K3 SURFACES WITH PICARD NUMBER 2, SALEM POLYNOMIALS AND PELL EQUATION

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ABSTRACT. If an automorphism of a projective K3 surface with Picard number 2 is of infinite order, then the automorphism corresponds to a solution of Pell equation. In this paper, by solving this equation, we determine all Salem polynomials of symplectic and anti-symplectic automorphisms of projective K3 surfaces with Picard number 2.

1. Introduction

A compact complex surface $X$ is called a K3 surface if it is simply connected and has a nowhere vanishing holomorphic 2-from $\omega_X$. By the global Torelli theorem the automorphism group of a K3 surface is determined, up to subquotient of finite index, by its Picard lattice. Suppose that a K3 surface $X$ is projective and has Picard number 2. Galluzzi, Lombardo and Peters [GLP] applied the classical theory of binary quadratic forms (cf. [D, J]) to prove that the automorphism group $\text{Aut}(X)$ is finite cyclic, infinite cyclic or infinite dihedral.

Suppose that $\varphi \in \text{Aut}(X)$ has infinite order and is either symplectic or anti-symplectic, where we have $\varphi^* \omega_X = \varepsilon \omega_X$ and $\varphi$ is said to be symplectic (resp. anti-symplectic) if $\varepsilon = 1$ (resp. $\varepsilon = -1$). In the present paper, we determine all Salem polynomials of such automorphisms. The Salem polynomial of $\varphi$ is defined as the characteristic polynomial of $\varphi^*|_{S_X}$ in this case. Here $S_X$ denotes the Picard lattice of $X$. Since $\det(\varphi^*|_{S_X}) = 1$, it is enough to determine $\text{trace}(\varphi^*|_{S_X})$, and the Salem polynomial of $\varphi$ is given as

$$x^2 - tx + 1, \quad t = \text{trace}(\varphi^*|_{S_X}). \tag{1.1}$$

We construct an automorphism $\varphi$ whose trace on $S_X$ is as in the condition (2) in Main Theorem below. We use a result on class numbers of real quadratic fields for the construction.

We apply the theory of binary quadratic forms to study actions on $S_X$ (cf. [GLP]). The (induced) action $\varphi^*|_{S_X}$ of $\varphi$ on $S_X$ as above corresponds to an integer solution $(u, v)$ to the Pell equation

$$u^2 - Dv^2 = 4, \quad D := -\text{disc}(S_X) > 0. \tag{1.2}$$

More precisely, let

$$\begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}, \quad a, b, c \in \mathbb{Z}, \tag{1.3}$$

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be the intersection matrix of $S_X$ (under some basis). Then $D = b^2 - 4ac$ and $\varphi^*|_{S_X}$ is represented as the matrix

$$g_{u,v} = \begin{pmatrix} (u - bv)/2 & -cv \\ av & (u + bv)/2 \end{pmatrix},$$

(1.4)

where $(u, v)$ is an integer solution to the Pell equation (1.2) with $u > 0$ (see Proposition 2.2). Moreover, we show

$$(u, v) = (\alpha^2 - 2\varepsilon, \alpha\beta), \quad \alpha^2 - D\beta^2 = 4\varepsilon$$

(1.5)

for some integers $\alpha, \beta$ (see Proposition 4.1). In particular, we have $g_{u,v} = g_{\alpha,\beta}$. Our goal is to determine all possible values of

$$\text{trace}(\varphi^*|_{S_X}) = u = \alpha^2 - 2\varepsilon.$$  

(1.6)

The result is stated as follows:

**Main Theorem.** For $\varepsilon \in \{\pm 1\}$ and $u \in \mathbb{Z}$, the following conditions are equivalent:

1. $u = \text{trace}(\varphi^*|_{S_X})$ for some automorphism $\varphi$ of a projective K3 surface $X$ with Picard number 2 such that $\text{ord}(\varphi) = \infty$ and $\varphi^*\omega_X = \varepsilon\omega_X$.
2. $u = \alpha^2 - 2\varepsilon$ for some $\alpha \in A_\varepsilon$, where

$$A_\varepsilon = \begin{cases} \mathbb{Z}_{\geq 4} \quad &\text{if } \varepsilon = 1, \\ \mathbb{Z}_{\geq 4} \setminus \{5, 7, 13, 17\} &\text{if } \varepsilon = -1. \end{cases}$$

(1.7)

**Remark 1.1.** (i) For a given trace $u$ as in Main Theorem, we have only finitely many pairs $(X, \varphi)$ with $\text{trace}(\varphi^*|_{S_X}) = u$, up to equivariant deformation. Indeed, for a given $u$ there are only finitely many $D$’s satisfying the Pell equation (1.2) for some integer $v$, and for a given $D$ there are only finitely many lattices of rank 2 with discriminant $-D$ up to isomorphism (see [CS, Chap. 15]). Each of such lattices corresponds to a connected moduli space (of dimension $20 - \text{rank } S_X = 18$) of $(X, \varphi)$ such that $S_X$ is isomorphic to the given lattice. (ii) An automorphism of infinite order of a projective K3 surface $X$ with Picard number 2 always acts on the Picard lattice $S_X$ with determinant 1.

As an application, we show that every projective K3 surface with Picard number 2 admitting a fixed-point-free automorphism must have the same Picard lattice as the K3 surfaces with the Cayley–Oguiso automorphism [OFGvGL], and that the action of the automorphism on the Picard lattice is the same as that of the Cayley–Oguiso automorphism (see Section 5).

The structure of this paper is as follows: We recall some results on lattices and their isometries in Section 2 and K3 surfaces and their automorphisms in Section 3. We prove Main Theorem in Section 4. Finally, in Section 5, a few applications of our result are discussed.

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2. Lattices

A lattice is a free \( \mathbb{Z} \)-module \( L \) of finite rank equipped with a symmetric bilinear form \( \langle , \rangle : L \times L \to \mathbb{Z} \). If \( x^2 := \langle x, x \rangle \in 2\mathbb{Z} \) for any \( x \in L \), a lattice \( L \) is said to be even. We fix a \( \mathbb{Z} \)-basis of \( L \) and identify the lattice \( L \) with its Gram matrix \( Q_L \) under this basis. The discriminant \( \text{disc}(L) \) of \( L \) is defined as \( \text{det}(Q_L) \), which is independent of the choice of basis. A lattice \( L \) is called non-degenerate if \( \text{disc}(L) \neq 0 \) and unimodular if \( \text{disc}(L) = \pm 1 \). For a non-degenerate lattice \( L \), the signature of \( L \) is defined as \( (s_+, s_-) \), where \( s_+ \) (resp. \( s_- \)) denotes the number of the positive (resp. negative) eigenvalues of \( Q_L \). An isometry of \( L \) is an isomorphism of the \( \mathbb{Z} \)-module \( L \) preserving the bilinear form. The orthogonal group \( O(L) \) of \( L \) consists of the isometries of \( L \). We consider \( L \) as \( \mathbb{Z}^n \) (column vectors) with Gram matrix \( Q_L \) and use the following identification:

\[
O(L) = \{ g \in GL_n(\mathbb{Z}) \mid g^T \cdot Q_L \cdot g = Q_L \}, \quad n = \text{rank } L. \tag{2.1}
\]

For a non-degenerate lattice \( L \), the discriminant group \( A(L) \) of \( L \) is defined by

\[
A(L) := L^*/L, \quad L^* := \{ x \in L \otimes \mathbb{Q} \mid \langle x, y \rangle \in \mathbb{Z} \ (\forall y \in L) \}. \tag{2.2}
\]

Let \( K \) be a sublattice of a lattice \( L \), that is, \( K \) is a \( \mathbb{Z} \)-submodule of \( L \) equipped with the restriction of the bilinear form of \( L \) to \( K \). If \( L/K \) is torsion-free as a \( \mathbb{Z} \)-module, \( K \) is said to be primitive. For a non-degenerate lattice \( L \) of signature \((1, k)\) with \( k \geq 0 \), we have the decomposition

\[
\{ x \in L \otimes \mathbb{R} \mid x^2 > 0 \} = C_L \sqcup (-C_L) \tag{2.3}
\]

into two disjoint cones. Here \( C_L \) and \( -C_L \) are connected components. We define

\[
O^+(L) := \{ g \in O(L) \mid g(C_L) = C_L \}, \quad SO^+(L) := O^+(L) \cap SO(L), \tag{2.4}
\]

where \( SO(L) \) is the subgroup of \( O(L) \) consisting of isometries of determinant 1. The group \( O^+(L) \) is a subgroup of \( O(L) \) of index 2.

**Lemma 2.1.** Let \( L \) be a non-degenerate even lattice of rank \( n \). For \( g \in O(L) \) and \( \varepsilon \in \{ \pm 1 \} \), \( g \) acts on \( A(L) \) as \( \varepsilon \cdot \text{id} \) if and only if \( (g - \varepsilon \cdot I_n) \cdot Q_L^{-1} \) is an integer matrix.

**Proof.** As in (2.1), we consider \( L \) as \( \mathbb{Z}^n \) with Gram matrix \( Q_L \). Then \( L^* \) is generated by the columns of \( Q_L^{-1} \). This implies the assertion. \( \square \)

In the rest of this section, we recall a few results related to lattices of signature \((1,1)\), or indefinite binary quadratic forms.

**Proposition 2.2.** \([D, U]\) Let \( L \) be an even lattice of signature \((1,1)\):

\[
L := \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}, \quad D := -\text{disc}(L) = b^2 - 4ac > 0. \tag{2.5}
\]

Then \( SO^+(L) \) consists of the elements of the form

\[
g = \begin{pmatrix} (u - bv)/2 & -cv \\ av & (u + bv)/2 \end{pmatrix}. \tag{2.6}
\]

Here \((u, v)\) is any solution of the positive Pell equation

\[
u^2 - Dv^2 = 4 \tag{2.7}
\]

with \( u, kv \in \mathbb{Z} \) and \( u > 0 \), where \( k := \gcd(a, b, c) \). If \( D \) is a square number (resp. not a square number), then \( SO^+(L) \) is isomorphic to a trivial group (resp. \( \mathbb{Z} \)).
Remark 2.3. In Proposition 2.2, $D$ is a square number if and only if there exists $v \in L$ with $v^2 = 0$.

Remark 2.4. Pell equation is usually considered as a Diophantine equation, that is, only integer solutions are admitted. However, in Proposition 2.2, we need non-integer solutions to find all elements in $SO^+(L)$. Of course we have usual Pell equation by replacing $a, b, c$ and $D$ by $a/k, b/k, c/k$ and $D/k^2$.

Remark 2.5. In Proposition 2.2, the eigenvalues of $g$ are $(u \pm v\sqrt{D})/2$ and the solution $(u', v')$ of (2.7) corresponding to $g^k (k \in \mathbb{Z})$ satisfies
\[
\frac{u' + v'\sqrt{D}}{2} = \left(\frac{u + v\sqrt{D}}{2}\right)^k. \tag{2.8}
\]

Let $(u_0, v_0)$ be the smallest positive solution of (2.7), that is, $(u_0, v_0)$ is the solution with $u_0, v_0 > 0$ which minimizes $u_0$ (or $v_0$). Then $SO^+(L)$ is generated by $g_0$ corresponding to $(u_0, v_0)$.

We define
\[
\mathbb{L}_{(1,1)} := \text{the set of even lattices of rank } 2 \text{ and signature } (1,1), \tag{2.9}
\]
\[
\mathbb{L}^*_{(1,1)} := \{ L \in \mathbb{L}_{(1,1)} \mid v^2 \not\in \{0, -2\} \ (\forall v \in L) \}. \tag{2.10}
\]

Lemma 2.6. Let $K = \mathbb{Q}(\sqrt{D})$ be a real quadratic field of discriminant $D (> 0)$. Then there exists $L \in \mathbb{L}^*_{(1,1)}$ with disc($L$) = $-D$ if and only if $h^+(D) > 1$. Here $h^+(D)$ denotes the narrow class number of $K$.

Proof. It is classically known that there is a natural bijection between the set of isomorphism classes of $L \in \mathbb{L}_{(1,1)}$ with disc($L$) = $-D$ and the narrow ideal class group of $K$ (see e.g. [FT] Section VII.2, Theorem 58).

Let $L \in \mathbb{L}_{(1,1)}$. There exists $v \in L$ with $v^2 = 0$ if and only if $-\text{disc}(L)$ is a square number. Hence, for any $L$ such that disc$(L) = -D$ and $L \not\in \mathbb{L}^*_{(1,1)}$, there exists $v \in L$ with $v^2 = -2$ and
\[
L \cong L_0 := \left(\frac{-2}{\delta} \delta \quad (D - \delta^2)/2\right), \tag{2.11}
\]
where $\delta \in \{0, 1\}$ and $D \equiv \delta \ (\text{mod } 2)$. Therefore the existence of $L \in \mathbb{L}^*_{(1,1)}$ with disc$(L) = -D$ is equivalent to the existence of $L \in \mathbb{L}_{(1,1)}$ with disc$(L) = -D$ and $L \not\cong L_0$, which is also equivalent to the condition $h^+(D) > 1$. \hfill \Box

3. K3 surfaces

A compact complex surface $X$ is called a K3 surface if it is simply connected and has a nowhere vanishing holomorphic 2-from $\omega_X$ (see [BHPV] for details). We consider the second integral cohomology $H^2(X, \mathbb{Z})$ with the cup product as a lattice. It is known that $H^2(X, \mathbb{Z})$ is an even unimodular lattice of signature $(3, 19)$, which is unique up to isomorphism and is called the K3 lattice. We fix such a lattice and denote it by $\Lambda_{K3}$. The Picard lattice $S_X$ and transcendental lattice $T_X$ of $X$ are defined by
\[
S_X := \{ x \in H^2(X, \mathbb{Z}) \mid \langle x, \omega_X \rangle = 0 \}; \tag{3.1}
\]
\[
T_X := \{ x \in H^2(X, \mathbb{Z}) \mid \langle x, y \rangle = 0 \ (\forall y \in S_X) \}. \tag{3.2}
\]
Here $\omega_X$ is considered as an element in $H^2(X, \mathbb{C})$ and the bilinear form on $H^2(X, \mathbb{Z})$ is extended to that on $H^2(X, \mathbb{C})$ linearly. The Picard group of $X$ is naturally isomorphic to $S_X$. It is known that $X$ is projective if and only if $S_X$ has signature $(1, \rho - 1)$, where $\rho = \text{rank } S_X$ is the Picard number of $X$.

Let $X$ be a projective K3 surface. Since $H^2(X, \mathbb{Z})$ is unimodular and $S_X$ is non-degenerate, we have the following natural identification:

$$A(S_X) = A(T_X) = H^2(X, \mathbb{Z}) / (S_X \oplus T_X)$$

(3.3)

(see [N2] for details). By the global Torelli theorem for K3 surfaces [PS, BR], the following map is injective:

$$\text{Aut}(X) \ni \varphi \mapsto (g, h) := (\varphi^*|_{S_X}, \varphi^*|_{T_X}) \in O(S_X) \times O(T_X).$$

(3.4)

Moreover, $(g, h) \in O(S_X) \times O(T_X)$ is the image of some $\varphi \in \text{Aut}(X)$ by the map (3.4) if and only if (1) the linear extension of $g$ (resp. $h$) preserves the ample cone $C_X$ of $X$ (resp. $\mathbb{C}\omega_X$) and (2) the actions of $g$ and $h$ on $A(S_X) = A(T_X)$ coincide.

In this paper, we study projective K3 surfaces $X$ with Picard number 2, for which we have $S_X \in \mathbb{L}_{(1,1)}$. Conversely, the following holds:

**Proposition 3.1.** For any $L \in \mathbb{L}_{(1,1)}$, there exists a (projective) K3 surface $X$ such that $S_X \cong L$.

**Proof.** Let $L \in \mathbb{L}_{(1,1)}$. The K3 lattice $\Lambda_{K3}$ contains a primitive sublattice $S$ which is isomorphic to $L$ [N1 Theorem 1.14.4]. (In fact, such an $S$ is unique up to $O(\Lambda_{K3})$.) The surjectivity of the period map for K3 surfaces [T] implies that there exists a K3 surface $X$ such that $S_X \cong L$. \hfill $\square$

Let $X$ be a projective K3 surface with Picard number 2. We are interested in automorphisms $\varphi \in \text{Aut}(X)$ of infinite order such that $\varphi^*\omega_X = \varepsilon \omega_X$ with $\varepsilon \in \{\pm 1\}$.

By applying the global Torelli theorem, we obtain the following:

**Proposition 3.2.** Let $X$ be a projective K3 surface with Picard number 2 and $(g, h) \in O(S_X) \times O(T_X)$. Then $(g, h) = (\varphi^*|_{S_X}, \varphi^*|_{T_X})$ for some $\varphi \in \text{Aut}(X)$ such that $\text{ord}(\varphi) = \infty$ and $\varphi^*\omega_X = \varepsilon \omega_X$ with $\varepsilon \in \{\pm 1\}$ if and only if the following conditions are satisfied:

1. $S_X \in \mathbb{L}^*_{(1,1)}$;
2. $1 \neq g \in SO^+(S_X)$;
3. $g$ acts on $A(S_X)$ as $\varepsilon \cdot \text{id}$;
4. $h = \varepsilon \cdot \text{id}$.

**Proof.** Assume that $(g, h) = (\varphi^*|_{S_X}, \varphi^*|_{T_X})$ for some $\varphi$ as in the statement. Since $\text{Aut}(X)$ is infinite if and only if $S_X \in \mathbb{L}^*_{(1,1)}$ [GLP Corollary 3.4], the condition (1) holds. Moreover, the ample cone $C_X$ of $X$ coincides with $C_{S_X}$ or $-C_{S_X}$ (see [2.3]). Since $\varphi^*\omega_X = \varepsilon \omega_X$, $h$ acts on $T_X$ as $\varepsilon \cdot \text{id}$ [N1 Theorem 3.1]. Thus the condition (4) holds. This implies the condition (3) because the actions of $g$ and $h$ on $A(S_X) = A(T_X)$ coincide. If $\det(g) = -1$, then the action of $g$ on $C_X$ is a reflection, and hence $g^2 = 1$. Since $h^2 = 1$, we have $\varphi^2 = 1$ by the injectivity of the map (3.4), which is a contradiction. Therefore $g \in SO^+(S_X)$. Similarly, we have $g \neq 1$. Hence the condition (2) holds.

Conversely, if the conditions (1)–(4) are satisfied, then there exists $\varphi \in \text{Aut}(X)$ such that $(g, h) = (\varphi^*|_{S_X}, \varphi^*|_{T_X})$ by the global Torelli theorem. The condition (4)
implies that $\varphi^*\omega_X = \varepsilon\omega_X$ because $\omega_X \in T_X \otimes \mathbb{C}$. Since $SO^+(S_X)$ is isomorphic to $\mathbb{Z}$ (Proposition 2.2), $\varphi$ is of infinite order.

\[\square\]

Remark 3.3. A non-projective K3 surface with Picard number 2 may have an automorphism of infinite order (cf. [M]).

4. Proof of Main Theorem

4.1. Preparation. In order to prove Main Theorem, we show the following:

Proposition 4.1. Let $X$ be a projective K3 surface with Picard number 2. For $\varepsilon \in \{\pm 1\}$ and $g \in O(S_X)$, the following conditions are equivalent:

1. $g = \varphi^*|_{S_X}$ for some $\varphi \in \text{Aut}(X)$ such that $\text{ord}(\varphi) = \infty$ and $\varphi^*\omega_X = \varepsilon\omega_X$.
2. $S_X \in \mathbb{L}^*_{\langle 1,1 \rangle}$ and $g$ is given by

$$g = g_{u,v} := \begin{pmatrix} (u-bv)/2 & -cv \\ av & (u+bv)/2 \end{pmatrix},$$

where

$$S_X = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}, \quad (u,v) = (\alpha^2 - 2\varepsilon, \alpha\beta),$$

and $\alpha, \beta$ are nonzero integers satisfying

$$\alpha^2 - D\beta^2 = 4\varepsilon, \quad D := -\text{disc}(S_X) = b^2 - 4ac > 0.$$ (4.3)

Remark 4.2. Assume that $D$ is not a square number and $\gcd(a, b, c) = 1$ in Proposition 4.1. Then Proposition 2.2 implies that

$$SO^+(S_X) = \{g_{u,v} \mid u, v \in \mathbb{Z}, u > 0, u^2 - Dv^2 = 4\} \cong \mathbb{Z}.\quad (4.4)$$

Under the condition (2) in Proposition 4.1, we have

$$\left(\frac{\alpha + \beta\sqrt{D}}{2}\right)^2 = \frac{\alpha^2 - 2\varepsilon + \alpha\beta\sqrt{D}}{2} = \frac{u + v\sqrt{D}}{2}.\quad (4.5)$$

Hence $g_{u,v} = g_{\alpha,\beta}^2$ (see Remark 2.5), and thus

$$\Gamma := \{g \in SO^+(S_X) \mid g = \varphi^*|_{S_X}, \varphi^*\omega_X = \varepsilon\omega_X \ (\exists \varphi \in \text{Aut}(X))\}$$ (4.6)

$$= \{g_{\alpha,\beta}^2 \mid \alpha, \beta \in \mathbb{Z}, \alpha^2 - D\beta^2 = \pm 4\}.\quad (4.7)$$

Therefore $\Gamma$ is a subgroup of $SO^+(S_X)$ of index 1 or 2 according to whether the equation $\alpha^2 - D\beta^2 = -4$ has an integer solution $(\alpha, \beta)$ or not.

We obtain Proposition 4.3 below from Proposition 4.1 because any lattice $L \in \mathbb{L}^*_{\langle 1,1 \rangle}$ is realized as the Picard lattice $S_X$ of a K3 surface $X$ by Proposition 3.1.

Proposition 4.3. For $\varepsilon \in \{\pm 1\}$ and $u \in \mathbb{Z}$, the following conditions are equivalent:

1. $u = \text{trace}(\varphi^*|_{S_X})$ for some automorphism $\varphi$ of a projective K3 surface $X$ with Picard number 2 such that $\text{ord}(\varphi) = \infty$ and $\varphi^*\omega_X = \varepsilon\omega_X$.
2. $u = \alpha^2 - 2\varepsilon$ for some $\alpha \in \mathbb{Z}_{>0}$ such that there exist $L \in \mathbb{L}^*_{\langle 1,1 \rangle}$ and $\beta \in \mathbb{Z}_{>0}$ satisfying $\text{disc}(L) = -(\alpha^2 - 4\varepsilon)/\beta^2$.

Proposition 4.1 is a direct conclusion of Proposition 3.2 and the following:
Lemma 4.4. Let \( L \in \mathbb{I}_{(1,1)} \):
\[
L = \begin{pmatrix} 2a & b \\ b & 2c \end{pmatrix}, \quad D := -\text{disc}(L) = b^2 - 4ac > 0,
\] (4.8)
and let \( g \) be a nontrivial isometry contained in \( \text{SO}^+(L) \) (see Proposition 2.2):
\[
g = \begin{pmatrix} (u - bv)/2 & -cv \\ av & (u + bv)/2 \end{pmatrix}, \quad u^2 - Dv^2 = 4, \quad u > 2.
\] (4.9)
For \( \varepsilon \in \{\pm 1\} \), the isometry \( g \) acts on \( A(L) \) as \( \varepsilon \cdot \text{id} \) if and only if we have
\[
(u, v) = (\alpha^2 - 2\varepsilon, \alpha\beta), \quad \alpha^2 - D\beta^2 = 4\varepsilon
\] (4.10)
for some nonzero integers \( \alpha, \beta \).

Proof. Recall that we have \( u, kv \in \mathbb{Z} \) and \( u > 0 \), where \( k := \gcd(a, b, c) \). Moreover, since \( g \neq 1 \), it follows that \( u > 2 \). We have
\[
\det(g - \varepsilon \cdot I_2) = \det(g) - \varepsilon \cdot \text{trace}(g) + 1 = -\varepsilon u + 2
\] (4.11)
and
\[
(g - \varepsilon \cdot I_2) \cdot Q_L^{-1} = -\frac{1}{D} \begin{pmatrix} (u - bv)/2 - \varepsilon & -cv \\ av & (u + bv)/2 - \varepsilon \end{pmatrix} \begin{pmatrix} 2c & -b \\ -b & 2a \end{pmatrix}
\] (4.12)
\[
= -\frac{1}{D} \begin{pmatrix} c(u - 2\varepsilon) & (Dv - b(u - 2\varepsilon))/2 \\ -(Dv + b(u - 2\varepsilon))/2 & a(u - 2\varepsilon) \end{pmatrix}.
\] (4.13)
Assume that \( g \) acts on \( A(L) \) as \( \varepsilon \cdot \text{id} \). By Lemma 2.1, \( (g - \varepsilon \cdot I_2) \cdot Q_L^{-1} \) is an integer matrix, and hence \( D \) divides \( \det(g - \varepsilon \cdot I_2) = -\varepsilon u + 2 \). Thus \( u - 2\varepsilon = mD \) for some \( m \in \mathbb{Z}_{>0} \). Therefore
\[
v^2 = \frac{1}{D}(u^2 - 4) = m(mD + 4\varepsilon).
\] (4.14)
In particular, we have \( v \in \mathbb{Z} \).

Claim. \( m \) is a square number.

This is shown by applying (4.14) as follows. Suppose that \( p \) is a prime number and \( p^e \mid m \) with \( e \in \mathbb{Z}_{>0} \) (that is, \( p^e \mid m \) and \( p \nmid (m/p^e) \)). It is enough to show that \( e \) is always even. (i) If \( p \) is odd, then \( p^e \mid m(mD + 4\varepsilon) = v^2 \). Hence \( e \) is even. (ii) In the case \( p = 2 \), if \( e \geq 3 \), then \( mD + 4\varepsilon \equiv 4 \pmod{8} \). Hence \( 2^{e+2} \mid m(mD + 4\varepsilon) = v^2 \), and thus \( e \) is even. If \( e = 1 \), then \( (v/2)^2 = (m/2)^2D + \varepsilon m \equiv 2 \) or \( 3 \pmod{4} \) because \( D = b^2 - 4ac \equiv 0 \) or \( 1 \pmod{4} \). On the other hand, \( v \) is even and \( (v/2)^2 \equiv 0 \) or \( 1 \pmod{4} \), which is a contradiction. This completes the proof of Claim.

By Claim, we have \( m = n^2 \) for some \( n \in \mathbb{Z}_{>0} \) and
\[
u^2 = \frac{1}{D}(u^2 - 4) = m(mD + 4\varepsilon).
\] (4.15)
By (4.14), we have \( v^2 = n^2(D + 4\varepsilon) \). Thus \( v = \alpha\beta \) and \( n^2D + 4\varepsilon = \alpha^2 \) for some \( \alpha \in \mathbb{Z}_{>0} \) and \( \beta \in \{\pm 1\} \). This implies (4.10).

Conversely, assume that \( D \) and \((u, v)\) satisfy (4.10) for some nonzero integers \( \alpha, \beta \). In order to show that \( g \) acts on \( A(L) \) as \( \varepsilon \cdot \text{id} \), it is enough to check that the matrix (4.13) is an integer matrix by Lemma 2.1. By (4.10), we have
\[
\frac{u - 2\varepsilon}{D} = \beta^2, \quad \frac{Dv \pm b(u - 2\varepsilon)}{2D} = \frac{v \pm \beta^2}{2}.
\] (4.16)
If \( v \pm b\beta^2 \) is odd, then
\[
1 \equiv v \pm b\beta^2 \equiv \alpha\beta + b\beta = (\alpha + b)\beta, \quad \alpha + b \equiv \beta \equiv 1 \quad (\text{mod } 2),
\]
and thus
\[
\alpha + 1 \equiv b \equiv b^2 - 4ac = D = (\alpha^2 - 4\varepsilon)/\beta^2 \equiv \alpha^2 - 4\varepsilon \equiv \alpha \quad (\text{mod } 2),
\]
which is a contradiction. Hence \( v \pm b\beta^2 \) is even and the matrix (4.13) is an integer matrix. Therefore \( g \) acts on \( A(L) \) as \( \varepsilon \cdot \text{id} \). \( \square \)

Thanks to Proposition [1.3] the proof of Main Theorem is reduced to showing
\[
A_\varepsilon := \{ \alpha \in \mathbb{Z}_{>0} \mid \text{disc}(L) = -(\alpha^2 - 4\varepsilon)/\beta^2 \ (\exists \beta \in \mathbb{Z}_{>0}, \ \exists L \in \mathbb{L}_{(1,1)}^*) \}
\]
\[
= \begin{cases} 
\mathbb{Z}_{\geq 4} & \text{if } \varepsilon = 1, \\
\mathbb{Z}_{\geq 4} \setminus \{5, 7, 13, 17\} & \text{if } \varepsilon = -1.
\end{cases}
\]

4.2. Symplectic case. For \( \alpha \in A_{+1}, \) we have \( \alpha \geq 3. \) If \( \alpha = 3, \) then there exists \( L \in \mathbb{L}_{(1,1)}^* \) such that \( \text{disc}(L) = -5. \) However, it follows from the table of indefinite binary quadratic forms [CS Table 15.2] that
\[
L \cong \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \notin \mathbb{L}_{(1,1)}^*,
\]
which is a contradiction. Hence \( \alpha \geq 4. \)

Conversely, for \( \alpha \in \mathbb{Z}_{\geq 4}, \) we set
\[
L := \begin{pmatrix} 2 & \alpha \\ \alpha & 2 \end{pmatrix} \in \mathbb{L}_{(1,1)}, \quad \text{disc}(L) = -(\alpha^2 - 4).
\]
By [GLP Example 4], we have \( L \in \mathbb{L}_{(1,1)}^* \), and thus \( \alpha \in A_{+1}. \)

4.3. Anti-symplectic case. Now we consider the case \( \varepsilon = -1. \) Let \( \alpha \in \mathbb{Z}_{>0}. \)

4.3.1. Case of odd \( \alpha. \) Suppose that \( \alpha \) is odd.

First we assume that \( \alpha^2 + 4 \) is not square-free, that is, \( D := \alpha^2 + 4 = n^2D_0 \) with \( n > 1. \) Since \( D \equiv 1 \ (\text{mod } 4), \) we have \( D_0 \equiv 1 \ (\text{mod } 4). \) We define an even lattice \( L \) by
\[
L := \begin{pmatrix} 2n & n \\ n & -n(D_0 - 1)/2 \end{pmatrix}, \quad \text{disc}(L) = -n^2D_0 = -D.
\]
Since \( D \) is not a square number, it follows that \( L \in \mathbb{L}_{(1,1)}^*. \) Hence \( \alpha \in A_{-1} \) in this case.

Next we assume that \( D := \alpha^2 + 4 \) is square-free. Then \( D \) is the discriminant of the real quadratic field \( \mathbb{Q}(\sqrt{D}) \). By Lemma [2.6] there exists \( L \in \mathbb{L}_{(1,1)}^* \) with \( \text{disc}(L) = -D \) if and only if \( h^+(D) > 1. \) Therefore, in this case, the conditions \( \alpha \in A_{-1} \) and \( \alpha \notin \{1, 3, 5, 7, 13, 17\} \) are equivalent by Theorem 4.5 and Remark 4.6 below.

**Theorem 4.5.** ([13]) For an odd integer \( k > 0 \) such that \( D := k^2 + 4 \) is square-free, the class number \( h(D) \) of \( \mathbb{Q}(\sqrt{D}) \) is 1 if and only if \( k \in \{1, 3, 5, 7, 13, 17\}. \)

**Remark 4.6.** In Theorem 4.5, since the equation \( u^2 - Dv^2 = -4 \) has an integer solution (e.g. \( (u, v) = (k, 1) \)), we have \( h^+(D) = h(D). \)
4.3.2. Case of even $\alpha$. Suppose that $\alpha$ is even and $\alpha \in A_{-1}$. If $\alpha = 2$, then there exists $L \in \mathbb{L}_{(1,1)}$ such that $\text{disc}(L) = -8$ or $-2$. By [CS, Table 15.2], we have

$$L \cong \begin{pmatrix} -2 & 0 \\ 0 & 4 \end{pmatrix} \notin \mathbb{L}_{(1,1)}^*,$$

which is a contradiction. Therefore $\alpha \geq 4$.

Conversely, for even $\alpha \geq 4$, we define an even lattice $L$ by

$$L := \begin{pmatrix} \alpha & 2 \\ 2 & -\alpha \end{pmatrix}, \quad \text{disc}(L) = -(\alpha^2 + 4).$$

(4.25)

We show $L \in \mathbb{L}_{(1,1)}^*$. It is easy to check this for the case that $\alpha$ is divisible by 4. In the case $\alpha \equiv 2 \pmod{4}$, we consider the following Gram matrix of $L$:

$$P^T \cdot \begin{pmatrix} \alpha & 2 \\ 2 & -\alpha \end{pmatrix} \cdot P = \begin{pmatrix} 4 & 0 \\ 0 & -(\alpha^2/4 + 1) \end{pmatrix}, \quad \text{where} \quad P = \begin{pmatrix} 1 & (\alpha - 2)/4 \\ 1 & (\alpha + 2)/4 \end{pmatrix}.$$

(4.26)

We can apply Theorem 4.7 below for $\Delta = \alpha^2/4 + 1$, $a = 2$ and $b = (\alpha^2/4 + 1)/2$ to show $L \in \mathbb{L}_{(1,1)}^*$. Indeed, we have the continued fraction expansion

$$\sqrt{\alpha^2/4 + 1} = [\alpha/2, \alpha, \alpha, \ldots].$$

(4.27)

The length of the period of this continued fraction is 1.

**Theorem 4.7.** ([Mo]) Suppose that an integer $\Delta > 2$ is not a square number. Then the length of the period of the continued fraction expansion of $\sqrt{\Delta}$ is even if and only if one of the following holds.

1. There exists a factorization $\Delta = ab$ with $1 < a < b$ such that

$$ax^2 - by^2 = \pm 1$$

has an integer solution.

2. There exists a factorization $\Delta = ab$ with $1 \leq a < b$ such that

$$ax^2 - by^2 = \pm 2$$

has an integer solution with $xy$ odd.

5. Applications

In this section we discuss applications of our result.

First we give direct consequences of Main Theorem. Let $\varphi$ be as in Main Theorem, that is, $\varphi$ is an automorphism of a projective K3 surface $X$ with Picard number 2 such that $\text{ord}(\varphi) = \infty$ and $\varphi^*\omega_X = \epsilon \omega_X$ ($\epsilon \in \{\pm 1\})$. Then $\text{trace}(\varphi^*|_{S_X}) = \alpha^2 - 2\epsilon$ for some $\alpha \in A_2$. By [O], each fixed point of $\varphi$ on $X$ is isolated. By the topological Lefschetz fixed point formula, the number of fixed points (with multiplicity) of $\varphi$ is given as

$$\sum_{i=0}^{4}(-1)^i \text{trace}(\varphi^*|_{H^i(X,\mathbb{C})}) = 1 + (20\epsilon + \text{trace}(\varphi^*|_{S_X})) + 1 = \alpha^2 + 18\epsilon + 2.$$  

(5.1)

As another consequence, we determine the spectral radius of $\varphi^*|_{H^2(X,\mathbb{C})}$, which is defined as the maximum absolute value $|\lambda|$ of eigenvalues $\lambda$ of $\varphi^*|_{H^2(X,\mathbb{C})}$. This plays
an important role in the study of complex dynamics of K3 surfaces (see [M]). In our case, the spectral radius is given as

$$
\frac{u + \sqrt{u^2 - 4}}{2} = \frac{\alpha^2 - 2\varepsilon + \alpha\sqrt{\alpha^2 - 4\varepsilon}}{2},
$$

where \( u = \alpha^2 - 2\varepsilon \).

Next we show that a fixed-point-free automorphism of a projective K3 surface with Picard number 2 is nothing but the following Cayley–Oguiso automorphism:

**Theorem 5.1.** ([O]) Any K3 surface \( X \) with

$$
S_X \cong \begin{pmatrix} 4 & 2 \\ 2 & -4 \end{pmatrix}
$$

admits a fixed-point-free automorphism \( \varphi \) of positive entropy with

$$
\varphi^*|_{S_X} = \begin{pmatrix} 5 & 8 \\ 8 & 13 \end{pmatrix}, \quad \varphi^*|_{T_X} = -\text{id}.
$$

**Remark 5.2.** In Theorem 5.1, \( \text{Aut}(X) \) is generated by \( \varphi \) [FGvGL, Theorem 1.1].

**Theorem 5.3.** Any fixed-point-free automorphism \( \varphi \) of a projective K3 surface \( X \) with Picard number 2 is of the form in Theorem 5.1.

**Proof.** By [O], we have \( \text{trace}(\varphi^*|_{S_X}) = 18 \) and \( \varphi^*\omega_X = -\omega_X \). Hence \( \text{disc}(S_X) = -20 \) or \(-5\) by Proposition 4.1. From [CS, Table 15.2], we find that a lattice belonging to \( \mathbb{L}_{(1,1)} \) with discriminant \(-20\) or \(-5\) is isomorphic to one of the following:

$$
\begin{pmatrix} 4 & 2 \\ 2 & -4 \end{pmatrix}, \quad \begin{pmatrix} -2 & 0 \\ 0 & 10 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}.
$$

(5.5)

Since \( S_X \in \mathbb{L}^*_{(1,1)} \) by Proposition 3.2 we have

$$
S_X \cong \begin{pmatrix} 4 & 2 \\ 2 & -4 \end{pmatrix}, \quad \varphi^*|_{S_X} = U \text{ or } U^{-1}, \quad U := \begin{pmatrix} 5 & 8 \\ 8 & 13 \end{pmatrix}
$$

(5.6)

by Proposition 4.1. Here \( U \) and \( U^{-1} \) correspond to \((u, v) = (18, \pm 4)\) satisfying \( u^2 - 20v^2 = 4 \). Since \( O^+(S_X) \) is isomorphic to the infinite dihedral group, \( U \) and \( U^{-1} \) are conjugate in \( O^+(S_X) \). In fact, we have

$$
V := \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \in O^+(S_X), \quad V^{-1} \cdot U^{-1} \cdot V = U.
$$

(5.7)

Hence we may assume \( \varphi^*|_{S_X} = U \) by changing basis of \( S_X \) if necessary. We have \( \varphi^*|_{T_X} = -\text{id} \) by Proposition 3.2. \( \square \)

**Remark 5.4.** One can show that the action of a Cayley–Oguiso automorphism (as in Theorem 5.1) on the K3 lattice is essentially unique by applying Nikulin’s lattice theory [N2]. Combined with Theorem 5.3, this implies that a pair \((X, \varphi)\) of a K3 surface \( X \) with Picard number 2 and a fixed-point-free automorphism \( \varphi \in \text{Aut}(X) \) is unique up to equivariant deformation. Moreover, \( X \) is realized as a determinantal quartic surface and \( \varphi \) is constructed by using cofactor matrix. See [FGvGL] for details.
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