On refined ramification filtrations in the equal characteristic case

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Abstract
Let \( k \) be a complete discretely valued field of equal characteristic \( p > 0 \). We introduce the differential refined conductors for a representation of the Galois group \( G_k \) with finite local monodromy. we prove that the differential refined Swan conductors coincide with the ones defined by Saito. Also, we study its relation with the toroidal variation of Swan conductors.

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Introduction

The ramification theory for a complete discretely valued field $k$ with possibly imperfect residue field $\kappa$ is initiated by Kato [Kat89], where he used the étale cohomology and some Milnor $K$-theory to study the ramification of a character of the Galois group $G_k$. Abbes and Saito [AS02, AS03] extended Kato’s work to the general case, by providing $G_k$ ramification filtrations $\text{Fil}^a G_k$ and $\text{Fil}^\log G_k$ satisfying certain properties. Later, Saito [Sai09] defined refined Swan conductors for log-ramification filtrations as a homomorphism for $a \in \mathbb{Q}_{>0}$.

\[ \text{rsw} : \text{Hom}(\text{Fil}^a \log G_k/\text{Fil}^{a+} G_k, \mathbb{F}_p) \to \Omega^1_{\mathcal{O}_k}(\log) \otimes_{\mathcal{O}_k} \pi^{-a} \kappa, \]

where $\mathcal{O}_k$ and $\pi_k$ are ring of integers and a uniformizer of $k$, and $\Omega^1_{\mathcal{O}_k}(\log)$ is the logarithmic differential. This provides some further information about the ramification groups.

Along a different path, Christol, Dwork, Matsuda, Mebkhout, and their collaborators gave an interpretation of classical Swan conductors using $p$-adic differential modules. Kedlaya [Ked07] generalized this approach to the case when the residue field may be imperfect. The author [Xia11] verified that this definition coincides with Abbes and Saito’s definition. An important consequence of this result is the Hasse-Arf theorem for the (log-)ramification filtrations [Xia11] Theorem 4.4.1.

In this paper, we give an interpretation of the refined Swan conductors using $p$-adic differential modules, as well as introducing the nonlog counterpart. We now describe the basic idea of the definition. For simplicity, we assume that $\kappa$ has a finite $p$-basis $\bar{b}_1, \ldots, \bar{b}_m$. Let $K$ be the fraction field of the Cohen ring of $\kappa$ with respect to $\bar{b}_1, \ldots, \bar{b}_m$; let $b_1, \ldots, b_m$ denote the canonical lifts of the $p$-basis. Let $A^1_K(\eta_0, 1)$ be the annulus over $K$ with radii in $(\eta_0, 1)$ for some $\eta_0 \in (0, 1)$ and with coordinate $T$. By standard theory of $p$-adic differential modules, a finite image $p$-adic representation of $G_k$ can be converted into a differential module $\mathcal{E}$ over $A^1_K(\eta_0, 1)$ for differential operators $\partial_0 = \partial/\partial T$ and $\partial_1 = \partial/\partial B_1, \ldots, \partial_m = \partial/\partial B_m$, where $B_1, \ldots, B_m$ are liftings of $b_1, \ldots, b_m$. Let $\pi = -p^{1/(p-1)}$ denote a Dwork pi and let $K' = K(\pi)$. When $\rho$ has pure ramification break $b$, i.e. when $\rho(\text{Fil}^b G_k)$ is trivial, one expects the following picture: there exists a basis of $\mathcal{E} \otimes K'$, on which the differential operators act as

\[ \partial_0 = \pi T^{-b} N_0, \quad \partial_1 = \pi T^{-b} N_1, \quad \ldots, \partial_m = \pi T^{-b} N_m, \]

where $N_0, \ldots, N_m$ are matrices in $\mathcal{O}_{K'}[T]$. Modulo $(\pi, T)$, these matrices commute. If we use $\{\theta_{i,j}, \ldots, \theta_{d,j}\}_{j=0, \ldots, m}$ to denote the common eigenvalues of the reductions of $N_j$, viewed as elements in $\kappa_{\text{alg}}$, the refined Swan conductors of $\rho$ is defined to be $\{\pi^{-b}(\theta_{i,0} \frac{d \bar{b}_k}{\pi_k} + \theta_{i,1} \bar{b}_1 + \cdots + \theta_{i,m} \bar{b}_m)\}_{i=1, \ldots, d} \subset \Omega^1_{\mathcal{O}_k}(\log) \otimes \pi^{-b} \kappa_{\text{alg}}$. Of course, this might be too good to be true as a result over the annulus $A^1_{K'}(\eta_0, 1)$. In practise, we need the following two technical arguments to "read off" the refined Swan conductors.

(a) The above picture can be better described over a field. Namely, we first define the refined radii, an analog of refined Swan conductors, for differential modules over a field, and then we study how refined radii vary over a one-dimensional space. (From this, we may not be able to conclude what $\partial_j$ looks like over the annulus, but we know what it looks like when tensored with the completion of $K'(T)$ with respect to any $\eta$-Gauss norm for $\eta \in (\eta_0, 1)$.)

(b) When the spectral norms of the differential operators are smaller than their operator norms over the base field, we may not be able to read off the refined radii directly. Instead, we
need to use Frobenius antecedents. (In practise, we start off defining refined radii using an equivalent way and show that it is well-behaved when taking Frobenius antecedents.)

We remark that there is a non-logarithmic analog of the refined Swan conductors, which leads to refined Artin conductors.

Part of the content of differential refined Swan conductors is already included in the author’s thesis [X-Thesis]. However, we feel it fits better the content of this paper. Also, we fill in some gaps in [X-Thesis].

To compare this differential refined Swan conductors with the ones defined by Saito, we essentially reduce to a computation of Dwork isocrystals, in which case, both refined Swan conductors can be computed explicitly.

We also remark that when \( k \) is an \( n \)-dimensional higher local field of characteristic \( p \), the refined conductors induce ramification filtrations on \( G_k \) indexed by \( \mathbb{Q}^n \) with lexicographic order. This is compatible with a filtration on the Milnor \( K \)-groups via Kato’s class field theory.

At last, we study the relation of refined Swan conductors with the variation of intrinsic radii over a polyannulus. We prove that the valuation of the refined Swan conductors at a vertex of the polygon associated to the polyannulus encodes some information about the slopes of the log-affine function of the intrinsic radii at that vertex. For the precise statement, please consult Proposition 4.3.13.

Plan of the paper

Section 1 is devoted to develop the theory of differential refined radii, the analog of refined conductors over fields. In the first two subsections, we set up notation and recall some basic result of differential modules from [KX10]. We define the refined radii in Subsection 1.3 and prove the decomposition Theorem 1.3.26. In Subsection 1.4 we consider the case where we allow multiple derivations to interact. In Subsection 1.5 we study how the refined radii vary on an annulus or a disc, when the radii is a log-affine function. Then we deduce the refined conductors for solvable differential modules over annulus in Subsection 1.6.

In Section 2, we apply the theory of refined conductors for solvable differential modules to define the refined conductors for Galois representations. The first two subsections recall the construction of differential modules following [Ked07], and deduce some basic properties. In subsection 2.3 we define the homomorphism of refined conductors. We also insert Subsection 2.4 to briefly discuss the application to the higher local fields.

In Section 3, we compare our definition to that of Saito. In Subsection 3.1 we review Saito’s definition, to the level that we can proceed with the proof. In Subsection 3.2 we lift the Abbes-Saito spaces over \( k \) to spaces over \( K \). In Subsection 3.3 we do a crucial calculation of Dwork isocystals to determine their refined radii, which is the heart of the comparison theorem. We finally wrap up to prove the comparison Theorem 3.4.1 in Subsection 3.4.

In Section 4, we focus on the role of refined Swan conductors in the toroidal variation of Swan conductors. A few technical lemmas are discussed in Subsection 4.2, and the main theorems are proved in Subsection 4.3.
1 Theory of differential modules

1.1 Setup

We first set up some notation.

Notation 1.1.1. By a multiset $S$, we mean a set where we allow elements to have multiplicity. For $s \in S$, the multiplicity of $s$ in $S$ is denoted by $\text{multi}_s(S)$. When $S$ consists of a single element (with multiplicity), we call it pure.

Notation 1.1.2. For a field, we fix an algebraic closure $K^{\text{alg}}$ and let $K^{\text{sep}}$ denote the separable closure inside $K^{\text{alg}}$. Denote $G_K = \text{Gal}(K^{\text{sep}}/K)$. For a finite Galois extension $L/K$ (inside $K^{\text{sep}}$), we denote the Galois group by $G_{L/K} = \text{Gal}(L/K)$.

Notation 1.1.3. By a nonarchimedean field, we mean a field $K$ equipped with a nonarchimedean norm $\| \cdot \| = \| \cdot \|_K : K^\times \to \mathbb{R}_+^\times$. A subring of $K$ (with the induced norm and topology) is called a nonarchimedean ring.

For $K$ a nonarchimedean field, denote the ring of integers and the maximal ideal by $\mathcal{O}_K = \{ x \in K \mid |x| \leq 1 \}$ and $\mathfrak{m}_K = \{ x \in K \mid |x| < 1 \}$, respectively; the residue field of $K$ is denoted by $\kappa_K = \mathcal{O}_K/\mathfrak{m}_K$. We reserve the letter $p$ for the characteristic of $\kappa_K$. If $\text{char} \kappa_K = p > 0$ and $\text{char} K = 0$, we normalize the norm on $K$ so that $|p| = 1/p$. For an element $a \in \mathcal{O}_K$, we denote its reduction in $\kappa_K$ by $\bar{a}$. In case $K$ is discretely valued, let $\pi_K$ denote a uniformizer of $\mathcal{O}_K$ and let $v_K(\cdot)$ be the corresponding valuation on $K$, normalized so that $v_K(\pi_K) = 1$.

For a nonarchimedean field $K$ and $s \in \mathbb{R}$, we set

$$m_K^{(s)} = \{ x \in K \mid |x| \leq e^{-s} \}, \quad m_K^{(s)+} = \{ x \in K \mid |x| < e^{-s} \}, \quad \kappa_K^{(s)} = m_K^{(s)}/m_K^{(s)+}.$$ 

If $s \in -\log|K^\times|$, we have a non-canonical isomorphism $\kappa_K \simeq \kappa_K^{(s)}$. For $a \in K$ with $|a| \leq e^{-s}$, we sometimes denote its image in $\kappa_K^{(s)}$ by $\bar{a}^{(s)}$. In particular, $\kappa_K^{(0)} = \kappa_K$ and $\bar{a}^{(0)} = \bar{a}$ if $v(a) \geq 0$.

Notation 1.1.4. Let $J$ be an index set. We write $e_J$ for a tuple $(e_j)_{j \in J}$. For another tuple $u_J$, write $u_J^{e_J} = \prod_{j \in J} u_j^{e_j}$, if all but finitely many $e_j = 0$. We also use $\sum_{e_J = 0}$ to mean the sum over $e_j \in \{0, 1, \ldots, n\}$ for each $j \in J$ and $e_j \neq 0$ for only finitely many $j$; for notational simplicity, we may suppress the range of the summation when it is clear. If $J$ is finite, write $|e_J| = \sum_{j \in J} |e_j|$ and $(e_J)!$ for $\prod_{j \in J} (e_j)!$.

Convention 1.1.5. Throughout this paper, all derivations on topological modules will be assumed to be continuous; in particular, $\Omega_R^1$ will denote the continuous differentials on the topological ring $R$. We may suppress the base ring from the module of continuous differentials when it is $\mathbb{F}_p$, $\mathbb{Z}$ or $\mathbb{Z}_p$. Moreover, all derivations on nonarchimedean rings will be assumed to be bounded (i.e., to have bounded operator norms). All connections considered will be assumed to be integrable.
Notation 1.1.6. For a matrix $A = (A_{ij})$ with coefficients in a nonarchimedean ring, we use $|A|$ to denote the supremum norm over entries.

Hypothesis 1.1.7. For the rest of this subsection, we assume that $K$ is a complete nonarchimedean field.

Notation 1.1.8. Let $I \subset [0, +\infty)$ be an interval and let $n \in \mathbb{N}$. Let

$$A^n_K(I) = \{(x_1, \ldots, x_n) \in K^n \mid |x_i| \in I \text{ for } i = 1, \ldots, n\}$$

denote the polyannulus of dimension $n$ with radii in $I$. (We do not impose any rationality condition on the endpoints of $I$, so this space should be viewed as an analytic space in the sense of Berkovich [Berk90].) If $I$ is written explicitly in terms of its endpoints (e.g., $[\alpha, \beta]$), we suppress the parentheses around $I$ (e.g., $A^n_K(\alpha, \beta]$).

Notation 1.1.9. Let $0 \leq \alpha \leq \beta < +\infty$. We define

$$K^{(\alpha/t, t/\beta)} = \left\{ \sum_{n \in \mathbb{Z}} a_n t^n \mid |a_n|^{\eta n} \to 0 \text{ as } n \to +\infty, \text{ for any } \eta \in [\alpha, \beta] \right\},$$

$$K^{(\alpha, t/\beta]} = \left\{ \sum_{n \in \mathbb{Z}} a_n t^n \mid |a_n|^{\eta n} \to 0 \text{ as } n \to +\infty, \text{ for any } \eta \in [\alpha, \beta] \right\},$$

$$K^{\{\alpha/t, t/\beta\}}_0 = \left\{ \sum_{n \in \mathbb{Z}} a_n t^n \mid |a_n|^{\eta n} \to 0 \text{ and } |a_n|^{\beta n} \text{ is bounded, as } n \to +\infty, \text{ for any } \eta \in (\alpha, \beta) \right\}.$$

When $\alpha = 0$, we simply use $K^{(t/\beta)}$ and $K^{\{t/\beta\}}$ to denote the first two rings above, respectively. Also, we write

$$K^{[t/\beta]}_0 = \left\{ \sum_{n=0}^{\infty} a_n t^n \mid |a_n|^{\beta n} \text{ is bounded as } n \to \infty \right\}.$$

Put $I = \{1, \ldots, n\}$. For a nonarchimedean ring $R$, we use $R^{(u_I)}$ to denote the Tate algebra, consisting of formal power series $\sum_{e_I \in \mathbb{Z}_{\geq 0}} a_{e_I} u_I^{e_I}$ with $a_{e_I} \in R$ and $|a_{e_I}| \to 0$ as $|e_I| \to +\infty$. For $(\eta_i)_{i \in I} \in (0, +\infty)^n$, the $\eta_I$-Gauss norm on $R^{(u_I)}$ is the norm $| \cdot |_{\eta_I}$ given by

$$\left| \sum_{e_I} a_{e_I} u_I^{e_I} \right|_{\eta_I} = \max_{e_I} \left\{ |a_{e_I}| \cdot \eta_I^{e_I} \right\};$$

this norm extends uniquely to $\text{Frac}(R^{(u_I)})$, and (if $|\eta_i| \leq 1$ for any $i \in I$) to $R^{(u_I)}$.

For $\eta \in [\alpha, \beta]$ and $\eta \neq 0$, the $\eta$-Gauss norm on $K^{[t]}$ extends to $K^{(\alpha/t, t/\beta)}$, (if $\eta \neq \beta$) $K^{(\alpha, t/\beta]}$, (if $\eta \neq \alpha$) $K^{(\alpha/t, t/\beta)}_0$, and $K^{[t/\beta]}_0$.

1.2 Differential modules and radii of convergence

In this subsection, we review the theory of differential modules over a field, following [KX10, Section 1]. We will focus on the case when the differential operators are of rational type.

Definition 1.2.1. Let $K$ be a ring equipped with a derivation $\partial$. Let $K\{T\}$ denote the (noncommutative) ring of twisted polynomials over $K$ [Ore33]; its elements are finite formal sums $\sum_{i \geq 0} a_i T^i$ with $a_i \in K$, multiplied according to the rule $Ta = aT + \partial(a)$ for $a \in K$. 


A \( \partial \)-differential module over \( K \) is a finite projective \( K \)-module \( V \) equipped with an action of \( \partial \) (subject to the Leibniz rule); any \( \partial \)-differential module over \( K \) inherits a left action of \( K\{T\} \) where \( T \) acts via \( \partial \). The rank of \( V \) is the rank of \( V \) as a \( K \)-module. The module dual \( V' = \text{Hom}_K(V,K) \) of \( V \) may be viewed as a \( \partial \)-differential module by setting \( (\partial f)(v) = \partial(f(v)) - f(\partial(v)) \). We say \( V \) is free if \( V \) as a module is free over \( K \).

For \( V \) a \( \partial \)-differential module free of rank \( d \) over \( K \), we say \( v \in V \) is a cyclic vector if \( v, \partial v, \ldots, \partial^{d-1}v \) form a basis of \( V \). A cyclic vector defines an isomorphism \( V \cong K\{T\}/K\{T\}P \) of \( \partial \)-differential modules for some twisted polynomial \( P \in K\{T\} \) of degree \( d \), where the \( \partial \)-action on \( K\{T\}/K\{T\}P \) is the left multiplication by \( T \). If \( K \) is a differential field of characteristic 0, \( V \) always has a cyclic vector (See, e.g., \cite[Theorem III.4.2]{DGS94} or \cite[Theorem 5.4.2]{Ked10}.)

For a \( \partial \)-differential module \( V \), we write \( H^0_\partial(V) = \text{Ker} \partial \).

**Hypothesis 1.2.2.** For the rest of this subsection, we assume that \( K \) is a complete nonarchimedean field of characteristic zero, equipped with a derivation \( \partial \) with operator norm \( |\partial|_K < \infty \), and that \( V \) is a nonzero \( \partial \)-differential module over \( K \).

**Definition 1.2.3.** Let \( p \) denote the residual characteristic of \( K \); we conventionally write

\[
\omega = \begin{cases} 
1 & p = 0 \\
 p^{-1/(p-1)} & p > 0 .
\end{cases}
\]

The spectral norm of \( \partial \) on \( V \) is defined to be \( |\partial|_{sp,V} = \lim_{n \to \infty} |\partial^n|_V^{1/n} \) for any fixed \( K \)-compatible norm \( |\cdot|_V \) on \( V \). Define the generic \( \partial \)-radius of \( V \) to be \( R_\partial(V) = \omega|\partial|_{sp,V}^{-1} \); note that \( R_\partial(V) > 0 \). Let \( V_1, \ldots, V_d \) be the Jordan-Hölder constituents of \( V \) as \( K\{T\} \)-modules. We define the (extrinsic) subsidiary \( \partial \)-radii, to be the multiset \( \mathfrak{R}_\partial(V) \) consisting of \( R_\partial(V_i) \) with multiplicity \( \dim V_i \) for \( i = 1, \ldots, d \). Let \( R_\partial(V; 1) \leq \cdots \leq R_\partial(V; \dim V) \) denote the elements of \( \mathfrak{R}_\partial(V) \) in increasing order. We say that \( V \) has pure \( \partial \)-radii if \( \mathfrak{R}_\partial(V) \) consists of \( \dim V \) copies of \( R_\partial(V) \).

**Definition 1.2.4.** Let \( R \) be a complete \( K \)-algebra and let \( V \) be a \( \partial \)-differential module over \( K \). For \( v \in V \) and \( x \in R \), define the \( \partial \)-Taylor series to be

\[
\mathbb{T}(v; \partial; x) = \sum_{n=0}^{\infty} \frac{\partial^n(v)}{n!} x^n \in V \otimes_K R, \quad (1.2.5)
\]

in case this series converges. When \( V = K \), \( (1.2.5) \) gives a homomorphism \( K \to R \) of rings, if it converges. For general \( V \), \( (1.2.5) \) gives a homomorphism of \( K \)-modules \( V \to V \otimes_K R \) respecting the aforementioned ring homomorphism, if both homomorphisms converge.

**Lemma 1.2.6.** Let \( V, V_1, V_2 \) be nonzero \( \partial \)-differential modules over \( K \).

(a) For \( 0 \to V_1 \to V \to V_2 \to 0 \) exact, we have \( \mathfrak{R}_\partial(V) = \mathfrak{R}_\partial(V_1) \cup \mathfrak{R}_\partial(V_2) \).

(b) We have \( \mathfrak{R}_\partial(V^\vee) = \mathfrak{R}_\partial(V) \).

(c) We have \( R_\partial(V_1 \otimes V_2) \geq \min \{ R_\partial(V_1), R_\partial(V_2) \} \). If \( V_1 \) is irreducible and \( R_\partial(V_1) < R_\partial(V_2) \), then \( V_1 \otimes V_2 \) has pure \( \partial \)-radii \( R_\partial(V_1) \).

(d) Let \( f : K \to K\{\partial\} \) be the homomorphism given by \( f(x) = \mathbb{T}(x; \partial, T) \). Then \( f^*V = V \otimes_K fK\{\partial\} \) is a \( \partial \)-differential module over \( K\{\partial\} \). For \( r \in (0, R_\partial(K)) \), \( R_\partial(V) \geq r \) if and only if \( f^*(V) \) restricts to a trivial \( \partial \)-differential module over \( A_K^1[0, r] \).
Proof. As in [Ked10] Lemma 6.2.8, [Ked10] Corollary 6.2.9, and [KX10] Proposition 1.2.14. □

**Definition 1.2.7.** For $P(T) = \sum a_i T^i \in K[T]$ or $K \{T\}$ a nonzero (twisted) polynomial, define the Newton polygon of $P$ as the lower convex hull of the set $\{(-i, -\log |a_i|)\}$ in $\mathbb{R}^2$.

**Proposition 1.2.8** (Christol-Dwork). Suppose that $V \simeq K\{T\}/K\{T\}P$, and let $s$ be the lesser of $-\log |\partial|_K$ and the least slope of the Newton polygon of $P$. Then, we have $\max \{ |\partial|_K, |\partial|_{sp,V} \} = e^{-s}$. In particular, the multiplicity of any $s' < -\log |\partial|_K$ as a slope of the Newton polygon of $P$ coincides with the multiplicity of $\omega s'$ in $\mathfrak{R}_\partial(V)$.

Proof. See [Ked10] Theorem 6.5.3. □

**Definition 1.2.9.** A derivation $\partial$ on $K$ is of rational type if there exists $u \in K$ such that the following conditions hold. (In this case, we call $u$ a rational parameter for $\partial$.)

(a) We have $\partial(u) = 1$ and $|\partial|_K = |u|^{-1}$.

(b) For each positive integer $n$, $|\partial^n/n!|_K \leq |\partial|^n_K$.

In fact, if $\partial$ is of rational type, we have equality in (b), and $|\partial|_{sp,K} = \omega |\partial|_K$; see [KX10] Definition 1.4.1.

**Lemma 1.2.10.** Let $\partial$ be a derivation on $K$ of rational type with rational parameter $u$ and let $L/K$ be a finite tamely ramified extension. Then the unique extension of $\partial$ to $L$ is of rational type (with $u$ again as a rational parameter).

Proof. See [KX10] Lemma 1.4.5. □

**Remark 1.2.11.** We sometimes need to replace $K$ by the completion of $K(x)$ with respect to $\eta$-Gauss norm for some $\eta \in \mathbb{R}_{>0}$, where $x$ is transcendental over $K$ and we set $\partial x = 0$. The derivation $\partial$ is again of rational type on the new field.

**Definition 1.2.12.** When $\partial$ is of rational type, it is more convenient to consider differently normalized $\partial$-radii, as follows. For $V$ a $\partial$-differential module, we define the intrinsic $\partial$-radii of $V$ to be $IR_\partial(V) = |\partial|_{sp,K}/|\partial|_{sp,V} = |\partial|_K \cdot R_\partial(V)$. We define the intrinsic subsidiary $\partial$-radii to be $\mathfrak{R}_\partial(V) = |\partial|_K \cdot \mathfrak{R}_\partial(V)$. We write $IR_\partial(V;i) = |\partial|_K \cdot R_\partial(V;i)$ for $i = 1, \ldots, \dim V$.

**Hypothesis 1.2.13.** From now on, we assume that $K$ is a complete nonarchimedean field of characteristic zero and residual characteristic $p$, equipped with a derivation $\partial$ of rational type. We fix $u \in K$ a rational parameter of $\partial$. We also assume $p > 0$ unless otherwise specified.

**Construction 1.2.14.** We construct the $\partial$-Frobenius as follows. If $K$ contains a primitive $p$-th root of unity $\zeta_p$, we may define an action of the group $\mathbb{Z}/p\mathbb{Z}$ on $K$ using $\partial$-Taylor series:

$$x^{(i)} = \mathbb{T}(x; \partial; (\zeta_p^i - 1)u), \quad (i \in \mathbb{Z}/p\mathbb{Z}, x \in K);$$

in particular, $u^{(i)} = \zeta_p^i u$. This gives an isometric action of $\mathbb{Z}/p\mathbb{Z}$ on $K$. Let $K^{(\partial)}$ be the fixed subfield of $K$ under this action; in particular, $u^p \in K^{(\partial)}$. Hence, we have a Galois extension $K/K^{(\partial)}$ generated by $u$ with Galois group $\mathbb{Z}/p\mathbb{Z}$. (If $K$ does not contain a primitive $p$-th root of unity, we may still define $K^{(\partial)}$ using Galois descent. Then, $K/K^{(\partial)}$ will not be Galois.)
We call the inclusion $\varphi^{(\partial)*} : K^{(\partial)} \hookrightarrow K$ the $\partial$-Frobenius morphism. We view $K^{(\partial)}$ as being equipped with the derivation $\partial' = \partial/(pu^{p-1})$; it is a derivation on $K^{(\partial)}$ because a simple calculation shows that $(\partial(x))^{(i)} = c^i \partial(x^{(i)})$ for any $x \in K$, yielding that $\partial'(x)$ is invariant under the $\mathbb{Z}/p\mathbb{Z}$-action if $x \in K^{(\partial)}$. By [KX10] Lemma 1.4.9], $\partial'$ is of rational type on $K^{(\partial)}$.

We sometimes use $\varphi^{(\partial,n)} : K^{(\partial,n)} \hookrightarrow K$ to denote the $p^n$-th $\partial$-Frobenius obtained by applying the above construction $n$ times; if $K$ contains a primitive $p^n$-th root of unity $\zeta_{p^n}$, this is the same as the fixed field for the natural action of $\mathbb{Z}/p^n\mathbb{Z}$ on $K$ given by $x^{(i)} = T(x; \partial; (\zeta_{p^n} - 1))u$ for $i \in \mathbb{Z}/p^n\mathbb{Z}$.

**Remark 1.2.15.** We point out that the definitions of $\partial$-Frobenius and $K^{(\partial)}$ depend on the choice of the rational parameter $u$.

**Lemma 1.2.16.** The residue field $\kappa_{K^{(\partial)}}$ contains $\kappa^p_K$.

**Proof.** We know that $K$ is generated by $u$ over $K^{(\partial)}$. If $|u| \notin |K^{(\partial)}|$, $K^{(\partial)}$ will have the same residue field as $K$ does. If $|u| \in |K^{(\partial)}|$, let $x \in K^{(\partial)}$ be an element such that $|x| = |u|$. Then $\kappa_{K^{(\partial)}}$ is generated over $\kappa_{K^{(\partial)}}$ by $u/x$, whose $p$-th power lies in $\kappa_{K^{(\partial)}}$. The statement follows.

**Definition 1.2.17.** Given a $\partial'$-differential module $V'$ over $K^{(\partial)}$, the $\partial$-Frobenius pullback is $\varphi^{(\partial)*}V' = V' \otimes K^{(\partial)} K$, viewed as a $\partial$-differential module over $K$ by setting

$$\partial(v' \otimes x) = pu^{p-1} \partial'(v') \otimes x + v' \otimes \partial(x) \quad (v' \in V', x \in K).$$

For a $\partial$-differential module $V$ over $K$, define the $\partial$-Frobenius descendant of $V$ as the $K^{(\partial)}$-module $\varphi^{(\partial)}V$ obtained from $V$ by restriction along $\varphi^{(\partial)*} : K^{(\partial)} \rightarrow K$, viewed as a $\partial'$-differential module over $K^{(\partial)}$ with differential $\partial' = \partial/pu^{p-1}$.

Let $V$ be a $\partial$-differential module over $K$ such that $IR_{\partial}(V) > p^{-1/(p-1)}$. A $\partial$-Frobenius antecedent of $V$ is a $\partial'$-differential module $V'$ over $K^{(\partial)}$ such that $V \simeq \varphi^{(\partial)*}V'$ and $IR_{\partial'}(V') > p^{-n/(p-1)}$.

**Lemma 1.2.18.** The $\partial$-Frobenius pullbacks and descendants have the following properties.

(a) For $V'$ a $\partial'$-differential module over $K^{(\partial)}$, $IR_{\partial'}(\varphi^{(\partial)*}V') \geq \min\{IR_{\partial'}(V')^1/p, pIR_{\partial'}(V')\}$. Moreover, if $IR_{\partial'}(V') \neq p^{-n/(p-1)}$, the above inequality is in fact an equality.

(b) For $V$ a $\partial$-differential module over $K$, there is a canonical isomorphism $\varphi^{(\partial)*} \varphi^{(\partial)}V \simeq V^{\oplus p}$.

(c) For $i = 0, \ldots, p - 1$, let $W_i^{(\partial)}$ be the $\partial'$-differential module over $K^{(\partial)}$ with one generator $v_i$ (a proxy of $u^i$) such that $\partial'(v_i) = \frac{1}{p}u^{-p}v_i$; we have $IR_{\partial'}(W_i^{(\partial)}) = p^{-n/(p-1)}$ for $i = 1, \ldots, p - 1$. Then for $V$ a $\partial$-differential module over $K$, there are canonical isomorphisms $\iota_i : (\varphi^{(\partial)}V) \otimes W_i^{(\partial)} \simeq \varphi^{(\partial)}V$ for $i = 0, \ldots, p - 1$. Moreover, a submodule $U$ of $\varphi^{(\partial)}V$ is itself the $\partial$-Frobenius descendant of a submodule of $V$ if and only if $\iota_i(U \otimes W_i^{(\partial)}) = U$ for $i = 0, \ldots, p - 1$.

For $V_1$ and $V_2$ $\partial$-differential modules over $K$, we have

$$\varphi^{(\partial)}V_1 \otimes \varphi^{(\partial)}V_2 = (\varphi^{(\partial)}(V_1 \otimes V_2))^{\oplus p}.$$ 

For $V'$ a $\partial'$-differential module over $K^{(\partial)}$, we have $\varphi^{(\partial)}V' \simeq V' \oplus \bigoplus_{i=1}^{p-1} V' \otimes W_i^{(\partial)}$. 

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(d) (Christol-Dwork) Let $V$ be a $\partial$-differential module over $K$ such that $\mathcal{IR}_\partial(V) > p^{-1/(p-1)}$. Then there exists a unique $\partial$-Frobenius antecedent $V'$ of $V$. Moreover, $\mathcal{IR}_\partial(V') = \mathcal{IR}_\partial(V)^p$.

(e) Let $V$ be a $\partial$-differential module over $K$. Then

$$\mathcal{IR}_\partial(\varphi_*(\psi)_{\partial} V) = \bigcup_{r \in \mathcal{IR}_\partial(V)} \begin{cases} \{ r^p, p^{-p/(p-1)} (p-1 \text{ times}) \} & r > p^{-1/(p-1)} \\ \{ p^{-1} r (p \text{ times}) \} & r \leq p^{-1/(p-1)} \end{cases}$$

In particular, $\mathcal{IR}_\partial(\varphi_*(\psi)_{\partial} V) = \min\{ p^{-1} \mathcal{IR}_\partial(V), p^{-p/(p-1)} \}$.

Proof. For (a), see [KX10, Lemma 1.4.11] and [KX10 Corollary 1.4.20]. (b) and (c) are straightforward. For (d), see [Ked10, Theorem 10.4.2]. For (e), see [KX10, Theorem 1.4.19].

Remark 1.2.19. As in [Ked10, Theorem 10.4.4], one can form a version of Lemma 1.2.18(d) for differential modules over discs or annuli.

For the following theorem, we do not assume $p > 0$.

Theorem 1.2.20. Let $V$ be a $\partial$-differential module over $K$. Then there exists a decomposition

$$V = \bigoplus_{r \in (0,1]} V_r,$$

where every subquotient of $V_r$ has pure intrinsic $\partial$-radii $r$. Moreover, $V_r = 0$ if $r \notin |K^\times|Q$.

Proof. For the decomposition, see [KX10, Theorem 1.4.21]. The rationality on $r$ follows from Proposition 1.2.8 when $r < \omega$ and from $\partial$-Frobenius antecedent in the general case.

Notation 1.2.21. We call $\oplus_{r \in (0,\omega]} V_r$ the visible part of $V$ and $\oplus_{r \in [\omega,1]} V_r$ the non-visible part of $V$. If $V$ consists of only visible part, we say $V$ has visible (intrinsic) $\partial$-radii; similarly, if $V$ consists of only non-visible part, we say $V$ has non-visible (intrinsic) $\partial$-radii.

Remark 1.2.22. Let $V$ be a $\partial$-differential module over $K$ with pure intrinsic $\partial$-radii $\mathcal{IR}_\partial(V) > p^{-1/(p-1)}$. By Lemma 1.2.18(d), $V$ has a $\partial$-Frobenius antecedent $V'$. By Lemma 1.2.18(c),

$$\varphi_*(\psi)_{\partial} V = \varphi_*(\psi)_{\partial} \psi V' \simeq V' \oplus \bigoplus_{i=1}^{p-1} V' \otimes W_i^{(\partial)}.$$ 

This decomposition coincides with the decomposition we obtained by applying Theorem 1.2.20 to $\varphi_*(\psi)_{\partial} V$.

1.3 Refined radii

In this subsection, we study the refined $\partial$-radii for a $\partial$-differential module of pure $\partial$-radii; this is a secondary information we can extract from the differential module. We will always assume that $\partial$ is of rational type. There are some part of the theory that works for general differential operators and differential modules with visible radii; for these, one may consult [X-Thesis].

For this subsection, we do not assume $p > 0$ unless otherwise specified.
Hypothesis 1.3.1. In this subsection, let $K$ be a complete nonarchimedean field of characteristic zero and residual characteristic $p$, equipped with a derivation of rational type. We fix $u \in K$ a rational parameter of $\partial$. We assume that $V$ is a $\partial$-differential module of rank $d$ over $K$ of pure intrinsic $\partial$-radii $IR_{\Theta}(V) < 1$. Denote $s = -\log(\omega R_{\Theta}(V)^{-1}) = -\log|\partial|_{sp,V}$.

Notation 1.3.2. For $P(T) = T^d + a_1 T^{d-1} + \cdots + a_d \in K[T]$ a polynomial whose Newton polygon has pure slope $s$, the reduced roots of $P$ are the reductions of the roots in $k^{(s)}_{K_{\text{alg}}}$. If $P$ is the characteristic polynomial of a matrix $A \in \text{Mat}(m_{(s)}^{(s)}_K)$, we call the reduced roots of $P$ the reduced eigenvalues of $A$.

Notation 1.3.3. We define $\lambda = \lambda_{\partial}(V)$ and $r = r_{\partial}(V)$ as follows.

(a) When $V$ has pure visible intrinsic $\partial$-radii $IR_{\partial}(V) < \omega$, we let $\lambda_{\partial}(V) = 0$ and $r_{\partial}(V) = 1$.

(b) When $V$ has pure non-visible $\partial$-radii, we must have $p > 0$ and $IR_{\partial}(V) \in [p^{-1/(p-1)}, 1)$. Let $\lambda_{\partial}(V)$ denote the unique positive integer such that $IR_{\partial}(V) \in [p^{-1/p^{\lambda_{\partial}(V)}(p-1)}, p^{-1/p^{\lambda_{\partial}(V)}(p-1)}]$, and denote $r_{\partial}(V) = p^{\lambda_{\partial}(V)}$.

Definition 1.3.4. A norm $| \cdot |$ on $V$ is called good if it has an orthogonal (not necessarily orthonormal) basis and

(a) when $V$ has visible $\partial$-radii, $|\partial|_V \leq \omega R_{\partial}(V)^{-1}$;

(b) when $V$ has non-visible $\partial$-radii (and hence $p > 0$), we have

$$|\frac{\partial^i}{p^j}|_V \leq |\partial|_K^i, \quad \text{for } i = 1, \ldots, r - 1, \quad |\frac{\partial^p}{p^1}|_V \leq p^{-1/(p-1)} R_{\partial}(V)^{-r}. \quad (1.3.5)$$

One may summarize the conditions (a) and (b) by writing

(c) $|\frac{\partial^i}{i!}|_V \leq \max\{|\partial|_K^i, |\partial|_{sp,V}/i!\}$ for $i = 1, \ldots, r$.

Indeed, the maximum above equals to $|\partial|_K^i$ when $i < r$ and to $p^{-1/(p-1)} R_{\partial}(V)^{-r}$ when $i = r$.

We will see in Lemma 1.3.8 below that, for $V$ as in Hypothesis 1.3.1 there exists an good norm on $V$.

We now define the refined $\partial$-radii, denoted by $\Theta_{\partial}(V)$, as follows. Enlarge the value group of $K$ in the sense of Remark 1.2.11 so that $V$ admits an orthonormal basis. Let $N_{\partial}$ be the matrix of $\partial^r$ on the chosen basis. If $\alpha_1, \ldots, \alpha_d$ are the reduced eigenvalues of $N_{\partial}$, viewed as elements in $k^{(rs)}_{K_{\text{alg}}}$, we denoted $\Theta_{\partial}(V, | \cdot |) = \{\alpha_1^{1/r}, \ldots, \alpha_d^{1/r}\} \subset k^{(s)}_{K_{\text{alg}}}$. (Note that there is no ambiguity of taking $r$-th roots for elements in $k^{(rs)}_{K_{\text{alg}}}$ when $p > 0$.) We will see in Lemmas 1.3.11 and 1.3.12 that the refined $\partial$-radii is independent of the choices of the good norm and the orthonormal basis of $V$. After these lemmas, we will abbreviate $\Theta_{\partial}(V)$ for $\Theta_{\partial}(V, | \cdot |)$.

We remark that $\Theta_{\partial}(V)$ does not depend on the choice of the rational parameter $u$. But it is sometimes convenient to use intrinsic refined $\partial$-radii $\mathcal{I}_u \Theta_{\partial}(V) = u \Theta_{\partial}(V)$ for a fixed rational parameter $u \in K$. 

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Remark 1.3.6. In the definition of refined \( \partial \)-radii, we first enlarged \( K \) to \( K' \), the completion of \( K(x_1, \ldots, x_n) \) for some \((\eta_1, \ldots, \eta_n)\)-Gauss norm. However, the refined \( \partial \)-radii \( \Theta_\partial(V, | \cdot |) \) is still a (multi)subset of \( \kappa^{(s)}_{K_{alg}} \). Indeed, since the construction is canonical, for any \( \theta \in \Theta_\partial(V, | \cdot |) \), \( g\theta \in \Theta_\partial(V, | \cdot |) \) for any automorphism \( g \) of \( K' \) fixing \( K \). But \( \Theta_\partial(V, | \cdot |) \) is a finite set. So it can consist only of elements in \( \kappa^{(s)}_{K_{alg}} \). Alternatively, we can work this out more carefully in the computation of reduced eigenvalues to cancel the new variables we introduced.

Remark 1.3.7. For a good norm, one expects that the inequalities in (1.3.5) are in fact equalities. Since we will not use this result later, we leave it as a question for interested readers. (See [Ked10, Lemma 6.2.4] for a proof of similar flavor.)

Lemma 1.3.8. For any \( V \) in Hypothesis [1.3.1] it has a good norm.

Proof. We first assume that \( IR_\partial(V) \leq \omega \). We take a cyclic vector \( \nu \in V \) and then the Newton polygon of the associated twisted polynomial \( P \) has slope \( \geq s \). Then we can define a good norm on \( V \) by taking the orthogonal basis to be \( \nu, \partial \nu, \ldots, \partial^{d-1} \nu \) with \( |\partial^i \nu| = e^{-is} \) for \( i = 0, \ldots, d-1 \).

For general \( V \) with \( IR_\partial(V) \in (\omega, 1) \) and \( p > 0 \), let \( n = \lambda - 1 \) if \( IR_\partial(V) = p^{-1/p^{\lambda-1}(p-1)} \) and \( n = \lambda \) otherwise. In other words, \( n \) is the unique nonpositive integer such that \( IR_\partial(V) \in (p^{-1/p^{\lambda-1}(p-1)}, p^{-1/p^{\lambda-1}(p-1)}] \). Let \( \varphi^{(\partial,n)} : K^{(\partial,n)} \rightarrow K \) be the \( p^n \)-th \( \partial \)-Frobenius and let \( \tilde{\partial} = \partial/(p^nu^{n-1}) \) be the corresponding derivation on \( K^{(\partial,n)} \). By repeatedly applying Lemma 1.2.18(d), we obtain an \( n \)-fold \( \partial \)-Frobenius antecedent \( W \) over \( K^{(\partial,n)} \); it has intrinsic \( \tilde{\partial} \)-radii \( IR_{\tilde{\partial}}(W) = IR_\partial(V)p^n \in (p^{-p/(p-1)}, p^{-1/(p-1)}) \). In particular, \( W \) has a good norm by the argument above in the case \( IR_\partial(V) \leq \omega \); we have

\[
|u^{p^n} \tilde{\partial}|_W = p^{-1/(p-1)} IR_{\tilde{\partial}}(W)^{-1} \in [1, p) \\
\Rightarrow |u\partial|_W = p^n |u^{p^n} \tilde{\partial}|_W \begin{cases} < p^{-n} \cdot p = p^{\lambda-1} & \text{when } n = \lambda, \\
= p^{-n} \cdot 1 = p^{\lambda-1} & \text{when } n = \lambda - 1. \end{cases}
\]

This norm on \( W \) gives rise to a \( K \)-norm \( | \cdot |_V \) on \( V \), which we will show is good. By (1.3.9), we have \( |u \partial - i|_V = |u \partial - i|_W \leq |i| \) for \( i = 1, \ldots, p^n - 1 \). Hence, we have, for \( i = 1, \ldots, p^n \),

\[
\begin{align*}
|u^i \partial^j|_V &= |u^i \partial^j|_W = \left| u \partial(u \partial - 1) \cdots (u \partial - (i - 1)) \right|_W \leq \left| u \partial \right|_i, \\
= \left| \frac{u^{p^n} \partial^j}{i} \right|_W \begin{cases} \leq 1 & \text{if } i = 1, \ldots, p^n - 1, \\
= p^{-1/(p-1)} IR_{\tilde{\partial}}(W)^{-1} & \text{if } i = p^n \text{ and } n = \lambda, \\
= p^{-p/(p-1)} IR_{\tilde{\partial}}(V)^{-1} & \text{if } i = p^n \text{ and } n = \lambda - 1.
\end{cases}
\end{align*}
\]

This verifies (1.3.5).

\[\square\]

Remark 1.3.10. In the proof above, when \( IR_\partial(V) = \omega \), Proposition 1.2.8 did give us a twisted polynomial with slope greater than or equal to \( -\log|\partial|_K \) and it provides a good norm on \( V \). However, one cannot compute the refined \( \partial \)-radii by taking the reduced roots of this twisted polynomial.

Lemma 1.3.11. Let \( | \cdot | \) be a good norm on \( V \). Then the refined \( \partial \)-radii \( \Theta_\partial(V, | \cdot |) \) are well-defined.

Proof. By possibly enlarging \( K \) in the sense of Remark 1.2.11, we have two orthonormal bases \( \vec{e} \) and \( \vec{e}' \) for \( | \cdot |_V \) such that \( \vec{e}' = \vec{e} A \) for a transition matrix \( A \in GL_d(\mathcal{O}_K) \). For \( i = 1, \ldots, r \), denote
and with this basis. The corollary follows.

Proof. By possibly enlarging $K$ as in Remark 1.3.14, we may choose orthonormal bases $e$ and $f$ of $| \cdot |_1$ and $| \cdot |_2$, respectively, so that $e_A = f$ with $A = \text{Diag}\{a_{11}, \ldots, a_{dd}\}$.

Let $N_i$ for the matrix of $\partial^i$ acting on $e$; by (1.3.5), we have $|N_i/i!| \leq |\partial|_K^i$ for $i = 1, \ldots, r - 1$. Then, we have

$$\frac{\partial^i(e')}{r!} = \frac{\partial^i(eA)}{r!} = \sum_{i=0}^{r} \frac{\partial^i(e)}{i!} \frac{\partial^{-i}(A)}{(r-i)!} = e' A^{-1} \left( \sum_{i=0}^{r} \frac{N_i \partial^{-i}(A)}{i!} \right).$$

If $A^{-1} M A$ denote the matrix of $\partial^r/r!$ acting on $e'$, we have

$$M = \frac{N_r}{r!} + \sum_{i=0}^{r-1} \frac{N_i \partial^{-i}(A) A^{-1}}{i!}.$$

Note that $|N_i/i!| \leq |\partial|_K^i$ and $|\partial^{-i}(A) A^{-1}/(r-i)!| \leq |\partial|_K^{-i} |A|^{-1} \leq |\partial|_K^{-i}$ imply that $|M - N_r/r!| \leq |\partial|_K^{-i} < \omega R (V)^{-r}$, which is smaller than any singular value of $N_r/r!$. By [Ked10, Theorem 4.2.2], the reduced eigenvalues of $N_r/r!$ coincide with those of $A^{-1} M A$. Therefore, $\Theta_\partial(V)$ does not depend on the choice of good norms $| \cdot |$ on $V$.

Lemma 1.3.12. Let $V$ be as above and let $| \cdot |_1$ and $| \cdot |_2$ be two good norms on $V$. Then, we have $\Theta_\partial(V, | \cdot |_1) = \Theta_\partial(V, | \cdot |_2)$.

Proof. By possibly enlarging $K$ as in Remark 1.3.14, we may choose orthonormal bases $e$ and $f$ of $| \cdot |_1$ and $| \cdot |_2$, respectively, so that $e_A = f$ with $A = \text{Diag}\{a_{11}, \ldots, a_{dd}\}$.

Let $N_i$ for the matrix of $\partial^i$ acting on $e$; by (1.3.5), we have $|N_i/i!| \leq 1$ for $i = 1, \ldots, r - 1$. Then, we have

$$\frac{\partial^i(f)}{r!} = \frac{\partial^i(eA)}{r!} = \sum_{i=0}^{r} \frac{\partial^i(e)}{i!} \frac{\partial^{-i}(A)}{(r-i)!} = f A^{-1} \left( \sum_{i=0}^{r} \frac{N_i \partial^{-i}(A)}{i!} \right).$$

It suffices to show that $N_r/r!$ has the same reduced eigenvalues as $\sum_{i=0}^{r} \frac{N_i \partial^{-i}(A)}{(r-i)!} A^{-1}$. This is true by [Ked10, Theorem 4.4.2] since

$$\left| \frac{N_i \partial^{-i}(A)}{i!} A^{-1} \right| = \left| \frac{N_i}{i!} \cdot \text{Diag}\left( \frac{\partial^{-i}(a_{11})}{(r-i)!} a_{11}^{-1}, \ldots, \frac{\partial^{-i}(a_{dd})}{(r-i)!} a_{dd}^{-1} \right) \right| \leq |\partial|_K^i \cdot |\partial|_K^{-i} < \omega R (V)^{-1},$$

for $i = 0, \ldots, r - 1$.

Corollary 1.3.13. Assume $V$ has pure visible $\partial$-radii. For any cyclic vector $v \in V$, the reduced roots of the twisted polynomial associated to $v$ are exactly the refined $\partial$-radii of $V$. In particular, they are nonzero in $K^{(s)}$.

Proof. (To echo Remark 1.3.10, we emphasize that the case $IR_\partial(V) = \omega$ is not included in the statement.) We can construct the good norm using the twisted polynomial as in Lemma 1.3.8. This twisted polynomial is then exactly the characteristic polynomial of the matrix of $\partial$ acting on this basis. The corollary follows.

Lemma 1.3.14. Keep $s, r, R_\partial(V)$, and $\lambda$ as before. Let $V'$ be another $\partial$-differential module of rank $d$, equipped with a basis $e$, on which the action of $\partial$ satisfies the conditions in Definition 1.3.4. Assume that the reduced eigenvalues of the matrix $N_r \in \text{Mat}(\mathfrak{m}_K^{(rs)})$ of $\partial^r$ on $V'$ are all nonzero in $K^{(rs)}$. Then $V'$ has pure $\partial$-radii $\omega e^a$. As a consequence, $\Theta_\partial(V')$ is exactly the set of reduced eigenvalues of $N$. 

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Proof. Since $N_r \in \text{Mat}(m^{(rs)}_K)$, we have $R_\partial(V') \geq R_\partial(V)$. Suppose that $V'$ is not of pure $\partial$-radii $R_\partial(V)$. Applying Theorem 1.2.20 and Lemma 1.3.8 to $V'$ and its factors, by possibly enlarging $K$ as in Remark 1.2.11 there is a basis $f$ on which, the conditions in Definition 1.3.4 are still fulfilled but the matrix $\tilde{N}_r \in \text{Mat}(m^{(rs)}_K)$ of $\partial f$ is degenerate modulo $m^{(rs)+}_{K^{alg}}$ (when identifying $\kappa^{(rs)}_K$ with $\kappa_K$). However, the same argument in Lemma 1.3.12 implies that $N_r$ and $\tilde{N}_r$ must have same reduced eigenvalues. This leads to a contradiction. Hence, $V'$ is of pure $\partial$-radii $R_\partial(V)$. The last statement follows from Definition 1.3.3.

Lemma 1.3.15. For any $V$ in Hypothesis 1.3.7, we have $\Theta_\partial(V^\vee) = -\Theta_\partial(V) = \{-\theta \mid \theta \in \Theta_\partial(V)\}$.

Proof. Straightforward.

We would like to obtain a decomposition by refined $\partial$-radii; this is established in Theorem 1.3.26. But we need some preparation first.

We start by discussing some basic properties of refined $\partial$-radii when the $\partial$-radii is visible. In fact, one may prove the following two lemmas directly without the visibility assumption. But we find it more elegant to use $\partial$-Frobenius antecedents and descendants, as proved in Proposition 1.3.19.

Lemma 1.3.16. Let $V$ and $W$ be two $\partial$-differential modules over $K$ of pure and visible $\partial$-radii $R_\partial(V) = R_\partial(W)$. The following two statements are equivalent.

(a) The refined $\partial$-radii of $V$ and $W$ are distinct, i.e., $\Theta_\partial(V) \cap \Theta_\partial(W) = \emptyset$.

(b) The tensor product $V \otimes W^\vee$ has pure $\partial$-radii $R_\partial(V)$.

Moreover, if either statement holds, we have $\Theta_\partial(V \otimes W^\vee) = \{\theta_1 - \theta_2 \mid \theta_1 \in \Theta_\partial(V), \theta_2 \in \Theta_\partial(W)\}$ as multisets. As a corollary, we have

(1) If $\Theta_\partial(V) \cap \Theta_\partial(W) = \emptyset$, any homomorphism $f : W \to V$ of $\partial$-differential modules is zero.

(2) If $\Theta_\partial(W)$ consists of only one element $\theta \in \kappa^{(s)}_{K^{alg}}$ (with multiplicity), then $\theta \in \Theta_\partial(V)$ if and only if $V \otimes W^\vee$ is not of pure $\partial$-radii $R_\partial(V)$.

(3) If $\Theta_\partial(V)$ and $\Theta_\partial(W)$ both consist of only one element $\theta \in \kappa^{(s)}_{K^{alg}}$, then $R_\partial(V \otimes W^\vee) > R_\partial(V)$.

Proof. By Lemma 1.3.15 above, $\Theta_\partial(W^\vee) = -\Theta_\partial(W)$. We may enlarge $K$ as in Remark 1.2.11 so that we have good norms on $V$ and $W^\vee$ given by orthonormal bases. Consider $V \otimes W^\vee$ with the norm given by the tensors of elements in the orthonormal bases of the two modules. Let $N_0, N_1 \in \text{Mat}(m^{(s)}_{K^{alg}})$ be the corresponding matrices of $\partial$ on $V$ and $W^\vee$, respectively. Since $N_0$ has reduced eigenvalues $\Theta_\partial(V)$ and $N_1$ has reduced eigenvalues $-\Theta_\partial(W)$, the matrix $N = N_0 \otimes 1 + 1 \otimes N_1$ would have reduced eigenvalues the same as $\{\theta_1 - \theta_2 \mid \theta_1 \in \Theta_\partial(V), \theta_2 \in \Theta_\partial(W)\}$.

If (a) holds, $N$ has nonzero reduced eigenvalues and hence $|N^n| = e^{-ns}$ for all $n \in \mathbb{N}$ with full rank when working modulo $m^{(ns)+}_{K^{alg}}$ (and when identifying $\kappa^{(ns)}_{K^{alg}}$ with $\kappa_{K^{alg}}$). Therefore, $V \otimes W^\vee$ has pure $\partial$-radii $R_\partial(V)$ by Lemma 1.3.14.

If (b) holds, the tensor product norm is a good norm on $V \otimes W^\vee$ already and the reduced eigenvalues of $N$ should give the refined $\partial$-radii of $V \otimes W^\vee$. By Corollary 1.3.13, $0 \notin \Theta_\partial(V \otimes W^\vee)$. This implies (a).

Now, we prove (1). Since $V \otimes W^\vee$ has pure $\partial$-radii $R_\partial(V) < \omega$, we have $H^0_\partial(V \otimes W^\vee) = 0$, which parameterizes all homomorphisms of $\partial$-differential modules from $W$ to $V$. 13
Proposition 1.3.18. Let $R$ be a good norm on $W$.\]

\[\begin{align*}
&\text{(a) We take a good norm on } V, \text{ since a good norm on } V, \text{ where } V\text{ and } W\text{ are orthonormal bases of } V. \text{ It is more convenient to work with refined intrinsic } \partial \text{-radii. Let } R_\partial(V) = R_\partial(W). \\
&\text{(b) By Theorem 1.2.20, we may assume that } W\text{ has pure } \partial \text{-radii. By Lemma 1.3.8, we may find a good norm on } W\text{ for which } |\partial|_W \leq \max\{\omega IR_\partial(W)^{-1}, 1\} < \omega IR_\partial(V)^{-1}. \\
&\text{(c) We proceed as in Lemma 1.3.16. Now, if } N_0, N_1 \in \text{Mat}(K)\text{ denote the matrices of } \partial \text{ on orthonormal bases of } V \text{ and } W, \text{ respectively, then } N_1 \in \text{Mat}(K)\text{, and } N_0 \text{ has reduced eigenvalues } \Theta_\partial(V). \text{ Hence } N_0 \otimes 1 + 1 \otimes N_1 \text{ has the same reduced eigenvalues as } N_0 \text{ but with multiplicity multiplied by dim } W. \text{ The lemma follows.}
\end{align*}\]

Now, we study how refined $\partial$-radii interact with pushing forward and pulling back along $\partial$-Frobenius. It is more convenient to work with refined intrinsic $\partial$-radii. For the following proposition, we assume $p > 0$ for the moment.

Proposition 1.3.18. Let $\varphi^{(\partial)} : K^{(\partial)} \rightarrow K$ be the $\partial$-Frobenius (with respect to $u$).

(a) If $IR_\partial(V) > p^{-1/(p-1)}$, by Lemma 1.2.18(d), $V = \varphi^{(\partial)*}W$ for some $\partial'$-differential module $W$ on $K^{(\partial)}$ such that $IR_{\partial'}(W) = IR_\partial(V)^{p'}$. Then we have $\Theta_\partial(V) = \{ - (p\theta')^{1/p} \mid \theta' \in \Theta_{\partial'}(W) \}.$

(b) If $IR_\partial(V) = p^{-1/(p-1)}$, $\varphi^{(\partial)}_\mathcal{V}(V)$ has pure intrinsic $\partial'$-radii $p^{-p/(p-1)}$. Then the elements in $I \Theta_{\partial'}(\varphi^{(\partial)}_\mathcal{V}(V))$ can be grouped into $p$-tuples $(\theta^{p}, \theta^{p-1}, \ldots, \theta^{p-p+1})$ (with some multiplicity), where $\theta \in \kappa_{K_{alg}}$, and $I \Theta_\partial(V)$ is the multiset consisting of $\theta^{p} - \theta^{1/p} \in \kappa_{K_{alg}}$ for each $p$-tuple above.

(c) If $IR_\partial(V) < p^{-1/(p-1)}$, then $I \Theta_{\partial'}(\varphi^{(\partial)}_\mathcal{V}(V)) = \{ p^{-1}(p \text{ times}) \theta \mid \theta \in I \Theta_\partial(V) \}.$

Proof. (a) We take a good norm on $W$ constructed as in Lemma 1.3.8. By possibly enlarging $K$ in the sense of Remark 1.2.11, we can take an orthonormal basis $e$ on $W$. Then, it naturally gives a good norm on $V$, since a good norm on $V$ constructed in Lemma 1.3.8 would come from the same (multi-folded) $\partial$-Frobenius antecedent. Let $\lambda$ and $r$ be as in Notation 1.3.3. We have

\[\begin{align*}
&u^{\lambda^p} \partial^{\lambda^p} = u^{\lambda p} \partial u^{\lambda - 1} \cdots (u \partial - \lambda^p + 1) = pu^{\lambda^p} \partial^{\lambda^p - 1} (pu^{\lambda^p} \partial^{\lambda^p - 1} - 1) \cdots (pu^{\lambda^p} \partial^{\lambda^p - 1} - p^\lambda + 1) \\
&= pu^{\lambda^p - 1} \partial^{p^{\lambda^p - 1}} \prod_{i=1, p|\lambda}^{p^{\lambda^p - 1}} (pu^{\lambda^p} \partial^{\lambda^p} - i)
\end{align*}\]

it also acts on $W$. Since $|u^{\lambda^p} \partial^{\lambda^p}|_W \leq \max\{1, p^{-1/(p-1)}IR_\partial(W)\} < p$, we can bound the norm

\[|u^{\lambda^p} \partial^{\lambda^p} - p^{\lambda^p - 1}((-1) \cdots (p + 1))^{\lambda^p - 1} u^{\lambda^p} \partial^{\lambda^p} |_W < |u^{\lambda^p} \partial^{\lambda^p} |_W.\]
Hence, the matrix of $\partial^\lambda$ acting on $e$ is congruent to the matrix of $(-1)^{p^{\lambda-1}(p-1)}(p!)^{p^{\lambda-1}}\partial^{p^{\lambda-1}}$ modulo $m_{p^{\lambda+1}}$. Hence, we must have
\[ \Theta_\partial(V, | \cdot |) = \{(\cdot \cdot \cdot | \cdot \cdot \cdot) \} = \left\{ (-1)^{p^{\lambda-1}(p-1)}(p!)^{1/p} \theta^1 \in \Theta_\partial(W) \right\}. \]

(b) When $IR_\partial(V) = p^{-1/(p-1)}$, Lemma 1.2.18(c) implies that $\varphi^{(\partial)}_V$ has pure intrinsic $\partial'$-radii $p^{-p/(p-1)}$. By Lemmas 1.2.18(e) and 1.3.16 the elements in $\mathcal{I}_\Theta_\partial(\varphi^{(\partial)}_V)$ can be grouped into $p$-tuples $\theta$ with some multiple, where $\theta \in \kappa_{K^{\alg}}$. By possibly enlarging $K$ in the sense of Remark 1.2.11, we may assume that $\varphi^{(\partial)}_V$ has a good norm with orthonormal basis. Let $N$ be the matrix of $pu^p\partial'$ with respect to this basis. Then on $\varphi^{(\partial)}_V = V^{\partial'}$, $u^p\partial'$ acts as
\[ u\partial(u\partial - 1) \cdots (u\partial - p + 1) = pu^p\partial'(pu^p\partial' - 1) \cdots (pu^p\partial' - p + 1). \]

It is congruence to $N(N - 1) \cdots (N - p + 1)$ modulo $p\mathcal{O}_{K^{(p)}}$ since $|pu^p\partial'|_{K^{(p)}} = p^{-1}$. Hence, its reduced eigenvalues is the multiset consisting of $\theta^p - \theta$ with multiple $p$ for each tuple $\left( \frac{\theta}{p}, \frac{\theta + 1}{p}, \ldots, \frac{\theta + p - 1}{p} \right)$ in the reduced eigenvalues of $N$. The statement follows.

(c) By Lemma 1.2.18(e), $\varphi^{(\partial)}_V$ has pure intrinsic $\partial'$-radii $p^{-1}IR_\partial(V) \leq p^{-p/(p-1)}$. Since $u^p\partial' = u\partial/p$, we can take a good norm of $\varphi^{(\partial)}_V$ and deduce $\mathcal{I}_\Theta_\partial(\varphi^{(\partial)}_V) = \frac{1}{p}\mathcal{I}_\Theta(\varphi^{(\partial)}_V)\varphi^{(\partial)}_V$, which in turn equals to $\frac{1}{p}\mathcal{I}_\Theta_\partial(V^{\partial'})$ by Lemma 1.2.18(b). The statement follows.

**Proposition 1.3.19.** Lemmas 1.3.16 and 1.3.17 hold without assuming that $V$ or $W$ to have visible $\partial$-radii.

**Proof.** We are left with the case when $p > 0$ and $IR_\partial(V) \geq p^{-1/(p-1)}$. If $IR_\partial(V) > p^{-1/(p-1)}$, the statements follow from the statements about the $\partial$-Frobenius antecedents by Proposition 1.3.18(a). If $IR_\partial(V) = p^{-1/(p-1)}$, the statements follow from the statements about the $\partial$-Frobenius descendants by Proposition 1.3.18(b) and Lemma 1.2.18(c).

Now, we give an example of $\partial$-differential modules with pure refined $\partial$-radii. We will use Gothic letter $s$ instead of $s$ when dealing with its logarithmic variant and we will never use them together.

**Example 1.3.20.** Let $s \in -\log|K^{\times}| \mathbb{Q}$ such that $s < 0$ if $p = 0$ and $s < \frac{1}{p}\log p$ if $p > 0$. Let $\theta \in \kappa_{K^{(s)}}$ be a nonzero element.

(a) If $p = 0$, then $s \in -\log|K^{\times}| \mathbb{Q}$ and $\theta \in \kappa_{K^{(s)}}$ for some finite tamely ramified extension $K'/K$.

Let $x \in m_{K^{(s)}}$ be a lift of $\theta$. Denote $d = 1$ and $n = 0$.

(b) If $p > 0$, there exists $n \in \mathbb{N}$ such that $\theta^{p^n} \in (\kappa_{K^{(s)}})^n$ with $p^{n-1}s \in -\log|K^{\times}|$ for some finite tamely ramified extension $K'/K$. By Lemma 1.2.16 we may find a lift $x \in u^{-p^n}m_{K^{(s)}}$ of $u^{-p^n}\theta^{p^n}$, where the extra $u^{-p^n}$ reflects the different normalizations of refined intrinsic $\partial$-radii and refined $\partial$-radii. Denote $d = p^n$.

Let $\mathcal{L}_{x, (a)}$ denote be the $\partial$-differential module over $K'$ of rank $d$ with basis $\{e_1, \ldots, e_d\}$, on which $\partial$ acts as $\partial e_i = e_{i+1}$ for $i = 1, \ldots, d - 1$ and $\partial e_d = xe_1$. 

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Remark 1.3.21. When \( p > 0 \), we observe that \( s < \frac{1}{p} \log p \) also includes non-visible range. The restriction \( s < \frac{1}{p} \log p \) in Example 1.3.20 is linked with the choice \( x \in u^{-p^n} m(p^n) \). In order to extend \( s \) to \( (-\infty, (\frac{1}{p-1} - \frac{1}{p} s)(\log p)) \) for some \( c \in N \), we need to be able find \( x \in u^{-p^n} m(p^n) \) lifting \( u^{-p^n} \theta p^n \), for some \( n \in N \) and a finite tamely ramified extension \( K'/K \). However, as \( c \) gets larger, \( n \) might need to take accordingly a bigger value to guarantee the existence of such lift \( x \). This is why we cannot essentially remove this restriction.

Remark 1.3.22. In the non-visible case, one can construct a \( \partial \)-differential module with pure refined \( \partial \)-radii by simply pulling back a \( \partial' \)-differential module over \( K^{(\partial)} \) with appropriate refined \( \partial' \)-radii. However, the form of the differential module depends very much on how many Frobenius antecedent we take in the construction. Later when we study the one-dimensional variation of refined \( \partial \)-radii, we essentially need Example 1.5.7, the family version of Example 1.3.20, which has similar form in both visible and non-visible cases.

Lemma 1.3.23. Keep the notation as in Example 1.3.20. Then \( L_{x,(n)} \) has pure intrinsic \( \partial \)-radii \( IR_0(L_{x,(n)}) = \omega^s \) and pure refined intrinsic \( \partial \)-radii \( \theta \).

Proof. We replace \( K \) by the completion of \( K(z) \) with respect to the \( |u|^{-1}e^{-s} \)-Gauss norm (and set \( \partial z = 0 \)).

First, we assume that either \( p = 0 \) or \( p > 0 \) and \( s \in \left( 0, \frac{1}{p-1} \log p \right) \), i.e. we consider the visible \( \partial \)-radii case. We note that \( e_1, z^{-1}e_2, \ldots, z^{-(d-1)}e_d \) gives a good norm on \( L_{x,(n)} \); it is a straightforward computation to check that the refined \( \partial \)-radii is as stated.

Then, we tackle the case when \( p > 0 \) and \( s \in \left( \frac{1}{p-1} \log p, \frac{1}{p} \log p \right) \). For \( i = 1, \ldots, p \), we have

\[
\partial^i e_i = e_{i+l}, \quad i + l \leq p^n;
\partial^i e_{p^n-l} = \partial^{i-l}(xe_1), \quad i \geq l.
\]

We will show that \( \{ e_1, z^{-1}e_2, \ldots, z^{-(p^n-1)}e_{p^n} \} \) gives rise to a good norm on \( L_{x,(n)} \). Indeed, for \( i = 1, \ldots, p \), the matrix for \( \partial^i \) with respect to this basis is

\[
N_i = \begin{pmatrix}
0 & 0 & \cdots & z^i & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & z^i & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & \cdots & z^i \\
z^{-p^n+i}x & 0 & \cdots & 0 & 0 & \cdots & 0 \\
z^{-p^n+i}\partial x & z^{-p^n+i}x & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
z^{-p^n+i}\partial^{i-1}x & (i-1)z^{-p^n+i}\partial^{i-2}x & \cdots & 0 & 0 & \cdots & 0 \\
\end{pmatrix}
\]

(1.3.24)

Note that

\[ |\partial|_{K^{(\partial)}} = p^{-1}|u|^{p-1}|\partial|_{K^{(\partial)}} = p^{-1}|u|^{-1} \leq \omega|z| < |z| . \]

Hence, modulo \( m_{K}^{-\log |z|} \), the nonzero terms of \( N_i \) are the \( z^i \)'s and \( z^{-p^n+i}x \)'s in (1.3.24); they form a 2-by-2 block matrix

\[
N_i^{-\log |z|} = \begin{pmatrix}
0 & z^i \\
z^{-p^n+i}x & I_{(p^n-1) \times (p^n-1)} \\
\end{pmatrix} \in \text{Mat}_{p^n \times p^n}(K^{(-\log |z|)}).
\]

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Note that $|z^{-p^n+i}x| = |z|^i$. By Lemma 1.3.14 $IR_\partial(\mathcal{L}_{x,(n)}) = \omega e^\xi$ and this basis gives a good norm on $V$. Now, the reduced eigenvalues of $N_p$ are exactly $x^{1/p^n-1}$ for $p^n$-times. This implies that $I\Theta_\partial(V) = \{\theta \ (p^n \text{-times})\}$ by the choice of $x$ in Example 1.3.20.

Lemma 1.3.25. Let $V$ be a $\partial$-differential module over $K$ of pure visible $\partial$-radii $R_\partial(V) = \omega e^\xi$.

(i) Then, for any subquotient $V_0$ of $V$, its refined $\partial$-radii cannot have any element not contained in $\Theta_\partial(V)$.

(ii) For any $\theta \in \kappa_{K^{salg}}^{(s)}$, there is a unique maximal $\partial$-differential submodule of $V$ which has pure refined $\partial$-radii $\theta$.

Proof. For $\theta \in \kappa_{K^{salg}}^{(s)}$ such that $\theta \notin \Theta_\partial(V)$, let $\mathcal{L}_{x,(n)}$ denote the $\partial$-differential module constructed in Example 1.3.20. By Lemmas 1.3.23 and 1.3.16 we have $V \otimes \mathcal{L}_{x,(n)}^\vee$ is pure of $\partial$-radii $R_\partial(V)$, and hence so is $V_0 \otimes \mathcal{L}_{x,(n)}^\vee$. By the same lemmas again, we have $\theta \notin \Theta_\partial(V_0)$. (i) is proved. We remark that we, however, did not prove the inclusion $\Theta_\partial(V_0) \subseteq \Theta_\partial(V)$ as a multiset, which will become clear by Theorem 1.3.26 below.

Note that if two submodules $V_1$ and $V_2$ of $V$ both have pure refined $\partial$-radii $\theta$, so is their sum $V_1 + V_2$ as it is a quotient of $V_1 \oplus V_2$. (ii) follows.

Using $\mathcal{L}_{x,(n)}$, we can obtain a decomposition by refined $\partial$-radii as follows.

Theorem 1.3.26. Let $K$ and $V$ be as in Hypothesis 1.3.2. Then $V$ admits a canonical decomposition by refined $\partial$-radii as follows.

$$V = \bigoplus_{\{\theta\} \subseteq \kappa_{K^{salg}}^{(s)}} V_{\{\theta\}},$$

where the direct sum runs through all Galois conjugacy classes in $\kappa_{K^{salg}}^{(s)}$ and the refined $\partial$-radii of $V_{\{\theta\}}$ are exactly the Galois conjugacy class $\{\theta\}$ with same multiplicity on each element.

After making a finite tamely ramified extension $K'$ of $K$, one can obtain the canonical decomposition (1.3.27) without taking the conjugacy classes. In particular, $\Theta_\partial(V) \subseteq \cup_n (\kappa_{K'}^{(p^n s)})^{1/p^n}$.

Proof. We may replace $K$ by a finite tamely ramified extension $K'$ so that $\Theta_\partial(V) \subseteq \cup_n (\kappa_{K'}^{(p^n s)})^{1/p^n}$.

(By Lemma 1.2.10, $\partial$ is still a derivation of rational type.)

We first assume that $p = 0$, or $p > 0$ and $IR_\partial(V) < p^{-1/(p-1)}$. For each $\theta \in \Theta_\partial(V)$, we construct $\mathcal{L}_{x,(n)}$ as in Example 1.3.20 which is a rank $d$ $\partial$-differential module of pure $\partial$-radii $R_\partial(V)$ and pure refined radii $\partial$. By Lemma 1.3.16(2), $V \otimes \mathcal{L}_{x,(n)}^\vee$ is not of pure radii $R_\partial(V)$. By Theorem 1.2.20 we get a decomposition $V \otimes \mathcal{L}_{x,(n)}^\vee = W_0 \oplus W_1$, where $R_\partial(W_0) > R_\partial(V)$ and $W_1$ is of pure $\partial$-radii $R_\partial(V)$.

Denote $\widetilde{W}_0 = W_0 \otimes \mathcal{L}_{x,(n)}$ and $\widetilde{W}_1 = W_1 \otimes \mathcal{L}_{x,(n)}$. Now consider the following homomorphisms of $\partial$-differential modules

$$V \xrightarrow{i} V \otimes \mathcal{L}_{x,(n)}^\vee \otimes \mathcal{L}_{x,(n)} \xrightarrow{j} \widetilde{W}_0 \oplus \widetilde{W}_1,$$

where $i$ is induced by the diagonal embedding $K \hookrightarrow \mathcal{L}_{x,(n)}^\vee \otimes \mathcal{L}_{x,(n)}$ and $j$ is induced by the trace map $\mathcal{L}_{x,(n)}^\vee \otimes \mathcal{L}_{x,(n)} \rightarrow K$ such that $ji = id$. Let $p_0$ and $p_1$ be the projections from $V \otimes \mathcal{L}_{x,(n)}^\vee \otimes \mathcal{L}_{x,(n)}$
to the factors $\tilde{W}_0$ and $\tilde{W}_1$, respectively, viewed as submodules of the source. Hence $p_0^2 = p_0$, $p_1^2 = p_1$, and $p_0 + p_1 = 1$. We claim that $jp_0 i$ and $jp_1 i$ are projections on $V$ which give the desired decomposition.

By Lemma 1.3.16(3), $R_\theta(L^\vee_{x,(n)} \otimes L_{x,(n)}) > R_\theta(V)$. By Lemma 1.3.17 $V \otimes L^\vee_{x,(n)} \otimes L_{x,(n)}$ and hence $\tilde{W}_0$ and $\tilde{W}_1$ have pure $\partial$-radii $R_\theta(V)$. Also, by Lemma 1.3.17 $\Theta_\theta(\tilde{W}_0)$ consists of solely $\theta$, and by the “moreover” part of Lemma 1.3.16

$$\Theta_\theta(\tilde{W}_1) = \{\theta_1 + \theta \text{ (with multiplicity } d) \mid \theta_1 \in \Theta_\theta(W_1)\}.$$ 

In particular, $\theta \notin \Theta_\theta(\tilde{W}_1)$. Hence any homomorphism of $\partial$-differential modules between $\tilde{W}_0$ and $\tilde{W}_1$ has to be zero by Lemma 1.3.16(1). In particular, $p_1 i p_0 = p_0 i p_1 = 0$. Thus, we have

$$(jp_0 i)(jp_0 i) = j p_0 i j (1 - p_1) i = j p_0 i (ji) - j(p_0 i p_1) i = jp_0 i$$

$$(jp_1 i)(jp_1 i) = j p_1 i j (1 - p_1) i = j p_1 i (ji) - j(p_1 i p_0) i = j p_1 i$$

$$j p_0 i + j p_1 i = j (p_0 + p_1) i = ji = 1.$$ 

This proves that $V = j p_0 i (V) \oplus j p_1 i (V)$. Moreover, by Lemma 1.3.25(i), $\Theta_\theta(j p_0 i (V))$ consists of only $\theta$ since it is a quotient of $W_0$, and $\Theta_\theta(j p_1 i (V))$ does not contain $\theta$ since it is a quotient of $W_1$. Applying this process to each of $\theta \in \Theta_\theta(V)$ gives the desired decomposition (1.3.27).

By Lemma 1.3.25(ii), the decomposition (over $K'$) is canonical. We can easily get the decomposition over $K$ as stated in the theorem using Galois descent.

Now if $p > 0$ and $IR_\theta(V) = p^{-1/(p-1)}$, the decomposition (1.3.27) comes from the decomposition of its $\partial$-Frobenius descendent, via the relation described in Proposition 1.3.18(2). If $p > 0$ and $IR_\theta(V) > p^{-1/(p-1)}$, the decomposition (1.3.27) comes from the decomposition of its $\partial$-Frobenius antecedent, via the relation described in Proposition 1.3.18(b). □

For completeness, we include the basic properties for tensor products of $\partial$-differential modules of pure $\partial$-radii and pure refined $\partial$-radii. One can combine this with Theorems 1.2.20 and 1.3.26 to obtain corresponding results for general $\partial$-differential modules.

**Proposition 1.3.28.** Let $V$ and $W$ be two $\partial$-differential modules over $K$ of pure $\partial$-radii $R_\theta(V) = R_\theta(W)$ and pure refined $\partial$-radii $\theta_\mathcal{V}$ and $\theta_\mathcal{W}$, respectively. Then, we have

(a) $W^\vee$ has pure refine $\partial$-radii $-\theta_W$.  
(b) If $\theta_V = \theta_W$, $R_\theta(V \otimes W^\vee) > R_\theta(V)$.  
(c) If $\theta_V \neq \theta_W$, then $V \otimes W^\vee$ has pure $\partial$-radii $R_\theta(V)$ and pure refined $\partial$-radii $\theta_V - \theta_W$.  
(d) Moreover, if we do not assume that $V$ and $W$ has pure refined $\partial$-radii and let $U$ denote the maximal submodule of $V \otimes W^\vee$ that has $\partial$-radii strictly larger than $R_\theta(V)$, then we have

$$\dim U = \sum_{\theta \in K^{(s)}_{K^{ab}}} \text{mult}_\theta(\Theta_\theta(V)) \cdot \text{mult}_\theta(\Theta_\theta(W)).$$

**Proof.** (a) is straightforward, and (d) follows from (b) and (c) by the decomposition (1.3.27).
When \( IR_{\emptyset}(V) = IR_{\emptyset}(W) < \omega \), (b) follows from Lemma \ref{3.16} and (c) follows from Lemma \ref{3.16}. Also, when \( p > 0 \) and \( IR_{\emptyset}(V) = IR_{\emptyset}(W) > p^{-1/(p-1)} \), (b) and (c) for \( V \) and \( W \) follow from the same statement for the \( \partial \)-Frobenius antecedents of \( V \) and \( W \), via the relation described in Proposition \ref{3.18}(a). It suffices to prove (b) and (c) in the case when \( p > 0 \) and \( IR_{\emptyset}(V) = IR_{\emptyset}(W) = p^{-1/(p-1)} \).

Now, by Lemma \ref{2.18}(3), we have \( \varphi^{(\emptyset)}(V) \otimes (\varphi^{(\emptyset)} W)^{\vee} = (\varphi^{(\emptyset)} (V \otimes W^{\vee}))^{\otimes p} \). By Proposition \ref{3.18}(2), we know that the refined intrinsic \( \partial \)-radii of \( V \) (resp. \( W \)) are exactly the solutions to \( (\frac{x}{p})^p - \frac{x}{p} = u\theta V \) (resp. \( (\frac{x}{p})^p - \frac{x}{p} = u\theta W \)), with same multiplicity. If \( \theta V \neq \theta W \), by (c) in the visible case together with Theorem \ref{3.26}, the refined intrinsic \( \partial' \)-radii of \( \varphi^{(\emptyset)} V \otimes \varphi^{(\emptyset)} W \) consists of roots of \( (\frac{x}{p})^p - \frac{x}{p} = u(\theta V - \theta W) \), each with multiplicity \( p \dim V \dim W \). (c) follows from Proposition \ref{3.18}(b). If \( \theta V = \theta W \), by (b) in the visible case together with Theorem \ref{3.26}, \( \varphi^{(\emptyset)} V \otimes \varphi^{(\emptyset)} W \) has a submodule of dimension \( (p - 1) \dim V \dim W \) whose intrinsic \( \partial' \)-radii is strictly larger than \( p^{-p/(p-1)} \). By Lemma \ref{2.18}(e), this can happen only if \( IR_{\emptyset}(V \otimes W) > p^{-1/(p-1)} \). (b) is proved. \( \square \)

1.4 Multiple derivations

In this subsection, we introduce differential fields of higher order.

**Notation 1.4.1.** In this subsection, set \( J = \{1, \ldots, m\} \) for notational convenience.

**Definition 1.4.2.** Let \( K \) denote a differential ring of order \( m \), that is a ring equipped with \( m \) commuting derivations \( \partial_1, \ldots, \partial_m \). A \( \partial_J \)-differential module, or simply a differential module, is a finite projective \( K \)-module \( V \) equipped with commuting actions of \( \partial_1, \ldots, \partial_m \). We may apply the results above by singling out one of \( \partial_1, \ldots, \partial_m \).

**Definition 1.4.3.** Let \( K \) and \( V \) be as above. Let \( R \) be a complete \( K \)-algebra. For \( \mathbf{v} \in V \) and \( T_1, \ldots, T_m \in R \), define the \( \partial_J \)-Taylor series to be

\[
\mathbb{T}(\mathbf{v}; \partial_J; T_1, \ldots, T_m) = \sum_{e,j=0}^{\infty} \frac{\partial_J^e(\mathbf{v})}{(e,j)!} T_J^e \in V \otimes_K R,
\]

if it converges.

We will need the following tautological lemma later.

**Lemma 1.4.4.** Let \( \partial = \alpha_1 \partial_1 + \cdots + \alpha_m \partial_m \) be another derivation for some \( \alpha_1, \ldots, \alpha_m \in K \). To simplify notation, we formally write \( \alpha_j = \partial(u_j) \) for any \( j \in J \) (and one can check that the formula can be written with no reference to \( u_j \)). Then, for any \( x \in V \), we have

\[
\mathbb{T}(x; \partial_J; \mathbb{T}(u_1; \partial; \delta) - u_1, \ldots, \mathbb{T}(u_m; \partial; \delta) - u_m) = \mathbb{T}(x; \partial; \delta),
\]  

(1.4.5)

as formal power series in \( V \otimes_K K[\delta] \).

**Proof.** Note that (1.4.5) is a tautological statement, we may assume that \( K \) is \( \mathbb{Z} \)-torsion free. It suffices to show that (1.4.5) is true modulo \( \delta^n \) for any \( \partial_J \)-differential module \( V \) and any \( x \in V \), by induction on \( n \). It is clear for \( n = 1 \). Assume that we have proved this claim for \( n \) and we need to prove it for \( n + 1 \). It suffices to prove

\[
\frac{\partial}{\partial \delta} \mathbb{T}(x; \partial_J; \mathbb{T}(u_1; \partial; \delta) - u_1, \ldots, \mathbb{T}(u_m; \partial; \delta) - u_m) = \frac{\partial}{\partial \delta} \mathbb{T}(x; \partial; \delta) = \mathbb{T}(\partial(x); \partial; \delta),
\]
modulo $\delta^n$. (Note that the derivation brings the coefficients of $\delta^n$ to the coefficients of $\delta^{n-1}$.) We calculate the left hand side, as follows.

$$\frac{\partial}{\partial \delta} \left( x; \partial_f; T(u_1; \partial; \delta) - u_1, \ldots, T(u_m; \partial; \delta) - u_m \right)$$

$$= \sum_{e_j=0}^{\infty} \frac{\partial^{e_j}(x)}{(e_j)!} \frac{\partial}{\partial \delta} \left( (T(u_1; \partial; \delta) - u_1)^{e_1} \cdots (T(u_m; \partial; \delta) - u_m)^{e_m} \right)$$

$$= \sum_{e_j=0}^{\infty} \frac{\partial^{e_j}(x)}{(e_j)!} \left( \sum_{j \in J} e_j \cdot (T(u_1; \partial; \delta) - u_1)^{e_1} \cdots (T(u_j; \partial; \delta) - u_j)^{e_j} \cdots (T(u_m; \partial; \delta) - u_m)^{e_m} \cdot \frac{\partial}{\partial \delta} T(u_j; \partial; \delta) \right)$$

By induction hypothesis, modulo $\delta^n$, this is congruent to

$$\sum_{j \in J} T(\partial_j(x); \partial; \delta) \cdot \frac{\partial}{\partial \delta} T(u_j; \partial; \delta) = \sum_{j \in J} T(\partial_j(x); \partial; \delta) \cdot T(\partial(u_j); \partial; \delta)$$

$$= T(\sum_{j \in J} \partial_j(x) \partial(u_j); \partial; \delta) = T(\partial(x); \partial; \delta).$$

This proves the claim and hence the lemma.

**Definition 1.4.6.** Let $K$ be a complete nonarchimedean differential field of order $m$ and characteristic zero, and let $V$ be a nonzero $\partial_f$-differential module over $K$. Define the **intrinsic radius** of $V$ to be

$$\text{IR}(V) = \min_{j \in J} \left\{ |\partial_j|_{sp,K} \right\} = \min \left\{ \text{IR}(V; \partial_j) \right\}.$$

For $j \in J$, we say $\partial_j$ is **dominant** for $V$ if $\text{IR}(V; \partial_j) = \text{IR}(V)$. We define the **intrinsic subsidiary radii** $\text{IR}(V) = \{ \text{IR}(V; 1), \ldots, \text{IR}(V; \dim V) \}$ by collecting and ordering intrinsic radii from Jordan-Hölder constituents, as in Definition 1.2.3. We again say that $V$ has **pure intrinsic radii** if the elements of $\text{IR}(V)$ are all equal to $\text{IR}(V)$.

Similarly, we define the **extrinsic radius** $ER(V)$ to be the minimum of $R_{\partial_j}(V)$ and **extrinsic subsidiary radii** $\text{ER}(V) = \{ \text{ER}(V; 1), \ldots, \text{ER}(V; \dim V) \}$ by collecting and ordering extrinsic radii from Jordan-Hölder constituents.

**Definition 1.4.7.** Let $K$ be a complete nonarchimedean differential field of order $m$ and characteristic zero. We say that $K$ is of **rational type** with respect to a set of parameters $\{ u_j : j \in J \}$ if each $\partial_j$ is of rational type with respect to $u_j$, and $\partial_i(u_j) = 0$ for $i \neq j$ in $J$.

**Hypothesis 1.4.8.** Let $K$ be a complete nonarchimedean field of characteristic zero, equipped with derivations $\partial_j$ of rational type with respect to parameters $u_j$. Let $V$ be a $\partial_j$-differential module of pure $\partial_j$-radii for each $j \in J$. We assume moreover that $\text{IR}(V) < 1$.

**Notation 1.4.9.** For each $j$, denote $s_j = -\log(\omega R_{\partial_j}(V)^{-1})$, $\lambda_j = \lambda_{\partial_j}(V)$, and $r_j = r_{\partial_j}(V)$. (By Theorem 1.2.20 we have $s_j \in \mathbb{Q} \cdot \log|K^\times|$ for any $j$.)
Theorem 1.2.20. It suffices to show that given any norm $| \cdot |$ on $V = \bigoplus V_{\theta_j}$, where each direct summand $V_{\theta_j}$ has pure refined $\partial_j$-radii $\theta_j$ for any $j \in J$. Define the refined radii of $V$, denoted by $\Theta(V)$, to be $\vartheta = \sum_{j \in J} \partial_j u_j$ with multiplicity $\dim V_{\theta_j}$; it is an element in $\bigoplus_{j \in J} \mathcal{K}_{\mathcal{K}^\text{alg}}(s) du_j$. The reason that we write the refined radii in form of differentials will be justified later, in Theorem 1.4.19.

Proof. By the same argument as in Lemma 1.3.8 using Frobenius antecedent, it suffices to prove the lemma under the assumption that $\mathcal{I} \mathcal{V} R_{\partial_j}(V) = IR(V)$. We call it the refined intrinsic radii. Often, we view it as an element in $\bigoplus_{j \in J} \mathcal{K}_{\mathcal{K}^\text{alg}}^\mathcal{W}(s) du_j$ for $s = -\log(\omega IR(V)^{-1})$.

We will also consider cases where the derivations with large radii are ignored.

(a) Let $\mathcal{I} \mathcal{V} R_{\partial_j}(V)$ be the multiset consisting of elements $\sum_{j \in J} \partial_j u_j$ with multiplicity $\dim V_{\theta_j}$, where the sum is only taken over those $j$ such that $IR_{\partial_j}(V) = IR(V)$. We call it the refined intrinsic radii. Often, we view it as an element in $\bigoplus_{j \in J} \mathcal{K}_{\mathcal{K}^\text{alg}}^\mathcal{W}(s) du_j$ for $s = -\log(\omega IR(V)^{-1})$.

(b) Let $\mathcal{E} \mathcal{V} R_{\partial_j}(V)$ be the multiset consisting of elements $\sum_{j \in J} \partial_j u_j$ with multiplicity $\dim V_{\theta_j}$, where the sum is only taken over those $j$ such that $R_{\partial_j}(V) = R(V)$. We call it the refined extrinsic radii.

Definition 1.4.10. By Theorem 1.3.26, we may replace $K$ by a finite tamely ramified extension such that $V$ admits a direct sum decomposition $V = \bigoplus V_{\theta_j}$, where each direct summand $V_{\theta_j}$ has pure refined $\partial_j$-radii $\theta_j$ for any $j \in J$. Define the refined radii of $V$, denoted by $\Theta(V)$, to be $\vartheta = \sum_{j \in J} \partial_j u_j$ with multiplicity $\dim V_{\theta_j}$; it is an element in $\bigoplus_{j \in J} \mathcal{K}_{\mathcal{K}^\text{alg}}(s) du_j$. The reason that we write the refined radii in form of differentials will be justified later, in Theorem 1.4.19.

We will also consider cases where the derivations with large radii are ignored.

Remark 1.4.12. In contrast to the single derivation case, we do not know if a good norm exists in general, unless we assume that $K$ is discretely valued. This assumption on the discreteness of the valuation may not be necessary for some of the results in this subsection. One might get around this using some approximation process. Since we will deal with discretely valued field in most applications, we restrict ourself here to this case to simplify some argument.

Hypothesis 1.4.13. For the rest of this subsection, we assume that $K$ is discretely valued.

Lemma 1.4.14. The differential module $V$ admits a good norm.

Proof. By the same argument as in Lemma 1.3.8 using Frobenius antecedent, it suffices to prove the lemma under the assumption that $\mathcal{I} R_{\partial_j}(V) \leq \omega$ for any $j \in J$. Note that the $\partial_j$-Frobenius antecedent is compatible with $\partial_{j'}$ for $j' \neq j$. We may further replace $K$ by the completion of $K(x_j)$ with respect to the $e^{-s_j}$-Gauss norm, where we set $\partial_j(x_{j'}) = 0$ for all $j,j' \in J$ and $s_j = -\log(\omega R_{\partial_j}(V))$. (In particular, $K$ is still discretely valued since $e^{-s_j} \in [\mathcal{K}^\mathcal{W}]$ for any $j \in J$ by Theorem 1.2.20.) It suffices to show that given any norm $| \cdot |$ on $V$ with orthonormal basis $e_1, \ldots, e_d$, the submodule $M$ of $V$ generated by

$$\{ x_j^{a_j} \partial_j^{a_j} e_i | a_j \in \mathbb{Z}_{\geq 0} \text{ for any } j \in J \text{ and } i \in \{ 1, \ldots, d \} \}$$

over $\mathcal{O}_K$ is a finite $\mathcal{O}_K$-module; if so, $M$ gives rise to a norm on $V$, under which $| \partial_j | \leq | x_j | = e^{-s_j}$ verifies the condition of good norm in Definition 1.3.3. To prove that $M$ is a finite sub-$\mathcal{O}_K$-module, it suffices to prove that $| x_j^n \partial_j^n |$ is bounded for each $j$ as $n \to +\infty$. (Here, we used the fact that $K$ is discretely valued, otherwise, boundness may not imply finiteness.) It is enough to verify this boundness condition for any norm on $V$. In particular, for each of $\partial_j$, we can choose a good norm by Lemma 1.3.8 for which $| x_j^n \partial_j^n | \leq 1$. Thus, $M$ is finite over $\mathcal{O}_K$ and the lemma follows.

Remark 1.4.15. One may hope to find an analog of Example 1.3.20 for $\partial_j$-differential modules. This, however, amounts to carefully choosing the element $x$ in Example 1.3.20 so that the actions of $\partial_j$ commutes. This places some restriction on possible refined intrinsic radii. In other words, all possible refined intrinsic radii may form only a subset of $\bigoplus_{j \in J} \mathcal{K}_{\mathcal{K}^\text{alg}}(s) du_j$, where $s = -\log(\omega IR(V)^{-1})$.
Unfortunately, we do not know how to identify this subset in general. Proposition 1.4.16 below partly answers this question.

It would be interesting to know, when \( p > 0 \), if any element in \( \oplus_{j \in J} \kappa_{\text{Kalg}}^{(s)} \frac{du_j}{u_j} \) can appear as a refined intrinsic radii. The referee also pointed out that the reduction of \( \partial_J \) may give rise to a \( \mathcal{D} \)-module in characteristic \( p \). We do not know if this construction is independent of the choice of good norms. But we suspect that this is related to the reduction of arithmetic \( \mathcal{D} \)-modules when the differential module comes from an arithmetic \( \mathcal{D} \)-module.

**Proposition 1.4.16.** Assume that \( IR(V) < \omega \) and let \( s = -\log(\omega IR(V)^{-1}) \). Assume moreover that \( p = 0 \) or \( d = \text{rank} \, V = 1 \). Note that the action of \( u_j \partial_j \) on \( K \) induces a derivation on \( \kappa_{\text{Kunr}}^{(s)} \). If \( \partial = \sum_{j \in J} \theta_j \frac{du_j}{u_j} \in \mathcal{I}\Theta(V) \), then for \( i, j \in J \), we have \( u_i \partial_i \theta_j = u_j \partial_j \theta_i \) in \( \kappa_{\text{Kunr}}^{(s)} \).

**Proof.** By possibly replacing \( K \) by a finite tamely ramified extension, we reduce to the case when \( V \) is irreducible and has pure refined intrinsic radii \( \sum_{j \in J} \theta_j \frac{du_j}{u_j} \). By Proposition 1.4.14 we can find a good norm \( | \cdot |_V \), for which \( u_j \partial_j \) acts as a matrix \( N_j \in \text{Mat}_{d \times d}(\mathfrak{m}_K^{(s)}) \). Since \( \partial_i \) and \( \partial_j \) commute with each other for any \( i, j \in J \), we have

\[
N_iN_j + u_i \partial_i(N_j) = N_jN_i + u_j \partial_j(N_i). \tag{1.4.17}
\]

Taking the trace of (1.4.17), we have \( d \cdot u_i \partial_i \theta_j = d \cdot u_j \partial_j \theta_i \). Note that the condition on \( d \) yields that they are elements in \( \kappa_{\text{Kunr}}^{(s)} \). The proposition follows. \( \Box \)

Before proceeding, we need some notation to use in the theorem later.

**Notation 1.4.18.** For \( n \in \mathbb{N} \), write it in the form of \( n = a_0 + pa_1 + \cdots + p^k a_k \) with \( a_1, \ldots, a_k \in \{0, \ldots, p-1\} \), if \( p > 0 \), and \( 0 \) if \( p = 0 \). It is straightforward to check that \( \sigma_p(n_1 + n_2) \geq \sigma_p(n_1 + n_2) \) for \( n_1, n_2 \in \mathbb{N} \), and \( |n| = \omega^{n - \sigma_p(n)} \) for \( n \in \mathbb{N} \).

Now, we study how refined radii behave if we form a new derivation using the linear combination of \( \partial_J \). This explains why we wrote refined radii in form of differentials in Definition 1.4.10.

**Theorem 1.4.19.** Assume that \( V \) has pure refined \( \partial_J \)-radii \( \theta_j \in \kappa_{\text{Kalg}}^{(s_j)} \) for any \( j \in J \). Let \( K' \) be a complete discretely valued nonarchimedean field containing \( K \). Let \( \partial \) is a derivation on \( K' \), extending the action of \( \alpha_1 \partial_1 + \cdots + \alpha_m \partial_m \) on \( K \) to \( K' \), where \( \alpha_1, \ldots, \alpha_m \in K' \). Let \( \omega \) be the equality is achieved if and only if \( \theta \neq 0 \) in \( \kappa_{\text{Kalg}}^{(s)} \). Moreover, if \( \omega \) is a good norm with respect to \( \partial_J \), given by some orthonormal basis \( \mathfrak{g} \). Similarly to Notation 1.3.3 we define integers \( r \) and \( \lambda \) as follows.

(a) When \( |\theta|_{K'} \omega^s < \omega \) we denote \( \lambda = 0 \) and \( r = 1 \).

(b) When \( |\theta|_{K'} \omega^s \in [\omega, 1) \) and \( p > 0 \), let \( \lambda \) denote the unique nonnegative integer such that \( |\theta|_{K'} \omega^s \in \left[p^{-1/p^\lambda}, p^{-1/p^\lambda}(p-1)\right) \), and denote \( r = p^\lambda \).
We remark that in case (b), we have $(\partial|_{K'}\omega^s)^{\rho_k} \leq \omega$ for $k < \lambda$ and hence $(|\partial|_{K'}\omega^s)^i \leq \omega^{\sigma_p(i)}$ for $i = 1, \ldots, r - 1$.

For each $j \in J$, we have
\[
\left| \frac{\partial^i|}{i!} \right|_V \leq |\partial_j|^i_K, \text{ for } i = 1, \ldots, r_j - 1, \text{ and } |\partial^i_j|_V \leq |u_j|^{-r_j} e^{-r_j s_j}.
\]

For $i = 1, \ldots, r$, the term of $\partial^i$ on $e$ can be expressed in terms of the action of $\partial_j$ by the coefficients of $\delta$ on the left hand side of (1.4.5), applied to some $x \in e$. More precisely, for any $j \in J$ and $i \in \mathbb{N}$, the coefficients of $\delta^i$ in $T(u_j; \partial; \delta) - u_j$ has norm $\leq |\partial(u_j)|||\partial|_{K'}^{i-1} = |\alpha_j||\partial|_{K'}^{i-1}$. For the term coming from the $\partial_j$-Taylor series, if we write $e_j = c_j + d_j r_j$ with $c_j \in \{0, \ldots, r_j - 1\}$ and $d_j \in \mathbb{Z}_{\geq 0}$ for any $j \in J$, then we have
\[
\frac{\partial^i_j(x)}{(e_j)} \leq \prod_{j \in J} \frac{\partial^d_j r_j}{(d_j r_j)!} \cdot \prod_{j \in J} \frac{\partial^j_j(x)}{(e_j)} \leq |x|_V \cdot \prod_{j \in J} |\partial_j|^c_j \cdot \prod_{j \in J} (e^{-d_j r_j s_j} - d_j r_j + \sigma_p(d_j r_j))
\]

Putting all of these together, we see that if $i = 1, \ldots, r$, the coefficient of this $\delta^i$-term on the left hand side of (1.4.5) comes from the term that has $\frac{\partial^i_j(x)}{(e_j)}$ (which particularly implies that $i \geq e_1 + \cdots + e_m$, then its norm is smaller than or equal to
\[
|x||\partial|_{K'}^{i-1} - e_1 - \cdots - e_m \prod_{j \in J} |\alpha_j|^e_j \prod_{j \in J} |\partial_j|^c_j \cdot \prod_{j \in J} (e^{-d_j r_j s_j} - d_j r_j + \sigma_p(d_j r_j))
\]
\[
= |x||\partial|_{K'}^{i-1} - e_1 - \cdots - e_m \prod_{j \in J} (|\partial_j|_K |\alpha_j|^e_j) \cdot \prod_{j \in J} ((|\alpha_j|^e_j e^{-s_j} d_j r_j) - d_j r_j + \sigma_p(d_j r_j))
\]
\[
\leq |x||\partial|_{K'}^{i-1} - e_1 - \cdots - e_m \prod_{j \in J} |\partial_j|^c_j \cdot \prod_{j \in J} (e^{-d_j r_j s_j} - d_j r_j + \sigma_p(d_j r_j)) \quad \text{(note } |\partial|_{K'} \geq |\partial(u_j)||u_j|^{-1} = |\alpha_j||\partial_j|_K)
\]
\[
\leq |x||\partial|_{K'}^{i} - \sum_{j \in J} d_j r_j \sigma_p(\sum_{j \in J} d_j r_j),
\]

When $i = 1, \ldots, r - 1$, the coefficient of this $\delta^i$-term has norm $\leq |\partial|_{K'}^{i} |x|$ by the remark after condition (b). When $i = r$, the term will have norm $\leq |\partial|_{K'}^{r} (|\partial|_{K'}^{r} \omega^s)^{\rho_r} |x| = \omega^{-r+1} e^{-r s} |x|$; this can happen only when $\sum_j d_j r_j = r$ and $\sigma_p(\sum_j d_j r_j) = \sum_j \sigma_p(d_j r_j)$, which together yield $c_j = 0$ for some $j \in J_0$ and $c_j' = 0$ for $j \neq j$. In the latter case, the corresponding term is $\alpha_j^r \delta_j^{r-1}(x)/r!$. Hence, modulo the elements with norm smaller than $e^{-r s}$, the matrix of $\partial^r$ on $e$ is congruent to $\sum_{j \in J_0} \alpha_j^r \delta_j^r$; this is a sum of matrices with single eigenvalues $\alpha_j^r \delta_j^r$ for $j \in J_0$. By Lemma 1.3.14, $R_{\partial}(V) \leq \omega^s$ and the equality holds if and only if $\sum_{j \in J_0} \alpha_j^r \theta_j^r \neq 0$ in $K_{\text{alg}}^{s}$, which is equivalent to $\sum_{j \in J_0} \alpha_j^r \theta_j^r \neq 0$ in $K_{\text{alg}}^{s}$; note that $r$ is always 1 or a power of $p$. Moreover, if either condition is satisfied, $V$ has pure refined $\partial$-radii
\[
\left( \sum_{j \in J_0} \theta_j^r \alpha_j^r \right)^{1/r} = \sum_{j \in J_0} \theta_j \alpha_j = \theta \in K_{\text{alg}}^{s}.
\]
\[
\square
\]

**Corollary 1.4.20.** Let $V$ be a $\partial$-differential module over $K$ and let $f = T(-; \partial, T) : K \to K[T/u]_0$ and $f^* V$ be as in Lemma 1.2.6(d). For $\eta \in \{0, |u|\}$, let $F_\eta$ denote the completion of $K(T)$ with respect to the $\eta$-Gauss norm.
(a) If \( \eta \in (0, R_0(V)] \), \( f^*V \otimes F_\eta \) has pure intrinsic \( \partial_T \)-radius 1; if \( \eta \in (R_0(V), |u|) \), \( f^*V \otimes F_\eta \) has (extrinsic) \( \partial_T \)-radius \( R_0(V) \).

(b) When \( \eta \in (R_0(V), |u|) \), we have \( \Theta_{\partial_T}(f^*V \otimes F_\eta) = \Theta_0(V) \).

**Proof.** For any \( x \in V \), \( f^*(\partial(x)) = \partial_T(f^*(x)) \). (a) follows from this immediately, and (b) follows by Theorem [1.4.19] \( \Box \)

### 1.5 One-dimensional variation of refined radii

In this subsection, we first review some results from [Ked10 Chapter 11] and [KX10 Section 2] regarding the variation of radii of convergence on discs and annuli. When the radii vary as log-affine functions, we will characterize the variation of the corresponding refined radii.

**Hypothesis 1.5.1.** Throughout this section, we assume that \( K \) is a complete nonarchimedean field of characteristic zero and residual characteristic \( p \). We also assume that \( K \) is equipped with derivations \( \partial_1, \ldots, \partial_m \) of rational type with respect to \( u_1, \ldots, u_m \).

**Notation 1.5.2.** Denote \( J = \{1, \ldots, m\} \) and \( J^+ = J \cup \{0\} \). For \( \eta > 0 \), let \( F_\eta \) be the completion of \( K(t) \) under the \( \eta \)-Gauss norm \( |\cdot|_\eta \). Put \( \partial_0 = \frac{d}{dt} \) on \( K[t] \); it extends to \( F_\eta \) and ring of functions on discs or annuli. The field \( F_\eta \) is of rational type for the derivations \( \partial_{J^+} \).

**Notation 1.5.3.** Fix \( j \in J^+ \) and an interval \( I \subseteq [0, \infty) \). We say that \( I \) is an open interval in \([0, \infty)\) if it is of the form \([0, \beta)\) or \((\alpha, \beta]\), where \( 0 < \alpha < \beta \). Let \( \tilde{I} \) denote \( I \setminus \{0\} \). Let \( M \) be a \( \partial_J \)-differential module of rank \( d \) over \( A^K_1(I) \). For \( r \in -\log \tilde{I} \) and \( i \in \{1, \ldots, d\} \), define

\[
 f_i^{(j)}(M, r) = -\log R_{\partial_j}(M \otimes F_{e^{-r}; i}), \quad F_i^{(j)}(M, r) = f_i^{(j)}(M, r) + \cdots + f_i^{(j)}(M, r).
\]

**Theorem 1.5.4.** Fix \( j \in J^+ \) and an interval \( I \subseteq [0, \infty) \). Let \( M \) be a \( \partial_J \)-differential module of rank \( d \) over \( A^K_1(I) \).

(a) **(Linearity)** For \( i = 1, \ldots, d \), the functions \( f_i^{(j)}(M, r) \) and \( F_i^{(j)}(M, r) \) are continuous. They are piecewise affine on the locus where \( f_i^{(j)}(M, r) > -\log |u_j| \) if \( j \in J \); they are piecewise affine on whole \(-\log \tilde{I}\) if \( j = 0 \).

(b) **(Weak integrality)**

(i) Suppose \( p = 0 \) or \( j = 0 \). If \( i = d \) or \( f_i^{(j)}(M, r_0) < f_i^{(j)}(M, r_0) \) in some neighborhood of \( r = r_0 \) belong to \( \mathbb{Z} \). Consequently, the slopes of each \( f_i^{(j)}(M, r) \) and \( F_i^{(j)}(M, r) \) belong to \( \frac{1}{p} \mathbb{Z} \cup \cdots \frac{1}{d} \mathbb{Z} \).

(ii) Suppose \( p > 0 \) and \( j \in J \). If \( f_i^{(j)}(M, r_0) > \frac{1}{p^{(p-1)}} \log p - \log |u_j| \) for some \( n \in \mathbb{Z}_{\geq 0} \), then the slopes of \( F_i^{(j)}(M, r) \) in some neighborhood of \( r_0 \) belong to \( \frac{1}{p^{(p-1)}} \mathbb{Z} \).

(c) **(Monotonicity)** Assume \( 0 \in I \). Suppose either \( j \in J \), or \( j = 0 \) and \( f_i^{(0)}(M, r_0) > r_0 \). Then the slopes of \( F_i^{(j)}(M, r_0) \) are nonpositive in a neighborhood of \( r_0 \).

(d) **(Convexity)** For \( i = 1, \ldots, d \), the function \( F_i^{(j)}(M, r) \) is convex.
(c) (Decomposition) Assume that \( I \) is an open interval in \([0, +\infty)\). Suppose that for some \( i \in \{1, \ldots, d\} \), \( F_i^{(j)}(M, r) \) is affine and \( f_i^{(j)}(M, r) > f_{i+1}^{(j)}(M, r) \) for \( r \in -\log \hat{I} \). Then \( M \) admits a unique direct sum decomposition separating the first \( i \) subsidiary \( \partial_j \)-radii of \( M \otimes F_\eta \) for any \( \eta \in \hat{I} \).

**Proof.** See [KX10, Theorems 2.2.5, 2.2.6, and 2.3.5]; for simple presentation, we slightly weakened some of the statements. \( \square \)

**Notation 1.5.5.** Let \( I \subseteq [0, +\infty) \) be an interval and let \( M \) be a \( \partial_{J^+} \)-differential module of rank \( d \) on \( A^1_k(I) \). For \( r \in -\log \hat{I} \) and \( i \in \{1, \ldots, d\} \), denote

\[
\hat{f}_i(M, r) = -\log IR(M \otimes F_{u_i-\hat{r}}; i), \quad \hat{F}_i(M, r) = \hat{f}_1(M, r) + \cdots + \hat{f}_i(M, r).
\]

When \( I \subseteq [0, 1) \) and \( |u_j| = 1 \) for all \( j \in J \), we write

\[
\tilde{f}_i(M, r) = -\log ER(M \otimes F_{u_i-\hat{r}}; i), \quad \tilde{F}_i(M, r) = \tilde{f}_1(M, r) + \cdots + \tilde{f}_i(M, r).
\]

**Theorem 1.5.6.** Fix an interval \( I \subseteq [0, +\infty) \). Let \( M \) be a \( \partial_{J^+} \)-differential module of rank \( d \) on \( A^1_k(I) \).

(a) (Linearity) For \( i = 1, \ldots, d \), the functions \( f_i(M, r) \) and \( F_i(M, r) \) are continuous and piecewise affine.

(b) (Integrality) If \( i = d \) or \( f_i(M, r_0) > f_{i+1}(M, r_0) \), then the slopes of \( F_i(M, r) \) in some neighborhood of \( r_0 \) belong to \( \mathbb{Z} \). Consequently, the slopes of each \( f_i(M, r) \) and \( F_i(M, r) \) belong to \( \mathbb{Z} \).

(c) (Monotonicity) Suppose that \( 0 \in I \). Then the slopes of \( F_i(M, r) \) are nonpositive, and each \( F_i(M, r) \) is constant for \( r \) sufficiently large.

(d) (Convexity) For \( i = 1, \ldots, d \), the function \( F_i(M, r) \) is convex.

(e) (Decomposition) Assume that \( I \) is an open interval in \([0, +\infty)\). Suppose for some \( i \in \{1, \ldots, d-1\} \), the function \( F_i(M, r) \) is affine and \( f_i(M, r) > f_{i+1}(M, r) \) for \( r \in -\log \hat{I} \). Then \( M \) admits a unique direct sum decomposition separating the first \( i \) subsidiary extrinsic \( \partial_j \)-radii of \( M \otimes F_\eta \) for any \( \eta \in \hat{I} \).

(f) (Dichotomy) Assume that \( I \) is an open interval in \([0, +\infty)\) and \( M \) is not the direct sum of two nonzero \( \partial_{J^+} \)-differential submodules. If \( f_1(M, r) \) is affine for \( r \in -\log \hat{I} \), then, for each \( j \in J^+ \),

1. either \( M \otimes F_\eta \) has pure intrinsic \( \partial_j \)-radii which equal to \( IR(M \otimes F_\eta) \) for all \( \eta \in \hat{I} \), or
2. \( IR_\partial(M \otimes F_\eta) > IR(M \otimes F_\eta) \) for all \( \eta \in \hat{I} \).

Moreover, if \( |u_j| = 1 \) for any \( j \in J \) and \( I \subseteq [0, 1) \), then same statements above except (c) hold for \( \hat{f}_i(M, r) \) and \( \hat{F}_i(M, r) \) in place of \( f_i(M, r) \) and \( F_i(M, r) \), respectively. We need to modify the statement (c) as follows.

(c') (Monotonicity) Suppose that \( 0 \in I \). For \( i = 1, \ldots, d \), for any point \( r_0 \) where \( \hat{f}_i(M, r_0) > r_0 \), the slopes of \( \hat{F}_i(M, r) \) are nonpositive in some neighborhood of \( r_0 \). Also, \( \hat{f}_i(M, r) = r \) for \( r \) sufficiently large.
Proof. For the statements (a)-(e) for $f_i(M, r)$ and $F_i(M, r)$, see [KX10] Theorems 2.4.4 and 2.5.1. For the statements (a), (b), (e'), (d), and (e) for $\tilde{f}_i(M, r)$ and $\tilde{F}_i(M, r)$, we can argue similarly as follows.

Let $\tilde{K}$ denote the completion of $K(x_J)$ with respect to the $(1, \ldots, 1)$-Gauss norm. For $I = [\alpha, \beta] \subseteq [0, 1)$, as in [KX10] Notation 2.4.1, Taylor series gives rise to an injective homomorphism $\tilde{f}^* : K(\alpha/t, t/\beta) \to \tilde{K}(\alpha/t, t/\beta)$ such that $\tilde{f}^*(u_j) = u_j + x_jt$.

For $\eta \in (\alpha, \beta)$, we use $\tilde{F}_\eta$ to denote the completion of $\tilde{K}(t)$ with respect to the $\eta$-Gauss norm. Then $\tilde{f}^*$ extends to an injective isometric homomorphism $\tilde{f}^* : F_\eta \hookrightarrow \tilde{F}_\eta$.

Now, $\tilde{f}^* M$ becomes a $\partial_0$-differential module on $A^1_K[\alpha, \beta]$. Moreover, since $\partial_0|\tilde{j}^*M = \partial_0|M + \sum_{j \in J} x_j \partial_j|_M$, we have

$$R_{\partial_0}(M \otimes \tilde{F}_\eta) = \min_{j \in J} \{ R_{\partial_j}(M \otimes F_\eta) \} = ER(M \otimes F_\eta), \text{ for any } \eta \in [\alpha, \beta].$$

In other words, $f_i^{(0)}(\tilde{f}^* M, r) = \tilde{f}_i(M, r)$ for $r \in (-\log \beta, -\log \alpha)$. The theorem follows from Theorem [1.5.4].

Now, we prove (f) for intrinsic radii and the analog for extrinsic radii follows by exactly the same argument.

Assume that we are not in case (2). Then $IR_{\partial_j}(M \otimes F_\eta) = IR(M \otimes F_\eta)$ for some $\eta \in \tilde{I}$. By Theorem [1.5.4](d), the convexity of $f_i^{(j)}(M, r)$ forces $IR_{\partial_j}(M \otimes F_\eta) = IR(M \otimes F_\eta)$ for all $\eta \in \tilde{I}$. Now, if $IR_{\partial_j}(M \otimes F_\eta; 2) > IR(M \otimes F_\eta)$ for all $\eta \in (\alpha, \beta)$, the decomposition (e) would imply that $M$ is decomposable, which contradicts the assumption. Therefore $IR_{\partial_j}(M \otimes F_\eta; 2) = IR(M \otimes F_\eta)$ for some $\eta \in \tilde{I}$. By Theorem [1.5.4](d) again, we have the equality for all $\eta \in \tilde{I}$. Continuing this argument for the third and other $\partial_j$-radii leads us to case (1). \(\square\)

Next, we discuss the variation of refined $\partial_j$-radii of a $\partial_j$-differential module $M$ when $f_i^{(j)}(M, r) = \cdots = f_{\dim M}^{(j)}(M, r)$ is affine. Before proving general results, we first look at an example of pure refined $\partial_j$-radii. It is a 1-dimensional analog of Example [1.3.20].

**Example 1.5.7.** Let $j \in J^+$ and let $(\alpha, \beta) \subseteq (0, \infty)$ be an open interval. Let $\theta \in \kappa_{\alpha, \beta}$ for some $b \in -\log |K^\times|^Q$ and let $a \in Q$. Assume that

$$e^{-b} \alpha^a, e^{-b} \beta^a > \begin{cases} 1 & \text{if } p = 0, \\ p^{-1/p} & \text{if } p > 0. \end{cases} (1.5.8)$$

We will see that this includes non-visible radii. As noted in Remark [1.3.21] we cannot improve the restriction $p^{-1/p}$ to 1.

Let $e$ be the prime-to-$p$ part of the denominator of $a$. We have the following.

(a) If $p = 0$, then $b \in -\log|(K')^\times|$ and $\theta \in \kappa_{\alpha, \beta}$ for some finite tamely ramified extension $K'/K$.

Let $x \in \mathfrak{m}_{K'}^{(b)}$ be a lift of $\theta$. We set $n = 0$ and $d = 1$ in this case.

(b0) If $p > 0$ and $j = 0$, there exists $n \in N$ such that $\theta^n \in \kappa_{\alpha, \beta}$ with $p^n b \in -\log|(K')^\times|$ and $p^n e a \in pZ$, for some finite tamely ramified extension $K'/K$. Let $x \in \mathfrak{m}_{K'}^{(p^n b)}$ be a lift of $\theta^n$. We set $d = p^n$. 

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(b1) If \( p > 0 \) and \( j \in J \), there exists \( n \in \mathbb{N} \) such that \( \theta^{pn} \in (\kappa_{K'}^{(r-1)b})^p \) and \( p^n a \in \mathbb{Z} \) with 
\[
p^{n-1}b - \log(|K'|^r) \in \mathbb{Z}
\]
for some finite tamely ramified extension \( K' / K \). Let \( x \in \mathfrak{m}_{K'/(\partial_j)}^{(p^n b)} \) be a lift of \( \theta^{pn} \) in the fixed field of \( \partial_j \); this is possible by Lemma \[\text{1.2.16}].

Let \( A_{K'}^{1}(\alpha^{1/e}, \beta^{1/e}) \) be the open annulus with coordinate \( t^{1/e} \). Define \( L_{x,a,(n)}^{(j)} \) to be the \( \partial_j \)-differential module over \( A_{K'}^{1}(\alpha^{1/e}, \beta^{1/e}) \) of rank \( d \) with basis \( \{e_1, \ldots, e_d\} \), on which \( \partial_j \) acts as \( \partial_j e_i = e_{i+1} \) for \( i = 1, \ldots, d-1 \) and \( \partial_j e_d = xt^{a}u_{j}^{-d}e_1 \), if \( j \in J \) and \( \partial_j e_d = xt^{d(a-1)}e_1 \), if \( j = 0 \). The added \( u_{j}^{-d} \) and \( t^{-d} \) are to balance the different normalization on intrinsic \( \partial_j \)-radii.

**Lemma 1.5.9.** Keep the notation as in Example \[\text{1.5.7}]. If we denote \( \Theta_{\partial_j}(L_{x,a,(n)}^{(j)} \otimes F_{e-r}^{d'}) \) has pure intrinsic \( \partial_j \)-radius \( \omega^{a r + b} \) and \( \Theta_{\partial_j}(L_{x,a,(n)}^{(j)} \otimes F_{e-r}^{d'}) \) consists of only \( \theta^a \) with multiplicity \( d \).

**Proof.** Comparing this with Example \[\text{1.3.20} \] we have for any \( r \in (-\log \beta, -\log \alpha) \), \( L_{x,a,(n)}^{(j)} \otimes F_{e-r}^{d'} \) is isomorphic to \( L_{xt^{a}u_{j}^{-d},a,n'} \) if \( j \in J \) and \( L_{x^{t^{a}a-1},a,n'} \) if \( j = 0 \). Applying Lemma \[\text{1.3.20} \] to this \( \partial_j \)-differential module yields the result; note that the condition \[\text{1.5.8} \] corresponds to the condition on \( s \) in Example \[\text{1.3.20} \]. \[\square\]

**Theorem 1.5.10.** Fix \( j \in J^+ \). Let \( M \) be a \( \partial_j \)-differential module over an open annulus \( A_{K}^{1}(\alpha, \beta) \) such that \( M \otimes F_{e-r} \) has pure intrinsic \( \partial_j \)-radii \( \omega^{a r + b} < 1 \) for any \( r \in (-\log \beta, -\log \alpha) \). (This in particular says that \( f_{1}^{(j)}(M, r) = \cdots = f_{\dim M}^{(j)}(M, r) \) is an affine function of slope \(-a\).) Let \( e \) be the prime-to-p part of the denominator of \( a \) and let \( A_{K}^{1}(\alpha^{1/e}, \beta^{1/e}) \) be the open annulus with coordinate \( t^{1/e} \). Then, by replacing \( K \) by a finite tamely ramified extension, we have a canonical decomposition \( M = \bigoplus_{\theta \in K_{alg}^{(b)}} M_{\theta} \) of \( \partial_j \)-differential modules over \( A_{K}^{1}(\alpha^{1/e}, \beta^{1/e}) \), where \( M_{\theta} \otimes F_{\eta} \) has pure refined intrinsic \( \partial_j \)-radii \( \theta^a \) for any \( \eta \in (\alpha, \beta) \). Moreover, by Galois descent, we may obtain the decomposition over \( A_{K}^{1}(\alpha, \beta) \) by grouping conjugates of \( \theta \)’s.

**Proof.** First of all, since defining a \( \partial_j \)-differential module only needs finite data, we may assume that \( \mathbb{Q} \cdot \log |K^X| \neq \mathbb{R} \).

The decomposition if exists is determined by the decomposition at each radius \( e^{-r} \in (\alpha, \beta) \); it is canonical. Thus, we may replace \( K \) by any finite tamely ramified extension when we need (and then obtain the decomposition over \( K \) by Galois descent). Also, it suffices to obtain the decomposition in a neighborhood of each radius in \( (\alpha, \beta) \).

Let \( r_0 \in (-\log \beta, -\log \alpha) \) be a point. We first assume that \( IR_{\partial_j}(M \otimes F_{e-r_0}) < 1 \) when \( p = 0 \) and \( IR_{\partial_j}(M \otimes F_{e-r_0}) < p^{-1/p(p-1)} \) when \( p > 0 \). (Note that this restriction still allows some non-visible radii.) By shrinking \( (\alpha, \beta) \) to a smaller neighborhood of \( r_0 \), we may assume that the above condition at \( r_0 \) holds for all points in \((-\log \beta, -\log \alpha) \). Pick a point \( r_1 \in (-\log \beta, -\log \alpha) \) which does not lie in \( \mathbb{Q} \cdot \log |K^X| \).

Let \( \theta^a \in \mathbb{Q} \otimes \partial_j(M \otimes F_{e-r_1}) \) be a refined intrinsic \( \partial_j \)-radius, with multiplicity \( \mu \). Since \( M \otimes F_{e-r_1} \) has pure intrinsic \( \partial_j \)-radii \( \omega^{a r + b} \), we have \( \theta^a \in K_{alg}^{(b)} \simeq t^{a}K_{alg}^{(b)} \), where the latter isomorphism follows from our choice \( r_1 \notin \mathbb{Q} \cdot \log |K^X| \). Hence, \( \theta \in K_{alg}^{(b)} \). Now, applying the construction in Example \[\text{1.5.7} \] gives a \( \partial_j \)-differential module \( L_{x,a,(n)}^{(j)} \) over \( A_{K}^{1}(\alpha^{1/e}, \beta^{1/e}) \) of pure \( \partial_j \)-radii \( \omega^{a r + b} \) and pure intrinsic \( \partial_j \)-radii \( \theta^a \) at radius \( [t^{1/e}] = e^{-r/e} \) for \( r \in (-\log \beta, -\log \alpha) \), where the coordinate on the annulus is \( t^{1/e} \).
Denote \(N = M \otimes (\mathcal{L}^{(j)}_{\mathcal{L},a}(n))^{\vee}\); it is a \(\partial_j\)-differential module over \(A_{K'}^{1}(\alpha^{1/e}, \beta^{1/e})\). Denote \(F'_{e^{-r'}} = F_{e^{-r'}}(t^{1/e}) \otimes K'\). Then \(IR_{\partial_j}(N \otimes F'_{e^{-r'}}) \leq \omega e^{ar'+b}\) for \(r' \in (-\log \beta, -\log \alpha)\). By Proposition \[1.3.19\] and Theorem \[1.3.26\] we have
\[
f^{(j)}_1(M, r_1) = f^{(j)}_1(N, r_1) = f^{(j)}_{(\dim M - \mu)d}(N, r_1) > f^{(j)}_{(\dim M - \mu)d+1}(N, r_1).
\]
By Theorem \[1.5.6(d)\], the same has to be true for all \(r' \in (-\log \beta, -\log \alpha)\) in place of \(r_1\) because a convex function below a linear function is same as the linear function if the two functions touch at some point. By Theorem \[1.5.4(e)\], we have a decomposition of \(\partial_j\)-differential module \(N = N_0 \oplus N_1\), where \(N_0\) accounts for the first \((\dim M - \mu)d\) subsidiary \(\partial_j\)-radii and \(N_1\) accounts for the other \(\mu d\) subsidiary \(\partial_j\)-radii. In particular, \(N_0 \otimes F'_{e^{-r'}}\) has pure intrinsic \(\partial_j\)-radii \(\omega e^{ar'+b}\) and \(IR_{\partial_j}(N_1 \otimes F'_{e^{-r'}}) > \omega e^{ar'+b}\). By the same argument as in Theorem \[1.3.26\] this implies that \(M\) has a decomposition of \(\partial_j\)-differential modules over \(A_{K'}^{1}(\alpha^{1/e}, \beta^{1/e})\):
\[
M \otimes_{K^{(b)}}(\mathcal{L}^{(j)}_{\mathcal{L},a}(n))^{\vee} = N_1 \oplus M'.
\]
such that \(M_0 \otimes (\mathcal{L}^{(j)}_{\mathcal{L},a}(n))^{\vee} = N_1\) and \(M' \otimes (\mathcal{L}^{(j)}_{\mathcal{L},a}(n))^{\vee} = N_0\). By Proposition \[1.3.28\] and Lemma \[1.5.9\] for all \(r' \in (-\log \beta, -\log \alpha)\), \(M_0 \otimes F'_{e^{-r'}}\) has pure refined intrinsic \(\partial_j\)-radii \(\theta t^a\) and the refined intrinsic \(\partial_j\)-radii \(M' \otimes F'_{e^{-r'}}\) does not contain \(\theta t^a\). We obtain the decomposition by applying the above process to every \(\theta\).

Now, it suffices to deal with the case when \(p > 0\) and \(IR_{\partial_j}(M \otimes F_{e^{-r'}}) \in \mathbb{Z}^{p-1/p^(-1)}(1)\). In this case, we have \(\partial_j\)-Frobenius antecedent of \(M\) in a neighborhood of \(r\). The decomposition follows from the decomposition of the \(\partial_j\)-Frobenius antecedent or, more generally, iterative \(\partial_j\)-Frobenius antecedent (until the intrinsic \(\partial_j\)-radii fall in the range above.)

**Remark 1.5.11.** The artificial reduction to the case \(Q \cdot \log |K^x| \neq \mathbb{R}\) is to simplify the proof of \(\theta \in \kappa^{(b)}_{\text{alg}}\). This can be avoided using Newton polygons if the intrinsic radii is not \(\omega\). If the intrinsic radii is constantly \(\omega\), one may alternatively use Frobenius pushforward to reduce to the visible case.

**Theorem 1.5.12.** Let \(M\) be a \(\partial_{j+}\)-differential module over an open annulus \(A_{K'}^{1}(I)\) such that \(M \otimes F_{e^{-r}}\) has pure intrinsic radii \(\omega e^{ar+b} < 1\) for \(r \in -\log(I)\). Let \(e\) denote the prime-to-\(p\) part of the denominator of \(a\). Then for some finite tamely ramified extension \(K'/K\), there exists a canonical decomposition
\[
M = \bigoplus_{\vartheta} M_{\vartheta}
\]
over \(A_{K'}^{1}(1/e)\), where the direct sum runs through all \(\vartheta \in \bigoplus_{j \in I} \kappa^{(b)}_{\text{alg}} \oplus \kappa^{(b)}_{\text{alg}} \frac{du}{u} \oplus \kappa^{(b)}_{\text{alg}} \frac{dt}{T}\), such that \(M_{\vartheta} \otimes F_{\eta}^r\) has pure refined intrinsic radii \(t^a \vartheta\) for all \(\eta \in -\log I\).

We may obtain the decomposition \[1.5.13\] over \(K\) if we group Galois conjugates of \(\vartheta\)’s.

**Proof.** We first assume that \(0 \notin I\). We need to show that the relevant \(\partial_j\)-radii are log-affine functions. Without loss of generality, we assume that \(M\) is not a direct sum of two nonzero sub-\(\partial_{j+}\)-modules. Hence, we have the dichotomy stated in Theorem \[1.5.6(f)\]. We apply Theorem \[1.5.10\] to the \(\partial_j\) for which case (1) of Theorem \[1.5.6(f)\] holds for \(M\). The decompositions for different \(\partial_j\)’s are compatible. This gives the desired decomposition.

Now, we deal with the case when \(I = [0, \beta]\). Since, we have already proved theorem over \((\alpha, \beta)\) for any \(\alpha > 0\), it suffices to find the decomposition \[1.5.13\] over for \(I = [0, \alpha)\) with \(\alpha\) sufficiently
small. We assume that $IR_{\partial_0}(M \otimes F_\eta) = 1$. Thus, by making $\alpha$ a bit smaller, $M \otimes A^1_K[0, \alpha]$ is a trivial $\partial_0$-differential module and hence is the pull back of a $\partial_\Gamma$-differential module $M_0$ over $K$. The decomposition (1.5.13) follows from the decomposition of $M_0$ over $K$ by Theorem 1.3.26.

Similarly, we have the same result for refined extrinsic radii, but only over $A^1_K(I)$ since base changing to $A^1_K(I^{1/e})$ would change the extrinsic radii.

**Theorem 1.5.14.** Assume that $|u_j| = 1$ for all $j \in J$. Let $M$ be a $\partial_{J^+}$-differential module over an open annulus $A^1_K(I)$, where $I \subseteq [0, 1)$. Assume that $M \otimes F_{\eta^{-r}}$ has pure extrinsic radii $\omega^{ar+b} < e^{-r}$ for $r \in -\log(\hat{I})$. Let $e$ denote the prime-to-$p$ part of the denominator of $a$ and let $\mu_e$ be the set of $e$-th roots of unity. Then there exists a canonical decomposition

$$M = \bigoplus_{\{\hat{\partial}\}} M_{\{\hat{\partial}\}}$$

(1.5.15)

over $A^1_K(I)$, where the direct sum runs through all Galois conjugacy classes of $\mu_e \hat{\partial}$ in $\bigoplus_{j \in J} \kappa^{(b)}_{\partial_\Gamma} du_j \oplus \kappa_{\partial_\Gamma}$, such that the refined extrinsic radii $M_{\{\hat{\partial}\}} \otimes F_{\eta}$ is exactly the Galois conjugates of $\mu_e \hat{\partial}a$ with same multiplicity, for all $\eta \in I$.

**Proof.** The proof goes verbatim as in Theorem 1.5.12, except that we need to cite the decomposition after Galois descent. Note also that when we descent a $\partial_{J^+}$-differential module from $A^1_K(I^{1/e})$ to $A^1_K(I)$, we pick up the $e$-th roots of unity $\mu_e$. □

### 1.6 Refined differential conductors

For a solvable differential module over an annulus with outer radius 1, we define the notion of differential conductors, as well as refined differential conductors if the differential module has pure differential break.

We continue to assume Hypothesis 1.5.1. Moreover, we assume $p > 0$ in this subsection.

**Definition 1.6.1.** Let $M$ be a $\partial_{J^+}$-differential module of rank $d$ on $A^1_K(\eta_0, 1)$ for some $\eta_0 \in (0, 1)$. We say that $M$ is solvable if $IR(M \otimes F_{\eta}) \to 1$ as $\eta \to 1^-$. 

**Theorem 1.6.2.** Let $M$ be a solvable $\partial_{J^+}$-differential module of rank $d$ over $A^1_K(\eta_0, 1)$ for some $\eta_0 \in (0, 1)$. Then by making $\eta_0$ closer to 1, there exists a decomposition $M = \hat{M}_1 \oplus \cdots \oplus \hat{M}_{\gamma}$ over $A^1_K(\eta_0, 1)$ and nonnegative distinct rational numbers $b_1, \ldots, b_{\gamma}$ with $b_i \cdot \text{rank}(\hat{M}_i) \in \mathbb{Z}$, such that

$$IR(M_i \otimes F_{\eta}; j) = \eta^{b_i} \quad (i = 1, \ldots, \gamma; j = 1, \ldots, \text{rank}(M_i); \eta \in (\eta_0, 1)).$$

Keep the same hypothesis and assume moreover that $|u_j| = 1$ for all $j \in J$. By making $\eta_0$ closer to 1, there exists a decomposition $M = \hat{M}_1 \oplus \cdots \oplus \hat{M}_{\gamma}$ over $A^1_K(\eta_0, 1)$ and nonnegative distinct rational numbers $b_1, \ldots, b_{\gamma}$ with $b_i \cdot \text{rank}(\hat{M}_i) \in \mathbb{Z}$, such that

$$R(M_i \otimes F_{\eta}; j) = \hat{\eta}^{b_i} \quad (i = 1, \ldots, \gamma; j = 1, \ldots, \text{rank}(\hat{M}_i); \eta \in (\eta_0, 1)).$$

**Proof.** The two statements can be proved using the same argument, as follows. By Theorems 1.5.6(a)(b)(d), for $l = 1, \ldots, d$, the functions $dF_l(M, r)$ and $d\hat{F}_l(M, r)$ on $(0, -\log(\eta_0))$ are continuous, convex, and piecewise affine with integer slopes. By hypothesis, $dF_l(M, r) \to 0$ and hence $d\hat{F}_l(M, r) \to 0$ as


\[ r \to 0^+; \] because of this and the fact that \( d!F_l(M, r) \geq 0 \) and \( d!\hat{F}_l(M, r) \geq 0 \) for all \( r \), the slopes of \( F_l(M, r) \) and \( \hat{F}_l(M, r) \) are forced to be nonnegative. Hence there is a least such slope, that is, \( d!F_l(M, r) \) and \( d!\hat{F}_l(M, r) \) are linear in a right neighborhood of \( r = 0 \).

We can thus choose \( \eta_0 \to 1^- \) so that \( d!F_l(M, r) \) and \( d!\hat{F}_l(M, r) \) are linear on \((0, -\log \eta_0)\) for \( l = 1, \ldots, d \). We obtain the desired decompositions by Theorem 1.5.6(e), respectively; the integrality of \( b_i \cdot \text{rank}(M_i) \) and \( \hat{b}_i \cdot \text{rank}(\hat{M}_i) \) follows from the fact that \( F_{\text{dim}M_i}(M_i, r) \) and \( \hat{F}_{\text{dim}M_i}(\hat{M}_i, r) \) have integral slopes, again by Theorem 1.5.6(b).

**Definition 1.6.3.** Let \( M \) be a solvable \( \partial_{J^+} \)-differential module of rank \( d \) over \( A_{/K}(\eta_0, 1) \) for some \( \eta_0 \in (0, 1) \). Define the **differential log-breaks** of \( M \) to be the multiset consisting of \( b_i \) from Theorem 1.6.2 above with multiplicity rank \( (M_i) \); we use \( b_{\text{log}}(M; 1) \geq \cdots \geq b_{\text{log}}(M; d) \) to denote them in decreasing order. We define the **differential Swan conductor** of \( M \) to be the sum of the differential log-breaks, that is, \( \text{Swan}(M) = \sum_{i=1}^d b_i \cdot \text{rank}(M_i) \); it is a nonnegative integer by Theorem 1.6.2 above.

When \( M \) has pure differential log-breaks, we define the **refined Swan conductors** of \( M \), denoted by \( \mathcal{I}\Theta(M) \), to be the (multi)set of \( \vartheta \) in (1.5.13) with multiplicity rank \( (M_\vartheta) \).

Similarly, when \( \{u_j\} = 1 \) for all \( j \in J \), we define the **differential nonlog-breaks** to be the multiset consisting of \( \hat{b}_i \) from Theorem 1.6.2 above with multiplicity rank \( (M_i) \); we use \( b_{\text{nonlog}}(M; 1) \geq \cdots \geq b_{\text{nonlog}}(M; d) \) to denote them in decreasing order. We define the **differential Artin conductor** of \( M \) to be the sum of the differential nonlog-breaks; it is also a nonnegative integer by Theorem 1.6.2 above.

When \( M \) has pure differential nonlog-breaks, we define **refined Artin conductors** of \( M \), denoted by \( \mathcal{E}\Theta(M) \), to be the (multi)set of Galois conjugacy classes of \( \{\mu_\vartheta\} \) in (1.5.15) with multiplicity the same as that of \( M_{\{\vartheta\}} \otimes F_\eta \) for any \( \eta \in (\eta_0, 1) \).

## 2 Refined differential conductors for Galois representations

### 2.1 Construction of differential modules

We keep the notation from Subsection 1.1. Throughout this section, we assume that \( p > 0 \) is a prime number.

**Definition 2.1.1.** For a field \( \kappa \) of characteristic \( p > 0 \), a **\( p \)-basis** of \( \kappa \) is a set \( (b_j)_{j \in J} \subset \kappa \) such that the products \( b_j^{e_j} \), where \( e_j \in \{0, 1, \ldots, p - 1\} \) for all \( j \in J \) and \( e_j = 0 \) for all but finitely many \( j \), form a basis of the vector space \( \kappa \overline{\kappa}p \).

**Notation 2.1.2.** Let \( k \) be a complete discretely valued field of characteristic \( p > 0 \). Let \( \pi_k \) be a uniformizer of \( k \), generating the maximal ideal \( \mathfrak{m}_k \) in the ring of integers \( \mathcal{O}_k \). Let \( \kappa = \kappa_k \) denote the residue field. Let \( \bar{\kappa} = \kappa^{alg} \) denote an algebraic closure of \( \kappa \). We choose and fix a non-canonical isomorphism \( k \simeq \kappa((\pi_k)) \). We fix a \( p \)-basis \( \bar{b}_J \) of \( \bar{\kappa} \) and let \( b_J \) be the preimage of them via the isomorphism above. Then \( (b_J, \pi_k) \) gives a set of \( p \)-basis of \( k \). Let \( k_0 = \cap_{n \in \mathbb{N}}k^n = \cap_{n \in \mathbb{N}}k^{p^n} \). We know that \( d\pi_k \) and \( db_J \) for a basis of \( \Omega^1_{\mathcal{O}_k} \) over \( \mathcal{O}_k \).

Let \( \mathcal{O}_K \) denote the Cohen ring of \( \kappa \) with respect to \( \bar{b}_J \) and let \( B_J \subset \mathcal{O}_K \) be the canonical lifts of the \( p \)-basis. Denote \( \mathcal{K} = \text{Frac}\mathcal{O}_K \). Also, we use \( \mathcal{O}_{K_0} \) to denote the the ring of Witt vectors of \( k_0 \) and \( K_0 = \mathcal{O}_{K_0}(\frac{1}{p}) \).
Notation 2.1.3. For an extension $k'/k$ of complete discretely valued field, the (naïve) ramification degree of $k'/k$ is simply the index of the valuation of $k$ in that of $k'$.

We say that $k'/k$ is tamely ramified if $p 
mid e$ and the residue field extension $k_{k'}/k_k$ is separable, that is $k_{k'}$ is algebraic and separable over $k_k(x_{\alpha}; \alpha \in \Lambda)$ for some transcendental elements $x_{\alpha}$ and an index set $\Lambda$. If moreover, $e = 1$, we call $k'/k$ unramified.

Notation 2.1.4. By a representation of $G_k$, we mean a continuous homomorphism $\rho : G_k \rightarrow \text{GL}(V_\rho)$, where $V_\rho$ is a vector space over a (topological) field $F$ of characteristic zero. We say that $\rho$ is a $p$-adic if $F$ is a finite extension of $\mathbb{Q}_p$.

Let $F$ be a finite extension of $\mathbb{Q}_p$. Let $\mathcal{O}$ and $\mathbb{F}_q$ denote its ring of integers and residue field, respectively, where $q$ is a power of $p$. Write $\mathbb{Z}_q$ for the ring of Witt vectors $W(\mathbb{F}_q)$ and $\mathbb{Q}_q$ for its fraction field. By an $\mathcal{O}$-representation of $G_k$, we mean a continuous homomorphism $\rho : G_k \rightarrow \text{GL}(\Lambda_\rho)$ with $\Lambda_\rho$ a finite free $\mathcal{O}$-module.

For $\rho$ a $p$-adic representation or an $\mathcal{O}$-representation, we say that $\rho$ has finite local monodromy if the image of the inertia group $I_k$ is finite.

We always assume that $\mathbb{F}_q \subseteq k_0$. Denote $K' = K F$. Since $F/\mathbb{Q}_q$ is totally ramified, we have $O_{K'} \cong O_K \otimes_{\mathbb{Z}_q} \mathcal{O}$ for the ring of integers.

Notation 2.1.5. We write $\mathcal{R}^\eta_{K'} = K'\langle \eta/T, T \rangle$ for $\eta \in (0, 1)$ and use $\mathcal{R}_{K'} = \bigcup_{\eta \in (0, 1)} \mathcal{R}^\eta_{K'}$ to denote the Robba ring over $K'$. Let $\mathcal{R}_{K'}^{\text{int}}$ be the elements in $\mathcal{R}_{K'}$ whose 1-Gauss norm is bounded by 1; it is a Henselian discrete valuation ring, with residue field $k$, where the reduction of $T$ is $\pi_k$. For $\eta \in (0, 1)$, we use $F'_\eta$ to denote the completion of $K'(T)$ with respect to the $\eta$-Gauss norm.

A Frobenius lift $\phi$ is an endomorphism of $\mathcal{R}_{K'}^{\text{int}}$ which lifts the natural $q$-th power Frobenius on $k$. The Frobenius lift naturally extends to an action on $\mathcal{R}_{K'}$. A standard Frobenius lift is a Frobenius lift which sends $T$ to $T^q$ and $B_j$ to $B_j^q$ for any $j \in J$.

Also, $d B_j$ and $dT$ form a basis of differentials $\Omega^1_{\mathcal{R}_{K'}^{\text{int}}}$, $\Omega^1_{\mathcal{R}_{K'}}$ and $\Omega^1_{\mathcal{R}_{K'}^{\eta}}$ for any $\eta \in (0, 1)$. Denote their dual basis to be $\partial_0 = \partial/\partial T, \partial_j = \partial/\partial B_j$ for $j \in J$. Then a $\nabla$-module over $\mathcal{R}_{K'}$ is just a $\partial_{J^+}$-differential module.

Definition 2.1.6. Let $\phi$ be a Frobenius lift. Let $R = \mathcal{R}_{K'}, \mathcal{R}^\eta_{K'}$, or $\mathcal{R}_{K'}^{\text{int}}$. A $(\phi, \nabla)$-module $M$ over $R$ is a $\partial_{J^+}$-differential module together with an isomorphism $\Phi : \phi^* M \rightarrow M$ of $\partial_{J^+}$-differential modules.

Theorem 2.1.7. For any Frobenius lift $\phi$, we have an equivalence of categories between the category of $\mathcal{O}$-representations with finite local monodromy and the category of $(\phi, \nabla)$-modules over $\mathcal{R}_{K'}^{\text{int}}$. Moreover, all $(\phi, \nabla)$-modules can be realized over $\mathcal{R}^\eta_K$, for some $\eta \in (0, 1)$. This $(\phi, \nabla)$-module is independent of the choice of the $p$-basis.

Proof. We refer to [Ked07 Section 3] or [Xia11 Subsection 2.2] for the construction of the functor. \hfill $\square$

Definition 2.1.8. For a $p$-adic representation $\rho$ of $G_k$ with finite local monodromy, we choose an invariant $\mathcal{O}$-lattice $\Lambda_\rho$ of $V_\rho$, stable under the action of $G_k$; this gives an $\mathcal{O}$-representation of $G_k$. By Theorem 2.1.7 we obtain a $(\phi, \nabla)$-module over $\mathcal{R}_{K'}^{\text{int}}$. We call its base change to $\mathcal{R}_{K'}$ the differential module associated to $\rho$, denoted by $E_\rho$. This does not depend on the choice of the lattice $\Lambda_\rho$.

Hypothesis 2.1.9. Assume that $\kappa$ as a finite $p$-basis $\bar{B}_J$, where $J = \{1, \ldots, m\}$. 

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Notation 2.1.10. Let $J = \{1, \ldots, m\}$ for notational convenience. We save the notation $j$ and $m$ for indexing $p$-basis. We also use $J^+$ to denote $J \cup \{0\}$, where 0 refers to the uniformizer $\pi_K$.

Proposition 2.1.11. Let $\phi$ be the standard $q$th-power Frobenius lift on $\mathcal{R}_{\text{int}}^{\eta}$. Then the pull back by Frobenius $\phi : F_{\eta}^q \to F_{\eta}^q$ is the same as the iterative Frobenius pullback $\varphi(\delta_0, \lambda) \circ \cdots \circ \varphi(\delta_m, \lambda)$, where $q = p^\lambda$.

Proof. We may assume that $K'$ contains $\zeta_q$ a $q$-th root of unity. It suffices to show that the image $\phi(F_{\eta}^q)$ is stable under the action of $(\mathbb{Z}/q\mathbb{Z})^{m+1}$ in the sense of Construction [Ked10, Chap. 17] (one $\mathbb{Z}/q\mathbb{Z}$ from each $\partial_j$-Frobenius lift for $j \in J^+$) and that the degree of $F_{\eta}^q$ over $\phi(F_{\eta}^q)$ is $q^{m+1}$.

For $\tilde{\iota} = (i_0, \ldots, i_m) \in (\mathbb{Z}/q\mathbb{Z})^{m+1}$, we have $T^{\tilde{\iota}} = \zeta_q^{i_0}T$ and $(B_j)^{\tilde{\iota}} = \zeta_q^{i_j}B_j$ for $j \in J$. Hence, both the standard Frobenius lift $\phi$ and $(\cdot)^{\tilde{\iota}} \circ \phi$ are continuous homomorphism from $O_K[T]$ to itself sending $B_j$ to $B_j^q$ and $T$ to $T^q$. By the functoriality of Cohen rings (e.g., [Xia11] Proposition 2.1.8), they must be the same. Hence the image of $\phi$ is stable under the $(\mathbb{Z}/q\mathbb{Z})^{m+1}$-action.

To see that $F_{\eta}^q$ has degree $q^{m+1}$ over $\phi(F_{\eta}^q)$, it suffices to show that the degree of $K$ has degree $q^m$ over $\phi(K)$, because the $T$ part is obvious. Note that $\phi : O_K \to O_K$ is a flat homomorphism since $O_K$ is torsion free $O_K$-module via $\phi$. Hence the degree of $\phi : K \to K$ is the same as the degree of $\phi : k \to k$, which is $q^m$. This concludes the proposition. \hfill $\square$

Proposition 2.1.12. Let $\phi$ be the standard $q$th-power Frobenius lift on $\mathcal{R}_{\text{int}}^{\eta}$. Let $E$ be a $(\phi, \nabla)$-module over $A_{\text{K'}}^1[\eta_0, 1]$ for some $\eta_0 \in (0, 1)$. Then $E$ is solvable.

Proof. This is well-known to experts. However, we do not have a good reference in our case. We include the following proof. By Lemma [1.2.18(a)], we have

$$f_i(\phi^* M, r) = \max \{p^{-\lambda} f_i(M, qr), p^{1-\lambda} (f_i(M, qr) - \log p), \ldots, f_i(M, qr) - \lambda \log p\},$$

where $\lambda = \log_p q$. Since $\phi^* M \xrightarrow{\sim} M$, the function $g_i(M) = \limsup_{r \to 0^+} f_i(M, r)$ satisfies

$$g_i(M) = \max \{p^{-\lambda} g_i(M), p^{1-\lambda} (g_i(M) - \log p), \ldots, g_i(M) - \lambda \log p\}.$$ 

This forces $g_i(M)$ to be zero. By the continuity of $f_i(M, r)$ and the convexity of $F_i(M, r)$ in Theorem [1.5.5.6] $\lim_{r \to 0^+} f_i(M, r) = 0$. Hence, $E$ is solvable. \hfill $\square$

Proposition 2.1.13. Let $\phi$ be the standard $q$th-power Frobenius lift and let $\phi'$ be another Frobenius lift on $\mathcal{R}_{\text{int}}^{\eta}$. Assume that $E$ is a $(\phi, \nabla)$-module over $A_{\text{K'}}^1[\eta, 1]$ for some $\eta \in (0, 1)$. Then $E$, restricted to $A_{\text{K'}}^1[\eta, 1]$ for some $\eta \in [\eta_0, 1)$, is naturally equipped with a $(\phi', \nabla)$-module structure.

Proof. Define the Frobenius structure for $\phi'$ by Taylor series as follows. For $v \in E$,

$$\phi'(v) = \sum_{e_j+ = 0}^\infty \frac{(\phi'(T) - \phi(T))^e \prod_{j \in J} (\phi'(B_j) - \phi(B_j))^e_j}{(e_j+)!} \phi\left(\frac{\partial e_0 \partial e_1 \cdots \partial e_m}{\partial T e_0 \partial B_1 e_1 \cdots \partial B_m e_m}(v)\right).$$

Since $|\phi'(T) - \phi(T)|_1 < 1$ and $|\phi'(B_j) - \phi(B_j)|_1 < 1$ for all $j \in J$, we have the same inequality for $\eta$-Gauss norm when $\eta \in [\eta_0, 1)$ sufficiently close to 1. Hence the expression for $\phi'$ converges under $| \cdot |_\eta$ for $\eta \in (\eta_0, 1)$ sufficiently close to 1 and also for $| \cdot |_1$. \hfill $\square$

Remark 2.1.14. One may also approach the results of this subsection without using the standard Frobenius first but using a generalized version of Lemma [1.2.18(a)] for non-centered Frobenius. This point of view is taken in [Ked10] Chap. 17.
2.2 Differential conductors

In this subsection, we detailize the results about differential modules in [Xia11] and [Ked07] by analyzing the breaks associated to each element in the $p$-basis of $k$.

We do not assume Hypothesis 2.1.9 for now.

**Definition 2.2.1.** For a $p$-adic representation $\rho$ of $G_k$ of finite local monodromy, let $l$ be the extension corresponding to $\text{Ker} \rho$. We may choose the $p$-basis $\{c_J, \pi_l\}$ of $l$, such that $\pi_l$ is an uniformizer, $c_J \subset O_l^\times$, and for a finite subset $J_0 \subset J$, $c_J \backslash J_0 \subset O_k$. Let $k^\wedge$ be the completion of $k(c_1/n_n; n \in N)$ and $k^\wedge$ verifies Hypothesis 2.1.9. We define the nonlog-breaks (resp. log-breaks) of $\rho$ to be those of $\rho|_{G_{k^\wedge}}$. Their sums are called the Artin (resp. Swan) conductors of $\rho$, denoted by $\text{Art}(\rho)$ (resp. $\text{Swan}(\rho)$). These do not depend on the choice of $p$-basis or $J_0$, by [Ked07, Proposition 2.6.6].

**Definition 2.2.2.** Put $\text{Fil}^0 G_k = G_k$ and $\text{Fil}^a G_k = I_k$ for $a \in (0, 1]$. For $a > 1$, let $R_a$ be the set of finite image representations $\rho$ with nonlog-break less than $a$. Define $\text{Fil}^a G_k = \bigcap_{\rho \in R_a} (I_k \cap \ker(\rho))$ and write $\text{Fil}^{a+} G_k$ for the closure of $\bigcup_{b \geq 0} \text{Fil}^b G_k$. This defines a filtration on $G_k$ such that for all finite image representation $\rho$, $\rho(\text{Fil}^a G_k)$ is trivial if and only if $\rho \in R_a$.

Similarly, put $\text{Fil}^0_{\log} G_k = G_k$. For $a > 0$, let $R_{a, \log}$ be the set of finite image representations $\rho$ with log-break less than $a$. Define $\text{Fil}^a_{\log} G_k = \bigcap_{\rho \in R_{a, \log}} (I_k \cap \ker(\rho))$ and write $\text{Fil}^{a+}_{\log} G_k$ for the closure of $\bigcup_{b \geq 0} \text{Fil}^b_{\log} G_k$. This defines a logarithmic filtration on $G_k$ such that for all finite image representation $\rho$, $\rho(\text{Fil}^a_{\log} G_k)$ is trivial if and only if $R_{a, \log}$. For a finite Galois extension $l/k$, the above filtrations induce filtrations on the Galois group $G_{l/k}$ by $G_{l/(k, \log)} = G_{l/\log}(G_k/G_l)$ and $G_{l/(k, \log)}^{a+} = G_{l/\log}(G_k^{a+}/G_l)$, for $a \geq 0$. We define the (log-)ramification breaks to be the numbers $b$ for which $G_{l/(k, \log)}^b = G_{l/(k, \log)}^{b+}$. We order them as $b_{(\log)}(l/k; 1) \geq b_{(\log)}(l/k; 2) \geq \cdots$. In particular, if $\rho$ is a faithful representation of $G_{l/k}$, we have $b_{(\log)}(\rho) = b_{(\log)}(l/k)$.

**Theorem 2.2.3.** Differential conductors satisfy the following properties:

(a) For any representation $\rho$ of finite local monodromy,

$$\text{Art}(\rho) = \sum_{a \in \mathbb{Q}_{\geq 0}} a \cdot \dim (V_{\rho}^{\text{Fil}^{a+} G_k}/V_{\rho}^{\text{Fil}^a G_k}) \in \mathbb{Z}_{\geq 0},$$

$$\text{Swan}(\rho) = \sum_{a \in \mathbb{Q}_{\geq 0}} a \cdot \dim (V_{\rho}^{\text{Fil}_{\log}^{a+} G_k}/V_{\rho}^{\text{Fil}_{\log}^a G_k}) \in \mathbb{Z}_{\geq 0}.$$

(b) Let $k'/k$ be a (not necessarily finite) extension of complete discretely valued fields. If $k'/k$ is unramified, then $\text{Fil}^a G_{k'} = \text{Fil}^a G_k$ for $a > 0$. If $k'/k$ is tamely ramified with ramification index $e < \infty$, then $\text{Fil}^a_{\log} G_{k'} = \text{Fil}^a_{\log} G_k$ for $a > 0$.

(c) For $a > 0$, $\text{Fil}^{a+} G_k \subseteq \text{Fil}^a_{\log} G_k \subseteq \text{Fil}^a G_k$.

(d) For graded pieces, we have

$$\text{Fil}^a G_k/\text{Fil}^{a+} G_k = \begin{cases} 0 & a \notin \mathbb{Q} \\
 \text{an abelian group killed by } p & a \in \mathbb{Q} \end{cases}$$

$$\text{Fil}^a_{\log} G_k/\text{Fil}^{a+}_{\log} G_k = \begin{cases} 0 & a \notin \mathbb{Q} \\
 \text{an abelian group killed by } p & a \in \mathbb{Q} \end{cases}$$
For each Proposition 2.2.5.

By applying the same argument of Proposition 2.1.12 to intrinsic Proof.

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write direction for uniformizer is "generically dominant". This motivates the following lemma.

Definition 2.2.4. Let \( \rho \) be a representation of \( G_k \) with finite local monodromy. The **log-breaks** of \( \rho \) are the differential log-breaks of \( E_\rho \), as a solvable \( \partial_+ \)-differential module. We write \( b_{\log}(\rho; l) = b_{\log}(E_\rho; l) \) for \( l = 1, \ldots, \dim \rho \).

Similarly, The **nonlog-breaks** of \( \rho \) are the differential nonlog-breaks of \( E_{\rho/\rho^l} \) and 0 with multiplicity \( \dim \rho^l \), where \( \rho^l \) is the maximal subrepresentation of \( \rho \) on which \( I_k \) acts trivially. We write \( b_{\log}(\rho; l) = b_{\log}(E_{\rho/\rho^l}; l) \) for \( l = 1, \ldots, \dim(\rho/\rho^l) \), and \( b_{\log}(\rho; \dim(\rho/\rho^l) + 1) = \cdots = b_{\log}(\rho; \dim \rho) = 0 \).

For simplicity, we also write \( b_{\log}(\rho) = b_{\log}(\rho; 1) \) and \( b_{\log}(\rho) = b_{\log}(\rho; 1) \) for the highest nonlog-break and the highest log-break.

**Proposition 2.2.5.** For each \( j \in J^+ \), there is a ramification break \( b_j(\rho) \) associated to \( b_j \) \( (j \in J) \) or \( \pi_k \) \( (j = 0) \), such that \( R\partial_j(E_\rho \otimes F_\eta^j) = \eta^{b_j(\rho)} \) for all \( \eta \rightarrow 1^- \). Moreover,

\[
b_{\log}(\rho) = \max\{b_j(\rho)\}, \quad b_{\log}(\rho) = \max\{b_0(\rho) - 1; b_j(\rho) \text{ for } j \in J\}.
\]

**Proof.** By applying the same argument of Proposition 2.1.12 to intrinsic \( \partial_j \)-radii, we know \( I R\partial_j(E_\rho \otimes F_\eta^j) = I R\partial_j(E_\rho \otimes F_\eta^j \eta^g) \) for \( \eta \rightarrow 1^- \). Therefore, by the convexity given by Theorem 1.5.4(d), \( f_1^{(j)}(E_\rho, r) \) is affine as \( r \rightarrow 0^+ \). The proposition follows.

**Definition 2.2.6.** We call \( b_j(\rho) \) the breaks by \( p \)-basis of \( \rho \) with respect to the lifted \( p \)-basis \( b_j \) and the uniformizer \( \pi_k \).

When doing operation on \( k \), we want to understand the corresponding effect on the \( b_j(\rho) \).

**Lemma 2.2.7.** Fix \( j_0 \in J \). Let \( b'_j(\rho) \) be the breaks by \( p \)-basis of \( \rho \) with respect to the lifted \( p \)-basis \( \{b_j \setminus \{j_0\}, b_{j_0} + \pi_k\} \) and the uniformizer \( \pi_k \). Then \( b'_j(\rho) = b_j(\rho) \) for \( j \in J \) and

\[
b'_0(\rho) = \begin{cases} \max\{b_0(\rho), b_{j_0}(\rho)\} & \text{if } b_0(\rho) \neq b_{j_0}(\rho), \\ b_0(\rho) & \text{if } b_0(\rho) = b_{j_0}(\rho). \end{cases}
\]

**Proof.** Let \( \partial'_j \) denote the derivations dual to the basis \( dB \setminus \{j_0\}, dT, d(B_{j_0} + T) \) of \( \Omega^1_{\mathcal{X}_{\text{int}}} \). Then \( \partial' = \partial_j \) and \( \partial'_0 = \partial_0 - \partial_{j_0} \). The lemma follows immediately.

**Remark 2.2.8.** This lemma is in fact much stronger than it looks. Applying the same argument to \( b_{j_0} + \alpha \pi_k \) for \( \alpha \in \mathbb{k}_0 \), we find out that for all but possibly one \( \alpha \in \mathbb{k}_0 \), \( b'_0(\rho) \geq b_{j_0}(\rho) \). So, the direction for uniformizer is “generically dominant”. This motivates the following lemma.
**Lemma 2.2.9.** Fix $j_0 \in J$. Let $\tilde{k}$ be the completion of $k(x)$ with respect to the 1-Gauss norm, equipped with lifted $p$-basis $\{b_{J \setminus \{j_0\}}, b_{j_0} + x \pi_k, x\}$ and the uniformizer $\pi_k$. Let $\tilde{\rho}$ be the representation $G_\tilde{k} \to GL(V_\rho)$. Let $\tilde{b}_{J + \cup (m+1)}(\tilde{\rho})$ denote the breaks by $p$-basis with respect to the lifted $p$-basis and the uniformizer above, where $\tilde{b}_{J \setminus \{j_0\}}(\tilde{\rho})$ corresponds to $b_{J \setminus \{j_0\}}, \tilde{b}_{j_0}(\tilde{\rho})$ corresponds to $b_{j_0} + x \pi_k$, $\tilde{b}_0(\tilde{\rho})$ corresponds to $\pi_k$, and $\tilde{b}_{m+1}(\tilde{\rho})$ corresponds to $x$. Then we have $\tilde{b}_j(\rho') = b_j(\rho)$ for $j \in J$, $\tilde{b}_{m+1}(\tilde{\rho}) = b_{j_0}(\rho) - 1$, $\tilde{b}_0(\tilde{\rho}) = \max\{b_0(\rho), b_{j_0}(\rho)\}$. In particular, $\tilde{b}_\log(\tilde{\rho}) = b_\log(\rho)$.

**Proof.** Let $\tilde{K}'$ denote the completion of $K'(X)$ with respect to the 1-Gauss norm, where $X$ is the canonical lift of $x$. Let $f : A^{1}_{\tilde{K}'[\eta_0, 1]} \to A^{1}_{K'[\eta_0, 1]}$ be the natural morphism. Then $f^*E_\rho$ is the differential module associated to $\rho'$. Let $\tilde{d}_{J + \cup (m+1)}$ be the differential operators corresponding to the $p$-basis $(b_{J \setminus \{j_0\}}, b_{j_0} + x \pi_k, \pi_k)$. Then under the identification by $f^*$, we have

$$\tilde{d}_J = \partial_J, \quad \tilde{d}_{m+1} = T \partial_{j_0}, \quad \tilde{d}_0 = \partial_0 - X \partial_{j_0}.$$

(2.2.10)

The lemma follows because $X$ is transcendental over $K'$.

**Lemma 2.2.11.** Fix $j_0 \in J$. Let $k' = k(b^{1/p}_{j_0}; n \in \mathbb{N})$, equipped with lifted $p$-basis $\{b_{J \setminus \{j_0\}}, b_{j_0}^{1/p}\}$. Let $b'_j(\rho|_{G_\xi'})$ be the breaks by $p$-basis of $\rho|_{G_\xi'}$ with respect to the $p$-basis above. Then $b'_j(\rho|_{G_\xi'}) = b_j(\rho)$ for $j \in J' \setminus \{j_0\}$ and $b'_{j_0}(\rho|_{G_\xi'}) = \frac{1}{p} b_{j_0}(\rho)$.

**Proof.** Replacing $k$ by $k'$ is equivalent to use $\varphi(\partial_\xi)$ to pullback the differential module $E_\rho$. The lemma follows from Lemma 2.1.18(a) applying to $E \otimes F'_\eta$ when $\eta \to 1^-$. 

**Lemma 2.2.12.** Fix $j_0 \in J$. Let $k'$ denote the completion of $k(b^{1/p}_{j_0}; n \in \mathbb{N})$ equipped with lifted $p$-basis $b_{J \setminus \{j_0\}}$. Let $b'_j(\rho|_{G_\xi'})$ be the breaks by $p$-basis of $\rho|_{G_\xi'}$ with respect to this $p$-basis. Then $b'_j(\rho|_{G_\xi'}) = b_j(\rho)$ for $j \in J' \setminus \{j_0\}$.

**Proof.** This operation is equivalent to simply forgetting the $j_0$-direction.

**Situation 2.2.13.** Now, we study a particular case of base change, which will be useful in the comparison Theorem 3.1.4. This type of base change was first considered by Saito in [Sai09].

Fix $e \in \mathbb{N}$ possibly divisible by $p$. Let $k$ be as above, and let $k'$ be the completion of $k(x)$ with respect to the 1-Gauss norm, with uniformizer $\pi_\xi = \pi_k$. Denote $\tilde{k} = k'[u]/(u^e - x^{-1} \pi_k)$.

The residue field of $\tilde{k}$ is $k(\tilde{x})$; we choose its $p$-basis to be $(b_j, \tilde{x})$, and the uniformizer of $\tilde{k}$ to be $\pi_\tilde{k} = u$. We choose the isomorphism $\kappa(\tilde{x})(u) \simeq \tilde{k}$ to be the one that is compatible with Notation 2.1.2 and that sends $\tilde{x}$ to $x$; thus we obtain a $p$-basis for $\tilde{k}$ given by $b_j, x$ and $u$.

**Proposition 2.2.14.** Assume Hypothesis 2.1.9. The natural homomorphism $G_\tilde{k} \to G_k$ induces a homomorphism $Fil^a_{\log} G_\tilde{k} \to Fil^a_{\log} G_k$ for $a \in \mathbb{Q}_{\geq 0}$. Moreover, the induced homomorphism $Fil^a_{\log} G_\tilde{k}/Fil^{a+} G_\tilde{k} \to Fil^a_{\log} G_k/Fil^{a} G_k$ is surjective for $a \in \mathbb{Q}_{>0}$.

**Proof.** It suffices to show that, for a $p$-adic representation of $G_k$ with finite local monodromy and pure log-break $b_\log(\rho)$, the representation $\tilde{\rho} : G_\tilde{k} \to G_k \to GL(V_\rho)$ also has the same log-break. We employ the argument of breaks by $p$-basis. Let $\tilde{K}'$ be the completion of $K'(X)$ with respect to the 1-Gauss norm, where $X$ is the canonical lift of $\tilde{x}$. Then we have a natural map $f : A^{1}_{\tilde{K}'[\eta_{1/1}], 1} \to A^{1}_{K'[\eta, 1]}$ for $\eta \to 1^-$, sending $T$ to $XU^{e}$, where $U$ is the coordinate of the former annulus.
Let $\tilde{b}_0(\rho), \ldots, \tilde{b}_{m+1}(\rho)$ be the breaks by $p$-basis with respect to $b_J, x$ and the uniformizer $\pi_k = u$. Then $f^*E_\rho$ is the differential module associated to $\tilde{\rho}$, with the actions of $\tilde{\partial}_0 = \partial/\partial U, \tilde{\partial}_J = \partial/\partial B_J$, and $\tilde{\partial}_{m+1} = \partial/\partial X$. We have

$$\tilde{\partial}_J = \partial_J, \quad \tilde{\partial}_0 = eXU^{e-1}\partial_0, \quad \tilde{\partial}_{m+1} = U^{e}\partial_0. \quad (2.2.15)$$

By Theorem 1.4.19 we have $\tilde{b}_J(\rho) = eb_J(\rho), \tilde{b}_0(\rho) \leq eb_0(\rho) - (e - 1)$, and $\tilde{b}_{m+1}(\rho) = eb_0(\rho) - e$. (When $e$ is prime to $p$, the inequality becomes an equality.) In particular, $\tilde{b}_{m+1}(\rho) \geq \tilde{b}_0(\rho) - 1$.

Hence, we conclude that

$$b_{\log}(\rho) = \max \{\tilde{b}_0(\rho) - 1, \tilde{b}_J(\rho), \tilde{b}_{m+1}(\rho)\} = \max \{eb_J(\rho), eb_0(\rho) - e\} = eb_{\log}(\rho).$$

This proves the proposition. \(\square\)

### 2.3 Refined differential conductors

In this subsection, we define the refined differential conductors. This provides some information about the graded pieces of the ramification filtrations. We keep the notation as in previous subsections but we drop Hypothesis 2.1.9.

**Notation 2.3.1.** Fix a Dwork pi $\pi = (-p)^{1/(p-1)}$.

**Notation 2.3.2.** We denote $\Omega^1_{\Omega}(\log) = \Omega^1_{\Omega_k} + \mathcal{O}_k \frac{dx_k}{\pi_k} \subset \Omega^1_k$. If we choose a $p$-basis $\tilde{b}_J$ of $\kappa$ as in Notation 2.1.2, we have $\Omega^1_{\Omega_k}(\log) = \mathcal{O}_k \frac{dx_k}{\pi_k} \oplus \bigoplus_{j \in J} \mathcal{O}_k \log \frac{b_j}{\pi_k}$.

**Construction 2.3.3.** Let $\rho$ be a $p$-adic representation of $G_k$ of finite local monodromy and of pure break $b = b_{\log}(\rho)$ (resp. log-break $b = b_{\log}(\rho)$). Replace $k$ by the completion an inseparable extension as in Definition 2.2.1 and we may assume Hypothesis 2.1.9. Let $E_\rho$ denote the $(\phi, \nabla)$-module associated to $\rho$. By Theorem 1.5.6(e), there exists $\eta_0 \in (0, 1)$ such that $E_\rho \otimes F_{\eta_0}$ has pure extrinsic (resp. intrinsic) radii $\eta^b$ for $\eta \in [\eta_0, 1]$.

We define the set of **refined Artin conductors** of $\rho$ to be

$$\text{rar}(\rho) = \left\{ \frac{1}{\pi} \partial_{\pi_k}^{-b} \vartheta \in \mathcal{I}\Theta(E_\rho) \right\} \subset \Omega^1_{\Omega_k} \otimes \mathcal{O}_k \pi_k^{-b \kappa}.$$

Similarly, we define the set of **refined Swan conductors** of $\rho$ to be

$$\text{rsw}(\rho) = \left\{ \frac{1}{\pi} \partial_{\pi_k}^{-b} \vartheta \in \mathcal{I}\Theta(E_\rho) \right\} \subset \Omega^1_{\Omega_k}(\log) \otimes \mathcal{O}_k \pi_k^{-b \kappa}.$$

**Remark 2.3.4.** There is a unique primitive $p$-th roots of unity $\zeta_p$ such that $\pi \equiv (\zeta_p - 1) \mod (\zeta_p - 1)^2$. If we replace $\pi$ by this $\zeta_p - 1$ in the definition above, the definition does not change.

**Lemma 2.3.5.** In Construction 2.3.3 the definition of refined Artin and Swan conductors does not depend on the choices of the lifted $p$-basis of $k$ and the uniformizer $\pi_k$.

**Proof.** We may assume Hypothesis 2.1.9 since only finitely many elements in the $p$-basis show up in the refined Artin and Swan conductors.
For another choice of lifted $p$-basis and uniformizer, we will end up considering another set of differential operators $\partial'_j = \partial/\partial B'_j$ for $j \in J$ and $\partial_0' = \partial/\partial T'$. We write

$$dB_j = \sum_{j' \in J} \alpha_{j,j'}dB'_j + \alpha_{j,0}dT' \text{ for } j \in J, \quad dT = \sum_{j' \in J} \alpha_{0,j'}dB'_j + \alpha_{0,0}dT',$$

where $\alpha_{j,j'} \in \mathcal{O}_{K'}[T]$ for $j, j' \in J^+$. Better, we have $\alpha_{0,j} \in T \cdot \mathcal{O}_{K'}[T]$.

We may assume that $E_\rho$ has pure differential nonlog-break (resp. log-break) and has pure extrinsic (resp. intrinsic) radii $\eta^\rho$ for $\eta \in [\eta_0, 1)$ for some $\eta_0 \in (0, 1)$.

By applying Theorem [1.4.19] for $\eta \in (\eta_0, 1) \cap p\mathbb{Q}$ and for any $j \in J^+$ such that $R_{\partial_j'}(V \otimes F'_{\eta}) = ER(V \otimes F'_{\eta})$ (resp. $IR_{\partial_j'}(V \otimes F'_{\eta}) = IR(V \otimes F'_{\eta})$), we have

$$\Theta_{\partial'_j}(E_\rho \otimes F'_{\eta}) = \{ \pi T^{-b}(\alpha_{0,0}\theta_0 + \cdots + \alpha_{m,j}\theta_m) \mid \pi T^{-b}(\theta_0 dT + \theta_1 dB_1 + \cdots + \theta_m dB_m) \in \mathcal{E}(E_\rho \otimes F'_{\eta}) \} \quad \Theta_{\partial'_j}(E_\rho \otimes F'_{\eta}) = \{ \pi T^{-b}(\alpha_{0,0}\theta_0 + \cdots + \alpha_{m,j}\theta_m) \mid \pi T^{-b}(\theta_0 dT + \theta_1 dB_1 + \cdots + \theta_m dB_m) \in \mathcal{I}(E_\rho \otimes F'_{\eta}) \}$$

Note also that

$$(\alpha_{0,0}\theta_0 + \cdots + \alpha_{m,0}\theta_m) dT + \sum_{j \in J} (\alpha_{0,j}\theta_0 + \cdots + \alpha_{m,j}\theta_m) dB'_j = \theta_0 dT + \theta_1 dB_1 + \cdots + \theta_m dB_m.$$

Combining these two formulas, we conclude that $\mathcal{E}(V)$ (resp. $\mathcal{I}(V)$) for $\partial_{j^+}$ is the same as that for $\partial_{j^+}$. Hence, the refined Artin (resp. Swan) conductors are well-defined. \(\Box\)

**Lemma 2.3.6.** Let $k'/k$ be a tamely ramified extension of ramification degree $e = e_{k'/k}$ and let $\rho$ be a $p$-adic representation of $G_k$ with finite local monodromy which has pure log-break $b = b_{\log}(\rho)$. Hence, $\rho|_{G_{k'}}$ has pure log-break $eb$. Then, upon identifying $\Omega_{1_{O_{k'}}(log)} \otimes_{O_{k'}} \pi_{k'}^{-eb} \bar{\kappa}$ with $\Omega_{1_{O_{k'}}(log)} \otimes_{O_{k'}} \pi_{k'}^{-eb} \bar{\kappa}$, we have $\text{rsw}(\rho) = \text{rsw}(\rho|_{G_{k'}})$. We will sometimes identify them later on.

**Proof.** This follows immediately from the fact that $\mathcal{E}_{\rho|_{G_{k'}}}$ is just the base change of $\mathcal{E}_\rho$ along $A_{1_{O_{k'}}(\eta_1/e, 1)} \rightarrow A_{1_{O_{k'}}(\eta, 1)}$ where the coordinate for the first annulus is $t^{1/e}$.

**Theorem 2.3.7.** Let $k$ be a complete discretely valued field of equal characteristic $p > 0$.

(a) Let $\rho$ be a $p$-adic representation of $G_k$ with finite local monodromy which has pure log-break $b = b_{\log}(\rho)$. Then there exists a finite tamely ramified extension $k'/k$ of ramification degree $e$ such that, we have a canonical decomposition of representations of $G_{k'}$ over some finite extension $F'$ of $F$:

$$\rho|_{G_{k'}} \otimes F' = \bigoplus_{\vartheta \in \text{rsw}(\rho)} \rho_\vartheta,$$

where $\rho_\vartheta$ has pure refined Swan conductors $\vartheta \in \Omega_{1_{O_{k'}}(log)} \otimes_{O_{k'}} \pi_{k'}^{-eb} \bar{\kappa} = \Omega_{1_{O_{k'}}(log)} \otimes_{O_{k'}} \pi_{k'}^{-eb} \bar{\kappa}$. By Galois descent, we have a decomposition of $\rho = \bigoplus_{\vartheta \in \text{rsw}(\rho)} \rho_\vartheta$, where the direct sum runs through all Galois conjugacy classes in $\text{rsw}(\rho)$, and where $\text{rsw}(\rho_\vartheta)$ consists of only the Galois conjugacy class $\{ \vartheta \}$ (with same multiplicity on each element).\(\Box\)
(b) Choose the $p$-th root of unity $\zeta_p$ as in Remark $2.3.4$. Then there exists an injective homomorphism for $b \in \mathbb{Q}_{>0}$,

$$\text{rsw} : \text{Hom}(\text{Fil}^b \lim_{\to} G_k/\text{Fil}^{b+} \lim_{\to} G_k, \mathbb{F}_p) \to \Omega^1_{\mathcal{O}_k} (\log) \otimes_{\mathcal{O}_k} \pi_k^{-b} \mathbb{K},$$

(2.3.8)
such that, when viewing the left hand side as a subset of $\text{Hom}(\text{Fil}^b \lim_{\to} G_k/\text{Fil}^{b+} \lim_{\to} G_k, \mathbb{Q}_p(\zeta_p))$ via the identification $1 \in \mathbb{F}_p$ with $\zeta_p \in \mathbb{Q}_p(\zeta_p)$ for $\zeta_p$ as in Remark $2.3.4$, we have, for any $p$-adic representation $\rho$ of $G_k$ with finite local monodromy and pure log-break $b$, the images of the summands of $\rho|_{\text{Fil}^b \lim_{\to} G_k}$ under $\text{rsw}$ are exactly the refined Swan conductors of $\rho$. Moreover, the homomorphism (2.3.8) does not depend on the choice of the Duwork pi.

Proof. For both (a) and (b), we may assume that Hypothesis $2.3.9$ since only finitely many elements in a $p$-basis matter.

(a) Using the identification Lemma $2.3.6$, we may first replace $k$ and $\text{Frac}(O)$ by a tamely ramified extension of $k$ and a finite extension of $\text{Frac}(O)$ so that the decomposition in Theorem $1.5.12$ of $\mathcal{E}_\rho$ can be realized on $\mathcal{R}_{\text{int}}$, and $\mathcal{F}_q \subseteq k_0$. Since the decomposition is canonical, it is a decomposition for $(\phi, \nabla)$-modules. By the slope filtration $[\text{Ked07, Theorem 3.4.6}]$, the Frobenius action on each direct summand of $\mathcal{E}$ is of unit-root, yielding the decomposition of the representation by the equivalence of categories in Theorem $2.1.7$.

(b) We first recall the following corollary of Proposition $1.3.19$

(i) For any $p$-adic representations $\rho$ and $\rho'$ of $G_k$ with finite local monodromy, pure log-break $b$, and pure refined Swan conductor $\vartheta$, $\rho \otimes \rho'$ has smaller log-break.

(ii) For any $p$-adic representations $\rho$ and $\rho'$ of $G_k$ with finite local monodromy, pure log-break $b = b'$, and pure refined Swan conductor $\vartheta \neq \vartheta'$, respectively, $\rho \otimes \rho'$ has pure log-break $b$, and pure refined Swan conductor $\vartheta - \vartheta'$.

We also need an easy fact about Galois representations.

(iii) For any homomorphism $\chi : \text{Fil}^b \lim_{\to} G_k/\text{Fil}^{b+} \lim_{\to} G_k \to \mathbb{F}_p$, there is a representation $\rho_{\chi}$ of $G_k'$ of finite local monodromy, pure log-break $eb$, and pure refined Swan conductor, where $k'$ is some finite tamely ramified extension of $k$ of ramification degree $e$.

Proof: By the chosen $p$-th root of unity (identifying 1 with $\zeta_p$ in Remark $2.3.4$), a homomorphism $\chi : \text{Fil}^b \lim_{\to} G_k/\text{Fil}^{b+} \lim_{\to} G_k \to \mathbb{F}_p$ can be identified with a $p$-adic representation with coefficients in $\mathbb{Q}_p(\zeta_p)$, which we still denote by $\chi$. Since $G_k/\text{Fil}^{b+} \lim_{\to} G_k$ is a pro-finite group, there exists a normal subgroup $H$ of $G_k$ of finite index containing $\text{Fil}^{b+} \lim_{\to} G_k$, such that $\chi$ factors through $I = \text{Fil}^b \lim_{\to} G_k/H \cap \text{Fil}^b \lim_{\to} G_k$. Let $\rho' = \text{Ind}_{I}^{G_k/H} \chi$ be the induction of $\chi$ to a $p$-adic representation of $G_k$, for which, $\rho'|_{\text{Fil}^b \lim_{\to} G_k}$ contains $\chi$ as a direct summand. Then the desired representation $\rho_{\chi}$ can be recovered from the decomposition of this direct summand using (a).

We then define $\text{rsw}$ to be the morphism sending $\chi$ to the (unique) refined Swan conductor of $\rho_{\chi}$, which is an element of $\Omega^1_{\mathcal{O}_k'} (\log) \otimes_{\mathcal{O}_k'} \pi_k^{-eb} \mathbb{K} \simeq \Omega^1_{\mathcal{O}_k} (\log) \otimes_{\mathcal{O}_k} \pi_k^{-b} \mathbb{K}$, via the identification in Lemma $2.3.6$.

It is well-defined by (iv) below and its injectivity follows from (v).

(iv) For any two representations $\rho_{\chi}$ and $\rho'_{\chi}$ satisfying (iii), they must have the same refined Swan conductor.
This is because $\rho_\chi \otimes \rho'_\chi|_{G_k'}$ is trivial on $\text{Fil}_{\log}^{b}G_k'$ and hence have ramification break strictly smaller than $eb$. If $\rho_\chi$ and $\rho'_\chi$ had different (pure) refined Swan conductor, by (ii), $\rho_\chi \otimes \rho'_\chi|_{G_k'}$ would have pure log-break $eb$, which is a contradiction. This yields (iv).

(v) For two homomorphisms $\chi \neq \chi'$: $\text{Fil}_{\log}^{b}G_k/\text{Fil}_{\log}^{b}G_k \rightarrow \mathbb{F}_p$, we obtain $\rho_\chi$ and $\rho'_\chi$ satisfying the condition in (iii). Then $\rho_\chi$ and $\rho'_\chi$ have different refined Swan conductor.

Indeed, if $\chi \neq \chi'$, $\rho_\chi \otimes \rho'_\chi$ is not trivial on $\text{Fil}_{\log}^{b}G_k/\text{Fil}_{\log}^{b}G_k = \text{Fil}_{\log}^{b}G_k'/\text{Fil}_{\log}^{b}G_k'$. If $\rho_\chi$ and $\rho'_\chi$ had same pure refined Swan conductor, then by (i), $\rho_\chi \otimes \rho'_\chi$ would pure log-break strictly less than $eb$. This is a contradiction and hence (v) is proved.

Now, we discuss the independence on Dwork pi. If we choose another Dwork pi, we would need to use another primitive $p$-th root of unity $\zeta_p^i$. On one hand, the refined Swan conductor is multiplied by $\zeta_p^{-i-1} \equiv i \mod (\zeta_p-1)$. On the other hand, the $p$-adic representation $\text{Fil}_{\log}^{b}G_k/\text{Fil}_{\log}^{b}G_k \rightarrow \mathbb{Q}_p(\zeta_p)^\chi$ becomes $\chi^i$. Hence, we need to take $\rho \otimes \chi$ as our $p$-adic representation of $G_k'$; it has refined Swan conductor $\text{rsww}(\rho) = i \cdot \text{rsww}(\rho)$, which is the same as the refined Swan conductor of $\rho$ computed using the old Dwork pi.

**Remark 2.3.9.** It is interesting to point out that the choice of a Dwork pi is related to the choice of the Artin-Scheier $\ell$-adic sheaf in [Sai03]; they both amount to choosing a primitive $p$-th root of unity. The difference is that we consider it as an element in $\mathbb{Q}_p$ whereas Saito viewed it as an element in $\mathbb{Q}_\ell$.

**Proposition 2.3.10.** Let $k$ be a complete discretely valued field of equal characteristic $p > 0$. Then for $b \in \mathbb{Q}_{>0}$, the conjugation action of $\text{Fil}_{\log}^{b}G_k/\text{Fil}_{\log}^{b}G_k$ on $\text{Fil}_{\log}^{b}G_k/\text{Fil}_{\log}^{b}G_k$ is trivial. In other words, $\text{Fil}_{\log}^{b}G_k/\text{Fil}_{\log}^{b}G_k$ lies in the center of $\text{Fil}_{\log}^{b}G_k/\text{Fil}_{\log}^{b}G_k$.

**Proof.** This proposition is proved in [AS03 Theorem 1]. We hereby give an alternative proof using differential modules.

It suffices to show that for a $p$-adic representation $\rho$ of $G_k$ with finite local monodromy which has pure (differential) log-break $b$, if it is absolutely irreducible under any tamely ramified extension, then $\rho|_{\text{Fil}_{\log}^{b}G_k/\text{Fil}_{\log}^{b}G_k}$ is a direct sum of a single character $\chi : \text{Fil}_{\log}^{b}G_k/\text{Fil}_{\log}^{b}G_k \rightarrow \mathcal{O}^\times$. This is equivalent to showing that the action of $\text{Fil}_{\log}^{b}G_k$ on $\rho \otimes \rho'$ is trivial, and hence to showing that $\rho \otimes \rho'$ has smaller log-break.

Since only finitely many elements in a lifted $p$-basis can be dominant in $\mathcal{E}_\rho$, we may assume Hypothesis 2.1.9 By Theorem 2.3.7 (a), such condition implies that $\rho$ must have pure refined Swan conductor and hence $\rho \otimes \rho'$ must have smaller log-break.

**Proposition 2.3.11.** Keep the notation as in Proposition 2.2.14. Then the refined Swan conductor homomorphism $\text{rsww}_{k}$ for $k$ factors as

$$
\text{Hom}(\text{Fil}_{\log}^{b}G_k/\text{Fil}_{\log}^{b}G_k, \mathbb{F}_p) \rightarrow \text{Hom}(\text{Fil}_{\log}^{b}G_k/\text{Fil}_{\log}^{b}G_k, \mathbb{F}_p) \xrightarrow{\text{rsww}_{k}} \Omega^1_{\mathcal{O}_k}(\log) \otimes_{\mathcal{O}_k} \pi^{-eb}_{k_{\text{alg}}},
$$

(2.3.12)

**Proof.** Keep the notation as in Proposition 2.2.14 let $\tilde{F}_\eta'$ be the completion of $\tilde{K}'(U)$ with respect to the $\eta^{1/\epsilon}$. Let $\eta_0 \in (0, 1)$ be such that $IR(\mathcal{E}_{\rho} \otimes \tilde{F}_\eta') = \eta^b$ for $\eta \in [\eta_0, 1)$. Then (2.2.18) implies that,
for any $\eta \in [\eta_0, 1) \cap p^Q$ and for any $j \in \{0, \ldots, m + 1\}$ such that $IR\partial_j(f^*\mathcal{E}_\rho \otimes \tilde{F}_\eta') = IR(\mathcal{E}_\rho \otimes \tilde{F}_\eta')$, we have
\[
\Theta_{\partial_j}(f^*\mathcal{E}_\rho \otimes \tilde{F}_\eta') = \begin{cases} 
\Theta\partial_j(\mathcal{E} \otimes F'_\eta) & j \in J, \\
 eUXU^{-1}\Theta\partial_0(\mathcal{E} \otimes F'_\eta) & j = 0 \text{ and hence } p \nmid e, \\
 U^e\Theta\partial_{m+1}(\mathcal{E} \otimes F'_\eta) & j = m + 1.
\end{cases}
\]
Here, we used Theorem [1.1.9] to compute the refined radii. The proposition follows. \hfill \Box

One may want to prove analogs of Theorem [2.3.7] and Proposition [2.3.10] for refined Artin conductors. This however needs to take a bit more effort because there may not be a representation of $G_k$ with pure refined Artin conductor. Instead, we reduce to the classical case, where the results for refined Artin conductors follows from those for refined Swan conductors.

**Theorem 2.3.13.** Let $k$ be a complete discretely valued field of equal characteristic $p > 0$.

(a) Choose the $p$-th root of unity $\zeta_p$ as in Remark [2.3.4]. Then there exists an injective homomorphism for $b \in \mathbb{Q}_{\geq 1}$,
\[
\text{rar}: \text{Hom}(\text{Fil}^bG_k/\text{Fil}^bG_k, \mathbb{F}_p) \to \Omega^1_{\mathcal{O}_k} \otimes_{\mathcal{O}_k} \pi^{-b}\kappa,
\]
such that, when viewing the left hand side as a subset of $\text{Hom}(\text{Fil}^bG_k/\text{Fil}^bG_k, \mathbb{Q}_p(\zeta_p))$ via the identification $1 \in \mathbb{F}_p$ with $\zeta_p \in \mathbb{Q}_p(\zeta_p)$ for $\zeta_p$ as in Remark [2.3.4], we have, for any $p$-adic representation $\rho$ of $G_k$ with finite local monodromy and pure break $b$, the image of the summands of $\rho|_{\text{Fil}^bG_k}$ under $\text{rar}$ is exactly the refined Artin conductor of $\rho$. Moreover, this homomorphism does not depend on the choice of the Dwork $p_i$.

(b) For $b \in \mathbb{Q}_{\geq 1}$, the conjugation action of $\text{Fil}^{1+b}G_k/\text{Fil}^bG_k$ on $\text{Fil}^bG_k/\text{Fil}^bG_k$ is trivial. In other words, $\text{Fil}^bG_k/\text{Fil}^bG_k$ lies in the center of $\text{Fil}^{1+b}G_k/\text{Fil}^bG_k$.

**Proof.** For (a) and (b), we may assume Hypothesis [2.1.9] because only finitely many elements in $p$-basis matter. Moreover, that $J$ is not empty because otherwise we are in the classical case, and both (a) and (b) follow from their log-version counterpart: Theorem [2.3.7] and Proposition [2.3.10] respectively.

Let $\rho$ be a representation of $G_k$ with finite local monodromy and of pure nonlog-break $b$. We perform a base change similar to the one in Lemma [2.2.9]. Let $k'$ be the completion of $k(x_1, \ldots, x_m)$ with respect to the $(1, \ldots, 1)$-Gauss norm and let $\tilde{k}$ be the completion of $k'((b_j + x_j\pi_k)^{1/p^n}, x_j^{1/p^n}; n \in \mathbb{N}; j \in J)$, equipped with the uniformizer $\pi_{\tilde{k}} = \pi_k$. It is in fact a complete discrete valuation field with perfect residue field. Let $\tilde{\rho}$ be the representation $G_{\tilde{k}} \to G_k \xrightarrow{\rho} \text{GL}(V_{\rho})$. Let $K''$ denote the completion of $K'(X_J)$ with respect to the $(1, \ldots, 1)$-Gauss norm, where $X_j$ is a lift of $x_j$ for $j \in J$. Let $\tilde{K}$ denote the completion of $K''((B_j + X_jT)^{1/p^n}, X_j^{1/p^n}; n \in \mathbb{N}; j \in J)$. Let $f: A_{K}^1[\eta_0, 1) \to A_{K'}^1[\eta_0, 1)$ denote the natural morphism. Then $f^*\mathcal{E}_\rho$ is the differential module associated to $\tilde{\rho}$. Let $\tilde{\partial}$ denote the differential operators on $f^*\mathcal{E}_\rho$ dual to the basis $dT$. Similarly to (2.2.10), we have
\[
\tilde{\partial} = \partial_0 - X_1\partial_1 - \cdots - X_m\partial_m.
\]
If we let $\tilde{F}_\eta$ denote the completion of $\tilde{K}(T)$ with respect to the $\eta$-Gauss norm, we have
\[
R_{\tilde{\partial}}(f^*\mathcal{E} \otimes \tilde{F}_\eta') = \min_{j \in J^*} \{ R_{\partial_j}(\mathcal{E} \otimes F'_\eta) \}.
\]
Hence, \( \check{\rho} \) has pure nonlog-break \( b \) and, by Theorem 1.4.9, its refined Artin conductor is 

\[
\text{rar}(\check{\rho}) = \{(\theta_0 - X_1\theta_1 - \cdots - X_m\theta_m) d\pi_k | \theta_0 d\pi_k + \theta_1 db_1 + \cdots + \theta_m db_m \in \text{rar}(\rho)\}.
\]

As a consequence, we have the following commutative diagram.

\[
\begin{array}{ccc}
\text{Rep}(G_k) & \overset{\text{rar}}{\longrightarrow} & \Omega^1_{\mathbb{O}_K} \otimes \mathbb{O}_k \pi_k^{b}\kappa \\
\downarrow & & \downarrow \\
\text{Rep}(G'_k) & \overset{\text{rar}}{\longrightarrow} & \pi_k^{b}\kappa_{\text{alg}} d\pi_k \\
\end{array}
\]

This also implies that the natural homomorphism \( \text{Fil}^b G'_k / \text{Fil}^{b+} G'_k \to \text{Fil}^b G_k / \text{Fil}^{b+} G_k \) is surjective. Therefore, it suffices to prove (a) and (b) for \( G'_k \), which we have already known. \( \square \)

### 2.4 Multi-indexed ramification filtrations for higher local fields

When \( k \) has multi-indexed valuation, the refined Artin and Swan conductors give a more refined filtration on the Galois group \( G_k \). We restrict ourselves to the equal characteristic \( p > 0 \) case.

**Definition 2.4.1.** We say that a complete discretely valued field \( k \) of characteristic \( p > 0 \) is an \((m+1)\)-dimensional local field if there is a chain of fields \( k = k_{m+1}, k_m, \ldots, k_0 \), where \( k_{i+1} \) is a complete discretely valued field with residue field \( k_i \) for \( i = 0, \ldots, m \). Contrary to most literature, we do not assume that \( k_0 \) is perfect.

An \((m+1)\)-tuple of elements \( t_0, \ldots, t_m \in k \) is called a system of local parameters of \( k \) if \( t_i \in \mathbb{O}_k \) is a lift of a uniformizer of \( k_{m+1-i} \) all the way up to \( k \). Such a choice gives a (non-canonical) isomorphism \( k \cong k_0((t_m))((\cdots))((t_0)) \). In this case, we have

\[
\Omega^1_{\mathbb{O}'_k}(\log) = \bigoplus_{i=0}^{m} \mathbb{O}_{k_i} dt_i / t_i \oplus \bigoplus_{j \in J} \mathbb{O}_{k_j} db_j / b_j, \quad \Omega^1_{\mathbb{O}_k'} = \bigoplus_{i=0}^{m} \mathbb{O}_{k_i} dt_i \oplus \bigoplus_{j \in J} \mathbb{O}_{k_j} db_j.
\]

Equip \( \mathbb{Q}^{m+1} \) with the lexicographic order: \( \mathbf{i} = (i_1, \ldots, i_{m+1}) < \mathbf{j} = (j_1, \ldots, j_{m+1}) \) if and only if

\[
i_l < j_l, i_{l+1} = j_{l+1}, \ldots, i_{m+1} = j_{m+1} \text{ for some } l \leq m + 1.
\]

For \( a \in \mathbb{Q} \), we also use \( \mathbb{Q}^{m+1}_a \) to denote the subset of \( \mathbb{Q}^{m+1} \) consisting of \( \mathbf{i} = (i_1, \ldots, i_{m+1}) \) such that \( i_{m+1} > a \).

Given a system of local parameters, we define a multi-indexed valuation to be \( \mathbf{v} = (v_1, \ldots, v_{m+1}) : k^\times \to \mathbb{Z}^{m+1} \subset \mathbb{Q}^{m+1} \), where \( v_{m+1} = v_{k_{m+1}} \) and recursively we have, downwards from \( i = m + 1 \) to \( i = 1 \), that \( v_{i-1}(\alpha) = v_{k_{i-1}}(\alpha_{i-1}) \) with \( \alpha_{i-1} \) equal to the residue of \( \alpha_i t_{m+1-i} \) in \( k_{i-1} \). Note that the definition of \( \mathbf{v} \) depends on the choice of local parameters \( t_0, \ldots, t_m \).

**Definition 2.4.2.** For \( \lambda = \sum_{i=0}^{m} \alpha_i dt_i + \sum_{j \in J} \beta_j db_j \in \Omega^1_{\mathbb{O}_k} \otimes \mathbb{O}_k k \), define

\[
\mathbf{v}_{\text{nlog}}(\lambda) = \min\{\mathbf{v}(\alpha_0), \ldots, \mathbf{v}(\alpha_m), \mathbf{v}(\beta_j); j \in J\}.
\]

It naturally gives rise to a valuation on \( \Omega^1_{\mathbb{O}_k} \otimes \mathbb{O}_k \pi_k^{-i_{m+1}} \mathbb{K} \).
For $\lambda = \sum^{n}_{i=0} \alpha_i \frac{dt_i}{t_i} + \sum_{j \in J} \beta_j \frac{db_j}{b_j} \in \Omega^1_{O_k} \log \otimes O_k$, define

$$v_{\text{log}}(\lambda) = \min\{v(\alpha_0), \ldots, v(\alpha_m), v(\beta_j); j \in J\}.$$  

It naturally gives rise to a valuation on $\Omega^1_{O_k} \log \otimes O_k$.  

For $i = (i_1, \ldots, i_{m+1}) \in \mathbb{Q}_{>1}^{m+1}$, we define $\text{Fil}^i G_k$ to be the inverse image along the homomorphism

$$\text{Fil}^{i_{m+1}} G_k \rightarrow \text{Fil}^{i_{m+1}} G_k / \text{Fil}^{i_{m+1}+1} G_k \rightarrow \Omega^1_{O_k} \otimes O_k t_0^{-i_{m+1}}.$$  

of the elements whose image under $-v_{\text{log}}$ is greater than or equal to $i$.  

For $i = (i_1, \ldots, i_{m+1}) \in \mathbb{Q}_{>0}^{m+1}$, we define $\text{Fil}^{i_{m+1}}_{\text{log}} G_k$ to be the inverse image along the homomorphism

$$\text{Fil}^{i_{m+1}}_{\text{log}} G_k \rightarrow \text{Fil}^{i_{m+1}}_{\text{log}} G_k / \text{Fil}^{i_{m+1}+1}_{\text{log}} G_k \rightarrow \Omega^1_{O_k} \otimes O_k t_0^{-i_{m+1}}.$$  

of the elements whose image under $-v_{\text{log}}$ is greater than or equal to $i$.  

\textbf{Remark 2.4.3.} The ramification filtration does not depend on the choice of local parameters, but the indexing does.  

Let $O_K = \{x \in K | v(x) \geq (0, \ldots, 0)\}$. It might be more natural to index the above filtrations by “rational powers of fraction ideals of $K$” of the form $I^{1/n}$, where $I$ is an $O_K$-submodules of $K$ containing $O_K$, $n$ is an integer, and $I^{1/n}$ is equivalent to $I^{1/n'}$ if $I^{n'} = I^n$ as $O_K$-submodules of $K$.  

\textbf{Remark 2.4.4.} When $k_0$ is a finite field, this filtration is expected to be compatible with an easily defined filtration on the Milnor $K$-groups via the class field theory for higher local fields. This may be verified by the comparison with Kato’s refined Swan conductor, through Saito’s definition by Theorem 3.4.1 proved later and [AS09, Theorem 9.1.1]. For more along this line, the reader may refer to the recipe in Kato’s masterpiece [Kat89].  

\section{Comparison with Saito’s definition}

In this section, we compare our definition with the one given by Saito in [Sai09]. Since the readers who are only interested in one side of the picture may use this result (Theorem 3.4.1) as a black box, we present the proof, assuming that the readers are familiar with the definition of arithmetic ramification filtrations (see e.g. [Sai09, Section 1] and [Xia11]). However, this comparison has its own interest. The Abbes-Saito spaces which define the ramification filtrations are rigid analytic spaces over $k$. Saito’s definition essentially made use of the special fibers (over $\kappa_k$) of their stable models, while our definition via differential modules, by contrast, needs to lift the Abbes-Saito spaces to rigid spaces over an annulus (over $K$). So, we will see three different levels (over $\kappa_k$, $k$, and $K$) together in this section. The crucial calculation is done in Subsection 3.3 and it provides an example where we can compute the refined radii explicitly.

Since Saito’s definition of refined Swan conductor is of geometric nature, we will restrict ourselves with the geometric case. For a story of linking Abbes-Saito spaces and its lifts in the general case, one is recommended to read [Xia11].

We assume $p > 0$ is a prime number throughout this section.
3.1 Review of Saito’s definition

In this subsection, we review the definition of ramification filtrations and the refined Swan conductors, defined by Abbes and Saito in [AS02, AS03, Sai09]. Instead of introducing the general construction, we will focus on a special case which is used in the comparison theorem. For more details and a complete treatment, one may consult [Sai09].

We continue to use the notation from the previous section. Let \( l \) be a finite Galois extension of \( k \).

**Construction 3.1.1.** We consider a closed immersion \( \text{Spec} \mathcal{O}_l \to P \) into a smooth (affine) scheme \( P \) over \( \text{Spec} \mathcal{O}_k \). Let \( I = \text{Ker} \left( \mathcal{O}_P \to \mathcal{O}_l \right) \).

Let \( r = a/b \in \mathbb{Q}_{>0} \) with \( a, b > 0 \). Let \( P_{\mathcal{O}_k}^{[a/b]} \to P \) be the blowup at the ideal \( I^b + m_k^n \mathcal{O}_P \) and let \( P_{\mathcal{O}_k}^{[a/b]} \subset P_{\mathcal{O}_k}^{[a/b]} \) be the complement of the support of \( (I^n \mathcal{O}_P_{\mathcal{O}_k}^{[a/b]} + m_k^n \mathcal{O}_P_{\mathcal{O}_k}^{[a/b]})/m_k^n \mathcal{O}_P_{\mathcal{O}_k}^{[a/b]} \). Let \( P_{\mathcal{O}_k}^{(r)} \) be the normalization of \( P_{\mathcal{O}_k}^{[a/b]} \); it does not depend on \( a \) and \( b \) but only their ratio. Let \( P_k^{(r)} \) and \( P_\kappa^{(r)} \) denote the generic fiber and the special fiber of \( P_{\mathcal{O}_k}^{(r)} \), respectively. Let \( P_k^{(r)} \) denote the generic fiber of completing \( P_{\mathcal{O}_k}^{(r)} \) along \( P_\kappa^{(r)} \). The immersion \( \text{Spec} \mathcal{O}_l \to P \) is uniquely lifted to an immersion \( \text{Spec} \mathcal{O}_l \to P_{\mathcal{O}_k}^{(r)} \).

By the finiteness theorem of Grothendieck-Riemann cited in [AS03, THEOREM 1.10], there exists a finite separable extension \( k'/k \) of ramification degree \( e = e_{k'/k} \) such that the normalization \( P_{\mathcal{O}_{k'}}^{(e_{k'/k})} \) of \( P_{\mathcal{O}_k} \times_{\mathcal{O}_k} \mathcal{O}_{k'} \) has reduced geometric fibers over \( \text{Spec} \mathcal{O}_{k'} \), which we call a stable model of \( P_{\mathcal{O}_k}^{(r)} \). We put \( P_k^{(r)} = P_{\mathcal{O}_{k'}}^{(e_{k'/k})} \times_{\mathcal{O}_{k'}} \kappa; \) it is called the stable special fiber of \( P_{\mathcal{O}_k}^{(r)} \) and it does not depend on the choice of \( k' \).

We defer the discussion of the properties of this construction later when we have a concrete example at hand.

For the rest of this section, we assume the following geometric assumption.

**Hypothesis 3.1.2** (Geom). There exists an affine smooth variety \( X \) over \( k_0 \) and an irreducible divisor \( D \) smooth over \( k_0 \) with generic point \( \xi \) such that \( \mathcal{O}_k \simeq \mathcal{O}_X^h_{\xi} \), where the latter is the completion of the local ring at \( \xi \). In particular, Hypothesis 2.1.9 is fulfilled.

**Remark 3.1.3.** This Hypothesis (Geom) is essentially the same as the hypothesis (Geom) in [Sai09, P.786], except that our \( k \) is the completion of the Henselian local field considered in Saito’s paper.

**Construction 3.1.4.** After replacing \( X \) (and hence \( D \)) by an étale neighborhood of \( \xi \) if necessary, there exists a finite flat morphism \( f : Y \to X \) of smooth schemes over \( k_0 \) such that \( V = Y \times_X U \to U = X \setminus D \) is finite étale with Galois group \( G_{l/\xi} \) and that \( Y \times_X \text{Spec} \mathcal{O}_X^h_{\xi} = \text{Spec} \mathcal{O}_l \).

Let \( (X \times X)' \) be the blowup of \( X \times_{k_0} X \) along \( (X \times_{k_0} D) \cup (D \times_{k_0} X) \), and let \( (X \times X)^\sim \) denote the complement of the proper transforms of \( X \times_{k_0} D \) and \( D \times_{k_0} X \) in \( (X \times X)' \). The diagonal embedding \( \Delta_X : X \to X \times_{k_0} X \) naturally lifts to an embedding \( \hat{\Delta}_X : X \to (X \times X)^\sim \). Now, we pull back the picture along \( f : Y \to X \) and obtain the following commutative diagram, where \( (Y \times X)^\sim \).
is the fiber product of the big square and all parallelograms are Cartesian.

\[
(Y \times X) \sim \xrightarrow{f \times 1} (X \times X) \sim
\]

We base change this commutative diagram over to \( \text{Spec} \, \mathcal{O}_{\tilde{X}, \xi} = \text{Spec} \, \mathcal{O}_k \) along \( p_2 \), as follows. Let \( P = (X \times X) \sim \times_{p_2 \circ \pi_X \times X} \text{Spec} \, \mathcal{O}_{\tilde{X}, \xi} \) and \( Q = (Y \times X) \sim \times_{p_2 \circ \pi_X \circ (f \times 1) \times X} \text{Spec} \, \mathcal{O}_{\tilde{X}, \xi} \). We then obtain the following commutative diagram.

\[
\begin{array}{ccc}
\text{Spec} \, \mathcal{O}_l & \xrightarrow{\Delta_Y} & Q \\
\downarrow f & & \downarrow f \times 1 \\
\text{Spec} \, \mathcal{O}_k & \xrightarrow{\Delta_X} & P \\
\end{array}
\]

Let \( \mathcal{I} \) denote the ideal of the immersion \( \tilde{\Delta}_X \). We will view \( P \) and \( Q \) as a scheme over \( \mathcal{O}_k \) via \( p_2 \).

Now, we can apply Construction 3.1.1 to the embeddings \( \tilde{\Delta}_X \) and \( \Delta_Y \) to define \( P^{(er)} \), \( P^{(er)} \), \( \tilde{P}^{(er)} \), \( \tilde{P}^{(er)} \), \( \tilde{Q}^{(er)} \), \( Q^{(er)} \), \( \tilde{Q}^{(er)} \), \( Q^{(er)} \), respectively, where \( k'/k \) is a finite separable extension of ramification degree \( e \). We still use \( p_1 \) to denote the morphism \( P^{(er)} \to P \xrightarrow{p_1} \text{Spec} \, \mathcal{O}_k \). By functoriality of Construction 3.1.1, we have a morphism \( f^{(r)} : Q^{(er)} \to P^{(er)} \).

**Remark 3.1.7.** The field extension \( k' \) serves as the role of a “coefficient field”. Its only use is to provide a reasonable integral structure over \( \mathcal{O}_{k'} \), and, in particular, to make \( er \) an integer. We can make \( k' \) as large as we need. The reason we need \( k'/k \) to be separable is that we need to lift this change of scalar to characteristic zero, as extensions of \( \mathcal{R}^{\text{int}}_{k'} \).

In contrast, the extension \( l/k \) pulled back from \( p_1 \) encodes the arithmetic information.

We collect some properties of these spaces.

**Proposition 3.1.8.** Keep the notation as above. Let \( k'/k \) be a finite separable extension of ramification degree \( e \).

1. When \( er \) is an integer, the space \( P^{(er)}_{\mathcal{O}_{k'}} \) is defined by \( \sum_{i \geq 0} \pi^{-ier} \cdot \mathcal{I}^i \subset \mathcal{O}_P \otimes_{\mathcal{O}_k} k' \). Hence, it is smooth over \( \mathcal{O}_{k'} \), and its closed fiber \( P^{(er)}_{\mathcal{O}_{k'}} \) can be canonically identified with the \( k'-\text{vector} \) space \( \Omega^1_{\mathcal{O}_k} (\log) \otimes_{\mathcal{O}_k} \pi^{-(er)} \cdot \mathcal{O}_{k'} \). The rigid space \( P^{(er)}_{k'} \) is isomorphic to \( \text{Sp}(k'(\pi^{-(er)} \cdot \delta_0, \pi^{-(er)} \cdot \delta_i)) \), where \( \delta_0, \ldots, \delta_m \) form a dual basis of \( \Omega^1_{\mathcal{O}_k} \).
(2) For the generic fiber, we have $Q_k^{(cr)} = P_k^{(cr)} \otimes_{p, k} \mathbb{L}$. In particular, $Q_k^{(cr)}$ is finite and étale over $P_k^{(cr)}$ with Galois group $G_{l/k}$, and the same is true for $Q_k^{(cr)}$ over $P_k^{(cr)}$.

(3) Let $\text{Spf} \mathcal{O}_Q$ be the completion of $Q$ along $\text{Spec} \mathcal{O}_l$. If $cr$ is an integer, $Q_k^{(cr)}$ is the affinoid variety $X_l^j(\mathcal{O}_Q \to \mathcal{O}_l)_{k^r}$ defined in [AS03 Section 4.2] for $j = r$.

(4) If the highest log ramification break $b_{\log}(l/k) \leq r$, then $Q_k^{(r)}$ is an element in the category $(\mathcal{F}(\mathcal{E})/\mathcal{P}_k^{(r)})_{\text{alg}}$, defined later in Definition 3.1.9.

(5) The log ramification break $b_{\log}(l/k) < r$ if and only if the number of connected components of $Q_k^{(r)}$ is $[l : k]$.

Proof. For (1), see [Sai09 Lemma 1.10]. (2) follows from the fact that $f : V \to U$ is finite and étale of Galois group $G_{l/k}$. For (3), see [Sai09 Example 1.21]. (4) and (5) follow from [Sai09 Lemma 1.13 and Theorem 1.24].

Definition 3.1.9. For an $\kappa$-vector space $W$ of finite dimension, let $(\mathcal{F}(\mathcal{E})/W)_{\text{alg}}$ be the full subcategory of $(\mathcal{F}(\mathcal{E})/W)$ whose objects are finite étale morphisms $g : Z \to W$ such that there exists a structure of algebraic group scheme on $Z$ and that $g$ is a morphism of algebraic groups.

Remark 3.1.10. By the argument just before [Sai09 Lemma 1.23], the category $(\mathcal{F}(\mathcal{E})/W)_{\text{alg}}$ is a Galois category associated to the Galois group $\pi_1^{\text{alg}}(W)$, which is to a quotient of the fundamental group $\pi_1(W)$. The group can be identified with the Pontrjagin dual of the extension group $\text{Ext}^1(W, \mathbb{F}_p)$ in the category of smooth algebraic groups over $\kappa$. The map $W^* \to \text{Hom}_\kappa(W, \kappa) \to \text{Ext}^1(W, \mathbb{F}_p)$ sending a linear form $f : W \to A^1_\kappa$ to the pullback along $f$ of the Artin-Scheier sequence $0 \to \mathbb{F}_p \to A^1_\kappa \to A^1_\kappa \to 0$ is an isomorphism.

Proposition 3.1.11. We have a surjective homomorphism $\pi_1^{\text{alg}}(P_k^{(b)}) \to \text{Fil}^b_{\log} G_k/\text{Fil}^{b+}_{\log} G_k$; it induces an injective homomorphism

$$\text{rs}^{\prime} : \text{Hom}(\text{Fil}^b_{\log} G_k/\text{Fil}^{b+}_{\log} G_k, \mathbb{F}_p) \to \Omega^1_{\pi_k}(\log) \otimes_{\mathcal{O}_k} \pi^{-b}_{k, \kappa}.$$

Proof. For first half of the proposition, see [Sai09 Theorem 1.24]. The second half follows from Remark 3.1.10.

In the following particular case, we give a more detailed study of these spaces.

Situation 3.1.12. Let $l/k$ be a finite totally ramified Galois extension, which is not tamely ramified. Assume that the highest log ramification break $b = b_{\log}(l/k) \in \mathbb{N}$. Assume moreover that $\text{Fil}^b_{\log} G_k/(\text{Fil}^{b+}_{\log} G_k \cap G_l) = \mathbb{F}_p$; in particular, the second highest log-break $b_{\log}(l/k, 2) < b_{\log}(l/k) - 1$.

By Proposition 3.1.11, $Q_k^{(b)}$ consists of $[l : k]/p$ copies of a same Artin-Scheier cover of $P_k^{(b)}$, if we forget about the structure of algebraic group. Assume that this cover is given by

$$\tilde{z}^p - \tilde{z} + (\tilde{\alpha}_0 \pi_k^{-b-1} \delta_0 + \tilde{\alpha}_1 \pi_k^{-b} \delta_1 + \cdots + \tilde{\alpha}_m \pi_k^{-b} \delta_m) = 0,$$

where the coordinates of $P_k^{(b)}$ is given by $\pi_k^{-b-1} \delta_0, \pi_k^{-b} \delta_1, \ldots, \pi_k^{-b} \delta_m$, and $\tilde{\alpha}_j \in \kappa$. The elements $\tilde{\alpha}_0, \ldots, \tilde{\alpha}_m$ are determined up to multiplication by $i \in \mathbb{F}_p^\times$, in accordance with the choice of $\tilde{z}$ up to multiplication by the same $i \in \mathbb{F}_p^\times$. 45
Let $k'/k$ be a finite separable extension of ramification degree $e > 1$, such that $Q^{(eb)}_{\mathcal{O}_{k'}}$ is a stable model. By possibly enlarge $k'$, we may assume that $\bar{\alpha}_{f+} \in \kappa_{k'}$ and $Q^{(eb)}_{\kappa_{k'}}$ consists of $[l : k]/p$ copies of above Artin-Scheier cover of $P^{(eb)}_{\kappa_{k'}}$.

**Lemma 3.1.14.** The space $Q^{(eb)}_{\mathcal{O}_{k'}}$ is $[l : k]/p$ copies of a same space $R^{(eb)}_{\mathcal{O}_{k'}}$. Let $\hat{R}^{(eb)}_{\mathcal{O}_{k'}}$ denote the completion of $R^{(eb)}_{\mathcal{O}_{k'}}$ along its special fiber and let $\hat{R}^{(eb)}_{k'}$ denote the generic fiber, viewed as a rigid analytic space. Also, $Q^{(eb-1)}_{k'}$ is $[l : k]/p$ copies of a same space $R^{(eb-1)}_{k'}$, which is étale over $P^{(eb-1)}_{k'}$.

**Proof.** The connected components of $Q^{(eb)}_{\mathcal{O}_{k'}}$ gives rise to the connected components of $Q^{(eb)}_{\kappa_{k'}}$, they are isomorphic because of the action by Galois group $G_{l/k}$.

Since the second log-ramification break $b_{\log}(l/k; 2) < b_{\log}(l/k) - 1$, by [AS02] Remark 3.13, the number of connected components of $Q^{(eb-1)}_{k'}$ is $[l : k]/p$. Hence, each connected component of $Q^{(eb-1)}_{k'}$ is the normal closure of $\hat{P}^{(eb-1)}_{k'}$ in $\hat{R}^{(eb-1)}_{k'}$, which is $\hat{R}^{(eb-1)}_{k'}$ as stated in the lemma.

**Proposition 3.1.15.** Let $\alpha_{f+} \subset \mathcal{O}_{k'}$ lift $\bar{\alpha}_{f+} \subset \kappa_{k'}$. We can choose a lift $z$ of $\bar{z}$ in $\hat{R}^{(eb)}_{\mathcal{O}_{k'}}$, such that its minimal polynomial over $\hat{P}^{(eb)}_{\mathcal{O}_{k'}} = \text{Spf} \mathcal{O}_{k'} \langle \pi_{k'}^{-eb} - \delta_0, \pi_{k'}^{-eb} - \delta_1, \ldots, \pi_{k'}^{-eb} - \delta_m \rangle$ is

$$z^p - z + (\alpha_0 \pi_{k'}^{-eb} - \delta_0 + \alpha_1 \pi_{k'}^{-eb} - \delta_1 + \cdots + \alpha_m \pi_{k'}^{-eb} - \delta_m) = 0, \quad (3.1.16)$$

The element $z$ generates $\hat{R}^{(eb)}_{\mathcal{O}_{k'}}$ over $\hat{P}^{(eb)}_{\mathcal{O}_{k'}}$. Also, $z$ extends to a section over $\hat{R}^{(eb-1)}_{k'}$ and it generates $\hat{R}^{(eb-1)}_{k'}$ over $\hat{P}^{(eb-1)}_{k'}$.

**Proof.** We first pick any lift $z'$ of $\bar{z}$ over $\hat{R}^{(eb)}_{\mathcal{O}_{k'}}$, it must satisfy an equation of the form $z'^p + a_1 z'^{p-1} + \cdots + a_p = 0$, where $a_1, \ldots, a_p \in \mathcal{O}_{k'} \langle \pi_{k'}^{-eb} - \delta_0, \pi_{k'}^{-eb} - \delta_1, \ldots, \pi_{k'}^{-eb} - \delta_m \rangle$ and the reduction of this equation is exactly (3.1.13). For the given $\alpha_{f+} \subset \mathcal{O}_{k'}$, we have

$$\epsilon = z'^p - z' + (\alpha_0 \pi_{k'}^{-eb} - \delta_0 + \alpha_1 \pi_{k'}^{-eb} - \delta_1 + \cdots + \alpha_m \pi_{k'}^{-eb} - \delta_m) \in \pi_{k'} \mathcal{O}_{\hat{R}^{(eb)}_{\mathcal{O}_{k'}}}.$$

Now, $z = z' + \epsilon + e^\varphi + e^{2\varphi} + \cdots$ converges and satisfies (3.1.16).

Since $z$ generates a subalgebra of $\mathcal{O}_{\hat{R}^{(eb)}_{\mathcal{O}_{k'}}}$, which is finite and étale over $\mathcal{O}_{\hat{P}^{(eb)}_{\mathcal{O}_{k'}}}$ of the same degree $p$, this subalgebra has to equal to $\mathcal{O}_{\hat{R}^{(eb)}_{\mathcal{O}_{k'}}}$.

For the similar statement for $eb - 1$, we argue as follows. Since $\hat{R}^{(eb-1)}_{k'}$ is finite and étale over $\hat{P}^{(eb-1)}_{k'}$, it must be the normal closure of $\hat{P}^{(eb-1)}_{k'}$ in $\hat{R}^{(eb)}_{k'}$. In particular, $z$ extends to a section over $\hat{R}^{(eb-1)}_{k'}$, with the same minimal equation (3.1.16). Again, $z$ generates a subalgebra of $\mathcal{O}_{\hat{R}^{(eb-1)}_{k'}}$, which is finite and étale over $\mathcal{O}_{\hat{P}^{(eb-1)}_{k'}}$ of same degree; it has to generate the whole ring. The proposition is proved.

\[\boxed{}\]
### 3.2 Lifting rigid spaces

In order to compare Saito’s definition of refined Swan conductors with the definition using differential modules, we need to lift the picture from $k$ to the annulus $A^1_{K[\eta, 1]}$. This is the essential part of the construction of rigid cohomology of Berthelot. We will also relate some structure between the spaces over $k$ and their lifts over $A^1_{K[\eta, 1]}$, following [Xia11] Section 3.1.

Keep the notation as in previous subsection.

**Construction 3.2.1.** Replacing $X$ by a Zariski neighborhood of $\xi$ if necessary, there exists a finite morphism $f : Y \to X$ between two affine smooth formal schemes of topologically finite type over $O_{K_0}$, such that $f$ reduces to $f$ modulo $p$ and $f$ is finite étale over $Y \setminus f^{-1}(D) \to X \setminus D$ with Galois group $G_{l/k}$. In particular, this says that the special fibers of $X$ and $Y$ are $X$ and $Y$, respectively.

Let $\Delta_X : X \to X \times_{\text{Spf}O_{K_0}} X$ be the diagonal embedding. Let $\Delta_Y : Y \to Y \times_{\text{Spf}O_{K_0}} X$ be the morphism induced by identity and $f$ on both factors, respectively. Let $p_1$ and $p_2$ be the projection of $X \times_{\text{Spf}O_{K_0}} X$ to the first and the second factors, respectively.

Let $X^\wedge$ denote the completion of $X \times_{\text{Spf}O_{K_0}} X$ along the diagonal embedding $\Delta_X$; it can be identified with the completion of the cotangent bundle of $X$ along its zero section. Let $Y^\wedge = X^\wedge \otimes_{p_1, X} Y^\wedge$; it is the same as the completion of $Y \times_{\text{Spf}O_{K_0}} X$ along the embedding $\Delta_Y$.

For $\eta \in (0, 1)$, we write $R^\text{int}_{K, \eta}$ for the subring of $R^\text{int}_K$ consisting of elements having 1-Gauss norm $\leq 1$; it is complete with respect to the $\eta'$-Gauss norm for $\eta' \in [\eta, 1]$. On one hand, this ring does not give rise to a formal scheme; on the other, it is good to keep to keep the geometric intuition. Hence, we introduce the *geometric incarnation* $\text{Sp}R^\text{int}_{K, \eta}$ which is just a symbol. Any morphism between geometric incarnations should be thought of ring homomorphisms; in particular, fiber product is simply (completed) tensor products. We also remark that everything we deal with now are affine schemes and there is no question of gluing.

We may compare the following commutative diagram with (3.1.5).

\[
\begin{array}{ccc}
Y & \xrightarrow{f} & X \\
\downarrow{\Delta_Y} & & \downarrow{\Delta_X} \\
Y^\wedge & \xrightarrow{f \times 1} & X^\wedge \\
\downarrow{p_2} & & \downarrow{i} \\
\text{Sp}R^\text{int}_{K, \eta} & \xleftarrow{i} & \text{Sp}R^\text{int}_{K, \eta} \\
\downarrow{p_1} & & \\
Y & \xrightarrow{f} & X \\
\end{array}
\]

where $i : \text{Sp}R^\text{int}_{K, \eta} \to X$ is the geometric incarnation of the natural homomorphism $O_{X, \xi}^\wedge \to R^\text{int}_{K, \eta}$, for some $\eta \in (0, 1) \cap \mu^Q$. We have $\text{Sp}R^\text{int}_{K, \eta} \times_{X} Y = \text{Sp}R^\text{int}_{L, \eta^{1/q_{l/k}}} \times_{X} Y$ for $\eta$ sufficiently close to 1−.

Denote $P_\eta = X^\wedge \times_{p_2, X} \text{Sp}R^\text{int}_{K, \eta}$ and $Q_\eta = Y^\wedge \times_{p_2 \circ (f \times 1), X} \text{Sp}R^\text{int}_{K, \eta}$.

Again, $P_\eta$ and $Q_\eta$ should be thought of as geometric incarnations of $O_{P_\eta}$ and $O_{Q_\eta}$, the completed tensor product of corresponding ring of functions. Therefore, we have the following Cartesian...

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Lemma 3.2.4. The morphism $p_1 : \mathbf{P}_\eta \to \text{Sp}(\mathcal{R}^\text{int}_{K,\eta})$ is given by a continuous homomorphism $\psi : \mathcal{R}^\text{int}_{K,\eta} \to \mathcal{R}^\text{int}_{K,\eta}[\delta_0/T, \delta_1, \ldots, \delta_m]$, for which $\psi(T) = T + \delta_0$, $\psi(B_j) = B_j + \delta_j$ for $j \in J$. More precisely, for $x \in \mathcal{R}^\text{int}_{K,\eta}$, we have

$$\psi(x) = \sum_{e_j > 0}^{+\infty} \frac{\partial e_j^r(x)}{(e_j^r)!} \delta_j.$$

Proof. The first statement follows from the description of $\mathbf{X}^\wedge$ above and the second statement follows by the uniqueness of such homomorphism. \qed

Construction 3.2.5. Let $k'/k$ be a finite separable extension of ramification degree $e$. Since $\mathcal{R}^\text{int}_{K}$ is Henselian, there exists $\mathcal{R}^\text{int}_{K'}$ corresponding to the extension $k'/k$, where $K'$ is the fraction field of a Cohen ring of $K$. For $\eta$ sufficiently close to $1^{-}$, the extension $\mathcal{R}^\text{int}_{K'}/\mathcal{R}^\text{int}_{K}$ descends to a finite étale algebra $\mathcal{R}^\text{int}_{K'\eta^{1/e}}$ over $\mathcal{R}^\text{int}_{K,\eta}$. Fix such an $\eta$. Let $T'$ denote the coordinate of $\mathcal{R}^\text{int}_{K'\eta^{1/e}}$.

Let $r \in \mathbb{N}$ (be a proxy of $eb$ or $eb - 1$). Let $\mathbf{P}^{(r)}_{K',\eta} = \text{Sp}(\mathcal{R}^\text{int}_{K',\eta^{1/e}}[T'r - e\delta_0, T'r - e\delta_j])$ be the geometric incarnation of a closed polydisc over $\text{Sp}(\mathcal{R}^\text{int}_{K,\eta})$; it may be viewed as a subspace of $\mathbf{P}_\eta$ (in the sense of geometric incarnation). Let $Q^{(r)}_{K',\eta}$ be preimage (in the sense of geometric incarnation) of $\mathbf{P}^{(r)}_{K',\eta}$ under the morphism $Q_\eta \to \mathbf{P}_\eta$.

Proposition 3.2.6. Let $\rho$ be a $p$-adic representation of $G_{1/k}$. Let $\mathcal{F}_\rho = ((f \times 1)_* \mathcal{O}_{\mathbf{P}_\eta} \otimes V_p)^{G_{1/k}}$ be the differential module over $\mathbf{P}_\eta$ and for $r \in \mathbb{N}$, let $\mathcal{F}^{(r)}_{\rho,K'} = ((f \times 1)_* \mathcal{O}_{\mathbf{P}^{(r)}_{K',\eta}} \otimes V_p)^{G_{1/k}}$ be the differential module over $\mathbf{P}^{(r)}_{K',\eta}$. Then $\mathcal{F}_\rho$ and $\mathcal{F}^{(r)}_{\rho,K'}$ are the pullbacks of $\mathcal{E}_\rho$ along $p_1 : \mathbf{P}_\eta \to \text{Sp}\mathcal{R}^\text{int}_{K,\eta}$ and $p_1 : \mathbf{P}^{(r)}_{K',\eta} \to \text{Sp}\mathcal{R}^\text{int}_{K,\eta}$, respectively.

Proof. This follows from the following $G_{1/k}$-equivariant Cartesian diagram of geometric incarnated morphisms.

Corollary 3.2.7. For $a \in \mathbb{Q} \setminus \mathbb{Z}$ and $\eta \in (0,1) \cap \mathbb{Q}$, let $F_{\eta,a}$ denote the completion of $K(T, \delta_j)$ with respect to the $(\eta, \eta^{a+1}, \eta^a, \ldots, \eta^0)$-Gauss norm and let $F_{\eta,a}' = F_{\eta,a} \otimes_{\mathcal{R}^\text{int}_{K,\eta}} \mathcal{R}^\text{int}_{K'^{1/e},\eta}$. Assume that $\rho$ has pure log-break $b$ and pure refined Swan conductor $\vartheta = \pi^{-b} \left( \frac{\partial b}{\partial k} + \alpha_1\frac{\partial b}{b_1} + \cdots + \alpha_m\frac{\partial b}{b_m} \right)$.
where $\bar{\alpha}_{J^+} \in \bar{k}$. Then, if $r < ea < eb$ and $\eta$ is sufficiently close to $1^-$, $\mathcal{F}_\rho \otimes \mathcal{F}_{\eta,a} = \mathcal{F}_{\rho,K'}^{(r)} \otimes \mathcal{F}_{\eta,a}^{(r)}$ as a $\partial/\partial \delta_{J^+}$-differential module has pure intrinsic radius $\eta^b$ and pure refined intrinsic radius

\[ T^{-b}(\bar{\alpha}_0 d\delta_0 + \bar{\alpha}_1 d\delta_1 + \cdots + \bar{\alpha}_m d\delta_m). \]

Proof. By Lemma [3.2.1] and Proposition [3.2.6], $\mathcal{F}_{\rho,K'}^{(r)}$ is the pullback of $\mathcal{E}_\rho$ along the multidimensional analog of generic point homomorphism as in Corollary [1.4.20]. However, the calculation of refined $\partial_j$-radii can be computed independently for each of $\partial_j$. Hence, the statement follows from Corollary [1.4.20].

Before proceeding, we briefly recall the lifting construction in [Xia11, Section 1], which lifts a rigid analytic space over $\kappa_K^r$ to a rigid analytic space over $A_{K^r}^{1,\eta^{1/e},1}$ for $\eta \in \bar{p}^{\mathbb{Q}} \cap (0,1)$ sufficiently close to $1^-$. 

Construction 3.2.8. Let $Z$ be a rigid analytic space over $k'$ with ring of analytic functions $A_{k'} = k'(u_1, \ldots, u_s)/I_{k'}$. Let $I_{k'} \subset \mathcal{O}_{k'}(u_1, \ldots, u_s)((T'))$ be an ideal so that $\mathcal{O}_{k'}(u_1, \ldots, u_s)((T'))/I_{k'}$ is flat over $\mathcal{O}_{k'}$ and $I_{k'} \otimes \mathcal{O}_{k'} k' = I_{k'}$. We call $X_\eta = \text{Spf}(R_{K'}^{\text{int}}(u_1, \ldots, u_s)/I_{k'})$ a lifting space of $X$.

Proposition 3.2.9. Let $r \in \mathbb{N}$.

(i) The space $Q_{K',\eta}^{(r)}$ is a lifting space of $Q_{k'}^{(r)}$.

(ii) Assume moreover that $Q_{K',\eta}^{(r)}$ is a stable model and $r = eb$ or $eb - 1$. Then for $\eta$ sufficiently close to $1^-$, $Q_{K',\eta}^{(r)}$ has $[l:k]/p$ connected components, each of them is isomorphic to a formal scheme $R_{K',\eta}^{(r)}$ finite and étale over $P_{K',\eta}^{(r)}$ of degree $p$.

(iii) Fix a Dwork pi $\pi = (-p)^{1/(p-1)}$ and fix $\alpha_{J^+} \subset R_{K'(\pi),\eta^{1/e}}^{\text{int}}$ lifts of $\alpha_{J^+}$. By making $\eta$ closer to $1^-$ if needed, we may choose a lift $z$ of $z$ on $R_{K'(\pi),\eta}^{(r)}$ whose minimal polynomial over $P_{K',\eta}^{(r)}$ is of the form

\[ \frac{1}{p\pi}((1 + \pi z)^p - 1 - p\pi(\alpha_0 T^{r-eb} \delta_0 + \alpha_1 T^{r-eb} + \cdots + \alpha_m T^{r-eb})) = 0. \]  

Proof. The first statement follows from the construction. The second statement follows from [Xia11, Proposition 1.2.11]; the fact that they are all isomorphic to the same $R_{K'/(\pi),\eta}^{(r)}$ is a corollary of (iii), proved above.

Now, we prove the last statement. We pick a lift $z_1$ of $z$ over $R_{K'/(\pi),\eta}^{(r)}$ whose minimal polynomial reduces to $[3.1.16]$ modulo $\pi$. (Note that $K$ is absolutely unramified.) We define the following substitution process. Assume that we have defined $z_i$. We define

\[ \lambda_i = \frac{1}{p\pi}((1 + \pi z_i)^p - 1 - p\pi(\alpha_0 T^{r-eb} \delta_0 + \alpha_1 T^{r-eb} + \cdots + \alpha_m T^{r-eb})). \]

and set $z_{i+1} = z_i - \lambda_i$. Hence,

\[ \lambda_{i+1} = \frac{1}{p\pi}((1 + \pi z_i - \pi \lambda_i)^p - (1 + \pi z_i)^p + p\pi \lambda_i) \]

\[ = (1 - (1 + \pi z_i)^{p-1})\lambda_i + \sum_{n=2}^{p-1} \frac{1}{p\pi}((-\pi \lambda_i)^n + (-1)^{p-1} \lambda_i^n). \]
Since $|\lambda_1| \leq p^{-1/(p-1)}$, by continuity, $|\lambda_1|_\eta < 1$ for $\eta \in [\eta_0, 1]$ for some $\eta_0$ sufficiently close to $1^-$. Thus, 

$$|\lambda_{i+1}|_\eta \leq \max \left\{ p^{-1/(p-1)}|\lambda_i|_\eta, |\lambda_i|_\eta^p \right\} \quad \text{for } \eta \in [\eta_0, 1].$$

As a consequence, this substitution process converges with respect to all $\eta$-Gauss norms for $\eta \in [\eta_0, 1]$. The limit $z = \lim_{\eta \to 0^+} z_\eta$ satisfies (3.2.10). By the same argument as in Proposition 3.1.15, the limit $z$ generates $R^{(r)}_{K(\pi),\eta}$ over $P^{(r)}_{K(\pi),\eta}$ when $\eta$ is sufficiently close to $1^-$. 

### 3.3 Dwork isocrystals

In this subsection, we single out a calculation of refined radii. This is the heart of the comparison Theorem 3.1.1. We state it in a slightly general form to reduce some load of the notation; also, we think it has its own interest in the study of differential modules.

**Hypothesis 3.3.1.** Only in this subsection, let $K$ be a complete discretely valued field of characteristic zero, containing $\pi$. Let $\kappa$ denote its residue field, which has characteristic $p > 0$.

**Situation 3.3.2.** Let $P = \text{Spf}R^{\text{int}}_{K,\eta}(\delta_0, \ldots, \delta_m)$, where $T$ is the coordinate of $R^{\text{int}}_{K,\eta}$. Let $R$ be a finite extension of $P$ generated by $z$ satisfying the relation

$$(1 + \pi T)^p = 1 + p\pi T^{-r}(\alpha_0 \delta_0 + \cdots + \alpha_m \delta_m),$$

where $r \in \mathbb{N}$ and $\alpha_j \in R^{\text{int}}_{K,\eta}$ for $j = 1, \ldots, m$. Let $\alpha_j \in \kappa$ be the reduction of $\alpha_j$ for any $j$. We assume that not all $\alpha_j$ is zero. Let $f : R \to P$ be the natural morphism, which is finite and étale.

**Construction 3.3.3.** We reproduce a multi-dimensional version of the construction in [Ked05, Lemma 5.4.7]. The pushforward $f_*O_{P}$ decomposes as the direct sum of $p$ differential modules of rank 1, with respect to $\partial_j = \partial / \partial \delta_j$ for $j = 0, \ldots, m$.

Let $E_i$ be the differential module given by $(1 + \pi z)^i$ for $i = 1, \ldots, p - 1$. (The trivial submodule of $f_*O_{P}$ is not interesting to us.)

**Notation 3.3.4.** For $\eta \in (0, 1)$, let $F_\eta$ be the completion of $K(T, \delta_0, \ldots, \delta_m)$ with respect to $(\eta, 1, \ldots, 1)$-Gauss norm.

**Proposition 3.3.5.** For $\eta$ sufficiently close to $1^-$, the intrinsic radius $IR(E_i \otimes F_\eta) = \eta^r$ and the refined intrinsic radius of $E_i$ for $i = 1, \ldots, p - 1$ is given by

$$\Theta(E_i \otimes F_\eta) = \{ i\pi T^{-r}(\alpha_0 d\delta_0 + \cdots + \alpha_m d\delta_m) \}.$$

**Proof.** Since

$$p \frac{d(1 + \pi z)^i}{(1 + \pi z)^i} = i \frac{d(1 + p\pi T^{-r}(\alpha_0 \delta_0 + \cdots + \alpha_m \delta_m))}{1 + p\pi T^{-r}(\alpha_0 \delta_0 + \cdots + \alpha_m \delta_m)},$$

$E_i$ is isomorphic to a differential module give by

$$\nabla v = i\pi T^{-r}(1 + p\pi T^{-r}(\alpha_0 \delta_0 + \cdots + \alpha_m \delta_m))^{-1} v \otimes (\alpha_0 d\delta_0 + \cdots + \alpha_m d\delta_m).$$

Fix $j = 0, \ldots, m$. Using the proof of [?, Lemma 5.4.7], when $\eta$ is sufficiently close to $1^-$ (e.g., $\eta > p^{-1/r}$), viewed as a $\partial_j$-differential module, this is the same as

$$\partial_j w_j = i\pi \alpha_j T^{-r} w_j,$$

where $w_j$ is a section of $E_i$, dependent on $j$. Hence, $\partial_j^n (w_j) = (i\pi \alpha_j T^{-r})^n w_j$ and the proposition follows immediately. 

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3.4 Comparison

In this subsection, we assemble the results from previous subsections to prove the following comparison theorem.

**Theorem 3.4.1.** Assume Hypothesis (Geom). Then for \( b \in \mathbb{Q}_{>0} \), the homomorphism \( \text{rsw} : \text{Hom}(\text{Fil}_{\log}^{b}G_{k}/\text{Fil}_{\log}^{b+}G_{k}, \mathbb{F}_{p}) \rightarrow \Omega_{k}^{1}(\log) \otimes \pi_{k}^{b} \bar{\kappa} \) in Theorem 2.3.7 is the same as \( \text{rsw}' \) in Proposition 3.1.11.

**Proof.** Let \( \bar{k} \) be as in Proposition 2.2.14. By [Sai09, Lemma 1.22], \( \text{rsw}' \) for \( k \) factors as

\[
\text{Hom}(\text{Fil}_{\log}^{b}G_{k}/\text{Fil}_{\log}^{b+}G_{k}, \mathbb{F}_{p}) \rightarrow \text{Hom}(\text{Fil}_{\log}^{\epsilon_{k}/b}G_{k}/\text{Fil}_{\log}^{(\epsilon_{k}/b)+}G_{k}, \mathbb{F}_{p}) \xrightarrow{\text{rsw}'} \Omega_{k}^{1}(\log) \otimes \mathcal{O}_{\bar{k}}^{\epsilon_{k}/b} \pi_{k}^{b} \bar{\kappa}.
\]

The same factorization is also valid for \( \text{rsw} \) as in (2.3.12). Hence, we may choose \( \epsilon_{k}/b \) divisible by the denominator of \( b \) and reduce to the case when \( b \) is an integer. We also remark that, by the same reason, we may feel free to replace \( k \) by a finite tamely ramified extension.

Fix \( \zeta_{p} \) a \( p \)-th root of unity. We fix a (nontrivial) character \( \chi : \text{Fil}_{\log}^{b}G_{k}/\text{Fil}_{\log}^{b+}G_{k} \rightarrow \mathbb{F}_{p} \) and denote \( \text{rsw}'(\chi) = \pi_{k}^{-b}(\bar{\alpha}_{0} d\pi_{k} + \bar{\alpha}_{1} db_{1} + \cdots + \bar{\alpha}_{m} db_{m}) \), where \( \bar{\alpha}_{0}, \ldots, \bar{\alpha}_{m} \in \bar{\pi} \). By identifying \( 1 \in \mathbb{F}_{p} \) with \( \zeta_{p} \in \mathbb{Q}_{p}(\zeta_{p}) \), we get a homomorphism \( \text{Fil}_{\log}^{b}G_{k}/\text{Fil}_{\log}^{b+}G_{k} \rightarrow \mathbb{F}_{p} \rightarrow \mathbb{Q}_{p}(\zeta_{p})^{*} \); we still use \( \chi \) to denote the composition. By the argument in Theorem 2.3.7 and by possibly replacing \( k \) by a finite tamely ramified extension, we can find a \( p \)-adic representation \( \rho \) of \( G_{k} \) with finite image and pure log-break \( b \) such that \( \rho|_{\text{Fil}_{\log}^{b}G_{k}} \) is a direct sum of copies of \( \chi \). Moreover, we may assume that \( \rho \) is irreducible when restricted to any finite tamely ramified extension of \( k' \) of \( k \). The representation \( \rho \) factors exact through \( l/k \) a finite Galois extension. It must be true that \( \text{Fil}_{\log}^{b}G_{k}/G_{l} \cap \text{Fil}_{\log}^{b+}G_{k} \simeq \mathbb{F}_{p} \). By possibly making another tamely ramified extension of \( k \), we may assume that the second highest ramification break of \( l/k \) is strictly less than \( b - 1 \); thus, \( \text{Fil}_{\log}^{b-1}G_{k}/G_{l} \cap \text{Fil}_{\log}^{b-1}G_{k} \simeq \mathbb{F}_{p} \).

Now, we employ the results and notation from previous subsections. By Proposition 3.2.9 \( \mathbb{Q}_{K'}(e_{b-1}^{-1}) \) is disjoint union of \([l : k]/p \) copies of \( \mathbb{R}_{K', \eta}^{(e_{b-1}^{-1})} \), which is finite and étale over \( \mathbb{F}_{K'}^{(e_{b-1}^{-1})} \), generated by \( z \) with minimal polynomial \( 3.2.10 \). (Here, we made a choice of \( z \) and \( z \) in accordance with the algebraic group structure on \( Q_{K'}^{b} \), see the remarks after 3.1.13.) By Proposition 3.3.5 this implies that \( \mathcal{F}_{\rho,K'}^{e_{b-1}^{-1}} \otimes F_{\eta,b_1^{-1}/2e}^{e_{b-1}^{-1}} \) for \( \eta \rightarrow 1^{-} \) has pure refined intrinsic radii \( \pi T^{-b}(\bar{\alpha}_{0} d\pi_{k} + \bar{\alpha}_{1} db_{1} + \cdots + \bar{\alpha}_{m} db_{m}) \). (Here, we made a choice of Dwork pi \( \pi \) so that \( \pi \equiv \zeta_{p} - 1 \mod (\zeta_{p} - 1)^{2} \) as in Remark 2.3.1.) By Corollary 3.2.7, the refined Swan conductor of \( \mathcal{E}_{\rho} \) has to be \( \pi_{k}^{-b}(\bar{\alpha}_{0} d\pi_{k} + \bar{\alpha}_{1} db_{1} + \cdots + \bar{\alpha}_{m} db_{m}) \), same as \( \text{rsw} \).

**Remark 3.4.2.** By [AS09, Theorem 9.1.1], the above two refined Swan conductors are the same as Kato’s definition in [Kat89], when the representation is one-dimensional. So all three definitions agree. This result might also be implicitly contained in [CP09].

4 Refined Swan conductors and variation of intrinsic radii on polyannuli

In this section, we explain the relation between refined Swan conductors and the variation of intrinsic radii on polyannuli. We assume Hypothesis 1.5.1 and keep the notation as in Section 1. We do not force \( p > 0 \) in this section unless otherwise specified. We also assume that \( K \) is discretely valued.
4.1 Partial decomposition for differential modules

In Subsection [1.5] we deliberately restricted ourselves to the situation over open annuli. To understand the situation over a bounded analytic ring \( K\{\alpha/t, t\}_0 \), we need a technical lemma on partial decomposition of differential modules, which is not covered by [KX10].

We take \( \alpha \in (0, 1) \) for this subsection.

**Notation 4.1.1.** We define \( E \) to be the completion of \( \text{Frac}(K\{\alpha/t, t\}_0) \) with respect to the 1-Gauss norm; it is also the completion of \( R_K^{bd} \) with respect to the same norm and hence does not depend on \( \alpha \). Also, \( E \) contains \( F_1 \) as a subfield.

If \( s \in -\log|K^\times| \), we can find an element \( x \in K^\times \) with \( |x| = e^{-s} \). This \( x \) gives rise to an isomorphism \( \kappa_E^{(s)} \xrightarrow{\sim} \kappa_E \cong \kappa_K((t)) \). Hence, we have a canonical valuation \( \nu_s(\cdot) \) on \( \kappa_E^{(s)} \) given by the valuation on \( t \); this does not depend on the choice of \( x \in K^\times \). This valuation extends naturally to \( \kappa_{E_{alg}}^{(s)} \) for \( s \in \mathbb{Q} \cdot \log|K^\times| \).

**Notation 4.1.2.** Let \( j \in J^+ \). For \( M \) a \( \partial_j \)-differential module over \( K\{\alpha/t, t\}_0 \) of rank \( d \) and \( i \in \{1, \ldots, d\} \), define

\[
\begin{align*}
 f_i^{(j)}(M, 0) & = -\log R_{\partial_j}(M \otimes E; i), \\
 F_i^{(j)}(M, 0) & = f_1^{(j)}(M, 0) + \cdots + f_i^{(j)}(M, 0).
\end{align*}
\]

We similarly define \( f_i(M, 0) \) and \( F_i(M, 0) \) if \( M \) is a \( \partial_{j^+}\)-differential module over \( K\{\alpha/t, t\}_0 \).

**Proposition 4.1.3.** Fix \( j \in J^+ \). Let \( M \) be a \( \partial_j \) - (resp. \( \partial_{j^+} \) ) differential module of rank \( d \) over \( K\{\alpha/t, t\}_0 \). Then we have the following.

(a) The functions \( f_i^{(j)}(M, r) \) and \( F_i^{(j)}(M, r) \) (resp. \( f_i(M, r) \) and \( F_i(M, r) \)) are continuous at \( r = 0 \).

(b) Suppose for some \( i \in \{1, \ldots, d - 1\} \), the function \( F_i^{(j)}(M, r) \) (resp. \( F_i(M, r) \)) is affine and \( f_i^{(j)}(M, r) > f_{i+1}^{(j)}(M, r) \) (resp. \( f_i(M, r) > f_{i+1}(M, r) \)) for \( r \in (0, -\log \alpha) \). Then \( M \) admits a unique direct sum decomposition over \( K\{\alpha/t, t\}_0 \) separating the first \( i \) subsidiary \( \partial_j \)-radii (resp. intrinsic radii) of \( M \otimes F_\eta \) for any \( \eta \in (0, -\log \alpha) \) and of \( M \otimes E \).

**Proof.** See [KX10] Theorems 2.3.9, 2.5.5 and Remarks 2.3.11, 2.5.7].

Note that, as stated in (b) of the above proposition, it excludes the case when \( f_i^{(j)}(M, r) > f_{i+1}^{(j)}(M, r) \) or \( f_i(M, r) > f_{i+1}(M, r) \) for \( r \in (0, -\log \alpha) \) but \( f_i^{(j)}(M, 0) = f_{i+1}^{(j)}(M, 0) \) or \( f_i(M, 0) = f_{i+1}(M, 0) \). The rest of this subsection is devoted to extend the conclusion of (b) to this case.

**Notation 4.1.4.** Denote \( \mathcal{R} = \cap_{\alpha \in (0, 1)} K\{\alpha/t, t\} \) and \( \mathcal{R}^{bd} = \cap_{\alpha \in (0, 1)} K\{\alpha/t, t\}_0 \), where the latter can be identified with the subring of the former consisting of elements with finite 1-Gauss norm.

**Hypothesis 4.1.5.** For notational convenience, we suppose that \( |u_j| = 1 \) for \( j \in J \). We can always reduce to this case via replacing \( K \) by the completion of \( K\{x_1, \ldots, x_m\} \) with respect to \((|u_1|, \ldots, |u_m|)-Gauss norm and replacing \( u_j \) by \( u_j/x_j \), where \( \partial_j(x_j') = 0 \) for \( j, j' \in J \). Note that \( K \) is still discretely valued.

**Lemma 4.1.6.** The ring \( \mathcal{R}^{bd} \) is a field. A sequence \( (f_n)_{n \in \mathbb{N}} \subset K\{\alpha/t, t\}_0 \) is convergent if it is convergent for \( r \)-Gauss norm for all \( r \in (\alpha, 1) \) and is bounded for 1-Gauss norm.
Proof. The first statement is well-known; see for example [Ked05, Lemma 3.5.2]. We remark that it is important that $K$ being discretely valued for this to be true. To see the second statement, we observe that $(f_n)_{n\in\mathbb{N}}$ converges in $K\{\{\alpha/t, t\}\}$. The limit has bounded coefficients and hence lies in $K\{\{\alpha/t, t\}\}$.

\[\square\]

**Lemma 4.1.7.** Fix $j\in J^+$. Let $R^{bd}\{T\}$ be the ring of twisted polynomial as in Definition 1.2.1, where $T$ stands for $\partial_j$ if $j\in J$ and $\partial_0$ if $j=0$. Let $P=T^d+a_1T^{d-1}+\cdots+a_d\in R^{bd}\{T\}$ be a monic twisted polynomial whose Newton polynomial has pure slope $s<1$. Let $\{a_1,\ldots,a_r\}$ be the set of $\nu_s$-valuations of reduced roots of $P$ (not counting multiplicity, with either increasing or decreasing order), when we view $P$ as a twisted polynomial in $E\{T\}$. Then $P$ admits a unique factorization $P=Q_1\cdots Q_r$ as products of monic twisted polynomials such that the reduced roots of $Q_i$, when viewed as a twisted polynomial in $E\{T\}$, have pure $\nu_s$-valuations $a_i$.

Proof. The statement of the lemma is symmetric, we assume that $a_1,\ldots,a_n$ is an increasing sequence. It then suffices to show that we can write $P=QR$ as products of two monic polynomials such that the reduced roots of $Q$ (resp. $R$), when viewed as a twisted polynomial in $E\{T\}$, have pure $\nu_s$-valuations $a_n$ (resp. strictly less than $a_n$). We can also write it as $P=QR$ satisfying the same condition, but with different $Q$ and $R$ themselves. Indeed, by Lemma 4.1.6, the claim follows from [Ked09, Proposition 3.2.2] because the sequence $\{P_i\}$ and $\{Q_i\}$ there is bounded under the 1-Gauss norm. \[\square\]

**Lemma 4.1.8.** Fix $j\in J$. Let $M$ be a $\partial_j$-differential module of rank $d$ over $K\{\{\alpha/t, t\}\}_0$ such that $M\otimes E$ has pure intrinsic $\partial_j$-radii $\text{IR}_{\partial_j}(M\otimes E)<\omega$. By choosing a cyclic vector of $M\otimes R^{bd}$, we may identify $M\otimes R^{bd}$ with $R^{bd}\{T\}/R^{bd}\{T\}P$, where $P$ is a twisted polynomial in $R^{bd}\{T\}$. Then for $\eta$ sufficiently close to $1^-$, the slopes of Newton polygon of $P$ (for the $\eta$-Gauss norm) are the log of the subsidiary $\partial_j$-radii of $M\otimes F_{\eta}$ minus $\log_\omega$.

Proof. The identification $M\otimes R^{bd}\simeq R^{bd}\{T\}/R^{bd}\{T\}P$ descents to

$$M\otimes K\{\{\beta/t, t\}\}_0\simeq K\{\{\beta/t, t\}\}_0\{T\}/K\{\{\beta/t, t\}\}_0\{T\}P$$

for $\beta$ sufficiently close to $1^-$. Note that for $\eta$ sufficiently close to $1^-$, all $\partial_j$-radii of $M\otimes F_\eta$ are visible. The lemma follows from Proposition 1.2.8 \[\square\]

For the following Theorem, we do not assume Hypothesis 4.1.5.

**Theorem 4.1.9.** Fix $j\in J^+$. Let $M$ be a $\partial_j$- (resp. $\partial_{j+}$-) differential module of rank $d$ over $K\{\{\alpha/t, t\}\}_0$ such that $M\otimes E$ has pure intrinsic $\partial_j$-radii $\text{IR}_{\partial_j}(M\otimes E)<1$ (resp. intrinsic radii $\text{IR}(M\otimes E)<1$). Suppose for some $i\in\{1,\ldots,d-1\}$, the function $F^{(j)}_i(M,r)$ (resp. $F^{(j)}_i(M,r)$) is affine and $f^{(j)}_i(M,r)>f^{(j)}_{i+1}(M,r)$ (resp. $f^{(j)}_i(M,r)>f^{(j)}_{i+1}(M,r)$) for $r\in(0,\log_\alpha)$. Then $M$ admits a unique direct sum decomposition over $K\{\{\alpha/t, t\}\}_0$ separating the first $i$ subsidiary $\partial_j$-radii (resp. intrinsic radii) of $M\otimes F_\eta$ for any $\eta\in(0,\log_\alpha)$.

Proof. We first deduce the $\partial_j$-differential module case. By Theorem 1.5.4(e), it suffices to obtain the decomposition over $K\{\{\beta/t, t\}\}_0$ for $\beta\in(\alpha,1)$ sufficiently close to 1.

To start, we assume that $\text{IR}_{\partial_j}(M\otimes E)<\omega$. By making $\beta$ closer to 1, we may assume that $\text{IR}_{\partial_j}(M\otimes F_\eta)<\omega$ for all $\eta\in(\beta,1)$ too. Also, we may impose Hypothesis 4.1.5. Since $\mathcal{R}^{bd}$ is a field, we can find a cyclic vector and write $M\otimes \mathcal{R}^{bd}=\mathcal{R}^{bd}\{T\}/\mathcal{R}^{bd}\{T\}P$ for a monic twisted polynomial
As in Lemma 4.1.9, we may assume that $P$ is a polynomial $P$ as in Lemma 4.1.9. Applying Lemma 4.1.9 to $M \otimes \mathcal{R}^{bd}$ with $a_i$ decreasing, we have can find a submodule of $M$ accounting for the first $i$ subsidiary $\partial_j$-radii of $M \otimes F_\eta$ for any $\eta$ sufficiently close to $1^-$. Applying Lemma 4.1.9 again with $a_i$ increasing, we find a quotient of $M$ accounting for the first $i$ subsidiary $\partial_j$-radii of $M \otimes F_\eta$ for any $\eta$ sufficiently close to $1^-$. Therefore, we obtain a direct sum decomposition and the theorem is proved in this case.

Next, we assume that $p > 0$ and $IR_{\partial_j}(M \otimes E) = p^{-1/(p-1)}$. If $j \in J$, the $\partial_j$-Frobenius $\varphi(\partial_j)$: $K(\partial_j) \rightarrow K$ naturally extends to $\varphi(\partial_j)$: $K(\partial_j)\{\{\alpha/t, t\} \rightarrow K\{\{\alpha/t, t\}$; if $j = 0$, we have $\varphi(\partial_0) : K\{\{\alpha^p/t^p, t^p\} \rightarrow K\{\{\alpha/t, t\}$. Then the desired decomposition follows from the decomposition of $\varphi(\partial_j)$ $M$. Note that $\varphi(\partial_j)^* \varphi(\partial_j) M \simeq M^{\oplus p}$.

If $p > 0$ and $IR_{\partial_j}(M \otimes E) > p^{-1/(p-1)}$, we may assume that $IR_{\partial_j}(M \otimes F_\eta) > p^{-1/(p-1)}$ for all $\eta \in (\beta, 1)$, and the decomposition follows from that of $\partial_j$-Frobenius antecedent of $M$.

Finally, we show that the $\partial_j$-differential module case follows from the $\partial_j$-differential module case. By Proposition 4.1.3(a) and Theorem 4.1.9(a), there exists $\beta$ such that $f_1^{\beta}(M, r)$ is affine over $[1, -\log \beta]$ if $IR_{\partial_j}(M \otimes E)$; if $< 1$. By decompositions given by Proposition 4.1.3(b) and the current theorem for $\partial_j$, $M$ is the direct sum of $\partial_j$-differential modules $M_i$ such that $M \otimes F_\eta$ has pure $\partial_j$-radii for all $\eta \in (\beta, 1)$ and all $j \in J^+$ if $IR_{\partial_j}(M \otimes E) < 1$. Since we already know that $M \otimes E$ has pure intrinsic radii $< 1$, $\partial_j$ for which $IR_{\partial_j}(M \otimes E) = 1$ will not dominate in $M \otimes F_\eta$ if $\eta$ is sufficiently close to $1$, and it hence does not show up in intrinsic radii. Hence, the desired decomposition follows from regrouping factors of this direct sum decomposition.

Remark 4.1.10. The condition $IR_{\partial_j}(M \otimes E) < 1$ is crucial. As pointed out in [Ked10, Remark 12.5.4], one may give counterexamples in the case $IR_{\partial_j}(M \otimes E) = 1$ using the theory of crystals. However, in the presence of Frobenius, one may still get the decomposition. We plan to come back to this point in a future work.

4.2 Refined radii and the log-slopes of the radii

In this subsection, we study the relation between the refined radii at the boundary radii of a differential module and the log-slopes of the intrinsic radii. We continue to assume that

Theorem 4.2.1. Fix $j \in J^+$ and let $M$ be a $\partial_j$-differential module over $K\{\{\alpha/t, t\} of rank $d$. Assume that $M \otimes F_\eta$ and $M \otimes E$ all have pure $\partial_j$-radii and $f_1^{\beta}(M, r)$ is affine of slope $-\alpha$ for $r \in [0, -\log \alpha]$. Moreover, we assume that $R_{\partial_j}(M \otimes E) = \omega \epsilon^s < |u_j|^{-1} (1 if $j = 0$. Then the $\nu$-valuation of refined $\partial_j$-radii of $M \otimes E$ is exactly $a$.

Proof. First, we assume that $M \otimes E$ has pure visible intrinsic $\partial_j$-radii $IR_{\partial_j}(M \otimes E) < \omega$. Moreover, by making $\alpha$ closer to $1^-$, we may assume that $f_i^{(j)}(M, r) > -\log \omega$ and is affine over $[0, -\log \alpha]$.

As in Theorem 4.1.9, we may assume identify $M \otimes \mathcal{R}^{bd}_K$ with $\mathcal{R}^{bd}(T)/\mathcal{R}^{bd}(T)P$ for some twisted polynomial $P = T^d + a_1T^d-1 + \cdots + a_d \in \mathcal{R}^{bd}(T)$.

Since $M \otimes E$ has pure $\partial_j$-radii $\omega \epsilon^s$, the Newton polygon of $P$ with respect to $\nu$-Gauss norm has pure slope $s$ and $\Theta_{\partial_j}(M \otimes E)$ consists of its reduced roots, i.e., the roots of $\overline{P} = T^d + \overline{a}_1T^{d-1} + \cdots + \overline{a}_d$, where $\overline{a}_i \in \kappa_K^{(i)}(t))$. When $\eta$ is sufficiently close to $1^-$, the Newton polygon of $P$ with respect to the $\eta$-Gauss norm is determined by the Newton polygon of $\overline{P}$ in the following sense: it is the lower convex hull of the set $\{(-i, -\log |a_i| - \nu(\overline{a}_i^{(i)}) \log \eta)\}$. By Lemma 4.1.8, this implies that the slopes of $f_i^{(j)}(M, r)$ at
$r = 0$ are exactly the valuations $\nu_s$ of roots of $P$, which in turn equal to the valuations $\nu_s$ of the refined $\partial_j$-radii.

Now, it suffices to reduce to the case above, using $\partial_j$-Frobenius. Assume $p > 0$ from now on. It is easier to work with intrinsic radii and refined intrinsic radii. So, we denote $g_i(M, r) = f_i^{(j)}(M, r) + \log|u_j|$ if $j \in J$ and $g_i(M, r) = f_i^{(j)}(M, r) - r$ if $j = 0$. Moreover, we denote $s' = -\log(\omega IR_{\partial_j}(V)^{−1})$.

If $IR_{\partial_j}(M \otimes E) = \omega = p^{−1/(p−1)}$, let $M_1 = \varphi^{(\theta)}_M$. Then by Lemma [L.2.18](d), we have

$$
\{g'_i(M_1, 0)\} = \begin{cases}
\{pg'_i(M_0, 0) \text{ (d times)}, 0 \text{ (p − 1 times)}\} & \text{if } j \in J; \\
\{pg'_i(M_0, 0) \text{ (pd times)}\} & \text{if } j = 0.
\end{cases}
$$

By Proposition [L.3.18] $\mathcal{I}_\Theta_{\partial_j}(M_1 \otimes E^{(\partial_i)})$ can be grouped into p-tuples $(\frac{\theta + 1}{p}, \frac{\theta + 2}{p}, \ldots, \frac{\theta + p - 1}{p})$ and $\mathcal{I}_\Theta(M \otimes E)$ consists of $(\theta^p − \theta)^{−1/p}$ for each $\theta$ in the group, where $\theta \in \kappa_{E=\mathbb{R}}$.

- When $v_0(\theta) < 0$, $\nu_{-\log p}(\frac{\theta + 1}{p}) = v_0(\theta)$ for $l = 0, \ldots, p − 1$, and $v_0((\theta^p − \theta)^{−1/p}) = v_0(\theta)$;
- when $v_0(\theta) \geq 0$, $\nu_{-\log p}(\frac{\theta + 1}{p}) = 0$ for $l = 0, \ldots, p − 1$, and $v_0((\theta^p − \theta)^{−1/p}) = \frac{1}{p}v_0(\theta)$.

Hence the statement for $M_1$ with $\nu_{-\log p}$ implies that of $M$ with $v_0$.

If $IR_{\partial_j}(M \otimes E) > \omega$, by Lemma [L.2.18](d) and Remark [L.2.19] $M$ has a $\partial_j$-Frobenius antecedent $M_0$ if $\alpha$ is sufficiently close to $1^−$. By Lemma [L.2.18](d) and Proposition [L.3.18] we have

$$
g_i(M_0, r) = pg_i(M, r) \text{ for any } i, \text{ and } \mathcal{I}_\Theta_{\partial_j}(M_0 \otimes E^{(\partial_i)}) = \{(-\theta)^p/p | \theta \in \mathcal{I}_\Theta_{\partial_j}(M \otimes E)\}, \text{ if } j \in J;
$$

$$
g_i(M_0, pr) = pg_i(M, r) \text{ for any } i, \text{ and } \mathcal{I}_\Theta_{\partial_j}(M_0 \otimes E^{(\partial_i)}) = \{(-\theta)^p/p | \theta \in \mathcal{I}_\Theta_{\partial_j}(M \otimes E)\}, \text{ if } j = 0.
$$

Since $v_{(ps')^{−\log p}}((-\theta)^p/p) = pv_{s'}(\theta)$, the statement for $M_0$ with $v_{(ps')^{−\log p}}(\frac{1}{p}v_{(ps')^{−\log p}}$ if $j = 0$, since $\theta^p$ is the coordinate) implies that of $M$ with $v_{s'}(−\log p)$.

**Corollary 4.2.2.** Fix $j \in J^+$ and let $M$ be a $\partial_j$-differential module over $K\{[\alpha/t, t]\}$. Assume that $M \otimes E$ has pure $\partial_j$-radii $R_{\partial_j}(M \otimes E) = \omega e^\delta < |u_j|^{−1}$ (1 if $j = 0$). Then the following two multisets are the same:

(a) the valuations of refined $\partial_j$-radii of $M \otimes E$, i.e., $\{\nu_s(\theta) | \theta \in \Theta_{\partial_j}(M \otimes E)\}$, and

(b) the negatives of slopes of $f_i^{(j)}(M, r)$ at $r = 0$, for $i = 1, \ldots, d$.

**Proof.** It follows from combining Theorems [I.1.9] and [I.2.1].

**Notation 4.2.3.** The valuation $\nu_s$ on $\kappa^{(s)}_E$ extends to a valuation $\kappa^{(s)}_E \frac{dt}{t} \oplus \bigoplus_{j \in J} \kappa^{(s)}_E \frac{du_j}{u_j}$ by setting

$$
\nu_s(\theta_0 \frac{dt}{t} + \theta_1 \frac{du_1}{u_1} + \cdots + \theta_m \frac{du_m}{u_m}) = \min_{j \in J^+ \nu_s(\theta_j)}.
$$

**Corollary 4.2.4.** Let $M$ be a $\partial_j^+$-differential module over $K\{[\alpha/t, t]\}$. Assume that $M \otimes E$ has pure intrinsic radii $IR(M \otimes E) = \omega e^\delta < 1$. Then the following two multisets are the same:
(a) the valuations of refined intrinsic radii of $M \otimes E$, i.e., $\{\nu_\theta(\theta)|\theta \in \Theta(M \otimes E)\}$, and

(b) the negatives of slopes of $f_i(M,r)$ at $r=0$, for $i=1,\ldots,d$.

Proof. It follows from Theorem 1.5.10 immediately. $\square$

Similar to Theorem 1.5.10 we have the following decomposition by refined radii.

**Theorem 4.2.5.** Fix $j \in J^+$ and let $M$ be a $\partial_j$-differential module of rank $d$ over $K\{\alpha/t, t\}_0$. Assume that $M \otimes F_\eta$ for $\eta \in (\alpha,1)$ and $M \otimes E$ all have pure $\partial_j$-radii, and $f_j^*(M,r)$ is affine of slope $-a$ for $r \in [0,-\log a)$. Let $e$ be the prime-to-$p$ part of the denominator of $a$. Moreover, we assume that $R_{\alpha}(M \otimes E) = \omega^e < |u_j|^{-1}$ (1 if $j = 0$). Then there exists a finite tamely ramified extension $K'$ of $K$ and a unique decomposition

$$M \otimes K'\{\alpha^{1/e}/t^{1/e}, t^{1/e}\}_0 = \bigoplus_{\theta \in e_{\kappa_{\text{alg}}}^{(e)}} M_\theta$$

of $\partial_j$-differential modules such that

(a) $M_\theta \otimes F_\eta$ has pure refined $\partial_j$-radii $\theta t^a$ for all $\eta \in (\alpha,1)$, and

(b) All refined $\partial_j$-radii of $M_\theta \otimes E$ is congruent to $\theta t^a$ modulo elements with $\nu_\theta$-valuation bigger than $\nu_\theta(\theta t^a) = a$.

Moreover, by Galois descent, we may obtain the decomposition over $K\{\alpha/t, t\}_0$ by grouping Galois conjugates of $\theta$’s. By further grouping the $\mu_e$-orbits of $\theta$, we can obtain the decomposition over $K\{\alpha/t, t\}_0$.

Proof. The proof is identical to that of Theorem 1.5.10 except that we use decomposition Theorem 4.1.9 in place of Theorem 1.5.4. $\square$

**Theorem 4.2.6.** Let $M$ be a $\partial_{j^+}$-differential module of rank $d$ over $K\{\alpha/t, t\}_0$. Assume that $M \otimes F_\eta$ for $\eta \in (\alpha,1)$ and $M \otimes E$ all have pure intrinsic radii, and $f_{j^+}(M,r)$ is affine of slope $-a$ for $r \in [0,-\log a)$. Let $e$ be the prime-to-$p$ part of the denominator of $a$. Moreover, we assume that \( IR(M \otimes E) = \omega^e < 1 \). Then there exists a finite tamely ramified extension $K'$ of $K$ and a unique decomposition

$$M \otimes K'\{\alpha^{1/e}/t^{1/e}, t^{1/e}\}_0 = \bigoplus_{\theta \in e_{\kappa_{\text{alg}}}^{(e)}} M_\theta$$

of $\partial_{j^+}$-differential modules such that

(a) $M_\theta \otimes F_\eta$ has pure refined intrinsic radii $\partial t^a$ for all $\eta \in (\alpha,1)$, and

(b) All refined intrinsic radii of $M_\theta \otimes E$ is congruent to $\partial t^a$ modulo elements with $\nu_\theta$-valuation bigger than $\nu_\theta(\partial t^a) = a$.

Moreover, by Galois descent, we may obtain the decomposition over $K\{\alpha/t, t\}_0$ by grouping Galois conjugates of $\partial$’s. By further grouping the $\mu_e$-orbits of $\partial$, we can obtain the decomposition over $K\{\alpha/t, t\}_0$. 56
Proof. The proof is identical to that of Theorem \ref{thm:1.5.2} except that we use invoke Theorem \ref{thm:4.2.5} in place of Theorem \ref{thm:1.5.4}. \qed

**Corollary 4.2.7.** Let $M$ be a $\partial_J$-differential module of rank $d$ over $K\{\{\alpha/t, t\}_{[0]}\}$. Assume that $M \otimes E$ has pure intrinsic radii $IR(M \otimes E) = \omega \varphi < 1$ and $f_1(M, r)$ is affine over $[0, -\log \alpha)$ for all $i = 1, \ldots, d$. Then $M = \oplus_{a \in Q} M_a$ be the decomposition of $M$ over $A^1(K, (\alpha, 1))$ such that $f_1(M_a, r) = \cdots = f_{\dim M_a}(M_a, r)$ has slope $-a$. Then the following two multisets are the same.

(a) The (multi)set consists of the union of $\mathcal{I}\Theta(M_a \otimes F) \subset \oplus_{i \in J} t_i \mathcal{K}_{\mathcal{I}}^{(s)} \mathcal{K}_\mathcal{I} \mathcal{T}^{(s)} \mathcal{T}^{(d)} \mathcal{T}$ for all $a$ and for some $\eta \in (a, 0)$ (this does not depend on the choice of $\eta$);

(b) The (multi)set consists of $\vartheta$ for all $\vartheta \in \Theta_{\partial_J}(V)$, where $\vartheta$ is $\vartheta \in \oplus_{i \in J} t_i \mathcal{K}_{\mathcal{I}}^{(s)} \mathcal{K}_\mathcal{I} \mathcal{T}^{(s)} \mathcal{T}^{(d)} \mathcal{T}$ modulo elements with $\nu_{\vartheta}$-valuation bigger than $\nu_{\vartheta}(\vartheta)$.

**Proof.** It follows from the decomposition Theorems \ref{thm:4.1.9} and \ref{thm:4.2.6}. \qed

### 4.3 Variation over polyannuli

Variation of refined intrinsic radii over polyannuli is simply a corollary of the 1-dimensional case. We focus on the interaction between the refined intrinsic radii and the slopes of intrinsic radii, as suggested by Corollary \ref{cor:4.2.4}. In this subsection, we keep the same hypothesis as in previous subsection, namely, we assume Hypothesis \ref{hyp:1.5.1} and assume that $K$ is discretely valued.

**Definition 4.3.1.** A subset $C \subset \mathbb{R}^n$ is called nondegenerate if it contains an open subset of $\mathbb{R}^n$. Its interior is denoted by $C^{\text{int}}$.

An integral affine functional on $\mathbb{R}^n$ is a map $\lambda : \mathbb{R}^n \to \mathbb{R}$ of the form $\lambda(x_1, \ldots, x_n) = a_1 x_1 + \cdots + a_n x_n + b$ for some $a_1, \ldots, a_n \in \mathbb{Z}$ and $b \in -\log |K^\times|Q$.

A subset $C \subseteq \mathbb{R}^n$ is rational polyhedral (or RP for short) if it is bounded and there exist integral affine functionals $\lambda_1, \ldots, \lambda_r$ such that $C = \{x \in \mathbb{R}^n | \lambda_i(x) \geq 0 \text{ for } i = 1, \ldots, r\}$.

For $C \subseteq \mathbb{R}^n$ a RP subset of functionals $\lambda_1, \ldots, \lambda_r$ such that $f(x) = \max\{\lambda_1(x), \ldots, \lambda_r(x)\}$ for any $x \in C$.

**Remark 4.3.2.** Our definition here is different from the convention in [KX10], where RP subsets are not assumed to be bounded. However, some of the statements below are true for unbounded RP, but they are often simple corollaries of the statements with boundedness hypothesis. We leave this as an exercise to the readers.

**Notation 4.3.3.** Throughout this subsection, we put $I = \{1, \ldots, n\}$ for notational simplicity. We may use $\mathfrak{a}$ to denote the $n$-tuple $(a, \ldots, a)$.

**Definition 4.3.4.** For a subset $C \subset \mathbb{R}^n$, let $e^{-C}$ denote the subset $\{e^{-r_1} : r_1 \in C\} \subset (0, +\infty)^n$. A subset $S$ of $[0, +\infty)^n$ is log-RP if $S = e^{-C}$ for some RP subset $C$ of $\mathbb{R}^n$. It is nondegenerate if $C$ is so.

For $S$ a log-RP subset of $[0, +\infty)^n$, define $A_K(S^{\text{int}})$ to be the subspace of the (Berkovich) analytic $n$-space with coordinates $t_1, \ldots, t_n$ satisfying the condition $(t_1, \ldots, t_n) \in S^{\text{int}}$; denote the its ring of functions by $K\{\{S\}\}$. We write $K[S]_0$ to denote the subring of $K\{\{S\}\}$ consisting of functions that are bounded over $|t_i| \in S^{\text{int}}$.  

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Notation 4.3.5. Let $S$ be a nondegenerate log-RP subset of $[0, +∞)^n$ and let $R$ denote either $K\{\{S\}\}$ or $K[S]_0$. Let $M$ be a $(\partial_{t,I,J})$-differential module over $R$ of rank $d$, with respect to the derivations $\partial_1, \ldots, \partial_m$ and $\partial_{m+1} = \partial/\partial t_1, \ldots, \partial_{m+n} = \partial/\partial t_n$. For $\eta_I = (\eta_1, \ldots, \eta_n) \in S$ ($S^{\text{int}}$ if $R = K\{\{S\}\}$), let $F_{\eta_I}$ be the completion of $\text{Frac}(R)$ with respect to the $\eta_I$-Gauss norm. We remark that for $\eta_I$ on the boundary of $S$, $F_{\eta_I}$ “looks different” (more like $E$ than $F_{\eta_I}$ in the 1-dimensional case).

For $r_I \in -\log(S)$ (the $-\log(S^{\text{int}})$ if $R = K\{\{S\}\}$), write $f_i(M, r_I) = -\log IR(M \otimes F_{e^{-r_I}}; l)$ and $F_i(M, r_I) = f_i(M, r_I) + \cdots + f_i(M, r_I)$ for $l = 1, \ldots, d$.

Theorem 4.3.6. Keep the notation as above. We have the following.

(a) (Polyhedrality) The functions $d!F_i(M, r_I)$ for $l = 1, \ldots, d - 1$ and $F_d(M, r_I)$ are integral polyhedral functions.

(b) (Decomposition) Suppose that for some $l \in \{1, \ldots, d\}$, $F_i(M, r_I)$ is affine and $f_i(M, r_I) > f_{i+1}(M, r_I)$ for any $r_I \in -\log(S)$. Then, $M$ admits a unique direct sum decomposition separating the first $l$ subsidiary intrinsic radii of $M \otimes F_{\eta_I}$ for any $\eta_I$.

(c) (Refined radii) Assume that $R = K\{\{S\}\}$ and that $f_1(M, r_I) = \cdots = f_d(M, r_I) = -\log \omega - s - a_1r_1 - \cdots - a_nr_n$ are affine functions over $-\log(S^{\text{int}})$. Let $e_i$ denote the prime-to-$p$ part of the denominator of $a_i$ for all $i \in I$. Then there exists a finite tamely ramified extension $K'$ of $K$ and a multiset $I\Theta(M) \subset \bigoplus_{i \in I} K_{K'/K}^{(s)}(\frac{dt}{t_i}) + \bigoplus_{j \in J} K_{K'/K}^{(s)}(\frac{du}{u_j})$ such that we have a decomposition of differential modules

$$M \otimes R[t^{1/e_1}, \ldots, t^{1/e_n}] = \bigoplus_{\vartheta \in I\Theta(M)} M_\vartheta,$$

where $M_\vartheta \otimes F_{\eta_I}[t^{1/e_1}, \ldots, t^{1/e_n}]$ has pure refined intrinsic radii $t^{a_I}_I \vartheta$.

Proof. For (a) and (b), see [KX10] Theorems 3.3.9 and 3.4.4, and Remark 3.4.7. (c) follows from the same argument but using Theorem 4.3.12 as the decomposition tool.

We state the boundary case about $S \setminus S^{\text{int}}$ separately for a special case and leave it as an exercise of the readers to assemble to get the general case.

Situation 4.3.7. Let $C = \{(x_I) \subset \mathbb{R}^n | x_I \geq 0, x_1 + \cdots + x_n \leq 1\}$, $S = e^{-C}$, and $R = K[S]_0$. Assume moreover that $f_1(M, \underline{0}) = \cdots = f_d(M, \underline{0}) = -\log \omega - s$ with $s < 0$. We define the following two multisets.

(a) Choose $x \in m_K^{(s)} \setminus m_K^{(s)+}$ to identify $\kappa_{F_\perp}^{(s)} : x^{-1} \kappa_{F_\perp}$ and embed the latter into the higher local field $\kappa_K((t_1)) \cdots ((t_n))$, which is equipped with a multi-indexed valuation with respect to the parameters $(t_n, \ldots, t_1)$. This gives rise to a valuation $\nu_s : \kappa_{F_\perp}^{(s)} \to \mathbb{Z}^n \subset \mathbb{Q}^n$, where the latter is equipped with the lexicographical order; this does not depend on the choice of $x$ and it extends to a valuation on $\kappa_{F_\perp}^{(s)+}$ and then to $\bigoplus_{i \in I} K_{K^{\text{alg}}}^{(s)}(\frac{dt}{t_i}) \oplus \bigoplus_{j \in J} K_{K^{\text{alg}}}^{(s)}(\frac{du}{u_j})$ by taking the minimum of $\nu_s$ over the coefficients. We define the multiset $A = \{(v(\nu), \nu^{\vartheta}) | \vartheta \in I\Theta(M \otimes F_\perp)\}$, where $\vartheta$ is the reduction of $t^{a_I}_I \nu_s(\vartheta)$ in $\bigoplus_{i \in I} K_{K^{\text{alg}}}^{(s)}(\frac{dt}{t_i}) \oplus \bigoplus_{j \in J} K_{K^{\text{alg}}}^{(s)}(\frac{du}{u_j})$. 58
(b) By Theorem 4.3.6(a), there exists a RP subset $C'$ of $C$ that is adjacent to the cells $t_1 = \cdots = t_{n-1} = 0$ for $i = 1, \ldots, n-1$, and so that $f_i(M, r_I)$ is affine for all $r_I \in C'$ and $l = 1, \ldots, d$.

Then, over $e^{-C'_{\text{diff}}}$, we have a decomposition of differential modules $M = \bigoplus_{a_I \in \mathbb{Q}^n} M_{a_I}$ such that

$$f_1(M_{a_I}, r_I) = \cdots = f_{\dim M_{a_I}}(M_{a_I}, r_I) = -\log \omega - s - a_1 r_1 - \cdots - a_n r_n.$$ 

We define

$$B = \{(a_1, \ldots, a_n, \vartheta)| a_I \in \mathbb{Q}^n, t_1 a_1 \cdots t_n a_n \vartheta \in \mathcal{I} \Theta(M \otimes F_{\eta_I})\},$$

for any $\eta_I \in C'_{\text{diff}}$; this does not depend on the choice of $\eta_I$ by Theorem 4.3.6(c).

Choose integers $e_1, \ldots, e_n \in \mathbb{N}$ coprime to $p$ such that $e_i a_i \in \mathbb{Z}$ for all $i$ and all $(a_1, \ldots, a_n, \vartheta) \in B$. Denote $R' = K[[C']_{0}[t_1^{1/e_1}, \ldots, t_n^{1/e_n}]]$.

**Theorem 4.3.8.** The two multisets $A$ and $B$ are the same (for any $C'$ that satisfies the condition in (b)). Moreover, there exists a finite tamely ramified extension $K'/K$ and a unique decomposition $M \otimes R' \otimes K' = \bigoplus_{(u, \vartheta) \in B} M_{(u, \vartheta)}$ such that, if we write $F'_{e^{-r_I}} = F_{e^{-r_I}}[t_1^{1/e_1}, \ldots, t_n^{1/e_n}] \otimes K'$,

(i) for all $r_I \in C'_{\text{diff}}, M_{(u, \vartheta)} \otimes F'_{e^{-r_I}}$ has pure intrinsic radii $\omega e^{a_1 r_1 + \cdots + a_n r_n + s}$ and pure refined intrinsic radii $t_1^{u_1} \vartheta$, and

(ii) any element in $\mathcal{I} \Theta(M \otimes F'_1)$ is congruent to $t_1^{u_1} \vartheta$ modulo elements with $v_s$-valuation strictly bigger than $(a_1, \ldots, a_n)$.

We may obtain the decomposition before tensoring with $K'$ if we group the Galois conjugates of $\vartheta$.

**Proof.** We first prove that we have a decomposition that satisfies condition (i). For this, we may replace $K$ by a finite tamely ramified extension so that all $\vartheta$ appears in $B$ lies in $\bigoplus_{i \in I} K^{(s)}_{K} \frac{dt_i}{t_i} \oplus \bigoplus_{j \in J} K^{(s)}_{K} \frac{du_j}{u_j}$ for appropriate $s$. Then the decomposition is expected to be defined for $M \otimes R'$. The proof goes exactly the same as in [KX10, Theorem 3.4.4] by invoking Theorems 4.1.9 and 4.2.6 at appropriate places.

Now, we check condition (ii) which is equivalent to identify the sets $A$ and $B$ for each $M_{a_I, \vartheta}$. Note that we have already known that $M_{a_I, \vartheta} \otimes F'_{e^{-r_I}}$ has pure intrinsic radii $\omega e^{a_1 r_1 + \cdots + a_n r_n + s}$. For notational convenience, we write $M$ for $M_{a_I, \vartheta}$. We do the induction on the dimension $n$. When $n = 0$ there is nothing to prove. We assume that the theorem is proved for $n - 1$. Let $D$ denote the face $t_1 = 0$ of $C$. Let $\overline{C} = C \cap D, \overline{C}' = C' \cap D, \overline{S} = \overline{C} - \overline{C}'$, and $\overline{R} = \overline{K}[[\overline{S}]_{0}]$ with coordinates $t_2, \ldots, t_n$, where $\overline{K}$ is the completion of $\text{Frac}(K[[t_1]])$ with respect to the $1$-Gauss norm.

By applying induction hypothesis to $\overline{M} = M \otimes \overline{R}$, the set $A$ equals to

$$A' = \{(v_s(\vartheta'), a_2, \ldots, a_n, t_1^{-v_s(\vartheta')} \vartheta')|(a_2, \ldots, a_n) \in \mathbb{Q}^{n-1}, t_2 a_2 \cdots t_n a_n \vartheta' \in \mathcal{I} \Theta(M \otimes F_{\eta_I})\},$$

for any $(r_2, \ldots, r_n) \in \overline{C}'$, where $v_s$ is the valuation on $\bigoplus_{i \in I} K^{(s)}_{K} \frac{dt_i}{t_i} \oplus \bigoplus_{j \in J} K^{(s)}_{K} \frac{du_j}{u_j}$ as in Notation 4.2.3 and $t_1^{-v_s(\vartheta')} \vartheta'$ is the reduction of $t_1^{-v_s(\vartheta')} \vartheta'$ in $\bigoplus_{i \in I} K^{(s)}_{K} \frac{dt_i}{t_i} \oplus \bigoplus_{j \in J} K^{(s)}_{K} \frac{du_j}{u_j}$.

It suffices to identify the multisets $A'$ with $B$. When $r_I \in \mathbb{Q}^n \cap \overline{C}'$, this follows from applying Corollary 4.2.7 the line that is parallel to $t_1$-axis and passes $r_I$. In particular, this says that for any $\vartheta'$ above, $t_1^{-v_s(\vartheta')} \vartheta'$ is the same as $\vartheta$. When $r_I$ is not rational, the same statement follows from the “continuity” result in Theorem 4.3.6(c).
Remark 4.3.9. One can also describe the intrinsic radii of \( M_{\alpha, \beta} \) at the point \((r_1) \in C' \) with \( r_1 = \cdots = r_l = 0 \) for some \( l \in \{1, \ldots, d - 1\} \). However, this would make our notation even more complicated. We leave this as an exercise for interested readers.

Next, we consider the situation for solvable differential modules.

Definition 4.3.10. Let \( C = \{(x_I) \subset \mathbb{R}^n | x_I \geq 0, x_1 + \cdots + x_n = 1\} \). For \([\alpha, \beta] \in (0, 1)\), we denote \( S_{[\alpha, \beta]} = \{\rho C | \rho \in [\alpha, \beta]\} \) and \( R_{[\alpha, \beta]} = K[S_{[\alpha, \beta]}]_0 \). For \( \alpha \in (0, 1) \), we define \( R_{\alpha} = \bigcap_{\beta \in (\alpha, 1)} R_{[\alpha, \beta]} \).

Fix \( \alpha \in (0, 1) \). Let \( M \) be a differential module over \( R_{\alpha} \). Assume that \( M \) is solvable, that is, for each \((x_I) \in C\), we have \( f_1(M, \rho^2) \to 0 \) as \( \rho \to 1^{-} \).

By Theorem 1.6.2 for \((x_I) \in C\), there exists \( b_1(M, x_I), \ldots, b_d(M, x_I) \) such that \( f_l(M, -x_I \log \rho) = \rho^{b_l(M, x_I)} \) when \( \rho \to 1^{-} \), for \( l = 1, \ldots, d \). Write \( B_l(M, x_I) = b_1(M, x_I) + \cdots + b_l(M, x_I) \) for \( l = 1, \ldots, d \).

Proposition 4.3.11. Keep the notation as above. Then the functions \( d!B_l(M, x_I) \) and \( B_d(M, x_I) \) are integral polyhedral functions.

Proof. For (a), see [Ked11, Theorem 3.3.3]. It also follows from Theorem 4.3.6(a).

Construction 4.3.12. Keep the notation as above.

Let \( x = (0, \ldots, 1) \in C \). Let \( \tilde{\mathfrak{f}} \) be the completion of the fraction field of \( \mathcal{O}_K((t_1)) \cdots ((t_{n-1})) \); it is a higher dimensional local field. We have a natural embedding \( R_{\alpha} \hookrightarrow \tilde{\mathfrak{f}} \{\eta/t_n, t_n\} = \tilde{\mathfrak{f}}_{\eta} \), if \( \eta \in (\alpha, 1) \). This is exactly saying that we restrict ourself to the line \((0, \ldots, 0, \rho) \) for \( \rho \in (\eta, 1) \). We assume that \( M \otimes \tilde{\mathfrak{f}}_{\eta} \) has pure-log break \( b \).

Recall that, as in Situation 4.3.7 we have a valuation \( v : \oplus_{i \in I} \kappa_{\tilde{\mathfrak{f}}_{\eta}} \otimes_{\tilde{\mathfrak{f}}_{\eta}} \mathbb{Q}^n \to \mathbb{Q}^n \).

Proposition 4.3.13. Keep the notation as above. The following two multisets of \( n-1 \) tuples are the same.

(a) The valuations \( v \) of elements in \( \frac{1}{\pi} I \Theta(M \otimes \tilde{\mathfrak{f}}_{\eta}) \), where \( \pi \) is a Dwork pi.

(b) The slopes of \( b_l(M, x_I) \) for \( l = 1, \ldots, d \) on a RP subset of \( C \) that is adjacent to the cells \( \{t_1 = \cdots = t_i = 0, t_{i+1} + \cdots + t_n = 1\} \) for all \( i = 1, \ldots, n \).

Proof. It follows from Theorem 4.3.8.

Remark 4.3.14. One may interpret the above proposition geometrically, as in [Ked11]. We leave that as an exercise for the readers to complete and we will come back to this in a future work.

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