CONVERGENCE RATES OF NONLINEAR INVERSE PROBLEMS IN BANACH SPACES VIA HÖLDER STABILITY ESTIMATES

Ankik Kumar Giri and Gaurav Mittal

Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee, India;

ABSTRACT. In this paper, we analyze the convergence rates of Tikhonov regularization for solving the nonlinear ill-posed problems by using the Hölder stability estimates as the smoothness condition. We obtain the convergence rates via two different approaches. The first approach is the standard one which is to obtain the convergence rates in terms of Bregman distance and the second one is to obtain the convergence rates in weaker norms. The important aspect in the second approach is that the regularization is only used to constrain the regularized solutions to a set where stability holds.

1. Introduction

Let \( F : D(F) \subset U \rightarrow V : F(u) = v \) be a nonlinear operator between the Banach spaces \( U \) and \( V \) equipped with norms \( \| \cdot \|_U \) and \( \| \cdot \|_V \) respectively with domain \( D(F) \). Suppose \( U^* \) and \( V^* \) be the respective dual spaces of \( U \) and \( V \) and \( \langle \cdot , \cdot \rangle_{U^* \times U} \) be the dual pairing between \( U^* \) and \( U \). We consider the nonlinear ill-posed problems governed by the operator equation

\[
F(u) = v, \quad v \in V
\]

and our main motive is to find the exact solution \( u^\dagger \) of the above equation. Since in practice the exact data is not always available, we have to restrict ourselves to find the stable approximations of the exact solution. Usually, we use the variational regularization (in particular, Tikhonov regularization) and iterative methods for finding the stable approximations. For the Hilbert spaces, widely used method is

\[
\|F(u) - v'\|^2 + \alpha \|u - u_0\|^2, \quad \alpha > 0
\]

where \( v' \) is some noisy approximation of \( v \) and \( u_0 \) is some initial approximation of the exact solution. The best possible rates which can be achieved with Tikhonov regularization in Hilbert spaces was shown in [21]. Because of its tendency to smooth the solutions, Tikhonov regularization in Hilbert spaces does not yield satisfactory results especially if there are jumps or sparsity in the structure of the exact solution. So, other types of Tikhonov regularization were explored in the Banach spaces in recent years [2]. Further, besides the convergence and stability of the regularized solutions, the next and the challenging step is the determination of convergence rates which in particular demonstrates the speed of the convergence of regularized solutions to the exact solution. To obtain the estimates for the speed, some kind of smoothness of the exact
solution has to be employed. The determination of the abovesaid convergence rates have a very long tradition in inverse problems (see e.g. [17, 5, 7, 18, 10, 13, 11, 3, 2, 12, 8]). In this work, we are particularly interested in the convergence rates of regularization methods for solving the nonlinear ill-posed inverse problems in Banach spaces. We consider the following Tikhonov regularization method which consists in the minimization of the functional

$$T_\alpha(u, \delta) = \| F(u) - v^\delta \|^p + \alpha R(u)$$

(1.2)

with $1 < p < \infty$, $R : U \rightarrow [0, \infty)$ is a convex and proper stabilizing functional, $\alpha$ is the regularization parameter and $v^\delta$ is some noisy approximation of $v$ satisfying

$$\| v^\delta - v \| \leq \delta$$

(1.3)

The domain of $R$ is $D(R) = \{ u \in U : R(u) \neq \infty \}$ and $D(R) \neq \emptyset$ as $R$ is assumed to be a proper functional. An element $u^\dagger \in D = D(F) \cap D(R)$ is called an $R$-minimizing solution of (1.1) if it minimizes the functional $R$ and also satisfies (1.1), i.e.,

$$R(u^\dagger) = \min\{ R(u) : F(u) = v \}$$

In practice, convergence rates can be determined with two different approaches. First one is on the basis of source and non-linearity conditions. See for instance [13, 2, 12, 3] for variational regularization (in particular Tikhonov regularization) and see for instance [8, 12] for Iterative regularization. The other one is solely on the basis of stability estimates which have been derived in [4] for variational regularization methods and in [1] for iterative regularization (in particular Landweber iteration method) in Banach spaces. These two concepts, however, are interconnected. The series of existing stability estimates which have developed independently in the community of inverse problems is more reliable than the first one which is on the basis of source and non-linearity conditions: The results of Logarithmic stability results can be found in [6, 19], in particular in [9] for Electrical Impedance Tomography and results of Hölder type stability estimates can be found in [14, 15].

Further, for the theory of linear inverse problems, the source conditions are optimal [20] in the sense that Hölder type stability estimates between residuum and error of the regularized solution are equivalent to that the regularized solutions satisfy some source condition. For the non-linear case the optimality of source conditions is not obvious [13].

This paper is structured in the following way: All the elementary results and definitions required in our framework are accumulated in Section 2. The third Section comprises of our main result in the form of Theorem 3.1. All the required assumptions are also discussed in the same section and moreover a convergence rate in terms of weaker norm for the special case of Hölder’s estimate is discussed in Theorem 3.2. In the Section 4, we give an example where our results on the convergence rates in terms of weaker norms can be applied.
2. Preliminaries and assumptions

For the approximation methods, error estimates are analyzed with respect to the Bregman distances instead of Lyapunov functionals [20]. So, the notion of Bregman distance is very important for variational regularization methods in Banach spaces.

2.1. Bregman distance. Let $R : U \to [0, \infty]$ (it can take infinite value) be a convex and proper functional having subdifferential $\partial R$. Then the Bregman distance with respect to $R$ for two elements $u$ and $\bar{u}$ from $D(R)$ and $\zeta \in \partial R(u) \subset U^*$ is defined by

$$D_\zeta(\bar{u}, u) = R(\bar{u}) - R(u) - \langle \zeta, \bar{u} - u \rangle_{U^*, U} \quad (2.1)$$

where $\langle \cdot, \cdot \rangle_{U^*, U}$ denotes the dual pairing with respect to $U^*$ and $U$. From definition, it is clear that $D_\zeta(\bar{u}, u)$ is defined at $u$ only if $\partial R(u) \neq \emptyset$. Let $D_B$ denotes the Bregman domain and it consists of all the points $u \in D(R)$ where $\partial R(u) \neq \emptyset$.

Next lemma is a basic result which will be used in the proof of our main theorem. For proof see [2, lemma 3.20].

Lemma 2.1. For a normed space $U$ and $p \geq 1$, we have

$$\|u_1 + u_2\|^p \leq \|u_1\|^p + \|u_2\|^p$$

for $u_1, u_2 \in U$.

Now to assure the well-posedness, convergence, and stability of the Tikhonov regularized solutions of (1.2), we make the following assumptions.

2.2. Assumptions.

(1) $\tau_U$ and $\tau_V$ are the topologies associated with the Banach spaces and these are weaker than the norm topologies.

(2) The norm $\| \cdot \|_V$ is sequentially lower semi-continuous with respect to $\tau_V$.

(3) The functional $R : U \to [0, \infty]$ is convex as well as sequentially lower semi-continuous with respect to $\tau_U$.

(4) $F : D(F) \subset U \to V$ is sequentially lower semi-continuous with respect to $\tau_U$ and $\tau_V$, i.e., for $u_n, u \in D(F)$ and $u_n \rightharpoonup u$ implies $F(u_n) \rightharpoonup F(u)$.

(5) $D = D(F) \cap D(R) \neq \emptyset$.

(6) The level sets

$$M_\alpha(K) = \{u \in D : T_\alpha(u, 0) \leq K\} \quad (2.2)$$

for every $\alpha > 0$ and $K > 0$ are sequentially closed with respect to $\tau_U$, i.e. $u_n \rightharpoonup u$ with $u_k \in M_\alpha(K)$ and $u \in U$ implies $u \in M_\alpha(K)$.

The next lemma is a result about the existence, stability and convergence of the Tikhonov minimizers provided the assumption (2.2) holds.
Lemma 2.2. Let the assumption (2.2) holds. Then the minimizer of the Tikhonov functional (1.2) exists and is also stable. Further, if in addition to assumption (2.2), (1.1) has a solution in $D$ and $\alpha : (0, \infty) \to (0, \infty)$ satisfies

\[ \alpha(\delta) \to 0 \quad \text{and} \quad \frac{\delta^\rho}{\alpha(\delta)} \to 0, \quad \text{as} \quad \delta \to 0 \]

then the minimizers converges in the sense of theorem (3.26) in [2].

Proof. See [2, chapter 3].

Since it is already mentioned in the introduction section that some kind of smoothness of the exact solution has to be employed to get the convergence rates, so next we recall the variational inequality introduced in [16] for getting the convergence rates.

Definition 2.1. An $R$-minimizing solution $u^\dagger$ satisfies a variational inequality, if there exists a $\zeta \in \partial R(u^\dagger)$ and constants $\rho > 0$, $\alpha_{\text{max}} > 0$, $\beta_1 \in [0,1)$, $\beta_2 \geq 0$ and $k > 0$ such that

\[ \langle \zeta, u^\dagger - u \rangle_{U^*,U} \leq \beta_1 D_\zeta(u,u^\dagger) + \beta_2 \| F(u) - F(u^\dagger) \|^k \] (2.3)

holds for all $u \in M_{\alpha_{\text{max}}}(\rho)$ and $\rho > \alpha_{\text{max}}R(u^\dagger)$.

It was also shown in [2, Remark 3.36] that for Hilbert spaces and $R(u) = \| u - u_0 \|^2$, (2.3) with $k = 1$ is a generalization of the source-wise representation $(u^\dagger - u_0 = F'(u^\dagger)^* w$ for some $w \in V$) of the exact solution.

Now the next question was what if the variational inequality (2.3) is not satisfied. So, the good remedy for this problem was given in terms of the introduction of approximate variational inequality in [16]. The idea was to measure the violation of (2.3). Mathematically, if there exists at least one $u \in M_{\alpha_{\text{max}}}(\rho)$ with

\[ \langle \zeta, u^\dagger - u \rangle_{U^*,U} > \beta_1 D_\zeta(u,u^\dagger) + \beta_2 \| F(u) - F(u^\dagger) \|^k \]

See [16] for more details on approximate variational inequality. Flemming in [22] considered the above-said smoothness concepts (variational inequality, source conditions, and approximate variational inequality) and their cross connections in Hilbert spaces.

To end this section, we define the Hölder type stability of the inverse mapping between the Banach spaces. The idea is gathered from [1]. We will find the convergence rates with the help of these stability estimates in the next section.

Definition 2.2. An $R$-minimizing solution $u^\dagger$ satisfies a Hölder stability estimate, if there exists constants $C > 0$, $\rho > 0$ and $k > 0$ such that

\[ D_\zeta(u,u^\dagger) \leq C \| F(u) - F(u^\dagger) \|^k \quad \forall u, u^\dagger \in M_{\alpha_{\text{max}}}(\rho). \] (2.4)
3. Convergence rates result using Hölder stability estimates

In this section, we obtain the convergence rates results for Tikhonov regularization of the type (1.2) using the Hölder stability estimates (2.4) in terms of the Bregmann distance. The analysis is solely on the basis of Hölder continuity of the inverse mapping which opposes the standard approach of obtaining the convergence rates via source and non-linearity conditions.

3.1. Convergence rate in terms of Bregman distance.

**Theorem 3.1.** Let $F$ be a non-linear operator between the Banach spaces $U$ and $V$. Moreover, let the following assumptions holds:

1. $u^\dagger$ is an $R$-minimizing solution of (1.1).
2. Assumption (2.2) holds and $p > 1$.
3. For a given $\alpha_{\text{max}} > 0$ and a constant $c > 0$, set
   \[ \rho = \alpha_{\text{max}}(c + R(u^\dagger)) \] (3.1)
4. $F : D(F) = M_{\alpha_{\text{max}}}(\rho_1) \subseteq U \to V$ satisfies the Hölder stability estimate
   \[ D_\zeta(u_1, u_2) \leq C\|F(u_1) - F(u_2)\|^k \quad \forall u_1, u_2 \in M_{\alpha_{\text{max}}}(\rho_1). \] (3.2)
   where $C > 0$ is some constant and $0 < k \leq 1$ and $\rho_1 = 2^{p-1}\rho$.

Then, if $\alpha = \alpha(\delta)$ is such that $0 < \alpha(\delta) \leq \alpha_{\text{max}}$, \(\frac{2\delta p}{\alpha} \leq c\) and $\alpha(\delta) \sim \delta^p$, for $\delta \to 0$ we have
\[ D_\zeta(u^\delta_\alpha, u^\dagger) = O(\delta^k) \] (3.3)
where $u^\delta_\alpha$ is the minimizer of the Tikhonov functional $T_\alpha(u, \delta)$ (see (1.2)).

**Proof.** First of all, we claim that $u^\delta_\alpha$ defined in the theorem is in $M_{\alpha_{\text{max}}}(\rho_1)$ for $\rho_1 = 2^{p-1}\rho$. From the definition of $T_{\alpha_{\text{max}}}(u, 0)$, we have
\[ T_{\alpha_{\text{max}}}(u^\delta_\alpha, 0) = \|F(u^\delta_\alpha) - v\|^p + \alpha_{\text{max}}R(u^\delta_\alpha) \]
\[ \leq 2^{p-1}(\|F(u^\delta_\alpha) - v\|^p + \delta^p) + \alpha_{\text{max}}R(u^\delta_\alpha) \]
where the last inequality follows from the lemma 2.1 and (1.3). Further, as $p > 1$, above can also be written as
\[ T_{\alpha_{\text{max}}}(u^\delta_\alpha, 0) \leq 2^{p-1}(\|F(u^\delta_\alpha) - v\|^p + \alpha R(u^\delta_\alpha) + \delta^p + \alpha_{\text{max}}R(u^\delta_\alpha) - \alpha R(u^\delta_\alpha)) \]
\[ = 2^{p-1}(T_\alpha(u^\delta_\alpha, \delta) + \delta^p + \alpha_{\text{max}}R(u^\delta_\alpha) - \alpha R(u^\delta_\alpha)) \]
\[ \leq 2^{p-1}(T_\alpha(u^\dagger, \delta) + \delta^p + \alpha_{\text{max}}R(u^\delta_\alpha) - \alpha R(u^\delta_\alpha)) \] (3.4)
where the last inequality holds by definition of $u^\delta_\alpha$. Again, by definition of $T_\alpha(u^\dagger, \delta)$, (3.4) with (1.3) implies
\[ T_{\alpha_{\text{max}}}(u^\delta_\alpha, 0) \leq 2^{p-1}(2\delta^p + \alpha R(u^\dagger) + (\alpha_{\text{max}} - \alpha)R(u^\delta_\alpha)) \] (3.5)
Further, for estimating the right side of (3.5), quickly, we find an estimate for $R(u^\delta_\alpha)$. For that, as we know that $u^\delta_\alpha$ is a minimizer of (1.2), we get 

$$\|F(u^\delta_\alpha) - v^\delta\|^p + \alpha R(u^\delta_\alpha) \leq \|F(u^\dagger) - v^\delta\|^p + \alpha R(u^\dagger)$$

By using the non-negativity of norm and (1.3), we get 

$$\alpha R(u^\delta_\alpha) \leq \delta^p + \alpha R(u^\dagger)$$

(3.6)

Incorporating (3.6) in (3.5) yields 

$$T_{\alpha_{\max}}(u^\delta_\alpha, 0) \leq 2^{p-1}\left[2\delta^p + \alpha R(u^\dagger) + (\alpha_{\max} - \alpha) \left(\frac{\delta}{\alpha} + R(u^\dagger)\right)\right]$$

$$= 2^{p-1}\left(\delta^p + \alpha\frac{\delta}{\alpha} + \alpha_{\max} R(u^\dagger)\right)$$

$$= 2^{p-1}\alpha_{\max}\left(\frac{\delta}{\alpha} + \frac{\delta}{\alpha} + R(u^\dagger)\right)$$

Now as $\alpha_{\max} \geq \alpha$, above equation can also be written as 

$$T_{\alpha_{\max}}(u^\delta_\alpha, 0) \leq 2^{p-1}\alpha_{\max}\left[\frac{\delta}{\alpha} + R(u^\dagger)\right] \leq 2^{p-1}\alpha_{\max}\left[c + R(u^\dagger)\right]$$

Thus, by (3.1) our claim holds, i.e. 

$$u^\delta_\alpha \in M_{\alpha_{\max}}(\rho_1)$$

(3.7)

where $\rho_1 = 2^{p-1}\rho$. Further, observe that $u^\dagger \in M_{\alpha_{\max}}(\rho)$. To prove this argument, use the definition of Tikhonov functional $T_{\alpha}(u, 0)$ and (3.1) to get 

$$T_{\alpha_{\max}}(u^\dagger, 0) = \|F(u^\dagger) - v\|^p + \alpha_{\max} R(u^\dagger)$$

$$< \alpha_{\max}(c + R(u^\dagger)) = \rho$$

The arguments (3.7) and $u^\dagger \in M_{\alpha_{\max}}(\rho) \subset M_{\alpha_{\max}}(\rho_1)$ implies that (3.2) is applicable for $u_1 = u^\delta_\alpha$ and $u_2 = u^\dagger$. So (3.2) yields 

$$D^\zeta(u^\delta_\alpha, u^\dagger) \leq C\|F(u^\delta_\alpha) - F(u^\dagger)\|^k \leq 2^k C\left(\|F(u^\delta_\alpha) - y^\delta\|^k + \delta^k\right)$$

$$\leq 2^k\left(\|F(u^\delta_\alpha) - y^\delta\|^k + \delta^k\right)$$

(3.8)

where the above inequality is obtained by using estimate (1.3) and the inequality 

$$\|u + v\|^k \leq (2\max\{\|u\|, \|v\|\})^k = 2^k\max\{\|u\|^k, \|v\|^k\} \leq 2^k\{\|u\|^k + \|v\|^k\}$$

(3.9)

for $u, v \in U$ and $k \in (0, 1)$. So to obtain the estimate for $D^\zeta(u^\delta_\alpha, u^\dagger)$, it is clear from (3.8) that we need a bound on $\|F(u^\delta_\alpha) - y^\delta\|$. For this again using the definition of $u^\delta_\alpha$, (1.2) and (1.3) to get 

$$T_{\alpha}(u^\delta_\alpha, \delta) = \|F(u^\delta_\alpha) - v^\delta\|^p + \alpha R(u^\delta_\alpha) \leq \|F(u^\dagger) - v^\delta\|^p + \alpha R(u^\dagger)$$

$$\leq \delta^p + \alpha R(u^\dagger)$$

Now as $R$ is a non-negative functional, we obtain 

$$\|F(u^\delta_\alpha) - v^\delta\|^p \leq \delta^p + \alpha R(u^\dagger)$$
where the last inequality is obtained by using (3.1). Now putting (3.10) in (3.8) yields

\[ D_\zeta(u_\alpha^\delta, u^\dagger) \leq 2C\left(\|F(u_\alpha^\delta) - v^\delta\|^k + \delta^k\right) \leq 2C\left[\frac{\delta^p}{\alpha} + \frac{\rho}{\alpha_{\text{max}}}\right]^{\frac{k}{p}} + \delta^k \]

Further as \( \frac{\delta^p}{\alpha} \leq c \), we get

\[ D_\zeta(u_\alpha^\delta, u^\dagger) \leq 2C\left[\frac{c}{2} + \frac{\rho}{\alpha_{\text{max}}}\right]^{\frac{k}{p}} + \delta^k \]

Finally, for the a-priori choice of \( \alpha = \alpha(\delta) = \delta^p \), above implies

\[ D_\zeta(u_\alpha^\delta, u^\dagger) = O(\delta^k) \]

\[ \square \]

**Remark 3.1.** The convergence rates obtained in the theorem (3.1) are similar to the rates obtained in [2, theorem 3.42] for \( p = 1 \).

Now, it is phenomenal to compare the inequalities (2.3) and (2.4) and to find the relation between them (if there is any).

**Theorem 3.2.** Let \( u \in M_{\alpha_{\text{max}}}^\rho \) and it satisfies the estimate (2.3). Also, let \( \zeta \in \partial R(u^\dagger) \). Then \( u \) satisfies the estimate (2.4) with \( C = \frac{\beta_2}{1-\beta_1} \) provided \( R(u) \leq R(u^\dagger) \).

**Proof.** From the definition of Bregmann distance and (2.3), we get

\[ D_\zeta(u, u^\dagger) = R(u) - R(u^\dagger) - \langle \zeta, u - u^\dagger \rangle_{U^*, U} \]

\[ \leq R(u) - R(u^\dagger) + \beta_1 D_\zeta(u, u^\dagger) + \beta_2\|F(u) - F(u^\dagger)\|^k \]

Above can also be written as

\[ (1 - \beta_1)D_\zeta(u, u^\dagger) \leq R(u) - R(u^\dagger) + \beta_2\|F(u) - F(u^\dagger)\|^k \]

Thus, result holds under the assumed condition. \( \square \)

Next proposition checks the validity of the estimate \( R(u) \leq R(u^\dagger) \) for \( u \in M_{\alpha_{\text{max}}}^\rho \).

**Proposition 3.1.** Let \( u \in U \) is such that \( u \in M_{\alpha_{\text{max}}}^\rho \), \( u^\dagger \) is R-minimizing solution of (1.1) and further

(1) if \( u \) is the minimizer of the Tikhonov functional \( T_\alpha(u, 0) \) for some \( \alpha > 0 \), then the condition \( R(u) \leq R(u^\dagger) \) holds obviously.

(2) if \( u \) is the minimizer of the Tikhonov functional \( T_\alpha(u, \delta) \) for some \( \alpha > 0 \), then the condition \( R(u) \leq R(u^\dagger) \) holds provided \( \|F(u) - v\| \geq \delta \).
Proof. For the first part as \(u\) is the minimizer of the Tikhonov functional \(T_\alpha(u,0)\), then from (1.2), we get
\[
\|F(u) - v\|^p + \alpha R(u) \leq \|F(u^\dagger) - v\|^p + \alpha R(u^\dagger)
\]
and therefore
\[
\alpha(R(u) - R(u^\dagger)) \leq -\|F(u) - v\|^p \leq 0
\]
which means \(R(u) \leq R(u^\dagger)\). For the second part, from (1.2), we get
\[
\|F(u) - v^\delta\|^p + \alpha R(u) \leq \|F(u^\dagger) - v^\delta\|^p + \alpha R(u^\dagger)
\]
and therefore on using the estimate (1.3), we get
\[
\alpha(R(u) - R(u^\dagger)) \leq -\|F(u) - v\|^p + \delta^p
\]
Now, from above equation right side is non-positive provided \(\delta^p \leq \|F(u) - v\|^p\) which further implies \(\|F(u) - v\| \geq \delta\).

\[\square\]

From above proposition, we conclude that for the non-noisy data, condition (2.3) is stronger than Hölder stability estimates (2.4) in the sense that former always implies the latter. In the case of noisy data, robustness of (2.3) is dependent on the noise level \(\delta\).

3.2. In terms of weaker norms. In this subsection, we make an attempt to find the convergence rates in terms of appropriate weaker norm and not in terms of Bregman distance. Here, the only motive of using the regularization is to restrict the regularized solutions to a set where the given Hölder stability estimate holds. We would also give an example in the next section in support of our result and the example shows the importance of these kind of convergence rates in weaker norms. The method consists in minimization of the functional
\[
T_\alpha^1(u, \delta) = \|F(u) - v^\delta\|^p_V + \alpha \|u - u_0\|^p_U
\]
where \(v^\delta\) satisfies (1.3) and \(u_0 \in U\).

**Theorem 3.3.** Let \(U\) and \(V\) be the Banach spaces. Moreover, let the following assumptions holds:

1. \(u^\dagger\) is an \(R\)-minimizing solution of (1.1) where \(R(u) = \|u - u_0\|^p_U\).
2. Assumption (2.2) holds with \(R(u) = \|u - u_0\|^p_U\).
3. \(u_0\) is in some neighborhood of \(u^\dagger\), i.e. there exists some \(K_1 > 0\) such that
   \[
   \|u_0 - u^\dagger\|_U \leq K_1
   \]
4. \(F: \text{dom}(F) \subset U \to V\) satisfies the Hölder stability estimate
   \[
   \|u_1 - u_2\|_\eta \leq C\|F(u_1) - F(u_2)\|^k
   \]

where \(\|\cdot\|_U\) and \(\|\cdot\|_V\) are appropriate norms on \(U\) and \(V\) respectively.
for $0 < k \leq 1$ and $\|u_1 - u_2\|_U \leq M$, where $M$ is some constant and $\|\cdot\|_\eta$ is an appropriate norm which induces a topology coarser than the topology induced by the norm topology on $U$.

Assuming $\alpha = \alpha(\delta) \to 0$ is such that $\frac{\delta^p}{\alpha} \leq c$ for some $c > 0$ and $M$ is such that $(c + K^p_1)\frac{1}{p} + K_1 \leq M$, then we have

$$\|u_\alpha^\delta - u^\dagger\|_\eta = O(\alpha^k)$$

where $u_\alpha^\delta$ is the minimizer of (3.11).

**Proof.** By definition of $u_\alpha^\delta$ and using (1.3), (3.12), we get

$$\|F(u_\alpha^\delta) - v^\delta\|_V^p + \alpha\|u_\alpha^\delta - u_0\|_U^p \leq \|F(u^\dagger) - v^\delta\|_V^p + \alpha\|u^\dagger - u_0\|_U^p$$

$$\leq \delta^p + \alpha K^p_1$$

Consequently, we have

$$\|F(u_\alpha^\delta) - v^\delta\|_V^p \leq \alpha \left( \frac{\delta^p}{\alpha} + K^p_1 \right) \leq \alpha(c + K^p_1)$$

(3.14)

and

$$\|u_\alpha^\delta - u_0\|_U^p \leq c + K^p_1$$

(3.15)

Then (3.12) and (3.15) implies

$$\|u_\alpha^\delta - u^\dagger\|_U \leq \|u_\alpha^\delta - u_0\|_U + \|u_0 - u^\dagger\|_U \leq M$$

Above estimate implies that (3.13) is satisfied with $u_1 = u_\alpha^\delta$ and $u_2 = u^\dagger$. Therefore, (3.13), (3.9) and (1.3) implies

$$\|u_\alpha^\delta - u^\dagger\|_\eta \leq C\|F(u_\alpha^\delta) - F(u^\dagger)\|_k \leq 2^k C \|F(u_\alpha^\delta) - v^\delta\|_k + \delta^k$$

Now (3.14) and choice of $\alpha \sim \delta^p$ yields

$$\|u_\alpha^\delta - u^\dagger\|_\eta = O(\alpha^{\frac{k}{p}})$$

□

4. INVERSE PROBLEM OF DETERMINING THE POTENTIAL FUNCTION

The main aim of this section is to discuss an example to verify the results of the theorem 3.2.

**Example 4.1.** In [14], the inverse problem of determining the potential function $q = q(x)$ from the Neumann to Dirichlet map $\Delta_q$ of the wave equation $u_{tt} - \Delta u + qu = 0$ in $\Omega \times (0, T)$ with $u(x, 0) = u_t(x, 0) = 0$ where $\Omega$ is bounded open set in $\mathbb{R}^n$ having a smooth boundary $\partial \Omega$ for $n \geq 2$ and $T > \text{diameter}(\Omega)$, has been studied. There a Hölder stability estimate has been established for the inverse problem. We consider that Hölder stability estimate in our case and so we recall the above said problem for completeness.
Consider the wave equation
\[ u_{tt} - \Delta u + qu = 0 \quad \text{for all } (x,t) \in \Omega \times (0,T), \]
\[ u = u_t = 0 \quad \text{for all } x \in \Omega \text{ and } t = 0, \]
\[ \frac{\partial u}{\partial \nu} = f \quad \text{for all } (x,t) \in \partial \Omega \times (0,T) \]
(4.1)

Further, for the above problem the Neumann to Dirichlet map \( \Delta_q \) associated with (4.1) is defined as
\[ \Delta_q : f \to u|_{\partial \Omega \times (0,T)} \]

There it is shown that for any small \( \epsilon > 0 \), there exists \( \beta_0 > 0 \) such that
\[ \|q_1 - q_2\|_{L^\infty(\Omega)} \leq C\|\Lambda_{q_1} - \Lambda_{q_2}\|_\ast^{1-\epsilon} \]
(4.2)

when \( \|q_1 - q_2\|_{H^\beta(\mathbb{R}^n)} \leq M \) for some \( \beta > \beta_0 \) and \( M \) is some positive number. Moreover \( \| \cdot \|_\ast \) denotes the operator norm from \( L^2(\partial \Omega \times (0,T)) \) to \( H^1(\partial \Omega \times (0,T)) \) and \( H^\beta(\mathbb{R}^n) \) is the standard Sobolev space of order \( \beta \).

The inverse problem concerned with the problem (4.1) is to invert the map \( q \to \Lambda_q \)
(4.3)

i.e., the reconstruction of \( q(x) \) from given \( \Lambda_q \). For our convenience, we write the above inverse problem as
\[ F(q) = \Lambda_q \]
(4.4)

Let \( q^\dagger \) be the solution of (4.4) and also let \( \|q^\dagger - q_0\|_{H^\beta} \leq K_1 \) for some \( K_1 > 0 \), \( q_0 \) is the initial approximation of the exact solution.

Let \( q^\delta_\alpha \) be the regularized solution (we assume these exist) of the inverse problem (4.4) which is the minimizer of Tikhonov functional
\[ T_\alpha(q, \delta) = \|F(q) - \Lambda^\delta_q\|_\ast^p + \alpha\|q - q_0\|_{H^\beta}^p \]
(4.5)

where \( \Lambda^\delta_q \) is some approximation of \( \Lambda_q \) satisfying \( \|\Lambda^\delta_q - \Lambda_q\|_\ast \leq \delta \). Now by assumption as \( q^\delta_\alpha \) is the minimizer of Tikhonov functional (4.5), so
\[ \|F(q^\delta_\alpha) - \Lambda^\delta_q\|_\ast^p + \alpha\|q^\delta_\alpha - q_0\|_{H^\beta}^p \leq \|F(q^\dagger) - \Lambda^\delta_q\|_\ast^p + \alpha\|q^\dagger - q_0\|_{H^\beta}^p \]
\[ \leq \delta^p + \alpha K_1^p \]
(4.6)

where the last inequality holds by the choice of \( q^\dagger \). Assuming that the regularization parameter \( \alpha \) is such that \( \frac{\delta^p}{\alpha} \leq c \) (as assumed in the theorem 3.3) for some constant \( c > 0 \). Also assume \( K_1 \) and \( c \) satisfies \( K_1 + (c + K_1^p)^\frac{1}{p} \leq M \), then (4.6) implies
\[ \|F(q^\delta_\alpha) - \Lambda^\delta_q\|_\ast^p \leq \delta^p + \alpha K_1^p \leq \alpha(c + K_1^p) \]

and thus we get the estimate
\[ \|F(q^\delta_\alpha) - \Lambda^\delta_q\|_\ast \leq \alpha^{\frac{1}{p}}(c + K_1^p)^{\frac{1}{p}} \]
(4.7)
Similarly, (4.6) implies
\[
\|q_\alpha^\delta - q_0\|_{H^\beta} \leq \left( \frac{\delta^p}{\alpha} + K^p \right)^{\frac{1}{p}} \leq (c + K^p)^{\frac{1}{p}}
\]

From the above estimate and \(\|q^\dagger\|_{H^\beta} \leq K\), we have
\[
\|q_\alpha^\delta - q^\dagger\|_{H^\beta} \leq \|q_\alpha^\delta\|_{H^\beta} + \|q^\dagger\|_{H^\beta} \leq M
\]

Thus, we can apply the estimate (4.2) now. So it follows from the estimate (4.2), (3.9) and (4.7) that
\[
\|q_\alpha^\delta - q^\dagger\|_{L^\infty(\Omega)} \leq C\|A_{q_\alpha^\delta} - A_q\|_{1-\epsilon} = C\|F(q_\alpha^\delta) - F(q^\dagger)\|_{1-\epsilon}
\]
\[
\leq 2^{1-\epsilon}C\left[\|F(q_\alpha^\delta) - \Lambda_{q_\alpha^\delta}\|_{1-\epsilon} + \delta^{1-\epsilon}\right]
\]
\[
\leq 2^{1-\epsilon}C\left[(\alpha^\frac{1}{p} (c + K^p)^{\frac{1}{p}})^{1-\epsilon} + \delta^{1-\epsilon}\right]
\]

Further, for \(\alpha \sim \delta^p\), we obtain
\[
\|q_\alpha^\delta - q^\dagger\|_{L^\infty(\Omega)} = O(\delta^{1-\epsilon})
\]

REFERENCES

[1] M. V. de Hoop, L. Qiu, O. Scherzer, Local analysis of inverse problems: Hölder stability and iterative reconstruction, Inverse problems, 28, 2012.
[2] O. Scherzer, M. Grasmair, H. Grossauer, M. Haltmeier and F. Lenzen, Variational methods in imaging, Springer, New York, 2009.
[3] S. I. kabanikhin, Inverse and ill-posed problems, Theory and applications, De Gruyter, Berlin, New York, (2012).
[4] J. Cheng and Y. Yamamoto, One new strategy for a priori choice of regularizing parameters in Tikhonov’s regularization, Inverse Problems, 16(4) : L31 – L38, 2000.
[5] A. N. Tikhonov and V. Y. Arsenin, On the solutions of Ill-posed problems, New York, 1977.
[6] V. Isakov, Stability estimates for obstacles in inverse scattering, J. Comput. Appl. Math, 42 : 79 – 88, 1992.
[7] V. A. Morozov, Methods for solving incorrectly posed problems, Springer Verlag, New York, Berlin, Heidelberg, 1984.
[8] B Kaltenbacher, A. Neubauer and O. Scherzer, Iterative regularization methods for nonlinear ill-posed problems, Walter de Gruyter, 2008.
[9] G. Alessandrini, Open issues of stability for the inverse conductivity problem, J. Inverse Ill-Posed Probl., 15(5) : 451 – 460, 2007.
[10] A. B. Bakushinskii and A. V. Goncharskii, Ill posed problems: Theory and applications, Kluwer academic publishers, Dordrecht, Boston, London, 1994.
[11] A. B. Bakushinskii and M. Y. Kokurin, Iterative Methods for Approximate Solution of Inverse problems, Mathematics and its applications, Springer, Dordrecht, (577), 2004.
[12] T. Schuster, B. Kaltenbacher, B Hofmann and K.S. Kazimierski, Regularization Methods in Banach Spaces, De Gruyter Publishers, 2012.
[13] H.W. Engl, M. Hanke, A. Neubauer, *Regularization of Inverse Problems*, Kluwer Academic Publications, 1996.

[14] G. Bao and K. Yun, *On the stability of an inverse problem for the wave equation*, Inverse Probl., 25(4)(2009). 045003, 7 pp.

[15] H. Bellout, A. Friedman and V. Isakov, *Stability for an inverse problem in potential theory*, Trans. Amer. Math. Soc., 332 : 271 – 296, 1992.

[16] J. Flemming and B. Hofmann, *A new approach to source conditions in regularization with general residual term*, Nume. Func. Anal. Optim., 31(3) : 245 – 284, 2010.

[17] A. N. Tikhonov and V. V. Arsenin, *Solutions of Ill-Posed Problems*, John Wiley & Sons, Washington, D.C., 1977.

[18] A. N. Tikhonov, A. Goncharsky, V. Stepanov and A. Ygola, *Numerical methods for the solution of Ill-Posed Problems*, Kluwer, Dordrecht, 1995.

[19] V. Isakon, *New stability estimates for soft obstacles in inverse scattering*, Inverse Probl., 9 : 535 – 543, (1993).

[20] S. Osher, M. Burger, D. Goldfarb, J. Xu and W. Yin, *An iterative regularization method for total variation-based image restoration*, Multiscale Model. Simul., 4(2005), 460 – 489.

[21] A. Neubauer, *Tikhonov regularization for nonlinear ill-posed problems: optimal convergence and finite dimensional approximation*, Inverse Problems, 5(1989), 517 – 527.

[22] J. Flemming, *Solution smoothness of ill-posed equations in Hilbert spaces: four concepts and their cross connections*, Applicable Analysis, 91(5)(2012), 1029 – 1044.