On the number of plane partitions and non isomorphic subgroup towers of abelian groups

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Abstract

In this paper we study \( \alpha_{k,r}(n) \), defined as the number of \( k \times r \) matrices such that \( m_{i,j} \geq m_{i+1,j} \geq 0, \ m_{i,j} \geq m_{i,j+1} \), and \( m_{1,1} + \cdots + m_{1,r} = n \). We consider the generating function

\[
F_{k,r}(x) = \sum_{n=0}^{\infty} \alpha_{k,r}(n)x^n.
\]

We use Erhart reciprocity to prove that

\[
F_{k,r}(x) = (-1)^{kr}x^{r(r-1+2k)/2}F_{k,r}(x).
\]

For the special case \( k = 1 \) this result also follows from the classical theory of partitions, and for \( k = 2 \) it was proved in Andersson-Bhowmik [AB] with another method. We give an explicit formula for \( F_{k,r}(x) \) in terms of Young tableaux. We then study the corresponding zeta-function

\[
Z_{k,r}(s) = \prod_{p \text{ prime}} F_{k,r}(p^{-s})
\]

and give an application on the average orders of towers of abelian groups. In particular we prove that the number of isomorphism classes of “subgroups of subgroups of ... (\( k-1 \) times) ... of abelian groups” of order at most \( N \) is asymptotic to \( c_k N (\log N)^{k-1} \). This generalises results from Erdős-Szekeres [ES35] and Andersson-Bhowmik [AB] where the corresponding result was proved for \( k = 1 \) and \( k = 2 \).

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1 Introduction

From the classical theory of partitions (see e.g. Hardy-Wright \[HW79\] page 281), we have that if

\[ \alpha_{1,r}(n) = \# \{ q \in \mathbb{Z}^r, 0 \leq q_1 \leq q_i \leq q_{i+1}, \ q_1 + \cdots + q_r = n \} \]

denotes the number of partitions of \( n \) into at most \( r \) parts, then

\[ F_{1,r}(x) = \sum_{n=0}^{\infty} \alpha_{1,r}(n) x^n = \frac{1}{(1-x)(1-x^2) \cdots (1-x^r)}. \]

In particular this implies that we have the functional equation

\[ F_{1,r}(x) = (-1)^r x^{-r(r+1)/2} F_{1,r} \left( \frac{1}{x} \right). \tag{2} \]

In the paper Andersson-Bhowmik \[AB\] the authors considered the related problem of counting pairs \( \{ p_i, q_i \}_{i=1}^r \) such that \( 0 \leq p_i \leq q_i \), \( p_i \leq p_{i+1}, \ q_i \leq q_{i+1} \) and \( q_1 + \cdots + q_r = n \). A recursion formula was obtained to calculate its generation function, and with its help it was proved that if

\[ F_{2,r}(x) = \sum_{n=0}^{\infty} \alpha_{2,r}(n) x^n, \]

denotes its generating function, then it is a rational function that satisfies the functional equation

\[ F_{2,r}(x) = x^{-r(r+3)/2} F_{2,r} \left( \frac{1}{x} \right). \tag{3} \]

We see that these problems can be viewed in matrix terms as counting the number of \( 1 \times r \) integer matrices

\[
\begin{pmatrix}
q_r & \cdots & q_1
\end{pmatrix},
\begin{pmatrix}
0 \leq q_1 \leq \cdots \leq q_r \\
q_1 + \cdots + q_r = n
\end{pmatrix}
\]

and \( 2 \times r \) integer matrices

\[
\begin{pmatrix}
q_r & \cdots & q_1 \\
p_r & \cdots & p_1
\end{pmatrix},
\begin{pmatrix}
0 \leq p_1 \leq \cdots \leq p_r \\
0 \leq q_1 \leq \cdots \leq q_r \\
p_1 \leq q_1, \ q_1 + \cdots + q_r = n
\end{pmatrix}
\]

In this paper we will generalize these problems from \( 1 \times r \) and \( 2 \times r \) matrices to \( k \times r \) matrices.
Definition 1. Let \( \alpha_{k,r}(n) \) count the number of \( k \times r \) matrices such that \( m_{i,j} \geq m_{i+1,j}, \ m_{i,j} \geq m_{i,j+1}, \) and \( m_{1,1} + \cdots + m_{1,r} = n \). Let \( F_{k,r}(x) \) denote the generating function

\[
F_{k,r}(x) = \sum_{n=0}^{\infty} \alpha_{k,r}(n)x^n. \tag{4}
\]

We prove a functional equation for \( F_{k,r}(x) \) that generalizes equations (2) and (3).

Theorem 1. Let \( F_{k,r}(x) \) be defined by Definition 1. Then

\[
F_{k,r}(x) = (-1)^{kr}x^{-r(1+2k)/2}F_{k,r}\left(\frac{1}{x}\right). \]

Non-negative integer valued \( k \times r \) matrices with decreasing rows and decreasing columns (as the matrices counted in Definition 1) are also called plane partitions with \( k \) columns and \( r \) rows. Plane partitions were first studied in MacMohan \[Mac60\] (See also Stanley \[Sta71\]). As an example of a result from the theory: If we define \( q_{k,r}(n) \) as the number of plane partitions with \( k \) columns and \( r \) rows such that the sum over all elements in the matrix equals \( n \), then the generating function can be written as

\[
\sum_{n=1}^{\infty} q_{k,r}(n)x^n = \prod_{i=1}^{r} \prod_{j=1}^{k} (1-x^{i+j-1})^{-1}. \]

In our case we define \( \alpha_{k,r}(n) \) as the number of plane partitions with \( k \) columns and \( r \) rows such that the sum over the elements in the first row equals \( n \). In this case the problem will be more difficult. In fact already for \( k = 2 \) as shown in Andersson-Bhowmik \[AB\] there seems to be no simple expression for the generating function. We also study the limit case

\[
\overline{\alpha}_k(n) = \lim_{r \to \infty} \alpha_{k,r}(n), \quad \text{and} \quad \overline{F}_k(x) = \lim_{r \to \infty} F_{k,r}(x). \tag{5}
\]

The associated zeta functions

\[
Z_{k,r}(s) = \prod_{p \ \text{prime}} F_{k,r}\left(p^{-s}\right),
\]

and

\[
\overline{Z}_k(s) = \prod_{p \ \text{prime}} \overline{F}_k\left(p^{-s}\right),
\]

have interpretations in the context of counting subgroup towers of abelian groups (of rank at most \( r \) or arbitrary rank).
Definition 2. A subgroup tower of a group \( G \) of length \( k \) is defined as a \( k \)-tuple of groups \((G_1, \ldots , G_k)\) where \( G_1 = G \) and \( G_{j+1} \subseteq G_j \).

We say that two subgroup towers \( G \) and \( \tilde{G} \) are isomorphic if \( G_i \cong \tilde{G}_i \) for \( i = 1, \ldots , k \). We will use analytic properties of the zeta functions \( Z_{k,r}(s) \) to prove the following theorem.

Theorem 2. One has that

1. The number of isomorphism classes of subgroup towers of length \( k \) of abelian groups of order at most \( N \) and rank at most \( r \) is asymptotic to \( c_{k,r} N (\log N)^{k-1} \), where \( c_{k,r} \) is a constant.

2. The number of isomorphism classes of subgroup towers of length \( k \) of abelian groups of order at most \( N \) is asymptotic to \( c_k N (\log N)^{k-1} \).

This is a classical result of Erdős-Szekeres [ES35] for \( k = 1 \). For \( k = 2 \) it was proved in Andersson-Bhowmik [AB].

2 The generating function of plane partitions

2.1 The functional equation, P-partitions and Ehrhart reciprocity

Proof of Theorem. Let \( \mathcal{A}_{k,r} \) denote the set of \( k \times r \) integer matrices such that \( m_{i,j} \geq m_{i+1,j} \geq 0 \), \( m_{i,j} \geq m_{i,j+1} \), and let \( \mathcal{B}_{k,r} \) denote the set obtained by replacing all inequalities with strict ones. In particular, if \((m_{i,j})_{i,j} \in \mathcal{B}_{k,r} \) then \( m_{i,j} > 0 \).

We give an element \( M = (m_{i,j}) \in \mathcal{A}_{k,r} \) weight \( w(M) = \prod x_{i,j}^{m_{i,j}} \), and introduce the generating functions

\[
F_{k,r}(t_1,\ldots,t_k) = \sum_{M \in \mathcal{A}_{k,r}} w(M) \\
G_{k,r}(t_1,\ldots,t_k) = \sum_{M \in \mathcal{B}_{k,r}} w(M)
\]

(6)

Specializing

\[
t_{i,j} = \begin{cases} x & i = 1 \\ 1 & i > 1 \end{cases}
\]

(7)

in \( F_{k,r}(t_1,\ldots,t_k) \) we recover the counting function \( \mathcal{P} \).

Denote by \( C_m \) the \( m \) element chain, and by \( P = P_{k,r} = (C_k \times C_r) \) the \( k \) times \( r \) “grid poset”. We have that elements in \( \mathcal{A}_{k,r} \), which are plane
partitions, correspond to $P$-partitions, i.e., order-reversing maps from $P$ to $\mathbb{N}$, and that elements in $\mathfrak{B}_{k,r}$, which are a special type of plane partitions, correspond to strict $P$-partitions, i.e., strictly order-reversing maps from $P$ to $\mathbb{N}$. This correspondence is illustrated in Figure 1.

Figure 1: $P$-partitions of the poset $C_3 \times C_2$ correspond to $2 \times 3$ plane partitions

Hence, from the reciprocity theorem for $P$-partitions [Sta97, Thm 4.5.7] (a special case of Ehrhart reciprocity (see also [BR], [Ehr77])) we have that

$$G_{k,r}(t_{1,1}, \ldots, t_{k,r}) \prod_{i=1}^{k} \prod_{j=1}^{r} t_{i,j} = (-1)^{kr} F_{k,r} \left( \frac{1}{t_{1,1}}, \ldots, \frac{1}{t_{k,r}} \right). \quad (8)$$

Furthermore, the obvious bijection

$$\phi : \mathfrak{A}_{k,r} \to \mathfrak{B}_{k,r}$$

$$M \mapsto M + \begin{pmatrix} k + r - 1 & k + r - 2 & \cdots & k+1 & k \\ k + r - 2 & k + r - 3 & \cdots & k & k - 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ r + 1 & r & \cdots & 3 & 2 \\ r & r - 1 & \cdots & 2 & 1 \end{pmatrix} \quad (9)$$

shows that

$$G(t_{1,1}, \ldots, t_{k,r}) = F(t_{1,1}, \ldots, t_{k,r}) \prod_{i=1}^{k} \prod_{j=1}^{r} t_{i,j}^{i+j-1}. \quad (10)$$

Combining (8) and (10) we get that

$$F_{k,r}(t_{1,1}, \ldots, t_{k,r}) = (-1)^{kr} F_{k,r} \left( \frac{1}{t_{1,1}}, \ldots, \frac{1}{t_{k,r}} \right) \prod_{i=1}^{r} \prod_{j=1}^{k} t_{i,j}^{-i-j} \quad (11)$$
which, using the specialization (7) and the relation
\[ \sum_{j=1}^{r} (k + j - 1) = r(r - 1 + 2k)/2, \]
becomes
\[ F_{k,r}(x) = (-1)^{k} F_{k,r} \left( \frac{1}{x} \right) x^{-r(r-1+2k)/2}. \] (12)

2.2 An explicit formula

Since the total extensions of the poset \( P^* \) is enumerated by standard Young tableaux of shape \( r^k \), and since a descent corresponds to a box labeled \( \ell + 1 \) occurring in a higher row than the box labeled \( \ell \), we get, by using Theorem 4.5.4 in Stanley [Sta97], an explicit (but not very efficient) formula for \( F_{k,r}(x) \).

**Theorem 3.** Let \( T \) be a standard tableau with shape \( r^k = (r, r, \ldots, r) \), let \( T_\ell \) be the subtableau consisting of the boxes with labels \( \leq \ell \), and let \( c(T_\ell) \) be the number of boxes in the first row of \( T_\ell \). Let \( d(T_\ell) = c(T_\ell) \) if \( \ell < rk \) and if in \( T \) the box labeled \( \ell + 1 \) occurs in a higher row than the box labeled \( \ell \), and let \( d(T_\ell) = 1 \) otherwise. Then

\[ F_{k,r}(x) = \sum_{T \in \text{SYT}(r^k)} \prod_{\ell=1}^{kr} \frac{\ell^{d(T_\ell)}}{1 - t^{c(T_\ell)}} \] (13)

As an example, we take two rows and three columns. The hook-length formula [Ful97] shows that there are \( 6!/(4 \times 3 \times 3 \times 2 \times 2 \times 1) = 5 \) standard Young tableaux of the desired shape. They are tabulated below, together with their contribution to \( F_{2,3}(x) \).

| \( x \) | \( x^2 \) | \( x^3 \) |
|--------|--------|--------|
| \( x \) | \( x^2 \) | \( x^3 \) |

\[
\begin{align*}
1 2 3 4 5 6 & \quad \frac{1}{(1-x)(1-x^2)(1-x^3)^2} \\
1 2 5 3 4 6 & \quad \frac{x^2}{(1-x)(1-x^2)^3(1-x^3)^2} \\
1 3 4 2 5 6 & \quad \frac{x}{(1-x)^2(1-x^2)(1-x^3)^3} \\
1 3 5 2 4 6 & \quad \frac{x}{(1-x)^2(1-x^2)^2(1-x^3)^2} \\
1 8 1 5 2 4 6 & \quad \frac{x^3}{(1-x)^2(1-x^2)^2(1-x^3)^2} \\
\end{align*}
\]
For instance, the tableau

\[
T = \begin{array}{ccc}
1 & 2 & 4 \\
3 & 5 & 6
\end{array}
\]

contributes

\[
x^2 \frac{1}{(1-x)(1-x^2)^2(1-x^3)^3},
\]
as we see by studying the initial subtableux:

\[
\begin{array}{cccccc}
\times & \times & \times & \times & \times & \times \\
1 & 1 & 1 & 1 & 1 & 1
\end{array}
\]

By adding the five terms corresponding to the standard tableaux we get

\[
F_{2,3}(x) = \frac{(1 + 2x + 2x^2 + x^3 + x^4)(1 + x + 2x^2 + 2x^3 + x^4)}{(-1 + x)^6(1+x)^3(1+x+x^2)^4},
\]

which coincide with the result calculated in Andersson-Bhowmik [AB]. By similar reasoning we obtain

\[
F_{3,2}(x) = \frac{x^4 + 2x^3 + 4x^2 + 2x + 1}{(x-1)^6(x+1)^5},
\]

and

\[
F_{4,2}(x) = \frac{x^6 + 3x^5 + 9x^4 + 9x^3 + 9x^2 + 3x + 1}{(x-1)^8(x+1)^7}.
\]

We see that Theorem 3 implies that the generating function is a rational function.

**Corollary 1.** The generating function \( F_{k,r}(x) \) can be written as

\[
F_{k,r}(x) = \frac{Q_{k,r}(x)}{((1-x)\cdots(1-x^{kr}))^{kr}}
\]

where \( Q_{k,r}(x) \) is a polynomial of degree \((k-1)r(kr + r - 1)\).

**Proof.** This follows from Theorem 3. The degree of the polynomial \( Q_{k,r}(x) \) follows from Theorem 4.

**Remark 1.** The estimation for the degree of the denominator in Corollary 1 is an overestimation. In fact for e.g. \( k = 2 \) it can be shown (either by the recursion formula or by more careful consideration of the Young tableaux) that \( F_{2,r}(x) = P_{2,r}(x)/((1-x)^2(1-x)^3\cdots(1-x^{r+1})) \) where \( P_{2,r}(x) \) is a polynomial of degree \( r(-5 + 3r + 2r^2)/6 \).
2.3 Further properties of the generating function

We calculate the first few values of $\alpha_{k,r}(n)$:

**Lemma 1.** One has that

(i) $\alpha_{1,r}(n) = p_r(n)$.

(ii) $\alpha_{k,r}(0) = 1$.

(iii) $\alpha_{k,r}(1) = k$.

(iv) $\alpha_{k,r}(2) = \begin{cases} \frac{k(k+1)}{2}, & r = 1, \\ k(k+1), & r \geq 2, \end{cases}$

(v) $\alpha_{k,1}(n) = \binom{n+k-1}{k-1}$.

**Proof.** (i) This is just the classical restricted partition function (Hardy-Wright [HW79] page 281).

(ii) The only matrix that will contribute is the zero matrix.

(iii) The matrices that will contribute have the first $j$ rows $(1,0,\ldots,0)$ and $k-j$ rows $(0,\ldots,0)$, for $j = 0,\ldots,k-1$. There are $k$ such matrices.

(iv) The matrices that will contribute will either have

(a) $j_1$ rows $(2,0,\ldots,0)$, $j_2$ rows $(1,0,\ldots,0,1)$ and $j_3$ rows $(0,\ldots,0)$ such that $j_1 + j_2 + j_3 = k$ and $j_1 \geq 1$, or

(b) $j_1$ rows $(1,1,0,\ldots,0)$, $j_2$ rows $(1,0,\ldots,0)$ and $j_3$ rows $(0,\ldots,0)$ such that $j_1 + j_2 + j_3 = k$ and $j_1 \geq 1$.

The number in each case will be $\binom{k+1}{2}$ and in general we get the contribution $2\binom{k+1}{2} = k(k+1)$. If $r = 1$ only the first case will contribute and we will instead get just $\binom{k+1}{2}$.

(v) The matrices that will contribute will have the rows $(a_i)$ for $i = 1,\ldots,k$ and $n = a_1 \geq a_2 \geq \cdots \geq a_k \geq 0$. There are $\binom{n+k-1}{k-1}$ such matrices. \hfill \Box

We see that Lemma 1(v) implies that

$$F_{k,1}(x) = (1-x)^{-k}. \quad (17)$$
One may ask the following question: What happens with the generating function $F_{k,r}(x)$ when $r$ or $k$ tends to infinity. From Lemma 1 (iii) it follows that $\lim_{k \to \infty} F_{k,r}(x)$ is divergent. However in the case when $r \to \infty$ we can take the limit. We define

$$\overline{\alpha}_k(m) = \alpha_{k,m}(m).$$ \hspace{1cm} (18)

**Lemma 2.** One has that

$$0 \leq \alpha_{k,r}(m) \leq \alpha_{k,r+1}(m),$$

and in particular

$$\alpha_{k,r}(m) = \overline{\alpha}_k(m). \hspace{1cm} (r \geq m)$$ \hspace{1cm} (19)

**Proof.** It is clear that $\alpha_{k,r}(m) \geq 0$, since it is a counting function. That it increases in $r$ follows from the fact that every $k \times r$ matrix which is counted in $\alpha_{k,r}(m)$ will correspond to the $k \times (r+1)$ matrix where we adjoin a zero column as the last column, which is counted in $\alpha_{k,r+1}(m)$. That $\alpha_{k,r}(m) = \overline{\alpha}_k(m)$ for $(r \geq m)$ follows from the fact that we can have at most $m$ non zero columns under the given conditions (the maximum number of non zero columns will be attained exactly when the first row has $m$ ones and $(r - m)$ zeroes. \hfill \square

**Lemma 3.** One has that

$$q(m) \leq \overline{\alpha}_k(m) \leq ((m + 1)q(m))^k,$$

where $q(m)$ denote the classical partition function.

**Proof.** The lower bound is obtained by counting the matrices with the first row an arbitrary partition of $n$ and $k - 1$ rows identically zero. For the upper bound we use the fact that each row in a matrix that we count for $\alpha_{k,m}(m)$ will be a classical partition for some number $0 \leq j \leq m$. Hence we have the inequality

$$\alpha_{k,m}(m) \leq (q(0) + \cdots + q(m))^k$$

Since the classical partition function is an increasing function this implies that

$$\alpha_{k,m}(m) \leq ((m + 1)q(m))^k.$$

\hfill \square
We introduce the generating function
\[ F_k(x) = \sum_{n=1}^{\infty} \alpha_k(n)x^n. \] (20)
and we prove (The proof is essentially the same as the proof of Lemma 2 of Andersson-Bhowmik [AB]):

**Lemma 4.** With \( F_{k,r}(x) \) and \( F_k(x) \) defined as above one has that \( F_{k,r}(x) \) and \( F_k(x) \) are analytic functions in the unit disc with integer power series coefficients such that \( F_{k,r}(0) = F_k(0) = 1 \). Furthermore the function \( F_k(x) \) satisfies the inequality
\[ F_k(x) \geq \frac{1}{\prod_{k=1}^{\infty}(1-x^k)}. \quad (0 < x < 1) \]

**Proof.** The power series coefficients of \( F_{k,r}(x) \) and \( F_k(x) \) are integers since they are counting functions and by Lemma 1 (ii) and eq. (19) we have that \( \alpha_{k,r}(0) = 1 \) and \( F_k(0) = 1 \), which implies \( F_{k,r}(0) = F_k(0) = 1 \). By the well known generating function for the classical partition function
\[ \sum_{n=0}^{\infty} q(n)x^n = \frac{1}{\prod_{n=1}^{\infty}(1-x^n)}, \quad (0 < x < 1) \] (21)
and the lower bound in Lemma 3
\[ q(n) \leq \overline{\alpha}_k(n), \]
this gives us the lower bound in Lemma 3. Equation (21) also implies that the generating function of the partition function is analytic in the unit disc, and hence the classical partition function \( q(n) \) is of subexponential order. This implies that \(((n+1)q(n))^k\) is of subexponential order and by the upper bound in Lemma 3 so is \( \alpha_{k,r}(n) \), and also \( \overline{\alpha}_k(n) \) since \( 0 \leq \alpha_{k,r}(n) \leq \overline{\alpha}_k(n) \). This proves that \( F_k(x) \) and \( F_{k,r}(x) \) are analytic in the unit disc.

2.4 The polynomials \( \alpha_{k,r}(n) \)

**Proposition 1.** For fixed \( n, r \), the quantity \( \alpha_{k,r}(n) \) is a polynomial of degree \( n \) in \( k \), with leading coefficient \( \alpha_{1,r}(n)/n! \).

Similarly, for fixed \( n \), the quantity \( \overline{\alpha}_k(n) \) is a polynomial of degree \( n \) in \( k \), with leading coefficient \( \overline{\alpha}_1(n)/n! \).
Proof. A plane partition $M = (m_{ij})$ of dimension $k \times r$ can be regarded as a sequence of $k$ partitions $m_1 \leq m_2 \leq \cdots \leq m_k$ with at most $r$ parts, where each $m_i$ corresponds to the $k + 1 - i$'th row of $M$. Hence, $M$ can be viewed as a multichain (i.e., a chain with possible repetitions) of length $k$ in the restricted Young lattice $\mathcal{Y}_r$ of partitions with at most $r$ parts. If the last row of $M$ sums to $n$, then the associated multichain ends at level $n$ in the ranked lattice $\mathcal{Y}_r$.

Now consider the principal order ideal $I$ in $\mathcal{Y}_r$ generated by $m_k$. The whole multichain $m_1 \leq m_2 \leq \cdots \leq m_k$ is contained in $I$, and the number of such $k$-multichains in $I$ is given by $Z(I; k) = \zeta^k(0, m_k)$, where $\zeta$ is the (combinatorial) zeta function of $\mathcal{Y}_r$. It is well known that this expression is a polynomial in $k$; indeed, it is called the Zeta-polynomial of $I$, see for instance [Aig79, IV:2].

Clearly, we get the desired quantity $\alpha_{k,r}(m)$ by summing over all Zeta-polynomials of principal ideals of partitions of $n$ with at most $r$ parts,

$$\alpha_{k,r}(n) = \sum_{\lambda \vdash n, \ell(\lambda) \leq r} Z(I_{\lambda}; k)$$ (22)

This is a finite sum of polynomials in $k$, hence a polynomial in $k$.

It follows from standard properties of the Zeta-polynomial (see again [Aig79, IV:2]) that each $Z(I_{\lambda})$ is an integer-valued polynomial in $k$ of degree $n$ with non-negative coefficients, hence that $\alpha_{k,r}(n)$ and $\alpha_{k,n}(n)$ both have degree $n$. Evaluated at 1, this polynomial is $\alpha_{1,r}(n)$, the number of partitions of $n$ with at most $r$ parts. We get from the elementary theory of integer-valued polynomials [see for instance [Sta96, Corollary 1.3]) that the leading coefficient of the polynomial $\alpha_{k,r}(n)$ is $\alpha_{1,r}(n)/n!$.

To obtain the result for the unrestricted coefficient $\alpha_{k,n}(n)$, replace $\mathcal{Y}_r$ with $\mathcal{Y}$.

Furthermore:

**Proposition 2.** The polynomial $\alpha_{k,r}(n) \in \mathbb{Q}[k]$ is divisible by $(k + s)$ for all $s < \frac{n}{r+1}$. There is some $c$, independent of $n$, such that the polynomial $\alpha_{k,n}(n) \in \mathbb{Q}[k]$ is divisible by $(k + s)$ for all integers $s < c\sqrt{n}$.

**Proof.** Since $\mathcal{Y}_r$ is a locally finite distributive lattice, every interval is a finite distributive lattice. Hence, the Möbius function for an interval $[\lambda, \tau]$ is either 0 or $(-1)^{rk(\lambda) - rk(\tau)}$. Thus, we can apply a result of Stanley’s [Aig79, Proposition 4.9] which says that (putting $P = I_{\lambda} \subset \mathcal{Y}_r$ for some partition of $n$ with at most $r$ parts)

$$Z(P; -k) = (-1)^{rk(P)} Z(P; k)$$ (23)
where \( \mathcal{Z}(P; k) \) counts the number of \( k \)-multichains

\[
0 \leq z_1 \leq \cdots \leq z_k = 1 \quad \text{with} \quad \mu(z_{i-1}, z_i) \neq 0 \quad \text{for all } i.
\]

Furthermore, \( \mu(z_{i-1}, z_i) \neq 0 \) if and only if \([z_{i-1}, z_i]\) is a Boolean lattice, which happens if and only if the supremum of the points in the interval is a Boolean algebra.

If there is an upper bound \( L \) on the length of Boolean subintervals of \( P \), then it follows that \( \mu(\lambda, \tau) = 0 \) whenever \( \text{rk}(\lambda) - \text{rk}(\tau) > L \). Thus, the minimal length of a chain (24) is \( \frac{N}{L} \), where \( N \) is the length of \( P \), i.e., the length of the longest chain in \( P \). Hence \( \mathcal{Z}(P; k) = 0 \) for \( k < \frac{L}{1+L} \), hence, by (23), the same holds for \( Z(P; k) \). It follows that the polynomial \( Z(P; k) \) is divisible by \( k + s \) for \( s < \frac{N}{L+1} \).

We now turn to partitions with an unlimited number of parts. Let the truncated Young lattice \( \mathcal{Y}^{\leq n} \) consist of partitions of \( s \leq n \). To estimate, by the above method, how many zeroes at negative integers the polynomial \( \alpha_k(n) \) will have, we would need to bound the size of the intervals in \( \mathcal{Y}^{\leq n} \) that are Boolean algebras. Such an interval would look like \([\lambda, \tau]\) with \( \lambda = (\lambda_1, \ldots, \lambda_v) \), and \( \tau \) the supremum of the elements covering \( \lambda \). A partition \( \tilde{\lambda} \) covering \( \lambda \) will either be \( \tilde{\lambda} = (\lambda_1, \ldots, \lambda_v, 1) \) or else \( \tilde{\lambda} = (\lambda_1, \ldots, \lambda_i + 1, \ldots, \lambda_v, 1) \). The position \( i \) where the extra element is inserted must either be 1, or else \( \lambda_i < \lambda_i - 1 \). If there are \( \ell \) such positions (the position at the end included) then there are \( \ell \) elements covering \( \lambda \), the supremum \( \tau \) of these elements is a partition of \( |\lambda| + \ell \), so \([\lambda, \tau]\) is a Boolean algebra of length \( \ell \).

The partition \( \lambda = (s, s-1, \ldots, 1) \) is a partition of \( \binom{s+1}{2} \) and is covered by \( s + 1 \) partitions; the supremum is a partition of \( \binom{s+1}{2} + s + 1 = \binom{s+2}{2} \), which should be no larger than \( n \) for the interval to fit inside \( \mathcal{Y}^{\leq n} \).

This is maximal, so that any Boolean algebra inside \( \mathcal{Y}^{\leq n} \) have length less or equal to \( \sqrt{\frac{1+8n-3}{2}} \). The second assertion now follows.

Using (22) and a MAPLE package by Stembridge [Ste02], we can calculate \( \alpha_{k,r}(n) \) using the following simple commands:

```maple
with(SF);
with(posets);
read("young_lattice");
alphaPart := proc(part,varname)
    zeta(young_lattice(part),varname);
```

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Recall that $\alpha_{k,1}(n) = \binom{k+n-1}{n}$. We tabulate the polynomials $\overline{a}_k(n)$, $\alpha_{2,k}(n)$ and $\alpha_{2,k}(n)$ below.

$\overline{a}_k(1) = k$
$\overline{a}_k(2) = k(k + 1)$
$\overline{a}_k(3) = \frac{1}{6}k(k + 1)(4k + 5)$
$\overline{a}_k(4) = \frac{5}{12}k(k + 1)^2(k + 2)$
$\overline{a}_k(5) = \frac{1}{60}k(k + 1)(k + 2)(13k^2 + 36k + 21)$
$\overline{a}_k(6) = \frac{1}{180}k(k + 1)(k + 2)^2(19k^2 + 58k + 33)$
$\overline{a}_k(7) = \frac{1}{2520}k(k + 1)(k + 2)(k + 3)(116k^3 + 508k^2 + 688k + 263)$
$\overline{a}_k(8) = \frac{1}{10080}k(k + 1)(k + 2)(k + 3)(191k^4 + 1338k^3 + 3297k^2 + 3330k + 1084)$
and
\[ \alpha_{k,2}(1) = k \]
\[ \alpha_{k,2}(2) = k (k + 1) \]
\[ \alpha_{k,2}(3) = 1/2 k (k + 1)^2 \]
\[ \alpha_{k,2}(4) = 1/4 k (k + 2) (k + 1)^2 \]
\[ \alpha_{k,2}(5) = 1/12 k (k + 2)^2 (k + 1)^2 \]
\[ \alpha_{k,2}(6) = 1/36 k (k + 3) (k + 2)^2 (k + 1)^2 \]

and
\[ \alpha_{k,3}(1) = k \]
\[ \alpha_{k,3}(2) = k (k + 1) \]
\[ \alpha_{k,3}(3) = 1/6 k (4 k + 5) (k + 1) \]
\[ \alpha_{k,3}(4) = 1/24 k (9 k + 7) (k + 2) (k + 1) \]
\[ \alpha_{k,3}(5) = 1/120 k (k + 2) (k + 1) (21 k^2 + 52 k + 27) \]
\[ \alpha_{k,3}(6) = 1/240 k (k + 2) (k + 1) (17 k^3 + 82 k^2 + 125 k + 56) \]

### 2.5 The growth of the coefficients $\overline{\alpha}_k(n)$

For $k = 1$ we have that $\overline{\alpha}_1(n)$ equals the classical partition function $q(n)$. Thus $\overline{\alpha}_k(n)$ is a proper generalisation of the partition function $q(n)$. For $q(n)$ we have good asymptotics by a theorem of Hardy-Ramanujan \[\text{HR18}\]

\[ q(n) \sim e^{\pi \sqrt{2n/3}} \frac{\sqrt{n}}{4n^{3/4}}. \]  \hspace{1cm} (25)

This is a strong result and it seems difficult to obtain a similar formula for the general case. Equation (25) implies that

\[ \log q(n) \frac{\sqrt{n}}{\sqrt{n}} = \pi \sqrt{2/3} + o(1). \]  \hspace{1cm} (26)

We will thus study

\[ \log \overline{\alpha}_k(n) \frac{\sqrt{n}}{\sqrt{n}}. \]  \hspace{1cm} (27)

We improve on the lower bound in Lemma \[\text{3}\]

**Proposition 3.** One has that $\overline{\alpha}_k(kn) \geq q(n)^k$.  

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\[ \alpha^2(n) \log q(n), \ \text{grey line} \]

\[ \log \pi \sqrt{2n/3}. \ \text{Right graph: from top to bottom, } \log \alpha^k(n) \]

**Proof.** Let \( r = kn \). And let \( B = \{q_{i,j}\} \) be a \( k \times r \) matrix such that \( q_{i,j} \geq q_{i,j+1} \geq 0 \) and the sum of each row \( q_{i,1} + \cdots + q_{i,r} = n \) is a partition of \( n \). It is clear that there are exactly \( q(n)^k \) such matrices. For each matrix \( B \) of this type we can construct a matrix \( A \)

\[
A = \begin{pmatrix}
q_{1,j} + \cdots + q_{1,k} & \cdots & q_{r,1} + \cdots + q_{r,k} & 0 & \cdots & 0 \\
q_{1,j} + \cdots + a_{1,k-1} & \cdots & q_{r,1} + \cdots + q_{r,k-1} & 0 & \cdots & 0 \\
& \vdots & \vdots & \vdots & \vdots \\
q_{1,j} + q_{1,2} & \cdots & q_{r,1} + q_{r,2} & 0 & \cdots & 0 \\
q_{1,j} & \cdots & q_{r,1} & 0 & \cdots & 0
\end{pmatrix}
\]

This matrix will be counted in \( \alpha_{k,kn}(kn) \). Hence \( \pi_k(kn) = \alpha_{k,kn}(kn) \geq q(n)^k \).

Together with the upper bound in Lemma 3 and eq. (26) this implies the following Corollary:

**Corollary 2.** Suppose that \( k \geq 1 \) is an integer. Then

\[
\sqrt{k} + o(1) \leq \frac{\log \pi_k(n)}{\pi \sqrt{2n/3}} \leq k + o(1).
\]

The case \( k = 2 \) can be studied numerically by means of the recursion formula given as Proposition 2 in Andersson-Bhowmik [AB]. For \( k \geq 3 \) we can use the \( \alpha_{k,r}(n) \)-polynomials that we calculated, although the algorithm is less efficient than the recursion formula for the case \( k = 2 \) and it will be difficult to calculate \( \alpha_{k,r}(n) \) for \( n \geq 30 \). We show related plots in Figure 2 above.

Even though Corollary 2 is somewhat of an improvement to the trivial lower bound \( 1 + o(1) \) we would like to know better. The graphs in Figure 2 suggests that equation (27) might have a limit and we propose the following problem.

**Problem 1.** Find an asymptotic formula for \( \log \pi_k(n) \).
2.6 Nonisomorphic subgroup towers of abelian $p$-groups

Let $p$ be a prime. An abelian $p$-group is an abelian group of order $p^n$. Each abelian $p$-group of rank at most $r$ and order $p^n$ is isomorphic to a group

$$G = \mathbb{Z}/(p^{q_1}\mathbb{Z}) \oplus \cdots \oplus \mathbb{Z}/(p^{q_r}\mathbb{Z}) \quad (0 \leq q_1 \leq \cdots \leq q_r, \ q_1 + \cdots + q_r = n)$$

With the definition of subgroup towers (Definition 1) we see that each subgroup tower of length $k$ and maximal group of order $p^n$ and rank $r$ will be isomorphic to

$$(G_1, \ldots, G_k) \cong \left( \oplus_{j=1}^r \mathbb{Z}/(p^{m_{1,j}}\mathbb{Z}), \ldots, \oplus_{j=1}^r \mathbb{Z}/(p^{m_{k,j}}\mathbb{Z}) \right),$$

such that

$$m_{i,j} \geq m_{i+1,j} \geq 0, \quad m_{i,j} \geq m_{i,j+1} \quad \text{and} \quad m_{1,1} + \cdots + m_{1,r} = n.$$ 

These are exactly our plane partitions with $k$ rows and $r$ columns such that the sums of the elements in the first row equals $n$ that we already studied. Hence, we get the following Lemma:

**Lemma 5.** (i) The number of isomorphism classes of subgroup towers of length $k$ such that the maximal group has order $p^n$ and rank at most $r$ equals $\alpha_{k,r}(n)$.

(ii) The number of isomorphism classes of subgroup towers of length $k$ such that the maximal group has order $p^n$ equals $\overline{\alpha}_k(n)$.

We will see how this will give average orders for nonisomorphic subgroup towers in the next section.

3 The zeta function and nonisomorphic subgroup towers of finite abelian groups

3.1 Average orders

If $G$ is a finite abelian group of order $n$ we have by the fundamental theorem of finite abelian groups that

$$G \cong G_{p_1^{a_1}} \oplus \cdots \oplus G_{p_r^{a_r}}$$

where $n = p_1^{a_1} \cdots p_m^{a_m}$ and $G_{p_j^{a_j}}$ is a $p$-group of order $p^{a_j}$. By Lemma 5 we see that

**Proposition 4.** Suppose that $n = p_1^{a_1} \cdots p_m^{a_m}$. Then

(i) The number of isomorphism classes of subgroup towers of length $k$ such that the maximal group has order $n$ and rank at most $r$ equals

$$A_{k,r}(n) = \prod_{j=1}^m \alpha_{k,r}(a_j).$$
(ii) The number of isomorphism classes of subgroup towers of length \(k\) such that the maximal group has order \(n\) equals \(A_k(n) = \prod_{j=1}^{m} \alpha_k(a_j)\).

We see that \(A_{k,r}(n)\) and \(A_k(n)\) are multiplicative functions and as in Andersson-Bhownik \([AB]\) we can introduce the zeta functions

\[
Z_{k,r}(s) = \prod_{p \text{ prime}} F_{k,r}(p^{-s}) = \sum_{n=1}^{\infty} A_{k,r}(n)n^{-s},
\]

and

\[
Z_k(s) = \prod_{p \text{ prime}} F_k(p^{-s}) = \sum_{n=1}^{\infty} A_k(n)n^{-s}.
\]

and they will have interpretations in terms of nonisomorphic subgroup towers of abelian groups. By Lemma 4 and Dahlquist’s theorem \([Dah52]\) we obtain the Proposition.

**Proposition 5.** Let \(\epsilon > 0\). There exist a positive integer \(P\) such that

\[
Z_{k,r}(s) = \zeta(s)^k \times \left( \prod_{p < P} F_{k,r}(p^{-s})(1 - p^{-s})^k \right) \times \left( \prod_{m=2}^{\infty} \zeta_p(ms)^{\beta_{k,r}(m)} \right),
\]

and

\[
Z_k(s) = \zeta(s)^k \times \left( \prod_{p < P} F_k(p^{-s})(1 - p^{-s})^k \right) \times \left( \prod_{m=2}^{\infty} \zeta_p(ms)^{\beta_k(m)} \right),
\]

valid for \(\text{Re}(s) > \epsilon\), where

\[
\zeta_p(s) = \zeta(s) \times \left( \prod_{p < P} (1 - p^{-s}) \right) = \prod_{p \geq P} (1 - p^{-s})^{-1}.
\]

Furthermore

\[
\beta_{r,k}(m) = \sum_{d|m} \mu\left( \frac{m}{d} \right) \frac{d}{m} B_{r,k}(d), \quad \text{where} \quad \log F_{k,r}(x) = \sum_{m=1}^{\infty} B_{k,r}(m)x^m,
\]

\[
\beta_k(m) = \beta_{k,m}(m), \quad \text{and} \quad \beta_{k,r}(m) \quad \text{and} \quad \beta_k(m) \quad \text{are integers}.
\]
Proof. This follows from Lemma 4 and Dahlquist’s [Dah52] Lemma 2.

From this, the fact that $F_r(x)$ has no zeroes for $0 < x < 1$ (positive power series coefficients), and from the explicit values of $\alpha_{k,r}(1)$, $\alpha_{k,r}(2)$ given by Lemma 4 and 17, the following Corollary follows:

**Corollary 3.** One has that

\[(i)\]  
\[Z_{k,r}(s) = \begin{cases} 
\zeta(s)^k, & r = 1, \\
\zeta(s)^k \zeta(2s)^{k(k+1)/2} G_{k,r}(s), & r \geq 2,
\end{cases}\]

\[(ii)\]  
\[\overline{Z}_k(s) = \zeta(s)^k \zeta(2s)^{k(k+1)/2} \overline{G}_k(s),\]

where $G_{k,r}(s)$ and $\overline{G}_k(s)$ are Dirichlet series absolutely convergent and without real zeroes for $\text{Re}(s) > 1/3$.

The average order of the Dirichlet series coefficients $A_{k,r}(n)$ and $\overline{A}_k(n)$ which count the relevant subgroup towers (Lemma 4) will come from the pole of the corresponding zeta-functions at $s = 1$ and by a standard Tauberian argument [SG00, Theorem 4.20], Corollary 3 implies Theorem 2.

### 3.2 The polynomials $\beta_{k,r}(n)$ and analytic properties of the zeta-function

By the inequality in Lemma 4

\[F_k(x) \geq \frac{1}{\prod_{k=1}^{\infty} (1-x^k)} \quad (0 < x < 1)\]

it is clear that $F_k(x)$ can not be written as a finite product

\[F_k(x) = \prod_{j=1}^{m} (1-x^j)^{b_j}. \quad (b_j \in \mathbb{Z})\]

Hence Dahlquist’s theorem also implies the following proposition.

**Proposition 6.** The zeta-functions $\overline{Z}_k(s)$ can be meromorphically continued to $\text{Re}(s) > 0$ but not beyond the imaginary axis.

This problem can also be studied for $Z_{k,r}(s)$. For further analytic information about the zeta functions $\overline{Z}_k(x)$ and $Z_{k,r}(s)$ we need to study the coefficients $\beta_{k}(n)$ and $\beta_{k,r}(n)$. By their definition in Proposition 5 and the fact that $\alpha_{k,r}(n)$ are polynomials in $k$ of degree $n$ (Proposition 1) the following Proposition follows.

**Proposition 7.** For fixed $n,r$, the quantity $\beta_{k,r}(n)$ is a polynomial in $k$, as is $\beta_{k}(n)$.
We tabulate the first few polynomials.

\[
\begin{align*}
\beta_k(1) &= k \\
\beta_k(2) &= \frac{1}{2} k(k + 1) \\
\beta_k(3) &= \frac{1}{6} k(k + 1)(k + 2) \\
\beta_k(4) &= -\frac{1}{12} (k - 3) k(k + 1)(k + 2) \\
\beta_k(5) &= \frac{1}{120} k(k + 1)(2k + 1) (k^2 + k + 18) \\
\beta_k(6) &= \frac{1}{120} k(k + 1)(k + 2) (k^3 - 6k^2 - 4k + 29) \\
\beta_k(7) &= -\frac{(k - 3) k(k + 1)(k + 2) (8k^3 + 49k + 48)}{1260}
\end{align*}
\]

In Andersson-Bhowmik we calculated the first values for \( k = 2 \)

\[
\beta_2(1), \ldots, \beta_2(15) = 2, 3, 3, 4, 2, 6, 1, 4, 6, 2, 0, 12, -1, -2, 9, \ldots
\]

For \( k = 3, 4 \) we calculate

\[
\begin{align*}
\beta_3(1), \ldots, \beta_3(15) &= 3, 6, 6, 10, 0, 21, -5, 0, 51, -42, -6, -110, -100, 151, -492 \\
\beta_4(1), \ldots, \beta_4(13) &= 4, 10, 10, 20, -10, 57, -19, -72, 324, -370, -92, 1137, -2406
\end{align*}
\]

In general we see that the polynomials \( \beta_k(1), \beta_k(2) \) and \( \beta_k(3) \) are positive for \( k \geq 1 \) and the polynomial \( \beta_k(4) \) is negative for \( k \geq 4 \). This implies some analytical properties of the zeta function.

**Proposition 8.** One has that \( \mathcal{Z}_k(s) \) and \( \mathcal{Z}_{k,r}(s) \) for \( r \geq 4 \) have no poles for \( \Re(s) > 1/4 \) except for a pole of order \( k \) at \( s = 1 \), a pole of order \( k(k+1)/2 \) at \( s = 1/2 \) and a pole of order \( k(k+1)(k+2)/6 \) at \( s = 1/3 \). Under the Riemann hypothesis it follows that \( \mathcal{Z}_k(s) \) has no poles for \( \Re(s) > 1/8 \) except for possible poles at \( 1/4, 1/5, 1/6, 1/7 \).

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