ERGODIC ESTIMATORS OF DOUBLE EXPONENTIAL ORNSTEIN-UHLENBECK PROCESSES

YAOZHONG HU AND NEHA SHARMA

ABSTRACT. The goal of this paper is to construct ergodic estimators for the parameters in the double exponential Ornstein-Uhlenbeck process, observed at discrete time instants with time step size $h$. The existence and uniqueness, the strong consistency and the asymptotic normality of the estimators are obtained for arbitrarily fixed time step size $h$. A simulation method of the double exponential Ornstein-Uhlenbeck process is proposed and some numerical simulations are performed to demonstrate the effectiveness of the proposed estimators.

1. INTRODUCTION

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a right continuous family of increasing $\sigma$-algebras $(\mathcal{F}_t, t \geq 0)$ satisfying the usual condition ([12]). The expectation on this probability space is denoted by $E$. Motivated by the recent successful applications to finance (e.g. [3, 4] and references therein), we shall study in this work the parametric estimation problem for the double exponential Ornstein-Uhlenbeck process. To introduce this process let $(Y_n, n \geq 1)$ be a sequence of independent real valued random variables with the following probability density function

$$f_Y(x) = p\eta e^{-\eta x} I_{[x \geq 0]} + q\varphi e^{\varphi x} I_{[x < 0]}, \quad (1.1)$$

where the parameters $p, q, \eta, \varphi$ are positive and $p + q = 1$. Let $N_t$ be the Poisson process with rate $\lambda > 0$, independent of $\{Y_i, i = 1, 2, \ldots\}$. Then $Z_t = \sum_{i=1}^{N_t} Y_i$ is called the double exponential compound Poisson process, which is a particular Lévy process. The stochastic calculus with respect to this process falls in the framework of the stochastic calculus for general Lévy processes. For more details we refer to [14] whose results will be used freely.

Let us consider the following double exponential Ornstein-Uhlenbeck process given by the following Langevin equation driven by the double exponential compound Poisson process $Z_t$.

$$dX_t = -\theta X_t dt + \sigma dZ_t, \quad t \in [0, \infty), \quad X_0 = x_0. \quad (1.2)$$

Of course, this equation is interpreted as its integral form:

$$X_t = x_0 - \theta \int_0^t X_s ds + \sigma Z_t. \quad (1.3)$$

Key words and phrases. Double exponential compound Poisson process, double exponential Ornstein-Uhlenbeck process, discrete time observation, ergodic theorem, ergodic estimators, strong consistency, central limit theorem, exact simulation.

This work is supported by an NSERC discover fund and a start-up fund of University of Alberta.
This process $X_t$ depends on the parameters $\theta$, $\sigma$, $p$ (or $q$), $\eta$, $\lambda$, and $\varphi$. We assume that the process $\{X_t; t \geq 0\}$ can be observed at discrete time instants $t_j = jh$, where $h > 0$ is some observation time interval. We want to use the observation data set $\{X_{t_j}; j = 1, 2, \ldots, n\}$ to estimate the parameters $\theta$, $\sigma$, $p$, $\eta$, $\lambda$, and $\varphi$. To construct such estimators, we shall use the ergodic theorem

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{\infty} f(X_{t_j}) = \int_{\mathbb{R}} f(x) \mu(dx),$$

where $\mu$ is the limiting distribution of $x_t$. One may think that with appropriate choices of different $f$ we may be able to find all the parameters. However, the limiting distribution depends on the parameters in such a way (e.g. (2.7)) that one cannot decouple them. For this reason and motivated by [5], we get involved the ergodic theorem of the form

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{\infty} g(X_{t_j}, X_{t_j+1}) = \int_{\mathbb{R}^2} g(x, y) \nu(dx, dy),$$

where $\nu(dx, dy)$ is the limiting distribution of $(X_t, X_{t+h})$. After finding the distribution of $\mu$ and $\nu$ we use the moment functions to obtain appropriate equations so that the ergodic estimators satisfy.

An immediate problem after the obtention of the equations to identify the parameters is the well-posedness of such system: the existence, local uniqueness and global uniqueness of the system. We shall also address this elementary and challenging problem assuming $\sigma = \lambda = 1$. We shall prove that when the sample size is sufficiently large we shall have the existence and uniqueness of a local solution. With a further treatment, we reduce the problem of global uniqueness of the system to a problem of finding zero for a real valued function of one variable, where the mean value theorem can be used. The strong consistency and asymptotic normality of our ergodic estimators are also given.

To validate our approach we propose an exact decomposition simulation algorithm for our double exponential Ornstein-Uhlenbeck process. This algorithm allows us to write the distribution of $X_{t+h}$ given $X_t$ as a sum of deterministic function and a mixed compound Poisson process. After discussing the algorithm we simulate the data from (1.1) assuming some given values of $\theta$, $p$, $\eta$, and $\varphi$. Then we apply the estimators to estimate these parameters. The numerical results show that our estimators converge fast to the true parameters.

The paper is organized as follows. In Section 2, we give some preliminaries and some basic results for our double exponential Ornstein-Uhlenbeck process. We also obtain the explicit form of the characteristic functions of limiting distributions $\mu$ and $\nu$ mentioned earlier. In Section 3, we construct the ergodic estimators for all the parameters in the double exponential Ornstein-Uhlenbeck process. The local existence, uniqueness and the global uniqueness of the system of equations determining these ergodic estimators are discussed. In Section 4, the joint asymptotic normality of the the estimators is obtained. In Section 5, we discuss the exact decomposition algorithm for simulating the process. In Section 6 we perform some numerical simulations to validate our results which demonstrate the effectiveness of our estimators. Section 7 contains the computation of a covariance matrix appeared in our theorem.

2. Preliminaries

The equation (1.2) has a unique solution given by

$$X_t = e^{-\theta t} x_0 + \sigma \int_0^t e^{-\theta (t-s)} dZ_s.$$  (2.1)
If $\theta > 0$, then the double exponential Ornstein-Uhlenbeck process $X_t$ converges in law to the random variable $X_\infty = \sigma \int_0^\infty e^{-\theta s} dZ_s$. If the initial condition $X_0$ has the law of $X_\infty$, namely, if the process starts at the stationary distribution and if $X_0$ is independent of the process $Z_t$, then $X_t$ is a stationary process. It is well-known from [14, Theorem 17.5]) that the double exponential process $\{X_t, t \geq 0\}$ is ergodic.

Proposition 2.1. Let $f : \mathbb{R} \to \mathbb{R}$ be measurable such that $\mathbb{E}|f(X_\infty)| < \infty$. Then for any initial condition $x_0 \in \mathbb{R}$ and for any $h \in \mathbb{R}_+$, we have (denoting $t_j = jh$)

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} f(X_{t_j}) = \mathbb{E}(f(X_\infty)) \quad \text{a.s.} \quad (2.2)$$

The explicit form of the distribution of $X_\infty$ is hard to obtain. So, it is hard to compute $\mathbb{E}(f(X_\infty))$ for general $f$. But when $f$ has some particular form, namely, when $f(x) = e^{i\xi x}$, then the computation of $\mathbb{E}(f(X_\infty))$ is much simplified.

2.0.1. Evaluation of the limiting characteristic functions.

Proposition 2.2. Let $\{Z_t\}$ be the double exponential compound Poisson process and let $0 < s < t < \infty$. Then for any real valued continuous function $g(u)$ on $[s, t]$ we have

$$\mathbb{E}\left[ \exp \left( iz \int_s^t g(u) dZ_u(\omega) \right) \right] = \exp \left[ \int_s^t \Psi(g(u)z) du \right], \quad \forall \ z \in \mathbb{R}, \quad (2.3)$$

where

$$\Psi(z) = \log \hat{P}_{Z_1}(u) = \log \mathbb{E}\left[ e^{iuZ_1} \right] = \lambda \int_{\mathbb{R}} e^{iuy} f_Y(y) dy - \lambda \quad (2.4)$$

with $f_Y$ being given by (1.1).

Proof. We follow the idea of [14, Section 17]. Let us first compute the characteristic function of $Z_t$.

$$\hat{P}_{Z_t}(u) := \mathbb{E}\left[ e^{iuZ_t} \right] = \mathbb{E}\left[ e^{iu \sum_{j=1}^{N_t} Y_j} \right]$$

$$= \sum_{n=0}^{\infty} \mathbb{E}\left[ e^{iu \sum_{j=1}^{N_t} Y_j} | N_t = n \right] P(N_t = n)$$

$$= \sum_{n=0}^{\infty} \frac{(\lambda t)^n}{n!} e^{-\lambda} \left( \mathbb{E}(e^{iuY_1}) \right)^n$$

$$= e^{[\lambda \mathbb{E}(e^{iuY_1}) - \lambda]} = \exp \left[ \lambda t \int_{\mathbb{R}} e^{iuy} f_Y(y) dy - \lambda \right],$$

where $f_Y(y)$ is the double exponential density defined by (1.1). When $t = 1$ we have (2.4).
Now we are going to compute the characteristic function of the limiting distribution of \( X_o \). From Equation (2.1) and Proposition 2.2 it follows

\[
\mathbb{E}[e^{iuX_t}] = \exp \left[ ie^{-\theta t}x_0 u + \int_0^t \Psi(\sigma e^{-\theta s} u) ds \right]
\]

\[
= \exp \left[ ie^{-\theta t}x_0 u + \lambda \int_0^t e^{i\sigma e^{-\theta s} u} f_Y(y) dy - 1 \right] ds
\]

\[
= \exp \left[ ie^{-\theta t}x_0 u + \lambda I_{1,t} \right],
\]

where \( I_{1,t} = \int_0^t [I_{2,s} - 1] ds \) and \( I_{2,s} \) is defined and computed as follows.

\[
I_{2,s} = \int_{\mathbb{R}} e^{i\sigma e^{-\theta s} u} f_Y(y) dy
\]

\[
= \int_{\mathbb{R}} e^{i\sigma e^{-\theta s} u} \left[ p\eta e^{-\eta y} I_{[y \geq 0]} + q\varphi e^{\varphi y} I_{[y < 0]} \right] dy
\]

\[
= \frac{p\eta}{\eta - i\sigma u} e^{-\theta s} + \frac{q\varphi}{\varphi + i\sigma u} e^{-\theta s},
\]

where in the above second identity we used the explicit form of \( f_Y \) given by (1.1).

Thus

\[
I_{1,t} = \int_0^t [I_{2,s} - 1] ds
\]

\[
= \int_0^t \left( \frac{p\eta}{\eta - i\sigma u} e^{-\theta s} + \frac{q\varphi}{\varphi + i\sigma u} e^{-\theta s} - 1 \right) ds
\]

\[
= \frac{p}{\theta} \ln \left( \frac{\eta - i\sigma e^{-\theta t} u}{\eta - i\sigma u} \right) + \frac{q}{\theta} \ln \left( \frac{\varphi + ie^{-\theta t} \sigma u}{\varphi + i\sigma u} \right) - t
\]

\[
= \ln \left[ \left( \frac{\eta - i\sigma e^{-\theta t} u}{\eta - i\sigma u} \right)^{\frac{p}{\theta}} \cdot \left( \frac{\varphi + ie^{-\theta t} \sigma u}{\varphi + i\sigma u} \right)^{\frac{q}{\theta}} \right].
\]

This combined with (2.5) yields

\[
\lim_{t \to \infty} \mathbb{E}[e^{iuX_t}] = \left( \frac{\eta}{\eta - i\sigma u} \right)^{\frac{p}{\theta}} \left( \frac{\varphi}{\varphi + i\sigma u} \right)^{\frac{q}{\theta}}.
\]

In other words, we have

\[
\mathbb{E} \left[ e^{iuX_0} \right] = \lim_{t \to \infty} \mathbb{E}[e^{iuX_t}] = \left( \frac{1}{1 - iu \frac{\eta}{\theta}} \right)^{\frac{p}{\theta}} \left( \frac{1}{1 + iu \frac{\varphi}{\theta}} \right)^{\frac{q}{\theta}}.
\]

The probability distribution function of \( X_o \) is uniquely determined by the above characteristic function (2.8). This formula also means that the invariant random
variable \( X_0 \) depends on \( \frac{\sigma}{\eta}, \frac{\varphi}{\eta}, \frac{\lambda}{\eta} \) and then we cannot separate the parameters \( \theta, \sigma, \eta, \varphi, \lambda \) and \( p \).

Motivated by the works of [5, 6] we use the multi-time ergodic theorem to find more parameters. Our theoretical basis is the following general ergodic result, which is a consequence of [2, Theorem 1.1].

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} g(X_{t_j}, X_{t_j+h}) = \mathbb{E} \left[ g(X_0, X_h) \right] \tag{2.9}
\]

where \( X_t \) satisfies the Langevin equation (1.2) with the initial condition \( X_0 = X_0 \), namely, \( dX_t = -\theta X_t dt + \sigma dZ_t \) and \( X_0 \) has the invariant measure given by (2.8). The right hand side of (2.9) is hard to compute for general \( X \), where \( \eta \) and \( \lambda \), namely,

\[
dX_t = -\theta X_t dt + \sigma dZ_t \quad \text{and} \quad X_0 \text{ has the invariant measure given by (2.8)}.
\]

The right hand side of (2.9) is hard to compute for general \( g \). So we shall compute

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \exp \left[ iuX_{t_j} + ivX_{t_j+h} \right] = \mathbb{E} \left[ \exp \left( iuX_0 + ivX_h \right) \right] \tag{2.10}
\]

for arbitrary \( u, v \in \mathbb{R} \). In fact, we shall evaluate the above quantity by evaluating \( \lim_{t \to \infty} \mathbb{E}[e^{i(uX_t + vX_{t+h})}] \). We shall still use the formula (2.3) to do our computations. As we see we can assume \( X_0 = 0 \). Thus,

\[
X_t(\omega) = \sigma \int_0^t e^{-\theta(t-s)} dZ_s(\omega) ; \quad X_{t+h}(\omega) = \sigma \int_0^t e^{-\theta(t-h+s)} dZ_s(\omega).
\]

Therefore,

\[
uX_t(\omega) + vX_{t+h}(\omega) = \sigma \int_0^t \left( u e^{-\theta(t-s)} + v e^{-\theta(t+h-s)} \right) dZ_s + \sigma \int_t^{t+h} v e^{-\theta(t+h-s)} dZ_s . \tag{2.11}
\]

Because of the independent increment property of the double exponential compound Poisson process \( Z_t \), we have

\[
\mathbb{E} \left[ \exp \left( iuX_t + ivX_{t+h} \right) \right] = \mathbb{E} \left[ \exp \left( i \int_0^t \sigma e^{-\theta(t-s)}(u + v e^{-\theta h}) dZ_s \right) \right] = \mathbb{E} \left[ \int_t^{t+h} \sigma v e^{-\theta(t+h-s)} dZ_s \right] =: I_{3,t} \cdot I_{4,t} , \tag{2.12}
\]

where \( I_{3,t} \) and \( I_{4,t} \) denote the above first and second expectations. Similar to (2.6), we have

\[
I_{3,t} = \exp \left[ \frac{p \lambda}{\theta} \ln \left( \frac{\eta - i\sigma(e^{-\theta t} u + v e^{-\theta h})}{\eta - i\sigma(u + v e^{-\theta h})} \right) \right.
\]
\[
+ \frac{q \lambda}{\theta} \ln \left( \frac{\varphi + i\sigma(e^{-\theta t} u + v e^{-\theta h})}{\varphi + i\sigma(u + v e^{-\theta h})} \right) - \ln(e^{\lambda t}) \right] \tag{2.13}
\]

and

\[
I_{4,t} = \left( \frac{\eta - i\sigma \varphi}{\eta - i\sigma e^{-\theta h} \varphi} \right)^{\frac{p \lambda}{\theta}} \left( \frac{\varphi + ie^{-\theta h} \sigma \varphi}{\varphi + i\sigma \varphi} \right)^{\frac{q \lambda}{\theta}} . \tag{2.14}
\]
It may be a bit strange to see that $I_{4,t}$ is independent of $t$. But this is because of the independent increment property of the process $Z_t$. In fact, we see easily that

$$
\int_t^{t+h} \sigma v e^{-\theta(t+s)} dZ_s
$$

has the same law as that of

$$
\int_0^h \sigma v e^{-\theta(h-s)} dZ_s.
$$

It is easy to verify

$$
\lim_{t \to \infty} I_{3,t} = \left( \frac{\eta}{\eta - i \sigma (u + ve^{-\theta h})} \right) \frac{\lambda^{ \frac{\varphi}{\lambda} } }{ \lambda^{ \frac{-\varphi}{\lambda} } }.
$$

(2.15)

Hence, we have

$$
E \left[ \exp \left( iu X_0 + iv X_h \right) \right] = \lim_{t \to \infty} E \left[ \exp \left( iu X_t + iv X_{t+h} \right) \right]
$$

$$
= \left( \frac{\eta}{\eta - i \sigma (u + ve^{-\theta h})} \right) \frac{\lambda^{ \frac{\varphi}{\lambda} } }{ \lambda^{ \frac{-\varphi}{\lambda} } } \cdot \left( \frac{\eta - i \sigma e^{-\theta h} v}{\eta - i \sigma v} \right) \frac{\lambda^{ \frac{-\varphi}{\lambda} } }{ \lambda^{ \frac{\varphi}{\lambda} } }.
$$

(2.16)

We summarize (2.2), (2.8), (2.10), (2.16) as the following theorem.

**Theorem 2.3.** Let $X_t$ be the double exponential Ornstein-Uhlenbeck process with initial condition $x_0 \in \mathbb{R}$. Then for any $h \in \mathbb{R}^+, u, v \in \mathbb{R}$, we have almost surely

$$
\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} e^{iuX_{t_j}} = \left( \frac{\eta}{\eta - i u \sigma} \right)^{\frac{\lambda}{\varphi}} \left( \frac{\eta + i u \varphi}{\varphi} \right)^{\frac{\varphi}{\lambda}}
$$

$$
\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \exp \left[ iu X_{t_j} + iv X_{t_j+h} \right]
$$

$$
= \left( \frac{\eta}{\eta - i \sigma (u + ve^{-\theta h})} \right) \frac{\lambda^{ \frac{\varphi}{\lambda} } }{ \lambda^{ \frac{-\varphi}{\lambda} } } \cdot \left( \frac{\eta - i \sigma e^{-\theta h} v}{\eta - i \sigma v} \right) \frac{\lambda^{ \frac{-\varphi}{\lambda} } }{ \lambda^{ \frac{\varphi}{\lambda} } }.
$$

(2.17)

3. **Estimation of the parameters $\eta$, $\theta$, $\varphi$ and $p$**

Assume now that the double exponential Ornstein-Uhlenbeck process can be observed at discrete time so that the observation data $\{X_{t_j}, j = 1, \ldots, n\}$ are available to us, where $t_j = jh$ for some given observation time interval length $h$. Presumably Theorem 2.3 can be used to estimate all the parameters $\eta$, $\theta$, $\varphi$, $\lambda$, $\sigma$, and $p$ by replacing the limits in (2.17) by their the empirical characteristic functions $\hat{\Psi}_n(u)$ and $\Theta_n(u, v)$ defined as follows

$$
\hat{\Psi}_{1,n}(u, v) := \frac{1}{n} \sum_{j=1}^{n} \exp iuX_{t_j};
$$

$$
\hat{\Psi}_{2,n}(u, v) = \frac{1}{n} \sum_{j=1}^{n} \exp (iuX_{t_j} + ivX_{t_j+h}).
$$

(3.1)

For any given pair $(u, v)$ although $\hat{\Psi}_{1,n}(u, v)$ depends only on $u$ we write it as a function of $u, v$ for convenience. Since we have 6 parameters, it may be possible for
us to choose appropriately 6 pairs of \((u_k, v_k)\) such that the 6 parameters can be determined by

\[
\begin{align*}
\left( \frac{\eta}{\eta - i u_k \sigma} \right)^{\frac{q_k}{\sigma}} & = \frac{\eta}{\eta + i u_k \varphi}^{\frac{q_k}{\varphi}} = \hat{\Psi}_{1,n}(u_k, v_k), & k = 1, \ldots, m, \\
\left( \frac{\eta}{\eta - i \sigma(u_k + v_k e^{-\theta h})} \right)^{\frac{q_k}{\sigma}} & = \left( \varphi + i \sigma(u_k + v_k e^{-\theta h}) \right)^{\frac{q_k}{\varphi}} = \hat{\Psi}_{2,n}(u_k, v_k), \\
\left( \frac{\eta}{\eta - i \sigma v_k} \right)^{\frac{q_k}{\varphi}} & = \left( \varphi + i \sigma v_k \right)^{\frac{q_k}{\varphi}} = \hat{\Psi}_{3,n}(u_k, v_k), \\
\left( \eta - i \sigma v_k e^{-\theta h} \right)^{\frac{q_k}{\varphi}} & = \left( \varphi + i \sigma v_k e^{-\theta h} \right)^{\frac{q_k}{\varphi}} = \hat{\Psi}_{4,n}(u_k, v_k), \\
& \quad k = m + 1, \ldots, 6,
\end{align*}
\]

where \(m\) is some integer between 1 and 6. For any given pair \((u, v)\), the empirical characteristic functions \(\hat{\Psi}_{1,n}(u, v)\) and \(\hat{\Psi}_{1,n}(u, v)\) are known since we have the available observation data. Thus (3.2) is a system of function equations on the parameters \(\eta, \varphi, \lambda, \sigma, \) and \(p\). We believe that with appropriate choice of \((u_k, v_k)\) we should be able to use (3.2) to estimate all the above six parameters. However, it is still difficult for us to argue if this system of equations have a global unique solution or not although this system of nonlinear function equations (3.2) is explicit and appears to be quite simple as well. Since we want to deal with the global uniqueness of the system (3.2), we shall assume \(\lambda = \sigma = 1\) so that we now have only four parameters: \(\eta, \varphi, \) and \(p\). Supposedly we should be able to choose four different values of \((u_k, v_k)\) so that we obtain a system of four equations for the four unknowns. However, it is still difficult to argue the global uniqueness for the obtained system. So we are proposing an alternative method. Since (2.17) holds true for all \((u, v) \in \mathbb{R}\) we can obtain explicit formulas for the moments and then we use the moments to identity the parameters. Since \(\mathbb{E}[X_0]^m < \infty\) and \(\mathbb{E}[X_0 X_h]^m < \infty\) for all \(m\) we know that (2.2) and (2.9) hold true for moment functions, in particular, we shall choose \(f = x, x^2, x^3, g(x, y) = xy\). Thus the system of four equations we choose to obtain the estimators for \(\eta, \varphi, \) and \(p\) are

\[
\begin{align*}
\mathbb{E}[X_0] & = \mu_{1,n}, \quad \text{where } \mu_{1,n} := \frac{1}{n} \sum_{j=1}^{n} X_{t_j}, \\
\mathbb{E}[X_0^2] & = \mu_{2,n}, \quad \text{where } \mu_{2,n} := \frac{1}{n} \sum_{j=1}^{n} X_{t_j}^2, \\
\mathbb{E}[X_0^3] & = \mu_{3,n}, \quad \text{where } \mu_{3,n} := \frac{1}{n} \sum_{j=1}^{n} X_{t_j}^3, \\
\mathbb{E}[X_0 X_{t+h}] & = \mu_{4,n}, \quad \text{where } \mu_{4,n} := \frac{1}{n} \sum_{j=1}^{n} X_{t_j} X_{t_j+h}.
\end{align*}
\]

The right hand sides of (3.3) (namely, \(\mu_{i,n}, i = 1, 2, 3, 4\)) are known from the discrete time observations of the double exponential Ornstein-Uhlenbeck process \(X_t\). The left hand sides of (3.3) are functions of the parameters \(\eta, \varphi, \) and \(p\). We need first to find out how they depend on the four parameters explicitly and then solve this system to construct the ergodic estimators \(\hat{\eta}_{n}, \hat{\varphi}_{n}, \hat{p}_{n}\) for the parameters. Let us also emphasize that (3.3) are not equations for the true parameters but they are equations for the ergodic estimators.
Now let us find the explicit forms for the left hand sides of (3.2). Let $\rho = \frac{\sigma}{\eta}$ and $\xi = \frac{\sigma}{\eta}$. From the identities (2.8) and (2.16), we see by the expression of moments through characteristic function (e.g. Corollary 1 to Theorem 2.3.1 in [8])

\[
\begin{align*}
E[X_0] &= \frac{1}{i} \frac{\partial}{\partial u} E[e^{i(uX_0)}] \bigg|_{u=0} \\
&= \frac{1}{i} \frac{\partial}{\partial u} \left( \frac{1}{i - iu\rho} \right)^{\frac{\lambda}{\theta}} \left( \frac{1}{1 + iu\xi} \right)^{\frac{q\lambda}{\theta}} \bigg|_{u=0} \\
&= \frac{\lambda}{\theta} [pp - q\xi]; \\
E[X^2_0] &= \frac{1}{i^2} \frac{\partial^2}{\partial u^2} E[e^{i(uX_0)}] \bigg|_{u=0} \\
&= \frac{\lambda}{\theta} [pp^2 + q\xi^2] + \frac{\lambda}{\theta} [pp + q\xi] [pp - q\xi] \\
&= \frac{\lambda}{\theta} [pp^2 + q\xi^2] + E[X_0]^2; \\
E[X^3_0] &= \frac{1}{i^3} \frac{\partial^3}{\partial u^3} E[e^{i(uX_0)}] \bigg|_{u=0} \\
&= \frac{2\lambda}{\theta} [pp^3 - q\xi^3] + \left( \frac{\lambda}{\theta} [pp^2 + q\xi^2] + \frac{\lambda}{\theta} [pp - q\xi] \right) \left( \frac{\lambda}{\theta} [pp - q\xi] \right) - 2 \lambda [pp^3 - q\xi^3] + E[X_0]E[X_0] + 2E[X_0](E[X^3_0] - E[X_0]^2); \\
E[X_0X_h] &= \frac{1}{i^2} \frac{\partial}{\partial u} \frac{\partial}{\partial v} E \left[ \exp \left( iuX_0 + ivX_h \right) \right] \bigg|_{u=0,v=0} \\
&= \frac{1}{i^2} \frac{\partial}{\partial v} \frac{\partial}{\partial u} \left( \frac{1}{1 - iu(\rho + \xi v - \theta h)} \right)^{\frac{\lambda}{\theta}} \left( \frac{1}{1 + iu(\xi u + v - \theta h)} \right)^{\frac{q\lambda}{\theta}} \\
&\cdot \left( \frac{1 - i\rho - \theta h v}{1 + i\rho v} \right)^{\frac{\lambda}{\theta}} \left( \frac{1 + i\xi - \theta h v}{1 + i\xi v} \right)^{\frac{q\lambda}{\theta}} \bigg|_{u=0,v=0} \\
&= e^{-\theta h} \frac{\lambda}{\theta} [pp^2 + q\xi^2] + \frac{\lambda^2}{\theta^2} [pp - q\xi] [pp - q\xi] \\
&= e^{-\theta h} \frac{\lambda}{\theta} [pp^2 + q\xi^2] + E[X_0]^2 \\
\end{align*}
\]

(3.4)

An elementary simplification yields (noticing $\lambda = 1$)

\[
\begin{align*}
\frac{1}{\theta} [pp - q\xi] &= \mu_{1,n}, \\
\frac{1}{\theta} [pp^2 + q\xi^2] &= \mu_{2,n} - \mu_{1,n}^2, \\
\frac{2}{\theta} [pp^3 - q\xi^3] &= \mu_{3,n} - \mu_{2,n} \mu_{1,n} - 2 \mu_{1,n} (\mu_{2,n} - \mu_{1,n}^2), \\
\frac{1}{\theta} e^{-\theta h} [pp^2 + q\xi^2] &= \mu_{4,n} - \mu_{1,n}^2.
\end{align*}
\]

(3.5)  (3.6)  (3.7)  (3.8)
Thus we have the explicit form (3.5)-(3.8) for (3.3). Now we want to solve this system of function equations (e.g. (3.5)-(3.8)). Dividing (3.6) by (3.8) gives
\[
\hat{\theta}_n = \frac{1}{h} \ln \left( \frac{\mu_{2,n} - \mu_{1,n}^2}{\mu_{4,n} - \mu_{1,n}^4} \right).
\]
(3.9)

Now we use the three equations (3.5)-(3.7) to solve for the remaining three unknowns \( p, \rho, \xi \) (noticing \( q = 1 - p \)). Denote
\[
\begin{cases}
  f_1 = \hat{\theta}_n \mu_{1,n}, \\
  f_2 = \hat{\theta}_n (\mu_{2,n} - \mu_{1,n}^2), \\
  f_3 = \frac{\hat{\theta}_n}{2} (\mu_{3,n} - \mu_{2,n} \mu_{1,n} - 2\mu_{1,n}(\mu_{2,n} - \mu_{1,n}^2)).
\end{cases}
\]
(3.10)

Thus we have
\[
\begin{cases}
  pp - (1 - p)\xi = f_1, \\
  pp^2 + (1 - p)\xi^2 = f_2, \\
  pp^3 - (1 - p)\xi^3 = f_3.
\end{cases}
\]
(3.11)

The first equation in (3.11) yields
\[
\xi = \frac{pp - f_1}{1 - p}.
\]
(3.12)

Substituting to the second equation in (3.11) we have
\[
pp^2 - 2f_1 pp + f_1^2 - f_2(1 - p) = 0.
\]

Solving for \( \rho \), we have
\[
\rho = \frac{f_1 p \pm \sqrt{p(1-p)(f_2 - f_1^2)}}{p}.
\]
(3.13)

Recalling \( \sigma = 1 \), \( \rho = \frac{1}{\eta} \) and \( \xi = \frac{1}{\varphi} \) we have
\[
f_2 - f_1^2 = pp^2 + q\xi^2 - (pp - q\xi)^2 = p(1-p)\rho^2 + q(1-q)\xi^2 + 2pq\rho\xi > 0
\]
so the discriminant defining \( \rho \) (ie (3.13)) is nonnegative. Moreover, since \( \xi = \frac{1}{\varphi} \), we see from (3.12) that \( \frac{pp - f_1}{1 - p} = \frac{1}{\varphi} \) which means
\[
\rho = \frac{f_1}{p} + \frac{1 - p}{\varphi} > f_1.
\]

Thus in (3.13), we should take the positive sign to obtain
\[
\rho = \frac{f_1 p + \sqrt{p(1-p)(f_2 - f_1^2)}}{p}.
\]
(3.14)

Now we substitute \( \xi \) given by (3.12) into the third equation in (3.11) to obtain
\[
pp^3 - (1-p)\left(\frac{pp - f_1}{1 - p}\right)^3 = f_3.
\]

This means
\[
f_3(1-p)^2 = p(1-p)^2 \rho^3 + (f_1 - pp)^3.
\]
Finally we substitute $\rho$ in the above equation by (3.14) to obtain one function equation for only one unknown $p$:

$$(1 - p)^2 \left( f_1 p + \sqrt{p(1-p)(f_2 - f_1^2)} \right)^3 + p^3 \left( f_1 - f_1 p - \sqrt{p(1-p)(f_2 - f_1^2)} \right)^3 - f_3 p^2 (1-p)^2 = 0.$$ (3.15)

This equation depends on $f_1, f_2, f_3$ computed from the observation data of the double exponential Ornstein-Ulenbeck process. Although it is still hard to know if this function equation has a unique global solution or not. However, since it contain only one equation for one unknown we can plot the graph of the function (we consider the left hand side of (3.15) as a function $g(p), 0 < p < 1)$ to see if $g(p)$ has a unique solution on the interval $0 < p < 1$ or not. Or we can plot the derivative $g'(p)$ on $0 < p < 1$ to see if $g'(p)$ remains the same sign or not. If $g'(p) > 0$ (or $g'(p) < 0$) on $0 < p < 1$, then $g(p) = 0$ has at most one solution on $(0, 1)$ by the mean value theorem.

We summarize the above discussions as the following theorem about the existence and uniqueness of the parameter estimators and their strong consistency results.

**Theorem 3.1.** From the observation data, we denote $\mu_{k,n}, k = 1, 2, 3, 4$ by (3.3). Then $\hat{\theta}_n$ is given by (3.9), namely

$$\hat{\theta}_n = \frac{1}{n} \ln \left( \frac{\mu_{2,n} - \mu_{1,n}^2}{\mu_{4,n} - \mu_{1,n}^2} \right)$$ (3.16)

and $f_k, k = 1, 2, 3$ by (3.10). If (3.15) has a unique solution $\hat{p}_n$ on $(0, 1)$, namely,

$$(1 - \hat{p}_n)^2 \left( f_1 \hat{p}_n + \sqrt{\hat{p}_n(1-\hat{p}_n)(f_2 - f_1^2)} \right)^3 + \hat{p}_n^3 \left( f_1 - f_1 \hat{p}_n - \sqrt{\hat{p}_n(1-\hat{p}_n)(f_2 - f_1^2)} \right)^3 - f_3 \hat{p}_n^2 (1-\hat{p}_n)^2 = 0$$ (3.17)

and if $\hat{p}_n$ is a continuous function of $f_1, f_2, f_3$, then (3.5)-(3.8) has a unique solution $(\hat{\theta}_n, \hat{\xi}_n, \hat{\rho}_n, \hat{\tilde{p}}_n)$ given by (3.16), (3.18) and

$$\begin{cases}
\hat{\rho}_n = \frac{f_1 \hat{p}_n + \sqrt{\hat{p}_n(1-\hat{p}_n)(f_2 - f_1^2)}}{\hat{p}_n}, \\
\hat{\xi}_n = \frac{\hat{p}_n \hat{\rho}_n - f_1}{1 - \hat{p}_n}.
\end{cases}$$ (3.18)

Define

$$\hat{\eta}_n := \frac{1}{\hat{\rho}_n}, \quad \hat{\varphi}_n := \frac{1}{\hat{\xi}_n}.$$ (3.19)

If $(\theta, \eta, \varphi, p)$ are the true parameters, namely, if the double exponential process $X_t$ satisfies (1.2) with the above parameters and with $\alpha = \sigma = 1$, and if (3.15) has a unique solution when $f_1, f_2, f_3$ are replaced by their limits as $n \to \infty$, then when $n \to \infty, (\hat{\theta}_n, \hat{\eta}_n, \hat{\varphi}_n, \hat{\rho}_n, \hat{\tilde{p}}_n) \to (\theta, \eta, \varphi, p)$ almost surely.

**Proof.** For any fixed $n$, it is clear that $f_1, f_2, f_3$ are continuous function of $\mu_{k,n}, k = 1, 2, 3, 4$. So, $\hat{\theta}_n, \hat{\xi}_n, \hat{\rho}_n, \hat{\tilde{p}}_n$ are continuous functions of $\mu_{k,n}, k = 1, 2, 3, 4$. Since $\mu_{k,n}, k = 1, 2, 3, 4$ have limits as $n \to \infty$, we then see $(\hat{\theta}_n, \hat{\xi}_n, \hat{\rho}_n, \hat{\tilde{p}}_n)$ have
limits \((\hat{\theta}, \hat{\xi}, \hat{\rho}, \hat{p})\). However, by the above argument, for each \(n\), \(\hat{\theta}_n, \hat{\xi}_n, \hat{\rho}_n, \hat{p}_n\) satisfy (3.5)-(3.8). Taking the limits of this system of equations we see \((\hat{\theta}, \hat{\xi}, \hat{\rho}, \hat{p})\) satisfies

\[
\begin{align*}
\frac{1}{\theta} \hat{p} \hat{\rho} - (1 - \hat{p}) \hat{\xi} &= \lim_{n \to \infty} \mu_{1,n}, \\
\frac{1}{\theta} \hat{p} \hat{\rho}^2 + (1 - \hat{p}) \hat{\xi}^2 &= \lim_{n \to \infty} \left[ \mu_{2,n} - \mu_{1,n}^2 \right], \\
\frac{2}{\theta} \hat{p} \hat{\rho}^3 - (1 - \hat{p}) \hat{\xi}^3 &= \lim_{n \to \infty} \left[ \mu_{3,n} - \mu_{2,n} \mu_{1,n} - 2\mu_{1,n}(\mu_{2,n} - \mu_{1,n}^2) \right], \\
\frac{1}{\theta} e^{-\theta h} \hat{p} \hat{\rho}^2 + (1 - \hat{p}) \hat{\xi}^2 &= \lim_{n \to \infty} \left[ \mu_{4,n} - \mu_{1,n}^2 \right].
\end{align*}
\]

(3.20)

Since (3.15) has a unique solution when \(f_1, f_2, f_n\) are replaced by their limits as \(n \to \infty\), by the same argument as above we can show (3.20) has a unique solution. Obviously, \((\theta, \xi, \rho, p)\) satisfy (3.20). Thus \((\hat{\theta}, \hat{\xi}, \hat{\rho}, \hat{p}) = (\theta, \xi, \rho, p)\). This means that when \(n \to \infty\), \((\hat{\theta}_n, \hat{\xi}_n, \hat{\rho}_n, \hat{p}_n) \to (\theta, \xi, \rho, p)\) almost surely and hence we obtain that when \(n \to \infty\), \((\hat{\theta}_n, \hat{\xi}_n, \hat{\rho}_n, \hat{p}_n) \to (\theta, \xi, \rho, p)\) almost surely.

**Remark 3.2.** The estimators \((\hat{\theta}_n, \hat{\xi}_n, \hat{\rho}_n, \hat{p}_n)\) defined in the above theorem are called the ergodic estimators of the parameters \((\theta, \xi, \rho, p)\). The above theorem states that these ergodic estimators are uniquely determined and are strongly consistent.

### 4. Joint Asymptotic Behavior of All the Obtained Estimators

In this section, we shall prove the central limit theorem for our ergodic estimators \(\hat{\Theta}_n = (\hat{\theta}_n, \hat{\xi}_n, \hat{\rho}_n, \hat{p}_n)\). Our goal is to prove that \(\sqrt{n}(\hat{\Theta}_n - \Theta)\), where \(\Theta = (\theta, \eta, \varphi, p)\) converges in law to a mean zero normal vector and to find the asymptotic covariance matrix. Let

\[
\begin{align*}
g(x, y) &= (g_1(x, y), g_2(x, y), g_3(x, y), g_4(x, y))^T, \\
g_1(x, y) &= x, \\g_2(x, y) &= x^2, \\g_3(x, y) &= x^3, \\g_4(x, y) &= xy
\end{align*}
\]

and

\[
\mu = (\mu_1, \mu_2, \mu_3, \mu_4), \\
\mu_k = E[g_j(X_o, X_h)], \quad k = 1, 2, 3, 4.
\]

Denote

\[
\mu_n = (\mu_{1,n}, \mu_{2,n}, \mu_{3,n}, \mu_{4,n}),
\]

where \(\mu_{k,n}, k = 1, 2, 3, 4\) are defined by (3.3).

First, we have the following central limit result.

**Lemma 4.1.** Let \(\mu_n, \mu, g\) be defined as above. Then as \(n \to \infty\), we have

\[
\sqrt{n}(\mu_n - \mu) \overset{d}{\to} N(0, A),
\]

with the 4 × 4 covariance matrix \(A\) being given by

\[
A = (\sigma_{g, g})_{1 \leq i, j \leq 4},
\]

where \(\sigma_{g, g}, 1 \leq i, j \leq 4\) will be given in the appendix.

**Proof.** We shall use the Cramer-Wold device (e.g. [1, Theorem 29.4]). For any \(a = (a_1, a_2, a_3, a_4) \in \mathbb{R}^4\), consider \(a^T \mu_n = \sum_{k=1}^{4} a_k \mu_{k,n}\). By [11, Theorem 2.6] and [10], the double exponential Ornstein-Uhlenbeck process \(\{X_t\}\) is exponentially \(\beta\)-mixing. By the fact that the exponential \(\beta\)-mixing implies the exponential \(\alpha\)-mixing
We compute the partial derivatives of $\tilde{h}$ with respect to the parameters to obtain

\[
\begin{align*}
\frac{\partial h_1}{\partial p} &= \frac{1}{\theta} (p + \xi), \\
\frac{\partial h_2}{\partial p} &= \frac{1}{\theta} e^{-\theta h} (p^2 - \xi^2), \\
\frac{\partial h_3}{\partial p} &= \frac{1}{\theta} (3pp^2 - q^3), \\
\frac{\partial h_4}{\partial p} &= \frac{1}{\theta} e^{-\theta h}(2pp^2 + q^2). \\
\frac{\partial h_1}{\partial \theta} &= -\frac{1}{\theta} (p - q), \\
\frac{\partial h_2}{\partial \theta} &= -\frac{1}{\theta} (p^2 + q^2), \\
\frac{\partial h_3}{\partial \theta} &= -\frac{1}{\theta} (pp^3 - 3q^2), \\
\frac{\partial h_4}{\partial \theta} &= -\frac{1}{\theta} e^{-\theta h} (pp^2 + q^2), \\
\frac{\partial h_1}{\partial \mu_1} &= -2\mu_1, \\
\frac{\partial h_2}{\partial \mu_1} &= 1, \\
\frac{\partial h_3}{\partial \mu_1} &= -3\mu_1, \\
\frac{\partial h_4}{\partial \mu_1} &= 1, \\
\frac{\partial h_1}{\partial \mu_2} &= 1, \\
\frac{\partial h_2}{\partial \mu_2} &= 1, \\
\frac{\partial h_3}{\partial \mu_2} &= -3\mu_2, \\
\frac{\partial h_4}{\partial \mu_2} &= -3\mu_2, \\
\frac{\partial h_1}{\partial \mu_3} &= 1, \\
\frac{\partial h_2}{\partial \mu_3} &= 1, \\
\frac{\partial h_3}{\partial \mu_3} &= 1, \\
\frac{\partial h_4}{\partial \mu_3} &= 1, \\
\frac{\partial h_1}{\partial \mu_4} &= 1, \\
\frac{\partial h_2}{\partial \mu_4} &= 1, \\
\frac{\partial h_3}{\partial \mu_4} &= 1, \\
\frac{\partial h_4}{\partial \mu_4} &= 1.
\end{align*}
\]
Let us denote the matrix
\[
\nabla h(\Theta) = \begin{pmatrix}
\frac{\partial h_1}{\partial \theta} & \frac{\partial h_1}{\partial \eta} & \frac{\partial h_1}{\partial \phi} & \frac{\partial h_1}{\partial p} \\
\frac{\partial h_2}{\partial \theta} & \frac{\partial h_2}{\partial \eta} & \frac{\partial h_2}{\partial \phi} & \frac{\partial h_2}{\partial p} \\
\frac{\partial h_3}{\partial \theta} & \frac{\partial h_3}{\partial \eta} & \frac{\partial h_3}{\partial \phi} & \frac{\partial h_3}{\partial p} \\
\frac{\partial h_4}{\partial \theta} & \frac{\partial h_4}{\partial \eta} & \frac{\partial h_4}{\partial \phi} & \frac{\partial h_4}{\partial p}
\end{pmatrix}
\]

Then we have the following result.

**Theorem 4.2.** Denote \( \Theta = (\theta, \eta, \phi, p) \) and \( \hat{\Theta}_n = (\hat{\theta}_n, \hat{\eta}_n, \hat{\phi}_n, \hat{p}_n) \). If \( \hat{p}_n \) is a continuous function of \( f_1, f_2, f_3 \) and if (3.15) has a unique solution when \( f_1, f_2, f_3 \) are replaced by their limits as \( n \to \infty \), then as \( n \to \infty \) we have
\[
\sqrt{n}(\hat{\Theta}_n - \Theta) \overset{d}{\to} N(0, \Sigma)
\]
where
\[
\Sigma = (\nabla h)^{-1} \nabla h^T A (\nabla h)^{-1} \nabla h.
\]

**Proof.** It is easy to see that \( h, \hat{h} : \mathbb{R}^4 \to \mathbb{R}^4 \) defined as above are smooth mappings. Using these two mappings, we can write the system (3.5)-(3.8) to determine the ergodic estimators \( \Theta_n \)
\[
h(\Theta_n) = \hat{h}(\mu_n).
\]
From Theorem 3.1, it follows that \( h \) has inverse \( h^{-1} \) so that
\[
\Theta_n = (h^{-1} \circ \hat{h})(\mu_n).
\]
By Lemma 4.1 and the Delta method, we see that
\[
\sqrt{n}(\hat{\Theta}_n - \Theta) \overset{d}{\to} N(0, \Sigma)
\]
where
\[
\Sigma = (\nabla h)^{-1} \nabla h^T A (\nabla h)^{-1} \nabla h.
\]
This proves the theorem. ■

5. **Exact Simulation for the Double Exponential Ornstein-Uhlenbeck Process**

Before we give some numerical simulations to validate our ergodic estimators, in this section we propose a distributional decomposition to exactly simulate the double exponential Ornstein-Uhlenbeck process. We follow the idea of [9], where the exact simulation of Gamma Ornstein-Uhlenbeck process is studied. First, we have the following result. Without loss of generality we can assume \( \sigma = 1 \).

**Theorem 5.1.** Let \( X_t \) be the double exponential Ornstein-Uhlenbeck process given by (1.2). For any \( t, t_1 > 0 \), the Laplace transform of \( X_{t+t_1} \) conditioning on \( X_t \) is given by
\[
\mathbb{E}[e^{iuX_{t+t_1}} | X_t] = e^{-iuw X_t} \exp \left( -\frac{\lambda}{\theta} \int_0^\infty (1 - e^{-is}) \int_\frac{1}{w} \eta ve^{-s\eta} \frac{1}{v} dv ds \right. \\
- \frac{\lambda}{\theta} \int_{-\infty}^0 (1 - e^{-is}) \int_{\frac{1}{w}} \phi ve^{s\phi} \frac{1}{v} dv ds \bigg),
\]
where \( w = e^{-\theta t_1} \).
Proof. Recall the $\Psi$ defined by (2.4) and the formula (2.5). We can write the characteristic function of $X_{t+1}$ as

$$
E[e^{iuX_{t+1}}] = \exp \left[ \int_{t}^{t+1} \Psi(\sigma e^{-\theta(t+1-s)}u) ds \right]
$$

where

$$
\Psi(\sigma e^{-\theta(t+1-s)}u) = \int_{0}^{t} -\lambda \left( 1 - E(e^{i\sigma e^{-\theta(t+1-s)}uY_{1}}) \right) ds.
$$

(5.2)

Denote $\hat{h}(z) = E(e^zY_1)$. The Laplace transform of $X_{t+1}$ conditioning on $X_t$ is

$$
E[e^{iuX_{t+1}}|X_t] = e^{-iuwX_t} \exp \left[ \int_{t}^{t+1} -\lambda \left( 1 - \hat{h}(i\sigma e^{-\theta(t+1-s)}u) \right) ds \right]
$$

(5.3)

Let $ue^{-\theta t_1} = x$, then for $\sigma = 1$, we have

$$
\int_{0}^{t} \left( 1 - \hat{h}(ix) \right) ds = \frac{1}{\theta} \int_{u}^{ue^{-\theta s}} \left( 1 - \hat{h}(ix) \right) dx
$$

$$
= I_1 + I_2,
$$

where

$$
I_1 = \frac{1}{\theta} \int_{uw}^{u} \frac{1}{x} \int_{x}^{\infty} (1 - e^{-ixy}) \eta e^{-\eta y} dy dx,
$$

$$
I_2 = \frac{1}{\theta} \int_{uw}^{u} \frac{1}{x} \int_{-\infty}^{0} (1 - e^{-ixy}) \phi e^{\phi y} dy dx.
$$

The first term $I_1$ can be written

$$
I_1 = \frac{1}{\theta} \int_{0}^{\infty} \frac{(1 - e^{-isu})}{s} \int_{s}^{s/w} \eta e^{-\eta y} dy ds
$$

$$
= \frac{p}{\theta} \int_{0}^{\infty} \frac{(1 - e^{-isu})}{s} e^{-\eta s} - e^{-\eta s/w} dy
$$

$$
= \frac{p}{\theta} \int_{0}^{\infty} \frac{(1 - e^{-isu})}{s} \frac{\eta}{s} e^{-\eta v} dv ds
$$

$$
= \frac{p}{\theta} \int_{0}^{\infty} \frac{(1 - e^{-isu})}{s} \frac{1}{1/v} \eta e^{-\eta v} \frac{1}{v} dv ds.
$$

The second term $I_2$ can be written as

$$
I_2 = \frac{1}{\theta} \int_{-\infty}^{0} \frac{(1 - e^{-isu})}{s} \int_{s}^{s/w} \phi e^{-\eta y} dy ds
$$

$$
= \frac{q}{\theta} \int_{-\infty}^{0} \frac{(1 - e^{-isu})}{s} \phi e^{-\phi s} - e^{-\phi s/w} dy
$$

$$
= \frac{q}{\theta} \int_{-\infty}^{0} \frac{(1 - e^{-isu})}{s} \frac{\phi}{s} e^{\phi v} dv ds
$$

$$
= \frac{q}{\theta} \int_{-\infty}^{0} \frac{(1 - e^{-isu})}{s} \frac{1}{s} \phi e^{\phi v} \frac{1}{v} dv ds.
$$

This gives us (5.1), proving the theorem. ■
Since the second exponential factor on the right hand side of (5.1) is the characteristic function of the compound Poisson process we have

**Corollary 5.2 (Exact Simulation via Decomposition Approach).** Let $N$ be a Poisson random variable of rate $\lambda h$ and let $\{S_k\}_{k=1,2,...}$ be i.i.d random variables following a mixture of double exponential distribution

$$f_{S_k}(y) = p e^{\theta h U} e^{-\eta e^{\theta h U} y} I_{y \geq 0} + q e^{\varphi h U} e^{\phi e^{\theta h U} y} I_{y < 0},$$

$\forall \ k = 1, 2, \ldots$.

(5.4)

where $U \overset{d}{=} U[0,1]$ is the uniform distribution on $[0,1]$. Then

$$X_{t+h} \overset{d}{=} X_t e^{-\theta h} + \sum_{k=1}^{N} S_k.$$ (5.5)

The above formula (5.5) enables us to simulate the process $X_t$ by the exact decomposition approach.

6. **Numerical results**

To validate our estimators discussed in Section 4, we perform some numerical simulations. We choose the values of $\rho = 0.6$, $\eta = 1.2$, $\varphi = 1.6$ and $\theta = 2.0$ (and $\lambda = \sigma = 1$). With these parameters, we simulate the double exponential Ornstein-Uhlenbeck process using the exact decomposition algorithm given by (5.5). A simulated sample is displayed in Figure 1. Figures 2 and 3 plot the assumed values versus the values by the ergodic estimators. Table 1 lists the approximation of ergodic estimators to the true parameters as the time becomes larger. It demonstrates that the rate of convergence is quite faster.

![Figure 1](image.png)

**Figure 1.** Simulated sample path for a double exponential Ornstein-Uhlenbeck process with $T=20$, $N=50$, $h=0.02$ $\eta = 1.2$, $\varphi = 1.6$ and $\theta = 2.0$, $\sigma = \lambda = 1$
The table 1 shows the estimated values of the parameters $p$, $\eta$, $\phi$ and $\theta$ with different number of steps $N$ and fixed $h = 0.02$ and $T = Nh$

| Time | Number of steps | $p = 0.6$ | $\eta = 1.2$ | $\phi = 1.6$ | $\theta = 2.0$ |
|------|-----------------|-----------|--------------|--------------|--------------|
| 1    | 50              | 0.8421    | 1.31677      | 0.8995       | 7.4297       |
| 2    | 100             | 0.7070    | 1.3477       | 1.2816       | 3.6995       |
| 4    | 200             | 0.74925   | 1.2498       | 1.1164       | 2.9127       |
| 6    | 300             | 0.6928    | 1.2532       | 1.4803       | 2.6587       |
| 8    | 400             | 0.6804    | 1.2571       | 1.5397       | 2.3808       |
| 10   | 500             | 0.6812    | 1.2204       | 1.4793       | 2.2743       |
| 12   | 600             | 0.5500    | 1.2089       | 1.6546       | 2.2217       |
| 20   | 1000            | 0.6320    | 1.1836       | 1.5078       | 2.1066       |
| 40   | 2000            | 0.5635    | 1.1255       | 1.7866       | 2.0631       |
| 60   | 3000            | 0.6135    | 1.2112       | 1.5940       | 2.0128       |

Table 1. Assumed Values and Estimated values of the parameters with different number of steps $N$ and fixed $h = 0.02$ and $T = Nh$
7. Appendix: Covariance matrix $A$

In this section we give the expression of the covariance matrix $A$ in Lemma 4.1. It is very sophisticated to express the entries of this matrix in terms of the parameters of the equation (2.1). So, we keep them as expression of the invariant probability measure of $X_0$ and that of $X_{kh}$. First, we compute $\sigma_{g_1g_1}$.

$$
\sigma_{g_1g_1} = \text{Cov}(X_0, X_0) + 2 \sum_{j=1}^{\infty} [\text{Cov}(X_0, X_{jh})] \\
= \mathbb{E}(X_0^2) - \mathbb{E}(X_0)^2 + 2 \sum_{j=1}^{\infty} \left[ \mathbb{E}(X_0X_{jh}) - \mathbb{E}(X_0)\mathbb{E}(X_{jh}) \right] \\
= \mathbb{E}(X_0^2) - \mathbb{E}(X_0)^2 + 2 \sum_{j=1}^{\infty} \left[ \mathbb{E}(X_0X_{jh}) - [\mathbb{E}(X_0)]^2 \right], \tag{7.1}
$$

where we used $\mathbb{E}(X_{jh}) = \mathbb{E}(X_0)$. Now we compute $\sigma_{g_2g_2}$.

$$
\sigma_{g_2g_2} = \text{Cov}(X_0^2, X_0^2) + 2 \sum_{j=1}^{\infty} [\text{Cov}(X_0^2, X_{jh}^2)] \\
= \mathbb{E}(X_0^4) - \mathbb{E}(X_0^2)^2 + 2 \sum_{j=1}^{\infty} \left[ \mathbb{E}(X_0^2X_{jh}^2) - \mathbb{E}(X_0^2)^2 \right]. \tag{7.2}
$$

Similarly, we have

$$
\sigma_{g_3g_3} = \text{Cov}(X_0^3, X_0^3) + 2 \sum_{j=1}^{\infty} [\text{Cov}(X_0^3, X_{jh}^3)] \\
= \mathbb{E}(X_0^6) - \mathbb{E}(X_0^3)^2 + 2 \sum_{j=1}^{\infty} \left[ \mathbb{E}(X_0^3X_{jh}^3) - \mathbb{E}(X_0^3)^2 \right], \tag{7.3}
$$

and

$$
\sigma_{g_4g_4} = \text{Cov}(X_0^2X_h, X_0^2X_h) + 2 \sum_{j=1}^{\infty} [\text{Cov}(X_0^2X_h, X_{jh}X_{(j+1)h})] \\
= \mathbb{E}((X_0X_h)^2) - \mathbb{E}(X_0X_h)^2 \\
+ 2 \sum_{j=1}^{\infty} \left[ \mathbb{E}(X_0X_hX_{jh}X_{(j+1)h}) - \mathbb{E}(X_0X_h)\mathbb{E}(X_{jh}X_{(j+1)h}) \right]. \tag{7.4}
$$

$\sigma_{g_1g_2}$ is computed as follows.

$$
\sigma_{g_1g_2} = \text{Cov}(X_0, X_0^2) + \sum_{j=1}^{\infty} \left[ \text{Cov}(X_0, X_{jh}^2) + \text{Cov}(X_0^2, X_{jh}) \right] \\
= \mathbb{E}((X_0)^3) - \mathbb{E}(X_0)\mathbb{E}(X_0^2) + \sum_{j=1}^{\infty} \left[ \mathbb{E}(X_0X_{jh}^2) - \mathbb{E}(X_0)\mathbb{E}(X_{jh}^2) \right] \\
+ \mathbb{E}(X_0^2X_{jh}) - \mathbb{E}(X_0^2)\mathbb{E}(X_{jh}) \right]. \tag{7.5}
$$
In similar way we can get

\[ \sigma_{g_{1g_{2}}} = \text{Cov}(X_0, X_0^3) + \sum_{j=1}^{\infty} [\text{Cov}(X_0, X_{jh}^3) + \text{Cov}(X_0^3, X_{jh})] \]

\[ = E((X_0)^4) - E(X_0)E(X_0^3) + \sum_{j=1}^{\infty} \left[ E(X_0X_{jh}^3) - E(X_0)E(X_{jh}^3) \right] \]

\[ + E(X_0^3X_{jh}) - E(X_0^3)E(X_{jh}) \]  \hspace{1cm} (7.6)

and

\[ \sigma_{g_{1g_{4}}} = \text{Cov}(X_0, X_0X_h) + \sum_{j=1}^{\infty} [\text{Cov}(X_0, X_{jh}X_{(j+1)h}) + \text{Cov}(X_{jh}, X_0X_h)] \]

\[ = E(X_0^2X_h) - E(X_0)E(X_0X_h) + \sum_{j=1}^{\infty} \left[ E(X_0X_{jh}X_{(j+1)h}) - E(X_0)E(X_{jh}X_{(j+1)h}) \right] \]

\[ + E(X_0X_hX_{jh}) - E(X_{jh})E(X_0X_h) \]  \hspace{1cm} (7.7)

\[ \sigma_{g_{2g_{2}}} \text{ is similar to } \sigma_{g_{1g_{2}}}. \]

\[ \sigma_{g_{2g_{4}}} = \text{Cov}(X_0^2, X_0^3) + \sum_{j=1}^{\infty} [\text{Cov}(X_0^2, X_{jh}^3) + \text{Cov}(X_0^3, X_{jh}^2)] \]

\[ = E((X_0)^5) - E(X_0^2)E(X_0^3) + \sum_{j=1}^{\infty} \left[ E(X_0^2X_{jh}^3) - E(X_0^2)E(X_{jh}^3) \right] \]

\[ + E(X_0^3X_{jh}^2) - E(X_0^3)E(X_{jh}^2) \]  \hspace{1cm} (7.8)

Finally, we have

\[ \sigma_{g_{2g_{4}}} = \text{Cov}(X_0^2, X_0X_h) + \sum_{j=1}^{\infty} [\text{Cov}(X_0^2, X_{jh}X_{(j+1)h}) + \text{Cov}((X_{jh})^2, X_0X_h)] \]

\[ = E(X_0^2X_h) - E(X_0^2)E(X_0X_h) + \sum_{j=1}^{\infty} \left[ E(X_0^2X_{jh}X_{(j+1)h}) - E(X_0^2)E((X_{jh}X_{(j+1)h}) \right] \]

\[ + E(X_0X_hX_{jh}^2) - E(X_{jh})E(X_0X_h) \]  \hspace{1cm} (7.9)

and

\[ \sigma_{g_{3g_{4}}} = \text{Cov}(X_0^3, X_0X_h) + \sum_{j=1}^{\infty} [\text{Cov}(X_0^3, X_{jh}X_{(j+1)h}) + \text{Cov}((X_{jh})^3, X_0X_h)] \]

\[ = E(X_0^4X_h) - E(X_0^3)E(X_0X_h) + \sum_{j=1}^{\infty} \left[ E(X_0^4X_{jh}X_{(j+1)h}) - E(X_0^3)E((X_{jh}X_{(j+1)h}) \right] \]

\[ + E(X_0X_hX_{jh}^3) - E(X_{jh})E(X_0X_h) \]  \hspace{1cm} (7.10)
References

[1] Billingsley, P. Probability and measure. Anniversary edition. With a foreword by Steve Lalley and a brief biography of Billingsley by Steve Koppes. Wiley Series in Probability and Statistics. John Wiley & Sons, Inc., Hoboken, NJ, 2012.

[2] Billingsley, P. Statistical inference for Markov processes. Statistical Research Monographs, Vol. II. The University of Chicago Press, Chicago, Ill. 1961.

[3] Cai, N.; Chen, N. and Wan, X. Occupation times of jump-diffusion processes with double exponential jumps and the pricing of options. Math. Oper. Res. 35 (2010), no. 2, 412-437.

[4] Hackmann, D. Analytic techniques for option pricing under a hyperexponential Lévy model. J. Comput. Appl. Math. 342 (2018), 225–248.

[5] Cheng, Y., Hu, Y. and Long, H. Generalized moment estimators for \(\alpha\)-stable Ornstein-Uhlenbeck motions from discrete observations. Stat. Inference Stoch. Process. 23 (2020), no. 1, 53-81.

[6] Haress, E. M. and Hu, Y. Estimation of all parameters in the fractional Ornstein-Uhlenbeck model under discrete observations. Stat. Inference Stoch. Process. 24 (2021), no. 2, 327-351.

[7] Jones, G. L. On the Markov chain central limit theorem. Probab. Surv. 1 (2004), 299-320.

[8] Lukacs, E. Characteristic functions. Second edition, revised and enlarged. Hafner Publishing Co., New York, 1970.

[9] Qu, Y., A. Dassios, and H. Zhao. Exact Simulation of Gamma-driven Ornstein–Uhlenbeck Processes with Finite and Infinite Activity Jumps. Journal of the Operational Research Society, 2019: 1–14.

[10] Jongbloed, G.; van der Meulen, F. H.; van der Vaart, A. W. Nonparametric inference for Lévy-driven Ornstein-Uhlenbeck processes. Bernoulli 11 (2005), no. 5, 759-791.

[11] Masuda, H. Ergodicity and exponential \(\beta\)-mixing bounds for multidimensional diffusions with jumps. Stochastic Process. Appl. 117 (2007), no. 1, 35-56.

[12] Dellacherie, C. and Meyer, P. A. Probabilities and potential. North-Holland Mathematics Studies, 29. North-Holland Publishing Co., Amsterdam-New York, 1978.

[13] Ibragimov, I. A.; Linnik, Yu. V. Independent and stationary sequences of random variables. Wolters-Noordhoff Publishing, Groningen, 1971.

[14] Sato, K. Lévy processes and infinitely divisible distributions. Cambridge Studies in Advanced Mathematics, 68. Cambridge University Press, Cambridge, 2013.

[15] Meyn S.P and Tweedie R.L. Stability of Markovian processes. II. Continuous-time processes and sampled chains. Adv. in Appl. Probab., 25(3):487-517, 1993.

Department of Mathematical and Statistical Sciences, University of Alberta at Edmonton, Edmonton, Canada, T6G 2G1
Email address: yaozhong@ualberta.ca

Department of Mathematical and Statistical Sciences, University of Alberta at Edmonton, Edmonton, Canada, T6G 2G1
Email address: neha2@ualberta.ca