$) at $B = 3$ T, taken from the experiment [13] (squares) and calculated theoretically in the Hartree-Fock approximation (dashed lines), with taking into account the interaction screening (solid lines) and with the self-consistent screening (dotted lines). The dielectric constants $\varepsilon$, used in each approximation, are listed in the second line of Table [I].**
oxidized silicon, and the falloff is well reproduced. Nevertheless, the experimental points demonstrate an additional maximum at $\nu = 0$. This is not described by our approach, which predicts plateaus at $|\nu| < 2$, as discussed in Sec. III.

V. LANDAU LEVEL BROADENING

One more experiment where the filling-factor dependent transition energy was measured is Ref. [11]. In this earlier work, graphene layer lied directly on an oxidized silicon substrate and carrier mobility was one-two orders of magnitude lower than in the aforementioned works [12, 13] due to charged impurities in the substrate. The $T_1$ cyclotron resonance was studied at $B = 18$ T and the unusual W-shaped form of the transition energy vs. $\nu$ was found with the local maximum at $\nu = 0$ and two minima at integer Landau level fillings $\nu = \pm 2$.

To explain these results, we need to take into account disorder, because at mobilities of several thousands of $\text{cm}^2/\text{V} \cdot \text{s}$, reported in [11], the disorder-induced Landau level widths $\sim 20 \text{meV}$ become comparable with the energy scale $e^2/\varepsilon l_H$ of Coulomb interaction effects. The main mechanism of disorder effect on the transition energies is the following. Assume that Landau levels are broadened giving rise to Gaussian mini-bands in the density of states, as shown in Fig. 5. At partial filling of each level, its mini-band is partially filled, so the average energy of the filled (empty) electron states is lower (higher) than the band center where the unperturbed Landau level energy would be located. As a result, the average transition energy increases due to Landau level broadening in addition to interaction effects when either initial or final level is partially filled ($\nu \neq \pm 2$ in our case). The similar effect was discussed in [14] for a two-dimensional gas of massive electrons in the framework of self-consistent Born approximation.

To describe this effect, we assume the Gaussian spectral density $\rho_n(E) = (\sqrt{2\pi} \Gamma_n)^{-1} \exp[-(E - E_n)^2/2\Gamma_n^2]$ for each $n$th partially filled broadened level. Integrating it up to the Fermi level $\mu$ and assuming low temperature, we find the occupation number $f_n$, and, using [14], we get the relationship between $\nu$ and $\mu$: $\nu = 4n + 2\Phi([\mu - E_n]/\sqrt{2\Gamma_n})$, where $\Phi$ is the error function. The disorder-induced correction $\langle \Delta \Omega_n \rangle$ to the transition energy is a difference between the average energies (relative to the band centers) of empty states on a final Landau level and filled states on an initial level. It should be additionally weighted according to [20], when $-2 < \nu < 2$ and thus both transitions $T_1^\pm$ are present, resulting in:

$$\langle \Delta \Omega_n \rangle = \begin{cases} \sqrt{2} \Gamma_n^{-1} e^{-\frac{(\nu - \mu)^2}{2\Gamma_n^2}}, & \text{if } -6 < \nu < -2, \\ \sqrt{2} \Gamma_n^{-1} e^{-\frac{(\nu - \mu)^2}{2\Gamma_n^2}}, & \text{if } -2 < \nu < 2, \\ \sqrt{2} \Gamma_n^{-1} e^{-\frac{(\nu - \mu)^2}{2\Gamma_n^2}}, & \text{if } 2 < \nu < 6. \end{cases}$$

This dependence has a maximum at $\nu = 0$ and minima at $\nu = \pm 2$ in accordance with the experiment [11].

Another effect of the disorder is the presence of inter-level transitions when any $n$th Landau level is partially filled, which provide an extra contribution to the screening. In the simplest approximation, they lead to the polarizability of the Thomas-Fermi kind

$$\Pi_n^\text{TF}(q) = -gF_n(q)\rho_n(\mu),$$

which was used in [42] to study Landau level broadening in graphene.

We use the self-consistent Born approximation for a polarizability in magnetic field, which was originally developed in [14, 15] for a two-dimensional electron gas with short-range impurities. In our work, we assume the disorder to be long-ranged, because the main origin of disorder in graphene on a SiO$_2$ substrate are long-range charged impurities [46]. Introducing the mean square $(U^2)$ of the slowly varying disorder potential $U(r)$, we get the following polarizability of disordered graphene (see the similar formulas in [11, 15] obtained by summing an impurity ladder in a polarization loop):

$$\Pi_n(q, i\omega) = g \sum_{nn'} F_{nn'}(q) \times \sqrt{T} \sum_{\epsilon} \frac{G_n^D(i\epsilon + i\omega)G_{n'}^D(i\epsilon)}{1 - G_n^D(i\epsilon + i\omega)G_{n'}^D(i\omega)}.$$
the branch cut at $\text{Im}(i\epsilon) = 0$, we get in the limit $T \to 0$:

\[
\Pi^D(q,0) = -\frac{g}{\pi} \sum_{nn'} F_{nn'}(q) \times \int_{-\infty}^{0} \text{Im} \frac{G_n^D(z + i\delta)G_{n'}^D(z + i\delta)}{1 - (U^2)G_n^D(z + i\delta)G_{n'}^D(z + i\delta)}. \tag{25}
\]

This polarizability consists of two physically distinct parts. The first one is the contribution of interlevel transitions with $n \neq n'$. It does not differ too much from that in a clean system \[12\] if the widths of Landau levels $\Gamma_n$ are much smaller than interlevel separations. The second one is the contribution of intralayer transitions $n = n'$ arising when the $n$th layer is partially filled. Taking the disorder strength to be equal to the Landau level width $\sqrt{U^2} = \Gamma_n$, as follows from calculations of $G_n^D$ with the long-range disorder, we get the static polarizability of disordered graphene:

\[
\Pi^D(q,0) \approx \Pi(q,0) - gF_{nn}(q) \int_{-\infty}^{0} \text{Im} \frac{[G_n^D(z + i\delta)]^2}{1 - \Gamma_n^2[G_n^D(z + i\delta)]^2}. \tag{26}
\]

and use it in the following calculations.

FIG. 6 shows the examples of static polarizabilities calculated at half-fillings of 0th and ±1th Landau levels. In a clean graphene, $\Pi(q,0) \propto q^6$ at $q \to 0$ since the system becomes insulating in magnetic field, and the only source of the screening are gapped interlevel transitions. Disorder makes $\Pi^D(q,0)$ nonzero at $q \to 0$ due to intralevel transitions. The Thomas-Fermi approximation, by taking into account only the latter, provides a wrong short-wavelength asymptotic of the polarizability $\Pi_n^{TF}(q)$, which should tend to the polarizability of undoped graphene $\Pi(q,0) = -gq/16\hbar v_F[1]$.

We calculated the renormalized Fermi velocity, corresponding to the weighted energy \[20\] of the $T_1$ transition with taking into account the correction \[22\] and the screening \[26\] in the disordered system. For comparison, we carried out the same calculations for the clean system, as did in the previous section. The results of the fitting of experimental points from Ref. \[11\] at $v_F = 0.85 \times 10^6$ m/s in the three approximations for the interaction listed in Sec. \[14\] for clean or disordered system. For disordered system, the widths of 0th and ±1st Landau levels are also specified in the last two lines.

![FIG. 7: Renormalized Fermi velocity $v_F^*$ for the $T_1$ transition (0 → 1/−1 → 0) at $B = 18$ T, taken from the experiment \[11\] (squares) and calculated theoretically for (a) clean and (b) disordered graphene. The calculations are carried out in the Hartree-Fock approximation (dashed lines), with taking into account the interaction screening (solid lines) and with the self-consistent screening (dotted lines). The dielectric constants $\varepsilon$ used in each calculation are listed in Table \[11\].](image)

TABLE II: First two lines: dielectric constants of surrounding medium $\varepsilon$, which provide the best least-square fittings of the experimental data from Ref. \[11\] at $v_F = 0.85 \times 10^6$ m/s in the three approximations for the interaction listed in Sec. \[14\] for clean or disordered system. For disordered system, the widths of 0th and ±1st Landau levels are also specified in the last two lines.

| System | Interaction | Interaction | Screening |
|--------|-------------|-------------|-----------|
| Clean  | 7.26 | 2.82 | 3.85 |
| Disordered | 9.24 | 4.95 | 5.74 |
| $\Gamma_0$ (meV) | 22 | 25 | 23 |
| $\Gamma_{\pm 1}$ (meV) | 12 | 19 | 17 |

![FIG. 6: Dimensionless static polarizability of graphene in magnetic field $\Pi(q,0) = -(2\pi v_F l_H / g)\Pi(q,0)$, where $l_H = \sqrt{\hbar c / eH}$, calculated (a) when the 0th Landau level is half-filled, $\nu = 0$, (b) when the 1st or −1th level is half-filled, $\nu = \pm 4$. Solid lines: clean graphene \[12\], dashed lines: disordered graphene \[26\], dotted lines: the Thomas-Fermi approximation \[22\]. Calculation parameters are $v_F = 0.85 \times 10^6$ m/s, $B = 18$ T, $\Gamma_0 = \Gamma_{\pm 1} = 20$ meV.](image)
to those obtained in our earlier analysis \[29\] of cyclotron resonance data for graphene on SiO\(_2\). However the most drastic effects come from inclusion of disorder: while in the clean limit \(v_F^*\) has the plateau at \(|\nu| < 2\) and remain the same (or slightly increases due to suppression of the excitonic correction) at \(|\nu| > 2\), in the disordered system it has the parabolic-like maximum at \(\nu = 0\) and the sharp minima at \(\nu = \pm 2\), just as the experiment shows. The values of Landau level widths \(\Gamma_n\) obtained via the fitting procedure (15 – 25 meV) look realistic, since they are close to typical widths of spectral lines observed in the same experiment \[11\] and in other works on graphene on a SiO\(_2\) substrate \[48\].

VI. CONCLUSIONS

We present detailed calculations of the inter-Landau level cyclotron transition energies in graphene in strong magnetic fields taking into account Coulomb interaction between massless Dirac electrons. Calculating the optical conductivity and solving the vertex equation in the static random-phase approximation with the excitonic ladder, we found the many-body corrections to the transition energies coming from the self-energy and excitonic effects. We show that the cyclotron transition lines can be split in doped graphene for opposite circular polarizations because of the electron-hole asymmetry of exchange self-energies, although this splitting may be unobservable if these lines are sufficiently wide or either a linearly polarized or unpolarized light is used. By this reason, we calculate the weighted transition energy for both polarizations at once and convert it to the renormalized Fermi velocity \(v_F^*\) for each transition.

The dependence of \(v_F^*\) on the Landau level filling factor \(\nu\) is analyzed. In the mean-field approximation, \(v_F^*(\nu)\) has a plateau at \(-2 < \nu < 2\) due to a partial cancelation of the self-energy and excitonic effects and rapidly decreases at \(|\nu| > 2\) due to enhancement of the screening. Our calculations, carried out with the bare Fermi velocity \(v_F = 0.85 \times 10^6\) m/s and with the dielectric constant of surroundings \(\varepsilon\), treated as an adjustable parameter, showed good agreement with two recent experiments \[12, 13\] on high-mobility graphene samples, when the screening by graphene electrons is taken into account.

The obtained phenomenological \(\varepsilon\) describe the external dielectric screening not only by an underlaying substrate, but also by adjacent hexagonal boron nitride layers. The Hartree-Fock approximation, which neglects the density-dependent screening by graphene electrons, fails to explain the observed rapid decrease of \(v_F^*\) at \(|\nu| > 2\).

We also describe the data of the earlier cyclotron resonance experiment \[11\] with graphene sample on SiO\(_2\), where carrier mobility is much lower. In this case we take into account long-range disorder, which broadens Landau levels and thus shifts the resonant energy upward when initial or final level is partially filled, and induces the interlevel transitions contributing to the screening. Assuming the Gaussian spectral density for the 0th and ±1th broadened Landau levels, we achieved good agreement with the experiment and explained the main features of the \(v_F^*(\nu)\) dependence: the parabolic-like maximum at \(\nu = 0\) and the sharp minima at \(\nu = \pm 2\).

As shown, the combined action of exchange interaction, excitonic effects, interaction screening and disorder should be taken into account when considering graphene in strong magnetic field. However, some issues remain to be clarified. In particular, the mean-field approach does not describe the A-shaped maxima of \(v_F^*\) at \(\nu = 0\) observed in \[12, 13\] for \(T_1\) transitions, the minima at \(\nu = \pm 4\) observed for \(T_2\) at \(B = 8\) T in \[12\], and a possible splitting of the \(T_1\) transition line observed in \[12\].

All these features go beyond the mean-field theory for massless Dirac electrons and can be attributed to some unaccounted role of disorder, finite size effects, Moire superlattice potential from adjacent boron nitride layers \[49\], Landau level splitting \[4, 50\] or electron dynamics on a partially filled level \[51\]. Note that assumption of a substrate-induced band gap allowed to explain some features of the experimental data \[12\] in the recent work \[25\], so a further analysis in this direction with considering possible symmetry breakings and gap formation in a system of Dirac electrons together with the interaction, screening and disorder seems to be promising.

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Appendix A: Electron wave functions

Similarly to \[4, 37, 39\], we describe single-particle states of massless electrons in magnetic field \(H\) using the symmetric gauge \(A = \frac{1}{2} [H \times \mathbf{r}]\). In the absence of a valley splitting or intervalley transitions, it is sufficient to consider the electrons only in the \(K\) valley, where the Dirac Hamiltonian is

\[
H = v_F \left( \mathbf{p} - \frac{e}{c} \mathbf{A} \right) \cdot \mathbf{\sigma} = \frac{\hbar v_F}{l_H} \sqrt{2} \begin{pmatrix} 0 & a \\ a^+ & 0 \end{pmatrix}.
\]  

(A1)

Here \(l_H = \sqrt{\hbar c/|e|H}\) is the magnetic length (we assume \(e < 0\) in this section), \(a = l_H p_-/\hbar - ir_-/2l_H\) and \(a^+ = l_H p_+/\hbar + ir_+/2l_H\) are, respectively, lowering and raising operators obeying the commutation relation \([a, a^+] = 1\), and
\[ p_\pm = (p_x \pm ip_y)/\sqrt{2}, \quad r_\pm = (x \pm iy)/\sqrt{2}. \]

Introducing the complementary set of ladder operators \( b = l_H p_+ / h - ir_+ / 2l_H, \ b^+ = l_H p_- / h + ir_- / 2l_H, \) which obey \([b, b^+] = 1\) and commute with \( a, a^+\), we can construct the states of a two-dimensional oscillator \( |\phi_{nk}\rangle = (a^+)^n (b^+)^k |\phi_0\rangle / \sqrt{n!k!} \) with the wave functions in polar coordinates:

\[
\phi_{nk}(r, \varphi) = i^{n-k} / \sqrt{2\pi l_H} \sqrt{\min(n,k)! / \max(n,k)!} e^{-r^2 / 4l_H^2} \left( r / \sqrt{2l_H} \right)^{n-k} e^{i(n-k)\varphi / 2l_H} L_{n-k}^{1/2}(r^2 / 2l_H^2), \quad (A2)
\]

where \( L_n^m(x) \) are the associated Laguerre polynomials. The eigenfunctions of the Hamiltonian \( [11] \) are \( [4, 18, 21, 36-40] \)

\[
\psi_{nk} = (\sqrt{2})^{\delta_{n0} - 1} \left( s_n \phi_{|n-1, k\rangle} / \phi_{|n, k\rangle} \right), \quad (A3)
\]

and eigenvalues are \( [11] \). Here \( n = 0, \pm 1, \pm 2, \ldots \) is the Landau level number, \( k = 0, 1, 2, \ldots \) is the guiding center index responsible for Landau levels degeneracy, \( s_n \equiv \text{sign}(n) \), and we assume that \( \phi_{nk} = 0 \) if \( n \) or \( k \) is negative.

The bare electron Green function in the Matsubara representation \( G(r, r', \tau) = -\langle T_\tau \Psi(r, \tau) \Psi(r', 0) \rangle \) can be constructed from \( (A3) \):

\[
G_0(r, r', ic) = \sum_{nk} \psi_{nk}(r) \psi_{nk}^\dagger (r') / ic - E_n + \mu, \quad (A4)
\]

where \( \mu \) is the chemical potential; note \( G_0 \) is the \((2 \times 2)\) matrix in the sublattice space.

Using the table integral Eq. 2.20.16.10 from \( [52] \), we can present \( (A2) \) in Cartesian coordinates as

\[
H'[|n0\rangle = 2 \text{ and commute with } \left\{ \phi_{nk} \right\}, \quad \text{a lot of useful transformation rules for } \phi_{nk} \text{ can be obtained, for example, the summation formula}
\]

\[
\sum_{k=0}^{\infty} \phi_{n+1,k}(r_1) \phi_{n+2,k}(r_2) = \frac{e^{i(r_1 \cdot r_2) / 2l_H^2}}{\sqrt{2\pi l_H}} \phi_{n+2,k}(r_1 - r_2) \quad (A6)
\]

and the form-factor of Landau level wave functions (see also \( [39] \))

\[
\int dr e^{-iqr} \phi_{n+1,k}(r) \phi_{n+2,k}(r) = 2\pi l_H^2 \phi_{n+1,k}(a_\mathbf{q}) \phi_{n+2,k}(a_\mathbf{q}), \quad a_\mathbf{q} \equiv -l_H^2 [e_\mathbf{z} \times \mathbf{q}]. \quad (A7)
\]

**Appendix B: Exchange self-energies**

Exchange self-energy acquired by an electron in the state \( \psi_{nk} \) is given by the usual Fock expression

\[
\Sigma_{nk}^{\text{exch}} = -\sum_{n'k'} f_{n'k'} \int dr_1 dr_2 v(r_1 - r_2) \psi_{nk}^\dagger (r_1) \psi_{n'k'}^\dagger (r_1) \psi_{n'k'} (r_2) \psi_{nk} (r_2). \quad (B1)
\]

After the Fourier transform of the Coulomb interaction \( v(r) = (2\pi)^{-2} \int d\mathbf{q} v_\mathbf{q} e^{i\mathbf{qr}} \) and using \( (A3) \)--\( (A7) \), we get

\[
\Sigma_{nk}^{\text{exch}} = -\sum_{n' \mathbf{q}} f_{n' \mathbf{q}} \langle nn' | v | nn' \rangle \quad (B2)
\]

with the exchange matrix elements of Coulomb interaction defined as

\[
\langle nn' | v | nn' \rangle = 2^{s_{n0} + s_{n0}} \frac{12}{2\pi} \int d\mathbf{q} v_\mathbf{q} \left| s_{n0} s_{n0} \phi_{|n-1, n'| - 1} (a_\mathbf{q}) + \phi_{|n|n'} (a_\mathbf{q}) \right|^2. \quad (B3)
\]
We assumed that the occupation numbers do not depend on \( k' \), \( f_{n'k'} \equiv f_{nk} \), and the resulting \( \Sigma^{\text{exch}}_{nk} \) turns out to be also independent on \( k \), so the Landau level degeneracy is preserved. By replacing \( \psi_j \) with the statically screened interaction \( V(q) \), as depicted in Fig. II(c), we get the screened exchange energy (7)–(8) and the bare electron Green function (A3) becomes “dressed” with the interaction and turns into

\[
G(r, r', i\epsilon) = \sum_{nk} \frac{\psi_{nk}(r)\psi_{nk}^+(r')}{i\epsilon - E_n - \Sigma_n + \mu}. \tag{B4}
\]

**Appendix C: Vertex equation**

Introducing the Green function for currents \( G^j_{\alpha\beta}(r, r', \tau) = -(T_\tau \Psi^+(r, \tau)\sigma_\alpha \Psi(r, \tau)\Psi^+(r', 0)\sigma_\beta \Psi(r', 0)) \), we can write the conductivity (11) as

\[
\sigma_{\alpha\beta}(q, \omega) = \frac{ie^2q^2}{\hbar\omega S} \int drdr' e^{-iq(r-r')}G^j_{\alpha\beta}(r, r', \hbar\omega + i\delta), \tag{C1}
\]

where \( G^j_{\alpha\beta} \) can be calculated, as shown in Fig. II(a), from the \((2 \times 2)\) vertex matrix:

\[
G^j_{\alpha\beta}(r, r', i\omega) = T \sum_\epsilon \int dr_1dr_2 Tr[\sigma_{\alpha\beta}G(r, r_1, i\epsilon + i\omega)\Gamma_\beta(r_1, r_2, r', i\epsilon, i\omega)G(r_2, r, i\epsilon)]. \tag{C2}
\]

Here the sum is taken over the fermionic Matsubara frequencies \( \epsilon = \pi T(2n+1) \).

The vertex equation in the mean-field (or ladder) approximation, depicted in Fig. II(b), is written analytically as

\[
\Gamma_\beta(r_1, r_2, r', i\epsilon, i\omega) = \delta(r_1 - r')\delta(r_2 - r')\sigma_\beta - T \sum_\epsilon \int dr'_1dr'_2 V(r_1 - r_2, i\epsilon - i\epsilon')
\]

\[
\times G(r_1, r'_1, i\epsilon' + i\omega)\Gamma_\beta(r'_1, r'_2, r', i\epsilon', i\omega)G(r'_2, r, i\epsilon'). \tag{C3}
\]

To solve it, we can use the basis of magnetoexcitonic states of Dirac electrons in the symmetric gauge, which were described earlier in [37] in slightly different notation:

\[
\Psi_{P_{n1n2}}(r_1, r_2) = \frac{1}{2\pi} e^{iP(r_1+r_2)/2+i\sigma_{P_{n1n2}}/2\hbar} \phi_{n1n2}(r_1 - r_2 - \mathbf{a} P). \tag{C4}
\]

Here \( P \) is the conserved magnetic momentum of the electron-hole pair and

\[
\phi_{n1n2}(r) = \sqrt{2}^{\delta_{n1}+\delta_{n2}+1} \begin{pmatrix} s_{n1}s_{n2}\phi_{|n1|-1,|n2|-1}(r) & s_{n1}\phi_{|n1|-1,|n2|}(r) \\ s_{n2}\phi_{|n1|,|n2|-1}(r) & \phi_{|n1|,|n2|}(r) \end{pmatrix} \tag{C5}
\]

is the matrix wave function of relative motion of electron and hole written in the basis of their sublattices \( A, B \). Using (A2), the unitary transformations between the magnetoexcitonic states and the states (A3) of individual electron and hole can be derived:

\[
\psi_{nk_1}(r_1)\psi_{n'k_2}(r_2) = \hbar^2 \int dP \phi_{k_1k_2}^*(\mathbf{a} P)\Psi_{P_{n1n2}}(r_1, r_2), \quad \Psi_{P_{n1n2}}(r_1, r_2) = \hbar^2 \sum_{k_1k_2} \phi_{k_1k_2}(\mathbf{a} P)\psi_{nk_1}(r_1)\psi_{n'k_2}(r_2). \tag{C6}
\]

Projecting the vertex matrix \( \Gamma_\beta \) on the magnetoexcitonic states

\[
\Gamma_{\beta, P_{n1n2}}(r', i\epsilon, i\omega) = \int drdr_2 Tr[\Psi_{P_{n1n2}}^+(r_1, r_2)\Gamma_\beta(r_1, r_2, r', i\epsilon, i\omega)] \tag{C7}
\]

and using (B3), (C6), we get (C3) in the electron-hole pair (or magnetoexcitonic) representation:

\[
\Gamma_{\beta, P_{n1n2}}(r', i\epsilon, i\omega) = \Gamma^{(0)}_{\beta, P_{n1n2}}(r') - T \sum_{\epsilon' n'1n2} \int dP \frac{|\Psi_{P_{n1n2}}(\mathbf{i}\epsilon' + i\omega)|^2 |\Psi_{P_{n'1n2}}(r', i\epsilon', i\omega)|^2}{(i\epsilon' + i\omega - E_{n'1} - \Sigma_{n'1} + \mu)(i\epsilon' - E_{n'2} - \Sigma_{n'2} + \mu)} \tag{C8}
\]

Here the bare vertex is \( \Gamma^{(0)}_{\beta, P_{n1n2}}(r') = (e^{-|\mathbf{P} r'|/2\pi}) Tr[\Phi_{n1n2}(\mathbf{a} P)\sigma_\beta] \).
To solve Eq. \((C8)\), we use the static approximation \(V(r, ie^{-i\epsilon'}) = V(r)\) and neglect the mixing of different electron-hole pairs in the ladder diagrams, assuming \(n_1 = n_1\), \(n_2 = n_2\). Therefore the vertex matrix turns out to be independent on a relative energy of electron and hole \(\epsilon\):

\[
\Gamma_{\beta, \mathbf{p}n_1n_2}(r', \omega) = \frac{e^{-i\mathbf{r}'\mathbf{r}}}{2\pi} \left[ \Phi^+_n(\mathbf{r}) \sigma_n \left\{ 1 + \langle n_1n_2 | V | n_1n_2 \rangle \left( \frac{f_{n_2} - f_{n_1}}{i\omega + E_{n_2} + \Sigma_{n_2} - E_{n_1} - \Sigma_{n_1}} \right)^{-1} \right\} \right].
\]

The average interaction energies of magnetoeectrons are \(\langle n_1n_2 | V | n_1n_2 \rangle = \int dr V(r - \mathbf{a}) \left[ \Phi^+_n(r) \Phi_n(r) \right]\), their counterparts in usual 2D electron gas were extensively studied earlier \([53]\). Making the Fourier transform \(V(r) = (2\pi)^{-2} \int dq V(q)e^{i\mathbf{q}\mathbf{r}}\) and using \((A7)\), we obtain

\[
\langle n_1n_2 | V | n_1n_2 \rangle = \frac{i}{2\pi} \int dq V(q)e^{-i\mathbf{q}\mathbf{a}} \frac{\text{Tr} \left[ \Phi^+_n(q) \Phi_n(q) \right]}{\text{Tr} \left[ \Phi^+_n(q) \Phi_n(q) \right]}.
\]

The Green function for currents \((C2)\) can be found using \((B4)\), \((C4)\), \((C5)\), \((C7)\), and \((C9)\):

\[
G^\alpha_{\beta}(r, r', \omega) = \sum_{n_1n_2} \int \frac{d\mathbf{P}}{(2\pi)^2} e^{i\mathbf{P}(r-r')} \frac{\text{Tr} \left[ \Phi^+_n(\mathbf{P}) \sigma_\alpha \right] \text{Tr} \left[ \Phi^+_n(\mathbf{P}) \sigma_\beta \right]}{i\omega + E_{n_2} + \Sigma_{n_2} - E_{n_1} - \Sigma_{n_1} + (f_{n_2} - f_{n_1})}\langle n_1n_2 | V | n_1n_2 \rangle.
\]

Substituting it in \((C1)\) and taking \(\mathbf{P} = 0\) for optical transitions in \((C10)\), we finally obtain \((5) - (10)\).

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