COMPLETE MONOTONICITY OF SOME FUNCTIONS INVOLVING POLYGAMMA FUNCTIONS

FENG QI, SENLIN GUO, AND BAI-NI GUO

Abstract. In the present paper, we establish necessary and sufficient conditions for the functions $x^\alpha |\psi^{(i)}(x + \beta)|$ and $\alpha |\psi^{(i)}(x + \beta)| - x |\psi^{(i+1)}(x + \beta)|$ respectively to be monotonic and completely monotonic on $(0, \infty)$, where $i \in \mathbb{N}$, $\alpha > 0$ and $\beta \geq 0$ are scalars, and $\psi^{(i)}(x)$ are polygamma functions.

1. Introduction

1.1. Recall [27, Chapter XIII] and [54, Chapter IV] that a function $f(x)$ is said to be completely monotonic on an interval $I \subseteq \mathbb{R}$ if $f(x)$ has derivatives of all orders on $I$ and

$$0 \leq (-1)^k f^{(k)}(x) < \infty$$

holds for all $k \geq 0$ on $I$. This definition was introduced in 1921 by F. Hausdorff in [23], who called such functions “total monotone”.

The celebrated Bernstein-Widder Theorem [54, p. 161] states that a function $f(x)$ is completely monotonic on $(0, \infty)$ if and only if

$$f(x) = \int_0^\infty e^{-xs} \, d\mu(s),$$

where $\mu$ is a nonnegative measure on $[0, \infty)$ such that the integral (1.2) converges for all $x > 0$. This means that a function $f(x)$ is completely monotonic on $(0, \infty)$ if and only if it is a Laplace transform of the measure $\mu$.

The most important properties of completely monotonic functions can be found in [27, Chapter XIII], [54, Chapter IV], [10, 50] and the related references therein.

The completely monotonic functions have applications in different branches of mathematical sciences. For example, they play some role in combinatorics [8], numerical and asymptotic analysis [18, 55], physics [16, 17], potential theory [11], and probability theory [12, 17, 26].

1.2. It is well-known [1, 51, 52] that the classical Euler gamma function may be defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t} \, dt.$$  

The logarithmic derivative of $\Gamma(x)$, denoted by $\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$, is called psi function, and $\psi^{(k)}(x)$ for $k \in \mathbb{N}$ are called polygamma functions.

It should be common knowledge [1, 51, 52] that the special functions $\Gamma(x)$, $\psi(x)$ and $\psi^{(i)}(x)$ for $i \in \mathbb{N}$ are important and basic and that they have much extensive applications in mathematical sciences.

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1.3. In [2, Lemma 1], it was shown that the functions \( x^c |\psi^{(k)}(x)| \) for \( k \in \mathbb{N} \) and \( c \in \mathbb{R} \) are strictly decreasing (or strictly increasing, respectively) on \((0, \infty)\) if and only if \( c \leq k \) (or \( c \geq k + 1 \), respectively).

In [3, Theorem 4.14], it was obtained that the function \( x^c |\psi^{(k)}(x)| \) for \( k \in \mathbb{N} \) and \( c \in \mathbb{R} \) is strictly convex on \((0, \infty)\) if and only if either \( c \leq k \), or \( c = k + 1 \), or \( c \geq k + 2 \). In [3, Remark 4.15], it was pointed out that there does not exist a real number \( c \) such that the function \( x^c |\psi^{(k)}(x)| \) for \( k \in \mathbb{N} \) is concave on \((0, \infty)\).

In [3, Lemma 2.2] and [47, Lemma 5], the functions \( x^i |\psi^{(i)}(x)| \) for \( i \in \mathbb{N} \) are proved to be strictly increasing (or strictly decreasing, respectively) on \((0, \infty)\) if and only if \( \alpha \geq i \) (or \( \alpha \leq 0 \), respectively).

In [19, Lemma 2.1], the function \( x^i \psi'(x + a) \) is proved to be strictly increasing on \([0, \infty)\) for \( \alpha \geq 1 \).

Motivated by the above results, the first and third authors considered in [22] the monotonicity of a more general function \( x^\alpha |\psi^{(i)}(x + \beta)| \) and the complete monotonicity of several related functions as follows: For \( \alpha \in \mathbb{R} \), \( \alpha > 0 \) and \( \beta \geq 0 \),

1. the function \( x^\alpha |\psi^{(i)}(x + \beta)| \) is strictly increasing on \((0, \infty)\) if \( (\alpha, \beta) \in \{ \alpha \geq i, \frac{1}{2} \leq \beta < 1 \} \cup \{ \alpha \geq i, \beta \geq \frac{\alpha + i + 1}{2} \} \cup \{ \alpha \geq i + 1, \beta \leq \frac{\alpha + i + 1}{2} \} \) and only if \( \alpha \geq i \);
2. the function \( \frac{1}{x^i} |\psi^{(i)}(x)| - |\psi^{(i+1)}(x)| \) is completely monotonic on \((0, \infty)\) if and only if \( \alpha \geq i + 1 \);
3. the function \( \frac{1}{x^i} |\psi^{(i)}(x)| - \frac{1}{2} |\psi^{(i)}(x)| \) is completely monotonic on \((0, \infty)\) if and only if \( \alpha < i \);
4. the function \( \frac{1}{x^i} |\psi^{(i)}(x + 1)| - |\psi^{(i+1)}(x + 1)| \) is completely monotonic on \((0, \infty)\) if and only if \( \alpha \geq i \);
5. the function \( \frac{1}{x^i} |\psi^{(i)}(x + \beta)| - |\psi^{(i+1)}(x + \beta)| \) on \((0, \infty)\) is completely monotonic if \( (\alpha, \beta) \in \{ \alpha \geq i + 1, \beta \leq \frac{\alpha + i + 1}{2} \} \cup \{ \alpha \leq \frac{i}{2} + 1, \beta \leq \frac{i}{2} \} \) and only if \( \alpha \geq i \);
6. the function \( \alpha |\psi^{(i)}(x + \beta)| - x |\psi^{(i+1)}(x + \beta)| \) is completely monotonic on \((0, \infty)\) if \( (\alpha, \beta) \in \{ \alpha \geq i, i \leq \alpha \leq i + 1, \beta \geq \frac{\alpha + i + 1}{2} \} \cup \{ \alpha \geq i + 1, \beta \leq \frac{\alpha + i + 1}{2} \} \) and only if \( \alpha \geq i \).

1.4. The first aim of this paper is to present necessary and sufficient conditions for the function \( x^\alpha |\psi^{(i)}(x + \beta)| \) to be monotonic on \((0, \infty)\), which can be summarized as the following Theorem 1.

**Theorem 1.** Let \( i \in \mathbb{N} \), \( \alpha \in \mathbb{R} \) and \( \beta \geq 0 \).

1. The function \( x^\alpha |\psi^{(i)}(x)| \) is strictly increasing (or strictly decreasing, respectively) on \((0, \infty)\) if and only if \( \alpha \geq i + 1 \) (or \( \alpha \leq i \), respectively).
2. For \( \beta \geq \frac{1}{2} \), the function \( x^\alpha |\psi^{(i)}(x + \beta)| \) is strictly increasing on \([0, \infty)\) if and only if \( \alpha \geq i \).
3. Let \( \delta : (0, \infty) \to \left(0, \frac{1}{2}\right) \) be defined by
   \[
   \delta(t) = \frac{e^t (t - 1) + 1}{(e^t - 1)^2} \tag{1.4}
   \]
   and \( \delta^{-1} : \left(0, \frac{1}{2}\right) \to (0, \infty) \) stand for the inverse function of \( \delta \). If \( 0 < \beta < \frac{1}{2} \) and
   \[
   \alpha \geq i + 1 - \left[ \frac{e^{\delta^{-1}(-\beta)} - 1}{e^{\delta^{-1}(-\beta)} - 1 + \beta - 1} \right] \delta^{-1}(-\beta), \tag{1.5}
   \]
   then the function \( x^\alpha |\psi^{(i)}(x + \beta)| \) is strictly increasing on \((0, \infty)\).

As a by-product of the proof of Theorem 1, lower and upper bounds for infinite series whose coefficients involve Bernoulli numbers may be derived as follows.
Corollary 1. Let \(0 < \beta < \frac{1}{2}\) and \(\delta^{-1}\) be the inverse function of \(\delta\) defined by (1.4). Then the following inequalities holds:

\[
\frac{1}{2} > \sum_{k=1}^{\infty} B_{2k} \frac{t^{2k-1}}{(2k-1)!} > 0, \quad (1.6)
\]

\[
t > \frac{1}{2} > \sum_{k=0}^{\infty} B_{2k+2} \frac{t^{2k+2}}{(2k+2)!} > \max \left\{ 0, \frac{t}{2} - 1 \right\}, \quad (1.7)
\]

\[
\sum_{k=0}^{\infty} B_{2k+2} \frac{t^{2k+2}}{(2k+2)!} > \left( \frac{1}{2} - \beta \right) t + \left[ \frac{e^{\delta^{-1}(\beta)}}{e^{\delta^{-1}(\beta) - 1}} - \beta + 1 \right] \delta^{-1}(\beta) - 1, \quad (1.8)
\]

where \(t \in (0, \infty)\) and \(B_n\) for \(n \geq 0\) represent Bernoulli numbers which may be defined [1, 51, 52] by

\[
\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} x^n = 1 - \frac{x}{2} + \sum_{j=1}^{\infty} \frac{B_{2j} x^{2j}}{(2j)!}, \quad |x| < 2\pi. \quad (1.9)
\]

1.5. The second aim of this paper is to establish necessary and sufficient conditions for the function \(\alpha |\psi^{(i)}(x + \beta)| - x |\psi^{(i+1)}(x + \beta)|\) to be completely monotonic on \((0, \infty)\), which may be stated as the following Theorem 2.

Theorem 2. Let \(i \in \mathbb{N}\), \(\alpha \in \mathbb{R}\) and \(\beta \geq 0\).

(1) The function

\[
\alpha |\psi^{(i)}(x)| - x |\psi^{(i+1)}(x)| \quad (1.10)
\]

is completely monotonic on \((0, \infty)\) if and only if \(\alpha \geq i + 1\).

(2) The negative of the function (1.10) is completely monotonic on \((0, \infty)\) if and only if \(\alpha \leq i\).

(3) If \(\beta \geq \frac{1}{2}\), then the function

\[
\alpha |\psi^{(i)}(x + \beta)| - x |\psi^{(i+1)}(x + \beta)| \quad (1.11)
\]

is completely monotonic on \((0, \infty)\) if and only if \(\alpha \geq i\).

(4) If \(0 < \beta < \frac{1}{2}\) and the inequality (1.5) is valid, then the function (1.11) is completely monotonic on \((0, \infty)\).

As immediate consequences of Theorem 2, the following corollary is obtained.

Corollary 2. Let \(i \in \mathbb{N}\), \(\alpha \in \mathbb{R}\) and \(\beta \geq 0\).

(1) The function

\[
\frac{\alpha}{x} |\psi^{(i)}(x)| - |\psi^{(i+1)}(x)| \quad (1.12)
\]

is completely monotonic on \((0, \infty)\) if and only if \(\alpha \geq i + 1\).

(2) The negative of the function (1.12) is completely monotonic on \((0, \infty)\) if and only if \(\alpha \leq i\).

(3) If \(\beta \geq \frac{1}{2}\), then the function

\[
\frac{\alpha}{x} |\psi^{(i)}(x + \beta)| - |\psi^{(i+1)}(x + \beta)| \quad (1.13)
\]

is completely monotonic on \((0, \infty)\) if and only if \(\alpha \geq i\).

(4) If \(0 < \beta < \frac{1}{2}\) and the inequality (1.5) holds true, then the function (1.13) is completely monotonic on \((0, \infty)\).
2. Remarks

Before proving the above theorems and corollaries, we give several remarks about Theorem 1, Theorem 2 and their applications.

Remark 1. Since \( \lim_{\beta \to 0^+} [\beta \delta^{-1}(\beta)] = 0 \) and that the inverse function \( \delta^{-1} \) is decreasing from \((0, \frac{1}{2})\) onto \((0, \infty)\), we claim that

\[
0 < \left[ \frac{e^{\delta^{-1}(\beta)}}{e^{\delta^{-1}(\beta)} - 1} + \beta - 1 \right] \delta^{-1}(\beta) < 1, \quad \beta \in (0, 1). \tag{2.1}
\]

Indeed, if replacing \( \delta^{-1}(\beta) \) by \( s \), the middle term in (2.1) becomes \( \frac{s^2\delta'}{(s-1)^2} \) which is decreasing from \((0, \infty)\) onto \((0, 1)\). This implies that the condition (1.5) is not only sufficient but also necessary in Theorem 1 and Theorem 2.

Remark 2. As mentioned in Section 1.3, some conclusions in Theorem 1 have been applied in nearby fields.

(a) The first conclusion in Theorem 1 was utilized in [2, Theorem 2] to obtain a functional inequality concerning polygamma functions: For \( k \geq 1 \) and \( n \geq 2 \), the inequality

\[
|\psi^{(k)}(M_n^{[r]}(x_\nu; p_\nu))| \leq M_n^{[s]}(\psi^{(k)}(x_\nu); p_\nu) \tag{2.2}
\]

holds if and only if either \( r \geq 0 \) and \( s \geq \frac{r}{n-1} \) or \( r < 0 \) and \( s \geq \frac{r}{n} \), where \( x_\nu > 0 \) and \( p_\nu > 0 \) with \( \sum_{\nu=1}^{n} p_\nu = 1 \), and

\[
M_n^{[r]}(x_\nu; p_\nu) = \begin{cases} \left( \sum_{\nu=1}^{n} p_\nu x_\nu^t \right)^{1/t}, & t \neq 0 \\ \prod_{\nu=1}^{n} x_\nu^{p_\nu}, & t = 0 \end{cases} \tag{2.3}
\]

stands for the discrete weighted power means.

(b) Some applications of the first two conclusions in Theorem 1 were carried out in [3] as follows.

(a) The first conclusion in Theorem 1 was applied in [3, Theorem 4.9] to obtain that the inequalities

\[
\left(1 + \frac{\alpha}{x + s}\right)^n < \frac{\psi^{(n)}(x + s)}{\psi^{(n)}(x + 1)} < \left(1 + \frac{\beta}{x + s}\right)^n \tag{2.4}
\]

hold for \( n \in \mathbb{N} \) and \( s \in (0, 1) \) with the best possible constants

\[
\alpha = 1 - s \quad \text{and} \quad \beta = s \left( \frac{\psi^{(n)}(s)}{\psi^{(n)}(1)} \right)^{1/n} - s. \tag{2.5}
\]

(b) The special cases for \( \beta = 1 \) of the third conclusion in Theorem 1 was employed in [3, Theorem 4.8] to derive that the inequalities

\[
(n-1)! \exp \left[ \frac{\alpha}{x} - n\psi(x) \right] < |\psi^{(n)}(x)| < (n-1)! \exp \left[ \frac{\beta}{x} - n\psi(x) \right] \tag{2.6}
\]

hold for \( n \in \mathbb{N} \) and \( x > 0 \) if and only if \( \alpha \leq -n \) and \( \beta \geq 0 \).

(c) Moreover, the convexity of \( x^\alpha|\psi^{(k)}(x)| \) was used in [3, Theorem 4.16] to establish that the double inequalities

\[
\alpha \left(\frac{1}{x} - \frac{1}{y}\right) < x^n |\psi^{(n)}(x)| - y^n |\psi^{(n)}(y)| < \beta \left(\frac{1}{x} - \frac{1}{y}\right) \tag{2.7}
\]

hold for \( n \in \mathbb{N} \) and \( y > x > 0 \) with the best possible constants \( \alpha = \frac{n!}{2} \) and \( \beta = n! \).
(3) The special cases for $\beta = 1$ of the third conclusion in Theorem 1, the monotonic properties of the functions  
\[ \begin{align*}
    &x^{4i} \psi^{(2i)}(1 + x), \\
    &x^{4i+1} \psi^{(2i)}(1 + x), \\
    &x^{4i-1} \psi^{(2i-1)}(1 + x) \quad \text{and} \quad x^{4i} \psi^{(2i-1)}(1 + x)
\end{align*} \]
on $[0, \infty)$, were used in [47, 49] to establish the monotonic, logarithmically convex, completely monotonic properties of the functions  
\[ \frac{[\Gamma(1 + x)]^y}{\Gamma(1 + xy)} \quad \text{and} \quad \frac{\Gamma(1 + y)[\Gamma(1 + x)]^y}{\Gamma(1 + xy)} \]  
(2.8)
or their first and second logarithmic derivatives.

(4) The special case for $\beta \geq 1$ and $i = 1$ of the third conclusion in Theorem 1 was employed in [19, Theorem 3.1] to reveal the subadditive property of the function $\psi(a + e^z)$ on $(-\infty, \infty)$.

Remark 3. The former two conclusions in Theorem 1, which were ever circulated in the preprint [44], have been employed in the proofs of [41, Theorem 1.4] and [45, Theorem 2.2 and Theorem 2.3].

(1) The special cases for $\alpha = i$ and $\beta = 1$ of the second conclusion in Theorem 1 were made use of to procure the following theorem.

**Theorem 3 ([41, Theorem 1.4]).** The function  
\[ G_{s,t}(x) = \frac{[\Gamma(1 + tx)]^s}{[\Gamma(1 + sx)]^t} \]  
(2.9)for $x, s, t \in \mathbb{R}$ such that $1 + sx > 0$ and $1 + tx > 0$ with $s \neq t$ has the following properties:

(a) For $t > s > 0$ and $x \in (0, \infty)$, $G_{s,t}(x)$ is an increasing function and a logarithmically completely monotonic function of second order in $x$;
(b) For $t > s > 0$ and $x \in (-\frac{1}{t}, 0)$, $G_{s,t}(x)$ is a logarithmically completely monotonic function in $x$;
(c) For $s < t < 0$ and $x \in (-\infty, 0)$, $G_{s,t}(x)$ is a decreasing function and a logarithmically absolutely monotonic function of second order in $x$;
(d) For $s < t < 0$ and $x \in (0, -\frac{1}{t})$, $G_{s,t}(x)$ is a logarithmically completely monotonic function in $x$;
(e) For $s < 0 < t$ and $x \in (-\frac{1}{t}, 0)$, $G_{t,s}(x)$ is an increasing function and a logarithmically absolutely convex function in $x$;
(f) For $s < 0 < t$ and $x \in (0, -\frac{1}{t})$, $G_{t,s}(x)$ is a decreasing function and a logarithmically absolutely convex function in $x$.

(2) The first two conclusions in Theorem 1 were hired in [45], a simplified version of the preprint [43], to derive the following two theorems.

**Theorem 4 ([45, Theorem 2.2]).** For $b > a > 0$ and $i \in \mathbb{N}$, the function  
\[ \frac{[\Gamma(bx)]^i}{[\Gamma(ax)]^i} \]  
is $(2i+1)$-log-convex and $(2i)$-log-concave with respect to $x \in (0, \infty)$.

**Theorem 5 ([45, Theorem 2.3]).** For $b > a > 0$, $i \in \mathbb{N}$ and $\beta \geq \frac{1}{2}$, the function  
\[ \frac{[\Gamma(bx+\beta)]^i}{[\Gamma(ax+\beta)]^i} \]  
is $(2i + 1)$-log-concave and $(2i)$-log-convex with respect to $x \in (0, \infty)$.

For exact definitions of the terminologies such as “logarithmically completely monotonic function of $k$-th order”, “logarithmically absolutely monotonic function of $k$-th order”, “$k$-log-convex function” and “logarithmically absolutely convex function”, see [41, Definition 1.1, Definition 1.2 and Definition 1.3], or the first paragraph of [45], or related texts in [48].
Remark 4. The former two conclusions in Theorem 2, which were also issued in the preprint [44], have also been applied in the proofs of [42, Theorem 1] and [46, Theorem 1].

(1) The first result in Theorem 2 was utilized in [42, Theorem 1] to procure upper bounds for the ratio of two gamma functions and the divided differences of the psi and polygamma functions as follows.

**Theorem 6 ([42, Theorem 1]).** For \( a > 0 \) and \( b > 0 \) with \( a \neq b \), inequalities

\[
\left[ \frac{\Gamma(a)}{\Gamma(b)} \right]^{1/(a-b)} \leq e^{\psi(I(a,b))} \tag{2.10}
\]

and

\[
\frac{(-1)^n [\psi(n-1)(a) - \psi(n-1)(b)]}{a-b} \leq (-1)^n \psi(n)(I(a,b)) \tag{2.11}
\]

hold true, where \( n \in \mathbb{N} \) and

\[
I(a,b) = \frac{1}{e} \left( \frac{b}{a} \right)^{1/(b-a)} \tag{2.12}
\]

represents the identric or exponential mean.

(2) The first two results in Theorem 2 were employed in [46, Theorem 1] to acquire lower bounds for the ratio of two gamma functions and the divided differences of the psi and polygamma functions and to refine the inequality (2.11).

**Theorem 7 ([46, Theorem 1]).** For \( a > 0 \) and \( b > 0 \) with \( a \neq b \), the inequality

\[
(-1)^i \psi(i)(L_\alpha(a,b)) \leq \frac{(-1)^i}{b-a} \int_a^b \psi(i)(u) \, du \leq (-1)^i \psi(i)(L_\beta(a,b)) \tag{2.13}
\]

holds if \( \alpha \leq -i-1 \) and \( \beta \geq -i \), where \( i \) is a nonnegative integer and

\[
L_p(a,b) = \begin{cases} 
\left[ \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^{1/p}, & p \neq -1,0 \\
\frac{b-a}{\ln b - \ln a}, & p = -1 \\
I(a,b), & p = 0 
\end{cases} \tag{2.14}
\]

stands for the generalized logarithmic mean of order \( p \in \mathbb{R} \).

The topic of bounding the ratio of two gamma functions has a history of at least sixty years since [53]. For more information on its history, backgrounds, motivations and recent developments, please refer to, for example, [2, 3, 25, 28, 29, 30, 33, 38, 42, 46, 56, 57], especially to the expository and survey preprint [31] in which plentiful references are collected. For knowledge of mean values, please refer to the celebrated book [13] or the paper [32].

**Remark 5.** Recall [4, 6, 19] that a function \( f(x) \) is said to be subadditive on \( I \) if the inequality

\[
f(x + y) \leq f(x) + f(y) \tag{2.15}
\]

holds for all \( x, y \in I \) with \( x + y \in I \). If the inequality (2.15) is reversed, then \( f(x) \) is called superadditive on \( I \).

The subadditive and superadditive functions play important roles in the theory of differential equations, in the study of semi-groups, in number theory, in the theory of convex bodies, and the like. See [4, 5, 6] and the related references therein.
Some subadditive or superadditive properties of the gamma, psi and polygamma functions have been discovered as follows.

In [5], the function $\psi(a + x)$ is proved to be sub-multiplicative with respect to $x \in [0, \infty)$ if and only if $a \geq a_0$, where $a_0$ denotes the only positive real number which satisfies $\psi(a_0) = 1$.

In [6], the function $[\Gamma(x)]^\alpha$ was proved to be subadditive on $(0, \infty)$ if and only if $\frac{\ln x}{\ln 2} \leq \alpha \leq 0$, where $\Delta = \min_{x \geq 0} \frac{1}{\Gamma(x)}$.

In [3, Lemma 2.4], the function $\psi(e^x)$ was proved to be strictly concave on $\mathbb{R}$.

In [19, Theorem 3.1], the function $\psi(a + e^x)$ is proved to be subadditive on $(-\infty, \infty)$ if and only if $a \geq c_0$, where $c_0$ is the only positive zero of $\psi(x)$.

In [14, Theorem 1], among other things, it was presented that the function $\psi^{(k)}(e^x)$ for $k \in \mathbb{N}$ is concave (or convex, respectively) on $\mathbb{R}$ if $k = 2n - 2$ (or $k = 2n - 1$, respectively) for $n \in \mathbb{N}$.

By the aid of the monotonicity of the function $x^n |\psi^{(i)}(x + \beta)|$ in Theorem 1, the following subadditive and superadditive properties of the function $|\psi^{(i)}(e^x)|$ for $i \in \mathbb{N}$ were acquired recently.

**Theorem 8** ([37]). For $i \in \mathbb{N}$, the function $|\psi^{(i)}(e^x)|$ is superadditive on $(-\infty, \ln \theta_0)$ and subadditive on $(\ln \theta_0, \infty)$, where $\theta_0 \in (0, 1)$ is the unique root of the equation $2|\psi^{(i)}(\theta)| = |\psi^{(i)}(\theta^2)|$.

**Remark 6.** The second conclusion in Theorem 2 was cited in [56, Lemma 2.4] and [57, Remark 2.3].

**Remark 7.** Theorem 8 and the facts mentioned in Remark 3 to Remark 6 show the potential applicability of Theorem 1 and Theorem 2 convincingly.

**Remark 8.** In passing, we recollect the notion “logarithmically completely monotonic function” which is equivalent to the logarithmically completely monotonic function of 0-th order mentioned in Remark 3. A function $f(x)$ is said to be logarithmically completely monotonic on an interval $I \subseteq \mathbb{R}$ if it has derivatives of all orders on $I$ and its logarithm $\ln f(x)$ satisfies

$$0 \leq (-1)^k |\ln f(x)|^{(k)} < \infty$$

for $k \in \mathbb{N}$ on $I$. By looking through the database MathSciNet, we find that this phrase was first used in [7], but with no a word to explicitly define it. Thereafter, it seems to have been ignored by the mathematical community. In early 2004, this terminology was recovered in [36] and it was immediately referenced in [40], the preprint of the paper [39]. A natural question that one may ask is: Whether is this notion trivial or not? In [36, Theorem 4], it was proved that all logarithmically completely monotonic functions are also completely monotonic, but not conversely. This result was formally published when revising [35]. Hereafter, this conclusion and its proofs were dug in [9, 20, 21, 48] once and again. Furthermore, in the paper [9], the logarithmically completely monotonic functions on $(0, \infty)$ were characterized as the infinitely divisible completely monotonic functions studied in [24] and all Stieltjes transforms were proved to be logarithmically completely monotonic on $(0, \infty)$, where a function $f(x)$ defined on $(0, \infty)$ is called a Stieltjes transform if it can be of the form

$$f(x) = a + \int_0^\infty \frac{1}{s + x} \, d\mu(s)$$

for some nonnegative number $a$ and some nonnegative measure $\mu$ on $[0, \infty)$ satisfying $\int_0^\infty \frac{1}{s + x} \, d\mu(s) < \infty$. For more information, please refer to [9].

It is remarked that many completely monotonic functions founded in a lot of literature such as [25, 28, 50], [27, Chapter XIII] and the related references therein are actually logarithmically completely monotonic.
3. Lemmas

In order to verify Theorem 1, Theorem 2 and their corollaries in Section 1.4 and Section 1.5, we need the following lemmas, in which Lemma 1 is simple but has been validated in [29, 30, 33, 34] to be especially effectual in proving the monotonicity and (logarithmically) complete monotonicity of functions involving the gamma, psi and polygamma functions.

Lemma 1. Let \( f(x) \) be a function defined on an infinite interval \( I \) whose right end is \( \infty \). If \( \lim_{x \to \infty} f(x) = \delta \) and \( f(x) - f(x + \varepsilon) > 0 \) hold true for some given scalar \( \varepsilon > 0 \) and all \( x \in I \), then \( f(x) > \delta \).

Proof. By mathematical induction, for all \( x \in I \), we have
\[
f(x) > f(x + \varepsilon) > f(x + 2\varepsilon) > \cdots > f(x + k\varepsilon) \to \delta \quad \text{as} \quad k \to \infty.
\]
The proof of Lemma 1 is complete. \( \square \)

Lemma 2 ([1, 51, 52]). The polygamma functions \( \psi^{(k)}(x) \) may be expressed for \( x > 0 \) and \( k \in \mathbb{N} \) as
\[
\psi^{(k)}(x) = (-1)^{k+1} \int_0^\infty \frac{t^k e^{-xt}}{1 - e^{-t}} \, dt. \tag{3.1}
\]

For \( x > 0 \) and \( r > 0 \),
\[
\frac{1}{x^r} = \frac{1}{\Gamma(r)} \int_0^\infty t^{r-1} e^{-xt} \, dt. \tag{3.2}
\]

For \( i \in \mathbb{N} \) and \( x > 0 \),
\[
\psi^{(i-1)}(x + 1) = \psi^{(i-1)}(x) + \frac{(-1)^{i-1}(i-1)!}{x^i}. \tag{3.3}
\]

Lemma 3. For \( k \in \mathbb{N} \), the double inequality
\[
\frac{(k-1)!}{x^k} + \frac{k!}{2x^{k+1}} < (-1)^{k+1} \psi^{(k)}(x) < \frac{(k-1)!}{x^k} + \frac{k!}{x^{k+1}} \tag{3.4}
\]
holds on \( (0, \infty) \).

Proof. In [25, Theorem 2.1] and [34, Lemma 1.3], the function \( \psi(x) - \ln x + \frac{\alpha}{x} \) was proved to be completely monotonic on \( (0, \infty) \), i.e.,
\[
(-1)^i \left[ \psi(x) - \ln x + \frac{\alpha}{x} \right]^{(i)} \geq 0 \tag{3.5}
\]
for \( i \geq 0 \), if and only if \( \alpha \geq 1 \), so is its negative, i.e., the inequality (3.5) is reversed, if and only if \( \alpha \leq \frac{1}{2} \). In [15] and [28, Theorem 2.1], the function \( \frac{e^x \Gamma(x)}{x^\alpha} \) was proved to be logarithmically completely monotonic on \( (0, \infty) \), i.e.,
\[
(-1)^k \left[ \ln \frac{e^x \Gamma(x)}{x^\alpha} \right]^{(k)} \geq 0 \tag{3.6}
\]
for \( k \in \mathbb{N} \), if and only if \( \alpha \geq 1 \), so is its reciprocal, i.e., the inequality (3.6) is reversed, if and only if \( \alpha \leq \frac{1}{2} \). Considering the fact [39, p. 82] that a completely monotonic function which is non-identically zero cannot vanish at any point on \( (0, \infty) \) and rearranging either (3.5) for \( i \in \mathbb{N} \) or (3.6) for \( k \geq 2 \) leads to the double inequality (3.4) immediately. \( \square \)
4. Proofs of theorems and corollaries

Proof of Theorem 1. It is a standard argument to obtain that the function $\delta(t)$ is strictly decreasing from $(0, \infty)$ onto $(0, \frac{1}{2})$.

Let $g_{\alpha, \beta}(x) = x^\alpha |\psi^{(i)}(x + \beta)|$ on $(0, \infty)$. Direct calculation and rearrangement yields
\[
g'_{\alpha, \beta}(x) = \frac{\alpha x |\psi^{(i)}(x + \beta)| - x |\psi^{(i+1)}(x + \beta)|}{x^{\alpha-1}} = (-1)^{i+1} [\alpha \psi^{(i)}(x + \beta) + x \psi^{(i+1)}(x + \beta)].
\]
(4.1)

Making use of (3.4) in (4.1) gives
\[
\lim_{x \to \infty} \frac{g'_{\alpha, \beta}(x)}{x^{\alpha-1}} = 0
\]
(4.2)

for $i \in \mathbb{N}$, $\alpha \in \mathbb{R}$ and $\beta \geq 0$. In virtue of formulas (3.3), (3.2) and (3.1) in sequence, straightforward computation reveals
\[
g'_{\alpha, \beta}(x) - g'_{\alpha, \beta}(x + 1) = (-1)^{i+1} \{ \alpha \psi^{(i)}(x + \beta) - \psi^{(i)}(x + \beta + 1) \\
+ x [\psi^{(i+1)}(x + \beta) - \psi^{(i+1)}(x + \beta + 1)] - \psi^{(i+1)}(x + \beta + 1) \}
= \frac{\alpha x}{(x + \beta)^{i+1}} - \frac{(i + 1)! x}{(x + \beta)^{i+2}} - (i + 1)! \frac{(x + \beta + 1)}{(x + \beta)^{i+2}} + (i + 1)! \frac{(x + \beta + 1)}{(x + \beta)^{i+2}}
= (i + 1)! \frac{(x + \beta + 1)}{(x + \beta)^{i+2}} + (i + 1)! \frac{(\beta - 1)}{(x + \beta)^{i+2}}
= \int_0^\infty \left[ \frac{t}{e - t} + (\beta - 1) + (\alpha - i) - 1 \right] e^{-(x + \beta)t} dt
\]

(4.3)

For $\beta = 0$, easy differentiation shows that $h_{\alpha, 0}(t) = -\delta(t) < 0$, and so the function $h_{\alpha, 0}(t)$ is strictly decreasing from $(0, \infty)$ onto $(\alpha - i - 1, \alpha - i)$. Thus, if $\alpha \geq i + 1$, the functions $h_{\alpha, 0}(t)$ and
\[
g'_{\alpha, 0}(x) - g'_{\alpha, 0}(x + 1)
\]
are positive on $(0, \infty)$. Combining this with (4.2) and considering Lemma 1, it is deduced that the functions $g'_{\alpha, 0}(x)$ and $g'_{\alpha, 0}(x)$ are positive on $(0, \infty)$. Hence, the function $g_{\alpha, 0}(x)$ is strictly increasing on $(0, \infty)$ for $\alpha \geq i + 1$. Similarly, for $\alpha \leq i$, the function $g_{\alpha, 0}(x)$ is strictly decreasing on $(0, \infty)$.

For $\beta > 0$, it is easy to see that $h'_{\alpha, 0}(t) = -\delta(t) + \beta$, and $h'_{\alpha, \beta}(t)$ is strictly increasing from $(0, \infty)$ onto $(\beta - \frac{1}{2}, \beta)$. Consequently, if $\beta \geq \frac{1}{2}$, the function $h'_{\alpha, \beta}(t)$ is positive and $h_{\alpha, \beta}(t)$ is strictly increasing from $(0, \infty)$ onto $(\alpha - i, \infty)$. Accordingly, if $\alpha \geq i$ and $\beta \geq \frac{1}{2}$, the function $h_{\alpha, \beta}(t)$ is positive on $(0, \infty)$, that is,
\[
g'_{\alpha, \beta}(x) - g'_{\alpha, \beta}(x + 1) > 0
\]
(4.4)
on $(0, \infty)$. Combining this with Lemma 1 results in the positivity of $g'_{\alpha, \beta}(x)$ on $(0, \infty)$. Therefore, for $\alpha \geq i$ and $\beta \geq \frac{1}{2}$, the function $g_{\alpha, \beta}(x)$ is strictly increasing on $(0, \infty)$.

For $0 < \beta < \frac{1}{2}$, since $h'_{\alpha, \beta}(t)$ is strictly increasing from $(0, \infty)$ onto $(\beta - \frac{1}{2}, \beta)$, the function $h_{\alpha, \beta}(t)$ attains its unique minimum at some point $t_0 \in (0, \infty)$ with
\( \delta(t_0) = \beta \). As a result, the unique minimum of \( h_{i,\alpha,\beta}(t) \) equals
\[
\frac{\delta^{-1}(\beta) e^{\delta^{-1}(\beta)}}{e^{\delta^{-1}(\beta)} - 1} + (\beta - 1)\delta^{-1}(\beta) + \alpha - i - 1,
\]
where \( \delta^{-1} \) is the inverse function of \( \delta \) and is strictly decreasing from \((0, \frac{1}{2})\) onto \((0, \infty)\). Consequently, when the inequality (1.5) holds for \( 0 < \beta < \frac{1}{2} \), the function \( h_{i,\alpha,\beta}(t) \) is positive on \((0, \infty)\), which means that the inequality (4.4) holds true. Accordingly, making use of the limit (4.2) and Lemma 1 again yields that the function \( g_{i,\alpha,\beta}(x) \) is strictly increasing on \((0, \infty)\) if \( 0 < \beta < \frac{1}{2} \) and the inequality (1.5) is valid. The sufficiency is proved.

If \( g_{i,\alpha,0}(x) \) is strictly decreasing on \((0, \infty)\), then
\[
x^{i+1-\alpha} g'_{i,\alpha,0}(x) = \alpha x^i |\psi^{(i)}(x)| - x^{i+1} |\psi^{(i+1)}(x)| < 0. \tag{4.5}
\]
Applying (3.4) in (4.5) and letting \( x \to \infty \) lead to
\[
0 \geq \lim_{x \to \infty} x^{i+1-\alpha} g'_{i,\alpha,0}(x)
\geq \alpha \lim_{x \to \infty} x^i \left[ (i-1)! + \frac{i!}{2x^{i+1}} \right] - \lim_{x \to \infty} x^{i+1} \left[ \frac{i!}{x^{i+1}} + \frac{(i+1)!}{x^{i+2}} \right]
= (i-1)!(\alpha - i),
\]
which means \( \alpha \leq i \).

If \( g_{i,\alpha,0}(x) \) is strictly increasing on \((0, \infty)\), then
\[
x^{i+2-\alpha} g'_{i,\alpha,0}(x) = \alpha x^{i+1} |\psi^{(i)}(x)| - x^{i+2} |\psi^{(i+1)}(x)| > 0. \tag{4.6}
\]
Employing (3.3) and (3.4) in (4.6) and taking \( x \to \infty \) results in
\[
0 \leq \lim_{x \to 0^+} x^{i+2-\alpha} g'_{i,\alpha,0}(x)
= \lim_{x \to 0^+} \left\{ \alpha x^{i+1} |\psi^{(i)}(x)| - x^{i+2} \left[ |\psi^{(i+1)}(x+1)| + \frac{(i+1)!}{x+2} \right] \right\}
= \alpha \lim_{x \to 0^+} x^{i+1} |\psi^{(i)}(x)| - (i+1)!
\leq \alpha \lim_{x \to 0^+} x^{i+1} \left[ \frac{(i-1)!}{x^i} + \frac{i!}{x^{i+1}} \right] - (i+1)!
\leq \lim_{x \to 0^+} x^{i+2} \left[ \frac{i!}{(x+1)^{i+1}} + \frac{(i+1)!}{2(x+1)^{i+2}} \right]
= i!(\alpha - i - 1),
\]
thus, the necessary condition \( \alpha \geq i + 1 \) follows.

If the function \( g_{i,\alpha,\beta}(x) \) is strictly increasing on \((0, \infty)\) for \( \beta > 0 \), then
\[
x^{i+1-\alpha} g'_{i,\alpha,\beta}(x) = \alpha x^i |\psi^{(i)}(x + \beta)| - x^{i+1} |\psi^{(i+1)}(x + \beta)| > 0. \tag{4.7}
\]
Utilizing (3.4) in (4.7) and taking limit gives
\[
0 \leq \lim_{x \to \infty} x^{i+1-\alpha} g'_{i,\alpha,\beta}(x)
\leq \alpha \lim_{x \to \infty} x^i \left[ \frac{(i-1)!}{(x+\beta)^i} + \frac{i!}{(x+\beta)^{i+1}} \right] - \lim_{x \to \infty} x^{i+1} \left[ \frac{i!}{(x+\beta)^{i+1}} + \frac{(i+1)!}{2(x+\beta)^{i+2}} \right]
= (i-1)!(\alpha - i),
\]
which is equivalent to \( \alpha \geq i \). The proof of Theorem 1 is thus completed. \( \Box \)

**Remark 9.** The first two conclusions in Theorem 1 were ever proved by virtue of the convolution theorem for Laplace transforms in [2, Lemma 1] and [3, Lemma 2.2], so we supply a new and unified proof for them here.
notations, the functions
\( h \),
is completely monotonic on \((0, \infty)\) and that Bernoulli numbers
\( B_k \) are well-known \([1, 51, 52]\) that Bernoulli polynomials
\( B_k \) may be defined by
\[
B_k(x) = \frac{t e^{x t}}{e^t - 1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!}
\]  
(4.8)
and that Bernoulli numbers \( B_k \) and Bernoulli polynomials \( B_k(x) \) are connected by
\( B_k(1) = (-1)^k B_k(0) = (-1)^k B_k \) and \( B_{2k+1}(0) = B_{2k+1} = 0 \) for \( k \geq 1 \). Using these notations, the functions \( h_{i,\alpha,\beta}(t) \) and \( h'_{i,\alpha,\beta}(t) \) may be rewritten as
\[
h_{i,\alpha,\beta}(t) = \frac{t e^t}{e^t - 1} + (\beta - 1)t + \alpha - i - 1
\]
\[
= \alpha - i + \left( \beta - \frac{1}{2} \right) t + \sum_{k=2}^{\infty} B_k(1) \frac{t^k}{k!}
\]
\[
= \alpha - i + \left( \beta - \frac{1}{2} \right) t + \sum_{k=2}^{\infty} (-1)^k B_k \frac{t^k}{k!}
\]
\[
= \alpha - i + \left( \beta - \frac{1}{2} \right) t + \sum_{k=1}^{\infty} (-1)^{k+1} B_{k+1} \frac{t^k}{(k+1)!}
\]
\[
= \alpha - i + \left( \beta - \frac{1}{2} \right) t + \sum_{k=0}^{\infty} B_{2k+2} \frac{t^{2k+2}}{(2k+2)!}
\]
and
\[
h'_{i,\alpha,\beta}(t) = \beta - \frac{1}{2} + \sum_{k=1}^{\infty} B_{2k+} \frac{t^{2k-1}}{(2k-1)!}
\]
The proof of Theorem 1 shows that
1. \( h'_{i,\alpha,\beta}(t) < 0 \) on \((0, \infty)\);
2. \( h_{i,\alpha,\beta}(t) > 0 \) on \((0, \infty)\) if \( \alpha \geq i + 1 \);
3. \( h_{i,\alpha,\beta}(t) < 0 \) on \((0, \infty)\) if \( 0 < \alpha \leq i \);
4. \( h'_{i,\alpha,\beta}(t) > 0 \) on \((0, \infty)\) if \( \beta \geq \frac{i}{2} \);
5. \( h_{i,\alpha,\beta}(t) > 0 \) on \((0, \infty)\) if \( \alpha \geq i + 1 \) and \( \beta \geq \frac{1}{2} \);
6. \( h_{i,\alpha,\beta}(t) > 0 \) on \((0, \infty)\) if \( 0 < \alpha < \frac{1}{2} \) and inequality (1.5) holds true.
Basing on these and by standard argument, Corollary 1 is thus proved.

Proof of Theorem 2. If \( h_{i,\alpha,\beta}(t) \geq 0 \) on \((0, \infty)\), then the function
\[
\pm \int_{0}^{\infty} h_{i,\alpha,\beta}(t) t e^{-(x+\beta)t} \, dt
\]
is completely monotonic on \((-\beta, \infty)\), and so, by virtue of (4.3), it is derived that
\[
\pm \left[ g'_{i,\alpha,\beta} x - g'_{i,\alpha,\beta} (x+1) \frac{1}{(x+1)^{\alpha-1}} \right]
\]
is completely monotonic on \((0, \infty)\), that is,
\[
(-1)^j \left[ g'_{i,\alpha,\beta} x - g'_{i,\alpha,\beta} (x+1) \frac{1}{(x+1)^{\alpha-1}} \right]^{(j)}
\]
\[
= (-1)^j \left[ g'_{i,\alpha,\beta} x \right]^{(j)} - (-1)^j \left[ g'_{i,\alpha,\beta} (x+1) \frac{1}{(x+1)^{\alpha-1}} \right]^{(j)} \geq 0
\]
on \((0, \infty)\) for \( j \geq 0 \). Moreover, formulas (3.4) and (4.1) imply
\[
\lim_{x \to -\infty} \left[ g'_{i,\alpha,\beta} x \right]^{(j)} = \lim_{x \to -\infty} (-1)^j \left[ g'_{i,\alpha,\beta} x \right]^{(j)} = 0.
\]
Combining (4.9) and (4.10) with Lemma 1 concludes that
\[ (-1)^j \left[ \frac{g'_{i,\alpha,\beta}(x)}{x^{\alpha-1}} \right]^{(j)} \geq 0, \]
that is, the function
\[ \pm \frac{g'_{i,\alpha,\beta}(x)}{x^{\alpha-1}} = \pm [\alpha |\psi^{(i)}(x+\beta)| - x |\psi^{(i+1)}(x+\beta)|] \]
is completely monotonic on \((0, \infty)\), if \(h_{i,\alpha,\beta}(t) \geq 0\) on \((0, \infty)\). In the proof of Theorem 1, we have demonstrated that \(h_{i,\alpha,\beta}(t)\) is positive on \((0, \infty)\) if either \(\beta = 0\) and \(\alpha \geq i + 1\), or \(\beta \geq \frac{1}{2}\) and \(\alpha \geq i\), or \(0 < \beta < \frac{1}{2}\) and the inequality (1.5) is satisfied, and that \(h_{i,\alpha,\beta}(t)\) is negative on \((0, \infty)\) if \(\beta = 0\) and \(\alpha \leq i\). As a result, the sufficient conditions for the function \(\alpha |\psi^{(i)}(x+\beta)| - x |\psi^{(i+1)}(x+\beta)|\) to be completely monotonic on \((0, \infty)\) follow.

The derivation of necessary conditions is same as in Theorem 1. The proof of Theorem 2 is complete.

**Proof of Corollary 2.** It follows easily from Theorem 2 and the facts that
\[ \pm \left[ \frac{\alpha}{x} |\psi^{(i)}(x+\beta)| - |\psi^{(i+1)}(x+\beta)| \right] = \pm \frac{1}{x} \left\{ \alpha |\psi^{(i)}(x+\beta)| - x |\psi^{(i+1)}(x+\beta)| \right\}, \]
that the function \(\frac{1}{x}\) is completely monotonic on \((0, \infty)\), and that the product of any finite completely monotonic functions is also completely monotonic on the intersection of their domains. \(\square\)

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