ORBIFOLD–LIKE AND PROPER $g$–MANIFOLDS

FRANZ W. KAMBER† AND PETER W. MICHOR

Abstract. In [4, 5], we generalized the concept of completion of an infinitesimal group action $\zeta : g \to \mathfrak{X}(M)$ to an actual group action on a (non-compact) manifold $M$, originally introduced by R. Palais [11], and showed by examples that this completion may have quite pathological properties (much like the leaf space of a foliation). In the present paper, we introduce and investigate a tamer class of $g$–manifolds, called orbifold–like, for which the completion has an orbifold structure. This class of $g$–manifolds is reasonably well–behaved with respect to its local topological and smooth structure to allow for many geometric constructions to make sense. In particular, we investigate proper $g$–actions and generalize many of the usual properties of proper group actions to this more general setting.

1. Introduction

1.1. Graph foliations. Given smooth connected manifolds $B$, $F$, a smooth 1–form $\alpha : T(B) \to \mathfrak{X}(F)$ defines a smooth distribution $D$ on $B \times F$ transversal to the fibers $F$ by lifting tangent vectors on the base space $B$ by the formula

$$D_{(x,y)} = \{(v, \alpha(v)(y)) \mid v \in T_x(B)\}.$$

The distribution $D$ is integrable, that is it defines a foliation $\mathcal{F}_\alpha$ on $B \times F$, if and only if $\alpha$ satisfies the Cartan–Maurer equation

$$d\alpha + \frac{1}{2} [\alpha, \alpha] = 0.$$

The immersed leaves $\mathcal{L} \supseteq B \times F$ of $\mathcal{F}_\alpha$, that is the maximal connected integral manifolds of the integrable distribution $T\mathcal{F}_\alpha$, are étale over $B$ under composition with the projection $\pi_1 : B \times F \to B$.

If $\alpha : T(B) \to \mathfrak{X}(F)$ takes values in a subalgebra of complete vector fields, then the foliation $\mathcal{F}_\alpha$ can be integrated to a generalized flat bundle, that is there exists an equivalence of bundles

$$B \times F \cong \tilde{B} \times_{\Gamma, h_\alpha} F \xrightarrow{\pi} B, \quad \Gamma = \pi_1(b), \quad h_\alpha : \Gamma \to \text{Diff}(F)$$

such that the leaves of the foliation $\mathcal{F}_\alpha$ correspond to the immersions $\tilde{B} \to \tilde{B} \times_{\Gamma} F$ of the form $\tilde{x} \mapsto [\tilde{x}, y]$ for $y \in F$ fixed and therefore are coverings of $B$. The
homomorphism $h_α : Γ \to \text{Diff}(F)$ determines the holonomy group $h_α(Γ) \subset \text{Diff}(F)$ of the flat bundle, respectively the foliation $F_α$, and the leaves are of the form $\tilde{B}/h_α^{-1}(Γ_x)$, where $Γ_x \subset h_α(Γ)$ is the isotropy group at $x \in F$ under the action of $h_α(Γ)$. The leaf space $B \times F/F_α$ of $F_α$ is then given as the orbit space $h_α(Γ)\setminus F$. It is well–known that this space has the Hausdorff separation property, if the holonomy group $h_α(Γ)$ acts properly discontinuously (as a discrete group) on the fiber $F$. The isotropy groups $Γ_x \subset h_α(Γ)$ are then finite and the leaf space is locally Euclidean near $\tilde{x}$ if $Γ_x = 1$ and has a local orbifold structure near $\tilde{x}$ otherwise.

The main examples for the complete case occur when the fiber $F$ is either compact, or $F = G$ is a Lie group and $α : T(B) \to g$ takes values in the Lie algebra $g$ of left invariant vector fields on $G$. In the latter case the holonomy is given by a homomorphism $h_α : Γ \to G$.

1.2. $g$–manifolds. In [5], we considered a different situation, related to the integration of infinitesimal group actions, that is the base space $B = G$ is a Lie group and the fiber $F = M$ is a (non-compact) manifold. An infinitesimal action of $G$ on $M$ is given by a Lie homomorphism $ζ : g \to X(M)$, and this determines a 1–form $α_ζ : T(G) \to X(M)$ satisfying the Cartan–Maurer equation (1.1) simply by composition of $ζ$ with the Cartan–Maurer form of $G$. As the examples in [5] show, the completion $\tilde{G}M$ with respect to the simply connected Lie group $\tilde{G}$ associated to $g$ may behave quite badly, e.g. $\tilde{G}M$ need not even have the $\mathcal{T}_1$–separation property.

In the present paper, we investigate a class of $g$–manifolds $(M, ζ)$, called discrete or better orbifold–like, for which the completion $\tilde{G}M$ has an orbifold structure. The $\tilde{G}$–completion for this class of $g$–manifolds is reasonably well–behaved with respect to its local topological and smooth structure to allow for many geometric constructions to make sense, yet is far more general than the completions considered in [11]. Thus in the terminology of [11], the $g$–manifolds considered here need neither be univalent nor uniform.

2. The completion of a $g$–manifold

2.1. The graph of the pseudogroup. Let $M$ be a $g$–manifold, effective and connected, so that the action $ζ = ζ^M : g \to X(M)$ is injective. Recall from [1], 2.3 that the pseudogroup $Γ(g)$ consists of all diffeomorphisms of the form

$$F^{ξ_{X^n}}_t \circ \ldots \circ F^{ξ_{X^2}}_t \circ F^{ξ_{X^1}}_t \mid U$$

where $X_i \in g$, $t_i \in \mathbb{R}$, and $U \subset M$ are such that $F^{ξ_{X^n}}_t$ is defined on $U$, $F^{ξ_{X^2}}_t$ is defined on $F^{ξ_{X^1}}_t(U)$, and so on.

Now we choose a connected Lie group $G$ with Lie algebra $g$, and we consider the integrable distribution of constant rank ($= \dim(g)$) on $G \times M$ which is given by

$$(1) \quad \{ (L_X(g), ζ^M_X(x)) : (g, x) \in G \times M, X \in g \} \subset TG \times TM,$$

where $L_X$ is the left invariant vector field on $G$ generated by $X \in g$. This gives rise to a foliation $F_ζ$ on $G \times M$, which we call the graph foliation of the $g$–manifold $M$.

Note that the graph foliation is transversal to the fibers $\{ g \} \times M$ of $pr_1 : G \times M \to G$.

Consider the following diagram, where $L_ζ(g, x) = L(g, x)$ is the leaf through $(g, x)$ in $G \times M$, $O_ζ(x)$ is the $g$–orbit through $x$ in $M$, and where $W_x \subset G$ is the
image of the leaf \( L(e, x) \) in \( G \).

\[
(2) \quad \begin{array}{c}
L(e, x) \xrightarrow{pr_2} O_{g}(x) \\
\downarrow \quad \downarrow \\
G \times M \xrightarrow{pr_2} M \\
\downarrow \quad \downarrow \\
\text{open} \quad \text{open} \\
W_{x} \xrightarrow{pr_1} G
\end{array}
\]

Moreover we consider a piecewise smooth curve \( c : [0, 1] \to W_{x} \) with \( c(0) = e \) and we assume that it is liftable to a smooth curve \( \tilde{c} : [0, 1] \to L(e, x) \) with \( \tilde{c}(0) = (e, x) \). Its endpoint \( \tilde{c}(1) \in L(e, x) \) does not depend on small (i.e. liftable to \( L(e, x) \)) homotopies of \( c \) which respect the ends. This lifting depends smoothly on the choice of the initial point \( x \) and gives rise to a local diffeomorphism \( \gamma_{c} : U \to \{ e \} \times U \to \{ c(1) \} \times U' \to U' \), a typical element of the pseudogroup \( \Gamma(g) \) which is defined near \( x \). See [1], 2.3 for more information. Note, that the leaf \( L(g, x) \) through \((g, x)\) is given by

\[
L(g, x) = \{ (gh, y) : (h, y) \in L(e, x) \} = (\mu_{g} \times \text{Id})(L(e, x))
\]

where \( \mu : G \times G \to G \) is the multiplication and \( \mu_{g}(h) = gh = \mu^{h}(g) \).

2.2. The holonomy and fundamental groupoid. We briefly recall the construction of the holonomy groupoid of a foliation \((M, F)\). For \( x, y \in \mathcal{L} \), consider (piecewise) smooth curves \( c : x \to y \) in the leaf \( \mathcal{L} \), that is \( \tilde{c}(t) \in T(\mathcal{F})_{c(t)} \). By varying the initial data \((x, \tilde{c}(0))\) and considering small variations of \( c \), always staying in the leaves, respectively flow charts, one constructs a local diffeomorphism \( \gamma : T_{x} \to T_{y} \), \( x, y \in \mathcal{L} \), from a small transversal disc \( T_{x} \subset M \) to a small transversal disc \( T_{y} \subset M \). The germ \( \tilde{\gamma} \) of \( \gamma \) at \( x \) depends only on the homotopy class \( [c] \) of the path \( c \) inside the leaf, so that we may write \( \tilde{\gamma}_{c(\{e\}, y)} = [\gamma] \). The ‘smooth’ holonomy groupoid \( \mathcal{G}_{\mathcal{F}} \) consists of all triples \([x, c, y]\) where \( c : x \to y \) is a path inside a leaf and \([x, c, y] = [x', c', y']\) if \( x = x' \), \( y = y' \) and \( c \), \( c' \) define the same holonomy transformation at the germ level. The holonomy groupoid \( \mathcal{G}_{\mathcal{F}} \) is a manifold, which in general is not Hausdorff, and the source and range maps \( s, r : \mathcal{G}_{\mathcal{F}} \to M \) determine submersions onto the manifold \( M \), whose fibers are the holonomy coverings of the leaves of \( \mathcal{F} \).

The fundamental groupoid \( \Gamma_{\mathcal{F}} \) of the graph foliation \( \mathcal{F}_{\zeta} \) on \( G \times M \): In the case of the graph foliation \( \mathcal{F}_{\zeta} \) on \( G \times M \), the fundamental groupoid takes a special form, which reflects the ‘incompleteness’ of the infinitesimal group action. The composition \( L_{\zeta}(g, x) \ni g \times M \overset{pr_{2}}{\to} G \) is an etale map onto \( G \), whose range is a connected open subset \( W_{g, x} \subseteq G \), \( g \in W_{g, x} \). By equivariance, we have

\[
W_{g, x} = p_{1}(L_{\zeta}(g, x)) = p_{1}(gL_{\zeta}(e, x)) = gW_{x} \subseteq G , \quad W_{x} = W_{e, x}.
\]

The projection \( O_{g}(x) = \zeta(g)(x) = p_{2} L_{\zeta}(e, x) \subseteq M \) is the \( g \)-orbit through \( x \).

Since \( L_{\zeta}(e, x) \overset{pr_{1}}{\to} W_{x} \) is etale, a path (piecewise smooth curve) \( \tilde{c} : (e, x) \to (g, y) \) in \( L_{\zeta}(e, x) \), i.e., \( \tilde{c}(t) \in T(\mathcal{F})_{c(t)} \), is uniquely determined by its projection \( c = p_{1} \circ \tilde{c} \) in \( W_{x} \) and the initial value \( \tilde{c}(0) = (e, x) \in L_{\zeta}(e, x) \). Such paths \( c \) in \( W_{x} \) are called liftable at \((e, x)\). We denote by \( P_{\mathcal{F}_{\zeta}, e} \subset P(G, e) \times M \) the subset of liftable paths. This subset is open in \( P(G, e) \times M \), since a path \( c \) lifts to a unique path \( \tilde{c} \) in
$L_\zeta(e,x)$ starting at $(e,x)$, provided that $c$ lies in a small convex neighborhood $U_e$ of the point path $e \in W_\zeta$, that is $c$ stays in a small convex neighborhood $U_e$ of $e \in W_x \subseteq G$. Two paths $c, c' \in P_{\zeta,e}$ with the same endpoints are said to be related by a liftable homotopy, if their lifts $\tilde{c}, \tilde{c'}$ at $(e,x)$ are related by a leafwise homotopy (covered by a finite number of flow charts) in $G \times M$. This implies that the lifted paths $\tilde{c}, \tilde{c'}$ have the same endpoint $(g,y) \in G \times M$. Using the $G$–invariance of the graph foliation $\mathcal{F}_\zeta$ and its leaves, we may translate the previous construction to any initial point $(g,x)$ and so obtain the open subspace $P_{\zeta} \subset P(G) \times M$ of liftable paths:

$$P_{\zeta} = \bigcup_{g \in G} P_{\zeta,g} = \bigcup_{g \in G} g \cdot P_{\zeta,e}$$

Identifying paths in $P_{\zeta}$ which are related by liftable homotopies with fixed endpoints, we obtain exactly the fundamental groupoid $\Gamma_{\zeta} \to \mathcal{G}_{\zeta}$ of $\mathcal{F}_\zeta$. By construction, the subset $\Gamma_{\zeta}(g,x) \subset \Gamma_{\zeta}$ of classes of liftable paths starting at $(g,x)$ parametrizes the universal covering of the leaf $L_\zeta(g,x)$ and therefore the leaf $L_\zeta(g,x)$ by a covering map

$$\Gamma_{\zeta}(g,x) \cong \widetilde{L_\zeta(g,x)} \xrightarrow{\pi(g,x)} L_\zeta(g,x) \quad \pi(g,x)((c,x),x) = \{\tilde{c}\}(1) ;$$

(1) $L_\zeta(g,x) = \{(g', y) : ((c,x),x) \in \Gamma_{\zeta}(g,x), \pi((c,x),x) = (g', y)\}$.

Moreover, there is a commutative diagram

$$\begin{array}{ccc}
\Gamma_{\zeta}(g,x) & \xrightarrow{\pi(g,x)} & L_\zeta(g,x) \\
\tilde{p_1} & \downarrow & \downarrow p_1 \\
\tilde{W}_{g,x} & \xrightarrow{\ell_g \circ \pi \circ \ell_{g^{-1}}} & W_{g,x}
\end{array}$$

where $\tilde{p_1}((\{c,x\},x)) = \{c\}$ and the horizontal maps are universal covering maps.

Since the leafwise homotopy relation implies $[x, c, y] = [x, c', y]$ , there is a canonical surjection $\Gamma_{\zeta} \to \mathcal{G}_{\zeta}$ of the fundamental groupoid $\Gamma_{\zeta}$ to the holonomy groupoid $\mathcal{G}_{\zeta}$ of $\mathcal{F}$

The pseudogroup of local diffeomorphisms on $M$ and the corresponding smooth groupoid associated to the $\mathfrak{g}$–manifold $(M, \zeta)$ are exactly the holonomy pseudogroup of local diffeomorphisms on the transversal $M$ and the holonomy groupoid $\mathcal{G}_{\mathcal{F}_\zeta}$, respectively the fundamental groupoid $\Gamma_{\mathcal{F}_\zeta}$ of the graph foliation $\mathcal{F}_\zeta$ on $G \times M$.

As is true for any foliation, the leaves $L_\zeta(g,x)$ of $\mathcal{F}_\zeta$, viewed as immersed submanifolds of $G \times M$, are initial submanifolds. Since leaves are by definition maximal connected integral manifolds of the integrable distribution $T\mathcal{F}_\zeta$, two leaves which intersect must be identical, that is $L_\zeta(g_1,x_1) \cap L_\zeta(g_2,x_2) \neq \emptyset$ implies $L_\zeta(g_1,x_1) = L_\zeta(g_2,x_2)$.

2.3. Enlarging to group actions. In the situation of 2.1 let us denote by $G \times \mathfrak{g} \times M = G \times M / \mathcal{F}_\zeta$ the space of leaves of the foliation $\mathcal{F}_\zeta$ on $G \times M$, with the quotient topology. For each $g \in G$ we consider the mapping

$$j_g : M \xrightarrow{i_\zeta} \{g\} \times M \subset G \times M \xrightarrow{\pi} G \times \mathfrak{g} \times M.$$
Note that the submanifolds \( \{g\} \times M \subset G \times M \) are transversal to the graph foliation \( \mathcal{F}_\zeta \). The leaf space \( G \mathcal{M} \) of \( \mathcal{F}_\zeta \) admits a unique smooth structure, possibly singular and non-Hausdorff, such that a mapping \( f : G \mathcal{M} \to N \) into a smooth manifold \( N \) is smooth if and only if the compositions \( f \circ j_g : M \to N \) are smooth. For example we may use the the structure of a Frölicher space or smooth space in the sense of [9], section 23 on \( G \mathcal{M} = G \times_g M \). We write \( j_g(x) = [g, x] \in G \mathcal{M} \), where \([g, x]\) represents the leaf \( L_\zeta(g, x) \) of \( \mathcal{F}_\zeta \) through \((g, x)\). We shall also use the notation \( V_g = j_g(M) \), an open subset of \( G \mathcal{M} \). The leaf space \( G \mathcal{M} = G \times_g M \) is a smooth G–space by \( \ell_g[g', x] = g[g', x] = [gg', x] \), the action being induced by the left action of \( G \) on \( G \times M \). By construction, we have

\[
j_g' = \ell_{g'g^{-1}} \circ j_g,
\]

and therefore the relationships between the maps \( j_g \), \( g \in G \) are given by left translation in \( G \mathcal{M} \).

By construction, for each \( x \in M \) and for \( g'g^{-1} \) near enough to \( e \) in \( G \) there exists a curve \( c : [0, 1] \to W_x \) with \( c(0) = e \) and \( c(1) = g'g^{-1} \) and an open neighborhood \( U \) of \( x \) in \( M \) such that for the smooth transformation \( \gamma_x(c) \) in the pseudogroup \( \Gamma(g) \) we have

\[
j_g'|U = j_g \circ \gamma_x(c).
\]

Thus the mappings \( j_g \) may serve as a replacement for charts in the description of the smooth structure on \( G \mathcal{M} \) and we therefore call the maps \( j_g : M \to V_g \subset G \mathcal{M}, g \in G \) the charts of \( G \mathcal{M} \). Note that the mappings \( j_g \) are not injective in general. Even if \( g = g' \) there might be liftable smooth loops \( c \) in \( W_x \) such that (3) holds. Note also some similarity of the system of 'charts' \( j_g \) with the notion of an orbifold where one uses finite groups instead of pseudogroup transformations.

**Theorem.** See [5], theorems 5, 7, and 9. The \( G \)-completion \( G \mathcal{M} \) has the following properties:

4. Given any Hausdorff \( G \)-manifold \( N \) and \( g \)-equivariant mapping \( f : M \to N \) there exists a unique \( G \)-equivariant continuous mapping \( \hat{f} : G \mathcal{M} \to N \) with \( \hat{f} \circ j_e = f \). Namely, the mapping \( f : G \times M \to N \) given by \( \hat{f}(g, x) = g, f(x) \) is smooth and factors to \( \hat{f} : G \mathcal{M} \to N \).

5. If \( M \) carries a symplectic or Poisson structure or a Riemannian metric such that the \( g \)-action preserves this structure or is even a Hamiltonian action then the structure ‘can be extended to \( G \mathcal{M} \) such that the enlarged \( G \)-action preserves these structures or is even Hamiltonian’.

6. Suppose that the \( g \)-action on \( M \) is transitive and \( M \) is connected. Then there exists a (possibly non-closed) subgroup \( H \subset G \) such that the \( G \)-completion \( G \mathcal{M} \) is diffeomorphic to \( G/H \). In fact, \( H = \{ g \in G : (g, x_0) \in L(e, x_0) \} = \{ g \in G : L(g, x_0) = L(e, x_0) \} \), and the Lie algebra of the path component of \( H \) equals \( \mathfrak{g}_{x_0} = \{ X \in \mathfrak{g} : \xi X(x_0) = 0 \} \).

7. In general, the \( G \)-completion \( G \mathcal{M} \) is given as follows: Form the leaf space \( \mathcal{M}/\mathfrak{g} \), a quotient of \( M \) which may be non-Hausdorff and not \( T_1 \) etc. For each point \( z \in \mathcal{M}/\mathfrak{g} \), replace the orbit \( \pi^{-1}(z) \subset M \) by the homogeneous space \( G/H_x \) described in [5], theorem 7, where \( x \) is some point in the orbit \( \pi^{-1}(z) \subset M \), chosen locally in transversals to the \( g \)-orbits in \( M \).
2.4. Intersection sets. The transversal \( \{g^I\} \times (M) \subset G \times M \) intersects the leaf \( \iota : L_\zeta(g, x) \hookrightarrow G \times M \) in the set \( \{g^I\} \times I(g^I; g, x) \), that is
\[
\{g^I\} \times I(g^I; g, x) := (\{g^I\} \times M) \cap \iota L_\zeta(g, x) = \iota J(g^I; g, x)
\]
where \( J(g^I; g, x) \subset L_\zeta(g, x) \). Since \( L_\zeta(g, x) = (\mu_g \times \text{Id}) \ L_\zeta(e, x) \) we have
\[
\{g^I\} \times I(g^I; g, x) = ((g^I) \times M) \cap (\mu_g \times \text{Id}) L_\zeta(e, x)
\]
\[
= (\mu_g \times \text{Id}) (\{g^{-1}g^I\} \times M \cap L_\zeta(e, x))
\]
\[
= (\mu_g \times \text{Id}) (\{g^{-1}g^I\} \times I(g^{-1}g^I; e, x)),
\]
\[
\{g^I\} \times I(g^I; g, x) = (\mu_g \times \text{Id}) (\{e\} \times M \cap L_\zeta(g, x))
\]
\[
= (\mu_g \times \text{Id}) (\{e\} \times M \cap L_\zeta(g^{-1}g, x))
\]
and therefore
\[
I(g^I; g, x) = I(g^{-1}g^I; e, x) = I(e; g^{-1}g, x) \ , \ I(e; g, x) = I(g^{-1}; e, x) .
\]

In terms of intersection sets, the construction in \(2.2\) may now be written as
\[
\text{(2) } y \in I(e; g, x) = I(g^{-1}; e, x) \iff (e, y) = \{c\}_1 , \ (\{c\}, x) \in \mathcal{H}_x \ (e; g, x) .
\]

2.5. Recurrence sets. By definition, the intersection sets \( I(g^I; g, x) \subset M \) depend only on the leaf \( L_\zeta(g, x) \) and the transversal \( i_y(M) \), that is we have
\[
(1) \quad I(g; g_1, x_1) = I(g; g_2, x_2) , \text{ for } [g_1, x_1] = [g_2, x_2] .
\]

If \( y \in I(g; g_1, x_1) \cap I(g; g_2, x_2) \), then \( (g, y) \in L_\zeta(g_1, x_1) \cap L_\zeta(g_2, x_2) \) and \( [g, y] = [g_1, x_1] \neq [g_2, x_2] \). Therefore distinct leaves have disjoint intersection sets, that is
\[
(2) \quad I(g; g_1, x_1) \cap I(g; g_2, x_2) = \emptyset , \text{ for } [g_1, x_1] \neq [g_2, x_2] .
\]

The following properties are equivalent :

(3) \quad [g_1, x_1] = [g_2, x_2] ;
(4) \quad L_\zeta(g_1, x_1) = L_\zeta(g_2, x_2) ;
(5) \quad g_2^{-1}g_1 \ L_\zeta(e, x_1) = L_\zeta(e, x_2) ;
(6) \quad x_1 \in I(g_1; g_2, x_2) = I(e; g_1^{-1}g_2, x_2) , \ x_2 \in I(g_2; g_1, x_1) = I(e; g_2^{-1}g_1, x_1) .

Further the construction \(2.1\) implies the transitivity relation
\[
(7) \quad \begin{cases} x_1 \in I(g_1; g_2, x_2) = I(e; g_1^{-1}g_2, x_2) \\ x_2 \in I(g_2; g_3, x_3) = I(e; g_2^{-1}g_3, x_3) \end{cases} \quad \Rightarrow \ x_1 \in I(g_1; g_2, x_3) = I(e; g_1^{-1}g_3, x_3) .
\]

From \(2.4.1\), we have \( x \in I(g; g, x) = I(e; e, x) , \forall x \in M \). We call the sets \( I(g; g, x) \) recurrence sets, because they count the the points at which the leaf \( L_\zeta(g, x) \) through \( (g, x) \) returns to the transversal \( \{g\} \times M \). From \(7\) and the previous formula, it follows that the recurrence sets are in addition symmetric and transitive :
\[
y \in I(g; g, x) \iff x \in I(g; g, y) , \\
y \in I(g; g, x) , \ z \in I(g; g, y) \implies z \in I(g; g, x) .
\]

Therefore the relation
\[
(8) \quad x \sim y \iff y \in I(g; g, x) = I(e; e, x)
\]
is an equivalence relation on \( M \) , independent of \( g \in G \).
The fiber of \( j_g : x \mapsto (g, x) \mapsto [g, x] \) is exactly the recurrence set \( I(g; g, x) = I(e; e, x) \). Since the projection \( G \times M \to G M \) to the leaf space is an (open) identification map by definition, we see that \( j_g : M \to V_g \) is equivalent to the identification map given by the equivalence relation \((\ref{2.5})\). Further it follows from \((\ref{2.4})\) that the units \( \Gamma_{\mathcal{F}_e} (e; e, x_0) \) in the fundamental groupoid \( \Gamma_{\mathcal{F}} \), respectively the units \( \mathcal{H}_{\mathcal{F}_e} (e; e, x_0) \) in the holonomy groupoid \( \mathcal{H}_{\mathcal{F}} \), act locally near \( x_0 \in M \), with the intersection sets \( I(e; e, x) \) as orbits at \( x \in M \) near \( x \).

### 2.6. Uniformity of the recurrence sets.

It will be desirable to have some control on how the recurrence sets \( I(g; g, x) = I(e; e, x) \) change under variations of \( x \in M \) in a small transversal disc \( \{ g \} \times U_x, U_x \subset M \). We set \( g = e \) and represent the holonomy element \( \tilde{\gamma}(x_0, e, y_0) = [\gamma] \) by a suitable local diffeomorphism \( \gamma : U_{x_0} \to U_{y_0} \). Then \( y_0 = \gamma(x_0) \in I(e; e, x_0) \) and it follows from the holonomy construction that \( y = \gamma(x) \in I(e; e, x) \), \( \forall x \in U_{x_0} \). The open neighborhood \( U_{x_0} \subset M \) is not necessarily uniform in \( y_0 = \gamma(x_0) \in I(e; e, x_0) \) and other elements may enter or leave the recurrence set \( I(e; e, x) \) as we vary \( x \).

We say that the set \( \mathcal{H}_{\mathcal{F}_e} (e; e, x_0) \) in the holonomy groupoid, respectively the recurrence sets \( I(e; e, x) \) are uniform at \( x_0 \in M \), if the elements \( \tilde{\gamma} \in \mathcal{H}_{\mathcal{F}_e} (e; e, x_0) \) can be represented by a pseudogroup element \( \gamma : U_{x_0} \ni x_0 \to U_{y_0} = \gamma U_{x_0} \), with \( U_{x_0} \) being independent of \( \tilde{\gamma} \). If in addition \( U_{x_0} \) can be chosen such that \( \{ x_0 \} \to x_0 \) in \( U_{x_0} \) and \( \{ \gamma_n(x_n) \} \to y_0 \) implies \( \gamma_n = \gamma \) for a cofinal subsequence of \( \{ \gamma_n \} \), we say that \( \mathcal{H}_{\mathcal{F}_e} (e; e, x_0) \) acts properly discontinuously near \( x_0 \in M \). It follows in particular that the recurrence sets \( I(e; e, x) \subset M \) are discrete and closed and that the intersections \( U_{x_0} \cap \gamma(U_{x_0}) = \emptyset \), except for finitely many elements \( \gamma \). Therefore the isotropy group \( \Gamma_{x_0} \subset M \), given by the units in \( \mathcal{H}_{\mathcal{F}_e} (e; e, x_0) \) is finite and we may choose \( U_{x_0} \) invariant under \( \Gamma_{x_0} \). Given any \( y_0 \in I(e; e, x_0) \), we have now constructed a uniform continuous slice (or section) around \( y_0 \), that is

\[
\{ y = \gamma(x) \mid x \in U_{x_0}, \tilde{\gamma} \in \mathcal{H}_{\mathcal{F}_e} (e; e, x_0) \} = \bigcup_{x \in U_{x_0}} I(e; e, x).
\]

Equivalently, there is a saturated neighborhood \( U \) of \( L(e, x_0) \subset G \times M \) of the form

\[
U \cong \hat{L}(e, x_0) \times_{\Gamma_{x_0}} U_{x_0} \to \Gamma_{x_0} \setminus U_{x_0}.
\]

where \( \hat{L}(e, x_0) \to L(e, x_0) \) is a finite covering space of \( L(e, x_0) \) with group \( \Gamma_{x_0} \), that is the saturated neighborhood \( U \) of the leaf \( L(e, x_0) \) admits a Seifert fibration over \( \Gamma_{x_0} \setminus U_{x_0} \). If the above properties are valid for any \( x_0 \in M \), we use the terms without reference to the baseline point \( x_0 \).

As \( [e, x_0] \in V_e \) approaches the boundary \( \partial V_e = \overline{V_e} \setminus V_e \) of \( V_e \subset G M \), the intersection sets may change drastically. This leads to the notion of limit elements explained in \((\ref{2.8})\) below.

### 2.7. Hausdorff separation property for \( G M \).

Recall that \( V_g = j_g(M) \subset G M \). Suppose that the elements \( z = [g, x] \in V_g \), \( z' = [g', y] \in V_{g'} \), \( z \neq z' \) are not Hausdorff separated. Then both \( z \) and \( z' \) would have to be arbitrarily close to points \( z_n \in V_g \cap V_{g'} \), that is leaves which intersect both transversals \( \{ g \} \times M \) and \( \{ g' \} \times M \) in \( G \times M \), and we would have \( z_n = [g, x_n] = [g', y_n] \in V_g \cap V_{g'} \), for convergent sequences \( \{ x_n \} \to x \), \( \{ y_n \} \to y \). Therefore the Hausdorff property for the leaf space \( G M \) is equivalent to the following statement:
Suppose that \( \{x_n\} \to x \) and \( \{y_n\} \to y \) are convergent sequences in \( M \), such that \( (g', y_n) \in L_\xi(g, x_n) \). Then we have \( (g', y) \in L_\xi(g, x) \). Equivalently, \( y_n \in I(g'; g, x_n) \) implies \( y \in I(g'; g, x) \).

The Hausdorff separation property for \( V_e \subseteq G M \), and hence for any \( V_g \subseteq G M \), is equivalent to the following statement:

Suppose that \( \{x_n\} \to x \) and \( \{y_n\} \to y \) are convergent sequences in \( M \), such that \( (e, y_n) \in L_\xi(e, x_n) \). Then we have \( (e, y) \in L_\xi(e, x) \). Equivalently, \( y_n \in I(e; e, x_n) \) implies \( y \in I(e; e, x) \).

2.8. Limit elements. The incompleteness of the \( G \)-action must of course show up somewhere and this happens if we move towards the boundary \( \partial V_g = V_g \setminus V_{g'} \) of the open sets \( V_g \subseteq G M \). We say that an element \( \hat{z} = [\hat{g}, \hat{x}] \in G M \) is a limit element for the transversal \( i_{g_0}(M) \) if \( \hat{z} \in \partial V_{g_0} \). This means that the leaf \( L_\xi(\hat{g}, \hat{x}) \), representing \( \hat{z} \) does not intersect the transversal \( i_{g_0}(M) \), that is \( I(g_0; \hat{g}, \hat{x}) = \emptyset \), but is arbitrarily close to a leaf \( L_\xi(g_0, x_n) \) such that \( I(g_0; g_0, x_n) \neq \emptyset \). It is obviously sufficient to elaborate this property for \( g_0 = e \) and \( V_e \cap V_{\hat{g}} \neq \emptyset \).

The following statements are equivalent:

1. \( \hat{z} = [\hat{g}, \hat{x}] \in G M \) is a limit element for the transversal \( i_e(M) \), that is \( \hat{z} \in \partial V_e \setminus V_{\hat{g}} \);

2. There exist sequences \( \{x_n\} \) and \( \{y_n\} \) in \( M \), such that \( \{x_n\} \) converges to \( \hat{x} \) and \( y_n \in I(e; \hat{g}, x_n) = I(e; e, y_n) \), but \( I(e; \hat{g}, \hat{x}) = \emptyset \).

2.9. Separation of limit elements. If \( V_{\hat{g}} \cap V_e = \emptyset \), then any points \( z_0 \in V_e \) and \( \hat{z} \in V_{\hat{g}} \) are obviously Hausdorff separated. Suppose that \( V_e \cap V_{\hat{g}} \neq \emptyset \), \( \hat{g} \neq e \). If \( z_0 \notin V_{\hat{g}} \), then \( \hat{z} \) and \( z_0 \) are still Hausdorff separated by the disjoint open sets \( V_{\hat{g}} \) and \( V_e \setminus \{V_{\hat{g}} \in V_{\hat{g}}\} \). The same is true if \( \hat{z} \notin V_e \). For \( z_0 \), \( \hat{z} \notin V_e \setminus V_{\hat{g}} \), the Hausdorff property can therefore fail only for limit elements \( z_0 \in \partial V_{\hat{g}} \) of the form \( z_0 = [e, y_0] \), that is \( z_0 \in V_e \cap \partial V_{\hat{g}} \), and \( \hat{z} \in \partial V_e \) of the form \( \hat{z} = [\hat{g}, \hat{x}] \), that is \( \hat{z} \in \partial V_e \cap \partial V_{\hat{g}} \). Then \( \hat{z} \neq z_0 \), since \( (V_e \cap \partial V_{\hat{g}}) \setminus (\partial V_e \cap \partial V_{\hat{g}}) = (V_e \cap \partial V_{\hat{g}}) \) is \( \emptyset \). The set \( H \subseteq G M \times \Delta G M \) of pairs \( (z_0, \hat{z}) \) of distinct non–separable points is obviously invariant under the diagonal action of \( G \) and we may write \( H = \bigcup_{g \in G} g H_e \), where pairs \( (z_0, \hat{z}) \) in \( H_e \) satisfy \( z_0 \in V_e \). The preceding argument shows that the set \( H_e \) is restricted by

\[
H_e \subseteq V_e \times V_e \setminus \left\{ V_e \cap V_{\hat{g}} \neq \emptyset \cup \bigcup_{g \in G} V_e \cap \partial V_{\hat{g}} \times \partial V_e \cap V_{\hat{g}} \right\}.
\]

We say that pairs of limit points are Hausdorff separable if the situation described above does not occur, in which case we have \( H_e \subseteq V_e \times V_e \).

Non–separable limit points typically occur when a complete \( G \)-action on \( M \) is made incomplete on \( M_1 = M \setminus A \) by removing a closed subset \( A \subseteq M \) from \( M \), such that some orbits of the complete \( G \)-action on \( M \) become disconnected on \( M_1 \). Such an orbit gives rise to multiple orbits in the completion \( \overline{G} M_1 \) and non–separable pairs appear whenever there are nearby orbits in \( M \) which are still connected in \( M_1 = M \setminus A \) (compare the examples in \( \mathbb{H} \)). Then the canonical map \( \overline{G} M_1 \to \overline{G} M \cong M \) defines \( M \) as a canonical Hausdorff quotient of \( \overline{G} M_1 \).

2.10. Proposition. The Hausdorff separation property for the leaf space \( G M \) is characterized as follows:
(1). The leaf space $\mathcal{G}M$ is Hausdorff, if and only if the open set $V_e \subset \mathcal{G}M$ is Hausdorff and pairs of limit points are Hausdorff separable. Then also each completion of a $\mathfrak{g}$-orbit in $M$ in the sense of (2.3.7) is of the form $G/H$ for a closed subgroup $H$ of $G$.

Proof. From (2.3.2), we see that the open sets $V_g$, $g \in G$ have the Hausdorff property if and only if $V_e$ has the Hausdorff property. (1) follows then from the preceding analysis in 2.9. To prove (2), let $\{x_n\} \to x$ and $\{y_n\} \to y$ be convergent sequences in $M$, such that $y_n \in I(e; e, x_n)$. Since $\{x_n\} \to x$ and $y_n \in I(e; e, x_n)$, we can use the slice construction around $x$ in (2.6.1) to find an open neighborhood $U_x$ and a sequence $\gamma_n$ so that $x_n \in U(x)$, $\gamma_n(x_n) = y_n$. By assumption, the action of the holonomy groupoid is properly discontinuous near $x$, so we have $\gamma_n \to \gamma$ for a subsequence of $\{\gamma_n\}$. But then $y_n = \gamma(x_n)$ and this converges to $y = \gamma(x)$. Thus we have $y \in I(e; e, x)$. $\square$

2.11. Orbifold structure of $j_e : M \to V_e \subset \mathcal{G}M$. For the chart $j_e : M \to V_e$ to be étale, that is a local diffeomorphism with respect to the differentiable structure induced in $\mathcal{G}M$ (cf. 2.4), it is necessary that the recurrence sets $I(e; e, x) \subset M$, $x \in M$ are discrete, closed and uniform. If the recurrence set $I(e; e, x) \subset M$ has an accumulation point $y_0 \in M$, it is clear that $j_e : M \to V_e$ cannot be a local homeomorphism at $y_0$. A sufficient condition for the étale structure of $j_e : M \to V_e$ is that in addition the action of the holonomy $\mathcal{H}_{\mathcal{F}_e} (e; e, x)$ on $M$ at $x$ is properly discontinuous, without fixed points, for all $x \in M$. In the sequel, we will allow fixed points, which are necessarily of finite order, that is we will allow an orbifold structure on $V_e$, so that the description in 2.6. and in particular (2.6.1), (2.6.2) apply. By translation invariance (2.3.2), the same conditions will then hold for all charts $j_g : V_g \subset \mathcal{G}M = G \times _\mathfrak{g} M$, $g \in G$.

3. Orbifold–like $\mathfrak{g}$–manifolds

A priori, the local orbifold structure described in 2.11 depends on the choice of a Lie group $G$ with Lie algebra $\mathfrak{g}$, that is on the choice of a quotient group $G \to \tilde{G}$ of the simply connected Lie group $\tilde{G}$ associated to $\mathfrak{g}$.

3.1. Definition. The $\mathfrak{g}$–action on $M$ is discretely valued with respect to $G$, and $(M, \zeta)$ a orbifold–like $\mathfrak{g}$–manifold with respect to $G$, if the chart $j_e : M \to V_e \subset \mathcal{G}M$ defines an orbifold structure on $j_e : M \to V_e$ as described in 2.11. This means that the action of the holonomy $\mathcal{H}_{\mathcal{F}_e} (e; e, x)$ on $M$ at $x$ is properly discontinuous near $x$ for all $x \in M$.

We collect some immediate properties of orbifold–like $\mathfrak{g}$–manifolds in the Lemma below.

3.2. Lemma. Let $(M, \zeta)$ be an orbifold–like $\mathfrak{g}$–manifold with respect to $G$. Then the following statements hold:

(1) The $G$–completion $\mathcal{G}M = G \times _\mathfrak{g} M$ of $(M, \zeta)$ is a smooth orbifold, such that the orbifold structure is determined by the charts $j_g : M \to V_g \subset GM$. More precisely, if $U_x \subset M$ is a uniform neighborhood as in 2.6, then $j_e(U_x) \cong \Gamma_x \setminus U_x$, where $\Gamma_x$ is the finite isotropy group at $x \in M$. 

(2) In the absence of fixed points at \( x \), the charts \( j_g : M \to V_g \subset G M \) are \( \acute{e}tale \) near \( x \) and therefore local diffeomorphisms. Therefore \( G M \) carries a canonical structure of a smooth manifold.

(3) The open orbifolds \( V_g \subset G M \) are always Hausdorff. Moreover, the Hausdorff separation property may fail only for pairs of limit points.

(4) If \( i_x : M \to G \times M \) is a complete transversal of \( \mathcal{F} \), then the chart \( j_e : M \to G M \) is a map onto the Hausdorff orbifold \( \tilde{G} M \).

(5) All intersection sets \( I(g'; g, x) \subset M \) are discrete, closed, uniform and at most countably infinite.

(6) The recurrence sets \( I(g; g, x) \subset M \), that is the fibers of the charts \( j_g : M \to V_g \subset G M \), are discrete, closed, uniform and at most countably infinite.

(7) The immersed leaves \( \iota : \iota \subset G \times M \) of \( \mathcal{F}_\xi \) are closed and thus are submanifolds.

(8) The charts \( j_g : M \to G M \) are morphisms of \( g \)-orbifolds, that is the vector fields \( \zeta(X) \) on \( M \) and \( \bar{\zeta}(X) \) on \( G M \), \( X \in g \) are \( j_g \)-related.

(9) The orbit spaces \( X = g \setminus M \) and \( G \setminus G M \) are homeomorphic, with the homeomorphism

\[
\tilde{j} : g \setminus M \cong G \setminus G M
\]

induced by the chart \( j_e : M \to G M \).

**Proof.** The above statements follow from the constructions in 2.6 to 2.11 \( \Box \)

We note that there exist \( g \)-actions which are not discretely valued (cf. [5], Examples 8). There exist discretely valued \( g \)-actions, such that \( j_e : M \to G M \) is not injective, with the fibers \( I(e; e, x) \subset M \) being finite or countably infinite (cf. [5], Examples 4). There are examples of discretely valued \( g \)-actions, for which \( G \) is a non–Hausdorff manifold. The Hausdorff quotient of \( \tilde{G} M \) may or may not be a smooth manifold (cf. [4]).

**3.3. The \( \tilde{G} \)-completion of a \( g \)-manifold.** We say that \( \tilde{G} M \) is the completion of the \( g \)-manifold \( (M, \zeta) \), if the \( g \)-action is discretely valued, that is \( (M, \zeta) \) is orbifold–like, with respect to \( \tilde{G} \). Then \( \tilde{G} M \) is a smooth orbifold, possibly with a limited failure of the Hausdorff separation property for pairs of limit points, as described in 2.9 and Proposition 2.10.1. In this case, we omit the reference to \( \tilde{G} \) in Definition 3.2.

Recall that \( G = \tilde{G} / Z \), where \( Z = \pi_1(G) \subset \tilde{G} \) is a discrete central subgroup of the simply connected Lie group \( \tilde{G} \) with Lie algebra \( g \). Almost from the definition of \( G M = G \times_g M \), it follows that we have a canonical equivalence of identification spaces

\[
G M = G \times_g M \cong Z \setminus \tilde{G} M
\]

that is \( G M \) is the orbit space of the left action of \( Z \) on the \( \tilde{G} \)-space \( \tilde{G} M \). As is true for coverings, the units in the fundamental groupoid are related by an exact sequence

\[
0 \to \Gamma_{\mathcal{F}_\xi} (\hat{e}; \hat{e}, x) \to \Gamma_{\mathcal{F}_\xi} (e; e, x) \to Z \to 0
\]

which results in a corresponding relation between the \( \Gamma_{\mathcal{F}_\xi} \)-orbits, namely the intersection sets \( I_G(e; e, x) = \bigcup_{z \in Z} I_G(\hat{e}; z, x) \).

\[
I_G(e; e, x) = \bigcup_{z \in Z} I_G(\hat{e}; z, x).
\]
3.4. Proposition.

(1) If $\tilde{G}M$ is an orbifold, then $G\tilde{M} = Z \setminus \tilde{G}M$ is an orbifold, if and only if $Z = \pi_1(G)$ acts properly discontinuously on $\tilde{G}M$.

(2) If $\tilde{G}M$ is an orbifold, then $G\tilde{M}$ is an orbifold.

(3) If $\tilde{G}M$ is a manifold, then $G\tilde{M} = Z \setminus \tilde{G}M$ is a manifold, if and only if $Z = \pi_1(G)$ acts properly discontinuously and without fixed points on $\tilde{G}M$.

(4) If $G\tilde{M} = Z \setminus \tilde{G}M$ is a manifold, then $\tilde{G}M$ is a manifold and the canonical projection $\pi: \tilde{G}M \to G\tilde{M} \cong Z \setminus \tilde{G}M$ is a covering map.

(5) If $G\tilde{M} = Z \setminus \tilde{G}M$ is Hausdorff, then $\tilde{G}M$ is Hausdorff.

Proof. (1) to (3) follow from (3.3.1) and (3.3.2). (4) follows from the fact that the charts $j_g : M \to \tilde{G}M$ and $j_{2g} : M \to G\tilde{M}$ are related by $j_{2g} = \pi \circ j_g$. (5) follows from the fact that the canonical projection $\pi$ is an identification map and hence open.

Thus the finite valuation condition, respectively the orbifold–like condition is the weakest condition to impose for a $\mathfrak{g}$–manifold. From the preceding construction, it is easy to produce examples where $\tilde{G}M$ is a manifold, but $G\tilde{M}$ is not Hausdorff and not locally euclidean.

3.5. Complete $\mathfrak{g}$–actions. If the $\mathfrak{g}$–action $\zeta : \mathfrak{g} \to \mathcal{X}(M)$ is given by complete vector fields on $M$, then the $\mathfrak{g}$–action integrates to a $G$–action on $M$, see [11]. More precisely, we have then $W_x = G$, $\forall x \in M$, the projection $\psi_1$ in diagram (2.12) is a universal covering map and $F_\zeta$ determines a generalized flat bundle

$$G \times M \overset{\tilde{\psi}}{\cong} \tilde{G} \times M \overset{pr_1}{\to} G$$

as a fiber space over $G$. Here $\tilde{\psi}(g,x) = [g, gx]$ is the diffeomorphism induced by $\tilde{\psi} : \tilde{G} \times M \cong \tilde{G} \times M$, $\tilde{\psi}(g,x) = (g, gx)$. Note that $\tilde{\psi}$ transforms the right diagonal action $(g', x) \cdot g = (g'g, g^{-1}x)$ into the right action $(g', x)g = (g'g, x)$ and the left action $g(g', x) = (gg', x)$ into the left diagonal action $g \cdot (g', x) = (gg', gx)$. Then we have from (3.3.1)

$$G\tilde{M} = G \times \mathfrak{g} M \cong G \times \tilde{G} M \cong Z \setminus M$$

as $G$–spaces. The recurrence sets $I(g; g, x) = I(e; e, x) = \{ Zx \}$ are exactly the $Z$–orbits of $x$ and $G\tilde{M}$ is a manifold if and only if $Z = \pi_1(G)$ acts properly discontinuously and without fixed points on $M$. The foliation $F_\zeta$ is induced by the submersion $pr_1 \circ \tilde{\psi} : \tilde{G} \times M \to \tilde{G}$, and the $G$–action on $G\tilde{M}$ is induced by the $G$–action on $M$, that is by the composition $\tilde{\mu} = pr_2 \circ \tilde{\psi} : \tilde{G} \times M \to M$. For $G = \tilde{G}$, the flat bundle (2.112) is a product and (1), (2) take the form

$$\tilde{G} \times M \overset{\tilde{\psi}}{\cong} \tilde{G} \times M \overset{pr_1}{\to} \tilde{G} \implies \tilde{G} \tilde{M} \cong M$$

in agreement with (3.3.1). Then the intersection sets $I(e; g, x) = I(g^{-1}; e, x) = \{ y \}$ are singleton sets and the $G$–action is uniquely determined by $gx = y$, that is $(e, y) \in L_\zeta(g, x)$. The preceding statements hold in particular if $M$ is compact and $\mathcal{X}_c(M) = \mathcal{X}(M)$.
3.6. Example. The following example illustrates how orbifolds occur naturally during the completion process.

Let $g = \mathbb{R}$ with basis $X = \frac{d}{dt}$, let $M_2 = \{ z \in \mathbb{C} : |z| < 2 \}$ be the open disc of radius 2, and let $\zeta : g \to \mathcal{X}(M)$ be given by the complete vector field

$$\zeta_X(z) = \frac{2\pi n}{n} z \frac{\partial}{\partial \theta}, \quad z = r e^{2\pi i \theta}, \quad n \in \mathbb{N}, \quad n > 2,$$

with integral curves $z(t) = z_0 e^{\frac{2\pi n}{n} t}, \quad z_0 = r_0 e^{2\pi i \theta_0}, \quad z_0 \in M_2$. As $\zeta_X$ is complete on $M_2$, we have $g_{(1)}M_2 \cong M_2$ and $g_{(2)}M_2 \cong \mathbb{Z}/M_2$ from (3.3.5.1). For $S^1 \cong \mathbb{R}/\mathbb{Z} \subset \mathbb{C}$ the unit circle, $S^1_{(n)} = \mathbb{R}/n\mathbb{Z} \to S^1$ the $n$-fold covering determined by $z \mapsto z^n$ and $\Gamma = \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ the cyclic group of order $n > 2$, we have moreover from (3.5.1)

$$S^1 \times M_2 \cong \mathbb{R} \times M_2 \cong S^1_{(n)} \times \Gamma \to S^1.$$

It follows that the foliation $\mathcal{F}_\zeta$ on $S^1 \times M_2$ is determined by the flow lines in the mapping torus of $f = e^{\frac{2\pi n}{n}} : M_2 \to M_2$ and that the completion $\mathcal{F}_{\gamma_{(1)}}M_2 \cong \Gamma \backslash M_2$ is the standard orbifold. The leaves of the graph foliation on $S^1 \times M_2$ are closed, parametrized by $S^1_{(n)}$ away from the origin in $M_2$, while the leaf at $0 \in M_2$ is parametrized by $S^1$. The orbit space $M_2/g$ is given by the half open interval $[0, 2]$.

To make $\zeta_X$ incomplete, we remove the relatively closed annular wedge $A = \{(r, \theta) : 1 \leq r < 2, 0 \leq |\theta| \leq \frac{1}{n} \}$ of width $\frac{2}{n}$ around the $x$-axis from the open disc $M_2$, that is we set $M = M_2 \backslash A$. For $0 \leq r_0 < 1$, the integral curves (orbits) are defined for all values of the parameter $t$, while for $1 \leq r_0 < 2$, the integral curves are defined only for $1 < |t + r_0 n \theta| < n - 1, \quad z_0 \in M$. Since $S^1 \cong \mathbb{R}/\mathbb{Z}$ via the exponential map, we have $g_{(1)}M \cong \mathbb{Z}/M$ from (3.3.1), and it is sufficient to determine the $\mathbb{R}$-completion $\mathcal{F}_\zeta$ with the induced $\mathbb{R}$-action. By construction of the graph foliation $\mathcal{F}_\zeta$ in (2.1.2), the leaves of $\mathcal{F}_\zeta$ on $\mathbb{R} \times M$ are determined explicitly as follows. For a regular curve $\varphi = c(t)$ in $\mathbb{R}$ starting at $\varphi_0 = c(0)$, we have $\dot{\varphi}(t) = \dot{c}(t) X$, $c(t)>0$ and the lifted curve $(c(t), z(t))$ is in the leaf $L(c_0, z_0)$, $z_0 \in M$, if and only if it satisfies the first order ODE $(z(t), \dot{z}(t)) = \varphi(t) \zeta_X(z(t))$ with initial value $z(0) = z_0 = (r_0, \theta_0) \in M$. Substituting (1) into this equation, we obtain the linear equation $\dot{z}(t) = \frac{2\pi n}{n} \varphi(t) z(t)$ and therefore

$$z(\varphi) = z_0 e^{\frac{2\pi n}{n}(c(t) - \varphi_0)} = r_0 e^{\frac{2\pi n}{n}(\varphi(c(t) - \varphi_0))}.$$

This is independent of the parametrization $\varphi = c(t)$, except for initial values, so we may take $\varphi = t + \varphi_0$. For $0 \leq r_0 < 1$, the projection

$$pr_1 : L(\varphi_0, z_0) \to W(\varphi_0, z_0) = \mathbb{R}, \quad 0 \leq r_0 < 1$$

is a diffeomorphism for all values of $\varphi_0$.

The orbits of the $g$-action are determined by the leaf structure via $pr_2$ in diagram (2.1.2) and they look as follows: The point $0 \in M \subset \mathbb{C}$ is a fixed point. For $0 < r_0 < 1$, the orbits are circles, parametrized by $S^1_{(n)} = \mathbb{R}/n\mathbb{Z}$ as in the complete case and the orbit space of the open disc $M_1 = \{ z \in \mathbb{C} : |z| < 1 \} \subset M$ is the half open interval $[0, 1]$. In other words, the foliation $\mathcal{F}_\zeta$ on $S^1 \times M_1$ is again given by the mapping torus of $f = e^{\frac{2\pi n}{n}} : M_1 \to M_1$ as in (2), the $\mathbb{R}$-completion $g_{(1)}M_1 \cong M_1$ and the $S^1$-completion $g_{(2)}M_1 \cong \Gamma \backslash M_1$ is again the standard orbifold.

Incompleteness occurs whenever the integral curve $z(\varphi)$ approaches the wedge $A \subset M_2$ in finite time, that is $z(\varphi) \to (r_0, \pm \frac{1}{n}), \quad 1 \leq r_0 < 2$, or equivalently
\[ \varphi - (\varphi_0 - n\theta_0) \downarrow 1 \], respectively \[ \varphi - (\varphi_0 - n\theta_0) \uparrow n - 1 \]. It follows that the leaf \[ L(\varphi_0, z_0) \] is parametrized by \[ \varphi - (\varphi_0 - n\theta_0) \in (1, n-1) \] and that

\[ (5) \quad \text{pr}_1 : L(\varphi_0, z_0) \rightarrow W_{(\varphi_0, z_0)} = \varphi_0 + W_{z_0}, \quad W_{z_0} = n(\frac{1}{n} - \theta_0, 1 - \frac{1}{n} - \theta_0) \subset \mathbb{R}, \]

given by \((\varphi, z(\varphi)) \mapsto \varphi\), is a diffeomorphism for \(1 \leq r_0 < 2\), \(1 < n\theta_0 < n-1\). The intervals \(W_{z_0}\) of length \(n-2\) range from \((0, n-2)\) to \((2-n, 0)\), with \((1 - \frac{k}{n}, \frac{k}{n})\) for \(\theta_0 = \frac{k}{n}\). This reflects on the fact that the cut–open circles are intervals and therefore simply connected; the completion process simply adds in copies of \(M\), rotated by the parameter \(\varphi\). For \(1 \leq r_0 < 2\), we parametrize the space of leaves \(\tilde{\mathbb{R}}(M\setminus M_1)\) by eliminating the parameter \(\theta_0\). From (5), we see that \(L(\varphi_0', z_0') = L(\varphi_0, z_0)\), if and only if \(r_0' = r_0\), \(\varphi_0' - n\theta_0' = \varphi_0 - n\theta_0\).

Setting \(\theta_0' = \frac{1}{2}\), it follows from \((2,13)\) that

\[ (6) \quad L(\varphi_0, z_0) = L(\varphi_0' + \frac{n}{2}, (r_0, \frac{1}{2})) = \varphi_0' + L(\frac{n}{2}, (r_0, \frac{1}{2})); \quad \varphi_0' = \varphi_0 - n\theta_0. \]

Therefore the leaves of the form \(L(\varphi_0 + \frac{n}{2}, (r_0, \frac{1}{2})) = \varphi_0 + L(\frac{n}{2}, (r_0, \frac{1}{2}))\) are distinct for different values of \(\phi_0 \in \mathbb{R}\) and \(r_0 \in [1,2]\), that is \(\tilde{G} = \mathbb{R}\) acts without isotropy on \(\mathbb{R}(M\setminus M_1)\). From (6), we see that the \(\mathbb{R}\)-completion \(\mathbb{R}(M\setminus M_1)\) has a section over the orbit space \(\mathbb{R}(M\setminus M_1)/\tilde{g}\), given by \(r_0 \mapsto L(\frac{n}{2}, (r_0, \frac{1}{2}))\). Therefore \(\mathbb{R}(M\setminus M_1) \cong \mathbb{R} \times (\mathbb{R}(M\setminus M_1)/\tilde{g}) \cong \mathbb{R} \times [1,2]\) and \(\mathbb{S}(M\setminus M_1) \cong S^1 \times [1,2]\). This is consistent with \(r_0 \uparrow 1\) and thus the completions are given by \(\mathbb{R}M \cong \mathbb{R}M_2 \cong M_2\) and \(\mathbb{S}M \cong S^1M_2 \cong \Gamma\setminus M_2\) with orbit space \([0,2]\).

Note that the \(\mathbb{R}\)- and the \(S^1\)-completion above are Hausdorff and essentially consist in completing the orbits in the gap created by the removed wedges in the annulus and then dividing by the discrete central subgroup \(\Gamma = \mathbb{Z}_n\). This is due to the fact that the incomplete orbits in this example are still connected. If we remove in addition the half-open interval \(1 \leq r < 2\), \(\theta = \pi\), the \(\mathbb{R}\)- and the \(S^1\)-completion of the resulting manifold will not be Hausdorff anymore, but the \(S^1\)-completion will retain the orbifold structure at the center.

4. Proper \(\mathfrak{g}\)-manifolds

Let \((M, \zeta)\) be an orbifold–like \(\mathfrak{g}\)-manifold.

4.1. Definition. The orbifold–like \(\mathfrak{g}\)-manifold \((M, \zeta)\) is proper if the \(\tilde{G}\)-action on the generated \(\tilde{G}\)-manifold \(\tilde{\math parity}\) is proper, that is the mapping

\[ \tilde{\mu} : \tilde{G} \times \tilde{\mathfrak{g}}M \rightarrow \tilde{\mathfrak{g}}M \times \tilde{\mathfrak{g}}M, \]

given by \(\mu(g, z) = (gz, z)\) is proper in the sense that compact sets have compact inverse images (no assumption of Hausdorff).

4.2. Sequential Definition. The \(\tilde{G}\)-action on \(\tilde{\mathfrak{g}}M\) is proper, iff the following holds. Given sequences \(\{g_n\}\), \(\{z_n\}\) in \(\tilde{G}\), respectively \(\tilde{\mathfrak{g}}M\), such that the sequences \(\{z_n\}\), \(\{g_nz_n\}\) converge to \(z, \tilde{z}\) in \(\tilde{\mathfrak{g}}M\), the sequence \(\{g_n\}\) in \(\tilde{G}\) has a convergent subsequence. We will see below that the sequential definition involves convergent sequences only inside the charts \(j_g : M \rightarrow V_g \subset \tilde{\mathfrak{g}}M\), which are Hausdorff as a consequence of our assumption.

4.3. Proposition. The orbifold–like \(\mathfrak{g}\)-manifold \((M, \zeta)\) is proper, if and only if the following condition is satisfied :
For convergent sequences \( \{x_n\} \to x \), \( \{y_n\} \to y \) in \( M \), a sequence \( \{g_n \in \tilde{G}\} \) has a convergent subsequence \( g \in \tilde{G} \), provided that the following equivalent properties hold:

1. \((e, y_n) \in L_\zeta(g_n, x_n) = g_nL_\zeta(e, x_n) \), or \((g_n, x_n) \in L_\zeta(e, y_n) \),

2. \( y_n \in I(e; gn, x_n) \), or \( x_n \in I(gn; e, y_n) = I(e; g^{-1}n, y_n) \).

**Proof.** Without loss of generality, we may assume that \( z = [e, x] \in V_e \subseteq \tilde{G}M \). Then \( z_n \in V_e \), that is the leaves represented by \( z_n \) intersect the transversal \( i_e(M) \), at least for sufficiently large \( n \), and we may write \( z_n = [e, x_n] \), such that \( \{x_n\} \to x \) is convergent in \( M \). The same applies to the sequence \( \{y_n\} \to \tilde{z} \). Write \( \tilde{z} = [g_0, y] \in V_{g_0} \), \( [e, y] \in V \subseteq \tilde{G}M \). Then \( g_n z_n \in V_{g_0} \), that is the leaves represented by \( g_n z_n \) intersect the transversal \( i_{g_0}(M) \), at least for sufficiently large \( n \), and we may write \( g_n z_n = [g_0, y_n] = g_0[e, y_n] \), such that \( \{y_n\} \to y \) is convergent in \( M \). Finally, \( g_n z_n \) may be written as \( g_n z_n = [g_0, y_n] = g_n[e, x_n] = [g_n, x_n] \), that is \( (g_0, y_n) \in L_\zeta(g_n, x_n) \), or equivalently \( (g_n, x_n) \in L_\zeta(g_0, y_n) \). In terms of intersection sets, this may also be formulated as \( y_n \in I(g_0; g_n, x_n) = I(e; g_0^{-1}n, x_n) \), or equivalently \( x_n \in I(g_n; g_0, y_n) = I(e; g^{-1}n, g_0, y_n) \). Therefore the properness of the \( g \)-manifold \( M \) may be reformulated in terms of sequences as follows:

If \( \{x_n\} \to x \), \( \{y_n\} \to y \) are convergent sequences in \( M \) and \( \{g_n\} \) is a sequence in \( \tilde{G} \), then \( \{g_n\} \) has a convergent subsequence \( g \in \tilde{G} \), provided that the following equivalent properties hold:

3. There exists \( g_0 \in \tilde{G} \), such that \( (g_0, y_n) \in L_\zeta(g_n, x_n) \), or \( (g_n, x_n) \in L_\zeta(g_0, y_n) \),

4. There exists \( g_0 \in \tilde{G} \), such that \( y_n \in I(g_0; g_n, x_n) = I(e; g_0^{-1}n, x_n) \), or \( x_n \in I(g_n; g_0, y_n) = I(e; g^{-1}n, g_0, y_n) \).

For any convergent subsequence of \( \{g_n\} \to g \), we have then

\[ \{g_n z_n = [g_0, y_n] = [g_n, x_n]\} \to \tilde{z} = [g_0, y] = [g, x] = g \cdot [e, x] = g z . \]

The necessity of the conditions in Proposition 4.3 is now clear by taking \( g_0 = e \). Conversely, if the convergent sequence \( \{g_n z_n = [g_0, y_n]\} \) is of the form \( \{g_n z_n = [g_0, y_n]\} \), the modified sequence \( \{g'_n = g_0^{-1}g_0\} \) satisfies \( \{g'_n z_n = g_0^{-1}(g_n z_n) = [e, y_n]\} \) and hence the conditions in Proposition 4.3. Therefore it has a convergent subsequence \( \{g'_n\} \to g' \). Then the convergent subsequence \( \{g_n = g_0 g'_n\} \to g = g_0 g' \) has the required property.

Since a proper \( g \)-action on \( M \) is discretely valued, the Hausdorff separation property holds inside the open sets \( V_g \subseteq \tilde{G}M \) and can fail only for pairs of limit elements by Proposition 2.10.

**4.4. Lemma.**

1. If the \( g \)-action on \( M \) is proper, then \( \tilde{G}M \) is a proper \( H \)-manifold for any closed subgroup \( H \subseteq \tilde{G} \).

2. If the \( g \)-action on \( M \) is proper, and \( \Gamma \subseteq \tilde{G} \) is a discrete subgroup, then \( \Gamma \) acts properly discontinuously on \( \tilde{G}M \) and the isotropy groups \( \Gamma_z \) at \( z \in \tilde{G}M \) are finite.

3. If the \( g \)-action on \( M \) is proper, then the isotropy groups \( G_z \) at \( z = [g, x] \in \tilde{G}M \) are compact.

**Proof.** (1) and (2) are immediate. (3) follows from a familiar argument. For any sequence \( \{g_n\} \) in \( G_z \), we have \( g_n [g, x] = [g_n g, x] = [g, x] \) and therefore \( \{g_n\} \) must
have an accumulation point in $G$, which must be in $G_z$ since $G_z \subset \tilde{G}$ is closed. Equivalently, $G_z$ is given by $G_z \cong G_z \times \{z\} = \tilde{\mu}^{-1}(z, z)$, which is compact. □

**4.5. Proposition.** Let $(M, \zeta)$ be an orbifold–like $g$–manifold. Then the following holds:

1. If the $g$–action on $M$ is proper, then $\tilde{g}M$ is a smooth orbifold, the discrete central subgroup $Z = \pi_1(G) \subset \tilde{G}$ acts properly discontinuously on $\tilde{g}M$ and the orbit space $\tilde{g}M = G \times_g M \cong Z \setminus G M$ is a proper $G$–orbifold admitting $\tilde{g}M \to \tilde{G}M = Z \setminus \tilde{G}M$ as a (ramified) Galois covering.

2. Conversely, if the $G$–action on $\tilde{g}M$ is proper, then the $g$–action on $M$, that is the $\tilde{G}$–action on $\tilde{g}M$ is proper, if and only if $Z = \pi_1(G) \subset \tilde{G}$ acts properly discontinuously on $\tilde{g}M$.

**Proof.** (1) follows from (3.3.1), since $Z \subset \tilde{G}$ is a discrete central subgroup. (2) follows also from (3.3.1) by a similar argument. □

Thus the properness condition is strongest for the $\tilde{G}$–orbifold $\tilde{g}M$.

**4.6. The slice theorem for proper $g$–manifolds.** We recall the slice theorem for proper $G$–actions in a form convenient for our purposes cf.

For any $x \in M$, there exists a submanifold (slice) $x \in S_x \subset M$, satisfying the following properties:

1. $G_x \cdot S_x \subset S_x$, that is $S_x$ is invariant under the action of the stabilizer $G_x \subset G$ of $G$ at $x$;

2. There exists a $G$–equivariant isomorphism $G \times_{G_x} S_x \cong G \cdot S_x$, such that the diagram

$$
\begin{array}{ccc}
G \times_{G_x} S_x & \cong & G \cdot S_x \\
\downarrow \pi & & \downarrow \\
G / G_x & \cong & G(x)
\end{array}
$$

is commutative and the tube $G \cdot S_x \subset M$ is open in $M$;

3. If $g \cdot S_x \cap S_x \neq \emptyset$ for $g \in G$, then $g \in G_x$.

Note that (4) is a consequence of (1) and (2).

**4.7. Theorem.** Let $(M, \zeta^M)$ be a proper $g$–manifold. For any $x \in M$, there exists a submanifold (slice) $S_x \subset M$, satisfying the following properties:

1. $\zeta(g_x) \subset \mathfrak{X}(S_x)$;
2. $\zeta(g)(y) + T_y(S_x) = T_y(M)$, $y \in S_x$;
3. $\zeta(g)(x) + T_x(S_x) = T_x(M)$;
4. The tube $\Gamma(g)(S_x) \subset M$ is open in $M$;
5. If $X \in g$ and $\zeta_X(y) \in T_y(S_x)$ for some $y \in S_x$, then $X \in g_x$.

**Proof.** Since $\tilde{g}M = \tilde{G} \times_g M$ is a proper $\tilde{G}$–manifold, we may invoke the slice theorem for a proper $\tilde{G}$–action and obtain the above slice decomposition (4.6.1) to (4.6.4) for $\tilde{g}M$. This implies the infinitesimal properties (1) to (5) for $\tilde{g}M$. For
$x \in M$, we may choose the $G_{j(x)}$–invariant slice $\tilde{S}_{j(x)} \subset V$, since the stabilizer subgroup $G_{j(x)}$ is compact. Since the chart $j : M \to V \subset \tilde{G}M$ is a $\mathfrak{g}$–equivariant local diffeomorphism onto $V$, we obtain a $\mathfrak{g}_{\mathfrak{e}}$–invariant slice $S_x \subset M$ satisfying the infinitesimal properties (1) to (5). \hfill \Box

4.8. Existence of $\mathfrak{g}$–invariant metrics. Given a Riemannian metric $h$ on $M$, the following are equivalent:

(1) $\zeta : \mathfrak{g} \to \mathfrak{X}(M)$ takes values in the Lie algebra of Killing fields of $h$, that is $\zeta : \mathfrak{g} \to \mathfrak{K}_h(M) = \{X \in \mathfrak{X}(M) \mid L_X h = 0\}$;

(2) The normal bundle $Q_{\zeta}$ of $\mathcal{F}_\zeta$ admits a $G$–invariant metric $h_Q$, such that $L_\xi h_Q = 0$, $\forall \xi \in C^\infty(T\mathcal{F}_\zeta)$;

(3) The holonomy pseudogroup of $\mathcal{F}_\zeta$ acts by local isometries on $(M, h)$.

The second statement says of course that the $G$–equivariant foliation $(\mathcal{F}_\zeta, h_Q)$ is a $(G$–equivariant) Riemannian foliation.

The Riemannian metric $h$ on $M$ is complete, if and only if $(\mathcal{F}_\zeta, h_Q)$ is transversally complete.

It is well known that the existence of a $G$–invariant Riemannian metric implies the properness of a $G$–action on a manifold $M$. In our context this can be formulated as follows.

4.9. Proposition. Suppose that the $\mathfrak{g}$–action $\zeta$ on $M$ is effective and that $G\cdot M = G \times \mathfrak{g} M$ satisfies the Hausdorff property. If there exists a (complete) Riemannian metric $h$ on $M$, such that the injective Lie homomorphism $\zeta : \mathfrak{g} \to \mathfrak{X}(M)$ consists of Killing fields relative to $h$, then there exists a (complete) Riemannian metric $\tilde{h}$ on the leaf space $\tilde{G}M = G \times \mathfrak{g} M$, such that the following conditions hold:

1. $G$ acts by isometries on the leaf space $\tilde{G}M = G \times \mathfrak{g} M$, that is $G \subseteq \text{Iso}_h(\tilde{G}M)$;

2. The charts $j_g : M \to \tilde{G}M$ are isometries of $\mathfrak{g}$–manifolds.

3. The action of the closure $\tilde{G} \subseteq \text{Iso}_h(\tilde{G}M)$ of $G \subseteq \text{Iso}_h(\tilde{G}M)$ is proper.

4.10. Theorem. If the $\mathfrak{g}$–action $\zeta$ on $M$ is proper, then there exist (complete) Riemannian metrics $h$ on $M$ and $\tilde{h}$ on $\tilde{G}M = G \times \mathfrak{g} M$, such that the following conditions hold:

1. $\tilde{G}$ acts by isometries on the leaf space $\tilde{G}M = G \times \mathfrak{g} M$, that is $\tilde{G} \to \text{Iso}_h(\tilde{G}M)$;

2. The charts $j_g : M \to \tilde{G}M$ are isometries of $\mathfrak{g}$–manifolds;

3. $\zeta : \mathfrak{g} \to \mathfrak{K}_{\tilde{h}}(\tilde{G}M) \subset \mathfrak{X}(M)$.

Proof. The proof follows the by now familiar pattern by working on the proper $\tilde{G}$–manifold $\tilde{G}M = G \times \mathfrak{g} M$. From the slice theorem for proper $\tilde{G}$–actions we know that there exists a $\tilde{G}$–invariant Riemannian metric $\tilde{h}$ on $\tilde{G}M$, so that (1) is satisfied. Pulling back $\tilde{h}$ to a metric on $h = j^*\tilde{h}$ on $M$ via the local diffeomorphism $j : M \to V \subset \tilde{G}M$, property (2) is satisfied by (1) and the transition formula (2.32), that is $j_g = L_g \circ j_e = L_g \circ j$. (3) is a direct consequence of (1) and (2). \hfill \Box

4.11. Corollary. If $G\cdot M \cong Z \setminus G\cdot M$ is a manifold and the $G$–action on $G\cdot M$ is proper, then $\tilde{G}M$ is a proper $\tilde{G}$–manifold, that is the $\mathfrak{g}$–action on $M$ is proper.

4.12. The pseudogroup of local isometries.
5. $g$--vector bundles and equivariant vector bundles

5.1. For a $G$--equivariant vector bundle $E \xrightarrow{\pi} M$, the associated infinitesimal action $X \mapsto X^*$, $X \in g$ is projectable to the corresponding vector field $X^*$ determined by the $G$--action on $M$. Further, since the action $\ell_g : E_u \to E_{\ell_g(u)}$ is linear, the vertical component of the vector field $\tilde{X}^*$ with respect to a local trivialization of $E$ is a linear vector field on $E$. This notion is clearly independent of the choice of local trivialization. Thus we define the Lie algebra $X_{proj}(E)$ as the Lie algebra of vector fields $Y$ on $E$ satisfying the following conditions:

1. $Y$ is $\pi$--projectable;
2. The vertical component of $Y$ is a linear vector field.

Let now $(M,\zeta)$ be a $g$--manifold. A $g$--structure on a vector bundle $E \xrightarrow{\pi} M$ is given by a Lie algebra homomorphism $\tilde{\zeta} : g \xrightarrow{\pi} X_{proj}(E)$, such that $\tilde{\zeta}$ lifts the $g$--structure on $M$, that is the following diagram is commutative

$$
\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{\zeta} & X(M) \\
\downarrow & \searrow & \downarrow \\
X_{proj}(E) & \xrightarrow{\tilde{\zeta}} & \mathfrak{g}
\end{array}
$$

It is clear that the previous constructions, in particular 2.1, 2.3 apply to this situation as well. In particular, we obtain a $G$--equivariant foliation $\tilde{F}_{\tilde{\zeta}}$ on $G \times E$ which projects under $\tilde{\pi} = \text{id} \times \pi$ to $G \times M$, that is, $E$ carries the structure of a foliated vector bundle in the sense of [3].

5.2. Proposition. Let $(\tilde{\zeta}, E)$ be a $g$--structure on the vector bundle $E \xrightarrow{\pi} M$, satisfying condition (5.1.3) above. Then

1. There is a $G$--equivariant foliation $\tilde{F}_{\tilde{\zeta}}$ on $G \times E$ which projects under $\tilde{\pi} = \text{id} \times \pi$ to $G \times M$. This defines a $G$--equivariant foliated bundle

$$(G \times E, \tilde{F}_{\tilde{\zeta}}) \xrightarrow{\tilde{\pi}} (G \times M, F_{\tilde{\zeta}}).$$

2. The $G$--completions

$$GE \xrightarrow{\alpha_{\pi}} GM$$

define a $G$--equivariant vector bundle.

REFERENCES

[1] D. V. Alekseevsky and Peter W. Michor. Differential Geometry of g-manifolds. Differ. Geom. Appl., 5:371–403, 1995.
[2] Élie Cartan. La géométrie des espaces de Riemann. Gauthier-Villars, Paris, 1925. Mémorial des Sciences Math., Vol 9.
[3] A. Frölicher and A. Nijenhuis. Theory of vector valued differential forms. Part I. Indagationes Math, 18:338–359, 1956.
[4] Franz W. Kamber and Peter W. Michor. The flow completion of a manifold with vector field. Electron. Res. Announc. Amer. Math. Soc., 6:95–97, 2000. math.DG/0007173
[5] Franz W. Kamber and Peter W. Michor. Completing Lie algebra actions to Lie group actions. Electron. Res. Announc. Amer. Math. Soc., 10:1–10, 2004. math.DG/0310398
Franz W. Kamber and Philippe Tondeur. *Foliated bundles and characteristic classes*. Springer-Verlag, Heidelberg, Berlin, New York, 1975. Lecture Notes in Math., Vol 493.

S. Kobayashi and K. Nomizu. *Foundations of Differential Geometry. Vol. I. Vol. II*. J. Wiley - Interscience, 1963, 1969.

Ivan Kolář, Jan Slovák, and Peter W. Michor. *Natural operators in differential geometry*. Springer-Verlag, Heidelberg, Berlin, New York, 1993.

Ivan Kolář, Jan Slovák, and Peter W. Michor. *Natural operators in differential geometry*. Springer-Verlag, Heidelberg, Berlin, New York, 1993.

Andreas Kriegl and Peter W. Michor. *The Convenient Setting for Global Analysis*. AMS, Providence, 1997. ‘Surveys and Monographs 53’.

R. Palais. On the existence of slices for actions of non-compact Lie groups. *Ann. of Math. (2)*, 73:295–323, 1961.

Richard S. Palais. A global formulation of the Lie theory of transformation groups. *Mem. AMS*, 22, 1957.