Most Recent Match Queries in On-Line Suffix Trees
(with appendix)

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Abstract A suffix tree is able to efficiently locate a pattern in an indexed string, but not in general the most recent copy of the pattern in an online stream, which is desirable in some applications. We study the most general version of the problem of locating a most recent match: supporting queries for arbitrary patterns, at each step of processing an online stream. We present augmentations to Ukkonen’s suffix tree construction algorithm for optimal-time queries, maintaining indexing time within a logarithmic factor in the size of the indexed string. We show that the algorithm is applicable to sliding-window indexing, and sketch a possible optimization for use in the special case of Lempel-Ziv compression.

1 Introduction

The suffix tree is a well-known data structure which can be used for effectively and efficiently capturing patterns of a string, with a variety of applications [1, 2, 3]. Introduced by Weiner [4], it reached wider use with the construction algorithm of McCreight [5]. Ukkonen’s algorithm [6] resembles McCreight’s, but has the advantage of being fully online, an important property in our work. Farach [7] introduced recursive suffix tree construction, achieving the same asymptotic time bound as sorting the characters of the string (an advantage for large alphabets), but at the cost inherently off-line construction. The simpler suffix array data structure [8, 9] can replace a suffix tree in many applications, but cannot generally provide the same time complexity, e.g., for online applications.

Arguably the most basic capability of the suffix tree is to efficiently locate a string position matching an arbitrary given pattern. In this work, we are concerned with finding the most recent (rightmost) position of the match, which is not supported by standard suffix trees. A number of authors have studied special cases of this problem, showing applications in data compression and surveillance [10, 11, 12], but to our knowledge, no efficient algorithm has previously been presented for the general case. One of the keys to our result is recent advancement in online suffix tree construction by Breslauer and Italiano [13].

We give algorithms for online support of locating the most recent longest match of an arbitrary pattern P in \(O(|P|)\) time (by traversing \(|P|\) nodes, one of which identifies the most recent position). When a stream consisting of \(N\) characters is subject to search, the data structure requires \(O(N)\) space, and maintaining the necessary position-updated properties takes at most \(O(N \log N)\)
total indexing time. If only the last $W$ characters are subject to search (a sliding window), space can be reduced to $O(W)$ and time to $O(N \log W)$.

In related research, Amir, Landau and Ukkonen [10] gave an $O(N \log N)$ time algorithm to support queries for the most recent previous string matching a suffix of the (growing) indexed string. The pattern to be located is thus not arbitrary, and the data structure cannot support sliding window indexing.

A related problem is that of Lempel-Ziv factorization [14], where it is desirable to find the most recent occurrence of each factor, in order to reduce the number of bits necessary for subsequent encoding. For this special case, Ferragina et al. [11] gave a suffix tree based linear-time algorithm, but their algorithm is not online, and cannot index a sliding window. Crochemore et al. [12] gave an online algorithm for the rightmost equal cost problem, a further specialization for the same application. In section 5, we discuss a possible optimization of our algorithm for the special case of Lempel-Ziv factorization.

2 Definitions and Background

We study indexing a string $T = t_0 \cdots t_{N-1}$ of length $|T| = N$, characters $t_i \in \Sigma$ drawn from a given alphabet $\Sigma$. (We consistently denote strings with uppercase letters, and characters with lowercase letters.) $T$ is made available as a stream, whose total length may not be known. The index is maintained online, meaning that after seeing $i$ characters, it is functional for queries on the string $t_0 \cdots t_{i-1}$. Following the majority of previous work, we assume that $|\Sigma|$ is a constant.

The data structure supports queries for the most recent longest match in $T$ of arbitrary strings that we refer to as patterns. More specifically, given a pattern $P = p_0 \cdots p_{|P|-1}$, a match for a length-$M$ prefix of $P$ occurs in position $i$ iff $p_j = t_{i+j}$ for all $0 \leq j < M$. It is a longest match iff $M$ is maximum, and the most recent longest match iff $i$ is the maximum position of a longest match.

2.1 Suffix Tree Construction and Representation

By $ST$, we denote the suffix tree [2,4,5,6] over the string $T = t_0 \cdots t_{N-1}$. This section defines $ST$, and specifies our representation.

A string $S$ is a nonempty suffix (of $T$, which is implied) iff $S = t_i \cdots t_{N-1}$ for $0 \leq i < N$, and a nonempty substring (of $T$) iff $S = t_i \cdots t_j$ for $0 \leq i \leq j < N$. By convention, the empty string $\epsilon$ is both a suffix and a substring. Edges in $ST$ are directed, and each labeled with a string. Each point in the tree, either coinciding with a node or be located between two characters in an edge label, corresponds to the string obtained by concatenating the edge labels on the path to that point from the root. $ST$ represents, in this way, all substrings of $T$. We regard a point that coincides with a node as located at the end of the node’s edge from its parent, and can thus refer to the points of all represented strings.

It should be noted, however, that ours and previous algorithms can provide the same expected time bounds for non-constant alphabets using hashing, and only a very small worst-case factor higher using efficient deterministic dictionary data structures.
as located on edges. An **external** edge is an edge whose endpoint is a leaf; other edges are **internal**. The endpoint of each external edge corresponds to a suffix of \( T \), but some suffixes may be represented inside the tree. Note that the point corresponding to an arbitrary pattern can be located (or found non-existent) in time proportional to the length of the pattern, by scanning characters left to right, matching edge labels from the root down.

We do not require that \( T \) ends with a unique character, which would make each suffix correspond to some edge endpoint. Instead, we maintain points of implicit suffix nodes using the technique of Breslauer and Italiano [13] (section 3.5).

Following Ukkonen, we augment the tree with an auxiliary node \( \perp \) above the root, with a single downward edge to the root. We denote this edge \( \triangleright \) and label it with \( \epsilon \). (Illustration in figure 1.) Although the root of a tree is usually taken to be the topmost node, we shall refer to the node below \( \perp \) (the root of the unaugmented tree) as the root node of \( ST \).

Apart from \( \triangleright \), all edges are labeled with nonempty strings, and the tree represents exactly the substrings of \( T \) in the minimum number of nodes. This implies that each node is either \( \perp \), the root, a leaf, or a non-root node with at least two downward edges. Since the number of leaves is at most \( N \) (one for each suffix), the total number of nodes never exceeds \( 2N + 1 \).

We generalize the definition to \( ST_i \) over the string \( T = t_0 \cdots t_{i-1} \), where \( ST_N = ST \). In iteration \( i \), we execute Ukkonen’s update algorithm [6] to reshape \( ST_{i-1} \) into \( ST_i \), without looking ahead any further than \( t_{i-1} \). When there is no risk of ambiguity, we refer to the current suffix tree simply as \( ST \), implying that \( N \) iterations have completed.

For downward tree navigation, we maintain \( \text{down}(e, a) = f \) for constant-time access, where \( e \) and \( f \) are adjacent edges such that \( e \)'s endpoint coincides with \( f \)'s start node, and the first character in \( f \)'s label is \( a \). Note that \( a \) uniquely identifies \( f \) among its siblings. We define the string that marks \( f \) as the shortest string represented by \( f \) (corresponds to the point just after \( a \)). We also maintain \( \text{pred}(f) = e \) for constant-time upward navigation.

For linear storage space in \( N \), edge labels are represented indirectly, as references into \( T \). Among the many representations available, we choose the following: For any edge \( e \), we maintain \( \text{pos}(e) \), a position in \( T \) of the string corresponding to \( e \)'s endpoint, and for each **internal** edge \( e \), we maintain \( \text{slen}(e) \), the length of that same string. I.e., \( e \) is labeled with \( t_i \cdots t_j \), where \( i = \text{pos}(e) + \text{slen}(	ext{pred}(e)) \) and \( j = \text{pos}(e) + \text{slen}(e) \). External edges need no explicit \( \text{slen} \) representation, since their endpoints always correspond to suffixes of \( T \), so \( \text{slen}(e) \) for external \( e \) would always be \( N - \text{pos}(e) \). Note that \( \text{pos}(e) \) is not uniquely defined for internal \( e \). Algorithms given in the following sections update \( \text{pos} \) values to allow efficiently finding the most recent occurrence of a pattern.

Ukkonen’s algorithm operates around the **active point**, the point of the longest suffix that also appears earlier in \( T \). This is the deepest point where \( ST \) may need updating in the next iteration, since longer suffixes are located on external edges, whose representations do not change. In iteration \( i \), \( t_i \) is to be incorporated in \( ST \). If \( t_i \) is already present just below the active point, the tree already contains
Figure 1. Suffix tree over the string \textit{abcabda}. Dotted lines show edge-oriented suffix links.

all the suffixes ending at \( t_i \), and the active point simply moves down past \( t_i \). Otherwise, a leaf is added at the old active point, which is made into a new explicit node if necessary, and the active point moves to the point of the next shorter suffix. To make this move efficient, typically jumping to a different branch of the tree, the algorithm maintains a \textit{suffix link} from any node corresponding to \( aA \), for some character \( a \) and string \( A \), directly to the node for \( A \).

We choose a representation where suffix links are edge-oriented, rather than node-oriented as in McCreight’s and Ukkonen’s algorithms: for edges \( e \) and \( f \), we let \( \text{suf}(e) = f \) iff \( A \) marks \( f \), and \( aA \) and is the shortest string represented by \( e \) such that \( A \) marks an edge. (Illustrated in figure 1.) Furthermore, we define \( \text{rsuf} \) to denote the \textit{reverse suffix link}: \( \text{rsuf}(f, a) = e \). We leave \( \text{suf}(\perp) \) undefined. Note that \( aA \) is the string that marks \( e \), unless \( e \) is a downward edge of the root with an edge label longer than one character. We have \( \text{suf}(e) = \perp \) iff \( e \)'s endpoint corresponds to a string of length one. This variant of suffix links facilitates the description of our \textit{most recent match} scheme, but also has practical impact on runtime behavior, due to reduced branch lookup [15]. The change it implies in Ukkonen’s algorithm is relatively straightforward, and has no impact on its asymptotic time complexity. We omit the details in this work.

We refer to the path from the active point to \( \perp \), via suffix links and (possibly) downward edges, as the \textit{active path}. All suffixes that also appear as substrings elsewhere in \( T \) are represented along this path. We refer to those suffixes as \textit{active suffixes}. A key to the \( O(N) \) time complexity of Ukkonen’s algorithm is that the active path is traversed only in the forward direction.

3 Algorithm and Analysis

To answer a most-recent longest-match query for a pattern \( P' \), we first locate the edge \( e \) in \( ST \) that represents the longest prefix \( P \) of \( P' \). For an \textit{exact-match} query, we report failure unless \( P = P' \). The time required to locate \( e \), by traversing edges from the root, while scanning edge labels, is \( O(|P|) \).

In this section, we give suffix tree augmentations that allow computing the most recent match of \( P \) once its edge is located, while maintaining \( O(|P|) \) query time.
Separation of Cases The following identifies two cases in locating the most recent match of a pattern string \( P \), which we treat separately.

**Lemma 1.** Let \( e \) be the edge that represents \( P \), and let the string corresponding to \( e \)'s endpoint be \( PA \), \(|A| \geq 0 \). Precisely one of the following holds:

1. The position of the most recent occurrence of \( P \) is also the position of the most recent occurrence of \( PA \).
2. There exists a suffix \( PB \), \(|B| \geq 0 \) such that \(|B| < |A| \).

(Proof in appendix.) Sections 3.1–3.4 show how to deal with case 1, and section 3.5 with case 2.

### 3.1 Naive Position Updating

We begin with considering a naive method, by which we update \( pos(e) \) at any time when the string corresponding to \( e \)'s endpoint reappears in the input.

Observe that any string that occurs later in \( t_0 \cdots t_{N-1} \) than in \( t_0 \cdots t_{N-2} \), for some \( 0 \leq j \leq N - 1 \). Hence, in each iteration, we need update \( pos(e) \) only if \( e \)'s endpoint corresponds to an active suffix. This immediately suggests the following: after update iteration \( i \), traverse the active path, and for any edge \( e \) whose endpoint corresponds to a suffix, and \( pos-update \( e \), which we define as setting \( pos(e) \) to \( i - \text{slen}(e) \). Thereby, we maintain \( pos(e) \) as the most recent position for any non-suffix represented by \( e \), and whenever case 1 of lemma 3 holds, we obtain the most recent position of \( P \) directly from the \( pos \) value of its edge.

The problem with this naive method is that traversing the whole active path in every iteration results in \( \Omega(N^2) \) worst case time. The following sections describe how to reduce the number of pos-updates, and instead letting the query operation inspect \(|P| \) edges in order to determine the most recent position.

### 3.2 Position Update Strategy

To facilitate our description, we define the link tree \( \mathcal{LT} \) as the tree of \( ST \) edges incurred by the suffix links: edges in \( ST \) are nodes in \( \mathcal{LT} \), and \( f \) is the parent of \( e \) in \( \mathcal{LT} \) iff \( \text{suf}(e) = f \). The root of \( \mathcal{LT} \) is \( - \). In order to keep the relationship between \( ST \) edges and \( \mathcal{LT} \) nodes clear, we use the letters \( e, f, g, \) and \( h \) to denote them in both contexts.

We define \( \text{depth}_{\mathcal{LT}}(e) \) as the depth of \( e \) in \( \mathcal{LT} \). Because of the correspondence between \( \mathcal{LT} \) nodes and \( ST \) edges, we have \( \text{depth}_{\mathcal{LT}}(e) = \text{slen}(\text{pred}(e)) \).

By the current update edge in iteration \( i \), we denote the edge \( e \) such that \( \text{depth}_{\mathcal{LT}}(e) \) is maximum among the edges, if any, that would be updated by the naive update strategy (section 3.1) in that iteration: the maximum-depth \( \mathcal{LT} \) internal edge whose endpoint corresponds to an active suffix. Section 3.5 describes how the update edge can be located in constant time.

Our update strategy includes pos-updating only the update edge, leaving \( pos \) values corresponding to shorter active suffixes unchanged. When no update edge
exists, we pos-update nothing. We introduce an additional value \( \text{repr}(e) \) for each internal edge \( e \), for which we uphold the following property:

**Property 1.** For every node \( g \) in the suffix link tree, let \( e \) be the most recently pos-updated node in the subtree rooted at \( g \). Then an ancestor \( a \) of \( g \) exists such that \( \text{repr}(a) = e \).

By convention, a tree node is both an ancestor and a descendant of itself. For new \( \mathcal{LT} \) nodes \( e \) (without descendants), we set \( \text{repr}(e) \) to \( \top \). We proceed with first the algorithm that exploits property 1, then the algorithm to maintain it.

### 3.3 Most Recent Match Algorithm

Algorithm \texttt{mrm-find}(e) scans the \( \mathcal{LT} \) path from node \( e \) to the root in search for any node \( g \) such that \( f = \text{repr}(g) \) is a descendent of \( e \). For each such \( f \), it obtains the position \( q = \text{pos}(f) + \text{depth}_{\mathcal{LT}}(f) - \text{depth}_{\mathcal{LT}}(e) \), and the value returned from the algorithm is the maximum among the \( q \).

\texttt{mrm-find}(e):
1. Let \( p = \text{pos}(e) \), and \( g = e \).
2. If \( g \) is \( \top \), we are done, and terminate returning the value \( p \).
3. If \( \text{repr}(g) = \top \) (i.e., it has not been set), go directly to step 6.
4. Let \( f = \text{repr}(g) \). If \( e \) is not an ancestor of \( f \) in \( \mathcal{LT} \), go directly to step 6.
5. Let \( q = \text{pos}(f) + \text{depth}_{\mathcal{LT}}(f) - \text{depth}_{\mathcal{LT}}(e) \). If \( q > p \), update \( p \) to \( q \).
6. Set \( g \) to \( \text{suf}(g) \), and repeat from step 2.

The following lemma establishes that when property 1 is maintained, the most recent occurrence of the string corresponding to \( e \)'s endpoint is among the positions considered by \texttt{mrm-find}(e).

**Lemma 2.** For an internal edge \( e \), let \( A \) be the string corresponding to \( e \)'s endpoint, and \( t_i \cdots t_{i-1} \) the most recent occurrence of \( A \) in \( T \). Then \( e \) has a descendent \( f \) in \( \mathcal{LT} \) whose endpoint corresponds to \( BA \) for some string \( B \), and \( \text{pos}(f) = i - |B| - |A| \). (Proof in appendix.)

Since \texttt{mrm-find}(e) returns the maximum among the considered positions, this establishes its validity for finding the most recent position of the string corresponding to \( e \)'s endpoint. Under case 1 of lemma 1, this is the most recent position of any string represented by \( e \). Hence, given that \( e \) represents pattern \( P \), \texttt{mrm-find}(e) produces the most recent position of \( P \) in this case.

**Lemma 3.** Execution time of \texttt{mrm-find}(e), where \( e \) represents a string \( P \) can be bounded by \( O(|P|) \). (Proof in appendix.)
3.4 Maintaining Property

Since $\triangledown$ is an ancestor of all nodes in $LT$, we can trivially uphold property in relation to any updated node $e$ simply by setting $\text{repr}(\triangledown) = e$. But since this ruins the property in relation to other nodes (unless the previous value of $\text{repr}(\triangledown)$ was an ancestor of $e$) we must recursively push the overwritten $\text{repr}$ value down $LT$ to the root of the subtree containing those nodes.

More specifically, when $\text{repr}(r)$ is set to $e$, for some $LT$ nodes $r$ and $e$, let $f$ be the previous value of $\text{repr}(r)$. Then find $h$, the minimum-depth node that is an ancestor of $f$ but not of $e$, and recursively update $\text{repr}(h)$ to $f$. To find $h$, we first locate $g$, the lowest common ancestor of $e$ and $f$. Figure 2 shows the five different ways in which $e$, $f$, $g$, and $h$ can be located in relation to one another. In case a, $h$ lies just under the path between $e$ and the root, implying that we need to set $\text{repr}(h)$ to $f$. We find $h$ via a reverse suffix link from $g$. Cases b (where $f = h$) and c ($g = e$) are merely special cases of the situation in a, and are handled in exactly the same way. In case d ($g = f$), the overwritten $\text{repr}$ value points to an ancestor of $e$, and the process can terminate immediately. Case e is the special case of d where the old and new $\text{repr}$ values are the same.

The following details the procedure. It is invoked as $\text{repr-update}(e, \triangledown)$ in order to reestablish property where $e$ is the current update edge.

$\text{repr-update}(e, r)$:
1. Let $f$ be the old value of $\text{repr}(r)$, and set its new value to $e$.
2. If $f = \triangledown$ (i.e. $\text{repr}(r)$ has not been previously set), then terminate.
3. Let $g$ be the lowest common ancestor of $e$ and $f$.
4. If $g = f$, terminate.
5. Let $h = r\text{su}(g, t_j)$, where $j = \text{pos}(f) + \text{depth}_{LT}(f) - \text{depth}_{LT}(g) - 1$.
6. Recursively invoke $\text{repr-update}(f, h)$.

Correctness of $\text{repr-update}$ in maintaining property is established by the preceding discussion. We now turn to bounding the total number of recursive calls.
Lemma 4. Given a sequence $V = e_1, \ldots, e_N$ of nodes to be updated in a tree $T$ with $M$ nodes, there exists a tree $T'$ with at most $2N$ nodes, such that the depths of any two leaves in $T'$ differ by at most one, and a sequence of $T'$ nodes $V' = e'_1, \ldots, e'_N$, such that invoking \texttt{repr-update}(e', \text{root}(T')) for each $e' \in V'$ results in at least as many recursive \texttt{repr-update} calls as invoking \texttt{repr-update}(e, \text{root}(T)) for each $e \in V$.

Proof (sketch). $V$ can be replaced by a sequence $V'$ containing only leaves, and $T$ by a balanced binary tree $T'$ with at most $2N$ nodes, without increasing the number of recursive \texttt{repr-update} calls. (Extended proof in appendix.) $\square$

3.5 Maintaining Implicit Suffix Nodes and Main Result

To conclude our treatment, we discuss handling case 2 in lemma 1: finding the most recent match of a pattern that corresponds to a point in $ST$ with an implicitly represented suffix on the same edge. Once such an implicit suffix node is identified, the most recent pattern position is trivially obtained (the position of the corresponding suffix). Furthermore, identifying implicit suffix nodes has a known solution: Breslauer and Italiano [13] describe how Ukkonen’s algorithm can be augmented with a stack of band trees, whose nodes map top $ST$ edges, by which implicit suffix nodes are maintained for amortized constant-time access, linear-time suffix tree online construction. (Further details in appendix.)

The band stack scheme has one additional use in our scheme: in each $ST$ update operation, Breslauer and Italiano’s algorithm pops a number of bands from the stack, and keeps the node that is the endpoint of the last popped edge. This node is the first explicit node on the active path, and, equivalently, the edge is the maximum-depth $LT$ internal edge whose endpoint corresponds an active suffix. This coincides with our definition of the update edge in section 3.2. Thus, we obtain the current update edge in constant time.

Theorem 1. A suffix tree with support for locating, in an input stream, the most recent longest match of an arbitrary pattern $P$ in $O(|P|)$ time, can be constructed online in time $O(N \log N)$ using $O(N)$ space, where $N$ is the current number of processed characters.

Proof (sketch). By lemma 4, the number of \texttt{repr-update} calls is $O(N \log N)$, each of which takes constant time, using a data structure for constant-time lowest common ancestor queries. [10], which bounds the time for maintenance under case 1 in lemma 3 to $O(N \log N)$. In case 2 we achieve $O(N)$ time by the data structure of Breslauer and Italiano. (Extended proof in appendix.) $\square$

We assert that an adversarial input exists that results in $\Omega(N \log N)$ recursive calls, and hence this worst-case bound is tight. (Further details in appendix.)

4 Sliding Window

A major advantage of online suffix tree construction is its applicability for a sliding window: indexing only the most recent part (usually a fixed length) of the
input stream [17,18]. We note that our augmentations of Ukkonen’s algorithm can efficiently support most recent match queries in a sliding window of size $W$:

**Corollary 1.** A suffix tree with support for locating, among the most recent $W$ characters of an input stream, the most recent longest match of an arbitrary pattern $P$ in $O(|P|)$ time, can be constructed online in time $O(N \log W)$ using $O(W)$ space, where $N$ is the current number of processed characters.

*Proof (sketch).* The suffix tree is augmented for indexing a sliding window using $O(W)$ space with maintained time bound [17,18]. Deletion from the data structure for ancestor queries takes $O(1)$ time [19]. Node deletion from band trees takes $O(1)$ time using $pmerge$ [20]. Hence, a $O(N \log W)$ term obtained analogously to lemma [4] dominates. (Extended proof in appendix.)  

\[\Box\]

5 An Optimization for the Lempel-Ziv Case

While our data structure supports arbitrary most-recent-match queries, some related work has considered only the queries that arise in Lempel-Ziv factorization, i.e., querying $ST_i$ only for the longest match of $t_i \cdots t_N$. The desire for finding the most recent occurrence of each factor is motivated by an improved compression rate in a subsequent entropy coding pass.

Ferragina, Nitto, and Venturini [11] gave an $O(N)$ time algorithm for this case, which is not online, and hence cannot be applied to a sliding window. Crochemore, Landiu, and Mignosi [12] presented an online $O(N)$ time suffix tree data structure that, under additional assumptions, circumvents the problem by replacing queries for most recent match with queries for matches with lowest possible entropy-code length. An interesting question is whether the time complexity of our method can be improved if we restrict queries to those necessary for Lempel-Ziv factorization. We now sketch an augmentation for this case.

As characters of one Lempel-Ziv factor are incorporated into $ST$, we need not invoke repr-update for the update edge in each iteration. Instead, we push each update edge on a stack. After the whole factor has been incorporated, we pop edges and invoke repr-update for the reverse sequence, updating edge $e$ only if it would have increased $pos(e)$. In other words, we ignore any updates superseded by later updates during the same sequence of edge pops. In experiments we noted drastic reduction in recursive calls, but whether worst case asymptotic time is reduced is an open question. (Extended discussion in appendix.)

6 Conclusion

We have presented an efficient online method of maintaining most recent match information in a suffix tree, to support optimal-time queries. The question whether the logarithmic factor in the time complexity of our method can be improved upon is, however, still open. Furthermore, precise characteristics of application to restricted inputs or applications (e.g., Lempel-Ziv factorization) is subject to future research, as is the practicality of the result for, e.g., data compression use.
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Appendix

This appendix presents proofs (and extended proofs) omitted from the main text, as well as some extended discussions and details.

A.1 Lemma 1–5 with Full Proofs

**Lemma 1.** Let $e$ be the edge that represents $P$, and let the string corresponding to $e$’s endpoint be $PA$, $|A| \geq 0$. Precisely one of the following holds:

1. The position of the most recent occurrence of $P$ is also the position of the most recent occurrence of $PA$.
2. There exists a suffix $PB$, $|B| \geq 0$ such that $|B| < |A|$.

**Proof.** Since there is no branching node between $P$ and $PA$, we know that for any substring $PC$, $C$ and $A$ must match in the first $\min\{|A|, |C|\}$ characters. If $|C| \geq |A|$, any occurrence of $PC$ (including the most recent one) is an occurrence of $PA$, and case 1 holds. If $|C| < |A|$, then, since there is no branching node between $PC$ and $PA$, $PC$ is a prefix of a string $B$ that corresponds to an implicit suffix node on $e$, and we have case 2. Clearly, $PB$ occurs more recently than $PA$, since $PB$ is a suffix and $|PB| < |PA|$.

⊓ ⊔

**Lemma 2.** For an internal edge $e$, let $A$ be the string corresponding to $e$’s endpoint, and $t_i - |A| \cdots t_i - 1$ the most recent occurrence of $A$ in $T$. Then $e$ has a descendant $f$ in $LT$ whose endpoint corresponds to $BA$ for some string $B$, and $\text{pos}(f) = i - |BA| = i - |B| - |A|$.

**Proof.** $A$’s most recent occurrence appeared in iteration $i$. The update edge in iteration $i$ must consequently be an edge $f$ whose endpoint is $BA$ for some $B$, and the iteration updates $\text{pos}(f)$ to $i - |BA| = i - |B| - |A|$.

We now show that $e$ is an ancestor of $f$ in $LT$. Let $A = A_LA_A_R$ such that $A_L$ marks $e$. By the definition of $LT$, any $BA_L$ is represented by a descendant of $e$. Since there is no branching node between $A_L$ and $A_RA_R$, $A_L$ is never followed by a string different from $A_R$, and hence neither is $BA_L$. Consequently, $BA_L$ and $BA$ are both represented by $f$.

⊓ ⊔

**Lemma 3.** Execution time of $\text{mrm-find}(e)$, where $e$ represents a string $P$ can be bounded by $O(|P|)$.

**Proof.** Let $Q$ be the string that marks $e$. We have $|Q| \leq |P|$. Traversing the path from $e$ to the root via suffix links takes $|Q|$ steps, since following a suffix link implies navigating to the position of a shorter string. The ancestor query in step 4 can be supported in $O(1)$ time [16], and all other operations in $\text{mrm-find}$ are trivially constant-time.

⊓ ⊔

**Lemma 4.** Given a sequence $V = e_1, \ldots, e_N$ of nodes to be updated in a tree $T$ with $M$ nodes, there exists a tree $T'$ with at most $2N$ nodes, such that the depths of any two leaves in $T'$ differ by at most one, and a sequence of $T'$
nodes \( V' = e'_1, \ldots, e'_N \), such that invoking \( \text{repr-update}(e', \text{root}(T')) \) for each \( e' \in V' \) results in at least as many recursive \( \text{repr-update} \) calls as invoking \( \text{repr-update}(e, \text{root}(T)) \) for each \( e \in V \).

Proof. We start from \( T \) and \( V \), modifying them in a series of steps. The end result of all adjustments is \( T' \) and \( V' \).

First, we make \( V \) contain only leaves. Consider \( e = e_j \in V \), and let \( f \) and \( g \) be defined in relation to \( e \) according to \( \text{repr-update} \). Replace \( e \) in \( V \) with some leaf \( e' \) whose ancestor is \( e \), with the following restriction: If \( g = e \neq f \) (case c), \( e' \) must be in a subtree of \( g \) other than that which contains \( f \). Reversely, if \( g = f \neq e \) (case d), \( e' \) is in a subtree of \( g \) other than that which contains \( e \). In either case, if \( g \) has only one child, we add \( e' \) as a new leaf below \( g \). Thereby, we ensure that the transformation does not reduce the number of recursive calls, and contributes at most \( N \) nodes.

Next, we show that the tree can be balanced, by the following argument. Consider \( e, f, \) and \( g \) as defined in \( \text{repr-update} \). Given the precious transformation of \( V \), we can assume that \( e \) and \( f \) are leaves. Recursion in \( \text{repr-update} \) progresses iff \( e \) and \( f \) are in different subtrees of \( g \). Let \( k \) be the number of children of \( g \), \( m_i \) the number of leaves in \( g \)'s \( i \)th subtree, and \( m_g \) the total number of leaves in the subtree rooted at \( g \). The number of possibilities for choosing \( e \) and \( f \) is \( \prod_{i=1}^{k} \binom{m_i}{m_g} \), which is maximized if the number of leaves is as evenly distributed as possible among the subtrees of \( g \). We move nodes between subtrees to even out the number of leaves, without changing the number of leaves or internal nodes. Applying for all internal nodes yields a balanced tree, where the depth of leaves differ by at most one. We are not restricted in choosing nodes for the modified update sequence in any other way, and can make use of the full choice made possible by the restructuring in order to make sure that we do not reduce the number of recursive calls. Hence, for some sequence, the number of recursive calls is at least the same, which concludes the proof. \( \square \)

Lemma 5, leading up to theorem 1, is completely omitted from the main text, and presented only in this appendix.

**Lemma 5.** The time for maintaining \( \text{repr}(e) \) for each \( e \) so as to maintain property [7] during construction of a suffix tree over a string of length \( N \) can be bounded by \( O(N \log N) \).

**Proof.** By lemma 4, the total number of recursive calls in invoking \( \text{repr-update} \) is proportional to the maximum for a balanced tree, whose height is \( O(\log N) \), when the number of nodes is linear in \( N \). The time for each recursive call is constant, when a data structure for constant-time lowest common ancestor queries is employed. [16]. Consequently, total time is at most \( O(N \log N) \). \( \square \)

Note that locating the update edge, discussed in section 3.5, is not included in the time accounted for by lemma 5.
A.2 Description of Band Trees and Full Theorem and Corollary

Breslauer and Italiano [13] describe augmentations of Ukkonen’s algorithm by which implicit suffix nodes can be maintained for amortized constant-time access, while maintaining linear suffix tree construction time. Implicit suffix nodes on external edges have cyclicity properties that can be used for computing their positions without any extra storage. Implicit nodes on internal edges are maintained through the use of a stack of bands, where a band is a tree whose nodes map to $ST$ edges with equal edge labels, and whose edges correspond to suffix links (i.e., it is a part of $LT$). Breslauer and Italiano show that the band stack, and an implicit suffix node position for one representative of each band, can be maintained in amortized $O(1)$ time per $ST$ update iteration, and support $O(1)$ time implicit-node queries.

Theorem 1. A suffix tree with support for locating, in an input stream, the most recent longest match of an arbitrary pattern $P$ in $O(|P|)$ time, can be constructed online in time $O(N \log N)$ using $O(N)$ space, where $N$ is the current number of processed characters.

Proof. For case 1 in lemma 1, query correctness and $|P|$ time bound under maintenance of property 1 for pos-updating the update edge at each iteration, follow from lemmas 2 and 3. The method for maintaining property 1 is given in section 3.4 and its $O(N \log N)$ time bound given by lemma 5. Locating the update edge for pos-updating takes constant time, using the described data structure of Breslauer and Italiano, which also provides $O(N)$ maintenance time for case 2 in lemma 1. The space usage of all described data structures is bounded by $O(N)$.

Corollary 1. A suffix tree with support for locating, among the most recent $W$ characters of an input stream, the most recent longest match of an arbitrary pattern $P$ in $O(|P|)$ time, can be constructed online in time $O(N \log W)$ using $O(W)$ space, where $N$ is the current number of processed characters.

Proof. Our augmentations of the suffix tree in itself do not alter its structure, and consequently, existing techniques for augmenting the suffix tree algorithm for to index a sliding window in $O(1)$ amortized time, limiting space usage to $O(W)$ [17,18] are directly applicable. The additional data structures are:

- The data structure for ancestor queries used in $\text{mrm-find}$ in section 3.3.
  Deletions in $O(1)$ time are available [19], which can keep the space usage down to the $O(W)$ tree size.
- The band trees kept on a stack in order to be able to find the update edge in constant time. Breslauer and Italiano [13] do not discuss deleting nodes from the band trees, but we note that the data structures for dynamic nearest marked ancestors they use for achieving $O(1)$ amortized time operations do also support leaf deletions with the same time bound by means of a $\text{pmerge}$ operation [20], again allowing space usage to be asymptotically bounded by the $O(W)$ tree size.
Analogously to the proof of lemma 4, the number of recursive \texttt{repr-update} calls is at most proportional to \(N\) times the height of a perfectly balanced tree. For the sliding window suffix tree of size \(O(W)\), this contributes a dominating term of \(O(N \log W)\) to the time complexity.

\[\square\]

A.3 Discussion of Worst Case for General Case and Lempel-Ziv Optimization

Lemma 5 does not state that any input exists that results in \(\Omega(N \log N)\) recursive calls, but such an adversarial input \textit{does} exist, and hence our analysis is tight. We now give an informal elaboration on the nature of an adversarial input. (Since our main results do not depend on the lower bound, we do not consider a formal argument to support the existence of this input.)

With specified parameter \(d\), an adversary can choose symbol \(t_i\) considering the most recent previous occurrence of a string \(Aa\), where \(A = t_{i-d} \cdots t_{i-1}\), and let \(t_i \neq a\). This produces a pair of edge updates that reaches depth \(d\) in \(LT\). The resulting adversarial string is a sequence with cycle length \(2^d\), and the number of times \texttt{repr-update} reaches recursion depth \(\log N\) is close to \(N/2\). For \(d = 2\), one such sequence has cycle \(abaaabbb\); for \(d = 3\), the corresponding cycle is \(aaaaaababababab\).

In experiments, we have observed the worst case behavior for constructed adversarial inputs only, and neither for naturally occurring data nor random inputs.

Furthermore, we note that optimization in section 5 yields \(O(N)\) time for the Lempel-Ziv special case for the given adversarial input, as well as for any other string exhibiting a cycle of constant length. We have not found any adversarial input that produces \(\Omega(N \log N)\) time in this case, and as noted in 5, it is an open question whether it achieves total \(o(N \log N)\) time.