Discrete Sampling and Interpolation: Orthogonal Interpolation for Discrete Bandlimited Signals

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Dedicated to the memory of Sivatheja Molakala

Abstract—We study the problem of finding unitary submatrices of the discrete Fourier transform matrix. This problem is related to a diverse set of questions on idempotents on \( \mathbb{Z}_N \), tiling \( \mathbb{Z}_N \), difference graphs and maximal cliques. Each of these is related to the problem of interpolating a discrete bandlimited signal using an orthogonal basis.

Index Terms—Discrete Fourier transforms, Discrete time systems, Interpolation, Sampling methods

I. INTRODUCTION

The simplest form of the question we address is this:

Which submatrices of the discrete Fourier transform are unitary up to scaling?

For example, if from the 16 \( \times \) 16 Fourier matrix we select rows 0, 2, 8, 10 and columns 0, 1, 4, 5 then the resulting 4 \( \times \) 4 submatrix is unitary up to a factor of 4. How did we find it? See Section V-A. In fact, for the 16 \( \times \) 16 Fourier matrix there are exactly 4,352 unitary 4 \( \times \) 4 submatrices. How did we count them? See Section V.

This is not just a curiosity. The answers to the question, insofar as we know them, involve the structure of convolution idempotents, tiling the integers modulo a prime power, and maximal cliques and perfect graphs. It is moreover striking that these disparate topics intersect in the fundamental problem of sampling and interpolation for discrete signals, specifically when the interpolating basis, defined below, is orthogonal.

It is the sampling problem that motivated us. This work is a sequel to [1], where we considered other aspects of the sampling and interpolation problem. While we will refer to some of the results there we have tried to make the present paper self-contained.

We need a few definitions. Let \( N \) be a positive integer and let \( \omega_N = e^{2\pi i/N} \). For a discrete signal \( f \) the Fourier transform and its inverse are

\[
\mathcal{F} f(m) = \sum_{n=0}^{N-1} f(n) \omega_N^{-mn},
\]

\[
\mathcal{F}^{-1} f(m) = \frac{1}{N} \sum_{n=0}^{N-1} f(n) \omega_N^{mn}.
\]

With this definition \( \mathcal{F} \mathcal{F}^* = N I \) as matrices. Multiples of the identity also occur for the submatrices we look for, so to keep from qualifying every such statement from now on when we say “unitary” we mean “unitary up to scaling;” the multiple itself is not important. Also, the particular way our problems arise makes it more natural to seek unitary submatrices of \( \mathcal{F}^{-1} \) rather than of \( \mathcal{F} \), but the principles are the same.

We regard all discrete signals as mappings \( f : \mathbb{Z}_N \rightarrow \mathbb{C} \). Likewise we always consider index sets to be subsets of \( \mathbb{Z}_N \) so that algebraic operations on their elements are taken modulo \( N \). For \( \mathcal{J} \subseteq \mathbb{Z}_N \) let

\[
\mathbb{B}^\mathcal{J} = \{ f : \mathcal{F} f(n) = 0, n \in \mathbb{Z}_N \setminus \mathcal{J} \}.
\]

In words, \( \mathbb{B}^\mathcal{J} \) is the \( |\mathcal{J}| \)-dimensional subspace of signals whose frequencies are zero off \( \mathcal{J} \); there may be additional zeros but there are at least these. We do not assume that \( \mathcal{J} \) is a band of contiguous indices (mod \( N \)), but for short we still employ the term bandlimited for such signals. That is what is meant in the title of the paper.

If \( f \in \mathbb{B}^\mathcal{J} \) for some \( \mathcal{J} \) we now ask if all values of \( f \) can be interpolated from the sampled values \( f(i) \) with \( i \) drawn from an index set \( \mathcal{I} \subseteq \mathbb{Z}_N \). Call this the sampling problem for \( \mathbb{B}^\mathcal{J} \) with sample set \( \mathcal{I} \). If the answer is yes for a given \( \mathcal{I} \), which is not always so, then associated with \( \mathcal{I} \) is an interpolating basis \( \{ u_i : i \in \mathcal{I} \} \) of \( \mathbb{B}^\mathcal{J} \) for

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which

\[ f = \sum_{i \in \mathcal{I}} f(i)u_i \]

for any \( f \in \mathbb{B}^{\mathcal{J}} \). The point of an interpolating basis, as opposed to any other basis of \( \mathbb{B}^{\mathcal{J}} \), is that the coefficients are the components of \( f \) with respect to the natural basis of \( \mathbb{C}^N \). One has

\[ u_i(j) = \delta_{ij}, \quad i, j \in \mathcal{I}. \quad (1) \]

A basis of \( \mathbb{B}^{\mathcal{J}} \) is an interpolating basis if and only if it satisfies (1). See [1] for further general properties of interpolating bases. Every \( \mathbb{B}^{\mathcal{J}} \) has an interpolating basis but not every \( \mathbb{B}^{\mathcal{J}} \) has an orthogonal interpolating basis, and this is the starting point of our study.

It is easy to give a criterion for the sampling problem to have a solution, and this relates sampling to submatrices of the Fourier matrix. First, in general, for an index set \( \mathcal{K} \subseteq \mathbb{Z}_N \) let \( \mathbb{E}_K \) be the \( N \times |\mathcal{K}| \) matrix whose columns are the natural basis vectors of \( \mathbb{C}^N \) indexed, in order, by \( \mathcal{K} \). Then \( E_{\mathcal{I}}^Tf \) and \( E_{\mathcal{I}}^T(Ff) \) are respectively the column vectors of the samples of \( f \) on \( \mathcal{I} \) and the samples of \( Ff \) on \( \mathcal{J} \). Since when \( f \in \mathbb{B}^{\mathcal{J}} \) the values of \( Ff \) off \( \mathcal{J} \) are known to be zero, if we can recover \( E_{\mathcal{I}}^Tf \) in terms of \( E_{\mathcal{I}}^T(Ff) \) then \( \mathcal{I} \) solves the sampling problem for \( \mathbb{B}^{\mathcal{J}} \). One can check (see also [1])

\[ E_{\mathcal{I}}^Tf = (E_{\mathcal{I}}^T F^{-1} E_{\mathcal{J}})(E_{\mathcal{I}}^T F f). \quad (2) \]

Here \( E_{\mathcal{I}}^T F^{-1} E_{\mathcal{J}} \) is the submatrix of \( F^{-1} \) with rows indexed by \( \mathcal{I} \) and columns indexed by \( \mathcal{J} \). We conclude:

**Theorem 1 ([1])**: The sampling problem for \( \mathbb{B}^{\mathcal{J}} \) has a solution with a sample set \( \mathcal{I} \) if and only if the submatrix \( E_{\mathcal{I}}^T F^{-1} E_{\mathcal{J}} \) of \( F^{-1} \) is invertible.

In particular \( |\mathcal{I}| = |\mathcal{J}| \). Equation (2) also provides an interpolation formula,

\[ f = F^{-1}E_{\mathcal{J}}(E_{\mathcal{I}}^T F^{-1} E_{\mathcal{J}})^{-1} E_{\mathcal{I}}^T f = U(E_{\mathcal{I}}^T f), \]

expressing all the values of \( f \) in terms of the sampled values \( E_{\mathcal{I}}^T f \) on \( \mathcal{I} \). The columns \( \{u_1, u_2, \ldots \} \) of \( U \) are an interpolating basis of \( \mathbb{B}^{\mathcal{J}} \).

In [1] we were concerned primarily with universal sampling sets. These are the index sets \( \mathcal{I} \) for which \( E_{\mathcal{I}}^T F^{-1} E_{\mathcal{J}} \) is invertible for every index set \( \mathcal{J} \) of the same size as \( \mathcal{I} \). So \( \mathcal{I} \) is universal if having chosen rows in \( F^{-1} \) according to \( \mathcal{I} \), any choice of \( |\mathcal{I}| \) columns results in an invertible submatrix. In terms of the sampling problem, \( \mathcal{I} \) is a sampling set for any space \( \mathbb{B}^{\mathcal{J}} \); for a universal sampling set the interpolating basis for \( \mathbb{B}^{\mathcal{J}} \) changes with \( \mathcal{J} \), but where to sample does not.

When \( N \) is a prime power we found several necessary and sufficient conditions for \( \mathcal{I} \) to be universal. These results generalize the Chebotarev theorem, according to which any square submatrix of the \( N \times N \) matrix \( F \) is invertible when \( N \) is prime. We also had applications to uncertainty principles.

**Theorem [1]** only goes so far. We need to be able to solve (2) feasibly even in the presence of noise or numerical errors and universality does not guarantee such stable recovery. For this we need the matrix \( E_{\mathcal{I}}^T F^{-1} E_{\mathcal{J}} \) to be well conditioned, along the lines of stability conditions imposed by other models, [2], [3]. For stable recovery the sampling set \( \mathcal{I} \) needs to be such that the energy in the samples is a non-negligible fraction of the energy in the entire signal,

\[ \|E_{\mathcal{I}}^T f\| \geq \alpha \|f\|, \quad f \in \mathbb{B}^{\mathcal{J}}, \alpha > 0, \]

for a positive constant \( \alpha \). This constraint is typically imposed for discrete (not necessarily timelimited) signals, [3]. In the case we are investigating (timelimited discrete signals) we would typically want \( \alpha \) to be as large as possible, uniformly over \( \mathbb{B}^{\mathcal{J}} \). In other words,

\[ \min_{f \in \mathbb{B}^{\mathcal{J}}} \frac{\|E_{\mathcal{I}}^T f\|}{\|f\|} \geq \alpha, \quad f \neq 0, \]

is equivalent to a condition on the norm of the Fourier submatrix,

\[ \|E_{\mathcal{I}}^T F^{-1} E_{\mathcal{J}}\| \geq \alpha, \]

which requires that all the singular values of \( E_{\mathcal{I}}^T F^{-1} E_{\mathcal{J}} \) be at least \( \alpha \).

From this point of view, the opening question of this paper thus identifies the extreme case when \( E_{\mathcal{I}}^T F^{-1} E_{\mathcal{J}} \) has the largest possible norm, and we want to know:

- Given the frequency set \( \mathcal{J} \), is it possible to find a sampling set \( \mathcal{I} \) such that \( E_{\mathcal{I}}^T F^{-1} E_{\mathcal{J}} \) is unitary?

  If so we say that \( (\mathcal{I}, \mathcal{J}) \) are a unitary pair; rows of \( F^{-1} \) chosen according to \( \mathcal{I} \) and columns according to \( \mathcal{J} \). Of course, since \( (1/N) F^{-1} = F^{-1} \) we have that \( E_{\mathcal{I}}^T F^{-1} E_{\mathcal{J}} \) is unitary if and only if \( E_{\mathcal{J}}^T F^{-1} E_{\mathcal{I}} \) is, so being a unitary pair is symmetric in \( \mathcal{I} \) and \( \mathcal{J} \).

- It is equivalent to the preceding question to ask:

  Does \( \mathbb{B}^{\mathcal{J}} \) have an orthogonal interpolating basis?

  If the answer is yes then the set \( \mathcal{I} \) that indexes the orthogonal basis (that determines where to sample) will be called an orthogonal sampling set. Our emphasis is on finding orthogonal sampling sets.

  The only \( \mathbb{B}^{\mathcal{J}} \) having an orthonormal interpolating basis is all of \( \mathbb{C}^N \), while proper subspaces \( \mathbb{B}^{\mathcal{J}} \) having an orthogonal interpolating basis cannot be too big:

  **Proposition 1**: If \( \mathbb{B}^{\mathcal{J}} \) has an orthogonal interpolating basis then \( |\mathcal{J}| \leq N/2 \).
We can express the symmetry of \( \mathcal{I} \) and \( \mathcal{J} \) as a unitary pair, or as orthogonal sampling sets, as:

**Proposition 2:** An index set \( \mathcal{I} \) is an orthogonal sampling set for \( \mathbb{B}^\mathcal{J} \) if and only if \( \mathcal{J} \) is an orthogonal sampling set for \( \mathbb{B}^\mathcal{I} \).

See [1] for these results, and also Section [VI] for an interpretation (and proof) of Proposition [1]. The condition in Proposition [1] is not sufficient, but Theorem [2] in Section [III] provides a necessary and sufficient condition for \( \mathbb{B}^\mathcal{J} \) to have an orthogonal interpolating basis (sampling set) in the case that \( N \) is a prime power. Theorem [4] in Section [IV-A] provides a necessary and sufficient condition for \( \mathcal{I} \) and \( \mathcal{J} \) to be a unitary pair, again when \( N \) is a prime power. These are among the principal results of this paper and are considerably deeper than Propositions [1] and [2]. The restriction that \( N \) must be a prime power also came up, in different ways, in [1]. It is an unmistakeable challenge – not yet met – to surmount this.

We briefly outline other aspects of the paper. Our approach to the problem opens with an important connection between orthogonal sampling sets and the algebraic structure of the zeros of an idempotent \( h = \mathcal{F}^{-1} \mathcal{J} \) on \( \mathbb{Z}_N \) that comes from a given \( \mathbb{B}^\mathcal{J} \), where \( \mathcal{J} \) is the indicator function of \( \mathcal{J} \). The fundamental results are in the next section.

In Sections [III] and [IV] we introduce **digit-tables**. These are matrices that record the base \( p \) expansions of elements of an index set along with certain columns “marked” in accordance with the algebraic nature of the set of zeros of associated idempotents. Digit-tables have a relatively simple structure but we have found them to be an effective tool, and some of their properties are rather subtle. It is through digit-tables that we are able to prove the main results on existence of orthogonal sampling sets (Theorem [2]) and on unitary pairs (Theorem [4]), and to show constructively how they can be found. Theorem [4] on unitary pairs depends on the dual of a digit-table, defined in Section [IV-A]. A recursive property of digit-tables allows us in turn to count the number of orthogonal sampling sets and unitary pairs via generating functions.

Section [VIII] establishes a connection between orthogonal sampling sets, digit-tables, and a conjecture of Fuglede on tiling, in our case tiling \( \mathbb{Z}_N \). Section [VIII] introduces the **difference graph** of an idempotent, recasting the problem of finding orthogonal sampling sets in graph theoretic terms as a search for maximal cliques. This is of more than theoretical interest because we show in that section that the difference graphs we consider are perfect graphs, and hence the problem of finding a maximal clique is tractable.

In Section [VIII] we show how the methods in the present paper can be applied to find interpolation formulas for the discrete counterparts to the classical Nyquist-Shannon sampling theorem. Finally, Section [IX] an appendix to Section [VI] treats the question of finding idempotents with a prescribed zero-set. Of special interest is the use of Ramanujan’s sum from number theory.

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# II. Idempotents on \( \mathbb{Z}_N \) and their Zero-sets

There is a natural, one-to-one correspondence between index sets and idempotents for convolution, and properties of one can be used to study properties of the other. Given \( \mathcal{J} \subseteq \mathbb{Z}_N \) let \( 1_{\mathcal{J}} : \mathbb{Z}_N \rightarrow \{0,1\} \) be its indicator function. Since \( 1_{\mathcal{J}} 1_{\mathcal{J}} = 1_{\mathcal{J}} \) the function

\[
h_{\mathcal{J}} = \mathcal{F}^{-1} 1_{\mathcal{J}}
\]

is an idempotent, and \( h_{\mathcal{J}} \in \mathbb{B}^\mathcal{J} \) by definition of \( \mathbb{B}^\mathcal{J} \). We write simply \( h \) if the set \( \mathcal{J} \) is clear from the context. Conversely, if an idempotent \( h \) is given then its Fourier transform, having values in \( \{0,1\} \), is the indicator function for an index set.

The following lemma opens the way to a very diverse set of phenomena. A similar result, in a much different context, holds for discrete signals in \( l^2(\mathbb{Z}) \); see [4].

**Lemma 1:** An index set \( \mathcal{I} \subseteq \mathbb{Z}_N \) is an orthogonal sampling set for \( \mathbb{B}^\mathcal{J} \) if and only if \( |\mathcal{I}| = |\mathcal{J}| \) and \( h(i_1 - i_2) = 0 \) for \( i_1, i_2 \in \mathcal{I}, i_1 \neq i_2 \).

**Proof:** Recall that a matrix is circulant if and only if it is diagonalized by the Fourier transform. Consider the matrix

\[
H = (E_{\mathcal{J}}^T \mathcal{F}^{-1} E_{\mathcal{J}})(E_{\mathcal{J}}^T \mathcal{F}^{-1} E_{\mathcal{J}})^* = E_{\mathcal{J}}^T \left( \mathcal{F}^{-1} E_{\mathcal{J}} E_{\mathcal{J}}^T (\mathcal{F}^{-1})^* \right) E_{\mathcal{J}}.
\]

Since \( E_{\mathcal{J}} E_{\mathcal{J}}^T \) is a diagonal matrix with the diagonal equal to \( 1_{\mathcal{J}} \), it follows that the matrix \( G = \mathcal{F}^{-1} E_{\mathcal{J}} E_{\mathcal{J}}^T (\mathcal{F}^{-1})^* \) is circulant, with first column \( Nh \). Hence the matrix \( H \) is a submatrix of the circulant matrix \( G \), with rows and columns both indexed by \( \mathcal{I} \); in other words, the entries of the matrix \( H \) are given by \( Nh(i - j) \) where \( i, j \in \mathcal{I} \). Now if \( E_{\mathcal{J}}^T \mathcal{F} E_{\mathcal{J}} \) is unitary, then \( H \) is diagonal, which implies

\[
h(i_1 - i_2) = 0 \text{ whenever } i_1 \neq i_2 \in \mathcal{I}.
\]
In geometric terms, $h$ defines the orthogonal projection $K: \mathbb{C}^N \to \mathbb{B}^J$ via
\[ Kv = h * v. \]
The orthogonal complement to $\mathbb{B}^J$ is $\mathbb{B}^{J^c}$ where $J^c = \mathbb{Z}_N \setminus J$ is the complement of $J$.

The proof of Lemma [1] is in keeping with the question we asked at the beginning of the paper, but it is worth pointing out an essentially equivalent approach. Let $\tau: \mathbb{Z}_N \to \mathbb{Z}_N$ be the zero-set $\{ n \in \mathbb{Z}_N : h(n) = 0 \}$ of an orthogonal sampling set $h$. Any shift of $h$’s is
\[ (\tau h, \tau^2 h) = (h * h)(i - j) = h(i - j). \]

Any shift of $h$ is also in $\mathbb{B}^J$ and thus an index set $I \subseteq \mathbb{Z}_N$ having the property that $h(i_1 - i_2) = 0$ for $i_1, i_2 \in I$, $i_1 \neq i_2$ determines a set of $|I|$ orthogonal vectors in $\mathbb{B}^J$. When $I$ is an orthogonal sampling set we normalize as in \([1]\) to then obtain an orthogonal interpolating basis for $\mathbb{B}^J$ given by
\[ \{ h(0)^{-1} \tau^i h : i \in I \}. \]

This is the recipe for turning an orthogonal sampling set into an orthogonal interpolating basis. All vectors in an orthogonal interpolating basis have length $|J|/N = \|h\|^2$, and in particular $0$ is never in $\mathcal{Z}(h)$. We also observe the symmetry relation
\[ h(-m) = \overline{h(m)}, \]
so that both $m$ and $-m$ either are or are not in $\mathcal{Z}(h)$.

It is important for our work that the zero-set has a very particular algebraic structure. Let
\[ D_N = \{ a : a | N \text{ and } 1 \leq a < N \} \]
be the set of divisors of $N$ that are $< N$. Then:

**Lemma 2:** If $h$ is an idempotent then its zero-set $\mathcal{Z}(h)$ is the disjoint union
\[ \mathcal{Z}(h) = \bigcup_{k \in D(h)} \mathcal{A}_N(k) \]
for a set of divisors $\mathcal{D}(h) \subseteq D_N$, where
\[ \mathcal{A}_N(k) = \{ i \in \mathbb{Z}_N : (i, N) = k \}. \]

Here $(i, N)$ is the greatest common divisor of $i$ and $N$. The equivalence relation $m_1 \sim m_2$ if $(m_1, N) = (m_2, N)$ already partitions $\mathbb{Z}_N \setminus \{ 0 \}$ into the disjoint union
\[ \mathbb{Z}_N \setminus \{ 0 \} = \bigcup_{k \in D_N} \mathcal{A}_N(k). \]

We also have
\[ \mathcal{A}_N(k) = k(\mathbb{Z}_N/k)^\times, \]
where $(\mathbb{Z}_N/k)^\times$ is the multiplicative group of units in the ring $\mathbb{Z}_N/k$, so $\mathcal{A}_N(k)$ is $k$ times the elements in $\mathbb{Z}_N$ that are coprime to $k$. In brief, the lemma says that the zero-set of an idempotent is essentially the disjoint union of multiplicative groups.

**Proof:** We show that if $m \in \mathcal{Z}(h)$ and $(r, N) = 1$ then $h(mr) = 0$, in other words, if $h$ vanishes at one element of an $\mathcal{A}_N(k)$ then it vanishes on all of $\mathcal{A}_N(k)$. This will prove the result.

Introduce the polynomial
\[ p_\mathcal{J}(x) = \sum_{j \in \mathcal{J}} x^j. \]

To say that $m \in \mathcal{Z}(h)$ is to say that $p_\mathcal{J}(\omega^m_N) = 0$. The order of $\omega^m_N$ is $s = N/(m, N)$ and $\omega^m_N$ is a root of the cyclotomic polynomial
\[ \Phi_s(x) = \prod_{(l,s)=1} (x - \omega^l_s). \]

Since $\Phi_s(x)$ divides any monic polynomial that vanishes at a primitive $s$’th root of unity it divides $p_\mathcal{J}(x)$, and it follows that $p_\mathcal{J}(x)$ must vanish at all the $\omega^l_s$ with $(l, s) = 1$.

Now consider evaluating $h(mr)$ for $r$ coprime to $m$. We have
\[ h(mr) = \frac{1}{N} \sum_{j \in \mathcal{J}} (\omega^m_N)^j = \frac{1}{N} p_\mathcal{J}(\omega^m_N). \]

But $\omega^m_N$ is also a primitive $s$’th root of unity, hence a root of $\Phi_s(x)$ and in turn a root of $p_\mathcal{J}(x).$ \(\blacksquare\)

We note one quick corollary.

**Corollary 1:** If $N$ is prime then either $\mathcal{Z}(h) = \emptyset$ or $\mathcal{Z}(h) = (\mathbb{Z}_N)^\times$. In the latter case $\mathbb{B}^J = \mathbb{C}^N$.

**Proof:** If there is a $k \in D_N \cap \mathcal{Z}(h)$ then we must have $k = 1$ and $\mathcal{Z}(h) = (\mathbb{Z}_N)^\times$. In this case $h(m) = \delta_{m0}$ and $Fh = (1, 1, \ldots, 1)$, so $\mathcal{J} = [0 : N - 1].$ \(\blacksquare\)

We refer to $\mathcal{D}(h)$, which we now know to be $D_N \cap \mathcal{Z}(h)$, as the zero-set divisors of $h$. It is helpful to describe $\mathcal{Z}(h)$ all at once as
\[ \mathcal{Z}(h) = \{ i : (i, N) \in \mathcal{D}(h) \}. \]
and then to restate Lemma 1 as saying that \( I \) is an orthogonal sampling set for \( \mathbb{B}^J \) if and only if
\[
(i-j, N) \in D(h), \quad i \neq j \in I. \tag{10}
\]

Lemma 2 and the proof given here are in [5]. After that thesis and this paper were written, we found, to our chagrin though it probably should not have been to our surprise, that the result was also known to Tao. [6].

A. A converse to Lemma 2?

To study orthogonal sampling sets we will need both Lemma 2 and a converse. The converse would ask to find an idempotent whose zero-set is a prescribed disjoint union of multiplicative groups, and this cannot be done in all cases. For example, let \( N = 6 \) and \( Z = \{2, 3, 4\} \). The set \( Z \) can be presented in the form given in Lemma 2 namely \( Z = \{2, 4\} \cup \{3\} \), but an exhaustive search shows that there is no idempotent \( h \) on \( Z \) with \( Z(h) = Z \).

Nevertheless, in Section VI-B-A we will give a converse of Lemma 2 in the case when \( N \) is a prime power. We would know a great deal more if we knew what a more general converse might be.

B. Bracelets of index sets

As observed in [11], index sets for the sampling problem have some algebraic structure of their own. We already introduced the shift operator \( \tau : \mathbb{Z}_N \rightarrow \mathbb{Z}_N \), and we let \( \rho : \mathbb{Z}_N \rightarrow \mathbb{Z}_N \) be reversal, \( \rho(n) = -n \). We can apply these operations, in any combinations, to an index set \( I \) yielding the bracelet determined by \( I \). From Lemma 1 and equation (5) we deduce:

**Lemma 3:** If \( I \) is an orthogonal sampling set then so is any set in the bracelet determined by \( I \).

Beyond this particular fact, in all cases we consider if a result holds for an index set \( J \) it also holds for the bracelet of \( J \). This is useful in some of our arguments, for example allowing for a translation to assume that \( 0 \in J \).

III. DIGIT-TABLES AND ORTHOGONAL SAMPLING SETS

We assume a space of bandlimited signals \( \mathbb{B}^J \) is given together with the associated idempotent \( h = F^{-1}_{1,J} \) and its zero-set divisors \( D(h) \). To get an existence theorem we must assume that \( N \) is a prime power, so our understanding is incomplete. The main work in this section is to establish:

**Theorem 2:** Let \( N = p^M \). Then \( \mathbb{B}^J \) has an orthogonal sampling set if and only if \( |J| = p^{D(h)} \).

Armed with an understanding of zero-sets, our interest is in finding an index set \( I \) such that any nonzero difference of two elements of \( I \) lies in \( Z(h) \), or, recalling (10), such that \( (i-j, N) \in D(h), i \neq j \). It is the latter condition that we work with. To do so we use the base \( p \) expansions of the numbers in \( \mathbb{Z}_N \) and the relation between the digits of the expansions and the greatest common divisor. The elementary, technical result we need is

**Lemma 4:** For \( z \in \{0, 1, 2, \ldots, p^M - 1\} \) let \( z = \sum_{j=0}^{M-1} a_j p^j \), where \( a_j \in \{- (p-1) : p - 1\} \). Let \( i = \min\{j : a_j \neq 0\} \), i.e., \( i \) is the index of the first nonzero coefficient among the \( a_j \). Then \( (z, p^M) = p^i \).

This may not hold when the coefficients \( a_j \) are not constrained to lie in \( \{- (p-1) : p-1\} \). Observe in particular that if for some \( s \) the expansion of \( z \) does not involve the coefficient corresponding to \( p^s \), i.e. if \( a_s = 0 \), then \( (z, p^M) \neq p^s \).

**Proof:** Clearly, \( p^i \mid z \) and so \( (z, p^M) \geq p^i \). Suppose \( (z, p^M) = p^i \) for some \( t > i \) and write
\[
z = \sum_{j=0}^{M-1} a_j p^j = \sum_{j=0}^{t-1} a_j p^j + \sum_{j=t}^{M-1} a_j p^j. \tag{11}
\]

Now, \( p^i \) divides the second term in (11), but it does not divide the first term, for the absolute value of the first term is
\[
\left| \sum_{j=0}^{t-1} a_j p^j \right| \leq \sum_{j=0}^{t-1} |a_j| p^j \leq (p-1) \sum_{j=0}^{t-1} p^j = p^t - 1,
\]
and so is not divisible by \( p^t \). Thus we cannot have \( p^i \mid z \) for \( t > i \). \( \blacksquare \)

Notationally, just for the present, write the numbers in \( \mathbb{Z}_N(h) \) in bold as \( i_0, i_1, \ldots \) and the base \( p \) expansions as
\[
i_k = \sum_{m=0}^{M-1} i_{km} p^m.
\]

From Lemma 4 \( (i_k - i_\ell, N) = p^{n_{k\ell}} \) where \( m_{k\ell}^* \) is the smallest power of \( m \) such that \( i_{km} - i_{\ell m} \) is not zero, and, again, for \( i_k - i_\ell \) to be in \( Z(h) \) we need \( (i_k - i_\ell, N) \in D(h) \). To put these together, and to understand our approach to proving Theorem 2 it is helpful to turn to an example.

1) Examples: Let \( N = 2^4 \) and suppose \( D(h) = \{2^1\} \), a singleton. If we are to have \( i_k - i_\ell \in Z(h) \) then we need
\[
(i_k - i_\ell, 2^4) = 2, \quad \text{i.e.} \quad m_{k\ell}^* = 1.
\]

and if \( i_0, i_1, \ldots \) are to form an orthogonal sampling set then we need all such \( m_{k\ell}^* \) to equal 1. The situation is described more completely in the display.
where we mark (box) the column of digits corresponding to the zero-set divisors, in this case a single column. For \(i_0, i_1, i_2, \ldots\) to form an orthogonal sampling set we require that:

1) The digits in the first column are all identical: \(i_{00} = i_{10} = i_{20} = \ldots\). This is to ensure that the \(m^*\) are nonzero.

2) The digits in the second column are all distinct. This along with the previous requirement ensures that all the \(m^*\) are 1.

The digits are all either 0 or 1 so clearly there can be at most two rows, i.e., an orthogonal sampling set can only be of size 2, and this can be achieved.

Staying with base 2, we extend the example and suppose \(D(h) = \{2^1, 2^3\}\). This is illustrated by the table

\[
\begin{array}{cccc}
2^0 & 2^1 & 2^2 & 2^3 \\
\hline
i_0 & i_{00} & i_{01} & i_{02} \quad i_{03} \\
i_1 & i_{10} & i_{11} & i_{12} \quad i_{13} \\
i_2 & i_{20} & i_{21} & i_{22} \quad i_{23} \\
\vdots & \vdots & \vdots & \vdots \\
\end{array}
\]

This time, for \(i_0, i_1, \ldots\) to constitute an orthogonal sampling set we require all the \(m^*\) to be either 1 or 3 and so:

1) The digits in the zeroth column must be identical: \(i_{00} = i_{10} = i_{20} = \ldots\). As before, this is to ensure that the \(m^*\) are nonzero.

However, unlike when \(D(h) = \{2\}\) we do not need the digits in the \(2^1\) column to be all distinct. It is possible, for example, for \(i_{01}\) to be equal to \(i_{11}\), provided \(i_{02} = i_{12}\) and \(i_{03} \neq i_{13}\). This will ensure that \(m^*_{01}\) is 3.

In other words, what we do need is:

2) The tuples \((i_{01}, i_{03}), (i_{11}, i_{13}), (i_{21}, i_{23})\) to be all distinct, and

3) The digits in the columns between the marked columns to be completely dependent on the marked digits preceding them, e.g., as above if \(i_{01} = i_{11}\) then \(i_{02} = i_{12}\). (We will come back to this condition in Section III-B)

From 2) above, it is easy to see that the largest such set can be of size \(2^2 = 4\). One of these largest sets can be constructed by setting all digits in the first and third columns to 0 and filling up the remaining two columns with all possible (binary) digits. Here is an example.

\[
\begin{array}{cccc}
2^0 & 2^1 & 2^2 & 2^3 \\
\hline
i_0 & 0 & 0 & 0 \\
i_1 & 0 & 0 & 1 \\
i_2 & 0 & 1 & 0 \\
i_3 & 1 & 0 & 1 \\
\end{array}
\]

Incidentally, from work in Section [VI-A] below, for the first example we find that one possible \(B^J\) has \(J = \{0, 4\}\) with \(h(n) = 1 + e^{2\pi i n/4}\), and for the second example one possible \(B^J\) has \(J = \{0, 1, 4, 5\}\) and \(h(n) = (1 + e^{2\pi i n/4})(1 + e^{2\pi i n/16})\).

A. Digit-tables

The essential features of these examples can be captured more generally to prove Theorem [2] We continue to assume that \(N = p^M\) and we let

\[\mathcal{P} = [0 : p - 1],\]

thought of as the set of base \(p\) digits. We consider matrices with entries in \(\mathcal{P}\) and with \(M\) columns, numbered from 0 to \(M - 1\), and where a subset of the columns are marked (boxed, as in the examples) at the indices

\[\mathcal{L} = \{i_0, i_1, \ldots, i_{\log d - 1}\},\]

with \(0 \leq i_0 < i_1 < \ldots < i_{\log d - 1}\). The number \(d\) will be a power of \(p\) so it is natural to denote the number of marked columns by \(\log d\), using the logarithm base \(p\). We call an array with this data a marked digit-table and denote it by \(m(M, \mathcal{L})\), or just \(m\) when the arguments are implicit. The number of rows is not specified by the data, nor is an ordering though we will sometimes arrange the rows lexicographically.

As in the examples, the rows are to be thought of as base \(p\) expansions of elements of \(\mathbb{Z}_N\). Thus a given marked digit-table determines an index set \(\mathcal{I} \subseteq \mathbb{Z}_N\) and, conversely, specifying an index set \(\mathcal{I}\) and a set of markings determines a marked digit-table. For our applications the markings of the columns will come from the zero-set divisors, as illustrated below with \(D(h) = \{p^0, p^2\}\):

\[
\begin{array}{cccc}
p^0 & p^1 & p^2 & p^3 \\
\hline
i_0 & i_{00} & i_{01} & i_{02} \quad i_{03} \\
i_1 & i_{10} & i_{11} & i_{12} \quad i_{13} \\
i_2 & i_{20} & i_{21} & i_{22} \quad i_{23} \\
\vdots & \vdots & \vdots & \vdots \\
\end{array}
\]

The indices of the marked columns, in this case \(\mathcal{L} = \{0, 2\}\), are simply the powers of \(p\) involved in \(D(h)\). It is convenient to write \(\mathcal{L} = \log D(h)\).

Next, borrowing from matrix theory, we define the pivots of a marked digit-table \(m\) to be the columns which
contain the first nonzero entry in some difference of rows:

\begin{align*}
\text{pivots of } m &= \{ j : i_{rj} \neq i_{r'j} \text{ and } i_{rk} = i_{r'k} \} \\
&\quad \text{for all } k < j, \text{ for some rows } r, r'
\end{align*}

To identify the special properties we need, we further say that \( m(M, \mathcal{L}) \) is a valid digit-table if two conditions are met:

1) The number of rows in \( m(M, \mathcal{L}) \) is \( p^{|\mathcal{L}|} \), and
2) The set of pivots of \( m(M, \mathcal{L}) \) is \( \mathcal{L} \).

To indicate a valid digit-table with data \( M \) and \( \mathcal{L} \) we use the notation \( m(M, \mathcal{L}) \), i.e. with a bar to indicate validity.

Based on these definitions, from Lemma 4 we can deduce:

Lemma 5: A set \( \mathcal{I} \) is an orthogonal sampling set for \( \mathbb{B}^\mathcal{J} \) if and only if the marked digit-table \( m(M, \mathcal{L}) \) determined by \( \mathcal{I} \) and the markings \( \mathcal{L} = \log D(h) \) is valid.

**Proof:** By Lemma 4 Property 2) of \( m \) is equivalent to the elements of \( \mathcal{I} \) satisfying \((i_j - i_k, N) \in D(h), i_j \neq i_k \). It follows that \( \mathcal{I} \) determines a set of orthogonal vectors in \( \mathbb{B}^\mathcal{J} \) if and only if Property 2) holds for \( m \).

Furthermore, Property 1) for \( m \) ensures that \( \mathcal{I} \) cannot be made larger. For suppose we add any new \( i_{d+1} \), resulting in a set of orthogonal vectors of size larger than \( p^{|\mathcal{L}|} \). Since \( \{i_{(d+1)r}\} \in \mathcal{I} \) can take at most \( p^{|\mathcal{L}|} \) values we would then have some \( i_k \) already in \( \mathcal{I} \) with \( i_{(d+1)r} = i_{kr} \) for all \( r \in \mathcal{L} \), by the pigeonhole principle. However, this would cause Property 2) to fail. Hence we see that \( \mathcal{I} \) is an orthogonal sampling set for \( \mathbb{B}^\mathcal{J} \) if and only if \( m \) is a valid digit-table.

This proves the necessity in Theorem 2 if \( \mathbb{B}^\mathcal{J} \) has an orthogonal sampling set \( \mathcal{I} \) then we must have \( |\mathcal{J}| = |\mathcal{I}| = \text{Number of rows in } m = p^{|D(h)|} \).

Conversely, when \( |\mathcal{J}| = p^{|D(h)|} \), we find an orthogonal sampling set by setting all digits in unmarked columns of \( m(M, \log D(h)) \) to 0 and filling up the marked columns with all possible digits from \( \mathcal{P} \). (We could also then order the rows lexicographically.) This is as in the example (12) above. An orthogonal sampling set constructed in this way can be described concisely as

\[
\mathcal{I} = \bigcup_{s_1, s_2, \ldots \in \mathcal{P}} \left\{ \sum_{p^k \in D(h)} s_ip^k \right\}.
\]

This completes the proof of Theorem 2.

As a corollary of Theorem 2 we obtain an inequality for idempotents. As a bound regulating the size of an idempotent, we think of this as a kind of uncertainty principle.

**Corollary 2:** Let \( N = p^M \) and let \( h : \mathbb{Z}_N \to \mathbb{C}^N \) be an idempotent. Then

\[
|D(h)| \leq \log |\mathcal{F}h|_0,
\]

**Proof:** Construct a valid digit-table as above with marked columns \( \log D(h) \), and let \( \mathcal{I} \) be the corresponding index set. Note that \( \mathcal{I} \) is then of size \( p^{|D(h)|} \). As in the proof of Lemma 3, the set \( \mathcal{I} \) determines an orthogonal set of vectors in \( \mathbb{B}^\mathcal{J} \). Therefore

\[
p^{|D(h)|} \leq |\mathcal{J}|.
\]

and \( \log |\mathcal{J}| = \log |\mathcal{F}h|_0 \).

**B. Constructing valid digit-tables**

In the proof of sufficiency in Theorem 2 we used a valid digit-table of a special type, filling the unmarked columns with zeros. It is useful to have a way to build general valid digit-tables.

**Proposition 3:** The valid digit-tables are exactly the marked digit-tables for which:

1) The digits in marked columns are filled with all possible \( d \)-tuples. To be precise,

\[
\bigcup_{r=0}^{d-1} \{ (i_{r0}, i_{r1}, i_{r2}, \ldots) \} = \mathcal{P}^d.
\]

2) The entries in unmarked columns preceding \( l_0 \) are all filled with equal entries: For any column \( m \notin \mathcal{L}, m < l_0 \) and rows \( r_1, r_2 \), we pick \( i_{rm} = i_{r_2m} \).

3) The entries of all other unmarked columns are filled up so that they are completely dependent on the entries in the marked columns preceding them. By this we mean for any column \( m \notin \mathcal{L} \) that has \( k \) marked columns preceding it we pick a map \( f_m : \mathcal{P}^k \to \mathcal{P} \), and set

\[
i_{rm} = f_m(i_{r0}, i_{r1}, \ldots, i_{r_k}),
\]

for any \( r \in [0 : d - 1] \).

**Proof:** From the first condition the number of rows in a digit-table constructed this way is clearly \( p^{|\mathcal{L}|} \). The second and third conditions ensure that for an unmarked column \( m \),

\[
i_{r,m} = i_{r_2m} \quad \text{whenever } (i_{r_10}, i_{r_11}, \ldots, i_{r_1k}) = (i_{r_20}, i_{r_21}, \ldots, i_{r_2k}),(i_{r_10}, i_{r_11}, \ldots, i_{r_1k}),
\]

in other words the entries in the \( m \)th column are completely determined by the entries in the marked columns \( l_0, l_1, \ldots \) preceding \( m \). This condition ensures that the difference of rows is always zero in the columns indexed by \( \mathcal{L}^c \), the complement of \( \mathcal{L} \), so the pivots are all in \( \mathcal{L} \).

This proposition provides an alternate definition of valid marked digit-tables.
C. Divisibility conjecture

We want to record what we think is an interesting conjecture. In Theorem 2 the fact that \(|\mathcal{I}|\) is a prime power is directly related to tiling \(\mathbb{Z}_N\), as we will explain in Section VII. When \(N\) is not itself a prime power we expect divisibility:

**Divisibility Conjecture:** For any \(N\), if a band-limited space \(\mathbb{B}\mathcal{J}\) has an orthogonal sampling set then \(|\mathcal{J}|\) divides \(N\).

We do not have a proof, but extensive checks confirm the conjecture in many cases. See also Section VII-B.

IV. THE RECURSIVE STRUCTURE OF VALID DIGIT-TABLES

It is not apparent from the definition, but valid digit-tables have a recursive property. This leads first to a stronger version of Theorem 2 incorporating unitary pairs of index sets, and then ultimately to a method for counting the number of orthogonal sampling sets and the number of unitary pairs.

To present this we need some additional notation. Given a digit-table \(\mathbf{m}(M, \mathcal{L})\) denote by \(\mathcal{L}_1\) the smaller set of marked columns

\[
\mathcal{L}_1 = \{l_1 - l_0 - 1, l_2 - l_0 - 1, \ldots, l_{\log d - 1} - l_0 - 1\}. \tag{14}
\]

In Theorem 3 we will express a valid digit-table with marked columns \(\mathcal{L}\) in terms of valid digit-tables with marked columns \(\mathcal{L}_1\).

We also denote by \(e + \mathbf{m}\) the digit-table obtained by adding the row \(e\) to each row of \(\mathbf{m}\). Conceptually, this addition represents (but is not the same as) the translation of the index set \(\mathcal{I}\) by the integer whose digits constitute \(e\). We easily see that if \(\mathbf{m}\) is valid so is \(e + \mathbf{m}\), as the pivots do not change as a result of translation.

The following theorem relates \(\mathbf{m}(M, \mathcal{L})\) to \(\bar{\mathbf{m}}(M - l_0, \mathcal{L}_1)\).

**Theorem 3:** A lexicographically ordered digit-table \(\bar{\mathbf{m}}(M, \mathcal{L})\) is valid if and only if it is of the form

\[
\bar{\mathbf{m}}(M; \mathcal{L}) = c + \begin{pmatrix} 0 + \bar{\mathbf{m}}(M - l_0; \mathcal{L}_1) \\ e_{l_0} + \bar{\mathbf{m}}(M - l_0; \mathcal{L}_1) \\ 2e_{l_0} + \bar{\mathbf{m}}(M - l_0; \mathcal{L}_1) \\ \vdots \\ (p - 1)e_{l_0} + \bar{\mathbf{m}}(M - l_0; \mathcal{L}_1) \end{pmatrix},
\]

where \(e_{l_0}\) is a row of zeros except a 1 in the \(i_{l_0}\)th slot and \(c\) is a row vector, allowing for a translate.

Here \(\bar{\mathbf{m}}\) represents a generic valid digit-table, so the tables \(\bar{\mathbf{m}}(M - l_0; \mathcal{L}_1)\) in each block above could be different, even though we use the same symbol for each. The order of the blocks is because of the lexicographical ordering of the rows.

**Proof:** Suppose \(\bar{\mathbf{m}}(M, \mathcal{L})\) is in the given form, made up of blocks of smaller, valid digit-tables. Each of the \(\bar{\mathbf{m}}(M - l_0, \mathcal{L}_1)\) has \(p|\mathcal{L}_1|\) rows, by validity, so putting \(p\) of them together yields \(p|\mathcal{L}_1| + 1 = p|\mathcal{L}|\) rows for the big table.

Now consider the pivots. The pivots for two rows in the same block are in \(\mathcal{L}_1\) by definition, and the pivots for two rows from different blocks are in the first marked column. It follows that the pivots for \(\bar{\mathbf{m}}(M, \mathcal{L})\) are in \(\mathcal{L}\), whence that \(\bar{\mathbf{m}}(M, \mathcal{L})\) is valid.

For the converse, let \(c\) be the \(M\)-vector containing the first \(l_0 - 1\) digits of (any) row of \(\bar{\mathbf{m}}\):

\[
c = (i_{k0}, i_{k1}, \ldots, i_{k(l_0-1)}, 0, 0, \ldots), \text{ for any } k.
\]

The vector \(c\) is well defined because all the rows start with the same \(l_0 - 1\) digits, as the first \(l_0 - 1\) columns do not contain a pivot; this is property 2) in Proposition 3.

With the rows of \(\bar{\mathbf{m}}\) arranged lexicographically, we obtain blocks depending on the entry in the \(l_0\) column:

\[
c + \begin{pmatrix} 0 + \bar{\mathbf{m}}(M - l_0; \mathcal{L}_1) \\ e_{l_0} + \bar{\mathbf{m}}(M - l_0; \mathcal{L}_1) \\ 2e_{l_0} + \bar{\mathbf{m}}(M - l_0; \mathcal{L}_1) \\ \vdots \\ (p - 1)e_{l_0} + \bar{\mathbf{m}}(M - l_0; \mathcal{L}_1) \end{pmatrix}
\]

The \(\bar{m}_i\) are digit-tables with marked columns \(\mathcal{L}_1\) and we want to show that they are also valid. First, the number of rows in each \(\bar{m}_i\) is \(p|\mathcal{L}_1|/p = p|\mathcal{L}_1|\). Second, the pivots in each of the \(\bar{m}_i\) have to be in \(\mathcal{L}_1\), for otherwise some pivots of \(\bar{m}\) would lie outside \(\mathcal{L}\), contradicting the validity of \(\bar{m}\). Thus each of the \(\bar{m}_i\) are valid, and the proof is complete.

This structure theorem has a number of consequences, the first being simultaneously a sharpening of Proposition 2 and a strengthening of Theorem 2.

A. Dual digit-tables and unitary pairs

Proposition 2 expresses a type of duality between a sampling set \(\mathcal{I}\) and a frequency set \(\mathcal{J}\) vis-a-vis the spaces \(\mathbb{B}\mathcal{I}\) and \(\mathbb{B}\mathcal{J}\). We use the term “duality” on purpose because by introducing the dual of a digit-table we can be more precise.

We define the dual of \(\bar{\mathbf{m}}(M, \mathcal{L})\) to be \(\bar{\mathbf{m}}(M, \mathcal{L}^*)\) where the dual set of marked columns \(\mathcal{L}^*\) is

\[
\mathcal{L}^* = M - \mathcal{L} - 1.
\]

The dual is defined for an entire type rather than for a particular digit-table. Note that the dual of \(\bar{\mathbf{m}}(M, \mathcal{L}^*)\) is \(\bar{\mathbf{m}}(M, \mathcal{L})\). The \(i\)th marked column of \(\bar{\mathbf{m}}(M, \mathcal{L}^*)\), starting from the leftmost being \(i = 0\), is \(M - l_{\log d - i - 1} - 1\).
With this definition, we can state a stronger version of Theorem 2 that characterizes both the sampling set and the frequency set in terms of dual digit-tables.

**Theorem 4:** Index sets $(\mathcal{I}, \mathcal{J})$ determine a unitary pair if and only if the digit-tables for $\mathcal{I}$ and $\mathcal{J}$ are of dual types.

**Proof:** As always let $h_\mathcal{J} = \mathcal{F}^{-1} \mathcal{J}$ and $h_\mathcal{I} = \mathcal{F}^{-1} \mathcal{I}$, and denote the corresponding zero-set divisors by $\mathcal{D}(h_\mathcal{J})$ and $\mathcal{D}(h_\mathcal{I})$.

We already know, from Lemma 5, that $(\mathcal{I}, \mathcal{J})$ form a unitary pair if and only if $\mathcal{I}$ corresponds to a valid digit-table of the type $\mathcal{m}(M, \log \mathcal{D}(h_\mathcal{J}))$. From Proposition 2 this is equivalent to $\mathcal{J}$ corresponding to a valid digit-table of type $\mathcal{m}(M, \log \mathcal{D}(h_\mathcal{I}))$. To prove Theorem 4 then, we need to show that $\log \mathcal{D}(h_\mathcal{J})$ and $\log \mathcal{D}(h_\mathcal{I})$ are a dual set of markings.

For this it is enough prove $\log \mathcal{D}(h_\mathcal{I}) \supseteq \log \mathcal{D}(h_\mathcal{J})^*$. To see why this suffices, under either hypothesis in Theorem 4 we have that $|\mathcal{I}| = p^{\mathcal{D}(h_\mathcal{J})}$ and $|\mathcal{J}| = p^{\mathcal{D}(h_\mathcal{I})}$. Combined with $|\mathcal{J}| = \mathcal{D}(h_\mathcal{J})$ we get $|\log \mathcal{D}(h_\mathcal{I})| = |\log \mathcal{D}(h_\mathcal{J})| = |\log \mathcal{D}(h_\mathcal{J})^*|$. Together with $\log \mathcal{D}(h_\mathcal{I}) \supseteq \log \mathcal{D}(h_\mathcal{J})^*$, which we establish as a separate lemma, we have $\log \mathcal{D}(h_\mathcal{I}) = \log \mathcal{D}(h_\mathcal{J})^*$ and this proves the result.

**Lemma 6:** Let $\mathcal{I}$ be an index set, and $g = \mathcal{F}^{-1} \mathcal{I}$. If $\mathcal{I}$ corresponds to a valid digit-table of the type $\mathcal{m}(M, \mathcal{L})$, then

$$\log \mathcal{D}(g) \supseteq \mathcal{L}^*.$$ 

We split this off from Theorem 4 because we will use the lemma in this form later. For Theorem 4 we apply the lemma with $g = h_\mathcal{I}$ and $\mathcal{L} = \log \mathcal{D}(h_\mathcal{J})$.

**Proof:** We prove this by induction on the size of $\mathcal{L}$ via Theorem 3. Recall the notation for $\mathcal{L}$ in (13) and also the definition of $\mathcal{L}_1$ in (14).

In preparation for the induction we make several observations. The digit-table for $\mathcal{I}$ has a decomposition of the form in Theorem 3. We will assume, without loss of generality, that $c = 0$. Note that each of the $\mathcal{p}$ smaller digit-tables determine an index set in $[0 : p^{M-\log \mathcal{I}} - 1]$, so let $g_b : \mathbb{Z}_{p^{M-\log \mathcal{I}}} \rightarrow \mathbb{C}, b = 0, \ldots, p - 1$, be the idempotents corresponding to these index sets.

Next, we have

$$g(n) = \sum_{m \in \mathcal{I}} \omega^{nm}.$$ 

Substitute the base-$\mathcal{p}$ expansion of each $m \in \mathcal{I}$ from the digit-table and split the resulting sum by blocks (there are a total of $p$ blocks). We get

$$g(n) = \sum_{b=0}^{p-1} \sum_{m \text{ block } b} \omega^{(b'y + \sum_{k=0}^{M-\log \mathcal{I}} i_{mk}p_0^{l+1})}.$$ 

Here $i_{mk}$ are the digits at row $m$ and column $l_0 + k + 1$. Thus

$$g(n) = \sum_{b=0}^{p-1} \sum_{m} \omega^{n(b'y + \sum_{k=0}^{M-\log \mathcal{I}} i_{mk}p_0^{l+1})}$$

$$= \sum_{b=0}^{p-1} \omega^{nb'y} \sum_{m} \omega^{nbp_0^{l+1}(\sum_{k} i_{mk}p^k)}$$

(15)

To find the zero-set divisors of $h$, first we see that

$$g(e^{2\pi i ob/p}) = (d/p) \sum_{b=0}^{p-1} e^{2\pi i ob/p}.$$ 

Here we used the fact that $g_b$ is periodic with period $p^{M-\log \mathcal{I}}$ and that $g_0(0)$ is size of each block is $d/p$ for any $b$. Now this summation vanishes whenever $\alpha$ is coprime to $p$. So we have that

$$\log \mathcal{D}(g) \supseteq \{M - \log \mathcal{I} - 1\}. \quad (16)$$

This establishes Lemma 6 when $\mathcal{L} = \{\log \mathcal{I}\}$ is a singleton. Next, from the induction hypothesis

$$\log \mathcal{D}(g_b) \supseteq \mathcal{L}_1^* \quad \text{for each } b,$$

i.e. the zero set divisors of each of the $g_b$ include $\mathcal{p} \mathcal{L}_1^*$. From (15), it follows that $\log \mathcal{D}(g) \supseteq \mathcal{L}_1^*$. Combined with (16) this completes the induction and the lemma is proved.

The notion of a valid digit-table and its dual also provides a blueprint for the construction of any unitary submatrix of the Fourier matrix. For example – the example mentioned in the first paragraph of the introduction – let $\mathcal{N} = 2^4$, and

$$\mathcal{I} = \begin{cases} 0+ & 0 \cdot 2^1 + 0 \cdot 2^2 + 0 \cdot 2^3 \\ 0+ & 1 \cdot 2^1 + 0 \cdot 2^2 + 0 \cdot 2^3 \\ 0+ & 0 \cdot 2^1 + 0 \cdot 2^2 + 1 \cdot 2^3 \\ 0+ & 1 \cdot 2^1 + 0 \cdot 2^2 + 1 \cdot 2^3 \end{cases} = \{0, 2, 8, 10\}$$

We know that $\mathcal{I}$ is an orthogonal sampling set when $\mathcal{D}(h_\mathcal{J}) = \{2^1, 2^3\}$, because $\mathcal{I}$ represents a valid digit-table with marked columns $\{1, 3\}$. One situation in which such a $\mathcal{D}(h_\mathcal{J})$ arises is from the frequency set $\mathcal{J}$ constructed from the dual set of marked columns $\{0, 2\}$:

$$\mathcal{J} = \begin{cases} 0+ & 0 \cdot 2^1 + 0 \cdot 2^2 + 0 \cdot 2^3 \\ 0+ & 0 \cdot 2^1 + 0 \cdot 2^2 + 1 \cdot 2^3 \\ 1+ & 0 \cdot 2^1 + 0 \cdot 2^2 + 0 \cdot 2^3 \\ 1+ & 0 \cdot 2^1 + 1 \cdot 2^2 + 0 \cdot 2^3 \end{cases} = \{0, 1, 4, 5\}.$$ 

So the submatrix $E_T^T \mathcal{F}^{-1} E_J$ of the $16 \times 16$ Fourier matrix is unitary. Indeed, by substituting the values we see that
The columns are clearly orthogonal.

V. COUNTING ORTHOGONAL SAMPLING SETS AND UNITARY PAIRS

Theorems 3 and 4 allow us to count the number of orthogonal sampling sets and unitary pairs. We first find the number of orthogonal sampling sets for a given $\mathbb{B}^{|J|}$ with $|J| = d$, for this is the number of valid digit-tables with given marked columns, and Theorem 3 provides a recursion.

**Lemma 7:** The number of valid digit-tables of type $\mathfrak{m}(M; L)$ is

$$|\mathfrak{m}(M; L)| = p^{|L|} (|\mathfrak{m}(M - l_0; L_1)|)^p,$$

where $L_1$ is given in (14).

**Proof:** From Theorem 3 the number of valid digit-tables with $c = 0$ is $(|\mathfrak{m}(M - l_0; L_1)|)^p$. We count the total number by finding the distinct translates.

When $c = 0$ the first $l_0 - 1$ digits in each row of $\mathfrak{m}$ are zeros. Hence the integers in the index set $I$ determined by the rows are all multiples of $p^{l_0}$. Distinct translates are then simply obtained by translating each such set $I$ in turn by $0, 1, 2, \ldots, p^{l_0} - 1$. So the total number of valid digit-tables is $p^{l_0} (|\mathfrak{m}(M - l_0; L_1)|)^p$.

Expanding upon the recursion, we see that

$$|\mathfrak{m}(M; L)| = p^{|L|} (|\mathfrak{m}(M - l_0; L_1)|)^p$$

$$= p^{|L|} p^{l_0(l_1 - l_0 - 1)} (|\mathfrak{m}(M - l_1; L_2)|)^p$$

$$= \ldots = \sum_{i=0}^{|L|} p^{i(l_i - l_{i-1} - 1)} (|\mathfrak{m}(M - l_i; L_{i+1})|)^p,$$

where we take $l_{log d} = M$.

Now let

$$r_i = l_i - l_{i-1} - 1.$$

Then $r_i$ represents the number of columns between the $i^{th}$ and $(i+1)^{th}$ marked columns (we take $l_{-1} = -1$).

Note that

$$\sum_{i=0}^{log d} r_i = \sum_{i=0}^{log d} (l_i - l_{i-1} - 1) = M - log d.$$

So we have

**Lemma 8:** The number of orthogonal sampling sets for $\mathbb{B}^{|J|}$ is given by $|\mathfrak{m}(M; L)| = p^{\lambda(r)}$ where

$$\lambda(r) = \sum_{i=0}^{log d} r_i p^i, \quad r = (r_0, r_1, r_2, \ldots),$$

where $r_i = l_{i+1} - l_i - 1$ is the number of columns between the $i^{th}$ and $(i+1)^{th}$ marked columns in the digit-table for any orthogonal sampling set of $\mathbb{B}^{|J|}$.

We obtain the total number of orthogonal sampling sets of size $d$ by summing over all possible $(r_0, r_1, \ldots, r_{log d})$ that sum to $M - log d$:

**Number of orthogonal sampling sets of size $d$**

$$= \sum_{r_i = M - log d} p^{\lambda(r)}$$

$$= \sum_{r_i = M - log d} p^{\sum_{i=0}^{log d} r_i p^i}.$$

From the form of this expression we observe that the count is given by a $(log d)$-fold convolution:

**Number of orthogonal sampling sets of size $d$**

$$= (f_0 * f_1 * f_2 * \ldots * f_{log d})(M - log d),$$

where $f_i : \mathbb{Z} \rightarrow \mathbb{Z}$ is

$$f_i(r) = \begin{cases} p^{r p^i} & r \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

The generating function of $f_i$ is

$$F_i(x) = \sum_{r \geq 0} x^r f_i(r) = \sum_{r=0}^{\infty} p^{r p^i} x^r = \frac{1}{1 - x p^i},$$

so the generating function for $f_0 * f_1 * f_2 * \ldots * f_{log d}$ is the product of individual generating functions. In conclusion:

**Theorem 5:** For prime powers $N = p^M$ and $d = p_{log d}$, define the generating function

$$\Theta_d(x) = \prod_{i=0}^{log d} \frac{1}{1 - x p^i}. \quad (17)$$

The number of orthogonal sampling sets of size $d$ in $\mathbb{Z}_N$ is the coefficient of $x^{M - log d}$ in $\Theta_d(x)$.

To get an idea of the numbers, first recall from Proposition 3 that for all valid digit-tables the marked columns are filled in in the same way, by all possible $log d$ tuples. The difference is in the digits in the unmarked columns. For sampling sets of size $d$, there are $M - log d$ unmarked columns, and these can be filled up in at most $(p^{M - log d})^d$ ways. The marked columns themselves can be chosen in at most $\left(M \choose M/2 \right)$ ways, and this gives the upper bound

**Number of orth. sampling sets of size $d$**

$$\leq \left(M \choose M/2 \right) p^{(M - log d)d}.$$

To compare the counts, define

$$\theta_N(d) = \frac{1}{d} \log \text{(Number of orth. sampling sets of size $d$)},$$

$$\sum_{i=0}^{log d} r_i p^i, \quad r = (r_0, r_1, r_2, \ldots),$$

where $r_i = l_{i+1} - l_i - 1$ is the number of columns between the $i^{th}$ and $(i+1)^{th}$ marked columns in the digit-table for any orthogonal sampling set of $\mathbb{B}^{|J|}$. We obtain the total number of orthogonal sampling sets of size $d$ by summing over all possible $(r_0, r_1, \ldots, r_{log d})$ that sum to $M - log d$:

**Number of orthogonal sampling sets of size $d$**

$$= \sum_{r_i = M - log d} p^{\lambda(r)}$$

$$= \sum_{r_i = M - log d} p^{\sum_{i=0}^{log d} r_i p^i}.$$
so that
\[ \theta_N(d) \leq M - \log d + \log \left( \frac{M}{M/2} \right) \]

Figures 1 and 2 plot \( \theta_N(d) \) vs \( \log d \) for various prime powers \( N \). The function \( \theta_N(d) \) is not equal to the linear function that jumps out in the plots, but in the examples we have tried it appears to be remarkably close.

For an example of Theorem 5 when \( d = 4 \) and \( p = 2 \) we have
\[ \Theta_4(x) = \prod_{i=0}^{2} 1 - \frac{1}{x2^i} = \frac{1}{(1-x2^1)(1-x2^2)(1-x2^4)}. \]

The number of orthogonal sampling sets \( \mathcal{I} \) of size 4 in \( \mathbb{Z}_8 \) is the coefficient of \( x^1 \), which we find to be 22. This is out of a total of 70 index sets of size 4. The number of orthogonal sampling sets of size 4 in \( \mathbb{Z}_{16} \) is the coefficient of \( x^2 \), which is 380, and is out of 1820. There is an attractive algebro-geometric interpretation of this count in terms of sets that tile \( \mathbb{Z}_N \), to be discussed in Section VI.

Appealing next to Theorem 6, a similar argument can be used to count the number of unitary submatrices of \( \mathcal{F} \).

**Theorem 6:** For prime powers \( N = p^M \) and \( d = p^{\log_d} \), define the generating function
\[ \Phi_d(x) = \prod_{i=0}^{\log_d} \frac{1}{1 - x^{p^i+d/p^i}}. \]

Then the number of unitary pairs \( (\mathcal{I}, \mathcal{J}) \) with \( \mathcal{I} \) and \( \mathcal{J} \) of size \( d \) (or alternately the number of \( d \times d \) unitary submatrices of the \( N \times N \) Fourier matrix) is given by the coefficient of \( x^{M-\log d} \) in \( \Phi_d(x) \).

Note that we are counting the total number of unitary pairs \( (\mathcal{I}, \mathcal{J}) \). It might be possible for two distinct unitary pairs, choosing different rows and columns of the Fourier matrix, to yield the same submatrix \( E_{\mathcal{I}}^T \mathcal{F} E_{\mathcal{J}} \) after cancellations.

**Proof:** We know that \( (\mathcal{I}, \mathcal{J}) \) form a unitary pair if and only if the digit-tables from \( \mathcal{I} \) and \( \mathcal{J} \) are of dual type. Given \( \bar{m}(M, \mathcal{L}) \), we know that the number of sets \( \mathcal{I} \) with digit-table of this type is
\[ |\bar{m}(M, \mathcal{L})| = p^{\lambda(r)}, \]

with the same notation as earlier. For \( (\mathcal{I}, \mathcal{J}) \) to be a unitary pair, \( \mathcal{J} \) needs to be of type \( \bar{m}(M, \mathcal{L}^*) \), and so the number of possible \( \mathcal{J} \) is
\[ |\bar{m}(M, \mathcal{L}^*)| = p^{\lambda(r^*)}, \]

where \( r^* = (r_0^*, r_1^*, r_2^*, \ldots) \) are the numbers of columns between the marked columns in \( \bar{m}(M, \mathcal{L}^*) \). Now from the definition of \( r_i^* \) and \( \mathcal{L}^* \) we get
\[ r_i^* = (M - l_{\log d-i-1}) - (M - l_{\log d-i}) - 1 = l_{\log d-i} - l_{\log d-i-1} - 1 = r_i \]

So the number of unitary pairs \( (\mathcal{I}, \mathcal{J}) \) for which \( \mathcal{I} \) has a digit-table of type \( \bar{m}(M, \mathcal{L}) \) is given by

Total number of unitary pairs of size \( d \)
\[ = p^{\lambda(r) + \lambda(r^*)} = p^{\sum_{i=0}^{\log d} r_i(p^i + p^{\log d-i})} \]
\[ = p^{\sum_{i=0}^{\log d} r_i(p^i + d/p^i)}. \]

Continuing the argument as in the proof of Theorem 5, we obtain the generating function for the total number of unitary pairs as
\[ \Phi_d(x) = \prod_{i=0}^{\log d} \frac{1}{1 - x^{p^i+d/p^i}}. \]
As in the case of $\Theta_d$, we can obtain a loose upper bound on $\Phi$. Consider counting the total number of unitary pairs $(I, J)$: picking a set of $\log d$ marked columns for $I$ fixes the marked columns for $J$, so we see that the marked columns can be chosen in at most $\binom{M}{M/2}$ ways. Once we pick the marked columns for $I$, we see that the unmarked columns in the digit tables for both $I$ and $J$ can be filled in $\left(\binom{p\log d}{d}\right)^2$ ways, and so

\[
\text{Number of unitary pairs of size } d \times d \leq \binom{M}{M/2} p^{2(M-\log d)d}.
\]

As before, to compare the counts, define

\[
\phi_N(d) = \frac{1}{d} \log(\text{Number of unitary pairs of size } d \times d),
\]

so that

\[
\phi_N(d) \leq 2M - 2 \log d + \log \left(\frac{M}{M/2}\right).
\]

Figures 3 and 4 plot $\phi_N(d)$ vs $\log d$ for $N = 2^{10}$ and $N = 3^{10}$; less linear looking than $\theta_N(d)$.

For an example of Theorem 6 analogous to that of Theorem 5 when $d = 4$ and $p = 2$ we have

\[
\Phi_4(x) = \prod_{i=0}^{2} \frac{1}{1 - x^{2^i + 4/2^i}} = \frac{1}{(1-x^{2^5})(1-x^{2^4})(1-x^{2^6})},
\]

and the number of $4 \times 4$ unitary submatrices of the $8 \times 8$ Fourier matrix is the coefficient of $x^1$ in $\Phi_4(x)$. This is 80, out of a total of $\binom{8}{2}^2 = 4,900$ possible $4 \times 4$ submatrices. Similarly the number of $4 \times 4$ unitary submatrices of the $16 \times 16$ Fourier matrix is the coefficient of $x^2$ in $\Phi_4(x)$, which is 4,352; this is the example mentioned in the introduction. That’s out of 3,312,400.

VI. IDEMPOTENTS WITH PRESCRIBED ZERO-SETS AND TILING $\mathbb{Z}_N$

In this section we show that the existence of an orthogonal sampling set for a bandlimited space $\mathbb{B}^J$ is equivalent to the tiling of $\mathbb{Z}_N$ by $J$. A set $J \subseteq \mathbb{Z}_N$ tiles $\mathbb{Z}_N$ if there exists a set of translates $K$ such that every $i \in \mathbb{Z}_N$ can be written uniquely as $i = j + k \mod N$, with $j \in J$, $k \in K$. An equivalent formulation that fits naturally in the present context is that $J$ tiles $\mathbb{Z}_N$ if and only if

\[1_{\mathbb{Z}_N} = 1_J * 1_K.\]

Taking inverse Fourier transforms the condition appears as

\[
\frac{1}{N} \delta = hg
\]

for the idempotents $h = F^{-1}1_J$ and $g = F^{-1}1_K$.

In the prime power case we now have

**Theorem 7:** Let $N = p^M$. The space $\mathbb{B}^J$ has an orthogonal sampling set if and only if $J$ tiles $\mathbb{Z}_N$.

**Theorem 7** is an affirmative answer to a discrete form of a conjecture of Fuglede, as explained in Section VI-B below. The proof is quite short, if we can find an idempotent with a given zero-set.

**Proof:** Let $h = F^{-1}1_J$, as usual, with zero-set $\mathcal{Z}(h)$ and zero-set divisors $\mathcal{D}(h)$. Let $d = |J|$ and $\hat{d} = p^d |\mathcal{D}(h)|$. It follows from Corollary 2 that $\hat{d} \leq d$.

Guided by (18) we want to construct an idempotent $g$ such that $\mathcal{Z}(h) \cup \mathcal{Z}(g) = \{j : N - 1\}$. For this we define a proposed zero-set corresponding to the divisors not in $\mathcal{D}(h)$, i.e., we put

\[\mathcal{Z}' = \{j : (j, N) \notin \mathcal{D}(h)^c\}, \quad \mathcal{D}(h)^c = \mathcal{D}_N \setminus \mathcal{D}(h).
\]

By the results in Section VI-A below there is an idempotent $g$ with zero-set $\mathcal{Z}(g) = \mathcal{Z}'$ and a corresponding set $K \subseteq \mathbb{Z}_N$ with $g = F^{-1}1_K$. (In fact there can be
many such \( g \), and many such \( K \). Moreover, from an application of Corollary 2 any such \( K \) satisfies
\[
|K| \geq p^{|D(h)|^c} = p^{M-|D(h)|} = \frac{N}{d}.
\]

The idempotent \( g \) is constructed precisely to make
\[
\mathcal{Z}(h) \cup \mathcal{Z}^l = [1 : N - 1],
\]
and so
\[
hg = h(0)g(0)\delta = \frac{|J||K|}{N} \delta = \frac{dN}{Nd} \delta = \delta \geq \frac{1}{N} \delta.
\]

Then the relationship (18) holds if and only if there is equality everywhere, i.e.,
1) \(|K| = N/d = p^{|D(h)|^c}.
2) \(d = \bar{d} \), which happens if and only if \( \mathbb{B}^J \) has an orthogonal sampling set, by Theorem 2.

Not only do we know which sets tile \( \mathbb{Z}_N \), by means of Theorem 5 we can count the tiling sets of a given size. Referring to the examples after Theorem 5 there are 22 sets of some 4 that tile \( \mathbb{Z}_8 \) and 380 sets of size 4 that tile \( \mathbb{Z}_{16} \).

Theorem 7 easily implies Proposition 4 for a set of size strictly larger than \( N/2 \) can never tile \( \mathbb{Z}_N \) and so cannot be an orthogonal sampling set. One might wonder if anything special happens at the mid-dimension \( N/2 \), if \( N \) is even. A space \( \mathbb{B}^J \) of this dimension does not necessarily have an orthogonal sampling set. Take \( N = 8 \) and
\[
\mathcal{J} = \{0^+ \cdot 2^1 + 0 \cdot 2^2, 1^+ \cdot 2^1 + 0 \cdot 2^2, 1^+ \cdot 2^1 + 1 \cdot 2^2, 0^+ \cdot 2^1 + 1 \cdot 2^2\} = \{0, 1, 5, 6\}.
\]

Suppose \( \mathcal{J} \) is an orthogonal sampling set. It must then satisfy the blueprint in Section VIII-A for some set of marked columns. Note that (indexing the columns as zeroth, first and second) the first column cannot be marked, as it contains an odd number of 0’s. This means that both the zeroth and second columns have to be marked. However in that case the digits in the first column must be completely dependent on the digits in the zeroth, which is not true for the given \( \mathcal{J} \). Hence \( \mathcal{J} \) cannot be an orthogonal sampling set for any bandlimited space. A similar argument can be used to show that the complement \( \mathcal{J}^c \) also cannot be an orthogonal sampling set for any bandlimited space. On the other hand, taking \( \mathcal{J} \) to be a set of \( N/2 \) consecutive indices in \( \mathbb{Z}_N \) will yield a \( \mathbb{B}^J \) with an orthogonal sampling set, as per the work in Section VIII.

A. Idempotents with prescribed zero-sets

The problem of finding idempotents with a prescribed zero-set is an interesting one. As mentioned in Section II-A the straight converse to Lemma 2 is not true, and fully recognizing what we do not know we formulate the problem very generally:

Problem i(\( \mathcal{D} \)):

Let \( N \) be given, not necessarily a prime power. For a set of divisors \( \mathcal{D} \subseteq \mathcal{D}_N \) let
\[
\mathcal{Z} = \{i : (i, N) \in \mathcal{D}\} = \bigcup_{k \in \mathcal{D}} \mathcal{A}_N(k).
\]

Find an index set \( \mathcal{J} \) such that the idempotent \( h_\mathcal{J} = \mathcal{F}^{-1} \mathcal{J} \) has zero set \( \mathcal{Z} \).

Here \( \mathcal{D}_N \) and \( \mathcal{A}_N(k) \) are defined in (6) and (7). We do have a positive result:

Proposition 4:

Suppose the digit-table for an index set \( \mathcal{J} \subseteq [0 : p^M - 1] \) is of the form
\[
\begin{pmatrix}
\tilde{m}(M, \mathcal{L}^*) \\
\tilde{m}(M, \mathcal{L}^*) \\
\vdots
\end{pmatrix},
\]
where each of the \( \tilde{m} \) is a valid digit-table with marked columns \( \mathcal{L}^* \). Then \( h_\mathcal{J} = \mathcal{F}^{-1} \mathcal{J} \) is a solution to i(\( \mathcal{L}^* \)).

Proof:

Given the structure of the digit-table for \( \mathcal{J} \), we see that \( \mathcal{J} \) breaks down into a disjoint union
\[
\mathcal{J} = \bigcup_i \mathcal{J}_i,
\]
where each \( \mathcal{J}_i \) is the index set corresponding to a single block in (19).

From Lemma 6 we know that \( \log \mathcal{D}(h_\mathcal{J}) \geq \mathcal{L} \). So we see that \( h_i = \mathcal{F}^{-1} \mathcal{J}_i \) is a solution to i(\( \mathcal{L}^* \)), for each \( i \). Now since
\[
h = \mathcal{F}^{-1} \mathcal{J} = \sum_i \mathcal{F}^{-1} \mathcal{J}_i = \sum_i h_i,
\]
it follows that \( h \) is a solution to i(\( \mathcal{L}^* \)).

In particular this means that when \( N \) is a prime power the converse of Lemma 2 holds, in the sense that i(\( \mathcal{D} \)) always has a solution. Proposition 4 is thus sufficient to prove Theorem 7 but more is true. We can say in the prime power case that the only solutions to i(\( \mathcal{D} \)) are the ones obtained in Proposition 4. Namely:

Theorem 8:

Suppose \( N = p^M \) is a prime power.

The digit-table of \( \mathcal{J} \) for any solution to i(\( \mathcal{L}^* \)) is a concatenation of blocks of valid digit-tables:
\[
\begin{pmatrix}
\tilde{m}(M, \mathcal{L}^*) \\
\tilde{m}(M, \mathcal{L}^*) \\
\vdots
\end{pmatrix}.
\]
where each of the $\bar{m}$ is a valid digit-table with marked columns $L^*$. The proof we present is by induction and requires some preparation, including some properties of Ramanujan’s sum. We defer it to Section IX. For now, we note that the solution $h = F^{-1} 1_\mathcal{J}$ to $i(p, l)$ with $\mathcal{J}$ of the smallest possible size is obtained with only one block in Theorem 8. The digit table for such a $\mathcal{J}$ would be of the type $\bar{m}(M, L^*)$: one such digit-table could be constructed by setting the entries in the unmarked columns to zeros. The corresponding $\mathcal{J}$ is $J = \bigcup_{i \in P} \log \{ \log d - \sum_{k=0}^{d-1} i_k p^M - l_k - 1 \}$.

This case of Theorem 8 can be used to find the index set $J$ and the associated idempotent $h_J$ for the examples in Section III-1.

**B. Fuglede’s conjecture**

Fuglede’s Conjecture, also known as the spectral set conjecture, first appeared in [7] and asks, in other language, whether a result such as Theorem 7 holds in greater generality.

A spectral set is a domain $\Omega \subset \mathbb{R}^N$ for which there exists a spectrum $\{ \lambda_k \}_{k \in \mathbb{Z}} \subset \mathbb{R}^N$ such that $\{ e^{2\pi i \lambda_k x} \}_{k \in \mathbb{Z}}$ is an orthogonal basis for $L^2(\Omega)$. Then

**Conjecture** (Fuglede, [7]): A domain $\Omega \subset \mathbb{R}^N$ is a spectral set if and only if it tiles $\mathbb{R}^N$.

See also [6].

The original result of Fuglede contained a proof of this conjecture under the assumption that $\Omega$ is a lattice subset of $\mathbb{R}^N$. Since then the conjecture has been proved to be true under more restrictive assumptions on the domain $\Omega$, though it has also been disproved in general, at least in one direction, by Tao [8].

Laba [9] proves Fuglede’s conjecture for cyclic $p$-groups $\mathbb{Z}_N$ when $N$ is a prime power. The proof is by construction of an appropriate spectral set and is similar to our method of obtaining orthogonal sampling sets in Theorem 2; the techniques might be usefully compared.

**VII. DIFFERENCE GRAPHS AND MAXIMAL CLIQUES**

When $N$ is a prime power Theorem 2 gives a complete answer to the question of when a space $B^J$ has an orthogonal sampling set. We can formulate the problem in the language of graph theory, and this has significant consequences because of the special structure of the graphs involved.

Let $\mathcal{G}(h)$ be the graph with vertices from $\mathbb{Z}_N$, and with an edge between two vertices $i_1, i_2 \in \mathbb{Z}_N$ if $h(i_1 - i_2) = 0$. We call this the difference graph of $h$. We will show that $\mathcal{G}(h)$ is a perfect graph in several cases. Figures 5 and 6 are two pictures of difference graphs generated with Mathematica.
satisfies
\[ h(i_1 - i_2) = 0, \quad i_1 \neq i_2 \in \mathcal{I}' \].

This means that \( \mathcal{I}' \) is an orthogonal sampling set for \( \mathbb{B}^j \) which is a contradiction since \( |\mathcal{I}'| > |\mathcal{J}| \).

The following corollary, when combined with Lemma 9, is essentially a restatement of Theorem 2 in terms of cliques. It is interesting to put it this way because the statement pertains just to idempotents with no references to sampling, orthogonal bases, etc.

**Corollary 3:** Let \( N = p^M \) be a prime power and let \( h: \mathbb{Z}_N \rightarrow \mathbb{C}^N \) be an idempotent. Then any maximal clique in the difference graph \( \mathcal{G}(h) \) is of size \( p^{[D(h)]} \).

It is possible to restate our other results in terms of difference graphs and cliques. There are some conceptual advantages to this, though perhaps with less computational usefulness than digit-tables. For example, Theorem 3 on the recursive structure of digit-tables takes the form below. For this we modify the notation and denote a difference graph of an idempotent \( h \) by \( \mathcal{G}(p^r) \), using the zero-set divisors \( p^C \) of \( h \).

**Theorem 9:** Any maximal clique in \( \mathcal{G}(p^r) \) is a union of cliques from an associated difference graph, precisely of the form
\[
\bigcup_{\alpha=0}^{p-1} \{ \alpha p^b + C_\alpha \},
\]
where \( l_0 \) is the smallest element of \( \mathcal{L} \), and each of the \( C_\alpha \) are maximal cliques in \( \mathcal{G}(p^r(l_0)) \).

See [10] for other results restated in terms of difference graphs and cliques.

### A. Perfect difference graphs

Since orthogonal sampling sets correspond to maximal cliques in a difference graph, it is natural to relate the sampling problem to the graph-theoretic (and computational) question of finding maximal cliques. Finding cliques takes exponential time for generic graphs, but in our case the difference graphs have enough structure to solve the problem in polynomial time — the graphs are perfect when \( N \) is a prime power, and in two other cases that we know. Recall that a graph is perfect if for every induced subgraph the chromatic number is equal to the size of a maximal clique.

**Theorem 10:** Let \( h \) be an idempotent and let \( \mathcal{G}(h) \) be the associated difference graph. Then \( \mathcal{G}(h) \) is perfect when: (a) \( N = p^M \); (b) \( N = pq \), the product of two primes; (c) \( N \) and \( h \) are such that \( |D(h)| \leq 2 \) and \( |D(h)^c| \leq 2 \), where \( \mathcal{G}(h)^c = \mathcal{D}_N \setminus \mathcal{D}(h) \).

In all cases we prove that \( \mathcal{G}(h) \) is a *Berge graph*. The result then follows from the celebrated Strong Perfect Graph Theorem, [11], which states that a graph is perfect if and only if it is a Berge graph. Recall that a graph \( \mathcal{G} \) is a Berge graph if every odd cycle with five or more nodes in \( \mathcal{G} \) or in \( \mathcal{G}^c \) (the complement of \( \mathcal{G} \)) has a chord. Under each of the assumptions on \( N \) the proofs proceed via a series of case distinctions.

**Proof:** Part (a) \( N = p^M \): Our starting point is Lemma 2 writing the zero-set \( \mathcal{Z}(h) \) as
\[
\mathcal{Z}(h) = \bigcup_{p^C \in \mathcal{D}(h)} \mathcal{A}_N(p^k),
\]
for some set of divisors \( \mathcal{D}(h) \subseteq \{1, p, p^2, \ldots, p^{M-1}\} \).

Suppose that \( i_1, i_2, i_3, i_4, \ldots, i_{p^k} = p^{k+1}, \ldots, p^{k+2} \) are maximal cliques in \( \mathcal{G}(h) \). Then \( i_2 - i_1 \in \mathcal{Z}(h) \), or \( (i_2 - i_1, N) = p^{k_1} \), say, with \( p^{k_1} \in \mathcal{D}(h) \). Similarly \( (i_3 - i_2, N) = p^{k_2} \in \mathcal{D}(h) \). We can thus write \( i_2 - i_1 = p^{k_1} q_1 \) and \( i_3 - i_2 = p^{k_2} q_2 \) for some \( q_1, q_2 \) coprime to \( N \). Now consider the following cases.

**Case 1:** \( k_1 < k_2 \). In this case, \( i_3 - i_1 = p^{k_1}(q_1 + q_2p^{k_2-k_1}) \). Since \( k_2 - k_1 > 0 \), we have \( p \mid (q_1 + q_2p^{k_2-k_1}) \). This means that \( (i_3 - i_1, N) = p^{k_1} \in \mathcal{D}(h) \), and so we have a chord between \( i_3 \) and \( i_1 \).

**Case 2:** \( k_1 = k_2, p \mid (q_1 + q_2) \).

As in Case 1, we have \( i_3 - i_1 = p^{k_1}(q_1 + q_2) \). Since \( p \mid (q_1 + q_2) \) we have \( (i_3 - i_1, N) = p^{k_1} \in \mathcal{D}(h) \), and so there is a chord between \( i_3 \) and \( i_1 \).

**Case 3:** \( k_1 = k_2, p \nmid (q_1 + q_2) \).

Suppose we have \( (q_1 + q_2, N) = p^r, r > 0 \). Then \( i_3 - i_1 = p^{k_1}(q_1 + q_2) \) leads us to \( (i_3 - i_1, N) = p^{k_1+r} \), which need not be in \( \mathcal{D}(h) \). Hence there need not be an edge between \( i_1 \) and \( i_3 \).

Now, \( i_4 - i_3 \in \mathcal{Z}(h) \), and so \( i_4 - i_3 = p^{k_3} q_3 \) for some \( q_3 \) coprime to \( N \). If \( k_3 \neq k_2 \), then we have a chord between \( i_2 \) and \( i_4 \), by the same argument in Case 1 applied to \( i_2, i_3, i_4 \). This leaves us with the case \( k_3 = k_2 = k_1 \). Now we have \( i_4 - i_1 = p^{k_1}(q_1 + q_2 + q_3) \).

Since \( p \nmid (q_1 + q_2 + q_3) \) and \( q_3 \) is coprime to \( N \), it follows that \( p \nmid (q_1 + q_2 + q_3) \) and so \( (i_4 - i_1, N) = p^{k_1} \in \mathcal{D}(h) \). Hence we have an edge between \( i_1 \) and \( i_4 \).

In all cases the cycle has a chord.

A similar proof holds for cycles in \( \mathcal{G}(h)^c \), with \( \mathcal{D}(h) \) replaced by \( \mathcal{D}(h)^c = \mathcal{D}_N \setminus \mathcal{D}(h) \) which, in this case, is still a set of prime powers. Thus every odd cycle in \( \mathcal{G}(h) \) and in \( \mathcal{G}(h)^c \) with at least 5 nodes has a chord and we conclude that \( \mathcal{G}(h) \) is a Berge graph, hence perfect.

Part (b), \( N = pq \): Then \( \mathcal{D}(h) \subset \mathcal{D}_{pq} = \{1, p, q\} \). Take the extreme case, when \( \mathcal{D}(h) = \{1, p, q\} \). Then by Lemma 2 the zero-set is the set of all nonzero indices, \( \mathcal{Z}(h) = [1 : N - 1] \). So all the vertices in the difference graph are connected to each other and every cycle has a chord, while \( \mathcal{G}(h)^c = \emptyset \). The remaining cases to show that \( \mathcal{G}(h) \) is perfect are then covered by part (c).
Now move to part (c) of the theorem. In the following we will take a cycle of size five, but the argument can be generalized to any cycle of odd size. Assume that $i_1, i_2, i_3, i_4, i_5$ form a cycle in $\mathcal{G}(h)$. Let $r_k = (i_k - i_{k+1}, N)$. Since $\mathcal{D}(h)$ is at most of size 2 there are two cases.

Case 1: $\mathcal{D}(h) \neq \{1\}$, i.e., $\mathcal{D}(h)$ includes a prime.

In this case all the $r_k$ are divisible by either of at most two primes, say $p$ and $q$. For the cycle not to have a chord between vertices $i_s, i_t$, we need $i_s - i_t$ to be coprime to both $p$ and $q$. Now consider any two consecutive edges, say $i_1 - i_2$ and $i_2 - i_3$. Suppose both these edges correspond to divisibility by the same prime, i.e., both $r_1$ and $r_2$ are divisible by, say, $p$. Then it follows that $i_3 - i_1$ is divisible by $p$ as well, which means the edge $i_3 - i_1$ exists in $\mathcal{G}(h)^c$. Hence all consecutive edges must correspond to divisibility by different primes, as indicated in Fig. 7. However since the cycle is odd in size, this is impossible.

Fig. 7. Case 1: Assume $i_1, i_2, i_3, i_4, i_5, \ldots$ form a cycle in $\mathcal{G}(h)$. Each of the edges must correspond to divisibility by either $p$ or $q$. If the cycle does not have any chords, then consecutive edges must correspond to divisibility by different primes, as indicated. Since the cycle is odd in size, this is impossible to obtain.

Case 2: $\mathcal{D}(h) = \{1\}$. Since $|\mathcal{D}(h)^c| \leq 2$, we can assume that $\mathcal{D}(h)^c \subseteq \{p, q\}$. In this case all the $r_k$ are coprime to $N$. For a chord $i_s - i_t$ not to exist, $i_s - i_t$ must be divisible by either $p$ or $q$, and thus each chord falls into one of these two groups. Now consider two adjacent chords, say $i_6 - i_2$ and $i_6 - i_3$, as in Fig. 8. Then both $i_6 - i_2$ and $i_6 - i_3$ should be divisible by different primes, for otherwise $i_2 - i_3$ would be divisible by $p$ as well. But this too is impossible according to the following purely geometric observation:

Lemma 10: Consider a polygon $C$ with $n$ vertices, where $n$ is odd. We call two chords (diagonals) of $C$ adjacent if they form a triangle with one of the sides of $C$ (For example as in Fig. 8). Then it is impossible to divide the chords into two groups such that adjacent chords belong to different groups.

Proof: Once again we make some case distinctions.

Case 1: $n = 5$ Suppose the chords in each group are represented by drawing them dashed and dotted, respectively. Assuming the chord $i_4 - i_2$ to be dotted, we can dash/dot the remaining chords (see Fig. 9). We end up with chord $i_1 - i_4$ which has to be both dashed and dotted, a contradiction.

Fig. 8. Case 1 (b): Assume $i_1, i_2, i_3, i_4, i_5, \ldots$ form a cycle in $\mathcal{G}(h)$.

Case 2: $n \geq 7$ The argument is similar to the previous case. Assume $i_5 - i_4$ is dotted. Then we can mark the remaining chords as dashed and dotted, as in Fig. 9. There is no way to mark the chord $i_1 - i_4$ consistently.

With Lemma 10 the final case of Theorem 10 is settled.

B. Further comments on perfect difference graphs

Naturally it is an interesting question what the most general result like Theorem 10 on perfect difference graphs should be. We do not offer a conjecture, but here is an example. Let $N = 8 \times 9$ and let $\mathcal{Z} = \mathcal{A}_N(1) \cup \mathcal{A}_N(3) \cup \mathcal{A}_N(4) \cup \mathcal{A}_N(12)$ (all elements of $\mathcal{Z}_N$ whose greatest common divisor with $N$ is 1, 3, 4 or 12). We take $\mathcal{G}$ to be the difference graph determined by $\mathcal{Z}$, i.e., there is an edge between $i_k$ and $i_\ell$ if $i_k - i_\ell \in \mathcal{Z}$. Now consider the nodes 1, 4, 3, 31, 12. Figure 11 shows...
Fig. 10. Assume $i_6 - i_4$ is dotted. Then we can mark all the chords shown as dashed/dotted, except there is no way to mark $i_1 - i_4$ consistently.

Fig. 11. $N = 72$. For the “zero-set” $\mathcal{Z} = A_1 \cup A_3 \cup A_4 \cup A_{12}$, the cycle formed by the nodes $1, 4, 3, 31, 12$ is shown. It has no chords.

Since, according to Lemma 2, the zero-set can be written $\mathcal{Z}(h) = \{i : (i, N) \in \mathcal{D}(h)\}$, as in (9), our difference graphs are also what have been called GCD-graphs, see [12]. Furthermore, when $\mathcal{D}(h)$ is a singleton the graph $\mathcal{G}(h)$ becomes a unitary Cayley graph, [13], and such graphs are shown to be perfect in a much broader setting. The argument here for our case is more elementary.

Regarding the divisibility conjecture in Section III-C, the authors in [12] provide an example of a GCD graph with $N = 20$ vertices for which the maximal clique size does not divide $N$. The divisor set used to construct this graph is $\mathcal{D} = \{1, 4, 10\}$. It is possible to generalize this counterexample to the case when $N = pq$ is a product of primes, [10].

However, a brute force search shows that the divisor set $\mathcal{D} = \{1, 4, 10\}$ does not come from an idempotent, and hence this example does not necessarily serve as a counterexample to the divisibility conjecture.

In general, to investigate divisibility we need to know which divisors arise from idempotents when $N$ is not a prime power. A more encompassing converse to Lemma 2 or a more general version of Proposition 4 and Theorem 8 is crucial for this, and for much else.

VIII. EXAMPLES: CONSECUTIVE INDICES AND INDICES IN ARITHMETIC PROGRESSION

We close with some additional examples. A discrete analog of the usual Nyquist-Shannon sampling theorem would naturally be for bandlimited spaces $\mathbb{B}^\mathcal{J}$ where $\mathcal{J}$ is a set of consecutive indices (i.e., a genuine “band” of frequencies) or slightly more generally, where the elements in $\mathcal{J}$ are in arithmetic progression. In this section
we see how these examples fit into the framework of idempotents and zero-sets. There are other approaches, of course, but the results as presented here again link up to the divisibility conjecture.

We do not assume that $N$ is a prime power. Suppose $\mathcal{J} \subseteq \mathbb{Z}_N$ consists of consecutive indices, mod $N$, and as usual let $h_{\text{cons}} = F^{-1}I$. Let $d = |\mathcal{J}|$ and

$$J = (1/2)(\max J + \min J).$$

It is easy to compute

$$h_{\text{cons}}(m) = \begin{cases} \frac{d}{N}, & m \equiv 0, \\ \sin(\pi md/N) \omega^{mj/N}, & m \not\equiv 0. \end{cases} \tag{20}$$

Then

$$h_{\text{cons}}(m) = 0 \text{ if and only if } md \equiv 0 \text{ and } m \not\equiv 0,$$

so, in the first case, if $d$ divides $N$ the zero-set is

$$\mathcal{Z}(h_{\text{cons}}) = \{d', 2d', \ldots, (d-1)d'\}, \quad d' = N/d,$$

of size $d - 1$, and, in the second case, if $d$ is coprime to $N$ then the zero-set is empty. As a third, intermediate case let $d_0 = (d, N)$ and note that $md \equiv 0$ if and only if $m(d, N) \equiv 0$. Replacing $d$ by $d_0$ in the above we see that the zero-set is

$$\mathcal{Z}(h_{\text{cons}}) = \{d''', 2d''', \ldots, (d_0 - 1)d'''\}, \quad d''' = N/d_0,$$

of size $d_0 - 1$. Again, if $d_0 = 1$ then the zero-set is empty while if $d$ divides $N$ then $d_0 = d$ and the zero-set is of size, $d - 1$.

Now let us find orthogonal sampling sets for $\mathbb{B}\mathcal{J}$, when possible. First consider the case when $d$ divides $N$. Take

$$\mathcal{I} = \{0\} \cup \mathcal{Z}(h_{\text{cons}}).$$

Then

$$h_{\text{cons}}(i_1 - i_2) = 0, \quad \text{for } i_1 \neq i_2 \in \mathcal{I}.$$ 

and by Lemma [□] the set $\mathcal{I}$ is an orthogonal sampling set. In fact, since $\mathcal{Z}(h_{\text{cons}})$ is of size $d - 1$ the set $\mathcal{I}$, and the bracelet containing $\mathcal{I}$, are the only possible orthogonal sampling sets. In the case when $d_0 = (d, N) < d$ the zero-set $\mathcal{Z}(h_{\text{cons}})$ is of size $d_0 - 1 < d - 1$. If $\mathcal{I}$ is an orthogonal sampling set then, because we can work with any index set in the bracelet, we can assume without loss of generality that $0 \in \mathcal{I}$ and Lemma [□] requires that $\mathcal{I} \setminus \{0\} \subseteq \mathcal{Z}(h_{\text{cons}})$ so that $|\mathcal{I}|$ is at most $d_0 < d$, contradicting Lemma [□] that we must have $d_0 = d$.

After the fact, we can also check this result directly. Constructing the Fourier submatrix $E_\mathcal{I}^T FE_\mathcal{J}$, with

$$\mathcal{I} = \{0, d', 2d', \ldots, (d-1)d'\}, \quad \mathcal{J} = \{0, 1, 2, \ldots, d-1\}$$

The $(m, n)$ entry of this matrix is given by

$$\exp \left(2\pi imd'/N\right) = \exp \left(2\pi imn/d\right),$$

and we have recovered the $d \times d$ Fourier matrix, which is unitary (up to scaling).

The analysis for consecutive indices can be easily extended to the case when the frequency set $\mathcal{J}$ consists of indices in arithmetic progression. Suppose the common difference is $s$ and let $d = |\mathcal{J}|$ as before. Defining $h_{\text{arith}} = F^{-1}I$ this time we find that

$$h_{\text{arith}}(m) = \begin{cases} \frac{d}{N}, & ms \equiv 0, \\ \sin(\pi msd/N) \omega^{mj/N}, & ms \not\equiv 0. \end{cases} \tag{21}$$

Thus the zero-set $\mathcal{Z}(h_{\text{arith}})$ consists of multiples of $N/(sd, N)$ except those that are multiples of $N/(s, N)$. Let $s_0 = (s, N)$ and $d_0 = (d, N)$ in (21) to get

$$h_{\text{arith}}(m) = 0 \text{ if and only if } ms_0d_0 \equiv 0 \text{ and } ms_0 \not\equiv 0.$$

There are $s_0d_0$ values of $m$ that satisfy $ms_0d_0 \equiv 0$ and among these there are $s_0$ values of $m$ that satisfy $ms_0 \equiv 0$. Hence the size of the zero-set is

$$|\mathcal{Z}(h_{\text{arith}})| = s_0d_0 - s_0 = s_0(d_0 - 1).$$

Once again we see that the zero-set is empty if $d$ is coprime to $N$.

To find an orthogonal sampling set $\mathcal{I}$ we can begin by again assuming without loss of generality that $0 \in \mathcal{I}$. We need $\mathcal{I} \setminus \{0\} \subseteq \mathcal{Z}(h_{\text{arith}})$, i.e., the non-zero indices in $\mathcal{I}$ are all the multiples of $N/s_0d_0$ except those that are multiple of $N/s_0$:

$$\mathcal{I} \setminus \{0\} = \frac{N}{s_0d_0} \left(\{1, 2, 3, \ldots, s_0d_0 - 1\} \setminus \{d_0, 2d_0, 3d_0, \ldots\}\right) = \frac{N}{s_0d_0} (A \setminus B).$$

For two indices $i_1 \neq i_2 \in \mathcal{I}$ we need $i_1 - i_2 \in \mathcal{Z}(h_{\text{arith}})$, and we conclude that $A \setminus B$ can have at most one element from each congruence class mod $d_0$. For if elements $s_1, s_2 \in A \setminus B$ are in the same congruence class mod $d_0$ then their difference is a multiple of $d_0$. On multiplying by the scale factor $N/s_0d_0$ we end up with a multiple of $N/s_0$, which is not included in the zero-set. This means that the largest possible size of $\mathcal{I}$ is $d_0$, when we have one index from each congruence class. One example is

$$\mathcal{I} = \frac{N}{s_0d_0} \{0, 1, 2, \ldots, d_0 - 1\}.$$
Thus for $\mathcal{I}$ to be an orthogonal sampling set we need \( d = |\mathcal{J}| = d_0 \), and by definition of \( d_0 \) this is only possible when \( d \) divides \( N \). Note that the value of \( s \) is immaterial to the existence of an orthogonal sampling set.

In summary:

**Theorem 11:** If \( \mathcal{J} \subset \mathbb{Z}_N \) is a set of consecutive indices or a set of indices in arithmetic progression then \( \mathbb{B}^\mathcal{J} \) has an orthogonal sampling set if and only if \( |\mathcal{J}| \) divides \( N \).

This conforms with the divisibility conjecture, Section III-C. We obtain orthogonal interpolating bases in the two cases by shifts of \( h_{\text{cons}} \) and \( h_{\text{arith}} \), as in (4).

The nature of the orthogonal sampling sets is different in the two cases. For a set of consecutive indices any resulting orthogonal sampling set is in the bracelet of \( (N/|\mathcal{J}|)\mathcal{J} \). For a set in arithmetic progression the orthogonal sampling set depends on the elements in each congruence class mod \( d \). Let \( N = 16, s = 6, d = 4 \), and \( \mathcal{J} = \{0, 6, 12, 2\} \). The corresponding idempotent has zero-set specified by

\[
h_{\text{arith}}(m) = 0 \quad \text{if and only if} \quad 24m \equiv 0 \pmod{16} \text{ and } 6m \neq 0 \pmod{16},
\]

which reduces to \( (s_0 = 2, d_0 = 4) \)

\[
h_{\text{arith}}(m) = 0 \quad \text{if and only if} \quad 8m \equiv 0 \pmod{16} \text{ and } 2m \neq 0 \pmod{16}.
\]

So \( \mathcal{Z}(h_{\text{arith}}) \) consists of all even indices except multiples of 8, and an orthogonal sampling set is given by

\[
\mathcal{I} = \frac{16}{8}\{0, 1, 2, 3\} = \{0, 2, 4, 6\}.
\]

IX. APPENDIX: PROOF OF THEOREM 8

In this section we prove Theorem 8 giving the form of a solution to the problem \( i(D) \) on idempotents with a prescribed zero-set. We restate it here for the convenience of the reader:

**Theorem 8:** Suppose \( N = p^M \) is a prime power. The digit-table of any solution to \( i(p^L) \) is a concatenation of blocks of valid digit-tables:

\[
\begin{pmatrix}
   \bar{m}(M, \mathcal{L}^*) \\
   m(M, \mathcal{L}^*) \\
   \vdots
\end{pmatrix},
\]

where each of the \( \bar{m} \) is a valid digit-table with marked columns \( \mathcal{L}^* \).

The natural way to characterize an \( h \) with \( \mathcal{Z} \) as its zero-set is to specify that \( h \cdot 1_\mathcal{Z} = 0 \). So if \( h \) is an idempotent with \( h = \mathcal{F}^{-1}1_\mathcal{J} \) for an index set \( \mathcal{J} \) and \( c = \mathcal{F}^{-1}1_\mathcal{Z} \),

then \( 1_\mathcal{J} \ast c = 0 \) or

\[
\sum_{j \in \mathcal{J}} c(n + j) = 0, \tag{22}
\]

for any \( n \in \mathbb{Z}_N \). Here \( \mathcal{J}^- = \{-j : j \in \mathcal{J}\} \), and we used \( N\mathcal{F}^{-1}\mathcal{F}^{-1}1_\mathcal{J}(n) = 1_\mathcal{J}(-n) \). Finding solutions to \( i(D) \) is equivalent to finding index sets \( \mathcal{J} \) satisfying (22).

Assuming \( \mathcal{J} \) can be found satisfying (22), what is its structure? That is the upshot of Theorem 8.

We first consider the case when the set of divisors \( D = \{p^l\} \) is a singleton. Let

\[
\mathcal{Z} = \{i \in \mathbb{Z}_N : (i, N) = p^l\} = \bigcup_{(\alpha, N) = 1} \{\alpha p^l\} \tag{23}
\]

be the corresponding set of zeros. The proof of Theorem 8 will ultimately go by induction on the size of the zero-set divisors \( D \), and and we first need to understand this special case. It is through a connection to Ramanujan’s sum, from analytic number theory, that we are able to do so.

Ramanujan’s sum is

\[
c_q(k) = \sum_{n \in \mathbb{Z}_q \atop (n, q) = 1} \exp(2\pi ink/q). \tag{24}
\]

See, for example, the original paper [14]. We will need the following two properties:

1. **When** \( q = p^s \) **is a prime power,**

\[
c_{p^s}(n) = \begin{cases} 0 & \text{if } p^s-1 \nmid n, \\ -p^{s-1} & \text{if } p^{s-1} \mid n \text{ and } p^s \nmid n, \\ \phi(p^s) & \text{if } p^s \mid n, \end{cases}
\]

where \( \phi \) is the Euler totient function.

2. **For any divisor** \( d \) **of** \( q \),

\[
\sum_{n \in \mathbb{Z}_q \atop (n, q) = d} \exp(2\pi ink/q) = \sum_{m \in \mathbb{Z}_{q/d} \atop (m, d) = 1} \exp(2\pi imk/d') = c_{d'}(k).
\]

From (23) and from property (ii) with \( q = N = p^M \), \( d = p^l \), and \( d' = p^{M-1} \) we can identify \( c_{d'} \) as an inverse Fourier transform

\[
N\mathcal{F}^{-1}1_\mathcal{Z} = c_{d'}, \quad \text{where } d' = N/d.
\]

We can then rewrite (22) as

\[
\sum_{j \in \mathcal{K}} c_{d'}(j) = 0, \tag{25}
\]
where $\mathcal{K}$ is any index set in the bracelet of $\mathcal{J}$. We will use property (i) to deduce properties of index sets satisfying (25).

First, we can translate to assume that $0 \in \mathcal{J}$. Let $l' = \log d' = M - l$, and note that the values of $c_{d'}$ are all integers, from property (i), with \[c_{d'}(0) = \phi(d') = (p - 1)p^{l' - 1}.
\] Since $c_{d'}$ is periodic with period $d'$, we get \[(p - 1)p^{l' - 1} = c_{d'}(0) = c_{d'}(d') = c_{d'}(2d') = \ldots.\]
These are the only positive values in $c_{d'}$. Now, from property (i), since a negative value in $c_{d'}$ can only be $-p^{l' - 1}$, we need at least $p - 1$ negative values to cancel out one positive value, and each positive value occurring in $c_{d'}$ necessitates the occurrence of $p - 1$ negative values. At this point, we can say that the size of $\mathcal{J}$ has to be at least $rp$, where $r$ is the number of positive values in $c_{d'}$ that come from $\mathcal{J}$.

Property (i) also says that the negative values of $c_{d'}$ are at indices that are multiples of $p^{l' - 1}$ (excluding those that are multiples of $p^{l'}$). So $\mathcal{J}$ includes $r$ multiples of $p^{l'}$ (these are the indices where the value of $c_{d'}$ is positive) and $r(p - 1)$ multiples of $p^{l' - 1}$ that are not multiples of $p^{l'}$ (these are the indices where the value of $c_{d'}$ is negative).

In summary, we have now determined that $\mathcal{J}$ has to include certain indices all of which are multiples of $p^{l' - 1}$. Isolate these indices as a subset $\mathcal{J}_1$ of $\mathcal{J}$. In addition to these indices, $\mathcal{J}$ could contain indices $n$ such that $p^{l' - 1} \nmid n$, so that $c_{d'}(n) = 0$, and (25) is satisfied. In the latter case we can isolate all such $n$ into another subset $\mathcal{J}'$ of $\mathcal{J}$. In this context we make the following simple, general observation:

If $h = \mathcal{F}^{-1}_{\mathcal{J}}$ and $h_1 = \mathcal{F}^{-1}_{\mathcal{J}_1}$ are both solutions to $i(D)$, and if $\mathcal{J}_1 \subseteq \mathcal{J}$, then $h' = \mathcal{F}^{-1}_{\mathcal{J}\setminus\mathcal{J}_1}$ is also a solution to $i(D)$.

So we see that $\mathcal{J}' = \mathcal{J} \setminus \mathcal{J}_1$ itself gives a solution to (25). We can translate $\mathcal{J}'$ so that $0 \in \mathcal{J}'$, and repeat the above arguments with $\mathcal{J}'$ instead of $\mathcal{J}$. But then $\mathcal{J}'$ (appropriately translated) contains the indices described previously, and hence can be broken down further. Repeating this process, it follows that $\mathcal{J}$ breaks down into a disjoint union $\mathcal{J} = \mathcal{J}_1 \cup \mathcal{J}_2 \cup \mathcal{J}_3 \cup \ldots$, where each of the $\mathcal{J}_i$, when appropriately translated, contains only multiples of $p^{l' - 1}$.

For the remainder of the discussion we can thus assume that $\mathcal{J}$ contains only multiples of $p^{l' - 1}$; arbitrary solutions can be constructed by taking disjoint unions of (translates of) such sets.

It would seem that a natural way to understand $\mathcal{J}$ is by reducing it modulo $d'$ = $p^{l'}$. Write $\mathcal{J}/d'$ for the set of congruence classes of elements of $\mathcal{J}$ reduced mod $d'$ and $(\mathcal{J}/d')^\sim$ for the corresponding multiset where congruence classes are listed according to their multiplicity. Then $(\mathcal{J}/d')^\sim$ has $r$ zeros and $r(p - 1)$ non-zeros, where the non-zeros are multiples of $p^{l' - 1}$. Intricately, this must hold for any translate of $\mathcal{J}$. Using this, we will argue that $(\mathcal{J}/d')^\sim$ contains all non-zero elements in equal amounts: i.e. each nonzero multiple of $p^{l' - 1}$ in $(\mathcal{J}/d')^\sim$ also has multiplicity $r$.

Suppose the multiset is

\[(\mathcal{J}/d')^\sim = \left\{ \begin{array}{ll} 0 & \text{with mult. } r \\ \alpha p^{l' - 1} & \text{with mult. } r_1 \\ \vdots & \vdots \\ \end{array} \right\},\]

where $\alpha \in [1 : p - 1]$, and $\alpha p^{l' - 1}$ is the element with highest multiplicity. Note that there are $p - 1$ non-zero entries, and the multiplicities of the non-zero entries must add up to $r(p - 1)$. Now suppose we translate $\mathcal{J}$ by $\beta = -\alpha p^{l' - 1}$ (to obtain $\tau^{\beta} \mathcal{J}$). Then the multiset $(\tau^{\beta} \mathcal{J}/d')^\sim$ has 0 with multiplicity $r_1$, so the multiplicity of remaining elements must add up to $r_1(p - 1)$, which is impossible unless $r_1 = r$.

We summarize our analysis as follows:

**Lemma 11:** For any solution $\mathcal{J}$ to to the singleton problem $i(\{p^{l'}\})$ such that $0 \in \mathcal{J}$, we have

\[(\mathcal{J}/d')^\sim = p^{l' - 1}\{0, \ldots, 1, \ldots, 2, \ldots, p - 1, \ldots\}\] (26)

where all the multiplicities are equal (to $r$).

Interpreting (26) in terms of digit-tables, the digit-table for such a $\mathcal{J}$ is of the form

\[
\begin{array}{cccc}
p^0 & p^1 & \ldots & p^{l' - 1} \\
\vdots & \vdots & \ddots & \vdots \\
\end{array}
\]

where the digits in the ($l'$ th) column contains all the digits $\{0, 1, 2, \ldots, p - 1\}$ in equal amounts, and the digits from column $l'$ onwards are arbitrary. In other words, we can write $\mathcal{J}$ in a block form as in the table below.

1 We used such multisets in [11] and the notation is borrowed from there.
Stated differently, each block contains a valid digit-table with a single marked column \( l' - 1 \). It is possible to translate this set by any amount (so the first \( l' - 2 \) columns need not be all zeros), and it is possible to take a disjoint union with other sets of this type (Note that disjoint union of sets corresponds to simply concatenating the digit tables row-wise). In summary we know that the digit-table for \( \mathcal{J} \) has to be of the type

\[
\begin{pmatrix}
\bar{m}(M, \{l' - 1\}) \\
\bar{m}(M, \{l' - 1\}) \\
\vdots \\
\bar{m}(M, \{l' - 1\})
\end{pmatrix},
\]

i.e. a collection of blocks, with each block being a valid digit-table with marked column \( \{l' - 1\} \). This is precisely the form that Theorem 8 takes when \( L \) is a singleton, and is thus the first step in an induction.

In general, as before, we let \( L = \{l_0, l_1, \ldots, l_{\log d - 1}\} \), \( L' = \{l_0^*, l_1^*, \ldots, l_{\log d - 1}^*\} \), the indices of both sets labeled in increasing order. Also let \( L'_1 = \{l_1^*, l_2^*, \ldots\} \), where, as earlier, \( l_i^* = M - l_{\log d - i - 1} - 1 \).

Consider the digit-table for any solution to \( i(p^L) \). Arrange the rows of the digit-table lexicographically. The digit-table splits into blocks depending on the values in the first \( l_0^* - 1 \) columns, as in Table I.

Since each of these blocks in Table I is arranged lexicographically, they further split into blocks depending on the value in the \( l_0^* \) column. In the notation from Theorem 3 we may write block \( i \) from Table I as

\[
\tilde{a}_i = \begin{pmatrix}
0 \cdot e_{l_0^*} + m_{i0} \\
1 \cdot e_{l_0^*} + m_{i1} \\
2 \cdot e_{l_0^*} + m_{i2} \\
\vdots \\
(p - 1) \cdot e_{l_0^*} + m_{ip(p - 1)}
\end{pmatrix},
\]

where the digit tables \( m_{ij} \) contain only zeros in the first \( l_0^* \) columns. The row \( e_{l_0^*} \) is as defined before, a row of zeros with a 1 in the (indexing from 0) \( (l_0^* - 1) \)th slot, and the row \( \tilde{a}_i \) is obtained by appending 0’s to \( a_i \) to make it size \( M \):

\[
\tilde{a}_i = (a_i, 0, 0, 0).
\]

Note that any solution to \( i(p^L) \) is also a solution to \( i(p^{L'}) \) for any subset \( L' \) of \( L \). Therefore, applying the induction hypothesis to \( L' = L \setminus \{l_0^*\} = L'_1 \) and to \( L'' = \{l_0^*\} \) the digit-table for any solution to \( i(p^L) \) is both of the form

\[
\begin{pmatrix}
\bar{m}(M, L'_1) \\
\bar{m}(M, L'_1) \\
\vdots \\
\bar{m}(M, L'_1)
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
\bar{m}(M, \{l_0^*\}) \\
\bar{m}(M, \{l_0^*\}) \\
\vdots \\
\bar{m}(M, \{l_0^*\})
\end{pmatrix}.
\]

From this we can say the following:

1) Since Table II needs to split into blocks \( \bar{m}(M, \{l_0^*\}) \), each of the blocks in Table I must also split into blocks of valid digit-tables \( \bar{m}(M, \{l_0^*\}) \). (It is not possible to construct a valid digit-table \( \bar{m}(M, \{l_0^*\}) \) from rows in different blocks; recall that all the initial unmarked columns in any valid digit-table must contain the same digit.)

As a consequence, each of the \( m_{ij} \) in (27) is of the same size.

2) Similarly, since Table II also needs to split into blocks \( \bar{m}(M, L'_1) \), each of the \( m_{ij} \) in (27) must also split into blocks of valid digit-tables \( \bar{m}(M, L'_1) \).

From 1) since all the \( m_{ij} \) are equal in size, they split into the same number of valid sub-blocks. Let’s say each \( m_{ij} \) splits into \( s \) valid sub-blocks \( m_{ijk} \), for \( k = 0 \) to \( s \); where each \( m_{ijk} \) is valid table of type \( \bar{m}(M, L'_1) \).

Now it is a simple matter of rearranging block \( i \) in (27) to a form to which we can easily apply Theorem 3.
For any \( l \) by \( \bar{l} \) digit-table \( \bar{m} \), from Theorem 3, each of the sub-blocks above is a valid digit-table \( \bar{m} \), thus block \( i \) in Table II is of the required form. Since this holds for any \( i \) the induction is complete and so is the proof.

A. Further Comments on Theorem 8

There is an alternate approach to Theorem 8 one that we have not been able to see through to completion but which we believe has some interesting features. It is patterned on the ideas from the proof of Lemma 2.

Suppose \( h = \mathcal{F}^{-1}1_{J} \) is a solution to \( i(p^{L}) \) and consider the characteristic polynomial \( p_{J} \in \mathbb{Z}[x] \) defined by
\[
p_{J}(x) = \sum_{j \in J} x^{j}.
\]
For any \( l \in \mathcal{L} \), the hypothesis of Theorem 8 implies that \( p_{J}(x) \) vanishes whenever \( x = \omega_{N}^{k} = \omega_{p^{M-1}}^{k} \). Hence \( p_{J}(x) \) must be divisible by the corresponding cyclotomic polynomial \( \phi_{p^{M-1}} \). Since the cyclotomic polynomials are irreducible, we have the decomposition
\[
p_{J}(x) = q(x) \prod_{l \in \mathcal{L}} \phi_{p^{M-1}}(x), \text{ for some } q \in \mathbb{Z}[x].
\]
So any solution \( h \) to \( i(p^{L}) \) can be obtained by choosing a \( q \in \mathbb{Z}[x] \) such that the product above yields a characteristic polynomial (i.e. only has 0, 1 as coefficients), and vice versa. However, it is not clear which polynomials \( q \) have this property, and this motivated us to try the approach via Ramanujan’s sum.

Still, take the particular case \( q = 1 \). Recall that
\[
\phi_{p^{M-1}}(x) = \sum_{i \in \mathcal{P}} x^{ip^{M-1}}
\]
and \( \mathcal{L} = \{ l_{0}, l_{1}, \ldots, l_{\log_{d}d} \} \), so that the corresponding product is
\[
p_{J}(x) = \prod_{l \in \mathcal{L}} \phi_{p^{M-1}}(x)
\]
\[
= \sum_{i \in \mathcal{P}^{\log_{d}d}} x^{\sum_{k=0}^{\log_{d}d-1} ikp^{M-1-k-1}},
\]
where we write \( i \) for \( (i_{1}, i_{2}, \ldots, i_{\log_{d}d}) \).

Since the powers of \( x \) are all distinct sums of prime powers, each power occurs only once, whence \( p_{J} \) is a characteristic polynomial. The corresponding index set \( J \) can be read off as the set
\[
\bigcup_{i \in \mathcal{P}^{\log_{d}d}} \left\{ \sum_{k=0}^{\log_{d}d-1} ikp^{M-1-k-1} \right\}.
\]
As we observed in Section VI-A this solution is the special case of Theorem 8 when there is only one block, and the valid digit table in this block is obtained by filling all unmarked columns with zeros.

Vanishing sums of roots of unity is a subject in itself, see for example [15] and [16]. When \( N \) contains at most two prime factors, say \( p \) and \( q \), then, up to a rotation, the minimal sums of vanishing roots of unity are \( 1 + \omega_{p} + \omega_{p}^{2} + \ldots \) and \( 1 + \omega_{q} + \omega_{q}^{2} + \ldots \), as shown in [16]. This seems to be the essence of our approach, as in Lemma 11 and in particular (25). In the case when \( l' = M \), i.e., when the divisor set \( D = \{ 1 \} \),
\[
(J / d')^c = p^{M-1} \{ 0, \ldots, 1, \ldots, 2, \ldots, p-1, \ldots \}.
\]
Constructing the sums for \( h = \mathcal{F}^{-1}1_{J} \), we see that all the roots of unity involved are \( p^{th} \) roots of unity, and all of them are present in equal amounts. Conceptually, Theorem 8 thus looks to be allied to results in the literature on minimal sums of roots, but with a structure for a generalized solution \( J \) (especially useful in the context of the present paper) and a different style of proof.

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