LENSPERFECT: GRAVITATIONAL LENS MASS MAP RECONSTRUCTIONS YIELDING EXACT REPRODUCTION OF ALL MULTIPLE IMAGES

D. Coe,1,2,3 E. Fuselier,4 N. Benítez,3,5 T. Broadhurst,6 B. Frye,7 and H. Ford2

Received 2007 December 14; accepted 2008 March 13

ABSTRACT

We present a new approach to gravitational lens mass map reconstruction. Our mass map solutions perfectly reproduce the positions, fluxes, and shears of all multiple images, and each mass map accurately recovers the underlying mass distribution to a resolution limited by the number of multiple images detected. We demonstrate our technique given a mock galaxy cluster similar to Abell 1689, which gravitationally lenses 19 mock background galaxies to produce 93 multiple images. We also explore cases in which as few as four multiple images are observed. Mass map solutions are never unique, and our method makes it possible to explore an extremely flexible range of physical (and unphysical) solutions, all of which perfectly reproduce the data given. Each reconfiguration of the source galaxies produces a new mass map solution. An optimization routine is provided to find those source positions (and redshifts, within uncertainties) that produce the “most physical” mass map solution, according to a new figure of merit developed here. Our method imposes no assumptions about the slope of the radial profile or mass following light. However, unlike “nonparametric” grid-based methods, the number of free parameters that we solve for is only as many as the number of observable constraints (or slightly greater if fluxes are constrained). For each set of source positions and redshifts, mass map solutions are obtained “instantly” via direct matrix inversion by smoothly interpolating the deflection field using a recently developed mathematical technique. Our LensPerfect software is straightforward and easy to use, and is publicly available on our Web site.

Subject headings: dark matter — galaxies: clusters: general — gravitational lensing — methods: data analysis

Online material: color figures

1. INTRODUCTION

Simulations of structure formation in a ΛCDM universe have provided concrete predictions for the form of dark matter halos over a wide range of scales (e.g., Merritt et al. 2006; Bett et al. 2007; Diemand et al. 2007). These mass distributions, quantified in terms of radial profile, ellipticity, and level of substructure, are among the key predictions of ΛCDM theory. Uncertainties do persist, however, especially with regard to baryons, which are absent from most dark matter simulations. Baryons are found to alter the inner profiles and ellipticities of halos, especially on galactic scales (e.g., Kazantzidis et al. 2004; Gustafsson et al. 2006; Dutton et al. 2007).

Testing these predictions in detail observationally has proven challenging. Gravitational lensing provides us with our most direct tool for mapping the distributions of mass (predominantly dark matter) within and surrounding galaxies and galaxy clusters, but mass maps recovered by this method are of much lower resolution than simulated dark matter halos. Improvements in imaging quality both from the ground and space now allow for a more definitive measurement of lensing effects, motivating new techniques to take full advantage of this advance.
they already contain so many parameters that navigating the parameter space proves challenging. It may be impossible to find the best solution with so many free parameters, given our current computational capabilities. Advances in computing power may someday allow parametric models the freedom necessary to produce perfect fits. Allowing galaxy subhalos to drift in position and to vary individually in mass could dramatically improve the fits, while breaking free from the assumption that dark matter subhalos strictly follow the light distribution.

The degree to which dark matter follows light (and/or gas) is an important outstanding question. The exciting discoveries in this area are currently being made by weak gravitational lensing analyses that make no assumptions about the underlying mass distributions. The “Bullet Cluster” finding (Clowe et al. 2006) that gas has been stripped from two colliding dark matter halos would not have been possible had the authors assumed that mass follows light from the outset. A similar cluster collision observed along the line of sight appears to have left a dark matter ring around CL 0024. This ring was also detected by weak-lensing analysis (Jee et al. 2007).

For many years now, we have obtained assumption-free mass map reconstructions from direct inversion of weak gravitational lensing data (Tyson et al. 1990; Kaiser & Squires 1993). Here, we present the first method to do the same given strong gravitational lensing data (multiple images). Our method is not entirely assumption free, as a few basic assumptions about the distribution of mass can be helpful in selecting the most “physical” among the possible mass map solutions (§ 2.4).

Model-free mass map reconstructions have previously been obtained for Abell 1689 using strong-lensing data (although not via direct inversion). As these mass maps are more flexible than model-based solutions, they should produce better fits to the data, but this promise has yet to be fully realized. The SLAP method (Diego et al. 2005a) computes mass maps on an adaptive mesh grid, and fits the data to a desired level of scatter. However, when this level is set too low, the solutions become “biased” with “a lot of substructure.” Their best solution for A1689 leaves scatters of ~3σ in the source plane (Diego et al. 2005b; J. M. Diego 2007, private communication). Meanwhile, Saha et al. (2006) use a method called PixeLens to obtain pixel-based (fixed-grid) mass maps that perfectly reproduce all multiple-image positions. However, computational constraints prevent them from fitting more than 30 images at a time. The mass map we obtain for A1689 (to be presented in an upcoming paper) perfectly reproduces the positions of 135 multiple images (as in Fig. 1b) and thus has about twice the resolution as the PixeLens mass map in each spatial direction (as dictated by the density of multiple images).

Mass maps may be further improved by incorporating constraints beyond simple image positions. Images that are resolved and extended should be properly reproduced by the mass model (Warren & Dye 2003; Suyu et al. 2006; Koopmans et al. 2006). Even unresolved images yield the information encoded in their fluxes. A simple mass model that accurately reproduces “quad” image positions may or may not accurately reproduce the image flux ratios. Discrepant cases are known as “flux anomalies.” If other causes (microlensing, time delays, and dust extinction) can be ruled out or accounted for, then small substructure (~10^6 M☉ subhalos) is generally invoked as the most likely explanation for an observed anomaly.

However, this explanation has perhaps been invoked too often. The amount of substructure observed in simulations may not be sufficient to produce flux anomalies as often as observed (Metcalfe et al. 2004; Amara et al. 2006; Macciò & Miranda 2006; Diemand et al. 2007). One way to resolve this possible discrepancy is to obtain macro mass model solutions that reproduce the observed fluxes without resorting to smaller substructure.

To address this issue, Evans & Witt (2003) developed a direct inversion mass map reconstruction method capable of perfectly fitting the observed positions and fluxes of four multiple images. While it provided reasonable solutions for the lensed systems Q2237+0305 and PG1115+080, their solution for B1422+231 was clearly unphysical. Although it is unclear how exactly to quantify a mass map’s physicality, the authors characterized their B1422+231 model as simply too “wiggly” to be plausible. Developing the method a bit further, Congdon & Keeton (2005) produced a more reasonable solution for B1422+231, but were less successful with B2045+265 and B1933+503. For the unsuccessful cases, the authors argue that small-scale substructure is the most likely explanation for the observed flux anomalies, but perhaps their models were simply not flexible enough to obtain physical solutions for these systems. We will revisit this question in an upcoming paper.

Our new LensPerfect method uses direct inversion to obtain assumption-free mass map solutions that perfectly reproduce all multiple-image positions. Multiple-image knots and fluxes may

![Figure 1](https://example.com/figure1.png)

**Fig. 1.—** Left: Multiple images of object 1 from Broadhurst et al. (2005), delensed back to the source plane using their best-fit deflection field. The scatter in the source plane (~2”) is typical of all model-based mass map reconstruction methods used to date. Right: The same multiple-image positions all delensed back to a single source position. These deflections can be fit exactly by a LensPerfect solution. [See the electronic edition of the Journal for a color version of this figure.]
also be perfectly constrained. LensPerfect is made possible by a recent advance in the field of mathematics. The essential tool is a method that produces a curl-free interpolation of vectors given at scattered data points. A set of observed multiple-image positions and redshifts along with assumed source positions defines a deflection field at the image positions. Once we interpolate this deflection field across the entire field, we take the gradient, multiply by 2, and “instantly” obtain our perfect mass map solution.

The interpolation of data given at scattered points is a complex problem without a unique solution. One method of attacking such problems involves the use of radial basis functions (RBFs). RBFs have been used to interpolate scattered data since the early 1970s (Hardy 1971) and are used in many applications today. By lensing equations, in gravitational lensing analysis.

Finally, the curl-free analog of this method was developed last year (Fuselier 2006, 2007). Here, we apply this new method to gravitational lensing analysis.

We describe our method, along with the necessary gravitational lensing equations, in § 2. In § 3, we demonstrate applications of the method, including the recovery of a known mass map given 93 multiple images (§ 3.1). We also explore the solutions obtained when only a handful of multiple images are available (§ 3.4), and we demonstrate the gains made by constraining extra image knots in § 3.6. The method for adding flux and/or shear constraints is given in § 2.6. In § 4, we discuss the relative merits of model-based and model-free methods, along with the potential of a hybrid method among other techniques that may become possible with LensPerfect. Finally, we provide a summary in § 5. The LensPerfect software and more information are available on our Web site.8

2. METHOD

2.1. Image Deflection

Image deflection by a gravitational lens is governed by a few simple equations (e.g., Wambsganss 1998). The relativistic bending of light due to a mass $M$ at a distance $R$ away is twice that expected from Newtonian physics: $\alpha = 4GM/c^2R$, given Newton’s gravitational constant $G$ and the speed of light $c$. In a gravitational lens, it is generally safe to assume that all of the deflection occurs in the plane of the lens (this is known as the thin-lens approximation). Given the projected mass surface density distribution $\kappa(\theta)$ of the lens, we can derive the deflection of light $\alpha(\theta) = \theta - \beta$ from its true position on the sky, $\beta$, to that at which we observe it, $\theta$ (Fig. 2):

$$\alpha(\theta) = \frac{1}{\pi} \int d^2\theta' \kappa(\theta') \frac{\theta - \theta'}{|\theta - \theta'|^2},$$

with the corresponding less intimidating inverse relation,

$$\nabla \cdot \alpha = 2\kappa.$$  

The surface density $\kappa = \Sigma/\Sigma_{\text{crit}}$ is defined in units of the critical density at the epoch of the lens. The critical density is that generally required for multiple images to be produced. It is a function of source redshift as given by

$$\Sigma_{\text{crit}} = \frac{c^2}{4\pi G D_S D_L D_{LS}}.$$  

8 See http://www.its.caltech.edu/%7Ecoe/LensPerfect/.

![Image](https://example.com/lens.png)

**Fig. 2.—** Light ray deflection by a gravitational lens of mass $\kappa$. In the absence of $\kappa$, the galaxy would appear at its true, or “source,” position $\beta$. The intervening mass deflects its light by an amount $\alpha$ to position $\theta$. The deflection angle $\alpha$ on our sky is related to the actual bend angle $\tilde{\alpha}$ of the light ray via $\alpha D_S = \tilde{\alpha} D_{LS} D_S$. $D_S$, $D_{LS}$, and $D_L$ are measured as angular diameter distances. [See the electronic edition of the Journal for a color version of this figure.]

involving a ratio of the angular diameter distances from observer to source $D_S = D_A(0, z_S)$, from observer to lens $D_L = D_A(0, z_L)$, and from lens to source $D_{LS} = D_A(z_L, z_S)$. For a flat universe ($\Omega = \Omega_m + \Omega_\Lambda = 1$), angular diameter distances are calculated as follows (Fukugita et al. 1992, filled beam approximation; see also Hogg 1999):

$$D_A(z_1, z_2) = \frac{c}{1 + z_2} \int_{z_1}^{z_2} \frac{dz'}{H(z')} ,$$

where the Hubble parameter varies with redshift as

$$H(z) = H_0 \sqrt{\Omega_m (1 + z)^3 + \Omega_\Lambda}.$$  

Thus, the critical density $\Sigma_{\text{crit}}$ is a function of the source redshift. This follows because the deflection angle $\alpha$ is a function of source redshift. As source redshift decreases, the light bend angle $\tilde{\alpha}$ remains constant, which (imagine moving the galaxy in Fig. 2
inward along the top blue arrow) requires the image deflection to decrease by the distance ratio

$$\alpha = \left( \frac{D_{ls}}{D_s} \right) \alpha_\infty,$$

(6)

where $\alpha_\infty$ is the deflection for a source at infinite redshift.

Thus, the problem of mass map reconstruction can be reduced to determining the deflection field with all deflections scaled to a common redshift (e.g., $\alpha_\infty$), at which point we simply take the divergence and divide by 2 to obtain the mass map (eq. [2]). The deflection field $\alpha(\theta) = \theta - \beta$ may be measured at the multiple-image positions $\theta$ once source positions $\beta$ are determined (see § 2.4). However, in order to take its divergence, the deflection field must be solved for as a continuous function of position (or at least defined on a regular grid). Our interpolated deflection field must also be curl free:

$$\nabla \times \alpha = 0.$$  

(7)

This follows from equation (1) in the same way that we find that an electric field due to a static distribution of charge is also curl free. One way to demonstrate this is to use the substitution $\nabla \ln \theta = \theta/\theta^2$ to define the lensing potential:

$$\psi(\theta) = \frac{1}{\pi} \int d^2\theta' \kappa(\theta') \ln |\theta - \theta'|,$$

(8)

such that

$$\alpha = \nabla \psi.$$  

(9)

The fact that the deflection field $\alpha(\theta)$ may be written as the gradient of a scalar field $\psi(\theta)$ guarantees that it has no curl.

2.2. Curl-free Vector Interpolation

Fuselier (2006, 2007) has shown how to obtain a curl-free interpolation of a vector field given on scattered data points. The method is general enough to be applied to an arbitrary number of dimensions, but will be discussed here for the two-dimensional case. The interpolated vector field is constructed using RBFs. An RBF is a positive definite function with radial symmetry $\phi(R/R_0)$. The fact that it is positive definite guarantees that our interpolation matrix (below) will have a solution (Wendland 2005; Fuselier 2006). The scale length $R_0$ is input by the user, and as we show in § 3.4, its freedom is closely related to the classic “mass sheet” degeneracy.

In our case, we must choose smooth RBFs that are at least three times continuously differentiable, or “$C^3$,” to ensure that our final mass map has no discontinuities. Then, a set of two curl-free basis vectors are constructed at each data point (image position) by simply taking derivatives of the basis function:

$$V_a = \phi_{xx} x + \phi_{xy} y,$$

(10)

$$V_b = \phi_{yx} x + \phi_{yy} y,$$

(11)

where $\phi_{xx}$ is the second derivative of $\phi(R/R_0)$ with respect to $x$, etc. The vector at that data point is given as a sum over all data points (with coefficients to be solved for):

$$\alpha = \sum_{i=1}^N (c_a V_{ai} + c_b V_{bi}).$$

(12)

Rewriting this in matrix form, we have

$$[\alpha] = [\phi(\psi)] [c].$$

(13)

For example, in the case of two data points, $\alpha(\theta_1)$ and $\alpha(\theta_2)$,

$$\begin{bmatrix}
  \alpha_{x1} \\
  \alpha_{y1} \\
  \alpha_{x2} \\
  \alpha_{y2}
\end{bmatrix} =
\begin{bmatrix}
  \phi_{xxx1} & \phi_{xyx1} & \phi_{xx2} & \phi_{yy2} \\
  \phi_{xyx1} & \phi_{yyx1} & \phi_{xx2} & \phi_{yy2} \\
  \phi_{xxx2} & \phi_{xyx2} & \phi_{xx1} & \phi_{yy1} \\
  \phi_{xyx2} & \phi_{yyx2} & \phi_{xx1} & \phi_{yy1}
\end{bmatrix}
\begin{bmatrix}
  c_{a1} \\
  c_{b1} \\
  c_{a2} \\
  c_{b2}
\end{bmatrix},$$

(14)

where, e.g.,

$$\phi_{jx12} = \frac{\partial^2 \phi}{\partial y \partial x}_{R=R_{12}},$$

(15)

with the derivative being evaluated at $R = R_{12} = |\theta_1 - \theta_2|$. We may now simply solve for the coefficients via the matrix inverse:

$$[c] = [\phi(\psi)]^{-1} [\alpha].$$

(16)

We may guarantee that this matrix inversion yields a solution by selecting a positive definite function for $\phi$. Wendland functions are especially suitable choices, and here we use a “$C^6$,” or 6 times continuously differentiable, Wendland function known as $W_{7,3}(\xi)$:

$$\phi(\xi) = \begin{cases}
  (1 - |\xi|)^6 [32|\xi|^3 + 25|\xi|^2 + 8|\xi| + 1], & |\xi| \leq 1 \\
  0, & |\xi| > 1
\end{cases}$$

(17)

where $\xi = R/R_0$. The function is very similar to a Gaussian with a peak of 1 and FWHM $\sim 0.50R_0$ ($\sigma \sim 0.21R_0$).\(^{10}\)

Of relevance in this work are derivatives of this function, which serve as basis functions for our mass maps. Having solved for the deflection field $[\alpha] = [\phi(\psi)][c]$, the mass map may then be calculated analytically as

$$\kappa = \frac{1}{2} \nabla \cdot \alpha$$

(18)

$$= \frac{1}{2} \sum_i^n [c_a \phi_{xxx} + c_b \phi_{yy}],$$

(19)

The basis functions

$$\kappa_a = \frac{1}{2} (\phi_{xxx} + \phi_{yy}),$$

(20)

$$\kappa_b = \frac{1}{2} (\phi_{xxx} + \phi_{yy})$$

(21)

are shown in Figure 3. Different combinations of these two basis functions simply serve to rotate it about its axis and change its amplitude.

The basis functions, with amplitudes and orientations solved for, are placed at the positions of the multiple images. The resulting coefficients are generally many orders of magnitude greater than the amplitude of the mass map. These large-mass components all cancel out nearly perfectly, with the small “residuals” being the mass map solution. Together, these mass components

\(^{10}\) In principle, a Gaussian could be used in place of $W_{7,3}$. However, it is well known (to mathematicians) that the interpolation matrix is ill conditioned when a Gaussian is used. Further, it is easier for a computer to evaluate a polynomial than a Gaussian.
form a very flexible basis set. Although the basis functions are derived from radial basis functions, no symmetry conditions (radial or otherwise) are imposed on the total mass map solution.

We note that other RBFs may be used in place of $W_{7/3}$, including other Wendland functions, thin plate splines, multiquadrics, power laws, and even Gaussians (e.g., Fuselier 2006). Wendland functions are especially well suited for use in the interpolation matrix, but other functions may be explored in future work.

### 2.3. Other Observables

For reference, we provide expressions for other observables as functions of our basis function. The lensing potential (eq. [8]) is defined as a sum of derivatives of our basis function

$$\psi = c_a \phi_x + c_b \phi_y.$$  

Other observables may then be derived as derivatives of this potential. The deflection field is given as the gradient of the potential

$$\alpha = \psi_x \hat{x} + \psi_y \hat{y}.$$  

We also obtain surface mass density

$$\kappa = \frac{1}{2} (\psi_{xx} + \psi_{yy}),$$  

shear

$$\gamma_+ = \frac{1}{2} (\psi_{xx} - \psi_{yy}),$$

$$\gamma_\times = \psi_{xy},$$

$$\gamma^2 = \gamma_+^2 + \gamma_\times^2,$$

the inverse of the magnification

$$\frac{1}{\mu} = (1 - \kappa)^2 - \gamma^2 = 1 - \psi_{xx} - \psi_{yy} + \psi_{xx} \psi_{yy} - \psi_{xy}^2,$$

and time delays

$$\Delta \tau = \frac{(1 + z_L) D_S}{c D_L D_{LS}} \left[ \frac{1}{2} |\theta - \beta|^2 - \psi \right].$$
2.4. Source Positions and Mass Map Physicality

Image deflections are defined by a combination of image positions, source positions, and redshifts, but source positions are not known in practice, and redshift measurements may contain large uncertainties. In our method, these become our free parameters. Each set of source positions and redshifts will redefine the deflection field, yielding a different LensPerfect mass map solution. We iterate to find the most plausible solutions, including a single “best” solution, as we describe below. An ideal optimization procedure would lead us directly to the true source positions and redshifts. The optimization procedure we have developed may not be ideal, but we believe that it produces an accurate “best” solution (see § 3.1) similar to that obtained with the true source positions. We describe our optimization procedure in the next subsection (§ 2.5) after first describing our method for rating solutions.

LensPerfect will return a perfect mass map solution for any set of input source positions and redshifts, but only certain sets, those with positive mass everywhere within the convex hull, where our solutions are constrained, will yield physical solutions, and some sets will yield “more physical” solutions than others. An extreme example of a “less physical” solution would be a “donut” solution comprised of a high-density ring surrounding a lower density center. By developing a method to rate different solutions as more or less physical, we can iterate over possible source positions to find those that yield plausible solutions, including the most plausible or “most physical” mass map solution. Our goal is to discard unreasonable solutions without biasing our result toward our concept of what a mass map should look like (that it should have a Navarro-Frenk-White profile, for example).

In the PixeLens method, a series of rigid criteria are used to distinguish “physical” from “unphysical” mass maps. Saha & Williams (2004) define a physical mass map as one that is positive everywhere and satisfies restrictions on maximum pixel-to-pixel variation and direction of the mass map gradient. Additional constraints, inversion symmetry about the axis and a minimum radial slope, are optionally imposed, and early incarnations of PixeLens found those mass map solutions that most closely followed the light distribution (as advocated in Saha & Williams 1997; Abdelsalam et al. 1998).

After some experimentation, we have developed a new measure of physicality. Rather than imposing rigid arbitrary constraints, we assign a figure of merit to each mass map solution based on the following physicality traits:

1. Positive mass everywhere within the convex hull.
2. Low mass scatter in each radial bin, thus preferring azimuthal symmetry.
3. No “tunnels” (penalty for individual mass pixels within the convex hull being “too low” relative to others at similar radius).
4. Overall smoothness (minimal pixel-to-pixel variation within the convex hull).
5. Average mass in radial bins decreases outwards (penalty for increasing outward).

Below, we motivate this physicality list, which has been chosen for the purposes of this paper. Note that the user has the ability to modify this list rather straightforwardly with the freely available LensPerfect software.

The first trait, which is obviously required of any physical mass map, is our only rigid constraint. Note that our mass maps are only constrained within the convex hull. Our basis functions do necessarily yield negative mass outside this region, but this is not necessarily a concern. We note that we are able to “get the right answer” inside the convex hull (Figs. 9 and 10), and if negative mass lies in a symmetric ring outside the convex hull, then it will have no effect on the image positions inside.

We would like to avoid one or more large, isolated pockets of negative mass off to one side outside the convex hull. These may arise as “corrections” to a solution that is not quite accurate within the convex hull. Large positive mass clumps outside the convex hull are similarly undesirable. Our second physicality trait serves to beat down these clumps, along with other positive mass clumps and underdense pockets within the convex hull. We may worry that this biases against the presence of subhalos, but turning this around, if we are biasing against subhalos, then we can be more confident of any subhalos that do arise in our solutions. Furthermore, Occam’s razor would dictate that the mass map solution with the fewest subhalos is most likely. Note also that we are talking about subhalos in large subclumps. Smaller satellites generally cannot be resolved (our resolution is limited by the density of multiple images observed in the lens plane). In the future, we may wish to perform tests to determine exactly how well LensPerfect recovers different amounts of substructure in large subhalos, given different numbers of multiple images.

Underdense pockets incur an extra penalty via our third physicality trait. While a small overdense area may be due to substructure, a small underdense area is much less physical. As our mass map is integrated along the line of sight, an underdense pocket suggests that a “tunnel” has been carved through the entire mass distribution. This is not very plausible, and thus we penalize such tunnels more severely than overdensities. We further beat lumps and pockets out of our solutions by minimizing pixel-to-pixel variation of mass (our fourth physicality trait).

Finally, we penalize (without forbidding) mass profiles that increase with radius. Of course, this might bias against spectacular findings such as the ringlike structure in CL 0024 reported by Lee et al. (2007), but we believe that it is perfectly healthy to bias against such spectacular solutions. If a more pedestrian solution may be found that exhibits no ring structure, then it is likely that no ring structure exists, and if such a ringlike structure were to persist in our solution despite this bias, we could be much more confident of its existence. Additional tests could then be performed to show how well LensPerfect recovers ring structures in known mass maps and whether it might “recover” rings when they are not present. Finally, we note that in CL 0024, the ringlike structure is detected (and predicted in simulations of a cluster collision) to lie well outside the strong-lensing region.

Note that our physicality traits 2, 3, and 5 assume at least a rough azimuthal symmetry. We have yet to experiment with mass distributions that are strongly multimodal. While our current method will suffice for many clusters, including Abell 1689, we will probably need to revise our penalty functions to analyze highly asymmetric clusters such as Abell 2218.

Even having settled on a set of physicality traits, it is unclear how exactly to calculate penalty functions based on them. Furthermore, once these penalty functions are established, how should they be weighted relative to each other? The goal is for equally “offensive” mass maps to be penalized equally. Again this is highly subjective, but after much experimentation, we have devised a total penalty function that works well. The details are given in the Appendix.

2.5. Mass Map Optimization

With a good penalty function in hand, we may then proceed to find those source positions and redshifts that produce the most
Given many lensed galaxies, this is far from trivial. For example, 19 source positions \((x, y)\) yield 38 parameters that must be optimized over, plus up to 19 redshifts with their associated uncertainties, and for each iteration, we must obtain a low-resolution mass map to calculate our penalty function. This takes from less than a second to a few seconds as more images are added. Thus, we must choose a scheme that efficiently navigates our 38(+19)-dimensional parameter space, minimizing the number of iterations necessary. Fortunately, we may use a few tricks to converge to good solutions fairly quickly.

The first trick is commonplace in strong-lensing mass map reconstruction methods: we build the mass map one galaxy at a time (that is, we add the multiple images of each galaxy in turn as constraints to our model). However, we take this a step further, tearing our solution down and rebuilding it for every iteration. These rebuilds are extremely quick, as at each step, the deflection field solution need only be calculated at the new image positions.

As we add each galaxy to our model, we place its source position using the average of those predicted from the current solution (Fig. 4). Each image will delens back to a slightly different position (as these images have not yet been constrained). By taking the average of these (weighted, as discussed below), we can obtain a good initial guess as to its source position.

The true source positions for objects 1 and 2 are plotted as triangles. Given this source position, we will obtain a new mass map solution. But we can also perturb this source position and obtain a different mass map solution instead. Thus, we iterate and search for the source position that yields the most physical solution according to our penalty function. We perform this two-dimensional optimization using the Powell (1964) routine\(^{11}\) included in the SciPy Python package.\(^{12}\) If the redshift of this galaxy is

---

\(^{11}\) More elaborate routines are available, but these may actually be less efficient. Specifically, calculating gradients of the penalty in source-position space would require two extra and time-consuming function evaluations for every iteration.

\(^{12}\) See http://www.scipy.org/.
uncertain, we may also optimize the redshift at this time (including a penalty if it drifts too far from its expected value; for this one-dimensional optimization, we use the simple golden section search method, also included in Scipy). Once the source position and redshift have been optimized, we may proceed to adding the next galaxy.

Before proceeding to the next galaxy, however, we may choose to reoptimize all previous source positions and redshifts. This “reshuffling” can improve the overall solution, but once, for example, 10 galaxies have been placed, we may worry that our solution has been “locked in.” Attempting to reoptimize the third galaxy position will not be very effective, as it is now tightly constrained by the positions of the other nine galaxies.

Thus, we use a flexible parameterization in which the perturbations to the predicted source positions (as described above) constitute our free parameters to be optimized. That is, we maintain a list of source position offsets [one for each galaxy, initially set to (0, 0)], and it is this list that we optimize, rather than the source positions themselves. Every time we rebuild our mass map solution, we obtain each source position as shown in Figure 4, but then we offset it according to our list of offsets before placing it and proceeding to add the next galaxy. Note that each offset affects not only the source position of that galaxy, but also (as this modifies the solution) those of all galaxies that follow. In essence, source positions move and adjust with each other.

As mentioned above, we determine each new source position by taking weighted averages of those predicted. We assign more weight to a delensed position for which the image is close to an already established image. For example, in Figure 4, the source position of a red image near any black image position will be given more weight in the average. The idea is that if the deflection field has already been established at a (black) image position, then the deflection field at the nearby (red) point should be similar. We have found that this weighting scheme yields better source positions (closer to the true model positions) than straight averaging. We have also had some success giving more weight to sources with images at large radii (where there is less mass available to support rapid changes in the deflection field). In the end, the exact scheme is somewhat irrelevant, as offsets from these average positions will be perturbed and optimized. However, it is possible that a better scheme would yield quicker convergence.

Above we have described all the details of our optimization scheme, except for how we begin! That is, where do we put our first source position? Before any galaxies are placed, we have no mass map solution, and thus no predictions for the first source position. Fortunately, just about any choice will do for the first source position, as all yield identical (or nearly so) solutions within the convex hull. Even after additional sources are added, shifting the first source position has little effect on the overall solution. The entire source plane basically shifts along with it, yielding a nearly identical solution within the convex hull. This is a well-known lensing degeneracy, but certain shifts in the source plane do yield more physical solutions than others, as we demonstrate in § 3.5. Correct source positions yield a solution that is more symmetric outside the convex hull.

2.6. Constraining Image Fluxes and/or Shears

Image fluxes may provide additional constraints to the mass model. To constrain the fluxes (and shears) of lensed images, previous authors have added terms to their matrix equation relating source positions, lens parameters, and observables (Evans & Witt 2003; Congdon & Keeton 2005; Diego et al. 2007). However, we prefer not to interfere with our equation (14), which, given proper basis functions, is guaranteed to obtain a perfect interpolated solution of any input deflection vectors. Fortunately, relative image fluxes (magnifications) and shears are determined by local rates of change in the deflection field and thus may be easily and precisely constrained by adding deflection field constraints.

In this subsection, we describe how to add flux and/or shear constraints for multiple images (see also § 3.6). We do not yet have a method for incorporating weak-lensing shear or any other constraints from singly imaged galaxies (but see § 4.3).

To constrain the fluxes of multiply imaged galaxies, we begin by obtaining an initial mass map that reproduces the image positions. Next, we construct a small box around each multiple image and delens each back to the source plane. We measure the area of each delensed box (now a parallelogram in the source plane) and compare it to its original area in the image plane. The ratio of these areas yields the magnification for that image, at least as predicted for our initial model.13 Now, by modifying the deflection field at the corners of the box, we can adjust the relative sizes of the lensed and unlensed boxes so that they produce the proper (observed) magnifications (Fig. 5). Given these new deflection field inputs, which encode both position and flux constraints for the multiple images, we obtain a new mass map solution that perfectly reproduces the observed positions and fluxes.

We note that image shears could be constrained in a similar manner, if desired, by adjusting the relative shapes of the lensed and delensed boxes. Furthermore, observational uncertainties may be incorporated by adding multiple realizations of noise to the measurements and finding a perfect solution to each, as done by Evans & Witt (2003) and Congdon & Keeton (2005).

In practice, we find that rather than adding four constraints to the deflection field for each flux measurement, we need only

13 Of course, we can also calculate the magnification using eq. (28).

![Fig. 5.—Method used to constrain image fluxes. A small box is constructed around each image (bottom right), and it is delensed back to the source plane (top left) using the initial mass map solution (derived using the image positions only). The inset zooms in on the delensed positions (circles). The ratio of the area of this parallelogram to the ratio of the “box” in the source plane yields the magnification of this image. We then adjust this ratio by adjusting the size of the delensed image (modifying our deflection field at the corners of the box). In this case, we require a larger magnification, so we decrease the size of the source image to that defined by the triangles. Our image box is constrained to delens to this smaller source, thus adjusting its magnification. In practice, we find that rather than adding four constraints to the deflection field, we need only constrain two points: one above or below, and the other to the left or right of the image. [See the electronic edition of the Journal for a color version of this figure.]

![Image 323x556 to 569x747]
constrain two extra points, one above or below, and the other to the left or right of the image. This not only helps to reduce computing time, but it also keeps our number of free parameters more in line with the number of observable constraints. We could keep the two numbers exactly equal by adding a single constraint, but this fails to properly constrain the flux, and instead squeezes mass out to the sides, as when one sits on a balloon.

2.7. Computational Efficiency

Other strong-lensing analysis methods (with the exception of PixeLens) face a dilemma over whether to minimize scatter in the source or image plane. The former choice, while less robust and subject to possible biases, is often chosen for computational efficiency. LensPerfect does not have to make this choice, as both source and image positions are always perfectly constrained.

Furthermore, LensPerfect is computationally efficient. Once all source positions and redshifts have been established (along with image positions), our mass map solution coefficients are obtained “instantly” (in a fraction of a second) via direct matrix inversion, without the need for iterations. Evaluating this mass map solution on a grid, while not quite “instant,” is still very fast. On a Mac Powerbook G4 laptop, given \( N_i = 93 \) multiple images, the mass map can be evaluated on a \( N_p = 2500 = 50 \times 50 \) grid in 3 s, scaling with \( N_i N_p \).

3. APPLICATIONS

3.1. Mass Map Recovery Test

Here, we demonstrate our technique given a mock galaxy cluster with simulated gravitational lensing. More important than the ability to perfectly reproduce all multiple-image positions is the accuracy to which we can recover the true mass map distribution.

We would like to test LensPerfect for a case similar to that observed in Abell 1689. Thus, we create a mock galaxy cluster “Babell 1689,” which is very similar to Abell 1689. In fact, the mass map of Babell 1689 (Fig. 6) is actually a solution obtained by Broadhurst et al. (2005) from their analysis of Abell 1689, but for our purposes, Babell 1689 is just a mock galaxy cluster. We use it to gravitationally lens 19 mock galaxies at redshifts between 1 and 5.5, producing 93 multiple images (Fig. 7). This is similar to the number of multiple images (106) identified by Broadhurst et al. (2005).

We stress that we are not analyzing Abell 1689; our analysis of Abell 1689 and its multiple images will be published in an upcoming paper. Here we are analyzing the mock cluster Babell 1689, given its mock multiple images.

The 93 mock multiple-image positions, along with 19 source positions and redshifts, are fed into LensPerfect as input. Figure 8 shows the input deflection field scaled to a source redshift of infinity along with a LensPerfect curl-free interpolation. The
solution was obtained using the Wendland function given in equation (17), with a scale factor of $R_0 = 700$ in the units plotted. One half the divergence of this deflection field gives the LensPerfect mass map solution (Fig. 9). Note that it is basically a low-resolution interpolation of the input mass map. Note that the significant features are reproduced inside the convex hull (black line). Outside this line, the solution is very poorly constrained, and can be varied simply by changing $R_0$. Here, the mass outside goes a bit negative (we have neatly clipped this from our color map), but when we increase $R_0$ to 1500, we find that the mass is positive within the entire region plotted, while the mass map inside is barely affected. [See the electronic edition of the Journal for a color version of this figure.]

Note that the Broadhurst et al. (2005) solution may appear to resolve very fine details in the Abell 1689 mass map that are absent from our reconstruction of Babell 1689 presented here. However, this is a result of their assumption that some component of the dark matter traces light. LensPerfect makes no such assumption.

In practice, we will not have knowledge of the source positions, so we repeat our analysis, but without providing the source positions as input. Instead, we use the source position optimization method outlined in §§2.4 and 2.5, and detailed in the Appendix. The “most physical” mass map solution we find is shown in Figure 10. It is very similar to that obtained with the true source positions (Fig. 9). In this case, the only inputs we gave to LensPerfect were the image positions and redshifts, which we assumed to have been measured with perfect precision.14

Radial profiles of the three mass maps (the model, the map recovered with true source positions, and the map recovered with optimized source positions) are compared in Figure 11. Only points within the convex hull are considered. Agreement of the mean mass is excellent, except for some deviation in the center and slight deviations toward the outside of the convex hull. The fine structure of the mass peaks is not recovered in these modest-resolution (93 multiple-image) mass maps.

3.2. Source Position Recovery

How well do our optimized source positions match the true source positions? Inaccuracies in the source positions propagate to the deflection field and thus to our mass map. Modest shifts of the entire source plane are acceptable, however. This well-known degeneracy does not affect the solution inside the convex hull (see §3.5).

With this in mind, we plot (Fig. 12) our optimized and true source positions from the previous subsection. A constant shift of a few pixels has been added to the optimized positions to bring them in line (on average) with the true positions. This is justified

14 Redshift uncertainties vary greatly from one study to the next. Thus, rather than attempting to present a test that includes “typical” redshift uncertainties, we propose that such tests be performed on a case-by-case basis. We note that we have had success in modeling real-life data such as the actual Abell 1689 multiple images, complete with their redshift uncertainties (D. Coe et al. 2008, in preparation).
We are developing methods to thoroughly explore the solution in practice. This is simply an inherent degeneracy in the problem. A mass map solution (not only for our method, but for any method), as shifts in the source plane are a degeneracy in the problem. A mass map solution obtained with one shift is virtually identical to, and inside the convex hull no less accurate than, a solution obtained with a different shift (§ 3.5).

Applying this shift, we find that our recovered source positions are offset from the true source positions at the rate of 1.05 pixels on average for the 19 systems. If this were A1689, 1.05 pixels would translate to 0.42", but this value cannot be directly compared to any published results for A1689 (or any other cluster). Source position offsets are likely inherent to all methods, but they can never be measured in practice, since the true source positions are unknown. The only measurable quantity is the scatter of (delensed) source positions within each multiple-image system. For LensPerfect, this scatter is zero. In other methods, the errors due to scatter and offsets may be cumulative.

Our optimal source positions are also biased toward a solution with slightly greater magnification than the true solution. This is most likely a consequence of our physicality measure, which rewards smoother mass maps. A smoother mass map will have a shallower profile and thus higher magnification. While we have made every effort not to bias profile slope, some small bias may still remain. We will work to reduce both the offsets and this bias in the future. In the meantime, it is a simple exercise for the user to spread out the optimal source positions and explore solutions with lower magnifications. In fact, this should be a part of any comprehensive analysis.

We must keep in mind the ultimate goal in our analysis. The goal is not to obtain the single best mass map, but rather to determine the range of mass maps that produce reasonable solutions. Both of the mass maps presented in § 3.1 are perfectly valid solutions that perfectly reproduce the 93 multiple-image positions. We cannot know which is more accurate without knowledge of the source positions, and of course this knowledge is unattainable in practice. This is simply an inherent degeneracy in the problem. We are developing methods to thoroughly explore the solution space and will report on them in future work.

3.3. 1000 Multiple Images

The resolution of a LensPerfect mass map is dictated by the density of multiple images detected. Each multiple image samples the deflection field at a given location. In the gaps between these images, the deflection field must be interpolated, and the exact form of the mass map becomes less certain.

To demonstrate this, we consider the mass map solution that we may obtain given 1000 multiple images. Rather than performing mock lensing to produce multiple images as in the previous subsection, here we will simply sample the deflection field at 1000 points. We restrict these samples to an area of a size similar to that used before (a circle of radius 150 pixels), and we ensure that all samples are separated by 2 pixels or more. Given these 1000 samples, we then interpolate to find the deflection field elsewhere. Our mass map solution is shown in Figure 13. Comparing it with the input mass map (Fig. 6), we can see that very fine detail is faithfully resolved.

This setup is a bit idealized. The “multiple images” are spread fairly uniformly (randomly) about a circle within the field. In practice, we are more likely to find clumps and voids of multiple images due to the magnification pattern, obscuring foreground galaxies, and physically linked background galaxies. Also, we have assumed the equivalent of perfect knowledge of the source positions. Nevertheless, this mass map gives us a rough idea of the resolution that we may expect given observations deep enough to reveal 1000 multiple images.

In practice, we would need to find and optimize a large number of source positions, as 1000 multiple images may be produced by about 300 background galaxies. This saddles us with 600 free parameters. However, even in our 93 image system, we find that source positions added toward the end are already fairly well constrained by those added previously. Therefore, we can speculate that the next 200+ source positions may similarly “fall into place,” making the computational challenge more manageable.

3.4. Fewer Constraints and the Role of $R_0$

While it is useful that LensPerfect can produce such detailed mass maps when given so many multiple images, what happens
when LensPerfect is given far fewer constraints, for example, a single quadruply imaged galaxy? Does the sum of four of the oddly shaped basis functions depicted in Figure 3 yield a reasonable mass map solution?

Yes it does. Figure 14 (left) shows the mass map solution obtained given a sample with a symmetric “Einstein cross” four-image configuration. The solution is exactly the same (to within \( \kappa = 0.01 \)) if we form a more complete “Einstein ring” with 16 image constraints, as shown. However, this is just one possible solution, obtained with \( R_0 = 100 \) equal to 10 times the Einstein radius.

We begin to explore other solutions by varying the width \( R_0 \) of our basis function (eq. [17]). Radial profiles of these different solutions are shown in the right-hand panel. By changing this single parameter, we neatly demonstrate the “steepness” or “mass sheet” degeneracy (originally dubbed the “magnification transformation” in Gorenstein et al. 1988), which states that the mass everywhere may be replaced by

\[
1 - \kappa' = \lambda (1 - \kappa), \quad (31)
\]

\[
\kappa' = \lambda \kappa + (1 - \lambda) \quad (32)
\]

without affecting the observed image positions (unless multiple galaxies of different redshifts have been lensed). Note that the new mass \( \kappa' \) is steeper than the previous mass \( \kappa \) by a factor of \( \lambda \).

The different LensPerfect solutions follow the transformation in equation (31) very closely, although not exactly. We have rescaled the green \( R_0 = 100 \) curve via this transformation such that its peak aligns with the \( R_0 = 55 \) profile. It now aligns with the full \( R_0 = 55 \) curve very well within the Einstein radius, but not perfectly. Note that the Einstein radius is also the convex hull in this case; thus we do not expect to be able to constrain the solution outside. Also note in the zoomed subplot that the different solutions have \( \kappa = 1 \) at slightly different radii, another indication that the classic mass sheet degeneracy relation is being followed approximately, but not exactly. [See the electronic edition of the Journal for a color version of this figure.]

**3.5. Source Plane Shifts**

Theory tells us that given a single source or even multiple sources at the same redshift, the entire source plane can be shifted, and a new mass map solution can be found that does not affect the image positions. This degeneracy was originally named the “prismatic transformation” by Gorenstein et al. (1988), who explained that this shift can be accomplished by adding an infinitely long and thin mass stick to the solution offset to one side of the images. Of course, such mass sticks are not very physical. Thus, we can overcome this degeneracy by simply finding a solution that does not resort to the addition of mass sticks. This is why we “know” that Einstein rings are produced by an on-axis alignment of the lens and source galaxies: The on-axis configuration requires a simple symmetric mass map, while any off-axis source position would require a less physical mass map, perhaps requiring the addition of a mass stick.
We say “perhaps,” because LensPerfect is able to find solutions to such off-axis Einstein rings without resorting to mass sticks. (Fortunately, LensPerfect has no mass stick basis functions to work with.) One such solution is shown in Figure 15. This mass distribution, while more plausible than a mass stick, is nevertheless highly improbable. The proper amount of mass must be added off axis in the correct configuration to refocus the light just enough toward the center of the lens, where it then produces the Einstein ring. It is very unlikely that such a precisely tuned mass configuration would occur in nature. Much more likely is the on-axis alignment, in which case any simple symmetric mass configuration will do. Our optimization procedure (§§ 2.4, 2.5) quickly converges to the on-axis source position as being the most likely, as off-axis solutions are penalized for being asymmetric.

However, we should not be surprised to find modest source position shifts in our solution either (§ 3.2). The mass within the convex hull is extremely insensitive to these shifts. (A constant shift in the source plane does not affect the divergence of the deflection field at the multiple-image positions; thus the mass map within the enclosed region experiences only negligible changes.)

### 3.6. Constraining Image Shapes and Sizes

Additional constraints may be derived from gravitationally lensed images by requiring the mass model to correctly reproduce not only image positions, but also their sizes, shapes, orientations, and fluxes. A method to constrain the fluxes (and shears) of unresolved images was discussed in § 2.6, but observed fluxes (and especially shears) may be uncertain. More precise constraints may be obtained when the multiple images are resolved, and extra knots may be identified within them.

Figure 16 shows two of the four multiple images of a z = 4.92 galaxy produced by the galaxy cluster MS 1358 (Franx et al. 1997). Six knots are identified as common in both of these images (B & C), with three of the knots also being identified in image A (not shown). All of the images of each knot in turn are constrained to originate from the same source position. The deflection model is updated after each knot is added. The final deflection solution yields the source plane images in the right-hand panel (see also Fig. 17). Note the identical alignment of the two images (B & C). At this point, the delensed images may be coadded to obtain greater depth, if desired (e.g., Colley et al. 1996).

However, this capability allows us to do more than simply produce pretty delensed images. The extra constraints provided by image knots can add significant detail to the mass map solution. Figure 18 compares (left) the MS 1358 mass map solution obtained when using only a single image position (the brightest knot) for each object and (right) the solution obtained when constraining all knot positions. In the latter, the mass is more stretched toward a second cluster galaxy, which proves crucial to traditional parametric mass modeling.

### 4. DISCUSSION

We now describe the current limitations of the LensPerfect method and plans for future capabilities that LensPerfect may provide us with. We also discuss the relative merits of both “parametric” and “nonparametric” methods, elaborating on points made in the introduction.
4.1. Imperfect Perfect Solutions

The “perfect” in LensPerfect refers to the reproduction of multiple-image positions attained by its mass map solutions. However, these solutions may be perfectly wrong and even unphysical if incorrect source positions and/or redshifts (and/or multiple-image identifications) are provided. We have developed a method to find source positions and redshifts that produce reasonable solutions. We do not claim that this method is perfect; it appears to work well, but may be improved upon in the future.

LensPerfect mass map solutions do not guarantee against predictions of additional multiple images that do not exist. This is an issue common to many methods, and it has been alternately argued that it should be more or less of a concern. The argument against its importance is that the prediction of additional multiple images is very sensitive to local substructure. Slight modifications might be made to this substructure, it is argued, to make these extra predicted multiple images disappear. LensPerfect may finally give us a tool capable of easily demonstrating this. We may be able to add deflection field constraints that tweak the mass map just enough to eliminate unwanted images. This idea has not yet been tested. In the meantime, we will simply check each solution for the correct number of multiple images. We find that our models generally do not produce extra multiple images as long as reasonable source positions are assumed.

4.2. Imperfect Solutions Superior?

Would it help to loosen the restriction that our mass map solutions must perfectly reproduce all the multiple-image positions? That is, might we obtain more accurate solutions by allowing our solutions to be imperfect? There are two parts to this question.

First, we consider measurement uncertainties. We have already discussed how we deal with redshift uncertainties in § 2.5. We allow the redshift to vary within the uncertainties as we optimize the mass map. The same could be done for other uncertain measurements. Image position uncertainties are generally very small (about 1 pixel), but when adding flux/shear constraints, we should certainly include the corresponding uncertainties in our optimization. Evans & Witt (2003) and Congdon & Keeton (2005)
followed this approach in their studies of galaxy lenses. By including uncertainties, they noted modest improvement in their solutions.

The second part of the question is more fundamental. Could a less perfect mass map be more accurate? This does prove true in the SLAP method of Diego et al. (2005a), who purposely leave some scatter in the source plane to avoid solutions that are “biased” with “a lot of substructure.” However, this is likely a result of their formulation of the problem. They consider all pixels in each lensed image and find the mass map solution that minimizes the sizes of all the delensed images. Of course, the delensed sizes should not be zero, so the delensed image pixels are allowed some “scatter.” (They make no attempt to match internal features, as we demonstrate in § 3.6.) Meanwhile, the proponents of PixeLens do not report such problems with their solutions, which perfectly reproduce all image positions (Saha et al. 2006).

We performed a test to see what an imperfect solution would look like and whether it might be more accurate. We began with our perfect solution to the 93 multiple images in § 3.1 with optimized source positions. We then “reoptimized” the source positions (based on our physicality criteria) one by one for each individual image. Source positions were no longer constrained to a single point for each multiple-image system. In practice, they drifted by an average of 0.4 pixels. Our resulting imperfect mass map was not extremely different from our perfect mass map, but not any more accurate either. Some of the substructure features in the center were erased as the physicality measure guided the optimization toward a smoother solution. The radial profile did not significantly improve or deteriorate in accuracy.

We believe the “1000 points of light” mass map in Figure 13 is a convincing demonstration that perfect is best. When the deflection field is sampled at 1000 points, we are able to map fine detail in the mass map. This fine structure is encoded in the exact positions of the multiple images. Allowing for an imperfect solution would erase this fine structure and waste information obtained in the observations.

4.3. Weak Lensing, Extended Images, and Time Delays

It may be possible to directly incorporate measurements of both weak lensing and not-so-weak shear into the LensPerfect method. Lacking multiple identifiable image knots, we can constrain galaxy shapes using the same method we use to constrain fluxes (§ 2.6). Instead of altering the relative sizes of the source and lensed regions, we can alter the shear. Thus, we could constrain all delensed images to be round, or perhaps round with some scatter of ellipticity as measured in large-scale surveys. We do not develop this idea further here. Such a technique would require three constraints per galaxy, which, given 100 galaxies, would require significant computing time (although at this stage, iterations may not be required).

A more practical idea may be to compare measured shears to those predicted from our mass models and to include the disagreement as a penalty in our optimization procedure. Note, however, that this would only work well for galaxies within our convex hull, as our solution (and thus shears) would be very poorly constrained outside this region.

Beyond simple constraints of flux, shear, and image knots, all of the pixel information in each lensed image may be utilized in the model derivation (e.g., Warren & Dye 2003; Suyu et al. 2006; Koopmans et al. 2006). This is especially desirable in strong galaxy-quasar lensing, in which observable constraints are less plentiful than in cluster lensing. It is unclear how to implement this in LensPerfect, except as another penalty in our optimization routine.

Another constraint often used in strong galaxy-quasar lensing studies is time delays observed among different images. Time delays are not local functions of the deflection field, and as such, we cannot constrain them directly with LensPerfect as we can fluxes and shears. However, for a given solution, they can be calculated directly from derivatives of our basis functions and compared to those observed, yielding another penalty for our optimization routine.

4.4. “Parametric” versus “Nonparametric” Methods

Methods are often classified as “parametric” or “nonparametric,” depending on whether a clear physical parameterization is used to construct the proposed mass map solutions. “Grid-based” methods are often referred to as “nonparametric,” even though strictly speaking they do have parameters, namely the mass at every pixel on a grid. The real distinction to be made here is between “model-based” and “model-free” methods. The former construct mass halos as physical analytical forms, while the latter do not. To give examples, the fully model-based strong-lensing analyses of the A1689 ACS images published to date have Halkola et al. (2006, 2007) and Limousin et al. (2007). Meanwhile, Diego et al. (2005b), Saha et al. (2006), and Leonard et al. (2007) have obtained model-free mass maps based on these images. The Broadhurst et al. (2005) and Zekser et al. (2006) analyses included both model-based and model-free elements.

LensPerfect, while clearly parametric, is also model free. The LensPerfect solutions are given as sums of basis functions, but these basis functions have no physical interpretation, and this functional form is practically indiscernible in the final solutions. As discussed in § 2.2, each mass map solution is a residual of the sum of these basis functions.

However, while LensPerfect is model free, it does not have the large number of free parameters typical of grid-based methods. In fact, when fitting image positions only (including extra knots, but not fluxes), the number of free parameters solved for (the coefficients) is exactly equal to the number of constraints. Note that the source positions and redshifts are not solved for directly, but rather must be provided as input. Each flux measurement provides a single constraint, but two free parameters are required to constrain it in our models. A shear measurement provides two constraints, which can be reproduced with an equal number (two) of free parameters.

Which are superior overall, model-based or model-free methods? Individual researchers may have a decided preference for one over the other, but in fact, both types of methods have their strengths, and each serves a purpose.

Physical model-based mass maps allow us to test whether dark matter is distributed in certain ways, according to our assumptions. In principle, they allow for a more straightforward determination of meaningful physical parameters. Cluster mass models attempt to simultaneously constrain the forms of both the overall cluster halo and the individual galaxy halos (e.g., Halkola et al. 2007), but degeneracies are strong between the two additive components.

On the other hand, model-free mass map reconstruction methods allow us to directly test for the presence of dark matter free of assumptions about its distribution. In particular, these methods make no assumptions about mass following light. Some of our most important discoveries about dark matter have come from, and are expected to come from, cases in which mass does not follow light. Mass peaks offset from light peaks can provide constraints on the collisional nature of dark matter particles, as in the Bullet Cluster (Markevitch et al. 2004; Clowe et al. 2006). Furthermore, observations of dark substructure (not associated with
light) may someday vindicate CDM simulations that predict much more halo substructure than is visible (Diemand et al. 2007; Strigari et al. 2007, and references therein).

Model-free methods are also able to explore a wider range of possible mass map solutions, including (with the advent of LensPerfect) those which perfectly reproduce all 100+ multiple-image positions. Of course, exploring the full range of model-free solutions can be a computational challenge, and it is not entirely clear how to sort the physical solutions from those that are “less” physical or aphysical.

Meanwhile, too much model flexibility has been cited as a potential problem, as a flexible model may fit incorrect data without sounding any alarms. Limousin et al. (2007) claim that their model-based method and that of Halkola et al. (2006) were unable to fit some multiple-image systems that were incorrectly identified in the initial Broadhurst et al. (2005) work, which was a bit more model free. LensPerfect, while still more flexible, may actually be more unforgiving than all previous methods when it comes to incorrect multiple-image identifications. The reason is simple: “imperfect” mass map reconstructions experience some offset in each and every predicted image position. Thus, a misidentified multiple-image set may have a larger $\chi^2$ than average, but this may be more easily dismissed in the analysis. In LensPerfect, however, each multiple image puts a rigid constraint on the deflection field. Thus, a misidentified multiple image is more likely to cause the deflection field to become tangled, leading to a “less physical” (if not aphysical) solution. We stay alert for such ill-fitting multiple-image systems as we add each to our models; of course, we must take care to avoid misidentifying multiple images from the start whenever possible.

Another objection to model flexibility was raised by Kochanek et al. (2004), who take exception to the ability of the model-based but flexible Evans & Witt (2003) method to produce a mass model that accounts for the flux anomaly observed in the strong galaxy-quasar lens Q2237+0305, even though this flux anomaly has since been shown to have been due to a microlensing event that has now passed. We would argue that the mass model proposed by Evans & Witt (2003) is but one of several possible solutions to the observed flux anomaly. All possible solutions should be considered, and in this case, microlensing is proven to be the true solution.

In analyses of galaxy-quasar lensing, model-based methods enjoy even greater appeal than in cluster lensing. Perfect solutions are much easier to come by when there are fewer constraints (four image positions, for example), and with so few image constraints, a large range of solutions is possible. We demonstrated one way of probing this solution space in § 3.4, and the PixeLens method provides another, but many choose to slash the solution space by making well-motivated model-based assumptions (e.g., an isothermal profile).

A powerful technique would be to combine LensPerfect with a model-based method. We could find good (imperfect) model-based solutions that leave offsets between the observed and predicted image positions and then perfect these solutions with LensPerfect. The deflection offsets may be interpolated and this “offset solution” added to the imperfect solution to create a perfect solution. We implemented this idea, but the results appeared unruly. A more creative approach will be required if this capability is to be realized.

We mention one other advantage to our method: ease of use. If a simple mass model is required, LensPerfect’s speed is hard to beat. Given a single lensed galaxy, LensPerfect instantly obtains a perfect solution “out of the box,” and if given 30+ lensed galaxies, as in Abell 1689, LensPerfect still obtains solutions with minimal user input. Parametric methods, on the other hand, require the user to develop complex models capable of fitting so many multiple images well. Previous studies have identified and measured properties of many cluster galaxies in order to model a “galaxy component,” which would be added to a separate halo component in their solutions. LensPerfect, on the other hand, bypasses this parameterization process, obtaining a detailed mass map free of strong assumptions.

5. SUMMARY AND FUTURE WORK

We have presented a new approach to gravitational lens mass map reconstruction. Given image positions, source positions, and redshifts, a new mathematical technique is used to interpolate the deflection field via direct inversion. The resulting mass map (simply half the divergence of the deflection field) perfectly reproduces all of the observed image positions. In practice, source positions are unknown. We have devised a method that efficiently optimizes over different possible configurations of the source galaxies. Each configuration produces a different mass map solution, which is evaluated for “physicality” based on criteria developed as part of this work. Our criteria make only minimal assumptions about the form of the mass map. Specifically, they make no assumptions about the slope of the radial profile or mass following light.

We have demonstrated our method on mock gravitational lensing data. A known mass map was used to lens 19 mock galaxies to produce 93 multiple images. Our mass map solution perfectly reproduces all the multiple-image positions while accurately recovering the known lens mass distribution to a resolution limited by the density of the multiple images. We have demonstrated the improved accuracy and fine detail that we may expect from a mass map derived from 1000 multiple images.

We have also presented LensPerfect solutions based on far fewer constraints, such as four multiple images of a single galaxy. Image fluxes or extra knots may provide additional constraints that can also be perfectly fit by our models. The number of free parameters solved for is kept nearly equal to the number of observable constraints. The LensPerfect software is easy to use and is made publicly available at our Web site (see § 1).

In subsequent papers, we will apply this method to Abell 1689 and other galaxy clusters. We will also attempt to resolve some of the “flux anomalies” observed in galaxy lenses.

In § 4, we discussed some possible improvements and extensions of our method, including the incorporation of weak-lensing data. We also hope to better quantify the accuracy of our mass map recovery and compare our performance to that of other methods. The uncertainties in our solutions should be well determined by exploring the full range of physical solutions. We will also experiment with other radial basis functions and other recipes for measuring mass map physicality.

We would like to thank many for useful discussions during the development of LensPerfect including Justin Read, Rick White, Marijn Franx, Piero Rosati, Myungkook Jee, Aleksi Halkola, Stella Seitz, Ralf Bender, Jean-Paul Kneib, Bernard Fort, Genevieve Soucail, Tony Tyson, Chris Fassnacht, Leonidas Moustakas, and Maruša Bradač. We thank Arjen van der Wel for comments that helped improve the manuscript, and we especially thank our anonymous referee for a thorough reading of the manuscript and useful scientific contributions. This research is supported by the European Commission Marie Curie International Reintegration Grant 017288-BPZ and the PNAYA grant AYA2005-09413-C02.
To calculate our penalty function, we first evaluate the mass map solution on a $41 \times 41$ grid. This proves large enough to measure physicality, while taking less than a second to calculate given as many as 50 image positions. (As this evaluation is to be part of our iterative procedure to determine source positions, it should be kept as quick as possible.) Based on this $41 \times 41$ mass map, we calculate the total penalty as follows:

1. The sum of all negative pixels inside the convex hull is multiplied by $-100$. We could simply assign a penalty of infinity to negative mass maps, but this would create a large region of constant penalty in source coordinate space. Our finite and varying penalty serves better during the optimization to corral the source positions back toward those that yield positive solutions.

2. The mean mass $\langle \kappa \rangle$ and rms scatter $\Delta \kappa$ are measured in radial bins of 80 points each. The rms scatter is totaled and divided by 10. Within the convex hull, the mean and scatter are recalculated in radial bins of 40 points each. This inner scatter is totaled and multiplied by 4. Outside the convex hull, we rebin the mass map yet again in bins of 80 points each. Rather than the rms mass scatter, this time we penalize the peak-to-peak variation (that is, the maximum minus the minimum) in each bin. These variations are totaled, multiplied by 5, and divided by the number of bins.

3. Any outward increase in $\langle \kappa \rangle$ over $R$ is measured and multiplied by 10. This is calculated first for all points and then again for those within the convex hull. Both penalties are added.

4. Given the mean mass $\langle \kappa \rangle$ and rms scatter $\Delta \kappa$ calculated within the convex hull (see item 2 above), we interpolate the 1σ lower limit $\langle \kappa \rangle - \Delta \kappa$ for all points within the convex hull. Deviation below this limit is measured and divided by 2. Note that we certainly expect some points to fall below the 1σ lower bound (by definition, 16% of the points should fall below), but the idea is to minimize these deviations. A large deviation below indicates a “tunnel” or unphysical dip in the surface density.

5. Finally, we put a premium on smooth solutions within the convex hull, as these are the most likely. Numerical derivatives of the mass map are calculated at each pixel $(x,y)$: $d\kappa/dx = [\kappa(x+1,y) - \kappa(x-1,y)]/(2x)$, $d\kappa/dy = [\kappa(x,y+1) - \kappa(x,y-1)]/(2y)$. The absolute values of these are totaled, and the entire sum is divided by 5.

The above penalties and their respective weights were defined after much trial and error. We find that mass maps with the lowest total penalty as defined above appear to be the best behaved, or most physical. Again, other weights and penalty schemes are certainly possible.

REFERENCES

AbdelSalam, H. M., Saha, P., & Williams, L. L. R. 1998, MNRAS, 294, 754
Amara, A., Metcalf, R. B., Cox, T. J., & Ostriker, J. P. 2006, MNRAS, 367, 1367
Bett, P., Eke, V., Frenk, C. S., Jenkins, A., Helly, J., & Navarro, J. 2007, MNRAS, 376, 215
Broadhurst, T., Huang, X., Frye, B., & Ellis, R. 2000, ApJ, 534, L15
Broadhurst, T., et al. 2005, ApJ, 621, 53
Clowe, D., Bradac, M., Gonzalez, A. H., Markevitch, M., Randall, S. W., Jones, C., & Zaritsky, D. 2006, ApJ, 648, L109
Colley, W. N., Tyson, J. A., & Turner, E. L. 1996, ApJ, 461, L83
Conbdon, A. G., & Keeton, R. C. 2005, MNRAS, 364, 1459
Diego, J. M., et al. 2005a, MNRAS, 360, 477
Diego, J. M., Sandvik, H. B., Protopapas, P., & Tegmark, M. 2005b, MNRAS, 362, 1247
Diego, J. M., Tegmark, M., & Protopapas, P. 2007, MNRAS, 375, 958
Diemand, J., Kuhlen, M., & Madau, P. 2007, ApJ, 657, 262
Diemand, J., Zemp, M., Moore, B., Stadel, J., & Carollo, C. M. 2005, MNRAS, 364, 665
Dutton, A. A., van den Bosch, F. C., Dekel, A., & Courteau, S. 2007, ApJ, 654, 27
Evans, N. W., & Witt, H. J. 2003, MNRAS, 345, 1351
Ford, H. C., et al. 2003, Proc. SPIE, 4854, 81
Franx, M., Illingworth, G. D., Kelson, D. D., van Dokkum, P. G., & Tran, K.-V. 1997, ApJ, 486, L75
Fukugita, M., Futamase, T., Kasai, M., & Turner, E. L. 1992, ApJ, 393, 3
Fuselier, E. 2006, Ph.D. thesis, Texas A&M
———. 2007, Adv. Comput. Math., in press (DOI: 10.1007/s10444-007-9046-3)
Gorenstein, M. V., Shapiro, I. I., & Falco, E. E. 1988, ApJ, 327, 693
Gustafsson, M., Fairbairn, M., & Sommer-Larsen, J. 2006, Phys. Rev. D, 74, 123522
Halkola, A., Seitz, S., & Pannella, M. 2006, MNRAS, 372, 1425
———. 2007, ApJ, 656, 739
Hardy, R. L. 1971, J. Geophys. Res., 76, 1905
Hogg, D. W. 1999, preprint (astro-ph/9905116)
Jee, M. J., et al. 2007, ApJ, 661, 728
Kaiser, N., & Squires, G. 1993, ApJ, 404, 441
Kazantzidis, S., Kravtsov, A. V., Zentner, A. R., Allgood, B., Nagai, D., & Moore, B. 2004, ApJ, 611, L73
Keeton, C. R., Kochanek, C. S., & Seljak, U. 1997, ApJ, 482, 604
Kneib, J.-P., Ellis, R. S., Smail, I., Couch, W. J., & Sharles, R. M. 1996, ApJ, 471, 643
Kochanek, C. S., Schneider, P., & Wambsganss, J. 2004, in Part 2 of Gravitational Lensing, Strong, Weak, & Micro. Proceedings of the 33rd Saas-Fee Advanced Course, ed. G. Meylan, P. Jetzer & P. North (Berlin: Springer), astro-ph/0407232
Koopmans, L. V. E., Treu, T., Bolton, A. S., Burles, S., & Moustakas, L. A. 2006, ApJ, 649, 599
Leonard, A., Goldberg, D. M., Haaga, J. L., & Massey, R. 2007, ApJ, 666, 51
Limousin, M., et al. 2007, ApJ, 668, 643
Maccio, A. V., & Miranda, M. 2006, MNRAS, 368, 599
Markevitch, M., Gonzalez, A. H., Clowe, D., Vikhlinin, A., Forman, W., Jones, C., Murray, S., & Tucker, W. 2004, ApJ, 606, 819
Merritt, D., Graham, A. W., Moore, B., Diemand, J., & Terzi´ c, B. 2006, AJ, 132, 2685
Merritt, D., Navarro, J. F., Ludlow, A., & Jenkins, A. 2005, ApJ, 624, L85
Metcalf, R. B., Moustakas, L. A., Bunker, A. J., & Parry, I. R. 2004, ApJ, 607, 43
Nar cowich, F. J., & Ward, J. D. 1994, Math. Comput., 63, 661
Navarro, J. F., et al. 2004, MNRAS, 349, 1039
Powell, M. J. D. 1964, Computer J., 7, 155
Saha, P., Read, J. I., & Williams, L. L. R. 2006, ApJ, 645, L5
Saha, P., & Williams, L. L. R. 1997, MNRAS, 292, 148
———. 2004, AJ, 127, 2604
———. 2006, ApJ, 653, 936
Sérsic, J. L. 1968, Atlas de Galaxias Australes (Córdoba: Observatorio Astronomico)
Strigari, L. E., Bullock, J. S., Kaplinghat, M., Diemand, J., Kuhlen, M., & Madau, P. 2007, ApJ, 669, 676
Suyu, S. H., Marshall, P. J., Hobson, M. P., & Blandford, R. D. 2006, MNRAS, 371, 983
Tyson, J. A., Wenk, R. A., & Valdes, F. 1990, ApJ, 349, L1
Wambsganss, J. 1998, Living Rev. Relativ., 1, 12
Warren, S. J., & Dye, S. 2003, ApJ, 590, 673
Wendland, H. 2005, Scattered Data Approximation (Cambridge: Cambridge Univ. Press)
Zekser, K. C., et al. 2006, ApJ, 640, 639