The Hodge ring of varieties in positive characteristic

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Abstract

Let $k$ be a field of positive characteristic. We prove that the only linear relations between the Hodge numbers $h^{i,j}(X) = \dim H^j(X, \Omega^i_X)$ that hold for every smooth proper variety $X$ over $k$ are the ones given by Serre duality. We also show that the only linear combinations of Hodge numbers that are birational invariants of $X$ are given by the span of the $h^{i,0}(X)$ and the $h^{0,j}(X)$ (and their duals $h^{i,n}(X)$ and $h^{n,j}(X)$). The corresponding statements for compact Kähler manifolds were proven by Kotschick and Schreieder [KS13].

Introduction

The Hodge numbers $h^{i,j}(X) = \dim H^j(X, \Omega^i_X)$ of a compact Kähler manifold $X$ satisfy the relations $h^{i,j} = h^{n-i,n-j}$ (Serre duality) and $h^{i,j} = h^{j,i}$ (Hodge symmetry). Kotschick and Schreieder showed [KS13, Thm. 1, consequence (2)] that these are the only universal linear relations between the $h^{i,j}$, i.e. the only ones that are satisfied for every compact Kähler manifold $X$ of dimension $n$.

Serre constructed smooth projective varieties in characteristic $p > 0$ for which Hodge symmetry fails [Ser58, Prop. 16]; however Serre duality still holds. Thus, the natural analogue of the result of Kotschick–Schreieder is the following.

**Theorem 1.** The only universal linear relations between the Hodge numbers $h^{i,j}(X)$ of smooth proper varieties $X$ over an arbitrary field $k$ of characteristic $p > 0$ are the ones given by Serre duality.

This is proven in Theorem 6.7. The strategy is identical to that of [KS13]: compute the subring of $\mathbb{Z}[x,y,z]$ generated by the formal Hodge polynomials

$$h(X) = \left( \sum_{i,j} h^{i,j}(X) x^i y^j \right) z^{\dim X}.$$ 

The term $z^{\dim X}$ is new in [KS13], and its introduction ensures that varieties of different dimensions cannot be identified with each other. It defines a grading on $\mathbb{Z}[x,y,z]$ by dimension, i.e. $\mathbb{Z}[x,y,z]_n = \mathbb{Z}[x,y]z^n$. 

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In characteristic 0, Kotschick and Schreieder prove [KS13, Cor. 3] that the graded subring of $\mathbb{Z}[x, y, z]$ given in degree $n$ by the equations $h^{i,j} = h^{n-i,n-j}$ and $h^{i,j} = h^{j,i}$ is the free algebra $\mathbb{Z}[h(P^1), h(E), h(P^2)]$ on the Hodge polynomials of $P^1$, an elliptic curve $E$, and $P^2$.

We prove the natural analogue in characteristic $p$:

**Theorem 2.** Let $\mathcal{H} \subseteq \mathbb{Z}[x, y, z]$ be the graded subring defined by

$$\mathcal{H}_n = \left\{ \left( \sum_{i,j=0}^n h^{i,j} x^i y^j \right) z^n \mid h^{i,j} = h^{n-i,n-j} \text{ for all } i,j \right\}.$$  

Then $\mathcal{H}$ is generated by $h(P^1), h(E), h(P^2)$, and $h(S)$, where $E$ is any elliptic curve and $S$ is any surface with $h^{1,0}(S) - h^{0,1}(S) = \pm 1$. Moreover, $h(S)$ satisfies a monic quadratic relation over the free polynomial algebra $\mathbb{Z}[h(P^1), h(E), h(P^2)]$.

This is proven in Corollary 3.8. An example of a surface $S$ satisfying the hypothesis is given by Serre’s counterexample to Hodge symmetry [Ser58, Prop. 16], since $h^{0,1}(S) = 0$ and $h^{1,0}(S) = 1$. We recall the construction in Proposition 1.3.

There are “twice as many” possible Hodge polynomials in characteristic $p$ as in characteristic 0: the Hodge diamonds now only have a $\mathbb{Z}/2$-symmetry instead of a $(\mathbb{Z}/2)^2$-symmetry. Thus, a Hilbert polynomial calculation suggests that the Hodge ring of varieties in characteristic $p > 0$ should be a quadratic extension of the one in characteristic 0. This is indeed confirmed by Theorem 2.

Similar to [KS13, Thm. 2], we also get a statement on birational invariants:

**Theorem 3.** A linear combination of the Hodge numbers $h^{i,j}(X)$ is a birational invariant of the variety $X$ (over a field $k$ of characteristic $p$) if and only if it is in the span of the $h^{0,1} = h^{n,n-j}$ and the $h^{1,0} = h^{n-i,n}$.

This is proven in Theorem 4.4 below. The result in characteristic 0 is the same [KS13, Thm. 2], except that there $h^{1,0}$ is actually equal to $h^{0,1}$ by Hodge symmetry. The fact that the $h^{0,1}$ are birational invariants in characteristic $p > 0$ is due to Chatzistamatiou and Rülling [CR11, Thm. 1]; for $h^{1,0}$ this is classical.

Finally we address the failure of degeneration of the Hodge–de Rham spectral sequence:

**Theorem 4.** The only linear relations between the Hodge numbers $h^{i,j}(X)$ and the de Rham numbers $h^{i}_{\text{dR}}(X)$ of smooth proper varieties $X$ over $k$ are the relations spanned by

- Serre duality: $h^{i,j}(X) = h^{n-i,n-j}(X)$;
- Poincaré duality: $h^{i}_{\text{dR}}(X) = h^{2n-i}_{\text{dR}}(X)$;
- Components: $h^{0,1}(X) = h^{1,0}_{\text{dR}}(X)$;
- Euler characteristic: $\sum_{i,j} (-1)^{i+j} h^{i,j}(X) = \sum_{i} (-1)^i h^{i}_{\text{dR}}(X)$.

This is proven in Theorem 6.7.
OUTLINE OF THE PAPER

In Section 1 we recall Serre’s example of the failure of Hodge symmetry, and make slight improvements that we will use later. In Section 2 we recall an example of W. Lang of a similar flavour of a surface for which the Hodge–de Rham spectral sequence does not degenerate.

In Section 3 we define the Hodge ring of varieties as the graded subring of $\mathbb{Z}[x, y, z]$ where Serre duality holds, and show that it is generated by the Hodge polynomials of $\mathbb{P}^1$, any elliptic curve $E$, $\mathbb{P}^2$, and the surface $S$ of Section 1. This proves Theorem 1. In Section 4 we use this presentation to study birational invariants, proving Theorem 3.

In Section 5 we define the de Rham ring of varieties as the graded subring of $\mathbb{Z}[t, z]$ where Poincaré duality holds, and produce a similar presentation in terms of $\mathbb{P}^1$, $E$, $\mathbb{P}^2$, and $S$. Finally, in Section 6 we combine the information of the previous sections into a Hodge–de Rham ring of varieties, and we use the surface $S$ of Section 2 to show that the Hodge–de Rham ring can be generated by varieties. This gives Theorem 4.

NOTATION

Throughout, $p$ will be a prime number, and $k$ will be a field of characteristic $p$ (fixed once and for all). A variety over a field $k$ will be a geometrically integral, finite type, separated $k$-scheme. We write $\text{Var}_k$ for the set of isomorphism classes of smooth proper $k$-varieties.

For a variety $X$ over a field $k$, we will write $H^{i,j}(X) = H^j(X, \Omega^i_X)$, and denote its dimension by $h^{i,j}(X) = h^j(X, \Omega^i_X)$ (we avoid the usual $h^{p,q}(X)$ as it leads to a clash of notation). Similarly, the algebraic de Rham cohomology is denoted $H^i_{\text{dR}}(X)$, and its dimension is $h^i_{\text{dR}}(X)$. We warn the reader that $h^{i,j}(X)$ and $h^{j,i}(X)$ may differ, and the sum $\sum_{i+j=m} h^{i,j}(X)$ may not equal $h^m_{\text{dR}}(X)$.

The Picard functor $\text{Pic}_X = R^1\pi_*G_m$ of a smooth proper variety $\pi: X \to \text{Spec } k$ is representable by a scheme [SGA6, Exp. XII, Cor. 1.5(a)]. We denote its identity component by $\text{Pic}_0^0 X$, and the union of the torsion components by $\text{Pic}_X^\tau$. Note that neither is reduced in general.

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1. Failure of Hodge symmetry

Following Serre [Ser58, Prop. 16], we construct a smooth projective surface \( X \) over \( \mathbb{F}_p \) with \( h^0(X, \Omega^1_X) = 0 \) but \( h^1(X, \mathcal{O}_X) = 1 \). We do a little better than Serre’s example: our \( X \) is defined over the prime field \( \mathbb{F}_p \), admits a lift to \( \mathbb{Z}_p \), and we include the exact computation of \( h^1(X, \mathcal{O}_X) \).

We start with a well-known lemma that is useful for the constructions in this section and the next.

**Lemma 1.1.** Let \( X \) and \( Y \) be proper \( k \)-varieties, let \( G \) be a finite group scheme over \( k \), and let \( f: Y \rightarrow X \) be a \( G \)-torsor. Then there is a short exact sequence

\[
0 \rightarrow G^\vee \rightarrow \text{Pic}_X \rightarrow (\text{Pic}_Y)^G
\]

on the big flat site \((\text{Spec } k)_{\text{fppf}}\), where \( G^\vee \) is the Cartier dual of \( G \).

**Proof.** We have a commutative diagram

\[
\begin{array}{ccc}
Y & \xrightarrow{\pi_Y} & \text{Spec } k \\
\downarrow f & & \downarrow f^{\text{univ}} \\
X & \xrightarrow{g} & BG \xrightarrow{\pi_{BG}} \text{Spec } k
\end{array}
\]

whose left hand square is a pullback, where \( \pi_Y: \mathcal{X} \rightarrow \text{Spec } k \) denotes the structure map of an algebraic stack \( \mathcal{X} \) over \( k \). The Grothendieck spectral sequence for \( X_{\text{fppf}} \rightarrow BG_{\text{fppf}} \rightarrow (\text{Spec } k)_{\text{fppf}} \) gives a short exact sequence of low degree terms

\[
0 \rightarrow R^1\pi_{BG,\ast}(g_\ast G_m) \rightarrow R^1\pi_{X,\ast}G_m \rightarrow \pi_{BG,\ast}(R^1g_\ast G_m). \quad (1.1)
\]

The middle term is \( \text{Pic}_X \), and the pullback of \( R^1g_\ast G_m \) along \( f^{\text{univ}} \) is \( \text{Pic}_Y \). Since \( \pi_{BG,\ast}: BG_{\text{fppf}} \rightarrow (\text{Spec } k)_{\text{fppf}} \) is computed by pulling back along \( f^{\text{univ}} \) and taking \( G \)-invariants, the third term of \((1.1)\) is \((\text{Pic}_Y)^G\).

Since \( \pi_Y \) is proper with geometrically connected fibres, the same goes for \( g \). This forces \( g_\ast G_m = G_m \), so the first term of \((1.1)\) is \( R^1\pi_{BG,\ast}G_m = \text{Pic}_{BG} \). A line bundle on \( BG \) is trivialised on \( f^{\text{univ}}: \text{Spec } k \rightarrow BG \), so it gives a cocycle on \( \text{Spec } k \times_{BG} \text{Spec } k \cong G \). This is a function \( \phi: G \rightarrow G_m \), and the cocycle condition amounts to the condition that \( \phi \) is a homomorphism. We finally conclude that \( \text{Pic}_{BG} \cong \text{Hom}(G, G_m) = G^\vee \).

There are more elementary proofs of **Lemma 1.1** by trivialising a line bundle \( \mathcal{L} \) on \( X \) by pulling back to \( Y \), and using \( H^0(Y, \mathcal{O}_Y) = k \) to construct a morphism \( G \rightarrow G_m \). The advantage of our proof above is that it gives the obstruction to surjectivity of \( \text{Pic}_X \rightarrow (\text{Pic}_Y)^G \): the next term in the sequence is \( R^2\pi_{BG,\ast}G_m \). Jensen showed [Jen78] that this group vanishes for most of the groups we’re interested in, e.g. \( \mathbb{Z}/p, \alpha_p, \text{ and } \mu_p \), but we don’t need this.

**Corollary 1.2.** Let \( X \) be a proper \( k \)-variety, \( G \) a finite group scheme over \( k \), and \( Y \rightarrow X \) a \( G \)-torsor. If \( \text{Pic}_Y^G = 0 \), then \( \text{Pic}_X^G \cong G^\vee \).
Proof. The image of $\text{Pic}_X^\tau$ always lands in $\text{Pic}_Y^\tau$, which is 0 by assumption. The restriction of the short exact sequence of Lemma 1.1 to the respective torsion components gives the result.

This allows us to construct a surface for which Hodge symmetry fails, following Serre [Ser58].

**Proposition 1.3.** There exists a smooth projective surface $X$ over $F_p$ admitting a lift to $Z_p$ such that $H^0(X, \Omega^1_X) = 0$ but $h^1(X, \mathcal{O}_X) = 1$.

**Proof.** Let $G \to \text{Spec} Z_p$ be the constant étale group scheme $Z/p$ for some $N \gg 0$ such that the fixed point locus of the special fibre has codimension at least 3. For example, we can take 3 copies of the regular representation, and then projectivise. Form the quotient $Z = P^n_N/G$, which is smooth away from the image of the fixed locus.

By the codimension $\geq 3$ assumption, repeatedly applying Poonen’s Bertini theorem [Poo04] produces a projective surface $X$ over $Z_p$ such that $H^0(X, \Omega^1_X) = 0$ but $h^1(X, \mathcal{O}_X) = 1$. Its inverse image $Y$ in $P^n_N$ is a complete intersection, and $Y \to X$ is a $G$-torsor.

The special fibre $Y$ of $Y$ is smooth since $X$ and $G$ are, so $H^0(Y, \Omega^1_Y) = 0$, $H^1(Y, \mathcal{O}_Y) = 0$ and $\text{Pic}_Y^\tau = 0$ [SGA7 II, Exp. XI, Thm. 1.8]. Then Corollary 1.2 gives $\text{Pic}_X^\tau = (Z/p)^N = \mu_p$, so $H^1(X, \mathcal{O}_X) = \text{Lie}(\text{Pic}_X^\tau) = \text{Lie}(\mu_p)$ has dimension 1. On the other hand, $Y \to X$ is étale since it is a $(Z/p)$-torsor, so $H^0(X, \Omega^1_X) \to H^0(Y, \Omega^1_Y)$ is injective, forcing $H^0(X, \Omega^1_X) = 0$.

**Remark 1.4.** Because the surface $X$ constructed in Proposition 1.3 lifts to $Z_p$, the Hodge–de Rham spectral sequence degenerates [DI87]. For $H^0(X, \Omega^1_X)$ and $H^1(X, \mathcal{O}_X)$, this can also be checked by hand using cocycles (analogous to the proof of Proposition 2.4 below).

## 2. Failure of Hodge–de Rham Degeneration

Inspired by a construction of W. Lang [Lan95], we do a precise computation of $H^0(X, \Omega^1_X)$, $H^1(X, \mathcal{O}_X)$, and $H^1_{\text{dR}}(X)$ when $X$ is an $\alpha_p$-quotient of a (singular) complete intersection. Lang moreover shows that $X$ lifts to a degree 2 ramified extension of $Z_p$, but this plays no role for us.

A lot is written about structure theory and singularities of $\alpha_p$-torsors and more generally purely inseparable quotients [Eke87; KN82; AA86; DG70; Tzi17], but we only need elementary results.

**Remark 2.1.** If $A$ is a ring of characteristic $p$, then an $\alpha_p$-torsor is given by $A \to A[z]/(z^p - f)$ for some $f \in A$, and it is the trivial torsor if and only if $f$ is a $p$-th power. Indeed, this follows from the exact sequence

$$A(-)^p \to A \to H^1(\text{Spec} A, \alpha_p) \to 0.$$
For a general $\mathfrak{g}_p$-torsor $\pi: Y \to X$, there are affine opens $\mathrm{Spec} \, A_i = U_i \subseteq X$ and sections $z_i \in \mathcal{O}_Y(p^{-1}(U_i))$ and $f_i \in \mathcal{O}_X(U_i)$ such that $\mathcal{O}_X(U_i) \to \mathcal{O}_Y(p^{-1}(U_i))$ is given by $A_i \to A_i[z_i]/(z_i^p - f_i)$. We have $f_i - f_j = g_{ij}$ for some $g_{ij} \in \mathcal{O}_X(U_{ij})$, hence the 1-forms $\omega_i = df_i$ glue to a global 1-form $\omega$.

**Lemma 2.2.** Let $X$ be a normal integral scheme of characteristic $p$, and let $\pi: Y \to X$ be an $\mathfrak{g}_p$-torsor, with $f_i$, $z_i$, $g_{ij}$, $\omega_i$, and $\omega$ as in Remark 2.1. Then the following are equivalent:

1. $\pi$ is not the trivial torsor;
2. no $f_i$ is a $p$-th power;
3. $Y$ is integral;
4. $\omega$ is not zero in $H^0(X, \Omega^1_{X/F_p})$ (or even in $\Omega^1_{K(X)/F_p}$);

**Proof.** If one $f_i$ is a $p$-th power, then they all are, since they differ by $p$-th powers. Hence, $\pi$ is trivial in this case. This proves (1) $\Rightarrow$ (2); the converse is trivial. Since $X$ is normal, $f_i$ is a $p$-th power in $A_i$ if and only if it is so in $K = K(X)$. Thus, if no $f_i$ is a $p$-th power, then all the rings $K[z_i]/(z_i^p - f_i)$ are fields, so $A_i[z_i]/(z_i^p - f_i)$ is a domain. This proves (2) $\Rightarrow$ (3), and again the converse is trivial. Finally, (2) $\iff$ (4) is [Stacks, Tag 07P2].

**Lemma 2.3.** Let $k$ be a field of characteristic $p$, let $X$ and $Y$ be finite type $k$-schemes, and let $\pi: Y \to X$ be a morphism of $k$-schemes that is an $\mathfrak{g}_p$-torsor. If $Y$ is smooth, then so is $X$.

**Proof.** We may replace $k$ by $\overline{k}$ and ‘smooth’ by ‘regular’. Since $Y$ is regular, Kunz’s theorem [Kun69, Thm. 2.1] implies that $\mathrm{Fr}_Y: Y \to Y$ is flat. We have a factorisation

$$Y \xrightarrow{\pi} X \xrightarrow{\pi} Y \xrightarrow{\mathrm{Fr}_Y} X,$$

where $\pi$ and $\mathrm{Fr}_Y$ are flat. We conclude that $\mathrm{Fr}_X \circ \pi = \pi \circ \mathrm{Fr}_Y$ is flat, hence $\mathrm{Fr}_X$ is flat since $\pi$ is faithfully flat. Then Kunz’s theorem [Kun69, Thm. 2.1] implies that $X$ is regular.

**Proposition 2.4.** There exists a smooth projective surface $X$ over $\mathbb{F}_p$ for which $H^0(X, \Omega^1_{X/\mathbb{F}_p})$, $H^1(X, \mathcal{O}_X)$, and $H^1_{\text{dR}}(X)$ are all 1-dimensional. In particular, the Hodge–de Rham spectral sequence of $X$ does not degenerate.

**Proof.** Let $G \to \mathrm{Spec} \, \mathbb{F}_p$ be the group scheme $\mathfrak{g}_p$, and choose a linear action of $G$ on $\mathbb{P}^N_{\mathbb{F}_p}$ for some $N > 0$ such that the fixed point locus has codimension at least 3. For example, we take 3 copies of the regular representation, and then projectivise [Ray79, Lem. 4.2.2]. Form the quotient $Z = \mathbb{P}^N/G$, which is an $\mathfrak{g}_p$-torsor away from the image $Z^\text{fix} \subseteq Z$ of the fixed locus [Stacks, Tag 07S7]. In particular, $Z$ is smooth outside $Z^\text{fix}$ by Lemma 2.3. Since $\mathbb{P}^N$ is geometrically integral, the differential form $\omega$ of Remark 2.1 is nontrivial by Lemma 2.2.

Repeatedly applying Poonen’s Bertini theorem [Poo04] produces a smooth projective surface $X \subseteq Z \setminus Z^\text{fix}$, which we may choose such that $\omega|_X$ is not identically zero (for example by specifying a tangency condition at a point $x \in Z \setminus Z^\text{fix}$).
The inverse image $Y \subseteq \mathbb{P}^N$ of $X$ is a complete intersection, and $\pi : Y \to X$ is an $\alpha_p$-torsor since $Y \subseteq \mathbb{P} \setminus \mathbb{P}^{\text{fix}}$. Hence $Y$ is geometrically integral by Lemma 2.2, since $\omega|_{X_{\text{reg}}} \neq 0$. Since $Y$ is a geometrically integral complete intersection of dimension $\geq 2$, we find $H^1(Y, \mathcal{O}_Y) = 0$ and $H^0(Y, \Omega^1_Y) = 0$. In particular, $	ext{Pic}^0_Y = 0$, so Lemma 1.1 gives

$$\text{Pic}^0_X = \alpha_p^\vee = \alpha_p.$$ 

Thus, $H^1(X, \mathcal{O}_X) = \text{Lie}(\text{Pic}^0_X)$ is 1-dimensional. The short exact sequence

$$0 \to H^1(X, \alpha_p) \to H^1(X, \mathcal{O}_X) \xrightarrow{\pi^*} H^1(X, \mathcal{O}_X)$$

shows that $H^1(X, \mathcal{O}_X)$ is spanned by the image of the nontrivial $\alpha_p$-torsor $\pi$. In the notation of Remark 2.1, the generator of $H^1(X, \mathcal{O}_X)$ is given by the Čech 1-cocycle $(g_{ij})$. We claim that its image in $H^1(X, \Omega^1_X)$ is nonzero, i.e. $(dg_{ij})$ is not a Čech coboundary. Suppose it were; say $(dg_{ij}) = \eta_i - \eta_j$ for forms $\eta_i \in H^0(U_i, \Omega^1_{U_i})$. Then consider the 1-form $dz_i - \eta_i$ on $\pi^{-1}(U_i)$. We have

$$(z_i - z_j)^p = f_i - f_j = g_{ij}^p,$$

hence $z_i - z_j = g_{ij}$ since $Y$ is integral. Hence,

$$(dz_i - \eta_i) - (dz_j - \eta_j) = dg_{ij} - \eta_i + \eta_j = 0,$$

showing that the $dz_i - \eta_i$ glue to a global 1-form on $Y$, which is nonzero because $dz_i$ is not pulled back from $U_i$. This contradicts the vanishing of $H^1(Y, \Omega^1_Y)$. We conclude that the generator $(g_{ij})$ of $H^1(X, \mathcal{O}_X)$ does not survive the Hodge–de Rham spectral sequence.

On the other hand, the kernel of $\Omega^1_X \to \pi_* \Omega^1_Y$ is locally generated by $\omega_i$, since the cotangent complex of $\pi^{-1}(U_i) \to U_i$ is the complex

$$\mathcal{I}_i/\mathcal{I}_i^2 \xrightarrow{0} \Omega^1_{\pi^{-1}(U_i)/U_i},$$

where $\mathcal{I}_i$ is the ideal sheaf on $U_i \times \mathbb{A}^1$ given by $z_i^p - f_i$. We get an exact sequence

$$0 \to \omega \mathcal{O}_X \to \Omega^1_X \to \pi_* \Omega^1_Y,$$

so vanishing of $H^0(Y, \Omega^1_Y)$ gives $H^0(X, \omega \mathcal{O}_X) = H^0(X, \Omega^1_X)$, i.e. $H^0(X, \Omega^1_X)$ is 1-dimensional and generated by $\omega$. Note that $d\omega = 0$ since $\omega$ is locally exact, so $\omega$ survives the Hodge–de Rham spectral sequence. Hence, $h^1_{\text{dR}}(X) = 1$. 

The constructions of Proposition 1.3 and Proposition 2.4 are both covered by [Ray79, Prop. 4.2.3], but we needed a more detailed analysis of the Hodge numbers. Using Poonen’s Bertini theorems, we were also able to construct examples over the prime field $\mathbb{F}_p$. The smoothness claim for $\alpha_p$-quotients is not explained in [loc. cit.].

Remark 2.5. In general we cannot expect the complete intersection $Y \subseteq \mathbb{P}^N$ of the proof of Proposition 2.4 to be smooth. Indeed, the degree of $Y$ is large with respect to $p$ in order for $Y$ to be $\alpha_p$-invariant. But a smooth complete intersection of sufficiently high degree does not admit any infinitesimal automorphisms, so in particular cannot have a free $\alpha_p$-action.
3. The Hodge Ring of Varieties

We modify the definition of the Hodge ring of [KS13] to account for the failure of Hodge symmetry in characteristic $p > 0$.

**Definition 3.1.** Consider $\mathbb{Z}[x, y, z]$ as a graded ring where $x$ and $y$ both have degree 0 and $z$ has degree 1. The Hodge ring of varieties over $k$ is the graded subring $H_* \subseteq \mathbb{Z}[x, y, z]$ whose $n$-th graded piece is

$$H_n = \left\{ \left( \sum_{i,j=0}^{n} h^{i,j} x^i y^j \right) z^n \mid h_{n-i,n-j} = h^{i,j} \text{ for all } i, j \in \{0, \ldots, n\} \right\}.$$  

This is different from the notation of [KS13], where $H_*$ denotes the ring where moreover Hodge symmetry holds.

**Definition 3.2.** Write $h: \text{Var}_k \to H_*$ for the map that sends an $n$-dimensional variety $X$ to its (formal) Hodge polynomial

$$h(X) = \left( \sum_{i,j=0}^{n} h^{i,j}(X) x^i y^j \right) z^n,$$

where $h^{i,j}(X) = h^{i,j}(X, \Omega^i_X)$ as usual. The Künneth formula [Stacks, Tag 0BED] shows that $h$ preserves products, i.e. $h(X \times Y) = h(X) \cdot h(Y)$ for all $X, Y \in \text{Var}_k$.

To justify the name Hodge ring of varieties, we show in Corollary 3.9 that $H_*$ is the subring (equivalently, subgroup) of $\mathbb{Z}[x, y, z]$ generated by the image of $h$.

**Theorem 3.3.** Consider the graded ring $\mathbb{Z}[A, B, C, D]$ where $A$ and $B$ have degree 1, and $C$ and $D$ have degree 2. Then the map

$$\phi: \mathbb{Z}[A, B, C, D] \to H_*$$

given by

$$\phi(A) = (1 + xy)z, \quad \phi(C) = xy \cdot z^2,$$

$$\phi(B) = (x + y)z, \quad \phi(D) = (x + xy^2)z^2$$

is a surjection of graded rings, with kernel generated by

$$G := D^2 - ABD + C(A^2 + B^2 - 4C).$$

**Remark 3.4.** Kotschick and Schreieder show [KS13, Thm. 6] that $\mathbb{Z}[A, B, C]$ maps isomorphically onto the subring of $H_*$ where Hodge symmetry holds. Our proof is a modification and simplification of theirs.

**Lemma 3.5.** Let $\psi: M \to N$ be a homomorphism between free abelian groups of the same finite rank. Then $\psi$ is an isomorphism if and only if $\psi \otimes F_\ell$ is injective for all primes $\ell$.

**Proof.** The cokernel $C$ is nonzero if and only if $C \otimes F_\ell \neq 0$ for all primes $\ell$, so the result follows from right exactness of $- \otimes F_\ell$ and a dimension argument. \qed
Definition 3.6. Write \( r_n \) for the number
\[
\begin{align*}
  r_n &:= \begin{cases} 
    \frac{(n+1)^2+1}{2}, & n \equiv 0 \pmod{2}, \\
    \frac{(n+1)^2}{2}, & n \equiv 1 \pmod{2}.
  \end{cases}
\end{align*}
\]
Then \( \mathcal{H}_n \) is free of rank \( r_n \), with basis given by
\[
\begin{align*}
  & (x^i y^j + x^{n-i} y^{n-j}) z^n, & (i, j) \neq (n - i, n - j), \\
  & x^i y^j z^n, & (i, j) = \left( \frac{n}{2}, \frac{n}{2} \right).
\end{align*}
\]

Lemma 3.7. The degree \( n \) part of the algebra \( R = \mathbb{Z}[A, B, C, D]/(G) \) is free of rank \( r_n \).

Proof. It has a basis given by
\[
\begin{align*}
\{ A^i B^j C^k \mid i, j, k \in \mathbb{Z}_{\geq 0}, & \quad i + j + 2k = n \} \cup \{ A^i B^j C^k D \mid i, j, k \in \mathbb{Z}_{\geq 0}, & \quad i + j + 2k = n - 2 \}.
\end{align*}
\]
Moreover, a simple induction shows that
\[
\begin{align*}
  r'_n := \# \left\{ (i, j, k) \in \mathbb{Z}_{\geq 0}^3 \mid i + j + 2k = n \right\} &= \begin{cases} 
    \frac{(n+2)^2}{4}, & n \equiv 0 \pmod{2}, \\
    \frac{(n+2)^2-1}{4}, & n \equiv 1 \pmod{2}.
  \end{cases}
\end{align*}
\]
Thus, the result follows since \( \text{rk } R_n = r'_n + r'_{n-2} = r_n \).

Proof of Theorem 3.3. Clearly \( \phi \) is a homomorphism of graded rings, and one easily verifies that \( G \in \ker(\phi) \). Thus, we get an induced map
\[
\psi: R \to \mathcal{H}_n
\]
between graded rings that have the same rank in each degree by Definition 3.6 and Lemma 3.7. By Lemma 3.5 it suffices to show that \( \psi_\ell = \psi \otimes \mathbb{F}_\ell \) is injective for every prime \( \ell \).

Note that \( R \otimes \mathbb{F}_\ell = \mathbb{F}_\ell[A, B, C, D]/(G) \) is a 3-dimensional domain since \( G \) is irreducible (for example, its restriction modulo \( B \) is Eisenstein at \( (C) \subseteq \mathbb{F}_\ell[A, C] \) when viewed as a polynomial in \( D \)). Thus, to show injectivity of \( \psi_\ell \), it is enough to show that the image \( \text{im}(\psi_\ell) \subseteq \mathbb{F}_\ell[x, y, z] \) has Krull dimension 3. But the elements \( \psi_\ell(A), \psi_\ell(B), \psi_\ell(C) \) are algebraically independent, for example because the Jacobian
\[
\begin{align*}
  J = \left( \begin{array}{ccc}
    \frac{dA}{dx} & \frac{dA}{dy} & \frac{dA}{dz} \\
    \frac{dB}{dx} & \frac{dB}{dy} & \frac{dB}{dz} \\
    \frac{dC}{dx} & \frac{dC}{dy} & \frac{dC}{dz}
  \end{array} \right) = \begin{pmatrix}
    yz & xz & 1 + xy \\
    z & z & x + y \\
    yz^2 & xz^2 & 2xyz
  \end{pmatrix}
\end{align*}
\]
is invertible at \( (x, y, z) = (0, 1, 1) \).

Corollary 3.8. Let \( E \) be an elliptic curve, and let \( S \) be any surface for which \( h^{1,0}(S) - h^{0,1}(S) = \pm 1 \) (for example, the surface constructed in Proposition 1.3). Then \( \mathcal{H}_n \) is generated by \( \mathbb{P}^1, E, \mathbb{P}^2 \), and \( S \), subject only to a monic quadratic equation in \( S \) over \( \mathbb{Z}[\mathbb{P}^1, E, \mathbb{P}^2] \).
Proof. In the notation of Theorem 3.3, we have $A = h(P^1)$, $B = h(E) - h(P^1)$, and $C = h(P^1 \times P^1) - h(P^2)$. Finally, $D$ can be obtained from $\pm h(S)$ by adding a suitable linear combination of $A^2$, $AB$, $B^2$, and $C$, since $h^{1,0}(S) - h^{0,1}(S) = \pm 1$. This proves the first statement, and the second follows from Theorem 3.3 since $h(S)$ differs from $\pm D$ by a translation in $\mathbb{Z}[P^1, E, P^2]$.

Corollary 3.9. The Hodge ring of varieties over $k$ is generated by smooth projective varieties that are defined over $\mathbb{F}_p$, admit a lift to $\mathbb{Z}_p$, and for which the Hodge–de Rham spectral sequence degenerates.

Proof. Clearly $P^1$ and $P^2$ can be defined over $\mathbb{F}_p$ and lifted to $\mathbb{Z}_p$. For $E$ we may choose an elliptic curve over $\mathbb{F}_p$ and lift to $\mathbb{Z}_p$, and for $S$ we may choose the surface of Proposition 1.3. The degeneration claim is Remark 1.4.

We deduce from the above presentation the following theorems.

Theorem 3.10. The only universal congruences between the Hodge numbers $h^{i,j}(X)$ of smooth proper varieties $X$ of dimension $n$ over $k$ are those given by Serre duality.

Theorem 3.11. The only universal linear relations between the Hodge numbers $h^{i,j}(X)$ of smooth proper varieties $X$ of dimension $n$ over $k$ are those given by Serre duality.

That is, if $m, n \in \mathbb{Z}_{\geq 0}$ and $\lambda_{i,j} \in \mathbb{Z}$ for $i, j \in \{0, \ldots, n\}$ are such that

$$\sum \lambda_{i,j} h^{i,j}(X) \equiv 0 \pmod{m} \quad (3.1)$$

for every $X \in \text{Var}_k$ of dimension $n$, then for all $(i, j) \neq (\frac{n}{2}, \frac{n}{2})$ we have

$$\lambda_{i,j} \equiv -\lambda_{n-i,n-j} \pmod{m}, \quad (3.2)$$

and similarly if we take $\lambda_{i,j} \in \mathbb{Q}$ and consider equality instead of congruence mod $m$ in (3.1) and (3.2).

Remark 3.12. While this paper was in preparation, the results of [KS13] on linear relations between Hodge numbers in characteristic $0$ were improved to cover all polynomial relations [PS19]. In positive characteristic, this will appear in joint work between the first author of [PS19] and the present author [DBP20].

4. Birational invariants

The Hodge numbers $h^0(X, \Omega_X^i)$ are birational invariants of a smooth proper variety $X$, and Chatzistamatiou and Rülling proved the same for the numbers $h^i(X, \mathcal{O}_X)$ [CR11, Thm. 1]. We show that these are the only linear combinations of Hodge numbers that are birational invariants.

Definition 4.1. Let $\mathcal{I} \subseteq \mathcal{H}_*$ be the subgroup generated by differences $X - X'$ for $X, X' \in \text{Var}_k$ birational. Note that $\mathcal{I}$ is a (homogeneous) ideal, for if $X \dashrightarrow X'$ is a birational map, then so is $X \times Y \dashrightarrow X' \times Y$ for any $Y \in \text{Var}_k$. Write $\mathcal{I}_n$ for the degree $n$ part of $\mathcal{I}$. 

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Proposition 4.2. Consider the quotient map
\[ \phi : \mathcal{H}_* \to \mathbb{Z}[x, y, z]/(xy). \]
Then
1. The degree \( n \) part of \( \text{im} \phi \) is free of rank \( 2n \), with basis given by
   \[ \left\{ y^j z^n \mid 0 \leq j \leq n - 1 \right\} \cup \left\{ x^i z^n \mid 0 < i \leq n - 1 \right\} \cup \left\{ (x^n + y^n)z^n \right\}. \]
2. The kernel of \( \phi \) satisfies \( \ker \phi = (C) = \mathcal{I} \).

Proof. The first statement is obvious, by looking at the image of the basis
\[ (x^i y^j + x^{n-i} y^{n-j})z^n, \quad (i, j) \neq \left( \frac{n}{2}, \frac{n}{2} \right), \]
\[ x^i y^j \cdot z^n, \quad (i, j) = \left( \frac{n}{2}, \frac{n}{2} \right) \]
of \( \mathcal{H}_n \). To prove the second statement, note that \( C = xy \cdot z \in \ker \phi \). We have
\[ \mathcal{H}_*/(C) = \mathbb{Z}[A, B, D]/(D^2 - ABD), \]
so a basis for \( \mathcal{H}_*/(C) \) is given by \( A^i B^j \) and \( A^i B^j D \) for \( i, j \in \mathbb{Z}_{\geq 0} \). Thus, the degree \( n \) part of \( \mathcal{H}_*/(C) \) is free of rank \( (n + 1) + (n - 1) = 2n \). Since \( \mathcal{H}_*/(C) \to \text{im} \phi \) is surjective and the degree \( n \) parts of both sides are free of the same rank, we conclude that it is an isomorphism. Thus, \( (C) = \ker \phi \).

The Hodge numbers \( h^{i,0}(X) = h^0(X, \Omega_X^i) \) and \( h^{0,j}(X) = h^j(X, \Omega_X^0) \) are birational invariants (for the latter, see [CR11]). Since \( \phi \) only remembers the \( h^{p,0} \) and the \( h^{0,q} \), we get \( \mathcal{I} \subseteq \ker \phi \). Finally, note that \( C \in \mathcal{I} \) because \( C = \text{Bl}_{pt}(\mathbb{P}^2) - \mathbb{P}^2 \) (or \( \mathbb{P}^1 \times \mathbb{P}^1 - \mathbb{P}^2 \)). Thus, \( (C) \subseteq \mathcal{I} \), which finishes the proof.

We deduce the following theorem, which is the analogue of Theorem 2 of [KS13].

Theorem 4.3. The mod \( m \) reduction of an integral linear combination of Hodge numbers is a birational invariant of smooth proper varieties if and only if the linear combination is congruent mod \( m \) to a linear combination of the \( h^{i,0} \) and the \( h^{0,j} \) (and their duals \( h^{n-i,n} \) and \( h^{n,n-j} \)).

Theorem 4.4. A rational linear combination of Hodge numbers is a birational invariant of smooth proper varieties if and only if it is a linear combination of the \( h^{i,0} \) and the \( h^{0,j} \) (and their duals \( h^{n-i,n} \) and \( h^{n,n-j} \)).

5. The de Rham ring

In analogy with the Hodge ring, we define a de Rham ring \( \mathcal{DR}_* \), whose elements correspond to formal de Rham polynomials of varieties.

Definition 5.1. Consider \( \mathbb{Z}[t, z] \) as a graded ring where \( t \) has degree 0 and \( z \) has degree 1. The de Rham ring of varieties over \( k \) is the graded subring
\( \mathcal{DR}_* \subseteq \mathbb{Z}[t, z] \) whose degree \( n \) part is

\[
\mathcal{DR}_n = \left\{ \left( \sum_{i=0}^{2n} h^i t^i \right) z^n \mid \begin{align*}
&h^i = h^{2n-i} \text{ for all } i, \\
&h^n \text{ is even if } n \text{ is odd.} \end{align*} \right\}.
\]

This differs from the notation of [KS13], where \( \mathcal{DR}_* \) denotes the ring where moreover \( h^i \) is even for every odd degree \( i \).

**Definition 5.2.** Write \( \mathrm{dR}: \text{Var}_k \to \mathcal{DR}_* \) for the map sending an \( n \)-dimensional variety \( X \) to its (formal) de Rham polynomial

\[
\mathrm{dR}(X) = \left( \sum_{i=0}^{2n} h^i_{\mathrm{dR}}(X)t^i \right) z^n.
\]

Note that the cup product defines a perfect pairing [Ber74, Thm. VII.2.1.3]

\[
H^i_{\mathrm{dR}}(X) \times H^{2n-i}_{\mathrm{dR}}(X) \to k,
\]

showing that \( h^i_{\mathrm{dR}}(X) = h^{2n-i}_{\mathrm{dR}}(X) \). The pairing on \( H^i_{\mathrm{dR}}(X) \) is alternating if \( n \) is odd, so in that case \( h^n_{\mathrm{dR}}(X) \) is even, showing that \( \mathrm{dR}(X) \) lands in \( \mathcal{DR}_* \). The Künneth formula [Ber74, Cor. V.4.2.3] shows that \( \mathrm{dR}(X \times Y) = \mathrm{dR}(X) \cdot \mathrm{dR}(Y) \) for \( X, Y \in \text{Var}_k \).

**Remark 5.3.** There is a natural map \( s: \mathcal{H}_* \to \mathcal{DR}_* \) given by \( x, y \mapsto t \) and \( z \mapsto z \). The triangle

\[
\begin{array}{ccc}
\text{Var}_k & \xrightarrow{\mathrm{dR}} & \mathcal{DR}_* \\
\downarrow{h} & & \downarrow{s} \\
\mathcal{H}_* & \to & \mathcal{DR}_*
\end{array}
\]

does not commute, because \( s(h(X)) = \mathrm{dR}(X) \) if and only if the Hodge–de Rham spectral sequence of \( X \) degenerates.

However, by Corollary 3.9 the Hodge ring \( \mathcal{H}_* \) can be generated by varieties for which the Hodge–de Rham spectral sequence degenerates. We can use the presentation of \( \mathcal{H}_* \) from Theorem 3.3 to get a compatible presentation of \( \mathcal{DR}_* \). In particular, \( \mathcal{DR}_* \) is generated by the image of \( \mathrm{dR} \); see Corollary 5.5.

**Theorem 5.4.** Consider the graded ring \( \mathbb{Z}[A, B, C, D] \) where \( A \) and \( B \) have degree 1, and \( C \) and \( D \) have degree 2. Then the map

\[
\psi: \mathbb{Z}[A, B, C, D] \to \mathcal{DR}_*
\]

given by

\[
\begin{align*}
\phi(A) &= (1 + t^2)z, & \phi(C) &= t^2 \cdot z^2, \\
\phi(B) &= 2t \cdot z, & \phi(D) &= (t + t^3)z^2
\end{align*}
\]

is a surjection of graded rings with \( \psi = s \circ \phi \). The kernel of \( \psi \) is given by

\[
J = (A^2C - D^2, AB - 2D, B^2 - 4C, BD - 2AC).
\]
Proof. Note that $DR_n$ is free of rank $n + 1$, with basis given by
\[
\begin{align*}
(t^i + t^{2n-i})z^n & \quad 0 \leq i < n, \\
t^n z^n & \quad n \equiv 0 \pmod{2}, \\
2t^n z^n & \quad n \equiv 1 \pmod{2}.
\end{align*}
\]
Computing the image of $s$ on the basis of $H_n$ of Definition 3.6, we see that $s$ is surjective. Since $\phi$ is surjective by Theorem 3.3, we conclude that $\psi = s \circ \phi$ is surjective. On the other hand, one checks that $J \subseteq \ker \psi$, so we get a surjection
\[
R = Z[A, B, C, D]/J \rightarrow DR_*. 
\]
It suffices to show that the degree $n$ part of $R$ is generated by $n + 1$ elements. It is generated by $A^iB^jC^kD^\ell$ for $i + j + 2k + 2\ell = n$. By the relation $B^2 - 4C$ we may assume $j \leq 1$. The relation $AB - 2D$ then shows that we need only consider the monomials $A^iC^kD^\ell$ and $BC^kD^\ell$. Then $BD - 2AC$ allows us to restrict to $A^iC^kD^\ell$ and $BC^k$. Finally, the relation $A^2C - D^2$ shows that $R_n$ is generated by
\[
A^iD^\ell, \quad C^kD^\ell \quad (k > 0), \quad AC^kD^\ell \quad (k > 0), \quad BC^k.
\]
If $n$ is even, there are $\frac{n}{2} + 1, \frac{n}{2}, 0, 0$ monomials of these types in $R_n$ respectively, and if $n$ is odd there are $\frac{n+1}{2}, 0, \frac{n-1}{2}, 1$ monomials of these types in $R_n$. In both cases, they add up to $n + 1$ elements generating $R_n$.

Corollary 5.5. The de Rham ring of varieties over $k$ is generated by smooth projective varieties that are defined over $F_p$, admit a lift to $Z_p$, and for which the Hodge–de Rham spectral sequence degenerates.

Proof. This follows from the same statement in Corollary 3.9 and the surjectivity of $s : H_* \rightarrow DR_*$. 

6. The Hodge–de Rham Ring

Because the triangle in Remark 5.3 does not commute, the last thing to compute is the image of the diagonal map
\[
\text{Var}_k \rightarrow H_* \times DR_* \\
X \mapsto (h(X), dR(X)).
\]
Once again, we will define a subring containing the image, and construct enough varieties that generate this subring.

Definition 6.1. Define the ring homomorphisms
\[
\begin{align*}
\chi : H_* & \rightarrow Z[z] \quad h^{0,0} : H_* \rightarrow Z[z] \\
x, y & \mapsto -1, \quad x, y \mapsto 0, \\
t & \mapsto -1, \quad t & \mapsto 0.
\end{align*}
\]
Then the Hodge–de Rham ring of varieties over $k$ is the subring
\[
HDR_* = \left\{ (a, b) \in H_* \times DR_* \mid h^{0,0}(a) = h^{0,0}(b), \quad \chi(a) = \chi(b) \right\}.
\]
Note that \( (h(X), dR(X)) \in \mathcal{HDR}_n \) if \( X \) is a smooth and proper variety over \( k \) of dimension \( n \). Indeed, the Euler characteristic in any spectral sequence is constant between the pages, so \( \chi(h(X)) = \chi(dR(X)) \). Moreover, \( h^{0,0}(X) \) and \( h^0(X) \) agree since they both equal the number of geometric components of \( X \).

**Lemma 6.2.** The kernel of the morphism \( (\chi, h^0) : \mathcal{DR}_n \to \mathbb{Z}[z] \times \mathbb{Z}[z] \) given by \( t \mapsto (-1, 0) \) is generated by \((t + 2t^2 + t^3)z^2 \) and \((t^2 + 2t^3 + t^4)z^3 \).

**Proof.** Clearly \( I = ((t + 2t^2 + t^3)z^2, (t^2 + 2t^3 + t^4)z^3) \) is contained in \( \ker(\chi, h^0) \), so we get a quotient map

\[
R = \mathcal{DR}_n/I \to \mathbb{Z}[z] \times \mathbb{Z}[z].
\]

The image is generated in degree \( n \) by \((1, 1)\) if \( n = 0 \), by \((2z^3, 0)\) and \((0, z^n)\) if \( n \) is odd, and by \((z^n, 0)\) and \((0, z^n)\) if \( n > 0 \) is even. Thus, it suffices to show that \( R_n \) can be generated by 1 element if \( n = 0 \) and by 2 elements if \( n > 0 \).

Under the presentation \( \mathbb{Z}[A, B, C, D]/J \cong \mathcal{DR}_n \) of Theorem 5.4, the elements \( D + 2C \) and \( AC + BC \) map to the generators of \( I \), so

\[
R = \mathbb{Z}[A, B, C, D]/(J + (D + 2C, AC + BC)).
\]

Pass to the quotient \( \mathbb{Z}[A, B, C, D]/(D + 2C) \cong \mathbb{Z}[A, B, C] \), where the image of \( J + (D + 2C, AC + BC) \) is generated by \( B^2 - 4C, AB + 4C, \) and \( AC + BC \). Setting \( E = A + B \) (corresponding to the class of an elliptic curve), we finally have to compute

\[
R \cong \mathbb{Z}[E, B, C]/(B^2 - 4C, EB, EC).
\]

Then \( R_n \) is generated by \( E^iB^k \) for \( i + j + 2k = n \). Using \( B^2 - 4C \) we reduce to \( j \leq 1 \). The relation \( EB \) shows that for \( j = 1 \) we may take \( i = 0 \). By the relation \( EC \) we may assume \( i = 0 \) if \( k > 0 \). Thus, \( R_n \) is generated by

\[
E^i, \quad C^k \quad (k > 0), \quad BC^k,
\]
sitting in degree \( i, 2k, \) and \( 2k + 1 \) respectively. We see that \( R_n \) is generated by 1 element if \( n = 0 \) and by 2 elements if \( n > 0 \).

**Remark 6.3.** The ideal generated by \((t + 2t^2 + t^3)z^2 \) contains \((2t^2 + 4t^3 + 2t^4)z^3 \) and \((t + 2t^2 + 2t^3 + 2t^4 + t^5)z^3 \), so we may replace the second generator of \( I \) as in Lemma 6.2 by any element in \( I_3 \) for which \( h^2 \) is odd.

**Theorem 6.4.** Let \( \mathbb{Z}[A, B, C, D] \) as in Theorem 3.3 and Theorem 5.4. Let \( S \) be a surface for which \( h^{1,0}(S) + h^{0,1}(S) = h_{dR}^{1,0}(S) \) is odd, and let \( T \) be a threefold for which \( h^{2,0}(T) + h^{1,1}(T) + h^{0,2}(T) = h_{dR}^{2,0}(T) \) is odd. Define the map

\[
\tau : \mathbb{Z}[A, B, C, D][S, T] \to \mathcal{HDR}_* \subseteq H_* \times DR_*
\]
on \( \mathbb{Z}[A, B, C, D] \) by \((\phi, \psi)\), and by

\[
\tau(S) = (h(S), dR(S)), \quad \tau(T) = (h(T), dR(T)).
\]

Then \( \tau \) is surjective. In particular, \( \mathcal{HDR}_* \) is generated by varieties.
Example 6.5. For $S$ we may take the surface of Proposition 2.4. For $T$, we may take a sufficiently high degree smooth hypersurface in $S \times S$. Indeed, a Künneth computation shows that for $S \times S$ the difference $h_{2,0} + h_{1,1} + h_{0,2} - h_{2,2}^{\text{dR}}$ is odd. In characteristic 0, Nakano vanishing implies weak Lefschetz for algebraic de Rham cohomology. In arbitrary characteristic, replacing Nakano vanishing by Serre vanishing shows that for sufficiently high degree hypersurfaces $T \subseteq S \times S$, the Hodge and de Rham numbers in degree $i < \dim T$ agree with those of $S \times S$.

See for example [DBP20] for a proof of weak Lefschetz for algebraic de Rham cohomology using Nakano vanishing or Serre vanishing.

Proof of Theorem 6.4. By Corollary 3.9 the Hodge ring $\mathcal{H}_*$ is generated by varieties for which the Hodge–de Rham spectral sequence degenerates. Thus, given $(a,b) \in \mathcal{H}^{\text{dR}}_*$, we know that $(a,s(a)) \in \im(\tau)$. We reduce to the case $a = 0$, hence $b \in \ker(\chi, h^0)$, which is the ideal $I$ of Lemma 6.2. Again using that $(a,s(a)) \in \im(\tau)$, and by surjectivity of $s$ (Theorem 5.4), it suffices to let $b$ be one of the generators of $I$.

Since $(h(S), s(h(S)))$ is in the image of $\tau$, so is

$$S' = \left( h(S), s(h(S)) \right) - \tau(S) = \left( 0, s(h(S)) - \text{dR}(S) \right).$$

Writing $b = s(h(S)) - \text{dR}(S)$ as $b = (\sum b^i t^i) z^2$, we have $b^1 = 1$ by assumption. Since $b \in I$ we get $h^0(b) = 0$, hence $b^0 = 0$, so we get $b^2 = 0$, $b^3 = 1$ by Poincaré duality. Finally, $\chi(b) = 0$ gives $b^2 = 2$. We conclude that

$$S' = (0, (t + 2t^2 + t^3) z^2).$$

Replacing $S$ by $T$ in this argument, we get an element of $I_3$ for which $h^2$ is odd. By Remark 6.3 any such class can be taken as the second generator for $I$. □

This gives all linear congruences between Hodge numbers and de Rham numbers.

Theorem 6.6. The only universal congruences between the Hodge numbers $h^{i,j}(X)$ and the de Rham numbers $h_{\text{dR}}^{i,j}(X)$ of smooth proper varieties $X$ of dimension $n$ over $k$ are the congruences spanned by

- Serre duality: $h^{i,j}(X) = h^{n-i,n-j}(X)$;
- Poincaré duality: $h_{\text{dR}}^{i,j}(X) = h_{\text{dR}}^{2n-i-j}(X)$;
- Components: $h^{1,0}(X) = h_{\text{dR}}^{1,0}(X)$;
- Euler characteristic: $\sum_{i,j} (-1)^{i+j} h^{i,j}(X) = \sum_{i} (-1)^i h_{\text{dR}}^{i}(X)$;
- Parity of middle cohomology: $h_{\text{dR}}^{n/2}(X) \equiv 0 \text{ (mod 2)}$ if $n$ is odd. □

Theorem 6.7. The only universal linear relations between the Hodge numbers $h^{i,j}(X)$ and the de Rham numbers $h_{\text{dR}}^{i,j}(X)$ of smooth proper varieties $X$ of dimension $n$ over $k$ are the relations spanned by

- Serre duality: $h^{i,j}(X) = h^{n-i,n-j}(X)$;
- Poincaré duality: $h_{\text{dR}}^{i,j}(X) = h_{\text{dR}}^{2n-i-j}(X)$;
- Components: $h^{0,0}(X) = h_{\text{dR}}^{0,0}(X)$;
- Euler characteristic: $\sum_{i,j} (-1)^{i+j} h^{i,j}(X) = \sum_{i} (-1)^i h_{\text{dR}}^{i}(X)$.
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