Algebraic and Differential Star Products on Regular Orbits of Compact Lie Groups

R. Fioresi, A. Levrero and M. A. Lledó

Abstract

In this paper we study a family of algebraic deformations of regular coadjoint orbits of compact semisimple Lie groups with the Kirillov Poisson bracket. The deformations are restrictions of deformations on the dual of the Lie algebra. We prove that there are non isomorphic deformations in the family. The star products are not differential, unlike the star products considered in other approaches. We make a comparison with the differential star product canonically defined by Kontsevich’s map.

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1 Introduction

Coadjoint orbits of Lie groups are symplectic manifolds that can be used to model physical systems that have a continuous group of symmetries. The Kirillov-Kostant orbit principle allows in many cases to associate canonically a unitary representation to the orbit. The Hilbert space of the representation can then be thought as the Hilbert space of the quantum theory. A quantization map which takes a class of functions on the phase space to operators in such Hilbert space can be constructed. This is the approach of geometric quantization (see Ref. [1] for a review).

On the other hand, the pioneering work by Bayen et al. [2] on deformation quantization raised the problem of quantizing the coadjoint orbits with a radically different method. However, being based on the same physical principles, it is natural to expect a relation between the two approaches. In fact, it was thought that deformation quantization, which “forgets” about the Hilbert space on which the quantum algebra is represented, could nevertheless throw light on the Kirillov-Kostant orbit principle [3]. The algebra that appears in geometric quantization is defined as the quotient of the enveloping algebra by a prime ideal which is contained in the kernel of the corresponding representation [1]. The method of geometric quantization is however more general than the Kirillov-Kostant orbit principle. A comparison with deformation quantization for the case of $\mathbb{R}^{2n}$ with the standard symplectic structure was done in Ref. [4].

In the work of Bayen et al. [2] only flat symplectic manifolds were studied. The existence of a deformation quantization of general symplectic manifolds was first established by De Wilde and LeComte [4], and using different methods by Omori, Maeda and Yoshioka [4] and by Fedosov [7]. For a comparison between the methods of De Wilde and Le Comte and Fedosov, see Ref. [20]. In Ref. [8], the existence of tangential deformations for any regular Poisson manifold was proven. In Ref. [9], Kontsevich settled the fundamental question of the existence of deformations for arbitrary (formal) Poisson manifolds. In all these works the deformations are taken to be differential, that is, the product structure in the deformed algebra is defined through bidifferential operators.

Explicit star products for non flat manifolds are not easy to construct. In Ref. [10], Gutt constructed a star product on the cotangent bundle of a Lie group. In Ref. [11], Cahen and Gutt constructed a deformation of the algebra of polynomials on the regular coadjoint orbits of compact semisimple
groups, using the fact that the universal enveloping algebra is a deformation of the algebra of polynomials on the dual of the Lie algebra [12]. They showed that, although the deformation on the whole space is differential, the one induced on the orbit is not. Moreover, in Ref. [13] they show that for semisimple groups “tangential” deformations (that is, deformations on the ambient space that restrict well to the orbits) that are at the same time differential and that extend over the origin do not exist. Deformations of coadjoint orbits were also studied in [14] in terms of a polarization of the orbit (also used in geometric quantization). The resulting star product is covariant. More generally, deformations of Kähler manifolds were studied in [15].

In Ref. [16] a family of star products on coadjoint orbits of semisimple Lie groups was constructed as a quotient of the enveloping algebra by a suitable ideal. With a certain choice inside the family of deformations one obtains the same star product as in Ref. [11]. For another choice, in the special case of SU(2), the deformed algebra turns out to be the one of geometric quantization [16]. In this case we can associate to the deformation quantization a unitary representation in the spirit of Berezin [17].

In the present work we further study the properties of this family of deformations. The organization of the paper is as follows. In Section 2, we review the construction of the algebraic star products on the orbit [16] and show that there exist non equivalent products associated with a given algebraic Poisson bracket. We also show that the ideal used to quotient the enveloping algebra is prime. In fact, in geometric quantization the quantum algebra is the enveloping algebra modulo a prime ideal; this ideal is contained in the kernel of the representation. In Section 3, we use Kontsevich’s theorem on differential star products to show that the Kirillov Poisson structure on the dual space of the Lie algebra of a semisimple Lie group has only one possible deformation. In Section 4, we study the algebraic star products on the orbit and show that they are not, in general, differential. In Section 5, we show different ways of constructing star products on the orbit.
2 Deformation of the polynomial algebra of a regular orbit

In this section we review the results of Ref. [16] where a family of deformations of the polynomial algebra of a regular coadjoint orbit of a semisimple Lie group was constructed. We show that the different deformations in the family are not necessarily equivalent by exhibiting a counterexample.

Let $G$ be a complex semisimple Lie group of dimension $n$ and rank $m$, $G$ its Lie algebra and $U$ the enveloping algebra of $G$. Let $T_C(G)$ be the full tensor algebra of $G$ over $C$. Consider the algebra $T_C(G)[[h]]$ and its proper two sided ideal

$$
\mathcal{L}_h = \sum_{X,Y \in G} T_C(G)[[h]] \otimes (X \otimes Y - Y \otimes X - h[X,Y]) \otimes T_C(G)[[h]].
$$

We define $U[[h]] = T_C(G)[[h]]/\mathcal{L}_h$.

**Definition 2.1** An associative algebra $A[[h]]$ over $C[[h]]$ is a formal deformation of a Poisson algebra $(A, \{\ , \ \})$ over $C$ if there exists an isomorphism of $C[[h]]$-modules $\psi : A[[h]] \rightarrow A[[h]]$ satisfying the following properties

a. $\psi(f_1 f_2) = \psi(f_1) \psi(f_2) \mod(h)$.

b. $\psi(f_1)\psi(f_2) - \psi(f_2)\psi(f_1) = h\psi(\{f_1, f_2\}) \mod(h^2)$.

Because of its relation with the problem of quantization, $A[[h]]$ is sometimes called a deformation quantization of $A$.

Notice that in the above definition we can substitute $C[[h]]$ by $C[h]$. The algebra is then a module over $C[h]$ which will be denoted as $A_h$. We will say that $A_h$ is a $C[h]$-deformation of $A$. Notice that a $C[h]$-deformation extends to a formal deformation, but the converse is not always true. Also, a $C[h]$-deformation can be specialized to any value of the parameter $h$, since the ideal generated by the element $h - h_0$ is proper in $A_h$. One obtains then a complex algebra, $A_{h_0} = A_h/(h - h_0)$.

It is well known that $U[[h]]$ is a formal deformation of $C[G^*][[h]]$ equipped with the Kirillov Poisson bracket $\{\ , \ \}$.

We denote by $p_i$, $i=1,\ldots,m$ the algebraically independent homogeneous generators of the subalgebra of invariant polynomials on $G^*$,

$$
I = \{ p \in C[G^*] \mid p(Ad^*(g)\xi) = p(\xi) \ \forall \xi \in G^*, \ g \in G \} = C[p_1, \ldots, p_m],
$$

(2)
given by Chevalley’s theorem. If \( S(G) \) is the algebra of symmetric tensors on \( G \), we can identify canonically \( \text{Pol}(G^*) = \mathbb{C}[G^*] \approx S(G) \).

Let \( \{X_1, \ldots, X_n\} \) be a basis for \( G \) and let \( \{x_1, \ldots, x_n\} \) be the corresponding generators of \( \mathbb{C}[G^*] \). Then the symmetrizer map \( \text{Sym} : \mathbb{C}[G^*] \to T_C(G) \) is given by

\[
\text{Sym}(x_1 \cdots x_p) = \frac{1}{p!} \sum_{s \in S_p} X_{s(1)} \otimes \cdots \otimes X_{s(p)}
\]

where \( S_p \) is the group of permutations of order \( p \). The composition of the symmetrizer with the natural projection \( T_C(G) \to U \) is a linear isomorphism that gives the identification \( \mathbb{C}[G^*] \approx U \). Moreover, it sends the invariant polynomials \( I \) isomorphically into the center of \( U \) (see for example Ref. [18]).

We can extend the symmetrizer map as \( \text{Sym} : \mathbb{C}[G^*][[h]] \to T_C(G)[[h]] \), and the projection \( \pi_h : T_C(G)[[h]] \to U[[h]] \). Then \( P_i = \pi_h \circ \text{Sym}(p_i) \) are also central elements. Note that \( \pi_h \circ \text{Sym} \) can be used as the isomorphism \( \psi \) in Definition 2.1, \( \psi : \mathbb{C}[G] \to U[[h]]. \)

We consider now the compact real form of \( G \) with \( G^r \) the real Lie algebra (\( G \) still denotes the complex Lie algebra). The coadjoint orbits are algebraic manifolds given by the constrains,

\[
p_i(x_1, \ldots x_n) = c_i^0, \quad c_i^0 \in \mathbb{R}, \quad i = 1, \ldots m.
\]

There is a one to one correspondence from the set of orbits with the elements of a Weyl chamber in the Cartan subalgebra. Regular orbits are the orbits of elements in the interior of a Weyl chamber, and they have maximal dimension. Non regular orbits are given by constants \( c_i^0 \) satisfying some constrains. This means that if \( c_i^0 \) define a regular orbit, there is a neighborhood of the orbit that is foliated with regular orbits. We will use this property in the next section.

Let \( \Theta^r \) be a regular orbit. Then, the ideal of polynomials in \( \mathbb{R}[G^{r*}] \) that vanish on \( \Theta^r \) is generated by the elements \( p_i - c_i^0 \) \([19]\), so we can define

\[
\mathcal{I}_0 = (p_i - c_i^0, \ i = 1, \ldots, m) \subset \mathbb{R}[G^{r*}],
\]

and the algebra of restrictions of polynomials to the orbit is \( \mathbb{R}[[\Theta^r]] = \mathbb{R}[G^{r*}]/\mathcal{I}_0 \). We take the complexification of this algebra \( \mathbb{C}[G^{r*}]/\mathcal{I}_0 \) (we denote still by \( \mathcal{I}_0 \) the ideal in the complexified algebra), which is the algebra of polynomials.
on the complex orbit $\Theta$, $\mathbb{C}[\Theta]$. Consider a regular orbit and define the two sided ideal in $U[[h]]$ generated by the elements $P_i - c_i(h)$ $i = 1, \ldots, m$

$$\mathcal{I}_{[h]} = (P_i - c_i(h), \ i = 1, \ldots, m) \subset U_{[h]},$$

where $c_i(h) \in \mathbb{C}[[h]]$ is such that $c_i(0) = c^0_i$ and $P_i = \text{Sym}(p_i)$. In [16] it was shown the following

**Theorem 2.1** The algebra $U_{[h]}/\mathcal{I}_{[h]}$ is a formal deformation of $\mathbb{C}[\Theta] = \mathbb{C}[G^*]/I_0$. $U_{h}/\mathcal{I}_{h}$ is a $\mathbb{C}[h]$-deformation of $\mathbb{C}[\Theta] = \mathbb{C}[G^*]/I_0$.

Regularity is a technical assumption to show that $U_{[h]}/\mathcal{I}_{[h]}$ is a free module isomorphic to $\mathbb{C}[\Theta][[h]]$.

The ideal $\mathcal{I}_0 \subset \mathbb{C}[G^*]$ is prime since the corresponding algebraic variety is irreducible. We want to show now that the ideal $\mathcal{I}_h \in U_h$ is prime. We define first a grading in $U_h$. If $\{X_i, i = 1, \ldots, n\}$ is a basis of $G$ we set $\deg(X_i) = \deg(h) = 1$. This is a set of generators for $T_\mathbb{C}(G)[h]$. Notice that the relations in $\mathcal{L}_h$ are homogeneous with respect to this grading, so a grading is defined on $U_h$. The degree of an inhomogeneous element in $U_h$ is the maximal degree occurring in all of its monomials. Let us restrict to modules over $\mathbb{C}[h]$.

**Proposition 2.1** Assume that $\deg(c_i(h)) \leq \deg(P_i)$. Then if $FG \in \mathcal{I}_h$, either $F \in \mathcal{I}_h$ or $G \in \mathcal{I}_h$. Hence $\mathcal{I}_h$ is prime.

**Proof.** Consider first the projection $\rho : U_h \to U_h/hU_h \approx \mathbb{C}[G^*]$. One has that for any $F \in U_h$, $\deg(\rho(F)) \leq \deg(F)$.

Since $\rho(P_i) = p_i$, $i = 1, \ldots, m$, we have $\rho(\mathcal{I}_h) = \mathcal{I}_0$. Since $\deg(c_i(h)) \leq \deg(P_i)$, if $f \in \mathcal{I}_0$ there exists $F' \in \mathcal{I}_h$ with $\rho(F') = f$ and $\deg(F') = \deg(f)$. If $f = f^{i}(p_i - c^0_i)$ one can take for example $F = \pi_h \circ \text{Sym}(f^i)(P_i - c^i(h))$.

Assume that $FG \in \mathcal{I}_h$. Then

$$\rho(FG) = \rho(F)\rho(G) =: fg \in \mathcal{I}_0,$$

where we denote the projections by small case letters. Since $\mathcal{I}_0$ is a prime ideal, either $f \in \mathcal{I}_0$ or $g \in \mathcal{I}_0$.

Assume that $f \in \mathcal{I}_0$. Then there exists $F' \in \mathcal{I}_h$ with $\rho(F') = f$ and $\deg(F') = \deg(f) \leq \deg(F)$. Denote $F - F' = h\Delta F$; it is clear that
deg(ΔF) < deg(F). If ΔF ∈ ℐ_h then F itself is in ℐ_h and we are through; otherwise observe that
\[ hΔFG = FG - F'G \in ℐ_h. \]
Since \( U_h/ℐ_h \) is without torsion ([14]), we can “divide” by \( h \), and it follows that \( ΔFG \in ℐ_h \).

We can now proceed to show that either \( ΔF \) or \( G \) is in \( ℐ_h \). But notice that we have reduced the total degree. We can apply the argument again until we arrive to the situation that one of the factors has degree zero (it is a number). Then it follows that the other factor is in \( ℐ_h \) and eventually that \( F \) or \( G \) are in \( ℐ_h \), as we wanted to prove.

We now want to show that there exists two different \( \mathbb{C}[h] \)-deformations on the same orbit that are not isomorphic. We consider \( G = \text{sl}_2(\mathbb{C}) \). Let
\[ ℐ_h = (P - \mu^0), \quad ℐ'_h = (P - \mu^0 - \sqrt{2}h), \]
where \( P \) is the quadratic Casimir. Assume that \( U_h/I_h \cong U_h/I'_h \). Since any isomorphism will send the ideal \( (h - 1) \) into the ideal \( (h - 1) \), the quotient of both algebras by \( (h - 1) \) must be isomorphic. But the algebra \( U/(P - \mu^0) \) has finite dimensional representations only for certain values of \( \mu^0 \). In particular, for \( \mu^0 \) irrational, it has no finite dimensional representations [18]. It is enough to take \( \mu^0 \) such that \( U_h/I_h \) has finite dimensional representations and we reach a contradiction.

The same is true for formal deformations. In fact, with the same reasoning as in Ref. [18] we have that \( U[[h]]/(P - \mu(h)) \) admits finite dimensional representations only for appropriate \( \mu(h) = \mu^0 \).

### 3 Star products and equivalence

**Definition 3.1** Given \( \mathcal{A}_{[[h]]} \), a formal deformation of a Poisson algebra \( \mathcal{A} \) and a \( \mathbb{C}[[h]] \)-module isomorphism \( \psi : \mathcal{A}[[h]] \rightarrow \mathcal{A}_{[[h]]} \) as in Definition 2.1, we say that the associative product in \( \mathcal{A}[[h]] \) defined by
\[ a \star b = \psi^{-1}(\psi(a) \cdot \psi(b)), \quad a, b \in \mathcal{A}[[h]] \]
is a star product on \( \mathcal{A}[[h]] \).
It follows from property \(a\) in Definition 2.1 that a star product can always be written as

\[
a \star b = ab + \sum_{n>0} h^n B_n(a,b)
\]

where \(B_n\) are bilinear operators and by juxtaposition \(ab\) we denote the commutative product in \(\mathcal{A}[[h]]\). Property \(b\) in Definition 2.1 implies that 

\[
\{a,b\} = B_1^{-1}(a,b) := B_1(a,b) - B_1(b,a).
\]

For a given \(\mathcal{A}[[h]]\) there are many choices of the isomorphism \(\psi\) (it is not canonical). Once \(\psi\) is given, the star product \(\star\) is defined and one regards \(\mathcal{A}[[h]]\) as an associative non commutative \(\mathbb{C}[[h]]\)-algebra. Let \(\star\) and \(\star'\) be different star products corresponding to the same deformation, defined by the maps

\[
\psi : \mathcal{A}[[h]] \to \mathcal{A}[[h]], \quad a \star b = \psi^{-1}(\psi(a) \cdot \psi(b)),
\]

\[
\psi' : \mathcal{A}[[h]] \to \mathcal{A}[[h]], \quad a \star' b = \psi'^{-1}(\psi'(a) \cdot \psi'(b)).
\]

They define isomorphic algebras. The isomorphism \(T : \mathcal{A}[[h]] \to \mathcal{A}[[h]]\) is given by

\[
T = \psi'^{-1} \circ \psi, \quad T(a \star b) = T(a) \star' T(b).
\]

\(T\) can also be expressed as a power series

\[
T = \sum_{n \geq 0} h^n T_n.
\]

in terms of the linear operators \(T_n\). It is easy to show that \(T_0\) is an automorphism of the commutative algebra \(\mathcal{A}[[h]]\)

\[
T_0(ab) = T_0(a)T_0(b), \quad a, b \in \mathcal{A}[[h]]
\]

and of the Poisson algebra \(\mathcal{A}\)

\[
T_0\{a,b\} = \{T_0(a),T_0(b)\}, \quad a, b \in \mathcal{A}.
\]

**Definition 3.2** If \(\star\) and \(\star'\) are two isomorphic star products on \(\mathcal{A}[[h]]\), the isomorphism being \(T : \mathcal{A}[[h]] \to \mathcal{A}[[h]]\) as in (6), we say that they are gauge equivalent if \(T_0 = Id\).
A star product is differential if $\mathcal{A} = C^\infty(M)$ for a smooth manifold $M$, and the operators $B_n$ in (3) are bidifferential operators. An example of differential star product is the one induced on $\mathcal{G}^*$ by the map (3). It is in principle defined on polynomials, but it can be extended to $C^\infty(M)$ through operators $B_i$ that are bidifferential. It was shown in Ref. [20] that with a gauge transformation any differential star product can be brought to a form under which the bilinear operators $B_n$ are null on the constants (that is, the zero degree doesn’t appear).

One can consider gauge equivalence inside the class of differential star products by considering only differential maps $T$. For this case, it was shown by Kontsevich in [9] the following important theorem,

**Theorem 3.1** The set of gauge equivalence classes of differential star products on a smooth manifold $M$ can be naturally identified with the set of equivalence classes of Poisson structures depending formally on $\hbar$,

$$\alpha = \hbar \alpha_1 + \hbar^2 \alpha_2 + \cdots$$

modulo the action of the group of formal paths in the diffeomorphism group of $M$, starting at the identity isomorphism.

In particular, for a given Poisson structure $\alpha_1$, we have the equivalence class of differential star products canonically associated to $\hbar \alpha_1$.

We explain briefly the concept of formal paths in the diffeomorphism group of $M$. For further details we refer to Ref. [9, 21]. Let $\mathfrak{m}$ be the maximal ideal in $\mathbb{R}[[t]]$. Consider $\mathcal{L}$ the algebra of polyvector fields with the Schouten-Nijenhuis bracket. It is a differential graded Lie algebra with zero differential. We recall that a Poisson structure is a bivector field such that its Schouten-Nijenhuis bracket with itself is zero. Let $\mathcal{L}_0$ be the algebra of vector fields on $M$. They are the 0-cochains of the complex. Consider $\mathcal{L}_0 \otimes \mathfrak{m}$. The exponential of this algebra is the group of formal paths in the diffeomorphism group starting with the identity. $\mathcal{L}_1$ is the set of (skew-symmetric) bivector fields. $\mathcal{L}_0$ acts on $\mathcal{L}_1$ with the Schouten-Nijenhuis bracket,

$$Z(B)(f_1, f_2) = Z(B(f_1, f_2)) - B(Z(f_1), f_2) - B(f_1, Z(f_2)),$$

$Z \in \mathcal{L}_0$, $B \in \mathcal{L}_1$, $f_1, f_2 \in C^\infty(M)$

and this action can be exponentiated to the group.
3.1 Uniqueness of the deformation of the Kirillov Poisson structure

We want to determine whether the equivalence class of the Kirillov Poisson bracket in $G^*$ is the only class of formal Poisson structures whose first order term is the Kirillov Poisson bracket (as it happens, for example, in any flat symplectic manifold [4]). This is actually the case, at least for algebraic Poisson structures (we say that a Poisson structure $\beta$ is algebraic if $\beta(p,q)$ is a polynomial whenever $p$ and $q$ are polynomials) and $G$ semisimple.

**Proposition 3.1** Let $G$ be a real semisimple algebra. Let $\beta$ be an algebraic differential formal Poisson structure

$$\beta = \sum_{i=0}^{\infty} h^{i+1} \beta_i,$$

such that $\beta_0$ is the Kirillov Poisson structure in $G^*$. Then $\beta$ is equivalent to $\beta_0$.

**Proof.** The Jacobi identity at first order is satisfied since $\beta_0$ itself is a Poisson structure and at second order it implies that $\beta_1$ is a two-cocycle in the Chevalley cohomology of $\beta_0$. If $\beta_1$ is a coboundary, then

$$\beta_1(f_1, f_2) = \delta Z(f_1, f_2) = Z(\beta_0(f_1, f_2)) - \beta_0(Z(f_1), f_2) - \beta_0(f_1, Z(f_2)),$$

with $Z$ a 0-cochain. Then a gauge transformation (formal path in the diffeomorphism group) of the form

$$\varphi = \text{Id} + hZ + \cdots$$

shows that $\beta$ is equivalent to a formal Poisson structure without term of order $h^2$, i.e., we can assume that $\beta_1 = 0$. But then $\beta_2$ is a cocycle and we can proceed recursively. Hence, to prove that $\beta \sim \beta_0$ it is actually sufficient to show that the Chevalley cohomology of $\beta_0$ is zero. Since $\beta$ is a bivector field and it is algebraic, it is sufficient to check that there is no non trivial algebraic two cocycle with order of differentiability (1,1). We will show that this is the case.

Such a cocycle is an antisymmetric bidifferential map, null on the constants and with polynomial coefficients,

$$C^2 : \text{Sym}(G) \otimes \text{Sym}(G) \longrightarrow M$$
where \( M = \text{Sym}(\mathcal{G}) \) is a left (Lie algebra) \( \text{Sym}(\mathcal{G}) \)-module, with the action given by the Poisson bracket \( \beta_0 \). If \( C^2 \) has order of differentiability \((1,1)\), we can restrict \( C^2 \) non trivially to first order polynomials. We denote that restriction by
\[
\hat{C}^2 : \mathcal{G} \otimes \mathcal{G} \rightarrow \text{Sym}(\mathcal{G}).
\]
Then \( \hat{C}^2 \) is a cocycle in the Lie algebra cohomology of order two of \( \mathcal{G} \) with values in \( \text{Sym}(\mathcal{G}) \). Since \( \mathcal{G} \) is semisimple, as a consequence of Whitehead’s lemma, this cohomology is zero (see for example Ref. [22]). Hence \( \hat{C}^2 \) is trivial, i.e., there exists a 1-cochain \( \hat{C}^1 : \mathcal{G} \rightarrow \text{Sym}(\mathcal{G}) \) such that \( \hat{C}^2 = \delta \hat{C}^1 \).

If \( \hat{C}^1 \) is given on a basis of \( \mathcal{G} \) by \( \hat{C}^1_i \), this means that
\[
\hat{C}^2(X_i, X_j) = \delta \hat{C}^1 = \hat{C}^1([X_i, X_j]) - \beta_0(X_i, \hat{C}^1_j) - \beta_0(\hat{C}^1_i, X_j),
\]
and \( \hat{C}^1 \) can be extended to a 1-cochain in the Chevalley complex by
\[
C^1(f) = \hat{C}^1_k \frac{\partial f}{\partial x^k}, \quad f \in \text{Sym}(\mathcal{G}).
\]
We then have \( C^2 = \delta C^1 \), showing that \( C^2 \) is trivial. Hence \( \beta \) is equivalent to \( \beta_0 \), as we wanted to show. \( \blacksquare \)

Using Theorem 3.1, we conclude that there is only one equivalence class of star products whose first order term is the Kirillov Poisson bracket. All these star products give algebra structures on the polynomials on \( \mathcal{G}^* \) isomorphic to \( U_{[\hbar]} \).

### 4 Star products on the orbit

A star product on the orbit \( \Theta \) associated to the deformation of Theorem 2.1 is given by a linear isomorphism
\[
\tilde{\psi} : \mathbb{C}[\Theta]\lbrack \hbar \rbrack \rightarrow U_{[\hbar]} / \mathcal{I}_{[\hbar]}.
\]
In particular, if \( \{ x_{i_1} \cdots x_{i_k}, (i_1, \ldots, i_k) \in S \} \) is a basis of \( \mathbb{C}[\Theta] \) for some set of multiindices \( S \), then \( \{ X_{i_1} \cdots X_{i_k}, (i_1, \ldots, i_k) \in S \} \) is a basis of \( U_{[\hbar]} / \mathcal{I}_{[\hbar]} \). This defines a particular isomorphism \( \tilde{\psi}(x_{i_1} \cdots x_{i_k}) = X_{i_1} \cdots X_{i_k} \) and the corresponding star product.

This star product can be seen as the restriction to the orbit of a star product on \( \mathbb{C}[\mathcal{G}^*] \). We have only to extend the map \( \tilde{\psi} \) to an isomorphism
Clearly, $\psi$ is an isomorphism if the corresponding Casimir is $p$-basis of $C^*_G$. The subalgebra of invariant polynomials on $\Theta$ is $\mathcal{I}_0$, and $\mathcal{I}_0$ and $\mathcal{I}_{[\hbar]}$ are also isomorphic as $\mathbb{C}[[\hbar]]$-modules.

We have then that the following diagram

\[
\begin{array}{ccc}
\mathbb{C}[\mathcal{G}^*][[\hbar]] & \xrightarrow{\psi} & U_{[\hbar]} \\
\downarrow \pi & & \downarrow \pi_{\hbar} \\
\mathbb{C}[\Theta][[\hbar]] & \xrightarrow{\psi} & U_{[\hbar]}/\mathcal{I}_{[\hbar]}
\end{array}
\] (7)

commutes. In general, we say that a star product on $\mathcal{G}^*$ is *tangential* to the orbit $\Theta$ if it defines a star product on $\Theta$ by restriction. So the star product in (7) is tangential.

**Example 4.1 Star product on an orbit of SU(2).**

Consider the Lie algebra of SU(2),

\[
[X, Y] = Z, \quad [Y, Z] = X, \quad [Z, X] = Y.
\]

The subalgebra of invariant polynomials on $\mathcal{G}^*$ is generated by $p = x^2 + y^2 + z^2$, so the corresponding Casimir is $P = X^2 + Y^2 + Z^2$. We consider the orbit $p = c^2$, $c \in \mathbb{R}$, $c \neq 0$. A basis of $\mathcal{I}_0$ is $B_1 = \{x^r y^s z^t(p - c^2), \ r, s, t = 0, 1, 2, \ldots \}$ and one can complete it to a basis in $\mathcal{G}^*$ by adding $B_2 = \{x^r y^s z^\nu, \ \nu = 0, 1, \ r, s = 0, 1, 2, \ldots \}$. The equivalence classes of the elements in $B_2$ are a basis of $\mathbb{C}[[\mathcal{G}^*]]/\mathcal{I}_0$.

Let $\mathcal{I}_{[\hbar]}$ be the ideal in $U_{[\hbar]}$ generated by $P - c^2$. We can define the isomorphism $\psi : \mathbb{C}[\mathcal{G}^*][[\hbar]] \rightarrow U_{[\hbar]}$ as

\[
\psi(x^r y^s z^t(p - c^2)) = X^r Y^s Z^t(P - c^2), \quad r, s, t = 0, 1, 2, \ldots
\]

\[
\psi(x^r y^s z^\nu) = X^r Y^s Z^\nu, \quad \nu = 0, 1, \ r, s = 0, 1, 2, \ldots
\] (8)

Clearly $\psi(\mathcal{I}_0) = \mathcal{I}_{[\hbar]}$, so the star product defined by $\psi$ is tangential to the orbit. It is easy to check that if we move to a neighboring orbit, $p = c^2$, then $\psi$, as defined in (8) doesn’t preserve the new ideal, that is, $\psi(\mathcal{I}_0) \neq \mathcal{I}_{[\hbar]}$.

One can construct a star product that is tangential to all the orbits in a neighborhood of the regular orbit (this is in fact the definition of “tangential star product” given in [3]). If $p_i = c_i^0, \ i = 1, \ldots m$ define the regular orbit $\Theta(c_1^0, \ldots c_m^0)$ with ideal $\mathcal{I}(c_1^0, \ldots c_m^0)$ one can construct a map $\psi$ such that

\[
\psi(\mathcal{I}(c_1, \ldots c_m)) = \mathcal{I}(c_1, \ldots c_m, \hbar)
\] (9)
for \((c_1, \ldots, c_m)\) in a neighborhood of \((c_1^0, \ldots, c_m^0)\) and \(\mathcal{I}_{(c_1, \ldots, c_m),h}\) an ideal in \(U_h\) of the type required in Theorem 2.1. The construction follows similar lines to the one in [11]. We consider the decomposition \(C^*[G^*] = I \otimes H\) where \(I\) is the subalgebra of invariant polynomials as in (2) and \(H\) is the set of harmonic polynomials (this result is due to Kostant [19]). Harmonic polynomials are in one to one correspondence with the polynomials on the orbit, so we have in fact

\[
C^*[G^*] \approx I \otimes C[\Theta(c_1^0, \ldots, c_m^0)].
\]

Consider now the basis in \(I\) \(\{(p_{i_1} - c_{i_1})(p_{i_k} - c_{i_k}), i_1 \leq \cdots \leq i_k\}\) and the basis in \(C[\Theta(c_1^0, \ldots, c_m^0)]\) as before, \(\{x_{j_1} \cdots x_{j_l}, (j_1, \ldots, j_l) \in S\}\). We define the \(\mathbb{C}[[h]]\)-module isomorphism

\[
\psi((p_{i_1} - c_{i_1})(p_{i_k} - c_{i_k}) \otimes x_{j_1} \cdots x_{j_l}) = (P_{i_1} - c_{i_1}(h)) \cdots (P_{i_k} - c_{i_k}(h)) \otimes (X_{j_1} \cdots X_{j_l}).
\]

(10)

It is obvious that it preserves the ideal, \(\psi(\mathcal{I}(c_1^0, \ldots, c_m^0)) = \mathcal{I}(c_1^0, \ldots, c_m^0), h\). A closer look reveals that, in fact \(\psi(\mathcal{I}(c_1, \ldots, c_m)) = \mathcal{I}(c_1, \ldots, c_m), h\), and then \(\tilde{\psi}\) in (7) is well defined for any \((c_1, \ldots, c_m)\) in a neighborhood of \((c_1^0, \ldots, c_m^0)\). Consequently we have a star product that is tangential to all the orbits in a neighborhood of the regular orbit.

In [11] it is shown that for SU(2) a star product of this type (with \(c_{i_1}(h) = c_{i_1}^0\)) is not differential. More generally, it was shown in [13] the following theorem

**Theorem 4.1** If \(G\) is a semisimple Lie algebra there is no differential star product on any neighborhood of the origin in \(G^*\) which is tangential to the coadjoint orbits.

The only property of tangential star products that is used in the proof of this theorem is that if \(f\) is a function that is constant on the orbits (in particular, the quadratic Casimir \(p_1\)), then, \(g \star f = gf\). It is easy to show that the tangential star products defined by (10) satisfy this property on all \(G^*\) (in particular in a neighborhood of 0), so they are not differential.

On any regular Poisson manifold there exists a star product that is tangential and differential [3]. But on all of \(G^*\), which is not regular, Theorem 4.1 states that a star product with both properties does not exist. To induce a star product on a particular orbit, it is enough to assume that the
star product on $\mathcal{G}^*$ is tangent to only such orbit. One can find star products on $\text{Pol}(\mathcal{G}^*)$ isomorphic to $U_h$ that restrict well to only one orbit (in the sense of (7)). Example 4.2 shows one of such star products for $\mathcal{G}=\text{su}(2)$. We prove that it is not differential, so at least in this case, the relaxation of the tangentiality condition does not allow in general for differentiability. In Section 5 we will investigate how these deformations are related to differential deformations on the orbit.

Example 4.2 Non differential star product on $\mathcal{G}^*=\text{su}(2)^*$. 

Consider again the Lie algebra of SU(2), with the same notation, and the orbit $\Theta^r$ given by $p=1$. It is a 2-sphere in $\mathbb{R}^3$. Fix the star product $\ast$ on $\Theta$ by choosing the $C[[h]]$-isomorphism

$$\tilde{\psi} : C[\Theta][[h]] \longrightarrow U[[h]]/I[[h]]$$

$$x^n y^m z^\nu \mapsto X^n Y^m Z^\nu, \quad \nu = 0, 1, m, n = 0, 1, 2, \ldots$$

We regard the cartesian coordinates $x$ and $y$ as functions on the sphere and let $V$ be an open set in $\Theta$ where $(x, y)$ are coordinates. On this open set $V$ the 1-forms $dx$ and $dy$ form a basis for the module of 1-forms. Let $\partial_x$ and $\partial_y$ be the elements of the dual basis, that is, $\partial_x$ and $\partial_y$ are vector fields on $V$ such that

$$\langle \partial_x, dx \rangle = \langle \partial_y, dy \rangle = 1, \quad \langle \partial_y, dx \rangle = \langle \partial_x, dy \rangle = 0.$$

Any differential operator on $V$ is an element of the algebra generated by functions and by $\partial_x$ and $\partial_y$. The advantage of $\partial_x$ and $\partial_y$ is that they behave well on polynomials in $x$ and $y$. We have

$$0 = \partial_x(1) = \partial_x(x^2 + y^2 + z^2) = 2x + 2z\partial_x(z),$$

hence $\partial_x(z) = -\frac{z}{2}$ and $\partial_y(z) = -\frac{z}{2}$. Observe that $\partial_x$ and $\partial_y$ commute.

Assume that $\ast$ is differential,

$$f \ast g = \sum_{i \geq 0} h^i B_i(f, g)$$

where $B_i$ are bidifferential operators. To determine $B_i$ it is enough to compute them on the monomials $x$ and $y$. With the following lemma we compute $B_1$. 

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Lemma 4.1 Let \( p_1, p_2 \) be two polynomials in \( x \) and \( y \), then we have:

\[
p_1 \star p_2 = p_1 p_2 - h z \partial_y (p_1) \partial_x (p_2) \mod(h^2)
\]

Proof. It is enough to show it for \( p_1, p_2 \) monomials. Let \( p_1 = x^m y^n, p_2 = x^r y^s \).

We use induction on \( N = m + r \). For \( N = 0 \) it is clear. Let \( N > 0 \). By the definition of \( \star \),

\[
p_1 \star p_2 = x^m y^n \star x^r y^s = x^n (y^m \star x^r) y^s = x^n [(y^m \star x^{r-1}) \star x] y^s.
\]

By induction we have:

\[
p_1 \star p_2 = x^n [(x^{r-1} y^m - h z m (r-1)) y^{m-1} x^{r-2}) \star x] y^s \mod(h^2),
\]

and by induction again we have:

\[
p_1 \star p_2 = x^n [x^{r-1} y^m - h z m (r-1) y^{m-1} x^{r-2}] y^s \mod(h^2)
= x^n [x^{r-1} y^m - h z m y^{m-1} x^{r-2}] y^s \mod(h^2)
= x^{n+r} y^{m+s} - h z m y^{m-1+s} x^{r-1+n} \mod(h^2),
\]

which is what we wanted to prove. \(\blacksquare\)

According to the previous lemma

\[
z \star z = z^2 - h \frac{x y}{z} \mod(h^2) = 1 - x^2 - y^2 - h \frac{x y}{z} \mod(h^2),
\]

on the other hand, by definition,

\[
z \star z = \tilde{\psi}^{-1} (\tilde{\psi}(z) \tilde{\psi}(z)) = \tilde{\psi}^{-1} (Z^2) = 1 - x^2 - y^2,
\]

a contradiction that shows that \( \star \) cannot be differential.

5 Algebraic and differential star products on the regular orbit.

Let us consider the regular orbit \( \Theta^r \) as a symplectic manifold. By Theorem 3.1 we can associate to the Poisson structure a (equivalence class of) differential star product. It was already known \([3, 4]\) that a star product exists for any symplectic manifold. In fact, differential star products are not in general
unique. The space of equivalence classes of differential star products such that
\[ f \star g - g \star f = h\{f, g\}, \]
being \( \{\cdot, \cdot\} \) a symplectic Poisson bracket, are classified by the sequences \( \{\omega_n\}_{n \geq 1} \) of de Rham cohomology classes in \( H^2(M) \) such that \( \omega_1 \) is the symplectic form associated to the Poisson bracket. In fact, the space of equivalence classes of star products is a principal homogeneous space under the group \( H^2(M[[h]]) \). \[ \text{[6, 20]} \]

The symplectic two form is not defined in arbitrary Poisson manifolds, so the natural structure to consider is the Poisson bivector. We want to describe the space of equivalence classes of star products for symplectic manifolds in terms of the Poisson bivector, being this approach closer to the one of Kontsevich’s theorem for arbitrary manifolds. Let \( M \) be a symplectic manifold and consider \( \omega_h = \sum_{j \geq 0} h^j \omega_j \in H^2(M[[h]]) \), where \( \omega_0 \) is the original symplectic two form and \( \omega_j \) are closed two forms. Since \( \omega_0 \) is non degenerate, \( \omega_h \) defines an invertible map between tangent and cotangent vector fields in the usual way,
\[ \mu_h : \Gamma(TM[[h]]) \rightarrow \Gamma(T^*M[[h]]), \]
which can be extended to tensors. In fact, by closedness of \( \omega_h \), the map
\[ (f, g) \mapsto \{f, g\}_h = h \omega_h(\mu_h^{-1}(df), \mu_h^{-1}(dg)) \]
is a formal Poisson structure in the sense of Kontsevich and this formal Poisson structure is gauge equivalent to zero (the gauge group is the group of formal paths in the diffeomorphism group starting with the identity) if and only if all the \( \omega_j \) are exact. We have then that the set of equivalence classes in \( H^2(M[[h]]) \) is in one to one correspondence with the set of formal Poisson structures modulo the action of the gauge group. So for symplectic manifolds both descriptions, in terms of the symplectic form or in terms of the Poisson bivector are equivalent.

Coadjoint orbits of compact groups are an example of manifolds that admit inequivalent quantizations. In fact they have a nontrivial de Rham cohomology \( H^2(\Theta^r) \). In particular, the symplectic form is a closed, non exact form, so we have many inequivalent deformations.

Let \( \Theta^r \) be the orbit defined by
\[ p_i(x_1, \ldots x_n) = c_i^0, \quad c_i^0 \in \mathbb{R}, \quad i = 1, \ldots m, \]
Θᵣ is regular if and only if the differentials \( dp_i \) are independent. One can consider all the regular orbits given by the constraints \( p_i = c_i, \, i = 1, \ldots, m \) with \((c_1, \ldots, c_m)\) in a neighborhood of \((c^0_1, \ldots, c^0_m)\) where the differentials are still independent. The set of these points is a neighborhood \( \mathcal{N}_{Θᵣ} \approx Θᵣ × \mathbb{R}^m \) of the regular orbit. \( \mathcal{N}_{Θᵣ} \) is a regular Poisson manifold. The Poisson structure on \( \mathcal{N}_{Θᵣ} \) can be seen as a symplectic structure on \( Θᵣ \) which depends on certain parameters, the invariant polynomials \( p_i \), which determine the leaf of the foliation in \( \mathcal{N}_{Θᵣ} \).

We now want to examine various star products that can be defined on the open set \( \mathcal{N}_{Θᵣ} \). We can consider the star product \( *_S \) induced by \( U_h \) by means of the map \( (3) \). It is differential, but not tangential. It was shown in Ref. [9] that the canonical deformation of the Kirillov Poisson structure on \( \mathcal{G}^* \) is isomorphic to \( U_h \).

We can also consider the quantization of the Kirillov symplectic structure on the orbit given by Kontsevich’s theorem. From the local expression of Kontsevich’s quantization, one can see that it is a smooth with respect to the parameters \( p_i \). Interpreting the parameters as transverse coordinates, Kontsevich’s theorem applied on \( Θᵣ \) gives indeed a star product on \( \mathcal{N}_{Θᵣ} \) that is tangential and differential. We denote it by \( *_T \).

Finally we can consider the star product \( *_P \) on \( \mathcal{N}_{Θᵣ} \), induced by a map \( ψ \) as in formula \( (10) \). \( *_P \) is tangential to the orbit, but, in general, not differential. To sum up we get Table 1.

| \( *_S \) | Isomorphic to \( U_h \) (on the polynomials), induced by Sym. | Defined on all \( \mathcal{G}^* \), differential, not tangential. |
| \( *_P \) | Isomorphic to \( U_h \), induced by a map \( ψ \) like \( (10) \). | Defined on all \( \mathcal{G}^* \), not differential (only given on polynomials), tangential to the orbits. |
| \( *_T \) | Gluing Kontsevich construction on the leaves. | Defined on \( \mathcal{N}_{Θᵣ} \), differential, tangential. |

Table 1: Star products on \( \mathcal{N}_{Θᵣ} \)

The relation among these star products on \( \mathcal{N}_{Θᵣ} \) and the corresponding star products induced on the orbit \( Θᵣ \) will be studied in [24].
6 Summary

In this paper we consider different methods of quantization for regular orbits of compact semisimple Lie groups. From the algebraic point of view, one can obtain non isomorphic deformations of the same Poisson structure. These deformations can be compared with geometric quantization since the formulation is in terms of a certain prime ideal in the enveloping algebra. The comparison with differential deformations becomes more difficult since the polynomials are "global" objects, very different from the "local" $C^\infty$ functions, and in fact we see that the star products obtained are not differential in general. At the end we define three star products on a regularly foliated neighborhood of the orbit.

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References

[1] D. Vogan, *The Orbit Method and Unitary Representations for Reductive Lie groups*, in "Algebraic and Analytic Methods in Representation Theory". Perspectives in Mathematics, Vol 17. Academic Press. (1996).

[2] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, D. Sternheimer, *Deformation Theory and Quantization. I and II*. Ann. Phys., 111 Vol. 1 61-151 (1978).

[3] C. Fronsdal *Some ideas about Quantization*. Reports on Math. Phys. 15, Vol. 1 111-145 (1978).

[4] J. M. García Bondía and J. C. Varilly, *From Geometric Quantization to Moyal Quantization*. J. Math. Phys. 36 2691-2701 (1995).

[5] H. Omori, Y. Maeda and A. Yoshhioka, *Weyl Manifolds and Deformation Quantization*. Adv. in Math. 85 224-255 (1991).

[6] M. De Wilde, P. B. A. Lecomte, *Existence of Star Products and of Formal Deformations of the Poisson Lie Algebra of Arbitrary Symplectic Manifolds*. Lett. Math. Phys., 7 487-49, (1983).
[7] B. Fedosov, *A Simple Geometric Construction of Deformation Quantization*. J. Diff. Geom. **40** Vol. 2 213-238, (1994).

[8] M. Masmoudi, *Tangential deformation of Poisson bracket and tangential star-products on a regular Poisson manifold*. J. Geom. Phys. **9** 155-171 (1992).

[9] M. Kontsevich, *Deformation Quantization of Poisson Manifolds*. math.QA/9709040.

[10] S. Gutt *An explicit *-product on the cotangent bundle of a Lie group*. Lett. Math. Phys. **7** 249-258 (1983).

[11] M. Cahen, S. Gutt, *Produits * sur les Orbites des Groupes Semi-Simples de Rang 1*, C.R. Acad. Sc. Paris **296** (1983), série I, 821-823; *An Algebraic Construction of * Product on the Regular Orbits of Semisimple Lie Groups*. In “Gravitation and Cosmology”. Monographs and Textbooks in Physical Sciences. A volume in honor of Ivor Robinson, Bibliopolis. Eds W. Rundler and A. Trautman, (1987); *Non Localité d’une Déformation Symplectique sur la Sphère S²*. Bull. Soc. Math. Belg. **36** B 207-221 (1987).

[12] J. Hoppe, *Quantum Theory of a Massless Relativistic Surface and a Two Dimensional Bound State Problem*. MIT PhD thesis, (1982).

[13] M. Cahen, S. Gutt, J. Rawnsley, *On Tangential Star Products for the Coadjoint Poisson Structure*. Comm. Math. Phys, **180** 99-108 (1996).

[14] D. Arnal, J. Ludwig and M. Masmoudi *Déformations covariantes sur les orbites polarisées d’un groupe de Lie*. J. Geom. Phys. **14** 309-331 (1994).

[15] M. Cahen, S. Gutt and J. Rawnsley, *Quantization of Kaehler manifold I to IV*. J. Geom. Phys. **7** Vol.1 45-62 (1990); Trans. Am. Math. Soc. **337** Vol.1 (1993) 73-98; Lett. Math. Phys. **30** 291-305 (1994) ; Lett. Math. Phys. **34** Vol.2 159-168 (1995).

[16] R. Fioresi and M. A. Lledó, *On the deformation Quantization of Coadjoint Orbits of Semisimple Lie Groups*. To appear in the Pacific J. of Math. math.QA/9906104.

[17] F. A. Berezin, *Quantization*. Math. USSR Izvestija, **8** 1109-1165 (1974).
[18] V. S. Varadarajan, Lie groups, Lie Algebras and their Representations. Springer Verlag, (1984).

[19] B. Kostant, Lie Group Representations on Polynomial Rings. Am. J. Math. 85 327 (1978).

[20] P. Deligne, Déformations de l’Algèbre des Fonctions d’une Variété Symplectique: Comparaison entre Fedosov et De Wilde, Lecomte. Selecta Math. New series 1 No.4 667-697 (1995).

[21] E. Arbarello, G. Bini, P. de Bartolomeis, A. Canocaco, F. De Vita, D. Fiorenza, G. Gaiffi, M. Grassi, M. Manetti, M. Polito, R. Silvotti, Notes of Seminars Held at the Scuola Normale di Pisa (1998-1999).

[22] C. Kassel, Quantum Groups. Springer-Verlag (1995).

[23] B. Fedosov, Proof of the index theorem for deformation quantization, in Boundary Value Problems, Schrödinger Operators, Deformation Quantization, Advances in Partial Differential Equations, Akademie Verlag, Berlin, (1995).

[24] R. Fioresi and M.A. Lledó, in preparation.