STANDARD FINITE ELEMENTS FOR THE NUMERICAL RESOLUTION OF THE ELLIPTIC MONGE-AMPÈRE EQUATION:
MIXED METHODS

GERARD AWANOU

Abstract. We prove the convergence of a mixed finite element method for the elliptic Monge-Ampère equation to its weak solution in the sense of Aleksandrov. The unknowns in the formulation are the scalar variable and the Hessian matrix.

1. Introduction

Let $\Omega$ be a convex polygonal domain of $\mathbb{R}^d$ for $d = 2, 3$ with boundary $\partial \Omega$. Let $f \in L^\infty(\Omega), g \in C(\partial \Omega)$ with $f \geq c_0 > 0$ for a constant $c_0 > 0$. We assume that $g$ can be extended to a function $\tilde{g} \in C(\overline{\Omega})$ which is convex in $\Omega$. We are interested in a mixed finite element method for the nonlinear elliptic Monge-Ampère equation: find a continuous convex function $u$ such that

$$
\det(D^2 u) = f \text{ in } \Omega \\
u = g \text{ on } \partial \Omega.
$$

(1.1)

The expression $\det D^2 u$ should be interpreted in the sense of Aleksandrov. We review the notion of Aleksandrov solutions in section 3.1. For $u \in C^2(\Omega)$, $\det D^2 u$ is the determinant of the Hessian matrix $D^2 u = \left( \frac{(\partial^2 u)}{(\partial x_i \partial x_j)} \right)_{i,j=1,\ldots,d}$.

We consider a mixed formulation with unknowns the scalar variable $u$ and the Hessian $D^2 u$. The scalar variable and the components of the Hessian are approximated by Lagrange elements of degree $k \geq d$. The method considered in this paper was analyzed from different point of views in [19] and [8] for smooth solutions of (1.1) and for $k \geq 3$. The case of quadratic elements in two dimension was handled in [6]. But the convergence of the discretization for non smooth solutions was not understood. That issue is resolved in this paper. We prove the uniform convergence on compact subsets of the discrete scalar variable to the Aleksandrov solution of (1.1).

For more 10 years, why mixed finite element type approximations of (1.1) indicate numerically convergence for non smooth solutions, has been a mystery for numerical analysts. This long standing open problem is resolved in this paper by regularizing the data and considering Monge-Ampère equations with smooth solutions on a uniformly convex smooth exhaustion of the domain. See section 3.

The author was partially supported by NSF DMS grant No 1319640.
Monge-Ampère equations arise in several applications of increasing importance, e.g., optimal transportation and reflector design. In fact, in optimal transportation problems, the approach through Aleksandrov solutions is more natural as it allows to treat discontinuous right hand sides.

In \[12, 15, 13, 18\], it was suggested that the issue could be approached through the notion of viscosity solution of \((1.1)\). Both notions of viscosity and Aleksandrov solutions for \((1.1)\) coincide for \(f > 0\) and continuous on \(\Omega \setminus \{0\}\). The investigation of numerical methods for \((1.1)\) through the notion of viscosity solutions is still an active research area.

A proven convergence method for \((1.1)\) through the notion of viscosity solution was obtained through monotone finite difference schemes, see for example \[14\]. For an approach through the notion of Aleksandrov solution for the two dimensional problem, we refer to \[20\]. The geometric approach taken in \[20\] is different from the approach taken in this paper. We use an analytical definition of the Aleksandrov solution of \((1.1)\), following \[21\], which is based on approximation by smooth functions. The techniques used in this paper were successfully implemented in the context of standard finite difference discretizations in \[5\] and in the context of standard finite element methods in \[3\]. The general methodology consists in

1- Prove the convergence of the discretization when \((1.1)\) has a smooth solution.
2- Verify that the numerical method is robust enough to handle the standard tests for non smooth solutions.
3- Consider a sequence of functions \(f_m, g_m \in C^\infty(\Omega)\) such that \(0 < c_1 \leq f_m\) for a constant \(c_1 > 0\), \(f_m\) converges uniformly to \(f\) on \(\Omega\) and \(g_m\) converges uniformly to \(\tilde{g}\) on \(\Omega\). Consider then the Monge-Ampère equations (with solutions not necessarily smooth)

\[ \det D^2 u_m = f_m \text{ in } \Omega, \quad u_m = g_m \text{ on } \partial \Omega. \]

It follows from \[23, \text{ Proposition 2.4 }\], see also \[3\] for details, that \(u_m\) converges to the Aleksandrov solution \(u\) of \((1.1)\) uniformly on compact subsets of \(\Omega\).

4- Consider a sequence of smooth uniformly convex domains \(\Omega_s\) increasing to \(\Omega\) \[9\], and the problems with smooth solutions \[24\]

\[ \det D^2 u_{ms} = f_m \text{ in } \Omega_s, \quad u_{ms} = g_m \text{ on } \partial \Omega_s. \]

Again from \[23, \text{ Proposition 2.4 }\], see also \[3\], \(u_{ms}\) converges uniformly on compact subsets of \(\Omega\) to \(u_m\) as \(s \to \infty\).

5- Establish that the discrete approximation \(u_{ms,h}\) of the smooth function \(u_{ms}\) converges uniformly to \(u_{ms}\) on \(\Omega_s\) as \(h \to 0\). This usually takes the form of an error estimate with constants depending on derivatives of \(u_{ms}\).

6- Use a subsequence argument to conclude that there exists a finite element function \(u_{m,h}\) such that \(u_{ms,h}\) converges uniformly on compact subsets of \(\Omega\) to \(u_{m,h}\) as \(s \to \infty\) and in turn \(u_{m,h}\) converges uniformly on compact subsets of \(\Omega\) to \(u_m\).

7- One must then show that \(u_{m,h}\) solve the discrete equations and \(u_{m,h}\) converges to \(u_m\) uniformly on compact subsets of \(\Omega\). For the mixed methods, an additional difficulty is to handle appropriately the discrete Hessian. One can then conclude from step 3 that there exists a subsequence \(u_{m_l,h_l}\) which converges
uniformly to \( u \) on compact subsets of \( \Omega \). Again a subsequence argument allows to conclude that there exists a finite element function \( u_h \), which has to be shown to solve the finite element equations, such that \( u_{m,h} \) converges uniformly on compact subsets of \( \Omega \) to \( u_h \) as \( m \to \infty \) and in turn \( u_h \) converges uniformly on compact subsets of \( \Omega \) to \( u \).

Given the above program, the main technical difficulties consist in completing steps 1 and 7. For steps 1 and 2, in this paper, we build on the work done in [19, 18, 8, 6]. For each type of discretization we have considered, step 7 requires new ideas.

Our focus on standard discretizations is motivated by the need to allow the efficient tools developed for computational mathematics such as adaptive mesh refinements and multigrid methods to be transferred seamlessly to the context of the Monge-Ampère equation. We expect that the general strategy of this paper can be adapted to the various mixed methods proposed in [12, 13, 19]. The main difficulty is an error analysis for smooth solutions of (1.1) with piecewise linear finite element approximations. We consider these and some lower order elements in [4].

We organize the paper as follows. In the second section we introduce some notation and preliminaries. In the last section we prove the convergence of the discretization for non smooth solutions.

2. Notation and Preliminaries

We use the usual notation \( L^p(\Omega), 2 \leq p \leq \infty \) for the Lebesgue spaces and \( H^s(\Omega), 1 \leq s < \infty \) for the Sobolev spaces of elements of \( L^2(\Omega) \) with weak derivatives of order less than or equal to \( s \) in \( L^2(\Omega) \). We recall that \( H^1_0(\Omega) \) is the subset of \( H^1(\Omega) \) of elements with vanishing trace on \( \partial \Omega \). We also recall that \( W^{s, \infty}(\Omega) \) is the Sobolev space of functions with weak derivatives of order less than or equal to \( s \) in \( L^\infty(\Omega) \). For a given normed space \( X \), we denote by \( X^d \) the space of vector fields with components in \( X \) and by \( X^{d\times d} \) the space of matrix fields with each component in \( X \).

The norm in \( X \) is denoted by \( ||.||_X \) and we omit the subscript \( \Omega \) and superscripts \( d \) and \( d \times d \) when it is clear from the context. The inner product in \( L^2(\Omega), L^2(\Omega)^d \), and \( L^2(\Omega)^{d\times d} \) is denoted by \( \langle , \rangle \) and we use \( \langle , \rangle \) for the inner product on \( L^2(\partial \Omega) \) and \( L^2(\partial \Omega)^d \). For inner products on subsets of \( \Omega \), we will simply append the subset notation. We will need the broken Sobolev norm

\[
||v||_{H^k(\mathcal{T}_h)} = \left( \sum_{K \in \mathcal{T}_h} ||v||^2_{H^k(K)} \right)^{\frac{1}{2}}.
\]

We denote by \( n \) the unit outward normal vector to \( \partial \Omega \). We recall that for a matrix \( A, A_{ij} \) denote its entries and the cofactor matrix of \( A \), denoted \( \text{cof} \ A \), is the matrix with entries \( (\text{cof} A)_{ij} = (-1)^{i+j} \text{det}(A)^{ij} \) where \( \text{det}(A)^{ij} \) is the determinant of the matrix obtained from \( A \) by deleting its \( i \)th row and its \( j \)th column. For two matrices \( A = (A_{ij}) \) and \( B = (B_{ij}) \), \( A : B = \sum_{i,j=1}^n A_{ij} B_{ij} \) denotes their Frobenius inner product. A quantity which is constant is simply denoted by \( C \).
For a scalar function \( v \) we denote by \( Dv \) its gradient vector and recall that \( D^2v \) denotes the Hessian matrix of second order derivatives. The divergence of a matrix field is understood as the vector obtained by taking the divergence of each row.

2.1. **Discrete variational problem.** We denote by \( \mathcal{T}_h \) a triangulation of \( \Omega \) into simplices \( K \) and assume that \( \mathcal{T}_h \) is quasi-uniform. We denote by \( V_h \) the standard Lagrange finite element space of degree \( k \geq d \) and denote by \( \Sigma_h \) the space of symmetric matrix fields with components in the Lagrange finite element space of degree \( k \geq d \).

Let \( I_h \) denote the standard Lagrange interpolation operator from \( C(\Omega) \), the space of continuous functions on \( \Omega \), into the space \( V_h \). We use as well the notation \( I_h \) for the matrix version of the Lagrange interpolation operator mapping \( C(\Omega)^{d \times d} \) into \( \Sigma_h \).

We consider the problem: find \( (u_h, \sigma_h) \in V_h \times \Sigma_h \) such that

\[
\begin{align*}
(\sigma_h, \tau) + (\text{div} \tau, Du_h) - \langle Du_h, \tau n \rangle &= 0, \forall \tau \in \Sigma_h, \\
(\det \sigma_h, v) &= (f, v), \forall v \in V_h \cap H^1_0(\Omega), \\
u_h &= g_h \text{ on } \partial \Omega,
\end{align*}
\]

where \( g_h = I_h \tilde{g} \).

2.2. **Properties of the Lagrange finite element spaces.** We recall some properties of the Lagrange finite element space of degree \( k \geq 1 \) that will be used in this paper. They can be found in [10]. We have

- **Interpolation error estimates.**
  \[
  ||v - I_h v||_{L^\infty} \leq C h^{k+1} |v|_{H^{k+1}}, \forall v \in H^{k+1}(\Omega).
  \]

- **Inverse inequalities**
  \[
  \begin{align*}
  ||v||_{L^\infty} &\leq C h^{-\frac{d}{2}} ||v||_{L^2}, \forall v \in V_h, \\
  ||v||_{H^1} &\leq C h^{-\frac{1}{2}} ||v||_{L^2}, \forall v \in V_h, \\
  ||v||_{W^{1,\infty}(\mathcal{T}_h)} &\leq C h^{-\frac{d}{2}} ||v||_{H^1}, \forall v \in V_h.
  \end{align*}
  \]

- **Scaled trace inequality**
  \[
  ||v||_{L^2(\partial \Omega)} \leq C h^{-\frac{d}{2}} ||v||_{L^2}, \forall v \in V_h.
  \]

2.3. **Error analysis of the mixed method for smooth solutions.** Let us assume that the unique convex solution \( u \) of \( (1.1) \) is in \( H^3(\Omega) \) and put \( \sigma = D^2u \). Then \( u \) satisfies the following mixed problem: find \( (u, \sigma) \in H^2(\Omega) \times H^1(\Omega)^{d \times d} \) such that

\[
\begin{align*}
(\sigma, \tau) + (\text{div} \tau, Du) - \langle Du, \tau n \rangle &= 0, \forall \tau \in H^1(\Omega)^{d \times d}, \\
(\det \sigma, v) &= (f, v), \forall v \in H^1_0(\Omega), \\
u &= g \text{ on } \partial \Omega.
\end{align*}
\]

It is proved in [8] that the above variational problem is well defined. We will assume without loss of generality that \( h \leq 1 \). We define for \( \rho > 0 \),

\[
B_h(\rho) = \{(w_h, \eta_h) \in V_h \times \Sigma_h, \|w_h - I_h u\|_{H^1} \leq \rho, \|\eta_h - I_h \sigma\|_{L^2} \leq h^{-1} \rho\}
\]
\[(2.8)\]

\[
Z_h = \{ (w_h, \eta_h) \in V_h \times \Sigma_h, w_h = g_h \text{ on } \partial \Omega, \]

\[
(\eta_h, \tau) + (\text{div } \tau, Dw_h) - (Dw_h, \tau \cdot n) = 0, \forall \tau \in \Sigma_h \} \text{ and }
\]

\[(2.9)\]

\[
B_h(\rho) = \bar{B}_h(\rho) \cap Z_h.
\]

We recall that, [8, Lemma 3.5], the ball \(B_h(\rho) \neq \emptyset\) for \(\rho = C_0 h^k\) for a constant \(C_0 > 0\). It follows from the analysis in [19, 8, 6] that (2.1) is well-posed for \(k \geq d\) and for \((u, \sigma) \in H^{k+3}(\Omega) \times H^{k+1}(\Omega)^{d \times d}\) we have the error estimates

\[
\begin{align*}
\|u - u_h\|_{H^1} &\leq C h^k, \\
\|\sigma - \sigma_h\|_{L^2} &\leq C h^{k-1}.
\end{align*}
\]

2.4. Algebra with matrix fields. We collect in the following lemma some properties of matrix fields, the proof of which can be found in [8, 2].

**Lemma 2.1.** We have for two matrix fields \(\eta\) and \(\tau\)

\[
\det \eta - \det \tau = \text{cof}(r \eta + (1 - r) \tau) : (\eta - \tau),
\]

for some \(r \in [0, 1]\).

For \(d = 2\) and \(d = 3\), and two matrix fields \(\eta\) and \(\tau\)

\[
\|\text{cof}(\eta) : \tau\|_{L^2} \leq C \|\eta\|_{L^\infty}^{d-1} \|\tau\|_{L^2},
\]

2.5. Continuity of the eigenvalues of a matrix as a function of its entries.

Let \(\lambda_1(A)\) and \(\lambda_2(A)\) denote the smallest and largest eigenvalues of the symmetric matrix \(A\). We have

**Lemma 2.2.** Assume that \(u \in C^2(\overline{\Omega})\). Then there exists constants \(r, R > 0\) independent of \(h\) and a constant \(C_{\text{conv}} > 0\) independent of \(h\) such that for all \(v_h \in V_h\) with \(v_h = g_h\) on \(\partial \Omega\) and \(v_h = I_h u\) on \(\partial \Omega\) and 

\[
\|v_h - I_h u\|_{H^1} < C_{\text{conv}} h^2,
\]

we have

\[
r \leq \lambda_1(D^2 v_h(x)) \leq \lambda_2(v_h(x)) \leq M, \forall x \in K, K \in T_h.
\]

**Proof.** The result is a consequence of the assumptions \(f \geq c_0 > 0\) and the continuity of the eigenvalues of a matrix as a function of its entries. See for example [7, Lemma 3.1].

3. Convergence of the discretization to the Aleksandrov solution

3.1. The Aleksandrov solution. We denote by \(K(\Omega)\) the cone of convex functions on \(\Omega\) and by \(B(\Omega)\) the space of Borel measures on \(\Omega\). Let \(M\) denote the mapping

\[
M : C^2(\Omega) \cap K(\Omega) \to B(\Omega)
\]

\[
M[v](B) = \int_B \det D^2 v(x) \, dx, \text{ for a Borel set } B.
\]

We equip \(K(\Omega)\) with the topology of compact convergence, i.e. for \(v_m, v \in K(\Omega), v_m\) converges to \(v\) if and only if \(v_m\) converges to \(v\) uniformly on compact subsets of \(\Omega\). We endow \(B(\Omega)\) with the topology of weak convergence of measures. Recall that
Definition 3.1. A sequence $\mu_m$ of Borel measures converges weakly to a Borel measure $\mu$ if and only if
\[ \int_{\Omega} p(x) \, d\mu_m \to \int_{\Omega} p(x) \, d\mu, \]
for every continuous function $p$ with compact support in $\Omega$.

If the measures $\mu_m$ have density $f_m$ with respect to the Lebesgue measure, and $\mu$ has density $f$ with respect to the Lebesgue measure, we have

Definition 3.2. Let $f_m, f \geq 0$. The sequence $f_m$ converges weakly to $f$ as measures if and only if
\[ \int_{\Omega} f_m p \, dx \to \int_{\Omega} f p \, dx, \]
for all continuous functions $p$ with compact support in $\Omega$.

It can be shown that the mapping $M$ extends uniquely to a continuous operator on $K(\Omega)$, [21, Proposition 3.1]. Thus we have

Lemma 3.3. Let $v_m$ be a sequence of convex functions in $\Omega$ such that $v_m \to v$ uniformly on compact subsets of $\Omega$. Then the associated Monge-Ampère measures $M[v_m]$ tend to $M[v]$ weakly.

We can now define the notion of Aleksandrov solution of (1.1).

Definition 3.4. A convex function $u \in C(\Omega)$ is an Aleksandrov solution of (1.1) if only if $u = g$ on $\partial \Omega$ and $M[u]$ has density $f$ with respect to the Lebesgue measure.

It is shown in [21, Proposition 3.4] that the extension of the mapping $M$ to $K(\Omega)$ coincides with the definition of Monge-Ampère measure as curvature measure. For the purposes of this paper, only the weak convergence result Lemma 3.3 and the definition of $M$ for $C^2$ functions will be needed. Thus we do not discuss further the notion of Aleksandrov measure. However we will need an existence and uniqueness result for (1.1).

Theorem 3.5 (Theorem 1.1 [17]). Let $\Omega$ be a bounded convex domain of $\mathbb{R}^d$ and assume that $g$ can be extended to a function $\tilde{g} \in C(\Omega)$ which is convex in $\Omega$. Then if $f \in L^1(\Omega)$, (1.1) has a unique convex Aleksandrov solution in $C(\Omega)$ which assumes the boundary condition in the classical sense.

3.2. Properties of convex functions. We will often use the following lemma

Lemma 3.6. A uniformly bounded sequence $u_j$ of convex functions on a convex domain $\Omega$ is locally uniformly equicontinuous and thus has a pointwise convergent subsequence.

Proof. For $p \in \partial u_j(x)$ and $x \in \Omega$, we have by [16, Lemma 3.2.1]
\[ |p_j| \leq \frac{|u_j(x)|}{d(x, \partial \Omega)} \leq \frac{C}{d(x, \partial \Omega)}, \]
for a constant $C$ independent of $j$. We conclude that the sequence $u_j$ is uniformly Lipschitz and hence equicontinuous on compact subsets of $\Omega$. The result then follows from the Arzela-Ascoli theorem.

\[ \square \]

We also note

**Lemma 3.7** ([22] Theorem 25.7). Let $\Omega$ be an open convex set, and let $u$ be a convex function which is finite and differentiable on $\Omega$. Let $u_m$ be a sequence of convex functions finite and differentiable on $\Omega$ such that $u_m$ converges pointwise to $u$ on $\Omega$. Then

\[ \lim_{m \to \infty} Du_m(x) = Du(x), \forall x \in \Omega. \]

Moreover, the mappings $Du_m$ converge to $Du$ uniformly on every closed bounded subset of $\Omega$.

### 3.3. Smooth and polygonal exhaustions of the domain.

Let $\Omega_s$ denote a sequence of smooth uniformly convex domains increasing to $\Omega$, i.e. $\Omega_s \subset \Omega_{s+1} \subset \Omega$ and $d(\partial \Omega_s, \partial \Omega) \to 0$ as $s \to \infty$. Here $d(\partial \Omega_s, \partial \Omega)$ denotes the distance between $\partial \Omega_s$ and $\partial \Omega$. The existence of the sequence $\Omega_s$ follows for example from the approach in [9].

We recall that $f_m$ and $g_m$ are $C^\infty(\overline{\Omega})$ functions such that $0 < c_2 \leq f_m \leq c_3, f_m \to f$ and $g_m \to \tilde{g}$ uniformly on $\overline{\Omega}$. It follows from [11] that the problem

\[ (3.1) \]

\[
\det D^2 u_{ms} = f_m \text{ in } \Omega_s
\]

\[ u_{ms} = g_m \text{ on } \partial \Omega_s, \]

has a unique convex solution $u_{ms} \in C^\infty(\overline{\Omega_s})$. It follows from [23, Proposition 2.4 ], see also [3] for details, that the sequence $u_{ms}$ converges uniformly on compact subsets of $\Omega$ to the unique convex solution $u_m \in C(\overline{\Omega})$ of the problem (2.7).

Let

\[ (3.2) \]

\[ \Omega_{sh} = \bigcup_{K \in \mathcal{T}_h} K \cap \Omega_s. \]

For simplicity we will assume that $\mathcal{T}_h$ consists of a conforming simplicial decomposition of $\Omega$, [10]. For elements $K$ such that $K \cap \partial \Omega_s \neq \emptyset, K \cap \Omega_s$ has a ”curved side”. However, since no computation will be done on $\Omega_s$, there is no need to introduce curved elements. By mapping $K, K \in \mathcal{T}_h$ to a reference element and using standard scaling arguments and the quasi uniformity assumption of $\mathcal{T}_h$, it is not difficult to see that the estimates of section 2.2 also hold on $\Omega_s$. Note also that as $s \to \infty, \Omega_{sh}$ is a sequence of convex domains increasing to $\Omega$ and $\Omega_{sh} = \Omega_s$.

Put $\sigma_{ms} = D^2 u_{ms}$. We consider the subdomain problem, analogue of (2.7): find $(u_{ms}, \sigma_{ms}) \in H^2(\Omega_s) \times H^1(\Omega_s)^{d \times d}$ such that

\[
(\sigma_{ms}, \tau) + (\text{div } \tau, Du_{ms}) - \langle Du_{ms}, \tau n_s \rangle = 0, \forall \tau \in H^1(\Omega_s)^{d \times d}
\]

\[
(\det \sigma_{ms}, v) = (f_m, v), \forall v \in H^1_0(\Omega_s)
\]

\[ u_{ms} = g_m \text{ on } \partial \Omega_s, \]

where we denote by $n_s$ the outward normal to $\partial \Omega_s$. 
We define \( V_{sh} = \{ v_h|_{\Omega_s}, v_h \in V_h \} \) and \( \Sigma_{sh} = \{ \tau_h|_{\Omega_s}, \tau_h \in \Sigma_h \} \). We then consider the analogue of (2.1): find \((u_{ms,h}, \sigma_{ms,h}) \) \( \in V_{sh} \times \Sigma_{sh} \) such that

\[
(s_{ms,h}, \tau) + (\text{div} \tau, Du_{ms,h}) - \langle Du_{ms,h}, \tau_n \rangle = 0, \forall \tau \in \Sigma_{sh}
\]

in \( V_{sh} \) and \( \Sigma_{sh} \). We have

\[
(\det \sigma_{ms,h}, v) = (f_m, v), \forall v \in V_{sh} \cap H^1(\Omega_s)
\]

and \( u_{ms,h} = g_{m,h} \) on \( \partial \Omega_s \).

It follows from the results of section 2.3 and the discussion above, that (3.3) has a solution which satisfies for sufficiently small

\[
||u_{ms} - u_{ms,h}||_{H^1} \leq C_{ms} h^k
\]

and

\[
||\sigma_{ms} - \sigma_{ms,h}||_{L^2} \leq C_{ms} h^{k-1},
\]

for a constant \( C_{ms} \). Moreover, by Lemma 2.2, \( u_{ms,h} \) is piecewise strictly convex.

### 3.4. Convergence of the discretization.

For a subset \( D \) of \( \Omega \), we denote by \(|D|\) the Lebesgue measure of \( D \). To be able to pass to the limit in (3.3), we need the following lemmas

**Lemma 3.8.** Let \( \Omega_s \) denote a smooth exhaustion of \( \Omega \) and assume that \( p_s \in L^2(\Omega) \) converges uniformly on compact subsets of \( \Omega \) to \( p \in L^2(\Omega) \). Then for \( v \in L^2(\Omega) \) we have

\[
\int_{\Omega_s} p_s v \, dx \to \int_{\Omega} pv \, dx, \text{ as } s \to \infty.
\]

**Proof.** We have

\[
\int_{\Omega} pv \, dx - \int_{\Omega_s} p_s v \, dx = \int_{\Omega_s} (p - p_s)v \, dx + \int_{\Omega \setminus \Omega_s} pv \, dx.
\]

Let \( \epsilon > 0 \) be given. Since \( \Omega_s \to \Omega \), there exists \( s_0 \) such that \( \forall s \geq s_0 \), we have \( |\Omega \setminus \Omega_s| < \epsilon \). On the compact set \( \Omega \setminus \Omega_{s_0} \), \( p_s \) converges to \( p \) uniformly. Thus, there exists \( s_1 \geq s_0 \) such that \( \forall s \geq s_1 \), \( ||p - p_s||_{L^\infty(\Omega_{s_0})} \leq \epsilon \). We then have for \( s \geq s_1 \)

\[
\left| \int_{\Omega} pv \, dx - \int_{\Omega_s} p_s v \, dx \right| \leq |\Omega \setminus \Omega_{s_0}| ||p||_{L^\infty} ||v||_{L^2} + \epsilon ||v||_{L^1},
\]

which proves the result.

**Lemma 3.9.** Let \( \Omega_s \) denote a smooth exhaustion of \( \Omega \) and assume that \( p_s, p \in W^{1,\infty}(\Omega) \) and \( D p_s \) converges uniformly on compact subsets of \( \Omega \) to \( D p \). Then for \( \tau \in L^\infty(\Omega)^{d \times d} \) we have

\[
\int_{\partial \Omega_s} (D p_s) \cdot (\tau n_s) \, dx \to \int_{\partial \Omega} (D p) \cdot (\tau n) \, dx, \text{ as } s \to \infty.
\]
Proof. We have

\[ \int_{\partial \Omega} (Dp) \cdot (\tau n) \, dx - \int_{\partial \Omega_s} (Dp_s) \cdot (\tau n_s) \, dx = \left( \int_{\partial \Omega} (Dp) \cdot (\tau n) \, dx - \int_{\partial \Omega_s} (Dp) \cdot (\tau n_s) \, dx \right) + \int_{\partial \Omega_s} D(p - p_s) \cdot (\tau n_s) \, dx. \]

Since \( \partial \Omega_s \to \partial \Omega \), we have

\[ \int_{\partial \Omega} (Dp) \cdot (\tau n) \, dx - \int_{\partial \Omega_s} (Dp) \cdot (\tau n_s) \, dx \to 0, \quad s \to \infty. \]

On the other hand, for \( s_0 \) fixed, \( \partial \Omega_s \subset \Omega \setminus \Omega_{s_0} \) and by assumption \( Dp_s \) converges uniformly to \( Dp \) on the compact subset \( K = \Omega \setminus \Omega_{s_0} \). It follows that

\[ \left| \int_{\partial \Omega_s} D(p - p_s) \cdot (\tau n_s) \, dx \right| \leq C|\partial \Omega_s| \| p - p_s \|_{W^{1, \infty}(\Omega)} \| \tau \|_{L^\infty(\Omega)}. \]

Since \( \Omega \) is bounded, \( |\partial \Omega_s| \leq C \) with \( C \) independent of \( s \) and this proves the result. \( \square \)

We can now state our main theorem

**Theorem 3.10.** Problem (2.1) has a local solution \((u_h, \sigma_h)\) for \( h \) sufficiently small, with \( u_h \) a piecewise convex function. Moreover \( u_h \) converges uniformly on compact subsets of \( \Omega \) to the Aleksandrov solution of (1.1).

**Proof.** It remains to complete steps 6 and 7 of the program outlined in the introduction.

**Part 1:** The subsequence argument for the existence of a limit \( u_{m,h} \).

By (2.4) and the inverse estimate (2.5), we have

\[ \| u_{ms} - u_{ms,h} \|_{W^{1, \infty}} \leq C_{ms} h^{\frac{d}{2}} \| u_{ms} - u_{ms,h} \|_{H^1} \leq C_{ms} h^{k - \frac{d}{2}}. \]

We recall that \( d = 2, 3 \) and \( k \geq 2 \). We conclude that for fixed \( m \) and \( s \), \( u_{ms,h} \) converges uniformly to \( u_{ms} \) on compact subsets of \( \Omega \) as \( h \to 0 \). Recall that for a fixed \( m \), \( u_{ms} \) converges uniformly to \( u_m \) on compact subsets of \( \Omega \) as \( s \to \infty \). Now, let \( K \) denotes a compact subset of \( \Omega \). There exists \( s_0 \) such that \( \forall s \geq s_0, K \subset \Omega_s \). For \( l \geq 1 \), choose \( s_l \geq s_0 \) such that

\[ |u_m(x) - u_{ms}(x)| \leq \frac{1}{2l}, \forall x \in K. \]

Choose also \( h_l \) such that for all \( h \leq h_l \),

\[ |u_{ms}(x) - u_{ms,h}(x)| \leq \frac{1}{2l}, \forall x \in K \cap \Omega_s. \]

We then obtain a subsequence \( u_{ms_l,h_l} \) which converges uniformly on \( K \) to \( u_m \) as \( l \to \infty \). The same arguments shows that any sequence \( u_{ms_l,h_l} \) has a further subsequence which converges uniformly on \( K \) to \( u_m \) as \( l \to \infty \). We conclude that for a fixed \( m \), \( u_{ms,h} \) converges uniformly on \( K \) to \( u_m \) as \( s \to \infty \) and \( h \to 0 \).

The above argument shows that if \( v_m \) is a sequence of piecewise convex functions which converges to \( v \) uniformly on compact subsets of \( \Omega \) and \( v_{ms} \) is a sequence of piecewise convex functions converging to \( v_m \) uniformly on compact subsets of \( \Omega \) as
$s \to \infty$, then the double sequence $v_{ms}$ converges to $v$ uniformly on compact subsets of $\Omega$ and there exists a function $v^s$ such that as $m \to \infty$ $v_{ms}$ converges to $v^s$ uniformly on compact subsets of $\Omega$ and in turn $v^s$ converges to $v$ uniformly on compact subsets of $\Omega$. We will refer to this argument as a subsequence argument.

We now assume that $h$ is fixed. For a fixed $m$, the family $u_{ms,h}$ is uniformly bounded on each compact subset of $\Omega$, since it is uniformly convergent. Thus on each compact subset, the sequence in $s$ of piecewise convex functions $u_{ms,h}$ is uniformly bounded, and hence by Lemma 3.6 has a convergent subsequence also denoted by $u_{ms,h}$ which converges pointwise to a function $u_{m,h}$. The argument has to be done in the interior on each element of $T_h$ and on inter elements where $u_{ms,h}$ is also piecewise convex. The function $u_{m,h}$ is piecewise convex as the pointwise limit of piecewise convex functions.

Next, we note that for a fixed $h$, $u_{ms,h}$ is a piecewise polynomial in the variable $x$ of fixed degree $k$ and convergence of polynomials is equivalent to convergence of their coefficients. Thus $u_{m,h}$ is a piecewise polynomial of degree $k$. Moreover, the continuity conditions on $u_{ms,h}$ are linear equations involving its coefficients. Thus $u_{m,h}$ has the same continuity property as $u_{ms,h}$. In other words $u_{m,h} \in V_h$.

Finally, since $u_{m,h}$ is a piecewise convex polynomial, it is continuous up to the boundary and thus we have on $\partial\Omega$, $u_{m,h} = I_h(u_m) = I_h(g_m)$.

Using the same subsequence argument above, it follows that for any sequence $h_l \to 0$, $u_{m,h_l}$ has a further subsequence which converges uniformly on compact subsets of $\Omega$ to $u_m$. We conclude the uniform convergence on compact subsets of $\Omega$ of $u_{m,h}$ to $u_m$ as $h \to 0$.

By the interpolation property (2.2), $I_h(g_m)$ converges uniformly to $g_m$ on $\partial\Omega$ as $h \to 0$. Therefore $g_{m,h} \to g_m$ uniformly on $\partial\Omega$.

**Part 2:** The subsequence argument for the existence of a limit $u_h$.

It is easily seen that for any family $u_{m,h_k}$ where $h_k \to 0$ as $k \to \infty$, any subsequence of $u_{m,h_k}$ has a further subsequence which converges uniformly on compact subsets of $\Omega$ to the solution $u$. Thus $u_{m,h}$ converges uniformly on compact subsets of $\Omega$ to $u$.

It follows, by Lemma 3.6, that there is a function $u_h$ such that as $m \to \infty$, $u_{m,h}$ converges to $u_h$ uniformly on compact subsets of $\Omega$ as $m \to \infty$.

Again, by the continuity of the eigenvalues of a matrix as a function of its Hessian, the limit $u_h$ is also piecewise convex.

By a subsequence argument as in Part 1, in turn $u_h$ converges uniformly on compact subsets of $\Omega$ to $u$ as $h \to 0$. Similarly $g_{m,h}$ converges as $m \to \infty$ pointwise to a function $g_h$ on $\partial\Omega$ which in turns converges to $g$ pointwise on $\partial\Omega$ as $h \to 0$. To see that $g_h = I_h g$ on $\partial\Omega$, recall that the coefficients of $g_h$ are obtained by solving an equation $Gx = b$, for an invertible matrix $G$ and where the coefficients of $b$ are obtained from the values of $g$ at the Lagrange points. The result then follows since $g_m$ converges to $g$ on $\partial\Omega$.

As in Part 1, $u_h \in V_h$.

**Part 3:** Equations solved by the limit functions $u_{m,h}$. 

We have for \( \tau \in \Sigma_h \)
\[
(\text{div} \tau, Du_{ms,h}) - \langle Du_{ms,h}, \tau n_s \rangle = \sum_{K \in T_h} \int_{K \cap \Omega_s} (Du_{mh,s}) \cdot \text{div} \tau \, dx \\
- \int_{K \cap \partial \Omega_s} (Du_{mh,s}) \cdot (\tau n_s) \, dx.
\]

Since \( u_{mh,s} \) is strictly convex on \( K \cap \Omega_s \) and converges uniformly on compact subsets of \( K \cap \Omega_s \) to \( u_{m,h} \), we have by Lemmas 3.8 and 3.9
\[
\int_{K \cap \Omega_s} (Du_{mh,s}) \cdot \text{div} \tau \, dx \to \int_{K} (Du_{m,h}) \cdot \text{div} \tau \, dx
\]
and
\[
\int_{K \cap \partial \Omega_s} (Du_{mh,s}) \cdot (\tau n_s) \, dx \to \int_{K \cap \partial \Omega} (Du_{m,h}) \cdot (\tau n) \, dx.
\]

We conclude that as \( s \to \infty \)
\[
(\text{div} \tau, Du_{ms,h}) - \langle Du_{ms,h}, \tau n_s \rangle \to F_{m,h}(\tau) := (\text{div} \tau, Du_{m,h}) - \langle Du_{m,h}, \tau n \rangle.
\]

For \( m \) and \( h \) fixed, we have by (2.4) and (2.6)
\[
|F_{m,h}(\tau)| \leq C(||\tau||_{H^1}||Du_{m,h}||_{L^2} + ||\tau||_{L^2(\partial \Omega)}||Du_{m,h}||_{L^2(\partial \Omega)})
\]
\[
\leq C(h^{-1}||Du_{m,h}||_{L^2} + h^{-\frac{1}{2}}||Du_{m,h}||_{L^2(\partial \Omega)}) ||\tau||_{L^2}.
\]

Thus \( F_{m,h} \) is continuous on \( \Sigma_h \) and by the Riesz representation theorem, there exists a unique \( \sigma_{m,h} \in \Sigma_h \) such that
\[
F_{m,h}(\tau) = (\sigma_{m,h}, \tau).
\]

In other words \( (u_{m,h}, \sigma_{m,h}) \in V_h \times \Sigma_h \) solves

\[
(\sigma_{m,h}, \tau) + (\text{div} \tau, Du_{m,h}) - \langle Du_{m,h}, \tau n \rangle = 0, \forall \tau \in \Sigma_h.
\]

It remains to show that
\[
(\det \sigma_{m,h}, v) = (f_m, v), \forall v \in V_h \cap H^1_0(\Omega).
\]

Since \( (\sigma_{ms,h}, \tau) \to (\sigma_{m,h}, \tau) \) as \( s \to \infty \) for all \( \tau \in \Sigma_h \), \( \sigma_{ms,h} \) converges weakly to \( \sigma_{m,h} \) and hence there exists a constant \( C_{m,h} \) independent of \( s \) such that
\[
||\sigma_{ms,h}||_{L^2} \leq C_{ms}.
\]

We first prove (3.7) for \( v \in V_h \) and with compact support in \( \Omega \). There exists \( t \) such that \( \text{supp} \ v \subset \Omega_s \) for all \( s \geq t \). By assumption \( \sigma_{ms,h} \in H^1(\Omega_t) \) and by (2.4)
\[
||\sigma_{ms,h}||_{H^1(\Omega_t)} \leq C h^{-1} ||\sigma_{ms,h}||_{L^2(\Omega_t)} \leq C_{ms} h^{-1}.
\]

Since \( \Omega_t \) has a smooth boundary, by the Rellich-Kondrachov theorem, the sequence in \( s \), \( \sigma_{ms,h} \) has a convergent subsequence, also denoted \( \sigma_{ms,h} \), in \( L^2(\Omega_t) \). If we denote by \( \sigma_{m,h} \) its limit we have
\[
(\hat{\sigma}_{m,h}, \tau) = (\sigma_{m,h}, \tau) \forall \tau \in \Sigma_h.
\]
We have by \((2.12)\) and \((3.9)\)
\[
\int_{\Omega} (\det \sigma_{m,h}) v \, dx = \int_{\Omega_s} (\det \sigma_{m,h}) v \, dx
\]
\[
= \int_{\Omega_s} (\det \sigma_{m,h} - \det \sigma_{ms,h}) v \, dx + \int_{\Omega_s} (\det \sigma_{ms,h}) v \, dx
\]
\[
= \int_{\Omega_s} (\det \sigma_{m,h} - \det \sigma_{ms,h}) v \, dx + \int_{\Omega_s} f_m v \, dx + \int_{\Omega_t} f_m v \, dx
\]
\[
= \int_{\Omega_t} \text{cof}(r \sigma_{m,h} + (1 - r) \sigma_{ms,h}) : (\sigma_{m,h} - \sigma_{ms,h}) \, dx + \int_{\Omega_t} f_m v \, dx
\]
\[
= \int_{\Omega_t} P_{\Sigma_h} (v \, \text{cof}(r \sigma_{m,h} + (1 - r) \sigma_{ms,h})) : (\sigma_{m,h} - \sigma_{ms,h}) \, dx + \int_{\Omega_t} f_m v \, dx
\]
\[
= \int_{\Omega_t} P_{\Sigma_h} (v \, \text{cof}(r \sigma_{m,h} + (1 - r) \sigma_{ms,h})) : (\sigma_{m,h} - \sigma_{ms,h}) \, dx + \int_{\Omega_t} f_m v \, dx,
\]
for some \(r \in [0, 1]\) and we denote by \(P_{\Sigma_h}\) the \(L^2\) projection into \(\Sigma_h\). We conclude using \((2.13), (2.3)\) and \((3.8)\)
\[
\left| \int_{\Omega} (\det \sigma_{m,h}) v \, dx - \int_{\Omega_t} f_m v \, dx \right| \leq C \|P_{\Sigma_h} (v \, \text{cof}(r \sigma_{m,h} + (1 - r) \sigma_{ms,h}))\|_{L^2}
\]
\[
\leq C \|v \, \text{cof}(r \sigma_{m,h} + (1 - r) \sigma_{ms,h})\|_{L^2} \|\sigma_{m,h} - \sigma_{ms,h}\|_{L^2}
\]
\[
\leq C \|v\|_{L^\infty} (\|\sigma_{m,h}\|_{L^\infty} + \|\sigma_{ms,h}\|_{L^\infty})^{d-1} \|\sigma_{m,h} - \sigma_{ms,h}\|_{L^2}
\]
\[
\leq C \|v\|_{L^\infty} (h^{-\frac{d}{2}} \|\sigma_{m,h}\|_{L^2} + Ch^{-\frac{d}{2}})^{d-1} \|\sigma_{m,h} - \sigma_{ms,h}\|_{L^2}.
\]
We recall that \(s \geq t\) and \(\|\sigma_{m,h} - \sigma_{ms,h}\|_{L^2} \to 0\) as \(s \to \infty\). Taking limits \(s, t \to \infty\), we obtain \((3.7)\) for \(v \in V_h\) and with compact support in \(\Omega\).

Finally let \(v \in V_h \cap H^1(\Omega)\) and choose \(v_l \in H^1(\Omega)\) such that \(\|v_l - v\|_{H^1} \to 0\) as \(l \to \infty\). Put \(G_{m,h} = (\det \sigma_{m,h})v - f_m\). We have using \(I_h v = v\)
\[
\int_{\Omega} G_{m,h} v \, dx = \int_{\Omega} G_{m,h} I_h v_l \, dx + \int_{\Omega} G_{m,h} (I_h (v - v_l)) - (v - v_l) \, dx + \int_{\Omega} G_{m,h} (v - v_l) \, dx
\]
\[
= 0 + \int_{\Omega} G_{m,h} (I_h (v - v_l)) - (v - v_l) \, dx + \int_{\Omega} G_{m,h} (v - v_l) \, dx.
\]
Thus
\[
\int_{\Omega} G_{m,h} v \, dx \leq C \|G_{m,h}\|_{L^2} \|I_h (v - v_l) - (v - v_l)\|_{L^2} + \|G_{m,h}\|_{L^2} \|v - v_l\|_{L^2}
\]
\[
\leq C \|G_{m,h}\|_{L^2} \|v - v_l\|_{H^1} \to 0 \text{ as } l \to \infty.
\]
This completes the proof of \((3.7)\).

**Part 4:** Equations solved by the limit functions \(u_h\).

Applying Lemmas \(3.8\) and \(3.9\) with \(\Omega_s = \Omega\) and with the same reasoning as above applied to \(u_{m,h}\), we obtain the existence of \(\sigma_h \in \Sigma_h\) such that the pair \((u_h, \sigma_h)\) solves
and thus
\[ C \] and there exists a constant \( C \) such that
\[ \|\sigma_{m,h}\|_{L^2} \leq C. \]
Using (2.4), we obtain the uniform boundedness of \( \sigma_{m,h} \) in \( H^1(\Omega) \). Since \( d = 2,3 \), by the compactness of the embedding of \( H^1(\Omega) \) into \( L^4(\Omega) \), up to a subsequence \( \sigma_{m,h} \) converges to a function \( \hat{\sigma}_h \) in \( L^4(\Omega) \). As in Part 3, we have \( \sigma_h = P_{\Sigma_h}(\hat{\sigma}_h) \). We have for some \( r \in [0,1] \)

\[
\int_{\Omega} (\det \sigma_{m,h} - \det \sigma_h) v \, dx = \int_{\Omega} \cof(r\sigma_{m,h} + (1-r)\sigma_h) : (\sigma_{m,h} - \sigma_h) v \, dx
= \int_{\Omega} P_{\Sigma_h}(v \cof(r\sigma_{m,h} + (1-r)\sigma_h)) : (\sigma_{m,h} - \sigma_h) v \, dx
= \int_{\Omega} P_{\Sigma_h}(v \cof(r\sigma_{m,h} + (1-r)\sigma_h)) : (\sigma_{m,h} - \hat{\sigma}_h) v \, dx
\]
and thus
\[
\left| \int_{\Omega} (\det \sigma_{m,h} - \det \sigma_h) v \, dx \right| \leq C \|v\|_{L^\infty} \|\cof(r\sigma_{m,h} + (1-r)\sigma_h)\|_{L^\infty} \|\sigma_{m,h} - \hat{\sigma}_h\|_{L^1}
\leq C \|v\|_{L^\infty} (\|\sigma_{m,h}\|_{L^\infty} + \|\sigma_h\|_{L^\infty})^{d-1} \|\sigma_{m,h} - \hat{\sigma}_h\|_{L^1}
\leq C \|v\|_{L^\infty} (h^{-\frac{d}{2}}\|\sigma_{m,h}\|_{L^2} + h^{-\frac{d}{2}}\|\sigma_h\|_{L^2})^{d-1} \|\sigma_{m,h} - \hat{\sigma}_h\|_{L^4}
\leq C \|v\|_{L^\infty} (Ch^{-\frac{d}{2}} + h^{-\frac{d}{2}}\|\sigma_h\|_{L^2})^{d-1} \|\sigma_{m,h} - \hat{\sigma}_h\|_{L^4}
\to 0 \text{ as } m \to \infty.
\]
On the other hand, we readily have \((f_m,v) \to (f,v)\) as \( m \to \infty \). It follows that (3.7) holds, i.e. since \( u_h = g_h \) on \( \partial \Omega \), the pair \((u_h,\sigma_h)\) solves (2.1). The proof is complete. \( \square \)

**Remark 3.11.** Let \( x_0 \in \Omega \). We may assume that the solution \( u_h \) is strictly convex by identifying \( u_h \) with \( u_h + \epsilon |x-x_0|^2 \), where \( |.| \) denotes the Euclidean norm in \( \mathbb{R}^d \) and \( \epsilon \) is taken to be close to machine precision. The arguments in [6] can then be repeated to show that the solution of (2.1) given in the previous theorem is locally unique. Using the notation of [6] for the 2D problem, one only needs to take the rescaling parameter \( \alpha \) equal to \( \nu h^{k+2} \).

**Remark 3.12.** The only reason we assume that \( k \geq 3 \) in three dimension is that the numerical solution in the case \( k = 2 \) is much closer to the Lagrange interpolant than what can be observed numerically using the approximation property of the Lagrange finite element spaces. The solution is to use a rescaled version of the equation. The same argument applies to the analysis in [7]. It also shows that one just have to use a rescaled version of the equation if one does not have a sufficiently close initial guess for the Newton’s iterations.

**Remark 3.13.** As pointed out in [3], a different approach consists in regularizing the exact solution [1]. Our technique for handling the discrete Hessian can be combined
with the approach which consists in a regularization of the exact solution as outlined in [3] section 7.3].

REFERENCES

[1] Awanou, G.: Smooth approximations of the Aleksandrov solution of the Monge-Ampère equation. To appear in Comm. Math. Sc., 2014
[2] Awanou, G.: Pseudo transient continuation and time marching methods for Monge-Ampère type equations (2013). http://homepages.math.uic.edu/~awanou/up.html
[3] Awanou, G.: Standard finite elements for the numerical resolution of the elliptic Monge-Ampère equation: Aleksandrov solutions (2013). http://homepages.math.uic.edu/~awanou/up.html
[4] Awanou, G.: Low order mixed finite element approximations of the Monge-Ampère equation (2014). Manuscript
[5] Awanou, G.: On standard finite difference discretizations of the elliptic Monge-Ampère equation (2014). http://homepages.math.uic.edu/~awanou/up.html
[6] Awanou, G.: Quadratic mixed finite element approximations of the Monge-Ampère equation in 2 D (2014). http://homepages.math.uic.edu/~awanou/up.html
[7] Awanou, G.: Standard finite elements for the numerical resolution of the elliptic Monge-Ampère equation: classical solutions (2014). To appear in IMA J. of Num. Analysis
[8] Awanou, G., Li, H.: Error analysis of a mixed finite element method for the Monge-Ampère equation. Int. J. Num. Analysis and Modeling 11, 745–761 (2014)
[9] Blocki, Z.: Smooth exhaustion functions in convex domains. Proc. Amer. Math. Soc. 125(2), 477–484 (1997)
[10] Brenner, S.C., Scott, L.R.: The mathematical theory of finite element methods, Texts in Applied Mathematics, vol. 15, second edn. Springer-Verlag, New York (2002)
[11] Caffarelli, L., Nirenberg, L., Spruck, J.: The Dirichlet problem for nonlinear second-order elliptic equations. I. Monge-Ampère equation. Comm. Pure Appl. Math. 37(3), 369–402 (1984)
[12] Dean, E.J., Glowinski, R.: Numerical solution of the two-dimensional elliptic Monge-Ampère equation with Dirichlet boundary conditions: an augmented Lagrangian approach. C. R. Math. Acad. Sci. Paris 336(9), 779–784 (2003)
[13] Feng, X., Neilan, M.: Error analysis for mixed finite element approximations of the fully nonlinear Monge-Ampère equation based on the vanishing moment method. SIAM J. Numer. Anal. 47(2), 1226–1250 (2009)
[14] Froese, B.D., Oberman, A.M.: Convergent filtered schemes for the Monge-Ampère partial differential equation. SIAM J. Numer. Anal. 51(1), 423–444 (2013)
[15] Glowinski, R.: Numerical methods for fully nonlinear elliptic equations. In: ICIAM 07—6th International Congress on Industrial and Applied Mathematics, pp. 155–192. Eur. Math. Soc., Zürich (2009)
[16] Gutiérrez, C.E.: The Monge-Ampère equation. Progress in Nonlinear Differential Equations and their Applications, 44. Birkhäuser Boston Inc., Boston, MA (2001)
[17] Hartenstine, D.: The Dirichlet problem for the Monge-Ampère equation in convex (but not strictly convex) domains. Electron. J. Differential Equations pp. No. 138, 9 pp. (electronic) (2006)
[18] Lakkis, O., Pryer, T.: A finite element method for nonlinear elliptic problems. SIAM J. Sci. Comput. 35(4), A2025–A2045 (2013)
[19] Neilan, M.: Finite element methods for fully nonlinear second order PDEs based on a discrete Hessian with applications to the Monge–Ampère equation. J. Comput. Appl. Math. 263, 351–369 (2014)
[20] Oliker, V.I., Prussner, L.D.: On the numerical solution of the equation $(\partial^2 z/\partial x^2)(\partial^2 z/\partial y^2) - ((\partial^2 z/\partial x\partial y))^2 = f$ and its discretizations. I. Numer. Math. 54(3), 271–293 (1988)
[21] Rauch, J., Taylor, B.A.: The Dirichlet problem for the multidimensional Monge-Ampère equation. Rocky Mountain J. Math. 7(2), 345–364 (1977)
[22] Rockafellar, R.T.: Convex analysis. Princeton Mathematical Series, No. 28. Princeton University Press, Princeton, N.J. (1970)
[23] Savin, O.: Pointwise $C^{2,\alpha}$ estimates at the boundary for the Monge-Ampère equation. J. Amer. Math. Soc. 26(1), 63–99 (2013)
[24] Trudinger, N.S., Wang, X.J.: Boundary regularity for the Monge-Ampère and affine maximal surface equations. Ann. of Math. (2) 167(3), 993–1028 (2008)

DEPARTMENT OF MATHEMATICS, STATISTICS, AND COMPUTER SCIENCE, M/C 249. UNIVERSITY OF ILLINOIS AT CHICAGO, CHICAGO, IL 60607-7045, USA

E-mail address: awanou@uic.edu

URL: http://www.math.uic.edu/~awanou