A COMPACTNESS RESULT IN $\text{GSBV}^p$ AND APPLICATIONS TO $
abla$-CONVERGENCE FOR FREE DISCONTINUITY PROBLEMS

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Abstract. We present a compactness result in the space $\text{GSBV}^p$ which extends the classical statement due to Ambrosio [2] to problems without a priori bounds on the functions. As an application, we revisit the $\nabla$-convergence results for free discontinuity functionals established recently by Cagnetti, Dal Maso, Scardia, and Zeppieri [12]. We investigate sequences of boundary value problems and show convergence of minimum values and minimizers.

1. Introduction

Since the pioneering work of Griffith [37], the propagation of crack is viewed as the result of a competition between elastic energy stored in the uncracked region of a body and dissipation related to an infinitesimal increase of the crack. It is the fundamental idea in the approach to quasistatic crack evolution by Francfort and Marigo [31] and has led to a variety of variational models, where the displacements and the (a priori unknown) crack paths are determined from an energy minimization principle. (Among the vast body of literature, we mention here only the brittle fracture models for small strains [7, 16, 30, 35, 36] and finite strains [26, 27, 28], and the cohesive models [15, 23, 29].) Problems of this form may be formulated in the frame of free discontinuity functionals

$$E(u) = \int_{\Omega} f(x, \nabla u(x)) \, dx + \int_{J_u} g(x, [u](x), \nu_u(x)) \, dH^{d-1}(x).$$

(1)

Here, $\Omega \subset \mathbb{R}^d$ denotes the reference configuration, $\nabla u$ the deformation gradient, and $J_u$ the crack surface. The energy density $f$ accounts for elastic bulk terms for the unfractured region of the body, whereas the surface term assigns energy contributions on the crack paths comparable to the $(d-1)$-dimensional Hausdorff measure $H^{d-1}(J_u)$ of the crack.

In its simplest formulation, the density $g$ is a constant, called toughness of the material, which is given by Griffith’s criterion of fracture initiation (see [37]). Densities $g$ depending explicitly on the crack opening $[u]$ allow for modeling fracture problems of cohesive-type [8]. Finally, the presence of the normal $\nu_u$ to the jump set $J_u$ and the material point $x$ take into account possible anisotropy and inhomogeneities in the body.

A basic and important question is to prove the existence of minimizers for (1) under appropriate Dirichlet boundary conditions. This requires a weak formulation of the problem in the space of special functions of bounded variation ($\text{SBV}$) (see [5] Section 4)). In [2] 3], lower semicontinuity for functionals of the form (1) is characterized in terms of quasiconvexity for $f$
and $BV$-ellipticity [4] for $g$. Compactness of sequences with bounded energy is guaranteed by an \textit{a priori} bound on the functions in $L^\infty$, see [1].

The drawback of this compactness result is that it is unfortunately difficult to obtain such uniform bounds for a minimizing sequence, even if lower order terms are present in the energy. Only in the antiplane case [30] (namely when the displacement $u$ is scalar and $f$ is of the form $f(x, \xi) = |\xi|^2$), $L^\infty$-bounds may be obtained by truncation, assuming that also the prescribed boundary values are bounded in $L^\infty$. If the boundary datum is only in some $L^p$ space or $f(x,0) > \min_\xi f(x,\xi)$, which is typically the case in finite elasticity, a truncation may change the boundary values or increase the energy.

This issue may be partially overcome by formulating the problem in the larger space of \textit{generalized special functions of bounded variation} (GSBV). In this setting, one can rely on the compactness result for GSBV with respect to convergence in measure (see [2, 5]): it requires only a very mild control on the functions of the form $\int_\Omega \psi(|u|) \, dx \leq C$ for some nonnegative and continuous $\psi$ with $\lim_{t \to \infty} \psi(t) = +\infty$. Adding a lower order \textit{fidelity term} of this kind to the energy, compactness and eventually the existence of minimizers are guaranteed.

Let us mention that similar compactness issues arise when dealing with a sequence of free discontinuity problems $(E_k)_k$ of the form (1). A classical example for this situation is the case of periodic homogenization. Here, the densities are of the form $f_k(x, \xi) = f(x/\varepsilon_k, \xi)$ and $g_k(x, \zeta, \nu) = g(x/\varepsilon_k, \zeta, \nu)$, where $f, g$ are periodic in the first variable and $\varepsilon_k$ describes the microscopical scale of a microstructure. The effective asymptotic behavior for such a sequence of fracture models in the finite strain framework was studied by Braides, Defranceschi, and Vitali [11] by means of $\Gamma$-convergence [10, 24]. In particular, they show convergence of minimum values and minimizers for boundary value problems under an \textit{a priori} $L^\infty$-bound on the deformations.

Very recently, a generalization of these results for sequences of densities $f_k$ and $g_k$ without any periodicity assumptions and under more general growth conditions has been derived by Cagnetti, Dal Maso, Scardia, and Zeppieri [12]. (Actually, their work is motivated by studying the case of stochastic homogenization [13].) Here, besides the size of a microstructure, the parameter $k$ may also have other interpretations, such as the scale of a regularization of the energy or the ratio of the contrasting value of the mechanical response in a high-contrast medium. The convergence of minimizers is shown by including an $L^p$-fidelity term $\|u - h\|_{L^p(\Omega)}$ in the energy for a suitable datum $h$.

We emphasize that, in contrast to the case of image reconstruction, a fidelity term is in general not appropriate in fracture mechanics. An investigation of the problem (1) only involving boundary conditions, without a priori bounds on the configurations or applied body forces, is desirable and in accordance with the original formulation of the problem [31, Section 2]. The main difficulty lies in the fact that, for configurations with finite energy $u$, small pieces of the body could be completely disconnected from the bulk part by the jump set $J_u$ and the function $u$ could take arbitrarily large values on such small components. Eventually, this may rule out measure convergence for minimizing sequences. It seems that only including a fidelity term in the energy can exclude such a phenomenon.

The issue of compactness results in variational fracture was recently tackled from a slightly different direction, namely via models in linearized elasticity. They are formulated in the space of \textit{generalized special functions of bounded deformation} (GSBD) introduced by Dal Maso [23]. Although in this setting only the symmetric part $e(u) = \frac{1}{2}(\nabla u)^T + \nabla u$ of the strain is controlled, similar compactness results under a priori $L^\infty$-bounds or mild fidelity terms have been established in [9] and [25], respectively. Nevertheless, the problem is more severe with
respect to the $SBV$-case since truncation methods are not applicable and thus already the simple situation $f(x, \nabla u) = |e(u)|^2$ with boundary data in $L^\infty$ is a delicate problem.

The recent paper [35] provides the first compactness and existence result in $GSBD$ for the Griffith energy in dimension two without any a priori bounds or fidelity terms. A related result [32] has been obtained in the passage from nonlinear-to-linear energies in brittle fracture by means of $\Gamma$-convergence (see also [34] for a discrete-to-continuum analysis). As discussed before, arbitrary minimizing sequences are typically not compact when (small) pieces are completely disconnected by the jump set. The compactness result relies on the idea that a control on a sequence of functions can always be ensured by subtracting suitable piecewise rigid motions. Using a piecewise Korn inequality [33, 35], it can be shown that such a modification can be performed without essentially increasing the energy of the configurations.

Very recently, a related compactness result in $GSBD$ in arbitrary space dimensions has been derived by Chambolle and Crisante [20]. Their strategy relies on a Korn-Poincaré inequality for functions with small jump set [17] together with arguments in the spirit of Rellich’s type compactness theorems. In contrast to [35], no passage to modifications of a sequence $(u_k)_k$ is necessary, at the expense of the fact that convergence to a limiting function $u$ is only guaranteed outside $\Omega := \{ x \in \Omega : |u_k(x)| \to \infty \}$. On the other hand, by setting $u = 0$ on $\Omega$ (or affine), this is enough to identify $u$ as a minimizer for certain fracture problems, including Griffith energies [18, 21, 35] or approximations à la Ambrosio-Tortorelli [6, 19]. On the other hand, this strategy is not expedient if $\arg\min \xi f(x, \xi)$ is $x$-dependent and therefore excludes a variety of interesting energies, e.g., models for composite materials. Moreover, this method is not adapted for applications to $\Gamma$-convergence where in general sequences are supposed to converge on the whole domain to a limiting function.

The main goal of the present paper is to derive a compactness result in the space $GSBV^p$, $p \in (1, \infty)$, without any a priori bounds or fidelity terms, see Theorem 3.1. We show that for a sequence of energies $(E_k)_k$ of the form (1), and for functions $(u_k)_k \subset GSBV^p(\Omega, \mathbb{R}^m)$ with $\sup_{k \in \mathbb{N}} E_k(u_k) < +\infty$ (possibly satisfying boundary conditions), one can find a subsequence, modifications $(y_k)_k \subset GSBV^p(\Omega, \mathbb{R}^m)$ (with the same boundary data as $(u_k)_k$) satisfying

$$E_k(y_k) \leq E_k(u_k) + \frac{1}{k}, \quad \mathcal{L}^d(\{ \nabla y_k \neq \nabla u_k \}) \leq \frac{1}{k},$$

and a limiting function $u \in GSBV^p(\Omega, \mathbb{R}^m)$ such that

(i) $y_k \to u$ in measure on $\Omega$,

(ii) $\nabla y_k \rightharpoonup \nabla u$ weakly in $L^p$,

(iii) $\mathcal{H}^{d-1}(J_u) \leq \liminf_{k \to \infty} \mathcal{H}^{d-1}(J_{y_k})$.

Properties (ii) and (iii) also hold for the original sequence $(u_k)_k$. As explained above, it is in general indispensable to pass to modifications $(y_k)_k$ to ensure property (i). The class of admissible energies is very general: we only require standard growth conditions in $GSBV^p$ together with a mild monotonicity condition on $g$ used in [12]. (For details, see assumptions (f1)-(f2) and (g1)-(g4) in Section 3.)

As applications, we prove existence of minimizers for energies of the form (1) under Dirichlet boundary data. Moreover, we revisit the $\Gamma$-convergence result for free discontinuity problems established recently in [12]. We show convergence of minimum values and minimizers for a sequence of boundary value problems without any fidelity term.

To prove the main compactness result, we follow the strategy devised in [32, 35]: given a sequence of functions, we pass to suitable modifications whose energies coincide with the original ones up to an error of order $\theta$. Subsequently, we let $\theta \to 0$ and apply carefully a diagonal
sequence argument (see Section 3.3). In contrast to the GSBD setting where piecewise rigid motions have to be subtracted, in the present context of $GSBV^p$ functions we can work with piecewise translated configurations. Accordingly, the piecewise Korn inequality [33] is replaced by a suitable piecewise Poincaré inequality (see Section 3.3), which is based on a careful use of the coarea formula in $BV$ (see [5] Theorem 3.40). Let us note that the coarea formula has been largely employed to approximate $BV$ functions by piecewise constant functions, particularly to prove lower semicontinuity [2] and $\Gamma$-convergence results [11, 12] in $SBV$, as well as the existence of quasistatic evolutions [20, 30, 39]. Compared to [35], the passage to modifications is more delicate due to the more general energies which may depend explicitly on the crack opening. At this point, we draw some ideas from truncation methods in [12] and use a mild monotonicity assumption on $g$ (see Section 3.2).

One of the main motivations for the compactness result is an application to $\Gamma$-convergence for free discontinuity problems. We extend the analysis in [12] by deriving a version of the $\Gamma$-convergence result including Dirichlet boundary data. To this end, we follow the strategy in [36, Lemma 7.1]. This eventually allows us to prove the convergence of minima and minimizers along a sequence of boundary value problems.

The paper is organized as follows. In Section 2 we first fix the notation and recall some basic properties. Section 3 contains the formulation of the main compactness result and its proof. In Section 4 we finally provide two applications: an existence result for functionals of the form (1) under Dirichlet boundary data and a convergence result for a sequence of functionals by means of $\Gamma$-convergence.

2. Notation and preliminaries

In this section we fix the notation and recall some basic tools.

**Basic notation:** We use the notations $\mathbb{R}^m_0 = \mathbb{R}^m \setminus \{0\}$, $\mathbb{S}^{d-1} = \{v \in \mathbb{R}^d : |v| = 1\}$, and $\mathbb{R}_+ = [0, +\infty)$. For $\Omega \subset \mathbb{R}^d$ open and bounded, we denote by $\mathcal{A}(\Omega)$ the open subsets of $\Omega$. We use the symbol $\Delta$ for the symmetric difference of two sets in $\mathbb{R}^d$. $\mathcal{L}^d$ denotes the Lebesgue measure on $\mathbb{R}^d$ and $\mathcal{H}^{d-1}$ the $(d-1)$-dimensional Hausdorff measure. By $L^0(\Omega; \mathbb{R}^m)$ we indicate the space of $\mathcal{L}^d$-measurable functions $u : \Omega \to \mathbb{R}^m$, endowed with the topology of convergence in measure. We observe that this convergence is metrizable. For $x \in \mathbb{R}^d$ and $\rho > 0$ we denote by $B_\rho(x)$ the open ball with center $x$ and radius $\rho$. We denote the indicator function of $E \subset \Omega$ by $\chi_E$.

We will use the following measure-theoretical result. (See [32, Lemma 4.1, 4.2] and note that the statement in fact holds in arbitrary space dimensions.)

**Lemma 2.1.** Let $\Omega \subset \mathbb{R}^d$ with $\mathcal{L}^d(\Omega) < \infty$. Then for every sequence $(u_n)_n \subset L^1(\Omega; \mathbb{R}^m)$ with

$$\mathcal{L}^d\left(\bigcap_{n \in \mathbb{N}} \bigcup_{m \geq n} \{|u_m - u_n| > 1\}\right) = 0$$

there exist a subsequence (not relabeled) and an increasing concave function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t \to +\infty} \psi(t) = +\infty$ such that

$$\sup_{n \geq 1} \int_{\Omega} \psi(|u_n|) \, dx < +\infty.$$

**BV functions:** For the general notions on $SBV$ and $GSBV$ functions and their properties we refer to [5]. For $u \in GSBV^p(\Omega; \mathbb{R}^m)$, $\Omega \subset \mathbb{R}^d$ open, we denote by $\nabla u$ the density of the absolutely continuous part of $Du$ with respect to the Lebesgue measure $\mathcal{L}^d$. $J_u$ stands for the set of approximate jump points of $u$ and $\nu_u$ denotes the measure-theoretic normal to $J_u$. 
The symbols $u^\pm$ denote the one-sided approximate limits of $u$ at a point of $J_u$ and we write $[u] = u^+ - u^-$. We will also use the notation

$$\text{GSBV}_p^p(\Omega; \mathbb{R}^m) = \{ u \in \text{GSBV}_p(\Omega; \mathbb{R}^m) : \| \nabla u \|_{L^p(\Omega)}^p + \mathcal{H}^{d-1}(J_u) \leq M \}. \quad (2)$$

The following compactness result in $\text{GSBV}_p$ due to Ambrosio [2] will be a key ingredient for our result.

**Theorem 2.2.** Let $\Omega \subset \mathbb{R}^d$ be open, bounded. Let $(u_k)_k$ be a sequence in $\text{GSBV}_p(\Omega; \mathbb{R}^m)$. Suppose that there exists a continuous function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t \to +\infty} \psi(t) = +\infty$ such that

$$\sup_{k \in \mathbb{N}} \left( \int_{\Omega} \psi(|u_k|) \, dx + \int_{\Omega} |\nabla u_k|^p \, dx + \mathcal{H}^{d-1}(J_{u_k}) \right) < +\infty.$$  

Then there exists a subsequence, still denoted by $(u_k)_k$, and a function $u \in \text{GSBV}_p(\Omega; \mathbb{R}^m)$ such that $u_k \to u$ in measure on $\Omega$, $\nabla u_k \rightharpoonup \nabla u$ weakly in $L^p(\Omega; \mathbb{R}^{m \times d})$, and $\mathcal{H}^{d-1}(J_u) \leq \liminf_{k \to \infty} \mathcal{H}^{d-1}(J_{u_k})$.

**Caccioppoli partitions:** We say that a partition $\mathcal{P} = (P_j)_j$ of an open set $\Omega \subset \mathbb{R}^d$ is a Caccioppoli partition of $\Omega$ if $\sum_j \mathcal{H}^{d-1}(\partial^* P_j) < +\infty$, where $\partial^* P_j$ denotes the essential boundary of $P_j$ (see [5] Definition 3.60). We say a partition is ordered if $\mathcal{L}^d(P_i) \geq \mathcal{L}^d(P_j)$ for $i \leq j$. The local structure of Caccioppoli partitions can be characterized as follows (see [5] Theorem 4.17).

**Theorem 2.3.** Let $(P_j)_j$ be a Caccioppoli partition of $\Omega$. Then

$$\bigcup_j (P_j)^1 \cup \bigcup_{i \neq j} (\partial^* P_i \cap \partial^* P_j)$$

contains $\mathcal{H}^{d-1}$-almost all of $\Omega$.

Here $(P)^1$ denote the points where $P$ has density one (see again [5] Definition 3.60). Essentially, the theorem states that $\mathcal{H}^{d-1}$-a.e. point of $\Omega$ either belongs to exactly one element of the partition or to the intersection of exactly two sets $\partial^* P_i, \partial^* P_j$. We now state a compactness result for ordered Caccioppoli partitions. (See [5] Theorem 4.19, Remark 4.20 or [35] Theorem 2.8) for the slightly adapted version presented here.)

**Theorem 2.4.** Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Let $\mathcal{P}_i = (P_{j,i})_j$, $i \in \mathbb{N}$, be a sequence of ordered Caccioppoli partitions of $\Omega$ with

$$\sup_{i \geq 1} \sum_j \mathcal{H}^{d-1}(\partial^* P_{j,i}) < +\infty.$$  

Then there exists a Caccioppoli partition $\mathcal{P} = (P_j)_j$ of $\Omega$ and a subsequence (not relabeled) such that $\sum_j \mathcal{L}^d(P_{j,i}) \to 0$ as $i \to \infty$.

The proof in [5] shows that the result still holds if the assumption of ordered partitions is replaced by the weaker assumption that for fixed $j_0 \in \mathbb{N}$ only $(P_{j,i})_{j \geq j_0}$ are ordered, i.e., $\mathcal{L}^d(P_{j,i}) \geq \mathcal{L}^d(P_{k,i})$ for all $j_0 \leq j \leq k$ and $i \in \mathbb{N}$.

The starting point for the construction of piecewise translated configurations will be the following approximation of $\text{GSBV}$ functions by piecewise constant functions, which can be seen as a piecewise Poincaré inequality.

**Theorem 2.5.** Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Let $m \in \mathbb{N}$. Then there exists a constant $C_0 = C_0(m) \geq 1$ such that for each $u \in (\text{GSBV}(\Omega; \mathbb{R}))^m$ with $\| \nabla u \|_{L^1(\Omega)} + \mathcal{H}^{d-1}(J_u) < $
There exists a Caccioppoli partition \((P_j)_{j=1}^{\infty}\) of \(\Omega\) and corresponding translations \((b_j)_{j=1}^{\infty} \subset \mathbb{R}^m\) such that \(v := u - \sum_{j=1}^{\infty} b_j \chi_{P_j} \in SBV(\Omega; \mathbb{R}^m) \cap L^\infty(\Omega; \mathbb{R}^m)\) and
\[
\begin{align*}
(i) & \quad \sum_{j=1}^{\infty} \mathcal{H}^{d-1}(\partial^* P_j) \leq 2 \mathcal{H}^{d-1}(J_u \cup \partial \Omega) + 1, \\
(ii) & \quad \|v\|_{L^\infty(\Omega)} \leq C_0 \|\nabla u\|_{L^1(\Omega)}.
\end{align*}
\]

This result essentially relies on the coarea formula in \(BV\) (see \[5, Theorem 3.40\]), where the sets \(P_j\) are chosen as the intersection of suitable level sets of the components \(u_i, i = 1, \ldots, m\). For the proof we refer to \[23, Theorem 2.3\], but we also mention that the argument can be found in previous literature, e.g., in \[2, Theorem 3.3\] and \[11, Proposition 6.2\].

3. Compactness result in \(GSBV^p\)

In this section we formulate and prove the main compactness result.

3.1. Formulation of the main compactness result. Throughout the paper we fix the constants \(p \in (1, \infty), 0 < c_1 \leq c_2 < +\infty, 1 \leq c_3 < +\infty\), and \(0 < c_4 < c_5 < +\infty\). We will consider integral functionals with bulk densities \(f : \Omega \times \mathbb{R}^{m \times d} \to \mathbb{R}_+\) satisfying the conditions
\[
\begin{align*}
(f1) & \quad \text{(measurability)} \quad f \text{ is Borel measurable on } \Omega \times \mathbb{R}^{m \times d}, \\
(f2) & \quad \text{(lower and upper bound)} \quad \text{for every } x \in \Omega \text{ and every } \xi \in \mathbb{R}^{m \times d}, \quad c_1 |\xi|^p \leq f(x, \xi) \leq c_2 (1 + |\xi|^p)
\end{align*}
\]
and surface densities \(g : \Omega \times \mathbb{R}_0^m \times \mathbb{S}^{d-1} \to \mathbb{R}_+\) satisfying the conditions
\[
\begin{align*}
(g1) & \quad \text{(measurability)} \quad g \text{ is Borel measurable on } \Omega \times \mathbb{R}_0^m \times \mathbb{S}^{d-1}, \\
(g2) & \quad \text{(estimate for } c_3|\zeta_1| \leq |\zeta_2|) \quad \text{for every } x \in \Omega \text{ and every } \nu \in \mathbb{S}^{d-1} \text{ we have} \\
& \quad \quad \quad g(x, \zeta_1, \nu) \leq g(x, \zeta_2, \nu) \\
(g3) & \quad \text{(lower and upper bound)} \quad \text{for every } x \in \Omega, \zeta_1, \zeta_2 \in \mathbb{R}^m, \text{ and } \nu \in \mathbb{S}^{d-1} \text{ we have} \\
& \quad \quad \quad c_4 \leq g(x, \zeta_1, \nu) \leq c_5, \\
(g4) & \quad \text{(symmetry)} \quad \text{for every } x \in \Omega, \zeta_1, \zeta_2 \in \mathbb{R}^m, \text{ and } \nu \in \mathbb{S}^{d-1} \text{ we have} \\
& \quad \quad \quad g(x, \zeta_1, \nu) = g(x, -\zeta, -\nu).
\end{align*}
\]

We let \(E_\Omega = E_\Omega(\Omega, c_1, c_2, c_3, c_4, c_5, p)\) be the collection of all integral functionals \(E : L^0(\Omega; \mathbb{R}^m) \times \mathcal{A}(\Omega) \to [0, +\infty]\) defined by
\[
E(u, A) = \int_{\Omega} f(x, \nabla u(x))\, dx + \int_{J_u \cap A} g(x, |u|(x), \nu_u(x))\, d\mathcal{H}^{d-1}(x) \quad \text{if } u|_A \in GSBV^p(A; \mathbb{R}^m), \\
+\infty \quad \text{else},
\]
where \(f : \Omega \times \mathbb{R}^{m \times d} \to \mathbb{R}_+\) satisfies \((f1)-(f2)\) and \(g : \Omega \times \mathbb{R}_0^m \times \mathbb{S}^{d-1} \to \mathbb{R}_+\) satisfies \((g1)-(g4)\).

(The dependence of \(E\) on subsets of \(\Omega\) will be convenient for our applications in Section 3.2.) For simplicity, we write \(E(u, \Omega) = E(u)\).

We remark that, apart from \((g2)\), the assumptions on the bulk and surface densities are standard. In particular, the symmetry condition \((g4)\) ensures that \(E\) is well defined since \(|u|\) is reversed if the orientation of \(\nu_u\) is reversed. Assumption \((g2)\) was used in \[12\]. Among others, it includes the case of densities that are ‘monotonic’ in the jump height \(|\zeta|\), see \[12, Remark 3.2\] for further details. In the proof of the main compactness result, this condition is necessary to
pass to piecewise translated configurations without essentially increasing the energy, see Section 3.2 for details.

The following theorem is the main result of the paper.

**Theorem 3.1** (Compactness in $\text{GSBV}^p$). Let $\Omega \subset \Omega' \subset \mathbb{R}^d$ be bounded Lipschitz domains. Let $(E_k)_k \subset \mathcal{E}_{\Omega'}$ and let $(h_k)_k \subset W^{1,p}(\Omega'; \mathbb{R}^m)$ converging in $L^p(\Omega'; \mathbb{R}^m)$ to some $h \in W^{1,p}(\Omega'; \mathbb{R}^m)$ such that $(|\nabla h_k|^p)_k$ are equi-integrable. Consider $(u_k)_k \subset \text{GSBV}^p(\Omega'; \mathbb{R}^m)$ with $u_k = h_k$ on $\Omega' \setminus \overline{\Omega}$ and $\sup_{k \in \mathbb{N}} E_k(u_k) < +\infty$.

Then we find a subsequence (not relabeled), modifications $(y_k)_k \subset \text{GSBV}^p(\Omega'\setminus \Omega)$ satisfying

$$y_k = h_k \text{ on } \Omega' \setminus \overline{\Omega}, \quad E_k(y_k) \leq E_k(u_k) + \frac{1}{k}, \quad \mathcal{L}^d(\{\nabla y_k \neq \nabla u_k\}) \leq \frac{1}{k},$$

and a limiting function $u \in \text{GSBV}^p(\Omega'; \mathbb{R}^m)$ with $u = h$ on $\Omega' \setminus \overline{\Omega}$ such that

(i) $y_k \rightharpoonup u$ in measure on $\Omega'$,

(ii) $\nabla y_k \rightharpoonup \nabla u$ weakly in $L^p(\Omega'; \mathbb{R}^m \times \mathbb{R}^d)$.

Moreover, $\nabla u_k \rightharpoonup \nabla u$ weakly in $L^p(\Omega'; \mathbb{R}^m \times \mathbb{R}^d)$, and

$$\mathcal{H}^{d-1}(J_u) \leq \liminf_{k \to \infty} \mathcal{H}^{d-1}(J_{y_k}) \leq \liminf_{k \to \infty} \mathcal{H}^{d-1}(J_{u_k}).$$

We emphasize that in general it is indispensable to replace the functions $(u_k)_k$ by certain modifications $(y_k)_k$. Consider, e.g., the sequence $u_k = k\chi_U$ for some set $U \subset \Omega$ of finite perimeter. Then $E_k(u_k) \leq c_3 \mathcal{H}^{d-1}(\partial^* U) + c_2 \mathcal{L}^d(\Omega')$ by $(f2)$ and $(g3)$ which is uniformly controlled. However, $u_k$ does not converge in measure on $U$.

The idea in the proof is to construct $y_k$ from $u_k$ by subtracting a function which is piecewise constant (up to a set of small measure). This prevents that the functions ‘escape to infinity’ on subsets which are completely disconnected from the rest of the domain by the jump set. The construction also implies that $\nabla y_k$ coincides with $\nabla u_k$ outside of a small set whose measure vanishes for $k \to \infty$. Thus, $\nabla u_k \rightharpoonup \nabla u$ weakly in $L^p$ also holds for the original sequence $(u_k)_k$. Moreover, by this construction the jump set is asymptotically not increased, see (6).

The result is proved in the following three subsections. In Section 3.2 we first construct piecewise translated configurations $(v_k^\theta)_k$ which are bounded in $L^\infty$ by a constant $C_\theta$ depending on $\theta$ with $C_\theta \to 0$ as $\theta \to 0$. Their energies coincide with the ones of $(u_k)_k$ up to a (small) error of order $\theta$. This construction exploits the monotonicity assumption $(g2)$ and relies on a suitable piecewise Poincaré inequality which is proved in Section 3.3. Finally, in Section 3.4 we define the sequence $(v_k^\theta)$ by letting $\theta \to 0$ and choosing a diagonal sequence in $(v_k^\theta)_k$. The choice of the latter is quite delicate since the $L^\infty$-control $C_\theta$ blows up for $\theta \to 0$. Additional arguments involving Lemma 2.1 are necessary to show that we can apply Theorem 3.1 on $(y_k)_k$.

### 3.2. Piecewise translated configurations

Recall the definition of $\text{GSBV}^p_M(\Omega; \mathbb{R}^m)$ in (2). The goal of this section is to prove the following result.

**Theorem 3.2** (Piecewise translated configurations). Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Let $M > 0$ and $0 < \theta < 1$. Then there exist constants $C_M = C_M(M, \Omega, \{c_j\}_j, p) > 0$ and $C_{\theta,M} = C_{\theta,M}(M, \theta, \Omega, \{c_j\}_j, p) > 0$ such that the following holds: for each $u \in \text{GSBV}^p_M(\Omega; \mathbb{R}^m)$ we find a finite Caccioppoli partition $\Omega = \bigcup_{j=1}^J P_j \cup R$ as well as translations $(t_j)_j$ such that
\[ v := \sum_{j=1}^{J} (u - t_j) \chi_{P_j} \in SBV^p(\Omega; \mathbb{R}^m) \text{ and we have} \]
\[
\begin{align*}
(i) & \quad E(v) \leq E(u) + C_M \theta, \\
(ii) & \quad \mathcal{H}^{d-1}(J_v) \leq \mathcal{H}^{d-1}(J_u) + C_M \theta, \\
(iii) & \quad \|v\|_{L^\infty(\Omega)} \leq C_{\theta,M}, \\
(iv) & \quad \mathcal{L}^d(R) \leq C_M \theta, \\
(v) & \quad \sum_{j=1}^{J} \mathcal{H}^{d-1}(\partial^* P_j) + \mathcal{H}^{d-1}(\partial^* R) \leq C_M
\end{align*}
\]
for all energies \( E \in \mathcal{E}_\Omega \). Moreover, we have \( \{v = 0\} \supset \{u = 0\} \) (up to a set of negligible measure). Finally, for each collection \((t'_j)_{j=1}^{J}\) with \( |t_j - t'_j| \leq \theta^{-1}\|v\|_{L^\infty(\Omega)} \) for \( j = 1, \ldots, J \), the function \( v' := \sum_{j=1}^{J} (u - t'_j) \chi_{P_j} \) also satisfies
\[
E(v') \leq E(u) + C_M \theta \quad \text{for all } E \in \mathcal{E}_\Omega, \quad \mathcal{H}^{d-1}(J_{v'}) \leq \mathcal{H}^{d-1}(J_u) + C_M \theta. \tag{8}
\]

Outside the rest set \( R \), \( v \) arises from \( u \) by subtracting a piecewise constant function. Therefore, we call \( v \) a piecewise translated configuration. The rest set is related to a piecewise Poincaré inequality, see Lemma 5.3 below and the comments thereafter.

A similar result has been derived in [35, Theorem 4.1] for a two-dimensional Griffith model in SBD where piecewise rigid motions are subtracted to obtain uniformly bounded functions. If the density \( g \) in [34] is constant (as in [35]), property (7)(i) follows essentially from (7)(ii). If, however, \( g \) depends explicitly on the jump height, the energy is in general affected by passing to piecewise translated configurations. In this case, the proof is much more delicate: the components \((P_j)_{j=1}^{J}\) and the constants \((t_j)_{j=1}^{J}\) have to be chosen in a careful way, and one needs to use (g2) to ensure the energy estimate (7)(i). This is subject of Lemma 5.5 below which is a refinement of Theorem 2.4. In the proof we will combine the strategy in [35] with ideas inspired by a truncation method for GSBV functions [12].

We remark that truncations, as used in [11,12], also yield a uniform bound of the form (7)(iii). In that case, however, in the energy estimate (7)(i), an additional term \( C_2 \mathcal{L}^{d,\Omega}(\{|u| \geq \lambda\}) \) occurs, where \( \lambda \) represents the level of truncation (see, e.g., [12, Lemma 4.1]). Along a sequence \((u_k)_{k=1}^{\infty}\) from Theorem 3.1, we cannot expect that \( \mathcal{L}^{d,\Omega}(\{|u_k| \geq \lambda\}) \to 0 \) as \( k \to \infty \). Thus, truncations could perturb the energy significantly and are thus not expedient in the present context.

We now formulate a version of Theorem 3.2 for functions satisfying boundary conditions.

**Corollary 3.3** (Piecewise translated configurations with boundary conditions). Let \( \Omega \subset \Omega' \subset \mathbb{R}^d \) be bounded Lipschitz domains. Let \( M > 0, 0 < \theta < 1 \). Then there exist constants \( C_{M} = C_{M}(M,\Omega',\{c_i\}_i,p) > 0 \) and \( C_{\theta,M} = C_{\theta,M}(M,\theta,\Omega',\{c_i\}_i,p) > 0 \) such that the following holds: for each \( h \in W^{1,p}(\Omega';\mathbb{R}^m) \) with \( \|\nabla h\|^p_{L^p(\Omega')} \leq M \) and each \( u \in GSBV^p_M(\Omega';\mathbb{R}^m) \) with \( u = h \) on \( \Omega' \setminus \overline{\Omega} \) we find a finite Caccioppoli partition \( \Omega' = \bigcup_{j=1}^{J} P_j \cup R \) as well as translations \((t_j)_{j=1}^{J}\) such that \( v := h \chi_R + \sum_{j=1}^{J} (u - t_j) \chi_{P_j} \in SBV^p(\Omega';\mathbb{R}^m) \) satisfies \( v = h \) on \( \Omega' \setminus \overline{\Omega} \) and
\[
\begin{align*}
(i) & \quad E(v) \leq E(u) + C_M \theta + C_M \|\nabla h\|^p_{L^p(\Omega')}, \\
(ii) & \quad \mathcal{H}^{d-1}(J_v) \leq \mathcal{H}^{d-1}(J_u) + C_M \theta, \\
(iii) & \quad \|v - h\|_{L^\infty(\Omega')} \leq C_{\theta,M}, \\
(iv) & \quad \mathcal{L}^d(R) \leq C_M \theta, \\
(v) & \quad \sum_{j=1}^{J} \mathcal{H}^{d-1}(\partial^* P_j) + \mathcal{H}^{d-1}(\partial^* R) \leq C_M
\end{align*}
\]
for all energies $E \in \mathcal{E}_{Q'}$. Moreover, for each collection $(t'_j)_{j=1}^J$ with $|t_j - t'_j| \leq \theta^{-1} \|v - h\|_{L^\infty(\Omega)}$ for $j = 1, \ldots, J$, the function $v' := h_{\chi_R} + \sum_{j=1}^J (u - t'_j)\chi_{P_j}$ also satisfies

$$E(v') \leq E(u) + CM\theta + CM\|\nabla h\|_{L^p(\Omega')}^p, \quad \mathcal{H}^{d-1}(J_{\nu^*}) \leq \mathcal{H}^{d-1}(J_u) + CM\theta$$

for all $E \in \mathcal{E}_{Q'}$. Finally, there is at most one component $P_j$ intersecting $\Omega' \setminus \overline{\Omega}$.

The idea in the proof is to apply Theorem 3.2 on $u - h$. The property that at most one component intersects $\Omega' \setminus \overline{\Omega}$ can be seen as follows: for each $P_j$ intersecting $\Omega' \setminus \overline{\Omega}$, we have $t_j = 0$ since $u = v = h$ on $\Omega' \setminus \overline{\Omega}$. Thus, if different components intersected $\Omega' \setminus \overline{\Omega}$, they could simply be combined to just one component. We again remark that truncations [11, 12] can not be applied here since they in general do not preserve boundary conditions.

Corollary 3.4 (Approximation by $L^p$ functions). Let $\Omega \subset \Omega' \subset \mathbb{R}^d$ be bounded Lipschitz domains. Let $h \in W^{1,p}(\Omega'; \mathbb{R}^m)$ and $E \in \mathcal{E}_{Q'}$. Then for each $u \in GSBV^p(\Omega'; \mathbb{R}^m)$ with $u = h$ on $\Omega' \setminus \overline{\Omega}$ we find a sequence $(u_k) \subset GSBV^p(\Omega'; \mathbb{R}^m) \cap L^p(\Omega'; \mathbb{R}^m)$ with $u_k \to h$ on $\Omega' \setminus \overline{\Omega}$ such that $u_k \to u$ in measure on $\Omega'$ and $\limsup_{k \to \infty} E(u_k) \leq E(u)$.

A key ingredient for the proof of Theorem 3.2 Corollary 3.4 will be the following result, which we will use in Section 4.2.

**Lemma 3.5** (Piecewise Poincaré inequality with additional control on translations). Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Let $\alpha \geq 1$ and $0 < \theta < 1$. Then there exist constants $C_{\Omega} = C_{\Omega}(\Omega) \geq 1$ and $C_{\theta, \alpha} = C_{\theta, \alpha}(\theta, \alpha) > 0$ such that the following holds: for each $u \in GSBV^p(\Omega; \mathbb{R}^m)$ we find a finite Caccioppoli partition $\Omega = \bigcup_{j=1}^J P_j \cup R_1 \cup R_2$ with

(i) $\mathcal{L}^d(R_1 \cup R_2) \leq C_{\Omega}\theta \mathcal{H}^{d-1}(J_u \cup \partial \Omega),$  

(ii) $\mathcal{H}^{d-1}(\partial^* R_1) \leq C_{\Omega}\theta \mathcal{H}^{d-1}(J_u \cup \partial \Omega),$  

(iii) $\sum_{j=1}^J \mathcal{H}^{d-1}(\partial^* P_j) + \mathcal{H}^{d-1}(\partial^* R_2) \leq C_{\Omega} \mathcal{H}^{d-1}(J_u \cup \partial \Omega)$ \hspace{1cm} (11)

as well as translations $(b_j)_{j=1}^J$ and $\lambda_{\theta, \alpha} \in [1, C_{\theta, \alpha}]$ such that

(i) $\|u - b_j\|_{L^\infty(P_j)} \leq \lambda_{\theta, \alpha} \|\nabla u\|_{L^1(\Omega)}$ \hspace{1cm} for $1 \leq j \leq J,$

(ii) $\min_{1 \leq j \leq J} \inf \{|u(x) - b_j| : x \in R_2\} \geq \alpha \lambda_{\theta, \alpha} \|\nabla u\|_{L^1(\Omega)},$

(iii) $|b_i - b_j| > \alpha \lambda_{\theta, \alpha} \|\nabla u\|_{L^1(\Omega)} \hspace{1cm} for \hspace{0.5cm} 1 \leq i < j \leq J.$ \hspace{1cm} (12)

We briefly comment on the statement of Lemma 3.5. Property (12)(i) is an estimate of Poincaré-type on the components $P_j$. In contrast to Theorem 2.5 the estimate has the additional property that the difference of the translations can be controlled from below in terms of the parameter $\alpha$, see (12)(iii). The choice $\alpha \geq 1$ then implies that the values of $u$ on different components $(P_j)_{j=1}^J$ are ‘well separated’, see (12)(i),(iii). This will eventually allow us to exploit (g2) in the proof of Theorem 3.2 and to show the energy estimate (17)(i).

The main idea to achieve (12)(i),(iii) is as follows: note that the components and translations given by Theorem 2.5 (or even just subsets of them) do possibly not satisfy (12)(iii). The strategy is to sort the indices into different groups by means of Lemma 3.7 below such that (a) the translations in each group are close to each other (in terms of a constant $\lambda_{\theta, \alpha}$), and (b) the translations in different groups differ very much (in terms of $\alpha \lambda_{\theta, \alpha}$). Then a new partition is defined by combining the components of each group and by defining new translations accordingly.
We point out that the grouping of the indices and the explicit choice of \( \lambda_{\theta, \alpha} \) depend on \( u \), but \( \lambda_{\theta, \alpha} \) always lies in the interval \([1, C_{\theta, \alpha}]\) independent of \( u \).

Note that this refined Poincaré estimate comes at the expense of two rest sets \( R_1 \) and \( R_2 \). For \( R_2 \) we have (11)(ii) which again means that the values of \( u \) on each component \( P_j \) and \( R_2 \) are 'well separated'. Finally, for \( R_1 \) we will exploit that the \( H^{d-1} \)-measure of its boundary is small in terms of \( \theta \), cf. (11)(ii). We remark that the necessity of rest sets is obvious if one considers functions with dense image in \( \mathbb{R}^m \): in fact, the image of \( u \) restricted to \( \bigcup_{j=1}^J P_j \) is contained in \( \bigcup_{j=1}^J B_r(b_j) \) with \( r = \lambda_{\theta, \alpha} \| \nabla u \|_{L^1(\Omega)} \) which does not cover \( \mathbb{R}^m \) for \( \alpha \geq 2 \), cf. (12)(i),(iii).

We defer the proof of Lemma 3.5 to Section 3.3 and proceed with the proofs of Theorem 3.2 Corollary 3.4.

**Proof of Theorem 3.2**. We apply Lemma 3.5 on \( u \) for \( \alpha = 8\theta^{-1}c_3 + 6 \) to obtain a partition of \( \Omega \), consisting of the sets \( (P_j)' \) and \( \bigcup_{j=1}^J P_j \) and \( (b_j)' \) and to get translations \( (b_j)' \) such that (11) hold. Then (11) and the fact that \( u \in \text{GSBV}^p(\Omega; \mathbb{R}^m) \) imply (7)(iv) and (7)(v). We define \( t_j = b_j \), if \( \| b_j \| > \lambda_{\theta, \alpha} \| \nabla u \|_{L^1(\Omega)} \) and \( t_j = 0 \) else. Note that at most one \( t_j \) is zero. Indeed, \( t_j = t_j' = 0 \) for \( j \neq j' = 2 \) would imply

\[
|b_j - b_j'| < 2 \lambda_{\theta, \alpha} \| \nabla u \|_{L^1(\Omega)}.
\]

In view of \( \alpha \geq 2 \), however, this contradicts (12)(iii). Define \( v := \sum_{j=1}^J (u - t_j) \chi_{P_j} \). We show \( \mathcal{L}^d(\{ v = 0 \}) = 0 \). Since \( v = 0 \) on \( R \), it suffices to show \( \mathcal{L}^d(\{ v = 0 \} \cap P_j) = 0 \) for each \( j = 1, \ldots, J \). Suppose that \( \mathcal{L}^d(\{ v = 0 \} \cap P_j) > 0 \). Then \( u_j = \lambda_{\theta, \alpha} \| \nabla u \|_{L^1(\Omega)} \) by (12)(i), i.e., \( t_j = 0 \). This implies \( v = u \) on \( P_j \) and thus \( v = 0 \) \( \cap P_j = \{ u = 0 \} \cap P_j \).

By (12) and the fact that \( |t_i - b_j| < \lambda_{\theta, \alpha} \| \nabla u \|_{L^1(\Omega)} \) for \( j = 1, \ldots, J \) we obtain

\[
\begin{align*}
(1) \quad & ||v||_{L^\infty(\Omega)} \leq 2 \lambda_{\theta, \alpha} \| \nabla u \|_{L^1(\Omega)} \\
(2) \quad & \min_{1 \leq j \leq J} \text{ess inf} \{ |u(x) - t_j| : x \in R \} \geq (\alpha - 1) \lambda_{\theta, \alpha} \| \nabla u \|_{L^1(\Omega)}, \\
(3) \quad & |t_i - t_j| \geq (\alpha - 2) \lambda_{\theta, \alpha} \| \nabla u \|_{L^1(\Omega)} \quad \text{for } 1 \leq i < j \leq J.
\end{align*}
\]

Note that \( \| \nabla u \|_{L^1(\Omega)} \leq C \| \nabla u \|_{L^p(\Omega)} \leq CM^{1/p} \) by Hölder's inequality for a constant \( C \) depending on \( \Omega \). This along with (12)(i) and \( \lambda_{\theta, \alpha} \leq C_{\theta, \alpha} \) yields (13)(iii) for \( C_{\theta, \alpha} \) sufficiently large. The fact that \( u \in \text{GSBV}^p(\Omega; \mathbb{R}^m) \), (13)(iii), and (7)(v) yields \( v \in \text{GSBV}^p(\Omega; \mathbb{R}^m) \cap L^\infty(\Omega; \mathbb{R}^m) \).

It remains to show (7)(i),(ii) and (8). Fix \( E \in \mathcal{E}(\Omega) \). For the bulk integral we obtain by (f2), (7)(iv), and the fact that \( \nabla v = \nabla u \) on \( \Omega \setminus R \)

\[
\int_{\Omega} f(x, \nabla v) \, dx = \int_{\Omega \setminus R} f(x, \nabla v) \, dx + \int_{R} f(x, 0) \, dx \leq \int_{\Omega \setminus R} f(x, \nabla u) \, dx + c_2 \mathcal{L}^d(R)
\]

\[
\leq \int_{\Omega} f(x, \nabla u) \, dx + C M \theta.
\]

(As usual, the generic constant \( C_M \) may vary from step to step.) For brevity we define \( \Gamma := \left( \bigcup_{j=1}^J \partial^* P_j \cup \partial^* R \right) \cap \Omega \). We can split the surface integral into

\[
\int_{J \setminus \Gamma} g(x, [v], \nu_v) \, d\mathcal{H}^{d-1} = T_1 + T_2 := \int_{J \setminus \Gamma} g(x, [v], \nu_v) \, d\mathcal{H}^{d-1} + \int_{J \setminus \Gamma} g(x, [v], \nu_v) \, d\mathcal{H}^{d-1}.
\]
To estimate $T_2$, we split $\Gamma$ into the sets (a) $\partial^* P_i \cap \partial^* P_j$, $1 \leq i < j \leq J$, (b) $\partial^* P_j \cap \partial^* R_2$, $1 \leq j \leq J$, and (c) $\partial^* P_j \cap \partial^* R_1$, $1 \leq j \leq J$.

(a) First, (12)(i),(iii) show that $J_u \supset \partial^* P_i \cap \partial^* P_j$ up to an $H^{d-1}$-negligible set. We choose the orientation of $\nu(x)$ for $x \in \partial^* P_i \cap \partial^* P_j$ such that $u^+(x)$ coincides with the trace of $u_1 \chi_{P_i}$ at $x$ and $u^-(x)$ coincides with the trace of $u_1 \chi_{P_j}$ at $x$. (The traces have to be understood in the sense of [3] Theorem 3.77.) Moreover, we suppose that $\nu = \nu_\alpha$ on $J_v \cap \partial^* P_i \cap \partial^* P_j$. Then we obtain by definition

$$[v](x) = u^+(x) - v^-(x) = (u^+(x) - t_i) - (u^-(x) - t_j) = [u](x) - (t_i - t_j)$$
for $H^{d-1}$-a.e. $x \in J_v \cap \partial^* P_i \cap \partial^* P_j$. By (13)(i),(iii) we get

$$||v||_1 = ||u(x) - (t_i - t_j)|| \leq 2 ||v||_{L^\infty(\Omega)} \leq 4 \lambda_{\theta,\alpha} ||\nabla u||_{L^1(\Omega)},$$

$$||u||_1 \geq |t_i - t_j| - 2 ||v||_{L^\infty(\Omega)} \geq (\alpha - 6) \lambda_{\theta,\alpha} ||\nabla u||_{L^1(\Omega)}.$$ Using $\alpha = 8 \theta^{-1} c_3 + 6$ and again (13)(i), we derive for $H^{d-1}$-a.e. $x \in J_v \cap \partial^* P_i \cap \partial^* P_j$

$$||v(x)|| \leq ||v||_1 + 2 \theta^{-1} ||v||_{L^\infty(\Omega)} \leq 8 \theta^{-1} \lambda_{\theta,\alpha} ||\nabla u||_{L^1(\Omega)} \leq \frac{8 \theta^{-1}}{\alpha - 6} ||u||_1 = \frac{1}{c_3} ||u||_1.$$ (We include an additional addend $\frac{\theta}{\alpha - 6} ||v||_{L^\infty(\Omega)}$ since this will be convenient for the proof of (8).)

(b) Similarly as before, (12) (i),(ii) show that $J_u \supset \partial^* P_j \cap \partial^* R_2$ up to an $H^{d-1}$-negligible set. We choose the orientation of $\nu(x)$ for $x \in \partial^* P_j \cap \partial^* R_2$ such that $u^+(x)$ coincides with the trace of $u_1 \chi_{P_i}$ at $x$ and $u^-(x)$ coincides with the trace of $u_1 \chi_{R_2}$ at $x$. Moreover, we suppose that $\nu = \nu_\alpha$ on $J_v \cap \partial^* P_j \cap \partial^* R_2$. Since $v = 0$ on $R_2$, we then obtain

$$[v](x) = u^+(x) - u^-(x) - t_j$$
for $H^{d-1}$-a.e. $x \in J_v \cap \partial^* P_j \cap \partial^* R_2$. By (13) we get

$$||v||_1 = ||u^+(x) - t_j|| \leq ||v||_{L^\infty(\Omega)} \leq 2 \lambda_{\theta,\alpha} ||\nabla u||_{L^1(\Omega)},$$

$$||u||_1 \geq |u^-(x) - t_j| - |u^+(x) - t_j| \geq |u^-(x) - t_j| - 2 ||v||_{L^\infty(\Omega)} \geq (\alpha - 3) \lambda_{\theta,\alpha} ||\nabla u||_{L^1(\Omega)}.$$ Recalling $\alpha = 8 \theta^{-1} c_3 + 6$, we deduce for $H^{d-1}$-a.e. $x \in J_v \cap \partial^* P_j \cap \partial^* R_2$

$$||v||_1 \leq ||v||_1 + \theta^{-1} ||v||_{L^\infty(\Omega)} \leq 4 \theta^{-1} \lambda_{\theta,\alpha} ||\nabla u||_{L^1(\Omega)} \leq \frac{4 \theta^{-1}}{\alpha - 3} ||u||_1 = \frac{1}{c_3} ||u||_1.$$ (As before, the additional addend $\theta^{-1} ||v||_{L^\infty(\Omega)}$ will be needed for the proof of (8).)

(c) Finally, for $\partial^* R_1$ we use (11)(ii) and $H^{d-1}(J_u) \leq M$ to find

$$H^{d-1}(\partial^* R_1) \leq C_M \theta.$$ (19)

We are now in a position to show (7)(i). From (17) - (18) we get that $c_3 ||v||_1 \leq ||u||_1$ for $H^{d-1}$-a.e. $x \in (\Gamma \cap J_v) \setminus \partial^* R_1$. Using this, $\nu = \nu_\alpha H^{d-1}$-a.e. on $(\Gamma \cap J_v) \setminus \partial^* R_1$, and (19), we
derive by (g2) and (g3)

\[ T_2 = \int_{(J_u \cap F) \cap \partial^* R_1} g(x, [v], \nu_v) \, d\mathcal{H}^{d-1} \leq \int_{(J_u \cap F) \cap \partial^* R_1} g(x, [v], \nu_v) \, d\mathcal{H}^{d-1} + c_5 C_M \theta \leq \int_{J_u \cap F} g(x, [v], \nu_v) \, d\mathcal{H}^{d-1} + C_M \theta. \]

This along with (15), (16) yields

\[ \int_{J_u} g(x, [v], \nu_v) \, d\mathcal{H}^{d-1} \leq \int_{J_u} g(x, [u], \nu_u) \, d\mathcal{H}^{d-1} + C_M \theta. \quad (20) \]

Now (14) and (20) give (i). Choosing specifically \( \vartheta = c_4 \), (20) also yields (ii). Finally, the same calculation can be repeated for \( v' := \sum_{j=1}^J (u - t_j') \chi_{P_j} \), where \( (t_j')_{j=1}^J \) satisfy \( |t_j - t'_j| \leq \theta^{-1} \|v\|_{L^\infty(F)} \) for \( j = 1, \ldots, J \). Indeed, in this case we still have \( c_3 |v'(x)| \leq \|u(x)\| \) for \( \mathcal{H}^{d-1}\)-a.e. \( x \in (F \cap J_{v'}) \setminus \partial^* R_1 \), see (17) and (18).

\[ \square \]

**Remark 3.6.** We recall from the proof that at most one translation \( t_j \) is zero. Say, without restriction, \( t_j = 0 \). By (13)(i), (iii) we then find for all \( j \geq 2 \) and almost all \( x \in P_j \)

\[ |u(x)| = |u(x) - t_1| \leq |t_j - t_1| - |u(x) - t_j| \geq (\alpha - 4) \lambda_{\theta, \alpha} \|\nabla u\|_{L^1(F)} \geq c_3 \theta^{-1} \|\nabla u\|_{L^1(F)} \]

where the last step follows from \( \alpha \geq 4 + c_3/\theta \) and \( \lambda_{\theta, \alpha} \geq 1 \) (see Lemma 3.5).

**Proof of Corollary 3.3.** As \( u \in GSBV^p_M(\Omega'; \mathbb{R}^m) \) and \( \|\nabla h\|_{L^p(\Omega')} \leq M \), we observe that \( u - h \in GSBV^p_M(\Omega'; \mathbb{R}^m) \). We apply Theorem 3.2 on \( u - h \) and find \( \bar{v} : = \sum_{j=1}^J (u - h - t_j) \chi_{P_j} \) such that (7)(ii)-(v) hold with \( \bar{v} \) in place of \( v \). We also note that (20) is satisfied with \( \bar{v} \) in place of \( v \) since \( J_{u-h} = J_u \) and \( |u - h| = |u| \) on \( J_u \). As \( u - h = 0 \) on \( \Omega' \setminus \overline{\Omega} \) and \( \{\bar{v} = 0\} \supseteq \{u - h = 0\} \), we get \( \bar{v} = 0 \) on \( \Omega' \setminus \overline{\Omega} \). This implies that \( t_j = 0 \) for each \( P_j \) intersecting \( \Omega' \setminus \overline{\Omega} \). As at most one \( t_j \) is zero, see Remark 3.6, at most one component \( P_j \) intersects \( \Omega' \setminus \overline{\Omega} \).

We define \( v = \bar{v} + h = h \chi_R + \sum_{j=1}^J (u - t_j) \chi_{P_j} \in GSBV^p(\Omega'; \mathbb{R}^m) \). Clearly, \( v = h \) on \( \Omega' \setminus \overline{\Omega} \) as \( \bar{v} = 0 \) on \( \Omega' \setminus \overline{\Omega} \). Properties (9)(ii)-(v) follow directly from (7)(ii)-(v) (with \( \bar{v} \) in place of \( v \)). To see (9)(i), we compute by (f2), (9)(iv), and (20) (with \( \bar{v} \) in place of \( v \))

\[
E(v) = \int_{\Omega'} f(x, \nabla v) \, dx + \int_{J_u} g(x, [v], \nu_v) \, d\mathcal{H}^{d-1} \\
\leq \int_{\Omega'} f(x, \nabla u) \, dx + \int_R f(x, \nabla h) \, dx + \int_{J_u} g(x, [\bar{v}], \nu_v) \, d\mathcal{H}^{d-1} \\
\leq \int_{\Omega'} f(x, \nabla u) \, dx + c_2(L^4(R) + \|\nabla h\|_{L^p(\Omega')}^p) + \int_{J_u} g(x, [u], \nu_u) \, d\mathcal{H}^{d-1} + C_M \theta \\
\leq E(u) + C_M \theta + M \|\nabla h\|_{L^p(\Omega')}^p
\]

for all \( E \in \mathcal{E}_{\Omega'} \). Similarly, also (10) follows as (20) is still applicable in this case.

\[ \square \]

**Proof of Corollary 3.4.** Let \( M \) large enough such that \( \|\nabla h\|_{L^p(\Omega')} \leq M \), \( u \in GSBV^p_M(\Omega'; \mathbb{R}^m) \). We apply Corollary 3.3 for \( u \) and \( \theta_k = 1/k \) for each \( k \in \mathbb{N} \) to obtain functions \( u_k := h \chi_{R_k} + \sum_{j=1}^k (u - t^k_j) \chi_{P^k_j} \). They satisfy \( u_k = h \) on \( \Omega' \setminus \overline{\Omega} \) and \( u_k \in SBV^p(\Omega'; \mathbb{R}^m) \cap L^p(\Omega'; \mathbb{R}^m) \) by (9)(iii). Moreover, we have \( \limsup_{k \to \infty} E(u_k) \leq E(u) \) by (9)(i) and (9)(iv).

We need to check that \( u_k \to u \) in measure on \( \Omega' \). For \( k \) sufficiently large such that \( L^4(R_k) < L^4(\Omega' \setminus \overline{\Omega}) \), exactly one component intersects \( \Omega' \setminus \overline{\Omega} \), say without restriction \( P_{j_k} \). As \( u_k = h \)
on \(\Omega \setminus \overline{\Omega}\), this implies \(t_k^i = 0\) and thus \(u_k = u\) on \(P_k^i\). By Remark 3.6 (applied on \(u - h\)) and \(\theta_k = 1/k\) we then get \(|u(x) - h(x)| \geq c_3 k ||\nabla u - \nabla h||_{L^1(\Omega')}\) for a.e. \(x \in \Omega' \setminus (R^k \cup P_k^i)\). As \(u - h\) is finite almost everywhere, we find \(L^d(\Omega' \setminus (R^k \cup P_k^i)) \to 0\). (Note that, possibly slightly modifying \(h\) inside \(\Omega\), it is not restrictive to suppose \(\|\nabla u - \nabla h\|_{L^1(\Omega')} > 0\).) This along with \(L^d(R^k) \to 0\) by (8)(iv) yields \(L^d(\Omega' \setminus P_k^i) \to 0\). As \(u_k = u\) on \(P_k^i\), we conclude \(u_k \to u\) in measure on \(\Omega'\).

### 3.3. Piecewise Poincaré inequality

This section is devoted to the proof of Lemma 3.5. The reader may wish to skip this section on first reading and to proceed directly with the proof of Theorem 3.1 in Section 3.4. As a preparation, we state the following elementary property.

**Lemma 3.7** (Covering with balls). Let \(N \in \mathbb{N}, \gamma \geq 2\), and \(R_0 > 0\). Then each set of points \(\{x_1, \ldots, x_n\} \subset \mathbb{R}^m, n \leq N\), can be covered by finitely many pairwise disjoint balls \(\{B_{r_k}(y_k)\}_{k=1}^M\), \(M \leq N\), \((y_k)_{k=1}^M \subset \mathbb{R}^m\), satisfying

\[
r_k \in [R_0, (2\gamma)^N R_0]\quad \text{for } k = 1, \ldots, M, \quad |y_i - y_j| \geq \gamma \max_{k=1,\ldots,M} r_k \quad \text{for } 1 \leq i < j \leq M.
\]

**Proof.** We prove the lemma by induction. Suppose that in step \(l \in \mathbb{N}_0\) there exist finitely many balls \(\{B_{r_k}(y_k)\}_{k=1}^{M_l}, M_l \leq N - l\), which cover \(\{x_1, \ldots, x_n\}\) and satisfy \(r_k \in [R_0, (2\gamma)^l R_0]\). For step \(l = 0\), we can take the balls centered at \(\{x_1, \ldots, x_n\}\) with radius \(R_0\).

If in some iteration step \(l \leq N - 1\) we have

\[
|y_i - y_j| > \gamma \max_{k=1,\ldots,M_l} r_k^i \quad \text{for } 1 \leq i < j \leq M_l,
\]

we have found a collection of balls covering \(\{x_1, \ldots, x_n\}\) and satisfying (21). We also observe that (21) and \(\gamma \geq 2\) induce that the balls are pairwise disjoint. Otherwise, it is not restrictive to suppose that \(|y_i - y_j| \leq \gamma \max_{k=1,\ldots,M_l} r_k\). Letting \(r = 2\gamma \max_{k=1,\ldots,M_l} r_k \leq (2\gamma)^{l+1} R_0\), we observe that \(B_{r_k}(y_k) \supset B_{r_k}(y_k) \cup B_{r_k}(y_k)\). We let \(B_{r_k}(y_k)\) and \(B_{r_k}(y_k)\), \(3 \leq k \leq M_l\), be the collection of balls in iteration step \(l + 1\), whose number is \(M_l - 1\) and thus at most \(N - (l + 1)\).

Now we observe that after at most \(N - 1\) iteration steps we have found a collection of balls such that (22) holds. Indeed, in step \(N - 1\), the collection consists only of one ball. \(\square\)

We now proceed with the proof of Lemma 3.5.

**Proof of Lemma 3.5.** Let \(u \in \text{GSBV}^p(\Omega; \mathbb{R}^m), \alpha \geq 1\), and \(0 < \theta < 1\) be given. Let \(C_0 \geq 1\) be the constant from Theorem 2.5. Define \(\beta = 6 \alpha (4\alpha)^{\theta-d}\) for brevity. We first use Theorem 2.5 to define an auxiliary partition and corresponding translations such that estimates of type (12)(i),(ii) are already satisfied (Step 1). Subsequently, we apply Lemma 3.7 to pass to a coarser partition and we define the translations suitably to ensure also (12)(iii) (Step 2).

**Step 1 (Auxiliary partition).** The goal of this step is to find two disjoint rest sets \(R_1, R_2 \subset \Omega\) satisfying

\[
(i) \quad L^d(R_1 \cup R_2) \leq C_{\Omega} \theta \mathcal{H}^{d-1}(J_u \cup \partial \Omega),
\]

\[
(ii) \quad \mathcal{H}^{d-1}(\partial^* R_1) \leq C_{\Omega} \theta \mathcal{H}^{d-1}(J_u \cup \partial \Omega), \quad \mathcal{H}^{d-1}(\partial^* R_2) \leq C_{\Omega} \mathcal{H}^{d-1}(J_u \cup \partial \Omega),
\]

(23)
a finite Caccioppoli partition $\Omega = \bigcup_{j=1}^{J_{\alpha}} P_{i}^a \cup R_1 \cup R_2$ for an index $J_{\alpha} \in \mathbb{N}$ with $J_{\alpha} \leq \theta^{-d}$, and corresponding translations $(b_{i}^j)^{J_{\alpha}}_{j=1}$ such that we have with $v_{i} := \sum_{j=1}^{J_{\alpha}} (u - b_{i}^j) \chi_{P_{i}^a}$

\[(i) \quad \sum_{j=1}^{J_{\alpha}} \mathcal{H}^{d-1}(\partial^s P_{i}^a) \leq C_{\Omega} \mathcal{H}^{d-1}(J_{\alpha} \cup \partial \Omega),
\]

\[(ii) \quad \|v_{i}\|_{L^\infty(\Omega)} \leq 4C_{\Omega} \beta K_a \|
abla u\|_{L^1(\Omega)}, \]

\[(iii) \quad \min_{1 \leq i \leq J_{\alpha}} \essinf\{|u(x) - b_{i}^j| : x \in R_2\} \geq 2C_{\Omega} \beta K_a + 1 \|
abla u\|_{L^1(\Omega)} \quad (24)\]

for some $K_{\theta} \in \mathbb{N}$, $K_{\theta} \leq \theta^{-1}$. Here, $C_{\Omega} > 0$ is a constant only depending on $\Omega$.

**Proof of Step 1.** We apply Theorem 2.3 on $u$ to find an ordered Caccioppoli partition $(P_{i}^j)^{J_{\alpha}}_{j=1}$ of $\Omega$ and corresponding translations $(b_{i}^j)^{J_{\alpha}}_{j=1} \subset \mathbb{R}^m$ such that

\[(i) \quad \sum_{j=1}^{\infty} \mathcal{H}^{d-1}(\partial^s P_{i}^j) \leq 2\mathcal{H}^{d-1}(J_{\alpha} \cup \partial \Omega) + 1 \leq C_{\Omega} \mathcal{H}^{d-1}(J_{\alpha} \cup \partial \Omega),
\]

\[(ii) \quad \|u - b_{i}^j\|_{L^\infty(P_{i}^j)} \leq C_{\Omega} \|
abla u\|_{L^1(\Omega)} \quad \text{for all } j \in \mathbb{N}, \quad (25)\]

where $C_{\Omega}$ depends only on $\Omega$. Let $J_{\alpha} \in \mathbb{N}$ be the largest index such that $\mathcal{L}^d(P_{i}^j) \geq \theta^d \mathcal{L}^d(\Omega)$. Then $J_{\alpha} \leq \theta^{-d}$. (Recall that the partition is assumed to be ordered.) By the isoperimetric inequality and (23) we have

\[
\sum_{j > J_{\alpha}} \mathcal{L}^d(P_{i}^j) \leq (\theta^d \mathcal{L}^d(\Omega))^{1/d} \sum_{j > J_{\alpha}} \mathcal{L}^d(P_{i}^j)^{1-1/d} \leq C_{\Omega} \theta \sum_{j \geq 1} \mathcal{H}^{d-1}(\partial^s P_{i}^j)
\]

\[
\leq C_{\Omega} \theta \mathcal{H}^{d-1}(J_{\alpha} \cup \partial \Omega). \quad (26)
\]

We now introduce a decomposition of the components $(P_{i}^j)_{J_{\alpha} > J_{\alpha}}$ according to the difference of the translations: for $k \in \mathbb{N}$ we define the sets of indices

\[
\mathcal{J}^0 = \{ j > J_{\alpha} : \min_{1 \leq i \leq J_{\alpha}} \|b_{i}^j - b_{i}^j\| \leq 3C_{\Omega} \beta \|
abla u\|_{L^1(\Omega)} \},
\]

\[
\mathcal{J}^k = \{ j > J_{\alpha} : 3C_{\Omega} \beta \|
abla u\|_{L^1(\Omega)} < \min_{1 \leq i \leq J_{\alpha}} \|b_{i}^j - b_{i}^j\| \leq 3C_{\Omega} \beta \|
abla u\|_{L^1(\Omega)} \}. \quad (27)
\]

Let $s_k = \sum_{j \in \mathcal{J}^k} \mathcal{H}^{d-1}(\partial^s P_{i}^j)$ for $k \in \mathbb{N}_0$. In view of (25) (i), we find some $K_{\theta} \in \mathbb{N}$, $K_{\theta} \leq \theta^{-1}$, such that $s_{K_{\theta}} \leq C_{\Omega} \theta \mathcal{H}^{d-1}(J_{\alpha} \cup \partial \Omega)$. Define

\[
R_1 := \bigcup_{j \in \mathcal{J}^{K_{\theta}}} P_{i}^j, \quad R_2 = \bigcup_{k > K_{\theta}} \bigcup_{j \in \mathcal{J}^k} P_{i}^j. \quad (28)
\]

The choice of $K_{\theta}$, (25) (i), and (26) show (23). We introduce a Caccioppoli partition $(P_{i}^j)^{J_{\alpha}}_{j=1}$ of $\Omega \setminus (R_1 \cup R_2)$ by combining different components of $(P_{i}^j)_{J_{\alpha} > 1}$: we decompose the indices in $\bigcup_{k=0}^{K_{\theta}-1} \mathcal{J}^k$ into sets $\mathcal{I}_i$ with $\bigcup_{i=1}^{J_{\alpha}} \mathcal{I}_i = \bigcup_{k=0}^{K_{\theta}-1} \mathcal{J}^k$ according to the following rule: an index $j \in \mathcal{J}^k$ is assigned to $\mathcal{I}_i$ when $i$ is the smallest index in $\{1, \ldots, J_{\alpha}\}$ such that the minimum in (27) is attained. Let $P_{i}^j = P_{i}^j \cup \bigcup_{i \in \mathcal{I}_i} P_{i}^j$ for $1 \leq i \leq J_{\alpha}$ and observe that the sets form a partition of $\Omega \setminus (R_1 \cup R_2)$.

We now show (24). First, (24) (i) holds by (25) (i). We define $b_{i}^j = b_{i}^j$ for $1 \leq i \leq J_{\alpha}$. Let $v' := u - \sum_{j \geq 1} b_{i}^j \chi_{P_{i}^a}$ and $v_{a} := \sum_{i=1}^{J_{\alpha}} (u - b_{i}) \chi_{P_{i}^a}$. We find by the definition of $\mathcal{I}_i \subset \bigcup_{k=0}^{K_{\theta}-1} \mathcal{J}^k$ and (27)

\[
\|v_{a} - v'\|_{L^\infty(P_{i}^a)} \leq 3C_{\Omega} \beta K_a \|
abla u\|_{L^1(\Omega)}
\]

for $i = 1, \ldots, J_{\alpha}$. By (26) (ii) we compute for each $i = 1, \ldots, J_{\alpha}$

\[
\|v_{a}\|_{L^\infty(P_{i}^a)} \leq \|v_{a} - v'\|_{L^\infty(P_{i}^a)} + \|v'\|_{L^\infty(P_{i}^a)} \leq 3C_{\Omega} \beta K_a \|
abla u\|_{L^1(\Omega)} + C_{\Omega} \|
abla u\|_{L^1(\Omega)} \leq 4C_{\Omega} \beta K_a \|
abla u\|_{L^1(\Omega)}.
\]
This yields (24)(ii). Finally, we show (24)(iii). Fix \(1 \leq i \leq J_\alpha\). For \(L^d\)-a.e. \(x \in R_2\), we choose \(j > J_\alpha\) such that \(x \in P_j^a \subset R_2\) (recall (28)). Then we compute by (25)(ii), (27), and the fact \(j \in \bigcup_{k > K_\alpha} J^k\)

\[
|u(x) - b'_i| \geq |b''_i - b'_i| - |u(x) - b'_i| = |b'_i - b'_j| - |u(x) - b'_j| \\
\geq 3C_0\beta K_\alpha + 1 \|\nabla u\|_{L^1(\Omega)} - C_0\|\nabla u\|_{L^1(\Omega)} \geq 2C_0\beta K_\alpha + 1 \|\nabla u\|_{L^1(\Omega)}.
\]

This concludes the proof of Step 1.

Step 2 (Passage to coarser partition). We now pass to a coarser Caccioppoli partition: there exists a partition \(\Omega = \bigcup_{j=1}^J P_j \cup R_1 \cup R_2\) with \(J \leq J_\alpha \leq \theta^{-d}\) and \(\bigcup_{j=1}^J \partial^* P_j \subset \bigcup_{i=1}^{J_\alpha} \partial^* P_i^a\) up to an \(H^{d-1}\)-negligible set, as well as corresponding translations \((b_j)^i_{j=1}\) such that for some \(\lambda_{\theta, \alpha} > 0\)

\[
(i) \quad \sum_{j=1}^J \mathcal{H}^{d-1}(\partial^* P_j) \leq C_\Omega \mathcal{H}^{d-1}(J_u \cup \partial \Omega), \\
(ii) \quad \|u - b_j\|_{L^\infty(P_j)} \leq \lambda_{\theta, \alpha} \|\nabla u\|_{L^1(\Omega)} \quad \text{for } 1 \leq j \leq J, \\
(iii) \quad \min_{1 \leq j \leq J} \text{ ess inf}\{|u(x) - b_j| : x \in R_2\} \geq \lambda_{\theta, \alpha} \|\nabla u\|_{L^1(\Omega)}, \\
(iv) \quad |b_i - b_j| > \alpha \lambda_{\theta, \alpha} \|\nabla u\|_{L^1(\Omega)} \quad \text{for } 1 \leq i < j \leq J. \quad (29)
\]

**Proof of Step 2.** We apply Lemma 5.7 on the points \((b_i^a)^i_{i=1}\) for \(R_0 = 4C_0\beta K_\alpha \|\nabla u\|_{L^1(\Omega)}\) and \(\gamma = 2\alpha\). We obtain finitely many pairwise disjoint balls \(\{B_r(y_j)\}^j_{j=1}\), \(J \leq J_\alpha \leq \theta^{-d}\), which cover \((b_i^a)^i_{i=1}\) and satisfy

\[
r_j \in [4C_0\beta K_\alpha \|\nabla u\|_{L^1(\Omega)}, 4C_0\beta K_\alpha \|\nabla u\|_{L^1(\Omega)}(4\alpha)^J] \quad \text{for } j = 1, \ldots, J \quad (30)
\]

as well as

\[
|y_i - y_k| > 2\alpha \max_{j=1, \ldots, J} r_j \quad \text{for } 1 \leq i < k \leq J. \quad (31)
\]

We set

\[
\lambda_{\theta, \alpha} = 2 \|\nabla u\|_{L^1(\Omega)}^{-1} \max_{j=1, \ldots, J} r_j \quad (32)
\]

and note that \(8C_0\beta K_\alpha \leq \lambda_{\theta, \alpha} \leq 8\delta C_0\beta K_\alpha\) by (30), where for brevity we set \(\delta := (4\alpha)^\theta\). As the balls are pairwise disjoint, each \(b_i^a\) is contained in exactly one ball. We define \(b_j = y_j\) for \(j = 1, \ldots, J\) and introduce the sets

\[
\mathcal{L}_j = \{i : b_i^a \in B_r(y_j)\}, \quad P_j = \bigcup_{i \in \mathcal{L}_j} P_i^a. \quad (33)
\]

Then the components \((P_j)_{j=1}^J\) form a Caccioppoli partition of \(\Omega \setminus (R_1 \cup R_2)\) which is coarser than \((P_i^a)_{i=1}^{J_\alpha}\). Note that (29)(ii) holds by (24)(i).

We now show (29)(ii)-(iv). First, (31) and the definition of \(\lambda_{\theta, \alpha}\) show (29)(iv). Fix \(P_j\) and \(P^a\) with \(P^a \subset P_j\). Then by (24)(ii), (32), (33), and the fact that \(4C_0\beta K_\alpha \leq \frac{1}{2} \lambda_{\theta, \alpha}\)

\[
\|u - b_j\|_{L^\infty(P_j \cap P^a)} \leq \|u\|_{L^\infty(P_j \cap P^a)} + |b_i^a - b_j| \leq 4C_0\beta K_\alpha \|\nabla u\|_{L^1(\Omega)} + r_j \leq \lambda_{\theta, \alpha} \|\nabla u\|_{L^1(\Omega)}.
\]
As $P^a \subset P^j$ was arbitrary, we get (29) (ii). Recalling the definition of $\beta$ and $\delta$ we have $\beta = 6\alpha(4\alpha)^{-d - 6\delta}$. This along with (21) (iii), (23) - (25) and $\lambda_{\theta, \alpha} \leq 8\delta C_0 \beta^{K^*}$ yields
\[
\min \frac{\text{ess inf}}{1 \leq j \leq J} \{|u(x) - b_j : x \in R_2\} \geq \min \frac{\text{ess inf}}{1 \leq j \leq J} \{|u(x) - b_i' : x \in R_2\} - \max \frac{\max}{1 \leq j \leq J \ i \in L_j} |b_j - b_i'|
\geq 2C_0\beta^{K^* + 1}\|\nabla u\|_{L^1(\Omega)} - \max r_j
\geq (2C_0\beta^{K^* + 1} - \lambda_{\theta, \alpha}/2)\|\nabla u\|_{L^1(\Omega)}
\geq \delta C_0\beta^{K^*}(12\alpha - 4)\|\nabla u\|_{L^1(\Omega)} \geq 8\alpha\delta C_0 \beta^{K^*} \|\nabla u\|_{L^1(\Omega)}
\geq \alpha \lambda_{\theta, \alpha} \|\nabla u\|_{L^1(\Omega)}.
\]
This shows (29) (iii) and concludes Step 2. Recall that we have $\lambda_{\theta, \alpha} \geq 8C_0\beta^{K^*} \geq 1$ and $\lambda_{\theta, \alpha} \leq 8\delta C_0 \beta^{K^*}$. Thus, $\lambda_{\theta, \alpha} \leq C_{\theta, \alpha} := 8(4\alpha)^{d - 4}C_0(6\alpha(4\alpha)^{-d})^{1/\theta}$.

The statement of the lemma now follows from (29) and (21). \hfill \Box

3.4. Proof of Theorem 3.1. The proof of Theorem 3.1 essentially relies on the following result.

**Theorem 3.8** (Existence of function $\psi$). Let $\Omega \subset \Omega' \subset \mathbb{R}^d$ be bounded Lipschitz domains. Let $(E_k)_k \subset E_{\Omega'}$ and let $(h_k)_k \subset W^{1,p}(\Omega'; \mathbb{R}^m)$ converging in $L^p(\Omega'; \mathbb{R}^m)$ to some $h \in W^{1,p}(\Omega'; \mathbb{R}^m)$ such that $\{\nabla h_k\}^p_k$ are equi-integrable. Consider $(u_k)_k \subset GSBV^p(\Omega'; \mathbb{R}^m)$ with $u_k = h_k$ on $\Omega' \setminus \overline{T}$ and $\sup_{k \in \mathbb{N}} E_k(u_k) < +\infty$.

Then we find a subsequence (not relabeled), modifications $(y_k)_k \subset GSBV^p(\Omega'; \mathbb{R}^m)$ with $y_k = h_k$ on $\Omega' \setminus \overline{T}$, and a continuous function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t \to +\infty} \psi(t) = +\infty$ such that

(i) $E_k(y_k) \leq E_k(u_k) + \frac{1}{k}$,
(ii) $\mathcal{H}^{d-1}(J_{y_k}) \leq \mathcal{H}^{d-1}(J_{u_k}) + \frac{1}{k}$,
(iii) $\mathcal{L}^d(\{\nabla y_k \neq \nabla u_k\}) \leq \frac{1}{k}$. \hfill (34)

Indeed, once Theorem 3.8 is proved, Theorem 3.1 follows directly from Theorem 2.2 and (f2), (g3), apart from the property that $\nabla u_k \rightharpoonup \nabla u$ weakly in $L^p(\Omega'; \mathbb{R}^{m \times d})$. To see the latter, we note by (f2) that we find $Z \in L^p(\Omega'; \mathbb{R}^{m \times d})$ such that $\nabla u_k \rightharpoonup Z$ weakly in $L^p(\Omega'; \mathbb{R}^{m \times d})$ (possibly up to a further subsequence). It suffices to check $Z = \nabla u$. To this end, we show that $\nabla u_k \rightharpoonup \nabla u$ weakly in $L^1(\Omega'; \mathbb{R}^{m \times d})$. Indeed, we have $\nabla y_k \rightharpoonup \nabla u$ weakly in $L^1(\Omega'; \mathbb{R}^{m \times d})$ and for each $\varphi \in L^\infty(\Omega'; \mathbb{R}^{m \times d})$ we compute by (33) (iii) and Hölder’s inequality

\[
\int_{\Omega'} (\nabla u_k - \nabla y_k) : \varphi \ dx \leq \|\varphi\|_{\infty} \int_{\{\nabla y_k \neq \nabla u_k\}} |\nabla u_k - \nabla y_k| \ dx
\leq (\mathcal{L}^d(\{\nabla y_k \neq \nabla u_k\}))^{1 - 1/p} \|\nabla u_k - \nabla y_k\|_{L^p(\Omega')} \to 0.
\]

We now proceed with the proof of Theorem 3.8. We point out that the result does not simply follow from Corollary 3.3 to construct modifications $(y_k)_k$ satisfying (34) (i). Corollary 3.3 has to be applied along a sequence $\theta \to 0$ to obtain piecewise translated configurations $(v^\theta_k)_{k, \theta}$. As $\theta \to 0$, unfortunately the uniform bound (35) (iii) blows up, and the definition of the function $\psi$ is not immediate. As a remedy, we first pass to a limit $v^\theta$ for each fixed $\theta$ as $k \to \infty$, and then we show that $(v^\theta_k)_k$ is close to each other in a certain sense on the bulk part of the domain. This allows us to apply Lemma 4.1 and to obtain the function $\psi$. Then, $(y_k)_k$ can be chosen as a suitable diagonal sequence in $(v^\theta_k)_{k, \theta}$. In this strategy, we follow closely [35, Theorem 6.1] and [25, Theorem 2.2]. Note, however, that some delicate adaptations are necessary due to the fact that the energies may depend on the crack opening.
Proof of Theorem 3.8. Consider a sequence \((E_k)_k \subset \mathcal{E}_{\Omega'}\). Let \((h_k)_k \subset \text{W}^{1,p}(\Omega';\mathbb{R}^m)\) converging in \(L^p(\Omega';\mathbb{R}^m)\) to some \(h \in \text{W}^{1,p}(\Omega';\mathbb{R}^m)\) such that \((|\nabla h_k|^p)\) are equi-integrable. Let \((u_k)_k \subset \text{GSBV}^p(\Omega';\mathbb{R}^m)\) with \(u_k = h_k\) on \(\Omega' \setminus \overline{\Omega}\) and \(\sup_{k \in \mathbb{N}} E_k(u_k) \leq C_* < +\infty\). Setting

\[
M := \frac{C_* c_1}{C_1} + \frac{C_*}{c_4} + \sup_{k \in \mathbb{N}} ||\nabla h_k||_{L^p(\Omega')},
\]

we find \(||\nabla h_k||_{L^p(\Omega')}^p \leq M\) and \(u_k \in \text{GSBV}^p(\Omega';\mathbb{R}^m)\) by \((f)\) and \((g.3)\). Define the decreasing sequence \(\theta_l = 2^{-l}\) for \(l \in \mathbb{N}\). As we will pass to subsequences (not relabeled) several times in the proof, we emphasize that we will eventually only have the inequality

\[
\theta_l \leq 2^{-l}.
\]

Step 1 (Application of Corollary 3.3). We apply Corollary 3.3 for \(\theta_l\) and \(M\) on the functions \(u_k\) and the boundary data \(h_k\). We find (finite) Caccioppoli partitions \(\Omega' = \bigcup_{j \geq 1} P_j \cup R_l\), and piecewise translated functions \((v^k_j)_k \subset \text{GSBV}^p(\Omega';\mathbb{R}^m)\) defined by

\[
v^k_j := h_k + \sum_{j \geq 1} (u_k - t_{j,k}^{l}) \chi_{P_{j,k}^l} + \sum_{j \geq 1} (u_k - t_{j,k}^{l}) \chi_{P_{j,k}^l},
\]

where \((t_{j,k}^{l})_{j \geq 1} \subset \mathbb{R}^m\) are suitable translations. For notational convenience, we will also use the notation \(P_j^k = R_k\) for each \(j \geq 1\) is a partition of \(\Omega'\). From Corollary 3.3 we have \(v^k_j = h_k\) on \(\Omega' \setminus \overline{\Omega}\) for all \(l,k \in \mathbb{N}\) and from \((11), (35), (37)\) we get

\[
\begin{align*}
(i) & \quad ||v^k_j - h_k||_{L^\infty(\Omega')} \leq C_{0,l,M}, \\
(ii) & \quad ||\nabla v^k_j||_{L^p(\Omega')} \leq ||\nabla u_k||_{L^p(\Omega')}^p + ||\nabla h_k||_{L^p(\Omega')}^p \leq 2M, \\
(iii) & \quad L^d(R^l_k) \leq C_{\theta_l}, \\
(iv) & \quad \mathcal{H}^{d-1}(J_{\partial^s P_j^l}) \leq \mathcal{H}^{d-1}(J_{u_k}) + C_{M} \theta_l \leq M + C_{M} \theta_l, \\
(v) & \quad \mathcal{H}^{d-1}\left(\bigcup_{j \geq 0} \partial^s P_j^l\right) \leq C_{M}.
\end{align*}
\]

By the fact that \((|\nabla h_k|^p)\) are equi-integrable and \((\text{iii) we find a decreasing sequence } \eta_l \to 0 \text{ as } l \to \infty \text{ such that}

\[
||\nabla h_k||_{L^p(P_j^l)}^p \leq \eta_l \quad \text{for all } k,l \in \mathbb{N}.
\]

From \((11) (i)\) we thus obtain

\[
E_k(v^l_j) \leq E_k(u_k) + C_M(\theta_l + \eta_l).
\]

For later purposes, we remark that for each collection \((t_{j,k}^{l})_{j \geq 1}\) with \(|t_{j,k}^{l} - t_{j,k}^{l}|| \leq \theta_l^{-1}||v^l_k - h_k||_{L^\infty}\), for all \(j\), the functions \(v^l_j = h_k \chi_{R^l_k} + \sum_{j \geq 1} (u_k - t_{j,k}^{l}) \chi_{P_{j,k}^l}\) also satisfy

\[
E_k(v^l_k) \leq E_k(u_k) + C_M(\theta_l + \eta_l), \quad \mathcal{H}^{d-1}(J_{\partial^s P_j^l}) \leq \mathcal{H}^{d-1}(J_{u_k}) + C_{M} \theta_l,
\]

see \((10)\). We also observe that it is not restrictive to assume that

\[
||v^l_k - h_k||_{L^\infty(\Omega')} \geq ||v^l_k - h_k||_{L^\infty(\Omega')} \quad \text{for all } l,k \in \mathbb{N}.
\]

In fact, otherwise we may replace the function \(v^l_k\) defined in \((37)\) for index \(l\) by the function \(v^{l+1}_k\). Then \((35)-(40)\) still hold as the sequences \(\eta_l\) and \(\theta_l\) are decreasing and \((41)\) is trivially satisfied.

Step 2 (Limiting objects for each \(l\)). In view of \((35), (ii), (iv)\) and the fact that \(h_k\) converges to \(h\) in \(L^p(\Omega';\mathbb{R}^m)\), Ambrosio’s compactness result (Theorem 2.2) is applicable for fixed \(l \in \mathbb{N}\). Thus, using a diagonal argument we get a subsequence of \((k)_k \in \mathbb{N}\) (not relabeled) such that for
every $l \in \mathbb{N}$ we find a function $v^l \in GSBV^p(\Omega'; \mathbb{R}^m)$ with $v^l \rightarrow v^l$ pointwise a.e. in $\Omega'$ for $k \rightarrow \infty$. Clearly, by (38) we then also have

$$v_k \rightarrow v^l$$

in $L^1(\Omega'; \mathbb{R}^m)$. (42)

Likewise, we can establish a compactness result for the Caccioppoli partitions as follows: in view of (38) (iii), for a suitable subsequence of $(l)_{l \in \mathbb{N}}$ (not relabeled) we may suppose that

$$\mathcal{L}^d(R_k^l) < \mathcal{L}^d(\Omega' \setminus \overline{\Omega})$$

(43)

for all $k, l \in \mathbb{N}$. Recall from Corollary 3.3 that for each partition at most one component $(P^k_{j,l})_{j \geq 1}$ intersects $\Omega' \setminus \overline{\Omega}$. (We emphasize that the rest set $R_k^l$ is not counted among the components here.) In view of (43), exactly one of these components intersects $\Omega' \setminus \overline{\Omega}$. We may reorder the components of the partitions such that $P^k_0 = R_k^l$, such that

$$\text{exactly } P^k_{1,l} \text{ intersects } \Omega' \setminus \overline{\Omega},$$

(44)

and $(P^k_{j,l})_{j \geq 2}$ are ordered for all $k, l \in \mathbb{N}$. By (38) (v), Theorem 2.3 and the comment thereafter we find for each $l \in \mathbb{N}$ a partition $(P^l_j)_{j \geq 0}$ such that $\sum_{j \geq 0} \mathcal{H}^{d-1}(\partial^+ P^l_j) \leq C_M$ such that for a suitable subsequence of $(k)_{k \in \mathbb{N}}$ one has $\sum_{j \geq 0} \mathcal{L}^d(P^k_{j,l} \triangle P^l_j) \rightarrow 0$ for $k \rightarrow \infty$. (Here, we again use a diagonal argument.)

Since $\sum_{j \geq 0} \mathcal{H}^{d-1}(\partial^+ P^l_j) \leq C_M$ for all $l \in \mathbb{N}$, we can repeat the arguments and get a partition $(P^l_j)_{j \geq 0}$ such that $\sum_{j \geq 0} \mathcal{L}^d(P^l_j \triangle P^l_j) \rightarrow 0$ for $l \rightarrow \infty$ after extracting a further suitable subsequence. Thus, using a diagonal argument, we can choose a (not relabeled) subsequence of $(l)_{l \in \mathbb{N}}$ and afterwards of $(k)_{k \in \mathbb{N}}$ such that

$$\sum_{j \geq 0} \mathcal{L}^d(P^l_j \triangle P^l_j) \leq 2^{-l}, \quad \sum_{j \geq 0} \mathcal{L}^d(P^k_{j,l} \triangle P^l_j) \leq 2^{-l} \quad \text{for all } k \geq l.$$ 

(45)

Our goal is to obtain the desired function $\psi$ by using Lemma 2.3 for the limiting sequence $(v^l)$. We will now show that, by redefining the translations on the components of the partitions appropriately (cf. (37)), we can indeed construct this sequence (which we will denote by $(\hat{v}^l)_l$ for better distinction) in such a way that

$$\mathcal{L}^d \left( \bigcap_{n \in \mathbb{N}} \bigcup_{m \geq n} \{ |\hat{v}^n - \hat{\tilde{v}}^m | > 1 \} \right) = 0.$$ 

(46)

Then Lemma 2.3 is applicable.

Step 3 (Redefinition of translations). We now come to the details how to choose the translations. Fix $k \in \mathbb{N}$. We describe an iterative procedure to redefine $t^k_{j,l}$ for all $l, j \in \mathbb{N}$. Let $\hat{v}^l_k = v^l_k$ as defined in (37). Assume that $(\hat{v}^k_{j,l})_j$ (which may differ from $(\hat{v}^k_{j,l})_j$) and the corresponding $\hat{v}^l_k$ (see (37)) have been chosen such that

$$\| \hat{v}^l_k - h_k \|_{L^\infty(\Omega')} \leq 2 \sum_{l=1}^{l-1} \| v^l_k - h_k \|_{L^\infty(\Omega')}$$

and $\hat{v}^l_k = h_k$ on $\Omega' \setminus \overline{\Omega}$. Clearly, these assumptions hold for $l = 1$.

Consider some $P^k_{j,l+1}$. If $\mathcal{L}^d(P^k_{j,l} \cap P^k_{j,l+1}) > 0$, we define $\hat{v}^k_{j,l+1} = t^k_{j,l}$. Otherwise, we set $\hat{v}^k_{j,l+1} = t^k_{j,l+1}$. In the first case, noting that $t^k_{j,l+1} = u_k - t^k_{j,l} + 1$ and $\hat{v}^l_k = u_k - t^k_{j,l+1}$ on
\[ P^{k,l}_j \cap P^{k,l+1}_j \text{ by (37), we obtain by the triangle inequality and (47)} \]
\[ |t^{k,l+1}_j - t^{k,l}_j| = \|u_k - t^{k,l+1}_j - (u_k - t^{k,l}_j)|_{L^\infty(P^{k,l}_j \cap P^{k,l+1}_j)} = \|v^{l+1}_k - \hat{v}^{l+1}_k|_{L^\infty(P^{k,l}_j \cap P^{k,l+1}_j)} \]
\[ \leq \|v^{l+1}_k - h_k\|_{L^\infty(\Omega')} + \|\hat{v}^{l+1}_k - h_k\|_{L^\infty(\Omega')} \]
\[ \leq 2 \sum_{\ell=1}^l \|v^{\ell}_k - h_k\|_{L^\infty(\Omega')} + \|v^{l+1}_k - h_k\|_{L^\infty(\Omega')} \].

(48)

We define \( \hat{v}^{l+1}_k \) as in (37) replacing \( t^{k,l+1}_j \) by \( t^{k,l+1}_j \) and derive by the previous calculation
\[ \|v^{l+1}_k - h_k\|_{L^\infty(\Omega')} \leq 2 \sum_{\ell=1}^l \|v^{\ell}_k - h_k\|_{L^\infty(\Omega')} \]

i.e., (47) holds for \( l+1 \). We also have \( \hat{v}^{l+1}_k = h_k \) on \( \Omega' \setminus \Omega \). In fact, by (44) only \( P^{k,l+1}_k \) intersects \( \Omega' \setminus \Omega \). But then (37) and \( v^{l+1}_k = \hat{v}^{l+1}_k = \hat{v}_k = u_k = h_k \) on \( \Omega' \setminus \Omega \) imply \( t^{k,l+1}_1 = t^{k,l+1}_1 = t^{k,l}_1 = \hat{v}^{l+1}_k = 0 \) and thus \( \hat{v}^{l+1}_k = h_k \) on \( \Omega' \setminus \Omega \).

By (45) and (44) we observe
\[ |t^{k,l+1}_j - t^{k,l}_j| \leq (2l + 1) \|v^{l+1}_k - h_k\|_{L^\infty(\Omega')} \leq \theta^{-1}_{l+1} \|v^{l}_k - h_k\|_{L^\infty(\Omega')} \]

where the last step follows from (36). Thus, in view of the remark before (40), also the newly constructed functions \( \hat{v}^{l+1}_k \) satisfy the energy bound (40).

By (38) and (47) we also have \( \|\hat{v}^{l}_k - h_k\|_{L^\infty(\Omega')} \leq 2 \sum_{\ell=1}^l C_{\theta_1,M} \). Thus, repeating the argument in (42), we find some \( \hat{v} \in GSBV^p(\Omega'; \mathbb{R}^m) \) such that
\[ \hat{v} \to \hat{v}^{l}_k \text{ in } L^1(\Omega'; \mathbb{R}^m). \]

(49)

Step 4 (Proof of (40)). Having redefined the piecewise translated functions, we are now in a position to show that (40) holds. To this end, we set \( A^n_{k,l} = \bigcup_{n \leq m \leq l} (\hat{v}^m_k = \hat{v}_k^n) \) for all \( n \in \mathbb{N} \) and \( k \geq l \geq n \). If we show
\[ \mathcal{L}^d (\Omega' \setminus A^n_{k,l}) \leq c 2^{-n} \]

(50)

for \( c = c(C_M) \), then (40) follows. In fact, for each \( l \geq n \) we can choose \( K = K(l) \geq l \) so large that \( \mathcal{L}^d \left( \{ |\hat{v}^m_k - \hat{v}_k^n| > \frac{1}{2} \} \right) \leq 2^{-m} \) for all \( n \leq m \leq l \) as \( \hat{v} \to \hat{v}^m_k \) in measure for \( k \to \infty \) (see (49)). This implies
\[ \mathcal{L}^d \left( \bigcup_{n \leq m \leq l} \{ |\hat{v}^m_k - \hat{v}_k^n| > \frac{1}{2} \} \right) \leq \mathcal{L}^d (\Omega' \setminus A^n_{K,l}) + \sum_{n \leq m \leq l} \mathcal{L}^d \left( \{ |\hat{v}^m_k - \hat{v}_k^n| > \frac{1}{2} \} \right) \leq c 2^{-n} \]

(Here, the constant \( c \) may vary from step to step.) Passing to the limit \( l \to \infty \) we find \( \mathcal{L}^d (\bigcup_{n \leq m \leq l} \{ |\hat{v}^m_k - \hat{v}_k^n| > \frac{1}{2} \}) \leq c 2^{-n} \) and taking the intersection over all \( n \in \mathbb{N} \) we obtain (40), as desired.

We now show (50). First, observe that by (37), (38) (iii), and \( \theta_m \leq 2^{-m} \) (see (36))
\[ \mathcal{L}^d \left( \bigcap_{n \leq m \leq l} \{ T^n_k = T^m_k \setminus A^n_{k,l} \right) \leq \sum_{n \leq m \leq l} \mathcal{L}^d (P^{k,m}_{0,l}) = \sum_{n \leq m \leq l} \mathcal{L}^d (R^m_k) \leq C_M 2^{-n} \]

(51)

where \( T^m_k := \left| \sum_{j \geq 0} i^{k,m}_j P^{k,m}_{j} \right| \) and \( P^{k,m}_{0,l} = R^m_k \). Due to the above construction of the translations in Step 3, we get \( \{ T^n_k = T^m_k \} \cup \bigcup_{n \leq m \leq l} (P^{k,m+1}_j \cap P^{k,m}_j) \) for \( n \leq m \leq l - 1 \). From (45) we deduce \( \sum_{j \geq 0} \mathcal{L}^d (P^{k,m+1}_j \setminus P^{k,m}_j) \leq 3 \cdot 2^{-m} \). This along with (36) and (38) (iii) yields
\[ \mathcal{L}^d (\Omega' \setminus \{ T^n_k = T^m_k \}) \leq \mathcal{L}^d (R^m_k) + \sum_{j \geq 0} \mathcal{L}^d (P^{k,m+1}_j \setminus P^{k,m}_j) \leq C_M 2^{-m+1} + 3 \cdot 2^{-m} \]

We now sum over \( n \leq m \leq l - 1 \) and, in view of (51), we obtain (50). Thus, as already shown above, also (40) holds.
Step 5 (Conclusion). We observe that \((\hat{v}^i)_1 \subset L^1(\Omega'; \mathbb{R}^m)\) by (19). In view of (40), we can apply Lemma 2.1 to obtain a nonnegative, increasing, concave function \(\hat{\psi}\) with \(\lim_{t \to +\infty} \hat{\psi}(t) = +\infty\) such that (up to a subsequence)
\[
\sup_{t \geq 1} \int_{\Omega'} \hat{\psi}(|\hat{v}^i|) \, dx < +\infty.
\] (52)

Define \(\psi(t) = \min\{\hat{\psi}(t), t\}\) and observe that \(\psi\) has the properties stated in Theorem 3.8. We are now in a position to define the modifications \((y_k)_k\) with the desired properties. Recalling \(\hat{v}^i_k \to \hat{v}^i\) in \(L^1(\Omega'; \mathbb{R}^m)\) (see (19)) and (40), we can select a subsequence of \((u_k)_k\) and a diagonal sequence \((\hat{v}^i_k)_k \subset (\hat{v}^i_k)_k\) such that \(\|y_k - \hat{v}^i\|_{L^1(\Omega')} \leq 1\) for some \(\hat{v}^i\),
\[
E_k(y_k) \leq E_k(u_k) + \frac{1}{k}, \quad \mathcal{H}^{d-1}(J_{y_k}) \leq \mathcal{H}^{d-1}(J_{u_k}) + \frac{1}{k},
\]
and \(y_k = h_k\) on \(\Omega' \setminus \overline{\Omega}\). This yields (34). In fact, (i) follows from the previous equation and (iii) follows from (37) and (38)(iii). Finally, to see (ii), we observe that \(\psi\) is subadditive as concave function with \(\psi(0) = 0\). Then \(\|y_k - \hat{v}^i\|_{L^1(\Omega')} \leq 1\) implies

\[
\sup_{k \in \mathbb{N}} \int_{\Omega'} \psi(|y_k|) \, dx \leq \sup_{t \geq 1} \left( \int_{\Omega'} \hat{\psi}(|\hat{v}^i|) \, dx + \|y_k - \hat{v}^i\|_{L^1(\Omega')} \right) \leq \sup_{t \geq 1} \int_{\Omega'} \hat{\psi}(|\hat{v}^i|) \, dx + 1.
\]
By (52) this concludes the proof. \(\square\)

Remark 3.9. We close this section with the observation that Theorem 3.8 is much easier to prove if (g3) is replaced by a condition of the form
\[
c_4(1 + \varphi(|\zeta|)) \leq g(x, \zeta, \nu) \quad \text{for every } x \in \Omega', \zeta \in \mathbb{R}^m_0, \text{ and } \nu \in S^{d-1},
\] (53)
where \(\varphi : \mathbb{R}_+ \to \mathbb{R}_+\) is an increasing function satisfying \(\varphi(t) \leq t\) for all \(t \in \mathbb{R}_+\) and \(\lim_{t \to +\infty} \varphi(t) = +\infty\). Indeed, in this case no modifications have to be introduced, but (33)(ii) can be shown for the original sequence \((u_k)_k\). The strategy is to apply the (standard) Poincaré inequality in \(BV\) on a suitable composition of \(u_k\) with some \(\psi\), which allows to control uniformly the \(L^1\)-norm of the compositions and leads to (33)(ii).

Let us come to the details. Consider a sequence \((E_k)_k \subset E_{\Omega'}\) with densities \(f_k\) and \(g_k\) and \((u_k)_k \subset GSBV^p(\Omega'; \mathbb{R}^m)\) with \(\sup_{k \in \mathbb{N}} E_k(u_k) \leq C_s < +\infty\). As \(\varphi\) is increasing and satisfies \(\lim_{t \to \infty} \varphi(t) = +\infty\), we can find a smooth, increasing, concave function \(\psi : \mathbb{R}_+ \to \mathbb{R}_+\) with \(\psi \leq \varphi + 2\) and \(\lim_{t \to \infty} \psi(t) = +\infty\). (An elementary construction of such a function may be found in (52) Lemma 4.1) using an increasing sequence \((b_i)_i\) satisfying \(\varphi(b_i) \geq 2^{i+1}\) for \(i \in \mathbb{N}\). Observe that, as concave function with \(\psi(0) = 0\), \(\psi\) is subadditive. Our goal is to show that for each \(i = 1, \ldots, m\) we have
\[
\sup_{k \in \mathbb{N}} \int_{\Omega'} \psi(|u_k^i|) \, dx \leq C < +\infty \quad \text{for all } k \in \mathbb{N}.
\] (54)

Here and in the following, the superscript indicates the \(i\)-th component. Once (54) is established, we can conclude \(\int_{\Omega'} \psi(|u_k^i|) \, dx \leq C_m\) by the subadditivity of \(\psi\).

Let us now confirm (53). We define the function \(v_k^i = \psi(|u_k^i|) \in GSBV(\Omega')\) and note that \(\|
abla v_k^i\| \leq \|
abla \psi\|_{\infty} \|
abla u_k^i\|\) \(\mathcal{L}^d\)-a.e. in \(\Omega'\). By (f2) and \(E_k(u_k) \leq C_s\) this implies
\[
\|
abla v_k^i\|_{L^p(\Omega')} \leq \|
abla \psi\|_{\infty} \|
abla u_k^i\|_{L^p(\Omega')} \leq \|
abla \psi\|_{\infty} c_1^{-1} \int_{\Omega'} f_k(x, \nabla u_k(x)) \, dx \leq \|
abla \psi\|_{\infty} c_1^{-1} C_s.
\] (55)

Moreover, for \(\mathcal{H}^{d-1}\)-a.e. point of \(J_{v_k^i}\), we find by the fact that \(\psi\) is increasing and subadditive \(\|v_k^i\| = |(v_k^i)^+ - (v_k^i)^-| = |\psi((v_k^i)^+) - \psi((v_k^i)^-)| \leq \psi((v_k^i)^+) - \psi((v_k^i)^-) = \psi(|u_k^i|) \leq \psi(|u_k|)\).
Using $\psi \leq \varphi + 2$, (53), and $E_k(u_k) \leq C_\ast$ we derive
\[
\int_{J_{\Omega_k}} ||v_k|| \, d\mathcal{H}^{d-1} \leq \int_{J_{\Omega_k}} (\varphi(||u_k||) + 2) \, d\mathcal{H}^{d-1} \leq \frac{2}{c_4} \int_{J_{\Omega_k}} g_k(x, [u_k], h_k) \, d\mathcal{H}^{d-1} \leq \frac{2}{c_4} C_\ast. \tag{56}
\]

Now Hölder’s inequality and (55)-(56) imply the bound $|Dv_k^i|(|\Omega') \leq C$ on the total variation, where $C = C(C_\ast, \Omega', c_1, c_2, p, ||\varphi'||_{\infty})$. By the Poincaré inequality in $BV$ (see [5, Remark 3.50]) we therefore find $b^1_k \in \mathbb{R}$ such that
\[
|v_k^i - b^1_k|_{L^1(\Omega')} \leq C|Dv_k^i|(|\Omega') \leq C.
\]

As $v_k^i = \psi(|h_k^i|)$ on $\Omega' \setminus \overline{\Omega}$, we also deduce $\psi(|h_k^i|) - b^1_k \in L^1(\Omega \setminus \Omega') \leq C$ and therefore
\[
|v_k^i|_{L^1(\Omega')} \leq C + C||\psi(|h_k^i|)||_{L^1(\Omega')}.
\]

Using $\psi(t) \leq \varphi(t) + 2 \leq t + 2$, we note that $\psi(|h_k^i|)_{L^1(\Omega')}$ is uniformly bounded in $k$. Recalling $v_k^i = \psi(|u_k^i|)$, this shows (54) and concludes the proof. \hfill \Box

4. Existence and $\Gamma$-convergence results for free discontinuity problems

In this section we provide some applications of the compactness result to boundary value problems. In the following, we suppose that there exist two bounded Lipschitz domains $\Omega' \subset \Omega$. We will impose Dirichlet boundary data on $\partial D\Omega := \partial \Omega' \cap \partial \Omega$. As usual for the weak formulation in the frame of $SBV$ functions, this will be done by requiring that configurations $u$ satisfy $u = h$ on $\Omega' \setminus \overline{\Omega}$ for some $h \in W^{1,p}(\Omega'; \mathbb{R}^m)$. We will first present an existence result and then address $\Gamma$-convergence for energies in the class $\mathcal{E}_\Omega$.

4.1. Existence. As a first application, we prove an existence result for energy functionals in the class $\mathcal{E}_{\Omega'}$ introduced in Section 3.1

**Theorem 4.1** (Existence result for free discontinuity problems in $GSBV^p$). Let $\Omega \subset \Omega' \subset \mathbb{R}^d$ be bounded Lipschitz domains. Let $E \in \mathcal{E}_{\Omega'}$ be lower semicontinuous in $L^0(\Omega'; \mathbb{R}^m)$ and let $h \in W^{1,p}(\Omega'; \mathbb{R}^m)$. Then the minimization problem
\[
\inf_{u \in L^0(\Omega'; \mathbb{R}^m)} \{ E(u) : u = h \text{ on } \Omega' \setminus \overline{\Omega} \}
\]

admits solutions.

**Proof.** The result follows from Theorem 3.1 and the direct method. Indeed, choosing a minimizing sequence $(u_k)_k$, we find another minimizing sequence $(y_k)_k$ converging in measure to some $u \in GSBV^p(\Omega'; \mathbb{R}^m)$ with $u = h$ on $\Omega' \setminus \overline{\Omega}$. The lower semicontinuity of $E$ with respect to convergence in measure then yields that $u$ is a minimizer. \hfill \Box

Without going into details, let us just briefly mention that in [2],[3], lower semicontinuity for functionals $E \in \mathcal{E}_{\Omega'}$ with respect to measure convergence is ensured (under the assumption that $g$ is continuous) by quasiconvexity for the bulk density $f$ and $BV$-ellipticity [4] for the surface density $g$.

Clearly, the minimizer of the problem is independent of the definition of $f(x, \xi)$ for $x \in \Omega' \setminus \Omega$ and independent of $g(x, \zeta, \nu)$ for $x \in \Omega \setminus \overline{\Omega}$. The value of $g(x, \zeta, \nu)$ for $x \in \partial \Omega$, however, may affect the minimization problem. Indeed, it might be energetically favorable if the crack runs alongside $\partial \Omega$. In this case, the boundary datum is not attained in the sense of traces, at the expense of a crack energy. Below in Section 4.2, we will present a variant where the minimizer is determined only by $g(x, \zeta, \nu)$ for $x \in \Omega$, see Remark 4.5.
4.2. $\Gamma$-convergence. We now revisit the $\Gamma$-convergence result for free discontinuity problems established recently in [12]. There, for minimization problems involving an $L^p$-perturbation of the energy functionals $\mathcal{E}$, convergence of minimum values and minimizers is proved. In the present contribution, we treat boundary value problems without any $L^p$-perturbation instead.

For the application to $\Gamma$-convergence results, we need some further assumptions on the bulk density $f : \Omega \times \mathbb{R}^{m \times d} \to \mathbb{R}_+$ and the surface density $g : \Omega \times \mathbb{R}_0^n \times S^{d-1} \to \mathbb{R}_+$, see [12]. Let $c_1, \ldots, c_5$ be the constants in the definition of $\mathcal{E}_\Omega$ in Section 3. Moreover, we let $\sigma_1, \sigma_2 : \mathbb{R}_+ \to \mathbb{R}_+$ be two nondecreasing continuous functions with $\sigma_1(0) = \sigma_2(0) = 0$. By $\mathcal{E}_\Omega' \subset \mathcal{E}_\Omega$ we denote the collection of integral functionals $\mathcal{E}$ where additionally the following holds:

(f3) (continuity in $\xi$) for every $x \in \Omega$ we have

$$|f(x, \xi_1) - f(x, \xi_2)| \leq \sigma_1(|\xi_1 - \xi_2|)(1 + f(x, \xi_1) + f(x, \xi_2))$$

for every $\xi_1, \xi_2 \in \mathbb{R}^{n \times d}$,

(g5) (estimate for $|\xi_1| \leq |\xi_2|$) for every $x \in \Omega$ and every $\nu \in S^{d-1}$ we have

$$g(x, \xi_1, \nu) \leq c_3 g(x, \xi_2, \nu)$$

for every $\xi_1, \xi_2 \in \mathbb{R}^n$ with $|\xi_1| \leq |\xi_2|$,

(g6) (continuity in $\xi$) for every $x \in \Omega$ and every $\nu \in S^{d-1}$ we have

$$|g(x, \xi_1, \nu) - g(x, \xi_2, \nu)| \leq \sigma_2(|\xi_1 - \xi_2|)(g(x, \xi_1, \nu) + g(x, \xi_2, \nu))$$

for every $\xi_1, \xi_2 \in \mathbb{R}^n$.

Besides the two continuity conditions, in [12] additionally (g5) is required which is a kind of ‘monotonicity condition’ for the jump height $|\xi|$. We refer to [12] Remark 3.2, 3.3 for more details. We include (g5) here only for the reader’s convenience to ease reference to the assumptions in [12]. Actually, the condition already follows (with different constants) from (g3).

In the following we denote by $\mathcal{A}(\Omega)$ the open subsets of $\Omega$.

**Theorem 4.2** (Compactness of $\Gamma$-convergence, see [12]). Let $(E_k)_k$ be a sequence in $\mathcal{E}_\Omega'$ with densities $(f_k)_k$ and $(g_k)_k$. Then there exists a subsequence (not relabeled) and $f : \Omega \times \mathbb{R}^{m \times d} \to \mathbb{R}_+$, $g : \Omega \times \mathbb{R}_0^n \times S^{d-1} \to \mathbb{R}_+$ such that for all $A \in \mathcal{A}(\Omega)$

$$E_k(\cdot, A) \Gamma\text{-converges to } E(\cdot, A) \text{ in } L^0(\Omega; \mathbb{R}^m),$$

where $E : L^0(\Omega; \mathbb{R}^m) \times \mathcal{A}(\Omega) \to [0, +\infty]$ is given by (4) and lies in $\mathcal{E}_\Omega'$. Moreover, we have

$$E_k^p(\cdot, A) \Gamma\text{-converges to } E^p(\cdot, A) \text{ in } L^p(\Omega; \mathbb{R}^m),$$

where $E^p_k$ and $E^p$ denote the restriction of $E_k$ and $E$ to $L^p(\Omega; \mathbb{R}^m) \times \mathcal{A}(\Omega)$, respectively.

For a general theory of $\Gamma$-convergence we refer the reader to [24]. The limiting bulk density $f$ and surface density $g$ associated to $E$ can be expressed in terms of the densities $f_k$ and $g_k$ via specific asymptotic cell formulas, see [12] Theorem 3.5, Theorem 5.2. The crucial point is that the problems for the volume and surface integrals are decoupled, i.e., $f$ depends only on the sequence $(f_k)_k$ while $g$ depends only on the sequence $(g_k)_k$. In particular, for $A \in \mathcal{A}(\Omega)$ and a sequence $(u_k)_k$ with $\sup_{k \in \mathbb{N}} E_k(u_k, A) < +\infty$ converging to $u$ in measure on $A$, we have

$$\int_A f(x, \nabla u(x)) \, dx \leq \liminf_{k \to \infty} \int_A f_k(x, \nabla u_k(x)) \, dx,$$

$$\int_{J_u \cap A} g(x, [u], \nu_u) \, dH^{d-1} \leq \liminf_{k \to \infty} \int_{J_{u_k} \cap A} g_k(x, [u_k], \nu_{u_k}) \, dH^{d-1}. \quad (57)$$

A proof of this fact may be found in [25] Proposition 4.3. Now suppose that $(u_k)_k$ is a recovery sequence for $u$ with respect to the $L^p(\Omega; \mathbb{R}^m)$-convergence. We will use the following general
fact several times: if \( A \in \mathcal{A}(\Omega) \) with \( E(u, \partial A) = 0 \), then \((u_k)_k\) is also a recovery sequence with respect to \( E_k(\cdot, A) \), see [36, Remark 3.6]. Thus, if \( E(u, \partial A) = 0 \), we find by (57)

\[
\int_A f(x, \nabla u(x)) \, dx = \lim_{k \to \infty} \int_A f_k(x, \nabla u_k(x)) \, dx,
\]

\[
\int_{J_u \cap A} g(x, [u], \nu_u) \, d\mathcal{H}^{d-1} = \lim_{k \to \infty} \int_{J_{u_k} \cap A} g_k(x, [u_k], \nu_{u_k}) \, d\mathcal{H}^{d-1}.
\]

(58)

Consider again bounded Lipschitz domains \( \Omega' \supset \Omega \) and suppose that also \( \Omega' \setminus \overline{\Omega} \) has Lipschitz boundary. To treat non-attainment of the boundary data (in the sense of traces) as internal jumps, we introduce energy functionals defined on \( \Omega' \). We set

\[
f'_k(x, \xi) := \begin{cases} f_k(x, \xi) & \text{if } x \in \Omega, \\
 c_1 |\xi|^p & \text{otherwise.} \end{cases}
\]

(59)

and

\[
g'_k(x, \zeta, \nu) := \begin{cases} g_k(x, \zeta, \nu) & \text{if } x \in \Omega, \\
 c_5 + 1 & \text{otherwise.} \end{cases}
\]

(60)

According to Theorem 4.2, the functionals \( E'_k \in E'_{\Omega'} \), with densities \( f'_k \) and \( g'_k \), \( \Gamma \)-converge in \( L^0(\Omega'; \mathbb{R}^m) \) (up to a subsequence) to some \( E' \in E'_{\Omega'} \), with densities \( f' \) and \( g' \). (Strictly speaking, we consider here the class \( E'_{\Omega'} \) with \( c_5 + 1 \) instead of \( c_5 \).) Then we clearly have

\[
f'(x, \xi) = \begin{cases} f(x, \xi) & \text{if } x \in \Omega, \\
 c_1 |\xi|^p & \text{otherwise}
\end{cases}
\]

and \( g'(x, \zeta, \nu) = g(x, \zeta, \nu) \) for \( x \in \Omega \). Below in Remark 4.4 we will see that \( g'(x, \zeta, \nu) \) for \( x \in \partial D \Omega \) is completely determined by the sequence \( (g_k)_k \).

We now prove the following version of the \( \Gamma \)-convergence result that takes boundary data into account.

**Lemma 4.3** (\( \Gamma \)-convergence with boundary data). Suppose that \( \Omega' \setminus \overline{\Omega} \) has Lipschitz boundary. Let the sequence of functionals \((E'_k)_k \subset E'_{\Omega'}\) with densities \((f'_k)_k\), \((g'_k)_k\) and the limiting functional \( E' \in E'_{\Omega'} \) with densities \( f' \), \( g' \) be given as above. Suppose that \((h_k)_k \subset W^{1,p}(\Omega'; \mathbb{R}^m)\) converges strongly to \( h \) in \( W^{1,p}(\Omega'; \mathbb{R}^m) \). Then the sequence of functionals

\[
\tilde{E}'(u) = \begin{cases} E'(u) & \text{if } u = h \text{ on } \Omega' \setminus \overline{\Omega}, \\
 +\infty & \text{otherwise}
\end{cases}
\]

\( \Gamma \)-converges in \( L^0(\Omega'; \mathbb{R}^m) \) to

\[
\begin{cases} E'(u) & \text{if } u = h \text{ on } \Omega' \setminus \overline{\Omega}, \\
 +\infty & \text{otherwise.}
\end{cases}
\]

**Proof.** We follow the proof in [36 Lemma 7.1]. In particular, we highlight the necessary adaptations in our setting which are related to the fact that (a) the surface densities also depend on the crack opening and (b) we prove that \( g' \) is determined completely by \((g_k)_k\), see Remark 4.4.

First, the \( \Gamma \)-liminf is immediate from the \( \Gamma \)-convergence of \( E'_k \) to \( E' \) and the fact that the constraint is closed under the convergence in measure. We now address the \( \Gamma \)-linsup. Due to a general approximation argument in the theory of \( \Gamma \)-convergence together with Corollary 3.4.
it suffices to construct recovery sequences for $u \in GSBV^p(\Omega'; \mathbb{R}^m) \cap L^p(\Omega'; \mathbb{R}^m)$ with $u = h$ on $\Omega' \setminus \overline{\Omega}$.

By Theorem $4.2$ there exists a recovery sequence $(u_k)_k$ for $u$ with respect to $L^p$-convergence, i.e., $\|u_k - u\|_{L^p(\Omega')} \to 0$ and $\lim_{k \to \infty} E'_k(u_k) = E'(u)$. We note that $(57)$-$(68)$ hold (for the densities defined in $(59)$-$(60)$). We claim that

(i) $u_k - h_k \to 0$ strongly in $L^p(\Omega' \setminus \overline{\Omega}; \mathbb{R}^m)$,
(ii) $\nabla u_k - \nabla h_k \to 0$ strongly in $L^p(\Omega' \setminus \overline{\Omega}; \mathbb{R}^m \times d)$,
(iii) $\mathcal{H}^{d-1}(J_{u_k} \cap (\Omega' \setminus \Omega)) \to 0$.

We defer the proof of these properties to the end of the proof.

**Definition of the recovery sequence:** We can find a neighborhood $U \supset \supset \Omega' \setminus \overline{\Omega}$ and an extension $(y_k)_k \subset GSBV^p(U; \mathbb{R}^m)$ satisfying $y_k = u_k - h_k$ on $\Omega' \setminus \overline{\Omega}$ such that in view of $(61)$

$$\|y_k\|_{L^p(U)} + \|\nabla y_k\|_{L^p(U)} + \mathcal{H}^{d-1}(J_{y_k} \cap U) \to 0$$

as $k \to \infty$. This can be done, e.g., as in $(14$, Theorem 3.1, Theorem 8.1) with $GSBV^p \cap L^p$ in place of $SBV^2 \cap L^2$. (In both cases, the problem can be reduced to more regular functions by approximation $(22)$.

Let $\varepsilon > 0$ and choose $V$ open with $V \supset \overline{\partial \Omega}$, $V \subset U$, $E'(u, \partial(V \cap \Omega')) = 0$, $\mathcal{L}^d(V) \leq \varepsilon$, and $\int_{V \setminus \Omega'} f'(x, \nabla u(x)) \, dx < \varepsilon$. Then by $(68)$ we also get

$$\limsup_{k \to \infty} \int_{V \cap \Omega'} f_k'(x, \nabla u_k(x)) \, dx < \varepsilon.$$  

Choose $W \subset \mathbb{R}^d$ open such that $\Omega' \setminus \overline{\Omega} \subset W$ and $W \cap \overline{\Omega \setminus V} = \emptyset$. Let $\psi \in C^\infty(\Omega')$ with $0 \leq \psi \leq 1$, $\psi = 0$ on $\Omega \setminus V$ and $\psi = 1$ on $W \cap \Omega'$. Define $\varphi_k \in GSBV^p(\Omega'; \mathbb{R}^m)$ by $\varphi_k = \psi y_k$ on $U \cap \Omega'$ and $\varphi_k = 0$ else. Note by $(22)$ that

$$\|\varphi_k\|_{L^p(\Omega')} + \|\nabla \varphi_k\|_{L^p(\Omega')} + \mathcal{H}^{d-1}(J_{\varphi_k} \cap \Omega') \to 0.$$  

Now we set $\tilde{u}_k := u_k - \varphi_k$. Then $\tilde{u}_k = u_k - y_k$ on $W \cap \Omega'$ and thus $\tilde{u}_k = h_k$ on $\Omega' \setminus \overline{\Omega}$. Moreover, $\tilde{u}_k = u_k$ on $\Omega \setminus V$. We also observe that $\tilde{u}_k \to u$ in $L^p(\Omega'; \mathbb{R}^m)$ by $(64)$. We now estimate $E'_k(\tilde{u}_k)$. As $\mathcal{H}^{d-1}(J_{\varphi_k} \cap \Omega') \to 0$, we find by $(g3)$

$$\limsup_{k \to \infty} \int_{J_{u_k}} g_k'(x, [\tilde{u}_k], \nu_{\tilde{u}_k}) \mathcal{H}^{d-1} \leq \limsup_{k \to \infty} \int_{J_{u_k}} g_k'(x, [u_k], \nu_{u_k}) \mathcal{H}^{d-1}.$$  

Moreover, $(59)$ implies

$$\int_{\Omega'} |f_k'(x, \nabla u_k) - f_k'(x, \nabla \tilde{u}_k)| \, dx \leq \int_{\Omega \setminus \Omega'} \left( f_k(x, \nabla u_k) + f_k(x, \nabla \tilde{u}_k) \right) + c_1 \int_{\Omega' \setminus \Omega} \|\nabla u_k\|^p - \|\nabla h_k\|^p.$$  

The rightmost term converges to zero for $k \to \infty$ by $(61)(ii)$. By using the growth conditions $(f2)$, $(63)$-$(64)$, and $\mathcal{L}^d(V) \leq \varepsilon$ we find

$$\limsup_{k \to \infty} \int_{V \cap \Omega} \left( f_k(x, \nabla u_k) + f_k(x, \nabla \tilde{u}_k) \right) \, dx \leq c_2 \mathcal{L}^d(V) + 2^{p-1} c_2 \limsup_{k \to \infty} \int_{V \cap \Omega} |\nabla \varphi_k|^p \, dx$$

$$+ (1 + 2^{p-1} c_2 c_1^{-1}) \limsup_{k \to \infty} \int_{V \cap \Omega} f_k(x, \nabla u_k) \, dx \leq c_2 \varepsilon + (1 + 2^{p-1} c_2 c_1^{-1}) \varepsilon.$$
By \((65)\) and the fact that \(E'_k(u_k) \rightarrow E'(u) = \tilde{E}'(u)\), we then derive
\[
\limsup_{k \to \infty} \tilde{E}'_k(\tilde{u}_k) \leq \limsup_{k \to \infty} E'_k(u_k) + c_2 \varepsilon + (1 + 2p^{-1}c_2c_1^{-1}) \varepsilon \leq \tilde{E}'(u) + c_2 \varepsilon + (1 + 2p^{-1}c_2c_1^{-1}) \varepsilon.
\]
Since \(\varepsilon\) was arbitrary, using a diagonal argument we have proved the \(\Gamma\)-limsup inequality.

**Proof of \((61)\):** To conclude, it remains to show \((61)\). First, to see (i), we recall \(u_k \to h\) in \(L^p(\Omega' \setminus \overline{\Omega}; \mathbb{R}^m)\) as \((u_k)_k\) is a recovery sequence in \(L^p\). Then it suffices to use that \(h_k \to h\) in \(L^p(\Omega'; \mathbb{R}^m)\). We now address (ii). Let \(A \in \mathcal{A}(\Omega'), \overline{A} \subset \Omega' \setminus \overline{\Omega}\) with \(E'(u, \partial A) = 0\). Then \((58)\) and \((59)\) imply
\[
\nabla u_k \to \nabla h \quad \text{in} \quad L^p(A; \mathbb{R}^{m \times d}).
\]
For \(\varepsilon > 0\) consider \(V\) open with \(V \supset \partial_D \overline{\Omega}\) such that \(E'(u, \partial(V \cap \Omega')) = 0\), \(\mathcal{L}^d(V) < \varepsilon\), and
\[
\int_{V \cap \Omega'} f'(x, \nabla u(x)) \, dx < \varepsilon, \quad \int_{V \cap \Omega'} f'(x, \nabla h_k(x)) \, dx < \varepsilon \quad \text{for all } k \in \mathbb{N}.
\]
(The latter is possible by \((f2)\) and the fact that \(\nabla h_k \to \nabla h\) strongly in \(L^p(\Omega'; \mathbb{R}^{m \times d})\).) For \(k\) large enough, we also have \(\int_{V \cap \Omega'} f_k'(x, \nabla u_k(x)) \, dx < \varepsilon\) by \((58)\). Then we calculate by \((f2)\)
\[
\int_{\Omega' \setminus \overline{\Omega}} |\nabla u_k - \nabla h_k|^p \, dx = \int_{\Omega' \setminus (\Omega' \cup V)} |\nabla u_k - \nabla h_k|^p \, dx + \int_{V \cap \Omega'} |\nabla u_k - \nabla h_k|^p \, dx
\]
\[
\leq \int_{\Omega' \setminus (\Omega' \cup V)} |\nabla u_k - \nabla h_k|^p \, dx + \frac{2p^{-1}}{c_1} \int_{V \cap \Omega'} (f_k'(x, \nabla u_k) + f_k'(x, \nabla h_k)) \, dx.
\]
Then \((66)\) and the fact that \(\|\nabla h_k - \nabla h\|_{L^p(\Omega')} \to 0\) yield
\[
\limsup_{k \to \infty} \int_{\Omega' \setminus \overline{\Omega}} |\nabla u_k - \nabla h_k|^p \, dx \leq 2p^{-1} c_1^{-1} \varepsilon.
\]
Since \(\varepsilon\) was arbitrary, we obtain (ii). We finally prove (iii). Up to a subsequence we have
\[
\mu_k := \mathcal{H}^{d-1}(J_{u_k} \cap \Omega' \setminus \overline{\Omega}) \rightharpoonup^* \mu \quad \text{weakly* in} \quad \mathcal{M}_b(\Omega').
\]
By \((58)\) we observe \(\mathcal{H}^{d-1}(J_{u_k} \cap U) \to 0\) for all \(U \in \mathcal{A}(\Omega'), \overline{U} \subset \Omega' \setminus \overline{\Omega}\), and \(E'(u, \partial U) = 0\). Consequently, to conclude the proof of (iii), it suffices to show \(\mu(\partial_D \Omega) = 0\). We argue by contradiction. Let us assume that \(\mu(\partial_D \Omega) > 0\). Then there exists a cube \(Q_\rho\) with center \(x \in \partial_D \Omega\) and sidelength \(2\rho\) such that \(Q_\rho \subset \Omega', E'(u, \partial Q_\rho) = 0\), and \(\mu(Q_\rho) > \sigma > 0\). We may also suppose that \(Q_{4\rho} \subset \Omega'\), where \(Q_{4\rho}\) denotes the cube with center \(x\) and sidelength \(8\rho\). For \(k\) large enough we also have
\[
\mathcal{H}^{d-1}(J_{u_k} \cap (Q_\rho \setminus \Omega)) = \mu_k(Q_\rho) > \sigma > 0. \quad (67)
\]
Following the proof of \((30)\) Lemma 7.1, one can modify the sequence \((u_k)_k\) by a reflection method and move the jump set inside \(\Omega\). This will lead to a contradiction as we assumed that \((u_k)_k\) is a recovery sequence, but inside \(\Omega\) the surface energy is much less than in \(\Omega' \setminus \Omega\). In contrast to \((30)\), the construction is a bit more delicate here since the surface densities also depend on the crack opening. Possibly after passing to a smaller \(\rho\) (not relabeled), we can assume that in a suitable coordinate system
\[
\Omega \cap Q_{4\rho} = \{ (x', y) : x' \in (-4\rho, 4\rho)^{d-1}, y \in (-4\rho, \tau(x')) \}
\]
for a Lipschitz function \(\tau\) with \(\|\tau\|_{\infty} \leq \rho\). We choose \(\eta \in (2\rho, 3\rho)\) such that
\[
V_\rho := \{ (x', y) : x' \in (-\rho, \rho)^{d-1}, y \in (\tau(x') - \eta, \tau(x') + \eta) \}
\]
satisfies $E'(u, \partial V_\rho) = 0$. Note that $Q_\rho \subset V_\rho$ since $\eta > 2\rho$. Let $\hat{u}$ be the function defined on $V_\rho$ by reflecting $u$ at $\tau(x')$, $x' \in (-\rho, \rho)^{d-1}$, i.e.,

$$
\hat{u}(x', y) = \begin{cases} 
  u(x', y) & y > \tau(x'), \\
  u(x', 2\tau(x') - y) & y < \tau(x').
\end{cases}
$$

Clearly $\hat{u} \in W^{1,p}(V_\rho; \mathbb{R}^m)$ as $u \in W^{1,p}(\Omega', \mathbb{R}^m)$. In a similar fashion, we define $\hat{u}_k$ on $V_\rho$ by

$$
\hat{u}_k(x', y) = \begin{cases} 
  u_k(x', y) & y > \tau(x') - \lambda_k, \\
  u_k(x', 2(\tau(x') - \lambda_k) - y) & y < \tau(x') - \lambda_k,
\end{cases}
$$

where $0 < \lambda_k \leq 1/k$ is chosen such that

$$
\mathcal{H}^{d-1}\left( \left\{ (x', y) \in J_{u_k} : x' \in (-\rho, \rho)^{d-1}, \ y \in (\tau(x') - \lambda_k, \tau(x')) \right\} \right) \leq \frac{1}{k}. \tag{68}
$$

We note that the functions are well defined since $Q_{4\rho} \subset \Omega'$, $\|\tau\|_\infty \leq \rho$, and $\eta < 3\rho$. We now introduce the sequence

$$
w_k := u_k + \hat{u} - \hat{u}_k \in GSBV^p(V_\rho; \mathbb{R}^m).
$$

The definition and $\lambda_k \to 0$ implies that $w_k \to u$ in measure on $V_\rho$. Moreover, we find

(i) $\mathcal{H}^{d-1}(J_{w_k} \cap (V_\rho \setminus \Omega)) = 0$,

(ii) $\mathcal{H}^{d-1}(J_{w_k} \setminus \Gamma_k) \leq \mathcal{H}^{d-1}\left( \left\{ (x', y) \in V_\rho \cap J_{u_k} : \ y > \tau(x') - \lambda_k \right\} \right). \tag{69}
$$

Here, with the choice $\nu_{w_k} = \nu_{u_k}$ $\mathcal{H}^{d-1}$-a.e. on $J_{w_k} \cap J_{u_k}$, $\Gamma_k$ is defined by

$$
\Gamma_k := \{ x \in J_{w_k} \cap J_{u_k} : [u_k](x) = [w_k](x) \}.
$$

In particular, the jump of $w_k$ lies inside $\Omega$. By (g3) and (69)(i) we now find

$$
G(w_k) := \int_{J_{w_k} \cap V_\rho} g_k(x, [w_k], \nu_{w_k}) \, d\mathcal{H}^{d-1} \leq \int_{J_{u_k} \cap \Gamma_k} g_k(x, [u_k], \nu_{u_k}) \, d\mathcal{H}^{d-1} + c_5 \mathcal{H}^{d-1}(J_{w_k} \setminus \Gamma_k).
$$

Then by (68) and (69)(ii) we derive

$$
G(w_k) \leq \int_{J_{u_k} \cap \Gamma_k} g_k(x, [u_k], \nu_{u_k}) \, d\mathcal{H}^{d-1} + c_5/\sigma + c_5 \mathcal{H}^{d-1}(J_{u_k} \cap (V_\rho \setminus \Omega)).
$$

Therefore, by (60), (67), $\Gamma_k \subset J_{u_k} \subset \Omega \cap V_\rho$, and $Q_\rho \subset V_\rho$ we get

$$
G(w_k) \leq \int_{J_{u_k} \cap \Gamma_k} g_k(x, [u_k], \nu_{u_k}) \, d\mathcal{H}^{d-1} + c_5/k + (c_5 + 1) \mathcal{H}^{d-1}(J_{u_k} \cap (V_\rho \setminus \Omega)) - \sigma
\leq \int_{J_{u_k} \cap V_\rho} g_k(x, [u_k], \nu_{u_k}) \, d\mathcal{H}^{d-1} + c_5/k - \sigma. \tag{70}
$$

On the other hand, recalling that $w_k \to u$ in measure on $V_\rho$, we have by (67)

$$
\int_{J_u \cap V_\rho} g'(x, [u], \nu_u) \, d\mathcal{H}^{d-1} \leq \liminf_{k \to \infty} \int_{J_{u_k} \cap V_\rho} g_k'(x, [w_k], \nu_{w_k}) \, d\mathcal{H}^{d-1} = \liminf_{k \to \infty} G(w_k).
$$

Moreover, since $(u_k)_k$ is a recovery sequence for $u$ and $E'(u, \partial V_\rho) = 0$, (68) yields

$$
\int_{J_u \cap V_\rho} g'(x, [u], \nu_u) \, d\mathcal{H}^{d-1} = \lim_{k \to \infty} \int_{J_{u_k} \cap V_\rho} g_k'(x, [u_k], \nu_{u_k}) \, d\mathcal{H}^{d-1}.
$$

The previous two equations contradict (70). This concludes the proof of (iii).
Remark 4.4. Recalling the definition of the recovery sequence \( \tilde{u}_k = u_k - \varphi_k \) below equation (64), we find \( H^{d-1}(J_{\tilde{u}_k} \setminus \Omega) \to 0 \) by (61) (iii) and (64), i.e., except for an asymptotically vanishing part, the jump set is contained in \( \Omega \). This shows that the surface density \( g'(x, \xi, \nu) \) for \( x \in \partial_D \Omega \) is completely determined by \( (g_k)_k \), where \( g_k : \Omega \times \mathbb{R}^m \times S^{d-1} \to \mathbb{R}_+ \). In particular, it is independent of the choice of \( \Omega' \) and of the constant value \( c' \) of \( g'_k \) on \( \Omega' \setminus \Omega \) as long as \( c' > c_5 \).

Remark 4.5. Consider the situation of Theorem 4.1 for \( E \in \mathcal{E}_\Omega' \), with densities \( f, g \) such that \( E(\cdot, A) \) is lower semicontinuous in \( L^0(\Omega'; \mathbb{R}^m) \) for all \( A \in \mathcal{A}(\Omega') \). Consider the corresponding constant sequence \( \tilde{E}'_k \) defined in Lemma 4.3 with densities given in (59)-(60). Let \( f', g' \) be the densities of the \( \Gamma \)-limit \( \tilde{E}' \). One can show that \( f(x, \xi) = f'(x, \xi) \) and \( g(x, \xi, \nu) = g'(x, \xi, \nu) \) for \( x \in \Omega \). The surface densities, however, may differ on \( \partial_D \Omega \) since \( g' \) is completely determined by the restriction of \( g \) on \( \Omega \) in the first variable, cf. Remark 4.4.

Consider, e.g., the densities \( f(x, \xi) = c_1 |\xi|^p \) and
\[
g(x, \xi, \nu) = \begin{cases} c_5 & x \in \Omega \\ c_4 & \in \Omega' \setminus \Omega, \end{cases}
\]
where \( c_4 < c_5 \). Then \( g(x, \xi, \nu) = c_4 \) and \( g'(x, \xi, \nu) = c_5 \) for \( x \in \partial_D \Omega \).

We close with a result about convergence of minimizers.

Theorem 4.6 (Convergence of minimizers). Consider a sequence of functionals \( (\tilde{E}'_k)_k \) and the limiting energy \( \tilde{E}' \) given by Lemma 4.3 for boundary data \((h_k)_k \in W^{1,p}(\Omega'; \mathbb{R}^m) \) which converge strongly in \( W^{1,p}(\Omega'; \mathbb{R}^m) \) to \( h \). Then
\[
\inf_{v \in L^p(\Omega'; \mathbb{R}^m)} \tilde{E}'_k(v) \to \min_{v \in L^p(\Omega'; \mathbb{R}^m)} \tilde{E}'(v) \quad \text{(71)}
\]
for \( k \to \infty \). Moreover, for each sequence \((u_k)_k \) with
\[
\tilde{E}'_k(u_k) \leq \inf_{v \in L^p(\Omega'; \mathbb{R}^m)} \tilde{E}'_k(v) + \varepsilon_k \quad \text{(72)}
\]
for some \( \varepsilon_k \to 0 \), there exist a subsequence (not relabeled), modifications \((y_k)_k \) satisfying \( L^d(\{ \nabla y_k \neq \nabla u_k \}) \to 0 \) as \( k \to \infty \), and \( u \in \text{GSBV}^p(\Omega'; \mathbb{R}^m) \) with \( y_k \to u \) in measure on \( \Omega' \) such that
\[
\lim_{k \to \infty} \tilde{E}'_k(y_k) = \lim_{k \to \infty} \tilde{E}'_k(u_k) = \tilde{E}'(u) = \min_{v \in L^p(\Omega'; \mathbb{R}^m)} \tilde{E}'(v).
\]

Proof. The statement follows in the spirit of the fundamental theorem of \( \Gamma \)-convergence, see [10] Theorem 1.21]. Given \((u_k)_k \in \text{GSBV}^p(\Omega'; \mathbb{R}^m) \) satisfying (72), we apply Theorem 3.1 on the functionals \((\tilde{E}'_k)_k \) and find a subsequence (not relabeled), \((y_k)_k \in \text{GSBV}^p(\Omega'; \mathbb{R}^m) \) with \( L^d(\{ \nabla y_k \neq \nabla u_k \}) \to 0 \) and
\[
\liminf_{k \to \infty} \tilde{E}'_k(y_k) = \liminf_{k \to \infty} \tilde{E}'_k(u_k) = \liminf_{k \to \infty} \tilde{E}'_k(u_k) = \liminf_{v \in L^p(\Omega'; \mathbb{R}^m)} \tilde{E}'_k(v).
\]
Here, the first equality holds as \( y_k = h_k \) on \( \Omega' \setminus \overline{\Omega} \). By Theorem 3.1 we also get \( u \in \text{GSBV}^p(\Omega'; \mathbb{R}^m) \) satisfying \( u = h \) on \( \Omega' \setminus \overline{\Omega} \) with \( y_k \to u \) in measure on \( \Omega' \). Thus, by the \( \Gamma \)-liminf inequality in Lemma 4.3 we derive
\[
\tilde{E}'(u) \leq \liminf_{k \to \infty} \tilde{E}'_k(y_k) \leq \liminf_{k \to \infty} \tilde{E}'_k(u_k) \leq \liminf_{v \in L^p(\Omega'; \mathbb{R}^m)} \tilde{E}'_k(v).
\]
Again by Lemma 4.3 for each \( w \in L^0(\Omega'; \mathbb{R}^m) \) we find a recovery sequence \((w_k)_k \) converging to \( w \) in measure satisfying \( \lim_{k \to \infty} \tilde{E}'_k(w_k) = \tilde{E}'(w) \). This implies
\[
\limsup_{k \to \infty} \inf_{v \in L^p(\Omega'; \mathbb{R}^m)} \tilde{E}'_k(v) \leq \lim_{k \to \infty} \tilde{E}'_k(w_k) = \tilde{E}'(w).
\]
By combining \((73)-(74)\) we find
\[
\tilde{E}'(u) \leq \liminf_{k \to \infty} \inf_{v \in L^p(\tilde{\Omega}'; \mathbb{R}^m)} \tilde{E}_k'(v) \leq \limsup_{k \to \infty} \inf_{v \in L^p(\tilde{\Omega}'; \mathbb{R}^m)} \tilde{E}_k'(v) \leq \tilde{E}'(w). \tag{75}
\]
Since \(w \in L^p(\tilde{\Omega}'; \mathbb{R}^m)\) was arbitrary, we get that \(u\) is a minimizer of \(\tilde{E}'\). The statement follows from \((72)\) and \((75)\) with \(w = u\). In particular, the limit in \((71)\) does not depend on the specific choice of the subsequence and thus \((71)\) holds for the whole sequence. \(\square\)

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