Intermittency in turbulence

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Abstract

We derive from the Navier-Stokes equation an exact equation satisfied by the dissipation rate correlation function,

\[ \langle \epsilon(\vec{x} + \vec{r}, t + \tau)\epsilon(\vec{x}, t) \rangle, \]

which we study in the equal time limit. We exploit its mathematical similarity to the corresponding equation derived from the 1-dimensional stochastic Burgers equation to show that the main intermittency exponents are

\[ \mu_1 = 2 - \zeta_6 \]

and

\[ \mu_2 = 2 \tilde{z}_4 - \zeta_4, \]

where the \( \zeta \)'s are exponents of velocity structure functions and \( \tilde{z}_4 \) is a dynamical exponent characterizing the 4th order structure function. We discuss the role of sweeping and Galilean invariance in determining the intermittency exponents.

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1 Introduction

The correlations of the local energy dissipation rate per unit mass, \( \epsilon(\vec{x}, t) \), whose behavior in the inertial range can be written as

\[ \langle \epsilon(\vec{x} + \vec{r})\epsilon(\vec{x}) \rangle \sim \langle \epsilon \rangle^2 \left( \frac{r}{L} \right)^{-\mu}. \tag{1} \]

play an important role in the study of high Reynolds number turbulence\[1, 2\]. The reason is that the algebraic decay of the correlations characterized by a positive value of \( \mu \) signals intermittent behavior. In Eqn. (1) \( \langle \epsilon \rangle \) is the mean value of the dissipation rate per unit mass, \( r \) belongs to the inertial range, and \( L \) is the integral scale or system size. We have assumed that the turbulence is homogeneous and isotropic. When \( \mu = 0 \) there are no fluctuations at large distances and scaling holds. A positive intermittency exponent \( \mu \) signals the breakdown of Kolmogorov scaling and, experimentally this is indeed the case; there now appears to be agreement\[3\] on a value of \( \mu = 0.25 \pm 0.05 \).

The standard description of the statistical properties of turbulence in terms of velocities leads to the introduction of (longitudinal) structure functions, \( S_p(r) \), whose behavior in the inertial range is given by

\[ S_p(r) \equiv \langle [(u(\vec{r}) - u(\vec{0})) \cdot \hat{r}]^p \rangle \sim r^{\zeta_p}. \]
A scaling argument (of which a more sophisticated version is the refined similarity hypothesis of Kolmogorov and Obukhov\cite{2}) has been employed to relate the exponent $\mu$ characterizing dissipation to the exponent, $\zeta_6$, which describes the behavior of the 6th order structure function. In the naive scaling argument, since $\epsilon$ has dimensions of velocity$^3$/length, one expects that $\epsilon$, when it occurs in correlation functions, behaves as $(\Delta u)^3/r$, where $\Delta u(r) \equiv \langle \vec{u}(\vec{x} + \vec{r}) - \vec{u}(\vec{x}) \rangle \cdot \hat{r}$ is the longitudinal velocity difference. Therefore, the dissipation correlation in (1) can be expected to decay as $S_6(r)/r^2$. This reasoning leads to the well-known identification\cite{1}

$$\mu = 2 - \zeta_6 \quad .$$

(2)

Experimental measurements yield $\zeta_6 \approx 1.8$ (its scaling value is 2); the value of $\mu$ which is then obtained from Eqn. (2) is consistent with the experimental result of $\mu = 0.25 \pm 0.05$ quoted above. Because of this agreement many discussions concerning the intermittency exponent have been limited to a discussion of the non-scaling behavior of $\zeta_6$, although other expressions for $\mu$ have been proposed. In particular, the relation

$$\mu = 2\zeta_2 - \zeta_4$$

(3)

has been proposed.\cite{4, 5} It leads to a value for $\mu$ of $\mu \simeq 0.1$ using data in the literature.\cite{6} For $r << L$, the larger value of $\mu$ would dominate.

Indeed one should generally expect that in an appropriate equation satisfied by the dissipation correlation different terms will lead to different possible values for $\mu$ from which the dominant behavior in the inertial range can be extracted. As far as we know a systematic treatment of this question does not exist. The aim of this paper is to provide one. We present an exact equation satisfied by the dissipation correlations, analyze it, and in particular, show how the two aforementioned values of $\mu$ arise. We should stress here that the only way we have found to derive the equations for the (experimentally measured) equal time, spatial correlations is to start from correlations in both space and time, and then impose the equal time limit.\cite{7}

We will treat both the Navier-Stokes equation and the one-dimensional stochastic Burgers equation.\cite{8} The mathematical structure of equations derived from the latter, while simpler algebraically, is similar to that encountered for the Navier-Stokes equation, as we have pointed out previously.\cite{8, 10, 11, 13} The stochastic Burgers equation thus provides a gateway to understanding issues of fully developed turbulence and our results confirm the usefulness of this approach. It enables one to go beyond generalized scaling arguments which essentially embody the correctness (up to intermittent effects) of the original Kolmogorov argument.

In Section 2 of the paper we present the exact equations for dissipation correlations, both for the Navier-Stokes equation and the stochastic Burgers equation. The details of the straightforward derivation are relegated to Appendix A. Our analysis discussed in Sec. 3 shows that there are two main contributions in the inertial range: the dominant one characterized by the exponent $\mu = 2 - \zeta_6$, and a second which depends on the dynamical behavior of the 4th order structure function. The mathematical similarity of the analysis for the Burgers problem and that for the Navier-Stokes equation is discussed. In section 3.4 we turn to a discussion of some technical points, in particular, the role played by Galilean invariance of the two fundamental equations, and its relationship to “sweeping” effects. A summary of our results and conclusions follow. Technical points are taken up in Appendices B, C and D.
2 The fundamental equations and their derivation

In addition to the Navier-Stokes equation we will also investigate, as mentioned in the Introduction, the one-dimensional, stochastic Burgers equation describing a compressible fluid, given by

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} + \eta(x,t)$$  \hspace{1cm} (4)

where $\nu$ is the kinematic viscosity, and $\eta$ a long-ranged, Gaussian random noise with zero mean and correlations in $k$ space given by

$$\langle \hat{\eta}(k,t)\hat{\eta}(k',t') \rangle = 2 D(k) \delta_{k+k',0} \delta(t-t')$$  \hspace{1cm} (5)

where $D(k) = D_0 |k|^\beta$. The exponent $\beta$ determines the range of the forcing and we focus on $\beta \in (0, -1]$. We have shown previously [9, 10] that for these values of $\beta$ the system exhibits a rich multifractal structure, going from a near scaling regime for $\beta$ close to zero to an extreme multicritical regime characterized by the presence of strong shocks, with $\zeta_n \rightarrow 1$ for $n \geq 3$ when $\beta \rightarrow -1$.

We will derive from the stochastic Burgers equation and analogously in the Navier-Stokes case, an equation satisfied by the correlations of the local dissipation rate. We emphasize that it is convenient to derive an equation for the time-dependent quantity

$$\tilde{C}(r, \tau) = \langle \epsilon(x+r, t+\tau) \epsilon(x, t) \rangle$$  \hspace{1cm} (6)

where the local dissipation rate is defined by

$$\epsilon = \frac{\nu}{2} (\partial_i u_j + \partial_j u_i)(\partial_i u_j + \partial_j u_i)$$  \hspace{1cm} (7)

in the Navier-Stokes case and by $\epsilon = \nu (\partial u/\partial x)^2$ in the one-dimensional Burgers problem. We have employed the notation $\partial_i \equiv \partial/\partial x_i$. The equal time correlation function can be obtained by taking the limit $\tau \rightarrow 0$.

Denote the space-time coordinates of two points by $\vec{x}$ and $\vec{x}' = \vec{x} + \vec{r}$ and $t$ and $t' = t + \tau$ and the various fields by primed and unprimed variables, for example $\epsilon' \equiv \epsilon(\vec{x}', t')$, $\epsilon \equiv \epsilon(\vec{x}, t)$ and similarly for the velocities and other fields. We consider the Navier-Stokes equation driven by a random force $\tilde{\eta}(\vec{k}, t)$ concentrated around a small wavevector; this enables us to define average values most simply. It is straightforward to derive (see Appendix A) the following equation describing the spatio-temporal behavior of the correlations of the dissipation rate:

$$\langle \epsilon' \epsilon \rangle = \frac{1}{4} \frac{\nu}{2} \langle \epsilon u_i u_j + \epsilon u_i' u_j' \rangle$$

$$+ \frac{\nu}{4} \nabla_r^2 \langle \epsilon u_i u_j + \epsilon u_i' u_j' \rangle$$

$$+ \frac{\nu}{2} \nabla_r \nabla_r' \langle \epsilon u_i u_j + \epsilon u_i' u_j' \rangle$$

where $p$ is the pressure divided by the mass density. This exact equation is one of the fundamental equations in this paper.

Similarly one can derive (see Appendix A) the corresponding equation for $\epsilon(x,t)$ in the $1d$ stochastic Burgers equation. We will denote the two space-time points by $x_1 = x + \frac{1}{2} r$, $x_2 = x - \frac{1}{2} r$. 

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\[ t_1 = t + \frac{1}{2} \tau, \quad x_2 = x - \frac{1}{2} \tau, \text{and} \quad t_2 = t - \frac{1}{2} \tau \] respectively and use the notation \( u_1 \equiv u(x_1, t_1), \) etc., for the various fields. The equation satisfied by \( \tilde{C}(r, \tau) \equiv \langle \epsilon \epsilon_2 \rangle \) is

\[ \langle \epsilon \epsilon_2 \rangle = \frac{1}{4} \partial_r \langle \epsilon_1 u_2^2 - \epsilon_2 u_1^2 \rangle + \frac{1}{6} \partial_r \langle \epsilon_1 u_2^3 - \epsilon_2 u_1^3 \rangle \]

\[ + \frac{\nu^2}{4} \partial_r^2 \langle \epsilon_1 u_2^2 + \epsilon_2 u_1^2 \rangle \]

\[ + \frac{1}{2} \langle \epsilon_1 u_2 \eta_2 + \epsilon_2 u_1 \eta_1 \rangle . \]

(9)

We draw attention to the close mathematical similarity of the two equations, in particular, to the first two terms in each equation, schematically \( \partial \langle \epsilon u^2 \rangle / \partial \tau \) and \( \partial \langle \epsilon u^3 \rangle / \partial r, \) which yield the dominant contributions to the behavior of the dissipation correlation. The second line in each equation contains terms proportional to \( \nu \) and these do not contribute in the large Reynolds number limit. We emphasize that the \( \nu \)–dependent terms that are multiplied by singular operators have to be handled carefully to extract the innocuous terms in the second line (see Appendix A). The difference in the last lines is evidently due to the pressure term in the Navier-Stokes equation. We hasten to add that physically the nature of turbulence is not the same in the two systems. For example, vortex stretching is important in the three-dimensional problem. In addition, the energy flux through a wavenumber \( k_0, \) \( \Pi(k_0), \) is proportional to \( (D_0/L) k_0^{1+\beta} \) in the stochastic Burgers problem while it is a constant in the Navier-Stokes case. Nevertheless, the mathematical similarity permits a parallel analytical investigation. Thus the main features of Eqn. (8), namely the expressions for the dominant intermittency exponents quoted in the Introduction, can be extracted by learning to analyze the equation in the one-dimensional stochastic Burgers equation and employing a similar strategy in the Navier-Stokes case. These equations for the dissipation rate correlations depend crucially on including time derivatives and the variation of the temporal fluctuations contribute to the potential intermittent behavior of \( \langle \epsilon \epsilon' \rangle. \)

3 Analysis of the fundamental equation in the 1d Burgers case

We will begin with the mathematically simpler one-dimensional case; in the rest of the paper we restrict our attention to the limit \( \tau \to 0. \) In this limit the last term in Eqn. (8) which depends on the random force is equal to the product \( \langle \epsilon_1 \rangle \langle \epsilon_2 \rangle, \) apart from terms that vanish in the limit \( \nu \to 0. \) This is shown most easily in \( k \) space, given the correlation of the random force in Eqn. (5), using Novikov’s result for Gaussian random processes[13] and the result \( \langle \epsilon \rangle = \frac{1}{L^2} \sum_k D(k). \) This leads in the \( \tau \to 0 \) limit to

\[ C(r) \equiv \langle \epsilon_1 \epsilon_2 \rangle - \langle \epsilon \rangle^2 = \frac{\nu}{4} \left[ \partial_r^2 \langle \epsilon_2 u_1^2 + \epsilon_1 u_2^2 \rangle + \partial_r S_2 \left( \frac{1}{6} \partial_r^2 S_3 - \nu \partial_r^3 S_2 \right) \right] \]

\[ + \frac{1}{6} \partial_r \langle \epsilon_2 u_1^3 - \epsilon_1 u_2^3 \rangle + \frac{1}{4} \partial_r \langle \epsilon_2 u_1^2 - \epsilon_1 u_2^2 \rangle . \quad (10) \]

All the terms on the right-hand side are evaluated in the \( \tau \to 0 \) limit.
3.1 Outline of the argument and main results

We analyze the fundamental equation in the Burgers case, Eqn. (10), for $r$ in the inertial range and extract the dominant terms. In the zero-viscosity limit it is easy to see that the first line in Eqn. (10) does not contribute: clearly, terms that do not depend explicitly on $\epsilon$ are finite and hence, are negligible as $\nu \to 0$. Note that in contrast to the case of fully developed turbulence where $<\epsilon>$ is a constant, in the stochastic Burgers equation one has $<\epsilon> \propto \delta^{-1-\beta}$ where $\delta$ is an inner cutoff scale (shock thickness) and since $\nu \propto \delta^{1-\beta/3}$ the other terms proportional to $\nu$ vanish for $\beta < 0$.\[14\]

The crux of the argument for extracting the possible values of $\mu$ depends on the observations that the dominant behavior of $\epsilon_1 u_2^3 - \epsilon_2 u_1^4$ is determined by $\frac{\partial S_6(r, \tau)}{\partial r}$ and that of $\epsilon_1 u_2^2 - \epsilon_2 u_1^2$ primarily by $\frac{\partial^2 S_4}{\partial \tau^2}$ (we discuss this later) and thus the dominant singular behavior of the dissipation correlation contains

$$\frac{\partial^2 S_6}{\partial r^2} \quad \text{and} \quad \left(\frac{\partial^2 S_4}{\partial \tau^2}\right)_{\tau=0}.$$ 

Since $S_6(r) \sim r^{\zeta_6}$ in the inertial range the first term immediately yields the well-known intermittency exponent

$$\mu_1 = 2 - \zeta_6$$

for the large distance behavior of $C(r)$.

The second term leads to an intermittency exponent

$$\mu_2 = 2\tilde{\zeta}_4 - \zeta_4$$

where the exponent $2\tilde{\zeta}_4$ determines the dynamical behavior of the second derivative of the fourth order structure function when $\tau \to 0$. If one assumes naive dynamic scaling\[15\]

$$S_4(r, \tau) \approx |r|^{\zeta_4} S_4(|r|/|\tau|^z_4)$$

one has $\tilde{\zeta}_4 = z_4$ and thus

$$\mu_2 = 2z_4 - \zeta_4.$$ \[14\]

We will show later that similar results obtain in the Navier-Stokes case also. Accepting this for now and using the fact that in the scaling limit, there is only one dynamical exponent $z = z_4$ which has the same numerical value as $\zeta_2 = 2/3$ in the Navier-Stokes case\[1\], the second intermittency exponent can be written as

$$\mu_2 = 2\zeta_2 - \zeta_4$$

which is Eqn. (3). This is precisely the result obtained in reference\[4\] using scaling arguments for static expressions. Our analysis shows that the expression in Eqn. (3) is an approximation, in the scaling limit, of what is actually a dynamical result. In the Navier-Stokes case if Kolmogorov scaling holds $\mu_1 = \mu_2 = 0$. Returning to the stochastic Burgers equation, for small $\beta$ where all the relevant structure functions scale $\mu_1 = \mu_2 = 2 + 2\beta$; in the limit $\beta \to -1$ where $\zeta_2 = 2/3$, the Kolmogorov value in 3d turbulence, the scaling values of both $\mu_1$ and $\mu_2$ vanish as in the Kolmogorov case. Note, however, that $\mu_1 \geq 1$ in the range $\beta \in (0, 1]$. As $\beta \to -1$ and multiscaling occurs for $p \geq 4$, $\mu_1 = 2 - \zeta_6$ approaches unity
from above and dominates the behavior of $C(r)$. We draw attention to the fact that for the dynamical exponent in Eqn. (12) we have not used the value of the kinematic exponent $z = 1$ that arises from sweeping but the intrinsic dynamical exponent. In Sec. 3.4 we provide a more detailed justification for this.

3.2 Contribution of the $<\epsilon_1u_2^3 - \epsilon_2u_1^3>$ term

We will now argue that $<\epsilon_1u_2^3 - \epsilon_2u_1^3>$ behaves as $\partial_r S_6$ for $\tau = 0$ as claimed above. In an earlier paper[11] we have presented the equations satisfied by the (low-order) equal-time structure functions; in particular, the sixth-order structure function, $S_6$, satisfies the equation

$$\frac{\partial S_6(r, \tau = 0)}{\partial r} = 3\nu \partial_r^2 S_5(r) - 30 <(\epsilon_1 + \epsilon_2)(u_1 - u_2)^3> + \frac{15}{2} <(u_1 - u_2)^4(\eta_1 - \eta_2)>$$

$$= 3\nu \partial_r^2 S_5(r) - 30 <(\epsilon_1 + \epsilon_2)(u_1 - u_2)^3> - <\epsilon_1 + \epsilon_2 > S_3] + 5S_3(r) dS_3/dr$$

(15)

In obtaining the second equality, the noise term has been simplified using Novikov’s theorem and the von-Karman-Howarth relation. The contribution $S_3 dS_3/dr$ that arises from the noise term can clearly be identified as the scaling contribution to $S_6$ which yields for the exponent describing the inertial-range behavior $\zeta_6 = 2\zeta_3 = -2\beta$. The $\nu$–dependent contribution vanishes as $\nu \to 0$ since it is multiplied by a term which is finite in the inertial range and thus we have, generically, the result that terms of the form $<\epsilon_1u_2^3>$ are proportional to $dS_6/dr$. This immediately yields the result $\mu_1 = 2 - \zeta_6$ as stated earlier. We emphasize the feature that the only term that can possibly lead to non-scaling behavior of $S_6$ is the term which is a product of the dissipation rate and velocities. This result, derived from the exact equation for $S_6$, agrees with the field-theoretic point of view in which the non-scaling behavior arises from short-distance singularities which are determined by the limit $\nu \to 0$ and the existence of an energy cascade.

3.3 Contribution of the $<\epsilon_1u_2^2 - \epsilon_2u_1^2>$ term

In contrast to the contribution of $\partial_r <\epsilon_1u_2^2 - \epsilon_2u_1^2>$ to the dissipation correlation function, the term $\partial_r <\epsilon_1u_2^2 - \epsilon_2u_1^2> |_{\tau = 0}$ is an intrinsically dynamical object. In order to study it we consider the temporal evolution of $S_4(r, \tau) \equiv <(u(x_1, t_1) - u(x_2, t_2))^4>$ where $\tau = t_1 - t_2$ and $r = x_1 - x_2$. It can be shown to satisfy the equation

$$\partial S_4/\partial \tau = -\frac{1}{2} \partial_r <(u_2 - u_1)^4(u_2 + u_1)> + 2 <(u_1 - u_2)^3(\eta_1 + \eta_2)>$$

$$-6 <(\epsilon_1 - \epsilon_2)(u_2 - u_1)^2>$$

(16)

The derivation of the general equation satisfied by the time evolution of the generating function for $S_4(r, \tau)$ is given in Appendix B. The above equation follows directly from it as a special case. This equation contains a term of the generic form $<\epsilon u^2>$ whose derivative with respect to $\tau$ occurs in the basic equation for $<\epsilon_1 \epsilon_2>$. The analysis of this equation is somewhat more subtle; however, once again we expect that the non-scaling behavior of the left-hand side is determined by correlations that involve products of the dissipation rate and
velocities, i.e., the last term on the right-hand side. As in the previous subsection we expect the noise term to yield a scaling result: to see this we first note that in the limit $\tau \to 0$ the noise term yields a discontinuity since it gives a different contribution depending on whether $\tau \to 0^+$ or $\tau \to 0^-$. A careful analysis yields, for $\tau > 0$, a contribution

$$-\frac{1}{2} S_2(r) \frac{dS_3(r)}{dr}$$

which dominates the behavior of $\partial_\tau S_4|_{\tau=0}$. In the range of $\beta$ we are interested in both $S_2$ and $S_3$ exhibit scaling behavior. Thus, the noise term again yields scaling behavior for the dynamics of the first derivative in the $\tau \to 0$ limit. The first term in Eqn. (16) ensures Galilean invariance of the equation and contains only sweeping effects as will be discussed below. Therefore, the behavior of terms of the form $[\partial_\tau \epsilon_1 u_2^2]_{\tau=0}$ in the equation for $< \epsilon_1 \epsilon_2 >$ will reflect the non-trivial singularity in $\partial_\tau^2 S_4|_{\tau=0}$, which then yields the second intermittency exponent, expression (12), namely $\mu_2 = 2z_4 - \zeta_4$.

### 3.4 Technical remarks

In this section we discuss some of the technical points that were not addressed in the earlier sections. There are two related difficulties: the first is the more obvious one; while our equation for $< \epsilon_1 \epsilon_2 >$ (see Eqn. (10)) involves velocities, our equations for $\partial_\tau S_6$ and $\partial_\tau S_4$ involve velocity differences and the precise combinations of the terms do not match exactly. Second, if we carry out a straightforward analysis of the term $\partial_\tau < \epsilon_1 u_2^2 - \epsilon_2 u_1^2 >$ we obtain terms of the form $\partial_\tau^2 < (u_1 - u_2)^4(u_1^2 + u_2^2) >$. A naive factorization of the expectation value leads to an exponent $2 - \zeta_4$ and not the value of $\mu_2$ claimed earlier. In particular, this corresponds to the kinematic value $z_4 = 1$ in the expression for $\mu_2$ (Eqn. (14)) and not to the dynamical value of $z_4$, which in the scaling limit is numerically equal to $\zeta_2$. While we have not been able to provide a mathematically complete analysis of all these points we present below convincing arguments in support of our results based on general considerations.

The first observation we make is that while $< \epsilon_1 \epsilon_2 >$ is manifestly Galilean invariant the invariance of the right hand side of Eqn. (9) is not evident; it can be checked explicitly (see Appendix C). Therefore, we expect a (Galilean-invariant) operator product expansion analysis of the terms of the form $\partial_\tau < \epsilon_1 u_2^2 - \epsilon_2 u_1^2 >$ to pick out precisely the terms that we have used in our analysis since the terms will involve Galilean invariant combinations of operators such as, for example, velocity differences. Physically, the potentially troublesome terms, such as $2 - \zeta_4$ mentioned above, arise from the effects of sweeping. For example, we have discussed in a previous paper how behavior such as

$$< (u_1 - u_2)^4(u_1^2 + u_2^2) > \sim < (u_1 - u_2)^4 > < (u_1^2 + u_2^2) >$$

arises. For nonzero values of $\tau$ such a Galilean non-invariant term is known to contain effects of sweeping. That our formalism gives rise to such effects is not surprising since we have adopted an Eulerian point of view. As pointed out by Tennekes large-scale energy containing structures advect inertial-range information past an Eulerian observer. This allows one to note that the dominant term in $<(u_1 - u_2)^4(u_1^2 + u_2^2)>$ is proportional to
\( < u^2 > < (u_1 - u_2)^4 > \) where we have exploited the independence of the large-scale flow characteristics encapsulated in \( < u^2 > \) and the information about the inertial range contained in \( < u_1 - u_2)^4 > \). Good numerical support for the correctness of such an argument has been presented in ref. [12]. In other words, in the Eulerian picture the dynamical behavior of the structure functions at finite values of the time difference \( \tau \) is dominated by sweeping effects which are characterized by an effective dynamical exponent of \( z = 1 \). (The second intermittency exponent, \( \mu_2 \), shown earlier to be \( 2 \tilde{\mu} = 2 - \zeta_4 \) would then reduce precisely to \( 2 - \zeta_4 \) as discussed in the previous paragraph.) However here we are dealing with the limit \( \tau \to 0 \). Physically, equal time correlations do not show sweeping effects. Therefore, only the intrinsic, i.e., non-sweeping, dynamical behavior contributes to \( < \epsilon_1 \epsilon_2 > \). In support of this, we also point out that we have shown previously [12] that intrinsic dynamical fluctuations determine the behavior of \( \partial_\tau S_2 |_{\tau=0} \) and yield the exponent \( z_2 = 1 + \beta/3 \), while the correlation \( S_4 (r = 0, \tau) \), where one observes fluctuations at a given point as a function of time picks up the sweeping contribution of exponent \( z = 1 \).

Experimentally, given the validity of Taylor’s frozen turbulence hypothesis, especially in situations with a large, externally imposed flow, one can determine equal time correlations by making one point measurements as a function of time. In fact, the transition from spatial to temporal variables depends on the kinematic exponent \( z = 1 \). Once, the equal time dissipation rate correlations are thus determined, albeit approximately, their behavior is given by our analysis.

One other term in \( C(r) \) not considered thus far is the \( \tau \to 0 \) limit of

\[
\frac{\partial}{\partial \tau} < (u_1 - u_2)^3 (\eta_1 + \eta_2) >
\]

. This term results from taking the \( \partial/\partial \tau \) of Eqn. [16]. The term \( < (u_1 - u_2)^3 (\eta_1 + \eta_2) > \) can be shown to be related to a fourth-order response function and a naive argument yields inertial-range behavior characterized by an exponent that is the scaling value of \( \mu_2 \).

Thus, the behavior of \( C(r) \) in the \( \tau \to 0 \) limit is determined by appropriate Galilean invariant combinations of spatial and temporal derivatives of structure functions, such as \( \frac{\partial^2}{\partial \tau^2} \) or \( \partial^2 < (u_1 - u_2)^4 > + \partial_\tau \partial_r < (u_1 - u_2)^4 (u_1 + u_2) > + (1/4) \partial^2 < (u_1 - u_2)^4 (u_1 + u_2)^2 > \), and they give rise to the dominant exponents discussed above. Of course, we have not provided a first-principles determination of the dynamical exponent \( z_4 \), or for that matter, of \( \zeta_6 \). We aimed only at establishing relations for the possible value of the intermittency exponent \( \mu \).

4 Analysis of the Fundamental equation for the Navier-Stokes case

We will follow in the three-dimensional Navier-Stokes case the strategy outlined earlier for the stochastic Burgers equation. The calculations are much more involved to carry out in the same detail. We will point out the mathematical correspondences between terms in the Navier-Stokes and the Burgers cases: we obtain the vectorial or tensorial generalizations of the scalar terms in the 1d Burgers problem that are allowed by rotational invariance. For example, the analysis of the crucial \( < \epsilon (\delta u)^3 > \) term in Eqn. [10] that gave rise to \( \mu = 2 - \zeta_6 \)
was based on the equation for $\partial_r S_6$. Thus we are led to consider the corresponding equation for an appropriate sixth-order structure function in the Navier-Stokes case. We use the notation

$$\delta \tilde{u} \equiv \tilde{u}(\vec{x}',t') - \tilde{u}(\vec{x},t) \text{ and } \delta \tilde{\eta} \equiv \tilde{\eta}(\vec{x}',t') - \tilde{\eta}(\vec{x},t) .$$

for velocity and noise differences. We have derived the following equation

$$\frac{\partial}{\partial r_j} < (\delta \tilde{u} \cdot \delta \tilde{u})^2 \delta u_j \delta u_k > = \frac{\nu}{2} \nabla^2_r < [\delta \tilde{u} \cdot \delta \tilde{u}]^2 \delta u_k >$$

$$-4 < \delta \tilde{u} \cdot \delta \tilde{u} \delta u_k [\epsilon + \epsilon'] > -6 < \delta \tilde{u} \cdot \delta \tilde{u} \delta u_i [\epsilon_{ki} + \epsilon'_{ki}] > -2 < \delta u_i \delta u_j \delta u_k [\epsilon_{ij} + \epsilon'_{ij}] >$$

$$< [\delta \tilde{u} \cdot \delta \tilde{u}]^2 \delta \eta_k + 4\delta u_k \delta \tilde{u} \cdot \delta \tilde{u} \cdot \delta \eta >$$

$$-2 < [\delta \tilde{u} \cdot \delta \tilde{u}]^2 \nabla \delta (p + p') > -4 < \delta \tilde{u} \cdot \delta \tilde{u} \delta u_k \delta u_i \frac{\partial}{\partial r_i} (\delta u_k + \delta u_i) >$$

In the preceding we have used the definitions for a (symmetric) tensor with the dimensions of the energy dissipation rate,

$$\epsilon_{jk} = \nu \partial_r u_j \partial_r u_k .$$

The analogy of Eqn. (17) with the corresponding equation in the Burgers case (see eqn. (13)) is striking. The first terms in both equations have the same form and vanish in the $\nu \rightarrow 0$ limit. The driving term $< (\delta u)^4 \delta \eta >$ in the Burgers problem generalizes to the two vector terms in the third line of Eqn. (17). The scalar term $< (\delta u)^3 (\epsilon + \epsilon') >$, which is crucial for identifying $\mu_1$ in the Burgers problem, generalizes in the Navier-Stokes case to the three terms in the second line of Eqn. (17) and involves the tensor generalization of $\epsilon$ defined above. It is reasonable to expect that terms on the right-hand side generically of the form $\nu \epsilon^3$ which involve the dissipation rate lead to the intermittent behavior of $\frac{\partial}{\partial r_j} < [\delta \tilde{u} \cdot \delta \tilde{u}]^2 \delta u_j \delta u_k >$ since the dissipation rate contains the singular behavior of operators in the $\nu \rightarrow 0$ limit. The other difference with the Burgers case arises from the pressure terms in the last line of Eqn. (17) and we do not expect it to yield the dominant intermittent behavior of the right-hand side since it does not depend explicitly on $\epsilon$; we expect it to yield scaling contributions. Straightforward tensor analysis shows that the left hand side of Eqn. (17) effectively includes the contribution of the sixth-order longitudinal structure function. We do not enter into the issue of whether longitudinal and transverse structure functions yield the same exponents; if they do differ as has been claimed [19] then our results should be modified appropriately. The situation is simpler if as is argued by L’vov et al. [20] there is a single exponent.

We have also derived the equation for the temporal derivative of an appropriate fourth-order structure function. The details of the derivation are presented in Appendix D; the equation is given by

$$\partial_r < [\delta \tilde{u} \cdot \delta \tilde{u}]^2 > = -\frac{1}{2} \partial_{r_j} < (u_j + u'_j)[\delta \tilde{u} \cdot \delta \tilde{u}]^2 > + 2< [\delta \tilde{u} \cdot \delta \tilde{u}]^2 \delta \tilde{u} \cdot (\tilde{\eta} + \tilde{\eta}') >$$

$$-2 < (\epsilon - \epsilon') \delta \tilde{u} \cdot \delta \tilde{u} > -4 \epsilon_{ik} - \epsilon'_{ik} \delta u_i \delta u_k > -$$

$$+ 2\nu < \delta \tilde{u} \cdot \delta \tilde{u} \nabla \delta p >$$

$$- 4 < \delta \tilde{u} \cdot \delta \tilde{u} \frac{\partial}{\partial r_i} (\delta p \delta u_i) >$$

(19)
Again we note the similarity to the structure of the equation in the 1d case, Eqn. (16), which leads to the second intermittency exponent $\mu_2$. The first term on the right-hand side which ensures the Galilean invariance of the equation is the vector version of the first term on the right-hand side of Eqn. (16) as is the noise term. The term that has the generic form $<\delta \epsilon (\delta u)^2>$ which is a scalar in the Burgers case involves the tensor generalization of $\epsilon$ defined earlier. It is easy to see that the term proportional to $\nu$ is negligible in the high Reynolds number limit. The last term in Eqn. (19) which depends on the pressure has no counterpart in the Burgers case; as argued earlier we do not expect this term to dominate the intermittent behavior of the left-hand side. On physical grounds it is reasonable to expect that the leading intermittent behavior of $\partial \tau <\delta \vec{u} \cdot \delta \vec{u}>^2$ is determined by terms that involve the dissipation rate on the right hand side, of the form $<\delta \epsilon (\delta u)^2>$, as in the Burgers case; thus the Galilean-invariant contribution of $\partial \tau <\epsilon 'u' \cdot u' - \epsilon 'u' \cdot \bar{u}>$ to the dissipation correlations in Eqn. (3) will be given by $\partial^2 S_4/\partial \tau^2|_{\tau=0}$.

We see that the mathematical similarity of the various equations in the Burgers and Navier-Stokes cases allows us to carry out a similar analysis and deduce the same relations for the leading intermittency exponents in terms of the exponents that characterize the static and dynamic behavior of the velocity structure functions. We have discussed the implications of this already in Sec. 3.1.

5 Discussion of Results

It is remarkable that the exponents which characterize the power law decay of the (equal time) dissipation rate correlations obey the same relations for the Burgers equation as for the Navier-Stokes equation. The leading exponent $\mu_1 = 2 - \zeta_6$ derives from the presence of $\partial^2 S_6$ in the equation for dissipation fluctuations while the second exponent $\mu_2$ arises from the term $\partial^2 S_4|_{\tau=0}$. Clearly, the origin of the second exponent (expression (14)) is dynamical, and coincides, in the Navier-Stokes case, with the static expression in Eqn. (3) (see Ref. [4]) in the scaling limit, where the only dynamical exponent $z$ is numerically equal to $\zeta_2$.

We have derived the exact equation satisfied by $C(\vec{r}, \tau)$ and analyzed it to obtain our results. Recently, L’vov and Procaccia [21, 22] have found general fusion rules for multipoint correlations in the Navier-Stokes problem and used them to show that $\mu = 2 - \zeta_6$; they also discuss the possibility of another scenario (see Eqn. (15) in Ref. [21]) in which the static version of $\mu_2$, i.e., $\mu_2 = 2\zeta_2 - \zeta_4$ can occur. [4] Our approach finds these as possible terms from an exact equation and provides a dynamical expression for the second exponent.

In our analysis we have had to disentangle sweeping effects from the dynamics of internal evolution, both of which are present in our Eulerian equations. We did this by exploiting the Galilean invariance of the internal dynamics, noting that sweeping which leads to a kinematic value of $z = 1$ yields effects that break Galilean invariance. It is clear that for the measurable quantity we are considering, namely dissipation rate correlations in the inertial range at equal times, sweeping effects do not matter. As we have shown previously [14] for the Burgers equation, for correlations involving velocities at different times sweeping effects can dominate the dynamic behavior: the structure functions, for example, $S_2$ satisfy a wave equation with a characteristic velocity of the order of the rms fluctuations of velocity.

While we have not proved that $\mu_1$ and $\mu_2$ are the only dominant exponents, especially in the case of the Navier-Stokes equation, our equations are exact, and the identifications we have
made in the 1-dimensional Burgers equation are transparent. The mathematical similarity of the equations derived from the Navier-Stokes equation to those in the Burgers case gives one confidence that one has indeed made progress in the derivation of the intermittency exponents in the Navier-Stokes case.
A Derivation of Eqns. (8) and (9)

Starting from the Burgers equation for $u = u(x, t)$ and multiply by $u$ to obtain

$$u_1 \partial u_1 / \partial t_1 = \nu u_1 \partial^2 u_1 / \partial x_1^2 - \frac{1}{3} \partial u_3 / \partial x_1 + u_1 \eta_1 .$$ (20)

The first term on the right-hand side can be rewritten using elementary manipulations as

$$\nu u_1 \partial^2 u_1 / \partial x_1^2 = \nu \left[ \partial (u_1 \partial u_1 / \partial x_1) / \partial x_1 - (\partial u_1 / \partial x_1)^2 \right]$$

$$= \frac{\nu}{2} \partial^2 u_1^2 / \partial x_1^2 - \epsilon_1 .$$ (21)

Note that this is a key step since it allows us to disentangle the terms that are finite in the $\nu \to 0$ limit, due to the singular behavior of the derivative terms, from those that vanish.

Using this result and multiplying Eqn. (20) by $\epsilon_2$ yields

$$\frac{1}{2} \partial \left[ \epsilon_2 u_1^2 \right] / \partial t_1 = \nu \frac{\partial [\epsilon_2 u_1^2]}{\partial x_1^2} - \epsilon_1 \epsilon_2 - \frac{1}{3} \partial \left[ \epsilon_2 u_1^3 \right] / \partial x_1 + \epsilon_2 u_1 \eta_1 .$$ (22)

where we have used the fact that $\epsilon_2$ does not depend on the coordinates $x_1$ and $t_1$. We compute next the average of the above equation with respect to the noise ensemble described by Eqn. (5). We assume that we have a spatially homogeneous and temporally steady state which implies that

$$\partial < ... > / \partial t = 0 \text{ and } \partial < ... > / \partial x = 0 .$$ (23)

Using the definitions $x_1 = x + r/2$ and $x_2 = x - r/2$, rudimentary calculus yields

$$\partial / \partial x_1 = \frac{1}{2} \partial / \partial x + \partial / \partial r \equiv \frac{1}{2} \partial_x + \partial_r \text{ and } \partial / \partial x_2 = \frac{1}{2} \partial / \partial x - \partial / \partial r \equiv \frac{1}{2} \partial_x - \partial_r .$$ (24)

and similar results for the time variable $t_1$ and $t_2$. Averaging Eqn. (22) yields

$$\frac{1}{2} \partial_r < \epsilon_2 u_1^2 > = \frac{\nu}{2} \partial^2 r < \epsilon_2 u_1^2 > - \epsilon_1 \epsilon_2 - \frac{1}{3} \partial_j < \epsilon_2 u_1^3 > + < \epsilon_2 u_1 \eta_1 > .$$ (25)

Writing a similar equation with the subscripts 1 and 2 interchanged and adding yields the symmetric form displayed in Eqn. (9).

The equation for the dissipation rate correlations in the Navier-Stokes case Eqn.(8) can be obtained analogously taking care to keep track of the Cartesian coordinate subscripts and the pressure term. One finds instead of Eqn. (25), changing to primed variables as is done in the main text, the following equation

$$\frac{1}{2} \partial_r < \epsilon' \bar{u} \cdot \bar{u} > = \frac{\nu}{2} \nabla_r^2 < \epsilon' \bar{u} \cdot \bar{u} > + \nu \nabla_r \nabla_j < \epsilon' u_i u_j >$$

$$- < \epsilon' \epsilon > - \frac{1}{2} \partial_j < \epsilon' \bar{u} \cdot \bar{u} u_j >$$

$$+ < \epsilon' \bar{u} \cdot \bar{\eta} > - \partial_j < \epsilon' p u_i > .$$ (26)

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The equations for the Burgers and the Navier-Stokes case correspond to each other term by term (if one makes allowance for the fact that the one deals with one dimensional the other with three dimensional quantities) as is evident from the comparison of equations (25) and (26). The two additional terms in the Navier-Stokes case are due to the presence of pressure and the use of the full three dimensional definition of $\epsilon$, rather than its expression along one direction which is the only quantity usually measured [3].

B Derivation of Eqn. (16)

The general derivation of the equation satisfied by $S_q(r, \tau)$ is most easily carried out using generating functions. We use the usual notation $u_1 = u(x_1, t_1)$ and $u_2 = u(x_2, t_2)$, $x_1 = x + r/2$, etc. Consider the generating function $\exp(a(u_1 - u_2))$ and compute its derivative with respect to $\tau$. Starting from the stochastic Burgers equation for $u_1$ and $u_2$ and using $\partial/\partial \tau = (1/2)(\partial/\partial t_1 - \partial/\partial t_2)$, we obtain

$$\frac{\partial e^{a(u_1-u_2)}}{\partial \tau} = \frac{a}{2}(\partial_1 u_1 + \partial_2 u_2) e^{a(u_1-u_2)}$$

$$= \frac{a}{2} e^{a(u_1-u_2)} (u \partial_1^2 u_1 - u_1 \partial_1 u_1 + \eta_1 + \nu \partial_2^2 u_2 - u_2 \partial_2 u_2 + \eta_2) . \quad (27)$$

The following identities that are easily verified are useful:

$$e^{a(u_1-u_2)} u_1 \partial_1 u_1 = \frac{1}{a^2} \partial_1 [ e^{a(u_1-u_2)}(au_1 - 1) ]$$

$$e^{a(u_1-u_2)} \partial_1^2 u_1 = \frac{1}{a} [ \partial_1^2 e^{a(u_1-u_2)} - a^2 e^{a(u_1-u_2)}(\partial_1 u_1)^2 ] . \quad (28)$$

We use these and the corresponding equations for derivatives with respect to $x_2$ and take average values over the Gaussian noise ensemble. We use Eqn. (24) and assuming a spatially homogeneous, temporally steady-state which implies Eqn. (23), we can evaluate $< \partial e^{a(u_1-u_2)}/\partial \tau >$ and obtain

$$\frac{\partial}{\partial \tau} \langle e^{a(u_1-u_2)} \rangle = - \frac{1}{2} \partial_\tau \langle - \frac{a^2}{2} e^{a(u_1-u_2)}(u_1 + u_2) \rangle + \langle e^{a(u_1-u_2)}(\epsilon_1 - \epsilon_2) \rangle$$

$$+ \frac{a}{2} \langle e^{a(u_1-u_2)}(\eta_1 - \eta_2) \rangle . \quad (29)$$

The coefficient of $a^4$ yields Eqn. (16) used in Sec. 3.3.

C Galilean invariance of Eqn. (9)

In this Appendix we show the Galilean invariance of the righthand side of Eqn. (9) explicitly. A similar derivation can be carried out in the Navier-Stokes case. In the one-dimensional problem the Galilean transformation to a frame moving with a relative velocity $V$ is accomplished by letting

$$x \to x + V t \quad \text{and} \quad t \to t$$

and

$$\partial/\partial x \to \partial/\partial x \quad \text{and} \quad \partial t \to \partial/\partial t - V \partial/\partial x .$$
Analogous results apply for the relative coordinates \( r = x_1 - x_2 \) and \( \tau = t_1 - t_2 \). Making these substitutions in the righthand side of Eqn. (9) we find that terms proportional to \( V^3 \) and \( V^2 \) vanish as can be easily checked. The term of order \( V \) is given by

\[
\frac{1}{2} \frac{\partial}{\partial \tau} < \epsilon_1 u_2 - \epsilon_2 u_1 > + \frac{1}{4} \frac{\partial}{\partial r} < \epsilon_1 u_2^2 - \epsilon_2 u_1^2 > + \frac{1}{2} < \epsilon_1 \eta_2 + \epsilon_2 \eta_1 > + \frac{\nu}{2} \frac{\partial^2}{\partial r^2} < \epsilon_1 u_2 + \epsilon_2 u_1 >.
\]

We now demonstrate an identity that shows that this term vanishes. Multiplying Burgers equation for \( u_1 \) by \( \epsilon_2 \) we have

\[
\frac{\partial}{\partial t} \left[ \epsilon_2 u_1 \right] = \nu \frac{\partial^2}{\partial R_i} \left[ \epsilon_2 u_1 \right] + \epsilon_2 u_2 \frac{\partial \delta u_1}{\partial r_j} - \frac{1}{2} \frac{\partial}{\partial t} \left[ \epsilon_2 u_1^2 \right] / \partial x_1 + \epsilon_1 \eta_1.
\]

Adding the analogous equation with the subscripts 1 and 2 interchanged to the above equation and using Eqns. (24) yields the required identity immediately.

**D Derivation of Eqn. (19)**

We derive the equation satisfied by \( < (\delta \vec{u} \cdot \delta \vec{u})^2 > \) where \( \delta \vec{u} = u(\vec{x}, t) - \vec{u}(\vec{x}', t') \equiv \vec{u} - \vec{u}' \). We use as usual

\[
\vec{x} = \vec{R} + \frac{1}{2} \vec{r} \quad \text{and} \quad \vec{x}' = \vec{R} - \frac{1}{2} \vec{r}
\]

and

\[
t = T + \frac{1}{2} \tau \quad \text{and} \quad t' = T - \frac{1}{2} \tau
\]

leading to the identities

\[
\frac{\partial}{\partial x_i} = \frac{1}{2} \frac{\partial}{\partial R_i} + \frac{\partial}{\partial r_i} \quad \text{and} \quad \frac{\partial}{\partial t} = \frac{1}{2} \frac{\partial}{\partial T} + \frac{\partial}{\partial \tau}
\]

and the corresponding equations for the primed variables. Using these and the Navier-Stokes equations for \( u_i \) and \( u'_i \) and adding them yields the basic equation

\[
2 \frac{\partial \delta u_i}{\partial \tau} = \nu \left[ \nabla^2 u_i + \nabla'^2 u'_i \right] - (u_j + u'_j) \frac{\partial \delta u_i}{\partial r_j} - \delta u_j \frac{\partial (u_i + u'_i)}{\partial r_j} - 2 \frac{\partial \delta p}{\partial r_i} + \eta_i + \eta'_i
\]

where \( \delta p = p(\vec{x}, t) - p(\vec{x}', t') \). Multiplying this equation by \( \delta \vec{u} \cdot \delta \vec{u} \delta u_i \), summing over \( i \) and taking averages over the spatially homogeneous, temporally steady state of fully developed turbulence leads to the equation we wish to establish after straightforward manipulations. Terms that are proportional to \( \nu \) need careful consideration. It is most convenient to extract the terms that are finite as \( \nu \to 0 \) by using Leibniz’s rule:

\[
< \delta \vec{u} \cdot \delta \vec{u} \delta u_i \nabla^2 \delta u_i > = \partial_j < \delta \vec{u} \cdot \delta \vec{u} \delta u_j \delta u_i > - < \delta \vec{u} \cdot \delta \vec{u} \delta u_j \delta u_i > - 2 < \delta u_i \delta u_j \delta u_k \delta u_k >
\]

and similar equation for the primed variable. The following identity that can be derived easily from incompressibility and Navier-Stokes equation

\[
\nabla^2 p = - \frac{\partial u_i}{\partial x_j} \frac{\partial u_j}{\partial x_i}
\]
and the obvious result that follows from it
\[ \epsilon \equiv \frac{\nu}{2} (\partial_i u_j + \partial_j u_i)^2 = \nu (\partial_i u_j)^2 - \nu \nabla^2 p \]
are useful in rewriting the viscous terms.

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By the von-Karman-Howarth relation one has $S_3 \propto |r|^{-\beta}$ yielding the scaling value for $\zeta_3$. There is very good numerical support in the case of $S_2$ that $\zeta_2$ has the scaling value $-2\beta/3$; the result is also supported by heuristic arguments.

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