Instantaneous flywheel torque of IC engine grey-box identification

A Milašinović, D Knežević, Z Milovanović and J Škundrić
Faculty of Mechanical Engineering, University of Banja Luka, Stepe Stepanovića 71
78000 Banja Luka, Bosnia & Herzegovina

E-mail: aleksandar.milasinovic@unibl.rs

Abstract. In this paper a mathematical model developed for the identification of excitation torque acting on the IC engine flywheel is presented. The excitation torque gained through internal combustion of the fuel in the IC engine is transmitted from the flywheel to the transmission. The torque is not constant but variable and is a function of the crank angle. The verification of the mathematical model was done on a 4-cylinder 4-stroke diesel engine for which the in-cylinder pressure was measured in one cylinder and the instantaneous angular speed of the crankshaft at its free end. The research was conducted on a hydraulic engine brake. Inertial forces of all rotational parts, from flywheel to the turbine wheel of the engine brake, are acting on the flywheel due to the nonuniform motion of the flywheel. It is known from the theory of turbomachinery that the torque on the hydraulic brake is a quadratic function of angular speed. Due to that and the variable angular speed of the turbine wheel of the engine brake, the torque during one engine cycle is also variable. The motivation for this research was the idea (intention) to determine the instantaneous torque acting on the flywheel as a function of the crank angle with a mathematical model without any measuring and based on this to determine the quality of work of specific cylinders of the multi-cylinder engine. The crankshaft was considered elastic and also its torsional vibrations were taken into account.

1. Introduction
The internal combustion (IC) engine is still a dominant drive unit of automobiles. Due to economic and ecological aspects, the optimization and diagnostics of the quality of IC engine work is very important. There are two methods to determine the instantaneous torque acting on the flywheel of an IC engine: the direct method (by measurement) and the indirect method of parameter identification of a mathematical model. The direct method implies the installation of another complex device on the engine and its connection to the engine electronics. The indirect method of determination of the torque on the flywheel would be (imply) a software solution which would require a more powerful computer support.

The motion of the crankshaft is the result of excitation torques due to the combustion of fuel in the cylinder, inertial torques of the slider-crank mechanism, elastic deformations of the crankshaft and crankshaft motion resistance torque which is transmitted over the flywheel to the power consumers (engine brake). Under stationary engine operation all torques, acting on the flywheel, could be considered periodical and as such may be expressed in a Fourier series. The total motion of the crankshaft can be considered as a sum of the motions caused by individual harmonics.

It is known from theoretical dynamics that there are two approaches considering mechanical system motion: the first, where forces and torques acting on the system are known while the law of
motion has to be determined and the second approach where the law of motion is known while the forces and torques causing this motion have to be determined. In this particular case the situation is more complex because the law of motion of the free end of the crankshaft is known (gained through measurement), but the law of motion of particular sections of the equivalent dynamical system of the crankshaft is not. The law of motion of the crankshaft depends both on excitation torques and dynamical characteristics of the crankshaft (reduced moment of inertia for the axis of rotation, torsional stiffness, friction, etc.). Accurate knowledge of these properties and a mathematical model which does not include assumptions that could lead to significant deviation between the actual and modeled system can lead to a good-quality simulation model of the dynamical behavior of the crankshaft. It is usually assumed that the torque on the brake is constant [1].

2. Crankshaft dynamics

Figure 1 shows an equivalent schematics of a dynamical model of elastic crankshaft. This approach of conversion of a real dynamical system into an equivalent system has proven to be a good solution in praxis [1-18].

![Figure 1. Equivalent schematics of a dynamical model of elastic crankshaft [19]](image)

The differential equations of motion for holonomic systems which are subject to geometric (holonomic) constraints are the Lagrange equations of the second kind:

\[
\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\alpha}} \right) - \frac{\partial T}{\partial \alpha} = - \frac{\partial V}{\partial \alpha} - \frac{\partial \Phi}{\partial \dot{\alpha}} + \mathbf{t}_I, \tag{1}
\]

where \(\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\alpha}} \right)\) is the row vector of generalized inertial forces, \(\frac{\partial T}{\partial \alpha}\) the row vector of generalized conservative forces, \(\frac{\partial \Phi}{\partial \dot{\alpha}}\) the row vector of generalized dissipative forces and \(\mathbf{t}_I\) the row vector of generalized external excitation forces. Here, external load refers to a vector comprising excitation forces on each lumped mass, regardless of externally applied force or forces of constraint. For example, if we observe the lumped mass model of the crankshaft, there is no external force acting
on the first lumped mass - the flywheel, but there is a generalized force, which is the result of the
constraint force acting between the flywheel and the brake. Frictional forces are not included in the
vector of external excitation forces (they are included in the stiffness matrix through external
damping) [20].

A crankshaft with \( n \) lumped masses, rotating at steady state operating conditions while the lumped
masses are moving variably (nonuniformly) around the mean angular velocity, is observed. The
angular position of each lumped mass is defined by a column vector of generalized coordinates
\[ \mathbf{a} = [\alpha_1, \alpha_2, \ldots, \alpha_n]^T. \]

\[ T = \frac{\mathbf{a}^T \mathbf{J}(\mathbf{a}) \mathbf{a}}{2}, \]  

(2)

where \( \mathbf{J}(\mathbf{a}) \) is the moments of inertia matrix of the lumped mass model of the crankshaft.

The potential energy can be expressed as

\[ V = \frac{\mathbf{a}^T \mathbf{C} \mathbf{a}}{2}, \]  

(3)

where \( \mathbf{C} \) is the stiffness matrix of the lumped mass model of the crankshaft.

If it is assumed that frictional resistance during the oscillation of the crankshaft is comprised of
external and internal (resistance within the shaft), the dissipation function has the following form:

\[ \Phi = \frac{\mathbf{a}^T \mathbf{K} \mathbf{a}}{2}, \]  

(4)

where \( \mathbf{K} \) is the damping matrix, both external and internal.

After substituting particular functional derivatives defined by (2)-(4) into the Lagrange equation
(1), the equation of motion as a function of the crank angle is obtained:

\[ \mathbf{J}(\mathbf{a}) \ddot{\mathbf{a}} + \frac{1}{2} \frac{d\mathbf{J}(\mathbf{a})}{d\mathbf{a}} \dot{\mathbf{a}}^2 + \mathbf{K} \mathbf{a} + \mathbf{C} \mathbf{a} = \mathbf{t}. \]  

(5)

If the motion of the lumped mass model of the crankshaft is observed as being composed of the
motion of all lumped masses with constant speed \( \Omega_b \) together and the relative motion of every lumped
mass in relation to the common crank angle of uniform motion \( \Omega_b t \), and also by defining the row
vector of angle of nonuniform motion \( \varphi \) of the lumped mass model of the crankshaft in relation to
\( \Omega_b t \), the following relation, regarding angular position of the lumped masses can be obtained:

\[ \mathbf{a}(t) = \Omega_b t + \varphi(t), \]  

(6)

where \( \mathbf{a}_n = [\Omega_b, \ldots, \Omega_b]^T \) is a column vector of mean angular speed, \( t \) is time (a scalar quantity) while
\( \varphi(t) = [\varphi_1(t), \ldots, \varphi_n(t)]^T \) is a column vector of the angle of nonuniform motion of the lumped mass
model of the crankshaft. Quantities which are functions of the vector of the angle of motion \( \mathbf{J}(\mathbf{a}) \),
\( \frac{d\mathbf{J}(\mathbf{a})}{d\mathbf{a}} \) and \( \mathbf{t}(\mathbf{a}) \) may be approximated by a Taylor series [19].

After expanding into Taylor series and omitting \( \varphi^2 \), \( \varphi^2 \), \( \phi \varphi \) and \( \phi \phi \) as lower-order terms, a
system of differential equations describing the motion of an IC engine crankshaft in the form of a
matrix is gained:
\[ J(\omega,t)\dot{\phi} + K\phi(t) + {\frac{dJ(\omega,t)}{d(\omega,t)}}\omega_0\phi + \frac{1}{2}\frac{d^2J(\omega,t)}{d(\omega,t)^2}\omega^2_0\phi + C\phi = t_f(\omega,f) - \frac{1}{2}\frac{dJ(\omega,t)}{d(\omega,t)}\omega^2_0 - Ko_s, \quad (7) \]

After analyzing equation (7), it can be noticed that the excitation is composed of the excitation torque referring to external torque acting on all lumped masses \( t_f(\omega,f) \) and the torque due to the motion of the slider-crank mechanism \( \frac{1}{2}\frac{dJ(\omega,t)}{d(\omega,t)}\omega^2_0 \). The constant friction torque \( Ko_s \) represents engine losses that occur as a result of absolute damping. The moment of inertia and its first and second derivative are periodic functions of the crank angle of uniform motion \( \omega_0t \). For the given system of differential equations, they are variable coefficients, i.e., the system is nonlinear. The term \( \frac{1}{2}\frac{d^2J(\omega,t)}{d(\omega,t)^2}\omega^2_0\phi \) takes into consideration the excitation variation caused by the masses of the slider-crank mechanism, with the angle of nonuniform motion \( \phi(t) \) of the lumped mass model of the crankshaft. The equation (7) takes into consideration the variability of the moment of inertia as well as the variability of excitation caused by the motion of the slider-crank mechanism with the angle of nonuniform motion.

3. **Parameter identification for the dynamical model of elastic crankshaft**

The dynamic behavior of a multibody system can be described by a set of ordinary differential equations. A relative coordinate formulation is applied such that no algebraic equations due to kinematic joints arise [21]

\[
M(q,b)\ddot{q} = Q^A(q,\dot{q},b,t),
q(t_0) = q_0, \quad \dot{q}(t_0) = \dot{q}_0.
\quad (8)
\]

In equation **Error! Reference source not found.** \( M \) is mass matrix, \( Q^A \) is generalized applied force, and \( q(t_0) \) and \( \dot{q}(t_0) \) are initial conditions. The column of generalized forces includes externally applied forces and torques.

The mathematical model of the dynamical system of equivalent crankshaft contains a large number of the unknown or approximately known parameters that significantly affect the dynamic motion of the crankshaft. Parameter identification of the unknown or approximately known parameters, from experimental tests of IC engine, is one of the key problems to achieve a good agreement of results obtained from the mathematical model and the experimental measurements. On the other hand, based on the identified parameters a better understanding can be obtained in those parts of the process where modeling is more or less formalized, and checks made assumptions done. The model parameters whose exact measurement in the working conditions of the engine impossible can be determined.

The goal of parameter identification is to determine the parameters of a fixed mathematical model so that the output from the model agrees with the results obtained by measuring. It should be noted that there is a clear distinction between the parameter identification (grey box model) and the system identification (black box model). Grey box model still has a number of unknown parameters which should be defined. The parameter identification calculates the mathematical model parameters that best fit a block of measured data. The least-squares parameter identification finds the parameter vector that minimizes the sum of squared prediction errors.

Equation **Error! Reference source not found.** represents differential equations of motion of the crankshaft. A common approach is to first determine all the parameters of the equivalent system (moments of inertia matrix \( J \), stiffness matrix \( C \), damping matrix \( K \), and torque vector \( t_f \)) and then solve differential equations – obtain vector \( \phi \). However, determining the parameters of the equivalent system is not simple. We can quite accurately determine some of the parameters such as...
moment of inertia (Computer Aided Design-CAD software), the gas-pressure torque (measuring the gas pressure in the cylinder) and the torque due to the motion of the slider-crank mechanism 

\[ \frac{1}{2} \frac{dJ(\omega_i,t)}{dt} \omega_i^2. \]

Determination of other parameters of the system such as the stiffness of particular parts of the crankshaft \( C \), damping matrix \( K \) and the reaction force acting between the flywheel and the brake. However, we can determine the boundaries within some of the unknown parameters range. In [15] a procedure for determining the stiffness crank of the crankshaft is given. It is important to determine as precisely as possible the border of the unknown parameter range. The mathematical model gives all the solutions. However, some of the solutions of the system are impossible from the standpoint of mechanics, e.g. negative stiffness crank of the crankshaft [15].

An integrated design approach for dynamic systems has to support all steps from problem formulation to problem solution by optimization, see Figure 1. Firstly, the technical system to be optimized has to be transformed to a mathematical model. Therefore, parameters of the mathematical model have to be classified either as design variables whose values can be chosen within given bounds or as system constants whose values remain fixed during optimization. In the presence of several criteria, multicriteria optimization strategies may be used to find optimal compromise solutions.

![Figure 2. Components of an integrated design concept [22]](image)

Typically in the parameter identification process [15], experimental data in the form \( \{(t_i, y_i)\}_{i=1,...,n_m} \) have been collected, and the design parameters \( b \in \mathbb{R}^p \) for the given model \( M(t,b) \) must be selected such that \( y_m(t) \approx y_i \), where

\[ y_m(t) = M(t,b), \quad (9) \]

Generally, there are far more data points than parameters \( n_m > p \).

In equation Error! Reference source not found., the mathematical model is an arbitrary nonlinear model. The system will include a vector of \( p \) unknown parameters \( b \). These parameters represent physical properties of the rigid body mechanism, such as masses, inertias, spring coefficients, damping coefficients. Note that any of the terms in equation Error! Reference source not found. can be a function of the unknown parameters \( b \).

The general approach to the solution of the parameter identification problem is to minimize a cost function of the model residuals, defined as follows [23-25]
\[ r_i(b) = y_i - y_m(t_i), \]  
(10)

The cost function is given as the quadratic form

\[ F(b) = \frac{1}{2} \sum_{i=1}^{n} r_i^2(b), \]  
(11)

The estimate of the design parameters \( b \), based on \( n_m \) data points, is then defined as

\[ b^* = \min_b \left[ F(b) \right], \]  
(12)

By assembling the model residuals into a vector, given as

\[ \mathbf{r}(b) = [r_1(b), r_2(b), \ldots, r_{n_m}(b)]^T, \]  
(13)

The design parameters must be chosen so that the residual vector is as small as possible, in some sense. The quadratic performance function \( F(b) \) becomes

\[ F(b) = \frac{1}{2} \mathbf{r}^T(b)\mathbf{r}(b), \]  
(14)

4. Result analysis

In order to study the dynamic characteristics of the IC engine, an experimental setup was made whose scheme is shown in Figure 3. A turbocharged low-speed diesel engine with direct fuel injection was experimentally investigated. The main characteristics of the tested engine are presented in Table 1. The following parameters of interest for the engine operation have been measured: instantaneous angular velocity of the crankshaft free end, cylinder pressure in the cylinder next to the flywheel and the torque on the engine dynamometer. All these parameters were measured simultaneously. The measurements were carried out at different engine speeds and loads. The task of experimental measurements in the context of this paper is to get the data necessary to perform parameter identification of the dynamical model of the crankshaft. Since the parameter identification of the dynamical model is based on a comparison of actual and modelled angular speed of the free end of the crankshaft, the basic measurement item is the instantaneous angular speed of the pulley.

| Property                  | Symbol | Value | Unit  |
|---------------------------|--------|-------|-------|
| Piston head area          | \( A_k \) | 0.123 | m\(^2\) |
| Crank radius              | \( r \) | 0.0725 | m     |
| Length of the connection rod | \( l \) | 0.237 | m     |
| Firing order              | ---    | 1-3-4-2 |       |
| Reciprocating mass        | \( m_{rec} \) | 4.6 | kg    |
| Rotating mass             | \( m_{rot} \) | 2.6 | kg    |
| Cylinder capacity         | \( V \) | 7000 | cm\(^3\) |
| Bore diameter             | \( d \) | 125 | mm    |

Table 1. Characteristics of the tested engine: TAM BF 4 L 515 C
Figure 3. Experimental setup a) Schematic of measuring installations for measuring dynamic characteristics of the IC engine b) The lumped mass model

As function $J(a)$ and $t_j(a)$ that exist in the mathematical model are periodic $-4\pi$, we can decompose it on a Fourier basis. Harmonic analysis and synthesis is used to solve optimization problems. The first harmonic is a constant and it causes uniform movement of the crankshaft with a constant angular speed. For the test engine, the pressure is measured in the fourth cylinder. However, it was assumed that the curve of change of pressure as a function of the angle of the crankshaft in the other cylinders (the first, second and third) the same as the curve obtained by measuring the pressure in the fourth cylinder only is phase shifted. The aforementioned curve of change of pressure in the first, second and third cylinders is used as the initial iteration in the optimization process. When an engine is working properly, this assumption is possible. The torque due to motion of the crank-slider mechanism masses is the same for all cylinders, but is phase shifted as a function of the crankshaft angle position. Any possible error in such an assumption is negligible.

The moment of inertia is a function of both mass and position. Therefore, in the case of a crank-slider mechanism, where the geometry changes, the moment of inertia will also vary, hence the term
varying inertia will be used for this. The moments of inertia a parts of crank-slider mechanism were determined using SolidWorks CAD (Computer Aided Design) package. Determination of the crankshaft torsional stiffness coefficient can be reduced to determining the torsional stiffness coefficient of simple cylindrical parts, and determining the torsional stiffness coefficient for a single crank of the crankshaft [15]. Torsional stiffness is determined by using CAE-Computer Aided Engineering software packages Catia and SolidWorks.

Figure 4 shows identified instantaneous torque that is a function of the crank angle. Figure 4 also shows identified torques of the cylinder in which the pressure is not measured (1-cyl., 2-cyl. and 3-cyl). Four peaks can be seen on the curve identified torque of the flywheel which are caused by the action of individual engine cylinders. Changes in torque at the flywheel during one cycle are significant. Moment of inertia brakes, angular acceleration of the parts of the brake and torque that is generated in the brake affect the torque at the flywheel.

![Figure 4](image_url)

**Figure 4.** Identified torque of the flywheel and torques of the cylinder in which the pressure is not measured at 1000 rpm speed of rotation of the crankshaft.

Figure 5 shows the instantaneous calculated and measured angular velocity of the free end of the crankshaft at 1000 rpm speed of rotation of the crankshaft. It could be noticed that the simulated speed follows very closely the measured speed. Amplitude of the unevenness of rotation of the crankshaft is large, at 1000 rpm, and torsional oscillations are small.
Figure 5. The instantaneous calculated and measured angular velocity of the free end of the crankshaft at 1000 rpm speed of rotation of the crankshaft

Parameter identification is done on the basis of harmonic analysis. Eight parameters are identified for a single harmonic: three amplitudes and three-phase pressure in the cylinders in which the pressure is not measured and amplitude and phase harmonica at the flywheel. "Trust-region methods" were used for the identification [26]. Changing the starting point of search for the optimal solution leads to a change in the number of iterations.

The given mathematical model of movement of the crankshaft, which is used to identify parameters, ends with flywheel. The impact of the brakes on the flywheel is replaced by reactive torque acting on the flywheel.

Figure 6. Angular acceleration of the pulley and flywheel in the function of the angle position of the crankshaft at 1000 rpm engine speed
Figure 6 shows the curve of change of angular acceleration of the flywheel and pulleys in the function of angle position of the crankshaft. Angular acceleration of the pulley has a dominant harmonics of higher orders. This phenomenon is caused by torsion vibration of the crankshaft.

5. Conclusions
Based on the established mathematical model and the analysis presented in this paper, the following conclusions can be made:

- The mathematical model must be as simple as possible and with minimum errors in determining the parameters of the model. Taking into account the variability moment of inertia of the system in the mathematical modelling of the dynamics of the crankshaft in the identification parameters of the dynamic system is not necessary. A greater mistake is made when estimating the friction or torsional stiffness in the system rather than neglecting the variability moment of inertia of the system.
- If some parameters of the mathematical model cannot be determined accurately then the segment can be determined in which the parameter ranges and the method of identifying parameters determine its real value.
- The instantaneous value of torque at the flywheel of IC engines is difficult to measure and such a sensor is not installed in the vehicle. By using the method of identifying parameters, it is possible to determine the value of the torque at the flywheel and this information can be used for diagnostic purposes.
- A common simplification that is created when the crankshaft is modelled as a rigid body leads to the inaccurate determination of excitation torque. The mathematical model of the elastic crankshaft allows the addition of detection of inadequate work of each cylinder and determines which cylinders are not working properly.

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APPENDIX: SYMBOLS

\( T \) - Kinetic energy of the lumped mass model of the crankshaft

\( V \) - Potential energy of the lumped mass model of the crankshaft or cylinder capacity

\( \Phi \) - Dissipation function (Rayleigh’s dissipation function)

\( \alpha \) - Column vector of generalized coordinates for each lumped mass

\( \dot{\alpha} \) - Column vector of generalized velocities of equivalent masses of crankshaft dynamic system

\( t \) - Time

\( \mathbf{t}_e \) - Row vector of excitation forces in generalized coordinates

\( \mathbf{J} \) - Moments of inertia matrix of the lumped mass model of the crankshaft or Jacobian matrix

\( \mathbf{C} \) - Stiffness matrix of the lumped mass model of the crankshaft

\( \mathbf{K} \) - Damping matrix of the lumped mass model of the crankshaft
\( \omega_0 \) - Mean angular speed of the crankshaft
\( \varphi \) - Column vector of the angle of nonuniform motion
\( J_{\text{const}} \) - Moments of inertia matrix of equivalent crankshaft system with constant terms not being functions of crankshaft position \( \omega_f \)

\( i \) - Imaginary unit \( \sqrt{-1} \)
\( i \) - Unit column vector
\( M \) - Matrix of inertial coefficients of the system
\( Q^A \) - Column vector of generalized forces (all external forces, torques and Coriolis force)
\( q \) - Vector of generalized coordinates of global dynamic system
\( b \) - Vector of unknown parameters of global dynamic system
\( d \) - Step while searching for optimal solution

\( \mathcal{M}(r, b) \) - Mathematical model of global dynamic system
\( p \) - Number of unknown parameters of the system (parameters which have been identified)
\( y_m \) - Exit from mathematical model (for instance, angular velocity of the pulley acquired by calculation based on mathematical model \( \mathcal{M} \))

\( y_i \) - Results acquired by measuring (for instance, measured angular velocity of the pulley)
\( r_i \) - Difference between measured and mathematically modelled value
\( r \) - Vector of differences between measured and mathematically modeled values
\( n_m \) - Number of measured values
\( F \) - Function of goal