A GALE-BERLEKAMP PERMUTATION-SWITCHING PROBLEM IN HIGHER DIMENSIONS

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Abstract. Let an \( n \times n \) array \((a_{ij})\) of lights be given, each either on (when \( a_{ij} = 1 \)) or off (when \( a_{ij} = -1 \)). For each row and each column there is a switch so that if the switch is pulled \((x_i = -1\) for row \( i \) and \( y_j = -1\) for column \( j \)) all of the lights in that line are switched: on to off or off to on. The unbalancing lights problem (Gale-Berlekamp switching game) consists in maximizing the difference between the lights on and off. We obtain the exact parameters for a generalization of the unbalancing lights problem in higher dimensions.

1. Introduction

We begin by presenting a combinatorial game, sometimes called Gale-Berlekamp switching game or unbalancing lights problem (for a presentation we refer, for instance to the classical book of Alon and Spencer [1]). Let an \( n \times n \) array \((a_{ij})\) of lights be given, each either on (when \( a_{ij} = 1 \)) or off (when \( a_{ij} = -1 \)). Let us also suppose that for each row and each column there is a switch so that if the switch is pulled \((x_i = -1\) for row \( i \) and \( y_j = -1\) for column \( j \)) all of the lights in that line are switched: on to off or off to on. The problem consists in maximizing the difference between the lights on and off.

A probabilistic approach (using the Central Limit Theorem) to this problem (see [1]) provides the following asymptotic estimate:

**Theorem 1.1** ([1], Theorem 2.5.1). Let \( a_{ij} = \pm 1 \) for \( 1 \leq i, j \leq n \). Then there exist \( x_i, y_j = \pm 1 \), \( 1 \leq i, j \leq n \), such that

\[
\sum_{i,j=1}^{n} a_{ij} x_i y_j \geq \left( \frac{\sqrt{2/\pi}}{n^{3/2}} + o(1) \right) n^{3/2},
\]

and the exponent 3/2 is optimal. In other words, for any initial configuration \((a_{ij})\) it is possible to perform switches so that the number of lights on minus the number of lights off is at least \( \left( \frac{\sqrt{2/\pi}}{n^{3/2}} + o(1) \right) n^{3/2} \).

In higher dimensions (cf. mathoverflow.net/questions/59463/unbalancing-lights-in-higher-dimensions, by A. Montanaro) the unbalancing lights problem is stated as follows:

Let an \( n \times \cdots \times n \) array \((a_{i_1 \cdots i_m})\) of lights be given each either on (when \( a_{i_1 \cdots i_m} = 1 \)) or off (when \( a_{i_1 \cdots i_m} = -1 \)). Let us also suppose that for each \( i_j \) there is a switch so that if the switch is pulled \((x_{i_j} = -1)\) all of the lights in that line are “switched”: on to off or off to on. The goal is to maximize the difference between the lights on and off.

It is a well known consequence of the Bohnenblust–Hille inequality [8] that there exist \( x_{i_j}^{(k)} = \pm 1 \), \( 1 \leq j \leq n \) and \( k = 1, \ldots, m \), and a constant \( C \geq 1 \), such that

\[
\sum_{i_1, \ldots, i_m=1}^{n} a_{i_1 \cdots i_m} x_{i_1}^{(1)} \cdots x_{i_m}^{(m)} \geq \frac{1}{C^m n^{\frac{m+1}{2}}}
\]

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and that the exponent $\frac{m+1}{2}$ is sharp. A step further suggested by A. Montanaro is to investigate if the term $C^m$ can be improved. Using recent estimates of the Bohnenblust–Hille inequality (see [6]) it is plain that there exist $x_{ij} = \pm 1$, $1 \leq j \leq n$ and a constant $C > 0$ such that

$$\sum_{i_1, \ldots, i_m=1}^n a_{i_1 \ldots i_m} x_{i_1} \cdots x_{i_m}^{(m)} \geq \frac{1}{1.3 m^{0.365} n^{\frac{m+1}{2}}},$$

and the exponent $\frac{m+1}{2}$ is sharp. It is still an open problem if the term $1.3 m^{0.365}$ (here and henceforth $1.3 m^{0.365}$ is just a simplification of $\kappa m^{2 - \frac{2 + \log 2 - \gamma}{2}}$, where $\gamma$ is the Euler–Mascheroni constant) can be improved to a universal constant.

Some variants of the unbalancing lights problem have been already investigated (see [9]). In this paper we consider a more general problem:

**Problem 1.2.** Let $(a_{i_1 \ldots i_m})$ be an $n \times \cdots \times n$ array of (real or complex) scalars such that $|a_{i_1 \ldots i_m}| = 1$. For $p \in [1, \infty]$, maximize

$$g(p) = \left\{ \sum_{i_1, \ldots, i_m=1}^n a_{i_1 \ldots i_m} x_{i_1}^{(1)} \cdots x_{i_m}^{(m)} : \left\| (x^{(j)})_{i=1}^n \right\|_p = 1 \text{ for all } j = 1, \ldots, m \right\}.$$

When $p = \infty$ with real norm-one scalars is precisely the classical unbalancing lights problem in higher dimensions ([14]).

The main result of this paper, in particular, gives sharp exponents for the unbalancing lights problem for $p \geq 2$:

- If $p \in [2, \infty]$, then

$$g(p) \geq \frac{1}{1.3 m^{0.365} n^{\frac{mp+p-2m}{2p}}}$$

and the exponents $\frac{mp+p-2m}{2p}$ are sharp.

2. Results

A first partial solution to Problem 1.2 is a straightforward consequence of the Hardy–Littlewood inequalities. The Hardy–Littlewood inequalities [10] [12] [18] for $m$–linear forms assert that for any integer $m \geq 2$ there exist constants $C^K_{m,p}, D^K_{m,p} \geq 1$ such that

$$\left| \left( \sum_{j_1, \ldots, j_m=1}^n |T(e_{j_1}, \ldots, e_{j_m})|^{\frac{2m}{p}} \right)^{\frac{p-m}{p}} \right|^{\frac{p}{p-m}} \leq D^K_{m,p} \|T\| \text{ for } m < p \leq 2m,$$

$$\left| \left( \sum_{j_1, \ldots, j_m=1}^n |T(e_{j_1}, \ldots, e_{j_m})|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} \right| \leq C^K_{m,p} \|T\| \text{ for } p \geq 2m,$$

for all $m$–linear forms $T : \mathbb{R}^m \times \cdots \times \mathbb{R}^m \to \mathbb{K}$, all positive integers $n$.

The optimal constants $C^K_{m,p}, D^K_{m,p}$ are unknown; even the asymptotic behaviour of these constants is unknown. Up to now, the best estimates for $C^K_{m,p}$ can be found in [3] [4]:

$$C^K_{m,p} \leq \left( \sqrt{2} \right)^{\frac{2m(m-1)}{p}} (1.3 m^{0.365})^{\frac{p-2m}{p}}.$$

For $p > 2m(m-1)^2$ we also know from [3] that $C^K_{m,p} \leq 1.3 m^{0.365}$; it is not known if, in general, the same estimate is valid for the other choices of $p$. The notation of $C^K_{m,p}, D^K_{m,p}$ as the optimal constants of the Hardy–Littlewood inequalities will be kept all along the paper.

By (2.1) we easily have the following:
Proposition 2.1. Let \( m, n \) be positive integers and \( p \in (m, \infty] \). There are positive constants \( C_{m,p}^K, D_{m,p}^K \) such that

\[
g(p) \geq \frac{1}{D_{m,p}^K} n^{\frac{m(p-m)}{p}} \text{ for } m < p \leq 2m, \\
g(p) \geq \frac{1}{C_{m,p}^K} n^{\frac{mp+p-2m}{2p}} \text{ for } p \geq 2m.
\]

Among other results, the main result of the present paper shows that the above estimates are far from being precise. We will show that:

- The exponent \( \frac{m(p-m)}{p} \) can be replaced by \( \frac{mp+p-2m}{2p} \) in the case \( m < p \leq 2m \);
- The constants \( \frac{1}{C_{m,p}^K} \) and \( \frac{1}{D_{m,p}^K} \) can be replaced by \( 1.3m^{0.365} \);
- The inequality is also valid for \( 2 \leq p \leq m \) with the same constants and exponents \( \frac{mp+p-2m}{2p} \);
- The above exponents \( \frac{mp+p-2m}{2p} \) are optimal.

Recently (see [2]), it has been shown that the constants \( D_{m,p}^K \) have essentially a very low growth but since we now improve the associated exponents, the estimates of \( D_{m,p}^K \) are not useful here.

To achieve our goals, we begin by revisiting the Kahane–Salem–Zygmund inequality. It is a probabilistic result that furnishes unimodular multilinear forms with “small” norms. This result is fundamental to the proof of the optimality of the exponents of the Hardy–Littlewood inequality. For \( p \geq 1 \), the Kahane–Salem–Zygmund asserts that there exists a \( m \)-linear form \( A : \ell^m_p \times \cdots \times \ell^m_p \to \mathbb{K} \) of the form

\[
A \left( x^{(1)}, \ldots, x^{(m)} \right) = \sum_{i_1, \ldots, i_m=1}^n \delta_{i_1 \cdots i_m} x_{i_1}^{(1)} \cdots x_{i_m}^{(m)},
\]

with \( \delta_{i_1 \cdots i_m} \in \{-1, 1\} \), such that

\[
\|A\| \leq C_m n^{\frac{1}{2} + m \left( \frac{1}{2} - \frac{1}{p} \right)}.
\]

However, for \( 1 \leq p \leq 2 \) a better estimate can essentially be found in [5]. So, we have the following:

Theorem 2.2 (Kahane–Salem–Zygmund inequality). Let \( n, m \) be positive integers and \( p \geq 1 \). Then there exists a \( m \)-linear form \( A : \ell^m_p \times \cdots \times \ell^m_p \to \mathbb{K} \) of the form

\[
A \left( x^{(1)}, \ldots, x^{(m)} \right) = \sum_{i_1, \ldots, i_m=1}^n \delta_{i_1 \cdots i_m} x_{i_1}^{(1)} \cdots x_{i_m}^{(m)},
\]

with \( \delta_{i_1 \cdots i_m} \in \{-1, 1\} \), such that

\[
\|A\| \leq C_m n^{\max \left\{ \frac{1}{2} + m \left( \frac{1}{2} - \frac{1}{p} \right), 1 - \frac{1}{p} \right\}}.
\]

We shall show that (2.1) can be significantly improved when dealing with unimodular forms. It is easy to see that our main result is a consequence of the following theorem (see Figure 1).

Before presenting the next result, let us introduce some required definitions for their proof. Let \( B_{\mathbb{K}^*} \) be the closed unit ball of the topological dual of \( E \). For \( s \geq 1 \) we represent by \( \ell^w_s(E) \) the linear space of the sequences \( (x_j)_{j=1}^\infty \in E \) such that \( (\varphi(x_j))_{j=1}^\infty \in \ell_s \) for every continuous linear functional \( \varphi : E \to \mathbb{K} \). For \( (x_j)_{j=1}^\infty \in \ell^w_s(E) \), the expression \( \sup_{\varphi \in B_{\mathbb{K}^*}} \left( \sum_{j=1}^\infty |\varphi(x_j)|^s \right)^{\frac{1}{s}} \)
defines a norm on $\ell^w_s(E)$. For $p, q \in [1, +\infty)$, a multilinear operator $T : E_1 \times \cdots \times E_m \to \mathbb{K}$ is multiple $(q; p)$-summing if there exist a constant $C > 0$ such that

$$\left( \sum_{j_1, \ldots, j_m = 1}^{\infty} |T(x^{(1)}_{j_1}, \ldots, x^{(m)}_{j_m})|^q \right)^{\frac{1}{q}} \leq C \left( \sup_{\varphi \in B_{E^*}} \left( \sum_{j=1}^{\infty} |\varphi(x^{(k)}_{j})|^p \right) \right)^{\frac{1}{p}}$$

for all $(x^{(k)}_{j})_{j=1}^{\infty} \in \ell^w_p(E_k)$. For recent results of multiple summing operators we refer to [17].

**Theorem 2.3.** If $m, n$ are positive integers and $p \in \left( \frac{2m}{m+1}, +\infty \right)$, then

$$\left( \sum_{j_1, \ldots, j_m = 1}^{n} |T(e_{j_1}, \ldots, e_{j_m})|^\frac{2m}{m+1} \right)^{\frac{m+1}{2m}} \leq 1.3m^{0.365} \|T\| \left( \sup_{\varphi \in B_{E^*}} \sum_{j=1}^{n} |\varphi_j| \right)^{m}$$

for all $m$-linear forms $T : \ell^n_p \times \cdots \times \ell^n_p \to \mathbb{K}$. Moreover, the exponent is sharp for $p \geq 2$. For $1 < p \leq \frac{2m}{m+1}$ the optimal exponent is not smaller than $\frac{mp}{p-1}$ and for $\frac{2m}{m+1} < p \leq 2$ the optimal exponent belongs to $\left[ \frac{mp}{p-1}, \frac{2mp}{mp+p-2m} \right]$.

**Proof.** Using the isometric characterization of the spaces of weak $1$-summable sequences on $c_0$ (see [11]) we know that every continuous $m$-linear form is multiple $\left( \frac{2m}{m+1}; 1 \right)$-summing with constant dominated by $1.3m^{0.365}$.

Thus

$$\left( \sum_{j_1, \ldots, j_m = 1}^{n} |T(e_{j_1}, \ldots, e_{j_m})|^\frac{2m}{m+1} \right)^{\frac{m+1}{2m}} \leq 1.3m^{0.365} \|T\| \left( \sup_{\varphi \in B_{E^*}} \sum_{j=1}^{n} |\varphi_j| \right)^{m}$$

for all $m$-linear forms

$$T : \ell^n_p \times \cdots \times \ell^n_p \to \mathbb{K}.$$ 

Hence

$$\left( n^{m} \right)^{\frac{m+1}{2m}} \leq 1.3m^{0.365} \|T\| \left( \frac{n}{n^{1/p}} \right)^{m}$$

and finally

$$\|T\| \geq \frac{1}{1.3m^{0.365} n^{\frac{mp+p-2m}{2p}}}$$

and this means that

$$\left( \sum_{j_1, \ldots, j_m = 1}^{n} |T(e_{j_1}, \ldots, e_{j_m})|^\frac{2mp}{mp+p-2m} \right)^{\frac{mp+p-2m}{2mp}} \leq 1.3m^{0.365} \|T\|.$$

Let us prove the optimality of the exponents for $p \geq 2$. Suppose that the theorem is valid for an exponent $r$, i.e.,

$$\left( \sum_{j_1, \ldots, j_m = 1}^{n} |T(e_{j_1}, \ldots, e_{j_m})|^r \right)^{\frac{1}{r}} \leq 1.3m^{0.365} \|T\|.$$ 

Since $p \geq 2$, from the Kahane–Salem–Zygmund inequality (Theorem 2.2) we have

$$n^{m} \leq 1.3m^{0.365} C_m n^{\frac{1}{2} + m \left( \frac{1}{2} - \frac{1}{p} \right)} = C_m 1.3m^{0.365} n^{\frac{mp+p-2m}{2p}},$$

and thus, making $n \to \infty$, we obtain $r \geq \frac{2mp}{mp+p-2m}$. 
For $1 < p \leq 2$, if the inequality holds for a certain exponent $r$, from the Kahane–Salem–Zygmund inequality (Theorem 2.2) we have

$$n^\frac{m}{r} \leq C n^{1 - \frac{1}{p}} = C n^{\frac{p}{p-1}}$$

and thus, making $n \to \infty$, we obtain $r \geq \frac{mp}{p-1}$.

The determination of the unknown exponents rely in an open result on the interpolation of certain multilinear forms, which seems to be open for a long time: every continuous $m$-linear form from $\ell_1 \times \cdots \times \ell_1$ to $\mathbb{K}$ is multiple $(1,1)$-summing and every continuous $m$-linear operators from $\ell_2 \times \cdots \ell_2$ to $\mathbb{K}$ is multiple $(2m-1,1)$-summing. What about intermediate results for $\ell_p$. The natural result would be, for $1 \leq p \leq 2$ that every continuous $m$-linear operators from $\ell_p \times \cdots \ell_p$ to $\mathbb{K}$ is multiple $(\frac{mp}{m+p-2m},1)$-summing. Even in the linear case, similar vector-valued problems remain open (see [7]).

We conjecture the following optimal result:

**Conjecture 2.4.** If $m, n$ are positive integers and $p \in [1, \infty]$, then there is a constant $K_m$ such that

$$\left( \sum_{j_1, \ldots, j_m=1}^n |T(e_{j_1}, \ldots, e_{j_m})|^{\frac{mp}{mp+2m}} \right)^{\frac{mp}{mp+2m}} \leq K_m \|T\| \quad \text{for } 1 \leq p \leq 2,$$

$$\left( \sum_{j_1, \ldots, j_m=1}^n |T(e_{j_1}, \ldots, e_{j_m})|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} \leq 1.3m^{0.365} \|T\| \quad \text{for } p \geq 2,$$

for all unimodular $m$-linear forms $T : \ell_p^m \times \cdots \times \ell_p^m \to \mathbb{K}$ and the exponents are sharp.
3. Revisiting the classical unbalancing lights problem

3.1. The classical unbalancing lights problem. In this section we prove a non asymptotic version of \((\ref{1.1})\) showing the only situations in which the minimum estimate is achieved.

**Theorem 3.1.** Let \(a_{ij} = \pm 1\) for \(1 \leq i, j \leq n\). Then there exist \(x_i, y_j = \pm 1, 1 \leq i, j \leq n\), such that
\[
\sum_{i,j=1}^{n} a_{ij} x_i y_j \geq 2^{-1/2} n^{3/2},
\]
and the equality happens if, and only if, \(n = 2\) and
\[
(a_{ij}) = \pm \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{or} \quad \pm \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.
\]

In other words, for any initial configuration \((a_{ij})\) it is possible to perform switches so that the number of lights on minus the number of lights off is at least \(2^{-1/2} n^{3/2}\) and the equality happens if and only if \((a_{ij})\) is as in \((\ref{3.1})\).

**Proof.** Littlewood’s 4/3-inequality asserts that
\[
\left( \sum_{j,k=1}^{n} |T(e_j, e_k)|^3 \right)^{\frac{1}{3}} \leq \sqrt{2} \sup_{\|x\|, \|y\| \leq 1} |T(x, y)|,
\]
for all continuous bilinear forms \(T : \ell_\infty^n \times \ell_\infty^n \to \mathbb{R}\) and all positive integers \(n\). It is not difficult to prove that the supremum in the right-hand-side of \((\ref{3.2})\) is achieved in the extreme points of the closed unit ball of \(\ell_\infty^n\). Since these extreme point are precisely those with the entries 1 or \(-1\), we conclude that there exist \(x_i, y_j = \pm 1, 1 \leq i, j \leq n\), such that
\[
\sum_{i,j=1}^{n} a_{ij} x_i y_j \geq 2^{-1/2} n^{3/2}.
\]

It remains to prove that the equality happens if and only if \((a_{ij})\) is as in \((\ref{3.1})\). To prove this we recall the following result of [16]:

- A bilinear form \(T\) is an (norm-one) extreme of Littlewood’s 4/3 inequality if and only if \(T\) is written as
  \[
  T(x, y) = \pm 2^{-1/2} (x_{i_1} y_{i_2} + x_{i_2} y_{i_3} + x_{i_3} y_{i_4} - x_{i_4} y_{i_1}),
  \]
  \[
  T(x, y) = \pm 2^{-1/2} (x_{i_1} y_{i_2} + x_{i_2} y_{i_3} - x_{i_4} y_{i_2} + x_{i_4} y_{i_3}),
  \]
  \[
  T(x, y) = \pm 2^{-1/2} (x_{i_1} y_{i_2} - x_{i_1} y_{i_3} + x_{i_4} y_{i_2} + x_{i_4} y_{i_3}),
  \]
  \[
  T(x, y) = \pm 2^{-1/2} (-x_{i_1} y_{i_2} + x_{i_1} y_{i_3} + x_{i_4} y_{i_2} + x_{i_4} y_{i_3})
  \]
for \(i_1 \neq i_4\) and \(i_2 \neq i_3\).

From the above theorem we conclude that when we deal with bilinear forms with coefficients 1 or \(-1\), the equality in \((\ref{3.2})\) happens if and only if \(n = 2\) and
\[
T(x, y) = \pm (x_1 y_1 + x_1 y_2 + x_2 y_1 - x_2 y_2),
\]
\[
T(x, y) = \pm (x_1 y_1 + x_1 y_2 - x_2 y_1 + x_2 y_2),
\]
\[
T(x, y) = \pm (x_1 y_1 - x_1 y_2 + x_2 y_1 + x_2 y_2),
\]
\[
T(x, y) = \pm (-x_1 y_1 + x_1 y_2 + x_2 y_1 + x_2 y_2)
\]
and the proof is done. \(\square\)
3.2. The classical unbalancing lights problem in higher dimensions. The next result provides an asymptotic variant of (1.2) in the lines of (1.1):

**Theorem 3.2.** Let $m$ be a positive integer and $a_{i_1 \cdots i_m} = \pm 1$ for all $i_1, \ldots, i_m$. Then, for all $k = 1, \ldots, m$, there exist $x_i^{(k)} = \pm 1$, $1 \leq j \leq n$, such that

\[
\sum_{i_1, \ldots, i_m = 1}^n a_{i_1 \cdots i_m} x_i^{(1)} \cdots x_i^{(m)} \geq \left(2^{1-\psi(m+1)}-\gamma \left(\prod_{k=2}^m \frac{\Gamma \left(\frac{3k-2}{2}\right)}{\Gamma \left(\frac{3k}{2}\right)}\right) + o(1)\right) n^{\frac{m+1}{2}},
\]

where $\psi$ is the digamma function and $\gamma$ is the Euler–Mascheroni constant.

We begin by recalling some useful technical results:

**Lemma 3.3** (Minkowski). If $0 < p < q < \infty$, then

\[
\left(\sum_{j=1}^n \left(\sum_{i=1}^n |a_{ij}|^p\right)^{\frac{q}{p}}\right)^{\frac{1}{q}} \leq \left(\sum_{i=1}^n \left(\sum_{j=1}^n |a_{ij}|^q\right)^{\frac{1}{q}}\right)^{\frac{1}{p}}
\]

for all positive integers $n$ and all scalars $a_{ij}$.

**Lemma 3.4** (Haagerup, see [15]). Let $1 \leq p \leq 2$. For all sequence of real scalars $(a_i)$ we have

\[
\left(\sum_{i=1}^n |a_i|^2\right)^{1/2} \leq \left(\frac{2^{\frac{p+1}{2}} \Gamma \left(\frac{p+1}{2}\right)}{\Gamma \left(\frac{p}{2}\right)}\right)^{-1} \left(\int_0^1 \left|\sum_{k=1}^n r_i(t)a_i\right|^p dt\right)^{\frac{1}{p}}.
\]

The next lemma is a well-known consequence of the Krein–Milman Theorem:

**Lemma 3.5.** For all $m$-linear forms $A : \ell_\infty^n \times \cdots \times \ell_\infty^n \to \mathbb{R}$ we have

\[
\|A\| = \max \left|A \left(x^{(1)}, \ldots, x^{(m)}\right)\right|,
\]

where $x^{(j)}$ has all entries equal to $1$ or $-1$, for all $j = 1, \ldots, m$.

Now we are able to begin the proof. Let

\[
f(p) := \left(\frac{2^{\frac{p+1}{2}} \Gamma \left(\frac{p+1}{2}\right)}{\Gamma \left(\frac{p}{2}\right)}\right)^{-1}.
\]

Consider the $m$-linear form

\[
A \left(x^{(1)}, \ldots, x^{(m)}\right) = \sum_{i,j=1}^n a_{i_1 \cdots i_m} x_i^{(1)} \cdots x_i^{(m)}.
\]

For bilinear forms, using Lemma 3.3 we have

\[
\sum_{j=1}^n \left(\sum_{i=1}^n |a_{ij}|^2\right)^{1/2} = \sum_{j=1}^n \left(\sum_{i=1}^n |A(e_i, e_j)|^2\right)^{1/2}
\]

\[
\leq f(1) + o(1) \sum_{j=1}^n \left[\sum_{i=1}^n r_i(t)A(e_i, e_j)\right] dt
\]

\[
\leq f(1) + o(1) \sup_{t \in [0,1]} \left[\sum_{j=1}^n \left|A \left(\sum_{i=1}^n r_i(t)e_i, e_j\right)\right|\right]
\]

\[
\leq f(1) + o(1) \|A\|.
\]
and, by symmetry and by Lemma 3.3 we have

\[(3.5) \quad \left( \sum_{j=1}^{n} \left( \sum_{i=1}^{n} |a_{ij}|^2 \right)^{1/2} \right)^{2} \leq \left( 2^{-1/2} \frac{\Gamma(1)}{\Gamma(3/2)} \right)^{-1} + o(1) \|A\|.

By the Hölder inequality for mixed sums combined with (3.4) and (3.5), we have

\[
\left( \sum_{i,j=1}^{n} |a_{ij}|^{4/3} \right)^{3/4} \leq (f(1) + o(1)) \|A\|.
\]

For trilinear forms we have

\[(3.6) \quad \left( \sum_{k, i=1}^{n} \left( \sum_{j=1}^{n} |a_{ijk}|^2 \right)^{1/2} \right)^{2/3} \leq (f(1) f(4/3) + o(1)) \|A\|.
\]

Using symmetry and Lemma 3.3 we have

\[(3.7) \quad \left( \sum_{k, i=1}^{n} \left( \sum_{j=1}^{n} |a_{ijk}|^{4/3} \right)^{3/4} \right)^{1/2} \leq (f(1) f(4/3) + o(1)) \|A\|
\]

and

\[(3.8) \quad \left( \sum_{k=1}^{n} \left( \sum_{i=1}^{n} \left( \sum_{j=1}^{n} |a_{ijk}|^2 \right)^{1/2} \right)^{1/2} \right)^{2/3} \leq (f(1) f(4/3) + o(1)) \|A\|.
\]

By the Hölder inequality for mixed sums and (3.6), (3.7), (3.8) we get

\[
\left( \sum_{i,j,k=1}^{n} |a_{ijk}|^{3/2} \right)^{2/3} \leq (f(4/3) + o(1)) (f(1) + o(1)) \|A\|
\]

Following this vein, for the general case we have

\[
\left( \sum_{i_1, \ldots, i_m=1}^{n} |a_{i_1 \ldots i_m}|^{2m/(m+1)} \right)^{m+1/2m} \leq \prod_{k=2}^{m} f\left( \frac{2(k-1)}{k} \right) + o(1) \|A\|
\]

\[
= \left( \prod_{k=2}^{m} f\left( \frac{2(k-1)}{k} \right) \right) + o(1) \|A\|.
\]
Theorem 4.1. Precisely, we prove the following result:

\[
\sum_{i_1, \ldots, i_m=1}^n a_{i_1 \cdots i_m} x_{i_1}^{(1)} \cdots x_{i_m}^{(m)} \geq \left( \prod_{k=2}^m \left( \frac{2(k-1)}{k} \right)^{-1} + o(1) \right) n^{m+1} \]

\[
= \left( \prod_{k=2}^m \left( 2^{1/k} \Gamma(\frac{3k-2}{2k}) \right) + o(1) \right) n^{m+1} \]

\[
= \left( 2^{1-\psi(m+1)-\gamma} \prod_{k=2}^m \Gamma\left( \frac{3k-2}{2k} \right) + o(1) \right) n^{m+1},
\]

where \( \psi \) is the digamma function and \( \gamma \) is the Euler-Mascheroni constant. The optimality of the exponent \( m+1 \) can be proved, as usual, using the Kahane–Salem–Zygmund inequality.

Observing that Lemma 3.4 holds for all sequence of real scalars \( (a_i) \), the argument of the previous section can be adapted to prove the following version, with asymptotic constants, of the Bohnenblust–Hille inequality:

**Theorem 3.6.** For all continuous \( m \)-linear forms \( T : c_0 \times \cdots \times c_0 \to \mathbb{R} \) we have

\[
(3.9) \quad \left( \sum_{i_1, \ldots, i_m=1}^n |T(e_{i_1}, \ldots, e_{i_m})|^{\frac{2m}{m+1}} \right)^{\frac{m+1}{2m}} \leq \left( \frac{1}{2^{1-\psi(m+1)-\gamma}} \prod_{k=2}^m \Gamma\left( \frac{3k-2}{2k} \right) + o(1) \right) \|T\|
\]

| Value of \( \left( \frac{1}{2^{1-\psi(m+1)-\gamma}} \prod_{k=2}^m \Gamma\left( \frac{3k-2}{2k} \right) \right) \) |
|------------------|
| \( m = 2 \) | \( \sqrt{\pi/2} \approx 1.2533 \) |
| \( m = 5 \) | 1.9895 |
| \( m = 10 \) | 3.0555 |
| \( m = 100 \) | 15.2457 |
| \( m = 1000 \) | 81.1974 |

From (3.9) and repeating the proof of Theorem 2.3 we have:

**Theorem 3.7.** Let \( p \in [2, \infty] \). For all unimodular \( m \)-linear forms \( T : \ell_p^n \times \cdots \times \ell_p^n \to \mathbb{R} \) we have

\[
\left( \sum_{i_1, \ldots, i_m=1}^n |T(e_{i_1}, \ldots, e_{i_m})|^{\frac{2mp}{mp+p-2m}} \right)^{\frac{mp+p-2m}{2mp}} \leq \left( \frac{1}{2^{1-\psi(m+1)-\gamma}} \prod_{k=2}^m \Gamma\left( \frac{3k-2}{2k} \right) + o(1) \right) \|T\|.
\]

4. **Blow up rate for the Hardy–Littlewood inequalities for unimodular forms**

In this section we provide the blow up rate for the constants in Theorem 2.3 as \( n \) grows when the \( \ell_{mp+p-2m} \)-norm in the left-hand-side is replaced by an \( \ell_r \)-norm with \( 0 < r < \infty \). More precisely, we prove the following result:

**Theorem 4.1.** If \( m \) is a positive integers and \( (r, p) \in (0, \infty) \times \left( \frac{2m}{m+1}, \infty \right) \) then

\[
\left( \sum_{j_1, \ldots, j_m=1}^n |T(e_{j_1}, \ldots, e_{j_m})|^r \right)^{\frac{1}{r}} \leq 1.3m^{0.365} n^{\max\left\{ \frac{2mp+2mp-mp-rp}{4pr}, 0 \right\}} \|T\|
\]

for all unimodular \( m \)-linear forms \( T : \ell_p^n \times \cdots \times \ell_p^n \to \mathbb{K} \) and all positive integers \( n \). Moreover, for \( (r, p) \) belonging to \( \left( \left( 0, \frac{2mp}{mp+p-2m} \right) \times [2, \infty] \right) \cup \left( \left[ \frac{2mp}{mp+p-2m}, \infty \right) \times \left( \frac{2m}{m+1}, \infty \right) \right) \) the power
In fact, suppose that the inequalities are valid for an exponent $G$. ARAÚJO AND D. PELLEGRINO
Therefore, if $(r, p)$ belongs to $\left(0, \frac{2mp}{mp + p - 2m}\right) \times \left(\frac{2m}{m + 1}, 2\right)$ the optimal exponent of $n$ belongs to the interval $\left[\max\left\{\frac{mp + p - 2m}{2pr}, 0\right\}, \frac{2mr + 2mp - mp + pr}{2pr}\right]$. 

Proof. For $p > \frac{2n}{m + 1}$ we know from Theorem 2.3 that

$$\left( \sum_{j_1, \ldots, j_m = 1}^n |T(e_{j_1}, \ldots, e_{j_m})|^{2mr} \right)^{\frac{1}{r}} \leq 1.3m^{0.365} \|T\|.$$  

Therefore, if $(r, p) \in \left(0, \frac{2mp}{mp + p - 2m}\right) \times \left(\frac{2m}{m + 1}, \infty\right)$, from Hölder’s inequality and (4.1) we have

$$\left( \sum_{j_1, \ldots, j_m = 1}^n |T(e_{j_1}, \ldots, e_{j_m})|^r \right)^{\frac{1}{r}} \leq 1.3m^{0.365} \|T\|.$$

Let us prove the optimality of the exponents for $(r, p) \in \left[0, \frac{2mp}{mp + p - 2m}\right] \times [2, \infty]$. Suppose that the theorem is valid for an exponent $s$, i.e.,

$$\left( \sum_{j_1, \ldots, j_m = 1}^n |T(e_{j_1}, \ldots, e_{j_m})|^s \right)^{\frac{1}{s}} \leq 1.3m^{0.365} \|T\|.$$

Since $p \geq 2$, from the Kahane–Salem–Zygmund inequality (Theorem 2.2) we have

$$n^{\frac{m}{m + 1}} \leq 1.3m^{0.365} n^{s} C_{m,n} \frac{1}{s} \left(\frac{1}{s} - \frac{1}{r}\right) = C_{m} 1.3m^{0.365} n^{s + \frac{mp + p - 2m}{2pr}}$$

and thus, making $n \to \infty$, we obtain $s \geq \frac{2mr + 2mp - mp + pr}{2pr}$.

If $(r, p) \in \left[\frac{2mp}{mp + p - 2m}, \infty\right) \times \left(\frac{2m}{m + 1}, \infty\right)$ we have $\frac{2mr + 2mp - mp + pr}{2pr} \leq 0$ and

$$\left( \sum_{j_1, \ldots, j_m = 1}^n |T(e_{j_1}, \ldots, e_{j_m})|^s \right)^{\frac{1}{s}} \leq \left( \sum_{j_1, \ldots, j_m = 1}^n |T(e_{j_1}, \ldots, e_{j_m})|^{\frac{2mr + 2mp - mp + pr}{2pr}} \right)^{\frac{2mr + 2mp - mp + pr}{2pr}} \leq 1.3m^{0.365} \|T\|$$

In this case the optimality of the exponent $\max\left\{\frac{2mr + 2mp - mp + pr}{2pr}, 0\right\}$ is immediate, since no negative exponent of $n$ is possible.

If $(r, p) \in \left(0, \frac{2mp}{mp + p - 2m}\right) \times \left(\frac{2m}{m + 1}, 2\right)$, we just have an estimate for the optimal exponent of $n$. In fact, suppose that the inequalities are valid for an exponent $s \geq 0$, i.e.,

$$\left( \sum_{j_1, \ldots, j_m = 1}^n |T(e_{j_1}, \ldots, e_{j_m})|^s \right)^{\frac{1}{s}} \leq 1.3m^{0.365} n^{s} \|T\|.$$
Since $1 \leq \frac{2m}{m+1} < p \leq 2$, from the Kahane–Salem–Zygmund inequality (Theorem 2.2) we have
\[
n^\frac{m}{r} \leq 1.3m^{0.365}n^sC_m n^{1 - \frac{1}{p}} = 1.3m^{0.365}C_m n^{s + \frac{p - 1}{p}}
\]
and thus, making $n \to \infty$, we obtain $s \geq \frac{mp + r - pr}{pr}$.

If Conjecture 2.3 is correct, using the same ideas of the proof of the previous theorem it is possible to improve it to the following optimal result:

**Conjecture 4.2.** If $m$ is a positive integers and $(r, p) \in (0, \infty) \times (1, \infty]$ then there is a constant $K_m$ such that
\[
\left( \sum_{j_1, \ldots, j_m = 1}^n |T(e_{j_1}, \ldots, e_{j_m})|^r \right)^{\frac{1}{r}} \leq K_m n^{\max\left\{ \frac{mp + r - pr}{pr}, 0 \right\}} \|T\| \text{ for } 1 < p \leq 2, \\
\left( \sum_{j_1, \ldots, j_m = 1}^n |T(e_{j_1}, \ldots, e_{j_m})|^r \right)^{\frac{1}{r}} \leq 1.3m^{0.365}n^{\max\left\{ \frac{2mr + 2mp - mp - pr}{2pr}, 0 \right\}} \|T\| \text{ for } p \geq 2,
\]
for all unimodular $m$-linear forms $T : \ell^m_p \times \cdots \times \ell^m_n \to \mathbb{K}$ and all positive integers $n$. Moreover, the exponents $\max\left\{ \frac{2mr + 2mp - mp - pr}{2pr}, 0 \right\}$ and $\max\left\{ \frac{mp + r - pr}{pr}, 0 \right\}$ are sharp.

In fact, the novelty is the case $1 < p \leq 2$. Supposing that Conjecture 2.3 is true, if $(r, p) \in \left( 0, \frac{mp}{p - r} \right) \times (1, 2)$, from Hölder’s inequality we have
\[
\left( \sum_{j_1, \ldots, j_m = 1}^n |T(e_{j_1}, \ldots, e_{j_m})|^r \right)^{\frac{1}{r}} \leq K_m n^\frac{mp + r - pr}{pr} \|T\|.
\]
On the other hand, if the above inequalities are valid for an exponent $s$ instead of $\frac{mp + r - pr}{pr}$, since $1 < p \leq 2$, from the Kahane–Salem–Zygmund inequality (Theorem 2.2) we have
\[
n^\frac{m}{r} \leq C n^s n^{1 - \frac{1}{p}} = C n^{s + \frac{p - 1}{p}}
\]
and
\[
s \geq \frac{mp + r - pr}{pr}.
\]
If $(r, p) \in \left[ \frac{mp}{p - r}, \infty \right) \times (1, 2]$ we have $\frac{mp + r - pr}{pr} \leq 0$ and, in this case, the optimality of the exponent $\max\left\{ \frac{mp + r - pr}{pr}, 0 \right\}$ is immediate, since no negative exponent of $n$ is possible.

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