Symmetry-protected topological phases and transition in a frustrated spin-$\frac{1}{2}$ XXZ chain

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Frustrated spin-$1/2$ XXZ zigzag chains are revisited in the light of symmetry-protected topological phases. Using a density-matrix renormalization group method for infinite systems, we identify projective representations for four time-reversal invariant gapped phases; two parity-symmetric dimer phases near the Heisenberg and XX limits and two parity-broken vector-chiral dimer phases in between. A small bond alternation in the nearest-neighbor exchange coupling induces a topological phase transition between the two vector-chiral dimer phases that belong to distinct nontrivial SPT phases. The observed entanglement entropy scaling supports the criticality of the $c = 1$ conformal field theory. A possible relevance to Rb$_2$Cu$_2$Mo$_3$O$_{12}$ and related systems is discussed.

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Topological orders and the quantum entanglement provide novel notions for classifying gapped quantum states beyond the conventional Landau theory \cite{1}. These notions are indispensable for distinguishing between gapped ground states of the same symmetry group that are not adiabatically connected. The entanglement remains short-range if the gapped ground state can be described as a direct (and thus unentangled) product of wavefunctions of finite-size blocks, and is long-range otherwise \cite{1}. Long-range entangled states may show nontrivial long-range topological orders either without any spontaneous symmetry breaking, as in Z$_2$ quantum spin liquids \cite{2}, or with a symmetry breaking, as in topological superconductors \cite{3}. Short-range entangled (SRE) states can be transformed into each other without closing the energy gap. However, this transformation may necessarily break a certain symmetry. Then, this symmetry protects a topological distinction between the two SRE states. Such phases are referred to as symmetry-protected topological (SPT) phases. Well-known examples include the Haldane phase \cite{4,5} of spin-1 chains having the time-reversal, dihedral, inversion symmetries and time-reversal invariant topological insulators \cite{3}. The topological structure of an SPT phase with a symmetry group $G$ is characterized by an algebra of the projective representation of $G$ for the SRE ground state, and can thus be classified according to the group cohomology \cite{3,10}. Some one-dimensional (1D) interacting cases including the Haldane spin chain \cite{6,7} and spin-1/2 ladders \cite{11} have been demonstrated numerically.

To the best of our knowledge, however, a topological transition between distinct nontrivial SPT phases has not been reported yet. This motivates us to study a simple yet more nontrivial case of a frustrated spin-$1/2$ chain \cite{12,13}:

$$\hat{\mathcal{H}} = J_1 \sum_i (1 - (-1)^i) \delta \left[ \hat{S}_i^x \hat{S}_{i+1}^x + \hat{S}_i^y \hat{S}_{i+1}^y + \Delta \hat{S}_i^z \hat{S}_{i+1}^z \right]$$

$$+ J_2 \sum_i \left[ \hat{S}_i^x \hat{S}_{i+2}^x + \hat{S}_i^y \hat{S}_{i+2}^y + \Delta \hat{S}_i^z \hat{S}_{i+2}^z \right],$$

with a spin-$1/2$ operator $\hat{S}_i$ at a site $i$, where $J_1 < 0$ and $J_2 > 0$ represent the nearest-neighbor ferromagnetic and second-neighbor antiferromagnetic exchange couplings, while $\delta$ and $\Delta$ represent the relative amplitude of the nearest-neighbor bond alternation and the XXZ-type easy-plane exchange anisotropy. Equation (1) with $\delta = 0$ provides a minimal model for understanding the emergence of a long-range order (LRO) of the vector spin chirality, $\langle \hat{\kappa}^z \rangle = \frac{1}{N} \sum_i \langle \hat{S}_i^z \times \hat{S}_{i+1}^z \rangle \neq 0$ with $N$ being the number of spins \cite{14,17}, and the associated ferroelectric polarization in various quasi-1D spin-1/2 cuprate Mott insulators \cite{16,18,23}. A vital role of nonzero $\delta$ \cite{24} has been proposed for a gapped vector-chiral (VC) dimer state without a quasi-LRO of a spin spiral, in accordance with experiments on Rb$_2$Cu$_2$Mo$_3$O$_{12}$ which has a weak crystallographic dimerization \cite{25,26}. This induces two pairs of time-reversal and translation invariant gapped phases with and without the inversion symmetry, each of which belong to the same symmetry group \cite{24}.

In this Letter, using the infinite-size density matrix renormalization group (iDMRG) \cite{27} method, we classify these four gapped phases of this $J_1$-$J_2$ frustrated spin-$1/2$ XXZ chain model in terms of SPT phases. We also analyze the criticality of an SPT transition between two VC dimer phases, which supports the conformal field theory (CFT) \cite{28} of the central charge $c = 1$.

The ground-state phase diagram of Eq. (1) was revealed numerically in a wide range of parameters $\Delta$ and $J_1/J_2$ for $\delta = 0$ \cite{16,17} and $\delta \neq 0$ \cite{21} and has also been reproduced by our present iDMRG calculations. In particular, the following distinct ground
other components $\langle \hat{D}^x \rangle = \langle \hat{D}^y \rangle$ are zero for $\delta = 0$ (gapless VC phase) [14, 16, 24], but are finite for $\delta \neq 0$ (critical VCD$_0$ state) [24]. For $\delta \neq 0$, the condition of $\langle \hat{D}^z \rangle = 0$ for the VCD$_0$ state is satisfied only at a single direct transition point between VCD$_\pm$ phases, although a possibility that it extends to a narrow gray hatched region in Fig. [1b] has not been ruled out.

v) Even-parity dimer (D$_-$) state — This has $\langle \hat{D}^x \rangle / \langle \hat{D}^z \rangle < 0$, while the vector spin chirality eventually vanishes, i.e., $\langle \hat{k}^z \rangle = 0$.

The D$_\pm$ phases belong to the same symmetry group $G$ as that of the Hamiltonian, $G_{2\text{c}}$, whose coset $H$ by $(U(1)/C_2)\tilde{z} \times T$ is given by $D_{2\text{h}} \times \{E, \Theta\}$ where $D_{2\text{h}} = D_2 \times \{E, I\}$. Henceforth, $T$ is a group of translations by integer multiples of two sites, $D_2$ is the dihedral group, $E$ is the identity, $I$ is the inversion about a bond center, and $\Theta$ is the time-reversal. The VCD$_\pm$ and VCD$_0$ also have a common symmetry group with the coset $H = C_{2\text{v}} \times \{E, \Theta\}$. Clearly, the $D_+\text{---VCD}_+$ and $D_-\text{---VCD}_-$ transitions are symmetry-breaking transitions, whose criticality is described with the $c = 1/2$ CFT [24]. In particular, it breaks the $I$ symmetry while preserving the mirror symmetry including the $z$ axis, e.g., $I R_{2\text{a}}$ with the $\pi$ rotation $R_{2\text{a}}$ about the $i$ axis. In contrast, the VCD$_+\text{---VCD}_-$ transition is not if it really occurs as a direct transition at the VCD$_0$ criticality. This VCD$_+\text{---VCD}_-$ transition has been probed not only by the sign of $\langle \hat{D}^x \rangle / \langle \hat{D}^z \rangle$ but also by the string order parameters [26, 27],

$$O^z_n = - \lim_{r \to \infty} \left( \langle \hat{S}^z_{2j+n} \rangle \hat{S}^z_{2j+n+1} \right) \exp(i\pi \sum_{k=2j+n+2}^{2(j+r)+n-1} \hat{S}^z_k) \times \left( \hat{S}^z_{2(j+r)+n} \hat{S}^z_{2(j+r)+n+1} \right) (n = 1, 2). \quad (2)$$

Only $\langle O^z_{2j+n} \rangle$ with a pair of sites $2j+n$ and $2j+n+1$ belonging to different dimer units (see Table I) and thus forming a strong (weak) bond becomes long-range in the D$_{2\text{h}}$ and VCD$_0$ phases, as shown in Fig. [1b] and in the previous work [24].

This change in the string order parameters is consistent with a change in the degeneracy of the lowest entanglement spectrum. In two rightmost columns of Table I we show the degeneracy $n_s$ ($n_w$) of the lowest bipartite entanglement spectrum, or in other words, that of the entanglement Hamiltonian $\mathcal{H}_s$ ($\mathcal{H}_w$) obtained through iDMRG calculations under the condition that the whole spin chain is divided at a strong (weak) bond: $n_u = 1$ and $n_w = 2$ for the D$_+$ and VCD$_+$ phases, while $n_u = 2$ and $n_w = 1$ for the D$_-$ and VCD$_-$ phases. This topological change occurring only at the VCD$_+\text{---VCD}_-$ transition indicates that the $D_\pm$ phases are not adiabatically connected and neither are the VCD$_\pm$ phases, as long as the symmetry of these phases is respected.

Nature of these gapped phases can be captured by classifying them as SPT phases, according to the 1D repre-
TABLE I: (Color online) Ten $\mathbb{Z}_2$ indices for the projective representation of $G_{2\mathbb{C}}$ in $D_+\mathbb{C} \mathbb{V}$, $\Lambda_{\mathbb{S}}\mathbb{C}$, and $\mathbb{V}^{\mathbb{C}}$ ground states, the degeneracy $n_s/n_w$, of the lowest entanglement spectrum $\mathbb{S}_0 = -\log \omega_0$ and the schematic picture of the ground state of $\mathbb{S}_s/\mathbb{S}_w$ when dividing the system at a stronger/weaker (left/right panel) bond. The emergence of $-1 \in \beta, \gamma$ and/or $\pm \omega$ points to a double topological degeneracy in the lowest entanglement spectrum. Orange, green and pink pairs indicate antisymmetric $\langle (| \uparrow \uparrow \rangle - | \downarrow \downarrow \rangle)/\sqrt{2}$, symmetric $\langle (| \uparrow \downarrow \rangle + | \downarrow \uparrow \rangle)/\sqrt{2}$ and mixed $\langle (| \uparrow \downarrow \rangle \uparrow) + e^{-i\theta/2} \downarrow | \downarrow \rangle\rangle/\sqrt{2}$ units of dimers which show $\langle D_j \rangle \langle D_j \rangle > 0$ and $\langle D_j \rangle \langle D_j \rangle < 0$, respectively. These parity symmetries are broken in pink pairs due to presence of vector-chiral order. The twofold Kramers degeneracy arising from the edge is denoted by a pair of black up and down arrows.

| Phase | $p$ | $\alpha(p)$ | $\beta(p)$ | $\gamma(p, h)$ | $\gamma(\Theta, h)$ | Degeneracy $n_s/n_w$ of the ground state of $\mathbb{S}_s/\mathbb{S}_w$ |
|-------|-----|-------------|-------------|-----------------|-----------------|-----------------|
| $D_+$ | $I$  | $-1$        | $+1$        | $+1$            | $\pm 1$         | $1$             |
| $D_-$ | $I$  | $+1$        | $-1$        | $+1$            | $\mp 1$         | $2$             |
| $\Lambda_{\mathbb{S}}\mathbb{C}$ | $IR_{2s}$ | $-1$ | $0$ | $+1$ | $\mp 1$ | $0$ |
| $\Lambda_{\mathbb{S}}\mathbb{C}$ | $IR_{2s}$ | $-1$ | $0$ | $+1$ | $\mp 1$ | $0$ |
| $\mathbb{V}^{\mathbb{C}}$ | $IR_{2s}$ | $-1$ | $0$ | $+1$ | $\mp 1$ | $1$ |

The two-fold Kramers degeneracy arising from the edge is denoted by a pair of black up and down arrows.

\[
\begin{align*}
\sum_{j j'} (T_1(h_p))_{i i'} (D_p)_{j j'} &= \alpha(p)(D_p)_{i i'}, \\
\sum_{j j'} (T_\Theta(h_\Theta))_{i i'} (D_\Theta)_{j j'} &= \alpha(\Theta)(D_\Theta)_{i i'}, \\
\sum_{j j'} (T(h))_{i i'} (\mathbb{U}_h)_{j j'} &= \alpha(h)(\mathbb{U}_h)_{i i'}, \\
\beta(p) &= \text{Tr}[D_p (D_p^{\dagger})^{-1}]/m, \\
\beta(\Theta) &= \text{Tr}[D_\Theta D_\Theta^{\dagger}]/m, \\
\gamma(p, h) &= \text{Tr}[\mathbb{U}_h \mathbb{D}_p \mathbb{D}_p^{\dagger}]/m, \\
\gamma(\Theta, h) &= \text{Tr}[\mathbb{U}_h \mathbb{D}_\Theta \mathbb{D}_\Theta^{\dagger}]/m, \\
\omega(h', h) &= \text{Tr}[\mathbb{U}_h \mathbb{U}_h \mathbb{U}_h^{\dagger}]/m,
\end{align*}
\]

where $h$ is taken from a minimal set of generators of the local unitary subgroup $H_{LU}$ of the whole symmetry group $G$, $p = h_p$ is a direct product of the inversion $I$ and $h_p = E$ or $R_{2s} \in H_{LU}$, $\Theta = h_\Theta K$ is a direct product of the complex conjugate operator $K$ and $h_\Theta = R_{2y} \in H_{LU}$. We also have introduced transfer matrices for a unit cell including two spins
\[
\begin{align*}
(T_1(h))_{ii'j j'} &= \sum_{s_1s_2s'_1s'_2} (A^{s_1s_2})_{ij}(U_h)_{s_1s_2s'_1s'_2}(A^{s'_1s'_2})_{i'j'}, \\
(T_\Theta(h))_{ii'j j'} &= \sum_{s_1s_2s'_1s'_2} (A^{s_1s_2})_{ij}(U_h)_{s_1s_2s'_1s'_2}(A^{s'_1s'_2})_{i'j'}, \\
(T(h))_{ii'j j'} &= \sum_{s_1s_2s'_1s'_2} (A^{s_1s_2})_{ij}(U_h)_{s_1s_2s'_1s'_2}(A^{s'_1s'_2})_{i'j'},
\end{align*}
\]

where the $m \times m$ matrix $A^{s_1s_2}$ represents in the Schmidt bases the state within the translation unit having the two-spin degrees of freedom, $(s_1, s_2)$, in the translation-invariant matrix product state (MPS) $\sum_{s_1s_2} (A^{s_1s_2})_{ij}(s_1s_2) \otimes | \Psi \rangle_j$ of the entanglement Hamiltonian, satisfying the orthonormal condition $\langle \Psi | \Psi \rangle_j = \delta_{ij}$. Right eigenvectors of transfer matrices in Eqs. (3), (4), and (5) are the representation matrices of $I$, $\Theta$, and $h$, respectively, in the Schmidt bases. (See Supplementary materials.) The arbitrary phases of $\mathbb{U}_{R_{2s}}$ and $\mathbb{U}_{R_{2s}}$ are fixed by $\mathbb{U}_{R_{2s}}^2 = \mathbb{U}_{R_{2s}} = \mathbb{I}$. Note that for $\Theta$-invariant states, $\alpha(h) = 1$, $\alpha(\Theta)$ just takes arbitrary $U(1)$ phase depending on that of $A^{s_1s_2}$ and thus is not important. The results are summarized in Table I. Because of the unbroken $U(1)_z$ symmetry, the 1D representation $\alpha(R_{2z}) = 1$ leading to $R_{2z}$-even states is rather obvious in all the phases shown in Table I and thus is not particularly mentioned below.

From two 1D representations $\alpha(I)$ and $\alpha(R_{2z})$, the $D_+$ ground state of the whole spin chain is $I$-odd and $R_{2z}$-even. All the other $\mathbb{Z}_2$ indices take the same value; $\beta(I) = \beta(\Theta) = \gamma(I, h) = \gamma(\Theta, h) = \omega(R_{2z}, R_{2z}) = +(-)1$ with $h = R_{2z}, R_{2z}$ if the spin chain is cut at a strong (weak) bond. This is consistent with the nondegeneracy $n_s = 1$ and the twofold degeneracy $n_w = 2$ in
the entanglement spectrum, and indicates that this SPT phase is protected by \( I, \Theta, \) and \( D_2 \) symmetries \(^2\). This phase has the same \( Z_2 \) indices as the Affleck-Kennedy-Lieb-Tasaki (AKLT) state \(^3\). The \( D_2 \) phase is \( I \)-even (\( \alpha(I) = +1 \)) and \( R_{2x} \)-odd (\( \alpha(R_{2x}) = -1 \)). Whichever bond the spin chain is cut at, \( \beta(I) = \gamma(I, R_{2x}) = +1 \), indicating that the \( I \) symmetry no longer protects the topological degeneracy. All the other indices take the same value; \( \beta(\Theta) = \gamma(I, R_{2x}) = \gamma(\Theta, h) = \omega(R_{2x}, R_{2z}) = -(+1) \) if the spin chain is cut at a strong (weak) bond. This is consistent with \( n_x = 2 \) and \( n_w = 1 \), and indicates that this SPT phase is protected by \( \Theta \) and \( D_2 \) symmetries. This phase has the same \( Z_2 \) indices as a director product of the even-parity dimer state, \( (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)/\sqrt{2} \(^4\). Let us proceed to the VCD\( \pm \) phases. These states respect the \( IR_{2x} \) symmetry and are \( IR_{2x} \)-odd, while they break the \( I \) and \( R_{2x} \) symmetries, as seen from \( \alpha(IR_{2x}) = -1 \) and \( \alpha(R_{2x}) = 0 \). (Note that the \( D_\pm \) states are also \( IR_{2x} \)-odd as \( \alpha(IR_{2x}) = \alpha(I)\alpha(R_{2x}) = -1 \).) The VCD\( \pm \) states have the same topological degeneracy in the entanglement spectrum as \( D_\pm \), respectively, but they are no longer protected by the \( I \) and \( D_2 \) symmetry, and not even by the \( IR_{2x} \) symmetry since \( \beta(I, R_{2x}) = \gamma(I, R_{2x}) = +1 \) always holds. The sign of \( \beta(\Theta) = \gamma(\Theta, R_{2x}) \) depends on the way of dividing the spin chain and are opposite between the VCD\( + \) an VCD\( - \) phases, and the minus sign appears when the entanglement spectrum is twofold degenerate. Hence, VCD\( \pm \) phases are classified into distinct SPT phases, whose distinction is protected by the \( \Theta \) symmetry. Indeed, once, the Néel LRO is realized in addition to the VCD orders, the \( \Theta \) symmetry is broken \(^2\) and the topological degeneracy is lost completely (Table I).

The nature of this VCD\( + \)—VCD\( - \) SPT phase transition can be understood from the entanglement entropy as a function of the correlation length \( \xi^{-1} = -\log(|w_1/w_0|) \), where \( w_n \) is the \((n + 1)\)th-largest (in terms of absolute value) eigenvalue of the transfer matrix. Figure 2 (a) shows the dependence of \( \xi \) on the dimension of the Schmidt bases in the vicinity of VCD\( + \)—VCD\( - \) transition. This indicates the strongest enhancement of \( \xi \) at \( \Delta = 0.88 \), suggesting that it is close to the criticality in reasonable agreement with the sign change of \( \langle \hat{D}^z \rangle \) at \( \Delta = 0.87 \pm 0.001 \). The scaling behavior of the entanglement entropy versus \( \xi \) in the form of \( S = \xi \log \xi + \text{const.} \) shown in Fig. 2 (b) is consistent, within the numerical accuracy, with the \( c = 1 \) CFT \(^2\).

Since the model parameters are at least close to those for the spin-gapped spin-1/2 chain compound Rb\(_2\)Cu\(_4\)Mo\(_4\)O\(_{12}\) \(^2\) it would be possible to experimentally find these SPT phases and the SPT transition by probing a gap closing under physical and/or chemical pressure.

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FIG. 2: (Color online) Scaling of the entanglement entropy \( S \) as a function of \( m \) (a) and correlation length \( \xi \) (b) in the vicinity of the VCD\( + \)—VCD\( - \) phase boundary. The solid line represents the \( c = 1 \) line.
The AKLT state on the $S = 1/2$ chain is described by a product state of the following translation-unit matrix $A_w^{(s_1 s_2)}$ or $A_u^{(s_1 s_2)}$ in the Schmidt bases obtained by dividing the whole system at a weak or strong $J_1$ bond; $A_w^{(\uparrow \downarrow)} = \sqrt{\frac{1}{2}} \sigma^x$, $A_w^{(\downarrow \uparrow)} = -\sqrt{\frac{1}{2}} \sigma^x$ and $A_w^{(\uparrow \uparrow)} = A_w^{(\downarrow \downarrow)} = -\sqrt{\frac{1}{2}} \sigma^x$. $A_u^{(s_1 s_2)}$ is readily obtained by applying the singular value decomposition to $A_w^{(s_1 s_2)}$; using singular vectors and values in $(A_w^{(s_1 s_2)})_{\alpha_1 \alpha_2} = \sum_{\beta_1} X_{\alpha_1 \beta_1, \beta} W_{\beta, \beta} Y_{\beta_2, \beta_2}$, we can take $(A_u^{(s_1 s_2)})_{\beta_1 \beta_2} = \sum_{\alpha} W_{\beta_1, \alpha} Y_{\beta_1, \alpha} X_{\alpha_1, \alpha_2 \beta_2}.$

$A_u^{(s_1 s_2)}$ of the direct product state of $(|\uparrow \downarrow\rangle + |\downarrow \uparrow\rangle)/\sqrt{2}$ can be written as $A_u^{(\uparrow \downarrow)} = A_u^{(\downarrow \uparrow)} = (0)$, $A_u^{(\uparrow \uparrow)} = A_u^{(\downarrow \downarrow)} = \left(\begin{smallmatrix}1\end{smallmatrix}\right)$.

$A_u^{(s_1 s_2)}$ of this state can be obtained in the same way as explained in [27]. These $A_u^{(s_1 s_2)}$ lead to the same $Z_2$ indices as in the D$_+$ state. $A_u^{(s_1 s_2)}$ of the direct product state of $(|\uparrow \downarrow\rangle - |\downarrow \uparrow\rangle)/\sqrt{2}$, namely, the Majumdar-Ghosh state [39], can be obtained by rotating every other spins about the $z$-axis by $\pi$. This resulting $Z_2$ indices are the same as in the D$_+$ state, except with the interchange of "w" and "u".

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Supplementary materials for “Symmetry-protected topological phases and transition in a frustrated spin-½ XXZ chain”

We provide details of a correspondence between the right eigenvector of the transfer matrix in Eq. (6) and the representation matrix in the Schmidt bases. Schmidt bases of a finitely correlated ground state for a uniform one-dimensional system in the thermodynamic limit can be represented by an infinite matrix product state with sufficiently large dimension $m$:

$$|\Psi\rangle_i = \sum_{\{s_k\}} \left( \prod_{k=1}^{+\infty} A^{(s_k)} v \right) |\{s_k\}\rangle,$$

(A.13)

where $A^{(s)}$ and $v$ are an $m$-dimensional matrix and vector. The gauge of $A^{(s)}$ can be chosen to be $\sum_s A^{(s)} A^{(s)} = 1$. The vector $v$ is determined satisfying the orthonormal condition, $i\langle\Psi|\Psi\rangle_j = \delta_{ij}$. The bases have a translation symmetry given by $|\Psi\rangle_i = \sum_s (A^{(s)})_{ij} |s\rangle \otimes |\Psi\rangle_j$.

Let’s consider a representation matrix of a local unitary operation, $\hat{U} = \prod_{k=1}^{+\infty} \sum_{s_k s_k'} (U_h)_{s_k s_k'} |s_k\rangle \langle s_k'|$, in the Schmidt bases. In other words, we calculate $(U_h)_{ii'} = i\langle\Psi|\hat{U}|\Psi\rangle_{i'}$:

$$i\langle\Psi|\hat{U}|\Psi\rangle_{i'} = \sum_{jj'} \left( \prod_{k=1}^{+\infty} \left( \sum_{s_k s_k'} (U_h)_{s_k s_k'} A^{(s_k)} \otimes A^{(s_k')} \right) \right)_{ii'jj'} (v^*)_j (v)_j',$$

where the definition of the transfer matrix $T(h)$ is given by Eq. (12). If the ground state is invariant under the unitary operation and is not a cat state, the norm of dominant eigenvalue $\alpha(h)$ of the transfer matrix $T(h)$ becomes unity and unique. In this case, $\prod_{k=1}^{+\infty} T(h)$ can be decomposed as $u_h (\prod_{k=1}^{+\infty} \alpha(h)) v_h$, where $u_h$ ($v_h$) is the right (left) eigenvector of $T(h)$ corresponding to $\alpha(h)$. Using this relation, we obtain

$$i\langle\Psi|\hat{U}|\Psi\rangle_{i'} = (u_h)_{ii'} \left( \prod_{k=1}^{+\infty} \alpha(h) \right) \left[ \sum_{jj'} (v_h^*)_{jj'} (v^*)_j (v)_j' \right] = e^{i\phi} (u_h)_{ii'},$$

(A.15)

with $e^{i\phi} = \left( \prod_{k=1}^{+\infty} \alpha(h) \right) \left[ \sum_{jj'} (v_h^*)_{jj'} (v^*)_j (v)_j' \right]$. This phase can be removed by redefining of $u_h$ and $v_h$, because there is an arbitrary property in the biorthogonal condition of $v_h^* u_h = 1$. Thus, we can obtain the representation matrix by reshaping the right eigenvector, as $(U_h)_{ii'} = (u_h)_{ii'}$. 

