ON THE WARING RANK OF BINARY FORMS

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Abstract. The $K$-rank of a binary form $f$ in $K[x, y]$, $K \subseteq \mathbb{C}$, is the smallest number of $d$-th powers of linear forms over $K$ of which $f$ is a $K$-linear combination. We provide lower bounds for the $\mathbb{C}$-rank (Waring rank) and for the $\mathbb{R}$-rank (real Waring rank) of binary forms depending on their factorization. We completely classify binary forms of Waring rank 3.

1. Introduction

The main results of this work concern symmetric tensor decomposition which is also known as the Waring problem for forms. Tensors have a rich history and they have recently become ubiquitous in signal processing, statistics, data mining and machine learning.

Let $K[x, y]_d$ denote the vector space of binary forms of degree $d$ with coefficients in the field $K \subseteq \mathbb{C}$. Given a binary form $f \in K[x, y]_d$, the $K$-rank of $f$, $L_K(f)$, is the smallest $r$ for which there exist $\lambda_j, \alpha_j, \beta_j \in K$ such that

\begin{equation}
    f(x, y) = \sum_{j=1}^{r} \lambda_j (\alpha_j x + \beta_j y)^d.
\end{equation}

A representation such as (1.1) is honest if the summands are pairwise distinct; that is, if $\lambda_i \lambda_j (\alpha_i \beta_j - \alpha_j \beta_i) \neq 0$ whenever $i \neq j$. Any representation in which $r = L_K(f)$ is necessarily honest. In case $K = \mathbb{C}$ or $\mathbb{R}$, the $K$-rank is commonly called the Waring rank or the real Waring rank. Sylvester [17, 18] presented an algorithm to compute $L_C(f)$ in 1851 and gave a lower bound for the real Waring rank in 1864. The Waring rank of binary forms has been studied extensively [1, 2, 6, 13, 15]. Recently the real Waring rank of binary forms has been investigated [3, 5, 7, 8, 9]. The relative Waring rank of binary forms over some intermediate fields of $\mathbb{C}/\mathbb{Q}$ was analyzed in [15, 16].

It has been known for a long time that $L_C(f) \leq \text{deg}(f)$. This still holds when the underlying field varies, that is, $f \in K[x, y]$ implies $L_K(f) \leq \text{deg}(f)$ for any $K \subseteq \mathbb{C}$ [15, Theorem 4.10]. The relation between the number of real roots and the real Waring rank of binary forms has also received substantial attention. Extending the work of Sylvester, Reznick showed that if $f(x, y)$ is a binary form of degree $d$, not a $d$-th power, with $\tau$ real roots (counting multiplicities), then $L_R(f) \geq \tau [15]$. 

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Theorem 3.2]: if $f$ is hyperbolic, that is $\tau = d$, then $L_R(f) = d$ \cite{15}. The converse was conjectured and proved for $d \leq 4$ in \cite{15}. Causa and Re \cite{8} and Comon and Ottaviani \cite{9} showed that the conjecture holds for any square-free binary form, and recently Blekherman and Sinn \cite{3} proved that the conjecture is true for any binary form. Reznick presented a classification of binary forms of Waring rank 1 and 2 \cite{15}.

In this paper, we provide a lower bound for the Waring rank of binary forms based on their factorization over $\mathbb{C}$ (Theorem 3.1). This result also improves the above-mentioned lower bound for the real Waring rank of binary forms (Corollary 3.4). Additionally, we also give the complete classification of binary forms of Waring rank 3 and provide supporting examples.

We now outline the remainder of the paper.

In Section 2, after briefly discussing apolarity, we recall Sylvester’s 1851 Theorem (Theorem 2.1) and the Apolarity Lemma (Theorem 2.2). We prove that if $f \in K[x, y]_d$ and $k < \frac{4d+2}{2}$, then $(f^\perp)_k$ is generated by a projectively unique form in $K[x, y]$ (unless it is empty). Let $f$ be a binary form of degree $d$ and $L_K(f) = d$, we say that $f$ has full $K$-rank. We include a well-known result on the binary forms of full Waring rank (Theorem 2.4) and a recent result on the real case (Theorem 2.5). We conclude this section by recalling the classification of binary forms of Waring rank 1 and 2 which was given in \cite{15}. We apply these theorems and observations in Section 3 and Section 4.

In Section 3, we first show that if $f$ is a binary form of degree $d$, not a $d$-th power, and $\alpha_i$ is a root of multiplicity $m_i$ of $f$, then $L_C(f) \geq m_i + 1$ (Theorem 3.1). It directly follows that $L_C(\ell_0^{d-2}\ell_1\ell_2) = L_R(\ell_0^{d-2}p) = d-1$ where $\ell_i$’s are distinct binary linear forms and $p$ is an irreducible quadratic (Corollaries 3.2 and 3.3). Theorem 3.1 combines with \cite{15} Theorem 3.2 into Corollary 3.4: if $f$ is a real binary form of degree $d$, not a $d$-th power, with $\tau$ real roots (counting multiplicities), and $\alpha_i$ is a root of multiplicity $m_i$ of $f$, then $L_R(f) \geq \max(\tau, m_i + 1)$. We then show that if $f_\lambda(x, y) = x^{2k} + \left(\frac{2k}{k}\right)x^k y^k + y^{2k}$, $\lambda \neq 0$, $k \geq 2$, then $L_R(f_\lambda) \in \{2k-2, 2k-1\}$ (Theorem 3.5). The minimal representations of $f_\lambda$ are parameterized in Theorem 3.6.

In Section 4, we completely classify binary forms of Waring rank 3 (Theorem 4.1). We then look at the special case when the underlying field $K$ is a real closed field (Corollary 4.2).

In Section 5, we give examples for each case considered in Theorem 4.1 and an additional example of a binary quartic form with infinitely many minimal representations of length 3 (Example 5.4).

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2. Tools and Background

Suppose \( p(x, y) = \sum_{i=1}^{d} a_i x^{d-i} y^i \in \mathbb{C}[x, y]_d \). The differential operator associated to \( p \) is given by

\[
p(D) = \sum_{i=1}^{d} a_i \frac{\partial^d}{\partial x^{d-i} \partial y^i}.
\]

The *apolar ideal* of \( p \), which is denoted by \( p^\perp \), is the set of all binary forms whose differential operator kills \( p \), that is,

\[
p^\perp = \{ h \in \mathbb{C}[x, y] \mid h(D)p = 0 \}.
\]

This is a homogeneous ideal with the decomposition

\[
p^\perp = \bigoplus_{k \geq 0} (p^\perp)_k,
\]

\[
(p^\perp)_k = \{ h \in \mathbb{C}[x, y]_k \mid h(D)p = 0 \}.
\]

The following theorem is proved in [15], and for \( K = \mathbb{C} \), is due to Sylvester [17, 18] in 1851.

**Theorem 2.1.** [15, Theorem 2.1, Corollary 2.2] Suppose \( K \subseteq \mathbb{C} \) is a field,

\[
p(x, y) = \sum_{i=1}^{d} \binom{d}{i} a_i x^{d-i} y^i \in K[x, y]_d
\]

and suppose \( r \leq d, \alpha_j, \beta_j \in K \) and

\[
h(x, y) = \sum_{t=0}^{r} c_t x^{r-t} y^t = \prod_{j=1}^{r} (-\beta_j x + \alpha_j y)
\]

is a product of pairwise distinct linear factors. Then there exist \( \lambda_k \in K \) such that

\[
p(x, y) = \sum_{k=1}^{r} \lambda_k (\alpha_k x + \beta_k y)^d
\]

if and only if

\[
\begin{pmatrix}
  a_0 & a_1 & \ldots & a_r \\
  a_1 & a_2 & \ldots & a_{r+1} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{d-r} & a_{d-r+1} & \ldots & a_d
\end{pmatrix}
\begin{pmatrix}
  c_0 \\
  c_1 \\
  \vdots \\
  c_r
\end{pmatrix}
= \begin{pmatrix}
  0 \\
  0 \\
  \vdots \\
  0
\end{pmatrix};
\]

that is, if and only if
If (2.4) holds and \( h \) is square-free, then we say that \( h \) is a Sylvester form of degree \( r \) for \( p \). Note that \( L_K(p) = r \) if and only if there is a Sylvester form of degree \( r \) for \( p \) which splits over \( K \).

It is known that any bivariate apolar ideal is a complete intersection ideal and the converse also holds.

**Theorem 2.2.** \cite{[11]} Theorem 1.44(iv) \ Let \( p(x, y) \in \mathbb{C}[x, y]_d \), then \( p^\perp \) is a complete intersection ideal over \( \mathbb{C} \), i.e. \( p^\perp = \langle f, g \rangle \) such that \( \deg(f) + \deg(g) = d + 2 \) and \( V_\mathbb{C}(f, g) = \emptyset \). Also, any two such binary forms \( f, g \) generate an ideal \( p^\perp \) for a binary form \( p \) of degree \( \deg(f) + \deg(g) - 2 \).

**Corollary 2.3.** Let \( p(x, y) \) be a nonzero binary form in \( \mathbb{K}[x, y]_d \), not a \( d \)-th power, and \((p^\perp)_k \neq \emptyset \) for \( k < \frac{d+2}{2} \). Then there exists a projectively unique binary form \( h(x, y) \in \mathbb{K}[x, y]_k \) such that \((p^\perp)_k = \langle h \rangle \). Thus, \( p(x, y) \) has at most one minimal representation of length \( k \).

**Proof.** We first prove uniqueness: If \( g(x, y) \) is a binary form which is apolar to \( p \) and non-proportional to \( h \), then \( \deg(g) > k \) by Theorem 2.2. It follows that \((p^\perp)_k \) has a unique element (up to a scalar multiple).

We now prove that \( h \in \mathbb{K}[x, y]_k \): If we take \( r = k \), then the linear system in (2.4) has at least one nonzero solution over \( \mathbb{C} \), since \( h(x, y) \) corresponds to a solution. Thus, it must have a solution over \( K \) as well and by uniqueness \( h(x, y) \in \mathbb{K}[x, y]_k \).

The following theorem gives all the binary forms of degree \( d \geq 3 \) with Waring rank \( d \). It is well known that \( L_\mathbb{C}(x^d - 1 y) = d \); however, to the best of our knowledge, the converse has been proven only later \cite{[2]} Corollary 3 and \cite{[10]} Ex.11.35.

**Theorem 2.4.** If \( d \geq 3 \), then \( L_\mathbb{C}(f) = d \) if and only if \( f = \ell_0^d - 1 \ell_1 \).

**Theorem 2.5.** \cite{[3]} Theorem 2.2 \ Let \( f(x, y) \in \mathbb{R}[x, y]_d \) be a binary form of degree \( d \geq 3 \) and suppose that \( f \) is not a \( d \)-th power. The real Waring rank of \( f \) is \( d \) if and only if \( f \) is hyperbolic.

The next tool is an application of Descartes’ Rule of Signs \cite{[14]} Question 49, pp.43.

**Theorem 2.6.** Let \( a_0 \neq 0, a_n \neq 0 \), and assume that \( 2m \) consecutive coefficients of the polynomial \( a_0 + a_1 x + \ldots + a_n x^n \) vanish, where \( m \) is an integer, \( m \geq 1 \). Then the polynomial has at least \( 2m \) non-real zeros.

Binary forms with Waring rank 1 and Waring rank 2 were studied by Reznick \cite{[15]}.

**Theorem 2.7.** \cite{[15]} Theorem 4.1 \ If \( p(x, y) \in \mathbb{K}[x, y] \), then \( L_\mathbb{K}(p) = 1 \) if and only if \( L_\mathbb{C}(p) = 1 \).
Theorem 2.8. [15, Theorem 4.6] Let \( p(x, y) \) be a nonzero binary form of degree \( d \geq 3 \), and not a \( d \)-th power, with \( \lambda_i, \alpha_i, \beta_i \in \mathbb{C} \) so that

\[
(2.6) \quad p(x, y) = \lambda_1(\alpha_1 x + \beta_1 y)^d + \lambda_2(\alpha_2 x + \beta_2 y)^d \in K[x, y].
\]

If \( (2.6) \) is honest and \( L_K(p) > 2 \), then there exists \( u \in K \) with \( \sqrt{u} \notin K \) so that \( L_K(\sqrt{u})(p) = 2 \). The summands in \( (2.6) \) are conjugates of each other in \( K(\sqrt{u}) \).

Example 2.1. Suppose there exists \( \gamma \in \mathbb{Q} \) with \( \sqrt{\gamma} \notin \mathbb{Q} \) so that

\[
(2.7) \quad p_d(x, y) = \sum_{0 \leq 2i \leq d} \left( \binom{d}{2i} \right) \gamma^i x^{d-2i} y^{2i}, \ d \geq 3.
\]

Then \( p_d(x, y) \) is a rational binary form of Waring rank 2 with the following projectively unique representation:

\[
(2.8) \quad p_d(x, y) = \frac{1}{2}(x + \sqrt{\gamma} y)^d + \frac{1}{2}(x - \sqrt{\gamma} y)^d.
\]

Notice that the summands in \( (2.8) \) are conjugates of each other in \( \mathbb{Q}(\sqrt{\gamma}) \). It follows from Corollary 2.3 that \( p \) has a unique representation of length 2, therefore \( L_K(p_d) = 2 \) if and only if \( \sqrt{\gamma} \in K \).

3. A Lower Bound for the Rank of Binary Forms

In this section we give a lower bound for the Waring rank of binary forms based on their factorization over \( \mathbb{C} \). We also improve the lower bound for the real Waring rank of binary forms.

Theorem 3.1. Let \( f(x, y) \) be a nonzero binary form of degree \( d \) with the factorization

\[
(3.1) \quad f(x, y) = \prod_{i=0}^{r} \ell_i(x, y)^{m_i}
\]

where \( r \geq 1 \) and \( \ell_i \)'s are distinct linear forms. Then \( L_{\mathbb{C}}(f) \geq \max(m_0, \ldots, m_r) + 1 \).

Proof. Set \( m = \max(m_0, \ldots, m_r) \). We use the fact that rank is invariant under invertible linear change of variables. Assume that \( m_0 = m \) and let \( \ell_0 \to y \), then we have

\[
(3.2) \quad \tilde{f}(x, y) = y^m g(x, y) \text{ such that } y \nmid g(x, y).
\]
The first \(m\) coefficients of \(\tilde{f}\) are zero, i.e. \(a_0 = \ldots = a_{m-1} = 0\) and \(a_m \neq 0\). Note that \(\deg(\tilde{f}) \geq m + 1\), so by setting \(r = m\), (2.4) becomes:

\[
(3.3) \quad \begin{pmatrix} 0 & 0 & \ldots & 0 & a_m \\ 0 & 0 & \ldots & a_m & a_{m+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & * & * \\
\end{pmatrix} \cdot \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ c_m \\
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\
\end{pmatrix} \implies a_mC_m = a_mC_{m-1} + a_{m+1}c_m = 0.
\]

Thus, \(c_{m-1} = c_m = 0\) and every apolar form of degree \(m\) is divisible by \(x^2\) and \(L_C(f) \geq m + 1\) by Theorem 2.1.

Landsberg and Teitler \[13\] Corollary 4.5 and Boij, Carlini and Geramita \[5\] have both shown that \(L_C(x^ay^b) = \max(a + 1, b + 1)\) if \(a, b \geq 1\).

**Corollary 3.2.** Let \(f(x, y) = \ell_0(x, y)d^2\ell_1(x, y)\ell_2(x, y)\) such that \(d \geq 3\) and \(\ell_i\)'s are distinct binary linear forms. Then \(L_C(f) = d - 1\).

**Proof.** It follows from Theorem 3.1 that \(d - 1 \leq L_C(f)\) and \(L_C(f) \leq d - 1\) by Theorem 2.4. Thus, \(L_C(f) = d - 1\). \(\square\)

**Corollary 3.3.** Suppose \(f(x, y) = \ell(x, y)d^2p(x, y)\) is a real binary form of degree \(d \geq 3\) where \(\ell(x, y)\) is a real linear form and \(p(x, y)\) is an irreducible real quadratic form. Then \(L_R(f) = d - 1\).

**Proof.** The Waring rank of \(f\) is \(d - 1\) by Corollary 3.2; therefore, \(d - 1 \leq L_R(f)\). On the other hand, it follows from Theorem 2.5 that \(L_R(f) \leq d - 1\). \(\square\)

We can determine the Waring rank of binary cubics based on their factorization \[15\] Theorem 5.2.

**Remark 3.1.** For \(d = 4\), we can assign a unique Waring rank to binary forms with repeating roots based on their factorization. Assume that \(\ell_i\)'s are distinct binary linear forms. The following table follows from Theorem 2.4, Theorem 2.5 and Theorem 3.1.

| \(p(x, y)\) | \(L_C(p(x, y))\) |
|----------------|------------------|
| \(\ell_0(x, y)^4\) | 1 |
| \(\ell_0(x, y)^3\ell_1(x, y)\) | 4 |
| \(\ell_0(x, y)^2\ell_1(x, y)^2\) | 3 |
| \(\ell_0(x, y)^2\ell_1(x, y)\ell_2(x, y)\) | 3 |
| \(\ell_0(x, y)\ell_1(x, y)\ell_2(x, y)\ell_3(x, y)\) | 2,3 |

Notice that we can not assign a unique Waring rank to square-free binary quartics; for example, \(L_C(x^4 + y^4) = L_C(8x^3y + 36x^2y^2 + 36xy^3) = 2\) and \(L_C(x^4 + 4x^2y^2 + y^4) = L_C(4x^3y + 6x^2y^2 + 4xy^3) = 3\).

It can be checked from the above table and Theorem 2.5 that \(L_R(p(x, y)^2) = 3\) where \(p(x, y)\) is an irreducible real quadratic. This result is a consequence of the known fact: \(L_C(p(x, y)^k) = L_R(p(x, y)^k) = k + 1\) \[15\] Corollary 5.6.
Corollary 3.4. Let \( f(x, y) \) be a nonzero real binary form of degree \( d \) and not a \( d \)-th power with the factorization

\[
(3.4) \quad f(x, y) = \prod_{i=0}^{r} \ell_i(x, y)^{m_i} \prod_{k=0}^{s} p_k(x, y)^{n_k}
\]

where \( \ell_i \)'s are distinct real binary linear forms and \( p_k \)'s are distinct irreducible real quadratic forms. Then \( L(\ell_j, \lambda) \) provide a lower bound for \( L(f, \lambda) \) where \( \lambda \neq 0 \) and \( k \geq 2 \).

Proof. The result follows from [15, Theorem 3.1, 3.2] and Theorem 3.1. □

Let \( f_\lambda(x, y) = x^{2k} + \binom{2k}{k} \lambda x^k y^k + y^{2k}, \lambda \neq 0 \). If \(|\lambda| \binom{2k}{k} < 2 \), then \( f_\lambda \) is a square-free definite form; therefore, Corollary 3.4 does not suggest a lower bound for the real Waring rank of \( f_\lambda \). In the following theorem, arguments employing Descartes' Rule of Signs provide a lower bound for \( L(\mathbb{R}, f_\lambda) \).

Theorem 3.5. Let \( f_\lambda(x, y) = x^{2k} + \binom{2k}{k} \lambda x^k y^k + y^{2k} \) where \( \lambda \neq 0 \) and \( k \geq 2 \). Then \( L(\mathbb{R}, f_\lambda) \in \{2k - 2, 2k - 1\} \).

Proof. We let \( r = k + j, 0 \leq j \leq k \) and look for the Sylvester form of degree \( r \). If \( k = 4, j = 1 \), then (2.4) becomes:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & \lambda & 0 \\
0 & 0 & \lambda & 0 & 0 \\
0 & 0 & \lambda & 0 & 0 \\
0 & \lambda & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
c_0 \\
c_1 \\
\vdots \\
c_5
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}
\implies (c_0, c_1, c_2, c_3, c_4, c_5) = (-\lambda c_4, c_1, 0, 0, c_4, -\lambda c_1).
\]

Instead if \( k = 5, j = 2 \), then \((c_0, c_1, c_2, c_3, c_4, c_5, c_6, c_7) = (-\lambda c_5, c_1, c_2, 0, 0, c_5, c_6, -\lambda c_2)\) is the solution of the linear system:

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 \\
0 & 0 & 0 & \lambda & 0 & 0 & 0 \\
0 & 0 & \lambda & 0 & 0 & 0 & 0 \\
0 & \lambda & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
c_0 \\
c_1 \\
\vdots \\
c_7
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}.
\]

In general for \( r = k + j \), we can see that if \((c_0, c_1, \ldots, c_{k+j})\) is a solution for (2.4), then

\[
c_i = 0, \quad j + 1 \leq i \leq k - 1,
\]

\[
c_0 = -\lambda c_k, \quad c_{k+j} = -\lambda c_j.
\]

Therefore, \( h_{k+j}(x, y) \), the corresponding Sylvester form of degree \( k + j \), has at least \( k - j - 1 \) consecutive missing coefficients. If \( h_{k+j} \) splits over \( \mathbb{R} \), then \( k - j \leq 2 \) by Theorem 2.6; thus, \( 2k - 2 \leq L(\mathbb{R}, f_\lambda) \). In addition, it follows from Theorem 2.5 that \( L(\mathbb{R}, f_\lambda) \leq 2k - 1 \). □

The following theorem gives a parametrization for a \( \mathbb{C} \)-minimal representation of \( f_\lambda(x, y) \) as \( \lambda \) varies over all nonzero complex numbers. Recall that \( \zeta_d = e^{\frac{2\pi i}{d}} \).
Theorem 3.6. Suppose \( f_\lambda(x, y) = x^{2k} + \lambda \binom{2k}{k} x^k y^k + y^{2k}, \lambda \neq 0. \) Then \( L_C(f_\lambda) = k \) if \( \lambda = \pm 1 \) and \( k + 1 \) otherwise. The following is a minimal representation of \( f_\lambda \), which is unique for \( \lambda = \pm 1 \),

\[
(3.5) \quad x^{2k} + \binom{2k}{k} \lambda x^k y^k + y^{2k} = (1 - \lambda^2) y^{2k} + \frac{1}{k} \sum_{i=0}^{k-1} (x + \lambda^i \xi_i y)^{2k}.
\]

**Proof.** We first evaluate the right-hand side of (3.5):

\[
(3.6) \quad (1 - \lambda^2) y^{2k} + \frac{1}{k} \sum_{i=0}^{k-1} (x + \lambda^i \xi_i y)^{2k} = (1 - \lambda^2) y^{2k} + \frac{1}{k} \sum_{j=0}^{2k} \binom{2k}{j} x^{2k-j} y^j \lambda^j \left( \sum_{i=0}^{k-1} \xi_{ij} \right).
\]

The sum \( \sum_{i=0}^{k-1} \xi_{ij} = 0 \) unless \( k \mid j \), in which case it equals to \( k \). The only multiples of \( k \) in the set \( \{j : 0 \leq j \leq 2k\} \) are 0, \( k, 2k \). The right-hand side of (3.6) reduces to left-hand side of (3.5). If we let \( r = k - 1 \), then the linear system in (2.4) has only the trivial solution, so \( k \leq L_C(f_\lambda) \leq k + 1 \).

If \( \lambda \in \{1, -1\} \) then the first summand in (3.5) is zero, therefore \( f_\lambda \) has a unique minimal representation which is given by (3.5) and \( L_C(f_\lambda) = k \).

Let \( \lambda \neq \pm 1 \) and \( r = k \), then the matrix in (2.4) is nonsingular, so \( L_C(f_\lambda) = k + 1 \). Then the minimal representation given by (3.5) is not necessarily unique. \( \square \)

4. Binary Forms of Waring Rank 3

**Remark 4.1.** Suppose \( d \geq 5 \) and there exist nonzero \( \lambda_i, \alpha_i, \beta_i \in \mathbb{C} \) so that

\[
(4.1) \quad f(x, y) = \lambda_1(\alpha_1 x + \beta_1 y)^d + \lambda_2 x^d + \lambda_3 y^d \in K[x, y].
\]

Then \( L_K(f) = 3 \), and (4.1) is the projectively unique representation of \( f \) of length 3.

**Proof.** The Sylvester form corresponding to (4.1) is \( h(x, y) = (\beta_1 x - \alpha_1 y)yx \) by Theorem 2.1. It follows from Corollary 2.3 that \( h \in K[x, y] \); thus, \( h \) splits over \( K \) and \( L_K(p) = 3 \). \( \square \)

**Theorem 4.1.** Suppose \( d \geq 5 \) and there exist \( \lambda_i, \alpha_i, \beta_i \in \mathbb{C} \) so that

\[
(4.2) \quad p(x, y) = \lambda_1(\alpha_1 x + \beta_1 y)^d + \lambda_2(\alpha_2 x + \beta_2 y)^d + \lambda_3(\alpha_3 x + \beta_3 y)^d \in K[x, y]
\]

is a honest representation and \( L_K(p) > 3 \). Then there exist \( (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{C}^3 \setminus K^3, \ u \in K \) such that

1. If \( \gamma_i \not\in K, \ i \in \{1, 2, 3\}, \) and \( \sqrt{u} \not\in K, \) then \( L_{K(\gamma_i)}(p) = 3, \)
2. If \( \gamma_i \not\in K, \ i \in \{1, 2, 3\}, \) and \( \sqrt{u} \not\in K, \) then \( L_{K(\gamma_i, \sqrt{u})}(p) = 3, \)
3. If there is exactly one \( \gamma_i \in K, \) then \( L_{K(\sqrt{u})}(p) = 3. \) The representation in (4.2) has a summand in \( K \) and a pair of conjugate summands in \( K(\sqrt{u}). \)
Proof. The above remark guarantees that at most one of \( \{ \alpha_i, \beta_i \} \) equals zero. After change of variables \( x \leftrightarrow y \) if necessary, we may assume that \( \alpha_i \neq 0 \) for \( i = 1, 2, 3 \).

Let \( \gamma_i = \frac{\beta_i}{\alpha_i} \), then \( h(x, y) = (y - \gamma_1 x)(y - \gamma_2 x)(y - \gamma_3 x) \in (p^1)_3 \) by Theorem 2.1. Since \( d \geq 5 \), we must have \( h(x, y) \in K[x, y] \) and projectively unique by Corollary 2.3.

By the hypothesis \( h \) does not split over \( K \), so at most one of \( \gamma_i \in K \). Let \( u \) be the discriminant of \( h \), where \( \tilde{h} = h(x, 1) \). Then \( u \in K \) and the rest follows from the Galois groups of cubics [12].

Case 1 and 2: Assume that \( \gamma_i \notin K \) for \( i = 1, 2, 3 \). Then the splitting field of \( h \) is \( K(\gamma_i) \) if \( \sqrt{u} \in K \) and \( K(\sqrt{u}, \gamma_i) \) otherwise.

Case 3: If only one of \( \gamma_i \in K \), then \( h \) splits over \( K(\sqrt{u}) \); therefore, the other two \( \gamma_i \)'s are conjugates of each other in \( K(\sqrt{u}) \). Note that every field automorphism which fixes \( K \) permutes the summands in (4.2). If we consider the conjugation with respect to \( \sqrt{u} \), then (4.2) has two summands which are conjugates of each other in \( K(\sqrt{u}) \) and a summand in \( K \).

□

The following theorem concerns a special case of Theorem 4.1 where \( K \) is a real closed field.

**Corollary 4.2.** Suppose \( d \geq 5 \), \( K \subseteq \mathbb{C} \) is a real closed field and there exist \( \lambda_i, \alpha_i, \beta_i \in \mathbb{C} \) such that

\[
(4.3) \quad p(x, y) = \lambda_1(\alpha_1 x + \beta_1 y)^d + \lambda_2(\alpha_2 x + \beta_2 y)^d + \lambda_3(\alpha_3 x + \beta_3 y)^d \in K[x, y]
\]

is a honest representation and \( L_K(f) > 3 \). Then there exists \( u \in K \) with \( \sqrt{u} \notin K \) such that \( L_K(\sqrt{u})(p) = 3 \). One of the summands in (4.3) is in \( K[x, y] \) whereas the other two summands are conjugates of each other in \( K(\sqrt{u}) \).

**Proof.** We can assume that \( \alpha_i \neq 0 \) for \( i = 1, 2, 3 \) as in Theorem 4.1. Let \( \gamma_i = \frac{\beta_i}{\alpha_i} \), then \( h(x, y) = (y - \gamma_1 x)(y - \gamma_2 x)(y - \gamma_3 x) \); by the hypothesis it does not split over \( K \). Since \( h \) is an odd degree form over a real closed field, it must have a root in \( K \) (see [4] Theorem 1.2.2]). Let \( u = (\gamma_1 - \gamma_2)^2(\gamma_1 - \gamma_3)^2(\gamma_2 - \gamma_3)^2 \), the discriminant of \( h(x, 1) \), then the rest follows from the Case 3 of Theorem 4.1. □

5. Examples

In this section, we give examples for each case of Theorem 4.1 and an additional one showing that if \( d \leq 5 \), there can be infinitely many representations of length 3.

**Example 5.1.** **Case 1:** Let \( f(x, y) = -15x^5 + 90x^4y - 30x^3y^2 + 60x^2y^3 + 3y^5 \). If we set \( r = 2 \), then the solution to the linear system in (2.4) is trivial, so \( L_C(f) \geq 3 \).

If we set \( r = 3 \) in (2.4), then up to a scalar multiple \( h(x, y) = x^3 - 3xy^2 + y^3 \). We can factorize \( h(x, y) \) by using the trigonometric identity \( 4 \cos^3(\theta) - 3 \cos(\theta) = \cos(3\theta) \):

\[
h(x, y) = (x - 2 \cos \frac{\pi}{3} y)(x - 2 \cos \frac{4\pi}{3} y)(x - 2 \cos \frac{8\pi}{3} y).
\]
Therefore, $L_K(f) = 3$ if and only if $\mathbb{Q}(\cos \frac{2\pi}{3}) \subseteq K$ with the $\mathbb{C}$-minimal representation:

$$f(x, y) = (y + 2x \cos \frac{2\pi}{3})^5 + (y + 2x \cos \frac{4\pi}{3})^5 + (y + 2x \cos \frac{8\pi}{3})^5.$$

**Example 5.2.** **Case 2:** Let $f(x, y) = 3x^7 + 210x^4y^3 + 84xy^6$. Then, $(f^{\perp})_2$ is empty by Theorem 2.1; thus, the Waring rank of $f$ is at least 3. The Sylvester form of degree 3 is

$$h(x, y) = y^3 - 2x^3 = (y - \sqrt{2}x)(y - \sqrt{2}\omega x)(y - \sqrt{2}\omega^2 x), \quad \omega = e^{\frac{2\pi i}{3}}.$$

Note that $h$ splits over $\mathbb{Q}(\sqrt{2}, \sqrt{-3})$. The $\mathbb{C}$-minimal representation of $f$ is given by

$$f(x, y) = (x + \sqrt{2}y)^7 + (x + \sqrt{2}\omega y)^7 + (x + \sqrt{2}\omega^2 y)^7.$$

**Example 5.3.** **Case 3:** Let $f(x, y) = (1 + 2\sqrt{2})x^5 - 25x^4y + (60\sqrt{2} + 10)x^3y^2 - 170x^2y^3 + (90\sqrt{2} + 5)xy^4 - 53y^5$. First, with $r = 2$, we see that the matrix from (2.4) is nonsingular, hence $L_{\mathbb{C}}(f) \geq 3$. On taking $r = 3$, we get the Sylvester form:

$$h(x, y) = 3x^3 - 3x^2y - xy^2 + y^3 = (y - \sqrt{3}x)(y + \sqrt{3}x)(y - x).$$

Thus, we arrive the following conclusion: $L_K(f) = 3$ if and only if $\mathbb{Q}(\sqrt{2}, \sqrt{3}) \subseteq K$ with the corresponding representation:

$$f(x, y) = (\sqrt{2} + \sqrt{3})(x - \sqrt{3}y)^5 + (\sqrt{2} - \sqrt{3})(x + \sqrt{3}y)^5 + (x + y)^5.$$

Notice that the above representation has two summands which are conjugates of each other under the conjugation with respect to $\sqrt{3}$ in $\mathbb{Q}(\sqrt{2})$ and a summand in $\mathbb{Q}(\sqrt{2})[x, y]$.

If the degree of a binary form is less than 5, then the Sylvester form of degree 3 does not need to be unique. There can be infinitely many representations of length 3.

**Example 5.4.** Let $f(x, y) = (x^2 + y^2)^2$, then by [15, Corollary 5.6] $L_K(f) = 3$ if and only if $\sqrt{3} \in K$ with the minimal representations

$$(x^2 + y^2)^2 = \frac{1}{18} \sum_{i=0}^{2} (\cos(\frac{i\pi}{3} + \theta)x + \sin(\frac{i\pi}{3} + \theta)y)^4, \quad \theta \in \mathbb{C}.$$

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