METHODS OF ENUMERATING TWO VERTEX MAPS OF ARBITRARY GENUS

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Abstract. This paper provides an alternate proof to parts of the Goulden-Slofstra formula \cite{goulden2016} for enumerating two vertex maps by genus, which is an extension of the famous Harer-Zagier formula \cite{harer1999} that computes the Euler characteristic of the moduli space of curves. This paper also shows a further simplification to the Goulden-Slofstra formula. Portions of this alternate proof will be used in a subsequent paper \cite{csh2017}, where it forms the basis for a more general result that applies for a certain class of maps with an arbitrary number of vertices.

1. Introduction

Let \( S \) be a set of even cardinality. A pairing \( \mu \) of \( S \) is a partition of \( S \) into disjoint subsets of size 2. In the context of permutations, \( \mu \) can be seen as a fixed-point free involution, where every cycle of \( \mu \) is of size 2. Now, let \( p = 2q \) be a positive integer. We use \( [p] \) to denote the set \( \{1, \ldots, p\} \), and \( \mathcal{P}_p \) to be the set of all pairings of \( [p] \). If \( \gamma_p \) is the canonical cycle permutation of \( [p] \), given by \( \gamma_p = (1, \ldots, p) \), we have the following theorem by Harer-Zagier on the Euler characteristic of the moduli space of curves.

Theorem 1. (Harer-Zagier \cite{harer1999}) Let \( q \) be a positive integer, and \( A^{(q)}_L \) be the subset of pairings of \( \mathcal{P}_{2q} \) such that for \( \mu \in A^{(q)}_L \), \( \mu^{-1} \gamma_q \) has exactly \( L \) cycles. If we let \( a^{(q)}_L = \left| A^{(q)}_L \right| \), then the generating series for \( a^{(q)}_L \) is given by

\[
A^{(q)}(x) = (2q - 1)!! \sum_{k \geq 1} 2^{k-1} \binom{q}{k-1} \binom{x}{k}
\]

where \((2k - 1)!! = \prod_{j=1}^{k} (2j - 1)\) is the double factorial.

There are numerous proofs of this formula in the literature, both algebraic and combinatorial. A selection of the proofs can be found in the papers by Goulden and Nica \cite{goulden2016}, Itzykson and Zuber \cite{itzykson1980}, Jackson \cite{jackson2003}, Kerov \cite{kerov1998}, Kontsevich \cite{kontsevich1997}, Lass \cite{lass2003}, Penner \cite{penner1987}, and Zagier \cite{zagier1985}. As seen in Lando and Zvonkin \cite{lando2004}, the Harer-Zagier formula enumerates 1-celled embeddings on an orientable surface by genus, which are equivalent to one vertex maps with \( q \) loop edges. The original proof of Harer-Zagier uses matrix integration, and there are numerous other algebraic proofs for this same result. Some subsequent proofs used purely combinatorial approaches, such as the use of Eulerian tours by Lass, and the use of trees by Goulden and Nica.

Next, we will set up the terminology for the Goulden and Slofstra result, which is an extension of the Harer-Zagier formula. Let \( p, n \geq 1 \). We use \( [p]^k \) to denote the set \( \{1^k, 2^k, \ldots, p^k\} \), whose elements \( i^k \), \( i = 1, \ldots, p \), are regarded as a labelled version of the integer \( i \), labelled by the “\( n \)” in the superscript position. Then, suppose \( p_1 \) and \( p_2 \) are positive integers, we let \([p_1, p_2]\) to be the set \([p_1]^k \cup [p_2]^k\). For example, \([3, 5]\) is the set \( \{1^2, 2^2, 3^2, 1^3, 2^3, 3^3, 4^2, 5^2\} \). Furthermore, if \( p_1 + p_2 \) is even, then the set of all pairings of \([p_1, p_2]\) is denoted as \( \mathcal{P}_{p_1, p_2} \). Now, if \( \mu \) is a pairing of \([p_1, p_2]\), then a pair \( \{x^2, y^2\} \) in \( \mu \) is a mixed pair if \( i \neq k \), and a non-mixed pair otherwise. To describe the number of mixed and non-mixed pairs in a pairing \( \mu \), we introduce the parameters \( q_1, q_2, \) and \( s \). Let \( q_1, q_2 \geq 0 \) and \( s > 0 \) such that \( p_i = 2q_i + s \) for \( i = 1, 2 \). We define \( \mathcal{P}^{(q_1, q_2; s)} \subseteq \mathcal{P}_{p_1, p_2} \) to be the subset of the pairing such that for \( \mu \in \mathcal{P}^{(q_1, q_2; s)} \), \( \mu \) has \( q_i \) non-mixed pairs of the form \( \{x^2, y^2\} \) and \( s \) mixed pairs. If \( \gamma_{p_1, p_2} \) is the canonical cycle permutation of \([p_1, p_2]\), given by \( \gamma_{p_1, p_2} = \left(1^2, \ldots, p_1^2\right) \left(1^2, \ldots, p_2^2\right) \), then the series that enumerates the number of two vertex maps according to the genus is given as follows.

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Theorem 2. (Goulden-Slofstra [5]) Let \( q_1 \) and \( q_2 \) be non-negative integers, and \( s \) be a positive integer. Let \( A_{L}^{(q_1,q_2;s)} \) be the subset of pairings of \( \mathcal{P}^{(q_1,q_2;s)} \) such that for \( \mu \in A_{L}^{(q_1,q_2;s)} \), \( \mu^{-1}_{p_1,p_2} \) has exactly \( L \) cycles. If we let \( a_{L}^{(q_1,q_2;s)} = |A_{L}^{(q_1,q_2;s)}| \), then the generating series for \( a_{L}^{(q_1,q_2;s)} \) is given by

\[
A^{(q_1,q_2;s)}(x) = p_1!p_2! \sum_{d=1}^{d-1} \sum_{i=0}^{d-1} \sum_{j=0}^{d-1} \frac{1}{2^{i+j}d!} \binom{d-i-j}{k-1} \Delta_k^{(q_1,q_2,s)}
\]

where \( p_1 = 2q_1 + s \), \( p_2 = 2q_2 + s \), \( d = q_1 + q_2 + s \), and

\[
\Delta_k^{(q_1,q_2,s)} = \frac{k-1}{q_1-i} \binom{k-1}{q_2-j} - \frac{k-1}{q_1+s-i} \binom{k-1}{q_2+s-j}
\]

In this expression, \( p_1 \) and \( p_2 \) are the degrees of vertices 1 and 2, respectively, and \( d \) is the total number of pairs in the pairing. Similar to the Harer-Zagier formula, the Goulden-Slofstra formula counts the number of combinatorial maps with 2 vertices by genus, where there are \( q_1 \) and \( q_2 \) loop edges on vertices 1 and 2, and \( s \) edges between the two vertices. To represent these maps, Goulden and Slofstra used a combinatorial object called the paired surjections, which we will define in the next section.

2. Definitions and Terminology of Paired Arrays

In this section, we will mostly follow the methodology of Goulden and Slofstra [5]. For that reason, we will not be providing proofs for the results stated, and skip over some of their constructions. However, we will be defining some terminology of our own, so that we can extend their approach later. Note that our notation in this paper is generally different from that of Goulden and Slofstra, as it makes it easier to refer to the results in the follow up paper [3] that covers multiple vertices.

Definition 3. Let \( K, s \geq 1 \), \( q_1, q_2 \geq 0 \), and \( p_i = 2q_i + s \) for \( i = 1, 2 \). An ordered pair \((\mu, \pi)\) is a paired surjection if \( \mu \in \mathcal{P}^{(q_1,q_2;s)} \) and \( \pi: [p_1,p_2] \to [K] \) is a surjection satisfying

\[
\pi(\mu(v)) = \pi(\gamma_{p_1,p_2}(v)) \quad \text{for all } v \in [p_1,p_2]
\]

We denote the set of paired surjection satisfying the parameters \( K, q_1, q_2, \) and \( s \) as \( \mathcal{F}^{K,(q_1,q_2,s)} \), and we let \( f_{K}^{(q_1,q_2,s)} = |\mathcal{F}^{K,(q_1,q_2,s)}| \).

We can then express the generating series \( A^{(q_1,q_2,s)}(x) \) using paired surjections as follows.

Proposition 4. (Goulden-Slofstra [5]) For \( q_1, q_2 \geq 0 \) and \( s \geq 1 \), we have

\[
A^{(q_1,q_2,s)}(x) = \sum_{K \geq 1} f_{K}^{(q_1,q_2,s)} \binom{x}{K}
\]

Now, paired surjections can be represented graphically with a combinatorial object called the labelled array. This is an \( 2 \times K \) array of cells arranged in a grid. Each element \( x^k \) of \( \mu \) is represented as a vertex, where the vertex labelled \( x^k \) is placed into cell \((i,j)\) if \( \pi(x^k) = j \). The vertices are arranged horizontally within a cell, in increasing order of the labels. Furthermore, for each pair \( \{x^k, y^l\} \) in \( \mu \), an edge is drawn between their corresponding vertices.

For example, let \( (\mu, \pi) \in \mathcal{F}^{(1,1,4)} \), with \( \mu \) and \( \pi \) given by

\[
\mu = \{1^1, 4^1, 2^1, 3^1, 5^1, 7^1, 8^1, 10^1, 2^2, 6^2\}
\]

\[
\pi^{-1}(1) = \{2^4, 4^4\} \quad \pi^{-1}(2) = \{3^4, 5^4, 7^4, 9^4\} \quad \pi^{-1}(3) = \{1^5, 9^5, 10^5, 5^2\}
\]

Then, the labelled array representing \( (\mu, \pi) \) is given by Figure 1.

Note that a \( 2 \times K \) array with paired and labelled vertices as described above uniquely represents a pairing \( \mu \in \mathcal{P}^{(q_1,q_2;s)} \) and a function \( \pi: [p_1, \ldots, p_n] \to [K] \). Furthermore, we can strip the labels and define paired
arrays as abstract combinatorial objects, with conditions that allow for a bijection between paired arrays and labelled arrays.

**Definition 5.** Let \( K, s \geq 1, q_1, q_2 \geq 0 \), and \( 1 \leq R_1, R_2 \leq K \). We define \( \mathcal{PA}^{(q_1,q_2;\mathbf{s})}_{K;R_1,R_2} \) to be the set of paired arrays, which are arrays of cells and vertices subject to the following conditions.

- A paired array is an \( 2 \times K \) array of cells, such that each cell \((i,j)\) contains an ordered list of vertices, arranged left to right, so that row \( i \) contains \( p_i := 2q_i + s \) vertices for \( i = 1, 2 \).
- Each vertex \( u \) is paired with exactly one other vertex \( v \). Exactly \( 2q_i \) vertices of row \( i \) are paired with other vertices of row \( i \), and exactly \( s \) vertices of row \( i \) are paired with vertices of the other row. Graphically, the pairings are denoted as edges between vertices.
- Each row \( i \) has exactly \( R_i \) marked cells, which are denoted by marking the cell with a box in its upper or lower right corner.
- A pair of vertices \( \{u,v\} \) is a mixed pair if \( u \) and \( v \) belong to different rows. The vertices \( u \) and \( v \) are called mixed vertices.

Generally, we use \( \alpha \in \mathcal{PA}^{(q_1,q_2;\mathbf{s})}_{K;R_1,R_2} \) to denote a paired array. Before introducing the conditions used in Goulden and Slofstra, we will first introduce a number of useful notations and conventions.

**Convention 6.** For notational convenience, we introduce the following:

- We use calligraphic letters to denote columns or sets of columns. For generic columns or sets of columns, we use the letters \( \mathcal{X}, \mathcal{Y}, \) and \( \mathcal{Z} \).
- For each calligraphic letter, we use the corresponding upper case letter to denote the number of columns in the set. For example, \( X = |\mathcal{X}| \).
- For each calligraphic letter, we use the corresponding lower case letter, subscripted by the row number, to denote the total number of vertices in those columns for a given row. For example, \( x_i \) is the total number of vertices in row \( i \) of the columns of \( \mathcal{X} \).
- We generally use \( i, j, k, \ell \) as index variables, with \( i \) and \( k \) for rows, and \( j \) and \( \ell \) for columns. Furthermore, we use cell \((i,j)\) to denote the cell in row \( i \), column \( j \) of the array.
- We use \( K \) to denote the set of all columns, and \( K \) to denote the total number of columns.
- We use \( \mathcal{R}_i \) to denote the set of columns that are marked in row \( i \), and \( R_i \) to denote the number of columns that are marked in row \( i \).
- We use \( \mathcal{F}_i \) to denote the set of columns that have at least one vertex in row \( i \), and \( F_i \) to denote the number of columns that are marked in row \( i \).
- We use \( w_{i,j} \) to denote the number of vertices in cell \((i,j)\), and \( \mathbf{w} \) to denote a matrix of \( w_{i,j} \) describing the number of vertices in each cell of row \( i \).

With these conventions, we are ready to define the three conditions that allow us to create a bijection between labelled arrays and paired arrays.

**Definition 7.** Let \( \alpha \in \mathcal{PA}^{(q_1,q_2;\mathbf{s})}_{K;R_1,R_2} \) be a paired array.

- \( \alpha \) is said to satisfy the non-empty condition if each column \( j \) contains at least one object.
Corollary 10. (Goulden-Slofstra [5]) Let \( n, K, s \geq 1 \) and \( q_1, q_2 \geq 0 \). We have

\[
A^{(q_1, q_2; s)}(x) = \sum_{K \geq 1} \binom{x}{K} \frac{p_1!p_2!}{2^{t_1+t_2}t_1!t_2!(s_1+q_1-t_1)!(s_2+q_2-t_2)!} \cdot v_{K; q_1-t_1+1, q_2-t_2+1}^{(s)}
\]

Finally, by decomposing the canonical array in Figure 2, we can obtain the vertical array in Figure 3. Then, by combining the theorems we have so far, we can write the generating series in terms of the number of vertical arrays.

A paired array is proper if it satisfies the non-empty, balance, and forest conditions. We denote the set of proper vertical arrays as \( \mathcal{PVA}_{K; R_1, R_2} \) and the set of proper vertical arrays as \( \mathcal{PVA}_{K; R_1, R_2} \). Finally, we let \( \psi_{1, q_1, q_2; s}^{(K; R_1, R_2)} \).

Note that we will generally not work directly with paired arrays that do not satisfy the forest condition. However, as vertical arrays not satisfying the forest condition are vital for extending paired arrays, we have separated the forest condition from the definition of vertical arrays itself. Next, we give a formula for relating the number of canonical arrays to the number of vertical arrays.

Theorem 8. (Goulden-Slofstra [5]) For \( K, s \geq 1 \) and \( q_1, q_2 \geq 0 \), we have \( \psi_{1, q_1, q_2; s}^{(K; R_1, R_2)} = c_{K}^{(q_1, q_2; s)} \).

To obtain a canonical array from a labelled array, we simply marked the cells that contain 1 in both rows, then delete the labels. Applying this to the labelled array in Figure 1 gives us the canonical array in Figure 2.

Here, the problem of enumerating maps on surfaces reduces to that of enumerating canonical arrays. To solve the latter problem, we will first decompose canonical arrays by removing all non-mixed pairs using the following theorem.

Theorem 9. (Goulden-Slofstra [5]) Let \( n, K, s \geq 1 \) and \( q_1, q_2 \geq 0 \). We have

\[
\psi_{1, q_1, q_2; s}^{(K; R_1, R_2)} = \sum_{t_1, t_2 \geq 0} \prod_{i=1}^{n} 2^{t_1+t_2}t_1!t_2!(s_1+q_1-t_1)!(s_2+q_2-t_2)! \cdot v_{K; q_1-t_1+1, q_2-t_2+1}^{(s)}
\]

For example, by decomposing the canonical array in Figure 2, we can obtain the vertical array in Figure 3.
Remark 11. While Theorem 9 is proved in Goulden and Slofstra using the forest completion algorithm, we can in fact use the techniques developed in this paper to bypass this requirement if we so desire. This alternate approach can be found in [2].

3. Definitions and Terminology of Arrowed Arrays

In this section, we will extend paired arrays by the addition of arrows, which represent hypothetical vertices used in the forest condition. This will allow us to decouple the forest condition with the vertex pairings, which allows for the deletion of vertices and pairings from paired arrays.

Definition 12. Let $K \geq 1$, $s \geq 0$, and $1 \leq R_1, R_2 \leq K$. An arrowed array is a pair $(\alpha, \phi)$, where $\alpha \in \mathcal{V}_A^{(s)}_{K,R_1,R_2}$ is a two-row vertical array, and $\phi : \mathcal{K} \setminus R_1 \rightarrow \mathcal{K}$ is a partial function from $\mathcal{H} \subseteq \mathcal{K} \setminus R_1$ to $\mathcal{K}$, with $R_1$ being the set of marked columns in row 1 of $\alpha$. Graphically, $\phi$ is denoted by arrows drawn above row 1, where an arrow from $j$ to $j'$ is drawn if $j \in \mathcal{H}$ and $\phi(j) = j'$. For convenience, the two ends of the arrow belonging to columns $j$ and $j'$ are called the arrow-tail and arrow-head respectively, and column $j$ is said to point to column $j'$. Furthermore, both the arrow-tail and arrow-head belong to row 1 of their respective columns.

With the generalization of paired arrays to arrowed arrays, there are corresponding generalizations of the terms and conventions used to describe paired arrays. These generalizations will be compatible with the conventions for paired arrays if the partial function $\phi$ is empty.

- An object of $(\alpha, \phi)$ refers to either a vertex, a box, or an arrow-tail. If a cell both contains vertices and a box, or vertices and an arrow-tail, either the box or the arrow-tail is to be taken as the rightmost object of the cell.
- A vertex $v$ of an arrowed array is critical if it is the rightmost vertex of a cell, and the cell it belongs to is neither marked nor contains an arrow-tail. A pair $\{u, v\}$ that contains a critical vertex is a critical pair.
- $(\alpha, \phi)$ is said to satisfy the non-empty condition if for each column $j$, there exists at least one cell that contains an object.
- $(\alpha, \phi)$ is said to satisfy the balance condition if for each column $j$, the number of vertices in cell $(1, j)$ is equal to the number of vertices in cell $(2, j)$.
- Let $\mathcal{F}_1$ be the set of columns in row $i$ that contain at least one vertex. The forest condition function $\psi_1 : (\mathcal{H} \cup \mathcal{F}_1) \setminus R_1 \rightarrow \mathcal{K}$ for row 1 is defined as follows: For each column $j \in \mathcal{F}_1 \setminus (\mathcal{H} \cup R_1)$, if the rightmost vertex $v$ is paired with a vertex $u$ in column $j'$, let $\psi_1(j) = j'$. The forest condition function $\psi_2$ for row 2 is defined to be the same as the one for paired arrays in Definition 7. $(\alpha, \phi)$ is said to satisfy the forest condition if the functional digraph of $\psi_1$ on the vertex set $\mathcal{H} \cup \mathcal{F}_1 \cup \psi_1(\mathcal{H} \cup \mathcal{F}_1) \cup R_1$ is a forest with root vertices $R_1$, and the functional digraph of $\psi_2$ on the vertex set $\mathcal{F}_2 \cup \psi_2(\mathcal{F}_2) \cup R_2$ is a forest with root vertices $R_2$. That is, for each column $j \in (\mathcal{H} \cup \mathcal{F}_1) \setminus R_1$, there exists some positive integer $t$ such that $\psi_1^t(j) \in R_1$, and for each column $j \in \mathcal{F}_2 \setminus R_2$, there exists some positive integer $t$ such that $\psi_2^t(j) \in R_2$. 

![Figure 3. Proper vertical array from the decomposition of Figure 2](image)

*Remark 11.* While Theorem 9 is proved in Goulden and Slofstra using the forest completion algorithm, we can in fact use the techniques developed in this paper to bypass this requirement if we so desire. This alternate approach can be found in [2].
Additionally, \( (\alpha, \phi) \) is said to satisfy the full condition if every cell contains at least one object.

The set of arrowed arrays that satisfy the forest condition is denoted \( \mathcal{AR}^{(s)}_{K; R_1, R_2} \).

Notice in particular that a cell cannot contain both an arrow-tail and be marked at the same time. Furthermore, a vertex is critical if and only if it contributes to the forest condition function. Unless otherwise stated, we will continue to use the conventions for paired arrays defined in Convention 6 for arrowed arrays.

As with paired arrays, we will always include the columns \( R_i \) in the vertex set for the functional digraph of \( \psi_i \), regardless of whether they are in the range of \( \psi_i \). Note that permuting the columns of an arrowed array does not change whether the array satisfies the balance or forest conditions, as all this action does is to relabel the vertices of the functional digraph. An example of an arrowed array that satisfies the forest condition can be found in Figure 4.

While the parameters used for defining the set of arrowed arrays is natural with respect to paired arrays, it does not easily lend itself to a formula. To make it manageable for summation, we need to partition the set of arrowed arrays by adding further constraints.

**Definition 13.** Let \( K \geq 1 \), \( s \geq 0 \), and \( 1 \leq R_1, R_2 \leq K \). A substructure \( \Theta \) of \( \mathcal{AR}^{(s)}_{K; R_1, R_2} \) is a set of constraints that defines a subset of \( \mathcal{AR}^{(s)}_{K; R_1, R_2} \). For convenience, an arrowed array \( (\alpha, \phi) \) is said to satisfy \( \Theta \) if \( (\alpha, \phi) \) satisfies the constraints given by \( \Theta \). In particular, let \( w \) be a non-negative matrix of size \( 2 \times K \), \( R_1, R_2 \) be \( R_1 \) and \( R_2 \) subsets of \( K \), and \( \phi \) be a partial function from \( H \subseteq K \setminus R_1 \) to \( K \). The substructure \( \Gamma = (w, R_1, R_2, \phi) \) is defined to be the subset of \( \mathcal{AR}^{(s)}_{K; R_1, R_2} \), such that for each pair \((\alpha', \phi') \in \mathcal{AR}^{(s)}_{K; R_1, R_2} \), the marked cells in row 1 and 2 of \( \alpha' \) are \( R_1 \) and \( R_2 \) respectively, \( \alpha' \) contains \( w_{i,j} \) vertices in cell \((i, j)\), and \( \phi' = \phi \).

Note that knowing \( w, R_1, R_2 \) and \( \phi \) is enough to determine whether an arrowed array satisfies the balance, non-empty, or full conditions. It is also sufficient to determine whether a vertex is critical, regardless of the actual pairing of the vertices. Therefore, we can use these terms, and terms such as arrow-head, arrow-tail, and points to with respect to \( \Gamma \).

Next, we will lay the groundwork for the enumeration of arrowed arrays satisfying a given substructure \( \Gamma \). This involves introducing several lemmas that limit the number of possibilities we have to consider, as well as lemmas that allow us to remove pairings from arrowed arrays. This allows us to categorize \( \Gamma \) based on a number of parameters that serve as invariants for the number of arrowed arrays that satisfy \( \Gamma \).

**Lemma 14.** Let \( \Gamma = (w, R_1, R_2, \phi) \) be a substructure of \( \mathcal{AR}^{(s)}_{K; R_1, R_2} \), and suppose that \( \phi \) contains a column \( X \) that points to a column \( Y \), with cell \((1, Y)\) marked. Let \( \Gamma' = (w, R_1 \cup \{X\}, R_2, \phi') \) be a substructure of
By applying the arrow simplification procedure to the left figure, we arrive at the right figure. R1 and R2 can be arbitrary in whether they are marked, but they must be the same between the two figures.

**Figure 5. Arrow Simplification 1**

\[
\mathcal{AR}_{K; R_1+1, R_2}, \text{ such that }
\]

\[
\phi'(j) = \begin{cases} 
\text{undefined} & j = X \\
\phi(j) & j \in \mathcal{H}\setminus\mathcal{X} 
\end{cases}
\]

that is, instead of pointing to \(\mathcal{Y}\), we mark cell \((1, \mathcal{X})\) of \(\Gamma'\). Then, the number of arrowed arrays satisfying \(\Gamma\) and the number of arrowed arrays satisfying \(\Gamma'\) are equal. Furthermore, \(\Gamma\) satisfies the balance, non-empty, and full conditions if and only if \(\Gamma'\) satisfies them, respectively.

Proof. Let \(\alpha \in \mathcal{VA}_{K; R_1, R_2}^{(s)}\) be a two-row vertical array, and \(\alpha'\) be a vertical array otherwise identical to \(\alpha\), but with cell \((1, \mathcal{X})\) marked. As we have not changed the vertex pairings, \(\psi_2\) remains unchanged between \((\alpha, \phi)\) and \((\alpha', \phi')\). The only change to the functional digraph of \(\psi_1\) is that \(\mathcal{X}\) is also a root vertex, instead of simply pointing to one. Therefore, \((\alpha, \phi)\) satisfies the forest condition if and only if \((\alpha', \phi')\) does, so the number of arrowed arrays satisfying \(\Gamma\) and \(\Gamma'\) are equal. As we have not changed the number of objects in each cell, we see that \(\Gamma\) satisfies the balance, non-empty, and full conditions if and only if \(\Gamma'\) satisfies them, respectively. \(\square\)

**Lemma 15.** Let \(\Gamma = (w, \mathcal{R}_1, \mathcal{R}_2, \phi)\) be a substructure of \(\mathcal{AR}_{K; R_1, R_2}^{(s)}\), and suppose that \(\phi\) contains a column \(\mathcal{X}\) that points to a column \(\mathcal{Y}\), and the column \(\mathcal{Y}\) points to another column \(\mathcal{Z}\). Let \(\Gamma' = (w, \mathcal{R}_1, \mathcal{R}_2, \phi')\) be a substructure of \(\mathcal{AR}_{K; R_1, R_2}^{(s)}\) such that

\[
\phi'(j) = \begin{cases} 
\mathcal{Z} & j = X \\
\phi(j) & j \in \mathcal{H}\setminus\mathcal{X} 
\end{cases}
\]

that is, instead of pointing to \(\mathcal{Y}\), \(\mathcal{X}\) now points to \(\mathcal{Z}\) in \(\phi'\). Then, the number of arrowed arrays satisfying \(\Gamma\) and the number of arrowed arrays satisfying \(\Gamma'\) are equal. Furthermore, \(\Gamma\) satisfies the balance, non-empty, and full conditions if and only if \(\Gamma'\) satisfies them, respectively.

Proof. Let \(\alpha \in \mathcal{VA}_{K; R_1, R_2}^{(s)}\) be a two-row vertical array. Again, as we have not changed the vertex pairings, \(\psi_2\) remains unchanged between \((\alpha, \phi)\) and \((\alpha', \phi')\). The only change to the functional digraph of \(\psi_1\) is that \(\mathcal{X}\) now points to \(\mathcal{Z}\), instead of pointing to \(\mathcal{Y}\). This is the same as detaching the subtree rooted at \(\mathcal{X}\) from \(\mathcal{Y}\), and attaching it elsewhere on the same tree. Therefore, \((\alpha, \phi)\) satisfies the forest condition if and only if \((\alpha', \phi')\) does, so the number of arrowed arrays satisfying \(\Gamma\) and \(\Gamma'\) are equal. Again, as we have not changed the number of objects in each cell, we see that \(\Gamma\) satisfies the balance, non-empty, and full conditions if and only if \(\Gamma'\) satisfies them, respectively. \(\square\)
By applying the arrow simplification procedure to the top figure, we arrive at the bottom figure. R1, R2, R3, and R4 can be arbitrary in whether they are marked, but they must be the same between the two figures. The same holds for the optional arrow with \( Z \) as its tail.

**Figure 6. Arrow Simplification 2**

Collectively, Lemma 14 and Lemma 15 are the **arrow simplification lemmas**, and pictures describing the applications of these lemmas can be found in Figure 5 and Figure 6. Furthermore, applying these lemmas to the array in Figure 4 gives us Figure 7. Note that these lemmas can be applied repeatedly to simplify a substructure, until either all arrow-heads are in cells that are unmarked and have no arrow-tails, or an arrow-head is in the same cell as its own arrow-tail. We are only interested in the former, as the latter implies that there is a cycle in the functional digraph of \( \phi \), which violates the forest condition. This gives rise to the following definition.

**Definition 16.** A substructure \( \Gamma = (w, \mathcal{R}_1, \mathcal{R}_2, \phi) \) is **irreducible** if the functional digraph of \( \phi \) is acyclic, and \( \Gamma \) cannot be further simplified with the application of the arrow simplification lemmas. Any cell of an irreducible substructure containing an arrow-head must be unmarked in row 1, and cannot contain an arrow-tail. Furthermore, it follows from definition that if an irreducible substructure satisfies the full condition, then any cell containing an arrow-head must also contain a critical vertex in row 1.

If \( \Gamma = (w, \mathcal{R}_1, \mathcal{R}_2, \phi) \) is an irreducible substructure, then we can categorize the columns of \( \Gamma \) as follows: Let \( A, B, C, D \) be a partition of the columns of \( \mathcal{K} \setminus \mathcal{H} \), where

- Columns in \( A \) have both row 1 and row 2 unmarked
- Columns in \( B \) have row 1 marked and row 2 unmarked
- Columns in \( C \) have row 1 unmarked and row 2 marked
- Columns in \( D \) have both row 1 and row 2 marked
Figure 7. Simplification of the arrowed array in Figure 4 into an irreducible array

Figure 8. Column types and variables for the number of vertices

Furthermore, if $\mathcal{X}$ is a column or a set of columns, let $\overline{\mathcal{X}}$ and $\overline{\overline{\mathcal{X}}}$ be the sets of columns that have arrows pointing to $\mathcal{X}$, and that have row 2 unmarked and marked, respectively. In particular, $\overline{\mathcal{A}}$ and $\overline{\overline{\mathcal{A}}}$ denotes the sets of columns pointing to $\mathcal{A}$, and $\overline{\mathcal{C}}$ and $\overline{\overline{\mathcal{C}}}$ denotes the sets of columns pointing to $\mathcal{C}$, with row 2 unmarked and marked, respectively. These sets of columns implicitly defined by $\Gamma$ are referred to as column types, and a diagram with all the column types can be found in Figure 8.

These eight column types form a partition of $K$ on irreducible substructures, and knowing the number of columns and the number of vertices for each column type of $\Gamma$ is sufficient to count the number of arrowed arrays satisfying it. However, before proving the theorem for the number of arrowed arrays satisfying $\Gamma$, we will need another two lemmas for simplifying arrowed arrays that contain a fixed pair of vertices.

Lemma 17. (column pointing) Let $\Gamma = (w, R_1, R_2, \phi)$ be a substructure of $\mathcal{AR}^{(e)}_{K,R_1,R_2}$, $v$ be a critical vertex in cell $(1, \mathcal{X})$, $u$ be a non-critical vertex in cell $(2, \mathcal{Y})$, and $\mathcal{X} \neq \mathcal{Y}$. Let the substructure $\Gamma_{vu}$ be the set of arrowed arrays that satisfies $\Gamma$ and contains the pair $\{v, u\}$, and $\Gamma' = (w', R_1, R_2, \phi')$ be a substructure of
\[ \mathcal{AR}^{(s-1)}_{K;R_1,R_2} \] such that

\[
w'_{i,j} = \begin{cases} 
  w_{i,j} - 1 & \text{cell (i,j) contains } u \text{ or } v \\
  w_{i,j} & \text{otherwise}
\end{cases}
\]

\[
\phi'(j) = \begin{cases} 
  \phi(j) & j \in \mathcal{H} \\
  \mathcal{Y} & j = \mathcal{X}
\end{cases}
\]

Note that \( \phi' \) contains one more element in its domain than \( \phi \). Then, the number of arrowed arrays satisfying \( \Gamma_{vu} \) and the number of arrowed arrays satisfying \( \Gamma' \) are equal. Furthermore, \( \Gamma_{vu} \) satisfies the non-empty and full conditions if and only if \( \Gamma' \) satisfies them.

**Proof.** To prove that the number of arrowed arrays are equal, we provide a bijection between arrowed arrays satisfying \( \Gamma_{vu} \) and arrowed arrays satisfying \( \Gamma' \). Let \( (\alpha, \phi) \) be an arrowed array that satisfies \( \Gamma \) and contains the pair \( \{v, u\} \). As \( u \) is not critical, removing the pair \( \{v, u\} \) does not affect \( \psi_2 \). Therefore, we can obtain an arrowed array \( (\alpha', \phi') \) by removing \( \{v, u\} \) and replacing it by an arrow pointing from \( \mathcal{X} \) to \( \mathcal{Y} \), while keeping all the other pairs intact. This reduces the number of vertices in \( (1, \mathcal{X}) \) and \( (2, \mathcal{Y}) \) by 1, and leaves \( \psi_1 \) unchanged. Hence, the forest condition is preserved, and \( (\alpha', \phi') \) satisfies \( \Gamma' \).

Conversely, given an arrowed array \( (\alpha', \phi') \) that satisfies \( \Gamma' \), we can remove the arrow pointing from \( \mathcal{X} \) to \( \mathcal{Y} \) and replace it by the pair \( \{v, u\} \) given by \( \Gamma_{vu} \). Since the positions of \( v \) and \( u \) are fixed in \( \Gamma_{vu} \), there is no ambiguity as to where to add them. Again, the forest condition is preserved as \( \psi_1 \) and \( \psi_2 \) are unchanged by this substitution. Finally, both cells \( (1, \mathcal{X}) \) and \( (2, \mathcal{Y}) \) contain at least one object in both \( \Gamma_{vu} \) and \( \Gamma' \). Cell \( (1, \mathcal{X}) \) contains either a critical vertex or an arrow-tail, and cell \( (2, \mathcal{Y}) \) contains at least one other object as \( u \) is not critical. Since all other cells remain unchanged, \( \Gamma_{vu} \) satisfies the non-empty and full conditions if and only if \( \Gamma' \) satisfies them. \( \square \)

**Lemma 18.** (column merging) Let \( \Gamma = (w, \mathcal{R}_1, \mathcal{R}_2, \phi) \) be a substructure of \( \mathcal{AR}^{(s)}_{K;R_1,R_2} \), \( v \) be a critical vertex in cell \( (1, \mathcal{X}) \), \( u \) a critical vertex in cell \( (2, \mathcal{Y}) \), and \( \mathcal{X} \neq \mathcal{Y} \). Suppose that \( \Gamma \) satisfies the full condition, and without loss of generality, assume that \( \mathcal{Y} \) is the last column of \( \Gamma \) for purposes of column indexing. Let the substructure \( \Gamma_{vu} \) be the set of arrowed arrays that satisfies \( \Gamma \) and contains the pair \( \{v, u\} \), and \( \Gamma' = (w', \mathcal{R}_1', \mathcal{R}_2', \phi') \) be a substructure of \( \mathcal{AR}^{(s-1)}_{K-1;R_1,R_2} \) such that

\[
\mathcal{R}_i' = \begin{cases} 
  \mathcal{R}_i \cup \mathcal{X}' \backslash \mathcal{Y} & \mathcal{Y} \in \mathcal{R}_i \\
  \mathcal{R}_i & \text{otherwise}
\end{cases}
\]

\[
w'_{i,j} = \begin{cases} 
  w_{i,j} + w_{1,\mathcal{Y}} - 1 & j = \mathcal{X}' \\
  w_{i,j} & \text{otherwise}
\end{cases}
\]

\[
\phi'(j) = \begin{cases} 
  \phi(\mathcal{Y}) & j = \mathcal{X}', \phi(\mathcal{Y}) \text{ is defined} \\
  \mathcal{X} & j \in \mathcal{H}, \phi(j) = \mathcal{Y} \\
  \phi(j) & j \in \mathcal{H}, \phi(j) \neq \mathcal{Y}
\end{cases}
\]

Then, the number of arrowed arrays satisfying \( \Gamma_{vu} \) and the number of arrowed arrays satisfying \( \Gamma' \) are equal. Furthermore, \( \Gamma' \) also satisfies the full condition.

**Proof.** To prove that the number of arrowed arrays are equal, we provide a bijection between arrowed arrays satisfying \( \Gamma_{vu} \) and arrowed arrays satisfying \( \Gamma' \). The idea behind this bijection is to merge the columns \( \mathcal{X} \) and \( \mathcal{Y} \) in such a way that keeps the rightmost objects of cell \( (2, \mathcal{X}) \) and \( (1, \mathcal{Y}) \) intact. As all other cells remain unchanged, \( \Gamma' \) satisfies the full condition.

Let \((\alpha, \phi)\) be an arrowed array that satisfies \( \Gamma \) and contains the pair \( \{v, u\} \). To obtain \( \alpha' \), we take the vertices of cell \( (2, \mathcal{Y}) \) and place them in cell \( (2, \mathcal{X}) \) in order, before the vertices originally in \( (2, \mathcal{X}) \). Then, for any column \( j \) that points to \( \mathcal{Y} \), we change them to point to \( \mathcal{X} \) instead. Similarly, we take the vertices of cell \( (1, \mathcal{Y}) \) and place them in cell \( (1, \mathcal{X}) \), but after the vertices originally in \( (1, \mathcal{X}) \). Furthermore, we mark cell \( (1, \mathcal{X}) \) if cell \( (1, \mathcal{Y}) \) is marked, and make \( \mathcal{X} \) point to a column \( \mathcal{Z} \) if column \( \mathcal{Y} \) points to \( \mathcal{Z} \) originally. Finally, we remove the pair \( \{v, u\} \) and the column \( \mathcal{Y} \). Conversely, given an arrowed array \( (\alpha', \phi') \) that satisfies \( \Gamma' \),
By applying the column pointing procedure to the top figure, we arrive at the bottom figure. Here, $u = p_{1,x_1}$ and $v = q_{2,1}$. R1, R2, and R3 can be arbitrary in whether they are marked, but they must be the same between the two figures. The same holds for the optional arrow with $\mathcal{Y}$ as its tail.

Figure 9. Column pointing

we can recover $(\alpha, \phi)$ by simply reversing the steps. As the arrows in row 1 and the number of vertices in each cell is given by $\Gamma$, the reverse is unambiguous.

By construction, $(\alpha, \phi)$ satisfies $\Gamma_{vu}$ if and only if $(\alpha', \phi')$ satisfies $\Gamma'$, with the possible exception of the forest condition. Now, the critical pair $\{u, v\}$ gives the edge $(\mathcal{X}, \mathcal{Y})$ in the functional digraph of $\psi_1$, and the edge $(\mathcal{Y}, \mathcal{X})$ in the functional digraph of $\psi_2$. By merging these two columns, we are contracting these two edge in their respective functional digraph. Therefore, $\psi_i$ satisfies the forest condition if and only if $\psi_i'$ satisfies it, for $i = 1, 2$. This shows that the numbers of arrowed arrays satisfying $\Gamma_{vu}$ and $\Gamma'$ are equal. □

The application of Lemma 17 to replace $\Gamma_{vu}$ with $\Gamma'$ is called the column pointing procedure, and a diagram of this procedure can be found in Figure 9. Similarly, the application of Lemma 18 to replace $\Gamma_{vu}$ with $\Gamma'$ is called the column merging procedure, and a diagram of this procedure can be found in Figure 10. After applying either procedure, we can apply the arrow simplification lemmas to $\Gamma'$ to further simplify the substructure.

Note that unlike the other simplification lemmas, column merging requires the substructure to satisfy the full condition. In particular, it requires each cell of the columns being merged to be non-empty. Otherwise, the resulting column will completely drop out of the forest condition, which can break the bijection.
Theorem 19. Given an irreducible substructure \( \Gamma = (w, R_1, R_2, \phi) \) that satisfies the full condition with \( s \geq A + 2 \), the number of arrowed arrays \((\alpha, \phi) \in AR_{K,R_1,R_2}^{(s)}\) that satisfy \( \Gamma \) is given by the formula

\[
T(\Gamma) = (s - 1)! \left[ \frac{(b_2 + d_2)(\bar{a}_1 + c_1 + \bar{c}_1 + d_1)}{s - A} + \frac{b_1 (c_2 + \bar{c}_2 + c_2) - r_1 (b_2 + d_2)}{(s - A)(s - A - 1)} \right]
\]

In the case where \( s = A + 1 \), the formula reduces to

\[
T(\Gamma) = (s - 1)! (b_2 + d_2)(\bar{a}_1 + c_1 + \bar{c}_1 + d_1)
\]
By the convention set out in Convention 6, we let a lower case variable $x_i$ represent the total number of points in row $i$ of the columns of type $\mathcal{X}$, and $A$ represent the number of columns of type $\mathcal{A}$.

**Proof.** We prove this via induction on the total number of vertices, and tiebreak by the number of critical vertices in row 2. There are two base cases and three inductive cases to consider, depending on whether $\Gamma$ contains a column of type $\mathcal{A}$, a column of type $\mathcal{C}$ and no columns of type $\mathcal{A}$, or no columns of type $\mathcal{A}$ or $\mathcal{C}$. Also, we will only do the proof for $s \geq A + 2$. In the case where $s = A + 1$, the proof is the same, but we have to use the second formula to avoid division by zero.

**Base case 1:**
Suppose $\Gamma$ has no critical vertex. As $\Gamma$ is irreducible, each cell must either be marked or have an arrow-tail. However, the latter cannot happen as an arrow-head of an irreducible substructure must be in an unmarked cell. Hence, every cell of $\Gamma$ must be marked, so the forest condition is trivially satisfied. Therefore, there are $s!$ ways to pair the vertices of the array. By substituting $d_1 = d_2 = s$ into $T(\Gamma)$, and setting all other variables to 0, we see that $T(\Gamma) = s!$ as desired.

**Base case 2:**
If $s = 2$, $\mathcal{A} = \emptyset$, and $\mathcal{C} \neq \emptyset$, then

$$T(\Gamma) = \frac{[b_2 + d_2] (c_1 + \bar{c}_1 + d_1) + b_1 (c_2 + \bar{c}_2) + 2}{2}$$

by substituting in $2 = b_1 + \bar{c}_1 + c_1 + d_1$. This case is needed as the inductive step for $\Gamma$ containing no columns of type $\mathcal{A}$ but at least one column of type $\mathcal{C}$ requires that $T(\Gamma)$ be true for $s - 1$. However, if $s = 1$, then $s < A + 2$, and this creates a zero in the denominator of our formula. The formula can be proved by checking all possible positions of the vertices in row 1. The details are omitted as it is tedious and not enlightening.

**Case 1:**
Suppose $\Gamma$ contains at least one column of type $\mathcal{A}$, and $\mathcal{X}$ is one such column. Let $\mathcal{X}$ and $\mathcal{X}$ be columns pointing to $\mathcal{X}$ as defined in Definition 16, and note that they are columns of type $\mathcal{A}$ and $\bar{\mathcal{A}}$, respectively. Then, the critical vertex $v$ of cell $(1, \mathcal{X})$ must be paired with some vertex $u$ in a cell $(2, \mathcal{Y})$. To satisfy the forest condition for row 1, $\mathcal{Y}$ cannot be a column of $\mathcal{X}$, $\mathcal{X}$, or $\mathcal{X}$. By fixing $u$, we can pair vertices $u$ and $v$ to obtain the substructure $\Gamma_{uv}$. Then, we simplify $\Gamma_{uv}$ using the column pointing and column merging procedures described in Lemma 17 and Lemma 18, which makes the columns of $\mathcal{X}$, $\mathcal{X}$, and $\bar{\mathcal{X}}$ point to $\mathcal{Y}$. Now, $\mathcal{Y}$ cannot point to $\mathcal{X}$, $\mathcal{X}$, or $\mathcal{X}$, as that would either imply that $\mathcal{Y} \in \mathcal{X} \cup \bar{\mathcal{X}}$, or that $\Gamma$ is not irreducible. Therefore, $\mathcal{Y}$ must either not contain an arrow-tail, or be pointing to some other column $\mathcal{Z}$ that has a critical vertex in row 1. Therefore, the functional digraph of $\phi$ is acyclic, and by using the arrow simplification procedures described in Lemma 14 and Lemma 15, we obtain an irreducible substructure $\Gamma'$ that has one less vertex per row than $\Gamma$. Furthermore, both $s$ and $A$ decrease by 1, so the inequality $s \geq A + 2$ holds. Depending on the column type of $\mathcal{Y}$ and whether $u$ is critical, we can use the inductive hypothesis to determine $T(\Gamma')$ in terms of existing parameters given by the column types of $\Gamma$.

For example, let $\mathcal{Y}$ be a column of type $\mathcal{D}$. Then, after applying the column pointing procedure, $\mathcal{X}$ becomes a column of type $\mathcal{B}$, the columns of $\mathcal{X}$ become columns of type $\mathcal{B}$, and the columns of type $\bar{\mathcal{X}}$ become columns of type $\mathcal{D}$. Hence, in the resulting substructure $\Gamma' = \Gamma_{AD}$ after simplification, we have

$$T(\Gamma_{AD}) = (s - 2)! \frac{[\frac{(b_2 + x_2 + \bar{x}_2 + d_2 + \bar{x}_2 - 1) (\bar{a}_1 + \bar{c}_1 + \bar{c}_1 + \bar{d}_1)}{s - A} + \frac{(b_1 + x_1 + \bar{x}_1 - 1) (c_2 + \bar{c}_2 + \bar{c}_2) - \bar{c}_1 (b_2 + x_2 + \bar{x}_2 + d_2 + \bar{x}_2 - 1)}{(s - A)(s - A - 1)}]}{13}$$
Similarly, we define \( T(Γ_{AA}) \) and \( T(Γ_{AC}) \) to be the number of arrowed arrays satisfying substructure \( Γ' \) if \( v \) is in a column of type \( A \) and \( C \), respectively. Then, we repeat this computation for the remaining possible column types of \( Y \), and whether \( u \) is critical. These are given by the column types \( A, \overline{A}, \overline{A}, B, C, \overline{C}, C \), and \( \overline{C} \). In the cases of \( A, \overline{A}, B, \) and \( \overline{C} \), the particular substitutions are dependent on whether \( v \) is also critical, even though the formulas for \( T(Γ') \) are the same. Furthermore, these can all be expressed in terms of \( T(Γ_{AA}), T(Γ_{AC}), \) and \( T(Γ_{BD}) \). By letting \( u \) range across all vertices of row 2, we obtain all possible pairings of the critical vertex \( v \) in column \( X \). Therefore, by counting the number of vertices of each column type, we obtain the number of occurrences of each \( Γ' \). Adding everything together, we have

\[
T(Γ) = (a_2 - x_2 + \overline{a}_2 - \overline{x}_2) T(Γ_{AA}) + (c_2 + \overline{c}_2 - \overline{x}_2) T(Γ_{AC}) + (b_2 + d_2) T(Γ_{BD})
\]

By substituting in \( s = a_i + \overline{a}_i + c_i + \overline{c}_i + b_i + d_i \) and simplifying, we can show that \( T(Γ) \) satisfies the inductive hypothesis. This proves the case where \( Γ \) contains a column of type \( A \).

**Case 2:**

Suppose that \( Γ \) does not contain any column of type \( A \), but contains at least one column of type \( C \). The formula simplifies to

\[
T(Γ) = (s - 1)! \left[ \frac{(b_2 + d_2)(c_1 + \overline{c}_1 + d_1)}{s} + \frac{b_1(c_2 + \overline{c}_2 - \overline{c}_1)(b_2 + d_2)}{s(s - 1)} \right]
\]

While the formula is simpler in this case, the proof is slightly more involved. Let \( X \) be a fixed column of type \( C \), and let \( \overline{X} \) and \( \overline{X} \) be columns pointing to \( X \) as defined in Definition 16. Note that they are columns of type \( \overline{C} \) and \( \overline{C} \), respectively. As in Case 1, the critical vertex \( v \) of cell \((1, X)\) must be paired with some vertex \( u \) in a cell. Again, to satisfy the forest condition for row 1, \( Y \) cannot be a column of \( X, \overline{X} \) or, \( \overline{X} \). Therefore, we pair \( u \) and \( v \) to obtain the substructure \( Γ_{uv} \), which we simplify using the same lemmas used in Case 1 to obtain an irreducible substructure \( Γ' \). As the case \( s = 2 \) is already handled, we can assume \( s \geq 3 \), so \( s \geq A + 2 \) still holds. Depending on the column type of \( Y \) and whether \( u \) is critical, we can use the inductive hypothesis to determine \( T(Γ') \) in terms of existing parameters given by column types of \( Γ \). The major difference in this case is that if \( u \) is a critical vertex, then both \( X \) and \( Y \) become columns of a different type, so we must introduce the parameters \( y_i \) for the number of vertices in column \( i \) of \( Y \).

As in Case 1, we define \( T(Γ_{CB}) \) and \( T(Γ_{CC}) \) to be the number of arrowed arrays satisfying substructure \( Γ' \) if \( v \) is in a column of type \( B \) and \( C \), respectively. However, we also need the correction terms \( T_{CBc} \) and \( T_{CCc} \), for the cases of \( B \) and \( \overline{C} \), depending on whether the vertex \( v \) is critical. Then, we can compute \( T(Γ') \) for all possible column types of \( Y \), and whether \( u \) is critical. These are given by the column types \( B, C, \overline{C}, \overline{C}, \overline{D}, \) and \( D \), and can all be expressed in terms of \( T(Γ_{AB}), T(Γ_{AC}), T_{CBc}, \) and \( T_{CCc} \).

By letting \( u \) range across all vertices of row 2, we obtain all possible pairings of the critical vertex \( v \) in column \( X \). Notice that as we pair \( v \) each vertex of \( B \), we add \( y_1 T_{CBc} \) if and only if \( u \) is the rightmost vertex of \( Y \). Since each column of \( B \) has exactly one rightmost vertex, \( \sum_{y \in B} y_1 = b_1 \). Similarly, \( \sum_{y \in \overline{B}} y_1 = c_1 - c_1 \). Therefore, by counting the number of vertices of each column type, we obtain the number of occurrences of each \( Γ' \). Adding everything together, we have

\[
T(Γ) = (c_2 - x_2 + \overline{c}_2 - \overline{x}_2) T(Γ_{CC}) + (\overline{c}_1 - \overline{c}_1) T_{CBc} + (b_2 + d_2) T(Γ_{BD}) + b_1 T_{CBc}
\]

By substituting in \( s = c_2 + \overline{c}_2 + b_2 + d_2 \) and simplifying, we can show that \( T(Γ) \) satisfies the inductive hypothesis. This proves the case where \( Γ \) contains a column of type \( C \), but no columns of type \( A \).

**Case 3:**

If \( Γ \) does not contain any column of type \( A \) or \( C \), then every cell in row 1 is marked, leaving us only with columns of type \( B \) and \( D \). In this case, the formula simplifies to

\[
T(Γ) = d_1 (s - 1)!
\]

as \( s = b_2 + d_2 \). Since \( Γ \) does not contain any arrows, we can switch the two rows and invert the roles of \( B \) and \( D \) to obtain \( Γ' \). Furthermore, at least one cell in row 2 is unmarked, as otherwise we would have the base case. Therefore, the number of critical vertices in row 2 decreases in \( Γ' \), and we can continue the induction.
using Case 2. Furthermore, neither $s$ nor $A$ changed, so $s \geq A + 2$ still holds. Now, $\Gamma'$ only have columns of type $\mathcal{C}$ and $\mathcal{D}$, so by the inductive hypothesis,

$$T(\Gamma') = d_2(s - 1)!$$

as $s = c_1 + d_1$ in $\Gamma'$. This completes the induction and proves our formula for $T(\Gamma)$.

Note that if $\Gamma$ satisfies the full condition and $s \leq A$, then $T(\Gamma) = 0$, as each column of type $A$ requires one critical vertex for each row. Furthermore, as those vertices can only be paired with each other, $\psi_i(\mathcal{X}) \in \mathcal{A}$ for all $\mathcal{X} \in \mathcal{A}$. This violates the forest condition for row $i$.

**Corollary 20.** Given a substructure $\Gamma = (w, R_1, R_2, \phi)$ where $\phi$ is empty and $s \geq A + 2$, the number of arrowed arrays $(\alpha, \phi) \in \mathcal{A}R^{(s)}_{K; R_1, R_2}$ that satisfy $\Gamma$ is given by the formula

$$T(\Gamma) = (s - 1)! \left[ \left( \frac{(b_2 + d_2)(c_1 + d_1)}{s - A} + \frac{b_1 c_2}{(s - A)(s - A - 1)} \right) \right]$$

where $A$ is the number of columns that contains no marked cells and at least one vertex in each row. In the case where $s = A + 1$, the formula simplifies to

$$T(\Gamma) = (s - 1)! \left[ \frac{(b_2 + d_2)(c_1 + d_1)}{s - A} \right]$$

Note that Corollary 20 holds even if the full condition is not satisfied, and the definition of $A$ has been adjusted to match this. This stems from the fact that we can remove columns with no arrows or vertices without impacting the forest condition.

5. Enumerating Proper Vertical Arrays

Finally, we are ready to compute the formula for $\psi^{(s)}_{K; R_1, R_2}$ using arrowed arrays. As proper vertical arrays are arrowed arrays that satisfies the non-empty, balance, and forest conditions that contain no arrows, we can take $\phi$ to be empty and $w$ to be a vector of size $K$. To enumerate proper vertical arrays, we will define a coarser substructure, which we will compute the formula for using our formula of $T(\Gamma)$.

**Definition 21.** Let $w$ be a non-negative vector. The substructure $\Omega = (w)$ is defined to be the subset of $\mathcal{PVA}^{(s)}_{K; R_1, R_2}$ that satisfies the non-empty and balance conditions, such that for each pair $\alpha \in \mathcal{PVA}^{(s)}_{K; R_1, R_2}$, $\alpha$ contains $w_i$ vertices in both cells $(1, j)$ and $(2, j)$. For a given substructure $\Omega = (w)$ and $A \geq 0$, we define $\Omega_A$ to be the substructure that describes the set of arrowed arrays that satisfies $\Omega$, and have exactly $A$ (non-empty) columns of type $A$. For convenience, we say a substructure $\Gamma$ is a refinement of another substructure $\Omega$ if the set of arrowed arrays satisfying $\Gamma$ is a subset of the arrowed arrays satisfying $\Omega$. We denote it as $\Gamma \hookrightarrow \Omega$. Furthermore, if $\Gamma_1, \ldots, \Gamma_t$ is a set of substructures that are refinements of a substructure $\Omega$, we say that $\Gamma_1, \ldots, \Gamma_t$ partitions $\Omega$ if the sets of arrowed arrays satisfying the $\Gamma_i$’s are mutually disjoint, and their union is the set of arrowed arrays that satisfy $\Omega$.

By considering all possible $R_1$-subsets $R_1$ and $R_2$-subsets $R_2$, we see that the set of substructures of the form $\Gamma = ([w, w], R_1, R_2, \emptyset)$ partition the substructure $\Omega$. Furthermore, the subset of substructures with exactly $A$ columns of type $A$ partitions $\Omega_A$, which in turn partitions $\Omega$ by taking $A$ from 0 to $s - 1$. With the substructure $\Omega = (w)$ defined, we will now provide a formula for it, which we will use to decompose vertical arrays into arrowed arrays.

**Theorem 22.** Let $R_1, R_2 \geq 1$, and let $\Omega = (w)$ be a substructure with $F$ columns that contains vertices, denoted $\mathcal{F}$. Then, the number of vertical arrays $\alpha \in \mathcal{PVA}^{(s)}_{K; R_1, R_2}$ satisfying the substructure $\Omega$ is given by the formula

$$T(\Omega) = s! \sum_{A=0}^{s-1} \frac{s}{s - A} \left( \frac{F - 1}{A} \right) \left( \frac{K - A - 1}{K - A - R_1, K - A - R_2, R_1 + R_2 - K + A - 1} \right)$$

where $\binom{a+b+c}{a,b,c} = \frac{(a+b+c)!}{a!b!c!}$ is the multinomial coefficient.
Proof: To prove this theorem, we sum $T(\Omega)$ over all substructures $\Gamma = ([w, w_1], R_1, R_2, \emptyset)$ that are refinements of $\Omega$. Note that $T(\Gamma)$ as given in Corollary 20 only depends on the number non-empty of columns of type $A$, even though it depends on the number of vertices of other column types. Therefore, we first sum over all $\Gamma$ with $A$ non-empty columns of type $A$ to obtain $T(\Omega_A)$, then we sum $A$ from 0 to $s - 1$ to obtain $T(\Omega)$. As $\Omega$ satisfies the balance condition, so must all $\Gamma$ that are refinements of $\Omega$. This implies that we can drop the subscripts from $T(\Gamma)$. For convenience, we will refer to the number of vertices of row 1 in a set of column $X$ simply as the number of vertices in $X$, as that number is the same between row 1 or row 2.

Now, let $\Gamma$ be a refinement of $\Omega$, and suppose $\Gamma$ have $A$, $B$, $C$, and $D$ columns of type $A$, $B$, $C$, and $D$, respectively. Then, as the columns marked in row 1 are type $B$ and $D$, and the columns marked in row 2 are type $C$ and $D$, we have

$$B = K - A - R_2$$

$$C = K - A - R_1$$

$$D = R_1 + R_2 - K + A$$

Therefore, there are

$$\left(\begin{array}{c}
F \\
A
\end{array}\right)
\left(\begin{array}{c}
K - A \\
K - A - R_2, K - A - R_1, R_1 + R_2 - K + A
\end{array}\right)$$

substructures $\Gamma$ that are refinements of $\Omega_A$. Note that the columns of type $A$ must be non-empty, as they must be a subset of the columns of $F$.

Now, we can rewrite $T(\Gamma)$ as

$$T(\Gamma) = (s - 1)\left[\frac{bd}{s - A} + \frac{cd}{s - A} + \frac{d^2}{s - A} + \frac{bc}{s - A - 1}\right]$$

for $0 \leq A \leq s - 2$. For $A = s - 1$, we let $T_4(\Gamma) = bc = 0$, as $A = s - 1$ means there are $s - 1$ columns of type $A$, which means that the remaining non-empty column cannot be both type $B$ and type $C$ at the same time. As the substructures $\Gamma$ with $A$ columns of type $A$ partitions $\Omega_A$, we can let $T_i(\Omega_A) = \sum_{\Gamma \hookrightarrow \Omega_A} T_i(\Gamma)$ for $i = 1, 2, 3, 4$, which gives us

$$T(\Omega) = (s - 1)\left(\sum_{A=0}^{s-1} T_1(\Omega_A) + \sum_{A=0}^{s-1} T_2(\Omega_A) + \sum_{A=0}^{s-1} T_3(\Omega_A) + \sum_{A=0}^{s-2} T_4(\Omega_A)\right)$$

Now, let $\{v, u\}$ be a pair of vertices such that $v$ and $u$ are in cell $(1, X)$ and cell $(2, Y)$, respectively. If $X$ and $Y$ are distinct columns, then $(v, u)$ contributes to $T_1(\Gamma)$ if and only if $X$ is of type $C$ and $Y$ is of type $B$. To have $A$ columns of $A$, exactly $A$ of the remaining $F - 2$ columns of $F$ must be unmarked. Then, the remaining columns must be of $B$, $C$, and $D$, which can be arbitrarily chosen from the remaining $K - A - 2$ columns. If we let $w = w_1^2 + \cdots + w_K^2$ be the squares of the number of vertices in each column, then we have

$$T_1(\Omega_A) = \frac{s^2 - w}{s - A} \left(\begin{array}{c}
F - 2 \\
A
\end{array}\right) \left(\begin{array}{c}
K - A - 2 \\
K - A - R_1, K - A - R_2 - 1, R_1 + R_2 - K + A - 1
\end{array}\right)$$

Similar calculations give us

$$T_2(\Omega_A) = \frac{s^2 - w}{s - A} \left(\begin{array}{c}
F - 2 \\
A
\end{array}\right) \left(\begin{array}{c}
K - A - 2 \\
K - A - R_1 - 1, K - A - R_2, R_1 + R_2 - K + A - 1
\end{array}\right)$$

$$T_3(\Omega_A) = \frac{s^2 - w}{s - A - 1} \left(\begin{array}{c}
F - 2 \\
A
\end{array}\right) \left(\begin{array}{c}
K - A - 2 \\
K - A - R_1 - 1, K - A - R_2 - 1, R_1 + R_2 - K + A
\end{array}\right)$$

To obtain $T_3(\Omega_A)$, we break it up into 2 cases, depending on whether $X = Y$. If $X = Y$, we have

$$T_{3a}(\Omega_A) = \frac{w}{s - A} \left(\begin{array}{c}
F - 1 \\
A
\end{array}\right) \left(\begin{array}{c}
K - A - 2 \\
K - A - R_1, K - A - R_2, R_1 + R_2 - K + A - 1
\end{array}\right)$$

Otherwise, we get

$$T_{3b}(\Omega_A) = \frac{s^2 - w}{s - A} \left(\begin{array}{c}
F - 2 \\
A
\end{array}\right) \left(\begin{array}{c}
K - A - 2 \\
K - A - R_1, K - A - R_2, R_1 + R_2 - K + A - 2
\end{array}\right)$$
To sum over $A$, we shift the index of $T_A (\Omega_A)$ by 1, and observe that for $1 \leq A \leq s - 1$, we have
\[
T_1 (\Omega_A) + T_2 (\Omega_A) + T_3 (\Omega_A) = s^2 \frac{F-1}{A} \frac{K-A-1}{s-A} \left( K-A-R_1, K-A-R_2, R_1 + R_2 - K + A - 1 \right)
\]
which is independent of $w$. Furthermore, for $A = 0$, we have
\[
T_1 (\Omega_0) + T_2 (\Omega_0) + T_3 (\Omega_0) = s \left( K - 1 \right)
\]
which is in agreement with the previous sum. Therefore, we have the formula for $T (\Omega)$ as
\[
T (\Omega) = \sum_{s=0}^{s-1} \frac{s \cdot s!}{s-A} \left( \begin{array}{c} F-1 \\ A \end{array} \right) \left( K-A-1 \right) \left( \begin{array}{c} K-A-R_1, K-A-R_2, R_1 + R_2 - K + A - 1 \end{array} \right)
\]

To obtain the formula for the number of vertical arrays in Goulden and Slofstra, we need to sum over all possible ways of placing $s$ points into $K$ columns. Doing so gives us the following theorem.

**Theorem 23.** Let $s, K, R_1, R_2 \geq 1$. Then,
\[
v_{K;R_1,R_2}^{(s)} = \frac{(s+R_1-1)! (s+R_2-1)!}{(s+R_1+R_2-2)!} \times \left[ \frac{K-1}{s+R_1-1} \left( \begin{array}{c} K-1 \\ s+R_1-1 \end{array} \right) - \frac{K-1}{s+R_2-1} \left( \begin{array}{c} K-1 \\ s+R_2-1 \end{array} \right) \right]
\]

**Proof.** For $F \geq 0$, there are $\binom{K}{F}$ to choose $F$ columns so that each of them contains at least one vertex, and there are $\binom{s-1}{F}$ ways to distribute $s$ vertices into those columns. Hence, the number of proper vertical arrays satisfying the non-empty condition is
\[
v_{K;R_1,R_2}^{(s)} = \sum_{F=0}^{s-1} \binom{K}{F} \frac{s!}{(s-A)} \left( \begin{array}{c} s-A-1 \\ s-F \end{array} \right) \left( K-A-1 \right) \left( \begin{array}{c} K-A-R_1, K-A-R_2, R_1 + R_2 - K + A - 1 \end{array} \right)
\]
by the Chu-Vandermonde identity (pg. 67 of [1]). As the binomial coefficient $\binom{s}{A}$ implies that the natural upper bound of the sum is $A$, we can rewrite this sum as
\[
v_{K;R_1,R_2}^{(s)} = \sum_{A=0}^{s-1} \binom{s}{A} \left( \begin{array}{c} s+K-A-1 \\ K-A-R_1 \end{array} \right) \left( \begin{array}{c} s+K-A-1 \\ K-A-R_2 \end{array} \right) \left( \begin{array}{c} s+K-A-1 \\ R_1 + R_2 - K + A - 1 \end{array} \right)
\]
Now, by the Pfaff-Saalschütz identity (pg. 69 of [1]), we can rewrite the first part as
\[
\sum_{A=0}^{s-1} \binom{s}{A} \left( \begin{array}{c} s+K-A-1 \\ K-A-R_1 \end{array} \right) \left( \begin{array}{c} s+K-A-1 \\ K-A-R_2 \end{array} \right) \left( \begin{array}{c} s+K-A-1 \\ R_1 + R_2 - K + A - 1 \end{array} \right)
\]
}\]
\[
3F_2 \left( -s,-K+R_1,-K+R_2; R_1 + R_2 - K, -s + K + 1 \right) \left( \begin{array}{c} s+K-A-1 \\ K-A-R_1 \end{array} \right) \left( \begin{array}{c} s+K-A-1 \\ K-A-R_2 \end{array} \right) \left( \begin{array}{c} s+K-A-1 \\ R_1 + R_2 - K + A - 1 \end{array} \right)
\]
\[
 \frac{s!}{(R_1+s-1)! (R_2+s-1)!} \frac{s!}{(R_1+s-1)! (R_2+s-1)!} \frac{s!}{(R_1+s-1)! (R_2+s-1)!} \frac{s!}{(R_1+s-1)! (R_2+s-1)!}
\]
\[
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\]
where we use as an upper bound for Slofstra using Pfaff’s identity. We start by rewriting Theorem 2 as $A_{g_1, q_2; (s)}^3 (x) = g_1 - g_2$ using our notation, where

$$
g_1 = \sum_{k=1}^{d+1} \sum_{t_1 \geq 0} \sum_{t_2 \geq 0} \frac{(2q_1 + s)! (2q_2 + s)! (k-1)!}{(q_1 - t_1)! (k - q_1 + t_1 - 1)!} \left( \frac{x}{k} \right) \times \frac{1}{(d - t_1 - t_2 - k + 1)!} \left( \frac{x}{t_1 + t_2} \right) \left( k - q_1 \right) \left( k - q_2 + t_2 - 1 \right)!
$$

and

$$
g_2 = \sum_{k=1}^{d+1} \sum_{t_1 \geq 0} \sum_{t_2 \geq 0} \frac{(2q_1 + s)! (2q_2 + s)! (k-1)!}{(q_1 - t_1)! (k - q_1 + t_1 - 1)!} \left( \frac{x}{k} \right) \times \frac{1}{(d - t_1 - t_2 - k + 1)!} \left( \frac{x}{t_1 + t_2} \right) \left( k - q_1 \right) \left( k - q_2 + t_2 - 1 \right)!
$$

with $d = q_1 + q_2 + s$ as in the original theorem. Note that we have removed the upper bounds for $t_1$ and $t_2$, as the summation terms can only be non-zero if both $t_1 \leq q_1$ and $t_2 \leq q_2$ hold. To reduce the number of sums in $g_1$ and $g_2$, we manipulate them separately with the same transforms. We first use Pfaff’s identity to transform the sum involving $t_1$, then use the Chu-Vandermonde identity to eliminate $t_2$. Afterwards, we make the summation variables symmetric by making a substitution for $k$, before combining the results together. For reference, the identities used for this procedure can be found in pg. 67 and pg. 69 of [1].

By rewriting the $t_1$ sum of $g_1$ using the standard notation for hypergeometric series and using Pfaff’s identity, we have

$$
g_1 = \sum_{k=1}^{d+1} \sum_{t_1 \geq 0} \frac{1}{2q_1 + t_2} \left( \frac{x}{k} \right) \left( \frac{x}{2} \right) \left( \frac{1}{2} \right) \left( k - q_1 \right) \left( k - q_2 + t_2 - 1 \right)!
$$

While there is no upper bound for $t_1$, the term $(d - t_1 - t_2 - k + 1)!$ in the denominator causes the sum to terminate. Furthermore, for the summation term to be non-zero, we must have $d - t_1 - t_2 - k + 1 \geq 0$ and $k - q_2 + t_2 - 1 \geq 0$ at the same time. Combining these inequalities together gives us $t_1 \leq q_1 + s$, which can be used as an upper bound for $t_1$. Next, we rewrite the $t_2$ sum as a hypergeometric series, and note that it

6. Further Reduction to the Goulden-Slofstra Formula

In this section, we will show a method of reducing the number of sums in the formula of Goulden and Slofstra using Pfaff’s identity. We start by rewriting Theorem 2 as $A_{g_1, q_2; (s)}^3 (x) = g_1 - g_2$ using our notation, where

$$
g_1 = \sum_{k=1}^{d+1} \sum_{t_1 \geq 0} \sum_{t_2 \geq 0} \frac{(2q_1 + s)! (2q_2 + s)! (k-1)!}{(q_1 - t_1)! (k - q_1 + t_1 - 1)!} \left( \frac{x}{k} \right) \times \frac{1}{(d - t_1 - t_2 - k + 1)!} \left( \frac{x}{t_1 + t_2} \right) \left( k - q_1 \right) \left( k - q_2 + t_2 - 1 \right)!
$$

and

$$
g_2 = \sum_{k=1}^{d+1} \sum_{t_1 \geq 0} \sum_{t_2 \geq 0} \frac{(2q_1 + s)! (2q_2 + s)! (k-1)!}{(q_1 - t_1)! (k - q_1 + t_1 - 1)!} \left( \frac{x}{k} \right) \times \frac{1}{(d - t_1 - t_2 - k + 1)!} \left( \frac{x}{t_1 + t_2} \right) \left( k - q_1 \right) \left( k - q_2 + t_2 - 1 \right)!
$$

with $d = q_1 + q_2 + s$ as in the original theorem. Note that we have removed the upper bounds for $t_1$ and $t_2$, as the summation terms can only be non-zero if both $t_1 \leq q_1$ and $t_2 \leq q_2$ hold. To reduce the number of sums in $g_1$ and $g_2$, we manipulate them separately with the same transforms. We first use Pfaff’s identity to transform the sum involving $t_1$, then use the Chu-Vandermonde identity to eliminate $t_2$. Afterwards, we make the summation variables symmetric by making a substitution for $k$, before combining the results together. For reference, the identities used for this procedure can be found in pg. 67 and pg. 69 of [1].

By rewriting the $t_1$ sum of $g_1$ using the standard notation for hypergeometric series and using Pfaff’s identity, we have

$$
g_1 = \sum_{k=1}^{d+1} \sum_{t_1 \geq 0} \frac{1}{2q_1 + t_2} \left( \frac{x}{k} \right) \left( \frac{x}{2} \right) \left( \frac{1}{2} \right) \left( k - q_1 \right) \left( k - q_2 + t_2 - 1 \right)!
$$

While there is no upper bound for $t_1$, the term $(d - t_1 - t_2 - k + 1)!$ in the denominator causes the sum to terminate. Furthermore, for the summation term to be non-zero, we must have $d - t_1 - t_2 - k + 1 \geq 0$ and $k - q_2 + t_2 - 1 \geq 0$ at the same time. Combining these inequalities together gives us $t_1 \leq q_1 + s$, which can be used as an upper bound for $t_1$. Next, we rewrite the $t_2$ sum as a hypergeometric series, and note that it

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satisfies the Chu-Vandermonde identity. This yields,

\[
g_1 = \sum_{k=1}^{d+1} \sum_{t_1=0}^{q_1+s} \frac{1}{2^{d-k+1}t_1!} \cdot \binom{x}{k} \binom{-q_2, -d + t_1 + k - 1}{k - q_2} \times
\frac{(2q_1 + s)!(2q_2 + s)!(k + t_1 - 1)!}{(d - t_1 - k + 1)!q_1!(k - q_1 + t_1 - 1)!q_2!(k - q_2 - 1)!}
\]

Note that the term \((d - t_1 - k + 1)!\) in the denominator means that for \(k > d - t_1 + 1\), the summation term is zero. Therefore, we can switch the two sums and lower the upper bound of \(k\) to \(d - t_1 + 1\). Next, the terms \((k - q_1 + t_1 - 1)!\) and \((k - 1)!\) in the denominator means that for the summand to be non-zero, we have \(k \geq \max\{q_1 - t_1 + 1, 1\}\). Hence, we can change the lower bound of \(k\) to \(q_1 - t_1 + 1\). As \(k + t_1 - 1 > q_1 \geq 0\) with this new lower bound, the factorial term in the numerator remains non-negative. After changing the bounds, we can reverse the sum with the substitution \(k = d - t_1 - t_2 + 1\). This gives us the formula

\[
(1) \quad g_1 = \sum_{t_1=0}^{q_1+s} \sum_{t_2=0}^{q_2+s} \frac{(d - t_1)! (d - t_2)! (2q_1 + s)!(2q_2 + s)!}{2^{d+t+2}t_1!t_2!(d - t_1 - t_2)!} \cdot \binom{x}{d - t_1 - t_2 + 1} \times \frac{1}{q_1!q_2!(s + q_1 - t_1)!(s + q_2 - t_2)!}
\]

which is symmetric between \(t_1\) and \(t_2\).

We now apply the same transformations to \(g_2\). However, instead of changing the upper bound to \(t_1 \leq q_1 + s\), we have \(t_1 \leq q_1\). Then, after applying the Chu-Vandermonde identity, we can tighten the bounds of \(k\) to \(q_1 + s - t_1 + 1 \leq k \leq d - t_1 + 1\). Finally, we can reverse the sum with the substitution \(k = d - t_1 - t_2 + 1\). This gives us the formula

\[
(2) \quad g_2 = \sum_{t_1=0}^{q_1} \sum_{t_2=0}^{q_2} \frac{(d - t_1)! (d - t_2)! (2q_1 + s)!(2q_2 + s)!}{2^{d+t+2}t_1!t_2!(d - t_1 - t_2)!} \cdot \binom{x}{d - t_1 - t_2 + 1} \times \frac{1}{(q_1 + s)!(q_2 + s)!(q_1 - t_1)!(q_2 - t_2)!}
\]

which is again symmetric in \(t_1\) and \(t_2\).

As we have \((q_1 - t_1)\) and \((q_2 - t_2)\) in the denominator of \(g_2\), we can actually increase the bounds of \(t_1\) and \(t_2\) to \(q_1 + s\) and \(q_2 + s\) without changing the sum, matching the bounds of \(g_1\). Finally, we can put (1) and (2) together and obtain

\[
A_2^{(q_1,q_2,s)}(x) = g_1 - g_2
\]

where \(d = q_1 + q_2 + s\).

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