A NOTE ON RESIDUALLY FINITE KAZHDAN GROUPS.

MASATO MIMURA

Abstract. Given a countable residually finite group, we construct a residually finite group with Kazhdan’s property (T) that contains an isomorphic copy of it. In combination with Osajda’s construction of residually finite non-exact groups, there exists a residually finite non-exact group with property (T).

1. Introduction

A (countable) group $G$ is residually finite if there exists a chain $(N_m)_{m \in \mathbb{N}}, N_{m+1} \triangleleft N_m$ for every $m$, of finite index normal subgroups of $G$ such that $\cap_{m} N_m = \{e_G\}$. This property plays a fundamental role in study of profinite groups and profinite actions. Further, from a (finitely generated) residually finite group, we can construct a box space, which serves as one of powerful devices to construct examples of metric spaces with noteworthy coarse geometric properties; see introduction of [Osa18] and [NY12, 4.4] for instance.

Given these backgrounds, it is a natural and important task to construct a residually finite group with a specified property. The problem we address here is to embed a given (countable) residually finite group into a residually finite group with Kazhdan’s property (T) (see [BdlHV08] on property (T)). Our main theorem states that it is always possible; see Remark 3.5 for stronger properties of the $\Lambda$ below.

Theorem 1.1. Given a countable residually finite group $G$, there exists a residually finite group $\Lambda$ with property (T) that contains an isomorphic copy of $G$.

Osajda [Osa18] constructed a residually finite non-(C*-exact) group $G$. For countable groups, exactness is equivalent to Yu’s property $A$, which may be regarded as the counterpart of amenability of groups in coarse geometry; see [Oza00] and [NY12], respectively, for exactness and property $A$. Non-exactness of groups is regarded as a pathological property. For instance, every countable subgroup of $\text{GL}(n, A)$ for $n \in \mathbb{N}_{\geq 2}$ and for a unital commutative (associative) ring $A$ is exact; see [GTY13, Theorem 5.2.2 and Theorem 4.6].

As a byproduct of Theorem 1.1 and of the result above of Osajda, we obtain:

Corollary 1.2. There exists a residually finite non-exact group with property (T).

Indeed, embed such a non-exact $G$ by [Osa18] into $\Lambda$; note that non-exactness passes to overgroups. Concerning this corollary, see also Remarks 3.1 and 3.2.
As an application of Corollary 1.2, by taking a box space, we may construct expanders with geometric property (T) from a non-exact group; see [WY14 Theorem 1.1.(4)].

2. The proof of Theorem 1.1

For \( n \in \mathbb{N}_{\geq 1} \), let \([n]\) denote the set \{1, 2, \ldots, n\}. Hereafter, we always assume rings to be associative; we exclude the zero ring from unital rings.

Our construction of \( \Lambda \) as in Theorem 1.1 employs elementary groups over a non-commutative ring; compare with the aforementioned result of [GTY13]. Let \( R \) be a unital ring and \( n \in \mathbb{N}_{\geq 2} \). Then the elementary group \( E(n, R) \) of degree \( n \) over \( R \) is defined by the subgroup of \( \text{GL}(n, R) \) generated by elementary matrices \( e_{i,j}^r \), \( i \neq j \in [n] \), \( r \in R \). Here for \( k, l \in [n] \), \( (e_{i,j}^r)_{k,l} \) equals 1 if \( k = l \), \( r \) if \((k, l) = (i, j)\), and 0 otherwise. The key to the proof of Theorem 1.1 is the following:

Lemma 2.1. Assume that a group \( G \) admits a subset \( S \) such that every element of \( S \) is of order 2 and that \( S \) generates \( G \) as a group. Then for every \( n \in \mathbb{N}_{\geq 2} \) and for every unital ring \( A \), the elementary group \( E(n, A[G]) \) contains an isomorphic copy of \( G \). Here \( A[G] \) denotes the group ring of \( G \) over \( A \).

Proof. Since \( E(n, R) \cong E(n + 1, R) \) for every \( n \) and \( R \), we prove the assertion for \( n = 2 \). For a unital ring \( R \) and for \( r \in R^{\times} \), we have that

\[
\begin{pmatrix}
  r & 0 \\
  0 & r^{-1}
\end{pmatrix} = e_{1,2}^r e_{2,1}^{r^{-1}} e_{1,2}^{-1} e_{2,1}^{-1}
\]

is in \( E(2, R) \). Write the left-hand side of this equality as \( D(r, r^{-1}) \). Now set \( R = A[G] \). Let \( D_S = \{ D(r, r^{-1}) : r = \delta_s, s \in S \} \), where \( \delta_s \in A[G] = R \), and \( \Gamma \) be the subgroup of \( E(2, R) \) generated by \( D_S \). Since \( s = s^{-1} \) for every \( s \in S \) by assumption, we conclude that the map \( \Gamma \ni D(\delta_g, \delta_g) \mapsto g \in G \) for \( g \in G \) gives an isomorphism. \( \square \)

Proof of Theorem 1.1. Let \( G \) be a countable residually finite group. By Wilson’s theorem [Wil80], \( G \) embeds into a residually finite group \( G_1 \) that is 2-generated. By [Mim18b] Lemma 5.1 and Remark 5.2, \( G_1 \) embeds into a residually finite group \( G_2 \) that admits a four-point set \( S \subseteq G_2 \) such that every element in \( S \) is of order 2 and that \( \langle S \rangle \cong G_2 \). For a prime \( p \), let \( \mathbb{F}_p \) denote the finite field of order \( p \).

We claim that for every \( n \in \mathbb{N}_{\geq 3} \) and for every prime \( p \), the group \( \Lambda = E(n, \mathbb{F}_p[G_2]) \) fulfills all conditions of Theorem 1.1. Indeed, by Lemma 2.1, the \( \Lambda \) above contains an isomorphic copy of \( G_2(\geq G) \). We will show that \( \Lambda \) is residually finite. Take a chain \((N_m)_m\), as in Introduction, of normal subgroups of \( G_2 \). For every \( m \in \mathbb{N} \), let \( \pi_m : G_2 \cong G_2/N_m \) be the natural projection. Then the map \( \mathbb{F}_p[G_2] \ni \delta_g \mapsto \delta_{\pi_m(g)} \in \mathbb{F}_p[G_2/N_m] \) induces a group quotient map

\[
\Lambda \cong E(n, \mathbb{F}_p[G_2/N_m]).
\]

Let \( \tilde{N}_m \) be the kernel of the map above. Then \((\tilde{N}_m)_{m\in\mathbb{N}}\) provides a desired chain of normal subgroups of \( \Lambda \). Finally, by the celebrated theorem by Ershov and Jaikin-Zapirain [EJZ10 Theorem 1.1], \( \Lambda \) has property (T); see also [Mim18a] for an alternative short proof of this statement. \( \square \)
In the proof above, the field $\mathbb{F}_p$ may be replaced with several other rings; for instance, an arbitrary unital finite ring, including all finite fields, and the ring $\mathbb{Z}$.

3. Remarks

Remark 3.1. In Theorem 1.1 it is rather standard to embed a given $G$ into a group with property (T) in the following way. Take an infinite hyperbolic group $\tilde{H}$ with property (T). Then by SQ-universality of $\tilde{H}$ [Ol'95], there exists a quotient group $H$ of $\tilde{H}$ such that $G$ embeds into $H$; note also that property (T) passes to group quotients. However, even if $\tilde{H}$ is residually finite, the procedure $\tilde{H} \twoheadrightarrow H$ may spoil the residual finiteness.

Remark 3.2. The original construction of residually finite non-exact groups by Osajda [Osa18] made essential use of the work of Wise [Wis11] and Agol [Ago13] on virtually special groups to obtain residual finiteness; these groups are direct limits of certain virtually special groups. All virtually special groups are known to have the Haagerup property, which may be seen as a strong negation of property (T) for infinite countable groups. It then follows that the originally constructed non-exact groups above (or their variant in [Mim18b, Remark 3.2]) never have property (T).

In the following three remarks, let $\Lambda = E(n, \mathbb{F}_p[G_2])$ be the group as in the proof of Theorem 1.1 in Section 2.

Remark 3.3. The group $\Lambda$ is always 5-generated if $n$ above is odd: it is generated by $e_{1,2}$, $s \in S$, and the matrix $\tau$ associated with a cyclic permutation on $[n]$. Indeed, for $s \in S$, $\tau e_{1,2}s \tau^{-1} = e_{2,3}s$ holds. Then by the following commutator relation for $r_1, r_2 \in R = \mathbb{F}_p[G_2]$:

\[(\#) \quad e_{i,j}^{r_1} e_{i,k}^{r_2} = e_{i,j}^{r_1 r_2}\]

it holds that $[e_{1,2}, e_{2,3}] = e_{1,3} e_{1,3}^{-1}$. Then by taking conjugations of $e_{1,3}$ by powers of $\tau$ and by (\#), we may obtain all elements of the form $e_{i,j}^{r_1}$, $i \neq j \in [n]$. Again by (\#), we may express every matrix of the form $e_{i,j}^r$, $i \neq j \in [n]$ and $r \in R$, as some product of the five elements $e_{1,2}, s \in S$, and $\tau$; note also that $e_{i,j}^{r_1} e_{i,j}^{r_2} = e_{i,j}^{r_1 + r_2}$.

Remark 3.4. In fact, in Theorem 1.1 every countable residually finite group embeds into a residually finite group $\Lambda_1$ with property (T) that is 2-generated. More concretely, for $\Lambda = E(n, \mathbb{F}_p[G_2])$ for $n$ odd, we may set $\Lambda_1$ as

$$\Lambda_1 = \Lambda \rtimes (\mathbb{Z}l\mathbb{Z}) \cong (\bigoplus_{\mathbb{Z}l\mathbb{Z}} \Lambda) \rtimes (((\bigoplus_{\mathbb{Z}l\mathbb{Z}} \Lambda)),$$

where $l$ is an integer bigger than $2^8$. Indeed, note that $\Lambda$ is perfect by (\#). Moreover, each element $e_{1,2}^\delta$ and $\tau$ as in Remark 3.3 can be written as a single commutator; we may thus obtain a seven-point subset $S$ of $\Lambda$ such that the set $\{[g_1, g_2] : g_1, g_2 \in S\}$ generates $\Lambda$. Then a Hall-type argument shows that the $\Lambda_1$ above is 2-generated; compare with [Mim18b, proof of Lemma 4.9]. Property (T) and residual finiteness of $\Lambda_1$ both follow because $\bigoplus_{\mathbb{Z}l\mathbb{Z}} \Lambda$ is a finite index subgroup of $\Lambda_1$.

A similar construction to one above applies to the case where $n$ is even.
Remark 3.5. By [Mim15], if we take $n$ from $\mathbb{N}_{\geq 4}$, then the groups $\Lambda$ and $\Lambda_1$ above, in fact, enjoy the fixed point property with respect to $L_q$-spaces for all $q \in (1, \infty)$. See also [Opp17, Subsection 5.2.2 and Remark 5.7] for other fixed point properties for these groups.

Acknowledgments

A part of this work has been done during the two-year stay of the author in the École Polytechnique Fédérale de Lausanne supported by Grant-in-Aid for JSPS Overseas Research Fellowships. The author wishes to express his gratitude to Professor Nicolas Monod and Mrs. Marcia Gouffon at the EPFL for their hospitality and help on his stay. He thanks Sven Raum for comments.

References

[Ago13] Ian Agol, *The virtual Haken conjecture*, Doc. Math. **18** (2013), 1045–1087, With an appendix by Agol, Daniel Groves, and Jason Manning. MR 3104553

[BdlHV08] Bachir Bekka, Pierre de la Harpe, and Alain Valette, *Kazhdan’s property (T)*, New Mathematical Monographs, vol. 11, Cambridge University Press, Cambridge, 2008. MR 2415834

[EJZ10] Mikhail Ershov and Andrei Jaikin-Zapirain, *Property (T) for noncommutative universal lattices*, Invent. Math. **179** (2010), no. 2, 303–347. MR 2570119

[GTY13] Erik Guentner, Romain Tessera, and Guoliang Yu, *Discrete groups with finite decomposition complexity*, Groups Geom. Dyn. **7** (2013), no. 2, 377–402. MR 3054574

[Mim15] Masato Mimura, *Upgrading fixed points without bounded generation*, forthcoming version (v3) of the preprint on arXiv:1505.06728 (2015).

[Mim18a] ________, *An alternative proof of Kazhdan property for elementary groups*, Topology and Analysis of Discrete Groups and Hyperbolic Spaces (RIMS, 2016), RIMS Kôkyûroku, vol. 2062, 2018, pp. 79–87.

[Mim18b] ________, *Amenability versus non-exactness of dense subgroups of a compact group*, preprint, arXiv:1805.01398v3 (2018).

[NY12] Piotr W. Nowak and Guoliang Yu, *Large scale geometry*, EMS Textbooks in Mathematics, European Mathematical Society (EMS), Zürich, 2012. MR 2986138

[O1’95] A. Yu. Ol’shanskiĭ, *SQ-universality of hyperbolic groups*, Mat. Sbornik. **186** (1995), no. 8, 119–132. MR 1357360

[Opp17] Izhar Oppenheim, *Averaged projections, angles between groups and strengthening of Banach property (T)*, Math. Ann. **367** (2017), no. 1-2, 623–666. MR 3606450

[Osa18] Damian Osajda, *Residually finite non-exact groups*, Geom. Funct. Anal. **28** (2018), no. 2, 509–517. MR 3788209

[Oza00] Narutaka Ozawa, *Amenable actions and exactness for discrete groups*, C. R. Acad. Sci. Paris Sér. I Math. **330** (2000), no. 8, 691–695. MR 1763912

[Wil80] John S. Wilson, *Embedding theorems for residually finite groups*, Math. Z. **174** (1980), no. 2, 149–157. MR 592912

[Wis11] Daniel T. Wise, *The structure of groups with quasiconvex hierarchy*, 2011.

[WY14] Rufus Willett and Guoliang Yu, *Geometric property (T)*, Chin. Ann. Math. Ser. B **35** (2014), no. 5, 761–800. MR 3246936

Masato Mimura, Mathematical Institute, Tohoku University, Japan

E-mail address: mimura-mas@m.tohoku.ac.jp