Higher-order behaviour of two-point current correlators

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Abstract Estimates of higher-order contributions for perturbative series in QCD, in view of their asymptotic nature, are delicate, though indispensable for a reliable error assessment in phenomenological applications. In this work, the Adler function and the scalar correlator are investigated, and models for Borel transforms of their perturbative series are constructed, which respect general constraints from the operator product expansion and the renormalisation group. As a novel ingredient, the QCD coupling is employed in the so-called $C$-scheme, which has certain advantages. For the Adler function, previous results obtained directly in the $\overline{\text{MS}}$ scheme are supported. Corresponding results for the scalar correlation function are new. It turns out that the substantially larger perturbative corrections for the scalar correlator in $\overline{\text{MS}}$ are dominantly due to this scheme choice, and can be largely reduced through more appropriate renormalisation schemes, which are easy to realise in the $C$-scheme.

1 Introduction

In quantum field theory, perturbative expansions in the coupling are generally assumed to be divergent, being asymptotic at best. For QCD this is backed up by studies of models and calculations in the large-$\beta_0$ approximation [1, 2]. In phenomenological applications, two-point current correlation functions, like the Adler function [3] or the scalar correlator, play an important role. Reliable estimates of uncertainties resulting from higher-order corrections necessitate a good understanding of the behaviour of those quantities at large perturbative order.

For an asymptotic expansion, within a radius of convergence, the Borel transform in the coupling turns out to be convergent. For this reason, models for the Borel transform which are based on physical constraints, like analyticity or the renormalisation group (RG), can be important tools for gaining a better insight into the large-order behaviour of basic QCD quantities, like the current correlators. For the case of the QCD Adler function such a model was first proposed in Ref. [4], and further investigated in more detail in ref. [5].

In this contribution, the basic form of Borel models for the Adler function, which are based on the renormalon structure, are reviewed. As a new ingredient, the presentation shall be explicated in the $C$-scheme [6, 7], which, as we shall see, has certain advantages. As a further novel addition, in this article also Borel models for the scalar correlation function shall be introduced and discussed.

To this end, in Sect. 2, the QCD coupling and quark mass in the $C$-scheme shall be defined. Section 3 then summarises our current knowledge of the perturbative expansions of Adler function and scalar correlator. Next, in Sect. 4, the general form of renormalon pole terms for the Borel transform of current correlators are derived from the RG and further ingredients, that are required for the phenomenological investigation, are introduced. Central results of the phenomenological applications are then presented in Sect. 5, and in Sect. 6, some concluding remarks summarise this presentation.

2 The $C$-scheme

To keep this article self-contained, in the next two subsections, useful relations for the QCD coupling as well as the quark mass in the $C$-scheme are collected.

2.1 QCD coupling in the $C$-scheme

The $C$-scheme coupling in QCD was introduced as well as applied to the vector correlator in Ref. [6], and will be employed in the following. A second application to the scalar correlation function was worked out in Ref. [7]. The defining relation for the $C$-scheme coupling $\hat{\alpha}_C$
takes the form
\[
\frac{1}{\beta_1} \ln \hat a_Q^C - \frac{\beta_2}{2} \beta_3 + \beta_4 \ln \frac{Q}{\Lambda^C} = \frac{1}{\beta_1} \ln \frac{Q}{\Lambda^C} + \beta_2 \ln \hat a_Q^C - \beta_1 \beta_2 \int_0^{a_Q^C} \frac{da}{\beta_2(a)}.
\]
where on the right-hand side \(a_Q^C \equiv a_Q^C(Q)/\pi\) is the QCD coupling in a particular renormalisation scheme RS, for example the \(\overline{\text{MS}}\) scheme [8], and
\[
\frac{1}{\beta(a)} \equiv \frac{1}{\beta_{\text{RS}}(a)} - \frac{1}{\beta_1 \beta_2}.
\]
In our conventions, the QCD \(\beta\)-function, together with its perturbative expansion, is defined by
\[
-Q^2 \frac{da_Q^C}{dQ} = \beta_Q^C(a_Q^C) = \beta_1 (a_Q^C)^2 + \beta_2 (a_Q^C)^3 + \beta_3 (a_Q^C)^4 + \cdots.
\]
Note that the first two \(\beta\)-coefficients \(\beta_1\) and \(\beta_2\) are independent of the renormalisation scheme. Next, \(A_{\text{RS}}\) denotes the \(\Lambda\)-parameter of QCD in the RS scheme. It coincides with the \(\Lambda\)-parameter of the \(a_Q^C\) coupling in the C-scheme with \(C = 0\), which in the following shall be abbreviated by \(a_Q \equiv a_Q^{C=0}\). Even though the C-scheme coupling \(a_Q^C\) also depends on the initial renormalisation scheme of \(a_Q^C\), for notational simplicity we refrain from making this dependence explicit. The respective dependence can always be compensated by a corresponding trivial shift in \(C\).

An explicit solution of \(a_Q^C\) as a function of the \(A\)-parameter can be found in terms of the Lambert W-function [9]. Defining
\[
A_C = A_{\text{RS}} e^{-C/2} \quad \text{and} \quad z = -\frac{\beta_1}{\beta_2} \exp\left(-\frac{\beta_1^2}{\beta_2^2} \ln \frac{Q}{A_C}\right) = -\frac{\beta_1}{\beta_2} \left(\frac{A_C}{Q}\right)^{\beta_1/\beta_2},
\]
one obtains
\[
\hat a_Q^C = -\frac{\beta_1}{\beta_2} W_0(z),
\]
with \(W(z)\) being a solution of the equation \(W \exp(W) = z\) and the index \(\text{“}-1\text{”}\) signifies the proper branch of \(W(z)\) to yield a physical coupling with the correct properties. For further details the reader is referred to Ref. [9].

From Eq. (1), renormalisation group equations with respect to both scale and scheme transformations can readily be derived. Simply taking the derivatives with respect to \(Q\) or \(C\), one obtains
\[
-Q^2 \frac{\beta_Q^C}{dQ} = \hat \beta(Q^C) = \frac{\beta_1 (Q^C)^2}{1 - \frac{\beta_2}{\beta_1} (Q^C)} = -2 \frac{d\hat a_Q^C}{dC},
\]
where \(\hat \beta(Q^C)\) is the simple, scheme-invariant \(\beta\)-function\(^\ddagger\) corresponding to the coupling \(a_Q^C\), which can be given in closed form. From Eq. (6), one also observes that a change in the renormalisation scheme is equivalent to an evolution in the renormalisation scale. A shift in the scale from \(Q\) to \(\mu\) can be compensated by a transformation in the scheme from \(C\) to \(C'\), satisfying the relation
\[
\frac{Q^2}{\mu^2} = e^{C-C'}.\]
In the \(C\)-scheme both transformations are interchangeable and completely equivalent.

This opens the possibility to arrive at the perturbative expansion in the \(C\)-scheme at arbitrary \(C\) by first computing the expansion in \(a_Q^C\) at \(C = 0\), and then employing the evolution equation to arrive at an arbitrary \(C\). Once again, this is completely analogous to the possibility of rederiving the scale logarithms from the RGE in the renormalisation scale. Then, from Eq. (1) the perturbative relation between the coupling \(a_Q^C\) in an arbitrary scheme, for example the \(\overline{\text{MS}}\) scheme, and \(a_Q\) is found to be
\[
a_Q^C = a_Q + \left(\frac{\beta_1}{\beta_2} - \frac{\beta_2}{\beta_1}\right) a_Q^3 + \left(\frac{\beta_4}{2 \beta_1} - \frac{\beta_3}{2 \beta_2}\right) a_Q^4 + \frac{\beta_5}{6 \beta_1^2} a_Q^5 + \frac{\beta_6}{6 \beta_1^2} a_Q^6 + \mathcal{O}(a_Q^7).
\]

The higher \(\beta\)-function coefficients \(\beta_n^\text{RS}\) with \(n \geq 3\) on the right-hand side are to be taken in the scheme corresponding to \(a_Q^C\). The relation between the coupling \(a_Q^C\) at arbitrary \(C\) and \(a_Q\) can be deduced by integrating the RGE (6), which results in the perturbative expansion
\[
a_Q = a_Q^C + \left(\frac{\beta_1}{2} C + \frac{\beta_2}{4} C^2\right) (a_Q^C)^3 + \left(\frac{\beta_3}{2 \beta_1} C + \frac{5 \beta_1 \beta_2}{8} C^2 + \frac{3 \beta_2}{8} C^3\right) (a_Q^C)^4 + \left(\frac{\beta_4}{2 \beta_1} C + \frac{9 \beta_1 \beta_2}{8} C^2 + \frac{13 \beta_2^2}{24} C^3 + \frac{\beta_3}{16} C^4\right) (a_Q^C)^5 + \mathcal{O}(a_Q^C)^6.
\]
\(\ddagger\) It only depends on the scheme-invariant \(\beta\)-function coefficients \(\beta_1\) and \(\beta_2\).
Equations (8) and (9) complete the set of relations which are needed in order to rewrite perturbative QCD expansions in terms of the $C$-scheme coupling $\hat{a}_Q^C$.

### 2.2 Quark mass in the $C$-scheme

In minimal subtraction schemes, analogously to the scale dependence, also the scheme dependence of the quark mass only originates from the QCD coupling. Hence, the scheme evolution of a generic running $C$-scheme quark mass $m_Q^C \equiv m^C(Q)$ is found to be

$$\frac{1}{m_Q^C} \frac{dm_Q^C}{dC} = \frac{d\hat{a}_Q^C}{dC} \frac{1}{m_Q^C} \frac{dm_Q^C}{dQ} = -\frac{1}{2} \frac{\hat{m}}{C} \frac{d}{d\hat{a}_Q^C} \left( \frac{\hat{m}}{C} \frac{d}{d\hat{a}_Q^C} \right).$$

(10)

The definition of the mass anomalous dimension, together with its first five coefficients in the $\overline{\text{MS}}$-scheme, is provided in appendix A. Because both scale and scheme dependences of the $C$-scheme coupling are given by the $\beta$-function, this dependence cancels. Hence, similarly to the scheme evolution of the coupling $\hat{a}_Q^C$ which is given by the $\beta$-function, scheme dependence of $m_Q^C$ is just governed by the quark-mass anomalous dimension, expressed as a function of $\hat{a}_Q^C$.

This entails the additional finding that in the $C$-scheme an expression involving the quark mass that is scale invariant, automatically also is scheme invariant. Let us demonstrate that explicitly for the case of the invariant quark mass $\hat{m}$, which is defined by

$$\hat{m} = m_Q^C \left[ \hat{a}_Q^C(Q) \right]^{-\hat{m} / \beta_1} \exp \left\{ \int_0^{\hat{a}_Q^C} \left[ \frac{\gamma_m^{(1)}}{\beta_1 \hat{a}_Q^C} - \frac{\gamma_m(\hat{a}_Q^C)}{\beta(\hat{a}_Q^C)} \right] \right\}.$$

(11)

Taking the derivative of $\hat{m}$ with respect to $C$, employing product and chain rules, yields

$$\frac{d\hat{m}}{dC} = \left\{ -\frac{1}{2} \frac{\hat{m}}{C} \frac{d}{d\hat{a}_Q^C} \left( \frac{\hat{m}}{C} \frac{d}{d\hat{a}_Q^C} \right) \right\} \frac{d\hat{a}_Q^C}{dC}.$$

(12)

which proves the claimed scheme independence of $\hat{m}$.

In order to relate the coefficients of the quark-mass anomalous dimension in the $C$-scheme and another minimal renormalisation scheme, for example $\overline{\text{MS}}$, we still have to fix the global normalisation of the quark mass. This can be done by assuming that the quark mass at a particular $C_m$ coincides with the running quark mass $m_Q$ of a generic quark flavour in the $\overline{\text{MS}}$-scheme, that is

$$m_Q^{\overline{\text{MS}}} \equiv m_Q^{C_m}.$$

(13)

This should always be possible since with a scheme evolution in $C$, any value of the quark mass can be reached. Now equating the invariant mass in the $\overline{\text{MS}}$-scheme and in the $C$-scheme, as well as employing the normalisation of eq. (13), we can extract the coefficients of the quark-mass anomalous dimension in the $C$-scheme. In particular, the first three coefficients are found to be:

$$\gamma_m^{(1)} = \gamma_m^{(1)}, \quad \gamma_m^{(2)} = \gamma_m^{(2)} + \frac{\beta_1}{2} \gamma_m^{(1)} C_m,$$

$$\gamma_m^{(3)} = \gamma_m^{(3)} + \left( \frac{\beta_2}{\beta_1} - \frac{\beta_2^2}{\beta_1^2} \right) \gamma_m^{(1)} + \frac{1}{2} \left( 2 \beta_2 \gamma_m^{(1)} + 2 \beta_1 \gamma_m^{(2)} \right) \times C_m + \frac{\beta_2^2}{4} \gamma_m^{(1)} C_m^2.$$

(14)

The first relation is evident as the leading coefficient of the mass anomalous dimension is scheme invariant. The renormalisation group coefficients without hat are taken to be in the $\overline{\text{MS}}$-scheme. Since the choice of $C_m$ is arbitrary, we can simplify the relations by assuming equality with the $\overline{\text{MS}}$ mass at $C_m = 0$. Analogously to the coupling, denoting the corresponding anomalous dimension with a bar, one obtains

$$\bar{\gamma}_m^{(1)} = \gamma_m^{(1)}, \quad \bar{\gamma}_m^{(2)} = \gamma_m^{(2)}, \quad \bar{\gamma}_m^{(3)} = \gamma_m^{(3)}.$$

(15)

Given the mass anomalous dimension together with the normalisation (13), the quark mass in the $C$-scheme is unambiguously defined.

### 3 QCD two-point correlation functions

In the following two sub-sections, our present knowledge of the perturbative expansions of the Adler function and the scalar correlation function shall be summarised.

#### 3.1 The vector two-point correlator

The vector correlation function $\Pi_{\mu\nu}(p)$ in momentum space is defined as

$$\Pi_{\mu\nu}(p) = i \int dx e^{ipx} \langle \Omega | T \{ j_\mu(x), j_\nu(0) \} | \Omega \rangle,$$

(16)

where $| \Omega \rangle$ denotes the physical QCD vacuum. In order to avoid complications with so-called singlet diagrams, arising from self-contractions of the quarks in the current, the normal-ordered current $j_\mu(x)$ is taken to be flavour non-diagonal with the particular choice

$$j_\mu(x) = : \bar{u}(x) \gamma_\mu s(x) :,$$

(17)
which for example plays a role in hadronic decays of the \( \tau \) lepton into strange final states. The correlator \( \Pi_{\mu \nu}(p) \) admits the Lorentz decomposition

\[
\Pi_{\mu \nu}(p) = (p_{\mu} p_{\nu} - g_{\mu \nu} p^2) \Pi^{(1)}(p^2) + p_{\mu} p_{\nu} \Pi^{(0)}(p^2) = (p_{\mu} p_{\nu} - g_{\mu \nu} p^2) \Pi^{(1+0)}(p^2) + g_{\mu \nu} p^2 \Pi^{(0)}(p^2),
\]

(18)

where the superscripts denote the components corresponding to angular momentum \( J = 1 \) (transversal) and \( J = 0 \) (longitudinal) in the hadronic rest frame. In the second way of writing eq. (18), the Lorentz-scalar correlators \( \Pi^{(1+0)}(p^2) \) and \( p^2 \Pi^{(0)}(p^2) \) are free from kinematical singularities at \( p^2 = 0 \). These are the correlation functions that will be investigated in the following.

The correlator \( \Pi^{(0)}(s) \) with \( s \equiv p^2 \) turns out to be proportional to the quark masses, and hence vanishes in the Adler function follows as

\[
\Pi^{(0)}(s) = \frac{N_c}{12\pi^2} \sum_{n=1}^{\infty} a_{n}^{(1)} \sum_{k=0}^{n+1} L_k \equiv \ln \frac{-s}{\mu^2},
\]

(19)

with \( a_{\mu} \equiv \alpha(\mu^2) \equiv \alpha_s(\mu)/\pi \) and \( \mu \) the renormalisation scale. \( \Pi^{(1+0)}(s) \) itself is not a physical quantity in the sense that it contains a renormalisation scale and scheme dependent subtraction constant. This subtraction constant can either be removed by taking the imaginary part which corresponds to the spectral function \( \rho^{(1+0)}(s) \equiv \Im \Pi^{(1+0)}(s + i0)/\pi \), or by taking a derivative with respect to \( s \), which leads to the Adler function

\[
D(s) = -s \frac{d}{ds} \Pi^{(1+0)}(s).
\]

(20)

Both, the spectral function and the Adler function are physical in the above mentioned sense. However, let us remark that the natural domain of the spectral function is for real and Minkowskian \( s > 0 \), while that of the Adler function is for Euclidean \( s < 0 \). Nonetheless, apart from the cut on the positive real axis, the Adler function can be continued into the whole complex \( s \)-plane, and the two functions are related by

\[
\rho^{(1+0)}(s) = \frac{1}{2\pi i} \int_{s+i0}^{s-i0} D(s') \frac{ds'}{s-s'},
\]

(21)

From Eq. (19), the general perturbative expansion of the Adler function follows as

\[
D_{PT}(s) = \frac{N_c}{12\pi^2} \sum_{n=0}^{\infty} a_{n}^{(1)} \sum_{k=1}^{n+1} k c_{n,k} L^{k-1}.
\]

(22)

In this expression, only the coefficients \( c_{n,1} \) have to be considered as independent. The coefficients \( c_{n,k} \) with \( k = 2, \ldots, n \) can be related to the \( c_{n,1} \) and \( \beta \)-function coefficients by means of the renormalisation group equation (RGE), while the coefficients \( c_{n,0} \) do not appear in measurable quantities and \( c_{n,n+1} = 0 \) for \( n \geq 1 \). Up to order \( \alpha_s^3 \), the respective RG constraints are provided in eq. (2.11) of Ref. [4].

Since the Adler function \( D(s) \) satisfies a homogeneous RGE, the logarithms in Eq. (22) can be summed with the choice \( \mu^2 = -s \equiv Q^2 \), leading to the simple expression

\[
D_{PT}(Q^2) = \frac{N_c}{12\pi^2} \sum_{n=0}^{\infty} c_{n,1} a_Q^n.
\]

(23)

where \( a_Q \equiv \alpha_s(Q)/\pi \). The independent coefficients \( c_{n,1} \) are known analytically up to order \( \alpha_s^4 \) \([10–12]\). At \( N_c = 3 \) and \( N_f = 3 \), in the \( \overline{\text{MS}} \)-scheme, they read:

\[
\begin{align*}
    c_{0,1} &= 1, \quad c_{1,1} = 1, \quad c_{2,1} = \frac{299}{24} - 9\zeta_3 = 1.640, \\
    c_{3,1} &= \frac{58057}{288} - \frac{779}{4} \zeta_3 + \frac{75}{2} \zeta_5 = 6.371, \\
    c_{4,1} &= \frac{78631453}{20736} - \frac{1704247}{432} \zeta_3 \\
    &+ \frac{4185}{8} \zeta_3 + \frac{34165}{96} \zeta_5 - \frac{1995}{16} \zeta_7 = 49.076.
\end{align*}
\]

(24)

As a final preparation for the subsequent analysis, employing Eq. (8), we rewrite the expansion of the Adler function in terms of the \( C \)-scheme coupling \( a_Q \) at \( C = 0 \). It will furthermore be convenient to introduce the reduced Adler function \( D_{PT}(a_Q) \) in the following way:

\[
D_{PT}(a_Q) \equiv 4\pi^2 D_{PT}(a_Q) - 1 = \sum_{n=1}^{\infty} c_{n,1} a_Q^n \equiv \sum_{n=1}^{\infty} \tilde{c}_{n,1} a_Q^n
\]

\[
= a_Q + 1.640 a_Q^2 + 7.682 a_Q^3 + 61.06 a_Q^4 + \cdots,
\]

(25)

which also defines the independent perturbative coefficients \( \tilde{c}_{n,1} \). Only the coefficients \( \tilde{c}_{3,1} \) and \( \tilde{c}_{4,1} \) turn out different from the \( \overline{\text{MS}} \) coefficients, and analytically as well as numerically read:

\[
\begin{align*}
    \tilde{c}_{3,1} &= \frac{262955}{1296} - \frac{779}{4} \zeta_3 + \frac{75}{2} \zeta_5 = 7.682, \\
    \tilde{c}_{4,1} &= \frac{357259199}{93312} - \frac{1713103}{432} \zeta_3 + \frac{4185}{8} \zeta_5 \\
    &+ \frac{34165}{96} \zeta_5 - \frac{1995}{16} \zeta_7 = 61.060.
\end{align*}
\]

(26)

The corresponding expansion in terms of the \( C \)-scheme coupling \( a_Q \) for arbitrary \( C \) can be obtained by inserting the relation (9) into Eq. (25). The resulting expres-
As the second correlation function, the scalar two-point correlator $\Psi(p^2)$ is introduced, which is defined by
\[
\Psi(p^2) \equiv i \int dx \, e^{ipx} \langle O(T \{ j(x) j^\dagger(0) \}) \rangle \Omega. \tag{27}
\]
For our application, the scalar current $j(x)$ is chosen to arise from the divergence of the vector current,
\[
j(x) = \partial^\mu :u(x)\gamma_\mu s(x): = i(m_u - m_s):\bar{u}(x)s(x) \tag{28}
\]
This choice has the advantage of an additional factor of the quark masses, which makes the currents $j(x)$ renormalisation group invariant (RGI).

The purely perturbative expansion of $\Psi(p^2)$ is known up to order $\alpha_s^4$ and takes the general form
\[
\Psi_{\text{PT}}(s) = -\frac{N_c}{8\pi^2} m_\mu^2 s \sum_{n=0}^\infty a_n^s \sum_{k=0}^{n+1} d_{n,k} L^k. \tag{29}
\]
To simplify the notation, we have introduced the generic mass factor $m_\mu$ which stands for the combination $(m_u(\mu) - m_s(\mu))$. The running quark masses and the QCD coupling are renormalised at the common scale $\mu$, which enters in the logarithm $L = \ln(-s/\mu^2)$. As a matter of principle, different scales could be introduced for the renormalisation of coupling and quark masses. We shall return to a discussion of this aspect below.

At each perturbative order $n$, the only independent coefficients $d_{n,k}$ are the $d_{n,1}$. The coefficients $d_{n,0}$ depend on the renormalisation prescription and do not contribute in physical quantities, while all remaining coefficients $d_{n,k}$ with $k > 1$ can again be obtained by means of the renormalisation group equation (RGE).

To order $\alpha_s^4$ they are listed in eq. (A.6) of Ref. [7]. The normalisation in Eq. (29) is chosen such that $d_{0,1} = 1$. Again, setting $N_c = 3$ and $N_f = 3$, as well as employing the $\overline{\text{MS}}$-scheme, the coefficients $d_{n,1}$ up to $O(\alpha_s^4)$ were found to be [14–16]:
\[
\begin{align*}
    d_{0,1} &= 1, \quad d_{1,1} = \frac{17}{3} = 5.6667, \\
    d_{2,1} &= \frac{9631}{144} - \frac{35}{2} \zeta_3 = 45.846, \\
    d_{3,1} &= \frac{4748953}{5184} - \frac{91519}{216} \zeta_3 - \frac{5}{2} \zeta_4 \\
    &\quad + \frac{715}{12} \zeta_5 = 465.85, \\
    d_{4,1} &= \frac{7055935615}{497664} - \frac{46217501}{5184} \zeta_3 \\
    &\quad + \frac{192155}{216} \zeta_3^2 - \frac{17455}{576} \zeta_4 \\
    &\quad + \frac{455725}{432} \zeta_5 - \frac{625}{48} \zeta_6 - \frac{52255}{256} \zeta_7 = 5588.7. \tag{30}
\end{align*}
\]

Like the vector correlator, the correlator $\Psi(s)$ itself is not related to a measurable quantity. It grows linearly with $s$ as $s$ tends to infinity, and hence has two unphysical subtraction constants which can be removed by taking two derivatives with respect to $s$, such that $\Psi''(s)$ is independent of the renormalisation prescription. Let us also remark that $\Psi(s)$ is related to $\Pi^{(0)}(s)$ through a Ward identity [17], leading to
\[
s \Pi^{(0)}(s) = \frac{1}{s} \left[ \Psi(s) - \Psi(0) \right]. \tag{31}
\]

Employing (29), the general perturbative expansion of $\Psi''(s)$ reads
\[
\Psi''_{\text{PT}}(s) = -\frac{N_c}{8\pi^2} \frac{m_\mu^2}{s} \sum_{n=0}^\infty a_n^s \sum_{k=1}^{n+1} d_{n,k} k \left[ L^{k-1} + (k-1) L^{k-2} \right]. \tag{32}
\]

Like for the Adler function, being a physical quantity, $\Psi''(s)$ satisfies a homogeneous RGE, and therefore the logarithms can be summed with the scale choice $\mu^2 = -s \equiv Q^2$, resulting in the compact expression
\[
\Psi''_{\text{PT}}(Q^2) = \frac{N_c}{8\pi^2} \frac{m_Q^2}{Q^2} \left\{ 1 + \sum_{n=1}^\infty (d_{n,1} + 2 d_{n,2}) a_n^Q \right\}. \tag{33}
\]

In this way, both the running quark mass as well as the running QCD coupling are to be evaluated at the renormalisation scale $Q$. The dependent coefficients $d_{n,2}$ can be calculated from the RGE. A closed expression is provided in Eq. (97) of appendix A, together with the coefficients of the QCD $\beta$-function and mass anomalous dimension. Numerically, at $N_f = 3$, the perturbative coefficients $d''_{n,1} \equiv d_{n,1} + 2d_{n,2}$ of Eq. (33) take the $\overline{\text{MS}}$ values
\[
\begin{align*}
    d''_{1,1} &= 3.6667, \quad d''_{2,1} = 14.179, \\
    d''_{3,1} &= 77.366, \quad d''_{4,1} = 511.83. \tag{34}
\end{align*}
\]

It is observed that the coefficients (34) for the physical correlator are substantially smaller than the $d_{n,1}$ of Eq. (30), but still much larger than the $c_{n,1}$ (24) for the Adler function.

For the ensuing investigation it will be advantageous to remove the running effects of the quark mass from...
the remaining perturbative series. This can be achieved by rewriting the running quark masses $m_q(\mu)$ in terms of RGI quark masses $\hat{m}_q$ of Eq. (11), this time expressed in terms of $\overline{\text{MS}}$ masses:

$$m_q(\mu) \equiv \hat{m}_q [\alpha_s(\mu)]^{\gamma_m(1)/\beta_1} \exp \left\{ \int_0^{\mu_0} \frac{d\alpha}{\beta(\alpha)} \left[ \frac{\gamma_m(\alpha)}{\beta(\alpha)} - \frac{\gamma_m(1)}{\beta_1} \right] \right\}.$$  

(35)

Accordingly, we define a modified perturbative expansion with new coefficients $r_n$,

$$\Psi_{\text{PT}}^\mu(Q^2) = \frac{N_c}{8\pi^2} \hat{m}_Q^2 [\alpha_s(Q)]^{2\gamma_m(1)/\beta_1} \left\{ 1 + \sum_{n=1}^{\infty} r_n a_Q^n \right\},$$

(36)

which now contain contributions from the exponential factor in Eq. (11). At $N_f = 3$ and in the $\overline{\text{MS}}$-scheme, the coefficients $r_n$ take the values

$$r_1 = \frac{442}{81} = 5.4568, \quad r_2 = \frac{2449021}{52488},$$

$$r_3 = \frac{24657869923}{51018336} - \frac{678901}{1944} \zeta_3,$$

$$+ \frac{18305}{324} \zeta_5 = 122.10,$$

$$r_4 = \frac{378986482023877}{66119763456} - \frac{21306070549}{37791360} \zeta_3 + \frac{601705}{648} \zeta_3^2,$$

$$+ \frac{445}{96} \zeta_4 + \frac{3836150}{6561} \zeta_5 - \frac{3285415}{20736} \zeta_7 = 748.09.$$  

(37)

The order $\alpha_s^4$ coefficient $r_4$ depends on quark-mass anomalous dimensions as well as $\beta$-function coefficients up to five-loops which for the convenience of the reader in our conventions have been collected in appendix A. Let us remark that the $\zeta_4$ term that is present in $d_{3,1}$ as well as $d^\prime_{4,1}$, and the $\zeta_6$ term being present in $d_{4,1}$ as well as $d^\prime_{3,1}$, have been cancelled by the additional contribution. The respective cancellation has also been observed in Ref. [18] for a related quantity.

As the last step, similarly to the preceding subsection, we reexpress the QCD coupling in terms of $\hat{\alpha}_s$. The perturbative expansion of $\Psi_{\text{PT}}^\mu$ then assumes the form

$$\Psi_{\text{PT}}^\mu(Q^2) = \frac{N_c}{8\pi^2} \hat{m}_Q^2 [\hat{\alpha}_s(Q)]^{2\gamma_m(1)/\beta_1} \left\{ 1 + \sum_{n=1}^{\infty} r_n \hat{a}_Q^n \right\},$$  

(38)

defining the coefficients $\hat{r}_n$, which take the particular values

$$\hat{r}_1 = \frac{442}{81} = 5.4568, \quad \hat{r}_2 = \frac{2510167}{52488},$$

$$\hat{r}_3 = \frac{12763567259}{25509168} - \frac{673561}{1944} \zeta_3,$$

$$+ \frac{18305}{324} \zeta_5 = 142.44,$$

$$\hat{r}_4 = \frac{49275071521973}{8264970432} - \frac{10679302931}{1889568} \zeta_3 + \frac{601705}{648} \zeta_3^2,$$

$$+ \frac{117947335}{3285415} \zeta_5 - \frac{3285415}{20736} \zeta_7 = 932.71.$$  

(39)

It is amusing to observe that now even the $\zeta_4$ term remaining in $r_4$ got cancelled by a corresponding contribution in $\beta_5$, originating from the global $\hat{\alpha}_s$ prefactor, such that only odd-integer $\zeta$-function contributions persist. Even though we have only provided results for $N_f = 3$, we have convinced ourselves that the cancellation of even $\zeta$ values is in fact independent of the number of flavours. This observation is discussed in substantially more detail and generalised to other quantities in Refs. [19–21]. In the following sections, we shall furthermore investigate rewriting the prefactor and the remaining perturbative expansion into a general C-scheme coupling $\hat{a}_Q^C$, which provides an interesting handle on reshuffling the series.

## 4 Borel transforms in the $\hat{\alpha}$ coupling

### 4.1 General renormalon-pole structure

This chapter contains an extension of the discussion performed in section 5 of ref. [4], but closely follows the material already presented there and in part in the review [2]. Consider the OPE of a physical quantity $D(Q)$ which is assumed to be defined such that it is dimensionless. Physical means that it does not depend on renormalisation scale nor renormalisation scheme. Particular examples would be the Adler function of Eq. (20) or the physical scalar correlator $\Psi''(s)$ of Eq. (32). The general structure of the OPE for $D(Q)$, expressed in the coupling $\hat{a}_Q$, can be written as

$$D(Q) = C_\downarrow(\hat{a}_Q) + \sum_{O_d} \hat{C}_{O_d}(\hat{a}_Q) \frac{\langle \hat{O}_d \rangle}{Q^{d}}.$$  

(40)

$C_\downarrow$ corresponds to the purely perturbative term, or unit operator, and the higher-dimensional contributions are expressed in terms of renormalisation group invariant

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3 For notational simplicity, in the ensuing discussion we have dropped the superscript $C$ in the C-scheme coupling $\hat{a}_Q^C$. 

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(RGI) operators $\hat{O}_d$. If at a given dimension a set of operators contributes, the basis should be chosen such that the leading-order anomalous dimension matrix is diagonal. The structure of dimension-6 operator contributions to vector and axialvector correlators is, for example, examined in Ref. [22].

Expressing the generic structure of the different contributions to $D(Q)$ requires some explanation. In general, a given correlation function may admit global powers $\delta$ of $a_Q$. As was already encountered above, for example $\delta$ is non-vanishing for the scalar correlation function with $\delta = 2\gamma^{(1)}_\eta / \beta_1$, see Eq. (38), or the scalar gluonium correlation function [23] (not discussed in this work) for which $\delta = 2$. Hence, we choose to factor out this global prefactor and write the perturbative part $C_1(\hat{a}_Q)$ as

$$C_1(\hat{a}_Q) = C_1^{(0)}[\hat{a}_Q]^\delta \left[ 1 + \tilde{C}_1(\hat{a}_Q) \right].$$  \hspace{1cm} (41)

Regarding the OPE, a generic higher-dimensional term takes the form

$$\tilde{C}_{O_d}(\hat{a}_Q) \frac{\langle \hat{O}_d \rangle}{Q^d} = C_{O_d}^{(0)} \times [\hat{a}_Q]^\delta [\hat{a}_Q]^{\gamma} \left[ 1 + \tilde{C}_{O_d}^{(1)} \hat{a}_Q + \tilde{C}_{O_d}^{(2)} \hat{a}_Q^2 + \cdots \right] \frac{\langle \hat{O}_d \rangle}{Q^d},$$  \hspace{1cm} (42)

where again we have explicitly factored out $[\hat{a}_Q]^\delta$ such that this factor can be considered global to the full correlation function. The remaining exponent $\eta$ in general receives contributions from explicit powers $[\hat{a}_Q]^\eta$ which may be present at leading order, and the leading-order anomalous dimensions $\gamma^{(1)}_{O_d}$ of the operators. Lastly, the scale-invariant operator $\hat{O}_d$ in Eq. (42) is defined by

$$\hat{O}_d \equiv O_d(\mu) \exp \left\{ - \int \frac{\gamma_{O_d}(\hat{a}_\mu)}{\beta(\hat{a}_\mu)} \, d\hat{a}_\mu \right\},$$  \hspace{1cm} (43)

where the anomalous dimension $\gamma_{O_d}$ of the operator $O_d$ is given by

$$- \mu \frac{d}{d\mu} O_d(\mu) \equiv \gamma_{O_d}(\hat{a}_\mu) O_d(\mu) = \left[ \gamma^{(1)}_{O_d} \hat{a}_\mu + \gamma^{(2)}_{O_d} \hat{a}_\mu^2 + \gamma^{(3)}_{O_d} \hat{a}_\mu^3 + \cdots \right] O_d(\mu).$$  \hspace{1cm} (44)

Since the operators (43) will only be needed up to a multiplicative factor, we do not have to specify the constant of integration, and without loss of generality, it can be assumed to be zero.

Next, we rewrite the $Q$-dependence of a higher-dimensional OPE contribution in terms of $\hat{a}_Q$. From Eq. (1), we find

$$\left( A^{RS} \right)^d = e^{-\frac{d}{\pi^2} a_Q^2} \hat{C}^{(1)} \left[ a_Q \right]^{-\frac{d}{\pi^2}}$$  \hspace{1cm} \text{or}

$$\left( A^{C} \right)^d = e^{-\frac{d}{\pi^2} a_Q^2} \left[ a_Q \right]^{-\frac{d}{\pi^2}},$$  \hspace{1cm} (45)

where in the second relation the $C$-dependence has been absorbed into the $A$-parameter according to the definition in Eq. (4). Employing the second equation, the $Q$-dependent part of the operator contribution reads

$$\tilde{C}_{O_d}(\hat{a}_Q) = \tilde{C}_{O_d}(\hat{a}_Q) \frac{e^{-\frac{d}{\pi^2} a_Q^2}}{\left( A^{C} \right)^d} \left[ a_Q \right]^{\eta} \left[ a_Q \right]^{-\frac{d}{\pi^2}} \times \left[ 1 + \tilde{C}_{O_d}^{(1)} \hat{a}_Q + \tilde{C}_{O_d}^{(2)} \hat{a}_Q^2 + \cdots \right].$$  \hspace{1cm} (46)

The aim is to compare the latter structure to potential exponentially suppressed terms in the perturbative part. To proceed we express $\tilde{C}_1(\hat{a}_Q)$, defined in Eq. (41), by means of a Borel transformation,

$$\tilde{C}_1(\hat{a}_Q) \equiv \int_0^\infty dt \, e^{-t/\hat{a}_Q} B[\tilde{C}_1](t)$$

$$= \frac{2}{\beta_1} \int_0^\infty du \, e^{-\frac{2u}{\pi^2} \beta} B[\tilde{C}_1](u),$$  \hspace{1cm} (47)

with $t = 2u/\beta_1$. If $\tilde{C}_1(\hat{a}_Q)$ admits the perturbative expansion

$$\tilde{C}_1(\hat{a}_Q) = \sum_{n=1}^\infty \tilde{C}_1^{(n)} \hat{a}_Q^n,$$

the expansion of the Borel transform $B[\tilde{C}_1](t)$ is given by

$$B[\tilde{C}_1](t) = \sum_{n=0}^\infty \tilde{C}_1^{(n+1)} t^n / n!.$$

To find the Borel integral (47) by the principal-value prescription, and employing eq. (A.8) of ref. [4], the
imaginary ambiguity corresponding to the Borel integral of $B(C_{1,p}^{IR}(u))$ is found to be:

$$\text{Im}\left[ C_{1,p}^{IR}(\hat{a}_Q) \right] = \pm \left( \frac{2}{\beta_1} \right)^\gamma \frac{d_p^{IR}}{\beta_1} \sin(\pi \nu) \Gamma(1-\gamma) e^{-\hat{a}_Q(\hat{a}_Q)^{1-\gamma}} \times \left[ 1 + b_1 \frac{\beta_1}{2} (\gamma-1) \hat{a}_Q + b_2 \frac{\beta_2}{3} (\gamma-1)(\gamma-2) \hat{a}_Q + \cdots \right].$$ (51)

Assuming that this ambiguity gets cancelled by a corresponding ambiguity in the definition of the operator matrix elements, as well as comparing Eqs. (50) and (51), one readily deduces:

$$p = \frac{d}{2}, \quad \gamma = 1 - \eta + 2p \frac{\beta_2}{\beta_1^2}, \quad b_1 = \frac{2\tilde{C}_1^{(1)}}{\beta_1(\gamma-1)}, \quad b_2 = \frac{4\tilde{C}_2^{(2)}}{\beta_2^2 (\gamma-1)(\gamma-2)}.$$ (52)

Taylor expanding the ansatz (50) in $u$ and performing the Borel integral term by term yields the perturbative series:

$$C_{1,p}^{IR}(\hat{a}_Q) = \frac{d_p^{IR}}{\beta^\gamma \Gamma(\nu)} \sum_{n=0}^{\infty} \Gamma(n+\gamma) \left( \frac{\beta_1}{2p} \right)^n (\hat{a}_Q)^{n+1} \times \left[ 1 + 2p \frac{\tilde{C}_1^{(1)}}{\beta_1 (n+\gamma-1)} + \left( \frac{2p}{\beta_1} \right)^2 \frac{\tilde{C}_2^{(2)}}{(n+\gamma-1)(n+\gamma-2)} + \mathcal{O} \left( \frac{1}{n^3} \right) \right].$$ (53)

Equation (53) expressed in the coupling $\hat{a}_Q$ extends the corresponding \overline{MS} coupling eq. (3.51) of ref. [2] to include terms of order $1/n^2$ in the large-order behaviour of the perturbative series.

### 4.2 Modified Borel transform

Employing a conventional definition of the Borel transformation like in Eq. (47) in full QCD entails that the Borel transform $B[\tilde{C}_1](t)$ has a non-trivial dependence on the renormalisation scheme for the coupling. In the past, this motivated the introduction of a so-called “modified” Borel transform [24,25], which is based on the QCD $\Lambda$-parameter, and which shares with it the simple transformation properties under scheme changes.

We concentrate here on a definition of the modified Borel transform which is inspired by the work of Brown, Yaffe and Zhai [24] (see also Ref. [26]), and takes the form

$$B[\tilde{C}_1](t) = \frac{1}{(1-t/R)^\gamma} \sum_{n=0}^{\infty} \frac{\Gamma(n+\gamma)}{n!} \left( \frac{t}{R} \right)^n,$$ (60)
4.3 Vector and scalar correlators in large-$\beta_0$

For a further discussion of the general structure of the Borel transform, we briefly digress to the large-$\beta_0$ approximation, in which the Borel transforms for Adler function and scalar correlator are known in closed form to all orders in perturbation theory. Furthermore, in large-$\beta_0$ we have the particular situation that the scheme dependence of the coupling only originates from the fermion loop, and hence for each renormalisation scheme RS a constant $\tilde{C}$ exists, such that

$$ \frac{1}{A_Q^{(0)}} = \frac{1}{\hat{a}_Q} = \frac{1}{a_Q^{(0)}} + \frac{\beta_0}{2} \frac{1}{a_Q^{(0)}} = \frac{1}{a_Q^{(0)}} - \frac{5}{3} \frac{\beta_0}{2} \left( \frac{A^{(0)}}{Q} \right) $$

(63)

is invariant \[2,7\]. $A_Q^{(0)}$ can therefore be considered a scheme-independent coupling in the large-$\beta_0$ approximation.

The conventional Borel transform (47) for the reduced Adler function of Eq. (25) with respect to $A_Q^{(0)}$,

$$ \hat{D}_{\beta_0}(Q^2) = \frac{2}{\beta_1} \int_0^\infty du \, e^{-2u/(\beta_0 A_Q^{(0)})} \hat{B}[\hat{D}_{\beta_0}](u), \quad (64) $$

has been calculated in Refs. \[27,28\], with the finding

$$ \hat{B}[\hat{D}_{\beta_0}](u) = \frac{8}{(2-u)} \sum_{k=2}^\infty \frac{(-1)^k k}{[k^2 - (1-u)]^2}. \quad (65) $$

Obviously, as both the correlator $\hat{D}_{\beta_0}(Q^2)$, as well as the coupling $A_Q^{(0)}$, are independent of the scheme, also the Borel transform $\hat{B}[\hat{D}_{\beta_0}](u)$ has to be scheme invariant, which is reflected in the explicit expression (65), and denoted by the hat on $B$.

Matters are a little more complicated for the scalar correlation function. For this case, the large-$\beta_0$ approximation has been investigated in detail in Ref. \[29\]. From these results, the Borel transform was then extracted explicitly in Ref. \[7\]. As can be observed from Eq. (3.16) of \[7\], besides the Borel integral an additional logarithmic term is present which depends on the scheme of the global coupling prefactor. The structure of this term, however, is precisely such that it gets cancelled when the prefactor is expressed in terms of the invariant coupling $A_Q^{(0)}$. At leading order in large-$\beta_0$ the physical scalar correlation function $\Psi ''(Q^2)$ can therefore be written in the following compact form:

$$ \Psi ''_{\beta_0}(Q^2) = \frac{N_c}{8\pi^2} \frac{m^2}{Q^2} \left( \pi A^{(0)}_{\beta_0} \right)^{2\gamma_m(1)/\beta_1} \left[ 1 + 2 \int_0^\infty du \, e^{-2u/(\beta_1 A^{(0)}_{\beta_0})} \hat{B}[\hat{\Psi }''_{\beta_0}](u) \right]. \quad (66) $$

The corresponding Borel transform $\hat{B}[\hat{\Psi }''_{\beta_0}](u)$ was found to be \[7,29\]

$$ \hat{B}[\hat{\Psi }''_{\beta_0}](u) = \frac{3}{2} C_F \left[ (1-u) G_D(u) - 1 \right], \quad (67) $$

with the function $G_D(u)$ given by

$$ G_D(u) = \frac{2}{1-u} - \frac{1}{2-u} + \frac{2}{3} \sum_{p=3}^\infty \frac{(-1)^p}{(p-u)^2} \sum_{p=1}^\infty \frac{(-1)^p}{(p+u)^2}, \quad (68) $$

explicitly displaying the infrared and ultraviolet renormalon poles at $u = 2, 3, 4, \ldots$ and $u = -1, -2, -3, \ldots$ respectively, in analogy to the case of the Adler function.

As is observed from Eq. (66) above, the physical scalar current correlation function in the large-$\beta_0$ approximation, $\Psi ''_{\beta_0}(Q^2)$, can only be written in the form proportional to $[1 + \text{“Borel integral”}]$, if the global coupling prefactor is expressed in terms of the invariant coupling $A_Q^{(0)}$. If this would not be done, besides the Borel integral, additional polynomial terms related to the anomalous dimensions of the involved currents would arise. As the reader can convince oneself with the help of the results of Ref. \[23\], this finding also holds true for the physical scalar gluonium correlator.

In full QCD, due to the substantially more complex structure, most probably the concept of a universal scheme-invariant coupling does not exist. Hence, generally the perturbative structure of a physical correlator,
expressed in terms of a Borel integral, reads
\[
C_1(\hat{a}_Q) \sim [\hat{a}_Q]^{\delta} \left[ 1 + \text{"Borel integral"} \right. \\
\left. + \text{"polynomial terms"} \right],
\]
where all involved components, coupling prefactor, Borel integral and polynomial terms, depend on the renormalisation scheme of the QCD coupling. Typically, as far as the perturbative expansion in the coupling is concerned, the Borel integral will lead to an asymptotic series, while presumably the polynomial contribution, even in full QCD, remains convergent.

As shall be discussed in more detail in the phenomenological section below, depending on the employed scheme, the contribution of the polynomial terms can be numerically significant, or even dominating over the contribution of the Borel integral, at least for low orders, before the asymptotic behaviour, governed by the renormalon singularities, sets in. This is for example the case for the scalar correlator in the MS-scheme or the C-scheme at \( C = 0 \). The task therefore will be to identify schemes for which the polynomial terms are small and the Borel integral provides the dominant contribution already for lower orders. It will turn out that the corresponding schemes lead to a QCD coupling close to the invariant coupling in large-\( \beta_0 \).

### 4.4 Gluon condensate contributions to correlation functions

In order to deduce the general renormalon structure of a given correlation function, we require the higher-dimensional operator corrections in the framework of the OPE. As we have seen from Eq. (52), an infrared renormalon pole at location \( u = p \) is related to operators of dimension \( d = 2p \). The lowest-dimensional gauge-invariant operator arising in the OPE of the correlators under consideration in this work is the gluon condensate. Therefore, we shall discuss those contributions for Adler function and scalar correlator in some detail.

As was also indicated in Sect. 4.1, it is convenient to work with an RGI basis of operators. Neglecting contributions from the quark condensate and quartic mass corrections which are irrelevant for the renormalon structure, the gluon condensate \( \langle O_{G^2} \rangle \equiv \langle G_{\mu\nu}^a G^{\mu\nu a} \rangle \) can be reexpressed in terms of the scale-invariant gluon condensate \( \langle \hat{O}_{G^2} \rangle \)

\[
\langle \hat{O}_{G^2} \rangle = \frac{\beta(aQ)}{\beta_1 aQ} \langle O_{G^2} \rangle \equiv \left( 1 + \frac{\beta_2}{\beta_1} aQ + \ldots \right) \langle a_Q G_{\mu\nu}^a G^{\mu\nu a} \rangle + \ldots.
\]

Employing the next-to-leading order result of Ref. [31] (eq. (4.7)) for the gluon-condensate contribution to the Adler function, and rewriting the QCD coupling in terms of the C-scheme coupling \( \hat{a}_Q \), the Wilson coefficient \( \hat{C}_G^2(Q^2) \) for the scale invariant gluon condensate is found to be

\[
\hat{C}_G^2(Q^2) = \frac{T_F}{3} \left[ 1 + \left( \frac{C_A}{2} - \frac{C_F}{4} - \frac{\beta_2}{\beta_1} \right) a_Q + \ldots \right].
\]

To compare renormalon pole residues in the next section below, it will be convenient to have available the Wilson coefficient function in the normalisation where the leading-order perturbative coefficient is taken out as a global factor, such that the OPE perturbatively starts with a “1”. Employing the parton model result

\[
C_1^{(0)D} = \frac{N_c}{12\pi^2},
\]
we finally obtain

\[
\hat{C}_G^2(Q^2) = \frac{2\pi^2}{N_c} \left[ 1 + \left( \frac{C_A}{2} - \frac{C_F}{4} - \frac{\beta_2}{\beta_1} \right) a_Q + \ldots \right],
\]

where \( T_F = 1/2 \) has been used.

Likewise, from the result of Ref. [31] for the scalar correlator (eq. (5.7)), and rewriting the QCD coupling in the global prefactor resulting from the running quark mass once again in terms of the coupling \( \hat{a}_Q \), the Wilson coefficient \( \hat{C}_G^{\psi''}(Q^2) \) reads

\[
\hat{C}_G^{\psi''}(Q^2) = \frac{T_F \hat{m}^2}{2 Q^2} \times \left[ a_s(Q) \right]^{2\gamma(1)} \left[ 1 + \hat{C}_G^{(1)\psi''} a_Q + \ldots \right],
\]

with

\[
\hat{C}_G^{(1)\psi''} = \frac{3}{2} C_A + \frac{3}{4} C_F - \frac{\beta_2}{\beta_1} + \left( \frac{3}{2} + C \right) \gamma_m^{(1)} \\
+ 2 \left( \frac{\gamma_m^{(2)}}{\beta_1} - \frac{\beta_2 \gamma_m^{(1)}}{\beta_1^2} \right).
\]

In this case, the ratio with the leading order perturbative result turns out to be

\[
\frac{\hat{C}_G^{\psi''}(Q^2)}{C_{\chi}^{(0)\psi''}} = \frac{2\pi^2}{N_c} \left[ 1 + \hat{C}_G^{(1)\psi''} a_Q + \ldots \right].
\]

It is observed that with \( 2\pi^2 / N_c \) at leading order, this ratio is the same for Adler function, Eq. (73), and scalar correlator, Eq. (76). As will be explained in the following section, this fact will imply that also the corresponding renormalon pole residues are identical.
4.5 Relation between renormalon-pole residues

In Sect. 4.1 above, we had assumed that an inherent ambiguity in the definition of the operators exists, such that they cancel against corresponding ambiguities in the resummation of the perturbative series. This lead to the relations of Eq. (52). Now we intend to investigate the cancellation of ambiguities a little further.

Employing the principal-value prescription for the integration over the renormalon singularities in the Borel sum, the ambiguity on the perturbative side should be purely imaginary. Without loss of generality, we can hence take the ambiguity in the operator $O_d$ to be of the form $\pm i\Delta_p^{IR}(\mathcal{X})^d$. To ensure cancellation of the two ambiguities, besides (52), we must then have the relation

$$C^{(0)}_{O_d} \Delta_p^{IR} = C^{(0)}_1 \left( \frac{2}{\beta_1} \right)^\gamma \sin(\pi \gamma) \Gamma(1 - \gamma) d_p^{IR},$$

or equivalently

$$\left( \frac{\beta_1}{2} \right)^\gamma \frac{\Delta_p^{IR}}{\sin(\pi \gamma) \Gamma(1 - \gamma)} = \frac{C^{(0)}_1}{C^{(0)}_{O_d}} d_p^{IR}. \tag{78}$$

The left-hand side of the last equation only depends on the operator considered. Thus it should be universal, that is, independent of the correlation function under investigation. Comparing two different correlators $D_A(Q)$ and $D_B(Q)$, we hence find

$$\frac{C^{(0)}_1(A)}{C^{(0)}_{O_d}(A)} d_p^{IR}(A) = \frac{C^{(0)}_1(B)}{C^{(0)}_{O_d}(B)} d_p^{IR}(B). \tag{79}$$

The relation (79) can be tested for different correlators that are available in the large-$\beta_0$ approximation. Let us begin with the Adler function. Employing the principal-value prescription for the $\hat{C}^{\alpha\beta\gamma} \cdots$ term in the Adler function, the ratio of leading-order coefficients was already calculated in Eq. (76) in the last sub-section, with the result

$$\frac{C^{(0)}_1}{C^{(0)}_{GG}} = \frac{N_c}{2\pi^2}, \tag{83}$$

identical to the one for the Adler function, for this reason the residue for the renormalon pole at $u = 2$ should coincide with Eq. (81). Inspection of Eq. (67) confirms that this is indeed the case. As an additional test, one can check that the relation (82) is also satisfied by the corresponding results for the scalar gluonium correlation function in the large-$\beta_0$ approximation [23].

5 Borel models

The aim in this section is to construct models for the Borel transform of the two correlation functions under investigation, along the lines of the work of Ref. [4], for the Adler function, but employing the $C$-scheme coupling $\hat{a}_Q$.

Let us begin with outlining the general philosophy of the Borel models. At large orders, the perturbative series for both correlators will be dominated by the renormalon lying closest to $u = 0$. In the cases at hand this is the leading UV renormalon with a pole at $u = -1$. At intermediate orders, low lying IR renormalons, the lowest one being at $u = 2$ related to the gluon condensate, and at $u = 3$ connected with dimension-6 operators, should provide significant contributions. Finally, at the very lowest orders, in general, renormalon dominance of only a few poles cannot yet be expected to be realised.

The relative importance of different renormalon contributions, however, also depends on the renormalisation scheme. Since the structure of the $u = 2$ IR renormalon is known best, it is desirable to work in a scheme where the last analytically known perturbative orders, typically the third and fourth, receive a sizeable contribution from this and perhaps the next renormalon. Below it shall be argued that this is achieved in the $\overline{\text{MS}}$ scheme, or the $C$-scheme with $C = 0$, which is close to $\overline{\text{MS}}$. For the scalar correlator, on the other hand, as was discussed in Sect. 4.3, a scheme close to the invariant one in large-$\beta_0$ should be chosen for the global $\alpha_s$-prefactor, in order to remove large polynomial contributions related to anomalous dimensions.

5.1 Borel model for the Adler function

Regarding the Borel model for the Adler function, we closely follow the work of Ref. [4], and the reader is referred to this article, in particular Sect. 6, for further details. The model for the Borel transform of $\tilde{C}_1(\hat{a}_Q)$ that we advocate here takes the form

$$\frac{C^{(0)}_1}{C^{(0)}_{GG}} d_2^{IR} = \frac{3}{8\pi^2} (N_c^2 - 1). \tag{82}$$
\[ B[\bar{C}_1](u) = B[\bar{C}_{1,2}^{IR}](u) + B[\bar{C}_{1,3}^{IR}](u) + B[\bar{C}_{1,1}^{UV}](u) + d_0^{PO}. \]  

The general structure of an IR renormalon pole at position \( p \) was provided in Eq. (50). In the spirit of Ref. [4], we have included the lowest lying IR renormalon pole at \( u = 2 \), for which most information is available, as well as the next-to-leading IR pole at \( u = 3 \), together with a UV pole at \( u = -1 \) which dominates the perturbative series at large orders. Regarding the polynomial contribution which should take care of very low orders, we have only included the constant \( d_0^{PO} \). In Ref. [4] it was found that this is sufficient to obtain compatible models, and it has the advantage that all unknown parameters, the three residues of the renormalon poles and the constant \( d_0^{PO} \) can be adjusted such that the four analytically known perturbative coefficients \( \bar{c}_{1,1} \) to \( \bar{c}_{4,1} \) of Eqs. (24) and (26) are reproduced.

Let us discuss the explicit structure of the included renormalon poles in some more detail. The OPE term corresponding to the gluon condensate was discussed in Sect. 4.4, and the Wilson coefficient for the Adler function was provided in Eq. (71). From this result we can deduce the parameters required in Eq. (52), namely \( \eta = 0 \) and the next-to-leading order correction \( b_1 \), hence fixing the \( u = 2 \) renormalon pole up to NLO apart from the residue \( d_2^{IR} \). For the IR pole at \( u = 3 \), matters are substantially more complicated because several dimension-6 operators, four-quark condensates and the triple-gluon condensate, contribute. A detailed discussion of the dimension-6 NLO Wilson coefficient for the Adler function was given in Ref. [22]. As it is impossible to include several poles with the corresponding anomalous dimensions, due to the many additional unknown residue parameters, we have decided to only include the strongest pole, that is, the one with the highest power in the exponent. For the dimension-6 four-quark operators, the contribution \( \eta = 1 \) is largely cancelled by the anomalous-dimension term, such that with \( \gamma \approx 3.1 \) the exponent is close to the one already employed in Ref. [4]. Still, it is much stronger than in the large-\( \beta_0 \) case where at most quadratic renormalon divergences are found. Since the \( u = 3 \) pole is only included in an effective sense, we have not incorporated a NLO correction, even though they are available. Finally, the general structure of a UV renormalon pole was provided in Ref. [2] and section 5 of Ref. [4]. Compared to the latter work we have also included the contribution of the anomalous dimension which yields the strongest pole, leading to \( \gamma \approx 1.6 \). However, because up fourth order the leading UV renormalon only has very little influence on the perturbative coefficients, the changes in our numerics turn out to be minor.

The last issue before being able to apply our model numerically is the question which renormalisation scheme is most adequate. In the \( C \)-scheme, this can be easily investigated by tuning the scheme-parameter \( C \). Performing this exercise, it is found that around \( C \approx -0.7 \) the second order coefficient \( \bar{c}_{2,1} \) turns negative as the first of the analytically known coefficients. This indicates that at such and even more negative \( C \), dominance of the leading UV renormalon sets in at rather low orders, because only the UV renormalons contribute in a sign-alternating fashion. On the other hand, if \( C \) is chosen positive and too large, the lowest IR renormalon at \( u = 2 \) does not yet dominate at the last known perturbative orders. We shall advocate here that the choice \( C = 0 \), being close to the MS scheme, appears most adequate. This will also be corroborated by our numerical analysis to which we turn next.

Adjusting the four unknown parameters of our model, the three renormalon-pole residues \( d_2^{IR}, d_3^{IR} \) and \( d_1^{UV} \), as well as the constant \( d_0^{PO} \), to the four known perturbative coefficients \( \bar{c}_{1,1} \) to \( \bar{c}_{4,1} \) of eqs. (24) and (26), we obtain

\[
\begin{align*}
d_2^{IR} &= 2.67, & d_3^{IR} &= -7.23, \\
d_1^{UV} &= -1.35 \times 10^{-2}, & d_0^{PO} &= 0.273.
\end{align*}
\]

The values of these four parameters turn out similar to the results of Ref. [4]. The Borel model now allows to predict still higher-order coefficients. To be conservative, in this work we restrict ourselves to only quote the next unknown, fifth order coefficient which assumes the value \( c_{5,1} = 310.4 \). The \( C \)-scheme result at \( C = 0 \) can be converted to an MS-scheme value with the finding \( c_{5,1} = 245.0 \). This \( c_{5,1} \) turns out only about 15% smaller than the corresponding coefficient advocated in Ref. [4], which might give an idea of the uncertainties induced by reasonable assumptions about the Borel model. Our result can be compared to other recent predictions for \( c_{5,1} \) from Padé approximants [32], \( c_{5,1} = 277 \pm 51 \), or conformal mappings [33], \( c_{5,1} = 287 \pm 40 \), finding good agreement. Still, to be substantially more conservative, in phenomenological applications the obtained \( c_{5,1} \) might be employed with 100% uncertainty as an estimate for yet unaccounted higher orders.

To conclude this sub-section, we investigate how large the contribution of the three renormalon poles is to a given perturbative coefficient. Taking the last included fourth order coefficient \( \bar{c}_{4,1} \), the contributions of the \( u = 2 \) and \( u = 3 \) IR poles, and the \( u = -1 \) UV pole, are respectively 122%, -34% and 12%. This nicely confirms that the coefficient is dominated by the lowest lying IR renormalon pole at \( u = 2 \), with a tolerable contribution of the next IR pole at \( u = 3 \), and a still small correction of the leading UV pole at \( u = -1 \), such that the hierarchy of contributions is realised as asserted above. The last observation provides further confidence in the applicability of models for the Borel transform of perturbative series as an estimator for as yet unknown higher orders.

### 5.2 Borel model for the scalar correlator

In Sect. 4.3 above, it was already observed that the generic form of a physical, RG invariant, two-point correlator, for which the initial current has an anomalous
dimension, like in the case of the scalar correlator, can suggestively be expressed as:

$$\Psi_{PT}(Q^2) \sim \left[ a_{Cm}^{C} \right]^{\delta} \left\{ 1 + \text{“Borel integral”} \right\} \left( a_{Q}^{C} \right) + \text{“polynomial terms”}(a_{Q}^C).$$

(87)

In writing Eq. (87), it is already indicated, that the global factor, and the terms in curly brackets, can be expressed in $C$-scheme couplings at different $C$, namely $C_m$ and $C_a$. The motivation behind this choice is as follows: if we aim at fitting a Borel model for the scalar correlator, we only have at our disposal the first four known perturbative coefficients. However, the central Borel model that was advocated for the Adler function contains four free parameters. Therefore, it is impossible to extract additional information on the polynomial contribution, and we have to try to make it as small as possible. This can be achieved by tuning the global scheme parameter $C_m$. In the large-$\beta_0$ approximation, for a particular choice of $C$, it was possible to make the polynomial contribution vanish, and then the global factor was expressed in terms of the invariant coupling $A_Q^{\infty}$. As presumably no universal, invariant coupling exists in full QCD, we can not hope to completely remove the polynomial contribution by varying $C_m$. Nonetheless, it should be possible to make it substantially smaller than the contribution from the Borel integral, such that we have a chance to extract a reasonable Borel model, also for the scalar correlation function.

In addition, in Sect. 4.5 we have seen that the combination (79) of Wilson coefficient functions and renormalon-pole residues is universal. Furthermore, the ratio of Wilson coefficient normalisations was even found identical for Adler function and scalar correlator in the case of the gluon condensate contribution. Hence, the residue of the gluon condensate renormalon pole is no longer a free parameter, but can be taken from the Borel model for the Adler function. This allows to introduce one additional free parameter which can be fixed from the four available perturbative coefficients. The idea now is to employ $C_m$ as this parameter, and to fix it together with the remaining three parameters of the Borel model, namely $d_{IR}^{C}$, $d_{UV}^{C}$ and $d_{PO}^{C}$. In this way, one might get an idea which range of scales is most compatible with a domination of the Borel integral contribution, and a negligible polynomial term.

Proceeding in this fashion, the central Borel model for the Adler function is also assumed for the Borel contribution of the scalar correlation function. It only remains to replace the NLO contribution for the gluon-condensate renormalon pole according to the results of the Wilson coefficient function of Sect. 4.4, and to fix a scheme choice for the coupling in the Borel integral. For simplicity, here $C_a = 0$ is taken, in analogy to the analysis for the Adler function. Adjusting all parameters such that the perturbative coefficients of Eq. (39) are reproduced, one obtains:

$$d_{2}^{IR} = 2.67, \quad d_{3}^{IR} = -43.69, \quad d_{1}^{UV} = -4.68 \cdot 10^{-2},$$

(88)$$d_{0}^{PO} = 0.400, \quad C_{m} = -1.603.$$  

(89)

Also inspecting the coefficients of the perturbative expansion within the curly brackets of Eq. (87), one finds

$$\Psi_{PT}(Q^2) \sim \left[ a_{Cm}^{C} \right]^{\delta} \left\{ 1 + 2.25 a_{Q} + 1.62 a_{Q}^{2} + 2.72 a_{Q}^{3} + 72.4 a_{Q}^{4} + 155 a_{Q}^{5} + \cdots \right\}. \quad (90)$$

Several observations can be made on the basis of these results. First of all it is interesting to note that the preferred value of $C_m$ is rather close to $C_m = -5/3$, in large-$\beta_0$ leading to the invariant coupling $A_Q^{\infty}$. This suggests that a large fraction of the perturbative correction to the scalar correlator in the $\overline{\text{MS}}$ scheme is due to the polynomial contribution, and can be resumed into the prefactor through an appropriate scheme choice. Then, the behaviour of the remaining perturbative corrections, that dominantly originate from the Borel integral, are much more “Adler function”-like.

Regarding the parameters of the Borel model, the constant $d_{0}^{PO}$ turns out small which indicates that even the lowest order correction already receives a substantial contribution from the renormalon poles. The residue of the $u = 3$ term is found very different from the Adler function. On the one hand, however, the dimension-6 contributions in the scalar correlator are different from the Adler function, and furthermore this pole also effectively parametrises all missing IR renormalon poles that are not included in the model. Therefore, no conclusions can be drawn on the basis of this residue. Finally, the residue of the leading UV pole, $d_{1}^{UV}$ is found larger than the one for the Adler function, which entails that the scalar correlator reaches its asymptotic behaviour at lower orders than the Adler function. In fact, investigating even higher orders to Eq. (90), it is found that the 7th order term is negative, while for the Adler function series in the $C = 0$ scheme the first negative coefficient is found at the 9th order.

As a prediction of the model, we are now in a position to compute the fifth order coefficient $d_{5,1}^{C}$ to the scalar correlation function in the $\overline{\text{MS}}$ scheme, which supplements Eq. (34). Performing all necessary scheme transformations, one arrives at

$$d_{5,1}^{C} = 3201 - 9.88 \times 10^{-3} \beta_0 - 8.89 \times 10^{-2} \gamma_6.$$  

(91)
which, however, also depends on the as yet unknown RG coefficients $\beta_0$ and $\gamma_6$. Therefore, in order to make definite predictions for the perturbative correction of the scalar correlator at the fifth order, the 6th order RG coefficients are required as well. Also making use of the dependent coefficient $d_{5,2} = -36913.5$, which can be calculated from Eq. (97), one finally finds the Borel model expectation

$$d_{5,1} = 77028 - 9.88 \times 10^{-3} \beta_0 - 8.89 \times 10^{-2} \gamma_0.$$  

(92)

Hence, we see that the $\overline{\text{MS}}$ coefficient $d_{5,1}$ is largely dominated by the contribution from $d_{5,2}$. A more detailed investigation of Borel models for the scalar correlator, and other two-point correlation functions based on currents with non-vanishing anomalous dimension, will be relegated to a forthcoming publication in the future [34].

### 6 Summary

Information on the higher-order behaviour of perturbative series is important in order to obtain reliable error estimates for phenomenological predictions. As the perturbative expansions in full QCD most probably only lead to asymptotic series, its Borel transforms, which have convergent expansions, are interesting objects to be studied. Making use of constraints originating from the operator product expansion and the renormalisation group, simple models for the Borel transforms of two-point correlation functions in QCD can be written down.

In this work, a Borel model for the vector correlation function, or Adler function, that has already been presented in the literature [4], was reviewed. A novel ingredient here is that the Borel model has been expressed in terms of the so-called $C$-scheme coupling [6]. The $C$-scheme coupling has several advantages: on the one hand, the $\beta$-function of the $C$-scheme coupling only depends on the invariant coefficients $\beta_1$ and $\beta_2$, and assumes a simple geometric form. Hence it is known to all orders in perturbation theory. For this reason, the general form of a certain renormalon pole in the $C$-scheme turns out simpler than for example in the $\overline{\text{MS}}$ scheme. Finally, variations of the renormalisation group scheme are easily realised through changes in the scheme parameter $C$.

On the phenomenological side, the Borel model allows to predict yet unknown higher-order coefficients in the perturbative series. To be conservative, in this article only an estimate of the next, fifth-order coefficient has been presented. For the Adler function this was discussed in detail in Sect. 5.1, with the result $c_{5,1} = 310.4$ in the $C$-scheme. Transforming this value back into the $\overline{\text{MS}}$ scheme, one obtains $c_{5,1} = 245.0$, which may be confronted by future analytical computations. It turns out similar to the results already obtained in Refs. [4,32,33] directly in the $\overline{\text{MS}}$ scheme, thereby providing support to these analyses.

As a second two-point correlator, the scalar correlation function was investigated, and a Borel model for its perturbative series has been presented in Sect. 5.2. The corresponding construction is new and has as yet not been presented elsewhere. Due to the fact that the scalar current carries an anomalous dimension, its structure is substantially more complicated. It turns out that the global factor of $\alpha_s$ entails that, besides the Borel integral, additional contributions are present that depend on the anomalous dimension of the initial current. Depending on the scheme choice, these contributions can be large, even dominating the series, which seems to be the case for the scalar correlator in the $\overline{\text{MS}}$ scheme. Extracting a Borel model necessitates that the additional, polynomial contribution is largely removed by a scheme change. Interestingly enough, the required scheme turns out close to the invariant coupling in large-$\beta_0$. On the phenomenological side, in Eq. (92) an estimate was also given for the unknown, fifth-order coefficient $d_{5,1}$ of the scalar correlator. This prediction, however, depends on the so far uncalkulated RG coefficients $\beta_5$ and $\gamma_6$, so that it remains unclear, when it might be tested against a fully analytical calculation of the fifth perturbative order for the scalar correlator.

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### Appendix A: Renormalisation group functions and dependent coefficients

In our notation, the QCD $\beta$-function and mass anomalous dimension are defined as:

$$-\mu \frac{da}{d\mu} \equiv \beta(a) = \beta_1 a^2 + \beta_2 a^3 + \beta_3 a^4 + \beta_4 a^5 + \cdots,$$

(93)

$$-\mu \frac{dm}{d\mu} \equiv \gamma_m(a) = \gamma_m^{(1)} a + \gamma_m^{(2)} a^2 + \gamma_m^{(3)} a^3 + \gamma_m^{(4)} a^4 + \cdots.$$  

(94)

It is assumed that we work in a mass-independent renormalisation scheme and in this study throughout the modified minimal subtraction scheme $\overline{\text{MS}}$ is used. To make the presentation self-contained, below the known coefficients of the $\beta$-function and mass anomalous dimension in the given conventions shall be provided. Numerically, for $N_c = 3$ and $N_f = 3$, the first five coefficients of the $\beta$-function are given

$$\beta_0 = -2.8442 \quad \text{and} \quad \beta_1 = 13.7547 \quad \text{and} \quad \beta_2 = 53.7326 \quad \text{and} \quad \beta_3 = 143.0790 \quad \text{and} \quad \beta_4 = 369.1345.$$
by [35–38]
\[
\beta_1 = \frac{9}{2}, \quad \beta_2 = 8, \quad \beta_3 = \frac{3863}{192},
\]
\[
\beta_4 = \frac{140599}{2304} + \frac{445}{16} \zeta_3,
\]
\[
\beta_5 = \frac{139857733}{603552} + \frac{11059213}{534385} \zeta_4 + \frac{1512}{1336} \zeta_5,
\]
(95)
and the first five for \( \gamma_m(a) \) are found to be [39, 40]
\[
\gamma_m^{(1)} = 2, \quad \gamma_m^{(2)} = \frac{91}{12}, \quad \gamma_m^{(3)} = \frac{8885}{288} - 5 \zeta_3,
\]
\[
\gamma_m^{(4)} = \frac{2977517}{20736} - \frac{9295}{216} \zeta_3 + \frac{135}{8} \zeta_4 - \frac{125}{6} \zeta_5,
\]
\[
\gamma_m^{(5)} = \frac{156500815}{248832} - \frac{23663747}{62208} \zeta_3 + \frac{170}{128} \zeta_4 + \frac{23765}{1875} \zeta_5,
\]
\[
= \frac{22625464}{31104} \zeta_5 + \frac{1875}{16} \zeta_6 + \frac{118405}{288} \zeta_7.
\]
(96)

The dependent perturbative coefficients \( d_{n,k} \) with \( k > 1 \) can be expressed in terms of the independent coefficients \( d_{n,1} \), and coefficients of the QCD \( \beta \)-function and mass anomalous dimension. In particular, the coefficients \( d_{n,2} \), which are required in Eq. (33), take the form
\[
d_{n,2} = -\frac{1}{2} \gamma_m^{(n)} d_{0,1} - \frac{1}{4} \sum_{k=1}^{n-1} (2 \gamma_m^{(n-k)} + k \beta_{n-k}) d_{k,1}.
\]
(97)

**Appendix B: General scheme-invariant structure**

In this appendix, the general scheme-invariant structure of a two-point correlation function in the \( C \)-scheme will be provided, which for example has to be obeyed by the polynomial contribution in Eq. (69). Denoting the structure by \( P(\hat{a}_Q) \), it takes the general form
\[
P(\hat{a}_Q) = [\hat{a}_Q]^\delta \left\{ 1 + \sum_{n=1}^{\infty} (\hat{a}_Q)^n \sum_{k=0}^{n} y_{n,k} \hat{C}^k \right\},
\]
(98)
where \( \hat{C} \equiv \beta_3/2C \). Relations between the coefficients \( y_{n,k} \) can be obtained from the RG equation (6). Up to order \( \hat{a}_Q^2 \), and setting \( \lambda = \beta_2/\beta_1 \), those relations read:
\[
y_{1,1} = \delta, \quad y_{2,2} = \frac{\delta}{2} (\delta + 1),
\]
\[
y_{2,1} = (\delta + 1) y_{1,0} + \lambda \delta,
\]
\[
y_{3,3} = \frac{\delta}{6} (\delta + 1) (\delta + 2), \quad y_{3,2} = \frac{1}{2} (\delta + 1) (\delta + 2) y_{1,0} + \lambda \delta (2 \delta + 3),
\]
\[
y_{3,1} = (\delta + 2) y_{2,0} + \lambda (\delta + 1) y_{1,0} + \lambda^2 \delta,
\]
\[
y_{4,4} = \frac{\delta}{24} (\delta + 1) (\delta + 2) (\delta + 3),
\]
\[
y_{4,3} = \frac{1}{6} \left[ (\delta + 1) (\delta + 2) (\delta + 3) y_{1,0} + \lambda (3 \delta^2 + 12 \delta + 11) \right],
\]
y_{4,2} = \frac{1}{6} \left[ (\delta + 2) (\delta + 3) y_{2,0} + \lambda (\delta + 1) (2 \delta + 5) \times y_{1,0} + 3 \lambda^2 \delta (\delta + 2) \right],
\]
y_{4,1} = (\delta + 3) y_{3,0} + \lambda (\delta + 2) y_{2,0} + \lambda^2 (\delta + 1) y_{1,0} + \lambda^3 \delta.
\]
(99)

If still higher orders are required, it is an easy matter to compute them from the RG equation. Like for the two-point correlators, the coefficients \( y_{0,0} \) cannot be determined from the renormalisation group, and can be considered independent.

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