Prediction of Lateral-Vibration Responses of A Full Size Cantilever Beam by Using Those of Its Scale Models

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Abstract. The main objective of this article is to study and calculate lateral vibration responses of the cantilever beam by using a scale model and the associated scaling laws. Lateral vibration responses of a cantilever beam are analyzed theoretically and to validate results using a method of finite element to calibrate vibration response of a similar beam. We get the results and then we had compared both scale model and full-size cantilever beam model. Initially derive the solution of free response and steady-state response for the full and scaled model. Later, the scaling-laws for all the applicable parameters concerning the cantilever beam in the size of full and its scale beam of free vibration are get from the Euler’s equation of motion of the last two systems. Next, by dimensional analysis find the scaling factors for applicable dimensions and properties of required two systems. And in Finite element method introduce property matrices and equations of motion of cantilever beam system and also systematic procedure to get element property matrices of the beam is shown. In the end it is solved by Method of Eigen value problem with the help of Mat lab software. Lastly the impact of specimen dimensions on the similarity basis, natural frequencies of cantilever beam specimen is investigated.

1. Introduction
The aim of this project is to introduce the problems of measuring instruments due to its cost and access where we used to vibrating bodies of huge in size. In fields of structural and mechanical-engineering application, the lateral-vibration responses of cantilever beams, for instance those of supporting beams, etc. are the essential data for engineers. Researchers have considered the vibration responses in machine members. Firoozian and Zhu [1] have applied the transfer-matrix method to estimate the lateral vibration responses of the rotor bearing model. Besides, Wu and Yang [2], Qing and Cheng [3] and Al-Bedoor [4, 5] have perform on the connected rotor-bearing system to calculate torsional-lateral vibration responses using methods of transfer matrix, finite element, respectively. Rezaeezapazhand et al. [6, 7] and Simitses and Rezaeezapazhand [8] have examine the buckling and free vibration responses of full size laminated shell and develop scale down models. Wu [9] has applied dynamic loads on a scale flat plate and compare with those of flat plate in full dimensions to predict dynamic responses by using respected scaling-laws. Further, Wu [10] has applied the lateral vibration responses of a scale rotor bearing arrangement
Properties & Scalesize beam

| Properties       | Fullsize beam | Scalesize beam |
|------------------|---------------|----------------|
| Mass(m)(Kg)      | 2             | 2 × 0.5³       |
| Damping(c)(N-s/m)| 4             | 4 × 0.5²       |
| Stiffness(K)     | 8             | 8 × 0.5        |
| Force(F)         | 3 × sin(5t)   | 3 × 0.5² sin(5 × 0.5⁻¹ × t) |
| Frequency(ω)     | 5             | 5 × 0.5⁻¹      |
| Natural Frequency(ωn) | 2        | 2 × 0.5⁻¹      |

Table 1: Physical parameters of full size and scale size vibration system

and to predict those of the rotor-bearing at the place of application (full size) due to free and forced vibrations by using corresponding scaling laws.

Initially, the article delivers the scaling-factors for all appropriate parameters concerned to cantilever beam. For validation, the lateral-vibration responses of a full size cantilever beam and its scale beam specimen are calculated theoretically and numerically and achieve valuable information on results. And the formulation of a cantilever beam in the finite element method of the equation of motion is solved by the help of mat lab Eigen value problem.

2. Steady & Free Vibration

The properties of full scale and scale size beam is shown in the given table.

2.1. Free vibration solution for scale model

The equation of motion is given by

\[ m\ddot{x} + c\dot{x} + kx = 0 \] (1)

For steady-state response

\[ x(t) = X \times e^{\xi \omega_n t} \times \sin(\sqrt{1 - \xi^2} \times \omega_n \times t + \phi) \] (2)

\[ c_c \] is the critical damping of the system, \( \phi \) is the phase angle, damping ratio

\[ \xi = \frac{c}{c_c} = \frac{c}{2 \times \sqrt{k \times m}} \] (3)

\[ x(0) = 0; \dot{x}(0) = 0, \] by sloving eq(2) and eq(4), we get \( \tan(\phi) = 1.732 \)

\[ X = \frac{1}{\sin(60)} = 1.1547, \phi = 60^0 \] (4)

\[ x(t) = 1.1547 \times \sin(1.732 \times t \times 60^0) \times e^{-t} \] (5)
2.2. Free vibration solution for full model

\[ x(t) = X \times e^{-\xi \omega_n t} \times \sin(\sqrt{1 - \xi^2} \times \omega_n \times t + \phi) \]  \hspace{1cm} (6)

\[ \xi = \frac{c}{c_c} = \frac{c}{2 \times \sqrt{k \times m}} = 0.5 \]  \hspace{1cm} (7)

\[ x(0) = 0.5; \dot{x}(0) = 0, \]

by solving eq(6) and eq(7), we get \( \tan(\phi) = 1.732 \)

\[ x(t) = 0.57735 \times \sin(3.4641 \times t + 60^\circ) \times e^{-2t} \]  \hspace{1cm} (8)

From Equations (5), (8), frequency is doubled and amplitude is halved.

![Figure 3: Amplitude v/s Time graph for full and scale model in free vibration](image)

2.3. Forced vibration for scale model

For steady state response,

\[ x(t) = \frac{F_e}{\sqrt{((k - m\omega^2)^2) + (c\omega)^2}} \times \sin(\omega t - \phi) \]  \hspace{1cm} (9)

\[ \phi = \tan^{-1}\left(\frac{c \times \omega}{k - m(\omega)^2}\right) = \tan^{-1}\left(\frac{4 \times 5}{8 - 2(5)^2}\right) = -25.4633 \]  \hspace{1cm} (10)

\[ x(t) = 0.064490037 \times \sin(5t + 25.4633) \]  \hspace{1cm} (11)

![Figure 4: Amplitude v/s Time graph for full and scale model in forced vibration](image)
2.4. Forced vibration for full model

\[ x(t) = 0.03224 \times \sin(10t + 25.4633) \]  \hspace{1cm} (12)

3. Determination of scaling-laws & scaling-factors for physical dimensions of a cantilever beam

The scaling study is to get terminology of the scaling-laws and scaling-factors for the dynamic or static-similitude between the full size beam specimen and its scale-size beam specimen.

3.1. Scaling laws between the cantilever beam in full size and its scale size beam

For a free-vibrating system, based on Euler-theory, we substituted following parameters in the equation of motion.

Let \( m(x) = \) distributed mass/unit length

\( L = \) length of beam-specimen

\( y = \) deflection of beam

\( \omega = \) angular velocity of the beam-specimen

\( E = \) Young’s modulus of the beam-specimen

\( I = \) moment of inertia of the beam-specimen

\( \lambda = \) scaling variable of the beam

Dynamic load per unit length

\[ \frac{d^2}{dx^2} EI(X) \frac{d^2 Y(x)}{dx^2} \]

Centrifugal force per unit length = \( m(x)Y(x)\omega^2 \)

\[ \frac{d^2}{dx^2} EI(X) \frac{d^2 Y(x)}{dx^2} = m(x)Y(x)\omega^2 \]  \hspace{1cm} (14)

After solving, we get

\[ EI(X) \frac{d^4 Y(x)}{dx^4} = \omega^2 m(x)Y(x), \lambda = \frac{d^2}{d^4} (d = m, y, I, \omega, \eta) \]

\[ \left( \frac{\lambda^3}{\lambda \eta} \right)^2 \frac{d^2}{dx^2} EI(X) \frac{d^4 Y(x)}{dx^4} = \left( 1, \frac{\lambda_2}{\lambda \eta} \right)^2 \frac{d^2}{dx^2} EI(X) \frac{d^4 Y(x)}{dx^4} = 1 \]  \hspace{1cm} (16)

the above equation is called the scaling laws and it is the requirement for the similitude between the cantilever beam specimen in the size of full and its scaled specimen.

3.2. Scaling factors between the cantilever beam specimen in the size of full and its scale - size

For length and breadth, depth use as fundamental parameter respectively

\[ \lambda_l = \lambda_y, \lambda_b = \lambda_y, \lambda_h = \lambda_y \]

- The displacement scaling factor is denoted by \( \lambda_y \)
- A mass scaling factor of the beam \( \lambda_m = \lambda^3_y \)
- Moment of inertia of scaling factor for cantilever beam \( I = \frac{bh^3}{12} = \left( \frac{\lambda^3}{\lambda \eta} \right)^2 \frac{bh^3}{12} = \lambda^4 \)

For the same material following scaling factors of full and scale cantilever beam does not change.

- The ratio of Young’s Modulus of scale and full-size cantilever beam is equal to 1
- The ratio of the density of scale and full-size cantilever beam is equal to 1
- Stiffness scaling factor of the cantilever beam \( \lambda_k = \lambda_m \times \lambda^3_y, \lambda_k = \lambda^3_y \times \frac{1}{\lambda_2} = \lambda^4 \)
4. Free vibration of a cantilever beam Mathematical Analysis

If a cantilever beam specimen subjected to free vibration, then the specimen arrangement is taken as a continuous system. In this case the mass of a beam is considered as a well propagated with respect to stiffness of the beam specimen. The free vibration equation can be taken from earlier equation (13). Where, \( E \) modulus of rigidity of beam, \( I \) -moment of inertia of the rectangular cross sectional beam specimen, \( Y(x) \) is deflection, \( \omega \) -circular natural frequency, \( m(x) \)-mass per unit length, \( m(x) = \rho A(x) \), \( \rho \) - density of specimen, \( x \)-linear distance calculated from the fixed end.

We have following boundary conditions of a cantilever beam specimen (Fig. 5)

\[
@x = 0 \Rightarrow Y(x) = 0, \quad \frac{dY(x)}{dx} = 0 \quad \text{&} \quad @x = L \Rightarrow \frac{d^2Y(x)}{dx^2} = 0, \quad \frac{d^3Y(x)}{dx^3} = 0 \quad (17)
\]

For a homogeneous cross sectional beam specimen subjected to natural vibration from equation (13), Can form following equation

\[
\frac{d^4Y(x)}{dx^4} - \beta^4 Y(x) = 0 \quad \beta^4 = \frac{\omega^2 m}{EI} \quad (18)
\]

The general equation of mode shapes for a structural beam specimen is written as a closed form of the natural frequency \( \omega_{nf} \), after applying boundary conditions to above equation, final form be written as below.

\[
f(x) = A_n ((\sin \beta_n L - \sinh \beta_n L)(\sin \beta_n x - \sin \beta_n x) + (\cos \beta_n L - \cosh \beta_n L)(\cos \beta_n x - \cosh \beta_n x)) \quad (19)
\]

\[
\beta_n L = n\pi \quad \omega_{nf} = \alpha_n^2 \sqrt{\frac{EI}{mL^4}} \quad (20)
\]

where \( \alpha_n = 1.875, 4.694, 7.895 \) & \( n = 1, 2, 3, \ldots \).
| Material         | Density | E   |
|------------------|---------|-----|
| Steel Beam       | 7850 Kg/m³ | 2.1e11 |

Table 2: Material properties of steel

| Length(L) | Width (b) | Height(d) |
|-----------|-----------|-----------|
| 0.45 m    | 0.02 m    | 0.003 m   |

Table 3: Geometry of the beam

\[ \omega_{nf1} = 1.875^2 \sqrt{\frac{EI}{mL^4}} \]
\[ \omega_{nf2} = 4.697^2 \sqrt{\frac{EI}{mL^4}} \]
\[ \omega_{nf3} = 7.855^2 \sqrt{\frac{EI}{mL^4}} \] (21)

Figure 6: First three un-damped natural frequencies, mode shapes of cantilever beam.

The relation between natural frequency, circular natural frequency is written as
\[ f = \frac{\omega_{nL}}{2\pi} \text{ & } I = \frac{bd^3}{12} \]

In the above formula, I referred as moment of inertia of the rectangular specimen.

Example: find out the un-damped natural frequency of a cantilever beam with the above dimensions. Based on scaling ratio for scale cantilever beam of length, breadth and depth dimensions are halved & following results can be obtained.

| Frequency                  | Full Model | Scale Model | Scale Factor |
|----------------------------|------------|-------------|--------------|
| First Natural Frequency    | 12.37      | 24.74       | 2.00         |
| Second Natural Frequency   | 77.53      | 155.06      | 2.00         |
| Third Natural Frequency    | 434.22     | 868.44      | 2.00         |

Table 4: Theoretical results on scale and full size cantilever beam

5. Finite Element Formulation

Finite element method is a numerical method for solving differential equations and also estimate the second natural frequency of a cantilever beam by the method of finite element formulation. The basic method is explained here.

For a cross sectional uniform beam, the stiffness matrix and the consistence mass matrix are given below.

\[ [K]^e = \frac{EI}{l^3} \begin{bmatrix} 12 & 6l & -12 & 6l \\ 6l & 4l^2 & -6l & 2l^2 \\ -12 & -6l & 12 & -6l \\ 6l & 2l^2 & -6l & 4l^2 \end{bmatrix} \text{ and } [M]^e = \frac{\rho Al}{420} \begin{bmatrix} 156 & 22l & 54 & -13l \\ 22l & 4l^2 & 13l & -3l^2 \\ -13l & -3l^2 & 156 & -22l \\ -22l & 4l^2 & \end{bmatrix} \] (22)
Figure 7: A cantilever beam

Figure 8: Discretization of beam into 2 elements

**Step 1:** Discrete the experimental beam arrangement into three small parts as shown in Fig.8. In Fig. 8, numbers 1, 2, 3 refer to the nodes.

**Step 2:** Arrange the elemental equations. For the first part, from equation (22), we have

\[
\frac{\rho Al}{420} \begin{bmatrix}
156 & 22l & 54 & -13l \\
22l & 4l^2 & 13l & -3l^2 \\
54 & 13l & 156 & -22l \\
-13l & -3l^2 & -22l & 4l^2
\end{bmatrix}
\ddot{x} + \frac{EI}{l^3} \begin{bmatrix}
12 & 6l & -12 & 6l \\
6l & 4l^2 & -6l & 2l^2 \\
-12 & -6l & 12 & -6l \\
6l & 2l^2 & -6l & 4l^2
\end{bmatrix} x = F
\] (23)

Here X refer to the vector node variables (i.e. at the first and second nodes). In this scenario node variables are slope and displacement, the vector F be a symbol of force.

Again we get the second step that also has the same elemental equation as equation (23).

**Step 3:** combine above first and second element matrices to form a global matrix. The dimensions of the beam are multiplied with a scaling factor 0.5.

**Step 4:** The final step is to find out the solution by using following equations

\[
\begin{bmatrix}
0.0066 & 0 & 0.0011 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0.001 & 0 & 0.006 & 0 & 0.0011 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.0011 & 0 & 0.003 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\ddot{x} + \begin{bmatrix}
3.36 & 0 & -1.68 & 0.063 & 0 & 0 \\
0 & 0.0063 & -0.063 & 0.0016 & 0 & 0 \\
-1.68 & -0.063 & 3.36 & 0 & -1.68 & 0.063 \\
0.063 & 0.0016 & 0 & 0.0063 & 1.68 & -0.06 \\
0 & 0 & -1.68 & -0.063 & 1.68 & -0.06 \\
0 & 0 & 0.0063 & 0.0016 & -0.063 & 0.003
\end{bmatrix} x = F
\] (24)

And for free vibration

\[
\ddot{x} = -\omega_{nf}^2 x
\]

Hence, above equations can be written as

\[(|M|\omega_{nf}^2 + [K])x = 0 \] (25)

6. **Eigen Value Problem of Cantilever Beam Solve by Mat Lab Program for Scale Model**

Here the dimensions of the cantilever beam are replaced with half of the dimensions of full size cantilever beam, then we get following results as seen in the MATLAB software.

**End Result**
| Frequency                  | Full Model | Scale Model | Scale Factor |
|----------------------------|------------|-------------|--------------|
| First Natural Frequency    | 12.40      | 24.755      | 2.00         |
| Second Natural Frequency   | 77.80      | 155.1375    | 2.00         |
| Third Natural Frequency    | 219.90     | 434.4331    | 2.00         |

Table 5: shows the results of Scale and Full Size Cantilever Beam by Using FEA

The above end results obtained at the end of Eigen value problem in Mat Lab Program by the finite element analysis coincide with results arriving from theoretical analysis.

7. Conclusion

- In this paper, one of the most popular technique scale-down model tests is used to analyse the theoretical results of a large size structural beam specimens.
- Numerical results in a scale model are calculated through the mathematical equations by presenting the scaling laws and the corresponding scaling factors, and these results are utilized to predict the responses of free lateral vibrations of a cantilever beam in size of full.
- To validate the theoretical results and developed mathematical analysis of cantilever beam to find out natural frequencies, the effect of various dimensions of component member on the similarity of a free lateral vibrations and the influence of external loads are induced which creates unbalance force on the free end of cantilever beam have been studied.

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