Tight bounds from multiple-observable entropic uncertainty relations

Alberto Riccardi and Giovanni Chesi
INFN Sezione di Pavia, Via Agostino Bassi 6, I-27100 Pavia, Italy

Chiara Macchiavello and Lorenzo Maccone
Dipartimento di Fisica, Università degli Studi di Pavia, Via Agostino Bassi 6, I-27100, Pavia, Italy
INFN Sezione di Pavia, Via Agostino Bassi 6, I-27100, Pavia, Italy

We investigate the additivity properties for both bipartite and multipartite systems by using entropic uncertainty relations (EUR) defined in terms of the joint Shannon entropy of probabilities of local measurement outcomes. In particular, we introduce state-independent and state-dependent entropic inequalities. Interestingly, the violation of these inequalities is strictly connected with the presence of quantum correlations. We show that the additivity of EUR holds only for EUR that involve two observables, while this is not the case for inequalities that consider more than two observables or the addition of the von Neumann entropy of a subsystem. We apply them to bipartite systems and to several classes of states of a three-qubit system.

Entropic uncertainty relations (EUR) are inequalities that express preparation uncertainty relations (UR) as sums of Shannon entropies of probability distributions of measurement outcomes. First introduced for continuous variables systems [14], they were then generalized for pair of observables with discrete spectra [59] (see [10] for a review of the topic). Conversely to the most known URs defined for product of variances [11] [12], which are usually state-dependent, EURs provide lower bounds, which quantify the knowledge trade-off between the different observables, that are state-independent. Variance-based URs for the sum of variances [13] in some cases also provide state-independent bounds [14] [18], but EURs, due to their simple structure, allow to consider URs for more than two observables in a natural way by simply adding more entropies, a task that is not straightforward for URs based on the product of variances. However, tight bounds for multiple-observable EURs are known only for small dimensions and for restricted sets of observables, typically for complementary observables [19] [25], namely the ones that have mutually unbiased bases as eigenbases, and for angular momentum observables [23] [26].

Besides their importance from a fundamental point of view as preparation uncertainty relations, EURs have recently been used to investigate the nature of correlations in composite quantum systems, providing criteria that enable to detect the presence of different types of quantum correlations, both for bipartite and multipartite systems. Entanglement criteria based on EURs were defined in [27] [31], while steering inequalities in [33] [38]. Almost all of these criteria are based on EURs for conditional Shannon entropies, where one tries to exploit, in the presence of correlations, side information about some subsystems to reduce global uncertainties, while only partial results for joint Shannon entropies are known [27] [32]. Moreover, it has been recently proven in [59] that if one considers EURs defined for the joint Shannon entropy and only pairs of observables, then it is not possible to distinguish between separable and entangled states since in this case additivity holds, namely, the lower bound on the sum of the joint Shannon entropies $H(A_1, B_1)$ and $H(A_2, B_2)$ is given by the sum of the lower bounds on the sum of the entropies $H(X_1)$ and $H(X_2)$, with $X = A, B$.

In this paper we show that if we consider EURs for more than two observables the additivity of EURs does no longer hold. This result implies that it is possible to define criteria that certify the presence of entanglement by using the joint Shannon entropy for both the bipartite and the multipartite case. We investigate which criteria can be derived from EURs based on the joint Shannon entropy and their performance. We then provide some examples of entangled states that violate our criteria. This paper is organized as follows: in Section I we briefly review some concepts of single system EURs, in particular we discuss the case of multiple observables. In Section II we establish the entanglement criteria for bipartite systems and in Section III we address the problem in the multipartite scenario. In the Appendix, we consider some examples of entangled states that are detected by these criteria, in particular we focus on the multi-qubit case.

I. ENTROPIC UNCERTAINTY RELATIONS: A BRIEF REVIEW

The paradigmatic example of EUR for observables with a discrete non-degenerated spectrum is due to Maassen and Uffink [7], and it states that for any two observables $A_1$ and $A_2$, defined on a $d$-dimensional system, the following inequality holds:

$$H(A_1) + H(A_2) \geq -2 \log_2 c = q_{MU},$$

(1)

where $H(A_1)$ and $H(A_2)$ are the Shannon entropies of the measurement outcomes of two observables $A_1 = \sum_j a_j^1 |a_j^1\rangle \langle a_j^1|$ and $A_2 = \sum_j a_j^2 |a_j^2\rangle \langle a_j^2|$, namely $H(A_1) = -\sum_j p(a_j^1) \log p(a_j^1)$ being $p(a_j^1)$ the probability of obtaining the outcome $a_j^1$ of $A_1$, and $c = \max_{j,k} \|a_j^1|a_k^2\|_1$ is the maximum overlap between their eigenstates. The bound in Eq. (1) is known to be tight if $A_1$ and $A_2$ are complementary observables [10].
remind that two observables $A_1$ and $A_2$ are said to be complementary if their eigenvalues are mutually unbiased, namely if $\langle a_i^1 | a_j^2 \rangle = \frac{1}{\sqrt{d}}$ for all eigenstates, where $d$ is the dimension of the system (see [40] for a review on MUBs). In this case $q_{MU} = \log_2 d$, hence we have:

$$H(A_1) + H(A_2) \geq \log_2 d.$$  

(2)

The above relation has a clear interpretation as UR: let us suppose that $H(A_1) = 0$, which means that the state of the system is an eigenstate of $A_1$, then the other entropy $H(A_2)$ must be maximal, hence if we have a perfect knowledge of one observable the other must be completely undetermined. For arbitrary observables stronger bounds, that involve the second largest term in $\langle | \langle a_j | b_k \rangle |^2 \rangle$, were derived in [8, 9].

An interesting feature of EURs is that they can be generalized to an arbitrary number of observables in a straightforward way from Maassen and Uffink’s EUR. Indeed, let us consider for simplicity the case of three observables $A_1$, $A_2$ and $A_3$, which mutually satisfy the following EURs:

$$H(A_i) + H(A_j) \geq q_{MU}^i,$$  

(3)

where $i, j = 1, 2, 3$ labels the three observables. Then, we have:

$$\sum_{k=1}^{3} H(A_k) = \frac{1}{2} \sum_{k=1}^{3} \sum_{j=k+1}^{3} [H(A_k) + H(A_j)] \geq \frac{1}{2} (q_{MU}^{12} + q_{MU}^{13} + q_{MU}^{23})$$  

(4)

where we have applied (3) to each pair. If we have $L$ observables, the above inequality becomes:

$$\sum_{k=1}^{L} H(A_k) \geq \frac{1}{L(L-1)} \sum_{i<j} q_{MU}^{ij},$$  

(5)

where $t$ takes values in the set $T_2$ of labels of all the possible $L(L-1)/2$ pairs of observables. For example if $L = 4$, then $T_2 = \{12, 13, 14, 23, 24, 34\}$. However, EURs in the form (5) are usually not tight, i.e. in most cases the lower bounds can be improved. Tight bounds are known only for small dimensions and for complementary or angular momentum observables. For the sake of simplicity, henceforth all explicit examples will be discussed only for complementary observables. The maximal number of complementary observables for any given dimension is an open problem [10], which finds its roots in the classification of all complex Hadamard matrices. However, if $d$ is a power of a prime then $d+1$ complementary observables always exist. For any $d$, even if it is not a power of a prime, it is possible to find at least three complementary observables [10]. The method that we will define in the next Section can be therefore used in any dimension. The qubit case, where at most three complementary observables exist, which are in correspondence with the three Pauli matrices, was studied in [20], while for systems with dimension three to five tight bounds for an arbitrary number of complementary observables were derived in [25]. For example in the qubit case, where the three observables $A_1, A_2$ and $A_3$ correspond to the three Pauli matrices $\sigma_x, \sigma_y$ and $\sigma_z$, we have:

$$H(A_1) + H(A_2) + H(A_3) \geq 2,$$  

(6)

and the minimum is achieved by the eigenstates of one of the $A_i$. In the case of a qutrit, where four complementary observables exist, we instead have:

$$H(A_1) + H(A_2) + H(A_3) + H(A_4) \geq 3,$$  

(7)

$$H(A_1) + H(A_2) + H(A_3) + H(A_4) \geq 4.$$  

(8)

The minimum values are achieved by:

$$\frac{e^{i\sigma_3/2} |0\rangle + |1\rangle}{\sqrt{2}}, \frac{e^{i\sigma_z/2} |0\rangle + |2\rangle}{\sqrt{2}}, \frac{e^{i\sigma_x/2} |1\rangle + |2\rangle}{\sqrt{2}},$$  

(9)

where $\varphi = \frac{\pi}{3}, \frac{2\pi}{3}$. Another result, for $L < d+1$, can be found in [22], where it has been shown that if the Hilbert space dimension is a square, that is $d = r^2$, then for $L < r+1$ the inequality (5) is tight, namely:

$$\sum_{i=1}^{L} H(A_i) \geq \frac{L}{2} \log_2 d = q_{BW}.$$  

(10)

In order to have a compact expression to use, we express the EUR for $L$ observables in the following way:

$$\sum_{i=1}^{L} H(A_i) \geq f(A, L),$$  

(11)

where $f(A, L)$ indicates the lower bound, which can be tight or not, and it depends on the set $A = \{A_1, \ldots, A_L\}$ of $L$ observables considered. Here we also point out in the lower bound how many observables are involved. When we refer explicitly to tight bounds we will use the additional label $T$, namely $f^T(A, L)$ expresses a lower bound that we know is achievable via some states.

II. BIPARTITE ENTANGLEMENT CRITERIA

In this Section, we discuss bipartite entanglement criteria based on EURs, defined in terms of joint Shannon entropies. The framework consists in two parties, say Alice and Bob, who share a quantum state $\rho_{AB}$, and they want to have a criterion defined in terms of the joint Shannon entropies $H(A_i, B_j)$ which certifies the presence of entanglement. As a reminder, in a bipartite scenario we say that the state $\rho_{AB}$ is entangled iff it cannot be expressed as a
convex combination of product states, which are represented by separable states, namely iff:

$$\rho_{AB} \neq \sum_i p_i \rho_A^i \otimes \rho_B^i,$$

where \( p_i \geq 0, \sum_i p_i = 1 \), and \( \rho_A^i, \rho_B^i \) are Alice and Bob’s states respectively.

**Proposition 1.** If the state \( \rho_{AB} \) is separable, then the following EUR must hold:

$$\sum_{i=1}^L H(A_i, B_i) \geq f(A, L) + f(B, L),$$

where \( f(A, L) \) and \( f(B, L) \) are the lower bounds of the single system EUR, namely

$$\sum_{i=1}^L H(A_i) \geq f(A, L),$$

$$\sum_{i=1}^L H(B_i) \geq f(B, L).$$

**Proof.** Let us focus first on \( H(A_i, B_i) \) which, for the properties of the Shannon entropy, can be expressed as:

$$H(A_i, B_i) = H(A_i) + H(B_i | A_i).$$

We want to bound \( H(B_i | A_i) \) which is computed over the state \( \rho_{AB} = \sum_j p_j \rho_A^j \otimes \rho_B^j \). Through the convexity of the relative entropy, one can prove that the conditional entropy \( H(B | A) \) is concave in \( \rho_{AB} \). Then we have:

$$H(B_i | A_i) \sum_j p_j \rho_A^j \otimes \rho_B^j \geq \sum_j p_j H(B_i | A_i) \rho_A^j \otimes \rho_B^j,$$

thus, since the right-hand side of the above Equation is evaluated on a product state, we have:

$$H(B_i | A_i) \sum_j p_j \rho_A^j \otimes \rho_B^j \geq \sum_j p_j H(B_i) \rho_B^j.$$

Therefore, considering \( \sum_{i=1}^L H(A_i, B_i) \), we derive the following:

$$\sum_{i=1}^L H(A_i, B_i) \geq \sum_i H(A_i) + \sum_j p_j \sum_i H(B_i) \rho_B^j.$$

Then we can observe that \( \sum_{i=1}^L H(A_i) \geq f(A, L) \) and \( \sum_i H(B_i) \rho_B^j \geq f(B, L) \), the latter holding due to EUR being state-independent bounds. Therefore we have:

$$\sum_{i=1}^L H(A_i, B_i) \geq f(A, L) + \sum_j p_j f(B, L)$$

$$= f(A, L) + f(B, L),$$

since \( \sum_j p_j = 1 \).\qed

Any state that violates the inequality \( \sum_{i=1}^L H(A_i, B_i) \geq f(A, L) + f(B, L) \) must be therefore entangled. If we consider the observables \( A_i \otimes B_i \) as ones of the bipartite system then they must satisfy an EUR for all states, even the entangled ones, which can be expressed as:

$$\sum_{i=1}^L H(A_i, B_i) \geq f(AB, L),$$

where the lower bound now depends on the observables \( A_i \otimes B_i \), while \( f(A, L) \) and \( f(B, L) \) depend respectively on \( A_i \) and \( B_i \) individually. In order to have a proper entanglement criterion then we should have that

$$f(AB, L) < f(A, L) + f(B, L),$$

which means that the set of entangled states that violate the inequality is not empty. In the case \( L = 2 \), this is not sufficient to have a proper entanglement criterion. Indeed, as it was shown in [39], for \( L = 2 \), we have \( f(AB, 2) = f(A, 2) + f(B, 2) \) for any observables, which expresses the additivity of EURs for pairs of observables. A counterexample of this additivity property for \( L > 3 \) is provided by the complete set of complementary observables for two qubits, indeed we have:

$$H(A_1, B_1) + H(A_2, B_2) + H(A_3, B_3) \geq 3,$$

and the minimum is attained by the Bell states while \( f(A, 3) + f(B, 3) = 4 \), which provides the threshold that enables entanglement detection in the case of two qubits. Let us now clarify the difference of this result with respect to those defined in terms of EURs based on conditional entropies, in particular to entropic inequalities. Indeed, if one looks at the proof of Proposition 1, it could be claimed that there is no difference at all since we used the fact that \( \sum_i H(B_i | A_i) \geq f(B, L) \), which is a steering inequality, namely a violation of it witnesses the presence of quantum steering from Alice to Bob. However, the difference is due to the symmetric behavior of the joint entropy, which contrasts with the asymmetry of quantum steering. To be more formal, the joint Shannon entropy \( H(A_i, B_i) \) can be rewritten in two forms:

$$H(A_i, B_i) = H(A_i) + H(B_i | A_i)$$

$$= H(B_i) + H(A_i | B_i),$$

then:

$$\sum_i H(A_i, B_i) = \sum_i [H(A_i) + H(B_i | A_i)],$$

and

$$\sum_i H(A_i, B_i) = \sum_i [H(B_i) + H(A_i | B_i)].$$

If now the state is not steerable from Alice to Bob, we have \( \sum_i H(B_i | A_i) \geq f(B, L) \), which implies
\[
\sum_{i=1}^{L} H(A_i, B_i) \geq f(A, L) + f(B, L),
\]
Note that in this case if we look at \( \sum_i H(A_i|B_i) \) no bound can be derived, apart from the trivial bound \( \sum_i H(A_i|B_i) \geq 0 \), since there are no assumptions on the conditioning from Bob to Alice. Conversely, if the state is not steerable from Bob to Alice, i.e. we exchange the roles, we have \( \sum_i H(B_i|A_i) \geq 0 \) and \( \sum_i H(A_i|B_i) \geq f(A, L) \), which implies again \( \sum_{i=1}^{L} H(A_i, B_i) \geq f(B, L) \). Therefore, if we just look at the inequality in Eq. \( [13] \), we cannot distinguish between entanglement and the two possible forms of quantum steering, but, since the presence of steering, for bipartite systems, implies entanglement, it is more natural to think about Eq. \( [13] \) as an entanglement criterion, while if we want to investigate steering properties of the state we should look at the violation of the criteria \( \sum_i H(B_i|A_i) \geq f(B, L) \) and \( \sum_i H(A_i|B_i) \geq f(A, L) \). In other words, Proposition 1 only detects two-way steerable states.

State-dependent bounds

A stronger entanglement criteria can be derived by considering the state-dependent EUR:

\[
\sum_{i=1}^{L} H(A_i) \geq f(A, L) + S(\rho_A),
\]

or the corresponding version for Bob’s system \( \sum_{i=1}^{L} H(B_i) \geq f(B, L) + S(\rho_B) \), where \( S(\rho_A) \) and \( S(\rho_B) \) are the Von Neumann entropies of the marginal states of \( \rho_{AB} \).

**Proposition 2.** If the state \( \rho_{AB} \) is separable, then the following EUR must hold:

\[
\sum_{i=1}^{L} H(A_i, B_i) \geq f(A, L) + f(B, L) + \max(S(\rho_A), S(\rho_B)).
\]

**Proof.** The proof is the same of Proposition 1 where we use \( [27] \) in \( [19] \), instead of the state-dependent bound \( [11] \). The same holds if we use the analogous version for Bob. Then, aiming at the strongest criterion, we can take the maximum between the two Von Neumann entropies.

The edge in using these criteria, instead of the one defined in Proposition 1, is such that even for \( L = 2 \) the bound is meaningful. Indeed a necessary condition to the definition of a proper criterion is that:

\[
f^T(AB, 2) < f(A, 2) + f(B, 2) + S(\rho_X),
\]

where \( X = A, B \) with the additional requirement that the bound on the left is tight, i.e. there exist states the violate the criterion. As an example, we can consider a two-qubit system, the observables \( X_{AB} = \sigma_A^X \otimes \sigma_B^X \) and \( Z_{AB} = \sigma_A^Z \otimes \sigma_B^Z \), for which all states of the whole system satisfy \( H(X_{AB}) + H(Z_{AB}) \geq 2 \), and the Bell state \( \rho_{AB} = |\phi^+\rangle \langle \phi^+| \). In this scenario, the entanglement criterion reads:

\[
H(X_{AB}) + H(Z_{AB}) \geq 3,
\]

which is actually violated since the left-hand side is equal to 2. Note that in general the condition \( f^T(AB, L) < f(A, L) + f(B, L) + S(\rho_X) \) is necessary to the usefulness of the corresponding entanglement criteria.

III. MULTIPARTITE ENTANGLEMENT CRITERIA

We now extend the results of Propositions 1 and 2 for multipartite systems, where the notion of entanglement has to be briefly discussed since it has a much richer structure than the bipartite case. Indeed, we can distinguish among different levels of separability. First, we say that a state \( \rho_{V_1,...,V_n} \) of \( n \) systems \( V_1,...,V_n \) is fully separable if it can be written in the form:

\[
\rho_{V_1,...,V_n} = \sum_i p_i \rho_{V_i}^{V_1} \otimes ... \otimes \rho_{V_i}^{V_n},
\]

with \( \sum_i p_i = 1 \), namely it is a convex combination of product states of the single subsystems. As a case of study, we will always refer to tripartite systems, where there are three parties, say Alice, Bob and Charlie. In this case a fully separable state can be written as:

\[
\rho_{ABC}^{FS} = \sum_i p_i \rho_A^{V_i} \otimes \rho_B^{V_i} \otimes \rho_C^{V_i}.
\]

Any state that does not admit such a decomposition contains entanglement among some subsystems. However, we can define different levels of separability. Hence, we say that the state \( \rho_{V_1,...,V_n} \) of \( n \) systems is separable with respect to a given partition \( \{I_1,...,I_k\} \), where \( I_i \) are disjoint subsets of the indices \( I = \{1,...,n\} \), such that \( \cup_{j=1}^k I_j = I \), if it can be expressed as:

\[
\rho_{V_1,...,V_n}^{1,...,k} = \sum_i p_i \rho_A^{I_1} \otimes ... \otimes \rho_B^{I_k},
\]

where \( \rho_A^{I_\alpha} \) is the state of the system \( \{V_i : i \in I_\alpha\} , \alpha = 1,...,k \). Namely, some systems share entangled states, while the state is separable with respect to the partition considered. For tripartite system we have three different possible bipartitions: \( 1|23, 2|13 \) and \( 3|12 \). As an example, if the state \( \rho_{ABC} \) can be expressed as:

\[
\rho_{ABC}^{1|23} = \sum_i p_i \rho_A \otimes \rho_{BC},
\]

then there is no entanglement between Alice and Bob+Charlie, while these last two share entanglement.
If a state does not admit such a decomposition, it is tangled with respect to this partition. Finally, we say that $\rho_{V_1,\ldots,V_n}$ of $n$ systems can have at most $m$-system entanglement if it is a mixture of all states such that each of them is separable with respect to some partition $\{I_1,\ldots,I_k\}$, where all sets of indices $I_k$ have cardinality $N \leq m$. For tripartite systems this corresponds to the notion of biseparability, namely the state can have at most 2-system entanglement. A biseparable state can be written as:

$$\rho_{ABC} = \sum_i p_i \rho_A^i \otimes \rho_B^i + \sum_j q_j \rho_B^j \otimes \rho_{AC}^j + \sum_k m_k \rho_C^k \otimes \rho_{AB}^k,$$

with $\sum_i p_i + \sum_j q_j + \sum_k m_k = 1$. For $n = 3$ a state is then said to be genuine tripartite entangled if it is 3-system entangled, namely if it does not admit such a decomposition.

**Full separability**

Let us clarify the scenario: In each system $V_i$ we consider a set of $L$ observables $V_i^1,\ldots,V_i^L$ that we indicate as $V_i$. The single-system EUR is expressed as:

$$\sum_{j=1}^L H(V_i^j) \geq f(V_i, L).$$

(36)

We are interested in defining criteria in terms of $\sum_{j=1}^L H(V_i^1,\ldots,V_i^n)$. A first result regards the notion of full separability.

**Proposition 3.** If the state $\rho_{V_1,\ldots,V_n}$ is fully separable, then the following EUR must hold:

$$\sum_{j=1}^L H(V_i^j,\ldots,V_n^j) \geq \sum_{i=1}^n f(V_i, L).$$

(37)

Proof. Let us consider the case $n = 3$. For a given $j$ we have:

$$H(V_1^j, V_2^j, V_3^j) = H(V_1^j) + H(V_2^j|V_1^j).$$

(38)

Since the state is separable with respect to the partition $23|1$, due to concavity of the Shannon entropy, we have:

$$H(V_2^j|V_1^j) \geq \sum_i p_i H(V_2^j|V_1^j)_{\rho^i_2 \otimes \rho^i_1}.$$  

(39)

By using the chain rule of the Shannon entropy, the above right-hand side can be rewritten as:

$$\sum_i p_i H(V_2^j|V_1^j)_{\rho^i_2 \otimes \rho^i_1} = \sum_i p_i H(V_2^j)_{\rho^i_2} + \sum_{i,j} f_i H(V_1^j|V_2^j)_{\rho^i_2 \otimes \rho^i_1},$$

(40)

where the last term can be lower bounded by exploiting the separability of the state and the concavity of the Shannon entropy, namely:

$$\sum_i p_i H(V_2^j|V_1^j)_{\rho^i_2 \otimes \rho^i_1} \geq \sum_i p_i H(V_2^j)_{\rho^i_2}.$$  

(41)

By summing over $j$ we arrive at the thesis:

$$\sum_{j=1}^L H(V_1^j, V_2^j, V_3^j) \geq \sum_{i=1}^n f(V_i, L).$$

(42)

since $\sum_j H(V_i^j) \geq f(V_i, L)$, $\sum_i p_i \sum_j H(V_j^i)_{\rho^i_2} \geq f(V_2, L)$ and $\sum_i p_i \sum_j H(V_j^i)_{\rho^i_1} \geq f(V_3, L)$ because of the state-independent EUR. The extension of the proof to $n$ systems is straightforward.

The following proposition follows directly by considering the state-dependent bound:

$$\sum_{j=1}^L H(V_i^j) \geq f(V_i, L) + S(\rho_i).$$

(43)

**Proposition 4.** If the state $\rho_{V_1,\ldots,V_n}$ is fully separable, then the following EUR must hold:

$$\sum_{j=1}^L H(V_i^j,\ldots,V_n^j) \geq \sum_{i=1}^n f(V_i, L) + \max(S(\rho_1),\ldots,S(\rho_n)).$$

(44)

Note that only the von Neumann entropy of one system is present in the above inequality. This is due to the fact that we use only Eq. (43) in the first step of the proof, otherwise we would end with criteria that require the knowledge of the decomposition in Eq. (31).

**Genuine multipartite entanglement**

We now analyze the strongest form of multipartite entanglement in the case of three systems, say Alice, Bob and Charlie. We make the further assumptions that the three systems have the same dimension and in each system the parties perform the same set of measurements, which implies that there is only one bound $F_1(L)$ given by the single-system EURs $\sum_{j=1}^L H(V_j^i) \geq F_1(L)$, with $V = A, B, C$. Similarly, we indicate the bound on a pair of systems as $F_2(L)$, namely $\sum_{j=1}^L H(V_i^j, W_j^i) \geq F_2(L)$, with $V \neq W = A, B, C$. Then, the criterion defined in Proposition 3 for three systems reads as $\sum_{j=1}^L H(A_j, B_j, C_j) \geq 3F_1(L)$, and must be satisfied by all fully separable states.

**Proposition 5.** If $\rho_{ABC}$ is not genuine multipartite entangled, namely it is biseparable, then the following EUR must hold:

$$\sum_{j=1}^L H(A_j, B_j, C_j) \geq \frac{5}{3} F_1(L) + \frac{1}{3} F_2(L).$$

(45)
Proof. Let us assume that \( \rho_{ABC} \) is biseparable, that is:

\[
\rho_{ABC} = \sum_i p_i \rho_A^i \otimes \rho_{BC}^i + \sum_l q_l \rho_B^l \otimes \rho_{AC}^l + \sum_k m_k \rho_C^k \otimes \rho_{AB}^k
\]

with \( \sum_i p_i + \sum_l q_l + \sum_k m_k = 1 \). The joint Shannon entropy \( H(A_j, B_j, C_j) \) can be expressed as:

\[
H(A_j, B_j, C_j) = \frac{1}{3} [H(A_j) + H(B_j, C_j|A_j)] + \frac{1}{3} [H(B_j) + H(A_j, C_j|B_j)] + \frac{1}{3} [H(C_j) + H(A_j, B_j|C_j)].
\]

By using the concavity of Shannon entropy and the fact that the state is biseparable we find the relations

\[
H(B_j, C_j|A_j) \geq \sum_i p_i H(B_j, C_j|A_j)_{\rho_{BC}^i} + \sum_l q_l H(B_j|C_j)_{\rho_B^l} + \sum_k m_k H(C_j|A_j)_{\rho_C^k}
\]

\[
H(A_j, B_j|C_j) \geq \sum_i p_i H(A_j, B_j|C_j)_{\rho_A^i} + \sum_l q_l H(B_j|C_j)_{\rho_B^l} + \sum_k m_k H(A_j|B_j)_{\rho_{AB}^k}
\]

\[
H(A_j, C_j|B_j) \geq \sum_i p_i H(A_j, C_j|B_j)_{\rho_A^i} + \sum_l q_l H(A_j, C_j|B_j)_{\rho_A^l} + \sum_k m_k H(C_j|B_j)_{\rho_C^k}.
\]

Then, by considering the sum over \( j \) of the sum of the above entropies, and using the EURs related to single systems and to pairs of systems, we find:

\[
\sum_j H(B_j, C_j|A_j) + H(A_j, B_j|C_j) + H(A_j, C_j|B_j) \geq 2F_1(L) + F_2(L).
\]

The thesis (45) is now implied by combining the expression above, Eq. (47), and the following EUR:

\[
\sum_j H(A_j) + H(B_j) + H(C_j) \geq 3F_1(L).
\]

Proposition 6. If \( \rho_{ABC} \) is not genuine multipartite entangled, namely it is biseparable, then the following EUR must hold:

\[
\sum_{j=1}^{L} H(A_j, B_j, C_j) \geq \frac{5}{3} F_1(L) + \frac{1}{3} F_2(L) + \frac{1}{3} \sum_{x=A,B,C} S(\rho_X).
\]

The above Proposition follows from the proof of Proposition 5 where we consider the single-system state-dependent EUR.

Note that, in principle, Propositions 5 and 6 can be extended to account for a larger number of systems and observables by iteratively exploiting the bounds provided by EURs related to smaller numbers of parties, as we did to prove Proposition 5.

IV. CONCLUSIONS

In conclusion, we derived and characterized EURs defined in terms of the joint Shannon entropy, whose violation implies the presence of entanglement. In the case of bipartite systems, we found that EUR entanglement criteria for the joint Shannon entropies require at least three different observables, namely \( L > 2 \), or, if one considers only two measurements, the addition of the von Neumann entropy of a subsystem, thus showing that the additivity character of the state-independent EURs retrieved in Ref. [39] holds only for two measurements. We extended our criteria to the case of multipartite systems, which enable us to discriminate between different types of multipartite entanglement. In particular, we established EURs that allow to certify entanglement among an arbitrary number of parties; then, we focused on the case of three systems and derived the EURs that imply the strongest multipartite criteria. In the Appendix, we showed how these criteria perform for both bipartite and multipartite systems, providing several examples of states that are detected by the proposed criteria.

This material is based upon work supported by the U.S. Department of Energy, Office of Science, National Quantum Information Science Research Centers, Superconducting Quantum Materials and Systems Center (SQMS) under contract number DEAC02-07CH11359 and by the EU H2020 QuantERA ERA-NET Cofund in Quantum Technologies project QuICHE.

Appendix A

Here we discuss our criteria for bipartite and multipartite systems. We will mainly focus on pure states and multi-qubit systems. We inspect in detail how many entangled states and which levels of separability can be detected with the different criteria derived from the EURs. We point out that, if one focuses just on the entanglement-detection efficiency, bounds retrieved from EURs based on the joint Shannon entropy are not as good as some existing criteria. On the other hand, note that the experimental verification of our EUR-based criteria may require less measurements. For instance, if we want to detect the entanglement of a multipartite state through the PPT method, we need to perform a tomography of the state, which involves the measurement of \( d^4 \) observables. On the contrary, the evaluation of the
entropies just needs the measurements of the observables involved in the EUR and its number can be fixed at 3 independently of the dimension d.

1. Bipartite systems

Let us start with the simple case of two qubits. In this scenario we will consider complementary observables and the single-system tight EURs in Refs. [20][25]. Hence, in this case the criteria proved in Section II read

\[ H(A_1, B_1) + H(A_2, B_2) < 2 + \max(S(\rho_A), S(\rho_B)), \]

\[ \sum_{i=1}^{3} H(A_i, B_i) < 4, \]

where \( A_1 = Z_1 (B_1 = Z_2), A_2 = X_1 (B_2 = X_2) \) and \( A_3 = Y_1 (B_3 = Y_2) \) being \( Z_1, X_1, Y_1 \) and \( Z_2, X_2, Y_2 \) the usual Pauli matrices for the first (second) qubit. In the case of two qubits we have already shown in Section II that maximally entangled states are detected by the above criteria.

We can then consider the family of entangled two-qubit states given by:

\[ |\psi_\epsilon\rangle = \epsilon |00\rangle + \sqrt{1-\epsilon^2} |11\rangle, \]

where \( \epsilon \in (0,1) \). We first note that for this family we have \( S(\rho_A) = S(\rho_B) = -\epsilon^2 \log_2 \epsilon^2 - (1-\epsilon^2) \log_2 (1-\epsilon^2) \), which is equal to \( H(A_1, B_1) \). Conversely, we have instead \( H(A_2, B_2) = -\frac{1}{2}(1-\epsilon^2) \log_2 (\frac{1}{2})(1-\epsilon^2) - \frac{1}{2}(1+\epsilon^2) \log_2 (\frac{1}{2})(1+\epsilon^2) \), with \( \epsilon = 2\sqrt{1-\epsilon^2} \) and \( H(A_2, B_2) = H(A_3, B_3) \). The family of states in Eq. (A4) is then completely detected by the criteria in Eqs. (A1) and (A3), since \( H(A_2, B_2) < 2 \), while the inequality in Eq. (A2) fails to detect all the states parametrizes by \( \epsilon \) in Eq. (A4), as shown in Fig. 1.

Let us consider now the entangled two-qubit states given by:

\[ |\psi_\lambda\rangle = \sum_{i=0}^{d-1} \lambda_i |ii\rangle, \]

where \( \sum_{i} \lambda_i^2 = 1 \) and \( 0 < \lambda_i < 1 \) \( \forall i \in [0,d] \). As an entanglement criterion we consider:

\[ H(A_1, B_1) + H(A_2, B_2) < 2 \log_2 d + \max(S(\rho_A), S(\rho_B)), \]

where the observables \( A_1 \) and \( B_1 \) are the computational bases, while \( A_2 \) and \( B_2 \) the corresponding Fourier transforms. First, we observe that for these states we have

\[ S(\rho_A) = S(\rho_B) = -\sum_i \lambda_i^2 \log_2 \lambda_i^2 = H(A_1, B_1). \]

Hence, the entanglement condition becomes:

\[ H(A_2, B_2) < 2 \log_2 d. \]

However, for any two-qudit states we have \( H(A_2, B_2) \leq 2 \log_2 d \) and the maximum is achieved by states that give uniform probability distributions for \( A_2 \otimes B_2 \). Being \( A_2 \) and \( B_2 \) the Fourier transform of the computational bases, the family of states in Eq. (A5) cannot give a uniform probability distribution. The maximum value could be attained only by states of the form \( |ii\rangle \), hence by separable states. Thus, our criterion in Eq. (A6) detects all two-qudit entangled states belonging to the family in Eq. (A5).

2. Multiparticle systems

As an example of a multipartite system we focus on the case of three-qubit systems. In this case a straightforward generalization of the Schmidt decomposition is not available. However, the pure states can be parameterized and classified in terms of five real parameters:

\[ |\psi\rangle = \lambda_0 |000\rangle + \lambda_1 e^{i\phi} |100\rangle + \lambda_2 |011\rangle + \lambda_3 |110\rangle + \lambda_4 |111\rangle, \]

where \( \sum_{i=0}^{d} \lambda_i^2 = 1 \). In particular, we are interested in two classes of entangled states: the GHZ states, given by

\[ |GHZ\rangle = \lambda_0 |000\rangle + \lambda_4 |111\rangle, \]

and the W-states, which are

\[ |W\rangle = \lambda_0 |000\rangle + \lambda_2 |101\rangle + \lambda_3 |110\rangle. \]

The three observables considered in each system are the Pauli matrices, i.e. \( A_1 = Z_1, A_2 = X_1 \) and \( A_3 = Y_1 \) and the same for the other subsystems. The criteria for
Figure 2. GHZ states. In the left panel, the continuous line represents $H(A_i, B_i, C_i)$ while the dashed line is the bound according to Eq. (A11). Note that this criterion does not identify any entangled state. In the right panel, the continuous line shows $\sum_{i=1}^{3} H(A_i, B_i, C_i) - \max(S(\rho_A), S(\rho_B), S(\rho_C))$, while the dashed line represents the bound in Eq. (A12), and the lower dotted line the bound in Eq. (A14), since for this class of states we have $\max(S(\rho_A), S(\rho_B), S(\rho_C)) = \frac{1}{3} \sum_{x=A,B,C} S(\rho_x)$. By using state-dependent bounds, we correctly detect all non-separable GHZ states, but we cannot detect which states are genuine multipartite entangled.

Figure 3. Tripartite W states. The contour plot shows in which areas in the plane $\lambda_0 \times \lambda_2$ the sum of the three entropies $\sum_{i=1}^{3} H(A_i, B_i, C_i)$ is below the state-independent bounds in Eqs. (A11) and (A13). The green area represents non-separable states, the blue one states that are not identified by these criteria. No states are identified as genuine multipartite entangled.

detecting the presence of entanglement, namely states that are not fully separable, in this case are

$$\sum_{i=1}^{3} H(A_i, B_i, C_i) < 6$$  \hspace{1cm} (A11)$$

and

$$\sum_{i=1}^{3} H(A_i, B_i, C_i) < 6 + \max(S(\rho_A), S(\rho_B), S(\rho_C)),$$

while the criteria for genuine multipartite entanglement are

$$\sum_{i=1}^{3} H(A_i, B_i, C_i) < \frac{13}{3}$$  \hspace{1cm} (A13)$$

Figure 4. W-state non-separability. The contour plot shows the effectiveness of the state-dependent criterion (A12) on the W states. Indeed, almost all states are correctly detected (green area) as non-separable. Only a small area (blue) close to the origin is not identified.

Figure 5. W-state genuine multipartite entanglement. The contour plot shows the performance of the state-dependent criterion in Eq. (A14) on the W states. A small set of these states is identified as genuine multipartite entangled (red area).
and
\[
\sum_{i=1}^{3} H(A_i, B_i, C_i) < \frac{13}{3} + \frac{1}{3} \sum_{x=A,B,C} S(\rho_X). \quad (A14)
\]

For the class of GHZ states the sum of the three entropies \(\sum_{i=1}^{3} H(A_i, B_i, C_i)\) is plotted as a function of \(\lambda_0\) with respect to the state-independent and dependent bounds. We can see that in this case the state-independent bounds fail to detect even the weakest form of entanglement. Conversely, the state-dependent bounds identify all states as non-separable but none as genuine multipartite entangled.

For the class of the W states the effectiveness of our criteria is shown in Figs. 3 and 4. Since the W states depend on two parameters, here we use contour plots in the plane \(\lambda_0 \times \lambda_2\) showing which subsets of W states are detected as non-fully separable or genuine multipartite entangled. As we can see, the state-independent bounds (Fig. 3) detect the non-separable character for a large subset of W states. Conversely, no state is identified as genuine multipartite entangled. By using the state-dependent bounds (Figs. 4 and 5) we are able to detect almost all non separable W states and, above all, we can also identify a small subset of W states as genuine multipartite entangled.

[1] I. I. Hirschman Jr., A note on entropy, Amer. J. Math. 79 (1957), 152–156 (1957)
[2] W. Beckner, Inequalities in Fourier analysis, Ann. of Math. (2) 102, no. 1, 159–182 (1975).
[3] I. Białynicki-Birula, J. Mycielski, Uncertainty relations for information entropy in wave mechanics, Commun. Math. Phys. 44, 129 (1975).
[4] I. Białynicki-Birula, Entropic uncertainty relations, Phys. Lett. A 103, 253 (1984).
[5] D. Deutsch, Uncertainty in Quantum Measurements, Phys. Rev. Lett. 50, 631 (1983).
[6] K. Kraus, Complementary observables and uncertainty relations, Phys. Rev. D 35, 3070 (1987).
[7] H. Maassen, J.B.M. Uffink, Generalized entropic uncertainty relations, Phys. Rev. Lett. 60, 1103 (1988).
[8] P. J. Coles, M. Piani, Improved entropic uncertainty relations and information exclusion relations, Phys. Rev. A 89, 022112 (2014).
[9] L. Rudnicki, Z. Puchała, K. Życzkowski, Strong majorization entropic uncertainty relations, Phys. Rev. A 89, 052115 (2014).
[10] P. J. Coles, M. Berta, M. Tomamichel, S. Wehner, Entropic uncertainty relations and their applications, Rev. Mod. Phys. 89, 015002 (2017).
[11] W. Heisenberg, Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik, Zeitschrift für Physik 43 (3-4), 172 (1927).
[12] H. P. Robertson, The Uncertainty Principle, Phys. Rev. 34 (1), 163 (1929).
[13] L. Maccone, A. K. Pati, Stronger Uncertainty Relations for All Incompatible Observables, Phys. Rev. Lett. 113, 260401 (2015).
[14] L. Dammeier, R. Schwonnek, R.F. Werner, Uncertainty relations for angular momentum, New J. Phys. 17 (9), 093046 (2015).
[15] R. Schwonnek, L. Dammeier, R.F. Werner, State-independent uncertainty relations and entanglement detection in noisy systems, Phys. Rev. Lett. 119 (17), 170404 (2017).
[16] H. de Guise, L. Maccone, B.C. Sanders, N. Shukla, State-independent uncertainty relations, Phys. Rev. A 98 (4), 042121 (2018).
[17] P. Giorda, L. Maccone, A. Riccardi, State-independent uncertainty relations from eigenvalue minimization, Phys. Rev. A 99 (5), 052121 (2019).
[18] K. Szymański, K. Życzkowski, Geometric and algebraic origins of additive uncertainty relations, J. Phys. A: Math. Theor. 53 015302 (2019).
[19] I. D. Ivanovic, An inequality for the sum of entropies of unbiased quantum measurements, J. Phys. A: Math. Gen. 25 (7), 363 (1992).
[20] J. Sanchez, Improved bounds in the entropic uncertainty and certainty relations for complementary observables, Phys. Lett. A 201, 125 (1995).
[21] A. Azarchs, Entropic uncertainty relations for incomplete sets of mutually unbiased observables, arXiv:quant-ph/0412083 (2004).
[22] M.A. Ballester, S. Wehner, Entropic uncertainty relations for more than two observables, Phys. Rev. A. 75, 022319 (2007).
[23] S. Wu, S. Yu, K. Mølner, Entropic uncertainty relation for mutually unbiased bases, Phys. Rev. A 79, 022104 (2009).
[24] S. Liu, L. Mu, H. Fan, Entropic uncertainty relations for multiple measurements, Phys. Rev. A 91, 042133 (2015).
[25] A. Riccardi, C. Macchiavello, L. Maccone, Tight entropic uncertainty relations for systems with dimension three to five, Phys. Rev. A 95 (3), 032109 (2017).
[26] K. Abdelkhalak, R. Schwonnek, H. Maassen, F. Furrer, J. Duhme, P. Raynal, B-G. Englert, R. F. Werner, Optimality of entropic uncertainty relations, Int. J. Quantum Inf. 13, 1550045 (2015).
[27] O. Gühne, M. Lewenstein, Entropic uncertainty relations and entanglement, Phys. Rev. A 70, 022316 (2004).
[28] L. Maccone, D. Bruss, C. Macchiavello, Complementarity and correlations, Phys. Rev. Lett. 114, 130401 (2015).
[29] D. Sauerwein, C. Macchiavello, L. Maccone, B. Krauss, Multiparticle correlations in mutually unbiased bases, Phys. Rev. A 95, 042315 (2017).
[30] Y. Huang, Entanglement criteria via concave-function uncertainty relations, Phys. Rev. A 82, 012335 (2010).
[31] A. E. Rastegin, On uncertainty relations and entanglement detection with mutually unbiased measurements, Open Sys. & Inf. Dyn., Vol. 22, 1550005 (2015).
[32] Z. Jia, Y. Wu, G. Guo, Characterizing nonlocal correlations via universal uncertainty relations, Phys. Rev. A 96, 032122 (2017).
[33] J. Schneeloch, C. J. Broadbent, S. P. Walborn, E. G. Cavalcanti and J. C. Howell, Einstein-Podolsky-Rosen steering inequalities from entropic uncertainty relations, Phys. Rev. A 87, 062103 (2013).
[34] J. Schneeloch, P.B. Dixon, G.A. Howland, C.J. Broadbent and J.C. Howell, Violation of continuous-variable Einstein-Podolsky-Rosen steering with discrete measurements, Phys. Rev. Lett. 110 (13), 130407 (2013).

[35] J. Schneeloch, G.A. Howland, Quantifying high-dimensional entanglement with Einstein-Podolsky-Rosen correlations, Phys. Rev. A 97, 042338 (2018).

[36] A. Riccardi, C. Macchiavello, L. Maccone, Multipartite steering inequalities based on entropic uncertainty relations, Phys. Rev. A 97, 052307 (2018).

[37] A Costa, R Uola, O Gühne, Entropic Steering Criteria: Applications to Bipartite and Tripartite Systems, Entropy 20 (10), 763 (2018).

[38] T. Krivachy, F. Frowis, N. Brunner, Tight steering inequalities from generalized entropic uncertainty relations, arXiv:1807.09603 (2018).

[39] R. Schwonnek, Additivity of entropic uncertainty relations, Quantum 2, 59 (2018).

[40] T. Durt, B. Englert, I. Bengtsson and K. Zyczkowski, On mutually unbiased bases, Int. J. Quantum Inform. 08, 535 (2010).

[41] A. Acín, A. Andrianov, L. Costa, E. Jané, J. I. Latorre, and R. Tarrach, Generalized Schmidt Decomposition and Classification of Three-Quantum-Bit States, Phys. Rev. Lett. 85, 1560 (2000).