Contractibility of the Space of Opers for Classical Groups

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Abstract
The geometric Langlands program is an exciting direction of research, although a lot progress has been made there are still open questions and gaps. One of the necessary steps for this program is the proof of the existence and the cohomological triviality (more precisely, the $\mathcal{O}$-contractibility) of the space of rational opers. In this paper we prove the homotopical contractibility of the space of rational opers for classical groups.

1 Introduction

One basic problem in the representation theory of reductive groups $G$ over local fields, and in the theory of automorphic forms, arises from the absence of Whittaker functionals for some cuspidal representations in the case where $G$ is not a general linear group. One expects that the categorical Langlands program will help us “see” the missing Whittaker functionals. A necessary step for this program is the proof of the $\mathcal{O}$-contractibility of the space of rational opers. In this paper we prove the contractibility of the space of rational opers for classical groups. Although we don’t get $\mathcal{O}$-contractibility but only “homotopical contractibility” we believe that this can be a useful step in attacking the problem and that the methods used are of independent interest. In particular we show that the same methods give rise to a certain “higher” version of Tsen’s theorem on rational points.

1.1 Opers
Let $k$ be an algebraically closed field of characteristic 0 (fixed for the remainder of the paper). Given $G$ a reductive $k$-group and $X$ a smooth projective $k$-curve, A. Beilinson and V. Drinfeld defined the notion of a $G$-oper which was proven to be a useful tool in the Geometric Langlands program. The definition of a $G$-oper below is taken from [BD05]. In what follows, $k$ will be an algebraically closed field of characteristic zero (the reader will not lose much by assuming that $k = \mathbb{C}$). Throughout this paper, $X/k$ will be a smooth connected curve, and $G$ a connected reductive group over $k$ with a fixed Borel subgroup $B \subset G$ and a fixed Cartan subgroup $H \subset B$. Let $\mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$ be the corresponding Lie
algebras and $\Gamma \subset h^*$ be the set of simple roots with respect to $b$. There is a unique Lie algebra grading $g = \bigoplus g_k$ such that $g_0 = h$, $g_1 = \bigoplus_{\alpha \in \Gamma} g^\alpha$ and $g^{-1} = \bigoplus_{\alpha \in \Gamma} g^{-\alpha}$. The corresponding filtration

$$g^k = \bigoplus_{r \geq k} g_r$$

is $B$-invariant.

Let $K$ be an algebraic group with Lie algebra $\mathfrak{k}$ and let $F$ be a $K$-bundle on $X$. Consider the $K$-equivariant map $T\mathcal{F} \to \mathcal{F}$. Taking the quotient by $K$ we get a map

$$\mathbb{L}_\mathcal{F} := (T\mathcal{F})/K \to \mathcal{F}/K = X.$$  

$\mathbb{L}_\mathcal{F}$ is a vector bundle on $X$ of dimension $\dim X + \dim K$ that sits in the Atiyah short exact sequence:

$$0 \to \mathfrak{k}_\mathcal{F} \to \mathbb{L}_\mathcal{F} \to TX \to 0$$

where $\mathfrak{k}_\mathcal{F}$ denotes the $\mathcal{F}$-twist of $\mathfrak{k}$.

Let $\mathcal{F}$ be a $B$-bundle on $X$. We denote by $\mathcal{F}^G := \mathcal{F} \times_B G \to E$ the associated $G$-bundle. We have a commutative diagram

$$\begin{array}{ccc}
0 & \to & \mathfrak{k}_\mathcal{F} \\
\downarrow & & \downarrow \\
0 & \to & \mathbb{L}_\mathcal{F} \\
\downarrow & & \downarrow \\
0 & \to & \mathbb{L}_{\mathcal{F}G} \\
\downarrow & & \downarrow \\
0 & \to & TX \\
\downarrow & & \downarrow \\
0 & \to & 0
\end{array}$$

with exact rows. Since the filtration above is $B$-invariant, we have an induced filtration $\mathfrak{g}^\mathbb{L}_\mathcal{F} \subset g_\mathcal{F}$. We now define a filtration on $\mathbb{L}_{\mathcal{F}G}$ by defining $\mathbb{L}^k_{\mathcal{F}G} \subset g_\mathcal{F}$ to be the pre-image of $\mathfrak{g}^k_\mathcal{F}/\mathfrak{b}_\mathcal{F} \subset \mathfrak{g}_\mathcal{F}/\mathfrak{b}_\mathcal{F} = \mathbb{L}_{\mathcal{F}G}/\mathbb{L}_\mathcal{F}$. Note that

$$\mathbb{L}^{-1}_\mathcal{F}/\mathbb{L}_\mathcal{F} = \mathbb{L}^{-1}/\mathbb{L} = \mathfrak{g}^{-1}_\mathcal{F}/\mathfrak{g}_\mathcal{F} = \bigoplus_{\alpha \in \Gamma} \mathfrak{g}^{-\alpha}_\mathcal{F}.$$ 

Here $\mathfrak{g}^{-\alpha}_\mathcal{F}$ denotes the $\mathcal{F}$-twist of the $B$-module $\mathfrak{g}^{-\alpha}$ (the action of $B$ on $\mathfrak{g}^{-\alpha}$ is defined to be the composition $B \to H \to \text{Aut} \mathfrak{g}^{-\alpha}$).

Now let $\mathcal{E} = (E, \nabla)$ be a $G$-local system on $X$, that is, a $G$-bundle $E$ with a connection

$$\nabla : TX \to \mathbb{L}_E.$$ 

**Definition 1.1.** Let $\mathcal{E} = (E, \nabla)$ be a $G$-local system on $X$; an **oper** is a pair

1. a $B$-bundle $F$,
2. an isomorphism of $G$-bundles $\theta : F \times_B G \to E$,

such that

1. $\nabla(TX) \subset \mathbb{L}^{-1}_\mathcal{F} \subset \mathbb{L}_E$,
2. for each $\alpha \in \Gamma$ the composition

$$TX \xrightarrow{\nabla} \mathbb{L}^{-1}_\mathcal{F} / \mathbb{L}_\mathcal{F} \to \mathfrak{g}^\alpha_\mathcal{F}$$

is an isomorphism.
The notion of oper plays a role in an outline of a proof of the geometric Langlands conjecture envisioned by D. Gaitsgory. Specifically, one of the two major remaining gaps in the proof ([Gai13c, 10.2.8]) is the existence of an oper for every local system. Further in [Gai13c, 10.5.7] is the conjecture that the space of opers is “$\mathcal{O}$-contractible”. Without getting into detail, this notion of contractibility is version of “homological”-contractibility which takes into account so called “Hodge-theoretic” information. Although the contractibility we show does not imply this “$\mathcal{O}$-contractibility” we believe that it is of interest.

Outline The main result of this paper is the aforementioned contractibility for any $G$-local system for $G$ a classical group. To make this statement precise we need first to give a precise definition of the pre-sheaf of generic opers (which we will do in Section 1.2). Next, in Section 1.3 we will define the notions of contractibility relevant to our result. This will allow us to formally state our main result at the end of Section 1.3 (Theorem 1.9). In Section 3 and Section 2 we present the desired pre-sheafs that appear in this paper as quotients by a monoid of some filtered pre-sheafs. In Section 4 we describe how to use this description to compute the homotopical realisation. Finally, in Section 5 we apply a theorem of Oka to obtain contractibility. This last step is the one that makes use of the fact that we work in the case of a classical group.

Known results The question of existence of opers for all groups was resolved by D. Arinkin in [Ari16] using different methods from those presented here. This problem also has a local variant, where the curve is replaced with an infinitesimal disc. In this case the problem was settled by E. Frenkel and X. Zhu in [FZ08].

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1.2 The pre-sheaf of generic opers

In this section we describe a pre-sheaf classifying generic opers on a $G$-local system. Throughout this paper all schemes mentioned are over $k$ unless otherwise stated. Given two schemes $S$ and $T$ we denote the product over $\text{Spec} \, k$ by $S \times T$. A map of schemes $f : T \to S$ gives rise to an adjunction $f^* \dashv f_*$ between the corresponding categories of quasi-coherent sheaves. We denote the unit (resp. counit) of this adjunction $\Psi^f$ (resp. $\Phi^f$). Given a scheme $S$ we denote by $\Gamma_S : S \to \text{Spec} \, k$ the structure map. For typographical reasons we denote $\Gamma_S^k := (\Gamma_S)_*$.

Let $\mathcal{E} = (E, \nabla)$ be a $G$-local system on $X$. Given an affine $k$-scheme $S$, we say that an open subset

$$U \subset S \times X$$


is a *domain* if $U$ surjects on $S$ by the projection, that is, if all the fibers of the induced map

$$U \rightarrow S$$

are dense in $X$. To denote the fact that $U$ is a domain we use the notation

$$U \subseteq S \times X.$$  

We also fix a notation for the projections

$$\pi_U : U \rightarrow X,$$

$$\rho_U : U \rightarrow S.$$  

If $U \subseteq S \times X$ is a domain and $U' \subset U$ is an open subset, we say that $U' \subset U$ is a *sub-domain* if $U' \Subset S \times X$ is also a domain. We denote $U' \Subset U$ to say that $U'$ is sub-domain of $U$.

Let $S$ be a $k$-scheme and let $F \subset E$ be an inclusion of algebraic vector bundles on $S$. We say that $F$ is a *sub-bundle* of $E$, and denote

$$F \prec E,$$  

if $E/F$ is also an algebraic vector bundle.

Let $S$ be a $k$-scheme, let $E$ be a vector bundle on $S$, and let $f : U \rightarrow S$ be a map of $k$-schemes. When the map $f$ is clear from context we denote

$$E|_U := f^* E$$

or

$$E_U := f^* E.$$  

One common use for this notation is when $U \subseteq S$ is an open embedding, or for the map

$$\pi_U : U \rightarrow X$$

for $U$ a domain.

**Definition 1.2.** Let $E = (E, \nabla)$ be a $G$-local system on $X$, let $S$ be an affine scheme, and let $U \Subset S \times X$ be a domain. An *$S$-family of opers in $E$ supported on $U$* is an oper structure on $E_U$. We denote the set of isomorphism classes of such opers by $\mathcal{P}_E(U, S)$.

If $V \Subset U$, then we have a restriction map

$$\mathcal{P}_E(U, S) \rightarrow \mathcal{P}_E(V, S).$$

We denote

$$\mathcal{P}_E(S) := \operatorname{colim} \mathcal{P}_E(U, S).$$
1.3 Homotopical realizations

Many of the notations and definitions in this section are taken from [Bar12]. We denote by $\text{Aff}$ the category of finite type affine schemes over $k$. By an $\infty$-category we mean an $(\infty,1)$-category, for example, in the sense of Joyal and Lurie [Lur09]. We denote by $S$ the $\infty$-category of spaces (i.e., $\infty$-groupoids). We denote by $\text{Pro}(S_f)$ the $\infty$-category of pro-finite spaces (i.e., the $\infty$-category of pro-objects in $S_f$ - the $\infty$-category of pro-finite spaces). We denote by $\text{Vect}_k$ the stable $\infty$-category of chain complexes of vector spaces over $k$, mod quasi-isomorphism (whose homotopy category is equivalent to the derived category of the ordinary category of $k$-vector spaces). Since throughout most of the paper the field $k$ is fixed, we usually use the notation $\text{Vect} = \text{Vect}_k$.

Given a $k$-scheme $S$, we denote by $\text{Dmod}(S)$ the DG-category of D-modules on $S$. Note that we have a natural identification $\text{Dmod}(\text{Spec}(k)) \cong \text{Vect}_k$.

We denote by $\text{Set}$ the category of sets. For a category $C$, we let $\mathcal{Psh}(C)$ denote the $\infty$-category of pre-sheaves, i.e., functors $C^{\text{op}} \to S$. We can think of $\text{Aff}$ as a site equipped with the Zariski topology. We denote the corresponding $\infty$-category of Zariski $\infty$-sheaves by $\mathcal{S}_{hv}^{\text{Zar}}(\text{Aff})$. We shall also spend some time working in the category $\mathcal{Psh}^0(\text{Aff})$ of pre-sheafs with values in the category $\text{Set}$. Given a pre-sheaf of sets $F \in \mathcal{Psh}^0(\text{Aff})$ we denote its corresponding pre-sheaf of spaces by $\tilde{F} \in \mathcal{Psh}(\text{Aff})$, obtained by post-composing with the inclusion $\text{Set} \to S$. This extra level of care will be important, as the functor $
abla: \mathcal{Psh}^0(\text{Aff}) \to \mathcal{Psh}(\text{Aff})$ does not preserve colimits.

Let $C$ be a co-complete symmetric monoidal $(\infty,1)$-category such that the monoidal structure is compatible with all colimits. Let $H: \text{Aff} \to C$ be a symmetric monoidal functor where the symmetric monoidal structure on $\text{Aff}$ is the cartesian one. Let $S \in \text{Aff}$ be an affine scheme and let $U_\bullet \to S$ be a Zariski hyper-cover. Let $H(U_\bullet)$ be the corresponding simplicial object in $C$, and let $\text{colimit}_{\Delta^{op}} H(U)$ be its geometric realization. We have a natural map $H_{U,S}: \text{colimit}_{\Delta^{op}} H(U) \to H(S)$.

**Definition 1.3.** We say that $H$ satisfies **Zariski descent** if for every affine scheme $S$ and every Zariski hyper-cover $U_\bullet \to S$ $H_{U,S}: \text{colimit}_{\Delta^{op}} H(U) \to H(S)$ is an equivalence in $C$. 


Denote by $Y$ the $(\infty, 1)$-Yoneda embedding

$$Y : \mathsf{Aff} \to \mathcal{P}sh(\mathsf{Aff}) = \mathcal{S}^{\mathsf{Aff}^{op}}.$$  

We denote the left Kan extension of $H$ along $Y$ by

$$\hat{H} := \text{Lan}_Y : \mathcal{P}sh(\mathsf{Aff}) \to C.$$  

By [Lur, Proposition 4.8.1.10.] $\hat{H}$ is also symmetric monodial. Recall the following classical fact:

**Proposition 1.4.** Let $H : \mathsf{Aff} \to C$ be a symmetric monoidal functor satisfying Zariski descent and let $a : F \to G \in \mathcal{P}sh(\mathsf{Aff})$ be a map of pre-sheaves that is an equivalence after Zariski sheafification; then

$$\hat{H}(a) : \hat{H}(F) \to \hat{H}(G)$$

is an equivalence in $C$.

In this paper we are mainly interested in the following three examples:

**Definition 1.5.** Let $\mathsf{Vect}$ be the $\infty$-category of $k$-complexes and let

$$\mathbb{H} : \mathsf{Aff} \to \mathsf{Vect}$$

be the functor

$$S \mapsto \Gamma^S_! (\omega_S),$$

where

$$\Gamma^S_! : \mathsf{Dmod}(S) \to \mathsf{Dmod}(\text{Spec}(k)) \cong \mathsf{Vect}$$

is the functor corresponding to the structure map

$$\Gamma_S : S \to \text{Spec}(k).$$

**Definition 1.6.** Let $\mathsf{Pro}(\mathcal{S}_f)$ be the $(\infty, 1)$-category of pro-finite spaces and let

$$\mathbb{Ê}_{\mathsf{Et}} : \mathsf{Aff} \to \mathsf{Pro}(\mathcal{S}_f)$$

be the pro-finite étale topological type functor. ($\mathbb{Ê}_{\mathsf{Et}}$ is defined by first applying the étale topological type as defined in [Tri16] or [BS16] based on the work in [AM06], and then applying pro-finite completion.)

**Definition 1.7.** Let $\sigma : k \to \mathbb{C}$ be an embedding. Let $\mathcal{S}$ be the $(\infty, 1)$-category of spaces and let

$$\mathbb{Cl}_{\mathcal{C}} : \mathsf{Aff} \to \mathcal{S}$$

be the functor taking the $k$-scheme $S$ to the space $S_\mathcal{C}(\mathbb{C})$ with the classical complex topology, where $S_\mathcal{C}$ is the base change of $S$ over $\sigma$.  

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Note that \( \mathbb{H}, \hat{\text{Et}}, \text{and Cl} \) satisfy Zariski descent. They are also all symmetric monoidal where the monoidal structure on \( S \) and \( \text{Pro}(S_f) \) is the cartesian one, and on \( \text{Vect} \), the tensor product.

**Remark 1.8.** The different realization functors described above are related to each other. Given an embedding \( \sigma: k \to \mathbb{C} \), we have

\[
\mathbb{H}(S) \otimes_k \mathbb{C} \cong H^\text{top}_\bullet(\text{Cl}_\sigma(S), \mathbb{C}) \in \text{Vect}_\mathbb{C}
\]

(this appears, for example, in [Bar12]). Similarly,

\[
\hat{\text{Et}}(S) \cong P\text{f}(\text{Cl}_\sigma(S)),
\]

where \( P\text{f} \) is the pro-finite completion functor (see [AM06, 12.9]).

We are now ready to state the main result of the paper.

**Theorem 1.9.** Let \( E \) be a \( G \)-local system for a classical group \( G \); then for any of the realization functors \( H \) above we have

\[
\hat{H}(\mathcal{P}_E) \cong H(*),
\]

where \( * \) is the constant pre-sheaf.

Informally, we can say that the “space” (that is, the functor of points) of generic oper structures over \( E \) is contractible, and therefore for every such local system there exists an essentially unique generic oper.

### 2 Reinterpreting the space of opers

In this section we shall describe different pre-sheaves we shall use to reinterpret the pre-sheaf of opers and to analyse it’s realization. Let \( E \) be a vector bundle on \( X \), let \( S \in \mathfrak{M} \) and let \( U \subset S \times X \). We denote \( \mathcal{L}_{E(U,S)} \) the set of 1-dimentional sub-bundles \( W \prec E_U \). We define the pre-sheaf of generic lines in \( E \) as the functor

\[
\mathcal{L}_E: \mathfrak{M} \to \text{Set}
\]

\[
\mathcal{L}_E(S) := \text{colimit}_{U \subset S \times X} \mathcal{L}_{E(U,S)}.
\]

Given \( E_1, E_2 \) vector bundles on \( X \) we shall make use of the map of presheaves

\[
\mathcal{O}_{E_1, E_2} : \mathcal{L}_{E_1} \times \mathcal{L}_{E_2} \to \mathcal{L}_{E_1 \otimes E_2},
\]

which for a \( k \)-scheme \( S \) sends the pair \((L_1 \prec (E_1)_U, L_2 \prec (E_2)_U)\) to

\[
(L_1)_{U_1 \cap U_2} \otimes (L_2)_{U_1 \cap U_2} \prec ((E_1)_{U_1 \cap U_2} \otimes (E_2)_{U_1 \cap U_2}).
\]

From now on we consider only the case where the group \( G \) is classical. In this case \( G \)-local systems and opers can be easily reinterpreted as more explicit
structures. First, when $G = GL_n$ a $G$-local system is just a pair $\mathcal{E} = (E, \nabla)$ where $E$ is an $n$-dimensional vector bundle on $X$ and

$$\nabla: E \to E \otimes \omega_X$$

is a map satisfying the Leibniz rule. For an affine $S$ and a domain $U \subseteq S \times X$, an oper $o \in \mathcal{P}_E(U, S)$ is then given by a sequence of sub-bundles

$$0 = F_0 \prec F_1 \prec \cdots \prec F_{n-1} \prec E_U$$

that satisfies the following conditions:

1. $\text{rank}(F_i) = i$ for $0 \leq i \leq n - 1$,
2. $\nabla_U$ restricts to a map
   $$\nabla_U: F_i \to F_{i+1} \otimes \pi_U^* \omega_X,$$
3. the induced map
   $$\nabla: F_{i+1}/F_i \to F_{i+2}/F_{i+1} \otimes \pi_U^* \omega_X$$
   is an isomorphism.

More generally, for an integer $1 \leq r \leq n - 1$, we call a collection

$$0 = F_0 \prec F_1 \prec \cdots \prec F_r \prec E_U$$

satisfying the above conditions a partial oper of rank $r$ and denote the set of all such partial opers by $\mathcal{P}^{[r]}_E(U, S)$ and

$$\mathcal{P}^{[r]}_E(S) = \text{colimit}_{U \subseteq S \times X} \mathcal{P}^{[r]}_E(U, S).$$

Similarly, we can give a “classical” description of local systems and opers for other classical groups. In this paper we deal only with the case where $G = Sp(2n)$, keeping in mind that essentially the same proof works for $G = SO(n)$ as well. We shall explain how in more details in Section 5.2 Recall that an $Sp(2n)$-local system can be defined as a triple $\mathcal{E} = (E, \nabla, \langle \cdot, \cdot \rangle)$ where $(E, \nabla)$ is a $GL(2n)$-local system on $X$ and $\langle \cdot, \cdot \rangle$ is a non-degenerate bilinear alternating form

$$\langle \cdot, \cdot \rangle: \wedge^2 E \to O_X$$

such that for every $\mu \in TX$

$$\mu(\langle a, b \rangle) = \langle \nabla_\mu(a), b \rangle + \langle a, \nabla_\mu(b) \rangle.$$

Given an $Sp(2n)$-local system $\mathcal{E} = (E, \nabla, \langle \cdot, \cdot \rangle)$ we denote by

$$\mathcal{E} = (E, \nabla)$$
the underlying GL(2n)-local system. For $G = Sp(2n)$, as in the case of GL(n), it is easy to describe opers. Indeed, for $S \in \text{Aff}$ and a domain $U \subset S \times X$, an oper $o \in \mathcal{P}_{\mathcal{E}(U,S)}$ is given by a partial oper

$$0 = F_0 \prec F_1 \prec \cdots \prec F_n \prec E_U$$

in $\mathcal{P}^{[n]}_{\mathcal{E}(U,S)}$ such that $(\langle , \rangle)$ is 0 when restricted to $F_n$. Specifically, we get that

$$\mathcal{P}_{\mathcal{E}(U,S)} \subset \mathcal{P}^{[n]}_{\mathcal{E}(U,S)}.$$ 

Now note that the definition of opers in $\mathcal{P}_{\mathcal{E}(S)}$ is generic, that is, for any open non-empty subset $X' \subset X$, $\mathcal{P}_{\mathcal{E}(S)} = \mathcal{P}_{\mathcal{E}|_{X'}}(S)$. Thus we can make a few simplifying assumptions. First, we assume that $X$ is affine. We denote the smooth compactification of $X$ by $\bar{X}$. We denote by $g$ the genus of $\bar{X}$. We denote by $D$ the divisor

$$D := \bar{X} \setminus X.$$ 

Given an integer $d \geq 0$, we denote

$$k^d[X] := \Gamma_*^\bar{X}(\mathcal{O}_{\bar{X}}(dD)) = \Gamma(\bar{X}, \mathcal{O}_{\bar{X}}(dD)).$$

Note that $k^d[X]$ is a finite-dimensional $k$-vector space. In fact, for $d \gg 0$ we have

$$\dim_k k^d[X] = d|D| - g + 1.$$ 

We also have

$$k[X] = \bigcup_d k^d[X].$$

We can also assume that $\omega_X$ is a trivial line-bundle, and choose a fixed generator $\nu^\vee \in \omega_X$. We get that the tangent line-bundle to $X$

$$TX := \text{Hom}(\omega_X, \mathcal{O}_X),$$

is also trivial and denote by $\nu$ the dual generator to $\nu^\vee$.

Note that $\nu$ may be viewed as a derivation

$$\nu : k[X] \to k[X].$$

It is easy to see that there exists some $d_\nu \geq 0$ such that for every $d \geq 0$ we have:

$$\nu(k^d[X]) \subset k^{d+d_\nu}[X].$$

Given a vector bundle $E$ on $X$ with a connection $\nabla$, we denote by

$$\nabla_\nu : E \to E$$

the parallel transport along $\nu$.

Finally, we can assume that the vector bundle $E$ appearing in our GL(n) or $Sp(2n)$ local system is trivial.
Lemma 2.1. Let $\mathcal{E} = (E, \nabla)$ be an irreducible local system on $X$ of rank $n$, and let $2 \leq r \leq n - 1$, the obvious forgetful map

\[ F^{[r]}_\mathcal{E} : \mathcal{P}^{[r]}_\mathcal{E} \to \mathcal{P}^{[r-1]}_\mathcal{E} \]

is an isomorphism of pre-sheaves.

Proof. Let $S$ be an affine scheme, we need to show that the map

\[ F^{[r]}_\mathcal{E}(S) : \mathcal{P}^{[r]}_\mathcal{E}(S) \to \mathcal{P}^{[r-1]}_\mathcal{E}(S) \]

is a bijection. It is enough to show that $F^{[r]}_\mathcal{E}(S)$ is surjective. For $U \subseteq S \times X$ let

\[ 0 = F_0 \prec F_1 \prec \cdots \prec F_{r-1} \prec E_U \]

be a partial oper in $\mathcal{E}$ of rank $r - 1$ supported on $U$. Denote

\[ L_{r-1} := F_{r-1}/F_{r-2}, \]

and let

\[ \nabla_{\nu, L_{r-1}} : L_{r-1} = F_{r-1}/F_{r-2} \to E/F_{r-1} \]

be the map induced by $\nabla_\nu$. Note that although $\nabla_\nu$ is not $\mathcal{O}_U$-linear, $\nabla_{\nu, L_{r-1}}$ is. If $f$ and $v$ are local sections of $\mathcal{O}_U$ and $F_{r-1}$, respectively, then

\[ \nabla_\nu(f \cdot v) = \nabla_\nu(f)v + f\nabla_\nu(v) = f\nabla_\nu(v) (\text{mod } F_{r-1}). \]

Denote

\[ M := (E/F_{r-1})/\nabla_{\nu, L_{r-1}}(L_{r-1}). \]

We see that $M$ is a coherent sheaf on $U$ but not necessarily a vector bundle. Since $L_{r-1}$ is a line-bundle, the fibers of $M$ over points of $U$ can have only rank $n - (r - 1)$ or $n - (r - 1) - 1$. By the upper semi-continuity theorem $U' \subset U$ is open. We show that $U' \supseteq U$. Let $s_0 \in S(k)$ and denote $X' := \rho_U^{-1}(s_0) \subset \{s_0\} \times X$. Assume that $X'' \cap U' = \emptyset$; then we have that

\[ \nabla_{\nu, L_{r-1}}|_{X''} = 0. \]

Thus, $F_{r-1}|_{X''} \prec E_{X''}$ is a proper sub-bundle of $E_{X''}$ that is invariant under $\nabla_\nu$. We get that $E_{X''}$ is thus a reducible local system on $X''$. However since the map

\[ \pi_1(X'') \to \pi_1(X) \]

is surjective this stands in contradiction to the irreducibility of $\mathcal{E}$. Thus we get that $X'' \cap U' \neq \emptyset$ and $U' \subseteq U$. Now note that $M_{U'}$ is a vector bundle on $U'$ and so is

\[ F_r := \text{Ker}[E_{U'} \to M_{U'}]. \]

Now it is easy to see that

\[ 0 = F_0|_{U'} \prec F_1|_{U'} \prec \cdots \prec F_{r-1}|_{U'} \prec F_r \prec E_{U'} \]

is a partial oper in $\mathcal{P}^{[r]}_{\mathcal{E}(U', S)}$. \qed
Now let $\mathcal{E} = (E, \nabla, \langle \cdot, \cdot \rangle)$ be an $Sp(2n)$-local system such that $\mathcal{E}$ is irreducible. In light of Lemma 2.1, we get that

$$\mathcal{P}_E(S) \simeq \mathcal{P}_E^{[1]}(S) \simeq \mathcal{L}_E(S),$$

keeping in mind that $\mathcal{L}_E$ classifies generic families of lines in $E$. We now proceed to describe the sub-pre-sheaf

$$\mathcal{P}_E \subset \mathcal{P}_E \cong \mathcal{L}_E.$$

**Proposition 2.2.** Let $\mathcal{E} = (E, \nabla, \langle \cdot, \cdot \rangle)$ be an $Sp(2n)$-local system on $X$ and assume that $\mathcal{E}$ is irreducible. Let $S$ be affine and

$$L = F_1 < E_U$$

be some element in $\mathcal{L}_E(S)$. Then $L \in \mathcal{P}_E$ iff for every $0 \leq i < n - 1$,

$$\langle \nabla^{(i)} \nu, \nabla^{(i+1)} \nu \rangle = 0.$$

**Proof.** It is clear that $L \in \mathcal{P}_E$ iff

$$\langle \nabla^{(i)} \nu(L), \nabla^{(i)} \nu(L) \rangle = 0 \quad \forall i, j \quad 0 \leq i, j \leq n - 1.$$

Thus, the only if direction is clear. For the other direction we proceed by induction on $|i - j|$. We assume that this is true for $|i - j| \leq m$, and show it for $|i - j| = m + 1$. By symmetry, let us assume that $j = i + m + 1$, and we know that

$$\langle \nabla^{(i)} \nu(L), \nabla^{(i+m)} \nu(L) \rangle = 0.$$

By applying $\nu$ to both sides we get

$$\langle \nabla^{(i+1)} \nu(L), \nabla^{(i+m+1)} \nu(L) \rangle + \langle \nabla^{(i)} \nu(L), \nabla^{(i+m+1)} \nu(L) \rangle = 0.$$

Since $\langle \nabla^{(i+1)} \nu(L), \nabla^{(i+m)} \nu(L) \rangle = 0$, we have that $\langle \nabla^{(i)} \nu(L), \nabla^{(i+m+1)} \nu(L) \rangle = 0$. $\square$

### 3 The homotopy type of $\mathcal{L}$ and its sub-pre-sheaves

In this section we start proving the main theorem.

#### 3.1 Auxiliary pre-sheaves

Let us now fix $\mathcal{E} = (E, \nabla, \langle \cdot, \cdot \rangle)$ a $Sp(2n)$-local system such that $(E, \nabla)$ is irreducible as an $GL(2n)$-local system.

We denote $\mathcal{L} = \mathcal{L}_E$ and $\mathcal{P} = \mathcal{P}_E$. Note that $\mathcal{P} \subset \mathcal{L}$ is a sub-pre-sheaf.

We want to show

**Theorem 3.1.** Let $H$ be one of the functors:

1. $\mathbb{H} : \text{Aff} \to \text{Vect}$,
(2) $\mathbb{E}t: \mathbb{A}ff \to Pro(\mathcal{S}_f)$,

(3) $\mathbb{C}l_r: \mathbb{A}ff \to \mathbb{S}$ for some embedding $k \to \mathbb{C}$.

Then

$$\hat{H}(\tilde{\mathcal{P}}) \cong \hat{H}(\tilde{\mathcal{L}}) \cong *$$

is contractible.

Remark 3.2. In fact, when proving that $\hat{H}(\tilde{\mathcal{L}})$ is contractible, we do not use at all the fact that the $GL(2n)$-local system $\mathcal{E}$ comes from an $Sp(2n)$-local system $\mathcal{E}$. Thus, while proving Theorem 3.1, we give an alternative proof of the result for $GL(n)$. This case can be also obtained by a general machinery from [Gai13a, Bar12] which allows to show contractibility for connected schemes which can be covered by open schemes $U_\alpha$, each of which is isomorphic to an open sub-scheme of the affine space $\mathbb{A}^n$. (See [Gai13a, 1.8.2]).

Fix $H$ to be one of the functors above; for notational convenience, we write for a pre-sheaf $F \in \mathcal{P}_{sh}(\mathbb{A}ff)$

$$\langle \langle F \rangle \rangle := \hat{H}(\tilde{F}).$$

Thus we are interested in proving that $\langle \langle \mathcal{L} \rangle \rangle$ and $\langle \langle \mathcal{P} \rangle \rangle$ are contractible.

Let $V$ be a finite dimensional $k$-vector space, there is a nice description for the functor of points of $\mathcal{P}(V)$. Specifically, for a $k$-scheme $S$ the set $\text{Hom}(S, \mathcal{P}(V))$ is in bijection with the set of 1-dimensional sub-bundles $W \prec \Gamma^*_S V$, where $V$ is considered as a coherent sheaf on $\text{Spec} k$. We say that a bilinear map of $k$-vector spaces

$$\alpha: V_1 \otimes V_2 \to V_3$$

is non-degenerate if $\alpha(v_1 \otimes v_2) = 0$ implies that either $v_1 = 0$ or $v_2 = 0$. Such an $\alpha$ gives rise to a Veronese embedding

$$\mathcal{V}_{\alpha}: \mathbb{P}(V_1) \times \mathbb{P}(V_2) \to \mathbb{P}(V_3).$$

The corresponding map for a scheme $S$

$$\text{Hom}(S, \mathbb{P}(V_1)) \times \text{Hom}(S, \mathbb{P}(V_2)) \to \text{Hom}(S, \mathbb{P}(V_3))$$

sends the pair $W_i \prec \Gamma^*_S V_i$ for $i = 1, 2$ to the image of

$$W_1 \otimes W_2 \prec \Gamma^*_S V_1 \otimes \Gamma^*_S V_2 \equiv \Gamma^*_S (V_1 \otimes V_2),$$

by the map $\Gamma^*_S(\alpha)$.

Now let $E$ be a vector bundle over $\tilde{X}$. We have that $\Gamma^*_S(E)$ is a finite-dimensional $k$-vector space. We shall construct a map of pre-sheaves:

$$F_E: \mathbb{P}(\Gamma^*_S(E)) \to \mathcal{L}_{E_X},$$
where $\mathcal{P}(\Gamma^S_*(E))$ is interpreted as the corresponding functor of points. Let $S$ be scheme, for a point $f \in \text{Hom}(S, \mathcal{P}(\Gamma^S_*(E)))$ let
\[ W_f \simeq \Gamma^S_*\Gamma^\mathcal{X}_*(E) \]
be the corresponding 1-dimensional sub-bundle. We denote by
\[ e_f : W_f \hookrightarrow \Gamma^S_*\Gamma^\mathcal{X}_*(E) \]
the inclusion map. Now consider the commutative square:
\[
\begin{array}{ccc}
S \times \mathcal{X} & \xrightarrow{\pi} & \mathcal{X} \\
\rho \downarrow & & \downarrow \rho_X \\
S & \xrightarrow{\Gamma_S} & \text{Spec} \ k
\end{array}
\]
and the composition:
\[ r_f : \rho^* W_f \xrightarrow{\rho^* e_f} \rho^* \Gamma^S_*\Gamma^\mathcal{X}_*(E) = \pi^* \Gamma^S_* \Gamma^\mathcal{X}_*(E) \to \pi^* E \]
where the last map induced from the counit of the adjunction. In other words we have:
\[ r_f := \pi^*(\Phi^\mathcal{X}_E) \circ \rho^*(e_f). \]
Denote by $U'_f \subset S \times \mathcal{X}$ the locus on which the $r_f$ is an injection and $U_f = U'_f \cap (S \times X)$. We shall show (Lemma 3.3) that $U_f \subset S \times X$ and thus we can take
\[ F_E(f) := \text{Image}(r_f|_{U_f}) \sim (E_X)_{U_f}. \]

**Lemma 3.3.** In the notation above we have $U_f \subset S \times X$.

**Proof.** We need to show that for every closed point $s \in S$
\[ U_f \cap \{s\} \times X \neq \emptyset \]
is non-empty. Equivalently, let $j : \mathcal{X} \to S \times \mathcal{X}$ be the closed embedding that sends $x$ to $(s, x)$, we need to show that the locus on which
\[ j^* r_f : j^* \rho^* W_f \to j^* \pi^* E = E \]
is an embedding is non-empty. Since $W_f$ is a line bundle, this is equivalent to $j^* r_f \neq 0$. To see this it is enough to show that $\Gamma^S_* j^* \pi^* E \neq 0$. Now denote by $i : \text{Spec} \ k \to S$ the embedding of the closed point $s \in S$. We get the following commutative diagram:
\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{j} & S \times \mathcal{X} & \xrightarrow{\pi} & \mathcal{X} \\
\Gamma_X \downarrow & & \rho \downarrow & & \Gamma_X \\
\text{Spec} \ k & \xrightarrow{i} & S & \xrightarrow{\Gamma_S} & \text{Spec} \ k.
\end{array}
\]
We now have:
\[
\Gamma^X_* j^* r_f = \Gamma^X_* j^* \left( \pi^* (\Phi_{E}^X) \circ \rho^*(e_f) \right) = \\
= \Gamma^X_* j^* \pi^* (\Phi_{E}^X) \circ \Gamma^X_* j^* \rho^*(e_f) = \\
= \Gamma^X_* (\Phi_{E}^X) \circ \Gamma^X_* \Gamma^X_* i^*(e_f).
\]
Note that
\[
i^*(e_f): i^* W_f \to i^* \Gamma^X_* \Gamma^X_* E = \Gamma^X_* E
\]
is non-zero since \(e_f\) is an embedding of a sub-bundle. Now to see that \(\Gamma^X_* j^* r_f \neq 0\) it is enough to show that
\[
\Gamma^X_* (\Phi_{E}^X) \circ \Gamma^X_* \Gamma^X_* i^*(e_f) \circ \Psi_{i^* W_f} \neq 0.
\]
We now have:
\[
\Gamma^X_* (\Phi_{E}^X) \circ \Gamma^X_* \Gamma^X_* i^*(e_f) \circ \Psi_{i^* W_f} = \\
= \Gamma^X_* (\Phi_{E}^X) \circ \Psi_{i^* (e_f)} = i^*(e_f) \neq 0,
\]
where the first equality follows from \(\Psi_{\Gamma^X_*}\) being a natural transformation and the second equality follows from the zig-zag identity of the adjunction \(\Gamma^X_* \dashv \Gamma^X_*\).

It’s worth noting that \(F_E\) behaves well with respect to tensor product.

**Lemma 3.4.** Let \(E_1, E_2\) be vector bundles on \(\tilde{X}\). Since \(\tilde{X}\) is irreducible the map
\[
\alpha: \Gamma^X_* (E_1) \otimes \Gamma^X_* (E_2) \to \Gamma^X_* (E_1 \otimes E_2)
\]
is non-degenerate and thus gives rise to a map
\[
\mathbb{P}(\Gamma^X_* (E_1) \otimes \Gamma^X_* (E_2)) \to \mathbb{P}(\Gamma^X_* (E_1 \otimes E_2)).
\]
We have a commutative diagram of presheaves:

\[
\begin{array}{ccc}
\mathbb{P}(\Gamma^X_* (E_1)) \times \mathbb{P}(\Gamma^X_* (E_2)) & \xrightarrow{\nu_*} & \mathbb{P}(\Gamma^X_* (E_1 \otimes E_2)) \\
| & | & |
F_{E_1} \times F_{E_2} & \downarrow & F_{E_1 \otimes E_2} \\
\mathcal{L}_{E_1} \times \mathcal{L}_{E_2} & \xrightarrow{\Omega_{E_1 \times E_2}} & \mathcal{L}_{E_1 \otimes E_2}.
\end{array}
\]

**Proof.** The map \(\alpha\) is adjoint to the map
\[
\Phi_{E_1}^X \otimes \Phi_{E_2}^X : \Gamma^X_* (\Gamma^X_* (E_1) \otimes \Gamma^X_* (E_2)) \cong \Gamma^X_* \Gamma^X_* (E_1) \otimes \Gamma^X_* \Gamma^X_* (E_2) \to E_1 \otimes E_2
\]
Thus the following diagram commutes:

\[
\begin{array}{ccc}
\Gamma^X_* (\Gamma^X_* (E_1) \otimes \Gamma^X_* (E_2)) & \xrightarrow{\Gamma^X_*(\alpha)} & \Gamma^X_* \Gamma^X_* (E_2) \\
\downarrow & & \downarrow \Phi_{E_1}^X \otimes \Phi_{E_2}^X \\
\Gamma^X_* (\Gamma^X_* (E_1 \otimes E_2)) & \xrightarrow{\Phi_{E_1}^X \otimes \Phi_{E_2}^X} & E_1 \otimes E_2
\end{array}
\]
The claim now follows from an easy diagram chase using the fact that $\pi^*$, $\rho^*$ and $\Gamma_S^*$ are symmetric monoidal functors. \hfill \Box

Now recall that 

$$D := \tilde{X} \setminus X.$$ 

Let $d \geq 0$ be an integer; we denote by $E_d$ the vector bundle $O_{\tilde{X}}(dD)^{2n}$. Note that $(E_d)_X \cong O_{\tilde{X}}^{2n} \cong E$. We denote the representable pre-sheaf of $\mathbb{P}(\Gamma_{\tilde{X}}^*(E_d))$ by $L_d$. Note that we have natural embeddings $L_d \hookrightarrow L_{d+1}$. We denote by 

$$L_\infty := \colim L_d$$ 

the resulting ind-scheme. We use the different $L_d$’s to approximate $L$. We also denote 

$$F_d := F_{E_d}: L_d \to L = L_E.$$ 

Remark 3.5. Let $W_\tilde{f} \subset \Gamma_{\tilde{X}}^*E_{\tilde{d}}$ be a 1-dimensional sub-bundle such that $W_\tilde{f} \cong O_S$. It is worth noting that in this case $F_d(f)$ has a “coordinate based” description. Take $g \in \Gamma_{\tilde{X}}^*E_{\tilde{d}}$ a generator of $W_\tilde{f}$. For every $s \in S$ we can think of $g(s)$ as an $n$-tuple of regular functions on $X$, $g_i \in \Gamma(\tilde{X}, O_{\tilde{X}}(dD))$, not all zero, choosing a different generator changes the $g_i(s)$ by a common scalar from $k$. We denote by $\Gamma_g \subset S \times X$ the closed subset whose fiber over $s \in S$ is the finite set of common zeros of $g_1(s), \ldots, g_{2n}(s)$. We denote 

$$U_g := S \times X \setminus \Gamma_g.$$ 

Note that we have $U_g \subset S \times X$. Now consider the inclusion $i_g : O_{\tilde{U}_g} \hookrightarrow O_{\tilde{U}_g}^{2n}$ defined by sending 1 to $g_i$ in the $i$’th coordinate (considered as a regular function on $S \times X$ by the exponential law). We have that, over $U_g$, the map $i_g : O_{\tilde{U}_g} \times O_{\tilde{U}_g}^{2n}$ is a sub-bundle and we get $F_d(f) = \Image(i_g)$ (note that changing all the $g_i$’s by a scalar from $k$ does not change $\Image(i_g)$). 

It is clear that the different $F_d$’s are compatible with the inclusions $L_d \hookrightarrow L_{d+1}$ and thus all the $F_d$’s fit together into a map 

$$F_\infty : L_\infty \to L.$$ 

More generally, let $i : B \subset L$ be a sub-pre-sheaf and for $0 \leq d \leq \infty$ denote by $B_d$ the pullback 

$$
\begin{array}{ccc}
B_d & \xrightarrow{F_d} & B \\
\downarrow g_d & & \downarrow i \\
L_d & \xrightarrow{F_d} & L.
\end{array}
$$ 

Note that we have $B_\infty = \colim B_d$ and that $B_d$ is a sub-pre-sheaf of $L_d$. We denote by $\mathbb{P}\infty$ the infinite projective space as an ind-scheme. 

Proposition 3.6. Let $L_\infty, L, B, B_\infty$ be as above; then
(1) $\langle \langle \mathcal{L}_\infty \rangle \rangle$ is weakly equivalent to $\langle \langle \mathbb{P}^\infty \rangle \rangle$.

(2) If the map
$$\langle \langle \mathcal{B}_\infty \rangle \rangle \to \langle \langle \mathcal{L}_\infty \rangle \rangle$$

is an equivalence, then so is the map
$$\langle \langle \mathcal{B} \rangle \rangle \to \langle \langle \mathcal{L} \rangle \rangle ,$$

(3) $\langle \langle \mathcal{L} \rangle \rangle$ is contractible.

(1) is trivial as $\langle \langle \mathcal{L}_\infty \rangle \rangle$ is indeed a colimit of projective spaces of increasing dimension. The rest of this section is devoted to proving (2) and (3).

### 3.2 Presenting $\mathcal{L}$ as a quotient

**Lemma 3.7.** The map of pre-sheaves
$$F_\infty : \mathcal{L}_\infty \to \mathcal{L}$$
is a local epimorphism in the Zariski topology.

**Proof.** For a $k$-scheme $Y$ denote by
$$A_Y := \Gamma(Y, \mathcal{O}_Y)$$
the $k$-algebra of regular functions. Let $S$ be an affine scheme and $l \in \mathcal{L}(S)$. We wish to show that for every $s_0 \in S$ there exist some neighborhood $s_0 \in O \subset S$ and $m \in \mathcal{L}_\infty(O)$ such that $F_\infty(m) = l|_O$.

Note that $l$ consists of the data of a domain $U \subset S \times X$ and a 1-dimensional sub-bundle
$$L_U \subset \mathcal{O}_U^{2 \times n}.$$First we show that we may assume that $U$ is affine and $L_U \cong \mathcal{O}_U$. Choose some point $y_0 = (s_0, x_0) \in U$ in the fiber above $s_0$ and choose some open affine neighborhood $U' \subset U$ of $y_0$ such that $L_U|_{U'} \cong \mathcal{O}_{U'}$. We define $O' \subset S$ as the open neighborhood of $s_0$ that is in the image of $U'$ under the projection to $S$, and $O \subset O'$ to be an open affine neighborhood of $s_0$. Write
$$U'' := U' \cap (O \times X)$$and so
$$U'' \subset O \times X.$$Thus, without loss of generality, we may assume from now on that $S = O$, $U = U''$, and $L_U \cong \mathcal{O}_U$.

Since $U \subset S \times X$ is open and $y_0 = (s_0, x_0) \in U$, there is a function
$$C = \sum_{i=1}^r a_i \otimes b_i \in A_S \otimes_k A_X = A_{S \times X}$$
such that
\[ C(y_0) = \sum_{i=1}^{r} a_i(s_0)b_i(x_0) \neq 0, \]
and $C$ is identically zero on $(S \times X) \setminus U$. We denote
\[ C_{x_0} := \sum_{i=1}^{r} b_i(x_0)a_i \in A_S. \]
Note that $C_{x_0}(s_0) \neq 0$, and so we can invert $C_{x_0}$ to get an open neighborhood of $s_0$
\[ s_0 \in O := S_{C_{x_0}} = \text{Spec } A_S \left[ \frac{1}{C_{x_0}} \right] \subset S. \]
Similarly, we denote
\[ y_0 \in U' := (O \times X)_C = \text{Spec } A_{O \times X} \left[ \frac{1}{C} \right] \subset U \cap (O \times X). \]
Note that we have
\[ O \times \{ x_0 \} \subset U' \subset O \times X. \]
Indeed, if $y = (s, x_0) \in O \times \{ x_0 \}$, we have that $C(y) = C(s, x_0) = C_{x_0}(s) \neq 0$
and so $y \in U'$. Thus all the fibers of the projection $U' \to O$ are non-empty, and we have
\[ U' \subset O \times X. \]
Since $U'$ is affine and $L_{U'} \cong \mathcal{O}_{U'}$, we get that $L_{U'}$ is generated as an $A_{U'}$ submodule by a vector
\[ (f_1, \ldots, f_{2n}) \in A_{U'}^{2n} \]
such that for every $y \in U'$ not all the values $f_i(y)$ are zero. Since $A_{U'} = A_{O \times X} \left[ \frac{1}{C} \right]$ there exists some number $N \geq 0$ such that for every $1 \leq i \leq n$ we can write $f_i = \frac{g_i}{C^N}$ for some $g_i \in A_{O \times X}$. Since $C$ is invertible on $U'$, we may assume that $N = 0$ and $f_i \in A_{O \times X}$ for all $1 \leq i \leq n$. Thus we can write
\[ f_i = \sum_{j=1}^{r_i} a_{i,j} \otimes_k b_{i,j} \in A_O \otimes_k A_X = A_{O \times X}. \]
Let $d \geq 0$ be a big enough integer such that $b_{i,j} \in k^d[X] = \Gamma(X, \mathcal{O}_X(d))$ for all $b_{i,j}$. We can thus consider the vector $(f_1, \ldots, f_{2n})$ as an element in the $A_O$-module
\[ (A_O \otimes_k k^d[X])^{2n} \cong \Gamma^*_O \Gamma^X(E_d). \]
Let $W \subset \Gamma^*_O \Gamma^X(E_d)$ be the submodule generated by $(f_1, \ldots, f_{2n})$. We wish to show that $W$ is in fact a sub-bundle. But to do so, we need to show that for all $s \in O$ not all $f_1(s), \ldots, f_{2n}(s)$ are zero. Indeed, we have that $(s, x_0) \in O \times \{ x_0 \} \subset U'$ and thus the $f_1(s, x_0), \ldots, f_{2n}(s, x_0)$ cannot all be zero.
We take $m \in L_d(O)$ to be the point corresponding to $W \prec \Gamma^*_X \Gamma^*_E d$. An easy diagram chase shows that $F_d(m) = l$. Note that we have

$$\rho^*(W) \cong \rho^* (O_O) \cong O_{O \times X}$$

so $F_d(m)$ gives a submodule generated by a single element.

Note that the map

$$F_\infty : L_\infty \to L$$

is not injective, however we shall show that it is an isomorphism after dividing by an action of a monoid, and sheafification.

For an integer $d \geq 0$ we denote $M_d := O_{\bar{X}} (dD)$ and

$$\mathcal{M}_d := \mathbb{P}(\Gamma^*_X (M_d)) = \mathbb{P}(k^d[X]).$$

Note that we have natural embeddings $\mathcal{M}_d \hookrightarrow \mathcal{M}_{d+1}$. We denote by

$$\mathcal{M}_\infty := \text{colimit } \mathcal{M}_d$$

the resulting ind-scheme. We have a special point $1 \in \mathcal{M}_0$ that corresponds to the constant function $1$ on $X$. Multiplication of regular functions on $X$ induces natural maps

$$\mathcal{M}_d \times \mathcal{M}_d \hookrightarrow \mathcal{M}_{2d}.$$ 

The resulting map

$$\mathcal{M}_\infty \times \mathcal{M}_\infty \to \mathcal{M}_\infty,$$

together with $1 \in \mathcal{M}_\infty$, turns $\mathcal{M}_\infty$ into a commutative monoid. Similarly, by multiplying regular functions “coordinate-wise” we have an isomorphism

$$M_d \otimes E_e \cong E_{d+e}.$$

Thus we get a compatible collection of Veronese maps

$$\mathcal{M}_d \times \mathcal{L}_e \to \mathcal{L}_{d+e}$$

that gives an action of $\mathcal{M}_\infty$ on $\mathcal{L}_\infty$.

**Lemma 3.8.** The following diagram

$$\mathcal{M}_\infty \times \mathcal{L}_\infty \xrightarrow{p} \mathcal{L}_\infty \xrightarrow{F_\infty} \mathcal{L}$$

commutes, where $a$ is the action map and $p$ is the projection map. In other words the action of $\mathcal{M}_\infty$ respects the fibers of the map

$$F_\infty : \mathcal{L}_\infty \to \mathcal{L}.$$
Proof. By Lemma 3.4 we have for every $e, d$ a commutative diagram:

\[
\begin{array}{ccc}
M_d \times L_e & \overset{a}{\longrightarrow} & L_{e+d} \\
\downarrow_{F_{M_d} \times F_e} & & \downarrow_{F_{e+d}} \\
L_{M_d} \times L & \overset{\varepsilon_1, \varepsilon_2}{\longrightarrow} & L.
\end{array}
\]

Since $M_d$ is a line bundle, $L_{M_d}(S)$ is a singleton for every scheme $S$, and the claim follows.

We have showed that the diagram

\[
\begin{array}{ccc}
M_\infty \times L_\infty & \overset{p}{\longrightarrow} & L_\infty \\
\downarrow_{a} & & \downarrow \\
\end{array}
\]

where $a$ is the action map and $p$ is the projection map, commutes. We would like to show that this is, in fact, a colimit diagram (after Zariski sheafification): i.e., $L$ is the coequalizer of $p$ and $a$. To put it differently, we have an isomorphism (after Zariski sheafification)

\[L_\infty/M_\infty \cong L.\]

Concretely, it is enough to prove the following:

**Lemma 3.9.** Let $S$ be an affine $k$-scheme and let $d \geq 0$. If $f, g \in L_d(S)$ are such that

\[F_\infty(f) = F_\infty(g),\]

then for every $s_0 \in S$ there exist an open neighborhood $O \subset S$ and some $m_1, m_2 \in M_\infty(O)$ such that

\[m_1 \cdot (f|_O) = m_2 \cdot (g|_O) \in L_\infty(O).\]

**Proof.** Let $d \geq 0$ and $f, g \in L_d(S)$ be represented by

\[W_f, W_g \prec \Gamma^X_\mathfrak{G} E_d\]

such that

\[F_\infty(f) = F_\infty(g).\]

As in the proof of Lemma 3.7 we can assume that $S$ is small enough such that $W_f, W_g \cong \mathcal{O}_S$. We can use the description of the map $F_d$ as in Remark 3.5. There is some domain

\[U \subset U_f \cap U_g,\]

where $U \subset S \times X$ is such that $\text{Image}(r_f|_U) = \text{Image}(r_g|_U)$. Thus there is an automorphism

\[a_h : O_U \to O_U\]

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defined by multiplying with some invertible function \(h \in \mathcal{O}_U^\times\) such that
\[
rf\mid_U = rg\mid_U \circ ah : \mathcal{O}_U \to \mathcal{O}_U^{2n}.
\]
We get that for \(1 \leq i \leq n\), we have
\[
f_i = g_i \cdot h.
\]
Let \(s_0 \in S\). Since \(f \in \mathcal{L}_d(S)\) there exists some \(1 \leq i \leq n\) for which \(f_i(s_0)\) is not the zero function (on \(X\)). We claim that \(g_i(s_0)\) is not the zero function either. Indeed, if \(g_i(s_0) = 0\), we have that for every \(1 \leq j \leq n\),
\[
g_j(s_0)f_i(s_0) = g_j(s_0)g_i(s_0) \cdot h(s_0) = g_i(s_0)f_j(s_0) = 0.
\]
But \(f_i(s_0)\) is not zero and so we get that \(g_j(s_0) = 0\) for all \(j\), which is not allowed.

Now there exists some open neighborhood \(s_0 \in O \subset S\) on which \(f_i\) (and hence \(g_i\)) are not the zero function. Define
\[
m_1 := g_i\mid_O, \ m_2 := f_i\mid_O \in \mathcal{M}_d(O).
\]
We see that on
\[
U' := U \cap (O \times X) \subset O \times X
\]
and for all \(1 \leq j \leq n\),
\[
f_j\mid_O m_1 = f_j\mid_O g_i\mid_O = g_j\mid_O g_i\mid_O h = g_j\mid_O f_i\mid_O = g_j\mid_O m_2;
\]
note that since \(U'\) is dense in \(O \times X\) we obtain the equality
\[
m_1 \cdot f_i\mid_O = m_2 \cdot g_i\mid_O \in \mathcal{L}_d(O).
\]

4 Realising Quotients by Monoids

By Lemma 3.9, \(\mathcal{L}\) is a quotient of \(\mathcal{L}_\infty\) by an action \(\mathcal{M}_\infty\). We would like to show that this action is “well behaved” in some sense.

**Definition 4.1.** Let \(M\) be a monoid and let \(A\) be a set with an \(M\)-action. We say that \(M\) acts **freely** on \(A\) if for all \(m, n \in M\) and \(a \in A\),
\[
ma = na \Rightarrow m = n.
\]

**Lemma 4.2.** For every \(S \in \mathfrak{Aff}\) the action of \(\mathcal{M}_\infty(S)\) on \(\mathcal{L}_\infty(S)\) is free.

**Proof.** The claim that the action is free is equivalent to the claim that the diagram of ind-schemes

\[
\begin{tikzcd}
\mathcal{M}_\infty \times \mathcal{L}_\infty \ar[r, \Delta_{\mathcal{M}_\infty} \times \Delta_{\mathcal{L}_\infty}] & \mathcal{M}_\infty \times \mathcal{M}_\infty \times \mathcal{L}_\infty \ar[rr, a_2 = a \circ (\pi_2 \times \pi_3)] & & \mathcal{L}_\infty \ar[ll, a_1 = a \circ (\pi_1 \times \pi_3)]
\end{tikzcd}
\]

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presents $\mathcal{M}_\infty \times \mathcal{L}_\infty$ as the equalizer of $a_1$ and $a_2$, where $a$ is the action of $\mathcal{M}_\infty$ on $\mathcal{L}_\infty$ and the $\pi_i$ are the obvious projections.

Since filtered colimits commute with finite limits, and the pre-sheaves $\mathcal{M}_d$ and $\mathcal{L}_d$ are representable for $d < \infty$, it is enough to show that for every $d < \infty$, the diagram

$$
\begin{aligned}
\mathcal{M}_d \times \mathcal{L}_d & \to \mathcal{M}_d \times \mathcal{M}_d \times \mathcal{L}_d \\
\Delta_{\mathcal{M}_d \times \mathcal{L}_d} & \to \mathcal{L}_d
\end{aligned}
$$

is an equalizer diagrams of schemes. The claim is easy to verify (note that all the $\mathcal{M}_d$’s and $\mathcal{L}_d$’s are just projective spaces of various dimensions). \hfill \square

We now turn to deal with a general sub-pre-sheaf $i: B \subset \mathcal{L}$.

**Proposition 4.3.** Given a sub-pre-sheaf $i: B \subset \mathcal{L}$, we have

1. $i^B_d: B_d \subset \mathcal{L}_d$ is a sub-pre-sheaf,
2. $B_\infty = \text{colimit} B_d$,
3. the map $F^B_\infty: B_\infty \to B$ is a Zariski local epimorphism (Zariski surjection),
4. the action $\mathcal{M}_\infty$ on $\mathcal{L}_\infty$ restricts to an action of $\mathcal{M}_\infty$ on $B_\infty$,
5. the action of $\mathcal{M}_\infty$ on $B_\infty$ respects the fibers of $F^B_\infty$,
6. the action of $\mathcal{M}_\infty(S)$ on $B_\infty(S)$ is free for every $S \in \mathfrak{Aff}$,
7. we have an isomorphism (after Zariski sheafification)

$$
B_\infty / \mathcal{M}_\infty \cong B.
$$

**Proof.** We already proved all the results above for $B = \mathcal{L}$ and the general case easily follows from the properties of sub-pre-sheafs and pullbacks. \hfill \square

For every $S \in \mathfrak{Aff}$ there is an element $\phi_s \in \mathcal{L}(S)$ corresponding to the line-bundle on $S \times X$ that is generated by the first coordinate in the trivial vector bundle of rank $n$ over $X \times S$. We thus get that $\mathcal{L}(S)$ is naturally pointed. So we have a sub-pre-sheaf $\mathcal{C} \subset \mathcal{L}$ such that $\mathcal{C}(S) = \{\phi_s\}$.

We would like to understand the pullback

$$
\begin{array}{ccc}
\mathcal{C} & \to & \mathcal{C} \\
\downarrow & & \downarrow \\
\mathcal{L} & \to & \mathcal{L}
\end{array}
$$

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It is easy to see that the inclusion
\[ C_d \subset L_d = \mathbb{P}(\Gamma(X, \mathcal{O}_X(dD)^{2n})) \]
is represented by the closed sub-scheme defined by setting the last \( n-1 \) functions in \( \mathbb{P}(\Gamma(X, dD)^{2n}) \) to zero. Thus we get an \( \mathcal{M}_\infty \) equivariant isomorphism
\[ \mathcal{M}_\infty \cong C_\infty. \]

4.1 The bar construction for a monoid action

In the previous subsection we presented the pre-sheafs in which we are interested as a quotient of an ind-scheme by the action of a monoid. In this subsection we study a way to describe this kind of quotient by a suitable bar construction.

**Definition 4.4.** Let \( M \) be a commutative monoid and let \( A \) be a set with an \( M \)-action. We say that \( M \) acts \textit{w-transitively} on \( A \) if for all \( a, b \in A \), there exist \( m, n \in M \) such that \( ma = nb \).

**Lemma 4.5.** Let \( M \) be a commutative monoid and let \( A \) be a set with an \( M \)-action. Consider the simplicial set \( B(M, A) \) defined by
\[ B(M, A)_n = M^n \times A, \]
with face and degeneracy maps as in the classical bar construction for groups. Assume further that \( M \) acts freely on \( X \). Then the natural map
\[ B(M, A) \to A/M \]
induces a weak equivalence after geometric realization.

**Proof.** First we can write \( A \) as a coproduct of \( M \)-sets on which \( M \) acts \( w \)-transitively. Since geometric realization commutes with coproduct, we can assume that \( M \) acts \( w \)-transitively on \( A \) and that \( A \) is non-empty. In this case we need to show that \( B(M, A) \) is contractible. Note that \( B(M, A) \) is exactly the nerve of the category \( A//M \) when
\[ \text{Ob} A//M := A, \]
\[ \text{Mor}_{A//M}(a, b) := \{ m \in M | ma = b \}. \]
Note that since \( M \) acts freely we have \( |\text{Mor}_{A//M}(a, b)| \leq 1 \) for all \( a, b \in A//M \); that is, \( A//M \) is a poset. Further, by \( w \)-transitivity—together with non-emptyness—\( A//M \) is a directed poset. Thus \( B(M, A) \) is contractible, as it is the nerve of a directed poset. \( \square \)

**Corollary 4.6.** Let \( F : \mathfrak{Aff} \to \text{Set} \) be a pre-sheaf of sets that has an action of \( \mathcal{M}_\infty \) (as a pre-sheaf of sets) and assume that the action of \( \mathcal{M}_\infty(S) \) on \( F(S) \) is free for every \( S \in \mathfrak{Aff} \). Then we have
\[ \tilde{(F/\mathcal{M}_\infty)} \sim \text{colimit}_{\Delta_{\text{op}}} \tilde{\mathcal{M}_\infty^n} \times \tilde{F}. \]
Note that the weak equivalence above is true when evaluated on every \( S \in \mathfrak{M} \) and thus in any Grothendieck topology as well.

In light of Corollary 4.6 and Proposition 4.3, we get that for every sub-pre-sheaf \( \mathcal{B} \subset \mathcal{L} \) we have an equivalence after Zarisky sheafification

\[
\tilde{\mathcal{B}} \simeq \text{colimit}_{\Delta^\text{op}} \tilde{\mathcal{M}}_\infty^n \times \tilde{\mathcal{B}}_\infty.
\]

Since \( \langle \langle \cdot \rangle \rangle \) commutes with homotopy colimits and is symmetric monoidal, we also have

\[
\langle \langle \mathcal{B} \rangle \rangle \sim \text{colimit}_{\Delta^\text{op}} \langle \langle \mathcal{M}_\infty \rangle \rangle^n \otimes \langle \langle \mathcal{B}_\infty \rangle \rangle.
\]

Thus to prove Proposition 3.6 (2), note that if the map

\[
\langle \langle \mathcal{B}_\infty \rangle \rangle \to \langle \langle \mathcal{L}_\infty \rangle \rangle
\]

is a weak equivalence, then so is the map

\[
\langle \langle \mathcal{B} \rangle \rangle \sim \text{colimit}_{\Delta^\text{op}} \langle \langle \mathcal{M}_\infty \rangle \rangle^n \otimes \langle \langle \mathcal{B}_\infty \rangle \rangle \sim \text{colimit}_{\Delta^\text{op}} \langle \langle \mathcal{M}_\infty \rangle \rangle^n \otimes \langle \langle \mathcal{L}_\infty \rangle \rangle \sim \langle \langle \mathcal{L} \rangle \rangle.
\]

To prove Proposition 3.6 (3), consider the case of \( \mathcal{B} = \mathcal{C} \cong * \). In this case we have

\[
\langle \langle \mathcal{C} \rangle \rangle \simeq \text{colimit}_{\Delta^\text{op}} \langle \langle \mathcal{M}_\infty \rangle \rangle^{n+1}.
\]

Now in order to prove the contractibility of \( \mathcal{L} \) (proven already in [Gal13b]) we can consider the diagram

\[
\begin{array}{ccc}
\langle \langle \mathcal{C} \rangle \rangle & \sim & \text{colimit}_{\Delta^\text{op}} \langle \langle \mathcal{M}_\infty \rangle \rangle^n \otimes \langle \langle \mathcal{M}_\infty \rangle \rangle \\
\downarrow & & \downarrow \\
\langle \langle \mathcal{L} \rangle \rangle & \sim & \text{colimit}_{\Delta^\text{op}} \langle \langle \mathcal{M}_\infty \rangle \rangle^n \otimes \langle \langle \mathcal{L}_\infty \rangle \rangle.
\end{array}
\]

Note that the map \( \langle \langle \mathcal{M}_\infty \rangle \rangle \to \langle \langle \mathcal{L}_\infty \rangle \rangle \) is a weak equivalence. Indeed, both spaces are homotopic to \( \mathbb{C} \mathbb{P}^\infty \) and so it is enough to check that the induced map

\[
\pi_2(\langle \langle \mathcal{M}_\infty \rangle \rangle) \to \pi_2(\langle \langle \mathcal{L}_\infty \rangle \rangle)
\]

in an isomorphism. This follows from the fact that an inclusion of a hyperplane in \( \mathbb{P}^n \) induces an isomorphism on \( \pi_2 \). Thus the right vertical map is a weak equivalence, therefore so is the left vertical map.
5 The case of opers and Oka’s theorem

5.1 Oka theorem’s and contractibility

To complete the proof of Theorem 3.1 we would like to show that the map
\[
\langle P_{\infty}\rangle \to \langle L_{\infty}\rangle
\]
is a weak equivalence. It will suffice to prove

**Proposition 5.1.** There exists some constant \( C \in \mathbb{N} \) (depending on \( E \)) such that for every \( d \gg 0 \) the map
\[
\langle P_d\rangle \to \langle L_d\rangle
\]
is \((2|D|d-C)\)-connected. When \( H = \mathbb{H} \), connectivity in \textbf{Vect} should be considered with respect to the standard t-structure.

**Proof.** First note that for \( d \gg 0 \) we have that
\[
L_d \cong \mathbb{P}^{2n-d|D|+O(1)}.
\]
The sub-pre-sheaf \( P_d \) is now given as a closed sub-scheme of \( L_d \) by a set of equations. Note that since the vector bundle \( E \cong \mathbb{O}^{2n}_{\mathbb{P}^n} \) is trivial, we can write the alternating form
\[
\langle , \rangle : E \cong \mathbb{O}^{2n}_X \wedge \mathbb{O}^{2n}_X \to \mathbb{O}_X
\]
as a matrix
\[
M_{\langle , \rangle} \in M_{2n}(k[X]).
\]
There is some constant \( C_1 \) such that all the entries of \( M_{\langle , \rangle} \) lie in \( k^{C_1}[X] \). Moreover, since \( \omega_X \cong \mathbb{O}_X \) is generated by \( \nu^\vee \), the non-\( \mathbb{O}_X \) linear map
\[
\nabla_\nu : \mathbb{O}^{2n}_X \to \mathbb{O}^{2n}_X
\]
can be written as
\[
\nabla_\nu(f_1, \ldots, f_{2n}) = (\nu(f_1), \ldots, \nu(f_{2n})) + (f_1, \ldots, f_{2n}) \cdot A
\]
for some matrix
\[
A \in M_{2n}(k[X]).
\]
Note that there is some constant \( C_2 \) such that all the entries of \( A \) lie in \( k^{C_2}[X] \). Recall that there is some constant \( d_\nu \) such that the map
\[
\nu : k[X] \to k[X]
\]
restricts for every \( d \) to a map of the form
\[
\nu : k^{d}[X] \to k^{d+d_\nu}[X].
\]
Thus, given a vector \( v \in k^d[X]^{2n} \), there exists a constant \( C_3 \) such that all the elements
\[
\langle v, \nabla_\nu(v) \rangle, \ldots, \langle \nabla_\nu^{(n-2)}v, \nabla_\nu^{(n-1)}v \rangle \in k[X]
\]
lie in \( k^{2d+C_3}[X] \). So, by Proposition 2.2 \( \mathcal{P}_d \) is defined inside \( \mathcal{L}_d \) by \( (n - 1) \cdot 2 \cdot d \cdot |D| + O(1) \) equations. This gives us a dimension of \( 2d|D| + O(1) \). If \( \mathcal{P}_d \) is a smooth complete intersection inside \( \mathcal{L}_d \), then by the celebrated Lefschetz hyperplane theorem we are done. In the case where \( \mathcal{P}_d \) is not smooth, we turn to Oka’s following generalization of the Lefschetz hyperplane theorem, which holds in \( \mathbb{P}^m \) without any assumptions on the defining equations.

**Lemma 5.2** ([Kat77], [Oka73]). Let \( V \) be a complex algebraic set in complex projective \( N \)-space. Suppose that \( V \) is defined by \( r \) homogeneous polynomials. Then the pair \( (\mathbb{P}^N, V) \) is \( (N - r) \)-connected.

**Corollary 5.3.** Let \( V \) be a \( k \)-algebraic set in projective \( N \)-space over \( k \) and let \( H \) be one of our three realization functors as above. Suppose that \( V \) is defined by \( r \) homogeneous polynomials. Then the map
\[
H(V) \to H(\mathbb{P}^N)
\]
is \( (N - r) \)-connected.

**Proof.** First note that since only finitely many coefficients appear in the definition of \( k \) we can assume that \( k \) is of finite transcendence degree and thus admits an inclusion \( \sigma : k \to \mathbb{C} \) and then we can apply Lemma 5.2 directly for \( H = \text{Cl}_\sigma \). Now the cases of \( H = \mathbb{H} \) follows from the case \( H = \text{Cl}_\sigma \), Remark 1.8 and the Hurwitz Theorem. The case \( H = \text{Et} \) follows from the case \( H = \text{Cl}_\sigma \), Remark 1.8 and [AM06, 6.2].

**5.2 Other groups**

Only in last step of the proof the main theorem, while using Oka’s lemma, do we actually use the fact that our group is classical. The crux of the matter is that the condition to be an oper consists of \( n - 1 \) equations of degree 2 in a space of dimension \( 2n \). Since
\[
2 \cdot (n - 1) < 2n,
\]
the connectivity of our embedding goes to \( \infty \) as \( d \to \infty \). Similarly, in analogy to Proposition 2.2 we have:

**Proposition 5.4.** Let \( \mathcal{E} = (E, \nabla, \langle, \rangle) \) be an \( SO(2n) \)-local or \( SO(2n + 1) \)-local system on \( X \) and assume that \( \mathcal{E} \) is irreducible. Let \( S \) be affine and
\[
L = F_1 \prec E_U
\]
be some element in \( \mathcal{L}_\mathcal{E}(S) \). Then \( L \in \mathcal{P}_\mathcal{E} \) iff for every \( 0 \leq i \leq n - 2 \),
\[
\langle \nabla_\nu^{(i)}L, \nabla_\nu^{(i)}L \rangle = 0.
\]
The idea of the proof is very similar, proving that $\langle \nabla_i^{(i)} L, \nabla_j^{(j)} L \rangle = 0$ by induction on $|i-j|$. Thus we get that for $SO(2n)$ we have again $n-1$ equations of degree 2 in a space of dimension $2n$. For $SO(2n+1)$ the situation is even better, as we have $n-1$ equations of degree 2 in a space of dimension $2n+1$. However, we failed to get a suitable set of equations for non-classical groups. For example, for flags in $G_2$ one can use the analysis in [And09] to produce two quadratic equations and one cubic equation in a space of dimension 7. Alas, as $2+2+3=7$ this strategy fails.

5.3 A higher Tsen’s theorem

Note that the method given in the proof above is quite general. We will demonstrate that it gives us a homotopical strengthening of Tsen’s theorem about rational points over function fields of curves over an algebraically closed field. Recall that Tsen’s theorem is as follows.

**Theorem 5.5 ([Tse33]).** Let $k$ be an algebraically closed field and let $\bar{X}$ be a smooth projective curve defined over $k$. Let

$$Y_{k(X)} \subset \mathbb{P}^n_{k(X)}$$

be a closed sub-variety defined by an intersection of $r$ homogeneous forms of degrees $d_1, \ldots, d_r$. Assume further that

$$\sum d_i \leq n.$$

Then

$$Y(k(X)) \neq \emptyset.$$

We would like to show that $Y(k(X))$ is not only non-empty, but also contractible. To make sense of this claim, choose some proper map

$$\mathcal{Y} \to \bar{X}$$

with the generic fibre of the map isomorphic to $Y$.

**Definition 5.6.** Let $S \in \mathcal{A}ff$ be an affine scheme and let $U \subset S \times X$ be a domain. An $S$-family of sections of $\mathcal{Y} \to \bar{X}$ supported on $U$, denoted by $\text{Sec}_\mathcal{Y}(S,U)$, is a map $U \to \mathcal{Y}$ such that the diagram

$$
\begin{array}{ccc}
\mathcal{Y} & \to & \mathcal{Y} \\
\downarrow & & \downarrow \\
U & \xrightarrow{\pi_U} & X
\end{array}
$$

commutes.
We also denote
\[
\Sec_Y(S) := \colim_{U \in S \times X} \Sec_Y(S, U).
\]
Note that \(\Sec_Y(S)\) depends only on \(Y\) and not on \(Y\). Then
\[
\Sec_Y : \affop \to \sets
\]
is a pre-sheaf. Following the same method of proof as before, we obtain

**Theorem 5.7.** Let \(k\) be an algebraically closed field and let \(X\) be a smooth projective curve defined over \(k\). Let
\[
Y_{k(X)} \subset \mathbb{P}^n_{k(X)}
\]
be a closed sub-variety defined by an intersection of \(r\) homogenous forms of degrees \(d_1, \ldots, d_r\). Assume further that
\[
\sum d_i \leq n.
\]
Then
\[
\langle \langle \Sec_Y \rangle \rangle \sim \ast.
\]

Again, we can present \(\Sec_Y\) as a quotient of some \((\Sec_Y)_\infty\) by the monoid \(\mathcal{M}_\infty\). The case \(Y = \mathbb{P}^n_{k(X)}\) is trivial, as one naturally gets that \((\Sec_{\mathbb{P}^n_{k(X)}})_\infty\) is isomorphic to \(\mathbb{P}_\infty\) as well. We now filter \((\Sec_Y)_\infty\) by a filtration \((\Sec_Y)_e\) which corresponds to sections with poles of degree \(e\) at some fixed point \(\infty \in X\). We then see that
\[
(\Sec_Y)_e \subset (\Sec_{\mathbb{P}^n_{k(X)}})_e
\]
is given as a zero locus of \(r(e) = e(\sum d_i) + O(1)\) equations in a space of dimension \(N(e) = e(n + 1) + O(1)\). Since \(N(e) - r(e)\) goes to infinity as \(e\) does, we can use Corollary 5.3 and get the desired contractibility.

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