Research Article

Mixed Variational Principles in Nondissipative Coupled Thermoelasticity

Francesco Marotti de Sciarra

Department of Structures for Engineering and Architecture, University of Naples Federico II, Via Claudio, 21-80125 Napoli, Italy

Correspondence should be addressed to Francesco Marotti de Sciarra; marotti@unina.it

Received 14 March 2014; Accepted 13 May 2014; Published 9 June 2014

Academic Editor: Zhaowei Zhong

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This paper is concerned with a general framework to consistently derive all the variational formulations with different combinations of the state variables for a thermoelastic structural model based on the nonlinear thermoelasticity of type II proposed by Green and Naghd. Comparisons with existing formulations are also given.

1. Introduction

Thermal stresses play an important role in structural mechanics and several theories for describing the heat conduction have been presented in the literature. Generally, the heat problem is first solved to get the three-dimensional distribution of the temperature, and then the temperature distribution is used, via an interface, as a thermal load in the structural problem; see, for example, Argyris and Tenek [1], Hetnarski and Ignaczak [2, 3], and Rolfes et al. [4].

It is well-known that the classical linear theory of heat conduction is based on Fourier’s law for the thermal flux and predicts that a thermal effect at a point of a body is felt instantly at other points of the body. Therefore in past years several alternative theories of heat conduction have been proposed; see for a survey Hetnarski and Eslami [5], Ignaczak and Ostoja-Starzewski [6].

In this paper the nonclassical theory of thermoelasticity developed by Green and Naghd [7, 8] is considered. Such a model incorporates the approach based on Fourier’s law (referred to as type I), the theory without energy dissipation (type II), and a theory which allows finite wave propagation as well as energy dissipation (type III).

Since the publication of the works of Green and Naghd (GN), several authors have been interested in studying the implications of these theories. However, it is apparent that further mathematical and physical studies are still needed to clarify their applicability.

Contributions on the GN approach can be found, among others, in [9, 10], an energy method is employed to derive uniqueness theorems in [11], continuous dependence results are established in [12], spatial-decay theorems are proved in [13], questions of existence are investigated in [14], and thermodynamic relations between the entropy flux and the heat flux are addressed in [15, 16].

An analytical treatment of coupled thermoelastic problems is complex so that the development of alternative methods of analysis plays a central role. Accordingly a framework which allows one to derive all the admissible variational formulations of such problems is of great interest from a theoretical point of view and from a computational standpoint since suitable variational formulations are the basis to develop mixed finite elements; see, for example, [17, 18].

Variational formulations of classical thermoelastic problems are discussed, among others, in [19–24]. Variational formulations for the GN theory of type II are addressed in [25, 26].

In this paper the nonclassical thermoelastic problem following the GN model of type II is analysed. One of the basic items in this theory is the thermal displacement which, on the macroscopic scale, is regarded as representing a mean thermal displacement magnitude on the molecular scale. Variational formulations for the GN theory of type II have received little attention, and thus our contribution represents
2. The Thermoeelastic Structural Problem

In small strain analysis the theory of thermoeelasticity without energy dissipation as described in [8] is considered and the body is regarded as the compositions of a mechanical and a thermal material [32]. A compact notation is adopted throughout the paper with bold-face letters associated with vectors and tensors; a superimposed dot means differentiation with respect to time.

Green and Naghdi’s theory is based on the introduction of a scalar thermal displacement \( \alpha \) defined as

\[
\alpha (\mathbf{x}, t) = \int_{0}^{t} \theta (\mathbf{x}, \tau) \, d\tau + \alpha_{o}(\mathbf{x}, 0),
\]

where \( \mathbf{x} \) is a point pertaining to the thermoelastic body defined on a regular bounded domain \( \Omega \) of an Euclidean space, \( \theta = \theta - \theta_{r} \) represents the temperature variation from the uniform reference temperature \( \theta_{r} \), and \( \alpha_{o}(\mathbf{x}, 0) \) is the initial value of \( \alpha \) at the time \( t = 0 \). As a consequence the time derivative of the thermal displacement is the temperature variation; that is, \( \dot{\alpha} = \theta - \theta_{r} \).

Assuming that the internal energy \( e \) [33] depends on strain \( \varepsilon \), entropy \( \eta \), and gradient of the thermal displacement \( \nabla \alpha \), the constitutive relations are

\[
\sigma = d_{e}e(\varepsilon, \eta, \nabla \alpha), \quad \theta = d_{\eta}e(\varepsilon, \eta, \nabla \alpha),
\]

\[
-p = d_{\varepsilon}e(\varepsilon, \eta, \nabla \alpha),
\]

where \( \sigma \) is the stress tensor and \( p = q/\theta \) is the entropy flux vector with \( q \) being the heat flux.

Using the well-known relation between the internal energy \( e \) and the Helmholtz free energy \( \varphi \), the constitutive relations (2) can be equivalently expressed in the following form:

\[
\sigma = d_{\varepsilon}\varphi (\varepsilon, \theta, \nabla \alpha),
\]

\[
-\eta = d_{\varphi}e(\varepsilon, \theta, \nabla \alpha),
\]

\[
-p = d_{\varepsilon}\varphi (\varepsilon, \theta, \nabla \alpha).
\]

It is worth noting that it is not necessary to consider a peculiar expression of the internal energy or to split it in an additive form for the development of the variational theory. Hence a nonlinear elastic anisotropic behaviour and a temperature dependency of the thermoelastic parameters are encompassed in the model.

Further issues such as thermoelastic models which depend on two distinct temperatures, that is, the conductive temperature and the thermodynamic temperature, where the difference between these two temperatures is proportional to the heat supply, see [34], dependence of the stress on the strain rate, or heat and mass transfer within the body to get a chemohygrothermomechanical model, see [35] for concrete, are definitively out the scope of this paper.

Let \( \mathcal{D} \) denote the linear space of strain tensors \( \varepsilon (\mathbf{x}, t) \) and let \( \mathcal{S} \) denote the dual space of stress tensors \( \sigma (\mathbf{x}, t) \). The scalar product between dual quantities (simple or double index saturation operation between vectors or tensors) is denoted by \( \ast \) and the inner product in the dual spaces \( \langle \cdot, \cdot \rangle \) has the mechanical meaning of the internal virtual work; that is,

\[
\langle \sigma, \varepsilon \rangle = \int_{\Omega} \sigma (\mathbf{x}, t) \ast \varepsilon (\mathbf{x}, t) \, d\Omega. \tag{4}
\]

The linear space of thermal displacement \( \alpha(\mathbf{x}, t) \) is \( \mathcal{A} \). To each thermal displacement field \( \alpha(\mathbf{x}, t) \in \mathcal{A} \), there corresponds a boundary field \( \Gamma \alpha(\mathbf{x}, t) \in \partial \mathcal{A} \). The dual space \( \partial \mathcal{A} \) of \( \partial \mathcal{A} \) is given by the boundary heat fluxes \( \mathcal{F}(\mathbf{x}, t) \) and the external heat sources \( r(\mathbf{x}, t) \) belong to the space \( \mathcal{H} \subset \mathcal{A} \) of square integrable fields. The external thermal forces are collected in the set \( \ell_{h} = \{ \mathcal{F}, -r/\theta_{r} \} \in \mathcal{A} \), where \( r \) is the rate of heat flow into the body by a heat source. Reactive thermal forces \( \gamma(\mathbf{x}, t) \) of the external thermal constraints

\[
\mathcal{F}(\mathbf{x}, t) = \int_{\Gamma} \theta(\mathbf{x}, t) \, d\Gamma, \quad r(\mathbf{x}, t) = \int_{\partial \Omega} r(\mathbf{x}, t) \, d\Omega.
\]
belong to the subspace $\mathcal{H}_0$ where $\mathcal{H}_0$ is the subspace of conforming thermal displacement fields which satisfy the homogeneous boundary conditions. To any $y(x, t) \in \mathcal{H}_0$, there corresponds a uniquely defined boundary thermal reaction system $\rho_0(x, t) \in (\mathcal{H}_0)^\perp \subset \partial \mathcal{A}'$ such that $\langle y, \alpha \rangle = \langle \rho_0, \Gamma \alpha \rangle$ for any $\alpha(x, t) \in \mathcal{A}'$.

The kinematic thermal operator $G \in \text{Lin}(\mathcal{A}', \mathcal{F})$ is a bounded linear operator and a thermal gradient $g \in \mathcal{F}$ is said to be thermally compatible if there exists an admissible thermal displacement field $\alpha$ such that $g = G\alpha$. The dual operator of $G$ is the equilibrium thermal operator $G^* \in \text{Lin}(\mathcal{F}', \mathcal{A}')$ which turns out to be defined by the equalities [36]

$$\langle G^* p, \alpha \rangle = \langle G^*_0 p, \alpha \rangle + \langle H p, \Gamma \alpha \rangle = \langle p, G\alpha \rangle$$

$$\forall p \in \mathcal{F}', \forall \alpha \in \mathcal{A}'$$

where $G^*_0 \in \text{Lin}(\mathcal{F}', \mathcal{H})$ is the formal adjoint of the kinematic thermal operator such that $G^*_0 p = -r_0 + \eta$. The boundary flux associated with the entropy flux vector $p$ is $H p = p \cdot n \in \partial \mathcal{A}'$, where $n$ denotes the outward normal to the boundary $\partial \Omega$.

External thermal forces and the rate of entropy in equilibrium with a entropy flux vector belong to the image of the equilibrium thermal operator $\text{Im} G^* = \{ p \in \partial \mathcal{A}', r_0, \eta \in \mathcal{H}, \eta \in (\mathcal{H}_0)^\perp : G^* p = -r_0 + \eta, H p = \overline{p} + \rho_0 \}$ (6)

Let $\overline{\mathcal{A}}(x) \in \partial \mathcal{A}$ be a prescribed thermal displacement field on $\partial \Omega$. A pair $\{ g(x, t), \overline{\alpha}(x) \} \in \mathcal{F} \times \partial \mathcal{A}$ is thermally compatible if there exists a thermal displacement field $\alpha(x, t) = v(x, t) + \overline{\alpha}(x)$ such that $G\alpha(x, t) = g(x, t)$ and $\Gamma \alpha(x, t) = \overline{\alpha}(x)$, where $v(x, t) \in \mathcal{H}_0$ and $\overline{\alpha}(x) \in \mathcal{A}'$ is a thermal displacement field which fulfills nonhomogeneous boundary thermal conditions.

Constraint conditions can be fit in field equations by noting that the external relation between reactive thermal forces and thermal displacements is provided by the relations

$$y \in \partial \Pi(\alpha) \iff \alpha \in \partial \Pi^*(y) \iff \Pi(\alpha) + \Pi^*(y) = \langle \gamma, \alpha \rangle$$

with $\Pi, \Pi^*$ being conjugate convex functionals and the relation (7)$_1$ represents Fenchel's equality. The symbol $\partial$ denotes the sub(super)differentiable of convex (concave) functions [37].

For completeness, the mechanical model is briefly reported hereafter [38, 39] and the time derivative of the displacement field is neglected. Let $\mathcal{U}$ denote the linear space of displacements $u(x, t)$. The linear space of forces is denoted by $\mathcal{F}$ and is placed in separating duality with $\mathcal{U}$ by a nondegenerate bilinear form $\langle \cdot, \cdot \rangle$ which has the physical meaning of external virtual work. For avoiding proliferation of symbols, the internal and external virtual works are denoted by the same symbol. Conforming displacement fields satisfy homogeneous boundary conditions and belong to a closed linear subspace $\mathcal{L}_0 \subset \mathcal{U}$.

The kinematic operator $B$ is a bounded linear operator from $\mathcal{U}$ to the space of square integrable strain fields $\varepsilon \in \mathcal{D}$. The subspace of external forces $\mathcal{F}$ is dual of $\mathcal{U}$. The continuous operator $B^*$ from $\mathcal{D}$ to $\mathcal{U}$ is dual of $B$ and represents the equilibrium operator. Let $\ell = [t, b] \in \mathcal{F}$ be the load functional where $t$ and $b$ denote the tractions and the body forces.

The equilibrium equation and the compatibility condition are

$$\ell + r = B^* r, \quad \varepsilon = B u,$$

(8)

The external relation between reactions $r$ and displacements $u$ is assumed to be given by the equivalent relations

$$r \in \partial Y(u) \iff u \in \partial Y^*(r) \iff Y(u) + Y^*(r) = \langle r, u \rangle$$

(9)

with $Y$ being a concave functional. The concave functional $Y^*$ represents the conjugate of $Y$ and the relation (9)$_2$ represents Fenchel's equality.

Accordingly the relations governing the thermoelastic structural problem without energy dissipation for given mechanical and thermal loads $\ell$ and $\ell_0$ in the time interval of interest $I = [0, t]$ are

$$B^* \sigma = \ell + r \quad \text{equilibrium},$$

$$Bu = \varepsilon \quad \text{compatibility},$$

$$G \alpha = g \quad \text{thermal compatibility},$$

$$G^* p = \eta + \ell_0 + \gamma \quad \text{thermal balance equation},$$

$$\begin{bmatrix}
\sigma = d_\varepsilon(\varepsilon, \eta, g) \\
\alpha = d_\eta(\varepsilon, \eta, g) \end{bmatrix} \quad \text{constitutive relations},$$

$$u \in \partial Y^*(r) \quad \text{external relation},$$

$$\alpha \in \partial \Pi^*(\gamma) \quad \text{thermal external relation}.$$

The initial thermal conditions are considered in the form

$$\alpha(x, 0) = \alpha_0(x), \quad \dot{\alpha}(x, 0) = \theta_0(x), \quad \text{with } x \in \Omega,$$

(11)

where $\alpha_0$ and $\theta_0$ are, respectively, the prescribed initial thermal displacement and the temperature in $\Omega$.

A solution of the thermoelastic structural problem is usually achieved by a finite element approach [40, 41] which can be obtained by starting from a suitable mixed variational formulation. Hence the definition of a general and consistent framework to derive variational principles with different combinations of the state variables, without ad hoc procedures, plays a central role.

The time integral of the thermal balance equation (10)$_4$ in the interval $I$ is given by

$$\int_I G^* p \, dt = \int_I (\ell_0 + \gamma) \, dt + \eta - \eta_0,$$

(12)

where $\eta_0 = \eta(x, 0)$. It is worth noting that $\eta_0$ can be obtained from the constitutive relation (3)$_2$ considering $\varepsilon = \varepsilon(x, 0) = \varepsilon_0, \theta = \theta_0(x) = \theta_0$, and $g = g(x, 0) = g_0$. 

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Moreover the time integral of the constitutive relation (10)_0 in the interval $I$ is given by

$$\alpha - \alpha_0 = \int_I d_s e \left( \varepsilon, \eta, g \right) d \tau,$$

where $\alpha_0 = \alpha_0(x, 0)$.

Accordingly the thermoelastic structural problem is reported in Table I and the initial conditions (11) are incorporated into the field equations.

### 3. Mixed Variational Formulations

Introducing the product space $\mathcal{W} = \mathcal{H} \times \mathcal{S} \times \mathcal{D} \times \mathcal{A}^t \times \mathcal{A} \times \mathcal{F} \times \mathcal{G} \times \mathcal{A}^t \times \mathcal{H} \times \mathcal{G} \times \mathcal{A}^t \times \mathcal{A}$ and its dual space $\mathcal{W}^* = \mathcal{F} \times \mathcal{D} \times \mathcal{A} \times \mathcal{A}^t \times \mathcal{H} \times \mathcal{G} \times \mathcal{A}^t \times \mathcal{A}$, the thermoelastic structural problem without dissipation reported in Table I can be collected in terms of the global multivalued structural operators $S, \bar{S}, S : \mathcal{W} \to \mathcal{W}^*$ governing the whole problem

$$0 \in \int_I \left[ S(w) + w_0 \right] dt + \bar{S}(w) + \bar{w}_0, \quad w \in \mathcal{W}, \quad w_0, \bar{w}_0 \in \mathcal{W}^*.$$  
(14)

The expressions of the vectors $w, w_0$, and $\bar{w}_0$, are

$$w^T = \left[ u \, \sigma^n \, \varepsilon \, \eta \, g \, r \, \alpha \, p \, \gamma \right]^T,$$
$$w_0^T = \left[ -\ell \, 0 \, 0 \, 0 \, 0 \, 0 \, -\ell^0 \, 0 \, 0 \right]^T,$$
$$\bar{w}_0^T = \left[ 0 \, 0 \, \sigma_0 \, 0 \, 0 \, \eta_0 \, 0 \, 0 \, 0 \right]^T.$$  
(15)

The expression of the structural operator $S$ is

$$S = \begin{bmatrix} 0 & B' & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -I_{g^{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & Q & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -I_g' & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$  
(16)

where the suboperator $Q$ is given by

$$Q(\varepsilon, \eta, g) = d_{11}(x) e \times \left( -d_{12} e \right),$$

and, finally, the expression of $\bar{S}$ is

$$\bar{S} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$  
(18)

The operators $S$ and $\bar{S}$ turn out to be integrable by virtue of the duality existing between the operators $B, G, I_{g}, I_{g^2}, I_{g'},$ and $I_{g'}$ and $B', G', I_{g'}, I_{g}, I_{g'}$, and $I_{g'}$; the conservativity of $Q$; and the conservativity of the super(sub)differentials $\partial Y^*$ and $\partial \Pi^*$.

The related potential can be evaluated by a direct integration [42] along a straight line in the space $\mathcal{W}$ starting from its origin to get

$$M(u, \sigma, \varepsilon, \eta, g, r, \alpha, p, \gamma)$$

$$= \int_I \left[ \int_0^1 \left( S(\xi \omega), \omega \right) d\xi - \left( \ell, u \right) - \left( \ell_0, \alpha \right) \right] d\tau + \int_0^1 \left[ \bar{S}(\xi \omega), \omega \right] d\xi + \left( g_0, \alpha_0 + \left( \eta_0, \alpha_0 \right) \right).$$

Hence the new potential in the complete set of state variables is given by

$$M(u, \sigma, \varepsilon, \eta, g, r, \alpha, p, \gamma)$$

$$= \int_I \left[ e \left( \varepsilon, \eta, g \right) + Y^* (r) + \Pi^* (\gamma) \right]$$

$$- \left( \sigma, \varepsilon \right) + \left( \sigma, Bu \right) - \left( \ell + r, u \right)$$

$$- \left( p, g \right) + \left( p, G \alpha \right) - \left( \ell_0 + \gamma, \alpha \right)$$

$$- \left( \eta \alpha - \alpha_0 + \left( \eta_0, \alpha_0 \right) \right),$$

which turns out to be linear in $(u, \sigma, \varepsilon, \eta, \gamma)$, jointly convex with respect to the state variables $(\varepsilon, \eta, \gamma)$, and jointly concave with respect to $(g, r)$. The following statement then holds.

**Proposition 1.** The set of state variables $(u, \sigma, \varepsilon, \eta, g, r, \alpha, p, \gamma)$ is a solution of the saddle problem

$$\min_{\varepsilon, \eta, g, r, \alpha, p, \gamma} \max_{\sigma, \varepsilon, \eta, g, r, \alpha, p, \gamma} \quad M(u, \sigma, \varepsilon, \eta, g, r, \alpha, p, \gamma),$$

if and only if it is a solution of the thermoelastic structural problem without dissipation reported in Table I.

The stationary condition of $M$, enforced at the point $(u, \sigma, \varepsilon, \eta, r, \alpha, p, g, \gamma)$, yields back the thermoelastic structural problem. In fact the stationarity of $M$ is

$$\left( 0, 0, 0, 0, 0, 0, 0, 0 \right) \in \partial M(u, \sigma, \varepsilon, \eta, g, r, \alpha, p, \gamma)$$

(22)
which can be rewritten in the following form:

\[
0 = d_w M (u, \sigma, \varepsilon, \eta, r, \alpha, p, g, y),
0 = d_w M (u, \sigma, \varepsilon, \eta, r, \alpha, p, g, y),
0 = d_w M (u, \sigma, \varepsilon, \eta, r, \alpha, p, g, y),
0 = d_w M (u, \sigma, \varepsilon, \eta, r, \alpha, p, g, y),
0 = d_w M (u, \sigma, \varepsilon, \eta, r, \alpha, p, g, y),
0 = d_w M (u, \sigma, \varepsilon, \eta, r, \alpha, p, g, y),
0 = d_w M (u, \sigma, \varepsilon, \eta, r, \alpha, p, g, y).
\]

Hence, performing the super(sub)differentials appearing in (23), the thermoelastic structural problem in Table 1 is recovered.

By enforcing the fulfillment of the thermoelastic relations reported in Table 1, the state variables appearing in the variational formulation stated in Proposition 1 can be alternatively eliminated and new equivalent mixed variational formulations are obtained. All the functionals, obtained following the proposed procedure, attain the same value at the solution point of the GN thermoelastic problem.

An example of the methodology to recover new variational formulations with different state variables for the GN thermoelastic structural problem is reported in the next section. Note that not all the possible combinations of the state variables are admissible in the variational formulations.

4. Comparison with Existing Variational Formulations

Quite a few variational formulations have been proposed for the GN thermoelastic structural problem; see, for example, [25, 26].

In the sequel it is shown how variational principles with a reduced number of state variables can be obtained following a general procedure. The peculiar feature of the proposed procedure is to provide a general methodology to obtain all the admissible variational formulations associated with the GN thermoelastic structural problem without ad hoc reasoning.

In order to show the effectiveness of the procedure, four variational principles, among all the admissible variational formulations, are provided and, then, they are specialized in recovering existing variational principles in the literature. Furthermore, three original potentials are contributed in the appendix to be used as auxiliary functionals for the proposed procedure.

4.1. Seven-Field Mixed Variational Formulation. A mixed seven-field variational principle in \((u, \sigma, \varepsilon, \eta, \alpha, p, g)\) can be obtained by imposing on the expression of the potential \(M_3\) (the appendix) the thermal external constraint (Table 1—(9)) in terms of Fenchel's equality (7). The variational formulation is reported in the next statement.

**Proposition 2.** The set of state variables \((u, \sigma, \varepsilon, \eta, \alpha, p, g)\) is a solution of the saddle problem

\[
\min \max \text{stat } M_1 (u, \sigma, \varepsilon, \eta, \alpha, p, g),
\]

where

\[
M_1 (u, \sigma, \varepsilon, \eta, \alpha, p, g) = \int_I [\bar{e} (\varepsilon, \eta, g) - \gamma (u) - \Pi (\alpha)]
- \langle \sigma, \varepsilon \rangle + \langle \sigma, Bu \rangle - \langle \ell, u \rangle - \langle \eta, p \rangle
+ \langle p, G \alpha \rangle - \langle \theta, \eta \rangle \right] d \tau - \langle \eta, \alpha - \alpha_0 \rangle + \langle \eta_0, \alpha \rangle,
\]

if and only if it is a solution of the thermoelastic structural problem without dissipation reported in Table 1.

To specialize the potential \(M_1\), the expression of the internal energy \(\bar{e}\) for the Cauchy model having a GN linear coupled thermoelastic behaviour is

\[
\bar{e} (\varepsilon, \eta, g) = \frac{1}{2} \langle \bar{E} \varepsilon, \varepsilon \rangle + \frac{\theta_r}{2c_V} \langle (M \otimes M) \varepsilon, \varepsilon \rangle + \frac{\theta_r}{2c_V} \langle \eta, \eta \rangle
+ \frac{1}{2} \langle K g, g \rangle - \frac{\theta_r}{c_V} \langle \eta M, \varepsilon \rangle + \langle \theta, \eta \rangle.
\]

The parameter \(c_V(\theta)\) is the temperature-dependent specific heat at constant strain, \(\bar{E}\) is the isothermal elastic moduli fourth-order tensor, and \(K\) is the symmetric tensor of conductivity moduli, both symmetric and positive definite. The second-order thermal expansion tensor \(M(\theta)\) is self-adjoint, that is, \(M^T = M\), and describes the thermoelastic coupling arising from thermal dilatations depending on the linear coefficient of thermal dilatation \(\alpha(\theta)\).

Accordingly, the potential \(M_1\) can be particularized in the form

\[
M_1 (u, \sigma, \varepsilon, \eta, \alpha, p, g) = \int_I \left[ H^{TH} (u, \sigma, \varepsilon) + H^{TH} (\eta, \alpha, p, g) \right] d \tau
- \int_I H^{TH} (\varepsilon, \eta) d \tau - H^{TH} (\eta, \alpha) + H^{TH} (\alpha),
\]

where

\[
H^{TH} (u, \sigma, \varepsilon) = \frac{1}{2} \int_\Omega \bar{E} \varepsilon \ast \varepsilon \, dx + \int_\Omega \sigma \ast (Bu - \varepsilon) \, dx
- \int_\Omega b \ast u \, dx - \int_{\Omega} t \ast \Gamma u \, dx - \int_{\Omega} g \, (\mathbf{u}),
\]
is the three-field Hu-Washizu mixed functional in classical elasticity [43] in which the elastic stiffness is replaced by the isentropic elastic stiffness \( E \), the functionals

\[
H_{TH}^{T} (\eta, \alpha, \mathbf{p}, \mathbf{g}) = -\frac{1}{2\theta_r} \int_{\Omega} \mathbf{K} \cdot \mathbf{g} \, d\mathbf{x} + \int_{\Omega} \mathbf{p} \cdot (\mathbf{G} \alpha - \mathbf{g}) \, d\mathbf{x} + \frac{\theta_r}{2\nu'} \int_{\Omega} \eta^2 \, d\mathbf{x} + \frac{1}{\theta_r} \int_{\Omega} \mathbf{r} \cdot \alpha \, d\mathbf{x} - \int_{\partial \Omega} \mathbf{q} \cdot \Gamma \alpha \, d\mathbf{x},
\]

\[
H_{TH}^{H} (\eta, \alpha) = \int_{\Omega} \eta \cdot (\alpha - \alpha_0) \, d\mathbf{x},
\]

are the mixed thermal parts of the potential, and the functionals

\[
H_{THC}^{T} (\varepsilon, \eta) = \frac{\theta_r}{\nu'} \int_{\Omega} \eta \mathbf{M} \cdot \varepsilon \, d\mathbf{x},
\]

\[
H_{THC}^{H} (\alpha) = \int_{\Omega} \left( \mathbf{M} \cdot \varepsilon_0 + \frac{\nu'}{\theta_r} \right) \alpha \, d\mathbf{x},
\]

take into account the thermoelastic coupling.

The variational formulation reported in Proposition 2, replacing \( M_1 \) with the functional \( M_1' \), yields the variational principle provided in [26] for the Cauchy model with a linear GN coupled thermoelastic behaviour if, therein, the velocity field \( \mathbf{u} \) vanishes.

### 4.2. Four-Field Mixed Variational Formulation

A mixed four-field variational principle in \( (\mathbf{u}, \sigma, \alpha, \mathbf{p}) \) can be obtained by imposing the following relations on the expression of the potential \( M_2 \) (see the appendix): (a) the constitutive relations (Table 1—(3) \( \oplus \) (5)) in terms of Fenchel’s equality (A.1); (b) the thermal external constraints (Table 1—(9)) in terms of Fenchel’s equality (7). Hence the related variational formulation is reported in the next proposition.

**Proposition 3.** The set of state variables \( (\mathbf{u}, \sigma, \alpha, \mathbf{p}) \) is a solution of the saddle problem

\[
\min_{\mathbf{u}, \mathbf{p}} \max_{\alpha, \sigma} M_2 (\mathbf{u}, \sigma, \alpha, \mathbf{p}),
\]

where

\[
M_2 (\mathbf{u}, \sigma, \alpha, \mathbf{p}) = \int_{\Omega} \left( -\varepsilon^* (\sigma, \alpha - \alpha_0, \mathbf{p}) - \mathbf{Y} (\mathbf{u}) - \Pi (\alpha) \right)
\]

\[
+ \langle \sigma, \mathbf{B} \mathbf{u} \rangle - \langle \xi, \mathbf{u} \rangle
\]

\[
+ \langle \mathbf{p}, \mathbf{G} \alpha \rangle - \langle \xi_0, \alpha \rangle \rangle \, d\tau + \langle \eta_0, \alpha \rangle,
\]

if and only if it is a solution of the thermoelastic structural problem without dissipation reported in Table 1.

In order to express the potential \( M_2 \) in terms of the Cauchy model for the GN linear coupled thermoelastic problem, the conjugate \( \varepsilon^* \) of the internal energy (26) has the following expression:

\[
\varepsilon^* (\sigma, \alpha - \alpha_0, \mathbf{p}) = \frac{1}{2} \left( \sigma, \varepsilon^* \right) + \frac{\nu'}{2\theta_r} \left( \alpha - \alpha_0, \alpha - \alpha_0 \right)
\]

(33)

\[
+ \left( \sigma - \sigma_0 \right) \mathbf{M} \varepsilon^* \sigma + \frac{\theta_r}{2} \left( \mathbf{p}, \mathbf{K}^{-1} \mathbf{p} \right),
\]

with \( \varepsilon^* = \nu' + \theta_r \mathbf{M} \varepsilon^* \sigma \) being the specific heat at constant stress at the reference state temperature \( \theta_r \). Hence the potential \( M_2 \) can be rewritten for the linear GN coupled thermoelastic model in the following form:

\[
M_2' (\mathbf{u}, \sigma, \alpha, \mathbf{p}) = \int_{\Omega} [M_{TH} (\mathbf{u}, \sigma) + M_{TH}^T (\alpha, \mathbf{p})] \, d\tau
\]

\[
- \int_{\Omega} M_{THC} (\sigma, \alpha) \, d\tau + H_{THC}^T (\alpha),
\]

where

\[
M_{TH} (\mathbf{u}, \sigma)
\]

\[
= -\frac{1}{2} \int_{\Omega} \sigma \cdot \varepsilon^* \sigma \, d\mathbf{x} + \int_{\Omega} \sigma \cdot \mathbf{B} \mathbf{u} \, d\mathbf{x} - \int_{\Omega} \mathbf{b} \cdot \mathbf{u} \, d\mathbf{x}
\]

\[
- \int_{\partial \Omega} \mathbf{t} \cdot \Gamma \mathbf{u} \, d\mathbf{x} - \int_{\partial \Omega} (\sigma - \sigma_0) \Gamma \mathbf{u} \, d\mathbf{x},
\]

(35)

is the classical Hellinger-Reissner functional, the potential

\[
M_{TH}^T (\alpha, \mathbf{p}) = \frac{\theta_r}{2} \int_{\Omega} \mathbf{p} \cdot \mathbf{K}^{-1} \mathbf{p} \, d\mathbf{x} + \int_{\Omega} \mathbf{p} \cdot \mathbf{G} \alpha \, d\mathbf{x}
\]

\[
- \frac{\nu'}{2\theta_r} \int_{\Omega} (\alpha - \alpha_0)^2 \, d\mathbf{x} + \frac{1}{\theta_r} \int_{\Omega} \mathbf{r} \cdot \alpha \, d\mathbf{x}
\]

(36)

represents the mixed thermal contribution, and the functionals

\[
M_{THC}^T (\alpha) = \int_{\Omega} \left( \mathbf{M} \cdot \varepsilon_0 + \frac{\nu'}{\theta_r} \right) \alpha \, d\mathbf{x},
\]

\[
H_{THC}^T (\alpha) = \int_{\Omega} \left( \mathbf{M} \cdot \varepsilon_0 + \frac{\nu'}{\theta_r} \right) \alpha \, d\mathbf{x},
\]

(37)

take into account the thermoelastic coupling where \( H_{THC}^T \) is the one appearing in the expression of the functional \( M_1' \).

The variational formulation reported in Proposition 3, replacing \( M_2 \) with the functional \( M_2' \), yields the four-field variational principle in terms of displacement, temperature, stress, and heat flux proved in [17] in the framework of the classical Fourier linear thermoelasticity.
4.3. Three-Field Mixed Variational Formulation. A three-field variational principle in terms of displacement, thermal displacement, and entropy flux can be obtained by imposing the following relations on the expression of the potential $M_z$ (the appendix): (a) the constitutive relations which provide the thermal displacement $\alpha$ and the entropy flux vector $\mathbf{p}$ (Table 1—(4), (5)) in terms of Fenchel’s equality (A.2); (b) the thermal external constraints (Table 1—(9)) in terms of Fenchel’s equality (7). Hence the following mixed three-field variational formulation is recovered.

**Proposition 4.** The set of state variables $(\mathbf{u}, \alpha, \mathbf{p})$ is a solution of the saddle problem

$$
\min_{\mathbf{u}} \max_{\alpha, \mathbf{p}} M_z(\mathbf{u}, \alpha, \mathbf{p}),
$$

where

$$
M_z(\mathbf{u}, \alpha, \mathbf{p}) = \int_{\Omega} \left( -e_1^* (B \mathbf{u}, \alpha - \alpha_0, \mathbf{p}) - Y(\mathbf{u}) - \Pi(\alpha) - \langle \mathbf{e}, \mathbf{u} \rangle + \langle \mathbf{p}, \mathbf{G} \alpha \rangle - \langle \mathbf{e}_0, \alpha \rangle \right) d\tau + \langle \mathbf{f}_0, \alpha \rangle,
$$

if and only if it is a solution of the thermoelastic structural problem without dissipation reported in Table 1.

For the Cauchy model with linear GN coupled thermoeconomic behaviour, the partial conjugate of the elastic energy $e$ with respect to the state variables $(\eta, \mathbf{g})$ is the functional

$$
e_1^*(e, \alpha - \alpha_0, \mathbf{p}) = \frac{1}{2} \langle \mathbf{e}, \mathbf{e} \rangle + \frac{\partial \alpha}{2c_r^*} \langle (\mathbf{M} \otimes \mathbf{M}) \mathbf{e}, \mathbf{e} \rangle + \langle (\alpha - \alpha_0), \mathbf{M} \mathbf{e} \rangle
+ \frac{c_r^*}{2\partial_r^*} \langle \alpha - \alpha_0, \alpha - \alpha_0 \rangle + \frac{\partial \alpha}{2} \langle \mathbf{p}, \mathbf{K}^{-1} \mathbf{p} \rangle.
$$

Accordingly the potential $M_z$ can be rewritten for the Cauchy linear GN coupled thermoelastic model in the following form:

$$
M_z^C(\mathbf{u}, \alpha, \mathbf{p}) = \int_{t} \left[ M_z^U(\mathbf{u}) + M_z^{TH}(\alpha, \mathbf{p}) \right] dt
$$

$$
= \int_{t} \left[ M_z^{THC}(\mathbf{u}, \alpha) dt + H_z^{THC}(\alpha) \right],
$$

where

$$
M_z^U(\mathbf{u}) = \frac{1}{2} \int_{\Omega} \mathbf{E} \mathbf{u} * \mathbf{B} \mathbf{u} - \int_{\Omega} \mathbf{B} * \mathbf{u} d\mathbf{x}
- \int_{\Omega} t * \Gamma \mathbf{u} d\mathbf{x} - \iota_{\mathfrak{S}^*}(\mathbf{u}),
$$

is the total potential energy in which the elastic stiffness is replaced by the isentropic one, the functional

$$
M_z^{TH}(\alpha, \mathbf{p}) = \frac{\partial \alpha}{2} \int_{\Omega} \mathbf{p} * \mathbf{K}^{-1} \mathbf{p} d\mathbf{x} + \int_{\Omega} \mathbf{p} * \mathbf{G} \alpha d\mathbf{x}
$$

$$
= - \frac{c_r^*}{2\partial_r} \int_{\Omega} (\alpha - \alpha_0)^2 d\mathbf{x} + \frac{1}{2} \int_{\Omega} r * \alpha d\mathbf{x}
$$

$$
+ \frac{\partial \alpha}{2} \int_{\Omega} \mathbf{p} * \mathbf{K}^{-1} \mathbf{p} d\mathbf{x},
$$

is the mixed thermal functional differing from $M_z^{TH}$, see (36), by the presence of the specific heat at constant strain $c_r^*$, and the functionals

$$
M_2^{THC}(\mathbf{u}, \alpha) = \int_{\Omega} (\alpha - \alpha_0) \mathbf{M} * \mathbf{B} \mathbf{u} d\mathbf{x},
$$

$$
H_2^{THC}(\alpha) = \int_{\Omega} \left( \mathbf{M} * \mathbf{e}_0 + \frac{c_r^*}{2\partial_r^*} \mathbf{e} \right) d\mathbf{x},
$$

take into account the thermoelastic coupling where $H_2^{THC}$ is the same functional appearing in the expressions of $M_z^C$ and $M_z^M$.

The variational formulation reported in Proposition 4, replacing $M_z$ with the functional $M_z^C$, coincides with the three-field variational principle in terms of displacement, temperature, and heat flux proposed by Lebon and Lambergmont [44], Lebon [20] for the classical Fourier model.

4.4. Two-Field Mixed Variational Formulation. A two-field variational principle can be obtained by imposing the following relations on the expression of the potential $M_z$ (the appendix): (a) the thermal external constraints (Table 1—(9)) in terms of Fenchel’s equality (7); (b) the mechanical compatibility condition (Table 1—(2)); the thermal compatibility condition (Table 1—(8)). Hence the following proposition holds.

**Proposition 5.** The set of state variables $(\mathbf{u}, \alpha)$ is a solution of the saddle problem

$$
\min_{\mathbf{u}} \max_{\alpha} P(\mathbf{u}, \alpha),
$$

where

$$
P(\mathbf{u}, \alpha)
= \int_{t} \left[ -e_1^* (B \mathbf{u}, \alpha - \alpha_0, \mathbf{Ga}) - Y(\mathbf{u}) - \Pi(\alpha) - \langle \mathbf{e}, \mathbf{u} \rangle - \langle \mathbf{p}, \mathbf{G} \alpha \rangle - \langle \mathbf{e}_0, \alpha \rangle \right] d\tau + \langle \mathbf{f}_0, \alpha \rangle,
$$

if and only if it is a solution of the thermoelastic structural problem without dissipation reported in Table 1.

For the Cauchy model with linear GN coupled thermoeconomic behaviour, the partial conjugate of the internal energy
of the isentropic elastic energy. The functional

\[ \mathcal{E}_1^*(\epsilon, \alpha - \alpha_o, g) \]

which, after some computations, has the following expression:

\[
\begin{align*}
\mathcal{E}_1^* (\epsilon, \alpha - \alpha_o, g) &= -\frac{1}{2} \langle \mathcal{E}_1, \epsilon \rangle + \frac{\partial}{2c'_{\gamma}} \langle \mathbf{M} \otimes \mathbf{M} \epsilon, \epsilon \rangle \\
&\quad + \langle \mathbf{M} \epsilon, \epsilon \rangle + \frac{c'_{\gamma}}{2\partial_r} (\alpha - \alpha_o, \alpha - \alpha_o) \\
&\quad + \frac{1}{2\partial_r} \langle \mathbf{Kg}, g \rangle.
\end{align*}
\]

Accordingly the potential \( P \) can be rewritten for the Cauchy linear GN coupled thermoelastic model in the following form:

\[
P^\prime (u, \alpha) = \int_\mathcal{E}_1^* (\epsilon, \alpha - \alpha_o, g)
\]

\[
+ \int_t \mathcal{M}_2^\text{THC} (u, \alpha) dt + \mathcal{H}_2^\text{THC} (\alpha),
\]

where

\[
P^\prime (u, \alpha) = \frac{1}{2} \int_\Omega \mathbf{E} \mathbf{u} \ast \mathbf{u} d\mathbf{x} - \int_\Omega \mathbf{b} \ast \mathbf{u} d\mathbf{x}
\]

\[
- \int_\Omega \mathbf{t} \ast \mathbf{u} d\mathbf{x} - \mathcal{I}_{\alpha} (u),
\]

is the classical total potential energy. Note that the three-field potential \( \mathcal{M}_2^\star \) depends on the total potential energy in terms of the isentropic elastic energy. The functional

\[
P^\text{TH} (\alpha) = -\frac{1}{2\partial_r} \int_\Omega \mathbf{Kg} \ast \mathbf{g} d\mathbf{x} - \frac{c'_{\gamma}}{2\partial_r} \int_\Omega (\alpha - \alpha_o)^2 d\mathbf{x}
\]

\[
+ \frac{1}{\partial_r} \int_\Omega \mathbf{r} \ast \mathbf{a} d\mathbf{x} - \mathcal{I}_{\alpha} (u) - \int_\Omega \mathbf{q} \ast \mathbf{g} d\mathbf{x},
\]

\[
\text{(47)}
\]

is the thermal part of the potential and the thermoelastic coupling functionals

\[
\mathcal{M}_2^\text{THC} (u, \alpha) = \int_\Omega (\alpha - \alpha_o) \mathbf{M} \ast \mathbf{u} d\mathbf{x}
\]

\[
\mathcal{H}_2^\text{THC} (\alpha) = \int_\Omega \left( \mathbf{M} \ast \epsilon_o + \frac{c'_{\gamma}}{2\partial_r} \alpha \right) d\mathbf{x}
\]

\[
\text{(51)}
\]

coincide with the ones appearing in the expression of \( \mathcal{M}_3^\star \).

The variational formulation reported in Proposition 5, replacing \( P \) with the functional \( P^\prime \), yields the two-field variational principle in terms of displacement and temperature proposed by Ieșan \[45\], Nickell and Sackman \[46\] for the classical Fourier model.

5. Closure

A consistent framework to develop variational formulations for the GN thermoelasticity without energy dissipation is presented.

Mechanical and thermal boundary conditions are incorporated into the field equations so that a thermoelastic structural operator is built up in order to perform a direct derivation of the new general variational formulation in the complete set of state variables. The novelty of the proposed analysis consists in casting the GN thermoelastic model without energy dissipation in a general framework which can be used to derive variational formulations with different combinations of state variables which, as is well-known, turn out to be useful from both the theoretical and computational point of view.

Original variational formulations with a reduced number of state variables are provided using the proposed methodology and existing variational principles are obtained starting from the general formulations.

Future developments will concern the consistent derivation of a suitable mixed variational formulation to develop a FE analysis of thermoelastic problems so that numerical examples for isotropic and anisotropic structures will be provided in a forthcoming paper.

Appendix

Performing the partial or total conjugates of the internal energy \( e \) with respect to the state variables \( (\epsilon, \eta, g) \), seven functions are obtained. In particular, in this paper, it is necessary to assess the conjugate \( e^\star (\sigma, \alpha - \alpha_o, p) \) of the internal energy with respect to all the state variables, the conjugate \( e^\star (\epsilon, \alpha - \alpha_o, p) \) of the internal energy with respect to the pair \( (\eta, g) \), and the conjugate \( e^\star (\epsilon, \alpha - \alpha_o, g) \) of the internal energy with respect to \( \eta \).

Hence the constitutive relations (Table 1—(3) + (5)) are expressed in terms of Fenchel’s equality in the form

\[
\int_t e^\star (\sigma, \alpha - \alpha_o, p) d\tau
\]

\[
= \int_t \left( \langle \sigma, \epsilon \rangle + \langle p, g \rangle - e (\epsilon, \eta, g) \right) d\tau + \langle \eta, \alpha - \alpha_o \rangle.
\]

\[
\text{(A.1)}
\]

The constitutive relations which provide the thermal displacement \( \alpha \) and the entropy flux vector \( p \) (Table 1—(4), (5)) are expressed in terms of Fenchel’s equality in the form

\[
\int_t e^- (\epsilon, \alpha - \alpha_o, p) d\tau
\]

\[
= \langle \eta, \alpha - \alpha_o \rangle + \int_t \left[ \langle p, g \rangle - e (\epsilon, \eta, g) \right] d\tau.
\]

\[
\text{(A.2)}
\]

The constitutive relation which provides the thermal displacement \( \alpha \) (Table 1—(4)) is expressed in terms of Fenchel’s equality in the form

\[
\int_t e^\star (\epsilon, \alpha - \alpha_o, g) d\tau = \langle \eta, \alpha - \alpha_o \rangle - \int_t e (\epsilon, \eta, g) d\tau.
\]

\[
\text{(A.3)}
\]

In order to obtain the variational formulations reported in Section 4, the following auxiliary functionals are provided. It is worth noting that this original procedure can be used to...
obtain many other functionals with different combination of the state variables depending on the problem at hand.

(i) Enforcing the constitutive relation (Table 1—(4)) in terms of Fenchel’s equality (A.3) in the expression of the functional $M$, the result is

$$M_4(u, \sigma, \varepsilon, r, \alpha, p, g, \gamma) = \int_I \left[ -c^2_2(\varepsilon, \alpha - \alpha_0, g) + Y^*(r) + \Pi^*(\gamma) \right. $$

$$- \langle \sigma, \varepsilon \rangle + \langle \sigma, Bu \rangle - \langle \ell, r, u \rangle - \langle p, g \rangle + \langle p, G\alpha \rangle$$

$$- \langle \ell_0 + \gamma, \alpha \rangle d\tau + \langle \eta_0, \alpha \rangle .$$

(A.4)

(ii) Enforcing the external constraint (Table 1—(6)) in terms of Fenchel’s equality (9), in the expression of the functional $M$, it follows that

$$M_5(u, \sigma, \varepsilon, \eta, \alpha, p, g, \gamma) = \int_I \left[ c(\varepsilon, \eta, g) - Y(u) + \Pi^*(\gamma) \right. $$

$$- \langle \sigma, \varepsilon \rangle + \langle \sigma, Bu \rangle - \langle \ell, u \rangle$$

$$- \langle p, g \rangle + \langle p, G\alpha \rangle - \langle \ell_0 + \gamma, \alpha \rangle d\tau - \langle \eta, \alpha - \alpha_0 \rangle + \langle \eta_0, \alpha \rangle .$$

(A.5)

(iii) Enforcing the external constraint (Table 1—(6)) in terms of Fenchel’s equality (9), in the expression of the functional $M_4$, the result is

$$M_6(u, \sigma, \varepsilon, \alpha, p, g, \gamma) = \int_I \left[ -c^2_2(\varepsilon, \alpha - \alpha_0, g) - Y(u) + \Pi^*(\gamma) \right. $$

$$- \langle \sigma, \varepsilon \rangle + \langle \sigma, Bu \rangle - \langle \ell, u \rangle - \langle p, g \rangle$$

$$+ \langle p, G\alpha \rangle - \langle \ell_0 + \gamma, \alpha \rangle d\tau + \langle \eta_0, \alpha \rangle .$$

(A.6)

(iv) To specialize the general variational formulations reported in Section 4, it is necessary to consider external frictionless bilateral constraints with nonhomogeneous boundary conditions. The admissible set of displacements is the subspace $\mathcal{L} = \bar{u} + \mathcal{L}_0$ and the subspace of the external constraint reactions is $\mathcal{L}_0^\perp$. Then the functional $Y$ turns out to be the indicator of $\mathcal{L}_0$ defined in the form

$$Y(u) = \Pi_{\mathcal{L}_0}^\perp(u - \bar{u}) = \begin{cases} 0 & \text{if } u - \bar{u} \in \mathcal{L}_0 \\ -\infty & \text{otherwise}, \end{cases}$$

(A.7)

and a direct evaluation shows that its conjugate $Y^*$ is given by

$$Y^*(r) = \langle r, \bar{u} \rangle + \Pi_{\mathcal{L}_0^\perp}(r) = \langle r, \bar{u} \rangle + \begin{cases} 0 & \text{if } r \in \mathcal{L}_0^\perp \\ -\infty & \text{otherwise}. \end{cases}$$

(A.8)

Similarly, nonhomogeneous constraints on thermal displacement fields are formulated by considering a prescribed thermal displacement field $\bar{u} \in \mathcal{A}$ and thermal displacement fields belonging to the affine set $\mathcal{H} = \bar{u} + \mathcal{H}_0$ are said to be admissible. Reactive thermal forces belong to the orthogonal complement $\mathcal{H}_0^\perp$ of $\mathcal{H}_0$. Then the functional $\Pi$ is the indicator of $\mathcal{H}_0^\perp$:

$$\Pi(\alpha) = \Pi_{\mathcal{H}_0^\perp}(\alpha - \bar{u}) = \begin{cases} 0 & \text{if } \alpha - \bar{u} \in \mathcal{H}_0 \\ +\infty & \text{otherwise}, \end{cases}$$

(A.9)

and its conjugate $\Pi^*$ is given by

$$\Pi^*(\gamma) = \langle \gamma, \bar{u} \rangle + \Pi_{\mathcal{H}_0^\perp}(\gamma) = \langle \gamma, \bar{u} \rangle + \begin{cases} 0 & \text{if } \gamma \in \mathcal{H}_0^\perp \\ +\infty & \text{otherwise}. \end{cases}$$

(A.10)

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The research work is funded by Project FARO IV Tornata, 2012, University of Naples Federico II, Polo delle Scienze e delle Tecnologie, and Compagnia di San Paolo.

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