The Jeans instability is analyzed for dense magnetohydrodynamic plasmas with intrinsic magnetization, the latter due to collective electron spin effects. Furthermore, effects of electron tunneling as well as the Fermi pressure are included. It is found that the intrinsic magnetization of the plasma will enhance the Jeans instability, and can significantly modify the structure of the instability spectra. Implications and limitations of our results are discussed, as well as possible generalizations.

PACS numbers: 52.25.Xz, 52.35.Bj, 51.60.+a

I. INTRODUCTION

Whenever the internal pressure of a gas or plasma cloud is too weak to balance the self-gravitational force of a mass density perturbation, there will be a collapse. The study of such mechanisms was pioneered by Jeans, and it is well established that the Jeans instability plays a crucial role in the mechanism responsible for the formation of structures in the universe [1].

For lengthscales comparable to the de Broglie wavelength of the charge carrier, tunneling effects become important and give rise to dispersion. This is particularly important for low temperature and/or high density plasmas. A characteristic condition for this effect to be of significance is when the de Broglie wavelength is comparable or larger than the mean separation of particles. The plasma then behaves as a Fermi gas, obeying Fermi-Dirac statistics rather than Boltzmann-Maxwell used in classical plasmas. These effects can be captured within the magnetohydrodynamic (MHD) model through a modified equation of state for the pressure [2], and by including the Bohm potential [2, 1, 3, 4, 7] describing quantum forces due to non-locality effects. Quantum plasmas was first studied by Pines in the 1960’s [8, 9], and many studies has appeared since then [10], e.g., kinetic models of the quantum electrodynamical properties of nonthermal plasmas [11] and covariant Wigner function descriptions of relativistic quantum plasmas [12]. The study of quantum plasmas in recent years have been motivated by e.g. possible applications to nanoscale technology [13], new developments in microelectronics [14], the discovery of ultracold plasmas [15, 16, 17] and the experimental demonstration [18] of collective modes in ultra cold plasmas, new laser plasma/solid-matter interaction regimes offered by the next generation of high intensity light sources [19, 20, 21, 22], as well as developments in laser fusion [23]. Quantum effects are also believed to be of importance in the interiors of compact astrophysical objects [24, 25, 26] such as white dwarfs, neutron stars, magnetars and supernovas, where the density can reach ten orders of magnitude that of ordinary solids. Furthermore, in some astrophysical environments, such as in the vicinity of pulsars [27, 28] and magnetars [29], strong external magnetic fields may be present. In such dense and/or strong magnetic field environments a quantum description of the plasma which incorporates the spin of the particle is needed. A great deal of interest has been directed toward finding such quantum plasma descriptions [30, 31, 32, 33, 34, 35, 36]. In the context of the Jeans instability, quantum effects have been studied in Ref. [37].

The objective of this paper is to investigate the Jeans instability in a magnetized self-gravitating spin–1/2 plasma. Starting with the MHD equations, Poisson’s equation for the gravitational potential, and an expression for the magnetization of the plasma due to the electron spin, we derive a dispersion relation for plasma modes with arbitrary directions of propagation. The overall stability of the system is investigated, and in particular we study the stabilizing/destabilizing effect of the Bohm potential and the electron spin. Finally, we discuss the implications and limitations of our model.

II. GOVERNING EQUATIONS

The governing equations for self-gravitating spin–1/2 ideal MHD plasmas [8, 32, 38, 39] are the mass density conservation law

\[ \partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0, \]  

and the momentum equation

\[ \rho (\partial_t + \mathbf{v} \cdot \nabla) \mathbf{v} = -\rho \nabla \phi - \nabla p - \nabla \left( \frac{B^2}{2\mu_0} - \mathbf{M} \cdot \mathbf{B} \right) + \mathbf{B} \cdot \nabla \left( \frac{\mathbf{B}}{\mu_0} - \mathbf{M} \right) + \frac{\hbar^2}{2m_e m_i} \nabla \left( \frac{\nabla^2 \sqrt{p}}{\sqrt{p}} \right), \]
the idealized Ohm’s law
\[ \partial_t \mathbf{B} = \nabla \times (\mathbf{v} \times \mathbf{B}), \tag{3} \]
and Poisson’s equation for the gravitational potential
\[ \nabla^2 \phi = 4\pi G \rho. \tag{4} \]
Here \( \rho \) is the mass density, \( \mathbf{v} \) is the fluid velocity, \( \mathbf{B} \) is the magnetic field, \( \mathbf{M} \) is the magnetization due to the electron spin, \( \phi \) is the gravitational potential determined by Poisson’s equation, \( \hbar \) is Planck’s constant divided by \( 2\pi \), \( m_{e,i} \) is the electron and ion mass respectively, and \( G \) is the gravitational constant. The momentum equation, eq. (2), has been modified to include non-locality effects, such as tunneling, described by the Bohm potential. These effects are important for dense and/or low temperature plasma systems and for short wavelengths.

This system of equations is closed with an equation of state for the pressure and an expression for the magnetization. The equation of state for the pressure is written as, \( \nabla P = c_s^2 \nabla \rho \), where \( c_s = (dp/d\rho_0)^{1/2} \) is the sound speed. The ion-acoustic velocity can be written as
\[ c_s^2 = v_{ti}^2 + \frac{m_i}{m_e} \left( v_{te}^2 + \frac{3}{5} v_{Fe}^2 \right), \tag{5} \]
where \( v_{ti} \) and \( v_{te} \) are the ion and electron thermal velocities and \( v_{Fe} = \hbar (3\pi^2 n_e)^{1/3} / m_e \) is the electron Fermi velocity.

For temperatures well below the Fermi temperature, the thermal velocity is given by \( v_{ts} = C(k_B T / h n_s^{1/3}) \), where \( C \) is a dimensionless constant of order unity, whereas for temperatures much higher than the Fermi velocity we have \( v_{ts} = (k_B T / m_s)^{1/2} \). Here \( n_s \) is the number density of particle species \( s \), \( k_B \) is Boltzmann’s constant and \( T \) is the temperature.

Multifluid equations including the effect of the magnetization due to the electron spin has been derived in Ref. [39] and further developed for MHD regimes in Ref. [32]. It is then showed that for dynamics on a time scale much slower than the spin precession frequency, the magnetization is given by
\[ \mathbf{M} = \frac{\mu_B \rho}{m_i} \tanh \left( \frac{\mu_B B}{k_B T} \right) \hat{\mathbf{B}}, \tag{6} \]
where \( \hat{\mathbf{B}} = \mathbf{B} / |\mathbf{B}| \), and \( |\mathbf{B}| \) is the magnetic field magnitude. Here it has been assumed that the spin orientation has reached the thermodynamic equilibrium state in response to the magnetic field, which accounts for the \( \tanh(\mu_B B / k_B T) \)-factor, where \( \mu_B = e\hbar / 2m_e \) is the Bohm magneton. Furthermore, on a time scale shorter than the spin relaxation time scale, the individual electron spins are conserved, and thus we can take \( \tanh(\mu_B B / k_B T) \) as constant for an initially homogeneous plasma.

Assuming perturbations on a homogeneous background (equilibrium values denoted by index 0) we write \( \rho = \rho_0 + \delta \rho \), \( \phi = \phi_0 + \delta \phi \), \( \mathbf{v} = \delta \mathbf{v} \), \( \mathbf{B} = B_0 \hat{\mathbf{z}} + \delta \mathbf{B} \), and \( \mathbf{M} = M_0 \hat{\mathbf{z}} + \delta \mathbf{M} \) and linearize our governing equations. Making a harmonic decomposition, we obtain
\[ \omega \delta \rho - \rho_0 \mathbf{k} \cdot \delta \mathbf{v} = 0, \tag{7} \]
\[ -i \omega \rho_0 \delta \mathbf{v} = -i \mathbf{k} \left( \rho_0 \delta \phi + c_s^2 \delta \rho + \frac{B_0 \delta B_z}{\rho_0} - M_0 \delta B_z - B_0 \delta M_z \right) + i k_z B_0 \left( \frac{\delta \mathbf{B}}{\rho_0} - \delta \mathbf{M} \right) - \frac{i \hbar^2 k^2 \mathbf{k}}{4m_e m_i} \delta \rho, \tag{8} \]
\[ -\omega \delta \mathbf{B} = B_0 \mathbf{k} \times (\delta \mathbf{v} \times \hat{\mathbf{z}}), \tag{9} \]
\[ -k^2 \delta \phi = 4\pi G \delta \rho, \tag{10} \]
and
\[ \delta \mathbf{M} = M_0 \left( \frac{\delta \mathbf{B}_\perp}{B_0} + \frac{\delta \rho}{\rho_0} \hat{\mathbf{z}} \right), \tag{11} \]
where \( \delta \mathbf{B}_\perp = (\delta B_x, \delta B_y, 0) \), and \( M_0 = (B_0 / \mu_0) \chi / (1 + \chi) \). Here
\[ \chi = \frac{B_0 \mu_B}{m_i C_s^2 \tanh \left( \frac{\mu_B B_0}{k_B T} \right)} \tag{12} \]
is the magnetic susceptibility and \( C_A = (B_0^2/\rho_0) \) is the Alfvén velocity.

Eq. (11) already gives us an expression for \( \delta \mathbf{v} \) in terms of \( \delta v \) and we can solve eq. (7) for \( \delta \rho \), and then solve eq. (10) for \( \delta \phi \). Using these results in eq. (8) we find an equation for \( \delta v \). The parallel and perpendicular component of \( \delta v \) are given by

\[
-\omega^2 \delta v_z = -\left( -\frac{4\pi G\rho_0}{k^2} + c_s^2 + \frac{\hbar^2 k^2}{4m_e m_i} \right) k_z \cdot \delta \mathbf{v} + \frac{\chi}{1 + \chi} C_A^2 k_z v_s \cdot \delta \mathbf{v}_\perp
\]  

and

\[
-\omega^2 \delta v_\perp = -k_\perp \left( \frac{4\pi G\rho_0}{k^2} k + c_s^2 k + \frac{1}{1 + \chi} C_A^2 k_\perp - \frac{\chi}{1 + \chi} C_A^2 k \right) \cdot \delta \mathbf{v} - \frac{1}{1 + \chi} C_A^2 k_\perp \delta \mathbf{v}_\perp.
\]

Eqs. (13) and (14) can be written in matrix form as \( D^{ab} v_b = 0 \), and the dispersion relation is obtained from \( ||D^{ab}|| = 0 \),

\[
\left( \omega^2 - \frac{1}{1 + \chi} C_A^2 k_\perp^2 \right) \left\{ \left[ \omega^2 - \frac{1}{1 + \chi} C_A^2 k^2 - \left( V^2 - \frac{\chi}{1 + \chi} C_A^2 \right) k_\perp^2 \right] \left( \omega^2 - V^2 k_\perp^2 \right) - k_\perp^2 \frac{\hbar^2 k_\perp^2}{4m_e m_i} \left( V^2 - \frac{\chi}{1 + \chi} C_A^2 \right)^2 \right\} = 0.
\]

Here we have used the notation,

\[
V^2 = -\frac{4\pi G\rho_0}{k^2} + c_s^2 + \frac{\hbar^2 k^2}{4m_e m_i},
\]

The first factor of the dispersion relation describes the shear Alfvén wave, which is always stable. Focusing on the stability properties, this mode is not considered below. The second factor describes the fast and the slow magnetosonic modes modified by a gravitational potential, non-locality effects due to the Bohm potential, and the spin magnetization. Our dispersion relation agrees with that found in Refs. [32] and [35] in the appropriate limits if we correct for a sign error in those papers.

In order to gain some understanding of the stability properties, we first study the special cases of parallel and orthogonal propagation to the background magnetic field. Initially we discard the contribution \( \frac{\hbar^2}{4m_e m_i} \) in (10) due to the Bohm potential, which only becomes important for short wavelengths. For parallel propagation, the instability condition then simply becomes the ordinary Jeans instability condition,

\[
\frac{4\pi G\rho_0}{k^2} > c_s^2.
\]

Formally, this means that our system is always unstable for sufficiently long wavelengths. However, we may interpret the condition (17) such as to give a required size of the system for an instability of this type to be possible. In particular, for the above calculation to be applicable, the system must be approximately homogeneous over length scales of the order of the Jeans length, \( \lambda_J = 2\pi/k_J \), where \( k_J = (4\pi G\rho_0/c_s^2)^{1/2} \). Next we consider orthogonal propagation, the instability condition becomes,

\[
V^2 + \frac{1}{1 + \chi} C_A^2 < 0.
\]

In this case the system may be unstable even if the gravitational effect is omitted, provided the condition

\[
\frac{\chi}{1 + \chi} > \frac{1}{2} \left( 1 + \frac{c_s^2}{C_A^2} \right)
\]

is fulfilled. The condition (19) then suggests that the plasma is unstable for all length scales. In particular even in the short wavelength regime, where our calculation is of special interest if we assume that the plasma system of study has a limited extension. For short wavelengths, on the other hand, we expect the Bohm potential to be of importance. Noting from (16) and (18) that the Bohm potential acts stabilizing for sufficiently short wavelengths, this term is therefore kept below.

Next we normalize the parameters according to

\[
\tilde{\omega} = \frac{\omega}{k_J c_s}, \quad \tilde{k} = \frac{k}{k_J}, \quad \alpha = \frac{\chi}{1 + \chi}, \quad \tilde{C}_A = C_A/c_s, \quad \tilde{Q} = \frac{\hbar^2 k_J^2}{4m_e m_i c_s^2}.
\]
such that the dispersion relation can be written as
\[ \tilde{\omega}^4 - \tilde{\omega}^2 \left[ \left( \frac{-1}{k^2} + 1 + \tilde{Q}k^2 + C_A^2 (1 - a) \right) \tilde{k}^2 - \alpha \tilde{C}_A^2 \tilde{k}_\perp^2 \right] + \tilde{k}_z^2 \tilde{C}_A^2 \left[ \left( \frac{-1}{k^2} + 1 + \tilde{Q}k^2 \right) (\tilde{k}^2 - \alpha \tilde{k}_z^2) - \alpha^2 \tilde{C}_A^2 \tilde{k}_\perp^2 \right] = 0. \] (20)

The growth rate \( \gamma = \text{Im} \tilde{\omega} \) as a function of \( \tilde{k}_z \) and \( \tilde{k}_\perp \) has been illustrated in Fig. 1 for (a) the classical Jeans instability and (b) the Jeans instability modified by the presence of an external magnetic field. The external magnetic field will decrease the instability in the direction orthogonal to the magnetic field. This will generate an oblate spheroidal collapse. This is expected since the magnetic forces are anisotropic so the magnetic pressure primarily work in directions perpendicular to \( \mathbf{B} \). While the oblate shape of the plasma cloud is an effect that occur in the later (nonlinear) stage of the collapse, we note that the stabilizing influence of the magnetic field can be seen already in the initial linear stage of the instability. In particular the dependence of the growth rate on the wavenumber shows that perturbations perpendicular to the magnetic field is less unstable, as is seen by comparing Fig. (1c) and Fig. (1b).

Furthermore, we note that the electron non-locality effects described by the Bohm potential has an over all stabilizing effect of the system which is illustrated in Fig. (1d).

Effects due to the electron spin has a destabilizing effect on the system, in particular in directions almost, but not quite, perpendicular to the magnetic field. This instability stems from the magnetic attraction of individual spins (i.e. magnetic dipole moments) and has been investigated in Ref. [33]. The effect of electron spin on the Jeans instability is demonstrated in Fig. (1d). In this parameter regime the spin magnetization is significant, but the plasma would still be stable if gravitational effects were disregarded. This is in contrast to Fig. (1e) in which the spin contribution alone is sufficient for collapse. Here, stability is regained for short wavelengths through the influence of the Bohm potential. For comparison, the spin instability alone, i.e. without the gravitational Jeans contribution, has been illustrated in Fig. (1f) with the same parameter values as in Fig. (1e).

The maximum value of \( |k| = k_{\text{max}} \) for which we have instability occur at an angle to \( k_\perp \), as can be seen in e.g. Fig. (1d). Our investigation is of particular interest when \( k_{\text{max}} > k_f \). Especially when considering a system of finite size, with inhomogeneity scale length shorter than the Jeans length, i.e. for \( \lambda_f \left( \nabla n_0 / n_0 \right) \) larger than unity. In this regime there will typically be no collapse due to the Jeans instability alone, but as shown from the above analysis a collapse for short wavelengths (which is well described by the homogeneous background model) can still occur when gravitational and magnetization effects are combined.

III. SUMMARY AND DISCUSSION

In the present study the effect of magnetization due to the electron spin has been combined with Newtonian gravity, in order to describe the stability properties of an initially homogeneous plasma. We note that a more complete treatment of the Jeans instability would require taking cosmological effects into account [41]. As expected, the gravitational effects tend to dominate on the large scales, i.e. for long wavelengths. For systems of a limited size, background inhomogeneities not included in our model can stabilize the longest wavelengths, in which case more significance is given to the short wavelength properties of the dispersion relation. Furthermore, for a system of moderate temperature and high density, it is seen that the magnetization of the plasma can contribute significantly to the instability in the short wavelength regime. For short wavelengths, we also note that the Bohm potential can be of importance, providing a stabilizing influence. A case where the present study is of special interest is when the long wavelength perturbations are stabilized by inhomogeneities, and the magnetization of the plasma is significant but not sufficient to cause an instability by itself. For such a plasma, the combined effects of gravitation and magnetization is needed to fulfill the instability condition.

In the present model we have not included the effect of the Hall current [42], or other finite Larmor radius effects [43] that is often introduced to improve the ideal MHD equations. We note that such modifications introduce dispersive properties of the waves, similar to the effects of the Bohm potential. However, as far as the stability properties is concerned, the moderate temperature high density regime is of most interest, since it is in this regime that the spin effects can give a significant contribution to the instability, as seen above. Furthermore, for such parameters the effects of the Bohm potential is more important than those of the Hall current.

Naturally there are several processes that can be expected in a physical collapse that are not captured within the low-frequency MHD limit. For instance, electrostatic repulsion, that could have a stabilizing effect on the system, is not included due to the assumption of quasineutrality. Furthermore, since our ideal MHD equations have an infinite conductivity, magnetic field lines will be frozen-in during collapse. In reality, there will be particle collisions within the plasma and thus the conductivity will always remain finite. The finite conductivity will prevent strict flux-freezing,
and as a result the magnetic field will not grow as fast as MHD predicts. This will also reduce the effects of spin magnetization somewhat.

As noted above, magnetization effects as well as electron tunneling and exclusion principle effects, are particularly important in low temperature high density systems. In particular only electrons with energies larger than the Fermi energy will be free to contribute to the magnetization process. Thus, the magnetization will need to contain an effective density of electrons, reduced by the number of electron below the Fermi surface. As such, a different effective scaling of the magnetization as a function of density will be obtained. Moreover, correlation effects in dense plasmas is of importance, as well as relativistic effects. These are examples of possible modifications that could be introduced in modeling of such quantum plasmas in the future.

The analysis here has been performed through a linearization of the perturbed parameters. More sophisticated methods exists which can give qualitative descriptions of the non-linear dynamics of a collapse. The use of integral virial theorems, for instance, has proven to often give a more reliable global view of the interaction of magnetic and gravitational fields in the non-linear domain [44]. Still, for a more complete understanding of a physically realistic collapse, the use of advanced computer simulations is often an necessity.

[1] J. H. Jeans, *Astronomy and cosmology* (Cambridge University Press, Cambridge, 1929).
[2] G. Manfredi, Fields Inst. Commun. 46, 263 (2005).
[3] D. Shaikh, and P. K. Shukla, Phys. Rev. Lett. 99, 125002 (2007).
[4] F. Haas, G. Manfredi, and M. R. Feix, Phys. Rev. E 62, 2763 (2000).
[5] F. Haas, Phys. Plasmas 12, 062117 (2005).
[6] L. G. Garcia, F. Haas, L. P. L. de Oliveira, and J. Geodert, Phys Plasmas 12, 012302 (2005).
[7] P.K. Shukla, Phys. Lett. A 352, 242 (2006).
[8] D. Pines, J. Nucl. Energy C: Plasma Phys. 2, 5 (1961).
[9] D. Pines, *Elementary Excitations in Solids* (Westview Press, 1999).
[10] D. Kremp, M. Schlanges, W. D. Kraeft, *Quantum Statistics of Nonideal Plasmas* (Springer, 2005).
[11] B. Bezerides, D. F. DuBois, Ann. Phys. (N.Y.) 70, 10 (1972).
[12] R. Hakim, J. Heyvaerts, Phys. Rev. A 18, 1250 (1978).
[13] H. G. Craighead, Science 290, 1532 (2000).
[14] P. A. Markovich, C. A. Ringhofer, and C. Schmeiser, *Semiconductor Equations* (Springer-Verlag, New York, 1990).
[15] M. P. Robinson, B. Laburthe Tolra, M. W. Noel, T. F. Gallagher, and P. Pillet, Phys. Rev. Lett. 85, 4466 (2000).
[16] T. C. Kilian, Nature (London) 441, 298 (2006).
[17] W. Li, P. J. Tanner, and T. F. Gallagher, Phys. Rev. Lett. 94, 173001 (2005).
[18] R. S. Fletcher, X. L. Zhang, and S. L. Rolston, Phys. Rev. Lett. 96, 105003 (2006).
[19] D. Kremp, Th. Bornath, M. Bonitz and M. Schlanges, Phys. Rev. E 60, 4725 (1999).
[20] A. V. Andreev, JETP Lett. 72, 238 (2000).
[21] M. Marklund, and P. K. Shukla, Rev. Mod. Phys. 78, 591 (2006).
[22] G. A. Mourou, T. Tajima, and S. V. Bulanov, Rev. Mod. Phys. 78, 309 (2006).
[23] S. H. Glentzer, O. L. Landen, P. Neumayer, R. W. Lee, K. Widmann, S. W. Pollaine, R. J. Wallace, G. Gregori, A. Hill, T. Bornath, R. Thiele, V. Schwartz, W.-D. Kraeft, and R. Redmer, Phys. Rev. Lett. 98, 065002 (2007).
[24] Y.F. Jung, Phys. Plasmas 8, 3842 (2001).
[25] M. Opher, L.O. Silva, D.E. Danger, V.K. Decyk and J.M. Dawson, Phys. Plasmas 8, 2454 (2001).
[26] G. Chabrier, F. Douchin, and A. Y. Potekhin, J. Phys. Condens. Matter 14, 9133 (2002).
[27] V.I. Beskin et al, *Physics of the Pulsar Magnetosphere* (Cambridge University Press, Cambridge, 1993).
[28] E. Asseo, Plasma Phys. Control. Fusion 45, 853 (2003).
[29] C. Kouveliotou et al, Nature 393, 235 (1998).
[30] M. G. Baring, P. L. Gonthier and A. K. Harding, Astrophys. J. 630 430 (2005).
[31] A. K. Harding and D. Lai, Rep. Prog. Phys. 69 2631 (2006).
[32] G. Brodin and M. Marklund, New J. Phys. 9, 277 (2007).
[33] G. Brodin, M. Marklund, Phys. Plasmas 14, 112107 (2007).
[34] P. K. Shukla, and B. Eliasson, Phys. Rev. Lett. 96, 245001 (2006).
[35] G. Brodin, and M. Marklund, Phys. Rev. E 76, 055403(R) (2007).
[36] M. Marklund, B. Eliasson, P. K. Shukla, Phys. Rev. E 76, 067401 (2007).
[37] P. K. Shukla, and L. Stenflo, Physics Letters A 355, 378 (2006) 055403(R) (2007).
[38] C. Gardner, SIAM J. Appl. Math. 54, 409 (1994).
[39] M. Marklund and G. Brodin, Phys. Rev. Lett. 98, 025001 (2007).
[40] G. Brodin, M. Marklund and G. Manfredi, Phys. Rev. Lett. 100, 175001 (2008).
[41] P. J. E. Peebles, *Principles of Physical Cosmology* (Princeton University Press, 1993)
[42] G. Brodin and L. Stenflo, Contr. Plasma Phys. 30, 413 (1990).
[43] G. Brodin and L. Stenflo, Phys. Scripta, 37, 89 (1988).
[44] L. Mestel, *Stellar Magnetism* (Clarendon Press, Oxford, 2003).
FIG. 1: The growth rate has been plotted for (a) the classical Jeans instability and (b) the Jeans instability modified by the presence of an external magnetic field ($C_A = 2$ in Fig. (b) – (f)). In (c) we have added non-locality effects of the electrons described by the Bohm potential ($Q = 0.5$), and (d) includes electron spin effects in the plasma description ($\alpha = 0.4$). In (e) both spin and non-locality effects are included in the model in a regime ($\alpha = 0.75, Q = 0.5$) where the plasma would be unstable even without gravitational effects, Fig. (f).