ENERGY SCALING LAW FOR THE REGULAR CONE

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Abstract. Consider a thin elastic sheet in the shape of a disk whose reference metric is that of a singular cone. I.e., the reference metric is flat away from the center and has a defect there. This setting is called the regular cone, since one expects that minimizers of the elastic energy are approximately conical. We want to find upper and lower bounds for the minimum elastic energy that have the same scaling with respect to the thickness of the sheet \( h \). Establishing a suitable lower bound is supposed to be hard, the main difficulty lying in the vast amount of potential candidates for energy minimization supplied by the constructions by Nash and Kuiper of \( C^1 \)-isometric immersions. Isometric immersions have zero membrane energy – hence one needs a general principle that shows that the maps occurring in the constructions by Kuiper are associated to high bending energy. This is also the main problem in proving lower bounds for the so-called confinement or crumpling problem, which consists in fitting a thin elastic sheet into a container whose size is smaller than the diameter of the sheet.

Here we prove such a lower bound for the regular cone. We work with two simplifying assumptions: Firstly, we think of the deformed sheet as an immersed 2-dimensional Riemannian manifold in Euclidean 3-space and assume that the exponential map at the origin (the center of the sheet) supplies a coordinate chart for the whole manifold. Secondly, we consider an energy functional that penalizes the difference between the induced metric and the reference metric in \( L^\infty \) (instead of, as is usual, in \( L^2 \)). Under these assumptions, we show that the elastic energy per unit thickness of the regular cone in the leading order of \( h \) is given by \( C^* h^2 |\log h| \), where the value of \( C^* \) is given explicitly.

1. Introduction

The problem of paper crumpling can be formulated as follows: How much elastic energy is needed to pack an elastic sheet of diameter \( R \) and thickness \( h \) into a container of diameter \( r < R \)? In experiments, one observes that for \( r \) small enough, the sheet develops an intricate pattern of edges and vertices which are the regions where the elastic energy focuses. A rigorous or even semi-rigorous analysis of the whole pattern seems to be out of reach for now. However, there have been a number of authors from the physics community who analyzed its "building blocks", i.e., a single edge or a single vertex, see \([3, 10, 12, 21, 22, 24, 25]\), and \([34]\) for an extensive review of the topic and many more references.

In a simplified setting, the elastic energy of a thin sheet consists of two terms, "membrane" and "stretching" energy; the former is non-convex, and the latter can be viewed as a singular perturbation. By the non-convexity of the energy, there can be many local minimizers. One approach to a rigorous study of the problem, in the context of nonlinear elasticity first used in \([19, 20]\), is to establish upper and lower bounds for the minimal energy in terms of the singular perturbation parameter. This is the approach we will take in the present paper. The perturbation parameter will be the sheet thickness \( h \).

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Concerning results in the mathematical literature, it has been shown that the elastic energy per unit thickness of a “single fold” scales with $h^{5/3}$ [11], building on results from [33]. In [8, 28], the following has been proved: Consider an elastic sheet in the shape of a disc, and a conical deformation with the apex of the cone at the center of the sheet. Now restricting to non-singular configurations of the sheet that agree with the singular one on the boundary and at the center, the elastic energy per unit thickness scales with $h^2 |\log h|$. These latter two results share a deficiency that needs to be overcome in order to be able to obtain good models for crumpling. The boundary conditions in both problems exclude short maps, i.e., functions that map paths in the reference configuration to shorter paths in the image. This is more than a coincidence. In both settings, the whole sheet can be covered by line segments such that a) the values of the deformation at both ends of the line segments are prescribed by the boundary conditions and b) the distance between the endpoints in the deformed configuration is the same as in the reference configuration. The basic idea behind the lower bound proofs is more or less this: If the deformation is non-singular (i.e., has finite bending energy), then the aforementioned line segments have to be longer than in the reference configuration after deformation, resulting in non-zero membrane energy. The optimal balance between membrane and bending energy yields the lower bound in the elastic energy. The assumptions a) and b) above exclude short maps. The method of proof breaks down if this class is included in the set of allowed configurations.

It is the very nature of the crumpling problem that short maps should be candidates for energy minimization. It seems difficult to come up with a rigorous argument that (parts of) a crumpled sheet should satisfy properties a) and b) above.

The relation between short maps and the difficulty of the crumpling problem can be understood as follows. The deformation $y : U \to \mathbb{R}^3$ of the sheet $U \subset \mathbb{R}^2$ induces a metric $g_y = y^* e^3$ on $U$, where $y^* e^3$ denotes the pullback under $y$ of the Euclidean metric on $\mathbb{R}^3$. The normal to the surface $y(U)$ at a point $y(x)$ is denoted by $\nu_y(x)$. The elastic energy of the sheet of thickness $h \ll 1$ is given by two terms, the non-convex membrane energy, and the bending energy:

$$I_h(y) = E_{\text{membrane}} + E_{\text{bending}} = \int_U \|g_y - g_0\|^2 + h^2 |D\nu_y|^2 \, dx$$

$$= \|g - g_0\|_{L^2(U)}^2 + h^2 \|D\nu_y\|_{L^2(U)}^2,$$

where $g_0$ is a given reference metric and $\|\cdot\|$ is an appropriate norm on the space of metrics. This functional can be derived from 3-dimensional nonlinear elasticity by a Kirchhoff-Love ansatz, where the deformation of the three-dimensional sheet $U_h := U \times [-h/2, h/2]$ is given by $\psi : U_h \to \mathbb{R}^3$,

$$\psi(x_1, x_2, x_3) = y(x_1, x_2) + x_3 \nu_y(x_1, x_2).$$

We are going to make some further comments on the relation of $I_h$ with energy functionals from non-linear 3-dimensional elasticity towards the end of the introduction. By the results by Nash [29] and Kuiper [23], every short immersion $\tilde{y} : U \to \mathbb{R}^3$ of $(U, g_0)$ can be approximated arbitrarily well in the $C^0$-norm by $C^1$-isometric immersions. This is an instance of Gromov’s famous $h$-principle [18]. Looking at the elastic energy functional,
the Nash-Kuiper construction yields maps with arbitrarily small membrane energy $C^0$-close to a given short immersion $\tilde{y}$. Hence, there is an abundance of maps with arbitrarily small membrane energy. The fundamental difficulty in the crumpling problem is how to eliminate these many degrees of freedom, that we will call “Nash-Kuiper degrees of freedom” in the sequel. There are two ways how to “eliminate” them – either by restricting the set of allowed configurations (as in $[3,11,28]$) or showing that they have large bending energy. Of course, only the latter option could be viewed as a step forward in the understanding of the crumpling problem.

In $[27]$, a setting related to the crumpling problem has been analyzed, the so-called regular cone. To be precise, the relation to the crumpling problem consists in the fact that the elimination of the Nash-Kuiper degrees of freedom is the main problem for the construction of a lower bound. The regular cone is obtained by cutting out a sector from a circular sheet and gluing the resulting edges back together. Contrary to the setting from $[28]$, the corresponding variational problem does not include any boundary conditions. However, in $[27]$, a different (rather strong) condition has been assumed. Namely, only radially symmetric configurations were considered. This reduces the problem from two to one dimensions, and allows for the use of ODE methods to deduce the scaling of the elastic energy per unit thickness.

In the present contribution, we again consider the regular cone, and apply to this problem what we believe is a more general way of proving lower bounds for variational problems for thin elastic sheets, in particular crumpling-related problems. This method focuses on intrinsically defined geometric objects in the Riemannian manifold $(U, g_y)$. Therefore, before we present the main idea, let us briefly recall the method of moving frames (see e.g. $[32]$):

Given a two-dimensional Riemannian manifold $(M, g)$, a moving frame $(X_1, X_2)$ is a pair of vector fields satisfying $g(X_i, X_j) = \delta_{ij}$, where $\delta_{ij}$ is the Kronecker delta. The dual one forms $(\theta_1, \theta_2)$ are defined by $\theta_i(X_j) = \delta_{ij}$. The connection one form $\omega$ is defined by $d\theta_1 = -\omega \wedge \theta_2$, $d\theta_2 = \omega \wedge \theta_1$. Its differential is the curvature form $d\omega = K dA$, where $K$ is the Gauss curvature, and $dA$ the volume element of $(M, g)$. In integral form, $\int_V K dA = \int_{\partial V} \omega$ for every domain $V \subset M$ with suitably smooth boundary.

The idea for obtaining lower bounds is as follows:

The metric $g_y$ is controlled by the membrane term in $[11]$, and the $L^1$-integral of the Gauss curvature $K_y$ of $(U, g_y)$, $\int_U |K_y| dA_y$, is controlled by the bending energy. The crucial point is that estimates for a connection one-form $\omega_y$ for $g_y$ can be obtained by interpolation between $g_y$ and $K_y$. For the moment, we only give a heuristic argument why that should be so: One may think of the connection one-form as a first derivative of the metric, and of the Gauss curvature as a second derivative. Thus, the connection belongs to an

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1The reader may be more familiar with specifying a connection in a Riemannian manifold via the Christoffel symbols $\Gamma^i_{jk}$. In $n$ dimensions, the connection form carries two indices, $\omega = \omega^i_j$, and is related to the Christoffel symbols $\Gamma^i_{jk}$ as follows: For an orthonormal frame $(X_1, \ldots, X_n)$ with dual one-forms $(\theta_1, \ldots, \theta_n)$, we have $\nabla_x X_j = \sum_k \omega^i_j (X_k) \theta_i$, where the connection $\nabla$ is defined in terms of the Christoffel symbols by $\nabla \theta_i \theta_j = \sum_k \Gamma^i_{jk} \theta_k$, where $\theta_i$ denotes the vector field defined by differentiation with respect to the coordinate function $x_i$. The connection forms $\omega^i_j$ are antisymmetric with respect to the indices $i, j$; hence, in dimension 2, there is only one non-vanishing component.
appropriate intermediate space between the space of metrics and the space of curvatures. We will come back to this point later in the introduction.

The connection one-form can be used to compute the amount of curvature contained in $V \subset U$ for any $V$ with smooth enough boundary, namely

$$\int_V K_y dA_y = \int_{\partial V} \omega_y.$$  \hfill (2)

The integral on the left hand side gives a lower bound for the area of $\nu_y(V)$. We now use the isoperimetric inequality on the sphere, and translate the lower bound for $\nu_y(V)$ into a lower bound for $\int_{\partial V} |D\nu_y|$. This in turn can be translated into a lower bound for $\int_{\partial V} |D\nu_y|^2$ by a straightforward application of Jensen’s inequality. By a suitable covering of $U = \cup_i \partial V_i$, this yields a lower bound for the bending energy.

This ansatz seems fairly general and should be applicable to other variational problems for thin elastic sheets. To the best of our knowledge, it has not been suggested before.

To support our claim that control over the metric and $\int_U |K_y| dA_y$ is the natural way of obtaining lower bounds in crumpling-related problems, we want to point out the relation of the latter to rigidity theorems for isometric immersions.

We have already mentioned the results by Nash and Kuiper on $C^1$-isometric immersions. On the other hand, there is the classical rigidity in the Weyl problem. This result states that any isometric immersion of a given Riemannian manifold $(S^2, g)$ in $C^2(S^2; \mathbb{R}^3)$ is rigid, i.e., unique up to a rigid motion.

The striking contrast between the results for $C^1$- and $C^2$ isometric immersions naturally leads to the question how the situation looks like for $C^{1,\alpha}$ isometric immersions. In [6], Borisov announced that if $g$ is analytic, the $h$-principle holds for isometric immersions in $C^{1,\alpha}(M; \mathbb{R}^3)$ for $\alpha < \frac{1}{2}$. A proof can be found in [7]. Another result by Borisov [4, 5] states that for $\alpha > \frac{2}{3}$, any isometric embedding $y \in C^{1,\alpha}(S^2; \mathbb{R}^3)$ of a $C^2$-metric $g$ on $S^2$ with positive Gauss curvature is rigid.

For our purposes, the latter is relevant. The proof of the statement consists in showing that the imbedded manifold $M$ is of bounded extrinsic curvature, i.e.,

$$\sup \left\{ \sum_{i=1}^N \mathcal{H}^2(\nu_y(E_i)) : N \in \mathbb{N}, \{E_i\}_{i=1,\ldots,N} \text{ a collection of closed disjoint subsets of } M \right\} < \infty,$$

which is sufficient to obtain the rigidity result, using classical results by Pogorelov [30]. We see that quantitative control over extrinsic curvature (the left hand side above) allows for the elimination of the Nash-Kuiper degrees of freedom. This agrees with the fact that extrinsic curvature diverges in the Nash-Kuiper approximation scheme. Since we will be dealing with smooth maps here, the extrinsic curvature is just the $L^1$-norm of the Gauss curvature, $\int_U |K_y| dA_y$, which is controlled by the bending energy.

We come to the question of how to obtain control over a connection one-form $\omega_y$ by interpolation between the metric and the Gauss curvature. This certainly requires more comment, and in fact, we will need extra assumptions to make this part of the ansatz
work. In general, \( \omega_y \) will be some non-linear function of the metric \( g_y \) and its derivatives, \( \omega_y = f(g_y, Dg_y) \). The curvature form \( K_y \) is just the differential \( d\omega_y \). For proving lower bounds of an elastic energy, we need to interpolate in some suitable \( L^p \) spaces. However there is no straightforward way\(^2\) of interpolating between \( g_y \) and \( K_y \) of the above form in \( L^p \) spaces.

We solve this problem here by introducing an additional assumption on \( y \) and using a particular set of coordinates. Denoting by \( B_d \) the open ball of radius \( d \) around the origin in \( \mathbb{R}^2 \), \( y : B_1 \to \mathbb{R}^3 \) the deformation of the elastic sheet \( U = B_1 \subset \mathbb{R}^2 \), the assumption is as follows:

**Assumption 1.** When viewing \((y(B_1), y^* e^3)\) as a Riemannian manifold immersed into \( \mathbb{R}^3 \), we assume that the exponential map at the origin \( \exp_0 \) supplies a diffeomorphism of some subset \( N \) of \( T_0 B_1 \) to \( B_1 \).

This guarantees that we may use the exponential map as a global coordinate system. It turns out that in these coordinates, the interpolation between \( g_y \) and \( K_y \) is straightforward. For our ansatz to work, it still remains the problem that the elastic energy is given in the coordinates of the reference configuration and the interpolation argument requires the use of the coordinate chart supplied by the exponential map. We will need to show that smallness of the membrane term implies that these two sets of coordinates differ only by little (in an \( L^\infty \) sense), in order to be able to translate energy estimates in one set of coordinates into the other.

Unfortunately, we have not been able to do so under the assumptions stated thus far. More precisely, in the construction of the lower bound, we will have to assume that the metric error \( g - g_0 \) is small in \( L^\infty \), and not only in \( L^2 \), as would be the result of an upper bound for \( I_h \) (cf. Remark \( \square \)). We are lead to work instead with the energy functional \( I_h^\infty : W^{2,2}(B_1; \mathbb{R}^3) \to \mathbb{R}^3 \),

\[
I_h^\infty(y) = E_m^h + E_{\text{bending}} = \|g_y - g_0\|_{L^\infty(B_1 \setminus B_h)} + \|D\nu\|_{L^2(B_1)}^2.
\]

Using Assumption \( \square \) and smallness of \( \|g_y - g_0\|_{L^\infty(B_1 \setminus B_h)} \), it will not be difficult to show that the coordinates of the reference configuration and the ones supplied by the exponential map are close to one another in a suitable sense (see Lemma \( \square \)), and our ansatz can be put to work.

The choice of \( I_h^\infty \) as the membrane energy requires further comment. First, in its definition, we excluded a ball of radius \( h \) from the domain so that our upper bound construction (see Lemma \( \square \)), that we believe to capture the qualitative shape of the regular cone, satisfies the “right” energy bound \( C^h h^2 \|\log h\| + Ch^2 \) with respect to both energy functionals, \( I_h(y) \) and \( I_h^\infty(y) \) (see also Remark \( \square \)). Second, the change from \( I_h \) to \( I_h^\infty \) does not eliminate the Nash-Kuiper degrees of freedom, since the approximants of isometric immersions in \( \square \) yield an induced metric that is close to the reference metric in \( C^0 \) (and hence, obviously, not only in \( L^2 \), but also in \( L^\infty \)).

\(^2\)By “straightforward”, we mean an interpolation of the kind \( \|Df\|_{L^p} \leq C\|f\|^\theta_{L^p}\|D^2 f\|^{1-\theta}_{L^p} \) for \( f \in W^{2,p}, \theta \in (0,1) \), where both \( \|f\|_{L^p} \) and \( \|D^2 f\|_{L^p} \) are controlled by the elastic energy.
In summary, we will prove that under Assumption $\[1\]$ the elastic energy per unit thickness $I_h^{\infty}$ of the regular cone in the leading order of the thickness $h$ is given by $C^* h^2 |\log h|$, where $C^*$ is some explicit constant given by the geometry of the problem.

It is of course desirable to, firstly, prove the analogous claim for the energy functional $I_h$, and secondly, remove Assumption $[1]$. Nevertheless, we hope to be able to convince the reader of our opinion that the essence of the method presented here – namely, combining the control over the metric error and the $L^1$-norm of the Gauss curvature via interpolation – is the natural way of eliminating the Nash-Kuiper degrees of freedom. One might say that this elimination is achieved by imposing Assumption $[1]$ rather than by showing that their elastic energy is large. Indeed, we do not believe that the approximating sequences for isometric immersions in $[23]$ satisfy Assumption $[1]$. Nevertheless, they can be considered to almost fulfill it: Assume the domain of the approximants $y_h$ is $B_1$. Let $L_h(x)$ be the length w.r.t. $g_{y_h}$ of a geodesic connecting the origin with $x \in B_1$. Then $L_h(x)$ converges uniformly to the length of the geodesic between 0 and $x$ in the reference configuration (i.e., $|x|$). This convergence might be enough to use $L_h$ just like a coordinate function, and carry out the argument similarly to the way we do it here.

Before we come to the structure of the present paper, we are going to briefly explain the relation of the 2-dimensional models discussed here to 3-dimensional nonlinear elasticity. The typical energy from 3-dimensional nonlinear elasticity is given by

$$E_h : W^{1,2}(U_h; \mathbb{R}^3) \to \mathbb{R},$$

$$y \mapsto \int_{U_h} W(\nabla y),$$

where as before, $U \subset \mathbb{R}^2$ is some Lipschitz domain, $U_h = U \times [-h/2, h/2]$, and the energy density $W(\cdot)$ is normalized ($W(\text{Id}) = 0$), frame invariant and coercive ($W(F) \geq C \text{dist}^2(F, \text{SO}(3))$). The simplest example of such an energy density is $W(F) = \text{dist}^2(F, \text{SO}(3))$. There are a number of instances in the literature where a lower bound for the “reduced” model

$$\tilde{I}_h : W^{2,2}(U; \mathbb{R}^3) \to \mathbb{R},$$

$$y \mapsto \int_U |Dy^T Dy - \text{Id}|^2 + h^2 |D^2 y|^2,$$

has been translated into lower bounds for the three-dimensional model, see $[8, 11]$ and the recent work $[31]$ where a general way of obtaining lower bounds for $E_h$ from lower bounds for $\tilde{I}_h$ has been found. The difference between $I_h$ and $\tilde{I}_h$ should not be a major issue when it comes to deriving lower bounds for $E_h$. However, lower bounds for $I_h^{\infty}$ do not translate into lower bounds for $E_h$. So the next step on the road to proving lower bounds for 3-dimensional nonlinear elasticity starting from the results in the present work would be the passage from $I_h^{\infty}$ to $I_h$.

Such lower bounds would be relevant for the rigorous derivation of plate theories as $\Gamma$-limits of three-dimensional models. These models can be classified by the assumed scaling of the energy per unit thickness $E_h$ in the underlying 3d theory. Assuming $E_h \sim h^\beta$, the $\Gamma$-limit for $\beta = 2$ is nonlinear bending theory $[16]$. The parameter choice $2 < \beta < 4$ results in “von-Kármán-like” plate theories, see $[15]$. For $\beta < \frac{5}{3}$, it has been shown in $[11]$ that...
the $\Gamma$-limit is equal to 0 for short maps, and $\infty$ otherwise. The derivation of the $\Gamma$-limit for the range $\beta \in \left(\frac{\pi}{3}, 2\right)$ is an open problem. A lower bound for $I_0$ or $\tilde{I}_0$ with the right scaling can serve as a starting point for the compactness part of such a $\Gamma$-convergence result.

This paper is structured as follows: In Section 2 we define our setting and give the main result Theorem 1. In Section 3, we establish $L^\infty$-estimates for the exponential map, that in the sequel will allow us to translate bounds on the elastic energy into estimates of the metric and the Gauss curvature in the coordinate chart supplied by the exponential map, see Lemma 3. In Section 4, we use these results to interpolate between the metric and the Gauss curvature, as in the ansatz that we have proposed in this introduction. One more ingredient is needed to carry out the ansatz, namely an appropriate isoperimetric inequality on the sphere, which is proved in Section 5. In Section 6, we combine these items to prove our main result.

Notation. For $\rho > 0$, let $B_\rho = \{x \in \mathbb{R}^2 : |x| < 1\}$. We denote by $e^3$ the Euclidean metric in $\mathbb{R}^3$. For a manifold $M$ and $x \in M$, $T_x M$ denotes the tangent space at $x$, and $T^*_x M$ its dual. For a finite dimensional vector space $V$, the space of alternating $k$-linear forms $V \otimes^k \rightarrow \mathbb{R}$ is denoted by $\wedge^k V$. A $k$-form on a manifold $M$ is a map that associates to every $x \in M$ an element of $\wedge^k T_x M$. If $M$ is a subset of $\mathbb{R}^2$, the gradient $Df$ of a function $f : M \rightarrow \mathbb{R}^m$ is defined via duality to $df$ with the standard Euclidean metric,

$$Df = (\partial_1 f, \partial_2 f)^T,$$

even if $M$ is equipped with a Riemannian metric that is different from the Euclidean one. The oscillation of a function $u : U \rightarrow \mathbb{R}$ is denoted by $\text{osc}_U u = \sup_U u - \inf_U u$. The space of Lipschitz functions from $U \subset \mathbb{R}^n$ to some metric space $V$ is denoted by $\text{Lip}(U; V)$; if $V = \mathbb{R}$, we simply write $\text{Lip}(U)$.

Unless stated otherwise, the symbol “$C$” will be used as follows: The statement “$f \leq Cg$” is short-hand for “There exists a constant $C > 0$, that only depends on $m_0$, with the property $f \leq Cg$”. The value of $C$ may change from line to line. Sometimes it will be convenient to be able to refer to the same constant later on in the text; in this case, we denote it by $C_i, i = 1, 2, \ldots$, and it will be fixed.

2. The main result

For an immersion $y \in W^{2,2}(B_1; \mathbb{R}^3)$, define the metric $g_y = y^* e^3$, the Riemannian manifold $M_y := (B_1, g_y)$, and the Gauss map

$$\nu_y = \frac{\partial_1 y \wedge \partial_2 y}{|\partial_1 y \wedge \partial_2 y|}.$$

On $B_1$, we introduce polar coordinates $\rho, \vartheta$ by

$$(\rho, \vartheta) : B_1 \rightarrow (0, 1) \times S^1,
\quad x \mapsto (|x|, x/|x|).$$

The symbols $\partial_\rho, \partial_\vartheta$ will at the same time denote partial differentiation with respect to $\rho, \vartheta$ as well as the vector fields $B_1 \rightarrow TB_1$, defined by their actions on functions $f : B_1 \rightarrow \mathbb{R}$,

$$\left(\partial_\rho f\right)(x) = \frac{x}{|x|} \cdot Df(x), \quad \left(\partial_\vartheta f\right)(x) = x^\perp \cdot Df(x),$$
where \( x^T = (x^2, -x_1)^T \) for \( x = (x_1, x_2)^T \). Now let \( 0 < m_0 < 1 \) and define the reference metric \( g_0 : B_1 \to T^*B_1 \otimes T^*B_1 \) by
\[
g_0 = d\rho \otimes d\rho + m_0^2 \rho^2 d\vartheta \otimes d\vartheta .
\]
Further, define the energy functional
\[
I^\infty_h(y) := \| g_y - g_0 \|^2_{L^\infty(B_1 \setminus B_h)} + h^2 \| D\nu_y \|^2_{L^2(B_1)} .
\]
Here, the \( L^\infty \)-norm on the space of metrics on \( B_1 \) is defined as follows. For \( x \in B_1 \setminus \{0\} \), we define a norm on inner products \( p : T_xB_1 \times T_xB_1 \to \mathbb{R} \) on the tangent space at \( x \): For
\[
p = p_1 d\rho \otimes d\rho + p_2 \rho (d\rho \otimes d\vartheta + d\vartheta \otimes d\rho) + p_3 \rho^2 d\vartheta \otimes d\vartheta
\]
we set
\[
\| p \|^2 = p_1^2 + 2p_2^2 + p_3^2 .
\]
This makes \( \| \cdot \| \) well defined since \( d\rho(x), \rho d\vartheta(x) \) span \( T^*_x B_1 \) for every \( x \in B_1 \setminus \{0\} \). Then we define the \( L^2 \)-norm of a metric \( g \) on \( B_1 \) by
\[
\| g \|_{L^2(B_1)} = \text{esssup}_{x \in B_1} \| g(x) \| .
\]
We introduce the following piece of notation for configurations \( y \) that satisfy our Assumption [1]

**Definition 1.** Let \( \mathcal{A} \) be the set of smooth immersions \( y : B_1 \to \mathbb{R}^3 \) such that there exists a subset \( N_y \subset T_0B_1 \) of the tangent space at zero, such that the exponential map \( (\exp_0)_y : N_y \to B_1 \) is a diffeomorphism.

Furthermore we define the constant
\[
C^* = 2\pi \left( 1 - m_0^2 \right) .
\]
In this contribution, we prove

**Theorem 1.** There exists a constant \( C > 0 \) depending only on \( m_0 \) with the following property: For all \( h < e^{-1} \),
\[
C^* \log h - \frac{3}{2} \log \log h - C \leq h^{-2} \inf_{y \in \mathcal{A}} I^\infty_h(y) \leq C^* \log h + C .
\]
Before we start with the proof of the theorem, we collect some more notation that we are going to use in the sequel.

The objects \( N_y, (\exp_0)_y, g_y, \nu_y \) as well as the maps that we will introduce in the following all depend on \( y \) – for notational convenience we will not indicate this dependence anymore from now on.

As already stated above, we denote by \((\rho, \vartheta)\) the radial coordinates on \( B_1 \). Next we introduce radial coordinates \((r, \varphi)\) with respect to \( N \subset T_0B_1 \cong \mathbb{R}^2 \),
\[
(r, \varphi) : B_1 \setminus \{0\} \to (0, \infty) \times S^1 \quad x \mapsto (|x|, \tilde{x}/|\tilde{x}|) ,
\]
where \( \tilde{x} = (\exp_0)^{-1}(x) \).

The vector fields \( \partial_r, \partial_\varphi \) are defined on \( N \) analogously to the definition of \( \partial_\rho, \partial_\vartheta \) on \( B_1 \).
above; and using the push forward \( \exp_s^* \), we will view them as vector fields on \( B_1 \) from now on.

By Assumption 1 the map \( (\rho(x), \vartheta(x)) \mapsto (r(x), \varphi(x)) \) is regular on its domain and hence we may write

\[
\begin{pmatrix} dr \\ r d\varphi \end{pmatrix} = \Gamma \begin{pmatrix} d\rho \\ \rho d\vartheta \end{pmatrix},
\]

where \( \Gamma : B_1 \setminus \{0\} \to \mathbb{R}^{2 \times 2} \) is defined by

\[
\Gamma = \begin{pmatrix} \partial_r \rho r & \rho^{-1} \partial_r \rho \\ r \partial_r \varphi & r \rho^{-1} \partial_r \varphi \end{pmatrix}.
\]

Note that

\[
rdr \wedge d\varphi = \det \Gamma \rho d\rho \wedge d\vartheta.
\]

By Assumption 1 \( rdr \wedge d\varphi \) is a nowhere vanishing two-form on \( B_1 \setminus \{0\} \), and hence

\[
det \Gamma > 0.
\]

We also introduce the following notation for the inverse of \( \Gamma \): Let \( \hat{\Gamma} : B_1 \setminus \{0\} \to \mathbb{R}^{2 \times 2} \) be defined by

\[
\hat{\Gamma} = \begin{pmatrix} \partial_r \rho & r^{-1} \partial_r \rho \\ \rho \partial_r \varphi & r^{\rho^{-1}} \partial_r \varphi \end{pmatrix}.
\]

An obvious consequence of this definitions is

\[
\Gamma(x) = \left( \hat{\Gamma}(x) \right)^{-1} \quad \text{for all } x \in B_1 \setminus \{0\}
\]

and hence

\[
\Gamma = \frac{1}{\det \hat{\Gamma}} \begin{pmatrix} \rho r^{-1} \partial_r \varphi & -\rho \partial_r \varphi \\ -r^{-1} \partial_r \rho & \partial_r \rho \end{pmatrix}.
\]

Let \( K : B_1 \to \mathbb{R} \) denote the Gauss curvature, and \( dA \) the volume 2-form of \((B_1, g)\).

We are now going to define three functions \( \Omega, \bar{\Omega}, G : B_1 \to \mathbb{R} \), and we will make these definitions with respect to the exponential polar coordinates \((r, \varphi)\). By Assumption 1 this will make them well defined as functions on \( B_1 \).

We set

\[
\Omega(0, \varphi) = \bar{\Omega}(0, \varphi) = 1 - G(0, \varphi) = 0 \quad \text{for all } \varphi \in S^1
\]

and

\[
\begin{align*}
\Omega(r, \varphi) &= \int_0^r K(s, \varphi) dA_{(s, \varphi)}(\partial_r, \partial_{\varphi}) ds \\
\bar{\Omega}(r, \varphi) &= \frac{1}{r} \int_0^r \Omega(s, \varphi) ds \\
G &= 1 - \bar{\Omega}.
\end{align*}
\]
3. Passage to “exponential polar coordinates”

The proof of Theorem 1 will be an application of the ansatz that we proposed in the introduction – however, to make the interpolation argument work, we need to pass from the polar coordinates in the reference configuration \((ρ, θ)\) to the “exponential polar coordinates” \((r, ϕ)\). This is what we will do in the present section.

The starting point is to note that in the \((r, ϕ)\) coordinates, the metric takes a particularly simple form. The following lemma can be found as Proposition 3 and Remark 1 in Chapter 4 of [13], or in Chapter 3 of [32]. For the convenience of the reader, we include the proof here. In this proof, we denote the Levi-Civita connection of \(g\) by \(\nabla\).

**Lemma 1.** The metric \(g : B_1 \to T^*B_1 \otimes T^*B_1\) is given by

\[
g = dr \otimes dr + G^2 r^2 dϕ \otimes dϕ.
\]

**Proof.** First, we have \(g(∂_r, ∂_r) = 1\) by the definition of the exponential map. Secondly, \(g\) and its first derivatives are identical to the Euclidean metric at the origin, by basic properties of the exponential map. Hence

\[
\lim_{r \to 0} g|(r,ϕ)(∂_ϕ, ∂_r) = 0 \quad \text{for all } ϕ.
\]

Next, we compute the derivative of \(g(∂_ϕ, ∂_r)\) along geodesics,

\[
∂_r g(∂_ϕ, ∂_r) = g(∇_r ∂_ϕ, ∂_r) + g(∂_ϕ, ∇_r ∂_r) = 0
\]

\[
= 1/2 ∂_ϕ g(∂_r, ∂_r)
\]

\[
= 0
\]

In the second equality above, we used that the Lie brackets of the coordinate vector fields vanish:

\[
[∂_r, ∂_ϕ] = ∇_r ∂_ϕ − ∇_ϕ ∂_r = 0,
\]

and that ∂_r is a geodesic vector field, ∇_r ∂_r = 0. The initial conditions together with \(\frac{∂}{∂r}\) prove that \(g(∂_ϕ, ∂_r) = 0\). It remains to show that \(g(r^{-1}∂_ϕ, r^{-1}∂_r) = G^2\). For vector fields \(X, Y\) let \(R(X, Y) = [∇_X, ∇_Y] − ∇_{[X,Y]}\) denote the Riemann curvature tensor. We compute

\[
∇^2_r ∂_ϕ = ∇_r (∇_ϕ ∂_r) + ∇_r [∂_ϕ, ∂_r]
\]

\[
=R(∂_r, ∂_ϕ)∂_r + ∇_ϕ ∇_r ∂_r
\]

\[
= −K∂_ϕ.
\]

The value of \(dA\) evaluated on the vector fields ∂_r, ∂_ϕ and the first derivatives of these values w.r.t. \(r\) at the origin are the same as in Euclidean space, by the defining property of the exponential map. Hence

\[
\begin{align*}
\lim_{r \to 0} dA|(r,ϕ)(∂_ϕ, ∂_r) &= 0 \\
\lim_{r \to 0} ∂_r dA|(r,ϕ)(∂_ϕ, ∂_r) &= 1
\end{align*}
\]

for all \(ϕ\).
We compute the second derivative of $dA(\partial_r, \partial_\varphi)$ along the radial curves parametrized by $r$:

$$
\partial_r^2 dA(\partial_r, \partial_\varphi) = dA(\partial_r, \nabla_r^2 \partial_\varphi)) = -KdA(\partial_r, \partial_\varphi)
$$

Together with the initial conditions (9), this defines an initial value problem, and thus $dA(\partial_r, \partial_\varphi)$ satisfies

$$
dA|_{(r,\varphi)}(\partial_r, \partial_\varphi) = r - \int_0^r ds \int_0^s dt \, KdA|_{(t,\varphi)}(\partial_r, \partial_\varphi) = rG(r, \varphi)
$$

By the definition of the volume form,

$$
(dA(\partial_r, r^{-1}\partial_\varphi))^2 = g(\partial_r, \partial_r)g(r^{-1}\partial_\varphi, r^{-1}\partial_\varphi) - (g(r^{-1}\partial_\varphi, \partial_r))^2
$$

The last three equalities show $g(r^{-1}\partial_\varphi, r^{-1}\partial_\varphi) = G^2$. \hfill \Box

The following lemma serves two purposes: On the one hand, it supplies the right hand side in the statement of Theorem 1. On the other hand, it assures that the assumptions in the subsequent lemmas make sense.

**Lemma 2.**

$$
\inf_{y \in A} I_h^\infty(y) \leq C^* h^2 |\log h| + Ch^2.
$$

**Proof.** We define the following $\varphi$-dependent orthonormal frame in $\mathbb{R}^3$:

$$
e_\rho = (\cos \vartheta, \sin \vartheta, 0), \quad e_\varphi = (\sin \vartheta, \cos \vartheta, 0), \quad e_z = (0, 0, 1)
$$

Further, we set $e_{m_0} := m_0 e_\rho + \sqrt{1-m_0^2} e_z$. Let $\psi \in C^\infty(\mathbb{R}^+)$ such that $\psi(r) = 0$ for $r \leq 1/2$, $\psi(\rho) = 1$ for $\rho \geq 1$ and $|\psi'(\rho)| \leq 4$, $|\psi''(\rho)| \leq 8$ for all $\rho$. We claim that the upper bound is satisfied by the map defined by (using polar coordinates on $B_1$)

$$
y(\rho, \vartheta) := (\psi(\rho/h) + (1 - \psi(\rho/h)) \rho e_\rho.
$$

We calculate

$$
D y = \partial_\rho y \otimes e_\rho + \frac{1}{\rho} \partial_\varphi y \otimes e_\varphi
$$

Hence the pullback of the Euclidean metric in $\mathbb{R}^3$ under $y$ is given by

$$
g_y = y^* e^3 = \left(\psi + \frac{\rho}{h} \psi'\right)^2 + \left(1 - \psi - \frac{\rho}{h} \psi'\right)^2 + 2 \left(\psi + \frac{\rho}{h} \psi'\right) \left(1 - \psi - \frac{\rho}{h} \psi'\right) d\rho \otimes d\rho
$$

Hence the pullback of the Euclidean metric in $\mathbb{R}^3$ under $y$ is given by

$$
g_y = y^* e^3 = \left(\psi + \frac{\rho}{h} \psi'\right)^2 + \left(1 - \psi - \frac{\rho}{h} \psi'\right)^2 + 2 \left(\psi + \frac{\rho}{h} \psi'\right) \left(1 - \psi - \frac{\rho}{h} \psi'\right) d\rho \otimes d\rho
$$

The surface normal is given by

$$
\nu_y = \frac{\psi + \frac{\rho}{h} \psi'}{f(\rho)} e_{m_0} + \left(1 - \psi - \frac{\rho}{h} \psi'\right) e_z.
$$
Thus we get the radial coordinate \( \rho \) are approximately mapped to circles in \( N \). We prove a certain “rigidity” of the exponential map \( \exp \) from (11) and (12), and using the properties of \( \psi \), we see that

\[
\| g_\gamma - g_0 \|^2 = 0 \quad \text{for } \rho \geq h, \\
h^2 |Dg_\gamma|^2 \leq C h^2 \rho^{-2} (1 - m_0^2) \quad \text{for } \rho < h.
\]

Thus we get

\[
I_\gamma^\infty(\bar{y}) = \sup_{B_1 \setminus B_h} \| g_\gamma - g_0 \|^2 + h^2 \int_{B_h} |Dg_\gamma|^2 + h^2 \int_{B_1 \setminus B_h} |Dg_\gamma|^2 \\
\leq C h^2 + h^2 \int_0^{2\pi} d\theta \int_h^1 \frac{d\rho}{\rho} (1 - m_0^2) \\
= C^* h^2 |\log h| + C h^2.
\]

This proves the present lemma. \( \square \)

In the following Lemma, we use the smallness of the membrane term \( \| g - g_0 \|_{L^\infty(B_1 \setminus B_h)} \) to prove a certain “rigidity” of the exponential map \( \exp_h \). Namely, we prove that circles in \( N \) are approximately mapped to circles in \( B_1 \) in an \( L^\infty \) sense. This allows us to pass from the radial coordinate \( \rho \) in the reference configuration to the radial coordinate \( r \) in \( N \).

**Lemma 3.** Let \( y \in A \) with \( I_\gamma^\infty(\bar{y}) < 2C^* h^2 |\log h| \). Then there exists \( r_0 > 0 \) such that

\[
\sup_{2h \leq \rho \leq 1} |r(\rho, \bar{y}) - r_0 - \rho| \leq C_1 h |\log h|^{1/2}.
\]

**Proof.** Let \( d_g, d_{g_0} \) denote the distance functions with respect to \( g, g_0 \) respectively, i.e.,

\[
d_g(x, x') = \min \left\{ \int_0^1 g|\gamma(t), \gamma'(t)|, \gamma(0) = x, \gamma(1) = x' \right\}.
\]

The minimum is achieved by a geodesic connecting \( x \) and \( x' \). The function \( d_{g_0} \) is defined analogously.

Let \( x \in B_1 \setminus B_h \). Recall that \( r(x) \) is the distance from the origin with respect to the induced metric \( g \),

\[
r(x) = d_g(x, 0).
\]

We claim that

\[
\sup_{\partial B_{2h}} \frac{1}{B_{2h}} \frac{1}{r \, \rho} \leq C h,
\]

where

\[
f(\rho) = \sqrt{1 - m_0^2} \left( \psi + \frac{\rho}{h} \psi' \right)^2 + (m_0 + 1 - \psi - \frac{\rho}{h} \psi')^2.
\]

We compute the derivative of the normal,

\[
Dv_g = \left( \frac{2}{h} \psi' + \frac{\rho}{h^2} \psi'' \right) \left[ \left( \frac{e^1_{\mu_0} - e_z}{f(\rho)} - [2 - m_0^2] \left( \psi + \frac{\rho}{h} \psi' \right) - (1 + m_0) \right) \times \left( \frac{e^1_{\mu_0} + (1 - \psi - \frac{\rho}{h} \psi') e_z}{f(\rho)} \right) \right] \otimes e_\rho
\]

\[
- \frac{1}{\rho} \left( \frac{\psi + \frac{\rho}{h} \psi'}{f(\rho)} \right) \sqrt{1 - m_0^2} e_\varphi \otimes e_\varphi
\]

From (11) and (12), and using the properties of \( \psi \), we get

\[
\| g_\gamma - g_0 \|^2 = 0 \quad \text{for } \rho \geq h, \\
h^2 |Dg_\gamma|^2 \leq C h^2 \rho^{-2} (1 - m_0^2) \quad \text{for } \rho < h
\]

Thus we get

\[
I_\gamma^\infty(\bar{y}) = \sup_{B_1 \setminus B_h} \| g_\gamma - g_0 \|^2 + h^2 \int_{B_h} |Dg_\gamma|^2 + h^2 \int_{B_1 \setminus B_h} |Dg_\gamma|^2 \\
\leq C h^2 + h^2 \int_0^{2\pi} d\theta \int_h^1 \frac{d\rho}{\rho} (1 - m_0^2) \\
= C^* h^2 |\log h| + C h^2.
\]

This proves the present lemma. \( \square \)
Indeed, let \(x, x' \in \partial B_{2h}\). There exists a curve \(\gamma \in \text{Lip}(\{0, 1\}; B_1 \setminus B_h)\) with \(\gamma(0) = x, \gamma(1) = x'\), and \(g_0(\gamma'(t), \gamma'(t)) \leq Ch^2\) for all \(t \in [0, 1]\). Hence
\[
\int_0^1 g|_{\gamma(t)}(\gamma'(t), \gamma'(t))^{1/2}dt \leq \int_0^1 (g_0|_{\gamma(t)}(\gamma'(t), \gamma'(t))(1 + \|g - g_0\|_{L^\infty(B_1 \setminus B_h)})^{1/2}dt \leq Ch. \]

Hence, by the triangle inequality for \(d_g\), \(|d_g(x, 0) - d_g(x', 0)| \leq d_g(x, x') \leq Ch\), which proves (14).

We set
\[r_0 := \sup_{\partial B_{2h}} r.\]

Now let \(x \in B_1 \setminus B_{2h}\), and \(\gamma\) a geodesic connecting 0 and \(x\) with \(\gamma(0) = 0\) and \(\gamma(1) = x\). There exists some \(t_0 \in (0, 1)\) such that \(|\gamma(t_0)| = 2h\) and \(|\gamma(t)| > 2h\) for all \(t \in (t_0, 1]\). We write \(x_0 = \gamma(t_0)\), and note \(\|x - d_{g_0}(x, x_0)\| \leq Ch\).

Now we have
\[
|v(x) - r_0 - |x|| \leq |d_g(x, x_0) - d_{g_0}(x, x_0) + d_g(0, x_0) - r_0| + Ch \leq |d_g(x, x_0) - d_{g_0}(x, x_0)| + Ch, \tag{15}
\]
where we have used (14) in the second inequality.

Let \(\gamma_0\) be a curve connecting \(x\) and \(x_0\) with \(\gamma_0(0) = x_0, \gamma_0(1) = x\) and
\[
d_{g_0}(x, x_0) = \int_0^1 g_0|_{\gamma_0(t)}(\gamma'_0(t), \gamma'_0(t))^{1/2}dt.
\]

Then
\[
d_g(x, x_0) = \int_0^1 g|_{\gamma(t)}(\gamma'_0(t), \gamma'_0(t))^{1/2}dt \leq \int_0^1 \left((1 + Ch|\log h|^{1/2})g_0|_{\gamma_0(t)}(\gamma'_0(t), \gamma'_0(t))\right)^{1/2}dt \leq (1 + Ch|\log h|^{1/2})d_{g_0}(x, x_0). \tag{16}
\]

On the other hand, for any curve \(\gamma\) connecting \(x\) and \(x_0\) with \(\gamma_0(0) = x_0, \gamma_0(1) = x\), we have
\[
\int_0^1 g|_{\gamma(t)}(\gamma'(t), \gamma'(t))^{1/2}dt \geq \int_0^1 \left((1 - Ch|\log h|^{1/2})g_0|_{\gamma(t)}(\gamma'(t), \gamma'(t))\right)^{1/2}dt \geq (1 + Ch|\log h|^{1/2})d_{g_0}(x, x_0).
\]

Hence we also get \(d_g(x, x_0) \geq (1 - Ch|\log h|^{1/2})d_{g_0}(x, x_0)\). Using this last inequality and (14) in (15), the claim of the lemma is proved. \(\square\)

**Remark 1.** If one works with the energy functional \(I_h\) coming from the Kirchhoff-Love ansatz instead, it is possible to show an analogous claim if one additionally assumes that
\[
\|\nabla_{g_0}\mu - e_\mu\|_{L^2} \leq C\|g - g_0\|_{L^2}, \tag{17}
\]
where $\nabla_{g_0}$ denotes the covariant derivative w.r.t. $g_0$, and $e_\rho$ denotes the unit vector in $\rho$ direction in the reference configuration.

We believe that (17) does hold true, since

$$\|g - g_0\|^2 \leq C \text{dist}^2 \left( \left( \nabla_{g_0} r \right)_{\text{Gr}} \right), \text{SO}(2) \quad \text{(pointwise)}, \quad (18)$$

cf. equation (21) in the proof of Lemma 4 below. This looks quite similar to the situation in the Geometric Rigidity Theorem by Friesecke, James and Müller [16]. However, we have not been able to adapt these ideas to deduce (17) from (18).

If one is able to show (17), then one can deduce more or less the same statement as in Lemma 3 through a combination of arguments as in the proof of Morrey’s inequality $\|f\|_{C^{0,1-n/p}} \leq C \|f\|_{W^{1,p}}$ and Assumption 7. Then one would also be able to show the same statement for $I_h$ as in Theorem 1 (with different coefficients in front of the lower order terms).

4. Interpolation between metric and Gauss curvature

Now we are in position to follow the ansatz from the introduction and interpolate between the metric and the Gauss curvature in the ($r, \varphi$) coordinates. The interpolation will be of the standard type

$$\|\partial_r f\|_{L^1} \leq C \|f\|_{L^1}^{1/2} \|\partial^2_r f\|_{L^1}^{1/2} \quad (19)$$

(see e.g. [17], Theorem 7.28), where

$$f(r) = \frac{r}{m_0} \int_{S^1} d\varphi G(r, \varphi) + \text{linear terms in } r.$$ 

Since $G$ is defined via the double integral of the Gauss curvature (cf. (7)), the second derivative $\partial^2_r f(r)$ can be bounded in $L^1$ by the bending energy.

In order to establish smallness of $\|f\|_{L^1}$ from the smallness of the metric error $g - g_0$, we will need the following lemma.

Lemma 4. There exists a constant $C = C(m_0)$ with the following property:

(i) For every $y \in \mathcal{A}$,

$$\left( \frac{G \det \Gamma}{m_0} - 1 \right)^2 \leq C \|g - g_0\|^2 \quad \text{(pointwise)},$$

and hence for every $y \in \mathcal{A}$ with $I_h^\infty(y) \leq 2 C^* h^2 |\log h|$,

$$\left| \frac{G \det \Gamma}{m_0} - 1 \right| \leq C h |\log h|^{1/2}.$$ 

(ii) For every $y \in \mathcal{A}$ with $I_h^\infty(y) \leq 2 C^* h^2 |\log h|$, every $R$ with $2h \leq R \leq 1$, and every $\vartheta \in S^1$,

$$\int_{2h}^R |\partial_r r(\rho, \vartheta) - 1| \, d\rho \leq C h |\log h|^{1/2}.$$ 

Proof. First we introduce the notation

$$\tilde{m}_0 = \begin{pmatrix} 1 & 0 \\ 0 & m_0 \end{pmatrix}, \quad \tilde{G} = \begin{pmatrix} 1 & 0 \\ 0 & G \end{pmatrix}.$$
By the definition of $g_0$ and Lemma 1,
\[ g_0 = (d\rho \rho d\vartheta) \otimes \bar{m}_0^2 \left( \frac{d\rho}{\rho d\theta} \right), \quad g = (dr, r d\varphi) \otimes \bar{G}^2 \left( \frac{dr}{r d\varphi} \right). \]

By definition of the matrix norm $\| \cdot \|$ (cf. section 2), we get
\[ \| g - g_0 \|^2 = |\Gamma^T \bar{G}^2 \Gamma - \bar{m}_0^2 |^2, \]
where on the right hand side, $| \cdot |$ denotes the usual Euclidean matrix norm. By multiplying $\Gamma^T \bar{G}^2 \Gamma - \bar{m}_0^2$ from the left and right with $\bar{m}_0^{-1}$, we alter the norm of this expression at most by a factor $m_0^2$, and get
\[ |Q^T Q - \text{Id}|^2 \leq C \| g - g_0 \|^2, \]
where
\[ Q = \bar{G} \Gamma \bar{m}_0^{-1}, \]
and the constant on the right hand side only depends on $m_0$. Now we write down the spectral decomposition of the symmetric matrix $Q^T Q$,
\[ Q^T Q = R^T \bar{D} R, \]
where $\bar{D} = \text{diag}(d_1, d_2)$ is some diagonal matrix and $R \in SO(2)$. By $\| A \|^2 = \| R^T A R \|$ for all $A \in \mathbb{R}^{2 \times 2}$ and $R \in SO(2)$, and $\det Q^T Q = d_1 d_2$ we easily deduce
\[ |\sqrt{\det Q^T Q} - 1|^2 \leq |Q^T Q - \text{Id}|^2 \]
and thus
\[ \left( \frac{\det \Gamma}{m_0} - 1 \right)^2 \leq C \| g - g_0 \|^2. \tag{20} \]
This proves the first claim of (i), and the second claim of (i) follows trivially.

Next we note
\[ |Q^T Q - \text{Id}|^2 = |(\sqrt{Q^T Q} - \text{Id})(\sqrt{Q^T Q} + \text{Id})|^2 \]
\[ \geq |\sqrt{Q^T Q} - \text{Id}|^2 = \text{dist}^2(Q, O(2)) \]
where the inequality holds by $\sqrt{Q^T Q} + \text{Id} \geq \text{Id}$ in the sense of positive definite matrices. Hence we have shown
\[ \text{dist}^2(Q, O(2)) \leq C \| g - g_0 \|^2. \]
In fact, since $\det Q > 0$, we even have
\[ \text{dist}^2(Q, SO(2)) \leq C \| g - g_0 \|^2. \tag{21} \]
Explicitly, $Q$ reads
\[ Q = \left( \begin{array}{cc} \partial_\rho r & m_0^{-1} \rho^{-1} \partial_\varphi r \\ rG \partial_\rho \varphi & rGm_0^{-1} \rho^{-1} \partial_\varphi \end{array} \right) . \tag{22} \]
We introduce the notation $(\cdot)_+ = \max(\cdot, 0)$, $(\cdot)_- = \max(-\cdot, 0)$. From (21) and (22), we get in particular
\[ (\partial_\rho r - 1)_+ \leq C \| g - g_0 \| \leq C h \log h^{1/2}. \]
As in the proof of Lemma 3, we set \( r_0 := \sup_{\partial B_{2h}} r \) and note that \( \text{osc} \, r \leq Ch \). Obviously, we have \( \partial_r (r - \rho - r_0) = \partial_r r - 1 \). Hence, for every \( R \) with \( 2h \leq R \leq 1 \), and fixed \( \vartheta \in S^1 \) (the arguments \( \rho, \vartheta \) being omitted in the notation),

\[
\int_{2h}^R (\partial_r r - 1)_- \, d\rho \leq \sup |r - \rho - r_0| + \text{osc} \, r + \int_{2h}^R (\partial_r r - 1)_+ \, d\rho \\
\leq Ch |\log h|^{1/2}
\]

Hence we get

\[
\int_{2h}^R |\partial_r r - 1| \, d\rho = \int_{2h}^R ((\partial_r r - 1)_+ + (\partial_r r - 1)_-) \, d\rho \\
\leq Ch |\log h|^{1/2}.
\]

This proves (ii) and completes the proof of the present lemma.

We will need to distinguish balls in \( B_1 \) and in \( N \subset T_0 B_1 \), and hence we write

\[
B_R = \{ x \in B_1 : \rho(x) < R \} \\
\tilde{B}_R = \{ x \in B_1 : r(x) < R \}.
\]

Also, for a measurable set \( V \subset M \), let

\[
\mathcal{K}(V) = \int_V K \, dA.
\]

In the statement of the following proposition let, as in the proof of Lemma 4,

\[
r_0 = \sup_{\vartheta \in S^1} r(2h, \vartheta).
\]

As a further piece of notation, we set

\[
r^* := 1 - 2h + r_0 - C_1 h |\log h|^{1/2},
\]

where \( C_1 \) is the constant from the right hand side of (13). By this choice, \( \tilde{B}_r \subset B_1 \).

**Proposition 1.** Let \( y \in \mathcal{A} \) with \( I_h(y) \leq 2C^* h^2 |\log h| \). Then for \( R \in [C_1 h |\log h|^{1/2}, 1 - 2h - C_1 h |\log h|^{1/2}] \),

\[
\int_{2h}^{2h+R} d\rho |\mathcal{K}(B_\rho) - 2\pi(1 - m_0)| \leq CR^{1/2} h^{1/2} |\log h|^{3/4}.
\]

**Proof.** Let

\[
\rho = \rho(r, \varphi), \quad \vartheta = \vartheta(r, \varphi)
\]

be understood as functions of the coordinates \( (r, \varphi) \). By Lemma 3

\[
\sup_{r \in [r_0, r^*], \varphi \in S^1} |\rho(r, \varphi) + r_0 - r| \leq Ch |\log h|^{1/2}. \tag{23}
\]

Now define

\[
f : [r_0, r^*] \to \mathbb{R} \\
r \mapsto \left( r \int_{\partial B_r} \frac{G}{m_0} \right) - 2\pi (r - r_0).
\]
We may rewrite

\[ f(r) = r \int_{\partial B_r} d\varphi \left( \frac{G}{m_0} - \frac{r - r_0}{r} \partial_\varphi \theta \right). \]

Note that

\[ f'(r) = \left( \frac{1}{m_0} \int_{\partial B_r} d\varphi \Omega \right) - 2\pi = \left( \frac{1}{m_0} \int_{B_r} KdA \right) - 2\pi \]

\[ f''(r) = \frac{1}{m_0} \int_{\partial B_r} d\varphi KGr. \]

By the Gauss equation,

\[ K = \frac{\det D\nu^T D\nu}{\det D\nu^T D\nu}, \]

and since for all \( x \in B_1 \), the image of \( D\nu|_x \) is contained in the image of \( D\nu|_x \), we have as a consequence

\[ |K|dA = \frac{\det D\nu^T D\nu}{\sqrt{\det D\nu^T D\nu}} dx = \sqrt{\det D\nu^T D\nu} dx. \]

Now we can estimate \( \|f''\|_{L^1(r_0, r^*)} \) as follows,

\[
\begin{align*}
\|f''\|_{L^1(r_0, r^*)} &= \int_{r_0}^{r^*} drd\varphi |KGr| \\
&= \int_{\hat{B}_r \setminus \hat{B}_{r_0}} |K|dA \\
&= \int_{\hat{B}_r \setminus \hat{B}_{r_0}} dx \sqrt{\det D\nu^T D\nu} \\
&\leq \int_{B_1} dx |D\nu|^2 \\
&\leq C|\log h|,
\end{align*}
\]

where we have used the upper bound for the bending energy in the last inequality.

For the application of the standard interpolation inequality \([19]\), we need to estimate the \( L^1 \) norm of \( f \). More precisely, we will estimate \( \|f\|_{L^1(r_0, r_0 + R)} \):

\[
\begin{align*}
\int_{r_0}^{r_0 + R} dr|f| &\leq \int_{r_0}^{r_0 + R} rdr \int d\varphi \left( \frac{G}{m_0} - \frac{r - r_0}{r} \partial_\varphi \theta \right) \\
&\leq \int_{r_0}^{r_0 + R} rdr \int d\varphi \left( \frac{G}{m_0} - \frac{r - r_0}{r} \partial_\varphi \theta \right) \left( \frac{\det \Gamma}{\rho} \right) \\
&\leq \int_{r_0}^{r_0 + R} rdr \int d\varphi \left( \frac{G}{m_0} - \frac{r - r_0}{r} \partial_\varphi \theta \right) \left| \frac{\det \Gamma}{\rho} - 1 \right| \\
&\quad + \left| \frac{r - r_0}{\rho} \right| \left| \partial_\varphi \theta - 1 \right| + \left| \frac{r - r_0}{\rho} \right| \right),
\end{align*}
\]
where we used (6) in the second estimate to obtain the relation \( \partial_\varphi \vartheta = \det \tilde{\Gamma} r \rho^{-1} \partial_\rho r \). We will estimate the three terms in the integrand on the right hand side separately. Before we do so, note that by Lemma 3
\[
\tilde{B}_{r_0 + R} \subset B \left( 0, R + 2h + Ch |\log h|^{1/2} \right)
\]
and hence
\[
\mathcal{L}^2(\tilde{B}_{r_0 + R}) \leq C \left( R + h |\log h|^{1/2} \right)^2 
\leq CR^2.
\]
Now we estimate the terms on the right hand side in (25). Firstly,
\[
\int_{r_0}^{r_{0} + R} r dr d\varphi \det \tilde{\Gamma} \left| \frac{G \det \Gamma}{m_0} - 1 \right| \leq \left\| \frac{G \det \Gamma}{m_0} - 1 \right\|_{L^\infty(B_1 \setminus B_h)} \mathcal{L}^2(\tilde{B}_{r_0 + R}) 
\leq Ch |\log h|^{1/2} R^2,
\]
where in the second inequality, we have used the assumption \( I_h^\infty (y) \leq 2C^* h^2 |\log h| \) and Lemma 4 (i). Secondly, setting \( \tilde{R}_h = R + 2h + C_1 h |\log h|^{1/2} \),
\[
\int_{r_0}^{r_{0} + R} r dr d\varphi \det \tilde{\Gamma} \left| \frac{r - r_0}{\rho} \right| |\partial_\rho r - 1| 
\leq \sup_{r \geq r_0} \left| \frac{r - r_0}{\rho} \right| \int_{2h}^{\tilde{R}_h} \rho d\rho \int_{S^1} d\vartheta |\partial_\rho r - 1| 
\leq C \tilde{R}_h \int_{2h}^{\tilde{R}_h} d\rho |\partial_\rho r - 1| 
\leq C R h |\log h|^{1/2},
\]
where in the last inequality, we have used Lemma 4 (ii). Thirdly,
\[
\int_{r_0}^{r_{0} + R} r dr d\varphi \det \tilde{\Gamma} \left| \frac{r - r_0 - \rho}{\rho} \right| 
\leq \int_{2h}^{\tilde{R}_h} \rho d\rho \int_{S^1} d\vartheta \left| \frac{r - r_0 - \rho}{\rho} \right| 
= C \int_{2h}^{\tilde{R}_h} d\rho |r - r_0 - \rho| 
\leq C \tilde{R}_h \sup |r - r_0 - \rho| 
\leq C \tilde{R}_h |\log h|^{1/2} R.
\]
Inserting the preceding estimates into (25), we get
\[
\|f\|_{L^1(r_0, r_0 + R)} \leq Ch |\log h|^{1/2} R.
\]
Using (24) and (27) in the interpolation inequality (19) we obtain
\[
\|K(\tilde{B}_\rho) - 2\pi(1 - m_0)\|_{L^1(r_0, r_0 + R)} = \|f'\|_{L^1(r_0, r_0 + R)} \leq Ch^{1/2} R^{1/2} |\log h|^{3/4}.
\]
Comparing this last estimate with the statement of the present proposition, we see that what remains to be done is a change of domain for the integration of KdA, from \( \tilde{B}_\rho = \)
\{x \in B_1 : r(x) < \bar{r}\} \text{ to } B_\bar{r} = \{x \in B_1 : \rho(x) < \bar{\rho}\}. \text{ We have}

\[
\int_{2h}^{2h+R} d\rho |K(B_\rho) - 2\pi(1 - m_0)| \leq \int_{2h}^{2h+R} d\rho \left( |K(B_\rho) - K(\tilde{B}_{\rho + r_0 - 2h})| + |K(\tilde{B}_{\rho + r_0 - 2h}) - 2\pi(1 - m_0)| \right). \tag{29}
\]

In the following estimate we will use the notation \(A \Delta B = (A \setminus B) \cup (B \setminus A)\) for the symmetric difference of sets \(A, B\), and the characteristic function of a set \(A\) is denoted by \(\chi_A\). The first term on the right hand side of (29) can be estimated as follows,

\[
\int_{2h}^{2h+R} d\rho \left| K(B_\rho) - K(\tilde{B}_{\rho + r_0 - 2h}) \right| \leq \int_{2h}^{2h+R} d\rho \int_{\Delta B_\rho} |K| Gr d\rho d\varphi \leq C h |\log h|^{3/2}. \tag{30}
\]

where in the first estimate, we have used the definition of \(K\), in the second line we have used Fubini’s Theorem, and in the last inequality, we have used Lemma 3 and (24). The second term on the right hand side of (29) can be estimated by a trivial change of variables and (28)

\[
\int_{2h}^{2h+R} d\rho \left| K(\tilde{B}_{\rho + r_0 - 2h}) - 2\pi(1 - m_0) \right| = \int_{r_0}^{r_0+R} dr \left| K(\tilde{B}_r) - 2\pi(1 - m_0) \right| \leq C h^{1/2} R^{1/2} |\log h|^{3/4}. \tag{31}
\]

Inserting (30) and (31) into (29) concludes the proof of the present proposition. \(\square\)

5. An isoperimetric inequality on the sphere

In the proof of our main theorem, we are going to need a certain isoperimetric inequality for the image of the Gauss map. We will need several items from the literature to prove our claim in Section 5.4, Lemma 7 below; they will be collected in Sections 5.1–5.3.

5.1. The isoperimetric inequality for sets on the sphere. Let \(S^2 = \{x \in \mathbb{R}^3 : |x| = 1\}\). We define \(F : [0, 4\pi] \to \mathbb{R}\) by

\[
F(x) = \sqrt{4\pi x - x^2}.
\]

Note that \(F\) is concave. The isoperimetric inequality for sets on the sphere is as follows:

**Theorem 2** ([3]). Let \(A \subset S^2\) be open. Then

\[
\mathcal{H}^1(\partial A) \geq F(\mathcal{H}^2(A)).
\]
5.2. The Brouwer degree. The statement of the isoperimetric inequality will involve the Brouwer degree. We mention some basic facts about this object – for a more thorough exposition with proofs of the claims made here, see e.g. [14]. Let \( M \) be a paracompact oriented manifold of dimension \( n \), and \( U \) a bounded subset of \( \mathbb{R}^n \). Further, let \( u \in C^\infty(\overline{U}; M) \). Assume that \( y \in M \setminus u(\partial U) \), and let \( \mu \) be a \( C^\infty \) \( n \)-form on \( M \) with support in the same connected component of \( M \setminus u(\partial U) \) as \( y \), such that \( \int_M \mu = 1 \). Then the degree is defined by

\[
\deg(y, U, u) = \int_U u^*(\mu),
\]

where \( u^* \) is the pull-back by \( u \). It can be shown that this definition is independent of the choice of \( \mu \). Further, \( \deg(\cdot, U, u) \) is constant on connected components of \( M \setminus u(\partial U) \) and integer valued. Moreover, it is invariant under homotopies, i.e., given \( H \in C^\infty([0, 1] \times \overline{U}, M) \) such that \( y \notin H([0, 1], \partial U) \), we have

\[
\deg(y, U, H(0, \cdot)) = \deg(y, U, H(1, \cdot)).
\]

Using these facts, one can go on to define the degree for \( u \in C^0(\overline{U}, M) \) by approximation. Since the reader may be more familiar with a different way of defining the degree, we mention that for points \( y \in M \setminus u(\partial U) \) that satisfy \( \det Du(x) \neq 0 \) for all \( x \in u^{-1}(\{y\}) \), it may be shown that

\[
\deg(y, U, u) = \sum_{x \in u^{-1}(\{y\})} \text{sgn} \det Du(x).
\]

This can also be used as a starting point to define the degree. If \( u \in C^1(\overline{U}; M) \), and \( \mathcal{L}^n(\partial U) = 0 \), then it follows from the defining formula for the degree and approximation by smooth functions that

\[
\int_M \deg(y, U, u) \mu = \int_U u^*(\mu)
\]

for any \( n \)-form \( \mu \) that can be written as

\[
\mu(y) = \varphi(y) dy_1 \wedge \cdots \wedge dy_n
\]

in local coordinates \((y_1, \ldots, y_n)\), with \( \varphi \in L^\infty(M) \). If \( \mu \) is an exact form, i.e.,

\[
\mu = d\omega
\]

for some \( n-1 \) form \( \omega \) on \( M \), then

\[
u^*(d\omega) = d(u^*\omega).
\]

If \( U \) has smooth boundary, this implies, by Stoke’s Theorem,

\[
\int_M \deg(y, U, u) d\omega = \int_U d(u^*\omega)
\]

\[
= \int_{\partial U} u^*\omega.
\]
5.3. The space $BV(S^2)$. In the following, we are going to consider the space $L^1(S^2) \equiv L^1(S^2, \mathcal{H}^2)$, where $\mathcal{H}^2$ is the 2-dimensional Hausdorff measure. For $f \in L^1(S^2)$, let

$$
\|Df\|_2(S^2) := \inf \left\{ \liminf_{\varepsilon \to 0} \int_{S^2} |Df_x| \, d\mathcal{H}^2 : f_\varepsilon \in \text{Lip} (S^2), f_\varepsilon \to f \in L^1(S^2) \right\}. \tag{34}
$$

Then $BV(S^2)$, the space of functions of bounded variation on the 2-sphere, is the set of $f \in L^1(S^2)$ with $\|Df\|_2(S^2) < \infty$ (see [1, 26], where functions of bounded variation are defined for much more general measure spaces).

We will need some basic facts about $BV(S^2)$, that we state in Lemmas 5, 6 below. We first need to collect some notation for the statement of the first part of Lemma 5.

For $x \in S^2$, the space $T_x S^2$ is naturally equipped with the standard metric $\bar{g}$ on $S^2$:

$$
\|v\|_{T_x S^2} = \sqrt{\bar{g}(v,v)} \quad \text{for } v \in T_x S^2.
$$

The space $\land^1 T_x S^2$ is equipped with the dual norm

$$
\|\omega\|_{\land^1 T_x S^2} = \sup \{ |\omega(v)| : v \in T_x S^2, \|v\|_{T_x S^2} \leq 1 \} \quad \text{for } \omega \in \land^1 T_x S^2.
$$

Now define $C^1(S^2; \land^1 T S^2)$, the space of $C^1$-one forms on $S^2$, to be the set of functions $\omega$ that associate to every $x \in S^2$ an element of $\land^1 T_x S^2$, with the property that for any coordinate chart $\psi : S^2 \supset U \to \mathbb{R}^2$, we have

$$
x \mapsto (\psi^* \omega)|_x(e_i) \in C^1(U) \quad \text{for } i = 1, 2,
$$

where $e_i$ denotes the unit vector in $i$-th direction in $\mathbb{R}^2$. On $C^1(S^2; \land^1 T S^2)$, we introduce the norm $\| \cdot \|_{0,1}$ by setting

$$
\|\omega\|_{0,1} := \sup_{x \in S^2} \|\omega|_x\|_{\land^1 T_x S^2}.
$$

Lemma 5. (i) For all $f \in L^1(S^2)$,

$$
\|Df\|_2(S^2) = \sup \left\{ \int_{S^2} f \, d\omega : \omega \in C^1(S^2; \land^1 T S^2), \|\omega\|_{0,1} \leq 1 \right\}. \tag{34}
$$

(ii) If $A \subset S^2$ is open and has Lipschitz boundary, then

$$
\|D\chi_A\|_2(S^2) = \mathcal{H}^1(\partial A),
$$

where $\chi_A$ denotes the characteristic function of the set $A$.

Proof. For all $\tilde{f} \in BV(\mathbb{R}^2)$,

$$
\|D\tilde{f}\|_2(\mathbb{R}^2) = \sup \left\{ \int_{\mathbb{R}^2} f \, \text{div} \, h \, dx : h \in C^1(\mathbb{R}^2; \mathbb{R}^2), \|h\|_0 \leq 1 \right\},
$$

where $\| \cdot \|$ denotes the $C^0$-norm. For a proof of this statement, see e.g. [2]. The statement (i) follows using a smooth atlas on $S^2$ and a subordinate partition of unity. Statement (ii) follows from (i) and an application of the Gauss-Green Theorem. \hfill \Box

The following coarea formula for $BV$ functions is taken from [26], where it is proved for much more general measure spaces.

Lemma 6 (26). For all $f \in L^1(S^2)$,

$$
\|Df\|_2(S^2) = \int_{-\infty}^{\infty} ds \|D\chi_{\{f \geq s\}}\|_2(S^2),
$$

where $\chi_{\{f \geq s\}}$ denotes the characteristic function of the set $\{ x \in S^2 : f(x) > s \}$. 

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Next, we claim that for $u \in C^1(B_R; S^2)$, the Brouwer degree $\deg(\cdot, B_R, u)$ is in $BV(S^2)$. Indeed, we first note that $\mathcal{H}^2(u(\partial B_R)) = 0$ and hence $\deg(\cdot, B_R, u)$ is defined $\mathcal{H}^2$-almost everywhere on $S^2$ and from (32), one can easily derive that it is in $L^1(S^2)$. Using (33),

$$\int_{S^2} \deg(\cdot, B_p, u) d\omega = \int_{B_p} u^* d\omega = \int_{B_R} d(u^* \omega) = \int_{\partial B_p} u^* \omega,$$

where, in the second equality, we have used the fact that pull-back and exterior derivative commute, and in the last equality, we have used Stoke’s Theorem. Hence, by Lemma 5 (i), we have

$$\|D(\deg(\cdot, B_p, u))\|(S^2) = \int_{\partial B_p} |Du| d\mathcal{H}^1 \text{ for all } u \in C^1(B_R; S^2).$$

(35)

5.4. An isoperimetric inequality for the Brouwer degree. As a last piece of notation before we make the main statement of the present section, we introduce the metric $d_{4\pi Z}$ on the real numbers given by

$$d_{4\pi Z}(x) = \text{dist}(x, 4\pi \mathbb{Z}).$$

Lemma 7. Let $B_R \subset \mathbb{R}^2$, $\nu \in C^1(\overline{B_R}; S^2)$ and

$$\mathcal{K} = \int_{S^2} \deg(x, B_R, \nu) d\mathcal{H}^2(x).$$

Then

$$\int_{\partial B_R} |D\nu| d\mathcal{H}^1 \geq F(d_{4\pi Z}(\mathcal{K})).$$

(36)

Proof. We set

$$A_s := \{x \in S^2 : \deg(x, B_R, \nu) \geq s\},$$

and note that $A_s$ is open and has Lipschitz boundary. Combining Lemmas 5 (ii) and 6 with equation (35), we get

$$\int_{\partial B_R} |D\nu| d\mathcal{H}^1 = \int_{-\infty}^{\infty} \mathcal{H}^1(\partial A_s) ds = \sum_{j=-\infty}^{\infty} \mathcal{H}^1(\partial A_j).$$

By Theorem 2 we get

$$\int_{\partial B_R} |D\nu| \geq \sum_{j=-\infty}^{\infty} F(\mathcal{H}^2(A_j))$$

(37)
Now
\[ K = \int_{S^2} \deg(x, B_R, \nu) d\mathcal{H}^2(x) \]
\[ = \sum_{j \geq 0} \mathcal{H}^2(A_j) - \sum_{j < 0} \mathcal{H}^2(S^2 \setminus A_j) \]
\[ \equiv \sum_{j \in \mathbb{Z}} \mathcal{H}^2(A_j) \pmod{4\pi}. \]  
(38)

By the triangle inequality for \( \mathcal{H}^2 \),
\[ d_{4\pi \mathbb{Z}}(K) \leq \sum_{j \in \mathbb{Z}} d_{4\pi \mathbb{Z}}(\mathcal{H}^2(A_j)). \]

By the concavity of \( F \), we obtain
\[ F \circ d_{4\pi \mathbb{Z}}(K) \leq \sum_{j \in \mathbb{Z}} F \circ d_{4\pi \mathbb{Z}}(\mathcal{H}^2(A_j)). \]

Since \( \mathcal{H}^2(A_j) \in [0, 4\pi] \) for all \( j \) and \( F = F \circ d_{4\pi \mathbb{Z}} \) on \([0, 4\pi]\), this can be rewritten as
\[ F(d_{4\pi \mathbb{Z}}(K)) \leq \sum_{j \in \mathbb{Z}} F(\mathcal{H}^2(A_j)). \]

Combining this last inequality with (37) proves the claim of the lemma. \( \square \)

6. Proof of the Main Theorem

Proof of Theorem 7: The upper bound has been proved in Lemma 2. Let \( y \) satisfy the upper bound. For any \( R \in [C_1 h \log h/|1/2], 1 - 2h - C_1 h \log h/|1/2], \) we have
\[ \int_{2h}^{2h+R} \rho(\mathcal{K}(B_{\rho}) - 2\pi(1 - m_0)) \leq C h^{1/2} R^{1/2} |\log h|^{3/4} \]
by Proposition 1. We set \( \tilde{F} = F \circ \text{dist}_{4\pi \mathbb{Z}} \) with \( F \) as in Lemma 7. Note that on \( I_{m_0} := (\frac{1-m_0}{2}, 1 - \frac{m_0}{2}) \), \( \tilde{F} \) is Lipschitz with a Lipschitz constant that only depends on \( m_0 \). Moreover, \( \tilde{F} \) is bounded by \( 4\pi \). Hence,
\[ \left| \tilde{F}(x) \right|^2 - \left( \tilde{F}(2\pi(1 - m_0)) \right)^2 \leq \begin{cases} C(m_0)|x - 2\pi(1 - m_0)| & \text{if } x \in I_{m_0} \\ \frac{16\pi^2}{1} & \text{else} \end{cases}. \]

This implies
\[ \int_{2h}^{2h+R} \rho \left( \left( \tilde{F}(\mathcal{K}(B_{\rho})) \right)^2 - \left( \tilde{F}(2\pi(1 - m_0)) \right)^2 \right) \leq C \int_{2h}^{2h+R} \rho |\mathcal{K}(B_{\rho}) - 2\pi(1 - m_0)| \leq C h^{1/2} R^{1/2} |\log h|^{3/4}. \]  
(39)

Now we may estimate the bending term, first using Jensen’s inequality:
\[ \int_D dx |D\nu|^2 \geq 2\pi \int \rho dx \left( \frac{\int_{\partial B_{2\rho}} |D\nu(x)| d\mathcal{H}^1(x)}{2\pi \rho} \right)^2. \]
On the right hand side, we may apply Lemma 7 and obtain
\[
\int_D \mathrm{d}x |D\nu|^2 \geq \frac{1}{2\pi} \int_0^1 \frac{\mathrm{d}\rho}{\rho} \tilde{F}(K(B_{\rho}))^2.
\] (40)

We set \( h_1 = 2C_1 h |\log h|^{3/2} \), and choose \( J \in \mathbb{N} \) such that
\[
2^J h_1 \leq 1 - C_1 h |\log h|^{1/2} < 2^{J+1} h_1.
\]

Note that this choice implies in particular
\[
\log 2^J \geq |\log h| - \frac{3}{2} \log |\log h| - C.
\]

Using this in (40), we get
\[
\int_D \mathrm{d}x |D\nu|^2 \geq \frac{1}{2\pi} \int_{h_1}^{2^J h_1} \frac{\mathrm{d}\rho}{\rho} \tilde{F}(K(B_{\rho}))^2
\geq C^* \log 2^J - \frac{1}{2\pi} \int_{h_1}^{2^J h_1} \frac{\mathrm{d}\rho}{\rho} \left| \tilde{F}(K(B_{\rho}))^2 - \tilde{F}(2\pi(1 - m_0))^2 \right|
\geq C^* \left( |\log h| - \frac{3}{2} \log |\log h| - C \right) - \text{error term}.
\] (41)

Letting \( R_j = 2^j h_1 \), the error term on the right hand side can be estimated using (40),
\[
\int_{h_1}^{2^J h_1} \frac{\mathrm{d}\rho}{\rho} \left| \tilde{F}(K(B_{\rho}))^2 - \tilde{F}(2\pi(1 - m_0))^2 \right| \leq \sum_{j=0}^J \int_{R_j}^{2R_j} \frac{\mathrm{d}\rho}{\rho} \left| \tilde{F}(K(B_{\rho}))^2 - \tilde{F}(2\pi(1 - m_0))^2 \right|
\leq \sum_{j=0}^J R_j^{-1} C h^{1/2}(2R_j)^{1/2} |\log h|^{3/4}
\leq C,
\]
where the last estimate is just the summation of a geometric series. This proves the theorem.

**Remark 2.** It is apparent from the proof that we would have been able to prove the same lower bound (with different coefficients in front of the lower order terms) if instead of working with the membrane term \( \|g - g_0\|_{L^\infty(B_1 \setminus B_{\tilde{h}})} \), we had worked with \( \|g - g_0\|_{L^\infty(B_1 \setminus B_{\tilde{h}})}^{1/2} \), where \( \tilde{h} = Ch|\log h|^p \) for some \( p > 0 \).

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