On DoF of Active RIS-Assisted MIMO Interference Channel With Arbitrary Antenna Configurations: When Will RIS Help?
Shuo Zheng, Bojie Lv, Tong Zhang, Yinfei Xu, Gaojie Chen, Senior Member, IEEE, Rui Wang, Member, IEEE, and P. C. Ching, Life Fellow, IEEE

Abstract—An active reconfigurable intelligent surface (RIS) has been shown to be able to enhance the sum-of-degrees-of-freedom (DoF) of a two-user multiple-input multiple-output (MIMO) interference channel (IC) with equal number of antennas at each transmitter and receiver. However, for any number of receive and transmit antennas, when and how an active RIS can help to improve the sum-DoF are still unclear. This paper studies the sum-DoF of an active RIS-assisted two-user MIMO IC with arbitrary antenna configurations. In particular, RIS beamforming, transmit zero-forcing, and interference decoding are integrated together to combat the interference problem. In order to maximize the achievable sum-DoF, an integer optimization problem is formulated to optimize the number of eliminating interference links by RIS beamforming. As a result, the derived achievable sum-DoF can be higher than the sum-DoF of two-user MIMO IC, leading to a RIS gain. Furthermore, a sufficient condition of the RIS gain is given in terms of the relationship between the number of RIS elements and the antenna configuration.

Index Terms—DoF, RIS, MIMO interference channel.

I. INTRODUCTION
Reconfigurable intelligent surface (RIS) emerges as a revolutionary tool to ameliorate the communication environment [1], [2], [3]. Specifically, the active RIS can adjust both the amplitude and phase of each RIS element to reflect the incident signal [4]. Unfortunately, the exact channel capacity of RIS-assisted multi-user communications is, however, hard to obtain. Therefore, the degrees-of-freedom (DoF), i.e., the maximal number of interference-free channels, which is a first-order approximation of the channel capacity in a high signal-to-noise ratio (SNR) regime, has become an alternative tractable solution.

The DoF for RIS-assisted multi-user communications has been investigated in [5], [6], [7], [8], [9], [10]. The authors in [5] and [6] showed if the data stream is available at a passive RIS and can be modulated through the adjustable phases at the passive RIS, the sum-DoF can be significantly elevated in the point-to-point multiple-input multiple-output (MIMO) channel. In [7], the DoF gain was characterized by allowing a passive RIS to transmit phase-modulated symbols in non-coherent communications without channel state information at the transmitter (CSIT). In [8], an active RIS-assisted MIMO wiretap channel was investigated, which showed that secure DoF can be elevated via leakage signal cancellation by RIS. For the RIS-assisted K-user interference channel (IC), the role of RIS assistance was studied in [9] and [10]. With a single antenna at each transmitter and receiver, the authors of [9] derived lower and upper bounds for the sum-DoF of K-user IC with an active RIS. Recently, with an equal number of antennas at each transmitter and receiver, the authors of [10] showed that the sum-DoF of K-user IC can be largely elevated, compared with the case without RIS. However, all existing works on applying RIS to MIMO communications are restricted by the specific requirement of the transmit and receive antenna configurations.

In this paper, in order to overcome the above limitation, we investigate the achievable sum-DoF of an active RIS-assisted two-user MIMO IC with arbitrary antenna configurations (i.e., the number of antennas of each transmitter or receiver can be arbitrary). Unlike previous works which only considered RIS beamforming [5], [6], [7], [8], [9], [10], we further apply transmit zero-forcing and receive interference decoding techniques to tackle the challenges of asymmetric antenna configurations. Compared to [11], an active RIS is exploited to elevate the sum-DoF. The achievable sum-DoF maximization problem is formulated to obtain the optimal number of eliminating interference links by RIS beamforming. Subsequently, an optimized achievable sum-DoF together with a sufficient condition for obtaining the RIS gain are also derived. It is also shown that the RIS gain actually depends on the relationship between the number of RIS elements and the antenna configuration. Furthermore, the RIS gain in closed-form is derived in terms of symmetric antenna configurations.

II. SYSTEM MODEL AND DEFINITION
As illustrated in Fig. 1, we consider an active RIS-assisted two-user MIMO IC, where two transmitters with \( M_1 \) and \( M_2 \) antennas are denoted by \( T_{x1} \) and \( T_{x2} \) and two receivers with \( N_1 \) and \( N_2 \) antennas are denoted by \( R_{x1} \) and \( R_{x2} \), respectively. It is worth mentioning that \( M_1, M_2, N_1 \) and \( N_2 \) can be arbitrary feasible value, which is defined as arbitrary antenna configurations. Without loss of generality, we assume \( \max\{M_1, N_1\} \geq \max\{M_2, N_2\} \). The communication between transmitters and receivers is assisted by an active RIS equipped with \( R \) reflecting elements, which can adjust the amplitude and phase shift in

Authorized licensed use limited to the terms of the applicable license agreement with IEEE. Restrictions apply.
each element to the incident signal. Transmitted signals of Tx1 and Tx2 are denoted by $P_1x_1 \in \mathbb{C}^{M_1 \times i}$ and $P_2x_2 \in \mathbb{C}^{M_2 \times i}$ respectively, where $x_1$ and $x_2$ contain the messages for Rx1 and Rx2, respectively, and $P_1$ and $P_2$ are transmit beamforming matrices. The received signals at Rx1 and Rx2 are expressed as follows:

$$\begin{align*}
y_1 &= (H_{11} + G_1\Psi D_1)P_1x_1 + (H_{21} + G_1\Psi D_2)P_2x_2 + z_1, \\
y_2 &= (H_{12} + G_2\Psi D_2)P_2x_2 + (H_{22} + G_2\Psi D_1)P_1x_1 + z_2,
\end{align*}$$

respectively, where $H_{ij} \in \mathbb{C}^{N_j \times M_i}, i, j = 1, 2$ denotes the direct channel matrix between transmitter Tx$i$ and receiver Rx$j$, $D_i \in \mathbb{R}^{R \times M_i}$ denotes the channel matrix between transmitter Tx$i$ and RIS, $G_i \in \mathbb{C}^{N_i \times R}$ denotes the channel matrix between receiver Rx$i$ and RIS, and $\Psi = \text{diag}(\psi_1, \ldots, \psi_R) \in \mathbb{C}^{R \times R}$ denotes the active RIS reflection matrix, and $z_i, i = 1, 2$ denotes the additive white Gaussian noise (AWGN) at the receiver Rx$i$. Without loss of generality, the channel matrices are assumed to be full-rank and known at both transmitters, both receivers, and the RIS.

In the IC, each Tx$i$, $i = 1, 2$ has independent message $W_i$, $i = 1, 2$ desired by Rx$i$, $i = 1, 2$, respectively. The rate of each message is given by $R_i(P) = \frac{\log_2 |W_i(P)|}{n}$, where $P$ is the power, $|W_i(P)|$ denotes the cardinality of the message set, and $n$ represents the number of channel uses. A rate tuple $(R_1(P), R_2(P))$ is said to be achievable if the decoding error probability of each message can be made arbitrarily small as channel uses tend to infinity. The capacity region $C(P)$ is the closure of all achievable rates. The sum-capacity $C_d(P) = \sup_{(R_1(P), R_2(P)) \in C(P), R_1(P) + R_2(P)}$ is the supremum of all achievable sum-rates. Therefore, sum-DoF is defined as $d_1 + d_2 = \lim_{P \to \infty} \frac{C_d(P)}{\log_2 P}$.

### III. PROPOSED ACHIEVABLE SUM-DOF

Similar to [11], due to $\max\{M_1, N_1\} \geq \max\{M_2, N_2\}$, we can divide all antenna configurations into 3 cases (please refer to Table I for the case division and the achievable sum-DoF), and present the achievable schemes of 3 cases below. Note that the union of 3 cases includes all antenna configurations satisfying $\max\{M_1, N_1\} \geq \max\{M_2, N_2\}$.

#### A. Case-1 (Row-Row Elimination): $M_1 \geq \{M_2, N_1, N_2\}$ and $M_2 \geq N_1$

First, the active RIS is used to cancel the interference links, i.e., $H_{31} \in \mathbb{C}^{N_1 \times M_1}$, $H_{32} \in \mathbb{C}^{N_2 \times M_2}$. Due to $M_2 \geq N_1$ and $M_1 \geq N_2$, to reduce the rank of $H_{31}$, $H_{32}$, we thus eliminate the first $f_1 \in [0, N_1]$ rows of $H_{31}$ ($f_1$ interference links) and the first $f_2 \in [0, N_2]$ rows of $H_{32}$ by RIS beamforming. This is why we call it row-row elimination. This is done by solving the following set of equations

$$\begin{align*}
h_{21} + G_1 \Psi D_2 &= \begin{bmatrix} 0_{f_1 \times M_2} \\ \bar{H}_{21} \end{bmatrix}, \\
h_{12} + G_2 \Psi D_1 &= \begin{bmatrix} 0_{f_2 \times M_1} \\ \bar{H}_{12} \end{bmatrix},
\end{align*}$$

where $\bar{H}_{12} \in \mathbb{C}^{(N_1 - f_1) \times M_2}$ and $\bar{H}_{21} \in \mathbb{C}^{(N_2 - f_2) \times M_1}$ are arbitrary matrices with full-rank. In order to solve (1), we can vectorize it as

$$\begin{align*}
h_{21} + \Gamma_1 \psi &= \begin{bmatrix} 0_{f_1 \times M_2} \\ h_{21} \end{bmatrix}, \\
h_{12} + \Gamma_2 \psi &= \begin{bmatrix} 0_{f_2 \times M_1} \\ h_{12} \end{bmatrix},
\end{align*}$$

where $\psi \in \mathbb{C}^{R \times 1} \triangleq [\psi_1, \psi_2, \ldots, \psi_R]^T$, $h_{21} \in \mathbb{C}^{N_1 \times M_2 \times 1} \triangleq \text{vec}(H_{21}^T)$, $\Gamma_1 \in \mathbb{C}^{N_1 \times M_1 \times R} \triangleq [\Gamma_1^{[1]} \times \Gamma_2^{[1]} \ldots \Gamma_1^{[R]} \times \Gamma_2^{[R]}]^T$, and $\Gamma_2 \in \mathbb{C}^{N_2 \times M_2 \times R} \triangleq [\tilde{H}_{21}^T, \tilde{H}_{22}^T, \ldots, \tilde{V}_{N_2}^T]^T$ with $[\tilde{H}_{21}^{[k]}]_{i,j} \triangleq [G_2]_{k,j}$ and $[\tilde{V}_{N_2}^{[k]}]_{i,j} \triangleq [G_1]_{k,j}$. Since we require the specific outputs (interference cancellation) at the first $f_1M_2$ rows in the first equation of (2) and the first $f_2M_1$ rows of the second equation of (2), the non-zero solutions (RIS beamforming) of (2) exists if $f_1M_2 + f_2M_1 \leq R$. By this way, the rank of $H_{31}$ reduces to $N_1 - f_1$ and the rank of $H_{32}$ reduces to $N_2 - f_2$. We further denote the equivalent matrices after the RIS beamforming by $\bar{H}_{12} \triangleq H_{12} + G_1 \Psi D_1$, $\bar{H}_{21} \triangleq [0_{M_1 \times f_1}, H_{21}^T]^T$, $\bar{H}_{22} \triangleq H_{22} + G_2 \Psi D_2$, and $\bar{H}_{12} \triangleq [0_{M_2 \times f_2}, H_{12}^T]^T$.

In the following, by adopting the zero-forcing and interference decoding in [11], we aim at eliminating the residual interference after RIS beamforming. The input-output relationships after the aforementioned RIS beamforming are given as

$$\begin{align*}
y_1 &= \Pi_1P_1x_1 + \Pi_2P_2x_2 + z_1, \\
y_2 &= \Pi_2P_2x_2 + \Pi_1P_2x_1 + z_2.
\end{align*}$$
Second, we consider the transmit zero-forcing design. The null space of $\mathbf{H}_{31}$ and $\mathbf{H}_{12}$ is $M_2 - (N_1 - f_1)$ and $M_1 - (N_2 - f_2)$, respectively. After that, the remaining vector space at Rx$_1$ and Rx$_{\bar{2}}$ is $N_1 + N_2 - M_1 - f_1$ and $N_1 + N_2 - M_2 - f_2$, respectively. Finally, we utilize the interference decoding, and send symbols in the vector space of $\min\{N_1 + N_2 - M_1 - f_2, N_1 + N_2 - M_2 - f_1\}$ with the corresponding transmitter. Thus, the created interference at another receiver can be decoded. By adding $M_2 - (N_1 - f_1)$, $M_1 - (N_2 - f_2)$ and $\min\{N_1 + N_2 - M_1 - f_2, N_1 + N_2 - M_2 - f_1\}$ up, we obtain $\min\{M_2 + f_1, M_1 + f_1\}$. The achievable sum-DoF is thus written as $\min\{M_2 + f_1, M_1 + f_1, M_1 + N_2, M_2 + N_2\}$, where $N_1 + N_2$ is due to the limitation of receiver space and $M_1 + M_2$ is due to the limitation of transmitter space. Since $M_1 + M_2 \geq \{M_2 + f_1, N_1 + N_2\}$ in this case, we can simplify the expression of the achievable sum-DoF as $\min\{M_2 + f_1, M_1 + f_1, N_1 + N_2\}$.

An integer optimization problem for optimal $f'_1$ and $f'_2$ is formulated as

$$\mathcal{P}_1 : \max_{f'_1, f'_2 \in \mathbb{N}} \min\{M_2 + f_1, M_1 + f_1, N_1 + N_2\}$$

s.t. $f_1 M_2 + f_2 M_1 \leq R, f_i \in [0, N_i], i = 1, 2.$

When $R \geq M_1 + M_2^2$, due to min function, once if $M_2 + f_1 = M_1 + f_2$, the objective reaches the maximum. Specifically, by exhausting $R$, we aim to solve the following two equations

$$\begin{cases}
M_2 + f_1 = M_1 + f_2, \\
f_1 M_2 + f_2 M_1 = R.
\end{cases}$$

(5)

By solving (5) with $f_1 \in [0, N_1], f_2 \in [0, N_2], f_1, f_2 \in \mathbb{N}$, the optimal $f'_1$ and $f'_2$ are given by

$$(f'_1, f'_2) = (\min\{\lceil \gamma_1 \rceil, N_1\}, \min\{\lceil \gamma_2 \rceil, N_2\}),$$

(6)

where $\gamma_1 = \frac{R - M_1 M_2}{M_1 + M_2}$ and $\gamma_2 = \frac{R - M_1 M_2}{M_1 + M_2}$. Substituting (6) into the objective function of Problem $\mathcal{P}_1$ gives $\min\{\lceil \frac{R + M_2^2 + M_2^2}{M_1 + M_2} \rceil, M_2 + N_1, N_1 + N_2\}$. When $R < M_1 M_2 - M_2^2$, however, this equivalence relationship, i.e., $M_2 + f_1 = M_1 + f_2$, cannot hold. Due to $M_1 > M_2$, RIS is devoted to elevate bottleneck term $M_2 + f_1$ in min function, leading to

$$\begin{cases}
f_1 M_2 + f_2 M_1 = R, \\
M_2 + f_1 < M_1 + f_2, \\
f_2 = 0.
\end{cases}$$

(7)

By solving (7) with $f_1 \in [0, N_1], f_2 \in [0, N_2], f_1, f_2 \in \mathbb{N}$, the optimal $f'_1$ and $f'_2$ are given by

$$(f'_1, f'_2) = (\min\{\lceil R/M_2 \rceil, N_1\}, 0).$$

(8)

Substituting (8) into the objective function of Problem $\mathcal{P}_1$ gives $\min\{M_2 + \lceil R/M_2 \rceil, M_2 + N_1, N_1 + N_2\}$. Therefore, it can be seen that Table I Case-1 is achieved, where $R = M_1 M_2 - M_2^2$ satisfies $\lceil \frac{R + M_2^2 + M_2^2}{M_1 + M_2} \rceil = M_2 + \lceil R/M_2 \rceil$.

This value can be negative, however, this does not change the achievable DoF expression, compared with that by introducing any restrictions. Due to page limitation, please refer to [12] for the details.

B. Case-2 (Column-Row Elimination): $M_1 \geq N_2$ and $N_1 \geq M_2$

First, the active RIS is used to cancel the interference links, i.e., $\mathbf{H}_{31} \in \mathbb{C}^{N_1 \times M_1}, \mathbf{H}_{12} \in \mathbb{C}^{N_2 \times M_2}$. Due to $N_1 \geq M_2$ and $M_1 \geq N_2$, to reduce the rank of $\mathbf{H}_{31}, \mathbf{H}_{12}$, we thus eliminate the first $f_1 \in [0, M_2]$ columns of $\mathbf{H}_{31}$ and the first $f_2 \in [0, N_2]$ rows of $\mathbf{H}_{12}$ by RIS beamforming. This is why we call it column-row elimination. This is done by solving the following set of equations

$$\begin{cases}
\mathbf{H}_{21} + \mathbf{G}_1 \Psi \mathbf{D}_2 = [0_{N_1 \times f_1}, \mathbf{H}_{31}], \\
\mathbf{H}_{12} + \mathbf{G}_2 \Psi \mathbf{D}_1 = [0_{f_2 \times M_2}, \mathbf{H}_{12}],
\end{cases}$$

(9)

respectively, where $\mathbf{H}_{31} \in \mathbb{C}^{N_1 \times (M_1 - f_1)}$ and $\mathbf{H}_{12} \in \mathbb{C}^{(N_2 - f_2) \times M_2}$ are arbitrary matrices with full-rank. To solve (9), we can vectorize them as

$$\begin{cases}
\mathbf{h}_{21} + \mathbf{G}_1 \mathbf{\psi} = [0_{N_1 \times f_1}], \\
\mathbf{h}_{12} + \mathbf{G}_2 \mathbf{\psi} = [0_{f_2 \times M_2}],
\end{cases}$$

(10)

where $\mathbf{\psi} \in \mathbb{C}^{R \times 1}$, i.e., $\mathbf{\psi} = [\psi_1, \psi_2, \ldots, \psi_R]^T$, $\mathbf{h}_{21} \in \mathbb{C}^{N_1 \times M_1}$, $\mathbf{h}_{12} \in \mathbb{C}^{N_2 \times M_2}$.

By this way, the rank of $\mathbf{H}_{31}$ reduces to $M_2 - f_1$ and the rank of $\mathbf{H}_{12}$ reduces to $N_2 - f_2$. We further denote the equivalent matrices after the RIS beamforming by $\mathbf{\tilde{H}}_{31} = \mathbf{H}_{31} + \mathbf{G}_1 \mathbf{\psi} \mathbf{D}_1$, $\mathbf{\tilde{H}}_{21} = [0_{N_1 \times f_1}, \mathbf{H}_{31}]$, $\mathbf{\tilde{H}}_{12} = \mathbf{H}_{12} + \mathbf{G}_2 \mathbf{\psi} \mathbf{D}_2$, $\mathbf{\tilde{H}}_{32} = [0_{M_2 \times f_2}, \mathbf{H}_{32}]^T$. The input-output relationships after the above RIS beamforming are the same as (3) and (4).

Second, we consider the zero-forcing design. The null space of $\mathbf{H}_{31}$ and $\mathbf{H}_{12}$ is $f_1$ and $M_1 - (N_2 - f_2)$, respectively. After that, the remaining vector space at Rx$_1$ and Rx$_{\bar{2}}$ is $N_1 + N_2 - M_1 - f_2$ and $N_2 - f_2$, respectively. Finally, we utilize the interference decoding, and send symbols in the vector space of $\min\{N_2 - f_1, N_1 + N_2 - M_1 - f_2\}$ with the corresponding transmitter. Thus, the created interference at another receiver can be decoded. By adding $f_1$, $M_1 - (N_2 - f_2)$ and $\min\{N_2 - f_1, N_1 + N_2 - M_1 - f_2\}$ up, we obtain $\min\{N_1 + f_1, M_1 + f_2\}$. The achievable sum-DoF is thus written as $\min\{N_1 + f_1, M_1 + f_2, M_1 + M_2, N_1 + N_2\}$.

An integer optimization problem for optimal $f'_1$ and $f'_2$ is formulated as

$$\mathcal{P}_2 : \max_{f'_1, f'_2 \in \mathbb{N}} \min\{N_1 + f_1, M_1 + f_1, M_1 + M_2, N_1 + N_2\}$$

s.t. $f_1 N_1 + f_2 M_1 \leq R, f_1 \in [0, M_2], f_2 \in [0, N_2]$.

When $M_1 \geq N_1$ and $R \geq M_1 N_1 - N_1^2$, or $N_1 > M_1$ and $R \geq M_1 N_1 - M_1^2$, due to min function, once if $N_1 + f_1 = M_1 + f_2$, the objective reaches the maximum. Specifically, by exhausting $R$, we aim to solve the following two equations

$$\begin{cases}
N_1 + f_1 = M_1 + f_2, \\
f_1 N_1 + f_2 M_1 = R.
\end{cases}$$

(11)
By solving (11) with \( f_1 \in [0, M_2], f_2 \in [0, N_2] \), the optimal \( f_1^* \) and \( f_2^* \) are given by
\[
(f_1^*, f_2^*) = \left( \min \{ [\gamma_1, M_2], \min \{ [\gamma_4, N_2] \} \} \right),
\]
where \( \gamma_3 = \frac{R - M_1 N_1 + M_2^2}{M_2 N_2 + N_2^2} \) and \( \gamma_4 = \frac{R - M_1 N_1 + N_2^2}{M_2 N_2 + N_2^2} \). Substituting (12) into the objective function of Problem \( P_2 \) gives
\[
\min \left\{ \frac{R + M_1 N_1 + M_2^2}{M_2 N_2 + N_2^2} \right\} .
\]
To characterize the achievable sum-DoF more precisely, we divide Case-2 into two sub-cases: Case-2-1 and Case-2-2.

1) Case-2-1 (\( M_1 \geq N_1 \)): When \( M_1 \geq N_1 \), the achievable sum-DoF derived above is simplified as \( \min \left\{ \frac{R + M_1 N_1 + M_2^2}{M_2 N_2 + N_2^2} \right\} \) since \( M_1 + N_2 \geq N_1 + N_2 \) and \( M_1 + M_2 \geq M_1 + N_2 \). When \( R < M_1 N_1 - N_2^2 \), however, this equivalence relationship, i.e., \( N_1 + f_1 = M_1 + f_2 \), cannot hold. Due to \( M_1 > N_1 \), RIS is devoted to elevate bottleneck term \( N_1 + f_1 \) in min function, leading to
\[
\begin{aligned}
f_1 N_1 + f_2 M_1 &= R, \\
N_1 + f_1 &= M_1 + f_2, \\
f_2 &= 0,
\end{aligned}
\]
By solving (13) with \( f_1 \in [0, M_2], f_2 \in [0, N_2] \), the optimal \( f_1^* \) and \( f_2^* \) are given by
\[
(f_1^*, f_2^*) = \left( \min \{ [R/M_1], N_2 \} \right) \cdot 0.
\]
Substituting (14) into the objective function of Problem \( P_2 \) gives
\[
\min \{ N_1 + [R/M_1], M_2 + N_1 + N_2 \}.
\]
Therefore, it can be seen that Table I Case-2-1 is achieved, where \( R = M_1 N_1 - N_2^2 \) satisfies
\[
\frac{R + M_1 N_1 + N_2^2}{M_2 N_2 + N_2^2} = N_1 + [R/M_1].
\]

2) Case-2-2 (\( N_1 > M_1 \)): When \( N_1 > M_1 \), the achievable sum-DoF derived above is simplified as \( \min \left\{ \frac{R + M_1 N_1 + M_2^2}{M_2 N_2 + N_2^2} \right\} \) since \( N_1 + N_2 > M_1 + N_2 \) and \( M_1 + M_2 > M_1 + N_2 \). When \( R < M_1 N_1 - M_2^2 \), however, this equivalence relationship, i.e., \( N_1 + f_1 = M_1 + f_2 \), cannot hold. Due to \( M_1 > N_1 \), RIS is devoted to elevate bottleneck term \( M_1 + f_2 \) in min function, leading to
\[
\begin{aligned}
f_1 N_1 + f_2 M_1 &= R, \\
N_1 + f_1 &= M_1 + f_2, \\
f_1 &= 0,
\end{aligned}
\]
By solving (15) with \( f_1 \in [0, M_2], f_2 \in [0, N_2] \), the optimal \( f_1^* \) and \( f_2^* \) are given by
\[
(f_1^*, f_2^*) = \left( \min \{ [R/M_1], N_2 \} \right) \cdot 0.
\]
Substituting (16) into the objective function of Problem \( P_2 \) gives
\[
\min \{ M_1 + [R/M_1], M_1 + N_2 + M_2 \}.
\]
Therefore, it can be seen that Table I Case-2-2 is achieved, where \( R = M_1 N_1 - M_2^2 \) satisfies
\[
\frac{R + M_1 N_1 + N_2^2}{M_2 N_2 + N_2^2} = M_1 + [R/M_1].
\]

C. Case-3 (Column-Column Elimination):
\( N_1 \geq \{ M_1, M_2, N_2 \} \) and \( N_2 \geq M_1 \)

First, the active RIS is used to cancel the interference links, i.e., \( H_{21} \in \mathbb{C}^{N_1 \times M_1}, H_{12} \in \mathbb{C}^{N_2 \times M_2} \). Due to \( N_1 \geq M_2 \) and \( N_2 \geq M_1 \), to reduce the rank of \( H_{21}, H_{12} \), we thus eliminate the first \( f_1 \in [0, M_2] \) columns of \( H_{21} \) and the first \( f_2 \in [0, M_1] \) columns of \( H_{12} \) by RIS beamforming. This is why we call it column-column elimination. This is done by solving the following set of equations
\[
\begin{aligned}
H_{21} + G_1 \Psi D_2 &= 0_{N_1 \times f_1}, \\
H_{12} + G_2 \Psi D_1 &= 0_{N_2 \times f_2},
\end{aligned}
\]
where \( G_1 = \frac{R - N_1 N_2 + N_2^2}{N_1 N_2 + N_2^2} \) and \( G_2 = \frac{R - N_1 + N_2^2}{N_1 + N_2} \). Substituting (20) into the objective function of Problem \( P_3 \) gives
\[
\min \{ \frac{R + N_1 N_2 + N_2^2}{N_1 + N_2} \}.
\]
When \( R < N_1 N_2 - N_2^2 \), however, this equivalence relationship, i.e., \( N_1 + f_1 = N_2 + f_2 \), cannot hold. Due to \( N_1 > N_2 \), RIS is devoted to elevate bottleneck term \( N_2 + f_2 \) in min function,
leading to

$$f_1, N_1 + f_2 N_2 = R,$$
$$N_1 + f_1 > N_2 + f_2$$

(21)

By solving (21) with $f_1 \in [0, M_2], f_2 \in [0, M_1]$, $f_1, f_2 \in \mathbb{N}$, the optimal $f_1^*$ and $f_2^*$ are given by

$$(f_1^*, f_2^*) = (0, \min\{\lfloor R/N_2 \rfloor, M_1 \}).$$

(22)

Substituting (22) into the objective function of Problem $P_1$ gives min $\{N_2 + \lfloor R/N_2 \rfloor, M_1 + N_2, M_1 + M_2\}$. Therefore, it can be seen that Table 1 Case-3 is achieved, where $R = N_1 N_2 - N_2^2$ satisfies \[\frac{R + N_1^2 + N_2^2}{N_1 + N_2} = N_2 + \lfloor R/N_2 \rfloor\].

D. Discussion

Remark 1: Fig. 2 shows that increasing $R$ can significantly elevate the sum-DoF of two-user MIMO IC (given in [11] or $R = 0$ in Table 1). This exhibits the superiority of RIS in assisting the two-user MIMO IC. Furthermore, all interference is cancelled out and the sum-DoF is 2 min $\{M, N\}$ if $R \geq 2MN$, where the optimal sum-DoF is attained. Fig. 2 shows that in our settings, results of [10] is a special case of ours. Additionally, our achievable sum-DoF can perfectly match the result in [9, Theorem 4], when both Txs and Rxs are single-antenna and the number of users is 2 in [9].

Proposition 1 (When RIS will help): When $R \geq M_2 + M_2 \ell(M_1 = M_2), M_2 < N_1 + N_2$ for Case-1; $R \geq N_1 + N_2 \ell(M_1 = N_1)$ for Case-2; $R \geq M_1$ for Case-2-1; $R \geq M_1$ for Case-2-2; and $R \geq N_2, N_2 < M_1 + M_2$ for Case-3, the derived achievable sum-DoF in Table 1 for an active RIS-assisted two-user MIMO IC is higher than the sum-DoF of two-user MIMO IC, where $\ell(\cdot)$ denotes the indicator function.

Proof: Please refer to Appendix A.

Remark 2: With an active RIS, it is shown that the existence of DoF gain depends on the relationship of $R$ and $(M_1, M_2, N_1, N_2)$. If there is no DoF gain, rate gain will be marginal if SNR is high enough.

Proposition 2 (RIS gain for symmetric antenna configurations): The RIS gain is defined as the difference between achievable sum-DoF of an active RIS-assisted two-user MIMO IC and sum-DoF of two-user MIMO IC. Thus, based on Table 1, for symmetric antenna configurations, i.e., $M_1 = M_2 = M$ and $N_1 = N_2 = N$, the RIS gain is given below

$$\text{RIS Gain} = \begin{cases} \min(\lfloor \frac{R}{MN} \rfloor, 2N - M), & N \leq M < 2N, \\ \min(\lfloor \frac{R}{MN} \rfloor, 2M - N), & M < N < 2M, \\ 0, & 2N \leq M, 2M \leq N. \end{cases}$$

IV. Conclusion

In conclusion, we obtained an achievable sum-DoF of an active RIS-assisted two-user MIMO IC with arbitrary antenna configurations. A new achievable scheme by integrating RIS beamforming, transmit zero-forcing, and interference decoding was given. We proved that the RIS gain regrading DoF is possible when the sufficient condition is satisfied. The RIS gain was further derived for symmetric antenna configurations.

APPENDIX A

PROOF OF PROPOSITION 1

Case-1 ($M_1 \geq \{M_2, N_1, N_2\}$ and $M_2 \geq N_1$): In this case, re-calling that the sum-DoF of two-user MIMO IC is $\min(M_2, M_1, N_1 + N_2)$ [11, Theorem 2]. We analyze the sufficient condition of RIS gain as follows:

1) If $M_2 \geq N_1 + N_2$, $\min\{M_2, M_1, N_1 + N_2\} = N_1 + N_2$, which indicates the sum-DoF is restricted to the number of receive antennas. RIS cannot provide a DoF gain.

2) If $M_2 < \{M_2, N_1, N_2\}$, $\min\{M_2, M_1, N_1 + N_2\} = M_2$, where for $R \geq M_2$, RIS can cancel at least one interference link, thereby RIS can provide a DoF gain; for $R < M_2$, RIS cannot provide a DoF gain.

3) If $M_2 = M_1 < N_1 + N_2$, $\min\{M_2, M_1, N_1 + N_2\} = M_1$, where for $R \geq 2M_2$ RIS can cancel at least two interference links, thereby RIS can provide a DoF gain; for $R < 2M_2$, RIS cannot provide a DoF gain.

Case-2 ($M_1 \geq N_2$ and $N_1 \geq M_2$): In this case, re-calling that the sum-DoF of two-user IC is $\min\{M_1, N_1, M_2 + M_1, N_1 + N_2\}$ [11, Theorem 2]. We analyze the sufficient condition of RIS gain as follows:
If \( N_1 < M_1 \) (Case-2-1), \( \min\{N_1, M_1, M_1 + M_2, N_1 + N_2\} = N_1 \), where for \( R \geq N_1 \), RIS can cancel at lease one interference link, thereby RIS can provide a DoF gain; for \( R < N_1 \), RIS cannot provide a DoF gain.

2) If \( N_1 = M_1 \) (Case-2-1), \( \min\{N_1, M_1, M_1 + M_2, N_1 + N_2\} = N_1 = M_1 \), where for \( R \geq 2N_1 \), RIS can cancel at least two interference links, thereby RIS can provide a DoF gain; for \( R < 2N_1 \), RIS cannot provide a DoF gain.

3) If \( N_1 > M_1 \) (Case-2-2), \( \min\{N_1, M_1, M_1 + M_2, N_1 + N_2\} = M_1 \), where for \( R \geq M_1 \), RIS can cancel at lease one interference link, thereby RIS can provide a DoF gain; for \( R < M_1 \), RIS cannot provide a DoF gain.

Case-3. \( \{N_1, M_2, N_2\} \) and \( N_2 \geq M_1 \). In this case, recalling that the sum-DoF of two-user MIMO IC is \( \min\{N_1, N_2, M_1 + M_2\} \) [11, Theorem 2]. We analyze the sufficient condition of RIS gain as follows:

1) If \( N_2 \geq M_1 + M_2 \), \( \min\{N_1, N_2, M_1 + M_2\} = M_1 + M_2 \), which indicates the sum-DoF is restricted to the number of transmit antennas. RIS cannot provide a DoF gain.

2) If \( N_2 < M_1 + M_2 \), \( \min\{N_1, N_2, M_1 + M_2\} = N_2 \), where for \( R \geq N_2 \), RIS can cancel at least one interference link, thereby RIS can provide a DoF gain; for \( R < N_2 \), RIS cannot provide a DoF gain.

3) If \( N_2 = N_1 < M_1 + M_2 \), \( \min\{N_1, N_2, M_1 + M_2\} = N_1 \), where for \( R \geq 2N_1 \), RIS can cancel at least two interference links, thereby RIS can provide a DoF gain; for \( R < 2N_1 \), RIS cannot provide a DoF gain.

**REFERENCES**

[1] W. Mei, B. Zheng, C. You, and R. Zhang, “Intelligent reflecting surface-aided wireless networks: From single-reflection to multireflection design and optimization,” *Proc. IEEE*, vol. 110, no. 9, pp. 1380–1400, Sep. 2022.

[2] Y. Han, N. Li, Y. Liu, T. Zhang, and X. Tao, “Artificial noise aided secure NOMA communications in STAR-RIS networks,” *IEEE Wireless Commun. Lett.*, vol. 11, no. 6, pp. 1191–1195, Jun. 2022.

[3] M. Di Renzo et al., “Smart radio environments empowered by reconfigurable intelligent surfaces: How it works, state of research, and the road ahead,” *IEEE J. Sel. Areas Commun.*, vol. 38, no. 11, pp. 2450–2525, Nov. 2020.

[4] Y. Zhang, C. You, and B. Zheng, “Multi-active multi-passive (MAMP)-RIS aided wireless communication: A multi-hop beam routing design,” *IEEE J. Sel. Areas Commun.*, vol. 41, no. 8, pp. 2497–2513, Aug. 2023.

[5] H. V. Cheng and W. Yu, “Multiplexing gain of modulating phases through reconfigurable intelligent surface,” in *Proc. IEEE Int. Symp. Inf. Theory*, 2021, pp. 2346–2351.

[6] K. G. Seddik, “On the degrees of freedom of IRS-assisted non-coherent MIMO communications,” *IEEE Commun. Lett.*, vol. 26, no. 5, pp. 1175–1179, May 2022, doi: 10.1109/LCOMM.2022.3153556.

[7] K. G. Seddik, “On the degrees of freedom of IRS-assisted non-coherent MIMO communications,” *IEEE Commun. Lett.*, vol. 26, no. 5, pp. 1175–1179, May 2022.

[8] M. Nafea and A. Yener, “Secure communication in a multi-antenna wiretap channel with a reconfigurable intelligent surface,” in *Proc. 17th Int. Symp. Wireless Commun. Syst.*, 2021, pp. 1–6.

[9] A. H. A. Bafghi, V. Jamali, M. Nasiri-Kenari, and R. Schober, “Degrees of freedom of the k-user interference channel assisted by active and passive IRSs,” *IEEE Trans. Commun.*, vol. 70, no. 5, pp. 3063–3080, May. 2022.

[10] S. H. Chae and K. Lee, “Cooperative communication for the rank-deficient MIMO interference channel with a reconfigurable intelligent surface,” *IEEE Trans. Wireless Commun.*, vol. 22, no. 3, pp. 2099–2112, Mar. 2023.

[11] A. Jafar and M. I. Fahad, “Degrees of freedom for the MIMO interference channel,” *IEEE Trans. Inf. Theory*, vol. 53, no. 7, pp. 2637–2642, Jul. 2007.

[12] S. Zheng et al., “On DoF of active RIS-assisted MIMO interference channel with arbitrary antenna configurations: When will RIS help?,” 2022. [Online]. Available: https://arxiv.org/abs/2211.11951