FIELD IDENTIFICATION IN NONUNITARY DIAGONAL COSETS

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ABSTRACT

We study the nonunitary diagonal cosets constructed from admissible representations of Kač-Moody algebras at fractional level, with an emphasis on the question of field identification. Generic classes of field identifications are obtained from the analysis of the modular \( S \) matrix. These include the usual class related to outer automorphisms, as well as some intrinsically nonunitary field identifications. They allow for a simple choice of coset field representatives where all field components of the coset are associated with integrable finite weights.

1. Introduction

Most unitary rational conformal field theories can be described in terms of coset models. The properties of the conformal theories can thus be extracted from those of the building blocks of the coset. Unitarity of the conformal field theories is guaranteed by restricting the levels of Kač-Moody algebras forming the cosets to the set of positive integers.

However, unitary solvable models in conformal field theory constitute a small class among all solvable models. The characteristic property of solvable models is modular invariance. As observed recently [1-9], nonunitary conformal models can also be described by cosets. The price to pay for the non-unitarity is the presence in the coset of Kač-Moody algebras at fractional level. Nevertheless, even if the level is fractional, there are admissible representations which are covariant with respect to modular transformations [1-3]. This is the root of modular covariance for the related conformal field theories.
In this paper we will specifically study diagonal cosets of the form
\[ \frac{\hat{g}_m \oplus \hat{g}_l}{g_{m+l}} \]  
where
\[ m = \frac{t}{u}, \quad l \in \mathbb{N}. \]  
Here \( \hat{g}_m \) is some affine Lie algebra at level \( m \) (whose finite restriction is denoted \( \bar{g} \)). The corresponding central charge is
\[ c = \frac{l \dim \bar{g}}{l + g} \left\{ 1 - \frac{g(g + l)(p' - p)^2}{l^2} \right\} \]
where we have introduced the integers \( p \) and \( p' \) defined by
\[ m + g = \frac{lp}{(p' - p)}, \quad p' - p = lu \]
\((g)\) is the dual Coxeter number of \( \bar{g} \). For \( l = 1 \), these cosets describe minimal models of \( W\bar{g} \) conformal algebras [10]. For \( l > 1 \) the chiral algebra is known only for very few cases. However, when \( \bar{g} = su(N) \), those cosets are in the universality class of the fused RSOS models introduced in [9]. The most famous nonunitary coset model is the Yang-Lee singularity [11].

We will be concerned mainly with field identification for diagonal nonunitary cosets, i.e. with the problem of determining the classes of coset fields which share the same characters, and which are thus undistinguishable. Our main result will be that it is always possible to choose representatives of those classes constructed solely from weights, whose finite parts are integrable. The discussion will be illustrated with many examples.

2. Kač-Moody algebras at fractional level.

In this section we review the results of Kač and Wakimoto concerning Kač-Moody algebras at fractional level. This is preceded by a brief review of general results on Kač-Moody algebras and their relation to conformal theories through WZNW models, which also fixes the notation.

2.1 A brief review of Kač-Moody algebras.

We will only consider untwisted affine Kač-Moody algebras, that is those that are the central extensions of loop algebras of finite Lie algebras. The generators \( J_{a,n} \) of the Kač-Moody algebra \( \hat{g} \) obey the following commutation rules:
\[ [J_{a,n}, J_{b,p}] = if_{ab}^c J_{c,n+p} + mK_{ab}\delta_{n+p} \]
where \( f_{ab}^c \) are the structure constants of a finite Lie algebra \( \bar{g} \) of rank \( r \), \( K_{ab} \) is the corresponding Killing form, and \( m \) is the level. Each generator \( J_{a,n} \) carries a Lie
index \( a \) and a Fourier mode index \( n \in \mathbb{Z} \). The zero-modes \( J_{a,0} \) generate a finite Lie algebra \( \hat{\mathcal{g}} \subset \mathcal{g} \). To give (2.1) properties similar to those of finite Lie algebras, we introduce two additional generators \( \hat{m} \) and \( \hat{n} \) such that

\[
[\hat{n}, J_{a,p}] = pJ_{a,p} \quad \text{and} \quad [\hat{m}, J_{a,p}] = [\hat{m}, \hat{n}] = 0 \quad (2.2)
\]

The Cartan subalgebra is therefore generated by \( \hat{m} \), \( \hat{n} \) and \( J_{h,0} \), where the index \( h \) refers to the Cartan sub-algebra of \( \hat{\mathcal{g}} \). Thus, the roots of \( \hat{\mathcal{g}} \) have the following form:

\[
\alpha = (\bar{\alpha}, 0, n) \quad (2.3)
\]

where \( \bar{\alpha} \) is a root of \( \bar{\mathcal{g}} \), or 0, and where \( n \) is the grade of \( \alpha \) in the root lattice of \( \hat{\mathcal{g}} \). The 0 in the last entry simply means that \( \hat{m} \) commutes with everything, and just gives the level when applied to a state (\( \hat{m} \) was introduced in order to treat weights with different levels within the same framework). The roots are at level 0, and do not depend on the central extension of (2.1). The \( r + 1 \) simple roots are taken to be

\[
\begin{align*}
\alpha_i &= (\bar{\alpha}_i, 0, 0) \quad i = 1, \ldots, r \\
\alpha_0 &= (-\bar{\theta}, 0, 1) \quad (2.4)
\end{align*}
\]

where the \( \bar{\alpha}_i \) are the simple roots of \( \bar{\mathcal{g}} \) and \( \bar{\theta} \) is the highest root of \( \bar{\mathcal{g}} \). The simple root \( \alpha_0 \) was chosen such as to make all roots \( (\bar{\alpha}, 0, n) \) with \( n > 0 \) positive. Given two weights \( \lambda = (\bar{\lambda}, m, n) \) and \( \lambda' = (\bar{\lambda}', m', n') \), one extends the inner product defined on the weight space of \( \bar{\mathcal{g}} \) to the following inner product:

\[
(\lambda, \lambda') := (\bar{\lambda}, \bar{\lambda}') + nm' + n'm \quad (2.5)
\]

This allows us to define for \( \hat{\mathcal{g}} \) quantities taken from finite Lie algebras.

Corresponding to a root \( \alpha \), we define the coroot \( \alpha^\vee \):

\[
\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)} = r_{\alpha} \alpha \quad (2.6)
\]

For long roots, \( (\alpha, \alpha) = 2 \) by convention, and the coroots then coincide with the roots \( (r_{\alpha} = 1) \). For short roots (absent from simply laced algebras) the coroots are integral multiples of the roots \( (r_{\alpha} = 2 \text{ or } 3) \). The fundamental weights \( \omega^\mu \) are defined as

\[
(\omega^\mu, \alpha^\vee) = \delta^\mu_{\nu} \quad (2.7)
\]

By convention, Greek indices will run from 0 to \( r \) and pertain to affine quantities, whereas Latin indices will run from 1 to \( r \) and pertain to the finite restrictions; thus we can identify \( \alpha_i \) and \( \bar{\alpha}_i \). Any affine weight \( \lambda = (\bar{\lambda}, m, 0) \) of grade zero can be uniquely decomposed with respect to the basis \( \{\omega^\mu\} \):

\[
\lambda = \sum_{\mu=0}^{r} \lambda^\mu \omega^\mu, \quad (2.8)
\]
and may also be represented by the vector \( \lambda = [\lambda_0, \lambda_1, \ldots, \lambda_r] \) of its Dynkin labels. For those weights, only the finite parts are important in inner products:
\[
(\lambda, \mu) = (\bar{\lambda}, \bar{\mu}).
\]
The finite Dynkin labels coincide with those of \( \bar{\lambda} \): \( \lambda_i = \bar{\lambda}_i \). One also defines the marks and comarks \( k_\mu \) and \( k_\mu^\vee \) by
\[
\tilde{\theta} = \sum_{i=1}^r k_i \alpha_i = \sum_{i=1}^r k_i^\vee \alpha_i^\vee,
\]
with \( k_0 = k_0^\vee = 1 \) and \( r_\mu = k_\mu/k_\mu^\vee \). The marks and comarks are always positive integers. For \( \hat{su}(N) \) they are all equal to 1. The level \( m(\lambda) \) of a weight is
\[
m(\lambda) = \sum_{\mu=0}^r \lambda_\mu k_\mu^\vee
\]
Once the level \( m \) of an affine weight \( \lambda = (\bar{\lambda}, m, 0) \) is fixed, there is a one-to-one correspondence between affine and finite weights.

The Weyl group of \( \hat{g} \) is denoted \( W \), and is generated by the \( r \) primitive (simple) reflections \( s_i \) associated with the simple roots \( \alpha_i \). Their action on a weight \( \lambda \) is
\[
s_i \lambda = \lambda - \lambda_i \alpha_i.
\]
Each element \( w \) of \( W \) has a parity \( \epsilon(w) \) equal to 1 (-1) if \( w \) is expressible as an even (odd) number of primitive reflections. The shifted action of \( w \in W \) is defined by
\[
w.\lambda := w(\lambda + \rho) - \rho
\]
with \( \rho \) given by
\[
\rho = \sum_{\mu=0}^r \omega^\mu = [1, 1, \ldots, 1]
\]
Appendix A briefly describes the roots, coroots and Weyl groups of the four classical algebras. For further details, the reader is referred to [12,13].

### 2.2 Kač-Moody algebras at integer level and WZNW models.

A weight \( \lambda \) is integral if all its Dynkin labels \( \lambda_\mu \) are integers. Further, if \( \lambda_\mu \geq 0 \), then \( \lambda \) is the highest weight of an integrable representation of \( \hat{g} \). These representations are unitary and can be integrated to representations of the Kač-Moody group. Since the comarks are positive integers, we see from (2.10) that the level of integrable highest weights is always integer. The set of integrable highest weights at fixed level \( m \) is denoted \( P_+^m \).

The current algebra of a WZNW model based on \( \hat{g} \) is exactly (2.1) for some positive integer level [14]. Each primary field of the WZNW model creates states filling out an integrable highest weight representation of \( \hat{g} \) at level \( m \) [15]. Thus WZNW primary fields and integrable highest weights are in one-to-one correspondence and the latter may be used to label the primary fields.

The conformal weight of the primary field \( \lambda \) is
\[
h_\lambda = \frac{(\lambda, \lambda + 2\rho)}{2(m + g)} = \frac{|\lambda + \rho|^2 - |\rho|^2}{2(m + g)}
\]
where \( \rho \) is given by (2.12) and \( g \) is the dual Coxeter number of \( \bar{g} \), the level of \( \rho \):

\[
g = \sum_{\mu=0}^{r} k_{\mu}^{\nu}
\]

(2.14)

The Sugawara construction

\[
L_n = \frac{1}{2(m+g)} \sum_{a,b} K^{ab} : J_{a,n+i} J_{b,-i} : \quad - \delta_{n,0} \frac{c_{\bar{g}}}{24}
\]

(2.15)

associates with \( \hat{g} \) the Virasoro algebra

\[
[L_n, L_p] = (n-p)L_{n+p} + \frac{c_{\bar{g}}}{12} n(n^2-1) \delta_{n+p,0}
\]

(2.16)

with the central charge

\[
c_{\bar{g}} = \frac{m \dim \bar{g}}{m+g}
\]

(2.17)

The colons in (2.15) indicate the standard normal ordering.

One defines the characters of the integrable representation as

\[
\chi_{\lambda}(\tau, z) = \text{Tr}_{\lambda} \exp \left\{ 2\pi i \tau \left[ L_0 - \frac{c}{24} + J_0 \cdot z \right] \right\}
\]

(2.18)

where \( J_0 \cdot z = \sum h z_h J_{h,0} \). The specialized characters are obtained by setting \( z = 0 \):

\[
\chi_{\lambda}(\tau) = \text{Tr}_{\lambda} \exp \left\{ 2\pi i \tau \left[ L_0 - \frac{c}{24} \right] \right\}
\]

(2.19)

The characters are the building blocks of the partition functions of WZNW models on the torus, and consequently have simple modular properties. Under the modular transformation \( S : \tau \mapsto -1/\tau \), The characters transform amongst each other [16]:

\[
\chi_{\lambda}(-1/\tau) = \sum_{\mu} S_{\lambda\mu} \chi_{\mu}(\tau)
\]

(2.20)

where

\[
S_{\lambda\mu} = F_m \sum_{w \in W} \epsilon(w) \exp \left\{ - \frac{2\pi i}{m+g} (w(\lambda + \rho), \mu + \rho) \right\}
\]

(2.21)

The constant of proportionality \( F_m \) is independent of \( \lambda, \mu \) and it can be fixed by the unitarity of the matrix \( S \):

\[
F_m = i^{1/2} |\bar{\Delta}_+| |M^*/(m+g)M|^{-\frac{1}{2}}
\]

(2.22)

where \( |\bar{\Delta}_+| \) is the number of positive roots of \( \bar{g} \), \( M \) is the lattice generated by the long roots of \( \bar{g} \), and \( M^* \) is its dual. (For simply laced algebras, \( M \) is just the root
lattice.) Under the transformation $\tau \mapsto \tau + 1$ (the other generator of the modular group) the characters simply acquire a phase factor:

$$\chi_\lambda(\tau + 1) = \exp \left\{ 2\pi i (h_\lambda - c/24) \right\} \chi_\lambda(\tau) \quad (2.23)$$

From the modular $S$ matrix one defines the charge conjugation matrix $C$ by [17]

$$CS = S^* \quad \text{or} \quad C = S^2 \quad (2.24)$$

A direct calculation yields $C_{\lambda,\lambda'} = \delta_{C\lambda,\lambda'}$, where the integrable weight $C\lambda$ is the charge conjugate of $\lambda$, given by [18]

$$C\lambda = -w^0.\lambda \quad (2.25)$$

$w^0$ being the longest element of $W$. Charge conjugation is related to the reflection symmetry of the Dynkin diagram of $\hat{g}$. For $su(N)$, charge conjugation of an integrable weight amounts to reversing the order of the finite Dynkin labels $\lambda_i$.

### 2.3 Admissible representations of Kač-Moody algebras at fractional level.

Consider a fractional level

$$m = \frac{t}{u}$$

where $t \in \mathbb{Z}$, $u \in \mathbb{N}$ and $(t, u) = 1$. When $u \neq 1$ (fractional level) representations of affine Kač-Moody algebras are necessarily nonunitary. But some of these still have modular properties analogous to those of the integer-level unitary integrable representations. Kač and Wakimoto [1-3] discovered a class of rational level highest weight representations that are modular invariant and include the integrable unitary representations. These they called admissible representations. Their highest weights may be described as follows.

A fractional weight $\lambda$ at level $m$ will be admissible if it satisfies the following two conditions:

1) \((\lambda + \rho, \alpha^\vee) \not\in -\mathbb{Z}_+ \quad \forall \alpha^\vee \in R_+ \quad (2.26)\)

2) \(QR^\lambda = Q\Pi \quad (2.27)\)

where $\Pi := \{\alpha_0^\vee, \alpha_1^\vee, \ldots, \alpha_r^\vee\}$ is the set of simple coroots, $R_+$ is the set of positive real coroots of $\hat{g}$, and where

$$R^\lambda := \{\alpha^\vee | (\lambda, \alpha^\vee) \in \mathbb{Z}\} \quad (2.27)$$

The second equation simply states that the rational span of the set $R^\lambda$ should be the same as the rational span of the simple coroots. However, we will use a different and more constructive characterization of admissible weights, also used by Kač and Wakimoto [1], whose compatibility with the above definition will be shown in appendix B: To every element $y$ of the Weyl group $W$ is associated a set
of possible admissible highest weights $\lambda$. Furthermore, each of these weights may be broken up into an integer (I) and a fractional (F) part:

$$
\lambda = y \cdot (\lambda^I - (m + g)\lambda^{F,y})
$$

(2.28)

where $\lambda^I$ and $\lambda^{F,y}$ are both integral weights. The level of the integer part $\lambda^I$ is

$$
m^I = u(m + g) - g \geq 0
$$

(2.29)

while that of the fractional part $\lambda^{F,y}$ is

$$
m^F = u - 1 \geq 0
$$

(2.30)

In addition, the integer part $\lambda^I$ is the highest weight of an integrable highest weight representation,

$$
\lambda^I \in P^m_+
$$

(2.31)

The characterization of the fractional part is more complicated. The Dynkin labels of $\lambda^{F,y}$ must satisfy

$$
\lambda^{F,y}_\mu \in r_\mu \mathbb{Z}
$$

(2.32)

and be such that

$$
\tilde{c}\lambda^{F,y}_\mu + y(\alpha^\vee_\mu) \in R_+
$$

(2.33)

where $\tilde{c}$ is the canonical central element

$$
\tilde{c} = \sum_{\mu=0}^r k^\vee_\mu \alpha^\vee_\mu = (0,1,0)
$$

(2.34)

Note that $k^\vee_\mu / k_\mu^\vee$ is always an integer, most often 1 (see appendix A). Thus given $y \in W$ one can determine the possible values of $\lambda^{F,y}_\mu$ at a given level $m^F$ and then construct the admissible weights $\lambda$ of level $m$ corresponding to the choice of $y$. This set of admissible highest weights for a fixed $y$ will be denoted $P^m_y$. The set of all admissible highest weights at level $m$ is just the union of these:

$$
P^m = \bigcup_{y \in W} P^m_y
$$

(2.35)

When $u = 1$, we find $P^m = P^m_+$. It turns out that not all $y \in W$ need be considered. An admissible $\lambda$ may have more than one decomposition of the form (2.28), each one corresponding to a different $y$. In fact, as shown in [4] we can restrict $y$ to a subgroup of the finite Weyl group $W$, namely $W/W(A)$, where $W(A)$ is the subgroup of $W$ that is isomorphic to the outer automorphism group $O(\hat{g})$ of $\hat{g}$. Recall that outer automorphisms permute the simple roots of $\hat{g}$ in ways that preserve the Dynkin diagram (or the
Cartan matrix). Their effect on weights is to permute the Dynkin labels. To each $A \in O(\hat{g})$ there corresponds an element $w_A$ of $W$ according to the relation [19]

$$A(\lambda - m\omega^0) = w_A(\lambda - m\omega^0)$$  \hspace{1cm} (2.36)

The outer automorphisms of specific algebras are described in appendix A.

Finally let us mention some properties of the set of admissible weights at fractional level. Consider the proper fractional part of an admissible weight:

$$\lambda^F := y(\lambda^{F,y})$$ \hspace{1cm} (2.37)

These fractional parts satisfy an additivity property modulo $u$, i.e., given two admissible fractional parts $\lambda^F$ and $\mu^F$, there exists an admissible fractional part $\nu^F$ such that\(^a\)

$$\lambda^F + \mu^F = \nu^F + u\xi$$  \hspace{1cm} (2.38)

where $\xi$ is an integer weight. This means that the finite weights $u\lambda^F$ belong to $\Gamma/(u\Gamma)$, where $\Gamma$ is the lattice of integral finite weights. The number of distinct fractional part is therefore $u^r$. Since the number $N_i(m^I)$ of admissible integral parts $\lambda^I$ depends only on the algebra and the integer level $m^I$, we conclude that the number of admissible weights at level $m = t/u$ is

$$N(m) = u^r N_i(m^I)$$ \hspace{1cm} (2.39)

Notice that $N_i(0) = 1$, and that, for $su(r + 1)$,

$$N_i(m^I) = \frac{(m^I + r)!}{r! m^I!}$$ \hspace{1cm} (2.40)

This additivity property modulo $u$ for fractional parts is useful when considering fusion rules [4]. Let us also mention the obvious fact that the set of admissible weights at level $m$ is symmetric with respect to all outer and inner automorphisms of the algebra $\hat{g}$. For $\hat{su}(3)$ at $m^I = 0$, this leads to a pattern of admissible weights ressembling a star of David.

2.4 Examples of admissible representations.

To make the above discussion more specific, we will illustrate it for $\hat{su}(2)$, $\hat{su}(3)$ and $\hat{so}(5)$. In the first two cases all marks and comarks are 1, and hence $\lambda^F_{\mu,y} \in \mathbb{Z}$. For $\hat{su}(2)$ we will use the following notation:

$$\lambda^I = [m^I - n, n], \quad \lambda^{F,y} = [m^F - k, k]$$ \hspace{1cm} (2.41)

where $n$ and $k$ are integers. One first fixes an element $y \in W/W(A)$ and derives the corresponding restrictions on the values of $k$ from (2.33). Since $W = W(A)$ in

\(^a\) This property is actually a conjecture, and has been checked on many examples.
the case of $\hat{su}(2)$, the only necessary $y$ is the identity. Then (2.33) reduces to the following two requirements:

$$
\begin{align*}
(m^F - k + 1)\alpha_0^\vee + (m^F - k)\alpha_1^\vee &\in R_+ \\
k\alpha_0^\vee + (k + 1)\alpha_1^\vee &\in R_+ 
\end{align*}
$$  \quad (2.42)

The coefficients of the coroots must be greater than or equal to zero, with at least one being positive. This forces $m^F - k \geq 0$ and $k \geq 0$. Therefore the two Dynkin labels must be positive definite, and any $\hat{su}(2)$ admissible weight is of the form $\lambda = \lambda^I - (m + g)\lambda^F$ with $\lambda^I \in P^{m^I}$ and $\lambda^F \in P^{m^F}$.

For $\hat{su}(3)$ the finite Weyl group is $\{1, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1\}$. The elements of $W(A)$ are $\{1, s_1s_2, s_2s_1\}$, corresponding respectively to the outer automorphisms 1, $a$ and $a^2$, where $a$ is a cyclic permutation of the affine Dynkin labels. Therefore one can restrict $y$ to the set $\{1, s_1\}$. For $y = 1$, (2.33) yields $\lambda_{\mu, 1}^{F, 1} \geq 0$. On the other hand, for $y = s_1$, the inequalities are $\lambda_{0, 2}^{F, s_1} \geq 0$ and $\lambda_1^{F, s_1} \geq 1$. For the specific example of level $-\frac{3}{2}$ ($u = 2$, $m^F = 1$ and $m^I = 0$), the allowed $\lambda^{F, y}$ are

$$
\lambda^{F, 1} : [1, 0, 0], [0, 1, 0], [0, 0, 1] \\
\lambda^{F, s_1} : [0, 1, 0].
$$  \quad (2.43)

Therefore there are 4 admissible highest weights:

$$
[-\frac{2}{3}, 0, 0], [0, -\frac{2}{3}, 0], [0, 0, -\frac{2}{3}], [\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}]
$$  \quad (2.44)

where the first three are obtained with $y = 1$ and the last one with $y = s_1$.

Let us now consider $\hat{so}(5)$. The simple roots are $\alpha_0 = [2, 0, -2]$, $\alpha_1 = [0, 2, -2]$ and $\alpha_2 = [-1, -1, 2]$, the last one being the short root. All marks and comarks are equal to 1 except $k_2 = 2$. This implies that $\lambda_{0, 1}^{F, y} \in \mathbb{Z}$ and $\lambda_{2}^{F, y} \in 2\mathbb{Z}$. The Weyl group is $\{1, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1, s_2s_1s_2, s_1s_2s_1s_2\}$ and its subgroup $W(A)$ is $\{1, s_1s_2s_1\}$. Therefore it is sufficient to consider $y$ among the set $\{1, s_1, s_2, s_1s_2\}$. The constraints on $\lambda^{F, y}$ are easily found to be

$$
\lambda_{\mu, 1}^{F, 1} \geq 0 \\
\lambda_{0, 2}^{F, s_1} \geq 0; \lambda_1^{F, s_1} \geq 1; \\
\lambda_{0, 1}^{F, s_2} \geq 0; \lambda_2^{F, s_2} \geq 1; \\
\lambda_0^{F, s_1s_2} \geq -1; \lambda_{1}^{F, s_1s_2} \geq 0; \lambda_2^{F, s_1s_2} \geq 2.
$$  \quad (2.45)

Notice that, in fact, $\lambda_{2}^{F, s_2} \geq 2$, and since the other two Dynkin labels must be positive, the sector $y = s_2$ is allowed only for $u \geq 3$.

2.5 The associated Weyl subgroup.

Given an admissible weight $\lambda$, we may define the so-called associated Weyl subgroup $W^{\lambda} \subset W[1-3]$. It is spanned by the reflections with respect to all positive
roots $\tilde{\alpha}$ such that $(\lambda, \tilde{\alpha}^\vee) \in \mathbb{Z}$. If the finite Dynkin labels are integers, it coincides with the full Weyl group. On the other hand, if some of the finite Dynkin labels are not integers, $W^\lambda$ will be a proper subgroup of $W$. For instance, the associated Weyl subgroup corresponding to the $su(3)$ weight $(\frac{1}{2}, -\frac{3}{2})$ is $\{1, s_1s_2s_1 = s_0\}$. As another example, consider the $su(4)$ weights $(-\frac{1}{2}, 0, -\frac{5}{2})$ and $(-\frac{1}{2}, -\frac{7}{2}, -\frac{7}{2})$. In the first case the two positive roots such that $(\lambda, \tilde{\alpha}^\vee) \in \mathbb{Z}$ are $\alpha_2$ and $\theta$, and thus $W^\lambda = \{1, s_2, s_1s_2s_3s_1s_2s_1\}$. For the second case, the corresponding set of positive roots is $\{\alpha_1 + \alpha_2\}$ and $W^\lambda = \{1, s_1s_2s_1\}$.

Given an admissible weight $\lambda$, it is generically possible to find elements $w$ of the Weyl group such that $w.\lambda$ is also admissible. In fact, these elements belong to the coset $W/W^\lambda$. In other words, for any $w \in W$, there is a unique element $\tilde{w} \in W^\lambda$ such that $w.\lambda$ is also admissible. The proof is given in appendix D. If $n(W)$ is the order of the group $W$, then admissible weights occur in sets comprising $n(W)/n(W^\lambda)$ elements, related by shifted actions. Within each set the conformal dimension is the same, since (2.13) is invariant with respect to shifted actions of $W$. If $\lambda$ and $w.\lambda$ are admissible weights, they both admit a decomposition of the form (2.28), with elements $y$ and $y'$ of $W$ respectively. In general $y$ and $y'$ are not unique, but we can choose them such that $y' = wy$. This assertion is also proven in Appendix D. If we insist on picking $y$ and $y'$ from a fixed set of representatives of $W/W(A)$, then the relation between $y$ and $y'$ is rather $wy = y'w_A$, for some $w_A \in W(A)$.

2.6 Modular properties of characters for admissible representations.

Let us now discuss the modular transformation properties of the characters of the highest weight admissible representations. For the untwisted Kač-Moody algebras, Kač and Wakimoto [1-3] found that the characters transformed according to the following modular $S$ matrix:

\[
S_{\lambda\mu} = F_m \epsilon(yyyy') \exp \left\{ 2\pi i \left[ (\lambda \mid + \rho, \mu^F) + (\lambda^F, \mu^I + \rho) - (m + g)(\lambda^F, \mu^F) \right] \right\} \\
\times \sum_{w \in W} \epsilon(w) \exp \left\{ \frac{-2\pi i}{m + g} (w(\lambda \mid + \rho), \mu^I + \rho) \right\}
\]

(2.46)

where

\[
F_m = \left| \frac{M^*}{w^2(m + g)\mathcal{M}} \right|^{-\frac{1}{2}}
\]

(2.47)

Here $\lambda \in P^m_y$ and $\mu \in P^m_y$ (the fractional parts $\lambda^F$ and $\mu^F$ are defined as in (2.37)). This matrix is unitary (this corrects statements to the contrary made in [4]).

\[b\] In the case of non-simply laced algebras, this formula is only valid when $\mu$ is not a multiple of the ratio $r_{\lambda\mu}$ corresponding to the short roots. Thus, this formula should not be used when $\mu$ is even for $B$, $C$, and $F_4$, and when $\mu \in 3\mathbb{Z}$ for $G_2$: The corresponding admissible sets are ill-defined from the point of view of modular transformations.
Notice that the summand depends only on the integer part \( \lambda^I \). This almost immediately implies that the fusion rules are essentially determined by the integer part (up to sign, in fact. e.g. see [4]). When \( m^F = 0 \), we only have to consider \( y = 1 \), and the modular matrix \( S \) reduces to the one found by Kač and Peterson [16] in the integrable case (see Eq.(2.21)).

For manipulations of the phase factors in front of the summand, the following result is useful [4]: If \( \zeta \) and \( \nu \) are integrable weights, then for any \( w \in W \) we have

\[
((w-1)\zeta, \nu) = 0 \pmod{1} ,
\]

(2.48)

provided

\[
\zeta \mu \in r \mu \mathbb{Z} \text{ or } \nu \mu \in r \mu \mathbb{Z} ,
\]

(2.49)

a condition satisfied by the fractional parts of admissible weights. With this result, the modular \( S \) matrix may be rewritten in the following form:

\[
S_{\lambda \mu} = F_m \epsilon(y) \exp \left\{ 2\pi i (\lambda^F, \mu + \rho) \right\} \sum_{w \in W} \epsilon(w) \exp \left\{ \frac{-2\pi i}{m + g} (w(\lambda^I + \rho), \mu + \rho) \right\}.
\]

(2.50)

As an example, let us consider the \( \hat{su}(2) \) modular matrix. For \( \hat{su}(2) \) we only need to consider \( y = 1 \). In the notation (2.41), the modular matrix \( S \) is easily seen to be

\[
S_{\lambda \lambda'} = \left[ \frac{2}{u^2(m + 2)} \right]^\frac{1}{2} (-1)^{k(n'+1)+k'(n+1)} e^{-\pi i m k k'} \sin \left[ \frac{\pi(n+1)(n'+1)}{m + 2} \right]
\]

(2.51)

Other examples are presented in the following sections.

Notice that when the level is fractional there are admissible representations with negative conformal weights, in contrast with unitary models, where the identity primary field has the lowest conformal weight (zero). In other words, the conformal weight formula (2.13) is not positive semi-definite, and the identity is no longer the true vacuum. Of course, since the number of representations is finite, there still exists a field (in general not unique) with minimum conformal weight.

2.7 Charge conjugation.

Let us now turn to a description of charge conjugation for weights at fractional level. The charge conjugation matrix \( C \) was calculated by Kač and Wakimoto from the general formula \( C = S^2 \). Let us denote by \( w^\lambda \) the longest element of \( W^\lambda \), the Weyl subgroup associated with \( \lambda \). \( C \) was found to be [3]

\[
C_{\lambda \lambda'} = \epsilon(w^0) \epsilon(w^\lambda) \delta_{C \lambda \lambda'}
\]

(2.52)

where

\[
C \lambda := -w^\lambda . \lambda
\]

(2.53)
Obviously charge conjugation in the fractional case is not related to inner automorphisms of the algebra. For $\mathfrak{su}(2)$, it takes a particularly simple form. In that case, $\check{\lambda} = n - (m + 2)k$, which is always fractional if $k \neq 0$. If $k = 0$, $W^\lambda = W = \{1, s_1\}$. But $-s_1, \lambda = \lambda$. Therefore, $C$ is the identity in the $k = 0$ sector. However, if $k \neq 0$, $W^\lambda = \{1\}$ and $-1, \lambda = -\lambda - 2\rho$, so that

$$C \lambda = -n + (m + 2)k - 2 = m' - n - (m + 2)(u - k) \quad (2.54)$$

Since $\epsilon(\omega^0)\epsilon(\omega^\lambda) = -1$, all entries in the $k \neq 0$ sector are $-1$'s. This result is easily checked directly from the expression (2.51) for the $\mathfrak{su}(2)$ modular $S$ matrix. The fact that for $\mathfrak{su}(2)_m$ all weights with $k \neq 0$ come as conjugate pairs with the same conformal dimensions was noticed also in [5,20]. (From (2.53), it is obvious that $|C\lambda + \rho|^2 = |\lambda + \rho|^2$, which implies that $h_{C\lambda} = h_\lambda$).

Let us give another example of eq.(2.53) for $\mathfrak{su}(4)$. The longest element of $W$ is $w^\mu = s_1s_2s_3s_8s_2s_1$, with $\epsilon(\omega^\mu) = 1$. The element $w^\lambda$ corresponding to $\lambda = (-\frac{1}{3}, 0, -\frac{2}{3})$ is also $w^0$ so that

$$C(-\frac{1}{3}, 0, -\frac{2}{3}) = -w^0(-\frac{1}{3}, 0, -\frac{2}{3}) = (-\frac{1}{3}, 0, -\frac{1}{3}) \quad (2.55)$$

Similarly, for the weight $(-\frac{1}{3}, -\frac{2}{3}, -\frac{5}{3})$, $w^\lambda = s_1s_2s_1$: $\epsilon(\omega^0)\epsilon(\omega^\lambda) = -1$ and

$$C(-\frac{1}{3}, -\frac{2}{3}, -\frac{5}{3}) = -w_1s_2s_1(-\frac{1}{3}, -\frac{2}{3}, -\frac{5}{3}) = (-\frac{1}{3}, -\frac{4}{3}, -\frac{1}{3}) \quad (2.56)$$

| label | $\lambda$ | $\lambda^{F,0}$ | $y$ | $w^\lambda$ | $C$ |
|-------|---------|---------------|----|------------|----|
| 1     | $[-\frac{9}{4}, 0, 0]$ | $[3, 0, 0]$ | 1 | $s_1s_2s_1$ | 1 |
| 2     | $[-\frac{3}{4}, 0, -\frac{2}{3}]$ | $[2, 0, 1]$ | 1 | $s_1$ | 4 |
| 3     | $[-\frac{1}{4}, 0, -\frac{4}{3}]$ | $[1, 0, 2]$ | 1 | $s_1$ | 3 |
| 4     | $[0, 0, -\frac{3}{4}]$ | $[0, 0, 3]$ | 1 | $s_1$ | 2 |
| 5     | $[-\frac{3}{2}, -\frac{3}{2}, 0]$ | $[2, 1, 0]$ | 1 | $s_2$ | 7 |
| 6     | $[-\frac{1}{2}, -\frac{3}{4}, 0]$ | $[1, 2, 0]$ | 1 | $s_2$ | 6 |
| 7     | $[0, -\frac{3}{2}, 0]$ | $[0, 3, 0]$ | 1 | $s_2$ | 5 |
| 8     | $[-\frac{3}{4}, -\frac{1}{4}, -\frac{1}{2}]$ | $[2, 0, 1]$ | $s_2$ | $s_1s_2s_1$ | 13 |
| 9     | $[-\frac{3}{4}, -\frac{1}{2}, -\frac{1}{2}]$ | $[1, 0, 2]$ | | $s_2$ | $s_1s_2s_1$ | 9 |
| 10    | $[-\frac{1}{2}, -\frac{1}{2}, -\frac{5}{3}]$ | $[1, 1, 1]$ | | $s_2$ | 1 | -16 |
| 11    | $[-\frac{1}{4}, -\frac{3}{4}, -\frac{2}{3}]$ | $[1, 1, 1]$ | | 1 | -15 |
| 12    | $[0, -\frac{1}{4}, -\frac{3}{4}]$ | $[0, 1, 2]$ | | 1 | -14 |
| 13    | $[-\frac{5}{4}, -\frac{1}{2}, -\frac{1}{2}]$ | $[0, 0, 3]$ | | $s_2$ | $s_2$ | 8 |
| 14    | $[-\frac{1}{3}, -\frac{3}{4}, -\frac{1}{2}]$ | $[0, 1, 2]$ | | $s_2$ | 1 | -12 |
| 15    | $[\frac{1}{4}, -\frac{3}{4}, -\frac{1}{2}]$ | $[0, 2, 1]$ | | $s_2$ | 1 | -11 |
| 16    | $[0, -\frac{3}{4}, -\frac{1}{2}]$ | $[0, 2, 1]$ | | 1 | -10 |
2.8 An example of ‘WZNW’ model at fractional level.

To conclude this section, we illustrate all previous results with a complete ex-  
ample of ‘WZNW model’ at fractional level: \( su(3) \) at level \(-\frac{9}{4} \). The list of all  
admissible weights is given in table 1, together with the corresponding values of \( \lambda \),  
\( \lambda^{F,y} \) and \( y \), and the label of the charge conjugated weight.  
The crucial observation of GKO [21] is that the energy-momentum tensor \( T \)  
and \( T_{\text{eff}} \) of the WZNW energy-momentum tensors. The Virasoro modes are then simple differences:  

\[
L_{n}^{(\hat{g}/\hat{h})} = L_{n}^{(\hat{g})} - L_{n}^{(\hat{h})}  
\]

where \( L_{n}^{(\hat{g})} \) and \( L_{n}^{(\hat{h})} \) are given by the Sugawara construction (2.15). This implies  
that the coset central charge is  

\[
c_{\hat{g}/\hat{h}} = c_{\hat{g}} - c_{\hat{h}}  
\]

where \( c_{\hat{g}} \) is given by (2.17).
The primary fields of the coset are labelled by the admissible (or integrable, if the level is an integer) representations of \( \hat{g} \) and \( \hat{h} \). Let \( \Lambda \) and \( \lambda \) be admissible weights of \( \hat{g} \) and \( \hat{h} \) respectively. We will denote coset primary fields (or coset fields, for short) by \( \{ \Lambda ; \lambda \} \). Not all combinations of \( \Lambda \) and \( \lambda \) are allowed. There are selection rules which reflect the existence of relations between the centers of the covering groups of \( \hat{g} \) and \( \hat{h} \) \([22-25]\). The allowed combinations of \( \Lambda \) and \( \lambda \) are in fact specified by the branching functions \( b^\lambda_\Lambda \) which tell how the character \( \chi_\Lambda \) can be decomposed into characters \( \chi_\lambda \) for admissible \( \lambda \in \hat{h} \):

\[
\chi_\Lambda = \sum_\lambda \chi_\lambda b^\lambda_\Lambda
\]  

(3.3)

Generically the branching functions \( b^\lambda_\Lambda \) are the characters of primary fields in the coset models (conditions under which this identification fails will be discussed below). The selection rules imply a restriction on the above summation.

Although the generic relation between the characters of the coset primary fields and those of the coset building blocks is not a simple ratio, it turns out to be so for the modular matrices \( S \) and \( T \):

\[
S^{(\hat{g}/\hat{h})}_{\{\Lambda,\lambda\}\{\Lambda',\lambda'\}} = S^{(\hat{g})}_{\Lambda\Lambda'} S^{(\hat{h})}_{\lambda\lambda'}^\ast
\]

(3.4)

\[
T^{(\hat{g}/\hat{h})}_{\{\Lambda,\lambda\}\{\Lambda',\lambda'\}} = T^{(\hat{g})}_{\Lambda\Lambda'} T^{(\hat{h})}_{\lambda\lambda'}^\ast
\]

(3.5)

(Because of the unitarity and symmetry of \( S \) and \( T \), we replaced the inverses of \( S \) and \( T \) by complex conjugates.)

A direct consequence of (3.1) is that

\[
h_{\{\Lambda,\lambda\}} = h_\Lambda - h_\lambda \quad (\text{mod 1}) \quad ,
\]

(3.6)

since \( h \) is the eigenvalue of \( L_0 \). (The reason for the \( \text{(mod 1)} \) is that coset primary fields may be built from descendent WZNW fields). This implies that the phase factor acquired by the character \( \chi_{\{\Lambda,\lambda\}} \) under the modular transformation \( \tau \mapsto \tau + 1 \) is

\[
\exp \left\{ 2\pi i \left( h_{\{\Lambda,\lambda\}} - \frac{1}{2\pi} c_{\hat{g}/\hat{h}} \right) \right\}
\]

(3.7)

Let us now discuss in more detail the selection rules which prevent some combinations of weights \( \Lambda \) and \( \lambda \) from being coset primary fields, and the related question of field identification. Here we will restrict ourselves to the unitary case, leaving to the next section a detailed analysis of the complications brought by non-unitarity.

As mentioned above, the selection rules reflect the existence of certain relations between the centers of the covering groups of \( \hat{g} \) and \( \hat{h} \). Denote by \( G \) and \( H \) the respective covering groups of \( \hat{g} \) and \( \hat{h} \), and by \( B(G) \) and \( B(H) \) their centers. There is an isomorphism between \( B(G) \) and the group \( O(\hat{g}) \) of outer isomorphisms of \( \hat{g} \). To \( A \in O(\hat{g}) \) corresponds an element \( \alpha \) of \( B(G) \) whose eigenvalue on \( \Lambda \), the highest
weight of a \(\bar{g}\) representation, is \(\exp[2\pi i (A\omega^0, \Lambda)]\). This action extends uniquely to the affine case, with the result

\[
\alpha\Lambda = A e^{2\pi i (A\omega^0, \bar{\Lambda})}
\]  

(3.8)

Similarly, let \(\check{\alpha} \in B(H)\) be related to \(\check{A} \in O(\hat{h})\). Then the elements \(\alpha\) and \(\check{\alpha}\) of the centers can be identified if and only if

\[
(A\omega^0, \Lambda) = (\check{A}\omega^0, \lambda) \pmod{1}
\]  

(3.9)

If this is not satisfied, the coset primary field \(\{\Lambda; \lambda\}\) does not appear.

Let us now turn to field identifications. The isomorphism between \(B(G)\) and \(O(\hat{g})\) and the above selection rule imply that the characters of \(\{\Lambda; \lambda\}\) and \(\{A\Lambda; \check{A}\lambda\}\) are equal. At the level of the \(S\) matrix, this implies [22-25]

\[
S_{\{\Lambda; \lambda\}}\{\Lambda'; \lambda'\} = S_{\{A\Lambda; \check{A}\lambda\}}\{\Lambda'; \lambda'\}
\]  

(3.10)

for all fields \(\{\Lambda'; \lambda'\}\). Therefore the fields \(\{\Lambda; \lambda\}\) and \(\{A\Lambda; \check{A}\lambda\}\) are not distinct, and should be identified.

If there are no fixed points, i.e. no coset field \(\{\Lambda; \lambda\}\) and no \(A \in O(\hat{g})\) such that \(\{\Lambda; \lambda\} = \{A\Lambda; A\lambda\}\) (strict equality), then the string of field identification has a constant length. For such cases, the branching functions are the characters of the coset primary fields. If there are fixed points this is no longer true.\(^c\) The fixed points have to be resolved (for a general discussion and many examples of fixed point resolution, see [26,27]). The modular matrix as given in (3.4) always describes how the branching functions transform under modular transformations. Happily, fixed points do not prevent us from identifying fields using this matrix.

4. Nonunitary diagonal cosets.

Let us now turn to diagonal cosets of the type (1.1) with \(m\) fractional. In the following we will denote a diagonal coset primary field by \(\{\gamma, \lambda; \lambda'\}\), where \(\gamma \in P_+^l\), \(\lambda \in P_y^m\) and \(\lambda' \in P_{y'}^{m+l}\).

Fixed points are ignored throughout. Hence we freely identify branching functions with characters of the coset fields. This is certainly not true when there are fixed points. However, as far as field identification is concerned, this is immaterial.

4.1 Character decomposition.

As already mentioned, not all combinations of admissible weights produce coset primary fields. Allowed triplets are specified by the tensor product decomposition of the characters:

\[
\chi_\gamma^{(l)} \chi_\lambda^{(m)} = \sum_{\lambda'} \chi_{\{\gamma, \lambda; \lambda'\}} \chi_{\lambda'}^{(m+l)}
\]  

(4.1)

\(^c\) In the event of fixed points, the formula (3.4) yields an \(S^\dagger S\) different from the identity, but only in some of its diagonal elements which are then positive integers different from 1. The same is true of \(C = S^2\).
with the following conditions on $\lambda'$:  
\[ \gamma + \lambda - \lambda' \in Q \quad , \quad \lambda' \in P_{y}^{m+l} \quad , \quad \lambda^F = \lambda'^F , \]  
(4.2)

where $Q$ is the root lattice of $\hat{g}$. Notice that for a diagonal coset, (3.9) becomes \((A\omega_{0}, \gamma + \lambda - \lambda') \in \mathbb{Z}, \forall A \in O(\hat{g})\), which is satisfied when $\gamma + \lambda - \lambda'$ lies in the root lattice. However, if $\lambda^F = \lambda'^F$, then $\gamma + \lambda - \lambda'$ is an integrable weight, and the condition $\gamma + \lambda - \lambda' \in Q$ is equivalent to (3.9). \(\chi_{(\gamma, \lambda; \lambda')}\) is the character of the primary field \(\{\gamma, \lambda; \lambda'\}\) (at this stage we ignore the question of fixed points). Eq. (4.1) has been proven by Kac and Wakimoto for $l = 1$, and it is a conjecture of these authors in the general case [1-3]. (It was also established in [9] for $\hat{g} = su(N)$, under the hypothesis that one can restrict oneself to the $y = 1$ sector and ignore the fractional part. The validity of these two assumptions is justified a posteriori by our analysis of field identifications.)

Let us stress some of the salient features of the decomposition (4.1):

i) $\lambda$ and $\lambda'$ are associated with the same Weyl group element $y$, i.e., $y' = y$.

ii) The fractional parts of $\lambda$ and $\lambda'$ are equal. This identification is made possible because $\lambda^F$ and $\lambda'^F$ have the same level $(u - 1)$, despite the fact that the levels of $\lambda$ and $\lambda'$ are different. The fractional part appears then as a conserved charge under tensor product decomposition.

iii) The condition constraining the sum (4.1) can be rewritten as  
\[ \gamma + \lambda^I + l\lambda^F: y = \lambda'^I \quad (\mod Q) \]  
(4.3)

Indeed, since $\lambda^F: y = \lambda'^F: y$, one has  
\[ \gamma + \lambda - \lambda' = \gamma + y(\lambda^I - \lambda'^I + \lambda^F: y) \]  
(4.4)

The invariance of $Q$ under the action of $y$ yields (4.3) from (4.2).

The dimensions of the coset primary fields can be extracted from the expansion of the branching function in powers of $q = e^{2\pi i \tau}$. However, since the latter is not known in general, we can only write the fractional part of $h$, since  
\[ h = h_{\gamma} + h_{\lambda} - h_{\lambda'} \quad (\mod 1) \]  
(4.5)

This can be written in the form  
\[ h = \frac{|p(\lambda' + \rho) - p'(\lambda + \rho)|^2 - (p - p')^2|\rho|^2 + (\gamma, \gamma + 2\rho)}{2(l + g)} - \frac{|\lambda - \lambda'|^2}{2l} \quad (\mod 1) \]  
(4.6)

in terms of the coprime numbers $p$ and $p'$ introduced in (1.4). Notice that  
\[ p = m^I + g \quad , \quad p' = m^I + g + lu \quad , \]  
(4.7)
which fixes the bounds $p \geq g$ and $p' \geq g + lu$. Notice that in terms of the integral and fractional parts of the weights, (4.6) can be rewritten as

$$h = \frac{|p(\lambda^I + \rho) - p'(\lambda^I + \rho)|^2 - (p - p')^2|\rho|^2}{2lp'} + \frac{\gamma + \gamma + 2\rho}{2(l + g)} - \frac{|\lambda^I - \lambda^I + l\lambda^F,\gamma|^2}{2l} \pmod{1}, \quad (4.8)$$

For $l = 1$ and simply laced algebras, the last two terms in (4.6) cancel. This can be seen as follows. One first rewrites the last two term under the form (with $A\mu$ integer).

For simply laced algebras $(\gamma, \beta) \in \mathbb{Z}$ and $|\beta|^2 \in 2\mathbb{Z}$. Therefore, modulo 1, the last two terms of the above disappear when $l = 1$. On the other hand, a characteristic feature of simply laced algebras is that any $\gamma \in P^I_+$ can be written as $A\omega^0$ for some $A$. Now, using (2.36), one finds that

$$\frac{|A\omega^0 + \rho|^2 - |\rho|^2}{2(g + 1)} = \frac{|A\omega^0|^2}{2} = (A\omega^0, w_A\rho) + \frac{g}{2}|A\omega^0|^2 \quad (4.10)$$

Now, since $w_A\rho = A\rho - g(A\omega^0 - \omega^0)$, this becomes

$$\frac{g}{2}|A\omega^0|^2 + (A\omega^0, \rho) \quad (4.11)$$

But it is true in general (i.e., for all Lie algebras) that the above expression is an integer.

$4.2$ Modular $S$ matrix for the coset.

In terms of the coset components, the coset modular $S$ matrix is

$$S_{\gamma, \lambda, \lambda'}(\xi, \mu; \mu') = S^{(l)}_{\gamma \xi} S^{(m)}_{\lambda \mu} \left[ S^{(m+l)}_{\lambda' \mu'} \right]^* \quad (4.12)$$

Up to a numerical factor which can be recovered from unitarity, the r.h.s. is

$$S^{(l)}_{\gamma \xi} e^{2\pi i(\lambda^I - \lambda^I, \mu^I) + (\mu^I - \mu^I, \lambda^F) + (\lambda^F, \mu^F)} \times \sum_{w \in W} \epsilon(w) e^{-2\pi i(w(\lambda^I + \rho), \mu^I + \rho)/(m + g)} \sum_{w' \in W} \epsilon(w') e^{2\pi i(w'(\lambda^I + \rho), \mu^I + \rho)/(m + g + l)} \quad (4.13)$$

Since $\lambda^F, \gamma \in r_\mu\mathbb{Z}$, all the factors of $y$ in (4.12) can be eliminated, i.e., we can write $\lambda^F, \gamma$ instead of $\lambda^F$. Next, using the fact that $(\nu + \beta, \sigma) = (\nu, \sigma) \pmod{1}$ if $\beta \in Q$ and if $\sigma \in r_\mu\mathbb{Z}$ for all $\mu$, one can eliminate $\lambda^F, \gamma$ and $\mu^F, \gamma$ by means of (4.7), i.e.

$$\lambda^F, \gamma = \lambda^I - \lambda^I - \gamma \pmod{Q} \quad (4.14)$$

$$\mu^F, \gamma = \mu^I - \mu^I - \xi \pmod{Q}$$
Then (4.13) can be expressed as

\[ S_{\gamma \xi}^{(l)} e^{2\pi i (\gamma, \xi)} e^{-2\pi i (\lambda^I - \lambda^I, \mu^I - \mu^I)} \phi^{(m+g)}_{\lambda^I, \mu^I} \phi^{(m+g+l)}_{\lambda^I, \mu^I} \] (4.15)

where we have defined

\[ \phi^{(k)}_{\lambda \mu} := \sum_{w \in W} \epsilon(w) e^{-2\pi i (w(\lambda + \rho), \mu + \rho) / k} \] (4.16)

For \( l = 1 \) and the simply-laced algebras \( A_r \) and \( D_r \), this result can be further simplified. Indeed, in that case

\[ S_{\gamma \xi}^{(1)} e^{2\pi i (\gamma, \xi)} = F_1 \] (4.17)

where \( F_1 \) is defined in (2.22). The clue to the proof of this identity is again the observation that, for those algebras, any level one integrable weight can be written as \( A\omega^0 \) for a suitable outer automorphism \( A \). Thus, writing \( \gamma = A\omega^0 \) and \( \xi = A'\omega^0 \), and using twice the result

\[ S_{A\lambda^I, \mu^I} = e^{-2\pi i (A\omega^0, \mu^I)} S_{\lambda^I, \mu^I} \] (4.18)

one easily finds

\[ S_{\gamma \xi}^{(1)} e^{2\pi i (\gamma, \xi)} = e^{2\pi i (A\omega^0, A'\omega^0)} S_{A\omega^0, A'\omega^0}^{(1)} = S_{\omega^0, \omega^0}^{(1)} = F_1 \] (4.19)

Hence, up to a multiplicative constant, the modular \( S \) matrix for \( A_r \) and \( D_r \) cosets at \( l = 1 \) is

\[ S_{\{\gamma, \lambda^I \} \{\xi, \mu^I \}} \propto e^{2\pi i [(\lambda^I + \rho, \mu^I + \rho) + (\lambda^I + \rho, \mu^I + \rho)]} \phi^{(p/p')}_{\lambda^I, \mu^I} \phi^{(p'/p)}_{\lambda^I, \mu^I} \] (4.20)

This reproduces the result found by Fateev and Lykyanov for the minimal models of \( Wsu(N) \) and \( Wso(2N) \) conformal algebras, obtained by means of the Feigin-Fuchs representation [10]. The expression (4.20) is independent of \( y \) and depends on \( \gamma \) and \( \lambda^F,y \) only through the condition (4.3) relating \( \lambda^I \) to \( \lambda^I \). For simply laced algebras, for any choice of \( \lambda^I \) there exists a \( \gamma \) and a \( \lambda^F,y \) yielding any possible \( \lambda^I \in D^m_{\pm} + u \).

Hence one can equivalently describe a coset field by a doublet \( (\lambda^I | \lambda^I) \) where the two integrable weights are independent, the natural standpoint in the Feigin-Fuchs description of \( \hat{W}_g \) minimal models.

### 4.3 Field Identification

Two coset primary fields \( \phi \) and \( \phi' \) can be identified if their modular properties are identical, i.e. if, for any field \( \psi \), we have [23]

\[ S_{\phi \psi} = S_{\phi' \psi} \] (4.21)
and if their conformal dimensions are the same. This guarantees that the characters \( \chi_\phi(\tau, z) \) and \( \chi_{\phi'}(\tau, z) \) are identical. Let us now describe generic classes of field identifications.

4.3.1 Field identification from outer automorphisms.

As in the unitary case one expects the action of any element \( A \) of the group of outer automorphisms of \( \hat{g} \) to yield a field identification. Define the action of \( A \) on the coset field by

\[
A\{\gamma, \lambda; \lambda'\} = \{A\gamma, A\lambda; A\lambda'\}
\]

(4.22)

A straightforward calculation shows that

\[
AS^{(m)}_{\lambda, \mu} = S^{(m)}_{A\lambda, A\mu} e^{-2\pi i (A\omega^0, \mu) - 2\pi i ((wA-1)\lambda^F, \mu + \rho)}.
\]

(4.23)

Therefore, for the coset \( S \) matrix given by (4.12), one has

\[
AS\{\gamma, \lambda; \lambda'\}{\xi, \mu; \mu'} = S\{A\gamma, A\lambda; A\lambda'\}{\xi, \mu; \mu'} = \alpha S\{\gamma, \lambda; \lambda'\}{\xi, \mu; \mu'},
\]

(4.24)

where

\[
\alpha = e^{-2\pi i [(A\omega^0, \xi + \mu - \mu') + ((wA-1)\lambda^F, \mu + \rho) - ((wA-1)\lambda^F, \mu' + \rho)]}.
\]

(4.25)

Since \( \lambda^F = \lambda'^F \), the last two terms in the phase become \( ((wA-1)\lambda^F, \mu - \mu') \). Furthermore, the branching rules demand that \( \mu - \mu' \) be an integral weight, in which case \( ((wA-1)\lambda^F, \mu - \mu') = 0 \pmod{1} \). Similarly, \( \xi + \mu - \mu' \) is an element of the \( \bar{g} \) root lattice, whose inner product with \( A\omega^0 \) is necessarily an integer. Therefore \( \alpha = 1 \) and \( A \) indeed produces a field identification.

It is also straightforward to check that the action of \( A \) on coset fields does not affect (4.6). Indeed, from (2.13) and (2.36), one has

\[
h_{A\lambda} = h_\lambda + (A\omega^0, wA(\lambda + \rho)) + \frac{1}{2}(m + g)|A\omega^0|^2
\]

(4.26)

so that

\[
h_{A\gamma} + h_{A\lambda} - h_{A\nu} = h_\gamma + h_\lambda - h_\nu + (A\omega^0, wA(\gamma + \lambda - \lambda' + \rho)) + \frac{g}{2}|A\omega^0|^2
\]

(4.27)

Since \( \gamma + \lambda - \lambda' = \beta \in Q \), since \( Q \) is invariant under \( W \), and since \( (A\omega^0, \beta) \in Z \), the last two terms reduce to (4.11), i.e., to an integer.

The above calculation illustrates the dual relation between selection rules and field identifications. If \( \alpha \neq 1 \) the branching function vanishes and, on the other hand, \( \alpha = 1 \) implies that fields related by an outer automorphism must be identified. However, the situation is not as simple as in the integer level case, where \( \alpha \) did not depend upon the \( \lambda \) fields. Then \( \alpha \) was directly related to a difference between elements of the covering group for the coset numerator and denominator. Only the first part of the phase (4.25) appeared. Nevertheless, we fall back to this situation once we know that the fractional part is conserved in tensor product decompositions.
For unitary coset models, this is the whole story on field identifications. However, in the nonunitary case, the problem is much richer, as we will now see.

4.3.2 Field identification in the fractional sector.

The expression (4.20) for the modular S matrix depends only on the integer parts of the weights, i.e. it does not explicitly depend on $y$ or on $\lambda_{F,y}$. This is an immediate source of field identification: Two coset fields $\{\gamma, \lambda; \lambda'\}$ and $\{\xi, \mu; \mu'\}$ will be identified if

$$\begin{align*}
g = \xi \\
\lambda = \mu \\
\lambda' = \mu' \tag{4.28}
\end{align*}$$

Equality of the fractional parts modulo the coroot lattice $Q^\vee$ instead of the root lattice is required in order to preserve the conformal dimension. This can be seen from (4.8): a shift of $\lambda_{F,y}$ by an element of $Q^\vee$ (without modifying the other fields) does not change $h$ modulo $\mathbb{Z}$. Such a shift does not affect the branching condition since $Q^\vee \subset Q$.

A large class of field identifications can be obtained by assuming that $\lambda_{F,y} = \mu_{F,y}'$ (strict equality) and $y \neq y'$. Then $\mu = w.\lambda$ and $\mu' = w.\lambda'$, where $w = y'y'^{-1}$. Since $\lambda$ and $\lambda'$ have the same fractional part, they share the same set $R_{\lambda}$ and the same associated subgroup $W_{\lambda}$. Thus if $w.\lambda$ is admissible, so is $w.\lambda'$. Furthermore, it is clear that if $\gamma, \lambda$ and $\lambda'$ satisfy the branching condition, this condition is also satisfied by the weights $\gamma, w.\lambda$ and $w.\lambda'$. Finally, the conformal dimension (4.5) of the coset field is not affected by a simultaneous shifted action of the Weyl group on any weight of the coset field. Thus, one has the identification

$$\{\gamma, \lambda; \lambda'\} \sim \{\gamma, w.\lambda; w.\lambda'\}$$

This identification may be seen more easily from the following relation:

$$S_{\mu,\lambda}^{(m)} S_{\mu',\lambda'}^{(m+l)} = \phi_{\mu,\lambda}^{(m+g)} \phi_{\mu',\lambda'}^{(m+g+l)} e^{2\pi i (\mu, \lambda - \lambda')} \tag{4.29}$$

which is manifestly invariant under the simultaneous actions $\mu \rightarrow w.\mu$ and $\mu' \rightarrow w.\mu'$.

Field identifications obtained by the shifted action of the Weyl group arise only in the fractional sector (i.e., the sector with non-integer finite weights). Indeed, we have shown that for a given admissible weight $\lambda$, the number of elements $w$ such that $w.\lambda$ is also admissible is $n(W)/n(W^\lambda)$ (including $w = 1$). For integrable weights $W^\lambda = W$, hence this yields no field identification. Thus, coset fields occur in sets of $n(W)/n(W^\lambda)$ fields identifiable with shifted actions of the Weyl group.

This class of field identification, together with identification from outer automorphisms, appears to perform all necessary field identifications for the classical algebras $A_r$, $B_r$, $C_r$ and $D_r$. This belief is based on the analysis of a large number of examples, performed by computer. For the algebras $A_r$ and $D_r$, we notice that all
these identifications occur within the \( y = 1 \) sector, while this isn’t true for \( B_r \) and \( C_r \). On the other hand, for the exceptional algebra \( G_2 \), it is only within the \( y = 1 \) sector that all identifications are of the above type. Otherwise, field identifications of the type (4.28) with \( \lambda^{F,y} \neq \mu^{F,y'} \) are necessary. For all algebras studied, there is always a coset field representative in the \( y = 1 \) sector with \( \bar{\lambda}^F = 0 \) (see next subsection).

A special case of identifications by shifted action of \( W \) is particularly useful for \( su(N) \) cosets. Let us associate with each element \( A \in O(g) \) an operator \( B_A \) whose action on a weight \( \lambda \) is defined by

\[
B_A \lambda := \begin{cases} 
y w_A y^{-1} \lambda & \text{if the result is admissible with the same } y \\
\lambda & \text{otherwise}
\end{cases}
\]  

(4.30)

Notice that

\[
y w_A y^{-1} \lambda = y.(A\lambda^I - (m + g)(A\lambda^{F,y} + \omega^0) - \omega^0)
\]  

(4.31)

This particular shifted action of \( W \), together with outer automorphisms, will be sufficient to perform all field identifications for \( A_r \).

4.3.3 A class of coset primary field representatives.

The results of the last subsection suggest that it is possible to find a proper set of coset primary fields characterized by a vanishing finite fractional part, i.e., fields of the form

\[
\{ \gamma, \lambda^I; \lambda'^I \} \quad \text{with} \quad \bar{\gamma} + \bar{\lambda}^I - \bar{\lambda}'^I = 0 \pmod{Q},
\]

(4.32)

modulo the action of the outer automorphism group. For \( su(N) \), it turns out that it is always possible to choose inequivalent \( y \)'s such that all \( \lambda^{F,y} \in P_u^{-1} \) (see appendix C). If \( y \neq 1 \), some Dynkin labels must satisfy a stronger constraint than \( \lambda^{F,y} \geq 0 \). As a result, for \( su(N) \), the set of admissible \( \lambda^{F,y} \) for \( y \neq 1 \) is a proper subset of the set of admissible \( \lambda^{F,1} \). From (4.28), it is therefore manifest in this case that all fields from the \( y \neq 1 \) sectors can be identified with fields of the \( y = 1 \) sector, and it is sufficient to consider this sector only. Moreover, in the \( y = 1 \) sector, all fields with \( \bar{\lambda}^{F,1} \neq 0 \) (the finite part) can be related to fields with \( \bar{\lambda}^{F,1} = 0 \) by using the operators \( A \) and \( B_A \). This will be illustrated in the next section where canonical chains of field identifications are constructed.

For other classical Lie algebras, the group of outer automorphisms is not sufficiently large to relate all fields with \( \bar{\lambda}^{F,y} \neq 0 \) to those with \( \bar{\lambda}^{F,y} = 0 \), even in the \( y = 1 \) sector. Furthermore, we cannot choose representative \( y \)'s in \( W/W(A) \) such that \( \lambda^{F,y} \in P_u^{-1} \), so that a priori we have no reason to restrict ourselves to the \( y = 1 \) sector. Nevertheless, we can still use Eq.(4.28) to identify coset fields, and for all examples we have considered it is possible to pick a set of primary field
representatives by restricting the search to \( y = 1 \) and \( \bar{\lambda}^{F,1} = 0 \).

5. Examples.

5.1 \( \mathfrak{su}(2) \) with \( l = 1 \) and \( m = -4/3 \).

This coset model has central charge \( c = -\frac{22}{3} \) and, as shown by Cardy [11], describes the Yang-Lee singularity. It is the simplest nonunitary diagonal coset. Here \( m^l = 0 \) so \( \lambda^l \) vanishes. The coset fields are then constructed from the triplets

\[
\gamma_1 = i, \quad \lambda_1 = -(m + g) \lambda_1^F = -\frac{2}{3}k \quad \text{and} \quad \lambda_1' = \lambda_1^H - (m + g + 1) \lambda_1^F = n' - \frac{2}{3}k
\]

which satisfy the relation \( i + k - n' = 0 \) (mod 2) (here \( 0 \leq i \leq l = 1, 0 \leq k \leq u - 1 = 2 \) and \( 0 \leq n' \leq m^l + u = 3 \)). There are 12 coset fields. They can be grouped in two sets according to their conformal dimensions, given by (4.5). Actually, \( h_{\{\gamma, \lambda; \lambda'\}} \geq h_{\gamma} + h_{\lambda} - h_{\lambda'} \), and generically the correct conformal dimension is the maximum of the latter expression, taken over a set of identified coset fields. Using the notation \( \{\gamma, \lambda; \lambda'\} \), the coset fields of dimension 0, corresponding to the identity, are

\[
h = 0 : \{0,0;0\} , \{1,-\frac{2}{3};-\frac{5}{3}\} , \{1,-\frac{4}{3};-\frac{7}{3}\} , \{0,-\frac{2}{3};-\frac{10}{3}\} , \{1,0;3\}
\]

(5.1)

All other fields have dimension \( h = -\frac{1}{3} \) (mod 1):

\[
h = -\frac{1}{3} : \{1,0;1\} , \{0,-\frac{2}{3};-\frac{5}{3}\} , \{0,-\frac{4}{3};-\frac{7}{3}\} , \{1,-\frac{2}{3};-\frac{5}{3}\} , \{1,-\frac{4}{3};-\frac{7}{3}\} , \{0,0;2\}
\]

(5.2)

The equality of their conformal dimensions suggests that the fields in each set can be identified (this is of course a necessary but not sufficient condition for field identification). The identification of the fields within each set can be established by the action of the operators \( a \) [\( a[\lambda_0, \lambda_1] = [\lambda_1, \lambda_0] \)] and \( b := B_a \) (see Eq. (4.30)). It is very simple to check that, in each of the above two sets and in the order shown, the fields are related by the following chain of operators: \( ababa \). For example,

\[
\{1,-\frac{2}{3};-\frac{5}{3}\} = ba\{0,0;0\} , \{1,0;3\} = ababa\{0,0;0\}
\]

(5.3)

Notice that the above chain of identifications starts with a coset field with \( \bar{\lambda}^{F} = k = 0 \) and ends up with another field with \( k = 0 \), on which a further application of \( b \) is neutral. Since the chain of operators has 5 elements, one obtains 6 (= 2n) field identifications.

5.2 \( \mathfrak{su}(2) \) with \( l = 1 \) and \( m = -1/2 \).

In this example \( u = 2 \) and \( m^l = 1 \). One finds 16 fields which can be grouped into four sets according to their conformal dimensions modulo 1:

\[
h = 0 : \{0,0;0\} , \{1,-\frac{1}{2};\frac{1}{2}\} , \{1,-\frac{3}{2};-\frac{1}{2}\} , \{0,1;3\}
\]

\[
h = -\frac{1}{3} : \{1,0;1\} , \{0,-\frac{1}{2};-\frac{1}{2}\} , \{0,-\frac{3}{2};-\frac{1}{2}\} , \{1,1;2\}
\]

\[
h = -\frac{1}{2} : \{0,0;2\} , \{1,-\frac{1}{2};\frac{1}{2}\} , \{1,-\frac{3}{2};-\frac{1}{2}\} , \{0,1;1\}
\]

\[
h = -\frac{1}{6} : \{1,0;3\} , \{1,-\frac{1}{2};\frac{1}{2}\} , \{0,-\frac{3}{2};\frac{1}{2}\} , \{1,1;0\}
\]

(5.4)
One verifies that within each set all the fields can be identified according to the chain of operators \(aba\). Again, starting with a field with \(k = 0\), one terminates on another field with \(k = 0\). Here we have \(4 (= 2u)\) field identifications.

5.3 \(\s(u(3))\) with \(l = 1\) and \(m = -3/2\).

This is the simplest \(\s(u(3))\) diagonal nonunitary coset model, with \(u = 2\) and \(m^l = 0\). For \(su(3)\), the condition that \(\bar{\mu} = (\mu_1, \mu_2) \in Q\) is equivalent to

\[
2\mu_1 + \mu_2 = 0 \pmod{3} \quad \mu_1 + 2\mu_2 = 0 \pmod{3} \tag{5.5}
\]

By applying the above conditions one finds 18 coset fields in the \(y = 1\) sector \((24\) in all sectors\)\(^d\) which are grouped into two sets of conformal dimensions \(0\) and \(-\frac{1}{2}\) (modulo 1). In the notation \(\{\gamma, \lambda; \bar{\lambda}\}\), the states with \(h = 0\) (mod 1) are

\[
\begin{align*}
\{(0, 0), (0, 0); (0, 0)\} & , \{(1, 0), (-\frac{3}{2}, 0); (-\frac{1}{2}, 0)\} & , \{(0, 1), (0, -\frac{3}{2}); (0, -\frac{1}{2})\} \\
\{(0, 1), (-\frac{3}{2}, 0); (-\frac{1}{2}, 0)\} & , \{(0, 0), (0, -\frac{3}{2}); (2, -\frac{3}{2})\} & , \{(1, 0), (0, 0); (0, 2)\} \\
\{(0, 0), (-\frac{3}{2}, 0); (-\frac{1}{2}, 2)\} & , \{(1, 0), (0, -\frac{3}{2}); (0, -\frac{3}{2})\} & , \{(0, 1), (0, 0); (2, 0)\}
\end{align*}
\]

They can be identified (in the above order) with the following chain:

\[a^2ba^4b = [(a^2b)a^2]^2 \tag{5.6}\]

(Recall that \(a[\lambda_0, \lambda_1, \lambda_2] = [\lambda_2, \lambda_0, \lambda_1]\) and \(b := B_0\).) This chain contains a loop since two fields in the middle of the chain are reproduced twice \((a^3 = 1)\). This is due to the fact that the action of \(b\) is neutral on \(a^2ba^2\Lambda\), where \(\Lambda\) is a state with \(\bar{\lambda}^{F;1} = 0\). However, it is not neutral on \(aba^2\Lambda\), and in order not to miss the state \(a^2ba^2\Lambda\) a loop has to be introduced. The chain contains 8 distinct elements, so that 9 fields can be identified. The same analysis holds for the fields with \(h = -\frac{1}{5}\). We therefore found another coset model with two independent fields (with \(h = 0\) and \(h = -\frac{1}{5}\)). Its central charge is again \(c = -\frac{22}{5}\), and it yields another representation of the Yang-Lee singularity.

5.4 Remark on the KNS duality for \(\s(u(N))\) diagonal cosets with \(l = 1\) and \(m^l = 0\).

The fact that the Yang-Lee singularity can be described either by an \(\s(u(2))\) or an \(\s(u(3))\) diagonal coset has first been observed in [9]. It is the simplest example of a general duality between \(u\) and \(N\) for \(\s(u(N))\) diagonal cosets with \(l = 1\) and \(m^l = 0\). For such models the central charge is

\[c = -\frac{(N - 1)(u - 1)(N + u + Nu)}{N + u} \tag{5.8}\]

\(^d\) We have shown in appendix C that, for \(su(N)\), we can always choose the \(y's\) such that \(\lambda^{F,y} \in P^u_{+}u^{-1}\), and, as argued in section 4.3.4, this allows us to concentrate on the \(y = 1\) sector.
which makes manifest the $u \leftrightarrow N$ duality [9,28]. Hence $\hat{su}(N)$ diagonal cosets with $l = 1$ and $m = N/u - N$ are equivalent to $\hat{su}(u)$ diagonal cosets with $l = 1$ and $m = u/N - u$. Note that the condition $m' = 0$ fixes $t$ to be $t = -N(u - 1)$. Since $u$ and $t$ must be coprime, $u$ and $N$ must also be coprime. The Yang-Lee singularity corresponds to the pair $(N,u) = (2,3)$. The next simplest case is $(N,u) = (3,4)$, with $c = -\frac{114}{7}$. (We have analyzed the corresponding two cosets and checked the equivalence of the two models directly, from their modular $S$ matrices). This duality has been proved under the assumption that it is possible to consider only the $y = 1$ sector with weights such that $\bar{\lambda}F,1 = 0$. That assumption is now justified by our analysis of field identifications.

5.5 Remark on the number of field identifications for $\hat{su}(N)$ cosets with $l = 1$.

As already pointed out, $\hat{su}(N)$ diagonal cosets with $l = 1$ provide an alternate description of minimal $W_N^{(p,p')}$, models, with $p = m' + N$ and $p' = p + u$. By showing how both descriptions yield the same modular $S$ matrix, we have established this result rigorously for all values of $p$ and $p'$. In the Feigin-Fuchs representation, the primary fields are described by two independent integrable weights $\lambda^I$ and $\lambda'^I$ of levels $m^I$ and $m^I + u$ respectively, and the fields are identified according to

$$(\lambda^I | \lambda'^I) \sim (a^i \lambda^I | a^i \lambda'^I) , \quad 1 = 1, \ldots, N$$

where $a$ denotes a cyclic permutation of the affine Dynkin labels. Since the number of integrable fields at level $m^I$ is $(m^I + N - 1)!/(m^I!(N - 1)!)$, the total number of inequivalent fields (obtained by multiplying the above number by similar expression with $m^I \to m^I + u$ and dividing by the order of the outer-automorphism group) is

$$\frac{(m^I + N - 1)!(m^I + N - 1 + u)!}{m^I!(m^I + u)!N!(N - 1)!} \quad (5.10)$$

In the coset description, there are $N$ possible values of $\gamma$, but (4.2) reduces the number of $\lambda'^I$ fields by a factor of $N$. Thus, in the $y = 1$ sector, there are

$$\frac{(m^I + N - 1)!(m^I + N - 1 + u)!(m^F + N - 1)!}{m^F!m^I!(m^I + u)![(N - 1)!)^3} \quad (5.11)$$

fields. Comparison of (5.10) and (5.11) shows that there must be

$$\frac{(u + N - 2)!N}{(u - 1)!(N - 1)!} \quad (5.12)$$

field identifications.

For $u = 1$, this number reduces to $N$, as it should, being the order of the automorphism group. For $\hat{su}(2)$, it reduces to $2u$ (compare with sections 5.1 and 5.2) while for $\hat{su}(3)$ it is $\frac{3}{2}u(u + 1)$ (compare with section 5.3).
The above argument is specific to \( l = 1 \), since only then are there \( N \) different \( \gamma \)'s, and only then does the constraint (4.2) amount to restrict the number of fields by a factor of \( N \). Furthermore, there are no fixed points if \( l = 1 \).

5.6 Canonical chains of field identifications for \( \hat{su}(N) \) diagonal cosets.

It is straightforward to generalize the chains of field identifications obtained before for \( \hat{su}(2) \) with \( u = 2, 3 \) and \( l = 1 \). For a general value of \( u \) it is simply \((ab)^{u-1}a\), starting and ending on states with \( k = 0 \). This yields \( 2u \) fields identified, in agreement with (5.12). Notice that apart from the extremities of the chain, there are no states with \( k = 0 \). This means that the coset fields with \( k = 0 \) are always identified according to

\[
\{ \gamma, \lambda^I; \lambda'^I \} \sim (ab)^{u-1}a\{ \gamma, \lambda^I; \lambda'^I \}
\] (5.13)

Since on \( \lambda^I \) and \( \lambda'^I \) the action of \( b \) is the same as that of \( a \), this can be rewritten as

\[
\{ \gamma, \lambda^I; \lambda'^I \} \sim \{ a^u\gamma, a\lambda^I; a\lambda'^I \}
\] (5.14)

Therefore, it is only for odd \( u \) (and in particular for the unitary case \( u = 1 \)) that the naive field identification \( \{ \gamma, \lambda^I; \lambda'^I \} \sim a\{ \gamma, \lambda^I; \lambda'^I \} \) is sufficient. This simple remark shows that it would not have been possible to understand properly the field identifications by restricting ourselves from the onset to fields with \( \lambda^F = 0 \).

It is also simple to work out the canonical chain of field identifications for \( \hat{su}(3) \) for arbitrary values of \( u \). The result is

\[
\prod_{i=1}^{u-1} (a^2b)^{i}a^2 \left( a^2b\right)^{u-1}a^2\Lambda
\] (5.15)

where again we denote by \( \Lambda \) a field with \( \bar{\lambda}^{F,1} = 0 \). For \( u = 2 \) this reduces to the chain found in section 5.3. To count the number of distinct elements, one multiplies by 3 the number of factors of \( a^2b \), and adds 3 (2 for the first factor of \( a^2 \) and 1 for the initial state). The result is \( \frac{3}{2}u(u+1) \), in agreement with (5.12). In each chain there are two \( \Lambda \) fields apart from the initial one, namely \((a^2b)^{u-1}a^2\Lambda\) and \([(a^2b)^{u-1}a^2]^2\Lambda\). For these fields the analog of (5.14) is

\[
\{ \gamma, \lambda^I; \lambda'^I \} \sim \{ a^u\gamma, a\lambda^I; a\lambda'^I \} \sim \{ a^{2u}\gamma, a^2\lambda^I; a^2\lambda'^I \}
\] (5.16)

As a final remark we display the canonical chain of field identification for \( \hat{su}(N) \) with \( u = 2 \):

\[
[(a^rb)a^r]^{\bar{r}}\Lambda \quad , \quad (r = N - 1)
\] (5.17)

This yields \( N^2 \) field identifications, in agreement with (5.12).

It should be stressed that \( \hat{su}(N) \) canonical chains depend only upon \( u \), and not on \( m^I \) or \( l \). For special values of \( l \) and \( m^I \) there could be fixed points of \( a^I \) (for some \( i \)), but this simply means that some chains will be truncated, without affecting
their completeness. Such truncations can only arise from $a^l$ fixed points. Indeed, as is easily seen, states with $\lambda^F \neq 0$ cannot be neutral under the action of $b$ (this argument is specific to $su(N)$). A necessary condition for an $su(N)$ coset field to be fixed under the action of $b$ is that both $m^t$ and $u$ be multiples of $N$ (the dual Coxeter number for $su(N)$). But this violates the requirement that $t$ and $u$ be coprime.

5.7 $\hat{su}(2)$ with $l = 2$ and $m = -4/3$.

This is the simplest example of a non-unitary coset with $l \neq 1$. When $l$ is even the $\hat{su}(2)$ branching conditions are independent of $\lambda^F$. All the coset fields can be grouped into six families as in table 2.

All the fields within each set can be identified, according to the chain $(ab)^2a$, in the order above. However, in the last set, it is truncated to $ab$, the state $\{1, -\frac{4}{3}; \frac{2}{3}\}$ being a fixed point of $a$. This is possible because $l$, $m^t$ and $u - 1$ are all multiples of 2, the dual Coxeter of $su(2)$.

Table 2: Coset fields of $\hat{su}(2)$ at $m = -4/3$ ($y = 1$).

| $h$         | $\frac{1}{2}$ | $\frac{1}{4}$ | $-\frac{1}{4}$ | $-\frac{3}{32}$ | $-\frac{7}{32}$ |
|-------------|---------------|----------------|-----------------|------------------|-----------------|
| $\{0,0;0\}$ | $\{2,0;0\}$  | $\{0,0;2\}$   | $\{2,0;2\}$    | $\{1,0;1\}$     | $\{1,0;3\}$    |
| $\{2,-\frac{4}{3}; \frac{2}{3}\}$ | $\{2,-\frac{4}{3}; -\frac{1}{3}\}$ | $\{0,-\frac{4}{3}; -\frac{1}{3}\}$ | $\{0,-\frac{4}{3}; -\frac{4}{3}\}$ | $\{1,-\frac{4}{3}; -\frac{1}{3}\}$ | $\{1,-\frac{4}{3}; -\frac{7}{3}\}$ |
| $\{2,-\frac{2}{3}; -\frac{5}{3}\}$ | $\{0,-\frac{2}{3}; -\frac{5}{3}\}$ | $\{2,-\frac{2}{3}; -\frac{2}{3}\}$ | $\{0,-\frac{2}{3}; -\frac{2}{3}\}$ | $\{1,-\frac{2}{3}; -\frac{5}{3}\}$ | $\{1,-\frac{2}{3}; -\frac{7}{3}\}$ |
| $\{0,-\frac{2}{3}; \frac{10}{3}\}$ | $\{2,-\frac{2}{3}; \frac{10}{3}\}$ | $\{0,-\frac{2}{3}; \frac{10}{3}\}$ | $\{2,-\frac{2}{3}; \frac{10}{3}\}$ | $\{1,-\frac{2}{3}; \frac{7}{3}\}$ | $\{1,-\frac{2}{3}; \frac{7}{3}\}$ |
| $\{0,-\frac{4}{3}; -\frac{10}{3}\}$ | $\{2,-\frac{4}{3}; -\frac{10}{3}\}$ | $\{0,-\frac{4}{3}; -\frac{10}{3}\}$ | $\{2,-\frac{4}{3}; -\frac{10}{3}\}$ | $\{1,-\frac{4}{3}; -\frac{13}{3}\}$ | $\{1,-\frac{4}{3}; -\frac{13}{3}\}$ |
| $\{2,0;6\}$ | $\{0,0;6\}$ | $\{2,0;4\}$ | $\{0,0;4\}$ | $\{1,0;5\}$ | $\{1,0;5\}$ |

$\hat{su}(2)$ diagonal cosets with $l = 2$ describe minimal superconformal models, for which

$$c = \frac{3}{2} \left( 1 - 2 \frac{(p' - p)^2}{pp'} \right)$$

and

$$h_{r,s} = \frac{(rp' - sp)^2 - (p' - p)^2}{8pp'} + \frac{1 - (-)^r s}{32}$$

with $p' - p = 2u$, $m + 2 = 2p/(p' - p)$, $1 \leq r \leq p - 1$ and $1 \leq s \leq p' - 1$. In the present case $p = 2, p' = 8$, and the superconformal fields have dimensions

$$h_{11} = 0, \quad h_{13} = -\frac{1}{4}, \quad h_{12} = -\frac{3}{32}, \quad h_{14} = -\frac{7}{32}$$
The last two lie in the Ramond sector \((r + s \text{ odd})\) while the first two lie in the Neveu-Schwarz sector \((r + s \text{ even})\). Fields in the NS sector are actually superfields whose component fields have dimensions \(h\) (given above) and \(h + \frac{1}{2}\). In this way we recover the spectrum listed in table 2.

It is interesting to notice that the component fields in a given multiplet are related to each other by the action of \(a\) restricted to the field \(\gamma\). The two coset fields \(\{\gamma, \lambda; \lambda'\}\) and \(\{a\gamma, \lambda; \lambda'\}\) form a NS multiplet if \(a\gamma \neq \gamma\), while fields in the Ramond sector are characterized by the condition \(a\gamma = \gamma\).

5.8 \(\mathfrak{so}(8)\) with \(l = 1\) and \(m = -5/2\).

For this coset one finds 24 distinct fields, while the full \(y = 1\) sector contains 384 fields. There are 16 field identifications, without any fixed points. All these fields can be related by the action of \(A\) and \(B\). With

\[
\begin{align*}
a[\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4] &= [\lambda_1, \lambda_0, \lambda_2, \lambda_4, \lambda_3] \\
\tilde{a}[\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4] &= [\lambda_4, \lambda_3, \lambda_2, \lambda_1, \lambda_0]
\end{align*}
\]

(5.21)

and \(b := B_a, \tilde{b} := B_{\tilde{a}}\), the canonical chain of identification is

\[
a\tilde{a}a\tilde{a}b\tilde{a}a\tilde{a}aa
\]

(5.22)

It starts and ends on an integral state (i.e., a state with \(\bar{\lambda}^F = 0\)). Furthermore, there are two intermediate integral states: \(a\tilde{a}a\tilde{a}a\Lambda\) and \(a\tilde{a}b\tilde{a}b\tilde{a}b\tilde{a}a\Lambda\), for a total of four. The integral states corresponding to the vacuum are

\[
\begin{align*}
\{(0,0,0,0),(0,0,0,0);(0,0,0,0)\} \\
\{(0,0,0,0),(1,0,0,0);(3,0,0,0)\} \\
\{(0,0,0,0),(0,0,1,0);(0,0,3,0)\} \\
\{(0,0,0,0),(0,0,0,1);(0,0,0,3)\}
\end{align*}
\]

(5.23)

They are related to each other by the chain \(a\tilde{a}\) acting on the integer parts \(\lambda^I\) and \(\lambda'^I\) only. In other words, coset fields in the \(\bar{\lambda}^F = 0\) sector can be identified according to

\[
\{\gamma, \lambda^I; \lambda'^I\} \sim \{A\gamma, A\lambda^I; A\lambda'^I\}
\]

(5.24)

5.9 \(\mathfrak{so}(5)\) with \(l = 1\) and \(m = -12/5\).

This is again a coset with \(m^I = 0\). Since \(B_2\) is not simply laced there is a restriction on \(u\): here \(u \not\in 2\mathbb{Z}\) (c.f. footnote b). If, for simplicity’s sake, we insist on having \(m^I = 0\), then \(u\) and \(g = 3\) must be coprime, and the simplest example is then \(u = 5\). One finds 18 distinct fields whose representatives can all be chosen with \(y = 1\) and \(\bar{\lambda}^F = 0\). These are given in table 3.

The first 12 fields can be grouped into multiplets with \(\Delta h = \frac{1}{2}\). In fact, the two fields in a multiplet are related by the action of \(a\) on \(\gamma\) (for \(B_r\), the only non-trivial
outer automorphism is \( a[\lambda_0, \lambda_1, \ldots, \lambda_r] = [\lambda_1, \lambda_0, \ldots, \lambda_r] \). Similarly, the remaining 6 fields are characterized by the property \( a\gamma = \gamma \). One thus finds NS and R sectors, exactly as in the superconformal case [10].

Table 3: Distinct coset fields of \( \tilde{s}o(5) \) at \( m = -12/5 \).

| \( h \) | \( \{\tilde{\gamma}, \tilde{\lambda}^I, \tilde{\lambda}'^I\} \) | \( \tilde{\gamma} \) | \( \{\tilde{\gamma}, \tilde{\lambda}^I, \tilde{\lambda}'^I\} \) |
|-------|-----------------|-------|-----------------|
| 0     | \{(0, 0), (0, 0); (0, 0)\} | \( -\frac{1}{4} \) | \{(1, 0), (0, 0); (1, 2)\} |
| \( \frac{1}{2} \) | \{(1, 0), (0, 0); (0, 0)\} | \(-1\) | \{(1, 0), (0, 0); (0, 4)\} |
| \(-\frac{9}{2} \) | \{(0, 0), (0, 0); (1, 0)\} | \( -\frac{1}{2} \) | \{(1, 0), (0, 0); (0, 4)\} |
| \(-\frac{15}{2} \) | \{(1, 0), (0, 0); (1, 0)\} | \(-\frac{7}{32} \) | \{(0, 1), (0, 0); (1, 1)\} |
| \(-\frac{27}{2} \) | \{(0, 0), (0, 0); (0, 2)\} | \(-\frac{23}{32} \) | \{(0, 1), (0, 0); (0, 5)\} |
| \(-\frac{33}{2} \) | \{(1, 0), (0, 0); (0, 2)\} | \(-\frac{31}{32} \) | \{(0, 1), (0, 0); (0, 3)\} |
| \(-\frac{39}{2} \) | \{(0, 0), (0, 0); (2, 0)\} | \(-\frac{37}{32} \) | \{(1, 0), (0, 0); (1, 3)\} |
| \(-\frac{43}{2} \) | \{(1, 0), (0, 0); (1, 2)\} | \(-\frac{39}{32} \) | \{(0, 1), (0, 0); (2, 1)\} |

Table 4: Coset fields identified to \{(1, 0), (0, 0); (1, 0)\}.

| \{(1, 0), (0, 0); (1, 0)\} | \{(0, 0), (0, -\frac{4}{5}); (0, 0)\} | \{(0, 0), (0, -\frac{12}{5}); (0, 0)\} |
| \{(0, 0), (0, -\frac{4}{5}); (0, 0)\} | \{(1, 0), (0, -\frac{4}{5}); (0, 0)\} | \{(0, 0), (0, -\frac{12}{5}); (0, 0)\} |
| \{(1, 0), (0, -\frac{4}{5}); (0, 0)\} | \{(0, 0), (0, -\frac{12}{5}); (0, 0)\} | \{(0, 0), (0, -\frac{12}{5}); (0, 0)\} |
| \{(0, 0), (0, -\frac{12}{5}); (0, 0)\} | \{(1, 0), (0, -\frac{12}{5}); (0, 0)\} | \{(0, 0), (0, -\frac{12}{5}); (0, 0)\} |
| \{(1, 0), (0, -\frac{12}{5}); (0, 0)\} | \{(0, 0), (0, -\frac{12}{5}); (0, 0)\} | \{(0, 0), (0, -\frac{12}{5}); (0, 0)\} |

Let us now turn to the question of field identification. In the \( y = 1 \) sector, the field \{(1, 0), (0, 0); (1, 0)\} is identified to 17 other fields. (This is true of all NS fields;
in the R sector, there are fixed points.) The 18 identified fields can be divided into three groups, as shown in table 4.

Within each group, the fields are identified using a and \( b = B_a \). The chains of field identification are respectively \((ab)^4a, (ab)^2a\) and \( a \). Now, the different groups can be identified according to (4.28). Indeed, the last coset field of the second group, and the first of the third group have the same \( \bar{\gamma}, \bar{\lambda}^I \) and \( \bar{\lambda}'^I \), and their \( \bar{\lambda}^{F,1} \) differ by \((0, 2) = 2\alpha_2 + \alpha_1 \in Q'\). Therefore they are not distinct coset fields. The same thing applies to the last field of the first group and the first of the last group, with \( \Delta \bar{\lambda}^{F,1} = (0, 4) \in Q' \). Here again we observe that in the \( \bar{\lambda}^F = 0 \) sector, fields are identified according to (5.24).

5.10 A remark on \( \hat{B}_r \) diagonal cosets.

For diagonal \( \hat{B}_r \) (or \( \hat{so}(2r + 1) \)) cosets with \( l = 1 \), the splitting into NS and R sectors is a general feature. The central charge is

\[
c = (r + \frac{1}{2}) \left\{ 1 - \frac{2r(2r - 1)}{(m + 2r)(m + 2r - 1)} \right\}
\]

and we recover the superconformal minimal series for \( r = 1 \) (set \( m + 1 = p/(p' - p) \)). This however is purely formal since \( \hat{B}_1 \) does not really make sense. On the other hand, for \( r \geq 2 \), although one still has the analog of NS and R sectors, these sectors are characterized by the value of \( \gamma \). When \( l = 1 \), \( \gamma \) can take only three values, namely \( \omega^0, \omega^1 \) and \( \omega^r \). The first two are related by the action of \( a \) and correspond to the NS sector. Since the \( r \)th Dynkin label of any root is even, the condition (4.2) implies \( \lambda_r + \lambda'_r \in 2\mathbb{Z} \) in the NS sector (where \( \gamma_r = 0 \), while \( \lambda_r + \lambda'_r \in 2\mathbb{Z} + 1 \) in the R sector (where \( \gamma_r = 1 \)).

6. Conclusion.

Highest weights for admissible representations of Kać-Moody algebras at fractional level \( m = t/u \) are of the form \( y, (\lambda^I - (m + g)\lambda^{F,y}) \) with \( \lambda^I \in P^m, m^I = u(m + g) - g \) and \( \lambda^{F,y} \in r_v\mathbb{Z} \). For diagonal cosets with one factor in the numerator being at integer level, field identifications allow us to restrict the choice of coset field representatives to a fundamental sector: \( y = 1 \) and \( \bar{\lambda}^{F,y} = 0 \). This fact, demonstrated for \( \hat{su}(N) \), has been verified for other algebras with a large number of examples. The independence of the coset modular matrix \( S \) on \( y \) and \( \bar{\lambda}^F \) ensures that fields in the fundamental sector transform among themselves. In the fundamental sector, fields can be identified according to

\[
\{\gamma, \lambda^I; \bar{\lambda}'^I\} \sim \{A^u\gamma, A\lambda^I; A\bar{\lambda}'^I\}
\]

where \( \gamma \in P^u, \lambda^I \in P^m, \bar{\lambda}'^I \in P^{m^I + tu} \) and where \( A \) is any element of the outer automorphism group. The branching condition is \( \bar{\gamma} + \bar{\lambda}^I - \bar{\lambda}'^I \in Q \). This has been

\[<e>\] The calculations have been carried out with the help of a computer program performing field identifications with the modular \( S \) matrix. Examples from \( B_{2-4}, C_{3-4}, D_4 \) and \( G_2 \) have been analyzed.
pointed out in all the examples displayed. The following proof makes it a general result: It is simple to see that
\[
AS^{(m)}_{\overline{\lambda}, \overline{\mu}} = e^{-2\pi i(A\omega^0, u\mu^I + (u-1)\rho) S^{(m)}_{\overline{\lambda}, \overline{\mu}}} \tag{6.2}
\]
Therefore, acting respectively with \(A\), \(A'\) and \(A''\) on the product \(S^{(l)}_{\gamma, \xi} S^{(m)}_{\overline{\lambda}, \overline{\mu}} S^{(m+l)}_{\overline{\lambda}', \overline{\mu}'}\) produces the phase
\[
\exp -2\pi i[(A\omega^0, \xi) + (A'\omega^0, u\mu^I + (u-1)\rho) - (A''\omega^0, u\mu'^I + (u-1)\rho)] \tag{6.3}
\]
Canceling the factors containing \(\rho\) requires \(A' = A''\). Then, using \(\xi + \overline{\lambda}' - \overline{\lambda}' \in \mathbb{Q}\), one needs \(A = (A')^n\) in order to get a vanishing phase.

That nonunitary diagonal cosets can be described solely in terms of weights whose finite parts are integrable, with fields identified according to (5.24), is the main result of this paper.

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is to transpose the components $\lambda$ and the outer automorphisms associated with the four classical algebras $A_r$, $B_r$, $C_r$, and $D_r$. In each case it is useful to introduce an orthogonal basis $\{e_i\}$ for weight space. The description of the roots and of the action of the Weyl group $W$ is thus simplified. The conventional ordering defined on the roots becomes a simple lexicographic ordering in this basis. The group of outer automorphisms is isomorphic to the symmetry group of the affine Dynkin diagrams.

### Appendix A

In this appendix we briefly describe the simple roots, coroots, the Weyl groups and the outer automorphisms associated with the four classical algebras $A_r$, $B_r$, $C_r$, and $D_r$. In each case it is useful to introduce an orthogonal basis $\{e_i\}$ for weight space. The description of the roots and of the action of the Weyl group $W$ is thus simplified. The conventional ordering defined on the roots becomes a simple lexicographic ordering in this basis. The group of outer automorphisms is isomorphic to the symmetry group of the affine Dynkin diagrams.

#### A.1 $A_r$ or $su(r + 1)$

The orthogonal basis spans a space of dimension $r + 1$, and the weight space of $A_r$ is orthogonal to the vector $e_1 + e_2 + \ldots + e_{r+1}$. The positive roots are $e_i - e_j$, with $i < j$. The simple roots are $\alpha_i = e_i - e_{i+1}$, with $i = 1, 2, \ldots, r$. The highest root is

$$\tilde{\theta} = e_1 - e_{r+1} = \sum_{i=1}^{r} \alpha_i$$

(A.1)

Therefore all the marks $k_i$ are equal to 1. The algebra is simply laced, and the coroots coincide with the roots, the comarks $k_i$ with the marks. On a weight $\lambda = \sum_{i=1}^{r+1} \lambda^i e_i$, the effect of a Weyl reflection with respect to $\tilde{\alpha} = e_i - e_j$ is to transpose the components $\lambda^i$ and $\lambda^j$. The Weyl group generated by these reflections is the permutation group of the $r + 1$ components $\lambda^i$.

The outer automorphisms are cyclic permutations of the simple roots, generated by $a(\alpha_i) = \alpha_{i+1}$, $a(\alpha_r) = a_0$. From the expression for $\tilde{\alpha}$ and $\tilde{\theta}$, we see that the corresponding Weyl group element is $w_0$ such that $w_0(e_i) = e_{i+1}$, $w_0(e_{r+1}) = e_1$. The subgroup $W(A)$ is isomorphic to $\mathbb{Z}_{r+1}$.

#### A.2 $B_r$ or $so(2r + 1)$

The orthogonal basis spans a space of dimension $r$. The positive roots are $e_i \pm e_j$, with $i < j$, and $e_j$. The simple roots are $\alpha_i = e_i - e_{i+1}$, with $i = 1, 2, \ldots, r - 1$, and $\alpha_r = e_r$. The highest root is

$$\tilde{\theta} = e_1 + e_2 - 2\alpha_2 + 2\alpha_3 + \ldots + 2\alpha_r$$

(A.2)

Therefore all the marks $k_i$ are equal to 2, except $k_1 = 1$. The algebra is not simply laced, and $\alpha_i = \tilde{\alpha}_i$ except for $\alpha_r = 2\tilde{\alpha}_r$. Similarly, $k_i = k_i$ except for $k_r = 1$. As before, the effect of a Weyl reflection with respect to $\tilde{\alpha} = e_i - e_j$ is to transpose the components $\lambda^i$ and $\lambda^j$. But a Weyl reflection with respect to
\( \tilde{\alpha} = e_i + e_j \) transposes the components \( \lambda' \) and \( \lambda' \) and changes their sign, while a Weyl reflection with respect to \( \tilde{\alpha} = e_i \) only changes the sign of \( \lambda' \). The Weyl group generated by these reflections is therefore the group of \( r! \) permutations of the \( r \) components \( \lambda' \), associated with the \( 2^r \) possible sign changes of the components, for a total of \( 2^r r! \) elements.

The only non-trivial outer automorphism is a transposition of the first two affine simple roots: \( a(\alpha_\mu) = \alpha_\mu \) except for \( a(\alpha_0) = \alpha_1 \) and \( a(\alpha_1) = \alpha_0 \). The corresponding Weyl group element is \( w_a \) such that \( w_a(e_i) = e_i \) except for \( w_a(e_1) = -e_1 \). The subgroup \( W(A) \) is isomorphic to \( \mathbb{Z}_2 \).

A.3 \( C_r \) or \( sp(r) \).

The Cartan matrix of \( C_r \) is just the transpose of that of \( B_r \), i.e. the long roots and the short roots are interchanged. The positive roots are \((e_i \pm e_j)/\sqrt{2}, \) with \( i < j \), and \( \sqrt{2} e_i \). The simple roots are \( \tilde{\alpha}_i = (e_i - e_{i+1})/\sqrt{2}, \) with \( i = 1, 2, \ldots r - 1 \), and \( \tilde{\alpha}_r = \sqrt{2} e_r \). The highest root is

\[
\tilde{\theta} = \sqrt{2} e_1 = 2\tilde{\alpha}_1 + \ldots + 2\tilde{\alpha}_{r-1} + \tilde{\alpha}_r \tag{A.3}
\]

Therefore all the marks \( k_i \) are equal to 2, except \( k_r = 1 \). The algebra is not simply laced, and \( \tilde{\alpha}' = 2\tilde{\alpha} \) except for \( \tilde{\alpha}' = \tilde{\alpha}_r \). Similarly, \( k'_i = 1 \), for all \( i \). The Weyl group is exactly the same as for \( B_r \).

The only non-trivial outer automorphism amounts to reversing the order of the affine simple roots: \( a(\alpha_\mu) = \alpha_{r-\mu} \). From the expression for \( \tilde{\alpha}_i \) and \( \tilde{\theta} \), we see that the corresponding Weyl group element is \( w_a \) such that \( w_a(e_i) = -e_{r+1-i} \). The subgroup \( W(A) \) is again isomorphic to \( \mathbb{Z}_2 \).

A.4 \( D_r \) or \( so(2r) \).

The orthogonal basis spans a space of dimension \( r \). The positive roots are \( e_i \), with \( i < j \). The simple roots are \( \tilde{\alpha}_i = e_i - e_{i+1}, \) with \( i = 1, 2, \ldots r - 1 \), and \( \tilde{\alpha}_r = e_{r-1} + e_r \). The highest root is

\[
\tilde{\theta} = e_1 + e_2 = \tilde{\alpha}_1 + 2\tilde{\alpha}_2 + \ldots + 2\tilde{\alpha}_{r-2} + \tilde{\alpha}_{r-1} + \tilde{\alpha}_r \tag{A.4}
\]

Therefore all the marks \( k_i \) are equal to 2, except \( k_1 = k_{r-1} = k_r = 1 \). The algebra is simply laced, and \( \tilde{\alpha}' = \tilde{\alpha}_i, k'_i = k_i \) for all \( i \). The Weyl group is similar to that of \( B_r \), except that the sign changes must always occur in pairs, since the vectors \( e_i \) are not roots. The order of the Weyl group is therefore \( 2^{r-1} r! \).

The group of outer automorphisms is not the same depending if \( r \) is even or odd. A generator common to the two cases is \( a \), which transposes \( \alpha_0 \) with \( \alpha_1 \), and \( \alpha_r \) with \( \alpha_{r-1} \). The corresponding Weyl group element is \( w_a(e_i) = e_i \) except for \( w_a(e_1) = -e_1 \) and \( w_a(e_r) = -e_r \). The number of sign changes is thus even. If \( r \) is even, we define \( \tilde{a} \) as \( \tilde{a}(\alpha_\mu) = \alpha_{r-\mu} \), and the corresponding Weyl group element is \( w_{\tilde{a}}(e_i) = -e_{r+1-i} \). Note that the number of sign changes is even, and that \( \tilde{a}^2 = a^2 = 1 \). The subgroup \( W(A) \) is thus isomorphic to \( \mathbb{Z}_2 \times \mathbb{Z}_2 \).

If \( r \) is odd, we define \( \tilde{a} \) as \( \tilde{a}(\alpha_\mu) = \alpha_{r-\mu} \), except for \( \tilde{a}(\alpha_0) = \alpha_{r-1}, \tilde{a}(\alpha_1) = \alpha_r, \tilde{a}(\alpha_{r-1}) = \alpha_1 \) and \( \tilde{a}(\alpha_r) = \alpha_0 \). The corresponding Weyl group element is \( w_{\tilde{a}} \) such that \( w_{\tilde{a}}(e_i) = -e_{r+1-i} \), except for \( w_{\tilde{a}}(e_1) = e_r \). In this case the number of sign changes is still even, with \( \tilde{a}^4 = 1 \) and \( \tilde{a}^2 = a \). The subgroup \( W(A) \) is thus isomorphic to \( \mathbb{Z}_4 \).
Appendix B

In this appendix we demonstrate that admissible weights satisfying the conditions (2.28-2.33) automatically satisfy the defining condition (2.26).

Firstly, let us show that (2.26 ii) is satisfied. The coroot $\alpha^\vee = (\bar{\alpha}_{\mu}^\vee, n, 0)$ ($n \in r_\mu \mathbb{Z}$) will belong to $R^\lambda$ if
\[(\alpha^\vee, \lambda) \in \mathbb{Z} \quad \text{or} \quad nt + j \in u\mathbb{Z} \quad (m = t/u) \quad (B.1)\]
where $(\lambda, \bar{\alpha}_{\mu}^\vee) \equiv j/u$. According to (2.28-2.33), $j$ is an integer in all cases. Moreover, $j \in r_\mu \mathbb{Z}$ if $(r_\mu, u) \neq 1$, as we can check from (2.28). $j$ being an integer, it is always possible to find an integer solution $n$ to (B.1), and this because $(t, u) = 1$. I The set of solutions to (B.1) is then $n + u\mathbb{Z}$. If $(r_\mu, u) = 1$, there will always be a member $n'$ of that set such that $n' \in r_\mu \mathbb{Z}$, and then the coroot $(\bar{\alpha}_{\mu}^\vee, n', 0) \in R^\lambda$. If, on the other hand, $(r_\mu, u) \neq 1$, i.e. if $u = ar_\mu$ with $a \in \mathbb{Z}$, then $j = j'r_\mu$ and (B.1) can be rewritten as
\[(n/r_\mu)t + j' \in a\mathbb{Z} \quad (B.2)\]
and there is always an integer solution $(n/r_\mu)$ to this equation. Therefore, if $\lambda$ satisfies the conditions (2.28-2.33), the set $R^\lambda$ contains coroots of the form $(\bar{\alpha}_{\mu}^\vee, n, 0)$ ($n \in r_\mu \mathbb{Z}$) for all $\mu$, and consequently the rational span of $R^\lambda$ coincides with the rational span of the simple roots. Note that this proof is considerably simplified if all roots are long roots, i.e. if $r_\mu = 1$ for all $\mu$.

Secondly, let us show that (2.26 i) is satisfied. Consider all the positive coroots given by $\alpha^\mu_\lambda = \lambda^{F:y}c + y(\alpha^\mu_\bar{\lambda})$. This set of $r + 1$ roots has the same Cartan matrix as the $r + 1$ original simple coroots. However, the sum $\sum_\mu k^\lambda_\mu \alpha^\mu_\lambda$ is equal to $uc$, i.e. the corresponding canonical central element is a multiple of $c$. Therefore, any positive coroot $\alpha^\vee$ is of the form $\sum_\mu l_\mu \alpha^\lambda_\mu = k\bar{c}$, where $0 \leq k < u$ and $l_\mu \in \mathbb{Z}_+$. If we substitute this form into the condition (2.26 i), we obtain
\[(\lambda + \rho, \alpha^\vee) = \sum_\mu (1 + \lambda^\mu_\lambda)l_\mu - (m + g)k \quad (B.3)\]
If $k = 0$, this is certainly not an element of $-\mathbb{Z}_+$ for any root $\alpha$, since $\lambda^\mu_\lambda \geq 0$, $\forall \mu$. If $0 < k < u$, then this is not an integer. Therefore, any weight satisfying (2.28-2.33) also satisfies the conditions (2.26).

Appendix C

In this appendix we show that, in the case of $su(N)$, it is possible to choose a representative $y$ in each class of $W(\bar{g})/W(A)$ such that $\lambda^{F:y} \in P^m_+$. Let us rewrite the condition (2.33):
\[\bar{c}\lambda^{F,y}_\mu + y(\alpha^\vee_\mu) \in R_+ \quad (C.1)\]
Since positive coroots have a non-negative grade, it is clear that the only negative value of $\lambda^{F,y}_\mu$ allowed is $-1$, and this only if $\mu = 0$. In that case the l.h.s. of (C.1) is simply $-y(\bar{\theta})$. If $y(\bar{\theta}) \in R_+$, then the value $\lambda^{F,y}_0 = -1$ is forbidden and $\lambda^{F,y}_\mu \in P^m_+$. If $y(\bar{\theta}) \notin R_+$, the question is if we can find an element $w_\alpha \in W(A)$ such that $at + bu = 1$.

\[\text{Recall that if } (t, u) = 1, \text{ then } 2a, b \in \mathbb{Z} \text{ such that } at + bu = 1.\]
such that \(yw_a(\tilde{\theta}) \in R_+\). In the case of \(\text{su}(N)\), \(\tilde{\theta} = e_1 - e_n\) (orthogonal basis, \(n = r + 1\)), and \(y(\tilde{\theta}) = e_n(e_1) - e_n(e_n)\), where the action of \(y\) on indices is defined by the corresponding permutation. If \(y(1) > y(n)\), then \(y(\tilde{\theta}) \notin R_+\). However, there is a suitable cyclic permutation \(w_a\) such that \(yw(1) < yw(n)\), i.e., such that \(yw(\tilde{\theta}) \in R_+\). Therefore, by right application of a suitable \(w_a \in W(A)\), we can make sure that \(\lambda_{F,y} \in P^w_+\). It is easy to verify that this argument works uniquely in the case of \(\text{su}(N)\). The outer automorphism groups of other classical algebras are not large enough to allow for such representative \(y\)'s.

Appendix D

In this appendix we show that if \(\lambda\) is admissible, then for any \(w \in W\), there is a unique \(\tilde{w} \in W^\lambda\) such that \(ww.\lambda\) is also admissible. Let \(R^\lambda_+ \subset R_+\) be the set of positive roots also part of \(R^\lambda\) (see (2.27)). Consider an element \(w \in W\) such that \(w(R^\lambda_+) \subset R_+\). Since \(w.\lambda\) is admissible (2.26) is satisfied, and consequently

\[
(w.\lambda + \rho, w(\tilde{\alpha}^\vee)) + mn \notin \mathbb{Z}_+	ag{D.1}
\]

where \(\alpha^\vee = (\tilde{\alpha}^\vee, 0, n)\) is a positive coroot \((n > 0)\), or \(n = 0\) with \(\tilde{\alpha}^\vee > 0\). If \(\alpha^\vee \in R^\lambda_+\), then by hypothesis \(w(\alpha^\vee) \notin R^\lambda_+\) and \(w.\lambda\) passes the test of admissibility for these coroots. To prove that \(w.\lambda\) satisfies (2.26), it remains to consider the case where \(n = 0\), \(\alpha^\vee \notin R^\lambda_+\) and \(w(\tilde{\alpha}^\vee) \in R_-\). But in this case \(-w(\tilde{\alpha}^\vee) \in R_+\) and

\[
-(w.\lambda + \rho, w(\tilde{\alpha}^\vee)) = -(\lambda + \rho, \tilde{\alpha}^\vee) \notin \mathbb{Z}	ag{D.2}
\]

and therefore (2.26) is also satisfied. Thus we have shown that \(w.\lambda\) is admissible. (The second condition of (2.26) is trivially satisfied).

Next, we will show that for any \(w \in W\) there is a unique element \(\tilde{w} \in W^\lambda\) such that \(ww.\lambda \in R_+\). Let us denote by \(\pi\) the projection that takes \(R\) into \(R^\lambda\). Then consider the set \(\pi w^{-1}(R_+) \subset R^\lambda\). This set is the image of \(R^\lambda_+\) by some unique element \(\tilde{w} \in W^\lambda\):

\[
\pi w^{-1} R_+ = \tilde{w} \pi R_+ = \pi w^{-1} R_+ \tag{D.3}
\]

Then \(\tilde{w}(R^\lambda_+) \subset w^{-1}(R_+)\) and our assertion follows.

Finally, let us show that if \(\lambda\) is an admissible weight admitting a decomposition (2.28) with an element \(y\) of \(W\), and if \(w.\lambda\) is also admissible, then \(w.\lambda\) admits a decomposition (2.28) with \(y\) replaced by \(wy\), and \(\lambda_{F,y} = \lambda_{F,wy}\). This almost seems obvious, but we must check that (2.33) implies

\[
\tilde{c}\lambda_{F,y} + wy(\alpha_\mu^\vee) \in R_+\tag{D.4}
\]

A problem might arise only when \(\lambda_{F} = (\lambda_{F,y}(\alpha_i)) = 0\), or when \(\lambda_{F,w} = (\lambda_{F}, y(\alpha_0)) = -1\). In both cases one easily checks that \(y(\alpha_\mu) \in R^\lambda_+\), and so is \(wy(\alpha_i)\) (by hypothesis). Hence (2.33) is satisfied with \(y \rightarrow wy\).