THE BERNARDI PROCESS AND TORSOR STRUCTURES ON SPANNING TREES

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ABSTRACT. Let $G$ be a ribbon graph, i.e., a connected finite graph $G$ together with a cyclic ordering of the edges around each vertex. By adapting a construction due to Olivier Bernardi, we associate to any pair $(v, e)$ consisting of a vertex $v$ and an edge $e$ adjacent to $v$ a bijection $\beta_{(v, e)}$ between spanning trees of $G$ and elements of the set $\text{Pic}^g(G)$ of degree $g$ divisor classes on $G$, where $g$ is the genus of $G$ in the sense of Baker-Norine. We give a new proof that the map $\beta_{(v, e)}$ is bijective by explicitly constructing an inverse. Using the natural action of the Picard group $\text{Pic}^0(G)$ on $\text{Pic}^g(G)$, we show that the Bernardi bijection $\beta_{(v, e)}$ gives rise to a simply transitive action $\beta_e$ of $\text{Pic}^0(G)$ on the set of spanning trees which does not depend on the choice of $e$. A plane graph has a natural ribbon structure (coming from the counterclockwise orientation of the plane), and in this case we show that $\beta_e$ is independent of $v$ as well. Thus for plane graphs, the set of spanning trees is naturally a torsor for the Picard group. Conversely, we show that if $\beta_e$ is independent of $v$ then $G$ together with its ribbon structure is planar. We also show that the natural action of $\text{Pic}^0(G)$ on spanning trees of a plane graph is compatible with planar duality.

These findings are formally quite similar to results of Holroyd et al. and Chan-Church-Grochow, who used rotor-routing to construct an action $\rho_v$ of $\text{Pic}^0(G)$ on the spanning trees of a ribbon graph $G$, which they show is independent of $v$ if and only if $G$ is planar. It is therefore natural to ask how the two constructions are related. We prove that $\beta_e = \rho_v$ for all vertices $v$ of $G$ when $G$ is a planar ribbon graph, i.e. the two torsor structures (Bernardi and rotor-routing) on the set of spanning trees coincide. In particular, it follows that the rotor-routing torsor is compatible with planar duality. We conjecture that for every non-planar ribbon graph $G$, there exists a vertex $v$ with $\beta_v \neq \rho_v$.

1. Introduction

If $G$ is a connected graph on $n$ vertices, the Picard group $\text{Pic}^0(G)$ of $G$ (also called the sandpile group, critical group, or Jacobian group) is a finite abelian group whose cardinality is the determinant of any $(n - 1) \times (n - 1)$ principal sub-minor of the
Laplacian matrix of $G$. By Kirchhoff’s Matrix-Tree Theorem, this quantity is equal to the number of spanning trees of $G$. There are several known families of bijections between spanning trees and elements of Pic$^0(G)$, which we think of as giving bijective proofs of Kirchhoff’s Theorem – see for example [BST13] and the references therein. However, such bijections depend on various auxiliary choices, and there is no canonical bijection in general between Pic$^0(G)$ and the set $S(G)$ of spanning trees of $G$ (see [CCG13, p. 2]).

Jordan Ellenberg asked if it might be the case that, under certain conditions, $S(G)$ is naturally a torsor for Pic$^0(G)$. This question was thoroughly studied in the paper [CCG13], where it was established via the rotor-routing process that given a plane graph there is a canonical simply transitive action of Pic$^0(G)$ on $S(G)$. More generally, if one fixes a ribbon structure on $G$ and then chooses a root vertex $v$, rotor-routing produces a simply transitive action $\rho_v$ of Pic$^0(G)$ on $S(G)$ which is shown in [CCG13] to be independent of $v$ if and only if $G$ together with its ribbon structure is planar.

The first author learned of the results of [CCG13] at a 2013 AIM workshop on “Generalizations of Chip Firing”, and at the same workshop he learned about an interesting family of bijections due to Olivier Bernardi [Ber08] between spanning trees, root-connected out-degree sequences, and recurrent sandpile configurations. Bernardi’s bijections depend on choosing a ribbon structure, a root vertex $v$, and an edge $e$ adjacent to $v$. It was natural to ask whether Bernardi’s bijections became torsor structures upon forgetting the edge $e$, and whether the resulting torsor was again independent of $v$ in the planar case. The present paper answers these questions affirmatively. Specifically, given a ribbon graph $G$, we show that:

1. (Theorem 4.1) Fixing a vertex $v$, the Bernardi process defines a simply transitive action $\beta_v$ of Pic$^0(G)$ on $S(G)$.
2. (Theorems 5.1 and 5.2) The action $\beta_v$ is independent of $v$ if and only if the ribbon graph $G$ is planar.
3. (Theorem 7.1) If $G$ is planar, the Bernardi and rotor-routing torsors coincide.

We also give an example which shows that if $G$ is non-planar then $\beta_v$ and $\rho_v$ do not in general coincide. In fact, we conjecture that if $G$ is non-planar then there always exists a vertex $v$ such that $\beta_v \neq \rho_v$.

We also investigate the relationship between the Bernardi process and planar duality. It is known that if $G$ is a planar graph then Pic$^0(G)$ and Pic$^0(G^*)$ are canonically isomorphic, and there is a canonical bijection between $S(G)$ and $S(G^*)$. It is thus natural to ask whether the Bernardi torsor is compatible these isomorphisms. We prove that this is indeed the case:

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$^1$A torsor for a group $H$ is a set $S$ together with a simply transitive action of $H$ on $S$. 

If $G$ is planar, the following diagram is commutative:

$$\begin{array}{ccc}
\text{Pic}^0(G) \times S(G) & \xrightarrow{\beta} & S(G) \\
\downarrow_{\Psi \times \sigma} & & \downarrow_{\sigma} \\
\text{Pic}^0(G^*) \times S(G^*) & \xrightarrow{\beta^*} & S(G^*)
\end{array}$$

Combined with (3), this proves that the rotor-routing torsor is also compatible with planar duality (which the first author had conjectured at the 2013 AIM workshop). An independent proof (not going through the Bernardi process) of the compatibility of the rotor-routing torsor with planar duality has recently been given by [CGM+14].

A key technical insight which we use repeatedly is an interpretation of Bernardi’s bijections in terms of (integral) break divisors in the sense of [ABKS13]. Break divisors are in canonical bijection with elements of $\text{Pic}^g(G)$, where $g$ is the genus of $G$ in the sense of [BN07], and $\text{Pic}^g(G)$ is canonically a torsor for $\text{Pic}^0(G)$. Among other things, we use break divisors to give a simpler proof than the one in [Ber08] that Bernardi’s maps are in fact bijections.\footnote{Break divisors allow us to sidestep treating the “cyclic part” of a $q$-connected orientation. Understanding how to standardize this cyclic part is one of the more subtle parts of [Ber08].} In particular, we are able to give a new family of bijective proofs of the Matrix-Tree Theorem.

The Bernardi and rotor-routing processes are defined quite differently, so it seems rather miraculous that the two stories parallel one another so closely, intertwining in the planar case and diverging in general. It is a strangely beautiful tale which still appears to hold some mysteries.

The plan of this paper is as follows. In Section 2 we review the necessary background material, including Picard groups, break divisors, ribbon graphs, rotor-routing, and the Bernardi process. In Section 3 we give our new proof that Bernardi’s maps are bijections, and in Section 4 we define the action $\beta_v$ and establish (1). In Section 5 we investigate planarity and establish (2), and in Section 6 we study the relationship between planar duality and the Bernardi process and establish (4). Finally, in Section 7 we relate the Bernardi and rotor-routing torsors in the planar case and establish (3).

2. Background

2.1. Graphs, divisors, and linear equivalence. Let $G$ be a graph (by which we mean a finite, connected, undirected multigraph without loop edges) with vertex set $V(G)$ and edge set $E(G)$. We denote by $\vec{e}$ an edge $e \in E(G)$ together with an orientation, and by $(\vec{e})^{\text{op}}$ the edge $e$ with the opposite orientation.
Following [BN07], we define the group of divisors on $G$, denoted $\text{Div}(G)$, to be the free abelian group on $V(G)$. The degree of a divisor $D = \sum_{v \in V(G)} a_v(v)$ is defined to be $\sum a_v$, and the set of divisors of degree $d$ is denoted $\text{Div}^d(G)$.

Let $M(G)$ be the set of functions $f : V(G) \to \mathbb{Z}$. We define the group of principal divisors on $G$, denoted $\text{Prin}(G)$, to be \{\(\Delta f : f \in M(G)\}\}, where

$$\Delta f = \sum_{v \in V(G)} \left( \sum_{e = vw} (f(v) - f(w)) \right)(v)$$

is the Laplacian of $f$ considered as a divisor on $G$. We say that two divisors $D$ and $D'$ are linearly equivalent, written $D \sim D'$, if $D - D' \in \text{Prin}(G)$.

For each $d \in \mathbb{Z}$ we let $\text{Pic}^d(G)$ be the set of linear equivalence classes of degree $d$ divisors on $G$. In particular, $\text{Pic}^0(G) = \text{Div}^0(G)/\text{Prin}(G)$ is a group which acts simply and transitively on each $\text{Pic}^d(G)$. By basic linear algebra, the cardinality of $\text{Pic}^0(G)$ is the determinant of any $(n-1) \times (n-1)$ principal sub-minor of the Laplacian matrix of $G$.

We denote by $[D] \in \text{Pic}^d(G)$ the linear equivalence class of a divisor $D \in \text{Div}^d(G)$.

2.2. Break divisors. Let $G$ be a graph, and let $g = g_{\text{comb}}(G)$ be the combinatorial genus of $G$, defined as $#E(G) - #V(G) + 1$. If $T$ is a spanning tree of $G$, then there are exactly $g$ edges $e_1, \ldots, e_g$ of $G$ not belonging to $T$. A divisor of the form $D = \sum_{i=1}^g (v_i)$, where $v_i$ is an endpoint of $e_i$, is called a $T$-break divisor. A break divisor $[D]$ is a $T$-break divisor for some spanning tree $T$. We denote by $B(G)$ the set of break divisors on $G$. We will sometimes say that $D$ is compatible with $T$ to mean that $D$ is a $T$-break divisor.

The following important fact is proved in [ABKS13] (see [Bac14] for an alternate proof):

**Theorem 2.1.** Every element of $\text{Pic}^0(G)$ is linearly equivalent to a unique break divisor.

2.3. Ribbon graphs. A ribbon graph is a finite graph $G$ together with a cyclic ordering of the edges around each vertex. A ribbon structure on $G$ gives an embedding of $G$ into a canonical (up to homeomorphism) closed orientable surface $S$. (The surface $S$ is obtained by first thickening $G$ to a compact orientable surface-with-boundary $R$ and then gluing a disk to each boundary component of $R$.) Conversely, every such embedding gives rise to a ribbon structure on $G$. Ribbon structures are therefore sometimes called combinatorial embeddings. (A good reference for basic combinatorial properties of ribbon graphs is [Tho95].) We refer to the genus of $S$ as the topological genus $g_{\text{top}}(G)$ of $G$, and say that $G$ is planar if $g_{\text{top}}(G) = 0$.

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3In [ABKS13], these are called integral break divisors.
Equivalently, a planar ribbon graph is one which can be embedded in the Euclidean plane $\mathbb{R}^2$ without crossings in such a way that the ribbon structure on $G$ is induced by the natural counterclockwise orientation on $\mathbb{R}^2$.

Every closed orientable surface $S$ can be cut along a collection of loops to give a polygon $P$, and conversely by identifying certain pairs of sides of $P$ one can recover the surface $S$. In fact, as is well-known (see e.g. [Hen94, p. 126]), one can arrange for the labeling of the edges of $P$ as one traverses the perimeter counterclockwise to have the form

\begin{equation}
P = a_1b_1a_1^{-1}b_1^{-1} \cdots a_gb_ga_g^{-1}b_g^{-1}.
\end{equation}

(In this labeling, $a_i$ and $a_i^{-1}$ get glued together with opposite orientations, and similarly for $b_i$ and $b_i^{-1}$.) We define the genus of $P$ to be the integer $g$, which is also the genus of $S$.

In particular (by avoiding the vertices), every ribbon graph $G$ can be drawn inside a fundamental polygon $P$ whose boundary is glued as in (2.2), with all vertices of $G$ lying on the interior of $G$. One may take the genus of $P$ to be the topological genus of $G$.

### 2.4. Planar duality.

A planar graph is a graph $G$ which can be embedded in the Euclidean plane $\mathbb{R}^2$ with no edges crossing. A plane graph is a graph $G$ together with such an embedding.

It is well-known that every plane graph has a planar dual $G^*$ whose vertices correspond to faces of $G^*$ and whose edges are dual to edges of $G$. Duality for plane graphs has the following well-known properties:

1. There is a canonical isomorphism $G^{**} \cong G$.
2. There is a canonical bijection $\psi : S(G) \to S(G^*)$ sending a spanning tree $T$ of $G$ to the spanning tree $T^*$ whose edges are dual to the edges of $G$ not in $T$.
3. There is a isomorphism of groups $\Psi = \Psi_\mathcal{O} : \text{Pic}^0(G) \xrightarrow{\sim} \text{Pic}^0(G^*)$ depending on the choice of an orientation $\mathcal{O}$ of the plane.

The isomorphism in (3) is obtained as follows. The choice of $\mathcal{O}$ allows us to identify directed edges of $G$ with directed edges of $G^*$ in a natural way: if $\vec{e}$ is a directed edge of $G$ then locally, near the crossing of $e$ and $e^*$, one obtains an orientation on $e^*$ by following the given orientation of the plane. This identification affords an isomorphism between the lattice $C_1$ of integral 1-chains on $G$ and the lattice $C_1^*$ of integral 1-chains on $G^*$. By [EldHN97, Proposition 8], there is also a canonical isomorphism

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4There are two ways to orient $\mathbb{R}^2$, and for most of this paper we will implicitly or explicitly work with the counterclockwise orientation. However, for planar duality it is important to consider both orientations.

5In general $G^*$ will have loop edges, which we exclude in this paper, so for duality considerations we assume that $G$ has no bridges (edges whose removal disconnects the graph).
between the lattice of integer flows $Z_I$ for $G$ and the lattice of integer cuts $B^*_I$ for $G^*$, and vice-versa. And by [Big97, Proposition 28.2], there is a canonical isomorphism $	ext{Pic}^0(G) \sim \to \frac{C_I}{Z_I \oplus B_I}$. We thus obtain an isomorphism $\Psi : \text{Pic}^0(G) \sim \to \text{Pic}^0(G^*)$ induced by the composition

$$
\text{Pic}^0(G) \sim \to \frac{C_I}{Z_I \oplus B_I} \sim \to \frac{C^*_I}{Z^*_I \oplus B^*_I} \sim \to \text{Pic}^0(G^*).
$$

If $G$ is a planar ribbon graph (with respect to some orientation $\mathcal{O}$ of the plane), the natural way to define a dual planar ribbon graph $G^*$ is to use the opposite orientation $\mathcal{O}^{\text{op}}$ to define the cyclic ordering around each vertex of the dual graph. It is a standard fact this is well-defined, i.e., does not depend on the choice of a particular plane embedding. The above facts translate as follows to planar ribbon graphs:

1. There is a canonical isomorphism $G^{**} \cong G$.
2. There is a canonical bijection $\psi : S(G) \to S(G^*)$.
3. There is a canonical isomorphism of groups $\Psi : \text{Pic}^0(G) \sim \to \text{Pic}^0(G^*)$.

The takeaway points here are that when we consider planar ribbon graphs as opposed to planar graphs:

(a) The isomorphisms in (1) and (2) do not depend on a choice of a particular planar embedding.

(b) The isomorphism in (3) is canonical.

Moreover, it is essential for (1) that we use $\mathcal{O}^{\text{op}}$ to define the cyclic ordering around each vertex of the dual graph.

2.5. **Rotor-routing.** We give a quick summary of some basic facts about rotor-routing from [HLM+08] and [CCG13] which will be needed for our proof of Theorem 7.1.

Let $G$ be a ribbon graph, and choose a sink vertex $y$ of $G$. The **rotor-routing model** is a deterministic process on states $(\rho, x)$, where $\rho$ is a rotor configuration (i.e., an assignment of an outgoing edge $\rho[z]$ to each vertex $z \neq y$ of $G$) and $x$ is a vertex of $G$, which one thinks of as the position of a chip which moves along the graph. Each step of the rotor-routing process consists of replacing $(\rho, x)$ with a new state $(\rho', x')$, where $\rho'$ is obtained from $\rho$ by rotating the rotor $\rho[x]$ to the next edge $\widehat{\rho[x]}$ in the cyclic order at $x$ and $x'$ is the other endpoint of $\widehat{\rho[x]}$. We think of the chip as moving from $x$ to $x'$ along the line $\rho[x]$ in the process.

Given a root vertex $y$, a divisor $D \in \text{Div}^0(G)$, and a spanning tree $T$, one defines a new spanning tree $((x) - (y))_y(T)$ as follows. Orienting the edges of $T$ towards $y$ gives a rotor configuration $\rho_T$ on $G$. Place a chip at the initial vertex $x$ and iterate the rotor-routing process starting with the pair $(\rho, x)$ until the
chip first reaches \(y\) (which it always does, see \([\text{HLM}^+08\text{, Lemma 3.6}]\)). Call the resulting pair \((\sigma, y)\). Denote the pairs at each step of the rotor-routing process by \((\rho_0, x_0) = (\rho, x), (\rho_1, x_1), \ldots, (\rho_k, x_k) = (\sigma, y)\). Define \(S_0 = T\), and for \(i = 0, \ldots, k - 1\) define a subset \(S_{i+1} \subset E(G)\) by \(S_{i+1} = S_i \setminus \rho_i[x_i] \cup \rho_i[x_i]\). Although it is not in general true that each \(S_{i+1}\) is a spanning tree, it is proved in \([\text{HLM}^+08]\) that \(S_k\) is a spanning tree \(T'\), and we define \(( (x) - (y) )_y(T) = T'\). (See Figure 1 for an example.) Extending linearly, this defines an action of \(\text{Div}^0(G)\) on \(S(G)\) which by \([\text{HLM}^+08]\) is trivial on \(\text{Prin}(G)\) and descends to a simply transitive action \(\rho_y\) of \(\text{Pic}^0(G)\) on \(S(G)\).

The following result is proved in \([\text{CCG13}]\):

**Theorem 2.3.** The action \(\rho_y\) is independent of the root vertex \(y\) if and only if the ribbon graph \(G\) is planar.

An important ingredient in the proof of Theorem 2.3 is the relationship between rotor-routing and unicycles. A unicycle is a state \((\rho, x)\) such that \(\rho\) contains exactly one directed cycle \(C(\rho)\) and \(x\) lies on this cycle. By \([\text{HLM}^+08\text{, Lemma 3.3}]\), the rotor-routing process takes unicycles to unicycles. Suppose \(G\) has \(m\) edges and \((\rho, x)\) is a unicycle on \(G\). By \([\text{HLM}^+08\text{, Lemma 4.9}]\), if we iterate the rotor-routing process \(2m\) times starting at the state \((\rho, x)\), the chip traverses each edge of \(G\) exactly once in each direction, each rotor makes exactly one full turn, and the stopping state is \((\rho, x)\).

Using this, one sees that the relation on unicycles defined by \((\rho, x) \sim (\rho', x')\) iff \((\rho', x')\) can be obtained from \((\rho, x)\) by iterating the rotor-routing process some number of times is an equivalence relation.

Given a unicycle \((\rho, x)\), denote by \(\bar{\rho}\) the configuration obtained from \(\rho\) by reversing the edges of \(C\) and keeping all other rotors the same. One says that the unicycle \((\rho, x)\) is reversible if \((\rho, x) \sim (\bar{\rho}, x)\). By \([\text{CCG13\text{, Proposition 7}]\), the notion of reversibility is intrinsic to the directed cycle \(C(\rho)\), and does not actually depend on \(\rho\) or \(x\); in other words, if \((x, \rho)\) and \((x', \rho')\) are unicycles with \(C(\rho) = C(\rho')\), then \((x, \rho)\) is reversible iff \((x', \rho')\) is. It therefore makes sense to talk about reversibility of directed cycles in \(G\). The importance of this concept stems from \([\text{CCG13\text{, Proposition 9}]\), which asserts that the ribbon graph \(G\) is planar if and only if every directed cycle of \(G\) is reversible.
2.6. **The Bernardi Process.** Let $G$ be a ribbon graph, and fix a pair $(v,e)$ (which we refer to as the *initial data*) consisting of a vertex $v$ and an edge $e$ adjacent to $v$. In this section, we recall the *tour* of $G$ which Bernardi [*Ber08*, §3.1] associates to the initial data $(v,e)$ together with a spanning tree $T$, and describe how to associate a break divisor to this tour.

Let $T$ be a spanning tree of $G$. The tour $\tau_{(v,e)}(T)$ is a traversal of $T$ which begins and ends at $v$. Informally, the tour is obtained by walking along edges belonging to $T$ and cutting through edges not belonging to $T$, beginning with $e$ and proceeding according to the ribbon structure. (See Figure 2.)

More formally, the tour is a sequence

$$\tau_{(v,e)}(T) = (v_0, \vec{e}_1, v_1, \vec{e}_2, \ldots, \vec{e}_k, v_k)$$

where each $v_i$ is a vertex of $G$ and $\vec{e}_i$ is a directed edge of $G$ leading to $v_i$. We set $v_0 = v$ and let $e_0$ (which is not part of the tour) be the edge preceding $e$ in the (given) cyclic ordering around $v$. The $(v_i, e_i)$ for $1 \leq i \leq k$ are defined inductively by declaring that $e_i = (v_{i-1}, v_i)$ is the first edge after $e_{i-1}$ belonging to $T$ in the cyclic ordering of the edges around $v_{i-1}$. The tour stops when each edge of $T$ has been included twice among the $e_i$ with $1 \leq i \leq k$, once with each orientation; necessarily we will have $v_k = v$.

The break divisor $\beta_{(v,e)}(T)$ associated to the tour is obtained by dropping a chip at the corresponding vertex each time the tour first cuts through an edge not belonging to $T$. In other words, for each edge $e'$ not in the spanning tree, let $\{\vec{e}_i, \vec{e}_j\}$ be the two oriented edges whose underlying unoriented edge is $e'$, with $i < j$, and set $\eta(e') := v_{i-1}$. We define

$$\beta_{(v,e)}(T) := \sum_{e' \not\in T} (\eta(e')).$$
The amazing fact implicitly discovered by Bernardi is that the association $T \mapsto \beta_{(v,e)}(T)$ gives a bijection between spanning trees of $G$ and break divisors. In the next section, we give a proof which is different from Bernardi’s that $\beta_{(v,e)}$ is bijective by explicitly constructing an inverse (or, more precisely, two one-sided inverses).

3. The Bernardi map is bijective

Let $S(G)$ denote the set of spanning trees of $G$ and let $B(G)$ denote the set of break divisors of $G$, which is canonically isomorphic to $\text{Pic}^g(G)$. As in the previous section, we fix a pair $(v,e)$ consisting of a vertex $v$ and an edge $e$ adjacent to $v$. In this section we give a new proof of the fact that the map $\beta := \beta_{(v,e)} : S(G) \to B(G)$ is bijective by explicitly constructing a left inverse $\alpha_L$ and a right inverse $\alpha_R$ to $\beta$. It follows a posteriori that $\alpha_L = \alpha_R$ and both are two-sided inverses to $\beta$, which is therefore a bijection.

Bernardi’s proof in [Ber08] that $\beta$ is bijective requires his difficult Proposition 34. We believe that our new proof is enlightening to the extent that it makes the inverse map completely explicit.

3.1. The right inverse. Let $D$ be a break divisor. To define the map $\alpha_R : B(G) \to S(G)$, we will inductively reconstruct a spanning tree $T$ with $\beta(T) = D$, along with a corresponding Bernardi tour which traverses $T$. Since the tour is obtained by walking along edges belonging to $T$ and cutting through edges not belonging to $T$, but in this case we don’t know $T$, our challenge is to use the break divisor $D$ to figure out which edges to walk along and which to cut through. The solution is that we will cut through an edge $e'$ if removing that edge from the graph and subtracting a chip from $D$ at the current vertex gives a break divisor $D'$ on the resulting graph $G'$; otherwise we walk along $e'$ and add it to the spanning tree which we’re building.

More formally, $\alpha_R(D)$ is defined to be the output of Algorithm 1 below.

**Proposition 3.1.** Let $G$ be a graph and $D$ a break divisor on $G$. Then $\alpha_R(D)$ is a spanning tree of $G$ and $\beta(\alpha_R(D)) = D$.

**Proof.** We will show that Algorithm 1 terminates (and hence $\alpha_R$ is well-defined), and that $\alpha_R(D)$ is a spanning tree. The fact that $\beta(\alpha_R(D)) = D$ is then clear from the definitions of $\alpha_R$ and $\beta$.

The fact that Algorithm 1 terminates follows easily from the fact that $G'$ is always connected at every stage. To prove that $S := \alpha_R(D)$ is a spanning tree, we must show that $S$ is connected, contains every vertex of $G$, and does not contain a cycle. The fact that $S$ is connected follows easily from the fact that $G'$ is always connected at every stage of Algorithm 1.

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6Bernardi phrases his result (Theorem 41(5)) in terms of out-degree sequences of orientations, but in view of the results of [ABKS13] the two points of view are equivalent.
**Input:** A connected graph $G$ and a break divisor $D \in \text{Div}(G)$.

**Output:** A spanning tree $T$.

Set $i := 0$, $T := \emptyset$, $v' := v$, and $e' := e$.

while $T \neq G$ do

  Let $w'$ be the other endpoint (besides $v'$) of $e'$.
  Let $G'$ be the graph obtained from $G$ by deleting the edge $e'$.
  Define $D' := D - (v')$.

  if $G'$ is connected and $D'$ is a break divisor on $G'$ then
   
  Replace $G$ by $G'$, $D$ by $D'$, and $e'$ by the edge following it in the (induced) cyclic ordering around $v'$ on $G'$.

  else
   
  Replace $T$ by $T \cup \{e'\}$, $v'$ by $w'$, and $e'$ by the edge following it in the cyclic ordering around $w'$ on $G'$.

end

end

Output $T$.

**Algorithm 1:** Right inverse to the Bernardi map

Suppose for the sake of contradiction that $S$ contains a cycle $C$. At the stage of running Algorithm 1 where the cycle $C$ has just been created, we have a break divisor $\tilde{D}$ on a subgraph $\tilde{G}$ of our original graph containing $C$. Let $\tilde{T}$ be a spanning tree of $\tilde{G}$ which is compatible with $\tilde{D}$. Let $f$ be an edge of $C$ not belonging to $\tilde{T}$. By construction, $\tilde{D}$ must contain a chip at some endpoint $w$ of $f$. Let $\tilde{w}$ be the value of $v'$ when Algorithm 1 decides to include the edge $f$ in $S$. By construction, $\tilde{w}$ is an endpoint of $f$. We have two cases to consider, depending on whether or not $w = \tilde{w}$:

- If $w = \tilde{w}$, then $\tilde{D} - (w)$ is a break divisor on $\tilde{G} \setminus f$ corresponding to the spanning tree $\tilde{T}$. This contradicts the fact that when running Algorithm 1 we did not cut through $f$.

- If $w \neq \tilde{w}$, let $f'$ be the other edge of $C$ having $w$ as an endpoint. Then $\tilde{D} - (w)$ is a break divisor on $\tilde{G} - f'$ corresponding to the spanning tree $\tilde{T}' := \tilde{T} \setminus f' \cup f$. This contradicts the fact that when running Algorithm 1 we did not cut through $f'$.

Finally, suppose that $S$ does not contain every vertex of $G$. Since we have just proved that $S$ is a tree, it is easy to see that for every vertex $w \in S$ and every edge $f$ adjacent to $w$, Algorithm 1 either includes $f$ in $S$ or cuts through $f$. It follows that at some stage of Algorithm 1 we must have cut through an edge $f$ whose removal disconnects $G$, a contradiction. \[\square\]

**Corollary 3.2.** The Bernardi map $\beta : S(G) \to B(G)$ is surjective.
Remark 3.3. There is an efficient (polynomial-time) algorithm for deciding whether or not a given divisor on a graph is a break divisor; see [Bac14].

3.2. The left inverse. It is not clear how to prove directly that the map $\alpha_R$ constructed in the previous subsection is injective. Instead, we prove injectivity by defining a different map $\alpha_L$ which is constructed in exactly the same way as $\alpha_R$ but **touring the graph in the opposite direction** and subtracting a chip from the endpoint of $e'$ opposite $v'$ when standing at $v'$ and deciding whether to walk along $e'$ or cut through it. More formally, we define $\alpha_L(D)$ to be the output of Algorithm 2 below.

**Algorithm 2:** Left inverse to the Bernardi map

| Input: A connected graph $G$ and a break divisor $D \in \text{Div}(G)$. |
| Output: A spanning tree $T$. |
| Let $e_0$ be the edge preceding $e$ in the cyclic ordering around $v$. |
| Set $i := 0$, $T' := \emptyset,v' := v$, and $e' := e_0$. |
| **while** $T \neq G$ **do** |
| Let $w'$ be the other endpoint (besides $v'$) of $e'$. |
| Let $G'$ be the graph obtained from $G$ by deleting the edge $e'$. |
| Define $D' := D - (w')$. |
| **if** $G'$ is connected and $D'$ is a break divisor on $G'$ **then** |
| Replace $G$ by $G'$, $D$ by $D'$, and $e'$ by the edge preceding it in the (induced) cyclic ordering around $v'$ on $G'$. |
| **else** |
| Replace $T$ by $T \cup \{e'\}$, $v'$ by $w'$, and $e'$ by the edge preceding it in the cyclic ordering around $w'$ on $G'$. |
| **end** |
| **end** |
| Output $T$. |

**Proposition 3.4.** Let $G$ be a graph.

1. If $D$ is a break divisor on $G$ then $\alpha_L(D)$ is a spanning tree of $G$.
2. If $T$ is a spanning tree of $G$ then $\alpha_L(\beta(T)) = T$.

**Proof.** It suffices to prove (2), since we already know by Corollary 3.2 that $\beta : S(G) \to B(G)$ is surjective.

To establish (2), let $T$ be a spanning tree and let $A_1, \ldots, A_k \subset \text{V}(G) \setminus \{v\}$ be the connected components of the complement of $v$ (the starting vertex for the Bernardi process) in $T$. We may order these components so that $A_i$ is connected to $v$ by an edge $a_i \in T$ and $a_1, \ldots, a_k$ follow the cyclic ordering of edges at $v$, with $a_1$ the first one visited by the Bernardi process. Thus $A_k$ will be the last component visited by
the Bernardi process, and any edge connecting $A_i$ and $A_k$ will be cut at the endpoint in $A_i$ before the Bernardi process reaches $A_k$.

Let $\psi$ be the inverse process described by Algorithm 2 (which induces the function $\alpha_L$). By induction, it suffices to prove that:

(a) The inverse process $\psi$ does not cut through the edge $a_k$.

(b) The inverse process $\psi$ does cut through each edge preceding $a_k$ in the reverse cyclic ordering at $v$ (starting with the edge $e_0$).

Indeed, when the inverse process $\psi$ goes along the edge $a_k$ to the next vertex $v'$, we can set the base vertex for the Bernardi process to be $v'$ and continue as before.

For (a), note that by deleting any edge before $a_k$ in this reverse cyclic ordering, the spanning tree induced by $T$ on the resulting graph $G'$ is compatible with the resulting break divisor $D'$. So the inverse process $\psi$ will cut through these edges.

For (b), denote by $D'$ and $G'$ the new break divisor and new graph obtained by cutting all edges before $a_k$ in the process $\psi$. We identify $T$ with its image in $G'$. Let $G_k'$ be the induced subgraph on $A_k$ in $G'$, which by assumption is connected, and let $g_k'$ be its genus (number of edges minus number of vertices plus one). By our above remarks, any edge of $G'$ not in $T$ which connects $A_k$ with its complement is cut through before the Bernardi process visits $A_k$. It follows that $\deg(D'|_{A_k}) = g_k'$. On the other hand, it is not hard to see that for any connected subgraph $G^*$ of $G'$ containing $A_k$ and any break divisor $D^*$ on $G^*$, we must have $\deg(D^*|_{A_k}) \geq g_k'$. If the edge $a_k$ can be cut through, then by deleting $a_k$ and subtracting 1 from $D'$ at the endpoint of $a_k$ belonging to $A_k$, we obtain a new break divisor $D''$ with $\deg(D''|_{A_k}) = g_k' - 1$, a contradiction. □

**Corollary 3.5.** The Bernardi map $\beta : S(G) \to B(G)$ is bijective.

**Proof.** Proposition 3.4 implies that $\beta$ is injective, and by Corollary 3.2 it is also surjective. □

**Remark 3.6.** The proof of Corollary 3.5 combined with the results of [ABKS13] and [Bac14], provides another ‘efficient bijective proof’ of Kirchhoff’s Matrix-Tree Theorem in the spirit of [BS13], as well as a new algorithm for choosing a uniformly random spanning tree of $G$. Indeed, by [BS13] the cardinality of $\Pic^0(G)$ equals the number of spanning trees of $G$, and by [ABKS13] there is a canonical bijection between $\Pic^0(G)$ and the set of break divisors on $G$. We have just proved that the Bernardi map $\beta_{(v,e)}$ gives a bijection between break divisors and spanning trees of $G$. The results of [Bac14] show that the resulting bijection between $\Pic^0(G)$ and $S(T)$ is efficiently computable in both directions. See [BS13] for an explanation of how such a bijection can be used to find random spanning trees.
4. The Bernardi torsor

In this section, we show how to associate a simply transitive action $\beta_v$ of $\Pic^0(G)$ on $S(G)$ to a pair $(G,v)$ consisting of a ribbon graph $G$ and a vertex $v$ of $G$. For a divisor $D \in B(G)$, we write $[D]$ for the linear equivalence class of $D$ in $\Pic^0(G)$. Recall that $\Pic^0(G)$ acts simply and transitively on $\Pic^0(G)$ in a natural way, and that the map $D \mapsto [D]$ gives a canonical bijection from $B(G)$ to $\Pic^0(G)$. From this we get a canonical simply transitive action of $\Pic^0(G)$ on $B(G)$ which we write as $\gamma \cdot D$.

**Theorem 4.1.** Let $v$ be a vertex of $G$.

1. Let $e_1, e_2$ be edges incident to $v$, and let $\beta_1 = \beta_{(v,e_1)}$ and $\beta_2 = \beta_{(v,e_2)}$ be the corresponding Bernardi bijections. Then there exists an element $\gamma_0 \in \Pic^0(G)$ such that $\beta_2(T) = \gamma_0 \cdot \beta_1(T)$ for all $T \in S(G)$.

2. The action $\beta_v : \Pic^0(G) \times S(G) \to S(G)$ defined by $\gamma \cdot T := \beta_{(v,e)}^{-1}(\gamma \cdot \beta_{(v,e)}(T))$ for any edge $e$ incident to $v$, depends only on $v$ and not on the choice of $e$.

**Proof.** Assuming (1) for the moment, we verify that (2) holds. We need to prove that if $\beta_1 = \beta_{(v,e_1)}$ and $\beta_2 = \beta_{(v,e_2)}$ are as in (1), then

$$\beta_2^{-1}(\gamma \cdot \beta_1(T)) = \beta_2^{-1}(\gamma \cdot \beta_2(T)).$$

To see this, observe that (1) implies $\beta_2^{-1}(x) = \beta_1^{-1}(\gamma_0^{-1} \cdot x)$ for all $x \in \Pic^0(G)$. Thus

$$\beta_2^{-1}(\gamma \cdot \beta_2(T)) = \beta_1^{-1}(\gamma_0^{-1} \gamma_0 \cdot \beta_1(T)) = \beta_1^{-1}(\gamma \cdot \beta_1(T))$$

as claimed.

For (1), it suffices to prove that $\beta_1(T) - \beta_2(T)$ and $\beta_1(T') - \beta_2(T')$ are linearly equivalent in $\Div^0(G)$ for any two spanning trees $T, T'$ of $G$. To do this we first derive a useful formula for $\beta_1(T) - \beta_2(T)$. By definition, we have

$$\beta_1(T) - \beta_2(T) = \sum_{f \not\in T} \delta(f),$$

where $\delta(f) := \eta_{(v,e_1)}(f) - \eta_{(v,e_2)}(f)$ (considered as a divisor on $G$). Thus it will suffice to find a formula for $\delta(f)$ when $f \not\in T$.

Let the cyclic ordering of the edges around $v$, starting with $e_1$, be

$$(e_1, a_1, \ldots, a_k, e_2, b_1, \ldots, b_t).$$

Let $I = \{e_1, a_1, \ldots, a_k\}$ and $J = \{e_2, b_1, \ldots, b_t\}$. Removing $v$ from $T$ partitions the set $V(G) \setminus \{v\}$ disjoint sets $A$ and $B$, where $A$ (resp. $B$) is the union of all vertices lying in the same connected component of $T \setminus v$ as some edge in $I$ (resp. $J$).

One sees easily that the Bernardi tours $\tau_{(v,e_1)}(T)$ and $\tau_{(v,e_2)}(T)$ are cyclic shifts of each other, the difference being that $\tau_{(v,e_1)}(T)$ traverses the $A$-components of $T$...
followed by the $B$-components, while the reverse is true for $\tau_{(v,e_2)}(T)$. This shows that $\delta(f) = 0$ (i.e., the Bernardi tours $\tau_{(v,e_1)}(T)$ and $\tau_{(v,e_2)}(T)$ cut through $f$ from the same vertex) when $f \in E(G) \setminus T$ is any one of the following:

- An edge whose endpoints both belong to $A$ or both belong to $B$.
- An edge $va \in I$ with $a \in A$, or an edge $vb \in J$ with $b \in B$.

On the other hand, the following kind of edges of $G \setminus T$ make a non-trivial contribution to the difference $\delta(f) := \eta_{(v,e_1)}(f) - \eta_{(v,e_2)}(f)$ (considered as a divisor on $G$):

- If $f = ab$ with $a \in A$ and $b \in B$ then $\delta(f) = (a) - (b)$.
- If $f = va' \in J$ with $a' \in A$ then $\delta(f) = (a') - (v)$.
- If $f = vb' \in I$ with $b' \in B$ then $\delta(f) = (v) - (b')$.

Summarizing, we obtain the following formula, where the sum is over edges not in $T$:

\[
\beta_1(T) - \beta_2(T) = \sum_{f=ab \atop a \in A, b \in B} (a) - (b) + \sum_{f=va' \in J \atop a' \in A} (a') - (v) + \sum_{f=vb' \in I \atop b' \in B} (v) - (b').
\]

See Figure 3 for an example.

We now use Equation 4.2 to prove that $\beta_1(T) - \beta_2(T) \sim \beta_1(T') - \beta_2(T')$ for any two spanning trees $T, T'$ of $G$. It is well-known, and follows easily from the fact that spanning trees of $G$ form the independent sets of a matroid, that the graph $\Sigma(G)$, whose vertices are spanning trees of $G$ and whose edges correspond to pairs $\{T, T'\}$ with $T' = (T \setminus f) \cup f'$ for some edges $f \in T \setminus T'$ and $f' \in T' \setminus T$, is connected. We may therefore assume that $T' = (T \setminus f) \cup f'$ as above.

In this case, we can give an explicit formula for the degree zero divisor

$\delta_{T,T'} := (\beta_1(T) - \beta_2(T)) - (\beta_1(T') - \beta_2(T'))$

which certifies that it is principal. To state the formula, we need to set some notation. For a spanning tree $T$ and edge $e \in T$, we let $e_T(e) \in E(G)$ be the endpoint other than $v$ of the unique edge $e' \in T$ adjacent to $v$ such that $e$ and $e'$ are in the same connected component of $T \setminus \{v\}$, and we let $K_T(e) \subset V(G) \setminus \{v\}$ be the corresponding connected component.

Applying this to our situation, where $T' = T \setminus f \cup f'$ with $f \in T \setminus T'$ and $f' \in T' \setminus T$, let $\chi$ be the characteristic function of $K_T(f) \cap K_{T'}(f')$. Then a tedious but straightforward case-by-case analysis yields the following formula for $\delta_{T,T'}$:

$\delta_{T,T'} = \begin{cases} 
0 & \text{if } e_T(f), e_{T'}(f') \in A' \text{ or } e_T(f), e_{T'}(f') \in B'. \\
-\Delta \chi & \text{if } e_T(f) \in B, e_{T'}(f') \in A'. \\
\Delta \chi & \text{if } e_T(f) \in A, e_{T'}(f') \in B'.
\end{cases}$

See Figure 4 for an example.
Figure 3. (a) The spanning tree $T$ is shown in red. The edges which contribute non-trivially to $\beta_1(T) - \beta_2(T)$ are shown in green. (b) The Bernardi tours $\tau_1 = \tau_{(v,e_1)}(T)$ and $\tau_2 = \tau_{(v,e_2)}(T)$ differ by a cyclic shift: $\tau_1$ begins at $s_1$ and $\tau_2$ begins at $s_2$. (c) The difference $\beta_1(T) - \beta_2(T)$. 
Figure 4. (a) The spanning tree $T'$ is shown in red. The edges which contribute non-trivially to $\beta_1(T') - \beta_2(T')$ are shown in green. The degree-zero divisor $\beta_1(T') - \beta_2(T')$ is also shown. (b) The divisor $\delta_{T,T'}$ is shown. Here $\epsilon_T(f) = \epsilon_{T'}(f') = w$ and $K_T(f) = K_{T'}(f') = \{w\}$. Thus $\chi = \chi_{\{w\}}$ and $\Delta \chi = \delta_{T,T'}$. Note that $w \in B \cap A'$.

It follows in all cases that $\delta_{T,T'} \in \Prin(G)$ as desired. □

Remark 4.3. From the proof, we see that the element $\gamma_0 \in \Pic^0(G)$ in the statement of Theorem 4.1 is the linear equivalence class of the right-hand side of (4.2) for any choice of $T$. We do not know if there is a simpler formula for $\gamma_0$.

Corollary 4.4. If $G$ is a ribbon graph and $v$ is a vertex of $G$, then the action $\beta_v$ defined above makes the set of spanning trees of $G$ into a torsor for $\Pic^0(G)$. 
5. Planarity and the dependence of the Bernardi torsor on the base vertex

Given a ribbon graph $G$, we prove that the action $\beta_v$ defined in the previous section is independent of the vertex $v$ if and only if $G$ is planar.

First, we deal with the case where $G$ is planar:

**Theorem 5.1.** If $G$ is a planar ribbon graph, then the action $\beta_v$ is independent of $v$, and hence defines a canonical action $\beta$ of $\text{Pic}^0(G)$ on $S(G)$.

**Proof.** Since $G$ is connected by assumption, it suffices to prove that $\beta_{v_1} = \beta_{v_2}$ whenever $v_2$ is a neighbor of $v_1$. Without loss of generality, we may assume that the ribbon structure corresponds to the counterclockwise orientation of the plane.

Let $e_1$ be an edge connecting $v_1$ and $v_2$, and let $e_2$ be the edge following $e_1$ in the cyclic ordering around $v_2$. (If $\deg(v_2) = 1$ then we will have $e_1 = e_2$.) By Theorem 4.1 which allows us to pick whichever edges we want in our initial data, it suffices to prove that for each spanning tree $T$ we have $\beta_{(v_1,e_1)}(T) = \beta_{(v_2,e_2)}(T)$.

If $e_1 \in T$, this is clear, so we may assume that $e_1 \notin T$. In this case, $T \cup e_1$ contains a unique simple cycle $C = C_{T,e_1}$, called the fundamental cycle associated to $T$ and $e_1$. Since $G$ is planar, the edges other than $e_1$ which are not in the spanning tree $T$ can be partitioned into two disjoint subsets: the edges $E_{in}$ lying inside $C$ and the edges $E_{out}$ lying outside $C$.

The Bernardi process associated to the initial data $(v_1, e_1)$ will cut through $e_1$, then cut through all of the edges in $E_{in}$, then cut through all the edges in $E_{out}$, touring around $T$ in the process. The Bernardi process associated to the initial data $(v_2, e_2)$ will cut through all of the edges in $E_{out}$, then cut through $e_1$, then cut through all the edges in $E_{in}$, touring around $T$ in the process. (See Figure 5 for an example.)
It follows that the tours \( \tau_{(v_i,e_i)} \) for \( i = 1, 2 \) are the same up to a cyclic shift, and they cut through edges not in \( T \) in exactly the same way. In particular, \( \beta_{(v_1,e_1)} = \beta_{(v_2,e_2)} \). \( \square \)

Next, we treat the non-planar case:

**Theorem 5.2.** If \( G \) is a non-planar ribbon graph, then there are vertices \( v, v' \) of \( G \) with \( \beta_v \neq \beta_{v'} \).

**Proof.** By the discussion in Section 2.3, \( G \) has a polygonal representation inside a fundamental polygon

\[
P = a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}
\]

which we may assume to have minimal genus among all such representations.

Let \( a = a_1, b = b_1 \). Since \( G \) is non-planar, we may assume that there are edges \( e \) and \( e' \) of \( G \) which intersect boundary edges \( a, a^{-1} \) and \( b, b^{-1} \) of \( P \), respectively.

Let \( G_0 \) be the complement in \( G \) of all edges which pass through \( a \) or \( b \). Then \( G_0 \) is connected, since otherwise one could redraw \( G \) by changing the relative position of the components of \( G_0 \) and obtain a polygonal representation inside a fundamental polygon

\[
P' = a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}
\]

of genus \( g - 1 \), contradicting the minimality of \( P \).

Since \( G_0 \) is connected, there exists a spanning tree \( T_1 \) of \( G \) contained in \( G_0 \). Let \( C = C_{T_1,e} \), let \( e^* \) be an edge of \( T_1 \cap C \) (so in particular \( e^* \) does not intersect \( a \) or \( b \)), and let \( T_2 = T_1 \cup e \setminus e^* \). Without loss of generality, we may orient \( C \) and label the endpoints of \( e, e^* \) so that \( e = (x, y) \) and \( (e^*)^* = (x^*, y^*) \) are oriented consistently in \( C \). Let \( e^{**} \) be the edge following \( e^* \) in the cyclic orientation around \( y^* \) and consider the Bernardi maps \( \beta_1 \) and \( \beta_2 \) arising from the initial data \( (x^*, e^*) \) and \( (y^*, e^{**}) \), respectively.

Since \( e^* \in T_1 \), we know that

\[
\beta_1(T_1) - \beta_2(T_1) = 0.
\]

On the other hand, for \( T_2 \) we have

\[
\beta_1(T_2) - \beta_2(T_2) = \sum_i \left( (c_i) - (d_i) \right),
\]

where \( (c_i, d_i) \) are the edges of \( G \) passing through \( b \) and \( b^{-1} \), oriented so that \( c_i \) lies on the path from \( x^* \) to \( y^* \) in the Bernardi tour of spanning tree \( T_2 \) with initial data \( (x^*, e^*) \). (See Figure 6 for an example.)

We claim that the divisor \( D := \sum_i \left( (c_i) - (d_i) \right) \) is not linearly equivalent to 0. To see this, suppose for the sake of contradiction that

\[
(5.3) \quad \sum_i \left( (c_i) - (d_i) \right) = \Delta f
\]
Figure 6. An example illustrating the proof of Theorem 5.2. In this example, \( \beta_1(T_2) - \beta_2(T_2) = (c) - (d) \).

with \( f \in M(G) \). Since \( e' \) is one of the edges \((c_i,d_i)\), we know that \( D \neq 0 \) and thus \( f \) is non-constant.

Let \( A \) be the set of vertices where \( f \) achieves its maximum value, and let \( H \) be a connected component of the induced subgraph \( G[A] \). It follows easily from (5.3) that every edge connecting \( H \) to its complement in \( G \) is of the form \((c_i,d_i)\), and thus the set of edges of the form \((c_i,d_i)\) separates \( H \) from its complement. But in this case we can redraw the embedding of \( G \) by moving \( H \), obtaining a polygonal representation inside a fundamental polygon

\[
P' = a_2b_2a_2^{-1}b_2^{-1} \cdots a_gb_ga_g^{-1}b_g^{-1}
\]

of genus \( g - 1 \), which as before contradicts the minimality of \( P \).

\[\square\]

6. Compatibility of the Bernardi torsor with planar duality

If \( G \) is a planar ribbon graph, we show that the natural action of \( \text{Pic}^0(G) \) on \( S(G) \) is compatible with planar duality:

**Theorem 6.1.** Let \( G \) be a bridgeless planar ribbon graph. Then the natural actions of \( \text{Pic}^0(G) \) on \( S(G) \) and of \( \text{Pic}^0(G^*) \) on \( S(G^*) \) defined by the Bernardi process are identified with one another via the canonical isomorphism \( \Psi : \text{Pic}^0(G) \cong \text{Pic}^0(G^*) \) and the canonical bijection \( \sigma : S(G) \to S(G^*) \) defined in Section 2.4. In other words, the following diagram is commutative:

\[
\begin{array}{ccc}
\text{Pic}^0(G) \times S(G) & \xrightarrow{\beta} & S(G) \\
\downarrow{\Psi \times \sigma} & & \downarrow{\sigma} \\
\text{Pic}^0(G^*) \times S(G^*) & \xrightarrow{\beta^*} & S(G^*)
\end{array}
\]

*Proof.* Fix some arbitrary initial data for the Bernardi processes on \( G \) and \( G^* \), and denote by \( \beta : S(G) \to B(G) \) and \( \beta^* : S(G^*) \to B(G^*) \) the corresponding Bernardi
maps. We need to prove that for any spanning trees $T_1$ and $T_2$ in $S(G)$ and their corresponding dual spanning trees $T_1^*$ and $T_2^*$ in $S(G^*)$, we have

$$\Psi([\beta(T_2) - \beta(T_1)]) = [\beta^*(T_2^*) - \beta^*(T_1^*)].$$

Without loss of generality, we may assume that $T_2$ is obtained from $T_1$ by adding an edge $e_1 \not\in T_1$ and deleting an edge $e_2 \in T_1$ from the fundamental cycle $C = C_{T_1,e_1}$. One checks easily that $T_2^*$ is obtained from $T_1^*$ by adding $e_2^*$ and deleting $e_1^*$.

We may assume without loss of generality that the ribbon structure on $G$ corresponds to the counterclockwise orientation on the plane. Orient the cycle $C$ counterclockwise, and let $\vec{e}_1 = (x_1, y_1)$ and $\vec{e}_2 = (x_2, y_2)$ be the orientations of $e_1$ and $e_2$, respectively, which are compatible with the orientation of $C$.

By Theorem 5.1, we may assume without loss of generality that the initial data for the Bernardi process on $G$ are $(v, e) = (x_1, e_1)$.

Deleting the edge $e_2$ from $T_1$ (or, alternatively, deleting the edge $e_1$ from $T_2$) defines a partition of $V(G)$ into disjoint subsets $A$ and $B$, where $A$ consists of all vertices on the path from $y_1$ to $x_2$ which follow the orientation of the cycle and $B$ consists of all vertices on the path from $y_2$ to $x_1$. Let $(a_1, b_1), \ldots, (a_k, b_k)$ be the oriented edges of $G$ not belonging to $T_1 \cup T_2$ which connect vertices in $A$ to vertices in $B$ and lie on the inside of the cycle $C$. We claim that

$$\beta(T_2) - \beta(T_1) = \sum_i ((a_i) - (b_i)) + (x_2) - (x_1).$$

Indeed, we can write the tour $\tau_{(x_1, e_1)}(T_1)$ as

$$\tau_{(x_1, e_1)}(T_1) = (\alpha_1, \vec{e}_2)^{op}, \alpha_2, \alpha_3, \vec{e}_2, \alpha_4),$$

where $\alpha_1$ goes from $x_1$ to $y_2$ inside $C$, $\alpha_2$ goes from $x_2$ to $y_1$ inside $C$, $\alpha_3$ goes from $y_1$ to $x_2$ outside $C$, and $\alpha_4$ goes from $y_2$ to $x_1$ outside $C$. An examination of the Bernardi process shows that

$$\tau_{(x_1, e_1)}(T_2) = (\vec{e}_1, \alpha_3, \alpha_2, (\vec{e}_1)^{op}, \alpha_1, \alpha_4),$$

and (6.2) follows easily. See Figure 7 for an example.

We can perform a similar calculation on the dual side. Orient $e_1^*$ so that $(\vec{e}_1)^* = (x_1^*, y_1^*)$ is obtained from $\vec{e}_1$ by a counterclockwise rotation, and orient the fundamental cycle $C^* = C_{T_1^*, e_2^*}$ consistently with $(\vec{e}_1)^*$. Let $(\vec{e}_2)^* = (x_2^*, y_2^*)$ be the induced orientation on $e_2^*$.

**Case 1:** $C^*$ is oriented clockwise.

Deleting the edge $e_2^*$ from $T_1^*$ defines a partition of $V(G)$ into disjoint subsets $A^*$ and $B^*$ with $x_2^*, y_1^* \in A^*$ and $x_1^*, y_2^* \in B^*$. Let $(a_1^*, b_1^*), \ldots, (a_k^*, b_k^*)$ be the oriented edges of $G^*$ not belonging to $T_1^* \cup T_2^*$ which connect vertices in $A^*$ to vertices in $B^*$.
Figure 7. (a),(b): The Bernardi tours $\tau_{(x_1,e_1)}(T_1)$ and $\tau_{(x_1,e_1)}(T_2)$ associated to two different spanning trees (shown in red), and their associated break divisors. (c): The difference $\beta_{(x_1,e_1)}(T_2) - \beta_{(x_1,e_1)}(T_1)$ between the break divisors associated to $T_2$ and $T_1$. The fundamental cycle $C = C_{T_1,e_1}$ is shown in red.

and which lie on the inside of $C^\ast$. Then the exact same calculation as above, with initial data $(x_1^\ast, e_1^\ast)$, shows that

$$\beta^\ast(T_2^\ast) - \beta^\ast(T_1^\ast) = \sum_i ((b_i^\ast) - (a_i^\ast)) + (x_1^\ast) - (x_2^\ast).$$

See Figure 8 for an illustration of the situation.

Define $\lambda = \tilde{e}_1 + \mu + \nu \in C_I$, where $\mu$ is the sum of all the oriented edges along the directed path in $C$ from $y_1$ to $x_2$ and $\nu$ is the sum of all the oriented edges $(b_i, a_i)$. By (6.2), we have $\delta(\lambda) = \beta(T_2) - \beta(T_1)$.

Under the natural duality isomorphism from $C_I$ to $C_I^\ast$, one checks easily that:

- $\mu^\ast$ is the sum of all the oriented edges $(a_i^\ast, b_i^\ast)$ and
- $\nu^\ast$ is the sum of all the oriented edges along the directed path in $C^\ast$ from $x_2^\ast$ to $y_1^\ast$. 
By (6.3), we have
\[ \delta(\lambda^*) = \delta((x_1^*, y_1^*) + \mu^* + \nu^*) = \sum_i ((b_i^*) - (a_i^*)) + (x_1^*) - (x_2^*) = \beta^*(T_2^*) - \beta^*(T_1^*), \]
which finishes the proof of Case 1.

**Case 2:** \( C^* \) is oriented *counterclockwise*. The proof is omitted, since it is quite similar to Case 1. \( \square \)

**Remark 6.4.** The first author conjectured the analogue of Theorem 6.1 for the rotor-routing process at an AIM workshop in July 2013. Together with Theorem 7.1 in the next section, Theorem 6.1 affirms our conjecture. Recently, Chan et. al. [CGM+14] have independently proved the compatibility of the rotor-routing torsor with planar duality using a different (more direct) method.

### 7. Comparison between the Bernardi and rotor-routing torsors

We show that given a ribbon graph \( G \) and a vertex \( v \) of \( G \), the Bernardi torsor \( \beta_v \) defined in this paper and the rotor-routing torsor defined in [HLM+08] and [CCG13] are equal when \( G \) is planar. In particular, the canonical torsor structures on \( S(G) \) defined by the Bernardi and rotor-routing processes are *the same* for planar ribbon graphs. We also give an example which shows that \( \beta_v \) is no longer independent of \( v \) when \( G \) is non-planar.
7.1. The planar case.

**Theorem 7.1.** Let $G$ be a planar ribbon graph. Then the Bernardi and rotor-routing processes define the same $\text{Pic}^0(G)$-torsor structure on $S(G)$.

**Proof.** Let $\beta$ be the Bernardi bijection associated to some initial data $(v, e)$, and let $T$ be a spanning tree of $G$. Since $\text{Pic}^0(G)$ is generated by the linear equivalence classes of divisors of the form $(x) - (y)$, with $x, y \in V(G)$, it suffices to prove that if $T' = ((x) - (y))_y T$ then $\beta(T') - \beta(T) \sim (x) - (y)$ for all $x, y \in V(G)$.

By Theorem 2.3 we may assume that the root vertex for the rotor-routing process is $y$. Let $T^*$ be the first spanning tree after $T$ which appears during the rotor-routing process $((x) - (y))_y$ from $T$ to $T'$, and let $x^*$ be the vertex to which the chip is sent when we reach $T^*$. By induction on the number of rotor-routing steps, it is enough to show that $\beta(T^*) - \beta(T) \sim (x) - (x^*)$.

**Case 1:** $T^*$ is obtained from $T$ in just one step of rotor-routing.

In this case, $T^*$ is obtained from $T$ by deleting an edge $e'$ incident to $x$ and adding an edge $e^*$ from $x$ to $x^*$. By Theorems 4.1 and 5.1, we may assume without loss of generality that the initial data for the Bernardi process are $(x^*, e^*)$. In this case, one easily checks that

$$\beta(T^*) - \beta(T) = (x) - (x^*).$$

**Case 2:** $T^*$ is obtained from $T$ in more than one step of rotor-routing. (See Figure 9 for an example.)

In this case (referring back to the notation from Section 2.5), if $S_0 = T$ then $S_1$ will consist of 2 connected components $A$ and $B$ such that $x \in A$, $A$ contains a unique directed cycle $C$, and $B$ contains no directed cycle. Since $G$ is planar, $C$ is reversible. Consider the rotor-routing process which takes $(\rho_{t_1}, x_{t_1}) := (\rho_1, x_1)$ to $(\rho_{t_2}, x_{t_2}) = (\tilde{\rho}_1, x_1)$. By [CCG13, Proposition 6], the set $L_C$ of vertices $v \notin C$ which are visited by this process depends only on $C$ and is contained in $A$. In particular, $T^*$ does not show up during this process (i.e., $S_i \neq T^*$ for $i = \ell_1, \ldots, \ell_2$).

In the next step of rotor-routing, the chip will be sent to $x_{t_2+1}$. If $x_{t_2+1} \notin A$ then $S_{t_2+1} = T^*$ is a spanning tree. Otherwise, $S_{t_2+1}$ will again contain a unique directed cycle $C'$ and the next several steps of rotor-routing will reverse this directed cycle. This process will continue a finite number of times until we reach some $\ell_t$ such that $x_{t\ell_t+1} = x^*$ and $S_{t\ell_t+1} = T^*$.

It follows that, during the entire process of rotor-routing from $T$ to $T^*$, $T^*$ can be obtained from $T$ by deleting an edge $e'$ incident to $x$ and the component $B$ and adding an edge $e^*$ from $x_{t_1}$ to $x^*$. As in Case 1, we may assume that the initial data for the Bernardi process are $(x^*, e^*)$, and one then checks that

$$\beta(T^*) - \beta(T) = (x) - (x^*)$$
7.2. The non-planar case. We now give an example which shows that for non-planar ribbon graphs, the torsors $\beta_v$ and $\rho_v$ can be different. In Figure 10, the sink vertex is $x^*$ and the chip begins at $x$. After one step of rotor-routing, the chip is sent to $x^*$ and the spanning tree $T$ is transformed into $T^*$.

If $\beta_{x^*} = \rho_{x^*}$, then we would have

$$\beta_{x^*}(T^*) - \beta_{x^*}(T) \sim (x) - (x^*).$$

However, setting the initial data for the Bernardi process as $(x^*, (x^*, x))$, we find that

$$(\beta_{x^*}(T^*) - \beta_{x^*}(T)) - ((x) - (x^*)) = (z) - (y)$$

which is not linearly equivalent to 0.

We conclude this paper with the following conjecture, one direction of which is Theorem 7.1.

**Conjecture 7.2.** The Bernardi and rotor-routing torsors $\beta_v$ and $\rho_v$ agree for all $v$ if and only if the ribbon graph $G$ is planar.
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