Convergence of Recursive Stochastic Algorithms using Wasserstein Divergence∗

Abhishek Gupta † and William B. Haskell ‡

Abstract. This paper develops a unified framework, based on iterated random operator theory, to analyze the convergence of constant stepsize recursive stochastic algorithms (RSAs) in machine learning and reinforcement learning. RSAs use randomization to efficiently compute expectations, and so their iterates form a stochastic process. The key idea is to lift the RSA into an appropriate higher-dimensional space and then express it as an equivalent Markov chain. Instead of determining the convergence of this Markov chain (which may not converge under constant stepsize), we study the convergence of the distribution of this Markov chain. To study this, we define a new notion of Wasserstein divergence. We show that if the distribution of the iterates in the Markov chain satisfy certain contraction property with respect to the Wasserstein divergence, then the Markov chain admits an invariant distribution. Inspired by the SVRG algorithm, we develop a method to convert any RSA to a variance reduced RSA that converges to the optimal solution with in almost sure sense or in probability. We show that convergence of a large family of constant stepsize RSAs can be understood using this framework. We apply this framework to ascertain the convergence of mini-batch SGD, forward-backward splitting with catalyst, SVRG, SAGA, empirical Q value iteration, synchronous Q-learning, enhanced policy iteration, and MDPs with a generative model. We also develop two new algorithms for reinforcement learning and establish their convergence using this framework.

Key words. Iterative Random Maps, Wasserstein Divergence, Stochastic Gradient Descent, Empirical Dynamic Programming, Constant step-size Q-learning, Stochastic Monotone Inclusion.

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1. Introduction. Over the past two decades, there has been explosive growth in new randomized algorithms for doing complex optimization tasks in machine learning and reinforcement learning. Many of these algorithms are essentially recursions of certain mappings that depend on stochastic parameters. Such algorithms are collectively called ‘recursive stochastic algorithms’ (RSAs) [28, 17, 26].

From a computational viewpoint, RSAs with constant stepsizes (also called ‘learning rate’ in some contexts) enjoy many benefits compared to RSAs with decaying stepsizes that converge to zero [5]. Constant stepsize RSAs often converge much faster to a neighborhood of the desired solution. This phenomenon has been observed in off-policy temporal difference methods [44], temporal difference with function approximation [27], tracking problem [25], gradient descent [1], among others. Furthermore, the size of this neighborhood is usually small if the stepsize is small (so too large a stepsize may not be beneficial) [9, 4]. Accordingly, in practice, researchers often use a constant stepsize for a certain number of steps, and then if needed, rerun the algorithm with a smaller (constant) stepsize.

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†Electrical and Computer Engineering, The Ohio State University, Columbus, OH, USA. (gupta.706@osu.edu).
‡Supply Chain and Operations Management, Krannert School of Management, Purdue University, West Lafayette, IN, USA. (whaskell@purdue.edu).
The practical success of constant stepsize RSAs compared to decaying stepsize RSAs has spurred significant interest in finding supporting theory. The existing convergence analyses are tailored to each specific algorithm and do not readily extend to other algorithms. In this paper, we unify these analyses by highlighting the features common to all of them. We find that: (i) constant stepsize RSAs can be modeled as Markov chains and (ii) their convergence is connected to a form of contraction with respect to a new notion of divergence on probability distributions. We call this new notion the Wasserstein divergence since it is based on the classical Wasserstein distance.

We show that many constant stepsize RSAs in machine learning and reinforcement learning are contractions with respect to the Wasserstein divergence. This perspective gives us new insight into the nature of the convergence of these algorithms. In particular, the central new notion of the Wasserstein divergence serves as a unifying feature for the convergence analysis of different families of RSAs. This perspective also suggests the design of some new constant stepsize RSAs.

1.1. Prior Work. The study of RSAs enjoys a rich history. Early work on RSAs was for solving regression problems, where certain stepsize parameters converge to zero as the iteration index grows; see [33, 24, 43]. These algorithms and their convergence are studied under the umbrella of stochastic approximation theory [26, 8]. It was soon realized that stochastic approximation theory can ascertain convergence of a wide variety of optimization and learning algorithms. To understand this method, let us consider the iteration $x_{k+1} = x_k + \beta_k d_k$, where $\{\beta_k\}$ are stepsizes that satisfy

$$\sum_k \beta_k = \infty, \quad \sum_k \beta_k^2 < \infty,$$

and $d_k$ is an unbiased noisy estimate of some operator $F$ evaluated at $x_k$. The key assumption is that the point of interest (say the desired optimal solution), $x^*$, satisfies $F(x^*) = 0$. In some cases, it is the unique point satisfying this condition. The convergence guarantee for such algorithms is very strong under some reasonable conditions that are typically met in practice. However, despite strong convergence guarantees, the rate of convergence is very slow for stochastic approximation type algorithms with decaying stepsizes.

Parallel to the development of stochastic approximation theory, some authors considered constant stepsizes, where all $\beta_k$s are equal and sufficiently small. In this case, \{x_k\} forms a Markov chain under suitable assumptions on $d_k$. This constant stepsize recursion is given by $x_{k+1} = f(x_k, w_k)$ for a suitably defined function $f$ that takes as input $x_k$ and i.i.d. noise $w_k$. The convergence guarantee for this class of algorithms is usually weak — the iterates may not converge to the desired solution $x^*$, but instead will do a random walk in some neighborhood of it. This class of recursive algorithm has been studied under the name of iterated random function systems [15, 2, 13, 16, 39] and stochastic approximation with constant stepsize; see [9, 8, 4, 34] and the references therein for related discussions.

Stochastic optimization algorithms play a pivotal role in large-scale machine learning as well as data-driven/simulation-based learning problems [10]. For instance, stochastic gradient descent is unarguably the most important class of algorithms for many machine learning tasks. At the same time, simulation-based reinforcement learning has received significant attention
In both families of the algorithms, randomization and sampling techniques are used in specialized ways to compute expectations that are otherwise expensive or intractable.

Constant stepsize RSAs for optimization problems (in particular large-scale optimization problems) are widely studied. Constant stepsize stochastic gradient descent (SGD) is studied with Markov chain methods in [14] with respect to the Wasserstein distance. It is shown that an invariant distribution exists, the distribution of the iterates converges geometrically to this invariant distribution. Further, it presents a formula for the concentration of the invariant distribution around the desired solution. Stochastic variance reduced gradient descent (SVRG) was designed to improve upon SGD (by using a variance reduced correction term), and it enjoys a linear convergence rate in expectation [22]. The stochastic average gradient algorithm (SAGA) is proposed in [12] which offers an alternative variance reduction scheme. A hybrid algorithm that combines the features of SVRG and SAGA is developed in [32]. SVRG and SAGA are extended to solve monotone inclusion problems in [31] (which contains function minimization as a special case). We focus on the more general monotone inclusion setting in this paper because the role of contractions is more prominent.

Within the reinforcement learning literature, constant stepsize stochastic approximation algorithms have been developed to compute the (approximately) optimal value function, Q value function, evaluating the performance (total discounted or average cost) of a stationary policy using temporal differences, and the different versions of these algorithms using function approximators. For finite-state finite-action discounted cost MDPs, constant step-size empirical value iteration (EVI) was studied in [20]. The convergence guarantee of the algorithm was derived by constructing a Markov chain over a finite space and using a stochastic dominance argument to bound the error in the iterates. This methodology significantly departed from the ODE approach usually taken for proving convergence of constant stepsize stochastic approximation algorithms in [9, 8, 4]. This approach was later extended to average cost MDPs with empirical relative value iteration in [19], where the dominating Markov chain constructed was over the space of natural numbers. Error bounds for constant step-size synchronous and asynchronous Q-learning algorithm was studied in [3] by employing union bound and good old triangle inequality. Finite-time bounds for temporal difference method for evaluating stationary policies with constant stepsize have been obtained in [38, 7] under a variety of assumptions.

1.2. Contributions. We summarize our key contributions as follows:

1. We model constant stepsize RSAs as abstract iterated random function systems. Typically, a measure of distance, such as a metric or a Lyapunov function, between the iterates of the RSA and $x^*$ is shown to have some kind of one-step contraction property (e.g. $E[||x_k - x^*||^2]$ is a common measure of distance for first-order optimization algorithms). Across RSAs, we noted that the measure of distance satisfies positive definiteness and symmetry – two well-known properties enjoyed by a metric. However, it may not satisfy the triangle inequality. Thus, we have a Markov chain whose distribution satisfies some contraction property with respect to a certain divergence.

2. We assume that, starting from two arbitrary initial distributions, the marginal distributions of the iterates of these RSAs contract with respect to the Wasserstein divergence. As a consequence, we demonstrate the existence of a unique invariant
distribution for the RSA, and we show that the marginal distribution of the iterates converges to this invariant distribution geometrically with respect to the Wasserstein divergence. These results let us make conclusions about the performance of the algorithm, which may include how quickly the RSA converges, how far the iterate is from $x^*$ after a sufficiently large runtime, etc.

3. We bound the concentration of the invariant distribution of the RSAs around $x^*$. For the usual Wasserstein distance (which is defined in terms of a metric), we show that the concentration of the invariant distribution depends on the action of the RSA on the desired solution (in expectation).

4. We show that variance reduction algorithms map the desired solution to itself almost surely. In this case, the invariant distribution for variance reduced RSAs is concentrated at $x^*$. This idea is used to establish the convergence of variance reduced algorithms such as SVRG, SAGA, HSAG, asynchronous enhanced policy iteration, etc.

5. We show how to construct new variance reduction algorithms based on existing RSAs. This construction is based on adding a ‘correction’ term to ensure that the modified RSA maps the desired solution back to itself. We demonstrate this construction by introducing a constant step-size variance reduced Q-learning algorithm and determine its convergence.

1.3. Outline of the Paper. This paper is organized as follows: In Section 2, we frame our problem in terms of iterated random contraction operators and introduce the notion of Wasserstein divergence, which is our key tool. We introduce our three main technical results in Section 3. We then proceed to establish the properties of the Wasserstein divergence in Section 4. The detailed proofs of our three main results are presented in Section 5. We then study the convergence of several constant stepsize RSAs in machine and reinforcement learning in detail in Section 6, as well as develop some new algorithms and show how their convergence can be readily established in our framework. We conclude in Section 7 and present some directions for future research.

2. Problem Formulation.

2.1. Preliminaries. Let $\mathcal{X}$ be a vector space with a metric $\rho$ so that $(\mathcal{X}, \rho)$ is a complete and separable (Polish) space. Examples include: Euclidean spaces for any $p$-norm (where $p \geq 1$), Euclidean space with a weighted max norm, separable Banach and Hilbert spaces (for example, $\ell_p$ spaces for $p \in [1, \infty]$), the space of continuous functions (in the supremum norm) over compact Hausdorff spaces, etc. Let $T : \mathcal{X} \to \mathcal{X}$ be a contraction operator with contraction coefficient $\alpha \in [0, 1)$ and (unique) fixed point $x^*$. The solution of each of our canonical problems is the fixed point $x^*$ of some such $T$.

We introduce another vector space $\mathcal{S}$ in addition to $\mathcal{X}$ (we will interpret $\mathcal{S}$ as a “lifting” of $\mathcal{X}$). Many variance reduction algorithms augment the original state space $\mathcal{X}$ with additional information (e.g. the proxy terms in SAGA which store past gradient evaluations). The lifting $\mathcal{S}$ allows us to cover these algorithms. For SAGA, $\mathcal{S} = \mathcal{X} \times \mathcal{Y}$ where $\mathcal{Y}$ is the space of proxies for all past gradient evaluations. For SVRG, we just have $\mathcal{S} = \mathcal{X}$. We frame the rest of our discussion on $\mathcal{S}$ to allow enough generality to cover all these cases. Abusing notation, we also
let $\rho$ denote a metric on $S$ so that $(S, \rho)$ is a Polish space (the metric on $S$ is usually based on the metric on $\mathcal{X}$ anyway).

When we operate on $S$, we are interested in computing $s^*$, which is an appropriate lifting of the desired fixed point $x^*$. For SAGA, $s^*$ is the concatenation of $x^*$ and all the gradient evaluations at $x^*$. For SVRG, $s^*$ is just $x^*$. In any case, we can recover $x^*$ from $s^*$.

The (classical) Wasserstein distance is defined next. Note that this definition is in terms of the metric $\rho$ on $S$. Throughout, we let $\mathcal{P}(\cdot)$ denote the collection of all probability measures on a given set.

**Definition 2.1.** Let $p \in [1, \infty)$.

(i) $\mathcal{P}_p(S)$ is the set of all probability measures on $S$ with finite $p^{th}$-order moment, i.e., those $\mu \in \mathcal{P}(S)$ for which there exists $s_0 \in S$ such that

$$\int_S \rho(s, s_0)^p \mu(ds) < \infty.$$ 

(ii) For $\mu_1, \mu_2 \in \mathcal{P}_p(S), C(\mu_1, \mu_2)$ is the collection of all $\xi \in \mathcal{P}(S \times S)$ with marginals $\mu_1$ and $\mu_2$, i.e., $\xi(B \times S) = \mu_1(B)$ and $\xi(S \times B) = \mu_2(B)$ for all $B \in \mathcal{B}(S)$.

(iii) For $\mu_1, \mu_2 \in \mathcal{P}_p(S)$, the $p-$Wasserstein distance is

$$W_p(\mu_1, \mu_2) \triangleq \left( \inf_{\xi \in C(\mu_1, \mu_2)} \int_{S \times S} \rho(s, s')^p d\xi(s, s') \right)^{1/p}.$$ 

**2.2. Wasserstein Divergence.** We now extend the previous definition (which depends on the metric $\rho$ on $S$) to accommodate Lyapunov functions on $S$. We need this extension because the convergence analyses of many algorithms are done with respect to a Lyapunov function which is not a metric. We work with the following class of Lyapunov functions in this paper.

**Definition 2.2.** A function $V : S \times S \to [0, \infty)$ is a Lyapunov function if $V$ is continuous and the following conditions hold:

(i) (Positive definiteness) $V(s_1, s_2) = 0$ if and only if $s_1 = s_2$.

(ii) (Symmetry) $V(s_1, s_2) = V(s_2, s_1)$ for all $s_1, s_2 \in S$.

(iii) (Inf-Compactness) For any $q \in \mathbb{R}$ and compact set $\mathcal{K} \subset S$, there exists a compact set $\mathcal{L} \subset S$ such that

$$\inf_{(s_1, s_2) \in \mathcal{L} \times \mathcal{K}} V(s_1, s_2) \geq q.$$ 

The above inf-compactness condition is automatically satisfied if $S$ is a compact set and we have $\mathcal{L} = S$ (since the infimum over an empty set is $\infty$).

Note that, by the above definition, $V(s_1, s_2) < \infty$ for all $s_1, s_2 \in S$. Some examples of Lyapunov functions satisfying the above conditions follow:

1. $S = \mathbb{R}^n$ and $V(s_1, s_2) = \rho(s_1, s_2)$ for any metric $\rho$.

2. $S = \mathbb{R}^n$ and $V(s_1, s_2) = \rho(s_1, s_2)^a \phi(\rho(s_1, s_2))$ where $\phi : [0, \infty) \to [0, \infty)$ is a monotonically increasing function satisfying $\phi(0) = 0$ and $a \geq 0$. If $a > 0$, then we can take $\phi$ to be any non-decreasing positive function.
3. For a simplex $S \subset \mathbb{R}^n$, define $V(s_1, s_2) = D_{KL}(s_1, s_2) + D_{KL}(s_2, s_1)$, where $D_{KL}$ is a Kullback-Liebler divergence.

4. $S$ is a compact subset of a Polish space and $V$ is induced from a divergence on this space. For instance, let $Y \subset \mathbb{R}^n$ be a compact set and define $S = C_b(Y)$ to be the set of Hölder continuous and bounded functions:

$$
S = \{ s : \mathbb{R}^n \to \mathbb{R} : |s(y_1) - s(y_2)| \leq L\rho(y_1, y_2)^a, \|s\|_\infty \leq M \},
$$

where $a, L$, and $M$ are positive constants. By the Arzela-Ascoli Theorem, $S$ is a compact Polish space under the metric induced by the sup norm. We then define

$$
V(s_1, s_2) = \|s_1 - s_2\|_\infty \text{ or } V(s_1, s_2) = \left( \int_Y |s_1(y) - s_2(y)|^p d\mu(y) \right)^{1/p},
$$

where $p > 0$ and $\mu$ is a measure with full support.

We use the Lyapunov function defined above to define the $(V, p)$-Wasserstein divergence.

**Definition 2.3.** Let $V$ be a Lyapunov function satisfying all the conditions of Definition 2.2 and $p \in [1, \infty)$.

(i) $\mathcal{P}_{V, p}(S)$ is the set of all probability measures on $S$ with finite $p^{th}$-order moment with respect to $V$, i.e., those $\mu \in \wp(S)$ for which there exists $s_0 \in S$ such that

$$
\int_S V(s, s_0)^p \mu(ds) < \infty.
$$

(ii) For $\mu_1, \mu_2 \in \mathcal{P}_{V, p}(S)$, the $(V, p)$-Wasserstein divergence is

$$
W_{V, p}(\mu_1, \mu_2) \triangleq \left( \inf_{\xi \in C(\mu_1, \mu_2)} \int_{S \times S} V(s, s')^p d\xi(s, s') \right)^{1/p}.
$$

### 2.3. Iteration of random operators.

We express constant stepsize RSAs as iteration of random operators on $S$, which we now formalize. Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\{\mathcal{F}_k\}_{k=0}^\infty$. Random operators $\hat{T}_k : \Omega \times S \to S$ are operator-valued random variables that are $\mathcal{F}_k$-adapted as follows.

**Assumption 2.4.** For any $\omega \in \Omega$ and $s \in S$, $\{\hat{T}_k(\omega, s)\}_{k \in \mathbb{N}}$ is a sequence of i.i.d. random variables.

Under Assumption 2.4, we have a Markov chain $\{s_k\}_{k \geq 0}$ produced by:

$$
s_{k+1} = \hat{T}_k(s_k) \triangleq \hat{T}_k(\omega, s_k), \forall k \geq 0,
$$

where we usually leave the dependence on $\omega$ implicit. We let $\{\mathcal{F}_k\}_{k \in \mathbb{N}}$ denote the filtration corresponding to $\{s_k\}_{k \in \mathbb{N}}$.

There is a transition kernel $\Omega$ on $S$ corresponding to Eq. (2.1) which satisfies:

$$
\Omega(s_k, B) \triangleq \Pr\{s_{k+1} \in B \mid s_k\}, \forall B \in \mathcal{B}(S).
$$
We recursively define the $k$–step transition kernels:

$$
\Omega^{k+1} (s_0, B) \triangleq \int_{\mathcal{S}} \Omega^k (s_0, ds) \Omega (s, B), \quad \forall B \in \mathcal{B}(\mathcal{S}),
$$

for all $k \geq 0$. We let $\mu_k \in \varphi(\mathcal{S})$ denote the marginal distribution of $s_k$ which satisfies $\mu_k = \mu_0 \Omega^k$, for all $k \geq 0$.

When we characterize the evolution of $\{s_k\}_{k \geq 0}$, we can talk about either dynamic $s_{k+1} = \hat{T}_k (s_k)$ or $\mu_{k+1} = \mu_k \Omega$. Both concepts are equivalent. When we characterize the evolution of the distribution of $s_k$, then the dynamic $\mu_{k+1} = \mu_k \Omega$ is more useful.

2.4. Epoch-Based Algorithms. Epoch-based algorithms (e.g. SVRG) deserve special comment. In epoch-based algorithms, each $\hat{T}_k$ is itself the composition of other random operators. We thus call $k \geq 0$ an “epoch” in this case (instead of an iteration), because it calls an inner loop of iterations of other random operators. The lengths of the epochs are given by a sequence of i.i.d. stopping times $\{\tau_k\}_{k \in \mathbb{N}}$ (we may simply take all $\tau_k = M \geq 1$). Epoch $k \geq 0$ will consist of $\tau_k$ inner iterations, starting with the current iterate $s_k \in \mathcal{S}$:

1. Do a full operator/expectation evaluation at $s_k$, that is, compute $T(s_k)$ exactly.
2. Initialize the inner loop with $\tilde{s}_0 = s_k$. For each $m = 0, 1, \ldots, \tau_k - 1$, define auxiliary random operators $\bar{T}_m(\tilde{s}_m, s_k)$ and compute $\tilde{s}_{m+1} = \bar{T}_m(\tilde{s}_m, s_k)$. Within an epoch, we let $\{\bar{T}_m\}_{m \in \mathbb{N}}$ denote the filtration corresponding to $\{\tilde{s}_m\}_{m \in \mathbb{N}}$.
3. Return $s_{k+1} = \tilde{s}_{\tau_k}$.

This procedure produces a sequence $\{s_k\}_{k \in \mathbb{N}}$ via:

$$(2.2) \quad s_{k+1} = \hat{T}_k (s_k) = \bar{T}_{\tau_k-1} (\cdot, s_k) \circ \bar{T}_{\tau_k-2} (\cdot, s_k) \circ \cdots \circ \bar{T}_0 (s_k, s_k), \quad \forall k \geq 0.$$ 

The resulting random operators $\{\hat{T}_k\}_{k \geq 0}$ (formed by the composition of $\{\bar{T}_m\}_{m \in \mathbb{N}}$) are i.i.d. by construction.

3. Overview and Main Results. We will encode the constant step size RSAs under study in the form of Eq. (2.1) (or Eq. (2.2) for epoch-based RSAs). Then, we will analyze the behavior of $\{s_k\}_{k \geq 0}$ in two steps:

1. (Contraction) Show that there exist a Lyapunov function $V, p \in [1, \infty)$, and $\alpha \in [0, 1)$ such that the relationship

$$(3.1) \quad W_{V, p}^2 (\mu_1 \Omega, \mu_2 \Omega) \leq \alpha W_{V, p}^2 (\mu_1, \mu_2) \quad \text{for all } \mu_1, \mu_2 \in \mathcal{P}_{V, p} (\mathcal{S}),$$

holds. Eq. (3.1) is effectively a contraction in the Wasserstein divergence. It captures the essential contraction property of the random operators $\{\hat{T}_k\}_{k \geq 0}$. This property implies the existence of an invariant distribution $\vartheta$ for $\{s_k\}_{k \geq 0}$. It also implies a geometric convergence rate of the marginal distributions $\mu_k = \mu \Omega^k$ to $\vartheta$ with respect to $W_{V, p}^2$, for any initial distribution $\mu \in \mathcal{P}_{V, p} (\mathcal{S})$.

2. (Concentration) Demonstrate the concentration of $\vartheta$ around the desired $s^\ast$.

Many RSAs satisfy the above contraction property, but are not concentrated around $s^\ast$ (e.g. SGD as we will show). So, both of these steps are essential.

In addition, we use the theory developed to develop a class of variance reduction algorithms, and show in the sequel that the convergence of existing variance reduction algorithms can be determined using our framework.
3.1. Contraction in Wasserstein divergence. We first discuss the implications of contraction in the Wasserstein divergence. Our main assumption is formalized next, it characterizes the key contraction property of \( \{\hat{T}_k\}_{k \geq 0} \).

**Assumption 3.1.** (i) There exists a Lyapunov function \( V, p \in [1, \infty) \), and \( \alpha \in (0, 1) \) such that:

\[
W_{p}^{V}(\mu_{1} \Omega, \mu_{2} \Omega) \leq \alpha W_{V,p}(\mu_{1}, \mu_{2}), \forall \mu_{1}, \mu_{2} \in \mathcal{P}_{V,p}(S).
\]

(ii) For any \( \mu \in \mathcal{P}_{V,p}(S) \), \( W_{V,p}(\mu \Omega^{k}, \mu) < \infty \) for all \( k \geq 0 \).

Assumption 3.1(i) states that the marginal distributions of the sequences starting from two different initial distributions mix at a geometric rate. We show in Section 4 that this assumption is trivially satisfied if the operator \( \hat{T}_k \) is itself a contraction with respect to the Lyapunov function \( V \). Assumption 3.1(ii) is satisfied if the random operators do not blow up in finite time, that is, all \( s_k \) are bounded almost surely. We do not require a uniform bound across time; all we need is the iterates to be bounded for any \( k \in \mathbb{N} \), and different sample paths can have different bounds. All the RSAs studied in this paper enjoy this property.

Under Assumption 3.1, there is an invariant distribution \( \vartheta \) for \( \{s_k\}_{k \geq 0} \), and the marginal distributions of \( s_k \) converge geometrically to \( \vartheta \) with respect to the \((V, p)\)-Wasserstein divergence.

**Theorem 3.2.** Suppose Assumption 3.1 holds and let \( \{s_k\}_{k \in \mathbb{N}} \) be produced by Eq. (2.1).

(i) For all \( k \geq 0 \), we have

\[
W_{V,p}(\mu_{1} \Omega^{k}, \mu_{2} \Omega^{k}) \leq \alpha^k W_{V,p}(\mu_{1}, \mu_{2}), \forall \mu_{1}, \mu_{2} \in \mathcal{P}_{V,p}(S).
\]

(ii) There exists a unique invariant distribution \( \vartheta \) for \( \{s_k\}_{k \geq 0} \) satisfying \( \vartheta \Omega = \vartheta \).

(iii) For all \( k \geq 0 \), we have

\[
W_{V,p}(\mu \Omega^{k}, \vartheta) \leq \alpha^k W_{V,p}(\mu, \vartheta), \forall \mu \in \mathcal{P}_{V,p}(S).
\]

**Proof:** See Subsection 5.1.

Theorem 3.2 is analogous to the Banach fixed point theorem, albeit with respect to a divergence rather than a metric. When \( \{\hat{T}_k\}_{k \in \mathbb{N}} \) satisfy Assumption 3.1, then Theorem 3.2 establishes the existence of a “fixed point” (in the sense of the invariant distribution \( \vartheta \)) and it also establishes a linear convergence rate of the sequence of distributions of the sequence \( \{s_k\}_{k \in \mathbb{N}} \) to this invariant measure (with respect to the \((V, p)\)-Wasserstein divergence). While Assumption 3.1(i) is the usual contraction condition and is the only assumption required for the Banach contraction mapping theorem to hold (in a Polish space), Assumption 3.1(ii) allows us to establish this result when the space may not be a metric space.

3.2. Concentration. Next we see that the concentration of \( \vartheta \) around \( s^* \) directly depends on the action of \( \{\hat{T}_k\}_{k \in \mathbb{N}} \) on \( s^* \). For RSAs like SGD and empirical Q-value iteration, the spread of \( \vartheta \) around \( s^* \) depends on the distribution of \( \{\hat{T}_k(s^*)\}_{k \in \mathbb{N}} \).
Theorem 3.3. If Assumption 3.1 holds with $V = \rho$ and $p = 1$ (in which case $W_{V,p} = W_1$), then
\[
W_1(\mu \Omega^k, \mathbb{I}_{\{s^*\}}) \leq \alpha^k W_1(\mu, \mathbb{I}_{\{s^*\}}) + \frac{1 - \alpha^k}{1 - \alpha} \mathbb{E} \left[ \rho(\hat{T}_1(s^*), s^*) \right].
\]
This implies $W_1(\vartheta, \mathbb{I}_{\{s^*\}}) \leq \frac{1}{1 - \alpha} \mathbb{E} \left[ \rho(\hat{T}_1(s^*), s^*) \right]$. 

Proof: See Subsection 5.2.

While the proof of the above theorem is straight-forward, it informs us about the concentration of the invariant distribution $\vartheta$ around $s^*$. In particular, if we expect $\mathbb{E} \left[ \rho(\hat{T}_1(s^*), s^*) \right]$ to be small (that is, $\hat{T}_1(s^*)$ is not too far from $s^*$ for most realizations of $\hat{T}_1$), then the support of $\vartheta$ is a small neighborhood of $s^*$. This reasoning leads to the following assumption which lets us get an even stronger result.

Assumption 3.4. For all $k \geq 0$, $\hat{T}_k(s^*) = s^*$ almost surely.

This assumption is satisfied for many variance reduction algorithms (e.g. SVRG, SAGA, HSAG, etc.). Assumption 3.4 is equivalent to saying that $\mathbb{I}_{\{s^*\}} \Omega = \mathbb{I}_{\{s^*\}}$, or that the transition kernel $\Omega$ always maps $s^*$ back to itself with probability one. It must then be that $\vartheta = \mathbb{I}_{\{s^*\}}$, since the invariant distribution of $\Omega$ is unique, and we get the following conclusion. We note that the following theorem does not require $V$ to be a metric.

Theorem 3.5. Suppose Assumptions 3.1 and 3.4 hold, then $\lim_{k \to \infty} W_{V,p}^p(\mu \Omega^k, \mathbb{I}_{\{s^*\}}) = 0$, that is, $\vartheta = \mathbb{I}_{\{s^*\}}$. Moreover, the rate of convergence is geometric in the $(V, p)$-$\text{Wasserstein}$ divergence.

Proof: From Assumption 3.4, we know that $\mathbb{I}_{\{s^*\}} \Omega = \mathbb{I}_{\{s^*\}}$. Moreover, from Assumption 3.1, we have $\vartheta\Omega = \vartheta$. This yields
\[
W_{V,p}^p(\mathbb{I}_{\{s^*\}}, \vartheta) = W_{V,p}^p(\mathbb{I}_{\{s^*\}} \Omega, \vartheta \Omega) \leq \alpha W_{V,p}^p(\mathbb{I}_{\{s^*\}}, \vartheta),
\]
where $\alpha < 1$. This immediately implies that $W_{V,p}^p(\mathbb{I}_{\{s^*\}}, \vartheta) = 0$, and so $\mathbb{I}_{\{s^*\}}$ is the invariant distribution (this follows from Proposition 4.2 (ii) which is proved later).

3.3. Epoch-Based Variance Reduction Algorithms. We now show how to construct epoch-based variance reduction algorithms of the form in Subsection 2.4. Recall that $\{\tau_k\}_{k \in \mathbb{N}}$ is a collection of i.i.d. stopping times representing the length of epochs. For a fixed epoch, let $\{\hat{G}_i\}_{i \in \mathbb{N}}$ be i.i.d. operators approximating the contraction operator $T$. Define
\[
\hat{T}_i(\tilde{s}, s) \triangleq \beta_0 \left( \hat{G}_i(\tilde{s}) - \hat{G}_i(s) \right) + \beta_1 T(s),
\]
where $\beta_0, \beta_1 \geq 0$ are free parameters. We make the following additional assumptions for the analysis of this algorithm.

Assumption 3.6. (i) There exists $\alpha \in (0, 1)$ such that:
\[
\rho(\hat{T}_i(\tilde{s}, s), \hat{T}_i(\tilde{s}', s)) \leq \alpha \rho(\tilde{s}, \tilde{s}') \quad \text{for all } \tilde{s}, \tilde{s}' \in \mathcal{S}, s \in \mathcal{S},
\]
for $P$–almost all $\omega \in \Omega$.

(ii) There exists $\kappa \in [0, 1 - \alpha)$ such that $\rho\left(\hat{T}_i(s^*, s^*), s^*_i\right) \leq \kappa \rho(s, s^*)$ for all $i \geq 0$ and for $P$–almost all $\omega \in \Omega$.

One can choose $\beta_0, \beta_1$ so that the above assumption is satisfied by $\hat{T}_i$. To see this, first suppose $S$ is a vector space with a norm $\|\cdot\|$ that induces a metric $\rho$. Let $\gamma$ denote the contraction coefficient of $T$ (with respect to $\rho$). Then suppose that $\hat{G}_i$ is almost surely $\beta$–Lipschitz with respect to $\rho$. Consequently, for $P$–almost all $\omega \in \Omega$, we have

$$\|\hat{T}_i(\hat{s}, s) - \hat{T}_i(\hat{s}', s)\| = \|\beta_0(\hat{G}_i(\hat{s}) - \hat{G}_i(\hat{s}'))\| \leq \beta_0 \beta \|\hat{s} - \hat{s}'\|.$$ 

In addition, we have

$$\|\hat{T}_i(s^*, s) - s^*\| \leq \|\beta_0(\hat{G}_i(s^*) - \hat{G}_i(s))\| + \|\beta_1 T(s) - T(s^*)\| \leq (\beta_0 \alpha + \beta_1 \gamma) \|s - s^*\| + |1 - \beta_1| \|T(s)\|.$$ 

Pick $\beta_1 = 1$ and $\beta_0 < (1 - \gamma)/(2 \beta)$ to immediately yield $\alpha = \beta_0 \beta = (1 - \gamma)/2 < 1$ and $\kappa + \alpha = 2\beta_0 \beta + \gamma < 1$. Depending on the problem, one could take a higher value of $\beta_0$ without violating Assumption 3.6. For instance, in SVRG, we take $\beta_0 = \beta_1 = 1$. Our main convergence result for this class of variance reduced algorithms follows.

**Theorem 3.7.** Let $\{s_k\}_{k \geq 0}$ be produced by Eq. 2.2 and define $\xi_m \triangleq \alpha^m + \kappa(1 - \alpha^m)(1 - \alpha)$. Suppose that Assumption 3.6 holds, then $\xi_m < 1$ for any $m \in \mathbb{N}$ and

$$\rho(s_k, s^*) \leq \left(\prod_{i=0}^{k} \xi_m\right) \rho(s_0, s^*) \quad \text{for all } k \in \mathbb{N}.$$ 

**Proof:** See Subsection 5.3.

Based on Theorem 3.7, we see that $\{s_k\}_{k \geq 0}$ converges geometrically to $s^*$ over epochs for any choice of epoch lengths $\{\tau_k\}_{k \in \mathbb{N}}$, where this convergence is in the almost sure sense.

We now consider variance reduction schemes in a separable Hilbert space (e.g. $\mathbb{R}^d$ equipped with the inner product and the induced $\ell_2$–norm). In this setting, we need $\hat{G}_i$ to yield unbiased estimates of the underlying contraction operator $T$. Define:

$$T_i(\hat{s}, s) \triangleq \hat{G}_i(\hat{s}) - \hat{G}_i(s) + T(s), \forall s, \hat{s} \in \mathcal{S}.$$ 

We make the following assumptions on inner iteration of this random operator within each epoch.

**Assumption 3.8.** Suppose epoch $k \geq 0$ is fixed.

(i) We have $E\left[\hat{G}_m(\hat{s})\right] = T(\hat{s})$ for all $\hat{s} \in \mathcal{S}$.

(ii) There exists $\alpha \in (0, 1)$ such that:

$$E\left[\|\hat{G}_m(\hat{s}) - \hat{G}_m(\hat{s}')\|^2\right] \leq \alpha \|\hat{s} - \hat{s}'\|^2, \forall \hat{s}, \hat{s}' \in \mathcal{S}.$$ 

(iii) There exists $\kappa \in [0, 1 - \gamma)$ such that

$$E\left[\|\hat{G}_m(s) - \hat{G}_m(s')\|^2\right] \leq \kappa \|s - s'\|^2, \forall s, s' \in \mathcal{S}.$$
Our main convergence result in expectation follows.

**Theorem 3.9.** Let \( \{s_k\}_{k \geq 0} \) be produced by Eq. 2.2, suppose Assumption 3.8 holds, and define \( \xi_m \triangleq \alpha^m + \kappa (1 - \alpha^m)/(1 - \alpha) \). Then \( \xi_m < 1 \) for any \( m \in \mathbb{N} \) and

\[
\mathbb{E} \left[ \|s_k - s^*\|^2 \right] \leq \mathbb{E} \left[ \prod_{i=0}^{k} \xi_{\tau_i} \right] \|s_0 - s^*\|^2, \quad \text{for all } k \in \mathbb{N}.
\]

**Proof:** See Subsection 5.4.

**Remark 3.10.** In Theorem 3.7, we show that the sequence \( \{s_k\} \) converges almost surely to \( s^* \). On the other hand, Theorem 3.9 implies that the convergence is in 2-Wasserstein metric:

\[
W_2(\mu^{Q_k}, 1_{\{s^*\}}) \leq \sqrt{\mathbb{E} \left[ \prod_{i=0}^{k} \xi_{\tau_i} \right] W_2(\mu, 1_{\{s^*\}})}.
\]

Since convergence in Wasserstein metric implies weak convergence, we conclude that the result of Theorem 3.9 implies convergence of the sequence \( \{s_k\} \) to \( s^* \) in probability.

**4. Preliminary Results on Wasserstein Divergence Based Convergence.** The Wasserstein divergence is a generalization of the Wasserstein metric, but it enjoys many properties of the same properties. In particular, a metric separates distinct points in the space, and we show this to be true for the Wasserstein divergence as well. We also show that a sequence of measures converging to a measure in the Wasserstein divergence also converges in the weak* topology (this is true for the Wasserstein metric as well). We also establish that a Cauchy sequence under the Wasserstein divergence converges to a unique limit – this fact is used to prove our main results in Section 5.

**4.1. Properties of Wasserstein Divergence.** We start by discussing the basic properties of the Wasserstein divergence. First we establish that \( W_{V, p} \) is a weak* lower semicontinuous function of the Lyapunov function.

**Lemma 4.1.** The map \( \xi \mapsto \int V^p(s_1, s_2) d\xi \) is weak* lower semicontinuous on \( \mathcal{P}(S \times S) \).

**Proof:** See [41, Lemma 4.3, p. 43].

The next result exploits the structure of our Lyapunov functions, as outlined in Definition 2.2, to show that the Wasserstein divergence is induced by an optimal coupling, separates distinct points, and is symmetric. These properties justify calling \( W_{V, p} \) a divergence.

**Proposition 4.2.** Let \( V \) be a Lyapunov function satisfying Definition 2.2, and let \( p \in [1, \infty) \). Pick \( \mu_1, \mu_2 \in \mathcal{P}(S) \). Then, the following statements hold:

(i) If \( V \) is continuous, then there exists an optimal coupling \( \xi^* \in C(\mu_1, \mu_2) \) in the definition of \( W_{V, p} \). Consequently, there exists a pair of random variables \( (s_1, s_2) \) on \( S \) such that

\[
\mathbb{E} [V(s_1, s_2)^p] = W_{V, p}^p(\mu_1, \mu_2).
\]

(ii) If \( V \) is positive definite, then \( W_{V, p}(\mu_1, \mu_2) = 0 \) if and only if \( \mu_1 = \mu_2 \).

(iii) If \( V \) is symmetric, then \( W_{V, p}(\mu_1, \mu_2) = W_{V, p}(\mu_2, \mu_1) \).

(iv) If \( V = \rho \), then \( W_{V, p} \) coincides with \( W_p \) (the usual Wasserstein distance), and is a metric. Furthermore, the space of probability measures is complete under this metric.
The first result follows because $C(\mu_1, \mu_2)$ is a weak* compact set in the space of probability measures ([41, Lemma 4.4, p. 44]) and Lemma 4.1. Consequently, we can apply the Weierstrass extreme value theorem to demonstrate the existence of $\xi^*$ that achieves the infimum in the definition of $W_{V,p}$ (which is the desired optimal coupling). Since an optimal coupling $\xi^*$ exists, by Strassen’s theorem [40], there exists a pair of random variables $(s_1, s_2)$ on $S$ which has the joint distribution $\xi^*$. For this pair, the equality in (4.1) holds.

For the second statement, if $\mu_1 = \mu_2$, by Strassen’s theorem [40], we can define random variables $s_1, s_2$ such that $s_1 = s_2$ almost surely. As a result, $W_{V,p}(\mu_1, \mu_2) = 0$. To prove the converse, if $W_{V,p}(\mu_1, \mu_2) = \inf_{\xi \in C(\mu_1, \mu_2)} \mathbb{E}[V(s_1, s_2)^p] = 0$, then it must be that $V(s_1, s_2) = 0, \xi^*$-almost surely. Consequently, we must have $\mu_1 = \mu_2$ since $\xi^* \in C(\mu_1, \mu_2)$.

We now establish the third statement. If $V$ is symmetric, then

$$W_{V,p}(\mu_1, \mu_2) = \inf_{\xi \in C(\mu_1, \mu_2)} \mathbb{E}[V(s_1, s_2)^p] = \inf_{\xi \in C(\mu_2, \mu_1)} \mathbb{E}[V(s_2, s_1)^p] = W_{V,p}(\mu_2, \mu_1),$$

where the second equality follows from symmetry of $V$. The fourth statement follows from [41, Theorem 4.3].

The next two results together establish that if a sequence of probability measures converging to another measure $\theta$ in $W_{V,p}$, then the limit point is also a probability measure, and the sequence converges to this probability measure in the weak* topology. This is a well-known result for the Wasserstein metric (note that the topology induced by the Wasserstein metric is stronger than the weak* topology). We first need the following lemma.

**Lemma 4.3.** Let $\{\mu_k\}_{k \in \mathbb{N}}$ be a sequence of probability measures converging to $\theta$ in the weak* topology. Pick any $\nu \in \varphi(S)$ and let $\xi^*_k \in C(\mu_k, \nu)$ be an optimal coupling in $W_{V,p}(\mu_k, \nu)$ for all $k \in \mathbb{N}$. Then, the set of probability measures $\{\xi^*_k\}_{k \in \mathbb{N}}$ is tight and there exists a subsequence $\{\xi^*_k\}_{k \in \mathbb{N}}$ that converges to $\xi \in C(\theta, \nu)$.

**Proof:** Since $\{\mu_k\}_{k \in \mathbb{N}}$ converges to $\theta$, the set $\{\mu_1, \mu_2, \ldots, \theta\}$ is tight. Consequently, for any $\epsilon > 0$ there is a compact set $K \subset S$ such that $\mu_k(K) \geq 1 - \epsilon$ for all $k \in \mathbb{N}$ and $\theta(K) \geq 1 - \epsilon$.

Let $L \subset S$ be another compact set such that $\nu(L) \geq 1 - \epsilon$. Then, for all $k \in \mathbb{N}$, $\xi^*_k$ satisfies

$$\xi^*_k(K \times L^c) = \xi^*_k(K^c \times L) + \xi^*_k(S \times L^c) \leq \mu_k(K^c) + \nu(L^c) < 2\epsilon,$$

since $\xi^*_k \in C(\mu_k, \nu)$. As a result, the set of probability measures $\{\xi^*_k\}_{k \in \mathbb{N}}$ is tight. Now pick a convergent subsequence $\{\xi^*_k\}_{k \in \mathbb{N}}$, and let $\xi$ denote its weak* limit.

Recall that the push forward of a measure with respect to a projection is a continuous mapping. The marginals of $\xi^*_k$ are $\mu_k$ and $\nu$, $\mu_k$ converges to $\theta$, and $\nu$ trivially converges to $\nu$. We conclude that the marginals of $\xi$ are $\theta$ and $\nu$, which implies $\xi \in C(\theta, \nu)$ as desired.

The preceding lemma leads to the following result.

**Theorem 4.4.** Let $V$ be a Lyapunov function satisfying Definition 2.2. Let $\{\mu_k\}_{k \in \mathbb{N}}$ be a sequence of probability measures and let $\theta$ be another measure (not necessarily a probability measure) such that $\limsup_{k \to \infty} W_{V,p}(\mu_k, \theta) = 0$. Then, $\theta$ is also a probability measure and $\mu_k$ converges to $\theta$ in the weak* topology.

**Proof:** Choose $K \in \mathbb{N}$ such that $W_{V,p}(\mu_k, \theta) \leq 1$ for all $k \geq K$. Since $V$ satisfies condition (iv) in Definition 2.2, by [18, Lemma 7.13, p. 107] we know that the set $\{\mu_k\}_{k \geq K}$ is a
weak* compact set of probability measures. Consequently, there exists a weak* convergent subsequence \( \{ \mu_k \}_k \) converging to a weak* limit, say \( \bar{\theta} \), which is a probability measure. Since \( \limsup_{k \to \infty} W_{V,p}(\mu_k, \theta) = 0 \), we conclude that \( \limsup_{l \to \infty} W_{V,p}(\mu_k, \theta) = 0 \), and so we must have \( \bar{\theta} = \theta \).

Let \( \xi_k^* \in C(\mu_k, \theta) \) be an optimal coupling in \( W_{V,p}(\mu_k, \theta) \). Since \( W_{V,p}(\mu_k, \theta) \to 0 \) and \( \mu_k \to \bar{\theta} \) in the weak* topology, by Lemma 4.3 we conclude that there exists a further subsequence, which we denote by \( \xi_k^* \), by a slight abuse of notation, that converges to a limit \( \xi^* \in C(\bar{\theta}, \theta) \).

Since \( V \) is continuous (but possibly unbounded), Lemma 4.1 shows us that

\[
\int V(s_1, s_2)\xi^*(ds_1, ds_2) \leq \liminf_{l \to \infty} \int V(s_1, s_2)\xi_k^*(ds_1, ds_2) = 0 \implies W_{V,p}(\tilde{\theta}, \theta) = 0.
\]

Since \( V \) is positive definite, we conclude that \( \tilde{\theta} = \theta \) by Proposition 4.2.

The above argument also implies that \( \{ \mu_k \}_k \) is tight and that the set of weak* limit points of \( \{ \mu_k \}_k \) is a singleton \( \{ \theta \} \). Since there is a unique limit point of the sequence \( \{ \mu_k \}_k \), the entire sequence must converge to \( \theta \), and hence the proof is complete.

**Example 4.5.** We show by example that \( V \) must satisfy condition (iii) in Definition 2.2 for the above result to hold. Let \( \mathcal{S} = \mathbb{R} \) and let \( V(s_1, s_2) = |s_1 - s_2| \exp(-|s_1 - s_2|) \). It is clear that \( V \) is a positive definite Lyapunov function. Pick \( \mu_k = 1_{\{k\}} \) and \( \theta = 0 \) (the zero measure). We readily have \( W_{V,p}(\mu_k, \theta) = k \exp(-k) \to 0 \) as \( k \to \infty \). In other words, the \((V, p)\)-Wasserstein limit of \( \mu_k \) is not a probability measure. The above theorem shows that this situation will not arise if \( V \) satisfies certain growth conditions.

It is well-known that a Cauchy sequence in a large class of metric spaces converges. We next show that a similar property is enjoyed by the Wasserstein divergence. This is the first key property that we need for our proof of Theorem 3.2 in Section 5.

**Proposition 4.6.** Let \( V \) be a Lyapunov function satisfying Definition 2.2. Let \( \{ \mu_k \}_k \) be a sequence of probability measures such that, for \( \epsilon > 0 \) and \( l \in \mathbb{N} \), there exists \( K_\epsilon \in \mathbb{N} \) and constant \( c_1 < \infty \) with \( W_{V,p}(\mu_k, \mu_{k+l}) < c_1 \epsilon \) for all \( k \geq K_\epsilon \). Furthermore, there exists a probability measure \( \theta \) such that \( \mu_k \to \theta \) in the weak* limit.

**Proof:** Pick \( \epsilon > 0 \) and fix \( l \in \mathbb{N} \). Since \( V \) satisfies condition (iii) in Definition 2.2, by [18, Lemma 7.13, p. 107], the set \( \{ \mu_k \}_{k \geq K_\epsilon} \) is a weak* precompact set of measures. Thus, there exists a weak* convergent subsequence \( \{ \mu_{k_m} \} \) such that \( \mu_{k_m} \to \theta \), where \( \theta \) is a probability measure. We claim that \( \lim_{m \to \infty} W_{V,p}(\mu_{k_m}, \mu_{k_{m+l}}) = 0 \).

Let \( \xi_{kl}^* \) be an optimal coupling in \( W_{V,p}(\mu_k, \mu_{k+l}) \). Since \( \mu_{k_m} \) converges to \( \theta \), we use Lemma 4.3 to conclude that there exists a subsequence \( \xi_{k_{m+l}}^* \) that converges to some \( \xi_{l}^* \in C(\theta, \mu_{k_{m+l}}) \).

Using Lemma 4.1, we conclude that

\[
W_{V,p}(\theta, \mu_{k_{m+l}}) \leq \int V d\xi_{l}^* \leq \liminf_{n \to \infty} \int V d\xi_{k_{m+n}}^* = \liminf_{n \to \infty} W_{V,p}(\mu_{k_{m+n}}, \mu_{k_{m+n}+l}) < c_1 \epsilon.
\]

From Proposition 4.2 (iii), we know that \( W_{V,p}(\theta, \mu_{k_{m+l}}) = W_{V,p}(\mu_{k_{m+l}}, \theta) \). Thus, for every \( \epsilon > 0 \), there exists \( K_\epsilon \in \mathbb{N} \) such that \( W_{V,p}(\mu_{k_{m+l}}, \theta) < c_1 \epsilon \) for all \( k \geq K_\epsilon \). Since \( \epsilon > 0 \) is arbitrary, we must have \( \lim_{k \to \infty} W_{V,p}(\mu_{k_{m+l}}, \theta) = 0 \). By Theorem 4.4, we conclude that \( \mu_k \) converges to \( \theta \) in the weak* sense.
In the preceding proof, we invoked symmetry of the Wasserstein divergence for the first time. In some cases, the Lyapunov function \( V \) may not be symmetric and our main results would not be applicable. However, one could potentially verify that the above result still holds, perhaps by exploiting the specific structure of the Lyapunov function in question.

We now consider two sequences of measures that converge with respect to the Wasserstein divergence. If the divergence between the elements of these two sequences approaches zero, then it is reasonable to expect that the limits of the two sequences should be the same. This property of the Wasserstein divergence is established in the next proposition, and it is the second key result leading to the proof of Theorem 3.2 in Section 5.

**Proposition 4.7.** Let \( V \) be a Lyapunov function satisfying Definition 2.2. Let \( \{\mu_k\}_{k\in\mathbb{N}} \) and \( \{\nu_k\}_{k\in\mathbb{N}} \) be two convergent sequences of probability measures converging to \( \theta \mu \) and \( \theta \nu \), respectively, in the \((V,p)\)-Wasserstein divergence. Then,

\[
\lim_{k \to \infty} W_{V,p}(\mu_k, \nu_k) = 0 \implies \theta \mu = \theta \nu.
\]

**Proof:** From Theorem 4.4, we conclude that \( \{\mu_k\}_{k\in\mathbb{N}} \) and \( \{\nu_k\}_{k\in\mathbb{N}} \) converge to \( \theta \mu \) and \( \theta \nu \) in the weak* topology, respectively. Let \( \xi^*_n \) be an optimal coupling in \( W_{V,p}(\mu_k, \nu_k) \). By essentially the same argument as in Lemma 4.3, we conclude that \( \{\xi^*_n\}_{n\in\mathbb{N}} \) is a tight set of measures. Therefore, it includes a convergent subsequence \( \{\xi^*_n\}_{n\in\mathbb{N}} \) which converges to some \( \xi \in C(\theta \mu, \theta \nu) \) in the weak* sense. Further, this coupling \( \xi \) satisfies

\[
W_{V,p}(\theta \mu, \theta \nu) \leq \int V d\xi \leq \lim_{n \to \infty} \int V d\xi^*_n = \lim_{n \to \infty} W_{V,p}(\mu_k, \nu_k) = 0,
\]

by Lemma 4.1. By Proposition 4.2(i), the above expression immediately yields \( \theta \mu = \theta \nu \), completing the proof.

4.2. Some sufficient conditions. Assumption 3.1 is the key to our analysis, but it can be hard to verify. Here we provide some sufficient conditions that are easier to check. We start with two sufficient conditions for Assumption 3.1(i).

**Assumption 4.8.** There exists \( \alpha \in (0,1) \) such that:

\[
\rho \left( \hat{T}_k (s), \hat{T}_k (s') \right) \leq \alpha \rho (s, s'), \forall s, s' \in \mathcal{S},
\]

for all \( k \geq 0 \), almost surely.

**Theorem 4.9.** Suppose Assumption 4.8 holds. Then

\[
W_1 (\mu_1 \Omega, \mu_2 \Omega) \leq \alpha W_1 (\mu_1, \mu_2), \forall \mu_1, \mu_2 \in \mathcal{P}_1 (\mathcal{S}).
\]

**Proof:** There exist \( s, s' \), independent of \( \{\hat{T}_k\}_{k\geq0} \), such that \( \mathbb{E} [\rho (s, s')] = W_1 (\mu_1, \mu_2) \), by Proposition 4.2(i). We then have:

\[
W_1 (\mu_1 \Omega, \mu_2 \Omega) \leq \mathbb{E} [\rho (\hat{T}_0 (s), \hat{T}_0 (s'))] \leq \alpha \mathbb{E} [\rho (s, s')] = W_1 (\mu_1, \mu_2),
\]

where the first inequality follows by definition of \( W_1 (\mu_1 \Omega, \mu_2 \Omega) \) and the second inequality follows by Assumption 4.8.
The next sufficient condition is with respect to the conditional expectation of a Lyapunov function.

**Assumption 4.10.** Given $V$, there exists $p \in [1, \infty)$ and $\alpha \in (0, 1)$ such that:

$$
\mathbb{E} \left[ V \left( \hat{T}_k (s), \hat{T}_k (s') \right)^p \right] \leq \alpha V (s, s')^p, \forall s, s' \in S,
$$

for all $k \geq 0$.

**Theorem 4.11.** Suppose Assumption 4.10 holds. Then

$$
W_{p,V} (\mu_1 \Omega, \mu_2 \Omega) \leq \alpha W_{p,V} (\mu_1, \mu_2), \forall \mu_1, \mu_2 \in \mathcal{P}_{V,p}(S).
$$

**Proof:** There exist random variables $s, s'$, independent of $\{\hat{T}_k\}_{k \in \mathbb{N}}$, such that

$$
\mathbb{E} \left[ V (s, s')^p \right] = W_{p,V} (\mu_1, \mu_2), \text{ by Proposition 4.2(i).}
$$

We then have

$$
W_{p,V} (\mu_1 \Omega, \mu_2 \Omega) \leq \mathbb{E} \left[ V \left( \hat{T}_0 (s), \hat{T}_0 (s') \right)^p \right] \leq \alpha \mathbb{E} \left[ V (s, s')^p \right] = W_{p,V} (\mu_1, \mu_2),
$$

where the first inequality follows by definition of $W_{p,V} (\mu_1 \Omega, \mu_2 \Omega)$ and the second inequality follows by Assumption 4.10. $\blacksquare$

Now we give some sufficient conditions for Assumption 3.1(ii).

**Assumption 4.12.** $S$ is compact.

Under Assumption 4.12, we must have $W_{V,p}(\mu, \nu) < \infty$ for any two $\mu, \nu \in \wp(S)$. In particular, Assumption 3.1(ii) is automatically satisfied whenever the iterates $\{s_k\}_{k \in \mathbb{N}}$ are almost surely bounded. This is the case for solving monotone inclusions restricted to a compact set or for solving finite state/action MDPs (since both problems enjoy a priori bounds on the optimal solutions).

The next sufficient condition applies when we cannot a priori bound all of the iterates.

**Assumption 4.13.** There is a constant $0 \leq C < \infty$ such that $\rho(\hat{T}_k (s), s) \leq C$ almost surely, for all $k \in \mathbb{N}$ and any $s \in S$.

Assumption 4.13 essentially says that $\{\hat{T}_k\}_{k \in \mathbb{N}}$ cannot map any $s \in S$ too far from its current position. Under this condition, we must have $W_{V,p}(\mu \Omega^k, \mu) < \infty$ for all finite $k \geq 0$.

### 4.3. Concentration around Dirac mass.

In some cases we can show that the invariant distribution $\vartheta$ has all of its mass concentrated at the desired solution $s^*$. In particular, we find this to be the case when $\hat{T}_k (s^*) = s^*$ almost surely, for all $k \geq 0$ (e.g. this condition is satisfied for variance reduction algorithms like SVRG and SAGA). The condition $\hat{T}_k (s^*) = s^*$ is essential, since it is not satisfied by SGD which conforms to the observed high variance of SGD around $s^*$ in steady state.

In the following result, we characterize the Wasserstein divergence between a measure and a Dirac mass at $s^* \in S$.

**Lemma 4.14.** For any $\mu \in \wp(S)$, we have $C(\mu, \mathbb{1}_{\{s^*\}}) = \{\mu \mathbb{1}_{\{s^*\}}\}$, i.e., there is only one coupling between $\mu$ and $\mathbb{1}_{\{s^*\}}$. 


Proof: Any coupling $\xi \in C(\mu, \mathbb{I}_{(s^*)})$ can be disintegrated as

$$\xi(ds_1, ds_2) = \xi(ds_1|s_2)\xi(ds_2) = \xi(ds_1|s_2)\mathbb{I}_{(s^*)}(ds_2).$$

The marginal measure on $S$ is $\mu(ds_1) = \xi(ds_1|s^*)$. Thus, any coupling $\xi \in C(\mu, \mathbb{I}_{(s^*)})$ must satisfy $\xi(\cdot|s^*) = \mu(\cdot)$.

Two couplings $\xi, \tilde{\xi} \in C(\mu, \mathbb{I}_{(s^*)})$ are the same if $\xi(\cdot|s_2)$ and $\tilde{\xi}(\cdot|s_2)$ differ on a set of measure zero. As a result, any $\xi \in C(\mu, \mathbb{I}_{(s^*)})$ is the same as $\tilde{\xi} = \mu\mathbb{I}_{(s^*)}$ (since $\xi$ and $\tilde{\xi}$ only differ on a set of measure zero).

Corollary 4.15. Let $s$ be a random variable on $S$ with probability distribution $\mu \in \mathcal{P}(S)$. Then, $W_{V,p}(\mu, \mathbb{I}_{(s^*)}) = E[V(s, s^*)^p]$. 

Proof: This result is a direct consequence of Lemma 4.14.

5. Proofs of The Main Results.

5.1. Proof of Theorem 3.2. (i) This part follows immediately by iterating the recursion in Assumption 3.1.

(ii) Choose any $\mu \in \mathcal{P}_{V,p}(S)$ and define the sequence $\mu_k = \mu \Omega^k$ for all $k \in \mathbb{N}$. Now pick any $l \in \mathbb{N}$. We then have $W_{V,p}(\mu_k, \mu_{k+l}) = W_{V,p}(\mu \Omega^k, \mu \Omega^{k+l}) \leq \alpha^k W_{V,p}(\mu, \mu \Omega^l)$ by Assumption 3.1. Consequently, we can apply Proposition 4.6 to conclude that there exists a probability measure $\bar{\nu}$ such that $\mu_k \to \bar{\nu}$ in the weak* sense. By the same argument as in Lemma 4.3, it follows that $\lim_{k \to \infty} W_{V,p}(\mu_k, \bar{\nu}) = 0$.

Now we show that this limit is the same for all initial $\mu \in \mathcal{P}_{V,p}(S)$. Choose any $\mu, \nu \in \mathcal{P}_{V,p}(S)$ and define the sequences $\mu_k = \mu \Omega^k$ and $\nu_k = \nu \Omega^k$ for all $k \in \mathbb{N}$. By the previous argument, $\mu_k$ converges to some $\bar{\mu}$, and $\nu_k$ converges to some $\bar{\nu}$ (also in the weak* sense). Furthermore,

$$W_{V,p}(\mu_k, \nu_k) = W_{V,p}(\mu \Omega^k, \nu \Omega^k) \leq \alpha^k W_{V,p}(\mu, \nu).$$

By Proposition 4.7, it follows that $\bar{\mu} = \bar{\nu}$ and so the limit must be the same for all initial $\mu \in \mathcal{P}_{V,p}(S)$. We denote this limit as $\bar{\nu}$. Note that this limit is unique since, for any initial condition, the limiting measure is always $\bar{\nu}$.

To complete the proof, we show that $\bar{\nu}$ is invariant with respect to $\Omega$. As before, choose any $\mu \in \mathcal{P}_{V,p}(S)$ and define the sequence $\mu_k = \mu \Omega^k$ for all $k \in \mathbb{N}$. We have just shown that $\lim_{k \to \infty} W_{V,p}(\mu_k, \bar{\nu}) = 0$. Since $W_{V,p}(\mu_k \Omega, \bar{\nu} \Omega) \leq \alpha W_{V,p}(\mu_k, \bar{\nu})$, we see that $\lim_{k \to \infty} W_{V,p}(\mu_k \Omega, \bar{\nu} \Omega) = 0$ and so the sequence $\{\mu_k \Omega\}_{k \in \mathbb{N}}$ must converge to $\bar{\nu} \Omega$. By Proposition 4.7, the limits of the sequences $\{\mu_k\}_{k \in \mathbb{N}}$ and $\{\mu_k \Omega\}_{k \in \mathbb{N}}$ are equal, and thus $\bar{\nu} = \bar{\nu} \Omega$.

(iii) Follows from invariance of $\bar{\nu}$ with respect to $\Omega$, i.e., $\bar{\nu} = \bar{\nu} \Omega^k$ for all $k \in \mathbb{N}$.

5.2. Proof of Theorem 3.3. Since all $\hat{T}_k$ have contraction coefficient $\alpha$ with respect to the metric $V = \rho$, we may apply the triangle inequality to get:

$$V(\hat{T}_k(s), s^*) \leq V(\hat{T}_k(s), \hat{T}_k(s^*)) + V(\hat{T}_k(s^*), s^*) \leq \alpha V(s, s^*) + V(\hat{T}_k(s^*), s^*).$$
Iterating this inequality gives:

\[ V(s_k, s^*) \leq \alpha^k V(s_0, s^*) + \sum_{j=0}^{k-1} \alpha^j V(\hat{T}_j(s^*), s^*), \forall k \geq 0. \]

The desired conclusion then follows by taking expectations in the above inequality and using Lemma 4.14, since all \( \{\hat{T}_k\}_{k \in \mathbb{N}} \) are mutually independent.

5.3. Proof of Theorem 3.7. Since \( \kappa < (1 - \alpha) \), we conclude that

\[ \xi_m = \alpha^m + \frac{\kappa(1 - \alpha^m)}{1 - \alpha} < \alpha^m + (1 - \alpha^m) = 1, \]

for any \( m \in \mathbb{N} \). Consequently, \( \xi_{\tau_i} < 1 \) almost surely. Now, consider the \( i \)th epoch and recall that we initialize with \( \tilde{s}_0 = s_i \). By the triangle inequality,

\[ \rho(\tilde{s}_m, s^*) = \rho(\tilde{T}^m(\tilde{s}_0, s_i), s^*) \leq \rho(\tilde{T}^m(\tilde{s}_0, s_i), \tilde{T}^m(s^*, s_i)) + \rho(\tilde{T}^m(s^*, s_i), s^*). \]

The first summand on the right side is upper bounded by \( \alpha^m \rho(s_i, s^*) \) by Assumption 3.6 (i). Now consider the second summand on the right side. We can upper bound it with

\[ \rho(\tilde{T}^m(s^*, s_i), s^*) \leq \rho(\tilde{T}^m(s^*, s_i), \tilde{T}^{m-1}(s^*, s_i)) + \ldots + \rho(\tilde{T}(s^*, s_i), s^*) \]

\[ \leq \sum_{i=1}^{m-1} \alpha^i \rho(\tilde{T}_i(s^*, s_i), s^*) \]

\[ \leq \sum_{i=1}^{m-1} \alpha^i \kappa \rho(s_i, s^*) = \kappa \frac{(1 - \alpha^m)}{(1 - \alpha)} \rho(s_i, s^*), \]

where the first and the second inequalities follow from Assumption 3.6 (i) and (ii), respectively. Collecting these inequalities yields

\[ \rho(\tilde{s}_m, s^*) \leq \alpha^m \rho(s_i, s^*) + \kappa \frac{(1 - \alpha^m)}{(1 - \alpha)} \rho(s_i, s^*) = \left[ \alpha^m + \kappa \frac{(1 - \alpha^m)}{(1 - \alpha)} \right] \rho(s_i, s^*). \]

It follows immediately that \( \rho(s_{i+1}, s^*) \leq \xi_{\tau_i} \rho(s_i, s^*) \), where \( \xi_{\tau_i} \) is almost surely strictly less than 1. The desired result follows by iterating this recursion.

5.4. Proof of Theorem 3.9. Let \( \{s'_k\}_{k \in \mathbb{N}} \) be another iteration of Eq. 2.2 with corresponding inner iterations \( \{\tilde{s}'_m\}_{m \in \mathbb{N}} \). Let \( \mathcal{F}_m \) be the \( \sigma \)-algebra generated by the random variables \( \{\tilde{s}_0, \ldots, \tilde{s}_m, \tilde{s}_0', \ldots, \tilde{s}_m'\} \). As a consequence of Assumption 3.8 (by expanding the squared-norm and using conditional unbiasedness), we have:

\[ \mathbb{E} \left[ \|\tilde{s}_{m+1} - \tilde{s}'_{m+1}\|_2^2 | \mathcal{F}_m \right] \leq \alpha \|\tilde{s}_m - \tilde{s}'_m\|_2^2 + \kappa \|s_k - s'_{k}\|_2^2, \forall m \geq 0. \]

Recall that we initialize epoch \( k \geq 0 \) with \( \tilde{s}_0 = s_k \) (and \( \tilde{s}'_0 = s'_k \), then we get

\[ \mathbb{E} \left[ \|\tilde{s}_{m+1} - \tilde{s}'_{m+1}\|_2^2 | \mathcal{F}_m \right] \leq \xi_m \|s_k - s'_{k}\|_2^2, \forall m \geq 0. \]

Taking \( s'_k = s^* \) for all \( k \geq 0 \) (which holds by construction of the variance-reduced operator if we initialize with \( s'_0 = s^* \)) gives the desired result.
6. Algorithms: Old and New. In this section, we recover many existing algorithms within our framework, and also create some new algorithms. The content of this section can be grouped into three families:

1. Empirical algorithms: use sample average approximation to estimate expectations. Subsection 6.1 considers mini-batch stochastic gradient descent and Subsection 6.2 examines empirical Q-value iteration.

2. Epoch-based algorithms: evaluate expectations exactly at fixed intervals. Subsection 6.3 discusses SVRG and Subsection 6.4 develops a new variance-reduced version of Q-value iteration.

3. Proxy-based algorithms: use proxy terms for all nonlinearities which are dynamically updated. Subsections 6.5 and 6.6 consider proxy-based forward-backward splitting and Subsections 6.7 and 6.8 consider proxy-based policy iteration (the empirical variant is new).

6.1. Constant stepsize Mini-batch Stochastic Gradient Descent. We first consider mini-batch SGD (see e.g. [30, 14]). We frame this problem in the monotone operator setting on $\mathcal{X} \subset \mathbb{R}^d$, with the following ingredients:

(i) a monotone operator $A : \mathbb{R}^d \to \mathbb{R}^d$;

(ii) a finite collection $\{B_n\}_{n=1}^N : \mathbb{R}^d \to \mathbb{R}^d$ of $c$–strongly monotone, $L$–Lipschitz operators;

(iii) their average $B \triangleq (1/N) \sum_{n=1}^N B_n$, and $B$ is $c$–strongly monotone and $L$–Lipschitz.

The resulting finite sum monotone inclusion problem is:

(6.1) Compute $x^\ast$ s.t. $0 \in A(x^\ast) + B(x^\ast) \equiv A(x^\ast) + \frac{1}{N} \sum_{n=1}^N B_n(x^\ast)$.

The corresponding contraction operator is $T = (I + \eta A)^{-1} (I - \eta B)$ for stepsize $\eta > 0$. This problem can be solved by fixed point iteration of this $T$ (which is known as forward-backward splitting). We define the contraction coefficient $\gamma(\eta) \triangleq 1 - 2\eta c + \eta^2 L^2$ of $T$ to be used throughout.

We want to avoid exact computation of $T$ due to the potentially large sum $B$, so we use randomization to compute tractable approximations of $T$. To this end, we define random operators $\{\hat{T}_k\}_{k \in \mathbb{N}}$ corresponding to mini-batch SGD for batch size $n \geq 1$ by

$$\hat{T}_k(x_k) = (I + \eta A)^{-1} \left( x_k - \frac{\eta}{n} \sum_{l=1}^n B_{l,k,l}(x_k) \right), \ \forall k \geq 0, $$

where $\{I_{k,l}\}_{l=1}^n$ is a sequence of i.i.d. uniform samples from $\{1, \ldots, N\}$. The following lemma expresses the key contraction property of $\hat{T}_k$ (and the usual stepsize selection rule for forward-backward splitting).

**Lemma 6.1.** For all $k \geq 0$, $\hat{T}_k : \mathbb{R}^d \to \mathbb{R}^d$ satisfies

$$\|\hat{T}_k(x) - \hat{T}_k(x')\|_2 \leq \sqrt{\gamma(\eta)} \|x - x'\|_2, \ \forall x, x' \in \mathbb{R}^d,$$

almost surely. For $\eta \in (0, 2 c/L^2)$, we have $\gamma(\eta) < 1$. 

Based on the Hoeffding inequality, for any \( n \geq 1 \) we define the term

\[
f(n, \epsilon) \triangleq (1 - p(n, \epsilon))\epsilon + 2\eta p(n, \epsilon)\|B(x^*)\|_2
\]

where

\[
p(n, \epsilon) \triangleq d \exp \left( -\frac{n\epsilon^2}{2d\|B(x^*)\|_2^2} \right).
\]

This term appears in our following concentration bound for mini-batch SGD.

**Corollary 6.2.** (i) (Contraction) There exists \( \vartheta \in \mathcal{P}_1(\mathcal{X}) \) such that, for all \( \mu \in \mathcal{P}_1(\mathcal{X}) \),

\[
W_1\left( \mu \Omega^k, \vartheta \right) \leq \sqrt{\gamma(\eta)^k} W_1(\mu, \vartheta), \forall k \geq 1.
\]

(ii) (Concentration) For any \( \mu \in \mathcal{P}_1(\mathcal{X}) \), \( n \geq 1 \), and \( \epsilon > 0 \),

\[
W_1\left( \mu \Omega^k, \mathbb{1}_{\{x^*\}} \right) \leq \sqrt{\gamma(\eta)^k} W_1(\mu, \mathbb{1}_{\{x^*\}}) + \frac{1 - \gamma^k}{1 - \gamma} \left( \frac{f(n, \epsilon)}{\epsilon} \right), \forall k \geq 1.
\]

**Proof:** Part (i) follows by Theorem 4.9. For Part (ii), we use Theorem 3.3 and the upper bound \( \mathbb{E}\left[ \|\hat{T}_0(x^*) - T(x^*)\|_2 \right] \leq f(n, \epsilon) \) which holds by construction.  

### 6.2. Q-Value Iteration with a Generative Model

We consider infinite horizon discounted MDPs with:

(i) finite state and action spaces \( \mathcal{Z} \) and \( \mathcal{A} \);

(ii) transition kernel \( P(\cdot \mid z, a) \);

(iii) cost function \( c : \mathcal{Z} \times \mathcal{A} \rightarrow \mathbb{R} \);

(iv) discount factor \( \gamma \in (0, 1) \).

We want to solve (for a given fixed initial state \( z_0 \in \mathcal{Z} \))

\[
(6.2) \quad \min_{\pi \in \Pi} \mathbb{E}\left[ \sum_{t=0}^{\infty} \gamma^t c(z_t, a_t) \mid a_t = \pi(z_t) \text{ for all } t \right],
\]

where II is the class of deterministic Markov policies \( \pi : \mathcal{Z} \rightarrow \mathcal{A} \).

There are three major algorithms for solving Problem 6.2 – value iteration, policy iteration, and Q-value iteration. These algorithms are all based on fixed point iteration of an appropriate contraction operator. We focus on Q-value iteration here and optimize over \( Q \)-functions \( Q = \{Q(z, a)\}_{(z, a) \in \mathcal{Z} \times \mathcal{A}} \subseteq \mathbb{R}^{\mathcal{Z} \times \mathcal{A}} \), which encode a value for every state-action pair (see [23, 11]).

The \( Q \)-value operator \( T : \mathbb{R}^{\mathcal{Z} \times \mathcal{A}} \rightarrow \mathbb{R}^{\mathcal{Z} \times \mathcal{A}} \) is defined by

\[
T(Q)(z, a) \triangleq c(z, a) + \gamma \sum_{z' \in \mathcal{Z}} P(z' \mid z, a) \min_{a' \in \mathcal{A}} Q(z', a') \quad \forall (z, a) \in \mathcal{Z} \times \mathcal{A},
\]

and the optimal \( Q \)-function \( Q^* \) satisfies the fixed point equation \( Q^* = T(Q^*) \).

The empirical \( Q \)-value operator in iteration \( k \geq 0 \) for batch size \( n \geq 1 \), denoted \( \hat{T}_k : \mathbb{R}^{\mathcal{Z} \times \mathcal{A}} \rightarrow \mathbb{R}^{\mathcal{Z} \times \mathcal{A}} \), is defined by:

\[
\hat{T}_k(Q)(z, a) = c(z, a) + \frac{\gamma}{n} \sum_{i=1}^{n} \min_{a' \in \mathcal{A}} Q\left(X^{(i), a}_{k,b} \right), \quad \forall (z, a) \in \mathcal{Z} \times \mathcal{A},
\]
where $X^{(z,a)}_{k,1}, \ldots, X^{(z,a)}_{k,n}$ are i.i.d. samples from $P(\cdot | z, a)$ for every $(z, a) \in \mathbb{Z} \times A$. The convergence and finite time properties of algorithm has been studied in [20]. We next confirm that $\{\tilde{T}_k\}_{k\in\mathbb{N}}$ are almost surely contractions.

**Lemma 6.3.** For all $k \geq 0$, $\tilde{T}_k : \mathbb{R}^{\mathbb{Z} \times A} \to \mathbb{R}^{\mathbb{Z} \times A}$ satisfies

$$\|\tilde{T}_k (Q) - \tilde{T}_k (Q')\|_\infty \leq \gamma \|Q - Q'\|_\infty, \forall Q, Q' \in \mathbb{R}^{\mathbb{Z} \times A},$$

almost surely.

Based on the Hoeffding inequality, for any $n \geq 1$ we define:

$$f(n, \epsilon) \triangleq (1 - p(n, \epsilon)) \epsilon + 2 \gamma p(n, \epsilon) \|Q^*\|_\infty,$$

where

$$p(n, \epsilon) \triangleq 2 |Z||A| \exp\left(\frac{-n \epsilon^2}{2 \gamma^2 \|Q^*\|_\infty}\right).$$

**Corollary 6.4.** (i) (Contraction) There exists $\vartheta \in \mathcal{P}_1 (\mathcal{X})$ such that, for all $\mu \in \mathcal{P}_1 (\mathcal{X})$,

$$W_1 (\mu \Omega^k, \vartheta) \leq \gamma^k W_1 (\mu, \vartheta), \forall k \geq 1.$$

(ii) (Concentration) For any $\mu \in \mathcal{P}_1 (\mathcal{X})$, $n \geq 1$, and $\epsilon > 0$,

$$W_1 (\mu \Omega^k, 1_{\{Q^*\}}) \leq \gamma^k W_1 (\mu, 1_{\{Q^*\}}) + \left(1 - \gamma^k\right) \left(\frac{f(n, \epsilon)}{1 - \gamma}\right), \forall k \geq 1.$$

**Proof:** Part (i) is due to Theorem 4.9. For Part (ii), we use Theorem 3.3 and the upper bound $\mathbb{E} \left[\|\tilde{T} (Q^*) - T (Q^*)\|_\infty\right] \leq f(n, \epsilon)$ which holds by construction. \hfill \blacksquare

### 6.3. Stochastic variance reduced gradient descent.

Since Problem (6.1) is finite sum, we can apply variance reduction techniques. We first consider SVRG, which is an epoch-based variance reduction algorithm (see [31, 21]). Suppose all epochs consist of a fixed number $M \geq 1$ of inner iterations. To characterize the inner iterations, we define auxiliary random operators $T_m : \mathcal{X} \to \mathcal{X}$ via

$$\tilde{T}_m (\tilde{x}_m; x_k) = (I + \eta A)^{-1} (\tilde{x}_m - \eta (B_{1m} (\tilde{x}_m) - B_{1m} (x_k) + B (x_k))).$$

In each epoch $k \geq 0$, starting with the current iterate $x_k \in \mathcal{X}$, we perform the following steps:

1. Compute $B (x_k)$ exactly (which is possible because $B$ is a finite sum).
2. For each $m = 0, 1, \ldots, M - 1$, compute $\tilde{x}_{m+1} = \tilde{T}_m (\tilde{x}_m, x_k)$.
3. Return $x_{k+1} = \tilde{x}_M$.

The iterates $\{x_k\}_{k\in\mathbb{N}}$ of SVRG then evolve according to:

$$\tilde{T}_k (x_k) = \tilde{T}_{M-1} (\cdot, x_k) \circ \tilde{T}_{M-2} (\cdot, x_k) \circ \cdots \circ \tilde{T}_0 (x_k, x_k), \forall k \geq 0.$$

The next lemma gives the basic contraction properties of SVRG that we will need. We recall that $\{\mathcal{F}_m\}_{m \geq 0}$ is the filtration for $\{\tilde{x}_m\}_{m=0}^{M}$ within a single epoch $k$. 


Lemma 6.5. (i) For all \( m = 0, 1, \ldots, M - 1 \), \( \tilde{T}_m \) satisfies

\[
E \left[ \| \tilde{T}_m (\bar{x}_m, x_k) - \tilde{T}_m (\bar{x}_m', x_k') \|_2^2 | \bar{F}_m \right] \leq \gamma(\eta) \| \bar{x}_m - \bar{x}_m' \|_2^2 + \eta^2 L^2 \| x_k - x_k' \|_2^2.
\]

(ii) For all \( k \geq 0 \), \( \tilde{T}_k : \mathbb{R}^d \to \mathbb{R}^d \) satisfies

\[
E \left[ \| \tilde{T}_k (x_k) - \tilde{T}_k (x_k') \|_2^2 | \bar{F}_k \right] \leq \left( \gamma(\eta)^M + \frac{\eta^2 L^2 (1 - \gamma(\eta)^M)}{1 - \gamma(\eta)} \right) \| x_k - x_k' \|_2^2.
\]

Proof: (i) For any \( m = 0, 1, \ldots, M - 1 \), we have

\[
E \left[ \| \tilde{T}_m (\bar{x}_m, x_k) - \tilde{T}_m (\bar{x}_m', x_k') \|_2^2 | \bar{F}_m \right] \leq \gamma(\eta) \| \bar{x}_m - \bar{x}_m' \|_2^2 + \eta^2 \mathbb{E} \left[ \| - B_{1m} (x_k) + B_{1m} (x_k') + B (x_k) - B (x_k') \|_2^2 | \bar{F}_m \right] \leq \gamma(\eta) \| \bar{x}_m - \bar{x}_m' \|_2^2 + \eta^2 \mathbb{E} \left[ \| - B_{1m} (x_k) + B_{1m} (x_k') \|_2^2 | \bar{F}_m \right] \leq \gamma(\eta) \| \bar{x}_m - \bar{x}_m' \|_2^2 + \frac{\eta^2 L^2}{2} \| x_k - x_k' \|_2^2,
\]

where the first inequality uses unbiasedness of the SVRG gradient estimator, and the second is because the variance of a random variable is less than its second-order moment.

(ii) Recall that \( \tilde{T}_k \) is the composition of \( T_{M-1}, T_{M-2}, \ldots, T_0 \). Then, iterating the recursion in Part (i) for \( X_{M-1}, X_{M-2}, \ldots, X_0 \) gives:

\[
E \left[ \| \tilde{T}_k (x_k) - \tilde{T}_k (x_k') \|_2^2 | \bar{F}_k \right] \leq \left( \gamma(\eta)^M \| x_k - x_k' \|_2^2 + \eta^2 L^2 \left( \sum_{m=0}^{M-1} \gamma(\eta)^m \right) \| x_k - x_k' \|_2^2,\right.
\]

since each epoch is initialized with \( x_k \) and \( x_k' \), respectively. It then follows that

\[
E \left[ \| \tilde{T}_k (x_k) - \tilde{T}_k (x_k') \|_2^2 | \bar{F}_k \right] \leq \left( \gamma(\eta)^M + \frac{\eta^2 L^2 (1 - \gamma(\eta)^M)}{1 - \gamma(\eta)} \right) \| x_k - x_k' \|_2^2.
\]

Now,

\[
\gamma(\eta)^M + \frac{\eta^2 L^2}{1 - \gamma(\eta)} (1 - \gamma(\eta)^M) < \gamma(\eta)^M + 1 - \gamma(\eta)^M = 1,
\]

for any \( M \geq 1 \) since \( \eta^2 L^2 / (1 - \gamma(\eta)) < 1 \), and the desired conclusion follows. 

Based on the preceding lemma, we can characterize the convergence of SVRG with respect to the 2–Wasserstein distance.

**Corollary 6.6.** Choose a step size \( \eta \in (0, c/L^2) \) satisfying \( \gamma(\eta) < 1 \) and \( \eta^2 L^2 / (1 - \gamma(\eta)) < 1 \), and suppose each epoch consists of \( M \geq 1 \) iterations.

(i) (Contraction) For all \( \mu_1, \mu_2 \in \mathcal{P}_2 (\mathcal{S}) \),

\[
W_2^2 \left( \mu_1 \mathcal{Q}^k, \mu_2 \mathcal{Q}^k \right) \leq \left[ \gamma(\eta)^M + \frac{\eta^2 L^2 (1 - \gamma(\eta)^M)}{1 - \gamma(\eta)} \right]^k W_2^2 \left( \mu_1, \mu_2 \right), \forall k \geq 1.
\]

(ii) (Concentration) \( \vartheta = \delta_{\mathcal{Q}^k} \).
6.4. SVRG Q-value iteration. We now develop a new Q-value iteration algorithm based on SVRG (see [35, 42]). This is also an epoch based algorithm where all epochs consist of $M \geq 1$ iterations. Each epoch $k \geq 0$ consists of the following steps, starting with the current iterate $Q_k$:

1. Compute $T(Q_k)$ and set $\hat{Q}_0 = Q_k$.
2. For each $m = 0, 1, \ldots, M - 1$, define
   \[ \hat{T}_m (\hat{Q}_m) (z, a) = c (z, a) + \min_{a' \in \mathcal{A}} \hat{Q}_m (X_m^{(z, a)}, a') , \]
   \[ \hat{T}_m (Q_k) (z, a) = c (z, a) + \min_{a' \in \mathcal{A}} Q_k (X_m^{(z, a)}, a') , \]
   for all $(z, a) \in \mathcal{Z} \times \mathcal{A}$ where $X_m^{(z, a)} \sim P (\cdot | z, a)$, define
   \[ \hat{T}_m (\hat{Q}_m, Q_k) = \beta_0 \hat{T}_m (\hat{Q}_m) - \beta_1 \hat{T}_m (Q_k) + \beta T (Q_k) , \]
   for weights $\beta_0, \beta_1, \beta \geq 0$ to be determined, and compute $\hat{Q}_{m+1} = \hat{T}_m (\hat{Q}_m, Q_k)$.
3. Return $Q_{k+1} = \hat{Q}_M$.

Epoch $k \geq 0$ is defined as the composition:
\[ \hat{T}_k (Q_k) = \hat{T}_{M-1} (\cdot , Q_k) \circ \hat{T}_{M-2} (\cdot , Q_k) \circ \cdots \circ \hat{T}_0 (Q_k, Q_k) , \]
where we emphasize that $Q_k$ is both the proxy for epoch $k$ and the first iterate of epoch $k$. We may then write the algorithm as $Q_{k+1} = \hat{T}_k Q_k$ for all $k \geq 0$ where $\{\hat{T}_k\}_{k \geq 0}$ is an i.i.d. sequence of random operators.

We will show that we have a contraction over epochs.

Lemma 6.7. Suppose $\beta_0 < 1/\gamma$, $\beta_0 = \beta_1$, and $\beta = 1$. Then,
\[ \| \hat{T}_k (Q) - \hat{T}_k (Q') \|_\infty \leq \left[ (\beta_0 \gamma)^M + \gamma (1 + \beta_0) \left( \frac{1 - (\beta_0 \gamma)^M}{1 - \beta_0 \gamma} \right) \right] \| Q - Q' \|_\infty , \]
for all $Q, Q' \in \mathbb{R}^{\mathcal{Z} \times \mathcal{A}}$ and $k \geq 0$.

We now frame SVRG Q-value iteration within the 1–Wasserstein distance with respect to $\| \cdot \|_\infty$:
\[ W_1 (\mu_1, \mu_2) \triangleq \inf_{\xi \in (\mu_1, \mu_2)} \int_{\mathcal{Q} \times \mathcal{Q}} \| Q - Q' \|_\infty d\xi (Q, Q') , \]
where $\mathcal{Q} = \mathbb{R}^{\mathcal{Z} \times \mathcal{A}}$ is the space of all Q value functions.

Corollary 6.8. Suppose $\beta_0 < 1/\gamma$, $\beta_0 = \beta_1$, and $\beta = 1$.
(i) (Contraction) For all $\mu_1, \mu_2 \in \mathcal{P}_1 (\mathcal{S})$,
\[ W_1 (\mu_1 \Omega^k, \mu_2 \Omega^k) \leq \left[ (\beta_0 \gamma)^M + \gamma (1 + \beta_0) \left( \frac{1 - (\beta_0 \gamma)^M}{1 - \beta_0 \gamma} \right) \right]^k W_1 (\mu_1, \mu_2) , \]
for all $k \geq 1$.
(ii) (Concentration) $\vartheta = 1_{\{Q^*\}}$. 
6.5. Variance reduction algorithms for solving monotone inclusions. SVRG is part of a larger family of variance reduction algorithms including SAGA (see [12]) and HSAG (see [32]). These algorithms include “proxies” that store past information about gradient computations. A subset $S \subset \{1, \ldots, N\}$ of the proxies follow SAGA-type updates (and are updated with some probability in every iteration) and the rest $S^C$ of the proxies follow SVRG-type updates (and are updated at the beginning of each new epoch). We define the proxies $y_k = (y_{kn})_{n \in S}$ on $\mathcal{Y} = \mathcal{X}^S$ for the gradients that follow a SAGA-type update, and we define $\mathcal{S} = \mathcal{X} \times \mathcal{Y}$ to be the corresponding augmented state space. In this setup, we have $s^* = (x^*, (B_n(x^*))_{n \in S})$. When $S = \emptyset$ (e.g. for SVRG), we just have $\mathcal{S} = \mathcal{X}$ and $s^* = x^*$.

Each epoch consists of iteration of a sequence of auxiliary random operators. The inner iteration uses proxies for every gradient term $y_m = (\bar{y}_{mn})_{n=1}^N$ (in contrast to the original sequence which only has proxies for $S$). The primal update is given by the auxiliary random operator $\bar{T}_m : \mathcal{S} \rightarrow \mathcal{X}$ defined by

$$\bar{T}_m(\bar{x}_m, \bar{y}_m) = (I + \eta A)^{-1} \left( \bar{x}_m - \eta \left( B_{\bar{T}_m}(\bar{x}_m) - \bar{y}_m, \bar{T}_m + \frac{1}{N} \sum_{n=1}^N \bar{y}_{mn} \right) \right),$$

which updates $\{\bar{x}_m\}_{m \geq 0}$. We also update the proxies following $\bar{U}_m : \mathcal{S} \rightarrow \mathcal{Y}$, which is

$$\bar{U}_m(\bar{x}_m, \bar{y}_m) = \left( B_{\bar{U}_m}(\bar{x}_m), (\bar{y}_{mn})_{n \neq I_m} \right)$$

if $I_m \in S$, otherwise it is $\bar{U}_m(\bar{x}_m, \bar{y}_m) = \bar{y}_m$. Then the entire inner iteration is captured by the concatenation $\bar{T}_m : \mathcal{S} \rightarrow \mathcal{S}$ defined by $\bar{T}_m(\bar{x}_m, \bar{y}_m) = \left( \bar{T}_m(\bar{x}_m, \bar{y}_m), \bar{U}_m(\bar{x}_m, \bar{y}_m) \right)$.

Recall that $\{\tau_k\}_{k \geq 1}$ are i.i.d. stopping times that give the epoch lengths. Each epoch $k \geq 0$ consists of $\tau_k \geq 1$ iterations, starting with the current iterate $(x_k, y_k)$:

1. Set $\bar{y}_{0,n} = y_{k,n}$ for all $n \in S$ and $\bar{y}_{0,n} = B_n(x_k)$ for all $n \in S^C$.
2. For each $m = 0, 1, \ldots, \tau_k - 1$, generate $I_m$ and compute $(\bar{x}_{m+1}, \bar{y}_{m+1}) = \bar{T}_m(\bar{x}_m, \bar{y}_m)$.
3. Return $(x_{k+1}, y_{k+1}) = (\bar{x}_{\tau_k}, (\bar{y}_{\tau_k,n})_{n \in S})$.

When $S \neq \emptyset$, we define the Lyapunov function

$$V_{b, S}(s, s') \triangleq \|x - x'\|^2 + b \sum_{n \in S} \|y_n - y'_n\|^2,$$

for $b > 0$. Again, when $S = \emptyset$ (as in SVRG) we just take $V(s, s') = \|x - x'\|^2$. We collect our assumptions next.

**Assumption 6.9.** For $S \subset \{1, \ldots, N\}$ with $S \neq \emptyset$ and $\eta > 0$, there are constants $\alpha = \alpha(\eta) \in (0, 1)$ and $b = b(\eta) > 0$ such that

$$\mathbb{E} \left[ V_{b, S}(s_{k+1}, s'_{k+1}) \mid \mathcal{F}_k \right] \leq \alpha V_{b, S}(s_k, s'_k), \quad \forall s_k, s'_k \in \mathcal{S}.$$

We now frame this class of algorithms within the modified Wasserstein distance:

$$W_{V_{b, S}}(\mu_1, \mu_2) \triangleq \inf_{\xi \in C(\mu_1, \mu_2)} \int_{\mathcal{S} \times \mathcal{S}} V_{b, S}(s, s') d\xi(s, s').$$

The next result is immediate.
Corollary 6.10. Suppose Assumption 6.9 holds.
(i) (Contraction) For all $\mu_1, \mu_2 \in \mathcal{P}_2(S)$,
\[ W_{k,S} \left( \mu_1 \Omega^k, \mu_2 \Omega^k \right) \leq \alpha^k W_{k,S} (\mu_1, \mu_2), \quad \forall k \geq 1. \]

(ii) (Concentration) $\vartheta = \delta_\ast$.

We now verify Assumption 6.9 for some specific algorithms.

Example 6.11. (i) (Random epoch SVRG) Let $S = \emptyset$ and choose a step size $\eta \in (0, c/L^2)$ satisfying $\gamma (\eta) < 1$ and $\eta^2 L^2 / (1 - \gamma (\eta)) < 1$. Then, we have the contraction coefficient
\[ \alpha (\eta) = \mathbb{E} \left[ \gamma (\eta) \tau_k + \frac{\eta^2 L^2 (1 - \gamma (\eta) \tau_k)}{1 - \gamma (\eta)} \right], \]
for the Lyapunov function $V(s, s') = ||x - x'||^2_2$.

(ii) (SAGA) Let $S = \{1, \ldots, N\}$ and $\tau_k = 1$ for all $k \geq 0$. Choose a step size $\eta \in (0, c/L^2)$. Choose a constant $b > 0$ such that $\eta^2 < b$ and $\gamma (\eta) + b L^2 < 1$. Then, we have the contraction coefficient
\[ \alpha (\eta) = \max \left\{ \gamma (\eta) + b L^2, \frac{\eta^2 / b + N - 1}{N} \right\}, \]
for the Lyapunov function $V_{b,S}$.

(iii) (Random epoch SAGA) Let $S = \{1, \ldots, N\}$ and $\{\tau_k\}_{k \geq 0}$ be a sequence of i.i.d. stopping times. Choose a step size $\eta \in (0, c/L^2)$. Choose a constant $b > 0$ such that $\eta^2 < b$ and $\gamma (\eta) + b L^2 < 1$. Then, we have the contraction coefficient $\alpha (\eta) = (K (\eta) (1 + b N L))$ for the Lyapunov function $V_{b,S}$, where
\[ K (\eta) = \mathbb{E} \left[ \max \left\{ \gamma (\eta) + b L^2, \frac{\eta^2 / b + N - 1}{N} \right\} \tau_k \right]. \]

(iv) (HSAG) Let $S \subset \{1, \ldots, N\}$ and $\tau_k = M$ for all $k \geq 0$. Choose a step size
\[ \eta \in \left(0, \frac{2 c}{(1 + |S|/N) L^2}\right). \]
Choose a constant $b > 0$ such that $\eta^2 < b$ and $\gamma (\eta) + b |S| L^2 / N < 1$. Then, we have the contraction coefficient
\[ \alpha (\eta) = K (\eta)^M + \frac{\eta^2 L^2 |S^C|}{N (1 - K (\eta))} \left(1 - K (\eta)^M\right) \]
for the Lyapunov function $V_{b,S}$, where
\[ K (\eta) = \max \left\{ \gamma (\eta) + b |S| L^2 / N, \frac{\eta^2 / b + N - 1}{N} \right\}. \]
6.6. Accelerated forward-backward splitting with catalyst. In this section we consider another approach for solving the monotone inclusion problem from Subsection 6.1. We can accelerate the variance reduction algorithms from the previous subsection by using a catalyst as in [31]. These accelerated algorithms are all epoch-based.

To begin, let $\tau \geq 0$ be an acceleration parameter. In epoch $k \geq 0$, the outer iterate $x_k$ is fixed. Within epoch $k \geq 0$, the (accelerated) auxiliary random operator is:

$$
\tilde{T}_m(\tilde{x}_m; x_k) \triangleq (I + \eta A)^{-1} \left( \tilde{x}_m - \eta \left( \tau \tilde{x}_m + BI_m(\tilde{x}_m) - \bar{y}_m, I_m + \frac{1}{N} \sum_{n=1}^{N} \bar{y}_{m,n} - \tau x_k \right) \right),
$$

and the inner iteration in epoch $k \geq 0$ is

$$(6.3) \quad \tilde{x}_{m+1} = \tilde{T}_m(\tilde{x}_m; x_k), \ m \geq 0,$$

where we initialize with $\tilde{x}_0 = x_k$. Note that iteration (6.3) computes $(\tau I + A + B)^{-1}(\tau x_k)$ rather than $(A + B)^{-1}(0)$. The catalyst increases the strong monotonicity of $B$ (which is $c$-strongly monotone) to $\tau I + B$ (which is $(c + \tau)$-strongly monotone). We note that it also increases the Lipschitz parameter of $B$ from $L$ to $L + \tau$.

The following assumption details the improved convergence rate from acceleration.

**Assumption 6.12.** (i) We initialize epoch $k$ with $\tilde{x}_0 = x_k$ for all $k \geq 0$.

(ii) There is a constant $\alpha = \alpha(\eta, \tau) \in (0, 1)$ such that

$$
E \left[ \|\tilde{x}_m - (\tau I + A + B)^{-1}(\tau x_k)\|^2 \right] \leq \alpha^m \|x_k - (\tau I + A + B)^{-1}(\tau x_k)\|^2, \forall m \geq 0.
$$

The next lemma gives our error bound in expectation for catalyst-based acceleration (using any variance reduction algorithm as a subroutine). The following lemma is the main error bound for catalyst-based acceleration.

**Lemma 6.13.** Suppose Assumption 6.12 holds and that each epoch consists of $M \geq 1$ iterations. For all $k \geq 0$, we have

$$
\left( E \left[ \|x_{k+1} - x^*\|^2 \right] \right)^{1/2} \leq \left( \alpha^{M/2} + \frac{\tau}{\tau + c} \right) \left( E \left[ \|x_k - x^*\|^2 \right] \right)^{1/2}.
$$

**Proof:** For $x^* \in (A + B)^{-1}(0)$, we have

$$
x^* \in (A + B)^{-1}(0) \Leftrightarrow 0 \in (A + B)(x^*)
\Leftrightarrow \tau x^* \in (\tau I + A + B)(x^*)
\Leftrightarrow x^* \in (\tau I + A + B)^{-1}(\tau x^*).
$$

By Minkowski’s inequality [29, Theorem 4, p.33], we have:

$$
\left( E \left[ \|x_{k+1} - x^*\|^2 \right] \right)^{1/2} \leq \left( E \left[ \|x_{k+1} - (\tau I + A + B)^{-1}(\tau x_k)\|^2 \right] \right)^{1/2}
+ \left( \| (\tau I + A + B)^{-1}(\tau x_k) - x^* \|^2 \right)^{1/2}.
$$
By Assumption 6.12, we have

\[
\mathbb{E} \left[ \|x_{k+1} - (\tau I + A + B)^{-1}(\tau x_k)\|_2^2 \right] \\
\leq \alpha^M \|x_k - (\tau I + A + B)^{-1}(\tau x_k)\|_2^2 \\
\leq \alpha^M \|x_k - x^* - (\tau I + A + B)^{-1}(\tau x_k) + (\tau I + A + B)^{-1}(\tau x^*)\|_2^2 \\
= \alpha^M \|x_k - x^* - (I + \tau^{-1}(A + B))^{-1}(x_k) + (I + \tau^{-1}(A + B))^{-1}(x^*)\|_2^2 \\
\leq \alpha^M \|x_k - x^*\|_2^2,
\]

where the last inequality uses the fact that \( I - (I + \tau^{-1}(A + B)) \) is 1-Lipschitz continuous. Next, we have

\[
\| (\tau I + A + B)^{-1}(\tau x_k) - x^*\|_2^2 = \| (\tau I + A + B)^{-1}(\tau x_k) - (\tau I + A + B)^{-1}(\tau x^*)\|_2^2 \\
= \| (I + \tau^{-1}(A + B))^{-1}(x_k) - (I + \tau^{-1}(A + B))^{-1}(x^*)\|_2^2 \\
\leq \left( \frac{\tau}{\tau + c} \right)^2 \|x_k - x^*\|_2^2.
\]

Combing these two inequalities gives the desired result.

The following corollary is an immediate consequence.

**Corollary 6.14.** Suppose Assumption 6.12 holds and that each epoch consists of \( M \geq 1 \) iterations. For all \( \mu \in \mathcal{P}_2(\mathcal{S}) \),

\[
W_2^2 \left( \mu \Omega^k, \delta_{x^*} \right) \leq \left( \alpha^{M/2} + \frac{\tau}{\tau + c} \right)^{2k} W_2^2 (\mu, \delta_{x^*}), \ \forall k \geq 1.
\]

We now illustrate the specifics with SVRG.

**Example 6.15.** (Accelerated SVRG) The accelerated contraction coefficient is \( \gamma(\eta, \tau) \triangleq 1 - 2\eta(c + \tau)/L^2 \). The accelerated SVRG subroutine then satisfies:

\[
(6.4) \quad \mathbb{E} \left[ \|\bar{x}_m - (\tau I + A + B)^{-1}(\tau x_k)\|_2^2 \right] \leq \alpha(\eta, \tau)^m \|\bar{x}_0 - (\tau I + A + B)^{-1}(\tau x_k)\|_2^2,
\]

where

\[
\alpha(\eta, \tau) = \gamma(\eta, \tau)\eta^2L^2 \frac{1 - \gamma(\eta, \tau)^M}{1 - \gamma(\eta, \tau)}.
\]

In particular, \( \gamma(\eta, \tau) \) is optimized for step size \( \eta^* = (c + \tau)/(L + \tau)^2 \). The corresponding optimal contraction factor is \( \gamma(\eta^*, \tau) = 1 - (c + \tau)^2/(L + \tau)^2 \), compared to the contraction factor \( 1 - c^2/L^2 \) for the unaccelerated iteration. If \( c < L \), then \( \gamma(\eta^*, \tau) \) is monotonically decreasing in \( \tau \in [0, \infty) \).
6.7. Epoch-Based Deterministic Asynchronous Enhanced Policy Iteration. Consider the discounted cost MDP introduced in Subsection 6.2. Empirical Q-value iteration determines a nearly optimal solution with high probability. In [6], enhanced policy iteration (EPI) was proposed to avoid some of the limitations of asynchronous policy iteration. In this work, the authors studied asynchronous and stochastic iterative versions of the algorithm in the finite state space setting. Our goal here is to recover the convergence results for asynchronous deterministic EPI within our framework.

We first define two new operators. Let \( \nu \) be a randomized stationary policy. Define \( F_{\nu} : \mathbb{R}^{S \times A} \times \mathbb{R}^{S \times A} \to \mathbb{R}^{S \times A} \) and \( M : \mathbb{R}^{S \times A} \to \mathbb{R}^{S} \) as

\[
F_{\nu}(J, Q)(z, a) = c(z, a) + \gamma \sum_{z' \in S} P(z'|z, a) \left( \sum_{a' \in A} \nu(a'|z') \min \left\{ J(z'), Q(z', a') \right\} \right),
\]

\[
M(Q)(z) = \min_{a \in A} Q(z, a).
\]

We consider two versions of the algorithm: one with sequential updates and the other with parallel updates. The algorithm with sequential updates runs as follows. Pick \( m \in \mathbb{N} \) arbitrarily and define \((J_{k+1}, Q_{k+1})\) via:

\[
\text{Sequential EPI:} \quad Q_{k+1} = F_{\nu}^m(J_k, Q_k), \quad J_{k+1} = M(Q_{k+1}).
\]

Under parallel updates, \((J_{k+1}, Q_{k+1})\) is defined via:

\[
\text{Parallel EPI:} \quad Q_{k+1} = F_{\nu}(J_k, Q_k), \quad J_{k+1} = M(Q_k).
\]

We can now put these algorithms in an operator theoretic framework. Define \( S = \mathbb{R}^S \times \mathbb{R}^{S \times A} \). Then, let \( T_{\nu, s} : S \to S \) and \( T_{\nu, p} : S \to S \) denote the composition of the operators for the sequential and parallel EPI algorithms in (6.7) and (6.8), respectively. Let \((J^*, Q^*)\) denote the optimal value function and the corresponding optimal \( Q \)-function for the MDP being solved. We have the following result from [6].

**Theorem 6.16.** Choose any randomized stationary policy \( \nu \).

1. Endow \( S \) with the sup norm \( \| \cdot \|_{\infty} \) defined by:

\[
\| (J, Q) \|_{\infty} := \max\{ \| J \|_{\infty}, \| Q \|_{\infty} \}.
\]

Then, for any \( m \in \mathbb{N} \), \( T_{\nu, s} \) is a contraction with respect to \( \| \cdot \|_{\infty} \) with contraction coefficient \( \gamma \).

2. Endow \( S \) with the norm \( \| \cdot \|_{\sqrt{\gamma}} \) defined by:

\[
\| (J, Q) \|_{\sqrt{\gamma}} := \max\left\{ \frac{\| J \|_{\infty}}{\sqrt{\gamma}}, \| Q \|_{\infty} \right\}.
\]

Then, \( T_{\nu, p} \) is a contraction with respect to \( \| \cdot \|_{\sqrt{\gamma}} \) with contraction coefficient \( \sqrt{\gamma} \).

Both operators have a unique fixed point \( s^* := (J^*, Q^*) \).
Proof: For the first result, see Proposition 2.1 and Lemma 2.3 in [6, p. 69, p. 71]. For the second result, see Proposition 4.1 in [6, p. 78].

In [6], the authors also propose and establish convergence of deterministic asynchronous versions of EPI. Our framework also applies in this setting. Pick a random sequence \( \{R_k := (R_{1,k}, R_{2,k})\}_{k \in \mathbb{N}} \), where \( R_{1,k} \subset S \times A \) and \( R_{2,k} \subset S \), such that all \((z,a)\) pairs appear infinitely often in \( \{R_{1,k}\}_{k \in \mathbb{N}} \) and all states \( s \) appear infinitely often in \( \{R_{2,k}\}_{k \in \mathbb{N}} \). The deterministic asynchronous version of EPI updates \((J_{k+1}, Q_{k+1})\) following:

\[
Q_{k+1}(z,a) = \begin{cases} 
F_k(J_k, Q_k)(z,a) & \text{if } (z,a) \in R_{1,k} \\
Q_k(z,a) & \text{if } (z,a) \notin R_{1,k}
\end{cases}
\]

\[
J_{k+1}(z) = \begin{cases} 
M(Q_k)(z) & \text{if } z \in R_{2,k} \\
J_k(z) & \text{if } s \notin R_{2,k}
\end{cases}
\]

Let \( \hat{G}_k \) denote the combined operator acting on the space \( S = \mathbb{R}^S \times \mathbb{R}^{S \times A} \). Let \( \tau_0 = 0 < \tau_1 < \tau_2 < \cdots \) be a strictly increasing sequence of stopping times such that for any \( i = 0, 1, \ldots, \)

\[
\mathbb{S} = \bigcup_{k=\tau_2+1}^{\tau_2+1} R_{1,k}, \quad \mathbb{S} = \bigcup_{k=\tau_2+1}^{\tau_2+2} R_{2,k}.
\]

Note that by construction of the sequence \( \{R_k := (R_{1,k}, R_{2,k})\}_{k \in \mathbb{N}} \), such stopping times exist. Define the operator \( \hat{T}_i \) via:

\[
\hat{T}_i(J, Q) = \hat{G}_{\tau_{2i+2}} \circ \cdots \circ \hat{G}_{\tau_{2i+1}}(J, Q).
\]

We have the following result from [6].

**Theorem 6.17.** Endow \( \mathbb{S} \) with the sup norm, defined as follows:

\[
\| (J, Q) \|_\infty = \max \{|J|, |Q|\}.
\]

Every realization of \( \hat{T}_i \) is a contraction map satisfying:

\[
\| \hat{T}_i(J_1, Q_1) - \hat{T}_i(J_2, Q_2) \|_\infty \leq \gamma \| (J_1, Q_1) - (J_2, Q_2) \|_\infty, \quad \hat{T}_i(J^*, Q^*) = (J^*, Q^*).
\]

**Proof:** See Proposition 3.1 in [6, p. 74].

We immediately arrive at the following result.

**Theorem 6.18.** Let \( \{ (\hat{J}_k, \hat{Q}_k) \}_{k \in \mathbb{N}} \) be generated by the deterministic asynchronous EPI algorithm. Then, \( (\hat{J}_k, \hat{Q}_k) \to (J^*, Q^*) \) as \( k \to \infty \).

**Proof:** The proof follows directly from Theorem 3.5.

6.8. *Empirical Enhanced Policy Iteration.* Consider the enhanced policy iteration algorithm that was proposed in [6] and reviewed in Subsection 6.7. We now propose a novel empirical version of the algorithm. Define the random operator \( \hat{F}_{\nu,k} \) as follows:

\[
\hat{F}_{\nu,k}(J, Q)(z,a) = c(z,a) + \frac{2}{n} \sum_{i=1}^{n} \left( \sum_{a' \in A} \nu(a') X_{k,i}^{(z,a)} \min \left\{ J(X_{k,i}^{(z,a)}), Q(X_{k,i}^{(z,a)}, a') \right\} \right),
\]

\[
M(Q)(z) = \min_{a \in A} Q(z,a),
\]
where \( X^{(z,a)}_{k,1}, \ldots, X^{(z,a)}_{k,n} \) is a collection of i.i.d. samples from \( P(\cdot | z, a) \), for all \((z, a) \in Z \times A\). We let \( \{ T_{\nu,s,k} \}_{k \in \mathbb{N}} \) and \( \{ T_{\nu,p,k} \}_{k \in \mathbb{N}} \) denote, respectively, the combined empirical operator for the empirical sequential EPI and empirical parallel EPI acting onto the space \( S \). We next confirm that these two empirical operators are almost surely contractions.

**Theorem 6.19.** The following statements hold for any randomized Markov policy \( \nu \).

1. Endow \( S \) with the sup norm, defined as follows:
   \[
   \|(J, Q)\|_\infty = \max\{\|J\|_\infty, \|Q\|_\infty\}. 
   \]
   Then, for any choice of \( m \in \mathbb{N} \), the operator \( \hat{T}_{\nu,s,k} \) is a contraction under the above norm with contraction coefficient \( \gamma \).

2. Endow \( S \) with the norm \( \| \cdot \|_\gamma \), defined as follows:
   \[
   \|(J, Q)\|_\gamma = \max\left\{ \frac{\|J\|_\infty}{\sqrt{\gamma}}, \frac{\|Q\|_\infty}{\sqrt{\gamma}} \right\}. 
   \]
   Then, the operator \( \hat{T}_{\nu,p,k} \) is a contraction under the above norm with contraction coefficient \( \sqrt{\gamma} \).

**Proof:** The proof is essentially the same as the corresponding proofs in [6]. To see this, define the empirical state transition kernel as
\[
\hat{P}_{n,k}(z'|z, a) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{z' = X^{(z,a)}_{k,i}\}} \text{ for all } (z', z, a) \in Z \times Z \times A.
\]
Then, the definition of empirical operator \( \hat{F}_{\nu,k} \) is the same as the definition of \( F_\nu \) in (6.5), but with \( P \) replaced with \( \hat{P}_{n,k} \). Since the contraction properties of the operators \( T_{\nu,s} \) and \( T_{\nu,p} \) under appropriate norms in [6] do not depend on the transition kernel, we conclude that the same property is inherited by each realization of the operators \( \hat{T}_{\nu,s,k} \) and \( \hat{T}_{\nu,p,k} \). This observation completes the proof of the result.

As a consequence of the theorem above, we conclude the following.

**Theorem 6.20.** Consider the empirical sequential/parallel EPI and let \( \{(\hat{J}_k, \hat{Q}_k)\}_{k \in \mathbb{N}} \) be the output of the algorithm. Let \( V(z_1, z_2) \) be defined as
\[
V(z_1, z_2) = \|z_1 - z_2\|_\infty \text{ for empirical sequential EPI,} \\
V(z_1, z_2) = \|z_1 - z_2\|_\gamma \text{ for empirical parallel EPI.}
\]
Then, \( V \) satisfies all the conditions listed in Definition 2.2 and there exists a unique invariant distribution of the Markov chain \( (\hat{J}_k, \hat{Q}_k) \).

**Proof:** The proof of \( V \) satisfying all conditions of Definition 2.2 is immediate since \( S \) is a finite dimensional Euclidean space and \( V \) is induced from some norm on this space. The proof of existence of a unique invariant distribution follows from Theorem 3.2.
7. Conclusion. Many algorithms in optimization and MDPs are based on contraction maps, where the convergence of these algorithms can often be established by invoking the contraction mapping theorem. Due to the emergence of new data-driven paradigms, we are now seeing proliferation of randomized variants of these algorithms. Our present paper conceptually unifies their convergence analysis using the notion of Wasserstein divergence.

We show that operators that are contractions with respect to the Wasserstein divergence enjoy stability properties. In cases where the random operator maps the optimal solution back to itself, we show that iteration of random operators convergences to the optimal solution in probability. We work out the details of this analysis for the convergence of several example randomized algorithms. The intuition developed here also suggests a way to design new variance reduced algorithms.

Our framework is very general and can be applied to other situations where a contraction operator is the central feature, e.g. zero-sum minimax dynamic programming and dynamical systems. Our analysis also holds when the underlying spaces, over which the random operators are defined, are Polish spaces. Thus, our framework can potentially be applied to randomized algorithms for continuous state/action MDPs.

We are hopeful that this study will facilitate further efforts in using random operators to analyze RSAs. The benefits come from new perspectives on the nature of the convergence of RSAs, and frameworks to design new RSAs. In future research, we will extend our present framework to cover algorithms for large-scale MDPs with function approximation and random state/action sampling.

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