Perturbations of eigenvalues embedded at threshold: I. One- and three-dimensional solvable models*

Claudio Cacciapuoti\(^1\), Raffaele Carlone\(^2\) and Rodolfo Figari\(^3\)

\(^1\) Hausdorff Center for Mathematics, Institut für Angewandte Mathematik, Bonn Universität, Endenicher Allee 60, 53115 Bonn, Germany
\(^2\) Dipartimento di Fisica e Matematica, Università degli Studi Insubria, Via Valleggio 11, 22100 Como, Italy
\(^3\) Dipartimento di Scienze Fisiche, Istituto Nazionale di Fisica Nucleare (INFN), Sezione di Napoli, Università di Napoli Federico II, Via Cintia 80126 Napoli, Italy

E-mail: cacciapuoti@him.uni-bonn.de, raffaele.carlone@me.com and figari@na.infn.it

Received 31 May 2010, in final form 22 September 2010
Published 9 November 2010
Online at stacks.iop.org/JPhysA/43/474009

Abstract

We examine perturbations of eigenvalues and resonances for a class of multi-channel quantum mechanical model Hamiltonians describing a particle interacting with a localized spin in dimension \(d = 1, 3\). We consider unperturbed Hamiltonians showing eigenvalues and resonances at the threshold of the continuous spectrum and we analyze the effect of various types of perturbations on the spectral singularities. We provide algorithms to obtain convergent series expansions for the coordinates of the singularities.

PACS numbers: 02.30.Tb, 02.30.Sa, 02.30.Mv
Mathematics Subject Classification: 81Q10, 30B40, 35B34

1. Introduction

An extensive recent literature presenting different rigorous approaches to the analysis of perturbations of energy eigenvalues embedded in the continuous part of the spectrum of Schrödinger operators is now available (see e.g. [6–8, 14, 19, 21] and references therein).

Less is known about the case of eigenvalues embedded at the threshold of the continuous spectrum.

To the best of our knowledge the main results on the topic are contained in [5, 15]. Related results on the behavior of isolated eigenvalues absorbed into the continuous spectrum are analyzed in [9, 11–13, 20] and references therein.

In [15] the authors consider the case of a Schrödinger operator

\[
H(\varepsilon) = -\Delta + V + \varepsilon W \equiv H + \varepsilon W
\]

* In memory of Pierre Duclos.
on $L^2(\mathbb{R}^n)$ with $n$ odd. The unperturbed Hamiltonian $H$ is assumed to have a non-degenerate eigenvalue at zero energy and suitable hypotheses on $V$ guarantee that the essential spectrum of $H$ is purely absolutely continuous and fills the half-line $[0, \infty)$. The self-adjoint operator $W$ is assumed to have a strictly positive expectation value in the eigenvector of $H$ at zero energy, in such a way that the singularity is ‘pushed up’ by the perturbation.

Under some technical assumptions on the properties of $(H - z)^{-1}$ for complex $z$, close to the origin, the authors prove that, as effect of the perturbation, the zero eigenvalue develops into a resonance.

They examine the migration of the singularity from the real axis outside the analyticity region of the resolvent and find a behavior like $\varepsilon^{2\nu/2}$, with $\nu$ odd integer $\nu \geq -1$ for the imaginary part of its position (proportional to the inverse of the resonance lifetime). In several examples of one- and two-channel Schrödinger operators in dimensions 1 and 3, the authors detail the computation of the leading order in $\varepsilon$ of the lifetime of the resonance, showing explicit cases where $\nu \geq 1$ and others where $\nu = -1$.

The survival probability of a resonant state arising from a threshold eigenvalue, has been studied in a specific model (in odd dimensions) in [5]. Once more the authors show that the decay rate of a metastable state depends on the dimension and on the spectral properties of the unperturbed Hamiltonian at the threshold.

We remind that the lifetime of resonances generated by a large class of perturbation of eigenvalues strictly embedded in the continuum shows in any dimension a universal dependence on the perturbation strength (corresponding to $\nu = 0$). Results in [15] and [5] indicate that in the threshold case there is no universal dependence of the lifetime on the perturbation strength.

In order to investigate formation of resonances by perturbation of threshold eigenvalues, we intend to make use of a family of Hamiltonians characterizing different dynamical models for a quantum particle moving in an array of localized spins (see [2] for details). By changing the geometrical and dynamical parameters the spectral structure of the Hamiltonians can be adapted to show isolated or embedded eigenvalues as well as eigenvalues and resonances at any threshold of the continuous spectrum. In particular, we examine in this paper two-channel Hamiltonians in dimensions 1 and 3, for one particle and one spin with eigenvalues and/or resonances in the upper channel at the continuum threshold.

Inasmuch as we examine specific models our analysis lacks some generality. However, we want to point out that zero-range interaction Hamiltonians are particularly versatile models reproducing all the qualitative dynamical features typical of a large class of Schrödinger Hamiltonians (see, e.g., [1, 4]). In particular point interaction Hamiltonians, for any number of channels and any number of centers, have a resolvent expansion around the origin of the type assumed by Jensen and Nenciu to prove their results. Conversely, any Schrödinger operator for which such resolvent expansion around the origin holds true can be approximated by (many-center) point interaction Hamiltonians, see [10].

We also want to emphasize the high degree of computability typical of the models examined in this paper. It allows us to write down explicitly convergent series expansions for the coordinates of resonances and eigenvalues.

In this paper we will only consider multichannel Hamiltonians in order to have a more direct comparison with the results obtained for embedded eigenvalues in [3]. One channel Hamiltonians, possibly with multiple point scatterers, also show very rich spectral configurations with resonances and eigenvalues at some continuous threshold. We plan to examine those cases in further work.

We will not state here our results in a time-dependent framework, limiting ourselves to the so-called spectral form of the Fermi golden rule. The investigation of the ‘survival probability’
of the resonant state is a fundamental step in order to investigate reality and time range of validity of the expected exponential behavior on which the very notion of lifetime relies. We mention that the complete knowledge of the Hamiltonian generalized eigenfunctions allows in our case a very detailed analysis of the time evolution of the resonant state as it was done in the case of embedded eigenvalues [3].

The paper is organized as follows. In section 2 we introduce notation and basic definitions. In section 3 we state and prove our results. Within this section we split in subsections the analysis of different ranges of the perturbation parameters. A final section consists of a summary of results together with further comments.

2. Basic definitions and results

For \( d = 1, 3 \), we consider the Hilbert space \( \mathcal{H} := L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d) \). We denote by \( \Psi \) the generic (column) vector in \( \mathcal{H} \):

\[
\Psi = \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix}, \quad \psi_j \in L^2(\mathbb{R}^d), \quad j = 0, 1.
\]

\( \mathcal{H} \) is the state space of a quantum particle in \( \mathbb{R}^d \) in the presence of a two-level quantum system (a spin) localized at the origin. \( \psi_0 \) (respectively \( \psi_1 \)) represents the state of the particle in the channel where the spin (more precisely, a particular component of the vector spin operator) has value 0 (respectively 1). In what follows we consider Hamiltonians in \( \mathcal{H} \) belonging to the family of the self-adjoint extensions of the symmetric operator \( S \) defined by \( D(S) := C^\infty(\mathbb{R}^d \setminus \{0\}) \oplus C^\infty(\mathbb{R}^d \setminus \{0\}) \), \( S \Psi := (-\Delta \psi_0, (-\Delta + 1)\psi_1) \). We do not detail here how to characterize the whole family of self-adjoint extensions of \( S \), as this was done, in a more general setting and with slightly different notation, elsewhere, see [2]. According to the definition of \( S \) the 0-channel (respectively the 1-channel) will be referred to as the lower (respectively the upper) channel.

For \( z \in \mathbb{C} \setminus \mathbb{R}^+ \), we denote with \( G^z(x) \) the fundamental solution in \( \mathbb{R}^d \) of Helmholtz’s equation: \((-\Delta - z)G^z = \delta\); explicitly

\[
G^z(x) = \begin{cases} \frac{e^{-i\sqrt{z}|x|}}{2\sqrt{z}} & d = 1 \\ \frac{e^{-i\pi x}}{4\pi |x|} & d = 3 \end{cases} \quad \text{with} \quad z \in \mathbb{C} \setminus \mathbb{R}^+; \quad \text{Im}(\sqrt{z}) > 0.
\]

Note that, for \( \text{Im}(\sqrt{z}) > 0 \), \( G^z(x) \in L^2(\mathbb{R}^d) \) for \( d = 1, 3 \), a property which does not hold in dimension higher than 3. This is a crucial feature in the definition of our model Hamiltonians and it is the reason why point interaction Hamiltonians are trivial in dimensions bigger than 3.

We denote by \( H_0 \) the Hamiltonian in \( \mathcal{H} \) given in the following.

**Definition 1.** Let \( \theta_0 \in \mathbb{R} \) and

\[
\theta_1 = \begin{cases} 2 & d = 1 \\ -4\pi & d = 3 \end{cases}
\]

\( H_0 : D(H_0) \subset \mathcal{H} \rightarrow \mathcal{H} \) is the self-adjoint operator:

\[
D(H_0) := \{ \Psi = \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} \in \mathcal{H} \mid \psi_0 = \phi_0^0 + q_0 G^z; \quad \psi_1 = \phi_1^0 + q_1 G^{z-1} \}
\]

\( \phi_0^0, \phi_1^0 \in H^2(\mathbb{R}^d); \quad z \in \mathbb{C} \setminus \mathbb{R}; \quad q_0 = \theta_0 f_0; \quad q_1 = \theta_1 f_1; \)
The spin evolution is not affected by the particle [3]. The two channel Hamiltonian 

\[ H_0(\psi_0, \psi_1) := \begin{cases} 
-\Delta \phi_0^2 + z q_0 G^2 & \text{for } j = 0, 1, d = 3; \\
- (\Delta + 1) \phi_1^2 + z q_1 G^{2-1} & \text{for } j = 0, 1, d = 3. 
\end{cases} \]

For any choice of \( \theta_0 \in \mathbb{R} \), \( H_0 \) belongs to a sub-family of the self-adjoint extensions of the operator \( S \) defined above. Each Hamiltonian of the sub-family generates a dynamics where the spin evolution is not affected by the particle [3]. The two channel Hamiltonian \( H_0 \) is often formally written as

\[ H_0 = \begin{pmatrix} -\Delta + \alpha_0 \delta & 0 \\
0 & -\Delta + 1 + \alpha_1 \delta \end{pmatrix} \]

to stress that it acts as \( S \) on \( D(S) \). To match our notation with the one used in the monograph [1] on point interactions one should take into account the following correspondence rules: \( \alpha_j = -\theta_j \) for \( d = 1 \); \( \alpha_j = 1/\theta_j \) for \( d = 3 \); \( j = 0, 1 \).

The spectrum of \( H_0 \) can be obtained directly from the spectrum of the operator ‘\(-\Delta + \alpha_j \delta\)’ (see [1]). The main results are collected in the following.

**Proposition 1.** For \( d = 1, 3 \), the essential spectrum of \( H_0 \) fills the positive real line, \( \sigma_{\text{ess}}(H_0) = [0, \infty) \) and \( 0 \in \sigma_p(H_0) \), the point spectrum of \( H_0 \). The bound state (of unit norm) corresponding to the zero-energy eigenvalue is \( \Phi_0 = (0, \Phi_1^0) \) with

\[ \phi_1^0(x) = \begin{cases} 
e^{-|x|} & d = 1 \\
\sqrt{2} ne^{-|x|} / 4\pi |x| & d = 3. 
\end{cases} \]

Moreover:

- For \( d = 1 \) and \( \theta_0 > 0 \), \( \sigma_p(H_0) = \{-\theta_0^2/4, 0\} \), while for \( \theta_0 \leq 0 \), \( \sigma_p(H_0) = \{0\} \).
- For \( d = 3 \) and \( \theta_0 < 0 \), \( \sigma_p(H_0) = \{-4\pi / \theta_0^2, 0\} \), while for \( \theta_0 \geq 0 \), \( \sigma_p(H_0) = \{0\} \).

As noted before, \( H_0 \) describes a two independent channel system. In each channel the particle ‘feels’ a point interaction placed in the origin whose strength may depend on the channel (equivalently on the spin state). Among all the self-adjoint extensions of \( S \) one can find a large class of Hamiltonians coupling the two channels. To the aim of examining the behavior of the eigenvalue of \( H_0 \) at the threshold of the essential spectrum, when the lower and upper channels are weakly coupled, we choose a suitable Hamiltonian, \( H_\varepsilon \), belonging to that class and close, in a sense that will be made precise in the following, to \( H_0 \).

**Definition 2.** Let us take \( \theta_0, b, c \in \mathbb{R} \) and let \( \varepsilon > 0 \). For \( d = 1, 3 \) we set

\[ \theta_\varepsilon^i = \begin{cases} 
2 + c \varepsilon & d = 1 \\
-4\pi + c \varepsilon & d = 3; 
\end{cases} \]

\( H_\varepsilon : D(H_\varepsilon) \subset \mathcal{H} \rightarrow \mathcal{H} \) is the self-adjoint operator:

\[ D(H_\varepsilon) := \left\{ \psi_0, \psi_1 \in \mathcal{H} \mid \begin{array}{l} 
\psi_0 = \phi_0^\varepsilon + q_0 G^2, \psi_1 = \phi_1^\varepsilon + q_1 G^{2-1}; \\
\phi_0^\varepsilon, \phi_1^\varepsilon \in H^2(\mathbb{R}^d); z \in \mathbb{C}\setminus \mathbb{R}; \\
q_0 = \theta_0 f_0 + bf_1, q_1 = bf_0 + \theta_\varepsilon^i f_1; \\
f_j = \psi_j(0) j = 0, 1, d = 1; 
\end{array} \right\} \]
\[
D_{\varepsilon}(z) = \lim_{|z| \to 0} \left[ \psi_j(x) \frac{q_j}{4\pi|x|} \right] \quad j = 0, 1, d = 3
\]

\[
H_{\varepsilon} \left( \begin{array}{c} \psi_0 \\ \psi_1 \end{array} \right) := \left( \begin{array}{c} -\Delta \phi_0^2 + zq_0 G(z) \\ -\Delta + 1 \phi_1^2 + zq_1 G(z) \end{array} \right); \quad \left( \begin{array}{c} \psi_0 \\ \psi_1 \end{array} \right) \in D(H_{\varepsilon}).
\]

For all \( z \in \mathbb{C} \setminus \mathbb{R} \) we denote by \( R_{\varepsilon}(z) := (H_{\varepsilon} - z)^{-1} \) the resolvent of \( H_{\varepsilon} \). An explicit formula for \( R_{\varepsilon}(z) \) can be obtained by using the theory of self-adjoint extensions of symmetric operators (see [2] and [3]). We summarize the result in the following formula:

\[
R_{\varepsilon}(z) = \left( \begin{array}{cc} (-\Delta - z)^{-1} & 0 \\ 0 & (-\Delta + 1 - z)^{-1} \end{array} \right) + \frac{1}{D_{\varepsilon}(z)} \left( \begin{array}{cc} \Gamma_{\varepsilon,11}(z)(G^0z \cdot \cdot)G(z) & \Gamma_{\varepsilon,12}(z)(G^0z \cdot \cdot)G(z) \\ \Gamma_{\varepsilon,21}(z)(G^0z \cdot \cdot)G(z) & \Gamma_{\varepsilon,22}(z)(G^0z \cdot \cdot)G(z) \end{array} \right),
\]

where the function \( G(z) \) was defined in equation (2.1) and

\[
D_{\varepsilon}(z) = \left[ \begin{array}{cc} b^2\varepsilon^2 - (\theta_0 + 2i\sqrt{z}) [2(1 + i\sqrt{z - 1}) + c\varepsilon] \\ 1 - i \frac{\theta_0}{4\pi \sqrt{z}} \end{array} \right] 1 + i \left( 1 - \frac{c\varepsilon}{4\pi} \right) \sqrt{z - 1} + \left( \frac{b\varepsilon}{4\pi} \right)^2 \sqrt{z - 1}\sqrt{z} \quad d = 3,
\]

where the matrix elements \( \Gamma_{\varepsilon,ij}(z) \) read

for \( d = 1 \)

\[
\begin{align*}
\Gamma_{\varepsilon,11}(z) &= -2i\sqrt{z}[-b^2\varepsilon^2 + \theta_0(2 + 2i\sqrt{z - 1} + c\varepsilon)] \\
\Gamma_{\varepsilon,12}(z) &= \Gamma_{\varepsilon,21}(z) = 4b\varepsilon \sqrt{z - 1} \\
\Gamma_{\varepsilon,22}(z) &= -2i\sqrt{z - 1} [-b^2\varepsilon^2 + (2 + c\varepsilon)(2i\sqrt{z} + \theta_0)],
\end{align*}
\]

for \( d = 3 \)

\[
\begin{align*}
\Gamma_{\varepsilon,11}(z) &= \theta_0 \left[ 1 + i \left( 1 - \frac{c\varepsilon}{4\pi} \right) \sqrt{z - 1} \right] - b^2\varepsilon^2 \sqrt{z - 1} \frac{1 + \sqrt{z}}{4\pi i} \\
\Gamma_{\varepsilon,12}(z) &= \Gamma_{\varepsilon,21}(z) = b\varepsilon \\
\Gamma_{\varepsilon,22}(z) &= -4\pi + c\varepsilon \left[ 1 - i \frac{\theta_0}{4\pi \sqrt{z}} \right] - b^2\varepsilon^2 \sqrt{z} \frac{1 + \sqrt{z}}{4\pi i},
\end{align*}
\]

For \( d = 1, 3 \) the explicit form of the resolvent of \( H_{\varepsilon} \), \( R_{\varepsilon}(z) := (H_{\varepsilon} - z)^{-1} \), can be obtained from formulas (2.3)–(2.6) by setting \( \varepsilon = 0 \).

**Remark 1.** For \( \theta_0 = 0 \) and \( d = 1 \) the resolvent \( R_{\varepsilon}(z) \) has a singularity in \( z = 0 \) on both channels. In the upper channel there is a polar singularity corresponding to the eigenvalue, while in the lower channel there is a singularity of order \( z^{-1/2} \). This is easily checked analyzing the behavior around \( z = 0 \) of \( (-\Delta - z)^{-1} \) (see, e.g., [17, 18]). A precise statement, obtained examining the integral kernel \( G^0(z)(x - y) \) of \( (-\Delta - z)^{-1} \), gives the following expansion:

\[
(-\Delta - z)^{-1} = \begin{cases} \frac{i}{2\sqrt{z}} + O(1) & d = 1 \\
O(1) & d = 3,
\end{cases}
\]

where \( O(1) \) denotes an operator on some suitable weighted \( L^2 \) space, whose norm remains bounded uniformly in \( z \). A possible choice for the weighted space is for example \( L^2(R^d, (1 + |x|)^{-\delta} \, dx) \) for some \( \delta \) large enough.

Finally we note that \( H_{\varepsilon} \) is a small perturbation of \( H_0 \) in the resolvent sense, i.e. \( \forall z \in \mathbb{C} \setminus \mathbb{R} \) there exists \( \varepsilon_0 \) such that for all \( 0 < \varepsilon < \varepsilon_0 \)

\[
\| R_{\varepsilon}(z) - R_0(z) \|_{\mathbb{C}^d, \mathbb{C}^d} \leq \varepsilon C,
\]

where the norm \( \mathbb{C}^d, \mathbb{C}^d \) denotes the usual norm of \( \mathbb{C}^d \times \mathbb{C}^d \).
where $C$ is a positive constant independent on $\varepsilon$ and $\|\cdot\|_{\mathcal{B}(\mathcal{H},\mathcal{H})}$ is the operator norm in the vector space $\mathcal{B}(\mathcal{H},\mathcal{H})$ of bounded linear operators on $\mathcal{H}$.

3. Results

In this section we analyze the spectral structure of $H_\varepsilon$ to examine the effect of the coupling between the lower and the upper channels on the spectrum of the Hamiltonian $H_0$ and in particular on its zero-energy eigenvalue. We denote by $\sigma_p(H_\varepsilon)$, $\sigma_{\text{ess}}(H_\varepsilon)$ and $\sigma_{\text{ac}}(H_\varepsilon)$ the point, essential and absolutely continuous spectrum of $H_\varepsilon$ respectively.

It will be clear from the proofs that value and sign of the parameter $b$ in definition 2 do not affect our results in any substantial way. For this reason we set $b = 1$.

We use sometimes the phrase ‘for $\varepsilon$ small enough ...’ as a short version of ‘there exists $\varepsilon_0$ such that for all $0 < \varepsilon < \varepsilon_0$ ...’.

3.1. Positive perturbations: $c < 0$

In this section we study the behavior of the threshold eigenvalue when the parameter $c$ in the Hamiltonian $H_\varepsilon$ is negative. This choice corresponds to a positive perturbation of the Hamiltonian $H_0$ in the sense that for small $\varepsilon$ the threshold eigenvalue is pushed inside the continuum as a consequence of the perturbative term $c\varepsilon$ in $\theta_1^\varepsilon$.

In order to make this statement more precise let us consider the case $b = 0$ in definition 2. The Hamiltonian $H_{\varepsilon}^{b=0}$ is a perturbation (in resolvent sense) of the Hamiltonian $H_0$ for which the channels 0 and 1 are not coupled. For all $\varepsilon$ small enough the Hamiltonian $H_{\varepsilon}^{b=0}$ has an eigenvalue $E_{\varepsilon}^{b=0}$ in a ball of radius $\varepsilon$ around the origin, and

$$E_{\varepsilon}^{b=0} = 1 - \frac{c^2 \varepsilon^2}{4} d = 1$$

$$E_{\varepsilon}^{b=0} = 1 - \frac{(4\pi)^2}{\theta_1^2} - \frac{c\varepsilon}{2\pi} + O(\varepsilon^2) d = 3.$$ (3.1)

Looking at $H_\varepsilon$ as perturbation of $H_{\varepsilon}^{b=0}$, we expect that the zero-energy eigenvalue of $H_0$ will be driven in a resonance as it happens for embedded eigenvalues.

We analyze first the cases $d = 1$ and $\theta_0 \neq 0$.

**Theorem 1.** Let $d = 1$ and assume that $c < 0$ and $\theta_0 \neq 0$. Then there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the essential spectrum of $H_\varepsilon$ fills the positive real line and is only absolutely continuous:

$$\sigma_{\text{ess}}(H_\varepsilon) = \sigma_{\text{ac}}(H_\varepsilon) = [0, +\infty);$$

there exists a positive constant $C$ such that the Hamiltonian $H_\varepsilon$ has no isolated eigenvalues in $(-C, 0)$; the analytic continuation of the resolvent $R_\varepsilon(z)$ through the positive real axis from the semi-plane $\text{Im}z > 0$ has a simple pole (resonance) in $z = E_\varepsilon^\varepsilon$ where $\text{Im}(E_\varepsilon^\varepsilon) < 0$ and

$$E_\varepsilon^\varepsilon = |c|\varepsilon + \left(\frac{1}{\theta_0} - \frac{c^2}{4}\right)\varepsilon^2 - \frac{2\sqrt{|c|}}{\theta_0}\varepsilon^{3/2} + O(\varepsilon^4).$$ (3.2)

**Remark 2.** Theorem 1 does not characterize the entire spectral structure of the Hamiltonian $H_\varepsilon$. Statements only concern spectral singularities in a small region of the complex plane around the origin. In particular for $\theta_0 > 0$ in $d = 1$ there is one negative eigenvalue close to
the corresponding eigenvalue of $H^b_{\epsilon=0}$. The associated singularity of the resolvent of $H_\epsilon$ is bounded away from the origin unless $\theta_0$ is very small.

Looking at the dependence on $\theta_0 \rightarrow 0$ of the coordinates of the resonance in $d = 1$ given in (3.4) one realizes that the case $\theta_0 = 0$ has to be treated independently.

**Proof.** In formula (2.3) the term $(-\Delta - z)^{-1}$ has a singularity of order $z^{-1/2}$ in $z = 0$ (see remark 1). That singularity is exactly canceled by an opposite singularity arising from the coefficient $\Gamma_{e,11}/D_\epsilon$. In fact an explicit calculation gives

$$\frac{\Gamma_{e,11}(z)(G^2, \cdot G^2)}{D_\epsilon(z)} = -\frac{i}{2\sqrt{z}} + O(1)$$

for all $\epsilon > 0$. Again the equality has to be intended in some weighted $L^2(\mathbb{R})$ space, where the weight depends on the number of terms of the expansion one considers (see [17] for details). A similar remark holds true for the singularity in $z = 1$ of order $(z - 1)^{-1/2}$ in the upper channel. This singularity, arising from the term $(-\Delta + 1 - z)^{-1}$ in formula (2.3), is compensated by an opposite singularity in the term $\Gamma_{r,zz}/D_\epsilon$. Then the singularities of the resolvent on the real axes coincide with the zeroes of the function $D_\epsilon(z)$ in equation (2.3).

We prove first statement (3.3), showing that $H_\epsilon$ has no embedded eigenvalues or eigenvalues at the threshold. Let us set $z = \lambda > 0$. Trivially $D_\epsilon(1) \neq 0$ and $D_\epsilon(0) \neq 0$. For $\lambda \in [0, 1)$ a direct calculation shows that equations $\text{Im}[D_\epsilon(\lambda)] = 0$ and $\text{Re}[D_\epsilon(\lambda)] = 0$ are not compatible. We deduce that there are no solutions to the equation $D_\epsilon(\lambda) = 0$ for $\lambda \in [0, 1)$. For $\lambda > 1$, taking real and imaginary parts of the equation $D(\lambda) = 0$ we get

$$(2 + c\epsilon)\sqrt{\lambda} = -\theta_0 \sqrt{\lambda + 1}$$

and

$$\epsilon^2 + 4\sqrt{\lambda} \sqrt{\lambda + 1} = \theta_0(2 + c\epsilon).$$

For $\theta_0 > 0$ equation (3.5) has no solutions in $(1, +\infty)$ and for $\theta_0 < 0$ equation (3.6) has no solutions in $(1, +\infty)$.

Next we prove that there are no isolated eigenvalues in some suitable neighborhood of $z = 0$. Let us set $z = -\lambda$; then equation $D_\epsilon(-\lambda) = 0$ gives

$$-\theta_0 + 2\sqrt{\lambda} = -\frac{\epsilon^2}{2(1 - \sqrt{\lambda + 1}) - |c|\epsilon}.$$  

(3.7)

For $\lambda \in (0, +\infty)$ the right-hand side of the equation is a strictly positive and decreasing function which equals $\epsilon/|c|$ for $\lambda = 0$. While the left-hand side of the equation is a strictly increasing function which equals $-\theta_0$ for $\lambda = 0$. Then for $\epsilon$ small enough there are no solutions of (3.7) when $\theta_0 < 0$. For $\theta_0 > 0$ the lhs has a zero in $\lambda = \theta_0^2/4$ and it is negative in $(0, \theta_0^2/4)$ and positive in $(\theta_0^2/4, +\infty)$. Then for $\theta_0 > 0$ there is only one solution to equation (3.7), say $\lambda_{0,\epsilon}$, and $\lambda_{0,\epsilon} > \theta_0^2/4$. It follows that for $\theta_0 \neq 0$ there are no isolated eigenvalues in $(-\theta_0^2/4, 0)$.

To find a solution to the equation $D_\epsilon(z) = 0$ we make use of the following recursive procedure. We first note that the equation $D_\epsilon(z) = 0$ can be written as

$$i\sqrt{z - 1} = -1 - \frac{\epsilon\epsilon}{2} + \frac{\epsilon^2}{2(\theta_0 + 2i\sqrt{z^*})}.$$  

We look then for a fixed point of the recurrence relation

$$z^{(0)} = 0$$

$$z^{(k+1)} = 1 - \left[1 - \frac{|c|\epsilon}{2} - \frac{\epsilon^2}{2(\theta_0 + 2i\sqrt{z^{(k)}})}\right]^2 \quad k = 0, 1, 2, \ldots.$$  

(3.9)
To prove convergence of the sequence $z^{(k)}$ we proceed by induction. Assume that $|z^{(k-1)}| \leq C\varepsilon$; then equation (3.9) implies that for $\varepsilon$ small enough $|z^{(k)}| < C\varepsilon$. Since $z^{(1)} = |c|\varepsilon + O(\varepsilon^2)$, $|z^{(2)}| \leq C\varepsilon$ for all $k$. Moreover, let us set $z^{(k)} = |c|\varepsilon(1 + w^{(k)})$, $k = 1, 2, 3, \ldots$. Trivially $|w^{(1)}| \leq C\varepsilon$ and by equation (3.9)

$$
|w^{(k+1)} - w^{(k)}| = \frac{1}{|c|\varepsilon} \left| 2 \left( 1 - \frac{|c|\varepsilon}{2} \right) \left( 2(\theta_0 + 2i\sqrt{|c|\varepsilon(1 + w^{(k)})}) \right) \right.
- \left. \frac{\varepsilon^2}{2(\theta_0 + 2i\sqrt{|c|\varepsilon(1 + w^{(k)})})^2} \right| \leq C\varepsilon^{1/2} |w^{(k)} - w^{(k+1)}|.
$$

(3.10)

Since the function $(\theta_0 + 2i\sqrt{|c|\varepsilon(1 + w)})^{-1}$ is analytic for $w$ in a ball of radius $\varepsilon$ around the origin,

$$
\frac{1}{2(\theta_0 + 2i\sqrt{|c|\varepsilon(1 + w^{(k)})})^2} - \frac{1}{2(\theta_0 + 2i\sqrt{|c|\varepsilon(1 + w^{(k-1)})})^2} \leq C\varepsilon^1 |w^{(k)} - w^{(k-1)}|.
$$

The second term on the rhs of equation (3.10) can be treated in a similar way. Then $|w^{(k+1)} - w^{(k)}| \leq C\varepsilon^{3/2} |w^{(k)} - w^{(k-1)}|$ for all $k = 2, 3, \ldots$ which in turns implies that $|z^{(k+1)} - z^{(k)}| \leq C\varepsilon^{3/2} |z^{(k)} - z^{(k-1)}|$ for all $k = 2, 3, \ldots$; the sequence $\{z^{(k)}\}$ converges in a ball of radius $\varepsilon$ and

$$
z^{(2)} = |c|\varepsilon + \left( \frac{1}{\theta_0} - \frac{c^2}{4} \right) \varepsilon^2 - \frac{2\sqrt{|c|}}{\theta_0} \varepsilon^2 + O(\varepsilon^4).
$$

\[\Box\]

In the next theorem we analyze the behavior of the zero-energy eigenvalue when $\theta_0 = 0$, $c < 0$ for $d = 1$. In addition to the embedded eigenvalue in (3.1), $H_0^{b=0}$ has in this case a resonance at zero energy in the lower channel. The coupling between channels will turn the resonance into a negative eigenvalue.

**Theorem 2.** Let $d = 1$; assume that $c < 0$ and set $\theta_0 = 0$. Then there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the essential spectrum of $H_0$ fills the positive real line and is only absolutely continuous:

$$
\sigma_{\text{ess}}(H_0) = \sigma_{\text{ac}}(H_0) = [0, +\infty); \quad (3.11)
$$

the analytic continuation of the resolvent $R_0(z)$ through the positive real axis from the semiplane $\text{Im} z > 0$ has a simple pole (resonance) in $z = E_0^r$ where $\text{Im}(E_0^r) < 0$ and

$$
E_0^r = |c|\varepsilon - \frac{i}{2\sqrt{|c|}} \varepsilon^2 + O(\varepsilon^4); \quad (3.12)
$$

the Hamiltonian $H_0$ has an isolated eigenvalue $E_0 < 0$ and

$$
\frac{\varepsilon^2}{4|c|^2} < E_0 < 0. \quad (3.13)
$$

**Remark 3.** The emergence of the eigenvalue (3.13) reveals the effect of the channel coupling on the zero-energy resonance in the lower channel, whereas the resonance (3.12) appears as the effect of perturbing the eigenvalue in the upper channel.

**Proof.** We first note that also in this case, similar to the case $\theta_0 \neq 0$ (see theorem 1), the singularities in $z = 0$ and $z = 1$, arising from $(-\Delta - z)^{-1}$ and $(-\Delta + 1 - z)^{-1}$ in equation (2.2), are canceled by opposite singularities in the two terms $\Gamma_{c,11}/D_\varepsilon$ and $\Gamma_{c,22}/D_\varepsilon$, respectively.
Statement (3.11) can be proved as it was done in theorem 1. A straightforward analysis shows in fact that the equation \( D_\varepsilon(z) = 0 \) has no solutions for \( z \in [0, \infty) \).

We discuss now the existence of isolated eigenvalues. Let us set \( z = -\lambda, \lambda > 0 \). The equation \( D_\varepsilon(-\lambda) = 0 \) can be written as

\[
2\sqrt{\lambda} = \frac{\varepsilon^2}{2(\sqrt{\lambda + 1} - 1) + |c|\varepsilon} \tag{3.14}
\]

The left-hand side in the last equation is a strictly positive, increasing function which equals zero for \( \lambda = 0 \). The right-hand side of the equation is a strictly decreasing, positive function which equals \( \varepsilon/|c| \) for \( \lambda = 0 \). It is obvious that there is only one solution \( \lambda_\varepsilon \) to equation (3.14) and that \( 0 < \lambda_\varepsilon < \varepsilon^2/(4|c|^2) \) which in turn implies the estimate (3.13).

We investigate now the presence of poles of the resolvent in the ‘unphysical’ Riemann sheet. Similar to what was done in the previous theorem we use a recursive procedure. We rewrite the equation \( D_\varepsilon(z) = 0 \) as

\[
i\sqrt{z - 1} = -1 + \frac{|c|\varepsilon}{2} + \frac{\varepsilon^2}{4i\sqrt{\varepsilon}}, \tag{3.15}
\]

which implies

\[
z = 1 - \left[ 1 - \frac{|c|\varepsilon}{2} - \frac{\varepsilon^2}{4i\sqrt{\varepsilon}} \right]^2.
\]

We set \( z = |c|\varepsilon(1 + w) \) and define the recursive procedure

\[
w^{(0)}(z) = 0
\]

\[
w^{(k+1)}(z) = \frac{\varepsilon^{1/2}}{2|c|i\sqrt{|c|(1 + w^{(k)})}} - \frac{|c|\varepsilon}{4} - \frac{\varepsilon^{3/2}}{4I\sqrt{|c|(1 + w^{(k)})}} + \frac{\varepsilon^2}{16|c|^2(1 + w^{(k)})} \quad k = 0, 1, 2, \ldots;
\]

then \( z^{(k)} = |c|\varepsilon(1 + w^{(k)}) \) for all \( k = 0, 1, 2, \ldots \). By induction it is easy to prove that for all \( k = 0, 1, 2, \ldots \) and for \( \varepsilon \) small enough, \( |w^{(k)}| \leq C\varepsilon^{1/2} \) which in turns implies \( |z^{(k)}| \leq C\varepsilon \). Moreover, for all \( k \), \( |w^{(k+1)}(z) - w^{(k)}(z)| \leq C\varepsilon^{1/2}|w^{(k)} - w^{(k-1)}| \). Then \( |z^{(k+1)} - z^{(k)}| \leq C\varepsilon^{1/2}|z^{(k)} - z^{(k-1)}| \); consequently the sequence \( \{z^{(k)}\} \) is convergent in a ball of radius \( C\varepsilon \) and the solution of equation (3.15) can be written as \( E_\varepsilon^r = z^{(\infty)} \). By a straightforward computation it is easy to see that

\[
E_\varepsilon^r = |c|\varepsilon - i \frac{\varepsilon}{2\sqrt{|c|\varepsilon}} + |c|^2 \varepsilon^2 + O(\varepsilon^3).
\]

In the next theorem we analyze the behavior of the zero-energy eigenvalue when \( c < 0 \) for \( d = 3 \).

**Theorem 3.** Let \( d = 3 \) and assume that \( c < 0 \); then there exists \( \varepsilon_0 > 0 \) such that for all \( 0 < \varepsilon < \varepsilon_0 \) the essential spectrum of \( H_\varepsilon \) fills the positive real line and is only absolutely continuous:

\[
\sigma_{ess}(H_\varepsilon) = \sigma_{ac}(H_\varepsilon) = [0, +\infty); \tag{3.16}
\]

there exists a positive constant \( C \) such that the Hamiltonian \( H_\varepsilon \) has no isolated eigenvalues in \((-C, 0)\); the analytic continuation of the resolvent \( R_\varepsilon(z) \) through the positive real axis from the semi-plane \( \text{Im} z > 0 \) has a simple pole (resonance) in \( z = E_\varepsilon^r \) where \( \text{Im}(E_\varepsilon^r) < 0 \) and

\[
E_\varepsilon^r = \frac{|c|\varepsilon}{2\pi} - \frac{3|c|^2\varepsilon^2}{16\pi^2} - i \frac{1}{8\pi^3} \sqrt{\frac{|c|}{2\pi}} \varepsilon^{5/2} + O(\varepsilon^3). \tag{3.17}
\]
Proof. Using the classical results we summarized in remark 1 and the explicit expressions for the components of the matrix \( \Gamma_c \) (see equation (2.6)) one can conclude that for \( d = 3 \) the singularities of the resolvent \( R_c(z) \) coincide with the zeroes of the function \( D_c(z) \) defined in equation (2.4).

By direct analysis one can see that the equation \( D_c(\lambda) = 0 \) has no solutions for \( \lambda > 0 \) from which statement (3.16) directly follows.

The eigenvalues of the Hamiltonian \( H_c \) are given by the solutions of the equation \( D_c(-\lambda) = 0 \) for \( \lambda > 0 \). Such equation can be rearranged as

\[
1 + \frac{\theta_0}{4\pi} \sqrt{\lambda} = \left[ \left( 1 + \frac{\theta_0}{4\pi} \sqrt{\lambda} \right) \left( 1 - \frac{c\varepsilon}{4\pi} \right) + \frac{\varepsilon^2}{(4\pi)^2} \sqrt{\lambda} \right] \sqrt{1 + \lambda}, \quad \lambda > 0. \tag{3.18}
\]

For \( c < 0 \) and \( \theta_0 \geq 0 \) equation (3.18) has no solutions while for \( c < 0 \) and \( \theta_0 < 0 \) there is only one solution in a neighborhood of radius \( \varepsilon \) of the point \((4\pi/\theta_0)^2\). Then for \( c < 0 \) and \( \varepsilon \) small enough there are no isolated eigenvalues in some suitable neighborhood of the origin.

It remains to analyze the existence of solutions of the equation \( D_c(z) = 0 \) in the unphysical Riemann sheet in a neighborhood of the origin. The equation \( D_c(z) = 0 \) can be rearranged in the following way:

\[
\left[ 1 + i \left( 1 - \frac{c\varepsilon}{4\pi} \right) \sqrt{z - 1} \right] = -\frac{\varepsilon^2}{(4\pi)^2} \left( 1 - i \frac{\theta_0}{4\pi} \sqrt{z} \right)^{-1} \sqrt{z - 1} \sqrt{z}, \tag{3.19}
\]

from which we define the sequence \( \{z^{(k)}\} \): \( z^{(0)} = 0 \):

\[
z^{(k+1)} = \frac{-c\varepsilon/(2\pi) + c^2\varepsilon^2/(4\pi)^2}{[1 - c\varepsilon/(4\pi)]^2} - \frac{2\varepsilon^2}{(4\pi)^2} \frac{\sqrt{z^{(k)}} - 1}{\sqrt{z^{(k)}}} \\
- \frac{\varepsilon^4}{(4\pi)^4} \frac{(z^{(k)} - 1)z^{(k)}}{(1 - c\varepsilon/(4\pi))^2(1 - i\theta_0\sqrt{z^{(k)}}/(4\pi))}, \quad k = 0, 1, 2, \ldots,
\]

where the formula for \( z^{(k+1)} \) was obtained by solving for \( z \) the lhs of equation (3.19). By induction one can prove that \( |z^{(k)}| \leq C\varepsilon \) for all \( k = 0, 1, 2, \ldots \). Moreover, by direct computation one can see that \( |z^{(k+1)} - z^{(k)}| \leq C\varepsilon^{3/2}|z^{(k)} - z^{(k-1)}| \). Then the series \( \{z^{(k)}\} \) converges in a ball of radius \( \varepsilon \) and \( E_c^\prime \equiv z^{(\infty)} \); by direct computation one can prove expansion (3.17).

3.2. Pure off-diagonal perturbations. \( c = 0 \)

The case \( c = 0 \) marks the boundary between two different behaviors of the threshold eigenvalue under perturbation: the evolution toward a proper eigenvalue \( (c > 0) \) and the evolution toward a resonance \( (c < 0) \). The model shows, in this case, peculiar features, strongly depending on the spatial dimension, which we will analyze for \( d = 1, 3 \) separately.

If in addition to \( c = 0 \) we set \( \theta_0 = 0 \) the Hamiltonian \( H_0 \) shows, for \( d = 1 \), both a zero-energy eigenvalue in the upper channel and a zero-energy resonance in the lower one. Once more the case \( \theta_0 = 0 \) requires a distinct analysis.

As it was done in the previous sections we set \( b = 1 \).

We study first the case \( d = 1 \).
Theorem 4. Let $d = 1$ and assume that $c = 0$. Then there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$, the essential spectrum of $H_\varepsilon$ fills the positive real line and is only absolutely continuous:

$$\sigma_{\text{ess}}(H_\varepsilon) = \sigma_{\text{ac}}(H_\varepsilon) = [0, +\infty).$$

Moreover for $\varepsilon$ small enough

- if $\theta_0 > 0$, the analytic continuation of the resolvent $R_\varepsilon(\lambda)$ through the positive real axis from the semi-plane $\text{Im} \, z > 0$ has a simple pole (resonance) in $z = E_\varepsilon^r$ and

  $$E_\varepsilon^r = \frac{\varepsilon^2}{\theta_0} - i \frac{2}{\theta_0^5/2} \varepsilon^3 + \mathcal{O}(\varepsilon^4),$$

  there exists a positive constant $C$ such that the Hamiltonian $H_\varepsilon$ has no isolated eigenvalues in $(-C, 0)$;

- if $\theta_0 = 0$, the Hamiltonian $H_\varepsilon$ has an isolated eigenvalue in $z = E_\varepsilon$ and

  $$E_\varepsilon = -\frac{\varepsilon^2}{2^{7/3}} + \mathcal{O}(\varepsilon^{8/3});$$

- if $\theta_0 < 0$ the Hamiltonian $H_\varepsilon$ has an isolated eigenvalue in $z = E_\varepsilon$ and

  $$E_\varepsilon = -\frac{\varepsilon^2}{|\theta_0|} + \mathcal{O}(\varepsilon^3).$$

Remark 4. When $\theta_0 = 0$ the Hamiltonian $H_\varepsilon$ also shows two singularities at the same distance from the origin of the eigenvalue (3.22) and with $\text{arg}(z = -5\pi/3)$ and $\text{arg}(z = -\pi/3)$.

Proof. Similar to the case $c < 0$, see theorems 1 and 2, for any $\theta_0 \in \mathbb{R}$ the singularities in $z = 0$ and $z = 1$ arising from $(-\Delta - z)^{-1}$ and $(-\Delta + 1 - z)^{-1}$, are compensated by the terms $\Gamma_{1,1}/D_\varepsilon$ and $\Gamma_{1,2}/D_\varepsilon$, respectively. Then statement (3.20) is a consequence of the fact that the equation $D_\varepsilon(\lambda) = 0$ has no solutions for $\lambda > 0$ and the singularities of the resolvent coincide with the roots of the equation $D_\varepsilon(z) = 0$.

For $\theta_0 > 0$ the equation $D_\varepsilon(-\lambda) = 0$, for $\lambda > 0$, has only one solution in $(\theta_0^2/4, \infty)$; then the Hamiltonian $H_\varepsilon$ has only one isolated eigenvalue in $(-\infty, -\theta_0^2/4)$.

The existence of a pole in the ‘unphysical’ Riemann sheet can be proven by making use of a recursive procedure. The equation $D_\varepsilon(z) = 0$ can be rearranged as

$$i \sqrt{z - 1} + 1 = \frac{\varepsilon^2}{2(\theta_0 + 2i\sqrt{2})}.$$

To find the solution of the last equation we define the recursive procedure

$$z^{(0)} = 0$$

$$z^{(k+1)} = \frac{\varepsilon^2}{\theta_0 + 2i\sqrt{z^{(k)}}} - \frac{\varepsilon^4}{4(\theta_0 + 2i\sqrt{z^{(k)}})^2} \quad k = 0, 1, 2, \ldots.$$

By using techniques similar to the ones used in the proof of theorem 1 one can see that the sequence $(z^{(k)})$ is convergent in a ball of radius $\varepsilon^2$ around the origin and that the estimate (3.21) holds.

For $\theta_0 = 0$ the equation $D_\varepsilon(z) = 0$ reads

$$4i \sqrt{(1 + i \sqrt{z - 1})} - \varepsilon^2 = 0.$$  

(3.24)

To find the isolated eigenvalue we set $z = -\lambda$, $\lambda > 0$, and rearrange equation (3.24) as

$$4(\sqrt{1 + \lambda} - 1) = \frac{\varepsilon^2}{\sqrt{\lambda}} \quad \lambda > 0.$$
Obviously the last equation has only one solution. The recursive procedure
\[ \lambda^{(0)} = \frac{\varepsilon^{4/3}}{2^{2/3}} \]
\[ \lambda^{(k+1)} = \frac{\varepsilon^2}{2\sqrt{\lambda^{(k)}}} + \frac{\varepsilon^4}{16\lambda^{(k)}} \quad k = 0, 1, 2, \ldots \]
converges to the solution and can be used to prove the estimate (3.22).

For \( \theta_0 < 0 \) the existence of an isolated isolated eigenvalue and the estimate (3.23) can be proven by writing the equation \( D_\varepsilon(-\lambda) = 0, \lambda > 0, \) as
\[ 2(\sqrt{1+\lambda} - 1) = \frac{\varepsilon^2}{2\sqrt{\lambda} + |\theta_0|} \]
and by using the recursive procedure defined by
\[ \lambda^{(0)} = 0 \]
\[ \lambda^{(k+1)} = \frac{\varepsilon^2}{2\sqrt{\lambda^{(k)}} + |\theta_0|} + \frac{\varepsilon^4}{4(2\sqrt{\lambda^{(k)}} + |\theta_0|)^2} \quad k = 0, 1, 2, \ldots. \]

We consider now the case \( d = 3 \).

**Theorem 5.** Let \( d = 3 \) and assume that \( c = 0 \). Then for all \( \varepsilon > 0 \) the essential spectrum of \( H_\varepsilon \) fills the positive real line and is only absolutely continuous. Moreover the Hamiltonian \( H_\varepsilon \) has a zero-energy resonance, i.e. the resolvent \( R_\varepsilon(z) \) has a singularity of order \( z^{-1/2} \) in \( z = 0 \).

**Remark 5.** For \( d = 3 \), as opposed to the cases \( c > 0 \) and \( c < 0 \), when \( c = 0 \) the singularity of the resolvent \( R_\varepsilon(z) \) in \( z = 0 \) does not move from the origin. The perturbation parameter \( \varepsilon \) affects only the character of the singularity, turning the embedded eigenvalue into a zero-energy resonance.

Similar to what happens in \( d = 1 \), when \( \theta_0 = 0 \), there are two solutions of \( D_\varepsilon(z) = 0 \), one of them is in a neighborhood of the origin. Both of them are in this case on the negative real axis of the second Riemann sheet of \( \sqrt{z} \).

We also note that for \( \theta_0 < 0 \) the Hamiltonian \( H_\varepsilon \) has an isolated eigenvalue in a neighborhood of order \( \varepsilon^2 \) of the point \(-4(\pi)^3/\theta_0^2\).

**Proof.** For \( c = 0 \), the equation \( D_\varepsilon(z) = 0 \) reads
\[ D_\varepsilon(z) = \left[ 1 - i \frac{\theta_0}{4\pi} \sqrt{z} \right] \left[ 1 + i \sqrt{z} - 1 \right] + \left( \frac{\varepsilon}{4\pi} \right)^2 \sqrt{z} - 1 = 0. \]

By a direct analysis one sees that the last equation has no solutions on the real positive axes, \( z = \lambda > 0 \). It is easy to verify that \( z = 0 \) is a solution of the last equation for all \( \varepsilon > 0 \). More precisely one can see that the function \( D_\varepsilon(z) \) can be expanded around \( z = 0 \) as \( D_\varepsilon(z) = i(\varepsilon^2/(4\pi)^3)\sqrt{z} + O(|z|) \). Correspondingly, for all \( \varepsilon > 0 \), the resolvent \( R_\varepsilon(z) \) has the following expansion around \( z = 0 \):
\[ R_\varepsilon(z) = \frac{A_\varepsilon}{\sqrt{z}} + B_\varepsilon + O(|z|), \quad (3.25) \]
where \( A_\varepsilon \) is the matrix-valued operator with integral kernel
\[ A_\varepsilon(x', x) = \begin{pmatrix} \frac{i}{4\pi} & 1 & 1 & \frac{i}{4\pi} e^{-|x'|} \\ \frac{i}{4\pi} |x| & |x'| & \varepsilon |x| & 4\pi |x| e^{-|x'|} \\ i e^{-|x|} & 1 & \frac{i}{4\pi} e^{-|x'|} \\ \varepsilon |x| & |x'| & \varepsilon |x| & 4\pi |x| e^{-|x'|} \end{pmatrix}. \]
According to standard results on the low-energy expansion of resolvents of Schrödinger operators in dimension 3 (see, e.g., [16] and [17]), the presence of a singularity of order $1/2$ is the signature of a zero-energy resonance. In the following we give the explicit form of the resonant state without making use of expansion (3.25). For this reason, we will not specify any suitable topology in order to give equality (3.25) a rigorous meaning.

Let us show that, for all $\varepsilon \neq 0$, there exists a distributional solution of the equation $H_\varepsilon \Phi = 0$. Consider the state $\Phi = \begin{pmatrix} \phi_0 \\ \phi_1 \end{pmatrix} = \begin{pmatrix} \frac{\varepsilon}{4\pi |r|} \\ \frac{e^{-|r|}}{|r|} \end{pmatrix},$ where $N$ is an inessential multiplicative constant which we set equal to 1. Let us verify that $\Phi$ is a zero-energy resonance for $H_\varepsilon$. The function $\phi_0 \in L^2_{\text{loc}}(\mathbb{R}^3)$ but $\phi_1 / \notin L^2_{\text{loc}}(\mathbb{R}^3)$; then $\Phi \in D(H_\varepsilon).$ Nevertheless using formulas given in definition 2, it is possible to compute the charges $q_0$ and $q_1$ and the regular parts $f_0$ and $f_1$ associated with $\Phi.$ A simple calculation gives $q_0 = -\varepsilon,$ $q_1 = 4\pi,$ $f_0 = 0$ and $f_1 = -1.$ Since the conditions $q_0 = \varepsilon f_1$ and $q_1 = -4\pi f_1$ are satisfied, one has that for all $\Psi \in C_0^\infty(\mathbb{R}^3) \oplus C_0^\infty(\mathbb{R}^3)$ the equation $(\Psi, H_\varepsilon \Phi) = 0$ is satisfied. □

3.3. Negative perturbations. $c > 0$

In this section we study the spectral structure of the Hamiltonian $H_\varepsilon$ in the vicinity of the origin, when the parameter $c$ is positive. It is clear from (3.1) and (3.2) that this choice corresponds to perturbations of the Hamiltonian $H_0$ for which the threshold eigenvalue is pushed toward negative energies by the perturbative term $c\varepsilon$ in $\theta_1.$

**Theorem 6.** Let $d = 1, 3$ and assume that $c > 0.$ Then there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0,$ the essential spectrum of $H_\varepsilon$ fills the positive real line and is only absolutely continuous; the Hamiltonian $H_\varepsilon$ has an isolated eigenvalue in $z = E_\varepsilon$ and

\[
E_\varepsilon = -c\varepsilon + O(\varepsilon^{3/2}) \quad d = 1
\]

\[
E_\varepsilon = -\frac{c\varepsilon}{2\pi} + O(\varepsilon^2) \quad d = 3.
\]

**Remark 6.** Note that when $c > 0$ the eigenvalue in the upper channel moves toward negative energies and is smoothly perturbed by the channel coupling (compare equations (3.26) and (3.27) with equations (3.1) and (3.2)). For $d = 1,$ when $\theta_0 = 0,$ the analytic continuation of $D_\varepsilon(z)$ from the semi-plane $\text{Im } z > 0$ through the positive real axis has zeroes close to the origin. As in the case $c = 0,$ see remark 4, in dimension 1, one of the roots of $D_\varepsilon(z) = 0$ has positive real part. The channel coupling produces this resonance as perturbation of the zero-energy resonance in the lower channel.

The order in $\varepsilon$ of the remainder in expansion (3.26) is exact for $\theta_0 = 0.$ It is possible to prove that for $\theta_0 \neq 0$ the remainder in equation (3.26) is indeed $O(\varepsilon^2).$

In both cases $d = 1$ and $d = 3$ the proof of theorem 6 comes straight from the analysis of the equation $D_\varepsilon(-\lambda) = 0, \lambda > 0,$ and we omit details.
4. Conclusions

We investigated the spectral properties of model Hamiltonians describing a quantum particle interacting with a localized spin via zero-range forces. Parameters were adjusted in such a way that the unperturbed Hamiltonian showed spectral singularities at the continuum threshold (chosen to be the zero-energy point).

All the unperturbed Hamiltonians we considered had a zero-energy eigenvalue. In addition, for particular values of parameters, a zero-energy resonance was also present. Our aim has been to characterize the effect of perturbations on the spectral structure around the threshold.

We defined perturbed Hamiltonians introducing a coupling between the two channels, associated with the two possible values of one component of the spin, together with potential-like perturbations of the upper-channel unperturbed Hamiltonian.

Direct, sometimes lengthy, calculations bring us to results which agree with the ones in [15] for positive perturbations. In fact the family of Hamiltonians we define have resolvents which are finite-rank perturbations of the unperturbed resolvent. Roughly, the great part of the considerable work done in [15], in a quite general setting, was to prove asymptotic expansions of the resolvent where only finite-rank operators appear. This property holds true by construction for all our Hamiltonians reducing significantly analytic difficulties and enhancing explicit computability.

The simplification mentioned above enables us to investigate also purely off-diagonal perturbations, where the perturbing term has no explicit bias to move singularities toward larger or lower values of energy. Moreover, we prove that all the expansions for the singularity coordinates are convergent and we give easy recurrent procedure to compute each term in the expansions.

Of special interest is the one-dimensional pure off-diagonal case ($c = 0$ and $\theta_0 = 0$ in our notation), when two singularities are present in the unperturbed Hamiltonian spectrum. Our results show a peculiar spectral structure of the corresponding Hamiltonian. As expected, there is no continuity of the spectral properties in the origin of the parameter space. In particular for $\theta_0 = 0$ and $c < 0$ we find an isolated eigenvalue at a distance from the origin of order $\varepsilon^2$, see equation (3.13) in theorem 2. The result is in agreement with the one found in ([20], table I) under quite different assumptions.

In dimension 3 the leading term of the displacement of the threshold eigenvalue is linear in the perturbation parameter $\varepsilon$, when $c \neq 0$. The result is in agreement with what is found in ([11], theorems 5 and 6, case III), see also ([20], theorem 2.3, case B).

The two-dimensional case shows an even richer structure and will be analyzed in a forthcoming paper.

In this paper we analyzed only multi-channel Hamiltonians. As we mentioned in the introduction, one-channel Schrödinger operators describing a quantum particle interacting with many point scattering centers show very rich spectral structures and can suitably approximate Hamiltonians with any kind of smooth potentials. Moreover their resolvents are finite-rank perturbations of the Laplacian resolvent for any (finite) number of scattering centers. In our opinion such kind of Hamiltonians are good candidates to examine spectral properties of a vast class of Schrödinger operators.

Acknowledgments

This work started when two authors, CC and RC, were employed at the Doppler Institute (Czech Republic). It was partially supported by the institute grant (LC06002). One of the
authors, CC, is also grateful to the Hausdorff Institute for Mathematics for the financial support.

References

[1] Albeverio S, Gesztesy F, Høegh-Krohn R and Holden H 2005 Solvable Models in Quantum Mechanics 2nd edn (Providence, RI: AMS)
[2] Cacciapuoti C, Carlone R and Figari R 2007 Spin-dependent point potentials in one and three dimensions J. Phys. A: Math. Theor. 40 249–61
[3] Cacciapuoti C, Carlone R and Figari R 2009 Resonances in models of spin-dependent point interactions J. Phys. A: Math. Theor. 42 035202
[4] Demkov Y N and Ostrovskii V N 1988 Zero-Range Potentials and their Applications in Atomic Physics (New York: Plenum)
[5] Dinu V, Jensen A and Nenciu G 2009 Nonexponential decay laws in perturbation theory of near threshold eigenvalues J. Math. Phys. 50 013516
[6] Duclos P, Exner P and Meller B 2001 Open quantum dots: resonances from perturbed symmetry and bound states in strong magnetic fields Rep. Math. Phys. 47 253–67
[7] Duclos P, Exner P and Štovíček P 1995 Curvature-induced resonances in a two-dimensional Dirichlet tube Ann. Inst. Henri Poincaré A 62 81–101
[8] Exner P 1991 A solvable model of two-channel scattering Helv. Phys. Acta 64 592–609
[9] Fassari S and Klaus M 1998 Coupling constant thresholds of perturbed periodic Hamiltonians J. Math. Phys. 39 4369–416
[10] Figari R, Holden H and Tata A 1988 A law of large numbers and a central limit theorem for the Schrödinger operator with zero-range potentials J. Stat. Phys. 51 205–14
[11] Gesztesy F and Holden H 1987 A unified approach to eigenvalues and resonances of Schrödinger operators using Fredholm determinants J. Math. Anal. Appl. 123 181–98
[12] Gesztesy F and Holden H 1987 A unified approach to eigenvalues and resonances of Schrödinger operators using Fredholm determinants J. Math. Anal. Appl. 123 181–97
[13] Gesztesy F and Holden H 1988 J. Math. Anal. Appl. 132 309 (addendum)
[14] Hunziker W 1990 Resonances, metastable states and exponential decay laws in perturbation theory Commun. Math. Phys. 132 177–88
[15] Jensen A and Nenciu G 2006 The Fermi golden rule and its form at thresholds in odd dimensions Commun. Math. Phys. 261 693–727
[16] Jensen A and Kato T 1979 Spectral properties of Schrödinger operators and time decay of the wave functions Duke Math. J. 46 583–611
[17] Jensen A and Nenciu G 2001 A unified approach to resolvent expansions at thresholds Rev. Math. Phys. 13 717–54
[18] Jensen A and Nenciu G 2001 A unified approach to resolvent expansions at thresholds Rev. Math. Phys. 13 717–54
[19] King C 1991 Exponential decay near resonance, without analyticity Lett. Math. Phys. 23 215–22
[20] Klaus M and Simon B 1980 Coupling constant thresholds in nonrelativistic quantum mechanics: I. Short-range two-body case Ann. Phys. 130 251–81
[21] Soffer A and Weinstein M I 1998 Time-dependent resonance theory Geom. Funct. Anal. 8 1086–128