HTL quasiparticle picture of the thermodynamics of QCD

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Starting from a nonperturbative expression for entropy and density obtained from \( \Phi \)-derivable two-loop approximations to the thermodynamic potential, a quasiparticle model for the thermodynamics of QCD can be developed which incorporates the physics of hard thermal loops and leads to a reorganization of the otherwise ill-behaved thermal perturbation theory through order \( \alpha^3/2 \). Some details of this reorganization are discussed and the differences to simpler quasiparticle models highlighted. A comparison with available lattice data shows remarkable agreement down to temperatures of \( \sim 2.5 T_c \).

1. Entropy and density in \( \Phi \)-derivable approximations

In the so-called \( \Phi \)-derivable approximations, obtained by truncating the skeleton functional \( (\Phi) \) in the Luttinger-Ward representation of the thermodynamic potential, the expression for the entropy density takes a remarkable form: for example in a scalar theory \( g\varphi^3 + g^2\varphi^4 \) it can be written as

\[
S = -\int \frac{d^4k}{(2\pi)^4} \frac{\partial n_{BE}(\omega)}{\partial T} \text{Im} \log D^{-1}(\omega, k) + \int \frac{d^4k}{(2\pi)^4} \frac{\partial n(\omega)}{\partial T} \text{Im} \Pi(\omega, k) \text{Re} D(\omega, k) + S' \tag{1}
\]

where \( D \) and \( \Pi \) are self-consistent propagator and self-energy, resp., and one finds

\[
S' = O(3\text{-loop}) = O(g^4). \tag{2}
\]

The fact that \( S' = 0 \) in 2-loop \( \Phi \)-derivable approximations, which has been observed first in Fermi-liquid theory by Riedel \[3\], turns out to hold rather generally \[4\]. In particular, the 2-loop contribution of Dirac fermions to the entropy density reads

\[
S_f = -2 \int \frac{d^4k}{(2\pi)^4} \frac{\partial n_{FD}(\omega)}{\partial T} \text{tr} \left\{ \text{Im} \log(\gamma_0 S^{-1}) \right\} - \text{Im} (\gamma_0 \Sigma) \text{Re} (S\gamma_0) \tag{3}
\]

For nonvanishing chemical potentials \( \mu_i \), this can be extended to the fermion number densities \( N_i \), which are obtained by replacing the explicit derivative \( \partial / \partial T \) in (3) by \( \partial / \partial \mu_i \), with

\[
S'_f = O(3\text{-loop}) = N'_f \tag{4}
\]

Explicit interactions at 2-loop order are thus completely encoded in the dressed (quasiparticle) propagators \( D, S, \ldots \), which makes entropy and density a preferred starting point for a quasiparticle description of thermodynamic quantities—in fact any thermodynamic quantity, since apart from a single integration constant (which in QCD corresponds to the inherently nonperturbative bag constant) the grand canonical potential can be reconstructed from entropy and density.

If a quasiparticle picture in the sense of Landau is applicable, that is, if a large fraction of the possibly strong interactions can be accounted for by ‘propagator renormalization’, the above dressed 2-loop expressions should provide a reasonable approximation to thermodynamic quantities and the residual quasiparticle interactions \( \sim g^4 \) described by \( S' \) or \( N' \) should be comparatively weak.

Technically, an important point turns out to be that the above expressions for entropy and density are manifestly ultraviolet finite, as all integrals involve \( \partial n / \partial T \) or \( \partial n / \partial \mu \) which go to zero exponentially for both positive and negative frequencies. In relativistic field theories, this is a

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great advantage, because it allows one to use these functionals in a nonperturbative manner once finite approximations for the propagators and self-energies have been obtained, whereas usually the renormalization programme requires systematic expansions (and truncations) in powers of the coupling constant.

At finite temperature, truncated perturbative series turn out to have extremely poor apparent convergence behaviour \[1\] and this problem arises with the appearance of odd powers in the coupling such as \(g^3\) (the so-called plasmon effect\[^2\]). In QCD this plasmon effect spoils apparent convergence up to extremely high temperatures \(\gg 10^5 T_c\). However, this breakdown of perturbation theory does not necessarily have anything to do with the nonabelian character of QCD, as a similar phenomenon occurs in virtually any field theory for coupling constants where \(T = 0\) perturbation theory still looks applicable.

On the other hand, in expressions such as \(
\begin{align*}
\hat{\Pi}_L(\omega, k) &= m_D^2 \left[ 1 - \frac{\omega + k}{2k} \log \frac{\omega + k}{\omega - k} \right], \\
\hat{\Pi}_T(\omega, k) &= \frac{1}{\delta} \left[ \hat{m}_D^2 + \frac{\omega^2 - k^2}{k^2} \hat{\Pi}_L \right], \\
\hat{\Sigma}_{\pm}(\omega, k) &= \frac{M^2}{k} \left( 1 - \frac{\omega \mp k}{2k} \log \frac{\omega + k}{\omega - k} \right),
\end{align*}
\)

\[2\] Caused in fact by the appearance of screening (Debye) masses as opposed to the dynamical plasmon mass.
with
\[ \tilde{m}_D^2 = (N + \frac{N_f}{2} \frac{g^2 T^2}{3} + \sum_f \frac{g^2 \mu^2_f}{2 \pi^2}), \]
\[ \tilde{M}_f^2 = \frac{g^2 C_F}{8} (T^2 + \frac{\mu^2_f}{\pi^2}). \]

The gauge boson propagator involves two different propagators, a spatially transverse one,
\[ \hat{D}_T(\omega, k) = [-\omega^2 + k^2 + \hat{\Pi}_T(\omega, k)]^{-1}, \]
and one that only appears in the thermal medium,
\[ \hat{D}_L(\omega, k) = [-k^2 + \hat{\Pi}_L(\omega, k)]^{-1}. \]

Similarly, the fermion propagator has a “normal” branch \( S_+ \propto [\omega - (k + \Sigma_+)]^{-1} \), and one that is a purely collective effect, \( S_- \propto [\omega + (k + \Sigma_-)]^{-1} \), where the label \((-\)) refers to the fact that chirality and helicity have opposite signs.

These HTL propagators have poles at real frequencies \( \omega \) for all \( k \), corresponding to (undamped) quasiparticles with momentum-dependent thermal masses, with the additional branches disappearing from the spectrum for \( k/gT \gg 1 \) because of exponentially vanishing residues. The normal branches on the other hand, approach constant asymptotic masses \( m_\infty^2 = \frac{1}{2} \tilde{m}_D^2 \) and \( M_\infty^2 = 2 \tilde{M}_D^2 \), resp., in this limit.

At hard momenta, the above HTL expressions are actually invalid, except for \( |\omega^2 - k^2| \ll T^2 \).

This is however the region where all the poles of the HTL propagators lie, so that the latter still give the correct dispersion law of quasiparticles at leading order, with next-to-leading order corrections calculable within standard HTL perturbation theory.

### 2.1. Leading-order terms in entropy and density

Let us now inspect how the leading-order interaction terms of the grand canonical potential arise in the self-consistent entropy (and density) functionals \( (1), (3) \), which at two-loop order are determined by interaction-free quasiparticle propagators.

Since \( \Pi \propto g^2 \), these must be reproduced by linearizing entropy and density in \( \Pi \). In the pure glue case, the relevant terms in the entropy read
\[ S_2 = -2N_g \int \left\{ \frac{d^4k}{(2\pi)^4} \frac{\partial n}{\partial T} \left[ \text{Im} \frac{\Pi^{(2)}_T}{\omega^2 - k^2} \right] + \text{Re} \frac{\Pi^{(2)}_T}{\omega^2 - k^2} \right\}, \]
\[ = 2N_g \int \frac{d^4k}{(2\pi)^4} \frac{\partial n}{\partial T} \text{Re} \frac{\Pi^{(2)}_T}{\omega^2 - k^2}, \]
where only the \( g^2 \) part of the transverse polarization tensor, \( \Pi^{(2)}_T \) is needed. The latter is seen to contribute only its light-cone value
\[ \Pi^{(2)}_T(\omega^2 = k^2) = g^2 NT^2/6 \equiv m_\infty^2, \]
yielding \( S_2 = -N_g m_\infty^2 T/6 \) with \( N_g = N^2 - 1 \).

Analogous results are obtained for the fermionic contributions, and also in the case of fermion densities. The order-\( g^2 \) terms are simply and generally given by the (leading-order) thermal masses of hard particles according to
\[ S_2 = -T \left\{ \sum_B \frac{m_B^2}{12} + \sum_F \frac{M_F^2}{24} \right\}, \]
\[ N_2 = -\frac{1}{8\pi^2} \sum_F \mu_F M_F^2, \]
where the sums run over all the bosonic (\( B \)) and fermionic (\( F \)) degrees of freedom (explicitly counting spin degrees of freedom).

### 2.2. Plasmon terms

In conventionally resummed perturbation theory, the so-called plasmon term \( \propto g^3 \) in the thermodynamic potential arises from the appearance of the Debye mass in the electrostatic propagator. Only the Matsubara zero modes are able to contribute an odd power of the Debye mass (and thus an odd power in the coupling) through ring resummation according to
\[ P_3 = -N_g T \int \frac{d^3k}{(2\pi)^3} \left\{ \log \left( 1 + \frac{\tilde{m}_D^2}{k^2} \right) - \frac{\tilde{m}_D^2}{k^2} \right\}, \]
\[ = N_g \tilde{m}_D^3 T/12\pi. \]
longer suffices to resum the Debye mass for zero
modes. Instead, there are soft as well as hard
contributions to order \( g^3 \).

At soft momenta, these contributions come
from HTL propagators and self-energies in the
form (for pure glue and with implicit sums over
colour and polarization states)

\[
S_3^{\text{soft}} = - \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega} \left\{ \text{Im} \left[ \log(1 + D_0 \Pi) - \Pi D_0 \right] \\
- \text{Im} \bar{\Pi} \text{Re} (\bar{D} - D_0) \right\}
\]

\[
= \frac{\partial P_3}{\partial T} \bigg|_{\tilde{m}_D} + \Delta S_3,
\]

with

\[
\Delta S_3 = N_g \int \frac{d^3k}{(2\pi)^3} \frac{1}{\omega} \left\{ 2 \text{Im} \bar{\Pi} \text{Re} (\bar{D}_T - D_T^{(0)}) \\
- \text{Im} \bar{\Pi} \text{Re} (\bar{D}_L - D_L^{(0)}) \right\}
\]

\[
\equiv \Delta S_T^{(3)} + \Delta S_L^{(3)}.
\]

We found numerically that \( \Delta S_3 = 0 \) by cancella-
tions in more than 8 significant digits, without
being able to gain more fundamental insight into
the peculiar sum rule \( \Delta S_T^{(3)} = -\Delta S_L^{(3)} \), which
emerges only after carrying out both, the fre-
quency and the momentum integrations in (18).

In the pure-glue case we thus have

\[
S_3^{\text{soft}} = \frac{\partial P_3}{\partial T} \bigg|_{\tilde{m}_D} = \frac{1}{4} S_3.
\]

At hard momenta, the only possibility for con-
tributions \( \sim g^3 \) is through NLO corrections to the
spectral properties of the hard excitations. In-
deed, one can show that the remaining three
quarters of the plasmon effect are provided by

\[
S_3^{\text{hard}} = -N_g \int \frac{d^3k}{(2\pi)^3} \frac{1}{k} \frac{\partial n(k)}{\partial T} \text{Re} \delta \Pi_T (\omega = k),
\]

\[\text{Re} \delta \Pi_T (\omega = k)\] is a momentum-dependent cor-
rection \( \sim g^3 T^2 \) to the asymptotic thermal mass
of transverse gluons, and it is calculable from
standard HTL perturbation theory without be-
ing afflicted by the IR sensitivity that arises in
the imaginary part.

Clearly, the self-consistent entropy involves a
“massive” reorganization of thermal perturbation
theory. Instead of coming exclusively from a mod-
ification of the static propagators, the plasmon
effect is now spread over the entire spectral prop-
erties of quasiparticles as given by HTL propaga-
tors at soft momenta and HTL masses plus NLO
corrections at hard momenta. While this seems
to be just an extravagant complication from the
point of view of perturbation theory, with regard
to the self-consistent entropy it is gratifying that
the plasmon effect is encoded in the spectral data
of (HTL) quasiparticles in a more uniform man-
ner, with the bulk of the plasmon effect coming in
fact from the dominant hard degrees of freedom.

All this can be extended in a straightforward
manner to \( N_f \neq 0 \). In the case of finite chemical
potential but high enough temperature \( T \gg \tilde{m}_D \),
one finds that more than 1/4 of \( S_3 \) is coming from
the soft momentum region, whereas all of the
plasmon term in the density, \( N_3 \), is due to NLO
corrections of the asymptotic thermal fermion
masses.

3. Nonperturbative usage

As mentioned above, conventional thermal per-
turbation theory exhibits very poor apparent con-
vergence as soon as collective phenomena such
as the plasmon effect are included, which involve
odd powers in the coupling. The self-consistent
entropy and density functionals at two-loop or-
der also contain the plasmon effect, but together
with higher-order terms that ordinarily would
have been discarded by a strictly perturbative
expansion in \( g \) because otherwise the renormal-
ization programme could not have been carried
out. By contrast, the two-loop entropy and den-
sity functional is UV finite and can be evalu-
ated in a nonperturbative manner using the high-
temperature/density approximation of the gluon
and quark propagators, which are finite at lead-
ing and next-to-leading order in HTL perturba-
tion theory.

3.1. HTL/HDL approximation

As a first approximation let us consider the
two-loop entropy and density functionals evalu-
ated completely using HTL (HDL) propagators.
3.1.1. Entropy

In the case of the entropy, this takes care of all contributions of order $g^2$, but only part (1/4) of the plasmon term $\sim g^3$. However, it also contains infinitely many higher-order terms which despite being incomplete may help to get rid of the pathological behaviour of the perturbation series truncated at low orders in $g$. Among such higher-order contributions is for instance a $g^4$ contribution involving a $g^4 \log(1/g)$-term, whose coefficient in pure-gluon QCD is $1/12$ of that of the complete perturbative result. By contrast, simple massive quasiparticle models such as those used in Refs. 1, 2 do not have a $g^4 \log(1/g)$-term in the entropy at all.

In Fig. 1 the numerical result for $S_{HTL}/S_{SB}$ in the case of pure glue is given as function of $m_D/T$, which is the only independent parameter. The HTL result is given by the full line and is found to be a monotonically decreasing function of $m_D/T$. If this were expanded in powers of $m_D/T$ and truncated beyond $(\tilde{m}_D/T)^3 \sim g^3$ (dashed line in Fig. 1), this property would have been lost at $m_D/(2\pi T) \approx 1/3$, where one might still expect a sufficiently clear separation of hard and soft scales which is a prerequisite of the HTL approximation.

The numerical result for $S_{HTL}$ is in fact very close to a simple massive quasiparticle model with entropy $2N_g S_{SB}(m_\infty)$, represented by the dotted line in Fig. 1. Remarkably, one has $S_{HTL} < 2N_g S_{SB}(m_\infty)$ even though the latter contains 30\% less of the plasmon term $\sim g^3$, which would be expected to work in the opposite direction. This is further emphasized by comparison with its perturbative approximation given by the dash-dotted line.

The rather small difference to a simple massive quasiparticle model is further inspected in Fig. 1 where it is shown that there are large cancellations between the longitudinal and transverse contributions to the HTL entropy.

In Fig. 2 the HTL entropy of 1 quark degree of freedom at zero chemical potential is displayed as a function of the fermionic plasma frequency $\tilde{M}/T$. Like in the pure-gluon case, this contains also an (incomplete) $g^4 \log(1/g)$ contributions that is not present in simpler quasiparticle models:

$$S_{f,HTL}^{(4)} = NN_f \frac{\tilde{M}^4}{\pi^2 T} \left( \log \frac{T}{\tilde{M}} + 0.22 \ldots \right)$$

$S_{f,HTL}$ does not contain any contributions to the plasmon term $\sim g^3$, which entirely come from NLO corrections to $M_\infty$. There is therefore no big deviation of the full numerical result (solid line) from the perturbative one truncated beyond $g^2$ (dashed line).

Compared to the entropy of a simple massive fermionic quasiparticle model $S_{f}^{(0)}(M_\infty)$ (dotted line in Fig. 2), one finds extremely good numerical agreement, which again takes place only after

\[ S_{HTL}= -N_g \frac{\tilde{m}_D^4}{16\pi^2 T} \left( \log \frac{T}{\tilde{m}_D} + 1.55 \ldots \right) \]
adding up all the quasiparticle ((+) and (−)) and Landau-damping contributions, however.

3.1.2. Quark density
In the case of zero temperature but high chemical potential, the quantity of interest is the quark density. This does not contain any plasmon term ∼g³, but rather ∼g⁴log(1/g). The HDL approximation does contain some though not all of this term:

\[ \mathcal{N}^{(4)}_{HDL} = NN \frac{M^4}{\pi^2 \mu} \left( \log \frac{\mu}{M} + 0.35 \ldots \right) \] (23)

Order-g²log g contributions to the asymptotic masses of the quark and gluon quasiparticles, still within the framework of the 2-loop Φ-derivable approximation, are responsible for the remaining contribution to the coefficient of the g⁴log g-term, while the coefficient under the logarithm also receives 3-loop contributions.

In Fig. 3 the numerical result for \( \mathcal{N}_{HDL} \) is given as a function of \( M/\mu \) (full line) and compared to that of a simple quasiparticle model

\[ \mathcal{N}_0(M_\infty) \bigg|_{T=0} = \frac{1}{3\pi^2} (\mu^2 - M^2_\infty)^2 \theta(\mu - M_\infty) \] (24)

as well as to a perturbative approximation truncated beyond \( (M/\mu)^2 \sim g^2 \). Remarkably, the full HDL result drops to zero at almost the same value as the simple quasiparticle model. However, the former becomes negative thereafter, showing that the HDL approximation is breaking down at \( M_\infty/\mu > 1 \) at the latest.

3.2. Next-to-leading approximations
The plasmon term ∼g³ at high temperatures \( T \gg m_\rho \) becomes complete only after inclusion of the next-to-leading correction to the asymptotic thermal masses \( m_\infty \) and \( M_\infty \). These are determined in standard HTL perturbation theory through

\[ \delta m^2_\infty(k) = \text{Re} \delta \Pi_T(\omega = k) \]
\[ = \text{Re} (\ldots + \ldots + \ldots + \ldots)|_{\omega = k} \] (25)

where thick dashed and wiggly lines with a blob represent HTL propagators for longitudinal and transverse polarizations, respectively. Similarly,

\[ \frac{1}{N} \delta M^2_\infty(k) = \delta \Sigma_+(\omega = k) \]
\[ = \text{Re} (\ldots + \ldots)|_{\omega = k} \] (26)

The explicit proof that these contributions indeed restore the correct plasmon term is given in
The free quark density of a quark with constant perturbative order-to-its free value (solid line), the corresponding \( \delta m \) functions contribute in the averaged form \( \delta m^2_{\text{pl.}}/\bar{m}_{\text{pl.}}^2 \approx -0.18 \sqrt{N} g \), which is only about a third of \( \delta m^2_{\infty}/\bar{m}_{\infty}^2 \); the NLO correction to the nonabelian Debye mass on the other hand is even positive for small coupling and moreover logarithmically enhanced \( \delta m^2_{\text{pl.}}/\bar{m}_{\text{pl.}}^2 = +\sqrt{3N}/(2\pi) \times g \log(1/g) \).

For an estimate of the effects of a proper incorporation of the next-to-leading order corrections we have therefore proposed to include the latter only for hard excitations and to define our next-to-leading approximation (for gluons) through

\[
S_{\text{NLA}} = S_{\text{HTL}} \bigg|_{\text{soft}} + S_{\bar{m}_{\infty}} \bigg|_{\text{hard}},
\]

where \( \bar{m}_{\infty} \) includes (29). To separate soft \( (k \sim \bar{m}_D) \) and hard \( (k \sim 2\pi T) \) momentum scales, we introduce the intermediate scale \( \Lambda = \sqrt{2\pi T \bar{m}_{\text{BE}}} \) and consider a variation of \( c_\Lambda = \frac{1}{2} \ldots 2 \) as part of our theoretical uncertainty.

Another crucial issue concerns the definition of the corrected asymptotic mass \( \bar{m}_{\infty} \). For the range of coupling constants of interest \( (g > 1) \), the correction \( \bar{\delta m}_{\infty}^2 \) is greater than the LO value \( \bar{m}_{\infty}^2 \), leading to tachyonic masses if included in a strictly perturbative manner.

However, this problem is not at all specific to QCD. In the simple \( g^2 \varphi^4 \) model, one-loop resummed perturbation theory gives

\[
m^2 = g^2 T^2 (1 - \frac{3}{\pi}g)
\]

which also turns tachyonic for \( g > 1 \). On the other hand, the solution of the corresponding simple scalar gap equation is a monotonic function in \( g \), and it turns out that the first two terms in a \( (m/T) \)-expansion of this gap equation,

\[
m^2 = g^2 T^2 - \frac{3}{\pi}g^2 T \bar{m}
\]

which is perturbatively equivalent to (32) has a solution that is extremely close to that of the full gap equation (for \( \overline{\text{MS}} \) renormalization scales \( \bar{\mu} \approx 2\pi T \)).

In QCD, the (non-local) gap equations are way too complicated to be attacked directly. We instead consider perturbatively equivalent expressions for the corrected \( \bar{m}_{\infty} \) which are monotonic functions in \( g \). Besides the solution to a quadratic

\[
\delta m^2_{\text{pl.}}/\bar{m}_{\text{pl.}}^2 \approx -0.18 \sqrt{N} g
\]

of the plasmon term is concerned, these corrections are known to differ substantially from (29) or (31). For instance, the gluonic plasma frequency at \( k = 0 \) reads \( \bar{m}_{\text{pl.}}^2/\bar{m}_{\text{pl.}}^2 \approx (\sqrt{3N}/(2\pi) \times g \log(1/g)).

Figure 4. HDL quark density at \( T = 0 \) normalized to its free value (solid line), the corresponding perturbative order-\( g^2 \) result (dashed line), and the free quark density of a quark with constant mass \( M_{\infty} \) (dotted line).

Ref. [1].

These corrections to the asymptotic thermal masses are, in contrast to the latter, nontrivial functions of the momentum, which can be evaluated only numerically. However, as far as the generation of the plasmon term is concerned, these functions contribute in the averaged form

\[
\bar{\delta m}^2_{\infty} = \frac{\int dk k n'_{\text{BE}}(k) \text{Re} \delta \Pi_T(\omega = k)}{\int dk k n''_{\text{BE}}(k)} \quad (27)
\]

(cp. eq. [1]) and similarly

\[
\bar{\delta M}^2_{\infty} = \frac{\int dk k n'_{\text{FD}}(k) \text{Re} 2k \delta \Sigma(\omega = k)}{\int dk k n''_{\text{FD}}(k)} \quad (28)
\]

These averaged asymptotic thermal masses turn out to be given by the remarkably simple expressions

\[
\bar{\delta m}^2_{\infty} = -\frac{1}{2\pi} g^2 N T \bar{m}_D, \quad (29)
\]

\[
\bar{\delta M}^2_{\infty} = -\frac{1}{2\pi} g^2 C_f T \bar{m}_D, \quad C_f = N_g/(2N). \quad (30)
\]

These corrections only pertain to the hard excitations; corrections to the various thermal masses of soft excitations are known to differ substantially from (31) or (32). For instance, the gluonic plasma frequency at \( k = 0 \) reads \[ \bar{m}_{\text{pl.}}^2/\bar{m}_{\text{pl.}}^2 \approx (\sqrt{3N}/(2\pi) \times g \log(1/g)).

For an estimate of the effects of a proper incorporation of the next-to-leading order corrections we have therefore proposed to include the latter only for hard excitations and to define our next-to-leading approximation (for gluons) through

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\[
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\]
equation analogous to (33) we have tried the simplest Padé approximant $m^2 = g^2 T^2 / (1 + \frac{3}{2} g)$, which also gives a greatly improved approximation to the solution of scalar gap equations. In QCD, our final results do not depend too much on whether we use the Padé approximant [5,6] or a quadratic gap equation [7].

The main uncertainty rather comes from the choice of the renormalization scale which determines the magnitude of the strong coupling constant when this is taken as determined by the renormalization group equation (2-loop in the following).

In Fig. 5, the numerical results for the HTL entropy and the NLA one are given as a function of $T/T_c$ with $T_c$ chosen as $T_c = 1.14 \Lambda_{\overline{MS}}$. The full lines show the range of results for $S_{HTL}$ when the renormalization scale $\mu$ is varied from $\pi T$ to $4\pi T$; the dash-dotted lines mark the corresponding results for $S_{NLA}$ with the additional variation of $c_\Lambda$ from $1/2$ to $2$. The dark-gray band are lattice data from Ref. [20]; the more recent results from Ref. [21] are consistent with the former within error bars and centered around the upper boundary of the gray band for $T \approx 3T_c$ and somewhat flatter around $2T_c$. Evidently, there is very good agreement for $T > 2.5T_c$.

Figure 5. Comparison of the lattice data for the entropy of pure-glue SU(3) gauge theory of Ref. [20] (gray band) with the range of $S_{HTL}$ (solid lines) and $S_{NLA}$ (dash-dotted lines) for $\mu = \pi T \ldots 4 \pi T$ and $c_\Lambda = 1/2 \ldots 2$.

In Fig. 6, $N_f$ massless quarks are included and compared with a recent estimate [22] of the continuum limit of lattice results for $N_f = 2$ (gray band), but now with $S_{HTL}$ and $S_{NLA}$ evaluated for the central choice of $\mu = 2\pi T$ and $c_\Lambda = 1$ (with unchanged $T_c/\Lambda_{\overline{MS}}$). When $N_f$ is increased, there are competing effects of larger thermal masses versus slower running of $\alpha_s$, which result into a rather weak dependence of our results on $N_f$ as a function of $T/\Lambda_{\overline{MS}}$ as it is in Fig. 6.

From the above results for the entropy density, one can recover the thermodynamic pressure by simple integration,

$$P(T) - P(T_0) = \int_{T_0}^T dT' S(T').$$

(34)

The integration constant $P(T_0)$, however, is a strictly nonperturbative input. It cannot be fixed by requiring $P(T = 0) = 0$, as this is in the confinement regime. It is also not sufficient to know that $\lim_{T \to \infty} P = P_{\text{free}}$ by asymptotic freedom. In fact, the undetermined integration constant in $P(T)/P_{\text{free}}(T)$ when expressed as a function of $\alpha_s(T)$ corresponds to a term $C \exp \left(-\frac{1}{\alpha_s} [4\beta_0^{-1} + O(\alpha_s)] \right)$

(35)
Figure 7. Comparison of our results for the pressure of QCD with \( N_f = 2 \) with the extrapolated lattice data of Ref. [22].

which vanishes for \( \alpha_s \rightarrow 0 \) with all derivatives and thus is not fixed by any order of perturbation theory. It is, in essence, the nonperturbative bag constant, which can be added on to standard perturbative results, too. However, in \( P(T)/P_{\text{free}}(T) \) this term becomes rapidly unimportant as the temperature is increased, as it decays like \( T^{-4} \).

In Fig. 7 which shows the HTL and NLA pressure in the case of \( N_f = 2 \), we have fixed this integration constant by choosing \( P(T_c) = 0 \) as \( T = T_c \) is beyond the range of applicability of any form of perturbation theory anyway. As with the entropy, there is good agreement with lattice data for \( T > 2.5T_c \), and this agreement does not depend on the precise value of \( P(T_c) \), which enters only at the percent level at such temperatures.

4. Discussion and outlook

Starting from the self-consistent two-loop approximations to entropy and density, we have been able to give a quasiparticle description of the thermodynamics of QCD that agrees well with available lattice data for \( T > 2.5T_c \), and which incorporate more of the perturbatively accessible details of gluonic and fermionic quasiparticles than previously proposed quasiparticle models \([16,17,23]\) involving simply free massive scalars and fermions. In contrast to the latter, however, we refrained from introducing phenomenological functions to fit the data, the only input being \( T_c/\Lambda_{\text{MS}} \) as gleaned from the lattice. Put conservatively, we obtained a lower bound on \( T/T_c \) for which HTL quasiparticles may be an appropriate description of hot QCD. This lower bound is however extremely low when compared with what is needed for conventional thermal perturbation theory to avoid an obvious breakdown, raising the hope that the latter just needs further resummations to become applicable at temperatures of physical interest in QCD.

There exist in fact alternative proposals for such resummations. One particularly interesting approach is an extension of so-called screened perturbation theory \([24,25]\), where thermal masses are introduced as variational parameters, to non-abelian gauge theories, using the HTL effective action as gauge invariant thermal mass term \([26]\).

It should be noted, however, that the latter is introduced as a mere technical device, whereas in the entropy-based approach the emphasis is on a description in terms of HTL quasiparticles and their perturbatively accessible modifications at NLO. Indeed, in the two-loop entropy it turned out that at hard momentum scales only the region close to the light-cone matters, where the HTL approximations remain accurate even at hard momenta. By contrast, in HTL-screened perturbation theory, the HTL's contribute throughout all of phase space. Moreover, the thus modified UV structure entails new UV problems (together with new scheme dependences).

To avoid these UV problems, Peshier \([27]\) has proposed a somewhat contrived way of HTL resummation directly on the thermodynamic pressure, which appears to come very close to what we have obtained in the HTL approximation in the entropy. Peshier's pressure is perturbatively equivalent to ours in the HTL approximation through at least order \( g^3 \), though not obviously identical to it.

Beyond the HTL approximation, it is not clear how the formulae of Ref. \([27]\) could be extended. In the present HTL-based quasiparticle approach, the aim is to describe these quasiparticles in as full detail as HTL perturbation theory permits. In this respect we have so far taken only a first
step by considering averaged asymptotic thermal masses. We intend to study the full momentum dependence of the asymptotic thermal masses and how this enters into the thermodynamic quantities in a future work.

Another extension of the present work will be to consider in more detail the dependence on chemical potential, for example by integrating entropy and density for all $T$ and $\mu$ in the deconfined phase, similarly to what has been carried out in simple quasiparticle models \[28\]. Using the NLA, this will include a complete plasmon term, and requires only one integration constant, $P(T_0)|_{\mu=0}$, that can be taken from the lattice.

A first (infinitesimal) step towards finite $\mu$ can be studied on the lattice in the form of quark number susceptibilities $\partial N/\partial \mu|_{\mu=0}$ at high $T$ \[29,30\]. Here conventional thermal perturbation theory is as ill-behaved as in the case of the pressure, whereas our approach can be applied in a straightforward manner, as will be presented in a forthcoming paper \[31\].

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