Eigenvalue-free interval for threshold graphs

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Abstract

This paper deals with the eigenvalues of the adjacency matrices of threshold graphs for which $-1$ and $0$ are considered as trivial eigenvalues. We show that among threshold graphs on a fixed number of vertices, the unique connected anti-regular graph has the smallest positive eigenvalue and the largest non-trivial negative eigenvalue. It follows that threshold graphs have no non-trivial eigenvalues in the interval $\left[\frac{-1 - \sqrt{2}}{2}, \frac{-1 + \sqrt{2}}{2}\right]$. These results confirm two conjectures by Aguilar, Lee, Piato, and Schweitzer.

Keywords: Threshold graph, Eigenvalue, Anti-regular graph

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1 Introduction

A threshold graph is a graph that can be constructed from a one-vertex graph by repeated addition of a single isolated vertex to the graph, or addition of a single vertex that is adjacent to all other vertices. An equivalent definition is the following: a graph is a threshold graph if there are a real number $S$ and for each vertex $v$ a real vertex weight $w(v)$ such that two vertices $u, v$ are adjacent if and only if $w(u) + w(v) > S$. This justifies the name “threshold graph” as $S$ is the threshold for being adjacent. Threshold graphs also can be defined in terms of forbidden subgraphs, namely they are \{$P_4$, $2K_2$, $C_4$\}-free graphs. Note, if a threshold graph is not connected then (since $2K_2$ is forbidden) at most one of its components is non-trivial (others are trivial, i.e. isolated vertices). For more information see [4] [9].
In this paper we deal with eigenvalues of the adjacency matrices of threshold graphs for which \(-1\) and \(0\) are considered as trivial eigenvalues. In \([8]\) it was shown that threshold graphs have no eigenvalues in \((-1, 0)\). This result was extended in \([6]\) by showing that a graph \(G\) is a cograph (i.e. a \(P_4\)-free graph) if and only if no induced subgraph of \(G\) has an eigenvalue in the interval \((-1, 0)\). A distinguished subclass of threshold graphs is the family of anti-regular graphs which are the graphs with only two vertices of equal degree. If \(G\) is anti-regular it follows easily that the complement graph \(\overline{G}\) is also anti-regular. Up to isomorphism, there is only one connected anti-regular graph on \(n\) vertices and its complement is the unique disconnected \(n\)-vertex anti-regular graph \([3]\). The unique connected anti-regular graph on \(n \geq 2\) vertices is denoted by \(A_n\).

In this paper we show that among threshold graphs on a fixed number of vertices, the unique connected anti-regular graph has the smallest positive eigenvalue and the largest non-trivial negative eigenvalue. It follows that threshold graphs have no non-trivial eigenvalues in the interval \([(-1 - \sqrt{2})/2, (-1 + \sqrt{2})/2]\). These results confirm two conjectures of \([1]\) and improve the aforementioned result of \([8]\).

2 Preliminaries

In this section we introduce the notations and recall a basic fact which will be used frequently. The graphs we consider are all simple and undirected. For a graph \(G\), we denote by \(V(G)\) the vertex set of \(G\). For two vertices \(u, v\), by \(u \sim v\) we mean \(u\) and \(v\) are adjacent. If \(V(G) = \{v_1, \ldots, v_n\}\), then the adjacency matrix of \(G\) is an \(n \times n\) matrix \(A(G)\) whose \((i, j)\)-entry is 1 if \(v_i \sim v_j\) and 0 otherwise. By eigenvalues of \(G\) we mean those of \(A(G)\). The multiplicity of an eigenvalue \(\lambda\) of \(G\) is denoted by \(\text{mult}(\lambda, G)\). For a vertex \(v\) of \(G\), let \(N_G(v)\) denote the open neighborhood of \(v\), i.e. the set of vertices of \(G\) adjacent to \(v\) and \(N_G[v] = N_G(v) \cup \{v\}\) denote the closed neighborhood of \(v\); we will drop the subscript \(G\) when it is clear from the context. Two vertices \(u\) and \(v\) of \(G\) are called duplicates if \(N(u) = N(v)\) and called coduplicates if \(N[u] = N[v]\). Note that duplicate vertices cannot be adjacent while coduplicate vertices must be adjacent. A subset \(S\) of \(V(G)\) such that \(N(u) = N(v)\) for any \(u, v \in S\) is called a duplication class of \(G\). Coduplication classes are defined analogously. We will make use of the interlacing property of graph eigenvalues which we recall below (see \([5\), Theorem 2.5.1\]).

**Lemma 1.** Let \(G\) be a graph of order \(n\), \(H\) be an induced subgraph of \(G\) of order \(m\), \(\lambda_1 \geq \cdots \geq \lambda_n\) and \(\mu_1 \geq \cdots \geq \mu_m\) be the eigenvalues of \(G\) and \(H\), respectively. Then

\[
\lambda_i \geq \mu_i \geq \lambda_{n-m+i} \quad \text{for } i = 1, \ldots, m.
\]

In particular, if \(m = n - 1\), then

\[
\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n.
\]
Figure 1: A threshold graph: $V_i$’s are cliques, $U_i$’s are cocliques, each thick line indicates the edge set of a complete bipartite subgraph on some $U_i, V_j$

### 3 Extremal properties of the eigenvalues of anti-regular graphs among threshold graphs

In this section we present the main results of the paper. We start by the following remark on the structure of threshold graphs.

**Remark 2.** As it was observed in [10] (see also [2, 7]), the vertices of any connected threshold graph $G$ can be partitioned into $h$ non-empty coduplication classes $V_1, \ldots, V_h$ and $h$ non-empty duplication classes $U_1, \ldots, U_h$ such that the vertices in $V_1 \cup \cdots \cup V_i$ form a clique and

$$N(u) = V_1 \cup \cdots \cup V_i \quad \text{for any } u \in U_i, \quad 1 \leq i \leq h.$$  

(It turns out that $U_1 \cup \cdots \cup U_h$ form a coclique.) Accordingly, a connected threshold graph is also called *nested split graph* (or NSG for short). If $m_i = |U_i|$ and $n_i = |V_i|$ for $1 \leq i \leq h$, then we write

$$G = \text{NSG}(m_1, \ldots, m_h; n_1, \ldots, n_h)$$

For an illustration of this structure with $h = 5$, see Figure 1. It follows that a threshold graph $G$ of order $n$ is anti-regular if and only if $n_1 = \cdots = n_h = m_1 = \cdots = m_h = 1$ (in case $n$ is even) or $n_1 = \cdots = n_h = m_1 = \cdots = m_{h-1} = 1$ and $m_h = 2$ (in case $n$ is odd).

In any graph $G$ if we add a new vertex duplicate (coduplicate) to $u \in V(G)$, then the multiplicity of 0 (of $-1$) increases by 1. That’s why the eigenvalues 0 and $-1$ are treated as trivial eigenvalues in threshold graphs. The following can be deduced in a similar manner.
Lemma 3. Let $G = \text{NSG}(m_1, \ldots, m_h; n_1, \ldots, n_h)$ be a connected threshold graphs. Then

$$\text{mult}(0, G) = \sum_{i=1}^{h} (m_i - 1),$$

$$\text{mult}(-1, G) = \sum_{i=1}^{h} (n_i - 1) + \begin{cases} 1 & \text{if } m_h = 1, \\ 0 & \text{if } m_h \geq 2. \end{cases}$$

Lemma 4. Let $G$ be threshold graph which is not an anti-regular graph. Then there is some vertex $v$ of $G$ such that for $H = G - v$ we have either

(i) $\text{mult}(0, G) = \text{mult}(0, H) + 1$, $\text{mult}(-1, G) = \text{mult}(-1, H)$; or

(ii) $\text{mult}(0, G) = \text{mult}(0, H)$, $\text{mult}(-1, G) = \text{mult}(-1, H) + 1$.

Proof. Let $G = \text{NSG}(m_1, \ldots, m_h; n_1, \ldots, n_h)$. First assume that $n_1 = \cdots = n_h = m_1 = \cdots = m_{h-1} = 1$. As $G$ is not an anti-regular graph, we have $m_h \geq 3$. Let $v \in U_h$ and $H = G - v$. Then $H = \text{NSG}(m_1, \ldots, m_{h-1}, m_h - 1; n_1, \ldots, n_h)$. So by Lemma 3

$$\text{mult}(0, G) = 2 = \text{mult}(0, H) + 1, \quad \text{mult}(-1, G) = 0 = \text{mult}(-1, H),$$

and so we are done. So we may assume that $n_k \geq 2$ for some $1 \leq k \leq h$ or $m_j \geq 2$ for some $1 \leq j \leq h - 1$. If $n_k \geq 2$, let $v \in V_k$ and $H = G - v$. Then

$$H = \text{NSG}(m_1, \ldots, m_{h-1}, m_h - 1; n_1, \ldots, n_k, 1, n_k+1, \ldots, n_h).$$

So by Lemma 3

$$\text{mult}(0, G) = \text{mult}(0, H), \quad \text{mult}(-1, G) = \text{mult}(-1, H) + 1.$$

If $m_j \geq 2$ for some $1 \leq j \leq h - 1$, the result follows similarly. \hfill \Box

By $\eta_+(G)$ we denote the smallest positive eigenvalue of $G$ and by $\eta_-(G)$ we denote the largest eigenvalue of $G$ less than $-1$.

Lemma 5. (1) For anti-regular graphs we have

$$\eta_-(A_{n-1}) < \eta_-(A_n) \quad \text{and} \quad \eta_+(A_n) < \eta_+(A_{n-1}).$$

We are now in a position to prove the main result of the paper.

Theorem 6. Let $G$ be a threshold graph of order $n$ which is not an anti-regular graph. Then

$$\eta_-(G) < \eta_-(A_n) \quad \text{and} \quad \eta_+(A_n) < \eta_+(G).$$
Proof. We proceed by induction on \(n\), the order of \(G\). With no loss of generality we may assume that \(G\) is connected. The assertion holds if \(n \leq 3\), so we assume \(n \geq 4\). Let \(v\) and \(H = G - v\) be as given in Lemma 4. Let \(\lambda_1 \geq \cdots \geq \lambda_n\) and \(\mu_1 \geq \cdots \geq \mu_{n-1}\) be the eigenvalues of \(G\) and \(H\), respectively. We may suppose that for some \(t\),

\[
\mu_{t-\ell} > \mu_t = \cdots = \mu_{t+1} = \cdots = \mu_{t+j} = -1 > \mu_{t+j+1},
\]

(1)

It is possible that either \(j = 0\) or \(\ell = 0\) but we have \(j + \ell \geq 1\). By the induction hypothesis

\[
\mu_{t-\ell} = \eta_+ (H) \geq \eta_+ (A_{n-1}) \quad \text{and} \quad \mu_{t+j+1} = \eta_- (H) \leq \eta_- (A_{n-1}).
\]

By interlacing, from (1) we have

\[
\lambda_{t-\ell} \geq \lambda_{t-\ell+1} = \cdots = \lambda_t = 0 \geq \lambda_{t+1} \geq \lambda_{t+2} = \cdots = \lambda_{t+j} = -1 \geq \lambda_{t+j+1},
\]

\[
\lambda_{t-\ell} \geq \mu_{t-\ell} \quad \text{and} \quad \mu_{t+j+1} \geq \lambda_{t+j+2}.
\]

As \(G\) is not an anti-regular graph, the case (i) or (ii) of Lemma 4 occurs. If the case (i) occurs, then \(\text{mult}(0, G) = \ell + 1 = \text{mult}(0, H) + 1 = j = \text{mult}(-1, G) = j = \text{mult}(-1, H)\). This is only possible if \(\lambda_{t-\ell} = \lambda_{t+1} = 0\) and \(\lambda_{t+j+1} = -1\). If the case (ii) occurs, then \(\text{mult}(0, G) = \ell = \text{mult}(0, H)\), \(\text{mult}(-1, G) = j + 1 = \text{mult}(-1, H) + 1\) which implies that \(\lambda_{t-\ell} = 0\) and \(\lambda_{t+1} = \lambda_{t+j+1} = -1\).

It turns out that

\[
\eta_+(G) = \lambda_{t-\ell-1} \geq \mu_{t-\ell-1} = \eta_+ (H) \geq \eta_+ (A_{n-1}),
\]

\[
\eta_-(G) = \lambda_{t+j+2} \leq \mu_{t+j+1} = \eta_- (H) \leq \eta_- (A_{n-1}).
\]

The result now follows from Lemma 5. \(\square\)

In [1] it is shown that for any \(n\), \(\eta_- (A_n) < (-1 - \sqrt{2})/2\) and \((-1 + \sqrt{2})/2 < \eta_+ (A_n)\). This result and Theorem 9 imply the following corollary follows.

Corollary 7. Other than the trivial eigenvalues \(-1, 0\), the interval \([(-1 - \sqrt{2})/2, (-1 + \sqrt{2})/2]\) does not contain an eigenvalue of any threshold graph.

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