Multiplicativity of the maximal output 2-norm for depolarized Werner-Holevo channels.

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We study the multiplicativity of the output 2-norm for depolarized Werner-Holevo channels and show that multiplicativity holds for a product of two identical channels in this class. Moreover, it is shown that the depolarized Werner-Holevo channels do not satisfy the entrywise positivity condition introduced by C. King and M.B. Ruskai, which suggests that the main result is non-trivial.

**THE SETUP AND MAIN RESULT**

The $d$-dimensional Werner-Holevo channel $W_d(\rho) = \frac{1}{d^2}((\text{Tr}(\rho))|1d-\rho^T\rangle\langle 1d|)$ is known \cite{[2]} to give a counterexample to the multiplicativity of the maximal output $p$-norm for $p > 4.79$, when $d = 3$. Nevertheless, it has been shown \cite{[2],[3]} that $W_d(\rho)$ satisfies multiplicativity for $1 \leq p \leq 2$. It is natural then to study the output $p$-norm of depolarized Werner-Holevo channels

$$W_{\lambda,d}(\rho) = \lambda \rho + (1 - \lambda)W_d(\rho),$$

and ask if those channels satisfy multiplicativity for $p$-norms with $p \leq 2$.

We focus our attention to the study of the output 2-norm for the tensor product channel $W_{\lambda,d} \otimes W_{\lambda,d}$ acting on bipartite states in $M_d(\mathbb{C}) \otimes M_d(\mathbb{C})$ and show that multiplicativity is satisfied for this norm.

A direct computation of the eigenvalues of $W_{\lambda,d} \otimes W_{\lambda,d}(|\psi_12\rangle\langle \psi_12|)$ turns out to be much harder for $0 < \lambda < 1$, than for the boundary cases $\lambda = 0, 1$. The reason is that the output consists of a combination of the input state and its transpose/partial transpose, which in general do not share a common eigenbasis. To work around this difficulty, we compute explicitly the output 2-norm of $W_{\lambda,d} \otimes W_{\lambda,d}(|\psi_12\rangle\langle \psi_12|)$ and the maximal output 2-norm of $W_{\lambda,d}$ and study the difference

$$D_{\lambda,d}(\psi_12) = (\|W_{\lambda,d}|\psi_12\rangle\langle \psi_12|\|)^2 - (\|W_{\lambda,d} \otimes W_{\lambda,d}(|\psi_12\rangle\langle \psi_12|)\|)^2$$

(1)

We show that $D_{\lambda,d} \geq 0$ for all input states and $\lambda \in [0, 1]$, $d \geq 2$. We begin with the computation of $\|W_{\lambda,d}|\psi_12\rangle\langle \psi_12|\|^2$ in the following Lemma.

**Lemma 1.** The (squared) maximal output 2-norm of $W_{\lambda,d}$ is given by

$$\|W_{\lambda,d}|\psi\rangle\langle \psi|\|_2^2 = \frac{(d - 2)\lambda^2 + 1}{d - 1}$$

Proof. It is easy to check that $\|W_{\lambda,d}|\psi\rangle\langle \psi|\|_2^2$ is

$$\text{Tr}(W_{\lambda,d}|\psi\rangle\langle \psi|)$$

$$= \lambda^2 + \frac{2(1 - \lambda)(1 - |\psi\rangle\langle \psi|)|^2}{d - 1} + \frac{(1 - \lambda)^2}{d - 1}$$

$$\leq \frac{(d - 2)\lambda^2 + 1}{d - 1},$$

where $|\psi\rangle$ denotes the complex conjugate of $|\psi\rangle$ in the standard basis. Taking $|\psi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$, with $|0\rangle, |1\rangle$ two standard basis vectors, we see that equality can be achieved in the above expression and the result follows.

We now turn our attention to the more complicated output 2-norm of $W_{\lambda,d} \otimes W_{\lambda,d}(|\psi_12\rangle\langle \psi_12|)$.

**Lemma 2.** The (squared) output 2-norm $\|W_{\lambda,d} \otimes W_{\lambda,d}(|\psi_12\rangle\langle \psi_12|)\|^2$ is given by:

$$\|W_{\lambda,d} \otimes W_{\lambda,d}(|\psi_12\rangle\langle \psi_12|)\|^2 \leq S_{\lambda}^2(\|\psi_12\rangle\langle \psi_12|\|^2)$$

$$- 2(S_{\lambda} + R_{\lambda}^2)(S_{\lambda} + (d - 2)Q_{\lambda}^2)(1 - \|\rho_1\|^2)$$

$$- S_{\lambda}\|W_{\lambda,d}\|^2\|\text{Tr}(\rho_1 \rho_1^T + \rho_2 \rho_2^T),$$

where $Q_{\lambda} = \frac{1}{1 - \lambda}$, $R_{\lambda} = \lambda - Q_{\lambda}$, $S_{\lambda} = 2\lambda Q_{\lambda}$, $\rho_1 = \text{Tr}_2|\psi_12\rangle\langle \psi_12|$, $\rho_2 = \text{Tr}_1|\psi_12\rangle\langle \psi_12|$ and $T$ denotes transposition.

Proof. It is easy to check that

$$W_{\lambda,d} \otimes W_{\lambda,d}(|\psi_12\rangle\langle \psi_12|) = \lambda^2|\psi_12\rangle\langle \psi_12|$$

$$+ Q_{\lambda}R_{\lambda}(\rho_1 \otimes |1d\rangle\langle 1d| + |1d\rangle\langle 1d| \otimes \rho_2)$$

$$+ Q_{\lambda}^2(|1d\rangle\langle 1d| + |1d\rangle\langle 1d| \otimes |1d\rangle\langle 1d|$$

$$- S_{\lambda}\|\psi_12\rangle\langle \psi_12|\|T_T$$

$$- S_{\lambda}\|\psi_12\rangle\langle \psi_12|\|T_T$$

where $T_1, T_2$ denote partial transposition w.r.t. the 1st, 2nd tensor factor, respectively. Taking the trace after squaring the above expression and noting that $\text{Tr}_1|\psi_12\rangle\langle \psi_12|\langle \psi_12|\langle \psi_12|)^T_k = \text{Tr}(\rho_k \rho_k^T)$ for $k = 1, 2$ (which one can show using the Schmidt decomposition of $|\psi_12\rangle$), we get the desired result.

The following general inequality will be very useful in the proof of the main theorem, so we state it here as a lemma.

**Lemma 3.** Let $\sigma_1 \leq \sigma_2 \leq \ldots \leq \sigma_d$ be non-negative numbers that sum up to 1. Then, the following inequality holds:

$$\sigma_d \geq \sum_{a=1}^d \sigma_a^2$$
Proof. The r.h.s. of the inequality can be thought of as the expected value of the random variable $X$ given by $\Pr(X = \sigma \alpha) = \sigma \alpha$. The upper bound then follows immediately.

PROOF OF THE MAIN RESULT

In this section, we will show that the difference $D_{\lambda,d}$ defined in (1) is always non-negative, which is equivalent to multiplicity of the output 2-norm for $W_{\lambda,d}$. We state this as a theorem:

Theorem 4. For the depolarized Werner-Holevo channel $W_{\lambda,d}$, we have for $\lambda \in [0,1], d \geq 2$:

$$\|W_{\lambda,d} \otimes W_{\lambda,d}\|_2 = \|W_{\lambda,d}\|_2^2$$

Proof. From Lemma 6 we see that the condition $D_{\lambda,d}(\langle \psi \rangle_2) \geq 0$ is equivalent to

$$S_\lambda^2(\langle \psi_2 | \langle \psi_2 \rangle_2 \rangle)^2 \leq 2(S_\lambda^2 + P_\lambda^2)(1-\|\rho_1\|^2_2) + S_\lambda\|W_{\lambda,d}\|_2^2 \Tr\left(\rho_1 T_1 + \rho_2 T_2^\dagger\right),$$

where $P_\lambda^2 = [Q_\lambda^2 + (d-2)R_\lambda^2]S_\lambda + (d-2)Q_\lambda^2 R_\lambda^2 \geq 0$. Using Lemma 1 to write $\|W_{\lambda,d}\|_2^2$ as $(1+\sqrt{d-1})S_\lambda + (\lambda-\sqrt{1-\lambda^2})^2$, we see that it is sufficient to prove the following inequality

$$\|\psi_2 | \langle \psi_2 \rangle_2 \rangle^2 \leq 2(1-\|\rho_1\|^2_2) + (1+\sqrt{d-1}) \Tr\left(\rho_1 T_1 + \rho_2 T_2^\dagger\right)$$

(2)

(the boundary cases $\lambda = 0,1$ follow from $1 \geq \|\rho_1\|_2^2$).

We will now make use of the Schmidt decomposition of the output state $|\psi_2 \rangle |\langle \psi_2 \rangle_2 \rangle$, given by $|\psi_2 \rangle = \sum \alpha \sqrt{\sigma_\alpha} |\alpha_1 \rangle \otimes |\alpha_2 \rangle$, where $\{|\alpha_1 \rangle\}, \{|\alpha_2 \rangle\}$ are orthonormal sets in $M_d(\mathbb{C})$. We have that $\|\rho_1\|_2^2 = \sum_{\alpha=1}^d \sigma_\alpha^2$, where some of the $\sigma_\alpha$ may be zero. Applying Lemma 3 (and borrowing its notation w.l.o.g.), it follows that $\|\rho_1\|_2^2 \leq \sigma_d$. Moreover, it becomes clear now that in order to prove (2), it is sufficient to show:

$$\|\psi_2 | \langle \psi_2 \rangle_2 \rangle^2 \leq 2(1-\sigma_d) + (1+\sqrt{d-1}) \Tr\left(\rho_1 T_1 + \rho_2 T_2^\dagger\right)$$

(3)

for $\sigma_d \geq 1/2$, since $\|\psi_2 | \langle \psi_2 \rangle_2 \rangle \leq 1$ and $\Tr\left(\rho_1 T_1 + \rho_2 T_2^\dagger\right) \geq 0$. We now use the triangle inequality to get an estimate for the l.h.s. of (3):

$$\|\psi_2 | \langle \psi_2 \rangle_2 \rangle^2 = \|\sum_{\alpha, \beta} \sqrt{\sigma_\alpha \sigma_\beta} |\alpha_1 \rangle |\beta_1 \rangle \langle \beta_2 | \langle \alpha_2 \rangle_2 \rangle^2 \leq \sum_{\alpha, \beta} \sqrt{\sigma_\alpha \sigma_\beta} |\alpha_1 \rangle |\beta_1 \rangle \langle \beta_2 | \langle \alpha_2 \rangle_2 \rangle^2$$

(4)

We will need to treat dimensions $d \leq 4$ and $d \geq 5$ separately. For $d \leq 4$ we use Cauchy-Schwarz to get the following estimate for

$$\left(\sum_{\alpha, \beta} \sqrt{\sigma_\alpha \sigma_\beta} |\alpha_1 \rangle |\beta_1 \rangle \langle \beta_2 | \langle \alpha_2 \rangle_2 \rangle^2 \right)^2 \leq \left(\sum_{\alpha, \beta} \sigma_\alpha \sigma_\beta |\langle \alpha_1 | \beta_1 \rangle|^2 \right) \left(\sum_{\alpha, \beta} |\langle \alpha_2 | \beta_2 \rangle|^2 \right)$$

$$\leq d \sum_{\alpha, \beta} \sigma_\alpha \sigma_\beta |\langle \alpha_1 | \beta_1 \rangle|^2 \leq d \sum_{\alpha, \beta} \sigma_\alpha \sigma_\beta |\langle \alpha_1 | \beta_1 \rangle|^2$$

(5)

where we have used Parseval’s identity in the last inequality. The same inequality is, of course, true for the second tensor factor. Using (3) and (4) along with the fact that

$$\Tr\left(\rho_k T_k^\dagger\right) = \sum_{\alpha, \beta} \sigma_\alpha \sigma_\beta |\langle \alpha_k | \beta_k \rangle|^2$$

(6)

we see from estimate (5) that it is sufficient to show that $d \leq 2(1 + \sqrt{d-1})$, which is true for $d \leq 4$. We now turn our attention to the case $d \geq 5$. We will need a different estimate than the one given in (5), since we need to make use of the assumption that $\sigma_d \geq 1/2$ in order to lower the factor $d$ in (5). We start by using Cauchy-Schwarz to get the following upper bound:

$$\left(\sum_{\alpha, \beta} \sqrt{\sigma_\alpha \sigma_\beta} |\langle \alpha_1 | \beta_1 \rangle||\langle \alpha_2 | \beta_2 \rangle|^2 \right)^2 \leq 3(I_1^2 + I_2^2 + I_3^2)$$

(7)

where

$$I_1 = \sum_{\alpha = d, \beta} \sqrt{\sigma_\alpha \sigma_\beta} |\langle \alpha_1 | \beta_1 \rangle||\langle \alpha_2 | \beta_2 \rangle|$$

$$I_2 = \sum_{\alpha \neq d, \beta = d} \sqrt{\sigma_\alpha \sigma_\beta} |\langle \alpha_1 | \beta_1 \rangle||\langle \alpha_2 | \beta_2 \rangle|$$

$$I_3 = \sum_{\alpha \neq d, \beta \neq d} \sqrt{\sigma_\alpha \sigma_\beta} |\langle \alpha_1 | \beta_1 \rangle||\langle \alpha_2 | \beta_2 \rangle|$$

A further application of Cauchy-Schwarz on $I_1, I_2, I_3$ will give us the desired result. We start with an estimate for $I_1$, since $I_2$ is very similar. Noting that one of the summation indices is fixed to $d$, we get

$$I_1^2 = \left(\sum_{\alpha = d, \beta} \sqrt{\sigma_\alpha \sigma_\beta} |\langle \alpha_1 | \beta_1 \rangle||\langle \alpha_2 | \beta_2 \rangle| \right)^2 \leq \left(\sum_{\alpha = d, \beta} \sigma_\alpha \sigma_\beta |\langle \alpha_1 | \beta_1 \rangle|^2 \right) \left(\sum_{\alpha = d, \beta} |\langle \alpha_2 | \beta_2 \rangle|^2 \right)$$

$$\leq \sum_{\alpha = d, \beta} \sigma_\alpha \sigma_\beta |\langle \alpha_1 | \beta_1 \rangle|^2$$

$$\leq \Tr\left(\rho_1 T_1^\dagger\right)$$

Similarly, we see that $I_2^2 \leq \Tr(\rho_2 T_2^\dagger)$. Since $1 + \sqrt{d-1} \geq 3$ for $d \geq 5$, we see from (3) and (7) that it remains to show $3 I_3^2 \leq 2(1 - \sigma_d)$. We have

$$I_3^2 = \left(\sum_{\alpha \neq d, \beta \neq d} \sqrt{\sigma_\alpha \sigma_\beta} |\langle \alpha_1 | \beta_1 \rangle||\langle \alpha_2 | \beta_2 \rangle| \right)^2 \leq \left(\sum_{\alpha \neq d, \beta \neq d} \sigma_\alpha |\langle \alpha_1 | \beta_1 \rangle|^2 \right) \left(\sum_{\alpha \neq d, \beta \neq d} \sigma_\beta |\langle \alpha_2 | \beta_2 \rangle|^2 \right)$$

$$\leq \left(\sum_{\alpha \neq d} \sigma_\alpha \right)^2 = (1 - \sigma_d)^2$$
It remains to show that \(3(1 - \sigma_d)^2 \leq 2(1 - \sigma_d) \iff (1 - \sigma_d)(3\sigma_d - 1) \geq 0\), which follows from our assumption that \(\sigma_d \in [\frac{1}{2}, 1]\).

**DISCUSSION**

We have shown that for depolarized Werner-Holevo channels the maximum output 2-norm is multiplicative. For \(\lambda \in (0, 1)\) and \(d \geq 3\), the depolarized Werner-Holevo maps do not satisfy the entrywise-positivity (EP) condition introduced by C. King and M.B. Ruskai in [4, 5]. This suggests that some elements of the above proof may be useful when tackling the multiplicativity of the maximal output 2-norm for arbitrary channels.

**Proposition 5.** The depolarized Werner-Holevo channels \(\mathcal{W}_{\lambda,d}\) with \(\lambda \in (0,1), d \geq 3\) do not satisfy the entrywise-positivity (EP) condition:

\[
\text{Tr} \mathcal{W}_{\lambda,d}(|e_i\rangle\langle e_i|) \mathcal{W}_{\lambda,d}(|e_j\rangle\langle e_k|) \geq 0, \quad \forall i,j,k,l,
\]

and \(|\{e_i\}\rangle^d_{i=1}\) some orthonormal basis of \(C^d\).

**Proof.** One can check that \(\text{Tr} \mathcal{W}_{\lambda,d}(|e_i\rangle\langle e_i|) \mathcal{W}_{\lambda,d}(|e_j\rangle\langle e_k|)\) is given by:

\[
\begin{align*}
\left[\lambda^2 + \frac{(1 - \lambda)}{d - 1}\right] \delta_{i,j} \delta_{k,l} + & \frac{2\lambda(1 - \lambda)}{d - 1} + (d - 2)\left(\frac{1 - \lambda}{d - 1}\right)^2 \delta_{i,l} \delta_{j,k} - \frac{2\lambda(1 - \lambda)}{d - 1} \langle e_i|e_{\overline{k}}\rangle \langle e_{\overline{l}}|e_j\rangle
\end{align*}
\]

where \(|e_{\overline{k}}\rangle\) denotes the complex conjugate of \(|e_k\rangle\), as before. Now, taking \(i = j, k \neq l\) in the above expression, we see that the EP condition implies:

\[
\langle e_i|e_{\overline{k}}\rangle \langle e_{\overline{l}}|e_i\rangle \leq 0, \quad \forall i,k \neq l.
\]

Summing over \(i\) in the above inequality gives us 0, which implies that:

\[
\langle e_i|e_{\overline{k}}\rangle \langle e_{\overline{l}}|e_i\rangle = 0, \quad \forall i,k \neq l. \tag{8}
\]

Fixing \(l\), we choose \(i = \pi(l)\) such that \(|\overline{\pi(l)}\rangle \neq 0\) (we can always find such a \(\pi(l)\), since otherwise \(|\pi(l)\rangle = 0\); a contradiction to \(|\overline{\pi(l)}\rangle\) being an orthonormal basis vector).

Condition (8) then implies that \(\langle e_{\pi(l)}|e_k\rangle = 0, \forall k \neq l\). Since the \(|\pi(l)\rangle\) form an orthonormal basis, it follows that \(|e_{\pi(l)}\rangle = |\overline{\pi(l)}\rangle, \forall l\). We may now rewrite the EP condition as:

\[
\begin{align*}
\left[\lambda^2 + \frac{(1 - \lambda)}{d - 1}\right] \delta_{i,j} \delta_{k,l} + & \frac{2\lambda(1 - \lambda)}{d - 1} + (d - 2)\left(\frac{1 - \lambda}{d - 1}\right)^2 \delta_{i,l} \delta_{j,k} - \frac{2\lambda(1 - \lambda)}{d - 1} \langle e_i|e_{\overline{k}}\rangle \langle e_{\overline{l}}|e_j\rangle
\end{align*}
\]

Choosing \(i = \pi(k), j = \pi(l)\) and \(k \neq l\), the above condition becomes:

\[
\left[\frac{2\lambda(1 - \lambda)}{d - 1} + (d - 2)\left(\frac{1 - \lambda}{d - 1}\right)^2\right] \delta_{\pi(k),l} \delta_{\pi(l),k} \geq \frac{2\lambda(1 - \lambda)}{d - 1}
\]

The EP condition forces \(\pi(l) = k, \forall k \neq l\) (note that \(\pi(k) = l\) then follows from the definition of \(\pi(k)\), which is impossible for \(d \geq 3\). For \(d = 2\), choosing \(|e_1\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}\) satisfies the EP condition.  

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