First order dipolar phase transition in the Dicke model with infinitely coordinated frustrating interaction

S. I. Mukhin\textsuperscript{1,2} and N. V. Gnezdilov\textsuperscript{1}
\textsuperscript{1}Instituut-Lorentz, Universiteit Leiden, P.O. Box 9506, 2300 RA Leiden, The Netherlands
\textsuperscript{2}Theoretical Physics and Quantum Technologies Department, NUST “MISIS”, Leninski Avenue 4, 119991 Moscow, Russia

(Dated: February 2, 2022)

We found analytically a first order quantum phase transition in the Cooper pair box array of $N$ low-capacitance Josephson junctions capacitively coupled to a resonant photon in a microwave cavity. The Hamiltonian of the system maps on the extended Dicke Hamiltonian of $N$ spins one-half with infinitely coordinated antiferromagnetic (frustrating) interaction. This interaction arises from the gauge-invariant coupling of the Josephson junctions phases to the vector potential of the resonant photon field. In $N \gg 1$ semiclassical limit, we found a critical coupling at which ground state of the system switches to the one with a net collective electric dipole moment of the Cooper pair boxes coupled to superradiant equilibrium photonic condensate. This phase transition changes from the first to second order if the frustrating interaction is switched off. A self-consistently ‘rotating’ Holstein-Primakoff (RHP) representation for the Cartesian components of the total superspin is proposed, that enables to trace both the first and the second order quantum phase transitions in the extended and standard Dicke models respectively.

I. INTRODUCTION

Realization of the equilibrium photonic condensates is of great interest for the fundamental study of a new states of light strongly coupled to quantum metamaterials\textsuperscript{1–6}. In particular, cavity quantum electrodynamics of superconducting qubits is crucial for the quantum computation perspectives\textsuperscript{7–10}. In quantum optics, e.g. in the cavity QED described by the famous Dicke model\textsuperscript{11}, the ‘no-go’ theorems made those perspectives gloomy\textsuperscript{12–14}, and only dynamically driven condensates are considered\textsuperscript{15–20}. Nevertheless, it was found that in the equilibrium circuit QED systems the ‘no-go’ theorems may not hold\textsuperscript{1,5}. In particular, an array of capacitively coupled Cooper pair boxes to a resonant cavity was proven to disobey the ‘no-go’ theorem for an equilibrium superradiant quantum phase transition\textsuperscript{5}. Nevertheless, another complication was found in this case, i.e. it was demonstrated\textsuperscript{6,21,22}, that allowance for the gauge invariance with respect to the electromagnetic vector potential of the photon field causes Hamiltonian of the system to map on the extended Dicke model Hamiltonian of (pseudo)spins one-half, adding to the standard Dicke model a frustrating infinitely coordinated antiferromagnetic interaction between the spins. Lately, a numerical diagonalization results for small clusters with $N$ spins were reported\textsuperscript{5} to behaved differently depending on the parity of the number of spins $N$.

Motivated by the above history of the extended Dicke model exploration, we present in this paper analytic description of the superradiant equilibrium quantum phase transition in the an array of $N \gg 1$ Cooper pair boxes strongly coupled to a resonant cavity. The plan of the present paper is as follows.

First, we reproduce derivation\textsuperscript{6,21} of the extended Dicke Hamiltonian with infinitely coordinated antiferromagnetic (frustrating) term. Next, we confirm the absence of the zero modes in the spectrum of the bosonic excitations, as was found in\textsuperscript{6}. Then, we introduce a new representation for the operators of Cartesian components of the total spin (‘superspin’) of $N$ spins $1/2$: a self-consistently rotating Holstein-Primakoff (RHP) representation. After that, we demonstrate that RHP method applied to extended Dicke Hamiltonian reveals the first order quantum phase transition for the Dicke Hamiltonian without frustrating interaction term, found earlier by other methods\textsuperscript{23,24}. We discuss a drastic difference between the critical values of the coupling strength $g_c$ in the $N \rightarrow \infty$ limit for the 1st and 2nd order phase transitions. In the Summary we present some evaluations of the parameters of a possible experimental setup for a validation of our theoretical predictions for Cooper pair box arrays in a microwave cavity.

II. DICKE HAMILTONIAN FOR A COOPER PAIR BOXES ARRAY

In this section we present a derivation of the extended Dicke model Hamiltonian. We consider a single mode electromagnetic resonant cavity of a linear dimension $L$ coupled to the array of $N$ independent dissipationless Josephson junctions. It is assumed that the wavelength $\lambda$ of the cavity’s resonant photon is much greater than the inter-junction distance: $\lambda \gg L/N$. The vector potential of the electromagnetic field related with the photon is expressed in the second quantized form:

\[ \vec{A} = \sqrt{\frac{2\hbar}{\omega V}} (\hat{a} + \hat{a}^\dagger) \vec{\epsilon}, \]

where $\vec{\epsilon}$ is the unit vector in the direction of the electric field of the photon.
where \( h \) - is Planck's constant, \( \omega \) is bare photon frequency, the photon creation and annihilation bosonic operators are \( \hat{a}^\dagger, \hat{a} \), \( \hat{e}' \) is polarization of the electric field, \( c \) is velocity of light, and \( V \) is the volume of the cavity.

The Hamiltonian of the Cooper pair box array in the cavity then reads:

\[
\hat{H} = \hat{H}_{ph} + \hat{H}_{JJ},
\]

\[
\hat{H}_{ph} = \frac{1}{2} \left( \hat{p}^2 + \phi^2 \right),
\]

\[
\hat{H}_{JJ} = E_C \sum_{i=1}^{N} \hat{n}_i^2 - E_J \sum_{i=1}^{N} \cos \left( \phi_i - \frac{g}{\hbar} \hat{q} \right),
\]

where the coupling constant is \( g = 2el\sqrt{\pi}/\sqrt{V}, \) and \( l \) is of the order of a penetration depth of an electric field into the superconducting islands forming the Josephson junction, thus, giving the effective thickness across of \( \pi/22 \).

For simplicity, we consider all the junctions being identical, with electric field polarization \( \hat{e}' \) aligned across a Josephson junction. Here the two mutually commuting sets of the conjugate variables are introduced: \( [\hat{p}, \hat{q}] = -i\hbar \) and \( [\hat{n}_i, \phi_i] = -i \). The second quantized (harmonic oscillator) variables of the photonic field are:

\[
\hat{p} = i\sqrt{\frac{\hbar \omega}{2}} (\hat{a}^\dagger - \hat{a}) \quad \text{and} \quad \hat{q} = \sqrt{\frac{\hbar \omega}{2}} (\hat{a}^\dagger + \hat{a}),
\]

where \([\hat{a}, \hat{a}^\dagger] = 1\). An operator 2\( e\hat{n}_i = 2e(\hat{n}_i^R - \hat{n}_i^L)/2 \) stands for half of a charge difference at the \( i \)-th junction, and equals half the difference of the number of Cooper pairs populating the left and right islands of a Josephson junction accordingly, multiplied by the elementary charge \( 2e \) of the Cooper pair. The quantum of the charging energy of a single junction is \( E_C = (2e)^2/2C \).

Following\(^{21}\) we make a canonical transformation:

\[
\hat{\phi}'_i = \hat{\phi}_i - \frac{g}{\hbar} \hat{q} \quad \text{and} \quad \hat{n}'_i = \hat{n}_i
\]

for the JJ’s variables and

\[
\hat{p}' = \hat{p} + g \sum_{i=1}^{N} \hat{n}_i \quad \text{and} \quad \hat{q}' = \hat{q}
\]

for photonic variables, so that \([\hat{p}', \hat{\phi}'_i] = 0\) and the other commutation relations between all the operators remain intact. The Hamiltonian (2) becomes

\[
\hat{H} = \frac{1}{2} \left( \hat{p}'^2 + \phi'^2 \right) - g \hat{p} \sum_{i=1}^{N} \hat{n}_i + \frac{g^2}{2} \left( \sum_{i=1}^{N} \hat{n}_i \right)^2
\]

\[
+ \sum_{i=1}^{N} \left( E_C \hat{n}_i^2 - E_J \cos \hat{\phi}_i \right),
\]

where the primes in the new variables are omitted for brevity. Thus, the infinitely coordinated interaction term \( \propto g^2 \) has appeared in (8) after the canonical transformation of the Hamiltonian (2).

We restrict ourselves to the Cooper pair box limit\(^{25}\), when the charging energy \( E_C \) is large in comparison with Josephson coupling \( E_J \) and the eigenstates of the Hamiltonian (4) in the zero order approximation can be chosen as the eigenstates of the charge difference operators \( \hat{n}_i \). The lowest bare energy level corresponding to the quantum states \( |n_i = -\frac{1}{2} \rangle \) and \( |n_i = \frac{1}{2} \rangle \) is thus twofold degenerate with respect to the direction of the Cooper pair box dipole moment \( \vec{d}_i = 2el\vec{n}_i \hat{e}' \) (\( l \) - is effective thickness of the \( i \)-th JJ). This double-degenerate level is separated from the levels with the greater charge differences by the \( E_C \) gap. The Josephson tunnelling term \( \sim E_J \) lifts the degeneracy and opens a gap between the energy levels of the two states that differ by the wave-function parity \( \pm 1 \) with respect to inversion of the Cooper pair box dipole’s direction. Thus, formed two-level system is naturally described by the Pauli matrices \( \hat{\sigma}_i^\alpha \). On the subset of these lowest energy states the initial Hamiltonian (4) of the Cooper pair box array of \( N \) Josephson junctions is represented by a Hamiltonian of interacting \( N \) spins half:

\[
\hat{H}_{JJ} = \sum_{i=1}^{N} \left( E_C \hat{n}_i^2 - E_J \cos \hat{\phi}_i \right) \approx
\]

\[
= \frac{E_C}{4} \sum_{i=1}^{N} \left( \left| \frac{1}{2} \right\rangle \left\langle \frac{1}{2} \right| + \left| -\frac{1}{2} \right\rangle \left\langle -\frac{1}{2} \right| \right) - \frac{E_J}{2} \sum_{i=1}^{N} \left( \left| \frac{1}{2} \right\rangle \left\langle -\frac{1}{2} \right| + \left| -\frac{1}{2} \right\rangle \left\langle \frac{1}{2} \right| \right) \]

\[
= \frac{NE_C}{4} - \frac{E_J}{2} \sum_{i=1}^{N} \hat{\sigma}_i^x.
\]

Here charge and phase difference operators \( \hat{n}_i \) and \( \cos \hat{\phi}_i \) are projected on \( \hat{s}_i^x \) and \( \hat{s}_i^x \) correspondingly, where \( \hat{s}_i^x = \frac{1}{2} \hat{\sigma}_i^x \) are spin-1/2 operators expressed via the Pauli matrices. As a result, initial Hamiltonian (8) reduces to the following spin-boson Hamiltonian, modulo energy shift \( NE_C/4 \):

\[
\hat{H} = \frac{1}{2} \left( \hat{p}'^2 + \phi'^2 \right) - g \hat{p} \sum_{i=1}^{N} \hat{s}_i^x
\]

\[
- E_J \sum_{i=1}^{N} \hat{s}_i^x + \frac{g^2}{2} \left( \sum_{i=1}^{N} \hat{s}_i^x \right)^2.
\]

It is important to clarify here the meaning of the spin-boson interaction term in (10), that had emerged when canonical transformation (6, 7) of the initial gauge-invariant Hamiltonian (2) was performed:

\[
-g \hat{p} \hat{s}_i^z = -i \sqrt{\frac{\hbar \omega}{2V}} (\hat{a}^\dagger - \hat{a}) 2e\sqrt{4\pi} \hat{s}_i^z = -\vec{\partial} \vec{d}_i
\]

and represents the energy of the dipole in the electric
field. The electric field operator in (11) is given by
\[ \hat{\mathbf{E}} = i\sqrt{\frac{\hbar \omega}{V}} (\hat{a}^\dagger - \hat{a}) \varepsilon \] (12)
and the dipole moment of the single junction is
\[ \hat{d}_i = 2e\varepsilon \hat{z}_i \varepsilon \].
(13)
The total dipole moment is then
\[ \hat{\mathbf{d}} = \sum_{i=1}^{N} \hat{d}_i = 2e \varepsilon \sum_{i=1}^{N} \hat{z}_i \varepsilon \].
(14)

For convenience of the further calculations we perform a unitary transformation \( U^\dagger \hat{H} U \), where \( U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \) that interchanges operators of the Cartesian components of spin half: \( \hat{s}_z \rightarrow -\hat{s}_y \), \( \hat{s}_y \rightarrow -\hat{s}_z \) and \( \hat{s}_x \rightarrow \hat{s}_x \).

Hence, the final Cooper pair box Hamiltonian, that we are going to explore, becomes:
\[ \hat{H} = \frac{1}{2} (\hat{p}^2 + \omega^2 \hat{q}^2) + g \hat{p} \hat{S}^y - E_J \hat{S}^2 + \frac{g^2}{2} (\hat{S}^y)^2 \] (15)
where we have introduced operators \( \hat{S}^x = \sum_i \hat{s}_x \), \( \hat{S}^y = \sum_i \hat{s}_y \), \( \hat{S}^z = \sum_i \hat{s}_z \) of the total spin components. The total spin \( \hat{S}^2 \) is conserved, because it commutes with (15): \( [\hat{S}^2, \hat{H}] = 0 \). Cooper pairs tunnelling is represented by \( -E_J \hat{S}_z \) term, \( g \hat{p} \hat{S}^y \) is a dipole coupling strength between Cooper pair box and photonic field, and \( (g^2/2) (\hat{S}^y)^2 \) stands for the infinitely coordinated ‘antiferromagnetic’ frustrating term.

III. DIAGONALIZATION OF THE FRUSTRATED DICKE MODEL

A. Tunnelling regime

In this section, we consider the frustrated Dicke Hamiltonian (15) and first assume that at small coupling strength \( g \) the Josepson tunneling term \( -E_J \hat{S}_z \) dominates at zero temperature. Then the superspin is in the large \( S \) sector and hence, one is allowed to use the Holstein-Primakoff transformation\(^{26} \) (HP) in the form:
\[ \hat{S}^x = \hat{b}^\dagger \hat{b}, \]
(16)
\[ \hat{S}^y = i \sqrt{\frac{S}{2}} \left( \hat{b}^\dagger \sqrt{1 - \frac{\hat{b}^\dagger \hat{b}}{2S}} - \sqrt{1 - \frac{\hat{b}^\dagger \hat{b}}{2S}} \hat{b} \right) \simeq i \sqrt{\frac{S}{2}} (\hat{b}^\dagger - \hat{b}), \]
(17)
\[ \hat{S}^z = \sqrt{\frac{S}{2}} \left( \hat{b}^\dagger \sqrt{1 - \frac{\hat{b}^\dagger \hat{b}}{2S}} + \sqrt{1 - \frac{\hat{b}^\dagger \hat{b}}{2S}} \hat{b} \right) \simeq \sqrt{\frac{S}{2}} (\hat{b}^\dagger + \hat{b}), \]
(18)
where \([\hat{b}^\dagger, \hat{b}] = 1 \). The substitution of (16, 17) into (15) gives Hamiltonian of the two linearly coupled harmonic oscillators:
\[ \hat{H} = \omega (\hat{a}^\dagger \hat{a} + \frac{1}{2}) - E_J (S - \hat{b}^\dagger \hat{b}) \]
\[ - \frac{g^2 S \omega}{2} (\hat{a}^\dagger - \hat{a}) (\hat{b}^\dagger - \hat{b}) - \frac{g^2 S}{4} (\hat{b}^\dagger - \hat{b})^2 \].
(19)
This model also arises in the case of ultrastrong light-matter coupling regime with polariton dots\(^{27} \). Here and in what follows we put \( \hbar = 1 \). With the help of the usual linear Bogoliubov’s transformation of the creation/annihilation operators (see appendix A) we obtain diagonalized Hamiltonian:
\[ \hat{H} = -E_J (S + \frac{1}{2}) + \frac{1}{2} (\varepsilon_1 + \varepsilon_2) \]
\[ + \varepsilon_1 \hat{c}_1^\dagger \hat{c}_1 + \varepsilon_2 \hat{c}_2^\dagger \hat{c}_2 \]
(20)
with the excitations spectrum described by the new oscillator frequencies:
\[ 2\varepsilon_{1,2}^2 = E_J (E_J + g^2 S) + \omega^2 \]
\[ \pm \sqrt{(E_J (E_J + g^2 S) - \omega^2)^2 + 4\omega^2 g^2 SE_J} \],
(21)
where the frequencies \( \varepsilon_{1,2} \) have to be chosen positive to keep the hermiticity of the initial operators \( \hat{a}, \hat{g}, \hat{S}_y \). In contrast with the Dicke model without frustration\(^{23} \) both energy branches are real in the whole range of the coupling constants \( g \), but with a caveat. Namely, the ground state energy equals:
\[ E_0(S) = -E_J \left( S + \frac{1}{2} \right) + \frac{1}{2} (\varepsilon_1 + \varepsilon_2). \]
(22)

This ground state is stable as long as the ground state energy \( E_0(S) \) (22) has global minimum as a function of the superspin \( S \) at the end of the interval \([0, N/2]\). One can find value of the coupling strength \( g = \tilde{g} \), at which the minimum becomes double degenerate via solving equation \( E_0(S = N/2, g = \tilde{g}) = E_0(S = 0, g = \tilde{g}) = \omega/2 \):
\[ \tilde{g} \simeq \sqrt{2E_J N} + (E_J + \omega) \sqrt{\frac{2}{NE_J}}, \]
(23)
which can be easily derived from large \( g \) asymptotic expression of \( E_0(S) \):
\[ E_0 \simeq -E_J \left( S + \frac{1}{2} \right) + \frac{g}{2} \sqrt{E_J S}. \]
(24)
For \( g > \tilde{g} \) the minimum of \( E_0(S) \) migrates from \( S = N/2 \) to \( S = 0 \), see Figure 1. This ‘jump’ of the minimum obviously makes ground state \( S = N/2 \) unstable and leads to an inapplicability of the quasi-classical HP approximation. Thus, our large \( S \) ground state description (22) is justified for \( g < \tilde{g} \).
FIG. 1: Ground state energy as a function of the superspin \( S \) at fixed dimensionless coupling constant \( \lambda = g\sqrt{N/2E_f} \). The blue dashed line shows the double degenerate minimum of the ground state at the coupling strength \( \lambda = \lambda_c \) (23).

B. Rotating Holstein-Primakoff representation

In order to make continuation of the theory into the stronger coupling strength region outside the interval \( g < g_c \) we substitute in the Hamiltonian (15) the \( y, z \) components of the total spin operators with a generalised expression of the Holstein-Primakoff representation in a coordinate frame rotated by an angle \( \theta \) in the \( y \)-\( z \) plane:

\[
\begin{aligned}
\hat{S}^z &= \hat{J}^z \cos \theta - \hat{J}^y \sin \theta \\
\hat{S}^y &= \hat{J}^z \sin \theta + \hat{J}^y \cos \theta
\end{aligned}
\]  

(25)

Here the set of operators of the Cartesian projections of the total spin \( \hat{J}^x,y,z \) are

\[
\begin{aligned}
\hat{J}^z &\approx S - \hat{b}^\dagger \hat{b} \\
\hat{J}^y &\approx i \sqrt{\frac{S}{2}} (\hat{b}^\dagger - \hat{b}) \\
\hat{J}^x &\approx \sqrt{\frac{S}{2}} (\hat{b}^\dagger + \hat{b})
\end{aligned}
\]  

(26)

To find \( \theta \neq 0 \) solution that diagonalises (15) we introduce a shift, \( i \sqrt{\alpha} \), of the photon creation/annihilation operators, similar to\(^{23} \), in the following way:

\[
\begin{aligned}
\hat{a}^\dagger &= \hat{c}^\dagger - i\sqrt{\alpha} \\
\hat{a} &= \hat{c} + i\sqrt{\alpha}
\end{aligned}
\]  

(27)

thus, envisaging formation of a superradiant state. One substitutes (25 - 27) into (15) and the Hamiltonian, quadratic in operators \( c, c^\dagger, b, b^\dagger \), becomes:

\[
\hat{H} = \omega \left( \hat{c}^\dagger \hat{c} + \frac{1}{2} \right) - E_f \cos \theta \left( S - \hat{b}^\dagger \hat{b} \right) - \frac{g \cos \theta \sqrt{\omega}}{2} \left( \hat{c}^\dagger - \hat{c} \right) \left( \hat{b}^\dagger - \hat{b} \right) - \frac{g^2 \cos^2 \theta}{4} S \left( \hat{b}^\dagger - \hat{b} \right)^2,
\]  

(28)

where an elimination of the linear in \( \hat{c}^\dagger - \hat{c} \) and \( \hat{b}^\dagger - \hat{b} \) terms in the Hamiltonian introduces a system of the two equations:

\[
\begin{aligned}
\sqrt{2} \omega \alpha + g \sin \theta \left( S - \langle \hat{b}^\dagger \hat{b} \rangle \right) &= 0; \\
E_f \sin \theta + g \cos \theta \sqrt{2} \omega \alpha + g^2 \cos \theta \sin \theta \left( S - \langle \hat{b}^\dagger \hat{b} \rangle - \frac{1}{2} \right) &= 0.
\end{aligned}
\]  

(29)

We have also made in (28) a mean-field decoupling of the products that are higher than quadratic in \( b, b^\dagger \)-operators: \( \hat{b}^\dagger \hat{b} \). The total of these three terms proves to be zero. This \( \alpha \)-independent cancellation, actually, stems from the degeneracy of the energy minima of the diagonal in spin operators part of the extended Dicke Hamiltonian (15) with respect to \( 2S + 1 \) different \( \hat{S}^y \) projections and classical part \( \sim \sqrt{\alpha} \) of the photonic operator \( \hat{p} \).

C. Superradiant dipolar regime

The structure of (28) is the same as (19), though with coefficients renormalised with prefactor \( \cos \theta \) due to RHP rotation by an angle \( \theta \). Hence, after the Bogoliubov’s transformation similar to the one already described in the Appendix A, the diagonalized Hamiltonian expressed
with the positive eigenvalues $\tilde{\varepsilon}_{1,2}$

$$2\tilde{\varepsilon}_{1,2} = \frac{g^4 \cos^4 \theta}{2} \left(S + \frac{1}{2}\right) + \omega^2$$

$$\pm \sqrt{\left(\frac{g^4 \cos^4 \theta}{2} \left(S + \frac{1}{2}\right) - \omega^2\right)^2 + 2S \omega^2 g^4 \cos^4 \theta}$$

and the ground state energy

$$\bar{E}_0(S) = -\frac{g^4 \cos^2 \theta}{2} \left(S + \frac{1}{2}\right) + \frac{1}{2} (\tilde{\varepsilon}_1 + \tilde{\varepsilon}_2).$$

The stability of the large $S$ state in this regime is provided by the negative slope of $\bar{E}_0(S)$ as a function of $S$ (see Figure 2) in the strong coupling limit:

$$\left.\bar{E}_0(S)\right|_{g \to +\infty} \simeq \frac{\omega}{2} - \frac{2E_j^2}{g^2} \left(S + \frac{1}{2}\right).$$

![Figure 2: Ground state energy $\bar{E}_0$ as a function of the superspin $S$ at fixed dimensionless coupling constant $\lambda = g\sqrt{N/2E_j}$.](image)

In turn, the spin operators are expressed via $\hat{e}_i$, $\hat{e}^\dagger_i$, $i = 1, 2$ as well:

$$\hat{J}_z = S \left(1 - \frac{\langle \hat{j}^\dagger_1 \hat{j}_1 \rangle}{S}\right) \simeq S,$$

$$\hat{J}_y = -i \frac{E_J \sqrt{S} \sin \delta}{g \sqrt{\tilde{\varepsilon}_1}} \left(\hat{e}_1^\dagger - \hat{e}_1\right) + i \frac{E_J \sqrt{S} \cos \delta}{g \sqrt{\tilde{\varepsilon}_2}} \left(\hat{e}_2^\dagger - \hat{e}_2\right),$$

$$\hat{J}_x = -i \frac{E_J \sqrt{S} \sin \delta}{g \sqrt{\tilde{\varepsilon}_1}} \left(\hat{e}_1^\dagger + \hat{e}_1\right) + i \frac{E_J \sqrt{S} \cos \delta}{g \sqrt{\tilde{\varepsilon}_2}} \left(\hat{e}_2^\dagger + \hat{e}_2\right).$$

The Bogoliubov’s ‘angle’ $\delta$ can be found from the consistency relation:

$$\tan 2\delta = \frac{2\sqrt{2S} \omega g^2 \cos^2 \theta}{g^4 \cos^4 \theta \left(S + \frac{1}{2}\right) - 2\omega^2}. \quad (43)$$

We find for $g \geq \sqrt{2E_j}$ the following non-zero expectation values in the ground state of Hamiltonian (34). For the electric field $\tilde{E}$:

$$\left\langle \hat{\delta} \cdot \hat{\epsilon} \right\rangle \sqrt{\frac{V}{4\pi}} = \left\langle \hat{p} \right\rangle = \sqrt{2\omega \alpha} \simeq gS \sin \theta = \mp gS \sqrt{1 - \frac{4E_j^2}{g^4}}. \quad (44)$$

for the modulus of the Josephson tunnelling energy of the Cooper pairs (it decreases):

$$-E_J \langle \hat{S}^z \rangle = -E_J \langle \hat{J}^z \rangle \cos \theta \simeq -S \frac{2E_j^2}{g^2}; \quad (45)$$

and a for the emergent finite mean value of the dipole moment:

$$\langle \hat{d} \rangle = 2e\langle \hat{S}^y \rangle = 2e\langle \hat{J}^z \rangle \sin \theta \simeq \pm 2eS \sqrt{1 - \frac{4E_j^2}{g^4}.} \quad (46)$$

Hence, results (44) and (46) indicate that upon an increase of the coupling strength $g > \sqrt{2E_j}$ there is a state with the energy given in (36), which is characterized by an emergent superradiant electromagnetic field $\langle \hat{p} \rangle \neq 0$ in the cavity together with a finite dipole moment of the Cooper pair boxes: $\langle \hat{d} \rangle \neq 0$. The latter means that rotation angle $\theta$ introduced in (25) regulates an extent of a Cooper pair wave function between the superconducting islands forming each Josephson junction in the Josephson junction array, see Figure 3. Namely, when $\theta$ progressively deviates from zero, the Cooper pairs become localized in one of the two superconducting islands constituting a given Josephson junction, and as a result, the latter acquires a dipole moment.
d=0, θ=0                          d=el, θ=π/2                          d=-el, θ=-π/2

FIG. 3: Schematic layout of the amplitude distributions of the Cooper pair’s wave function in the adjacent islands of a single JJ and corresponding dipole moment values depending on the rotation angle θ, see text and equations (25).

IV. FIRST ORDER DIPOLAR PHASE TRANSITION

In this section we calculate a critical coupling $g_c$, at which a first order phase transition between the tunnelling and dipolar states described in Sections III A and III C takes place.

In Figure 4 we plotted ground state energies calculated for tunnelling and dipolar states as functions of coupling $g$: $E_0(S)$ and $E_0(S)$, see (22) and (36) correspondingly. A dimensionless coupling constant $\lambda = g\sqrt{N/2E_J}$ is used. In the strong coupling limit, $g \gg \sqrt{2E_J}$, the $g$ dependence of the both branches of energy is very well approximated by (24) and (37).

Hence, in the thermodynamic limit $N \to \infty$, solution of $E_0 = \tilde{E}_0$ gives the critical value $g_c$ of the coupling constant:

$$g_c \simeq \sqrt{2E_JN + \omega \sqrt{\frac{2}{NE_J}}}.$$  (47)

Here a crucial difference with respect to (23) is that the critical point corresponds to $\lambda_c \approx N$ and not 1 as in the standard Dicke model without frustration. Hence, transition now is size-dependent, where the ‘size’ of the system is the total number $N$ of Cooper pair box’s inside the microwave cavity.

At $\lambda = \sqrt{N}$, i.e. $g = \sqrt{2E_J}$, the ground state becomes degenerate and a dipolar branch $\tilde{E}_0(S)$ first appears. For $\sqrt{N} < \lambda < N$, i.e. $\sqrt{2E_J} < g < g_c$, the dipolar state minimal energy $\tilde{E}_0(S)$ is higher then the tunnelling ground state energy $E_0(S)$. Hence, the system remains in the tunnelling state (i.e. dipolar disordered). At $\lambda = \lambda_c$ the ground state energy $E_0(S)$ crosses the dipole state energy branch $\tilde{E}_0(S)$ for the second time and goes above $E_0(S)$. At the critical coupling $g = g_c$ (i.e. $\lambda = \lambda_c$) the first order phase transition from the tunnelling state to dipolar ordered state takes place. It is, indeed, a first order transition, since at $g = g_c$ dipole moment in the dipolar state is already finite: $\langle \hat{d} \rangle \approx \pm 2e\lambda S$, see (46), while in the tunnelling state it equals zero. Namely, the first order phase transition results in

$$\langle \hat{p} \rangle = -\frac{Sg}{g_c} \sin \theta = \begin{cases} 0, & g < g_c \\ \pm \frac{Sg}{g_c} \sqrt{1 - 4E_J^2/g^2}, & g \geq g_c \end{cases}$$

see Figure 5, and

$$-E_J \langle \hat{S}^z \rangle = -SE_J \cos \theta = \begin{cases} -SE_J, & g < g_c \\ -E_J^2 2S/g^2, & g \geq g_c \end{cases},$$

$$\langle \hat{d} \rangle = 2e\lambda S \sin \theta = \begin{cases} 0, & g < g_c \\ \pm 2e\lambda S \sqrt{1 - 4E_J^2/g^2}, & g \geq g_c \end{cases}.$$  (50)

The collective dipole moment (50) is defined by the angle $\theta$, which is shown in the Figure 6.

It is important to mention here, that, comparison of (47) with (23) gives: $\tilde{g} - g_c = \sqrt{2E_J/N} > 0$. Hence, we have found the first order phase transition in the region of validity (i.e. $g < \tilde{g}$) of the large superspin limit $S = N/2 \gg 1$, that justifies the use of the HP approach. In the limit $g \to +\infty$ the dipolar ordered ground state
 FIG. 6: The angle $\theta$, that characterizes rotation of HP, as a function of dimensionless coupling constant $\lambda = g\sqrt{N/2E_J}$. The colour scheme is chosen the same as for the Figure 5.

The energy $\tilde{E}_0$ approaches from below the ground state energy of a free resonant photon, $\omega/2$. Simultaneously, at $g = g_c$ the ground state energy: $\tilde{E}_0(S) = 0 < \omega/2$. Hence, our semiclassical description indicates that after the dipole transition the system gradually approaches decoupled state $\tilde{E}_0(S) = \omega/2$, but with saturated value of the collective dipole moment $\propto \langle \hat{S}_y \rangle \rightarrow N/2$. It is not possible to decide in the framework of our semiclassical approach whether a crossover to a state $\langle \hat{S}_y \rangle = 0$ happens in the $g \rightarrow +\infty$ limit. The latter state was predicted numerically in finite, even $N$ spin-1/2 cluster realization of the extended Dicke model.

The excitations branches (35) of the diagonalized Hamiltonian are shown on the Figures 7, 8.

FIG. 7: Excitation branches $\varepsilon_1$, $\hat{\varepsilon}_1$ (21, 35) as a function of dimensionless coupling constant $\lambda = g\sqrt{N/2E_J}$. The vertical axis is shown in the logarithmic scale. At the critical coupling $\lambda_c \approx N$ the frequency $\varepsilon_1$ jumps to $\hat{\varepsilon}_1 \approx \omega$. The color scheme is the same as in Figure 4.

The branch $\varepsilon_1$, that grows with the increase of the coupling, goes to the initial photon’s frequency $\omega$ after the first order transition. $\varepsilon_2$ approaches zero in the strong coupling limit.

Combining together (47) and expression $g = 2el\sqrt{4\pi/(\sqrt{V})}$, one may formulate the condition for the dipolar quantum phase transition as:

$$4el\sqrt{\pi/V} = \sqrt{2NE_J},$$

where $l$ is a penetration depth of electric field into Cooper pair box superconducting island and $V$ is volume of the microwave cavity. Taking into account, that charging energy $E_C = (2e)^2/2C$ is of order: $E_C = 2e^2/l$, one may rewrite (51) in the following form:

$$L \approx l \frac{E_C l^2}{E_J N \Sigma},$$

where $\Sigma$ and $L$ are wave-guide (microwave cavity) cross-section area and length respectively. Assuming $L \approx Nl$ we finally find the following condition:

$$N^2 \approx E_C l^2 \frac{E_J}{E_J \Sigma}.$$  

Hence, we come to a similar conclusion (see Figure 9) as was already made in, that in order to achieve strong coupling limit $g \geq g_c$ for a Cooper pair box array of a 'thermodynamic size' $N \approx 10^2$ inside a microwave resonator, a coplanar geometry with one-dimensional superconducting transmission line (stripline resonator) should be used, thus providing inequality $S/l^2 \ll 1$, and Cooper pair box should have charging energy much greater than Josephson coupling energy: $E_C \gg E_J$.

V. CONCLUSIONS

In summary, we have demonstrated that strong enough capacitive coupling of the Cooper pair box array of low-capacitance Josephson junctions to a microwave resonant photon may lead to a first order quantum phase transition. As a result, a dipolar ordered state of Cooper pairs is formed, coupled to emerged coherent photonic condensate. The phase transition is of the first order due to infinitely coordinated antiferromagnetic (frustrating)
interaction, that arises between Cooper pair dipoles of different Cooper pair boxes. This frustrating interaction is induced by the gauge-invariant coupling of the Josephson junctions to the resonant photon vector-potential in the microwave cavity. The strength of the coherent electromagnetic radiation field that emerges under the phase transition is proportional to the number $N$ of the Cooper pair boxes in the array and is reminiscent of the superradiant state of the Dicke model without frustrating term found previously \(^{23}\). Nevertheless, the phase transition into the latter state is of second order \(^{23}\) (see also Figure 10 in the Appendix B).

The analytical description of the first order quantum phase transition in the Dicke model with infinitely coordinated antiferromagnetic frustrating interaction has become possible due to a new analytic tool: self-consistently ‘rotating’ Holstein-Primakoff representation for the Cartesian components of the total spin, which is described in this paper. Our approach enables, as a by-product, description of the second order quantum phase transition in the Dicke model without frustrating antiferromagnetic interaction, explored previously by other authors \(^{23}\). Nevertheless, ‘rotating’ Holstein-Primakoff representation remains to be semiclassical ($S \to \infty$). Therefore, the region of ‘spin liquid’ with $S \sim 1$ is not attainable within this method.

VI. ACKNOWLEDGMENTS

The authors acknowledge illuminating discussions with Carlo Beenakker, Konstantin Efetov and Bernard van Heck during the course of this work. This research was supported by the Netherlands Organization for Scientific Research (NWO/OCW), an ERC Synergy Grant, the Russian Ministry of Education and Science via the Increase Competitiveness Program of NUST MISiS grant No. K2-2017-085, and ‘Goszadaniye’ grant 3.3360.2017/PH.

---

1. Olivier Viehmann, Jan von Delft, and Florian Marquardt, Superradiant Phase Transitions and the Standard Description of Circuit QED, Phys. Rev. Lett. 107, 113602 (2011).
2. S. I. Mukhin and M. V. Fistul, Generation of non-classical photon states in superconducting quantum metamaterials, Supercond. Sci. Technol. 26, 084003 (2013).
3. M. A. Iontsev, S. I. Mukhin, and M. V. Fistul, Double-resonance response of a superconducting metamaterial: Manifestation of nonclassical states of photons, Phys. Rev. B 94, 174510 (2016).
4. Motoaki Bamba, Kunihiro Inomata, and Yasunobu Nakamura, Superradiant Phase Transition in a Superconducting Circuit in Thermal Equilibrium, Phys. Rev. Lett. 117, 173601 (2016).
5. Pierre Nataf and Cristiano Ciuti, No-go theorem for superradiant quantum phase transitions in cavity QED and counter-example in circuit QED, Nature Communications 1, 72 (2010).
6. Tuomas Jaako, Ze-Liang Xiang, Juan Jos Garcia-Ripoll, and Peter Rabl, Ultrastrong-coupling phenomena beyond the Dicke model, Phys. Rev. A 94, 033850 (2016).
7. Alexandre Blais, Ren-Shou Huang, Andreas Wallraff, S. M. Girvin, and R. J. Schoelkopf, Cavity quantum electrodynamics for superconducting electrical circuits: An architecture for quantum computation, Phys. Rev. A 69, 062320 (2004).
8. A. Wallraff, D. I. Schuster, A. Blais, L. Frunzio, R.-S. Huang, J. Majer, S. Kumar, S. M. Girvin, and R. J. Schoelkopf, Strong coupling of a single photon to a superconducting qubit using circuit quantum electrodynamics, Nature 431, 162-167 (2004).
9. J. M. Raimond, M. Brune, and S. Haroche, Manipulating quantum entanglement with atoms and photons in a cavity, Rev. Mod. Phys. 73, 565 (2001).
10. L. DiCarlo, M. D. Reed, L. Sun, B. R. Johnson, J. M. Chow, J. M. Gambetta, L. Frunzio, S. M. Girvin, M. H. Devoret, and R. J. Schoelkopf, Preparation and measurement of three-qubit entanglement in a superconducting circuit, Nature 467, 574-578 (2010).
11. R. H. Dicke, Coherence in Spontaneous Radiation Processes, Phys. Rev. 93, 99 (1954).
12. K. Rzażewski, K. Wódkiewicz, and W. Żakowicz, Phase Transitions, Two-Level Atoms, and the $A^2$ Term, Phys. Rev. Lett. 35, 432 (1975).
13. Iwo Bielanyicki-Birula and Kazimierz Rzążewski No-go theorem concerning the superradiant phase transition in atomic systems, Phys. Rev. A 19, 301 (1979).
14. Jonathan Keeling, Coulomb interactions, gauge invariance, and phase transitions of the Dicke model, Journal of Physics: Condensed Matter 19, 29 (2007).
15. M. C. Cassidy, A. Bruno, S. Rubbert, M. Irfan, J. Kammlhuber, R. N. Schouten, A. R. Akhmerov, and L. P. Kouwenhoven, Demonstration of an ac Josephson junction laser, Science 355, 6328, 939-942 (2017).
16. Hidehiro Asai, Sergey Savel’ev, Shiro Kawabata, and Alexandre M. Zagoskin, Effects of lasing in a one-dimensional quantum metamaterial, Phys. Rev. B 91.
Appendix A: Bogoliubov’s transformation for the frustrated Hamiltonian

Below we show in detail a diagonalization procedure of the Hamiltonian (19). Let’s introduce

\[
p_x = i \frac{1}{\sqrt{2\omega}} (a^\dagger - a) \quad \text{and} \quad \hat{x} = \sqrt{\frac{\omega}{2}} (a^\dagger + a) \tag{A1}
\]

and rewrite (19) in terms of (A1, A2):

\[
\hat{H} = -E_J \left( \frac{1}{2} S + \frac{1}{2} \right) + \frac{1}{2} \left( \hat{x}^2 + \omega^2 \hat{p}_x^2 \right) + \frac{1}{2} \left( \hat{y}^2 + E_J \frac{g^2}{2} \hat{p}_y^2 \right) + \\
\omega g \frac{SE_J}{2} \hat{p}_x \hat{y} + \frac{g^2 SE_J}{2} \frac{E_J}{2} - \frac{g^2 SE_J}{2} \hat{p}_y^2 = -E_J \left( \frac{1}{2} S + \frac{1}{2} \right) + \frac{1}{2} \hat{K}_{xy} + \frac{1}{2} \hat{K}_{p_x p_y}, \tag{A3}
\]

where

\[
\hat{K}_{xy} = \hat{x}^2 + \hat{y}^2, \tag{A4}
\]

\[
\hat{K}_{p_x p_y} = \omega^2 \frac{g^2}{2} + E_J \left( \frac{1}{2} S + \frac{1}{2} \right) \frac{g^2}{2} + 2\omega g \frac{SE_J}{2} \hat{p}_x \hat{y}. \tag{A5}
\]

We diagonalize (A3) by performing a linear transformation of the quantum operators:

\[
\begin{pmatrix}
\hat{p}_x \\
\hat{p}_y
\end{pmatrix} = \begin{pmatrix}
\cos \gamma & \sin \gamma \\
-\sin \gamma & \cos \gamma
\end{pmatrix}
\begin{pmatrix}
\hat{p}_1 \\
\hat{p}_2
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
\hat{x} \\
\hat{y}
\end{pmatrix} = \begin{pmatrix}
\cos \gamma & \sin \gamma \\
-\sin \gamma & \cos \gamma
\end{pmatrix}
\begin{pmatrix}
\hat{q}_1 \\
\hat{q}_2
\end{pmatrix}. \tag{A6}
\]

Then

\[
\hat{K}_{xy} = \hat{q}_1^2 + \hat{q}_2^2, \tag{A7}
\]

and

\[
\begin{align*}
\hat{K}_{p_x p_y} &= \left( \omega^2 \frac{g^2}{2} + E_J \left( \frac{1}{2} S + \frac{1}{2} \right) \sin 2\gamma - 2\omega g \frac{SE_J}{2} \sin \gamma \cos \gamma \right) \hat{p}_1^2 + \\
&+ \left( \omega^2 \sin^2 \gamma + E_J \left( \frac{1}{2} S + \frac{1}{2} \right) \cos^2 \gamma + 2\omega g \frac{SE_J}{2} \sin \gamma \cos \gamma \right) \hat{p}_2^2 + \\
&+ \left( \omega^2 - E_J \left( \frac{1}{2} S + \frac{1}{2} \right) \sin 2\gamma + 2\omega g \frac{SE_J}{2} \cos 2\gamma \right) \hat{p}_1 \hat{p}_2. \end{align*} \tag{A8}
\]
The diagonalization condition that eliminates the cross-term $\sim \hat{p}_1 \hat{p}_2$, is:
\[
\tan 2\gamma = \frac{2\gamma \sqrt{SE_J}}{E_J (E_J + g^2 S) - \omega^2}.
\] (A9)

So, diagonalized operator $\hat{K}_{p_x p_y}$ becomes:
\[
\hat{K}_{p_x p_y} = \varepsilon_1^2 \hat{p}_1^2 + \varepsilon_2^2 \hat{p}_2^2,
\] (A10)

where:
\[
\begin{align*}
2\varepsilon_1^2 &= E_J (E_J + g^2 S) + \omega^2 - (E_J (E_J + g^2 S) - \omega^2) \cos 2\gamma - 2\gamma \sqrt{SE_J} \sin 2\gamma, \\
2\varepsilon_2^2 &= E_J (E_J + g^2 S) + \omega^2 + (E_J (E_J + g^2 S) - \omega^2) \cos 2\gamma + 2\gamma \sqrt{SE_J} \sin 2\gamma.
\end{align*}
\] (A11, A12)

Substitution of (A9) to (A11) and (A12) gives the eigenvalues:
\[
2\varepsilon_{1,2}^2 = E_J (E_J + g^2 S) + \omega^2 \pm \sqrt{(E_J (E_J + g^2 S) - \omega^2)^2 + 4\omega^2 g^2 SE_J}.
\] (A13)

The transformation:
\[
\hat{p}_{1,2} = i \sqrt{\frac{2\varepsilon_{1,2}}{1}} \left( \hat{c}_{1,2}^\dagger - \hat{c}_{1,2} \right) \quad \text{and} \quad \hat{q}_{1,2} = \sqrt{\frac{\varepsilon_{1,2}}{2}} \left( \hat{c}_{1,2}^\dagger + \hat{c}_{1,2} \right)
\] (A14)

finally gives the diagonal Hamiltonian (20).

The initial operators $\hat{a}$ and $\hat{b}$ are expressed via the new operators $\hat{c}_{1,2}^\dagger$ as:
\[
a = \sqrt{\frac{\omega}{\varepsilon_1}} \cos \gamma \hat{c}_1 + \sqrt{\frac{\omega}{\varepsilon_2}} \sin \gamma \hat{c}_2
\] (A15)

and
\[
b = -\sqrt{\frac{E_J}{\varepsilon_1}} \sin \gamma \hat{c}_1 + \sqrt{\frac{E_J}{\varepsilon_2}} \cos \gamma \hat{c}_2,
\] (A16)

where $\gamma$ is defined in (A9).

**Appendix B: Second order quantum phase transition in the Dicke model within the RHP method**

We consider the standard Dicke Hamiltonian\ref{11,23} (modulo our notations)
\[
\hat{H} = \frac{1}{2} (\hat{p}^2 + \omega^2 \hat{q}^2) + g \hat{p} \hat{S}^y - E_J \hat{S}^z
\] (B1)

at small coupling $g$. We apply (16, 17) to B1:
\[
\hat{H} = -E_J \left( S + \frac{1}{2} \right) + \omega \left( \hat{a}^\dagger \hat{a} + \frac{1}{2} \right) + E_J \left( \hat{b}^\dagger \hat{b} + \frac{1}{2} \right) - \frac{g \sqrt{SE_J} \omega}{2} (\hat{a}^\dagger - \hat{a}) (\hat{b}^\dagger - \hat{b}).
\] (B2)

The Bogoliubov’s transformation, similar to those in the Appendix A, gives:
\[
\hat{H} = -E_J \left( S + \frac{1}{2} \right) + \varepsilon_1 \left( \frac{1}{2} + \hat{c}_{1}^\dagger \hat{c}_1 \right) + \varepsilon_2 \left( \frac{1}{2} + \hat{c}_{2}^\dagger \hat{c}_2 \right),
\] (B3)

with the excitations spectrum described by the new oscillator frequencies:
\[
2\varepsilon_{1,2}^2 = E_J^2 + \omega^2 \pm \sqrt{(E_J^2 - \omega^2)^2 + 4\omega^2 g^2 SE_J}.
\] (B4)
The ground state energy equals:

\[ E_0(S) = -E_J \left( S + \frac{1}{2} \right) + \frac{1}{2} (\varepsilon_1 + \varepsilon_2) \]  

(B5)

One can check that as a function of \( S \) the energy \( E_0(S) \) has minimum at \( S = N/2 \), i.e. at the end of the interval of all possible total spin values \( 0 \leq S \leq N/2 \). This fact justifies the Holstein-Primakoff approach (16-18) valid in the large spin limit.

However, the lowest branch of excitations becomes imaginary when \( g > g_c = \sqrt{E_J/S} \):

\[ \varepsilon_2 = \sqrt{\frac{E_J^2 + \omega^2}{2} - \sqrt{\left( \frac{E_J^2 - \omega^2}{2} \right)^2 + 4\omega^2 g J E_J}} \]  

(B6)

Thus, the ground state described above is unstable in the interval \( g > g_c \), compare(23).

The method described in Section III B (25-27) transforms the Hamiltonian \((B2)\) into:

\[
\hat{H} = \omega \left( \hat{c}^{\dagger} \hat{c} + i \sqrt{\alpha} (\hat{c}^{\dagger} - \hat{c}) + \alpha + \frac{1}{2} \right) - E_J \cos \theta \left( \hat{S} - \hat{b}^{\dagger} \hat{b} \right) + \\
+ E_J \sin \theta i \sqrt{\frac{S}{2}} \left( \hat{b}^{\dagger} - \hat{b} \right) - \frac{g \cos \theta \sqrt{S \omega}}{2} (\hat{c}^{\dagger} - \hat{c}) \left( \hat{b}^{\dagger} - \hat{b} \right) + g \cos \theta \sqrt{2 \omega \alpha} i \sqrt{\frac{S}{2}} \left( \hat{b}^{\dagger} - \hat{b} \right) + \\
+ g \sin \theta i \sqrt{\frac{\omega}{2}} (\hat{c}^{\dagger} - \hat{c}) \left( S - \hat{b}^{\dagger} \hat{b} \right) + g \sin \theta \sqrt{2 \omega \alpha} \left( S - \hat{b}^{\dagger} \hat{b} \right). 
\]  

(B7)

Here we have decoupled cubic in \( \hat{c}, \hat{b} \) operators terms in a mean-field approximation. Conditions for vanishing of the linear terms \( \propto (\hat{c}^{\dagger} - \hat{c}) \) and \( \left( \hat{b}^{\dagger} - \hat{b} \right) \) in the Hamiltonian \((B7)\) are:

\[
\sqrt{2 \omega \alpha} + g \sin \theta \left( S - \hat{b}^{\dagger} \hat{b} \right) = 0 \quad \text{(B8)}
\]

\[
E_J \sin \theta + g \cos \theta \sqrt{2 \omega \alpha} = 0 \quad \text{(B9)}
\]

Solving the system of equations \((B8)\) and \((B9)\) we find:

\[
\cos \theta = \frac{E_J}{S g^2} \left( 1 - \frac{\langle \hat{b}^{\dagger} \hat{b} \rangle}{S} \right) \approx \frac{E_J}{S g^2} \equiv \frac{g_c^2}{g^2}, \quad \text{(B10)}
\]

\[
\sqrt{\alpha} = -\frac{g S}{\sqrt{2 \omega}} \left( 1 - \frac{\langle \hat{b}^{\dagger} \hat{b} \rangle}{S} \right) \sin \theta \approx \frac{g S}{\sqrt{2 \omega J}} \sqrt{1 - \frac{g_c^2}{g^2}}, \quad \text{(B11)}
\]

where both the shift \( \sqrt{\alpha} \) and rotation angle \( \theta \) are non-zero when \( g > g_c \). Thus, using solutions \((B10)\) and \((B11)\) we obtain the initial Hamiltonian \((B7)\) in the form similar to \((B2)\), but renormalised with \( \cos \theta \) coefficients :

\[
\hat{H} = \frac{E_J S}{2 \cos \theta} \left( 1 - \cos^2 \theta \right) - \frac{E_J}{\cos \theta} \left( S + \frac{1}{2} \right) + \omega \left( \hat{c}^{\dagger} \hat{c} + \frac{1}{2} \right) + \frac{E_J}{\cos \theta} \left( \hat{b}^{\dagger} \hat{b} + \frac{1}{2} \right) - \\
- \frac{g \cos \theta \sqrt{S \omega}}{2} (\hat{c}^{\dagger} - \hat{c}) \left( \hat{b}^{\dagger} - \hat{b} \right). 
\]  

(B12)

Next, we perform Bogoliubov’s transformation that diagonalizes \((B12)\), by performing a linear transform of Bose-operators \( \hat{c}, \hat{b} \) into Bose-operators \( \hat{e}_{1,2} \), and obtain:

\[
\hat{H} = \frac{E_J S}{2 \cos \theta} \left( 1 - \cos^2 \theta \right) - \frac{E_J}{\cos \theta} \left( S + \frac{1}{2} \right) + \tilde{\varepsilon}_1 \left( \frac{1}{2} + \hat{e}_1 \right) + \tilde{\varepsilon}_2 \left( \frac{1}{2} + \hat{e}_2 \right) 
\]  

(B13)

with the eigenvalues:

\[
2\tilde{\varepsilon}_{1,2}^2 = \frac{E_J^2}{\cos^2 \theta} + \omega^2 \pm \sqrt{\left( \frac{E_J^2}{\cos^2 \theta} - \omega^2 \right)^2 + 4\omega^2 E_J^2}, 
\]  

(B14)
where both branches are now real for $g > g_c = \sqrt{E_J/S}$ due to renormalisation of the coefficients with $\cos \theta$ factors. We have expressed in (B14) the coupling constant $g$ via $\cos \theta$ using the self-consistency relation (B10). As is obvious from (B10) and (B11), both the shift $\sqrt{\alpha}$ and rotation angle $\theta$ progressively deviate from zero with increasing coupling strength $g$ in the interval $g > g_c$, thus, providing a description of the new stable phase of the system.

The ground state energy of the system is now:

$$\bar{E}_0(S) = -\frac{E_J}{2 \cos \theta} (S + 1) - \frac{E_J S}{2} \cos \theta + \frac{1}{2} (\tilde{\epsilon}_1 + \tilde{\epsilon}_2),$$

which always has a minimum at the end of the spin interval, at $S = N/2$, thus justifying the Holstein-Primakoff approximation at finite angles $\theta$.

Thus, we found the second order phase transition that is manifested by a gradual rotation of the total spin expectation value in the $y-z$ plane by an angle $\theta$:

$$\langle \hat{\rho} \rangle = -S g \sin \theta = \begin{cases} 0, & g < g_c \\ \mp S g \sqrt{1 - g_c^4/g^4}, & g \geq g_c \end{cases}$$

and

$$\langle \hat{d} \rangle = 2eS l \sin \theta = \begin{cases} 0, & g < g_c \\ \pm 2e S \sqrt{1 - g_c^4/g^4}, & g \geq g_c \end{cases}$$

where $g_c = \sqrt{2E_J/N}$ and $S = N/2$. The angle $\theta$, that describes the transition is plotted in the Figure 10.

![Graph of angle θ as a function of dimensionless coupling constant λ](image)

**FIG. 10:** The angle $\theta$ as a function of dimensionless coupling constant $\lambda = g \sqrt{N/2E_J}$ in the Dicke model without frustration term.