Degeneration of the Leray spectral sequence
for certain geometric quotients

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Abstract

We prove that the Leray spectral sequence in rational cohomology
for the quotient map $U_{n,d} \to U_{n,d}/G$ where $U_{n,d}$ is the affine variety
of equations for smooth hypersurfaces of degree $d$ in $\mathbb{P}^n(\mathbb{C})$ and $G$ is
the general linear group, degenerates at $E_2$. 

1 Introduction

We consider an affine complex algebraic group $G$ which acts on a smooth
algebraic variety $X$. Assume that a geometric quotient $f : X \to Y$ for the
action of $G$ on $X$ exists (cf. [11, Sect. 0.1]). We want to give geometric
conditions ensuring that the Leray spectral sequence degenerates at $E_2^{p,q} = H^p(Y, R^q f_* \mathbb{Q})$.

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The cohomology ring of $G$ is well known ([8], [2]). It is an exterior algebra with exactly one generator $\eta_i$ in certain odd degrees $2r_i - 1$, $i = 1, \ldots, r = r(G)$, the rank of $G$. So, if the Leray spectral sequence degenerates at $E_2$, knowing the cohomology of the source $X$ is equivalent to knowing that of the target $Y$. As an example of how this could be used, we point out that for any group $G$ acting with finite stabilizers on a topological space $X$ the equivariant cohomology $H^*_G(X, \Q)$ equals $H^*(X/G, \Q)$ ([3, §1, Remark 2]) and the former can often be calculated group theoretically. See [3] for examples. So, in these cases one knows $H^*(X, \Q)$.

We prove a general result (Theorem 3) giving sufficient geometric conditions for this to happen. These turn out to be satisfied for the group $\text{GL}_{n+1}(\C)$ acting on the affine variety $U_{n,d}$ of those homogeneous polynomials of degree $d$ in $(n + 1)$ variables which give smooth hypersurfaces in $\mathbb{P}^n$:

**Theorem 1** Let $d \geq 3$. Then the Leray spectral sequence in rational cohomology for the quotient map $U_{n,d} \to U_{n,d}/G$, $G = \text{GL}_{n+1}(\C)$ degenerates at $E_2$.

**Examples**

1. By results of Vassiliev [12] the map $H^*(U_{n,d}; \Q) \to H^*(\text{GL}_{n+1}(\C); \Q)$ is an isomorphism in the cases $(n, d) = (2, 3), (3, 3)$. Moreover Gorinov [3] has proved the same result for the cases $(n, d) = (4, 3), (2, 5)$. It follows that $M_{n,d}$ has the rational cohomology of a point in these cases.

2. For the case $(n, d) = (2, 4)$ it follows from [12] and Theorem 4 that the space $M_{2,4}$ has a cohomology group of dimension 1 in degrees 0 and 6 and has zero rational cohomology in other degrees. This agrees with a result of Looijenga [10] about the Poincaré-Serre polynomial of $M_{2,4}$:

$$H^6(M_{2,4}; \Q) \simeq \Q(-6)$$

and the other cohomology groups are those of a point.

**Remark** In [1] there is a description of $M_{3,3}$ using periods of threefolds. This moduli space turns out to be a certain explicitly described open subset of the quotient of complex hyperbolic 4-space by a certain discrete group. From this description it is quite unexpected that $M_{3,3}$ has the rational cohomology of a point. It is an interesting question to calculate the cohomology of the various compactifications of $M_{3,3}$ studied in loc. cit.
2 Generalizing the Leray-Hirsch theorem

The proof of the Leray-Hirsch theorem as given on [9, p. 229] is valid for a locally trivial fibration \( p : M \to B \). For cohomology with rational coefficients, the same proof applies to a slightly more general situation:

**Definition** A continuous map \( p : M \to B \) is a locally trivial fibration, say with fibre \( F \), in the *orbifold sense* if for every \( b \in B \) there exists a neighbourhood \( V_b \), a topological space \( U_b \), and a topological group \( G_b \) such that

1. \( G_b \) acts on \( U_b \) and on \( F \); the action on \( F \) is by homeomorphisms homotopic to the identity;
2. \( V_b \) is homeomorphic to \( U_b/G_b \);
3. \( p^{-1}V_b \) is homeomorphic to the quotient of \( U_b \times F \) by the product action of \( G_b \).

In this setting, composing the natural quotient map \( F \to F/G_b \) with the homeomorphism \( (F/G_b) \sim p^{-1}b \) and the inclusion \( p^{-1}b \hookrightarrow X \), defines the *orbifold fibre inclusion* \( r_b : F \to X \).

Indeed, in this setting the proof as given in loc. cit. applies starting from the observation that over the rationals we still have graded isomorphisms (replacement of the K"unneth formula)

\[
H^\bullet(p^{-1}V_b; \mathbb{Q}) \cong H^\bullet(U_b \times F; \mathbb{Q})^{G_b} \cong H^\bullet(U_b; \mathbb{Q})^{G_b} \otimes H^\bullet(F; \mathbb{Q})^{G_b} \\
\cong H^\bullet(V_b; \mathbb{Q}) \otimes H^\bullet(F; \mathbb{Q}),
\]

because \( g \in G_b \) acts trivially on \( H^q(F; \mathbb{Q}) \) since it is homotopic to the identity by assumption.

We thus arrive at:

**Theorem 2** Let \( p : M \to B \) be a fibration which is locally trivial in the orbifold sense. Suppose that for all \( q \geq 0 \) there exist classes \( e_1^{(q)}, \ldots, e_{n(q)}^{(q)} \in H^q(M; \mathbb{Q}) \) that restrict to a basis for \( H^q(F; \mathbb{Q}) \) under the map induced by the orbifold fibre inclusion \( r_b : F \to M \). The map \( a \otimes r_b^* (e_i) \mapsto p^*a \cup e_i, a \in H^\bullet(B; \mathbb{Q}) \) extends linearly to a graded linear isomorphism

\[
H^\bullet(B; \mathbb{Q}) \otimes H^\bullet(F; \mathbb{Q}) \sim H^\bullet(M; \mathbb{Q}).
\]
Example Let \( \phi : X \to Y \) be a geometric quotient for \( G \). Suppose that \( G \) is connected and that for all \( x \in X \), the identity component of the stabiliser \( S_x \) of \( x \) is contractible (e.g. when \( S_x \) is finite). For \( y \in Y \) we take for \( U_y \) any open slice for the action of \( G \) through \( x \in \phi^{-1} y \), i.e. a contractible submanifold through \( x \) which intersects \( Gx \) transversally at \( x \). Then, if \( gx \) is any other point in same orbit, \( gU_y \) is a slice through \( gx \) and \( gS_x g^{-1} = S_{gx} \) so that for all \( g \in G \), the quotient \( gU_y / S_{gx} \) gives the same neighbourhood \( V_y \) of \( y \). We have \((U_y \times G)/S_x = \phi^{-1}(V_y)\). The assumption that \( G \) is connected implies that multiplication by \( g \in G \) is homotopic to the identity in \( G \). So \( \phi \) is indeed locally trivial in the orbifold sense (with typical fibre \( G \)).

We study this example in more detail in the next section.

3 The case of a geometric quotient for a reductive group

We assume that \( G \) is a reductive complex affine group, that \( V \) is a representation space for \( G \) and that \( X \) is an open set of stable points with complement \( \Sigma = V \setminus X \). For \( x \in X \) the orbit map is denoted as follows

\[
o_x : G \to X
\]
\[
g \mapsto g(x),
\]

and the geometric quotient by

\[
\phi : X \to Y = X/G.
\]

Recall that \( H^\bullet(G) \) is an exterior algebra freely generated by classes \( \eta_i \in H^{2r_i - 1}(G) \). Note also that \( V \) being a vector space, we have isomorphisms

\[
H^{2r_i - 1}(X) \cong H^{2r_i}(V).
\]

We can now apply the variant of the Leray-Hirsch theorem as stated in the previous section to the geometric quotient \( \phi \) and we obtain:

Theorem 3 Suppose that there are schemes \( Y_i \subset \Sigma \) of pure codimension \( r_i \) in \( V \) whose fundamental classes map to a non-zero multiple of \( \eta_i \) under the composition

\[
H^{2r_i}(V) \to H^{2r_i}(V) \cong H^{2r_i - 1}(X) \overset{\phi}{\to} H^{2r_i - 1}(G).
\]
Denote the image of \([Y_i]\) in \(H^\bullet(X; \mathbb{Q})\) by \(y_i\); then the map \(a \otimes \eta_i \mapsto \phi^*a \cup y_i, a \in H^\bullet(X/G; \mathbb{Q})\) extends to an isomorphism of graded \(\mathbb{Q}\)-vector spaces

\[
H^\bullet(X/G; \mathbb{Q}) \otimes H^\bullet(G; \mathbb{Q}) \cong H^\bullet(X; \mathbb{Q}).
\]

4 Properties of fundamental classes

We collect some facts on fundamental classes that we need later on. We refer to [4] for the cohomology-version and [5] for the Chow-version.

1. For any connected submanifold \(Z\) of pure codimension \(c\) in a complex algebraic manifold \(X\), its fundamental class \([Z]\) is the image of \(1 \in H^0(Z)\) under the Thom-isomorphism \(H^\bullet(Z) \cong H^\bullet(X)[2c]\). For \(Z\) an irreducible subvariety, one still has a fundamental class as above, since restriction to the smooth part of \(Z\) induces isomorphisms between the relevant cohomology groups with support in \(Z\), respectively the smooth part of \(Z\). If \(Z = \sum_i n_i Z_i\) is a cycle of codimension \(c\) (with \(Z_i\) irreducible), with support \(|Z|\), there is a cycle class \([Z]\) \(\in H^\bullet_{|Z|}(X)\). More generally still, one may assume \(Z\) to be a complex subscheme of pure codimension \(c\) with irreducible components \(Z_i\) of multiplicity \(n_i\) in \(Z\) and define the fundamental class to be the fundamental class of the associated cycle \(\sum_i n_i Z_i\). There are natural maps \(H^\bullet_{Z_i} \rightarrow H^\bullet_{|Z|}\) and if we identify \([Z]\) with their images under these maps we have the equality

\[
[Z] = \sum_i n_i [Z_i].
\]

2. The fundamental classes behave functorially as follows. Let \(f : X \rightarrow Y\) be a holomorphic map between complex algebraic manifolds, \(Z \subset X, W \subset Y\) subschemes such that \(Z\) is contained in the scheme-theoretic inverse image \(f^{-1}W\). Then \(f\) induces \(H^\bullet_W(Y) \rightarrow H^\bullet_Z(X)\) and if moreover \(Z = f^{-1}W\) has the same codimension \(c\) as \(W\), then \(f^*[W] = [Z]\). In particular, if \(W\) is irreducible and the cycle associated to \(Z = f^{-1}W\) is \(\sum n_i Z_i\), we find

\[
f^*[W] = [f^{-1}W] = \sum n_i [Z_i] \in H^\bullet_{|Z|}(X).
\]

3. We can refine the fundamental class of \(Z\), a purely \(c\)-codimensional subscheme of \(X\) to a class in the Chow group \(A_{n-c}(X), n = \dim(X)\). The refinement works as follows. There is a cycle class map \(A_\bullet(X) \rightarrow H^{BM}_\bullet(X)\)
to Borel-Moore homology. One composes this map in degree \( n - c \) with Poincaré-duality for Borel-Moore homology, which reads

\[
H^\bullet_{\text{BM}}(X) \xrightarrow{\sim} H_Z^{2n-\bullet}(X).
\]

In sum, we get a cycle class map

\[
A_{n-c}(X) \to H^c_Z(X)
\]

sending the Chow cycle of \( Z \) to \([Z]\). Abusing notation, we denote the Chow cycle also by \([Z]\). This is especially useful if \( Z \) is the scheme of zeros of a section \( s \) of a vector bundle \( E \) over \( X \). In fact, if \( s : E \to X \) is the zero-section with image, say \([0]\), there is a Gysin isomorphism \( s^* : A_\bullet(E) \to A_\bullet(X)[-r] \) with the property

\[
A_n(E) \ni [\{0\}] \xmapsto{s^*} c_r(X) \in A_{n-r}(X).
\]

See \([5, \text{Example 3.3.2}]\). This Gysin map is in fact the inverse of the isomorphism

\[
\pi^* : A_{n-r}(X) \xrightarrow{\sim} A_n(E).
\]

5 The cohomology ring of the general linear group

We turn to \( G = G_n = \text{GL}_n(\mathbb{C}) \), \( n \geq 1 \). In this case, by \([2]\), \( H^\bullet(G) \) is the exterior algebra with generators \( \eta^{(n)}_{\ell} \) in all odd degrees \( 2\ell - 1 \), \( \ell = 1, \ldots, n \). In other words \( r_1 = 1, r_2 = 2, \ldots, r_n = n \). Since \( G_n \subset M_n = \text{Mat}_n(\mathbb{C}) \), a vector space, we have an identification of mixed Hodge structures

\[
H^\bullet(G) \xrightarrow{\sim} H^\bullet_{\text{DM}}(M_n)[1],
\]

where

\[
D_n = \{ A \in M_n \mid \det(A) = 0 \} = M_n \setminus G_n,
\]

and so \( \eta^{(n)}_{\ell} \) corresponds to some class in \( H^{2\ell}_{\text{DM}}(M_n) \). The goal is to find explicit descriptions of this class as fundamental class of the subvariety \( D_{n,\ell} \subset D_n \) to be defined below. This will turn out to be essential for the next section. We are going to show this by first defining classes \( \eta^{(n)}_{\ell} \) that clearly have this property. Then we prove that these classes do generate \( H^\bullet(G) \) as an exterior algebra.

We introduce the following notation:

\[
\text{...}
\]
• $D_{n,\ell} \subset D_n$: the subvariety consisting of those matrices for which the first $n + 1 - \ell$ columns are linearly dependent. Note that $D_{n,\ell}$ has codimension $\ell$ in $M_n$.

• $\tilde{D}_n = \{(A, p) \in D_n \times \mathbb{P}^{n-1}(\mathbb{C}) \mid A(p) = 0\}$ and $\pi_n : \tilde{D}_n \to D_n$ is the projection to the first factor.

• $Q_n = \{(x, y) \in \mathbb{C}^n \times \mathbb{C}^n \mid x \cdot y = 1\}$.

• $\alpha_n : M_{n-1} \to M_n$ is the inclusion which maps a matrix $A$ to $\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}$.

• $h$: the hyperplane class in $H^2(\mathbb{P}^n(\mathbb{C}))$.

Note that the projection to the second factor turns $\tilde{D}_n$ into a vector bundle of rank $n^2 - n$ over $\mathbb{P}^{n-1}(\mathbb{C})$, so $\tilde{D}_n$ is smooth and $\pi_n$ is a resolution of singularities of $D_n$.

**Lemma 4** Let $X$ be a smooth variety, $D \subset X$ a subvariety of codimension $k$ and $\pi : \tilde{D} \to D$ a resolution of singularities. Then there are natural Gysin maps $\beta_\ell : H^{\ell-2k}(\tilde{D})(\ell) \to H^\ell_D(X)$ which are morphisms of mixed Hodge structures.

**Proof** Let $n = \dim(X)$. Because $X$ is smooth, cap product with the fundamental class $[X] \in H^{BM}_{2(n-k)}(X)$ in Borel-Moore homology induces isomorphisms of mixed Hodge structures

$$H^\ell_D(X) \simeq H^{BM}_{2n-\ell}(D)(-n).$$

by [4, Sect. 19.1]. As Borel-Moore homology is covariant for proper morphisms we have natural maps

$$H^{BM}_j(\tilde{D}) \to H^{BM}_j(D).$$

As $\tilde{D}$ is smooth, cup product with the fundamental class $[\tilde{D}]$ induces an isomorphism

$$H^{\ell-2k}(\tilde{D}) \to H^{BM}_{2(n-k)-\ell}(\tilde{D})(k-n).$$

The map $\beta_\ell$ is obtained the isomorphism

$$H^{\ell-2k}(\tilde{D})(\ell) \to H^{BM}_{2(n+k)-\ell}(\tilde{D})(-n).$$
foolowed by the direct image map
\[ H^{BM}_{2(n+k)-\ell}(\tilde{D})(-n) \to H^{BM}_{2(n+k)-\ell}(D)(-n) \]
and the inverse of the isomorphism
\[ H^{\ell-2k}(X)(-k) \simeq H^{BM}_{2n+2k-\ell}(D)(-n). \]
□

Let us apply this to the situation of \( \tilde{D}_n \to D_n \hookrightarrow M_n \). We obtain maps
\[ \beta^{(n)}_{\ell}(1) : H^{2\ell-2}(\mathbb{P}^{n-1})(-1) \to H^{2\ell}_{D_n}(M_n) \simeq H^{2\ell-1}(G_n) \]
and define for \( \ell = 1, \ldots, n \):
\[ \eta^{(n)}_{\ell} := \beta^{(n)}_{\ell} \left( \frac{h^{\ell-1}}{2\pi i} \right) \in H^{2\ell-1}(G_n). \]

We observe that the class in \( H^{2\ell}_{D_n}(M_n) \) corresponding to \( \eta^{(n)}_{\ell} \) is indeed the fundamental class of \( D_{n,\ell} \subset D_n \).

**Lemma 5** The map \( \alpha : M_{n-1} \to M_n \) maps \( D_{n-1} \) and \( G_{n-1} \) to \( D_n \) and \( G_n \) respectively and \( \alpha^*(\eta^{(n)}_{\ell}) = \eta^{(n-1)}_{\ell} \) for \( \ell = 1, \ldots, n-1 \) while \( \alpha^*(\eta^{(n)}_{n}) = 0 \).

**Proof** Observe that \( \alpha^{-1}(D_{n,\ell}) = D_{n-1,\ell} \). One checks that this holds not only set theoretically, but even as schemes. Then the lemma follows from property 2) from section 4.

Because the classes \( \eta^{(n)}_{\ell} \) are of odd degree, they have square zero and anti-commute, so we have a homomorphism of graded algebras
\[ R_n : \Lambda(z_1, \ldots, z_n) \to H^*(G_n). \]
Here \( \Lambda(z_1, \ldots, z_n) \) is the exterior algebra on \( n \) generators \( z_1, \ldots, z_n \) with \( z_i \) of degree \( 2i - 1 \), and \( R_n(z_\ell) = \eta^{(n)}_{\ell} \).

**Theorem 6** The map \( R_n \) is an isomorphism. Moreover, the generators \( \eta^{(n)}_{\ell} \in H^{2\ell-1}(G_n) \) have pure type \( (\ell, \ell) \) and map to the fundamental classes \( D_{n,\ell} \) under the identification \( H^{2\ell-1}(G_n) \simeq H^{2\ell}_{D_n}(M_n) \).
Proof By induction on \( n \). For \( n = 1 \) everything is clear. Suppose the map \( R_{n-1} \) is an isomorphism. We consider the map

\[
\rho : G_n \to Q_n, \quad \rho(g) = (g(e_1), g^{-1}(e_1)).
\]

This is the orbit map of a transitive action of \( G_n \) on \( Q_n \) and \( \alpha(G_{n-1}) \) is the isotropy subgroup of \((e_1, e_1) \in Q_n\). Therefore, \( \rho \) is also the quotient map for the action of \( G_{n-1} \) on \( G_n \) by left translation via \( \alpha \). As the classes \( \eta^{(n-1)}_\ell \) generate the cohomology ring of \( G_{n-1} \) and are images of classes on \( G_n \), the restriction maps \( \alpha^* : H^i(G_n) \to H^i(G_{n-1}) \) are surjective. Hence by Theorem 2 we have an isomorphism

\[
H^*(Q_n) \otimes H^*(G_{n-1}) \simeq H^*(G_n).
\]

The variety \( Q_n \) is homotopy equivalent to a sphere of dimension \( 2n - 1 \) (in fact to its subvariety consisting of pairs \((x, y)\) with \( y = \overline{x}\)). Moreover, a generator of \( H^{2n-1}(Q_n) \) is mapped to a non-zero multiple of \( \eta^{(n)}_n \) by the map \( \rho^* \). This implies the surjectivity and hence bijectivity of \( R_n \). \( \Box \)

Remark For any Lie group \( G \), the map \( g \mapsto g^{-1} \) induces multiplication by \(-1\) on the Lie algebra, hence on \( H^k(G) \) it induces multiplication by \((-1)^k\). The involution \( \sigma : G_n \to G_n \) given by \( \sigma(g) = \overline{g}^{-1} \) has \( \sigma^*(\eta^{(n)}_\ell) = (-1)^\ell \eta^{(n)}_\ell \). Indeed, if we let \( \sigma : Q_n \to Q_n \) be given by \( \sigma(x, y) = (y, x) \) then \( \rho \) becomes equivariant, and it is an easy exercise to see that \( \sigma^* = (-1)^{n} \) on \( H^{2n-1}(Q_n) \). We conclude that transposition \( \tau \) on \( G_n \) induces \( \tau^*(\eta^{(n)}_\ell) = (-1)^{\ell-1} \eta^{(n)}_\ell \). As the inclusion \( G_{n-1} \to G_n \) commutes with transposition, we conclude that \( \tau^*(\eta^{(n)}_\ell) = (-1)^{\ell-1} \eta^{(n)}_\ell \) for all \( \ell \leq n \).

6 Moduli of smooth hypersurfaces

We let \( \Pi_{n,d} = \mathbb{C}[x_0, \ldots, x_n]_d \) denote the vector space of homogeneous polynomials of degree \( d \) in \( n+1 \) variables over \( \mathbb{C} \). We let

\[
\Sigma_{n,d} = \{ f \in \Pi_{n,d} \mid f \text{ has a critical point outside } 0 \}.
\]

There exists an irreducible polynomial \( \Delta \) in the coefficients of \( f \in \Pi_{n,d} \) such that \( f \in \Sigma_{n,d} \) if and only if \( \Delta(f) = 0 \). Moreover, \( \Delta \) is homogeneous of degree \((n+1)(d-1)^n\).
We let \( U_{n,d} = \Pi_{n,d} \setminus \Sigma_{n,d} \). The group \( \text{GL}_{n+1}(\mathbb{C}) \) acts on \( U_{n,d} \). For \( d \leq 2 \) or \( d = 3, n = 1 \) it acts transitively, but in the remaining cases it acts with finite isotropy groups and we have a geometric quotient \( M_{n,d} \) which is a coarse moduli space for non-singular projective hypersurfaces of degree \( d \) in \( \mathbb{P}^n(\mathbb{C}) \).

In our situation we fix a particular \( f = f_{n,d} \in U_{n,d} \), the Fermat hypersurface:
\[
f_{n,d} = x_0^d + \cdots + x_n^d,
\]
and the orbit map then extends to a map
\[
r_n : M_{n+1} \rightarrow \Pi_{n,d}
\]
\[
A \mapsto f_{n,d} \circ A.
\]
It induces maps for cohomology with supports:
\[
H^{2\ell}_{\Sigma_{n,d}}(\Pi_{n,d}) \xrightarrow{r_n^\ast} H^{2\ell}_{D_{n+1}}(M_{n+1}).
\]
Define for \( \ell = 1, \ldots, n+1 \)
\[
\Sigma^{(\ell)}_{n,d} = \{ f \in \Pi_{n,d} \mid V(f)^{\text{sing}} \cap [e_0, \ldots, e_{n-\ell+1}] \neq \emptyset \}.
\]
Then \( \Sigma^{(\ell)}_{n,d} \subset \Sigma_{n,d} \) has codimension \( \ell \) in \( \Pi_{n,d} \). Below we shall prove:

**Lemma 7** The class \( r_n^\ast([\Sigma^{(\ell)}_{n,d}]) \) is a non-zero multiple of \([D_{n+1,\ell}]\).

Recall from the previous section that \([D_{n+1,\ell}]\) corresponds to the generator \( \eta^{(n)}_\ell \in H^{2\ell-2}(G) \) and we now apply Theorem 3 to deduce:

**Theorem 8** Let \( d \geq 3 \). Then the Leray spectral sequence in rational cohomology for the quotient map \( U_{n,d} \rightarrow M_{n,d} \) degenerates at \( E_2 \).
term $x_0^d$ multiplies the Milnor numbers by $d - 1$. We obtain a commutative diagram

\[
\begin{array}{ccc}
M_n & \xrightarrow{r_n} & \Pi_{n-1,d} \\
\downarrow \alpha & & \downarrow \iota \\
M_{n+1} & \xrightarrow{r_n} & \Pi_{n,d}.
\end{array}
\]

We have a corresponding diagram in cohomology with supports

\[
\begin{array}{ccc}
H_{2\Sigma_{n,d}}^2(\Pi_{n,d}) & \xrightarrow{r_n^*} & H_{D_{n+1}}^{2\ell}(M_{n+1}) \\
\downarrow \iota^* & & \downarrow \alpha^* \\
H_{2\Sigma_{n-1,d}}^2(\Pi_{n-1,d}) & \xrightarrow{r_n^*} & H_{D_{n}}^{2\ell}(M_{n})
\end{array}
\]

Observe that $\iota^*(\Sigma_{n,d}) = \nu_{n,d}^{(\ell)}[\Sigma_{n-1,d}]$ where $\nu_{n,d}^{(\ell)}$ is the intersection multiplicity of $\Sigma_{n,d}^{(\ell)}$ with $\iota(\Pi_{n-1,d})$ in $\Pi_{n,d}$. In particular, $\nu_{n,d}^{(\ell)}$ is a positive integer.

We can now prove the lemma by induction on $n$ using the above diagram, provided we check the case $\ell = n + 1$ for each $n$.

The variety $S = \Sigma_{n,d}^{(n+1)}$ is the linear space of all polynomials singular at $e_0$. Its pre-image under $r_n$ has two irreducible components: one consists of the matrices whose first column is zero, i.e. with $A(e_0) = 0$; this component is exactly $T_1 = D_{n+1,n+1}$. The other component, $T_2$, which has the same dimension, consists generically of matrices $A$ mapping $e_0$ to some point $p$ with $f_{n,d}(p) = 0$ and such that the image of $A$ is contained in the tangent space to the hypersurface $V(f_{n,d}) \subset \mathbb{C}^{n+1}$ at this point. The component $T_2$ has multiplicity one, whereas $T_1$ has multiplicity $d(d - 1)^n$. We have the commutative diagram

\[
\begin{array}{ccc}
H_{S}^{2n+2}(\Pi_{n,d}) & \rightarrow & H_{\Sigma_{n,d}}^{2n+2}(\Pi_{n,d}) \\
\downarrow r_n^* & & \downarrow \\
H_{T_1\cup T_2}^{2n+2}(M_{n+1}) & \rightarrow & H_{D_{n+1}}^{2n+2}(M_{n+1})
\end{array}
\]

and therefore

\begin{equation}
(1) \quad r_n^*([S]) = d(d - 1)^n[T_1] + [T_2]
\end{equation}

by Property 1) in Sect. [4].

Claim: We have

\[ [T_2] = (-1)^n(1 - (1 - d)^n)[T_1] \] in $H_{D_{n+1}}^{2n+2}(M_{n+1})$. 

11
Combining the Claim with (1) we find:
\[ r_n^* [S] = d(d-1)^n [T_1] + [T_2] = ((d-1)^{n+1} + (-1)^n) [T_1] \neq 0, \]
which proves the Lemma.

It remains to prove the Claim. Let \( T_2' \) denote the image of \( T_2 \) under the transposition map \( \tau \). Then
\[ (2) \quad [T_2] = (-1)^n [T_2'] \]
in \( H^{2n+2}_{D_{n+1}}(M_{n+1}) \) by the Remark at the end of Sect. [3]. Let \( \tilde{T}_1 = T_1 \times \{ e_0 \} \subset \tilde{D}_{n+1} \).

Write \( X = V(f_{n,d}) \subset \mathbb{P}^n \) and let \( \gamma : X \to \mathbb{P}^n \) be the Gauss map, which associates to a point \( p \in X \) the coordinates of its tangent hyperplane, i.e. \( \gamma(p) = \nabla f_{n,d}(p) \).

The space \( D_{n+1} \) is the total space of a vector bundle \( E \) over \( \mathbb{P}^n \) of rank \( r = n(n+1) \). Let
\[ \tilde{T} := \{ (A, p) \in M_{n+1} \times X \mid (df_0)_p \circ t_A = 0 \text{ and } t_A(e_0) = p \}. \]
Then \( \tilde{T} \) is the total space of a vector bundle \( F \) over \( X \) of rank \( r - n + 1 \) which is a subbundle of \( \gamma^*(E) \), because \( (A, p) \in \tilde{T} \) implies that \( A(\gamma(p)) = 0 \). The projection of \( \tilde{T} \) in \( M_{n+1} \) is precisely \( T_2' \).

We will carry out our calculations in Chow groups instead of cohomology groups, using property 3) in Sect. [3]. Consider the diagram
\[
\begin{array}{ccc}
F & \hookrightarrow & \gamma^*(E) \\
\downarrow & & \downarrow \pi' \\
X & = & X \\
\end{array} \xrightarrow{\tilde{\gamma}} \begin{array}{ccc}
\gamma \in & \mathbb{P}^n \\
\end{array}
\]
We let \( s \) be the 0-section of \( E \), and \( s' \) that of \( \gamma^*E \) and recall from Sect. [3] 3) that these induce Gysin maps in Chow groups.

The strategy is to compare the classes \( \tilde{T}_1 \) and \( \tilde{T} \) by pushing them to \( \mathbb{P}^n \). We get two 0-cycles on \( \mathbb{P}^n \) whose degrees we compare. Clearly \( \deg s^*[\tilde{T}_1] = 1 \) and so it suffices to calculate the degree of
\[ s^* \tilde{\gamma}_*([F]) \in A_0(\mathbb{P}^n). \]
By [3, Proposition 1.7] we find that
\[ \tilde{\gamma}_* \pi'^* \alpha = \pi^* \gamma_* \alpha \in A_{i+r}(E) \]
for any $\alpha \in A_i(X)$. Applying this to $\alpha = s^*[F]$ we find

$$\tilde{\gamma}_s[F] = \pi^*\gamma_s s^*[F].$$

Next, applying $s^*$ to both sides and using that the Gysin map $s^*$ is in fact the inverse of the isomorphism induced by the bundle projection $\pi : E \to \mathbb{P}^n$, and similarly for $s'$, we get

$$s^*\tilde{\gamma}_s[F] = \gamma_{s'} s^*[F].$$

We next compute $s'^*[F] \in A_0(X)$. By [5, Example 3.3.2] applied to the vector bundle $\gamma^*(E)/F$ we get

$$s'^*[F] = c_{n-1}(\gamma^*(E)/F).$$

On $\mathbb{P}^n$ we have the exact sequence

$$0 \to E \to \mathcal{O}^{(n+1)^2} \to \mathcal{O}(1)^{n+1} \to 0$$

showing that $c(E) = (1+h)^{-n-1}$ where $h = c_1(\mathcal{O}(1))$. As $\gamma^*\mathcal{O}(1) = \mathcal{O}_X(d-1)$ we get

$$c(\gamma^*E) = (1 + (d-1)h_X)^{-n-1}$$

where $h_X = c_1(\mathcal{O}_X(1))$. For the bundle $F$ we have the exact sequences

$$0 \to F \to \mathcal{O}_X^{(n+1)^2} \to Q_X \oplus \mathcal{O}_X(d-1)^n \to 0$$

$$0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X^{n+1} \to Q_X \to 0$$

so $Q_X$ is the restriction of the universal quotient bundle to $X$. Hence we find

$$c(F) = (1 + (d-1)h_X)^{-n} c(Q_X)^{-1} = (1 + (d-1)h_X)^{-n} (1 - h_X)^{-1}$$

so

$$c(\gamma^*E/F) = (1 + (d-1)h_X)^{-1} (1 - h_X)^{-1}.$$  

We find

$$c_{n-1}(\gamma^*E/F) = \left( \frac{1 - (1 - d)^n}{d} \right) h_X^{n-1}$$

which has degree equal to $1 - (1 - d)^n$. Combining this with (2), the Claim then follows.
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