SOME EXPLICIT SOLUTIONS OF
THE LAMÉ AND BOURLET TYPE EQUATIONS

Alexander V. Razumov
Institute for High Energy Physics, 142284, Protvino, Moscow region, Russia

and

Mikhail V. Saveliev
Laboratoire de Physique Théorique de l’École Normale Supérieure
24 rue Lhomond, 75231 Paris CÉDEX 05, France

Abstract

Some special solutions to the multidimensional Lamé and Bourlet type equations are constructed in an explicit form.

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1 E-mail: razumov@mx.ihep.su
2 On leave of absence from the Institute for High Energy Physics, 142284, Protvino, Moscow region, Russia; e-mail: saveliev@mx.ihep.su
3 Unité Propre du Centre National de la Recherche Scientifique, associée à l’École Normale Supérieure et à l’Université de Paris-Sud
4 E-mail: saveliev@physique.ens.fr
1 Introduction

The classical differential geometry serves as an injector of many equations integrable in this or that sense. Among them, the Lamé and Bourlet equations play especially remarkable role. These equations arise, in particular, in the following way.

Let \((U, z_1, \ldots, z_n)\) be a chart on a Riemannian manifold \(M\), such that the metric tensor \(g\) has on \(U\) the form

\[
g = \sum_{i=1}^{n} \beta_i^2(z) \, dz_i \otimes dz_i. \tag{1.1}
\]

In such a situation the metric tensor \(g\) is said to be diagonal with respect to the coordinates \(z_i\). The functions \(\beta_i\) are called the Lamé coefficients. Define the so called rotation coefficients

\[
\gamma_{ij} = \frac{1}{\beta_i} \frac{\partial \beta_j}{\partial z_i}, \quad i \neq j. \tag{1.2}
\]

Here and henceforth we denote \(\partial_i = \partial/\partial z_i\). It can be shown that the Riemannian submanifold \(U\) of \(M\) is flat if and only if the rotation coefficients \(\gamma_{ij}\) satisfy the following system of partial differential equations

\[
\partial_i \gamma_{jk} = \gamma_{ji} \gamma_{ik}, \quad i \neq j \neq k, \tag{1.3}
\]

\[
\partial_i \gamma_{ij} + \partial_j \gamma_{ji} + \sum_{k \neq i, j} \gamma_{ki} \gamma_{kj} = 0, \quad i \neq j, \tag{1.4}
\]

where the notation \(i \neq j \neq k\) means that \(i, j, k\) are distinct. Equations (1.3), (1.4) are called the Lamé equations.

With the so called Egoroff property, \(\gamma_{ij} = \gamma_{ji}\), equations (1.4) are equivalent to the following ones:

\[
\left( \sum_{k=1}^{n} \partial/\partial_k \right) \gamma_{ij} = 0, \quad i \neq j.
\]

The corresponding solutions are represented in the form \(\beta_i^2 = \partial_i F\) where \(F\) is some function of the coordinates \(z_i\).

It can be also shown that the Riemannian submanifold \(U\) is of constant curvature \(1\) with the sectional curvature equal to 1 if and only if the rotation coefficients satisfy the equations

\[
\partial_i \gamma_{jk} = \gamma_{ji} \gamma_{ik}, \quad i \neq j \neq k, \tag{1.5}
\]

\[
\partial_i \gamma_{ij} + \partial_j \gamma_{ji} + \sum_{k \neq i, j} \gamma_{ki} \gamma_{kj} + \beta_i \beta_j = 0, \quad i \neq j. \tag{1.6}
\]

We call equations (1.3), (1.5) and (1.2) the Bourlet type equations. The Bourlet equations in the precise sense correspond to the case with \(\sum_{i=1}^{n} \beta_i^2 = 1\), see, for example, [4, 5].

Sometimes it is suitable to rewrite at least a part of these equations in a ‘Laplacian’ type form. Impose the condition

\[
\sum_{i=1}^{n} \beta_i^2 = c, \tag{1.7}
\]
where $c$ is a constant. It is convenient to allow the functions $\beta_i$, and hence the functions $\gamma_{ij}$, to take complex values. Therefore, we will assume that $c$ is an arbitrary complex number. One can easily get convinced by a direct check with account of (1.2) and (1.7) that there takes place the relation

$$\partial_i \beta_i = - \sum_{j \neq i} \gamma_{ij} \beta_j.$$ 

Now, using the same calculations as those in [4, 5], and introducing, as there, the operators

$$\Delta^{(i)} = \sum_{j \neq i} \partial_j^2 - \partial_i^2,$$

we obtain from equations (1.6)

$$\Delta^{(i)} \beta_i = \sum_{j \neq i} \beta_j[(\beta_j^{-1} \partial_j \beta_i)^2 - (\beta_i^{-1} \partial_i \beta_j)^2] - 2 \sum_{j \neq k \neq i} \beta_j \beta_k^{-2} (\partial_k \beta_i)(\partial_k \beta_j) + \beta_i (\beta_i^2 - c).$$

The integrability of the equations in question has been established for quite a long time ago; the general solution is defined by $n(n - 1)/2$ functions of two variables for the Lamé system, and by $n(n - 1)$ functions of one variable and $n$ constants for the Bourlet system. However, an explicit form of the solutions for higher dimensions remained unknown. In the beginning of eighties an interest to these equations was revived. In particular, it was shown that for $\sum_{i=1}^n \beta_i^2 = 1$, the completely integrable system (1.3), (1.6) provides the necessary and sufficient condition for a construction of an arbitrary local analytic immersion of the Lobachevsky space $L_n$ in $\mathbb{R}^{2n-1}$ [4], see also [6]. For $n = 2$ equations (1.3) and (1.6) are absent; equation (1.6) is reduced to the Liouville and sine-Gordon equations for $\beta_1^2 + \beta_2^2$ equals 0 and 1, respectively; while (1.4) is the wave equation. This is why for higher dimensions the Bourlet type equations with a nonzero constant $c$ in (1.7), with $c = 0$, and the Lamé equations sometimes are called multidimensional generalisations of the sine-Gordon, Liouville, and wave equations, respectively, see, for example, [4, 6, 5, 7]. In accordance with [7, 8], these systems can be integrated with the help of the inverse scattering method. Moreover, in the last paper it was shown that the problems of description of $n$-orthogonal surfaces and classification of Hamiltonians of hydrodynamic type systems are almost equivalent. It was also pointed out there that system (1.3) is a natural generalisation of the three wave system which is a relevant object in nonlinear optics. Finally notice that the Lamé equations also arise very naturally in the context of the Cecotti-Vafa equations describing topological-antitopological fusion, see [9] and references therein, and in those of the multidimensional generalisations of the Toda type systems [10]. In general, classification and description of diagonal metrics seems to be relevant for some modern problems of supergravity theories, in particular their elementary and solitonic supersymmetric $p$-brane solutions, see, for example, [11] and references therein.

In the present paper we obtain in an explicit and rather simple form some special class of the solutions to the Lamé equations and to the Bourlet type equations with and without condition (1.7). If one does not impose condition (1.7), then our solutions are determined by $n$ arbitrary functions of only one variable, while with the condition (1.7) the obtained solutions of the Bourlet equations are expressed as the products of the elliptic integrals of the first kind and are determined by $2n$ arbitrary constants.
The derivation of the solutions to both of these systems is given by using two different methods. One is based on the geometrical interpretation of the corresponding equations. Another approach uses the zero curvature representation which, for the Lamé equations, is different from [8], and for the Bourlet equations is different from [7].

2 Bourlet type equations

We begin with the description of the zero curvature representation of the Bourlet type equations following [5].

Let $M_{ab}$ be the elements of the Lie algebra $\mathfrak{o}(n + 1, \mathbb{R})$ of the Lie group $O(n + 1, \mathbb{R})$ defined as

$$(M_{ab})_{cd} = \delta_{ac}\delta_{bd} - \delta_{bc}\delta_{ad},$$

The commutation relations for these elements have the standard form

$$[M_{ab}, M_{cd}] = \delta_{ad}M_{bc} + \delta_{ac}M_{bd} - \delta_{ac}M_{bd} - \delta_{bd}M_{ac},$$

and any element $X$ of $\mathfrak{g}$ can be represented as

$$X = \sum_{a,b=1}^{n+1} x_{ab}M_{ab}.$$ 

Such a representation is unique if we suppose that $x_{ab} = -x_{ba}$.

In what follows we assume that the indices $a, b, c, \ldots$ run from 1 to $n + 1$, while the indices $i, j, k, \ldots$ run from 1 to $n$. Let $(U, z_1, \ldots, z_n)$ be a chart on some smooth manifold $M$. Consider the connection $\omega = \sum_{i=1}^{n} \omega_i dz^i$ on the trivial principal fibre bundle $U \times O(n + 1, \mathbb{R})$ with the components given by

$$\omega_i = \sum_{k=1}^{n} \gamma_{ki}M_{ik} + \beta_i M_{i,n+1}.$$ 

(2.1)

One can get convinced that the Bourlet type equations (1.2), (1.5) and (1.6) are equivalent to the zero curvature condition for the connection $\omega$, which, in terms of the connection components, has the form

$$\partial_i \omega_j - \partial_j \omega_i + [\omega_i, \omega_j] = 0.$$ 

(2.2)

Identify the Lie group $O(n, \mathbb{R})$ with the Lie subgroup of $O(n + 1, \mathbb{R})$ formed by the matrices $A \in O(n + 1, \mathbb{R})$, such that

$$A_{i,n+1} = 0, \quad A_{n+1,i} = 0, \quad A_{n+1,n+1} = 1.$$ 

Similarly, identify the Lie algebra $\mathfrak{o}(n, \mathbb{R})$ with the corresponding subalgebra of $\mathfrak{o}(n+1, \mathbb{R})$.

Let the connection $\omega$ with the components of form (2.1) satisfies the zero curvature condition (2.2). Suppose that $U$ is simply connected, then there exists a mapping $\varphi$ from $U$ to $O(n + 1, \mathbb{R})$, such that

$$\omega_i = \varphi^{-1}\partial_i \varphi.$$
Parametrise $\varphi$ in the following way

$$\varphi = \xi \chi,$$  \hfill (2.3)

where $\chi$ is a mapping from $U$ to $O(n, \mathbb{R})$ and the mapping $\xi$ has the form

$$\xi = e^{\psi_1 M_{12}} e^{\psi_2 M_{23}} \cdots e^{\psi_{n-1} M_{n-1,n}} e^{\psi_n M_{n,n+1}}.$$ \hfill (2.4)

Here $\psi_i$ are some functions on $U$ having the meaning of the generalised Euler angles [12]. For the connection components $\omega_i$ one obtains the expression

$$\omega_i = \chi^{-1} (\xi^{-1} \partial_i \xi) \chi + \chi^{-1} \partial_i \chi.$$  \hfill (2.9)

Relation (2.4) gives

$$\xi^{-1} \partial_i \xi = \sum_{j=1}^{n-1} \partial_i \psi_j \sum_{k=j+1}^{n} \mu_{jk}(\psi) M_{jk} + \sum_{j=1}^{n} \partial_i \psi_j \nu_j(\psi) M_{j,n+1},$$

where

$$\mu_{j-1,j}(\psi) = \cos \psi_j, \quad 1 < j \leq n,$$ \hfill (2.5)

$$\mu_{jk}(\psi) = \left( \prod_{l=j+1}^{k-1} \sin \psi_l \right) \cos \psi_k, \quad 1 < j + 1 < k \leq n,$$ \hfill (2.6)

$$\nu_{j}(\psi) = \prod_{l=j+1}^{n} \sin \psi_l, \quad 1 \leq j < n, \quad \nu_{n}(\psi) = 1.$$ \hfill (2.7)

Now, using the evident equalities

$$\chi^{-1} \partial_i \chi = \frac{1}{2} \sum_{j,k,l=1}^{n} \chi_{lj} \partial_i \chi_{lk} M_{jk}, \quad \chi^{-1} M_{i,n+1} \chi = \sum_{j=1}^{n} \chi_{ij} M_{j,n+1},$$ \hfill (2.8)

one comes to the expressions

$$\omega_i = \frac{1}{2} \sum_{j,k,l=1}^{n} \chi_{lj} \partial_i \chi_{lk} M_{jk}$$

$$+ \sum_{j,k=1}^{n} \sum_{l=1}^{n-1} \partial_i \psi_l \sum_{m=l+1}^{n} \mu_{lm}(\psi) \chi_{lj} \chi_{mk} M_{jk} + \sum_{j,l=1}^{n} \partial_i \psi_l \nu_l(\psi) \chi_{lj} M_{j,n+1}.$$ \hfill (2.9)

Comparing (2.9) and (2.1), we have, in particular,

$$\sum_{l=1}^{n} \partial_i \psi_l \nu_l(\psi) \chi_{lj} = \beta_i \delta_{ij}.$$ \hfill (2.10)

Note that the geometrical meaning of the functions $\beta_i$ do not allow them to take zero value. Therefore, from (2.10) it follows that for any point $p \in U$ we have

$$\det(\partial_i \psi_j(p)) \neq 0, \quad \nu_i(\psi(p)) \neq 0.$$ \hfill (2.11)
Since the matrix \((\chi_{ij})\) is orthogonal, one easily obtains
\[
\chi_{ij} = \frac{1}{\beta_j} \partial_j \psi_i \nu_i(\psi),
\]
and, using again the orthogonality of \((\chi_{ij})\), one sees that
\[
\beta_i^2 = \sum_{l=1}^{n} (\partial_i \psi_l \nu_l(\psi))^2.
\tag{2.12}
\]
Therefore, we have
\[
\chi_{ij} = \frac{\partial_j \psi_i \nu_i(\psi)}{\sqrt{\sum_{l=1}^{n} (\partial_j \psi_l \nu_l(\psi))^2}}.
\tag{2.13}
\]
Thus, the matrix \((\chi_{ij})\), and hence the mapping \(\chi\), is completely determined by the functions \(\psi_i\), and its orthogonality is equivalently realised by the relation
\[
\sum_{l=1}^{n} \partial_l \psi_l \nu_l^2(\psi) \partial_j \psi_l = 0, \quad i \neq j.
\tag{2.14}
\]
Suppose now that a set of functions \(\psi_i\) satisfies relations (2.11) and (2.14). Consider the mapping \(\varphi\) defined by (2.3) with the mapping \(\xi\) having form (2.4) and the mapping \(\chi\) defined by (2.13). Show that the mapping \(\varphi\) generates the connection with the components of form (2.1). First of all, with \(\beta_i\) of form (2.12) we can get convinced that in the case under consideration relation (2.10) is valid. Taking into account (2.14), one can write the relation
\[
\sum_{l=1}^{n} \partial_j \psi_l \nu_l^2(\psi) \partial_k \psi_l = \beta_j^2 \delta_{jk},
\]
whose differentiation with respect to \(z_i\) gives
\[
\sum_{l=1}^{n} \partial_j \psi_l \nu_l^2(\psi) \partial_l \partial_k \psi_l
\]
\[
= - \sum_{l=1}^{n} \partial_l \partial_j \psi_l \nu_l^2(\psi) \partial_k \psi_l - 2 \sum_{l=1}^{n} \partial_j \psi_l \nu_l(\psi) \partial_l \nu_l(\psi) \partial_k \psi_l + 2 \beta_j \partial_l \beta_j \delta_{lj}.
\]
Since the left hand side of this equality is symmetric with respect to the transposition of the indices \(i\) and \(k\), its right hand side must also be symmetric with respect to this transposition, and, therefore, we have
\[
\sum_{l=1}^{n} \partial_j \psi_l \nu_l^2(\psi) \partial_k \psi_l
\]
\[
= - \sum_{l=1}^{n} \partial_k \partial_j \psi_l \nu_l^2(\psi) \partial_i \psi_l - 2 \sum_{l=1}^{n} \partial_j \psi_l \nu_l(\psi) \partial_k \nu_l(\psi) \partial_i \psi_l + 2 \beta_j \partial_k \beta_j \delta_{ij}.
\]
Using this equality, it is not difficult to show that
\[
\sum_{l=1}^{n} \chi_{lj} \partial_l \chi_{lk} = \gamma_{kj} \delta_{ij} - \gamma_{jk} \delta_{ik}
\]
\[
- \frac{1}{\beta_j \beta_k} \sum_{l=1}^{n} [\partial_j \psi_l \nu_l(\psi) \partial_k \nu_l(\psi) \partial_l \psi_l - \partial_k \psi_l \nu_l(\psi) \partial_j \nu_l(\psi) \partial_l \psi_l],
\]  
(2.15)
where the functions $\gamma_{ij}$ are defined by (1.2).

Using the concrete form of the functions $\mu_{ij}(\psi)$ and $\nu_i(\psi)$, we can get convinced in the validity of the equalities
\[
\frac{\partial \nu_j(\psi)}{\partial \psi_i} = 0, \quad 1 \leq i \leq j, \quad \mu_{ij}(\psi) = \frac{1}{\nu_j(\psi)} \frac{\partial \nu_i(\psi)}{\partial \psi_j},
\]
which allow to show that
\[
\sum_{l=1}^{n-1} \partial_i \psi_l \sum_{m=l+1}^{n} \mu_{lm}(\psi) \chi_{lj} \chi_{mk} = \frac{1}{\beta_j \beta_k} \sum_{l=1}^{n} \partial_j \psi_l \nu_l(\psi) \partial_k \nu_l(\psi) \partial_l \psi_l.
\]  
(2.16)

Substituting (2.15), (2.16) and (2.10) into (2.9), we come to expression (2.1). Thus, any set of functions $\psi_i$ satisfying (2.11) and (2.14) allows to construct a connection of form (2.1) satisfying the zero curvature condition (2.2) which is equivalent to the Bourlet type equations. Therefore, the general solution to the Bourlet type equations is described by (2.12) where the functions $\psi_i$ satisfy (2.11) and (2.14). In the simplest case we can satisfy (2.14) assuming that
\[
\partial_i \psi_j = 0, \quad i \neq j;
\]  
(2.17)
in other words, each function $\psi_i$ depends on the corresponding coordinate $z_i$ only. In this case we obtain the following expressions for the functions $\beta_i$:
\[
\beta_i = \partial_i \psi_i \prod_{j=i+1}^{n} \sin \psi_j, \quad 1 \leq i < n, \quad \beta_n = \partial_n \psi_n.
\]  
(2.18)
The corresponding expressions for the functions $\gamma_{ij}$ can be easily found and we do not give here their explicit form.

There is a transparent geometrical interpretation of the results obtained above. Recall that solutions of the Bourlet type equations are associated with diagonal metrics in Riemannian spaces of constant curvature. Namely, let $(U, z_1, \ldots, z_n)$ be a chart on a manifold $M$, and we have a solution of the Bourlet type equations. Supply the open submanifold $U$ with metric (1.1); then $(U, g)$ is a Riemannian manifold of constant curvature with the sectional curvature equal to 1. From the other hand, let $(M, g)$ be a Riemannian manifold of constant curvature with the sectional curvature equal to 1, and $(U, z_1, \ldots, z_n)$ be such a chart on $M$ that the metric $g$ has on $U$ form (1.1). Then the Lamé and the corresponding rotation coefficients satisfy the Bourlet type equations.

The simplest example of a manifold of constant curvature is a unit $n$-dimensional sphere $S^n$ in $\mathbb{R}^{n+1}$ with the metric induced by the standard metric on $\mathbb{R}^{n+1}$. Denote the
spherical coordinates in $S^n$ by $z_1, \ldots, z_n$. The explicit expression for the metric on $S^n$ has the form

$$g = \sum_{l=1}^{n-1} \left( \prod_{m=l+1}^{n} \sin^2 z_m \right) dz_l \otimes dz_l + dz_n \otimes dz_n.$$  

So we have a diagonal metric. Note that it can be written in the form

$$g = \sum_{l=1}^{n} \nu_l^2 z_l dz_l \otimes dz_l,$$

(2.19)

where the functions $\nu_l$ are given by (2.7). Let $\psi$ be a diffeomorphism from $S^n$ to $S^n$. It is clear that $(S^n, \psi^*g)$ is also a Riemannian manifold of constant curvature with the sectional curvature equal to 1. Denoting $\psi^*z_i = \psi_i$, one gets

$$\psi^*g = \sum_{j,k,l=1}^{n} \partial_j \psi_l \nu_l^2(\psi) \partial_k \psi_l dz_j \otimes dz_k.$$  

Therefore, the metric $\psi^*g$ is diagonal with respect to the coordinates $z_i$ if and only if the functions $\psi_i$ satisfy relations (2.14). In particular, if the functions $\psi_i$ satisfy relations (2.17) we obtain the diagonal metrics with the Lamé coefficients given by (2.18).

In general, starting from some fixed diagonal metric in the space of constant curvature with the unit sectional curvature, one gets the family of explicit solutions to the Bourlet type equations parametrised by a set of $n$ functions each depending only on one variable. In terms of equations (1.2), (1.5) and (1.6) themselves, we formulate this observation as follows. Let the functions $\beta_i, \gamma_{ij}$ satisfy the Bourlet type equations; then for any set of functions $\psi_i$, such that

$$\partial_i \psi_j = 0, \quad i \neq j,$$

the functions

$$\beta_i'(z) = \beta_i(\psi(z)) \partial_i \psi_i(z), \quad \gamma_{ij}'(z) = \gamma_{ij}(\psi(z)) \partial_j \psi_j(z_j),$$

(2.20)

where $\psi(z)$ stands for the set $\psi_1(z), \ldots, \psi_n(z)$, also satisfy the Bourlet type equations.

Note that our considerations can be easily generalised to the case of complex metrics. In this case the zero curvature representation of the Bourlet type equations should be based on the Lie group $O(n+1, \mathbb{C})$.

Return to the consideration of solutions (2.18) to the Bourlet type equations. If one imposes condition (1.7) where $c$ is an arbitrary zero or nonzero constant, then the arbitrary functions $\psi_i(z_i)$ satisfy the equation

$$\sum_{l=1}^{n} \left( \prod_{m=l+1}^{n} \sin^2 \psi_m \right) (\partial_l \psi_l)^2 = c,$$

thereof for some constants $c_i, i = 0, \ldots, n$, such that $c_0 = 0$ and $c_n = c$, one gets

$$\partial_i \psi_i = \sqrt{c_i - c_{i-1} \sin^2 \psi_i}.$$  

(2.21)
Hence, solution (2.18) takes the form
\begin{equation}
\beta_i = \sqrt{c_i - c_{i-1}} \sin^2 \psi_i \prod_{j=i+1}^{n} \sin \psi_j, \tag{2.22}
\end{equation}

where the functions \( \psi_i \) are determined by the ordinary differential equations (2.21). Suppose that all constants \( c_i, i = 1, \ldots, n \), are different from zero. With appropriate conditions on the constants \( c_i \), in accordance with (2.21) one has
\begin{equation}
z_i + d_i = \int_{0}^{\psi_i} \frac{d\psi_i}{\sqrt{c_i - c_{i-1}} \sin^2 \psi_i},
\end{equation}
where \( d_i \) are arbitrary constants. Therefore,
\begin{equation}
\sqrt{c_i}(z_i + d_i) = F\left(\psi_i, \sqrt{\frac{c_{i-1}}{c_i}}\right),
\end{equation}
where \( F(\phi, k) \) is the elliptic integral of the first kind,
\begin{equation}
F(\phi, k) = \int_{0}^{\phi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}.
\end{equation}

Thus, using Jacobi elliptic functions, we can write
\begin{align*}
\sin \psi_i(z_i) &= \sn\left(\sqrt{c_i}(z_i + d_i), \sqrt{\frac{c_{i-1}}{c_i}}\right), \\
\cos \psi_i(z_i) &= \cn\left(\sqrt{c_i}(z_i + d_i), \sqrt{\frac{c_{i-1}}{c_i}}\right).
\end{align*}

Now, with the evident relation
\begin{equation}
\partial_i \psi_i(z_i) = \frac{\partial_i \sin \psi_i(z_i)}{\cos \psi_i(z_i)} = \sqrt{c_i} \dn\left(\sqrt{c_i}(z_i + d_i), \sqrt{\frac{c_{i-1}}{c_i}}\right),
\end{equation}
we write our solution as the product of elliptic functions,
\begin{equation}
\beta_i(z) = \sqrt{c_i} \dn\left(\sqrt{c_i}(z_i + d_i), \sqrt{\frac{c_{i-1}}{c_i}}\right) \prod_{j=i+1}^{n} \sn\left(\sqrt{c_j}(z_j + d_j), \sqrt{\frac{c_{j-1}}{c_j}}\right). \tag{2.23}
\end{equation}

The case when some of the constants \( c_i \) are equal to zero can be analysed in a similar way. Note that, taking into account the relations
\begin{align*}
\sn(u, 1) &= \tanh u, \\
\dn(u, 1) &= \frac{1}{\cosh u}, \\
\sn(u, 0) &= \sin u, \\
\dn(u, 0) &= 1,
\end{align*}
with an appropriate choice of the constants \( c_i \), we can reduce some of the elliptic functions entering the obtained solution to the trigonometric or hyperbolic ones.

It is clear from the solution in form (2.22) or (2.23), that it does not depend on the variable \( z_1 \) at all, since among the functions \( \beta_i \), only \( \beta_1 \) depends on \( \psi_1 \) and only as \( \partial_1 \psi_1 \), while \( \psi_1 = c_1 z_1 + d_1 \).

In the simplest case \( n = 2 \) and \( c_2 = 1 \) with the parametrisation \( \beta_1 = \cos(u/2) \), \( \beta_2 = \sin(u/2) \), system (1.2), (1.5) and (1.6) is reduced to the sine-Gordon equation
\begin{equation}
\partial_1^2 u - \partial_2^2 u + \sin u = 0,
\end{equation}
and one gets the evident solution \( \sin(u/2) = \dn(z_2 + d_2, \sqrt{c_1}) \).
3 Lamé equations

The zero curvature representation of the Lamé equations is based on the Lie group $G$ of rigid motions of the affine space $\mathbb{R}^n$. This Lie group is isomorphic to the semidirect product of the Lie groups $O(n, \mathbb{R})$ and $\mathbb{R}^n$, where the linear space $\mathbb{R}^n$ is considered as a Lie group with respect to the addition operation. The standard basis of the Lie algebra $\mathfrak{g}$ of the Lie group $G$ consists of the elements $M_{ij}$ and $P_i$ which satisfy the commutation relations

\[
[M_{ij}, M_{kl}] = \delta_{il}M_{jk} - \delta_{jk}M_{il} - \delta_{ij}M_{kl},
\]

\[
[M_{ij}, P_k] = \delta_{jk}P_i - \delta_{ik}P_j,
\]

\[
[P_i, P_j] = 0.
\]

Let $(U, z_1, \ldots, z_n)$ be a chart on the manifold $M$. Consider the connection $\omega = \sum_{i=1}^n \omega_i dz^i$ on the trivial principal fibre bundle $U \times G$ with the components given by

\[
\omega_i = \sum_{k=1}^n \gamma_{ik}M_{ik} + \beta_i P_i.
\]

(3.1)

It can be easily verified that equations (1.2)–(1.4) are equivalent to the zero curvature condition for the connection $\omega$. It is well known that the Lie algebra $\mathfrak{g}$ can be obtained from the Lie algebra $\mathfrak{o}(n+1, \mathbb{R})$ by an appropriate İnönü–Wigner contraction. Unfortunately, this fact does not give us a direct procedure for obtaining solutions of the Lamé equations from solutions of the Bourlet type equations. Therefore, we will consider the procedure for obtaining solutions of the Lamé equations independently.

Let the connection $\omega$ with the components of form (3.1) satisfies the zero curvature condition. Restricting to the case of simply connected $U$, write for the connection components $\omega_i$ the representation

\[
\omega_i = \varphi^{-1} \partial_i \varphi,
\]

where $\varphi$ is some mapping from $U$ to $G$. Parametrise $\varphi$ in the following way:

\[
\varphi = \xi \chi
\]

(3.2)

where $\chi$ is a mapping from $U$ to $O(n, \mathbb{R})$, and the mapping $\xi$ has the form

\[
\xi = e^{\psi_1 P_1} e^{\psi_2 P_2} \ldots e^{\psi_{n-1} P_{n-1}} e^{\psi_n P_n}.
\]

(3.3)

For the connection components $\omega_i$ one obtains the expression

\[
\omega_i = \frac{1}{2} \sum_{j,k,l=1}^n \chi_{ij} \partial_l \chi_{lk} M_{jk} + \sum_{j,l=1}^n \partial_l \psi_i \chi_{lj} P_j.
\]

(3.4)

From the comparison of (3.4) and (3.1) we see that

\[
\chi_{ij} = \frac{\partial_j \psi_i}{\sqrt{\sum_{l=1}^n (\partial_j \psi_l)^2}},
\]

(3.5)
and the functions $\psi_i$ satisfy the relation

$$\sum_{l=1}^{n} \partial_i \psi_l \partial_j \psi_l = 0, \quad i \neq j.$$  \hfill (3.6)

The functions $\beta_i$ are connected with the functions $\psi_i$ by the formula

$$\beta_i^2 = \sum_{l=1}^{n} (\partial_i \psi_l)^2,$$  \hfill (3.7)

and from the geometrical meaning of $\beta_i$ it follows that

$$\det(\partial_i \psi_j(a)) \neq 0.$$  \hfill (3.8)

Suppose now that a set of functions $\psi_i$ satisfies relations (3.6) and (3.8). Consider the mapping $\varphi$ defined by (3.2) with the mapping $\xi$ having form (3.3) and the mapping $\chi$ defined by (3.5). It can be shown that the mapping $\varphi$ generates the connection with the components of form (3.1). Here the functions $\beta_i$ are defined from (3.7), and the functions $\gamma_{ij}$ are given by (1.2). Thus, any set of functions $\psi_i$ satisfying (3.8) and (3.6) allows to construct a connection of form (3.1) satisfying the zero curvature condition which is equivalent to the Lamé equations, and in such a way we obtain the general solution.

Assuming that the functions $\psi_i$ satisfy (2.17), we have

$$\beta_i = \partial_i \psi_i.$$  \hfill (3.9)

It is clear that in this case $\gamma_{ij} = 0$. So one ends up with a trivial solution of the Lamé equations. To get nontrivial solutions one should consider different parametrisations of the mapping $\varphi$. For example, let us represent the mapping $\varphi$ in form (3.2) where the mapping $\chi$ again takes values in $O(n, \mathbb{R})$, while the mapping $\xi$ has the form

$$\xi = e^{\psi_1 M_{12}} e^{\psi_2 M_{23}} \ldots e^{\psi_{n-1} M_{n-1,n}} e^{\psi_n P_n}.$$  

With such a parametrisation of $\xi$, one gets

$$\omega_i = \frac{1}{2} \sum_{j,k,l=1}^{n} \chi_{lj} \partial_l \chi_{lk} M_{jk}$$

$$+ \sum_{j,k=1}^{n} \sum_{l=1}^{n-1} \partial_l \psi_l \sum_{m=l+1}^{n} \mu_{lm}(\psi) \chi_{lj} \chi_{mk} M_{jk} + \sum_{j,l=1}^{n} \partial_l \psi_l \nu_l(\psi) \chi_{lj} P_j,$$

where

$$\mu_{j-1,j}(\psi) = \cos \psi_j, \quad 1 < j < n; \quad \mu_{n-1,n}(\psi) = 1;$$  \hfill (3.10)

$$\mu_{jk}(\psi) = \left( \prod_{l=j+1}^{k-1} \sin \psi_l \right) \cos \psi_k, \quad 1 < j + 1 < k < n;$$  \hfill (3.11)

$$\mu_{jn}(\psi) = \prod_{l=j+1}^{n-1} \sin \psi_l, \quad 1 < j + 1 < n;$$  \hfill (3.12)

$$\nu_j(\psi) = \left( \prod_{k=j+1}^{n-1} \sin \psi_k \right) \psi_n, \quad 1 \leq j < n - 1; \quad \nu_{n-1}(\psi) = \psi_n; \quad \nu_n(\psi) = 1.$$  \hfill (3.13)
Using these relations we come to the following description of the general solution to the Lamé equations. Let functions $\psi_i$ satisfy the relations
\[
\sum_{l=1}^{n} \partial_i \psi_l \nu_l^2(\psi) \partial_j \psi_l = 0, \quad i \neq j,
\]
and for any point $p \in U$ one has
\[
\det(\partial_i \psi_j(p)) \neq 0, \quad \nu_i(\psi(p)) \neq 0.
\]
Then the functions $\beta_i$ determined from the equality
\[
\beta_i^2 = \sum_{l=1}^{n} (\partial_i \nu_l(\psi))^2,
\]
and the corresponding functions $\gamma_{ij}$ defined by (1.2) give the general solution of the Lamé equations. If the functions $\psi_i$ satisfy (2.17), we get the following expressions for the functions $\beta_i$
\[
\beta_i = \partial_i \psi_i \left( \prod_{j=i+1}^{n-1} \sin \psi_j \right) \psi_n, \quad 1 \leq i < n-1, \quad \beta_{n-1} = \partial_{n-1} \psi_{n-1} \psi_n, \quad \beta_n = \partial_n \psi_n. \quad (3.14)
\]

The geometrical interpretation of the obtained solutions to the Lamé equations is similar to one given in the previous section. Recall that solutions of the Lamé equations are associated with flat diagonal metrics in flat Riemannian spaces. The simplest case here is the standard metric in $\mathbb{R}^n$,
\[
g = \sum_{l=1}^{n} dz_l \otimes dz_l.
\]
Applying a diffeomorphism $\psi$ one gets the metric
\[
\psi^* g = \sum_{j,k,l=1}^{n} \partial_j \psi_l \partial_k \psi_l dz_j \otimes dz_k,
\]
which is diagonal if and only if the functions $\psi_i$ satisfy (3.6). Here the functions $\psi_i$ which obey (2.17) give the Lamé coefficients described by (3.9).

A more nontrivial example is provided by the metric arising after the transition to the spherical coordinates in $\mathbb{R}^n$. Denoting the standard coordinates on $\mathbb{R}^n$ by $x_i$ and the spherical coordinates by $r$ and $\theta_1, \ldots, \theta_{n-1}$, one has
\[
x_1 = r \prod_{k=1}^{n-1} \sin \theta_k, \quad x_i = r \cos \theta_{i-1} \prod_{k=i}^{n-1} \sin \theta_k, \quad 1 < i < n, \quad x_n = r \cos \theta_{n-1}.
\]

For the metric we obtain the expression
\[
g = r^2 \left[ \sum_{l=1}^{n-2} \left( \prod_{m=l+1}^{n-1} \sin^2 \theta_m \right) d\theta_l \otimes d\theta_l + d\theta_{n-1} \otimes d\theta_{n-1} \right] + dr \otimes dr.
\]
Denoting $z_i = \theta_i$, $i = 1, \ldots, n - 1$, and $z_n = r$, we rewrite this relation as
\[
g = z_n^2 \left[ \sum_{l=1}^{n-2} \left( \prod_{m=l+1}^{n-1} \sin^2 z_m \right) \, dz_l \otimes dz_l + \, dz_{n-1} \otimes dz_{n-1} \right] + \, dz_n \otimes dz_n.
\]
Therefore, the metric $g$ has form (2.19) with the functions $\nu_i$ given by (3.13). A diffeomorphism $\psi$ with the functions $\psi_i = \psi^* z_i$ satisfying (2.17) gives the metric with the Lamé coefficients (3.14).

In conclusion note that relations (2.20) describe the symmetry transformations not only of the Bourlet type equations, but also of the Lamé equations, and actually the existence of such transformations allows us to construct the solutions of the equations under consideration parametrised by $n$ arbitrary functions each depending on one variable.

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