Harmonic oscillator with minimal length, minimal momentum, and maximal momentum uncertainties in SUSYQM framework

M. Asghari, a P. Pedram, a and K. Nozari b

aDepartment of Physics, Science and Research Branch, Islamic Azad University, Tehran, Iran
bCenter for Excellence in Astronomy and Astrophysics (CEAAI-RIAAM)-Maragha, P. O. Box: 55134–441, Maragha, Iran

May 11, 2014

Abstract

We consider a Generalized Uncertainty Principle (GUP) framework which predicts a maximal uncertainty in momentum and minimal uncertainties both in position and momentum. We apply supersymmetric quantum mechanics method and the shape invariance condition to obtain the exact harmonic oscillator eigenvalues in this GUP context. We find the supersymmetric partner Hamiltonians and show that the harmonic oscillator belongs to a hierarchy of Hamiltonians with a shift in momentum representation and different masses and frequencies. We also study the effect of a uniform electric field on the harmonic oscillator energy spectrum in this setup.

Keywords: Harmonic Oscillator; Generalized Uncertainty Principle; Supersymmetric Quantum Mechanics.

1 Introduction

The assumption of the continuity of spacetime manifold may be broken at high energy limit. In this limit the effects of gravity are so important that would result in discreetness of the spacetime. In the context of quantum gravity, the Heisenberg uncertainty principle should be modified to the so-called Generalized Uncertainty Principle (GUP). This generalization leads to a nonzero minimal uncertainty in position measurements. In ordinary quantum mechanics we can make $\Delta x$ arbitrarily small by letting $\Delta p$ grow correspondingly, but in the GUP framework there exists a nonzero and minimal position uncertainty.

Various candidates of quantum gravity such as the string theory, loop quantum gravity, and quantum geometry [1-23] all indicate the existence of a minimal measurable length of the order of the Planck length $\ell_{pl} = \sqrt{\frac{G \hbar}{c^3}} \approx 10^{-35}$ m. Kempf et al. proposed a GUP proposal that implies a minimal length and length $\ell_{pl} = \sqrt{\frac{G \hbar}{c^3}} \approx 10^{-35}$ m. Kempf et al. proposed a GUP proposal that implies a minimal length and

*Email: p.pedram@srbiau.ac.ir
constructed the Hilbert space representation of quantum mechanics in this GUP framework \[12\]. On the other hand, since the curvature of spacetime is important at large distances and the notion of the plane wave does not hold on the curved spacetime, the existence of a nonzero minimal uncertainty in momentum is also inevitable \[23\]. The idea of a maximal observable momentum can also be incorporated into this scenario based on the Doubly Special Relativity (DSR) theories where the Planck energy (Planck momentum) is considered as an additional invariant other than the velocity of light \[25\]–\[27\]. Ali, Das and Vagenas have recently proposed the following commutation relation between position and momentum operators that implies minimal length and maximal momentum uncertainties:

$$[x, p] = \frac{i}{\hbar}(1 + 2\bar{\alpha}^2 p^2 - \bar{\alpha} p),$$  \hspace{1cm} (1)

where its Hilbert space representation is also constructed in Ref. \[23\].

Supersymmetric Quantum Mechanics (SUSYQM) is an application of the idea of supersymmetry to quantum mechanics. Supersymmetry proposes that to each fermion there exists a boson and vice versa. So, we can think of supersymmetry as proposing that a symmetry exists between bosons and fermions, and that in nature, there are equal numbers of fermion and boson states. SUSYQM involves pairs of Hamiltonians which share a particular mathematical relationship, which are called partner Hamiltonians. For every eigenstate of one Hamiltonian, its partner Hamiltonian has a corresponding eigenstate with the same energy (except possibly for zero energy eigenstates). This fact can be exploited to deduce many properties of the eigenstate spectrum. It is analogous to the original description of SUSY, which referred to bosons and fermions. One can imagine a \textit{bosonic Hamiltonian}, whose eigenstates are the various bosons of the theory. The SUSY partner of this Hamiltonian would be \textit{fermionic}, and its eigenstates would be the theory’s fermions. Each boson would have a fermionic partner of equal energy but, in the relativistic world, energy and mass are interchangeable, so we can just as easily say that the partner particles have equal mass. For more details and also machinery of SUSYQM see Refs. \[28\]–\[29\].

The problem of the harmonic oscillator has attracted much attention in various GUP frameworks. In Ref. \[12\] this problem is exactly solved in the presence of a minimal length. Quesne and Tkachuk studied
this problem with nonzero uncertainties in both position and momentum in the supersymmetric framework \[30\]. We can use SUSYQM to construct exact solutions of many quantum mechanical problems. A certain class of exactly solvable potentials have a property known as the shape invariance \[31–33\]. The potentials that their SUSY partner has the same spatial dependence with possibly altered parameters are shape invariant potentials. If a potential satisfies this condition, we can obtain the energy eigenvalues and eigenfunctions without solving the differential equation \[34, 35\]. The SUSYQM method plus the shape invariance is related to the factorization method developed by Schrödinger, Infeld and Hull \[36\].

In this Letter, we apply SUSYQM method and the notion of the shape invariance on the eigenvalue problem of the harmonic oscillator in a GUP framework that predicts maximal uncertainty in momentum and minimal uncertainties in both position and momentum. In this case \(x\) and \(p\) satisfy the following modified commutation relation

\[
[x,p] = i\hbar(1 + \bar{\rho}x^2 + 2\bar{\alpha}^2p^2 - \bar{\alpha}p),
\]

with \(\bar{\rho} \geq 0\), \(\bar{\alpha} \geq 0\), and \(\bar{\rho}\bar{\alpha} < \hbar^{-2}\). We obtain the supersymmetric partner Hamiltonian of the harmonic oscillator and show that the harmonic oscillator belongs to a hierarchy of Hamiltonians with a shift in momentum representation and with different masses and frequencies. Finally we study the effect of a uniform electric field on the harmonic oscillator energy spectrum in this setup.

2 One-dimensional harmonic oscillator in the GUP framework

In this section, using the SUSYQM method, we obtain the harmonic oscillator energy spectrum in the GUP framework which implies maximal uncertainty in momentum and minimal uncertainties in both position and momentum. First let us introduce dimensionless position and momentum operators, \(X = x/a\) and \(P = pa/\hbar\), where \(a = \sqrt{\hbar/(m\omega)}\) is known as the oscillator’s length. These dimensionless operators, satisfy the following deformed commutation relation

\[
[X, P] = i(1 + \rho X^2 + 2\alpha^2P^2 - \alpha P),
\]

where \(\rho = \bar{\rho}\hbar/(m\omega)\) and \(\alpha = \bar{\alpha}\sqrt{m\hbar\omega}\), are the dimensionless parameters.
Notice that, minimal length, minimal momentum and maximal momentum all stand for the uncertainties. These are not a test particle’s characteristics, but these are limitations on measurement of position or momentum for a test particle. In other words, we do not suppose a test particle that has minimal momentum and maximal momentum at the same time. Minimal momentum and maximal momentum are natural cutoffs on particle’s momentum measurement. Indeed, a test particle’s momentum cannot be less than the minimal momentum and cannot be larger than the maximal momentum. Note that it is supposed nontrivially that minimal uncertainty in momentum measurement leads to the existence of a minimal momentum for a test particle. Also a test particle’s momentum cannot be arbitrarily imprecise and therefore there is an upper bound for momentum fluctuations. We suppose, as a nontrivial assumption, that this upper bound for momentum fluctuation defines a maximal measurable momentum for a test particle. So, we can define a GUP that contains all possible natural cutoffs as the minimal length, minimal momentum and maximal momentum as bounds on measurement of position and momentum. This does not mean that a particle has these features simultaneously, but it means that measurement of position and momentum for a test particle is bounded to these natural cutoffs. From the mathematical grounds, the relation (3) gives these natural cutoffs simultaneously.

The expression

\[
h = \frac{H}{\hbar \omega} = \frac{1}{2} (P^2 + X^2),
\]

is a harmonic oscillator Hamiltonian in terms of the dimensionless parameters with the following eigenvalue problem

\[
h |\psi_n\rangle = e_n |\psi_n\rangle, \quad n = 0, 1, 2, ...
\]

where \( e_n = E_n / \hbar \omega \). In order to obtain the energy eigenvalues \( e_n \), first we show that the Hamiltonian is factorizable, then we prove that this factorized Hamiltonian satisfies the shape invariance condition. Now, we try to factorize the Hamiltonian as

\[
h_0 = B^+(g_0, s_0, \nu_0)B^-(g_0, s_0, \nu_0) + \epsilon_0,
\]
where
\[ B^\pm (g_0, s_0, \nu_0) = \frac{1}{\sqrt{2}} \left( s_0 X \mp ig_0 P \mp i\nu_0 \right), \tag{7} \]
and \( \epsilon_0 \) is the factorization energy. Note that a new parameter namely \( \nu_0 \), is introduced in the definition of \( B^\pm \) due to the existence of an additional momentum term in the modified commutation relation. By inserting \( B^\pm \) into the factorized Hamiltonian (6), we obtain the expression
\[ h_0 = \frac{1}{2} \left( (g_0^2 - 2\alpha^2 s_0 g_0)P^2 + (s_0^2 - \rho s_0 g_0)X^2 + (\alpha s_0 g_0 + 2\nu_0 g_0)P - s_0 g_0 + \nu_0^2 \right) + \epsilon_0. \tag{8} \]
So Eq. (4) implies the following four conditions:
\[ g_0^2 - 2\alpha^2 s_0 g_0 = 1, \tag{9} \]
\[ s_0^2 - \rho s_0 g_0 = 1, \tag{10} \]
\[ \alpha s_0 g_0 + 2\nu_0 g_0 = 0, \tag{11} \]
\[ \frac{1}{2} g_0 s_0 - \frac{1}{2} \nu_0^2 = \epsilon_0. \tag{12} \]
The solutions of the above equations are given by
\[ g_0 = s_0 k, \quad s_0 = \frac{1}{\sqrt{1 - \rho k}}, \quad \nu_0 = -\frac{\alpha}{2} s_0, \tag{13} \]
where
\[ k = \frac{1}{2}(2\alpha^2 - \rho) + \sqrt{1 + \frac{1}{4}(2\alpha^2 - \rho)^2}. \tag{14} \]
Therefore, we can write the dimensionless Hamiltonian \( h \) in the form of Eq. (6) with the factorization energy that is obtained in Eq. (12). We write a hierarchy of Hamiltonians as
\[ h_i = B^+(g_i, s_i, \nu_i)B^-(g_i, s_i, \nu_i) + \sum_{j=0}^{i} \epsilon_j, \tag{15} \]
where \( i = 0, 1, 2, ... \). In order to obtain the parameters \( s_i, g_i \) and \( \nu_i \), the shape invariance condition should be satisfied
\[ B^-(g_i, s_i, \nu_i)B^+(g_i, s_i, \nu_i) = B^+(g_{i+1}, s_{i+1}, \nu_{i+1})B^-(g_{i+1}, s_{i+1}, \nu_{i+1}) + \epsilon_{i+1}. \tag{16} \]
After inserting $B^\mp$ in the above equation and using the commutation relation, we have

\[
\frac{1}{2}[(g^2_i + 2\alpha^2 s_i g_i)P^2 + (s^2_i + \rho s_i g_i)X^2 + (2\nu_i g_i - \alpha s_i g_i)P + s_i g_i + \nu_i^2] = \frac{1}{2}[(g^2_{i+1} - 2\alpha^2 s_{i+1} g_{i+1})P^2 + (s^2_{i+1} - \rho s_{i+1} g_{i+1})X^2 + (2\nu_{i+1} g_{i+1} + \alpha s_{i+1} g_{i+1})P - s_{i+1} g_{i+1} + \nu_{i+1}^2] + \epsilon_{i+1}. \tag{17}
\]

So we find four additional conditions as:

\[
g^2_{i+1} - 2\alpha^2 s_{i+1} g_{i+1} = g^2_i + 2\alpha^2 s_i g_i, \tag{18}
\]

\[
s^2_{i+1} - \rho s_{i+1} g_{i+1} = s^2_i + \rho s_i g_i, \tag{19}
\]

\[
2\nu_{i+1} g_{i+1} + \alpha s_{i+1} g_{i+1} = 2\nu_i g_i - \alpha s_i g_i, \tag{20}
\]

\[
\frac{1}{2}(s_i g_i + s_{i+1} g_{i+1}) + \frac{1}{2}(s^2_i - s^2_{i+1}) = \epsilon_{i+1}. \tag{21}
\]

If we multiply Eqs. (18) and (19) respectively by $\rho$ and $2\alpha^2$, we obtain

\[
g^2_{i+1} - \gamma^2 s^2_{i+1} = g^2_i - \gamma^2 s^2_i, \tag{22}
\]

where $\gamma \equiv \sqrt{\frac{2\alpha^2}{\rho}}$. Now it is useful to combine the parameters $g_i$ and $s_i$ and introduce new parameters

\[
u_i = g_i + \gamma s_i, \quad v_i = g_i - \gamma s_i, \tag{23}
\]

Their inverse transformations are

\[
g_i = \frac{1}{2}(u_i + v_i), \quad s_i = \frac{1}{2\gamma}(u_i - v_i), \tag{24}
\]

and the assumption $g_i, s_i > 0$ implies $u_i > |v_i|$. Thus, using Eqs. (18) and (19), we obtain

\[
u^2_{i+1} + q v^2_{i+1} = u^2_i + q u^2_i, \tag{25}
\]

\[
u_{i+1} v_{i+1} = u_i v_i, \tag{26}
\]

where

\[
q = \frac{1 + \sqrt{2\alpha^2 \rho}}{1 - \sqrt{2\alpha^2 \rho}}. \tag{27}
\]

Based on Eqs. (25) and (26) let us introduce new parameters $f_i$ and $t_i$ as

\[
f_i = u_i v_i, \quad t_i = \frac{v_i}{u_i}, \tag{28}
\]
where they have the same sign as \( v_i \), and \( |t_i| < 1 \). Also we have

\[
f_i = f_0, \quad t_i = q^{-i} t_0, \quad u_i = q^{i/2} u_0, \quad v_i = q^{-i/2} v_0.
\] (29)

Using Eq. (20) and \( \nu_0 \) as given in Eq. (13) we obtain

\[
\nu_{i+1} = \frac{1}{g_{i+1}} \left[ -\alpha s_0 g_0 - \alpha \left( \sum_{j=1}^{i} s_j g_j + \frac{1}{2} s_{i+1} g_{i+1} \right) \right].
\] (30)

So we also have

\[
\nu_i = -\frac{\alpha}{g_i} \left( \sum_{j=0}^{i-1} s_j g_j + \frac{1}{2} s_i g_i \right).
\] (31)

Now Eqs. (24) and (29) lead to

\[
\nu_i = -\frac{2\alpha}{q^{i/2} u_0 (1 + \frac{t_0^2}{q^n})^2} \left\{ \frac{u_0^2}{4\gamma} \left[ \left( 1 - \frac{t_0^2}{q^n} \right) q_i + \frac{1}{2} \left( q_i - \frac{t_0^2}{q^n} \right) \right] \right\},
\] (32)

where we used the following definition

\[
[i]_q = \frac{q^i - 1}{q - 1}.
\] (33)

We write the eigenvalues of the Hamiltonian as

\[
e_n(q, t_0) = \sum_{i=0}^{n} \epsilon_i = \sum_{i=0}^{n-1} \epsilon_{i+1} + \epsilon_0 = \frac{1}{2} \sum_{i=0}^{n-1} (s_i g_i + s_{i+1} g_{i+1}) + \frac{1}{2} \sum_{i=0}^{n-1} (\nu_i^2 - \nu_{i+1}^2) + \epsilon_0.
\] (34)

By writing the first term in the right-hand side of the above equation as

\[
\sum_{i=0}^{n-1} s_i g_i - \frac{1}{2} \sum_{i=0}^{n-1} (s_i g_i - s_{i+1} g_{i+1}),
\] (35)

and using the relation \( \sum_{k=0}^{n-1} (a_k - a_{k+1}) = a_0 - a_n \), we obtain

\[
e_n(q, t_0) = \sum_{i=0}^{n-1} s_i g_i + \frac{1}{2} s_n g_n - \frac{1}{2} \nu_n^2.
\] (36)

Finally, after inserting the explicit values of the parameters, we find the energy eigenvalues as

\[
e_n(q, t_0) = \frac{u_0^2}{4\gamma} \left\{ \left( 1 - \frac{t_0^2}{q^n - 1} \right) [n]_q + \frac{1}{2} \left( q^n - t_0^2 \right) \right\} \left\{ 1 - \frac{2\alpha^2}{q^n u_0^2 (1 + \frac{t_0^2}{q^n})^2} \frac{u_0^2}{4\gamma} \left[ \left( 1 - \frac{t_0^2}{q^n - 1} \right) [n]_q + \frac{1}{2} \left( q^n - \frac{t_0^2}{q^n} \right) \right] \right\}.
\] (37)

The first term in the above equation agrees with the harmonic oscillator energy spectrum in the presence of the minimal uncertainties in both position and momentum measurements \[30]. Moreover, the shift in the energy spectrum due to the minimal length, minimal momentum, and maximal momentum is smaller than the shift in the presence of minimal length and minimal momentum.
2.1 Special Case (limit $\rho \to 0$)

Now we consider a special case where one of the GUP parameters, namely $\rho$ which corresponds to the minimal momentum tends to zero. In this case, we only have the minimal length and maximal momentum uncertainties [25]

$$[X, P] = i(1 + 2\alpha^2 P^2 - \alpha P).$$ \hfill (38)

Also, at this limit, $q$ takes the form

$$q \simeq 1 + 2\sqrt{2\alpha^2\rho} + O(\rho),$$ \hfill (39)

and

$$q^n \simeq 1 + 2n\sqrt{2\alpha^2\rho} + O(\rho), \quad [n]_q \simeq n + O(\sqrt{\rho}).$$ \hfill (40)

The terms in the brackets in the energy spectrum (37) become

$$\frac{1}{4\gamma} \left( u_0^2 - \frac{v_0^2}{q^{n-1}} \right) \simeq g_0 s_0 + \frac{1}{2}(2\alpha^2)s_0^2(n-1) + O(\sqrt{\rho}),$$ \hfill (41)

$$\frac{1}{8\gamma} \left( \frac{u_0^2 q^n - v_0^2}{q^n} \right) \simeq \frac{1}{2}g_0 s_0 + \frac{1}{2}(2\alpha^2)s_0^2 n + O(\sqrt{\rho}).$$ \hfill (42)

Moreover, the parameters $g_0$ and $s_0$ are given by

$$g_0 \simeq \frac{1}{2}(2\alpha^2) + \sqrt{1 + \frac{1}{4}(2\alpha^2)^2} + O(\rho), \quad s_0 \simeq 1 + O(\rho),$$ \hfill (43)

and

$$\frac{2\alpha^2}{q^n \left( u_0 + \frac{v_0}{q^n} \right)^2} \simeq \frac{2\alpha^2 \left( 1 - 2n\sqrt{2\alpha^2\rho} \right)}{\left[ 2g_0(1 - n\sqrt{2\alpha^2\rho}) + 2ns_0(2\alpha^2) \right]^2}.$$ \hfill (44)

Using these results, the energy spectrum takes the form

$$\varepsilon_n \simeq \left[ \left( n + \frac{1}{2} \right) \sqrt{1 + \frac{1}{4}(2\alpha^2)^2} + \frac{1}{2}(2\alpha^2) \left( n^2 + n + \frac{1}{2} \right) \right] \times \left\{ 1 - \frac{\alpha^2(1 - 2n\sqrt{2\alpha^2\rho})}{2 \left( \alpha^2 + \sqrt{1 + \alpha^2} \right) \left( 1 - n\sqrt{2\alpha^2\rho} \right) + 2n\alpha^2} \right\} \left[ \left( n + \frac{1}{2} \right) \sqrt{1 + \frac{1}{4}(2\alpha^2)^2} + \frac{1}{2}(2\alpha^2) \left( n^2 + n + \frac{1}{2} \right) \right],$$ \hfill (45)

where the first bracket is the energy spectrum of the harmonic oscillator in the presence of the minimal length ($\varepsilon_n^{\text{min}}$) if we replace $2\alpha^2$ with $\beta$ where $[X, P] = 1 + \beta P^2$ [30]. Notice that, the above equation can
be written as \( e_n = e_n^{\text{min}} - \Delta \) where \( \Delta \) is a positive term. Indeed, as it can be seen from figure 1, the presence of the minimal length and maximal momentum decreases the energy spectrum with respect to the existence of just the minimal length. This effect is also observed in the perturbative study of the problem [14].

![Figure 1: Comparing \( e_n \) for \( \rho = 10^{-4} \) and \( \alpha = 1 \) in three scenarios.](image)

2.2 Supersymmetric partner

Using Eq. (15) the Hamiltonians of the SUSYQM hierarchy can be written as

\[
 h_i = \frac{1}{2} \left[ (g_i^2 - 2\alpha^2 s_i g_i) P^2 + (s_i^2 - \rho g_i s_i) X^2 + (2g_i \nu_i + \alpha g_i s_i) P - g_i s_i + \nu_i^2 \right] + \sum_{j=0}^i \epsilon_j. \tag{46}
\]

It can also be expressed as

\[
 h_i = \frac{1}{2} (a_i P^2 + b_i X^2) + c_i P + d_i, \tag{47}
\]

where

\[
 a_i = g_i^2 - 2\alpha^2 g_i s_i = \frac{u_0^2}{2(q+1)} (q^i + t_0) \left( 1 + \frac{t_0}{q^{i-1}} \right), \tag{48}
\]

\[
 b_i = s_i^2 - \rho g_i s_i = \frac{u_0^2}{2\gamma^2 (q+1)} (q^i - t_0) \left( 1 - \frac{t_0}{q^{i-1}} \right), \tag{49}
\]

\[
 c_i = 2g_i \nu_i + \alpha g_i s_i = -u_0^2 \sqrt{\frac{\beta}{8}} \left( 1 - \frac{t_0^2}{q^{i-1}} \right) [i]_q, \tag{50}
\]

\[
 d_i = \sum_{j=0}^i \epsilon_j - \frac{1}{2} g_i s_i + \frac{1}{2} \nu_i^2 = \frac{u_0^2}{4\gamma} \left( 1 - \frac{t_0^2}{q^{i-1}} \right) [i]_q. \tag{51}
\]
So the Hamiltonian in terms of the variables with dimensions are given by

\[ H_i \equiv \hbar \omega_i = \frac{p^2}{2m_i} + \frac{1}{2} m_i \omega_i^2 x^2 + c_i \sqrt{\frac{\hbar \omega_i}{m_i}} p + d_i \hbar \omega_i, \quad (52) \]

where \( m_i = m/a_i \) and \( \omega_i = \sqrt{a_i b_i \omega} \). Equivalently we have

\[ H_i = \frac{1}{2m_i} \left( p + c_i \sqrt{\frac{m_i \hbar \omega_i}{a_i}} \right)^2 + \frac{1}{2} m_i \omega_i^2 x^2 - \frac{1}{2} \left( \frac{c_i^2}{a_i} - 2d_i \right) \hbar \omega_i, \quad (53) \]

Therefore, in the GUP framework, the harmonic oscillator belongs to a hierarchy of Hamiltonians with a shift in momentum space and with different masses and frequencies.

3 One-dimensional harmonic oscillator in a uniform electric field

Now let us study a particle with mass \( m \) and charge \( \bar{q} \) in the presence of the harmonic oscillator potential \( V(x) = \frac{1}{2} m \omega^2 x^2 \) and a uniform electric field \( \bar{\varepsilon} \). The Hamiltonian of this system is

\[ H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2 - \bar{q} \bar{\varepsilon} x, \quad (54) \]

which can be written in terms of the dimensionless operators as

\[ h = \frac{H}{\hbar \omega} = \frac{1}{2}(P^2 + X^2) - \varepsilon X, \quad (55) \]

where \( \varepsilon = \bar{q} \bar{\varepsilon} a / (\hbar \omega) \). Again, in order to find the exact energy spectrum we factorize the Hamiltonian as

\[ h_0 = B^+(g_0, s_0, \nu_0, r_0)B^-(g_0, s_0, \nu_0, r_0) + \epsilon_0, \quad (56) \]

where \( \epsilon_0 \) is the factorization energy and \( B^\pm \) are defined as

\[ B^\pm(g_0, s_0, \nu_0, r_0) = \frac{1}{\sqrt{2}}(s_0 X \mp ig_0 P \mp iv_0 + r_0). \quad (57) \]

Here \( r_0 \) is the new parameter that corresponds to the nonzero electric field. Now we have

\[ h_0 = \frac{1}{2} \left[ (g_0^2 - 2s_0^2 g_0 s_0) P^2 + (s_0^2 - \rho g_0 s_0) X^2 + (2g_0 \nu_0 + \alpha s_0 g_0) P + 2r_0 s_0 X - g_0 s_0 + r_0^2 + \nu_0^2 \right] + \epsilon_0. \quad (58) \]
This equation yields five conditions on the parameters which three of them are the same as Eqs. (9-11). The remaining conditions are

\begin{equation}
  r_0 s_0 = -\varepsilon, \quad (59)
\end{equation}

\begin{equation}
  \frac{1}{2} (g_0 s_0 - r_0^2 - \nu_0^2) = \varepsilon_0. \quad (60)
\end{equation}

Also the hierarchy of Hamiltonians is

\begin{equation}
  h_i = B^+(g_i, s_i, \nu_i, r_i) B^-(g_i, s_i, \nu_i, r_i) + \sum_{j=0}^{i} \epsilon_j, \quad (61)
\end{equation}

and the shape invariance condition reads

\begin{equation}
  B^- (g_i, s_i, \nu_i, r_i) B^+ (g_i, s_i, \nu_i, r_i) = B^+ (g_{i+1}, s_{i+1}, \nu_{i+1}, r_{i+1}) B^- (g_{i+1}, s_{i+1}, \nu_{i+1}, r_{i+1}) + \epsilon_{i+1}, \quad (62)
\end{equation}

Similar to the absence of the electric field, the above equation gives five conditions on the parameters which three of them are the same as Eqs. (18-20) and the other two conditions are given by

\begin{equation}
  r_{i+1} s_{i+1} = r_is_i, \quad (63)
\end{equation}

\begin{equation}
  \epsilon_{i+1} = \frac{1}{2} (s_i g_i + s_{i+1} g_{i+1}) + \frac{1}{2} (\nu_i^2 - \nu_{i+1}^2) + \frac{1}{2} (r_i^2 - r_{i+1}^2). \quad (64)
\end{equation}

Now following Ref. [37] parameters \( r_i \) read

\begin{equation}
  r_i = \frac{-2\gamma \varepsilon}{u_0} q^{-i/2} \left( 1 - \frac{t_0}{q^i} \right)^{-1}. \quad (65)
\end{equation}

In addition, Eq. (64) gives the energy spectrum as

\begin{equation}
  e_n(q, t_0, \varepsilon) = \sum_{i=0}^{n} \epsilon_i = \sum_{i=0}^{n-1} s_i g_i + \frac{1}{2} s_n g_n - \frac{1}{2} \nu_n^2 - \frac{1}{2} r_n^2 = e_n(q, t_0, 0) + \Delta e_n(q, t_0, \varepsilon), \quad (66)
\end{equation}

where \( e_n(q, t_0, 0) \) is the energy spectrum in the absence of the electric field and \( \Delta e_n(q, t_0, \varepsilon) \), is the shift due to the presence of the electric field

\begin{equation}
  \Delta e_n(q, t_0, \varepsilon) = \frac{-2\gamma^2 \varepsilon^2}{u_0^2} q^{-n} \left( 1 - \frac{t_0}{q^n} \right)^{-2}. \quad (67)
\end{equation}

Finally, the Hamiltonians of the SUSYQM hierarchy can be written as

\begin{equation}
  H_i \equiv \hbar \omega h_i = \frac{1}{2m_i} \left( p + c_i \sqrt{\frac{m_i \hbar \omega}{a_i}} \right)^2 + \frac{1}{2} m_i \omega_i^2 x^2 - \frac{1}{2} \left( \frac{c_i^2}{a_i} - 2d_i \right) \hbar \omega - \bar{q} \bar{x}. \quad (68)
\end{equation}
Thus, in the presence of a uniform electric field, the harmonic oscillator belongs to a hierarchy of Hamiltonians with a shift in momentum space and with different masses and frequencies but with the same electric field.

4 Conclusions

In this Letter, we have considered a GUP framework that admits maximal momentum uncertainty and nonzero minimal position and momentum uncertainties. We applied the supersymmetric quantum mechanics method and the shape invariance condition in order to find the exact GUP-corrected harmonic oscillator energy spectrum without solving the corresponding generalized Schrödinger equation. The results show that although the shift in the energy spectrum is positive, it is smaller with respect to the case with minimal position and momentum uncertainties. We obtained the supersymmetric partner Hamiltonians and showed that the GUP-corrected harmonic oscillator belongs to a hierarchy of Hamiltonians of the same type but with a shift in momentum space and with different masses and frequencies. Finally, we have studied the effects of a uniform electric field on this modified harmonic oscillator energy spectrum.

Acknowledgements

The work of K. Nozari has been supported financially by the Center for Excellence in Astronomy and Astrophysics (CEAAI - RIAAM), Maragha, Iran.

References

[1] D. Amati, M. Ciafaloni and G. Veneziano, Phys. Lett. B 216, 41 (1989).

[2] K. Konishi, G. Paffuti and P. Provero, Phys. Lett. B 234, 276 (1990).

[3] M. Maggiore, Phys. Lett. B 304, 65 (1993).

[4] F. Scardigli, phys. Lett. B 452, 39 (1999).
[5] M. Maggiore, Phys. Rev. \textbf{49}, 5182 (1994).

[6] S. Ghosh and S. Mignemi, Int. J. Theor. Phys. \textbf{50}, 1803 (2011).

[7] C. Castro, phys. Lett. \textbf{10}, 273 (1997).

[8] S. Hossenfelder, M. Bleicher, S. Hofmann, J. Ruppert, S. Scherer and H. Stoecker, Phys. Lett. B \textbf{575}, 85 (2003).

[9] C. Bambi and F.R. Urban, Class. Quant. Grav. \textbf{25}, 095006 (2008), \texttt{arXiv:0709.1965}.

[10] K. Nozari and B. Fazipour, Gen. Relativ. Grav. \textbf{38}, 1661 (2006).

[11] B. Vakili and H.R. Sepanji, Phys. Lett. B \textbf{651}, 79 (2007).

[12] A. Kempf, G. Mangano, R.B. Mann, Phys. Rev. D \textbf{52}, 1108 (1995).

[13] D.J. Gross and P.F. Mende, Nucl. Phys. B \textbf{303}, 407 (1988).

[14] P. Pedram, Euro. Phys. Lett. \textbf{89}, 50008 (2010).

[15] P. Pedram, Int. J. Mod. Phys. D \textbf{19}, 2003 (2010).

[16] P. Pedram, K. Nozari, and S.H. Taheri, JHEP \textbf{1103}, 093 (2011).

[17] P. Pedram, Physica A \textbf{391}, 2100 (2012).

[18] P. Pedram, Phys. Lett. B \textbf{714}, 317 (2012).

[19] P. Pedram, Phys. Rev. D \textbf{85}, 024016 (2012), \texttt{arXiv:1112.2327}.

[20] K. Nozari and T. Azizi, Int. J. Quant. Inf. \textbf{3}, 623 (2005).

[21] P. Pedram, Phys. Lett. B \textbf{710}, 478 (2012).

[22] K. Nozari and P. Pedram, Europhys. Lett. \textbf{92}, 50013 (2010), \texttt{arXiv:1011.5673}.

[23] K. Nozari, A. Etemadi, Phys. Rev. D \textbf{85}, 104029 (2012).
[24] A. Kempf, arXiv:hep-th/9405067.

[25] A.F. Ali, S. Das, and E.C. Vagenas, Phys. Lett. B 678, 497 (2009).

[26] S. Das, E.C. Vagenas, and A.F. Ali, Phys. Lett. B 690, 407 (2010).

[27] A.F. Ali, S. Das, and E.C. Vagenas, Phys. Rev. D 84, 044013 (2011).

[28] A. Bilal, Introduction to Supersymmetry, arXiv:hep-th/0101055.

[29] D.J. Fernandez C., AIP Conf. Proc. 1287, 3 (2010).

[30] C. Quesne and V.M. Tkachuk, J. Phys. A: Math. Gen. 36, 10373 (2003).

[31] F. Cooper, A. Khare and U. Sukhatme, Phys. Rep. 251, 267 (1995).

[32] F. Cooper, A. Khare and U. Sukhatme, Supersymmetry in Quantum Mechanics (Singapore: World Scientific) (2001).

[33] G. Junker, Supersymmetric Methods in Quantum and Statistical Physics (Berlin: Springer) (1996).

[34] L.E. Gendenshtein, Pisma Zh. Eksp. Teor. Fiz. 38, 299 (1983).

[35] J. Dabrowska, A. Khare and U. Sukhatme, J. Phys. A: Math. Gen. 21, L195 (1988).

[36] L. Infeld and T.E. Hull, Rev. Mod. Phys. 23, 21 (1951).

[37] C. Quesne and V.M. Tkachuk, J. Phys. A: Math. Gen. 37, 10095 (2004).