Note on a Lyapunov-type inequality for a fractional boundary value problem with Caputo-Fabrizio derivative

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Abstract

In this short note, we present a Lyapunov-type inequality that corrects the recently obtained result in [M. Kirane, B. T. Torebek: A Lyapunov-type inequality for a fractional boundary value problem with Caputo-Fabrizio derivative, J. Math. Inequal. 12, 4 (2018), 1005–1012].

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MSC (2010): 34A08, 34A40, 26A33.

1 Introduction

Recently, in [1] the authors discussed a Lyapunov-type inequality for the following linear fractional boundary value problem:

\[ \begin{array}{ll}
\text{CF}_{\alpha} D_{a}^{\alpha} u(t) + q(t)u(t) = 0, & 0 \leq a < t < b, \\
u(a) = u(b) = 0, & \end{array} \tag{1} \]

where \( \text{CF}_{\alpha} D_{a}^{\alpha} \) denotes the Caputo-Fabrizio derivative [2, 3] of order \( \alpha, (1 < \alpha \leq 2) \), \( q : [a, b] \rightarrow \mathbb{R} \) is a continuous function. And they included the following result:

**Theorem 1 ([1])** If the fractional boundary value problem (1) has a nontrivial solution, then

\[ \int_{a}^{b} |q(t)|ds > \frac{4(\alpha - 1)(b - a)}{[(\alpha - 1)(b - a) - 2 + \alpha]^2}. \tag{2} \]

We have noticed that, the denominator in the inequality (2) is equal zero when \( b - a = \frac{2 - \alpha}{\alpha - 1} \). So a mistake has been occurred during the previous result and some other results (Corrolary 3.4 and Corrolary 3.5 in [1]) are also incorrect. These mistakes come from the main wrong in (Lemma 3.2 in [1] related to the calculations for the maximum value of the Green's function of the problem (1)).

This work aims to show these mistakes and present the correct version of them.

We will also refer the interested reader in studying the Lyapunov-type inequalities for a fractional boundary value problems to the works compiled in chapter in the book [4], as well as some other papers published recently, for example see [5]–[10] and the references cited therein.

2 Main results

The fractional boundary value problem (1) is equivalent to the integral equation

\[ u(s) = \int_{a}^{b} G(t, s)q(s)u(s)ds, \tag{3} \]

where \( G(t, s) \) is called the Green's function of the problem (1) and it's defined by

\[ G(t, s) = \begin{cases} 
g_{1}(t, s) = \frac{b - t}{b - a}[(\alpha - 1)(s - a) - 2 + \alpha], & a \leq s \leq t \leq b, \\
g_{2}(t, s) = \frac{t - a}{b - a}[(\alpha - 1)(b - s) + 2 - \alpha], & a \leq t \leq s \leq b. 
\end{cases} \tag{4} \]
See [1] for more details, (note that this function in (Lemma 3.1, [1]) is written in wrong way. Although its proof is true).

The mistake alluded to in ([1], Section 3) is that the authors concluded that the maximum value of the function \( G(t, s) \) is obtained at the point

\[
t = s = \frac{1}{2} \left( b + a + \frac{2 - \alpha}{\alpha - 1} \right) := s^*,
\]

where \( s^* \) is defined by (equality (3.5) in [1]). However, this is wrong for \((t, s) \in [a, b] \times [a, b] \) with \( 1 < \alpha \leq 2 \), as we will show nextly.

Let us start to discuss the previous value of \( s^* \).

Note that, if \( b - a < \frac{2 - \alpha}{\alpha - 1} \), we have

\[
b - a < \frac{2 - \alpha}{\alpha - 1} \iff 2b < b + a + \frac{2 - \alpha}{\alpha - 1} \iff b < \frac{1}{2} \left( b + a + \frac{2 - \alpha}{\alpha - 1} \right) \iff b < s^*,
\]

thus \( s^* \notin [a, b] \). Then the maximum value of the function \( G(t, s) \) is not at \( s^* \) when \( b - a < \frac{2 - \alpha}{\alpha - 1} \).

Now, for \( a \leq s \leq t \leq b \), with \( b - a < \frac{2 - \alpha}{\alpha - 1} \), we have

\[
b - a < \frac{2 - \alpha}{\alpha - 1} \iff b < a + \frac{2 - \alpha}{\alpha - 1} \implies s < a + \frac{2 - \alpha}{\alpha - 1} \implies \alpha (s - a) - 2 + \alpha < 0,
\]

on other hand, we have

\[
\frac{b - t}{b - a} \leq \frac{b - s}{b - a}.
\]

By the inequalities (6) and (7) we get

\[
g_1(t, s) \geq \frac{b - s}{b - a}[(\alpha - 1)(s - a) - 2 + \alpha], \quad a \leq s \leq b < a + \frac{2 - \alpha}{\alpha - 1}.
\]

Observe that the inequality (8) is contrary to (the inequality (3.4) in [1]).

**Remark 2** Note that on a general interval \([a, b], 0 \leq a < b\), we have

\[
h_1(s) \leq g_1(t, s) \leq 0, \quad \text{for } a \leq s \leq t \leq a + \frac{2 - \alpha}{\alpha - 1},
\]

where the function \( h_1 \) is defined by

\[
h_1(s) = g_1(s, s) = \frac{b - s}{b - a}[(\alpha - 1)(s - a) - 2 + \alpha], \quad s \in [a, b]
\]

We differentiate the function \( h_1(s) \) to get

\[
h_1'(s) = -\frac{2(\alpha - 1)}{b - a} s + \frac{(\alpha - 1)(b + a) + 2 - \alpha}{b - a}.
\]

We have \( s^* \) is the unique solution of the equation \( h_1'(s) = 0 \), where \( s^* \) is given by (5) but the value of \( s^* \) in some cases does not belong to the interval \([a, b]\) as we have shown previously.

By the discussion above, we can conclude that the maximum value of the function \( G(t, s) \) lays in the following two cases:

**Case 1.** \( b - a < \frac{2 - \alpha}{\alpha - 1} \).
Because \( b - a < \frac{2 - \alpha}{\alpha - 1} \), so then \( s \leq b < s^* \) (here \( a + \frac{2 - \alpha}{\alpha - 1}, s^* \notin [a, b] \)), we obtain \( h_1'(s) \geq 0 \) and \( h_1(s) \leq 0 \) for \( s \leq b < a + \frac{2 - \alpha}{\alpha - 1} \). So by (9) and the continuity of the function \( h_1 \) we conclude that

\[
\max_{a \leq x \leq y \leq b \leq a + \frac{2 - \alpha}{\alpha - 1}} |g_1(t, s)| = \max_{a \leq x \leq y \leq b \leq a + \frac{2 - \alpha}{\alpha - 1}} |h_1(s)|
= -h_1(a)
= 2 - \alpha.
\] (12)

Next, for \( a \leq t \leq s \leq b \). Obviously,

\[
0 \leq g_2(t, s) \leq \frac{s - a}{b - a}[(\alpha - 1)(b - s) + 2 - \alpha].
\] (13)

We define a function \( h_2 \) by

\[
h_2(s) = g_2(s, s) = \frac{s - a}{b - a}[(\alpha - 1)(b - s) + 2 - \alpha], \; a \leq s \leq b.
\] (14)

Differentiating the function \( h_2(s) \)

\[
h_2'(s) = -\frac{2(\alpha - 1)}{b - a}s + \frac{(\alpha - 1)(a + b) + (2 - \alpha)}{b - a},
\] (15)

which implies that the function \( h_2(s) \) has a unique zero, at the point \( s^* \), but \( s^* > b \) (i.e. \( s^* \notin [a, b] \)). Because \( h_2'(s) > 0 \) for all \( s \in [a, b] \) and \( g_2(t, s) \geq 0 \) for \( a \leq t \leq s \leq b \), and \( h_2(s) \) is continuous function, then

\[
\max_{a \leq t \leq s \leq b \leq a + \frac{2 - \alpha}{\alpha - 1}} |g_2(t, s)| = \max_{a \leq t \leq s \leq b \leq a + \frac{2 - \alpha}{\alpha - 1}} h_2(s)
= h_2(b)
= 2 - \alpha.
\] (16)

By (16) and (12) we get

\[
\max_{a \leq t \leq s \leq b \leq a + \frac{2 - \alpha}{\alpha - 1}} |G(t, s)| = 2 - \alpha.
\] (17)

**Case 2.** \( b - a \geq \frac{2 - \alpha}{\alpha - 1} \).

From the inequality \( b - a \geq \frac{2 - \alpha}{\alpha - 1} \) we get \( a + \frac{2 - \alpha}{\alpha - 1} \in [a, b] \) and \( s^* \in [a + \frac{2 - \alpha}{\alpha - 1}, b] \), we obtain

\[
\begin{cases}
0 \leq g_1(t, s) \leq h_1(s), \; a + \frac{2 - \alpha}{\alpha - 1} \leq s \leq b, \\
h_1(s) \leq g_1(t, s) \leq 0, \; a \leq s \leq a + \frac{2 - \alpha}{\alpha - 1} < s^*.
\end{cases}
\] (18)

Because \( h_1(s) \) is continuous function, and \( h_1(a + \frac{2 - \alpha}{\alpha - 1}) = h_1(b) = 0 \), and \( h_1(a) = -(2 - \alpha) \), we conclude that

\[
\max_{a \leq t \leq s \leq a + \frac{2 - \alpha}{\alpha - 1} \leq b} |g_1(t, s)| = \max_{a \leq t \leq s \leq a + \frac{2 - \alpha}{\alpha - 1} \leq b} |h_1(s)|
= \max \{-h_1(a), h_1(s^*)\}
= \max \left\{2 - \alpha, \frac{[(\alpha - 1)(b - a) - 2 + \alpha]^2}{4(\alpha - 1)(b - a)} \right\}.
\] (19)

On other hand, we have

\[
\max_{a \leq t \leq s \leq a + \frac{2 - \alpha}{\alpha - 1} \leq b} |g_2(t, s)| = \max_{a \leq t \leq s \leq a + \frac{2 - \alpha}{\alpha - 1} \leq b} h_2(s)
= h_2(s^*)
= \frac{[(\alpha - 1)(b - a) + (2 - \alpha)]^2}{4(\alpha - 1)(b - a)}.
\] (20)
By (19) and (20) we get
\[
\max_{a \leq t, s \leq a + \frac{2-\alpha}{\alpha-1}} |G(t, s)| = \max \left\{ 2 - \alpha, \frac{[(\alpha - 1)(b-a) - (2 - \alpha)^2]}{4(\alpha - 1)(b-a)}, \frac{[(\alpha - 1)(b-a) + (2 - \alpha)^2]}{4(\alpha - 1)(b-a)} \right\}
\]
using the inequality \( \frac{(A+B)^2}{4} \geq AB \), with \( A = (\alpha - 1)(b-a) \) and \( B = (2 - \alpha) \), we obtain
\[
\max_{a \leq t, s \leq a + \frac{2-\alpha}{\alpha-1}} |G(t, s)| = \frac{[(\alpha - 1)(b-a) + (2 - \alpha)^2]}{4(\alpha - 1)(b-a)}.
\tag{21}
\]

Thus we conclude the following result.

**Proposition 3** The Green’s function \( G \) defined by (4), has the following properties:

i). If \( b - a < \frac{2-\alpha}{\alpha-1} \), then
\[
\max_{(t, s) \in [a, b] \times [a, b]} |G(t, s)| = 2 - \alpha,
\tag{22}
\]

ii). If \( b - a \geq \frac{2-\alpha}{\alpha-1} \), then
\[
\max_{(t, s) \in [a, b] \times [a, b]} |G(t, s)| = \frac{[(\alpha - 1)(b-a) + (2 - \alpha)^2]}{4(\alpha - 1)(b-a)}.
\tag{23}
\]

Hence we have the following Lyapunov-type inequality.

**Theorem 4** If the fractional boundary value problem (1) has a nontrivial solution. Then
\[
\int_{a}^{b} |q(t)|ds \geq \begin{cases} \frac{1}{2 - \alpha}, & \text{if } b - a < \frac{2-\alpha}{\alpha-1}, \\ \frac{4(\alpha - 1)(b-a)}{[(\alpha - 1)(b-a) + (2 - \alpha)^2]}, & \text{if } b - a \geq \frac{2-\alpha}{\alpha-1}. \end{cases}
\tag{24}
\]

**Proof.** Since the proof is well-known so that the reader can easily check it on, where it’s used in [1] but in here we get into details in two cases related the properties (22) and (23).

By using Theorem 4, the reader can smoothly correct (Corrolary 3.4 and Corrolary 3.5 in [1]), but should separate each of them in two cases \( b - a < \frac{2-\alpha}{\alpha-1} \), and \( b - a \geq \frac{2-\alpha}{\alpha-1} \).

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