Review

Quantum control of bosonic modes with superconducting circuits

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A R T I C L E I N F O

Article history:
Received 21 February 2021
Received in revised form 20 May 2021
Accepted 24 May 2021
Available online 31 May 2021

Keywords:
Bosonic modes
Circuit QED
Unitary dynamics
Quantum feedback control
Driven-dissipative processes
Holonomic quantum computation

A B S T R A C T

Bosonic modes have wide applications in various quantum technologies, such as optical photons for quantum communication, magnons in spin ensembles for quantum information storage and mechanical modes for reversible microwave-to-optical quantum transduction. There is emerging interest in utilizing bosonic modes for quantum information processing, with circuit quantum electrodynamics (circuit QED) as one of the leading architectures. Quantum information can be encoded into subspaces of a bosonic superconducting cavity mode with long coherence time. However, standard Gaussian operations (e.g., beam splitting and two-mode squeezing) are insufficient for universal quantum computing. The major challenge is to introduce additional nonlinear control beyond Gaussian operations without adding significant bosonic loss or decoherence. Here we review recent advances in universal control of a single bosonic mode proposed to date, including unitary control, quantum feedback control, driven-dissipative control and holonomic dissipative control. Various approaches to entangling different bosonic modes are also discussed.

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1. Introduction

Quantum computation holds the promise of solving some specific problems, such as factorization of large integers and simulation of quantum many-body problems [1], much faster than any known classical computer. To build such a quantum computer, the physical platform should work in the quantum regime with long coherence time, fast quantum operations and good scalability, which are daunting obstacles for current technologies. The promising strategies to overcome such obstacles are quantum error correction (QEC) [2–4] and fault-tolerant (FT) quantum computation [5], where the coherence time of the quantum memories can be extended and the quantum operations can tolerate some low-probability errors (including errors in the QEC circuit) below a certain threshold.

In the prototypical model for quantum computation – the quantum circuit model, a quantum bit of information (qubit) is encoded into a two-level system, called a physical qubit, and the usual approach for QEC is to encode a logical qubit into some subspace of multiple physical qubits, so that different error processes lead to distinguishable syndromes and can therefore be corrected. However, the increased number of physical qubits for a logical qubit introduces more decoherence for the system to correct. Moreover, the logical gate operations become quite complicated since multiple physical systems need to be addressed simultaneously. Hence, it is still an outstanding experimental challenge to build a more robust quantum register using multiple physical qubits.

An alternative scheme is to encode the quantum information into bosonic modes such as harmonic oscillators [6,7]. A single bosonic mode already provides an infinitely large Hilbert space, from which we choose a logical subspace for an error-correcting code [8–13]. Such bosonic QEC modes can be hardware-efficient compared to the conventional QEC codes based on multiple qubits. Moreover, the bosonic modes often have relatively simple decoherence processes (mainly bosonic excitation loss channel) during which the bosonic excitations are lost one by one [14]. There have been several error-correcting encoding schemes in a single bosonic mode proposed to date, including the Gottesman-Kitaev-Preskill (GKP) codes [10,15,16], cat codes [11,17,18], binomial codes [12], rotation-symmetric codes [19] and other variations [13,20]. The GKP codes, consisting of superpositions of highly squeezed states, are not only protected against small shifts in position but also have been shown to perform well against the more realistic amplitude damping channel [13]. The cat codes use superpositions of coherent states evenly distributed around a circle in phase space, which can be protected against (single or multiple) bosonic excitation.
loss and dephasing errors. The binomial codes exploit superpositions of Fock states weighted with binomial coefficients, which can exactly correct the bosonic excitation loss, gain and dephasing errors up to a specific degree.

For bosonic modes, the standard operations (e.g., displacement operation, phase rotation, one-mode squeezing, beam splitting, and two-mode squeezing) are all Gaussian operations, which can only transform Gaussian states into Gaussian states [6,21]. However, universal control of a single bosonic mode can be achieved by adding a single nonlinear operation [21]. When such a direct nonlinear operation is difficult to realize directly, it is still possible to implement an indirect nonlinear interaction by coupling the bosonic mode to a finite-level ancilla. Moreover, quantum nondemolition (QND) measurement of the ancilla enables measurement-based feedback control and therefore arbitrary operation on the bosonic mode. Here, we will review recent advances in the approaches for universal control and arbitrary operation of bosonic modes (Table 1), including unitary control, quantum feedback control, driven-dissipative control and holonomic control (Fig. 1a). In the first two approaches, an ancilla qubit is coupled to a single bosonic mode to introduce nonlinear interaction and feedback control, while in the remaining two approaches, a special coupling between the bosonic mode and reservoir or a special Hamiltonian of the bosonic mode is engineered to support some stabilized manifold, consisting of all coherent superpositions of multiple steady states that are free of any nonunitary effect caused by the reservoir.

The physical platform we consider is circuit quantum electrodynamics (circuit QED) [14,56–61], which is an analog of cavity QED [62] using superconducting circuits [63,64]. Cavity QED engineers the environment of the atoms by placing them in a cavity that suppresses the environment of the atoms by placing them in a cavity that suppresses the environment of the atoms by placing them in a cavity that suppresses the environment of the atoms by placing them in a cavity that suppresses the environment of the atoms by placing them in a cavity that suppresses the environment of the atoms by placing them in a cavity that suppresses the environment of the atoms by placing them in a cavity that suppresses the environment of the atoms by placing them in a cavity that suppresses the environment of the atoms by placing them in a cavity that suppresses the environment of the atoms by placing them in a cavity that suppresses the environment of the atoms by placing them in a cavity that suppresses the environment of the atoms by placing them in a cavity that suppresses the environment of the atoms by placing them in a cavity that suppresses the environment of the atoms by placing them in a cavity that suppresses the environment of the atoms by placing them in a cavity that suppresses the environment of the atoms by placing them in a cavity that suppresses the environment of the atoms by placing them in a cavity that suppresses the environment of the atoms by placing them in a cavity that suppresses the environment of the atoms by placing them in a cavity that suppresses the environment of the atoms by placing them in a cavity that suppresses the environment of the atoms by placing them in a cavity that suppresses the environment of the atoms by placing them in a cavity that suppresses the environment of the atoms by placing them in a cavity that suppresses the environment of the atoms by placing them in a cavity that suppresses the environment of the atoms by placing them in a cavity that suppresses the environment of the atoms by placing them in a cavity that suppresses

![Fig. 1.](image-url) (Color online) (a) Schematic of various approaches for controlling a quantum system: (i) unitary control on the system alone or both the system and an ancilla; (ii) quantum feedback control based on measurement of the ancilla; (iii) driven-dissipative control with either engineered dissipation or Hamiltonian engineering; (iv) holonomic quantum control based on only engineered dissipation. (b), (c) Schematic and device photograph of a circuit QED system modeled as a coupled qubit-oscillator system. The storage cavity with long coherence time is used to encode quantum information, the transmon qubit acts as an ancilla for universal control of the storage cavity, and the readout cavity with short coherence time is used for qubit readout. Both the storage cavity and transmon qubit can be addressed by microwave control fields. Reprinted with permission from Refs. [27,55].

Table 1
Recent theoretical and experimental advances in quantum control of bosonic modes in circuit QED.

| Unitary control | Unitary & feedback | Unitary & dissipation | Quantum adaptive control |
|-----------------|-------------------|----------------------|-------------------------|
| Ancilla-induced nonlinearity | Theory: | Theory: | Theory: |
| SNAP gate [22] | - SNAP gate [22] | - CPTP maps [27] | - Dissipative cat [32] |
| - Optimal control [23,24] | - Teleported gate [28] | - Teleported gate [28] | - Kerr cat [33] |
| - E-SWAP gate [25] | - ET gate [29,30] | - FT syndrome detection [34] | - Bias-preserving Kerr cat [35] |
| - CPHASE gate [26] | - PI gate [31] | - Bias-preserving dissipative cat [36] | - Holonomic gate [37] |

| Experiments | Experiments | Experiments |
|-------------|-------------|-------------|
| SNAP gate [38] | QEC [42–44] | - Dissipative cat [51–53] |
| - Optimal control [23] | CPTP simulation [45,46] | - Kerr cat [54] |
| - CNOT gate [39] | Teleported CNOT gate [47] | - Bias-preserving cat [55] |
| - CZ gate [40] | FT parity measurement [48] | - Holonomic gate [56] |
| - E-SWAP gate [41] | PI SNAP gate [49] | - Phase gate [50] |

This scheme uses only dissipation.
the transmon qubits can act as ancillas to aid universal control of the storage cavity (Fig. 1b, c). Such an encoding scheme can make use of various bosonic QEC codes, which are hardware-efficient compared to more standard qubit-based codes such as surface codes. Moreover, bosonic QEC codes often have specific noise resilience [36], and therefore can be concatenated with conventional QEC codes to reduce the hardware overhead [38,89,90]. Recently there has been significant experimental progress in bosonic QEC. QEC based on cat codes or binomial codes in superconducting cavities have reached or approached the break-even point [42,43], at which the lifetime of the logical qubit exceeds that of the single best physical qubit within the logical qubit. The encoding based on GKP codes has also been demonstrated in trapped-ion mechanical oscillators [91,92] and superconducting cavities [44]. However, compared to conventional one-qubit and two-qubit control, universal control of single and multiple bosonic modes requires the introduction of nonlinearity and therefore is more complex. This will be the main topic of this review.

This review is organized as follows. In Section 2, we review the universal control of a single bosonic mode with the aid of an ancilla qubit dispersively coupled to it. Then we introduce, in Section 3, the extension from the universal unitary control to quantum feedback control and arbitrary quantum channel construction for the bosonic mode by QND measurement of the ancilla. In Section 4, it is shown that reservoir engineering and Hamiltonian engineering can be promising strategies to realize universal quantum computation in some unitarily evolving subspace of the bosonic mode. In Section 5, the combination of reservoir engineering and holonomic quantum control is introduced to realize universal control of bosonic modes. Then in Section 6, we introduce the quantum control schemes to entangle different bosonic modes for universal quantum computation. In Section 7, we briefly summarize the review and outline some future directions for quantum control of the bosonic modes. For convenience, we take the reduced Plank constant as $\hbar = 1$ throughout this review.

2. Unitary quantum control

Quantum control of a single bosonic mode (typically a harmonic oscillator) can be achieved in the coupled qubit-oscillator system with a qubit as an ancilla. Many theoretical and experimental works were devoted to preparing arbitrary oscillator states assisted by an ancilla qubit with Jaynes-Cummings (JC) coupling [93–97]. However, it is more challenging to achieve universal control of the oscillator, which usually needs a multi-level ancilla [98], slow adiabatic evolutions [99] or a large number of control operations [100].

In circuit QED, the transmon (as an anharmonic oscillator) can act as an ancilla to aid the control of cavity bosonic modes (as harmonic oscillators). If the ancilla and a single oscillator are strongly off-resonant with the detuning much larger than their coupling strength, we arrive at the dispersive Hamiltonian [55,101]

$$H_0 = \sum_{j=0}^{\infty} (\mathcal{A}_j - j \chi a^\dagger a) |j\rangle \langle j| + \omega_0 a^\dagger a,$$  

where $|j\rangle$ labels the eigenstate of the ancilla, $\mathcal{A}_j$ is the eigenenergy, $\chi$ is the dispersive coupling strength, $\omega_0$ is the oscillator frequency, and $a$ ($a^\dagger$) is the annihilation (creation) operator of the oscillator excitation. Note that here we have neglected the weak anharmonicity of the cavity modes. The anharmonicity of the ancilla $(\mathcal{A}_j - \mathcal{A}_{j-1} \neq \mathcal{A}_{j+1} - \mathcal{A}_j)$ makes it possible to selectively drive specific ancilla transitions, so the infinite-dimensional ancilla can often be truncated to a finite-dimensional one. Below the lowest ancilla eigenstates $|0\rangle, |1\rangle, |2\rangle \cdots$ are denoted as $|g\rangle, |e\rangle, |f\rangle \cdots$, and the eigenenergy difference between $|e\rangle$ and $|g\rangle$ is denoted as $\omega_0 = \lambda_1 - \lambda_0$.

The dispersive Hamiltonian can be interpreted from two different perspectives (Fig. 2b). On the one hand, the oscillator frequency has a shift dependent on the ancilla state. This ancilla-state-dependent shift of the cavity leads to changes in the amplitude and phase of photons reflected from or transmitted through the cavity and therefore enables a QND measurement on the ancilla state [56,57]. On the other hand, the ancilla transition frequency has a shift proportional to the oscillator excitation number. In the strongly dispersive regime of circuit QED, the ancilla frequency shift is much larger than the cavity line width and ancilla line width, and therefore the ancilla spectrum is split into a series of separately resolved peaks, representing the distribution of photon numbers within the driven cavity [101]. Moreover, for quantum control of the oscillator, such a strongly-dispersive coupling regime makes it possible to selectively address the ancilla if and only if the oscillator is in a specific number state, hence providing new opportunities for universal control of the oscillator.

Typically we can achieve universal unitary control and quantum measurements of the ancilla, but only limited unitary control on the oscillator, so the key point is to use the ancilla to realize some other unitary control on the oscillator to achieve universal control. Below we introduce two schemes: the unitary control either separately acts on the ancilla or the oscillator and then is combined, or acts on the both of them simultaneously (Fig. 2a).

2.1. Displacement operations and SNAP gates

The first scheme for universal control of the oscillator is to separately apply unitary control on the subsystems (either the ancilla or the oscillator) and then combine them [22,38]. The unitary control on the ancilla may indirectly realize some unitary operations on the oscillator if we make appropriate pre-section and post-selection of the ancilla state. Then combining these indirect operations with the direct ones, we may realize universal control of the oscillator.

One common kind of direct unitary transformation on the oscillator is the displacement operation

$$D(x) = \exp(\chi a^\dagger - x^* a),$$

which can be generated by a linear drive on the cavity $H_C = \epsilon(t) e^{ix^* a^\dagger + H.c.}$ with $x = -i/\epsilon(t) dt$. However, the displacement operation alone is not universal, i.e. it cannot generate arbitrary operations on the oscillator. To see this, note that the displacement operation can only prepare a coherent state from the vacuum, i.e., $|x\rangle = D(x)|0\rangle = \exp(-|x|^2/2) \sum_n a^n e^{ix^* n/\sqrt{n!}} |n\rangle$, while universal control requires that any given target state can be prepared from any initial state including the vacuum state.

The dispersive coupling of the oscillator to the ancilla introduces a nonlinear term for the oscillator Hamiltonian, and makes it possible to realize indirect control on the oscillator, such as the selective number-dependent arbitrary phase (SNAP) gate,

$$S(\bar{\theta}) = \sum_{n=0}^{\infty} e^{i\phi_n |n\rangle \langle n|},$$

which imparts arbitrary phases $\bar{\theta} = \{\phi_n\}_{n=0}^{\infty}$ to the different Fock states of the oscillator. The original proposal to realize SNAP gates is to weakly drive the ancilla with multiple frequency components, $H_F = \epsilon(t) e^{i\epsilon(t) |e\rangle \langle e| + H.c.}$ with $\epsilon(t) = \sum_n \Omega e^{i\omega_n(t) - n\Omega t}$ and $\phi_n$ being time-dependent. If $\Omega < \chi$, the driving component with frequency $\omega_n - \Omega t$ induces a unitary evolution in the ancilla subspace $|n\rangle, |e\rangle, |n\rangle \pm |g\rangle \equiv |n\rangle$ with a negligible effect on the rest of the system, while the driving phases $\phi_n(t)$ depending on the oscillator excitation numbers can induce different evolution paths in
different ancilla subspaces, as shown in Fig. 2b. Suppose the initial state of the whole system is a product state, \(|\psi(0) = |g\rangle \otimes \sum_{n=0}^{N} c_n |n\rangle\) with \(N\) being the truncated oscillator excitation number, then we may let the ancilla undergo cyclic evolutions in each subspace \(|g, n\rangle, |e, n\rangle\) and return to \(|g, n\rangle\) at time \(\tau\). We can tune \(\phi_n(t)\) so that the final state accumulates different geometric phases \(\phi_n\) for different \(n\) [102], i.e., \(|\psi(\tau) = \sum_{n=0}^{N} c_ne^{i\phi_n} |g, n\rangle\). For example, we may set \(\phi_n(t) = 0\) for \(t \in [0, \tau/2]\) and \(\phi_n(t) = \phi_n\) for \(t \in [\tau/2, \tau]\) with \(\tau = \pi/\Omega\) being the Rabi period, and the unitary propagator on the whole system (in the interaction picture associated with the dispersive Hamiltonian in Eq. (1)) is

\[
U(\tau, 0) = |g\rangle \langle g| \otimes S(\phi) + |e\rangle \langle e| \otimes S(-\phi),
\]

which implies that the unitary gate on the oscillator is \(S(\phi)\) (\(S(-\phi)\)) if the initial ancilla state is \(|g\rangle\) (\(|e\rangle\)).

The original SNAP gate based on the geometric phases can be simplified by first decomposing the above propagator (Eq. (4)) as

\[
U(\tau, 0) = U(\tau, \tau/2)U(\tau/2, 0)
\]

where \(U(\tau, \tau/2) = (|g\rangle \langle e| + |e\rangle \langle g|) \otimes 1\),

\[
U(\tau, \tau/2) = |g\rangle \langle e| \otimes S(\phi) + |e\rangle \langle g| \otimes S(-\phi),
\]

where \(i\) is the identity operator for the oscillator. Note that the first half evolution \(U(\tau/2, 0)\) causes a flip of the ancilla state while leaving the oscillator state unchanged, and the second half evolution \(U(\tau/2, \tau/2)\) causes a further flip of the ancilla state and produces the SNAP gate on the oscillator at the same time (Fig. 2b). So we may simplify the SNAP gate by applying only the drive during the second half period, which we may call the simplified SNAP gate. Moreover, if the simplified SNAP gate is not completed, we have

\[
U(\tau/2 + \Delta \tau, \tau/2) = \text{cos} \theta (|g\rangle \langle g| + |e\rangle \langle e|) \otimes 1 - \text{isin} \theta (|g\rangle \langle e| \otimes S(\phi) + |e\rangle \langle g| \otimes S(-\phi)),
\]

where \(\Delta \tau \in [0, \tau/2]\) and \(\theta = \Omega \Delta \tau\). In this case, the pre-selection and post-selection of the ancilla state induces either the identity operation or SNAP gate on the oscillator. For example, \(P[U(\tau/2 + \Delta \tau, \tau/2)| = 1\) and \(P[U(\tau/2 + \Delta \tau, \tau/2)| = 1\) with \(P_m = \langle m|\langle m| \langle m = g, e)\). Note that in the above discussions, we consider the limiting case \(\Omega/\gamma \rightarrow 0\), while in practice \(\Omega/\gamma\) is finite and causes deviations from the ideal SNAP gates [22,38]. Nevertheless, it is possible to minimize such gate errors due to finite \(\Omega/\gamma\) by optimizing the detunings and pulse shapes of the multi-frequency drive on the ancilla [26]. Besides the resonant driving approach for SNAP gates, a photon-number dependent Hamiltonian of the oscillator can also be engineered by off-resonantly driving the ancilla with multiple frequencies [26]. It has been demonstrated that universal control of an oscillator can be achieved by combining the displacement operations \(D(\phi)\) and the SNAP gates \(S(\phi)\) [22], since the generators of \(D(\phi)\) and
S(\hat{\varphi}) and the commutators between these generators generate the full Lie algebra \(u(N)\) for any truncated oscillator space \(\{|0\}, \ldots, |N-1\rangle\)\footnote{As an example, we show in Fig. 2c that a Fock state \(|1\rangle\) of the oscillator can be created by applying the operation \(D(\beta_1)S(\hat{\varphi})D(\beta_2)\), where \(\hat{\varphi}\) is fixed to be \(\pi, 0, \ldots\) while the displacement parameters \(\beta_1, \beta_2\) are obtained by numerical optimization. A systematic method was presented in Ref. [22] to construct an arbitrary unitary operation in any truncated oscillator space. With this method, the number of operations to prepare the oscillator Fock state \(|n\rangle\) can be significantly decreased from \(O(n)\) to \(O(\sqrt{n})\). Recently a more efficient scheme by parameter optimization has been proposed to implement a broad range of cavity control with only 3 to 4 SNAP gates [103]. Nevertheless, it is still an open problem to find the optimal way for decomposing an arbitrary target unitary into displacement operations and SNAP gates.

2.2. Universal control by numerical optimization algorithms

The previous analytic approach based on displacement operations and SNAP gates implicitly assumes that the cavity drive \(e|c(t)\rangle\) and transmon drive \(e|\tau(t)\rangle\) are never applied simultaneously, which makes the evolution more tractable. However, to find more efficient control schemes, it is better to include the possibility of simultaneously driving both the ancilla and the oscillator. The arbitrary control field can take the form

\[
H_{CT} = e_\tau(t)e^{i\Omega t}a + e_\tau(t)e^{i\Omega t}|g\rangle\langle e| + H.c.,
\]

(7)

where \(e_\tau(t), e_\tau(t)\) are arbitrary complex-valued functions of time. The exact form of the control field can be obtained by numerical optimization algorithms [23], such as the Gradient Ascent Pulse Engineering (GRAPE) method [24,104]. The basic procedure of the GRAPE method is as follows: (1) specify the target unitary \(U\) and the evolution time \(\tau\); (2) discretize the total time \(\tau\) into \(M\) equal steps of duration \(\Delta t = \tau/M\), and during each step the control amplitudes are constant; (3) make an initial guess of the control amplitudes, then calculate the fidelity between the implemented unitary and target unitary, and also the gradient of the fidelity with respect to each variation of the control amplitude in each time step; (4) adapt the control amplitudes according to the fidelity gradient, and repeat step (3) until a local maximum of the gate fidelity is achieved.

Both \(e_\tau(t)\) and \(e_\tau(t)\) can be optimized with GRAPE to achieve universal control of the cavity. The numerical optimized pulses thus obtained have been extensively used in experiments to control superconducting cavity modes [39,42,43,105]. As an example, we show in Fig. 2d the control amplitudes \(e_\tau(t), e_\tau(t)\) to prepare the cavity state from the vacuum state \(|0\rangle\) to the Fock state \(|\beta\rangle\). With this approach, Heeres et al. [23] have also realized a universal set of gates on the logical qubit based on error-correcting cat codes in a cavity. Compared with the SNAP gate that takes a rather long time \(2\pi/\Omega\) due to \(\Omega/\gamma \ll 1\), the logical gates based on GRAPE algorithm take a much shorter time \(2\pi/\gamma\).

2.3. Other approaches

Besides the above schemes, there are various other approaches to control harmonic oscillators via ancilla-induced nonlinearity. One approach is called photon blockade control, in which the frequency of \(\epsilon_\tau(t)\) is set as \(\omega_\tau = N\Omega\) to drive resonantly the transition \(|g, N\rangle \rightarrow |e, N\rangle\), therefore blocking the population transfer between the cavity subspace \(\{|0\}, |1\}, \ldots, |N-1\rangle\) and the rest of the cavity Hilbert space [106]. Then universal control of the N-level qudit can be realized by optimizing \(\epsilon_\tau(t)\) with GRAPE [107], which has been experimentally demonstrated in Ref. [108]. In another approach, using a single transmon as the central processor, universal quantum operations have been realized between arbitrary eigenmodes of a linear array of coupled superconducting resonators, realizing a random access quantum information processor [109].

It should be mentioned that the weak point of the transmon as an ancilla is its small dispersive coupling strength with the cavity modes. This limits the control fidelity of both SNAP gates and blockade control. Such limitations may be overcome in the future by using better ancillas. For example, with an ancilla oscillator, a superconducting oscillator can have stimulated nonlinearity by a three-wave interaction, enabling control of the single-photon manifold at rates faster than the dispersive protocols [110]. Other possible better ancillas include the C-shunt flux qubit with large anharmonicity [111] and the fluxonium with millisecond coherence time [112,113].

3. Quantum feedback control

In the last section, the system (an oscillator and an ancilla) as a whole are assumed to be a closed system and therefore can be sufficiently described by unitary dynamics. However, the inevitable coupling of the system to the environment typically induces nonunitary evolutions of the system, which can be fully characterized by completely positive and trace preserving (CPTP) maps [1,114] (also called quantum operations or quantum channels). Hence, it is important to systematically extend the quantum control techniques from a closed system to an open quantum system. In this section, we will show that an arbitrary CPTP map of the system can be constructed by coupling the system to an ancilla qubit with QND readout and quantum feedback control.

Feedback control, where information about the system state is fed back to the controller for correction, is widely used in classical control theory. However, its extension to the quantum world is nontrivial [115], since a quantum measurement of the system will inevitably affect the quantum state of the system. Quantum feedback control generally falls into two categories: measurement-based feedback control [116] and coherent feedback control [117]. Below we will show that the measurement-based approach can be used to construct arbitrary CPTP maps and realize robust quantum operations.

3.1. Arbitrary CPTP map construction

A CPTP map can be described by the Kraus representation [114]

\[
\varepsilon(\rho) = \sum_{i=1}^{d^{2}} K_i \rho K_i^\dagger,
\]

(8)

where \(\rho\) is the density matrix of the system we consider and \(\{K_i\}_{i=1}^{d^{2}}\) is the set of Kraus operators satisfying \(\sum_{i=1}^{d^{2}} K_i K_i^\dagger = \mathbb{I}\) to preserve the trace of \(\rho\). The Kraus representation is not unique, since a new set of Kraus operators \(\{F_i\}_{i=1}^{N}\) can be constructed with any \(N \times N\) unitary matrix \(U, F_i = \sum_{j=1}^{N} U_j K_j^\dagger\), characterizing the same CPTP map. The minimum number of Kraus operators is called the Kraus rank of the CPTP map, and is no larger than \(d^2\) with \(d\) being the Hilbert space dimension of the system.

3.1.1. Construction of CPTP maps with arbitrary Kraus rank

For the construction of arbitrary CPTP maps, Lloyd and Viola [118] first showed that it is sufficient to repeatedly apply Kraus rank-2 channels in an adaptive fashion, but they did not consider efficient construction with a low-depth quantum circuit. Recently Shen et al. [27] have extended the binary-tree construction for arbitrary positive operator-valued measure (POVM) [119] to an...
efficient protocol for CPTP map construction. In this protocol, a CPTP map with Kraus rank-\(N\) can be constructed with an ancilla qubit by the lowest possible circuit depth \(L = \log N\), where each round of operation consists of one joint unitary of system and ancilla and one QND measurement on the ancilla qubit. Below we will briefly introduce such a binary-tree construction for CPTP maps.

Let us first consider the construction of a rank-2 CPTP map with Kraus operators \(\{K_i, K_j\}\), which can be achieved by only one round of operation: (1) initialize the ancilla qubit in \(|0\rangle\) (the qubit state basis being \(|0\rangle, |1\rangle\)); (2) perform a joint unitary operation \(U \in SU(2d)\) with \(d\) being the dimension of the system; (3) discard or trace over the ancilla qubit. The key point is to design \(U\) so that its \(d \times d\) submatrices satisfy \(|0/U|0\rangle = K_0, \langle1/U|0\rangle = K_1\).

The quantum circuit to implement a rank-\(N\) CPTP map with Kraus operators \(\{K_0, \ldots, K_N\}\) consists of \(L = \log N\) rounds of operations (Fig. 3a). Each round of operation includes: (1) initialization of the ancilla qubit in \(|0\rangle\); (2) joint unitary gate over the system and ancilla (conditional on the measurement outcomes from previous rounds); (3) QND readout of the ancilla, and (4) storage of the classical measurement outcome for later use. The \(\text{th}\) round unitary gate \(U_{\text{th}}\) is represented by the node of the binary tree \(b^{(d) = (b_1 b_2 \ldots b_d)} \in \{0, 1\}^d\) with \(l = 0, \ldots, L - 1\), while the Kraus operators \(K_{b^{(d)}}\) are associated with the leaves of the binary tree \(b^{(d)} \in \{0, 1\}^d\) (Fig. 3b). A systematic way is presented in Ref. [27] to design the nodes \(U_{\text{th}}\) so that the leaves of the binary tree are exactly the desired Kraus operators, \(K_{b^{(d)}} = K_l\). Arbitrary quantum channels can also be constructed in the quantum circuit model including controlled-not (CNOT), single-qubit gates and partial trace operations on the qubits and any ancilla, and with free single-qubit gates the minimum number of CNOT gates has been found in Ref. [120].

3.1.2. Physical implementation with circuit QED

Circuit QED in the dispersive regime is a promising platform to implement the arbitrary CPTP map construction. The transmon qubit acts as the ancilla (the transmon state \(|g/e\rangle\) corresponds to the ancilla state \(|0/1\rangle\) in the last subsection), and a \(d\)-dimensional subspace (e.g., the lowest \(d\) Fock states) of the storage cavity with high-quality-factor (high-Q) acts as the qubit. The QND readout of the transmon qubit can be realized by coupling a readout cavity with low-Q to the transmon. Then the readout result is fed back to a controller that induces an effector to implement the conditional control on the qubit (Fig. 3c).

Similar to the SNAP gates, we can implement the following entangling unitary gate for the whole system including the transmon and the cavity,

\[
U_{\text{ent}}(\vec{\theta}) = \prod_{i=0}^{d-1} \exp(-iY_{\theta_i}/2),
\]

where \(\vec{\theta} = (\theta_0, \ldots, \theta_l)\) and \(Y_n = -|g, n\rangle \langle e, n| + \text{H.c.}\) is the Pauli-\(Y\) operator for the two-dimensional subspace \(|g, n\rangle, |e, n\rangle\). The drive on the transmon for the above gate is \(H_{\text{ent}} = \sum_n \alpha_n e^{-i\theta_n X_n/2} |g, n\rangle \langle e, n| + \text{H.c.}\), where the driving amplitude \(\alpha_n\) and the gate time \(\tau\) should satisfy \(\theta_n = 2\alpha_n\tau\). This entangling gate produces a CPTP map with Kraus operators \(\{S_x, S_y\}\) with \(S_y = \text{diag}((\cos \theta_1, \ldots, \cos \theta_d))\) and \(S_x = \text{diag}(\sin \theta_1, \ldots, \sin \theta_d)\). If we precede \(U_{\text{ent}}\) with a unitary \(V\) acting on the qubit alone and perform an conditional unitary \(W = |g\rangle \langle g| \otimes W_x + |e\rangle \langle e| \otimes W_y\) after \(U_{\text{ent}}\), the entangling gate becomes \(U_{\text{ent}} = WU_{\text{ent}}V\) (Fig. 3d), which is known as the “cosine-sine” decomposition [121] that can decompose an arbitrary unitary into CNOT and single-qubit gates. The Kraus operators corresponding to \(U_{\text{ent}}\) are \(|gU_{\text{ent}}|g\rangle = W_y S_y V\), \(|eU_{\text{ent}}|g\rangle = W_x S_x V\), which are singular value decomposition of any operator for the qubit [11] and therefore can simulate any rank-2 CPTP map. Likewise we can use such entangling gates to simulate the CPTP map with any Kraus rank.

Recently there have been several experiments for quantum channel simulations in various platforms, including trapped ions [91], nuclear magnetic resonance (NMR) system [122] and IBM’s cloud computer [123]. In particular, using a scheme similar to the above one, Hu et al. [45] first realized arbitrary quantum channel simulation for a single photonic qubit in circuit QED. Although this experiment only simulates quantum channels with Kraus rank-2 for a 2-level qubit with one round of adaptive control, a recent experiment has extended the capability to simulate arbitrary rank-16 channels for a 4-level qubit with 4 rounds of adaptive control [46]. For the platforms other than circuit QED, the real-time adaptive control is often the main limitation: for trapped ions, it is quite challenging to avoid the recoil problem when performing adaptive measurement for trapped ions; for NMR systems, single-shot readout is not available, so many ancillas must be used.
to simulate the adaptiveness; the IBM’s cloud computer does not allow real-time adaptive control.

The ability to construct an arbitrary CPTP map may have various applications, such as QEC and quantum state initialization/stabilization. For example, the simulated quantum channel enabling QEC can help achieve the Heisenberg limit in quantum metrology [124,125], and dissipative quantum circuits consisting of sequences of quantum channels subject to specific constraints can lead to finite-time robust state stabilization [126].

3.2. Robust quantum operations with adaptive control

The measurement-based adaptive control can also help achieve robust quantum operations, such as FT quantum measurements and FT quantum gates. Below we show the recent theoretical and experimental advances of FT operations enabled by adaptive control in circuit QED.

In Section 2, we have shown that universal control of a bosonic mode can be achieved with the aid of an ideal ancilla. In addition, the ancilla can measure the even–odd excitation number parity of the bosonic mode (as an error syndrome detecting single excitation loss). This can be done by first preparing the ancilla in state \( |\gamma\rangle \pm |\epsilon\rangle \sqrt{2} \), evolving with the dispersive Hamiltonian (Eq. (1)) for a time \( \pi/\chi \), and finally performing Ramsey interferometry on the ancilla to determine its phase. However, ancilla systems are typically more vulnerable to environmental noise, e.g., the transmon coherence time (\( \sim 1 \mu s \)) is much shorter than the cavity mode coherence time (\( \sim 10 \mu s \)), so the ancilla errors (e.g., relaxation error \( |\gamma\rangle \langle \epsilon |e\rangle \langle e |-|\gamma\rangle \langle \epsilon |) \) and dephasing error \( |\epsilon\rangle \langle e |g\rangle \langle g |-|\epsilon\rangle \langle e |) \) during the operation time can propagate to the oscillator and corrupt the encoded information irreversibly. This drawback calls for new operation schemes that are FT to these ancilla errors.

A recent experiment shows that the parity measurement of a cavity mode in circuit QED can be made FT to the ancilla transmon errors by using three transmon levels and adaptive control [48]. The three-level transmon \( |\gamma\rangle, |\epsilon\rangle, |f\rangle \) is coupled to a cavity mode with

\[
H_{\text{dis}} = -\gamma a^\dagger a (|\epsilon\rangle \langle e| + |f\rangle \langle f|). \tag{10}
\]

Note that the dispersive coupling strength is the same for the transmon in \( |\epsilon\rangle \) or \( |f\rangle \) (\( \chi \)-matching condition), which can be realized with an engineered side-band drive [48]. The dispersive Hamiltonian commutes with the dominant ancilla relaxation error \( (|\epsilon\rangle \langle f|) \) and also any ancilla dephasing error \( (c_\epsilon |\gamma\rangle \langle g| + c_\epsilon |\epsilon\rangle \langle e| + c_\gamma |\epsilon\rangle \langle f| \) with \( c_\epsilon, c_\epsilon, c_\gamma \in \mathbb{C} \). This ancilla error during the measurement is equivalent to an ancilla error at the end, so although the measurement fails if the error happens, the cavity logical state is still well protected and the measurement errors can be overcome by majority voting.

With the same three-level ancilla satisfying the \( \chi \)-matching condition, the SNAP gates in Section 2.1 can be made FT to the dominant ancilla relaxation error and any dephasing error by adaptive control [31,49]. Such a SNAP gate is implemented by applying the Hamiltonian that drives the \( |\gamma\rangle \rightarrow |f\rangle \) transition instead of the \( |\gamma\rangle \rightarrow |e\rangle \) transition, with the effective Hamiltonian in the interaction picture as

\[
H_{\text{int}} = \Omega_\epsilon (|f\rangle \langle f| \otimes S(-\phi) + |g\rangle \langle g| \otimes S(\phi)), \tag{11}
\]

Without any ancilla error, the logical gate on the cavity with the ancilla going from \( |\gamma\rangle \) to \( |f\rangle \) is the ideal SNAP gate \( S(\phi) \). With a single ancilla relaxation error \( |\epsilon\rangle \langle f| \) during the control, the ancilla ends in \( |\epsilon\rangle \) and the final logical operation is still \( S(\phi) \). With a single ancilla dephasing error (e.g., \( |f\rangle \langle f|-|g\rangle \langle g| \)) and a projective measurement of the ancilla after the gate, the ancilla may end in \( |f\rangle \) with the logical gate still being \( S(\phi) \), or end in \( |g\rangle \) with the logical gate being the identity operation (Fig. 4a, b). Thus the control protocol can be repeated if the ancilla is measured in \( |g\rangle \) until the SNAP gate succeeds. Such error-corrected SNAP gates have recently been experimentally realized [49] with a reduction of the logical gate error by a factor of two in the presence of naturally occurring decoherence, a sixfold suppression of the gate error with increased transmon relaxation rates and a fourfold suppression with increased transmon dephasing rates (Fig. 4c, d).

Recent theory shows that the error-corrected SNAP gate belongs to a general class of FT gates on a logical system protected against Markovian ancilla errors, called path-independent (PI) quantum gates [31]. The PI principle requires that for given initial and final ancilla states, the logical system undergoes a unitary gate independent of the specific ancilla path induced by control drives and ancilla error events. With a certain initial ancilla state, the desired quantum gate on the logical system is successfully implemented for some final ancilla states, while the other final ancilla states herald a failure of the attempted operation, but the logical system still undergoes a deterministic unitary evolution without loss of coherence. So the PI gate on the central system can be repeated until it succeeds. A special class of the PI gates is the error-transparent (ET) gates for a QEC code, theoretically proposed in Refs. [29,30] and experimentally demonstrated [50] against a specific system error.

The FT measurement and PI gates belong to an interesting class of CPTP maps, called quantum instruments [27]. For quantum instruments, both the classical measurement outcomes and the post-measurement states of the quantum system are tracked, with the corresponding CPTP map

\[
e_{\text{meas}}(\rho) = \sum_{\mu=1}^{M} e_{\mu}(\rho) \otimes |\mu\rangle \langle \mu|, \tag{12}
\]

where \( \{|\mu\rangle \langle \mu|\}_{\mu=1}^{M} \) is a set of \( M \) orthogonal projections of the measurement device, and \( \{e_{\mu}\}_{\mu=1}^{M} \) is a set of completely positive and trace nonincreasing maps while \( \sum_{\mu=1}^{M} e_{\mu}(\rho) \) preserves the trace. For the PI parity measurement, \( \{e_{\mu}\} \) contains either the parity measurement channels or the identity channel, while for PI gates \( \{e_{\mu}\} \) is a set of unitary channels.

4. Driven-dissipative control

The inevitable coupling of a quantum system to the reservoir generally deteriorates the coherence and coherent control of the system. However, in some cases, the system can be driven into a unitarily evolving steady subspace, which can encode and process the quantum information while being largely immune to environmental noise. This can be achieved by either reservoir engineering (designing both the system Hamiltonian and the coupling to the reservoir) or Hamiltonian engineering (designing only the system Hamiltonian), which are both called driven-dissipative control in this paper. In this section, we will discuss the formation and control of stabilized manifolds of Schrödinger cat states in cavity bosonic modes with both approaches.

4.1. Reservoir engineering

Reservoir engineering is a powerful technique to realize steady state or subspace in condensed matter physics and quantum information processing [127–129], since the steady state is often an exotic phase of matter that is difficult to stabilize in nature [128], while the steady subspace may be used to store, protect and process quantum information [129]. In particular, when the quantum system is coupled to a Markovian reservoir, the time evolution of the system is governed by the Lindblad master equation [130].
\[ \dot{\rho} = \mathcal{L}\rho = -i[H, \rho] + \sum_{\beta} D[F_{\beta}]\rho, \]  

where the Liouvillian \( \mathcal{L} \) is a superoperator on the system, \( H \) is the Hamiltonian of the system including the driving term, \( D[F_{\beta}]\rho = 2F_{\beta}\rho F_{\beta}^\dagger - F_{\beta}^\dagger F_{\beta}\rho - \rho F_{\beta}^\dagger F_{\beta} \) is the Lindbladian dissipator with \( F_{\beta} \), being the dissipation-inducing jump operator that can depend on a parameter. The Markovian reservoir engineering refers to the design of the system Hamiltonian \( H \) and the jump operators \( F_{\beta} \), so that a stabilized manifold consisting of multiple steady states \([32,131,132]\) is formed to encode quantum information and even allow QEC.

### 4.1.1. Stabilized manifold with quantum information

#### Single-mode two-photon process

Consider that a single cavity mode is driven by an external field such that it can only absorb photons in pairs, and the energy decay of the mode also happens in pairs of photons, then the Lindbladian master equation describing such a two-photon driven-dissipative process is

\[ \dot{\rho} = [\epsilon_2 a^2 - \epsilon_4 a^4, \rho] + D[\sqrt{\kappa_2} a^\dagger] \rho = D[\sqrt{\kappa_2}(a^2 - a^4)]\rho, \]

where \( \epsilon_2 \) and \( \kappa_2 \) are the driving amplitude and decay rate, respectively. The second line of the above equation shows that the driven-dissipative dynamics can be described by a single Lindbladian dissipator \( D[\sqrt{\kappa_2}(a^2 - a^4)] \) with \( \alpha = \sqrt{2\epsilon_2}/\kappa_2 \). The stabilized manifold is determined by \( D[\sqrt{\kappa_2}(a^2 - a^4)]\rho = 0 \), and any state satisfying \( a^\dagger(\xi) = x^\dagger(\xi) \) or \( a(\xi) = x(\xi) \) is in this manifold. Such a stabilized manifold also forms a decoherence-free subspace \([4]\). The stabilized manifold for two-photon process is the two-dimensional Hilbert space spanned by two coherent states \( \{ |\alpha, -\alpha\rangle \} \) (Fig. 5a). For any initial state \( \rho(0) \), the cavity...
mode asymptotically converges to some pure or mixed state $\rho(\infty)$ in such a stabilized manifold. For example, if the initial state is the vacuum state $\ket{0}$ or the single-photon Fock state $\ket{1}$, the asymptotic state is the pure even ($\ket{C_+}$) or odd ($\ket{C_-}$) Schrödinger cat state with

$$\ket{C_{+}^{N}} = N_{2}^{N}(|\bar{x} \pm -\bar{x}|),$$

where $N_{2}^{N}$ is a normalization constant.

The logical qubit can be encoded into the even–odd Schrödinger cat states $\{\ket{C_{+}}, \ket{C_{-}}\}$ (with large $\bar{x}$ so that $|\braket{C_{+}|C_{+}}| \approx 0$) [32]. A qubit encoded in such a way is called the dissipative-cat qubit. For such a logical qubit, the dephasing error $D[\sqrt{\kappa_{a}a\dagger a}]$ can be largely suppressed when $\kappa_{a} \ll \kappa_{b}$, while the single photon loss error $D[\sqrt{\kappa_{b}b\dagger b}]$ causes a bit-flip error and therefore cannot be suppressed by the two-photon process. Experimentally Leghtas et al. [51] first successfully confined the quantum states of a superconducting cavity to the stabilized manifold spanned by the even–odd cat states.

**Single-mode four-photon process.** The four-photon process is described by letting both the absorption from the driving field and the energy decay into the bath happen through quadruples of photons,

$$\dot{\rho} = \left[ \epsilon_{a}a^{\dagger}a - \epsilon_{a}^{*}a^{\dagger}a, \rho \right] + D[\sqrt{\kappa_{a}a\dagger a}\rho] = D[\sqrt{\kappa_{a}a^{\dagger}a\rho}][\sqrt{\kappa_{a}a\dagger a}].$$

The stabilized manifold is the four-dimensional Hilbert space spanned by $\{\ket{\pm \beta}, \ket{\pm i\beta}\}$ with $\beta = (2\epsilon_{a}/\kappa_{a})^{1/4}$. When the cavity mode starts at initial Fock states $\{\ket{0}, \ket{1}, \ket{2}, \ket{3}\}$, it asymptotically converges to the pure states

$$\ket{C_{\beta}^{0\text{mod}4}} = \mathcal{N}_{4}(\ket{C_{\beta}} + \ket{C_{\beta}^{\dagger}}),$$

$$\ket{C_{\beta}^{1\text{mod}4}} = \mathcal{N}_{4}(\ket{C_{\beta}} - i\ket{C_{\beta}^{\dagger}}),$$

$$\ket{C_{\beta}^{2\text{mod}4}} = \mathcal{N}_{4}(\ket{C_{\beta}} - \ket{C_{\beta}^{\dagger}}),$$

$$\ket{C_{\beta}^{3\text{mod}4}} = \mathcal{N}_{4}(\ket{C_{\beta}} + i\ket{C_{\beta}^{\dagger}}),$$

which form the four-component subspace of the Schrödinger cat states.

To suppress the single photon loss error, which is usually the dominant error channel of the cavity modes, we can use the encoding scheme that can track the single-photon jump event and perform QEC. This can be achieved by encoding the qubit into the logical subspace spanned by the two cat states $\{\ket{C_{\beta}^{0\text{mod}4}}, \ket{C_{\beta}^{2\text{mod}4}}\}$ with even photon number parity. Then a single photon loss changes the photon number parity from even to odd. The photon number parity of the cavity mode can be monitored in a QND manner by a Ramsey experiment on an ancilla transmon qubit dispersively coupled to the cavity.
Single-mode d-photon process. The two-photon and four-photon processes can be generalized to d-photon processes \((d = 2, 4, 6, \ldots)\) being an even integer) with
\[
\hat{\rho} = \left[ \hat{c}_d a^d - \hat{c}_d a^d, \hat{\rho} \right] + D[\sqrt{\hat{K}_d a^d}] \hat{\rho} = D[\sqrt{\hat{K}_d}(a^d - \gamma^d)] \hat{\rho},
\]
(18)
with \(\gamma = (2\kappa_d/\kappa_0)^{\frac{1}{d}}\). The stabilized manifold is the \(d\)-dimensional Hilbert spanned by \(\{|\gamma \lambda_i\rangle\}\), with \(\lambda_i = \exp(2i\pi i/d)\) \((i = 0, 1, \ldots, d - 1)\), which are \(d\) coherent states lying equi-distinctly in the phase space. The asymptotic states or cat code are \(d\) different superpositions of such \(d\) coherent states \(\{|C_j^d \rangle \mod d \} \]
\[(\mu = 0, 1, \ldots, d - 1)\]
with \(|C_j^d \rangle \mod d \rangle = N_d^{2-1} \sum_{\nu=0}^{d-1} \hat{\gamma}^{\nu} |\gamma \lambda_i \rangle \),
(19)
which is a superposition of \(\mu\) modd \(d\) Fock states. The \(d\)-dimensional Hilbert cat space can be divided into \(d/2\) subspaces labeled by \(s = 0, 1, \ldots, d/2 - 1\), where the \(s\)-subspace is spanned by two states \(\{|C_s^d \rangle \mod d \}, \{|C_{s+d}^{d} \rangle \mod d \} \) and may encode a logical qubit [17,18]. After losing \(k\) photons, the \(s\)-subspace is mapped to the \(s - k\) subspace. Hence we can distinguish up to \(d/2 - 1\) photon losses without destroying the encoded logical states by projectively measuring the excitation number \(d/2\) (called the \(s - d/2\) measurement). We can also encode a qubit into the \(d/2\)-dimensional subspace \(\{|C_{s+d}^{d} \rangle \mod d \}, \{|C_s^d \rangle \mod d \}, \ldots, \{|C_{d-1}^d \rangle \mod d \} \) that can correct a single photon loss error.

Multimode processes. The driven-dissipative processes can be extended from a single cavity mode to two modes with operators \((a, b)\) [133]. Suppose that both modes simultaneously absorb energy from the driving field and release energy to the bath through pairs of photons,
\[
\hat{\rho} = \left[ \hat{c}_d b^2 a^2 - \hat{c}_d a^2 b^2, \hat{\rho} \right] + D[\sqrt{\hat{K}_d a^d}] \hat{\rho} = D[\sqrt{\hat{K}_d}(a^d - \beta^d)] \hat{\rho}.
\]
(20)
The stabilized manifold is spanned by the pair-coherent/Barut-Girardello states [134]. Quantum information encoded in a subspace of such a manifold is immune to the dephasing errors in both modes. Most interestingly, arbitrarily photon loss errors in either mode can be corrected by continuously monitoring the photon number difference between the two modes. The two-mode generalization above can also be extended to the multimode case, with the additional advantage of being able to correct for higher-weight products of losses or for photon losses and gains at the same time [133].

4.1.2. Quantum gates by quantum Zeno dynamics

We have shown that the logical qubit encoded in the stabilized manifold can be dynamically protected from the photo loss and dephasing errors and therefore act a good quantum memory. It is also possible to perform universal gates on such a logical qubit. The arbitrary rotations around \(x\)-axis of a single qubit and the two-qubit entangling gate can be generated by quantum Zeno dynamics.

When a quantum system is frequently measured to determine whether it is in the initial state, the system will always stay in the initial state, which is called the quantum Zeno effect [135]. But if frequent measurements are performed to see if it is in a multi-dimensional subspace, the system is not frozen but evolves according to an effective Hamiltonian obtained by projecting the initial Hamiltonian into the measurement subspace. Such dynamics are called Quantum Zeno dynamics [136]. The driven-dissipative processes act as a continuous measurement on the quantum system to see if it is in the multi-dimensional stabilized manifold, so if we apply another driving Hamiltonian \(H\), the effective driving Hamiltonian is \(H_{\text{eff}} = H_{\text{dr}} + H_{\text{PC}}\) with \(H_{\text{PC}}\) being the projector onto the stabilized manifold (Fig. 5b).

For the two-photon process with the logical qubit \(\{|C_j^2 \rangle \mod 2\} \), we may apply a linear drive on the oscillator, \(H_{\text{eff}} = \epsilon_a (a + a^\dagger)\). The two-photon process acts as a continuous measurement which projects the driving Hamiltonian onto an effective \(x\)-axis rotation Hamiltonian in the qubit space,
\[
P_C H_{\text{eff}} P_C = \Omega_x X,
\]
(21)
where \(P_C = \{|C_0^2 \rangle \mod 2\} \) and \(X = \{|C_+^2 \rangle \mod 2\} \). One can see that a population transfer between the even state \(|C_0^2 \rangle \mod 2\) and the odd cat state \(|C_+^2 \rangle \mod 2\) is enabled by a resonant single-photon drive on the system (Fig. 5d, e). Recently Touzard et al. [52] have experimentally observed such coherent oscillations between the even and odd cat states by tuning the desired dissipation rate (two-photon loss rate \(\kappa_2\)) to be 2 orders of magnitude larger than the undesired dissipation rate (single-photon loss rate \(\kappa_1\)) (Fig. 5c).

For the four-photon process with the logical qubit \(\{|C_{\pm 2}^2 \rangle \mod 4\} \), the population transfer between two logical states needs a two-photon drive \(H_{\text{eff}} = \epsilon_b (a^2 + a^{2\dagger})\) with the projected Hamiltonian in the stabilized manifold as
\[
P_C H_{\text{eff}} P_C = \Omega_{\pm 2} (X_{\pm 2} + X_0),
\]
(22)
where \(P_C = \sum_{s=0, \pm 2}|C_s^4 \rangle \mod 4\rangle \langle C_s^4 | \). In the above effective Hamiltonian have two driving components: one acting on the qubit subspace to drive the Rabi oscillation between \(|C_0^4 \rangle \mod 4\rangle\) and \(|C_{\pm 2}^4 \rangle \mod 4\rangle\) and the other one acting on the remaining subspace to drive the Rabi oscillation between \(|C_0^4 \rangle \mod 4\rangle\) and \(|C_{\pm 2}^4 \rangle \mod 4\rangle\) with the same driving amplitude. Such a gate has the additional advantage of being error-transparent to single-photon loss error, since in the stabilized manifold the single photon loss operator commutes with the effective Hamiltonian and therefore can be detected/corrected at the end of the gate without compromising the encoded quantum information [29,30]. For a general \(d\)-photon processes to a qubit, the \(x\)-axis rotation Hamiltonian is \(H_{\text{eff}} = \epsilon_x (a^d + a^{d\dagger})\).

In addition to the \(x\)-axis single-qubit gates, the two-qubit entangling gates can be realized by applying appropriate driving fields. To complete the set of universal gate, we may turn off the driven-dissipative control and apply a Kerr Hamiltonian to implement single-qubit \(\pi/2\)-rotation around the \(z\) axis [32]. The universal control of the qubits encoded in single modes can be extended to those encoded in multiple modes [133]. For example, an arbitrary \(x\)-axis rotation of the qubit encoded in double modes can be realized by the drive \(H_y = \epsilon_x (ab + b^{\dagger} a^{\dagger})\).

4.2. Hamiltonian engineering

Apart from reservoir engineering, it is also possible to form and process a stabilized manifold by only Hamiltonian engineering. The stabilized manifold can be chosen to be a degenerate eigenspace of the system with a designed Hamiltonian, which is typically decoupled to the remaining eigenspace by a large energy gap and therefore can be protected from specific system errors (Fig. 6a).

4.2.1. Formation of the Kerr-cat qubit

Consider the Hamiltonian of a Kerr-nonlinear resonator under the application of a single-mode squeezing drive [33,137], written in a frame rotating at the resonator frequency \(\omega_c\),
The subspace is separated from the rest of Hilbert space by a large energy gap, which is the degenerate eigenstates of this Hamiltonian. This cat state evolves to the states $|\text{vac}\rangle$ and $|\text{single-photon}\rangle$, which are two eigenstates with a large energy gap from the other eigenstates. (b) Photograph of the nonlinear resonator (purple frame) with a squeezing drive. The even- and odd-parity cat states $|C_n^+\rangle$ and $|C_n^-\rangle$ are the eigenstates with a large energy gap from the other eigenstates. (c) Schematic of the nonlinear resonator with pad offset $\delta$ to set the dispersive coupling to the readout cavity and spiral symbol representing the nonlinear inductor (SNAIL element). (d) Scanning electron micrograph of the SNAIL element consisting of four Josephson junctions in a loop threaded by an external magnetic flux. (e) Pulse sequence for initialization ($|C_0^+\rangle$), Rabi oscillation and readout of the Kerr-cat qubit. Here $\omega_0 - 2\omega_d$, and $\omega_d$ is the frequency of the readout cavity. The inherent nonlinearity of the Kerr-cat mode provides the ability to implement fast, high-fidelity gates. It also naturally provides the ability to parametrically engineer two-photon dissipation, which can be subsequently used for autonomous correction of possible leakage errors. Recently, the adiabatic preparation of Kerr-cat was experimentally demonstrated and the asymmetry in the bit- and phase-flip errors was also confirmed. Fig. 6b–d shows the device of the superconducting setup for realization of the Kerr-cat qubit.

$$H_{\text{Kerr}} = -K\alpha^2 \alpha + \omega_0 \alpha^2 + \omega_d \alpha^2, = -K(a^2 - \alpha^2)(a^2 - \alpha^2) + \frac{|\alpha|^2}{K}$$

Here $K$ is the strength of the nonlinearity and $\alpha = \sqrt{\frac{\omega_0}{\omega_d}}$. The second line makes it clear that the even- and odd-parity cat states $|C_n^+\rangle$ and $|C_n^-\rangle$ are the degenerate eigenstates of this Hamiltonian. This cat subspace is separated from the rest of Hilbert space by a gap $\omega_{\text{gap}} = 4K|\alpha|^2$ [33,137]. A qubit encoded in such a way is called the Kerr-cat qubit. Observe that as the strength of the two-photon drive decreases, that is $|\alpha| \to 0$ and hence $|\alpha| \to 0$, the states $|C_n^+\rangle$ and $|C_n^-\rangle$ continuously approach the vacuum and single-photon Fock state, respectively. In fact, in this limit the Kerr-cat qubit is essentially the well-known transmon which encodes a “Fock qubit” in the two-photon number states: vacuum and single-photon Fock state. It follows that, an initially undriven Kerr-nonlinear resonator (= Fock-qubit) prepared in vacuum or single-photon Fock state will respectively evolve to the states $|C_n^+\rangle$ or $|C_n^-\rangle$ as the amplitude of the squeezing drive is increased adiabatically. For the adiabatic condition to be satisfied, the rate of change of the two-photon drive must be slower than the minimum energy gap, $|\alpha_f(t)|/|\alpha_0(t)| < 2K$. So typically a large Kerr-nonlinearity results in faster cat state. Nevertheless, it is possible to apply counter-adiabatic two-photon drive to go faster than the adiabatic condition would allow [33].

Like the case of a dissipative-cat qubit (Eq. (14)), the probability of a bit-flip error (e.g., due to frequency fluctuations $\Delta f/\sqrt{K_0 d \bar{a}}$) is exponentially suppressed compared to a phase-flip error [for example due to single photon loss $\Delta f/\sqrt{\bar{a}}$] in the Kerr-cat qubit as well. While the dissipative-cat qubit is protected against bit-flip errors by a decoherence-free subspace enabled by engineered dissipation [36,53,138], the Kerr-cat qubit is protected from such errors by the underlying eigenspace structure of the two-photon driven Kerr-nonlinear resonator [34,35,54]. Interestingly, the Kerr- and dissipative-cat qubit realizations are completely compatible with each other and have complementary properties [33–35].
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freely evolves under the Kerr-nonlinearity and a
two-photon pump. When this pump is turned off, the cat-qubit
the driven Kerr-nonlinear resonator. Consequently, in order to
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error as a lin-

iments a X(θ) = exp(ıθX/2) operation [33–35], where θ = ΩT
T being the evolution time. Since r − r−1 ∼ 2e−2x2 in the limit
of large x, the Rabi oscillations around y-axis is exponentially sup-
pessed with x2. The Rabi oscillations of the Kerr-cat qubit were
demonstrated in a recent experiment [54], as shown in Fig. 6e–i.
Readout of the Kerr-cat qubit can be realized by coupling the
Kerr-cat cavity to a line resonator with a beam-splitter interaction
followed by a homodyne measurement of the line resonator [54].
Furthermore, it follows from Eq. (24) that a resonant beam-
splitter interaction, generated parametrically between two driven
nonlinear resonators, leads to an Ising coupling and realizes a
XX(θ) = exp(ıθX2X2) gate [33–35].

The Kerr-cat qubit has an asymmetric noise channel such that Y
and Z errors are exponentially suppressed. This asymmetry illus-
trates that the qubit couples to the environment predominantly
along the x-axis, while coupling along the y and z-axis is sup-
pressed. This coupling asymmetry, also evident from Eq. (24),
results from the Hilbert-space structure of the Hamiltonian of the
driven Kerr-nonlinear resonator. Consequently, in order to
allow coupling to the z-axis, it becomes necessary to turn-off the
two-photon pump. When this pump is turned off, the cat-qubit
freely evolves under the Kerr-nonlinear and a Z(π/2) gate is
realized after a duration π/2k [54,55,139]. After finishing the op-
eration, the two-photon pump can be turned on again in order to
recover the Kerr-cat qubit. It is important to note that unlike the
X(θ) and XX(θ) gates, a Z(π/2) rotation propagates a X error as a lin-
ear combination of X and Y errors and consequently destroys the
underlying asymmetric noise structure of the qubit [35].

Remarkably, recent theory shows that it is possible to realize a
two-qubit, controlled-Z (CZ) gate without turning off the
two-photon drive and thereby preserving the structure of the noise bias
(termed as CX gate in Ref. [35] due to the different bases adopted
there). The ability to implement a bias-preserving CZ gate makes
the Kerr-cat qubits desirable for efficient quantum error correction.

Moreover, the CZ gate can be implemented with parametric drives
and four-wave mixing via the inherent Kerr-nonlinearity in the
cat-qubit mode, which is very convenient to realize as no addi-
tional coupling elements are required.

5. Holonomic quantum control

In the last section, we have shown that through Markovian
reservoir engineering, the Lindbladian dynamics can be designed
to support a multi-dimensional stabilized manifold or
decoherence-free subspace to encode the quantum information
without suffering dissipation, and it is also possible to realize uni-
versal control of the states in the stabilized manifold with the aid
of the quantum Zeno dynamics. In this section, we will show that
the universal control of the states in the stabilized manifold can be
achieved by an alternative method – holonomic quantum control.

In holonomic quantum computation (HQC) [4,140,141], the
qubit states undergo adiabatic closed-loop parallel transport in
parameter spaces, acquiring Abelian Berry phases [142] or non-
Abelian adiabatic quantum holonomies [143] to achieve noise-
resistant universal computation. Recently Albert et al. [37,132]
introduced the idea of HQC to Markovian reservoir engineering
and found that universal computation of a quantum system con-
sisting of superpositions of well-separatated coherent states of single
or multiple harmonic oscillators can be achieved by three families
of adiabatic holonomic gates, including the loop gates, collision
gates for single oscillator mode and controlled-phase gates for
multiple harmonic oscillators. Below we will briefly introduce
the first two family of gates.

Consider the following Lindbladian for a single oscillator,
\[
\dot{\rho} = D \left[ \sum_{q} \left( a - \alpha_{q}(t) \right) \rho \right],
\]
which is a generalization of Eq. (18) that supports the stabilized
manifold spanned by a set of coherent states \(|\alpha(t)\rangle, \ldots, |\alpha_{k}(t)\rangle\). Note that different from the constant parameters \(|\alpha_{q}\rangle\n\]
in the last section, \(|\alpha(t)\rangle\) here is time-dependent and can be tuned
as external parameters. Then by adiabatically changing \(|\alpha(t)\rangle\n\]
through closed paths in phase space, the stable coherent states
\(|\alpha(t)\rangle\) also undergo the same adiabatic evolutions.

5.1. Loop gates

One type of the holonomic control is the loop gate, which can
accumulate tunable relative Berry phases over superpositions of
stabilized coherent states. First consider the simple case with
d = 2 (e.g., the single-mode two-photon process), the steady state
space is \(|\alpha_{0}(t)\rangle, |\alpha_{1}(t)\rangle\) with \(\alpha_{0}(t) = -\alpha_{1}(t) = \alpha (Fig. 7a)\). This sta-
bilized manifold holds the even–odd cat qubit \(|C_{0}\rangle, |C_{1}\rangle\).
Suppose that \(\alpha_{1}(t)\) undergoes an adiabatic variation through a closed path
while \(\alpha_{0}(t)\) is kept constant and well separated from \(\alpha_{1}(t)\n
(Fig. 7a,b), the state \(|\alpha_{1}(t)\rangle\) will accumulate a Berry phase \(\theta = 2\pi
A\) with A being the area enclosed by the closed path. Such an opera-

**Fig. 7.** (Color online) Holonomic gates for the logical subspace spanned by \(|C_{0}\rangle, |C_{1}\rangle\). (a) Wigner function sketch of the state before (top) and after (bottom)

(termed as CX gate in Ref.[35] due to the different bases adopted
there). The ability to implement a bias-preserving CZ gate makes
the Kerr-cat qubits desirable for efficient quantum error correction.
5.2. Collision gates

The other type of the holonomic control is the collision gate, which can coherently convert the population of a stabilized coherent state to another. To get the idea of collision gates, notice that there are two distinct parameter regimes for the even-odd cat states: \(\alpha \gg 1\) and \(\alpha \ll 1\). In the regime \(\alpha \gg 1\), the cat states \(|C^+\rangle\) are well separated and nearly orthogonal. However, in the regime \(\alpha \ll 1\), the cat states are reduced to the Fock states with \(|C^+\rangle \to |0\rangle\) and \(|C^+_\alpha\rangle \to |1\rangle\), so a bosonic rotation \(R_\alpha = \exp(i\phi \alpha a)\) will make the two Fock states \(|0\rangle\) and \(|1\rangle\) accumulate a relative phase \(\phi\). If we start with the even-odd cat qubit \(|C^+\rangle\) with large \(\alpha\), first reduce \(\alpha\) to 0, then apply bosonic rotation \(R_\alpha\), and finally drive back from 0 to \(\alpha\), the net result is that \(|C^+\rangle\) and \(|C^+_\alpha\rangle\) accumulate a relative phase \(\phi\) with the implemented unitary

\[
U_{\text{coll}} = \exp(i\phi/2) \exp[-i\phi(|C^+_\alpha\rangle \langle C^+_\alpha| - |C^+\rangle \langle C^+|)/2] = \exp(-i\phi/2) \exp[-i\phi(\langle x| - \langle z|)/2] \tag{27}
\]

which performs a z-axis rotation for the even-odd cat qubit (Fig. 7f). Denote the nonunitary driving from 0 to \(\alpha e^{\pi/2}\) as \(S_\alpha\), then the collision gate can also be represented as \(S_\alpha R_{\alpha} S_\alpha^{-1} = R_{\alpha}(R_\alpha^\ast S_\alpha R_{\alpha} S_\alpha^{-1} = R_\alpha S_\alpha S_\alpha^{-1}\). So an equivalent construction of the collision gate is reducing \(\alpha\) to 0, driving back to \(\alpha e^{\pi/2}\) and rotating back to \(\alpha\) (Fig. 7d, e). The generalization of the collision gate to an arbitrary \(d\) is straightforward: start with the \(|(x| \langle x| \rangle \cdots \rangle\) configuration with \(\lambda_\alpha = e^{2\pi i/d}\) and large enough \(\alpha\), tune \(\alpha\) to zero (or close to zero), pump back to a different phase \(\alpha e^{\pi/2}\), and rotate back to the initial configuration. In the cat state basis with \(\alpha \ll 1\), \(|C_{\alpha}^{\text{mod}}\rangle \to |j\rangle\) will gain a phase proportional to its mean photon number [37].

Besides the adiabatic HQC approaches above, it is also possible to implement nonadiabatic HQC based on shortcuts-to-adiabatic (STA) dynamics [144]. Recent theory shows that in the ultrastrong and deep-strong coupling regimes of the Rabi model [145], STA methods can generate arbitrary nonclassical bosonic states and induce fast nonadiabatic gates in tens of nanoseconds [146,147].

6. Multi-mode quantum control

In all the sections above, we have concentrated on the quantum control of single bosonic mode, although sometimes we briefly mention the extension to the multi-mode control. In this section, we will focus on the quantum control of multiple bosonic modes, mainly how to entangle two bosonic modes, which is a prerequisite for universal quantum computation. Recent experimental advances in circuit QED to entangle cavity modes include preparation of two-mode cat state [148], two-mode W state [108] and three-mode W state [108], on-demand state transfer and entanglement generation [105,149–152], and the realization of CNOT gate [39], controlled-\(Z\) (CZ) gate [40], exponential-SWAP (eSWAP) gate [41] and teleported CNOT gate [47]. Below we will introduce the eSWAP gate and teleported CNOT gate.

6.1. Exponential-SWAP gate

The eSWAP gate can coherently transfer the states between two bosonic modes, regardless of the choice of encoding [25]. To illustrate the eSWAP gate, we first introduce the unitary SWAP operation \(S_y\) between any two bosonic modes \(a, a\), defined as \(S_y a S_y = a\) and \(S_y a S_y = a\) (the same relation for \(a, a\)). Applying the SWAP operation twice results in \(S_y a S_y S_y a = a\), which is the identity operation \(I_\mu\) for the two bosonic modes. The eSWAP gate is defined as

\[
U_{\text{eSWAP}} = \exp(i\theta S_y) = \cos \theta I_\mu + \sin \theta S_y,
\]

which represents a superposition of the identity and the SWAP operation with the superposition coefficient tunable by the rotation angle \(\theta\). At \(\theta = \pi/2, \pi/4\), the eSWAP gate is reduced to the SWAP gate and \(\sqrt{\text{SWAP}}\) gate, respectively. One powerful feature of the eSWAP gate is that it can entangle two bosonic modes for any bosonic code. To see this, suppose the qubit code for the bosonic mode is \(|0\rangle/|1\rangle = f_{\text{SWAP}}(a)|0\rangle\langle 0|\) with \(f_{\text{SWAP}}(a)\) being a function of \(a\) and \(|0\rangle\), the vacuum state for the \(i\)th bosonic mode, and the initial state of the \(i\)th and \(j\)th bosonic modes is \(|0\rangle_i |1\rangle_j\), then applying the eSWAP gate results in \(U_{\text{eSWAP}}(0)|0\rangle_i |1\rangle_j = \cos \theta |0\rangle_i |1\rangle_j + \sin \theta |1\rangle_i |0\rangle_j\). (Note that \(S_y f_{\text{SWAP}}(a) S_y = f_{\text{SWAP}}(a)|j\rangle\) and \(S_y |0\rangle_i |1\rangle_j = |0\rangle_i |0\rangle_j\).

To implement the eSWAP operations, one can first use an ancilla qubit with states \(|\rangle\rangle = \{|0\rangle|e\rangle\rangle\) coupled to the two bosonic modes to realize the controlled-SWAP or Fredkin gate \(C_{\text{y}} = |g\rangle\langle g| \otimes |e\rangle\rangle \otimes |e\rangle\rangle \otimes |0\rangle\rangle\). The Fredkin gate can be decomposed as (Fig. 8b)

\[
C_{\text{y}} = e^{-\frac{2\pi i e}{\alpha} - \frac{\pi}{2} |e\rangle\langle 1| \otimes |e\rangle\rangle} e^{\frac{2\pi i e}{\alpha}, |e\rangle\rangle} e^{\frac{2\pi i e}{\alpha} |e\rangle\langle 1| |e\rangle\rangle}.
\]

where the first and last unitaries are 50:50 beam splitters and the middle one is the controlled-phase shift (CPS) of one bosonic mode conditioned on the ancilla state. The CPS operation can be achieved by a dispersive coupling between the ancilla and the bosonic mode (Eq. (1)). Then the eSWAP gate can be realized as (Fig. 8c)

\[
U_{\text{eSWAP}}(0)|+\rangle |\langle +| = C_{\text{y}} X_{\alpha} S_{\alpha} |+\rangle |\langle +|,
\]

where \(|+\rangle = (|g\rangle + |e\rangle)/\sqrt{2}\), \(X_{\alpha} = e^{\theta (|e\rangle\langle e| - |g\rangle\langle g|)}\) and \(|\langle +|\rangle\rangle\) is the wave-function for the two bosonic modes.

Recently Gao et al. [41] have experimentally implemented the eSWAP operations in three-dimensional (3D) circuit QED system (Fig. 8a) and demonstrated high-quality deterministic entanglement between two cavity modes with several different encodings including the Fock- and coherent-state coding schemes. As opposed to the eSWAP gate, a traditional CNOT gate between the multiphoton qubits in two cavities has also be realized by the mediation of a driven ancilla transmon, with the driving pulse obtained from GRAPE optimal control algorithm [39]. Moreover, a geometric method has been utilized for realizing CZ gates between two logical qubits encoded in two cavities [40].

6.2. Teleported CNOT gate

A promising strategy toward scalable quantum computation is to adopt a quantum modular architecture (Fig. 9a), which is a distributed network of modules that communicate with one another through quantum and classical channels [153,154]. Each module is composed of two functional subsystems (Fig. 9b): the data qubits that store and process quantum information and the communication qubits that mediate interactions between different modules. The intra-modular operations between the data and communication qubits are performed independently in each module so that the crosstalk and residual interactions between different modules are minimized even for a scaled-up system, while the inter-modular operations between the data qubits are enabled by distributing entanglement between communication qubits.

Due to the isolation between different modules, the multi-qubit operations between modules cannot depend on direct interactions but instead utilize quantum teleported gates [28,155–157] that are enabled by entanglement sharing, local operations and classical communications. Consider two modules with the data qubits (D1 and D2) and communication qubits (C1 and C2), the teleported
The CNOT gate between D1 and D2 can be implemented by the following steps (Fig. 9c): (1) generation of entanglement in the communication qubits C1 and C2, (2) local operations performed within each module entangle the data and communication qubits, (3) measurement of C1 in the Pauli-\(Z\) basis and C2 in the Pauli-\(X\) basis and (4) classical communication and feedforward operations.

Recently Chou et al. [47] have experimentally realized such a teleported CNOT gate in a deterministic way in circuit QED. The experimental architecture consists of two modules (Fig. 9d). Each module consists of a high-\(Q\) 3D electromagnetic cavity as the data qubit, a transmon qubit as the communication qubit and a Purcell-filtered, low-\(Q\) stripline resonator for readout of the transmon qubit. The local operations on the data cavity mode in each module were realized by the optimal control pulses obtained by GRAPE method (see Section 2.2). The communication channel was realized by an additional cavity mode that functions as a quantum bus coupling to both communication qubits in the two modules. With the first-level bosonic binomial quantum code [12], the teleported CNOT gate was implemented deterministically with the process fidelity reaching 79%.

7. Summary and outlook

Encoding quantum information in bosonic modes is a hardware-efficient approach to quantum computation, and universal quantum control of the bosonic modes is a crucial step towards this goal. Here we have given an extensive account of the recent
advances in universal control of the bosonic modes. Although the approaches in this review were initially developed in the context of circuit QED, they can be extended to various other platforms, such as cavity QED [16], trapped ions [97], nanophotonics [158] and Rydberg atoms [159] in the strongly dispersive regime [101].

We have shown that universal control of a single bosonic mode can be achieved with the aid of an ancilla qubit. The SNAP gates of a harmonic oscillator (cavity resonator) can be implemented by indirect control of a dispersively coupled ancilla (transmon qubit), and the SNAP gates combined with displacement operation are sufficient for universal control. We can even construct arbitrary quantum channels for the oscillator by QND readout of the ancilla and quantum feedback control. However, it is still an open problem to find the optimal control of the qubit–oscillator system with minimized expenditure of energy and resources [160]. Another problem with this qubit–oscillator system is that the ancilla qubit usually suffer relaxation and dephasing errors during the quantum gates and these ancilla errors may propagate to the logical qubits in the oscillator and corrupt the quantum information. We have shown recent theoretical and experimental advances in this respect, including the theoretical discovery of PI gates [31] and experimental realization of FT parity measurement [48] and PI SNAP gates [49] in circuit QED.

Universal quantum control can also be achieved in some noise-resilient subspace of the bosonic modes. With the aid of reservoir engineering or Hamiltonian engineering, the bosonic modes may support some multi-dimensional decoherence-free subspace to encode quantum information. Applying appropriate drive can implement the desired unitary on this stabilized manifold allowed by quantum Zeno dynamics. Universal control in such stabilized manifold can also be achieved by holonomic quantum control, where the external parameters are tuned so that the stable states undergo some adiabatic evolutions. Recent experimental advances include the formation of stabilized manifold in two-photon process [51,54], Rabi population oscillations in such a manifold [52,54] and the formation and control of a Kerr–cat qubit [54]. However, it is still challenging to experimentally generate desired engineered dissipation that is much stronger than the undesired dissipations. Moreover, it remain unsolved to systematically extract high-order nonlinear Hamiltonian of the oscillator, in order to support high-dimensional steady state subspaces.

Apart from universal control of single bosonic modes, coupling different bosonic modes is also needed for universal quantum computation. We have introduced two approaches to entangling two bosonic modes with recent experimental realizations: the eSWAP gate independent of the bosonic encoding [41] and the teleported CNOT gate for a modular architecture [47]. It is interesting to further design some robust generalization of eSWAP gates that are FT to the ancilla errors and bosonic loss errors. Moreover, the teleported CNOT gate has only been realized for adjacent modules, and it will be the next milestone to demonstrate the non-local teleported gates using spatially separate modules.

Conflict of interest

The authors declare that they have no conflict of interest.

Acknowledgments

We acknowledge support from the ARO (W911NF-18-1-0020 and W911NF-18-1-0212), ARO MURI (W911NF-16-1-0349), AFOSR MURI (FA9550-19-1-0399), NSF (EFMA-1640959, OMA-1936118, ECC-1941583), NTT Research, the Packard Foundation (2013-39273), and the Startup Foundation of Institute of Semiconductors, Chinese Academy of Sciences (E05EBB11).

Author contributions

Liang Jiang, Wenlong Ma and S. M. Girvin organized the manuscript. Wenlong Ma and Liang Jiang carried out the literature search and wrote most parts of the manuscript. Shruti Puri wrote the part in Section 4.2. S. M. Girvin, Michel H. Devoret and Robert J. Schoelkopf contributed to the manuscript revision. All authors contributed to the discussion.

Appendix A. Supplementary materials

Supplementary materials to this article can be found online at https://doi.org/10.1016/j.scib.2021.05.024.

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