BOREL MEASURES WITH A DENSITY ON A COMPACT SEMI-ALGEBRAIC SET

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Abstract. Let $K \subset \mathbb{R}^n$ be a compact basic semi-algebraic set. We provide a necessary and sufficient condition (with no \textit{a priori} bounding parameter) for a real sequence $y = (y_\alpha), \alpha \in \mathbb{N}^n$, to have a finite representing Borel measure absolutely continuous w.r.t. the Lebesgue measure on $K$, and with a density in $\cap_{p \geq 1} L_p(K)$. With an additional condition involving a bounding parameter, the condition is necessary and sufficient for existence of a density in $L_\infty(K)$. Moreover, nonexistence of such a density can be detected by solving finitely many of a hierarchy of semidefinite programs. In particular, if the semidefinite program at step $d$ of the hierarchy has no solution then the sequence cannot have a representing measure on $K$ with a density in $L_p(K)$ for any $p \geq 2d$.

1. Introduction

The famous Markov moment problem (also called the $L$-problem of moments) is concerned with characterizing real sequences $(s_n), n \in \mathbb{N},$ which are moment of a Borel probability measure $\mu$ on $[0, 1]$ with a bounded density with respect to (w.r.t.) the Lebesgue measure. It was posed by Markov and later solved by Hausdorff with the following necessary and sufficient condition:

\begin{equation}
1 = s_0 \quad \text{and} \quad 0 \leq s_{nj} \leq c/(n + 1), \quad \forall n, j \in \mathbb{N},
\end{equation}

for some $c > 0$, where $s_{nj} := (-1)^{n-j} \binom{n}{j} \Delta^{n-j} s_j$, and $\Delta$ is the forward operator $s_n \mapsto \Delta s_n = s_{n+1} - s_n$. Similarly, with $p > 1$, if in (1.1) one replaces the condition “$s_{nj} \leq c/(n + 1)$ for all $n, j \in \mathbb{N}$”, with the condition

\begin{equation}
\sup_n \left( \frac{1}{n + 1} \sum_{j=0}^{n} ((n + 1) s_{nj})^p \right)^{1/p} < c,
\end{equation}

then one obtains a characterization of real sequences having a representing Borel measure with a density in $L_p([0, 1])$ with $p$-norm bounded by $c$. For an illuminating discussion with historical remarks the reader is referred to Diaconis and Freedman [4] where the authors also make a connection with De Finetti’s theorem on exchangeable 0-1 valued random variables. In addition, Putinar [10, 11] has provided a characterization of extremal solutions of the two-dimensional $L$-problem of moments.

Observe that the above condition (1.1) is stated in terms of linear inequalities on the $s_j$’s. An alternative \textit{if and only if} characterization is via positive definiteness of some sequence $t_n(c)$ related to the sequence $(s_n)$, as described in Ahiezer and Krein

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Contribution: Consider a compact basic semi-algebraic set $K \subseteq \mathbb{R}^n$ of the form (1.3)\[ K := \{ x \in \mathbb{R}^n : g_j(x) \geq 0, \ j = 1, \ldots, m \}, \]for some polynomials $g_j \in \mathbb{R}[x], \ j = 1, \ldots, m$. For every, $p \in \mathbb{N}$, denote by $L_p(K)$ the Lebesgue space of functions such that $\int_K |f|^p \lambda(dx) < \infty$. We then provide a set of conditions (S) with no $\hat{a}$ priori bound $c$, and such that:

- A real sequence $y = (y_n), \alpha \in \mathbb{N}^n$, has a finite representing Borel measure with a density in $\cap_{p \geq 1} L_p(K)$, if and only if (S) is satisfied. In particular, if (S) is violated we obtain a condition (with no $\hat{a}$ priori bounding parameter $c$) for non existence of a density in $\cap_{p \geq 1} L_p(K)$ (hence no density in $L_\infty(K)$ either).

- A real sequence $y = (y_n), \alpha \in \mathbb{N}^n$, has a finite representing Borel measure with a density in $L_\infty(K)$ if and only if (S) and an additional condition (involving an $\hat{a}$ priori bound $c > 0$), are satisfied.

In addition, the conditions (S) consist of a hierarchy of Linear Matrix Inequalities (LMIs) (again in the spirit of [1, 6]) and so can be tested numerically via available semidefinite programming softwares. In particular, if a finite Borel measure $\mu$ does not have a density in $\cap_{p \geq 1} L_p(K)$ (and so no density in $L_\infty(K)$ either), it can be detected by solving $finitely$ many semidefinite programs in the hierarchy until one has no feasible solution. That is, it can be detected from $finitely$ many of its moments. This is illustrated on a simple example.
Conversely, if the semidefinite program at step $d$ of the hierarchy has no solution, then one may conclude that the real sequence $y$ cannot have a representing measure on $K$ with a density in $L_p(K)$, for any $p \geq 2d$.

So a distinguishing feature of our result is the absence of an à priori bound $c$ in the set of condition (S) to test whether $y$ has a density in $\cap_{p=1}^{\infty} L_p(K)$. Crucial for our result is a representation of polynomials that are positive on $K \times \mathbb{R}$, by Powers [9]; see also Marshall [7, 8].

2. Main result

2.1. Notation, definitions and preliminary results. Let $\mathbb{R}[x, t]$ (resp. $\mathbb{R}[x, t]_d$) denote the ring of real polynomials in the variables $x = (x_1, \ldots, x_n, t)$ (resp. polynomials of degree at most $d$), whereas $\Sigma[x, t]$ (resp. $\Sigma[x, t]_d$) denotes its set of sums of squares (s.o.s.) polynomials (resp. of s.o.s. of degree at most $2d$). For every $\alpha \in \mathbb{N}^n$ the notation $x^\alpha$ stands for the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and for every $d \in \mathbb{N}$, let $\mathbb{N}^{d+1}_d := \{ \beta \in \mathbb{N}^{d+1} : \sum_j \beta_j \leq d \}$ whose cardinal is $s(d) = \binom{n+d}{d+1}$. A polynomial $f \in \mathbb{R}[x, t]$ is written

$$(x, t) \mapsto f(x, t) = \sum_{(\alpha, k) \in \mathbb{N}^n \times \mathbb{N}} f_{\alpha k} x^\alpha t^k,$$

and $f$ can be identified with its vector of coefficients $f = (f_{\alpha k})$ in the canonical basis $(x^\alpha, t^k)$, $(\alpha, k) \in \mathbb{N}^n \times \mathbb{N}$, of $\mathbb{R}[x, t]$. But we can also write $f$ as

$$(2.1) \quad (x, t) \mapsto f(x, t) = \sum_{k \in \mathbb{N}} f_k(x) t^k,$$

for finitely many polynomials $f_k \in \mathbb{R}[x]$.

A real sequence $z = (z_{\alpha k}), (\alpha, k) \in \mathbb{N}^n \times \mathbb{N}$, has a representing measure if there exists some finite Borel measure $\nu$ on $\mathbb{R}^n \times \mathbb{R}$ such that

$$z_{\alpha k} = \int_{\mathbb{R}^{n+1}} x^\alpha t^k d\nu(x, t), \quad \forall (\alpha, k) \in \mathbb{N}^n \times \mathbb{N}.$$

Given a real sequence $z = (z_{\alpha k})$ define the linear functional $L_y : \mathbb{R}[x, t] \to \mathbb{R}$ by:

$$f = \left( \sum_{\alpha, k} f_{\alpha k} x^\alpha t^k \right) \mapsto L_y(f) = \sum_{\alpha, k} f_{\alpha k} z_{\alpha k}, \quad f \in \mathbb{R}[x, t].$$

Moment matrix. The moment matrix associated with a sequence $z = (z_{\alpha k}), (\alpha, k) \in \mathbb{N}^n \times \mathbb{N}$, is the real symmetric matrix $M_d(z)$ with rows and columns indexed by $\mathbb{N}^{d+1}_d$ and whose entry $(\alpha, \beta)$ is just $z_{\alpha + \beta}$, for every $\alpha, \beta \in \mathbb{N}^{d+1}_d$. Alternatively, let $v_d((x, t)) \in \mathbb{R}^{s(d)}$ be the vector $((x, t)^\alpha), \alpha \in \mathbb{N}^{d+1}_d$, and define the matrices $(B_\alpha) \subset S^{s(d)}$ by

$$(2.2) \quad v_d((x, t)) v_d((x, t))^T = \sum_{\alpha \in \mathbb{N}^{d+1}_d} B_\alpha(x, t)^\alpha, \quad \forall (x, t) \in \mathbb{R}^{n+1}.$$

Then $M_d(z) = \sum_{\alpha \in \mathbb{N}^{d+1}_d} z_\alpha B_\alpha$.

If $z$ has a representing measure $\nu$ then $M_d(z) \succeq 0$ because

$$\langle f, M_d(z)f \rangle = \int f^2 d\nu \geq 0, \quad \forall f \in \mathbb{R}^{s(d)}.$$
Localizing matrix. With \( z \) as above and \( g \in \mathbb{R}[x, t] \) (with \( g(x, t) = \sum_\gamma g_\gamma(x, t)^\gamma \)), the localizing matrix associated with \( z \) and \( g \) is the real symmetric matrix \( M_d(gz) \) with rows and columns indexed by \( \mathbb{N}_d^{n+1} \), and whose entry \((\alpha, \beta)\) is just \( \sum_\gamma g_\gamma z_{\alpha+\beta+\gamma} \), for every \( \alpha, \beta \in \mathbb{N}_d^{n+1} \). Alternatively, let \( C_\alpha \in \mathbb{S}(d) \) be defined by:

\[
g(x, t) v_d(x, t) v_d(x, t)^T = \sum_{\alpha \in \mathbb{N}_d^{n+1}} C_\alpha (x, t)^\alpha, \quad \forall (x, t) \in \mathbb{R}^{n+1}.
\]

Then \( M_d(gz) = \sum_{\alpha \in \mathbb{N}_d^{n+1}} z_\alpha C_\alpha \).

If \( z \) has a representing measure \( \nu \) whose support is contained in the set \( \{(x, t) : g(x, t) \geq 0\} \) then \( M_d(gz) \geq 0 \) because

\[
\langle f, M_d(gz)f \rangle = \int f^2 g \, d\nu \geq 0, \quad \forall f \in \mathbb{R}^d.
\]

With \( K \) as in (1.3), and for every \( j = 0, 1, \ldots, m \), let \( v_j := \lceil \deg g_j \rceil / 2 \).

**Definition 2.1.** With \( K \) as in (1.3) let \( P(g) \subset \mathbb{R}[x, t] \) be the convex cone:

\[
P(g) = \left\{ \sum_{\beta \in \{0, 1\}^m} \psi_{\beta}(x, t) g_1(x)^{\beta_1} \cdots g_m(x)^{\beta_m} : \psi_{\beta} \in \Sigma[x, t] \right\}.
\]

The convex cone \( P(g) \) is called a preordering associated with the \( g_j \)'s.

**Proposition 2.2.** Let \( K \) be as in (1.3). A polynomial \( f \in \mathbb{R}[x, t] \) is nonnegative on \( K \times \mathbb{R} \) only if \( f \) can be written as

\[
(x, t) \mapsto f(x, t) = \sum_{k=0}^{2d} f_k(x) t^k,
\]

for some \( d \in \mathbb{N} \) and where \( f_{2d} \geq 0 \) on \( K \).

**Proof.** Suppose that the highest degree in \( t \) is \( 2d+1 \) for some \( d \in \mathbb{N} \). Then \( f_{2d+1} \neq 0 \) and so by fixing an arbitrary \( x_0 \in K \), the univariate \( t \mapsto f(x_0, t) \) can be made negative, in contradiction with \( f \geq 0 \) on \( K \times \mathbb{R} \). Hence the highest degree in \( t \) is even, say \( 2d \). But then of course, for obvious reasons \( f_{2d} \geq 0 \) on \( K \). \( \Box \)

We have the following important preliminary result.

**Theorem 2.3** ([7, 9]). Let \( K \) as in (1.3) be compact and let \( f \in \mathbb{R}[x, t] \) be of the form \( f(x, t) = \sum_{k=0}^{2d} f_k(x) t^k \) for some polynomials \( (f_k) \subset \mathbb{R}[x] \), and with \( f_{2d} > 0 \) on \( K \). Then \( f \in P(g) \) if \( f > 0 \) on \( K \times \mathbb{R} \).

And so we can derive a version of the \( K \times \mathbb{R} \)-moment problem where for each \( \beta \in \mathbb{N}^m \), the notation \( g^\beta \) stands for the polynomial \( g_1^{\beta_1} \cdots g_m^{\beta_m} \).

**Corollary 2.4.** Let \( K \) as in (1.3) be compact. A real sequence \( z = (z_{\alpha k}) \), \((\alpha, k) \in \mathbb{N}^n \times \mathbb{N} \), has a representing measure on \( K \times \mathbb{R} \) if and only if

\[
M_d(z) \geq 0; \quad M_d(g^\beta z) \geq 0, \quad \beta \in \{0, 1\}^m,
\]

for every \( d \in \mathbb{N} \).
Proof. The only if part is straightforward from the definition of the moment and localizing matrix $M_d(z)$ and $M_d(g^\beta z)$, respectively.

The if part. Suppose that (2.6) holds true, and let $f \in \mathbb{R}[x,t]$ be nonnegative on the closed set $K \times \mathbb{R}$. Hence by Proposition 2.2, $f$ has the decomposition (2.5) for some integer $d \neq 0$. For every $\epsilon > 0$, the polynomial $(x,t) \mapsto f_\epsilon(x,t) := f(x,t) + \epsilon(1 + t^{2d})$ has the decomposition

$$f_\epsilon(x,t) = \sum_{k=0}^{2d} f_{\epsilon,k}(x) t^k,$$

with $f_{\epsilon,0} = f_0 + \epsilon$ and $f_{\epsilon,2d}(x) = f_{2d}(x) + \epsilon$. Therefore, $f_\epsilon$ is strictly positive on $K \times \mathbb{R}$, and $f_{\epsilon,2d} > 0$ on $K$. By Theorem 2.3, $f_\epsilon \in Q(g)$, i.e.,

$$f_\epsilon(x,t) = \sum_{\beta \in \{0,1\}^m} \psi_{\beta}(x,t) g(x)^\beta,$$

for some SOS polynomials $(\psi_{\beta}) \in \Sigma[x,t]$. Next, let $z$ satisfy (2.6). Then

$$L_x(f) + \epsilon L_y(1 + t^{2d}) = L_x(f_\epsilon) = \sum_{\beta \in \{0,1\}^m} L_x(\psi_{\beta} g^\beta) \geq 0$$

where the last inequality follows from

$$M_d(g^\beta z) \geq 0 \Leftrightarrow L_x(h^2 g^\beta) \geq 0, \ \forall h \in \mathbb{R}[x,t]_d,$$

for every $\beta \in \{0,1\}^m$. But since $L_x(1 + t^{2d}) \geq 0$ and $\epsilon > 0$ was arbitrary, one may conclude that $L_x(f) \geq 0$ for every $f \in \mathbb{R}[x,t]$ which is nonnegative on $K \times \mathbb{R}$. Hence by the Riesz-Haviland theorem (see e.g. [6, Theorem 3.1, p. 53]), $z$ has a representing measure on $K \times \mathbb{R}$. \hfill \square

2.2. Main result. Let $L_\infty(K)$ be the Lebesgue space of integrable functions on $K$ (with respect to the Lebesgue measure $\lambda$ on $K$, scaled to a probability measure) and essentially bounded on $K$. And with $1 \leq p < \infty$, let $L_p(K)$ be the Lebesgue space of integrable functions $f$ on $K$ such that $\int_K |f|^p \lambda(dx) < \infty$. A Borel measure $\mu$ absolutely continuous w.r.t. $\lambda$ is denoted $\mu \ll \lambda$.

**Theorem 2.5.** Let $K \subset \mathbb{R}^n$ be a nonempty compact basic semi-algebraic set of the form

$$K := \{ x \in \mathbb{R}^n : g_j(x) \geq 0, \ \ j = 1, \ldots, m \}$$

for some polynomials $(g_j) \subset \mathbb{R}[x]$, and recall the notation $g^\beta \in \mathbb{R}[x]$, with

$$x \mapsto g^\beta(x) := g_1(x)^{\beta_1} \cdots g_m(x)^{\beta_m}, \ x \in \mathbb{R}^n, \ \beta \in \{0,1\}^m.$$ 

Let $y = (y_\alpha), \ \alpha \in \mathbb{N}^n$, be a real sequence with $y_0 = 1$. Then the following two propositions (i) and (ii) are equivalent:

(i) $y$ has a representing Borel probability measure $\mu \ll \lambda$ on $K$, with a density in $\cap_{p \geq 1} L_p(K)$.

(ii) $M_d(y) \geq 0$ and $M_d(g^\beta y) \geq 0$ for all $\beta \in \{0,1\}^m$ and all $d \in \mathbb{N}$. In addition, there exists a sequence $z = (z_{\alpha k}), (\alpha, k) \in \mathbb{N}^n \times \mathbb{N}$, such that (2.6) holds, and

$$z_{\alpha0} = \int_K x^\alpha \lambda(dx); \ \ z_{\alpha1} = y_\alpha, \ \ \forall \alpha \in \mathbb{N}^n.$$
Moreover, if in (2.7) one includes the additional condition $\sup_k z_{0k} < \infty$, then (ii) is necessary and sufficient for $\mathbf{y}$ to have a representing Borel probability measure $\mu \ll \lambda$ on $K$, with a density in $L_\infty(K)$.

Proof. The (i) $\Rightarrow$ (ii) implication. As $\mathbf{y}$ has a representing Borel probability measure $\mu$ on $K$ with a density $f \in L_p(K)$ for every $p = 1, 2, \ldots$, one may write

$$\mu(A) = \int_A f(x) \lambda(dx), \quad \forall A \in B(\mathbb{R}^n).$$

Define the stochastic kernel $\varphi(B|x), B \in B(\mathbb{R}), x \in K$, where for almost all $x \in K$, $\varphi(\cdot|x)$ is the Dirac measure at the point $f(x)$. Next, let $\nu$ be the finite Borel measure on $K \times \mathbb{R}$ defined by

$$\nu(A \times B) := \int_A \varphi(B|x) \lambda(dx), \quad \forall A \in B(\mathbb{R}^n), B \in B(\mathbb{R}).$$

Let $z = (y_{\alpha k}), (\alpha, k) \in \mathbb{N}^n \times \mathbb{N}$, be the sequence of moments of $\nu$.

$$z_{\alpha k} = \int_{\mathbb{R}} x^\alpha t^k d\nu(x, t) = \int_K x^\alpha \left( \int_{\mathbb{R}} t^k \varphi(dt|x) \right) \lambda(dx),$$

$$\quad = \int_K x^\alpha f(x) \lambda(dx) \quad \text{(well defined as } f \in L_p(K) \text{ for all } p).$$

In particular, for every $\alpha \in \mathbb{N}^n$,

$$z_{\alpha 0} = \int_K x^\alpha \lambda(dx); \quad z_{\alpha 1} = \int_K x^\alpha f(x) \lambda(dx) = \int_K x^\alpha d\mu = y_\alpha.$$ Moreover, as $\nu$ is supported on $K \times \mathbb{R}$ then $M_d(y) \geq 0$ and $M_d(g^\beta y) \geq 0$ for all $d$ and all $\beta \in \{0, 1\}^m$. Hence (2.6)-(2.7) hold.

The (ii) $\Rightarrow$ (i) implication. Let $z = (z_{\alpha k})$ be such that (2.6)-(2.7) hold. By Corollary (2.4), $z$ has a representing Borel probability measure $\nu$ on $K \times \mathbb{R}$. One may disintegrate $\nu$ in the form

$$\nu(A \times B) = \int_{A \times K} \varphi(B|x) \psi(dx), \quad B \in B(\mathbb{R}), A \in B(\mathbb{R}^n),$$

for some stochastic kernel $\varphi(\cdot|x)$, and where $\psi$ is the marginal (probability measure) of $\nu$ on $K$. From (2.7) we deduce that

$$\int_K x^\alpha \psi(dx) = z_{\alpha 0} = \int_K x^\alpha \lambda(dx), \quad \forall \alpha \in \mathbb{N}^n,$$

which, as $K$ is compact, implies that $\psi = \lambda$. In addition, still from (2.7),

$$z_{\alpha 1} = \int_K x^\alpha t d\nu(x, t) = \int_K x^\alpha \left( \int_{\mathbb{R}} t \varphi(dt|x) \right) \lambda(dx) \quad \forall \alpha \in \mathbb{N}^n$$

$$\quad = \int_K x^\alpha f(x) \lambda(dx) \quad \forall \alpha \in \mathbb{N}^n,$$

where $f : K \rightarrow \mathbb{R}$ is the measurable function $x \mapsto \int_{\mathbb{R}} t \varphi(dt|x)$, and $\theta$ is the signed Borel measure $\theta(B) := \int_{K \cap B} f(x) \lambda(dx)$, for all $B \in B(\mathbb{R}^n)$.
But as $K$ is compact, by Schm"udgen's Positivstellensatz [12], the conditions

$$M_d(y) \geq 0, \quad M_d(g^\beta y) \geq 0, \quad \beta \in \{0, 1\}^m, \quad \forall d \in \mathbb{N},$$

imply that $y$ has a finite representing Borel probability measure $\mu$ on $K$. And so as $z_\alpha = y_\alpha$ for all $\alpha \in \mathbb{N}^n$, and measures on compact sets are moment determinate, one may conclude that $d\mu = d\theta = f \, d\lambda$, that is, $\mu \ll \lambda$ on $K$ (and $f \geq 0$ almost everywhere on $K$). Next, observe that for every even $p \in \mathbb{N}$, using Jensen's inequality,

$$z_{0p} = \int_K t^p d\nu(x,t) = \int_K \left( \int_\mathbb{R} t^p \varphi(dt|x) \right) \lambda(dx) \quad \forall \alpha \in \mathbb{N}^n,$$

and so $f \in L_p(K)$ for all even $p \geq 1$ (hence all $p \in \mathbb{N}$).

Finally consider (2.7) with the additional condition $\sup_p z_{0p} < \infty$. Then in the above proof of (i) $\Rightarrow$ (ii) and since now $y$ has a finite representing Borel measure with a density $f \in L_\infty(K)$, one has $\lim_{p \to \infty} \|f\|_p = \|f\|_\infty$ because $K$ is compact; see e.g. Ash [3, problem 9, p. 91]. And therefore since $z_{0p} = \int_K f(x)^p \lambda(dx)$, we obtain $\sup_p z_{0p} < \infty$.

Similarly, in the above proof of (ii) $\Rightarrow$ (i), $\sup_p z_{0p} < \infty$ implies $\sup_p \int_K f(x)^p \lambda(dx) = \sup_p \|f\|_p < \infty$. But this implies that $f \in L_\infty(K)$ since $K$ is compact. \hfill $\square$

**Computational procedure.** Let $\gamma = (\gamma_\alpha)$, $\alpha \in \mathbb{N}^n$, the moment of the Lebesgue measure on $K$, scaled to make it a probability measure. In fact, the (scaled) Lebesgue measure on any box that contains $K$ is fine. Let $y = (y_\alpha)$, $\alpha \in \mathbb{N}^n$, be a real given sequence, and with $K$ as in (1.3) let $v_j := [(\deg g_j)/2]$, $j = 1, \ldots, m$. To check the conditions in Theorem 2.5(ii), one solves the hierarchy of optimization problems, parametrized by $d \in \mathbb{N}$.

$$\begin{align*}
\rho_d &= \min_{\mathbf{z}} \quad \text{trace}(M_d(\mathbf{z})) \\
\text{s.t.} & \
M_d(\mathbf{z}) \succeq 0 \\
M_{d-v_j}(g^\beta \mathbf{z}) & \succeq 0, \quad \beta \in \{0, 1\}^m \\
z_{\alpha 0} &= \gamma_\alpha, \quad \alpha \in \mathbb{N}_d^n \\
z_{\alpha 1} &= y_\alpha, \quad (\alpha, 1) \in \mathbb{N}^{n+1}_d.
\end{align*}$$

Each problem (2.11) is a semidefinite program$^1$. Moreover, if (2.11) has a feasible solution then it has an optimal solution. This is because as one minimizes the trace of $M_d(\mathbf{z})$, the feasible set is bounded and closed, hence compact.

In (2.11) one may also include the additional constraints $z_{0k} < c$, $k \leq 2d$, for some fixed $c > 0$. Then by Theorem 2.5, $y$ has a representing Borel probability measure on $K$ with a density in $L_\infty(K)$ bounded by $c$, if and only if $\rho_d < \infty$ for all $d$.

$^1$A semidefinite program is a convex optimization problem that can be solved efficiently, i.e., up to arbitrary fixed precision it can be solved in time polynomial in the input size of the problem; see e.g. [2].
Each semidefinite program of the hierarchy (2.11), \( d \in \mathbb{N} \), has a dual which is also a semidefinite program and which reads:

\[
\rho_d^* = \max_{p,q,t,\sigma} \int_K p(x)\lambda(dx) + L_y(q) \quad \text{s.t.} \quad \sum_{(\alpha,k) \in \mathbb{N}_d^{n+1}} (x^\alpha t^k)^2 - (p(x) + tq(x)) = \sigma_0(x,t) + \sum_{j=1}^m \sigma_j(x,t)g_j(x) \\
\deg p \leq 2d; \quad \deg q \leq 2d - 1; \quad \sigma_j \in \Sigma[x,t]_{t-v_j}, \quad j = 0, \ldots, m,
\]

where \( v_0 = 0 \). In particular, if \( y \) is the sequence of a Borel measure on \( K \) then in (2.12) one may replace \( L_y(q) \) with \( \int_K q(x) d\mu(x) \).

2.3. On membership in \( L_p(K) \). An interesting feature of the hierarchy of semidefinite programs (2.11), \( d \in \mathbb{N} \), is that it can be used to detect if a given sequence \( y = (y_\alpha), \alpha \in \mathbb{N}^n \), cannot have a representing Borel measure on \( K \) with a density in \( L_p(K) \), \( p > 1 \).

**Corollary 2.6.** Let \( K \subset \mathbb{R}^n \) be as in (1.3) and let \( y = (y_\alpha), \alpha \in \mathbb{N}^n \), be a real sequence with \( y_0 = 1 \). If the semidefinite program (2.11) with \( d \in \mathbb{N} \), has no solution then \( y \) cannot have a representing finite Borel measure on \( K \) with a density in \( L_p(K) \), for any \( p \geq 2d \).

**Proof.** Suppose that \( y \) has a representing measure on \( K \) with a density \( f \in L_{2d}(K) \), and hence in \( L_k(K) \) for all \( k \leq 2d \). Proceeding as in the proof of Theorem 2.5, let \( \nu \) be the Borel measure on \( K \times \mathbb{R} \) defined in (2.8). Then from (2.9) one obtains

\[
z_{\alpha k} = \int_K x^\alpha f(x)^k \lambda(dx), \quad (\alpha,k) \in \mathbb{N}_d^{n+1},
\]

which is well-defined since \( K \) is compact (so that \( x^\alpha \) is bounded) and \( k \leq 2d \). And so the sequence \( z = (z_{\alpha k}), (\alpha,k) \in \mathbb{N}_d^{n+1}, \) is a feasible solution of (2.11) with \( d \).

Notice that again, the detection of absence of a density in \( L_p(K) \) is possible with no \( a \ priori \) bounding parameter \( c \). But of course, the condition is only sufficient.

**Example 1.** Let \( K := [0,1] \) and \( s \in [0,1] \). Let \( \lambda \) be the Lebesgue measure on \([0,1] \) and let \( \delta_s \) be the Dirac measure at \( s \). One wants to detect that the Borel probability measure \( \mu_a := a\lambda + (1-a)\delta_s \), with \( a \in (0,1) \) has no density in \( L_\infty(K) \). Then (2.7) reads

\[
z_{k0} = \frac{1}{k+1}, \quad k = 0, 1, \ldots; \quad z_{k1} = \frac{a}{k+1} + (1-a)s^k, \quad k = 0, 1, \ldots
\]

The set \( K \) is defined by \( \{ x : g(x) \geq 0 \} \) with \( x \mapsto g(x) := x(1-x) \). We have tested the conditions \( M_d(z) \geq 0 \) and \( M_d(gz) \geq 0 \) along with (2.7) where \( k \leq 2d \) (for \( z_{k0} \)) and \( k \leq 2d - 1 \) (for \( z_{k1} \)).

We have considered a Dirac at the points \( s = k/10, k = 1, \ldots, 10 \), and with weights \( a = 1 - k/10, k = 1, \ldots, 10 \). To solve (2.11) we have used the GloptiPoly software of Heurion et al. [5] dedicated to solving the generalized problem of moments. Results are displayed in Table 1 which should be read as follows:

- A column is parametrized by the number of moments involved in the conditions (2.7). For instance, Column “10” refers to (2.7) with \( d = 10/2 \), that is, the moment matrix \( M_d(z) \) involves moments \( z_{ij} \) with \( i + j \leq 10 \), i.e., moments up to order 10.
| $s$ \ moments | 8   | 10   | 12   | 14   |
|--------------|-----|------|------|------|
| 0.0          | $1-a \geq 0.3$ | $1-a \geq 0.1$ | $1-a \geq 0.1$ | $1-a \geq 0.1$ |
| 0.1          | $1-a \geq 1$    | $1-a \geq 0.3$ | $1-a \geq 0.2$ | $1-a \geq 0.2$ |
| 0.2          | $1-a \geq 1$    | $1-a \geq 0.3$ | $1-a \geq 0.1$ | $1-a \geq 0.1$ |
| 0.3          | $1-a \geq 1$    | $1-a \geq 0.5$ | $1-a \geq 0.2$ | $1-a \geq 0.1$ |
| 0.4          | $1-a \geq 1$    | $1-a \geq 0.5$ | $1-a \geq 0.2$ | $1-a \geq 0.1$ |
| 0.5          | $1-a \geq 1$    | $1-a \geq 0.5$ | $1-a \geq 0.2$ | $1-a \geq 0.1$ |
| 0.6          | $1-a \geq 1$    | $1-a \geq 0.5$ | $1-a \geq 0.2$ | $1-a \geq 0.1$ |
| 0.7          | $1-a \geq 1$    | $1-a \geq 0.5$ | $1-a \geq 0.2$ | $1-a \geq 0.1$ |
| 0.8          | $1-a \geq 1$    | $1-a \geq 0.5$ | $1-a \geq 0.2$ | $1-a \geq 0.1$ |
| 0.9          | $1-a \geq 1$    | $1-a \geq 0.5$ | $1-a \neq 0.4,0.5$ | $1-a \geq 0.1$ |
| 1.0          | $1-a \geq 0.4$  | $1-a \geq 0.1$ | $1-a \geq 0.1$ | $1-a \geq 0.1$ |

Table 1. Moments required for detection of failure; one Dirac

| $(s,s+0.1)$ \ moments | 10   | 12   |
|-----------------------|------|------|
| (0.1,0.2)             | $1-a \geq 0.4$ | $1-a \geq 0.2$ |
| (0.2,0.3)             | $1-a \geq 0.6$ | $1-a \geq 0.2$ |
| (0.3,0.4)             | $1-a \geq 0.5$ | $1-a \geq 0.2$ |
| (0.4,0.5)             | $1-a \geq 0.6$ | $1-a \geq 0.1$ |
| (0.5,0.6)             | $1-a \geq 0.6$ | $1-a \geq 0.2$ |
| (0.6,0.7)             | $1-a \geq 0.7$ | $1-a \geq 0.2$ |
| (0.7,0.8)             | $1-a \geq 0.6$ | $1-a \geq 0.3$ |
| (0.8,0.9)             | $1-a \geq 0.5$ | $1-a \geq 0.2$ |
| (0.9,1.0)             | $1-a \geq 0.1$ | $1-a \geq 0.1$ |

Table 2. Moments required for detection of failure; two Dirac

- Each row is indexed by the location of the Dirac $\delta_s$, $s \in [0,1]$ (with $\mu_a = a\lambda + (1-a)\delta_s$). The statement “$1-a \geq 0.5$” in row “$s = 0.3$” and column “10” means that (2.7) is violated whenever $1-a \geq 0.5$, i.e., when the weight associated to the Dirac $\delta_s$ is larger than 0.5.

One may see that no matter where the point $s$ is located in the interval [0, 1], if its weight $1-a$ is above 0.5 then detection of impossibility of a density in $L_\infty([0,1])$ occurs with moments up to order 10. If its weight $1-a$ is only above 0.1 then detection of impossibility occurs with moments up to order 12. So even with a small weight on the Dirac $\delta_s$, detection of impossibility does not require moments of order larger than 12.

**Example 2.** Still with $K = [0, 1]$, consider now the case where $\mu_a = a\lambda + (1-a)(\delta_s + \delta_{s+0.1})/2$, that is, $\mu_a$ is a $(a, 1-a)$ convex combination of the uniform probability distribution on [0, 1] with two Dirac measures at the points $s$ and $s + 0.1$ of $[0, 1]$, with equal weights. The results displayed in Table 2 are qualitatively very similar to the results in Table 1 for the case of one Dirac.
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