THE FUNDAMENTAL THEOREM OF
AFFINE AND PROJECTIVE GEOMETRIES

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Abstract

We deal with two natural generalizations of the Fundamental Theorem of Affine Geometry to the case of non bijective maps. These extensions were obtained by W. Zick in 1981, but his results were not published. Our aim is to expose simplified or new proofs of these extensions and discuss their context and naturality.

Key words: Fundamental Theorem, parallel morphisms, morphisms of geometries

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Introduction

Let \( A, A' \) be affine spaces of dimensions \( \geq 2 \) over division rings \( K, K' \), respectively (of orders \( \neq 2 \)). The classical Fundamental Theorem states that collineations \( A \to A' \) (bijections transforming lines into lines) are just semi-affinities.

A similar result, essentially equivalent, holds in Projective Geometry. Let \( \mathbb{P}(V) \) and \( \mathbb{P}(V') \) be projective spaces of dimensions \( \geq 2 \), associated to some vector spaces \( V \) and \( V' \) over division rings \( K \) and \( K' \). The Fundamental Theorem [1] states that collineations \( \mathbb{P}(V) \to \mathbb{P}(V') \) are just maps induced by semilinear isomorphisms \( V \to V' \).

It is natural to extend these results to the case of non necessarily bijective maps. For projective spaces, such extension was obtained by Faure and Frölicher [8] and Havlicek [12] in 1994. For affine spaces, W. Zick obtained in 1981 two different generalizations of the Fundamental Theorem. But his results have not been published and it is not easy to get his preprints [24], [25]. We learned of their existence from the papers [13], [14].

The aim of this paper is to expose such generalizations of the Fundamental Theorem, proving Zick’s results, and discussing the context where such extensions naturally fit.

Let us explain the content of this article.
Section 1. A preliminary problem, underlying our subject, is to determine the morphisms between affine and projective spaces that we should consider. That is to say, what is the category where we must place affine and projective spaces? Texts on affine and projective geometry ignore such question. For example, the basic fact that any affine space admits a natural embedding into a projective space is presented (see [2] or [3]) without proposing a definition of embedding.

We shall work in the category of closure spaces satisfying MacLane’s exchange axiom and a finitary condition. For the sake of brevity, following [9] and [11], such objects are simply named geometries. These objects have a natural notion of morphism. In the most common case (geometries generated by lines) a morphism of geometries \( \varphi : X \rightarrow X' \) is just a map transforming collinear points into collinear points and such that it is injective or constant on any line.

Sections 2-3. We shall prove the following extension of the Fundamental Theorem of Affine Geometry for non necessarily bijective maps,

**Fundamental Theorem (Zick [24]).** A map \( \kappa \rightarrow \kappa' \) between affine spaces, such that the image is not contained in a line, is a semiaffine morphism if and only if it is a parallel morphism.

Parallel morphisms are just morphisms of geometries preserving parallelism in a natural sense.

The proof is elementary and quite brief, so that we think that this version of the Fundamental Theorem should be included in textbooks, the classical version for bijective maps being a corollary.

The natural question of determining the morphisms of geometries (eventually non parallel) between affine spaces is solved by Zick’s second theorem that we comment below.

Section 4. In this section we discuss the role of the Fundamental Theorem in the equivalence between the algebraic and synthetic point of views in the affine geometry. Essentially there are two definitions of affine space: The algebraic definition (where an affine space is a set endowed with an action of a vector space) and the synthetic definition (based on incidence and parallelism axioms). Both definitions are essentially equivalent; the proof is not difficult, but it is laborious. Both definitions of affine space suggest in a natural way a notion of morphism between affine spaces. In the algebraic case, the notion of morphism corresponds to the concept of semiaffine morphism, while in the synthetic case it corresponds to the concept of parallel morphism. Zick’s Fundamental Theorem states the equivalence of both concepts of morphism, so confirming the equivalence between the algebraic and the synthetic points of view.

Sections 5-6. These sections deal with the following generalization of the classical Fundamental Theorem of Projective Geometry, obtained by Faure–Frölicher [8] and Havlicek [12]:

**Fundamental Theorem.** A map \( \phi : \mathbb{P}(V) \rightarrow \mathbb{P}(V) \), such that the image is not contained in a line, is a morphism of geometries if and only if it is induced by an injective semilinear map \( \Phi : V \rightarrow V' \).
Actually, the theorem was stated in more generality for partial maps defined on the complement of a subspace $E \subset \mathbb{P}(V)$, satisfying certain conditions. We shall not prove this result (an elementary proof is given in [7]); we limit ourselves to show that the conditions imposed on the partial map $\phi: \mathbb{P}(V) \to \mathbb{P}(V')$ correspond to a general notion of partial morphism between geometries.

Section 7. In this last section we prove another theorem of Zick, determining the morphisms of geometries, non necessarily parallel, between affine spaces.

Theorem (Zick [25]). Let $V$, $V'$ be vector spaces over division rings $K$, $K'$, respectively (with $|K| \geq 4$ or $|K| = 3 = \text{charact } K'$). Let $\varphi: V \to V'$ be a map such that $\varphi(0) = 0$ and the image is not contained in an affine line of $V'$. Then, $\varphi$ is a morphism of geometries if and only if $\varphi$ is a fractional semilinear map.

A map $\varphi: V \to V'$ is said to be a fractional semilinear map if there are semilinear maps $\Phi: V \to V'$, $\omega: V \to K'$, with the same associated morphism $K \to K'$, such that $\varphi(x) = \frac{1}{1+\omega(x)} \Phi(x)$.

We obtain this result as a consequence of the Faure–Frölicher–Havlicek theorem and an extension theorem stating that, under certain very general conditions, a morphism of geometries $A \to \mathbb{P}'$, between an affine space and a projective space, extends to the projective closure of $A$, defining a partial morphism of geometries $\mathbb{P} \to \mathbb{P}'$. This extension theorem seems to be unknown in the literature.

1 Geometries and morphisms

In this preliminary section we fix the category where we shall work.

The spaces that we shall consider have different names in the literature. For the sake of brevity we shall use the name geometry, also used in [9] and [11].

Definition. A geometry is a set $X$ (whose elements are named points) with a family $\mathcal{F}$ of subsets of $X$ (named subspaces or flats), satisfying the following four axioms:

G1. $\emptyset \in \mathcal{F}$, $X \in \mathcal{F}$ and $\{x\} \in \mathcal{F}$ for any $x \in X$.

The subspace $\{x\}$ will be simply denoted by $x$.

G2. Any intersection $\bigcap S_i$ of subspaces is a subspace.

This axiom implies that the set $\mathcal{F}$ of subspaces of $X$, with the inclusion, is a complete lattice: For any set $\{S_i\}$ of subspaces, the infimum is $\bigwedge S_i = \bigcap S_i$ and the supremum is $\bigvee S_i = \bigcup T_k$ where $T_k$ runs over the subspaces of $X$ containing $\bigcup S_i$.

G3 (exchange axiom). Given a subspace $S$ and a point $x \notin S$, there is no subspace $S'$ such that $S \subseteq S' \subseteq S \vee x$ (strict inclusions).

That is to say, if $x, y \notin S$ then $y \in S \vee x \iff x \in S \vee y$.

The closure of a subset $A \subseteq X$ is the least subspace $\overline{A}$ containing $A$.

G4 (finitary axiom). For any subset $A \subseteq X$ and any point $x \notin A$, we have $x \in \{a_1, \ldots, a_r\}$ for some finite subset $\{a_1, \ldots, a_r\} \subseteq A$. 

3
Note. A geometry is a particular case of a closure space: A set $X$ endowed
with a map $\mathcal{P}(X) \to \mathcal{P}(X), A \mapsto \overline{A}$, satisfying the axioms,

\begin{align*}
(i) & \quad A \subseteq \overline{A}, \\
(ii) & \quad A \subseteq B \Rightarrow \overline{A} \subseteq \overline{B}
\end{align*}

Axioms G3 and G4 enable us to settle a beautiful dimension theory that we
next resume.

Definitions. Given a geometry $X$, a subset $A \subseteq X$ is said to be independent
when for any $a \in A$ we have $a \notin \overline{A} \setminus a$.

Let $A$ be a subset of a subspace $S$. If $\overline{A} = S$ we say that $A$ generates $S$.
If $A$ is independent and generates $S$, we say that $A$ is a basis of $S$.

Some easy properties:

Any subset of an independent set is independent.

A subset $A \subseteq X$ is independent if and only if so is any finite subset of $A$.

Let $A \subseteq X$ be an independent subset and $p \in X$. If $p \notin \overline{A}$ then $A \cup p$ is
independent.

Theorem 1.1 Let $S$ be a subspace of a geometry $X$. Then:

- Any set generating $S$ contains a basis of $S$.
- Any independent subset of $S$ is contained in a basis of $S$.
- Any two bases of $S$ have equal cardinal.

For a proof in a more general context, see [19] or [9] Chap. IV.

Definition. The dimension of a subspace is the common cardinal of its bases
minus 1.

Points are just subspaces of dimension 0. Subspaces of dimension 1 are
named lines and subspaces of dimension 2 planes.

Corollary 1.2 Let $A = \{p_0, \ldots, p_r\} \subseteq X$ be an independent finite set. Then
$\overline{A} = p_0 \vee \cdots \vee p_r$ is a subspace of dimension $r$.

In particular, any two distinct points $p_0, p_1$ lie in a unique line $p_0 \vee p_1$, and
any three non collinear points $p_0, p_1, p_2$ lie in a unique plane $p_0 \vee p_1 \vee p_2$.

1.1 Morphisms of geometries

Proposition 1.3 Let $\varphi : X \to X'$ be a map between geometries. The following
statements are equivalent:

a) The preimage of a subspace $S' \subseteq X'$ is a subspace $\varphi^{-1}(S') \subseteq X$,

b) For any subset $A \subseteq X$ we have $\varphi(\overline{A}) \subseteq \overline{\varphi(A)}$, that is to say,

\[ x \in \overline{A} \Rightarrow \varphi(x) \in \overline{\varphi(A)} \]

c) For any finite subset $A \subseteq X$ we have $\varphi(\overline{A}) \subseteq \overline{\varphi(A)}$. 

4
Proof. $(a \Rightarrow b)$. Since $\varphi(A)$ is a subspace of $X'$, it follows that $\varphi^{-1}(\varphi(A))$ is a subspace of $X$. Hence the inclusion $A \subseteq \varphi^{-1}(\varphi(A))$ implies $A \subseteq \varphi^{-1}(\varphi(A))$, so that $\varphi(A) \subseteq \varphi(A)$.

$(b \Rightarrow a)$. Let $S' \subseteq X'$ be a subspace. By $b$ we have $\varphi(\varphi^{-1}(S')) \subseteq \varphi(\varphi^{-1}(S')) \subseteq S' = S'$, hence $\varphi^{-1}(S') \subseteq \varphi^{-1}(S')$, so that $\varphi^{-1}(S')$ is a subspace.

$(b \Rightarrow c)$. Let us put $A \subseteq X$ as a union of all finite subsets: $A = \bigcup A_i$. By axiom G4, $A = \bigcup A_i$, hence $\varphi(A) = \bigcup \varphi(A_i) \subseteq \bigcup \varphi(A_i) \subseteq \varphi(A)$.

□

**Definition.** A map $\varphi : X \to X'$, between geometries, is a morphism when it fulfills the equivalent conditions of 1.3. According to 1.4 we have:

1.4 A map $\varphi : X \to X'$, between geometries, is a morphism if and only if for any finite subset $\{x_0, \ldots, x_r\} \subseteq X$ we have

$$x_0 \in x_1 \lor \cdots \lor x_r \Rightarrow \varphi(x_0) \in \varphi(x_1) \lor \cdots \lor \varphi(x_r)$$

**Definition.** We say that a geometry $X$ is generated by lines when the subspaces of $X$ are just the subsets $S \subseteq X$ such that

$$x_1, x_2 \in S \Rightarrow x_1 \lor x_2 \subseteq S$$

That is to say, in a geometry generated by lines, subspaces are just subsets containing the line joining any two different points of such subset.

We say that a geometry $X$ is generated by lines and planes when the subspaces of $X$ are just the subsets $S \subseteq X$ such that

$$x_1, x_2, x_3 \in S \Rightarrow x_1 \lor x_2 \lor x_3 \subseteq S$$

That is to say, a subset $S$ is a subspace if it contains the line joining any two different points of $S$ and the plane defined by any three non collinear points of $S$.

We shall see that projective spaces are geometries generated by lines. Affine spaces over a division ring $K \neq \mathbb{Z}_2$ also are generated by lines, and affine spaces over $\mathbb{Z}_2$ are generated by lines and planes.

1.5 Let $\varphi : X \to X'$ be a map between geometries, where $X$ is generated by lines. The map $\varphi$ is a morphism if and only if

$$x_0 \in x_1 \lor x_2 \Rightarrow \varphi(x_0) \in \varphi(x_1) \lor \varphi(x_2)$$

(1)

for any $x_0, x_1, x_2 \in X$. This condition states that $\varphi$ transforms any three collinear points into collinear points and that the restriction of $\varphi$ to any line is injective or constant.

From (1) it follows easily (when $X$ is generated by lines) that the preimage of any subspace also is a subspace, hence $\varphi$ is a morphism. The converse holds by 1.4.

A similar argument proves the following statement.
1.6 Let $\varphi: X \to X'$ be a map between geometries, where $X$ is generated by lines and planes. The map $\varphi$ is a morphism if and only if

$$x_0 \in x_1 \lor x_2 \lor x_3 \Rightarrow \varphi(x_0) \in \varphi(x_1) \lor \varphi(x_2) \lor \varphi(x_3)$$

for any $x_0, x_1, x_2, x_3 \in X$.

**Definition.** A morphism of geometries $\varphi: X \to X'$ is an **isomorphism** if it is biyective and the inverse map $\varphi^{-1}$ also is a morphism.

A bijective morphism may be not an isomorphism. For example, given a geometry $X$ of dimension $n \geq 2$ and a natural number $m < n$, let $X'$ be the geometry (named truncation of $X$) with the same underlying set $X$ but such that the proper subspaces are just the subspaces of $X$ of dimension $< m$ (so that $\dim X' = m$). The identity $X \to X'$ is a bijective morphism but not an isomorphism. There are more interesting examples; Ceccherini [6] gives a bijective morphism between a 4-dimensional projective space and a non-arguesian projective plane.

**Note.** An isomorphism $\varphi: X \to X'$, between geometries generated by lines, is also said to be a **collineation**. That is to say, a collineation (between such geometries) is just a bijection transforming lines into lines.

**Proposition 1.7** Let $\varphi: X \to X'$ be a morphism of geometries. We have:

a) If $\varphi$ is surjective, then $\dim X \geq \dim X'$,

b) If $\varphi$ is surjective and $\dim X = \dim X' < \infty$ then $\varphi$ is an isomorphism.

**Proof.** a). Let $\{x'_i = \varphi(x_i)\}$ be a basis of $X'$. It is easy to see that $\{x_i\}$ is an independent set in $X$, hence $\dim X \geq \dim X'$.

b). Let $\{x_0, \ldots, x_r\}$ be a basis of $X$. We have

$$X' = \varphi(X) = \varphi(\{x_0, \ldots, x_r\}) \subseteq \{\varphi(x_0), \ldots, \varphi(x_r)\}$$

hence $\{\varphi(x_0), \ldots, \varphi(x_r)\}$ generates $X'$ and, since $\dim X' = \dim X = r$, we conclude that $\{\varphi(x_0), \ldots, \varphi(x_r)\}$ is a basis of $X'$.

Since $\varphi$ transforms bases into bases, it also transforms independent sets into independent sets. In particular, $\varphi$ is injective, hence bijective.

We have to prove that $\varphi^{-1}$ is a morphism, that is to say, if $S$ is a subspace of $X$ then $\varphi(S)$ is a subspace of $X'$. Let $\{s_0, \ldots, s_n\}$ be a basis of $S$. We have an inclusion

$$\varphi(S) = \varphi(s_0 \lor \cdots \lor s_n) \subseteq \varphi(s_0) \lor \cdots \lor \varphi(s_n) =: S'$$

This inclusion is an equality (so that $\varphi(S) = S'$ is a subspace of $X'$) because otherwise there is $x' \in S'$ such that $x := \varphi^{-1}(x') \notin S$, so that $\{s_0, \ldots, s_n, x\}$ is independent so contradicting that the image $\{\varphi(s_0), \ldots, \varphi(s_n), x' = \varphi(x)\}$ is dependent.

$\Box$
In the somewhat different context of linear spaces, the above proposition is proved in [17].

**Definition.** Given a geometry \( X \), any subset \( A \subseteq X \) is a geometry where subspaces are just subsets \( S \cap A \) with \( S \) a subspace of \( X \). Then we say that \( A \) is a subgeometry of \( X \).

The natural inclusion \( A \hookrightarrow X \) is a morphism.

**Definition.** A morphism \( \varphi: X \to X' \) is an embedding when \( \varphi: X \to \varphi(X) \) is an isomorphism, where \( \varphi(X) \) is considered as a subgeometry of \( X' \).

## 2 Affine Geometry

### 2.1 An affine space

An affine space is a set \( \mathbb{A} \) (its elements are named points) together with a (left) vector space \( V \) over a division ring \( K \) and a map \(+: \mathbb{A} \times V \to \mathbb{A}\), \((p,v) \mapsto p + v\), such that the following axioms are satisfied:

1. \((p + v_1) + v_2 = p + (v_1 + v_2)\) for all \( p \in \mathbb{A}, v_1, v_2 \in V \),
2. \(p + v = p \iff v = 0\) for all \( p \in \mathbb{A}, v \in V \),
3. Given two points \( p, \bar{p} \in \mathbb{A} \) there is a vector \( v \in V \) (necessarily unique) such that \( \bar{p} = p + v \).

An affine space \((\mathbb{A}, V, +)\) is simply denoted by \( \mathbb{A} \).

**Definition.** A non-empty subset \( S \subseteq \mathbb{A} \) is a subspace when it is \( S = p + W := \{p + w : w \in W\} \) where \( p \in \mathbb{A} \) is a point and \( W \subseteq V \) is a vector subspace. Then \( W \) is said to be the direction of \( S \).

We agree that the empty subset also is a subspace.

Remark that a subspace \( S \) of direction \( W \) is an affine space \((S, W, +)\).

### 2.2 It is easy to check that an affine space, with the family of subspaces, is a geometry.

Moreover, for any subspace \( S = p + W \) we have that the dimension of \( S \) coincides with the dimension of \( W \) as a vector space.

**Definition.** Two non-empty subspaces \( S = p + W \) and \( S' = p' + W' \) are said to be parallel when both have the same direction: \( S \parallel S' \iff W = W' \).

Note that parallel subspaces have the same dimension.

**Definition.** Let \( V, V' \) be vector spaces over division rings \( K, K' \), respectively. A map \( \varphi: V \to V' \) is said to be semilinear when:

1. It is additive: \( \varphi(v_1 + v_2) = \varphi(v_1) + \varphi(v_2) \) for all \( v_1, v_2 \in V \),
2. There is a ring morphism \( \sigma: K \to K' \) such that \( \varphi(\lambda v) = \sigma(\lambda) \varphi(v) \) for all \( \lambda \in K, v \in V \).
Remark that we do not require that the morphism $\sigma: K \to K'$ be surjective, as it is usual. If $\varphi$ is not the null map, then the morphism $\sigma$ is unique and it is said to be the **morphism associated** to $\varphi$.

A semilinear map $\varphi: V \to V'$ is said to be a **semilinear isomorphism** when $\varphi$ and $\sigma$ are bijective. In such case the inverse map $\varphi^{-1}: V' \to V$ also is semilinear.

**Definition.** Let $(A, V, +)$ and $(A', V', +)$ be affine spaces over division rings $K$ and $K'$, respectively. A map $\varphi: A \to A'$ is a **semiaffine morphism** when there is a semilinear map $\vec{\varphi}: V \to V'$ such that
\[
\varphi(p + v) = \varphi(p) + \vec{\varphi}(v) \quad \forall \ p \in A, \ v \in V
\]
The semilinear map $\vec{\varphi}$ is unique and it is named **differential** of $\varphi$.

A semiaffine morphism $\varphi: A \to A'$ is a **semiaffine isomorphism** or a **semiaffinity** when both $\varphi$ and the ring morphism $\sigma: K \to K'$ (associated to $\vec{\varphi}$) are bijective. In such case the inverse map $\varphi^{-1}: A' \to A$ also is a semiaffine morphism.

The prefix *semi* in the terms *semilinear*, *semiaffine*, *semiaffinity* is deleted when $K = K'$ and the associated ring morphism $\sigma: K \to K'$ is the identity.

2.3 Any vector space $V$ has an underlying structure of affine space $(A = V, V, +)$, where the map $+: A \times V \to A$ is just the addition of vectors,
\[
A \times V = V \times V \xrightarrow{+} V = A
\]

Conversely, given an affine space $(A, V, +)$ and a fixed point $p_0 \in A$ we have an affine isomorphism
\[
\begin{array}{ccc}
V & \xrightarrow{\varphi} & A \\
& \downarrow & \downarrow \\
& v & \mapsto p_0 + v
\end{array}
\]
Remark that $0 \mapsto p_0$. This isomorphism supports the colloquial statement that an affine space is a vector space where we have forgotten the origin; once we fix a point $p_0 \in A$ as the **origin** we have an identification $A = V$.

2.4 Let $V$ and $V'$ vector spaces, hence also affine spaces, over division rings $K$ and $K'$, respectively. A map $\varphi: V \to V'$ is a semiaffine morphism if and only if it is
\[
V \xrightarrow{\varphi} V' , \quad \varphi(v) = \vec{\varphi}(v) + a'
\]
where $\vec{\varphi}: V \to V'$ is a semilinear map and $a' := \varphi(0)$.

**Lemma 2.5** Let $V$ be a vector space over a division ring $K$ with $|K| \neq 2$. Let $W \subseteq V$ be a subset such that
- $0 \in W$,
- If $w_1, w_2 \in W$ then $(1 - t)w_1 + tw_2 \in W, \ \forall t \in K$ (that is to say, $W$ contains the affine line passing through the points $w_1, w_2$).

Then $W$ is a vector subspace.
Proof. It is enough to show that \( \langle w_1, w_2 \rangle \subseteq W \) whenever \( w_1, w_2 \in W \).

Remark that if \( w \in W \) then \( \langle w \rangle \subseteq W \): For any \( t \in K \) we have \( tw = (1 - t)0 + tw \in W \).

Now, given \( w_1, w_2 \in W \), for all \( x, y \in K \) we have \( xw_1, yw_2 \in W \), hence
\[
W \ni (1 - t)xw_1 + tyw_2 = \bar{x}w_1 + \bar{y}w_2, \quad \text{where } \bar{x} := (1 - t)x, \bar{y} := ty
\]
Taking \( t \neq 0, 1 \) (since \( |K| \neq 2 \)), the values of \( \bar{x}, \bar{y} \) are arbitrary, so that the vector \( \bar{x}w_1 + \bar{y}w_2 \) is any vector of \( \langle w_1, w_2 \rangle \).

\[\square\]

Let \( \mathbb{A} \) be an affine space over a division ring \( K \). Considering it as a geometry we have:

**Proposition 2.6** If \( |K| \neq 2 \), the affine space \( \mathbb{A} \) is generated by lines.

**Proof.** Let \( S \subseteq \mathbb{A} \) be a subset containing the line joining any two different points of \( S \). We have to show that \( S \) is a subspace. Fix \( p_0 \in S \) and consider the affine isomorphism \( V \simeq \mathbb{A}, v \mapsto p_0 + v \). Via this isomorphism, the subset \( S \) corresponds to a subset \( W \subseteq V \) fulfilling the conditions of the lemma, so that \( W \) is a vector (hence affine) subspace of \( V \) and, therefore, \( S \) is a subspace of \( \mathbb{A} \).

\[\square\]

**Remark 2.7** The lemma and the proposition do not hold on the field \( K = \mathbb{Z}_2 \), but it is easy to check that in such case the geometry \( \mathbb{A} \) is generated by lines and planes.

## 3 Fundamental Theorem of Affine Geometry

### 3.1 Parallel morphisms

The Fundamental Theorem of Affine Geometry gives a geometric characterization of semiaffine morphisms. Precisely, it states that such morphisms coincide with the following ones,

**Definition (Zick [24]).** A map \( \varphi: \mathbb{A} \to \mathbb{A}' \), between affine spaces, is a parallel morphism when
\[
(a \lor b) \parallel (c \lor d) \Rightarrow (\varphi(a) \lor \varphi(b)) \parallel (\varphi(c) \lor \varphi(d))
\]
for all \( a, b, c, d \in \mathbb{A} \).

Recall that any two parallel subspaces have equal dimension. Therefore the expression \( (a \lor b) \parallel (c \lor d) \) means that both \( a \lor b \) and \( c \lor d \) are parallel lines or both are points \( (a = b \text{ and } c = d) \).

**Lemma 3.1** Any parallel morphism \( \varphi: \mathbb{A} \to \mathbb{A}' \) fulfills the property
\[
x_0 \in x_1 \lor x_2 \Rightarrow \varphi(x_0) \in \varphi(x_1) \lor \varphi(x_2)
\]
for any \( x_0, x_1, x_2 \in \mathbb{A} \).
Proof. If \( x_1 = x_2 \) then \( x_0 = x_1 = x_2 \) and it is clear. Otherwise \( x_0 \) is not \( x_1 \) or \( x_2 \), let us assume that \( x_0 \neq x_2 \). We have \( x_1 \lor x_2 = x_0 \lor x_2 \), hence \( x_1 \lor x_2 \parallel x_0 \lor x_2 \), so that \( \varphi(x_1) \lor \varphi(x_2) \parallel \varphi(x_0) \lor \varphi(x_2) \), and therefore \( \varphi(x_1) \lor \varphi(x_2) = \varphi(x_0) \lor \varphi(x_2) \). □

As a consequence, the restriction of a parallel morphism \( \varphi: \mathbb{A} \to \mathbb{A}' \) to a line is constant or it is an injection into a line of \( \mathbb{A}' \). Now the next statement directly follows from the definition.

3.2 A map \( \varphi: \mathbb{A} \to \mathbb{A}' \), between affine spaces, is a parallel morphism if and only if it fulfills the following condition:

For any two parallel lines \( L_1, L_2 \subseteq \mathbb{A} \) the restrictions \( \varphi|_{L_1} \) and \( \varphi|_{L_2} \) are both constant or both injective and in such case the images \( \varphi(L_1), \varphi(L_2) \) lie in two parallel lines \( L_1', L_2' \subseteq \mathbb{A}' \).

Lemma 3.3 Any parallel morphism \( \varphi: \mathbb{A} \to \mathbb{A}' \) fulfills the property

\[
x_0 \in x_1 \lor x_2 \lor x_3 \implies \varphi(x_0) \in \varphi(x_1) \lor \varphi(x_2) \lor \varphi(x_3)
\]

for any \( x_0, x_1, x_2 \).

Proof. If \( x_0 \) is collinear with some pair of points in \( \{x_1, x_2, x_3\} \) we conclude by 3.1. Otherwise, we consider in the plane \( x_1 \lor x_2 \lor x_3 \) the parallelogram defined by the lines \( x_1 \lor x_2, x_1 \lor x_3 \) and the respective parallel lines passing through \( x_0 \). The image by \( \varphi \) of this parallelogram is a parallelogram in \( \mathbb{A}' \) (perhaps degenerate) and then it is immediate that \( \varphi(x_0) \in \varphi(x_1) \lor \varphi(x_2) \lor \varphi(x_3) \). □

Corollary 3.4 Any parallel morphism \( \varphi: \mathbb{A} \to \mathbb{A}' \) is a morphism of geometries.

Proof. Since \( \mathbb{A} \) is a geometry generated by lines and planes \( [2.6, 2.7] \), the above lemma and \( [1.6] \) imply that \( \varphi \) is a morphism of geometries. □

Proposition 3.5 Any semiaffine morphism \( \varphi: \mathbb{A} \to \mathbb{A}' \) is a parallel morphism.

Proof. Let \( L_1 = p_1 + \langle v \rangle, L_2 = p_2 + \langle v \rangle \) be two parallel lines of \( \mathbb{A} \). If \( \varphi(v) = 0 \) then \( \varphi(L_1) = \varphi(p_1) \) and \( \varphi(L_2) = \varphi(p_2) \). If \( \varphi(v) \neq 0 \) then \( \varphi \) embeds the lines \( L_i = p_i + \langle v \rangle \) into the lines \( L'_i = \varphi(p_i) + \langle \varphi(v) \rangle, (i = 1, 2) \), which are parallel. By [3.2] \( \varphi \) is a parallel morphism. □
3.2 Proof of the fundamental theorem

In this section, \( V, V' \) are vector spaces over division rings \( K, K' \), respectively.

**Lemma 3.6** \cite{Zick} Let \( \varphi, \phi : V \to V' \) be additive maps. If for any \( x \in V \) we have \( \phi(x) \in K' \cdot \varphi(x) \) and the image of \( \varphi \) contain two linearly independent vectors, then there is a scalar \( \lambda \in K' \) such that \( \phi = \lambda \cdot \varphi \).

**Proof.** For any \( x \in V \setminus \ker \varphi \) we have \( \phi(x) = \lambda_x \cdot \varphi(x) \) for a unique scalar \( \lambda_x \in K' \). We have to show that \( \lambda_x \) does not depend on \( x \). Let \( x, y \in V \setminus \ker \varphi \); we distinguish two cases.

1. \( \varphi(x) \) and \( \varphi(y) \) are linearly independent. Then \( x, y, x+y \in V \setminus \ker \varphi \) and the equality \( \phi(x+y) = \phi(x) + \phi(y) \) shows that \( \lambda_{x+y} \varphi(x+y) = \lambda_x \varphi(x) + \lambda_y \varphi(y) \), that is to say, \( \lambda_{x+y} \varphi(x) + \lambda_{x+y} \varphi(y) = \lambda_x \varphi(x) + \lambda_y \varphi(y) \), hence \( \lambda_x = \lambda_{x+y} = \lambda_y \).

2. \( \varphi(x) \) and \( \varphi(y) \) are linearly dependent. Take \( z \in V \) such that \( \varphi(z) \) is linearly independent of both vectors. According to the former case, we have \( \lambda_x = \lambda_z = \lambda_y \).

\( \square \)

**Proposition 3.7** \cite{Zick} Let \( \varphi : V \to V' \) be an additive map, such that \( \varphi(Kx) \subseteq K' \varphi(x) \) for all \( x \in V \), and such that the image contains two linearly independent vectors. Then \( \varphi : V \to V' \) is semilinear.

**Proof** \cite{Zick}. Given \( \lambda \in K \) we define the additive map \( \phi_\lambda(x) := \varphi(\lambda x) \). By the above lemma, there is a scalar \( \sigma(\lambda) \in K' \) such that \( \phi_\lambda = \sigma(\lambda) \varphi \), that is to say, \( \varphi(\lambda x) = \sigma(\lambda) \varphi(x) \). We have to check that \( \sigma : K \to K' \) is a ring morphism.

Taking \( x \in V \setminus \ker \varphi \) we have

\[ \varphi((\lambda_1 + \lambda_2)x) = \sigma(\lambda_1 + \lambda_2) \varphi(x) \]

and moreover

\[ \varphi((\lambda_1 + \lambda_2)x) = \varphi(\lambda_1 x + \lambda_2 x) = \sigma(\lambda_1) \varphi(x) + \sigma(\lambda_2) \varphi(x) = (\sigma(\lambda_1) + \sigma(\lambda_2)) \varphi(x) \]

so that \( \sigma(\lambda_1 + \lambda_2) = \sigma(\lambda_1) + \sigma(\lambda_2) \). Analogously we prove that \( \sigma(\lambda_1 \lambda_2) = \sigma(\lambda_1) \sigma(\lambda_2) \).

\( \square \)

Recall \cite{2.3} that a vector space \( V \) also is an affine space.

**Lemma 3.8** Let \( \varphi : V \to V', x \mapsto x', \) be a parallel morphism. If \( \varphi(0) = 0 \) and the image of \( \varphi \) contains two linearly independent vectors, then \( \varphi : V \to V' \) is additive.

**Proof.** Since \( \varphi \) transforms the parallelogram (eventually degenerated) with vertices \( 0, x, y, x+y \) into a parallelogram \( 0, x', y', (x+y)' \), we have

\[ (x+y)' = \lambda x' + y' = x' + \mu y' \]

for certain \( \lambda, \mu \in K' \).
When $x' \notin \langle y' \rangle$, then $(x + y)' = x' + y'$, because either $x' = 0$, so that we put $\lambda x' = x'$ en (2), or $x'$ and $y'$ are linearly independent (so that $\lambda = \mu = 1$). The case $y' \notin \langle x' \rangle$ is similar.

Otherwise we have $\langle x' \rangle = \langle y' \rangle$. By hypothesis, there exists $z \in V$ such that $z' \notin \langle x' \rangle = \langle y' \rangle$ and by (2) we also have $z' \notin \langle (x + y) \rangle$. By the former case, we have

$$
(x' + y' + z') = (x + y + z)' = (x + y)' + z'
$$

hence $x' + y' = (x + y)'$.

\[\square\]

**Lemma 3.9** Let $\varphi: V \to V'$ be a parallel morphism. If $\varphi(0) = 0$ then we have $\varphi(Kx) \subseteq K'\varphi(x)$ for all $x \in V$. 

*Proof.* By 3.1 we have $\varphi(x_1 \lor x_2) \subseteq \varphi(x_1) \lor \varphi(x_2)$, so that

$$
\varphi(Kx) = \varphi(0 \lor x) \subseteq \varphi(0) \lor \varphi(x) = 0 \lor \varphi(x) = K'\varphi(x)
$$

\[\square\]

**Proposition 3.10** Let $\varphi: V \to V'$ be a parallel morphism. If the image of $\varphi$ is not contained in an affine line, then $\varphi$ is a semiaffine morphism, that is to say, we have

$$
\varphi(x) = \varphi(x) + a'
$$

where $\varphi: V \to V'$ is a semilinear map and $a' \in V'$.

*Proof.* Composing $\varphi$ with a translation we may assume that $\varphi(0) = 0$. The above two lemmas show that $\varphi: V \to V'$ fulfills the hypotheses of proposition 3.7, hence $\varphi: V \to V'$ is semilinear.

\[\square\]

According to 2.3 any affine space is isomorphic (an affinity) to its direction: $A \simeq V$. Combining 3.5 and 3.10 we finally obtain

**Fundamental Theorem of Affine Geometry (Zick [24])** Let $\varphi: A \to A'$ be a map such that the image is not contained in a line. Then $\varphi$ is a semiaffine morphism if and only if it is a parallel morphism.

For bijective maps, we obtain the classical result,

**Corollary 3.11** Let $A, A'$ be affine spaces of dimensions $\geq 2$ over respective division rings $K, K'$, with $|K|, |K'| \neq 2$. A bijective map $\varphi: A \to A'$ is a semiaffinity if and only if it is a collineation.

*Proof.* ($\Rightarrow$). Any line $L = p + \langle v \rangle$ goes to a line $\varphi(L) = \varphi(p) + \langle \varphi(v) \rangle$.

($\Leftarrow$). Since $A, A'$ are generated by lines (2.6), any collineation $\varphi: A \to A'$ is an isomorphism of geometries, hence transforms planes into planes and then preserves parallelism, that is to say, $\varphi$ is a parallel morphism. By the
Fundamental Theorem, \( \varphi \) is a semiaffine morphism. Analogously, the inverse collineation \( \varphi^{-1} \) is a semiaffine morphism, hence \( \varphi \) is a semiaffinity.

\[\square\]

The classical result was first proved by E. Kamke [15]. More information about its history can be found in [16], p. 51-52.

4 The algebraic and synthetic points of view

Affine spaces may be defined in two very different ways. We have the algebraic definition given above, by means of certain algebraic operations, and on the other hand there is a synthetic definition, based on the incidence properties of subspaces. Each definition suggests certain definition of "morphism" between affine spaces. The equivalence between the algebraic and synthetic points of view means that both definitions of affine space are equivalent, and also that both concepts of morphism also are equivalent. This last equivalence is just the statement of the Fundamental Theorem, so that it should be understood as part of an equivalence of categories.

In this section we give a brief presentation of the synthetic concept of affine space and we discuss the notion of morphism.

In the literature there are several synthetic definitions of affine space. The definition that we take is due to Tamaschke [22]. We prefer this definition because it emphasizes parallelism as a primitive element in the concept of affine space (for a definition where parallelism is not involved, see [23]).

4.1 Definition. An affine space is a set \( \mathcal{A} \) (its elements are named points), with a family \( \mathcal{L} \) of subsets (named lines) endowed with an equivalence relation \( \parallel \) (named parallelism), satisfying the following axioms:

A1. Any two different points lie in a unique line,

A2. Any line has two points,

A3 (Parallel axiom). Given a line \( L \) and a point \( p \) there is a unique parallel line to \( L \) passing through \( p \),

A4 (Similar triangles axiom). Let \( a, b, c \) be three noncollinear points and let \( a', b' \) be two different points such that \( ab \parallel a'b' \). Then there exists a point \( c' \notin a'b' \) such that \( ac \parallel a'c' \) and \( bc \parallel b'c' \).

Definition In the context of the synthetic definition of affine space, a subset \( S \subseteq \mathcal{A} \) is said to be a subspace when it fulfills the following conditions:

a). The line joining any two different points of \( S \) is contained in \( S \),

b). For any line \( L \subseteq S \) and any point \( p \in S \), the parallel line to \( L \) passing through \( p \) also is contained in \( S \).

Condition (b) is superfluous when the lines of \( \mathcal{A} \) have at least three points.
Two non-empty subspaces $S$ and $S'$ are said to be parallel (we put $S \parallel S'$) when for any line $L \subseteq S$ there is a parallel line $L' \subseteq S'$ and, conversely, for any line $L' \subseteq S'$ there is a parallel line $L \subseteq S$.

It is an easy exercise to check that any affine space, in the sense of the algebraic definition \[2.1\] fulfills the synthetic definition \[4.1\].

The converse is not so easy. An essential role is played by Desargues’s theorem, which holds in any algebraic affine space and also in any synthetic affine space of dimension $\geq 3$. In dimension 2 there is a difference between the algebraic and synthetic definitions of affine space, since the first ones always are Desarguesian (Desargues’s theorem holds), while the second ones may be non-Desarguesian. With this exception, the algebraic and synthetic notions of affine space are equivalent:

**Theorem 4.2** Let $\mathbb{A}$ be a synthetic affine space of dimension $\geq 3$ (or of dimension 2 and Desarguesian). There exist, canonically associated to $\mathbb{A}$, a division ring $K$, a $K$-vector space $V$ and a map $\mathbb{A} \times V \rightarrow \mathbb{A}$ such that

- $(\mathbb{A}, V, +)$ is an algebraic affine space,
- Subspaces of the algebraic affine space $(\mathbb{A}, V, +)$ are just subspaces of the synthetic affine space $\mathbb{A}$.

For a proof, using different synthetic definitions, see \[2\], \[4\], \[22\].

**4.1 What are the morphisms between affine spaces?**

In the case of algebraic structures, such as group, ring or vector space, the (homo)morphisms are defined to be maps preserving the structure. Typically, an algebraic structure consists of some sets (and their direct products) with certain maps between them (named operations) satisfying certain identities (named axioms). A map between two structures of the same kind is said to preserve the structure when it is compatible with the operations in an obvious sense.

For example, a group is a set $G$ with operations $G \times G \rightarrow G$, $G \rightarrow 1$, $* \rightarrow G$, satisfying the usual axioms. A morphism between groups $\varphi: G \rightarrow G'$ is defined to be a map preserving the structure, in the sense that the following diagrams are commutative,

\[
\begin{array}{ccc}
G \times G & \xrightarrow{\varphi \times \varphi} & G' \times G' \\
\downarrow & & \downarrow \\
G & \xrightarrow{\varphi} & G'
\end{array}
\quad
\begin{array}{ccc}
G & \xrightarrow{\varphi} & G' \\
\downarrow & & \downarrow \\
1 & \xrightarrow{1} & 1
\end{array}
\quad
\begin{array}{ccc}
* & \xrightarrow{\varphi \times \varphi} & * \\
\downarrow & & \downarrow \\
* & \xrightarrow{\varphi} & *
\end{array}
\quad
\begin{array}{ccc}
G & \xrightarrow{\varphi} & G' \\
\downarrow & & \downarrow \\
G & \xrightarrow{\varphi} & G'
\end{array}
\quad
\begin{array}{ccc}
G & \xrightarrow{\varphi} & G' \\
\downarrow & & \downarrow \\
G & \xrightarrow{\varphi} & G'
\end{array}
\]

The commutativity of the first diagram states that $\varphi(g_1 \cdot g_2) = \varphi(g_1) \cdot \varphi(g_2)$ $\forall g_1, g_2 \in G$. The other two diagrams state that $\varphi(g^{-1}) = \varphi(g)^{-1}$ $\forall g \in G$ and $\varphi(1) = 1$ (in fact both follow from the former condition, due to the axioms of group). Hence a map $\varphi: G \rightarrow G'$ preserves the structure when it fulfills the condition $\varphi(g_1 \cdot g_2) = \varphi(g_1) \cdot \varphi(g_2)$ $\forall g_1, g_2 \in G$, which is the standard definition of group morphism.
Analogously, if we consider vector spaces \( V, V' \) over division rings \( K, K' \), it is easy to check that maps \((\varphi, \sigma): (V, K) \to (V', K')\) preserving the structure are just semilinear maps.

Analogously, given two algebraic affine spaces \((A, V, K)\) and \((A', V', K')\), maps \((\varphi, \vec{\varphi}, \sigma): (A, V, K) \to (A', V', K')\) preserving the structure are just semi-affine morphisms.

Now, the synthetic definition of affine space is not algebraic as they are the above structures, so that it is not evident what does it mean to say that a map \( \varphi: A \to A' \), between synthetic affine spaces, preserves the structure. Since the synthetic notion \[4.1\] of affine space is based on the relations of collinearity and parallelism, at first sight it seems reasonable to say that \( \varphi: A \to A' \) preserves the structure when

\[
(a \lor b) \parallel (c \lor d) \Rightarrow (\varphi(a) \lor (\varphi(b))) \parallel (\varphi(c) \lor \varphi(d))
\]

for all \( a, b, c, d \in A \). This is just the definition of parallel morphism. Hence it seems reasonable to consider the parallel morphisms as the morphisms preserving the structure in the synthetic case. This intuition is confirmed by the Fundamental Theorem, stating the equivalence between the semi-affine and the parallel morphisms.

5 Projective Geometry

Definition. The projectivization of a vector space \( V \), or the projective space associated to \( V \), is the set

\[
\mathbb{P}(V) := \{1\text{-dimensional vector subspaces } \langle v \rangle \text{ of } V\}
\]

Subspaces of \( \mathbb{P}(V) \) are defined to be projectivizations \( \mathbb{P}(W) \) of vector subspaces \( W \subseteq V \).

For any pair of subspaces \( \mathbb{P}(W_1), \mathbb{P}(W_2) \) of \( \mathbb{P}(V) \) we have

\[
\mathbb{P}(W_1) \lor \mathbb{P}(W_2) = \mathbb{P}(W_1 + W_2), \quad \mathbb{P}(W_1) \cap \mathbb{P}(W_2) = \mathbb{P}(W_1 \cap W_2)
\]

It is easy to check that the projective space \( \mathbb{P}(V) \), with the family of its subspaces, is a geometry.

Moreover, \( \dim \mathbb{P}(W) = \dim W - 1 \).

Lemma 5.1 Let us consider a subspace \( S \neq \emptyset \) and a point \( p \notin S \) in \( \mathbb{P}(V) \). The subspace \( S \lor p \) is just the union of all lines joining \( p \) with points of \( S \),

\[
S \lor p = \bigcup_{s \in S} (s \lor p)
\]
Proof. We have $S = \mathbb{P}(W)$, $p = \mathbb{P}(v)$, for some vector subspaces $W, \langle v \rangle \subset V$. The stated equality corresponds to the equality

$$W + \langle v \rangle = \bigcup_{(w) \subseteq W} ((w) + \langle v \rangle)$$

□

**Corollary 5.2** A hyperplane $H$ and a line $L \nsubseteq H$ in $\mathbb{P}(V)$ always meet at a point.

**Proposition 5.3** The projective geometry $\mathbb{P}(V)$ is generated by lines.

**Proof** Let $A \subseteq \mathbb{P}$ a subset containing the line joining any two points of $A$. We have to show that $A$ is a subspace. Let $S \subseteq A$ be a maximal subspace. If $S = A$ we conclude. Otherwise there is a point $p \in A$, $p \notin S$; then

$$S \lor p \mathop{=}^{\text{def}} \bigcup_{s \in S} (s \lor p) \subseteq A$$

so contradicting the maximal character of $S$. □

**Definition.** A semilinear map $\Phi: V \to V'$, $v \mapsto v'$, induces a map

$$\phi: \mathbb{P}(V) \longrightarrow \mathbb{P}(V') \quad , \quad \langle v \rangle \mapsto \langle v' \rangle$$

defined on the complement of the subspace $E = \mathbb{P}({\ker \Phi})$. We say that $\phi$ is the partial projective morphism associated to $\Phi$, and $E$ is said to be the exceptional subspace of $\phi$.

Remark that $\phi$ is globally defined ($E = \emptyset$) if and only if $\Phi$ is injective ($\ker \Phi = 0$). In such case, we say that $\phi: \mathbb{P}(V) \to \mathbb{P}(V')$ is a projective morphism. A projective morphism may be not injective.

A projective isomorphism is a projective morphism $\phi: \mathbb{P}(V) \to \mathbb{P}(V')$ associated to a semilinear isomorphism $\Phi: V \to V'$.

It seems to be an open problem, to know if each bijective morphism $\mathbb{P}(V) \to \mathbb{P}(V')$, with dimensions $\geq 2$, is an isomorphism. There are examples \cite{13} of injective morphisms $\mathbb{P}(V) \to \mathbb{P}(V')$, preserving non-collinearity, with $\dim \mathbb{P}(V) > \dim \mathbb{P}(V')$.

If two semilinear maps $\Phi, \Phi': V \to V'$ are proportional ($\Phi' = \lambda \Phi$ for some $\lambda \in K'$), then both induce the same partial projective morphism $\phi = \phi': \mathbb{P}(V) \longrightarrow \mathbb{P}(V')$. Conversely,

**Proposition 5.4** If two semilinear maps $\Phi, \Phi': V \to V'$ induce the same partial projective morphism $\phi = \phi': \mathbb{P}(V) \longrightarrow \mathbb{P}(V')$ and it is not constant, then $\Phi$ and $\Phi'$ are proportional: $\Phi' = \lambda \Phi$ for some $\lambda \in K'$.
Proof. Since $\phi' = \phi$, for all $v \not\in \ker \Phi = \ker \Phi'$ we have $\Phi'(v) = \lambda_v \Phi(v)$ for a unique $\lambda_v \in K'$. We have to show that this scalar $\lambda_v$ does not depend on $v$.

Take $v_1, v_2 \in V - \ker \Phi$ such that $\Phi(v_1), \Phi(v_2)$ are linearly independent. We have

$$\lambda_{v_1} \Phi(v_1) + \lambda_{v_2} \Phi(v_2) = \Phi'(v_1) + \Phi'(v_2) = \Phi'(v_1 + v_2)$$

hence $\lambda_{v_1} = \lambda_{v_1 + v_2} = \lambda_{v_2}$.

Take $v_1, v_2 \in V - \ker \Phi$ such that $\Phi(v_1)$ and $\Phi(v_2)$ are proportional. Let us consider a vector $v$ such that $\Phi(v)$ is not proportional to $\Phi(v_1)$ and $\Phi(v_2)$ (it exists because $\phi$ is not constant). By the former case, we obtain $\lambda_v = \lambda_v = \lambda_v$.

\[\square\]

5.1 Synthetic definition of projective space

Definition (Veblen–Young). A projective space is a set $P$ (its elements are named points) with a family $\mathcal{L}$ of subsets (named lines), satisfying the following axioms:

- P1. There is a unique line joining any two different points,
- P2. Any line has at least three different points,
- P3. If a line meets two sides of a triangle, not in the common vertex, then it meets the third side.

Even if it does not use the not yet defined notion of plane, axiom P3 essentially states that any two different lines in a plane meet at a point.

A subspace of $P$ is a subset $S$ containing the line joining any two different points of $S$.

It is easy to check that any projective space $\mathbb{P}(V)$, associated to a vector space, satisfies the axioms of the synthetic definition of projective space. The converse also holds ([1]), modulo the Desarguesian condition in dimension 2.

**Theorem 5.5** Let $P$ be a synthetic projective space, of dimension $\geq 3$ or of dimension 2 and Desarguesian. There is an isomorphism (collineation) $P \simeq \mathbb{P}(V)$ for some vector space $V$ over a division ring $K$.

The algebraic and synthetic definitions of projective space suggest two definitions of morphism. In the algebraic case, we consider projective morphisms $\mathbb{P}(V) \to \mathbb{P}(V')$, defined by injective semilinear maps $V \to V'$. In the synthetic case, since we deal with geometries, it is natural to consider the morphisms of geometries $P \to P'$. The Fundamental Theorem of Projective Geometry, below stated [5,7], says that both notions of morphism are equivalent.
5.2 Fundamental Theorem of Projective Geometry

The following generalization of the classical Fundamental Theorem of Projective Geometry is due to Faure and Fröhlicher [8] and Havlicek [12]. A brief and elementary proof may be found in [7].

5.6 Fundamental Theorem of the Projective Geometry ([8], [12]) Let $\phi: \mathbb{P} = \mathbb{P}(V) \rightarrow \mathbb{P}(V') = \mathbb{P}'$ be a map defined on the complement of a subspace $E \subset \mathbb{P}$ and such that the image is not contained in a line. The following statements are equivalent:

a). $\phi$ is a partial projective morphism,
b). $\phi$ satisfies the following two conditions

\begin{enumerate}
\item[$b_1$).] For any $x_0, x_1, x_2 \in \mathbb{P} - E$ we have $x_0 \in x_1 \vee x_2 \Rightarrow \phi(x_0) \in \phi(x_1) \vee \phi(x_2)$
\item[$b_2$).] For any $x_1, x_2 \in \mathbb{P} - E, x_1 \neq x_2$, we have $(x_1 \vee x_2) \cap E \neq \emptyset \Rightarrow \phi(x_1) = \phi(x_2)$
\end{enumerate}

If we assume in this theorem that $\phi$ is globally defined ($E = \emptyset$), then condition ($b_2$) is vacuous and condition ($b_1$) states that $\phi$ is a morphism of geometries (since projective spaces are generated by lines). Therefore

Corollary 5.7 Let $\phi: \mathbb{P}(V) \rightarrow \mathbb{P}(V')$ be a map such that the image is not contained in a line. The following statements are equivalent:

a). $\phi$ is a projective morphism,
b). $\phi$ is a morphism of geometries.

6 Quotients and partial morphisms

In this section we shall see that the partial maps considered in part (b) of the Fundamental Theorem [5.6] are a particular case of the following general notion.

Definition. Let $X$ and $X'$ be geometries. A partial morphism $\varphi: X \rightarrow X'$ is a morphism of geometries $\varphi: X - E \rightarrow X'$, defined on the complement of a subspace $E \subseteq X$, such that for all $x_1, x_2 \in X - E$ we have

\[ x_1 \vee E = x_2 \vee E \Rightarrow \varphi(x_1) = \varphi(x_2) \quad (3) \]

We say that $E$ is the exceptional subspace of $\varphi$.

In this definition, $X - E$ is considered as a subgeometry of $X$.

It is interesting to note that the dominion of a partial morphism in general may be not extended: Let $X$ be a geometry where the lines have at least three points. A non constant partial morphism $\varphi: X \rightarrow X'$, with exceptional subspace $E$, may be not extended to a morphism of geometries $X \supseteq U \rightarrow X'$ defined on a subgeometry $U$ strictly containing $X - E$. 

18
Partial morphisms are reduced to global morphisms due to the following notion.

**Definition.** Let $E$ be a subspace of a geometry $X$. The **quotient space** is the quotient set

$$X/E := (X-E)/\sim$$

where $\sim$ is the following equivalence relation on $X-E$,

$$x_1 \equiv x_2 \Leftrightarrow x_1 \lor E = x_2 \lor E$$

We name **subspace** of $X/E$ any subset $S/E$ where $S$ is a subspace of $X$ containing $E$.

6.1 For any subspace $S \subseteq X$, strictly containing $E$, we have

$$S = \bigcup_{s \in S-E} (s \lor E) = \bigcup_{[s] \in S/E} (s \lor E)$$

so that $S$ is fully determined by $S/E$. Therefore, we have a bijective correspondence (lattice isomorphism)

$$\begin{array}{ccc}
\{\text{subspaces of } X\} & \xrightarrow{\sim} & \{\text{subspaces of } X/E\} \\
\text{containing } E & & \quad S/E \\
\end{array}$$

6.2 It is easy to check that

The set $X/E$, with the family of subspaces, is a geometry.

6.3 The quotient map $\pi: X-E \to X/E$, $x \mapsto [x]$, is a morphism of geometries: For any subspace $S/E \subseteq X/E$ we have $\pi^{-1}(S/E) = S-E = S \cap (X-E)$, which is a subspace of $X-E$. That is to say,

The quotient map $\pi: X \to X/E$, $x \mapsto [x]$, is a partial morphism of geometries, with exceptional subspace $E$.

**Lemma 6.4** Let $X$ be a geometry where the lines have at least three points. Let $\pi: X-E \to X/E$ be the quotient morphism with respect to a subspace $E$ of $X$. If a non empty subspace $S_0$ of $X-E$ is a union of equivalence classes, then $\pi(S_0)$ is a subspace of $X/E$.

**Proof.** By definition of subgeometry, we have $S_0 = S \cap (X-E)$ for some subspace $S$ of $X$.

Let us see that $S \supset E$. Given $e \in E$, we take $s_0 \in S_0 \subseteq S$. The line $s_0 \lor e$ of $X$ meets $E$ at $e$. By hypothesis, there is a third point $s' \in s_0 \lor e$, different from $s_0$ and $e$. Since $s'$ is equivalent to $s_0$ (that is to say, $s_0 \lor E = s' \lor E$), we have $s' \in S_0 \subseteq S$, hence $e \in s_0 \lor s' \subseteq S$.

It is clear that $\pi(S_0) = S/E$, which is a subspace of $X/E$. 

$\square$
The partial morphism $\pi: X \to X/E$ has the usual universal property.

**Proposition 6.5** Let $X$ be a geometry where the lines have at least three points. Let $\pi: X \to X/E$ the partial quotient morphism with respect to a subspace $E$ of $X$.

Any partial morphism $\varphi: X \to X'$, with exceptional subspace $E$, uniquely factors $\varphi = \tilde{\varphi} \circ \pi$ through a morphism $\tilde{\varphi}: X/E \to X'$.

\[ X \xrightarrow{\varphi} X' \xrightarrow{\pi} X/E \xrightarrow{\tilde{\varphi}} X' \]

**Proof.** According to the universal property of the quotient set with respect to an equivalence relation, the map $\varphi: X - E \to X'$ uniquely factors $\varphi = \tilde{\varphi} \circ \pi$ for a unique map $\tilde{\varphi}: X/E \to X'$. We have to show that $\tilde{\varphi}$ is a morphism of geometries. Given a subspace $S'$ of $X'$, we have that $\varphi^{-1}(S')$ is a subspace of $X - E$ which is a union of equivalence classes. Applying the lemma to this subspace $\varphi^{-1}(S') = \pi^{-1}(\tilde{\varphi}^{-1}(S'))$, we obtain that its $\pi$-image $\pi(\varphi^{-1}(S')) = \tilde{\varphi}^{-1}(S')$ is a subspace of $X/E$, hence $\tilde{\varphi}$ is a morphism.

\[ \square \]

According to this proposition we have a bijective correspondence,

\[
\begin{align*}
\{ \text{partial morphisms } \varphi: X \to X' \} & \quad \leftrightarrow \quad \{ \text{morphisms } \tilde{\varphi}: X/E \to X' \} \\
\{ \text{with exceptional subspace } E \} & \quad \leftrightarrow \quad \{ \text{morphisms } \tilde{\varphi}: X/E \to X' \}
\end{align*}
\]

**Proposition 6.6** Any partial projective morphism $\phi: \mathbb{P}(V) \to \mathbb{P}(V')$ is a partial morphism of geometries.

**Proof.** Let $\Phi: V \to V'$ be the corresponding semilinear map and put $E = \mathbb{P}({\text{ker } \Phi})$.

For any subspace $\mathbb{P}(W') \subseteq \mathbb{P}(V')$ it is easy to check that $\phi^{-1}\mathbb{P}(W') = \mathbb{P}(\Phi^{-1}W') - E$, so that it is a subspace of $\mathbb{P}(V) - E$. Hence $\phi: \mathbb{P}(V) - E \to \mathbb{P}(V')$ is a morphism of geometries.

Put $x_1 = \mathbb{P}(v_1)$, $x_2 = \mathbb{P}(v_2) \in \mathbb{P}(V) - E$. If $\langle v_1 \rangle + \ker \Phi = \langle v_2 \rangle + \ker \Phi$, it is clear that $\Phi(v_1) = \Phi(v_2)$, that is to say, if $x_1 \lor E = x_2 \lor E$ then $\phi(x_1) = \phi(x_2)$.

\[ \square \]

**6.7** Let $E$ be a subspace of a projective space $\mathbb{P}$. For any pair of points $x_1, x_2 \notin E$, $x_1 \neq x_2$, the conditions

\[ x_1 \lor E = x_2 \lor E \quad \text{and} \quad (x_1 \lor x_2) \cap E \neq \emptyset \]

are equivalent. It follows easily from 5.1.

Therefore, property (b2) of the fundamental theorem 5.6 is equivalent to condition (3) of the definition of partial morphism.
Proposition 6.8 Let $\mathbb{P}(W)$ be a subspace of $\mathbb{P}(V)$. There is a canonical isomorphism

\[ \mathbb{P}(V)/\mathbb{P}(W) = \mathbb{P}(V/W) \]

Proof. Let us consider the natural projection $\Gamma: V \to V/W$, $v \mapsto [v]$, with kernel $W$. This linear map induces a partial projective morphism $\gamma: \mathbb{P}(V)\to\mathbb{P}(V/W)$, $[v] \mapsto ([v])$, with exceptional subspace $\mathbb{P}(W)$. By the universal property of the kernel $\mathbb{P}$, this partial morphism corresponds to a morphism in order to see that it is an isomorphism we construct the inverse morphism $\tilde{\gamma}$ just the composition $\tilde{\gamma} = \gamma \circ \pi$. Since $\varphi$, that is to say, $\varphi = \tilde{\varphi} \circ \pi$ of the natural projection $\pi: \mathbb{P}(V)\to\mathbb{P}(W)$ with a unique morphism $\tilde{\varphi}: \mathbb{P}(V/W)\to Y$.

Corollary 6.9 Any partial morphism of geometries $\varphi: \mathbb{P}(V)\to Y$, with exceptional subspace $E = \mathbb{P}(W)$, is the composition $\varphi = \tilde{\varphi} \circ \pi$ of the natural projection $\pi: \mathbb{P}(V)\to\mathbb{P}(W)$ with a unique morphism $\tilde{\varphi}: \mathbb{P}(V/W)\to Y$.

Proposition 6.10 A map $\phi: \mathbb{P} = \mathbb{P}(V)\to\mathbb{P}(V') = \mathbb{P}'$, defined on the complement of a subspace $E = \mathbb{P}(W)$, is a partial morphism of geometries if and only if it fulfills conditions $(b_1)$ and $(b_2)$ of Proposition 6.8.

$b_1$. For any $x_0, x_1, x_2 \in \mathbb{P} - E$ we have

\[ x_0 \in x_1 \lor x_2 \implies \varphi(x_0) \in \varphi(x_1) \lor \varphi(x_2) \]

$b_2$. For any $x_1, x_2 \in \mathbb{P} - E$, $x_1 \neq x_2$, we have

\[ (x_1 \lor x_2) \cap E \neq \emptyset \implies \varphi(x_1) = \varphi(x_2) \]

Proof. Let $W' \subset V$ be a complement of $W$, that is to say, $V = W \oplus W'$, so that $W' = V/W$. Remark that the projection $\pi: \mathbb{P}(V)\to\mathbb{P}/E = \mathbb{P}(V/W)$ induces the obvious isomorphism $\pi: \mathbb{P}(W')\to\mathbb{P}(V/W)$.

$(\Leftarrow)$. Since $\phi: \mathbb{P} - E \to \mathbb{P}'$ fulfills $(b_2)$, then by 6.7 it also fulfills condition of partial morphisms: $x_1 \lor E = x_2 \lor E \implies \phi(x_1) = \phi(x_2)$. Therefore, $\phi$ factors through the quotient set $\mathbb{P}/E = \mathbb{P}(V/W)$, that is to say, $\phi = \tilde{\phi} \circ \pi$ for a unique map $\tilde{\phi}: \mathbb{P}(V/W)\to\mathbb{P}'$.
We have to show that \((b_1)\) implies that \(\phi: \mathbb{P} - E \to \mathbb{P}'\) is a morphism of geometries (it is not obvious since, in general, the geometry \(\mathbb{P} - E\) is not generated by lines). Via the isomorphism \(\mathbb{P}(W') \cong \mathbb{P}(V/W)\), the restriction \(\phi|_{\mathbb{P}(W')}\) coincides with \(\tilde{\phi}\). Since \(\phi|_{\mathbb{P}(W')}\) is a morphism by \((b_1)\), then \(\tilde{\phi}\) also is a morphism, hence the composition \(\phi = \tilde{\phi} \circ \pi\) is a morphism.

\((\Rightarrow)\). Any morphism of geometries \(\phi: \mathbb{P} - E \to \mathbb{P}'\) fulfills \((b_1)\). By 6.7, a partial morphism also fulfills \((b_2)\).

\[\square\]

According to this proposition, theorem 5.6 may be restated as follows

\[6.11\] **Fundamental Theorem of Projective Geometry** Let \(\phi: \mathbb{P}(V) \to \mathbb{P}(V')\) be a map, defined on the complement of a subspace \(E \subset \mathbb{P}(V)\), such that the image is not contained in a line. The following statements are equivalent:

a). \(\phi\) is a partial projective morphism,

b). \(\phi\) is a partial morphism of geometries.

7 Morphisms of geometries between affine spaces

The natural question of determining all the morphisms of geometries \(\mathbb{A} \to \mathbb{A}'\), between affine spaces, is answered by a theorem of Zick. We shall deduce this result as a consequence of an extension theorem of such morphisms to partial projective morphisms \(\mathbb{P} \to \mathbb{P}'\), between the respective projective closures of the affine spaces.

7.1 Projective embedding of the affine space

**Definition.** Let \((\mathbb{A}, V, +)\) be an affine space over a division ring \(K\). We name **vectorial extension** of \(\mathbb{A}\) to a (left) \(K\)-vector space \(E\) with an injective affine morphism \(j: \mathbb{A} \to E\) defining an affinity between \(\mathbb{A}\) and an affine hyperplane of \(E\) not containing 0.

The differential of \(j: \mathbb{A} \to E\) is an injective \(K\)-linear map \(\vec{j}: V \to E\).

**7.1 Comments.** a). It is easy to construct a vectorial extension: Fix a point \(p_0 \in \mathbb{A}\), put \(E := Kp_0 \oplus V\) and let us consider the natural inclusion

\[\mathbb{A} = p_0 + V \xrightarrow{j} Kp_0 \oplus V = E\]

b). The vectorial extension is essentially unique: Given any two \(j, j': \mathbb{A} \to E\) and \(j': \mathbb{A} \to E'\), there is a unique \(K\)-linear isomorphism \(\Phi: E \to E'\) such that \(j' = \Phi \circ j\).

c). A **canonical** vectorial extension may be constructed as follows. Let \(F = \{\text{affine morphisms } j: \mathbb{A} \to K\}\); it is a right \(K\)-vector space: \((f \cdot \lambda)(p) = f(p)\lambda\). The dual space \(E = F^*\) is a left \(K\)-vector space and we have a canonical affine morphism \(j: \mathbb{A} \to E = F^*, p \mapsto \delta_p\), where \(\delta_p(f) := f(p)\).
Via the injective maps \( j : \mathcal{A} \hookrightarrow \mathcal{E} \) and \( \vec{j} : V \hookrightarrow \mathcal{E} \), we may identify \( \mathcal{A} \) with an affine hyperplane of \( \mathcal{E} \) and \( V \) with a vectorial hyperplane of \( \mathcal{E} \), both hyperplanes being parallel.

**Definition.** The projective space \( \mathbb{P} = \mathbb{P}(\mathcal{E}) \) is said to be the projective closure of \( \mathcal{A} \). The hyperplane \( H = \mathbb{P}(V) \) of \( \mathbb{P}(\mathcal{E}) \) is said to be the hyperplane at infinity.

**Theorem 7.2** The composition
\[
\mathcal{A} \xrightarrow{j} \mathcal{E} - \{0\} \xrightarrow{\pi} \mathbb{P}(\mathcal{E}) = \mathbb{P}
\]
defines an isomorphism of geometries \( \mathcal{A} = \mathbb{P} - H \).

See [3] for a proof. From now on we identify \( \mathcal{A} \) with the subgeometry \( \mathbb{P} - H \) of \( \mathbb{P} \).

We frequently use that the closure in \( \mathbb{P} \) of an affine line \( L \subseteq \mathcal{A} \), with direction \( \langle v \rangle \subseteq V \), is a projective line meeting the hyperplane at infinity \( H \) at a point \( p = \langle v \rangle \), named point at infinity of the affine line \( L \). Therefore,

*Two affine lines are parallel if and only if both have the same point at infinity.*

**7.3 Comments.** Vectorial extensions have the following basic property: *Any semiaffine morphism \( \phi : \mathcal{A} \to \mathcal{A}' \) uniquely extends to a semilinear map \( \Phi : \mathcal{E} \to \mathcal{E}' \) between their vectorial extensions.*

Now, this semilinear map defines a partial projective morphism \( \phi : \mathbb{P} = \mathbb{P}(\mathcal{E}) \dashrightarrow \mathbb{P}(\mathcal{E}') = \mathbb{P}' \). Therefore,

*Any semiaffine morphism \( \phi : \mathcal{A} \to \mathcal{A}' \) uniquely extends to a partial projective morphism between their projective closures*

\[
\mathcal{A} \xrightarrow{\phi} \mathcal{A}'
\]

\[
\mathbb{P} \xrightarrow{\phi} \mathbb{P}'
\]

**7.2 Projective extension of morphisms \( \mathcal{A} \to \mathbb{P} \)**

From now on, we consider an affine space \( \mathcal{A} \) over a division ring \( K \), its projective closure \( \mathbb{P} \) and the hyperplane at infinity \( H \subseteq \mathbb{P} \), so that \( \mathcal{A} = \mathbb{P} - H \).

Let us consider another projective space \( \mathbb{P}' \) over a division ring \( K' \).

We assume that \( |K| \geq 4 \) or \( |K| = 3 = \text{caract} K' \).

Finally, let us fix a morphism of geometries
\[
\varphi : \mathcal{A} \longrightarrow \mathbb{P}'
\]
such that the image is not contained in a line.
Notation. Since \( \varphi \) is a morphism, for any line \( L \) of \( \mathbb{A} \) we have that \( \varphi|_L \) is constant or \( \varphi|_L \) is injective and \( \varphi(L) \) is contained in a unique line of \( \mathbb{P}' \), denoted by \( L' \).

Lemma 7.4 Given a point \( p \in H \), let \( \{L_i\} \) be the family of parallel lines in \( \mathbb{A} \) whose point at infinity is \( p \). Let us consider the family \( \{L'_j\} \) of lines in \( \mathbb{P}' \), where \( L_j \) runs over the lines in \( \{L_i\} \) such that \( \varphi|_{L_j} \) is injective. One of the following alternatives holds:

i). \( \varphi \) is constant on any line in the family \( \{L_i\} \) (that is to say, \( \{L'_j\} = \emptyset \)),

ii). The union of the lines in the family \( \{L'_j\} \) contains \( \varphi(\mathbb{A}) \).

Proof. If (i) does not hold, then there is a line \( L_0 \) in the family \( \{L_i\} \) such that \( \varphi|_{L_0} \) is injective. Let us prove (ii). Given \( x \in \mathbb{A} \), let us see that \( \varphi(x) \in L'_j \) for some index \( j \). If \( \varphi(x) \in L'_0 \) it is obvious. Otherwise, \( \varphi(x) \) and \( \varphi(L_0) \) are not collinear. Consider the plane \( P = L_0 \lor x \) in the geometry \( \mathbb{A} \). Since \( \dim \varphi(P) = 2 \), we have that \( \varphi : P \rightarrow \varphi(P) \) is an isomorphism \( [1.7.b] \). Now let \( L_j \) be the line in the plane \( P \) parallel to \( L_0 \) passing through \( x \). Since \( \varphi|_P \) is injective, so is \( \varphi|_{L_j} \). Finally, since \( x \in L_j \) we have \( \varphi(x) \in \varphi(L_j) \subseteq L'_j \).

Points \( p \in H \) where (i) holds are said to be exceptional.

Proposition 7.5 The set \( E \subseteq H \) of exceptional points of \( \varphi : \mathbb{A} \rightarrow \mathbb{P}' \) is a subspace.

Proof. Let \( p_1, p_2 \in H \) be two different exceptional points. We must show that any point \( p \in p_1 \lor p_2 \subseteq H \) also is exceptional. Let \( L \) be an affine line in \( \mathbb{A} \) with the point \( p \) at infinity and let \( x \in L \). Let us consider the lines \( L_1, L_2 \) of \( \mathbb{A} \) passing through \( x \) with the points \( p_1, p_2 \) at infinity respectively. Then \( L_1 \lor L_2 \) is a plane of \( \mathbb{A} \) containing the line \( L \). Since \( p_1, p_2 \) are exceptional, we have \( \varphi(L_1) = \varphi(x) \), \( \varphi(L_2) = \varphi(x) \). Hence \( \varphi(L) \subseteq \varphi(L_1 \lor L_2) \subseteq \varphi(L_1) \lor \varphi(L_2) = \varphi(x) \lor \varphi(x) = \varphi(x) \), so that \( \varphi|_{L} \) is constant.

Lemma 7.6 If \( p \in H \) is not an exceptional point of \( \varphi : \mathbb{A} \rightarrow \mathbb{P}' \), then the lines of \( \mathbb{P}' \) in the family \( \{L'_j\} \) meet at a unique point \( p' \) of \( \mathbb{P}' \).

Proof. Let \( L'_1, L'_2 \) be two different lines in the family \( \{L'_j\} \). Since the lines \( L_1, L_2 \) of \( \mathbb{A} \) are coplanar, the lines \( L'_1, L'_2 \) of \( \mathbb{P}' \) so are coplanar, hence they intersect each other. Now we must prove that any three different lines \( L'_1, L'_2, L'_3 \) meet at a point.

Case 1: \( L_1, L_2, L_3 \) lie in a plane \( \mathbb{A}_2 \). In this case \( L'_1, L'_2, L'_3 \) are coplanar and \( \dim \varphi(\mathbb{A}_2) = 2 \), hence \( \varphi : \mathbb{A}_2 \rightarrow \varphi(\mathbb{A}_2) \) is an isomorphism, that is to say, \( \varphi \) induces an embedding of \( \mathbb{A}_2 \) into a projective plane \( \mathbb{P}'_2 \subseteq \mathbb{P}' \).

1a. If \( |K| \geq 4 \), there is a forth line \( L \subseteq \mathbb{A}_2 \), different and parallel to the other three lines. Let us pick different points \( x_1, x_1 \in L_1, z_1, z_1, z_1, z_1 \in L \)
and let us consider the points $x_2 = L_2 \cap (x_1 \lor z_{12})$, $\bar{x}_2 = L_2 \cap (\bar{x}_1 \lor z_{12})$, $x_3 = L_3 \cap (x_1 \lor z_{13})$, $\bar{x}_3 = L_3 \cap (\bar{x}_1 \lor z_{13})$. The vertices of the triangles $x_1 x_2 x_3$ and $\bar{x}_1 \bar{x}_2 \bar{x}_3$ lie in the parallel lines $L_1$, $L_2$, $L_3$, hence by Desargues’s theorem the point $z_{23} := (x_2 \lor x_3) \cap (\bar{x}_2 \lor \bar{x}_3)$ is collinear with $z_{12}$ and $z_{13}$, that is to say, $z_{23} \in L$.

Applying to these triangles the embedding $\varphi : A_2 \to P'_2$ we obtain two triangles in $P'_2$, with vertices in the lines $L'_1, L'_2, L'_3$, such that the corresponding sides meet at points in the line $L'$. By Desargues’s theorem, the lines $L'_1, L'_2, L'_3$ are concurrent. We take this argument from [21] Th 6.2.

1b. If $|K| = 3 = \text{caract.} K'$ then $A_2$ is an affine plane over $K = \mathbb{Z}_3$, with nine points and 12 lines. Therefore, the image of the embedding $\varphi : A_2 \to P'_2$ has 9 points as follows,

(In this figure, it is assumed that any side of the rhombus passes through the opposite vertex of the square.)
Let us fix the projective reference in $\mathbb{P}'_2$ defined by the four vertices of the square. Then the coordinates of the vertices of the rhombus are those in the figure. Since the two vertices of each side of the rhombus are collinear with the opposite vertex of the square, a direct calculation (using that caracter $K' = 3$) shows that:

\[ a = b = 2, \quad c = d = 1 \]

Now we check, using that caracter $K' = 3$, that the lines $L'_1, L'_2, L'_3$ are concurrent at the point $(1,0,1)$. We take this argument from [20] Th.2.

Case 2: $L_1, L_2, L_3$ generate a 3-space $\mathcal{A}_3$. We argue in a similar way to case 1a; the triangles being constructed as follows. Let $P \subset \mathcal{A}_3$ be a plane not parallel to the lines $L_1, L_2, L_3$, so that it meet them at the vertices of a triangle. Let $L \subset P$ be a line not parallel to any side of the former triangle (it exists because $|K| > 2$). Take another plane $\bar{P} \subset \mathcal{A}_3$, with the same property than $P$, intersecting $P$ at $L$. This plane $\bar{P}$ meets the lines $L_1, L_2, L_3$ at the vertices of a second triangle. By construction, the vertices of both triangles lie in the lines $L_1, L_2, L_3$ and the corresponding sides meet at points of $L$. We conclude as in case 1a.

The above lemma let us extend the morphism $\varphi: \mathcal{A} \to \mathbb{P}'$ to a partial map $\phi: \mathbb{P} \dashrightarrow \mathbb{P}'$ taking $\phi(p) := p'$ for any non exceptional point $p \in H$.

**Lemma 7.7** Let $p \in H$ be a non exceptional point and let $L_0$ be a line of $\mathcal{A}$ with the point $p$ at infinity.

a). If $\varphi|_{L_0}$ is constant then $\varphi(L_0) = p'$,

b). If $\varphi|_{L_0}$ is injective then $p' \notin \varphi(L_0)$.

**Proof.** a). Put $x' := \varphi(L_0)$. Given a line $L'_j$ in the family $\{L'_j\}$, let us consider the plane $\bar{P}$ of $\mathcal{A}$ containing the parallel lines $L_0$ and $L_j$. Since $\varphi(L_0) = x'$ we have $\dim \varphi(\bar{P}) \leq 1$, so that $x' = \varphi(L_0)$ and $\varphi(L_j)$ are collinear, that is to say, $x' \in L'_j$. Therefore, $x'$ is the point where the lines in the family $\{L'_j\}$ intersect, that is to say, $x' = p'$.

b). If $p' \in \varphi(L_0)$ we obtain a contradiction. Let $\varphi(x) \in \varphi(\mathcal{A})$ be a point, not collinear with $\varphi(L_0)$. Let us consider the plane $P \subseteq \mathcal{A}$ containing $L_0$ and $x'$; let $L_j \subset P$ be the parallel line to $L_0$ through $x$. Remark that $\dim \varphi(P) = 2$ because $\varphi(P) \supset \varphi(L_0), \varphi(x)$, so that $\varphi: P \to \varphi(P)$ is an isomorphism. It follows that $\varphi(L_j)$ is a line of $\varphi(P)$, hence $\varphi(L_j) = L'_j \cap \varphi(P) \ni p'$. Therefore, $\varphi(L_0)$ and $\varphi(L_j)$ meet at $p'$, so contradicting that $\varphi: P \to \varphi(P)$ is an isomorphism and $L_0 \cap L_j = \emptyset$.

**Lemma 7.8** For any $p_0, p_1, p_2 \in H - E$ we have

\[ p_0 \in p_1 \lor p_2 \Rightarrow p'_0 \in p'_1 \lor p'_2 \]
Proof. Case $p_1' = p_2' =: \bar{p}$. We have to show that $p_0' = \bar{p}$, that is to say, for any line $L_0 \subset \mathbb{A}$ with the point $p_0$ at infinity and such that $\varphi|_{L_0}$ is injective, we have $\bar{p} \in L_0'$. Given $x_0 \in L_0$, let $L_1, L_2 \subset \mathbb{A}$ be the lines through $x_0$ with the points at infinity $p_1, p_2$, respectively. If $\varphi|_{L_i}$ is constant, then $\bar{p} = p_i' = \varphi(L_i) \ni \varphi(x_0)$, so that $\bar{p} = \varphi(x_0) \in \varphi(L_0) \subseteq L_0'$. It is the same if $\varphi|_{L_i}$ is constant. Finally, if $\varphi$ is injective on $L_1$ and $L_2$, then $L_1' = L_2'$ because both lines pass through $\varphi(x_0)$ and $\bar{p}$; the inclusion $L_0 \subset L_1 \lor L_2$ implies $\varphi(L_0) \subseteq \varphi(L_1) \lor \varphi(L_2) \subseteq (L_1' = L_2')$, hence $L_0' = L_1' = L_2' \ni \bar{p}$.

Case $p_1' \neq p_2'$. We may assume that $p_0 \neq p_1, p_2$. Let us consider the plane $P \subseteq \mathbb{A}$ with the line $p_1 \lor p_2$ at infinity and passing through a point $x_0 \in \mathbb{A}$ such that $\varphi(x_0) \notin p_1' \lor p_2'$.

Let us see that $\dim \varphi(P) = 2$ (so that $\varphi : P \to \mathbb{P}'$ is an embedding). Let $L_i \subset P$ be the line through $x_0$ with the point $p_i$ at infinity ($i = 1, 2$). Remark that $\varphi|_{L_i}$ is injective since otherwise $p_i' = \varphi(L_i) \ni \varphi(x_0)$, against the choice of $x_0$. If $\dim \varphi(P) \leq 1$ then the subsets $\varphi(L_1), \varphi(L_2) \subseteq \varphi(P)$ lie in a line, hence $L_1' = L_2'$, so that $\varphi(x_0), p_1', p_2'$ are collinear, against the choice of $x_0$.

Now let us consider in the affine plane $P$ two perspective triangles such that the corresponding sides are parallel with the points $p_0, p_1, p_2$ at infinity (such triangles exist because $P$ has enough points when $|K| > 2$). Applying the embedding $\varphi : P \to \mathbb{P}'$ to such configuration, we obtain a pair of perspective triangles in $\mathbb{P}'$ such that the corresponding sides intersect at the points $p_0', p_1', p_2'$, hence these points are collinear by Desargues’s theorem.

□

Lemma 7.9 Let $p_1, p_2 \in H - E$ be two different points. Then

$$(p_1 \lor p_2) \cap E \neq \emptyset \quad \Rightarrow \quad p_1' = p_2'$$

Proof. Put $p = (p_1 \lor p_2) \cap E$. Let $\{I_i\}$ the family of parallel lines in $\mathbb{A}$ with point at infinity $p_i$ and let $\{I_i'\}$ be the corresponding family of lines in $\mathbb{P}'$, concurrent at $p_i'$. Given one of such lines $I'$, let us consider in $\mathbb{A}$ a plane $P \supset I'$, with the line at infinity $p_1 \lor p_2$. Let $J \subset P$ be lines with the points at infinity $p_2, p$ respectively. Since $p$ is exceptional we have that $\varphi|_J$ is constant, hence $\dim \varphi(P) \leq 1$ and therefore $\varphi(P) \subseteq I'$. We have $\varphi(J) \subseteq \varphi(P) \subseteq I'$; hence, if $\varphi|_J$ is injective, we have $J' = I'$ so that $p_2' \in I'$. If $\varphi|_J$ is constant, then $p_2' = \varphi(J) \subseteq I'$. In both cases, $p_2' \in I'$. We conclude that $p_2'$ is a common point of the lines in the family $\{I_i'\}$, so that $p_2' = p_1'$.

□

Theorem 7.10 Let $\mathbb{A}$ and $\mathbb{P}'$ be an affine space and a projective space over division rings $K$ and $K'$, respectively, with $|K| \geq 4$ or $|K| = 3 = \text{caract } K'$.

Any morphism of geometries $\varphi : \mathbb{A} \to \mathbb{P}'$, such that the image is not contained in a line, extends to a unique partial projective morphism $\phi : \mathbb{P} \dashrightarrow \mathbb{P}'$, where $\mathbb{P}$ is the projective closure of $\mathbb{A}$.
Proof. Let $E \subset H$ be the exceptional subspace of $\varphi: A \rightarrow P'$. We define $\phi: P - E \longrightarrow P'$ by the formula

$$\phi(x) := \begin{cases} \varphi(x) & \text{when } x \in A \\ x' & \text{when } x \in H - E \end{cases}$$

Let us see that $\phi: P \rightarrow P'$ is a partial projective morphism, with exceptional subspace $E$. We must check the conditions $(b_1)$ and $(b_2)$ of \[5.6\]

$(b_1)$. Let us prove that for all $x_0, x_1, x_2 \in P - E$ we have: $x_0 \in x_1 \lor x_2 \Rightarrow \phi(x_0) \in \phi(x_1) \lor \phi(x_2)$. There are several cases, according to the distribution of the points $x_0, x_1, x_2$ in $A$ and $H - E$. In any case the required property directly follows from the definition of $\phi$, except in the case $x_0, x_1, x_2 \in H - E$ (then we apply \[7.8\]) and the case $x_1, x_2 \in A, \varphi(x_1) = \varphi(x_2)$ and $x_0$ is the point at infinity of the line $x_1 \lor x_2$ (then we apply \[7.7\]).

$(b_2)$. Let us show that for all $x_1, x_2 \in P - E$, $x_1 \neq x_2$, we have $(x_1 \lor x_2) \cap E \neq \emptyset \Rightarrow \phi(x_1) = \phi(x_2)$.

If $x_1, x_2 \in H - E$ we apply \[7.9\].

If $x_1, x_2 \in A$, then the affine line $L$ passing through $x_1, x_2$ has an exceptional point at infinity $p \in E$, so that $\varphi|_L$ is constant, and $\phi(x_1) = \phi(x_2)$.

A related theorem is obtained in \[10\] (Satz 2) replacing $A$ by a non empty open set of $P$ with respect to a linear topology.

**Corollary 7.11** With the hypotheses of the theorem, if $\varphi: A \rightarrow P'$ is injective then the extension $\phi: P \rightarrow P'$ is globally defined.

**Proof.** We have $E = \emptyset$: If there is an exceptional point $p$, then for any line $L \subset A$ with the point $p$ at infinity we have that $\varphi|_L$ is constant, so contradicting that $\varphi$ is injective.

**□**

**Corollary 7.12** With the hypotheses of the theorem, if $\varphi: A \rightarrow P'$ is an embedding then the extension $\phi: P \rightarrow P'$ so is an embedding.

**Proof.** If $\dim A = n < \infty$, then we have $\dim \varphi(P) \leq \dim P = n$ and $\dim \phi(P) \geq \dim \varphi(A) = \dim A = n$, so that $\dim \phi(P) = n$. By \[1.7\], $\phi: P \rightarrow \varphi(P)$ is an isomorphism, that is to say, $\phi: P \rightarrow P'$ is an embedding.

In the general case we put $A$ as a union of affine subspaces of finite dimension: $A = \bigcup A_i$. Then $P = \bigcup P_i$, where $P_i$ is the projective closure of $A_i$. By the former case, $\phi: P_i \rightarrow \varphi(P_i)$ is an isomorphism for any index $i$, hence $\phi: P = \bigcup P_i \longrightarrow \bigcup \phi(P_i) = \phi(P)$ is an isomorphism.

**□**

This result was obtained in \[21\] in the case $\dim A = 2$ and in \[5\] in the case $\dim A < \infty$. It may be generalized replacing $A$ by an open set of $P$ with respect to a linear topology (see \[10\] Satz 1).
7.13 Counterexamples

- When $|K| = 3$, it is well known that the affine plane $A_2$ may be embedded in the complex projective plane $P'_2$ as the set of nine inflexion points of a non-singular cubic, since any line passing through two inflexion points also passes through a third one (this property is related with the group law of the cubic). This embedding may be not extended to the projective closure of $A_2$ because there are no homomorphisms $K \to C$.

- Let $A_3$ be the 3-dimensional affine space over $K = \mathbb{Z}_2$; this space has eight points $p_0, \ldots, p_7$ (vertices of a cube) and each line has two points. Let $P'_2$ be the projective plane over a field $K'$ with $|K'| \geq 7$; so that the non-singular conic $C \subset P'_2$ given by the homogeneous equation $x_1^2 - x_2x_3 = 0$ has $\geq 8$ points. Let us consider an octagon $X = \{p'_1, \ldots, p'_8\} \subseteq C$ inscribed in the conic. Since there are not three collinear vertices in the octagon, it is easy to check that the injective map $A_3 \to P'_2$, $p_i \mapsto p'_i$, is a morphism of geometries. If we order the points of $A_3$ so that the lines $p_1p_2$, $p_3p_4$ and $p_5p_6$ are parallel, and we take an octagon such that the sides $p'_1p'_2$, $p'_3p'_4$, $p'_5p'_6$ are not concurrent, then it is clear that the morphism $A_3 \to P'_2$ does not extend to the projective closure of $A_3$.

Now we see that theorem 7.10 let us express the morphisms of geometries $\varphi: A \to A'$ in terms of semilinear maps.

7.14 Let $(A, V, +)$ and $(A', V', +)$ be affine spaces over division rings $K$ and $K'$, respectively, such that $|K| \geq 4$ or $|K| = 3 = \text{caract } K'$. Let $\varphi: A \to A'$ be a morphism of geometries such that the image is not contained in a line.

Let us fix points $p_0 \in A$, $p'_0 = \varphi(p_0) \in A'$. Let us consider the morphism of geometries $\vec{\varphi}: V \to V'$ defined by the commutative square

\[
\begin{array}{ccc}
p_0 + v & \xrightarrow{\varphi} & p'_0 + v' \\
v & \xrightarrow{\vec{\varphi}} & v'
\end{array}
\]

that is to say,

\[\vec{\varphi}(v) = \varphi(p_0 + v) - \varphi(p_0)\]

The image of $\vec{\varphi}$ is not contained in a line and we have $\vec{\varphi}(0) = 0$. Let us consider the vectorial extensions

\[A = p_0 + V \subset Kp_0 \oplus V = E\]

\[A' = p'_0 + V' \subset K'p'_0 \oplus V' = E'\]

By theorem 7.10 the morphism $\varphi: A \to A' \subset P'$ extends to a partial projective morphism $\phi: \overline{P} = \overline{P(E)} \to \overline{P(E')} = \overline{P'}$, so that we obtain a commutative
The partial projective morphism $\phi$ is defined by a semilinear map $\Phi: E \to E'$, unique up to a scalar factor. The above commutative square shows that the following diagram

$$
\begin{array}{ccc}
A & \subset & P(E) \\
\downarrow \phi & & \downarrow \Phi \\
A' & \subset & P(E')
\end{array}
$$

is commutative modulo a scalar factor,

$$
\Phi(x) = \lambda'_x \cdot \varphi(x) \quad \text{with} \quad \lambda'_x \in K'
$$

for all $x \in A$. Replacing $\Phi$ by a proportional map, we may assume that $\Phi(p_0) = \varphi(p_0)$, that is to say, $\lambda'_{p_0} = 1$.

Let $\omega': E = K' p_0' \oplus V' \to K'$ be the projection onto the first factor, which is a $K'$-linear map. Remark that $\ker \omega' = V'$ and $\omega'|_{A'} = 1$. We define $\omega := \omega' \circ \Phi : E \to K'$; since $\omega'$ is linear, the semilinear maps $\Phi, \omega$ have the same associated homomorphism $\sigma: K \to K'$.

For all $x \in A$ we have

$$
\omega(x) = \omega'\Phi(x) = \lambda_x \omega' \varphi(x) \quad \text{with} \quad \lambda_x = 1
$$

so that (4) states that

$$
\Phi(x) = \omega(x) \cdot \varphi(x) \quad \forall \ x \in A
$$

(5)

Remark that $\omega(p_0) = 1$ because $\Phi(p_0) = \varphi(p_0)$.

Now let us calculate $\bar{\varphi}$,

$$
\bar{\varphi}(v) = \varphi(p_0 + v) - \varphi(p_0) = \varphi(p_0) + \frac{1}{\omega(p_0) + \omega(v)} \Phi(p_0 + v) - \frac{1}{\omega(p_0)} \varphi(p_0)
$$

$$
= \frac{1}{1 + \omega(v)} (\varphi(p_0) + \Phi(v)) - \varphi(p_0) = \frac{1}{1 + \omega(v)} (\Phi(v) - \omega(v) \varphi(p_0))
$$

Now we consider the map $\Psi: V \to E'$, $\Psi(v) := \Phi(v) - \omega(v) \varphi(p_0)$, which is a semilinear map with associated homomorphism $\sigma: K \to K'$. Then

$$
\bar{\varphi}(v) = \frac{1}{1 + \omega(v)} \Psi(v)
$$

Let us see that $\Psi$ takes values in $V'$: We have $\omega'(\Psi(v)) = \omega'\Phi(v) - \omega'(v) \varphi(p_0) = \omega'(v) - \omega'(v) \cdot 1 = 0$, so that $\Psi(v) \in \ker \omega' = V'$.

Finally, we obtain the following result,
Theorem 7.15 (Zick [25]) Let $V$ and $V'$ be vector spaces over division rings $K$ and $K'$, respectively, with $|K| \geq 4$ or $|K| = 3 = \text{caract} K'$.

Let $\varphi: V \to V'$ be a morphism of geometries, such that the image is not contained in an affine line of $V'$, and $\varphi(0) = 0$.

There are semilinear maps $\Psi: V \to V'$ and $\omega: V \to K'$, with the same associated homomorphism $\sigma: K \to K'$, such that $1 + \omega$ is non null at any point of $V$ and

$$\varphi(v) = \frac{1}{1 + \omega(v)} \Psi(v) \quad \forall \ v \in V$$

According to [25], the above theorem was proved by G. Martin when $K$ and $K'$ are commutative.

Maps $(1 + \omega(v))^{-1} \Psi(v)$, with $\Psi$ and $\omega$ as in the above theorem, are named fractional semilinear maps in [25]. The converse theorem (any fractional semilinear map $\varphi: V \to V'$ is a morphism of geometries) is easy to check.

Since any homomorphism $\sigma: \mathbb{R} \to \mathbb{R}$ is the identity, as a direct consequence of theorem 7.15 we have the following

Corollary 7.16 Let $\varphi: \mathbb{A} \to \mathbb{A}'$ be a morphism of geometries between real affine spaces, such that the image is not contained in a line. Then $\varphi$ is an affine morphism.

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