Degeneracy in Studying the Supranuclear Equation of State and Modified Gravity with Neutron Stars

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Abstract. It is generally acknowledged that an extrapolation in physics from a well-known scale to an unknown scale is perilous. This prevents us from using laboratory experience to gain precise information for the supranuclear matter inside neutron stars (NSs). With operating and upcoming astronomical facilities, NSs’ equation of state (EOS) is expected to be determined at a new level in the near future, under the assumption that general relativity (GR) is the correct theory for gravitation. While GR is a reasonable working assumption yet still an extrapolation, there could be a large uncertainty due to the not-so-well-tested strong gravitational field inside NSs. Here we review some recent theoretical efforts towards a better understanding of the degeneracy between the supranuclear EOS and alternative gravity theories.

INTRODUCTION

Neutron stars (NSs) provide the best celestial laboratory to study the coupling between the strong-field gravity and the matter fields \cite{1, 2, 3, 4, 5}. In Einstein’s general relativity (GR), it is postulated that such a coupling is minimal, and the dynamics of the spacetime and matter fields are derived from a simple and esthetically appealing action \cite{6, 7},

\[ S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left( R + S_{\text{matter}}[\psi; g_{\mu\nu}] \right), \tag{1} \]

where $g$ is the determinant of the metric $g_{\mu\nu}$, $R$ is the Ricci scalar, and $S_{\text{matter}}$ is the action for all the matter degrees which are collectively denoted as $\psi$. The strikingly neat insight in Eq. (1) is the universal coupling in $S_{\text{matter}}[\psi; g_{\mu\nu}]$ between the matter degrees and the spacetime metric. It follows from the principle of equivalence \cite{8, 2, 9}. Equipped with this principle, it is validate to couple matter fields to the spacetime via the principle of general covariance \cite{6} thus obtaining the quantum fields on a classically curved spacetime.

The field equation derived from the action is \cite{6},

\[ R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}^{\text{matter}}, \tag{2} \]

where $R_{\mu\nu}$ is the Ricci tensor and $T_{\mu\nu}^{\text{matter}}$ is the energy-momentum tensor for matters. The consequence of this equation is that, as nicely summarized by John A. Wheeler, “matter tells spacetime how to curve, and spacetime tells matter how to move.” In our everyday experience, the spacetime is hardly curved however, due to the smallness of the gravitational coupling constant “$G$”. Only with exotic objects like the NSs, the spacetime reacts to the dense matter (namely, the $T_{\mu\nu}$) in a highly significant way.

Despite its delicate beauty, there are reasons to question GR, including the observational facts like the “dark matter” and the “dark energy”, as well as the theoretical dilemmas like the unavoidable singularities and the black-hole information loss problem \cite{2, 10}. Naive attempts to modify the Einstein’s equation (2) are classified, unsurprisingly, into two categories: (i) modifying the left-hand side (i.e., the geometric property of spacetime) and (ii) modifying...
We consider a NS that is made of perfect fluid with an energy-momentum tensor $T_{\mu \nu}$. In studying such a degeneracy is still lacking, here we review some efforts along this line. Fortunately, degeneracy is not the whole story. If non-minimal couplings between spacetime and matters are allowed, there exist many more ways to modify the Einstein’s equation [14, 8, 2]. Nevertheless, with non-minimal couplings, the degeneracy between modified gravity and matter contents is not gone altogether. We emphasize that, especially when the supranuclear EOS for NSs [12, 13] is quite uncertain, and the alternative gravity theories in the strong field are not empirically examined thoroughly, the degeneracy should not be overlooked. Though a complete picture in studying such a degeneracy is still lacking, here we review some efforts along this line.

The paper is organized as follows. In the next section, we give the theoretical ingredients to obtain the NSs’ structure in GR. Then, keeping alternative gravity theories in mind, we review the work to extend the framework with perturbative approaches and non-perturbative approaches. Some discussions are presented at the end. Throughout the paper we use the unit system where the light speed $c = 1$.

**THE STRUCTURE OF A NS IN GR**

We consider a NS that is made of perfect fluid with an energy-momentum tensor $T^{\mu \nu} = (\epsilon + p) u^\mu u^\nu + pg^{\mu \nu}$, where $u^\mu$ is the fluid element’s four-velocity, $p$ and $\epsilon$ are pressure and energy density respectively. Under the assumption of spherical symmetry, Tolman-Oppenheimer-Volkoff (TOV) equations describe a fully relativistic NS in hydrostatic equilibrium [15, 16, 6],

$$\frac{dp}{dr} = -G\frac{\epsilon + p}{r^2}(1 - 2Gm/r)$$

and

$$\frac{dm}{dr} = 4\pi r^2 \epsilon,$$

where $m$, $\epsilon$, and $p$ are functions of the stellar radius $r$.

Given an EOS, $\epsilon = \epsilon(p)$, and a central energy density $\epsilon^{(c)}$, the above equations can be integrated up to the stellar surface where $r = R$ and $p(R) = 0$. For a variety of $\epsilon^{(c)}$ one obtains the mass-radius relation $M(R)$ where $M \equiv m(R)$ is the Schwarzschild mass (in alternative gravity theories it can be different from the Arnowitt-Deser-Misner mass).

In Figure 1 we present the calculation in GR for 9 EOSs [17] that have a maximum NS mass larger than 2 $M_\odot$. As we can see, the current observations, from PSR J0348+0432 [18] and GW170817 [19, 20], are not capable to definitely pin down the correct EOS for NSs yet. However, if the measurements from GW170817 are taken into account, though with significant uncertainties, EOSs H4, PAL1, and WFF1 are starting to be in tension with observations [20, 21].

**MODIFIED GRAVITY: PERTURBATIVE REGIME**

In alternative gravity theories, the TOV equations for NSs are modified. There are in general two approaches to study the NS structures with modified gravity: theory specific or generally parameterized.

- In the former case, one usually needs to work out field configurations, as well as the metric and its dependence on the spacetime profile of extra fields. Nevertheless, for a well-defined problem in a well-proposed gravity theory, one will obtain the fully relativistic nonlinear equations, similar to Eqs. (3–4) in GR; see the next section for an example.

- In the latter case, the goal is to use plausible parameterization to cover as many alternative gravity theories as possible. Due to the nonlinearity inherent for the gravitational interaction, it becomes very hard, if ever possible, to incorporate all sensible effects with a limited number of free parameters. Nevertheless, it has the advantage of being once for all, and being advantageous from data analysis point of view.

In this section, we will review some earlier work to parameterize the TOV equation, starting from the famous parameterized post-Newtonian (PPN) framework [8, 22, 23] and a recently improved version, the post-TOV formalism [24, 25]. Both of them belong to the catalog of perturbative approach.
Modified TOV equations in the PPN formalism

The PPN formalism was originally developed to test alternative gravity theories in the Solar System in a systematic way [26, 27, 8]. Due to the intrinsic weak-field and slow-motion characteristics for bodies in the Solar System, the formalism was naturally adopting a post-Newtonian (PN) approximation, and only terms at 1 PN order are included. Therefore, the PPN formalism captures (almost) all relativistic corrections to GR at the leading PN order in the Solar System. Nowadays, the free parameters in the PPN formalism are already well constrained by various observations [2, 1, 8, 9].

Based on the earlier work by Wagoner and Malone [22] and Ciufolini and Ruffini [23], Glampedakis et al. [24] used an “improved” gauge choice which enables an easier comparison with the TOV equations in GR. They obtained a set of modified TOV equations at 1 PN order,

\[
\frac{dp}{dr} = -\frac{Gm\rho}{r^2} \left[ 1 + \frac{P}{\rho} + (5 + 3\gamma - 6\beta + \zeta_2) \frac{Gm}{r} + (\gamma + \zeta_4) \frac{4\pi^2 \rho}{m} \right], \tag{5}
\]

\[
\frac{dm}{dr} = 4\pi^2 \rho \left[ 1 + (1 + \zeta_3) \Pi - \frac{1}{2} (11 + \gamma - 12\beta + \zeta_2 - 2\zeta_4) \frac{Gm}{r} \right]. \tag{6}
\]

In the above equations, \(\beta, \gamma, \text{ and } \zeta_i\) are PPN parameters (in GR, \(\beta = \gamma = 1\) and \(\zeta_i = 0\); see Refs. [8, 2] for details), \(\rho\) is the baryonic rest-mass density, and \(\Pi \equiv (\epsilon - \rho) / \rho\). Eq. (5) and Eq. (6), when using the GR values for the PPN parameters, recover Eq. (3) and Eq. (4) respectively, if the latter set of equations were expanded to the 1 PN order.
FIGURE 2. “Unphysical” mass-radius relation for NSs from integrating the TOV equations in GR, but expanded to the 1 PN order.

Therefore, Eqs. (5–6) are a generalization of the TOV equations at 1 PN order. They shall be general enough for bodies that equipped with weak field and slow motion.

However, NSs have strong gravitational fields inside. In Figure 2, we plot the mass-radius relation for NSs by integrating the TOV equations in GR, but expanded to 1 PN order, or equivalently, by integrating Eqs. (5–6) with PPN parameters set to their GR values. It is easily seen that the results are “unphysical” by a lot, due to the omission of higher-order PN terms. NSs are intrinsically strong-field objects! Therefore, searching for modified-gravity signals based on the 1 PN-expended TOV equations is not going to be useful.

The post-TOV formalism

Inspired by the PPN formalism [26, 8], and to go beyond the leading-order PN approximation, a PN-nonlinear hybrid framework to study the structure of NSs was developed by Glampedakis et al. [24]. It is dubbed as the post-TOV formalism [24, 25]. The framework collects the 1 PN terms in Eqs. (5–6) that also appear in the 1 PN expansion of Eqs. (3–4); these terms are resumed to the GR form to mimic nonlinear effects as much as possible. The remaining 1 PN terms are left as is (namely, 1 PN Taylor expanded), forming the 1 PN corrections to the TOV equations [24].

In addition to the 1 PN corrections, with some reasonable assumptions, Glampedakis et al. [24] also worked out the generic 2 PN terms for the post-TOV equations. Some simplifications were made, such as grouping self-similar 2 PN functionals and so on (see the original paper for details). The final post-TOV equations at 2 PN read [24, 25],

\[
\frac{dp}{dr} = -G \frac{\epsilon + p m + 4\pi r^3 p}{r^2} - \frac{G\rho}{r^2} (P_1 + P_2),
\]

\[
\frac{dm}{dr} = 4\pi r^2 \epsilon + 4\pi r^2 \rho (M_1 + M_2),
\]

where the corrections are all encoded in \(P_1\) and \(M_i\) \((i = 1, 2)\). These corrections are [24],

\[
P_1 = \delta_1 \frac{Gm}{r} + \delta_2 \frac{4\pi r^3 p}{m},
\]

\[
M_1 = \delta_3 \frac{Gm}{r} + \delta_4 \Pi,
\]

\[
P_2 = \pi_1 \frac{G^2 m^3}{r^3 \rho} + \pi_2 \frac{G^2 m^2}{r^2} + \pi_3 G r^2 p + \pi_4 \frac{\Pi p}{\rho},
\]
FIGURE 3. Mass-radius relation for NSs from integrating the TOV equations in GR (solid lines; same as panel (b) in Figure 1) and post-TOV equations with (1 PN parameter) $\delta_3 = 0.5$ (dotted lines; notice that such a value for $\delta_3$ was already excluded by Solar System and radio pulsar observations [2]); the other post-TOV parameters are set to zero.

The 1 PN correction terms, $\mathcal{P}_1$ and $\mathcal{M}_1$, are characterized by $\delta_i$'s which are simply linear combinations of the PPN parameters in Eqs. (5–6) [24],

$$
\mathcal{M}_2 = \mu_1 \frac{G^2 m^3}{r^5 \rho} + \mu_2 \frac{G^2 m^2}{r^2} + \mu_3 Gr^2 \rho + \mu_4 \frac{\Pi p}{\rho} + \mu_5 \Pi^3 \frac{r}{Gm},
$$

where $\delta_i (i = 1, \cdots, 4)$, $\pi_i (i = 1, \cdots, 4)$, and $\mu_i (1, \cdots, 5)$ are post-TOV parameters which vanish in GR.

Due to the tight constraints from Solar System and radio pulsars [1, 2, 28, 29, 30, 9], one has $\mathcal{P}_1 \ll 1$ and $\mathcal{M}_1 \ll 1$. In the limit that $\mathcal{P}_1 = \mathcal{M}_1 = 0$, the 2 PN corrections can be effectively described by a gravity-modified energy density $\epsilon_{\text{eff}} = \epsilon + \rho \mathcal{M}_2$.

Therefore, in the post-TOV formalism the EOS and the gravity theory become degenerate. This conclusion agrees with a similar one in Wen, Li, and Chen [12] when a Yukawa correction to the Newtonian potential is considered, though, the post-TOV formalism assumes no massive propagating gravitational modes, and it does not include exponentially suppressed correction like that of a Yukawa term at the first place (see Will’s monograph [8] for details).

In Figures 3 and 4, we plot illustrative cases where we have fixed (1 PN parameter) $\delta_3 = 0.5$ and (2 PN parameter) $\mu_1 = -0.5$ respectively.\(^1\) We see that, (i) compared with the naive PN expansion in the last subsection, now the behaviors of $M-R$ curves are much more regulated due to the inclusion of nonlinear effects from GR by the resummation; (ii) with varying post-TOV parameters, one is able to move the $M-R$ curves forwards (see Figure 3) or backwards (see Figure 4). Therefore, the degeneracy between modifying gravity and modifying EOS is evident.

\(^1\)The value $\delta_3 = 0.5$ was already excluded from the observations from Solar System and radio pulsars [2]. Here we only use it as an illustrative example.
FIGURE 4. Same as Figure 3, but with (2 PN parameter) $\mu_1 = -0.5$ for the post-TOV case; the other post-TOV parameters are set to zero.

MODIFIED GRAVITY: NON-PERTURBATIVE REGIME

While the post-TOV formalism covers a variety of modified gravity theories, it fails to describe the non-perturbative behaviors that could be triggered by the inner strong gravitational field of NSs in some alternative gravity theories [31, 32, 24, 33]. “Spontaneous scalarization” in the Damour & Esposito-Farèse (DEF) gravity is an outstanding example [31, 32, 33]. We will use NSs in the DEF gravity as an example for this section; for more examples, we refer the readers to the review by Doneva and Pappas [38] and references therein.

An example: NSs in the DEF scalar-tensor gravity

In the DEF gravity, a new scalar field, $\varphi$, is introduced with non-minimal couplings [39, 31]. For the current content, it is easier to discuss in the Einstein frame, where the action for the geometry takes the Hilbert-Einstein form,

$$S = \frac{1}{16\pi G_*} \int d^4x \sqrt{-g_*} \left[ R_* - 2g^{\mu\nu}_* \partial_\mu \varphi \partial_\nu \varphi - V(\varphi) \right] + S_{\text{matter}} \left[ \psi; A^2(\varphi) g^*_{\mu\nu} \right],$$

where “*” means that we are in the Einstein frame; $G_*$ is the bare gravitational constant, and $V(\varphi)$ is the potential for the scalar field and we take it to be zero for simplicity. The most notable point in the action (18) is the non-universal coupling of matter fields to the geometry (namely the metric $g^*_{\mu\nu}$) in $S_{\text{matter}} \left[ \psi; A^2(\varphi) g^*_{\mu\nu} \right]$. Here the conformal factor $A(\varphi)$ is a function of $\varphi$ which can be spacetime-dependent. Consequently, for objects that source the scalar field, equivalence principle breaks down, and these objects do not follow the geodesics of $g^*_{\mu\nu}$. It is a manifestation of the strong equivalence principle violation, and has a deeper impact to the nature of gravitation [2, 3, 9]. In addition, the divergence of $T^*_{\mu\nu}$ does not vanish.

For a spherically symmetric and stationary spacetime produced by a NS, the following metric was used [31] (here, and only in this place, $\varphi$ is a coordinate for the azimuthal angle, not to be confused with the scalar field),

$$\text{d}s^2 = -e^{\nu(r)} \text{d}t^2 + \frac{\text{d}r^2}{1 - 2G_* m(r)/r} + r^2 \left( \text{d}\theta^2 + \sin^2 \theta \text{d}\varphi^2 \right).$$

\(^2\)In binary NS systems, a related phenomenon called “dynamical scalarization” is closely relevant to binary NS mergers in the new field of gravitational-wave astrophysics [34, 35, 36, 37].
With the help of the field equations, a set of first-order differential equations were obtained by Damour and Esposito-Farèse [31] for the structure of NSs,

\[
\begin{align*}
\frac{d\phi}{dr} &= \psi, \\
\frac{d\psi}{dr} &= \frac{4\pi GA^4(\phi)}{1 - 2G_m/r} \left[ \alpha(\phi)(\epsilon - 3p) + r\psi(\epsilon - p) \right] - 2G_m/r \frac{1}{\epsilon - 2G_m/r} \\
\frac{dm}{dr} &= \frac{4\pi r^2 A^4(\phi)}{1 - 2G_m/r} + \frac{r^2 \psi^2}{2G_s}(1 - 2G_m/r), \\
\frac{dv}{dr} &= \frac{8\pi G_s rA^4(\phi)}{1 - 2G_m/r} + r^2 \epsilon + \frac{2G_m}{r^2 (1 - 2G_m/r)}, \\
\frac{dp}{dr} &= -\left(\epsilon + p\right)\left[ \frac{4\pi G_s rA^4(\phi)}{1 - 2G_m/r} + \frac{1}{2} r^2 \psi^2 + \frac{G_m}{r^2(1 - 2G_m/r)} + \alpha(\phi)\psi \right],
\end{align*}
\]

where \(\alpha(\phi)\) is defined by,

\[
\alpha(\phi) \equiv \frac{\partial \ln A(\phi)}{\partial \phi}.
\]

There is a subtle point in above equations (in contrast to those in GR). The energy density and pressure (\(\epsilon\) and \(p\)) are Jordan-frame/physical-frame variables, which were denoted as \(\tilde{\epsilon}\) and \(\tilde{p}\) in Refs. [31, 32]. It means that their values measured in laboratories should be used.

In the following we will restrict ourselves to the DEF parameterization \(\alpha(\phi) = \beta_0 \phi\), or equivalently,

\[
A(\phi) = \exp \left( \frac{1}{2} \beta_0 \phi^2 \right).
\]

This is the simplest parameterization that reproduces significant strong-field deviations from GR. We assume that the asymptotic value for \(\phi\) at spatial infinity is \(\phi_0\), and we denote \(\alpha_0 \equiv \alpha(\phi_0) = \beta_0 \phi_0 [31]\). Therefore, the DEF scalar-tensor gravity is only described by two extra parameters, \(\alpha_0\) and \(\beta_0\) (or equivalently, \(\phi_0\) and \(\beta_0\)). As we will see, \(\alpha_0\) only smooths the non-perturbative transition behaviors, while \(\beta_0\) is the real game player [32, 37] that controls the critical point where the “phase transition” of spontaneous scalarization happens [31].

The integration of the modified TOV equations in the DEF theory is similar to that in GR. Given a central energy density \(\epsilon(\phi)\) and the scalar field value at the center of a NS \(\phi(\phi)\), one easily solves the above first-order differential equations [32]. In the case that one wants to fix the asymptotic value for the scalar field \(\phi_0\), a shooting algorithm and some iterations are needed [32].

In addition to the metric and the matter distribution, the configuration for the scalar field is also obtained from solving the modified TOV equations. The effective scalar coupling for the NS is defined as,

\[
\alpha_{\text{eff}} \equiv \frac{\partial \ln M}{\partial \phi_0}.
\]

Damour and Esposito-Farèse [31] discovered that in the DEF theory, when \(\beta_0 \leq -4.5\), a non-perturbative phenomenon happens. After reaching a critical value of NS’s compactness, a sudden increase by orders of magnitude in the effective scalar coupling \(\alpha_{\text{eff}}\) is observed. Such an increase introduces a large gravitational-wave dipole radiation (in addition to the canonical quadrupole radiation in GR) in an asymmetric binary system, thus it can be well constrained by the observations of binary pulsars [1, 40] or binary NS inspirals [41, 33, 5]. In Figure 5 we show the effective scalar coupling of NSs as a function of their gravitational mass when \(|\alpha_0| = 10^{-5}\) and \(\beta_0 = -4.5\). The spontaneous scalarization is obvious for these curves. Nevertheless, it is interesting to observe that, the non-perturbative phenomenon happens at different masses for different EOSs [35, 33]. For example, the EOS WFF1 has spontaneous scalarization at a relatively low mass, while the EOS PAL1 has spontaneous scalarization at a quite high mass. Different binary pulsar systems have different strength to probe this phenomenon for different EOSs [33]. For NSs with a very large mass, the EOS becomes ultra-relativistic, and the effective scalar coupling decreases (see Damour and Esposito-Farèse [32] and Esposito-Farèse [42] for more details).

We show the mass-radius relation for NSs in the DEF theory in Figure 6 for \(|\alpha_0| = 10^{-5}\) and \(\beta_0 = -4.5\). It is interesting to observe that (1) for most of the parameter space, the mass-radius relation turns out to be very close to
that of GR; (2) there are “bumps” that appear for some masses for different EOSs. These bumps are resulting from the non-perturbative behaviors, and they are very hard, if ever possible, to be captured by the post-TOV equations mentioned in the last section [24]. The bumps make NSs with a given mass become larger in radius, compared with that of GR. This is a very distinct feature for spontaneously scalarized NSs. It can probably play a role that any changes in the EOS could not mimic.

DISCUSSIONS

The properties of the high-density low-temperature nuclear matters, that compose the NSs, carry very important knowledge for the community of nuclear physics and astrophysics. It will tell us profound answers related to the color-confinement in the quantum chromodynamics, the origin of mass, and the evolution of our universe spanning from large (early cosmology) to small (“frozen” stars).

However, it is challenging to constrain the supranuclear EOS above several times of the nuclear density from laboratory experiments [43]. Extrapolation in science to some unknown regime is not an easy task! The input from astrophysics, especially from NS observations, is valuable and complementary to what can be achieved on colliders. Most of valuable information comes from binary pulsar timing observations [1], X-ray observations for “hot spots” on the NS surface [44, 45], and, very recently, binary NS observations from gravitational waves and the subsequent electromagnetic followups [19, 46].

In this paper we discuss one caution when extracting the EOS information from NSs, namely, the degeneracy with the not-so-well-tested gravity theory in the strong gravitational field of NSs. As we learned from the past, scales matter in physics. The validity of GR at different scales, no matter length scales or field strength scales, should be tested empirically [47]. The validity of GR has been tested to some extent in the strong field but not yet fully [48], so we shall be careful to interpret the observations. As we showed in this paper, for example, the mass-radius relation of NSs could be different when the gravity is not GR. On the theoretical hand, unlike the quantum chromodynamics at low energy, GR elegantly makes definite predictions for the gravity behavior even in the not-so-well-tested strong-field regime. This gives us a lot reasonable confidence, alleviating some of our concerns. But still, in the spirit of physical science, empirical verification is needed eventually, because we still have alternative gravity theories that agree with existing observations while making different predictions from GR for NSs.

As a new era for next-generation astronomical facilities is coming close, we have a great hope to investigate both the NSs’ EOS and the strong-field gravity with new telescopes and observatories. For example, (1) the upcoming Square Kilometre Array (SKA) [49, 50, 51] will provide us much better timing sensitivity than ever before, and it
will allow a decent measurement (or even several measurements) of the moment of inertia for NSs at a good precision [52]. It encodes important information about the supranuclear EOS. (2) The enhanced X-ray Timing and Polarimetry mission (eXTP) [53, 45, 54, 55], led by the Chinese and European teams, will allow a precise measurement of NS radii via modeling the X-ray flux from the hot spots formed through the accretion. It will shrink the uncertain region in the mass-radius plots. (3) Last but not least, though the current searching for the remnant of GW170817 [56, 57] is still prevented from positive detections by the large detector noises, future gravitational-wave detectors will take us to identify the merger remnant and to identify the characteristic oscillation modes implicating the EOS of supranuclear matters. In all the above mentioned modeling, the possibility of a deviation from GR can, in principle, be included, to reflect our empirical uncertainty in the strong-field gravity. Such a deviation can be fit together with the EOS. Being said, there is still a lot theoretical work remaining to be done.

In summary, most of the current approaches to study the supranuclear matters use an implicit assumption that GR describes the strong gravitational field inside NSs. This is not empirically verified to a safe precision. Therefore, we shall at least keep a caution and work with this uncertainty. The knowledge of EOSs is to be earned in the hard way, and a deeper understanding will happen in the near future.

ACKNOWLEDGMENTS

We thank Nils Andersson, Zhoujian Cao, James Lattimer, Bao-An Li, Renxin Xu, and Bing Zhang for helpful discussions during the Xiamen-CUSTIPEN Workshop on the EOS of Dense Neutron-Rich Matter in the Era of Gravitational Wave Astronomy. We are grateful to James Lattimer for providing us with tabulated data for neutron-star equations of state. This work was supported by the National Science Foundation of China (11721303), and XDB23010200.

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