Warp Factors and Extended Sources in Two Transverse Dimensions

Alan Chodos and Erich Poppitz
alan.chodos@yale.edu, erich.poppitz@yale.edu

Department of Physics
Yale University
New Haven
CT 06520-8120, USA

Abstract

We study the solutions of the Einstein equations in \((d + 2)\)-dimensions, describing parallel \(p\)-branes \((p = d - 1)\) in a space with two transverse dimensions of positive gaussian curvature. These solutions generalize the solutions of Deser and Jackiw of point particle sources in \((2 + 1)\)-dimensional gravity with cosmological constant. Determination of the metric is reduced to finding the roots of a simple algebraic equation. These roots also determine the nontrivial “warp factors” of the metric at the positions of the branes. We discuss the possible role of these solutions and the importance of “warp factors” in the context of the large extra dimensions scenario.
1 Introduction and summary.

We study solutions of the Einstein equations of $d + 2$ dimensional gravity with a cosmological constant. The solutions represent $(d - 1)$-branes embedded in a space with two compact transverse dimensions with a positive curvature. They generalize the static solutions of Deser and Jackiw [1], describing point particles in $2 + 1$ dimensional gravity with cosmological constant, to the case of extended objects.

The solutions we find obey Einstein’s equations, as well as the equation of motion of the branes (the geodesic equation). It is remarkable that the metric, in appropriate coordinates, can be found without explicitly solving the Einstein equations. Analyzing the properties of the solutions is reduced to finding the roots of a simple algebraic equation.

From the point of view of applications to the recently proposed [2], [3] large extra dimensions scenario, the most important property of our solutions is that they are characterized by non-trivial “warp factors” that affect the values of the parameters of the world-volume theories [4]. In particular, the usual relation between the four dimensional Planck scale, the 6 dimensional “fundamental” scale of gravity, and the volume of the compactified space is affected by the presence of the warp factors. This difference can be important if the warp factors significantly deviate from unity (which, in the present framework can only be achieved by fine tuning various parameters) (for related papers with one transverse dimension, see [5]).

While our recipe for finding solutions is rather general, motivated by the large extra dimensions proposal, we discuss in some detail the solution that describes two 3-branes embedded in six dimensional space time with negative (de Sitter) cosmological constant. The two branes are located on the north and south poles of a “sphere”, with a “wedge” cut out, due to the deficit angle characteristic of pointlike sources in 2 dimensions. An important property of the solution is that the metric on the branes is not flat, but rather de Sitter. While the cosmological constant and the strength of gravity on the “visible” brane can be tuned to satisfy the experimental constraints, within our ansatz there is no solution with flat world volume metric.

This paper is structured as follows. In Section 2, we present the ansatz and derive the equations of motion. We show that they reduce to a simple differential equation in one real variable, similar to the case considered by Deser and Jackiw [1]. In Section 3, we give a general analysis of the solutions. We show that the form of the metric (in appropriate coordinates) as well as the interpretation of its singularities can be obtained without explicitly solving the differential equation. All information on the solution can be deduced by finding the appropriate roots of an algebraic equation. In particular, these roots determine the values of the warp factors on the branes. We derive expressions for the strength of the gravitational coupling, the cosmological constant on the various branes, and the size of the transverse space. Finally, in Section 4, we discuss an explicit example, with $d = 4$, which is of interest to the large extra dimension scenario. We show that the parameters allow fine tuning of the Planck scale and cosmological constant to values that do not contradict the observed ones.

#1 We note that our conventions for the signs of the curvature are those of [2]; in addition $\Lambda < 0$ in [1] corresponds to de Sitter space.
The ansatz and equations of motion.

The action we consider is that of gravity in \( D = d + 2 \) dimensions, with the usual Einstein action with cosmological term included:

\[
S_{\text{gravity}} = -M^{D-2} \int d^Dy \sqrt{g} \left( R - 2\Lambda \right) .
\]  

(1)

The indices \( M, N = 1, ..., D \), while \( \mu, \nu = 0, ..., d = D - 2 \) and \( i, j = 1, 2 \) span the rest of the space; the metric has signature \((-+, \ldots, +)\). The vacuum Einstein equations follow from the variation of the gravity action:

\[
\delta S_{\text{gravity}} = -M^{D-2} \int d^Dy \sqrt{g} \left( -R_{MN} + \frac{1}{2} g^{MN} R - g^{MN} \Lambda \right) \delta g_{MN} = 0 .
\]  

(2)

The “matter” is in the form of \( d - 1 \) dimensional branes, whose action is proportional to the area of the world surface they sweep in the \( d + 2 \) dimensional spacetime:

\[
S_{\text{matter}} = -\sum_a f_a^d \int d^Dy \int d^d\sigma \delta^D(y - X_a(\sigma)) \sqrt{\tilde{g}},
\]  

(3)

where \( X^M_a(\sigma) \) is the embedding of the world surface in space time (\( a \) runs over the various branes) and \( \tilde{g}_{\mu\nu} = X^M_{\mu} X^N_{\nu} g_{MN}(y) \) the induced metric and \( \tilde{g} \equiv -\det \tilde{g}_{\mu\nu} \) (hereafter we will omit the sum over the various sources, it will be implicit in all our formulae). The matter energy momentum tensor is defined by:

\[
\delta S_{\text{matter}} = \frac{1}{2} \int d^Dy \sqrt{g} \left( T^{MN}(y) \delta g_{MN}(y) \right)
\]

\[
= \frac{1}{2} \int d^Dy \int d^d\sigma \delta^D(y - X(\sigma)) \tilde{g}^{1/2} X^M_{\mu} X^N_{\nu} \tilde{g}^{\mu\nu} \delta g_{MN}(y) ,
\]  

(4)

hence

\[
\sqrt{g} T^{MN}(y) = -f^d \int d^d\sigma \delta^D(y - X(\sigma)) \tilde{g}^{1/2} X^M_{\mu} X^N_{\nu} \tilde{g}^{\mu\nu} .
\]  

(5)

The equation of motion of the branes (obtained by varying the action with respect to the brane coordinates \( X(\sigma) \)) is the “geodesic” equation:

\[
\nabla^2 X^M + \Gamma^M_{KL} \tilde{g}^{\mu\nu} X^K_{\mu\nu} X^L_{\nu} = 0 ,
\]  

(6)

where \( \nabla^2 = \tilde{g}^{-1/2} \partial_{\mu} \tilde{g}^{1/2} \tilde{g}^{\mu\nu} \partial_{\nu} \) is the covariant D’Alembertian in the induced metric.

Our ansatz for the metric includes a nontrivial warp factor \( e^{2\lambda} \), depending only on the coordinates transverse to the branes (denoted by \( x^i, i = 1, 2 \)):

\[
ds^2 = e^{2\lambda(x^i)} \left[ g_{\mu\nu}(x^\lambda) dx^\mu dx^\nu + e^{2K(x^i)} dx^i dx^i \right] .
\]  

(7)

We assume that the d-dimensional metric \( g_{\mu\nu}(x^\lambda) \) obeys the equation

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \alpha g_{\mu\nu} = 0 ,
\]  

(8)
so that $\alpha$ becomes a parameter of our solution.

The idea for finding a solution with delta-function sources is to first solve the source-free equations of motion. Then, admitting delta function like curvature singularities of the metric will allow us to include the effect of the sources. Following this strategy, in the remainder of this section we consider the source free equations.

The Einstein equation following from the variation of the action with respect to the $\mu, \nu$ components of the metric is:

$$d\lambda_{,ii} + K_{,ii} + \frac{d(d - 1)}{2} \lambda_{,i} \lambda_{,i} = e^{2\lambda + 2K} \Lambda - e^{2K} \alpha$$

This equation, away from the singularities, is a consequence of the other equations of motion. Static sources will lead to delta-function singularities on the r.h.s. of equation (9). These will appear, as we will show in Section 3, because of singularities of $K_{,ii}$.

The (trace of) the variation of the action w.r.t. the $i, j$ components of the metric gives the following equation:

$$\partial \bar{\partial} \lambda + d \partial \lambda \bar{\partial} \lambda = \frac{2}{d} e^{2\lambda + 2K} \Lambda - \frac{2}{d - 2} e^{2K} \alpha .$$

We have introduced complex coordinates, $z = (x^1 + ix^2)/2, \bar{z} = (x^1 - ix^2)/2$, and defined $\partial = \partial_{x_1} - i\partial_{x_2}, \bar{\partial} = \partial_{x_1} + i\partial_{x_2}$ such that $\partial z = 1, \partial \bar{z} = 0$. On the other hand, the traceless part of the equations of motion that follow from varying the action with respect to the $i, j$ components of the metric are nothing but the Cauchy-Riemann conditions for the function $V = e^{-2K - \lambda} \bar{\partial} \lambda$.

The function $V$, therefore is holomorphic, $\bar{\partial} V = 0$. This fact will play a central role in our ability to find explicit solutions.\(^\#2\)

We can now further simplify our equations, following [1]. Multiplying (10) by $V$ given by (11) and using $\bar{\partial} V = 0$, we obtain after integrating over $\bar{z}$,

$$V \partial e^{d\lambda} = \frac{2\Lambda}{d + 1} e^{(d+1)\lambda} - \frac{2d}{(d - 2)(d - 1)} \alpha e^{(d-1)\lambda} + \epsilon (z) ,$$

where $\epsilon (z)$ is an integration constant. Upon introducing the new complex variable

$$\xi = \int^z \frac{dw}{V(w)}$$

\(^\#2\)Before continuing with the analysis of solutions with nontrivial warp factors, we note that the source-free equations of motion (11) admit a simple solution with $K = -\log(1 + 4|z|^2/\rho^2)$, and $\lambda = \text{const.}$, provided the values of $\Lambda$ and $\alpha$ are related, so that the r.h.s. of (11) vanishes. These solutions are of the form $dS_{d-2} \times S^2$, with the radius of $S^2$, $\rho$, and the cosmological constant of the $dS_{d-2}$, $\alpha$, related by (11): $\alpha = -2\rho^{-2}(d - 2)$, $\Lambda = -2\rho^{-2}d$ (for $\lambda = 0$). These vacuum solutions of the $d + 2$ dimensional Einstein equations with cosmological constant generalize the Nariai solutions [7] to $d + 2$ dimensions. We thank N. Kaloper for pointing out to us the existence of ref. [6].
as well as denoting $e^\lambda = N$, we can rewrite the equation as:

$$\frac{\partial}{\partial \xi} N^d = \frac{2\Lambda}{d+1} N^{d+1} - \frac{2d}{(d-2)(d-1)} \alpha N^{d-1} + \epsilon .$$

(14)

Now upon taking the complex conjugate equation, we see that the reality condition for $N$ implies that $N$ is a function of the real part of $\xi$ only and that $\epsilon$ must be a real constant. Finally, introduce the variable

$$t = \frac{2\Lambda}{(d+1)d} (\xi + \xi^*) = \frac{2\Lambda}{(d+1)d} \left( \int z \frac{dw}{V(w)} + \text{h.c.} \right)$$

(15)

and write the equation in terms of the single real variable $t$ (13):

$$f(N) \, dN \equiv \frac{N^{d-1}}{P(N)} \, dN = dt ,$$

(16)

where $f(N)$ is defined by the first equality, and the polynomial $P(N)$ is:

$$P(N) = N^{d+1} - aN^{d-1} + b ,$$

(17)

with the coefficients $a, b$ defined as follows:

$$a = \frac{\alpha}{\Lambda} \frac{d(d+1)}{(d-2)(d-1)} , \quad b = \frac{\epsilon}{\Lambda} \frac{d+1}{2} .$$

(18)

This completes our discussion of the equations of motion. We saw that the equation of motion for the ansatz (7) reduced to a simple differential equation in one real variable (16); a general solution can be expressed in terms of the general solution of this equation and a holomorphic function $V$ (11). We also note that this provides also a solution to eq. (9)—it is easy to see that (10) together with the holomorphicity of $V$ implies (9), away from possible singularities (which, as we will see in the next section, appear in $K_{ii}$).

In the following section we present a general analysis of the solutions of (16).

3 General analysis of the solutions.

In this section, we discuss the properties of the general solution of the equation (16). To begin, we need to specify the range of the variable $t$ in (16). In the cases of interest to us $t$ will span the entire real axis—consider e.g. the case $V(z) = z/c$, so that $t \sim \log |z|$. In order to solve (16) we need to find a function $F(N)$ such that $F'(N) = f(N)$. Moreover, we should be able to invert $F(N)$ for all values of $t$—then $N(t) = F^{-1}(t)$ will satisfy (16).

Even though the equation is rather complicated and the explicit form of $F(N)$ is generally unknown, we will see that the quantities of interest of us do not require an explicit knowledge of the solution. In fact, the metric can, in appropriate coordinates, be expressed in terms of the polynomial $P(N)$, while the ranges of the variables are determined by appropriately chosen roots of the polynomial $P(N)$. 
The basic requirement is that we find a function $F(N)$ which monotonically changes from $-\infty$ to $\infty$ over some finite range of values of $N$. For that to be the case, the function $f(N)$ has to have the properties (recall that $F' = f$) that between two values of $N$ a) it does not change sign and b) its modulus approaches infinity at the two boundary values of $N$. Consider two “nearest neighbor” roots of the polynomial $P(N)$, $N_1$ and $N_2$. Assume that both $N_1$ and $N_2$ have the same sign. Now, at the roots of $P(N)$ $f(N)$ blows up; moreover, if $N_1$ and $N_2$ have the same sign, $f(N)$ does not change sign as $N$ varies between $N_1$ and $N_2$ (we are mostly interested in the case of even $d$). Then, between these two roots $F(N)$ is monotonic and approaches $\infty$ at one of the roots, say $N_1$, and $-\infty$ at the other root, $N_2$. Hence, we can invert the equation (16) and find the function $N(t) = F^{-1}(t)$ for all values of $t$ on the real axis; moreover, the function $N(t)$ approaches finite values—the roots of $P(N)$, $N_1$ and $N_2$—as $|t| \to \infty$.

Thus the problem of finding a solution of (16) is reduced to finding the condition that there are two real “nearest-neighbor” roots of $P(N)$ of the same sign. It is easily seen, e.g. for $d = 4$, that this requires that $a > 0$, i.e. $\alpha$ and $\Lambda$ have the same sign.

We assume now that the parameters $a, b$ (18) are such that two such roots, $N_1$ and $N_2$ are found (assume for simplicity that they are both positive). Let us now analyze the properties of the solution. The metric has the form (17); having found the function $N(t)$, we can now determine the $N^2 e^{2K}$ factor in the metric by using (11) and the equation (16):

$$N^2 e^{2K} = \frac{1}{V} \frac{\partial N}{\partial \rho} = \frac{2\Lambda}{d(d + 1)} \frac{1}{|V|^2} \frac{P(N)}{N^{d-1}}. \tag{19}$$

Now note that using

$$dt = \frac{2\Lambda}{d(d + 1)} \left[ \frac{dz}{V(z)} + \frac{d\bar{z}}{V(\bar{z})} \right] \tag{20}$$

$$d\theta = i\beta \left[ \frac{dz}{V(z)} - \frac{d\bar{z}}{V(\bar{z})} \right]$$

we can rewrite the metric (17) as:

$$ds^2 = N^2 ds^2_{(d)} + 4N^2 e^{2K} dz d\bar{z} \tag{21}$$

$$= N^2 ds^2_{(d)} + \frac{d(d + 1)}{2\Lambda} \frac{P(N)}{N^{d-1}} dt^2 + \frac{2\Lambda}{d(d + 1)} \frac{P(N)}{N^{d-1}} \frac{d\theta^2}{\beta^2}. \tag{22}$$

Finally, we can change variables from $t$ to $N$ using (16) and write the metric as:

$$ds^2 = N^2 ds^2_{(d)} + \frac{d(d + 1)}{2\Lambda} \frac{N^{d-1}}{P(N)} dN^2 + \frac{2\Lambda}{d(d + 1)} \frac{P(N)}{N^{d-1}} \frac{d\theta^2}{\beta^2}. \tag{23}$$

#3This requirement can be dropped for $d = 1$. 

5
where \( N \) now varies between the two roots, \( N_1 \) and \( N_2 \) of \( P(N) \). The region of variation of the angle \( \theta \) depends on the precise form of \( V(z) \) and the choice of coefficient \( \beta \); \( \beta \) in (20) is an arbitrary constant that can be reabsorbed in redefinition of \( \theta \).

Clearly the metric (23) has singularities at \( N = N_1, N_2 \). To find what these singularities imply for the physical interpretation of the solution, it is convenient to consider again the metric in the form (21), (19). Clearly the zeros of \( V(z) \) are singular points of the metric. Under the map \( z \rightarrow t \) (20) these zeros are mapped to \( t = \pm \infty \), which, finally, are mapped to the roots of the polynomial \( P(N) \), \( N_{1,2} \).

Now recall that the geodesic equation—the equation of motion of the branes (6) has also to be obeyed (for example, if this equation is not obeyed, an initially static brane will accelerate). In the static gauge, \( X^\mu = \sigma^\mu, X^i,\mu = 0 \), (6) implies that \( \sqrt{\tilde{g}} \Gamma^i_{\mu \nu} \tilde{g}^{\mu \nu} = 0 \) at the positions of the branes (note that in the static gauge the induced metric is equal to the metric of the embedding space restricted to the brane). For our ansatz (7) the geodesic equation in the static gauge amounts to the condition

\[
e^{(d-2)\lambda - 2K} \Lambda_i = 0 \rightarrow N^{d-1}V = 0 ,
\]

which should hold at the positions of the branes. The positions of the branes, therefore, correspond to zeros of the holomorphic function \( V \) (11), and hence to \( N_{1,2} \) in the coordinates (23).

In the following, consider the simple ansatz for \( V(z) \), \( V(z) = z/c \); note that since \( t = 2\Lambda (c \log z + c^* \log \bar{z})/(d + 1)d \), single valuedness of the map \( t(z, \bar{z}) \) requires \( c \) to be real. To see what kind of singularity this is, consider the \( \mu, \nu \) components of the Einstein equations in the static gauge:

\[
\sqrt{g} \left( \tilde{R}_{\mu \nu} - \frac{1}{2} \tilde{g}_{\mu \nu} \tilde{R} + \tilde{g}_{\mu \nu} \Lambda \right) = \frac{1}{2M^d} \delta^2(y - X_a) \tilde{g}^{1/2} \tilde{g}_{\mu \nu} ,
\]

where the hats indicate that the curvatures and metric are those of eq. (7). The metric in the \( z \)-coordinates (21) can be written in the form:

\[
d s^2 = N^2 d s^2_{(d)} + e^{2\kappa} dz d\bar{z} ,
\]

where

\[
\kappa = \frac{1}{2} \log \left[ \frac{c^2 P(N)}{|z|^2 N^{d-1} d(d + 1)} \right] ,
\]

Near the singularity, equation (23) becomes:

\[
- \frac{1}{2} e^{2\kappa} R_2 = - \partial \bar{\partial} \kappa = \frac{1}{2} \frac{f^d}{M^d} \delta^2 (z - X_a) ,
\]

where the equality holds up to terms that do not contain delta functions. To calculate the function \( \kappa \) (27) near \( z = 0, \infty \) note that these two points are mapped by (20) to \( t = \mp \infty \) (whether \( z = 0 \) is mapped to \( t = +\infty \) or \( -\infty \) depends on the signs of \( \Lambda \) and \( c \)). These points,

\[\#4 \text{In } d = 1, \alpha = 0, P(N) = N^2 + b, \text{ so upon changing variables } N = |b|^{1/2} \cos \Omega \text{ in (23) we recover the solution obtained in [1] by explicitly solving the equations of motion.}\]
on the other hand, correspond to the roots of the polynomial \( P, N_1 \) and \( N_2 \). Near \( z = 0 \) (assume, for concreteness, that \( z = 0 \) is mapped to \( N = N_1 \)), the dependence of \( N \) on \( t \) is easy to find upon solving eqn. (13) by keeping the most singular term:

\[
|N(z) - N_1| \simeq |z|^{2\delta_1},
\]

where

\[
\delta_1 = \frac{2\Lambda c}{d(d+1)} \frac{P'(N_1)}{N_1^{d-1}}.
\]

Therefore, near \( z = 0 \), the function \( \kappa \) from the metric (20) behaves as:

\[
\kappa \simeq -\log |z|^{1-\delta_1}.
\]

Upon comparison with eq. (28), noting that \( \partial \bar{\partial} \kappa = -2\pi(1-\delta_1)\delta^2(z) \) we can now find the tension of the brane at \( z = 0 \),

\[
f_1^d = M^d 4\pi (1-\delta_1).
\]

We can repeat the same analysis near \( z = \infty \) mapped to the other root, \( N = N_2 \). It is convenient to change variables to \( u = 1/z \). Note that the metric (26) (and the function \( \kappa \)) has the same form in the \( u \) variables. The behavior of \( N \) near \( u = 0 \) from eq. (16) is:

\[
|N(u) - N_2| \simeq |u|^{-2\delta_2},
\]

where

\[
\delta_2 = \frac{2\Lambda c}{d(d+1)} \frac{P'(N_2)}{N_2^{d-1}}.
\]

Consequently, \( \kappa \simeq -\log |u|^{1+\delta_2} \), and the tension of the brane near \( z = \infty \) is then

\[
f_2^d = M^d 4\pi (1+\delta_2).
\]

We note the conditions that \( 0 < \delta_1 < 1 \) and \( -1 < \delta_2 < 0 \). They follow from requiring positivity of the brane tensions (\( \delta_2 > -1 \) and \( \delta_1 < 1 \)). On the other hand, the distance between the two branes should be finite along any path in the \( z \)-plane, which requires \( \delta_1 > 0 \) and \( \delta_2 < 0 \); this follows from considering the interval (21) in the \( z \)-coordinates and requiring integrability of \( ds \) near \( z = 0 \) and \( z = \infty \). Finally, positivity of the metric (24) of the transverse space also requires that \( \Lambda P(N)N^{1-d} \) be positive as \( N \) varies over the interval \( N_1, N_2 \). These restrictions impose constraints on the various parameters in the action and the solution; these will be considered for our \( d = 4 \) example in Section 4.

We can now deduce a general formula for the Planck scale and cosmological constant on the \( a \)-th brane. The Einstein term in \( d \)-dimensional effective action will be:

\[
-M^d \int d^d x \ dN \frac{d\theta}{\beta} \sqrt{-\det(N^2 g_{(d)})} \ R_{(d)}(N^2 g_d)
\]

\[= - \int d^d x \sqrt{g_{(d)}} \ R_{(d)}(g_d) \ M^d \int_{N_1}^{N_2} N^{d-2} dN \ \beta^{-1} \int d\theta .
\]
Let the warp factor on the \(a\)-th brane be \(N_{(a)}\). The physical metric there is the induced metric \(N_{(a)}^2 g_{(d)}\). Hence the \(d\)-dimensional Planck constant—the coefficient in front of the curvature term in the action—will be:

\[
M_{P(a)}^{d-2} = \frac{M^d}{(d-1)} N_{(a)}^{2-d} \left( N_{2}^{d-1} - N_{1}^{d-1} \right) \beta^{-1} \int d\theta ,
\]

where the \(\theta\) integral is over the appropriate range that follows from (20) and depends on the choice of \(V\) and \(\beta\). Since the induced metric on the \(a\)-th brane is \(N_{(a)}^2 g_{(d)}\), and since \(g_{(d)}\) obeys the equation (8), the cosmological constant on the \(a\)-th brane, denoted by \(\Lambda_{(d),(a)}\), is

\[
\Lambda_{(d),(a)} = N_{(a)}^{-2} \alpha .
\]

Similarly, one can derive an expression for the proper distance between the branes at \(N_1\) and \(N_2\), which we will present for the case \(d = 4\) in the following section.

4 An explicit example and some speculations.

As an application of the general analysis of the previous section, let us now consider the physically interesting case \(d = 4\). The polynomial \(P(N)\) then is of degree 5. It is easy to see that if the coefficients \(a, b\) in \(P(N)\) obey\[^{\#7}\]

\[
a > 0 \text{ and } 0 < b < .18 a^{5/2}
\]

then there are two positive roots \(N_1\) and \(N_2\) of \(P(N)\); in addition, \(P(N) < 0\) as \(N\) varies between the two roots, while \(P'(N_1) < 0\) and \(P'(N_2) > 0\). Since \(P(N)\) is negative in the range of variation of \(N\), the metric \((27)\) is positive definite only for negative \(\Lambda\), corresponding to de Sitter space in our convention.\[^{\#8}\] Note that since \(a \sim \alpha/\Lambda > 0\), this implies that \(\alpha < 0\) as well and the four dimensional metric on the branes is also de Sitter (also note that a solution with \(\alpha = 0\) does not exist—then \(P(N)\) has a single real root and the equation (16) can not be solved for the desired interval of values of \(t\)).

We need to satisfy the conditions \(\delta_i > 0\) for the root \(N_i\) which is the image of \(z = 0\) and \(\delta_j < 0\) for the other root \(N_j\) (the image of \(z = \infty\)), so that the proper distance between the two roots is finite. Since \(\delta_i \sim \Lambda c P'(N_i)/N_i^3\), \((30, 34)\), for our choice of two positive roots, \(0 < N_1 < N_2\), we find that sign \(\delta_1 = \text{sign } c\) and sign \(\delta_2 = -\text{sign } c\) (since \(\Lambda < 0\)). On the other hand, since \(t \sim \Lambda c \log |z|\), if \(c > 0\), then \(z = 0\) is mapped to \(t = +\infty\) (and, since \(f(N) < 0\), the smaller root \(N_1\)) while \(z = \infty\) is mapped to \(t = -\infty\) and, hence, the larger root \(N_1\). Then the finite proper distance requirement is satisfied by the choice \(c > 0\), yielding \(\delta_1 > 0\), \(\delta_2 < 0\). The coordinate \(\theta\) \((20)\) changes then from \((0, 2\pi)\) (we chose \(\beta = -1/2\)). Expressing the parameters

\[^{\#7}\]More precisely, the numerical constant on the r.h.s. is \((3/5)^{3/2} - (3/5)^{5/2}\).

\[^{\#8}\]This conclusion is always true: in the case of two negative roots of \(P(N)\), which can be achieved by adjusting the value of \(b\), \(P > 0\) between the roots, but \(N < 0\) and positivity of the metric also requires negative \(\Lambda\).
\[ \delta_1 = \frac{|\Lambda| c}{10} \frac{5N_1^4 - 3aN_1^2}{N_1^3} = 1 - \frac{f_1^4}{4\pi M^4} \]
\[ |\delta_2| = \frac{|\Lambda| c}{10} \frac{5N_2^4 - 3aN_2^2}{N_2^3} = 1 - \frac{f_2^4}{4\pi M^4}. \quad (40) \]

The induced metric on the \( i \)-th brane has

\[ \Lambda_{4,i} = N_i^{-2} \alpha \quad \text{and} \quad M_{Pl,i}^2 = M^4 \frac{4\pi c}{3} N_i^{-2} (N_i^2 - N_1^2), \quad (41) \]
while the proper distance between the branes is:

\[ R = \sqrt{\frac{10}{\Lambda} \int_{N_1}^{N_2} \frac{N^{3/2} dN}{N^5 - aN^3 + b}^{1/2}}. \quad (42) \]

There are four parameters in the action with which we started: the 6 dimensional gravitational constant, \( M \), the cosmological constant, \( \Lambda \), and the tensions of the two branes, \( f_1, f_2 \). The solution involves three additional dimensionful parameters, the parameter \( \alpha \) of the ansatz for the d-dimensional metric (8), the integration constant \( \epsilon \) (14), and \( c \)—the "strength" of the zero of \( V(z) \). The equations (40) expressing the tensions for the branes through the other parameters of the solutions represent two constraints on the seven parameters \( M, \Lambda, f_1, f_2, \epsilon, \alpha, c \). We can use them to eliminate the tensions \( f_1, f_2 \) from the list of our parameters. The only importance of eqns. (40) then is that their left hand sides have to be smaller than 1, to ensure positivity of the brane tensions.

From the four dimensional effective theory point of view the quantities of interest are the four dimensional Planck constant, \( M_{Pl} \) and the cosmological constant, \( \Lambda_4 \). Of interest are also the mass of Kaluza-Klein excitations as well as the distance scale where a gravitational experiment on the brane will reveal deviations from Newton’s law. The last two quantities depend, of course on the size of the extra dimensions (and hence on the distance \( R \) (42)). However, due to the warped geometry, there can be also nontrivial dependence on the warp factors; we leave a detailed investigation of this issue for future work.

To get an idea as to what the effect of the nontrivial warp factors might be, consider first the case where the parameters \( a, b \) in \( P(N) \) (17), (18) satisfy \( 0 < b << a^{5/2} \). The advantage of this limit is that the roots can be easily approximated by:

\[ N_1 \approx \left( \frac{b}{a} \right)^{1/3} \approx \left( \frac{\epsilon}{\alpha} \right)^{1/3} \quad \text{and} \quad N_2 \approx a^{1/2} \sim \left( \frac{\alpha}{\Lambda} \right)^{1/2}. \quad (43) \]

Note that the case we are considering corresponds to \( N_2 \gg N_1 \)—the warp factors on the two branes can be (upon adjusting the parameters, of course) very different.

The conditions of positive tensions (40) then imply

\[ \frac{3|\Lambda| c}{10} a \left( \frac{a}{b} \right)^{1/3} < 1 \quad \text{and} \quad |\Lambda| c a^{1/2} < 1. \quad (44) \]
In the case we are discussing, the proper distance \((42)\) between the branes is easily seen to be approximately
\[
R \sim \frac{1}{\sqrt{\Lambda}} . \tag{45}
\]
The induced metrics on the two branes at \(N_1\) and \(N_2\) are de Sitter with cosmological constants:
\[
|\Lambda(4),1| = N_1^{-2}|\alpha| \sim |\alpha| \left( \frac{\alpha}{c} \right)^{2/3} \quad \text{and} \quad |\Lambda(4),2| = N_2^{-2}|\alpha| \sim |\Lambda| , \tag{46}
\]
while the corresponding Planck constants are:
\[
M_{Pl,1}^2 \sim M^4c a^{3/2}N_1^{-2} \sim M^4c \frac{|\alpha|^{13/6}}{|\Lambda|^{3/2}|c|^{2/3}} \quad \text{and} \quad M_{Pl,2}^2 \sim M^4c a^{3/2}N_2^{-2} \sim M^4c \frac{|\alpha|^{1/2}}{|\Lambda|^{1/2}} . \tag{47}
\]
The vacuum energy density on the brane is of order \(M^2_pl\Lambda(4)\). If one of the branes described the physical universe, the parameters should be tuned so that \(\Lambda(4),i/M^2_{Pl,i} \sim 10^{-120}\) (or to a value smaller than that; note also that this ratio is the same on both branes, since the warp factors cancel between numerator and denominator). It is not our objective to solve the cosmological constant problem; we will only ask whether our parameter space allows for at least a fine-tuned situation, subject to the constraints imposed by the existence of a solution.

Tuning the cosmological constant to the observed small value, therefore, requires that
\[
\frac{M^4}{\Lambda^2} (|\Lambda|c) a^{1/2} \sim 10^{120} . \tag{48}
\]
It is convenient to work in terms of the dimensionless ratios \(a, b, (M^2|\Lambda|^{-1}), (|\Lambda|c),\) and keep \(\alpha\) as the only dimensionful parameter.

Consider first the case where the observed universe is at the first brane. Then,
\[
M_{Pl,1}^2 \sim \left[ \frac{M^4}{|\Lambda|^2} (|\Lambda|c) a^{1/2} \right] |\alpha| \left( \frac{a}{b} \right)^{2/3} \sim 10^{120} |\alpha| \left( \frac{a}{b} \right)^{2/3} \sim 10^{54} \text{ eV}^2 .
\]
Hence,
\[
\alpha \sim \left( \frac{b}{a} \right)^{2/3} 10^{-66} \text{ eV}^2 . \tag{49}
\]
Then
\[
\frac{1}{R} \sim |\Lambda|^{1/2} \sim |\alpha|^{1/2} a^{-1/2} \sim \frac{b^{1/3}}{a^{5/6}} 10^{-33} \text{ eV} .
\]
This is a very large distance, unless \(b \gg a^{5/2}\), which violates \((33)\). If the observed universe is on the other brane, we obtain instead \(|\alpha| \sim a10^{-66} \text{ eV}^2\) and \(1/R \sim 10^{-33}\text{ eV}\)—also a large value for \(R\).

Therefore, if the deviations from Newton’s law reveal themselves to an experimentalist on the brane at a distance scale of order \(R\), then we can not accommodate the observed universe on one of our branes. If, on the other hand, the relevant scale of deviations is \(N_i R\), then it is possible to fine-tune the parameters to obtain a value for \(N_i R\) of order a millimeter, and
hence accommodate the observed world in our setup (admittedly with enormous fine-tuning of parameters!).

The above considerations apply also in a more general situation, where eq. (18) is replaced by:

\[
\frac{M^4}{\Lambda^2} \left( |\Lambda|c \right) \frac{N_2^3 - N_1^3}{a} \sim 10^{120},
\]

while (49) becomes:

\[
|\alpha| \sim N_i^2 \times 10^{-66} eV^2,
\]

provided that the observed universe is identified with the brane at \(N_i\). But then the inverse radius becomes:

\[
\frac{1}{R} \sim \frac{N_i}{a^{1/2}} I \times 10^{-33} eV,
\]

where \(I\) is the integral

\[
I = \int_{N_1}^{N_2} \frac{N^{3/2}dN}{|N^5 - aN^3 + b|^{1/2}},
\]

which is trivial in the limit \(b \to 0\) considered before.

Assuming that in our geometrical setup the deviations of gravity from Newton’s law will show up, to an observer on “our” brane, at a distance scale of order \(R\), an acceptable value of \(1/R > 10^{-3} eV\) requires \(N_i/a^{-1/2} > 10^{30}\) (which appears hard to achieve, similar to the example given above). If the deviations show up at a scale \(N_iR\), then the requirement is \(Ia^{-1/2} > 10^{30}\) (which is achievable via fine tuning). We leave the investigation of this issue for future work.

5 Acknowledgments.

We would like to thank N. Kaloper, R. Leigh, M. Luty, and V. Moncrief for helpful discussions and comments.

References

[1] S. Deser and R. Jackiw, Ann. Phys. 153 (1984) 405; S. Deser, R. Jackiw, and G. ’t Hooft, Ann. Phys. 152 (1984) 220.

[2] N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Phys. Lett. B429 (1998) 263; Phys. Lett. B436 (1998) 257; Phys.Rev.D59 (1999) 086004; I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Phys. Lett. B448 (1999) 257.

[3] N. Kaloper, I. Kogan, K. Olive, Phys. Rev. D57 (1998) 7340; ibid. D60 (1999) 049901; K. Benakli, Int. J. Mod. Phys. D8 (1999) 153; Phys. Lett. B447 (1999) 52; N. Kaloper, A. Linde, Phys. Rev. D59 (1999) 101303; D. Lyth, Phys. Lett. B448 (1999) 191; G. Dvali, S.-H. Tye, Phys. Lett B450 (1999) 72; C. Csaki, M. Graesser, J. Terning, Phys. Lett. B456 (1999) 16; G. Dvali, Phys. Lett. B459 (1999) 489; A. Lukas, B. Ovrut, D.
[4] L. Randall and R. Sundrum, [hep-th/9905221; hep-th/9906064].

[5] N. Kaloper, [hep-th/9905210; E. Halyo, [hep-ph/9905244; T. Nihei, [hep-ph/9905487; A. Kehagias, [hep-th/9906204; C. Csaki, M. Graesser, C. Kolda, J. Terning, [hep-ph/9906513; J. Cline, C. Grojean, G. Servant, [hep-ph/9906523; D. Chung, K. Freese, [hep-ph/9906542; P. Steinhard, [hep-th/9907080; N. Arkani-Hamed, S. Dimopoulos, G. Dvali, M. Gogberashvili, [hep-th/9907209; W. Goldberger, M. Wise, [hep-ph/9907218; G. Dvali, M. Shifman, [hep-ph/9907417; I. Oda, [hep-th/9907810; J. Lykken, L. Randall, [hep-th/9908070; T. Li, [hep-th/9908174; C. Csaki, Y. Shirman, [hep-th/9908186; K. Dienes, E. Dudas, T. Ghergetta, [hep-ph/9908530; A. Nelson, [hep-th/9909001; H.B. Kim, H.D. Kim, [hep-th/9909053; K. Behrndt, M. Cvetic, [hep-th/9909058; K. Skenderis, P. Townsend, [hep-th/9909070; H. Hatanka, M. Sakamoto, M. Tachibana, K. Takenaga, [hep-th/9909076; U. Ellwanger, [hep-th/9909103; E. Halyo, [hep-th/9909127; A. Chamblin, G.W. Gibbons, [hep-th/9909130; O. DeWolfe, D.Z. Freedman, S.S. Gubser, A. Karch, [hep-th/9909134].

[6] S. Weinberg, “Gravitation and Cosmology”, (John Wiley and Sons, 1972).

[7] H. Nariai, “On Some Higher Dimensional Space-Times”, Hiroshima U., RITP preprint RRK-86-44 (unpublished); H.Ishihara, K. Tomita, and H. Nariai, Prog.Theor.Phys.71 (1984) 859.