DECOUPLING OF CUBIC POLYNOMIAL MATRIX SYSTEMS

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ABSTRACT. The decoupling of polynomial matrix system is to diagonalize its system matrix. In this paper, decoupling problems for cubic polynomial matrix system are considered. The decoupling conditions for a class of cubic polynomial matrix systems are derived under strict equivalence transformation. By using linearization, isospectral decoupling method for cubic polynomial matrix system is proposed. To be specific, necessary and sufficient conditions of isospectral diagonalization for nonsingular cubic polynomial matrix are given. These results are extended to singular cubic polynomial matrix. Solving processes are given to obtain isospectral diagonal cubic polynomial matrix for nonsingular and singular cases. Finally, illustrating examples are provided to verify the main results.

1. Introduction. Polynomial matrix theory has wide research in the algebra field [4, 8, 21]. But the development of the theory is far from perfect, many problems remain unresolved, and some fundamental control problems of polynomial matrix system have not been even put forward clearly to study [4, 21, 20, 5]. We know that linear differential system model in essence is polynomial matrix system [6, 7, 13]. Therefore, the study of polynomial matrix system is useful for analysis and design of high order differential system [1, 2, 19]. For control purposes, it is desirable that system model is decoupled, or the corresponding polynomial matrix of system is diagonalizable.

First-order linear systems which contain state space systems and descriptor systems correspond to linear polynomial matrix systems [21]. There are perfect results of diagonalization for state space systems and descriptor systems which diagonalization can be realized by strict equivalence transformation [21, 3]. However, quadratic polynomial matrix systems can be decoupled only under restrictive hypotheses on systems matrices [22, 11, 23]. Most of quadratic polynomial matrices can not be diagonalized by strict equivalence transformation [11]. There are still some restrictive hypotheses even for triangularization of quadratic polynomial matrix in real number field [18, 14]. There are not common diagonalization methods for general polynomial matrix with degree more than 2.

The classical approach to study polynomial matrix eigenvalue problems is linearization, where a polynomial matrix is converted into a larger linear matrix pencil with the same eigenvalues [4, 12]. There have been many literatures concerning linearization of polynomial matrix [4, 12, 10, 15]. [4]
introduces companion forms of linearization which is termed as companion linearization. [12] gives the definition of strong linearization, and shows that strong linearizations preserve the finite and infinite eigenstructure of polynomial matrix. Therefore, strong linearization is also a structure preserving transformation [10]. [10, 15] investigate the problems of eigenvalue in infinity and minimal indices for polynomial matrix. [16] develops a framework that generalizes the fundamental notion of linearization in a direct and simple way from degree one to other low degrees. [17] shows how to deflate eigenvalues of quadratic polynomial matrix by structure preserving transformations. These linearization results will provide a method for analyzing eigenstructure of polynomial matrix.

The isospectral diagonalization is a more general diagonalization process [11, 23]. Necessary and sufficient conditions are given by P. Lancaster and I. Zaballa in [11] for a quadratic polynomial matrix with nonsingular leading coefficient to be isospectral to a quadratic diagonal polynomial matrix. This results are generalized by J. C. Zúñiga Anaya in [23] to cover quadratic polynomial matrix with singular leading coefficients. Inspired by the idea and method of [11] and [23], we will consider isospectral diagonalization of cubic polynomial matrix, and study the diagonalization conditions of cubic polynomial matrix by using strict equivalence transformation and unimodular matrix transformation, respectively.

The rest of this paper is organized as follows. In Section 2, we give preliminaries and some lemmas. Section 3 considers diagonalization problems for nonsingular cubic polynomial matrix without using linearization. In Section 4, we derive necessary and sufficient conditions of isospectral diagonalization for nonsingular cubic polynomial matrix by linearization method, then generalize isospectral diagonalization conditions of nonsingular cubic polynomial matrix to singular cubic polynomial matrix. Section 5 presents the solving processes of isospectral diagonal cubic polynomial matrix. In Section 6, we give some conclusions and describes possible future work.

2. Preliminaries. Consider the following cubic polynomial matrix system

\[ A(s)X(s) = BU(s) \]  

with

\[ A(s) = A_3s^3 + A_2s^2 + A_1s + A_0 \]  

where \( A_i \in \mathbb{C}^{n \times n}, \) \( A_3 \) is a nonzero matrix, and \( A(s) \) is a regular polynomial matrix, that is, \( detA(s) \) is not identically zero. Before giving research objective of this paper, some preliminary definitions are needed.

**Definition 2.1.** (Partial multiplicity [16]) Let \( A(s) \in \mathbb{C}^{n \times n} \) be a regular polynomial matrix. For any eigenvalue \( s_i, \) the invariant polynomials \( d_j(s) \) of \( A(s) \) for \( j = 1 : n, \) can each be uniquely factored as

\[ d_j(s) = (s - s_i)^{m_{ij}}p_j(s) \text{ with } m_{ij} \geq 0, \quad p_j(s_i) \neq 0 \]

The sequences \( m_{ij}, \) for \( j = 1 : n \) satisfy the condition \( 0 \leq m_{i1} \leq m_{i2} \leq \cdots \leq m_{in} \) by the divisibility chain property of the Smith canonical form. \( m_{ij} \) is called the partial multiplicity of \( A(s) \) associated with \( s_i. \)

The partial multiplicity sequences are obtainable via solving Smith canonical form of polynomial matrix (see [4]). The algebraic multiplicity of an eigenvalue is the sum of its partial multiplicities, while the geometric multiplicity is the number of nonzero terms in the partial multiplicities sequences (see [16]).

**Definition 2.2.** Two cubic polynomial matrices with form (2) are called isospectral if they share the same eigenvalues and the same partial multiplicities; The system (1) can be decoupled or, equivalently, \( A(s) \) is diagonalizable if \( A(s) \) admits an isospectral diagonal cubic polynomial matrix.

Generally, we say that \( A(s) \) admits an isospectral diagonalization if \( A(s) \) admits an isospectral diagonal cubic polynomial matrix. By Definition 2.1, partial multiplicity is the degree of elementary divisor. If two cubic polynomial matrices have same elementary divisors, then they are isospectral and vice versa. If the geometric multiplicity of \( s_i \) is \( \mu_i, \) then \( \mu_i \leq n. \) To remove the terms of \( m_{ij} = 0, \) the sequences satisfy \( 0 < m_{i1} \leq m_{i2} \leq \cdots \leq m_{i\mu_i}. \)

This paper will study decoupling problem of the system (1). The main work is to study isospectral decoupling of the system (1) or, equivalently, isospectral diagonalization of polynomial matrix (2). We will derive conditions of strict equivalence transformation decoupling and isospectral decoupling, respectively.

Linearization method is used to derive isospectral decoupling conditions in this paper. We will only give a few useful results for cubic polynomial matrix.
Definition 2.3. Let \( A(s) \in \mathbb{C}^{n \times n} \) be a polynomial matrix of degree 3. A matrix pencil \( L(s) = sE - A \in \mathbb{C}^{3n \times 3n} \) is called a linearization of \( A(s) \) if there exist unimodular polynomial matrices \( U(s), V(s) \) such that
\[
U(s)L(s)V(s) = \begin{bmatrix} A(s) & 0 \\ 0 & I_{2n} \end{bmatrix}
\]
To study infinite eigenvalue, we need to introduce the reversal of polynomial matrix:
\[
A_*(s) = s^3A(1/s) = A_0s^3 + A_1s^2 + A_2s + A_3
\]

Definition 2.4. \( A(s) \) is said to have an infinite eigenvalue if \( A_*(s) \) has a zero eigenvalue. The algebraic, geometric, and partial multiplicities of the infinite eigenvalue are defined to be the same as the corresponding multiplicity of the zero eigenvalue of \( A_*(s) \).

Obviously, if \( s_0 \) is a nonzero eigenvalue of \( A(s) \), then \( 1/s_0 \) is the eigenvalue of \( A_*(s) \). The algebraic, geometric, and partial multiplicities of \( s_0 \) and \( 1/s_0 \) are identical.

Definition 2.5. Let \( L_*\) and \( A_*\) be the reversal of \( L(s) \) and \( A(s) \), respectively. If \( L(s) \) is a linearization of \( A(s) \) and \( L_* \) is a linearization of \( A_* \), then \( L(s) \) is said to be a strong linearization of \( A(s) \).

Lemma 2.6. Let \( A_1(s), A_2(s) \in \mathbb{C}^{n \times n} \) be cubic polynomial matrices, and \( L_1(s), L_2(s) \in \mathbb{C}^{3n \times 3n} \) be matrix pencils. If \( L_1(s) \) and \( L_2(s) \) are, respectively, strong linearization of \( A_1(s) \) and \( A_2(s) \), then \( \text{diag}(L_1(s), L_2(s)) \) is a strong linearization of \( \text{diag}(A_1(s), A_2(s)) \).

Proof. By the definition of linearization, one can obtain that
\[
U_1(s)L_1(s)V_1(s) = \begin{bmatrix} A_1(s) & 0 \\ 0 & I_{2n} \end{bmatrix}, U_2(s)L_2(s)V_2(s) = \begin{bmatrix} A_2(s) & 0 \\ 0 & I_{2n} \end{bmatrix}
\]
It follows that
\[
\begin{bmatrix} U_1(s) & 0 \\ 0 & U_2(s) \end{bmatrix} \begin{bmatrix} L_1(s) & 0 \\ 0 & L_2(s) \end{bmatrix} \begin{bmatrix} V_1(s) & 0 \\ 0 & V_2(s) \end{bmatrix} = \begin{bmatrix} A_1(s) & I_{2n} \\ I_{2n} & A_2(s) \end{bmatrix}
\]
Then, there exists nonsingular matrix \( W_1 \) such that
\[
W_1 \begin{bmatrix} U_1(s) & 0 \\ 0 & U_2(s) \end{bmatrix} \begin{bmatrix} L_1(s) & 0 \\ 0 & L_2(s) \end{bmatrix} \begin{bmatrix} V_1(s) & 0 \\ 0 & V_2(s) \end{bmatrix} W_1^{-1} = \begin{bmatrix} A_1(s) & I_{2n} \\ I_{2n} & A_2(s) \end{bmatrix}
\]
So, matrix \( \text{diag}(L_1(s), L_2(s)) \) is a linearization of matrix \( \text{diag}(A_1(s), A_2(s)) \). Similarly, matrix \( \text{diag}(L_*\), \( A_* \)) is a linearization of matrix \( \text{diag}(A_1(s), A_2(s)) \). Thus, \( \text{diag}(L_1(s), L_2(s)) \) is a strong linearization of matrix \( \text{diag}(A_1(s), A_2(s)) \).

It is easy to know that Lemma 2.6 is a general result for arbitrary polynomial matrix when the degrees of \( A_1(s) \) and \( A_2(s) \) are identical. The following lemma reveals an important property of strong linearization, which is necessary for the main results in later sections.

Lemma 2.7. ([15]) If \( L(s) \) is a strong linearization of \( A(s) \), then the finite elementary divisors and infinite elementary divisors of \( L(s) \) and \( A(s) \) are identical, where infinite elementary divisors are defined as the elementary divisors of the correspondence reversal polynomial matrix in origin.
Lemma 2.7 shows that strong linearizations preserve the finite eigenstructure and infinite eigenstructure of original polynomial matrix. There are many different possibilities for linearizations, one of them has been the so-called companion forms or companion polynomials [4]. Letting
\[ E = \begin{bmatrix} I & I \\ I & A_3 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & I \\ A_0 & A_1 & A_2 \end{bmatrix} \] (4)
then (4) is a companion form of cubic polynomial matrix (2).

The companion form has many nice properties that make them attractive as linearization for polynomial matrix. One of them is that it is always a strong linearization for given polynomial matrix. Therefore, companion linearization preserves the eigenstructure of polynomial matrix. For diagonal cubic polynomial matrix, taking companion form according to diagonal elements, then resulted matrix is still a strong linearization by Lemma 2.6.

For the sake of arguments in following sections, we give Jordan canonical form of matrix pencil.

For given regular matrix pencil \((E, A)\), there exist nonsingular matrix \(P, Q\) such that
\[ P(EI - A)Q = \begin{bmatrix} sI - J_f & 0 \\ 0 & sN - I \end{bmatrix} \] (5)

where \(J_f\) is finite Jordan canonical form, \(N\) is nilpotent matrix and also characterizes infinite eigenstructure. If \(E\) is nonsingular, then \(sN - I\) vanishes in (5).

In this paper, assume that \(A(s)\) has distinct finite eigenvalues \(s_i\) for \(i = 1: t\), and infinite eigenvalue \(\mu_i, \bar{\mu}_i\) are, respectively, geometric multiplicity and algebraic multiplicity of finite eigenvalue \(s_i\), \(\mu_i, \bar{\mu}_i\) are, respectively, geometric multiplicity and algebraic multiplicity of infinite eigenvalue. Define \(0 < m_{i1} \leq m_{i2} \leq \cdots \leq m_{i\mu_i}\) as the partial multiplicities of finite eigenvalue \(s_i\), and \(0 < m_{\infty j} \leq m_{\infty 2} \leq \cdots \leq m_{\infty \mu_{\infty}}\) as the partial multiplicities of infinite eigenvalue.

If two systems are isospectral, then they have same eigenvalues, and \(m_{ij}, m_{\infty j}\) are identical for each \(s_i\) or infinite eigenvalue. Therefore, they have same \(\mu_i, \bar{\mu}_i, \mu_{\infty}, \bar{\mu}_{\infty}\) for corresponding eigenvalue. Define that
\[ J = \text{diag}(J_f, J_{\infty}) \]
is a canonical form with finite and infinite eigenstructure, where, \(J_f, J_{\infty}\) are finite eigenstructure and infinite eigenstructure, respectively. The infinite eigenstructure is defined as the zero eigenstructure of the reversal of polynomial matrix. Note that companion linearization is eigenstructure preserving transformation. Therefore, \(J\) can be rewritten as
\[ J = \text{diag}(J_f, N) \] (6)

where, \(J_f, N\) can be obtained from (5) with following forms
\[
\begin{cases}
J_f = \text{diag}(J_1, J_2, \cdots, J_t) \\
J_1 = \text{diag}(J_{s_1}, J_{s_2}, \cdots, J_{s_{\mu_1}}) \\
J_{ij} = \begin{bmatrix} s_i & \cdots & 1 \\ \cdots & \cdots & \cdots \\ 1 & \cdots & s_i \end{bmatrix} \\
N = \text{diag}(N_1, N_2, \cdots, N_{\mu_{\infty}}) \\
N_i = \begin{bmatrix} 0 & \cdots & 1 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 \end{bmatrix}
\end{cases}
\]

It is seen that \(N_i\) is nilpotent matrix, and nilpotent index is determined by the partial multiplicity of infinite eigenvalue. In order to differ from the previous Jordan canonical form, we call (6) generalized Jordan canonical form or generalized Jordan structure. By Lemma 2.7, there are same elementary divisors for polynomial matrix and its linearization. So, in this paper, we claim that Jordan canonical form of cubic polynomial matrix refers to the form (6). The following corollary can be used as an equivalent definition of isospectral polynomial matrix.

Corollary 1. Two cubic polynomial matrices with form (2) are called isospectral if and only if they share the same Jordan canonical form.

We consider only regular cubic polynomial matrix. For regular cubic polynomial matrix (2), \(\text{rank} A(s) = n\), and the following conclusion holds.

Lemma 2.8. ([23, 6]) Let \(A(s) \in \mathbb{C}^{n \times n}\) be a regular cubic polynomial matrix with \(n_f\) finite eigenvalues counting algebraic multiplicities, and with an eigenvalue at infinity of algebraic multiplicity \(n_{\infty}\), then
\[ n_f + n_{\infty} = 3n \] (7)
3. Decoupling without linearization. In this section, we will derive the conditions of diagonalization for nonsingular cubic polynomial matrix by strict equivalence transformation. These conditions are an extension of the results of quadratic polynomial matrix for diagonalization by strict equivalence transformation.

Lemma 3.1. ([11]) Let $A_0, A_3 \in \mathbb{C}^{n \times n}$ with $\det A_3 \neq 0$, assume that $sA_3 + A_0$ is semisimple (i.e. nondefective, [8]), and write a diagonal matrix of the eigenvalues of $sA_3 + A_0$ in the form

$\Lambda = \text{diag}(s_1I_1, s_2I_2, \ldots, s_tI_t)$

where $s_i \neq s_j$ for $i \neq j$. Then there exist two nonsingular matrix $U, V \in \mathbb{C}^{n \times n}$ such that

$UA_3V = I, UA_0V = \Lambda$

If $A = \text{diag}(A_1, A_2, \ldots, A_t)$ is nonsingular and $A_j$ has the size of $I_j$, then $U, V$ can be replaced by $A^{-1}U, VA$, respectively.

Theorem 3.2. Let $A_i \in \mathbb{C}^{n \times n}, i = 0, 1, 2, 3$, with $\det A_3 \neq 0$ and assume that $sA_3 + A_0$ has $n$ distinct eigenvalues. Then there exist nonsingular matrix $U, V \in \mathbb{C}^{n \times n}$ such that $UA_3V = I$ and

$UA_2V, UA_1V, UA_0V$

are diagonal matrices if and only if

$A_0A_3^{-1}A_2 = A_2A_3^{-1}A_0, \quad A_0A_3^{-1}A_1 = A_1A_3^{-1}A_0$

Proof. By Lemma 3.1, there exist matrices $U, V \in \mathbb{C}^{n \times n}$, such that $UA_3V = I$ and $UA_0V = \Lambda$.

Since

$A_0A_3^{-1}U^{-1}AV^{-1} = U^{-1}A_3V^{-1}U = U^{-1}A_0$

$A_3^{-1}A_0 = VUU^{-1}A_0V^{-1} = VAV^{-1}$

It follows from $A_0A_3^{-1}A_2 = A_2A_3^{-1}A_0$ that

$UA_2V = UA_2VA$

This implies that $UA_2V$ is commutative with a diagonal matrix. Note that $A$ has distinct diagonal entries. Thus, $UA_2V$ is a diagonal matrix. Similar arguments by using $A_0A_3^{-1}A_1 = A_1A_3^{-1}A_0$, one can obtain that $UA_1V$ is a diagonal matrix.

Conversely, if $UA_0V, UA_1V, UA_2V, UA_3V$ are diagonal matrices. Let $A_0 = UA_0V, A_1 = UA_1V, A_2 = UA_2V, A_3 = UA_3V$, then

$A_0A_3^{-1}A_2 = U^{-1}A_0V^{-1}V^{-1}A_3^{-1}U^{-1}A_2V^{-1} = U^{-1}A_0A_3^{-1}A_2V^{-1}$

$A_2A_3^{-1}A_0 = U^{-1}A_2V^{-1}A_3^{-1}U^{-1}A_3^{-1}V^{-1} = U^{-1}A_2A_3^{-1}A_0V^{-1}$

Notice that $A_0, A_1, A_2, A_3$ are diagonal matrices. Thus, $A_0A_3^{-1}A_2 = A_2A_3^{-1}A_0$. Similarly, $A_0A_3^{-1}A_1 = A_1A_3^{-1}A_0$.

Theorem 3.2 gives conditions of strict equivalence transformation decoupling for cubic polynomial matrix system. This decoupling method has restrictive hypotheses, and most cubic polynomial matrix systems don’t satisfy these hypotheses. In the following section, we will introduce isospectral decoupling method, that is, decoupling by using linearization, and give decoupling conditions.

4. Conditions for isospectral decoupling. Just like quadratic polynomial matrices, most cubic polynomial matrices cannot be diagonalized by strict equivalence transformation or congruence transformation. As mentioned above, isospectral decoupling may be a feasible decoupling approach for cubic polynomial matrix system. The isospectral decoupling problem of cubic polynomial matrix system is to diagonalize its system matrix. In this section, we generalize the results of quadratic polynomial matrix for isospectral diagonalization (see [11, 23]) to cubic polynomial matrix. We will give necessary and sufficient conditions of isospectral diagonalization for cubic polynomial matrix.
4.1. System with nonsingular $A_3$. In order to derive diagonalization conditions of cubic polynomial matrix, the spectrum of digonal cubic polynomial matrix with nonsingular $A_3$ will be analyzed. Then necessary and sufficient conditions for isospectral diagonalization will be obtained.

Assume that $\tilde{A}(s) = \tilde{A}_3 s^3 + \tilde{A}_2 s^2 + \tilde{A}_1 s + \tilde{A}_0$ is a diagonal polynomial matrix with nonsingular $\tilde{A}_3$, that is, $\tilde{A}_i$ for $i = 0, 1, 2, 3$ are diagonal matrices. The distinct eigenvalues of $\tilde{A}(s)$ are $s_i$ for $i = 1 : t$. Define $0 < m_{i1} \leq m_{i2} \leq \cdots \leq m_{i\mu_i}$ as the partial multiplicities of $s_i$. $\mu_i, \bar{\mu}_i$ are, respectively, geometric multiplicity and algebraic multiplicity of $s_i$. It is easy to see that

$$\sum_{j=1}^{\bar{\mu}_i} m_{ij} = \mu_i \leq 3n.$$  

The non-singularity of $\tilde{A}_3$ implies that $\tilde{A}(s)$ has no infinite eigenvalue, then by (7)

$$\sum_{i=1}^{t} \bar{\mu}_i = 3n$$  

(8)

Note that the degree of each entry of $\tilde{A}(s)$ is not more than 3. This implies the largest degree of elementary divisors at most 3, then,

$$1 \leq m_{ij} \leq 3, \ 1 \leq i \leq t, 1 \leq j \leq \mu_i$$  

and, in consequence

$$\bar{\mu}_i \leq 3\mu_i$$

Define each partial multiplicity of $s_i$ that

$$m_{ij} = \begin{cases} 
3, & j = 1, 2, \cdots \tau_i, \\
2, & j = \tau_i + 1, \cdots \tau_i + \rho_i, \\
1, & j = \tau_i + \rho_i + 1, \cdots \mu_i.
\end{cases}$$

Let

$$\sum_{i=1}^{t} \tau_i = p$$  

(10)

Considering the companion matrix of each diagonal entry, it is nonderogatory matrix (see [4]). Each cubic elementary divisor (i.e. $m_{ij}=3$) is associated with just one entry of $\tilde{A}(s)$. By (10), the total number of cubic elementary divisors is $p$. Then two cases are included in the remaining $n - p$ diagonal entries. a) The product of one quadratic elementary divisor and one distinct linear elementary divisor (i.e. $m_{ij}=2$, $m_{kl}=1$, $k \neq i$); b) The product of three distinct linear elementary divisors (i.e. $m_{ij}=1$, $m_{kl}=1$, $m_{ij}=1$, $i \neq k \neq l$). It follows immediately from a) that each quadratic elementary divisor matches one distinct linear elementary divisor. Thus, the number of quadratic elementary divisor associated with eigenvalue $s_i$ is not more than the total number of other linear elementary divisors associated with eigenvalue $s_j$ with $j \neq i$. Namely,

$$\rho_i \leq \sum_{j \neq i} (\mu_j - \tau_j - \rho_j)$$  

(11)

In either case, the number of each quadratic elementary divisor or linear elementary divisor in each entry of $\tilde{A}(s)$ is contained at the most 1. Therefore, $\mu_i - \tau_i - \rho_i \leq n - p$ and

$$\rho_i \leq n - p$$  

(12)

For these $\mu_i - \tau_i - \rho_i$ linear elementary divisors, some of them may match with some quadratic elementary divisors. Assume that there are $x_i$ divisors of $\mu_i - \tau_i - \rho_i$ linear elementary divisors to match with some quadratic elementary divisors. Then

$$0 \leq x_i \leq \min \left\{ \sum_{j \neq i} \rho_j, \mu_i - \tau_i - \rho_i \right\}$$  

(13)

The case b) implies that

$$\mu_i - \tau_i - \rho_i - x_i \leq n - p - \sum_{i=1}^{t} \rho_i$$  

(14)

That is, inequation (14) has solutions $x_i$ for $i = 1 : t$, where $x_i$ for $i = 1 : t$ are subject to (13). Hence, we obtain the following theorem.
Theorem 4.1. Assume that $A(s)$ is a regular cubic polynomial matrix with form (2) and nonsingular $A_{\lambda_i}$. $J$ is the Jordan canonical form of $A(s)$ with distinct eigenvalues $s_i$ for $i = 1 : t$. $\mu_i, \bar{\mu}_i$ are geometric and algebraic multiplicity of $s_i$, respectively, and $0 < m_{i1} \leq m_{i2} \leq \cdots \leq m_{i\mu_i}$ are partial multiplicities associated with $s_i$. Then $A(s)$ admits an isospectral diagonalization if and only if (8), (9), (11) and (12), and (14) has solutions. Let $\bar{A}(s)$ be an undetermined diagonal cubic polynomial matrix. The largest degree of the elementary divisors is 3, therefore, each of these cubic elementary divisors could be associated with one of the $n$ entries in $\bar{A}(s)$. The total number of these entries is $p$. Let

$$A_{11}, \ldots, A_{1r_1}, \ldots, A_{t1}, \ldots, A_{tr_t}$$

be their corresponding $3 \times 3$ companion matrices. The remaining elementary divisors are quadratic and linear. Consider that the largest degree for each entries of $\bar{A}(s)$ is 3. Next, we prove that the remaining divisors can be classified into two categories, case a) and case b) (in the proof of necessity). We first verify case a). It is seen from (11) that there must exist one linear elementary divisor to form an entry (i.e. $(s - s_i)^3(s - s_j), i \neq j$) of $A(s)$. Let

$$B_{11}, \ldots, B_{1p_1}, \ldots, B_{t1}, \ldots, B_{tp_t}$$

be the companion matrices with the form $(s - s_i)^3(s - s_j), i \neq j$, respectively. Then, we need to verify case b). The number of undetermined entries of $\bar{A}(s)$ is $n - p - \sum_{i=1}^{t} \rho_i$. To finish the proof, we need verify that the remaining linear divisors must be a multiple of 3. Note that

$$\bar{\mu}_i = 3r_i + 2\rho_i + \mu_i - \tau_i - \rho_i$$

and because of (8) and (10),

$$3n = 3p + \sum_{i=1}^{t} \rho_i + \sum_{i=1}^{t} (\mu_i - \tau_i)$$

Then

$$\sum_{i=1}^{t} (\mu_i - \tau_i - \rho_i) - \sum_{i=1}^{t} \rho_i = 3(n - p - \sum_{i=1}^{t} \rho_i)$$

The left-hand side of equation is the total number of remaining linear divisors. This is just a multiple of 3. Now, condition (14) ensures that the maximum number of remaining linear divisors associated with $s_i$ for $i = 1 : t$ is $n - p - \sum_{i=1}^{t} \rho_i$. In other words, the linear elementary divisors can be organized in three ordered lists, each with $(n - p - \sum_{i=1}^{t} \rho_i)$ eigenvalues. This implies that case b) is verified (i.e. $(s - s_i)(s - s_j)(s - s_k), i \neq j \neq k$). Thus, $n$ entries of $\bar{A}(s)$ have been determined. Let $C_{ij}, i = 1, 2, \ldots, c$, be the companion matrix of $(s - s_i)(s - s_j)(s - s_k), i \neq j \neq k$, where $c = n - p - \sum_{i=1}^{t} \rho_i$.

Finally, we will show that $A(s)$ and $\bar{A}(s)$ are isospectral. Let

$$J_0 = \text{diag}(A_{11}, \ldots, A_{1r_1}, \ldots, A_{t1}, \ldots, A_{tr_t}, B_{11}, \ldots, B_{1p_1}, \ldots, B_{t1}, \ldots, B_{tp_t}, C_1, \ldots, C_c)$$

then, it follows that $J_0$ has $J$ for its Jordan canonical form, and $sI - J_0$ is a strong linearization of $\bar{A}(s)$ by Lemma 2.6. Therefore, $J$ is also the Jordan canonical form of $\bar{A}(s)$. By Corollary 1, $A(s)$ and $\bar{A}(s)$ are isospectral. Thus, $A(s)$ admits an isospectral diagonalization.

4.2. System with singular $A_{\lambda_i}$. In this section, we consider system (2) with singular leading coefficient. This will be an extension of the previous section, and infinite eigenvalue is included in Jordan structure. Following the order of ideas of the previous, we start by analyzing the eigenstructure of diagonal cubic polynomial with singular leading coefficient. Note that infinite eigenvalue is defined by zero eigenvalue of the reversal of cubic polynomial matrix. Therefore, in this case, the diagonal cubic polynomial matrix could include not only cubic terms, but also quadratic, linear, and constant terms.

Assume that, in addition to the distinct eigenvalues $s_i$ for $i = 1 : t$, the diagonal cubic polynomial matrix $A(s)$ has an infinite eigenvalue with algebraic and geometric multiplicity $\bar{\mu}_\infty, \mu_\infty$. 

Proof. The necessity has already been established. To prove sufficiency, suppose that the Jordan structure satisfies (8), (9), (11) and (12), and (14) has solutions. Let $\bar{A}(s)$ be an undetermined diagonal cubic polynomial matrix. The largest degree of the elementary divisors is 3, therefore, each of these cubic elementary divisors could be associated with one of the $n$ entries in $\bar{A}(s)$. The total number of these entries is $p$. Let

$$A_{11}, \ldots, A_{1r_1}, \ldots, A_{t1}, \ldots, A_{tr_t}$$

be their corresponding $3 \times 3$ companion matrices. The remaining elementary divisors are quadratic and linear. Consider that the largest degree for each entries of $\bar{A}(s)$ is 3. Next, we prove that the remaining divisors can be classified into two categories, case a) and case b) (in the proof of necessity). We first verify case a). It is seen from (11) that there must exist one linear elementary divisor to form an entry (i.e. $(s - s_i)^3(s - s_j), i \neq j$) of $A(s)$. Let

$$B_{11}, \ldots, B_{1p_1}, \ldots, B_{t1}, \ldots, B_{tp_t}$$

be the companion matrices with the form $(s - s_i)^3(s - s_j), i \neq j$, respectively. Then, we need to verify case b). The number of undetermined entries of $\bar{A}(s)$ is $n - p - \sum_{i=1}^{t} \rho_i$. To finish the proof, we need verify that the remaining linear divisors must be a multiple of 3. Note that

$$\bar{\mu}_i = 3r_i + 2\rho_i + \mu_i - \tau_i - \rho_i$$

and because of (8) and (10),

$$3n = 3p + \sum_{i=1}^{t} \rho_i + \sum_{i=1}^{t} (\mu_i - \tau_i)$$

Then

$$\sum_{i=1}^{t} (\mu_i - \tau_i - \rho_i) - \sum_{i=1}^{t} \rho_i = 3(n - p - \sum_{i=1}^{t} \rho_i)$$

The left-hand side of equation is the total number of remaining linear divisors. This is just a multiple of 3. Now, condition (14) ensures that the maximum number of remaining linear divisors associated with $s_i$ for $i = 1 : t$ is $n - p - \sum_{i=1}^{t} \rho_i$. In other words, the linear elementary divisors can be organized in three ordered lists, each with $(n - p - \sum_{i=1}^{t} \rho_i)$ eigenvalues. This implies that case b) is verified (i.e. $(s - s_i)(s - s_j)(s - s_k), i \neq j \neq k$). Thus, $n$ entries of $\bar{A}(s)$ have been determined. Let $C_{ij}, i = 1, 2, \ldots, c$, be the companion matrix of $(s - s_i)(s - s_j)(s - s_k), i \neq j \neq k$, where $c = n - p - \sum_{i=1}^{t} \rho_i$.

Finally, we will show that $A(s)$ and $\bar{A}(s)$ are isospectral. Let

$$J_0 = \text{diag}(A_{11}, \ldots, A_{1r_1}, \ldots, A_{t1}, \ldots, A_{tr_t}, B_{11}, \ldots, B_{1p_1}, \ldots, B_{t1}, \ldots, B_{tp_t}, C_1, \ldots, C_c)$$

then, it follows that $J_0$ has $J$ for its Jordan canonical form, and $sI - J_0$ is a strong linearization of $\bar{A}(s)$ by Lemma 2.6. Therefore, $J$ is also the Jordan canonical form of $\bar{A}(s)$. By Corollary 1, $A(s)$ and $\bar{A}(s)$ are isospectral. Thus, $A(s)$ admits an isospectral diagonalization.


and \( \mu_\infty \leq n \). \( 0 < m_{31} \leq m_{32} \leq \cdots \leq m_{3n} \mu_\infty \) are the partial multiplicities of infinite eigenvalue. Obviously, \( \bar{\mu}_\infty \leq 3n \). It is obtained from (7) that,

\[
\sum_{i=1}^t \bar{\mu}_i + \bar{\mu}_\infty = 3n.
\]

If the degree of an entry in diagonal cubic polynomial matrix is 3, then the Jordan canonical form of companion matrix for this entry is same as previous section. As already mentioned, if the degree of an entry less than 3, then infinite eigenvalue is included in this entry. Thus, these terms may be

\[
(s - s_i)^2; (s - s_i)(s - s_j), i \neq j; s - s_i; 1
\]

After linearization according to (4), the canonical forms corresponding to (5) are, respectively,

\[
\begin{bmatrix}
  s - s_i & -1 & 0 \\
  0 & s - s_i & 0 \\
  0 & 0 & -1
\end{bmatrix}, \quad
\begin{bmatrix}
  s - s_i & 0 & 0 \\
  0 & s - s_j & 0 \\
  0 & 0 & -1
\end{bmatrix},
\]

\[
\begin{bmatrix}
  s - s_i & 0 & 0 \\
  0 & -1 & s \\
  0 & 0 & -1
\end{bmatrix}, \quad
\begin{bmatrix}
  -1 & s & 0 \\
  0 & 0 & -1 \\
  0 & 0 & -1
\end{bmatrix}
\]

Obviously, the partial multiplicities of infinite eigenvalue are, respectively, 1, 1, 2 and 3. Thus, \( J_f \) and \( N \) are easy to obtain.

Note that infinite eigenvalue and its multiplicities of \( \hat{A}(s) \) are given by zero eigenvalue of \( \hat{A}_s(s) \). So, by (9), we know \( 1 \leq m_{\infty j} \leq 3 \). The finite eigenvalues obviously satisfy condition (9). To sum up, that is,

\[
\begin{align*}
1 & \leq m_{ij} \leq 3, \quad i = 1 : t, j = 1 : \mu_i \\
1 & \leq m_{\infty j} \leq 3, \quad j = 1 : \mu_\infty
\end{align*}
\]

\( m_{\infty j} \in \{ 3, \quad j = 1, 2, \ldots, \tau_\infty, \quad 2, \quad j = \tau_\infty + 1, \ldots, \rho_\infty + \tau_\infty, \quad 1, \quad j = \tau_\infty + \rho_\infty + 1, \ldots, \mu_\infty \} \). (17)

Since, only the cubic entries in \( \hat{A}(s) \) have no infinite eigenvalue. This implies that

\[
\mu_\infty \leq n - p
\]

The number of constant entries in \( \hat{A}(s) \) is \( \tau_\infty \). In consequence, the numbers of quadratic elementary divisors and linear elementary divisors associated with \( s_1 \) are smaller than or equal to \( n - p - \tau_\infty \), and so is the number of quadratic infinite elementary divisors. Thus, \( \mu_i - \tau_i - \rho_i \leq n - p - \tau_\infty, \quad \mu_\infty - \tau_\infty - \rho_\infty \leq n - p - \tau_\infty \) and

\[
\rho_\infty \leq n - p - \tau_\infty
\]

\[
\rho_\infty \leq n - p - \tau_\infty
\]

\( m_{\infty j} \in \{ 3, \quad j = 1, 2, \ldots, \tau_\infty, \quad 2, \quad j = \tau_\infty + 1, \ldots, \rho_\infty + \tau_\infty, \quad 1, \quad j = \tau_\infty + \rho_\infty + 1, \ldots, \mu_\infty \} \). (17)

It is easy to see that each quadratic finite elementary divisor (i.e. partial multiplicity \( m_{ij} = 2 \)) matches one distinct linear finite elementary divisor (i.e. partial multiplicity \( m_{kj} = 1, k \neq i \)) or one linear infinite elementary divisor (i.e. partial multiplicity \( m_{\infty j} = 1 \)). Each quadratic infinite elementary divisor (i.e. partial multiplicity \( m_{\infty j} = 2 \)) matches one linear finite elementary divisor. So,

\[
\begin{align*}
\rho_i & \leq \mu_i - \tau_i - \rho_i + \sum_{j \neq i} (\mu_j - \tau_j - \rho_j) \\
\rho_\infty & \leq \sum_i (\mu_i - \tau_i - \rho_i)
\end{align*}
\]

\( m_{\infty j} \in \{ 3, \quad j = 1, 2, \ldots, \tau_\infty, \quad 2, \quad j = \tau_\infty + 1, \ldots, \rho_\infty + \tau_\infty, \quad 1, \quad j = \tau_\infty + \rho_\infty + 1, \ldots, \mu_\infty \} \). (17)

\[
\rho_i \leq \mu_i - \tau_i - \rho_i + \sum_{j \neq i} (\mu_j - \tau_j - \rho_j) + \sum_{i=1}^t \rho_i + \rho_\infty
\]

\[
0 \leq x_i \leq \min \left\{ \sum_{j \neq i} \rho_j + \rho_\infty, \mu_i - \tau_i - \rho_i \right\}
\]

\[
0 \leq x_\infty \leq \min \left\{ \sum_{i=1}^t \rho_i, \mu_\infty - \tau_\infty - \rho_\infty \right\}
\]

\( \sum_{i=1}^t x_i + x_\infty = \sum_{i=1}^t \rho_i + \rho_\infty \)
The remaining each entry of diagonal polynomial matrix is product of three distinct linear elementary divisors. Thus,
\[
\begin{align*}
\mu_i - \tau_i - p_i - x_i &\leq n - p - \tau_\infty - \sum_{i=1}^{t} \rho_i - \rho_\infty \\
\mu_\infty - \tau_\infty - p_\infty - x_\infty &\leq n - p - \tau_\infty - \sum_{i=1}^{t} \rho_i - \rho_\infty
\end{align*}
\] (22)

As a generalization of Theorem 4.1, we can obtain the following result by above analysis.

**Theorem 4.2.** Assume that \(A(s)\) is a regular cubic polynomial matrix with form (2) and singular \(A_\infty\). \(J\) is the generalized Jordan canonical form of \(A(s)\) with distinct finite eigenvalues \(s_i\) for \(i = 1, \ldots, t\), and infinite eigenvalue, \(\mu, \mu_i\) are algebraic and geometric multiplicity of \(s_i\), respectively, and \(0 < m_{\infty,1} \leq m_{\infty,2} \leq \cdots \leq m_{\infty,\mu_\infty}\) with \(\mu_\infty \leq n\) are algebraic and geometric multiplicities of infinite eigenvalue, respectively, and \(0 < m_{\infty,1} \leq m_{\infty,2} \leq \cdots \leq m_{\infty,\mu_\infty}\) are partial multiplicities associated with infinite eigenvalue. Then \(A(s)\) admits an isospectral diagonalization if and only if (15), (16), (18), (19) and (20) hold, and (22) has solutions under (21).

**Proof.** The necessity has already been established. To prove sufficiency, suppose that the Jordan structure satisfies (15), (16), (18), (19) and (20), and (22) has solutions. Assume that there is an undetermined diagonal cubic polynomial matrix. Each of elementary divisors with degree 3 could be associated with one of the \(n\) entries in the diagonal cubic polynomial matrix as a cubic entry or as a constant entry. The total number of these entries is \(p + \tau_\infty\). Let
\[
A_{11}, \ldots, A_{1\tau_1}, \ldots, A_{t1}, \ldots, A_{t\tau_1}, B_1, \ldots, B_{\tau_\infty}
\]
be companion linearizations of these elementary divisors with degree 3, respectively. Let
\[
L_1 = \text{diag}(A_{11}, \ldots, A_{1\tau_1}, \ldots, A_{t1}, \ldots, A_{t\tau_1}, B_1, \ldots, B_{\tau_\infty})
\]
The remaining elementary divisors need to construct \(n - p - \tau_\infty\) entries of the diagonal cubic polynomial matrix. Each of these entries is a product of some elementary divisors. It is sufficient to prove that these products can be classified into two categories: 1) The product of one elementary divisor with degree 2 and one distinct elementary divisor with degree 1. 2) The product of three distinct elementary divisors with degree 1. It is worth to note that these elementary divisors are finite or infinite. For case 1), conditions (19) and (20) ensure that there exists a distinct elementary divisor with degree 1 for each quadratic elementary divisor to form an entry of the diagonal polynomial matrix. The total number of these entries for case 1) is \(\rho_\infty + \sum_{i=1}^{t} \rho_i\). Let
\[
C_{11}, \ldots, C_{1\rho_1}, \ldots, C_{11}, \ldots, C_{1\rho_1}, D_1, \ldots, D_{\rho_\infty}
\]
be companion linearizations of these entries, respectively. Let
\[
L_2 = \text{diag}(C_{11}, \ldots, C_{1\rho_1}, \ldots, C_{11}, \ldots, C_{1\rho_1}, D_1, \ldots, D_{\rho_\infty})
\]
The degrees of remaining elementary divisors are equal to 1, and these divisors need to construct \(n - p - \tau_\infty - \rho_\infty - \sum_{i=1}^{t} \rho_i\) entries of the diagonal cubic polynomial matrix. We first prove that the number of these linear elementary divisors is just a multiple of 3. By relationship of multiplicities associated with \(s_i\) and infinite eigenvalue, we obtain
\[
\begin{align*}
\mu_i &= 3\tau_i + 2\rho_i + \mu_i - \tau_i - \rho_i \\
\mu_\infty &= 3\tau_\infty + 2\rho_\infty + \mu_\infty - \tau_\infty - \rho_\infty
\end{align*}
\] (23)

It follows from (15) that
\[
\sum_{i=1}^{t}(\mu_i + \rho_i + 2\tau_i) + \mu_\infty + \rho_\infty + 2\tau_\infty = 3n
\]
Subtracting \(3[\rho_\infty + \tau_\infty + \sum_{i=1}^{t}(\rho_i + \tau_i)]\) in both sides, and note (10) that the above equation can be rewritten as
\[
\sum_{i=1}^{t}(\mu_i - \tau_i - \rho_i) + (\mu_\infty - \tau_\infty - \rho_\infty) - \sum_{i=1}^{t} \rho_i - \rho_\infty = 3(n - p - \tau_\infty - \rho_\infty - \sum_{i=1}^{t} \rho_i)
\]
It is easy to know that the left-hand side of equation is the total number of remaining elementary divisors with degree 1. This is just a multiple of 3.

Now, we may prove that case 2) holds. If inequation (22) has solutions \(x_1, x_\infty\), where \(x_1, x_\infty\) are subject to (21), then three eigenvalues can be distinct from each other for each product. Hence, the
diagonal cubic polynomial matrix has been determined. Let \( D_i, i = 1, 2, \cdots, d \), be the companion linearizations of these entries for case 2). Let

\[
L_3 = \text{diag}\{D_1, D_2, \cdots, D_d\}
\]

where

\[
d = n - p - \tau_s - \rho_s - \sum_{i=1}^t \rho_i
\]

Let \( L = \text{diag}\{L_1, L_2, L_3\} \). Then \( L \) has \( J \) for its generalized Jordan canonical form. By Lemma 2.6, \( L \) is a strong linearization of the diagonal cubic polynomial matrix. Since companion linearization is a strong linearization, it preserves the eigenstructure of polynomial matrix by Lemma 2.7. Therefore, \( J \) is also the Jordan canonical form of the diagonal cubic polynomial matrix. \( \square \)

**Remark 1.** The companion linearization is an important tool for the proof of the main theorems. Companion matrix is a nonderogatory matrix \([4, 8]\), which is a strong linearization that preserves the eigenstructure of original polynomial matrix. In proof of Theorem 4.1 and Theorem 4.2, strong linearization and property of nonderogatory matrix are used to analyze eigenstructure of cubic polynomial matrix. Then, the diagonalization conditions for cubic polynomial matrix are derived.

**Remark 2.** The results of Theorem 4.1 and Theorem 4.2 generalized the results of \([11]\) and \([23]\). To compare with the results of \([11]\), conditions (8) and (9) are respectively direct extensions of conditions (10) and (13) of \([11]\), and condition (15) of \([11]\) is a special case of condition (14). Besides, to ensure isospectral diagonalization for nonsingular cubic polynomial matrix, conditions (11), (12) and (13) need to be added. Similarly, to compare with the results of \([23]\), conditions (15) and (16) are respectively direct extensions of conditions (12) and (13) of \([25]\), and condition (14) of \([23]\) is a special case of condition (22), and conditions (18), (19), (20) and (21) need to be added to ensure isospectral diagonalization for singular cubic polynomial matrix. Therefore, the diagonalization processes of cubic polynomial matrix need to consider more cases, not only a simple extension of results of the quadratic polynomial matrix.

Theorem 4.1 and Theorem 4.2 gave conditions of isospectral diagonalization for cubic polynomial matrix. The following result is an important application for isospectral diagonalization, and can be useful in investigating polynomial matrix systems for control.

**Theorem 4.3.** Let \( sE - A \) be a companion linearization of cubic polynomial matrix \( A(s) \in \mathbb{C}^{n \times n} \), and \( sE - \hat{A} \) be a companion linearization of cubic polynomial matrix \( \hat{A}(s) \in \mathbb{C}^{n \times n} \), then \( A(s) \) and \( \hat{A}(s) \) are isospectral if and only if \( sE - A \) and \( sE - \hat{A} \) are strictly equivalent, that is to say, there exist nonsingular matrices \( U, V \in \mathbb{C}^{n \times n} \) such that

\[
U(sE - A)V = sE - \hat{A}.
\]

*Proof.*** Sufficiency. If \( sE - A \) and \( sE - \hat{A} \) are strictly equivalent, then they have the same Jordan canonical form. Since companion linearization is eigenstructure preserving transformations, \( A(s) \) and \( \hat{A}(s) \) have same Jordan structure. Thus, they are isospectral.

Necessity. \( A(s) \) and \( \hat{A}(s) \) are isospectral. Thus, their linearizations \( sE - A \) and \( sE - \hat{A} \) are isospectral. It follows from the Kronecker reduction of regular pencils \([9]\) that the pencils are strictly equivalent to the same canonical form. So, \( sE - A \) and \( sE - \hat{A} \) have the same Jordan canonical form. Hence, \( sE - A \) and \( sE - \hat{A} \) are strictly equivalent. \( \square \)

5. Solving processes for decoupled systems. Theorem 4.1 and Theorem 4.2 also provide an approach to solve an isospectral diagonal polynomial matrix under the conditions of diagonalization. We will present the solving process in this section.

**Polynomial matrix with nonsingular \( A_3 \)**

Step 1. Find all the eigenvalues \( s_i \) for \( i = 1 : t \) and compute corresponding partial multiplicity \( m_{ij} \), geometric multiplicity \( \mu_i \) and algebraic multiplicity \( \bar{\mu}_i \), by solving Smith canonical form of \( A(s) \), go to next step.

Step 2. If \( 1 \leq m_{ij} \leq 3 \) for \( i = 1 : t, j = 1 : \mu_i \), denote eigenvalue \( s_i \) with \( m_{ij} = 3 \) by \( s_{3,ij} \). Let \( \tau_i \) be the total number of eigenvalue \( s_i \) with \( m_{ij} = 3 \), and \( p = \sum_{i=1}^t \tau_i \). Let

\[
\hat{A}_1(s) = \text{diag}\{(s - s_{3,11})^3, \cdots, (s - s_{3,1\tau_1})^3, \cdots, (s - s_{3,t1})^3, \cdots, (s - s_{3,t\tau_t})^3\}
\]

go to next step, else if there exist \( i, j \) such that \( m_{ij} > 3 \), go to step 6.
Step 3. Compute the number of partial multiplicities with \( m_{ij} = 2 \), denote by \( \rho_i \), and denote the corresponding eigenvalues by \( s_{2,ij} \). Then the number of partial multiplicities with \( m_{ij} = 1 \) is \( \mu_i - \tau_i - \rho_i \), denote the corresponding eigenvalues by \( s_{1,ij} \). If \( \rho_i \leq n - p \) and \( \mu_i - \tau_i - \rho_i \leq n - p \), go to next step, else, go to step 6.

Step 4. If \( \rho_i \leq \sum_{j \neq i}(\mu_j - \tau_j - \rho_j) \) for \( i = 1 : t \), and (13)-(14) have solutions. Let 
\[
\hat{A}_2(s) = \text{diag}(s - s_{2,11})^2(s - s_{1,11}), \ldots, (s - s_{2,t\rho_1})^2(s - s_{1,\rho_1}), \ldots,
\]
\[
(s - s_{2,11})^2(s - s_{1,11}), \ldots, (s - s_{2,t\rho_1})^2(s - s_{1,\rho_1}), s_{2,ij} \neq s_{1,ij}, i = 1 : t
\]
and go to next step, else if there exists \( i \) such that \( \rho_i > \sum_{j \neq i}(\mu_j - \tau_j - \rho_j) \), or (13)-(14) have no solutions, go to step 6.

Step 5. The remaining elementary divisors are linear. Let 
\[
\hat{A}_3(s) = \bigoplus_{i \neq j \neq k} (s - s_i)(s - s_j)(s - s_k)
\]
where \( \bigoplus \) represents a direct diagonal sum. Let 
\[
\hat{A}(s) = \text{diag}(\hat{A}_1(s), \hat{A}_2(s), \hat{A}_3(s))
\]
Thus, \( \hat{A}(s) \) is a diagonal cubic polynomial matrix. \( \hat{A}(s) \) and \( A(s) \) are isospectral.

Step 6. There doesn’t exist a diagonal cubic polynomial matrix isospectral to \( A(s) \).
The following examples can illustrate these different cases.

**Example 1** Consider cubic polynomial matrix 
\[
A(s) = \begin{bmatrix}
    s^3 - 1 & s & 0 \\
    1 & (s - 1)^3 & 1 \\
    1 - s & -s & s^3 - s
\end{bmatrix}
\]
The elementary divisors are \( s - 1, s - 1, (s - 1)^3, s, s^2 + s + 1 \). Let \( s_1 = 1, s_2 = 0, s_3 = -1, s^2 + s + 1 = (s - s_4)(s - s_5) \), then \( \tau_1 = 1, \tau_2 = 0 \) for \( i = 2 : 5 \) and \( p = 1 \). \( \rho_1 = 0 \) for \( i = 1 : 5 \). The conditions (8), (9), (11) and (12) hold. Furthermore, 
\[
2 - x_1 \leq 2.0 \leq x_1 \leq \min\{0, 2\}
\]
\[
1 - x_2 \leq 2.0 \leq x_2 \leq \min\{0, 1\}
\]
\[
1 - x_3 \leq 2.0 \leq x_3 \leq \min\{0, 1\}
\]
\[
1 - x_4 \leq 2.0 \leq x_4 \leq \min\{0, 1\}
\]
\[
1 - x_5 \leq 2.0 \leq x_5 \leq \min\{0, 1\}
\]
\[
x_1 + x_2 + x_3 + x_4 + x_5 = 0
\]
So, \( x_1 = x_2 = x_3 = x_4 = x_5 = 0 \) are the solutions of (13)-(14). \( A(s) \) admits an isospectral diagonal polynomial matrix by Theorem 4.1. By the solving processes, 
\[
\begin{bmatrix}
    (s - 1)^3 \\
    (s - 1)(s^2 + s + 1) \\
    s(s - 1)(s + 1)
\end{bmatrix}
\]
is a diagonal matrix isospectral to given matrix.

**Example 2** Consider a regular cubic polynomial matrix 
\[
\begin{bmatrix}
    (s - 1)^2 & 0 & s(s - 1)^2 \\
    0 & (s - 1)^2 & 0 \\
    (s - 1)^2(1 - 2s) & 0 & s(s - 1)^2
\end{bmatrix}
\]
with nonsingular highest-degree coefficient matrix, and the elementary divisors are \( s + 1, s^2, (s - 1)^2, (s - 1)^2, (s - 1)^2 \). We can’t find a diagonal cubic polynomial matrix that has same Jordan form with given polynomial matrix. That’s because the multiplicities of eigenvalues don’t satisfy condition (11).

**Example 3** Let \( s_1, s_2 \) and \( s_3 \) be arbitrary complex numbers and consider the following family of polynomials: 
\[
(s - s_1)^2, (s - s_1), (s - s_1), (s - s_2)^2, (s - s_2), (s - s_3), (s - s_3)
\]
Put 
\[
\gamma_1 = (s - s_1)(s - s_3), \gamma_2 = (s - s_1)(s - s_3), \gamma_3 = (s - s_1)^2(s - s_2)^2(s - s_3).
\]
Then \( \gamma_1 | \gamma_2 | \gamma_3 \) and \( \deg(\gamma_1) + \deg(\gamma_2) + \deg(\gamma_3) = 9 \). By Lemma 3.2 of [14] there is a triangular \( 3 \times 3 \) cubic polynomial matrix, \( A(s) \), with \( \gamma_1 | \gamma_2 | \gamma_3 \) as invariant polynomials. Thus the elementary
divisors of $A(s)$ are the polynomials in (24). However there is no diagonal $3 \times 3$ cubic polynomial matrix whose elementary divisors are the polynomials in (24) because the only possible distributions of those polynomials in products of degree 3 are:

\[
\begin{align*}
(1) & \quad (s - s_1)^2(s - s_1), (s - s_2)^2(s - s_1), (s - s_3)(s - s_3) \\
(2) & \quad (s - s_1)^2(s - s_3), (s - s_2)^2(s - s_3), (s - s_1)(s - s_3) \\
(3) & \quad (s - s_1)^2(s - s_3), (s - s_2)^2(s - s_3), (s - s_1)(s - s_3) \\
(4) & \quad (s - s_1)^2(s - s_3), (s - s_2)^2(s - s_3), (s - s_1)(s - s_3)
\end{align*}
\]

No diagonal matrix with these diagonal elements has the polynomials of (24) as elementary divisors. In other words there is no diagonal cubic polynomial matrix isospectral to $A(s)$. We will explain it by the conditions of Theorem 4.1. Obviously, $\mu_1 - \tau_1 - \mu_1 = 2, \mu_2 - \tau_2 - \mu_2 = 0, \mu_3 - \tau_3 - \rho_3 = 3, n - p - \sum_{i=1}^3 \rho_i = 1$. Thus, $x_2 = 0$ and

\[
\begin{align*}
2 - x_1 & \leq 1, 0 \leq x_1 \leq 1 \\
3 - x_3 & \leq 1, 0 \leq x_3 \leq 2 \\
x_1 + x_2 + x_3 & = 2
\end{align*}
\]

By (25) and (26), $x_1 = 1, x_3 = 2$, which are contradictory with (27).

**Polynomial matrix with singular $A_3$**

Step 1. Find all the eigenvalues $s_i$, and compute corresponding partial multiplicity $m_{ij}$, geometric multiplicity $\mu_i$, and algebraic multiplicity $\bar{\mu}_i$, $i = 1, \cdots, t, \infty$, by solving Smith canonical form of $A(s)$. Then go to next step.

Step 2. If $1 \leq m_{ij} \leq 3, i = 1; j = 1: \mu_i$, denote eigenvalue $s_i$ with $m_{ij} = 3$ by $s_{3,ij}$, and denote the number by $\tau_i$. Let $p = \sum_{i=1}^t \tau_i$ and

\[
\hat{A}_1(s) = \text{diag}\{(s - s_{3,1})^3, \cdots, (s - s_{3,\tau_1})^3, \cdots, (s - s_{3,1})^3\}
\]

go to next step, else if there exist $i, j$ such that $m_{ij} > 3$, go to step 7.

Step 3. Let $\tau_\infty$ be the number of infinite eigenvalue with $m_{\infty,j} = 3$. If $\tau_\infty = 0$, skip this step, else if $\tau_\infty \neq 0$, let

\[
\hat{A}_2(s) = \text{diag}\{1, 1, \cdots, 1\}
\]

and go to next step, where the number of 1 is $\tau_\infty$.

Step 4. Compute the number of partial multiplicities with $m_{ij} = 2$, denote the number of finite and infinite eigenvalues with $m_{ij} = 2$ by $\mu_i, \rho_\infty$, respectively. Denote the corresponding finite eigenvalues with $m_{ij} = 2$ by $s_{2,ij}$. Then the number of finite eigenvalues with $m_{ij} = 1$ is $\mu_i - \tau_i - \rho_i$, and the number of infinite eigenvalue with $m_{\infty,j} = 1$ is $\mu_\infty - \tau_\infty - \rho_\infty$. If $\rho_\infty \leq n - p - \tau_\infty, \rho_i \leq n - p - \tau_\infty$ and $\mu_i - \tau_i - \rho_i \leq n - p - \tau_\infty$, let

\[
\hat{A}_3(s) = \text{diag}\{s - s_{1,\cdots, s - \rho_\infty}\}
\]

go to next step, else, go to step 7.

Step 5. If $\rho_i \leq \mu_\infty - \tau_\infty - \rho_\infty + \sum_{\substack{j \neq i, \rho_j}} (\mu_j - \tau_j - \rho_j)$, $\rho_\infty \leq \sum_{i} (\mu_i - \tau_i - \rho_i)$, and (21)-(22) have solutions. Let

\[
\hat{A}_4(s) = \text{diag}\{(s - s_{2,1})^2(s - s_{1,1}), \cdots, (s - s_{2,k\rho_k})^2(s - s_{1,k\rho_k}), \cdots, s_{2,ij} \neq s_{1,ij}\}
\]

\[
\hat{A}_5(s) = \text{diag}\{(s - s_{2,1})^2, (s - s_{2,1})^2, \cdots\}
\]

go to next step, else if there exists $i$ such that $\mu_i > \mu_\infty - \tau_\infty - \rho_\infty + \sum_{\substack{j \neq i, \rho_j}} (\mu_j - \tau_j - \rho_j)$, or $\rho_\infty > \sum_{i} (\mu_i - \tau_i - \rho_i)$, or (21)-(22) have no solutions, go to step 7.

Step 6. The remaining each three distinct linear divisors could be associated with one of entries, and two cases are included, i.e., whether or not infinite eigenvalue is contained in these entries. Let

\[
\hat{A}_6(s) = \bigoplus_{i \neq j \neq k} (s - s_i)(s - s_j)(s - s_k)
\]

\[
\hat{A}_7(s) = \bigoplus_{i \neq j} (s - s_i)(s - s_j)
\]

Let

\[
\hat{A}(s) = \text{diag}\{\hat{A}_1(s), \hat{A}_2(s), \hat{A}_3(s), \hat{A}_4(s), \hat{A}_5(s), \hat{A}_6(s), \hat{A}_7(s)\}
\]

Thus, $\hat{A}(s)$ is an isospectral diagonal polynomial matrix of $A(s)$.

Step 7. There doesn’t exist a diagonal cubic polynomial matrix isospectral to $A(s)$. 

Remark 3. Since infinite linear elementary divisors are considered together with finite linear elementary divisors, any combinations for some terms of $\hat{A}_4(s), \hat{A}_5(s), \hat{A}_6(s)$ and $\hat{A}_7(s)$ are possible under the conditions of Theorem 4.2.

The following examples present the results of Theorem 4.2 and above solving processes that could arise.

Example 4 For singular cubic polynomial matrix

$$A(s) = \begin{bmatrix} s + 1 & s^3 \\ s & 1 \end{bmatrix},$$

the finite elementary divisor is $s^2$. The reversal polynomial matrix is

$$A_*(s) = s^3A(1/s) = \begin{bmatrix} s^3 + s^2 & 1 \\ s^3 & s \end{bmatrix},$$

and the elementary divisor of $A_*(s)$ in origin is $s^4$. Therefore, $A(s)$ has an infinite eigenvalue with partial multiplicity 4. Thus, $A(s)$ doesn’t admit a diagonalization.

Example 5 Consider the following singular cubic polynomial matrix

$$A(s) = \begin{bmatrix} s^3 + 1 & s & 0 \\ 0 & (s - 1)^3 & s + 2 \\ 0 & 0 & s + 1 \end{bmatrix}.$$  

$A(s)$ has finite elementary divisors $(s - 1)^3, (s + 1)^2, (s^2 - s + 1)$, and one infinite eigenvalue with partial multiplicity 2. Let $s_1 = 1, s_2 = -1, (s^2 - s + 1) = (s - s_3)(s - s_4)$ and $s_\infty$ be infinite eigenvalue. Then $\tau_i = 1, \tau_\infty = 0, \tau_i = 0$ for $i = 2, 4$ and $p = 1, \rho_2 = \rho_\infty = 1, \rho_1 = \rho_3 = \rho_4 = 0$. Obviously, the conditions (15), (16), (18), (19) and (20) hold. $\mu_i - \tau_i - \rho_i = 0$ for $i = 1, 2, \infty$ imply $x_1 = x_2 = x_\infty = 0$. Since

$$1 - x_3 \leq 0, 0 \leq x_3 \leq 1$$

$$1 - x_4 \leq 0, 0 \leq x_4 \leq 1$$

$$\sum_{i=1}^4 x_i + x_\infty = 2,$$

$x_1 = x_2 = x_\infty = 0, x_3 = x_4 = 1$ are the solutions of (21)-(22). $A(s)$ can be diagonalizable and diagonal polynomial matrix can be written as

$$\begin{bmatrix} (s - 1)^3 & 0 & 0 \\ 0 & (s + 1)^2(s + \frac{1+\sqrt{3}i}{2}) & 0 \\ 0 & 0 & s + \frac{1+i\sqrt{3}}{2} \end{bmatrix}.$$  

It is seen that lower-right entry contains an infinite eigenvalue with partial multiplicity 2.

6. Conclusions. In this paper, we investigated decoupling problems for cubic polynomial matrix systems. A direct decoupling approach under strict equivalence transformation was considered, and decoupling conditions were derived. We also presented a more general decoupling approach, named isospectral decoupling, which preserves the spectrum structure and degree of systems. Moreover, the isospectral decoupling conditions were derived by cubic polynomial matrix linearization, and the solving processes in two cases were given for obtaining isospectral diagonal polynomial matrix systems. The results show that not all cubic polynomial matrix systems can be decoupled in isospectral meaning. This study derived only diagonalization conditions on complex field. On the basic of the results, diagonalization problems for real field case may be considered.

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