SOBOLEV EXTENSION OPERATORS AND NEUMANN EIGENVALUES

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Abstract. In this paper we apply estimates of the norms of Sobolev extension operators to the spectral estimates of of the first nontrivial Neumann eigenvalue of the Laplace operator in non-convex extension domains. As a consequence we obtain a connection between resonant frequencies of free membranes and the smallest-circle problem (initially proposed by J. J. Sylvester in 1857).

1. Introduction

Let \( \Omega \subset \mathbb{R}^n \) be a bounded domain (an open connected set). Then \( \Omega \) is called a Sobolev \( L^2 \)-extension domain if there exists a continuous linear extension operator
\[
E: L^2(\Omega) \to L^2(\mathbb{R}^n)
\]
such that \( Eu|_{\Omega} = u \) for all \( u \in L^2(\Omega) \).

Lipschitz domains are examples of Sobolev \( L^2 \)-extension domains [31]. In [11] was given characterization of planar extension domains in terms of quasiconformal geometry, namely a simply connected planar domain \( \Omega \) is a Sobolev \( L^2 \)-extension domain if and only if \( \Omega \) is an image of the unit disc \( D \) under a \( K \)-quasiconformal mapping of the plane \( \mathbb{R}^2 \) onto itself (\( K \)-quasidisc) for some \( K \geq 1 \).

The aim of this article is estimate the Neumann eigenvalues of the Laplace operator defined in bounded domains \( \Omega \) in terms of the norms of the extension operators. By the Min–Max Principle (see, for example, [6]) the first non-trivial Neumann eigenvalue \( \mu_1(\Omega) \) for the Laplacian can be characterized as
\[
\mu_1(\Omega) = \min \left\{ \frac{\int |\nabla u(x)|^2 \, dx}{\int |u(x)|^2 \, dx} : u \in W^1_2(\Omega) \setminus \{0\}, \int_{\Omega} u \, dx = 0 \right\}.
\]

Hence \( \mu_1(\Omega)^{-\frac{1}{2}} \) is the best constant \( B_{2,2}(\Omega) \) in the following Poincaré inequality
\[
\inf_{c \in \mathbb{R}} \|u - c| L^2(\Omega)\| \leq B_{2,2}(\Omega) \|\nabla u| L^2(\Omega)\|, \quad u \in W^1_2(\Omega).
\]

It is well known (see, for example, [24]) that if \( \Omega \subset \mathbb{R}^n \) is the Sobolev \( L^2 \)-extension domain then the spectrum of Neumann-Laplace operator in \( \Omega \) is discrete and can be written in the form of a non-decreasing sequence
\[
0 = \mu_0(\Omega) < \mu_1(\Omega) \leq \mu_2(\Omega) \leq \ldots \leq \mu_n(\Omega) \leq \ldots.
\]

The main result of the article is

\[\text{Key words and phrases: elliptic equations, Sobolev spaces, extension operators.}\]

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**Theorem A.** Let $\Omega \subset \mathbb{R}^n$ be a Sobolev $L^1_2$-extension domain. Then the following inequality holds

$$\mu_1(\Omega) \geq \left( \frac{1}{\|E_\Omega\| \cdot R_\Omega} \right)^2,$$

where $R_\Omega$ is a radius of a minimum enclosing ball $B_\Omega$ of $\Omega$, $p_{n/2}$ denotes the first positive zero of the function $(t^{1-n/2} J_{n/2}(t))'$ and $\|E_\Omega\|$ denoted the norm of a continuous linear extension operator $E_\Omega : L^1_2(\Omega) \to L^1_2(B_\Omega)$.

**Remark 1.1.** If a domain $\Omega$ has a center of a symmetry then $R_\Omega = d(\Omega)/2$, where $d(\Omega) = \sup_{x,y \in \Omega} |x - y|$. In this case

$$\mu_1(\Omega) \geq \left( \frac{p_{n/2}}{d(\Omega)} \right)^2 \cdot \left( \frac{2}{\|E_\Omega\|} \right)^2.$$

The explicit value for the first non-trivial Neumann eigenvalue of the Laplace operator in a $n$-ball of radius $R$

$$\mu_1(B_R) = \left( \frac{p_{n/2}}{R} \right)^2,$$

where $p_{n/2}$ denotes the first positive zero of the function $(t^{1-n/2} J_{n/2}(t))'$ (see, for example [27, 35]). For some $n$, we give values of the parameter $p_{n/2}$ which represent first non-trivial Neumann eigenvalue in the unit ball in $\mathbb{R}^n$:

| $n$ | 2  | 3  | 4  | 5  | 6  | 7  | 8  |
|-----|----|----|----|----|----|----|----|
| $p_{n/2}$ | 1.841 | 2.081 | 2.299 | 2.501 | 2.688 | 2.864 | 3.031 |

The proof of Theorem A is based on

**Theorem B.** Let $\Omega \subset \mathbb{R}^n$ be a Sobolev $L^1_2$-extension domain. Then for any bounded Lipschitz domain $\tilde{\Omega} \supset \Omega$

$$\mu_1(\tilde{\Omega}) \leq \|E_\Omega\|^2 \cdot \mu_1(\Omega),$$

where $\|E_\Omega\|$ denoted the norm of a continuous linear extension operator $E_\Omega : L^1_2(\Omega) \to L^1_2(\tilde{\Omega})$.

In the space case general estimates of the norm of the extension operator (1.1) is an open and complicated problem and we use estimates from [25] for balls and star-shaped domains.

The construction of the extension operator which is based on Whitney decomposition [36] was studied by many authors (see, for example, [5, 21, 29, 30, 31]), but in this case estimates of the norms of extension operators is an open complicated problem.

Extension operators on Sobolev spaces in terms of a measure density condition were studied in [18, 19].
In the planar case we use the extension operator which was suggested in [11] and is based on quasiconformal reflections [1] in Ahlfors domains ($K$-quasidiscs). In this case

$$
\|E_\Omega(u)\|_{L^2(\Omega)} \leq \|E(u)\|_{L^2(\mathbb{R}^2)} \leq (1 + K)\|u\|_{L^2(\mathbb{R}^2)}, \quad \text{for all } u \in L^2(\Omega).
$$

Hence, $\|E_\Omega\| \leq 1 + K$ and Theorem A can be refined as

**Corollary A.** Let $\Omega$ be a $K$-quasidisc. Then

$$
\mu_1(\Omega) \geq \left( \frac{j_{1,1}^2}{R_\Omega} \right)^2 \left( \frac{1}{1 + K} \right)^2,
$$

where $R_\Omega$ is a radius of a minimum enclosing ball $B_\Omega$ of $\Omega$ and $j_{1,1}^* \approx 1.84118$ denotes the first positive zero of the derivative of the Bessel function $J_1$.

**Remark 1.2.** Corollary A gives a connection between resonant frequencies of free membranes and the smallest-circle problem. This problem was initially proposed by J. J. Sylvester in 1857.

The classical lower estimate for the first non-trivial eigenvalue

$$
\mu_1(\Omega) \geq \frac{\pi^2}{d(\Omega)^2} \tag{1.4}
$$

was proved in [26] (see, also [8, 10]) for convex domains.

**Remark 1.3.** If a domain $\Omega$ has a center of a symmetry and a norm of an extension operator $E_\Omega : L^2(\Omega) \rightarrow L^2(B_\Omega)$ satisfies to the following condition

$$
\|E_\Omega\| \leq \frac{2p_{n/2}}{\pi}
$$

then estimate (1.2) is better than the classical estimate (1.4).

**Example 1.4.** Consider $n$-dimensional half-ball $B^- = \{x \in \mathbb{R}^n : |x| < 1 \land x_n < 0\}$. Define the extension operator

$$
E_{B^-}(u) = \begin{cases} 
 u(x_1, \ldots, x_{n-1}, x_n), & \text{if } x \in B^-,
 u(x_1, \ldots, x_{n-1}, -x_n), & \text{if } x \in B^+,
\end{cases}
$$

where $B^+ = \{x \in \mathbb{R}^n : |x| < 1 \land x_n > 0\}$.

Then $E_{B^-} : L^2(B^-) \rightarrow L^2(B)$, $B = \{x \in \mathbb{R}^n : |x| < 1\}$, and $\|E_{B^-}\| = \sqrt{2}$. Hence by Theorem A

$$
\mu_1(B^-) \geq \frac{p_{n/2}^2}{2} > \frac{\pi^2}{4}, \quad \text{if } n \geq 4.
$$

So for $\mu_1(B^-)$ the estimate by Theorem A is better than classical estimates (1.4) for $n \geq 4$.

In the present work we suggest lower estimates in non-convex domains in the terms of extension operators. In the planar case these estimates can be reformulated in the terms of (quasi)conformal geometry of $\Omega$. Another approach to the lower spectral estimates in non-convex domains is based on the geometric theory of composition operators on Sobolev spaces [12, 13, 14, 15, 16].
Upper spectral estimates arise in works [33, 35] and were intensively studied in the recent decades, see, for example, [2, 3, 4, 7, 23].

2. Extension operators

Let \( \Omega \) be a domain in \( \mathbb{R}^n \), \( n \geq 2 \). For any \( 1 \leq p < \infty \) we consider the Lebesgue space \( L_p(\Omega) \) of measurable functions \( u : \Omega \to \mathbb{R} \) equipped with the following norm:

\[
\| u \|_{L_p(\Omega)} = \left( \int_{\Omega} |u(x)|^p \, dx \right)^{\frac{1}{p}} < \infty.
\]

The Sobolev space \( W_{p,1}^1(\Omega), 1 \leq p < \infty, \) is defined as a Banach space of locally integrable weakly differentiable functions \( u : \Omega \to \mathbb{R} \) equipped with the following norm:

\[
\| u \|_{W_{p,1}^1(\Omega)} = \left( \int_{\Omega} |u(x)|^p \, dx \right)^{\frac{1}{p}} + \left( \int_{\Omega} |
abla u(x)|^p \, dx \right)^{\frac{1}{p}}.
\]

Recall that the Sobolev space \( W_{p,1}^1(\Omega) \) coincides with the closure of the space of smooth functions \( C^\infty(\Omega) \) in the norm of \( W_{p,1}^1(\Omega) \).

We consider also the homogeneous seminormed Sobolev space \( L_{1,p}^{1}(\Omega), 1 \leq p < \infty, \) of locally integrable weakly differentiable functions \( u : \Omega \to \mathbb{R} \) equipped with the following seminorm:

\[
\| u \|_{L_{1,p}^{1}(\Omega)} = \left( \int_{\Omega} |\nabla u(x)|^p \, dx \right)^{\frac{1}{p}}.
\]

Remark 2.1. By the standard definition functions of \( L_{1,p}^{1}(\Omega) \) are defined only up to a set of measure zero, but they can be redefined quasieverywhere i.e. up to a set of \( p \)-capacity zero. Indeed, every function \( u \in L_{1,p}^{1}(\Omega) \) has a unique quasicontinuous representation \( \tilde{u} \in L_{1,p}^{1}(\Omega) \). A function \( \tilde{u} \) is called quasicontinuous if for any \( \varepsilon > 0 \) there is an open set \( U_\varepsilon \) such that the \( p \)-capacity of \( U_\varepsilon \) is less than \( \varepsilon \) and on the set \( \Omega \setminus U_\varepsilon \) the function \( \tilde{u} \) is continuous (see, for example [20, 24]).

Recall that a continuous linear operator

\[
E : L_{1,p}^{1}(\Omega) \to L_{1,p}^{1}(\mathbb{R}^n)
\]

satisfying the conditions

\[
Eu|_{\Omega} = u \quad \text{and} \quad \|E\| := \sup_{u \in L_{1,p}^{1}(\Omega)} \frac{\|Eu\|_{L_{1,p}^{1}(\mathbb{R}^n)}}{\|u\|_{L_{1,p}^{1}(\Omega)}} < \infty
\]

is called a continuous linear extension operator.

We say that \( \Omega \) is a Sobolev \( L_{p,1}^{1} \)-extension domain if there exists a continuous linear extension operator

\[
E : L_{p,1}^{1}(\Omega) \to L_{p,1}^{1}(\mathbb{R}^n).
\]

It is well known that existence of an extension operator from \( L_{p,1}^{1}(\Omega) \) to \( L_{p,1}^{1}(\mathbb{R}^n) \), \( k \in \mathbb{N}, n \geq 2 \), depends on the geometry of \( \Omega \). In the case of Lipschitz domains, Calderón [33] constructed an extension operator on \( L_{p,1}^{1}(\Omega) \) for \( 1 < p < \infty, \) \( k \in \mathbb{N} \). Stein [31] extended Calderón’s result to include the endpoints \( p = 1, \infty \). Jones [21] introduced an extension operator on locally uniform domains. This is a much larger class of domains that includes examples with highly non-rectifiable boundaries.
In [11] were obtained necessary and sufficient conditions for $L^1_2$-extensions from planar simply connected domains in terms of quasiconformal mappings and was obtained estimates of norms of extension operators. Necessary and sufficient conditions for $L^p_1$-extension operators for $p > 2$ from planar simply connected domains were obtained in [29, 30] and were claimed for $1 < p < 2$ in [22]. For case $p \neq 2$ estimates of norm of extension operators are unknown.

Using the extension operators theory on Sobolev spaces we estimates the weak domain monotonicity for the first non-trivial Neumann eigenvalues in a bounded domains $\Omega \subset \mathbb{R}^n$.

The property of domain monotonicity holds for Dirichlet eigenvalues, i.e., if $\Omega \subset \bar{\Omega}$ are a bounded domains, then we have $\lambda_1(\Omega) \geq \lambda_1(\bar{\Omega})$. This property does not holds for Neumann eigenvalues, even in the case of convex domains (see, for example, [17]). For Sobolev $L^1_2$-extension domain there exists something like a quasi-monotonicity property that it can sees from the following theorem:

**Theorem B.** Let $\Omega \subset \mathbb{R}^n$ be a Sobolev $L^1_2$-extension domain. Then for any bounded Lipschitz domain $\Omega \supset \bar{\Omega}$

$$\mu_1(\bar{\Omega}) \leq \|E_\Omega\|^2 \cdot \mu_1(\Omega),$$

where $\|E_\Omega\|$ denoted the norm of a continuous linear extension operator

$$E_\Omega : L^1_2(\Omega) \to L^1_2(\bar{\Omega}).$$

**Proof.** Because $\Omega$ is the $L^1_2$-extension domain then there exists a continuous linear extension operator

(2.1) $$E_\Omega : L^1_2(\Omega) \to L^1_2(\bar{\Omega})$$

defined by formula

$$(E_\Omega(u))(x) = \begin{cases} u(x) & \text{if } x \in \Omega, \\ \bar{u}(x) & \text{if } x \in \bar{\Omega} \setminus \Omega, \end{cases}$$

where $\bar{u} : \bar{\Omega} \setminus \Omega \to \mathbb{R}$ be an extension of the function $u$.

Hence for every function $u \in W^1_2(\Omega)$ we have

(2.2) $$\|u - u_\Omega| L^2(\Omega)\| = \inf_{c \in \mathbb{R}} \|u - c| L^2(\Omega)\| = \inf_{c \in \mathbb{R}} \|E_\Omega u - c| L^2(\Omega)\|$$

$$\leq \|E_\Omega u - (E_\Omega u)|\|_\bar{\Omega}| L^2(\bar{\Omega})\| \leq \|E_\Omega u - (E_\Omega u)|\|_\bar{\Omega}| L^2(\bar{\Omega})\|. $$

Here $u_\Omega$ and $(E_\Omega u)|\|_\bar{\Omega}$ are mean values of corresponding functions $u$ and $E_\Omega u$.

Because $\Omega$ is a Lipschitz domain, then taking into account the classical Poincaré inequality in Lipschitz domains [24]

$$\inf_{c \in \mathbb{R}} \|u - c| L^2(\bar{\Omega})\| \leq B_{2,2}(\bar{\Omega})\|E_\Omega u| L^2(\bar{\Omega})\|, \quad u \in W^1_2(\bar{\Omega}),$$

and the continuity of the linear extension operator [24], i.e.,

$$\|E_\Omega u| L^1_2(\bar{\Omega})\| \leq \|E_\Omega\| \cdot \|u| L^2_2(\Omega)\|,$$
we obtain
\begin{equation}
\|E_{\Omega} u - (E_{\Omega} u)_{\tilde{\Omega}} | L_2(\tilde{\Omega})\|
\leq B_{2,2}(\tilde{\Omega}) \cdot \|\nabla (E_{\Omega} u) | L_2(\tilde{\Omega})\|
\leq B_{2,2}(\tilde{\Omega}) \cdot \|E_{\Omega}\| \cdot \|\nabla u | L_2(\Omega)\|.
\end{equation}

Combining inequalities (2.2) and (2.3) we have
\[\|u - u_{\Omega} | L_2(\Omega)\| \leq B_{2,2}(\Omega) \cdot \|\nabla u | L_2(\Omega)\|,\]
where
\[B_{2,2}(\Omega) \leq B_{2,2}(\tilde{\Omega}) \cdot \|E_{\Omega}\|.
\]

By the Min-Max Principle \([6]\) \(\mu_1(\Omega)^{-1} = B_{2,2}^2(\Omega)\). Thus, we finally have
\[\mu_1(\Omega) \leq \|E_{\Omega}\|^2 \cdot \mu_1(\Omega).
\]

\[\square\]

Let \(B_{\Omega}\) be a minimum enclosing ball of \(\Omega\). Taking in Theorem B \(\tilde{\Omega} = B_{\Omega}\) we obtain the main result of the article:

**Theorem A.** Let \(\Omega \subset \mathbb{R}^n\) be a Sobolev \(L_2^1\)-extension domain. Then the following inequality holds
\[\mu_1(\Omega) \geq \left(\frac{1}{\|E_{\Omega}\| R_{\Omega}}\right)^2 \left(\frac{p_{n/2}}{R_{\Omega}}\right)^{2},\]
where \(R_{\Omega}\) is a radius of a minimum enclosing ball \(B_{\Omega}\) of \(\Omega\), \(p_{n/2}\) denotes the first positive zero of the function \((t^{1-n/2}J_{n/2}(t))'\) and \(\|E\|\) denoted the norm of continuous extension operator
\[E_{\Omega} : L_2^1(\Omega) \to L_2^1(B_{\Omega}).\]

### 2.1. Spectral estimates in planar domains.
In this section we give estimates of the norm of extension operators \((2.1)\) in Ahlfors domains \((K\)-quasidisics). Recall that a domain \(\Omega \subset \mathbb{R}^2\) is called a \(K\)-quasidisc if it is an image of the unit disc \(\mathbb{D}\) under a \(K\)-quasiconformal mapping of the plane \(\mathbb{R}^2\) onto itself. Quasidiscs represent large class domains that include some fractal type domains (like snowflakes).

Because there exists a Möbius transformation of the unit disc \(\mathbb{D} \subset \mathbb{R}^2\) onto the upper halfplane \(H^+\) we can replace in the definition of quasidisics \(\mathbb{D}\) to \(H^+\). By the Riemann Theorem there exists a conformal mapping \(\varphi\) of the upper halfplane \(H^+\) onto \(\Omega\). Since \(\Omega\) is a \(K\)-quasidisc there exists a quasiconformal extension of the conformal mapping \(\varphi\) to a \(K^2\)-quasiconformal homeomorphism \(\tilde{\varphi} : \mathbb{R}^2 \to \mathbb{R}^2\) (see, for instance, \([1]\)).

By \(\tilde{\Omega}\) a homeomorphism \(\psi : \Omega \to \tilde{\Omega}\) is a \(K\)-quasiconformal mapping if and only if \(\tilde{\psi}\) generates by the composition rule \(\psi^*(\tilde{u}) = \tilde{u} \circ \psi\) a bounded composition operator on Sobolev spaces \(L_2^1(\Omega)\) and \(L_2^1(\tilde{\Omega})\):
\[\|\psi^*(\tilde{u}) | L_2^1(\Omega)\| \leq K^+ \|\tilde{u} | L_2^1(\tilde{\Omega})\|\]
for any \(\tilde{u} \in L_2^1(\tilde{\Omega})\).
Consider the following diagram

\[
\begin{align*}
  L_1^2(\Omega) & \xrightarrow{\varphi^*} L_1^2(H^+) \\
  & \downarrow (\varphi^{-1})^* \circ \omega \circ \varphi^* \\
  L_1^2(\mathbb{R}^2) & \leftarrow L_1^2(\mathbb{R}^2)
\end{align*}
\]

where \( \omega \) is a continuous extension operator (a symmetry with respect to the real axis) that extend any function \( u \in L_1^2(H^+) \) to a function \( \tilde{u} \) from \( L_1^2(\mathbb{R}^2) \).

In accordance to this diagram we define the extension operator on Sobolev spaces \( E : L_1^2(\Omega) \to L_1^2(\mathbb{R}^2) \) by the formula

\[
(Eu)(x) = \begin{cases} 
  u(x) & \text{if } x \in \Omega, \\
  \tilde{u}(x) & \text{if } x \in \mathbb{R}^2 \setminus \Omega,
\end{cases}
\]

where \( \tilde{u} : \mathbb{R}^2 \setminus \Omega \to \mathbb{R} \) is defined as \( \tilde{u} = (\varphi^{-1})^* \circ \omega \circ \varphi^* u \).

Now, using the above diagram, it is easy to check that

\[
\| Eu \|_{L_2^1(\mathbb{R}^2)} = \left( \int_{\Omega} |\nabla u(x)|^2 \, dx \right)^{\frac{1}{2}} + \left( \int_{\mathbb{R}^2 \setminus \Omega} |\nabla \tilde{u}(x)|^2 \, dx \right)^{\frac{1}{2}} \leq (1 + K) \left( \int_{\Omega} |\nabla u(x)|^2 \, dx \right)^{\frac{1}{2}} = (1 + K) \| u \|_{L_2^1(\Omega)}.
\]

So, the norm of this extension operator \( \| E \| \leq 1 + K \).

Thus, taking into account Theorem A we have the following result:

**Corollary A.** Let \( \Omega \) be a \( K \)-quasidisc. Then

\[
\mu_1(\Omega) \geq \left( \frac{j_{1,1}}{R_\Omega} \right)^2 \cdot \left( \frac{1}{1 + K} \right)^2,
\]

where \( R_\Omega \) is a radius of a minimum enclosing ball \( B_\Omega \) of \( \Omega \) and \( j_{1,1} \approx 1.84118 \) denotes the first positive zero of the derivative of the Bessel function \( J_1 \).

As an example, we consider the following non-convex domain

\[
\Omega := \left\{ (x, y) \in \mathbb{R}^2 : -\frac{1}{2} < x < \frac{1}{2}, \quad \frac{1}{2} - |x| < y < \frac{3}{2} - |x| \right\}
\]

which is the image of the square

\[
Q := \left\{ (x, y) \in \mathbb{R}^2 : -\frac{1}{2} < x < \frac{1}{2}, \quad 0 < y < 1 \right\}
\]

under a \( K \)-quasiconformal mapping of the plane \( \mathbb{R}^2 \) onto itself with the coefficient of quasiconformality \( K = (3 + \sqrt{5})/2 \).

We show that domain \( \Omega \) is a \( K \)-quasidisc. For this we input the following notation:

\[
Q := \left[ -\frac{1}{2}, \frac{1}{2} \right] \times [0, 1], \quad S = \left[ -\frac{1}{2}, \frac{1}{2} \right] \times \mathbb{R},
\]

\[
Q^- = \left[ -\frac{1}{2}, 0 \right] \times [0, 1], \quad S^- = \left[ -\frac{1}{2}, 0 \right] \times \mathbb{R},
\]
Let mapping $\tilde{\varphi}_+ : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by the rule $\tilde{\varphi}_+ := \{(x, x + y + \frac{1}{2}) \}$ and mapping $\tilde{\varphi}_- : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by the rule $\tilde{\varphi}_- := \{(x, y - x + \frac{1}{2}) \}$. It is easy to see that $\tilde{\varphi}_+(0, y) = \{(0, y + \frac{1}{2}) \} = \tilde{\varphi}_-(0, y)$ and $\tilde{\varphi}_+\left(\frac{1}{2}, y\right) = \left\{-\frac{1}{2}, y \right\}$, $\tilde{\varphi}_-\left(\frac{1}{2}, y\right) = \left\{\frac{1}{2}, y \right\}$.

Let mapping $\varphi : Q \to \mathbb{R}^2$ is defined as $\varphi|Q^- := \varphi_+$ and $\varphi|Q^+ := \varphi_-$, and mapping $\varphi : S \to \mathbb{R}^2$ is defined as $\varphi|S^- := \varphi_+$ and $\varphi|S^+ := \varphi_-$, where $\varphi_+ := \tilde{\varphi}_+|Q^-$, $\varphi_+ := \tilde{\varphi}_+|S^-$, $\varphi_- := \tilde{\varphi}_-|Q^+$, $\varphi_- := \tilde{\varphi}_-|S^+$.

Then extended mapping $\tilde{\varphi} : \mathbb{R}^2 \to \mathbb{R}^2$ is defined as $\tilde{\varphi}|Q := \varphi$, $\tilde{\varphi}|S := \varphi$, and $\tilde{\varphi}|\mathbb{R}^2 \setminus S := \text{id}$.

Now we calculate the quasiconformality coefficient for the mapping $\varphi : Q \to \mathbb{R}^2$, using the following formula

$$K = \frac{\lambda}{J_\varphi(x, y)}.$$ 

Here $\lambda$ is the largest eigenvalue of the matrix $A = DD^T$, where $D = D\varphi(x, y)$ is Jacobi matrix of mapping $\varphi = \varphi(x, y)$ and $J_\varphi(x, y) = \det D\varphi(x, y)$ is its Jacobian.

Note that the Jacobi matrix corresponding to the mapping $\varphi = \varphi(x, y)$ on $S^-$, $S^+$ and outside of $S$, respectively, has the form

$$D = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad \text{and} \quad D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$ 

A straightforward calculation yields

$$J_\varphi(x, y) = 1 \quad \text{and} \quad \lambda = \frac{3 + \sqrt{5}}{2}.$$ 

Hence

$$K = \frac{\lambda}{J_\varphi(x, y)} = \frac{3 + \sqrt{5}}{2}.$$ 

Thus, we established that the domain $\Omega$ is $K$-quasidisc with the quasiconformality coefficient $K = (3 + \sqrt{5})/2$.

It is easy to show that

$$d(\Omega) = \sup_{x, y \in \Omega} |x - y| = \frac{\sqrt{10}}{2}.$$ 

Then by Corollary A we have

$$\mu_1(\Omega) \geq \left(\frac{j_{1,1}}{d(\Omega)}\right)^2 \cdot \left(\frac{2}{1 + K}\right)^2 \approx \frac{2}{5}.$$
As another application of Corollary A, we obtain the lower estimates of the first non-trivial eigenvalue on the Neumann eigenvalue problem for the Laplace operator in the star-shaped and spiral-shaped domains.

**Star-shaped domains.** We say that a domain \( \Omega^* \) is \( \beta \)-star-shaped (with respect to \( z_0 = 0 \)) if the function \( \varphi(z) \), \( \varphi(0) = 0 \), conformally maps a unit disc \( D \) onto \( \Omega^* \) and the condition satisfies [9]:

\[
\left| \frac{z\varphi'(z)}{\varphi(z)} \right| \leq \beta \pi/2, \quad 0 \leq \beta < 1, \quad |z| < 1.
\]

In [9] proved the following: the boundary of the \( \beta \)-star-shaped domain \( \Omega^* \) is a \( K \)-quasicircle with \( K = \cot^2(1 - \beta)\pi/4 \).

Then by Corollary A we have

\[
\mu_1(\Omega^*) \geq 4 \sin^4((1 - \beta)\pi/4) \cdot \left( \frac{j_{1,1}'}{d(\Omega^*)} \right)^2.
\]

For example, the diffeomorphism

\[
\varphi(z) = \tan z, \quad z = x + iy, \quad |z| < 1,
\]

conformally maps a unit disc onto \( \frac{1}{2} \)-star-shaped domain \( \Omega^* \):

**Figure 2.1.** Image of \( D \) under \( \varphi(z) = \tan z \).

By straightforward calculation we get

\[
\mu_1(\Omega^*) \geq 4 \sin^4(\pi/8) \cdot \left( \frac{j_{1,1}'}{d(\Omega^*)} \right)^2 \approx 4 \left( \frac{\sqrt{2} - \sqrt{2}}{2} \right)^4 \cdot \left( \frac{1.84118}{3.2} \right)^2 \approx \frac{1}{5}.
\]

**Spiral-shaped domains.** We say that a domain \( \Omega_s \) is \( \beta \)-spiral-shaped (with respect to \( z_0 = 0 \)) if the function \( \varphi(z) \), \( \varphi(0) = 0 \), conformally maps a unit disc \( D \) onto \( \Omega_s \) and the condition satisfies [28, 32]:

\[
\left| \operatorname{arg} e^{i\gamma} \frac{z\varphi'(z)}{\varphi(z)} \right| \leq \beta \pi/2, \quad 0 \leq \beta < 1, \quad |\gamma| < \beta \pi/2, \quad |z| < 1.
\]

In [28, 32] proved the following: the boundary of the \( \beta \)-spiral-shaped domain \( \Omega_s \) is a \( K \)-quasicircle with \( K = \cot^2(1 - \beta)\pi/4 \).
Then by Corollary A we have
[\mu_1(\Omega_s) \geq 4 \sin^4 \left( (1 - \beta)\pi/4 \right) \left( \frac{d(\Omega_s)}{d(\Omega_\alpha)} \right)^2].

### 2.2. Spectral estimates in space domains.

At the best of our knowledge estimates of norms of extension operators for $L^2_1$ in space domains are not known. We found only Mikhlin’s estimates [25] for balls and star-shaped domains. Let $B_1$ and $B_R$ be balls in Euclidean space $\mathbb{R}^n$ with a common center at the origin and with radii 1 and $R$, $R > 1$, respectively. In this case, Mikhlin [25] established that the norm of extension operator

$$E_R : W^1_2(B_1) \to W^1_2(B_R)$$

is

$$\|E_R\|^2 = 1 + \frac{I_\alpha(1)}{I_{\alpha+1}(1)} \cdot \frac{I_\alpha(R)K_{\alpha+1}(1) + K_\alpha(R)I_{\alpha+1}(1)}{I_\alpha(R)K_\alpha(1) - K_\alpha(R)I_\alpha(1)}.$$  

Here $\alpha = (n-2)/2$, $n > 2$ and $I_\nu(z)$, $K_\nu(z)$ are the Bessel functions of the imaginary argument:

$$I_\nu(z) = \sum_{m=0}^{\infty} \frac{1}{m!\Gamma(m+\nu+1)} \left( \frac{z}{2} \right)^{2m+\nu},$$

$$K_\nu(z) = \frac{\pi}{2} \cdot \frac{I_{-\nu}(z) - I_\nu(z)}{\sin(\nu\pi)}.$$  

For $n = 3$ and $R = 2$ or $R = 3$ we have the following values of the norm of extension operator (2.4):

$$\|E_2\|^2 \approx 8.38905 \quad \text{and} \quad \|E_3\|^2 \approx 7.50825.$$  

Let $u(x)$ be a nonnegative continuous function of class $W^1_\infty$ in some two-sided neighborhood of the unit sphere $S := \{x: |x| = 1\}$. We set $\rho = |x|$, $\theta = x/\rho$ and denote

$$M_1 = \min_{x \in S} u(x) > 0, \quad M_2 = \max_{x \in S} u(x), \quad M_3 = \sup_{x \in S} |\nabla u(x)|.$$  

Let $\Omega_1$ and $\Omega_R$, $R > 1$, be a star-shaped domains having the form

$$\Omega_\beta := \{x: \rho < \beta u(\theta)\}, \quad \beta > 0.$$  

Then in [25] obtained the estimate of the norm of extension operator

$$E^* : W^1_2(\Omega_1) \to W^1_2(\Omega_R)$$

which have the form

$$\|E^*\|^2 \leq 1 + \left( \frac{M_2}{M_1} \right)^2 (N_1 N_2)^2 \left( \|E_R\|^2 - 1 \right),$$

where

$$N_1^2 = \max \left\{ \frac{M_2^2 + (n-1)M_3^2}{M_1^2}, \frac{2}{M_2}, 1 \right\}$$

and

$$N_2^2 = \max \left\{ \frac{M_2^2 + 2(n-1)M_3^2}{M_1^2}, 2M_2^2, 1 \right\}.$$  

Hence, by Theorem A we have the following quasi-monotonicity result for Neumann eigenvalues in star-shaped domains:
Corollary B. Let $\Omega_1$ and $\Omega_R$, $R > 1$, be star-shaped domains in Euclidean space $\mathbb{R}^n$. Then the following quasi-monotonicity estimate

$$\mu_1(\Omega_R) \leq \mu_1(\Omega_1) \|E^*\|^2 \leq 1 + \left(\frac{M_2}{M_1}\right)^2 (N_1 N_2)^2 (\|E_R\|^2 - 1)$$

holds.

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