Dynamical control on the homotopy analysis method for solving nonlinear shallow water wave equation

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Abstract. In this paper, the nonlinear shallow water wave equation is illustrated. The famous semi-analytical method, homotopy analysis method (HAM) is applied for solving this equation. The main novelty of this study is to validate the numerical results using the stochastic arithmetic, the CESTAC method and the CADNA library. Based on this method, we can find the optimal iteration of the HAM, optimal approximation of the shallow water wave equation and optimal error. The main theorem of the CESTAC method is proved. Based on this theorem, we can show that the number of common significant digits for two successive approximations are almost equal to the number of common significant digits for exact and approximate solutions. Thus instead of traditional absolute error to show the accuracy of method we can apply the new termination criterion depends on two successive approximations. In order to find the convergence region of the HAM, several \(h\)-curves are demonstrated.

1. Introduction
Tsunamis are sea surface gravity waves generated by large-scale underwater disturbances. These long waves can be instigated by some phenomena [1]. Because of seismic and volcanic activity associated with tectonic plate boundaries along the Pacific Ring of Fire, tsunamis occur most frequently in the Pacific Ocean, but are a worldwide natural phenomenon. They are possible wherever large bodies of water are found, including inland lakes, where they can be caused by landslides and glacier calving. Very small tsunamis, non-destructive and undetectable without specialized equipment, occur frequently as a result of minor earthquakes and other events. Tsunami waves in recent years have caused extensive damage to coastal cities in various countries.

In 2018, a localised tsunami struck Palu, sweeping shore-lying houses and buildings on its way; the earthquake, tsunami and soil liquefaction killed at least 1,234 and injured over 600. The Indonesian Agency for Meteorology, Climatology and Geophysics (BMKG) confirmed that a tsunami occurred, with a height of between 1.5 to 2 metres (4.9 to 6.6 ft), striking the settlements of Palu, Donggala and Mamuju. Also, Anak Krakatau erupted and damaged local seismographic equipment though a nearby seismographic station detected continuous tremors. BMKG detected
a tsunami event at the western coast of Banten, but the agency had not detected any preceding tectonic events. It was confirmed via satellite data and helicopter footage that the southwest sector of the Anak Krakatau had collapsed which triggered the tsunami and the main conduit is now erupting from underwater producing Surtseyan style activity. The Indonesian National Board for Disaster Management initially reported 20 deaths and 165 injuries and the figure had been revised to 43 deaths, 584 injured, and 2 missing. Of the 43 recorded deaths, 33 were killed in Pandeglang, 7 in South Lampung, and 3 in Serang, with most of the injuries recorded (491) also occurring in Pandeglang. The areas of Pandeglang struck by the wave included beaches which are popular tourist destinations. Finally, the death toll had risen to 426, while the injured numbered 7,202 and the missing 24.

Therefore, providing numerical solutions to the governing equations of waves in a common way to more accurately predict waves and analyze the behavior of waves in different situations and conditions is of particular importance [2, 3, 4]. During recent years, the shallow water wave equations have been solved and analysed by many researchers applying numerical and semi analytical methods [5, 6, 7, 8].

Homotopy analysis method is one of important and flexible semi-analytical methods for solving linear and nonlinear problems. This method has been used for solving ill posed problems, integral and differential equations and many kinds of mathematical and engineering problems.

In all of the mentioned papers, the authors used the FPA to find the approximate solution without considering about optimal iteration, optimal approximation and the validation of numerical results. In these works, the numerical results are obtained for fixed and determined step size. In the FPA, the results may be false or with low accuracy without awareness of the user. The usual termination criterion based on the FPA is in form:

$$|Q - Q_J| ≤ ϵ, \quad \text{or} \quad |Q_J - Q_{J-1}| ≤ ϵ,$$  \hspace{1cm} (1)

where, $ϵ$ is an arbitrary positive value and $Q_J$ is the approximate of $Q$. This criterion may not be acceptable. Because for large value of $ϵ$ the process will be stopped quickly and the suitable approximation can not be produced and for small $ϵ$ many iterations can be generated without progressing the accuracy of the results. Also, the first criterion in (1) is traditional absolute error to be less or equal to $ϵ$ that depends on the exact solution.

In the SA, in place of (1), the following criterion is applied

$$|Q_J - Q_{J-1}| = @0.0,$$  \hspace{1cm} (2)

which depends on two successive approximations $Q_J$ and $Q_{J-1}$. Also, sign $@0.0$ is called the informatical zero as it includes the mathematical zero. It means that the difference between these two numerical results has not any significant digits and therefore the number of common significant digits (NCSDs) of $Q_J$ and $Q_{J-1}$ is the same.

The aim of this work, is to find the optimal iteration, error and approximation of G-LIR using the CESTAC method. Chesneaux in [9] presented the equality relations in scientific computing and also properties of the SA and he described the CADNA library as an ADA tool for round-off errors analysis and for numerical debugging. Finally, Lamotte et al in [10] presented a version of the CADNA for use with C or C++ programs. In recent years, the CESTAC method was applied to validate the numerical methods [11, 12, 13, 14, 15, 16, 17, 18, 19].

The CESTAC method applies the SA in its computations and it is more useful than the FPA. Also, in order to implement the CESTAC method, the CADNA library (http://cadna.lip6.fr) must be used instead of the common mathematical packages such as Matlab or Maple. In this library we prepare the programs using C/C++ or FORTRAN and they are performed on LINUX operating system. This method is able to eliminate the unessential iterations but the FPA does not have this ability.
2. Governing equations of non-linear wave propagation

Tsunamis are generally classified as long waves. Solitary waves or combinations of negative and positive solitary-like waves are often used to simulate the run-up and shoreward inundation of these catastrophic waves. The following equations display the specific case of the run-up of 2D long waves incident upon a uniform sloping beach connected to an open ocean with a uniform depth (figure 1).

![Figure 1. Definition Sketch for solitary wave run-up](image)

The related classical nonlinear shallow-water equations are shown as follows:

\[ \eta_t + (u(h + \eta))_x = 0, \]
\[ u_t + uu_x + g\eta_x = 0, \]

where \( \eta \) is wave amplitude, \( u \) is depth averaged velocity, \( h \) is variable depth, and \( g \) is acceleration of gravity. In addition, the initial condition of these wave is generally represented as

\[ \eta(x, 0) = H \text{sech}^2 \sqrt{\frac{3H}{4d}}x, \]
\[ u(x, 0) = \eta \sqrt{gd}, \]

where \( H \) and \( d \) denote the initial wave height and stationary elevation, respectively.

3. Main Idea

To show the basic idea of HAM, the following procedure is considered. At first, differential equation is considered as follows:

\[ \mathcal{N}[^\omega(x, t)] = 0, \]

where \( \mathcal{N} \) is a nonlinear operator, \( x \) and \( t \) represent the independent variables, and \( \omega \) is an unknown function. Then, all boundaries or initial conditions are ignored for simplicity, and the deformation equation which is so-called zeroth-order deformation equation is constructed in the following form

\[ (1 - q)\mathcal{L}[\phi(x, t; q) - \omega_0(x, t)] = q\hbar\mathcal{N}[\phi(x, t; q)], \]

where \( q \in [0, 1] \) is the embedding parameter, \( \hbar \neq 0 \) is an auxiliary parameter, \( \mathcal{L} \) is an auxiliary linear operator, \( \phi(x, t; q) \) is an unknown function, \( \omega_0(x, t) \) is an initial guess of \( \omega(x, t) \), and \( \phi(x, t; q) \) is an unknown function. It is obvious when the embedding parameter \( q \), equals to 0 and 1, then (6) can be written as

\[ \phi(x, t; 0) = \omega_0(x, t), \]
\[ \phi(x, t; 1) = \omega(x, t), \]

respectively. Thus, as \( q \) increases from 0 to 1, the solution varies from the initial guess \( \omega_0(x, t) \) to the solution \( \omega(x, t) \). Expanding \( \phi(x, t; q) \) in Taylor series when \( q \) equals to 1 we have:

\[ \phi(x, t; q) = \omega_0(x, t) + \sum_{m=1}^{\infty} \omega_m(x, t)q^m, \]
where \( \omega_m(x,t) = \left. \frac{1}{m!} \frac{\partial^m \phi(x,t;q)}{\partial q^m} \right|_{q=0} \) The convergence of the series in (8) depends upon the auxiliary parameter \( h \). If it is convergent at \( q = 1 \), then we get:

\[
\omega(x,t) = \omega_0(x,t) + \sum_{m=1}^{\infty} \omega_m(x,t),
\]

which must be one of the solutions of the original nonlinear equation, as proven by Liao. Then, \( \overline{\omega_m} \) is defined as follows:

\[
\overline{\omega_m} = \{\omega_0(x,t), \omega_1(x,t), \ldots, \omega_n(x,t)\}.
\]

Therefore, the following \( m \)-th order deformation equation is obtained by differentiating (5) \( m \)-times with respect to \( q \), dividing them by \( m! \), and finally setting \( q = 0 \). Then, we have:

\[
\mathcal{L}[\omega_m(x,t) - \chi_m \psi_{m-1}(x,t)] = hR_m(\overline{\omega_{m-1}}),
\]

where

\[
R_m(\overline{\omega_{m-1}}) = \left. \frac{1}{m!} \frac{\partial^{m-1} N[\phi(x,t;q)]}{\partial q^{m-1}} \right|_{q=0},
\]

and

\[
\chi_m = \begin{cases} 
0, & m \leq 1, \\
1, & m > 1.
\end{cases}
\]

It should be emphasized that \( \omega_m(x,t) \) for \( m \geq 1 \) is governed by the linear equation of 11 with linear boundary conditions coming from the original problem, which can be solved by the symbolic computation software such as Mathematica or Maple. To perform HAM, the following initial approximations are considered:

\[
\eta_0(x,0) = \eta(x,0) = Hsech^2 \sqrt{\frac{3H}{4d}} x, \\
u_0(x,0) = u(x,0) = \frac{\eta}{2} \sqrt{gd}.
\]

And the linear operator is defined as follows:

\[
\mathcal{L}_i[\phi_i(x,t; q)] = \frac{\partial \phi_i(x,t; q)}{\partial t} , i = 1, 2.
\]

According to (3) nonlinear operators \( \mathcal{N}_1 \) and \( \mathcal{N}_2 \) can be defined:

\[
\mathcal{N}_1[\phi_1(x,t; q), \phi_2(x,t; q)] = \frac{\partial \phi_1(x,t; q)}{\partial t} + \frac{\partial (\phi_1(x,t; q) \phi_2(x,t; q))}{\partial x} + \frac{\partial h(x) \phi_2(x,t; q)}{\partial x},
\]

\[
\mathcal{N}_2[\phi_1(x,t; q), \phi_2(x,t; q)] = \frac{\partial \phi_2(x,t; q)}{\partial t} + \phi_2 \frac{\partial \phi_2(x,t; q)}{\partial x} + g \frac{\partial \phi_1(x,t; q)}{\partial x}.
\]

Then, zeroth-order deformation equations are constructed in the following form:

\[
(1-q)\mathcal{L}_1[\phi_1(x,t; q) - \eta_0(x,t)] = qh_1 \mathcal{N}_1[\phi_1(x,t; q), \phi_2(x,t; q)],
\]

\[
(1-q)\mathcal{L}_2[\phi_2(x,t; q) - u_0(x,t)] = qh_2 \mathcal{N}_2[\phi_1(x,t; q), \phi_2(x,t; q)].
\]

Obviously, when \( q = 0 \) and \( q = 1 \), we have:

\[
\phi_1(x,t; 0) = (\eta_0(x,t), \phi_1(x,t; 1) = \eta(x,t),
\]

\[
\phi_2(x,t; 0) = u_0(x,t), \phi_2(x,t; 1) = u(x,t).
\]
Thus, as the embedding parameter \( q \) increases from 0 to 1, \( \phi_1(x, t; q) \) and \( \phi_2(x, t; q) \) vary from the initial approximations of \( \eta_0(x, t) \) and \( u_0(x, t) \) to \( \eta(x, t) \) and \( u(x, t) \) solutions, respectively. By expanding \( \phi_1(x, t; q) \) and \( \phi_2(x, t; q) \) in Taylor series with respect to \( q \), we have:

\[
\begin{align*}
\phi_1(x, t; q) &= \eta_0(x, t) + \sum_{m=1}^{\infty} \eta_m(x, t) q^m, \\
\phi_2(x, t; q) &= u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t) q^m,
\end{align*}
\]

where

\[
\eta_m(x, t) = \frac{1}{m!} \frac{\partial^m \phi_1(x, t; q)}{\partial q^m},
\]

\[
u_m(x, t) = \frac{1}{m!} \frac{\partial^m \phi_2(x, t; q)}{\partial q^m}.
\]

If the auxiliary linear operator, the initial approximations, and the auxiliary parameters \( h_1 \) and \( h_2 \) are so properly chosen, the above series converge at \( q = 1 \). Then, we get:

\[
\begin{align*}
\eta(x, t; q) &= \eta_0(x, t) + \sum_{m=1}^{\infty} \eta_m(x, t), \\
u(x, t; q) &= u_0(x, t) + \sum_{m=1}^{\infty} u_m(x, t),
\end{align*}
\]

which must be one of solutions of original system. Afterwards, the following vectors are defined:

\[
\eta_m = \{ \eta_0(x, t), \eta_1(x, t), \ldots, \eta_n(x, t) \},
\]

\[
u_m = \{ u_0(x, t), u_1(x, t), \ldots, u_n(x, t) \},
\]

and the \( m \)-th order deformation equation is obtained:

\[
\mathcal{L}_1[\eta_m(x, t) - \chi_m \eta_{m-1}(x, t)] = h_1 R_{1,m}(\eta_{m-1}, u_{m-1}).
\]

Now, the solution of the \( m \)-th order deformation equation (20), for \( m \geq 1 \) is:

\[
\begin{align*}
\eta_m(x, t) &= \chi_m \eta_{m-1}(x, t) + h_1 \int_0^1 R_{1,m}(\eta_{m-1}, u_{m-1}) ds, \\
u_m(x, t) &= \chi_m u_{m-1}(x, t) + h_2 \int_0^1 R_{2,m}(\eta_{m-1}, u_{m-1}) ds.
\end{align*}
\]

According to the initial conditions which are initially assumed, we have:

\[
\begin{align*}
\eta_m(x, 0) &= 0, u_m(x, 0) = 0
\end{align*}
\]

\( R_{1,m}(\eta_{m-1}, u_{m-1}) \) and \( \int_0^1 R_{2,m}(\eta_{m-1}, u_{m-1}) \) are:

\[
\begin{align*}
R_{1,m}(\eta_{m-1}, u_{m-1}) &= \frac{\partial \eta_{m-1}}{\partial t} + \frac{\partial}{\partial x} \left( \sum_{n=0}^{m-1} \eta_n u_{m-1-n} + h(x) \frac{\partial}{\partial x} u_{m-1} \right), \\
R_{2,m}(\eta_{m-1}, u_{m-1}) &= \frac{\partial u_{m-1}}{\partial t} + \left( \sum_{n=0}^{m-1} u_n \frac{\partial u_{m-1-n}}{\partial x} \right) + g \frac{\partial}{\partial x} \eta_{m-1}.
\end{align*}
\]

Obviously, the solution of the \( m \)-th order deformation equations (20), for \( m \geq 1 \) becomes:

\[
\begin{align*}
\eta_m &= \chi_m \eta_{m-1} + h_1 \mathcal{L}^{-1}[R_{1,m}(\eta_{m-1}, u_{m-1})], \\
u_m &= \chi_m u_{m-1} + h_2 \mathcal{L}^{-1}[R_{2,m}(\eta_{m-1}, u_{m-1})].
\end{align*}
\]

To make the solution method more simple, \( h_1 \) and \( h_2 \) are assumed equal to \( \bar{h} \).
3.1. CESTAC method-CADNA library

In this section, the CESTAC method is described and the algorithm of this method is presented. Also, a sample program of the CADNA library is demonstrated and finally advantages of the presented method based on the SA are investigated in comparison with the traditional FPA [10].

Assume that some representable values are produced by computer and they are collected in set $A$. Then $W \in A$ can be produced for $w \in R$ with $R$ mantissa bits of the binary FPA in the following form

$$W = w - \chi 2^{E-R} \xi,$$

(25)

where sign of $w$ showed by $\chi$, missing segment of the mantissa presented by $2^{-R} \xi$ and the binary exponent of the result characterized by $E$. Moreover, in single and double precisions $R = 24, 53$ respectively.

Assume $\xi$ is the casual variable that uniformly distributed on $[-1, 1]$. After making perturbation on final mantissa bit of $w$ we will have ($\mu$) and ($\sigma$) as mean and standard deviation for results of $W$ which they have important role in accuracy of $W$. Repeating this process $J$ times for $W_i, i = 1, ..., J$ we will have quasi Gaussian distribution for results. It means that $\mu$ for these data equals to the exact $w$. It is clear that we should find $\mu$ and $\sigma$ based on $W_i, i = 1, ..., J$. For more consideration, the following algorithm is presented where $\tau$ is the value of $T$ distribution as the confidence interval is $1 - \delta$ with $J - 1$ freedom degree.

In general form, in order to find the numerical results we need to apply the usual packages like Mathematica and Matlab. Here, instead of them we introduce the CADNA library as a new tool to implement the CESTAC method.

There are important advantages to apply the CESTAC method and the CADNA library instead of traditional packages which are based on the FPA. In this method, instead of applying (1) we present a criterion independence of absolute error and tolerance value like $\varepsilon$. Applying the CADNA library, we can find the optimal iteration, best approximation in the point of computational view and estimation the accuracy of numerical methods. Moreover, the numerical instabilities can be identified.

**Definition 1.** [9] For $\rho_1, \rho_2 \in R$, the NCSDs is defined as

(1) for $\rho_1 \neq \rho_2$,

$$C_{\rho_1, \rho_2} = \log_{10}\left|\frac{\rho_1 + \rho_2}{2(\rho_1 - \rho_2)}\right| = \log_{10}\left|\frac{\rho_1}{\rho_1 - \rho_2} - \frac{1}{2}\right|,$$

(26)

(2) $C_{\rho_1, \rho_1} = +\infty$.

**Theorem 1.** Let us $\eta_m = \sum_{j=0}^{m} \eta_j$ and $u_m = \sum_{j=0}^{m} \eta_j$ are the approximate solution of problem (3), obtained from (24). The number of common significant digits between the exact and approximate solutions are equal to the number of common significant digits between two successive approximations as

$$C_{\eta_N, \eta_{m+1}} = C_{\eta, \eta_m} + O\left(\frac{1}{m+1}\right),$$

$$C_{u_N, u_{m+1}} = C_{u, u_m} + O\left(\frac{1}{m+1}\right).$$

4. Numerical illustrations

In this section, the numerical results are presented based on three forms: 1- Semi-flat shores ($h(x) = 0.2x - 20$), 2- Moderate-slope shores ($h(x) = x - 100$), 3- Sharp-slope shores ($h(x) = 5x - 500$). Also we assumed that $H = 2$ and $d = 20$. We applied the mentioned HAM method for solving shallow water wave equation. We know that this method is flexible.
and applicable method because of embedding operators, parameters and functions to control the convergence region. For this aim, we should plot the $h$-curves to show the convergence regions and using these plots the parallel parts of $h$-curves with axis $x$ will be the convergence interval. In figures 2, 3 and 4 we can find the $h$-curves for $\eta(x,t)$ and $u(x,t)$ for $x,t = 1$. Based
Figure 6. The approximate solution for $\bar{h} = -0.3$ in the moderate-slope shores mode.

Figure 7. The approximate solution for $\bar{h} = -0.3$ in the sharp-slope shores mode.

on figure 2, the convergence regions for semi-flat shores are $-0.8 \leq h_u \leq 0, -0.9 \leq h_\eta \leq 0.1$, for moderate-slope shores are $-0.5 \leq h_u \leq 0.1, -1 \leq h_\eta \leq 0.2$ and for sharp-slope shores are $-0.2 \leq h_u \leq 0.1, -0.6 \leq h_\eta \leq 0.1$. Also, the graph of approximate solutions are demonstrated in figures 5, 6 and 7.

As we describe in Introduction, in order to show the accuracy of method we do not need to have the exact solution. Here instead of applying the stopping condition (1), the termination
Table 1. The numerical results of $u(x, t)$ applying the CESTAC method.

| $m$ | $u_m$ | $|u_m - u_{m+1}|$  |
|-----|-------|---------------------|
| 0   | 1.4006342777101624E+000 | 1.40063E+000       |
| 1   | 1.4016459511865995E+000 | 0.101167E-02       |
| 2   | 1.401774945040798E+000  | 0.128994E-03       |
| 3   | 1.4017742495070338E+000 | 6.95534E-07        |
| 4   | 1.40177424861038917E+000| 8.966E-10          |
| 5   | 1.40177424861038816E+000| 1.11E-15           |
| 6   | 1.40177424861038819E+000| 0.0.              |

Table 2. The numerical results of $\eta(x, t)$ applying the CESTAC method.

| $m$ | $\eta_m$ | $|\eta_m - \eta_{m+1}|$  |
|-----|----------|---------------------|
| 0   | 1.99990615793555E+000 | 1.99991E+000       |
| 1   | 1.9987297595799882E+000| 0.11764E-02       |
| 2   | 2.000801404783396E+000 | 0.207165E-02      |
| 3   | 2.000003289694038E+000 | 0.79811E-03       |
| 4   | 2.00000355522895E+000  | 2.9341E-06        |
| 5   | 2.00000045204226E+000  | 3.103E-07         |
| 6   | 2.000000000035452E+000 | 4.51E-08          |
| 7   | 2.0000000000000385E+000| @.0.              |

criterion (2) is applied. Thus the HAM is validated by applying the CESTAC method based on two successive approximations. The approximate solution is produced until $|u_{m+1} - u_m| \neq @.0$. If this condition equals to $@.0$, the mentioned method will be stopped. It means that the values $u_m$ and $u_{m+1}$ are equal stochastically and the number of iteration will be the optimal iteration of the HAM and shows by $m_{opt}$. Also, it is connected for $\eta$. In tables 1 and 2, the numerical results of the CESTAC method are presented. Based on these results we can find the optimal iteration of the HAM, optimal approximation and optimal error of method for solving shallow water wave equation. Thus the optimal approximation of $u(x, t)$ is $1.40177424861038819E+000$, the optimal step is $m_{opt} = 6$ and the optimal error $1.11E-15$. Also, for the approximate solution of $\eta(x, t)$ is $2.0000000000000385E+000$, the optimal step is $m_{opt} = 7$ and the optimal error $4.51E-08$.

5. Conclusions
The nonlinear shallow water wave equation, is among important and applicable problems to forecast the waves habits in different phenomenon. Thus solving and analysing this equation is interesting problem for researchers. The HAM was applied for solving the mentioned problem. Also, we used the CESTAC method and the CADNA library based on the stochastic arithmetic to validate the numerical results. Based on this method, not only the optima approximation of the shallow water wave equation can be obtained but also the optimal iteration of the HAM and optimal error can be found. For future researches, we will apply the collocation method with orthogonal basis functions with the CESTAC method to validate the results.
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