Kelvin–Helmholz instability in thermoviscous free shear flow

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Abstract. This paper is focused on the entrainment and mixing in the nonisothermal shear flow, wherein the velocity profile discontinuity arises due to viscosity-temperature relation, i.e. thermoviscosity. Kelvin–Helmholz instability generated at the interface paves the way to large-scale entrainment which is analysed in terms of perturbation growth rate and momentum thickness. The flow is simulated in the plane domain with periodic boundary conditions using second-order accuracy explicit CABARET numerical scheme in weakly compressible formulation. Initial instability evolution is fully governed by a universal scaling parameter $k_t$, composed of Reynolds number $Re$ and kinematic viscosity ratio $R_\nu$. There are seven flow configurations determined by nonlinear mode interaction, vorticity convection and diffusion and cascade vortex merger. The problem at issue appears to be analogues to the classical boundary layer theory that totally fortifies its remarkable features.

1. Introduction

The noted influence of large-scale motion on the fine-scale fluid dynamics is particularly strong in viscosity-stratified [1, 2] flows wherein depending on the initial conditions viscosity variation can have either stabilizing or destabilizing effect on the instability development. However, Kolmogorov K41 theory widely used for large Reynolds number flows assumes that [3] the statistical properties of turbulence in the inertial and dissipative intervals do not depend on the way of turbulence excitation and are universally determined by dissipation rate $\varepsilon$, kinematic viscosity $\nu$, and scale $l$ (hypothesis No. 1), while in the inertial interval the number of governing parameters reduces to dissipation rate and scale (hypothesis No. 2). In a view of aforementioned hypotheses a fluid viscosity appears to be an important actor only at small scales, and, therefore, should not have a decisive influence on the process of large-scale mixing. As a result, the overwhelming majority of papers consider either single-fluid or density-stratified flows.

Experimental investigations showed that in the co-current two-fluid flow with different kinematic viscosities the mixing process appears to be multiscale, whereas in constant viscosity flows (CVF) the conventional Kelvin–Helmholtz vortices are generated. Voivenel et al [4] proposed a phenomenological description of mixing, where the patches of the surrounding fluid are entrained into the jet due to various instabilities and create additional resistance, slowing the main fluid flow and causing pulsations of radial velocity.

Campbell and Turner [5] experimentally investigated the influence of viscosity discontinuity considered low-viscosity turbulent jet injected into a more viscous host liquid. Predetermined viscosity ratio $R_\nu = \nu_{\text{high}}/\nu_{\text{low}}$ varied in the range 1–400 allowed to control mixing progress: if the
viscosities of two liquids are approximately equal, then turbulent fluctuations in the immersed jet lead to intense mixing, generating a transitional layer with stratified viscosity. Conversely, if $R_\nu$ exceeds 400 any noticeable macroscopic engulfment does not take place. In the latter case, it turns out that the large impulse of the injected fluid does not propagate into a more viscous host media, as the perturbations at the interphase boundary are not large enough to overcome suppressing impact of the viscous stresses.

Numerical modeling of free shear flows in similar formulation [6] for two viscosity ratios $R_\nu = 1$ (CVF) and variable viscosity flow (VVF) $R_\nu = 9$ allows to provide the phenomenological and statistical description of the earliest stages of the time evolution of the mixing layer. Reynolds number based on the initial momentum thickness $\delta_{\theta,0}$ and the mean viscosity $\nu_{\text{ref}} = \nu_{\text{low}} + \nu_{\text{high}}$, takes the value $\text{Re}_{\delta_{\theta,0}} = U_0\delta_{\theta,0}/\nu_{\text{ref}} = 160$ for the CVF and $\text{Re}_{\delta_{\theta,0}} = 16$, 32 for the VVF case. For the latter viscosity variation affects both small scales and the averaged velocity profile that is distorted by simultaneous action of medium velocity and viscosity gradients. The VVF possess an accelerated transition to turbulence with ill-defined coherent periodic motions, and the whole temporal evolution of momentum thickness can described by two parameters—Reynolds number $\text{Re}$ and the viscosity ratio $R_\nu$.

Departing from main phenomenological results obtained by Campbell and Turner [5] indicating mixing being suppressed by viscosity rupture increase and scaled by $k_t = \text{Re}/R_\nu$ parameter in present paper we turn to the problem formulation (section 2) for a plane free shear single-fluid flow with a viscous stratification in an inhomogeneous temperature field, which arises from the dependence of dynamic viscosity on temperature $\mu = \mu(T)$. We analyze mixing in terms of the instability growth rate patterns $\gamma = \gamma(t, \text{Re}, R_\nu)$ in section 3 and momentum thickness $\delta_h = \delta_h(t, \text{Re}, R_\nu)$ in section 5 confirming governing $k_t$-dependence. There are also sections 4 and 6 specially devoted to the grid convergence of instability growth rate and momentum thickness correspondingly. In section 7 we provide a phenomenological description of entrainment and mixing regimes in a wide range of $R_\nu$ and $\text{Re}$. The results obtained allow to treat the problem under consideration as a special case of boundary layer theory.

2. Problem formulation

Mixing layer evolution intrinsically belongs to large variety of free turbulent flows originally studied by Brown and Roshko [7]. The flow is arranged so that the gradients of streamwise velocity and viscosity are oppositely directed: top and bottom flow regions of higher kinematic viscosity $\nu_{\text{high}}$ are at rest, while the core has a velocity $U_0$ and a lower viscosity $\nu_{\text{low}}$, thereby simulating the mixing process in an immersed laminar jet (figure 1). The longitudinal velocity shear is smoothed out using the tanh function:

$$
U = U_0/2(1 + \tanh(r(y - 1/4L_Y))), \quad y \in [0, L_Y/2],
$$

$$
U = U_0/2(1 + \tanh(r(3/4L_Y - y))), \quad y \in (L_Y/2, L_Y].
$$

(1)

Mixing layer initial thickness controlled by the parameter $r$ should be very small, thereby we set it to be 16 times higher ($r = 1280$) compared to value used in [8]. The initial condition for the longitudinal velocity is complemented by the transverse harmonic perturbation

$$
V = \delta \sin(2\pi k x), \quad x \in [0, L_X], \quad k = 6.
$$

(2)

In (2) we use the most unstable mode determined by Sandham and Reynolds [9] that allows reducing rated time interval. Periodic conditions are set at the boundaries of square computational domain. Considering thermoviscous fluid one can express the viscosity discontinuity at the interface of the mixing layer in terms of temperature difference between cold viscous host fluid (top and bottom regions) and the hot immersed jet (central region). Jet velocity remains unchanged within all series of calculations while Reynolds number and viscosity
Figure 1. Streamwise $U$ (a blue line) and spanwise $V$ (a red line) velocity distributions.

ratio are varying via reference viscosity $\mu_0$ and temperature of the cold liquid reservoir. Given the law of viscosity–temperature dependence

$$\mu = \mu_0 e^{-\beta(T - T_0)/T_0},$$

and also assuming that the jet temperature is equal to the reference value $T_0$, we obtain the temperature of the host fluid

$$T_{\text{host}} = T_0 \left(1 + \frac{1}{\beta} \ln R_\nu\right).$$

So, the transverse temperature distribution can be defined as

$$T = T_1 \tanh \left[r(y - 3/4L_Y)\right], \quad y \in [0, L_Y/2],$$

$$T = T_1 \tanh \left[r(3/4L_Y - y)\right], \quad y \in (L_Y/2, L_Y],$$

wherein $T_1 = 1/2 \left[(T_{\text{host}} + T_0) + (T_0 - T_{\text{host}})\right]$.

The initial velocity profile being inflectional both in the CVF and VVF cases, provides necessary conditions for Kelvin–Helmholtz instability.

We performed numerical simulation using in-house implementation of CABARET scheme in weakly compressible formulation [10], all necessary parameters are specified in table 1. In the updated version the heat transfer equation was added and the new temperature variable $T$ acts as an active scalar affecting the flow via variable viscosity. We used a set of refined square grids with $512 \times 512$ (i), $1024 \times 1024$ (ii), $2048 \times 2048$ (iii), $4096 \times 4096$ (iv) cells, the majority of calculations was performed on grids (i) and (ii), grids (iii) and (iv) were used for grid convergence tests. As already noted Reynolds number $\text{Re}$ and the viscosity ratio can be combined into one universal parameter

$$k_t = UL/\nu_{\text{high}} = UL/(R_\nu \nu_{\text{low}}) = \text{Re}/R_\nu,$$

with typical scale $L = L_Y/2$ accepted as the width of immersed jet in experimental investigations. Henceforward, we use the transverse size of computational domain $L = L_Y$ as a scale to define the corresponding Reynolds number $\text{Re}_{L_Y}$. 
| Parameter name                  | Notation | Set value |
|--------------------------------|----------|-----------|
| Sound speed                    | $c$      | 10.0      |
| Reference viscosity            | $\mu_0$ | 1         |
| Reference temperature          | $T_0$    | 127.0     |
| Exponential index              | $\beta$ | $-2.3$    |
| Channel length                 | $L_X$    | 1.0       |
| Channel width                  | $L_Y$    | 1.0       |
| Reference density              | $\rho_0$| 1000.0    |
| Thermal conductivity           | $\lambda$| 0.3       |
| Heat capacity                  | $C_p$    | 2000.0    |
| Grid size                      | $n_X \times n_Y$ | 512 $\times$ 512, 1024 $\times$ 1024, 2048 $\times$ 2048, 4096 $\times$ 4096 |
| Courant number                 | CFL      | 0.15      |
| Disturbance amplitude          | $\delta$| 0.001     |
| Mode number                    | $k$      | 6         |
| Centerline velocity            | $U_0$    | 1.0       |

In a number of mixing problems Re is usually defined basing either initial momentum thickness $\delta_{\theta,0}$ or initial vorticity layer $\delta_{\omega,0}$, that in our case yields $\delta_{\theta,0} \approx 3.86 \times 10^{-4}$ and $\text{Re}_{\delta_{\omega,0}} \approx 0.386$. However, generally accepted definition of Re given in [6] seems to be improper in certain cases if we consider the mixing process in terms of boundary-layer theory.

### 3. Instability increment of thermoviscous shear layer

According to experimental data [5], mixing process is completely suppressed, unless $k_t$ exceeds a certain threshold, and the entrainment and subsequent mixing strongly depends on $R_\nu$ for $k_t \in [7, 70]$. In this section the growth rate $\gamma$ being one of the substantial quantities of linear stability theory is determined from the solution of simultaneous Navier–Stokes and heat conduction equations using CABARET finite difference numerical method.

A very abrupt temporal decay of enstrophy integral

$$E(t) = \int_0^{L_X} \int_0^{L_Y} \omega(t)^2 dy dx,$$

where $\omega$ denotes vorticity, for all pairs of perturbation amplitudes $\delta_1$ and $\delta_2$ ($\delta_2 > \delta_1$) in the range $\delta_2/\delta_1 = 10^{-4}$ prevents using growth rate calculation approach [8] based on the time shift $\delta t$ of the enstrophy curve determined by the formula

$$\gamma = \frac{1}{\delta t} \ln \frac{\delta_2}{\delta_1}. \quad (7)$$

As an efficient alternative one can perform one-dimensional Fourier transform of (2) to calculate normalized growth rate curve in the vicinity of initial position of mixing layers at ordinates $y \approx L_Y/4$ and $y \approx 3/4L_Y$. Thus, the reduced amplitude at subsequent time instants can be determined as

$$V_6(t) = \max_y \frac{\delta_6[V(t)]}{\delta_6[V(0)]}, \quad y \in [1/5L_Y, 3/10L_Y] \cup [7/10L_Y, 4/5L_Y], \quad (8)$$
wherein \( \tilde{g}[V(0)] \) is Fourier transform of initial amplitude of perturbation mode \( k = 6 \) and \( \tilde{g}[V(t)] \) is the same value at arbitrary point of time.

In such problems, Reynolds number is determined resting upon the initial thickness of the vorticity layer

\[
\delta_{\omega,0} = \frac{\Delta U}{(\partial U/\partial y)_{\text{max},0}},
\]

where \( \Delta U \) stands for a total velocity difference and \( (\partial U/\partial y)_{\text{max},0} \) denotes maximum transverse gradient at a start time. Therefore, at large gradients, it can depend on the quality of the grid resolution, in particular for the grid \( 512 \times 512 \) cells we obtain \( \delta_{\omega,0} = 4.0 \times 10^{-3} \), while for the grid \( 1024 \times 1024 \) cells it grows to \( \delta_{\omega,0} = 7.8 \times 10^{-3} \). One can compare the ranges of Reynolds numbers based on the computational domain size \( L \) and the initial vorticity thickness on grid (ii) \( \text{Re}_{\delta_{\omega,0}} = 7.81-781 \) for the CVF case and also \( \text{Re}_{\delta_{\omega,0}} = 0.15-520 \) for the VVF case.

The amplitude growth curve \( V_6 = V_6(t) \) at the initial stages has completely different behavior depending on \( L \) and \( \nu \). In particular, in some regions, the exponential growth predicted by the linear theory is observed, that accelerates when approaching some time point. The linear growth time interval can have a different length, that complicates approximation procedure. Sometimes calculation sequence reveals some deviations from the predictions of linear theory, such as an amplitude growth deference, and parabolic or linear behavior in the wide vicinity of the neutral stability curve. These considerations forced us using exponential approximation

\[
V_6(t) = e^{\gamma t},
\]

along with parabolic

\[
V_6(t) = 1 + \gamma t + (\gamma t)^2/2 + O((\gamma t)^3)
\]

and linear

\[
V_6(t) = 1 + \gamma t + O((\gamma t)^2)
\]

Taylor series expansions of (10), always selecting that one yielding best fit of the amplitude in the initial time interval.

There are four different regions of the growth rate curve behavior shown in figure 2. In the transitional regimes, a limited growth of the mixing layer was noted, sometimes accompanied by the appearance of oval vortices “squeezed” in the mixing layer. The vorticity isolines in this case resemble phase trajectories of a nonlinear pendulum compressed vertically (separatrix and “cat’s eye” [11]). In the region of rapid growth, the instability increment is a two-parameter function of the Reynolds number \( R_{\nu} \), of the form \( \gamma = A_1 \left( R_{\nu} \right)^{B_1} \).

When increasing the ratio of viscosities \( \nu \), sharp growth section of \( \gamma \) shifts to the region of higher \( R_{\nu} \), this area becomes more and more stretched, and the exponent values rise from \( B_1 \approx 4 \) to \( B_1 \approx 7 \), while in the range \( R_{\nu} \gtrsim 50000 \) all dispersion curves are flattened with a slight decrease in \( \gamma \). The general dependence of \( \gamma = \gamma(R_{\nu}) \) at \( R = \text{const} \) confirms the experimental observations [5]: as \( R_{\nu} \) grows in the range \( 1-100 \), \( \gamma \) is actively suppressed at the initial stage of evolution described by the linearized equations. On the other hand, dispersion dependencies obtained refer exclusively to the initial interval \( t \approx 0.1 \) preceding the nonlinear stage that leads to a deviation from the law \( V_6 \propto e^{\gamma t} \).

It is worth noting that the results of perturbation growth in a wide range of parameters \( \nu \), \( R_{\nu} \) are limited: in many cases, a perturbation being in good progress at the linear stage is then suppressed at \( t \approx 1 \). Moreover, as it follows from equation (8) we consider fast growing \( (\gamma > 0) \) and minimally decreasing \( (\gamma < 0) \) disturbance of the selected mode. The most complex dynamics of \( \gamma = \gamma(t) \) was detected in the neighborhood of the marginal stability curve as the
Figure 2. Instability growth rate $\gamma$ as the function of Reynolds number $Re$ for various $R_{\nu}$ plotted in ($\gamma, Re_{LY}$)-scale. The numerals in the legend indicate the viscosity ratio $R_{\nu}$. Horizontal bars denote growth rate $\gamma$ error range, while solid black lines separate four growth rate regions: 1—the region of sharply negative values, 2—the neutral stability curve neighborhood, 3—the zone of sharp rise and 4—the region of asymptotic values at $Re_{LY} \gtrsim 50000$.

Linear or parabolic (not exponential) growth here is superimposed by fine-amplitude (0.001) periodic oscillations which are independent of grid density and sound speed. Probably, these oscillations can be associated with the influence of high-order terms neglected in linear stability analysis.

One can also compare the obtained values of $\gamma$ with the results of the classical linear theory of inviscid Kelvin–Helmholtz instability and findings of other researchers. Asymptotic values of $\gamma$ on all the curves in figure 2 correspond to $\gamma \sim 10$, whereas linear theory yields $\gamma = \frac{2\pi U_0}{\lambda} \approx 18$, that is almost twice higher compared to our results. Probably, the role of viscosity suppressing disturbance growth rate is significant in the chosen range of Reynolds numbers $Re_{LY} \lesssim 10^5$.

To match the obtained data with the results of [9], we plot the dependencies $\gamma = \gamma(Re_{LY})$ according to [6] versus Reynolds number

$$Re_{\delta, 0} = \frac{U\delta_{\omega, 0}}{\nu_{\text{ref}}}$$

and the scaled time $t^* = tU/\delta_{\omega, 0}$ (figure 3) The blue curve shows the growth rate $\gamma$ at Mach number $M = 0.2$ for the arbitrary shape disturbance (two-dimensional inviscid eigenfunction), the red one depicts an analogous value at $M = 0.4$. The instability calculated via the formula (8) grows faster than predicted by linear theory.
Figure 3. The set of growth rate curves at various viscosity ratios $R_\nu$ plotted in $(\gamma^*, \text{Re}_{\delta,0})$ scale. The results obtained on grid (i) are compared to the data of other researchers [9]: the blue curve was calculated for Mach number $M = 0.2$, the red one denotes results for $M = 0.4$. The numerals in legend correspond to the values of $R_\nu$.

Figure 4. The section of $V_\delta = V_\delta(t)$ obtained on the sequence of refined grids (i)–(iv).

4. Grid convergence and accuracy of growth rate values
The remarkable feature of problem at issue is that growth rate $\gamma$ and momentum thickness $\delta_{\theta,0}$ have different convergence rates, so that grid (i) is sufficient for adequate calculation of $\gamma$
Figure 5. Normalized momentum thickness for the VVF case $R_\nu = 1$ at various Reynolds numbers obtained on grid (ii).

(figure 4), while to integrate $\delta_\theta$ grid cells number should be not less than $1024 \times 1024$ (figure 11). To estimate the accuracy of $V_6 = V_6(t)$ and $\gamma$ in the mixing layer we perform a series of calculations on the sequence of refined grids (i)–(iv) at $R_\nu = 1, R e_L = 4000$. When passing to grid (ii), faster perturbation growth is observed ($\sigma_\gamma \approx 0.0051$), while further mesh refinement shows complete coincidence of growth rate curves in the mixing layer, in addition, more precise amplitude values of $V_6 = V_6(t)$ do not affect fitted $\gamma$ within an approximation error. The total calculation error $\varepsilon_\gamma^{\text{total}}$ for $\gamma$ can be estimated as the set of three relative errors $\sigma_\gamma^{\text{grid}}$, determined by the unfinished grid convergence, $\sigma_\gamma^{\text{comp}} \approx 0.01$ denoting actual accuracy of weakly compressible fluid model, and also by standard error of approximation SE of fitted $\gamma$:

$$\varepsilon_\gamma^{\text{total}} = \left\{ \left[ (\sigma_\gamma^{\text{grid}})^2 + (\sigma_\gamma^{\text{comp}})^2 \right] \gamma^2 + \text{SE}^2 \right\}^{1/2}.$$  

(15)

Compressibility error $\sigma_\gamma^{\text{comp}}$ is estimated [12] by Mach number $M$ as

$$\sigma_\gamma^{\text{comp}} = M^2 = \left[ \frac{\max_{t \in [0, t_{\text{final}}]} (U, V)}{c} \right]^2,$$  

(16)

where $t_{\text{final}}$ denotes calculation time. Since the curve $\gamma = \gamma(t)$ at the initial time interval $t \lesssim 0.2$ appears to be linear (or parabolic) in many modes, then the error due to Taylor series expansion (11), (12) is not considered. The error range of growth rate $\gamma$ denoted by horizontal bars is shown in figure 4.

5. Momentum thickness analysis

Momentum thickness $\delta_\theta$ has been used for many decades of precomputer era as a simple way to determine the drag force $W$ acting upon arbitrary shaped cylindrical body [13] in external flow:

$$W = \rho U_\infty^2 \delta_\theta.$$  

(17)
where $U_\infty$ is the velocity at an infinite distance from the body, and the momentum thickness calculated by the formula

$$\delta_\theta(t) = \frac{1}{U_0^2} \int_0^{L_Y} U(t,y) [U_0 - U(t,y)] dy. \quad (18)$$

In the present paper, $\delta_\theta$ is treated as a quantitative measure of the engulfment rate at free boundary layer between hot and cold liquids. Under the conditions of the algorithm $\delta_\theta$ is integrated as long as the nondecelerated layers (stream tubes) still exist, i.e. longitudinal velocity in the core flow does not differ from its initial condition. Start time value of the integral (18) is used to normalize momentum thickness obtained at subsequent instants:

$$\delta_\theta(t) = \frac{\delta_\theta(t)}{\delta_\theta(0)}. \quad (19)$$

To maintain the stability of the calculation procedure for different $R_\nu$ and $Re_{L_Y}$, the sound velocity should be tuned in the range $c = 10.0$–$80.0$. We used higher values in case of steep initial grid viscosity gradients at relatively small Reynolds numbers $Re_{L_Y} \gtrsim 1000$, while the main corpus of calculations was carried out at $c = 10.0$.

When mixing begins, $\delta_\theta$ increases monotonically as $\sqrt{t}$, while its growth rate falls quite abruptly with $Re_{L_Y}$ growing, as shown in figure 5 for the CVF case. The same pattern is also observed (figure 6) for VVF, which is formed as a result of the temperature drop between a moving and a stationary liquid. A remarkable feature is momentum thickness being suppressed with $R_\nu$ increasing (figure 7), though large viscous stresses at the interface provide the possibilities to rapid momentum transfer. In fact, $R_\nu$ growing leads to accelerated suppression of transverse harmonic perturbations (2), and after while hot less viscous fluid starts sliding over cold medium along the contact boundary. Thus, more viscous patches of liquid act as obstacles that inhibit flow of immersed jet during the mixing, which in our case leads to an accelerated growth of $\delta_\theta$. For the turbulent flow considered in [6] these patches increase transverse velocity fluctuations and accelerate energy transfer from the averaged flow to turbulence further followed by a rapid decay of the turbulent kinetic energy after a certain instant of time.

Plotting $\delta_\theta = \delta_\theta(t)$ in log-log scale, we introduce one-parameter approximating function satisfying the asymptotic behavior:

$$\lim_{t \to \infty} = 1$$

and also having a slope of “1/2” in logarithmic coordinates. A suitable fitting function is $\delta_\theta = A\sqrt{t + 1/A'}$, where the coefficient $A$ depends on other parameters $A = A(Re_{L_Y}, R_\nu, c)$. Assuming such a relation, we represent the curves of $A$ in the form $A = A(Re_{L_Y})$ for different values of $R_\nu$ and $c$ (figure 8). These graphs appear to have regular shape with the growth rate of $\delta_\theta$ being independent of sound speed (see magenta stars at $R_\nu = 100$ in figure 8). When presenting the obtained data in logarithmic coordinates, one can employ another function $A = B/\sqrt{Re_{L_Y}}$ suitable for approximation. The set of graphs can also be reviewed in $(A, [k_i]^{1/2})$-scale, so that $A = A(Re_{L_Y}, R_\nu)$ turns into linear functions $A = Bk_i^{-1/2}$ (figure 9). All of them pass approximately through the origin of coordinates $(0,0)$ ($Re \to \infty$) approaching it at different angles. It is instructive to consider the slope of $A$ as a function of $R_\nu$: if $R_\nu \to \infty$ ($R_\nu = 100$), then $B \to 1$, on the other hand, at $R_\nu \to 1$ (or at $R_\nu \to 0$, this case, however, is out of scope) $B$ should be a confined function (figure 10). As an appropriate fitting function satisfying asymptotic constraints we choose

$$B = 1 + \frac{C}{1 + R_\nu}, \quad (20)$$
Figure 6. Normalized momentum thickness $\delta$ as a function of time for the VVF case at $R_{\nu} = 50$ for various Reynolds numbers. The results are obtained on grid (ii).

Figure 7. Temporal dependence of normalized thickness $\delta$ at Reynolds number $Re_{LY} = 7500$ for various viscosity ratios $R_{\nu}$ calculated on grid (ii).

where $C$ is a constant. Thus, momentum thickness dependence $\delta = \delta(t, Re_{LY}, R_{\nu})$ appears to be well parametrized and can be represented in the final form as

$$\delta = \left( \frac{C + R_{\nu}}{1 + R_{\nu}} \right)^2 \frac{t}{k_t} + 1 \right]^{1/2},$$

(21)
Figure 8. Coefficient $A$ graphs versus Reynolds number $Re_{L\gamma}$ for various $R_{\nu}$.

Figure 9. The coefficient $A$ as a function of the dimensionless group $k_t$ at various $R_{\nu}$ denoted in the legend as ‘Rn’. Approximating curves are shown with red lines.

where $C_1 = C + 1$. Asymptotics $\delta_\theta \sim \sqrt{t}$ survives only at the initial stages ($t \ll 0.3 - 0.4$), and later momentum thickness reaches some constant value suggesting limited mixing in some modes. The coefficient $1/\sqrt{Re_{L\gamma}}$ in general dependence (21) arises due to intrinsic similarity of problem considered to the classical boundary-layer theory. In our case, liquid interface
Figure 10. Coefficient $B$ as the function of the viscosity ratio $R_\nu$ (the values are reduced to $B|_{R_\nu=100}$). The fitted function is marked in red.

with a stationary medium acts as rigid wall, so this coefficient corresponds to the total scale of transverse lengths in the general theory [13]. By the nature of the initial conditions, the simulated flow relates to transient fluid motions [14], while the evolution of a resting viscous layers resembles acceleration flow problem. Thus, the resulting dependence $\delta_\eta \propto \sqrt{t}$ is similar to a plane wall suddenly set in motion (the first Stokes problem). In this case Navier–Stokes equations are reduced to the velocity diffusion equation

$$\frac{\partial U}{\partial t} = \nu \frac{\partial^2 U}{\partial y^2},$$

where $\nu = \mu/\rho_0$ denotes kinematic viscosity, resolved by the error function $U = U_0 \text{erf}(\eta)$ with $\eta = y/(2t)$ as a dimensionless group. In the general theory the boundary layer spans up to $U = 0.01U_0$, that yields $\eta \approx 2$ and, consequently, $\delta \propto \sqrt{\eta t}$. Such a dependence probably persists both for the displacement thickness and $\delta_\theta$.

6. Grid convergence and momentum thickness accuracy

The section of curve $\delta_\theta = \delta_\theta(t)$ calculated on a sequence of square grids with a different number of cells in one direction—512, 600, 768, 1024, 2048, 4096 is shown in figure 11. With the number of cells increasing, the graph slightly shifts downwards. Moving to finer grid (ii) causes approximately 0.72-fold reduction of coefficient $A$ for all modes. However, it turns out that grid (ii) is sufficient to calculate $\delta_\theta$ with an accuracy of $\sigma_A^\text{grid} \approx 0.01187$ (relative to grid (iv)). Assuming the last-mentioned the error of the obtained values can be estimated as

$$\varepsilon_A = \left\{ \left( \sigma_A^{\text{grid}} \right)^2 + \left( \sigma_A^{\text{comp}} \right)^2 \right\}^{1/2} + \text{SE}^2,$$

wherein $\sigma_A^{\text{comp}} \approx 0.01$ denotes a weakly compressible fluid accuracy and SE is the standard approximation error of the values of $A$. 

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Figure 11. Temporal evolution of momentum thickness $\delta_\theta$ for a different number of grid nodes. The data are presented in a logarithmic scale.

Figure 12. The relative location of regions where mixing develops in a similar way. The region number corresponds to the variant of the process evolution.

7. Mixing process phenomenology
The whole diversity of interactions between immersed jet and resting fluid can be reduced to a countable, albeit not very small, number of scenarios. The evolution of the mixing processes is determined by various factors such as vorticity convection and diffusion, generation of the
6th and sometimes 12th harmonics, cascade vortex merger due to two-dimensional nature of the flow and by large-scale mode $k = 2$ also arising during the mixing. There are several regions in the parameter space ($R_\nu, Re_{LY}$), corresponding to different flow evolution histories. Long-term calculation was carried out for a limited number of cases noted in the following description and indicating approximate ranges of regions (figure 12).

Case 1 corresponds to the region of small Reynolds numbers expanding towards larger $Re_{LY}$ with increasing $R_\nu$ and including regimes $1 \lesssim R_\nu \lesssim 100$, $Re_{LY} = 1000$; $5 \lesssim R_\nu \lesssim 100$, $Re_{LY} = 5000$; $50 \lesssim R_\nu \lesssim 100$, $Re_{LY} = 10000$. The diffusion of the vorticity occurs either with the same rate for CVF case or at different rates in VVF in cross direction to the interface. The vorticity diffusion rate into the high-viscous fluid will be approximately $R_\nu$ times faster than in the low-viscous one. In this case, vortex street is not observed and perturbations of the transverse velocity turn out to be extinguished.

Case 2 belongs to the region of continuous (or weakly-stratified) viscosity at high Reynolds numbers $Re_{LY} \gtrsim 10000$, where vorticity convection predominates over the diffusion. As the viscosity is rather low, the initial vorticity forms two thin vortex sheets which begin to roll up into vortices with relatively slow broadening at $t = t_1$, this vortex layer has a thickness $h \approx 0.1$. The speed of further merger apparently depends on the Reynolds number: as it increases, the cascade fusion of the vortices accelerates. The second merger almost doubles mixing layer thickness ($h \approx 0.2$), the third one increases the mixing layer to $h \approx 0.3$. The fourth merger is accompanied by the destruction of the central zone of the jet. Figure 13 shows the decay time of the vortex sheet and the formation of vortices in rolling modes $k = 6$ and $k = 12$.

Moderate Reynolds numbers and viscosity discontinuities (case 3) refer to an intermediate region with incompletely defined boundaries, connecting regimes $(2, 5000)$ and $(10, 10000)$. Although the dynamics in this range is quite different, several stages can be distinguished. The value of the viscosity jump controls the entrainment depth of hot jet, a stronger discontinuity decreases depth size. At the time $t = t_1$, closed billows of low-viscosity liquid are formed enhancing active mixing. However, further vortex merging does not occur and the evolution is determined exclusively by vorticity diffusion. The formation of an intermediate mixing layer, perturbed by a harmonic $k = 2$, is shown in figure 14.

Case 4 refers to developed mixing of liquid layers with different viscosities and exists in quite narrow zone including points $(2, 10000)$ and $(5, 10000)$. At $t = t_1$ an engulfment occurs with
subsequent billows closure at $t = t_2$ forming an active mixing layer of a thickness $h \approx 0.007–0.025$ (figure 15). At the time $t = t_3$ large-scale mode $k = 2$ begins to distort mixed layer due to vortex merger, that leads to large-scale mixing following by the closure of the $k = 2$-mode billows and the growth of the layer thickness up to $h \approx 0.02–0.025$ with elliptical eddies. At the initial stage, the vortex sheet diffuses on both sides of the interface and while the vorticity drops in the diffusing layer, vortices of elliptical shape begin to appear. At the time $t = t_4$ they begin to “wind” the original vortex sheet while the new vortex size increases to $l \approx 0.08$. Approaching time $t = t_5$ a regular vorticity layer of constant thickness $h \approx 0.06–0.08$ is created, consisting of vortices of elliptical shape. Again the vortices merge in the vorticity layer at $t = t_6$ thereby deforming it. This large vortex also winds the remaining vorticity layer, that also leads to the appearance of a regular vortex street at $t = t_7$. Finally at the time $t = t_8$ the remaining pairs of vortices merger take place.
Case 5 depicts a rapid multi-stage merging starting from mode \( k = 12 \) and associated with the range of moderate \( R_\nu \) and large \( Re_{LY} \): \( 2 \lesssim R_\nu \lesssim 10 \), \( Re_{LY} \lesssim 10^5 \). At the time \( t = t_1 \) the small-scale entrainment of a low-viscosity jet into a stationary liquid as well as the coiling of the billows of the sixth harmonic begins, at this instant the vortex roll up overtakes the diffusion. In further four merger stages take place. Figure 16 shows the third merger and the beginning of large-scale mixing.

Case 6 exists only at large viscosity discontinuities in the range \( 50 \lesssim R_\nu \lesssim 100 \) and \( Re_{LY} \sim 50000 \) wherein the formation of a thin mixing layer is exposed to harmonic oscillations. At \( t = t_1 \), the \( k = 6 \) mode waves forms the billows closing in a mixing layer of a certain thickness. A further evolution is determined by diffusion of vorticity shown in figure 17.

Case 7 refers to the region of maximum viscosity discontinuities \( 50 \lesssim R_\nu \lesssim 100 \) and \( Re_{LY} \sim 10^5 \). The vortices of hot low-viscous liquid slide over the surface of steady layers.

**Figure 16.** Temperature \( T \) field at time \( t = 2.98 \) with \( R_\nu = 2 \), \( Re_{LY} = 50000 \) (case 5).

**Figure 17.** Vorticity \( \omega \) field at time \( t = 2.51 \) for \( R_\nu = 50 \), \( Re_{LY} = 50000 \) (case 6).
Figure 18. Vorticity $\omega$ field at time $t = 6.63$ with $R_\nu = 100$, $Re_{LY} = 100000$ (case 7).

Thin initial vortex sheet diffuses rapidly into the streamless region and is perturbed at the same time from the low-viscosity jet. At $t = t_1$ the crests of less viscous liquid overturn into host fluid then the billows are formed creating mixing layer, wherein liquid is homogenized. A new mixed layer is slightly distorted by mode $k = 2$. As the vorticity diffuses, patches of vorticity remain on the side of the low-viscosity liquid in the mixing layer (figure 18). Thus, small low-viscous eddies slide along the contact boundary without active mixing.

8. Conclusion

In present paper we have made an attempt to simulate the mixing of fluids with different viscosities calculating the plane layered thermoviscous fluid flow that allowed to reveal some interesting features. An immersed hot jet penetrating into cold host fluid appears to belong to a variety of boundary layer phenomena [13]. At the initial instant drag seizes outer regions of the jet, then the viscosity leads to the deceleration of the liquid particles located closer and closer to the centerline. Finally the entire jet turns into boundary layer wherein mixing processes begin to develop.

An increase of the viscosity ratio $R_\nu$, ceteris paribus, suppresses instability growth rate. Although the employed method of calculating $\gamma$ overestimates growth rate value, it is able to predict perturbation growth pattern at the linear stage. Momentum thickness being a convenient way of describing the mixing process generally depends on a single parameter introduced experimentally. Increasing viscosity discontinuity and Reynolds number enforces the suppression of disturbances at the contact boundary constraining growth of $\delta_\theta$ and inhibiting the mixing, so that for large viscosity ratios the central jet just slides along the contact boundary. From the phenomenological point of view, there are approximately 7 mixing regimes that pass each other at different $R_\nu$, $Re_{LY}$. Actually, such a variety is explained by the relation of the terms of vorticity equation. Thus, the flow evolution (vortex sheet roll-up or slow diffusion) is determined by either convective transfer or diffusion terms at a given time instant. Another feature that distinguishes two-dimensional calculation from experimental practice is the observed reverse enstrophy cascade expressed in subsequent merger of small vortices into larger ones.

It is worth to mention that numerical values of hydrodynamic quantities analyzed in this paper possess different convergence rates at the sequence of refined grids. Nevertheless our experience shows that when increasing number of cells the same stages of flow evolution are
observed, developing at a slightly slower rate (that as apparent from the dependence of $\delta_2$ on cell density) when moving from a $512 \times 512$ cells grid to a $1024 \times 1024$ cells grid. The latter is valid only if the spurious vortex [15] is not formed during the vorticity layer roll-up.

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