Designs toward synchronization of optical limit cycles based on coupled silicon photonic crystal microcavities

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We propose a novel nanophotonic device to realize synchronized optical limit cycle oscillations based on coupled silicon (Si) photonic crystal (PhC) microcavities. A driven high-Q Si microcavity is known to exhibit limit cycle oscillation originating from carrier-induced and thermo-optic non-linearities. Here, coupled limit cycle oscillators are realized by using coherently coupled Si PhC microcavities. Simulating coupled-mode equations, we theoretically demonstrate mutual synchronization (entrainment) of two limit cycles induced by coherent coupling. Furthermore, we interpret the numerically simulated synchronization in the framework of phase description. Since our proposed design is perfectly compatible with the current silicon photonic fabrication process, the synchronization of optical limit cycle oscillations can easily be implemented to future silicon photonic circuits.

1. INTRODUCTION

Synchronization is a universally observed phenomenon in nature [1]. In fact, the observation of synchronization has a long history, which may go back to the 17th century: Huygens’s discovery of synchronization of two pendulum clocks. In the 19th century, Lord Rayleigh reported the unison of sounds in acoustical systems. The first modern experiment of synchronization was performed by Appleton and van der Pol during the early 20th century using electrical and radio engineering techniques [2,3]. On the other hand, for clear understanding of synchronization, we have to wait for the late 20th century, during when phase description of limit cycles was developed by Winfree and Kuramoto [4,5]. Limit cycle oscillation emerges from nonlinear dissipative system, and well models various rhythm and self-pulsing phenomena. Since limit cycles have stable orbits, they are different from harmonic oscillations in conservative systems. The main idea of phase description is to describe limit cycle dynamics solely with a (generalized) phase degree of freedom. The phase description was found to be a powerful tool for understanding not only single limit cycle dynamics but also synchronization phenomena. In fact, for intuitive understanding of mutual synchronization (entrainment) of coupled limit cycles, phase description provides a powerful tool called the phase coupling function. Furthermore, phase description is not limited to two oscillators, and is able to analyze an ensemble of coupled oscillators, which is called Kuramoto model. Nowadays, the phase analysis of synchronization is an indispensable tool to understand various synchronization phenomena in physics, chemistry, biology, and physiology. In biology, there are numerous examples of synchronization phenomena ranging from the circadian rhythm to Firefly Synchronization [1]. In physics, only recently, synchronization phenomena have

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started to be discussed in several systems. The most famous example may be the Josephson
junction array, which is known to be described by the Kuramoto model [6–8]. In photonic
systems, synchronization has been demonstrated with coupled lasers, microcavity polar-
tons, and optomechanical oscillators [9–16]. Furthermore, very recently, a frequency comb
was interpreted in terms of synchronization [17].

In this paper, we propose a novel nanophotonic system that realizes synchronization of op-
tical limit cycles, which is based on standard silicon (Si) photonic crystal (PhC) technologies.
In our previous paper [18], we have experimentally investigated the detailed properties of
stochastic limit cycle oscillation (self-pulsing) in a single driven high-Q Si PhC microcavity.
We extended the previous study to coupled driven Si PhC microcavities. First, numerically
simulating coupled-mode equations, we numerically demonstrate that an introduction of co-
herent field coupling between two cavities gives rise to synchronization (entrainment) of two
limit cycle oscillations. Interestingly, we found that synchronization phase (for example, in-
and anti-phase synchronizations) can be controlled by the phase difference between two laser
inputs. Second, we qualitatively interpreted the numerically demonstrated synchronization
in the framework of the phase description (the phase reduction theory). For this purpose, we
calculated a phase coupling function, which plays a central role in phase description [5, 19].
The obtained phase coupling function intuitively explains the origin of synchronization and
the synchronization phase. Finally, we demonstrated synchronization in a realistic coupled
cavity device, which has moderately different cavity resonance frequencies.

For studying synchronization of optical limit cycles, Si PhC cavities are advantageous in
the standpoint of measurements and their controllability. In particular, for measurements,
the realtime dynamics of light outputs are easily obtained with conventional optical setups.
Meanwhile, the input power and frequency of a driving laser are easily controlled. Furthere-
more, since the proposed coupled Si PhC cavity device does not require any active material,
and are based on the standard Si fabrication technique, its integration to other Si photonic
devices is easy. Thus, the demonstrated limit cycle synchronization is easily implemented
in future silicon photonic information processing and optical communications [20]. Ulti-
mately, an array of Si PhC cavities will works as a one-dimensional nearest-neighbor coupled
Kuramoto model.

2. LIMIT CYCLE IN A SINGLE HIGH-Q SI PHC CAVITY

First, we review limit cycle oscillation emerging from a single high-Q Si PhC cavity, which
has been investigated in our previous paper [18]. We consider a single Si L3-type cavity with
two waveguides schematically shown in Fig. 1(a), which is the same as in Ref [18]. The PhC
slab is the two-dimensional hexagonal lattice, and the cavity is introduced by removing three
air hole. We note that, in the sample used in Ref. [18], several air-holes around the cavity
region are carefully modulated to achieve larger $Q$ value than that of the conventional L3
cavity [21]. The cavity, which has a resonance frequency $\omega_c$, is driven by a laser input with
a frequency $\omega_L$ and power $P$ through the input waveguide. When input power exceeds a
critical power, a light output exhibits limit cycle oscillation (self-pulsing) originating from
nonlinear field, carrier, and thermal dynamics.

Now, we write down coupled-mode equations describing field, carrier, and thermal dynam-
ics in the nonlinear Si PhC cavity, which were proposed in Ref. [22, 23], and have been also
used in our paper [18]. An electric field $\alpha$, normalized carrier density $n$, and normalized
thermal effect $\theta$ follow coupled-mode equations:

\[
\dot{\alpha} = \kappa \{ i(-\delta/\kappa - \theta + n) - (1 + f n) \} \alpha + \sqrt{P} \tag{1}
\]

\[
\dot{n} = -\gamma n + \kappa \xi |\alpha|^4 \tag{2}
\]

\[
\dot{\theta} = -\Gamma \theta + \kappa \beta |\alpha|^2 + \kappa \eta |\alpha|^2 n, \tag{3}
\]

where the detuning $\delta$ is defined as $\delta = \omega_L - \omega_c$. The thermal effect $\theta$ is proportional to a temperature difference between the inside and outside regions of the cavity. It is important to note that both $n$ and $\theta$ are normalized variables so that the nonlinear coefficients before $n$ and $\theta$ in Eq. (1) are units. The nonlinear coefficients $f$, $\xi$, $\beta$, and $\eta$ represent free-carrier absorption (FCA), two-photon absorption (TPA), heating with linear photon absorption, and FCA-induced heating, respectively. The small Kerr nonlinearity is neglected in the coupled-mode equations. In the rest of the paper, we use $f = 0.0244$, $\xi = 8.2$, $\beta = 0.0296$, 

FIG. 1. (a) The schematic of a high-Q Si PhC microcavity with two waveguides. (b) Self-pulsing (SP) and bistable (BS) regions as functions of laser input power $P$ and detuning $\delta(= \omega_L - \omega_c)$. (c) The input power $P$ and detuning $\delta$ dependence of the limit cycle oscillation frequency $\omega$. In (b) and (c), the blue and red filled circles represent parameters used for the cavity C1 and C2 in Fig. 2, respectively. (d,e) Time evolutions of the light output intensity $|\alpha(t)|^2$ (left), carrier $n(t)$ (right), and thermal component $\theta(t)$ (right) for $P = 0.6\kappa^2$ (d) and $1.0\kappa^2$ (e). In (d) and (e), we used $\delta = -2\kappa$. 
and $\eta = 0.0036$, which are the same as in Ref \[24\]. Although precise determination of the values of the nonlinear coefficients is difficult, exact values are not necessary, and we the qualitative reproduction of the observed limit cycle oscillation is sufficient. For the lifetimes of the three variables, we set $1/2\kappa = 300$ ps ($Q \sim 3.5 \times 10^3$), $1/\gamma = 200$ ps, and $1/\Gamma = 100$ ns. As discussed in Ref. \[24, 25\], in the L3-type PhC cavity, due to the small cavity region, fast carrier diffusion makes the carrier lifetime comparable to the field lifetime. The detail of our model is described in the Supplemental Material in Ref. \[18\].

We briefly discuss the steady-state properties of coupled-mode equations \eqref{equations}. Here, $\alpha_{ss}$, $n_{ss}$, and $\theta_{ss}$ represent the steady state values of the field, carrier, and thermal effect, respectively. Setting $\dot{\alpha} = 0$, $\dot{n} = 0$, and $\dot{\theta} = 0$ in Eqs \eqref{equations}, an algebraic equation for $I_{ss} = |\alpha_{ss}|^2$ is obtained as

$$0 = f_{ss}(I) \equiv I \left[ \left( \frac{\delta}{\kappa} - \frac{\kappa}{\gamma I} \right) - \frac{\kappa^2}{\gamma^2 I^3} \eta I^3 + \frac{\kappa}{\gamma} I f(\eta I) \right]^2 + \left( 1 + \frac{\kappa}{\gamma} I f(\eta I) \right)^2 - \frac{P}{\kappa^2}. \tag{4} $$

Depending on input power $P$ and detuning $\delta$, the algebraic equation \eqref{equation} has one or two solutions for $I$. We numerically solve Eq. \eqref{equation}, and obtain $I_{ss}$. Using $I_{ss}$, $n_{ss}$ and $\theta_{ss}$ are respectively calculated as

$$n_{ss} = \frac{\kappa}{\gamma} I_{ss}^2 \quad \text{and} \quad \theta_{ss} = \frac{\kappa}{\gamma} I_{ss} + \frac{\kappa^2}{\gamma^2} I_{ss} \eta I_{ss}^3. \tag{5} $$

Using $n_{ss}$ and $\theta_{ss}$, the complex electric filed $\alpha_{ss}$ can be written as

$$\alpha_{ss} = \sqrt{P} \left( 1 + fn_{ss} \right) + \frac{i(-\delta/\kappa - \theta_{ss} + n_{ss})}{(-\delta/\kappa - \theta_{ss} + n_{ss})^2 + (1 + fn_{ss})^2} \tag{6} \text{.}$$

Second, to check the stabilities of the steady states, we perform the linear stability analysis for coupled-mode equations \eqref{equations}. For this purpose, decomposing the complex field $\alpha$ as $\alpha = x + iy$, we rewrite Eqs \eqref{equations} as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) = \begin{pmatrix} f_x(x) & f_y(x) & f_n(x) & f_\theta(x) \\ f_x(y) & f_y(y) & f_n(n) & f_\theta(\theta) \\ f_x(n) & f_y(n) & f_n(n) & f_\theta(\theta) \\ f_x(\theta) & f_y(\theta) & f_n(\theta) & f_\theta(\theta) \end{pmatrix} = \begin{pmatrix} -\kappa(1 + fn)x - \kappa(-\delta/\kappa - \theta + n)y + \sqrt{P} \\ -\kappa(1 + fn)y + \kappa(-\delta/\kappa - \theta + n)x \\ -\gamma n + \kappa\xi(x^2 + y^2)^2 \\ -\xi n + \kappa\beta(x^2 + y^2) + \kappa\eta(x^2 + y^2)n \end{pmatrix}, \tag{7}$$

where the vector $\mathbf{x}$ is defined as $\mathbf{x} = (x, y, n, \theta)$. Now, a $4 \times 4$ Jacobian matrix corresponding the dynamical system Eq. \eqref{system} is given by

$$J(\mathbf{x}) = \begin{pmatrix} \frac{\partial f_x}{\partial x} & \frac{\partial f_x}{\partial y} & \frac{\partial f_x}{\partial n} & \frac{\partial f_x}{\partial \theta} \\ \frac{\partial f_y}{\partial x} & \frac{\partial f_y}{\partial y} & \frac{\partial f_y}{\partial n} & \frac{\partial f_y}{\partial \theta} \\ \frac{\partial f_n}{\partial x} & \frac{\partial f_n}{\partial y} & \frac{\partial f_n}{\partial n} & \frac{\partial f_n}{\partial \theta} \\ \frac{\partial f_\theta}{\partial x} & \frac{\partial f_\theta}{\partial y} & \frac{\partial f_\theta}{\partial n} & \frac{\partial f_\theta}{\partial \theta} \end{pmatrix} = \kappa \begin{pmatrix} -fn - 1 & \delta/\kappa - n + \theta & -fx - y & y \\ -\delta/\kappa + n - \theta & -fn - 1 & -fy + x & -x \\ 4\xi(x^2 + y^2) & 4\xi y(x^2 + y^2) & -\gamma/\kappa & 0 \\ 2\beta x + 2\eta nx & 2\beta y + 2\eta ny & \eta(x^2 + y^2) & -\Gamma/\kappa \end{pmatrix}. \tag{8}$$

Now, a small fluctuation $\delta \mathbf{x}$ follows $\dot{\delta \mathbf{x}} \simeq J \delta \mathbf{x}$, where $\delta \mathbf{x} \equiv \mathbf{x} - \mathbf{x}_{ss}$ with the steady state values $\mathbf{x}_{ss} = (x_{ss}, y_{ss}, n_{ss}, \theta_{ss})$. We calculate the eigenvalues of $J(\mathbf{x})$ at the steady states $\mathbf{x} = \mathbf{x}_{ss}$ for various input power $P$ and detuning $\delta$. When the pair of the eigenvalues of the Jacobian $J(\mathbf{x})$ have positive real values, the steady state $\mathbf{x}_{ss}$ becomes unstable, which leads
to limit cycle oscillation (the Hopf bifurcation) \cite{24}. In Fig. 1(b), we show nontrivial regions as functions of input power $P$ and detuning $\delta$. In Fig. 1(b), the bistable and limit cycle (self-pulsing) region are indicated by “BS” and “SP”, respectively. In the SP+BS region, a one steady state is stable, while the other is unstable. In this paper, since we are interested in limit cycle oscillation, we focus solely on the SP region. We also note that the Jacobian matrix Eq. (5) is used again in Section 4. Additionally, in Fig. 1(c), we show the input power $P$ and detuning $\delta$ dependence of the limit cycle’s frequency $\omega$, which were obtained with numerical time evolutions. Figure 1(c) indicates that the limit cycle’s frequency decreases with an increase in pump power $P$.

Now, we directly simulate coupled-mode equations (1)-(3). The realtime evolutions of light output $I(t) = |\alpha(t)|^2$ (left), carrier $n(t)$ (right), and $\theta(t)$ (right) are shown in Fig. 1(d), where the detuning is $\delta = -2\kappa$, and laser input powers are $P = 0.6\kappa^2$ (d) and $1.0\kappa^2$ (e). In Fig. 1(d), which is for $P = 0.6\kappa^2$, all the variables reach steady-states when $t \simeq 1000$ ns, and there is no self-pulsing. Meanwhile, for $P = 1.0\kappa^2$ [see Fig. 1(e)], all the variable clearly exhibits temporal periodic oscillations (limit cycle oscillation) with a frequency of $\omega/2\pi = 11$ MHz. In fact, in Fig. 1(b), the values $\delta = -2\kappa$ and $P = 1.0\kappa^2$ are represented as a blue filled circle in the SP region. Meanwhile, for $P = 0.6\kappa^2$, which is outside the SP region. In the rest of this paper, we show only light outputs $I(t) = |\alpha(t)|^2$, because the light output is the only measurable valuable in experiments.

Finally, we comment on the origin of limit cycle oscillation in Si PhC microcavities. In a minimum model that exhibits limit cycle oscillation, we set $\eta = 0$ and $f = 0$ in Eqs. (1) and (3), which have only quantitative effects. Furthermore, the exponent of the term $\kappa|\alpha|^4$ in Eq. (2) is not essential, because limit cycle oscillation appears even if this term is replaced with $\kappa|\alpha|^2$. In fact, limit cycle oscillation requires only the following three elements. First, the signs of the nonlinear energy shifts are opposite for carrier and thermal components. Second, the carrier lifetime is comparable to or even shorter than the photon lifetime, while the thermal lifetime is much longer than the photon lifetime. Third, $\beta$ is much smaller than $\xi$, approximately $\beta/\xi \simeq \gamma_\theta/\gamma_n$, to make carrier- and thermal- induced energy-shifts comparable.

3. COUPLED LIMIT CYCLE DYNAMICS

In this section, we propose coupled two Si PhC cavities. The proposed device is sketched in Fig. 2, where the two cavities are labelled as $C1$ and $C2$. Since coupling is introduced through evanescent coupling, a coupling strength $g$ depends on the distance between the two cavities. To drive the two cavities, we separate a single laser source to two inputs using an on-chip Si wire waveguides, instead of using two laser sources. This process is very important to temporally fix the relative phase difference $\phi_L$ between the two laser inputs. Actually, we show that the relative phase difference $\phi_L$ has a crucial impact on synchronization. The design shown in Fig. 2 has two output waveguides, which are used to measure light outputs from $C1$ and $C2$.

Coupled-mode equations (1)-(3) for a single Si PhC cavity are easily extended to the
f values of the nonlinear coefficients thus is not directly related to a limit cycle’s phase, which is introduced in Section 4. The detuning is defined as $\delta e (12)$, the term $\delta_1$ from Eq. (9)-(12) and Eqs. (12)-(14) represent dynamics for $C1$ and $C2$ coupled two cavities as

$$\dot{\alpha}_1 = \kappa_1 \{i(-\delta_1/\kappa_1 - \theta_1 + n_1) - (1 + fn_1)\} \alpha_1 + g\alpha_2 + \sqrt{P_1}$$

(9)

$$\dot{n}_1 = -\gamma_1 n_1 + \kappa_1 \xi |\alpha_1|^4$$

(10)

$$\dot{\theta}_1 = -\Gamma_1 \theta_1 + \kappa_1 \beta |\alpha_1|^2 + \kappa_1 \eta |\alpha_1|^2 n_1.$$  

(11)

$$\dot{\alpha}_2 = \kappa_2 \{i(-\delta_2/\kappa_2 - \theta_2 + n_2) - (1 + fn_2)\} \alpha_2 + g\alpha_1 + \sqrt{P_2}e^{i\phi_L}$$

(12)

$$\dot{n}_2 = -\gamma_2 n_2 + \kappa_2 \xi |\alpha_2|^4$$

(13)

$$\dot{\theta}_2 = -\Gamma_2 \theta_2 + \kappa_2 \beta |\alpha_2|^2 + \kappa_2 \eta |\alpha_2|^2 n_2.$$  

(14)

Equations (9)-(12) and Eqs. (12)-(14) represent dynamics for $C1$ and $C2$, respectively. The coherent field coupling between $C1$ and $C2$ is represented by the coupling strength $g$. In Eq. (12), the term $e^{i\phi_L}$ represents a phase factor originating from the phase difference between the two laser inputs. It is worth noting that $\phi_L$ is a phase associated with the field, and thus is not directly related to a limit cycle’s phase, which is introduced in Section 4. The values of the nonlinear coefficients $f$, $\xi$, $\beta$, and $\eta$ are the same as those in Fig. 1. The cavity detuning is defined as $\delta_{1,2} \equiv \omega_L - \omega_{1,2}$, where $\omega_{1,2}$ is the resonance frequency of the cavity.

To observe synchronization, there must be a small frequency difference in two limit cycles. However, in our proposal, the two cavities are designed to be identical, because natural disorders or unavoidable fabrication errors will introduce an intrinsic parameter and resonance frequency difference to the two cavities. In this section, for demonstration of synchronization, we consider a rather idealistic device. Namely, only the cavity resonance frequencies are slightly different: $\delta_1 = \omega_L - \omega_1 = -2\kappa_1$, while $\delta_2 = \omega_L - \omega_2 = -1.5\kappa_1$. The other parameters are the same for cavity $C1$ and $C2$: $1/2\kappa_1 = 1/2\kappa_2 = 300$ ps, $1/\gamma_1 = 1/\gamma_2 = 200$ ps, and $1/\Gamma_1 = 1/\Gamma_2 = 100$ ns. Additionally, we derive the two cavities with the same input powers, $P_1 = P_2 = 1\kappa_1$. In Section 5, we consider a more realistic device, where the resonance frequencies of the two cavities are moderately different.

Figure 3(a) shows time evolutions of the light output $|\alpha_{1,2}(t)|^2$ without $g = 0$ (top) and with coupling $g = 0.02\kappa_1$ (middle and bottom), which are the central results of this paper.
FIG. 3. (a) Simulated time evolutions of the light output intensity \(|\alpha_{1,2}(t)|^2\) without (top) and with coupling \(g = 0.02\kappa_{1}\) (middle and bottom). The figures in the middle and bottom row are for \(\phi_{L} = 0\) and \(\pi\), respectively. (b) The average frequencies \(\bar{\omega}_{1,2}\) of the two limit cycle oscillations \(|\alpha_{1,2}(t)|^2\) for \(\phi_{L} = 0\) (upper) and \(\pi\) (lower) as a function of the coupling strength \(g\). The critical coupling strengths of synchronization are \(g_{c} = 0.0115\) and \(0.011\) for \(\phi_{L} = 0\) and \(\pi\), respectively. In this figure, we used \(1/2\kappa_{1} = 300\) ps, \(1/\gamma_{1} = 200\) ps, \(1/\Gamma_{1} = 1/\Gamma_{2} = 100\) ns, \(P_{1} = P_{2} = 1\kappa_{1}\), \(\delta_{1} = \omega_{L} - \omega_{1} = -2\kappa_{1}\), and \(\delta_{2} = \omega_{L} - \omega_{2} = -1.5\kappa_{1}\). In Fig. 3(a), the middle and bottom figures are for the phase difference \(\phi_{L} = 0\) and \(\phi_{L} = \pi\), respectively. We note that this coupling strength \((g = 0.02\kappa_{1})\) is much smaller than the cavity decay rates \((g \ll \kappa_{1,2})\), the coupled two cavities are in the weak-coupling regime. Without coherent coupling \((g = 0)\) [See on the top in Fig. 3(a)], the two limit cycle oscillations are completely decoupled, and thus have their own frequencies: \(\omega_{1}/2\pi = 11\) and \(\omega_{2}/2\pi = 11.75\) MHz for \(C1\) and \(C2\), respectively. On the other hand, when small coupling \(g = 0.02\kappa_{1}\) is introduced, time evolution dramatically changes as shown on the middle and bottom in 3(a), where the two limit cycle oscillations are perfectly synchronized (entrained) with each other. Furthermore, on the middle and bottom in 3(a), we notice that synchronization is in-phase for \(\phi_{L} = 0\) (middle), while “anti-phase” for \(\phi_{L} = \pi\) (bottom). Interestingly, synchronization occurs just after the buildups of limit cycle oscillations, and thus the synchronization time is comparable to the period of the limit cycle oscillations. Let us compare the coupling strength \(g\) with the typical oscillation frequency of limit cycles. While the oscillation frequency of limit cycles are typically about \(\sim 10\) MHz in Fig. 3(a), the strength of coherent coupling \(g = 0.02\kappa_{1}\) (\(1/g = 1/0.02\kappa_{1} = 50\) ns) corresponds to 5 MHz. Thus, the coupling strength is comparable to the limit cycle’s oscillation frequency, which could be the reason why synchronization time is as short as one period of the limit cycle.

In addition to the time evolutions, in Fig. 3(b), we show the mean frequencies \(\bar{\omega}_{1,2}\) of the two limit cycles as a function of the coupling strength \(g\). In Fig. 3(b), the upper and lower figures are for \(\phi_{L} = 0\) and \(\pi\), respectively. In Fig. 3(b), we used the mean frequency of
limit cycle oscillation $\bar{\omega}_{1,2}$, because oscillations are not perfectly periodic when the coupling strength is smaller than a critical value of synchronization. Figure 3(b) clearly shows that as the coupling strength $g$ increases, the mean frequencies $\bar{\omega}_1$ and $\bar{\omega}_2$ approach each other, and merge when $g$ reaches the critical value $g_c = 0.0115 \kappa_1$ and $0.011 \kappa_1$ for $\phi_L = 0$ and $\pi$, respectively. Furthermore, Both for $\phi_L = 0$ and $\pi$, the frequencies of the synchronized limit cycles are the same: $\omega_1 = \omega_2$, which is called the 1:1 synchronization. Finally, we comment on the fact that the value of $g_c$ is not the same for $\phi_L = 0$ and $\pi$. We found that when the frequency difference of uncoupled limit cycles becomes smaller, synchronization occurs with the same critical values $g_c$ both for $\phi_L = 0$ and $\pi$, respectively.

In Appendix, we show simulation for an intermediate phase difference $\phi_L = 0.5 \pi$, which does not exhibit synchronization with the coupling strength $g = 0.02 \kappa_1$.

4. PHASE DESCRIPTION

In Section 3, we have demonstrated synchronization of limit cycle oscillations in coupled two cavities by directly simulating time evolutions. For the qualitative understanding of synchronization, the phase reduction theory provides a powerful tool called a phase coupling function [5, 19]. In particular, the phase coupling function is able to explain why the in- or anti-phase synchronization occurs depending on the phase difference of the two laser inputs. In this section, after a brief review of the phase reduction theory, we numerically derive the phase equation of motion and phase coupling function for coupled-mode equations (9)-(14).

4.1. General phase description for a single limit cycle

The key idea of the phase reduction theory is to describe limit cycle dynamics solely with a generalized phase degree of freedom. First, we consider phase description for general single limit cycle dynamics, and introduce a scalar “phase field” $\phi(x)$. Let us consider a general dynamical system that exhibits limit cycle oscillation:

$$\dot{x} = f(x),$$

where $f(x)$ is a general function. In phase description, the phase field $\phi(x)$ is defined in such a way that

$$\dot{\phi}(x) = \nabla_x \phi(x) \cdot f(x) = \omega,$$

where $\omega$ is the frequency of the limit cycle oscillation. If there is no perturbation, dynamics converge on the orbit of the limit cycle, and follow the very simple equation of motion: $\dot{\phi} = \omega$, where $\phi$ without any argument represents the phase on the limit cycle orbit. For simplicity, we denote the orbit of the limit cycle as $\chi(\phi)$. When the dynamical system [Eq. (15)] is perturbed by a force $p(x)$ as $\dot{x} = f(x) + p(x, t)$, equation of motion (16) is modified as

$$\dot{\phi}(x) = \omega + \nabla_x \phi(x) \cdot p(x, t).$$

(17)
If the perturbation $p(x)$ is sufficiently weak, $x$ is approximated as a point on the limit cycle’s orbit, $x \simeq \chi(\phi)$. With this approximation, Eq. (16) is further simplified as

$$\dot{\phi} = \omega + Z(\phi) \cdot P(\phi, t), \quad (18)$$

where $Z(\phi) \equiv \nabla_{\psi=\chi(\phi)} \phi(x)$ is called “sensitivity” [4]. Here the capital $P(\phi, t)$ is defined as $P(\phi, t) \equiv p(\chi(\phi), t)$. Equation (18) is called the phase equation of motion, and plays a central role in the phase reduction theory. Actually, with Eq. (18), the perturbed limit cycle dynamics are described solely by the phase degree of freedom $\phi$.

Therefore, our next step is to numerically determine the sensitivity $Z(\phi)$ for our dynamical system described by coupled-mode equations (1)-(3). To numerically obtain $Z(\phi)$, fortunately, we are able to use the “adjoint method” [27, 28], which employs the fact that $Z(\phi)$ is already given in Eq. (8). Since Eq. (19) is unstable for forward time integration due to the minus sign before $\chi^\top$, we need to perform backward time integration as $dZ(-\omega t')/dt' = J^\top(\chi(-\omega t'))Z(-\omega t')$ with $t' = -t$. Additionally, the numerically obtained $Z(\phi)$ was normalized as $Z(\phi) \cdot f(\chi(\phi)) = \omega$, which is equivalent to Eq. (15). Figure 4 shows numerically obtained $Z_i(\phi)$, where the index $i$ represents $x$, $y$, $n$, and $\theta$. Importantly, parameters used for calculating $Z(\phi)$ are the same as those in Fig. 1(e).

4.2. Phase coupling function

Here, we extend phase equation of motion (18) to coupled two limit cycles. Let us consider weakly coupled dynamical systems, where both dynamical systems exhibit limit cycle oscillations:

$$\begin{align*}
\dot{x}_1 &= f(x_1) + \delta f_1(x_1) + g_{12}(x_1, x_2) \\
\dot{x}_2 &= f(x_2) + \delta f_2(x_2) + g_{21}(x_2, x_1),
\end{align*} \quad (20)$$

where $\delta f_{1,2}(x_{1,2})$ is a deviation from the “standard” oscillator $f(x)$ [Eq. (15)], while $g_{12}(x_1, x_2)$ and $g_{21}(x_2, x_1)$ represent coupling between the two systems. Rewriting with the phase coordinate $\phi_{1,2}$ of the standard oscillator and taking the terms $\delta f_1(x_1)$, $g_{12}(x_1, x_2)$, and $g_{21}(x_2, x_1)$ as perturbations, the phase equations of motion corresponding to Eqs (20) and (20) are given by

$$\begin{align*}
\dot{\phi}_1 &= \omega + Z(\phi_1) \cdot \delta F_1(\phi_1) + Z(\phi_1) \cdot G_{12}(\phi_1, \phi_2) \\
\dot{\phi}_2 &= \omega + Z(\phi_2) \cdot \delta F_2(\phi_2) + Z(\phi_2) \cdot G_{21}(\phi_2, \phi_1),
\end{align*} \quad (22, 23)$$

where the capital symbols mean that they are the functions of the standard oscillator’s phase $\phi_{1,2}$, which is given by $\dot{x}_{1,2} = f(x_{1,2})$. For further simplification of Eqs. (22) and (23), we transform $\phi_{1,2}$ to the rotating frame of the standard oscillator as $\psi_{1,2} \equiv \phi_{1,2} - \omega t$, where $\omega$ is the standard oscillator’s oscillation frequency. Additionally, we perform approximation for the coupled phase equations of motion by averaging over the one period of the standard
oscillator. With these procedures, Eqs. (22) and (23) become

\[
\dot{\psi}_1 = \omega + \delta \omega_1 + \Gamma_{12}(\psi_1 - \psi_2) \\
\dot{\psi}_2 = \omega + \delta \omega_2 + \Gamma_{21}(\psi_2 - \psi_1).
\] (24)

Here, the frequency shift $\delta \omega_{1,2}$ and the phase coupling function $\Gamma_{ij}(\psi)$ are given by

\[
\delta \omega_{1,2} = \frac{1}{2\pi} \int_0^{2\pi} d\theta Z(\theta) \cdot \delta F_{1,2}(\theta) \\
\Gamma_{ij}(\psi) = \frac{1}{2\pi} \int_0^{2\pi} d\eta Z(\eta + \psi) \cdot G_{ij}(\eta + \psi, \eta),
\] (26)

respectively. Finally, the phase difference between the two oscillators, $\psi = \psi_2 - \psi_1$ follows the following simple equation:

\[
\dot{\psi} = \Delta \omega + \Gamma_a(\psi),
\] (28)

where $\Delta \omega \equiv \delta \omega_2 - \delta \omega_1$ and $\Gamma_a(\psi) \equiv \Gamma_{21}(\psi) - \Gamma_{12}(-\psi)$. In fact, $\Gamma_a(\psi)$ is the anti-symmetric part of the phase coupling function. A synchronization phase $\psi_{\text{sync}}$ is required to satisfy $\Gamma_a'(\psi_{\text{sync}}) = 0$ and $\Gamma_a'(\psi_{\text{sync}}) < 0$, where the prime represents the derivative. For example, if $\Gamma_a(0) = 0$ and $\Gamma_a'(0) < 0$, the phase difference $\psi$ is locked to $\psi = 0$ by negative feedback, which is in-phase synchronization. Therefore, the shapes of phase coupling function allow intuitive interpretation of a synchronization phase.

4.3. Phase coupling function for limit cycles in coupled Si PhC cavities

Now, we attempt to numerically calculate the phase coupling function for our dynamical system described by Eqs. (9)-(14). For this purpose, it is convenient to perform phase rotation for the variable $\alpha_2$ in Eq. (22) as $\alpha_2 e^{-i\phi_L} \rightarrow \alpha_2$. With this transformation, the phase difference of the two lasers $\phi_L$ appears as a coupling phase: $g_\alpha \alpha_2 \rightarrow g e^{i\phi_L} \alpha_2$ in Eq. (9) and $g_\alpha \alpha_1 \rightarrow g e^{-i\phi_L} \alpha_1$ in Eq. (12). The purpose of this phase rotation is to define a common standard oscillator and its common phase $\phi$ for the two limit cycles. Now, the coupling function $g_{12}(x_1, x_2)$ and $g_{21}(x_2, x_1)$ are given by

\[
g_{12}(x_1, x_2) = g_{12}(x_2) = g \begin{pmatrix} x_2 \sin \phi_L + y_2 \cos \phi_L \\
-x_2 \cos \phi_L + y_2 \sin \phi_L \\
0 \\
0 \end{pmatrix}
\] (29)

and

\[
g_{21}(x_2, x_1) = g_{21}(x_1) = g \begin{pmatrix} x_1 \sin(-\phi_L) + y_1 \cos(-\phi_L) \\
-x_1 \cos(-\phi_L) + y_1 \sin(-\phi_L) \\
0 \\
0 \end{pmatrix},
\] (30)
FIG. 4. (a) Numerically obtained sensitivity $Z(\phi)$ for coupled-mode equations (1)-(3). (b) The anti-symmetric parts of calculated phase coupling functions $\Gamma_a(\psi) \equiv \Gamma(\psi) - \Gamma(-\psi)$ as a function of the phase $\psi \equiv \psi_2 - \psi_1$ for $\phi_L = 0$ (left) and $\pi$ (right). The arrows indicate phase locking points. In this figure, to calculate $Z(\phi)$ and $\Gamma_a(\psi)$, we used the same values of the parameters as those in Fig. 1(e).

respectively. Additionally, for simplicity, we use the limit cycle in the cavity $C_1$ as a standard oscillator, and thus we put $\delta \omega_1 = 0$. Since the parameters for the standard oscillator are the same as those in Fig. 1(e), we are able to use the sensitivity $Z$ shown in Fig. 4(a). Representing the coupling function $g_{ij}(x_j)$ with the standard oscillator’s phase coordinate as $G_{ij}(\phi_j)$, we numerically integrate Eq. (27). Figure 4(b) and (c) show the anti-symmetric parts of the phase coupling function $\Gamma_a(\psi) \equiv \Gamma_{21}(\psi) - \Gamma_{12}(-\psi)$ for $\phi_L = 0$ and $\pi$, respectively. Here, the power of phase description is that the complex limit cycle dynamics represented by coupled-mode equations (9)-(14) are reduced to simple phase equation of motion (28). In fact, the origin of synchronization is understood only in this phase coordinate. Figure 4(b) and (c) clearly indicate that when $\phi_L = 0$ (b), $\Gamma_a(0) = 0$ and $\Gamma'_a(0) < 0$ hold, and thus in-phase locking occurs. Meanwhile when $\phi_L = \pi$ (c), $\Gamma_a(\pi) = 0$ and $\Gamma'_a(\pi) < 0$ hold, and thus anti-phase locking occurs. Here, Fig. 4(c) is the reflected image of Fig. 4(b) about the $x$-axis, which is intuitive because the signs of Eqs (29) and (30) are opposite for $\phi_L = 0$ and $\pi$. In our case, since the phase coupling function for $\phi_L = 0$ [see Fig. 4(b)] resemble the sine function, in- and anti-phase synchronizations will occur for $\phi_L = 0$ and $\pi$, respectively. The surprise is that although the two cavities are in the weak coupling regime ($g \ll \kappa_1, \kappa_2$), the coupling phase $\phi_L$ in Eqs (29) and (30) strongly modifies synchronization behaviors. In fact, since coherent coupling between fields has a (relative) phase degree of freedom, in coupled cavity system, it is always important to take the coupling phase into account.

The parameters used for calculating phase coupling functions in Fig. 4(b) and (c) are again the same as those in Fig. 1(e). We stress that both the sensitivity and the phase coupling functions shown in Fig. 4(b) will be completely different when we change the values of the parameters. Thus, anti-phase synchronization for $\phi_L = \pi$ is not a general result, which depends on models and parameters. Meanwhile, we found that in-phase synchronization for $\phi_L = 0$ seems to be general.

Finally, we discuss why the linear coherent coupling $g_{ij}(x_j)$ gives rise to the nonlinear phase coupling function $\Gamma_{1,j}(\phi_j)$ shown in Fig. 4. The mathematical answer is the transformation of the coordinate from the Cartesian coordinates $x$ to the phase coordinate of the limit cycle $\phi$. Namely, on the phase coordinate, the linear coupling $g_{ij}(x_j)$ appears as a nonlinear function $G_{ij}(\phi_j)$. Since limit cycle oscillation itself originates in a nonlinear dissipative system, the transformation from the Cartesian to the phase coordinate is also nonlinear. We are also able to interpret our synchronization phenomenon in analogous to injection locking [29] or
mutual injection locking \cite{30} in laser physics. In injection locking, coupling between slave and master lasers is usually provided by partially transmitting mirrors, which are definitely linear coupling. Therefore, although coupling itself is linear, synchronization occurs with the modulation of the slave laser’s field by the master laser. Similarly to injection locking, in our system, the coherent coupling $g$ allows the oscillating light in the cavity $C_1$ to modulate the light in the cavity $C_2$. Thus, synchronization is interpreted as a response of limit cycle in the cavity $C_2$ ($C_1$) to the modulation from $C_1$ ($C_2$).

5. SYNCHRONIZATION OF TWO MODERATELY DIFFERENT LIMIT CYCLES

Until now, we have considered synchronization in rather ideal systems, where the two limit cycles are almost identical, and only their cavity resonance frequencies are slightly different. In fact, it is still questionable whether or not realistic Si PhC cavity devices are able to exhibit synchronization of limit cycle oscillations. Even with the state-of-arts fabrication technology, fabrication errors or natural disorders introduce, for example, unavoidable resonance frequency difference to cavities. Therefore, in this section, we consider a more realistic device, where two cavities have a moderate resonance frequency difference.

First, we set photon lifetimes for the two cavities as $1/2\kappa_1 = 1/2\kappa_2 = 100$ ps, which correspond to $Q \sim 1.0 \times 10^5$. Importantly, comparing with the simulations in previous sections, here, we slightly decreased the photon lifetime from 300 to 100 ps. The reason why we decreased the photon lifetime is because it is technically easier to reduce the difference of cavities’ resonance frequencies for a shorter photon lifetime (a lower $Q$ value). We use the same values as in previous sections for the nonlinear coefficients: $f = 0.0244$, $\xi = 8.2$, $\beta = 0.0296$, and $\eta = 0.0036$. For these parameters, the SP and BS regions are represented by the diagram shown in Fig. 5(a). Additionally, we show the detuning and input power dependence of the limit cycles’ frequency $\omega$ in Fig. 5(b), which is more complicated than Fig. 1(c). In fact, the oscillation frequency does not monotonically decrease with an increase in input power, because there is an increase of oscillation frequency at $P \simeq 2.6\kappa_1^2$, and this jump might be related to the onset of fast photon-carrier oscillation \cite{31}. Second, we introduce a moderate difference to the cavity resonance frequencies as $\omega_2 - \omega_1 = -7\kappa_1$. Finally, we also set the value of the coupling strength as $g = 0.4\kappa_1$, which is much stronger than in Section 3.

We show the spectra of the two cavities in Fig. 5(c), which was obtained by sweeping the laser frequency $\omega_L$ from $\omega_c - 20\kappa_1$ to $\omega_c + 20\kappa_1$, and plotting the steady state outputs $|\alpha_1|^2$ and $|\alpha_2|^2$ with very low input power $P_1 = P_2 = 0.001\kappa_1^2$ not to induce any nonlinearity. In Fig. 5(c), the dashed curves are the spectra without coupling $g = 0$, while the solid blue and red curves are the spectra with coupling $g = 0.4\kappa_1$. Comparing the spectra with and without coupling, we notice that the large coupling strength ($g = 0.4\kappa$) does not strongly modify the spectral shapes of the system, and no normal mode splitting appears. Therefore, the system is still in the week-coupling regime, and we are able to consider coupling as perturbation.

We also note that this value of the resonance frequencies difference $\omega_2 - \omega_1 = -7\kappa_1$ is experimentally available with the state-of-arts fabrication technology \cite{32}.

To drive the cavities, we set the detuning values between the cavity resonance and laser frequency as $\delta_1 = \omega_1 - \omega_L = -1.0\kappa_1$, which leads to $\delta_2 = \omega_2 - \omega_L = -8.0\kappa_1$. The both cavities are driven by inputs with the same powers $P_1 = P_2 = 9.0\kappa_1^2$. These parameters are represented by the blue (C1) and red (C2) filled circles in the diagram in Fig. 5(a) and (b), which indicate that both cavities exhibit self-pulsing (limit cycle oscillation).
FIG. 5. Synchronization in realistic cavities with photon lifetime $1/2\kappa_1 = 1/2\kappa_2 = 100$ ps and moderate resonance frequency difference $\omega_2 - \omega_1 = -7\kappa_1$. (a) Self-pulsing (SP) and bistable (BS) regions as functions of laser input power $P$ and detuning $\delta = \omega_L - \omega_i$. (b) The input power $P$ and detuning $\delta$ dependence of the limit cycle’s frequency $\omega$. In (a) and (b), the blue and red filled circles represent parameters used for the cavity $C_1$ and $C_2$, respectively. (c) The transmission spectra of the coupled cavities obtained as steady state light output intensities $|\alpha_{1,2}(\omega_L)|^2$ as a function of the laser input frequency $\omega_L$. For obtaining the spectra, the laser input power was fixed as $P_1 = P_2 = 0.001\kappa_1^2$. (d) The simulated time evolutions of the light output intensity $|\alpha_{1,2}(t)|^2$ without $g = 0$ (upper) and with coupling $g = 0.4\kappa_1$ (lower). For simulating time evolutions, we used $P_1 = P_2 = 9\kappa_1^2$, $\delta_1 \equiv \omega_L - \omega_1 = -1.0\kappa_1$, and $\delta_2 \equiv \omega_L - \omega_2 = -8.0\kappa_1$.

Now, in the same way as in Fig. 3(a), we show time evolution of the light output intensity $|\alpha_{1,2}(t)|^2$ in Fig. 3(d) with (upper) and without coupling (lower). First, we discuss time evolution without coupling $g = 0$ shown on the upper row in Fig. 3(d). Without coupling, both cavities exhibit limit cycle oscillations with their own frequencies: $\omega_1/2\pi = 5.8$ and $\omega_2/2\pi = 9.2$ MHz for the cavity $C_1$ and $C_2$, respectively. Second, we show time evolution with coupling $g = 0.4\kappa_1$ on the lower row in Fig. 3(d), which clearly shows synchronization of the two oscillations. However, the profile of synchronized oscillations is very different from that in Fig. 3(a). For instance, the profile of $|\alpha_2(t)|^2$ is strongly modified by the introduction of the large coupling.

In the synchronized state [see the lower row in Fig. 3(d)], the oscillation period of the limit cycle oscillation for $|\alpha_1(t)|^2$ is easily identified as $T_1 = 210$ ns, which corresponds to $\omega_1/2\pi = 4.77$ MHz. On the other hand, the identification of the oscillation period for the limit cycle oscillation $|\alpha_2(t)|^2$ is not trivial. In a strict sense, the oscillation period of the limit cycle oscillation $|\alpha_2(t)|^2$ is $T_2 = 210$ ns [see the green bidirectional arrow in Fig.
which is the same as \( T_1 \), and corresponds to \( \omega_2/2\pi = 4.77 \text{ MHz} \). Meanwhile, if we take into account the sub peaks between the highest peaks, the oscillation period for the cavity \( |\alpha_2(t)|^2 \) is \( T_2 = 105 \text{ ns} \) [see the black bidirectional arrows in Fig. 5(c)], which leads to \( \omega_2/2\pi = 9.54 \text{ MHz} \), which is a twice of \( \omega_1/2\pi \). In the first interpretation, the synchronization is the 1:1 synchronization \( (\omega_1 = \omega_2 = 4.77) \). Meanwhile, in the second interpretation, the synchronization considered to be the 1:2 synchronization \( (\omega_1 : \omega_2 = 1 : 2) \). Here, we employ the second interpretation, because considering the original oscillation frequencies, the 1:2 synchronization \( (\omega_1/2\pi = 5.8 \rightarrow 4.77, \text{ and } \omega_1/2\pi = 9.2 \rightarrow 9.54 \text{ MHz}) \) might be easier than the 1:1 synchronization \( (\omega_1/2\pi = 5.8 \rightarrow 4.77, \text{ and } \omega_1/2\pi = 9.2 \rightarrow 4.77 \text{ MHz}) \). In fact, according to the general synchronization theory [1], when the frequency difference of two uncoupled limit cycles becomes large, the \( m : n \) synchronization generally occurs instead of the 1:1 synchronization. In the \( m : n \) synchronization, the oscillation frequencies of synchronized two limit cycles are locked as \( \omega_1 : \omega_2 = m : n \).

Finally, we comment on the phase difference of the two laser inputs \( \phi_L \). In this section, we have shown the simulation only for \( \phi_L = 0 \), because we found that synchronization occurs only for near-zero phase \( \phi_L \approx 0 \). Actually, when \( \phi_L \) is not close to zero, synchronization does not occur even with \( g > \kappa \). This result is related to the large frequency difference of the two uncoupled limit cycles \( (\omega_1 = 5.8 \text{ and } \omega_2/2\pi = 9.2 \text{ MHz}) \). In fact, if the parameters for the cavity \( C1 \) and \( C2 \) are closer, the frequency difference between uncoupled two limit cycles is smaller, and synchronization occurs both for \( \phi_L = 0 \) and \( \pi \). Therefore, to realize synchronization in a realistic coupled cavities with moderate frequency difference, it is important to adjust the phase difference of laser inputs to near-zero \( (\phi_L \neq 0) \), which will be achieved by adjusting optical path lengths, for example, with on-chip Si wire waveguides.

### 6. DISCUSSION AND FUTURE PERSPECTIVE

First, we argue that the proposed scheme of synchronization is not limited to Si PhC cavities, but is applicable to wide range of limit cycle oscillation in nanophotonic systems such as nanolasers [33, 34], semiconductor microcavities [35, 36], and microring resonators [22, 23, 37]. Actually, the coherent field coupling is easily implemented in these nanophotonic devices, which will lead to synchronization of optical limit cycles. In particular, since coupled-mode equations (1)-(3) were originally proposed for modelling optical limit cycles in Si microring resonators [22, 23], our synchronization scheme is easily applicable to them. In terms of the tunability of various physical parameters such as resonance frequencies, Si microring resonators may be advantageous over the PhC structures.

Second, we discuss a future perspective of limit cycle synchronization in Si PhC cavities. One can naturally imagine the extension of the coupled two Si PhC cavities to an array of coupled cavities as illustrated in Fig. 6. In principle, the coupled PhC cavity array illustrated in Fig. 6 is able to behave as a one-dimensional (1D) nearest-neighbor coupling (local) Kuramoto oscillator. The 1D local Kuramoto model has been theoretically investigated with numerical simulations [23] and renormalization group analysis [38], which have predicted various nontrivial collective phenomena including a synchronization state, phase slip at the onset of desynchronization, and coupling-induced chaos. In the standpoint of device application, the predicted chaotic state in the 1D local Kuramoto model may be employed for photonic reservoir computing [39].
7. CONCLUSION

In conclusion, we have theoretically demonstrated synchronization of optical limit cycles based on driven coupled silicon (Si) photonic crystal (PhC) cavities, where limit cycle oscillation emerges from carrier- and thermal-induced nonlinearities. An introduction of coherent field coupling between two cavities synchronizes (entrains) two limit cycle oscillations. First, we quantitatively demonstrated synchronization by directly simulating the time evolutions of coupled-mode equations. We found that synchronization phase depends on the phase difference of two laser inputs. Second, the numerically simulated synchronization was qualitatively interpreted in the framework of phase description. In particular, we calculated phase coupling functions, which intuitively explain why the synchronization phase depends on the phase difference of the two laser inputs. Finally, we discussed synchronization in a realistic coupled cavity device, where two cavities’ resonance frequencies are moderately different. Since our proposed design is perfectly compatible with the conventional Si fabrication processes, synchronization of optical limit cycles is easily implemented in future silicon photonic devices, and is able to be extended to an array of coupled cavities.

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APPENDIX: SIMULATIONS FOR $\phi_L = 0.5\pi$

In this appendix, we show simulations when the phase difference of laser inputs is $\phi_L = 0.5\pi$ in Fig. 6. In Section 3 in the main text, we have simulated only for $\phi_L = 0$ and $\pi$, which exhibited in- and anti-phase synchronization, respectively. Thus, it is of natural interest to discuss an intermediate case $\phi_L = 0.5\pi$. Figure 7(a) shows the time evolution of light output $|\alpha_1(t)|^2$ for $\phi_L = 0.5\pi$ with coherent coupling $g = 0.02\kappa_1$. In fact, in Fig. 7(a), all the parameters except for $\phi_L$ are the same as in Fig. 3(a). Surprisingly, even though the coupling strength is the same as in Fig. 3(a), no synchronization is observed in 7(a). We found that this result is explained in terms of a phase coupling function. Similarly to 4(b,c), we show the anti-symmetric part of the phase coupling function in Fig. 7(b). Interestingly, $\Gamma_a(\psi)$ for $\phi_L = 0.5\pi$ never crosses the zero axis, and thus there is no phase locking point. This
explains why phase synchronization does not occur for $\phi_L = 0.5\pi$ with the small coupling strength ($g = 0.02\kappa_1$).

Even for $\phi_L = 0.5\pi$, if the coupling strength is further increased, for example, to $g \simeq 0.2\kappa_1$, synchronization occurs (not shown). However, this synchronization with a large coupling strength may not be interpreted as the 1:1 synchronization, because, there is no smooth transition of limit cycles’ average frequencies from the independent to synchronized state for $\phi_L = 0.5\pi$. Thus, we cannot show a figure like Fig. 3(b). In summary, in the case for $\phi_L = 0.5\pi$, the 1:1 synchronization does not occur, but m:n synchronization can occur with a large value of coupling.

![FIG. 7. (a) Simulated time evolution of the light output intensity $|\alpha_1,2(t)|^2$ for $\phi_L = 0.5\pi$ with coupling $g = 0.02\kappa_1$. (b) The anti-symmetric part of a phase coupling function $\Gamma_a(\psi) \equiv \Gamma(\psi) - \Gamma(-\psi)$ as a function of the phase $\psi \equiv \psi_2 - \psi_1$ for $\phi_L = 0.5\pi$. In this figure, all the parameters except for $\phi_L$ are the same as in Fig. 3(a).]
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