Mathematical Modeling of Hydroelastic Oscillations of the Stamp and the Plate, Resting on Pasternak Foundation

L I Mogilevich¹, V S Popov¹, A A Popova¹ and A V Christoforova²

¹Yuri Gagarin State Technical University of Saratov, 77 Politechnicheskaya Street, Saratov, 410054, Russia
²Saratov State University, 83 Astrakhanskaya Street, Saratov, 410012, Russia

e-mail: vic_p@bk.ru

Abstract. The forced oscillations of the elastic fixed stamp and the plate, resting on Pasternak foundation are studied. The oscillations are caused by pressure pulsation in liquid layer between the stamp and the plate. Pasternak model is chosen as an elastic foundation. The laws of the stamp movement, the plate deflection and pressure in the liquid are discovered on the basis of hydroelasticity problem analytical solution. The functions of amplitude deflection distribution and liquid pressure along the plate are constructed, as well as the stamp amplitude-frequency characteristic. The obtained mathematical model allows to investigate the dynamics of hydroelastic interaction of the stamp with the plate, resting on elastic foundation, to define resonance frequencies of the plate and the stamp and corresponding deflections amplitudes, as well as liquid presser amplitudes. Keywords – elastic plate; hydroelastic oscillations; vibration stamp; viscous liquid, Pasternak foundation

1. Introduction

The investigation of the plate interaction with liquid is one of the important problem of contemporary mechanics for considering dynamic processes in complicated mechanical systems. For example, the problem of bending beam oscillations under interaction with an ideal liquid is solved in [1]. This solution made it possible to construct the mathematical model for defining the cylinder cavitations resource of the combustion engine with water cooling. Reference [2] deals with modeling of oscillations of a rectangular plate plunged into an ideal incompressible liquid with a free surface. Reference [3] studied the interaction of an ideal incompressible liquid with a plate, making forced oscillations due to its contact with a vibrating machine. The plane problem of acoustic wave’s radiation by vibrating plate is considered. The free oscillations of the cantilevered plates partly plunged into an ideal incompressible liquid with free surface are investigated in [4]. This investigation made it possible to evaluate the liquid movement inertia impact on the natural oscillations of the plate. Analogous study is carried out in [5] for cantilevered composite plate. The numerical investigation of free oscillations rectangular plates, fully plunged in a motionless ideal liquid or floating on its free surface, is made in [6]. Analogous investigation for rectangular plates, interacting with flow of ideal fluid, is considered in [7]. The paper’s authors carried out the analysis of the plate eigenfrequencies and found critical velocities corresponding to the loss of stability. Reference [8] studied chaotic oscillations of the plate, interacting with ideal incompressible fluid flow.
However, the liquid damping characteristics caused by its viscosity are excluded from the above mentioned investigations. Reference [9] considered bounding vibrations of infinitely long beam on the viscous liquid layer. Hydroelastic oscillations of cantilevered beam plunged into a viscous incompressible fluid are studied in [11]. Reference [10] is devoted to the investigation of transverse oscillations of elastically fixed rigid wall of the flat channel, the channel having finite sizes. The study of vibrating discs, one of them being elastic, with viscous incompressible liquid between them is made in [12]. The analogous problem for two vibrating plates is considered in [13]. Reference [14] studies hydroelastic beam oscillations in viscous liquid flow for piezoelectric elements. Reference [15] solves the problem of bending hydroelastic oscillations of the plate, forming the narrow channel wall under pulsating viscous liquid layer impact. The forced hydroelastic oscillations of three-layered circular plate, interacting with a viscous incompressible liquid layer are investigated in [16]. On the other hand, the investigation of the constructions elastic elements oscillations with consideration of foundation elastic pliability is of great importance for contemporary engineering [17]. For instance, references [18-21] consider oscillations and stability of multi-layered beams and plates resting on an elastic foundation under the impact of local and distributed loadings of various natures. Winkler and Pasternak models are used for foundation reactions. That is why, the evaluation of foundation elasticity impact on hydroelastic oscillations of the plate is important. There are few papers devoted to this subject matter: e.g. the oscillations of the membrane resting on Winkler foundation and situated on the bottom of a tank filled with an ideal incompressible liquid with a free surface are investigated in [22]. Hydroelastic oscillations of rectangular plates, resting on Pasternak foundation and interacting with an ideal incompressible liquid with a free surface, are studied in [23-25]. The oscillations of the plate, resting on Winkler foundation and interacting with viscous incompressible liquid layer, are studied in [26-29]. Reference [30] considers the oscillations of the shell interacting with an annular layer of strongly viscous liquid inside it and surrounded by Winkler elastic medium.

2. Setting the problem

This paper deals with the interaction dynamics of the stamp 1 with the elastic plate 2 through viscous liquid layer 3 (Fig. 1). The stamp oscillations and bending oscillations of the plate are caused by pressure pulsation in a thin layer of viscous incompressible liquid between them, the plate resting on Pasternak foundation. Let us connect Cartesian coordinate system Oxz with a medium surface of the plate in an unperturbed state.

![Figure 1](image_url)

**Figure 1.** The plate, resting on Pasternak foundation and interacting with the stamp through layer of pulsating viscous liquid

We consider the stamp to possess a spring suspension and to be able to move only in vertical direction. The plate’s length is 2ℓ, the thickness is h₀. The plate is simply supported on the edges. We assume the pressure pulsation at the edges acting in terms of the harmonic process. The viscous liquid layer possesses middle thickness δ₀<<ℓ, while the stamp oscillations amplitude z₀ and the plate deflection w₀ are considerably less than δ₀. The liquid at the edges freely leaks into the same liquid where the assumed law of pressure pulsation is maintained. The layer of viscous liquid between the stamp and the plate possesses strong damping characteristics. Therefore transitional processes in the considered oscillating system
quickly go out and the steady-state harmonic oscillations appear. Consequently, the initial conditions impact may be excluded from consideration, according to [31]. We will continue investigating the steady-state harmonic oscillation regime, taking into account this remark.

The assumed law of pressure pulsation at the edges takes the form of:

\[ p = p_0 + p'(\omega t), \quad p'(\omega t) = p_m f_p(\omega t), \quad f_p(\omega t) = \sin \omega t \]

where \( p_0 \) is the constant pressure, \( p_m \) is the amplitude of pressure pulsation, \( \omega \) is the frequency, \( t \) is the time.

The equation of the stamp oscillations takes the form of:

\[ \ddot{z}_s + n z_s = N \]

where \( z_s = z_{stf}(\omega t) \) is the law of stamp motion, \( z_{stf} \) is the stamp oscillations amplitude, \( m \) is the stamp mass, \( n \) is the spring constant, \( N \) is the force influencing the stamp due to viscous liquid.

The expression for force \( N \) takes the form of:

\[ N = -b \int q_z \, dx, \quad q_z = -p + 2 \rho \nu \frac{\partial u_z}{\partial z} \text{ at } z = \delta_0 + z_{stf} f(\omega t) \]

where \( q_z \) is the normal stress, \( u_z \) is the liquid velocity vector projection on a coordinate axis, \( p \) is the pressure; \( \rho \) is density of the liquid; \( \nu \) is the kinematic viscosity coefficient of liquid.

The equation of bending oscillations of the plate resting on Pasternak foundation can be written down as [29, 32]:

\[ D \dddot{w} + \kappa w - \eta \dddot{w} = \rho h_0 \dddot{z} = q_z \bigg|_{z = h_0/2 + w} \]

where \( w \) is the plate deflection, \( D \) is the flexural rigidity of the plate, \( \rho \) is the density of the plate material, \( \kappa \) is the foundation normal stiffness, \( \eta \) is the foundation shear coefficient.

The plate dynamic equation is added with the simply supported boundary conditions, i.e.:

\[ w = \dddot{w} = 0, \text{ at } x = \pm \ell. \]

The movement of viscous liquid layer between the stamp and the plate can be considered as a creeping one [33]. Thus, the liquid dynamic equations represent the Navier-Stokes equation, wherein the inertia members being excluded from the consideration. As a result we have:

\[ \frac{1}{\rho} \frac{\partial p}{\partial x} = \nu \left( \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial z^2} \right), \quad \frac{1}{\rho} \frac{\partial p}{\partial z} = \nu \left( \frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial z^2} \right) \]

\[ \frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} = 0, \]

where \( u_x, u_z \) are liquid velocity projections on the coordinate axis.

The boundary conditions of (6) include: no-slip conditions

\[ u_x = \dddot{u}_x \text{ at } z = h_0/2 + w, \quad u_z = \dddot{u}_z \text{ at } z = h_0/2 + \delta_0 + z_{stf} f(\omega t), \]

and pressure at the edges conditions
\[ p = p_0 + p_n f_n(\omega t) \quad \text{at} \quad x = \pm \ell. \]  

Here \( u, w \) are longitudinal movement and deflection plate laws.

3. The Theory

Let us introduce dimensionless variable and small parameters into consideration:

\[ \xi = x/\ell, \ \zeta = (z - 0.5h)/\delta_0, \ \tau = \omega t, \ u_\xi = z_m \omega U_\xi, \]

\[ u_\zeta = (z_m \omega)(\psi) U_\zeta, \ w = w_n W, u = u_\ell U, \ \lambda = z_m/\delta_0 << 1, \]

\[ p = p_0 + p^*(\tau) + P \nu z_n \omega \psi^2 /\delta_0, \ \psi = \delta_0 / \ell << 1. \]  

The problem of liquid dynamics in dimensionless variable (9) in zero approximation in \( \psi \) and \( \lambda \) written down as:

\[ \frac{\partial P}{\partial \xi} = \frac{\partial^2 U_\xi}{\partial \zeta^2}, \frac{\partial P}{\partial \zeta} = 0, \ \frac{\partial U_\xi}{\partial \zeta} + \frac{\partial U_\zeta}{\partial \xi} = 0, \]  

with boundary conditions:

\[ U_\xi = 0, U_\zeta = d f/\partial \tau \ \text{at} \ \zeta = 1, \]

\[ U_\zeta = 0, U_\xi = (w_n/z_m)\partial W/\partial \tau \ \text{at} \ \zeta = 0, \ P = 0 \ \text{at} \ \zeta = \pm 1. \]

Normal stress in the liquid in zero approximation in \( \psi \) takes the form:

\[ q_\xi = -p_0 - \rho \nu z_n \omega (\delta_0 \psi^2)^{-1} P, \]  

and the expression for force (3) takes the form:

\[ N = 2\ell b(p_0 + p^*(\tau)) + b(\rho \nu z_m \omega (\delta_0 \psi^2)^{-1}) \int_{-1}^{1} P d\xi. \]  

By solving (10) with boundary conditions (11) we get:

\[ U_\xi = z_m^2 - \zeta \frac{\partial P}{\partial \xi}, U_\zeta = \frac{w_n}{z_m} \frac{\partial W}{\partial \tau} - \frac{2 \zeta^2 - 3 \zeta^2}{12} \frac{\partial^2 P}{\partial \zeta^2} \]

and the equation for pressure:

\[ \frac{\partial^2 P}{\partial \xi^2} = 12(d f/\partial \tau - (w_n/z_m)\partial W/\partial \tau). \]  

Further, by solving (15) and satisfying the boundary conditions for the pressure at the edges (11) we get:

\[ P = 6(\xi^2 - 1) \frac{df}{d\tau} + 12 \frac{w_n}{z_m} \int_{-\zeta}^{\zeta} \partial W/\partial \tau d\zeta d\zeta + 6(\xi - 1) \frac{w_n}{z_m} \int_{-1}^{1} \partial W/\partial \tau d\zeta d\zeta. \]

According to the boundary conditions (5) we present the plate deflection dependence in the series of trigonometric functions:

\[ w = w_n W = \sum_{k=1}^{\infty} \left[ R_n^k (\tau) \cos((2k-1)z) / 2 \right], \]

here \( R_n^k (\tau) \) is the time harmonic function, \( R_n^0 \) is the constant.

By substituting (16), (17) in (12) and making disintegration of the remaining members in the series of the chosen trigonometric functions of longitudinal coordinate \( \zeta \), we get:
The solution of (19) for the regime of steady-state harmonic oscillations is written down as:

\[ q_m = (p_0 + p^*(\tau)) \sum_{\ell=1}^{\infty} \frac{4(-1)^\ell}{(2\ell-1)\pi} \cos \frac{2k-1}{2} \xi + \]

\[-\frac{\rho v_c^2}{\delta \nu^2} \sum_{\ell=1}^{\infty} \frac{2}{(2\ell-1)\pi} \left( \frac{dR_k}{d\tau} \cos \frac{2k-1}{2} \xi \right) + \]

\[ + 12 \frac{df}{d\tau} \sum_{\ell=1}^{\infty} \frac{4(-1)^\ell}{(2\ell-1)\pi} \left( \frac{2}{(2\ell-1)\pi} \cos \frac{2k-1}{2} \xi \right) \]

Equation (4) with consideration of (17), (18) can be written down as:

\[ \left( \frac{D}{\ell^2} \left( \frac{2k-1}{2} \right)^4 \right) + \kappa + \frac{\eta}{\ell^2} \left( \frac{2k-1}{2} \right)^2 \right) w_m(R_0 + R_k) + \]

\[ + \rho \delta h_0 \omega^2 w_m \frac{d^2 R_k}{d\tau^2} = -12 \frac{\rho \nu^2 \omega^2}{\delta \nu^2} \left( \frac{2}{(2k-1)\pi} \right)^2 w_m \frac{dR_k}{d\tau} + \]

\[ + 12 \frac{\rho \nu^2 \omega^2}{\delta \nu^2} \frac{4(-1)^k}{(2k-1)\pi} \left( \frac{2}{(2k-1)\pi} \right)^2 \left[ \frac{df}{d\tau} \left( p_0 + p^*(\tau) \right) \right] \]

Due to linearity of (19) for the constant pressure component we get the following expression:

\[ w_m R_k^0 = \frac{4(-1)^k}{(2k-1)\pi} \left( \frac{D}{\ell^2} \left( \frac{2k-1}{2} \right)^4 \right) + \frac{\eta}{\ell^2} \left( \frac{2k-1}{2} \right)^2 + \kappa \right)^{-1} \]

The solution of (19) for the regime of steady-state harmonic oscillations is written down as:

\[ w_m R_k = \frac{4(-1)^k}{(2k-1)\pi} \left[ a_m \left( C_{pk} \frac{df}{d\tau} + B_{pk} \frac{df}{d\tau} \right) + \zeta_m \left( C_{zk} \frac{df}{d\tau} + B_{zk} \frac{df}{d\tau} \right) \right] \]

\[ a_m = \frac{a_{3m}^2}{a_{k}^2 + a_{3k}^2}, \quad C_{pk} = -\frac{a_{3m}^2}{a_{k}^2 + a_{3k}^2}, \quad B_{pk} = \frac{a_{3m}^2 a_{2k}}{a_{k}^2 + a_{3k}^2}, \quad B_{zk} = \frac{a_{2m}^2 a_{3k}}{a_{k}^2 + a_{3k}^2}, \]

\[ a_{2m} = \frac{D_k}{\ell^2} \frac{2}{(2k-1)\pi} \left( \frac{2}{(2k-1)\pi} \right)^2 + \frac{\eta}{\ell^2} \left( \frac{2k-1}{2} \right) \right)^2 + \kappa \]

By substituting (21), (16) in the expression (13), and then taking into account the obtained expression in (2), we get:

\[ (m + M_r) \frac{d^2 \zeta}{dt^2} + (m + M_i) K \frac{d\zeta}{dt} = 2(b + M_r) p + K \frac{dp}{dt} \]

here we introduce the symbols:

\[ K_{m} = 8\ell b \frac{\rho v}{\delta \nu^2} \left[ 1 - 6 \sum_{\ell=1}^{\infty} \left( \frac{2}{(2k-1)\pi} \right)^4 B_{pk} \right], \quad K_{p} = -48\ell b \frac{\rho v}{\delta \nu^2} \sum_{\ell=1}^{\infty} \left( \frac{2}{(2k-1)\pi} \right)^4 B_{pk} \]
\[ M_z = 48lb \frac{pv}{\omega \delta \beta^{2}} \sum_{k=1}^{\infty} \left( \frac{2(2k-1)\pi}{\delta \beta} \right)^{4} C_{\delta}, \quad M_p = 48lb \frac{pv\omega}{\delta \beta^{2}} \sum_{k=1}^{\infty} \left( \frac{2(2k-1)\pi}{\delta \beta} \right)^{4} C_{\delta}. \]

The solution of (22) for the regime of steady-state harmonic oscillations takes the form of:

\[ z_k = \frac{2lb}{n} + p_m A_{(\omega)} \sin(\omega t + \varphi_{z}(\omega)), \quad A_{(\omega)} = \sqrt{C^2 + \tilde{B}^2}, \quad \varphi_{z}(\omega) = \arctg(\tilde{B}/\tilde{C}), \]

\[ \tilde{C} = e_1 d_1 - e_2 d_2, \quad \tilde{B} = e_1 d_2 + e_2 d_1, \quad d_1 = n-(m+M_{\omega})\omega^2, \quad d_2 = K_{\omega} \omega, \quad e_1 = 2b + M_{\rho}, \quad e_2 = K_{\rho} \omega. \]

Finally, with consideration of (20), (21) and (23) the law of the plate hydroelastic deflection (17) takes the form of:

\[ w = \frac{p_0}{D} \sum_{k=1}^{\infty} \frac{4(-1)^k}{2(2k-1)\pi} \left( \frac{(2k-1)\pi}{2} \right)^{4} + \frac{\eta^4}{D} \left( \frac{(2k-1)\pi}{2} \right)^{2} + \frac{\kappa^4}{D} \right)^{-1} \cos \left( \frac{2k-1}{2\ell} \pi \right) \]

\[ - p_m A_u(x, \omega) \sin(\omega t + \varphi_{\delta}(x, \omega)), \quad A_u(x, \omega) = \sqrt{T^2 + E^2}, \quad \varphi_{\delta}(x, \omega) = \arctg(E/T), \]

\[ T = \sum_{k=1}^{\infty} 4(-1)^k \left( B_{pk} - (B_{\delta} \tilde{C} + C_{\delta} \tilde{B}) \right) \cos \left( \frac{2k-1}{2\ell} \pi \right), \quad E = \sum_{k=1}^{\infty} 4(-1)^k \left( C_{pk} - (B_{\delta} \tilde{B} - C_{\delta} \tilde{C}) \right) \cos \left( \frac{2k-1}{2\ell} \pi \right). \]

By substituting (21) in (16) and taking into account (7), the pressure in the liquid layer between the stamp and the plate is written down as:

\[ p = p_0 + p_m \sin \omega t + p_m \Pi(x, \omega) \sin(\omega t + \varphi_{p}(x, \omega)), \quad \Pi(x, \omega) = \sqrt{S^2 + Q^2}, \quad \varphi_{p}(x, \omega) = \arctg(Q/S), \]

\[ Q = 6 \left( \frac{\pi}{\ell} \right)^2 \tilde{C} + 24 \sum_{k=1}^{\infty} \left( \frac{2}{2k-1}\pi \right)^{3} \left( -1 \right)^k \left( B_{pk} - C_{\delta} \tilde{B} - B_{\delta} \tilde{C} \right) \cos \left( \frac{2k-1}{2\ell} \pi \right), \]

\[ S = 6 \left( 1 - \frac{\pi}{\ell} \right)^2 \tilde{B} + 24 \sum_{k=1}^{\infty} \left( \frac{2}{2k-1}\pi \right)^{3} \left( -1 \right)^k \left( B_{\delta} \tilde{B} - C_{\delta} \tilde{C} - C_{pk} \right) \cos \left( \frac{2k-1}{2\ell} \pi \right). \]

4. Summary and Conclusion

The obtained analytical solution allows making the following conclusions. The function \( A_{(\omega)} \) of expression (23) is the stamp amplitude-frequency characteristic and \( \varphi_{z}(\omega) \) is the stamp phase response. The first element of expression (24) for the deflection presents the plate static deflection, conditioned by static pressure in liquid \( p_0 \), the second element is a dynamic deflection, conditioned by the plate and stamp interaction through a viscous liquid layer. The values of the given deflection component are defined by function \( A_u(x, \omega) \), which can be considered as frequency dependent function of deflection amplitude distribution along the plate.

The analogous remarks can be made to the law of pressure change in the channel (25). The first and the second components in the pressure law present the assumed static pressure and the law of pressure pulsation at the edges. The third component is dynamic pressure in the channel, conditioned by the stamp motion and oscillation of the plate resting on Pasternak foundation. The value of the third component is defined by function \( \Pi(x, \omega) \), which is the frequency dependent function of dynamic pressure amplitudes distribution along the channel. Function \( \varphi_{p}(x, \omega) \) is the frequency dependent function of the phase response.
distribution of plate deflection, and function $\varphi_p(x, \omega)$ is the frequency dependent function of the phase response distribution of pressure along the channel. The investigation of the mentioned above functions behavior makes it possible to study dynamic processes in the oscillation system under consideration. The analysis of the obtained expression for the plate deflection (24) allows to confirm, that the coefficients depending on Pasternak foundation reaction will affect as the static plate deflection, as the dynamic plate deflection. Also, the obtained solution provides transition from Pasternak foundation to Winkler one, i.e. in the case then the foundation shear coefficient $\eta$ equaling zero. Thus, the results of the considered study can be used for mathematical modeling and analyzing hydroelastic oscillations of the constructions elastic elements interacting with viscous liquid and elastic foundation.

Acknowledgments
This study was supported by the Russian Foundation for Basic Research grants 15-01-01604-a and 16-01-00175-a.

References
[1] Indeitsev D A, Polypanov I S, Sokolov S K 1994 Calculation of Cavitation Life-Time of Ship Engine Liner Problemy Mashinostræeniya i Nadeznos‘ti Mashin 4 pp 59-64.
[2] Haddara M R, Cao S A 1996 Study of the Dynamic Response of Submerged Rectangular Flat Plates Marine Structures 9(10) pp 913-933.
[3] Chapman C J, Sorokin S V 2005 The forced vibration of an elastic plate under significant fluid loading Journal of Sound and Vibration 281(3) pp 719-741.
[4] Ergin A, Ugurlu B 2003 Linear vibration analysis of cantilever plates partially submerged in fluid Journal of Fluids and Structures 17 pp 927-939.
[5] Kramer M R, Liu Z, Young Y L 2013 Free vibration of cantilevered composite plates in air and in water Composite Structures 95 pp 254-263.
[6] Kerboua Y, Lakis A A, Thomas M, Marcouiller L 2008 Vibration analysis of rectangular plates coupled with fluid Applied Mathematical Modelling 32(12) pp 2570-2586.
[7] Bochkarev S A, Lekomtsev S V, Matveenko V P 2016 Hydroelastic stability of a rectangular plate interacting with a layer of ideal flowing fluid Fluid Dynamics 51(6) pp 821-833.
[8] Avramov K V, Strelnikova E A 2014 Chaotic oscillations of plates interacting on both sides with a fluid flow International Applied Mechanics 50(3) pp 303-309.
[9] Önşay T 1993 Effects of layer thickness on the vibration response of a plate-fluid layer system Journal of Sound and Vibration 163(2) pp 231-259.
[10] Ageev R V, Mogilevich L I, Popov V S, Popova A A, Kondratov D V 2014 Mathematical Model of Pulsating Viscous Liquid Layer Movement in a Flat Channel with Elastically Fixed Wall Applied Mathematical Sciences 8(159) pp 7899-7908.
[11] Faria C T, Inman D J 2014 Modeling energy transport in a cantilevered Euler-Bernoulli beam actively vibrating in Newtonian fluid Mechanical Systems and Signal Processing 45(2) pp 317-329.
[12] Mogilevich L I, Popov V S 2011 Investigation of the interaction between a viscous incompressible fluid layer and walls of a channel formed by coaxial vibrating discs Fluid Dynamics 46(3) p 375-388.
[13] Mogilevich L I, Popov V S, Popova A A 2010 Dynamics of interaction of elastic elements of a vibrating machine with the compressed liquid layer lying between them Journal of Machinery Manufacture and Reliability 39(4) p 322-331.
[14] Akcabay D T, Young Y L 2012 Hydroelastic response and energy harvesting potential of flexible piezoelectric beams in viscous flow Physics of Fluids 24(5).
[15] Ageev R V, Kuznetsova E L, Kulikov N I, Mogilevich L I, Popov V S 2014 Mathematical model of movement of a pulsing layer of viscous liquid in the channel with an elastic wall PNRPU Mechanics Bulletin 3 pp 17-35.

[16] Ageev R V, Mogilevich L I, Popov V S 2014 Vibrations of the walls of a slot channel with a viscous fluid formed by three-layer and solid disks Journal of Machinery Manufacture and Reliability 43(1) pp 1-8.

[17] Wang Y H, Tham L G, Cheung Y K 2005 Beams and plates on elastic foundations: A review Progress in Structural Engineering and Materials 7(4) pp 174-182.

[18] Kudenko V D, Pleskachevskii Yu M, Starovoytov E I, Leonenko D V 2006 Natural vibration of a sandwich beam on an elastic foundation International Applied Mechanics 42(5) pp 541-547.

[19] Starovoytov E I, Leonenko D V 2012 Thermal impact on a circular plate on an elastic foundation Mechanics of Solids 47(1) pp 111-118.

[20] Starovoytov E I, Leonenko D V 2016 Vibrations of circular composite plates on an elastic foundation under the action of local loads Mechanics of Composite Materials 52(5) pp 665-672.

[21] Dash P R, Pradhan M and Pradhan P K 2016 Static and dynamic stability analysis of an asymmetric sandwich beam resting on a variable pasternak foundation subjected to thermal gradient Meccanica, 51(3) pp 725-739.

[22] Alekseev V V, Indeitsev D A, Mochalova Yu A 1999 Resonant oscillations of an elastic membrane on the bottom of a tank containing a heavy liquid Technical Physics 44(8) pp 903-907.

[23] Hosseini-Hashemi S, Karimi M, Hossein Rokni D T 2010 Hydroelastic vibration and buckling of rectangular Mindlin plates on Pasternak foundations under linearly varying in-plane loads Soil Dynamics and Earthquake Engineering, 30(12) pp 1487-1499.

[24] Ergin A, Kutlu A, Omurtag M H, Ugurlu B 2012 Dynamic response of Mindlin plates resting on arbitrarily orthotropic Pasternak foundation and partially in contact with fluid Ocean Engineering 42 pp 112-125.

[25] Ergin A, Kutlu A, Omurtag M H., Ugurlu B 2008 Dynamics of a rectangular plate resting on an elastic foundation and partially in contact with a quiescent fluid Journal of Sound and Vibration, 317(1-2) pp 308-328.

[26] Kuznetsova E L, Mogilevich L I, Popov V S, Rabinsky L N 2016 Mathematical model of the plate on elastic foundation interacting with pulsating viscous liquid layer Applied Mathematical Sciences 10(23) pp 1101-1109.

[27] Mogilevich L I, Popov V S and Popova A A 2017 Interaction dynamics of pulsating viscous liquid with the walls of the conduit on an elastic foundation Journal of Machinery Manufacture and Reliability, 46(1) pp 12-19.

[28] Mogilevich L I, Popov V S, Popova A A, Christoforova A V 2016 Mathematical modeling of highly viscous liquid dynamic interaction with walls of channel on elastic foundation IEEE Conference 2016 Dynamics of Systems, Mechanisms and Machines (Omsk, 2016).

[29] Mogilevich L I, Popov V S, Popova A A, Christoforova A V 2016 Mathematical modeling of hydroelastic walls oscillations of the channel on Winkler foundation under vibrations Vibroengineering PROCEEDIA 8 pp 294-299.

[30] Mogilevich L I and Popov V S 2016 Mathematical modeling of incompressible viscous liquid layer interaction dynamics in an annular slit with its wall, surrounded by elastic medium IEEE Conference 2016 Dynamics of Systems, Mechanisms and Machines (Omsk, 2016).

[31] Panovko Y G and Gubanova I I 1965 Stability and Oscillations of Elastic Systems (New York: Consultants Bureau Enterprises, Inc.).

[32] Pasternak P L 1954 On a New Method of Analysis of an Elastic Foundation by Means of Two Foundation Constants (Moscow: Gosudarstvennoe Izdatelstvo Literatury po Stroitelsvu i Architekture)

[33] Loitsyanskii L G 1966 Mechanics of Liquids and Gases (Oxford: Pergamon Press).
