A perturbation approach to studying sign-changing solutions of Kirchhoff equations with a general nonlinearity

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Abstract
By employing a nonlocal perturbation approach and the method of invariant sets of descending flow, this manuscript investigates the existence and multiplicity of sign-changing solutions to a class of semilinear Kirchhoff equations in the following form

$$-\left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 \right) \Delta u + V(x)u = f(u), \quad x \in \mathbb{R}^3,$$

where $a, b > 0$ are constants, $V \in C(\mathbb{R}^3, \mathbb{R})$, $f \in C(\mathbb{R}, \mathbb{R})$. The methodology proposed in the current paper is robust, in the sense that, neither the monotonicity condition on $f$ nor the coercivity condition on $V$ is required. Our result improves the study made by Deng et al. (J Funct Anal 269:3500–3527, 2015), in the sense that, in the present paper, the nonlinearities include the power-type case $f(u) = |u|^{p-2}u$ for $p \in (2, 4)$, in which case, it remains open in the existing literature whether there exist infinitely many sign-changing solutions to the problem above. Moreover, energy doubling is established, namely, the energy of sign-changing solutions is strictly larger than two times that of the ground state solutions for small $b > 0$.

Keywords Kirchhoff equation · Sign-changing solution · Nonlocal perturbation approach · Invariant sets of descending flow
1 Introduction

In the present paper, we are concerned with sign-changing solutions of the following Kirchhoff equation

\[- \left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 \right) \Delta u + V(x)u = f(u), \quad x \in \mathbb{R}^3, \quad u \in H^1(\mathbb{R}^3), \quad (K)\]

where \( V \in C(\mathbb{R}^3, \mathbb{R}) \), \( f \in C(\mathbb{R}, \mathbb{R}) \), and \( a, b > 0 \) are positive constants. Problem (K) arises in an interesting physical context. Indeed, as a special case, the following Dirichlet problem

\[
\begin{aligned}
- (a + b \int_{\Omega} |\nabla u|^2) \Delta u &= f(x, u), & \text{in } \Omega, \\
\quad u &= 0, & \text{in } \partial \Omega
\end{aligned}
\]

is the general form of the stationary counterpart of the hyperbolic Kirchhoff equation

\[
\rho \frac{\partial^2 u}{\partial t^2} = \left[ \frac{P_0}{h} + \frac{E}{2L} \int_0^L \left( \frac{\partial u}{\partial x} \right)^2 dx \right] \frac{\partial^2 u}{\partial x^2}.
\]

Such equations were proposed by Kirchhoff [26] as an extension of the classical D’Alembert’s wave equations for free vibration of elastic strings once one takes into account the changes in length of the string produced by transverse vibrations. In (1.2), \( u \) denotes the displacement, \( b \) is the initial tension while \( a \) is related to the intrinsic properties of the string (such as Young’s modulus). The nonlinearity \( f(x, u) \) stands for the external force. Besides, we also point out that Kirchhoff problems appear in other fields like biological systems, such as population density, where \( u \) describes a process which depends on its own average. For the further physical background, we refer the readers to [13, 17].

Due to the presence of the term \( \int_{\Omega} |\nabla u|^2 \), Eqs. (1.1) and (1.2) are no longer pointwise identities and therefore, Kirchhoff problems are viewed as being nonlocal. This observation brings mathematical challenges to the analysis, and at the same time, makes the study of such a problem particularly interesting. In the past decades, this kind of problems have been receiving extensive attention. Initiated by Lions [30], by using the variational methods, the solvability of Kirchhoff type Eq. (1.1) has been investigated by many researchers in the literature (see [2, 3, 5, 29, 40, 41, 43, 49, 55] and the references therein). For the case of unbounded domain, there also have been many interesting works about the existence of positive solutions, multiple solutions, ground states and semiclassical states to Kirchhoff type equation (K) via variational methods, see for instance [4, 6, 19, 21–23, 28, 33, 34, 38, 42, 47, 51, 53, 54] and the references therein. Compared to the local elliptic equation \(- \Delta u = f(x, u), \ x \in \Omega\), difficulty lies in showing the boundedness of Palais–Smale sequences.

In recent years, another interesting topic has been investigated, namely the existence of sign-changing solutions to Kirchhoff problems. Via the minimax approach and the method of invariant sets of descent flow, Zhang and Perera [55], see also Mao and Zhang [41] proved the existence of sign-changing solutions of (1.1) provided \( f \) satisfies the 4-supercilinear growth condition:

\[
\left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 \right) \Delta u + V(x)u = f(u), \quad x \in \mathbb{R}^3, \quad u \in H^1(\mathbb{R}^3),
\]
\[
\lim_{|t| \to +\infty} F(x, t)/t^4 = +\infty, \text{ uniformly for } x \in \Omega, \quad (4\text{-superlinear})
\]
where \(F(x, t) = \int_0^1 f(x, s)ds\). Meanwhile, the authors in [55] also considered the asymptotically 4-linear case:
\[
\lim_{|t| \to +\infty} f(x, t)/bt^3 = \kappa > 0, \text{ uniformly for } x \in \Omega
\]
and the 4-sublinear case:
\[
|f(x, t)| \leq C(1 + |t|^{p-1}), \ x \in \Omega, \ t \in \mathbb{R}, \ for \ some \ p \in (2, 4). \quad (4\text{-sublinear})
\]
Here, we should point out that the 4-sublinear case in [55] requires the following additional restriction on \(f\): \(F(x, t) \geq ct^2\) for \(|t|\) small, which rules out cases such as that \(f(x, t) = |t|^{p-2}t\) for \(p \in (2, 4)\).

Subsequently, by the constraint variational method, Shuai [44] obtained the existence of least energy sign-changing solutions to problem (1.1). The author also showed that the energy of any sign-changing solution is strictly larger than that of the ground state solutions of (1.1). Note that one key assumption in [44] is the Nehari-type monotonicity condition of \(f\):
\[
(\text{Ne}) \quad \frac{f(u)}{|u|} \text{ is increasing on } t \in (-\infty, 0) \cup (0, +\infty).
\]
Lu [39] investigated a more general Kirchhoff-type equation of the form
\[
-M \left( \int_{\Omega} |\nabla u|^2 \right) \Delta u = \lambda f(u). \quad (1.3)
\]

By virtue of Nehari manifold methods, ground state and least energy sign-changing solutions are obtained. In particular, the author shows that under more restricted assumptions on \(M\) and \(f\), the energy doubling property holds for all \(\lambda > 0\), that is, the energy of any sign-changing solution is strictly twice larger than that of the ground state solutions. A weaker condition than (Ne) was proposed by Tang and Cheng [50], based on which the authors established the existence of least energy sign-changing solutions to problem (1.1) and showed that the energy doubling property holds. We emphasize that there are also some existence results of sign-changing solutions to Kirchhoff-type problems on \(\mathbb{R}^N\). Deng et al. [18] established the existence and asymptotic behavior of nodal solutions to (K) with the nonlinear term \(f(|x|, t)\). Precisely, they obtained the existence of a sign-changing solution, which changes signs exactly \(k\) times for any \(k \in \mathbb{N}\). Their argument consists of reducing the original problem to a system of \((k + 1)\) equations where the unknowns have disjoint support. Then the nodal solution is constructed through gluing \(u_i\) by matching the normal derivative at each junction point. We highlight that the monotonicity condition like (Ne) plays a crucial role in [18]. Moreover, they further assumed that \(F(|x|, u)/u^4 \to +\infty\) as \(|u| \to \infty\). We should point out that the case \(f(t) = |t|^{p-2}t\) for \(p \in (2, 4)\) remains open in [18]. Recently, by virtue of the invariant sets method, Sun et al. [48] obtained infinitely many sign-changing solutions of problem (K) when \(f(t) \sim |t|^{p-2}t\) as \(t \to \infty\) for \(p \in (2, 4)\). However, \(V\) is required to be coercive, that is, \(\lim_{|x| \to \infty} V(x) = \infty\), which plays a key role in obtaining the boundedness of Palais–Smale sequences. Very recently, this case also was considered in [11] for a class of Kirchhoff type equations
\[
[1 + \lambda \int_{\mathbb{R}^3} |\nabla u|^2 + V(x)u^2][-\Delta u + V(x)u] = f(u) \text{ in } \mathbb{R}^3
\]
and where the coercivity condition was not imposed, and in addition, for \(\lambda > 0\) small, multiple sign-changing solutions were obtained.
The results obtained in [11, 18, 48] suggest the following open question.

**Problem 1** Does problem (K) admit sign-changing solutions without assuming condition (Ne) or dropping the coercivity condition of $V$? In particular, without the coercivity condition, do there exist infinitely many sign-changing solutions of (K) in the case $f(t) = |t|^{p-2}t$ for $p \in (2, 4)$?

The main interest of the present paper is to give an affirmative answer to this open question.

## 2 Main results

### 2.1 Variational setting

Throughout this paper, we assume the external Schrödinger potential $V \in C(\mathbb{R}^3, \mathbb{R})$ is radially symmetric and enjoys the following condition

\[
(V_1) \quad \inf_{x \in \mathbb{R}^3} V(x) := V_0 > 0,
\]

and $f \in C(\mathbb{R}, \mathbb{R})$ satisfies the following hypotheses

\[
(f_1) \quad f \in C(\mathbb{R}, \mathbb{R}) \text{ and } \lim_{t \to 0} \frac{f(t)}{t} = 0;
\]

\[
(f_2) \quad \limsup_{|t| \to \infty} \frac{|f(t)|}{|t|^{p-1}} < \infty \text{ for some } p \in (2, 6);
\]

\[
(f_3) \quad \text{there exists } \mu > 2 \text{ such that } tf(t) \geq \mu F(t) > 0 \text{ for } t \neq 0, \text{ where } F(t) = \int_0^t f(s)ds.
\]

**Remark 2.1** By $(f_2)$ and $(f_3)$, we have $2 < \mu \leq p < 6$. As a reference model, $f(u) = |u|^{p-2}u$ satisfies $(f_1)$–$(f_3)$ for $p \in (2, 6)$.

Consider the Hilbert space

\[
E = \left\{ u \in H^1_\sigma(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2 < \infty \right\}
\]

with the inner product

\[
\langle u, v \rangle = \int_{\mathbb{R}^3} a\nabla u \nabla v + V(x)uv
\]

and the norm

\[
\|u\| := \sqrt{\langle u, u \rangle} = \left( \int_{\mathbb{R}^3} a|\nabla u|^2 + V(x)u^2 \right)^{\frac{1}{2}}.
\]

Obviously, the embedding $E \hookrightarrow L^q(\mathbb{R}^3)$ is compact for $2 < q < 6$ (see Strauss [45]). The associated energy functional $I : E \to \mathbb{R}$ is given by
\[
I(u) = \frac{1}{2} \|u\|^2 + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla u|^2 \right)^2 - \int_{\mathbb{R}^3} F(u).
\]

It is a well-defined \(C^1\) functional in \(E\) and its derivative is given by

\[
I'(u)v = \langle u, v \rangle + b \int_{\mathbb{R}^3} |\nabla u|^2 \int_{\mathbb{R}^3} \nabla u \nabla v - \int_{\mathbb{R}^3} f(u)v, \ v \in E.
\]

**Definition 2.1** \(u\) is called a weak solution of (K), if \(u \in E\) satisfies

\[
I'(u)\varphi = \langle u, \varphi \rangle + b \int_{\mathbb{R}^3} |\nabla u|^2 \int_{\mathbb{R}^3} \nabla u \nabla \varphi - \int_{\mathbb{R}^3} f(u)\varphi = 0
\]

for all \(\varphi \in C^\infty_0(\mathbb{R}^3)\). Furthermore, if \(u^\pm \neq 0\), then \(u\) is called a sign-changing solution of (K), where \(u^\pm = \max\{\pm u, 0\}\).

Before stating our main results, we impose some additional hypotheses on \(V(x)\) as follows.

\((V_2)\) \(V(x) \in C(\mathbb{R}^3, \mathbb{R}^+)\) is differentiable and satisfies \((\nabla V(x), x) \in L^\infty(\mathbb{R}^3) \cup L^2(\mathbb{R}^3)\). Moreover, there exists \(\mu > 2\) such that

\[
\frac{\mu - 2}{\mu} V(x) - (\nabla V(x), x) \geq 0.
\]

Under these assumptions on \(V(x)\) and \(f(u)\), we establish the main results in terms of existence, multiplicity and energy doubling on sign-changing solutions, which are stated in the following three subsections.

### 2.2 Existence

Our first result reads as follows.

**Theorem 2.1** (Existence) If \((V_1)-(V_2)\) and \((f_1)-(f_3)\) hold, then problem (K) admits at least one radially symmetric least energy sign-changing solution.

Observe that if \(b = 0\), problem (K) reduces to the following local Schrödinger equation

\[
-a\Delta u + V(x)u = f(u), \ u \in H^1(\mathbb{R}^3),
\]

(2.1)

which does not depend on the nonlocal term \(\int_{\mathbb{R}^3} |\nabla u|^2\) any more. To look for sign-changing solutions of Eq. (2.1), several approaches were introduced in the literature. Based on the Nehari manifold technique, Cerami et al. [14] proved the existence of sign-changing solutions for elliptic problems involving critical exponent(see also [9, 10]). The heat flow method was explored in [15] to study the existence of sign-changing solutions. Morse theory can be also used to consider the existence of sign-changing solutions (see [16]). In finding sign-changing solutions of elliptic problems, the method of invariant sets of descending flow has been a powerful tool. Here we refer to [7, 8, 56] and the reference therein. However, we should address a remark on the case \(b > 0\).

**Remark 2.2** However, in contrast to problem (2.1), the non-locality leads problem (K) to be more complicated in seeking sign-changing solutions.
Now, we summarize the main difficulties and novelties developed in this study.

(1) In finding sign-changing solutions of (2.1), a crucial ingredient is the following decomposition: for any $u \in E$, $I_0(u) = I_0(u^+) + I_0(u^-)$, $\langle I_0'(u), u^\pm \rangle = \langle I_0'(u^\pm), u^\pm \rangle$. (2.2)

where $I_0$ is the energy functional of (2.1) defined by $I_0(u) = \frac{a}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{1}{2} \int_{\mathbb{R}^3} V(x)u^2 - \int_{\mathbb{R}^3} F(u)$.

However, for problem (K), due to the nonlocal term $\int_{\mathbb{R}^3} |\nabla u|^2$, one will get

\[
\begin{cases}
I(u) = I(u^+) + I(u^-) + \frac{b}{2} \int_{\mathbb{R}^3} |\nabla u^+|^2 \int_{\mathbb{R}^3} |\nabla u^-|^2, \\
\langle I'(u), u^\pm \rangle = \langle I'(u^\pm), u^\pm \rangle + b \int_{\mathbb{R}^3} |\nabla u^+|^2 \int_{\mathbb{R}^3} |\nabla u^-|^2,
\end{cases}
\]

(2.3) which do no longer satisfy the decomposition (2.2). So some classical methods cannot be used to deal with our problem directly. Motivated by [31, 32], we attempt to find sign-changing solutions for problem (K) by using the method of invariants sets of descending flow together with some minimax technique and nonlocal perturbation approach.

(2) For the case $f(t) = |t|^{p-2}t$, $p \in (2, 4]$, the effect of the nonlocal term $\int_{\mathbb{R}^3} |\nabla u|^2$ has two aspects. Firstly, it seems much more complicated to find an operator to construct invariants sets of descending flow associated with problem (K). A similar difficulty also arises in seeking sign-changing solutions of the Schrödinger–Poisson systems (see [12, 20, 32])

\[
\begin{cases}
-\Delta u + V(x)u + \phi u = |u|^{p-2}u \quad \text{in } \mathbb{R}^3, \\
-\Delta \phi = u^2 \quad \text{in } \mathbb{R}^3,
\end{cases}
\]

(2.4) where $p \in (3, 4)$. It is motivated by [32] that we introduce an auxiliary operator $A$ as the starting point to construct a pseudo-gradient vector field ensuring the existence of the desired invariant sets of the flow. Secondly, the so-called 4-(AR) condition fails, which makes tough to get the boundedness of (PS) sequences. In [32, 48], the authors adopt a perturbation approach together with the coercivity of $V(x)$ to recover such boundedness. However, without the coercivity condition, the method in [32, 48] is not applicable anymore. To overcome this obstacle, we introduce a nonlocal perturbation approach by adding a higher order term $|u|^{-2}u$ and another perturbation which is nonlocal. For the perturbed problems, by minimax arguments in the presence of invariant sets, we obtain sign-changing solutions. By passing to the limit, a convergence argument allows us to get sign-changing solutions of the original problem (K), which involves the case $f(t) = |t|^{p-2}t$, $p \in (2, 4)$. We note that this perturbation approach has been developed by [35–37] to deal with some nonlocal variational problems and should be of independent interest in other problems with the difficulty in verifying the boundedness of Palais–Smale sequences.
2.3 Multiplicity

Another aim of the paper is to prove the existence of infinitely many sign-changing solutions to problem (K) when \( f \) is odd.

**Theorem 2.2** (Multiplicity) If \((V_1)\)–\((V_2)\) and \((f_1)\)–\((f_3)\) hold, then problem (K) has infinitely many radially symmetric sign-changing solutions when \( f \) is odd.

**Remark 2.3** In the spirit of arguments of Theorem 2.1 and the multiple sign-changing critical point approaches developed by Liu et al [31], we obtain infinitely many sign-changing solutions \( u_{\lambda,\beta}^k, k = 1, 2, \ldots \) of the perturbed problem as approximation solutions for the original equation. Then by passing to the limit, sign-changing solutions of the original problem are obtained. However, the minimax values \( c_{\lambda,\beta}^k \) of the perturbed problem enjoy the different monotonicity properties on the two perturbation parameters \( \lambda, \beta \), so it is not easy to distinguish the limits of \( u_{\lambda,\beta}^k \) as \( \lambda, \beta \to 0 \). In our arguments, we use an auxiliary functional to show that \( c_{\lambda,\beta}^k \to \infty \) as \( k \to \infty \) uniformly for \( \lambda, \beta \). Based on this estimate, we obtain infinitely many radially symmetric sign-changing solutions for problem (K).

2.4 Energy doubling

The last investigation is to establish *energy doubling* of sign-changing solutions to problem (K) with \( f(u) = |u|^{p-2}u, \ p \in (2, 6) \). This fact has been proved for the *local* problem (2.1) when \( V(x) \) is a constant or a periodic function [52]. In particular, we denote the Nehari manifold associated with (2.1) by

\[
\mathcal{N} := \{u \in E \setminus \{0\} : \langle I_0(u), u \rangle = 0\},
\]

and

\[
c_0 = \inf \{I_0(u) : u \in \mathcal{N}\}. \quad (2.5)
\]

For any sign-changing solution \( w \in E \) of (2.1), it follows from the fact \( w^\pm \in \mathcal{N} \) that

\[
I_0(w) = I_0(w^+) + I_0(w^-) \geq 2c_0. \quad (2.6)
\]

In fact, the minimizer of (2.5) is indeed a ground state solution of (2.1), and \( c_0 > 0 \) is the ground state energy. If some sign-changing solution \( w \) of (2.1) satisfies

\[
I_0(w) > 2c_0,
\]

it was called in [1] (see also [52]) that \( w \) satisfies “energy doubling.” They showed that any sign-changing solution of (2.1) satisfies energy doubling in the case \( f(u) = |u|^{p-2}u \) for \( p \in (2, 6) \) and \( V(x) \) is a constant or a periodic function [52].

In the current study, we also estimate the energy of sign-changing solutions to problem (K). Analogous to problem (2.1), the definition of energy doubling corresponding to problem (K) is given as follows.

**Definition 2.2** Let \( w_h \in E \) be a sign-changing solution of problem (K), we say that \( w_h \) satisfies energy doubling if \( I(w_h) > 2c_h \), where
Let $w_b \in E$ be a sign-changing solution of problem (K). Since the interaction of the positive and negative parts of solutions cannot be neglected, it follows from (2.3) that

$$w_b^\pm \notin \mathcal{N}_b.$$  

Thus, a natural open question is whether energy doubling holds. Generally speaking, it is even not easy to compare $I(w_b)$ with $c_b$. To proceed, we impose the following additional assumptions on $V$.

$(V_3)$ $V \in C^2([0, +\infty), \mathbb{R}^+)$ is radially symmetric and

$$0 < \inf_{r>0} V(r) \leq \sup_{r>0} V(r) < \infty;$$

$(V_4)$ $\inf_{r>0} [(V''(r)r^2 + (3 + \tau)V'(r)r + 2\tau V(r)] > 0$ with $\tau = \frac{4(p-1)}{3p}$.

By using an approximation procedure, we give a partial positive answer for such an open problem, that is, energy doubling holds if $b > 0$ small. Precisely, we have the following result.

**Theorem 2.3** (Energy doubling) For $f(u) = |u|^{p-2}u$, $p \in (2, 6)$ and assume $V$ satisfies $(V_1)$–$(V_2)$ and $(V_3)$–$(V_4)$, then there exists $b^* > 0$ such that, for any sign-changing solution $w_b \in E$ of problem (K), we have $I(w_b) > 2c_b$ if $b < b^*$, i.e., energy doubling holds. Furthermore, for any sequence $\{b_n\}$ with $b_n \to 0$ as $n \to \infty$, up to a subsequence, $w_{b_n} \to w_0$ in $E$, where $w_0$ is a sign-changing solution of (2.1).

**Remark 2.4** The assumptions $(V_3)$–$(V_4)$ are imposed only to guarantee the uniqueness of positive solutions to Eq. (2.1) (see [25]), which we use in our argument.

The outline of the paper is as follows. An perturbation approach is introduced, by which we obtain the existence of sign-changing solutions to problem (K) in Sect. 3. Section 4 is devoted to proving Theorem 2.2 by the minimax theorem through invariant sets of descending flow. Finally, the energy doubling property is established for sign-changing solutions to problem (K) in Sect. 5. The notations used in this paper are summarized as follows.

**Notations.**

- $\|u\|_p := \left( \int_{\mathbb{R}^3} |u|^p \right)^{1/p}$ for $p \in [1, \infty)$.
- $\mathcal{C}$ will be used repeatedly to denote various positive constants which may change from line to line.
- $D^{1,2}(\mathbb{R}^3) := \{ |\nabla u| \in L^2(\mathbb{R}^3) : u \in L^6(\mathbb{R}^3) \}$.
- $S$ denotes the best Sobolev constant, i.e.,

$$S := \inf_{u \in D^{1,2}(\mathbb{R}^3) \setminus \{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2}{(\int_{\mathbb{R}^3} u^6)^{1/3}}.$$
3 Existence

In this section, we prove the existence of least energy sign-changing solutions to problem (K) when \( V(x) \) is a radially symmetric function.

3.1 A perturbed problem

Since we do not impose the 4-superlinear Ambrosetti–Rabinowitz condition, the boundedness of the Palais–Smale sequence becomes not easy to obtain. A perturbed problem is introduced to overcome this difficulty. Set \( \alpha \in (0, \frac{p+2}{3p+2}) \) and fix \( \lambda, \beta \in (0, 1) \) and \( r \in \left( \max\{p, \frac{3}{2}\}, 6 \right) \), we consider the modified problem

\[
-\left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 \right) \Delta u + V(x)u = f_{\lambda, \alpha, \beta}(u), \quad u \in E,
\]

where

\[
f_{\lambda, \alpha, \beta}(u) = f(u) + \beta |u|^{r-2}u - \lambda \left( \int_{\mathbb{R}^3} u^2 \right)^\alpha.
\]

The associated functional can be constructed as follows

\[
I_{\lambda, \beta}(u) = I(u) + \frac{\lambda}{2(1+\alpha)} \left( \int_{\mathbb{R}^3} u^2 \right)^{1+\alpha} - \frac{\beta}{r} \int_{\mathbb{R}^3} |u|^r.
\]

It is easy to show that \( I_{\lambda, \beta} \in C^1(E, \mathbb{R}) \) and

\[
I'_{\lambda, \beta}(u)v = I'(u)v + \lambda \left( \int_{\mathbb{R}^3} u^2 \right)^\alpha \int_{\mathbb{R}^3} uv - \beta \int_{\mathbb{R}^3} |u|^{r-2}uv, \quad u, v \in E.
\]

We will make use of the following Pohozaev type identity, whose proof is standard and can be found in [27, Theorem 29.4].

Lemma 3.1 Let \( u \) be a critical point of \( I_{\lambda, \beta} \) in \( E \) for \( (\lambda, \beta) \in (0, 1) \times (0, 1) \), then

\[
a \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{3}{2} \int_{\mathbb{R}^3} V(x)u^2 + \frac{1}{2} \int_{\mathbb{R}^3} (\nabla V(x), x)u^2
\]

\[
+ \frac{b}{2} \left( \int_{\mathbb{R}^3} |\nabla u|^2 \right)^2 + \frac{3\lambda}{2} \left( \int_{\mathbb{R}^3} u^2 \right)^{1+\alpha} - 3 \int_{\mathbb{R}^3} \left( F(u) + \frac{\beta}{r} |u|^r \right) = 0.
\]

As an application of the Lax–Milgram theorem, for each \( u \in E \), the following equation

\[
-\left( a + b \int_{\mathbb{R}^3} |\nabla u|^2 \right) \Delta v + V(x)v + \lambda \left( \int_{\mathbb{R}^3} u^2 \right)^\alpha v = f(u) + \beta |u|^{r-2}u
\]

has a unique weak solution \( v \in E \). In order to construct the descending flow for the functional \( I_{\lambda, \beta} \), we introduce an auxiliary operator \( T_{\lambda, \beta} : u \in E \mapsto v \in E \), where \( v = T_{\lambda, \beta}(u) \) is the unique weak solution of problem (3.1). Clearly, the fact that \( u \) is a solution of problem...
(3.1) is equivalent to that $u$ is a fixed point of $T_{\lambda, \beta}$, which is well defined based on the above arguments. Moreover, this operator is continuous and compact, as stated in the next lemma.

**Lemma 3.2** The operator $T_{\lambda, \beta}$ is continuous and compact.

**Proof** For the compactness, a similar proof can be found in [11]. In the following, we only show the continuity. Assume that $\{u_n\} \subset E$ with $u_n \to u$ strongly in $E$ as $n \to \infty$. Let $v = T_{\lambda, \beta}(u)$ and $v_n = T_{\lambda, \beta}(u_n)$, then we have

$$
\int_{\mathbb{R}^3} (a \nabla v_n \nabla w + V(x)v_n w) + b \int_{\mathbb{R}^3} |\nabla u_n|^2 \int_{\mathbb{R}^3} \nabla v_n \nabla w + \lambda \left( \int_{\mathbb{R}^3} u_n^2 \right)^{\alpha} \int_{\mathbb{R}^3} v_n w
= \int_{\mathbb{R}^3} f(u_n)w + \beta \int_{\mathbb{R}^3} |u_n|^{p-2}u_n w, \quad \forall w \in E
$$

(3.2)

and

$$
\int_{\mathbb{R}^3} (a \nabla v \nabla w + V(x)v w) + b \int_{\mathbb{R}^3} |\nabla u|^2 \int_{\mathbb{R}^3} \nabla v \nabla w + \lambda \left( \int_{\mathbb{R}^3} u^2 \right)^{\alpha} \int_{\mathbb{R}^3} v w
= \int_{\mathbb{R}^3} f(u)w + \beta \int_{\mathbb{R}^3} |u|^{p-2}uw, \quad \forall w \in E.
$$

(3.3)

We need to show $\|v_n - v\| \to 0$ as $n \to \infty$. Indeed, it follows from $(f_1)$ and $(f_2)$ that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$
|f(t)| \leq \varepsilon |t| + C_\varepsilon |t|^{p-1}.
$$

(4.4)

Testing with $w = v_n$ in (3.2) gives

$$
\|v_n\|^2 + b \int_{\mathbb{R}^3} |\nabla u_n|^2 \int_{\mathbb{R}^3} |\nabla v_n|^2 + \lambda \|u_n\|_2^{2\alpha} \int_{\mathbb{R}^3} v_n^2
\leq \int_{\mathbb{R}^3} (\varepsilon |u_n| + C_\varepsilon |u_n|^{p-1})|v_n| + \beta \int_{\mathbb{R}^3} |u_n|^{p-1} |v_n|,
$$

which, together with the Hölder inequality, imply that $\{v_n\}$ is a bounded sequence in $E$. Assume $v_n \rightharpoonup v^*$ in $E$ and $v_n \to v^*$ in $L^s(\mathbb{R}^3)$ for $s \in (2, 6)$ after extracting a subsequence, then by (3.2) we have

$$
\int_{\mathbb{R}^3} (a \nabla v^* \nabla w + V(x)v^* w) + b \int_{\mathbb{R}^3} |\nabla u|^2 \int_{\mathbb{R}^3} \nabla v^* \nabla w + \lambda \left( \int_{\mathbb{R}^3} u^2 \right)^{\alpha} \int_{\mathbb{R}^3} v^* w
= \int_{\mathbb{R}^3} f(u)w + \beta \int_{\mathbb{R}^3} |u|^{p-2}uw, \quad \forall w \in E.
$$

(3.5)

Hence $v^*$ is a weak solution of (3.1), which implies $v = v^*$ by the uniqueness. Moreover, taking $w = v_n - v$ in (3.2) and (3.3) and then subtracting, we have

\[ Springer \]
\[ \|v_n - v\|^2 + b \int_{\mathbb{R}^3} |\nabla u_n|^2 \int_{\mathbb{R}^3} |\nabla (v_n - v)|^2 + \lambda \|u_n\|^{2a} \int_{\mathbb{R}^3} |v_n - v|^2 \]

\[ = -b \int_{\mathbb{R}^3} (|\nabla u_n|^2 - |\nabla u|^2) \int_{\mathbb{R}^3} \nabla v \nabla (v_n - v) - \lambda (\|u_n\|^{2a} - \|u\|^{2a}) \int_{\mathbb{R}^3} v(v_n - v) \]

\[ + \int_{\mathbb{R}^3} (f(u_n) - f(u))(v_n - v) + \beta \int_{\mathbb{R}^3} (|u_n|^r - 2u_n - \|u\|^{r-2})u_n(v_n - v). \]

(3.6)

It follows from (3.5)–(3.6) and Sobolev’s embedding inequality that \( v_n \to v \) in \( E \) as \( n \to \infty \). Therefore, \( T_{\lambda,\beta} \) is continuous.

\[ \square \]

**Lemma 3.3**

1. \( I'_{\lambda,\beta}(u)(u - T_{\lambda,\beta}(u)) \geq \|u - T_{\lambda,\beta}(u)\|^2 \) for all \( u \in E \);
2. \( \|I''_{\lambda,\beta}(u)\| \leq \|u - T_{\lambda,\beta}(u)\|(1 + C_1 \|u\|^2 + C_2 \|u\|^{2a}) \) for all \( u \in E \), where \( C_1 \) and \( C_2 \) are two positive constants.

**Proof** Since \( T_{\lambda,\beta}(u) \) is the solution of Eq. (3.1), we have

\[ I'_{\lambda,\beta}(u)(u - T_{\lambda,\beta}(u)) = \|u - T_{\lambda,\beta}(u)\|^2 + b \int_{\mathbb{R}^3} |\nabla u|^2 \int_{\mathbb{R}^3} |\nabla (u - T_{\lambda,\beta}(u))|^2 \]

\[ + \lambda \|u\|^{2a} \int_{\mathbb{R}^3} |u - T_{\lambda,\beta}(u)|^2 , \]

which means \( I'_{\lambda,\beta}(u)(u - T_{\lambda,\beta}(u)) \geq \|u - T_{\lambda,\beta}(u)\|^2 \) for all \( u \in E \). Notice that for any \( \varphi \in C^0_0(\mathbb{R}^3) \),

\[ I'_{\lambda,\beta}(u)\varphi = \int_{\mathbb{R}^3} [a \nabla (u - T_{\lambda,\beta}(u))\nabla \varphi + V(x)(u - T_{\lambda,\beta}(u))\varphi] \]

\[ + b \int_{\mathbb{R}^3} |\nabla u|^2 \int_{\mathbb{R}^3} \nabla (u - T_{\lambda,\beta}(u))\nabla \varphi + \lambda \|u\|^{2a} \int_{\mathbb{R}^3} (u - T_{\lambda,\beta}(u))\varphi, \]

which implies \( \|I'_{\lambda,\beta}(u)\| \leq \|u - T_{\lambda,\beta}(u)\|(1 + C_1 \|u\|^2 + C_2 \|u\|^{2a}) \) for all \( u \in E \).

**Lemma 3.4** For fixed \( (\lambda, \beta) \in (0,1) \times (0,1) \) and for \( c < d \) and \( \tau > 0 \), there exists \( \delta > 0 \) (which depends on \( \lambda \) and \( \beta \)) such that \( \|u - T_{\lambda,\beta}(u)\| \geq \delta \) if \( u \in E \), \( I_{\lambda,\beta}(u) \in [c, d] \) and \( \|I''_{\lambda,\beta}(u)\| \geq \tau \).

**Proof** Step 1. We claim that for any \( a, b > 0 \) given and \( 2 < p < r < 6 \), there holds

\[ \inf_{u \in H^1(\mathbb{R}^3)} \left( \|u\|_r^p + a\|u\|_2^{2a+2} - b\|u\|_p^p \right) > -\infty. \]  

(3.7)

In fact, for any \( a, b > 0 \) and \( 2 < p < r \), there exists \( \kappa = \kappa(a, b, p, r) > 0 \) such that \( \inf_{t \geq 0} (t^r + a\kappa^{2a}t^2 - bt^p) \geq 0 \). It follows that

\[ \inf_{\|u\|_2 \geq \kappa} \left( \|u\|_r^p + a\|u\|_2^{2a+2} - b\|u\|_p^p \right) \geq 0. \]  

(3.8)
Moreover, if \( u \in H^1(\mathbb{R}^3) \) with \( \|u\|_2 \leq \kappa \), then by the interpolation inequality, there exists \( s \in (0, 1) \) with \( 1/p = s/2 + (1-s)/r \) such that
\[
\|u\|_p^p \leq \|u\|_2^{sp} \|u\|_r^{(1-s)p} \leq \kappa^{sp} \|u\|_r^{(1-s)p}.
\]

Noting that \( (1-s)p < r \), one can check that
\[
\inf_{u \in H^1(\mathbb{R}^3)} \left( \|u\|_r^r + a\|u\|_2^{2s+2} - b\|u\|_p^p \right) > -\infty.
\]

Then (3.7) follows from (3.8) and (3.9).

**Step 2.** Fixing \( r \in (4, r) \), then for \( u \in E \), we have
\[
I_{\lambda, \beta}(u) - \frac{1}{r} (u, u - T_{\lambda, \beta}(u)) = \frac{\gamma - 2}{2\gamma} \|u\|^2 + \frac{b}{\gamma} \int_{\mathbb{R}^3} |\nabla u|^2 \int_{\mathbb{R}^3} (\nabla u - \nabla T_{\lambda, \beta}(u)) \nabla u + \lambda \frac{\gamma - 2(1 + \alpha)}{2\gamma(1 + \alpha)} \|u\|_2^{2s+2} + \lambda \frac{2s}{\gamma} \int_{\mathbb{R}^3} u - T_{\lambda, \beta}(u)
\]
\[
+ \int_{\mathbb{R}^3} \left( \frac{1}{\gamma} f(u) - F(u) \right) + \frac{\gamma - 4}{4\gamma} b \left( \int_{\mathbb{R}^3} |\nabla u|^2 \right)^2 + \frac{(r - \gamma)\beta}{r\gamma} \int_{\mathbb{R}^3} |u|^r.
\]

It follows from (3.4) that for any \( \varepsilon > 0 \), there exists \( C_\varepsilon > 0 \) such that
\[
|I_{\lambda, \beta}(u)| + \frac{1}{r} \|u\|_r \|u - T_{\lambda, \beta}(u)\| \geq \left( \frac{\gamma - 2}{2\gamma} - \varepsilon C \right) \|u\|^2 + \frac{b}{\gamma} \int_{\mathbb{R}^3} |\nabla u|^2 \int_{\mathbb{R}^3} (\nabla u - \nabla T_{\lambda, \beta}(u)) \nabla u + \lambda \frac{\gamma - 2(1 + \alpha)}{2\gamma(1 + \alpha)} \|u\|_2^{2s+2}
\]
\[
- \frac{\alpha}{\gamma} \|u\|^2 + \lambda \frac{2s}{\gamma} \int_{\mathbb{R}^3} u - T_{\lambda, \beta}(u).
\]

Then,
\[
\|u\|^2 + b \left( \int_{\mathbb{R}^3} |\nabla u|^2 \right)^2 + \beta \|u\|_r^r + \lambda \|u\|_2^{2s+2} - C_\varepsilon \|u\|_p^p
\]
\[
\leq C \left( |I_{\lambda, \beta}(u)| + \|u\|_r \|u - T_{\lambda, \beta}(u)\| + \frac{\lambda}{\gamma} \|u\|_2^{2s+2} \int_{\mathbb{R}^3} \|u - T_{\lambda, \beta}(u)\| \right) \left( \frac{r}{r\gamma} \int_{\mathbb{R}^3} |\nabla u - \nabla T_{\lambda, \beta}(u)| \|\nabla u\| \right).
\]

By Hölder’s inequality and Sobolev’s inequality, we have
By (3.10) and (3.11) and Young’s inequality, we get
\[
\|u\|^2 + b \left( \int_{ \mathbb{R}^3 } |\nabla u|^2 \right)^2 + \beta \|u\|_p^p + \lambda \|u\|_2^{2a+2} - C_c \|u\|_p^p \leq C (|I_{\lambda, \beta}(u)| + \|u-T_{\lambda, \beta}(u)\| + \|u\|^2\|u-T_{\lambda, \beta}(u)\|^2 + \|u\|^{4a}).
\] (3.12)

In the following, we show that for any fixed \((\lambda, \beta) \in (0, 1) \times (0, 1), c < d\) and \(\tau > 0\),
\[
\inf \left\{ \|u-T_{\lambda, \beta}(u)\| : u \in E, I_{\lambda, \beta}(u) \in [c, d], \|I'_{\lambda, \beta}(u)\| \geq \tau \right\} > 0.
\]

Assume on the contrary that there exists \(\{u_n\} \subset E\) with \(I_{\lambda, \beta}(u_n) \in [c, d]\) and \(\|I'_{\lambda, \beta}(u_n)\| \geq \tau\) such that \(\|u_n-T_{\lambda, \beta}(u_n)\| \to 0\) as \(n \to \infty\), then it follows from (3.12) that
\[
\|u_n\|^2 + b \left( \int_{ \mathbb{R}^3 } |\nabla u_n|^2 \right)^2 + \beta \|u_n\|_p^p + \lambda \|u_n\|_2^{2a+2} - C_c \|u_n\|_p^p \leq C (1 + \|u_n\|^{4a})
\] (3.13)

for large \(n\). By (3.7) and \(4a < 2\), (3.13) yields that \(\{u_n\}\) is a bounded sequence in \(E\). This combined with Lemma 3.3 implies \(||I'_{\lambda, \beta}(u_n)|| \to 0\) as \(n \to \infty\), which is a contradiction. The proof is complete.

\[ \square \]

### 3.2 Invariant subsets of descending flow

In order to obtain sign-changing solutions, we define the positive and negative cones by
\[
P^+ := \{ u \in E : u \geq 0 \} \quad \text{and} \quad P^- := \{ u \in E : u \leq 0 \},
\]
respectively. For \(\epsilon > 0\), set
\[
P^+_\epsilon := \{ u \in E : \text{dist}(u, P^+) < \epsilon \} \quad \text{and} \quad P^-_\epsilon := \{ u \in E : \text{dist}(u, P^-) < \epsilon \},
\]
where \(\text{dist}(u, P^\pm) = \inf_{v \in P^\pm} \|u-v\|\). Clearly, \(P^-_\epsilon = -P^+_\epsilon\). Let \(W = P^+_\epsilon \cup P^-_\epsilon\). It is easy to check that \(W\) is an open and symmetric subset of \(E\) and \(E \setminus W\) contains only sign-changing functions.

We denote by \(K\) the set of critical points of \(I_{\lambda, \beta}\), that is, \(K = \{ u \in E : I'_{\lambda, \beta}(u) = 0 \}\) and \(E_0 := E \setminus K\). For \(c \in \mathbb{R}\), define \(K_c = \{ u \in E : I_{\lambda, \beta}(u) = c, I'_{\lambda, \beta}(u) = 0 \}\) and \(I'_{\lambda, \beta} = \{ u \in E : I_{\lambda, \beta}(u) \leq c \}\).

In the following, we will show that, for \(\epsilon\) small enough, all sign-changing solutions to \((K_{\lambda, \beta})\) are contained in \(E \setminus W\).

**Lemma 3.5** There exists \(\epsilon_0 > 0\) such that for \(\epsilon \in (0, \epsilon_0)\),
(1) $T_{\lambda,\beta}(\partial P^-_e) \subset P^-_e$ and every nontrivial solution $u \in P^-_e$ is negative,
(2) $T_{\lambda,\beta}(\partial P^+_e) \subset P^+_e$ and every nontrivial solution $u \in P^+_e$ is positive.

**Proof** We only prove $T_{\lambda,\beta}(\partial P^-_e) \subset P^-_e$, and the other case is similar. For $u \in E$, define $v := T_{\lambda,\beta}(u)$. Since $\text{dist}(v, P^-_e) \leq \|v^+\|$, by Sobolev’s inequality and $(f_1)-(f_2)$, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$\text{dist}(v, P^-_e)\|v^+\| \leq \|v^+\|^3 = \langle v, v^+ \rangle$$

$$\leq \int_{\mathbb{R}^3} f(u)v^+ - b \int_{\mathbb{R}^3} |\nabla u|^2 \int_{\mathbb{R}^3} \nabla v \nabla v^+ + \beta \int_{\mathbb{R}^3} |u|^{r-2}uv^+ - \lambda \|u\|_2^2 \int_{\mathbb{R}^3} v v^+$$

$$\leq \int_{\mathbb{R}^3} f(u)v^+ + \beta \int_{\mathbb{R}^3} |u|^{r-2}uv^+$$

$$\leq \int_{\mathbb{R}^3} (\varepsilon u^+ v^+ + C_\varepsilon |u|^p v^+) + \beta \int_{\mathbb{R}^3} |u^+|^{r-2} u^+ v^+$$

$$\leq C[\varepsilon \text{dist}(u, P^-_e) + C_\varepsilon \text{dist}(u, P^-_e)^{p-1} + \text{dist}(u, P^-_e)^{r-1}]\|v^+\|,$$

which further implies that

$$\text{dist}(v, P^-_e) \leq C[\varepsilon \text{dist}(u, P^-_e) + C_\varepsilon \text{dist}(u, P^-_e)^{p-1} + \text{dist}(u, P^-_e)^{r-1}].$$

There exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0),$

$$\text{dist}(T_{\lambda,\beta}(u), P^-_e) = \text{dist}(v, P^-_e) < \varepsilon.$$

Therefore, we have $T_{\lambda,\beta}(u) \in P^-_e$ for any $u \in P^-_e$.

Since the operator $T_{\lambda,\beta}$ is not locally Lipschitz continuous, we need to construct a locally Lipschitz continuous vector field which inherits its properties. Arguing as in the proof of Lemma 2.1 of [8] (see also Lemma 2.5 of [11]), we have

**Lemma 3.6** There exists a locally Lipschitz continuous operator $B_{\lambda,\beta} : E \to E$ such that

(i) $\langle I'_{\lambda,\beta}(u), u - B_{\lambda,\beta}(u) \rangle \geq \frac{1}{2}\|u - B_{\lambda,\beta}(u)\|^2;$

(ii) $\frac{1}{2}\|u - B_{\lambda,\beta}(u)\|^2 \leq \|u - T_{\lambda,\beta}(u)\|^2 \leq 2\|u - B_{\lambda,\beta}(u)\|^2;$

(iii) $B_{\lambda,\beta}(\partial P^+_e) \subset P^+_e, \forall \varepsilon \in (0, \varepsilon_0);$

(iv) if $I'_{\lambda,\beta}$ is even, then $B_{\lambda,\beta}$ is odd.

**Proof** The proof follows the line of [11, Lemma 2.5].

In what follows, we verify that the functional $I_{\lambda,\beta}$ satisfies (PS)-condition.

**Lemma 3.7** For any fixed $(\lambda, \beta) \in (0, 1] \times (0, 1]$, assume that there exist $\{u_n\} \subset E$ and $c \in \mathbb{R}$ such that $I_{\lambda,\beta}(u_n) \to c$ and $I'_{\lambda,\beta}(u_n) \to 0$ as $n \to \infty$, then there exists a convergence subsequence of $\{u_n\}$, denoted by $\{u_n\}$ for simplicity, such that $u_n \to u$ in $E$ for some $u \in E.$
Proof For $\gamma \in (4, r)$, we have
$$
\gamma I_{\lambda, \beta}(u_n) - \langle I'_{\lambda, \beta}(u_n), u_n \rangle
\geq \frac{\gamma - 2}{2} \|u_n\|^2 + \frac{b(\gamma - 4)}{4} \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 \right)^2
+ \lambda \gamma - 2(1 + \alpha) \|u_n\|^{2(1 + \alpha)} + \int_{\mathbb{R}^3} (f(u_n)u_n - \gamma F(u_n)) + \beta \frac{r - \gamma}{r} \int_{\mathbb{R}^3} |u_n|^r.
$$

It follows from $(f_1)$–$(f_2)$ and Sobolev’s inequality that for any $0 < \varepsilon < \frac{\gamma - 2}{2}$, there exists $C_{\varepsilon} > 0$ such that
$$
\gamma |I_{\lambda, \beta}(u_n)| + \|I'_{\lambda, \beta}(u_n)\| \|u_n\|
\geq \left( \frac{\gamma - 2}{2} - \varepsilon \right) \|u_n\|^2 + \frac{b(\gamma - 4)}{4} \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 \right)^2 - C_{\varepsilon} \|u_n\|^p
+ \lambda \gamma - 2(1 + \alpha) \|u_n\|^{2(1 + \alpha)} + \beta \frac{r - \gamma}{r} \|u_n\|^r.
$$

Similar as above, it follows from (3.7) that $\{u_n\}$ is bounded in $E$. Up to subsequence, we assume that there exists $u \in E$ such that
$$
u_n \rightharpoonup u \text{ weakly in } E, \quad \text{and} \quad u_n \to u \text{ strongly in } L^q(\mathbb{R}^3) \text{ for } q \in (2, 6).
$$

Note that
$$
\langle I'_{\lambda, \beta}(u_n) - I'_{\lambda, \beta}(u), u_n - u \rangle
= \|u_n - u\|^2 + \int_{\mathbb{R}^3} |\nabla u_n|^2 \int_{\mathbb{R}^3} |\nabla (u_n - u)|^2
+ b\left( \int_{\mathbb{R}^3} |\nabla u_n|^2 - \int_{\mathbb{R}^3} |\nabla u|^2 \right) \int_{\mathbb{R}^3} \nabla u \nabla (u_n - u) - \int_{\mathbb{R}^3} (f(u_n) - f(u))(u_n - u)
+ \lambda \|u_n\|^{2a} \int_{\mathbb{R}^3} (u_n - u)^2 + \lambda \|u_n\|^{2a} - \|u\|^{2a} \int_{\mathbb{R}^3} u(u_n - u)
- \beta \int_{\mathbb{R}^3} \|u_n\|^{-2} u_n - |u|^{-2} u (u_n - u).
$$

By the boundedness of $\{u_n\}$ in $E$, one has
$$
b\left( \int_{\mathbb{R}^3} |\nabla u_n|^2 - \int_{\mathbb{R}^3} |\nabla u|^2 \right) \int_{\mathbb{R}^3} \nabla u \nabla (u_n - u) \to 0,
\lambda \|u_n\|^{2a} - \|u\|^{2a} \int_{\mathbb{R}^3} u(u_n - u) \to 0, \text{ as } n \to \infty.
$$

Moreover, for any $\varepsilon > 0$, one has
$$
\int_{\mathbb{R}^3} (f(u_n) - f(u))(u_n - u) \leq \int_{\mathbb{R}^3} [\varepsilon \|u_n\| + |u|] + C_{\varepsilon} (\|u_n\|^{p-1} + |u_n|^{p-1}) \|u_n - u\|
\leq \varepsilon C + C_{\varepsilon} (\|u_n\|^{p-1} + \|u\|^{p-1}) \|u_n - u\| \to 0.
as $n \to \infty$. Similarly, we also have
\[
\beta \int_{\mathbb{R}^3} (|u_n|^{r-2} u_n - |u|^{r-2} u)(u_n - u) \to 0, \quad \text{as } n \to \infty.
\]

Based on the above facts, from (3.14) we deduce that
\[
(I'_{\lambda, \beta}(u_n) - I'_{\lambda, \beta}(u), u_n - u)
= \|u_n - u\|^2 + b \int_{\mathbb{R}^3} |\nabla u_n|^2 \int_{\mathbb{R}^3} |\nabla (u_n - u)|^2 + \lambda \|u_n\|^2 2 \int_{\mathbb{R}^3} (u_n - u)^2 + o_n(1).
\]
Noting that $(I'_{\lambda, \beta}(u_n) - I'_{\lambda, \beta}(u), u_n - u) \to 0$ as $n \to \infty$, we get that $u_n \to u$ in $E$ as $n \to \infty$.

Here, we state a deformation lemma for the functional $I_{\lambda, \beta}$ whose proof is almost the same as that of Lemma 3.6 in [32].

**Lemma 3.8** (Deformation lemma) Let $S \subset E$ and $c \in \mathbb{R}$ such that
\[
\forall u \in I_{\lambda, \beta}^{-1}(\{c - 2\varepsilon_0, c + 2\varepsilon_0\}) \cap S_{2\delta}, \quad \|I'_{\lambda, \beta}(u)\| \geq \varepsilon_0,
\]
where $\varepsilon_0$ was given in Lemma 3.5 and $S_{2\delta} := \{u \in S : \text{dist}(u, S) < 2\delta\}$. Then for $0 < \varepsilon < \varepsilon_1 < \varepsilon_0$ there exists $\eta \in C([0, 1] \times E, E)$ such that

(i) $\eta(t, u) = u$ if $t = 0$ or if $u \not\in I_{\lambda, \beta}^{-1}(\{c - 2\varepsilon_1, c + 2\varepsilon_1\})$;
(ii) $\eta(1, I_{\lambda, \beta}^{-\varepsilon} \cap S) \subset I_{\lambda, \beta}^{-\varepsilon}$;
(iii) $I_{\lambda, \beta}(\eta(\cdot, u))$ is non-increasing for all $u \in E$;
(iv) $\eta(t, P_{\varepsilon}^0) \subset P_{\varepsilon}^0$, $\eta(t, P_{\varepsilon}^-) \subset P_{\varepsilon}^-$, $\forall t \in [0, 1]$;
(v) if $I_{\lambda, \beta}(\cdot)$ is even, then $\eta(t, \cdot)$ is odd for any $t \in [0, 1]$.

Now, we introduce a critical point theorem (see [31]). For more details, we let $P, Q \subset E$ be open sets, $M = P \cap Q$, $\Sigma = \partial P \cap \partial Q$ and $W = P \cup Q$.

**Definition 3.1** (see [31]) \{P, Q\} is called an admissible family of invariant sets with respect to $J$ at level $c$, provided that the following deformation property holds: if $K_c \setminus W = \emptyset$, then, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, there exists $\eta \in C(E, E)$ satisfying

1. $\eta(\bar{P}) \subset \bar{P}$, $\eta(\bar{Q}) \subset \bar{Q}$;
2. $\eta|_{J^{r-\varepsilon}} = \text{id}$;
3. $\eta(J^{r+\varepsilon} \cup W) \subset J^{r-\varepsilon}$.

**Theorem 3.1** (see [31]) Assume that \{P, Q\} is an admissible family of invariant sets with respect to $J$ at any level $c \geq c_* := \inf_{u \in \Sigma} J(u)$ and there exists a continuous map $\psi_0 : \Delta \to E$ satisfying

1. $\psi_0(\partial_1 \Delta) \subset P$ and $\psi_0(\partial_2 \Delta) \subset Q$,
2. $\psi_0(\partial_1 \Delta) \cap M = \emptyset$,
3. $\sup_{u \in \psi_0(\partial_0 \Delta)} J(u) < c_*$.
where $\Delta = \{(t_1, t_2) \in \mathbb{R}^2 : t_1, t_2 \geq 0, t_1 + t_2 \leq 1\}$, $\partial_1 \Delta = \{0\} \times [0, 1]$, $\partial_2 \Delta = [0, 1] \times \{0\}$ and $\partial_0 \Delta = \{(t_1, t_2) \in \mathbb{R}^2 : t_1, t_2 \geq 0, t_1 + t_2 = 1\}$. Define

$$c = \inf_{\psi \in \Gamma} \sup_{u \in \psi(\Delta) \setminus W} J(u),$$

where $\Gamma := \{\psi \in C(\Delta, E) : \psi(\partial_1 \Delta) \subset P, \psi(\partial_2 \Delta) \subset Q, \psi|_{\partial_0 \Delta} = \psi_0|_{\partial_0 \Delta}\}$. Then $c \geq c_\ast$ and $K_c \setminus W \neq \emptyset$.

### 3.3 Proof of Theorem 2.1

In order to employ Theorem 3.1 to prove the existence of sign-changing solutions to problem $(K_{\lambda, \beta})$, we take $P = P^+, Q = P^-$ and $J = I_{\lambda, \beta}$. We need to prove that $\{P^+, P^-\}$ is an admissible family of invariant sets for the functional $I_{\lambda, \beta}$ at any level $c \in \mathbb{R}$. Moreover, $K_c \subset W$ if $K_c \setminus W \neq \emptyset$. Since the functional $I_{\lambda, \beta}$ satisfies the (PS)-condition, $K_c$ is compact. Thus, $2\delta := \text{dist}(K_c, \partial W) > 0$.

**Lemma 3.9** For $q \in [2, 6]$, there exists $m > 0$ independent of $\epsilon$ such that $\|u\|_q \leq m$ for $u \in M = P^+_\epsilon \cap P^-\epsilon$.

**Proof** For any fixed $u \in M$, we have

$$\|u^+\|_q = \inf_{v \in P^+} \|u - v\|_q \leq C \inf_{v \in P^+} \|u - v\| \leq C \text{dist}(u, P^\pm).$$

Then $\|u\|_q \leq m$ for $u \in M$. \qed

**Lemma 3.10** If $\epsilon > 0$ is small enough, then $I_{\lambda, \beta}(u) \geq \frac{\epsilon^2}{4}$ for all $u \in \Sigma = \partial P^+_\epsilon \cap \partial P^-\epsilon$, that is, $c_\ast \geq \frac{\epsilon^2}{4}$.

**Proof** For any fixed $u \in \partial P^+_\epsilon \cap \partial P^-\epsilon$, we have $\|u^+\| \geq \text{dist}(u, P^-) = \epsilon$. By Lemma 3.9 and $(f_1)$–$(f_2)$, for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$I_{\lambda, \beta}(u) = \frac{1}{2} \|u\|^2 + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla u|^2 \right)^2 + \frac{\lambda}{2(1 + \alpha)} \|u\|^{2(1 + \alpha)} - \int_{\mathbb{R}^3} F(u) - \frac{\beta}{r} \int_{\mathbb{R}^3} |u|^r$$

$$\geq \frac{1}{2} \epsilon^2 - \frac{\epsilon}{2} \|u\|^2 - \frac{C_\epsilon}{p} \|u\|^p - \frac{\beta}{r} \|u\|^r$$

$$\geq \frac{1}{2} \epsilon^2 - \frac{\epsilon}{2} \epsilon^2 - \frac{C_\epsilon}{p} \epsilon^p - \frac{\beta}{r} \epsilon^r \geq \frac{\epsilon^2}{4},$$

for $\epsilon$ and $\epsilon$ small enough. \qed

**Proof of Theorem 2.1** We use Theorem 3.1 to prove the existence of sign-changing solutions to problem $(K_{\lambda, \beta})$. Let $P = P^+_\epsilon, Q = P^-\epsilon$ and $J = I_{\lambda, \beta}$. Take $S = E \setminus W$ in Lemma 3.8, then we can easily deduce that $\{P^+_\epsilon, P^-\epsilon\}$ is an admissible family of invariant sets for the functional $I_{\lambda, \beta}$ at any level $c \in \mathbb{R}$.

In what follows, we divide three steps to complete the proof.

**Step 1** Choose $u, v \in C_0^\infty(B_1(0))$ such that $\text{supp}(u) \cap \text{supp}(v) = \emptyset$ and $u < 0, v > 0$, where $B_r(0) := \{x \in \mathbb{R}^3 : |x| < r\}$. For $(t, s) \in \Delta$, let...
\[\varphi_0(t, s) := R^2[\nu_1(Rt) + sv_2(Rs)],\]

where \( R > 0 \) will be determined later. Obviously, for \( t, s \in [0, 1] \), \( \varphi_0(0, s)(\cdot) = R^2sv_2(Rs) \in P_e^+ \) and \( \varphi_0(t, 0)(\cdot) = R^2tv_1(R) \in P_e^- \). It follows from Lemma 3.10 that, for small \( e > 0 \),

\[ I_{\lambda, \beta}(u) \geq \frac{e^2}{4} \quad \text{for all} \quad u \in \Sigma = \partial P_e^+ \cap \partial P_e^-, \quad (\lambda, \beta) \in (0, 1] \times (0, 1]. \]

Hence \( c_\ast = \inf_{u \in \Sigma} I_{\lambda, \beta}(u) \geq \frac{e^2}{4} \) for any \( (\lambda, \beta) \in (0, 1] \times (0, 1] \). Let \( u_t = \varphi_0(t, 1 - t) \) for \( t \in [0, 1] \). Observe that

\[ \rho = \min \{ \|tv_1 + (1 - t)v_2\|_2 : 0 \leq t \leq 1 \} > 0, \]

then \( \|u_t\|_2^2 \geq \rho R \) for \( u \in \varphi_0(\partial_0 \Delta) \). It follows from Lemma 3.9 that \( \varphi_0(\partial_0 \Delta) \cap P_e^+ \cap P_e^- = \emptyset \).

A direct computation shows that

\[
\int_{\mathbb{R}^3} |\nabla u_t|^2 = R^3 \int_{\mathbb{R}^3} (t^2 |\nabla v_1|^2 + (1 - t)^2 |\nabla v_2|^2) =: R^3 B(t),
\]

\[
\int_{\mathbb{R}^3} V(x)|u_t|^2 \leq R \max_{x \in B_t(0)} V(x) \int_{\mathbb{R}^3} (t^q |v_1|^q + (1 - t)^q |v_2|^q) =: RB_2(t),
\]

\[
\int_{\mathbb{R}^3} |u_t|^q = R^{2 - 3q} \int_{\mathbb{R}^3} (t^q |v_1|^q + (1 - t)^q |v_2|^q) =: R^{2q - 3} B_q(t) \quad \text{for} \quad q \in (2, 6],
\]

\[
\left( \int_{\mathbb{R}^3} |u_t|^2 \right)^{1 + \alpha} = R^{1 + \alpha} \left( \int_{\mathbb{R}^3} (t^2 |v_1|^2 + (1 - t)^2 |v_2|^2) \right)^{1 + \alpha} =: R^{1 + \alpha} B(t).
\]

(3.15)

Since \( F(t) \geq C_3 |t|^\mu - C_4 \) for any \( t \in \mathbb{R} \) and some positive constants \( C_3, C_4 \), we have

\[
I_{\lambda, \beta}(u_t) = \frac{a}{2} \int_{\mathbb{R}^3} |\nabla u_t|^2 + \frac{1}{2} \int_{\mathbb{R}^3} V(x)|u_t|^2 + \frac{1}{2(1 + \alpha)} \|u_t\|^{2(1 + \alpha)}
\]

\[
+ \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla u_t|^2 \right)^2 - \int_{B_{r-1}(0)} F(u_t) - \frac{\beta}{r} \int_{\mathbb{R}^3} |u_t|^r
\]

\[
< \frac{aR^3}{2} B(t) + \frac{R}{2} B_2(t) + \frac{bR^6}{4} B^2(t) + \frac{R^{1 + \alpha}}{2(1 + \alpha)} B(t)
\]

\[
- C_3 R^{2\mu - 3} B_\mu(t) + CC_4 R^{-3} - \frac{R^{2r - 3}}{r} B_r(t).
\]

Since \( r \in (\max \{p, \frac{9}{2} \}, 6) \), one sees that \( I_{\lambda, \beta}(u_t) \to -\infty \) as \( R \to +\infty \) for any fixed \((\alpha, \beta) \in (0, 1] \times (0, 1] \). Hence we can choose \( R \) large enough such that

\[
\sup_{u \in \varphi_0(\partial_0 \Delta)} I_{\lambda, \beta}(u) < c_\ast = \inf_{u \in \Sigma} I_{\lambda, \beta}(u).
\]

Since \( I_{\lambda, \beta} \) satisfies the assumptions of Theorem 3.1, the number

\[
c_{\lambda, \beta} = \inf_{\varphi \in \mathcal{I}} \sup_{u \in \varphi(\partial_0 \Delta) \setminus W} I_{\lambda, \beta}(u)
\]

is a critical value of \( I_{\lambda, \beta} \) satisfying \( c_{\lambda, \beta} \geq c_\ast \). Therefore, there exists \( u_{\lambda, \beta} \in E \setminus (P_e^+ \cup P_e^-) \) such that \( I_{\lambda, \beta}(u_{\lambda, \beta}) = c_{\lambda, \beta} \) and \( I'_{\lambda, \beta}(u_{\lambda, \beta}) = 0 \) for \((\lambda, \beta) \in (0, 1] \times (0, 1] \).
Step 2 Passing to the limit as $\lambda \to 0$ and $\beta \to 0$. According to the definition of $c_{\lambda, \beta}$, we see that for any $(\lambda, \beta) \in (0, 1) \times (0, 1)$,

$$c_{\lambda, \beta} \leq C_R := \sup_{u \in \mathcal{V}_0(\Delta)} I_{1,0}(u) < \infty,$$

(3.16)

where $C_R$ is independent of $(\lambda, \beta) \in (0, 1) \times (0, 1)$. Without loss of generality, we set $\lambda = \beta$. Choosing a sequence $\{\lambda_n\} \subset (0, 1]$ satisfying $\lambda_n \to 0^+$, then we find a sequence of sign-changing critical points $\{u_{\lambda_n}\}$ (still denoted by $\{u_n\}$ for simplicity) of $I_{\lambda_n, \beta_n}$ and $I_{\lambda_n, \beta_n}(u_n) = c_{\lambda_n, \beta_n}$. Now, we show that $\{u_n\}$ is bounded in $E$. By the definition of $I_{\lambda, \beta}$, we have

$$c_{\lambda_n, \beta_n} = \frac{a}{2} \int_{\mathbb{R}^3} |\nabla u_n|^2 + \frac{1}{2} \int_{\mathbb{R}^3} V(x)u_n^2 + \frac{\lambda}{2(1 + \alpha)} \|u_n\|^2_{2^{(1 + \alpha)}}$$

$$+ \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 \right)^2 - \int_{\mathbb{R}^3} F(u_n) - \frac{\beta}{r} \int_{\mathbb{R}^3} |u_n|^r$$

(3.17)

and

$$0 = a \int_{\mathbb{R}^3} |\nabla u_n|^2 + \int_{\mathbb{R}^3} V(x)u_n^2 + \lambda \|u_n\|^2_{2^{(1 + \alpha)}}$$

$$+ \frac{b}{2} \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 \right)^2 - \int_{\mathbb{R}^3} f(u_n)u_n - \beta \int_{\mathbb{R}^3} |u_n|^r.$$  

(3.18)

Moreover, from Lemma 3.1, the following identity holds

$$\frac{a}{2} \int_{\mathbb{R}^3} |\nabla u_n|^2 + \frac{3}{2} \int_{\mathbb{R}^3} V(x)u_n^2 + \frac{1}{2} \int_{\mathbb{R}^3} (\nabla V(x), x)u_n^2$$

$$+ \frac{b}{2} \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 \right)^2 + \frac{3\lambda}{2} \left( \int_{\mathbb{R}^3} u_n^2 \right)^{1 + \alpha} - 3 \int_{\mathbb{R}^3} (F(u_n) + \frac{\beta}{r} |u_n|^r) = 0.$$  

(3.19)

Multiplying (3.17), (3.18) and (3.19) by $4$, $-\frac{1}{\mu}$ and $-1$, respectively, and adding them up, we get

$$4c_{\lambda_n, \beta_n} = \frac{3\mu - 2}{2\mu} \int_{\mathbb{R}^3} |\nabla u_n|^2 + \frac{\mu - 2}{2\mu} \int_{\mathbb{R}^3} V(x)u_n^2 - \frac{1}{2} \int_{\mathbb{R}^3} (\nabla V(x), x)u_n^2$$

$$+ \frac{\mu - 2}{2\mu} b \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 \right)^2 + \lambda \frac{\mu - 2 - 3\mu \alpha - 2\alpha}{2\mu(1 + \alpha)} \|u_n\|^2_{2^{(1 + \alpha)}}$$

$$+ \int_{\mathbb{R}^3} \left( \frac{1}{\mu} f(u_n)u_n - F(u_n) \right) + \frac{\beta - \mu}{\mu r} \int_{\mathbb{R}^3} |u_n|^r.$$  

Since $\alpha < \frac{\mu^2}{2\mu + 2}$ and $\mu > 2$, it follows from (V2), (f) and (3.16) that

$$4C_R > a \frac{3\mu - 2}{2\mu} \int_{\mathbb{R}^3} |\nabla u_n|^2 + \frac{\mu - 2}{2\mu} b \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 \right)^2,$$

which implies that there exists $C > 0$ independent of $\lambda, \beta$ such that
Moreover, combining (3.16), (3.17) and hypotheses (V₁), (f₁) and (f₂), we obtain that for small $\epsilon > 0$, there exists $C_\epsilon > 0$ such that

$$C_R \geq \frac{a}{2} \int_{\mathbb{R}^3} |\nabla u_n|^2 + \frac{1}{2} \int_{\mathbb{R}^3} V(x)u_n^2 - \int_{\mathbb{R}^3} F(u_n) - \frac{\beta}{r} \int_{\mathbb{R}^3} |u_n|^r$$

$$\geq \frac{1 - \epsilon}{2} \int_{\mathbb{R}^3} V(x)u_n^2 - C_\epsilon \int_{\mathbb{R}^3} u_n^6 - \frac{1}{r} \int_{\mathbb{R}^3} |u_n|^r$$

$$> \frac{1 - \epsilon}{2} \int_{\mathbb{R}^3} V(x)u_n^2 - C_\epsilon S^{-3} \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 \right)^3 - \frac{1}{r} \int_{\mathbb{R}^3} |u_n|^r.$$  \tag{3.21}

From interpolation inequality, Sobolev’s inequality and Young’s inequality, we deduce that for $\epsilon > 0$, there exists $\tilde{C}_\epsilon > 0$ such that

$$\int_{\mathbb{R}^3} |u_n|^r \leq \left( \int_{\mathbb{R}^3} u_n^6 \right)^{\frac{6-r}{2}} \left( \int_{\mathbb{R}^3} |u_n|^6 \right)^{\frac{r-2}{2}}$$

$$\leq \epsilon \left( \int_{\mathbb{R}^3} u_n^6 \right)^{\frac{6-r}{2}} + \tilde{C}_\epsilon \left( \int_{\mathbb{R}^3} |u_n|^6 \right)^{\frac{r-2}{2}}$$

$$\leq \epsilon \left( \int_{\mathbb{R}^3} u_n^6 \right)^{\frac{6-r}{2}} + \tilde{C}_\epsilon S^{\frac{3(r-2)}{2}} \left( \int_{\mathbb{R}^3} |\nabla u_n|^2 \right)^{\frac{6(r-2)}{2}}.$$  \tag{3.22}

Combining (3.20), (3.21) and (3.22), we immediately see that $\{u_n\}$ is bounded in $E$. In view of (3.16) and Lemma 3.10, we have

$$\lim_{n \to \infty} I(u_n) = \lim_{n \to \infty} \left( I_{\lambda_n, \theta_n}(u_n) - \frac{\lambda_n}{2(1 + \alpha)} \|u_n\|_2^{2(1 + \alpha)} + \frac{\beta_n}{r} \int_{\mathbb{R}^3} |u_n|^r \right)$$

$$= \lim_{n \to \infty} c_{\lambda_n, \theta_n}^* = c_* > \frac{\epsilon^2}{4}.$$ 

Moreover, for any $\psi \in C_0^\infty(\mathbb{R}^3)$,

$$\lim_{n \to \infty} I'(u_n)\psi = \lim_{n \to \infty} \left( I'_{\lambda_n, \theta_n}(u_n)\psi - \lambda_n \|u_n\|_2^{2\alpha} \int_{\mathbb{R}^3} u_n \psi + \beta_n \int_{\mathbb{R}^3} |u_n|^{-2} u_n \psi \right) = 0.$$

That is to say, $\{u_n\}$ is a bounded Palais–Smale sequence for $I$ at level $c_*$. Thus, there exists $u^* \in E$ such that $u_n \rightharpoonup u^*$ weakly in $E$ and $u_n \to u^*$ strongly in $L^q(\mathbb{R}^3)$ for $q \in (2, 6)$. Similar argument of Lemma 3.7 lead to that $I'(u^*) = 0$ and $u_n \to u^*$ strongly in $E$ as $n \to 0$. Thus, the fact that $u_n \in E \backslash (P^+_e \cup P^-_e)$ yields $u^* \in E \backslash (P^+_e \cup P^-_e)$ and then $u^*$ is a sign-changing solution of (K).

**Step 3** Define

$$\bar{c} := \inf_{u \in \Theta} I(u), \quad \Theta := \{u \in E \backslash \{0\}, I'(u) = 0, u^* \neq 0\}.$$ 

Based on Step 2, we see that $\Theta \neq \emptyset$ and $\bar{c} \leq c_*$, where $c_*$ is given in the Step 2. By the definition of $\bar{c}$, there exists $\{u_n\} \subset E$ such that $I(u_n) \to \bar{c}$ and $I'(u_n) = 0$. Using the earlier arguments, we can prove that $\{u_n\}$ is bounded in $E$. Arguing as in Lemma 3.7, there
exists a nontrivial \( u \in E \) such that \( I(u) = \tilde{c} \) and \( I'(u) = 0 \). Furthermore, we deduce from \( \langle I'(u_n), u^\pm_n \rangle = 0 \) that for any \( \varepsilon > 0 \), there exists \( C_\varepsilon > 0 \) such that

\[
C(\|u^\pm_n\|_p^2 + \int_{\mathbb{R}^3} |u^\pm_n|^2) \leq \|u^\pm_n\|^2 \leq \int_{\mathbb{R}^3} f(u_n)u^\pm_n = \int_{\mathbb{R}^3} f(u^\pm_n)u^\pm_n \leq \varepsilon \int_{\mathbb{R}^3} |u^\pm_n|^2 + C_\varepsilon \int_{\mathbb{R}^3} |u^\pm_n|^p \leq \varepsilon \|u^\pm_n\|_2^2 + C_\varepsilon \|u^\pm_n\|_p^p,
\]

which, together with the boundedness of \( \{u_n\} \) in \( E \), implies that \( \|u^\pm_n\|_p \geq C \). Hence, \( \|u^\pm_n\|_p \geq C \), and then \( u \) is a least energy sign-changing solution of problem (K). The proof is complete.

\[\square\]

### 4 Multiplicity

In this section, we prove the existence of infinitely many sign-changing solutions to problem (K).

#### 4.1 Proof of Theorem 2.2 (multiplicity)

In order to obtain infinitely many sign-changing solutions, we exploit an abstract critical point approach developed by Liu et al [31], which we recall below. The notations from Sect. 2 are still valid. Assume \( G : E \to E \) is an isometric involution, that is, \( G^2 = \text{id} \) and \( d(Gx; Gy) = d(x; y) \) for \( x, y \in E \). A subset \( F \subset E \) is said to be symmetric if \( Gx \in F \) for any \( x \in F \). We assume \( J \) is \( G \)-invariant on \( E \) in the sense that \( J(Gx) = J(x) \) for any \( x \in E \). We also assume \( Q = GP \). The genus of a closed symmetric subset \( F \) of \( E \setminus \{0\} \) is denoted by \( \gamma(F) \).

**Definition 4.1** (see [31]) \( P \) is called a \( G \)-admissible invariant set with respect to \( J \) at level \( c \), if the following deformation property holds: there exist \( \varepsilon_0 > 0 \) and a symmetric open neighborhood \( N \) of \( K_c \setminus W \) with \( \gamma(\tilde{N}) < \infty \), such that for \( \varepsilon \in (0, \varepsilon_0) \) there exists \( \eta \in C(E, E) \) satisfying

1. \( \eta(\tilde{P}) \subset \tilde{P}, \eta(\tilde{Q}) \subset \tilde{Q} \); 
2. \( \eta \circ G = G \circ \eta \); 
3. \( \eta|_{J^c} = \text{id}; \)
4. \( \eta((J^c + \varepsilon)\setminus(N \cup W)) \subset J^c - \varepsilon \).

**Theorem 4.1** (see [31]) Assume that \( P \) is a \( G \)-admissible invariant set with respect to \( J \) at any level \( c \geq c_\varepsilon := \inf_{u \in \Sigma} J(u) \) and for any \( n \in N \), there exists a continuous map \( \psi_n : B_n := \{ x \in \mathbb{R}^n : |x| \leq 1 \} \to E \) satisfying

1. \( \psi_n(0) \subset M := P \cap Q \) and \( \psi_n(-t) = G\psi_n(t) \) for \( t \in B_n \); 
2. \( \psi_n(\partial B_n) \cap M = \emptyset \); 
3. \( \sup_{u \in \text{Fix}_G \setminus \text{Fix}_G(\partial B_n)} J(u) < c_\varepsilon \), where \( \text{Fix}_G := \{ u \in E; Gu = u \} \).
For $j \in \mathbb{N}$, define
\[
c_j = \inf_{B \in \Gamma_j} \sup_{u \in B \setminus W} J(u),
\]
where
\[
\Gamma_j := \{ B \mid B = \psi(B_n \setminus Y) \text{for some } \psi \in G_n, n \geq j, \text{ and open } Y \subset B_n \text{ such that } -Y = Y \text{ and } \gamma(Y) \leq n - j \}.
\]
and
\[
G_n := \{ \psi \mid \psi \in C(B_n, E), \psi(-t) = G\psi(t) \text{ for } t \in B_n, \psi(0) \in M \text{ and } \psi|_{\partial B_n} = \psi_n|_{\partial B_n} \}.
\]

Then for $j \geq 2$, $c_j \geq c_n$, $K_{c_j} \setminus W \neq \emptyset$ and $c_j \to \infty$ as $j \to \infty$.

In order to apply Theorem 4.1, we set $G = -id$, $J = I_{\lambda, \beta}$ and $P = P_\epsilon^+$. Then $M = P_\epsilon^+ \cap P_\epsilon^-$, $\Sigma = \partial P_\epsilon^+ \cap \partial P_\epsilon^-$, and $W = P_\epsilon^+ \cup P_\epsilon^-$. In this subsection, $f$ is assumed to be odd, and consequently, $I_{\lambda, \beta}$ is even. Now, we show that $P_\epsilon^+$ is a $G$-admissible invariant set for the functional $I_{\lambda, \beta}$ at any level $c$. Since $K_c$ is compact, there exists a symmetric open neighborhood $N$ of $K_c \setminus W$ such that $\gamma(N) < \infty$.

**Lemma 4.1** There exists $e_0 > 0$ such that for $0 < e < e' < e_0$, there exists a continuous map $\sigma : [0, 1] \times E \to E$ satisfying

1. $\sigma(0, u) = u$ for $u \in E$;
2. $\sigma(t, u) = u$ for $t \in [0, 1]$, $u \notin I_{\lambda, \beta}^{-1} [c - e', c + e']$;
3. $\sigma(t, -u) = -\sigma(t, u)$ for $(t, u) \in [0, 1] \times E$;
4. $\sigma(1, I_{\lambda, \beta}^{+e}(N \cup W)) \subset I_{\lambda, \beta}^{-e}$;
5. $\sigma(t, P_\epsilon^+) \subset P_\epsilon^+$, $\sigma(t, P_\epsilon^-) \subset P_\epsilon^-$ for $t \in [0, 1]$.

**Proof** The proof is similar to that of Lemma 3.8. Since $I_{\lambda, \beta}$ is even, $B$ is odd and thus $\sigma$ is odd in $u$. \qed

**Proof of Theorem 2.2 (Multiplicity)** We divide the proof into two steps.

**Step 1** Since $f$ is odd, it follows from Lemma 4.1 that $P_\epsilon^+$ is a $G$-admissible invariant set for the functional $I_{\lambda, \beta}$ for $\lambda, \beta \in (0, 1]$ at any level $c$. We are now constructing $\psi_n$, satisfying the hypotheses of Theorem 4.1. For any fixed $n \in \mathcal{N}$, we choose $\{v_i\}_{i=1}^n \subset C_{0, r}(\mathbb{R}^3) \setminus \{0\}$ such that $\text{supp}(v_i) \cap \text{supp}(v_j) = \emptyset$ for $i \neq j$. Define $\psi_n \in C(B_n, E)$ as
\[
\psi_n(t)(\cdot) = R_n^{\frac{2}{2} \sum_{i=1}^n t_i v_i(R_n \cdot)}, \quad t = (t_1, t_2, \ldots, t_n) \in B_n.
\]
Observe that
\[
\rho_n = \min\{ \| t_1 v_1 + t_2 v_2 + \cdots + t_n v_n \|_2 : \sqrt{\sum_{i=1}^n t_i^2} = 1 \} > 0,
\]
then $\| u \|_2^2 \geq \rho_n R_n$ for $u \in \psi_n(\partial B_n)$ and it follows from Lemma 3.9 that $\psi_n(\partial B_n) \cap P_\epsilon^{+e} \cap P_\epsilon^- = \emptyset$. Similar to the proof of Theorem 2.1 (existence part), we also have

\[ Springer \]
\[
\sup_{u \in \Psi_{j}(\partial B_{\epsilon})} I_{\lambda, \beta}(u) < 0 < \inf_{u \in \Sigma} I_{\lambda, \beta}(u).
\]

Clearly, \( \psi_{n}(0) = 0 \in P_{\epsilon}^{+} \cap P_{\epsilon}^{-} \) and \( \psi_{n}(-t) = -\psi_{n}(t) \) for \( t \in B_{\epsilon} \). For any fixed \( \beta \in (0, 1) \) and \( j \in \{1, 2, \ldots, n\} \), we define

\[
c_{j, \lambda, \beta} = \inf_{B \in \Gamma_{j}} \sup_{u \in \Sigma \cap W_{j}} I_{j, \lambda, \beta}(u),
\]

where \( W := P_{\epsilon}^{+} \cup P_{\epsilon}^{-} \) and \( \Gamma_{j} \) was defined in Theorem 4.1. In view of the definition of \( \Gamma_{j} \), it is easy to see that \( c_{j, \lambda, \beta} \) is independent of \( \epsilon \). Based on Lemma 3.10 and Theorem 4.1, for any fixed \( \beta \in (0, 1) \) and \( j \geq 2 \),

\[
\frac{e^{2}}{4} \leq \inf_{u \in \Sigma} I_{j, \lambda, \beta}(u) := c_{s} \leq c_{j, \lambda, \beta} \to \infty, \quad \text{as} \quad j \to \infty \tag{4.2}
\]

and there exists \( \{u_{j, \lambda, \beta}\} \subset E \setminus W \) such that \( I_{j, \lambda, \beta}(u_{j, \lambda, \beta}) = c_{j, \lambda, \beta} \) and \( I'(u_{j, \lambda, \beta}) = 0 \).

**Step 2** Using similar arguments to those in Theorem 2.1, for any fixed \( j \geq 2 \), \( \{u_{j, \lambda, \beta}\}_{\lambda, \beta \in (0, 1]} \) is bounded in \( E \), that is to say, there exists \( C > 0 \) independent of \( \lambda, \beta \) such that \( \|u_{j, \lambda, \beta}\| \leq C \).

Without loss of generality, we assume \( u_{j, \lambda, \beta} \to u_{s} \) weakly in \( E \) as \( \lambda, \beta \to 0^{+} \). By Lemma 3.10 and Theorem 4.1 we have

\[
\frac{e^{2}}{4} \leq \inf_{u \in \Sigma} I_{j, \lambda, \beta}(u) \leq c_{j, \lambda, \beta} \leq c_{R_{s}} := \sup_{u \in \Psi_{j}(\partial B_{\epsilon})} I_{1,0}(u),
\]

where \( c_{R_{s}} \) is independent of \( \lambda, \beta, \) and

\[
I_{1,0}(u) := \frac{1}{2} \int_{\mathbb{R}^{3}} (a|\nabla u|^{2} + V(x)u^{2}) + \frac{1}{2(1 + \alpha)} \left( \int_{\mathbb{R}^{3}} |u|^{2} \right)^{1 + \alpha} - \int_{\mathbb{R}^{3}} F(u).
\]

Assume \( c_{j, \lambda, \beta} \to c_{s} \) as \( \lambda, \beta \to 0^{+} \). Then, we can prove \( u_{j, \lambda, \beta} \to u_{s} \) strongly in \( E \) as \( \lambda, \beta \to 0^{+} \) and \( u_{s} \in E \setminus W \) such that \( I'(u_{s}) = 0 \) and \( I(u_{s}) = c_{s} \). We claim that \( c_{s} \to \infty \) as \( j \to \infty \).

Indeed, it follows from \((f_{1})\) and \((f_{2})\) that

\[
I_{j, \lambda, \beta}(u) \geq \frac{1}{2} \int_{\mathbb{R}^{3}} (a|\nabla u|^{2} + V(x)u^{2}) - \int_{\mathbb{R}^{3}} F(u) - \frac{1}{r} \int_{\mathbb{R}^{3}} |u|^{r} \\
\geq \frac{1}{2} \int_{\mathbb{R}^{3}} (a|\nabla u|^{2} + V(x)u^{2}) - \int_{\mathbb{R}^{3}} \left( \frac{V_{0}}{4} u^{2} + \frac{C_{V_{0}}}{r} |u|^{r} \right) - \frac{1}{r} \int_{\mathbb{R}^{3}} |u|^{r} \tag{4.3}
\]

where \( W(x) := V(x) - V_{0}/2 \) and \( C_{V_{0}}, C > 0 \) are constants. We observe that the boundedness of the Palais–Smale sequence is not hard to verify for functional \( L \). As a result, with some suitable modification, the arguments of functional \( I_{j, \lambda, \beta} \) are still valid for \( L \) without any perturbation. That is to say, the functional \( L \) satisfies all conditions of Theorem 4.1. So we can define

\[
d^{\epsilon} := \inf_{B \in \Gamma_{j}} \sup_{u \in \Sigma \cap W_{j}} L(u),
\]

where \( W := P_{\epsilon}^{+} \cup P_{\epsilon}^{-} \) and \( \Gamma_{j} \) was defined in Theorem 4.1, and \( d^{\epsilon} \) is independent of \( \epsilon \). And then, Theorem 4.1 gives \( d^{\epsilon} \to +\infty \) as \( j \to +\infty \). Combining (4.3) and (4.4), it follows
from the definition of $c^j_{\lambda, \beta} > d^j$. Taking $\lambda, \beta \to 0^+$, we immediately get $c^j_\star > b^j \to +\infty$ as $j \to +\infty$. Therefore, problem (K) has infinitely many sign-changing solutions. The proof is complete.

5 Energy doubling of sign-changing solutions

In view of Theorem 2.1, we know that problem (K) has always a ground state sign-changing solution $w_b \in E$ for any $b > 0$. We prove now that $I(w_b)$ is of an energy which is strictly large than $2c_b$ defined in (2.7) as $b > 0$ is small.

Proof of Theorem 2.3 For any $b > 0$, let $w_b \in E$ be a ground state sign-changing solution of problem (K) with $I(w_b) = m_b$, where $m_b$ satisfies

$$m_b := \inf_{u \in \Theta} I(u), \quad \Theta := \{u \in E \setminus \{0\}, I'(u) = 0, u^\pm \neq 0\}. \quad (5.1)$$

In view of the proof of Theorem 2.1, we can deduce from (3.16) that

$$m_b \leq C_R := \sup_{u \in \psi_0(\Delta)} \bar{I}_1(u) < \infty, \quad (5.2)$$

where $\psi_0(\Delta)$ was defined in the proof of Theorem 2.1 and $C_R$ is independent of $b \in (0, 1]$ and the functional $\bar{I}_1 : E \to \mathbb{R}$ is defined as

$$\bar{I}_1 := \frac{1}{2} \|u\|^2 + \frac{1}{4} \left( \int_{\mathbb{R}^3} |\nabla u|^2 \right)^2 + \frac{1}{2(1 + \alpha)} \|u\|^{2(1 + \alpha)} - \int_{\mathbb{R}^3} F(u).$$

We claim that, for any sequence $b_n \to 0$, $\{w_{b_n}\}$ is a bounded sequence in $E$. We first have

$$m_{b_n} = \frac{a}{2} \int_{\mathbb{R}^3} |\nabla w_{b_n}|^2 + \frac{1}{2} \int_{\mathbb{R}^3} V(x) w_{b_n}^2 + \frac{b_n}{4} \left( \int_{\mathbb{R}^3} |\nabla w_{b_n}|^2 \right)^2 - \int_{\mathbb{R}^3} F(w_{b_n}) \quad (5.3)$$

and

$$0 = a \int_{\mathbb{R}^3} |\nabla w_{b_n}|^2 + \int_{\mathbb{R}^3} V(x) w_{b_n}^2 + b_n \left( \int_{\mathbb{R}^3} |\nabla w_{b_n}|^2 \right)^2 - \int_{\mathbb{R}^3} f(w_{b_n}) w_{b_n}. \quad (5.4)$$

Moreover, from Lemma 3.1, the following identity holds

$$\frac{a}{2} \int_{\mathbb{R}^3} |\nabla w_{b_n}|^2 + \frac{3}{2} \int_{\mathbb{R}^3} V(x) w_{b_n}^2 + \frac{1}{2} \int_{\mathbb{R}^3} (\nabla V(x), x) w_{b_n}^2$$

$$+ \frac{b}{2} \left( \int_{\mathbb{R}^3} |\nabla w_{b_n}|^2 \right)^2 - 3 \int_{\mathbb{R}^3} F(w_{b_n}) = 0. \quad (5.5)$$

Multiplying (5.3), (5.4) and (5.5) by $4, -\frac{1}{\mu}$ and $-1$, respectively, and adding them up, we get
\[
4m_{b_n} = \frac{3\mu - 2}{2\mu} \int_{\mathbb{R}^3} |\nabla w_{b_n}|^2 + \frac{\mu - 2}{2\mu} \int_{\mathbb{R}^3} V(x)w_{b_n}^2 - \frac{1}{2} \int_{\mathbb{R}^3} (\nabla V(x), x)w_{b_n}^2 \\
+ \frac{\mu - 2}{2\mu} b_n \left( \int_{\mathbb{R}^3} |\nabla w_{b_n}|^2 \right)^2 + \int_{\mathbb{R}^3} \left( \frac{1}{\mu} f(w_{b_n})w_{b_n} - F(w_{b_n}) \right).
\]

Combining this with (5.2), we argue that \( \{w_{b_n}\} \) is a bounded sequence in \( E \). Up to a subsequence, we assume that \( w_{b_n} \to w_0 \) weakly in \( E \). Note that \( w_{b_n} \) is a ground state sign-changing solution to problem (K) with \( b = b_n \), then by the compactness of the embedding \( E \hookrightarrow L^q(\mathbb{R}^3) \) \((2 < q < 6)\), we deduce that \( w_{b_n} \to w_0 \) strongly in \( E \) and \( w_0 \) is a sign-changing solution of (2.1). Thus,

\[
m_{b_n} = I(w_{b_n}) = I_0(w_0) + o(1) \geq m_0 + o(1),
\]

where \( m_0 \) is given in (5.1) with \( b = 0 \).

We now claim \( m_0 > 2c_0 \). Indeed, (2.6) implies that \( m_0 \geq 2c_0 \). Assume \( m_0 = 2c_0 \) and \( w_0 \) is a ground state sign-changing solution of (2.1). Then, we immediately get \( w_0^+ \in \mathcal{N} \) and \( I_0(w_0^+) = c_0 \). That is, \( w_0^+ \) is a minimizer of the functional \( I_0 \) which is restricted at \( \mathcal{N} \). It is known that \( \mathcal{N} \) is a \( C^1 \) and natural constraint manifold. Hence, by lagrange multiplier principle, we can easily get that \( w_0^+ \) are critical points of the free functional \( I_0 \). This fact tells us that \( w_0^+ \) and \( w_0^- \) are two nonnegative solutions of (2.1). Using the strong maximum principle, we can obtain two different positive solutions of (2.1) corresponding respectively to \( w_0^+ \) and \( w_0^- \), contradicting to the uniqueness of positive solution. Therefore, the claim is true. In view of (5.6), we have

\[
m_{b_n} = I_0(w_0) + o(1) \geq m_0 + o(1) > 2c_0,
\]

for large \( n \).

We also claim that for each \( b > 0 \), there exists \( u_b \in \mathcal{N}_b \) such that \( I(u_b) = c_b \), and \( u_b \) is a positive ground state solution of problem (K), where \( \mathcal{N}_b \) and \( c_b \) have been given in (2.7). In fact, similar to that of [28, 33], we construct a modified energy functional satisfying the geometric conditions of monotonicity trick developed by Struwe and Jeanjean [24, 46] to obtain a bounded Palais–Smale sequence at a mountain-pass level. By the compactness of the embedding \( E \hookrightarrow L^q(\mathbb{R}^3) \) \((2 < q < 6)\), we can obtain a nontrivial critical point of the modified energy functional. With the aid of the corresponding Pohozaev identity, a convergence argument allows us to pass limit to the original problem (K) and then to obtain a nontrivial solution in \( E \). Without loss of generality, we may assume that the nontrivial solution is nonnegative, then the strong maximum principle implies that such a solution is positive. We also obtain a positive ground state solution \( u_b \in \mathcal{N}_b \) with \( I(u_b) = c_b \), when the above convergence argument is used to the minimal sequence \( \{u_n\} \subset \mathcal{N}_b \) satisfying \( I(u_n) \to c_b \) as \( n \to \infty \). The claim is proved.

Now taking \( b_n \to 0 \), we know that there exists \( u_0 \in E \) such that \( u_{b_n} \to u_0 \) strongly in \( E \) and \( u_0 \) is a positive solution of problem (2.1). Hence, by the uniqueness of positive solution of (2.1), we have

\[
c_{b_n} = I(u_{b_n}) = I_0(u_0) + o(1) = c_0 + o(1),
\]

where \( c_0 \) is given in (2.5). Combining (5.6) with (5.8), we have \( m_{b_n} > 2c_{b_n} \) for large \( n \in \mathcal{N} \).

Hence, by (4.1) there exists \( b^* > 0 \) such that \( m_b > 2c_b \) for any \( b \in (0, b^*) \). By the definition of \( c_b, m_b \) is strictly two times larger than that of the ground state energy. The proof is complete. \( \square \)
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References

1. Ackermann, N., Weth, T.: Multibump solutions of nonlinear periodic Schrödinger equations in a degenerate setting. Commun. Contemp. Math. 7, 1–30 (2005)
2. Alves, C., Corrêa, F., Figueiredo, G.: On a class of nonlocal elliptic problems with critical growth. Differ. Equ. Appl. 2, 409–417 (2010)
3. Alves, C., Corrêa, F., Ma, T.-F.: Positive solutions for a quasilinear elliptic equation of Kirchhoff type. Comput. Math. Appl. 49, 85–93 (2005)
4. Alves, C., Figueiredo, G.: Nonlinear perturbations of a periodic Kirchhoff equation in $\mathbb{R}^N$. Nonlinear Anal. 75, 2750–2759 (2012)
5. Arosio, A., Panizzi, S.: On the well-posedness of the Kirchhoff string. Trans. Am. Math. Soc. 348, 305–330 (1996)
6. Azzollini, A.: The elliptic Kirchhoff equation in $\mathbb{R}^N$ perturbed by a local nonlinearity. Differ. Int. Equ. 25, 543–554 (2012)
7. Bartsch, T., Liu, Z., Weth, T.: Sign-changing solutions of superlinear Schrödinger equations. Commun. Partial Differ. Equ. 29, 25–42 (2004)
8. Bartsch, T., Liu, Z., Weth, T.: Nodal solutions of a p-Laplacian equation. Proc. Lond. Math. Soc. 91, 129–152 (2005)
9. Bartsch, T., Willem, M.: Infinitely many radial solutions of a semilinear elliptic problem on $\mathbb{R}^N$. Arch. Ration. Mech. Anal. 124, 261–276 (1993)
10. Cao, D., Zhu, X.: On the existence and nodal character of semilinear elliptic equations. Acta Math. Sci. 8, 345–359 (1988)
11. Cassani, D., Liu, Z., Tarsi, C., Zhang, J.: Multiplicity of sign-changing solutions for Kirchhoff-type equations. Nonlinear Anal. 186, 145–161 (2019)
12. Cassani, D., Vilasi, L., Zhang, J.: Concentration phenomena at saddle points of potential for Schrödinger-Poisson systems. Commun. Pure Appl. Anal. 20, 1737 (2021). https://doi.org/10.3934/cpaa.2021039
13. Cavalcanti, M., Cavalcanti, V., Soriano, J.: Global existence and uniform decay rates for the Kirchhoff-Carrier equation with nonlinear dissipation. Adv. Differ. Equ. 6, 701–730 (2001)
14. Cerami, G., Solimini, S., Struwe, M.: Some existence results for superlinear elliptic boundary value problems involving critical exponents. J. Funct. Anal. 69, 289–306 (1986)
15. Chang, K.: Heat method in nonlinear elliptic equations. In: Methods, T. (ed.) Variational Methods and Their Applications (Taiyuan, 2002), pp. 65–76. World Sci Publ, River Edge (2003)
16. Chang, K., Jiang, M.: Dirichlet problem with indefinite nonlinearities. Calc. Var. Partial Differ. Equ. 20, 257–282 (2004)
17. Chipot, M., Lovat, B.: Some remarks on non local elliptic and parabolic problems. Nonlinear Anal. 30, 4619–4627 (1997)
18. Deng, Y., Peng, S., Shuai, W.: Existence and asymptotic behavior of nodal solutions for the Kirchhoff-type problems in $\mathbb{R}^3$. J. Funct. Anal. 269, 3500–3527 (2015)
19. Figueiredo, G., Ikoma, N., Junior, J.: Existence and concentration result for the Kirchhoff type equations with general nonlinearities. Arch. Ration. Mech. Anal. 213, 931–979 (2014)
20. Gu, L., Jin, H., Zhang, J.: Sign-changing solutions for nonlinear Schrödinger-Poisson systems with sub-quadratic or quadratic growth at infinity. Nonlinear Anal. 198, 111897 (2020)
21. He, X., Zou, W.: Existence and concentration behavior of positive solutions for a Kirchhoff equation in $\mathbb{R}^4$. J. Differ. Equ. 252, 1813–1834 (2012)
22. He, Y., Li, G.: Standing waves for a class of Kirchhoff type problems in $\mathbb{R}^3$ involving critical Sobolev exponents. Calc. Var. Partial Differ. Equ. 54, 3067–3106 (2015)
23. He, Y.: Concentrating bounded states for a class of singularly perturbed Kirchhoff type equations with a general nonlinearity. J. Differ. Equ. 261, 6178–6220 (2016)
24. Jeanjean, L.: On the existence of bounded Palais-Smale sequence and application to a Landesman-Lazer type problem set on $\mathbb{R}^N$. Proc. R. Soc. Edinb. Sect. A 129, 787–809 (1999)
25. Kabeya, Y., Tanaka, K.: Uniqueness of positive radial solutions of semilinear elliptic equations in $\mathbb{R}^N$ and Séré’s non-degeneracy condition. Commun. Partial Differ. Equ. 24, 563–598 (1999)
26. Kirchhoff, G.: Mechanik. Teubner, Leipzig (1883)
27. Kuzin, I., Pohozaev, S.: Entire Solutions of Semilinear Elliptic Equations. Birkhäuser, Boston (1997)
28. Li, G., Ye, H.: Existence of positive ground state solutions for the nonlinear Kirchhoff type equations in $\mathbb{R}^2$. J. Differ. Equ. 257, 566–600 (2014)
29. Liang, Z., Li, F., Shi, J.: Positive solutions to Kirchhoff type equations with nonlinearity having prescribed asymptotic behavior. Ann. Inst. H. Poincare Anal. Non Lineaire 31, 155–167 (2014)
30. Lions, J.: On some questions in boundary value problems of mathematical physics. North-Holland Math. Stud. 30, 284–346 (1978)
31. Liu, J., Liu, X., Wang, Z.-Q.: Multiple mixed states of nodal solutions for nonlinear Schrödinger systems. Calc. Var. Partial Differ. Equ. 52, 565–586 (2015)
32. Liu, Z., Wang, Z.-Q., Zhang, J.: Infinitely many sign-changing solutions for the nonlinear Schrödinger-Poisson system. Ann. Mat. 195, 775–794 (2016)
33. Liu, Z., Guo, S.: Existence of positive ground state solutions for Kirchhoff type problems. Nonlinear Anal. 120, 1–13 (2015)
34. Liu, Z., Squassina, M., Zhang, J.: Existence and multiplicity of sign-changing solutions for a static Schrödinger–Poisson–Slater equation. J. Differ. Equ. 266, 5912–5941 (2019)
35. Liu, Z., Zhang, Z., Huang, S.: Existence and nonexistence of positive solutions for a static Schrödinger–Poisson–Poisson–Slater equation. J. Differ. Equ. 266, 5912–5941 (2019)
36. Liu, Z., OuYang, Z., Zhang, J.: Existence and multiplicity of sign-changing standing waves for a gauged nonlinear Schrödinger equation in $\mathbb{R}^2$. Nonlinearity 32, 3082–3111 (2019)
37. Liu, Z., Siciliano, G.: A perturbation approach for the Schrödinger–Born–Infeld system: solutions in the subcritical and critical case. J. Math. Anal. Appl. 503, 125326 (2021)
38. Liu, Z., Luo, H., Zhang, J.: Existence and multiplicity of bound state solutions to a Kirchhoff type equation with a general nonlinearity (2021). arXiv:2102.13422v1
39. Lu, S.: Signed and sign-changing solutions for a Kirchhoff-type equation in bounded domains. J. Math. Anal. Appl. 432, 965–982 (2015)
40. Ma, T., Rivera, J.: Positive solutions for a nonlinear nonlocal elliptic transmission problem. Appl. Math. Lett. 16, 243–248 (2003)
41. Mao, A., Zhang, Z.: Sign-changing and multiple solutions of Kirchhoff-type problem without the P.S. condition. Nonlinear Anal. 70, 1275–1287 (2009)
42. Nie, J., Wu, X.: Existence and multiplicity of non-trivial solutions for Schrödinger–Kirchhoff-type equations with radial potential. Nonlinear Anal. 75, 3470–3479 (2012)
43. Perera, K., Zhang, Z.: Nontrivial solutions of Kirchhoff-type problems via the Yang index. J. Differ. Equ. 221, 246–255 (2006)
44. Shuai, W.: Sign-changing solutions for a class of Kirchhoff-type problem in bounded domains. J. Differ. Equ. 259, 1256–1274 (2015)
45. Strauss, W.: Existence of solitary waves in higher dimensions. Commun. Math. Phys. 55, 149–162 (1977)
46. Struwe, M.: On the evolution of harmonic mappings of Riemannian surfaces. Comment. Math. Helv. 60, 558–581 (1985)
47. Sun, D., Zhang, Z.: Existence and asymptotic behaviour of ground state solutions for Kirchhoff-type equations with vanishing potentials. Z. Angew. Math. Phys. 70, 37 (2019)
48. Sun, J., Li, L., Cencelj, M., Gabrovšek, B.: Infinitely many sign-changing solutions for Kirchhoff-type problems in $\mathbb{R}^3$. Nonlinear Anal. (2019). https://doi.org/10.1016/j.na.2018.10.007
49. Sun, J., Wu, T.: Existence and multiplicity of solutions for an indefinite Kirchhoff-type equation in bounded domains. Proc. R. Soc. Edinb. Sect. A 146, 435–448 (2016)
50. Tang, X., Cheng, B.: Ground state sign-changing solutions for Kirchhoff-type problems in bounded domains. J. Differ. Equ. 259, 2384–2402 (2016)
51. Wang, J., Tian, L., Xu, J., Zhang, F.: Multiplicity and concentration of positive solutions for a Kirchhoff type problem with critical growth. J. Differ. Equ. 253, 2314–2351 (2012)
52. Weth, T.: Energy bounds for entire nodal solutions of autonomous superlinear equations. Calc. Var. Partial Differ. Equ. 27, 421–437 (2006)
53. Wu, X.: Existence of nontrivial solutions and high energy solutions for Schrödinger–Kirchhoff-type equations in $\mathbb{R}^3$. Nonlinear Anal. RWA 12, 1278–1287 (2011)
54. Xue, Q., Ma, S., Zhang, X.: Positive ground state solutions for some non-autonomous Kirchhoff type problems. Rocky Mt. J. Math. 47, 329–350 (2017)
55. Zhang, Z., Perera, K.: Sign changing solutions of Kirchhoff type problems via invariant sets of descent flow. J. Math. Anal. Appl. 317, 456–463 (2006)
56. Zou, W.: Sign-Changing Critical Points Theory. Springer, New York (2008)

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