PULSE PROFILES FROM THERMALLY EMITTING NEUTRON STARS

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ABSTRACT

The problem of computing the pulse profiles from thermally emitting spots on the surface of a neutron star in general relativity is reconsidered. We show that it is possible to extend Beloborodov’s approach to include (multiple) spots of finite size in different positions on the star surface. The results for the pulse profiles are expressed by comparatively simple analytical formulae which involve only elementary functions.

Key words: gravitation – stars: neutron – X-rays: stars

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1. INTRODUCTION

X-ray emission from isolated neutron stars (NSs), first detected in radio pulsars (PSRs), is now increasingly observed in other classes of sources, most of which are radio-silent or have radio properties much at variance with those of PSRs. They include thermally emitting NSs (XDINSs; e.g., Turolla 2009), central compact objects in supernova remnants (e.g., De Luca 2008), magnetar candidates (SGRs and AXPs; e.g., Mereghetti 2008; Rea & Esposito 2011), and rotating radio transients (e.g., Burke-Spolaor 2013).

With the exception of some PSRs, the X-ray emission of which is dominated by a non-thermal component of magnetospheric origin, the spectra of all other X-ray emitting, isolated NSs exhibit one (or more) thermal component that most likely originates at the star surface. Since pulsations are observed, thermal X-ray photons either come from a localized, heated region, like in SGRs/AXPs and PSRs, or from the entire cooling surface with an inhomogeneous temperature distribution, like in XDINSs. In this respect, the analysis of the observed pulse profiles in different energy bands is bound to reveal much on the surface thermal map of the NS, particularly the physical size and position of the emitting region(s) (e.g., Zane & Turolla 2006; Albano et al. 2010).

The problem of modeling the pulse profiles of a rotating, thermally emitting NS, including the effects of gravitational ray bending, is an old one and has been thoroughly addressed in the literature (e.g., Pechenick et al. 1983; Page 1995; Page & Sarmiento 1996; Psaltis et al. 2000; Beloborodov 2002; Zane & Turolla 2006). In particular, in their classic paper, Pechenick et al. (1983) analyzed the emission from two antipodal, uniform, circular caps. Although their approach contains no inherent complexity, the treatment of photon propagation in a Schwarzschild spacetime leads to elliptic integrals and requires numerical evaluation. In general, resorting to a numerical approach is unavoidable every time a continuous surface temperature distribution, anisotropic emission, and/or an arbitrary shape of the emission regions must be accounted for. However, Beloborodov (2002), by means of a clever approximation, revealed that simple analytical expressions can be derived for the pulse profiles in full general relativity (Schwarzschild spacetime) for point-like spots.

In this paper, we make use of Beloborodov’s (2002) approximate treatment and extend his analysis to the case of finite, uniform, circular spots. Our results are valid for an arbitrary number of spots, regardless of their size, temperature, and mutual position on the star surface (e.g., two different, non-antipodal caps). Some more complex emission geometries (such as a cap surrounded by a corona) can also be easily accommodated. The expression for the total observed flux is analytical and this makes our approach both simple and fast for the evaluation and comparison of pulse profiles with observations.

2. OBSERVED FLUX

Let us consider a surface element \( dS \) on an NS with a radius \( R \) and mass \( M \) and let us assume that the Schwarzschild solution correctly describes the spacetime outside the star (in the following, \( R_S = 2GM/c^2 \) is the Schwarzschild radius). Let us further introduce a spherical coordinate system, \((r, \theta, \phi)\), centered on the star in such a way that the polar axis coincides with the line of sight (LOS; unit vector \( \hat{l} \)). The distance to the observer is \( D \gg R \).

Because photon trajectories are not straight lines, the ray from \( dS = R^2 \sin \theta \, d\theta \, d\phi \), which reaches the observer, leaves the surface with respect to the local normal at an angle \( \alpha \neq \theta \) (see Figure 1). The relation between \( \alpha \) and \( \theta \) is given implicitly by the two equations

\[
\begin{align*}
\theta &= \int_R^\infty \frac{dr}{r^2} \left[ \frac{1}{b^2} - \frac{1}{r^2} \left( 1 - \frac{R_S}{r} \right) \right]^{-1/2} \\
\sin \alpha &= b \left( 1 - \frac{R_S}{R} \right)^{1/2},
\end{align*}
\]

where \( b \) is the ray impact parameter (Beloborodov 2002).

The (monochromatic) flux \( dF_\nu \) from \( dS \) detected by the observer is then

\[
dF_\nu = \left( 1 - \frac{R_S}{R} \right) I_\nu \cos \alpha \frac{d \cos \alpha}{d \cos \theta} \frac{dS}{D^2},
\]

where \( \nu \) is the photon frequency and \( I_\nu \) the specific intensity, both measured by the static observer at \( r = R \). The total flux is obtained by integrating the previous expression over the visible part of the emitting region, \( S_V \). If the emission is Planckian at
the local (uniform) temperature $T$, $I_v = B_v(T)$, and this results in

$$F_v = \left(1 - \frac{R_S}{R}\right) B_v(T) \int_{S_v} \cos \alpha \frac{d \cos \alpha}{d \cos \theta} \, dS. \quad (4)$$

In Newtonian gravity, it is $\alpha = \theta$ and the flux is simply proportional to the area of the visible emitting region projected in the plane of the sky.

Beloborodov (2002) found that a simple, approximate expression can be used to link $\alpha$ and $\theta$, without the need to (numerically) solve Equations (1) and (2),

$$1 - \cos \alpha = (1 - \cos \theta) \left(1 - \frac{R_S}{R}\right). \quad (5)$$

Equation (5) is remarkably accurate and produces a fractional error $\lesssim 3\%$ for $R \gtrsim 3R_S$. Substituting $\cos \alpha$ and $d \cos \alpha/d \cos \theta$ into Equation (4), one obtains

$$F_v = \left(1 - \frac{R_S}{R}\right)^2 B_v(T) \int_{S_v} \left[ \frac{R_S}{R} + \left(1 - \frac{R_S}{R}\right) \cos \theta \right] \frac{dS}{D^2}. \quad (6)$$

The flux is then expressed by the sum of two contributions, the first proportional to the surface area and the second to the projected area of the visible part of the emitting region. The latter, besides the factor $(1 - R/R_S)$, is the analog of the Newtonian expression, while the former is a purely relativistic correction. The problem of computing the flux, once the geometry is fixed, is therefore reduced to that of determining $S_v$ and evaluating the two integrals

$$I_v = \int_{S_v} \cos \theta \sin \theta \, d\theta \, d\phi, \quad I_s = \int_{S_v} \sin \theta \, d\theta \, d\phi. \quad (7)$$

2.1. Single Circular Spot

In order to proceed further, we first consider the simplest case, in which the emitting region is a circular cap of semi-aperture $\theta_c$ with its center at $(R, \theta_o, 0)$. For the sake of simplicity, and also because this is the most common occurrence, we consider only the case* $\theta_c \leq \pi/2$. Moreover, we restrict to $0 \leq \theta_o \leq \pi$.

* Since the case $\pi \leq \theta_o \leq 2\pi$ is reduced to the previous one upon the substitution $\theta_o \to 2\pi - \theta_o$, given the axial symmetry around the LOS.

The $\phi$-integral in both $I_v$ and $I_s$ is immediate. By denoting with $\phi_b(\theta)$ the cap boundary $(0 \leq \phi_b \leq \pi)$, it is

$$I_v = 2 \int_{\phi_{\min}}^{\phi_{\max}} \cos \theta \sin \phi_b(\theta) \, d\theta,$$

$$I_s = 2 \int_{\phi_{\min}}^{\phi_{\max}} \sin \phi_b(\theta) \, d\theta, \quad (8)$$

where $\phi_{\min}, \phi_{\max}$ are the limiting values of the colatitude, which are discussed below.

The function $\phi_b$ can be readily found noting that a generic point on the cap boundary has the coordinates $(R, \theta, \phi_b)$. In a spherical coordinate system with the polar axis coincident with the cap axis (unit vector $\hat{\epsilon}$), its coordinates are $(R, \Theta \equiv \theta_c, \Phi)$; the latter system is rotated by an angle $\theta_o$ with respect to the former around an axis perpendicular to the $\hat{1}$-$\hat{\epsilon}$ plane. By exploiting the transformation between the (Cartesian) coordinates in the two systems, one gets

$$\cos \theta = \cos \theta_o \cos \theta_c - \sin \theta_o \sin \theta_c \cos \Phi, \quad (9)$$

$$\sin \theta \cos \phi_b = \cos \theta_o \sin \theta_c \cos \Phi + \sin \theta_o \cos \theta_c. \quad (10)$$

Solving the second for $\cos \Phi$ and substituting into the first one, one finally obtains

$$\cos \phi_b = \frac{\cos \theta_c - \cos \theta_o \cos \theta}{\sin \theta_o \sin \theta}. \quad (11)$$

$\phi_b$ is actually related to the function $h$ introduced by Pechenick et al. (1983). It is immediately verified to be $-1 \leq \cos \phi_b \leq 1$, and hence $\phi_b$ is defined in the range $|\phi_b - \phi_o| < \theta < \min [\theta_o + \phi_o, 2\pi - (\theta_o + \phi_o)]$, since by definition it must be $0 \leq \theta \leq \pi$. It is important to note that the $\theta$-range in which it is possible to define $\cos \phi_b$ is not sufficient to cover the entire cap when the LOS intersects the cap itself: this occurs either for $\theta_o - \theta_c < 0$ or $\theta_o + \phi_o > \pi$. Since it is $0 < \theta_o \leq \pi/2$ by assumption, and disregarding visibility, there is just one intersection at $\theta = 0$ or $\pi$. In these cases, the cap is fully covered only by adding the range $0 \leq \theta < |\theta_o - \theta_c|$ or $2\pi - (\theta_o + \phi_o) \leq \theta < \pi$, respectively. This accounts for the missing surface, which is a circular cap perpendicular to the LOS where $\phi$ spans the entire range $[0, 2\pi]$ (see Figure 2). Accordingly, the definition of $\phi_b(\theta)$ can be continuously extended as

$$\arccos \left[ \frac{\cos \theta_o - \cos \theta_c \cos \theta}{\sin \theta_o \sin \theta} \right]$$

$$\phi_b(\theta) = \begin{cases} \arccos \left[ \frac{\cos \theta_o - \cos \theta_c \cos \theta}{\sin \theta_o \sin \theta} \right] & \text{if } 0 < \theta < \min [\theta_o + \phi_o, 2\pi - (\theta_o + \phi_o)] \\ \pi & \text{if } \theta_o - \theta_c \leq 0 \\ 2\pi - (\theta_o + \phi_o) & \text{if } \theta_o + \phi_c \geq \pi \end{cases} \quad (12)$$

to include all cases.

At each visible point of the star surface, it has to be $\cos \alpha \geq 0$, and the terminator lies precisely at $\alpha = \pi/2$. From Equation (5) it follows that the terminator colatitude is given by

$$\cos \theta_F = \left(1 - \frac{R}{R_S}\right)^{-1}. \quad (13)$$
it is \( \cos \theta_F \leq 0 \) and \( \theta_F \geq \pi/2 \), as expected, since relativistic effects bring more than half the sphere into view.

In the case when the cap is entirely visible, i.e., it does not intersect the terminator, its colatitude is in the range\(^4\) \( \theta_{\text{min}} = \max(0, \theta_o - \theta_c) \leq \theta \leq \theta_{\text{max}} = \min[2\pi - (\theta_o + \theta_c), \theta_o + \theta_c] \). The presence of the terminator (at \( \pi/2 \leq \theta_F < \pi \)) is easily accounted for by replacing \( \theta_{\text{min}} (\theta_{\text{max}}) \), as defined above, with \( \theta_F \) every time it is \( \theta_{\text{min}} (\theta_{\text{max}}) < \theta < \theta_{\text{min}} (\theta_{\text{max}}) \). To summarize, it is

\[
\theta_{\text{min}} = \min \left[ \max(0, \theta_o - \theta_c), \theta_F \right]
\]

(14)

and

\[
\theta_{\text{max}} = \min (\theta_0 + \theta_c, \theta_F).
\]

(15)

Turning to the evaluation of \( I_{p,s} \), both integrals become trivial for \( \phi_0 = \pi \) and yield

\[
I_p = 2 \int_{\theta_1}^{\theta_2} \pi \cos \theta \sin \theta \, d\theta = \pi (\sin^2 \theta_2 - \sin^2 \theta_1),
\]

\[
I_s = 2 \int_{\theta_1}^{\theta_2} \pi \sin \theta \, d\theta = 2\pi (\cos \theta_2 - \cos \theta_1)
\]

(16)

for any pair of angles \( \theta_1, \theta_2 \). In the opposite case, the two indefinite integrals

\[
I_1 = 2 \int \cos \theta \sin \theta \arccos \left[ \frac{\cos \theta_c - \cos \theta_o \cos \theta}{\sin \theta_o \sin \theta} \right] \, d\theta,
\]

(17)

and

\[
I_2 = 2 \int \sin \theta \arcsin \left[ \frac{\cos \theta_c - \cos \theta_o \cos \theta}{\sin \theta_o \sin \theta} \right] \, d\theta
\]

(18)

must be calculated. It turns out that \( I_{1,2} \) can be evaluated analytically in terms of elementary functions (see the Appendix for more details)

\[
I_1 = \sin^2 \theta \arccos \left[ \frac{\cos \theta_c - \cos \theta_o \cos \theta}{\sin \theta_o \sin \theta} \right] + 2 \cos \theta \arcsin \left[ \frac{\cos \theta - \cos \theta_o \cos \theta}{\sin \theta_o \sin \theta} \right] + \text{sign}(\theta_o + \theta_c - \pi) \arcsin \left[ \frac{(\cos \theta_c \cos \theta + 1) \cos \theta + \sin^2 \theta_o - \cos^2 \theta_c - \cos \theta_o \cos \theta_c}{(1 + \cos \theta) \sin \theta_o \sin \theta} \right]
\]

\[- \text{sign}(\theta_o - \theta_c) \arcsin \left[ \frac{(\cos \theta_c \cos \theta - 1) \cos \theta + \sin^2 \theta_o - \cos^2 \theta_c - \cos \theta_o \cos \theta_c}{(1 - \cos \theta) \sin \theta_o \sin \theta} \right],
\]

(19)

(20)

where the arbitrary constant was set to zero. It is then \( I_{p,s} = I_{1,2}(\theta_{\text{max}}) - I_{1,2}(\theta_{\text{min}}) \). We note that \( I_{1,2} \) take a simple form for \( \theta = \theta_o \pm \theta_c \). In particular, if the cap is fully in view (see Figure 2, left), then it is \( I_p = \pi \cos \theta_o \sin^2 \theta_c \) and \( I_s = 2\pi (1 - \cos \theta_o) \), as it also follows from geometrical considerations. The complete form of Equations (19) and (20) is actually required only when evaluating the integrals at \( \theta_F \).

The flux (Equation (6)) is finally written as

\[
F_v = \left( 1 - \frac{R_S}{R} \right)^2 B_o(T) A_{\text{eff}}(\theta_c, \theta_o),
\]

(21)

where we introduced the “effective” area

\[
A_{\text{eff}}(\theta_c, \theta_o) = R^2 \left[ \frac{R_S}{R} I_s + \left( 1 - \frac{R_S}{R} \right) I_p \right].
\]

(22)

2.2. Multiple Spots and Other Geometries

Having computed the flux seen by a distant observer for a single circular spot, it is straightforward to generalize the result to an arbitrary number of spots. We stress that this is possible because, using Beloborodov’s approximation, the flux is proportional to the “effective” area of the cap, \( A_{\text{eff}} \), introduced in the previous section. Although we impose no restrictions on the parameters, the assumption that the spots do not intersect is understood. For the sake of simplicity, here we consider just two

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\( ^4 \) Note, however, that it cannot be \( \theta_{\text{min}} = 0 \) and \( \theta_{\text{max}} = 2\pi - (\theta_o + \theta_c) \) because this would imply \( \theta_c > \pi/2 \).
circular, uniform caps with semi-aperture $\theta_{c,i}$, and temperature $T_i$ ($i = 1, 2$). Let us further assume that the spots are aligned, in the sense that their centers lie on the same meridian (the more general case of misaligned caps will be discussed in the next section), and let $\theta_{d,i}$ be the relative angular displacement of the second one with respect to the first, the center of which is at $\theta_o$ (of course it is $\theta_{d,1} = 0$). The spot centers are then at $\theta_{o,i} = \theta_o + \theta_{d,i}$ and the total flux can be calculated by simply adding the two contributions

$$F_v^{\text{TOT}} = \left(1 - \frac{R_S}{R}\right)^2 \left[ B_v(T_1)A_{\text{eff}}(\theta_{c,1}, \theta_o) + B_v(T_2)A_{\text{eff}}(\theta_{c,2}, \theta_o + \theta_{d,2}) \right].$$

(23)

The case of a two-temperature cap, i.e., a cap at $T_1$ surrounded by a circular corona at $T_2$, is treated much in the same way by subtracting from the larger cap the contribution of the inner spot and adding the latter at the proper temperature

$$F_v^{\text{TOT}} = \left(1 - \frac{R_S}{R}\right)^2 \left[ B_v(T_1)[A_{\text{eff}}(\theta_{c,2}, \theta_o) - A_{\text{eff}}(\theta_{c,1}, \theta_o)] + B_v(T_2)A_{\text{eff}}(\theta_{c,1}, \theta_o) \right].$$

(24)

With the aid of the previous expressions, more configurations can be modeled. In particular, for an NS with a thermal map made of two (antipodal) caps at $T_1$ while the rest of the surface is at $T_2$, one obtains the flux by using twice Equation (24) with $\theta_{c,2} = \pi/2$, the second time replacing $\theta_o$ with $\theta_o + \pi$, and summing the two contributions. The similar case of a single cap at $T_1$ is handled by summing the flux given by Equation (24) with $\theta_{c,2} = \pi/2$ and that of Equation (21), with $\theta_c = \pi/2$, $\theta_o \rightarrow \theta_o + \pi$.

2.3. Pulse Profiles

In order to compute pulse profiles, we first consider the case of a single spot. Let $\mathbf{F}$ be a unit vector parallel to the rotation axis and $\Omega$ the star angular velocity, $\Omega = 2\pi/P$, where $P$ is the spin period. Observed periods in X-ray emitting INSs are in the range $\approx 0.1$–$10$ s, so the assumption of Schwarzschild spacetime previously introduced is fully justified. We also introduce the angles $\chi, \xi$ between the LOS, the cap axis, and the rotation axis, respectively, i.e., $\cos \chi = \mathbf{F} \cdot \hat{\mathbf{L}}$ and $\cos \xi = \mathbf{F} \cdot \hat{\mathbf{\hat{c}}}$.

Since the cap corotates with the star, the vector $\hat{\mathbf{\hat{c}}}$ rotates around $\mathbf{F}$, keeping $\xi$ constant. This implies that $\theta_c$ changes in time. Introducing the rotational phase $\gamma = \Omega t + \gamma_0$ ($\gamma_0$ is an arbitrary initial phase), from simple geometrical considerations it follows that

$$\cos \theta_o = \cos \chi \cos \xi - \sin \chi \sin \xi \cos \gamma.$$ 

(25)

Equation (21) then provides the phase-resolved spectrum once the previous expression is used for $\cos \theta_o$. The pulse profile in a given energy band is immediately obtained integrating over frequencies. Since $\int_0^{\infty} B_v(T) d\nu = \sigma T^4/\pi$ ($\sigma$ is the Stefan–Boltzmann constant), the pulse profile in the $[v_1, v_2]$ range is

$$F(v_1, v_2) = \left(1 - \frac{R_S}{R}\right)^2 C(v_1, v_2) \frac{\sigma T^4}{\pi} A_{\text{eff}}(\theta_c, \theta_o).$$

(26)

where $C(v_1, v_2) = (\pi/\sigma T^4) \int_{v_1}^{v_2} B_v(T) d\nu$. Similar expressions hold for other geometries.

Some examples are illustrated in Figures 3–5 where the pulse profiles are shown for seven values of the angles $\chi, \xi$ in the range $[0^\circ, 90^\circ]$, step $15^\circ$. Because Equation (25) is invariant by exchanging $\chi$ and $\xi$, only the 28 pulse profiles that are actually diverse are shown. In all cases it is $M = 1.4 M_{\odot}$ and $R = 15$ km ($R/R_S = 3.6$), corresponding to $\theta_F = 112^\circ$, and the pulse profiles refer to the bolometric flux (i.e., $C = 1$) normalized to $(F_{\text{max}} + F_{\text{min}})/2$. Figure 3 shows the case of a single spot for $\theta_c = 40^\circ$ (left) and $\theta_c = 3^\circ$ (right), small enough to be treated as point-like (see Section 3). The pulse profiles for two equal, antipodal ($\theta_{d,2} = 180^\circ$) spots are illustrated in Figure 4, again for $\theta_c = 40^\circ$ (left) and $\theta_c = 3^\circ$ (right). Figure 5 (left) refers to two non-antipodal ($\theta_{d,2} = 120^\circ$), different ($\theta_{c,1} = 30^\circ, \theta_{c,2} = 45^\circ, kT_1 = 0.4$ keV, $kT_2 = 1$ keV) caps, while the right panel illustrates the same case but with the second spot shifted in longitude by $45^\circ$. The latter is simply obtained by adding a constant phase-shift to $\gamma$ in Equation (25) when $\theta_o$ refers to the second spot.

3. DISCUSSION AND CONCLUSIONS

In this investigation, we revisited the problem of computing the pulse profiles from thermally emitting spots on the surface of an NS in general relativity. Our goal has been to develop a simple approach that can be readily used for a quantitative comparison of models with observations. Beloborodov (2002), by means of a suitable approximation, was able to derive analytical expressions for the pulse profiles in full GR for point-like, equal, antipodal spots. However, if more realistic thermal configurations are to be accounted for, going beyond the
point-like approximation becomes necessary. We have shown that it is possible to extend Beloborodov’s approach to include (multiple) spots of finite size in different positions on the star surface. The results for the pulse profiles are expressed by comparatively simple analytical formulae which involve only elementary functions.

A qualitative comparison between point-like and finite-size spots is provided by Figures 3 (single spot) and 4 (two equal, antipodal spots); since \( \theta_c = 0 \) produces a vanishing flux, \( \theta_c = 3° \) was used instead to simulate a (nearly) point-like spot (see below). Indeed, the pulse profiles in Figure 4 (right) appear very similar to those discussed by Beloborodov (2002, see his Figure 4) and the four “types” he introduced (classes I–IV) are clearly recognizable. This is better seen in Figure 6 (right), where the pulsed fraction, defined as \( \text{PF} = \frac{F_{\text{max}} - F_{\text{min}}}{F_{\text{max}} + F_{\text{min}}} \), is shown as a function of \( \chi \) and \( \xi \), together with Beloborodov’s analytical result (his Equation (8)). The two sets of contours are nearly indistinguishable and the maximum pulsed fraction, equal to \( \frac{R - 2RS}{R + 2RS} \) for point-like spots, is the same.

As expected, the pulse shape changes for larger caps, the “plateau” disappears, and the pulsed fraction decreases (Figure 4, left). Now the constant PF contours are quite different with respect to those of a point-like spot, as clearly shown in the left panel of Figure 6. The maximal pulsed fraction is \( \sim 30\% \) lower than \( \frac{R - 2RS}{R + 2RS} \) for \( \theta_c = 40° \). In general, we find that the point-like approximation is reliable up to \( \theta_c \sim 5° \).

Larger caps can be treated either using the approach described here or resorting to methods based on general relativistic ray-tracing. We believe that the former offers a number of advantages, since it involves no numerical integration, and allows for a great flexibility, so that diverse thermal configurations of the NS surface can be modeled. An obvious limitation is that only purely blackbody (or at any rate isotropic) emission can be treated. Although this simple model is often successfully used in fitting the thermal components of X-ray spectra, emission from the cooling surface of isolated NSs is expected to be more complicated, e.g., because the star is covered by an atmosphere, or because the emissivity is strongly suppressed at energies below the electron plasma frequency if the surface layers are in condensed form (see, e.g., Turolla 2009 and references therein). Realistic emission models predict, to a different extent, an angular dependence of the emitted intensity. While anisotropy is modest for non-magnetized atmospheric models (Zavlin et al. 1996), it becomes substantial in magnetized atmospheres (Pavlov et al. 1994) or in condensed surfaces (Turolla et al. 2004). In general, it would be impossible to analytically compute the analogs of \( I_{1,2} \) (see Section 2.1) for a non-isotropic radiation field. We point out, however, that in case the intensity depends on the angle \( \alpha \) only, i.e., \( I_\alpha = I_\alpha(\theta) \) by using Equation (5), all the considerations presented in Section 2.1 still hold, although now numerical integration is required to obtain the pulse profiles. This, being the integral just over a single variable, \( \theta \), adds only a modest complication and the present method still has advantages with respect to fully numerical ray-tracing. Clearly this is not the case if \( I_\alpha \) depends on both angles,

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5 The different levels of the flat portion of the pulse profiles, or “plateau,” are due to our different flux normalization.
\[ \alpha \] and the associated azimuth, since double integrals should be evaluated.

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\section*{APPENDIX}

\subsection*{INTEGRALS EVALUATION}

An integration by parts brings the first integral into the form

\[ I_1 = \sin^2 \theta \arccos \left[ \frac{\cos \theta_c - \cos \theta_o \cos \theta}{\sin \theta_o \sin \theta_c} \right] - \frac{1}{\sin \theta_o \sin \theta_c} \int \frac{dz}{\sqrt{1 - \left( \frac{\mu - \cos \theta_c \cos \theta_o \sin \theta}{\sin \theta_o \sin \theta_c} \right)^2}}, \]  
(A1)

where \( \mu = \cos \theta \). By introducing \( z = (\mu - \cos \theta_o \cos \theta_c) / (\sin \theta_o \sin \theta_c) \), the previous expression becomes

\[ I_1 = \sin^2 \theta \arccos \left[ \frac{\cos \theta_c - \cos \theta_o \cos \theta}{\sin \theta_o \sin \theta_c} \right] - \sin^2 \theta_c \cos \theta_o \int \frac{dz}{\sqrt{1 - z^2}} + \cos \theta_c \sin \theta_c \sin \theta_o \int \frac{z \, dz}{\sqrt{1 - z^2}}, \]  
(A2)

which, after some trivial manipulations, yields Equation (19).

\( I_2 \) is handled in a similar way. After integrating by parts, one obtains

\[ I_2 = -2 \cos \theta \arccos \left[ \frac{\cos \theta_c - \cos \theta_o \cos \theta}{\sin \theta_o \sin \theta_c} \right] + \frac{2}{\sin \theta_o \sin \theta_c} \int \frac{\mu \cos \theta_o - \cos \theta_o \cos \theta_c}{1 - \mu^2} \, d\mu \int \frac{dz}{\sqrt{1 - \left( \frac{\mu - \cos \theta_o \cos \theta_c}{\sin \theta_o \sin \theta_c} \right)^2}}, \]  
(A3)

Upon writing

\[ \frac{\mu \cos \theta_o - \cos \theta_o \cos \theta_c}{1 - \mu^2} = \cos \theta_c + \frac{1}{2} \left( -\frac{\cos \theta_o + \cos \theta_c}{1 + \mu} + \frac{\cos \theta_o - \cos \theta_c}{1 - \mu} \right), \]  
(A4)

Equation (A3) can be cast as

\[ I_2 = -2 \cos \theta \arccos \left[ \frac{\cos \theta_c - \cos \theta_o \cos \theta}{\sin \theta_o \sin \theta_c} \right] + 2 \cos \theta_c \int \frac{dz}{\sqrt{1 - z^2}} - (\cos \theta_o + \cos \theta_c) \int \frac{d\mu}{(\mu + 1)^{1/2}} + 2 \cos \theta_c \cos \theta_o \mu - \cos^2 \theta_c - \cos^2 \theta_o + 1 \]  

\times \int \frac{d\mu}{(\mu - 1)^{1/2}} + 2 \cos \theta_c \cos \theta_o \mu - \cos^2 \theta_c - \cos^2 \theta_o + 1. \]  
(A5)

The last two integrals in Equation (A5) are of the general type

\[ \int \frac{dx}{(x + p)\sqrt{ax^2 + bx + c}} = \frac{1}{\sqrt{b^2 - 4ac}} \arcsin \frac{(b - 2ap)x - bp + 2c}{(x + p)\sqrt{b^2 - 4ac}}, \]  
(A6)

\((ap^2 - bp + c < 0, b^2 - 4ac > 0)\); e.g., Prudnikov et al. (1992).

Note, however, that the previous expression is valid only if \( x + p > 0 \). If \( x + p < 0 \), as in the last integral in Equation (A5) where \( \mu - 1 < 0 \), then a minus sign must be placed in front of the result. Making use of Equation (A6) and after some algebra, Equation (20) is finally recovered.

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