I. INTRODUCTION

Stable magnetic flux tubes, or strings, occur in many high energy models. Fermionic zero modes on these strings can in some cases render them superconducting, and dramatically change their lifetime and interactions. In most of the examples that have been studied in a cosmological context the strings are topological, yet the effect of fermion zero modes on stable non–topological strings is equally powerful and interesting.

In this paper we investigate the fermionic zero modes on magnetic vortices in supersymmetric (SUSY) \( N=2 \) QED with two hypermultiplets of opposite charge. This model was proposed as a toy model for the low energy limit of type-II superstrings compactified on Calabi–Yau manifolds, where the possibility of magnetic vortices was first suggested in [1] (in the full low energy theory, there are sixteen hypermultiplets charged under fifteen U(1) groups, that is, fifteen copies of the model studied here).

In the absence of Fayet–Iliopoulos terms, it was proved in [2] that the strings appearing in that model are unstable. In [3] the present authors studied the case where a (gauge) symmetry breaking Fayet-Iliopoulos (FI) term was added. In this case we found BPS string solutions. We also found that out of all the possible vacua, only certain choices can give rise to static strings, due to a vacuum selection effect [4]. Even so, this effect was still not enough to obtain stable strings, unlike in the case of a comparable \( N=1 \) supersymmetric model with two chiral multiplets [3, 5]. The strings formed are semilocal strings [6], and in the BPS limit (equal scalar and gauge masses) they are only neutrally stable \([7, 8]\).

In the present work, a study of the possible fermionic zero modes arising in this model has been carried out, in order to see whether they stabilise the string by further augmenting the vacuum selection effect. If this were so, the semilocal strings could possibly turn into chiral cosmic strings [9, 10], long-lived vortices in which the fermions move in one direction only. However we have found that this is not the case. The fermions respect the same vacuum selection effect as the bosons, meaning that the underlying string remains semilocal. Moreover, the fermions move in both directions along the string, and therefore they could mix and leave the string. It is known that the fermion back-reaction can modify the stability of the background string [11]. We argue that this is not the case in the present context, because, unlike the systems studied in [11], the Bogomol'nyi bound is protected by the remnant unbroken SUSY. The “topological charge” is identical for all the vortices in the family, and we do not expect any member of the family to be singled out by the fermion back-reaction.

Nevertheless, we have found that, due to the bosonic zero mode of the semilocal string, one of the fermions is coupled to a boson which is not zero at the core of the string, contrary to what might be expected from related assertions in the literature. To our knowledge this is the first time that the effect of charged, non–winding scalars has been considered on fermion zero modes, and it is interesting to point out that our model falls outside the scope of a number of index theorems for vortex backgrounds in scalar–fermion systems.

In section II we present a brief review of the bosonic sector of this model; we then obtain fermionic zero modes by SUSY transformations of the bosonic background in section III. We check that these in fact satisfy the fermionic equations of motion in section IV, and derive the mass matrix. The results are then discussed in section V.
We will investigate an $N=2$ supersymmetric (SUSY) model consisting of two $N=2$ hypermultiplets $(h_{ai}, \psi_a, F_{ai})$, labelled by $a=1, 2$, with opposite charges $q_a = \pm q$, coupled to an $N=2$ abelian vector multiplet $(A_\mu, M, N, \lambda^i, \tilde{D})$ (in Wess-Zumino gauge), $i=1, 2$, plus a Fayet-Iliopoulos term $\vec{k} \cdot \tilde{D}$. This last term is responsible for breaking the $U(1)$ gauge symmetry, although it preserves the supersymmetry.

Using the conventions of [12], the Lagrangian we are interested in can be written as

$$\mathcal{L} = \frac{1}{2} D^\mu h_{ai} D_\mu h_{ai} + i \bar{\psi}_a \gamma^\mu D_\mu \psi_a + \frac{i}{2} \bar{\lambda}_i \gamma^\mu \partial_\mu \lambda^i$$

$$- \frac{1}{4} F_{\mu \nu} F^{\mu \nu} + i \bar{q}_a h_{ai} \gamma_\mu \lambda^i \psi_a - i \bar{q}_a \psi_a \lambda^i h_{ai}$$

$$- \frac{1}{2} \left( H_1^1 - H_2^1 - \frac{1}{2} \right)^2 - \frac{1}{2} \left( H_1^2 + H_2^2 \right)^2 - \frac{1}{2} \left( iH_2^1 - iH_1^1 \right)^2,$$  \hspace{1cm} (1)

where $\bar{q}_a = q_a/q$, $h_{ai} = h_{ai}^0$, $H_1^i = -(\bar{q}_a/2) h_{ai}^0$, $D_\mu = \partial_\mu + i \bar{q}_a A_\mu$ and $A_\mu$ is a U(1) gauge field. Auxiliary fields have been eliminated, all fields with zero vacuum expectation value (that is, $M$ and $N$), have been set to zero and suitable rescalings have been performed [3].

We can use a Bogomol’nyi argument to express the energy of the bosonic part of a static, straight vortex configuration as

$$E = \frac{1}{2} \int d^2 x \left[ |(D_1 + iD_2) h_{11}|^2 + |(D_1 - iD_2) h_{12}|^2 + |(D_1 + iD_2) h_{21}|^2 + |(D_1 - iD_2) h_{22}|^2 + \right.$$  

$$\left. \left[ B + (H_1^1 - H_2^1 + \frac{1}{2}) \right]^2 + (H_1^2 + H_2^2)^2 + (i H_2^1 - i H_1^1)^2 \right]$$

$$- \frac{1}{2} \int d^2 x B,$$  \hspace{1cm} (2)

where $B = \partial_1 A_2 - \partial_2 A_1$, and the integral in the last term is the magnetic flux, which is quantised in units of $2\pi$. The Bogomol’nyi equations can then be immediately read off

$$(D_1 + iD_2) h_{11} = 0; \hspace{1cm} (D_1 - iD_2) h_{12} = 0; \hspace{1cm} (D_1 + iD_2) h_{21} = 0; \hspace{1cm} (D_1 - iD_2) h_{22} = 0; \hspace{1cm} H_1^1 = H_2^1 = 0; \hspace{1cm} B + H_1^1 - H_2^1 + \frac{1}{2} = 0.$$

The only solutions to these equations have $h_{12} = h_{21} = 0$ (this was dubbed a “vacuum selection effect” in [3]); the remaining equations are those of a semilocal string [6, 7] in $(h_{11}, h_{22})$.

Up to global SU(2) transformations in the $(h_{11}, h_{22})$ space, the unit winding, cylindrically symmetric semilocal string solution can be expressed as

$$h_{11} = f(r)e^{i\theta}; \hspace{1cm} h_{22} = g(r); \hspace{1cm} A_\theta = a(r).$$  \hspace{1cm} (4)

with boundary conditions $f(0)=a(0)=g'(0)=0$ and $f(\infty)=1, g(\infty)=0$ and $a(\infty)=-1$. $g(r)$ is given by

$$g(r) = \kappa \frac{f(r)}{r}.$$  \hspace{1cm} (5)

where the constant $\kappa$ essentially measures the width of the string, ranging from $\kappa=0$ – the Nielsen-Olesen [13] string – to $\kappa=\infty$ – a CP$^1$ instanton [7]. Note the lack of winding in $h_{22}$ and furthermore that $h_{22}$ does not necessarily have to be zero at $r=0$. Note also that $B$ is always negative with this choice of boundary condition, and its total flux does not depend on $\kappa$. 
We are now interested in studying the fermionic zero mode solutions to this system to see whether they influence this vacuum selection effect. These solutions could be obtained by solving the explicit fermionic equations of motion, but we can also use the SUSY transformation to get the zero modes directly \[14\]. This will give us static configurations in the plane perpendicular to the string, and we will subsequently introduce $t$ and $z$ dependence on the solutions.

By zero mode solutions, we mean infinitesimal changes to the background configuration that preserve the action (for static configurations, that amounts to leaving the energy unchanged) and satisfy their equations of motion. We know that a SUSY transformation of a given configuration leaves the energy unchanged. Moreover, as we started with a static solution to the bosonic equations of motion, the fermions produced by this transformation must automatically satisfy their equations of motion.

The fermionic content of our system consists of two higgsinos (two Dirac fermions $\psi_1$ and $\psi_2$) coming from the hypermultiplets, and two gauginos (two symplectic Majorana fermions $\lambda^1$ and $\lambda^2$) coming from the gauge vector multiplet. Recall that the symplectic Majorana spinors are 4-component $\tilde{\psi}_{\alpha}^\dagger \equiv (1_{2 \times 2}, \sigma \cdot \vec{D})$.

Let us perform a SUSY transformation of the system in the background of the (bosonic) semilocal string obtained in the previous section. The bosonic fields do not transform. The higgsinos take the form

$$\delta \psi_{(a)} = -i \gamma^\mu D_\mu \epsilon^{(i)} h_{ai},$$

while the gauginos may be written as

$$\delta \lambda^{(i)} = -i \frac{\varepsilon^{12}}{2} \gamma^{\mu \nu} \epsilon^{(i)} F_{\mu \nu} - i \epsilon^{(j)} \tilde{\sigma}^{(i) \, j} \cdot \vec{B}.$$ 

Our conventions are \[12\]

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \frac{1}{2} i [\gamma^\mu, \gamma^\nu] = \sigma^{\mu \nu},$$

where, $\sigma^\mu = (1, \sigma)$, $\bar{\sigma}^\mu = (1, -\sigma)$ and $\sigma^i$ are the Pauli matrices.

The higgsinos and the gauginos can be written more explicitly as

$$\delta \psi_{(1)} = -i \frac{\varepsilon^{ij}}{2} \left[ (D_1 - i D_2) h_{11} (\gamma^1 + i \gamma^2) \epsilon^{(1)} + (D_1 + i D_2) h_{12} (\gamma^1 - i \gamma^2) \epsilon^{(2)} \right];$$

$$\delta \psi_{(2)} = -i \frac{\varepsilon^{ij}}{2} \left[ (D_1 + i D_2) h_{22} (\gamma^1 - i \gamma^2) \epsilon^{(2)} + (D_1 - i D_2) h_{21} (\gamma^1 + i \gamma^2) \epsilon^{(1)} \right];$$

$$\delta \lambda^{(1)} = \gamma^1 \gamma^2 \epsilon^{(1)} B - i \left( H_1^1 - H_2^2 + \frac{1}{2} \right) \epsilon^{(1)};$$

$$\delta \lambda^{(2)} = \gamma^1 \gamma^2 \epsilon^{(2)} B + i \left( H_1^1 - H_2^2 + \frac{1}{2} \right) \epsilon^{(2)},$$

where we have used the Bogomol’nyi equations \[3\].

We expect the strings to be \[1/2\]BPS saturated \[13\], so let us try to obtain the broken and unbroken part of the SUSY transformation, i.e., let us obtain the fermionic zero modes. In order to do so, we can use the following projectors

$$P_{\pm} = \frac{1}{2} \left( 1 \pm i \gamma^1 \gamma^2 \right),$$
which with our conventions, are given by

\[ P_+ \equiv \text{diag}(1, 0, 1, 0) , \quad P_- \equiv \text{diag}(0, 1, 0, 1) . \]

These projectors, besides \( P_\pm^2 = P_\pm = P_\pm \) and \( P_\pm P_\mp = 0 \), have the following properties:

\[
\begin{align*}
\gamma^1 P_+ &= P_+ \gamma^1 ; \\
\gamma^2 P_+ &= P_+ \gamma^2 ; \\
P_+ \gamma^1 &= \pm i P_+ \gamma^2 .
\end{align*}
\]

\( \gamma^1 \gamma^2 \) is essentially a two-dimensional version of \( \gamma^5 \), acting in the plane perpendicular to the string. Applying these projectors onto the fermions, we learn that

\[
P_+ \delta \psi(1) \equiv \delta \psi(1)_+ = -i (D_1 - iD_2) h_{11} \gamma^1 P_- \epsilon^{(1)} = -2i D_1 h_{11} \gamma^1 P_- \epsilon^{(1)} ; \\
P_- \delta \psi(1) \equiv \delta \psi(1)_- = -i (D_1 + iD_2) h_{12} \gamma^1 P_+ \epsilon^{(2)} = 0 ; \\
P_+ \delta \psi(2) \equiv \delta \psi(2)_+ = -i (D_1 - iD_2) h_{21} \gamma^1 P_- \epsilon^{(1)} = 0 ; \\
P_- \delta \psi(2) \equiv \delta \psi(2)_- = -i (D_1 + iD_2) h_{22} \gamma^1 P_+ \epsilon^{(2)} = -2i D_1 h_{22} \gamma^1 P_+ \epsilon^{(2)} ,
\]

and

\[
P_+ \delta \lambda^{(1)} \equiv \delta \lambda^{(1)}_+ = -i (B + H_1^1 - H_2^2 + 1) P_+ \epsilon^{(1)} = 0 ; \\
P_- \delta \lambda^{(1)} \equiv \delta \lambda^{(1)}_- = i (B - H_1^1 + H_2^2 - 1) P_- \epsilon^{(1)} = 2i B P_- \epsilon^{(1)} ; \\
P_+ \delta \lambda^{(2)} \equiv \delta \lambda^{(2)}_+ = -i (B - H_1^1 + H_2^2 - 1) P_+ \epsilon^{(2)} = -2i B P_+ \epsilon^{(2)} ; \\
P_- \delta \lambda^{(2)} \equiv \delta \lambda^{(2)}_- = i (B + H_1^1 - H_2^2 + 1) P_- \epsilon^{(2)} = 0 .
\]

Note that \( \delta \psi(1)_- \) and \( \delta \psi(2)_+ \) vanish on BPS states due to the vacuum selection effect \( h_{12} = h_{21} = 0 \). It is clear that \( P_- \epsilon^{(1)} \) and \( P_+ \epsilon^{(2)} \) generate the fermionic zero modes, whereas \( P_+ \epsilon^{(1)} \) and \( P_- \epsilon^{(2)} \) are the generators of the unbroken SUSY. Note that there are fermionic zero modes of both chiralities, since \( \mathbb{N}=2 \) SUSY is non–chiral.

**IV. FERMIONIC EQUATIONS OF MOTION**

In this section we investigate the structure of the fermionic zero modes by analysing the equations of motion directly, without reference to supersymmetry. We use two-spinor notation and define

\[
\begin{align*}
\delta \psi(1) &= \begin{pmatrix} \phi_{\alpha(1)} \\ \tilde{\phi}^{\dot{\alpha}(1)} \end{pmatrix} , & \delta \psi(2) &= \begin{pmatrix} \phi_{\alpha(2)} \\ \tilde{\phi}^{\dot{\alpha}(2)} \end{pmatrix} , & \delta \lambda^{(1)} &= \begin{pmatrix} -i \Lambda_{\alpha(2)} \\ \tilde{\Lambda}^{\dot{\alpha}(2)} \end{pmatrix} , & \delta \lambda^{(2)} &= \begin{pmatrix} i \Lambda_{\alpha(1)} \\ \tilde{\Lambda}^{\dot{\alpha}(1)} \end{pmatrix} .
\end{align*}
\]

The projector operators can be defined in two spinor notation as

\[
\sigma_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} ; \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} .
\]

In this notation the zero modes that we found previously are

\[
\begin{align*}
\phi_{\alpha(1)} &= -2i D_1 h_{11} \sigma^1 \sigma_- \tilde{\xi}^{\dot{\alpha}(1)} ; \\
\tilde{\phi}^{\dot{\alpha}(1)} &= -2D_1 h_{11} \sigma^1 \sigma_- \xi_{\alpha(2)} ; \\
\phi_{\alpha(2)} &= -2i D_1 h_{22} \sigma^1 \sigma_+ \tilde{\xi}^{\dot{\alpha}(2)} ; \\
\tilde{\phi}^{\dot{\alpha}(2)} &= 2D_1 h_{22} \sigma^1 \sigma_+ \xi_{\alpha(1)} ; \\
\Lambda_{\alpha(1)} &= -2i B \sigma_+ \xi_{\alpha(1)} ; \\
\Lambda_{\alpha(2)} &= 2i B \sigma_- \xi_{\alpha(2)} .
\end{align*}
\]
where $h_{11}$, $h_{22}$ and $B$ satisfy (3).

Consider the fermionic equations of motion derived from (1) in the bosonic background given by a solution to (3). Recall that $h_{12} = h_{21} = 0$, and all other fields are independent of $t$ and $z$:

\[
\begin{align*}
(\bar{\sigma}^\mu) D_\mu \phi_{(1)} - \bar{\Lambda}^{(1)} h_{11} &= 0; \\
(\sigma^\mu) D_\mu \bar{\chi}_{(1)} + i\Lambda_{(2)} h_{11} &= 0; \\
(\bar{\sigma}^\mu) D_\mu \phi_{(2)} + \bar{\Lambda}^{(2)} h_{22} &= 0; \\
(\sigma^\mu) D_\mu \bar{\chi}_{(2)} + i\Lambda_{(1)} h_{22} &= 0; \\
(\sigma^\mu) \partial_\mu \bar{\Lambda}^{(1)} + h_{11}^* \phi_{(1)} + i\chi_{(2)} h_{22} &= 0; \\
(\sigma^\mu) \partial_\mu \bar{\Lambda}^{(2)} - h_{22}^* \phi_{(2)} + i\chi_{(1)} h_{11} &= 0.
\end{align*}
\] (19)

It can be checked that the static, $z$–independent configurations (18) satisfy these equations. Including the $(t,z)$ dependence into (19) we learn that three of the fermions are functions of $(t - z)$ and therefore move in the positive $z$ direction, while the other three move in the negative $z$ direction:

\[
\begin{align*}
\phi_{\alpha(1)}, \chi_{\alpha(2)}, \Lambda_{\alpha(1)} &\to t + z; \\
\phi_{\alpha(2)}, \chi_{\alpha(1)}, \Lambda_{\alpha(2)} &\to t - z.
\end{align*}
\] (20)

The fermions move in opposite directions, as expected, since $N=2$ SUSY is intrinsically non-chiral and it has not been broken in this model. We are interested in the $(r,\theta)$ dependence of the zero modes.

The usual Nielsen-Olesen string is one of the possible configurations in the family of semilocal strings, and it corresponds to the narrowest string. One can characterise this string as having $h_{22}=0$, annihilating two out of the six Weyl fermions $\phi_{(2)} = \bar{\chi}_{(2)} = 0$ [16]. Moreover, if we remove one SUSY generator out of the two, we recover the Nielsen-Olesen string, but with chiral fermions [14] moving in the same direction. This is a good check of our results.

The situation is different when $h_{22}\neq0$. The fermions $\phi_{(2)}$ and $\chi_{(2)}$ are coupled to a field which is not zero at $r=0$, which might seem surprising. In terms of the general string ansatz [4], these fermions may be expressed in the form

\[
\begin{align*}
\phi_{(2)} &= -2i\kappa \partial_r \left( \frac{f(r)}{r} \right) e^{i\theta} \left( \begin{array}{c} 0 \\ \xi_{(2)} \end{array} \right); \\
\chi_{(2)} &= -2\kappa \partial_r \left( \frac{g(r)}{r} \right) e^{-i\theta} \left( \begin{array}{c} \xi_{(1)} \\ 0 \end{array} \right),
\end{align*}
\] (21)

and, as can be seen from figure 1, they tend to 0 at $r=0$. These two fermions are the only ones that wind.

![Figure 1: Profiles of the derivatives of the functions $f(r)$ and $g(r)$ given by equations (4) and (5), for different values of the parameter $\kappa$.](image)

In order to analyse the fermion zero modes, we convert the Dirac equations (10) into second order equations by acting with the operators $\sigma \cdot D$ and $\bar{\sigma} \cdot D$:
on constant bosonic backgrounds. As a simple check, far from the core we recover the fermion masses; indeed,
\[ \sigma \text{ where } \sigma \]
for any two-spinor \( \Psi \), where \( \sigma \) for any two-spinor \( \Psi \), where \( \sigma \) are all greater than zero everywhere, and thus there are no non-zero normalizable solutions to (23), as can be shown by simply multiplying by \( \bar{\Psi} \)
\[ (\bar{\Psi} \cdot D) \bar{\Psi} = 0 \]
Note that each set of equations correspond to fermions of one given chirality and coming from one of the supersymmetry generators
\[ \text{Half of the equations (22) correspond to the fermions coming from the unbroken SUSY symmetry:} \]
\[ \left\{ \begin{array}{l}
\Box - B + |h_{11}|^2 \phi_{(1)-} + i h_{11} h_{22} \chi_{(2)-} = 0; \\
\Box - B + |h_{22}|^2 \chi_{(2)-} - i h_{11}^* h_{22} \phi_{(1)-} = 0; \\
\Box + |h_{11}|^2 + |h_{22}|^2 \Lambda^{(1)}_{-} = 0; \\
\Box - B + |h_{22}|^2 \phi_{(2)-} - i h_{11} h_{22} \chi_{(1)+} = 0; \\
\Box - B + |h_{11}|^2 \chi_{(1)+} + i h_{11}^* h_{22} \phi_{(2)+} = 0; \\
\Box + |h_{11}|^2 + |h_{22}|^2 \Lambda^{(2)}_{+} = 0.
\end{array} \right. \]
(23)
We can think of these equations as giving position–dependent masses (squared) for the fermions, in the sense that after diagonalisation, all six equations are of the form \( \Box \Psi + M^2 \Psi = 0 \), where
\[ M^2 = \text{diag}(-B + |h_{11}|^2 + |h_{22}|^2, -B, |h_{11}|^2 + |h_{22}|^2, -B + |h_{11}|^2 + |h_{22}|^2, -B, |h_{11}|^2 + |h_{22}|^2). \]
(24)
Note that the matrix \( M^2 \) is not the same as the fermion mass matrix squared, due to the presence of derivative terms (in particular \( M^2 \) involves the magnetic field strength and is gauge invariant). Both matrices of course agree on constant bosonic backgrounds. As a simple check, far from the core we recover the fermion masses; indeed, \( B \to 0 \), \( h_{11} \to 1 \) and \( h_{22} \to 0 \) as \( r \to \infty \), and the diagonal terms become 1, 0, 1, the correct masses for one higgsino, one goldstino and one gaugino of each chirality.

Recall that we are using signature \((+,-,-,-)\). Since \( \Box = -\nabla^2 \), the laplacian in the plane perpendicular to the string, \( \Box \) is a set of coupled time–independent Schrödinger equations in \((r, \theta)\) and we are looking for the zero energy states. Obviously, if the “potential” is everywhere positive there are no zero energy eigenstates. The functions in \( M^2 \) are all greater than zero everywhere, and thus there are no non-zero normalizable solutions to \( M^2 \), as can be shown by simply multiplying by \( \bar{\Psi} \) and integrating by parts.

Thus \textit{all} of these fermionic components are automatically zero because of the equations of motion, and this agrees with the form of the solution obtained from the SUSY transformations \( U \).

The other half of the equations reads
\[ \left\{ \begin{array}{l}
\Box + B + |h_{11}|^2 \phi_{(1)+} + i h_{11} h_{22} \chi_{(2)+} - 2 \sigma^2 \Lambda^{(1)}_{-} D_2 h_{11} = 0; \\
\Box + B + |h_{22}|^2 \chi_{(2)+} - i h_{11}^* h_{22} \phi_{(1)+} + 2 i \sigma^2 \Lambda^{(1)}_{-} D_2 h_{22} = 0; \\
\Box + |h_{11}|^2 + |h_{22}|^2 \sigma^2 \Lambda^{(1)}_{-} + 2 \phi_{(1)+} D_2 h_{11} - 2 i \chi_{(2)+} D_2 h_{22} = 0; \\
\Box + B + |h_{22}|^2 \phi_{(2)-} - i h_{11} h_{22} \chi_{(1)-} + 2 \sigma^2 \Lambda^{(2)}_{+} D_2 h_{22} = 0; \\
\Box + B + |h_{11}|^2 \chi_{(1)-} + i h_{11}^* h_{22} \phi_{(2)-} + 2 i \sigma^2 \Lambda^{(2)}_{+} D_2 h_{11} = 0; \\
\Box + |h_{11}|^2 + |h_{22}|^2 \sigma^2 \Lambda^{(2)}_{+} + 2 \phi_{(2)-} D_2 h_{22} - 2 i \chi_{(1)-} D_2 h_{11} = 0,
\end{array} \right. \]
(25)
where, again, the Bogomol’nyi equations \( U \) have been used extensively.

These may be written as two sets of three coupled equations\(^1\) and we again seek to diagonalize the resulting mass matrices.

\(^1\) Note that each set of equations correspond to fermions of one given chirality and coming from one of the supersymmetry generators
corrections. The two fermionic zero modes obtained directly from supersymmetry are related to translational zero modes in the core of the string, and zero at infinity. After diagonalisation, the other mass squared terms are

\[
\frac{1}{2}(B + 2|h_{11}|^2 \pm \sqrt{B^2 + 16D_2h_{11}D_2h_{11}^*})
\]

which correspond to two (r-dependent) linear combinations of \(\phi(1)^+\) and \(\sigma^2_\Lambda(1)^-\). At \(r=0\), the signs of these masses are +, - respectively. At infinity these masses are 1, 1, and the mass states are simply the uncombined spinors. We can immediately see by the same reasoning as before that one combination of fermions is zero everywhere, since its mass squared is positive everywhere. The other one is a combination of the higgsino and the gaugino. 

For simplicity we just consider the first set of three equations. In the case of the Nielsen-Olesen string member of the semilocal family, for which \(\kappa=0\), \(h_{22}=0\), the spinors \(\chi(2)^+\) and \(\phi(2)^-\) do not couple to any spinors, and have mass squared B, which is negative in the core of the string, and zero at infinity. After diagonalisation, the other mass squared terms are

\[
\frac{1}{2}(B + 2|h_{11}|^2 \pm \sqrt{B^2 + 16D_2h_{11}D_2h_{11}^*})
\]

which correspond to two (r-dependent) linear combinations of \(\phi(1)^+\) and \(\sigma^2_\Lambda(1)^-\). At \(r=0\), the signs of these masses are +, - respectively. At infinity these masses are 1, 1, and the mass states are simply the uncombined spinors. We can immediately see by the same reasoning as before that one combination of fermions is zero everywhere, since its mass squared is positive everywhere. The other one is a combination of the higgsino and the gaugino. 

For the general case, \(\kappa \neq 0\), we have to diagonalize two sets of \(3 \times 3\) matrices. The signs of the “eigenvalues” of the \(M^2\) matrices are +, -, at \(r=0\) and +, 0, + at infinity, the “eigenvectors” being a combination of the three fermions. Once again, the “eigenvector” whose mass–squared function is always positive is zero, and thus we are left with two non-zero “eigenvectors”.

In all cases, the fermions at infinity have masses 1, 0, 1, 1, 0, 1, which agree with the masses of the fields \(h_{11}, h_{12}, A_\mu, h_{22}, h_{21}, M + iN\), as should be the case since supersymmetry is unbroken there.

V. DISCUSSION

We have studied the fermion zero modes arising in SUSY N=2 QED with two hypermultiplets and a FI term. It is known that in such a model there is a vacuum selection effect in the bosonic sector, and only some of all the possible vacua form a family of neutrally stable vortices \(\mathbb{R}^3\). We showed that the fermion zero modes do not improve the stability of the vortices. This is due to the fact that fermions obey the same vacuum selection effect as the vortices. On the other hand, as the SUSY is N=2, fermions with both chiralities are present in the vortex.

We do not expect the stability to be altered by the back-reaction of the fermions either\(^{14}\). A recent detailed calculation of quantum corrections for SUSY kinks shows that the Bogomol’nyi bound is preserved\(^{17}\), even if the mass receives corrections.

The model analysed in this paper contains a family of vortices that have the same topological charge, and all saturate the Bogomol’nyi bound. SUSY is half broken for all members of the family, and the unbroken SUSY will preserve the Bogomol’nyi bound in all cases, since the multiplets are shortened. Thus the BPS condition holds for all members of the family, and as they all have the same topological charge, the energy of all of them is also the same. Thus, no member of the family will be singled out by fermionic back-reaction.

The fermions in this system are also interesting because some are coupled to a scalar field which is not zero at the core of the vortex. It is sometimes stated in the literature that the reason why vortices support fermion zero modes in their core is because the fermion masses, which come from coupling to the scalars, are zero there. We have shown here that this heuristic argument is incorrect by calculating explicitly the zero modes that couple to the non-zero scalar at the core \(h_{22}\), if the string is in \(h_{11}\).

It is interesting to try to relate the zero modes derived from supersymmetry to the zero modes we see in the equations of motion. The two fermionic zero modes obtained directly from supersymmetry are related to translational zero modes in the bosonic sector.

We can see from the second order equations of motion that in fact there are two more fermionic zero modes: In the bosonic sector, there are zero modes corresponding to changes in the parameter \(\kappa\) of the semilocal string, expanding the string core. This bosonic zero mode corresponds to the fermionic zero modes that have negative mass at zero, and which are massless at infinity. One way to see this is to take the case where there is just one hypermultiplet \((h_i, \psi, F_i)\). The string formed would then be an ordinary Nielsen-Olesen string, which would not possess the bosonic zero mode.
Furthermore, the spinors $\phi_2$ and $\chi_2$ would disappear, and the remaining eigenstates would have signs $+,-,+,-$ for their masses squared at zero and $+,-,+,-$ at infinity, so that only two of the eigenstates are non-zero. When we re-introduce the second hypermultiplet, the extra fermionic zero modes correspond to the extra bosonic zero mode associated with $h_{22}$.

It is also possible to consider supersymmetric transformations of the bosonic zero mode to give the fermionic zero mode. We perturb the field $h_{22}$, adjusting the other fields so that we retain the Bogomol'nyi equations, and hence keep the same energy. In the limit $\kappa = 0$, the semilocal string has $h_{22} = 0$, and an infinitesimal perturbation of $h_{22}$ of the form $\delta h_{22} = a h_{11} (\text{background})/r$ does not modify the other fields. Hence, a supersymmetry transformation of this zero mode gives fermionic zero modes in $\phi_{(2)}$ and $\chi_{(2)}$ only.

At infinity the bosonic zero mode corresponding to changing the string width is a goldstone boson, and this agrees with what we have discovered from the fermionic equations of motion: that the eigenstates that have mass zero at infinity are pure $\phi_{(2)}$ or $\chi_{(2)}$ in the $\kappa=0$ case.

For a general semilocal string ($\kappa \neq 0$), all the bosonic fields will be affected by a perturbation in the string width, and so the corresponding fermionic zero modes are combinations of each set of three spinors.

We note that the index theorem of Davis, Davis and Perkins does not apply in the case of the semilocal string due to the fact that one of the fields doesn’t wind in the core – an assumption made in the derivation of the index theorem. Hence we can’t use it to ascertain the number of zero modes in this case. In the case where the index theorem does apply, namely the Nielsen–Olesen string (i.e. one hypermultiplet), our results agree with the index theorem. For the semilocal string there are two extra zero modes corresponding to the change of the string width; the fermion zero modes are those fermions corresponding to the bosonic zero mode. This has been constructed explicitly in the Nielsen–Olesen limit of the semilocal string. The model also falls outside the scope of the index theorem of Ganoulis and Lazarides, because $h_{22}$ is charged under $U(1)$ but goes to zero at infinity. Our results agree with Weinberg’s index theorem (see also [21]), since the new zero modes have opposite chiralities and therefore do not affect the counting of net (left minus right) fermion zero modes. This is in agreement with the fact that the magnetic flux measured at infinity is the same for all semilocal strings, including the Nielsen–Olesen string.

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