PARABOLIC COMPARISON REVISITED AND APPLICATIONS

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Abstract. We consider the Cauchy-Dirichlet problem

$$\partial_t u - F(t, x, u, Du, D^2u) = 0 \text{ on } (0, T) \times \mathbb{R}^n$$

in viscosity sense. Comparison is established for bounded semi-continuous (sub-/super-)solutions under structural assumption (3.14) of the User’s Guide plus a mild condition on $F$ such as to cope with the unbounded domain. Comparison on $(0, T]$, space-time regularity and existence are also discussed. Our analysis passes through an extension of the parabolic theorem of sums which appears to be useful in its own right.

1. Introduction

We recall some basic ideas of (second order) viscosity theory (Crandall, Ishii, Lions ... [10, 12]). Consider a real-valued function $u = u(x)$ with $x \in \mathbb{R}^n$ and assume $u \in C^2$ is a classical supersolution,

$$-G(x, u, Du, D^2u) \geq 0,$$

where $G$ is a (continuous) function, degenerate elliptic in the sense that $G(x, u, p, A) \leq G(x, u, p, A + B)$ whenever $B \geq 0$ in the sense of symmetric matrices, one also requires that $G$ is non-increasing in $u$; under these assumptions $G$ is called proper.

The idea is to consider a (smooth) test function $\varphi$ which touches $u$ from below at some point $\bar{x}$. Basic calculus implies that $Du(\bar{x}) = D\varphi(\bar{x})$, $D^2u(\bar{x}) \geq D^2\varphi(\bar{x})$ and, from degenerate ellipticity,

$$(1.1) -G(\bar{x}, \varphi, D\varphi, D^2\varphi) \geq 0.$$

This suggests to define a viscosity subsolution (at the point $\bar{x}$) to $-G = 0$ as a (upper semi-)continuous function $u$ with the property that (1.1) holds for any test function which touches $u$ from below at $\bar{x}$. Similarly, viscosity supersolutions are (lower semi-)continuous functions, defined via testfunctions touching $u$ from below and by reversing the inequality in (1.1); viscosity solutions are both super- and subsolutions (and hence continuous).

Observe that this definition covers (completely degenerate) first order equations as well as parabolic equations, e.g. by considering $\partial_u - F = 0$ where $F$ is proper. The resulting theory (existence, uniqueness, stability, ...) is without doubt one of most important recent developments in the field of partial differential equations. In particular, much is known about the Cauchy-Dirichlet problem

$$(1.2) \quad \partial_t u - F(t, x, u, Du, D^2u) = 0 \text{ on } (0, T) \times \Omega$$

with (nice) initial data, say $u_0 \in C(\Omega)$, on some bounded domain $\Omega$; see e.g. Theorem 8.2 in the User’s Guide [10]. Under structural assumptions on $F$ there

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is existence and uniqueness (in some class). In fact, uniqueness follows from a stronger property known as comparison: assume $u$ (resp. $v$) is a semicontinuous sub- (resp. super) solution and $u_0 \leq v_0$; then $u \leq v$ on $(0, T) \times \Omega$.

Surprisingly perhaps, much less has been written about the Cauchy-Dirichlet problem on unbounded domains. This seems to be particularly unfortunate since much of the recent applications from stochastics are naturally on unbounded domains. Let us be specific.

(i) We are unaware of a precise result that gives the simplest set of additional structural assumptions on $F$ such as to generalize the aforementioned Theorem 8.2. to, say, bounded solutions on $(0, T) \times \mathbb{R}^n$.
(ii) Comparison should be valid up to time $T$; after all $T \times \mathbb{R}^n$ is not part of the parabolic boundary.
(iii) When does bounded uniformly continuous initial data, $u_0 \in \text{BUC} (\mathbb{R}^n)$, lead to a modulus of continuity of $u(t, \cdot)$, uniformly in $t \in [0, T]$?
(iv) When do we have a space-time modulus or, say, a solution $u \in \text{BUC} ([0, T] \times \mathbb{R}^n)$?

There are partial answers to these things in the literature of course. Let us mention in particular [14] (towards (i) and (iii)) and [6] (and the references therein) concerning (ii). In the first order case, much can be found in the books [1, 2].

The contribution of this paper is to provide such results (with fully detailed proofs) in the generality of (1.2). While some ”general ideas” are without doubt part of the folklore of the subject (e.g. ”spatial modulus follows from comparison”, ”time modulus follows from spatial modulus”) their proper implementation is far from trivial. In particular, we were led to an extension of the parabolic theorem of sums which seems to be quite useful in its own right. To elaborate on this point, recall that almost every modern treatise of second order comparison relies in one way or another on the theorem of sums (TOS), also known as Crandall-Ishii lemma [9]. A parabolic version of the TOS on $(0, T) \times \Omega$ then underlies most second order (parabolic) comparison results; such as those in [10, Chapter 8] or [12, Chapter 5]. As is well-known, its application requires a barrier at time $T$; e.g. replace a subsolution $u$ by $u^\gamma := u - \gamma / (T - t)$ or so, followed by $\gamma \downarrow 0$ in the end. In many application this simple tricks works perfectly fine; sometimes, however, it makes life difficult. For instance, if $u$ is assumed to be bounded, the same is not true for $u^\gamma$ (although it is bounded from above); consequently one may have to introduce various localizations of the non-linearity to deal with the resulting unboundedness. (An example of the resulting complication is seen in [11].) Concerning the present paper, establishing a spatial modulus of solutions with the (standard) form of the parabolic theorem of sums would have led to a (apriori) dependence of the spatial modulus in time; establishing the (desired) uniformity in $t \in [0, T]$, cf. (iii) above, then entails a painstaking checking of uniformity in $\gamma$ for all double limits in the technical lemma [2] below. All these difficulties can be avoided by our extension of the (parabolic) TOS which remains valid for $t = T$. Perhaps, from a ”general point of view”, this is not surprising (after all, the elliptic TOS holds in great generality for locally compact domains and the parabolic TOS, in a sense, just

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1Leaving aside standard examples from stochastic control, let us mention 2BSDEs [1] and stochastic viscosity theory [12, 14, 15, 19], a related rough path point [20, 21, 13] was introduced in [2] and also relies on viscosity methods.

2The authors also point out various mistakes in previous papers in this context.
discards unwanted second order information related to the $t$ variable) but then, here again, a proper implementation with full details is quite involved.

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Assume that, in the sense of parabolic viscosity sub- and super-solutions,

\begin{equation}
\partial_t u - F(t, x, u, Du, D^2u) \leq 0 \leq \partial_t v - F(t, x, v, Dv, D^2v) \text{ on } (0, T) \times \mathbb{R}^n.
\end{equation}

Then the following statements hold true.

(i) The validity of (3.1) extends to \(Q := (0, T] \times \mathbb{R}^n\) (which reflects that \(\{T\} \times \mathbb{R}^n\) is not part of the parabolic boundary of \(Q\)).

(ii) If \(F\) satisfies the structural condition of the previous section, \(u_0 := u(0, \cdot), \ v_0 := v(0, \cdot) \in \text{BUC} (\mathbb{R}^n)\) and

\[ u_0 \leq v_0 \text{ on } \mathbb{R}^n \]

one has the key estimate, valid for all \((t, x, y) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n,\)

\[ u(t, x) - v(t, y) \leq \inf_{\alpha} \left[ \frac{\alpha}{2} |x - y|^2 + l(\alpha) \right] \]

where \(l(\alpha)\) tends to 0 as \(\alpha \uparrow \infty,\) uniformly in \(t \in [0, T].\)

\[ \text{Remark 1. Since } \hat{u}(t, x) = e^{-\gamma t} u(t, x) \text{ [resp. } \tilde{u}(t, x) = e^{-\gamma t} v(t, x) \] is a sub-[resp. super]-solution to \((\partial_t - \tilde{F}) \hat{u} + \gamma \hat{u} = 0\) with \(F(t, x, p, X) = e^{-\gamma t} F(t, x, e^{\gamma t} \hat{u}, e^{\gamma t} D\hat{u}, e^{\gamma t} D^2\hat{u})\)

we can always reduce to the case that \(\gamma > 0.\) In particular, we shall give the proof under this assumption.

\[ \text{Remark 2. The key estimate implies immediately comparison (take } x = y) \]

\[ u \leq v \text{ on } [0, T] \times \mathbb{R}^n. \]

By a 2e argument, it also yields a spatial modulus for any solution \(u;\) uniform in \(t \in [0, T].\) Indeed, for fixed \(t \leq T\) pick \(\alpha\) large enough so that \(l(\alpha) < \epsilon/2;\) for any \(x, y : |x - y|\) small enough (only depending on \(\alpha\) and hence \(\epsilon\)) we have \(u(t, x) - u(t, y) < \epsilon\). By switching the roles of \(x\) and \(y,\) if necessary, we see \(|u(t, x) - u(t, y)| < \epsilon.\)

4. Parabolic Comparison: Proof of (1)

Assume \(u \in \text{bUSC} ([0, T) \times \mathbb{R}^n)\) solves \(\partial_t u - F(t, x, u, Du, D^2u) \leq 0\) with "properness" \(\gamma \geq 0;\) with initial data \(u(0, \cdot) \in [0, T) \times \mathbb{R}^n.\) Extend \(u\) to \(\text{bUSC} ([0, T] \times \mathbb{R}^n)\) by setting

\[ u(T, x) = \lim_{t \uparrow T} \sup_{y \rightarrow x} u(t, y) \]

Assume \(u - \phi\) has a (strict) max at \((T, \bar{x}),\) relative to \([0, T] \times \mathbb{R}^n.\) (The Test function \(\phi\) is defined in an open neighbourhood of \([0, T] \times \mathbb{R}^n.)\) Claim that

\[ \partial_t \phi(T, \bar{x}) - F(T, \bar{x}, u(T, \bar{x}), D\phi(T, \bar{x}), D^2\phi(T, \bar{x})) \leq 0. \]

\[ \text{Proof: Take } (t^n, x^n) \in (0, T) \times \mathbb{R}^n \text{ s.t. } (t^n, x^n) \rightarrow (T, \bar{x}) \text{ and } u(t^n, x^n) \rightarrow u(T, \bar{x}). \text{ Set } \alpha_n := T - t^n \downarrow 0. \text{ Then take } \]

\[ (t_n, x_n) \in \text{arg max} \left( u - \phi - \frac{\alpha_n^2}{T - t} \right) = \text{arg max } \psi_n, \]

over \([0, T] \times \mathbb{R}^n.\) In order to guarantee that the sequence \((t_n, x_n) \in [0, T) \times \mathbb{R}^n\) remains in a compact, say \([T/2, T] \times B_1(\bar{x}),\) we make the assumption (without loss of generality) that \(\psi_n\) is expressed (equivalently) in terms of "touching" test-functions or in term of sub- and super-jets. We shall switch between these points without further comments.
of generality) that \( \phi(T, \bar{x}) = 0 \) and \( \phi(t, x) > 3|u|_\infty \) for \((t, x) \notin [T/2, T] \times \bar{B}_1(\bar{x})\); this implies \((t_n, x_n) \in [T/2, T] \times \bar{B}_1(\bar{x})\) for \(n\) large enough, as desired. By compactness, \((t_n, x_n) \to (\bar{t}, \bar{x})\) at least along a subsequence \(n(k)\). We shall run through the other sequence \((t^n, x^n)\) along the same subsequence and relabel both to keep the same notation. Note \(\psi_n(t_n, x_n)\) is non-decreasing and bounded, hence

\[
\psi_n(t_n, x_n) \to l.
\]

Since \(\psi_n(t_n, x_n) \leq (u - \phi)(t_n, x_n)\) it follows (using USC of \(u - \phi\)) that

\[
l \leq (u - \phi)(\bar{t}, \bar{x})
\]

On the other hand,

\[
\psi_n(t_n, x_n) \geq \psi_n(t^n, x^n) = (u - \phi)(t^n, x^n) - \frac{\alpha^2}{T - t^n}
\]

and hence \(l \geq (u - \phi)(T, \bar{x})\). Since \((T, \bar{x})\) was a strict maximum point for \(u - \phi\) conclude that \((\bar{t}, \bar{x}) = (T, \bar{x})\) is the common limit of the sequences \((t^n, x^n), (t_n, x_n)\). Now we note that

\[
(u - \phi)(t_n, x_n) \geq \psi_n(t_n, x_n) \geq (u - \phi)(t^n, x^n) - \alpha_n
\]

which implies that \((\alpha(1) \to 0 \text{ as } n \to \infty)\)

\[
u(t_n, x_n) \geq u(t^n, x^n) + o(1)
\]

By definition of a subsolution,

\[
\partial_t \phi(t_n, x_n) - F(t_n, x_n, u(t_n, x_n), D\phi(t_n, x_n), D^2\phi(t_n, x_n)) \leq 0
\]

and hence, using properness of \(F = F(u)\), omitting the other arguments, "with \(u = u(t_n, x_n)\) and \(v = u(t^n, x^n) + o(1)\)";

\[
-F(u(t_n, x_n)) \geq -F(u(t^n, x^n) + o(1)) + \gamma(u(t_n, x_n) - (u(t^n, x^n) + o(1)))
\]

also using uniform continuity of \(F\) as function of \(u\) over compacts, we obtain

\[
\partial_t \phi(t_n, x_n) - F(t_n, x_n, u(t^n, x^n), D\phi(t_n, x_n), D^2\phi(t_n, x_n)) \leq o(1).
\]

Sending \(n \to \infty\) yields (use continuity of \(\phi\) and \(F\))

\[
\partial_t \phi(T, \bar{x}) - F(T, \bar{x}, u(T, \bar{x}), D\phi(T, \bar{x}), D^2\phi(T, \bar{x})) \leq 0,
\]

as desired.

5. Parabolic Comparison: Proof of (ii)

Proof. By assumption, \(u(t, x) - v(t, y)\) is bounded on \([0, T] \times \mathbb{R}^n \times \mathbb{R}^n\). Let \((\tilde{t}, \tilde{x}, \tilde{y})\) be a maximum point of

\[
\phi(t, x, y) := u(t, x) - v(t, y) - \frac{\alpha}{2} |x - y|^2 - \varepsilon \left(|x|^2 + |y|^2\right)
\]

over \([0, T] \times \mathbb{R}^n \times \mathbb{R}^n\) where \(\alpha > 0\) and \(\varepsilon > 0\); such a maximum exists since \(\phi \in \text{USC}([0, T] \times \mathbb{R}^n \times \mathbb{R}^n)\) and \(\phi \to -\infty\) as \(|x|, |y| \to \infty\). (The presence \(\varepsilon > 0\) amounts to a barrier at \(\infty\) in space.) The plan is to show a "key estimate" of the form

\[
u(t, x) - v(t, y) \leq \inf_{\alpha} \left[\frac{\alpha}{2} |x - y|^2 + l(\alpha)\right],
\]
valid on \([0, T] \times \mathbb{R}^n \times \mathbb{R}^n\), where \(l(\alpha)\) tends to 0 as \(\alpha \uparrow \infty\). Thanks to the very definition of \((\hat{t}, \hat{x}, \hat{y})\) as \(\arg\max\) of \(\phi(t, x, y) = u(t, x) - v(t, y) - \frac{\alpha}{2} |x - y|^2 - \varepsilon \left(|x|^2 + |y|^2\right)\), we obtain the estimate
\[
   u(t, x) - v(t, y) \leq \frac{\alpha}{2} |x - y|^2 + \varepsilon \left(|x|^2 + |y|^2\right) + \phi(\hat{t}, \hat{x}, \hat{y}).
\]

Note that \((\hat{t}, \hat{x}, \hat{y})\) depends on \(\alpha, \varepsilon\). We shall consider the cases \(\hat{t} = 0\) and \(\hat{t} \in (0, T]\) separately. In the first case \(\hat{t} = 0\) we have
\[
   \phi(0, \hat{x}, \hat{y}) = \sup_{x,y} \left[u_0(x) - v_0(y) - \frac{\alpha}{2} |x - y|^2 - \varepsilon \left(|x|^2 + |y|^2\right)\right] =: A_{\alpha, \varepsilon}
\]
and lemma \([\text{I}]\) below asserts that \(A_{\alpha, \varepsilon} \to \sup_x [u_0(x) - v_0(x)] \leq 0\) as \((\varepsilon, \alpha) \to (0, \infty)\). The second case is \(\hat{t} \in (0, T]\) and we will show
\[
   \phi(\hat{t}, \hat{x}, \hat{y}) \leq B_{\alpha, \varepsilon} \text{ where } \left(\limsup_{\varepsilon \to 0} B_{\alpha, \varepsilon}\right) \to 0 \text{ as } \alpha \to \infty;
\]

it is here that we will use theorem of sums and viscosity properites. (Since
\[
   \phi(\hat{t}, \hat{x}, \hat{y}) \leq u(\hat{t}, \hat{x}) - v(\hat{t}, \hat{y})
\]
we can and will use the fact that it is enough to consider the case \(u(\hat{t}, \hat{x}) - v(\hat{t}, \hat{y}) \geq 0\).) Leaving the details of this to below, let us quickly complete the argument: our discussion of the two cases above gives \(\phi(\hat{t}, \hat{x}, \hat{y}) \leq A_{\alpha, \varepsilon} \lor B_{\alpha, \varepsilon}\) and hence
\[
   u(t, x) - v(t, y) \leq \frac{\alpha}{2} |x - y|^2 + \varepsilon \left(|x|^2 + |y|^2\right) + A_{\alpha, \varepsilon} \lor B_{\alpha, \varepsilon};
\]
we emphasize that this estimate is valid for all \(t, x, y \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n\) and \(\alpha, \varepsilon > 0\). Take now \(\limsup_{\varepsilon \to 0}\) on the right hand side, then optimize over \(\alpha > 0\), to obtain the key estimate
\[
   u(t, x) - v(t, y) \leq \inf_\alpha \left\{\frac{\alpha}{2} |x - y|^2 + l(\alpha)\right\}
\]
where we may take
\[
   l(\alpha) := \limsup_{\varepsilon \to 0} A_{\alpha, \varepsilon} \lor \limsup_{\varepsilon \to 0} B_{\alpha, \varepsilon},
\]
noting that \(l(\alpha)\) indeed tends to 0 as \(\alpha \to \infty\). It remains to prove the estimate \((5.3)\). To this end, rewrite \(\phi\) as
\[
   \phi(t, x, y) = u^\varepsilon(t, x) - v^\varepsilon(t, y) - \frac{\alpha}{2} |x - y|^2
\]
where \(u^\varepsilon(t, x) = u(t, x) - \varepsilon |x|^2\) and \(v^\varepsilon(t, y) = v(t, y) + \varepsilon |y|^2\). Since \(u^\varepsilon\) (resp. \(v^\varepsilon\)) are upper (resp. lower) semi-continuous we can apply the (parabolic) theorem of sums as given in the appendix at \((\hat{t}, \hat{x}, \hat{y}) \in (0, T] \times \mathbb{R}^n \times \mathbb{R}^n\) to learn that there are numbers \(a, b\) and \(X, Y \in S^n\) such that
\[
   (a, \alpha (\hat{x} - \hat{y}), X) \in \mathcal{P}^{2,+} u^\varepsilon(\hat{t}, \hat{x}), \quad (b, \alpha (\hat{x} - \hat{y}), Y) \in \mathcal{P}^{2,-} v^\varepsilon(\hat{t}, \hat{y})
\]
such that \(a - b \geq 0\) (equality if \(\hat{t} \in (0, T]\), although this does not matter), and such that one has the two-sided matrix estimate \((2.2)\). It is easy to see (cf. \([\text{I}0\] Remark 2.7)) that \((5.3)\) is equivalent to
\[
   (a, \alpha (\hat{x} - \hat{y}) + 2\varepsilon \hat{x}, X + 2\varepsilon I) \in \mathcal{P}^{2,+} u(\hat{t}, \hat{x}), \quad (b, \alpha (\hat{x} - \hat{y}) - 2\varepsilon \hat{y}, Y - 2\varepsilon I) \in \mathcal{P}^{2,-} v(\hat{t}, \hat{y}).
\]
Using the viscosity sub- and super-solution properties (and part (i) in the case that \( t = T \)) we then see that
\[
a - F \left( \hat{t}, \hat{x}, u \left( \hat{t}, \hat{x} \right), \alpha (\hat{x} - \hat{y}) + 2\varepsilon \hat{x}, X + 2\varepsilon I \right) \leq 0, \\
b - F \left( \hat{t}, \hat{y}, v \left( \hat{t}, \hat{y} \right), \alpha (\hat{x} - \hat{y}) - 2\varepsilon \hat{y}, Y - 2\varepsilon I \right) \geq 0.
\]

Note that (using \( a - b \geq 0 \))
\[
(5.5) \\
F \left( \hat{t}, \hat{y}, v \left( \hat{t}, \hat{y} \right), \alpha (\hat{x} - \hat{y}) - 2\varepsilon \hat{y}, Y - 2\varepsilon I \right) - F \left( \hat{t}, \hat{x}, u \left( \hat{t}, \hat{x} \right), \alpha (\hat{x} - \hat{y}) + 2\varepsilon \hat{x}, X + 2\varepsilon I \right) \leq 0
\]
Trivially, (recall it is enough to consider the case \( u \left( \hat{t}, \hat{x} \right) \geq v \left( \hat{t}, \hat{y} \right) \))
\[
\gamma \phi \left( \hat{t}, \hat{x}, \hat{y} \right) \leq \gamma (u \left( \hat{t}, \hat{x} \right) - v \left( \hat{t}, \hat{y} \right)) \\
\leq F \left( \hat{t}, \hat{y}, v \left( \hat{t}, \hat{y} \right), \alpha (\hat{x} - \hat{y}) - 2\varepsilon \hat{y}, Y - 2\varepsilon I \right) \\
- F \left( \hat{t}, \hat{y}, u \left( \hat{t}, \hat{x} \right), \alpha (\hat{x} - \hat{y}) - 2\varepsilon \hat{y}, Y - 2\varepsilon I \right) \\
\leq F \left( \hat{t}, \hat{y}, v \left( \hat{t}, \hat{y} \right), \alpha (\hat{x} - \hat{y}) - 2\varepsilon \hat{y}, Y - 2\varepsilon I \right) \\
- F \left( \hat{t}, \hat{y}, u \left( \hat{t}, \hat{y} \right), \alpha (\hat{x} - \hat{y}) - 2\varepsilon \hat{y}, Y - 2\varepsilon I \right)
\]
\[
\leq F \left( \hat{t}, \hat{x}, u \left( \hat{t}, \hat{x} \right), \alpha (\hat{x} - \hat{y}) + 2\varepsilon \hat{x}, X + 2\varepsilon I \right) \\
- F \left( \hat{t}, \hat{x}, v \left( \hat{t}, \hat{x} \right), \alpha (\hat{x} - \hat{y}) - 2\varepsilon \hat{y}, Y - 2\varepsilon I \right)
\]
where we used \((5.5)\) in the last estimate. If \( \varepsilon \) were absent (e.g. set \( \varepsilon = 0 \) throughout) we would estimate, with \( R := |u|_\infty \vee |v|_\infty \),
\[
F \left( \hat{t}, \hat{x}, u \left( \hat{t}, \hat{x} \right), \alpha (\hat{x} - \hat{y}), X \right) - F \left( \hat{t}, \hat{y}, u \left( \hat{t}, \hat{x} \right), \alpha (\hat{x} - \hat{y}), Y \right) \leq \theta_R \left( \alpha |\hat{x} - \hat{y}|^2 + |\hat{x} - \hat{y}| \right) =: B_\alpha
\]
and since \( \alpha |\hat{x} - \hat{y}|^2 + |\hat{x} - \hat{y}| \leq 2\alpha |\hat{x} - \hat{y}|^2 + 1/\alpha \to 0 \) as \( \alpha \to \infty \), thanks to \([10]\) lemma 3.1, we see that \( B_\alpha \to 0 \) with \( \alpha \to \infty \), which is enough to conclude. The present case where \( \varepsilon > 0 \) is essentially reduced to the case \( \varepsilon = 0 \) by adding/subtracting
\[
F \left( \hat{t}, \hat{x}, u \left( \hat{t}, \hat{x} \right), \alpha (\hat{x} - \hat{y}), X \right) - F \left( \hat{t}, \hat{y}, u \left( \hat{t}, \hat{x} \right), \alpha (\hat{x} - \hat{y}), Y \right),
\]
but we need some refined properties of \( \left( \hat{t}, \hat{x}, \hat{y} \right) \) as collected in lemma\([2]\) (a) \( p = \alpha (\hat{x} - \hat{y}) \) remains, for fixed \( \alpha \), bounded as \( \varepsilon \to 0 \), (b) \( 2\varepsilon |\hat{x}| \) and \( 2\varepsilon |\hat{y}| \) tend to zero as \( \alpha \to 0 \) for fixed (large enough) \( \alpha \); this follows from the fact, that for \( \alpha \) large enough we must have \( \limsup_{\varepsilon \to 0} \varepsilon |\hat{x}| = c_\alpha < \infty \) (after all, \( c_\alpha \) tends to zero with \( \alpha \to \infty \)) and by rewriting \( \limsup_{\varepsilon \to 0} \varepsilon |\hat{x}| \leq \sqrt{c_\alpha} \limsup_{\varepsilon \to 0} \sqrt{\varepsilon} = 0, \) (c) that \( \limsup_{\varepsilon \to 0} \left( \frac{2}{\varepsilon} |\hat{x} - \hat{y}|^2 + |\hat{x} - \hat{y}| \right) \to 0 \) as \( \alpha \to \infty \). We also note that \((2.2)\) implies (d): any matrix norm of \( X, Y \) is bounded by a constant times \( \alpha \), independent of \( \varepsilon \).
We can now return to the estimate of \( \phi \) and clearly have
\[
\phi \left( \hat{t}, \hat{x}, \hat{y} \right) \leq \frac{1}{\gamma} \left[ (i) + (ii) + (iii) \right] =: B_{\alpha, \varepsilon}
\]
where

\[(i) = |F(\hat{t}, \hat{x}, u(\hat{t}, \hat{x}), \alpha(\hat{x} - \hat{y}) + 2\varepsilon\hat{x}, X + 2\varepsilon I) - F(\hat{t}, \hat{x}, u(\hat{t}, \hat{x}), \alpha(\hat{x} - \hat{y}), X)|\]

\[(ii) = |F(\hat{t}, \hat{y}, u(\hat{t}, \hat{x}), \alpha(\hat{x} - \hat{y}) - 2\varepsilon\hat{y}, Y - 2\varepsilon I) - F(\hat{t}, \hat{y}, u(\hat{t}, \hat{x}), \alpha(\hat{x} - \hat{y}), Y)|\]

\[(iii) = \theta_R \left(\alpha |\hat{x} - \hat{y}|^2 + |\hat{x} - \hat{y}| \right).\]

From (a),(d) above the gradient and Hessian argument in \(F\) as seen in (i), (ii), i.e.

\[\alpha(\hat{x} - \hat{y}) \pm 2\varepsilon\hat{x} \text{ and } X + 2\varepsilon I, Y - 2\varepsilon I,\]

remain in a bounded set, for fixed \(\alpha\), uniformly as \(\varepsilon \to 0\). From (b) above and the assumed uniform continuity properties of \(F\), it then follows that for fixed (large enough) \(\alpha\)

\[(i), (ii) \to 0 \text{ as } \varepsilon \to 0.\]

On the other hand, continuity of \(\theta_R\) at 0+ together with (c) above shows that also

\[(iii) \to 0 \text{ as } \varepsilon << \frac{1}{\alpha} \to 0,\]

which implies \(\Box\), as desired. The proof is now finished. \(\square\)

**Lemma 1 (H Lemme 2.9).** Assume \(u_0, v_0 \in BUC(\mathbb{R}^n)\). Then

\[\sup_{x,y} \left[|u_0(x) - v_0(y) - \alpha |x - y|^2| - \varepsilon \left(|x|^2 + |y|^2\right)\right] \to \sup_x |u_0(x) - v_0(x)| \text{ as } (\varepsilon, \frac{1}{\alpha}) \to (0, 0).\]

**Proof.** Without loss of generality \(M := \sup_x |u_0(x) - v_0(x)| > 0\); for otherwise replace \(u_0\) by \(u_0 + 2 |M|\). Write \(M_{\alpha,\varepsilon}\) for the achieved maximum (at \(\hat{x}, \hat{y}\), say) of the left-hand-side. Obviously, \(u_0(x) - v_0(x) - 2\varepsilon |x|^2 \leq M_{\alpha,\varepsilon}\) for any \(x\) and so

\[M \leq \lim_{\varepsilon \to 0} \inf_{\alpha \to \infty} M_{\alpha,\varepsilon}.\]

(It follows that we can and will consider \(\varepsilon (\alpha)\) small (large) enough so that \(M_{\alpha,\varepsilon} > 0\.) On the other hand, \(|u_0|, |v_0| \leq R < \infty\) and so

\[0 \leq M_{\alpha,\varepsilon} \leq 2R - \frac{\alpha}{2} |\hat{x} - \hat{y}|^2 - \varepsilon \left(|\hat{x}|^2 + |\hat{y}|^2\right)\]

from which we deduce \(\frac{\alpha}{2} |\hat{x} - \hat{y}|^2 \leq 2R\), or \(|\hat{x} - \hat{y}| \leq \sqrt{4R/\alpha}\). By omitting the (positive) penalty terms, we can also estimate

\[M_{\alpha,\varepsilon} \leq u_0(\hat{x}) - v_0(\hat{y}) \leq u_0(\hat{x}) - v_0(\hat{y}) + \sigma_{v_0} \left(\sqrt{4R/\alpha}\right) \leq M + \sigma_{v_0} \left(\sqrt{4R/\alpha}\right)\]

where \(\sigma_{v_0}\) denotes the modulus of continuity of \(v_0\). It follows that

\[\limsup_{\varepsilon \to 0} \alpha \to \infty M_{\alpha,\varepsilon} \leq M\]

which shows that the lim \(M_{\alpha,\varepsilon}\) (as \(\varepsilon \to 0, \alpha \to \infty\)) exists and is equal to \(M\). \(\square\)
Lemma 2. Let \( u \in \text{bUSC} ([0, T] \times \mathbb{R}^n) \) and \( v \in \text{bLSC} ([0, T] \times \mathbb{R}^n) \). Consider a maximum point \((\hat{t}, \hat{x}, \hat{y}) \in (0, T) \times \mathbb{R}^n \times \mathbb{R}^n\) of

\[
\phi(t, x, y) = u(t, x) - v(t, y) - \frac{\alpha}{2} |x - y|^2 - \varepsilon \left( |x|^2 + |y|^2 \right).
\]

where \( \alpha, \varepsilon > 0 \). Then

(5.6) \( \limsup_{\varepsilon \to 0} \alpha (\hat{x} - \hat{y}) = C(\alpha) < \infty, \)

(5.7) \( \limsup_{\alpha \to \infty} \limsup_{\varepsilon \to 0} \varepsilon \left( |\hat{x}|^2 + |\hat{y}|^2 \right) = 0, \)

(5.8) \( \limsup_{\alpha \to \infty} \limsup_{\varepsilon \to 0} \left( \frac{\alpha}{2} |\hat{x} - \hat{y}|^2 + |\hat{x} - \hat{y}| \right) = 0. \)

Remark 3. A similar lemma (without \( t \) dependence) is found in Barles’ book [1] Lemme 4.3; the order in which limits are taken is important and suggests the notation

\[
\limsup_{\varepsilon \to 0 < \frac{\sqrt{\alpha}}{2} \to 0} \alpha (\hat{x} - \hat{y}) = C(\alpha) < \infty, \quad \liminf_{\varepsilon \to 0 < \frac{\sqrt{\alpha}}{2} \to 0} \alpha (\hat{x} - \hat{y}) = C(\alpha) < \infty.
\]

Proof. We start with some notation, where unless otherwise stated \( t \in [0, T] \) and \( x, y \in \mathbb{R}^n \),

\[
M_{\alpha, \varepsilon} := \sup_{t, x, y} \phi(t, x, y) = u(\hat{t}, \hat{x}) - v(\hat{t}, \hat{y}) - \frac{\alpha}{2} |\hat{x} - \hat{y}|^2 - \varepsilon \left( |\hat{x}|^2 + |\hat{y}|^2 \right);
\]

\[
M(h) := \sup_{t, x, y : |x - y| \leq h} [u(t, x) - v(t, y)] \geq \sup_{t, x} [u(t, x) - v(t, x)]
\]

\[
M' := \lim_{h \to 0} M(h).
\]

(As indicated, \( M' \) exists as limit of \( M(h) \), non-increasing in \( h \) and bounded from below.)

Step 1: Take \( t = x = y = 0 \) as argument of \( \phi(t, x, y) \). Since \( M_{\alpha, \varepsilon} = \sup \phi \) we have

\[
c = u(0, 0) - v(0, 0) \leq M_{\alpha, \varepsilon} = u(\hat{t}, \hat{x}) - v(\hat{t}, \hat{y}) - \frac{\alpha}{2} |\hat{x} - \hat{y}|^2 - \varepsilon \left( |\hat{x}|^2 + |\hat{y}|^2 \right)
\]

and hence, for a suitable constant \( C \) (e.g. \( C^2 := \sup u + \sup (-v) + c \))

\[
\frac{\alpha}{2} |\hat{x} - \hat{y}|^2 + \varepsilon \left( |\hat{x}|^2 + |\hat{y}|^2 \right) \leq C^2
\]

which implies

(5.9) \( |\hat{x} - \hat{y}| \leq C \sqrt{2/\alpha} \)

and hence \( \alpha |\hat{x} - \hat{y}| \leq \sqrt{2\alpha} C \) which is the first claimed estimate (5.6).

Step 2: We first argue that it is enough to show the (two) estimates

(5.10) \( \limsup_{\varepsilon \to 0 < \frac{\sqrt{\alpha}}{2} \to 0} [u(\hat{t}, \hat{x}) - v(\hat{t}, \hat{y})] \leq M' \leq \liminf_{\varepsilon \to 0 < \frac{\sqrt{\alpha}}{2} \to 0} M_{\alpha, \varepsilon}. \)
Indeed, from \( \frac{\alpha}{2} |\dot{x} - \dot{y}|^2 + \varepsilon \left( |\dddot{x}|^2 + |\dddot{y}|^2 \right) = u \left( \dot{t}, \dot{x} \right) - v \left( \dot{t}, \dot{y} \right) - M_{\alpha, \varepsilon} \) it readily follows that
\[
\limsup_{\varepsilon << \frac{1}{\alpha} \to 0} \frac{\alpha}{2} |\dot{x} - \dot{y}|^2 + \varepsilon \left( |\dddot{x}|^2 + |\dddot{y}|^2 \right) \leq \limsup_{\varepsilon << \frac{1}{\alpha} \to 0} \left[ u \left( \dot{t}, \dot{x} \right) - v \left( \dot{t}, \dot{y} \right) - M_{\alpha, \varepsilon} \right] = \limsup_{\varepsilon << \frac{1}{\alpha} \to 0} \left[ u \left( \dot{t}, \dot{x} \right) - v \left( \dot{t}, \dot{y} \right) \right] - \liminf_{\varepsilon << \frac{1}{\alpha} \to 0} M_{\alpha, \varepsilon} \leq 0 \text{ (and hence } = 0)\.
\]
This already gives (5.7) and also (5.8), noting that
\[
\left| \dddot{x} \right| \leq \lim \sup_{\varepsilon \to 0} \left| \dddot{x} \right| \leq \lim \sup_{\varepsilon \to 0} \left| \dddot{x} \right| \leq \lim \sup_{\varepsilon \to 0} \left| \dddot{x} \right| = 0 \text{ (and hence } = 0)\.
\]

We now turn to the second estimate in (5.10). From the very definition of \( M(h) \) applied with \( h = C \sqrt{2/\alpha} \),
\[
\limsup_{\varepsilon << \frac{1}{\alpha} \to 0} \left[ u \left( \dot{t}, \dot{x} \right) - v \left( \dot{t}, \dot{y} \right) \right] \leq \limsup_{\varepsilon << \frac{1}{\alpha} \to 0} M \left( \frac{\sqrt{2/\alpha}}{2} \right) = \lim_{\alpha \to \infty} M \left( \frac{\sqrt{2/\alpha}}{2} \right) = M'.
\]

We now turn to the second estimate in (5.10). From the very definition of \( M' \) as \( \lim_{h \to 0} M(h) \), there exists a family \( (t_h, x_h, y_h) \) so that
\[
(5.11) \quad |x_h - y_h| \leq h \text{ and } u(t_h, x_h) - v(t_h, x_h) \to M' \text{ as } h \to 0
\]
For every \( \alpha, \varepsilon \) we may take \( (t_h, x_h, y_h) \) as argument of \( \phi \); since \( M_{\alpha, \varepsilon} = \sup \phi \) we have
\[
(5.12) \quad u(t_h, x_h) - v(t_h, y_h) - \frac{\alpha}{2} h^2 - \varepsilon(|x_h|^2 + |y_h|^2) \leq M_{\alpha, \varepsilon}
\]
Take now \( \varepsilon = \varepsilon(h) \to 0 \) with \( h \to 0 \); fast enough so that \( \varepsilon(|x_h|^2 + |y_h|^2) \to 0 \); for instance \( \varepsilon(h) := h / (1 + (|x_h|^2 + |y_h|^2)) \) would do. It follows that
\[
M' = \lim_{h \to 0} u(t_h, x_h) - v(t_h, y_h) = \liminf_{h \to 0} u(t_h, x_h) - v(t_h, y_h) - \frac{\alpha}{2} h^2 - \varepsilon(|x_h|^2 + |y_h|^2) \leq \liminf_{h \to 0} \liminf_{\varepsilon \to 0} M_{\alpha, \varepsilon} \text{ by monotonicity of } M_{\alpha, \varepsilon} \text{ in } \varepsilon.
\]
Since this is valid for every \( \alpha \), we also have
\[
M' \leq \liminf_{\alpha \to \infty} \liminf_{\varepsilon \to 0} M_{\alpha, \varepsilon}.
\]
This is precisely the second estimate in (5.10) and so the proof is finished. \( \square \)

6. BUC \([0, T] \times \mathbb{R}^n\) viscosity solutions

If \( F \) satisfies the above structural condition with the further strengthening that \( F \) is bounded whenever \( u, p, X \) remain bounded, then any bounded viscosity solution with \( \text{BUC}(\mathbb{R}^n) \) initial data is in \( \text{BUC}([0, T] \times \mathbb{R}^n) \). More precisely,
Corollary 1. Assume $F$ satisfies the assumptions of section 2 with assumption 3 strengthened to

$$\forall R > 0 : F|_{[0,T] \times \mathbb{R}^n \times [-R, R] \times B_R \times M_R}$$
is bounded, uniformly continuous.

Let $u \in \mathcal{BC} ([0, T] \times \mathbb{R}^n)$ be a viscosity solution to $\partial_t - F = 0$ on $(0, T) \times \mathbb{R}^n$ with initial data $u_0 = u(0, \cdot) \in \mathcal{BUC} (\mathbb{R}^n)$. Then

$$u = u(t, x) \in \mathcal{BUC} ([0, T] \times \mathbb{R}^n).$$

Proof. We adapt the argument from [3, Lemma 9.1]. From theorem [4] there exists a spatial modulus $m$ for $u(t, \cdot)$, uniform over $t \in [0, T]$. Given $0 \leq t_0 < t \leq T$ and $x_0, x \in \mathbb{R}^n$ we now estimate, using the triangle inequality,

$$|u(t, x) - u(t_0, x_0)| \leq m(|x_0 - x|) + |u(t, x) - u(t_0, x_0)|.$$

We shall show that $|u(t, x_0) - u(t_0, x_0)|$ goes to zero as $t \to t_0$, uniformly in $x \in \mathbb{R}^n$ and $t_0 \in [0, T)$. We will show a little more. Fix $x_0 \in \mathbb{R}^n$ and $R \in (0, \infty)$; for instance $R = 1$ would do (and there is no need to track dependence in $R$). We claim that for every $\eta > 0$ one can find constants $C = C(\eta), K = K(\eta)$, not dependent on $x_0$ and $t_0$, such that, for all $x \in B_R(x_0)$ and $y \in B_R(x_0)$ and all $t \in [t_0, T]$

$$u(t, y) - u(t_0, x) \leq \eta + C|y - x|^2 + K(t - t_0)$$

and

$$u(t, y) - u(t_0, x) \geq -\eta - C|y - x|^2 - K(t - t_0).$$

(Choosing $x = y = x_0$ in these estimates shows that $|u(t, x_0) - u(t_0, x_0)| \leq \inf \{\eta + K(\eta) (t - t_0) : \eta > 0\}$ which immediately gives the desired uniform continuity in time, uniformly in $x_0$.) We only prove (6.2), (6.3) being proved in an analogous way. In the sequel, $x$ is fixed in $B_R(x_0)$. We write (6.2) as

$$u - \chi \leq 0 \text{ on } [t_0, T] \times B_R(x_0)$$

where $\chi(t, y) := u(t_0, x) + \eta + C|y - x|^2 + K(t - t_0)$. We shall see below we can find $C$, the choice of which only depends on $\eta$ (and in a harmless way on $|u|_{\infty;[t_0, T] \times \mathbb{R}^n \times R}$ and $m(\cdot)$ but not on $K$ and not on $x_0, t_0$), such that $u - \chi \leq 0$ on the parabolic boundary of $[t_0, T] \times B_R(x_0)$. The extension to the interior is then based on the maximum principle. More precisely, we can chose $K$ depending on $\eta$ (and again in a harmless way $|u|_{\infty;[t_0, T] \times \mathbb{R}^n \times R}$ and $m(\cdot)$) such that $\chi$ is a (smooth) strict supersolution of $\partial_t - F$ on $(t_0, T) \times B_R(x_0)$:

$$K - F(t, y, \chi(t, y), 2C(y - x), 2CI) > 0 \text{ on } (t_0, T) \times B_R(x_0).$$

Indeed, by properness we have

$$K - F(t, y, \chi(t, y), 2C(y - x), 2CI) > K - F(t, y, -|u|_{\infty}, 2C(y - x), 2CI);$$

noting $|y - x| \leq 2R$ so that $p := 2C(y - x), X := 2CI$ remain in a bounded set where size may depend on $\eta$ through $C$, it then follows by our structural assumption on the non-linearity that we can pick $K = K(\eta)$ large enough such as to achieve the claimed strict inequality. (Note that this choice of $K$ is uniformly in $t_0$ provided we can find $C$ with the correct dependences.) Since, on the other hand, $u$ is

\footnote{... notably boundedness of $F(\cdot, \cdot, y, p, X)$ when $y, p, X$ remain in a bounded set...}
Let $u$ be a viscosity solution (hence subsolution), it follows from the very definition of a subsolution that
\[
K - F (t, \hat{y}, \chi(t, y), 2C (\hat{y} - x), 2CI) \leq 0
\]
whenever $(t, \hat{y}) \in (t_0, T) \times B_R (x_0)$ is a maximum point of $u - \chi$. (Note that $t = T$ is possible here, we then rely on part (i) of theorem 1.) This contradiction shows that maximum points of $u - \chi$ over $[t_0, T] \times B_R (x_0)$ are necessarily achieved on the parabolic boundary
\[
(t, y) \in [t_0, T] \times \partial B_R (x_0) \cup \{t_0\} \times \bar{B}_R (x_0).
\]
The remainder of the proof is thus concerned with showing that $u - \chi \leq 0$ on this parabolic boundary. Consider first the case that $t \in [t_0, T]$ and $|y - x_0| = R$. Since $x \in B_{R/2} (x_0)$ we must have $|y - x| \geq R/2$ and it thus suffices to take $C \geq 8 |u|_{\infty, [0, T] \times \mathbb{R}^n} / R^2$ to ensure that
\[
 u(t_0, y) \leq u(t_0, x) + \eta + C |y - x|^2 + K (t - t_0)
\]
for all $t \in [t_0, T]$ and $y \in B_R (x_0)$, and any $\eta, K \geq 0$. The second case to be considered is $t = t_0$ and $y \in B_R (x_0)$. We want to see that for every $\eta$ there exists $C = C (\eta)$ such that
\[
 u(t_0, y) \leq u(t_0, x) + \eta + C |y - x|^2 \quad \text{for all } y \in B_R (x_0);
\]
but this follows immediately from the fact (cf. theorem 1) that $u(t_0, \cdot)$ has a spatial modulus $m$. Indeed: If there were $\eta > 0$ such that for all $C$ there are points $y_C$ so that $u(t_0, y) > u(t_0, x) + \eta + C |y - x|^2$, then $|y_C - x|^2 \leq 2 |u|_{\infty, [0, T] \times \mathbb{R}^n} / C \to 0$ with $C \to \infty$ and a contradiction to
\[
m(|y_C - x|) \geq \delta(0, y) - u(0, x) \geq \eta > 0.
\]
is obtained as soon as $C$ is chosen large enough and this choice depends only on $\eta, |\n|_{\infty, [0, T] \times \mathbb{R}^n}$ and $m$. Since all these quantities are independent of $t_0$, so is our choice of $C$.

7. Existence

At last, we discuss existence via Perron’s Method; the only difficulty in the proof is to produce subsolutions and supersolutions.

**Theorem 2.** Assume $F$ satisfies the assumptions of section 2 with assumption 3 strengthened to
\[
\forall R > 0 : F|_{[0, T] \times \mathbb{R}^n \times [-R, R] \times B_R \times \mathcal{M}_R} \text{ is bounded, uniformly continuous.}
\]
Let $u_0 \in \text{BUC} (\mathbb{R}^n)$. Then there exists $u = u(t, x) \in \text{BUC} ([0, T] \times \mathbb{R}^n)$ such that $u$ is a viscosity solution to the initial value problem
\[
\begin{align*}
\partial_t F &= 0 \text{ on } (0, T) \times \mathbb{R}^n, \\
u(0, \cdot) &= u_0.
\end{align*}
\]
(By Theorem 1 this solution is unique in the class of bounded viscosity solutions.)

**Proof.** Step 1: Assume $u_0$ is Lipschitz continuous with Lipschitz constant $L$. Define for $z \in \mathbb{R}^n, \varepsilon > 0$
\[
\psi_{\varepsilon, z} (x) := u_0(z) - L \left( |x - z|^2 + \varepsilon \right)^{1/2}.
\]
We will show that there exists $A_\varepsilon \leq 0$ (non-positive, yet to be chosen) such that

$$u_{\varepsilon,z}(t,x) := A_\varepsilon t + \psi_{\varepsilon,z}(x)$$

is a (classical) subsolution of $\partial_t - F = 0$. To this end we first note that $Du_{\varepsilon,z} = D\psi_{\varepsilon,z}$ and $D^2 u_{\varepsilon,z} = D^2 \psi_{\varepsilon,z}$ are bounded by $LC_\varepsilon$ where $C$ is a constant dependent on $\varepsilon$. We also note that (for any non-positive choice of $A_\varepsilon$)

$$u_{\varepsilon,z}(t,x) \leq u_{\varepsilon,z}(0,x) = \psi_{\varepsilon,z}(x) \leq u_0(z) - L|x-z| \leq u_0(x),$$

thanks to $L$-Lipschitzness of $u_0$. Since $F = F(t,x,u,p,X)$ is assumed to be proper, and thus in particular anti-monotone in $u$, we have

$$\partial_t u_{\varepsilon,z} - F(t,x,u_{\varepsilon,z},Du_{\varepsilon,z},D^2 u_{\varepsilon,z}) = A_\varepsilon - F(t,x,u_{\varepsilon,z},D\psi_{\varepsilon,z},D^2 \psi_{\varepsilon,z}) \leq A_\varepsilon - F(t,x,u_0,|u_0|_\infty,D\psi_{\varepsilon,z},D^2 \psi_{\varepsilon,z}).$$

Since $|u_0|_\infty < \infty$ and $|D\psi_{\varepsilon,z}|, |D^2 \psi_{\varepsilon,z}| \leq LC_\varepsilon$ we can use the assumed boundedness of $F$ over sets where $u,p,X$ remain bounded. In particular, we can pick $A_\varepsilon$ negative, large enough, such that

$$\partial_t u_{\varepsilon,z} - F(t,x,u_{\varepsilon,z},Du_{\varepsilon,z},D^2 u_{\varepsilon,z}) \leq \cdots \leq 0.$$

We now define the sup of all these subsolutions,

$$\hat{u}(t,x) := \sup_{\varepsilon \in (0,1],z \in \mathbb{R}^n} u_{\varepsilon,z}(t,x) \leq u_0(x) \leq |u_0|_\infty < \infty,$$

and note that

$$\hat{u}(0,x) = \sup_{\varepsilon \in (0,1],z \in \mathbb{R}^n} \psi_{\varepsilon,z}(x) = \sup_{\varepsilon \in (0,1]} u_0(x) - L\varepsilon^{1/2} = u_0(x).$$

The upper semicontinuous envelope $\underline{u}(t,x) := \hat{u}^*$ is then (cf. Proposition 8.2 in [6] for instance) also a subsolution to $\partial_t - F = 0$.

**Step 2:** We show that $\hat{u}(t,x)$ is continuous at $t = 0$; this implies that $\underline{u}(0,x) := \hat{u}(0,x) = u_0(x)$ and thus yields a sub-solution with the correct initial data. Let $(t^n,x^n) \to (0,x)$. First we show lower semicontinuity, i.e.

$$\lim_{n \to \infty} \inf \hat{u}(t^n,x^n) \geq \hat{u}(0,x).$$

Let $\delta > 0$. Choose $\tilde{\varepsilon},\tilde{z}$ such that

$$u_{\tilde{\varepsilon},\tilde{z}}(0,x) \geq \hat{u}(0,x) - \delta.$$

Let $M$ be a bound for $|Du_{\tilde{\varepsilon},\tilde{z}}|$ (and hence for $|D\psi_{\tilde{\varepsilon},\tilde{z}}|$). Choose $N$ such that for $n \geq N$

$$|t^n|, |x^n - x| \leq \min \left\{ \frac{\delta}{A_\varepsilon}, \frac{\delta}{M} \right\}.$$

Then

$$\hat{u}(t^n,x^n) \geq u_{\tilde{\varepsilon},\tilde{z}}(t^n,x^n)$$

$$= u_{\tilde{\varepsilon},\tilde{z}}(t^n,x^n) - u_{\tilde{\varepsilon},\tilde{z}}(0,x) + u_{\tilde{\varepsilon},\tilde{z}}(0,x)$$

$$= A_\varepsilon t^n + \psi_{\tilde{\varepsilon},\tilde{z}}(x^n) - \psi_{\tilde{\varepsilon},\tilde{z}}(x) + u_{\tilde{\varepsilon},\tilde{z}}(0,x)$$

$$\geq \hat{u}(0,x) - 3\delta,$$

which proves the lower semicontinuity.
For upper semicontinuity, notice that
\[
\begin{align*}
u_{\varepsilon,x}(s,y) &= A_\varepsilon s + \psi_{\varepsilon,x}(y) \\
&\leq A_\varepsilon s + u_0(y) \\
&\leq u_0(y),
\end{align*}
\]
where we have used that \(A_\varepsilon \leq 0\) and that \(\psi_{\varepsilon,x}(y) \leq u_0(y)\), as shown above. Hence, \(\hat{u}(s,y) \leq u_0(y)\), and then for \((t^n, x^n) \to (0, x)\), we have
\[
\limsup_{n} \hat{u}(t^n, x^n) \leq \limsup_{n} u_0(x^n) = u_0(x) = \hat{u}(0, x).
\]
Hence \(\hat{u}\) is also upper semicontinuous at \((0, x)\) and hence continuous at \((0, x)\).

Step 3: Similarly, one constructs a super-solution with correct (bounded, Lipschitz) initial data \(u_0\). Perron’s method then applies and yields a bounded viscosity solution to \(\partial_t - F = 0\) with bounded, Lipschitz initial data.

Step 4: Let now \(u_0 \in BUC(\mathbb{R}^n)\) and \(u^n_0\) be a sequence of bounded Lipschitz functions such that \(|u^n_0 - u_0|_{\infty} \to 0\). By the previous step there exists a bounded solution \(u^n\) to \(\partial_t - F = 0\) with initial data \(u^n(0, \cdot) = u^n_0\). (It is also unique by comparison.) Since \(F\) is proper \((\gamma \geq 0)\), the solutions form a contraction in the sense
\[
|u^n - u^m|_{\infty;[0,T] \times \mathbb{R}^n} \leq |u^n_0 - u^m_0|_{\infty;\mathbb{R}^n}
\]
(This follows immediately from comparison and properness.) Hence \(u^n\) is Cauchy in supremum norm and converges to a continuous bounded function \(u : [0, T] \times \mathbb{R}^n \to \mathbb{R}\). By Lemma 6.1 in the User’s Guide we then have that \(u\) is a bounded solution to \(\partial_t - F = 0\) with \(BUC(\mathbb{R}^n)\) initial data. By comparison, it is the unique (bounded) solution with this initial data. At last, corollary \(\S\) shows that the solution is \(BUC\) in time space.

\(\Box\)

8. Appendix I: Recalls on parabolic jets

If \(u : (0, T) \times \mathbb{R}^n \to \mathbb{R}\) its parabolic semijet \(P^{2,+}u\) is defined by \((b, p, X) \in \mathbb{R} \times \mathbb{R}^n \times S^n\) lies in \(P^{2,+}u(s, z)\) if \((s, z) \in (0, T) \times \mathbb{R}^n\) and
\[
u(t, x) \leq u(s, z) + b(t-s) + \langle p, x - z \rangle + \frac{1}{2} \langle X(z), x - z \rangle + o\left(\|t-s\| + |x-z|^2\right)
\]
as \((0, T) \times \mathbb{R}^n \ni (t, x) \to (s, z)\). Consider now \(u : Q \to \mathbb{R}\) where \(Q = (0, T) \times \mathbb{R}^n\). The parabolic semijet relative to \(Q\), write \(P^{2,+}_Q u\), as used in \(\S\) for instance, is defined by \((b, p, X) \in \mathbb{R} \times \mathbb{R}^n \times S^n\) lies in \(P^{2,+}_Q u(s, z)\) if \((s, z) \in (0, T) \times \mathbb{R}^n\) and
\[
u(t, x) \leq u(s, z) + b(t-s) + \langle p, x - z \rangle + \frac{1}{2} \langle X(z), x - z \rangle + o\left(\|t-s\| + |x-z|^2\right)
\]
as \(Q \ni (t, x) \to (s, z)\). Note that \(P^{2,+}_Q u(s, z) = P^{2,+}u(s, z)\) for \((s, z) \in (0, T) \times \mathbb{R}^n\). Note also the special behaviour of the semijet at time \(T\) in the sense that
\[
(b, p, X) \in P^{2,+}_Q u(T, z) \implies \forall b' \leq b : (b', p, X) \in P^{2,+}_Q u(T, z).
\]
Closures of these jets are defined in the usual way; e.g.
\[
(b, p, X) \in \overline{P^{2,+}_Q u(T, z)}
\]
if \(\exists (t_n, z_n; b_n, p_n, X_n) \in Q \times \mathbb{R} \times \mathbb{R}^n \times S^n : (b_n, p_n, X_n) \in \overline{P^{2,+}_Q u(t_n, z_n)}\) and
\[
(t_n, z_n; u(t_n, z_n); b_n, p_n, X_n) \to (T, z; u(T, z); b, p, X).
\]
9. Appendix 2: Parabolic theorem of sums revisited

**Theorem 3** ([9] Thm 7). Let \( u_1, u_2 \in \text{USC} \left( (0, T) \times \mathbb{R}^n \right) \) and \( w \in \text{USC} \left( (0, T) \times \mathbb{R}^{2n} \right) \) be given by

\[
w(t, x) = u_1(t, x_1) + u_2(t, x_2)\]

Suppose that \( s \in (0, T), z = (z_1, z_2) \in \mathbb{R}^{2n}, b \in \mathbb{R}, p = (p_1, p_2) \in \mathbb{R}^{2n}, A \in S^{2n}\) with

\[
(b, p, A) = \mathcal{P}^{2,+} w(s, z).
\]

Assume moreover that there is an \( r > 0 \) such that for every \( M > 0 \) there is a \( C \) such that for \( i = 1, 2 \)

\[
b_i \leq C \text{ whenever } (b_i, q_i, X_i) \in \mathcal{P}^{2,+} w(t, x),
\]

\[
|x_i - z| + |s - t| < r \text{ and } |u_i(t, x)| + |q_i| + \|X_i\| \leq M.
\]

Then for each \( \varepsilon > 0 \) there exists \( (b_i, X_i) \in \mathbb{R} \times S^n \) such that

\[
(b_i, p_i, X_i) \in \mathcal{P}^{2,+} u(s, z_i)
\]

and

\[
(9.3) \quad -\left(\frac{1}{\varepsilon} + \|A\|\right) I \leq \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \leq A + \varepsilon A^2 \text{ and } b_1 + b_2 = b.
\]

The proof of the above theorem is reduced (cf. Lemma 8 in [9]) to the case \( b = 0, z = 0, p = 0 \) and \( v_1(s, 0) = v_2(s, 0) = 0 \), where (in order to avoid confusion) we write \( v_i \) instead of \( u_i \). Condition (9.1) translates then to

\[
(9.4) \quad v_1(t, x_1) + v_2(t, x_2) - \frac{1}{2} \langle Ax, x \rangle \leq 0 \text{ for all } (t, x) \in (0, T) \times \mathbb{R}^{2n}.
\]

This also means that the left-hand-side as a function of \((t, x_1, x_2)\) has a global maximum at \((s, 0, 0)\). The assertion of the (reduced) theorem is then the existence of \((b_i, X_i) \in \mathbb{R} \times S^n \) such that \((b_i, 0, X_i) \in \mathcal{P}^{2,+} v_i(s, 0)\) for \( i = 1, 2 \) and (9.3) holds with \( b = 0 \).

**Theorem 4.** Assume that \( u_i \) has a finite extension to \((0, T] \times \mathbb{R}^n\), \( i = 1, 2 \), via its semi-continuous envelopes, that is,

\[
u_i(T, x) = \limsup_{(t, y) \in (0, T] \times \mathbb{R}^n: t \uparrow T, y \to x} u_i(t, y) < \infty.
\]

Then the above theorem remains valid at \( s = T \) if

\[
\mathcal{P}^{2,+} w(s, z) \text{ and } \mathcal{P}^{2,+} u(s, z_i)
\]

is replaced by

\[
\mathcal{P}^{2,+}_Q w(T, z) \text{ and } \mathcal{P}^{2,+}_Q u(T, z_i)
\]

and the final equality in (9.3) is replaced by

\[
(9.5) \quad b_1 + b_2 \geq b.
\]

**Remark 4.** If we knew (but we don’t!) that the final conclusion is \((b_i, p_i, X_i) \in \mathcal{P}^{2,+} u(T, z_i)\), rather than just being an element in the closure \( \overline{\mathcal{P}^{2,+}} u(T, z_i) \), then we could trivially diminish the \( b_i \)'s such as to have \( b_1 + b_2 = b \); cf. (8.7).
Proof. Step 1: We focus on the reduced setting (and thus write \( v_i \) instead of \( u_i \)) and (following the proof of Lemma 8 in [9]) redefine \( v_i(t_i, x_i) \) as \(-\infty\) when \(|x_i| > 1\) or \( t_i \not\in [T/2, T] \). We can also assume that (9.4) is strict if \( t < s = T \) or \( x \neq 0 \). For the rest of the proof, we shall abbreviate \((t_1, t_2), (x_1, x_2)\) etc by \((t, x)\). With this notation in mind we set
\[
 w(t, x) = v_1(t_1, x_1) + v_2(t_2, x_2) - \frac{1}{2} \langle Ax, x \rangle.
\]

By the extension via semi-continuous envelopes, there exist a sequence \((t^n, x^n) \in (0, T)^2 \times (\mathbb{R}^n)^2\), such that
\[
 (t^n, x^n) = (t_1^n, t_2^n, x_1^n, x_2^n) \to (T, T, 0, 0).
\]

We now consider \(w\) with a penalty term for \( t_1 \neq t_2 \) and a barrier at time \( T \) for both \( t_1 \) and \( t_2 \).
\[
 \psi_{m,n}(t, x) = w(t, x) - \left\{ \frac{m}{2} |t_1 - t_2|^2 + \sum_{i=1}^{2} (T - t_i^n)^2 / (T - t_i) \right\},
\]

indexed by \((m, n) \in \mathbb{N}^2\), say. By assumption \(w\) has a maximum at \((T, T, 0, 0)\) which we may assume to be strict (otherwise subtract suitable forth order terms ...).

Define now
\[
 (\hat{t}, \hat{x}) \in \arg\max \psi_{m,n} \text{ over } [T - r, T]^2 \times \hat{B}_r(0)^2
\]
where \( r = T/2 \) (for instance). When we want to emphasize dependence on \( m, n \) we write \((\hat{t}_{m,n}, \hat{x}_{m,n})\). We shall see below (Step 2) that there exists increasing sequences \( m = m(k) \), \( n = n(k) \) so that
\[
 (\hat{t}, \hat{x}) \big|_{m=m(k), n=n(k)} \to (T, T, 0, 0).
\]

Using the (elliptic) theorem of sums in the form of [9, Theorem 1] we find that there are
\[
 (b_i, p_i, X_i) \in \bar{P}^{2,+}v_i(\hat{t}_i, \hat{x}_i)
\]
where \( \hat{t}_i \to T, \hat{x}_i \to 0 \) as \( k \to \infty \) such that the first part of (9.3) holds and
\[
 A \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \quad b_i = m(t_i - t_{3-i}) + (T - t_i + \epsilon)^2 / (T - t_i)^2.
\]

for \( i = 1, 2 \). Note that
\[
 b_1 + b_2 = m(t_1 - t_2) + m(t_2 - t_1) + \text{(positive terms)} \geq 0;
\]

since each \( b_i \) is bounded above by the assumptions and the estimates on the \( X_i \) it follows that the \( b_i \) lie in precompact sets. Upon passing to the limit \( k \to \infty \) we obtain points
\[
 (b_i, p_i, X_i) \in \bar{P}^{2,+}v_i(T, 0), \quad i = 1, 2;
\]

with \( b_1 + b_2 \geq 0 \).

Step 2: We still have to establish (9.6). We first remark that for arbitrary (strictly) increasing sequences \( m(k), n(k) \), compactness implies that
\[
 \{ (\hat{t}_{m(k), n(k)}, \hat{x}_{m(k), n(k)}) : k \geq 1 \} \in \{ T - r, T \}^2 \times \hat{B}_r(0)^2
\]
has limit points. Note also $\hat{t}_1, \hat{t}_2 \in [T - r, T]$ thanks to the barrier at time $T$. The key technical ingredient for the remained of the argument is and we postpone details of these to Step 3 below:

\[(9.7)\]
\[w(\hat{t}, \hat{x}) - \psi_{m,n}(\hat{t}, \hat{x}) = \left\{ \frac{m}{2} |\hat{t}_1 - \hat{t}_2|^2 + \sum_{i=1}^2 (T - t^{i,n})^2 / (T - \hat{t}_i) \right\} \rightarrow 0 \text{ as } \frac{1}{n} < \frac{1}{m} \rightarrow 0.\]

In particular, for every $k > 0$ there exists $m(k)$ such that for all $m \geq m(k)$

\[\limsup_{n \to \infty} \{...\} < \frac{1}{k}.\]

By making $m(k)$ larger if necessary we may assume that $m(k)$ is (strictly) increasing in $k$. Furthermore there exists $n(m(k), k) = n(k)$ such that for all $n \geq n(k) : \{...\} < 2/k$. Again, we may make $n(k)$ larger if necessary so that $n(k)$ is strictly increasing. Recall $t^{1,n(k)} - t^{2,n(k)} \rightarrow T - T = 0$ as $k \rightarrow \infty$. For reasons that will become apparent further below, we actually want the stronger statement that

\[(9.8) \quad \frac{m(k)}{2} |t^{1,n(k)} - t^{2,n(k)}|^2 \rightarrow 0 \text{ as } k \rightarrow \infty\]

which we can achieve by modifying $n(k)$ such as to run to $\infty$ even faster. Note that the so-constructed $m = m(k), n = n(k)$ has the property

\[(9.9) \quad [w(\hat{t}, \hat{x}) - \psi_{m,n}(\hat{t}, \hat{x})]_{m=m(k), n=n(k)} = \{...\}_{m=m(k), n=(k)} \rightarrow 0 \text{ as } k \rightarrow \infty.\]

By switching to a subsequence $(k_i)$ if necessary we may also assume (after relabeling) that

\[(\hat{t}_{m(k), n(k)}, \hat{x}_{m(k), n(k)}) \rightarrow (\hat{t}, \hat{x}) \in [T - r, T]^2 \times \bar{B}_r(0)^2 \text{ as } k \rightarrow \infty.\]

In the sequel we think of $(\hat{t}, \hat{x})$ as this sequence indexed by $k$. We have

\[(9.10) w(\hat{t}, \hat{x}) \geq \limsup_{k \to \infty} w(\hat{t}, \hat{x})|_{m=m(k), n=(k)} \text{ by upper-semi-continuity} \]

\[= \limsup_{k \to \infty} \psi_{m,n}(\hat{t}, \hat{x})|_{m=m(k), n=(k)} \text{ thanks to (9.9).}\]

On the other hand, thanks to the particular form of our time-$T$ barrier,

\[\psi_{m,n}(\hat{t}, \hat{x}) \geq \psi_{m,n}(t^n, x^n)\]

\[= w(t^n, x^n) - \left\{ \frac{m}{2} |t^{1,n} - t^{2,n}|^2 + \sum_{i=1}^2 (T - t^{i,n}) \right\}.\]

Take now $m = m(k), n = n(k)$ as constructed above. Then

\[\psi_{m,n}(\hat{t}, \hat{x})|_{m=m(k), n=(k)} \geq w(t^{n(k)}, x^{n(k)})\]

\[= \left\{ \frac{m(k)}{2} |t^{1,n(k)} - t^{2,n(k)}|^2 + \sum_{i=1}^2 (T - t^{i,n(k)}) \right\}.\]
The first term in the curly bracket goes to zero (with \( k \to \infty \)) thanks to (9.8), the other term goes to zero since \( t^{1,n} \to T \) with \( n \to \infty \), and hence also along \( n (k) \).

On the other hand (recall \( x^{i,n} \to 0 \))

\[
w(t^{n(k)}, x^{n(k)}) \to v_1(T, 0) + v_2(T, 0) - \frac{1}{2} \langle A0, 0 \rangle = 0 \text{ as } k \to \infty.
\]

(In the reduced setting \( v_1(T, 0) = v_2(T, 0) = 0 \).) It follows that

\[
\lim_{k \to \infty} \inf_{m,n} \psi_{m,n} \left( \hat{t}, \hat{x} \right) |_{m=m(k), n=(k)} = 0.
\]

Together with (9.10) we see that \( w(\hat{t}, \hat{x}) \geq 0 \). But \( w(T, T, 0, 0) = 0 \) was a strict maximum in \([T - r, T]^2 \times B_r(0)^2\) and so we must have \((\hat{t}, \hat{x}) = (T, T, 0, 0)\).

**Step 3:** Set

\[
M (h) = \sup_{(t,x) \in [T - r, T]^2 \times B_r(0)^2} w(t_1, t_2, x_1, x_2) \text{ and } M' = \lim_{h \to 0} M (h)
\]

It is enough to show

\[
\limsup_{\frac{\hat{t}}{n} \to \frac{h}{m}} w\left( \frac{\hat{t}}{n}, \frac{\hat{x}}{m} \right) \leq M' \leq \liminf_{\frac{\hat{t}}{n} \to \frac{h}{m}} \psi_{m,n} \left( \frac{\hat{t}}{n}, \frac{\hat{x}}{m} \right).
\]

since the claimed

\[
w\left( \frac{\hat{t}}{n}, \frac{\hat{x}}{m} \right) - \psi_{m,n} \left( \frac{\hat{t}}{n}, \frac{\hat{x}}{m} \right) = \left\{ \frac{m}{2} |\hat{t}_1 - \hat{t}_2|^2 + \sum_{i=1}^{2} (T - t^{i,n})^2 / (T - t_i) \right\} \to 0 \text{ as } \frac{1}{n} \to \frac{1}{m} \to 0.
\]

follows from

\[
\limsup_{\frac{\hat{t}}{n} \to \frac{h}{m}} \left\{ \ldots \right\} \leq \limsup_{\frac{\hat{t}}{n} \to \frac{h}{m}} w\left( \frac{\hat{t}}{n}, \frac{\hat{x}}{m} \right) - \liminf_{\frac{\hat{t}}{n} \to \frac{h}{m}} \psi_{m,n} \left( \frac{\hat{t}}{n}, \frac{\hat{x}}{m} \right) \leq 0 \text{ (and hence = 0)}.
\]

Note that \( w\left( \frac{\hat{t}}{n}, \frac{\hat{x}}{m} \right) \) is bounded on \([T - r, T]^2 \times B_r(0)^2\) so that

\[
|\hat{t}_1 - \hat{t}_2|^2 = O (1/m) \implies w \left( \frac{\hat{t}}{n}, \frac{\hat{x}}{m} \right) \leq M \left( \text{const}/\sqrt{m} \right).
\]

On the other hand, from the very definition of \( M' \) as \( \lim_{h \to 0} M (h) \), there exists a family \((t_h, x_h)\) so that

\[
|t_{1,h} - t_{2,h}| \leq h \text{ and } w(t_h, x_h) \to M' \text{ as } h \to 0.
\]

For every \( m, n \) we may take \((t_h, x_h)\) as argument of \( \psi_{m,n} \) (which itself has a maximum at \( \hat{t}, \hat{x} \)); hence

\[
w(t_h, x_h) - \frac{m}{2} h_2^2 - \sum_{i=1}^{2} (T - t^{i,n})^2 / (T - t_{i,h}) \leq \psi_{m,n} \left( \frac{\hat{t}}{n}, \frac{\hat{x}}{m} \right).
\]

Take now a sequence \( n = n (h) \), fast enough increasing as \( h \to \gamma \) such that \( (T - t^{i,n})^2 / (T - t_{i,h}) \to 0 \) with \( h \to 0 \). It follows that

\[
M' = \lim_{h \to 0} w(t_h, x_h)
\]

\[
= \liminf_{h \to 0} \left( w(t_h, x_h) - \frac{m}{2} h^2 - \sum_{i=1}^{2} \left( T - t^{i,n(h)} \right)^2 / (T - t_{i,h}) \right)
\]

\[
\leq \liminf_{h \to 0} \psi_{m,n(h)} \left( \frac{\hat{t}}{n}, \frac{\hat{x}}{m} \right) = \liminf_{n \to \infty} \psi_{m,n} \left( \frac{\hat{t}}{n}, \frac{\hat{x}}{m} \right) \text{ by monotonicity of } \sup \psi_{m,n} \text{ in } n.
\]
(In the last equality we used that $t^{i,n} \uparrow T$; this shows that $\sup \psi_{m,n}$ is indeed monoton in $n$.) The proof is now finished.

\begin{flushright}
$\square$
\end{flushright}

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