The logic of pseudo-uninorms and their residua

SanMin Wang

Faculty of Science, Zhejiang Sci-Tech University, Hangzhou 310018, P.R. China
Email: wangsanmin@hotmail.com

Abstract

Our method of density elimination is generalized to the non-commutative substructural logic $G_{\text{psUL}}^\ast$. Then the standard completeness of $H_{\text{psUL}}^\ast$ follows as a lemma by virtue of previous work by Metcalfe and Montagna. This result shows that $H_{\text{psUL}}^\ast$ is the logic of pseudo-uninorms and their residua and answered the question posed by Prof. Metcalfe, Olivetti, Gabbay and Tsinakis.

Keywords: Density elimination, Pseudo-uninorm logic, Standard completeness of $H_{\text{psUL}}^\ast$, Substructural logics, Fuzzy logic

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1. Introduction

In 2009, Prof. Metcalfe, Olivetti and Gabbay conjectured that the Hilbert system $H_{\text{psUL}}$ is the logic of pseudo-uninorms and their residua [1]. Although $H_{\text{psUL}}$ is the logic of bounded representable residuated lattices, it is not the case, as shown by Prof. Wang and Zhao in [2]. In 2013, we constructed the system $H_{\text{psUL}}^\ast$ by adding the weakly commutativity rule

$$(WCM) \vdash (A \rightarrow t) \rightarrow (A \rightarrow t)$$

to $H_{\text{psUL}}$ and conjectured that it is the logic of residuated pseudo-uninorms and their residua [3].

In this paper, we prove our conjecture by showing that the density elimination holds for the hypersequent system $G_{\text{psUL}}^\ast$ corresponding to $H_{\text{psUL}}^\ast$. Then the standard completeness of $H_{\text{psUL}}^\ast$ follows as a lemma by virtue of previous work by Metcalfe and Montagna [4]. This shows that $H_{\text{psUL}}^\ast$ is an axiomatization for the variety of residuated lattices generated by all dense residuated chains. Thus we also answered the question posed by Prof. Metcalfe and Tsinakis in [5] in 2017.

In proving the density elimination for $G_{\text{psUL}}^\ast$, we have to overcome several difficulties as follows. Firstly, cut-elimination doesn’t holds for $G_{\text{psUL}}^\ast$. Note that $(WCM)$ and the density rule $(D)$ are formulated as

$\frac{G[\Gamma, \Delta \Rightarrow t]}{G[\Delta, \Gamma \Rightarrow t]}$ \hspace{1cm} $\frac{G[\Pi \Rightarrow p, \Gamma, p, \Delta \Rightarrow B]}{G[\Gamma, \Pi, \Delta \Rightarrow B]}$
in GpsUL*, respectively. Consider the following derivation fragment.

\[
\frac{\vdash \cdot}{G_1[\Gamma_1, t, \Delta_1 \Rightarrow A} \quad \frac{G_2[\Gamma_2, \Delta_2 \Rightarrow t]}{G_3[\Gamma_2, \Delta_2, \Gamma_1, \Delta_1 \Rightarrow A} (WCM) \quad \frac{G_1[\Gamma_1, \Delta_1 \Rightarrow A} {G_1[\Gamma_1, \Delta_1, \Delta_1 \Rightarrow A} (CUT).
\]

By the induction hypothesis of the proof of cut-elimination, we get that GpsUL* can’t be strengthened further in order to solve this difficulty we introduced in [6] in order to solve a longstanding open problem, i.e., the standard code-splicing operation, are introduced in Definition 2.1.

\[
\frac{\vdash \cdot}{G_1[\Gamma_1, \Pi_1, \Sigma_1 \Rightarrow A} \quad \frac{G_2[\Gamma_1, \Pi_1, \Pi_2, p, \Pi_2', \Sigma_1 \Rightarrow p]}{G_1[\Gamma_1, \Pi_1, \Pi_2, p, \Pi_2', \Sigma_1 \Rightarrow A} (COM) \quad \frac{G_1[\Gamma_1, \Pi_1, \Pi_2, p, \Pi_2', \Sigma_1 \Rightarrow p]}{G_1[\Gamma_1, \Pi_1, \Pi_2, p, \Pi_2', \Sigma_1 \Rightarrow A} (D).
\]

Here, the major problem is how to extend (D) such that it is applicable to \(G_2[\Gamma_2, \Pi_2', p, \Pi_2', \Sigma_2 \Rightarrow p\). By replacing \(p\) with \(t\), we get \(G_2[\Gamma_2, \Pi_2', p, \Pi_2', \Sigma_2 \Rightarrow t\) but there exists no derivation of \(G_1[\Gamma_1, \Pi_1', \Gamma_2, \Pi_1', \Sigma_1 \Rightarrow A_1\) from \(G_2[\Gamma_2, \Pi_2', \Sigma_2 \Rightarrow t\) and \(G_1[\Gamma_1, \Pi_1, \Sigma_1 \Rightarrow A_1\). Notice that \(\Pi_2', \Sigma_2\) in \(G_2[\Gamma_2, \Pi_2', \Sigma_2 \Rightarrow p\) and \(\Pi_2', \Sigma_2\) in \(G_1[\Gamma_1, \Pi_1', \Gamma_2, \Pi_1', \Sigma_1 \Rightarrow A_1\) which we can’t obtain by (WCM). It seems that (WCM) can’t be strengthened further in order to solve this difficulty. We overcome this difficulty by introducing a restricted subsystem GpsUL\(\Omega\) of GpsUL*. GpsUL\(\Omega\) is a generalization of GIUL\(\Omega\), which we introduced in [6] in order to solve a longstanding open problem, i.e., the standard completeness of IUL. Two new manipulations, which we call the derivation-splitting operation and derivation-splicing operation, are introduced in GpsUL\(\Omega\).

The third difficulty we encounter is that the conditions of applying the restricted external contraction rule (EC\(\Omega\)) become more complex in GpsUL\(\Omega\) because new derivation-splitting operations make the conclusion of the generalized density rule to be a set of hypersequents rather than one hypersequent. We continue to apply derivation-grafting operations in the separation algorithm of the multiple branches of GIUL\(\Omega\) in [6] but we have to introduce a new construction method for GpsUL\(\Omega\) by induction on the height of the complete set of maximal (pEC)-nodes other than on the number of branches.

2. GpsUL, GpsUL* and GpsUL\(\Omega\)

Definition 2.1. ([1]) GpsUL consist of the following initial sequents and rules:

Initial sequents

- \(A \Rightarrow A\) (ID)
- \(t \Rightarrow t\) (tr)
- \(\Gamma, \Delta \Rightarrow A\) (\(\Delta\t\))
- \(\Gamma \Rightarrow \top\) (\(\top\t\))
Definition 2.3. $\text{GpsUL}^*$ is $\text{GpsUL}$ plus the weakly commutativity rule

$$\frac{G|\Gamma \Rightarrow t}{G|\Delta \Rightarrow t}(\text{WCM}).$$

Definition 2.3. $\text{GpsUL}^{*D}$ is $\text{GpsUL}^*$ plus the density rule

$$\frac{G|\Pi \Rightarrow p|\Gamma, p, \Delta \Rightarrow B}{G|\Gamma, \Pi, \Delta \Rightarrow B}(D).$$

Lemma 2.4. $G \equiv B \lor ((D \rightarrow B) \lor C \lor (C \rightarrow D) \lor A \rightarrow A)$ is not a theorem in $\text{HpsUL}$. 

Proof. Let $A = \{\{0, 1, 2, 3, 4, 5\}, \land, \lor, \lor, \rightarrow, \rightarrow, 3, 0, 5\}$ be an algebra, where $x \land y = \min(x, y)$, $x \lor y = \max(x, y)$ for all $x, y \in \{0, 1, 2, 3, 4, 5\}$, and the binary operations $\land$, $\lor$ and $\rightarrow$ are defined by the following tables (See [2]).
By easy calculation, we get that $A$ is a linearly ordered $\text{HpsUL}$-algebra, where 0 and 5 are the least and the greatest element of $A$, respectively, and 3 is its unit. Let $v(A) = 3, v(B) = v(C) = 2, v(D) = 5$. Then $v(G) = 2 \lor (2 \otimes 2 \otimes 5 \otimes 3 \rightarrow 3) = 2 < 3$. Hence $G$ is not a tautology in $\text{HpsUL}$. Therefore it is not a theorem in $\text{HpsUL}$ by Theorem 9.27 in [1]. □

**Theorem 2.5.** Cut-elimination doesn’t holds for $\text{GpsUL}^∗$.

**Proof.** $G \equiv B \lor ((D \rightarrow B) \odot C \odot (C \rightarrow D) \odot A \rightarrow A)$ is provable in $\text{GpsUL}^∗$, as shown in Figure 1.

![Figure 1 A proof $τ$ of $G$](image)

Suppose that $G$ has a cut-free proof $ρ$. Then there exists no occurrence of $t$ in $ρ$ by its subformula property. Thus there exists no application of $(\text{WCM})$ in $ρ$. Hence $G$ is a theorem of $\text{GpsUL}$, which contradicts Lemma 2.4. □

**Remark 2.6.** Following the construction given in the proof of Theorem 53 in [4], $(\text{CUT})$ in the figure 1 is eliminated by the following derivation. However, the application of $(\text{WCM})$ in $ρ$ is invalid, which illustrates the reason why the cut-elimination theorem doesn’t hold in $\text{GpsUL}^∗$.

![Figure 2 A possible cut-free proof $ρ$ of $G$](image)

**Definition 2.7.** $\text{GpsUL}^{∗∗}$ is constructed by replacing $(\text{CUT})$ in $\text{GpsUL}^∗$ with

$$G_1[Γ, t, Δ \Rightarrow A ] G_2[Π \Rightarrow ] (\text{WCT}).$$

We call it the weakly cut rule and, denote by $(\text{WCT})$.

**Theorem 2.8.** If $ρ_{G_{\text{psUL}}} G$, then $ρ_{G_{\text{psUL}}^{∗∗}} G$.

**Proof.** It is proved by a procedure similar to that of Theorem 53 in [4] and omitted. □

**Definition 2.9.** ($[6]$) $\text{GpsUL}_\Omega$ is a restricted subsystem of $\text{GpsUL}^∗$ such that

(i) $p$ is designated as the unique eigenvariable by which we means that it does not be used to built up any formula containing logical connectives and only used as a sequent-formula.
(ii) Each occurrence of $p$ in a hypersequent is assigned one unique identification number $i$ in $\text{GpsUL}_\Omega$ and written as $p_i$. Initial sequent $p \Rightarrow p$ of $\text{GpsUL}_\Omega^*$ has the form $p_1 \Rightarrow p_1$ in $\text{GpsUL}_\Omega$. $p$ doesn’t occur in $A, \Gamma$ or $\Delta$ for each initial sequent $\Gamma, \Delta$. $A \Rightarrow A$ or $\Gamma \Rightarrow \Gamma$ in $\text{GpsUL}_\Omega$.

(iii) Each sequent $S$ of the form $\Gamma_0, p_1, \ldots, \Gamma_{i-1}, p_i, \Gamma_i \Rightarrow A$ in $\text{GpsUL}_\Omega^*$ has the form $\Gamma_0, p_i, \Gamma_1, \ldots, \Gamma_{i-1}, p_i, \Gamma_i \Rightarrow A$ in $\text{GpsUL}_\Omega$, where $p_i$ does not occur in $\Gamma_i$ for all $0 \leq i \leq \lambda$ and, $i_k \neq i_l$ for all $1 \leq k < l \leq \lambda$. Define $\nu_i(S) = \{i_1, \ldots, i_k\}$. $\nu_i(S) = \{j_1\}$ if $A$ is an eigenvariable with the identification number $j_1$ and, $\nu_i(S) = \emptyset$ if $A$ isn’t an eigenvariable.

Let $G$ be a hypersequent of $\text{GpsUL}_\Omega$ in the form $S_1 \cdots S_n$, then $\nu_i(S_k) \cap \nu_i(S_l) = \emptyset$ and $\nu_i(S_k) \cap \nu_i(S_l) = \emptyset$ for all $1 \leq k < l \leq n$. Define $\nu_i(G) = \bigcup_{k=1}^n \nu_i(S_k)$, $\nu_i(S) = \bigcup_{k=1}^n \nu_i(S_k)$.

(iv) A hypersequent $G$ of $\text{GpsUL}_\Omega$ is called closed if $\nu_i(G) = \nu_i(G)$. Two hypersequents $G'$ and $G''$ of $\text{GpsUL}_\Omega$ are called disjoint if $\nu_i(G') \cap \nu_i(G'') = \emptyset$, $\nu_i(G') \cap \nu_i(G'') = \emptyset$. $G'$ and $G''$ is a copy of $G'$ if they are disjoint and there exist two bijections $\sigma_1 : \nu_i(G') \rightarrow \nu_i(G'')$ and $\sigma_2 : \nu_i(G') \rightarrow \nu_i(G'')$ such that $G''$ can be obtained by applying $\sigma_1$ to antecedents of sequents in $G'$ and $\sigma_2$ to succedents of sequents in $G'$.

(v) A hypersequent $G|G_1|G_2$ can be contracted as $G|G_1$ in $\text{GpsUL}_\Omega$ under certain condition given in Construction 3.15, which we called the constraint external contraction rule and denote by $\frac{G'|G_1|G_2}{G'|G_1}(EC\Omega)$.

(vi) (EW) is forbidden in $\text{GpsUL}_\Omega$ and, (EC) and (CUT) are replaced with (EC\Omega) and (WCT), respectively.

(vii) Two rules ($\wedge$) and ($\vee$) of GL are replaced with $G_1|\Gamma_1 \Rightarrow A \quad G_2|\Gamma_2 \Rightarrow B \quad \frac{G_1|\Gamma_1 \Rightarrow A \wedge B|\Gamma_2 \Rightarrow A \wedge B}{G_1|\Gamma_1 \Rightarrow A \wedge B}(\wedge_{wv})$ and $G_1|\Gamma_1, A \vee B, \Delta_1 \Rightarrow C_1 \quad G_2|\Gamma_2, A \vee B, \Delta_2 \Rightarrow C_2 \quad \frac{G_1|\Gamma_1|\Delta_1 \Rightarrow C_1 \quad G_2|\Gamma_2|\Delta_2 \Rightarrow C_2}{G_1|\Gamma_1, A \vee B, \Delta_1 \Rightarrow C_1 \quad G_2|\Gamma_2, A \vee B, \Delta_2 \Rightarrow C_2}(\vee_{hv})$ in $\text{GpsUL}_\Omega$, respectively.

(viii) $G_1|\Sigma_1$ and $G_2|\Sigma_2$ are closed and disjoint for each two-premise inference rule $\frac{G_1|\Sigma_1 \quad G_2|\Sigma_2}{G_1|\Sigma_1 \cup \Sigma_2}(I)$ of $\text{GpsUL}_\Omega$ and, $G'|S'$ is closed for each one-premise inference rule $\frac{G'|S'}{G'|S'}(I)$.

**Proposition 2.10.** Let $\frac{G'|S'}{G'|S'}(I)$ and $\frac{G_1|\Sigma_1 \quad G_2|\Sigma_2}{G_1|\Sigma_1 \cup \Sigma_2}(I)$ be inference rules of $\text{GpsUL}_\Omega$ then $\nu_i(G'|S'') = \nu_i(G'|S') = \nu_i(G'|S')$ and $\nu_i(G_1|G_2|H') = \nu_i(G_1|G_2|H'') = \nu_i(G_1|G_2|H') = \nu_i(G_1|G_2|H''')$.

**Proof.** Although (WCT) makes $\tau$’s in its premises disappear in its conclusion, it has no effect on identification numbers of the eigenvariable $p$ in a hypersequent because $t$ is a constant in $\text{GpsUL}_\Omega$ and distinguished from propositional variables. □

**Definition 2.11** (1). Let $G$ be a closed hypersequent of $\text{GpsUL}_\Omega$ and $S \in G$. $[S]_G := \bigcap \{H : S \in H \subseteq G, \nu_i(H) = \nu_i(H)\}$ is called a minimal closed unit of $G$.

3. The generalized density rule (D) for $\text{GpsUL}_\Omega$

In this section, $\text{GL}_\Omega^\Omega$ is $\text{GpsUL}_\Omega$ without (EC\Omega). Generally, $A, B, C, \ldots$, denote a formula other than an eigenvariable $p_i$.

**Construction 3.1.** Given a proof $\tau^*$ of $H \equiv G|p_j, \Delta \Rightarrow p_j$ in $\text{GL}_\Omega^\Omega$, let $\hat{H}_{\tau^*}(p_j \Rightarrow p_j) = (H_0, \ldots, H_n)$, where $H_0 \equiv p_j \Rightarrow p_j, H_n \equiv H$. By $\hat{\Gamma}, p_j, \Delta_k \Rightarrow p_j$, we denote the sequent containing...
p_j in H_k. Then Γ_k = ∅, Δ_k = ∅, and Δ_k = Δ. Hyper sequents \( \langle H_k \rangle_j^\tau, \langle H_k \rangle_j^\tau \) and their proofs \( \langle \tau^* \rangle_j^\tau \langle (H_k) \rangle_j^\tau \) are constructed inductively for all 0 ≤ k ≤ n in the following such that \( \Gamma_k \Rightarrow t \in \langle H_k \rangle_j^\tau, \Delta_k \Rightarrow t \in \langle H_k \rangle_j^\tau \), and \( \langle (H_k) \rangle_j^\tau \{ \Delta_k \Rightarrow t \}|\langle H_k \rangle_j^\tau \{ \Gamma_k \Rightarrow t \} = H_k \{ \Gamma_k, p_j, \Delta_k \Rightarrow p_j \} \).

(i) \( \langle H_0 \rangle_j^\tau := \langle H_0 \rangle_j^\tau, \langle \tau^* \rangle_j^\tau \langle (H_0) \rangle_j^\tau \) are built up with \( \Rightarrow \).

(ii) Let \( G[S]^\tau G''[H']^\tau (II) \) or \( G[S]^\tau (I) \) be in \( \tau^* \). \( H_0 = G[S]^\tau \) and \( H_{k+1} = G[G''[H']^\tau \) (accordingly \( H_{k+1} = G[S]^\tau \) for (I)) for some 0 ≤ k ≤ n − 1. There are three cases to be considered.

Case 1 \( S' = \Gamma_k, p_j, \Delta_k \Rightarrow p_j \). If all focus formula(s) of \( S' \) is (are) contained in \( \Gamma_k \),

\( \langle H_{k+1} \rangle_j^\tau := (\langle H_k \rangle_j^\tau \{ \Gamma_k \Rightarrow t \}) \langle G''[H']^\tau \{ \Gamma_{k+1}, p_j, \Delta_{k+1} \Rightarrow p_j \} \{ \Gamma_{k+1} \Rightarrow t \} \).

The case of all focus formula(s) of \( S' \) contained in \( \Delta_k \) is dealt with by a procedure dual to above and omitted.

Case 2 \( S' \in \langle H_k \rangle_j^\tau \). \( \langle H_{k+1} \rangle_j^\tau := (\langle H_k \rangle_j^\tau \{ S' \}) \langle G''[H']^\tau \) (accordingly \( \langle H_{k+1} \rangle_j^\tau = (\langle H_k \rangle_j^\tau \{ S' \}) \langle G''[H']^\tau \) for (I)). \( \langle H_{k+1} \rangle_j^\tau := (\langle H_k \rangle_j^\tau \{ \tau^* \rangle_j^\tau \langle (H_k) \rangle_j^\tau \rangle_j^\tau \) is constructed by combining the derivation \( \langle \tau^* \rangle_j^\tau \langle (H_k) \rangle_j^\tau \rangle_j^\tau \) and \( \langle H_k \rangle_j^\tau \rangle_j^\tau \) \( (I) \) for (I) and, \( \langle \tau^* \rangle_j^\tau \langle (H_k) \rangle_j^\tau \rangle_j^\tau \) is constructed by combining \( \langle \tau^* \rangle_j^\tau \langle (H_k) \rangle_j^\tau \rangle_j^\tau \) and \( \langle H_k \rangle_j^\tau \rangle_j^\tau \) \( (I) \).

Case 3 \( S' \in \{ H_k \}_j^\tau \). It is dealt with by a procedure dual to Case 2 and omitted.

Definition 3.2. The manipulation described in Construction 3.1 is called the derivation-splitting operation when it is applied to a derivation and, the splitting operation when applied to a hyper-sequent.

Corollary 3.3. Let \( \vdash \text{GL}_A G[I], p_j, \Delta \Rightarrow p_j \). Then there exist two hyper sequents \( G_1 \) and \( G_2 \) such that \( G \models G_1 \lor G_2 \). \( \vdash \text{GL}_A G[I] \Rightarrow t \) and \( \vdash \text{GL}_A G_2[I] \Rightarrow t \).

Construction 3.4. Given a proof \( \tau^* \) of \( H \models G[I] \Rightarrow p_j[I], p_j, \Delta \Rightarrow A \) in \( \text{GL}_A \), let \( T_h(p_j \Rightarrow p_j) = (H_0, \cdots, H_n) \), where \( H_0 \equiv \Delta \Rightarrow p_j \) and \( H_n \equiv H \). Then there exists 1 ≤ m ≤ n such that \( H_m \) is in the form \( G[I]^\tau \Rightarrow p_j[I'], p_j, \Delta \Rightarrow A' \) and \( H_{m-1} \) is in the form \( G'[I^\tau, p_j, \Delta' \Rightarrow A' \Rightarrow A \). A proof of \( G[I], p_j, \Delta \Rightarrow A \) in \( \text{GL}_A \) is constructed by induction on \( n - m \) as follows.

- For the base step, let \( n - m = 0 \). Then
  \( H_{m+1} \equiv G[I]^\tau, p_j, \Delta', \Pi' \Rightarrow p_j \Rightarrow G'[I', \Pi', \Pi' \Rightarrow p_j[I', \Pi', p_j, \Delta', \Pi' \Rightarrow A \} \) (\( \text{COM} \)) \( \in \tau^* \), where \( G'[I'] = G \) and \( I', \Pi', \Pi' \Rightarrow I \) and \( \Pi' \Rightarrow I \).

It follows from Corollary 3.3 that
there exist $G'_1$ and $G'_2$ such that $G' = G'_1 \cup G'_2$. $G'_1 \cap G'_2 = \emptyset$, $\vdash_{GL_2} G'_1[\Pi', \Gamma'] \Rightarrow t$ and 
$\vdash_{GL_2} G'_2[\Delta', \Pi'''] \Rightarrow t$. Then $G[\Gamma, \Pi, \Delta] \Rightarrow A$ is proved as follows.

| $G'[\Pi', \Pi'', \Delta''] \Rightarrow A$ (\text{WCM}) |
|---------------------------------------------|
| $G'[\Pi', \Pi'', \Delta'] \Rightarrow A$ (\text{WCM}) |
| $G'[\Pi', \Pi', \Pi'', \Delta'', \Delta'''] \Rightarrow A$ |

- For the induction step, let $n - m > 0$. Then it is treated using applications of the induction hypothesis to the premise followed by an application of the relevant rule. For example, let $H_{n-1} = G'[\Pi, p_1, \Delta] \Rightarrow A$, $G[\Pi, \Delta] \Rightarrow A$. Define $D_j(H) = \{G[\Pi, \Delta] \Rightarrow A\}$, where $G_1$ and $G_2$ are determined by Corollary 3.3.

Definition 3.5. The manipulation described in Construction 3.4 is called the derivation-splicing operation when it is applied to a derivation and, the splicing operation when applied to a hypersequent.

Corollary 3.6. If $\vdash_{GL_2} G[\Pi, \Pi', \Delta] \Rightarrow A$, then $\vdash_{GL_2} G[\Gamma, \Pi, \Delta] \Rightarrow A$.

Definition 3.7. (i) Let $\vdash_{GL_2} H \equiv G[\Gamma, \Pi, \Delta] \Rightarrow p_j$. Define $(H)^+_j = G[\Gamma] \Rightarrow t$. (ii) Let $\vdash_{GL_2} H \equiv G[\Pi] \Rightarrow p_j, \Delta \Rightarrow A$. Define $D_j(H) = \{G[\Gamma, \Pi, \Delta] \Rightarrow A\}$.

Theorem 3.8. Let $\vdash_{GL_2} G$. Then $\vdash_{GL_2} H$ for all $H \in D(G)$.

Proof. Immediately from Corollary 3.3, Corollary 3.6 and Definition 3.7.

Lemma 3.9. Let $G'$ be a minimal closed unit of $G[G']$. Then $G'$ has the form $\Gamma' \Rightarrow A[\Pi_2] \Rightarrow p_{i_1} \cdots \mid \Pi_n \Rightarrow p_n$ if there exists some sequent $\Gamma \Rightarrow A \in G'$ such that $A$ is not an eigenvariable otherwise $G'$ has the form $\Gamma \Rightarrow A[\Pi_2] \Rightarrow p_{i_1} \cdots \mid \Pi_n \Rightarrow p_n$.

Proof. Define $G_1 \Rightarrow \Gamma \Rightarrow A$ in Construction 5.2 in [6]. Then $\emptyset = \nu_i(G_1) \subseteq \nu_i(G_1)$. Suppose that $G_1$ is constructed such that $\nu_i(G_1) \subseteq \nu_i(G_1)$. If $\nu_i(G_1) = \nu_i(G_1)$, the procedure terminates and $n = k$, otherwise $\nu_i(G_1) \nu_i(G_1) / \emptyset$ and define $i_{k+1}$ to be an identification number in $\nu_i(G_1) \nu_i(G_1)$. Then there exists $G_{i_{k+1}} \Rightarrow p_{i_{k+1}} \in G[G]$. Define $G_{i_1} = G_1[\Gamma_1] \Rightarrow p_{i_1}$, $G_{i_2} = G_1[\Gamma_1] \Rightarrow p_{i_1}$, $G_{i_{k+1}} = G_1[\Gamma_1] \Rightarrow p_{i_{k+1}}$ and $G_{i_{k+1}} = G_1[\Gamma_1] \Rightarrow p_{i_{k+1}}$. Hence there exists a sequence $i_1, i_2, \cdots, i_k$ of identification numbers such that $\nu_i(G_1) \subseteq \nu_i(G_1)$ for all $1 \leq k \leq n$, where $G_1 \Rightarrow \Gamma \Rightarrow A, G_k \Rightarrow \Gamma \Rightarrow A[\Pi_i] \Rightarrow p_{i_1} \cdots \mid \Pi_n \Rightarrow p_n$ for all $2 \leq k \leq n$. Therefore $G'$ has the form $\Gamma \Rightarrow A[\Pi_2] \Rightarrow p_{i_1} \cdots \mid \Pi_n \Rightarrow p_n$. 

\]
Definition 3.10. Let $G'$ be a minimal closed unit of $G|G'$. $G'$ is a splicing unit if it has the form $\Gamma \Rightarrow A[\Gamma_i] \Rightarrow p_{i1} \cdots \Gamma_{in} \Rightarrow p_{in}$. $G'$ is a splitting unit if it has the form $\Gamma_i \Rightarrow p_{1i} \cdots \Gamma_{in} \Rightarrow p_{in}$.

Lemma 3.11. Let $G'$ be a splicing unit of $G|G'$ in the form $\Gamma \Rightarrow A[\Gamma_i] \Rightarrow p_{i1} \cdots \Gamma_{in} \Rightarrow p_{in}$ and $K = \{i_1, \cdots, i_n\}$. Then $|D_K(G|G')| = 1$.

Proof. By the construction in the proof of Lemma 3.9, $i_k \in \nu(G_{k-1})$ for all $2 \leq k \leq n$. Then $p_{in} \in \Gamma$ and $D_k(G|G') = G[\Gamma[\Gamma_i]] \Rightarrow A[\Gamma_i] \Rightarrow p_{i1} \cdots \Gamma_{in} \Rightarrow p_{in}$, where $\Gamma[\Gamma_i]$ is obtained by replacing $p_i$ in $\Gamma$ with $\Gamma_i$. Then $p_{in} \in \Gamma[\Gamma_i]$. Repeatedly, we get $D_{i_{n-k}}(G|G') = D_K(G|G') = G[\Gamma[\Gamma_i]] \cdots \Gamma_{in} \Rightarrow A$. \hfill \Box

This shows that $D_K(G|G')$ is constructed by repeatedly applying splicing operations.

Definition 3.12. Let $G'$ be a minimal closed unit of $G|G'$. Define $V_{G'} = \nu(G')$, $E_{G'} = \{(i, j) \mid \Gamma, p_i, \Delta \Rightarrow p_j \in G'\}$ and, $j$ is called the child node of $i$ for all $(i, j) \in E_{G'}$. We call $\Omega_{G'} = (V_{G'}, E_{G'})$ the $\Omega$-graph of $G'$.

Let $G'$ be a splitting unit of $G|G'$ in the form $\Gamma_1 \Rightarrow p_1 \cdots \Gamma_n \Rightarrow p_n$. Then each node of $\Omega_{G'}$ has one and only one child node. Thus there exists one cycle in $\Omega_{G'}$ by $|V_{G'}| = n < \infty$. Assume that, without loss of generality, $(1, 2), (2, 3), \cdots, (1, 1)$ is the cycle of $\Omega_{G'}$. Then $p_1 \in \Gamma_2, p_2 \in \Gamma_3, \cdots, p_{i-1} \in \Gamma_i$ and $p_i \in \Gamma_1$. Thus $D_{i-1}(G|G') = G[\Gamma[\Gamma_i]] \cdots \Gamma_2 \Rightarrow p_1$ is in the form $G_{i-1}, p_{i-1}, \Delta \Rightarrow p_1$. By a suitable permutation $\sigma$ of $i + 1, \cdots, n$, we get $D_{i-2\sigma(i+1)\cdots n}(G|G') = G[\Gamma[\Gamma_i]] \cdots \Gamma_2 \Gamma_{\sigma(i+1)} \cdots \Gamma_{\sigma(n)} \Rightarrow p_1 = G[\Gamma, p_1, \Delta \Rightarrow p_1]$. This process also shows that there exists only one cycle in $\Omega_{G'}$. Then we introduce the following definition.

Definition 3.13. (i) $\Gamma_i \Rightarrow p_j$ is called a splitting sequent of $G'$ and $p_j$ its corresponding splitting variable for all $1 \leq j \leq i$.

(ii) Let $K = \{1, 2, \cdots, n\}$ and $D_1(G|G', p_i, \Delta \Rightarrow p_j) = G[\Gamma[\Gamma_i] \Rightarrow t, G_2[\Delta \Rightarrow t]$. Define $\langle G|G'\rangle_K = G[\Gamma[\Gamma_i] \Rightarrow t, \langle G|G'\rangle_{K_1} = G[\Gamma[\Gamma_i] \Rightarrow t, \langle G|G'\rangle_{K_2} = \langle G|G'\rangle_{K_1} \cdots \langle G|G'\rangle_{K_n} \rangle$. \hfill \Box

Lemma 3.14. If $G'$ be a splitting unit of $G|G'$, $K = \nu(G')$ and $k$ be a splitting variable of $G'$. Then $D_{G\setminus K(k)}(G|G')$ is constructed by repeatedly applying splicing operations and only the last operation $D_k$ is a splitting operation.

Construction 3.15. (The constrained external contraction rule)

Let $H = \nu(G') \cap \nu([S_H])_1 \cap \nu([S_H])_2 \cap \nu([S_H])_3$ be two copies of a minimal closed unit $[S_H]_1$, where we put two copies into $\{\}$ and $\{\}$ in order to distinguish them. For an arbitrary $S$-unit $\{S_H\}_3 \leq G'$, $\{S_H\}_1 \leq (H)_K$ or $\{S_H\}_1 \leq (S_H)_K$ where $K \neq \nu([S_H])$. Then $G''[\{S_H\}_1]$ is constructed by cutting off $\{S_H\}_2$ and some sequents in $G'$ as follows.

(i) If $\{S_H\}_1$ and $\{S_H\}_2$ are two splicing units, then $G'' \Rightarrow G'$:

(ii) If $\{S_H\}_1$ and $\{S_H\}_2$ are two splicing units and, $k$, $k'$ their splitting variables, respectively, $K = \nu(\{S_H\}_1), K' = \nu(\{S_H\}_2), D_{K \setminus \nu(\{S_H\}_1)}(G'') = G[\Gamma \Rightarrow t, G_1[\Delta \Rightarrow t] \cdots G'\_2[\Delta \Rightarrow t])$ or $D_{K \cup \nu(\{S_H\}_1)}(H) = G[\Gamma \Rightarrow t, G_1[\Delta \Rightarrow t] \cdots G'\_2[\Delta \Rightarrow t])$, where $G_1[\Delta \Rightarrow t, G_2[\Delta \Rightarrow t]$ is a copy of $G'_2$. Then $G'' \Rightarrow G_1[\Delta \Rightarrow t, G_2[\Delta \Rightarrow t]$. \hfill \Box

The above operation is called the constrained external contraction rule, denoted by $\langle EC_{G'_{\nu(\{S_H\}_1)}} \rangle$ and written as $G''[\{S_H\}_1] \cap (\{S_H\}_2 \cup \nu(\{S_H\}_1))$. \hfill \Box

Lemma 3.16. If $\nu(\{S_H\}_1) \neq H$ as above. Then $\nu(G_{\text{put}H'})$ for all $H' = \nu(\{S_H\}_1)$. \hfill \Box
4. Density elimination for $G_{\text{psUL}}^*$

In this section, we adapt the separation algorithm of branches in [6] to $G_{\text{psUL}}^*$ and prove the following theorem.

**Theorem 4.1.** Density elimination holds for $G_{\text{psUL}}^*$.

The proof of Theorem 4.1 runs as follows. It is sufficient to prove that the following strong density rule

$$
\frac{G_0 \equiv G'|\{\Gamma_i, p, \Delta_i \Rightarrow A_i\}_{i=1}^n \mid \{\Pi_j \Rightarrow p\}_{j=1}^m}{D_0(G_0) \equiv G'|\{\Gamma_i, \Pi_j, \Delta_i \Rightarrow A_j\}_{i=1}^n, j=1} (D_0)
$$

is admissible in $G_{\text{psUL}}^*$, where $n, m \geq 1$, $p$ does not occur in $G', \Gamma_i, \Delta_i, A_j, \Pi_j$ for all $1 \leq i \leq n$, $1 \leq j \leq m$.

Let $\tau$ be a proof of $G_0$ in $G_{\text{psUL}}^*$ by Theorem 2.8. Starting with $\tau$, we construct a proof $\tau'$ of $G|G^*$ in $G_{\text{psUL}}^*$ by a preprocessing of $\tau$ described in Section 4 in [6].

In Step 1 of preprocessing of $\tau$, a proof $\tau'$ is constructed by replacing inductively all applications of $(\wedge)$ and $(\vee)$ in $\tau$ with $(\wedge)_{\text{mw}}$ and $(\vee)_{\text{lw}}$ followed by an application of $(EC)$, respectively. In Step 2, a proof $\tau''$ is constructed by converting all $G'_{\text{psUL}}{S_j}^n(\text{EC}^*) \in \tau'$ into $G'_{\text{psUL}}{S_j}^m(I\Omega)\cup(I\Omega)$, where $G'_{\text{psUL}} \subseteq G''$. In Step 3, a proof $\tau'''$ is constructed by converting $G'_{\text{psUL}}{S_j}^m(I\Omega)\cup(I\Omega) \in \tau''$ into $G'_{\text{psUL}}{S_j}^m(\text{EC}^*) \in \tau''$. In Step 4, a proof $\tau''''$ is constructed by replacing some $G'|I', p, \Delta' \Rightarrow A' \in \tau''$ (or $G'|I' \Rightarrow p \in \tau''$) with $G'|I', \tau, \Delta' \Rightarrow A'$ (or $G'|I' \Rightarrow \bot$). In Step 5, a proof $\tau^*$ is constructed by assigning the unique identification number to each occurrence of $p$ in $\tau''''$. Let $H'_i = G'_{\text{psUL}}{S_j}^m$ denote the unique node of $\tau^*$ such that $H'_i \in G'_{\text{psUL}}{S_j}^m$ and $S_j^i$ is the focus sequent of $H'_i$ in $\tau^*$. We call $H'_i, S_j^i$ the $i$-th $(pEC)$-node of $\tau^*$ and $(pEC)$-sequent, respectively. If we ignore the replacements from Step 4, each sequent of $G$ is a copy of some sequent of $G_0$ and, each sequent of $G^*$ is a copy of some contraction sequent in $\tau^*$.

Now, starting with $G|G^*$ and its proof $\tau^*$, we construct a proof $\tau^\omega$ of $G^\omega$ in $G_{\text{psUL}}^\Omega$ such that each sequent of $G^\omega$ is a copy of some sequent of $G$. Then $G_{\text{psUL}}^\Omega \vdash (G^\omega)$ by Theorem 3.8 and Lemma 3.16. Then $G_{\text{psUL}}^\Omega \vdash D_0(G_0)$ by Lemma 9.1 in [6].

In [6], $G^\omega$ is constructed by eliminating $(pEC)$-sequents in $G|G^*$ one by one. In order to control the process, we introduce the set $I = \{H'_i, \ldots, H'_n\}$ of maximal $(pEC)$-nodes of $\tau^*$ (See Definition 4.2) and the set $I$ of the branches relative to $I$ and construct $G^\omega_{\text{psUL}}$ such that $G^\omega_{\text{psUL}}$ doesn’t contain the contraction sequents lower than any node in $I$, i.e., $S_j^i \in G^\omega_{\text{psUL}}$ implies $H'_j \parallel H'_i$ for all $H'_j \in I$. The procedure is called the separation algorithm of branches in [6].

The problem we encounter in $G_{\text{psUL}}^\Omega$ is that Lemma 7.11 of [6] doesn’t hold because new derivation-splitting operations make the conclusion of $(D)$-rule to be a set of hypersequents rather than one hypersequent. Then $G^\omega_{\text{psUL}}$ generally can’t be contracted to $G_1$ in Step 2 of Stage 1 in Main algorithm in [6] and, $\{G^\omega_{\text{psUL}}\}^{\omega,\nu}$ can’t be contracted to $G^\omega_{\text{psUL}}$ in Step 2 of Stage 2. Furthermore, we sometimes can’t construct some branches to $I$ in $G_{\text{psUL}}^\Omega$ before we construct $\tau^\omega$.

Therefore we have to introduce a new induction strategy for $G_{\text{psUL}}^\Omega$ and don’t perform the induction on the number of branches. First we give some primary definitions and lemmas.
**Definition 4.2.** A (pEC)-node $H_i^*$ is maximal if no other (pEC)-node is higher than $H_i^*$. Define $I_0$ to be the set of maximal (pEC)-nodes in $\tau^*$. A nonempty subset $I$ of $I_0$ is complete if $I$ contains all maximal (pEC)-nodes higher than or equal to the intersection node $H_i^*$ of $I$. Define $H_i^{\psi} = H_i^*$ if $I = \{ H_i^* \}$, i.e., the intersection node of a single node is itself.

**Proposition 4.3.** (i) $H_i^* \parallel H_j^*$ for all $i \neq j$, $H_i^*, H_j^* \in I_0$.

(ii) Let $I$ be complete and $H_i^* \geq H_j^*$. Then $H_i^* \not\subseteq H_j^*$ for some $H_i^* \in I$.

(iii) $I_0$ is complete and $\{ H_i^* \}$ is complete for all $H_i^* \in I_0$.

(iv) If $I \subseteq I_0$ is complete and $|I| > 1$, then $I_1$ and $I_2$ are complete, where $I_1$ and $I_2$ denote the sets of all maximal (pEC)-nodes in the left subtree and right subtree of $\tau^*(H_i^*)$, respectively.

**Proof.** (v) $I_1 \subseteq I_2$, $I_2 \subseteq I_1$ or $I_1 \cap I_2 = \emptyset$ holds by $H_i^{\psi} \leq H_i^*$, $H_i^{\psi} \leq H_j^*$ or $H_i^{\psi} \parallel H_j^*$, respectively.

**Definition 4.4.** A labeled binary tree $\rho$ is constructed inductively by the following operations.

(i) The root of $\rho$ is labeled by $I_0$ and leaves labeled $\{ H_i^* \} \subseteq I_0$.

(ii) If an inner node is labeled by $I$, then its parent nodes are labeled by $I_1$ and $I_2$, where $I_1$ and $I_2$ are defined in Proposition 4.3 (iv).

**Definition 4.5.** We define the height $o(I)$ of $I \in \rho$ by letting $o(\emptyset) = 1$ for each leave $I \in \rho$ and, $o(I) = \max\{o(I_1), o(I_2)\} + 1$ for any non-leaf node.

Note that in Lemma 7.11 in [6] only uniqueness of $G_{H_1, G_2}^\phi[S_2] \in G_{H_1}^\phi$ doesn’t hold in GpsUL$\Omega$. And the following lemma holds in GpsUL$\Omega$.

**Lemma 4.6.** Let $G_1[S_1] G_2[S_2] H_1 \equiv G_1 G_2[S_2] H_2 \equiv G_1[S_1] S_2 \in \tau^*$, $G_0[S_{i_1}] G_2[S_2] H_2 \equiv G_1[S_1] \in \tau^*$, $G_0[S_{i_1}] S_2 \in \tau^*$, $G_0[S_i] \in \tau^*$, $G_0[S_{i_1}] \in \tau^*$.

Then $H''$ is separable in $\tau^*_H$ and there are some copies of $G_{H_1, G_2}^\phi[S_2] \in G_{H_1}^\phi$.

**Lemma 4.7.** (New main algorithm for GpsUL$\Omega$) Let $I$ be a complete subset of $I_0$ and $\bar{T} = \{ H'_1 : H'_1 \not\subseteq H'_j \text{ for some } H'_j \in I \}$. Then there exist one close hypersequent $G_{T}^\phi \subseteq G_{\tau^*}$ and its derivation $\tau^*_T$ in GpsUL$\Omega$ such that

(i) $\tau^*_T$ is constructed by initial hypersequent $\widetilde{G_{\tau^*}}(\tau^*)$, the fully constraint contraction rules of the form $\overline{G_{\tau^*}}(EC_\Omega^\phi)$ and elimination rule of the form

$$
\overline{G_{\tau^*}}(EC_\Omega^\phi) \equiv \{ G_{I_w}, G_{I_w} S_{I_w}^c \ldots G_{I_w} S_{I_w}^c \} (\tau^*_I),
$$

where $1 \leq w \leq |I|$, $H'_{I_w} \equiv H'_{I_k} \text{ for all } 1 \leq k < l \leq w$, $I_1 = \{ H'_{I_1}, \ldots, H'_{I_k} \} \subseteq \bar{T}$, $\bar{T}_2 = \{ S_{I_w}^c, S_{I_w}^c, \ldots, S_{I_w}^c \}$, $G_{I_w} S_{I_w}^c$ is closed for all $1 \leq k \leq w$. Then $H'_{I_k} \not\subseteq H_{I_k}$ for each $S_{I_w}^c \in G_{I_2}^{\phi}$ and $H_{I_k} \in I$.

(ii) For all $H \in \tau^*_T$, let

$$
\partial_H^\phi := \begin{cases}
G_{\tau^*}^\phi H \text{ is the root of } \tau^*_T \text{ or } G_2 \in G_{I_1} \equiv \{EC_\Omega^\phi \text{ or } 1D_\Omega \} \in \tau^*_T, \\
H_{I_k} \in G_{I_2}^\phi \text{ in } \tau^*_T \text{ for some } 1 \leq k \leq w.
\end{cases}
$$
where, $\tau^\omega_I$ is the skeleton of $\tau^\omega_I$, which is defined by Definition 7.13 [6]. Then $\partial_{v^\omega_i}(G_{t_1}^{\omega}) \subseteq \partial_{v^\omega_i}(G_{b_0} | S^c_{b_0})$ for some $1 \leq k \leq w$ in $\tau^\omega_I$:

(iii) Let $H \in \tau^\omega_I$ and $G | G^* < \partial_{v^\omega_i}(H) \leq H^V_I$ then $G^\omega_{H^V_I} \in \tau^\omega_I$ and it is built up by applying the separation algorithm along $H^V_I$ to $H$, and is an upper hypercent of either $(EC^*_\omega_i)$ if it is applicable, or $(1D_{2k})$ otherwise.

(iv) $S^c_I \in G^\omega_I$ implies $H^V_I \parallel H$ for all $H^V_I \in I$ and, $S^c_I \in G^\omega_I$ for some $\tau^\omega_I$ in $\tau^\omega_I$.

**Proof.** $\tau^\omega_I$ is constructed by induction on $o(I)$. For the base case, let $o(I) = 1$, then $\tau^\omega_I$ is built up by Construction 7.3 and 7.7 in [6]. For the induction case, suppose that $o(I) \geq 2$, $\tau^\omega_I$ and $\tau^\omega_I$ are constructed such that Claims from (i) to (iv) hold.

Let $G^\omega|G^\omega[H'] \langle (II) \rangle \in \tau^\omega$, where $G^\omega[H'] = H^V_I$. Then $I_I$ and $I_I$ occur in the left subtree $\tau^\omega(G|S')$ and right subtree $\tau^\omega(G^\omega|S''$) of $\tau^\omega(H^V_I)$, respectively. Here, almost all manipulations of the new main algorithm are same as those of the old main algorithm. There are some caveats need to be considered.

Firstly, all leaves $\overline{G^\omega|G^\omega}(\tau^\omega) \in \tau^\omega_I$ are replaced with $\tau^\omega_I$ in Step 3 at Stage 1 in old main algorithm and, $\overline{G^\omega|G^\omega}(\tau^\omega) \in \tau^\omega_I$ are replaced with $\tau^\omega_I$ in Step 3 at Stage 2. Secondly, we abandon the definitions of branch to I and Notation 8.1 in [6] and then the symbol $I_I$ of the set of branches, which occur in $\tau^\omega_I$ in [6], is replaced with I in the new algorithm. We also replace $\Omega$ in $\tau^\omega_I$ with $\star$. Thirdly, under the new requirement that $I$ is complete, we prove the following property.

**Property (A) $G^\omega_I$ contains at most one copy of $G^\omega_{H'}|G''|S''.$**

**Proof.** Suppose that there exist two copies $\{(G^\omega_{H'}|G'')|S''_1\}$ and $\{(G^\omega_{H'}|G'')|S''_2\}$ of $G^\omega_{H'}|G''|S''$ in $G^\omega_I$, and, we put them into $\{1\}$ and $\{2\}$ in order to distinguish them. Let $[S]_{G^\omega_I}$ be a splitting unit of $G^\omega$ and $S$ its splitting sequent. Then $|v(S)| + |v(S)| \geq 2$. Thus $S$ is a ($pEC$)-sequent and has the form $S_I = [S]_{G^\omega_I} \subseteq G^\omega$. Then $[S]_{G^\omega_I} = [S_I]_{G^\omega_I}$. $H' \parallel H'$ for all $H' \in I$ and $S' \in G^\omega_{H'}$ for some $\tau^\omega_I$ by Claim (iv). Since $I$ is complete and $G^\omega|S' \leq H^V_I$, then $H^V_I \parallel G^\omega|S'$.

Let $\tau^\omega_I$ be in the form

$$G_{b_0}|S^c_{b_0} \cdot \cdots \cdot G_{b_0}|S^c_{b_0} \cdot G_{b_0}|S^c_{b_0} \cdot \{\tau^\omega_I\}$$

where $G_{b_0}|S^c_{b_0} \subseteq G_{b_0}|S^c_{b_0} \subseteq G_{b_0}|S^c_{b_0} \subseteq G_{b_0}|S^c_{b_0}$.

Figure 3. Then $H^V_I \parallel G_{b_0}|S^c_{b_0} \parallel G_{b_0}|S^c_{b_0} \parallel G_{b_0}|S^c_{b_0} \parallel G_{b_0}|S^c_{b_0} \parallel G_{b_0}|S^c_{b_0}$.

Since $S_2$ is separable in $G^\omega_I$ by $G^\omega|S' \leq H^V_I$, then $S^c_I \in G^\omega_I$ and $S^c_I$ is not $S_2$.  

11
Figure 3 A fragment of $\tau_{h_k}^2$

**Property (B)** The set of splitting sequents of $[S'_1]_{G^*_h}$ is equal to that of $[S'_1]_{G_2S_2}$.

*Proof.* Let $\frac{G_1[S'_1]}{H'_1} \equiv G_2[H'_2]$. Then $S'_1$ and $S'_2$ are separable in $G^*_h$. Thus $G_1^{(j)}(S'_1) \subseteq G^*_h$ is closed. Hence $G_2^{(j)}(S'_1) \subseteq G_1^{(j)}(S'_1) \cup G_2^{(j)}(S'_2)$ is closed, and the set of splitting sequents of $[S'_1]_{G^*_h}$ is equal to that of $[S'_1]_{G_2S_2}$, since each splitting sequent $S'' \in [S'_1]_{G^*_h}$ is a $(pEC)$-sequent by $|\nu(S''')| + |\nu(S''')| + 2$ and $S''' \in \nu(G^*_h)$. This completes the proof of Property (B).

We therefore assume that, without loss of generality, $S'_1$ is in the form $\Gamma, p_k, \Delta \Rightarrow p_k$ by Property (B), Lemma 3.16 and the observation that each derivation-splicing operation is local. There are two cases to be considered in the following.

**Case 1** $S'_1 \in (G_1[S_1])_{G_2S_2}$, for all $\tau_{G_2S_2}^j$, $G_1[S_1] \subseteq H_j \subseteq H'_1$. Then $G_1^{(j)}(S'_1) \subseteq G_2^{(j)}(S'_1) \subseteq G_1^{(j)}(S'_1) \cup G_2^{(j)}(S'_2) \subseteq \emptyset$. We assume that, without loss of generality, $(G_2[S'_2])_k = G_2^{(j)}(S'_2) \Rightarrow \Gamma, (G_2[S'_2])_k = G_2^{(j)}(S'_2) \Rightarrow \Gamma, (G_2[S'_2])_k = G_2^{(j)}(S'_2) \Rightarrow \Gamma$.

Then $\{G_2^{(j)}[S'_1]_{G_2S_2}\} \subseteq \{G_2^{(j)}[S'_1]_{G_2S_2}\} \subseteq \{G_2^{(j)}[S'_1]_{G_2S_2}\} \subseteq \{G_2^{(j)}[S'_1]_{G_2S_2}\} \subseteq \{G_2^{(j)}[S'_1]_{G_2S_2}\} \subseteq \{G_2^{(j)}[S'_1]_{G_2S_2}\}$.

This shows that any splitting unit $[S'_1]_{G_2S_2}$ outside $G_2^{(j)}(S'_1) \subseteq G_2^{(j)}(S'_1) \subseteq G_2^{(j)}(S'_1) \subseteq G_2^{(j)}(S'_1) \subseteq G_2^{(j)}(S'_1)$ doesn’t happen. We therefore assume that, without loss of generality, $S'_1$ is in the form $\Gamma, p_k, \Delta \Rightarrow p_k$ by Property (B), Lemma 3.16 and the observation that each derivation-splicing operation is local. There are two cases to be considered in the following.

**Case 2** $S'_1 \in (G_1[S_1])_{G_2S_2}$, for all $\tau_{G_2S_2}^j$, $G_1[S_1] \subseteq H_j \subseteq H'_1$. Then $G_2^{(j)}(S'_1) \subseteq G_2^{(j)}(S'_1) \cup G_2^{(j)}(S'_2) \subseteq \emptyset$. Hence $[S'_1]_{G_2S_2}$ does not happen. The case
of \( S' \in G'' \) is tackled with the same procedure as the following. Let \([S']_{G''} \subseteq \left\{ G_{H''_G'G''}^{\circ}(\overline{S''}) \right\}_1 \). Then there exists a copy of \([S']_{G''} \sqsubseteq \left\{ G_{H''_G'G''}^{\circ}(\overline{S''}) \right\}_2 \) and let \( \Gamma, p \rightarrow \Delta \Rightarrow p' \) be its splitting sequent. We put two splitting units into \( \{\} \) and \( \{\} \) in order to distinguish them. Then \( \{[S]_{G''}\}_k \subseteq \left\{ G_{H''_G'G''}^{\circ}(\overline{S''}) \right\}_1 \) and \( \{[S]_{G''}\}_\ell \subseteq \left\{ G_{H''_G'G''}^{\circ}(\overline{S''}) \right\}_2 \). We assume that, without loss of generality, \((G_2[S_2]_t = G_2') \Gamma \Rightarrow t, (G_2[S_2]_t = G_2') \Delta \Rightarrow t\). Then \( \left\{ G_{H''_G'G''}^{\circ}(\overline{S''}) \right\}_1 \subseteq \left\{ G_{H''_G'G''}^{\circ}(\overline{S''}) \right\}_2 \). Thus \( \{[S]_{G''}\}_\nu \subseteq \left\{ G_{H''_G'G''}^{\circ}(\overline{S''}) \right\}_2 \subseteq \left\{ G_{H''_G'G''}^{\circ}(\overline{S''}) \right\}_1 \). Let \( \Delta \Rightarrow t \) into \( \{\} \) and \( \{\} \) in order to distinguish them. Then \( \Gamma \Rightarrow t \in \left\{ G_{H''_G'G''}^{\circ}(\overline{S''}) \right\}_1 \). From \( \Gamma \Rightarrow t \in \left\{ G_{H''_G'G''}^{\circ}(\overline{S''}) \right\}_1 \), \( \Gamma \Rightarrow t \in \left\{ G_{H''_G'G''}^{\circ}(\overline{S''}) \right\}_1 \). \( \Gamma \Rightarrow t \in \left\{ G_{H''_G'G''}^{\circ}(\overline{S''}) \right\}_1 \). This completes the proof of Property (A).

With Property (A), all manipulations in the old main algorithm in [6] work well. This completes the construction of \( \tau'' \) and the proof of Theorem 4.1.

**Theorem 4.8.** The standard completeness holds for \( HvpsUL' \).

**Proof.** Let \( \leftrightarrow \) denote the \( i \)-th logical link of \( \text{if in the following.} \) \( \models_K \ A \) means that \( \nu(A) \geq t \) for every algebra \( A \) in \( K \) and valuation \( \nu \) on \( A \). Let \( psUL' \), \( LIN(psUL') \), \( psUL'^D \) and \( psUL' \) denote the classes of all \( psUL' \) algebras, \( psUL' \) chain, dense \( psUL' \) chain and standard \( psUL' \) algebras (i.e., their lattice reducts are \([0,1]\)) respectively. We have an inference sequence, as shown in Figure 4.

![Figure 4 Two ways to prove standard completeness](image)

Links from 1 to 4 show Jenei and Montagna’s algebraic method to prove standard completeness and currently, it seems hopeless to built up the link 3, see [7–10]. Links from 1 to 4 show Metcalfe and Montagna’s proof-theoretical method. Density elimination is at Link 2 in Figure 4 and other links are proved by standard procedures with minor revisions and omitted, see [1, 4].
References

[1] G. Metcalfe, N. Olivetti and D. Gabbay, *Proof Theory for Fuzzy Logics*, Springer Series in Applied Logic, Vol.36, 2009.

[2] S. M. Wang, B. Zhao, *HpsUL is not the logic of pseudo-uninorms and their residua* [J], Logic Journal of the IGPL, 17(4): 413C419, 2009.

[3] S.M. Wang, *Logics for residuated pseudo-uninorms and their residua* [J], Fuzzy Sets and Systems, 218 (2013) 24-31.

[4] G. Metcalfe, F. Montagna, *Substructural fuzzy logics* [J], Journal of Symbolic Logic 73(3)(2007), 834-864.

[5] G. Metcalfe, C. Tsinakis, *Density revisited* [J], Soft computing, 2017, 21(1): 175-189.

[6] S. M. Wang, *Density Elimination for Semilinear Substructural Logics*, arXiv preprint arXiv: 1509.03472, 2015.

[7] S. Jenei, F. Montagna, *A proof of standard completeness of Esteva and Godo’s monoidal logic MTL* [J], Studia Logica 70(2)(2002), 184-192.

[8] S. M. Wang, *Uninorm logic with the n-potency axiom* [J], Fuzzy Sets and Systems, 205 (2012) 116-126.

[9] S. M. Wang, *Involutive uninorm logic with the n-potency axiom* [J], Fuzzy Sets and Systems, 218(2013), 1-23.

[10] S. M. Wang, *The Finite Model Property for Semilinear Substructural Logics* [J], Mathematical Logic Quarterly, 59(4-5)(2013), 268-273.