Idempotent Divisor Graph of Commutative Ring

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Abstract
This work aims to introduce and to study a new kind of divisor graph which is called idempotent divisor graph, and it is denoted by \( I(R) \). Two non-zero distinct vertices \( v_1 \) and \( v_2 \) are adjacent if and only if \( v_1v_2 = e \), for some non-unit idempotent element \( e \in R \). We establish some fundamental properties of \( I(R) \), as well as it’s connection with \( J(R) \). We also study planarity of this graph.

Keywords: Idempotent Elements, Zero Divisor Graph, Idempotent Divisor Graph, Planar Graph.

1. Introduction
Let \( R \) be a finite commutative ring with unity \( 1 \neq 0 \). We denote \( Z(R) \), \( I(R) \), and \( U(R) \) the set of zero divisors, the set of idempotent elements and the set of unit elements respectively.

In [1], Beck introduced the idea that connects between ring theory and graph theory when studied the coloring of commutative ring. Later in [2], Anderson and Livingston modified this idea when studied the zero divisor graph \( J(R) \) that have vertices \( Z(R) \), edges if and only if \( v_1v_2 = e \). Many authors studied this notion see for examples [3], [4], [5] and [6]. Recently, there are other concepts of zero divisor graph, see for examples [7], [8], [9],and [10].

In graph theory “(v)” denotes by the eccentricity of a vertex \( v \) of a connected graph \( G \) which is the number \( \max_{u \in V(G)} d(u, v) \). That means \( e(v) \) is the distance between \( v \) and a vertex furthest from \( v \). The radius of \( G \), which is denoted by \( radG \), is \( \max_{u \in V(G)} d(u, v) \), while the diameter of \( G \) is the maximum eccentricity and it is denoted by \( diamG \). Consequently, \( diamG \) is the greatest distance between any two vertices of \( G \). Also, a graph \( G \) has radius 1 if and only if \( G \) contains a vertex \( u \) adjacent to all other vertices of \( G \). A vertex \( v \) is a central
vertex if \( e(v) = \text{rad}G \) and the center \( \text{Cent}(G) \) is the sub-graph of \( G \) that induced by its central vertices. The girth of a graph \( G \) is the length of a shortest cycle contained in \( G \), it is denoted by \( g(G) \). The neighborhood of \( x \) in a graph \( G \) denotes by \( N_G(x) \), is the set of all \( y \in V(G) \) such that \( y \) is adjacent to \( x \) in \( G \). In our graph in this case, \( N_G(x) = \{ y \in V(G) \setminus \{x\} \mid xy = 0 \} \). \( K_n \) \( K_{n,m} \) symbolized complete graph and complete bipartite graph respectively. \( K_{1,m} \) we call star graph. A clique number of \( G \) symbolized \( \omega(G) \) is greats complete sub-graph of \( G \). If a connected graph does not contain cycle, we call tree. Let \( H \) and \( G \) two graphs, \( G \cup H \) is a graph with \( V(G \cup H) = V(G) \cup V(H) \) and \( E(G \cup H) = E(G) \cup E(H) \), and for \( n \in \mathbb{Z}^+ \), \( nH = \bigcup_{i=1}^{n} H \). the graph \( G + H \) is a graph with \( V(G + H) = V(G) \cup V(H) \) and \( E(G + H) = E(G) \cup E(H) \cup \{\{u, v\} : u \in V(G), v \in V(H)\} \). A path graph of order \( n \) is denoted by \( P_n \) is a graph with \( V(P_n) = \{v_i : i = 1,2,...,n\} \) and \( E(P_n) = \{\{v_j, v_j + 1\} : j = 1,2,...,n-1\} \), so that \( C_n \) is a graph \( P_n + \{v_1, v_n\} \) and it called a cycle graph of order \( n \) for \( n \in \mathbb{Z}^+ \). For more details see for example” [11].

In ring theory, a ring \( R \) is said to be local if has exactly one maximal ideal. Also, if \( R \) finite local ring, then the cardinality of \( R \) symbolized \( |R| \) equal \( p^t \), where \( p \) prime number and \( t \in \mathbb{Z}^+ \), as well as the cardinality of maximal ideal \( M = p^r \), where \( 0 < r < t \). A ring \( R \) is called Boolean, if every element is an idempotent. We denote \( F_q \) is a field order \( q \). In section two we defined a new graph on the ring and prove some basic properties of about this graph and we give all possible graphs less than or equal 6 vertices. In section three, we give all graphs to be planer.

2. Examples and Basic Properties

In this section, we introduce a new class of divisor graph manly idempotent divisor graph, we give some of about this graph, and we also provide some examples.

**Definition 2.1:** The undirected graph is called idempotent divisor graph, and which is symbolized by \( \Pi(R) \) which a simple graph with vertices set in \( R' = R - \{0\} \), and two non-zero distinct vertices \( v_1 \) and \( v_2 \) are adjacent if and only if \( v_1v_2 = e \), for some non-unit idempotent element \( e \in R \) (i.e \( e \neq 0 \)). **Example 1:** Let \( R = \mathbb{Z}_6 \), since the idempotent elements \( I(R) = \{0,1,3,4\} \), then \( I(R) \) is:

![Graph](image)

**Figure 2.1**

**Remarks:**

1- If 0 idempotent element in \( R \), then \( I(R) \subseteq \Pi(R) \).
2- If \( R \) has only idempotent elements 0 and 1, then \( I(R) = \Pi(R) \). Consequently, when \( R \) local, then \( I(R) = \Pi(R) \).
3- If \( R \) finite non local ring, then \( R \cong R_1 \times R_2 \ldots \times R_n \). Since \((1,0,\ldots,0)^2 = (1,0,\ldots,0)\), then \( R \) has idempotent element distinct \( \{0,1\} \).
4- If $R$ non-local ring, then there are at greater than or equal two non-trivial idempotent elements in $R$. if $e^2 = e \neq 0$ or $1$, then $1 - e$ also idempotent and $e \neq 1 - e$ (because if $e = 1 - e$, then $e + e = 1$ and $e + e = (e + e) + (e + e) = (e + e)^2 = 1$ implies that $1 = 0$ which is a contradiction. Therefore, $e \neq 1 - e$). Hence if $u \in U(R)$, then $u$ adjacent with $u^{-1}e$, for every $e \in \mathcal{I}(R) - \{0, 1\}$, so that $V(\mathcal{I}(R)) = R^* = R - \{0\}$.

**Example 2:** We shall give all possible idempotent divisor graphs, with $\mathcal{I}(R) \leq 6$.

If $|\mathcal{I}(R)| = 1$, then $R$ is local and $|Z(R)| = 2$, so by [12] $R \cong Z_4$ or $F_2[Y]/(Y^2)$.

If $|\mathcal{I}(R)| = 2$, then $R$ is local and $|Z(R)| = 3$, so by [12] $R \cong Z_9$ or $F_3[Y]/(Y^2)$.

If $|\mathcal{I}(R)| = 3$, and $R$ is local, then $|Z(R)| = 4$, so that by [12].

If $|\mathcal{I}(R)| = 4$, then $R$ is local and $|Z(R)| = 5$, which implies $R \cong Z_{25}$ or $F_5[Y]/(Y^2)$.

If $|\mathcal{I}(R)| = 5$, then $R$ is non-local and $|R| = 6$. Hence $R \cong F_2 \times F_3$.

If $|\mathcal{I}(R)| = 6$, then $R$ is local with $|Z(R)| = 7$. So $R \cong Z_{49}$ or $F_7[Y]/(Y^2)$.

Table 2.1- Rings with $|\mathcal{I}(R)| \leq 6$

| Vertices | Ring(s) type | Graph |
|----------|-------------|-------|
| 1        | $Z_4$ or $F_2[Y]/(Y^2)$ | $K_1$ |
| 2        | $Z_9$ or $F_3[Y]/(Y^2)$ | $K_2$ |
| 3        | $Z_8, F_2[Y]/(Y^3), F_4[Y]/(Y^2)$ or $Z_4[Y]/(2Y, Y^2 - 2)$ | $K_3$ |
|          | $F_4[Y]/(Y^2 2), F_4[Y]/(2, Y^2), F_2[Y_1, Y_2]/(Y_1, Y_2)^2$ or $F_2 \times F_2$ |       |
| 4        | $Z_{25}$ or $F_5[Y]/(Y^2)$ | $K_4$ |
| 5        | $F_2 \times F_3$ | $K_5$ |
| 6        | $Z_{49}$ or $F_7[Y]/(Y^2)$ | $K_6$ |

Now, we give some basic properties of idempotent divisor graph.

**Theorem 2.2:** For any ring $R$, $\mathcal{I}(R)$ is connected graph. Moreover, $\text{diam}(\mathcal{I}(R)) \leq 3$.

**Proof:** Since if $R$ local ring, then $I(R) = \mathcal{I}(R)$, so by [2, Theorem 2.3 ] $R$ connected. Now we investigate the case when $R$ is non-local. Let $a, b \in \mathcal{I}(R)$. Since $R$ finite ring, then $R^* = Z(R)^* \cup U(R)$. So there are three cases:

**Case 1:** If $a, b \in Z(R)^*$. Since $0 \neq 1$ is an idempotent element in $R$, then by [2, Theorem 2.3] there exist a path between $a, b \in I(R)$ and $d_{I(R)}(a, b) \leq 3$. So there is a path between $a$ and $b$ in $\mathcal{I}(R)$ and $d_{\mathcal{I}(R)}(a, b) \leq 3$.

**Case 2:** If $a, b \in U(R)$, then there are $x, y \in U(R)$ such that $ax = by = 1$. Also for any idempotent element $e^2 = e \not{\in} \{0, 1\}$.

$a(e^2 x e) = e$ and $b(y(1 - e)) = 1 - e$. Since $e(1 - e) = 0$, then $a = xe = y(1 - e) = b$ is a path and $d_{\mathcal{I}(R)}(a, b) \leq 3$. 

Fig. 2.2- $\mathcal{I}(Z_8)$  
Fig. 2.3- $\mathcal{I}(F_2 \times F_3)$
**Case 3:** if \( a \in U(R) \) and \( b \in Z(R)^* \). First, if there exists \( e^2 = e \not\in \{0,1\} \) such that \( be = 0 \), then \( a-a^{-1}(1-e) \rightarrow b \) is a path. So \( d_{\Lambda(R)}(a,b) \leq 3 \). If for any \( e^2 = e \not\in \{0,1\} \), \( be \neq 0 \). Since \( b \in Z(R)^* \), then there is \( c \neq c^2 \) so that \( bc = 0 \). If \( ce = 0 \), then \( a-a^{-1}e \rightarrow c \rightarrow b \). So \( d_{\Lambda(R)}(a,b) \leq 3 \).

**Theorem 2.3:** For any ring \( R \), the \( g(\Lambda(R)) = 3 \) except the cases \( R \cong Z_9, F_3[Y]/(Y^2), Z_9, F_2[Y]/(Y^2) \) or \( Z_4 \), then \( g(\Lambda(R)) = \infty \).

**Proof:** Clearly. If \( R \cong Z_9, F_3[Y]/(Y^2), Z_9, F_2[Y]/(Y^2) \) or \( Z_4 \), then \( g(\Lambda(R)) = \infty \). Suppose \( R \) is non-isomorphic to \( Z_9, F_3[Y]/(Y^2), Z_9, F_2[Y]/(Y^2) \) or \( Z_4 \), then there are two cases:

**Case 1:** If \( R \) is local ring, then \( \Lambda(R) = \mathcal{I}(R) \). So there is \( z \in Z(R)^* \) adjacent with any elements in \( Z(R)^* \). Since \( R \) is non-isomorphic to \( Z_9, F_3[Y]/(Y^2), Z_9, F_2[Y]/(Y^2) \) or \( Z_4 \), then either \( \mathcal{I}(R) \) is star graph or has circle of length 3. If \( R \) is star graph which is a contradiction by [2, Theorem 2.5]. So \( \Lambda(R) = \mathcal{I}(R) \) has circle of length 3. Hence the \( g(\Lambda(R)) = 3 \).

**Case 2:** If \( R \) is non-local ring, then there exists \( e^2 = e \not\in \{0,1\} \) and \( 1 \rightarrow e \rightarrow (1-e) \rightarrow 1 \) is a circle of length 3. So \( g(\Lambda(R)) = 3 \).

**Corollary 2.4:** Let \( \Lambda(R) \) is an idempotent divisor graph of ring \( R \), then \( \Lambda(R) \) is tree if and only if \( R \cong Z_9, F_3[Y]/(Y^2), Z_9, F_2[Y]/(Y^2) \) or \( Z_4 \).

**Corollary 2.5:** For any non-local ring \( R \), \( \alpha(\Lambda(R)) \geq 3 \).

**Proposition 2.6:** If \( R \cong F_2 \times F_2 \times ... \times F_2 \) (n-times), then \( \Lambda(R) = K_{2^n-1} \).

**Proof:** Since every element in \( R \) is an idempotent, then every non-zero two elements are adjacent in \( \Lambda(R) \). Hence \( \Lambda(R) \) is complete and \( V(\Lambda(R)) = |R'| \), so \( \Lambda(R) = K_{2^n-1} \).

**Corollary 2.7:** \( \Lambda(R) \) is a complete graph if and only if \( R \) is a Boolean ring or local with \( Z(R)^2 = 0 \).

**Proof:** Suppose that \( \Lambda(R) \) is a complete, if \( R \) is local, then \( \Lambda(R) = \mathcal{I}(R) \) by [9, Theorem 2.5]. If \( R \) is a non-local ring, and for any \( a \neq 1 \) since \( a.1 = a \) and \( \Lambda(R) \) is a complete, then \( a \) is an idempotent element in \( R \). Therefore, \( R \) Boolean ring.

The converse is obvious.

**Proposition 2.8:** For every non-local ring \( R \), then \( deg_{\Lambda(R)}(u) = |I(R)| - 2, \) for every \( u \in U(R) \).

**Proof:** Let \( u \in U(R) \), then for every \( e \in I(R) - \{0,1\} \) we have \( u \rightarrow u^{-1}e \). Since \( u^{-1}e \neq u \), then \( u^{-1}e \in N_{\Lambda(R)}(u) \) and \( deg_{\Lambda(R)}(u) = |I(R)| - 2 \).

**Theorem 2.9:** For any non-local ring \( R \), if \( diam(\Lambda(R)) \leq 2 \), then \( Cent(\Lambda(R)) \subseteq I(R) \)

**Proof:** Since \( diam(\Lambda(R)) \leq 2 \), then \( rad(\Lambda(R)) = 0 \) or 1.

If \( rad(\Lambda(R)) = 0 \), then \( diam(\Lambda(R)) = 0 \), which is a contradiction since \( R \) is non-local.

If \( rad(\Lambda(R)) = 1 \), then either \( \Lambda(R) \) complete, so by Proposition 2.7 \( R \) is a Boolean ring and every element idempotent, therefore every element in \( \Lambda(R) \) is central, we are done. If \( \Lambda(R) \) not complete graph, then for any \( a \in Cent(\Lambda(R)) \), adjacent with every elements in \( R^* \) and \( a \rightarrow 1 \), therefore \( a.1 = a \) is an idempotent element in \( R - \{0,1\} \). So \( Cent(\Lambda(R)) \subseteq I(R) \).

**Theorem 2.10:** For any non-local ring \( R \), a graph \( \Lambda(R) \) has no end vertex.

**Proof:** For any \( a \in R^* \), there are three cases:

**Case 1:** If \( a \in U(R) \), since \( a \not\in \{a^{-1}e, a^{-1}(1-e)\} \), for every idempotent element \( e = e^2 \not\in \{0,1\} \) and \( a^{-1}e \neq a^{-1}(1-e) \), then \( \{a^{-1}e, a^{-1}(1-e)\} \subseteq N_{\Lambda(R)}(a) \). So \( deg_{\Lambda(R)}(a) \geq 2 \).

**Case 2:** If \( a \in I(R) - \{0,1\} \), then \( \{1-a,1\} \subseteq N_{\Lambda(R)}(a) \). So \( deg_{\Lambda(R)}(a) \geq 2 \).

**Case 3:** If \( a \in Z(R)^* - I(R) \). Since \( R \) finite, then either \( a = a^m \) or \( a^n = 0 \) for some \( n, m \in Z^+ \).
If \( a = a^m \), then there is \( k \in \mathbb{Z}^+ \) such that \( a^k \) idempotent element in \( R \) and since \( a \in \mathbb{Z}(R)^* \), then there are \( b \in \mathbb{Z}(R)^* - \{a\} \) so that \( ab = 0 \). Therefore \( \{b, a^{k-1}\} \subseteq \mathcal{N}_{\mathcal{L}(R)}(a) \). So \( \text{deg}_{\mathcal{L}(R)}(a) \geq 2 \).

If \( a^n = 0 \) and \( n = 2 \). But \( ab = 0 \) for some \( b \in \mathbb{Z}(R)^* - \{a\} \).

Therefore \( \{b, a - b\} \subseteq \mathcal{N}_{\mathcal{L}(R)}(a) \). So \( \text{deg}_{\mathcal{L}(R)}(a) \geq 2 \).

If \( n \geq 3 \), then \( a, a^{n-1} = 0 \). Which implies that \( a^{n-1}R = \{0, a^{n-1}\} \). Now for any idempotent element \( e \notin \{0, 1\} \). Either \( a^{n-1}e = 0 \) or \( a^{n-1} \) for all cases, there are idempotent element \( f \notin \{0, 1\} \) such that \( a^{n-1}f = 0 \). If \( a^{n-2}f \neq 0 \), then \( \{a^{n-1}, a^{n-2}f\} \subseteq \mathcal{N}_{\mathcal{L}(R)}(a) \). So \( \text{deg}_{\mathcal{L}(R)}(a) \geq 2 \). If \( a^{n-2}f = 0 \), then \( \{a^{n-2}, a^{n-3}f\} \subseteq \mathcal{N}_{\mathcal{L}(R)}(a) \). If we repeat this process, we can get \( af = 0 \). This means that there is at least two elements adjacent to \( a \).

3. Planarity and Cliques of Idempotent Divisor Graph

In this part, we investigate the planarity, and the clique number of the idempotent divisor graph.

**Proposition 3.1:** Suppose that \( R \cong K \times K' \), where \( K \) and \( K' \) are fields, then \( \omega(\mathcal{L}(R)) = 3 \).

**Proof:** Since \( R \cong K \times K' \), then the only idempotent elements in \( R \) are \( \{(0,0), (1,0), (0,1), (1,1)\} \). For any \( (a, b) \in R \). If \( a \) and \( b \neq 0 \), then \( (a, b) \) adjacent with only elements \( (a^{-1}, 0), (0, b^{-1}) \). So \( (a, b) \notin K_4 \). Also if \( a = 0 \) and \( b \neq 0 \), then \( (a, b) \) adjacent with only elements \( (x, b^{-1}) \), for every \( x \in \mathbb{K} \). But \( (x, b^{-1}) \) adjacent with only elements \( (x^{-1}, 0) \) or \( (0, b^{-1}) \) and non-adjacent with \( (0, b^{-1}) \). So \( (a, b) \notin K_4 \). Similarly if \( a \neq 0 \) and \( b = 0 \), then we have \( (a, b) \notin K_4 \) and hence \( \omega(\mathcal{L}(R)) = 3 \).

**Theorem 3.2:** If \( R \cong R_1 \times R_2 \), where \( R_1 \) and \( R_2 \) are local rings but not fields, then \( \omega(\mathcal{L}(R)) = 3 \) if \( R \cong Z_4 \times Z_4 \), \( Z_4 \times F_2[Y] / (Y^2) \) or \( F_2[Y] / (Y^2) \). Otherwise \( \omega(\mathcal{L}(R)) \geq 4 \).

**Proof:** If \( R \cong Z_4 \times Z_4 \), \( Z_4 \times F_2[Y] / (Y^2) \) or \( F_2[Y] / (Y^2) \), then \( \omega(\mathcal{L}(R)) = 3 \) see Fig 3.1. Suppose \( R \) is non-isomorphic \( Z_4 \times Z_4 \), \( Z_4 \times F_2[Y] / (Y^2) \) or \( F_2[Y] / (Y^2) \). Since \( R_1 \) and \( R_2 \) are local but not fields, then there exists \( (z_1, z_2) \in R \) with \( z_1 \in \mathbb{Z}(R_1)^* \) and \( z_2 \in \mathbb{Z}(R_2)^* \), thus there are \( a_1 \in \mathbb{Z}(R_1)^* - \{z_1\} \) and \( a_2 \in \mathbb{Z}(R_2)^* \) such that \( z_1a_1 = z_2a_2 = 0 \). Therefore the set \( \{(z_1, z_2), (a_1, 0), (0, a_2), (a_1, z_2)\} \) induced a sub-graph \( K_4 \). So \( \omega(\mathcal{L}(R)) \geq 4 \).

![Figure 3.1- \( \mathcal{L} (A_1 \times A_2) \), where \( A_1 \) and \( A_2 \cong Z_4 \) or](image-url)

Recall that “a graph \( G \) is said to be planar if it can be drawn in the plane in such a way that pairs of edges intersect only at vertices, if at all. If \( G \) has no such representation, \( G \) is called
non-planar. It we know that a graph $G$ is planar if and only if contained no sub-graph $K_5$ or $K_{3,3}$ “[11].

**Proposition 3.3:** For any local ring $R$, a graph $\mathcal{L}(R)$ is planar if and only if $R$ is isomorphic to one of the following table:

| Table 3.1- local rings with $|\mathcal{L}(R)|$ is planar |
|-----------------------------------|
| **Ring(s) type** | **Graph** |
| $Z_4$ or $F_2[Y]/(Y^2)$ | $K_1$ |
| $Z_6$ or $F_3[Y]/(Y^2)$ | $K_2$ |
| $F_2[Y_1,Y_2]/(Y_1^2,Y_2^2)$, $Z_4[Y]/(2Y,Y^2)$, or $F_4[Y]/(Y^2)$ | $K_3$ |
| $Z_4[Y]/(2Y,Y^2-2)$, $Z_6$ or $F_2[Y]/(Y^3)$ | $K_{1,2}$ |
| $Z_{25}$ or $F_5[Y]/(Y^2)$ | $K_4$ |
| $Z_{27}$, $F_3[Y]/(Y^3)$ or $Z_9[Y]/(3Y,Y^2±3)$ | $K_{2,6}$ |
| $Z_{16}$, $F_2[Y]/(Y^4)$, $Z_4[Y]/(Y^2)$, $Z_4[Y]/(2Y,Y^3-2)$, $Z_4[Y]/(2Y,Y^2-2)$ | $K_1 + (4K_1 ∪ K_2)$ |
| $F_2[Y]/(Y^4)$, $Z_4[Y]/(Y^2)$, $Z_4[Y]/(2Y,Y^3-2)$, $Z_4[Y]/(2Y,Y^2-2)$ | $K_1 + (K_2 ∪ C_4)$ |
| $Z_4[Y]/(Y^2)$, $Z_4[Y]/(Y^2)$, $Z_4[Y]/(Y^2)$, or $Z_4[Y]/(Y^2 + Y + 1)$ | $K_1 + (2K_1 ∪ C_4)$ |

**Proof:** Since $R$ local, then $\mathcal{L}(R) = \mathcal{I}(R)$. Therefore the prove follows by Propositions 2,3 and 4 in [13].

**Theorem 3.4:** If $R \cong F_{q_1} \times F_{q_2}$, then $\mathcal{L}(R)$ is a planar if and only if $F_{q_i} = F_2$ or $F_3$ for $i = 1,2$.

**Proof:** Without loss generality, let $F_{q_1} = F_2$ or $F_3$. First, if $F_{q_1} = F_2$, then $R \cong F_2 \times F_{q_2}$, since $\alpha(\mathcal{L}(R)) = 3$, by Proposition 3.1. Therefore, $\mathcal{L}(R)$ does not contain a sub-graph $K_5$.

Now we shall to prove $\mathcal{L}(R)$ does not contain a $K_{3,3}$ sub-graph. If not, then there exist disjoint two subsets $V_1 = \{(a_1,b_1),(a_2,b_2),(a_3,b_3)\}$ and $V_2 = \{(x_1,y_1),(x_2,y_2),(x_3,y_3)\}$ such that every element in $V_1$ adjacent with every element in $V_2$, and $a_1,a_2,a_3,x_1,x_2$ and $x_3 \in F_2$, and $b_1,b_2,b_3,y_1,y_2$ and $y_3 \in F_{q_2}$. Since $R$ have exactly idempotent elements $(0,0),(1,0),(0,1)$ and $(1,1)$, then $(a_i,b_1)(x_i,y_j) \in \{(0,0),(1,0),(0,1)\}$. So $b_iy_j = 0$ or 1, if $b_i \neq 0$ or 1 for all $i = 1,2,3$ then $y_j = 0$ or $b_i^{-1}$ for all $j = 1,2,3$. But $x_i \in F_2$, then we have $V_2 = \{(0,b_i),(1,b_i),0,1\}$. Therefore $V_1 = \{(0,b_i),(1,b_i),0,1\}$. But $(1,b_i)(1,b_i^{-1}) = (1,1)$ a contradiction. Also, if $b_i = 0$ or 1 for all $i = 1,2,3$ we get a contradiction. Therefore, $\mathcal{L}(R)$ does not contain a $K_{3,3}$ sub-graph and $\mathcal{L}(R)$ is a planar. Similarly, we can show that if $F_{q_1} \cong F_3$, then $\mathcal{L}(R)$ is a planar. Finally, if $F_{q_1} \neq F_2$ or $F_3$ for $i = 1,2$. Then there exist $a_1,a_2 \not\in F_{q_1} - \{0,1\}$ and $b_1,b_2 \not\in F_{q_2} - \{0,1\}$. Whence $V_1 = \{(a_1,0),(a_2,0),(1,0)\}$ and $V_2 = \{(0,b_2),(0,b_2),(0,1)\}$ are disjoint sub-sets induced $K_{3,3}$ sub-graph in $\mathcal{L}(R)$.

Therefor $R$ not planar.

**Theorem 3.5:** For any ring $R$, a graph $\mathcal{L}(R)$ is planar if and only if $R$ isomorphic one of the following rings in table 3.1 or $R$ isomorphic one of the following rings:

- $F_2 \times F_{q_2}$, $F_3 \times F_{q_2}$, $F_2 \times Z_4$, $F_2 \times F_{q_2}$, $F_2 \times Z_4$ or $F_2 \times F_{q_2}$
- $F_2 \times Z_4$ or $F_2 \times F_{q_2}$

**Proof:** If $R \cong R_1 \times R_2 \times \ldots \times R_n$, where $R_i$ local ring for all $i = 1,2,\ldots,n$ and $n \geq 3$. The set $\{(1,0,\ldots,0),(0,1,0,\ldots,0),(1,1,0,\ldots,0),(0,0,\ldots,1),(1,1,\ldots,1)\} \subseteq V(\mathcal{L}(R))$ so induced a sub-graph $K_5$, therefore $\mathcal{L}(R)$ is not planar. If $n = 2$, then $R \cong R_1 \times R_2$, where $R_1,R_2$ are local rings, there are three cases:
Case1: If $R_1$ and $R_2$ are fields, then by Theorem 3.4 $\mathcal{L}(R)$ is planar if and only if $R \cong F_2 \times F_2$ or $F_3 \times F_2$, where $F_q$ is a field of order $q$.

Case2: If $R_1$ and $R_2$ are not fields, then $|R_1|, |R_2| \geq 4$. Obviously $\mathcal{L}(R)$ not planar.

Case3: If $R_1$ is a field and $R_2$ not field. Let $R_1 = F_2$ or $F_3$ and $|Z(R_2)| = 2$, then $|R_2| = 4$, which implies that $R_2 \cong Z_4$ or $F_2[Y]/(Y^2)$, so $\mathcal{L}(R)$ is planar see Fig. 3.2. If $Z(R_2) \geq 3$, then there exists $a, b \in Z(R_2)$, so that $ab = 0$. Therefore the vertices $(1,0), (1,a), (1,b), (0,a), (0,b)$ are adjacent, whence $\mathcal{L}(R)$ induced a sub-graph $K_5$, therefore $\mathcal{L}(R)$ not planar. If $|R_1| \geq 4$, then it is easy to show that a graph $\mathcal{L}(R)$ is not planar. Finally, if $n = 1$, then $R$ is local and a complete proved it’s follow by proposition 3.3 and table 3.1.

![Figure 3.2](image)

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