On some cocycles which represent the Dixmier-Douady class in simplicial de Rham complexes

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Abstract

There is a cocycle in the simplicial de Rham complex which represents the Dixmier-Douady class. We explain that this cocycle coincides with a kind of transgression of the second Chern class when we consider a central extension of the loop group and a connection form due to J. Mickelsson and J-L. Brylinski, D. McLaughlin. After that we construct also a cocycle in a certain triple complex. Furthermore, we explain the relationship between our cocycle and the another description of the Dixmier-Douady class due to K. Behrend and P. Xu.

1 Introduction

It is well known that for any Lie group $G$, we can define a simplicial manifold $\{NG(*)\}$ and we can see the cohomology group of the classifying space $BG$ is isomorphic to the total cohomology of the double complex $\{\Omega^q(NG(p))\}$. See [3] [8] [13] for details. On the other hand, in [6] Carey, Crowley, Murray proved that when a Lie group $G$ admits a central $U(1)$-extension $1 \to U(1) \to \hat{G} \to G \to 1$, there exists a characteristic class of principal $G$-bundle $\pi : Y \to M$ which belongs to a cohomology group $H^3(M, \mathbb{Z})$. It is called the Dixmier-Douady (DD) class associated to the central $U(1)$-extension $\hat{G} \to G$. So there is a cocycle on $\Omega^*(NG(*))$ which represents the Dixmier-Douady class. We explain that this cocycle coincides with a kind of transgression of the second Chern class when we use the central extension $\hat{LSU}(2) \to LSU(2)$ and the connection form due to Mickelsson [12] and Brylinski, McLaughlin.
[1][3]. We consider also the case of semi-direct product \( LSU(2) \rtimes S^1 \) and construct a cocycle in a certain triple complex. After that, we explain the relationship between our cocycle and the another description of the DD class due to Behrend and Xu (see [1] [2] and related papers).

2 Dixmier-Douady class on the double complex

In this section we recall the relation between the simplicial manifold \( NG \) and the classifying space \( BG \), then we show that we can construct the cocycles on \( \Omega^*(NG(*)) \) which represents the DD class when \( G \) has a central \( U(1) \)-extension \( \pi : \hat{G} \to G \).

2.1 The double complex on simplicial manifold

First we define a simplicial manifold \( NG \) for a Lie group \( G \) as follows:

\[
NG(p) = G \times \cdots \times G \ni (g_1, \cdots, g_p):
\]

face operators \( \varepsilon_i : NG(p) \to NG(p-1) \)

\[
\varepsilon_i(g_1, \cdots, g_p) = \begin{cases} 
(g_2, \cdots, g_p) & i = 0 \\
(g_1, \cdots, g_i g_{i+1}, \cdots, g_p) & i = 1, \cdots, p - 1 \\
(g_1, \cdots, g_{p-1}) & i = p 
\end{cases}
\]

Then we recall how to construct a double complex associated to a simplicial manifold.

**Definition 2.1.** For any simplicial manifold \( \{X_\ast\} \) with face operators \( \{\varepsilon_\ast\} \), we define double complex as follows:

\[
\Omega^{p,q}(X) \overset{\text{def}}{=} \Omega^q(X_p)
\]

Derivatives are:

\[
d' := \sum_{i=0}^{p+1} (-1)^i \varepsilon_i^*, \quad d'' := (-1)^p \times \text{the exterior differential on } \Omega^*(X_p).
\]
For \( NG \) the following holds \([3][8][13]\).

**Theorem 2.1.** There exists a ring isomorphism

\[
H(\Omega^*(NG)) \cong H^*(BG).
\]

Here \( \Omega^*(NG) \) means the total complexes and \( BG \) means the classifying space of principal \( G \)-bundles.

**Remark 2.1.** To prove this theorem, they used the property that \( G \) is an ANR (absolute neighborhood retract) and the theorem of de Rham on \( G \) holds true.

2.2 The cocycle on the double complex

Let \( \pi : \hat{G} \to G \) be a central \( U(1) \)-extension of a Lie group \( G \). Following \([5][6]\), we recognize it as a \( U(1) \)-bundle. Using the face operators \( \{\varepsilon_i\} : NG(2) \to NG(1) = G \), we can construct the \( U(1) \)-bundle over \( NG(2) = G \times G \) as \( \delta\hat{G} := \varepsilon_0^*\hat{G} \otimes (\varepsilon_1^*\hat{G})^{\otimes -1} \otimes \varepsilon_2^*\hat{G} \). Here we define the tensor product \( S \otimes T \) of \( U(1) \)-bundles \( S \) and \( T \) over \( M \) as

\[
S \otimes T := \bigcup_{x \in M} (S_x \times T_x/(s,t) \sim (su, tu^{-1}), (u \in U(1))).
\]

**Lemma 2.1.** \( \delta\hat{G} \to G \times G \) is a trivial bundle.

**Proof.** See \([19]\). \( \square \)

**Remark 2.2.** \( \delta(\delta\hat{G}) \) is canonically isomorphic to \( G \times G \times G \times U(1) \) because \( \varepsilon_i \varepsilon_j = \varepsilon_{j-1} \varepsilon_i \) for \( i < j \).

For any connection \( \theta \) on \( \hat{G} \), there is the induced connection \( \delta\theta \) on \( \delta\hat{G} \) \([4][Brylinski]\).

**Proposition 2.1.** Let \( c_1(\theta) \) denote the 2-form on \( G \) which hits \( \left(\frac{-1}{2\pi i}\right)d\theta \in \Omega^2(\hat{G}) \) by \( \pi^* \), and \( \hat{s} \) a global section of \( \delta\hat{G} \) such that \( \delta\hat{s} := \varepsilon_0^*\hat{s} \otimes (\varepsilon_1^*\hat{s})^{\otimes -1} \otimes \varepsilon_2^*\hat{s} \otimes (\varepsilon_3^*\hat{s})^{\otimes -1} = 1 \). Then the following equations hold.

\[
(\varepsilon_0^* - \varepsilon_1^* + \varepsilon_2^*)c_1(\theta) = \left(\frac{-1}{2\pi i}\right)d(\hat{s}^*(\delta\theta)) \in \Omega^2(NG(2)).
\]

\[
(\varepsilon_0^* - \varepsilon_1^* + \varepsilon_2^* - \varepsilon_3^*)(\hat{s}^*(\delta\theta)) = 0.
\]
Proof. See [14],[15] or [19].

The propositions above give the cocycle \( c_1(\theta) - \left( \frac{-1}{2\pi i} \right) \hat{s}^*(\delta \theta) \in \Omega^3(NG) \) below.

\[
\begin{array}{c}
0 \\
\uparrow -d
\end{array}
\begin{array}{c}
c_1(\theta) \in \Omega^2(G) \\
\uparrow d
\end{array}
\begin{array}{c}
\Omega^2(G \times G) \\
\uparrow -d
\end{array}
\begin{array}{c}
- \left( \frac{-1}{2\pi i} \right) \hat{s}^*(\delta \theta) \in \Omega^1(G \times G) \\
\uparrow d
\end{array}
\begin{array}{c}
0
\end{array}
\]

Lemma 2.2. The cohomology class \([c_1(\theta) - \left( \frac{-1}{2\pi i} \right) \hat{s}^*(\delta \theta)] \in H^3(\Omega(NG))\) does not depend on \(\theta\).

Proof. See [19].

Errata 1. In [19], the author insisted that \(\hat{s}\) can be any section, but we must take \(\hat{s}\) as above. Besides, the author apologize that there are some confusions of the signature on the diagram of the double complex in [19]. The statement of Theorem 4.2 in it also must be modified to “The transgression map of the universal bundle \(EG \to BG\) maps the Dixmier-Douady class to \((-1)\) times of the first Chern class of \(\hat{G} \to G\).”

Now we consider what happens if we change the section \(\hat{s}\). There is a natural section \(\hat{s}_{nt}\) of \(\delta \hat{G}\) defined as:

\[
\hat{s}_{nt}(g_1, g_2) := (((g_1, g_2), \hat{g}_2), ((g_1, g_2), \hat{g}_1 \hat{g}_2) \otimes^{-1}, ((g_1, g_2), \hat{g}_1)).
\]

Then any other section \(\hat{s}\) such that \(\delta \hat{s} = 1\) can be represented by \(\hat{s} = \hat{s}_{nt} \cdot \varphi\) where \(\varphi\) is a \(U(1)\)-valued smooth function on \(G \times G\) which satisfies \(\delta \varphi = 1\). If we pull back \(\delta \theta\) by \(\hat{s}\), the equation \(\hat{s}^*(\delta \theta) = \hat{s}_{nt}^*(\delta \theta) + d\log \varphi\) holds. If there exists a \(U(1)\)-valued smooth function \(\varphi'\) on \(G\) which satisfies \(\delta \varphi' = \varphi\), the cohomology class \([- \left( \frac{1}{2\pi i} \right) d\log \varphi\] is equal to 0 in \(H^3(\Omega(NG))\). So we have the following proposition.

Proposition 2.2. Up to the cohomology class in the smooth cohomology \(H^2(G, U(1))\), the cohomology class \([c_1(\theta) - \left( \frac{-1}{2\pi i} \right) \hat{s}^*(\delta \theta)] \in H^3(\Omega(NG))\) is decided uniquely by the central \(U(1)\)-extension \(\hat{G} \to G\).
2.3 Dixmier-Douady class

We recall the definition of the Dixmier-Douady class, following [6]. Let \( \phi : Y \to M \) be a principal \( G \)-bundle and \( \{U_\alpha\} \) a Leray covering of \( M \). When \( G \) has a central \( U(1) \)-extension \( \pi : \hat{G} \to G \), the transition functions \( g_{\alpha \beta} : U_{\alpha \beta} \to G \) lift to \( \hat{G} \). i.e. there exist continuous maps \( \hat{g}_{\alpha \beta} : U_{\alpha \beta} \to \hat{G} \) such that \( \pi \circ \hat{g}_{\alpha \beta} = g_{\alpha \beta} \). This is because each \( U_{\alpha \beta} \) is contractible so the pull-back of \( \pi \) by \( g_{\alpha \beta} \) has a global section. Now the \( U(1) \)-valued functions \( c_{\alpha \beta \gamma} \) on \( U_{\alpha \beta \gamma} \) are defined as \( (\hat{g}_{\beta \gamma}(\hat{g}_{\alpha \beta} \hat{g}_{\beta \gamma})^{-1} \hat{g}_{\alpha \beta}) \cdot c_{\alpha \beta \gamma} := \hat{g}_{\beta \gamma} \hat{g}_{\alpha \gamma}^{\alpha \beta} \). Then it is easily seen that \( \{c_{\alpha \beta \gamma}\} \) is a \( U(1) \)-valued \( \check{\text{C}}ech \)-cocycle on \( M \) and hence defines a cohomology class in \( H^2(M,U(1)) \cong H^3(M,\mathbb{Z}) \). This class is called the Dixmier-Douady class of \( Y \).

Remark 2.3. Let \( s_{\alpha \beta \gamma} \) be a section of \( \hat{G}_{\alpha \beta \gamma} := g_{\beta \gamma}^* \hat{G} \otimes (g_{\alpha \gamma}^* \hat{G})^{-1} \otimes g_{\alpha \beta}^* \hat{G} \) such that \( \delta s_{\alpha \beta \gamma} := s_{\beta \gamma \delta} \otimes s_{\gamma \delta \Theta} \otimes s_{\alpha \beta \Theta} \otimes s_{\alpha \beta \Theta} = 1 \). This condition makes sense since \( \hat{G}_{\beta \gamma \delta} \otimes \hat{G}_{\alpha \gamma \Theta}^{-1} \otimes \hat{G}_{\alpha \beta \Theta} \otimes \hat{G}_{\alpha \beta \Theta} \) is canonically trivial. Then we can define a \( U(1) \)-valued \( \check{\text{C}}ech \)-cocycle \( c_{\alpha \beta \gamma}^s \) on \( M \) by the equation \( s_{\alpha \beta \gamma} \cdot c_{\alpha \beta \gamma}^s = \hat{g}_{\beta \gamma} \hat{g}_{\alpha \gamma}^{\alpha \beta} \). The cohomology class \( [c_{\alpha \beta \gamma}^s] \in H^2(M,U(1)) \cong H^3(M,\mathbb{Z}) \) can be also called the Dixmier-Douady class of \( Y \).

To obtain a non-torsion class \( G \) must be infinite dimensional (cf. for example [4] Ch.4 p.166) and we require \( G \) to have a partition of unity so that we can consider a connection form on the \( U(1) \)-bundle over \( G \). A good example which satisfies such a condition is the loop group of a finite dimensional compact Lie group and the restricted unitary group \( U_{res}(H) \) [4] [17] (We can also see that these groups are ANR and the theorem of de Rham holds on them [11] [16]).

We fix any section \( \hat{s} \) of \( \delta \hat{G} \) which satisfies \( \delta \hat{s} = 1 \). Since \( g_{\beta \gamma}^* \hat{G} \otimes (g_{\alpha \gamma}^* \hat{G})^{-1} \otimes g_{\alpha \beta}^* \hat{G} \) is the pull-back of \( \delta \hat{G} \) by \( (g_{\alpha \beta}, g_{\beta \gamma}) : U_{\alpha \beta \gamma} \to G \times G \), there is the induced section of \( g_{\beta \gamma}^* \hat{G} \otimes (g_{\alpha \gamma}^* \hat{G})^{-1} \otimes g_{\alpha \beta}^* \hat{G} \). So we can define the Dixmier-Douady class by using this section.

In [19], the theorem below was shown.

**Theorem 2.2.** The cohomology class \( [c_1(\theta) - (\frac{1}{2\pi}) \hat{s}^*(\delta \theta)] \in H^3(\Omega(NG)) \) represents the universal Dixmier-Douady class associated to \( \pi \) and a section
3 The String class

Using the idea of Brylinski, McLaughlin [5] and Murray, Stevenson [14][15], we discuss the case of a central $U(1)$-extension of a loop group.

3.1 In the case of special unitary group

It’s known that the second Chern class $c_2 \in H^4(BSU(2))$ of the universal $SU(2)$-bundle $ESU(2) \to BSU(2)$ is represented in $\Omega^4(NSU(2))$ as the sum of following $C_{1,3}$ and $C_{2,2}$ (see for example [10] or [18]):

$$C_{1,3} = \left(\frac{-1}{2\pi i}\right)^2 \frac{-1}{6} \text{tr}(h^{-1}dh)^3,$$

$$C_{2,2} = \left(\frac{-1}{2\pi i}\right)^2 \frac{1}{2} \text{tr}(h_2^{-1}h_1^{-1}dh_1dh_2).$$

Pulling back this cocycle by the evaluation map $ev : LSU(2) \times S^1 \to SU(2), (\gamma, z) \mapsto \gamma(z)$ and integrating along the circle, we obtain the cocycle in $\Omega^3(NLSU(2))$. Here $LSU(2)$ is the free loop group of $SU(2)$ and the map $\int_{S^1} ev^*$ is also called the transgression map.

Now we pose the following problem. Is there corresponding central extension of $LSU(2)$ and connection form on it such that the Dixmier-Douady class in $\Omega^3(NLSU(2))$ constructed previous section coincides with $\int_{S^1} ev^*(C_{1,3} + C_{2,2})$? In this section, we explain that the central extension and the connection form constructed by Mickelsson and Brylinski, McLaughlin in [4] [5] [12] meet such a condition.

To begin with, we recall the definition of the $U(1)$-bundle $\pi : Q(\nu) \to LSU(2)$ and the multiplication $m : Q(\nu) \times Q(\nu) \to Q(\nu)$ in [4] [5]. We fix any
based point \( x_0 \in SU(2) \) and denote \( \gamma_0 \in SU(2) \) the constant loop at \( x_0 \).

For any \( \gamma \in SU(2) \), we consider all paths \( \sigma_\gamma : [0, 1] \to SU(2) \) that satisfies \( \sigma_\gamma(0) = \gamma_0 \) and \( \sigma_\gamma(1) = \gamma \).

Then the equivalence relation \( \sim \) on \( \{\sigma_\gamma\} \times S^1 \) is defined as follows:

\[
(\sigma_\gamma, z) \sim (\sigma'_\gamma, z') \iff z = z' \cdot \exp \left( \int_{I^1 \times S^1} 2\pi i F^* \nu \right).
\]

Here \( F : I^1 \times S^1 \to SU(2) \) is any homotopy map that satisfies \( F(0, t, z) = \sigma_\gamma(t)(z) \), \( F(1, t, z) = \sigma'_\gamma(t)(z) \) and \( \nu \) is \( C_{1,3} = \left( \frac{-1}{2\pi^2} \right)^2 \frac{1}{6} \text{tr}(h^{-1} dh)^3 \).

It’s well-known \( \nu \in \Omega^3(SU(2)) \) is a closed, integral form hence this relation is well-defined. Now the fiber \( \pi^{-1}(\gamma) \) of \( Q(\nu) \) is defined as the quotient space \( \{\sigma_\gamma\} \times S^1 / \sim \).

We can adapt the same construction for any closed integral 3-form on \( SU(2) \).

Let \( \eta, \eta' \) be such forms and suppose there is a 2-form \( \beta \) with \( d\beta = \eta' - \eta \).

Then the isomorphism from \( Q(\eta) \) to \( Q(\eta') \) is constructed as:

\[
[(\sigma_\gamma, z)]_\eta \mapsto [(\sigma_\gamma, z \cdot \exp \left( \int_{I^1 \times S^1} 2\pi i \sigma_\gamma^* \beta \right))]_{\eta'}.
\]

Here we regard \( \sigma_\gamma \) as a map from \([0, 1] \times S^1 \) to \( SU(2) \).

For the face operators \( \{\varepsilon_i\} : SU(2) \times SU(2) \to SU(2) \) (we use the same notation for the face operators \( \text{LSU}(2) \times \text{LSU}(2) \to \text{LSU}(2) \)), we can check \( \varepsilon_0^* Q(\nu) \otimes \varepsilon_1^* Q(\nu)^{\otimes -1} \otimes \varepsilon_2^* Q(\nu) \) is isomorphic to \( Q(\varepsilon_0\nu - \varepsilon_1\nu + \varepsilon_2\nu) = Q(-dC_{2,2}) \) over \( \text{LSU}(2) \times \text{LSU}(2) \). The isomorphism from \( Q(0) \) to \( Q(-dC_{2,2}) \) is given by

\[
[(\sigma_{\gamma_1}, \sigma_{\gamma_2}, z)]_0 \mapsto [(\sigma_{\gamma_1}, \sigma_{\gamma_2}, z \cdot \exp \left( \int_{I^1 \times S^1} 2\pi i (\sigma_{\gamma_1}, \sigma_{\gamma_2})^* C_{2,2} \right))]_{-dC_{2,2}}.
\]

Now we can define the section \( s_L \) of \( \varepsilon_0^* Q(\nu) \otimes \varepsilon_1^* Q(\nu)^{\otimes -1} \otimes \varepsilon_2^* Q(\nu) \) over \( \text{LSU}(2) \times \text{LSU}(2) \) as:

\[
s_L(\gamma_1, \gamma_2) := [(\sigma_{\gamma_1}, \sigma_{\gamma_2}, \exp \left( \int_{I^1 \times S^1} 2\pi i (\sigma_{\gamma_1}, \sigma_{\gamma_2})^* C_{2,2} \right))]_{-dC_{2,2}}.
\]

The multiplication \( m : Q(\nu) \times Q(\nu) \to Q(\nu) \) is defined by the following equation

\[
s_L(\gamma_1, \gamma_2) = ([\sigma_{\gamma_1}, z_1]_{\varepsilon_0^* \nu}) \otimes (\gamma_1 \gamma_2), m([\sigma_{\gamma_1}, z_1]_{\nu}, [\sigma_{\gamma_2}, z_2]_{\nu}))^{\otimes -1} \otimes ([\sigma_{\gamma_2}, z_2]_{\varepsilon_2^* \nu}).
\]
Next we recall how Brylinski and McLaughlin constructed the connection on $Q(\nu)$. Let denote $P_1 SU(2)$ the space of paths on $SU(2)$ which starts from based point $x_0$ and $f: P_1 SU(2) \to SU(2)$ a map that is defined by $f(\gamma) = \gamma(1)$. It is well known that $f$ is a fibration. Then we define the 2-form $\omega$ on $P_1 SU(2)$ as:

$$\omega_\gamma(u, v) = \int_0^1 \nu \left( \frac{d\gamma}{dt}, u(t), v(t) \right) dt.$$ 

Note that $d\omega = f^*\nu$ holds. Let $U = \{U_i\}$ be an open covering of $SU(2)$. Since $SU(2)$ is simply connected, we can take $U$ such that each $U_i$ is contractible and $\{LU_i\}$ is an open covering of $LSU(2)$. For example, we take $U = \{U_x := SU(2) - \{x\}|x \in SU(2)\}$.

Now we quote the lemma from [5].

**Lemma 3.1** (Brylinski, McLaughlin [5]). (1) There exists a line bundle $L$ over each $f^{-1}(U_i)$ with a fiberwise connection such that its first Chern form is equal to $\omega|_{f^{-1}(U_i)}$. This line bundle is called the pseudo-line bundle.

(2) There exists a connection $\nabla$ on each pseudo-line bundle $L$ such that its first Chern form $R$ satisfies the condition that $R - \omega|_{f^{-1}(U_i)}$ is basic.

Let $K$ be a 2-form on $U_i$ which satisfies $f^*K = 2\pi i(R - \omega|_{f^{-1}(U_i)})$. Then the 1-form $\theta_i$ on $LU_i$ is defined by $\theta_i := \int_{S^1} ev^*K$. It is easy to see

$$\left(\frac{1}{2\pi i}\right) d\theta_i = (\int_{S^1} ev^*\nu)|_{LU_i}.$$

There is a section $s_i$ on $LU_i$ defined by $s_i(\gamma) := [\sigma_\gamma, H_{\sigma_i}(L, \nabla)]$. Here $H_{\sigma_i}(L, \nabla)$ is the holonomy of $(L, \nabla)$ along the loop $\sigma_\gamma: S^1 \to f^{-1}(U_i)$. We also have the corresponding local trivialization $\varphi_i: \pi^{-1}(U_i) \to U_i \times U(1)$.

From above, we have the connection form $\theta$ on $Q(\nu)$ defined by $\theta|_{\pi^{-1}(U_i)} := \pi^*\theta_i + d\log(pr_2 \circ \varphi_i)$. Its first Chern form $c_1(\theta)$ is $\int_{S^1} ev^*\nu$ and $d\delta\theta$ is equal to

$$(-2\pi i) \cdot \int_{S^1} ev^*(\varepsilon_1^* - \varepsilon_2^*) = (-2\pi i) \cdot \pi^* \int_{S^1} ev^*C_{2,2}$$

hence $\delta\theta + (-2\pi i) \cdot \pi^* \int_{S^1} ev^*C_{2,2}$ is a flat connection on $\delta Q(\nu)$. Since $LSU(2)$ is simply connected, it is trivial connection so $s_i^*(\delta\theta + (-2\pi i) \cdot \pi^* \int_{S^1} ev^*C_{2,2}) = 0$.

So as a reformulation of the Brylinski and McLaughlin’s result in [5], we obtain the proposition below.

**Proposition 3.1.** Let $(Q(\nu), \theta)$ be a $U(1)$-bundle on $LSU(2)$ with connection and $s_L$ be a global section of $\delta Q(\nu)$ constructed above. Then the cocycle
c_1(θ) - \left(\frac{1}{2\pi i}\right) s_1^* (δθ) on Ω^3(NLSU(2)) is equal to ∫_{S^1} ev^* (C_{1,3} + C_{2,2}), i.e. the map ∫_{S^1} ev^* sends the second Chern class c_2 ∈ H^4(BSU(2)) to the Dixmier-Douady class (associated to Q(ν)) in H^3(BLSU(2)). □

Remark 3.1. We explain what happens if we adapt this construction to the loop group of the unitary group. In the case of unitary group \( U(2) \), the second Chern class is represented as the sum of following \( C_{1,3}^U \) and \( C_{2,2}^U \) (see [18]):

\[
\begin{align*}
0 & \quad \text{d} \\
C_{1,3}^U & \in Ω^3(U(2)) \xrightarrow{\frac{e_0^* - e_1^* + e_2^*}{2}} Ω^3(U(2) \times U(2)) \\
& \quad \text{d} \\
C_{2,2}^U & \in Ω^2(U(2) \times U(2)) \xrightarrow{e_0^* - e_1^* + e_2^*} 0 \\
C_{1,3}^U & = \left(\frac{-1}{2\pi i}\right)^2 \frac{1}{6} \text{tr}(h^{-1} dh)^3 \\
C_{2,2}^U & = \left(\frac{-1}{2\pi i}\right)^2 \frac{1}{2} \text{tr}(h_2^{-1} h_1^{-1} dh_1 dh_2) - \left(\frac{-1}{2\pi i}\right)^2 \frac{1}{2} \text{tr}(h_1^{-1} dh_1) \text{tr}(h_2^{-1} dh_2).
\end{align*}
\]

We recognize \( U(2) \) as semi-direct group \( SU(2) \rtimes U(1) \). Let denote \( ΩU(1) \) the based loop group of \( U(1) \). Then any element \( γ \) in \( LU(2) \) is decomposed as \( γ = (γ_1, γ_2, z) ∈ LSU(2) \rtimes (ΩU(1) \rtimes U(1)) \). Each connected component of \( LU(2) \) is parametrized by the mapping degree of \( γ_2 \). We write \( ΩU(1)_n, LU(2)_n \) the connected component which includes a based loop \( γ_2 \) whose mapping degree is \( n \). We can see \( π_1(LU(2)_0) = π_1(LSU(2)) \oplus π_1(LU(1)_0) = π_1(ΩU(1)_0) \oplus π_1(U(1)) \cong \mathbb{Z} \). There is a homeomorphism from \( ΩU(1)_0 \) to \( ΩU(1)_n \) defined by \( γ \mapsto γ \cdot (e^{is} \mapsto e^{ins}) \) for any \( n \) so \( π_1(LU(2)_n) \) is also isomorphic to \( \mathbb{Z} \). The generator \( ψ_n \) of \( π_1(LU(2)_n) \cong H_1(LU(2)_n) \) is the map defined as \( ψ_n(e^t) := (e^{is} \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & e^{int} \end{smallmatrix}\right)) \) hence any cycle \( a ∈ Z_1(LU(2)_n) \) can be written as \( mψ_n + ∂φ \) for some 2-chain \( φ \).

Since \( LU(2) \) is not simply connected we need the differential character \( k \) to construct a principal \( U(1) \)-bundle over \( LU(2) \). Differential character is a homomorphism from \( Z_1(LU(2)) \) to \( U(1) \) such that there exists a specific 2-form
\[ \omega \text{ satisfying } k(\partial \omega) = \exp(\int_\rho 2\pi i \omega) \text{ for any 2-singular chains } \rho \text{ of } LU(2) \] (see also \cite{13}).

We set \( \Phi := \int_{S^1} ev^* C_{1,3}^U \). If we define \( k \) as \( k(a) := \exp(\int_0^a 2\pi i \Phi) \), this is well-defined since \( \Phi \) is integral and we obtain the \( U(1) \)-bundle \( Q^U \) over \( LU(2) \) by using this differential character \( k \) instead of \( \exp(\int_{\rho} 2\pi i \Phi) \) in section 3.1. But unfortunately \( k(a_1a_2) \) is not equal to \( \exp(\int_{(a_1,a_2)} 2\pi i \Phi) \) in general.

So in this way we can not obtain a section \( s^U_L \) of \( \delta Q^U \) nor a multiplication \( m^U : Q^U \times Q^U \to Q^U \).

4 Cocycle in the triple complex

In this section we deal with a semi-direct \( LG \rtimes S^1 \) for \( G = SU(2) \). Here we define a group action of \( S^1 \) on \( LG \) by the adjoint action of \( S^1 \) on \( SU(2) \), i.e. we fix a semi-direct product operator \( \cdot \) of \( LG \rtimes S^1 \) as \( \gamma, z) \cdot (\gamma', z') := (\gamma \cdot z \gamma' z^{-1}, z \gamma') \). So in this case \( LG \rtimes S^1 \) is a subgroup of \( LU(2) \).

First we define a bisimplicial manifold \( NLG(*) \rtimes NS^1(*) \). A bisimplicial manifold is a sequence of manifolds with horizontal and vertical face and degeneracy operators which commute with each other. A bisimplicial map is a sequence of maps commuting with horizontal and vertical face and degeneracy operators. We define \( NLG(*) \rtimes NS^1(*) \) as follows:

\[ NLG(p) \rtimes NS^1(q) := \underbrace{LG \times \cdots \times LG}_{p \text{-times}} \times \underbrace{S^1 \times \cdots \times S^1}_{q \text{-times}} \]

Horizontal face operators \( \varepsilon^L_i \) of \( NLG(p) \rtimes NS^1(q) \to NLG(p-1) \rtimes NS^1(q) \) are the same with the face operators of \( NLG(p) \). Vertical face operators \( \varepsilon^S_i \) of \( NLG(p) \rtimes NS^1(q) \to NLG(p) \rtimes NS^1(q-1) \) are

\[
\varepsilon^S_i (\gamma, z_1, \cdots, z_q) = \begin{cases} 
(\gamma, z_2, \cdots, z_q) & i = 0 \\
(\gamma, z_1, \cdots, z_i z_{i+1}, \cdots, z_q) & i = 1, \cdots, q - 1 \\
(z_q^{-1} z_{q-1}^{-1}, z_1, \cdots, z_{q-1}) & i = q
\end{cases}
\]

Here \( \gamma := (\gamma_1, \cdots, \gamma_p) \).

Then we define a bisimplicial map \( \rho_\gamma : PLG(p) \times PS^1(q) \to NLG(p) \rtimes NS^1(q) \) as \( \rho_\gamma (\gamma, z_1, \cdots, z_{q+1}) = (z_{q+1} \rho(\gamma), z_{q+1}, \rho(z_1, \cdots, z_{q+1})) \).

\( LG \rtimes S^1 \) acts on \( PLG(p) \times PS^1(q) \) by right as \( (\gamma, \hat{z}) \cdot (\gamma, z) = (z^{-1} \gamma, \gamma z, \hat{z}) \).

Since \( \rho_\gamma (\gamma, \hat{z}) = \rho_\gamma ((\gamma, \hat{z}), (\gamma, z)) \), one can see that \( \rho_\gamma \) is a principal \( (LG \rtimes S^1) \)-bundle. \( \| PLG(*) \times PS^1(*) \| \) is \( ELG \times ES^1 \) and \( \| NLG(*) \rtimes NS^1(*) \| \) is
homeomorphic to \((ELG \times ES^1)/(LG \times S^1)\), so \(NLG(\ast) \rtimes NS^1(\ast)\) is a model of \(B(LG \times S^1)\).

**Definition 4.1.** For a bisimplicial manifold \(NLG(\ast) \rtimes NS^1(\ast)\), we have a triple complex as follows:

\[
\Omega^{p,q,r}(NLG(\ast) \rtimes NS^1(\ast)) \overset{\text{def}}{=} \Omega^r(NLG(p) \rtimes NS^1(q))
\]

Derivatives are:

\[
d' = \sum_{i=0}^{p+1} (-1)^i (\varepsilon_i^{LG})^*, \quad d'' = \sum_{i=0}^{q+1} (-1)^i (\varepsilon_i^{S^1})^* \times (-1)^p
\]

\[
d''' = (-1)^{p+q} \times \text{the exterior differential on } \Omega^*(NLG(p) \rtimes NS^1(q)).
\]

The following proposition can be proved by adapting the same argument in the proof of Theorem 2.1 (See [20]).

**Proposition 4.1.** There exists an isomorphism

\[
H(\Omega^*(NLG \rtimes NS^1)) \cong H^*(B(LG \times S^1)).
\]

*Here \(\Omega^*(NLG \rtimes NS^1)\) means the total complex.*

Now we want to construct a cocycle in \(\Omega^3(NLG \rtimes NS^1)\) which coincides with \(c_1(\theta) - \left(\frac{1}{2\pi}\right) s_L^*(\delta\theta)\) when it is restricted to \(\Omega^3(NLG)\).

To do this, it suffice to construct a differential form \(\tau\) on \(\Omega^1(LG \times S^1)\) such that \(d\tau = (-\varepsilon_0^{S^1} + \varepsilon_1^{S^1})c_1(\theta)\) and \((\varepsilon_0^{LG} - \varepsilon_1^{LG} + \varepsilon_2^{LG})\tau = (\varepsilon_0^{S^1} - \varepsilon_1^{S^1})\left(\frac{1}{2\pi}\right) s_L^*(\delta\theta)\) and \((-\varepsilon_0^{S^1} + \varepsilon_1^{S^1} - \varepsilon_2^{S^1})\tau = 0\). We consider a trivial \(U(1)\)-bundle \((\varepsilon_0^{S^1} Q)^{\otimes -1} \otimes \varepsilon_1^{S^1} Q\) and the induced connection form \(\delta s\theta\) on it. We define a section \(s_\gamma : LG \rtimes S^1 \to (\varepsilon_0^{S^1} Q)^{\otimes -1} \otimes \varepsilon_1^{S^1} Q\) as \(s_\gamma(\gamma, z) := (\hat{\gamma}, z)^{\otimes -1} \otimes (z^{-1}\hat{\gamma} z, z)\) and set \(\tau := \left(\frac{1}{2\pi}\right) s_\gamma^*(\delta s\theta)\) then we can see that \(\tau\) satisfies the required conditions.

**Remark 4.1.** We can adapt the same argument to the case of \(LSU(2) \rtimes LS^1\) and \(LSU(2) \rtimes SU(2)\).
5 Another description of the DD class

On the other hand, there is a simplicial manifold $N\hat{G}$ and face operators $\hat{\varepsilon}_i$ of it. Using this, Behrend and Xu described the cocycle which represents the Dixmier-Douady class in another way.

**Proposition 5.1 (1, 2).** Let $\hat{G} \times \hat{G} \to G \times G$ be a $(U(1) \times U(1))$-bundle. Then the 1-form $(\hat{\varepsilon}_0^* - \hat{\varepsilon}_1^* + \hat{\varepsilon}_2^*)\theta$ on $\hat{G} \times \hat{G}$ is horizontal and $(U(1) \times U(1))$-invariant, hence there exists the 1-form $\chi$ on $G \times G$ which satisfies $(\pi \times \pi)^*\chi = (\hat{\varepsilon}_0^* - \hat{\varepsilon}_1^* + \hat{\varepsilon}_2^*)\theta$.

**Proof.** For example, see [1, G. Ginot, M. Stiénon].

Behrend and Xu proved the theorem below in [2].

**Theorem 5.1 (1, 2).** The cohomology class $[c_1(\theta) - (\frac{1}{2\pi i}) \chi] \in H^3(\Omega(NG))$ represents the universal Dixmier-Douady class.

Now we show our cocycle in section 2.2 satisfies the required condition in Proposition 5.1 when we choose a natural section $s_{nt} : G \times G \to \delta\hat{G}$.

**Theorem 5.2.** The equation $(\pi \times \pi)^*s_{nt}^*(\delta\theta) = (\hat{\varepsilon}_0^* - \hat{\varepsilon}_1^* + \hat{\varepsilon}_2^*)\theta$ holds.

**Proof.** Choose an open cover $\mathcal{V} = \{V_\lambda\}_{\lambda \in \Lambda}$ of $G$ such that all the intersections of open sets in $\mathcal{V}$ are contractible and such that there exist local sections $\eta_\lambda : V_\lambda \to \hat{G}$ of $\pi$. Then $\{\varepsilon_0^{-1}(V_\lambda) \cap \varepsilon_1^{-1}(V_\lambda) \cap \varepsilon_2^{-1}(V_\lambda)\}_{\lambda, \lambda', \lambda'' \in \Lambda}$ is an open covering of $G \times G$ and there are the induced local sections $\varepsilon_0^*\eta_\lambda \otimes (\varepsilon_1^*\eta_{\lambda'}) \otimes (\varepsilon_2^*\eta_{\lambda''})$ on that covering.

If we pull back $\delta\theta$ by these sections, the induced form on $\varepsilon_0^{-1}(V_\lambda) \cap \varepsilon_1^{-1}(V_\lambda) \cap \varepsilon_2^{-1}(V_\lambda)$ is $\varepsilon_0^*(\eta_\lambda^*\theta) - \varepsilon_1^*(\eta_{\lambda'}^*\theta) + \varepsilon_2^*(\eta_{\lambda''}^*\theta)$.

Then the $U(1)$-valued functions $\tau_{\lambda\lambda',\lambda''}$ on $\varepsilon_0^{-1}(V_\lambda) \cap \varepsilon_1^{-1}(V_\lambda) \cap \varepsilon_2^{-1}(V_\lambda)$ as $(\varepsilon_0^*\eta_\lambda \otimes (\varepsilon_1^*\eta_{\lambda'}) \otimes (\varepsilon_2^*\eta_{\lambda''})) \cdot \tau_{\lambda\lambda',\lambda''} = s_{nt}$.

Let $\bar{\varphi}_\lambda : \pi^{-1}(V_\lambda) \to V_\lambda \times U(1)$ be a local trivialization of $\pi$ and we define $\varphi_\lambda := \text{pr}_2 \circ \bar{\varphi}_\lambda : \pi^{-1}(V_\lambda) \to U(1)$. For any $\hat{g} \in \pi^{-1}(V_\lambda)$ the equation $\hat{g} = \eta_\lambda \circ \pi(\hat{g}) \cdot \varphi_\lambda(\hat{g})$ holds so we can see $\varepsilon_0^*(\varepsilon_1^*(\varepsilon_2^*\varphi_\lambda^{-1}d\varphi_\lambda) = \varepsilon_0^*(\varepsilon_1^*(\varepsilon_2^*\varphi_\lambda^{-1}d\varphi_\lambda) = (\pi \times \pi)^*\varepsilon_0^*(\varepsilon_1^*(\varepsilon_2^*\varphi_\lambda^{-1}d\varphi_\lambda)$. Then on $(\pi \times \pi)^*(\varepsilon_0^{-1}(V_\lambda) \cap \varepsilon_1^{-1}(V_\lambda') \cap \varepsilon_2^{-1}(V_\lambda''))$ there is a differential form $\tilde{\varepsilon}_0^*\theta - \tilde{\varepsilon}_1^*\theta + \tilde{\varepsilon}_2^*\theta = (\pi \times \pi)^*(\varepsilon_0^*(\varepsilon_1^*(\varepsilon_2^*\varphi_\lambda^{-1}d\varphi_\lambda) - \varepsilon_0^*(\varepsilon_1^*(\varepsilon_2^*\varphi_\lambda^{-1}d\varphi_\lambda) - \varepsilon_0^*(\varepsilon_1^*(\varepsilon_2^*\varphi_\lambda^{-1}d\varphi_\lambda) + \varepsilon_0^*(\varepsilon_1^*(\varepsilon_2^*\varphi_\lambda^{-1}d\varphi_\lambda)$. Then we have

$$
\bar{\varphi}_\lambda : \pi^{-1}(V_\lambda) \to V_\lambda \times U(1)
$$
Since \( \hat{\varepsilon}_i = (\eta_\lambda \circ \pi \circ \hat{\varepsilon}_i) \cdot \varphi_\lambda \circ \hat{\varepsilon}_i = (\eta_\lambda \circ \varepsilon_i \circ (\pi \times \pi)) \cdot \varphi_\lambda \circ \hat{\varepsilon}_i \), we can see that \( \hat{\varepsilon}_0 \otimes \hat{\varepsilon}_1^{\otimes -1} \otimes \hat{\varepsilon}_2 : \hat{G} \times \hat{G} \to \delta \hat{G} \) is equal to \( ((\varepsilon_0^* \eta_\lambda \otimes (\varepsilon_1^* \eta_\lambda'))^{\otimes -1} \otimes \varepsilon_2^* \eta_{\lambda''}) \circ (\pi \times \pi)) \cdot (\varphi_\lambda \circ \hat{\varepsilon}_0)((\varphi_{\lambda'} \circ \hat{\varepsilon}_1)^{-1}(\varphi_{\lambda''} \circ \hat{\varepsilon}_2)). \)

We have \( \tau_{\lambda \lambda'} \lambda'' \circ (\pi \times \pi) = (\varphi_\lambda \circ \hat{\varepsilon}_0)((\varphi_{\lambda'} \circ \hat{\varepsilon}_1)^{-1}(\varphi_{\lambda''} \circ \hat{\varepsilon}_2)) \) because \( s_{nt} \circ (\pi \times \pi) = \hat{\varepsilon}_0 \otimes \hat{\varepsilon}_1^{\otimes -1} \otimes \hat{\varepsilon}_2 \), so it follows that \( \theta = (\pi \times \pi)^* s_{nt}^* \delta \). This completes the proof. \( \square \)

Remark 5.1. This theorem gives another proof of Proposition 5.1.

Remark 5.2. Let \( \hat{\Gamma}_1 \to \Gamma_1 \Rightarrow \Gamma_0 \) be a central \( U(1) \)-extension of a groupoid and \( \theta \) be a connection form of the \( U(1) \)-bundle \( \hat{\Gamma}_1 \to \Gamma_1 \). Then we can use the same argument in section 2.2 and obtain the cocycle on \( \Omega^*(N(\Gamma(\ast))) \). In \cite{1,2} and related papers, they call \( \theta \) a pseudo-connection of a central \( U(1) \)-extension of a groupoid \( \hat{\Gamma}_1 \to \Gamma_1 \Rightarrow \Gamma_0 \) and when \( -\left( \frac{1}{2\pi} \right) s_{nt}^* (\delta \theta) \in \Omega^1(N(\Gamma(2))) \) vanishes they call \( \theta \) a connection of \( \hat{\Gamma}_1 \to \Gamma_1 \Rightarrow \Gamma_0 \). If the horizontal complex \( \Omega^1(N(\Gamma(1))) \xrightarrow{d} \Omega^1(N(\Gamma(2))) \xrightarrow{d} \Omega^1(N(\Gamma(3))) \) is exact, a connection of \( \hat{\Gamma}_1 \to \Gamma_1 \Rightarrow \Gamma_0 \) exists.

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