The classical action for a Bianchi $VI_h$ model

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Abstract

An estimate for the classical action $I_{\text{classical}}$ for a Bianchi $VI_h$ homogeneous spatially closed model with $h = -1/9$ is given by

$$\frac{I_{\text{classical}}}{\hbar} \approx \frac{3}{5} i \pi^2 (1 + \alpha_0) \left[ \left( \frac{r(t)}{L^*} \right)^2 \left( \frac{r(t)}{r_{m2}} \right)^{1/2} \right. $$

$$ - 2 \left( \frac{r(t)}{r_1} \right) \left( \frac{r(t)}{r_{m2}} \right)^{3/2} \left( \frac{L^*}{r_0} \right)^2 \left( \frac{b}{8a_0^2 L^*} \right)^2 \right],$$

where $b$ and $a_0$ are parameters of the model, $b$ is zero if and only if the relative rotation of inertial frames and matter is zero, $\alpha_0$ is an initial value of an anisotropy of the expansion rate at $r = r_0$, $L^*$ is the Planck length, and $r_{m2}$ is a constant of integration that gives the maximum size of the universe for the isotropic ($b = 0$, $a_0 = 0$) case.

It is assumed that the equation of state is $p = (\gamma - 1) \rho$, where $p$ is pressure and $\rho$ is density. It is assumed that $\gamma$ has a constant value of $\gamma = 4/3$ (to represent a relativistic early universe) for $r < r_1$ and a constant value of $\gamma = 1$ (to represent a matter-dominated late universe) for $r > r_1$. The approximation is valid for $b$ small enough that $|I_{\text{classical}}[b] - I_{\text{classical}}[b = 0]| < \hbar$.

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An explanation for why our inertial frame seems not to rotate relative to the stars is found in a straightforward application of semi-classical approximations to quantum cosmology. Application of a saddlepoint approximation leads to the result that only those classical geometries whose action $I_{\text{classical}}$ satisfies $|I_{\text{classical}} - I_{\text{saddlepoint}}| < \hbar$ contribute significantly to the integration to give the present value of the wave function. Using estimates for our universe implies that only those classical geometries for which the present relative rotation rate of inertial frames and matter are less than about $10^{-130}$ radians per year contribute significantly to the integration. This is well below the limit set by experiment. The result depends on the Hubble distance being much larger than the Planck length, but does not depend on the details of the theory of quantum gravity.

1 Introduction

Although Newton recognized that inertial frames seem not to rotate relative to the stars, he seems to have taken that as evidence for the existence of absolute space. Ernst Mach \cite{Mach1872, Mach1933} was probably the first physicist to recognize that such an apparent coincidence requires an explanation. He based his arguments on the observation that only relative positions of bodies are observable. Mach further suggested that matter might cause inertia.

Einstein \cite{Einstein1918} tried to include what he called “Mach’s Principle” in General Relativity, and although it is generally agreed that matter is a source of inertia in his theory, it is not the only source, because there are solutions (such as empty Minkowski space) that have inertia without matter. Further, solutions of Einstein’s field equations include cosmologies where inertial frames rotate relative to the bulk of matter in the universe, so that an explanation for why our inertial frame does not rotate relative to the stars is still needed.

It has been suggested (e.g. \cite{Wheeler1964}) that Mach’s principle be used as a boundary condition to eliminate those solutions of the field equations that have inertia from sources other than from matter. Similar suggestions (e.g. \cite{Al'tshuler1967, Lynden-Bell1967, Sciama1968, Gilman1970, Raine1973}) that such selection might take place automatically if a satisfactory integral formulation of Einstein’s field equations might be found.
would eliminate the need for explicit boundary conditions. Other suggestions in which gravitation (including inertia) is represented by a theory analogous to Newtonian gravitation or Maxwell theory (e.g. Sciama 1953, Lynden-Bell 1992) are also intriguing, and have the similar property that the gravitational field would be determined solely by the matter distribution.

Even if we could successfully find a theory along the lines of those mentioned above, in which matter somehow determined the gravitational field, it might not be satisfactory. Rather than explain why our inertial frame seems not to rotate relative to the stars, such a theory would simply impose that condition, and would take away all degrees of freedom from the gravitational field. We are not so restrictive with the electromagnetic field, for example. We allow arbitrary initial values on the electromagnetic field that are consistent with Maxwell’s equations. We don’t require that the electromagnetic field be completely determined by charges and currents. Just as the electromagnetic field should be as fundamental as charges and currents, the gravitational field should be as fundamental as matter [Earman 1993, Kuchař 1993, Raine 1993].

Allowed arbitrary initial conditions for the gravitational field (consistent with the field equations) is inconsistent with trying to explain why our inertial frame seems not to rotate relative to the stars, at least on the classical level.

On the quantum level, however, we might imagine that somehow the selection takes place automatically through wave interference, and this turns out to be the case. To show that requires calculating the action for a classical cosmology as a function of the appropriate parameters of the model, which is the goal here.

2 An example from ordinary wave mechanics

To help explain the ideas that follow, we first consider elementary wave mechanics as an example. If we have an initial single-particle state specified by an initial wave function $<x_1, t_1|\psi>$ at time $t_1$ then the wave function $<x_2, t_2|\psi>$ at time $t_2$ is [Feynman and Hibbs 1965, p. 57]

\[
<x_2, t_2|\psi> = \int_{-\infty}^{\infty} <x_2, t_2|x_1, t_1><x_1, t_1|\psi> dx_1,
\]

where $<x_2, t_2|x_1, t_1>$ is the propagator for the particle to go from $(x_1, t_1)$ to $(x_2, t_2)$. We consider the case where the semiclassical approximation for
the propagator is valid. That is, [Feynman and Hibbs 1965, p. 60]
\[
\langle x_2, t_2 | x_1, t_1 \rangle \approx f(t_1, t_2) e^{i\frac{\hbar}{\pi} I_{cl}[x_2, t_2; x_1, t_1]},
\]
(2)
where \( I_{cl}[x_2, t_2; x_1, t_1] \) is the action calculated along the classical path from \((x_1, t_1)\) to \((x_2, t_2)\). Thus, (1) becomes
\[
\langle x_2, t_2 | \psi \rangle \approx f(t_1, t_2) \int_{-\infty}^{\infty} e^{i\frac{\hbar}{\pi} I_{cl}[x_2, t_2; x_1, t_1]} \langle x_1, t_1 | \psi \rangle \, dx_1.
\]
(3)
Notice that because of the initial wave function we have an infinite number of classical paths contributing to each value of the final wave function.

There are two cases to consider. In the first, \( I_{cl} \) is not a sharply peaked function of \( x_1 \). In that case, there will be contributions to the wave function at \( t_2 \) from classical paths that differ greatly from each other.

In the second case, which we now consider, \( I_{cl} \) is sharply peaked about some value of \( x_1 \), say \( x_{sp} \). That is, we have
\[
\frac{\partial}{\partial x_1} I_{cl}[x_2, t_2; x_1, t_1]|_{x_1=x_{sp}} = 0.
\]
(4)
Thus, \( x_{sp} \) is a saddlepoint of the integral (3), and significant contributions to the integral are limited to values of \( x_1 \) such that
\[
|x_1 - x_{sp}|^2 < \left| \frac{2\hbar}{\frac{\partial^2}{\partial x_1^2} I_{cl}[x_2, t_2; x_1, t_1]|_{x_1=x_{sp}}} \right|.
\]
(5)
If \( \langle x_1, t_1 | \psi \rangle \) is nearly constant over that range, then we can take it outside of the integral. A saddlepoint evaluation of the integral then gives
\[
\langle x_2, t_2 | \psi \rangle \approx f(t_1, t_2) \langle x_{sp}, t_1 | \psi \rangle \left[ \frac{2\pi i\hbar}{\frac{\partial^2}{\partial x_1^2} I_{cl}[x_2, t_2; x_1, t_1]|_{x_1=x_{sp}}} \right]^{1/2} e^{i\frac{\hbar}{\pi} I_{cl}[x_2, t_2; x_{sp}, t_1]}.
\]
(6)
We notice from (4) that the momentum at \( t_1 \) at the saddlepoint is zero. That is,
\[
p_1|_{x_1=x_{sp}} = 0.
\]
(7)
However, for the paths that contribute significantly to the integral in (3), there is a range of momenta, namely

$$|p^2_1| < 2\hbar \left| \frac{\partial^2}{\partial x_1^2} I_{cl}[x_2, t_2; x_1, t_1]|_{x_1 = x_{sp}} \right|,$$

consistent with (5) and the uncertainty relation.

As a check, using a special case, we consider the free-particle propagator

[Feynman and Hibbs 1965, p. 42]

$$\langle x_2, t_2 | x_1, t_1 \rangle = \left[ \frac{m}{2\pi\hbar(t_2 - t_1)} \right]^{1/2} \exp \left[ \frac{im(x_2 - x_1)^2}{2\hbar(t_2 - t_1)} \right],$$

and we choose

$$\langle x_1, t_1 | \psi \rangle = \langle A, t_1 | \psi \rangle \exp[-B(x_1 - A)^2]$$

(10)

to represent a broad initial wave function. For this case, the integral in (4) or (3) can be evaluated exactly to give

$$\langle x_2, t_2 | \psi \rangle = \langle A, t_1 | \psi \rangle \left[ 1 + \frac{2B\hbar(t_2 - t_1)}{im} \right]^{-1/2} \exp \left[ \frac{-B(x_2 - A)^2}{1 - \frac{2B\hbar(t_2 - t_1)}{im}} \right].$$

The condition that the initial wave function is slowly varying is now

$$|B| \ll \left| \frac{m}{2\hbar(t_2 - t_1)} \right|,$$

so that (11) is approximately

$$\langle x_2, t_2 | \psi \rangle \approx \langle x_2, t_1 | \psi \rangle$$

(13)
in agreement with (4), since

$$I[x_2, t_2; x_1, t_1] = I_{cl}[x_2, t_2; x_1, t_1] = \frac{m}{2} \frac{(x_2 - x_1)^2}{t_2 - t_1},$$

(14)

and

$$x_{sp} = x_2.$$
Notice that it is the sharply peaked action that determines which classical paths in (3) dominate the integral in this case, not the maximum of the initial wave function.

The calculations can clearly be generalized to two or three dimensions. The main point is that whenever the initial wave function is broad and the action of the classical propagator is not a sharply peaked function of $x_1$, some very different classical paths may contribute to the wave function in the final state, even when a semiclassical approximation is valid for the propagator.

When the classical action is sharply peaked as a function of the coordinates of the initial state, however, only a narrow range of classical paths contribute significantly to the wave function in the final state. This is thus a mechanism for selecting classical paths in wave mechanics. As we shall argue in the next sections, this principle has broader application.

3 Quantum cosmology

In the case of quantum cosmology, we have a formula analogous to (1) to give the wave function over 3-geometries $g_2$ and matter fields $\phi_2$ on a 3-dimensional hypersurface $S_2$.

$$<g_2, \phi_2, S_2|\psi> = \int <g_2, \phi_2, S_2|g_1, \phi_1, S_1><g_1, \phi_1, S_1|\psi>D(g_1)D(\phi_1),$$

(16)

where $<g_1, \phi_1, S_1|\psi>$ is the wave function over 3-geometries $g_1$ and matter fields $\phi_1$ on a 3-dimensional hypersurface $S_1$, and $<g_2, \phi_2, S_2|g_1, \phi_1, S_1>$ is the amplitude to go from a state with 3-geometry $g_1$ and matter fields $\phi_1$ on a surface $S_1$ to a state with 3-geometry $g_2$ and matter fields $\phi_2$ on a surface $S_2$ [Hawking 1979]. $D(g_1)$ and $D(\phi_1)$ are the measures on the 3-geometry and matter fields. The integration is over all initial 3-geometries $g_1$ and matter fields $\phi_1$ for which the integral is defined.

4 Semiclassical approximation

As in section 2, we want to consider the case where the semiclassical approximation for the propagator is valid. That is, [Gerlach 1969]

$$<g_2, \phi_2, S_2|g_1, \phi_1, S_1> \approx f(g_2, \phi_2, S_2; g_1, \phi_1, S_1)e^{\frac{i}{\hbar}\int_{\Sigma_{g_2,\phi_2,S_2}} I_{cl}[g_2,\phi_2,S_2;g_1,\phi_1,S_1]},$$

(17)
where the function outside of the exponential is a slowly varying function and $I_{cl}$ is the action for a classical 4-geometry. Substituting (17) into (16) gives

$$<g_2, \phi_2, S_2|\psi> = \int f(g_2, \phi_2, S_2; g_1, \phi_1, S_1)e^{i\bar{I}_{cl}[g_2, \phi_2, S_2; g_1, \phi_1, S_1]}<g_1, \phi_1, S_1|\psi > D(g_1)D(\phi_1).$$

(18)

Each value of the integrand in (18) corresponds to one classical 4-geometry. As in (3), there will be an infinite number of classical 4-geometries that contribute to each value of the final wave function. Here, however, we do not have only one single integration, but an infinite number of integrations, because the integration is carried out over all possible 3-geometries and all matter fields on the initial surface.

In the simple example in Section 2, there were two cases to consider for the single integration being carried out. In the first case, the classical action was not a sharply peaked function. In the second case, the classical action was a sharply peaked function so that a saddlepoint approximation could be applied to the integration. Following that strategy, we would need to consider those two cases for each of the infinite number of integrations in (18).

Here, however, we consider only the case where $I_{cl}$ is a sharply peaked function of $g_1$ and matter fields $\phi_1$ for each of the infinite number of integrations in (18). We consider this case in the following section.

5 Saddlepoint approximation for the integral over initial states

We consider the case here where $I_{cl}$ is a sharply peaked function of $g_1$ and matter fields $\phi_1$ for each of the infinite number of integrations in (18). In that case, we can formally make the saddlepoint approximation for each of the integrations in (18). In analogy with (4), we have the saddlepoint condition

$$ \frac{\partial}{\partial g_1}I_{cl}[g_2, \phi_2, S_2; g_1, \phi_1, S_1]_{g_1=g_{sp}} = 0$$

(19)
and
\[ \frac{\partial}{\partial \phi_1} I_{cl}[g_2, \phi_2, S_2; g_1, \phi_1, S_1] \bigg|_{\phi_1 = \phi_{sp}} = 0, \tag{20} \]
where the derivatives in (19) and (20) are with respect to each parameter that defines the 3-geometry \( g_1 \) and matter fields \( \phi_1 \). We consider the case where there is only one solution to the saddlepoint conditions (19) and (20). In that case, (19) selects a single classical 4-geometry. However, there will be a range of classical 4-geometries in the neighborhood that contribute significantly to the integral in (18). These are determined by (e.g. Courant and Hilbert 1953)
\[ |I_{cl}[g_2, \phi_2, S_2; g_1, \phi_1, S_1] - I_{cl}[g_2, \phi_2, S_2; g_{sp}, \phi_{sp}, S_1]| < \hbar. \tag{21} \]

We can formally write the saddlepoint approximation to the integration in (18) as
\[ <g_2, \phi_2, S_2|\psi> = f(g_2, \phi_2, S_2; g_{sp}, \phi_{sp}, S_1) <g_{sp}, \phi_{sp}, S_1|\psi> \]
\[ \int f_1(g_2, \phi_2, S_2; g_{sp}, \phi_{sp}, S_1) e^{i \frac{1}{\hbar} \left[ I_{cl}[g_2, \phi_2, S_2; g_{sp}, \phi_{sp}, S_1] \right]} \tag{22} \]
where classical 4-geometries that contribute significantly to (22) (through the function \( f_1 \)) lie within a narrow range specified by (21).

Equation (19) requires that the momentum canonical to the initial 3-geometry for the classical 4-geometry at the saddlepoint be zero. That is
\[ \pi^{ij} \bigg|_{g_1 = g_{sp}} = 0. \tag{23} \]
(The extrinsic curvature on \( S_1 \) will therefore also be zero at the saddlepoint.) However, there will be a range of initial canonical momenta and a range of initial 3-geometries corresponding to the range of classical 4-geometries that satisfy (21), so that the uncertainty relations between initial 3-geometries and their canonical momenta are satisfied.

Whether there is a narrow or broad range of classical 4-geometries that satisfy (21) depends on the second derivative of the action with respect to the initial 3-geometry.

We can take the action to be
\[ I = \int (-g^{(4)})^{1/2}(L_{geom} + L_{matter})d^4x + \frac{1}{8\pi} \int (g^{(3)})^{1/2}Kd^3x, \tag{24} \]
where [York 1972] [Hawking 1979] show the importance of the surface term. [Hawking 1979] also points out a potential problem in that the action can be changed by conformal transformations, but suggests a solution.

\[ K = g^{(3)ij} K_{ij} \]  

(25)

is the trace of the extrinsic curvature. Although the extrinsic curvature is zero on \( S_1 \) at the saddlepoint, it will be nonzero in a region around the saddlepoint. The extrinsic curvature is given by

\[ K_{ij} = -\frac{1}{2} \frac{\partial g^{(3)}_{ij}}{\partial t}, \]  

(26)

where \( g^{(3)}_{ij} \) is the 3-metric. In this example, we take the Lagrangian for the geometry as

\[ L_{\text{geom}} = \frac{R}{16\pi}, \]  

(27)

where \( R \) is the scalar curvature, but we realize that a different Lagrangian might eventually be shown to be more appropriate in a correct theory of quantum gravity.

6 Perfect fluid models

For a perfect fluid, the energy momentum tensor is

\[ T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu}, \]  

(28)

where \( p \) is the pressure, \( \rho \) is the density, and \( u \) is the 4-velocity. For solutions to Einstein’s field equations for a perfect fluid, (27) becomes

\[ L_{\text{geom}} = \frac{1}{2} \rho - \frac{3}{2} p, \]  

(29)

and we can take the Lagrangian for the matter as [Schutz and Sorkin 1977]

\[ L_{\text{matter}} = \rho \]  

(30)

Thus, the classical action for perfect fluids is

\[ I_{cl} = \frac{3}{2} \int (g^{(4)})^{1/2} (\rho - p) d^4x - \frac{1}{16\pi} \int (g^{(3)})^{1/2} g^{(3)ij} \frac{\partial g^{(3)}_{ij}}{\partial t} d^3x. \]  

(31)
We can take
\[ p = (\gamma - 1)\rho \] (32)
for the equation of state, where \( 1 \leq \gamma < 2 \). Then (31) becomes
\[
I_{cl} = \frac{3}{2} \int (-g^{(4)})^{1/2}(2 - \gamma)\rho d^4x - \frac{1}{16\pi} \int (g^{(3)})^{1/2} g^{(3)ij} \frac{\partial g^{(3)}_{ij}}{\partial t} d^3x. \quad (33)
\]

Equation (33) diverges for a spatially open universe. The significance of that might be that only spatially closed universes make sense. On the other hand, it might be that the calculation of the action for the correct theory of quantum gravity will give a finite value for the action, even for a spatially open universe, but here, we shall restrict our calculation to the case of a spatially closed universe.

7 Spatially homogeneous spacetimes

The integration in (18) is an integration over functions \( g_1 \) and \( \phi_1 \) defined on \( S_1 \). In that sense, it is similar to a path integral. For example, there are six independent functions that define \( g_1 \). As in the integration for a path integral, there are approximations that can be made to reduce the number of integrations that must be performed.

Here, we want to consider matter distributions similar to that observed, at least for the large scale in our universe. Thus, we want to restrict the integration in (18) to classical spatially homogeneous 4-geometries that have a homogeneous matter distribution in calculating the classical action in the exponential. The integration in (18) would then be over the 3-geometries that form the boundary of those 4-geometries on \( S_1 \).

As an example, we shall use Einstein’s General Relativity for the classical 4-geometries, but the same calculations could be done for other classical gravitational theories, in case it turns out that General Relativity is not the correct theory of gravity. Thus, we want to consider the integration in (18) in which the classical 4-geometries used to calculate the action in the exponential are restricted to Bianchi cosmologies.

The appropriate calculation would be to consider the most general Bianchi model, with all of the parameters that describe that model, and carry out the integration over all of those parameters. We notice that the Bianchi
parameters (which are time independent) define the initial three geometry, and therefore are valid integration variables in (18). On the other hand, if it is suspected that the saddlepoint for the integration will correspond to the Friedmann-Robertson-Walker (FRW) model, then one can restrict consideration to only those Bianchi models that include the FRW model as a special case, and consider integration in (18) for only one Bianchi parameter at a time, holding the others fixed at the FRW value. Here, we do that for only one of the Bianchi models for illustration.

In choosing which Bianchi model to use, we would like one that has a parameter that can be varied continuously to give the FRW model. In addition, we would like to choose a parameter that represents rotation of inertial frames relative to the matter distribution. In that way, we could directly test the ability of quantum selection to implement Mach’s ideas about inertia.

So far, I have not been able to find a completely satisfactory example. Although the Bianchi IX cosmology is often used to represent anisotropy, it seems inappropriate for the present case because it is only a superposition of gravitational waves on a Friedman-Robertson-Walker background.

The Bianchi VI$_h$ model seems to be a better homogeneous model that has a parameter that represents an angular velocity of inertial frames relative to matter, and setting that parameter to zero seems to give the FRW metric. However, there seem to be some difficulties with the Bianchi VI$_h$ model being able to change continuously into the FRW model, and also a possible problem with the topology. Until I find a better example, however, I shall use this one.

We use the solution for the Bianchi VI$_h$ model from [Ellis and MacCallum 1969] with $h = -1/9$. This cosmological model is relevant here because it has a relative rotation of inertial frames with respect to the matter. Specifically,

$$\Omega(t) = \frac{b}{Y^2(t)Z(t)}$$

(34)

is the angular velocity in the rest frame of an observer moving with the fluid, of a set of Fermi-propagated axes with respect to a particular inertial triad. The parameter $b$ is an arbitrary constant of the model, and is zero if and only if there is no rotation of inertial frames relative to matter. Thus, we are interested to know the dependence of the classical action on $b$. 
After some algebra, we have

\[ I_{cl} = \frac{3\pi^2}{4a_0} \int_{t_0}^{t} \frac{Y(t)Z(t)}{X(t)} dt, \quad (35) \]

where the spatial part of the 4-volume integration has already been carried out, \( a_0 \) is a parameter of the model, \( t_0 \) corresponds to the surface \( S_1 \) in (14) and (18) and is enough larger than the Planck time \( T^* \) that the semiclassical approximation is valid, the upper limit in (35) corresponds to the surface \( S_2 \) in (16) and (18), and \( X(t), Y(t), \) and \( Z(t) \) are functions of the model that must be determined by differential equations given by [Ellis and MacCallum 1969]. As expected, the surface term in (33) has canceled.

If we define

\[ r^3(t) = X(t)Y(t)Z(t) \left( \frac{-3k}{3a_0^2 + q_0^2} \right)^{3/2} \quad (36) \]

and

\[ 1 + \alpha(t) = \frac{Y(t)^{2/3}Z(t)^{2/3}}{X(t)^{4/3}} \quad (37) \]

then the classical action in (33) becomes

\[ I_{cl} = \frac{3\pi^2}{4} \left( \frac{3 + q_0^2/a_0^2}{-3k} \right)^{1/2} \int_{r_0}^{\hat{r}} (1 + \alpha) r \frac{dr}{\hat{r}}, \quad (38) \]

where \( \hat{r} = dr/dt, r_0 \) is enough larger than the Planck length \( L^* \) that quantum effects can be neglected, and \( k = +1 \) for a closed universe.

For the \( h = -1/9 \) case, we have

\[ q_0 = -3a_0 \quad (39) \]

if and only if \( b \neq 0 \). Substituting (39) into (38) gives

\[ I_{cl} = \frac{3\pi^2}{2} \left( \frac{-1}{k} \right)^{1/2} \int_{r_0}^{\hat{r}} (1 + \alpha) r \frac{dr}{\hat{r}}, \quad (40) \]

The form of the equation of state in (32) allows one of the differential equations for the model to be integrated in closed form to give

\[ 8\pi \rho = 3r_m^{3\gamma-2}r^{-3\gamma} \quad (41) \]
where $r_m$ is a constant of integration that depends on the amount of matter in the universe and the speed of expansion relative to the gravitational attraction. Equation (11) shows that $r_m$ is a measure of the amount of matter in the universe for a given value of $r$. Therefore, we might expect Machian effects (inertial induction) to increase for larger values of $r_m$.

Using (41), we have

$$
\dot{r}^2 = \left( \frac{r_m}{r} \right)^{3\gamma - 2} - k - k\alpha - \frac{k}{(2a_0)^6} \frac{b^2}{3(1 + \alpha)^2 r^4} + \frac{r^2}{12} \left( \frac{\dot{\alpha}}{1 + \alpha} \right)^2.
$$

For the isotropic case, only the first two terms on the right hand side of (42) are nonzero. $r_m$ is the value of $r$ where those two terms are equal. For a closed universe for the isotropic case, $r_m$ is the maximum value of $r$.

Equation (42) can be written

$$
\dot{\alpha} = \frac{2V}{r^3},
$$

and the remaining differential equations to solve are

$$
\dot{V} = 3k(1 + \alpha)r - \frac{k}{(2a_0)^6} \frac{2b^2}{(1 + \alpha)^2 r^3}
$$

and

$$
\frac{\dot{\alpha}}{1 + \alpha} = \frac{2V}{r^3},
$$

where (15) is the definition of $V(t)$, and $r(t)$ represents the size of the universe. The angular rotation (34) of an inertial frame relative to local matter is given by

$$
\Omega(t) = \left( \frac{-k}{4a_0^3} \right)^{3/2} \frac{b}{[1 + \alpha(t)] r^3(t)}
$$

It is not possible to solve the differential equations exactly in closed form, but we can find approximate solutions. In the limit as $b$ approaches zero, it is valid to neglect all but the first term under the radical in (13). The appendix then gives

$$
1 + \alpha \approx (1 + \alpha_0) \exp \left\{ \frac{2r_0^3}{3\gamma - 6} \frac{\dot{\alpha}_0}{1 + \alpha_0} (r_0^{\frac{3}{2}\gamma - 3} - r_m^{\frac{3}{2}\gamma - 3}) r_m^{1 - \frac{\gamma}{2}} \right\}.
$$
This solution is valid for $r$ smaller than $r_m$ if $b$ is small enough.

We assume that $\gamma$ changes from $\gamma_1$ to $\gamma_2$ at $r = r_1$. To satisfy continuity of $\rho$ at $r = r_1$, we must have $r_m$ change from $r_{m1}$ to $r_{m2}$ at $r = r_1$, where

$$\frac{r_{m1}}{r_{m2}} = \left( \frac{r_1}{r_{m2}} \right)^{\frac{3\gamma_1 - 3\gamma_2}{\gamma_1 - \gamma_2}}. \tag{48}$$

Equation (47) applies for $r \leq r_1$ with $\gamma = \gamma_1$ and $r_m = r_{m1}$.

For $r > r_1$, (47) gives

$$1 + \alpha \approx (1 + \alpha_1) \exp \left\{ \frac{2r_1^3}{3\gamma_2 - 6} \frac{\dot{\alpha}_1}{1 + \alpha_1} (r_1^{\frac{\gamma_2 - 3}{\gamma_2}} - r_1^{\frac{\gamma_2 - 3}{\gamma_2}}) r_{m2}^{\frac{1}{\gamma_2}} \right\} +$$

$$12k \frac{(3\gamma_2 - 6)r^{3\gamma_2 - 2} - (6\gamma_2 - 4)r_1^{\frac{\gamma_2 + 1}{\gamma_2}} r_1^{3\gamma_2 - 3} + (3\gamma_2 + 2)r_1^{3\gamma_2 - 2}}{(3\gamma_2 + 2)(3\gamma_2 - 2)(3\gamma_2 - 6)r_m^{3\gamma_2 - 2}}$$

$$- 8k \frac{b^2}{3} \frac{r^{3\gamma_2 - 6} - 2r_1^{\frac{\gamma_2 - 3}{\gamma_2}} r_1^{3\gamma_2 - 3} + r_1^{3\gamma_2 - 6}}{(\gamma_2 - 2)(3\gamma_2 - 6)r_m^{3\gamma_2 - 2}} \right\}.$$
where continuity of $\alpha$ and $\dot{\alpha}$ at $r = r_1$ requires that

\[ 1 + \alpha_1 \approx (1 + \alpha_0) \exp \left\{ \frac{2r^3}{3\gamma_1 - 6} \left( \frac{\dot{\alpha}_0}{\alpha_0} \right) \left( r_1^{\frac{3}{\gamma_1} - 3} - r_0^{\frac{3}{\gamma_1} - 3} \right) r_{m1}^{1 - \frac{3}{\gamma_1}} + \frac{12k (3\gamma_1 - 6) r_1^{3\gamma_1 - 2} - (6\gamma_1 - 4) r_0^{\frac{3}{\gamma_1} + 1} r_1^{\frac{3}{\gamma_1} - 3} + (3\gamma_1 + 2) r_0^{3\gamma_1 - 2}}{(3\gamma_1 + 2)(3\gamma_1 - 2)(3\gamma_1 - 6) r_{m1}^{3\gamma_1 - 2}} \right\} \]

\[ \approx (1 + \alpha_0) \left\{ 1 + \frac{2r^3}{3\gamma_1 - 6} \left( \frac{\dot{\alpha}_0}{\alpha_0} \right) \left( r_1^{\frac{3}{\gamma_1} - 3} - r_0^{\frac{3}{\gamma_1} - 3} \right) r_{m1}^{1 - \frac{3}{\gamma_1}} + \frac{12k (3\gamma_1 - 6) r_1^{3\gamma_1 - 2} - (6\gamma_1 - 4) r_0^{\frac{3}{\gamma_1} + 1} r_1^{\frac{3}{\gamma_1} - 3} + (3\gamma_1 + 2) r_0^{3\gamma_1 - 2}}{(3\gamma_1 + 2)(3\gamma_1 - 2)(3\gamma_1 - 6) r_{m1}^{3\gamma_1 - 2}} \right\} \]

\[ \approx (1 + \alpha_0) \left\{ 1 + \frac{12k (3\gamma_1 + 2)(3\gamma_1 - 2) r_{m1}^{3\gamma_1 - 2}}{8k (3\gamma_1 - 6)(3\gamma_1 - 2) r_{m1}^{3\gamma_1 - 2}} \right\}. \]
Classical action for a Bianchi $VI_h$ model

$$\frac{24k}{(3\gamma_1 + 2)(3\gamma_1 - 2)(9\gamma_1 - 2)} \left( \frac{r_1}{r_m} \right)^{\frac{2}{\gamma_1} + 1} - \left( \frac{r_0}{r_m} \right)^{\frac{2}{\gamma_1} - 1} \right) \left( \frac{r_m}{r_m} \right)^2$$

$$\frac{-8k}{3} \left( \frac{r_1^2}{r_m} \right)^{\frac{3}{\gamma_1} - 2} - \left( \frac{r_0}{r_m} \right)^{\frac{3}{\gamma_1} - 1} \right) \left( \frac{r_m}{r_m} \right)^2$$

$$+ \left[ 1 + 12k \right] \left( \frac{r_1^3}{r_m} \right)^{\frac{2}{\gamma_1} - 3} \left( \frac{r_0}{r_m} \right)^{\frac{3}{\gamma_1} - 6} \left( \frac{r_m}{r_m} \right)^2$$

$$- \frac{8k}{3} \left( \frac{r_1^2}{r_m} \right)^{\frac{3}{\gamma_1} - 2\gamma_1} - \left( \frac{r_0}{r_m} \right)^{\frac{3}{\gamma_1} - 2\gamma_1} \right) \left( \frac{r_m}{r_m} \right)^2$$

$$+ \left[ 1 + 12k \right] \left( \frac{r_1^3}{r_m} \right)^{\frac{2}{\gamma_1} - 3} \left( \frac{r_0}{r_m} \right)^{\frac{3}{\gamma_1} - 6} \left( \frac{r_m}{r_m} \right)^2$$

$$+ \left[ 1 + 12k \right] \left( \frac{r_1^3}{r_m} \right)^{\frac{2}{\gamma_1} - 3} \left( \frac{r_0}{r_m} \right)^{\frac{3}{\gamma_1} - 6} \left( \frac{r_m}{r_m} \right)^2$$

$$+ \left[ 1 + 12k \right] \left( \frac{r_1^3}{r_m} \right)^{\frac{2}{\gamma_1} - 3} \left( \frac{r_0}{r_m} \right)^{\frac{3}{\gamma_1} - 6} \left( \frac{r_m}{r_m} \right)^2$$

Equation (51) neglects all but the first term under the radical in (43). In making the calculation, I actually included the other terms under the radical to first order, but then determined after the integration that they could be neglected.

Neglecting some small terms gives

$$I_{cl} = \frac{3\pi^2}{2} \left( \frac{r_m^2}{L^*} \right)^2 \left( \frac{r_m}{r_m} \right)^{\frac{2}{\gamma_1} + 1} \left( \frac{r_m}{r_m} \right)^2$$

Substituting (48) into (52) gives

$$I_{cl} = \frac{3\pi^2}{2} \left( \frac{r_m^2}{L^*} \right)^2 \left( \frac{r_m}{r_m} \right)^{\frac{2}{\gamma_1} + 1} \left[ 1 - \frac{8k}{3} \left( \frac{r_1^3}{r_m} \right)^{\frac{2}{\gamma_1} - 2\gamma_1} \right] \left( \frac{r_m}{r_m} \right)^2$$

To get a rough estimate, we take

$$\gamma_1 = 4/3$$

and

$$\gamma_2 = 1$$
to represent a matter-dominated late universe. Substituting (54) and (55) into (51) gives

\[
\frac{I_{cl}}{\bar{h}} = \frac{3\pi^2}{2} \left( \frac{1}{2} \right)^{1/2} \left( \frac{r_{m2}}{L^*} \right)^2 \left( 1 + \alpha_0 \right) \left\{ \frac{1}{3} \left( \frac{r_1}{r} \right)^3 \left( \frac{r}{r_{m1}} \right)^{5/2} \left( \frac{r}{r_1} \right)^{1/2} \right. \\
+ \left. \frac{2}{5} \left[ \left( \frac{r}{r_{m2}} \right)^2 - \left( \frac{r_1}{r} \right)^{5/2} \left( \frac{r}{r_{m2}} \right)^{5/2} \left( \frac{r}{r_1} \right)^{5/2} \right] \right\} \\
+ \frac{k}{5} \left[ \left( \frac{r}{r_{m2}} \right)^5 \left( \frac{r}{r_{m1}} \right)^5 - \left( \frac{r_0}{r_{m1}} \right)^2 \right] \left( \frac{r_{m1}}{r_{m2}} \right)^2 \\
- \frac{2\pi^2}{2} \left( \frac{r_{m1}}{r_{m2}} \right)^2 \left( \frac{r}{r_{m2}} \right)^{5/2} \left( \frac{r}{r_{m1}} \right)^3 \\
\left[ \frac{1}{5} \left( \frac{r_{m1}}{r_{m2}} \right)^2 \left[ \left( \frac{r_1}{r} \right)^3 \left( \frac{r}{r_{m1}} \right)^3 - \left( \frac{r_0}{r_{m1}} \right)^3 \right] \\
+ \frac{2}{5} \left( \frac{r}{r_{m2}} \right)^{5/2} \left( 1 - \left( \frac{r_0}{r_{m1}} \right)^{5/2} \right) \right\}.
\]

Neglecting some small terms, letting \( k = +1 \) for a closed universe, and using (18) gives

\[
\frac{I_{cl}}{\bar{h}} = \frac{3\pi^2}{2} \left( \frac{r_{m2}}{L^*} \right)^2 \left( 1 + \alpha_0 \right) \left\{ \frac{1}{3} \left( \frac{r_1}{r} \right)^3 \left( \frac{r}{r_{m2}} \right)^{5/2} \left( \frac{r}{r_1} \right)^{1/2} \right. \\
+ \left. \frac{2}{5} \left[ 1 + \left( \frac{r_1}{r} \right)^2 \left( \frac{r}{r_{m2}} \right)^2 \left( \frac{r}{r_{m2}} \right)^{2} \left( \frac{r}{r_1} \right)^{5/2} \left( \frac{r}{r_{m2}} \right)^{7/2} \left( \frac{r}{r_1} \right)^{3/2} \right] \\
- \frac{2}{5} \left( \frac{2\alpha_0}{r_{m2}^2} \right)^{5/2} \left( \frac{r}{r_{m2}} \right)^{5/2} \right\} \\
\approx \frac{3\pi^2}{2} \left( \frac{r_{m2}}{L^*} \right)^2 \left( 1 + \alpha_0 \right) \left\{ \frac{2}{5} \left( \frac{r}{r_{m2}} \right)^{5/2} - \frac{2}{5} \left( \frac{2\alpha_0}{r_{m2}^2} \right) \left( \frac{r}{r_{m2}} \right)^{5/2} \right\}.
\]

Because the parameter \( b \) is an initial value for the cosmology, it is one of the variables of integration in (18). In making the saddlepoint approximation for that integration, we need to locate the saddlepoint (that is, the value of \( b \) that makes the action in (52), (53), or (57) stationary. We see that the action is stationary with respect to variation of \( b \) at the isotropic case of
$b = 0,$ as expected. The range of values of $b$ that contribute significantly to the integral in (18) is given by (21). That is
\[ |I_{cl}(b) - I_{cl}(b = 0)| < 1. \] (58)

Thus, substituting (53) into (58) gives
\[ \frac{3\pi^2}{2} \left( \frac{r_{m2}}{L^*} \right)^2 \frac{(1 + \alpha_0)b^2}{(2a_0)^6 r_{m2}^4} \left( \frac{r}{r_{m2}} \right)^{2\gamma_2 - 3\gamma_1 + 1} \left( \frac{r_{m2}}{r} \right)^{3(\gamma_1 - \gamma_2)} \left( \frac{r_{m2}}{r_0} \right)^{6 - 3\gamma_1} < 1. \] (59)

The approximations made so far are valid whenever (59) holds.

Substituting (54) and (55) into (59) gives
\[ \frac{6\pi^2}{5} (1 + \alpha_0) \left( \frac{r}{r_1} \right) \left( \frac{r}{r_{m2}} \right)^{\frac{3}{2}} \left( \frac{L^*}{r_0} \right)^2 \frac{b^2}{(2a_0)^6 L^{*4}} < 1. \] (60)

This gives
\[ b < \frac{\sqrt{5}(2a_0)^3}{\pi \sqrt{6}} \left( \frac{r_1}{r} \right)^{1/2} L^{*2} \frac{1}{(1 + \alpha_0)^{1/2}} \left( \frac{r_{m2}}{r} \right)^{\frac{3}{4}} \left( \frac{r_0}{r} \right)^{1/2} \left( \frac{L^*}{L} \right). \] (61)

Thus, from (46), the rotation rate of inertial frames is
\[ |\Omega(t)| < \frac{\sqrt{5}}{\pi \sqrt{6}} \left( \frac{L^*}{r_{m2}} \right)^2 \frac{1}{(1 + \alpha_0)^{1/2}} \left( \frac{r_{m2}}{r(t)} \right)^{\frac{15}{4}} \left( \frac{r_1}{r} \right)^{1/2} \left( \frac{r_0}{r} \right). \] (62)

If we now take the Planck length $L^*$ to be $1.6 \times 10^{-33}$ cm, use the Hubble distance of $1.7 \times 10^{28}$ cm for $r_{m2}$, a tenth of that for $r$, neglect $\alpha$ and $\alpha_0$ compared to 1, and take
\[ r_1 = \frac{r}{100} \] (63)
as an estimate that the universe changed from radiation-dominated to matter-dominated when the universe was about one-hundredth of its present size [Weinberg 1972, Section 15.3, p. 481], then we get
\[ |\Omega| < 1.6 \times 10^{-130} \text{ radians per year}, \] (64)
which is much less than the bound set by experiment of $10^{-14}$ to $7 \times 10^{-17}$ radians per year if the universe is spatially closed [Hawking 1969].

The rotation rate in (62) and (64) is so small because the Planck length is so much smaller than the Hubble distance.

That the small value of allowed rotation rate depends mostly on the universe being much larger than a Planck length rather than on details of the model suggests that the result has some generality.

We notice also, that the selection criterion in (58) is so sharp that the initial wave function in the integration in (18) would have to be very sharply peaked to overcome it.

8 Discussion

We see that considerations of quantum cosmology show how a range of classical cosmologies can be selected that contribute significantly to the wave function in the final state. The effect enters through the action. Using semiclassical calculations gives results that should not depend on particular details of the theory of quantum gravity.

For our universe (which is much larger than the Planck length) the selection is very sharp. The initial wave function over 3-geometries would have to be extremely sharp (not a probable occurrence) to dominate over the effect of the action.

The selection process seems to occur very soon in the development of a cosmology. That is, for a broad wave function over 3-geometries in the initial state, the wave function becomes sharply peaked after the universe has become a few orders of magnitude larger than the Planck length.

A different choice than (30) [Schutz 1976] is

$$L_{\text{matter}} = p.$$  

The choice in (65) gives a third of (31) for the total action. A correct theory of quantum gravity will determine which (if either) of these two choices is correct, but for this illustration, a factor of three in the action makes little difference.

It appears likely now, however, that there is not enough matter to keep our universe from expanding forever (e.g. [Coles and Ellis 1994]). To accommodate that with a spatially closed universe within General Relativity would
require a positive cosmological constant (e.g. [Hawking 1998]). I shall try to include the cosmological constant in a future calculation.

9 Acknowledgment

I would like to thank Douglas Gough for first bringing the paper by [Sciama 1953] to my attention in 1967.

A Bianchi $VI_h$ Models

We start with Equations (6.7) of Ellis and MacCallum (1969). For the case of zero cosmological constant, $\Lambda$, these can be written as

$$\frac{\dot{\rho}}{\rho + p} = -\frac{3\dot{R}}{R}$$ (66)

$$4\pi(\rho - p) = R^{-3}\left(R^3\frac{\dot{X}}{X}\right) - \frac{2(a_0^2 + q_0^2)}{X^2} + \frac{2b^2}{Y^4Z^2}$$ (67)

$$4\pi(\rho - p) = R^{-3}\left(R^3\frac{\dot{Y}}{Y}\right) - \frac{2(a_0^2 + a_0q_0)}{X^2} - \frac{2b^2}{Y^4Z^2}$$ (68)

$$4\pi(\rho - p) = R^{-3}\left(R^3\frac{\dot{Z}}{Z}\right) - \frac{2(a_0^2 - a_0q_0)}{X^2}$$ (69)

$$8\pi \rho + \frac{3a_0^2 + q_0^2}{X^2} = \frac{1}{2} \left[ 9 \left(\frac{\dot{R}}{R}\right)^2 - \left(\frac{\dot{X}}{X}\right)^2 - \left(\frac{\dot{Y}}{Y}\right)^2 - \left(\frac{\dot{Z}}{Z}\right)^2 \right] + \frac{2b^2}{Y^4Z^2},$$ (70)

where

$$R(t)^3 = X(t)Y(t)Z(t),$$ (71)

$b$, $a_0$, and $q_0$ are constants that are parameters of the model, and the variation of $\rho$, $X$, $Y$, and $Z$ with time is determined by (66) through (70).

According to Ellis and MacCallum (1969), (70) is a first integral of the other equations. If I understand that correctly, then taking the derivative of (70) should be a combination of the other equations. When I try that, the result is close, but not quite correct. I have not been able to find a similar
equation that does work, although I have found one that works for the special case \((q_0 + 3a_0 = 0\) and \(XY = Z^2\)) that I use later. This is

\[
8\pi\rho + \frac{3a_0^2 + q_0^2}{X^2} = \frac{1}{2} \left[ 9 \left( \frac{\dot{R}}{R} \right)^2 - \left( \frac{\dot{X}}{X} \right)^2 - \left( \frac{\dot{Y}}{Y} \right)^2 - \left( \frac{\dot{Z}}{Z} \right)^2 \right] - \frac{b^2}{Y^4 Z^2}. \tag{72}
\]

We can add (67), (68), and (69) with coefficients \(A\), \(B\), and \(C\) to give

\[
(A+B+C) \left[ 4\pi(\rho - p) + \frac{2a_0^2}{X^2} \right] = R^{-3} \left( R^3 \frac{\dot{U}}{U} \right) - (B-C) \frac{2a_0q_0}{X^2} - 2A \frac{q_0^2}{X^2} + (A-B) \frac{2b^2}{Y^4 Z^2}, \tag{73}
\]

where

\[
U = X^4 Y^B Z^C. \tag{74}
\]

We can take special cases of \(A\), \(B\), and \(C\). Taking \(A = B = C = 1/3\) in (73) gives

\[
4\pi(\rho - p) = R^{-3}(R^2 \dot{R}) - \frac{2}{3} \frac{3a_0^2 + q_0^2}{X^2}, \tag{75}
\]

where I have used (71). For another special case, we take \(A = B = 1\) and \(C = -2\) in (73) to give

\[
R^{-3} \left( R^3 \frac{\dot{U}}{U} \right) = -2q_0 \frac{3a_0 + q_0}{X^2}, \tag{76}
\]

where

\[
U = \frac{XY}{Z^2}. \tag{77}
\]

For a third special case, we take \(A = -4/3\) and \(B = C = 2/3\) in (73) to give

\[
R^{-3} \left( R^3 \frac{\dot{U}}{U} \right) = \frac{4b^2}{Y^4 Z^2} - \frac{8q_0^2}{3X^2}, \tag{78}
\]

where

\[
1 + \alpha = \frac{Y^{2/3} Z^{2/3}}{X^{4/3}}. \tag{79}
\]

We can use (71), (77), and (79) to determine \(X\), \(Y\), and \(Z\) in terms of \(R\), \(U\), and \(\alpha\). This gives

\[
X = \frac{R}{(1 + \alpha)^{1/2}}, \tag{80}
\]
Classical action for a Bianchi $VI_h$ model

\[ Y = U^{1/3}(1 + \alpha)^{1/2}R, \]  

(81)

and

\[ Z = \frac{R}{U^{1/3}}. \]  

(82)

Using (80), (81), and (82) in (75), (76), (78), and (72) gives

\[ R^3 \left( R^2 \dot{R} \right)' = 4\pi(\rho - p) + \frac{2(3a_0^2 + q_0^2)(1 + \alpha)}{3R^2}, \]  

(83)

\[ R^3 \left( R^3 \dot{U} \right)' = -2q_0(3a_0 + q_0)(1 + \alpha) \frac{R^2}{R^2}, \]  

(84)

\[ R^3 \left( \frac{\dot{\alpha}}{1 + \alpha} \right)' = \frac{4b^2}{(1 + \alpha)^2U'^3R^6} - \frac{8q_0^2(1 + \alpha)}{3R^2}, \]  

(85)

and

\[ \left( \frac{\dot{R}}{R} \right)^2 = \frac{1}{3} \left[ \frac{1}{2} \left( \frac{\dot{\alpha}}{1 + \alpha} \right)^2 + (\frac{1}{2} \dot{U}) \frac{1}{3U} + \left( \frac{1}{3U} \right)^2 \right] \]

\[ + \frac{8\pi\rho}{3} + \frac{(3a_0^2 + q_0^2)(1 + \alpha)}{3R^2} + \frac{b^2}{3(1 + \alpha)^2U'^3R^6}. \]  

(86)

If we change variables from $R$ to $r$, where

\[ r = \sqrt{-\frac{3k}{3a_0^2 + q_0^2}R}, \]  

(87)

(and where $k$ is +1 for a closed universe and -1 for an open universe), then (83) through (86) become

\[ r^{-3}(r^2 \dot{r})' = 4\pi(\rho - p) - \frac{2k(1 + \alpha)}{r^2}, \]  

(88)

\[ r^{-3} \left( r^3 \dot{U} \right)' = \frac{6kq_0(3a_0 + q_0)(1 + \alpha)}{(3a_0^2 + q_0^2)r^2}, \]  

(89)

\[ r^{-3} \left( \frac{\dot{\alpha}}{1 + \alpha} \right)' = \left( \frac{-3k}{3a_0^2 + q_0^2} \right)^3 \frac{4b^2}{(1 + \alpha)^2U'^3r^6} + \frac{8kq_0^2(1 + \alpha)}{(3a_0^2 + q_0^2)r^2}, \]  

(90)
and
\[(r^2 \dot{r})^2 = \frac{1}{3} \left[ \left( \frac{1}{2} \frac{\dot{\alpha}}{1 + \alpha} \right)^2 + \left( \frac{1}{2} \frac{\dot{\alpha}}{1 + \alpha} \right) \left( \frac{1}{3} \frac{\dot{U}}{U} \right) + \left( \frac{1}{3} \frac{\dot{U}}{U} \right)^2 \right] + \frac{8\pi \rho}{3} - \frac{k(1 + \alpha)}{r^2} + \left( \frac{-3k}{3a_0^2 + q_0^2} \right)^3 \frac{b^2}{3(1 + \alpha)^2 U^{2/3} r^6}. \] (91)

If we define \(W, V, \) and \(K\) by
\[r^2 \dot{r} = W, \] (92)
\[\frac{\dot{\alpha}}{1 + \alpha} = \frac{2V}{r^3}, \] (93)
and
\[\frac{\dot{U}}{U} = \frac{3K}{r^3}, \] (94)
then (88) through (91) become
\[\dot{W} = 4\pi (\rho - p) r^3 - 2k(1 + \alpha) r, \] (95)
\[\dot{K} = \frac{2kq_0 (3a_0 + q_0)(1 + \alpha) r}{3a_0^2 + q_0^2}, \] (96)
\[\dot{V} = \left( \frac{-3k}{3a_0^2 + q_0^2} \right)^3 \frac{2b^2}{(1 + \alpha)^2 U^{2/3} r^3} + \frac{4kq_0^2 (1 + \alpha) r}{3a_0^2 + q_0^2}, \] (97)
\[W^2 = \frac{V^2 + VK + K^2}{3} + \frac{8\pi \rho r^6}{3} - k(1 + \alpha) r^4 + \left( \frac{-3k}{3a_0^2 + q_0^2} \right)^3 \frac{b^2}{3(1 + \alpha)^2 U^{2/3}}. \] (98)

If we take
\[p = (\gamma - 1) \rho \] (99)
for the equation of state, where
\[1 \leq \gamma < 2 \] (100)
is a constant, then (66) can be integrated to give

$$8\pi\rho = 3r_m^{3\gamma-2}r^{-3\gamma}$$  \hspace{1cm} (101)

where $r_m$ is a constant of integration. Substituting (99) and (101) into (98) and (97) gives

$$\frac{\dot{W}}{r_m^2} = 3(1 - \gamma/2)r_m^{3\gamma-2}r^{3-3\gamma} - 2k(1 + \alpha)r,$$  \hspace{1cm} (102)

$$W^2 = V^2 + VK + K^2 + r_m^{3\gamma-2}r^{6-3\gamma} - k(1 + \alpha)r^4 + \left(\frac{-3k}{3a_0^2 + q_0^2}\right)^3 \frac{b^2}{3(1 + \alpha)^2U^{2/3}}.$$  \hspace{1cm} (103)

Let us now convert to a set of dimensionless variables defined by

$$\tau = t/r_m$$  \hspace{1cm} (104)

$$x = r/r_m$$  \hspace{1cm} (105)

$$y = V/r_m^2$$  \hspace{1cm} (106)

$$z = W/r_m^2$$  \hspace{1cm} (107)

$$C = K/r_m^2$$  \hspace{1cm} (108)

$$D = b/r_m^2$$  \hspace{1cm} (109)

Let us also use $'$ for $d/d\tau$. Then (92) through (97), (102), and (103) become

$$x^2x' = z,$$  \hspace{1cm} (110)

$$\frac{\alpha'}{1 + \alpha} = \frac{2y}{x^3},$$  \hspace{1cm} (111)

$$\frac{U'}{U} = \frac{3C}{x^3},$$  \hspace{1cm} (112)

$$C' = \frac{2kq_0(3a_0 + q_0)(1 + \alpha)x}{3a_0^2 + q_0^2},$$  \hspace{1cm} (113)

$$y' = \left(\frac{-3k}{3a_0^2 + q_0^2}\right)^3 \frac{2D^2}{(1 + \alpha)^2U^{2/3}x^3} + \frac{4kq_0^2(1 + \alpha)x}{3a_0^2 + q_0^2},$$  \hspace{1cm} (114)
Classical action for a Bianchi V $I_h$ model

$$z' = 3(1 - \gamma/2) x^{3-3\gamma} - 2k(1+\alpha)x,$$

(115)

$$z^2 = x^{6-3\gamma} - k(1+\alpha)x^4 + \left(\frac{-3k}{3a_0^2 + q_0^2}\right)^3 \frac{D^2}{3(1+\alpha)^2 U^{2/3}} + \frac{y^2 + yC + C^2}{3}.\quad (116)$$

For the $h = -1/9$ case, we have

$$q_0 = -3a_0.\quad (117)$$

if and only if $b \neq 0$. However, the $b = 0$ case is only a single point. When integrating over $b$, the behavior for small $b$ is more important than exactly at $b = 0$. Therefore, we shall use (117) in any case. Therefore, from (113), we have that $C$ is a constant.

So both $b$ and $C$ are constants that are determined by initial conditions. There are two possibilities. Either they are connected by a relation, or they are independent. There seems to be no obvious connection between them from the equations, so we shall assume they are independent unless we shall find it necessary to make a connection to get a consistent solution to the equations. Therefore, we can assume that both $b$ and $C$ must be integrated over on the initial hypersurface. For this demonstration, however, it is sufficient to integrate over only one initial constant, $b$. Therefore, we shall fix $C$ at the value we would guess for the FRW case, which is zero. Therefore, we take $C$ to be zero.

That means from (112) that $U$ is constant. Again, we guess that $U$ is independent from $b$, and choose the isotropic value. In the isotropic case, we would have $X = Y = Z$, and therefore, from (11) we have $U = 1$ as the isotropic value.

Therefore, from (114) we have

$$y' = 3k(1+\alpha)x - \frac{k}{(2a_0)^6 (1+\alpha)^2 x^3},$$

(118)

and from (116) we have

$$z^2 = x^{6-3\gamma} - k(1+\alpha)x^4 - \frac{k}{(2a_0)^6 3(1+\alpha)^2} \frac{D^2}{3} + \frac{y^2}{3}.\quad (119)$$

We notice at this point that if we take the derivative of (119) and substitute from (110), (111), (118), and (115), that we get an identity, confirming the consistency of the equations at this point.
Combining (110) with (119) gives

\[ x' = \sqrt{x^{2-3\gamma} - k(1 + \alpha)} - \frac{k}{(2a_0)^6} \frac{D^2}{3(1 + \alpha)^2 x^4} + \frac{y^2}{3x^4}. \]  \hspace{1cm} (120)

Combining (118) with (120) gives

\[ \frac{dy}{dx} = \frac{3k(1 + \alpha)x - k}{(2a_0)^6 (1 + \alpha)^2 x^4} \frac{2D^2}{\sqrt{x^{2-3\gamma} - k(1 + \alpha)}} + \frac{y^2}{3x^4}. \]  \hspace{1cm} (121)

Both \( b \) and \( a_0 \) are constants that are determined by initial conditions on the initial hypersurface. They are either related or independent. We assume first that they are independent. In that case, the third term under the radical in (121) and (122) will get smaller as \( D \) gets smaller. We shall assume that we can neglect that term relative to the first term under the radical for all values of \( x \). We can check that assumption later. There is also the possibility of iterating later by assuming that this term is small instead of zero.

We shall also neglect the last term under the radical in (120) and (121). We can check that approximation later. There is also the possibility of iterating by substituting an approximate solution for \( y \) into (121). We shall see that it is permissible to neglect the fourth term for all values of \( x \) within the integration range if \( b \) is small enough.

Neglecting the third and fourth terms under the radical in (120) and (122) gives

\[ x' = \sqrt{x^{2-3\gamma} - k(1 + \alpha)}. \]  \hspace{1cm} (122)

and

\[ \frac{dy}{dx} = \frac{3k(1 + \alpha)x - k}{(2a_0)^6 (1 + \alpha)^2 x^4} \frac{2D^2}{\sqrt{x^{2-3\gamma} - k(1 + \alpha)}}. \]  \hspace{1cm} (123)

We shall assume that

\[ \alpha \ll 1. \]  \hspace{1cm} (124)

We can test that approximation from the solution later and iterate if necessary. In that case, (120) and (121) become

\[ x' = \sqrt{x^{2-3\gamma} - k}. \]  \hspace{1cm} (125)
and

\[
\frac{dy}{dx} = \frac{3kx - \frac{k}{(2a_0)^6} \frac{2D^2}{x^3}}{\sqrt{x^2 - 3\gamma} - k}.
\]  

(126)

We notice that (126) could be integrated numerically to obtain an approximate solution for \(y(x)\). For a closed universe (which is the case we are considering), we can get an estimate of that solution, at least for small \(x\). The second term under the radical in (125) and (126) is smaller than the first term under the radical except at the point of maximum expansion, when they are equal (when \(x = 1\)). Thus, before the universe gets too close to the point of maximum expansion, we can neglect the second term under the radical in (125) and (126) to give

\[
x' = x^{1 - \frac{\gamma}{2}}.
\]  

(127)

and

\[
\frac{dy}{dx} = 3kx^{\frac{3}{2}\gamma} - 2kE^2x^{\frac{3}{2}\gamma-4},
\]  

(128)

where

\[
E \equiv \frac{D}{(2a_0)^3} = \frac{b/r_m^2}{(2a_0)^3}.
\]  

(129)

We can integrate (128) to give

\[
y = y_0 + \frac{6k}{3\gamma + 2} (x^{\frac{3}{2}\gamma+1} - x_0^{\frac{3}{2}\gamma+1}) - \frac{4k}{3} E^2 (x^{\frac{3}{2}\gamma-3} - x_0^{\frac{3}{2}\gamma-3}),
\]  

(130)

where, without loss of generality, the constant of integration has been chosen such that \(y = y_0\) when \(x = x_0\).

We notice that on substituting (130) into (120) and (121) the terms it gives that do not involve \(b\) have a higher power of \(x\) than the dominant term. Therefore, for small \(b\) and small \(x\), it was justified to neglect the fourth term under the radical in (120) and (121), since the terms that involve \(b\) are proportional to a positive power of \(b\).

Using (130) in (111) gives

\[
\frac{\alpha'}{1 + \alpha} = 2y_0x^{-3} + \frac{12k}{3\gamma + 2} (x^{\frac{3}{2}\gamma-2} - x_0^{\frac{3}{2}\gamma+1} x^{-3}) - \frac{8k}{3} E^2 (x^{\frac{3}{2}\gamma-6} - x_0^{\frac{3}{2}\gamma-3} x^{-3}).
\]  

(131)
Combining (131) with (127) gives

\[
\frac{d \ln(1 + \alpha)}{dx} = 2y_0 x^{\frac{3}{2} \gamma - 3} + \frac{12k}{3 \gamma + 2} \left( x^{3 \gamma - 3} - x_0^{\frac{3}{2} \gamma + 1} \right) x^{\frac{3}{2} \gamma - 4} - \frac{8k}{3} \frac{E^2}{\gamma - 2} (x^{3 \gamma - 7} - x_0^{\frac{3}{2} \gamma - 3} x^{\frac{3}{2} \gamma - 4}).
\]

(132)

We can integrate (132) to give

\[
\ln \frac{1 + \alpha}{1 + \alpha_0} = \frac{4y_0}{3 \gamma - 6} \left( x^{\frac{3}{2} \gamma - 3} - x_0^{\frac{3}{2} \gamma - 3} \right) + \frac{12k}{3 \gamma + 2} \frac{(3 \gamma - 6) x^{3 \gamma - 2} - (6 \gamma - 4) x_0^{\frac{3}{2} \gamma + 1}}{3 \gamma - 6} + \frac{8k}{3} \frac{E^2}{\gamma - 2} \frac{x^{3 \gamma - 6} - 2 x_0^{\frac{3}{2} \gamma - 3} x^{\frac{3}{2} \gamma - 3} + x_0^{3 \gamma - 6}}{3 \gamma - 6},
\]

(133)

where, without loss of generality, the constant of integration has been chosen such that \( \alpha = \alpha_0 \) when \( x = x_0 \).

Writing (130) and (133) in original variables using (105), (106), (129), and (93) gives

\[
V \equiv \frac{r^3}{2} \frac{\dot{\alpha}}{1 + \alpha} = \frac{r_0^3}{2} \frac{\dot{\alpha}_0}{1 + \alpha_0} + \left[ \frac{12k}{3 \gamma + 2} \frac{r_0^{\frac{3}{2} \gamma + 1} - r_0^{\frac{3}{2} \gamma + 1}}{3 \gamma + 2} - \frac{4k}{3} \frac{b^2}{(2a_0)^6} \frac{r_0^{\frac{3}{2} \gamma - 3} - r_0^{\frac{3}{2} \gamma - 3}}{\gamma - 2} \right] r_m^{1 - \frac{3}{2} \gamma},
\]

(134)

\[
1 + \alpha \approx (1 + \alpha_0) \exp \left\{ \frac{2r_0^3}{3 \gamma - 6} \frac{\dot{\alpha}_0}{1 + \alpha_0} (r_0^{\frac{3}{2} \gamma - 3} - r_0^{\frac{3}{2} \gamma - 3}) r_m^{1 - \frac{3}{2} \gamma} + \frac{12k}{3 \gamma + 2} \frac{(3 \gamma - 6) r_0^{3 \gamma - 2} - (6 \gamma - 4) r_0^{\frac{3}{2} \gamma + 1} r_0^{\frac{3}{2} \gamma - 3} + (3 \gamma + 2) r_0^{3 \gamma - 2}}{(3 \gamma + 2)(3 \gamma - 2)(3 \gamma - 6)} r_m^{3 \gamma - 2} \right\} - \frac{8k}{3} \frac{b^2}{(2a_0)^6} \frac{r_0^{3 \gamma - 6} - 2 r_0^{\frac{3}{2} \gamma - 3} r_0^{\frac{3}{2} \gamma - 3} + r_0^{3 \gamma - 6}}{(\gamma - 2)(3 \gamma - 6)r_m^{3 \gamma - 2}},
\]

(135)

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