MINIMAL SURFACES AND CPE METRIC

BENEDITO LEANDRO

ABSTRACT. The critical points of the total scalar curvature functional, restricted to closed n-dimensional manifolds with constant scalar curvature metrics and unit volume, are termed CPE metrics. In 1987, Arthur L. Besse conjectured that CPE metrics are always Einstein. Using the theory of minimal surfaces, we prove the conjecture for three-dimensional manifolds.

1. INTRODUCTION AND MAIN STATEMENT

In mathematics, variational methods play an important role. David Hilbert, for example, proved that it is possible to recover the equations of general relativity through the action of the total scalar curvature functional (Einstein-Hilbert functional).

This functional can be very useful in geometric analysis. For instance, a natural way to prove the existence of Einstein metrics arises when we look at the critical points of this functional. Furthermore, this functional is a starting point to attack the Yamabe problem [1].

In this paper, we restrict the Einstein-Hilbert functional to a space of certain Riemannian metrics. Then, we look at the critical points of the total scalar curvature functional with this constraint.

The total scalar curvature functional is given by

$$S(g) = \int_M R dv,$$

where $R$ is the scalar curvature determined by the metric $g$. Here, $M$ is the set of smooth Riemannian structures on a closed oriented smooth manifold $M^n$ of volume 1.

The Euler-Lagrangian equation of Einstein-Hilbert functional (cf. [1, 5]) restricted to

$$\mathcal{C} = \{ g \in M \mid R \text{ is constant} \},$$

can be written as the following critical point equation

$$\frac{1}{1 + f} \hat{R}ic = \nabla^2 f + \frac{Rf}{n(n-1)} g,$$

where $f$ is a smooth function on $M^n$. Furthermore, $R$, $\hat{R}ic$, and $\nabla^2$ stand for the scalar curvature, Ricci traceless tensor, and the Hessian form for $g$. We will call $(M^n, g, f)$ a CPE metric.

Contracting the CPE metric (1.1), we obtain

$$-\Delta f = \frac{Rf}{n-1},$$

Date: January 22, 2025.

2020 Mathematics Subject Classification. 53C25.

Key words and phrases. Total scalar curvature functional, Critical point equation, Einstein metric.

Benedito Leandro was partially supported by CNPq/Brazil Grant 303157/2022-4 and 403349/2021-4.
Hence, \( f \) is an eigenfunction of the Laplacian for \( g \). Notice that the Laplacian has a non-positive spectrum. Then, we may conclude that \( R \) must be a positive constant.

Moreover, if \( f \) is a constant function, then (1.2) tells us that \( f \) must be identically zero. Einstein metrics are recovered when \( f = 0 \). However, non-trivial solutions for CPE metrics are a very strong condition. Nonetheless, the CPE metric was studied by Lafontaine in [13], where the conjecture was proved considering a conformally flat CPE metric. In [1] Remark 4.48, p. 128, Besse conjectured that

Conjecture 1. A CPE metric is always Einstein.

The only known solution with \( f \) not identically zero is that of the standard sphere (cf. [13]). We can see that if \( g \) is an Einstein metric from (1.1) we have

\[ \nabla^2 f = -\frac{Rf}{n(n-1)} g. \]

Thus, applying Obata’s theorem [15], we may conclude that \((M^n, g)\) is isometric to the standard round sphere \( S^n \) and \( f \) is the height function. The Einstein solutions for the CPE metric are called trivial. However, Conjecture 1 is still an open problem, even in the three-dimensional case (cf. [11]).

Several researchers have addressed this conjecture, notably Seungsu Hwang, who achieved partial results by exploring stable minimal surfaces (cf. [6, 7, 8, 9]). As we can see, the theory of minimal surfaces plays an important role in the attempt to solve Conjecture 1.

However, the standard spheres (with dimensions less than or equal to five) do not admit stable minimal hypersurfaces [3]. We are interested in proving a similar theorem for a three-dimensional CPE metric. Our idea relies on providing the bounds for the genus of a minimal surface in a three-dimensional CPE metric. Looking for minimal surfaces in the standard 3-sphere having a particular genus has great consequences (see [2] and the references therein).

Without further ado, we state our main results.

Theorem 1. There exists no embedded closed minimal surface in a non-trivial three-dimensional CPE metric.

Therefore, we can conclude the following theorem via [7].

Theorem 2. Assume that \( f_1 \) and \( f_2 \) are two distinct non-trivial solutions of the CPE on \((M^3, g)\). Then, \( M^3 \) is diffeomorphic to \( S^3 \) and the set \( \{ f_1 = f_2 \} \) is connected.

This theorem establishes Conjecture 1 at the topological level. Indeed, we proved that a three-dimensional CPE metric is a topological 3-sphere. Towards Conjecture 1 we can improve [9 Theorem 1.1]. The following theorem proves Conjecture B in [7] by a different method than the one used by [10 Theorem 1.1].

Theorem 3. Assume that \( f_1 \) and \( f_2 \) are two distinct non-trivial solutions of the CPE on \((M^3, g)\). Then, \((M^3, g)\) is isometric to the standard 3-sphere.

Another interesting consequence of Theorem 1 and Corollary 2.2] is that since we proved that there is no compact stable minimal surface on a CPE metric, it must satisfy Frankel’s property, i.e., any two closed minimal surfaces in the CPE metric must intersect.

Suppose we prove that any closed minimal surface in a three-dimensional CPE metric is contained in \( \{ 1 + f \geq 0 \} \). In that case, we can use [12, which says that the union of all closed, smooth, embedded, minimal surfaces of \( M^3 \) is dense in \( M^3 \). Then, from Lemma 1 in [6], we can infer that Conjecture 1 is valid.

Theorem 4. Let \((M^3, g, f)\) be a CPE metric. Then \( M^3 \) contains an infinite number of distinct closed, smooth embedded, minimal surfaces with genus at least one. Moreover, any
CPE METRIC

closed minimal surface must be contained at \( \{1 + f \geq 0\} \). Therefore, we may conclude that \( \{1 + f < 0\} \) is of isolated points.

Consequently, we prove Conjecture 1 for three-dimensional manifolds.

**Theorem 5.** Any three-dimensional CPE metric must be an Einstein manifold.

2. PROOF OF THE MAIN RESULT

Before presenting the proof of the main results, we need to remember some important facts about the CPE metric.

**Proposition 1.**

Let \((M^3, g, f)\) be a CPE metric and \(\Sigma^2 \subset M^3\) a compact stable minimal surface. Then,

(i) \(\Sigma\) is properly contained in \(\{1 + f < 0\}\).

(ii) \(\Sigma\) is totally geodesic.

The proof of Theorem 1 is a direct consequence of Theorem 6 and Theorem 7 below. Now we are ready to prove our main theorems concerning the genus of compact minimal surfaces of a CPE metric.

**Theorem 6.** Let \((M^3, g, f)\) be a non-trivial CPE metric and \(\Sigma^2 \subset M^3\) an embedded compact stable minimal surface. Then,

\[
g(\Sigma) < 1 - \frac{R}{8\pi} |\Sigma|,
\]

where \(g(\Sigma)\) and \(|\Sigma|\) stand for the genus and the area of \(\Sigma\), respectively.

**Proof.** Suppose a closed minimal hypersurface \(\Sigma\) exists in an \(n\)-dimensional non-trivial CPE metric. From the Gauss equation, we have

\[
\frac{R}{2} = \frac{R_\Sigma}{2} + \text{Ric}(\nu, \nu) + \frac{1}{2} |A|^2,
\]

where \(R_\Sigma\) is the scalar curvature of \(\Sigma\), and \(A\) its second fundamental form. Since,

\[
\Delta f = \Delta_{\Sigma} f + \nabla^2 f(\nu, \nu) + H(\nabla f, \nu) = 0
\]

from the CPE equations, we get

\[
-\frac{1}{n} Rf = \Delta_{\Sigma} f + (1 + f) \hat{\text{Ric}}(\nu, \nu),
\]

i.e.,

\[
(2.1) \quad \Delta_{\Sigma} f + (1 + f) \hat{\text{Ric}}(\nu, \nu) = \frac{R}{n}.
\]

So, combining (2.1) with the Gauss equation we have

\[
\frac{1}{2} (1 + f) R = (1 + f) \frac{R_\Sigma}{2} + (1 + f) \text{Ric}(\nu, \nu) + \frac{1}{2} (1 + f) |A|^2
\]

\[
= (1 + f) \frac{R_\Sigma}{2} + \frac{R}{n} - \Delta_{\Sigma} f + \frac{1}{2} (1 + f) |A|^2,
\]

i.e.,

\[
(2.2) \quad \Delta_{\Sigma} f - \frac{R}{n} = \frac{1}{2} (1 + f) \left[ R_\Sigma - R + |A|^2 \right].
\]

If \(\Sigma\) is stable we have \(\Sigma \subseteq \{1 + f < 0\}\) and \(A = 0\). Considering \(n = 3\) we get

\[
\Delta_{\Sigma} f - \frac{R}{3} = (1 + f) \left[ K - \frac{R}{2} \right],
\]
where $K$ is the Gaussian curvature of $\Sigma$. Hence,

$$\frac{\Delta \Sigma f}{(1 + f)} - \frac{R}{3(1 + f)} = K - \frac{R}{2}. $$

Integrating the above identity yields to

$$0 < \int_{\Sigma} \left[ \frac{|\nabla \Sigma f|^2}{(1 + f)^2} - \frac{R}{3(1 + f)} \right] = 4\pi(1 - g(\Sigma)) - \frac{R}{2} |\Sigma|,$$

i.e.,

$$R|\Sigma| < 8\pi(1 - g(\Sigma)).$$

□

The following theorem characterizes the topology of a given closed minimal surface in a three-dimensional CPE metric.

**Theorem 7.** Let $(M^3, g, f)$ be a non-trivial CPE metric and $\Sigma^2 \subset M^3$ a closed minimal surface. Then,

$$K = -\frac{1}{2} |A|^2,$$

where $K$ and $A$ stand for the Gauss curvature and the second fundamental form of $\Sigma$, respectively. In particular, the genus $g(\Sigma)$ of $\Sigma$ must be at least one, i.e.,

$$1 \leq g(\Sigma).$$

**Proof.** The decomposition of the Hessian operator on a given surface of a CPE metric is given by

$$\nabla_\Sigma^2 f(X, Y) + \langle \nabla f, \nu \rangle A(X, Y) = \nabla^2 f(X, Y) = (1 + f) \hat{\text{Ric}}(X, Y) - \frac{1}{n(n-1)} R fg(X, Y),$$

where $X, Y$ are any tangent vector fields in $\Sigma$. Here, $\nabla_\Sigma^2 f$ and $A$ stand for the Hessian of $f$ and the second fundamental form for the induced metric on $\Sigma$. On the other hand, in $\Sigma$ we have

$$\text{Ric}(X, Y) = \text{Ric}_\Sigma(X, Y) + \text{Rm}(\nu, X, \nu, Y) + A^2(X, Y),$$

where $A^2(X, Y) = \langle S^2(X), Y \rangle$, where $S$ stands for the shape operator. Here, $\text{Ric}_\Sigma$ and $\text{Rm}$ stand for the Ricci curvature of $\Sigma$ and the curvature operator, respectively.

On a Riemannian manifold $(M^n, g)$ we have the following decomposition formula for the curvature tensor $\text{Rm}$ (cf. [1, 1.116]):

$$\text{Rm}(X, Y, Z, L) = W(X, Y, Z, L) + \frac{1}{n-2} \left( \text{Ric}(X, Z)g(Y, L) + \text{Ric}(Y, L)g(X, Z) - \text{Ric}(X, L)g(Y, Z) - \text{Ric}(Y, Z)g(X, L) \right) - \frac{R}{(n-1)(n-2)} (g(Y, L)g(X, Z) - g(X, L)g(Y, Z)),$$

where $W$ stands for the Weyl tensor. Moreover, $X, Y, Z$ and $L$ are tangent vector fields in $M^n$. Thus,

$$\text{Rm}(\nu, X, \nu, Y) = W(\nu, X, \nu, Y) + \frac{1}{n-2} \left[ \text{Ric}(\nu, \nu)g(X, Y) + \text{Ric}(X, Y) \right] - \frac{R}{(n-1)(n-2)} g(X, Y).$$
Therefore, from (2.4) we have
\[
\frac{n - 3}{n - 2} \text{Ric}(X, Y) = \text{Ric}_\Sigma(X, Y) + W(\nu, X, \nu, Y) + \frac{1}{n - 2} \text{Ric}(\nu, \nu)g(X, Y)
- \frac{R}{(n - 1)(n - 2)} g(X, Y) + \mathcal{A}^2(X, Y)
\]
i.e.,
\[
(n - 3)\text{Ric}(X, Y) = \left[ (n - 2)W(\nu, X, \nu, Y) + \text{Ric}(\nu, \nu)g(X, Y) \\
- \frac{R}{(n - 1)} g(X, Y) + (n - 2)\text{Ric}_\Sigma(X, Y) + (n - 2)\mathcal{A}^2(X, Y) \right].
\]

So,
\[
(n - 3)\ddot{\text{Ric}}(X, Y) = \left[ (n - 2)W(\nu, X, \nu, Y) + \text{Ric}(\nu, \nu)g(X, Y) \\
- \frac{R}{(n - 1)} g(X, Y) + (n - 2)\text{Ric}_\Sigma(X, Y) + (n - 2)\mathcal{A}^2(X, Y) \right]
- (n - 3)\frac{R}{n} g(X, Y)
= \left[ (n - 2)W(\nu, X, \nu, Y) + \text{Ric}(\nu, \nu)g(X, Y) \\
- \left( \frac{n + (n - 1)(n - 3)}{n(n - 1)} \right) Rg(X, Y) + (n - 2)\text{Ric}_\Sigma(X, Y) \right. \\
+ (n - 2)\mathcal{A}^2(X, Y)].
\]

Hence, from (2.3) we get
\[
(n - 3)[\nabla^2_\Sigma f(X, Y) + \langle \nabla f, \nu \rangle \mathcal{A}(X, Y)]
= (1 + f)[(n - 2)W(\nu, X, \nu, Y) + \text{Ric}(\nu, \nu)g(X, Y) - \left( \frac{n + (n - 1)(n - 3)}{n(n - 1)} \right) Rg(X, Y)
+ (n - 2)\text{Ric}_\Sigma(X, Y) + (n - 2)\mathcal{A}^2(X, Y)] - \frac{(n - 3)}{n(n - 1)} Rfg(X, Y).
\]

Moreover, from (2.1) we have
\[
\Delta_\Sigma f + (1 + f)\text{Ric}(\nu, \nu) = \frac{R}{n}.
\]
Thus,
\[
(n - 3) \left( \nabla^2_\Sigma f(X, Y) + \langle \nabla f, \nu \rangle \mathcal{A}(X, Y) \right)
= (n - 2)(1 + f)W(\nu, X, \nu, Y) + \left( \frac{R}{n} - \Delta_\Sigma f \right) g(X, Y)
- (1 + f) \left( \frac{n + (n - 1)(n - 3)}{n(n - 1)} \right) Rg(X, Y) + (n - 2)(1 + f)\text{Ric}_\Sigma(X, Y)
+ (n - 2)(1 + f)\mathcal{A}^2(X, Y) - \frac{(n - 3)}{n(n - 1)} Rfg(X, Y).
\]

Consider \( n = 3 \) in (2.5) to obtain
\[
(1 + f)\mathcal{A}^2(X, Y) = \left[ \Delta_\Sigma f - \frac{R}{3} - (1 + f) \left( \frac{K - \frac{1}{2}R}{2} \right) \right] g(X, Y),
\]
where \( K \) stands for the Gaussian curvature. In contrast with (2.2) we get
\[
(1 + f) \left[ \mathcal{A}^2(X, Y) - \frac{1}{2} |A|^2 g(X, Y) \right] = 0.
\]
It is known that we can not have $f = -1$ everywhere in a minimal surface $\Sigma$ (see [4, proof of Theorem A] and [6, Equation 3]). Consider $B = \Sigma \cap f^{-1}(-1) \neq \emptyset$. Since $f^{-1}(-1)$ is closed, we can infer that $B$ is closed, and from (2.2) we also have

$$0 = \int_B \Delta_B f = \frac{R}{3} |B|.$$ 

Therefore, $B = \emptyset$.

On the other hand, the Cayley–Hamilton theorem says that

$$S^2 - \text{trace}(S)S + \det(S)I = 0,$$

where $I$ stands for the identity matrix. Here, the shape operator $S$ is such that $A(X, Y) = \langle S(X), Y \rangle$, i.e.,

$$\langle S^2(X), Y \rangle - \text{trace}(S)\langle S(X), Y \rangle + \det(S)\langle X, Y \rangle = 0,$$

see Theorem 5.3.3 in [16]. Therefore,

$$A^2(X, Y) - 2HA(X, Y) + Kg(X, Y) = 0.$$ 

Consequently,

$$(2.7) \quad K = -\frac{1}{2} |A|^2.$$ 

Considering $\Sigma$ compact, we may conclude by the Gauss-Bonnet theorem that $1 \leq g(\Sigma)$, where $g(\Sigma)$ stands for the genus of $\Sigma$. So, Theorem 5 implies that there is no stable minimal surface in a CPE metric.

**Theorem 8.** Any closed minimal surface in a three-dimensional CPE metric must be contained at $\{1 + f \geq 0\}$. Therefore, we may conclude that $\{1 + f < 0\}$ is of isolated points.

**Proof.** Combining (2.2) and (2.7) we get

$$|\Sigma| = \frac{3}{2} \int_\Sigma (1 + f).$$

In particular, $\Sigma$ cannot be contained in $\{1 + f < 0\}$.

Now, consider $B = \Sigma \cap f^{-1}(-1) \neq \emptyset$. Since $f^{-1}(-1)$ is closed, we can infer that $B$ is closed, and from (2.2) we also have

$$0 = \int_B \Delta_B f = \frac{R}{3} |B|.$$ 

Therefore, $B = \emptyset$.

Now, consider $C = \Sigma \cap \{1 + f < 0\}$. We can infer that $f$ has a global maximum, i.e., $-1 = \max_{\Sigma} f$, and from (2.2) we also have $\Delta_C f > 0$ (since (2.6) holds). Then, we can apply the strong maximum principle to conclude that $f$ is constant at $C$ which contradicts (2.2), i.e.,

$$0 = \Delta_C f = \frac{R}{3} - \frac{R}{2} (1 + f) > 0.$$ 

Therefore, $C = \emptyset$.

Hence, any closed minimal surface in a three-dimensional CPE metric must be contained at $\{1 + f \geq 0\}$. We can use [12], which says that the union of all closed, smooth, embedded, minimal surfaces of $M^3$ is dense in $M^3$. Therefore, we may conclude that $\{1 + f < 0\}$ is of isolated points. In fact, suppose that $p \in \{1 + f < 0\}$ and there exist $\varepsilon > 0$ such that $B_\varepsilon(p) \subset \{1 + f < 0\}$. On the other hand, there is a closed minimal surface $\Sigma$ such that $\Sigma \cap B_\varepsilon(p) \neq \emptyset$, which is a contradiction since for any minimal surface $\Sigma$ we must have $\Sigma \subset \{1 + f \geq 0\}$. 

□
Proof of Theorem 5. Now, since \( \{ 1 + f < 0 \} \) is of isolated points by continuity of \( f \) we can conclude that \( f \geq -1 \) in \( M^3 \). We conclude the proof by applying Lemma 1 in [6]. □

Acknowledgment: The author thanks Professor Fábio Reis for stimulating discussions and insightful conversations.

REFERENCES

[1] A. L. Besse - Einstein Manifolds. Spring-Verlag, Berlin, 1987.
[2] S. Brendle - Minimal surfaces in \( S^3 \): a survey of recent results, Bull. Math. Sci. (2013) 3:133-171.
[3] G. Catino; P. Mastrolia; A. Roncoroni - Two rigidity results for stable minimal hypersurfaces, Geom. Funct. Anal. 34 (2024), no. 1, 1-18.
[4] Y. Fang, and Y. Wei - Brown–York mass and positive scalar curvature II: Besse’s conjecture and related problems. Annals of Global Analysis and Geometry 56.1 (2019): 1-15.
[5] A. E. Fischer; J. E. Marsden - Deformations of the scalar curvature. Duke Math. J. 42.3 (1975), 519-547.
[6] S. Hwang - Critical points of the total scalar curvature functional on the space of metrics of constant scalar curvature, Manuscripta Math. 103 (2000), no. 2, 135-142.
[7] S. Hwang - The critical point equation on a three-dimensional compact manifold, Proceedings AMS. 131, No 10, (2003) 3221-3230.
[8] S. Hwang - Stable minimal hypersurfaces in a critical point equation, Commun. Korean Math. Soc. 20 (2005), No. 4, pp. 775-779.
[9] S. Hwang; J. Chang; G. Yun - Rigidity of the critical point equation. Math. Nachr. 283, 846–853 (2010).
[10] S. Hwang - Three dimensional critical point of the total scalar curvature, Bull. Korean Math. Soc. 50 (2013), no. 3, 867-871.
[11] S. Hwang; G. Yun - Critical Point Equation on three-dimensional manifolds and the Besse Conjecture, arXiv:2208.10887v2 [math.DG]
[12] K. Irie; F.C. Marques; A. Neves. - Density of minimal hypersurfaces for generic metrics. Ann. of Math. (2), 187 (2018), 963–972.
[13] J. Lafontaine - Sur la géométrie d’une généralisation de l’équation différentielle d’Obata, J. Math. Pures Appliquées, 62 (1983), 63-72.
[14] F. C. Marques - Abundance of minimal surfaces, Japan. J. Math. 14, 207-229 (2019). DOI: 10.1007/s11537-019-1839-x
[15] M. Obata - Certain conditions for a Riemannian manifold to be isometric with a sphere, J. Math. Soc. Japan Vol. 14, No. 3, 1962.
[16] P. Petersen - Differential Geometry. www.math.ucla.edu/~petersen/120a.1.11f/DGnotes.pdf

1 Departamento de Matemática, Universidade de Brasília, Brasília-DF, 70910-900, Brazil.
Email address: benedito.neto@unb.br

1