RANDOMIZED URYSOHN–TYPE INEQUALITIES

THOMAS HACK AND PETER PIVOVAROV

Abstract. As a natural analog of Urysohn’s inequality in Euclidean space, Gao, Hug, and Schneider showed in 2003 that in spherical or hyperbolic space, the total measure of totally geodesic hypersurfaces meeting a given convex body $K$ is minimized when $K$ is a geodesic ball. We present a random extension of this result by taking $K$ to be the convex hull of finitely many points drawn according to a probability distribution and by showing that the minimum is attained for uniform distributions on geodesic balls. As a corollary, we obtain a randomized Blaschke–Santaló inequality on the sphere.

§1. Introduction. Recent research on isoperimetry in Euclidean space has centered on the following theme: in the presence of convexity, isoperimetric principles often admit stronger stochastic versions. The expository article [35] outlines how work in stochastic geometry has motivated randomized versions of fundamental inequalities. As an illustrative example, Urysohn’s inequality for convex bodies $K \subseteq \mathbb{R}^n$ relates volume $V_n$ and mean width $w$, namely,

$$
\left( \frac{V_n(K)}{V_n(B)} \right)^{1/n} \leq \frac{w(K)}{w(B)},
$$

where $B$ is the Euclidean unit ball. When viewed as the isoperimetric principle that balls minimize mean width under prescribed volume, (1) affords a stochastic strengthening. Namely, each convex body $K \subseteq \mathbb{R}^n$ can be approximated from within by a random polytope $[K]_N = \text{conv}\{X_1, \ldots, X_N\}$, where $X_i$ are independent random vectors distributed uniformly in $K$. Then one has an “empirical” version of (1), that is,

$$
\mathbb{E} w([K]_N) \geq \mathbb{E} w\left(\left[ B^w_v K \right]_N \right),
$$

where $B^w_v$ is a Euclidean ball with $V_n(K) = V_n(B^w_v K)$, see, for example, [27, 36]. When $N \to \infty$, one recovers Urysohn’s inequality (1) in the limit. Volumetric inequalities for expected mean values have a long history in stochastic geometry and go back (at least) to Blaschke’s resolution of Sylvester’s four point problem [11], and its numerous generalizations, for example, arbitrary dimension [18, 19, 26], compact sets and other intrinsic volumes [27, 36], continuous distributions [32] (see also [39, Chapter 10]).

Alternatively, (1) also means that Euclidean balls maximize volume for a given mean width. This too admits a stochastic strengthening, by approximating $K$ from outside by intersections of halfspaces containing it, say $[K]^N = \bigcap_{i=1}^N H^+_i$; here the $H^+_i$ are independent random halfspaces with boundary $H_i$ meeting a neighborhood of $K$, sampled according to the (suitably normalized) motion invariant measure on such hyperplanes. Then

$$
\mathbb{E} V_n([K]^N)^{1/n} \leq \mathbb{E} V_n(\left[ B^w_K \right]^N)^{1/n},
$$
where $B_w^N$ is a Euclidean ball with $w(K) = w(B_w^N)$. Again, when $N \to \infty$, one obtains (1). The latter follows from work in [12], as explained in [34, § 5]. An altogether different approximation of $K$ connected to (1) by intersections of Euclidean balls with large radius, as opposed to halfspaces, was studied in [34].

As a natural analog of (1) on the sphere $S^n$, Gao, Hug, and Schneider [23] considered the functional $U_1$ defined on the unit sphere by

$$U_1(K) = \int_{S_{n-1}} \chi(K \cap S) \, dS,$$

where $K \subseteq S^n$ is a proper spherical convex body, $S_{n-1}$ the set of $(n-1)$-dimensional great subspheres in $S^n$, equipped with its rotation-invariant measure, and $\chi$ the Euler characteristic.

They showed that

$$U_1(K) \geq U_1(C_K),$$

where $C_K$ is a geodesic cap with the same spherical volume as $K$. The connection to (1) becomes clear in Euclidean space, when one parametrizes affine hyperplanes via their normal vector and (signed) distance to the origin: integration over parallel subspaces of the function $\chi(K \cap \cdot \cdot \cdot)$ yields precisely the width of $K$ in that direction.

While Urysohn’s inequality (1) is just one example from the rich theory of isoperimetry for intrinsic volumes, for example, [39], the situation on the sphere is wide open. In [23], three natural families of functionals that could be considered spherical intrinsic volumes are discussed and most isoperimetric inequalities are still conjectural. However, progress has been made in hyperbolic space [1, 41]. Beyond intrinsic volumes, there is much recent interest in extending fundamental notions from Euclidean space to the sphere and beyond, for example, floating bodies [9]; polytopal approximation [8]; and behavior of asymptotic mean values of functionals of random polytopes [5].

The randomized forms of Urysohn’s inequality (2) and (3) are just two examples of stochastic inequalities in $\mathbb{R}^n$. Other fundamental inequalities like those of Brunn–Minkowski [24] and Blaschke–Santaló [38] also admit stochastic strengthenings [21, 35]. Moreover, centroid bodies and their $L_p$ [29] and Orlicz extensions [30] also admit stochastic forms [32, 35]. Interest is driven in part by applications to high-dimensional probability, especially small-ball probabilities [33, 35]. In comparison, stochastic isoperimetry on the sphere is in early stages. Recently, spherical centroid bodies and their empirical analogs were introduced and stochastic isoperimetric inequalities were established in [7].

In this paper, we address an empirical version of the spherical Urysohn inequality (5). In fact, as in [23], our treatment will also include hyperbolic space. Throughout, we let $\mathbb{M}^n$ be either spherical, Euclidean, or hyperbolic space equipped with its isometry-invariant volume measure $\lambda_n$, and consider the natural analog of (4) in $\mathbb{M}^n$ (see Section 2 for precise definitions).

Our first main theorem then reads as follows.

**Theorem 1.1.** Let $N \in \mathbb{N}$, $f_1, \ldots, f_N : \mathbb{M}^n \to \mathbb{R}^+$ bounded, integrable, and with proper support, if $\mathbb{M}^n = S^n$. Set

$$I(f_1, \ldots, f_N) = \int_{\mathbb{M}^n} \ldots \int_{\mathbb{M}^n} U_1(\text{conv}\{x_1, \ldots, x_N\}) \prod_{i=1}^N f_i(x_i) \, dx_1 \ldots dx_N.$$

Then

$$I(f_1, \ldots, f_N) \geq I(\|f_1\|_\infty 1_{B_1}, \ldots, \|f_N\|_\infty 1_{B_N}),$$

(6)
where the $B_i$ are geodesic balls in $\mathbb{M}^n$ centered at a common point, satisfying

$$\lambda_n(B_i) = \frac{\|f_i\|_{L^1(\mathbb{M}^n)}}{\|f_i\|_\infty}.$$ 

By taking indicator functions on compact sets, we obtain the following corollary.

**Corollary 1.2.** Let $N \in \mathbb{N}, K \subseteq \mathbb{M}^n$ be compact, and proper in the case $\mathbb{M}^n = \mathbb{S}^n$. Then

$$\mathbb{E}U_1([K]_N) \geq \mathbb{E}U_1([B_K]_N),$$

where $B_K$ is a geodesic ball satisfying $\lambda_n(K) = \lambda_n(B_K)$.

The proof of Theorem 1.1 does not rely on (5). Rather, we first prove a rearrangement inequality for $I(f_1, \ldots, f_N)$ that reduces the problem to radially decreasing densities. This is similar to the route taken in the Euclidean setting [32, 35] but we use two-point symmetrization, for example, [6, 14, 42], as in [23], rather than Steiner symmetrization. In $\mathbb{R}^n$, a key tool behind stochastic isoperimetric inequalities is Steiner symmetrization and associated rearrangement inequalities [13, 20, 37]. These inequalities interface is well with fundamental tools in convex geometry, like shadow systems, Alexandrov’s theory of mixed volumes, Busemann’s convexity of intersection bodies (see [35] and the references therein). There is no comparable development on the sphere or in hyperbolic space. Going back to Baernstein and Taylor [4], two-point symmetrization has been used as an analytical tool on $\mathbb{M}^n$ for multiple integral rearrangement inequalities, see also [16, 17, 31]; more recently in isoperimetric inequalities [10]. However, such techniques have not yet been fused with stochastic convex geometry in $\mathbb{M}^n$. Theorem 1.1 is a first step in this direction.

As noted in [23], there is a special relationship between $U_1$ and spherical polar duality (see Proposition 2.1). In this way, (6) can also be reinterpreted as a spherical Blaschke–Santaló inequality in stochastic form.

**Theorem 1.3.** Let $N \in \mathbb{N}, f_1, \ldots, f_N: \mathbb{S}^n \to \mathbb{R}^+$ bounded, integrable, and with proper support. Set

$$\tilde{I}(f_1, \ldots, f_N) = \int_{\mathbb{S}^n} \ldots \int_{\mathbb{S}^n} \lambda_n(\text{conv}\{x_1, \ldots, x_N\}^*) \prod_{i=1}^N f_i(x_i) \, dx_1 \ldots dx_N.$$

Then

$$\tilde{I}(f_1, \ldots, f_N) \leq \tilde{I}(\|f_1\|_\infty \mathbb{1}_{C_1}, \ldots, \|f_N\|_\infty \mathbb{1}_{C_N}),$$

where the $C_i$ are spherical caps centered at a common point, satisfying

$$\lambda_n(C_i) = \frac{\|f_i\|_{L^1(\mathbb{S}^n)}}{\|f_i\|_\infty}.$$ 

Again, by taking indicator functions on compact sets, we obtain as a corollary.

**Corollary 1.4.** Let $N \in \mathbb{N}, K \subseteq \mathbb{S}^n$ be compact, and proper. Then

$$\mathbb{E}\lambda_n([K]_N^*) \leq \mathbb{E}\lambda_n([C_K]_N^*),$$

where $C_K$ is a spherical cap satisfying $\lambda_n(K) = \lambda_n(C_K)$.
A symmetric version of Corollary 1.4, where \([K]_{\nu}\) is replaced by the convex hull of the random points \(X_t\) and their reflections about some fixed origin, can be obtained by using a stochastic Blaschke–Santaló inequality in Euclidean space from [21], together with gnomonic projection. This strategy of relying on Euclidean techniques such as Steiner symmetrization in \(\mathbb{R}^n\) after projecting from the sphere has also been studied in detail for spherical centroid bodies in [7].

By using two-point symmetrization, we treat Euclidean, spherical, and hyperbolic space simultaneously. Note that in addition to Steiner symmetrization, which is used to prove (2), and Minkowski symmetrization for (3), this gives another method to obtain a randomized Urysohn inequality in \(\mathbb{R}^n\).

§2. Background material. In this section, we introduce our models for the spaces of constant curvature, along with certain isometry-invariant measures associated with them. Moreover, we give a short account of two-point symmetrization and rearrangements in these spaces and collect properties that we need later on. As a general reference for spherical convexity in particular, we refer the reader to [25] or [40]. Further information about two-point symmetrization can also be found in [2–4, 14, 15].

2.1. Spaces of constant curvature. We will use the following models \(\mathbb{M}^n\) for spherical, Euclidean, and hyperbolic geometry: Let \(e := (1, 0, \ldots, 0) \in \mathbb{R}^{n+1}\) and write \(x \cdot y := x_0y_0 + x_1y_1 + \cdots + x_ny_n\) and \((x, y) := x_0y_0 - x_1y_1 - \cdots - x_ny_n\) for the standard Euclidean and Minkowski scalar products. Then we define

- Spherical space: \(\mathbb{S}^n := \{x \in \mathbb{R}^{n+1} : x \cdot x = 1\}\),
- Euclidean space: \(\mathbb{R}^n := \{x \in \mathbb{R}^{n+1} : x \cdot e = 1\}\),
- Hyperbolic space: \(\mathbb{H}^n := \{x \in \mathbb{R}^{n+1} : x \cdot e > 0, (x, x) = 1\}\).

We think of \(\mathbb{R}^n\) as an affine subspace, but we will still refer to \(\mathbb{S}^{n-1}\) as \(\mathbb{S}^n \cap \{x_{n+1} = 0\}\). Topological interior and closure will always be understood relative to \(\mathbb{M}^n\) and written as int and cl, respectively. The isometry-invariant volume measure will be denoted by \(\lambda_n\) simultaneously for all three geometries. In each case, it is the restriction of \(n\)-dimensional Hausdorff measure on \(\mathbb{R}^{n+1}\). We will just write \(dx\) instead of \(d\lambda_n(x)\) when integrating over \(\mathbb{M}^n\).

Similarly, we will denote by \(\mu^n_{n-1}\) the motion-invariant measure on \(\mathcal{M}^n_{n-1}\), the collection of \((n-1)\)-dimensional totally geodesic submanifolds of \(\mathbb{M}^n\), that is,

\[\mathcal{M}^n_{n-1} = \{E \cap \mathbb{M}^n | E \text{ is an } n\text{-dimensional subspace of } \mathbb{R}^{n+1} \text{ that intersects } \mathbb{M}^n\}\].

The normalization is \(\mu^n_{n-1}(\mathcal{M}^n_{n-1}) = 1\), if \(\mathbb{M}^n = \mathbb{S}^n\), and such that \(\mu^n_{n-1}(\{M \in \mathcal{M}_{n-1}^n | M \cap B_1 \neq \emptyset\}) = 1\), if \(\mathbb{M}^n = \mathbb{R}^n\) or \(\mathbb{H}^n\), where \(B_1\) is the geodesic ball of radius 1 around \(e\). For every \(M \in \mathcal{M}^n_{n-1}\), we can find a vector \(u \in \mathbb{R}^{n+1}\), such that \(M = u \perp \cap \mathbb{M}^n\), where the orthogonal complement is taken either with respect to Euclidean or Minkowski scalar product in \(\mathbb{R}^{n+1}\). Again, we write \(dM\) instead of \(d\mu^n_{n-1}(M)\) when integrating over \(\mathcal{M}^n_{n-1}\).

Let \(K \subseteq \mathbb{M}^n\), \(K \neq \{x, -x\}\) if \(\mathbb{M}^n = \mathbb{S}^n\). If for any \(x, y \in K\), \(x \neq -y\) in the case \(\mathbb{M}^n = \mathbb{S}^n\), the shortest geodesic segment \([x, y]\) connecting \(x, y\) lies inside \(K\), we call \(K\) convex. Moreover, \(K\) is a convex body if it is convex and compact. The convex hull of a set \(K\), denoted by \(\text{conv}\ K\), is the intersection of all convex sets containing \(K\). A closed set \(K \subseteq \mathbb{S}^n\) is called proper, if it is contained in an open hemisphere. Accordingly, \(f : \mathbb{S}^n \to \mathbb{R}\) is said to have proper support, if \(\text{spt} f = \text{cl}\{x \in \mathbb{S}^n | f(x) \neq 0\}\) is proper.

For a set \(A \subseteq \mathbb{M}^n\), we define

\[\chi(A) := \begin{cases} 1, & \text{if } A \neq \emptyset, \\ 0, & \text{if } A = \emptyset. \end{cases}\] (7)
In polar coordinates, we have

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in Euler characteristic. In some parts, we will also allow densities having non-proper support in \( S^m \); hence, it seems more appropriate to use the above version of \( \chi \). We can now give a definition of \( U_1 \) on \( M^m \) analogous to (4):

\[
U_1(K) := \int_{M^m_{n-1}} \chi(K \cap M) \, dM,
\]

where \( K \subseteq M^m \) is compact and \( \chi \) as in (7).

**Polar coordinates.** We set

\[
R^M := \begin{cases} \pi, & \text{if } M^m = S^n, \\ \infty, & \text{if } M^m = \mathbb{R}^n \text{ or } \mathbb{H}^n. \end{cases}
\]

For \( t \in [0, R^M] \), we define

\[
\begin{align*}
\text{cs} \, t & := \begin{cases} \cos t, & \text{if } M^m = S^n, \\ 1, & \text{if } M^m = \mathbb{R}^n, \\ \cosh t, & \text{if } M^m = \mathbb{H}^n, \end{cases} \\
\text{sn} \, t & := \begin{cases} \sin t, & \text{if } M^m = S^n, \\ t, & \text{if } M^m = \mathbb{R}^n, \\ \sinh t, & \text{if } M^m = \mathbb{H}^n, \end{cases}
\end{align*}
\]

and introduce polar coordinates \( x(t, u) := e \text{cs} \, t + u \text{sn} \, t, \) for \( u \in S^{n-1} \) and \( t \in [0, R^M] \). The following transformation formula holds in all three geometries:

\[
\int_{M^n} f(x) \, dx = \int_{S^{n-1}} \int_0^{R^M} f(x(t, u)) \, \text{sn}^{n-1} t \, dt \, du,
\]

for any integrable function \( f : M^n \to \mathbb{R} \). For a proof, see [22, Sections 3.F, 3.H].

**Metric.** We write \( d_{M^n}(x, y) \) for the geodesic distance between \( x, y \in M^n \), that is,

\[
d_{M^n}(x, y) = \begin{cases} \arccos(x \cdot y), & \text{if } M^n = S^n, \\ ||x - y||, & \text{if } M^n = \mathbb{R}^n, \\ \arccosh(\langle x, y \rangle), & \text{if } M^n = \mathbb{H}^n. \end{cases}
\]

In polar coordinates, we have \( d_{M^n}(e, x(t, u)) = t, \) for \( u \in S^{n-1} \) and \( t \in [0, R^M] \).

**Polarity.** The following relation is special to the case \( M^n = S^n \): For \( K \subseteq S^n \), its polar set \( K^* \) is given by

\[
K^* = \{ x \in S^n \mid x \cdot y \leq 0 \text{ for all } y \in K \}.
\]

**PROPOSITION 2.1.** Let \( K \subseteq S^n \) be a convex body. Then

\[
\frac{U_1(K)}{\mu_{n-1}^n(\mathcal{M}_{n-1}^n)} + \frac{2\lambda_n(K^*)}{\lambda_n(S^n)} = 1.
\]

**Proof.** If \( K \) is proper the proof can be found in [23, eq. 20]. On the other hand, as soon as \( K \) contains antipodal points, we have \( U_1(K) = \mu_{n-1}^n(\mathcal{M}_{n-1}^n) \) and \( K^* = \emptyset \). \( \square \)

2.2. **Two-point symmetrization and rearrangements.** A hyperplane \( H \in \mathcal{M}_{n-1}^n \) divides \( M^n \) into two connected components, which we will call the closed halfspaces \( H^+ \) and \( H^- \) in such
a way that always $e \in H^+$. The associated orthogonal reflection about $H$ will be denoted by $\rho : M^n \to M^n$. If $H = u^+ \cap M^n$, for $u \in \mathbb{R}^{n+1}$, then $\rho$ is given by

$$
\rho x := \rho(x) = \begin{cases} 
  x - 2 \frac{x \cdot u}{u \cdot u} u, & \text{if } M^n = S^n, \\
  x - 2 \frac{1}{1 - (u \cdot e)^2} [u - (u \cdot e)e], & \text{if } M^n = \mathbb{R}^n, \\
  x - 2 \frac{(x, u)}{(u, u)} u, & \text{if } M^n = \mathbb{H}^n.
\end{cases}
$$

In view of the following, we make a note of the fact that for any $x, y \in H^+$, we have

$$
d_{M^n}(x, y) \leq d_{M^n}(x, \rho(y)).$$

Indeed, let $z$ be the intersection of the geodesic segment $[x, \rho y]$ with $H$. Then

$$
d_{M^n}(x, y) \leq d_{M^n}(x, z) + d_{M^n}(z, y) = d_{M^n}(x, z) + d_{M^n}(z, \rho y) = d_{M^n}(x, \rho y),$$

by the triangle inequality.

**Two-point symmetrization.** If we decompose a set $K \subseteq M^n$ as

$$
K = (K \cap \rho K) \cup (K \cap H^+) \setminus K^{sym} \cup (K \cap H^-) \setminus K^{sym},
$$

the two-point symmetrization $T = (H, \rho, T)$ with respect to $H$ (Figure 1) is given by

$$
TK = (K \cap \rho K) \cup (K \cap H^+) \setminus K^{sym} \cup \rho(K \cap H^-) \setminus K^{sym}.
$$

Figure 1 (colour online): A two-point symmetrization of $K$.

Note that all unions are disjoint up to sets of measure zero, which immediately shows that $\lambda_n(TK) = \lambda_n(K)$ for all measurable sets $K \subseteq M^n$. Intuitively, $T$ pushes as much mass as possible toward $e$ (that is, into $H^+$) without double-covering points.

Two-point symmetrization of a function $f : M^n \to \mathbb{R}^+$ is given by

$$
Tf(x) = \begin{cases} 
  \max\{f(x), f(\rho x)\}, & \text{if } x \in H^+, \\
  \min\{f(x), f(\rho x)\}, & \text{if } x \in H^-.
\end{cases}
$$

We have $T1_K = 1_{TK}$ for any set $K \subseteq M^n$. More generally, one can check that

$$
\{Tf > s\} = T\{f > s\} \quad (9)
$$

for all $s > 0$. For a continuous function $f \in C(M^n)$ and $\delta > 0$, denote by

$$
\omega(\delta, f) = \sup\{|f(x) - f(y)| : d_{M^n}(x, y) < \delta, \ x, y \in M^n\}
$$

the modulus of continuity of $f$. The proof of the next lemma can be found, for example, in [3, Proposition 1.37] for $M^n = R^n$ and translated verbatim to $S^n$ and $H^n$ (cf. [3, Section 7.1]).
**Lemma 2.2.** For every continuous function \( f \in C(\mathbb{M}^n) \), \( \delta > 0 \), and every two-point symmetrization \( T \), we have
\[
\omega(\delta, Tf) \leq \omega(\delta, f).
\]

**Rearrangements.** Let \( f : \mathbb{M}^n \to \mathbb{R}^+ \) be integrable so that in particular \( \lambda_n(\{ f > t \}) < \infty \) for all \( t > 0 \). We want to associate to \( f \) a function \( f^* \) whose sublevel sets are geodesic balls around \( e \in \mathbb{M}^n \). The symmetric decreasing rearrangement of \( f \) is the function \( f^* : \mathbb{M}^n \to \mathbb{R}^+ \) of \( x = x(t, u) \), depending only on \( t \), is non-increasing as \( t \) increases, and has the property
\[
\lambda_n(\{ f > s \}) = \lambda_n(\{ f^* > s \})
\]
for all \( s > 0 \). It is defined up to sets of measure zero and can be written explicitly as
\[
f^*(x(t, u)) = \inf \{ s \mid \lambda_n(\{ f > s \}) \leq \lambda_n(B_t) \},
\]
where \( B_t \) is the geodesic ball around \( e \) with radius \( t = d_{\mathbb{M}^n}(e, x) \).

The next lemma relates two-point symmetrization and symmetric decreasing rearrangements.

**Lemma 2.3.** Let \( f : \mathbb{M}^n \to \mathbb{R}^+ \) be integrable and \( T = (H, \rho, T) \) any two-point symmetrization. Then the following holds:
(i) For \( x \in H^+ \) we have \( f(x)f^*(x) + f(\rho x)f^*(\rho x) \leq T f(x)f^*(x) + T f(\rho x)f^*(\rho x) \).
(ii) \( \int_{\mathbb{M}^n} |f(x) - f^*(x)|^2 \, dx \geq \int_{\mathbb{M}^n} |Tf(x) - f^*(x)|^2 \, dx \).

**Proof.** For (i), we refer to [3, Theorem 2.8 (a)], whereas (ii) follows from integration over \( H^+ \) and the fact that \( T \) and \( * \) preserve \( L^2 \)-norms (cf. [3, eq. (2.13)]). \( \square \)

§3. **Auxiliary results.** In this section, we compile results from the literature that are necessary to complete the proof of our main result. We consider functionals of the form
\[
I_\Psi(f_1, \ldots, f_N) = \int_{\mathbb{M}^n} \ldots \int_{\mathbb{M}^n} \Psi(x_1, \ldots, x_N) \prod_{i=1}^N f_i(x_i) \, dx_1 \ldots dx_N,
\]
for a bounded, measurable function \( \Psi : (\mathbb{M}^n)^N \to \mathbb{R}^+ \) and bounded, integrable functions \( f_i : \mathbb{M}^n \to \mathbb{R}^+ \), \( 1 \leq i \leq N \). Sometimes, we will also consider the truncated functional
\[
I_\Psi^R(f_1, \ldots, f_N) = I_\Psi(1_{B_R}f_1, \ldots, 1_{B_R}f_N),
\]
where \( B_R \) is the geodesic ball of radius \( R > 0 \) around the origin \( e \in \mathbb{M}^n \). Clearly, in the case \( \mathbb{M}^n = \mathbb{S}^n \), we have \( I^R = I \) whenever \( R \geq \pi \). Integrals of type (10), involving monotone functions of pairwise distances \( d_{\mathbb{M}^n}(x_i, x_j) \), have already been considered by Morpurgo [31], Burchard and Schmuckenschläger [17], and Burchard and Hajaiej [16]. Before treating the functional \( \Psi(x_1, \ldots, x_N) = U_1(\text{conv}(x_1, \ldots, x_N)) \), we recall a general convergence result for iterated two-point symmetrizations.

3.1. **Achieving radial symmetry.** The following proposition is due to Baernstein and Taylor [4] in the case \( \mathbb{M}^n = \mathbb{S}^n \) (see also [3, Section 2.5]). We reproduce their proof at once in all three geometries.

**Proposition 3.1.** Let \( f_1, \ldots, f_N : \mathbb{M}^n \to \mathbb{R}^+ \) be bounded, integrable functions, and let \( \Psi(x_1, \ldots, x_N) : (\mathbb{M}^n)^N \to \mathbb{R}^+ \) be bounded and measurable. Furthermore, assume that
\[
I_\Psi(f_1, \ldots, f_N) \geq I_\Psi(T f_1, \ldots, T f_N),
\]


for every two-point symmetrization $T$ in $\mathbb{M}^n$. Then

$$I_\Psi(f_1, \ldots, f_N) \geq I_\Psi(f_1^*, \ldots, f_N^*).$$

**Proof.** We follow [4, Section 2]. We start with the following facts:

(i) $\mathbb{I}_{B_R} f_i \to f_i$ in $L^1(\mathbb{M}^n)$ as $R \to \infty$,

(ii) $|I_\Psi(f_1, \ldots, f_N)| \leq \|\Psi\|_\infty \prod_{i=1}^N \|f_i\|_{L^1(\mathbb{M}^n)}$,

(iii) the map $f \mapsto f^*$ is continuous in $L^1(\mathbb{M}^n)$,

(iv) there are sequences $(\phi_i^j)_{j \in \mathbb{N}}$ in $C_c(\mathbb{M}^n)$, $\operatorname{spt} \phi_i^j \subseteq \operatorname{spt} f_i$, such that $\phi_i^j \to f_i$ in $L^1(\mathbb{M}^n)$.

Indeed, (i) follows from dominated convergence, (ii) is a standard estimate, (iii) follows from the non-expansivity of symmetric decreasing rearrangements (see [28, Theorem 3.5]), and (iv) is an application of Urysohn’s lemma.

By (i) and (ii), we can assume without loss of generality that $\operatorname{spt} f_i \subseteq B_R$, and by (ii), (iii), and (iv), we can restrict further to continuous functions $f_i$ (supported in $B_R$) and denote this space by $C(B_R) \subseteq L^2(B_R)$. We define for $f \in C(B_R)$

$$S(f) := \{F \in C(B_R) : \omega(\cdot, F) \leq \omega(\cdot, f) \text{ and } F^* = f^* \}.$$

We show now that $S(f)$ is a compact subset of $L^2(B_R)$. Since $f$ is uniformly continuous, for each $\varepsilon > 0$ there exists $\delta > 0$, such that $\omega(\delta, F) \leq \omega(\delta, f) < \varepsilon$, for all $F \in S(f)$. Moreover, $\|F\|_\infty = \|F^*\|_\infty = \|f^*\|_\infty$. Hence, $S(f)$ is a uniformly equicontinuous, uniformly bounded family of continuous functions, thus relatively compact in $(C(B_R), \|\cdot\|_\infty)$ by the Arzelá–Ascoli theorem. Since the map $f \mapsto f^*$ is continuous also in the $(C(B_R), \|\cdot\|_\infty)$ topology (take $p \to \infty$ in [28, Theorem 3.5]) and the sets $\{\omega(\delta, F) \leq \omega(\delta, f)\}$ are closed (by the triangle inequality), the set $S(f)$ is compact in $(C(B_R), \|\cdot\|_\infty)$. It is also compact in $(C(B_R), \|\cdot\|_{L^2(B_R)})$, since by $\|f\|_{L^2(B_R)}^2 \leq \lambda_n(B_R) \|f\|_\infty^2$, the latter space has a coarser topology.

Next, by Lemma 2.2, $TS(f) \subseteq S(f)$ for every two-point symmetrization $T$. Since, by the Cauchy–Schwarz inequality, we have

$$|I_\Psi(f_1, \ldots, f_N)| \leq \lambda_n(B_R)^N \|\Psi\|_\infty \prod_{i=1}^N \|f_i\|_{L^2(B_R)},$$

for fixed $f_1, \ldots, f_N \in C(B_R)$, the set

$$\mathcal{P} := \{(F_1, \ldots, F_N) \in S(f_1) \times \cdots \times S(f_N) : I(f_1, \ldots, f_N) \geq I(F_1, \ldots, F_N)\}$$

is compact in $L^2(B_R) \times \cdots \times L^2(B_R)$. By assumption, $\mathcal{P}$ is closed under simultaneous two-point symmetrizations $(F_1, \ldots, F_N) \mapsto (T F_1, \ldots, T F_N)$. We are done, if we can show that $\mathcal{P}$ contains $(f_1^*, \ldots, f_N^*)$.

By compactness, there exist $(F_1^0, \ldots, F_N^0) \in \mathcal{P}$ such that

$$\sum_{i=1}^N \|F_i^0 - f_i^*\|_{L^2(B_R)} = \min \left\{ \sum_{i=1}^N \|F_i - f_i^*\|_{L^2(B_R)} : (F_1, \ldots, F_N) \in \mathcal{P} \right\}.$$

Without loss of generality, assume that $F_i^0 \neq f_i^*$, that is, there exists $t > 0$ such that

$$E_1 := \{F_i^0 > t\} \neq \{f_i^* > t\} =: E_2, \quad E_1, E_2 \subseteq B_R.$$

Since $\lambda_n(E_1) = \lambda_n(E_2)$, there exist $x \in \operatorname{int}(E_1 \setminus E_2)$ and $y \in \operatorname{int}(E_2 \setminus E_1)$. Let $H \in \mathcal{M}^{n-1}$ be the totally geodesic hypersurface that orthogonally bisects the geodesic segment $[x, y]$. Then, $\rho(t)$
reflection about $H$, and $T$ the associated two-point symmetrization. Then we can find a small geodesic ball $B$ around $y$, such that $B \subseteq (E_2 \setminus E_1)$ and $\rho B \subseteq (E_1 \setminus E_2)$.

For all $z \in B$, we then have $f_1^*(z) - t \geq f_1^*(\rho z)$ and $F_{i}^{0}(\rho z) - t \geq F_{i}^{0}(z)$, yielding

$$F_{i}^{0}(z)f_1^*(z) + F_{i}^{0}(\rho z)f_1^*(\rho z) < TF_{i}^{0}(z)f_1^*(z) + TF_{i}^{0}(\rho z)f_1^*(\rho z).$$

Since, by Lemma 2.3 (a), the same inequality holds with “$\leq$” for all $z \in B_R$, we get

$$\int_{B_R} F_{i}^{0}(z)f_1^*(z) \, dz \leq \int_{B_R} TF_{i}^{0}(z)f_1^*(z) \, dz,$$

and, since $T$ preserves $L^2$-norms,

$$\int_{B_R} |F_{i}^{0} - f_1^*|^2 = \int_{B_R} (F_{i}^{0})^2 - 2F_{i}^{0}f_1^* + (f_1^*)^2 > \int_{B_R} (TF_{i}^{0})^2 - 2TF_{i}^{0}f_1^* + (f_1^*)^2 = \int_{B_R} |TF_{i}^{0} - f_1^*|^2.$$

Moreover, by 2.3 (b), we have

$$\int_{B_R} |F_{i}^{0} - f_1^*|^2 \geq \int_{B_R} |TF_{i}^{0} - f_1^*|^2$$

for $2 \leq i \leq N$, hence,

$$\sum_{i=1}^{N} \| F_{i}^{0} - f_1^* \|_{L^2(B_R)} \geq \sum_{i=1}^{N} \| TF_{i}^{0} - f_1^* \|_{L^2(B_R)},$$

which, since $(TF_{1}^{0}, \ldots, TF_{N}^{0}) \in \mathcal{P}$, is a contradiction. \hfill $\square$

3.2. From rotation invariance to balls. We explain now how one can progress further from bounded radially symmetric densities to indicator functions of geodesic balls. In doing so, we make use of polar coordinates in $\mathbb{M}^n$. First, we need the following version of [32, Lemma 3.5].

**Lemma 3.2.** Let $f : [0, R^M] \to \mathbb{R}^+$ be bounded, measurable and assume that

$$\int_{0}^{R^M} f(t) \, t \, dt < \infty.$$

Define $A \subseteq [0, R^M] \setminus \{0\}$ such that

$$\int_{0}^{R^M} f(t) \, t \, dt = \int_{0}^{A} \| f \|_{\infty} \, t \, dt.$$

Then for any increasing function $\phi : [0, R^M] \to \mathbb{R}^+$,

$$\int_{0}^{R^M} \phi(t) f(t) \, t \, dt \geq \int_{0}^{R^M} \phi(t) \| f \|_{\infty} \, t \, dt.$$

**Proof.** By the monotonicity of $\phi$ and the positivity of $\text{sn}$ on $[0, R^M]$, we can estimate

$$\int_{0}^{R^M} \phi(t) f(t) \, t \, dt = \int_{0}^{A} \phi(t) f(t) \, t \, dt + \int_{A}^{R^M} \phi(t) f(t) \, t \, dt$$

$$\geq \int_{0}^{A} \phi(t) f(t) \, t \, dt + \phi(A) \int_{A}^{R^M} f(t) \, t \, dt.$$
which gives the statement. □

Changing to polar coordinates \( x(t, u) = \cos t + u \sin t, \ t \in [0, R^M], u \in S^{n-1} \) (see Section 2), we can formulate the next proposition. It is a variant of [32, Proposition 3.9].

**Proposition 3.3.** Let \( f_1, \ldots, f_N : \mathbb{M}^n \to \mathbb{R}^+ \) be bounded, integrable functions, and let \( \Psi(x_1, \ldots, x_N) : (\mathbb{M}^n)^N \to \mathbb{R}^+ \) be bounded, measurable, such that the function

\[
\phi(t_1, \ldots, t_N) = \int_{S^{n-1}} \cdots \int_{S^{n-1}} \Psi(x(t_1, u_1), \ldots, x(t_N, u_N)) \, du_1 \ldots du_N
\]

is increasing in each coordinate. Then

\[
I_\Psi(f^*_1, \ldots, f^*_N) \geq I_\Psi(\| f_1 \|_\infty B_1, \ldots, \| f_N \|_\infty B_N),
\]

where \( B_i \) is a geodesic ball around \( e \) such that \( \lambda_n(B_i) = \frac{\| f_i \|_{L^1(B_i, \mathbb{M}^n)}}{\| f_i \|_\infty} \).

**Proof.** We can assume without loss of generality, that already \( f_i = f^*_i \), since taking rearrangements neither changes \( L^1 \) nor \( L^\infty \) norms. Using (8), we obtain

\[
I_\Psi(f_1, \ldots, f_N) = \int_{\mathbb{M}^n} \cdots \int_{\mathbb{M}^n} \Psi(x_1, \ldots, x_N) \prod_{i=1}^N f_i(x_i) \, dx_1 \ldots dx_N,
\]

\[
= \int_{S^{n-1}} \cdots \int_{S^{n-1}} \int_{0}^{R^M} \cdots \int_{0}^{R^M} \Psi(x(t_1, u_1), \ldots, x(t_N, u_N))
\]

\[
\times \prod_{i=1}^N f_i(x(t_i, u_i)) \, s_n^{-1} t_1 dt_1 \ldots dt_N \, du_1 \ldots du_N.
\]

By the radial symmetry of \( f_i \), we can write \( \tilde{f}_i(t_i) = f_i(x(t_i, u_i)) \) to arrive at

\[
I_\Psi(f_1, \ldots, f_N) = \int_{0}^{R^M} \cdots \int_{0}^{R^M} \phi(t_1, \ldots, t_N) \prod_{i=1}^N \tilde{f}_i(t_i) \, s_n^{-1} t_1 dt_1 \ldots dt_N.
\]

Now, applying Lemma 3.2 successively to each coordinate and noticing that

\[
\lambda_n(B_i) = \int_{0}^{A_i} s_n^{-1} t \, dt = \frac{1}{\| f_i \|_\infty} \int_{0}^{R^M} \tilde{f}_i(t) \, s_n^{-1} t \, dt = \frac{\| f_i \|_{L^1(B_i, \mathbb{M}^n)}}{\| f_i \|_\infty},
\]

where the \( A_i \in \mathbb{R}^+ \) come from the lemma, yields the statement. □

§4. **Proof of main theorem.** We split the proof of Theorem 1.1 into two parts: first, we show how to pass from given functions to their symmetric decreasing rearrangements. In a second step, we further move from radially symmetric, decreasing functions to (multiples)
of indicators of geodesic balls. For positive, bounded, and integrable functions \( f_1, \ldots, f_N \) on \( \mathbb{M}^n \), write
\[
I(f_1, \ldots, f_N) = \int_{\mathbb{M}^n} \ldots \int_{\mathbb{M}^n} U_1(\text{conv}\{x_1, \ldots, x_N\}) \prod_{i=1}^N f_i(x_i) \, dx_1 \ldots dx_N.
\]

**Proposition 4.1.** Let \( f_1, \ldots, f_N : \mathbb{M}^n \to \mathbb{R}^+ \) bounded, integrable. Then
\[
I(f_1, \ldots, f_N) \geq I(f_1^*, \ldots, f_N^*).
\]

**Proof.** For bounded, measurable subsets \( K_1, \ldots, K_N \subseteq \mathbb{M}^n \), we set \( I(K_1, \ldots, K_N) := I(\mathbb{I}_{K_1}, \ldots, \mathbb{I}_{K_N}) \). Our first step is to show that \( I(K_1, \ldots, K_N) \geq I(TK_1, \ldots, TK_N) \) for every two-point symmetrization \((H, \rho, T)\). To this end, for \( M \in \mathcal{M}_{n-1}^n \), let
\[
I(K_1, \ldots, K_N; M) := \int_{K_1} \ldots \int_{K_N} \chi(\text{conv}\{x_1, \ldots, x_N\} \cap M) \, dx_1 \ldots dx_N.
\]
We want to investigate how the quantity \( I(K_1, \ldots, K_N; M) + I(K_1, \ldots, K_N; \rho M) \) changes, when the \( K_i \) are replaced by \( TK_i \). Note that we have
\[
I(K_1, \ldots, K_N; \rho M) = I(\rho K_1, \ldots, \rho K_N; M)
\]
by the \( \rho \)-invariance of \( \chi \). We begin by decomposing each \( K_i \) according to the symmetrization
\[
K_i = \frac{K_i \cap \rho K_i}{K_i^{\text{sym}}} \cup \left[ \frac{(K_i \cap H^+) \setminus K_i^{\text{sym}}}{K_i^{\text{fix}}} \right] \cup \left[ \frac{(K_i \cap H^-) \setminus K_i^{\text{sym}}}{K_i^{\text{mov}}} \right],
\]
that is, \( TK_i = K_i^{\text{sym}} \cup K_i^{\text{fix}} \cup \rho K_i^{\text{mov}} \). Now, let \( (x_1, \ldots, x_N) \in K_1 \times \cdots \times K_N \) and introduce the following labeling:
\[
\begin{align*}
\{a_1, \ldots, a_{N_0}\} := & \{x_i \mid x_i \in K_i^{\text{sym}}, 1 \leq i \leq N\} \\
\{b_1, \ldots, b_{N_1}\} := & \{x_i \mid x_i \in K_i^{\text{fix}}, 1 \leq i \leq N\} \\
\{c_1, \ldots, c_{N_2}\} := & \{x_i \mid x_i \in K_i^{\text{mov}}, 1 \leq i \leq N\},
\end{align*}
\]
where \( N_0 + N_1 + N_2 = N \). For brevity we will use the notation \( \tilde{x} := \rho x \) for \( x \in \mathbb{M}^n \) and consider the tuples
\[
D_1 := (a_1, \ldots, a_{N_0}, b_1, \ldots, b_{N_1}, c_1, \ldots, c_{N_2}), \\
D_2 := (a_1, \ldots, a_{N_0}, \tilde{b}_1, \ldots, \tilde{b}_{N_1}, \tilde{c}_1, \ldots, \tilde{c}_{N_2}),
\]
and
\[
E_1 := (a_1, \ldots, a_{N_0}, b_1, \ldots, b_{N_1}, \tilde{c}_1, \ldots, \tilde{c}_{N_2}), \\
E_2 := (a_1, \ldots, a_{N_0}, \tilde{b}_1, \ldots, \tilde{b}_{N_1}, c_1, \ldots, c_{N_2}).
\]
After a suitable reordering of elements, we have
\[
D_1 \in \times_{i=1}^N K_i, \quad D_2 \in \times_{i=1}^N \rho K_i, \quad E_1 \in \times_{i=1}^N TK_i, \quad E_2 \in \times_{i=1}^N \rho TK_i.
\]
(The actual order of elements in \( D_i \) and \( E_i, i = 1, 2 \), will be irrelevant in what follows.) Exchanging \( c_l \) with \( \tilde{c}_l \) for every \( 1 \leq l \leq N_2 \), yields the mapping
\[
D_1 \mapsto E_1, \quad D_2 \mapsto E_2,
\] (11)
whereas exchanging $b_k$ with $\bar{b}_k$ for every $1 \leq k \leq N_1$ induces
\[ D_1 \mapsto E_2, \quad D_2 \mapsto E_1. \] (12)

We claim that
\[ \sum_{i=1}^{2} \chi(\text{conv}\{D_i\} \cap M) \geq \sum_{i=1}^{2} \chi(\text{conv}\{E_i\} \cap M) \] (13)
for almost all $M \in \mathcal{M}_n^{\mathbb{R}}$, that is, we tacitly assume that $x_1, \ldots, x_N$ do not lie on $M$, see Figure 2. We will verify the claim by checking all possible positions of the points $a_j, b_k, c_l$ relative to $M$. In doing so, we mean by a pair of points a point and its reflection about $H$, that is, $x$ and $\bar{x} = \rho x$. We call a pair of points $x$ and $\bar{x}$ split if they lie on opposite sides of $M$. In other words, for every non-split $(x, \bar{x})$,
\[ \chi(\text{conv}\{x, F\} \cap M) = \chi(\text{conv}\{\bar{x}, F\} \cap M), \]
for all subsets $F \subseteq \mathbb{R}$.

- **Case 1:** None of the pairs of functions of $b$ are split. By (12), the terms on both sides of (13) are just a permutation of each other, thus there is equality in (13).
- **Case 2:** None of the pairs of functions of $c$ are split. By (11) and the same argument as in the first case, we have equality in (13).
- **Case 3:** There exist split pairs of functions of $b$ and split pairs of functions of $c$. Suppose that $\{b_k, \bar{b}_k\}$, and $\{c_l, \bar{c}_l\}$ are split for some $1 \leq k \leq N_1$ and $1 \leq l \leq N_2$. Since $b_k, \bar{c}_l \in H^+$ and $\bar{b}_k, c_l \in H^-$, the geodesic segments $[b_k, c_l]$ and $[\bar{b}_k, \bar{c}_l]$ intersect in $H$. As $M$ divides $\mathbb{R}$ into two connected components, $b_k$ and $\bar{c}_l$ must lie on one side of $M$, whereas $\bar{b}_k$ and $c_l$ must lie on the other side. Thus, the left-hand side of (13) equals 2 and the inequality holds.

Integrating the pointwise inequality (13) over $K_1 \times \cdots \times K_N$ yields
\[ I(K_1, \ldots, K_N; M) + I(K_1, \ldots, K_N; \rho M) \]
\[ = I(K_1, \ldots, K_N; M) + I(\rho K_1, \ldots, \rho K_N; M) \]
\[ \geq I(TK_1, \ldots, TK_N; M) + I(\rho TK_1, \ldots, \rho TK_N; M) \]
\[ = I(TK_1, \ldots, TK_N; M) + I(TK_1, \ldots, TK_N; \rho M), \]
that is, the quantity $I(K_1, \ldots, K_N; M) + I(K_1, \ldots, K_N; \rho M)$ decreases whenever the sets $K_1, \ldots, K_N$ are replaced by $TK_1, \ldots, TK_N$.

Our next step is to use the layer-cake formula to generalize the previous inequality to functions. Let $f_1, \ldots, f_N : \mathbb{R} \to \mathbb{R}$ be bounded integrable functions and set
\[ I(f_1, \ldots, f_N; M) := \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \chi(\text{conv}\{x_1, \ldots, x_N\} \cap M) \prod_{i=1}^{N} f_i(x_i) \, dx_1 \cdots \, dx_N. \]
Indeed, we have that
\[
I(f_1, \ldots, f_N; M) + I(f_1, \ldots, f_N; \rho M)
\]
\[
= \int_0^\infty \cdots \int_0^\infty I(\{f_1 > t_1\}, \ldots, \{f_N > t_N\}; M)
\]
\[
+ I(\{f_1 > t_1\}, \ldots, \{f_N > t_N\}; \rho M) dt_1 \ldots dt_N
\]
\[
\geq \int_0^\infty \cdots \int_0^\infty I(\{T f_1 > t_1\}, \ldots, \{T f_N > t_N\}; M)
\]
\[
+ I(\{T f_1 > t_1\}, \ldots, \{T f_N > t_N\}; \rho M) dt_1 \ldots dt_N
\]
\[
= \int_0^\infty \cdots \int_0^\infty I(\{T f_1 > t_1\}, \ldots, \{T f_N > t_N\}; M)
\]
\[
+ I(\{T f_1 > t_1\}, \ldots, \{T f_N > t_N\}; \rho M) dt_1 \ldots dt_N
\]
\[
= I(T f_1, \ldots, T f_N; M) + I(T f_1, \ldots, T f_N; \rho M).
\]

Here, we used the layer-cake representation \( f(x) = \int_0^\infty \mathbb{1}_{\{T f > t\}}(x) \, dt \), identity (9), and the above inequality for sets. We can now apply Proposition 3.1 to the bounded function \( \Psi_1(\cdot) \) to obtain

\[
I(f_1, \ldots, f_N; M) + I(f_1, \ldots, f_N; \rho M) \geq I(\{f_1^* \}, \ldots, \{f_N^* \}; M) + I(f_1^*, \ldots, f_N^*; \rho M).
\]

The proof of the inequality is now completed by integrating \( M \) over \( \mathcal{M}_n^{a-1} \). Note, that since Proposition 4.1 does not require the functions \( f_i \) to have proper support in the spherical case, Corollaries 1.2 and 1.4 also hold for non-proper sets \( K \subseteq \mathbb{S}^n \).

**Proposition 4.2.** Let \( N \in \mathbb{N} \), \( f_1, \ldots, f_N : \mathbb{M}^n \to \mathbb{R}^+ \) bounded, integrable, and with proper support, if \( \mathbb{M}^n = \mathbb{S}^n \). Then

\[
I(f_1^*, \ldots, f_N^*) \geq I(\|f_1\|_\infty \mathbb{1}_{B_1}, \ldots, \|f_N\|_\infty \mathbb{1}_{B_N}),
\]

where \( B_i \) is a geodesic ball around \( e \) such that \( \lambda_n(B_i) = \frac{\|f_i\|_1}{\|f_i\|_\infty} \).

**Proof.** We use polar coordinates around \( e \in \mathbb{M}^n \) (see Section 2),

\[
x(t, u) = e \cos t + u \sin t, \quad t \in [0, R^M], u \in \mathbb{S}^{n-1},
\]

and appeal to Proposition 3.3. In doing so, we will justify monotonicity in each coordinate of the following function:

\[
\phi(t_1, \ldots, t_N) = \int_{\mathbb{S}^{n-1}} \cdots \int_{\mathbb{S}^{n-1}} \chi(\text{conv}\{x(t_1, u_1), \ldots, x(t_N, u_N)\} \cap M)
\]

\[
+ \chi(\text{conv}\{x(t_1, u_1), \ldots, x(t_N, u_N)\} \cap M^c) \, du_1 \cdots du_N,
\]

where \( M \in \mathcal{M}_n^{a-1} \) is fixed and \( x^e := -x + (x \cdot e) e \) denotes the geodesic reflection of \( x \in \mathbb{M}^n \) about \( e \), that is, orthogonal reflection about span\{\( e \)\} in \( \mathbb{R}^{n+1} \).

Without loss of generality, we show that \( \phi \) is increasing in \( t_1 = t \). We fix \( t_2, \ldots, t_N \) and \( u_1, \ldots, u_N \) and write \( x(t) := x(t, u_1) \) and \( x_i := x(t_i, u_i), 2 \leq i \leq N \). Define

\[
\alpha_1(t) := \chi(\text{conv}\{x(t), x_2, \ldots, x_N\} \cap M), \quad \alpha_2(t) := \chi(\text{conv}\{x(t)^e, x_2, \ldots, x_N\} \cap M),
\]

\[
\alpha_3(t) := \chi(\text{conv}\{x(t), x_2^e, \ldots, x_N^e\} \cap M), \quad \alpha_4(t) := \chi(\text{conv}\{x(t)^e, x_2^e, \ldots, x_N^e\} \cap M),
\]
and set \( \alpha(t) := \alpha_1(t) + \alpha_2(t) + \alpha_3(t) + \alpha_4(t) \). Note that we have \( \alpha_1 = \alpha_4^e \) and \( \alpha_2 = \alpha_2^e \). Our goal is to show that the function \( \alpha : [0, R^M] \to \{0, 1, 2, 3, 4\} \) is increasing in \( t \). We denote by \( X := \text{conv}\{x_2, \ldots, x_N\} \) and consider the following cases:

- **Case 1**: \( e \in X \), and thus, \( e \in X^e \). For \( s \leq t \), we have \([e, x(s)] \subseteq [e, x(t)]\) as geodesic segments. Therefore,
  \[
  \alpha_1(s) = \chi(\text{conv}\{[e, x(s)] \cup X\} \cap M) \leq \chi(\text{conv}\{[e, x(t)] \cup X\} \cap M) = \alpha_1(t),
  \]
and similarly for \( \alpha_2, \alpha_3, \alpha_4 \), hence, \( \alpha(s) \leq \alpha(t) \).

- **Case 2**: \( e \notin X \), and thus, \( e \notin X^e \).
  
  - **Case 2a**: \( M \) meets both \( X \) and \( X^e \). Here, \( \alpha(t) \equiv 4 \).
  
  - **Case 2b**: \( M \) meets \( X \) but not \( X^e \). We first show that in this case, \( e \) must lie on the same side of \( M \) as \( X^e \). Assume the opposite, that is, \( e \) lies opposite of \( X^e \). Since \( M \) meets \( X \), there exist points of \( X \) on either side of \( M \). Therefore, we find \( y \in X \) lying opposite of \( e \). But then \( y^e \in X^e \), and thus, the segment \([y, y^e]\) also lies opposite of \( e \). This is a contradiction, as \( e \in [y, y^e] \) (here we use the assumption that, if \( M^n = S^n \), the functions \( f_1, \ldots, f_N \) have proper support, and thus, their rearrangements, \( f_1^*, \ldots, f_N^* \) are supported in \( \text{int} S^n^+ \)).

  Hence, for \( t \) small enough, we have
  \[
  \alpha_1(t) = 1, \quad \alpha_2(t) = 1, \quad \alpha_3(t) = 0, \quad \alpha_4(t) = 0,
  \]
  that is, \( \alpha(t) = 2 \). As \( t \) increases, as soon as either \( x(t) \) or \( x(t)^e \) cross \( M \), \( \alpha_3(t) = 1 \) or \( \alpha_4(t) = 1 \), that is, \( \alpha(t) = 3 \) (Figure 3).

  - **Case 2c**: \( M \) meets \( X^e \) but not \( X \) is similar to Case 2b.

  - **Case 2d**: \( M \) meets neither \( X \) nor \( X^e \). If \( X \) and \( X^e \) lie on opposite sides of \( M \), then \( \alpha(t) \equiv 2 \) is essentially constant, as \( \alpha_1(t) + \alpha_3(t) \equiv 1 \) and \( \alpha_2(t) + \alpha_4(t) \equiv 1 \) (except for at most one value of \( t \), where \( x_1 \) or \( x_1^e \) might lie on \( M \)). If \( X \) and \( X^e \) lie on the same side of \( M \), then so does \( e \), and thus, \( \alpha(t) = 0 \) for small \( t \), and \( \alpha(t) = 2 \), as soon as \( x(t) \) or \( x(t)^e \) cross \( M \), since then \( \alpha_1(t) = 1, \alpha_3(t) = 1 \) or \( \alpha_2(t) = 1, \alpha_4(t) = 1 \), respectively (Figure 3).

Since the above shows that \( \alpha(t) \) is either increasing or constant a.e. for every choice of \( u_1, \ldots, u_N \), integrating over \( S^{n-1} \times \cdots \times S^{n-1} \) yields that
\[
2\phi(t, t_2, \ldots, t_N) = \int_{S^{n-1}} \cdots \int_{S^{n-1}} \alpha(t) \, du_1 \cdots du_N
\]
is increasing in \( t \) as well. Hence, an application of Proposition 3.3 gives
\[
I(f_1^*, \ldots, f_N^*; M) + I(f_1^*, \ldots, f_N^*; M^e)
\geq I(\|f_1\|_1 \mathbb{I}_{B_1}, \ldots, \|f_N\|_1 \mathbb{I}_{B_N}; M) + I(\|f_1\|_1 \mathbb{I}_{B_1}, \ldots, \|f_N\|_1 \mathbb{I}_{B_N}; M^e).
\]
Once again, integrating \( M \) over \( M''_{n-1} \) concludes the proof.
Acknowledgements. T. Hack was supported by the European Research Council (ERC), Project number: 306445, and the Austrian Science Fund (FWF), Project numbers: Y603-N26 and P31448-N35. P. Pivovarov was supported by NSF grant DMS-1612936 and Simons Foundation grant #635531. We thank the referee for helpful comments that improved the presentation.

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Thomas Hack,  
Institut für Diskrete Mathematik und Geometrie,  
TU Wien,  
Wiedner Hauptstrasse 8-10/104,  
Vienna, A-1040,  
Austria  
Email: thomas.hack9@gmail.com

Peter Pivovarov,  
Mathematics Department,  
University of Missouri,  
202 Mathematical Sciences Bldg Columbia, MO 65211,  
U.S.A.  
Email: pivovarovp@missouri.edu