Parametric oscillations of viscoelastic orthotropic plates of variable thickness

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Abstract. Viscoelastic orthotropic rectangular plates of variable thickness are considered in the paper under the effect of periodic load. It is believed that under periodic load, the plates allow displacements commensurate with their thickness. Based on the Kirchhoff-Love hypothesis, a mathematical model of the problem of parametric oscillations of a viscoelastic orthotropic rectangular plate of variable thickness is constructed in a geometrically nonlinear statement. A method for solving the problem under consideration is proposed, based on the application of the Bubnov-Galerkin method with polynomial approximation of displacements and deflection, and on a numerical method based on the use of quadrature formulas. In calculations, the three-parameter Koltunov-Rzhanitsyn kernel is used as a weakly singular kernel. Based on the algorithm for solving the problem, a program was developed in the Delphi algorithmic language to solve the problem of parametric oscillations of viscoelastic orthotropic rectangular plates of variable thickness under the effect of an external periodic load. The effect of geometrical nonlinearity, viscoelastic properties of the material, physico-mechanical and geometrical parameters of a viscoelastic orthotropic plate on the areas of dynamic instability was investigated. The results obtained are in good agreement with the results of other authors.

1. Introduction

In modern technology and engineering, more and more complex structures are used; to ensure their strength, reliability and high efficiency is a very important issue. Without mathematical models that allow considering the maximum possible number of factors affecting the structure performance, their optimal design is impossible. At that, significant weight reduction is achieved, and geometrical and mechanical characteristics of structures are improved.

In published works, in mathematical modeling of a structure, its shape was considered as a set and invariable one. However, in recent years, an increasing importance is given to the search for optimal configuration, especially to the studies of vibrations of thin-walled structures (made of composite materials) of variable thickness. These tasks require new methods of mathematical and computer modeling [1,2].

In literature there are a number of papers devoted to the study of nonlinear vibrations and dynamic stability of elastic and viscoelastic thin-walled structures such as plates and shells of variable thickness.

In [3], the eigenfrequencies of composite doubly-curved shells of a variable thickness are
estimated.
Dynamic instability of composite plates of variable stiffness under various mechanical and geometrical parameters was studied in [4].
Using the finite element method, the dynamic stability of composite rectangular panels of variable thickness under compressive loads was studied in [5].
A special place in solving the problems of the dynamics of thin-walled structures such as plates and shells is occupied by the study of parametric vibrations of structures. There are many published works on this topic.
In [6], parametric vibrations of composite plates of variable stiffness induced by harmonic force were studied. The influence of geometrical imperfections on the oscillatory behavior of the plate was studied.
The study of dynamic instability of hinge-supported conical shells under periodic axial loads was the subject of [7]. The areas of dynamic instability are determined by the Bolotin method.
In [8,9], dynamic instability of composite plates and cylindrical shells under harmonic axial loads was studied.
In [10], based on a geometrically nonlinear theory, the influence of initial imperfections on the parametric vibrations of cylindrical shells was considered.
An experimental study of nonlinear dynamics of circular cylindrical shells under axial compressive static and periodic loads was reported in [11].
Most problems of the theory of viscoelasticity lead to the need for a numerical solution of boundary value problems for systems of integro-differential equations with partial derivatives. Moreover, systems of integro-differential equations can have a high order and variable coefficients. In addition, the nonlinearity of the simulated processes leads to the nonlinearity of the boundary value problems describing these processes.
Wide use of personal computers in the practice of calculations made it possible to develop and involve numerical analysis methods for solving the problems of hereditary theory of viscoelasticity and, thus, significantly expand the class of problems to be solved by this theory [12–19].
An analysis of the available published works showed that studies on nonlinear parametric vibrations of viscoelastic orthotropic plates and shells of variable thickness are almost not found in literature. In this paper, we consider the problems of parametric vibrations of viscoelastic orthotropic plates of variable thickness in a geometrically nonlinear statement.

2. Methods
Consider a rectangular viscoelastic orthotropic plate of a variable thickness \( h = h(x) \) with sides \( a \) and \( b \). Let the plate undergo dynamic loading along the side \( a \) under periodic load \( P(t) = P_0 + P_1 \cos \Theta t \) (\( P_0, P_1 = \text{const} \); \( \Theta \) - is the frequency of external periodic load), provided that the plate has initial deflections.

Under the accepted assumptions, taking into account periodic force \( P(t) \frac{\partial^2 w}{\partial x^2} \) and initial deflection, the mathematical model of this problem with respect to transverse deflection \( w = w(x, y, t) \) and displacements \( u = u(x, y, t) \), \( v = v(x, y, t) \) is described by the following system of equations [20]

\[
\begin{align*}
\frac{h}{B_{11}} \left[ B_{11} \left( 1 - \Gamma_{11}^* \right) \frac{\partial^2 e_x}{\partial x^2} + B_{12} \left( 1 - \Gamma_{12}^* \right) \frac{\partial^2 e_y}{\partial y^2} + 2B \left( 1 - \Gamma^* \right) \frac{\partial^2 e_{xy}}{\partial x \partial y} \right] + \\
+ \frac{\partial h}{\partial x} \left[ B_{11} \left( 1 - \Gamma_{11}^* \right) e_x + B_{12} \left( 1 - \Gamma_{12}^* \right) e_y \right] + 2B \frac{\partial h}{\partial y} \left( 1 - \Gamma^* \right) e_{xy} - \rho h \frac{\partial^2 u}{\partial t^2} = 0,
\end{align*}
\]
A mathematical model obtained using system (1) under the corresponding boundary and initial conditions takes into account the viscoelastic properties and the orthotropy of the material of the plate.

It should be noted here that while solving problems of the dynamics of viscoelastic systems in an isotropic statement, only one relaxation kernel is involved in the system of integro-differential equations, and in orthotropic statement five different kernels are involved.

Further, when calculating the problem, the weakly singular Koltunov-Rzhansityn kernel with three rheological parameters \((A, \beta, \text{and} \alpha)\) is taken as a relaxation kernel, in the form [21]:

\[
\Gamma(t) = Ae^{-\beta t}t^{\alpha - 1}, \quad (0 < \alpha < 1)
\]

Let’s approximate complete and initial deflections \(w\) and \(w_0\), displacements \(u, v\) in the resulting system using
\[ u(x,y,t) = \sum_{n=1}^{N} \sum_{m=1}^{M} u_{nm}(t) \varphi_{nm}(x,y), \quad v(x,y,t) = \sum_{n=1}^{N} \sum_{m=1}^{M} v_{nm}(t) \varphi_{nm}(x,y), \]

\[ w(x,y,t) = \sum_{n=1}^{N} \sum_{m=1}^{M} w_{nm}(t) \psi_{nm}(x,y), \quad w_0(x,y) = \sum_{n=1}^{N} \sum_{m=1}^{M} w_{0nm} \psi_{nm}(x,y) \]

Substituting (2) into the system of equations (1) and applying the Bubnov-Galerkin method, we introduce the following dimensionless quantities

\[
\begin{align*}
\frac{u}{h_0}, \quad \frac{v}{h_0}, \quad \frac{w}{h_0}, \quad \frac{\lambda}{h_0}, \quad \alpha = \frac{a}{b}, \quad \beta = \frac{b}{h_0}, \quad \frac{q}{E} \left( \frac{b}{h_0} \right)^4, \quad \Theta, \quad \omega
\end{align*}
\]

and maintaining previous notation to determine the unknowns \( w_{nm} = w_{nm}(t), \quad u_{nm} = u_{nm}(t), \quad v_{nm} = v_{nm}(t) \), we obtain the following system of basic resolving nonlinear integro-differential equations:

\[ \sum_{n=1}^{N} \sum_{m=1}^{M} a_{klnm} \dot{u}_{nm} - \eta_1 \left[ \sum_{n=1}^{N} \sum_{m=1}^{M} \left[ (1-G_{11}^*) \dot{u}_{klnm} + (1-G_{11}^*) \dot{u}_{2klnm} \right] \right] + \left[ (1-G_{12}^*) \dot{u}_{3klnm} + (1-G_{12}^*) \dot{u}_{4klnm} \right] \right] = 0, \]

\[ \sum_{n=1}^{N} \sum_{m=1}^{M} b_{klnm} \ddot{u}_{nm} - \eta_2 \left[ \sum_{n=1}^{N} \sum_{m=1}^{M} \left[ (1-G_{21}^*) \ddot{u}_{klnm} + (1-G_{21}^*) \ddot{u}_{2klnm} \right] \right] + \left[ (1-G_{22}^*) \ddot{u}_{3klnm} + (1-G_{22}^*) \ddot{u}_{4klnm} \right] \right] = 0, \]

\[ \sum_{n=1}^{N} \sum_{m=1}^{M} c_{klnm} \dddot{u}_{nm} + \eta_3 \sum_{n=1}^{N} \sum_{m=1}^{M} \left( 1-2 \mu_{klnm} \cos \Theta \right) \dot{w}_{nm} - \eta_4 \left[ \sum_{n=1}^{N} \sum_{m=1}^{M} \left[ (1-G_{11}^*) \dddot{w}_{klnm} + (1-G_{11}^*) \dddot{w}_{2klnm} \right] \right] + \left[ (1-G_{12}^*) \dddot{w}_{3klnm} + (1-G_{12}^*) \dddot{w}_{4klnm} \right] \right] = 0, \]

\[ u_{nm}(0) = u_{0nm}, \quad \dot{u}_{nm}(0) = \dot{u}_{0nm}, \quad v_{nm}(0) = v_{0nm}, \quad \dot{v}_{nm}(0) = \dot{v}_{0nm}, \quad w_{nm}(0) = w_{0nm}, \quad \dot{w}_{nm}(0) = \dot{w}_{0nm}, \]

where \( h_0 = h(0); \quad p_{klnm}^2 = f_{5klnm} + f_{6klnm} + f_{7klnm} + f_{8klnm} + f_{9klnm} - 4 \pi^2 \lambda^2 p_{klnm} \delta_0; \)
\[ \mu_{klnm} = \frac{2\pi^2 \lambda^2 p_{klnm}^*}{p_{klnm}} \delta_0; \quad \delta_0 = \frac{P_0}{P_{cr}}; \quad \delta_1 = \frac{P_1}{P_{cr}}; \quad P_{cr} \] is Euler static critical load; other coefficients used in this system are related to coordinate functions and their derivatives.

Integration (3) was performed using the numerical method proposed in [22]. The results of calculations on the computer are reflected in the graphs shown in Figs.1-3. The dependence of the thickness variation is selected in the form: \[ h = 1 + \alpha^* x \] (\( \alpha^* \) is the parameter of the thickness variation).

Here, unless otherwise specified, the following data were taken in calculations as the initial ones:
\[ \delta = 25; \quad w_0 = 0.01; \quad q = 0; \quad \lambda = 1; \quad \alpha^* = 0.5; \quad \delta_0 = 0.3; \quad \delta_1 = 0.5; \quad \Theta = 1.1. \]

3. Results and discussion

Figure 1 shows the results of thickness variation parameter effect on the behavior of a viscoelastic plate.

![Figure 1](image1.png)

**Figure 1.** Deflection dependence on time at \( \alpha^* = 0 \) (1); 0.4 (2); 0.8 (3)

It is clearly seen from the Figure 1 that the amplitude of oscillations increases with this parameter. Note that at the beginning of the oscillation process, the amplitudes slightly differ from the oscillation of plates of constant thickness.

The effect of inhomogeneous material properties on the plate behavior was studied (Fig. 2). As seen from the Figure 2, an increase in the parameter \( \Delta \) determining the degree of anisotropy (curve 1 - \( \Delta = 1 \); curve 2 - \( \Delta = 1.5 \) and curve 3 - \( \Delta = 2 \)) leads to a more rapid increase in the amplitude of oscillations.

![Figure 2](image2.png)

**Figure 2.** Deflection dependence on time at \( \Delta = 1 \) (1); 1.5 (2); 2 (3)
Figure 3 shows the results obtained by various theories. Here, curve 1 corresponds to the elastic case, curve 2 - to the results obtained taking viscosity into account only in shear directions \( A = 0.1, A_{ij} = 0, i, j = 1,2 \), and curve 3 - to the case when viscosity is taken into account in all directions \( A = A_{ij} = 0.05, i, j = 1,2 \).

As seen from the Figure 3, while the results corresponding to curves 1 and 2 at the initial points in time almost coincide, then over time the differences in results increase. However, in the case corresponding to curve 3, differences in the results with curves 1 and 2 arise at the beginning of the oscillations process, and at time \( t = 90 \) differ from them in amplitude values by 15-20 percent. This once again confirms the need to take into account the viscoelastic properties of the material not only in shear direction, but in other directions as well.

4. Conclusion

1. Based on the polynomial approximation of deflections, the dynamic behavior of orthotropic rectangular plates of variable thickness is estimated under the action of a periodic load.

2. The influence of the change in physico-mechanical and geometrical parameters on the amplitude-time characteristics of the plate is estimated.

3. It was stated that the results of viscoelastic problem obtained using the exponential relaxation kernel almost coincide with the results of the elastic problem; when using the Koltunov-Rzhanitsyn kernel, the difference is very significant and amounts to more than 40%.

4. The proposed method and algorithm make it possible to take into account the smooth variation in thickness of thin-walled elements and can be applied in their optimal design.

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