Energy and Angular Momentum Densities in a Gödel-Type Universe in the Teleparallel Geometry

A. A. Sousa*, R. B. Pereira and A. C. Silva
Instituto de Ciências Exatas e da Terra
Campus Universitário do Araguaia
Universidade Federal de Mato Grosso
78698-000 Pontal do Araguaia, MT, Brazil

December 8, 2009

Abstract

The main scope of this research consists in evaluating the energy-momentum (gravitational field plus matter) and gravitational angular momentum densities in the universe with global rotation, considering the Gödel-Obukhov metric. For this, we use the Hamiltonian formalism of the Teleparallel Equivalent of General Relativity (TEGR), which is justified for presenting covariant expressions for the considered quantities. We found that the total energy density calculated by the TEGR method is in agreement with the results reported by other authors in the literature using pseudotensors. The result found for the angular momentum density depends on the rotational parameter as expected. We also show explicitly the equivalence among the field equations of the TEGR and Einstein equations (RG), considering a perfect fluid and Gödel-Obukhov metric.

PACS NUMBERS: 98.80.-k, 04.20.-q, 04.20.Cv, 04.20.Fy

(*) E-mail: adellane@ufmt.br
1 Introduction

Determination of the energy, momentum, and angular momentum of the gravitational field is a long-standing problem in General Relativity (GR). In a general way, it is believed that the energy of the gravitational field is not localizable, that is, defined in a finite region of space. This problem is addressed in the literature by means of different approaches. The most common approach is based in the use of pseudotensors, since these gravitational physical quantities do not possess proper definitions in terms of tensorial equations [1].

In order to consider the material content of space-time, it is possible to define the energy-momentum (angular momentum) complex as the sum of the energy-momentum (angular momentum) pseudotensor of the gravitational field and the energy-momentum (angular momentum) tensor of the matter. These complexes appear with several names, such as Landau-Lifshitz, Bergman-Thompson, and Einstein, among others [2]. They differ from each other in the way in which they are constructed. These complexes have been applied to several configurations of the gravitational field, as Friedmann-Lemaître-Robertson-Walker (FLRW) [3]–[7], Bianchi I and II [8], [9], Gödel [10], and Gödel-type universes [11] and for a general non-static spherically symmetric metric of the Kerr-Schild class [12].

In an attempt to contour the localized gravitational energy problem, we consider the Weitzenböck space-time, a particular case of Riemann-Cartan space-time constrained to have zero curvature. In this space-time, the gravitational field is described in terms of the tetrad field $e^a_{\mu}$, instead of metric tensor $g_{\mu\nu}$. In the context of teleparallel gravity, the zero curvature condition means that the parallel transport of the orientation of the tetrads is path independent, that is, there is a global moving frame or teleparallelism. Mathematically, as a natural consequence of this definition, the Weitzenböck covariant derivative of the tetrad field vanishes identically: $\nabla_\nu e_{a\mu} \equiv \partial_\nu e_{a\mu} - \Gamma^\rho_{\nu\mu} e_{a\rho} = 0$, where $\Gamma^\rho_{\nu\mu}$ is the Weitzenböck connection given by $\Gamma^\rho_{\nu\mu} = e^a_\rho \partial_\nu e^a_{\mu}$. The gravitational interaction is attributed to the torsion tensor $T_{a\mu\nu} = \partial_\mu e_{a\nu} - \partial_\nu e_{a\mu}$. In Andrade & Pereira [13], the teleparallel gravity can indeed be understood as a gauge theory for the translation group. In that approach, the gravitational interaction is described by a force similar to the Lorentz force equation of electrodynamics, with torsion playing the role of force.

In 1961, Møller [14] showed that only in terms of tetrads can we obtain a
Lagrangian density that leads to a tensor of gravitational energy-momentum. This tensor, constructed from the first derivatives of the tetrads, does not annul any coordinate’s transformation.

In 2001, Maluf established the Hamiltonian formulation of the Teleparallel Equivalent of General Relativity (TEGR) without the Schwinger time gauge. In this formulation, the definition of the energy and gravitational angular momentum arises by suitably interpreting the integral forms of the constraints’ equations $C^a = 0$ and $\Gamma^{ab} = 0$, respectively [15], [16]. Several configurations of gravitational energies were investigated with success, such as the space-time configurations of de Sitter [17], conical defects [18], static Bondi [19], disclination defects [20], Kerr black hole (irreducible mass) [21], BTZ black hole [22], and Kerr anti-de Sitter [23]. The definition of gravitational angular momentum as the integral form of the constraint equation was applied satisfactorily for the gravitational field of a thin, slowly rotating mass shell [16] and for the rotational BTZ black hole [24].

On the other hand, the idea that the Universe is rotating was first mentioned by Gamow [25]. The exact solution of the Einstein equations for the model of a homogeneous, static universe with rotation was proposed for Gödel [26]. However, the Gödel equations presented serious problems, such as closed time-like curves (CTCs) that exhibit a causality violation in space-time. Several new models based on Gödel’s original idea were created in an attempt to address the problems in their model [27]–[29]. Among these ideas, we stress the nonstationary model proposed by Obukhov [30], which presents expansion as well as rotation. This generalization is known as the Gödel-Obukhov or Gödel-type anisotropic cosmological model [31]. This model would not be in conflict with any cosmological observations [32].

Despite advances in the theoretical understanding of these problems, the global rotation has not yet been detected. However, we should stress that there are weak observational evidences of global rotation of the universe [33].

In this work, we show explicitly the equivalence between GR and TEGR. More specifically, we consider the solution for an anisotropic and homogeneous universe described by the Gödel-Obukhov metric in Cartesian coordinates. The main reason for using these coordinates is to allow posterior comparison of our work with other literature results. We demonstrate that considering a perfect fluid, there is no solution that incorporates rotation and expansion simultaneously that is in accord with Ozsváth [34]. By making the rotational parameters equals to zero, we obtain the Friedmann equations of
FLRW for a spatially flat universe.

We also apply the Hamiltonian formulation and field equations of the TTEGR implemented by Maluf [15], [16] to find the energy-momentum (gravitational field plus matter) and gravitational angular momentum densities in the Gödel-Obukhov universe, irrespective of the equations of the state of the cosmic fluid. Our result for total energy density is in accord with that obtained by Rybníčková, differing by a constant factor, where the Komar superpotential was used [11]. Our result for gravitational angular momentum disagree with that obtained by Dabrowski & Garecki using the gravitational angular momentum pseudotensor of Bergman-Thomson [10].

The article is organized as follows. In Section 2 we review the Lagrangian and Hamiltonian formulation of the TTEGR. In Section 3, using the field equations of the TTEGR, we find the teleparallel version of the Friedmann equations. In Section 4, we calculate the total energy of the Gödel-Obukhov universe and compare it with those obtained from the pseudotensors. In Section 5, we find the total three-momentum of the universe. In Section 6, we also obtain the gravitational angular momentum density of this Gödel-type universe. Finally, in Section 7, we present our conclusions.

The notation is the following: space-time indices $\mu, \nu, \ldots$ and global SO(3, 1) indices $a, b, \ldots$ run from 0 to 3. Time and space indices are indicated by $\mu = 0, a = (0), (i)$. The flat, Minkowski space-time metric tensor raises and lowers tetrad indices and is fixed by $\eta_{ab} = e_{a\mu} e_{b\nu} g^{\mu\nu} = (- + ++)$. The determinant of the tetrad field is represented by $e = det(e_{a \mu})$. We use units in which $c = 1$, where $c$ is the light speed.

2 The Hamiltonian constraints equations as an energy and gravitational angular momentum equations

We will briefly recall both the Lagrangian and Hamiltonian formulations of the TTEGR. The Lagrangian density for the gravitational field in the TTEGR [35] with the cosmological constant $\Lambda$ is given by

$$L(e_{a\mu}) = -k' e \left( \frac{1}{4} T^{abc}_{\mu} T_{abc} + \frac{1}{2} T^{abc}_{\mu} T_{bac} - T^{a}_{\mu} T^a_{\mu} \right) - L_M - 2\epsilon k' \Lambda$$
\[ \equiv - k' e^{\Sigma abc} T_{abc} - L_M - 2 e k' \Lambda, \]  

(1)

where \( k' = 1/(16\pi G) \), \( G \) is the Newtonian gravitational constant, and stands for the Lagrangian density for the matter fields. As usual, tetrad fields convert space-time into Lorentz indices and vice-versa. The tensor \( \Sigma^{abc} \) is defined by

\[ \Sigma^{abc} = \frac{1}{4} (T^{abc} + T^{bac} - T^{cab}) + \frac{1}{2} (\eta^{ac} T^{b} - \eta^{ab} T^{c}), \]  

(2)

and \( T^{b} = T^{a\ b} \). The quadratic combination \( \Sigma^{abc} T_{abc} \) is proportional to the scalar curvature \( R(e) \), except for a total divergence. The field equations for the tetrad field read

\[ e_{a\lambda} e_{b\mu} \partial_\nu (e^{\Sigma b\lambda\nu}) - e \left( \Sigma^{b\nu} a T_{b\nu\mu} - \frac{1}{4} e_{a\mu} T_{bcd} \Sigma^{bcd} \right) + \frac{1}{2} e e_{a\mu} \Lambda = \frac{1}{4k'} e T_{a\mu}. \]  

(3)

where \( e T_{a\mu} = \delta L_M / \delta e^{a\mu} \). It is possible to prove by explicit calculations that the left hand side of Eq. (3) is exactly given by

\[ \frac{1}{2} e \left\{ R_{a\mu}(e) - \frac{1}{2} e_{a\mu} R(e) + e_{a\mu} \Lambda \right\}, \]  

(4)

and thus it follows that the field equations arising from the variation in \( L \) with respect to \( e^{a\mu} \) are strictly equivalent to Einstein’s equations in tetrad form.

The field equations (3) may be rewritten in the form

\[ \partial_\nu (e^{\Sigma a\lambda\nu}) = \frac{1}{4k'} e^{a\mu} \left( t^{\lambda\mu} + \tilde{T}^{\lambda\mu} \right), \]  

(5)

where

\[ t^{\lambda\mu} = k' \left( 4 \Sigma^{b\lambda\mu} T_{b\mu} - g^{\lambda\mu} \Sigma^{bcd} T_{bcd} \right), \]  

(6)

and

\[ \tilde{T}^{\lambda\mu} = T^{\lambda\mu} + 2k' g^{\lambda\mu} \Lambda, \]  

(7)

are interpreted as the gravitational energy-momentum tensor [36] and the matter energy-momentum tensor, respectively.
The Hamiltonian formulation of the TEGR is obtained by first establishing the phase space variables. The Lagrangian density does not contain the time derivative of the tetrad component. Therefore this quantity will arise as a Lagrange multiplier \[37\]. The momentum canonically conjugated to \(e_{ai}\) is given by \(\Pi^{ai} = \delta L/\delta \dot{e}_{ai}\). The Hamiltonian formulation is obtained by rewriting the Lagrangian density in the form \(L = p\dot{q} - H\), in terms of \(e_{ai}\), \(\Pi^{ai}\) and Lagrange multipliers. The Legendre transform can be successfully carried out, and the final form of the Hamiltonian density is expressed as \[15\]

\[H = e_{a0}C^a + \alpha_{ik}^*\Gamma^{ik} + \beta_k^*\Gamma^k,\]

plus a surface term. Here \(\alpha_{ik}\) and \(\beta_k\) are Lagrange multipliers that (after solving the field equations) are identified as \(\alpha_{ik} = 1/2(T^{0i}_{0k} + T^{0k}_{0i})\) and \(\beta_k = T^{00}_{0k}\). \(C^a\), \(\Gamma^{ik}\) and \(\Gamma^k\) are first class constraints. The Poisson bracket between any two field quantities \(F\) and \(G\) is given by

\[\{F, G\} = \int d^3x \left( \frac{\delta F}{\delta e_{ai}(x)} \frac{\delta G}{\delta \Pi^{ai}(x)} - \frac{\delta F}{\delta \Pi^{ai}(x)} \frac{\delta G}{\delta e_{ai}(x)} \right).\] \[9\]

The constraint \(C^a\) is written as \(C^a = -\partial_i\Pi^a_{0i} + h^a\), where \(h^a\) is an intricate expression of the field variables. The integral form of the constraint equation \(C^a = 0\) motivates the definition of the energy-momentum four-vector \[15\]

\[P^a = -\int_V d^3x \partial_i \Pi^a_{0i},\]

where \(V\) is an arbitrary volume of the three-dimensional space. In the configuration space we have

\[\Pi^a_{0i} = -4k' e\Sigma^a_{00i}.\]

The emergence of total divergences in the form of scalar or vector densities is possible in the framework of theories constructed out of the torsion tensor. Metric theories of gravity do not share this feature.

By making \(\lambda = 0\) in Eq. (5) and identifying \(\Pi^{ai}\) on the left hand side of the latter, the integral form of Eq. (5) can be written as

\[P^a = \int_V d^3x e e^a_{\mu} \left( t^{0\mu} + \tilde{T}^{0\mu} \right).\] \[12\]

This equation suggests that \(P^a\) is now understood as the total, gravitational, and matter fields (plus a cosmological constant fluid) energy-momentum \[36\].

6
The spatial components $P^{(i)}$ form a total three-momentum, while a temporal component $P^{(0)}$ is the total energy (gravitational field plus matter) [1].

It is important to rewrite the Hamiltonian density $H$ in the most simple form. It is possible to simplify the constraints which may be rewritten as a single constraint $\Gamma^{ab}$. The Hamiltonian density (8) may be written in the equivalent form [16]

$$H = e_{a0}C^a + \frac{1}{2}\lambda_{ab}\Gamma^{ab},$$

where $\lambda_{ab} = -\lambda_{ba}$ are Lagrange multipliers that are identified as $\lambda_{ik} = \alpha_{ik}$ and $\lambda_{0k} = -\lambda_{k0} = \beta_k$. The constraints $\Gamma^{ab} = -\Gamma^{ba}$ embody both constraints $\Gamma^{ik}$ and $\Gamma^k$ by means of relations

$$\Gamma^{ik} = e_a^i e_b^k \Gamma^{ab},$$

and

$$\Gamma^k \equiv \Gamma^{0k} = \Gamma^{ik} = e_a^0 e_b^k \Gamma^{ab}.$$  

It reads

$$\Gamma^{ba} = M^{ab} + 4k' e \left( \Sigma^{a0b} - \Sigma^{b0a} \right).$$

Similarly to the definition of $P^a$, the integral form of the constraint equation $\Gamma^{ab} = 0$ motivates the new definition of the space-time angular momentum. The equation $\Gamma^{ab} = 0$ implies

$$M^{ab} = -4k' e \left( \Sigma^{a0b} - \Sigma^{b0a} \right).$$

Therefore Maluf defines [16]

$$L^{ab} = \int_V d^3 x e^a_\mu e^b_\nu M^{\mu\nu},$$

where

$$M^{ab} = e^a_\mu e^b_\nu M^{\mu\nu} = -M^{ba}.$$
The quantities \( P^a \) and \( L^{ab} \) are separately invariant under general coordinate transformations of the three-dimensional space and under time reparametrizations, which is an expected feature since these definitions arise in the Hamiltonian formulation of the theory. Moreover these quantities transform covariantly under global \( \text{SO}(3,1) \) transformations.

### 3 Teleparallel version of Gödel-Obukhov equations

The line element of the Gödel-Obukhov or Gödel-type cosmological model is given by the interval [30]

\[
ds^2 = -dt^2 + 2a(t)\sqrt{\sigma e^{mx}} dt dy + a(t)^2 \left( dx^2 + ke^{2mx} dy^2 + dz^2 \right).\]  

where \( m, \sigma \) and \( k \) are the constant parameters and \( a(t) \) is the time-dependent cosmological scale factor. Here, we adopted the signature \((-+++\))

A general analysis [30], [38] of the kinematic properties of such a space-time shows that the closed time-like curves are absent in this manifold when \( k > 0 \) (\( \sigma > 0 \) by definition). We obtain the usual Gödel metric [26] by making \( a(t) = 1, \sigma = 1, m = 1 \) and \( k = -1/2 \) in the expression (20).

Using the relations

\[
g_{\mu\nu} = e^a_{\mu} e_{a\nu},
\]

and

\[
e_{a\mu} = \eta_{ab} e^b_{\mu},
\]

a set of tetrads fields that satisfy the metric is given by [39]

\[
e^a_{\mu} = \begin{pmatrix}
1 & 0 & -a(t)\sqrt{\sigma e^{mx}} & 0 \\
0 & a(t) & 0 & 0 \\
0 & 0 & a(t)\sqrt{k + \sigma e^{mx}} & 0 \\
0 & 0 & 0 & a(t)
\end{pmatrix}.
\]

This set of tetrads fields yields the velocity field given by \( e_{(0)}^\mu \) \((t, x, y, z) = (1, 0, 0, 0)\). According to the physical interpretation of Eq. (23), the latter is adapted to static observers in space-time [16].
Now, with the help of the inverse metric tensor $g^{\mu\nu}$, we can write the inverse tetrads \( e_a^\mu = g^{\mu\nu} e_a^\nu \),

\[
(24)
\]
as
\[
(25)
\]
where the determinant of \( e^a_\mu \) is
\[
(26)
\]
Before solving the field equations, it is necessary to consider the material content of the universe. We restrict our consideration here to the stress-energy-momentum tensor of a perfect fluid \([40]\) given by
\[
(27)
\]
where \( \rho = \rho(x) \) is the matter energy density, \( p \) is the matter pressure and \( u_\mu \) are velocity components. Here, we denote the velocity with respect to the comoving matter by
\[
u^a = \delta_a^\alpha,
\]
which in covariant notation is written as \( u_\mu = g_\mu\alpha \delta_\alpha^\mu \). Thus, we can write the nonzero components of the stress-energy-momentum tensor \( T_{\mu\nu} \) as
\[
(28)
\]
\[
(29)
\]
\[
(30)
\]
\[
(31)
\]
\[
(32)
\]
Now, it is convenient rewrite the field equations of the TEGR (3) as
\[
9
\]
\[ + \frac{1}{2} e e_{a\mu} \Lambda = \frac{1}{4k} e e_{a} \gamma T_{\gamma \mu}, \quad (33) \]

where:

\[
\Sigma^{abc} = \frac{1}{4} \left( \eta^{ad} e_{by} e^{cv} T_{d\mu\nu} + \eta^{bd} e_{ay} e^{cv} T_{d\mu\nu} - \eta^{cd} e_{ay} e^{bv} T_{d\mu\nu} \right) \\
+ \frac{1}{2} \left( \eta^{ac} e_{by} e^{dv} T_{d\mu\nu} - \eta^{ab} e_{cy} e^{dv} T_{d\mu\nu} \right), \quad (34)\]

The nonzero components of the torsion tensor \( T_{a\mu\nu} \) are given by

\[
T_{(1)01} = -T_{(1)10} = T_{(3)03} = -T_{(3)30} = \dot{a}, \quad (35) \\
T_{(2)02} = -T_{(2)20} = \dot{a} \sqrt{k + \sigma e^{mx}}, \quad (36) \\
T_{(0)02} = -T_{(0)20} = \dot{a} \sqrt{\sigma e^{mx}}, \quad (37) \\
T_{(0)12} = -T_{(0)21} = m \dot{a} \sqrt{\sigma e^{mx}}, \quad (38) \\
T_{(2)12} = -T_{(2)21} = m a \sqrt{k + \sigma e^{mx}}, \quad (39) \]

remembering that the torsion components are antisymmetrical under the exchange of the two last indexes.

After tedious but straightforward calculations, we obtain the nonzero components of the tensor \( \Sigma^{abc} \)

\[
\Sigma^{(0)(1)(0)} = -\Sigma^{(0)(0)(1)} = \frac{m}{2a}, \quad (40) \\
\Sigma^{(3)(3)(1)} = -\Sigma^{(3)(1)(3)} = \frac{m}{2a}, \quad (41) \\
\Sigma^{(0)(1)(2)} = -\Sigma^{(0)(2)(1)} = -\frac{m}{4a} \sqrt{\frac{\sigma}{k + \sigma}}, \quad (42) \\
\Sigma^{(1)(2)(0)} = -\Sigma^{(1)(0)(2)} = \frac{m}{4a} \sqrt{\frac{\sigma}{k + \sigma}}, \quad (43) \\
\Sigma^{(2)(1)(0)} = -\Sigma^{(2)(0)(1)} = -\frac{m}{4a} \sqrt{\frac{\sigma}{k + \sigma}}, \quad (44) \\
\Sigma^{(1)(1)(0)} = -\Sigma^{(1)(0)(1)} = -\frac{\dot{a}}{a}, \quad (45) \\
\Sigma^{(2)(2)(0)} = -\Sigma^{(2)(0)(2)} = -\frac{\dot{a}}{a}. \quad (46) \]
\[
\Sigma^{(3)(0)(3)} = -\Sigma^{(3)(0)(3)} = -\frac{\dot{a}}{a},
\]
(47)
\[
\Sigma^{(0)(2)(0)} = -\Sigma^{(0)(2)(0)} = -\frac{\dot{a}}{a} \sqrt{\frac{\sigma}{k + \sigma}},
\]
(48)
\[
\Sigma^{(1)(2)(1)} = -\Sigma^{(1)(2)(1)} = -\frac{\dot{a}}{a} \sqrt{\frac{\sigma}{k + \sigma}},
\]
(49)
\[
\Sigma^{(3)(3)(2)} = -\Sigma^{(3)(3)(2)} = \frac{\dot{a}}{a} \sqrt{\frac{\sigma}{k + \sigma}}.
\]
(50)

Next, we proceed to obtain the components of the field equations in the model of the Gödel-Obukhov universe. We mention here only the details used to write the components \( a = (0), \mu = 0 \) of field equations. We divide the field equations into five parts. Thus we rewrite the field equations as

\begin{align*}
\text{Part 1} &= e(0)_{\lambda} e_{\nu 0} \partial_\nu \left( e e^\lambda e^\nu \Sigma^{bcd} \right) \\
&= e(0)_{\nu} e_{\nu 0} \partial_\nu \left( e e(0)^0 e(1)^1 \Sigma^{(0)(0)(1)} \right) + \\
&+ e(0)_{\nu} e_{\nu 0} \partial_\nu \left( e e(1)^1 e(2)^0 \Sigma^{(0)(2)(1)} \right) + \\
&+ e(0)_{\nu} e_{\nu 0} \partial_\nu \left( e e(2)^2 e(0)^0 \Sigma^{(0)(2)(0)} \right) + \\
&+ e(0)_{\nu} e_{\nu 0} \partial_\nu \left( e e(1)^1 \Sigma^{(0)(2)(1)} \right) \\
&= m^2 a e^m x \left( \frac{2k + \sigma}{\sqrt{k + \sigma}} \right) - \frac{e^m x \sigma}{\sqrt{k + \sigma}} \left( a \dot{a}^2 + a^2 \ddot{a} \right) \\
&\quad : \\
\text{Part 2} &= -e e^\nu \eta_{\nu d} \Sigma^{bcd} T_{b d 0} \\
&= ee(1)^1 \Sigma^{(1)(1)(0)} T_{(1)10} + ee(2)^2 \Sigma^{(0)(2)(0)} T_{(0)20} + ee(2)^2 \Sigma^{(2)(2)(0)} T_{(2)20} + \\
&+ ee(3)^3 \Sigma^{(3)(3)(0)} T_{(3)30} \\
&= 3a^2 a \sqrt{k + \sigma} e^m x - \frac{a^2 a \sigma e^m x}{\sqrt{k + \sigma}} \\
&\quad : \\
\text{Part 3} &= \frac{1}{4} e e_{\mu \lambda} e^\lambda e^\nu T_{\nu \lambda \mu} \Sigma^{bcd} \\
&= \frac{1}{4} e e_{\mu \lambda} \left( 2 e(0)^0 e(1)^1 T_{(1)01} \Sigma^{(1)(0)(1)} + 2 e(0)^0 e(2)^2 T_{(2)02} \Sigma^{(2)(0)(2)} \right) +
\end{align*}
Substituting Eqs. (23), (25), (26), (28), and (35)–(50) into the above equation, and summing the parts, we obtain

\[
-\frac{2\sigma}{k + \sigma} \left( \frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} \right) + \frac{3\dot{a}^2}{a^2} \left( \frac{k}{k + \sigma} \right) - \frac{m^2 k}{a^2 (k + \sigma)} - \omega^2 - \Lambda = -8\pi G \varepsilon ,
\]

where the dot indicates the temporal derivative and \( \omega \) is the global angular velocity of the universe [31] given by

\[
\omega = \frac{m}{2a} \left[ \frac{\sigma}{(k + \sigma)} \right]^{1/2} .
\]

Repeating the procedure for the components \( a = (1), \mu = 0; a = (1), \mu = 1; a = (0), \mu = 2; a = (2), \mu = 0; \) and \( a = (3), \mu = 3; \) we obtain, respectively, the independent equations

\[
\frac{m \ddot{a}}{\sqrt{k + \sigma}} - \frac{\sigma}{\sqrt{k + \sigma}} = 0 ,
\]

\[
\frac{2a \ddot{a} + \ddot{a}^2}{a^2} \left( \frac{k}{k + \sigma} \right) - \omega^2 - \Lambda = -8\pi G p ,
\]

\[
\frac{2a \ddot{a} + \ddot{a}^2}{a^2} \left( \frac{k}{k + \sigma} \right) - \frac{m^2 k}{a^2 (k + \sigma)} - \omega^2 - \Lambda = 8\pi G \varepsilon ,
\]

\[
\frac{(\ddot{a} a - \dot{a}^2)}{a^2} \sqrt{\sigma} = 0
\]
\[
\frac{2a\ddot{a} + \dot{a}^2}{a^2} \left( \frac{k}{k + \sigma} \right) - \frac{m^2 k}{a^2 (k + \sigma)} - 3\omega^2 - \Lambda = -8\pi G p. \quad (57)
\]

Equation (53) reveals the impossibility of the simultaneous existence of rotation and expansion using a perfect fluid matter source for the Gödel-Obukhov metric.

The field equations (53)–(57) represent the Gödel-Obukhov universe in the context of the TEGR. Fixing \( m = 0, \sigma = 0 \) and \( k = 1 \), they reduce to the Friedmann equations for the flat universe

\[
\frac{3\dot{a}^2}{a^2} - \Lambda = 8\pi G \varepsilon, \\
\frac{2a\ddot{a} + \dot{a}^2}{a^2} - \Lambda = -8\pi G p. \quad (58)
\]

Moreover, Eqs. (53)–(57) are according to the works of Krechet & Panov [41], and Korothii & Obukhov [42], using the Einstein equations and a perfect fluid

### 4 Total energy of the Gödel-Obukhov model

Let us now calculate the total energy of the Gödel-Obukhov universe using the equations shown in Section 2. By making \( \lambda = 0 \), and substituting Eq. (5) in (12) and using

\[
\Sigma^{a\lambda\nu} = \Sigma^{abc} e_b^\lambda e_c^\nu, \quad (59)
\]

we have

\[
P^{a} = 4k' \int_V d^3x \partial_i \left( e \Sigma^{abc} e_b^0 e_c^i \right). \quad (60)
\]

As already observed, the temporal component represents the system energy. Therefore the energy will be given by

\[
P^{(0)} = 4k' \int_V d^3x \partial_i \left( e \Sigma^{(0)bc} e_b^0 e_c^i \right). \quad (61)
\]

Such a quantity can be written as

\[
P^{(0)} = 4k' \int_V d^3x \partial_i \left( ee^{(0)} e^{(1)} \Sigma^{(0)(0)(1)} + ee^{(2)} e^{(1)} \Sigma^{(0)(2)(1)} \right). \quad (62)
\]
By substituting Eqs. (25), (26), (40), and (42) in (62), we obtain

\[ P^{(0)} = -k'am^2 \frac{2k + \sigma}{\sqrt{k + \sigma}} \int_V d^3x e^{mx}. \]  

(63)

Simplifying the above expression, it follows that the energy density is given by

\[ \mathcal{E}^{(0)} = -\frac{am^2}{16\pi G} \frac{2k + \sigma}{\sqrt{k + \sigma}} e^{mx}. \]  

(64)

We note that \( \mathcal{E}^{(0)} \) is negative and clearly diverges when integrated in the entire space.

We can now compare our results with those obtained by Rybníčková [11] and Dabrowski & Garecki [10].

Rybníčková obtained the total energy density, \( \omega^0 \), using the Komar superpotential, as

\[ \omega^0 = -\frac{am^2e^{mx}}{16\pi \sqrt{k + \sigma}}. \]

Dabrowski and Garecki used the pseudotensor of Einstein for the stationary metric of the acausal Gödel model and arrived at the null result for the total energy density [10]. They found a negative total energy density for the causal Gödel model. Sharif [43] calculated the total energy density associated with a space-time homogeneous Gödel-type metric by using Einstein and Papapetrou energy-momentum complexes. Sharif’s results are not in agreement with our result and he found that the two definitions of energy-momentum complexes do not provide the same result for this type of metric.

Our result and that obtained by Rybníčková present the same dependence on \( a(t) \) and \( x \), but differ by a constant factor. Moreover, we see that \( \mathcal{E}^{(0)} = 0 \), in the case \( k = -1/2, m = \sigma = 1 \), corresponding to the energy density of the Gödel universe (rotation only). Then, our result is compatible with those of the works of Rybníčková and Dabrowski & Garecki for the acausal model. They also found a nonzero total density energy for the causal Gödel model.

In particular, fixing \( m = \sigma = 0 \) (without rotation) and \( k = 1 \) all these the results are identical to zero

\[ \mathcal{E}^{(0)} = \omega^0 = 0, \]

recovering the result for the flat FLRW universe.

These results for the flat universe were also found by Rosen [3], Cooperstock [4], Johri [5], and Garecki [6].
5 The total momentum of the Gödel-Obukhov model

Let us now consider the calculation of the total three-momentum (matter plus gravitational field) of the FLRW Universe. As seen in Section 2, it is noted that the total three-momentum is given by space components \( a = \{1\}, \{2\} \) and \( \{3\} \) of Eq. (60).

In order to obtain the space component \( a = \{1\} \) of the total momentum, we can write the quantity \( P^{(1)} \) as

\[
P^{(1)} = 4k' \int_V d^3x \partial_1 (ee_0^0 e_1^1 \Sigma^{(1)(0)(1)} + ee_2^0 e_1^1 \Sigma^{(1)(2)(1)}).
\]  

(65)

By substituting (25), (26), (45), and (49) in the previous equation, we obtain

\[
P^{(1)} = 4k' \dot{a} \alpha m k \sqrt{k + \sigma} \int_V d^3 x e^{m x}.
\]  

(66)

Thus, the spatial momentum density \( \wp^{(1)} \) it is given by

\[
\wp^{(1)} = \frac{\dot{a} \alpha m k}{4\pi G \sqrt{k + \sigma}} e^{m x}.
\]  

(67)

For the absolutely analogous calculation, we have

\[
\wp^{(2)} = \frac{m^2 a \sqrt{\sigma}}{16\pi G} e^{m x},
\]  

(68)

\[
\wp^{(3)} = 0.
\]  

(69)

We can observe that Eq. (67) is valid for any fluid matter. However, it shows that considering a perfect fluid, for a purely rotational or expansion universe, the spatial momentum density \( \wp^{(1)} \) is zero. Equation (68) shows that the spatial momentum density \( \wp^{(2)} \) is zero only for an expansion universe. We can note again that \( \wp^{(1)} \) and \( \wp^{(2)} \) diverge when integrated in the entire space.
6 The Gravitational angular momentum of the Gödel-Obukhov model

Let us verify the expression of gravitational angular momentum (18). By making use of (19) and (17), we can write (18) in the form

\[ L^{ab} = -4k' \int_V d^3x \epsilon \left( \Sigma^{a0b} - \Sigma^{b0a} \right). \]

(70)

By making use of \( \Sigma^{a0b} = e^c_0 \Sigma^{acb} \) and using the determinant of tetrads, Eq.(26), and the components of the tensor \( \Sigma^{abc} \), (40)–(50), we find that the components of the three-angular momentum densities \( L^{(1)(3)} \) and \( L^{(2)(3)} \) vanish. The unique nonzero gravitational three-angular momentum density component is given by

\[ L^{(1)(2)} = -L^{(2)(1)} = \frac{ma^2 \sqrt{\sigma}}{8\pi G} e^{mx}. \]

(71)

The above result demonstrates that there is only a direction to the gravitational angular momentum, as expected. The component \( L^{(1)(2)} \) found here diverges when integrated in the entire space.

The other components \( L^{(0)(i)} \) are all null except for

\[ L^{(0)(1)} = -L^{(1)(0)} = \frac{ma^2 \sqrt{k+\sigma}}{8\pi G} e^{mx}, \]

(72)

which, although it represents the component of the gravitational center of mass moment, does not possess physical meaning [44].

Our result Eq.(71) when fixing \( k = -1/2, m = \sigma = 1 \) is not in agreement with those obtained by Dabrowski & Garecki which used the gravitational angular momentum pseudotensor of Bergmann-Thomson to calculate the gravitational angular momentum density [10]. Indeed as they used the pseudotensor to calculate the gravitational angular momentum, their results are not coordinate invariant.

7 Conclusions

In this work, we show explicitly the equivalence among equations obtained with the TEGR and those obtained by the GR for the Gödel-Obukhov metric and also calculated the total energy-momentum and gravitational angular
momentum densities with the use of tensorial expressions of the TERG, irrespective of the equations of state of the cosmic fluid metric. In the case in which the rotation parameters equals to zero, we recovered the flat FLRW universe.

Our result for the energy density presents the same dependence on $a(t)$ and $x$ as that found by Rybníčková. Both results diverge when integrated in the entire space. Fixing $k = -1/2, m = \sigma = 1$ (parameters of the stationary Gödel metric), the energy density vanishes in accordance with the results obtained by Dabrowski & Garecki.

By analyzing the equations for the total three-momentum (matter plus gravitational field) we conclude that all these components vanish simultaneously in the universe without rotation.

As we used tetrads fields adapted to static observers in space-time, we found a unique nonzero component of the gravitational angular momentum density with physical meaning, $L^{(1)(2)}$, which reflects the preferred direction ($z$-axis) related to the cosmic rotation.

We finally conclude from this work that the TEGR obtained equivalent results to the GR with the great advantage of addressing covariantly the definitions of quantities as energy-momentum and angular momentum tensors of the gravitational field.

In order to find a solution that simultaneously permits rotation and expansion, we will consider others types of fluids as the Chaplygin gas. Efforts in this respect will be carried out.

Acknowledgements
One of us (A. C. S.) would like to thank the Brazilian agency CAPES by financial support.

References

[1] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon Press, Oxford, 1980).

[2] J. M. Aguirregabiria, A. Chamorro, K. S. Virbhadra, Gen. Rel. Grav. 28, 1393 (1996).

[3] N. Rosen, Gen. Rel. Grav. 26, 319 (1994).
[4] F. I. Cooperstock, Gen. Rel. Grav. **26**, 323 (1994).

[5] V. B. Johri, D. Kalligas, G. P. Singh and C. W. F. Everitt, Gen. Rel. Grav. **27**, 313 (1995).

[6] J. Garecki, Gen. Rel. Grav. **27**, 55 (1995).

[7] T. Vargas, Gen. Rel. Grav. **36**, 1255 (2004).

[8] N. Banerjee and S. Sen, Pramana J. Phys. **49**, 609 (1999); I. Radinschi, Acta Phys. Slov. **49**, 789 (1999); S. Xulu, Int. J. Theor. Phys. **39**, 1153 (2000)

[9] L. L. So, and T. Vargas, Chin. J. Phys. **43**, 901 (2005).

[10] M. P. Dabrowski and J. Garecki, Phys. Rev. **D70**, 043511 (2004).

[11] J. Rybnícková, *Energy of some Gödel type models*, Proceedings of the 8th International Conference on Differential Geometry and Its Applications, edited by O. Kowalski, D. Krupka and J. Slovak (Silesian University, Opava, 2001).

[12] K.S. Virbhadra, Phys. Rev. **D60**, 104041 (1999).

[13] V. C. Andrade and J. G. Pereira, Phys. Rev. **D56**, 4689 (1997).

[14] C. Möller, Annalen. Phys. **12**, 118 (1961).

[15] J. W. Maluf and J. F. da Rocha-Neto, Phys. Rev. **D64**, 084014 (2001).

[16] J. W. Maluf, S. C. Ulhoa, F. F. Faria and J. F. da Rocha-Neto, Class. Quantum Grav. **23**, 6245 (2006)

[17] J. W. Maluf, J. Math. Phys. **37**, 6293 (1996).

[18] J. W. Maluf, E. F. Martins and A. Kneip, J. Math. Phys. **37**, 6310 (1996).

[19] J. W. Maluf and J.F. da Rocha-Neto, J. Math. Phys. **40**, 1490 (1999).

[20] J. W. Maluf and A. Goya, Class. Quant. Grav. **18**, 5143 (2001).
[21] J. W. Maluf, J. F. da Rocha-Neto, T. M. L. Toribio and K. H. Castello-Branco, Phys. Rev. D65, 124001 (2002).

[22] A. A. Sousa and J. W. Maluf, Prog. Theor. Phys. 108, 457 (2002).

[23] J. F. da Rocha-Neto and K. H. Castello-Branco, JHEP 0311, 002 (2003).

[24] A. A. Sousa, R. B. Pereira and J. F. Rocha-Neto, Prog. Theor. Phys., 114, 1179 (2005).

[25] G. Gamow, Phys. Rev. 70, 573 (1946).

[26] K. Gödel, Rev. Mod. Phys. 21, 450 (1949).

[27] H. Stein, Phil. Sci. 37, 601 (1970).

[28] J. Pfarr, Gen. Relat. Grav. 13, 1091 (1981).

[29] L. Pimentel, A. Camacho, and A. Macias, Mod. Phys. Lett. A 9 3712, (1994).

[30] Y. N. Obukhov, On physical foundations and observational effects of cosmic rotation, e-print:astro-ph/0510803, (2000).

[31] M. J. Reboucas and J. Tiomno, Phys. Rev., D28, 1251 (1983).

[32] P. Jain, M. S. Modgil, and J. P. Ralston, Search for Global Metric Anisotropy in Type Ia Supernova Data, e-print:astro-ph/0008106, (2005).

[33] S. A. Gregory, L. A. Thompson, and W. G. Tifft, Astrophys. J. 243, 411 (1981); W. Godlowski, M. Szydlowski, P. Flin, and M. Biernacka, Gen. Rel. Grav. 35, 907 (2003); D. Palle, Nuovo Cim. B 119, 1124 (2004); B. Nodland, J. P. Ralston, Phys. Rev. Lett. 78, 3046 (1997); S. M. Carroll and G. B. Field, Phys. Rev. Lett. 79, 2397, (1997).

[34] I. Oszváth, J. Math. Phys. 6, 590 (1965).

[35] J. W. Maluf, J. Math. Phys. 35, 335 (1994).

[36] V.C. Andrade, L. C. T. Guilhen and J.G. Pereira, Phys. Rev. Lett. 84, 4533 (2000).
[37] P. A. M. Dirac, *Lectures on Quantum Mechanics* (Belfer Graduate School of Science (Monographs Series No 2), Yeshiva University, New York, 1964).

[38] S. Maitra, J. Math. Phys. 7, 1030 (1966).

[39] Y. N. Obukhov and T. Vargas, Phys. Lett. A, 327, 365 (2004).

[40] R. A. d’Inverno, *Introducing Einstein’s Relativity* (Clarendon Press, Oxford, 1992).

[41] V. G. Kretchet, and V. F. Panov, Astrophysics 28, 400 (1988).

[42] V. A. Korotkii, and Y. N. Obukhov, Russ. Phys. J. 36, 568 (1993).

[43] M. Sharif, Int. J. Mod. Phys. D13, 1028 (2004).

[44] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (John Wiley & Sons, 1972).