BPS STATES AND THE $P=W$ CONJECTURE

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Abstract. A string theoretic framework is presented for the work of Hausel and Rodriguez-Vilegas as well as de Cataldo, Hausel and Migliorini on the cohomology of character varieties. The central element of this construction is an identification of the cohomology of the Hitchin moduli space with BPS states in a local Calabi-Yau threefold. This is a summary of several talks given during the Moduli Space Program 2011 at Isaac Newton Institute.

1. Introduction

Consider an M-theory compactification on a smooth projective Calabi-Yau threefold $Y$. M2-branes wrapping holomorphic curves in $Y$ yield supersymmetric BPS states in the five dimensional effective action. These particles are electrically charged under the low energy $U(1)$ gauge fields. The lattice of electric charges is naturally identified with second homology lattice $H_2(Y, Z)$. Quantum states of massive particles in five dimensions also form multiplets of the little group $SU(2)_L \times SU(2)_R \subset Spin(4,1)$, which is the stabilizer of the time direction in $\mathbb{R}^5$. The unitary irreducible representations of $SU(2)_L \times SU(2)_R$ may be labelled by pairs of half-integers $(j_L, j_R) \in \left(\frac{1}{2}\mathbb{Z}\right)^2$, which are the left, respectively right moving spin quantum numbers. In conclusion, the space of five dimensional BPS states admits a direct sum decomposition

$$H_{BPS}(Y) \simeq \bigoplus_{\beta \in H_2(Y, Z)} \bigoplus_{j_L, j_R \in \frac{1}{2}\mathbb{Z}} H_{BPS}(Y, \beta, j_L, j_R).$$

The refined Gopakumar-Vafa invariants are the BPS degeneracies

$$N(Y, \beta, j_L, j_R) = \dim H_{BPS}(Y, \beta, j_L, j_R).$$

The unrefined invariants are BPS indices,

$$N(Y, \beta, j_L) = \sum_{j_R \in \frac{1}{2}\mathbb{Z}} (-1)^{2j_R+1}(2j_R + 1)N(Y, \beta, j_L, j_R).$$

String theory arguments [18] imply that BPS states should be identified with cohomology classes of moduli spaces of stable pure dimension sheaves on $Y$. More specifically, let $\mathcal{M}(Y, \beta, n)$ be the moduli space of slope (semi)stable pure dimension one sheaves $F$ on $Y$ with numerical invariants

$$ch_2(F) = \beta, \quad \chi(F) = n.$$ 

Suppose furthermore that $(\beta, n)$ are primitive, such that there are no strictly semistable points. If $\mathcal{M}(Y, \beta, n)$ is smooth, the BPS states are in one-to-one correspondence with cohomology classes of the moduli space. In this case there is a geometric construction of the expected $SL(2)_L \times SL(2)_R$ action on the BPS Hilbert
Let $O$ where $R$ of an ample class $\omega$ weight ($F$)

For sheaves $F$ proper support define the Hilbert polynomial of

is given by cup product with a relative ample class $\omega_h$, respectively the pull back of an ample class $\omega_B$ on the base. One then obtains a decomposition

$$H^*(M(Y, \beta, n)) \cong \bigoplus_{(j_L, j_R) \in \mathbb{Z}^2} R(j_L, j_R)^{\oplus d(j_L, j_R)}$$

where $R(j_L, j_R)$ is the irreducible representation of $SL(2)_s \times SL(2)_r$ with highest weight $(j_L, j_R)$. A priori the multiplicities $d(j_L, j_R)$ should depend on $n$ for a fixed curve class $\beta$. Since no such dependence is observed in the low energy theory, one is lead to further conjecture that the $d(j_L, j_R)$ are in fact independent of $n$, as long as the numerical invariants $(\beta, n)$ are primitive. Granting this additional conjecture, the refined BPS invariants are given by $N(Y, \beta, j_L, j_R) = d(j_L, j_R)$.

In more general situations no rigorous mathematical construction of a BPS cohomology theory is known. There is however a rigorous construction of unrefined GV invariants via stable pairs [45, 46] which will be briefly reviewed shortly. It is worth noting that the BPS cohomology theory would have to detect the scheme structure and the obstruction theory of the moduli space as is the case in [45, 46].

Concrete examples where the moduli space $M(Y, \beta, n)$ is smooth are usually encountered in local models, in which case $Y$ is a noncompact threefold.

**Example 1.1.** Let $S$ be a smooth Fano surface and $Y$ the total space of the canonical bundle $K_S$, $\pi : Y \to S$ the natural projection ad $\sigma : S \to Y$ the zero section. Let $O_S(1)$ be a very ample line bundle on $S$. For any coherent sheaf $F$ on $Y$ with proper support define the Hilbert polynomial of $F$ by

$$P_F(m) = \chi(F \otimes_Y \pi^* O_S(m)).$$

For sheaves $F$ with one dimensional support,

$$P_F(m) = mr_F + n_F, \quad r_F, n_F \in \mathbb{Z}, \quad r_F > 0.$$  

Such a sheaf will be called (semi)stable if

$$r_F n_{F'} \leq r_{F'} n_F$$

for any proper nontrivial subsheaf $0 \subset F' \subset F$. Note that if $(r_F, n_F)$ are coprime, any semistable sheaf with numerical invariants $(r_F, n_F)$ must be stable. Moreover, Lemma [23, Lemma 7.1] proves that any stable sheaf $F$ on $Y$ with Hilbert polynomial $P_F(m) = mr_F + n_F$, with $r_F > 0$, must be the extension by zero, $F = \sigma_* E$, of a stable sheaf $E$ on $S$. Furthermore [23, Lemma 7.2] proves that $\text{Ext}^2_S(E, E) = 0$.

Let $D$ be an effective divisor on $S$ of degree $d > 0$ with respect to the polarization $O_S(1)$. Let $n \in \mathbb{Z}$ such that $(d, n)$ are coprime. Let $M^s(Y, D, n)$ be the moduli space of stable dimension one sheaves $F$ on $Y$ with numerical invariants

$$\text{ch}_2(F) = \sigma_* \text{ch}(O_D), \quad \chi(F) = n, \quad n \in \mathbb{Z}.$$

and $M^s(S, D, n)$ the moduli space of stable dimension one sheaves $E$ on $S$ with numerical invariants

$$\text{ch}_1(E) = \text{ch}_1(O_D), \quad \chi(E) = n.$$

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Then [23, Lemma 7.1] implies that there is an isomorphism \( \mathcal{M}^*(Y, D, n) \simeq \mathcal{M}^*(S, D, n) \). The vanishing result in [23, Lemma 7.2] implies that \( \mathcal{M}^*(S, D, n) \) is smooth according to [24, Thm. 4.5.1]. Moreover there is a well defined morphism \( \mathcal{M}^*(S, D, n) \to |D| \) sending \( E \) to its determinant [51, Prop. 3.0.2]. Therefore \( \mathcal{M}^*(Y, D, n) \) is smooth projective and equipped with a morphism \( h : \mathcal{M}^*(Y, r, d) \to |D| \).

Example 1.2. Let \( X \) be a smooth projective curve and \( D \) an effective divisor on \( X \), possibly trivial. Let \( Y \) be the total space of the rank two bundle \( \mathcal{O}_X(−D) ⊕ K_X(D) \). Note that \( H_2(Y) \simeq \mathbb{Z} \) is generated by the class \( σ \) of the 0 section. Let \( \mathcal{O}_X(1) \) be a very ample line bundle on \( X \). For any dimension one sheaf \( F \) on \( Y \) with proper support define

\[
P_F(m) = \chi(F ⊗ Y \pi^∗ \mathcal{O}_X(1)) = m r_F + n_F,
\]

where \( \pi : Y \to X \) is the natural projection. Then \( F \) is (semi)stable if

\[
r_F n_F, (≤) r_F n_F
\]

for any proper nontrivial subsheaf \( 0 < F' \subset F \).

Let \( (d, n) \) be a pair of coprime integers, with \( d > 0 \). Then there is a quasi-projective moduli space \( \mathcal{M}(Y, d, n) \) of stable dimension one sheaves \( F \) on \( Y \) with proper support and numerical invariants

\[
\text{ch}_2(F) = dσ, \quad \chi(F) = n.
\]

Let \( \mathcal{H}_r^+(X) \) be the moduli space of rank \( r \geq 1 \), degree \( e \in \mathbb{Z} \) stable Hitchin pairs on \( X \). Then it is easy to prove the following statements.

a) If \( D = 0 \) and \( (d, n) = 1 \), there is an isomorphism

\[
\mathcal{M}(Y, d, n) \simeq \mathcal{H}_d^{n + d(g−1)}(X) \times \mathbb{C}.
\]

b) If \( D \neq 0 \) and \( (d, n) = 1 \), there is an isomorphism

\[
\mathcal{M}(Y, d, n) \simeq \mathcal{H}_d^{n + d(g−1)}(X).
\]

The proof is analogous to the proof of [12, Thm. 1.9], the details being omitted.

As mentioned above unrefined GV numbers can be defined via Donaldson-Thomas [37] or stable pair invariants [15]. For smooth projective Calabi-Yau threefolds such invariants are defined by integration of virtual cycles on a component of the Hilbert scheme of curves, respectively the stable pair moduli space. When \( Y \) is a non-compact Calabi-Yau threefold as in Example 1.2 one has to employ equivariant virtual integration as in [4, 14] because the moduli spaces are non-compact. The torus action used in this construction is a fiberwise action on \( Y \) with weights +1, −1 on the direct summands \( K_X(D), \mathcal{O}_X(−D) \) leaving the zero section pointwise fixed. Compactness of the fixed loci in Donaldson-Thomas theory was proven in [4, 14] while in stable pair theory in [12]. Moreover, the equivalence between reduced Donaldson-Thomas theory and stable pair theory has been proven in [9] for smooth projective Calabi-Yau threefolds. Certain versions of this result were also proven in [18, 19]. For the quasi-projective varieties in Example 1.2 this equivalence follows in principle combining the results of [4, 14], and [38, Section 5]. To explain this briefly, recall that the local GW, respectively DT theory of curves has been computed in [4, 14] using degenerations of \( Y \) to normal crossing divisors where each component is a rank two bundle over \( \mathbb{P}^1 \) and each component intersects at most three other along common fibers. Therefore in order to prove their equivalence it suffices to prove equivalence of the resulting relative local theories. The same strategy will
lead to a proof of DT/stable pair correspondence using the results of [38, Section 5] to prove the equivalence of relative GW and stable pair theories. The details have not been fully worked out anywhere in the literature, but this result will be assumed in this paper.

In a IIA compactification on $Y$, $Z_{DT}(Y, q, Q)$ is the generating function for the degeneracies of BPS states corresponding to bound states of one D6-brane and arbitrary D2-D0 brane configurations on $Y$ (see [11, Sect. 6].) According to [18, 13], M-theory/IIA duality yields an alternative expression for this generating function in terms of the five dimensional BPS indices $N(Y, \beta, j_L)$. Then

$$Z_{DT}(Y, q, Q) = \exp \left( F_{GV}(Y, q, Q) \right),$$

(1.1)

where

$$F_{GV}(Y, q, Q) = \sum_{k \geq 1} \sum_{\beta \in H_2(Y), \beta \neq 0} \sum_{j_L \in \frac{1}{2} \mathbb{Z}} \frac{Q^{k\beta}}{k} \left( -1 \right)^{2j_L} N(Y, \beta, j_L) \frac{q^{-2kj_L} \cdots + q^{2kj_L}}{(q^{k/2} - q^{-k/2})^2}. $$

(1.2)

Relation (1.1) can be either inferred from [18] relying on the GW/DT correspondence conjectured in [37], or directly derived on physical grounds from Type IIA/M-theory duality [13]. Note that the generating function $F_{GV}(Y, q, Q)$ in (1.1) may be rewritten in the form [31]

$$F_{GV}(Y, q, Q) = \sum_{g \geq 0} \sum_{\beta \neq 0} n_{g, \beta} u^{2g-2} \sum_{k \geq 1} 1 \left( \frac{\sin(\frac{k}{u})}{\frac{u}{\sin(\frac{k}{u})}} \right) 2^{g-2} Q^{k\beta}$$

(1.3)

where $q = -e^{-iu}$ and

$$N(Y, \beta, j_L) = \sum_{g \geq 2j_L} \left( \frac{2g}{q + 2j_L} \right) n_{g, \beta}.$$

In the mathematics literature relation (1.1) with $F_{GV}(Y, q, Q)$ of the form (1.3), where $n_{g, \beta} \in \mathbb{Z}$ is known as the strong rationality conjecture [45]. It was proven for irreducible curve classes on smooth projective Calabi-Yau threefolds in [46] and for general curve classes in [50] with a technical caveat concerning holomorphic Chern-Simons functions for perverse coherent sheaves. In all these cases the proof does not provide a cohomological interpretation of the invariants $n_{g, \beta}$.

According to [28], a similar relation is expected to hold between refined stable pair invariants and the GV numbers $N(Y, \beta, j_L, j_R)$. As explained in [14] refined stable pair invariants are obtained as a specialization of the virtual motivic invariants of Kontsevich and Soibelman [34]. Then one expects [28] a relation of the form

$$Z_{DT,Y}(q, Q, y) = \exp \left( F_{GV,Y}(q, Q, y) \right),$$

(1.4)

where

$$F_{GV,Y}(q, Q, y) = \sum_{k \geq 1} \sum_{\beta \in H_2(Y), \beta \neq 0} \sum_{j_L, j_R \in \frac{1}{2} \mathbb{Z}} \frac{Q^{k\beta}}{k} \left( -1 \right)^{2j_L+2j_R} N(Y, \beta, j_L, j_R) q^{-k} \left( \frac{q^{-2kj_L} \cdots + q^{2kj_L}}{(1 - (qy)^{-k})(1 - (qy^{-1})^{-k})} \right).$$

(1.5)

The expression (1.5) was written in [28] in different variables, $(q^{-1} y, q^{-1} y^{-1})$. 


The main goal of this note is to point out that the refined GV expansion \((1.4)\) for a local curve geometry is related via a simple change of variables to the Hausel-Rodriguez-Villegas formula for character varieties. There are a few conjectural steps involved in this identification. First, it relies on an explicit conjectural formula for the refined stable pair theory of a local curve derived in section \((3)\) from geometric engineering and instanton sums. In fact, it is expected that a rigorous construction of motivic stable pair theory of local curves should be possible following the program of Kontsevich and Soibelman \([34]\). A conjectural motivic formula generalizing equation \((3.4)\) has been recently written down by Mozgovoy \([42]\). Second, as explained in detail in section \((4)\), the refined GV invariants of the local curve are in fact perverse Betti numbers of the Hitchin moduli space. Therefore, the conversion of the HRV formula into a refined GV expansion relies on the identification between the weight filtration on the cohomology of character varieties and the perverse filtration on the cohomology of the Hitchin system conjectured by de Cataldo, Hausel and Migliorini \([5]\). This will be referred to as the \(P = W\) conjecture. The connection found here provides independent physics based evidence for this conjecture. Finally, note that further evidence for all the claims of the present paper comes from the recent rigorous results of \([12, 39, 40]\). In \([12]\) it is rigorously proven that the refined theory of the local curve implies the HRV conjecture for the Poincaré polynomial of the Hitchin system via motivic wallcrossing while \([39, 40]\) prove expansion formulas analogous to \((1.4)\) for families of irreducible reduced plane curves.

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2. \textbf{Hausel-Rodriguez-Villegas formula and} \(P = W\)

Let \(X\) be a smooth projective curve over \(\mathbb{C}\) of genus \(g \geq 1\), and \(p \in X\) an arbitrary closed point. Let \(\gamma_p \in \pi_1(X \setminus \{p\})\) be the natural generator associated to \(p\). For any coprime integers \(r \in \mathbb{Z}_{\geq 1}\), \(e \in \mathbb{Z}\), the character variety \(C^e_r(X)\) is the moduli space of representations

\[
\phi : \pi_1(X \setminus \{p\}) \to GL(r, \mathbb{C}), \quad \phi(\gamma_p) = e^{2i\pi e/r}I_r
\]

modulo conjugation. \(C^e_r(X)\) is a smooth quasi-projective variety, and its rational cohomology \(H^*(C^e_r(X))\) carries a weight filtration

\[
(2.1) \quad W^k_0 \subset \cdots \subset W^k_i \subset \cdots \subset W^k_{2k} = H^k(C^e_r(X)).
\]
According to [19], \( W_k^i = W_{2i+1}^k \) for all \( i = 0, \ldots, 2k \), hence one can define the mixed Poincaré polynomial
\[
W(C^e_r(X), z, t) = \sum_{i, k} \dim(W^k_i/W^k_{i-1}) t^k z^{i/2}.
\]
Moreover it was proven in [19] that \( W(C^e_r(X), z, t) \) is independent of \( e \) for fixed \( r \), with \((r, e)\) coprime. Therefore it will be denoted below by \( W_r(z, t) \). Obviously \( W_r(1, t) \) is the usual Poincaré polynomial. Note that one can equally well use compactly supported cohomology in (2.2), which is related to cohomology without support condition by Poincaré duality [19],
\[
H^k_c(C^n_r(X)) \times H^{2d-k}(C^n_r(X)) \to \mathbb{C}.
\]
The difference would be an irrelevant overall monomial factor. Using number theoretic considerations Hausel-Rodriguez-Villegas [19] derive a conjectural formula for the mixed Poincaré polynomials \( W_r(z, t) \) as follows.

2.1. Hausel-Rodriguez-Villegas formula. The conjecture formulated in [19] expresses the generating function
\[
F_{HRV}(z, t, T) = \sum_{r, k \geq 1} B_r(z, t) W_r(z^k, t^k) T^{kr}/k,
\]
where
\[
B_r(z, t) = \frac{(zt^2(1-g) r(r-1)}{(1-z)(1-zt^2)},
\]
(2.3)
\[
F_{HRV}(z, t, T) = \ln Z_{HRV}(z, t, T)
\]
where \( Z_{HRV}(z, t, T) \) is a sum of rational functions associated to Young diagrams. Given a Young diagram \( \mu \) as shown below

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let \( \mu_i \) be the length of the \( i \)-th row, \( |\mu| \) the total number of boxes of \( \mu \), and \( \mu^t \) the transpose of \( \mu \). For any box \( \square = (i, j) \in \mu \) let
\[
a(\square) = \mu_i - j, \quad l(\square) = \mu^t_j - i, \quad h(\square) = a(\square) + l(\square) + 1,
\]
be the arm, leg, respectively hook length. Then
\[
Z_{HRV}(z, t, T) = \sum_\mu \mathcal{H}_g^\mu(z, t) T^{|\mu|}
\]
where
\[
\mathcal{H}_g^\mu(z, t) = \prod_{\square \in \mu} \frac{(zt^2)^{(l(\square)+2)}(1-z^{h(\square)} t^2l(\square)+1)^2g}{(1-z^{h(\square)} t^2l(\square)+2)(1-z^{h(\square)} t^2l(\square))},
\]
The main observation in this note is that equation (2.3) can be identified with the expansion of the refined Donaldson-Thomas series of a certain Calabi-Yau threefold in terms of numbers of BPS states.

2.2. Hitchin system and P = W. Let \( \mathcal{H}_r^c(X) \) be the moduli space of stable Higgs bundles \((E, \Phi)\) on \(X\), where \(\Phi\) is a Higgs field with coefficients in \(K_X\). For coprime \((r, e)\) this is a smooth quasi-projective variety equipped with a projective Hitchin map

\[ h : \mathcal{H}_r^c(X) \rightarrow \mathcal{B} \]

to the affine variety

\[ \mathcal{B} = \sum_{i=1}^r H^0(K_X^i) \].

The decomposition of the derived direct image \( R_h.\mathbb{Q} \) into perverse sheaves yields \([9, 8]\) a perverse filtration

\[ 0 = P_0^k \subset P_1^k \subset \cdots \subset P_k^k = H^k(\mathcal{H}_r^c(X)) \]
on cohomology. Following the construction in \([8, \text{Sect. 1.4.1}]\), let \( H^k(\mathcal{B}, R_h.\mathbb{Q}) \) denote the \(k\)-th hypercohomology group and \(\tau_{\leq p} R_h.\mathbb{Q} \) denote the truncations of \( R_h.\mathbb{Q} \). Then set

\[ P_p H^k(\mathcal{B}, R_h.\mathbb{Q}) = \text{Im}(H^k(\mathcal{B}, \tau_{\leq p} R_h.\mathbb{Q}) \rightarrow H^k(\mathcal{B}, \tau_{\leq p} R_h.\mathbb{Q})) \]

and

\[ P_p^k = P_p H^{k-d}(\mathcal{B}, R_h.\mathbb{Q}[d]) \]

where \(d = \dim \mathcal{B}\).

It is well known that \( \mathcal{C}_r^c(X) \) and \( \mathcal{H}_r^c(X) \) are identical as smooth real manifolds. This result is due to \([20, 15]\) for rank \(r = 2\) and \([7, 47]\) for general \(r \geq 2\). Therefore there is a natural identification \( H^*(\mathcal{C}_r^c(X)) = H^*(\mathcal{H}_r^c(X)) \). Then it is conjectured in \([8]\) that the two filtrations \( W^k \), \( P^k \) coincide,

\[ W^k_{kj} = P^k_j \]

for all \(k, j\). This is proven in \([8]\) for Hitchin systems of rank \(r = 2\).

For future reference note that a relative ample class \(\omega\) with respect to \(h\) yields a hard Lefschetz isomorphism \([10]\)

\[ \omega^j : Gr_{d-j}^P H^k(\mathcal{H}_r^c(X)) \rightarrow Gr_{d+j}^P H^{k+2j}(\mathcal{H}_r^c(X)) \]

This is known under the name of relative hard Lefschetz theorem.

Note that granting the \(P = W\) conjecture, equation (2.3) yields explicit formulas for the perverse Poincaré polynomial of the Hitchin moduli space. In particular, by specialization to \(z = 1\) it determines the Poincaré polynomial of the Hitchin moduli space of any rank \(r \geq 1\).

3. Refined stable pair invariants of local curves

Let \(Y\) be the total space of the rank two bundle \(O_X(-D) \oplus K_X(D)\) where \(D\) is an effective divisor of degree \(p \geq 0\) on \(X\) as in Example (1.2). Note that \(H_2(Y) \simeq \mathbb{Z}\) is generated by the class \(\sigma\) of the zero section. Following \([45]\), stable pairs on \(Y\) are two term complexes \(P = (O_Y \rightarrow F)\) where \(F\) is a pure dimension one sheaf and \(s\) a generically surjective section. Since \(Y\) is noncompact, in the present case, it will
be also required that $F$ have compact support, which must be necessarily a finite cover of $X$. The numerical invariants of $F$ will be
\[ \text{ch}_2(F) = d\sigma, \quad \chi(F) = n. \]
Then according to [45], there is a quasi-projective fine moduli space $P(Y, d, n)$ of pairs of type $(d, n)$ equipped with a symmetric perfect obstruction theory. The moduli space also carries a torus action induced by the $\mathbb{C}^\times$ action on $Y$ which scales $\mathcal{O}_X(-D), K_X(D)$ with weights $-1, 1$. Virtual numbers of stable pairs can be defined by equivariant virtual integration by analogy with [4, 44]. On smooth projective Calabi-Yau threefolds, the virtual number of pairs is equal to the Euler characteristic of the moduli space weighted by the Behrend function [1]. The analogous relation,
\[ P(\beta, n) = \chi^B(P(Y, d, n)), \]
for equivariant residual invariants of local curves follows from [12 Thm. 1.9] and [6, Lemma 3.1]. Let
\[ Z_{\text{PT}}(Y, q, Q) = 1 + \sum_{d \geq 1} \sum_{n \in \mathbb{Z}} P(d, n) Q^d q^n. \]
Applying the motivic Donaldson-Thomas formalism of Kontsevich and Soibelman, one obtains a refinement $P^{\text{ref}}(d, n, y)$ of stable pair invariants modulo foundational issues. The $P^{\text{ref}}(d, n, y)$ are Laurent polynomials of the formal variable $y$ with integral coefficients. In a string theory compactification on $Y$ these coefficients are numbers of D6-D2-D0 bound states with given four dimensional spin quantum number. The resulting generating series will be denoted by $Z^{\text{ref}}_{\text{PT}}(Y, q, Q, y)$.

3.1. TQFT formalism. A TQFT formalism for unrefined Donaldson-Thomas theory of a local curve has been developed in [44], in parallel with a similar construction in Gromov-Witten theory. Very briefly, the final result is that the generating series of local invariants is obtained by gluing vertices corresponding to a pair of pants decomposition of the Riemann surface $X$. Each such vertex is a rational function $P_\mu(q)$ labelled by three partitions $\mu_i, i = 1, 2, 3$ corresponding to the three boundary components. In the equivariant Calabi-Yau case a nontrivial result is obtained only for identical partitions, $\mu_i = \mu$, $i = 1, 2, 3$, in which case
\[ P_\mu(q) = \prod_{\square \in \mu} (q^{h(\square)}/2 - q^{h(\square)}/2). \]
Then the generating function is given by
\[ Z_{\text{DT}}(Y, q, Q) = \sum_\mu (-1)^{p[\mu]} q^{-(g-1-p)\kappa(\mu)} (P_\mu(q))^{2g-2} Q^{[\mu]} \]
where
\[ \kappa(\mu) = \sum_{\square \in \mu} (i(\square) - j(\square)). \]

3.2. Refined invariants from instanton sums. Although the refined stable pair invariants are not rigorously constructed for higher genus local curves, string duality leads to an explicit conjectural formula for the series $Z^{\text{ref}}_{\text{PT}}(Y, q, Q, y)$. This follows using geometric engineering [32, 41, 25, 35] of supersymmetric five dimensional gauge theories.

For completeness, geometric engineering is a correspondence between local Calabi-Yau threefolds and five dimensional gauge theories with eight supercharges. Such
gauge theories are classified by triples \((G, R, p)\), where \(G\) is a compact semisimple Lie group and \(R\) a unitary representation of \(G\), and \(p \in \mathbb{Z}\). Physically \(R\) encodes the matter content of the theory, and \(p\) is the level of a five dimensional Chern-Simons term. For certain triples \((G, R, p)\) (but not all) there exists a noncompact smooth Calabi-Yau threefold \(Y_{(G,R,p)}\) such that the gauge theory specified by \((G, R, p)\) is the extreme infrared limit of M-theory in the presence of a gravitational background specified by \(Y_{(G,R,p)}\). Many such examples are known \[32, 33\], but the list is not exhaustive, and there is no known necessary and sufficient condition on \((G, R, p)\) guaranteeing the existence of \(Y_{(G,R)}\).

For example if \(G = SU(N)\), \(N \geq 2\), \(R\) is the zero representation, and \(p = 0\), the corresponding threefold \(Y_{SU(N),0,0}\) is constructed as follows. Let \(Y\) be the total space of the rank two bundle \(\mathcal{O}_2 \oplus \mathcal{O}_1(-2)\) and let \(\mu_N\) be the multiplicative group of \(N\)-th roots of unity. There is a fiberwise action \(\mu_N \times Y \to Y\) where the generator \(\eta = e^{2\pi i/N}\) acts by multiplication by \((\eta, \eta^{-1})\) on the two summands. The quotient \(Y/\mu_N\) is a singular toric variety. Then \(Y_{SU(N),0,0}\) is the unique toric crepant resolution of \(Y/\mu_N\).

Moreover, suppose \(Y\) is replaced in previous paragraph by a rank two bundle of the form \(\mathcal{O}_X(-D) \oplus \mathcal{O}_X(K_X + D)\), with \(X\) a curve of genus \(g \geq 1\), as in Example \[12\]. Then there exists a corresponding gauge theory, and it has gauge group \(G = SU(N)\), matter content \(R = ad(G)^{\oplus g}\), and level \(p = \text{deg}(D)\), where \(ad(G)\) denotes the adjoint representation.

An important mathematical prediction of this correspondence is an identification between a generating function of stable pair invariants of \(Y_{(G,R,p)}\) and the five dimensional equivariant instanton sum of the gauge theory \((G, R, p)\) defined by Nekrasov in \[43\]. Some care is needed in formulating a precise relation; since \(Y_{(G,R,p)}\) are noncompact, the stable pair invariants must be defined as residual equivariant invariants with respect to a torus action. In addition, this identification also involves a nontrivial change of formal variables which is known in many examples, but has no general prescription.

Therefore a more precise formulation of this conjecture would state that there exists a torus action on \(Y_{(G,R,p)}\) such that the residual equivariant stable pair theory is well defined, and its generating function equals the equivariant instanton sum of the gauge theory \((G, R, p)\) up to change of variables. Such statements have been formulated and proved in many examples where \(Y_{(G,R,p)}\) is a toric Calabi-Yau threefold in \[16, 20, 27, 17, 21, 33, 36, 28\]. Furthermore, a refined version of the geometric engineering conjecture is available due to the work of \[29\], where it has been checked for \(SU(N)\) with \(N = 2, 3\), and \((R, p) = (0, 0)\).

In the present case, the geometric engineering conjecture yields \[5\] an explicit prediction for the residual stable pair theory of the threefolds \(Y\) in Example \[12\]. Because of a subtlety of physical nature, this case was treated in \[5\] as a limit of \(SU(2)\) gauge theory with \(R = ad(G)^{\oplus 2}\) and level \(p = \text{deg}(D)\). Omitting the computations, which are given in detail in \[5\] Sect 3, note that the final result can be presented in terms of quivariant K-theoretic invariants of the Hilbert scheme of points in \(\mathbb{C}^2\) as follows.

Let \(\text{Hilb}^k(\mathbb{C}^2)\) denote the Hilbert scheme of length \(k \geq 1\) zero dimensional subschemes of \(\mathbb{C}^2\). It is smooth, quasi-projective and carries a \(G = \mathbb{C}^\times \times \mathbb{C}^\times\)-action induced by the natural scaling action on \(\mathbb{C}^2\). Let \(R_G\) denote the representation ring
of $G$, and $q_1, q_2 : G \to \mathbb{C}$ the characters defined by

$$q(t_1, t_2) = t_1, \quad q_2(t_1, t_2) = t_2.$$  

Let also $\text{ch} : R_G \to \mathbb{Z}[q_1, q_2]$ denote the canonical ring isomorphism assigning to any representation $R$ the character $\text{ch}(R)$.

Now let tautological vector bundle $\mathcal{V}_k$ on the Hilbert scheme whose fiber at a point $[Z]$ is the space of global sections $H^0(O_Z)$. For each pair of integers $(g, p) \in \mathbb{Z}^2$, $g \geq 0$, $p \geq 0$ let

$$\mathcal{E}_k^{g,p} = T^* \text{Hilb}^k(C^2)^{\otimes g} \otimes \det(\mathcal{V}_k)^{1-g-p}.$$  

By construction, $\mathcal{E}_k^{g,p}$ has a natural $G$-equivariant structure which yields a linear $G$-action on the sheaf cohomology groups $H^i(G^\dagger \mathcal{E}_k^{g,p})$ of its exterior powers. Moreover, as observed for example in [5], although these spaces are infinite dimensional, each irreducible representation of $G$ has finite multiplicity in the decomposition of $H^i(G^\dagger \mathcal{E}_k^{g,p})$. Therefore one can formally define the equivariant $\chi_g$-genus of $\mathcal{E}_k^{(g,p)}$,

$$\chi_g(\mathcal{E}_k^{(g,p)}) = \sum_{i,j} (-\tilde{y})^i (-1)^j \text{ch} H^i(G^\dagger \mathcal{E}_k^{(g,p)})$$

as an element of $\mathbb{Z}[[q_1, q_2]]$. The equivariant K-theoretic partition function is defined by

$$Z_{\text{inst}}(q_1, q_2, \tilde{Q}, \tilde{y}) = \sum_{k \geq 0} \chi_g(\mathcal{E}_k^{(g,p)}) \tilde{Q}^k.$$  

A fixed point theorem gives an explicit formula for $Z_{\text{inst}}(q_1, q_2, \tilde{Q}, \tilde{y})$ as a sum over partitions:

$$Z_{\text{inst}}(q_1, q_2, \tilde{Q}, \tilde{y}) = \sum_{\mu} \prod_{\square \in \mu} (q_1^{-\ell(\square)} q_2^{-a(\square)})^{g-1+p}$$

$$\cdot \frac{(1 - \tilde{y} q_1^{-\ell(\square)} q_2^{a(\square)+1})^g (1 - \tilde{y} q_1^{-\ell(\square)+1} q_2^{-a(\square)})^g \tilde{Q}^{|\mu|}}{(1 - q_1^{-\ell(\square)} q_2^{a(\square)+1}) (1 - q_1^{-\ell(\square)+1} q_2^{-a(\square)})}.$$

The resulting conjectural expression for the refined stable pair partition function is then [5]

$$Z_{\text{ref}}^{\text{inst}}(Y, q, \tilde{Q}, y) = Z_{\text{inst}}(q^{-1} y, q y, (-1)^g y^{-2 g} Q, y^{-1}).$$  

A straightforward computation shows that

$$Z_{\text{ref}}^{\text{inst}}(Y, q, \tilde{Q}, y) = \sum_{\mu} \Omega^\mu(q, y) Q^{|\mu|}$$

where

$$\Omega^\mu(q, y) = (-1)^{p(|\mu|)} \prod_{\square \in \mu} \left[ (q^{\ell(\square)-a(\square)})^{-1} (y^{-(\ell(\square)+a(\square))})^p (q y^{-1})^{2(\ell(\square)+1)(g-1)} \right]$$

$$\cdot \left( \frac{(1 - q^{-h(\square)} y^{\ell(\square)-a(\square)})}{(1 - q^{-h(\square)} y^{\ell(\square)-a(\square)-1}) (1 - q^{-h(\square)} y^{\ell(\square)-a(\square)+1})} \right)^{2g}.$$  

The change of variables in (3.3) does not have a conceptual derivation. This conjecture is supported by extensive numerical computations involving wallcrossing for refined invariants in [5]. Further supporting evidence for the formula (3.3) is
obtained by comparison with the unrefined TQFT formula (3.1) for local curves. Specializing the right hand side of (3.3) at \( y = 1 \), one obtains

\[
Z_{P T}^{ref}(Y, q, Q, 1) = \sum_{\mu} Q^{\mu} \prod_{\square \in \mu} (-1)^{p(\square)} q^{(g-1)p(\square)-a(\square)} (q^{h(\square)/2} - q^{-h(\square)/2})^{2g-2}.
\]

Agreement with (3.3) follows from the identity

\[
\sum_{\square \in \mu} (l(\square) - a(\square)) = \sum_{\square \in \mu} (j(\square) - i(\square)) = -\kappa(\mu).
\]

Finally, note that the expression (3.3) with \( p = 0 \) is related to the left hand side of the HRV formula by

\[
Z_{HRV}(z, t, T) = Z_{P T}^{ref}(Y, (zt)^{-1}, (zt)^{g-1}T, t).
\]

4. HRV FORMULA AS A REFINED GV EXPANSION

This section spells out in detail the construction of refined GV invariants of a threefold \( Y \) as in Example 1.2 with \( p = \deg(D) = 0 \) in terms of the perverse filtration on the cohomology of the Hitchin moduli space. In this case the generic fibers and the base of the Hitchin map \( h : \mathcal{H}^e_r(X) \to \mathcal{B} \) have equal complex dimension \( d \). Using the conjectural formula (3.3), it will be shown that equation (3.3) yields the HRV formula by a monomial change of variables. As observed in Example 1.2, the moduli space of slope stable pure dimension one sheaves \( F \) on \( Y \) with compact support and numerical invariants

\[
\text{ch}_2(F) = r \sigma, \quad \chi(F) = n
\]
is isomorphic to \( \mathbb{C} \times \mathcal{H}^{n+r(g-1)}(X) \) provided that \( (r, n) = 1 \). Therefore, following the general arguments in the introduction, one should be able to define refined GV invariants using the decomposition theorem for the Hitchin map \( h : \mathcal{H}^e_r(X) \to \mathcal{B} \), \( e = n + r(g-1) \). However, since the base of the Hitchin fibration is a linear space, there will not exist an \( SL(2)_L \times SL(2)_R \) action on cohomology as required by M-theory. In this situation one can only define an \( SL(2)_L \times \mathbb{C}^* \)-action where \( \mathbb{C}^* \) can be thought of as a Cartan subgroup of \( SL(2)_R \). This action can be explicitly described in terms of the perverse sheaf filtration constructed in [3] Sect. 1.4], which was briefly reviewed in Section 2.2.

Note that given a relative ample class \( \omega \) for the Hitchin map there is a preferred splitting

\[
H^k(\mathcal{H}^e_r(X)) \simeq \bigoplus_{p} Gr_p H^k(\mathcal{H}^e_r(X))
\]
of the perverse sheaf filtration presented in detail in [3] Sect 1.4.2, 1.4.3]. Moreover, the relative Lefschetz isomorphism

\[
\omega^j : Gr^P_{d-j} H^k(\mathcal{H}^e_r(X)) \cong Gr^P_{d+j} H^{k+2j}(\mathcal{H}^e_r(X)).
\]
yields a decomposition

\[
Gr^P H^k(\mathcal{H}^e_r(X)) \simeq \bigoplus_{i+j=k} Q^{i,j;k}, \quad Q^{i,j;k} = \omega^j \cap Q^{i,0;k-2j},
\]
where

\[
Q^{i,0;k} = \text{Ker}(\omega^d H^k_{d-i+1} : Gr^P_{i} H^k(\mathcal{H}^e_r(X)) \to Gr^P_{2d-i+1} H^{k+2(d-i+1)}(\mathcal{H}^e_r(X))).
\]
for all $0 \leq i \leq d$. Let $Q^{i,j} = \bigoplus_{k \geq 0} Q^{i,j;k}$. By construction, for fixed $0 \leq i \leq d$, there is an isomorphism
\[
\bigoplus_{j=0}^{d-i} Q^{i,j} \simeq R^{\oplus \dim(Q^{i,0})}_{(d-i)/2}
\]
where $R_{jL}$ is the irreducible representation of $SL(2)_L$ with spin $j_L \in \frac{1}{2}\mathbb{Z}$. The generator $J^L_+ \mid L$ is represented by cup-product with $\omega$, and $Q^{i,j}$ is the eigenspace of the Cartan generator $J^L_0 \mid L$ with eigenvalue $j - (d - i)/2$. Note that cup-product with $\omega$ preserves the grading $k - d - 2j$ therefore one can define an extra $\mathbb{C}^\times$-action on $H^*(\mathcal{H}_r^c(X))$ which scales $Q^{i,j;k}$ with weight $d + 2j - k$. This torus action will be denoted by $\mathbb{C}^\times \times H^*(\mathcal{H}_r^c(X)) \to H^*(\mathcal{H}_r^c(X))$. Note also that
\[
d + 2j - k \geq -d
\]
since $j \geq 0$ and $k \leq -2d$.

In conclusion, in the present local curve geometry the $SL(2)_L \times SL(2)_R$ action on the cohomology of the moduli space of D2-D0 branes is replaced by an $SL(2)_L \times \mathbb{C}^\times_R$ action. This is certainly puzzling from a physical perspective since the BPS states are expected to form five-dimensional spin multiplets. The absence of a manifest $SL(2)_R$ symmetry of the local BPS spectrum is due to noncompactness of the moduli space. This is simply a symptom of the fact that there is no well defined physical decoupling limit associated to a local higher genus curve as considered here in M-theory. In principle, in order to obtain a physically sensible theory, one would have to construct a Calabi-Yau threefold $\mathfrak{Y}$ containing a curve $X$ with infinitesimal neighborhood isomorphic to $Y$ so that the moduli space $\mathcal{M}_Y(r[X], n)$ is compact and there is an embedding $H^*(\mathcal{M}_Y(r, n)) \subseteq H^*(\mathcal{M}_Y(r[X], n))$. The cohomology classes in the complement would then provide the missing components of the five-dimensional spin multiplets. Such a construction seems to be very difficult, and it is not in fact needed for the purpose of the present paper.

Given the $SL(2)_L \times \mathbb{C}^\times_R$ action described in the previous paragraph, one can define the following local version of the refined Gopakumar-Vafa expansion (125).

\[
F_{GV,Y}(q, Q, y) = \sum_{k \geq 1} \sum_{r \geq 1} Q^{kr} k (-1)^{2j_L + i} N_r((j_L, l)) \frac{q^{-k} + \cdots + q^{2kj_L}}{1 - (qy)^{-k} (1 - (qy)^{-k})}
\]
\[
(4.2)
\]
\[
where
\]
\[
N_r((j_L, l)) = \dim(Q^{d-2j_L, 0, d+l}).
\]
The same change of variables as in equation (3.3) yields
\[
F_{GV,Y}((zt)^{-1}, (zt^2)^{g-1} T, t) = \sum_{k \geq 1} \sum_{r \geq 1} T_k \frac{B_r(z^k, t^k) P_r(z^k, t^k)}{k}
\]
\[
(4.3)
\]
where $B_r(z, t)$ is defined above equation (2.3) and
\[
P_r(z, t) = \sum_{j=0}^{d} \sum_{l=0}^{d} (-1)^{j+l} N_r((j - d)/2, l - d) t^l (1 + \cdots + (zt)^{2j}).
\]
Now it is clear that the change of variables
\[
(q, Q, y) = ((zt)^{-1}, (zt^2)^{g-1} T, t)
\]
identifies the HRV formula (2.3) with the refined GV expansion (1.4) for a local curve provided that

\[ P_r(z, t) = W_r(z, t). \]

However, given the cohomological definition of the refined GV invariants \( N_r(j_L, l) \), relation (4.3) follows from the \( P = W \) conjecture of [8]. This provides a string theoretic explanation as well as strong evidence for this conjecture.

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