On the level density of spin chains of Haldane–Shastry type

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We provide a rigorous proof of the fact that the level density of all \( su(m) \) spin chains of Haldane–Shastry type associated with the \( A_{N-1} \) root system approaches a Gaussian distribution as the number of spins \( N \) tends to infinity. Our approach is based on the study of the large \( N \) limit of the characteristic function of the level density, using the description of the spectrum in terms of motifs and the asymptotic behavior of the dispersion relation.

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A well-known conjecture on spin chains of Haldane–Shastry (HS) type \([1,10]\) states that their level density becomes Gaussian as the number of sites tends to infinity. Although this conjecture has been numerically verified for all chains of HS type whose spectrum has been computed in closed form \([8,9,11–17]\), a rigorous proof thereof is lacking, except in a few exceptional cases in which the partition function factorizes \([18]\). In this paper we settle the conjecture in the affirmative for spin \( \pi \)-s, \( \pi /2 \)-s, and \( \pi /4 \)-s, respectively.

Our result has implications in connection with two fundamental conjectures in the theory of quantum chaos that we shall now discuss. The first of these conjectures, due to Berry and Tabor \([19]\), asserts that the probability density of spacings between consecutive levels in the spectrum of a quantum system whose classical analog is integrable follows Poisson’s law \( p(s) = e^s \). The second conjecture, formulated by Bohigas, Giannoni, and Schmidt \([20]\), posits that for a fully chaotic quantum system this density is instead given by Wigner’s surmise \( p(s) = (\pi s/2) \exp(-\pi s^2/4) \), characteristic of the Gaussian orthogonal ensemble (GOE) in random matrix theory \([21]\).

Let us briefly recall the definition of the three spin chains we shall deal with in what follows. The Hamiltonian of the original \( su(m) \) Haldane–Shastry chain is defined as

\[
H = \frac{1}{2} \sum_{i<j} \frac{1 - \varepsilon S_{ij}}{\sin^2((\xi_i - \xi_j)/2)}, \quad \xi_k = \frac{k\pi}{N},
\]

where (as always hereafter, unless otherwise stated) the sum runs from 1 to the number of spins \( N \), and \( \varepsilon = 1 \) (resp. \( \varepsilon = -1 \)) for the ferromagnetic (resp. antiferromagnetic) chain. The operator \( S_{i,j} \) permutes the \( i \)th and \( j \)th spins, i.e., its action on an element \( |s_1, \ldots, s_i, \ldots, s_j, \ldots, s_N \rangle \) \((s_k \in \{1, \ldots, m\})\) of the canonical spin basis is given by

\[
S_{ij} | s_1, \ldots, s_i, \ldots, s_j, \ldots, s_N \rangle = | s_1, \ldots, s_j, \ldots, s_i, \ldots, s_N \rangle.
\]

The permutation operators \( S_{i,j} \) can be expressed in terms of the generators \( t^{ab}_k \) of the fundamental representation of \( su(m) \) at the \( k \)-th site as

\[
S_{ij} = 2 \sum_{a=1}^{m^2-1} t^{ab}_i t^{ba}_j + \frac{1}{m},
\]

with the normalization \( \text{tr}(t^{ab}_k t^{cb}_k) = \frac{1}{2} \delta^{ab} \). The chain \([1]\) is intimately connected with the Hubbard model. For instance, it can be obtained from the one-dimensional Hubbard model with long range hopping introduced in Ref. \([22]\) when the on-site interaction tends to infinity and the sites are half-filled. The rational version of the HS chain \([1]\) was subsequently introduced by Polychronakos \([3]\) and Frahm \([23]\). The Hamiltonian of the Polychronakos–Frahm (PF) chain can be taken as

\[
H = \sum_{i<j} \frac{1 - \varepsilon S_{ij}}{(|\xi_i - \xi_j|)^2},
\]

where the chain sites \( \xi_k \) are no longer equidistant, but are given by the zeros of the Hermite polynomial of degree \( N \). Finally, the hyperbolic version of the HS chain, known as

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\( a \in \{1, 2, \ldots, m\} \) and \( \eta \in \{1, 2, \ldots, m\} \) are the mean and standard deviation of the spectrum, the spacings density follows a characteristic distribution which is neither of Poisson’s nor of Wigner’s type \([9,11–17]\).
the Frahm–Inozemstov (FI) chain \([4]\), is defined by the Hamiltonian

\[
H = \frac{1}{2} \sum_{i<j} \frac{1 - \varepsilon S_{ij}}{\sinh^2(\xi_i - \xi_j)}.
\]

The numbers in this case are given by \(\xi_i = \frac{1}{2} \log \zeta_i\), where \(\zeta_i\) is the \(i\)th zero of the Laguerre polynomial \(L_{\alpha}^{\alpha-1}\) with \(\alpha > 0\). In particular, unlike the previous two chains, the sites of the FI chain depend on an essential parameter.

Each of the chains \([1]–[3]\) can be obtained from a corresponding spin dynamical model of Calogero–Sutherland type \([24, 26]\) by applying the so-called freezing trick \([3]\). As first shown by Polychronakos \([27]\), this connection can be exploited to derive closed-form expressions for the partition functions of the above chains. It turns out that these expressions can be rewritten in a remarkable unified way as \([11, 14, 17, 28]\)

\[
Z(q) = \sum_{\mathbf{k} \in \mathcal{P}_N} \prod_{i=1}^{r} d(k_i) \cdot q^{\sum_{i=1}^{r-1} \mathcal{F}(K_i)} \prod_{i=1}^{N-r} (1 - q^{\mathcal{F}(K'_i)}),
\]

where \(q \equiv e^{-1/(knT)}\), \(\mathbf{k} \equiv (k_1, \ldots, k_r)\) is an element of the set \(\mathcal{P}_N\) of partitions of \(N\) with order taken into account, and the spin degeneracy factor \(d(k_i)\) is given by

\[
d(k_i) = \begin{cases} 
\binom{m+k_i-1}{k_i}, & \varepsilon = 1 \\
\binom{m}{k_i}, & \varepsilon = -1.
\end{cases}
\]

The numbers \(K_i\) in Eq. \((4)\) are defined as \(K_i = \sum_{j=1}^{i} k_j \in \{1, \ldots, N - 1\}\), and \(\{K'_1, \ldots, K'_{N-r}\} = \{1, \ldots, N - 1\} \setminus \{K_1, \ldots, K_{r-1}\}\). The partition function \(Z\) depends on the chain under consideration only through its dispersion relation \(\mathcal{F}(i)\), given by

\[
\mathcal{F}(i) = \begin{cases} 
i(N - i), & \text{for the HS chain} \\
i, & \text{for the PF chain} \\
i(\alpha + i - 1), & \text{for the FI chain}.
\end{cases}
\]

Using Eq. \((4)\), Basu-Mallick et al. \([28, 29]\) derived a simple set of rules for generating the spectrum of the chains \([1]–[4]\) in terms of Young tableaux of certain irreducible representations of the Yangian \(Y(gl(m))\). It can be easily shown that these rules are equivalent to the explicit formula

\[
E_n = \sum_{i=1}^{N-1} \delta(n_i, n_{i+1}) \mathcal{F}(i), \quad n \equiv (n_1, \ldots, n_N),
\]

where the quantum numbers \(n_i\) independently take the values \(1, \ldots, m\). As to the function \(\delta\), it is given by

\[
\delta(j, k) = \begin{cases} 
1, & j < k \\
0, & j \geq k.
\end{cases}
\]

in the ferromagnetic case, whereas in the antiferromagnetic one it suffices to exchange 0 and 1 in Eq. \((7)\). The vectors \(\delta(n) \in \{0, 1\}^{N-1}\) with components \(\delta_i(n) = \delta(n_i, n_{i+1})\) are in fact the celebrated motifs introduced by Haldane et al. in Ref. \([30]\). It should be emphasized that the formula \((6)\) for the energies is obtained from the partition function and not vice versa, as is usually the case.

Equation \((6)\) shall be our starting point for establishing the asymptotically Gaussian character of the level density of the chains \([11]–[13]\). In fact, since the sum of the partition function equals 1 for \(n_{i+1} > n_i\), regardless of the values taken by the \(N - 2\) remaining quantum numbers \(n_k\), and is 0 otherwise. Thus

\[
\mu = m^{-N} \sum_{n_1, \ldots, n_N=1}^{m} \sum_{i=1}^{N-1} \delta(n_i, n_{i+1}) \mathcal{F}(i).
\]

By Eq. \((7)\), the coefficient of \(\mathcal{F}(i)\) in the previous expression equals 1 for \(n_{i+1} > n_i\), regardless of the values taken by the \(N - 2\) remaining quantum numbers \(n_k\), and is 0 otherwise. Thus

\[
\mu = m^{-N} \sum_{i=1}^{N-1} \left(\frac{m}{2}\right)^{N-2} \mathcal{F}(i) = \frac{1}{2} \left(1 - \frac{1}{m}\right) \sum_{i=1}^{N-1} \mathcal{F}(i).
\]

Similarly, the variance of the energy is given by \(\sigma^2 = \langle E_n^2 \rangle - \mu^2\), where

\[
\langle E_n^2 \rangle = m^{-N} \sum_{n_1, \ldots, n_N=1}^{m} E_n^2 \\
= m^{-N} \sum_{n_1, \ldots, n_N=1}^{m} \sum_{i,j=1}^{N-1} \delta_i(n)\delta_j(n) \mathcal{F}(i)\mathcal{F}(j).
\]

Taking into account that \(\delta_i^2 = \delta_i\) and proceeding as before we easily obtain

\[
\langle E_n^2 \rangle = \frac{1}{2} \left(1 - \frac{1}{m}\right) \sum_{i=1}^{N-1} \mathcal{F}(i)^2 \\
+ 2m^{-N} \sum_{n_1, \ldots, n_N=1}^{m} \sum_{i,j=1}^{N-1} \delta_i(n)\delta_j(n) \mathcal{F}(i)\mathcal{F}(j).
\]

If \(i < j - 1\), the coefficient of \(\mathcal{F}(i)\mathcal{F}(j)\) in the last sum equals 1 provided that \(n_i < n_{i+1}\) and \(n_j < n_{j+1}\), and is otherwise zero. Likewise, the coefficient of \(\mathcal{F}(j-1)\mathcal{F}(j)\) (\(j = 2, \ldots, N - 1\)) is 1 if \(n_{j-1} < n_j < n_{j+1}\), and vanishes
mean energy of the antiferromagnetic chains is given by

\[ \langle E_n^2 \rangle = \frac{1}{2} \left( 1 - \frac{1}{m} \right) \sum_{i=1}^{N-1} F(i)^2 \]

+ \frac{1}{2} m^{N-2} \sum_{i,j=1}^{N-1} \delta_{i,j} \frac{1}{2} m^{N-1} F(i) F(j)

+ \frac{1}{3} m^{N-3} \sum_{j=2}^{N-1} m^{N-1} F(j-1) F(j).

Using Eq. (8) for \( \mu \), after some straightforward algebra we obtain

\[ \sigma^2 = \left( 1 - \frac{1}{m} \right) \left[ \frac{1}{4} \sum_{i=1}^{N-1} F(i)^2 - \frac{1}{6} \sum_{i=2}^{N-1} F(i-1) F(i) \right]. \]

(9)

Since, up to an additive constant, the energies of the antiferromagnetic chains differ from those of their ferromagnetic counterparts by a sign change, it is clear that Eq. (9) is also valid in the antiferromagnetic case. As to Eq. (8), using the antiferromagnetic analog of Eq. (7) and reasoning as before it is immediate to show that the mean energy of the antiferromagnetic chains is given by

\[ \mu = \frac{1}{2} \left( 1 - \frac{1}{m} \right) \sum_{i=1}^{N-1} F(i). \]

(10)

It may be easily verified that the unified expressions differ from those of their ferromagnetic counterparts by a sign change. Indeed, consider the vectors \( \hat{v}(\omega) \) with components

\[ \hat{v}(\omega) = \left( \begin{array}{c} \omega^m \cdots \omega^m \\ \vdots \\ \vdots \\ 1 \cdots 1 \end{array} \right), \quad \omega \in \mathbb{C}, \]

(15)

and

\[ \omega_j(q) = q^{F(j)/m}. \]

(16)

The \( m \times m \) matrix can be easily diagonalized for arbitrary \( m \) and \( \omega \). Indeed, consider the vectors \( v^k(\omega) \) \((k = 1, \ldots, m)\) with components

\[ v^k(\omega) = \left[ \omega e^{2\pi ik/m} \right] \]

and

\[ \lambda_k(\omega) = \frac{1}{m} \sum_{l=0}^{m-1} \left[ \omega e^{2\pi ik/m} \right]^l. \]

Thus \( v^k(\omega) \) is an eigenvector of \( T(\omega) \) with eigenvalue \( \lambda_k(\omega) \). The above result is valid for arbitrary \( \omega \in \mathbb{C} \).
When $\omega$ is unimodular, the vectors $v^k(\omega)/\sqrt{m}$ ($k = 1, \ldots, m$) form an orthonormal basis of $\mathbb{C}^m$. Indeed, they are clearly of unit length, and their scalar product is given by

$$v^k \cdot v^{k'} = \sum_{l=1}^{m} \left[ e^{-2\pi ik/m} \cdot e^{2\pi ik'/m} \right]^{m-l} = \sum_{l=0}^{m-1} e^{2\pi i(k' - k)l/m} = 0, \quad k \neq k',$$

where we have used the fact that $\omega = \omega^{-1}$. In other words, when $|\omega| = 1$ we can write

$$T(\omega) = U(\omega)D(\omega)U^\dagger(\omega),$$

where

$$D(\omega) = \text{diag}\left(\lambda_1(\omega), \ldots, \lambda_m(\omega)\right)$$

and $U(\omega)$ is the unitary $m \times m$ matrix with entries

$$U_{nn'}(\omega) = \frac{1}{\sqrt{m}} \left[ \omega e^{2\pi in'/m} \right]_{m-n}.$$

For later convenience, we shall also evaluate the sum

$$\sum_{n,n'=1}^{m} U_{nk}(\omega)U_{n'k}(\omega)$$

$$= \frac{1}{m} \sum_{n,n'=1}^{m} e^{2\pi ik(m-n)/m} e^{-2\pi ik(m-n')/m}$$

$$= \frac{1}{m} \sum_{n,n'=1}^{m} e^{2\pi ik(n' - n)/m} = m \delta_{km}, \quad |\omega| = 1. \quad (21)$$

Let us now go back to the characteristic function (11). Since

$$\omega_j(e^{it}/\sigma) = e^{i\gamma_j t}, \quad \gamma_j \equiv \frac{F(j)}{\sigma}, \quad (22)$$

we can apply Eqs. (18)–(20) to the matrices $T_j(e^{it}/\sigma)$ to $T(e^{i\gamma_j t})$ in Eq. (13). We thus readily obtain

$$\hat{\phi}(t) = \frac{e^{-i\mu t/\sigma}}{m} \sum_{n,n'=1}^{m} M_{nn'}(t), \quad (23)$$

where the $m \times m$ matrix $M(t)$ is given by

$$M(t) = U(e^{i\gamma_1 t})D(e^{i\gamma_1 t})B_1(t) \cdots D(e^{i\gamma_{N-2} t})B_{N-2}(t) \times D(e^{i\gamma_{N-1} t})U^\dagger(e^{i\gamma_{N-1} t}) \quad (24)$$

with

$$B_j(t) = U^\dagger(e^{i\gamma_j t})U(e^{i\gamma_{j+1} t}). \quad (25)$$

From Eqs. (23)–(25) one can determine the large $N$ limit of the characteristic function $\hat{\phi}(t)$, as we shall next discuss. To this end, we note first of all that

$$\gamma_j = O(N^{-1/2}). \quad (26)$$

Indeed, for the HS and FI chains $F(j)$ is $O(N^2)$ by Eq. (5), while it can be checked that $\sigma$ is $O(N^{5/2})$ by substituting the corresponding expressions of $F$ in Eq. (19) into Eq. (30) (cf. the explicit formulas in Refs. 11 and 17). Likewise, for the PF chain $F(j)$ is $O(N)$, while $\sigma$ is $O(N^{3/2})$ [11]. By Eq. (20), this implies that

$$U(e^{i\gamma_j t}) = R + O(N^{-1/2}), \quad (27)$$

where

$$R \equiv U(1) \quad (28)$$

is a constant unitary matrix (independent of $N$). In order to estimate $B_j(t)$, note first that for all three chains (11–3) we have

$$\gamma_j - \gamma_{j+1} = O(N^{-3/2}); \quad (29)$$

indeed, $F(j) - F(j+1)$ is $O(N)$ (resp. $O(1)$) for the HS and FI (resp. PF) chains. Taking this into account and using Eq. (20) we immediately obtain

$$[B_j(t)]_{nn'} = \sum_{k=1}^{m} U_{kn}(e^{i\gamma_j t})U_{n'k}(e^{i\gamma_{j+1} t})$$

$$= \frac{1}{m} \sum_{k=1}^{m} \left[ e^{-2\pi ik/m} \right]^{m-k} \left[ e^{2\pi ik/2} \right]^{m-k}$$

$$= \frac{1}{m} \sum_{k=1}^{m} e^{i(\gamma_{j+1} - \gamma_{j})(m-k)t} e^{2\pi i(n'-n)(m-k)/m}$$

$$= \frac{1}{m} \sum_{k=1}^{m} e^{2\pi i(n'-n)(m-k)/m} + O(N^{-3/2})$$

$$= \delta_{nn'} + O(N^{-3/2}),$$

so that

$$B_j(t) = I + O(N^{-3/2}). \quad (30)$$

On the other hand, from Eq. (17) it easily follows that

$$|\lambda_k(e^{i\gamma_j t})| \leq 1, \quad k = 1, \ldots, m. \quad (31)$$

Equation (21) and the estimates (27), (30) and (31) immediately yield the asymptotic formula

$$M(t) = R \Lambda(t) R^\dagger + O(N^{-1/2}), \quad (32)$$

where $\Lambda(t)$ is the diagonal matrix with entries

$$\Lambda_k(t) = \prod_{j=1}^{N-1} \lambda_k(e^{i\gamma_j t}), \quad k = 1, \ldots, m. \quad (33)$$

Inserting Eqs. (32)–(33) into Eq. (23) and using the identity (21) with $\omega = 1$ we obtain the simple asymptotic estimate

$$\hat{\phi}(t) = e^{-i\mu t/\sigma} \Lambda_m(t) + O(N^{-1/2}). \quad (34)$$
In view of the latter equation, in order to complete our proof of Eq. (12) we just have to determine the asymptotic behavior as $N \to \infty$ of the eigenvalue $\lambda_m(\omega)$, with $\omega = e^{i\gamma t}$ unimodular. By Eq. (17), this eigenvalue is given by

$$\lambda_m(e^{i\gamma t}) = \frac{1}{m} \sum_{j=0}^{m-1} e^{i\gamma j t} = \frac{e^{im\gamma t} - 1}{m(e^{i\gamma t} - 1)}$$

provided that $t \notin (2\pi/\gamma j) \mathbb{Z}$. Note that for any fixed $t \neq 0$ the condition $mt \notin (2\pi/\gamma j) \mathbb{Z}$ is fulfilled for sufficiently large $N$ on account of Eq. (26), and implies the weaker condition $t \notin (2\pi/\gamma j) \mathbb{Z}$. Thus for all $t \neq 0$ Eq. (35) holds if $N$ is large enough, and $\lambda_m(e^{i\gamma t}) \neq 0$. The latter equation, together with the elementary Taylor expansion

$$\log \left[ \frac{e^{ix} - 1}{m(e^{ix} - 1)} \right] = \frac{1}{2} (m-1)ix - \frac{1}{24}(m^2 - 1)x^2 + O(x^4)$$

and the identity (3), easily yields the asymptotic formula

$$- \frac{i\mu t}{\sigma} + \log \Lambda_m(t) = -\frac{t^2}{24} (m^2 - 1) \sum_{j=1}^{N-1} \gamma_j^2 + O(N^{-1}).$$

In order to estimate the coefficient of $t^2$ in the previous formula, it suffices to note that Eq. (9) can be equivalently written as

$$\frac{1}{12}(m^2 - 1) \sum_{j=1}^{N-1} \gamma_j^2 = 1 + \frac{1}{6}(m^2 - 1) \sum_{j=2}^{N-1} (\gamma_{j-1} - \gamma_j) \gamma_j = 1 + O(N^{-1}),$$

where we have used Eqs. (26) and (29) for the last estimate. Hence

$$- \frac{i\mu t}{\sigma} + \log \Lambda_m(t) = -\frac{t^2}{2} + O(N^{-1}),$$

which obviously implies Eq. (12) in view of Eq. (34).

We shall conclude by summarizing the main result of this paper and presenting an outline of related future work. We have rigorously shown that for all spin chains of Haldane–Shastry type associated with the $A_{N-1}$ root system the level density (normalized to unity) approaches a Gaussian distribution as the number of sites tends to infinity. Our proof essentially relies on two key properties of these chains, namely Eq. (3) for the energies in terms of the motifs (7), and the estimates (26) and (29) involving the large $N$ behavior of the dispersion relation (6).

Our results admit several natural generalizations. For instance, one could consider the $su(n|n')$ supersymmetric extensions of the chains (11–13), some of which have already been studied in the literature [12–16]. It is straightforward to check that Eq. (7) for the motifs should be replaced by

$$\delta(j, k) = \begin{cases} 1, & j > k \text{ or } j = k > n \\ 0, & j < k \text{ or } j = k \leq n \end{cases},$$

where $j, k = 1, \ldots, n + n' = m$. As a consequence, the last $n'$ elements in the main diagonal of the transfer matrix (15) are replaced by $\omega^m$, so that the resulting matrix is no longer Toeplitz. Although this fact certainly complicates the explicit diagonalization of the transfer matrix, we believe that the main ideas behind our proof can still be applied to this case.

It is also natural to consider the generalization of our result to spin chains of HS type associated with other root systems, like $BC_N$ or $D_N$. The main difficulty in this respect is the fact that for these chains no description of the energies in terms of motifs akin to Eq. (6) is known so far. At least for the Sutherland spin chain of $BC_N$ type [3, 8], some preliminary results of our group indicate that such a description is possible, and that our proof can be suitably adapted to this case.

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