ON SOLUTIONS OF SEMILINEAR UPPER DIAGONAL INFINITE SYSTEMS OF DIFFERENTIAL EQUATIONS

JÓZEF BANAŚ *
Department of Nonlinear Analysis
Rzeszów University of Technology
Al. Powstańców Warszawy 8, 35-959 Rzeszów, Poland

MONIKA KRAJEWSKA
Institute of Economic and Management
State Higher School of Technology and Economics in Jarosław
ul. Czarnieckiego 16, 37-500 Jarosław, Poland

Dedicated to Professor Vicentiu Radulescu
on the occasion of his 60th anniversary

Abstract. The goal of the paper is to investigate the existence of solutions for semilinear upper diagonal infinite systems of differential equations. We will look for solutions of the mentioned infinite systems in a Banach tempered sequence space. In our considerations we utilize the technique associated with the Hausdorff measure of noncompactness and some existence results from the theory of ordinary differential equations in abstract Banach spaces.

1. Introduction. The principal goal of the paper is to study the solvability of some kind of infinite systems of differential equations. More precisely, we will investigate semilinear upper diagonal infinite systems of differential equations which are perturbed by nonlinear terms. Our investigations are located in the Banach space consisting or real sequences converging to zero if we temper them appropriately. Such an approach guarantees that we can use the technique of measures of noncompactness being very convenient and fruitful tool in the study of the existence of solutions of various types of operator equations (differential, integral, integrodifferential etc.) (cf. [1, 2, 3, 5, 8], for example).

It is worthwhile mentioning that infinite systems of differential equations can be considered as special cases of ordinary differential equations treated in abstract Banach spaces [5, 7]. Indeed, an infinite system of ordinary differential equations can be always considered as a special case of an ordinary differential equation in some Banach sequence space.

In our study conducted in the paper we will investigate infinite systems of differential equations in the Banach space $c_0^\beta$ consisting of real sequences converging to zero if we temper them by a suitable tempering sequence $\beta = (\beta_n)$. Such an approach is very convenient in applications. Indeed, choosing an appropriate

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* Corresponding author: Józef Banaś.
tempering sequence $\beta$ we can always “reduce” any Banach sequence space to the mentioned Banach sequence space $c_0^\beta$ containing sequences converging to zero after suitable temperation.

On the other hand, in the mentioned Banach tempered sequence space $c_0^\beta$ we can use the so-called Hausdorff measure of noncompactness being the most convenient measure in applications as far as we know (cf. [1, 3, 6, 9]).

It is worthwhile noticing that numerous real world problems encountered in mathematical physics, mechanics, engineering, the theory of branching processes, the theory of neural nets etc. lead to infinite systems of ordinary differential equations in some Banach sequence space (cf. [4, 7]). On the other hand some numerical methods of solving of partial differential equations also lead to investigations connected with ordinary differential equations in abstract Banach spaces [7].

The investigations of the present paper create the continuation and extension of the study conducted in paper [4], where we considered semilinear lower diagonal infinite systems of differential equations. Similarly as in the present paper, the considerations of the mentioned paper [4] were also located in Banach tempered sequence spaces. However, despite of the fact that methods applied in our research of this paper are similar to those from [4], the problem of the solvability of semilinear upper diagonal infinite systems of differential equations considered in this paper forced to develop some new methods and tools than those used in [4]. By these regards the results obtained in this paper create a generalization of those from [4].

2. Auxiliary facts. In this section we provide a few auxiliary facts which will be used in our further considerations.

At the beginning we establish some notation. Namely, by $\mathbb{R}$ we will denote the set of real numbers and we put $\mathbb{R}_+ = [0, \infty)$. The symbol $\mathbb{N}$ stands for the set of natural numbers (positive integers).

Next, if $E$ is a Banach space with the norm $\|\cdot\|$ and with the zero element $\theta$, we denote by $B(x, r)$ the closed ball in $E$ centered at $x$ and with radius $r$. We will write $B_r$ to denote the ball $B(\theta, r)$. If $X$ is a subset of $E$ then by $\overline{X}$, Conv$X$ we denote the closure and convex closure of $X$, respectively. Moreover, the symbols $X + Y$, $AX$ ($A \in \mathbb{R}$) stand for algebraic operations on sets $X$ and $Y$.

Further, let $\mathcal{M}_E$ denote the family of all nonempty and bounded subsets of the space $E$ and $\mathcal{N}_E$ its subfamily consisting of relatively compact sets.

For an arbitrary set $X \in \mathcal{M}_E$ we define number $\chi(X)$ by putting

$$\chi(X) = \inf \{\varepsilon > 0 : X \text{ has a finite } \varepsilon - \text{net in } E\}.$$ 

The quantity $\chi(X)$ defined in such a way is called the Hausdorff measure of noncompactness of the set $X$ [3]. It can be shown that this quantity has the following useful properties [3, 5]:

1° $\chi(X) = 0 \Leftrightarrow X \in \mathcal{N}_E.$

2° $X \subset Y \Rightarrow \chi(X) \leq \chi(Y).$

3° $\chi(X) = \chi(\text{Conv} X) = \chi(X).$

4° $\chi(X + Y) \leq \chi(X) + \chi(Y).$

5° $\chi(\lambda X) = |\lambda| \chi(X)$ for $\lambda \in \mathbb{R}.$

6° $\chi(X \cup Y) = \max\{\chi(X), \chi(Y)\}.$

7° $\chi(B(x, r)) = r.$
8° If \((X_n)\) is a sequence of closed sets from \(\mathcal{M}_E\) such that \(X_{n+1} \subset X_n\) for \(n = 1, 2, \ldots\) and \(\lim_{n \to \infty} \chi(X_n) = 0\) then the set \(X_\infty = \bigcap_{n=1}^{\infty} X_n\) is nonempty.

For other properties of the Hausdorff measure of noncompactness \(\chi\) we refer to \([1, 3]\).

Now, we recall some facts concerning the Banach sequence spaces. First of all, let us remind that by the symbol \(c_0\) we denote the classical Banach sequence space consisting of real sequences \(x = (x_n)\) converging to zero and normed via the classical norm

\[ ||x|| = ||(x_n)|| = \sup \{|x_n| : n = 1, 2, \ldots\} = \max\{|x_n| : n = 1, 2, \ldots\}. \]

Obviously, in nonlinear analysis we consider also other Banach sequence spaces denoted by \(c, l_\infty\). Since we will work in the space \(c_0\) only we will not remind details concerning those spaces.

Further, let us recall a very convenient formula expressing the Hausdorff measure of noncompactness in the space \(c_0\). Namely, for \(X \in \mathcal{M}_{c_0}\) we have \([3]\)

\[ \chi(X) = \lim_{n \to \infty} \left\{ \sup \{ \sup\{ |x_i| : i \geq n \} \} \right\}. \quad (1) \]

In what follows let us observe that the sequence space \(c_0\) is not very handy in applications since it is very easy to encounter a situation that investigated problems cannot be located in the space \(c_0\) (cf. \([4]\), for instance). It is caused by the fact that this space is rather small.

To overcome the indicated difficulty we can consider the so-called Banach tempered sequence spaces \([4]\).

To this end let us fix a real sequence \(\beta = (\beta_n)\) such that \(\beta_n\) is positive for \(n = 1, 2, \ldots\) and the sequence \((\beta_n)\) is nonincreasing. In practice we also assume that \(\beta_n \to 0\) as \(n \to \infty\), but such an assumption is not needed in general. Next, consider the set \(c_0^\beta\) consisting of all sequences \(x = (x_n)\) such that \(\beta_n x_n \to 0\) as \(n \to \infty\). It is easily seen that \(c_0^\beta\) forms a Banach space under the norm

\[ ||x|| = ||(x_n)|| = \sup \{ |\beta_n x_n| : n = 1, 2, \ldots\} = \max\{ |\beta_n x_n| : n = 1, 2, \ldots\}. \]

The essential property of the space \(c_0^\beta\) is connected with the fact that it is isometric to the classical space \(c_0\). Indeed, it is easy to check that the mapping \(J : c_0^\beta \to c_0\) defined in the following way

\[ J(x) = J((x_n)) = (\beta_n x_n) \]

is an isometry of the spaces \(c_0^\beta\) and \(c_0\) (cf. \([4]\)).

This observation enables us to define the Hausdorff measure of noncompactness in the space \(c_0^\beta\) \([4]\).

Indeed, for an arbitrary set \(X \in \mathcal{M}_{c_0^\beta}\), in view of \(1\) we have

\[ \chi(X) = \lim_{n \to \infty} \left\{ \sup \{ \sup\{ |\beta_i x_i| : i \geq n \} \} \right\}. \quad (2) \]

3. A few results from the theory of differential equations in Banach spaces. This section is devoted to collect a few results from the theory of ordinary differential equations in Banach spaces \([3, 7]\). We restrict ourselves to present those results which will be needed in our further investigations. Thus, let us assume
that \((E, \| \cdot \|)\) is a given real Banach space. Let \(x_0\) be a fixed element of \(E\) and, as earlier, the symbol \(B(x_0, r)\) denotes the ball in \(E\).

We will consider the differential equation

\[
x' = f(t, x)
\]

with the initial condition

\[
x(0) = x_0.
\]

We assume that \(f = f(t, x)\) is a given function acting from the set \([0, T] \times B(x_0, r)\) into \(E\). For simplicity we write \(I = [0, T]\). Assume also that \(\chi\) is the Hausdorff measure of noncompactness in the space \(E\).

Now, we recall a result concerning initial value problem 3-4 being not very general but sufficient and convenient for our purposes [3].

**Theorem 3.1.** Assume that the function \(f = f(t, x)\) is uniformly continuous on the set \([0, T] \times B(x_0, r)\) and \(\|f(t, x)\| \leq A\), where \(AT \leq r\). Moreover, for an arbitrary nonempty set \(X \subset B(x_0, r)\) and for almost all \(t \in I\) the following inequality holds

\[
\chi(f(t, X)) \leq p(t)\chi(X),
\]

where \(p(t)\) is an integrable function on the interval \(I\). Then problem 3-4 has at least one solution \(x = x(t)\) on the interval \(I_1 = [0, T_1]\).

The below given result is a special case of the above theorem and will be used in our further study [4].

**Theorem 3.2.** Let \(f\) be a function defined on \(I \times E\) with values in \(E\) such that

\[
\|f(t, x)\| \leq P + Q\|x\|
\]

for each \(t \in I = [0, T]\) and \(x \in E\), where \(P\) and \(Q\) are positive constants. Further, assume that \(f\) is uniformly continuous on the set \([0, T_1] \times B(x_0, r)\), where \(QT_1 < 1\) and

\[
r = \frac{(P + Q)T_1\|x_0\|}{1 - QT_1}.
\]

Moreover, we assume that \(f\) satisfies condition 5. Then, initial value problem 3-4 has a solution \(x = x(t)\) on the interval \(I_1 = [0, T_1]\).

**Remark 1.** It is worthwhile mentioning that in the case when the Banach space \(E\) is weakly compactly generated on the uniform continuity of the function \(f\) can be replaced by the weaker one requiring the continuity [10]. Particularly, we can show that any separable Banach space is weakly compactly generated [10]. Since the sequence space \(c_0^\beta\) is separable we can apply Theorems 3.1 and 3.2 with the assumption that the function \(f\) is continuous on the set \(I_1 \times B(x_0, t)\).

4. **Main results.** In this section we will consider the semilinear upper diagonal infinite system of differential equations which has the form

\[
x'_n = \sum_{i=1}^{k_n} a_{nn_i}(t)x_{n_i} + f_n(t, x_1, x_2, ...)
\]

with the initial value conditions

\[
x_n(0) = x_n^0, \quad \text{for} \quad n = 1, 2, ... .
\]

We will assume that for any fixed natural number \(n\) (\(n \in \mathbb{N}\)) the sequence \((n_1, n_2, ..., n_{k_n})\) satisfies the inequalities \(n \leq n_1 < n_2 < \cdots < n_{k_n}\).
Thus, in contrast to paper [4], the infinite system of differential equations 7 represents the semilinear upper diagonal system since all terms of the linear part of equations appearing in 7 are located upper the main diagonal of the matrix $(a_{nn,i})$. It is worthwhile mentioning that in [4] we considered also infinite system 7, but we assumed that $1 \leq n_1 < n_2 < \cdots < n_k \leq n$. Obviously in such a case an infinite system 7 represents the lower diagonal system.

In what follows we will also assume that there exists a natural number $K$ such that $k_n \leq K$ for $n = 1, 2, \ldots$. Such an assumption means that the linear part of each equation appearing in system 7 contains only finite number of nonzero terms and the number of those term does not exceed $K$. In the sequel of the paper infinite systems 7 satisfying the above constraint will be called infinite systems of differential equations with linear part of constant width (cf. also [4]).

In our further investigations of problem 7-8, apart of the assumption concerning the constant width of linear parts of equations in 7 we will also impose the following assumptions:

(i) The functions $a_{nn,i} = a_{nn,i}(t)$ are equicontinuous on the interval $I = [0, T]$ for $n = 1, 2, \ldots$ and for $i = 1, 2, \ldots, k_n$.
(ii) The functions $a_{nn,i}(t)$ are uniformly bounded on the interval $I$ by a positive constant $A$ i.e., $|a_{nn,i}(t)| \leq A$ for $t \in I$ and for $n = 1, 2, \ldots$, $i = 1, 2, \ldots, k_n$.
(iii) The sequence $(x^n_0)$ is a member of the space $c_0^\beta$.
(iv) For every fixed $n$ the function $f_n(t, x_1, x_2, \ldots) = f_n(t, x)$ acts from the set $I \times \mathbb{R}^\infty$ into $\mathbb{R}$. Moreover, the function $f_n : I \times c_0^\beta \rightarrow \mathbb{R}$ is continuous on $I \times c_0^\beta$.
(v) There exists a sequence $(p_n)$ of nonnegative terms such that $\beta_n p_n \rightarrow 0$ as $n \rightarrow \infty$ and such that $|f_n(t, x)| \leq p_n$ for $t \in I$, $x \in c_0^\beta$ and for $n = 1, 2, \ldots$.
(vi) There exists a positive constant $M$ such that $\beta_n / \beta_{n_k} \leq M$ for any $n = 1, 2, \ldots$.

Now, we are in a position to present our existence result concerning the initial value problem 7-8.

**Theorem 4.1.** Assume that 7 is an infinite sublinear upper diagonal system of differential equations with linear parts of constant width $K$, satisfying assumptions (i)-(vi). Then, initial value problem 7-8 has at least one solution $x(t) = (x_n(t)) = (x_1(t), x_2(t), \ldots)$ in the sequence space $c_0^\beta$ on interval $I_1 = [0, T_1]$, where $T_1 \leq T$ and $T_1 < 1/A \beta K M$.

**Proof.** To simplify the proof, for arbitrarily fixed $n \in \mathbb{N}$ let us denote

$$g_n(t, x) = g_n(t, x_1, x_2, \ldots) = \sum_{i=1}^{k_n} a_{nn,i}(t)x_{n_i} + f_n(t, x_1, x_2, \ldots),$$

where $t \in I$ and $x = (x_n) \in c_0^\beta$. Then, keeping in mind the imposed assumptions, we obtain:

$$\beta_n |g_n(t, x_1, x_2, \ldots)| \leq \beta_n \sum_{i=1}^{k_n} |a_{nn,i}(t)||x_{n_i}|$$

$$+ \beta_n |f_n(t, x_1, x_2, \ldots)| \leq \beta_n A \sum_{i=1}^{k_n} |x_{n_i}| + \beta_n p_n$$

$$= \beta_n A \left[ |x_{n_1}| + |x_{n_2}| + \cdots + |x_{n_k_n}| \right] + \beta_n p_n$$
Hence, replacing $n$ by $j$ and $j$ by $i$, we can rewrite the above inequality in the form

$$
\beta_j g_j(t, x_1, x_2, \ldots) \leq AKM \sup i \geq 1 \beta_i |x_i| + \beta_j p_j.
$$

(9)

Now, let us observe that from estimate 9 we derive the following inequality

$$
|g(t, x)| = \sup j \geq 1 \beta_j g_j(t, x_1, x_2, \ldots) \leq AKM \sup j \geq 1 \beta_j |x_j| + \sup j \geq 1 \beta_j p_j \leq AKM ||x|| + P,
$$

(10)

where the operator $g = g(t, x)$ is defined on the set $I \times c_0^\beta$ in the following way

$$
g(t, x) = (g_1(t, x), g_2(t, x), \ldots).
$$

Moreover, the constant $P$ is defined as

$$
P = \sup n \geq 1 \beta_n p_n.
$$

Obviously, $P < \infty$ in view of assumption (v).

In what follows we show that the operator $g$ acts continuously from the set $I \times c_0^\beta$ into the space $c_0^\beta$. To this end we represent the operator $g$ in the form

$$
g(t, x) = (Lx)(t) + f(t, x),
$$

where the operators $L$ and $f$ are defined as follows:

$$
(Lx)(t) = ((L_1x)(t), (L_2x)(t), \ldots),
$$

where

$$
(L_nx)(t) = \sum_{i=1}^{k_n} a_{n_n_i}(t)x_{n_i}
$$

$(n = 1, 2, \ldots)$, and

$$
f(t, x) = (f_1(t, x), f_2(t, x), \ldots).
$$

At first we show that $f$ is continuous on the set $I \times c_0^\beta$.

To realize this goal fix arbitrarily a number $\varepsilon > 0$ and $x \in c_0^\beta$, $t \in I$. Then, in view of assumption (v) we can find a natural number $n_0$ such that

$$
\beta_n p_n < \frac{\varepsilon}{2}
$$

(11)

for $n \geq n_0$. Further, in virtue of assumption (iv) we can find a number $\delta_i$ $(i = 1, 2, \ldots, n_0)$ such that for any $y \in c_0^\beta$ such that $||x - y|| \leq \delta_i$ and for $s \in I$ such that $|t - s| \leq \delta_i$ we have

$$
|f_i(t, x) - f_i(s, y)| \leq \frac{\varepsilon}{\beta_1}.
$$
Next, take \( \delta = \min\{\delta_1, \delta_2, \ldots, \delta_n\} \). Then, for arbitrary \( y \in c_0^\beta \) such that \( ||x - y|| \leq \delta \) and for \( s \in I \) such that \( |t - s| \leq \delta \), we get

\[
|f_i(t, x) - f_i(s, y)| \leq \frac{\varepsilon}{\beta_1}. \tag{12}
\]

Linking 11 and 12, for \( y \in c_0^\beta \) with \( ||x - y|| \leq \delta \) and for \( s \in I \) with \( |t - s| \leq \delta \), we obtain

\[
||f(t, x) - f(s, y)|| = \sup\{\beta_n |f_n(t, x) - f_n(s, y)| : n = 1, 2, \ldots\}
\]

\[
= \max\{\max\{\beta_n |f_n(t, x) - f_n(s, y)| : n = 1, 2, \ldots, n_0\}, \sup\{\beta_n |f_n(t, x) - f_n(s, y)| : n > n_0\}\}
\]

\[
\leq \max\{\max\{\beta_n |f_n(t, x) - f_n(s, y)| : n = 1, 2, \ldots, n_0\}, \sup\{\beta_n |f_n(t, x)| + |f_n(s, y)| : n > n_0\}\}
\]

\[
\leq \max\left\{\beta_n \frac{\varepsilon}{\beta_1}, \sup\{2\beta_n p_n : n > n_0\}\right\} = \varepsilon.
\]

This shows that the operator \( f \) is continuous on the set \( I \times c_0^\beta \).

Now, we show that the operator \( L \) is continuous on the set \( I \times c_0^\beta \). Similarly as previously, fix arbitrary \( x \in c_0^\beta, t \in I \) and a number \( \varepsilon > 0 \). Then, for \( y \in c_0^\beta \) with \( ||x - y|| \leq \varepsilon \), for \( s \in I \) with \( |t - s| \leq \varepsilon \) and for an arbitrarily fixed natural number \( n \), in virtue of our assumptions we get

\[
\beta_n |(L_n x)(t) - (L_n y)(s)|
\]

\[
= \beta_n \sum_{i=1}^{k_n} a_{nn_i}(t)x_{n_i} - \sum_{i=1}^{k_n} a_{nn_i}(s)y_{n_i}
\]

\[
\leq \beta_n \sum_{i=1}^{k_n} a_{nn_i}(t)x_{n_i} - \sum_{i=1}^{k_n} a_{nn_i}(s)x_{n_i}
\]

\[
+ \beta_n \sum_{i=1}^{k_n} a_{nn_i}(s)x_{n_i} - \sum_{i=1}^{k_n} a_{nn_i}(s)y_{n_i}
\]

\[
\leq \beta_n \sum_{i=1}^{k_n} |a_{nn_i}(t) - a_{nn_i}(s)||x_{n_i}| + \beta_n \sum_{i=1}^{k_n} |a_{nn_i}(s)||x_{n_i} - y_{n_i}|
\]

\[
\leq \beta_n \sum_{i=1}^{k_n} \omega(|t - s||x_{n_i}| + \beta_n A \sum_{i=1}^{k_n} |x_{n_i} - y_{n_i}|
\]

\[
\leq \omega(\varepsilon) \sum_{i=1}^{k_n} \beta_n |x_{n_i}| + \beta_n |x_{n_i} - y_{n_i}|
\]

where the symbol \( \omega = \omega(\varepsilon) \) denotes the common modulus of continuity of the functions \( a_{nn_i}(t) \) on the interval \( I \). Such a modulus exists in view of assumption
(i). Further, keeping in mind assumption (vi), we obtain
\[ \beta_n |(L_n x)(t) - (L_n y)(s)| \]
\[ \leq \omega(\varepsilon) \sum_{i=1}^{k_n} \frac{\beta_n}{\beta_n_i} |x_{n_i} - y_{n_i}| + A \sum_{i=1}^{k_n} \frac{\beta_n}{\beta_n_i} |x_{n_i} - y_{n_i}| \]
\[ \leq M \omega(\varepsilon) \sum_{i=1}^{k_n} \beta_n |x_{n_i}| + AM \sum_{i=1}^{k_n} \beta_n |x_{n_i} - y_{n_i}| \]
\[ \leq KM \omega(\varepsilon)\|x\| + AM \|x - y\| \leq KM \omega(\varepsilon) + AM \varepsilon. \]

The above obtained estimate allows us to infer that the operator \( L \) is continuous on the set \( I \times c_0^\beta \). Joining this fact with the continuity of the operator \( f \) established before, we deduce that the operator \( g \) is continuous on the set \( I \times c_0^\beta \).

Next, let us take a positive number \( T_1 \) such that \( T_1 \leq T \) and \( AKMT_1 < 1 \). Denote \( I_1 = [0, T_1] \). Keeping in mind the above established facts and Theorem 3.2 let us take the number
\[ r = \frac{(P + AKM)T_1 \|x_0\|}{1 - AKMT_1}. \]

Now, consider the ball \( B(x_0, r) \) and choose an arbitrary nonempty subset \( X \) of \( B(x_0, r) \). Then, for a fixed element \( x \in X \) and for an arbitrary number \( t \in I_1 \), in view of estimates 9 and 10, for arbitrarily fixed natural number \( n \), we obtain:
\[ \sup \{ \beta_j |g_j(t, x_1, x_2, ...) : j \geq n \} \]
\[ \leq \sup \{ AKM \sup \{ \beta_i |x_i| : i \geq j_1 \} : j \geq n \} + \sup \{ \beta_j p_j : j \geq n \} \]
\[ \leq AKM \sup \{ \sup \{ \beta_i |x_i| : i \geq m_1 \} , \sup \{ \beta_i |x_i| : i \geq (n + 1) \} , \sup \{ \beta_i |x_i| : i \geq n + 2 \} \} , \ldots + \sup \{ \beta_j p_j : j \geq n \}. \]

Consequently, we arrive at the following estimate:
\[ \sup \{ \sup \{ \beta_j |g_j(t, x_1, x_2, ...) : j \geq n \} \} \]
\[ \leq AKM \sup \{ \sup \{ \beta_i |x_i| : i \geq j_1 \} : j \geq n \} \]
\[ + \sup \{ \beta_j p_j : j \geq n \}. \]

Now, passing with \( n \to \infty \) and bearing in mind that \( j_1 \to \infty \) as \( j \to \infty \), in view of formula 2 expressing the Hausdorff measure of noncompactness in the space \( c_0^\beta \), we derive the following inequality
\[ \chi(g(t, X)) \leq AKM \chi(X). \]

Finally, gathering all the above stated facts, in view of Theorem 3.2 we complete the proof. 

In order to illustrate the result contained in Theorem 4.1 we consider the following example.
Example 1. Let us take into account the following infinite system of differential equations
\[
\begin{align*}
x_1' & = x_1 + \frac{1}{1+t} x_2 + \frac{x_1}{1+x_1^2}, \\
x_2' & = x_2 + \frac{1}{t^2} x_3 + \frac{1}{1+t} x_4 + \frac{2x_2}{1+x_2^2}, \\
x_3' & = x_3 + \frac{1}{3t^2} x_4 + \frac{1}{6+t} x_6 + \frac{2x_3}{1+x_3^2}, \\
& \vdots \\
x_n' & = x_n + \frac{1}{n+t} x_{n+1} + \frac{1}{2n+t} x_{2n} + \frac{nx_n}{1+x_n^2},
\end{align*}
\]
(13)
with initial conditions
\[
x_n(0) = 2n + 1 \quad \text{for} \quad n = 1, 2, \ldots
\]
(14)
Observe that 13 is a semilinear upper diagonal infinite system of differential equations with linear parts of constant width \( K = 3 \). Apart from this it is easily seen that system 13 is a particular case of system 7 if we take \( a_{nn}(t) = 1 \) for \( n \in \mathbb{N} \) and for \( t \in I_1 = [0, T_1] \), where \( T_1 \leq T \) is a number chosen according to assumptions of Theorem 3.2. Moreover, \( a_{nn}(t) = \frac{1}{n+t} \) for \( i = 2 \) and \( n_2 = n+1 \), while for \( i = 3 \) we have that \( n_3 = 2n \). Obviously \( |a_{nn}(t)| \leq 1 \) for \( t \in I_1 \) and \( n = 1, 2, \ldots; i = 1, 2, 3 \). This implies that the functions \( a_{nn}(t) \) satisfy assumption (ii) of Theorem 4.1.

On the other hand it is easy to check that the functions \( a_{nn}(t) \) satisfy the Lipschitz condition with the constant 1 for \( n = 1, 2, \ldots \) and \( i = 1, 2, 3 \). Hence we see that there is satisfied assumption (i) of our theorem.

In the sequel of our considerations let us take the tempering sequence \( \beta_n = \frac{1}{n^2} \) for \( n = 1, 2, \ldots \). Obviously we have that \( x_0 = (x^{(0)}_n) = (2n + 1) \in c^\beta_0 \), where \( \beta = (\beta_n) = (\frac{1}{n^2}) \). This proves that assumption (iii) is satisfied with the space \( c^\beta_0 \) for \( \beta \) stated above.

Further, let us observe that from the form of system 13 stems that
\[
f_n(t, x_1, x_2, \ldots) = \frac{nx_n}{1 + x_n^2}
\]
for \( n = 1, 2, \ldots \). It is clear that the function \( f_n = f_n(t, x) \) is continuous on the set \( I \times c^\beta_0 \). Apart from this, we have
\[
|f_n(t, x_1, x_2, \ldots)| \leq \frac{1}{2} n, \quad \text{for} \quad n = 1, 2, \ldots
\]
The above established facts allows us to deduce that functions \( f_n \) satisfy assumptions (iv) and (v) with \( p_n = \frac{1}{2} n \) for \( n = 1, 2, \ldots \). On the other hand we have that
\[
\frac{\beta_n}{\beta_{n+1}} = \frac{\beta_n}{\beta_{2n}} = 4.
\]
Thus we see that assumption (vi) is satisfied with \( M = 4 \).

Finally, in view of Theorem 4.1 we deduce that there exists at least one solution \( x(t) = (x_n(t)) \) of initial value problem 13-14 defined on some interval \( I_1 = [0, T_1] \) such that for any \( t \in I_1 \) the sequence \( (x_n(t)) \) belongs to the space \( c^\beta_0 \) with \( \beta = (\frac{1}{n^2}) \). Obviously, we can easily calculate that \( T_1 < \min\{T, 1/12\} \).

In what follows we provide an analogon of the result formulated in paper [4] as Theorem 6.3, for infinite semilinear upper diagonal system of differential equations.

Namely, we will consider problem 7-8 for infinite upper diagonal system of differential equations and we dispense with the assumption requiring that system 7 has linear parts with constant width.
More precisely, we replace that assumption and assumption (ii) by the following hypotheses:

(ii’) The sequence \( \left( \sum_{i=1}^{k_n} |a_{nn_i}(t)| \right) \) is uniformly bounded on the interval \( I_1 = [0, T_1] \), where \( T_1 \) is chosen according to Theorem 3.2. This means that there exists a constant \( A > 0 \) such that
\[
\sum_{i=1}^{k_n} |a_{nn_i}(t)| \leq A
\]
for any \( n = 1, 2, ... \) and for \( t \in I_1 \).

(ii”) The functions \( a_{nn_i}(t) \) are nondecreasing on the interval \( I_1 \) for \( n = 1, 2, ..., i = 1, 2, ..., k_n \) and the sequence \( \left( \sum_{i=1}^{k_n} a_{nn_i}(t) \right) \) is equicontinuous on the interval \( I_1 \).

Now we formulate the announced result.

**Theorem 4.2.** Under assumptions (i), (iii)-(vi) of Theorem 4.1 and assumptions (ii’), (ii”), initial value problem 7-8 for infinite upper diagonal system of differential equations has at least one solution \( x(t) = (x_n(t)) \) in the sequence space \( c_0^\beta \) defined on the interval \( I_1 = [0, T_1] \).

**Proof.** In the same way as in the proof of Theorem 4.1, for a fixed \( n \in \mathbb{N} \) let us denote:
\[
g_n(t, x) = \sum_{i=1}^{k_n} a_{nn_i}(t)x_{n_i} + f_n(t, x),
\]
\[
(L_n x)(t) = \sum_{i=1}^{k_n} a_{nn_i}(t)x_{n_i},
\]
where \( t \in I_1 \) and \( x = (x_n) \in c_0^\beta \). Further, let us put:
\[
g(t, x) = (g_1(t, x), g_2(t, x), ...),
\]
\[
(Lx)(t) = ((L_1 x)(t), (L_2 x)(t), ...),
\]
\[
f(t, x) = (f_1(t, x), f_2(t, x), ...).
\]

Next, keeping in mind the imposed assumptions, we get:
\[
\beta_n |g_n(t, x_1, x_2, ...)|
\leq \sum_{i=1}^{k_n} |a_{nn_i}(t)||x_{n_i}| + \beta_n |f_n(t, x_1, x_2, ...)|
\leq \sum_{i=1}^{k_n} \beta_n \beta_n |a_{nn_i}(t)||x_{n_i}| + \beta_n p_n
\leq \sum_{i=1}^{k_n} \beta_n \beta_n |a_{nn_i}(t)||x_{n_i}| + \beta_n p_n
\leq M \sum_{i=1}^{k_n} |a_{nn_i}(t)| \max \{ \beta_n |x_{n_i}| : i = 1, 2, ..., k_n \} + \beta_n p_n
\leq MA \sup \{ \beta_j |x_j| : j \geq n \} + \beta_n p_n.
\]
Hence we obtain

\[ \|g(t,x)\| = \sup \{ \beta_n |g_n(t,x_1,x_2,...) : n \in \mathbb{N} \} \leq AM\|x\| + P, \quad (16) \]

where we denoted \( P = \sup \{ \beta_n p_n : n \in \mathbb{N} \} \). Obviously \( P < \infty \). Estimate 16 shows that the operator \( g \) maps the set \( I_1 \times c_0^\beta \) into \( c_0^\beta \). Moreover, this estimate coincides with estimate 6 which is needed to apply Theorem 3.2.

In the sequel we are going to show that the operator \( g \) is continuous on the set \( I_1 \times c_0^\beta \). To this end let us observe that the proof of the continuity of the operator \( f \) runs exactly in the same way as in a suitable part of the proof of Theorem 4.1. Therefore, it is sufficient to prove the continuity of the operator \( L \) on the set \( I_1 \times c_0^\beta \).

In order to conduct this proof let us fix arbitrarily \( x \in c_0^\beta \), \( t \in I_1 \) and a number \( \varepsilon > 0 \). Next, take an arbitrary element \( y \in c_0^\beta \) with \( \|x - y\| \leq \varepsilon \) and a number \( s \in I_1 \) with \( |t - s| \leq \varepsilon \). Without loss of generality we may assume that \( s < t \). Then, for an arbitrary natural number \( n \) we obtain:

\[
\beta_n|L_n x(t) - (L_n y)(s)| \leq \beta_n \left| \sum_{i=1}^{k_n} a_{nn_i}(t)x_{n_i} - \sum_{i=1}^{k_n} a_{nn_i}(s)x_{n_i} \right| \\
+ \beta_n \left| \sum_{i=1}^{k_n} a_{nn_i}(s)x_{n_i} - \sum_{i=1}^{k_n} a_{nn_i}(s)y_{n_i} \right| \\
\leq \beta_n \left| \sum_{i=1}^{k_n} (a_{nn_i}(t) - a_{nn_i}(s))x_{n_i} \right| \\
+ \beta_n \sum_{i=1}^{k_n} |a_{nn_i}(s)||x_{n_i} - y_{n_i}| \\
\leq \sum_{i=1}^{k_n} |a_{nn_i}(t) - a_{nn_i}(s)| \beta_n |x_{n_i}| \\
+ \sum_{i=1}^{k_n} |a_{nn_i}(s)| \beta_n |x_{n_i} - y_{n_i}| \\
\leq M \sum_{i=1}^{k_n} |a_{nn_i}(t) - a_{nn_i}(s)| \sup \{ \beta_n |x_{n_i}| : i = 1, 2, ..., k_n \} \\
+ M \sum_{i=1}^{k_n} |a_{nn_i}(s)| \sup \{ \beta_n |x_{n_i} - y_{n_i}| : i = 1, 2, ..., k_n \} \\
\leq M \sum_{i=1}^{k_n} |a_{nn_i}(t) - a_{nn_i}(s)| \sup \{ \beta_j |x_j| : j = 1, 2, ... \} \\
+ M \sum_{i=1}^{k_n} |a_{nn_i}(s)| \sup \{ \beta_j |x_j - y_j| : j = 1, 2, ... \} \\
\leq M \|x\| \left| \sum_{i=1}^{k_n} a_{nn_i}(t) - \sum_{i=1}^{k_n} a_{nn_i}(s) \right|
\]
sequence (ii") instead of the requirement that the functions $a_{nn_i}(t)$ considered in Theorem 4.2. Indeed, as we stated above, infinite system of differential equations 17 is upper diagonal and the functions $a_{nn_i}(t)$ are nonincreasing on the interval $I_1$. Moreover, we will also assume that the following initial conditions are satisfied

\[
(I) \quad x_n(0) = n^2 \quad \text{for } n = 1, 2, ..., 
\]

for $n = 1, 2, ..., 18$ is upper diagonal and the functions $a_{nn_i}(t)$ appeared in 17 have the form

\[
a_{nn_i}(t) = \frac{t^{n_i-1}}{(n_i - 1)!}
\]
for \( n_i = n, n + 1, ..., \frac{3}{2} n - 1 \) (if \( n \) is even) or \( n_i = n, n + 1, ..., 3 \left[ \frac{n}{2} \right] \) (if \( n \) is odd). Observe that the functions \( a_{nn_i}(t) \) are equicontinuous on the interval \( I_1 = [0, T_1] \) since we have that

\[
a'_{nn_i}(t) = \frac{t^{n_i - 2}}{(n_i - 2)!} \leq e^{T_1}
\]

for \( n = 1, 2, ..., i = 1, 2, ..., k_n \) and \( t \in I_1 \). Thus, there is satisfied assumption (i).

Further, let us note that

\[
\sum_{i=1}^{k_n} |a_{nn_i}(t)| \leq 1 + t + \frac{t^2}{2!} + \cdots + \frac{t^{k_n-1}}{(k_n-1)!} \leq e^t
\]

for \( t \in I_1 \). This shows that assumption (ii') is satisfied with \( A = e^{T_1} \).

Moreover, it is obvious that the functions \( a_{nn_i}(t) \) are nondecreasing on the interval \( I_1 \). Further, for any fixed \( n \) we have:

\[
\sum_{i=1}^{k_n} a_{nn_i}(t) = 1 + t + \frac{t^2}{2!} + \cdots + \frac{t^{k_n-1}}{(k_n-1)!}.
\]

This implies that

\[
\left( \sum_{i=1}^{k_n} a_{nn_i}(t) \right) \leq e^{T_1}.
\]

Hence we infer that the functions \( \sum_{i=1}^{k_n} a_{nn_i}(t) \) satisfy the Lipschitz condition with the constant \( L = e^{T_1} \). From this we derive that the sequence \( \left( \sum_{i=1}^{k_n} a_{nn_i}(t) \right) \) is equicontinuous on the interval \( I_1 \). Thus there is satisfied assumption (ii').

In what follows let us take the tempering sequence having the form \( \beta = (\beta_n) = (\frac{1}{\beta}) \). Then the initial sequence \( (n_0) = (n^2) \) belongs to the tempered function space \( c_0^\beta \) and this assertion implies that assumption (iii) is satisfied.

Similarly as in the earlier mentioned paper \([4]\) it is not difficult to verify that the functions

\[
f_n(t, x) = f_n(t, x_1, x_2, ...) = n x_{n-1} + x_n \frac{x_{n-1} + x_n}{1 + x_{n-1}^2 + x_n^2}
\]

\((n = 2, 3, ...)\) are continuous on the set \( I_1 \times c_0^\beta \). Additionally, we have that \( |f_n(t, x)| \leq n \). Thus we infer that there are satisfied assumptions (iv) and (v) with \( p_n = n \), since \( \beta_n p_n = \frac{1}{\pi^2} \to 0 \).

Finally, taking into account that \( n_{k_n} = \frac{3}{2} n - 1 \) for \( n \) even and \( n_{k_n} = 3 \left[ \frac{n}{2} \right] \) for \( n \) odd, we get

\[
\frac{\beta_n}{\beta_{n_{k_n}}} = \left( \frac{\frac{3}{2} n - 1}{n} \right)^3 = \left( \frac{3}{2} - \frac{1}{n} \right)^3 \leq \frac{27}{8}
\]

if \( n \) is even. On the other hand, for \( n \) odd we have

\[
\frac{\beta_n}{\beta_{n_{k_n}}} = \left( \frac{\frac{n}{2}}{n} \right)^3 = \left( \frac{3}{2} \right)^3 \leq \frac{27}{8}.
\]

Thus we see that there is is satisfied assumption (vi) with \( M = 27/8 \).

From the above argumentations and Theorem 4.2 we conclude that problem 17-18 has at least one solution \( x(t) = (x_n(t)) \) in the space \( c_0^\beta \) defined on a suitable interval \( I_1 \).
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E-mail address: jbanas@prz.edu.pl
E-mail address: monika.krajewska@pwste.edu.pl