Study on nonlinear vibration of flexible electronic membrane engendered by high-precision imprinting system

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Abstract
In the process of guide roller transmission, the geometric nonlinearity caused by the lateral vibration of the flexible electronic membrane will result in the divergence and instability of the membrane velocity, thus affecting the printing accuracy. In this paper, in order to engineer the exact condition that affecting the printing accuracy, the Elliptic integral method, whose effectiveness has been verified for its solved result, is consistent with the one concluded from L-P method and Hé’s Frequency formula while solving the nonlinear vibration equation, respectively, on basis of Von Karman’s large deflection theory and Hamilton’s principle, is mainly applied, and thus provide a theoretical support for the design and manufacture of high-precision flexible electronic printing press.

Keywords
Flexible electronic membrane, geometric nonlinearity, L-P perturbation method, Hé’s frequency formula, elliptic integral method

Introduction
Flexible electronic membrane materials1–5 are widely used in electronic products and people’s demand for flexible electronic membrane materials is daily increasing. The flexible electronic membrane will be pre-tensioned by the guide roller inside the automatic printing machine during the transmission process (as shown in Figure 1).

Chinese scholar Chen LQ6–8 systematically studied the nonlinear vibration of string, beam, and ring. Nosier9 studied the influence of material properties and boundary conditions on the vibration characteristics of solid circular plate by establishing a dynamic model of functionally graded circular plate under transverse mechanical load. Huang JL10 proposed a new incremental harmonic balance method to study the quasi periodic motion of an axially moving beam, which is verified the effectiveness by comparing the frequency and amplitude. Tornabene F11 established the governing equation of the thin shell by using Hamilton’s principle and get it solved by using the generalized differential quadrature method. MH Ghayesh12 obtained the nonlinear dynamic equation of the forced motion of the axially moving plate based on the energy method. Li HY13 established the motion equation based on the classical thin plate theory, which is discretized by the Galerkin method and solved by using the multi-scale method to obtain the frequency response equation under the steady motion condition. RM Soares14 established a nonlinear static model of pre stretched hyperelastic annular membrane under finite deformation. Li MZ15 analyzed the free vibration of functionally graded plates with simply supported boundary conditions. Based on Hamilton principle, the governing differential equations and boundary conditions were derived. Ke LL16 studied the nonlinear free vibration of beam and obtained the influence of volume fraction, amplitude, and slenderness

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ratio on the nonlinear free vibration of beam. Feng C established the nonlinear free vibration equation of composite beam, and calculated the vibration frequency and amplitude of the beam by using Ritz method. Wang YB used Hamilton principle to obtain the equation of a beam under axial time-varying load. He Jihuan has done a lot of work on the stability of nonlinear vibration equations by using He-Laplace method and Homotopy Perturbation Method.

To sum up, previous studies rarely considered the large deformation of the middle surface of the flexible electronic membrane caused by the pre-tension of the roller, and as well failed to offer the actual motion speed of the flexible electronic membrane while dimensionless method is adopted to solve the motion equation.

Establishment of dynamic model

Figure 2 shows the dynamic model of flexible electronic membrane, where $N_{0x}$ and $N_{0y}$ show the pulling force in x and y, respectively, of the guide roller during its transmission process. $N_x$ and $N_y$ show the tension caused by the large deflection deformation because of the transverse vibration within membranes. $b$ denotes the longitudinal length of the membrane, $a$ denotes the width, and $v$ denotes the longitudinal transmission velocity.

According to Von Karman’s large deflection theory, the vibration equation and compatibility equation of flexible electronic membrane subjected to the pretension are

$$\rho \left( \frac{\partial^2 \sigma}{\partial t^2} + 2v \frac{\partial^2 \sigma}{\partial x \partial t} + \nu^2 \frac{\partial^2 \sigma}{\partial x^2} \right) - (N_x + N_{0x}) \frac{\partial^2 \sigma}{\partial x^2} - (N_y + N_{0y}) \frac{\partial^2 \sigma}{\partial y^2} = 0 \quad (1)$$

$$\frac{\partial^2 N_x}{\partial y^2} + \frac{\partial^2 N_y}{\partial x^2} - \mu \frac{\partial^2 N_x}{\partial x^2} - \mu \frac{\partial^2 N_y}{\partial y^2} - 2(1 + \mu) \frac{\partial^2 N_{0x}}{\partial x \partial y} = Eh \left[ \left( \frac{\partial^2 \sigma}{\partial x \partial y} \right)^2 - \frac{\partial^2 \sigma}{\partial x^2} \frac{\partial^2 \sigma}{\partial y^2} \right] \quad (2)$$

Substituting equations $N_x = \frac{\partial^2 \Phi}{\partial y^2}, N_y = \frac{\partial^2 \Phi}{\partial x^2}, N_{0x} = -\frac{\partial^2 \Phi}{\partial x \partial y}, N_{0y} = \sigma_{0x} N_{0y} / h = \sigma_{0y}, N_{0y} = \sigma_{0y}$ into (1) and (2), we have Separating variables, we obtain
\[
\rho \left( \frac{\partial^2 \omega}{\partial t^2} + 2v \frac{\partial \omega}{\partial x} + \nu \frac{\partial^2 \omega}{\partial x^2} \right) - \left( \frac{\partial^2 \Phi}{\partial y^2} + h \sigma_0 \right) \frac{\partial^2 \omega}{\partial x^2} - \left( \frac{\partial^2 \Phi}{\partial x^2} + h \sigma_0 \right) \frac{\partial^2 \omega}{\partial y^2} = 0 \quad (3)
\]

\[
\frac{\partial^4 \Phi}{\partial x^4} + \frac{\partial^4 \Phi}{\partial y^4} = Eh \left[ \left( \frac{\partial^2 \Phi}{\partial x^2} \right)^2 - \frac{\partial^2 \Phi}{\partial x^2} \frac{\partial \Phi}{\partial y^2} \right] \quad (4)
\]

\[
\overline{w}(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} T_{mn}(t) W_{mn}(x,y) \quad (5)
\]

\[
\Phi(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} U_{mn}(t) \phi_{mn}(x,y) \quad (6)
\]

Substituting equations (5) and (6) into equation (4) yields

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} U_{mn}(t) \frac{\partial^4 \Phi_{mn}(x,y)}{\partial x^4} + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} U_{mn}(t) \frac{\partial^4 \phi_{mn}(x,y)}{\partial y^4} = Eh \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[ \left( \frac{\partial^2 W_{mn}(x,y)}{\partial x^2} \right)^2 - \frac{\partial^2 W_{mn}(x,y)}{\partial x^2} \frac{\partial^2 W_{mn}(x,y)}{\partial y^2} \right] T_{mn}^2(t) \quad (7)
\]

Equation (7) can be simplified as follows:

\[
U_{mn}(t) = T_{mn}^2(t) \quad (8)
\]

\[
\Phi(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} T_{mn}^2(t) \phi_{mn}(x,y) \quad (9)
\]

Assume the mode shape functions is

\[
W_{mn}(x,y) = W(x,y) = W = \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad (10)
\]

Substituting equations (8) and (10) into equation (7) yields

\[
\frac{\partial^4 \phi}{\partial x^4} + \frac{\partial^4 \phi}{\partial y^4} = Eh \frac{m^4 \pi^4}{2a^4 b^4} \left( \cos \frac{2m\pi x}{a} + \cos \frac{2n\pi y}{b} \right) \quad (11)
\]

The solution of equation (11) can be expressed as

\[
\phi(x,y) = a \cos \frac{2m\pi x}{a} + b \cos \frac{2n\pi y}{b} + \gamma_1 x^3 + \gamma_2 x^2 + \gamma_3 y^3 + \gamma_4 y^2
\]

Substituting equation (12) into equation (11) yields
\[ a = \frac{a^2 n^2}{32 m^2 b^2} \quad \beta = \frac{b^2 m^2}{32 n^2 a^2} \quad \varepsilon = \frac{b^2 m^2}{32 n^2 a^2} \quad \varepsilon = \frac{b^2 m^2}{32 n^2 a^2} \]

The corresponding four edges simply boundary conditions are as follows:

\[ x = 0, 1 : \frac{\partial^2 \phi}{\partial y^2} = 1 \quad \frac{\partial^2 \phi}{\partial x \partial y} = 0, \quad \omega = 0 \]

\[ y = 0, 1 : \frac{\partial^2 \phi}{\partial x^2} = 1 \quad \frac{\partial^2 \phi}{\partial x \partial y} = 0, \quad \omega = 0 \]

Bring equation (12) into the boundary condition, we get \( \gamma_1 = \gamma_2 = n^2 \pi^2 / 16 b^2 \varepsilon \), \( \gamma_4 = m^2 \pi^2 / 16 a^2 \varepsilon \).

Then equation (12) is expressed as

\[ \phi(x,y) = \frac{a^2 n^2}{32 m^2 b^2} \varepsilon \cos \frac{2 \pi x}{a} + \frac{b^2 m^2}{32 n^2 a^2} \varepsilon \cos \frac{2 \pi y}{b} + \frac{n^2 \pi^2}{16 b^2} \varepsilon x^2 + \frac{m^2 \pi^2}{16 a^2} \varepsilon y^2 \]  

(12)

Substituting equations (5) and (6) into equation (3) yields

\[ \int_s \left[ \rho \left( \frac{\partial^2 T(t)}{\partial t^2} W + 2 \frac{\partial W}{\partial x} \frac{\partial T(t)}{\partial t} + \nu^2 \frac{\partial^2 W}{\partial x^2} T(t) \right) - \left( h \sigma_{x0} \frac{\partial^2 W}{\partial x^2} + h \sigma_{y0} \frac{\partial^2 W}{\partial y^2} \right) T(t) \right] W(x,y) ds = 0 \]

(13)

where

\[ A = \int_s \rho W^2 ds = \frac{ab}{4} \rho \]  

(14)

\[ B = \int_s 2 \rho \frac{\partial W}{\partial x} ds = 0 \]  

(15)

\[ C = \int_s \left[ \rho v^2 \frac{\partial^2 W}{\partial x^2} \left( h \sigma_{x0} \frac{\partial^2 W}{\partial x^2} + h \sigma_{y0} \frac{\partial^2 W}{\partial y^2} \right) \right] W(x,y) ds = \frac{\pi^2 h}{4 a^2} \left( \frac{m^2}{a^2} \sigma_{x0} + \frac{n^2}{b^2} \sigma_{y0} \right) - \frac{m v^2 b}{n a} (\cos(m \pi) - 1)(\cos(n \pi) - 1) \]  

(16)

\[ D = \int_s - \left( \frac{\partial^2 \phi}{\partial y^2} \frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 \phi}{\partial x^2} \frac{\partial^2 W}{\partial y^2} \right) W ds = \frac{3 \pi^4 \varepsilon h}{64 \left( \frac{n^2}{a^2} - \frac{n^2}{b^2} \right)} \]  

(17)

Then equation (13) is discretized as

\[ \frac{\partial^2 T(t)}{\partial t^2} + \left[ \frac{\pi^2 h}{\rho} \left( \frac{m^2}{a^2} \sigma_{x0} + \frac{n^2}{b^2} \sigma_{y0} \right) - \frac{4 m v^2}{a^2 h} (\cos(m \pi) - 1)(\cos(n \pi) - 1) \right] T(t) + \frac{3 \pi^4 \varepsilon h}{16 a b p} \left( \frac{m^2 b}{a^2} - \frac{n^2}{b^2} \right) T^3 = 0 \]  

(18)

**L-P Perturbation Method**

When \( \varepsilon = \left( \frac{\pi^2 h}{\rho} \right) \ll 1 \), then equation (18) can be expressed as

\[ \frac{d^2 T(t)}{dt^2} + \omega_0^2 \left( T(t) + \varepsilon \alpha_1 T^3(t) \right) = 0 \]

(19)

\[ \omega_0^2 = \frac{\pi^2 h}{\rho} \left( \frac{m^2}{a^2} \sigma_{x0} + \frac{n^2}{b^2} \sigma_{y0} \right) - \frac{4 m v^2}{a^2 h} (\cos(m \pi) - 1)(\cos(n \pi) - 1) \]

(20)
Let \( \tau = t \omega \), then equation (19) is expressed as
\[
\frac{\partial^2 T(t)}{\partial \tau^2} + \omega_0^2 T(t) = -\varepsilon \omega_0 \omega_1^2 T^3(t)
\]  
(22)

Expand \( \omega \) and \( T(t) \) into power series about \( \varepsilon \)
\[
T(t) = T_0(t) + \varepsilon T_1(t) + \varepsilon^2 T_2(t) + \ldots + \varepsilon^n T_n(t) + o(\varepsilon^{n+1})
\]
(23)
\[
\omega = \omega_0 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + \ldots + \varepsilon^n \omega_n + o(\varepsilon^n)
\]
(24)
Substituting equations (23) and (24) into equation (22), we obtain
\[
(\omega_0 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + \ldots)^2 \left( T_0(t) + \varepsilon T_1(t) + \varepsilon^2 T_2(t) + \ldots \right) + \omega_0^2 (T_0(t) + \varepsilon T_1(t) + \varepsilon^2 T_2(t) + \ldots) = -\varepsilon \omega_0 \omega_1^2 (T_0(t)
\]
(25)
Comparison \( \varepsilon^0 \):
\[
T_0(t) + T_0(t) = 0
\]
(26)
The general solution of equation (26) is as follows:
\[
T_0(t) = C_1 \cos(t) + C_2 \sin(t)
\]
(27)
where \( C_1 \) and \( C_2 \) are constants.
Let the initial condition be:
\[
T(0) = T_0(0) + \varepsilon T_1(0) + \varepsilon^2 T_2(0) + \ldots = a_0
\]
(28)
\[
\dot{T}(0) = T_0(0) + \varepsilon T_1(0) + \varepsilon^2 T_2(0) + \ldots = 0
\]
(29)
Bringing the initial conditions \( T_0(t) = a_0 \), \( \dot{T}_0(0) = 0 \) into equation (27), we obtain
\[
T_0(t) = a_0 \cos(t)
\]
(30)
Comparison \( \varepsilon^1 \):
\[
2 \omega_0 \omega_1 \dot{T}_0(t) + \omega_0^2 \dot{T}_1(t) + \omega_0^2 T_1(t) = -\omega_1 \omega_0^2 T_0^3(t)
\]
(31)
Substituting equation (30) into equation (31), we obtain
\[
\ddot{T}_1(t) + T_1(t) = -\frac{3}{4} \alpha_1 a_0^3 \cos(t) - \frac{1}{4} \alpha_1 a_0^3 \cos(3t) + 2 \frac{\omega_1}{\omega_0} a_0 \cos(t)
\]
(32)
In order to eliminate the internal resonance in equation (32), we have
\[
2 \frac{\omega_1}{\omega_0} a_0 - \frac{3}{4} \alpha_1 a_0^3 = 0
\]
(33)
\[
\omega_1 = \frac{3}{8} \omega_0 \alpha_1 a_0^3
\]
(34)
Then equation (32) becomes
\[
\ddot{T}_1(t) + T_1(t) = -\frac{1}{4} \alpha_1 a_0^3 \cos(3t)
\]
(35)
The solution of equation (35):

\[ T_1(\tau) = C_3 \cos(\tau) + C_4 \sin(\tau) + \frac{a_1a_0^3}{32} \cos(3\tau) \]  (36)

Supposing \( T_1(0) = 0, \dot{T}_1(0) = 0 \), and we bring them into equation (36)

\[ T_1(\tau) = \frac{a_1a_0^3}{32} (\cos(3\tau) - \cos(\tau)) \]  (37)

Substituting equations (37), (34), and (30) into equation (23) and (24) can be obtained

\[ T(\tau) = a_0 \cos(\tau) + \frac{3a_0a_1a_0^2e}{32} \left( \cos3\left(1 + \frac{3a_0a_1a_0^2e}{8}\right) + o\left(e^2\right) \right) \]  (38)

Suppose \( \tau = \omega t \)

\[ \tau = \left(\omega_0 + \frac{3a_0a_1a_0^2e}{8} + o\left(e^2\right)\right)t \]  (40)

Substituting equation (40) into equation (38) yields

\[ T(t) = a_0 \cos\left(1 + \frac{3a_0a_1a_0^2e}{8}\right) + a_0a_1a_0^2e \left( \cos3\left(1 + \frac{3a_0a_1a_0^2e}{8}\right) + o\left(1 + \frac{3a_0a_1a_0^2e}{8}\right) \right) \]  (41)

The frequency after adding nonlinear term is

\[ \omega = \omega_0 + \frac{3a_0a_1a_0^2e}{8} + o\left(e^2\right) \]  (42)

When \( a_0 = 0 \), it is a linear vibration. Substituting equations (10) and (41) into equation (5) yields

\[ \psi(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \left[ a_0 \cos\left(1 + \frac{3a_0a_1a_0^2e}{8}\right) + a_0a_1a_0^2e \left( \cos3\left(1 + \frac{3a_0a_1a_0^2e}{8}\right) + o\left(1 + \frac{3a_0a_1a_0^2e}{8}\right) \right) \right] \]  (43)

Substituting \( \psi(x,y,0) = W_0(x,y) \) in equation (43), we obtain

\[ W_0(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_0 \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \]  (44)

The modes corresponding to different frequencies are orthogonal, we multiply \( W_0ph \)

\[ a_0 = \frac{\int_{0}^{a} \int_{-b}^{b} \rho W_0(x,y) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \, dx \, dy}{\int_{0}^{a} \int_{-b}^{b} \rho \sin^2\frac{m\pi x}{a} \sin^2\frac{n\pi y}{b} \, dx \, dy} = \frac{4}{ab} \int_{0}^{a} \int_{0}^{b} W_0(x,y) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \, dx \, dy \]  (44)

When \( W_0(x,y) = W_0 \), we have

\[ a_0 = \begin{cases} 16W_0 \frac{m\pi u}{mn\pi}, & (m,n = 1,3,5\ldots) \\ 0, & (m,n = 2,4,6\ldots) \end{cases} \]
He’s Frequency Formula

Here, L-P perturbation method is compared with He’s frequency formula. The square of its frequency can be obtained, which is

\[ \omega_i^2 = \frac{df}{du} \left( u = \frac{iA}{N} \right), \quad i = 1, 2, 3, ..., N - 1 \]  

(45)

where \( A \) is the amplitude,

\[ \omega^2 = \sum_{i=1}^{N-1} \omega_i^2 \]  

(46)

when \( f(T(t)) = \omega_0^2 T(t) + \omega_0^2 \omega_0 T^3(t) \), the frequency of equation (19) is easily obtained as follows

\[ \omega^2 = f'(T(t)) \quad T = a_0 \left( 1 + \frac{3}{4} \epsilon \omega_0 a_0^2 \right) \]  

(47)

When \( \epsilon \ll 1 \), equation (47) is equivalent to that obtain by L-P Perturbation Method. When we increase the size of \( N \) in equation (46), the calculation accuracy of frequency can be improved.

When \( T(t) = a_0 \cos(\omega t) \), it can be obtained by integrating equation (19)

\[ T'(t) = - \int_0^t \left[ \omega_0^2 T(t) + \omega_0^2 \omega_0 T^3(t) \right] dt = - \left( \frac{\omega_0^2 a_0}{\omega} + \frac{2 \epsilon \omega_0^2 \omega_0 a_0^3}{3 \omega} \right) \sin(\omega t) - \frac{\epsilon \omega_0^2 \omega_0 a_0^3}{3 \omega} \sin(\omega t) \cos^2(\omega t) + C_1 \]  

(48)

We assume the initial conditions: \( T'(0) = 0 \), \( C_1 = 0 \). By integrating equation (48), we obtain

\[ T(t) = \int_0^t T'(t) dt = \int_0^t \left[ - \left( \frac{\omega_0^2 a_0}{\omega} + \frac{2 \epsilon \omega_0^2 \omega_0 a_0^3}{3 \omega} \right) \sin(\omega t) - \frac{\epsilon \omega_0^2 \omega_0 a_0^3}{3 \omega} \sin(\omega t) \cos^2(\omega t) \right] dt \]  

(49)

\[ dt = \left( \frac{\omega_0^2 a_0}{\omega^2} + \frac{2 \epsilon \omega_0^2 \omega_0 a_0^3}{3 \omega^2} \right) \cos(\omega t) + \frac{\epsilon \omega_0^2 \omega_0 a_0^3}{9 \omega^2} \left( \cos^3(\omega t) - 1 \right) + C_2 \]

When \( T(0) = a_0 \), We obtain

\[ C_2 = a_0 - \frac{\omega_0^2 a_0}{\omega^2} \frac{2 \epsilon \omega_0^2 \omega_0 a_0^3}{3 \omega^2} \]  

(50)

Substituting equations (50) into equation (49), We also get the solution of equation (19)

\[ T(t) = \frac{4 \left( 3a_0 + 2 \epsilon \omega_0 a_0^3 \right)}{3 \left( 4 + 3 \epsilon \omega_0 a_0^3 \right)} \left( \cos(\omega t) - 1 \right) + \frac{\epsilon \omega_0 a_0^3}{9 \left( 4 + 3 \epsilon \omega_0 a_0^3 \right)} \left( 3 \cos(\omega t) + \cos(3 \omega t) - \frac{1}{4} \right) \]  

(51)

Elliptic integral method

Simplified equation (18), we have

\[ \ddot{T} + k_a T(t) - k_b T^3(t) = 0 \]  

(52)

\[ k_a = \frac{\pi^2 h}{\rho} \left( \frac{m^2}{a^2 \sigma_0} + \frac{n^2}{b^2 \sigma_0} \right) - \frac{4 m v^2}{a^2 h} \left( \cos(m \pi) - 1 \right) \left( \cos(n \pi) - 1 \right) \]

\[ k_b = -\frac{3 \pi^4 E}{16 \rho h \mu_0} \left( \frac{m^4}{a^4} - \frac{n^4}{b^4} \right) \]
The integral of equation (52) is obtained as follows

$$\dot{T} + k_a T^2 - \frac{k_b}{2} T^4 = H$$

$$\varphi = \frac{T}{W_0}$$

Bring above in the following formula

$$\frac{d\varphi}{dt} = \sqrt{\frac{k_b}{2}} \sqrt{(1 - \varphi^2)(1 - n^2\varphi^2)}$$

When \( n^2 = \frac{1}{2} \), the vibration period is obtained

$$T = 4 \sqrt{\frac{2}{k_b W_0^2 \pi^2}} \int_0^{\varphi} \frac{d\varphi}{\sqrt{(1 - \varphi^2)(1 - n^2\varphi^2)}} = 4 \sqrt{\frac{2}{k_b W_0^2 \pi^2}} Z(n)$$

Then the vibration frequency is

$$\omega = \frac{\pi}{2} \sqrt{\frac{k_a - \frac{1}{2} k_b W_0}{Z(n)}}$$

$$Z(n) = \frac{\pi}{2} \left\{ 1 + \left( \frac{1}{2} \right)^2 n^2 + \left( \frac{1 \cdot 3}{2 \cdot 4} \right)^2 n^4 + \left( \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \right)^2 n^6 + \ldots \right\}$$

Namely,

$$\omega = \sqrt{\frac{k_a - \frac{1}{2} k_b W_0^2}{\frac{\pi}{2} \left\{ 1 + \left( \frac{1}{2} \right)^2 n^2 + \left( \frac{1 \cdot 3}{2 \cdot 4} \right)^2 n^4 + \left( \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \right)^2 n^6 + \ldots \right\}}}$$

$$n^2 = \frac{\frac{1}{2} k_b W_0^2}{k_a - \frac{1}{2} k_b W_0^2} = \frac{\left[ \frac{\pi^2}{4} \left( \frac{\pi^2}{L^2} \sigma_{xx} + \frac{\pi^2}{L^2} \sigma_{yy} \right) - \frac{2 \mu \pi^2}{\lambda \pi^2} \left( \cos(m\pi) - 1 \right) \left( \cos(n\pi) - 1 \right) \right] W_0^2}{\left[ \frac{2 \pi^2}{\rho} \left( \frac{\pi^2}{L^2} \sigma_{xx} + \frac{\pi^2}{L^2} \sigma_{yy} \right) - \frac{\mu \pi^2}{\lambda \pi^2} \left( \cos(m\pi) - 1 \right) \left( \cos(n\pi) - 1 \right) \right] W_0^2}$$

### Numerical calculation

Taking Shaanxi People B624 high-precision flexible electronic printing press as the research object, its parameters are shown in Table 1.

Use the formula (43) obtained by L-P perturbation method, taking \( t = 0.02s \), as well as \( a = 1m, b = 1m, N_{0x} = 50N/m \), and \( N_{0y} = 50N/m \) to calculate the first three modes of flexible electronic membrane.

The first mode shape: \( \phi(x,y,t) = (0.0891 \cos 325.18t + 2.596 \times 10^{-3} (\cos 975.54t - \cos 325.18t) \sin \pi x \sin \pi y) \).

### Table 1. Basic parameters of Shaanxi People B624 High-Precision Flexible Electronic Printing Press.

| Material | Membrane Length (m) | Membrane Width (m) | Density (Kg/m²) | Elastic Modulus (N/m²) | Maximum Transverse and Longitudinal Tension (N/m) |
|----------|---------------------|--------------------|-----------------|------------------------|-----------------------------------------------|
| TPU      | 2                   | 1.25               | 1.7             | 0.9x10⁷                | 120                                           |
The second mode shape:
\[ \omega(x,y,t) = (0.0297 \cos 673.98t + 5.65 \times 10^{-3}(\cos 2021.94t - \cos 673.98t) \sin \pi x \sin 3\pi y). \]

The third mode shape:
\[ \omega(x,y,t) = (0.0297 \cos 746.77t + 9.559 \times 10^{-3}(\cos 2240.3t - \cos 746.77t) \sin 3\pi x \sin \pi y). \]

From Figure 3, we can see that with the increase of mode order, the amplitude of flexible electronic membrane gradually decreases, and the total modes present the saddle surface. The amplitude of the first mode decreases compared with the second mode. The results show that the L-P perturbation method is effective. From Figure 4, we calculate the first three order vibration frequencies of flexible electronic membrane by using the L-P perturbation method and the Elliptic Integral method, respectively. The vibration frequency of the membrane decreases with the decrease of mode. The horizontal ordinate is the initial amplitude and the longitudinal coordinates is the frequency. By comparison, it can be seen that with the change of initial displacement, when \( W_0 \) tends to 0, the first three order frequencies obtained by the two methods will converge to 38.37 Hz, 85.79 Hz, and 85.79 Hz. The values obtained by both methods converge. It is proved that the properties of equation (42) and equation (49) are the same.
Analysis of the calculation results

In the process of membrane production, the guide roller length \( b \) and membrane density \( \rho \) have a great influence on the nonlinear vibration of membrane, so these two quantities are taken to study the nonlinear vibration of membrane on the basis of equation (49). When setting \( b \) as 0.2m, 0.4m, 0.6m, respectively, and other parameters \( N_{0x} = 120N/m, N_{0y} = 120N/m, E = 0.9 \times 10^9N/m^2, W_0 = 0m \), Figure 5 shows the relationship between density coefficient \( \rho \) and complex frequency \( \omega \). With the increase of membrane length \( b \), the real part of the complex frequency of the membrane becomes smaller, the imaginary part tends to 0, and the membrane remains in a stable working state. This shows that by increasing the membrane length \( b \), it will reduce the real part of the complex frequency of the system, so it is necessary to control the length \( b \).

Figure 6 shows the relationship between the complex frequency and the density coefficient, subject to the condition that the initial amplitude \( W_0 = 0.1 \) and other parameters remain unchanged. With the growing of the initial amplitude, the real part of the dimensionless complex frequency of the membrane tends to 0, the imaginary part diverges, and the membrane becomes unstable. This indicates, the increase of the initial amplitude will make the membrane into the unstable working state in advance, so the initial amplitude should be controlled.

Taking a further step, we continue to study the influence of membrane density on system vibration. As with \( \rho = 0.4kg/m^2, 0.6kg/m^2, 0.8kg/m^2 \) and other parameters as \( N_{0x} = 120N/m, N_{0y} = 120N/m, a = 0.9m, b = 1m, E = 0.9 \times 10^9N/m^2, W_0 = 0.3m \), the relation curve between complex frequency \( \omega \) and velocity \( c \) is shown in Figure 7.
When $\rho = 0.4\text{kg/m}^2$, $\rho = 0.6\text{kg/m}^2$, and $\rho = 0.8\text{kg/m}^2$, the maximum velocity of moving membrane is accordingly 19.239m/s, 15.53m/s, and 13.687m/s. With the increase of membrane density, the critical velocity of moving membrane decreases.

**Conclusion**
Considering the additional tension, the nonlinear dynamic model of flexible electronic membrane is established. The following conclusions are obtained:

1. The initial amplitude of the membrane should be controlled in the motion process. as the increase of the initial amplitude will make the flexible electronic membrane unstable, and the imaginary part of the complex frequency will always increase, which makes the membrane system divergent and unstable.
2. The increase of length $b$—the distance between guide rollers—will reduce the film complex frequency and make the system enter into the unstable working state in advance. Therefore, it is necessary to control this distance in the design of high-precision printing press.
3. The increase of the density of flexible electronic membrane will lead to the decrease of the working speed of the coater machine. When $N_{0x} = 120\text{N/m}$, $N_{0y} = 120\text{N/m}$, $W_0 = 0.3\text{m}$, $a = 1\text{m}$, $b = 1\text{m}$, $E = 0.9 \times 10^6\text{N/m}^2$, and $\rho = 0.4\text{kg/m}^2$, the coater machine can reach the maximum working speed of 19.239m/s.

This paper mainly discuss the influence of length $b$ and membrane density on the vibration of flexible electronic membrane, and thus determine the maximum working speed of flexible electronic printing equipment, which provides a theoretical basis for the structural design of flexible electronic printing press.

**Declaration of conflicting interests**
The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

**Funding**
The author(s) disclosed receipt of the following financial support for the research, authorship, and/or publication of this article: The author gratefully acknowledges the support of the National Natural Science Foundation of China (No. 52075435) and the Natural Science Foundation of Shaanxi Province (No. 2021JQ-480,2020JM-457).

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