Concentration of solutions for a singularly perturbed mixed problem in non smooth domains

Serena Dipierro
SISSA, Sector of Mathematical Analysis
Via Bonomea 265, 34136 Trieste, Italy
E-mail address: dipierro@sissa.it

Abstract

We consider a singularly perturbed problem with mixed Dirichlet and Neumann boundary conditions in a bounded domain \( \Omega \subset \mathbb{R}^n \) whose boundary has an \((n-2)\)-dimensional singularity. Assuming \( 1 < p < \frac{n+2}{n-2} \), we prove that, under suitable geometric conditions on the boundary of the domain, there exist solutions which approach the intersection of the Neumann and the Dirichlet parts as the singular perturbation parameter tends to zero.

Keywords: Singularly Perturbed Elliptic Problems, Finite-dimensional reductions, Local Inversion.

AMS subject classification: 35B25, 35B34, 35J20, 35J60

1 Introduction

In this paper we study the following singular perturbation problem with mixed Dirichlet and Neumann boundary conditions in a bounded domain \( \Omega \subset \mathbb{R}^n \) whose boundary \( \partial \Omega \) is non smooth:

\[
\begin{cases}
-\epsilon^2 \Delta u + u = u^p & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial_N \Omega, \\
u > 0 & \text{on } \partial_D \Omega, \\
u > 0 & \text{in } \Omega.
\end{cases}
\]

Here \( p \in \left(1, \frac{n+2}{n-2}\right) \) is subcritical, \( \nu \) denotes the outer unit normal at \( \partial \Omega \) and \( \epsilon > 0 \) is a small parameter. Moreover \( \partial_N \Omega, \partial_D \Omega \) are two subsets of the boundary of \( \Omega \) such that the union of their closures coincides with the whole \( \partial \Omega \), and their intersection is an \((n-2)\)-dimensional smooth singularity.

Problem (1) or some of its variants arise in several physical or biological models. Consider for instance the study of the population dynamics: suppose that a species lives in a bounded region \( \Omega \) whose boundary has two parts, \( \partial_N \Omega \), which is an obstacle that blocks the pass across, and \( \partial_D \Omega \), which is a killing zone for the population. Moreover (1) is a model of the heat conduction for small conductivity, when there is a nonlinear source in the interior of the domain, with combined isothermal and isolated regions at the boundary.

Concerning reaction-diffusion systems, this phenomenon is related to the so-called Turing’s instability. More precisely, for single equation with Neumann boundary conditions it is known
that scalar reaction-diffusion equations in a convex domain admit only constant stable steady state solutions; see [CH], [Mat]. On the other hand, as noticed in [Tu], reaction-diffusion systems with different diffusivities might generate non-homogeneous stable steady states. A well-known example is the Gierer-Meinhardt system, introduced in [GM] to describe some biological experiment. We refer to [Ni], [NTY] for more details.

Another motivation comes from the Nonlinear Schrödinger Equation

\[ i\hbar \frac{\partial \psi}{\partial t} = -\hbar^2 \frac{\Delta \psi}{2m} + V\psi - \gamma |\psi|^{p-2}\psi, \]

where \( \hbar \) is the Planck constant, \( V \) is the potential, and \( \gamma \) and \( m \) are positive constants. In fact, if we analyze standing waves and consider the semiclassical limit \( \hbar \to 0 \), we obtain a singularly perturbed equation; see for example [ABC], [AM], [AMS], [FW], and references therein.

Let us now describe some results which concern singularly perturbed problems with Neumann or Dirichlet boundary conditions, and specifically

\[
\begin{aligned}
-\epsilon^2 \Delta u + u &= u^p & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} &= 0 & \text{on } \partial\Omega, \\
u > 0 & \text{ in } \Omega,
\end{aligned}
\]

and

\[
\begin{aligned}
-\epsilon^2 \Delta u + u &= u^p & \text{in } \Omega, \\
u &= 0 & \text{on } \partial\Omega, \\
u > 0 & \text{ in } \Omega.
\end{aligned}
\]

The study of the concentration phenomena at points for smooth domains is very rich and has been intensively developed in recent years. The search for such condensing solutions is essentially carried out by two methods. The first approach is variational and uses tools of the critical point theory or topological methods. A second way is to reduce the problem to a finite-dimensional one by means of Lyapunov-Schmidt reduction.

The typical concentration behavior of solution \( U_{Q,\epsilon} \) to (1) is via a scaling of the variables in the form

\[
U_{Q,\epsilon}(x) \sim U \left( \frac{x - Q}{\epsilon} \right),
\]

where \( Q \) is some point of \( \bar{\Omega} \), and \( U \) is a solution of the problem

\[
-\Delta U + U = U^p \quad \text{in } \mathbb{R}^n \quad \text{(or in } \mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}),
\]

the domain depending on whether \( Q \) lies in the interior of \( \Omega \) or at the boundary. When \( p < \frac{n+2}{n+2} \) (and indeed only if this inequality is satisfied), problem (5) admits positive radial solutions which decay to zero at infinity; see [BL], [St]. Solutions of (1) with this profile are called \textit{spike-layers}, since they are highly concentrated near some point of \( \Omega \).

Consider first the problem with pure Neumann boundary conditions. Solutions of (2) with a concentration at one or more points of the boundary \( \partial\Omega \) as \( \epsilon \to 0 \) are called \textit{boundary-spike layers}. They are peaked near critical points of the mean curvature. In particular, it was shown in [NT1], [NT2] that mountain-pass solutions of (2) concentrate at \( \partial\Omega \) near global maxima of the mean curvature. One can see this fact considering the variational structure of the problem. In fact, solutions of (2) can be found as critical points of the following Euler-Lagrange functional

\[
I_{c,N}(u) = \frac{1}{2} \int_{\Omega} \epsilon^2 |\nabla u|^2 + u^2 \, dx - \frac{1}{p+1} \int_{\Omega} |u|^{p+1} \, dx, \quad u \in H^1(\Omega).
\]
Plugging into $I_{\varepsilon,N}$ a function of the form (4) with $Q \in \partial \Omega$ one sees that
\[
I_{\varepsilon,N}\left(U_{Q,\varepsilon}\right) = C_0 \varepsilon^n - C_1 \varepsilon^{n+1} \mathcal{H}(Q) + o\left(\varepsilon^{n+1}\right),
\]
where $C_0, C_1$ are positive constants depending only on $n$ and $p$, and $\mathcal{H}$ is the mean curvature; see for instance [AM], Lemma 9.7. To obtain this expansion one can use the radial symmetry of $U$ and parametrize $\partial \Omega$ as a normal graph near $Q$. From the above formula one can see that the bigger is the mean curvature the lower is the energy of this function: roughly speaking, boundary spike layers would tend to move along the gradient of $\mathcal{H}$ in order to minimize their energy. Moreover one can say that the energy of spike-layers is of order $\varepsilon^n$, which is proportional to the volume of their support, heuristically identified with a ball of radius $\varepsilon$ centered at the peak. There is an extensive literature regarding the search of more general solutions of (2) concentrating at critical points of $\mathcal{H}$; see [DFW], [Gr], [GPW], [Gu], [Li], [LNT], [NPT], [We].

Consider now the problem with pure Dirichlet boundary conditions. In this case spike layers with minimal energy concentrate at the interior of the domain, at points which maximize the distance from the boundary; see [LN], [NW]. The intuitive reason for this is that, if $Q$ is in the interior of $\Omega$ and if we want to adapt a function like (4) to the Dirichlet conditions, the adjustment needs an energy which increases as $Q$ becomes closer and closer to $\partial \Omega$. Following the above heuristic argument, we could say that spike layers are repelled from the regions where Dirichlet conditions are imposed.

Concerning mixed problem (1), in two recent papers [GMMP1], [GMMP2] it was proved that, under suitable geometric conditions on the boundary of a smooth domain, there exist solutions which approach the intersection of the Neumann and the Dirichlet parts as the singular perturbation parameter tends to zero. In fact, denoting by $u_{\varepsilon,Q}$ an approximate solution peaked at $Q$ and by $d_\varepsilon$ the distance of $Q$ from the interface between $\partial_N \Omega$ and $\partial_D \Omega$, then its energy turns out to be the following
\[
I_{\varepsilon}(u_{Q,\varepsilon}) = C_0 \varepsilon^n - C_1 \varepsilon^{n+1} \mathcal{H}(Q) + \varepsilon^n e^{-2\mathcal{H}(Q)(1+o(1))} + o(\varepsilon^{n+2}),
\]
where $I_{\varepsilon}$ is the functional associated to the mixed problem. Note that the first two terms in (7) are as in the expansion (6), while the third one represents a sort of potential energy which decreases with the distance of $Q$ from the interface, consistently with the repulsive effect which was described before for (3). In almost all the papers mentioned above the case of $\Omega$ smooth was considered. Concerning instead the case of $\Omega$ non smooth, in [Di] the author studied the concentration of solutions of the Neumann problem (2) at suitable points of the boundary of a non-smooth domain. Assuming for simplicity that $\Omega \subset \mathbb{R}^3$ is a piecewise smooth bounded domain whose boundary $\partial \Omega$ has a finite number of smooth edges, one can fix an edge $\Gamma$ on the boundary and consider the function $\alpha : \Gamma \rightarrow \mathbb{R}$ which associates to every $Q \in \Gamma$ the opening angle at $Q$, $\alpha(Q)$. Then it was proved that this function plays a similar role as the mean curvature $\mathcal{H}$ for a smooth domain. In fact, plugging into $I_{\varepsilon,N}$ a function of the form (4) with $Q \in \Gamma$, one obtains the analogous expression to (4) for this kind of domains, with $C_0 \alpha(Q)$ instead of $C_0$. Again, one can give an heuristic explanation considering the fact that in this case one has to intersect the ball of radius $\varepsilon$, which is identified with the support of the solution, with the domain, obtaining the dependence on the angle $\alpha(Q)$. We are interested here in finding boundary spike-layers for the mixed problem (1). We call $\Gamma$ the intersection of the closures of $\partial_N \Omega$ and $\partial_D \Omega$, and suppose that it is an $(n-2)$-dimensional smooth singularity. Moreover we denote by $\mathcal{H}$ the mean curvature of $\partial \Omega$ restricted to the closure of $\partial_N \Omega$, that is $\mathcal{H} : \overline{\partial_N \Omega} \rightarrow \mathbb{R}$.

The main result of this paper is the following:
Remark 1.2. \( s \) sufficiently small problem \( H_1 \) behavior of the solution element in \( H \) projection operator use the corresponding error turns out to be too large. Following the line of [GMMP1] and [NW], we will \( \partial \) functions are non zero on \( (2) \) (see Subsection 2.1). Theorem 1.1. Let \( \Omega \subset \mathbb{R}^n, n \geq 2, \) be a bounded domain whose boundary \( \partial \Omega \) has an \( (n-2) \)-dimensional smooth singularity, and \( 1 < p < \frac{n+2}{n-2} \) \( (1 < p < +\infty \) if \( n = 2 \). Suppose that \( \partial_N \Omega, \partial_D \Omega \) are disjoint open sets of \( \partial \Omega \) such that the union of the closures is the whole boundary of \( \Omega \) and such that their intersection \( \Gamma \) is the singularity. Suppose \( Q \in \Gamma \) is such that \( \alpha (Q) \neq 0 \) and \( H \mid \Gamma \) is critical and non degenerate at \( Q \), and that \( \nabla H (Q) \neq 0 \) points toward \( \partial_D \Omega \). Then for \( \epsilon > 0 \) sufficiently small problem \( (1) \) admits a solution concentrating at \( Q \).

Remark 1.2. (a) The non degeneracy condition in Theorem 1.1 can be replaced by the condition that \( Q \) is a strict local maximum or minimum of \( H \mid \Gamma \), or by the fact that there exists an open set \( V \) of \( \Gamma \) containing \( Q \) such that \( H (Q) > \sup_{\partial V} H \) or \( H (Q) < \inf_{\partial V} H \).

(b) With more precision, as \( \epsilon \to 0 \), the above solution possesses a unique global maximum point \( Q_\epsilon \in \partial_N \Omega \), and \( \text{dist} (Q_\epsilon, \Gamma) \) is of order \( \epsilon \log \frac{1}{\epsilon} \).

The general strategy for proving Theorem 1.1 relies on a finite-dimensional reduction; see for example the book [AM]. One finds first a manifold \( Z \) of approximate solutions to the given problem, which in our case are of the form \( (1) \), and solve the equation up to a vector parallel to the tangent plane of this manifold. To do this one can use the spectral properties of the linearization of \( (5) \), see Lemma 4.3. Then, see Theorem 2.2 one generates a new manifold \( \tilde{Z} \) close to \( Z \) which represents a natural constraint for the Euler functional of \( (1) \), which is

\[
\tilde{I}_\epsilon (u) = \frac{1}{2} \int_\Omega (\epsilon^2 |\nabla u|^2 + u^2) \, dx - \frac{1}{p+1} \int_\Omega |u|^{p+1} \, dx, \quad u \in H^1_D (\Omega),
\]

where \( H^1_D (\Omega) \) is the space of functions \( H^1 (\Omega) \) which have zero trace on \( \partial \Omega \). By natural constraint we mean a set for which constrained critical points of \( \tilde{I}_\epsilon \) are true critical points.

Now, we want to have a good control of the functional \( \tilde{I}_\epsilon \mid Z \). Improving the accuracy of the functions in the original manifold \( Z \), we make \( \tilde{Z} \) closer to \( Z \); in this way the main term in the constrained functional will be given by \( \tilde{I}_\epsilon \mid Z \), see Propositions 3.12, 3.14, 3.15. To find sufficiently good approximate solutions we start with those constructed in literature for the Neumann problem \( (2) \) (see Subsection 2.2 which reveal the role of the mean curvature. The problem is that these functions are non zero on \( \partial_D \Omega \), and even if one use cut-off functions to annihilate them the corresponding error turns out to be too large. Following the line of [GMMP1], [NW], we will use the projection operator in \( H^1 (\Omega) \), which associates to every function in this space its closest element in \( H^1_D (\Omega) \). To study the asymptotic behavior of this projection we will use the limit behavior of the solution \( U \) to \( (5) \):

\[
\lim_{r \to +\infty} \epsilon^r \frac{n-2}{2} U (r) = c_{n,p},
\]

where \( r = |x| \) and \( c_{n,p} \) is a positive constant depending only on the dimension \( n \) and \( p \), together with

\[
\lim_{r \to +\infty} \frac{U' (r)}{U (r)} = - \lim_{r \to +\infty} \frac{U'' (r)}{U (r)} = -1,
\]

as it was done in some previous works, see for instance [LN] and [We2]. Moreover, we will work at a scale \( \delta \simeq \epsilon \log \epsilon \), which is the order of the distance of the peak from \( \Gamma \), see Remark 1.2 (b). At this scale both \( \partial_N \Omega \) and \( \partial_D \Omega \) look flat; so we can identify them with the hypersurfaces of equations \( x_n = 0 \) and \( x_1 \tan \alpha + x_n = 0 \), and their intersection with the set \( \{ x_1 = x_n = 0 \} \). Note that \( \alpha = \alpha (Q) \) is the angle between \( x_1 \) and \( x_n \) at a fixed point \( Q \in \Gamma \). Then we can replace \( \Omega \) with a suitable domain \( \Sigma_D \), which in particular for \( 0 < \alpha \leq \pi \) is even with respect to the coordinate \( x_n \), see the beginning of Subsections 3.1 and 3.2. Now, studying the projections in this domain, we will find functions which have zero \( x_n \)-derivative on \( \{ x_n = 0 \} \setminus \partial \Sigma_D \), which mimics the...
Neumann boundary condition on $\partial N \Omega$. After analyzing carefully the projection in Subsections 3.1 3.2 we will be able to define a family of suitable approximate solutions to (1) which have sufficient accuracy for our analysis, estimated in Propositions 3.12 3.14 3.15.

We can finally apply the above mentioned perturbation method to reduce the problem to a finite-dimensional one, and study the functional constrained on $\tilde{Z}$. We obtain an expansion of the energy of the approximate solutions, which turns out to be

$$\tilde{I}_\epsilon(u_\epsilon, Q) = \tilde{C}_0 \epsilon^n - \tilde{C}_1 \epsilon^{n+1} H(Q) + \epsilon^n e^{-2 \frac{\sqrt{\tan \alpha(Q)}}{\sqrt{\tan \alpha(Q) + 1}}} (1 + o(1)) + o(\epsilon^{n+2}),$$

in the case $0 < \alpha < \frac{\pi}{2}$, and

$$\tilde{I}_\epsilon(u_\epsilon, Q) = \tilde{C}_0 \epsilon^n - \tilde{C}_1 \epsilon^{n+1} H(Q) + \epsilon^n e^{-2 \frac{\sqrt{\tan \alpha(Q)}}{\sqrt{\tan \alpha(Q) + 1}}} (1 + o(1)) + o(\epsilon^{n+2}),$$

in the case $\frac{\pi}{2} \leq \alpha < 2\pi$. As for (7), we have that the first two terms come from the Neumann condition, while the others are related to the repulsive effect due to the Dirichlet condition. Let us notice that, in the first case, in the terms related to the Dirichlet condition appears the opening angle $\alpha$, whereas in the second case it does not; this phenomenon comes from the fact that the distance of the point $Q$ from the Dirichlet part $\partial D \Omega$ depends on $\alpha$ only if $0 < \alpha < \frac{\pi}{2}$.

Concerning the regularity of the solution, following the ideas in [Gri], it is possible to say that it is influenced by the presence of the angle. In fact, the solution is at least $C^2$ in the interior of the domain, far from the angle; whereas, near the angle, one can split the solution into a regular part and a singular one, whose regularity depends on the value of $\alpha$. For more details about the regularity of solutions in non-smooth domains we refer the reader to the book [Gri].

The fact that the solution $u$ is $C^2$ in the interior of the domain allows to say also that it is strictly positive, by using the strong Maximum Principle. In fact, we have that $u \geq 0$ in the domain. Moreover, if there exists a point $x_0$ in the interior of the domain such that $u(x_0) = 0$, we can consider a ball centered at $x_0$ of small radius such that it is contained in the domain; since in the ball $u$ is $C^2$ we can conclude that $u$ cannot be zero in $x_0$.

The plan of the paper is the following. In Section 2 we collect some preliminary material: we recall the abstract variational perturbative scheme and some known results concerning the Neumann problem [2]. In Section 3 we construct a model domain to deal with the interface, analyze the asymptotics of projections in $H^1$ and then construct approximate solution to (1). Finally in Section 4 we expand the functional on the natural constraint, prove the existence of critical points and deduce Theorem 1.1.

Notation

Generic fixed constant will be denoted by $C$, and will be allowed to vary within a single line or formula. The symbol $o(t)$ will denote quantities for which $\frac{o(t)}{t}$ tends to zero as the argument $t$ goes to zero or to infinity. We will often use the notation $d(1 + o(1))$, where $o(1)$ stands for a quantity which tends to zero as $d \to +\infty$.

2 Preliminaries

We want to find solutions to (1) with a specific asymptotic profile, so it is convenient to make the change of variables $x \mapsto \epsilon x$, and study (1) in the dilated domain

$$\Omega_\epsilon := \frac{1}{\epsilon} \Omega.$$
Then the problem becomes

\[
\begin{cases}
-\Delta u + u = u^p & \text{in } \Omega_e, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \partial N \Omega_e, \\
 u = 0 & \text{on } \partial D \Omega_e, \\
 u > 0 & \text{in } \Omega_e,
\end{cases}
\]  

(10)

where \( \partial N \Omega_e \) and \( \partial D \Omega_e \) stand for the dilations of \( \partial N \Omega \) and \( \partial D \Omega \) respectively. Moreover we denote by \( \Gamma_{e} \) the intersection of the closures of \( \partial N \Omega_e \) and \( \partial D \Omega_e \).

Here \( \partial \) can be found in Subsection 2, and \( \partial \) can be parametrized by a map on \( B_1 \) whose \( C^3 \) norm is bounded by \( C \).

Moreover we are interested in functionals \( I_\varepsilon : H \to \mathbb{R} \) of class \( C^2; \gamma \) which satisfy the following properties:

i) there exists a smooth finite-dimensional manifold \( Z_\varepsilon \subseteq H \) and \( C, r > 0 \) such that for any \( z \in Z_\varepsilon \), the set \( Z_\varepsilon \cap B_r (z) \) can be parametrized by a map on \( B_1 \) whose \( C^3 \) norm is bounded by \( C \).

We set \( W = (T_z Z_\varepsilon)^\perp \), and look for critical points of \( I_\varepsilon \) in the form \( u = z + w \) with \( z \in Z_\varepsilon \) and \( w \in W \). If \( P_z : H \to W \) is as in iv), the equation \( I'_\varepsilon (z + w) = 0 \) is equivalent to the following system

\[
\begin{cases}
P_z I'_\varepsilon (z + w) = 0 & \text{(the auxiliary equation)}, \\
(Id - P_z) I'_\varepsilon (z + w) = 0 & \text{(the bifurcation equation)}.
\end{cases}
\]

(11)

**Proposition 2.1.** (See Proposition 2.1 in [GMMPI]) Let i) – iv) hold true. Then there exists \( \varepsilon_0 > 0 \) with the following property: for all \( \varepsilon < \varepsilon_0 \) and for all \( z \in Z_\varepsilon \), the auxiliary equation in (11) has a unique solution \( w = w_\varepsilon (z) \in W \), which is of class \( C^1 \) with respect to \( z \in Z_\varepsilon \) and such that \( \|w_\varepsilon (z)\| \leq C_1 f (\varepsilon) \) as \( \varepsilon \to 0 \), uniformly with respect to \( z \in Z_\varepsilon \). Moreover the derivative of \( w \) with respect to \( z \), \( w'_\varepsilon \), satisfies the bound \( \|w'_\varepsilon (z)\| \leq C C_1 f (\varepsilon) \) for all \( z \in Z_\varepsilon \).
We shall now solve the bifurcation equation in \([11]\). In order to do this, let us define the reduced functional \(I_e : Z_e \to \mathbb{R}\) by setting \(I_e(z) = L_e(z + w(z))\).

**Theorem 2.2.** (See Proposition 2.3 in [GMMPI]) Suppose we are in the situation of Proposition 2.1 and let us assume that \(I_e\) has, for \(\varepsilon\) sufficiently small, a stationary point \(z_\varepsilon\). Then \(u_\varepsilon = z_\varepsilon + w(z_\varepsilon)\) is a critical point of \(I_e\). Furthermore, there exist \(c, \tilde{r} > 0\) such that if \(u\) is a critical point of \(I_e\) with \(\text{dist}(u, Z_{\varepsilon, \tilde{r}}) < \tilde{r}\), where \(Z_{\varepsilon, \tilde{r}} = \{z \in Z_\varepsilon : \text{dist}(z, \partial Z_\varepsilon) > c\}\), then \(u\) has to be of the form \(z_\varepsilon + w(z)\) for some \(z_\varepsilon \in Z_\varepsilon\).

### 2.2 Approximate solutions for \([1]\) with Neumann conditions

In this subsection we introduce some convenient coordinates which stretch the boundary and we recall some results from [AM] and [GMMPI] concerning approximate solutions to the Neumann problem.

First of all it can be shown that near a generic point \(Q \in \Gamma\) the boundary of \(\Omega\) can be described by a coordinate system \(y = (y_1, \ldots, y_n)\) such that

\[
\begin{align*}
\text{(a)} \quad & \partial \chi \Omega \text{ coincides with } \{y_n = 0\}, \\
\text{(b)} \quad & \partial \alpha \Omega \text{ coincides with } \{y_1 \tan \alpha + y_n = 0\}, \text{ where } \alpha = \alpha(Q) \text{ is the opening angle of } \Gamma \text{ at } Q, \\
\text{(c)} \quad & \text{the corresponding metric coefficients are given by } g_{ij} = \delta_{ij} + o(\varepsilon). \\
\end{align*}
\]

For further details we refer the reader to [Di].

**Remark 2.3.**

(i) We stress that, in the new coordinates \(y\), the origin parametrizes the point \(Q\), and those functions decaying as \(|y| \to +\infty\) will concentrate near \(Q\).

(ii) It is also useful to understand how the metric coefficients \(g_{ij}\) vary with \(Q\). Notice that condition (c) says that the deviation from the Kronecker symbols is of order \(\varepsilon\), and we are working in a domain scaled of \(\frac{1}{\varepsilon}\); hence a variation of order 1 of \(Q\) corresponds to a variation of order \(\varepsilon\) in the original domain. Therefore, a variation of order 1 in \(Q\) yields a difference of order \(\varepsilon^2\) in \(g_{ij}\), and precisely

\[
\frac{\partial g_{ij}}{\partial Q} = o\left(\varepsilon^2 |y|^2\right),
\]

with a similar estimate for the derivatives of the inverse coefficients \(g^{ij}\). For more details see the end of Subsection 9.2 in [AM].

Suppose that this coordinate system \(y\) is defined in \(B_{\mu_0}(Q)\), with \(\mu_0 > 0\) sufficiently small. Now, in this set of coordinates we choose a cut-off function \(\chi_{\mu_0}\) with the following properties

\[
\begin{align*}
\chi_{\mu_0}(x) &= 1 \quad \text{in } B_{\frac{\mu_0}{2}}(Q), \\
\chi_{\mu_0}(x) &= 0 \quad \text{in } \mathbb{R}^n \setminus B_{\frac{\mu_0}{2}}(Q), \\
|\nabla \chi_{\mu_0}| + |\nabla^2 \chi_{\mu_0}| &\leq C \quad \text{in } B_{\frac{\mu_0}{2}}(Q) \setminus B_{\frac{\mu_0}{4}}(Q),
\end{align*}
\]

and we define the approximate solution \(\bar{u}_{\varepsilon, Q}\) as

\[
\bar{u}_{\varepsilon, Q}(y) := \chi_{\mu_0}(\varepsilon y) (U_Q(y) + \varepsilon w_Q(y)), \tag{12}
\]

where \(U_Q(y) = U(y - Q)\) and \(w_Q\) is a suitable function obtained in Subsection 2.2 of [GMMPI] by a small modifications of Lemma 9.3 in [AM], satisfying the following estimate

\[
|w_Q(y)| + |\nabla w_Q(y)| + |\nabla^2 w_Q(y)| \leq C\Omega (1 + |y|^K) e^{-|y|}, \tag{13}
\]

\[
\]
where $C_\Omega$ and $K$ are constants depending on $\Omega$, $H$, $n$ and $p$.

The next result collects estimates obtained following the same arguments of Lemmas 9.4, 9.7 and 9.8 in [AM].

**Proposition 2.4.** There exist $C, K > 0$ such that for $\epsilon$ small the following estimates hold

\[
\left| \frac{\partial \bar{u}_\epsilon,Q}{\partial y} \right|(y) \leq \begin{cases} 
C \epsilon^2 (1 + |y|^K) e^{-|y|} & \text{for } |y| \leq \frac{\mu_0}{4\epsilon}, \\
Ce^{-\frac{|y|}{\epsilon}} & \text{for } \frac{\mu_0}{4\epsilon} \leq |y| \leq \frac{\mu_0}{2\epsilon}; 
\end{cases}
\]

\[
-\Delta_y \bar{u}_\epsilon,Q + \bar{u}_\epsilon,Q - \bar{u}_{\epsilon,Q}^p \leq \begin{cases} 
C \epsilon^2 (1 + |y|^K) e^{-|y|} & \text{for } |y| \leq \frac{\mu_0}{4\epsilon}, \\
Ce^{-\frac{|y|}{\epsilon}} & \text{for } \frac{\mu_0}{4\epsilon} \leq |y| \leq \frac{\mu_0}{2\epsilon}; 
\end{cases}
\]

\[
I_{\epsilon,N}(\bar{u}_\epsilon,Q) = \tilde{C}_0 - \tilde{C}_1 \epsilon H'(\epsilon Q) + o(\epsilon^2); \quad \frac{\partial}{\partial Q} I_{\epsilon,N}(\bar{u}_\epsilon,Q) = -\tilde{C}_1 \epsilon^2 H'(\epsilon Q) + o(\epsilon^2),
\]

where

\[
\tilde{C}_0 = \left(\frac{1}{2} - \frac{1}{p + 1}\right) \int_{\mathbb{R}^n_+} U^{p+1} dy, \quad \tilde{C}_1 = \left(\int_0^\infty r^n U_r^2 dr\right) \int_{S^n_+} y_n |y'|^2 d\sigma.
\]

An immediate consequence of this proposition is that

\[
\|I'_\epsilon(\bar{u}_\epsilon,Q)\| \leq C \epsilon^2 \quad \text{for all } Q \in \partial_N \Omega_\epsilon \text{ such that } \text{dist}(Q, \Gamma_\epsilon) \geq \frac{\mu_0}{\epsilon},
\]

where $C > 0$ is some fixed constant and $\mu_0$ is as before.

### 3 Approximate solutions to (10)

To construct good approximate solutions to (10), we will start from a family of known functions which constitute good approximate solutions to (10) when we impose pure Neumann boundary conditions. Since we have to take into account the effect of the Dirichlet boundary conditions, we will modify these functions in a convenient way. Following the line of [GMMP1] and [NW], we will use the projection operator onto $H^1_D(\Omega_\epsilon)$, which associates to every element in $H^1(\Omega_\epsilon)$ its closest point in $H^1_D(\Omega_\epsilon)$. Explicitly, this is constructed subtracting to any given $u \in H^1(\Omega_\epsilon)$ the solution to

\[
\begin{aligned}
-\Delta v + v &= 0 \quad \text{in } \Omega_\epsilon, \\
v &= u \quad \text{on } \partial_D \Omega_\epsilon, \\
\frac{\partial v}{\partial \nu} &= 0 \quad \text{on } \partial_N \Omega_\epsilon.
\end{aligned}
\]

This solution can be found variationally by looking at the following minimum problem

\[
v = u \inf_{\partial_D \Omega_\epsilon} \left\{ \int_{\Omega_\epsilon} (|\nabla v|^2 + v^2) \, dx \right\}.
\]

Instead of studying (15) directly, it is convenient to modify the domain in order that the region of the boundary near $\Gamma_\epsilon$ becomes flat. We fix $Q \in \Gamma_\epsilon$ and consider the opening angle of $\Gamma_\epsilon$ at $Q$, $\alpha = \alpha(Q)$. Since the construction of this new domain is different for $0 < \alpha \leq \pi$ and $\pi < \alpha < 2\pi$, we will study separately the two cases in the following two subsections.
3.1 Case $0 < \alpha \leq \pi$

For technical reasons we construct a domain $\Sigma$ in the following way: we consider two hypersurfaces defined by the equations $x_1 \tan \alpha + x_n = 0$ and $x_1 \tan \alpha - x_n = 0$, which obviously intersect at $\{x_1 = x_n = 0\}$. Then we close the domain between the two hypersurfaces with $x_1 < 0$ if $0 < \alpha < \frac{\pi}{2}$ and with $x_1 > 0$ if $\frac{\pi}{2} \leq \alpha \leq \pi$ with a smooth surface, in such a way that the scaled domain

$$\Sigma_D = D\Sigma, \quad (16)$$

defined for a large number $D$, contains a sufficiently large cube. In $\Sigma_D$ we denote by $\Gamma_D$ the singularity, which lies on $\{x_1 = x_n = 0\}$. The following figure represents a section of the domain in the plane $x_1, x_n$.

The advantage of dealing with this set is that if we solve a Dirichlet problem in $\Sigma_D$ with data even in $x_n$, then for suitable boundary conditions the solution in the upper part $\Sigma_D \cap \{x_n > 0\}$ will be qualitatively similar to that of (15).

Our next goal is to consider the following problem

$$\begin{cases}
-\Delta \tilde{\phi} + \tilde{\phi} = 0 & \text{in } \Sigma_{dD}, \\
\tilde{\phi} = U(\cdot - dQ_0) & \text{on } \partial\Sigma_{dD},
\end{cases} \quad (17)$$

where $Q_0 = (-1, 0, \cdots, 0)$. By a scaling of variables, this problem is equivalent to

$$\begin{cases}
-\frac{1}{d^2} \Delta \varphi + \varphi = 0 & \text{in } \Sigma_D, \\
\varphi = U(d(\cdot - Q_0)) & \text{on } \partial\Sigma_D,
\end{cases} \quad (18)$$

3.1.1 Asymptotic analysis of \(18\)

First of all we need to know if (18) is solvable. It follows from Lemma 3.1 in GMMPT: in fact, making a modification of some arguments in Grl, they construct barrier functions for the operators $\Delta$ and $-\Delta + 1$ at all boundary points of the set $\Sigma$. This guarantees, via the classical Perron method, the existence of a solution for the problem (18).

If we consider the function $\phi = -\frac{1}{d} \log \varphi$, then $\phi$ satisfies

$$\begin{cases}
\frac{1}{d} \Delta \phi - |\nabla \phi|^2 + 1 = 0 & \text{in } \Sigma_D, \\
\phi = -\frac{1}{d} \log (U(d(\cdot - Q_0))) & \text{on } \partial\Sigma_D,
\end{cases} \quad (19)$$

Using the limit behavior of the function $U$ given by [8], it is easy to show the following:
Lemma 3.1. For any fixed constant $D > 0$ we have that
\[
- \frac{1}{d} \log(U(d(\cdot - Q_0))) \to |\cdot - Q_0| \quad \text{uniformly on } \partial \Sigma_D
\] (20)
as $d \to +\infty$.

Since Lemma 3.1 states that the boundary datum is everywhere close to the function $|x - Q_0|$, it is useful to consider the following auxiliary problem
\[
\begin{aligned}
\frac{1}{d} \Delta \phi - |\nabla \phi|^2 + 1 &= 0 \quad \text{in } \Sigma_D, \\
\phi &= |x - Q_0| \quad \text{on } \partial \Sigma_D.
\end{aligned}
\] (21)

Lemma 3.2. Let $D > 1$ be a fixed constant. Then, when $d \to \infty$, problem (21) has a unique solution $\phi^d$, which is everywhere positive, and which more precisely satisfies the estimates
\[
\frac{\tan \alpha}{\sqrt{\tan^2 \alpha + 1}} < \phi^d(x) < C \quad \text{in } \Sigma_D,
\] (22)
if $0 < \alpha < \frac{\pi}{2}$, and
\[
1 < \phi^d(x) < C \quad \text{in } \Sigma_D,
\] (23)
if $\frac{\pi}{2} \leq \alpha \leq \pi$, where $C$ depends only on $D$ and $\Sigma$.

Proof. Applying the transformation inverse to the one at the beginning of this subsection and using the existence of barrier functions for the operator $-\Delta + 1$, as shown in [GMMPT], Lemma 3.1, we get existence. Uniqueness and positivity of $\phi^d$ follows from the maximum principle.

To prove the estimates (22) and (23), we can reason as in [GMMPT], Lemma 3.4, or in [NW], Lemma 4.2. In the case $0 < \alpha < \frac{\pi}{2}$, we have that $\phi^d(x) \equiv \frac{\tan \alpha}{\sqrt{\tan^2 \alpha + 1}}$ in $\Sigma_D$ is a subsolution to (21), since $\text{dist}(Q_0, \partial \Sigma_D) = \frac{\tan \alpha}{\sqrt{\tan^2 \alpha + 1}}$; whereas, in the case $\frac{\pi}{2} \leq \alpha \leq \pi$, we have that $\text{dist}(Q_0, \partial \Sigma_D) = 1$, and then the subsolution is given by $\phi^d(x) \equiv 1$. Moreover, in both the two cases, the function $\phi^d_\alpha(x) = C + x_1$ is a supersolution for $C$ sufficiently large. Then our claim follows.

We next show some pointwise bounds on $\phi^d$, which in particular imply a control on the gradient within some region in the boundary of $\Sigma_D$. We obtain gradient bounds only near smooth parts of the boundary, away from the singularity $\Gamma_D$.

Lemma 3.3. Let $D > 1$ be as in Lemma 3.2. Then, there exists a constant $C > 0$ such that for any $\sigma > 0$ sufficiently small there exist $\delta > 0$ and $d_\sigma > 0$ so large that
\[
|\phi^d(x) - \phi^d(z_x)| \leq C|x - z_x|, \quad z_x \in \partial \Sigma_D, \text{dist}(z_x, D\Gamma_D) \geq \sigma, |x - z_x| \leq \delta, d \geq d_\sigma.
\]
In the above formula $z_x$ denotes the point in $\partial \Sigma_D$ closest to $x$.

Proof. Let us first consider the case $0 < \alpha < \frac{\pi}{2}$. Let us fix $\sigma > 0$ small and consider, for every $0 < \delta < \delta = \sigma \tan \alpha$, the points $x \in \Sigma_D$ of the form $z + \delta \nu(z)$, where $z \in \partial \Sigma_D$ and $\nu(z)$ is the inner unit normal at $z$. Note that there is no problem in the representation of $x$ if $\text{dist}(z, D\Gamma_D) \geq \sigma$; whereas if $\text{dist}(z, D\Gamma_D) < \sigma$, we follow the inner normal direction given by $\nu(z)$ and stop at $x_n = 0$ if we reach this hyperplane at a distance from the boundary smaller than $\delta$. Let us call $\Delta_\delta$ this set of points $x \in \Sigma_D$ at distance $\delta$ from the boundary. Note that the $\Delta_\delta$’s are all disjoint as $\delta$ varies in $[0, \delta]$. Now in $\Delta_\delta$ we can define the functions
\[
\phi_1(x) = |z_1(x) - Q_0| + M_\delta_1(x), \\
\phi_2(x) = |z_2(x) - Q_0| + M_\delta_2(x),
\]
and
where $z_1(x), z_2(x)$ are the points in $\partial \Sigma_D$ closest to $x$ with the $n$-th coordinate respectively positive and negative; $\delta_1(x), \delta_2(x)$ give the distance of $x$ from $z_1(x), z_2(x)$. If we set

$$\hat{\phi}_+^d(x) = \min \{ \phi_1(x), \phi_2(x) \},$$

we choose the constant $M$ so large that $\hat{\phi}_+^d(x) > \phi^d(x)$ when $x \in \{ z + \delta \nu(z) : z \in \partial \Sigma_D \}$. The existence of such constant $M$ is guaranteed by Lemma 3.2.

Next we consider a smooth function $\rho \in C_0^\infty(\mathbb{R}^n)$, such that $\text{supp}\rho \subset B_1(0)$, and $\int_{\mathbb{R}^n} \rho(x) \, dx = 1$. Moreover we define the function

$$\lambda(x) = -\frac{2}{\delta} \delta^2(x) + 2\delta(x), \quad \text{for } x \in \{ z + \delta \nu(z) : z \in \partial \Sigma_D, \delta \in [0, \delta] \}.$$

Then we construct a mollifiers

$$\rho_{\lambda(x)}(y) = \frac{1}{\lambda^n(x)} \rho\left(\frac{y}{\lambda(x)}\right),$$

in such a way that the support of each $\rho_{\lambda(x)}$ depends on the point $x$, and, in particular, it shrinks to a point when we are close to the boundary.

Finally we regularize $\hat{\phi}_+^d$ using the convolution with the mollifiers defined in (24). Then we obtain the following smooth function

$$\hat{\phi}_+^d(x) = \left( \phi_+^d * \rho_{\lambda(x)} \right)(x) = \int_{\mathbb{R}^n} \hat{\phi}_+^d(x - y) \rho_{\lambda(x)}(y) \, dy.$$

It is easy to see that, for $i = 1, \ldots, n - 1$,

$$\frac{\partial \hat{\phi}_1}{\partial x_i} < o(1) - \frac{1}{C} M, \quad (25)$$

$$\frac{\partial \hat{\phi}_2}{\partial x_i} < o(1) - \frac{1}{C} M. \quad (26)$$

Moreover, using (25) and (26), we have that

$$\frac{\partial \hat{\phi}_+^d}{\partial x_i} = \int_{\mathbb{R}^n} \frac{\partial \hat{\phi}_+^d}{\partial x_i}(x - y) \rho_{\lambda(x)}(y) \, dy + \int_{\mathbb{R}^n} \hat{\phi}_+^d(x - y) \frac{\partial \rho_{\lambda(x)}}{\partial \lambda}(y) \frac{\partial \lambda}{\partial x_i}(x) \, dy$$

$$\leq o(1) - \frac{1}{C} M + \int_{\mathbb{R}^n} \hat{\phi}_+^d(x - y) \frac{\partial \rho_{\lambda(x)}}{\partial \lambda}(y) \frac{\partial \lambda}{\partial x_i}(x) \, dy. \quad (27)$$

Now we need an estimate for the last term in (27), let us call it $A$. If we add and subtract $\hat{\phi}_+^d(x)$ in the integral, we obtain

$$A = \int_{\mathbb{R}^n} \hat{\phi}_+^d(x) \frac{\partial \rho_{\lambda(x)}}{\partial \lambda}(y) \frac{\partial \lambda}{\partial x_i}(x) \, dy + \int_{\mathbb{R}^n} \left[ \hat{\phi}_+^d(x - y) - \hat{\phi}_+^d(x) \right] \frac{\partial \rho_{\lambda(x)}}{\partial \lambda}(y) \frac{\partial \lambda}{\partial x_i}(x) \, dy$$

$$= \frac{\partial \lambda}{\partial x_i}(x) \int_{\mathbb{R}^n} \left[ \hat{\phi}_+^d(x - y) - \hat{\phi}_+^d(x) \right] \frac{\partial \rho_{\lambda(x)}}{\partial \lambda}(y) \, dy;$$

in the last step we have used the fact that $\hat{\phi}_+^d(x)$ and $\frac{\partial \lambda}{\partial x_i}(x)$ do not depend on $y$, and the fact that $\int_{\mathbb{R}^n} \frac{\partial \rho_{\lambda(x)}}{\partial \lambda}(y) \, dy = \frac{\partial}{\partial \lambda} \int_{\mathbb{R}^n} \rho_{\lambda(x)}(y) \, dy = 0$, since $\int_{\mathbb{R}^n} \rho_{\lambda(x)}(y) \, dy = 1$, for every $\lambda > 0$. Now, from (24), a simple computation yields

$$\frac{\partial \rho_{\lambda(x)}}{\partial \lambda}(y) = -n \lambda^{-n - 1}(x) \rho \left( \frac{y}{\lambda(x)} \right) - \lambda^{-n - 2}(x) y \nabla \rho \left( \frac{y}{\lambda(x)} \right).$$
Then, using the fact that $\frac{\partial \lambda}{\partial x_i}(x) \simeq -C x_i$, for some positive constant $C$, and making the change of variable $y = \lambda(x)z$, we have

$$A = C\lambda^{-1}(x)x_i \int_{\mathbb{R}^n} \left[ \hat{\phi}_+^d(x - \lambda(x)z) - \hat{\phi}_+^d(x) \right] : [\rho(z) + z\nabla \rho(z)] dz. \quad (28)$$

Since $\hat{\phi}_+^d$ is a Lipschitz function, from (28) we get that

$$A \leq C x_i \int_{\mathbb{R}^n} |z| [\rho(z) + z\nabla \rho(z)] dz,$$

and then $A \leq o(1)$. It follows that, for $M$ sufficiently large, the norm of $\nabla \phi_+^d$ can be arbitrarily big on its domain. By (22), if $M$ is large then $\phi_+^d$ is everywhere bigger than $\phi^d$ on $\Sigma_D \cap \{\text{dist}(\cdot, \partial\Sigma) = \bar{\delta}\}$, so $\phi_+^d$ is a supersolution of (21) in $\Sigma_D \cap \{\text{dist}(\cdot, \partial\Sigma) < \delta\}$.

On the other hand, we claim that the function $\phi_-^d(x) = [x - \bar{Q}_0]$ is a supersolution of (21) in $\Sigma_D \cap \{\text{dist}(\cdot, \partial\Sigma) < \delta\}$. In fact, if we consider the set $\Sigma_D \setminus B_{\delta(d)}(Q_0)$, where $\delta(d)$ is a small positive number depending on $d$, we can see by easy computation that here $\phi_+^d$ satisfies

$$\frac{1}{d} \Delta \phi_+^d - |\nabla \phi_+^d|^2 + 1 = \frac{n-1}{d|x - Q_0|}.$$

Moreover, since $\phi_+^d$ is positive, we can choose $\bar{\delta}(d)$ sufficiently small so that $\phi_+^d < \phi^d$. Hence we obtain that $\phi_+^d \leq \phi^d$ in the closure of $\Sigma_D \cap \{\text{dist}(\cdot, \partial\Sigma) < \delta\}$.

Finally, the conclusion follows from the fact that $\phi_+^d$ and $\phi_+^d$ coincide on the set

$$\{x \in \partial\Sigma_D : \text{dist}(x, D\Gamma_D) \geq \sigma\}$$

and that we have uniform bounds on the gradient here, independently on $d$.

In the case $\frac{n}{2} \leq \alpha \leq \pi$, we can repeat essentially the same construction of the proof of Lemma 3.5 in [GMP1] and obtain the same conclusion. \hfill \box

Using the same arguments as in Lemma 3.6 in [GMP1], we are able to extend the gradient estimate which follows from the previous lemma to a subset of the interior of the domain.

**Lemma 3.4.** Let $D > 1$ be as in Lemma 3.2. Then, there exists a constant $C > 0$ such that for any $\sigma > 0$ sufficiently small there exists $d_\sigma > 0$ so large that

$$|\nabla \phi^d(x)| \leq C \quad \text{in} \quad \{x \in \Sigma_D : \text{dist}(x, D\Gamma_D) \geq \sigma\}, \quad d \geq d_\sigma. \quad (29)$$

The next proposition is about the asymptotic behavior of the solutions of (21).

**Lemma 3.5.** Let $\phi^d$ be the solution of (21), then we have that

$$\phi^d(x) \to \bar{\phi}(x) := \inf_{z \in \partial\Sigma_D} (|x - z| + |z - Q_0|), \quad \text{as} \quad d \to \infty, \quad (30)$$

uniformly on the compact sets of $\Sigma_D$.

**Proof.** We will show (30) in two steps:

1) we prove that the function on the right-hand side of (30) is the supremum of all the elements of

$$F = \{v \in W^{1,\infty}(\Sigma_D) : v(x) \leq |x - Q_0| \text{ on } \partial\Sigma_D, |\nabla v| \leq 1 \text{ a.e. in } \Sigma_D\};$$

12
2) we prove that for any sequence \(d_k \to \infty\), there is a subsequence \(d_{k_i} \to \infty\) such that \(\phi^{d_{k_i}} \to \overline{\phi}\) uniformly on the compact sets of \(\Sigma_D\) as \(d_k \to \infty\). Then it follows that \(\phi^d \to \overline{\phi}\) uniformly on the compact sets of \(\Sigma_D\) as \(d \to \infty\).

We first prove 1). To begin we show that \(\overline{\phi} \in F\). If \(x_1, x_2 \in \Sigma_D\) and \(z_2 \in \partial \Sigma_D\) realizes the infimum for \(x_2\), we have

\[
|\overline{\phi}(x_1) - \overline{\phi}(x_2)| \leq |x_1 - z_2| + |z_2 - Q_0| - |x_2 - z_2| - |z_2 - Q_0| \leq |x_1 - x_2|.
\]

Then, taking \(x_1, x_2\) close, we get \(\overline{\phi} \in W^{1,\infty}(\Sigma_D)\) and \(|\nabla \overline{\phi}| \leq 1\ a.e.\ in \Sigma_D\). Moreover, it is easy to see that \(\overline{\phi}(x) = |x - Q_0|\) if \(x \in \partial \Sigma_D\). We next show that \(\overline{\phi}\) is the maximum element of \(F\). We construct a \(\delta\) neighborhood \(\Sigma_D^\delta\) of \(\Sigma_D\) in this way: consider \(Q_0 = (-1,0,\cdots,0)\) and, for every \(z \in \partial \Sigma_D\), the line from \(Q_0\) to \(z\). If \(\delta > 0\) is small enough, each point \(x \in \Sigma_D^\delta \setminus \Sigma_D\) is uniquely determined by the equation \(x = z + \delta r(z)\), where \(z \in \partial \Sigma_D\) is the intersection point of the line from \(Q_0\) to \(x\) with \(\partial \Sigma_D\), \(r(z)\) is the unit outer vector on the line, and \(0 < \delta < \frac{\delta}{\cos \theta(z)}\); here \(\theta(z)\) is the angle between \(r(z)\) and the unit outer normal at \(z\), \(\nu(z)\), in the plane generated by \(r(z)\) and \(\nu(z)\). Note that for the point on the boundary \(z \in \{z_1 = z_n = 0\}\) we can consider \(\nu(z)\) just taking the normal to the hypersurface defined by the equation \(x_1 \tan \alpha + x_n = 0\) or to the one defined by the equation \(x_1 \tan \alpha - x_n = 0\), and it is well defined since the angle \(\theta(z)\) is the same for those points. In addition, the map \(x \to (z, \overline{\phi})\) is continuous in \(\Sigma_D^\delta \setminus \Sigma_D\).

Now, we can extend every \(v \in F\) to a \(\tilde{v} \in W^{1,\infty}(\Sigma_D^\delta)\), taking \(v = \tilde{v}\) in \(\Sigma_D\) and \(\tilde{v}(x) = v(z)\) for \(x \in \Sigma_D^\delta \setminus \Sigma_D\). Moreover, if we consider the function

\[
\tilde{K}(x) = \begin{cases} 
1 & \text{in } \Sigma_D, \\
1 + C\overline{\phi} & \text{in } \Sigma_D^\delta \setminus \Sigma_D,
\end{cases}
\]

for some large constant \(C > 0\) independent of \(\delta\), we get \(|\nabla \tilde{v}| \leq \tilde{K}\) a.e. in \(\Sigma_D^\delta\). Now, we regularize \(\tilde{v}\) using the convolution with mollifiers, that is considering, for \(\lambda > 0\) small enough, \(v_\lambda := \tilde{v} \ast \rho_\lambda\), with \(\rho_\lambda(x) = \lambda^{-n} \rho(x/\lambda)\), \(\rho \in C_0^\infty(\mathbb{R}^n)\), \(supp \rho \subset B_1(0)\), \(\int_{\mathbb{R}^n} \rho(x) \, dx = 1\). Then we have

\[
|\nabla v_\lambda| \leq |\nabla \tilde{v}| \ast \rho_\lambda \leq \tilde{K} \ast \rho_\lambda \leq 1 + C\lambda
\]

on \(\Sigma_D\) and \(v_\lambda \to v\) in \(C(\Sigma_D)\) as \(\lambda \to 0\). Let now \(x, y \in \Sigma_D\) and consider the function \(\xi(t) = tx + (1-t)y\), for \(t \in [0,1]\); then we can estimate

\[
|v_\lambda(x) - v_\lambda(y)| \leq \int_0^1 \left| \nabla v_\lambda(\xi(t)) \right| \left| \frac{d\xi}{dt} \right| dt \leq \int_0^1 (1 + C\lambda) \cdot |x - y| \, dt \leq (1 + C\lambda) \cdot |x - y|.
\]

Letting \(\lambda \to 0\), we obtain \(|v(x) - v(y)| \leq |x - y|\). Hence \(v(x) \leq v(y) + |x - y|\), and \(v(x) \leq |y - Q_0| + |x - y|\) for all \(y \in \partial \Sigma_D\). So \(v \leq \overline{\phi}\).

We next prove 2). By gradient estimate and the Ascoli-Arzelà theorem we know that the \(\phi^d\)s admit limit \(\hat{\phi}\) in the whole closure of \(\Sigma_D\). Moreover it is easy to see that \(\hat{\phi}\) belongs to the set \(F\), hence \(\hat{\phi} \leq \overline{\phi}\). We need then to prove only \(\hat{\phi} \leq \overline{\phi}\). Let \(v \in F\). Similarly to 1), we extend \(v\) to \(\tilde{v}\) in \(\Sigma_D^\delta\) and regularize \(\tilde{v}\) to \(v_\lambda\) in such a way that we have \(\|v - v_\lambda\|_{L^\infty(\Sigma_D)} \leq C\lambda\) and \(|\nabla v_\lambda| \leq \tilde{K} \ast \rho_\lambda \leq \tilde{K}\). Hence as before we get \(|\nabla v_\lambda| \leq 1 + C\lambda\) on \(\Sigma_D\) and \(v_\lambda \to v\) in \(C(\Sigma_D)\) as \(\lambda \to 0\). By simple computation we obtain that \(v_\lambda\) satisfies

\[
\begin{cases} 
\frac{1}{2} \Delta v_\lambda - |\nabla v_\lambda|^2 + 1 + C\lambda + \frac{1}{d} A_\lambda \geq 0 & \text{in } \Sigma_D, \\
v_\lambda \leq |x - Q_0| + C\lambda & \text{on } \partial \Sigma_D,
\end{cases}
\]

where \(A_\lambda \geq 0\). If we define

\[
v_\lambda := \frac{v_\lambda}{\sqrt{1 + C\lambda + \frac{1}{d} A_\lambda}}
\]

for any sequence \(d_k \to \infty\), there is a subsequence \(d_{k_i} \to \infty\) such that \(\phi^{d_{k_i}} \to \overline{\phi}\) uniformly on the compact sets of \(\Sigma_D\) as \(d_k \to \infty\). Then it follows that \(\phi^d \to \overline{\phi}\) uniformly on the compact sets of \(\Sigma_D\) as \(d \to \infty\).
by comparison we deduce that
\[ \tilde{v}_\lambda \leq \phi^d \sqrt{1 + C\lambda + \frac{1}{d_k} A\lambda} + C\lambda. \] (31)
Choosing \( d = d'_{k_l} \) in (31) such that
\[ d_{k_l} = d'_{k_l} \sqrt{1 + C\lambda + \frac{1}{d_{k_l}} A\lambda}, \]
we see that
\[ \frac{v_\lambda}{\sqrt{1 + C\lambda}} \leq \hat{\phi} + C\lambda \]
as \( d'_{k_l} \to \infty \). Then, letting \( \lambda \to 0 \), we obtain \( v \leq \hat{\phi} \); in particular, \( \overline{\phi} \leq \hat{\phi} \). Hence \( \overline{\phi} = \hat{\phi} \).

Next we analyze the asymptotic behavior of the solutions of (19). From now on in this subsection we study separately the two cases \( 0 < \alpha < \frac{\pi}{2} \) and \( \frac{\pi}{2} \leq \alpha \leq \pi \). Let us consider the first case.

**Proposition 3.6.** Suppose that \( 0 < \alpha < \frac{\pi}{2} \). Let \( D \) be a large fixed constant and \( \Phi^d \) the solution of (19). Then we have
\[ \Phi^d (x) \to \min \{ d_1 (x), d_2 (x) \}, \quad \text{as } d \to \infty, \]
uniformly on the compact sets of \( \Sigma_D \cap \overline{B}_\frac{\pi}{4} (0) \), where
\[
d_1 (x) := \sqrt{\left( x_1 - \frac{\tan^2 \alpha - 1}{\tan^2 \alpha + 1} \right)^2 + |x'|^2 + \left( x_n - \frac{2\tan \alpha}{\tan^2 \alpha + 1} \right)^2}, \quad (32)
\]
\[
d_2 (x) := \sqrt{\left( x_1 - \frac{\tan^2 \alpha - 1}{\tan^2 \alpha + 1} \right)^2 + |x'|^2 + \left( x_n + \frac{2\tan \alpha}{\tan^2 \alpha + 1} \right)^2}. \quad (33)
\]

**Remark 3.7.** Note that \( d_1 \) and \( d_2 \) are the distance functions, respectively, from the point \( Q_1 = \left( \frac{\tan^2 \alpha - 1}{\tan^2 \alpha + 1}, 0, \ldots, 0, \frac{2\tan \alpha}{\tan^2 \alpha + 1} \right) \), which is the symmetrical point to \( Q_0 \) with respect to the hypersurface defined by the equation \( x_1 \tan \alpha + x_n = 0 \), and from the point \( Q_2 = \left( \frac{\tan^2 \alpha - 1}{\tan^2 \alpha + 1}, 0, \ldots, 0, -\frac{2\tan \alpha}{\tan^2 \alpha + 1} \right) \), which is the symmetrical point to \( Q_0 \) with respect to the hypersurface defined by the equation \( x_1 \tan \alpha - x_n = 0 \). So the function \( \tilde{\phi} (x) \) is even with respect to the coordinate \( x_n \) and a.e. differentiable. The problem is that it does not have zero \( x_n \)-derivative on \( \{ x_n = 0 \} \).

**Proof.** If \( \phi^d \) is a solution of (21), it is easy to see that \( \phi^d + \sup_{x \in \partial \Sigma_D} ||x - Q_0| + \frac{1}{\alpha} \log (U (d (x - Q_0))) | is a supersolution of (19) and \( \phi^d - \sup_{x \in \partial \Sigma_D} ||x - Q_0| + \frac{1}{\alpha} \log (U (d (x - Q_0))) | is a subsolution. Then \( \Phi^d \) must lie in between these two functions. Hence, by Lemma 3.8 it is sufficient to prove the analogous statement for \( \phi^d \). The proof of the latter fact is a consequence of Lemma 3.5 and the following Lemma 3.8. \( \square \)

**Lemma 3.8.** Suppose that \( 0 < \alpha < \frac{\pi}{2} \). If \( \overline{\phi} (x) \) is as in (30), then
\[ \overline{\phi} (x) = \min \{ d_1 (x), d_2 (x) \}, \quad x \in \overline{B}_{\frac{\pi}{4}} (0), \]
where \( d_1 \) and \( d_2 \) are as in (32) and (33).
Proof. Consider a point \( x = (x_1, \ldots, x_n) \) with \( x_n \geq 0 \). By construction of \( \Sigma_D \), the point \( z \in \partial \Sigma_D \) which realizes the infimum will necessarily belong to the set \( \{x_1 \tan \alpha + x_n = 0\} \cap \{x_1 < 0\} \). This implies that

\[
\bar{\phi}(x) = \inf_{z \in \{x_1 \tan \alpha + x_n = 0\} \cap \{x_1 < 0\}} (|x - z| + |z - Q_0|).
\]

Now we can reason as follows: given \( x \), the level sets of the function \( z \to |x - z| + |z - Q_0| \) are the axially symmetric ellipsoids with focal points \( x \) and \( Q_0 \). The smaller is the ellipsoid, the smaller is the value of this function; so we are reduced to find the smallest ellipsoid which intersects \( \{x_1 \tan \alpha + x_n = 0\} \cap \{x_1 < 0\} \). We note that if we fix \( x_1, x_n \) and vary only \( x'' \), the corresponding infimum \( z \) has the same \( z_1, z_n \) and different \( z'' \); so we can determine \( z_1, z_n \) in the simplest case \( x'' = (0, \ldots, 0) \), and obviously \( z'' = (0, \ldots, 0) \). Then we are reduced to consider the minimum problem

\[
\min_{(z_1, z_n) \in \{x_1 \tan \alpha + x_n = 0\} \cap \{x_1 < 0\}} \left( \sqrt{(x_1 - z_1)^2 + (x_n + \tan \alpha z_1)^2} + \sqrt{(z_1 + 1)^2 + \tan^2 \alpha z_1^2} \right).
\]

Deriving with respect to the variable \( z_1 \) we obtain that at a minimum point

\[
\frac{-(x_1 - z_1) + \tan \alpha (x_n + \tan \alpha z_1)}{\sqrt{(x_1 - z_1)^2 + (x_n + \tan \alpha z_1)^2}} + \frac{(z_1 + 1) + \tan^2 \alpha z_1}{\sqrt{(z_1 + 1)^2 + \tan^2 \alpha z_1^2}} = 0,
\]

which implies

\[
\begin{align*}
z_1 &= \frac{-2 \tan \alpha x_1 + (\tan^2 \alpha - 1) x_n}{(\tan^2 \alpha + 1) (\tan \alpha x_1 + x_n - \tan \alpha)}, \quad (34) \\
z_n &= \frac{2 \tan^2 \alpha x_1 - \tan \alpha (\tan^2 \alpha - 1) x_n}{(\tan^2 \alpha + 1) (\tan \alpha x_1 + x_n - \tan \alpha)}.
\end{align*}
\]

Now assume that \( x'' \neq (0, \ldots, 0) \) and \( x_1, x_n \) are as before. By the previous observation we know that the coordinates \( z_1, z_n \) of the corresponding infimum are given by \( (34) \) and \( (35) \). So we have to determine only \( z'' \). To do this let us consider the minimum problem

\[
\min_{z'' \in \mathbb{R}^{n-2}} \left( \sqrt{(x_1 - z_1)^2 + |x'' - z''|^2 + (x_n + \tan \alpha z_1)^2} + \sqrt{(z_1 + 1)^2 + |z''|^2 + \tan^2 \alpha z_1^2} \right). \quad (36)
\]

Again by differentiation we obtain that a minimum point must satisfy

\[
\frac{z'' - x''}{\sqrt{(x_1 - z_1)^2 + |x'' - z''|^2 + (x_n + \tan \alpha z_1)^2}} + \frac{z''}{\sqrt{(z_1 + 1)^2 + |z''|^2 + \tan^2 \alpha z_1^2}} = 0,
\]

which gives

\[
z'' = \frac{\sqrt{(z_1 + 1)^2 + \tan^2 \alpha z_1^2}}{\sqrt{(x_1 - z_1)^2 + (x_n + \tan \alpha z_1)^2} + \sqrt{(z_1 + 1)^2 + \tan^2 \alpha z_1^2}}. \quad (37)
\]

If we plug \( (34), (35) \) and \( (37) \) into \( (36) \), we obtain that \( \bar{\phi}(x) = d_1(x) \). Reasoning in the same way for points with \( x_n < 0 \), we have \( \bar{\phi}(x) = d_2(x) \). Then we get the conclusion.

\[ \square \]

**Remark 3.9.** Note that \( \bar{\phi} \) is a viscosity solution of the Hamilton-Jacobi equation \( |\nabla \phi|^2 = 1 \) in \( \Sigma_D \). In fact, what we have to show is that

\[ 15 \]
Lemma 3 together with the fact that in the case

\[
\text{Proof.}
\]

Proposition 3.10. Suppose that \( \tfrac{\pi}{2} \leq \alpha \leq \pi \). Let \( D \) be a large fixed constant and \( \Phi^d \) the solution of \((19)\). Then we have

\[
\Phi^d (x) \rightarrow \Phi (x), \quad \text{as } d \rightarrow \infty,
\]

uniformly on the compact sets of \( \Sigma_D \cap \bar{B}_r (0) \), where

\[
\Phi (x) = \begin{cases} 
\min \{ d_1 (x), d_2 (x) \}, & \text{if } \tan \alpha \leq \frac{x_1 - \sqrt{x_1^2 + x_n^2}}{x_n}, \\
\sqrt{\left( 1 + \sqrt{x_1^2 + x_n^2} \right)^2 + |x'|^2}, & \text{if } \tan \alpha \geq \frac{x_1 - \sqrt{x_1^2 + x_n^2}}{x_n}.
\end{cases}
\]

Proof. We can reason as in the proof of Proposition 3.6 obtaining that it is sufficient to show the convergence in \((40)\) for the function \( \phi^d \). To prove the latter assertion we have to use Lemma 3.5 together with the fact that in the case \( \tfrac{\pi}{2} \leq \alpha \leq \pi \) the function \( \Phi \) defined in \((30)\) is equal to that one defined in \((41)\). We can obtain this expression by mixing the arguments used in the proof of Lemma 3.8 and those used in Lemma 3.9 in \([GMM\&P2]\).
3.2 Case $\pi < \alpha < 2\pi$

In this case we construct the domain $\Sigma$ in the following way: we consider the set $\{x_n = 0\} \cap \{x_1 \leq 0\}$ and the hypersurface defined by the equation $x_1 \tan \alpha + x_n = 0$ with $x_n \leq 0$. Then we close the domain with a smooth surface; the following figure represents a section of the domain in the plane $x_1, x_n$.

We define the scaled domain $\Sigma_D$ as in (16) and denote by $\Gamma_D$ the singularity, which lies on $\{x_1 = x_n = 0\}$. As in the previous case, the solution of a Dirichlet problem in $\Sigma_D$ will be qualitatively similar to that of (15).

We have to study the asymptotic behavior of the solution of the problem

$$
\begin{align*}
&\begin{cases}
-\frac{1}{\pi} \Delta \varphi + \varphi = 0 & \text{in } \Sigma_D, \\
\varphi = U (d (\cdot - Q_0)) & \text{on } \partial \Sigma_D,
\end{cases}
\end{align*}
$$

To do this we consider the function $\phi = -\frac{1}{d} \log \varphi$, which satisfies

$$
\begin{align*}
&\begin{cases}
\frac{1}{d} \Delta \phi - |\nabla \phi|^2 + 1 = 0 & \text{in } \Sigma_D, \\
\phi = -\frac{1}{d} \log (U (d (\cdot - Q_0))) & \text{on } \partial \Sigma_D.
\end{cases}
\end{align*}
$$

Since the asymptotic analysis is very similar to that one made in Subsection 3.1.1 for $0 < \alpha \leq \pi$ we will not repeat the computations. What we obtain is the following result:

**Proposition 3.11.** Suppose that $\pi < \alpha < 2\pi$. Let $D$ be a large fixed constant and $\Phi^d$ the solution of (42). Then we have

$$
\Phi^d (x) \to \text{dist} (x, Q_0) = \sqrt{(x_1 + 1)^2 + |x'|^2}, \quad \text{as } d \to \infty,
$$

uniformly on the compact sets of $\Sigma_D \cap \bar{B}_D \frac{x}{D} \{0\}$.

3.3 Definition of the approximate solutions

In order to apply the theory in Subsection 2.1 in this subsection we construct a manifold of approximate solutions to (10). Since the limit function of the solutions of (19) is not the same for different angles $\alpha$, as we have seen in Subsections 3.1 and 3.2, we will distinguish the cases. We will give the precise construction only for $0 < \alpha < \frac{x}{\alpha}$, in fact in this case the computations are quite different from the flat case $\alpha = \pi$. In the other cases the estimates for the approximate solutions are the same (for $\frac{x}{\alpha} \leq \alpha \leq \pi$) or very similar (for $\pi < \alpha < 2\pi$) to that ones obtained in [GMMP1], Subsection 3.2, and then we will omit the proofs.
3.3.1 Case $0 < \alpha < \frac{\pi}{2}$

Since the function $\tilde{u}_{e,Q}$ defined in Subsection 2.2 is an approximate solution of (10) with pure Neumann boundary conditions, we need to modify it in the following way. If $\Phi^d$ the solution of (10), the function

$$\Xi^d (y) = e^{-d\Phi^d(y, Q_0)}$$

solves the problem

$$\begin{cases} -\Delta \Xi^d + \Xi^d = 0 & \text{in } d(\Sigma_D - Q_0), \\ \Xi^d = U(\cdot) & \text{on } d(\partial \Sigma_D - Q_0). \end{cases}$$

We can obtain a solution to (44) looking at the minimum problem

$$\inf_{v=U} \int_{d(\partial \Sigma_D - Q_0)} (|\nabla v|^2 + v^2) \, dy.$$  

From (45) we can derive norm estimate on $\Xi^d$. In fact, we can take a cut-off function $\chi_1 : d(\Sigma_D - Q_0) \to \mathbb{R}$ such that

$$\begin{cases} \chi_1(y) = 1 & \text{for } \text{dist}(y, d(\partial \Sigma_D - Q_0)) \leq \frac{1}{4}, \\ \chi_1(y) = 0 & \text{for } y \in d(\Sigma_D - Q_0), \text{dist}(y, d(\partial \Sigma_D - Q_0)) \geq 1, \\ |\nabla \chi_1(y)| \leq 4 & \text{for all } y, \end{cases}$$

and then consider the function $\bar{v}(y) = \chi_1(y) U(y)$. It is easy to see that $\|\bar{v}\|_{H^1(d(\Sigma_D - Q_0))} \leq e^{-d(1+o(1))}$, so by (45) we find that

$$\|\Xi^d\|_{H^1(d(\Sigma_D - Q_0))} \leq \|\bar{v}\|_{H^1(d(\Sigma_D - Q_0))} \leq e^{-d(1+o(1))}. \quad (46)$$

We can also obtain pointwise estimates on $\Xi^d$. In fact, from Proposition 3.6 we obtain that, as $d \to +\infty$,

$$\Xi^d (y) = \exp \left[ - \min \left\{ -\left( y_1 - d - \frac{d(\tan^2 \alpha - 1)}{\tan^2 \alpha + 1} \right) + |y'|^2 + \left( y_n + \frac{2d \tan \alpha}{\tan^2 \alpha + 1} \right)^2 \right\} \cdot e^{o(d)}, \right] \quad (47)$$

for $y \in d(V - Q_0)$, where $V$ is any set compactly contained in $\Sigma_D$. Finally, we have pointwise estimates for the gradient of $\Xi^d$. Indeed, using the uniform convergence in (20) and reasoning as in the proof of Lemmas 3.3 and 3.4 we obtain that (29) holds true also for $\Phi^d$. Then we can apply the arguments in LN (see in particular Proposition 1.4, Lemma 1.5 and Lemma B.1) to conclude that $\nabla \Phi^d \to \nabla \bar{\phi}$ uniformly as $d \to +\infty$ in any set compactly contained in $\Sigma_D$ on which $\nabla \bar{\phi}$ is defined. This convergence implies that, as $d \to +\infty$,

$$\nabla \Xi^d (y) = - \exp \left[ - \min \left\{ -\left( y_1 - d - \frac{d(\tan^2 \alpha - 1)}{\tan^2 \alpha + 1} \right) + |y'|^2 + \left( y_n + \frac{2d \tan \alpha}{\tan^2 \alpha + 1} \right)^2 \right\} \cdot e^{o(d)}, \nabla \bar{\phi} \left( \frac{y}{d} + Q_0 \right) + o(1) \right], \quad (48)$$

for $y \in d(V - Q_0)$, where $V$ is as before.
Now, we want to obtain similar bounds and estimates for $\frac{\partial \xi_d}{\partial d}$ and its gradient. Using the definition of $\xi_d (y) = \varphi \left( \frac{y}{d}, Q_0 \right)$ and the fact that also $\varphi$ depends on $d$, we have that

$$\frac{\partial \xi_d}{\partial d} (y) = \frac{\partial \varphi}{\partial d} \left( \frac{y}{d}, Q_0 \right) - \frac{y}{d^2} \cdot \nabla \varphi \left( \frac{y}{d}, Q_0 \right).$$

(49)

Since $\varphi$ is the solution of (18), we can differentiate (18) obtaining

$$\left\{ \begin{array}{l}
- \frac{1}{d^2} \Delta \frac{\partial \varphi}{\partial d} + \frac{\partial \varphi}{\partial d} = - \frac{2}{d^2} \Delta \varphi = - \frac{2}{d^2} \varphi \\
\frac{\partial \varphi}{\partial d} (x) = \nabla U \left( d(x - Q_0) \right) \cdot (x - Q_0) \quad \text{in } \Sigma_D,
\end{array} \right.$$  \quad (50)

Because of the asymptotic behavior of $U$ at infinity, there exists a positive constant $C_D$ such that for $d$ large we have

$$\frac{1}{C_D} U (d(x - Q_0)) \leq -\nabla U (d(x - Q_0)) \cdot (x - Q_0) \leq C_D U (d(x - Q_0)).$$  \quad (51)

Hence, from (18), (51), the fact that $\varphi > 0$ and the maximum principle we obtain that $\varsigma := - \frac{\partial \varphi}{\partial d} \geq \frac{1}{C_D} \varphi$ in $\overline{\Gamma}_D$. Moreover, as for (19), we can check that the function $Y_d := - \frac{d}{2} \log \varsigma$ satisfies

$$\left\{ \begin{array}{l}
\frac{1}{2} \Delta Y_d + |\nabla Y_d|^2 + 1 - \frac{\varphi}{\varsigma} = 0 \\
Y_d (x) = - \frac{1}{2} \log ( - \nabla U (d(x - Q_0)) \cdot (x - Q_0)) \quad \text{on } \partial \Sigma_D,
\end{array} \right.$$  \quad (52)

Since $\varphi$ stays bounded, $\frac{\partial \varphi}{\partial d}$ tends to zero as $d \to +\infty$. Moreover, using again the asymptotic behavior of $U$ at infinity, we can say that the boundary datum in (52) converges in every smooth sense (where $\partial \Sigma_D$ is regular) to $|x - Q_0|$ as $d \to +\infty$. As a consequence, the previous analysis adapts to $Y_d$ and allows to conclude that still

$$Y_d \to \overline{\varphi} \quad \text{and} \quad \nabla Y_d \to \nabla \overline{\varphi}$$  \quad (53)

uniformly as $d \to +\infty$ in any set compactly contained in $\overline{\Sigma}_D$ on which $\nabla \overline{\varphi}$ is defined.

From (50), reasoning as for (46), we have that

$$\left\| \frac{\partial \varphi}{\partial d} \left( \frac{y}{d}, Q_0 \right) \right\|_{H^1 (d(\Sigma_D - Q_0))} \leq e^{-d(1 + o(1))}.$$  \quad (54)

On the other hand, from (44) one finds that the function $\varpi := \frac{y}{d^2} \cdot \nabla \varphi \left( \frac{y}{d}, Q_0 \right) = \frac{y}{d} \cdot \nabla \xi_d (y)$ satisfies

$$- \Delta \varpi + \varpi = - \frac{2}{d} \overline{\varphi} \quad \text{in } d(\Sigma_D - Q_0).$$

To control the boundary value of $\varpi$ we divide $\partial d (\Sigma_D - Q_0)$ into its intersection with $\{y_n = 0\}$ and its complement. In the first region we have simply that $\varpi = \frac{y}{d} \cdot \nabla U (y)$. In the second instead the estimates in (47) and (48) hold true, which shows that the $L^2$ norm of the trace of $\varpi$ on $\partial d(\Sigma_D - Q_0)$ is of order $e^{-d(1 + o(1))} + e^{-d \left[ 1 + \frac{2 \tan \alpha}{\sqrt{1 - \tan^2 \alpha}} \right] (1 + o(1))}$. This fact and the latter formula imply that

$$\left\| \varpi \right\|_{H^1 (d(\Sigma_D - Q_0))} \leq e^{-d(1 + o(1))} + e^{-d \left[ 1 + \frac{2 \tan \alpha}{\sqrt{1 - \tan^2 \alpha}} \right] (1 + o(1))}.$$  \quad (55)

Then, from (54) and (55), we conclude that

$$\left\| \frac{\partial \xi_d}{\partial d} \right\|_{H^1 (d(\Sigma_D - Q_0))} \leq e^{-d(1 + o(1))} + e^{-d \left[ 1 + \frac{2 \tan \alpha}{\sqrt{1 - \tan^2 \alpha}} \right] (1 + o(1))}.$$  \quad (56)
Now, using the fact that \( \varphi \leq C_D \frac{\partial \varphi}{\partial d} \) and \([53]\), together with the Harnack inequality (which implies \( |\nabla \varphi| \leq C d \varphi \) in \( d ( V - Q_0 ) \)) one also finds

\[
\frac{\partial \Xi_d}{\partial d}(y) = -\exp \left[ - \min \left\{ \sqrt{\left( y_1 - \frac{2 d \tan^2 \alpha}{\tan^2 \alpha + 1} \right)^2 + \left| y'' \right|^2 + \left( y_n \mp \frac{2 d \tan \alpha}{\tan^2 \alpha + 1} \right)^2} \right\} \cdot e^{\alpha(d)} \cdot \left( 1 + o \left( \frac{|y|}{d} \right) \right) \right],
\tag{57}
\]

and

\[
|\nabla \frac{\partial \Xi_d}{\partial d}(y)| \leq \exp \left[ - \min \left\{ \sqrt{\left( y_1 - \frac{2 d \tan^2 \alpha}{\tan^2 \alpha + 1} \right)^2 + \left| y'' \right|^2 + \left( y_n \mp \frac{2 d \tan \alpha}{\tan^2 \alpha + 1} \right)^2} \right\} \cdot e^{\alpha(d)} \right],
\tag{58}
\]

for \( d ( V - Q_0 ) \) and \( d \to +\infty \).

After these preliminaries, we are now in position to introduce our approximate solutions. Let us define two smooth non negative cut-off functions \( \chi_D : \mathbb{R}^n \to \mathbb{R} \), \( \chi_0 : \mathbb{R} \to \mathbb{R} \) satisfying respectively

\[
\left\{ \begin{array}{ll}
\chi_D (y) = 1 & \text{for } |y| \leq \frac{d D}{\sqrt{D}}, \\
\chi_D (y) = 0 & \text{for } |y| \geq \frac{d D}{\sqrt{D}} , \\
|\nabla \chi_D | \leq \frac{\sqrt{2}}{d D^2} & \text{on } \mathbb{R}^n ,
\end{array} \right.
\tag{59}
\]

and

\[
\left\{ \begin{array}{ll}
\chi_0 (y) = 1 & \text{for } y \leq 0 , \\
\chi_0 (y) = 0 & \text{for } y \geq 1 , \\
\chi_0 \text{ is non increasing} & \text{on } \mathbb{R} .
\end{array} \right.
\tag{60}
\]

Now, using the new coordinates \( y \) in Subsection \([2.2]\) we define

\[
u_{\epsilon, Q}(y) := \chi_{\mu_0}(y) \left[ (U_Q (y) - \Xi_d (y)) \chi_D(y) + \epsilon w_Q(y) \chi_0(y_1 - d) \right].
\tag{61}
\]

Following the line of \([\text{GMMPP4}]\) we prove that the \( u_{\epsilon, Q} \)'s are good approximate solutions to \([10]\) for suitable conditions of \( Q \).

**Proposition 3.12.** Let \( \mu_0 \) be the constant appearing in Subsection \([2.2]\). Then there exists another constant \( C_\Omega \), independent of \( \epsilon \), such that, for \( C_\Omega \leq d \leq \frac{1}{\epsilon C^2_\Omega} \) and for \( D d < \frac{\mu_0}{\epsilon C^2_\Omega} \), the functions \( u_{\epsilon, Q} \) satisfy

\[
\| I' (u_{\epsilon, Q}) \| \leq C \left( e^{2 + \epsilon e^{-d(1+o(1))}} + e^{-d \left[ \frac{2}{d} \sqrt{\tan^2 \alpha + 1} + \frac{2}{\sqrt{\tan^2 \alpha + 1}} \right] (1+o(1))} \right) + C \left( e^{-d \frac{p+1}{2} (1+o(1))} + e^{-d \left( \frac{\mu_0}{d} + \frac{2}{\sqrt{\tan^2 \alpha + 1}} \right) (1+o(1))} \right),
\tag{62}
\]

for a fixed \( C > 0 \) and for \( \epsilon \) sufficiently small.

**Proof.** Using the coordinates \( y \), we can split \( u_{\epsilon, Q}(y) = \tilde{u}_{\epsilon, Q}(y) + \tilde{u}_{\epsilon, Q}(y) \), where \( \tilde{u}_{\epsilon, Q} \) is defined in \([12]\) and

\[
\tilde{u}_{\epsilon, Q}(y) = \chi_{\mu_0}(y)\left[ (\chi_D(y) - 1) U_Q (y) - \chi_D(y) \Xi_d (y) + \epsilon (\chi_0(y_1 - d) - 1) w_Q(y) \right].
\tag{63}
\]
Then, if we test the gradient of $I_{e}$ at $u_{e,Q}$ on any function $v \in H^{1}_{D}(\Omega_{e})$, we obtain

$$I'_{e}(u_{e,Q})[v] = \int_{\Omega_{e}} (\nabla g u_{e,Q} \nabla v + u_{e,Q} v) dy - \int_{\Omega_{e}} u^p_{e,Q} v dy$$

$$= \int_{\Omega_{e}} (\nabla g \tilde{u}_{e,Q} \nabla v + \tilde{u}_{e,Q} v) dy - \int_{\Omega_{e}} \tilde{u}^p_{e,Q} v dy$$

$$+ \int_{\Omega_{e}} (\nabla g \tilde{u}_{e,Q} \nabla v + \tilde{u}_{e,Q} v) dy - \int_{\Omega_{e}} (\tilde{u}^p_{e,Q} - u^p_{e,Q}) v dy$$

$$= I'_{e}(\tilde{u}_{e,Q})[v] + A_{1} + A_{2},$$

(64)

where

$$A_{1} = \int_{\Omega_{e}} (\nabla g \tilde{u}_{e,Q} \nabla v + \tilde{u}_{e,Q} v) dy; \quad A_{2} = \int_{\Omega_{e}} (\tilde{u}^p_{e,Q} - u^p_{e,Q}) v dy.$$ 

By Proposition 2.4 and in particular by (14) we have that $I'_{e}(\tilde{u}_{e,Q})[v]$ is of order at most $\epsilon^{2}$. Hence we only need to estimate $A_{1}$ and $A_{2}$ in the last line of (64).

To estimate $A_{1}$ we divide further $\tilde{u}_{e,Q} = \tilde{u}_{e,Q,1} + \tilde{u}_{e,Q,2} + \tilde{u}_{e,Q,3}$, where

$$\tilde{u}_{e,Q,1}(y) = \chi_{\mu_{0}}(\epsilon y) (\chi_{D}(y) - 1) U_{Q}(y); \quad \tilde{u}_{e,Q,2}(y) = \chi_{\mu_{0}}(\epsilon y) \chi_{D}(y) \Xi_{d}(y);$$

$$\tilde{u}_{e,Q,3}(y) = \chi_{\mu_{0}}(\epsilon y) \epsilon (\chi_{0}(y_{1} - d) - 1) w_{Q}(y).$$

Then we write $A_{1} = A_{1,1} + A_{1,2} + A_{1,3}$, with

$$A_{1,i} = \int_{\Omega_{e}} (\nabla g \tilde{u}_{e,Q,i} \nabla v + \tilde{u}_{e,Q,i} v) dy, \quad i = 1, 2, 3.$$ 

Since $\chi_{D}(y)$ is identically equal to 1 for $|y| \leq \frac{dD}{16}$ and since $\chi_{0}(y_{1} - d) - 1 = 0$ for $y_{1} \leq d$, from (6) and (13) we get

$$|A_{1,1}| \leq e^{-\frac{dD}{40} (1 + o(1))} \|v\|_{H^{1}_{D}(\Omega_{e})}; \quad |A_{1,3}| \leq C \epsilon (1 + |d| K) e^{-d} \|v\|_{H^{1}_{D}(\Omega_{e})}.$$ 

(65)

To control $A_{1,2}$ we write that

$$A_{1,2} = \int_{\Omega_{e}} (\nabla g \tilde{u}_{e,Q,2} \nabla v + \tilde{u}_{e,Q,2} v) dy = \int_{\Omega_{e}} (g^{ij} \partial_{i} \tilde{u}_{e,Q,2} \partial_{j} v + \tilde{u}_{e,Q,2} v) dy$$

$$= \int_{\Omega_{e}} (\nabla \tilde{u}_{e,Q,2} \nabla v + \tilde{u}_{e,Q,2} v) dy + \int_{\Omega_{e}} (g^{ij} - \delta^{ij}) \partial_{i} \tilde{u}_{e,Q,2} \partial_{j} v dy.$$ 

From the condition (c) in Subsection 2.2 we have that $|g^{ij} - \delta^{ij}| \leq C \epsilon |y|$; then

$$|A_{1,2}| \leq \int_{\Omega_{e}} (\nabla \tilde{u}_{e,Q,2} \nabla v + \tilde{u}_{e,Q,2} v) dy \leq C \epsilon \left( \int_{\Omega_{e}} |y|^{2} |\nabla \tilde{u}_{e,Q,2}|^{2} dy \right)^{\frac{1}{2}} \|v\|_{H^{1}_{D}(\Omega_{e})}.$$ 

Since the support of $\tilde{u}_{e,Q,2}$ is contained in the set $\{|y| \leq \frac{dD}{16}\}$, we obtain from the last formula and (46) that

$$|A_{1,2}| \leq C \epsilon d D e^{-d(1 + o(1))} \|v\|_{H^{1}_{D}(\Omega_{e})}.$$ 

Since $\Xi_{d}$ satisfies (14), we have

$$\int_{\Omega_{e}} (\nabla \tilde{u}_{e,Q,2} \nabla v + \tilde{u}_{e,Q,2} v) dy =$$

$$\int_{\Omega_{e}} (\nabla (\Xi_{d}(y) (\chi_{\mu_{0}}(\epsilon y) \chi_{D}(y) - 1)) \nabla v + \Xi_{d}(y) (\chi_{\mu_{0}}(\epsilon y) \chi_{D}(y) - 1) v) dy.$$ 

(66)
Since also $Dd < \frac{1}{c\epsilon^{2}}$, the function $\chi_{\mu_{0}}(\epsilon y)\chi_{D}(y) - 1$ is identically zero in the set $\{|y| \leq \frac{dD}{16}\}$ if $C_{1\eta}$ is sufficiently large. Then, using (47), (48) and the Hölder inequality, we find that (also for $D$ large)

$$\int_{\Omega_\epsilon} (\nabla (\Xi_{d}(y) (\chi_{\mu_{0}}(\epsilon y)\chi_{D}(y) - 1))\nabla v + \Xi_{d}(y) (\chi_{\mu_{0}}(\epsilon y)\chi_{D}(y) - 1)) v\,dy) \leq e^{-\frac{dD}{16} + \frac{2}{d^{2}}} \left( \frac{\tan\alpha}{\tan^{2} \alpha + 1} + \frac{2dC_{16\eta}}{\sqrt{\tan^{2} \alpha + 1}} \right) (1 + o(1)) \|v\|_{H_{\beta}^{1}(\Omega_\epsilon)}.$$  

(67)

The last three formulas imply

$$|A_{1,2}| \leq C\left( c\epsilon D e^{-d(1 + o(1))} + e^{-\frac{dD}{16} + \frac{2}{d^{2}}} \left( \frac{\tan\alpha}{\tan^{2} \alpha + 1} + \frac{2dC_{16\eta}}{\sqrt{\tan^{2} \alpha + 1}} \right) (1 + o(1)) \right)\|v\|_{H_{\beta}^{1}(\Omega_\epsilon)}.$$  

(68)

It remains to estimate $A_{2}$. First of all, let us recall that the following inequality holds:

$$|\bar{u}_{\epsilon,Q}^{p} - u_{\epsilon,Q}^{p}| \leq \begin{cases} C|\bar{u}_{\epsilon,Q}|^{p-1}|\bar{u}_{\epsilon,Q}| & \text{for } \bar{u}_{\epsilon,Q} \in (0, \frac{1}{2}] \bar{u}_{\epsilon,Q}, \\ C|\bar{u}_{\epsilon,Q}|^{p-1}|\bar{u}_{\epsilon,Q}| + C|\bar{u}_{\epsilon,Q}| & \text{otherwise,} \end{cases}$$  

(69)

for a fixed constant $C$ depending only on $p$. Moreover, using (8) and (13), we can say that there exists a small constant $c_{K,n}$ such that

$$\bar{u}_{\epsilon,Q}(y) \geq \frac{7}{8} \frac{e^{-|y|}}{1 + |y|^{\frac{1}{2} - \frac{1}{2}}} \quad \text{for } |y| \leq \frac{1}{c_{K,n}}.$$  

We divide next $\Omega_\epsilon$ into the two regions

$$B_{1} = \left\{ |y| < \min \left\{ \frac{d}{2}, \frac{1}{c_{K,n}} \right\} \right\}; \quad B_{2} = \Omega_\epsilon \setminus B_{1}.$$  

For $y \in B_{1}$ we have that $\chi_{\mu_{0}}(\epsilon y) \equiv 1$, $\chi_{D}(y) \equiv 1$, $\chi_{0}(y_{1} - d) \equiv 1$, and hence $\bar{u}_{\epsilon,Q}(y) \equiv -\Xi_{d}(y)$. By (47) we have also that $|\bar{u}_{\epsilon,Q}(y)| = |\Xi_{d}(y)| \leq e^{-\frac{d}{2} - \frac{2d\tan\alpha}{\sqrt{\tan^{2} \alpha + 1}} + o(1)} < \frac{1}{2} \bar{u}_{\epsilon,Q}$ for $y \in B_{1}$. This fact, (69) and the Hölder inequality yield

$$\int_{B_{1}} |\bar{u}_{\epsilon,Q}^{p} - u_{\epsilon,Q}^{p}|\,v\,dy \leq C \int_{B_{1}} |\bar{u}_{\epsilon,Q}|^{p-1}|\bar{u}_{\epsilon,Q}|\,|v|\,dy \leq Ce^{-\frac{d}{2} - \frac{2d\tan\alpha}{\sqrt{\tan^{2} \alpha + 1}} + o(1)} \|v\|_{H_{\beta}^{1}(\Omega_\epsilon)}.$$  

On the other hand, in $B_{2}$ we have that $|\bar{u}_{\epsilon,Q}^{p} - u_{\epsilon,Q}^{p}| \leq e^{-\frac{d}{2} + o(1)} + e^{-\frac{p-1}{2} + o(1)}$ and that $|\bar{u}_{\epsilon,Q}| \leq e^{-d + o(1)}$; therefore (69) and the Hölder inequality imply again

$$\int_{B_{2}} |\bar{u}_{\epsilon,Q}^{p} - u_{\epsilon,Q}^{p}|\,v\,dy \leq C \left[ e^{-\frac{d}{2} + o(1)} + e^{-\frac{p-1}{2} + o(1)} \right] e^{-d + o(1)} + e^{-d + o(1)} \|v\|_{H_{\beta}^{1}(\Omega_\epsilon)}.$$  

The last two formulas provide

$$|A_{2}| \leq C \left[ e^{-\frac{d}{2} - \frac{2d\tan\alpha}{\sqrt{\tan^{2} \alpha + 1}} + o(1)} + e^{-d + o(1)} + e^{-\frac{p-1}{2} + o(1)} e^{-d + o(1)} \right] \|v\|_{H_{\beta}^{1}(\Omega_\epsilon)}.$$  

(70)

Finally, we obtain the conclusion from (14), (64), (68) and (70).
We have next another estimate for the functional $I_\epsilon$, which allows to say that the condition \( ii \) in Subsection \( 2.1 \) holds true for $I_\epsilon$ and the manifold of the $u_{\epsilon,Q}$'s.

**Proposition 3.13.** Let $\mu_0$ be the constant appearing in Subsection \( 2.2 \). Then there exists another constant $C_{Q}$, independent of $\epsilon$, such that, for $C_{Q} \leq d \leq \frac{1}{2\epsilon C}$ and for $Dd < \frac{\mu_0}{\epsilon C}$, the functions $u_{\epsilon,Q}$ satisfy

\[
\frac{\mu_0}{\epsilon C}
\]

for some fixed $C > 0$ and for $\epsilon$ sufficiently small. In the above formula $q$ represents a vector in $H_1^T(\Omega_\epsilon)$ which is tangent to the manifold of the $u_{\epsilon,Q}$'s (when $Q$ varies).

**Proof.** Since the arguments are quite similar to those in the proof of Proposition \( 3.12 \) we will be rather quick. Using the fact that $\det (g^{ij}) = 1$ and the first line in \( 61 \), for any given test function $v \in H_1^T(\Omega_\epsilon)$ we can write that

\[
I'_\epsilon(u_{\epsilon,Q})[v] = \sum_{i,j} \int_{\mathbb{R}_+^n} (g^{ij} \partial_i u_{\epsilon,Q} \partial_j v + u_{\epsilon,Q} v) \, dy - \int_{\mathbb{R}_+^n} w_{\epsilon,Q}^p v \, dy.
\]

We want to differentiate next with respect to the parameter $Q$, taking first a variation $q_T$ of the point $Q$ for which $d$ stays fixed, namely we take the tangential derivative to the level set of the distance $d$ to the interface. Let us notice that in the above formula the dependence on $Q$ is in the metric coefficients $g^{ij}$ and in the function $w_{Q}$ appearing in the expression of $u_{\epsilon,Q}$ (see \( 61 \)). Therefore we obtain

\[
\frac{\partial}{\partial Q} I'_\epsilon(u_{\epsilon,Q})[v] = I''_\epsilon(u_{\epsilon,Q}) \left[ \frac{\partial u_{\epsilon,Q}}{\partial Q}, v \right] = \sum_{i,j} \int_{\mathbb{R}_+^n} \frac{\partial g^{ij}}{\partial Q} \partial_i u_{\epsilon,Q} \partial_j v \, dy + \sum_{i,j} \int_{\mathbb{R}_+^n} \left( g^{ij} \partial_i \frac{\partial u_{\epsilon,Q}}{\partial Q} \partial_j v + \frac{\partial u_{\epsilon,Q}}{\partial Q} v \right) \, dy - p \int_{\mathbb{R}_+^n} w_{\epsilon,Q}^p \frac{\partial u_{\epsilon,Q}}{\partial Q} v \, dy.
\]

From Remark \( 2.3 \) \( ii \) we have that $\frac{\partial g^{ij}}{\partial Q}$ is of order $\epsilon^2 |y|$. Moreover, computing the expression of $\frac{\partial u_{\epsilon,Q}}{\partial Q}$ we obtain $\frac{\partial^2 u_{\epsilon,Q}}{\partial Q^2} = \epsilon C_{Q_0} (\epsilon y) \chi_0 (y_1 - d) \frac{\partial w_Q}{\partial Q} = o \left( \epsilon^2 \left( 1 + |y|^k \right) e^{-|y|} \right)$, see Subsection \( 2.2 \) in \( [\text{GMMPT}] \). Reasoning as in the proof of Proposition \( 3.12 \) we then have

\[
\left\| \frac{\partial}{\partial Q} I'_\epsilon(u_{\epsilon,Q})[v] \right\| \leq C \epsilon^2 \left\| v \right\|_{H_1^T(\Omega_\epsilon)} \quad \text{for every } v \in H_1^T(\Omega_\epsilon).
\]

On the other hand, when we take a variation $q_d$ of $Q$ along the gradient of $d$, similarly to \( 72 \) we get

\[
\frac{\partial}{\partial Q_d} I'_\epsilon(u_{\epsilon,Q})[v] = I''_\epsilon(u_{\epsilon,Q}) \left[ \frac{\partial u_{\epsilon,Q}}{\partial Q_d}, v \right] = \sum_{i,j} \int_{\mathbb{R}_+^n} \frac{\partial g^{ij}}{\partial Q_d} \partial_i u_{\epsilon,Q} \partial_j v \, dy + \sum_{i,j} \int_{\mathbb{R}_+^n} \left( g^{ij} \partial_i \frac{\partial u_{\epsilon,Q}}{\partial Q_d} \partial_j v + \frac{\partial u_{\epsilon,Q}}{\partial Q_d} v \right) \, dy - p \int_{\mathbb{R}_+^n} w_{\epsilon,Q}^p \frac{\partial u_{\epsilon,Q}}{\partial Q_d} v \, dy.
\]
Concerning the derivatives of \( g^{ij} \) with respect to \( Q_d \) we can argue exactly as for \( Q_T \), to find
\[
\left| \sum_{i,j} \int_{\mathbb{R}_+^n} \frac{\partial g^{ij}}{\partial Q_d} \partial_t u_{r,Q_d} \partial_j v dy \right| \leq C e^2 \|v\|_{H^1_b(\Omega_t)}.
\]

Now, computing the derivative of \( u_{r,Q} \) with respect to \( Q_d \) is more complicated than the previous case, because \( \frac{\partial u_{r,Q}}{\partial Q_d} \) has a more involved expression. If we assume that the cut-off function \( \chi_D (y) \) is defined as \( \bar{\chi}_D \left( \frac{y}{\bar{y}} \right) \) for some fixed \( \bar{y} \), we obtain
\[
\frac{\partial u_{r,Q}}{\partial Q_d} = -\chi_{\mu_0} \chi_D \frac{\partial \Xi_d}{\partial d} + \frac{1}{\bar{y}^2} \chi_{\mu_0} (\Xi_d - U_Q) y \cdot \nabla \bar{\chi}_D \left( \frac{y}{\bar{y}} \right) + \epsilon \chi_{\mu_0} w_Q \frac{\partial \chi_0 (y_1 - d)}{\partial Q_d} + \epsilon \chi_{\mu_0} \chi_0 (y_1 - d) \frac{\partial w_Q}{\partial Q_d}.
\]
(75)

It is easy to see that the last two terms in the right hand side give a contribution to \([74]\) of order at most \( \epsilon e^{d(1+o(1))} \|v\|_{H^1_b(\Omega_t)} \) and \( e^2 \epsilon e^{d(1+o(1))} \|v\|_{H^1_b(\Omega_t)} \), respectively. Concerning the second one, we can use the fact that the support of \( \nabla \chi_D \) is contained in the set \( \{ |y| \geq \frac{dD}{16} \} \), together with \([47], [48]\) to see that the contribution of this term is at most of order
\[
\left( -\frac{d^2}{4D^2} + \frac{e^2}{2} \frac{\frac{\sqrt{2} \tan \frac{\alpha}{2}}{\tan^2 \alpha + 1} + \frac{2 \tan \alpha}{\tan^2 \alpha + 1}}{\sqrt{\tan^2 \alpha + 1}} (1+o(1)) \right) + C e^{d(1+o(1))} \|v\|_{H^1_b(\Omega_t)}.
\]
(76)

We can then focus on the first term in the right hand side of \([75]\), and consider the quantity
\[
-\sum_{i,j} \int_{\mathbb{R}_+^n} \left( g^{ij} \partial_t \left( \chi_{\mu_0} \chi_D \frac{\partial \Xi_d}{\partial d} \right) \right) \partial_j v + \chi_{\mu_0} \chi_D \frac{\partial \Xi_d}{\partial d} v dy + p \int_{\mathbb{R}_+^n} u_{r,Q}^{-1} \chi_{\mu_0} \chi_D \frac{\partial \Xi_d}{\partial d} v dy.
\]
(76)

Now, using condition (c) in Subsection \([2.2]\) and \([56]\), if we substitute the coefficients \( g^{ij} \) with the Kronecker symbols we find a difference of order \( (1 + \epsilon) e^{-d(1+o(1))} + (1 + \epsilon) e^{-d(1+o(1))} ) \). Next, since \( \Xi_d \) satisfies \( -\Delta \Xi_d + \Xi_d = 0 \), when we differentiate with respect to \( d \) we get the same equation for \( \frac{\partial \Xi_d}{\partial d} \), so reasoning as for \([66, 67]\), together with \([57], [58]\), we find
\[
\left| \int_{\mathbb{R}_+^n} \left( \nabla \left( \chi_{\mu_0} \chi_D \frac{\partial \Xi_d}{\partial d} \right) \cdot \nabla v + \chi_{\mu_0} \chi_D \frac{\partial \Xi_d}{\partial d} v \right) dy \right| \leq C e^{-d(1+o(1))} \left( e^{-d+e^{-d(1+o(1))}} + e^{-d(1+o(1))} \right) \|v\|_{H^1_b(\Omega_t)}.
\]
(77)

It remains to estimate the last term in \([76]\). Using \([56], [57]\) and the exponential decay of \( u_{r,Q} \) and reasoning with argument similar to those for \([70]\), we find that it is of order
\[
\left( e^{-d(1+o(1))} \right) \left( e^{-d+e^{-d(1+o(1))}} + e^{-d(1+o(1))} \right) \|v\|_{H^1_b(\Omega_t)}.
\]
(77)

All the above comments yield that
\[
\left\| \frac{\partial}{\partial Q_d} \Pi'_{r,Q} [v] \right\| \leq C \left( e^2 + \epsilon e^{-d(1+o(1))} + e^{-d(1+o(1))} \right) \|v\|_{H^1_b(\Omega_t)} + C \left( e^{-2d(1+o(1))} + e^{-d(1+o(1))} \right) \|v\|_{H^1_b(\Omega_t)}.
\]
(77)

From \([73]\) and \([77]\) we finally obtain the desired conclusion.
3.3.2 Case $\frac{\pi}{2} \leq \alpha \leq \pi$

In this subsection we introduce the manifold of approximate solutions in the case $\frac{\pi}{2} \leq \alpha \leq \pi$. Since the construction is substantially the same as in the previous subsection, we will be rather sketchy.

Let us consider the solution of (19), $\Phi^d$, and the function $\Xi_d$ defined in (43). Reasoning as at the beginning of the Subsection 3.3.1, we derive norm estimate for $\Xi_d$:

$$||\Xi_d||_{H^1(d(\Sigma_d-Q_0))} \leq e^{-d(1+o(1))}. $$

Moreover, from Proposition 3.10 we also obtain pointwise estimates for $\Xi_d$ and its gradient.

Now, using the cut-off functions (59), (60), we define, in the new coordinates $y$ introduced in Subsection 2.2, the functions

$$ u_{\epsilon,Q}(y) := \chi_{\mu_0}(\epsilon y) [U_Q(y) - \Xi_d(y)] \chi_D(y) + \epsilon w_Q(y) \chi_0(y_1-d). $$

Following the line of the Subsection 3.3.1, we prove that the $u_{\epsilon,Q}$'s are good approximate solutions to (10) for suitable conditions of $Q$. Since the computations in the following proposition are the same as in Proposition 3.12 and Proposition 3.13 in [GMP1], we will omit the proof.

**Proposition 3.14.** Let $\mu_0$ be the constant appearing in Subsection 2.2. Then there exists another constant $C_{\Omega_1}$, independent of $\epsilon$, such that, for $C_{\Omega_1} \leq d \leq \frac{\mu_0}{\rho_0 \alpha}$ and for $Dd < \frac{\mu_0}{\rho_0 \alpha}$, the functions $u_{\epsilon,Q}$ satisfy

$$ ||I'(u_{\epsilon,Q})|| \leq C \left( \epsilon^2 + \epsilon e^{-d(1+o(1))} + e^{-\frac{d\alpha+1}{2}(1+o(1))} + e^{-\frac{3}{4}d(1+o(1))} \right), $$

and

$$ ||I''(u_{\epsilon,Q})(q)|| \leq C \left( \epsilon^2 + \epsilon \exp^{-d(1+o(1))} + e^{-\frac{d\alpha+1}{2}(1+o(1))} + e^{-\frac{3}{4}d(1+o(1))} \right) ||q||, $$

for some fixed $C > 0$ and for $\epsilon$ sufficiently small. In (79) $q$ represents a vector in $H_D^1(\Omega_\epsilon)$ which is tangent to the manifold of the $u_{\epsilon,Q}$'s (when $Q$ varies).

3.3.3 Case $\pi < \alpha < 2\pi$

In this subsection we introduce the manifold of approximate solutions in the case $\pi < \alpha < 2\pi$. Also in this case we will be very quick, since the construction is the same as in the previous subsections.

Let us consider the solution of (19), $\Phi^d$, and the function $\Xi_d$ defined in (43). Reasoning as at the beginning of the Subsection 3.3.1, we derive norm estimate for $\Xi_d$:

$$ ||\Xi_d||_{H^1(d(\Sigma_d-Q_0))} \leq e^{-d(1+o(1))}. $$

Moreover, from Proposition 3.11 we also obtain pointwise estimates for $\Xi_d$ and its gradient.

Now, using the cut-off functions (59), (60), we define, in the new coordinates $y$ introduced in Subsection 2.2, the functions

$$ u_{\epsilon,Q}(y) := \chi_{\mu_0}(\epsilon y) [U_Q(y) - \Xi_d(y)] \chi_D(y) + \epsilon w_Q(y) \chi_0(y_1-d). $$

Following the line of the Subsection 3.3.1, we obtain that the $u_{\epsilon,Q}$'s are good approximate solutions to (10) for suitable conditions of $Q$. Since the computations in the following proposition are very similar to those in Proposition 3.12 and Proposition 3.13 in [GMP1], we will omit the proof.
Proposition 3.15. Let $\mu_0$ be the constant appearing in Subsection 2.2. Then there exists another constant $C_1$, independent of $\epsilon$, such that, for $C_1 \leq d \leq \frac{1}{\epsilon C_0}$ and for $Dd < \frac{1}{\epsilon C_0}$, the functions $u_{\epsilon,Q}$ satisfy

$$\|I'_\epsilon(u_{\epsilon,Q})\| \leq C \left( e^2 + \epsilon e^{-d(1+o(1))} + \epsilon^{-\frac{d(p+1)}{2}(1+o(1))} + e^{-\frac{C_1}{2}(1+o(1))} \right),$$

and

$$\|I''_\epsilon(u_{\epsilon,Q})[q]\| \leq C \left( e^2 + \epsilon \exp^{-d(1+o(1))} + \epsilon^{-\frac{d(p+1)}{2}(1+o(1))} + e^{-\frac{C_1}{2}(1+o(1))} \right)\|q\|,$n

for some fixed $C > 0$ and for $\epsilon$ sufficiently small. In (81), $q$ represents a vector in $H^1_0(\Omega_\epsilon)$ which is tangent to the manifold of the $u_{\epsilon,Q}$’s (when $Q$ varies).

4 Proof of Theorem 1.1

To prove our main Theorem we need to derive an expansion in terms of $Q$ and $\epsilon$ of the energy of approximate solutions. Then we can apply the abstract theory in Subsection 2.2 to obtain the existence result.

In the case $\frac{\pi}{2} \leq \alpha \leq \pi$ the energy expansions for the approximate solutions $u_{\epsilon,Q}$ are the same as in the case $\alpha = \pi$, see Proposition 4.1 and Proposition 4.2 in [GMMMP1]. Then also the definition of the critical manifold and the study of the reduced functional are the same. Therefore for the proof of Theorem 1.1 in the case $\frac{\pi}{2} \leq \alpha \leq \pi$ we refer the reader to Section 4 in [GMMMP1].

In the case $\pi < \alpha < 2\pi$, even if the approximate solutions are different from the previous case, the energy expansions turn out to be the same. Then also in this case we omit the proof of Theorem 1.1 and refer the reader to Section 4 in [GMMMP1].

In the case $0 < \alpha < \frac{\pi}{2}$ the energy expansions are quite different, so we will give the proof in the details.

From now on we will assume $0 < \alpha < \frac{\pi}{2}$.

4.1 Energy expansions for the approximate solutions $u_{\epsilon,Q}$

Here we expand $I_\epsilon (u_{\epsilon,Q})$ in terms of $Q$ and $\epsilon$, where $u_{\epsilon,Q}$ is the function defined in (61).

Proposition 4.1. For $\epsilon \to 0$ and $d = d(Q) \to +\infty$, the following expansion holds

$$I_\epsilon (u_{\epsilon,Q}) = \tilde{C}_0 - \tilde{C}_1 \epsilon H(\epsilon Q) + e^{-2d(1+o(1))} + e^{-\frac{d}{2}(\tan^{-1}(\epsilon Q))^{1+o(1)}} + o(\epsilon^2),$$

where $\tilde{C}_0$ and $\tilde{C}_1$ are the constants in Proposition 2.4.

Proof. As in the proof of Proposition 3.12, let us write $u_{\epsilon,Q}(y) = \tilde{u}_{\epsilon,Q}(y) + \bar{u}_{\epsilon,Q}(y)$, see (12) and (63). Then, using the coordinates $y$ introduced in Subsection 2.2, we find that

$$I_\epsilon (u_{\epsilon,Q}) = I_\epsilon (\tilde{u}_{\epsilon,Q}) + \int_{\Omega_\epsilon} (\nabla_g \tilde{u}_{\epsilon,Q} \nabla_g \tilde{u}_{\epsilon,Q} + \tilde{u}_{\epsilon,Q} \tilde{u}_{\epsilon,Q}) dy + \frac{1}{p+1} \int_{\Omega_\epsilon} (|\tilde{u}_{\epsilon,Q}|^{p+1} - |u_{\epsilon,Q}|^{p+1}) dy$$

$$+ \frac{1}{p+1} \int_{\Omega_\epsilon} (|\bar{u}_{\epsilon,Q}|^{p+1} - |\bar{u}_{\epsilon,Q}|^{p+1}) dy$$

Using condition (c) in Subsection 2.2 we have that

$$\left| \int_{\Omega_\epsilon} (\nabla_g \tilde{u}_{\epsilon,Q} \nabla_g \tilde{u}_{\epsilon,Q} + \tilde{u}_{\epsilon,Q} \tilde{u}_{\epsilon,Q}) dy - \int_{\mathbb{R}_+^2} (\nabla \tilde{u}_{\epsilon,Q} \nabla \tilde{u}_{\epsilon,Q} + \tilde{u}_{\epsilon,Q} \tilde{u}_{\epsilon,Q}) dy \right|$$

$$\leq C \epsilon \int_{\mathbb{R}_+^2} |\nabla \tilde{u}_{\epsilon,Q}| \|\nabla \tilde{u}_{\epsilon,Q}\| dy;$$
we find that
\[ v \]
from which we deduce that
\[ \text{Formulas (8) and (13) imply } \]
The last two formulas, (83), (84), (85) imply
\[ \text{(8), (13), (46), (47), (48) to find } \]
\[
\int_{R^n_+} \left( |\nabla g \tilde{u}_{e,Q}|^2 + \tilde{u}_{e,Q}^2 \right) dy - \int_{R^n_+} \left( |\nabla \tilde{u}_{e,Q}|^2 + \tilde{u}_{e,Q}^2 \right) dy \leq C \epsilon \int_{R^n_+} |y||\nabla \tilde{u}_{e,Q}|^2 dy. \quad (85)
\]
Concerning (84), we can divide the domain of integration into \( B_{\frac{2}{3}} (0) \) and its complement and use [9], (13), (46), (47), (48) to find
\[
C \epsilon \int_{R^n_+} |y||\nabla \tilde{u}_{e,Q}||\nabla \tilde{u}_{e,Q}|dy \leq C \epsilon \left( e^{-\frac{2}{d}(1+o(1))} + e^{-d \left( \frac{\sqrt{7} \tan \alpha}{\sqrt{\tan^2 \alpha + 1}} \right)(1+o(1))} \right).
\]
For (85), the same estimates yield
\[
C \epsilon \int_{R^n_+} |y||\nabla \tilde{u}_{e,Q}|^2 dy \leq C \epsilon \left( e^{-2d(1+o(1))} + e^{-d \left( \frac{\sqrt{7} \tan \alpha}{\sqrt{\tan^2 \alpha + 1}} \right)(1+o(1))} \right).
\]
The last two formulas, (83), (84), (85) imply
\[
I_e (u_{e,Q}) = I_e (\tilde{u}_{e,Q}) + \int_{R^n_+} (\nabla \tilde{u}_{e,Q} \nabla \tilde{u}_{e,Q} + \tilde{u}_{e,Q} \tilde{u}_{e,Q}) dy + \frac{1}{\sqrt{2}} \int_{R^n_+} (|\nabla \tilde{u}_{e,Q}|^2 + \tilde{u}_{e,Q}^2) dy
\]
\[
+ \frac{1}{p+1} \int_{\Omega_e} \left( |\tilde{u}_{e,Q}|^{p+1} - |u_{e,Q}|^{p+1} \right) dy + o \left( \epsilon \left( e^{-\frac{2}{d}(1+o(1))} + e^{-d \left( \frac{\sqrt{7} \tan \alpha}{\sqrt{\tan^2 \alpha + 1}} \right)(1+o(1))} \right) \right). \quad (86)
\]
Using the same notation as in the proof of Proposition 3.12 we write \( \tilde{u}_{e,Q} = \tilde{u}_{e,Q,1} + \tilde{u}_{e,Q,2} + \tilde{u}_{e,Q,3} \). Formulas (8) and (13) imply
\[
\left| \int_{R^n_+} (\nabla \tilde{u}_{e,Q} \nabla \tilde{u}_{e,Q,1} + \tilde{u}_{e,Q} \tilde{u}_{e,Q,1}) dy \right| \leq C e^{-\frac{4d}{\sqrt{p}}(1+o(1))};
\]
\[
\left| \int_{R^n_+} (\nabla \tilde{u}_{e,Q} \nabla \tilde{u}_{e,Q,3} + \tilde{u}_{e,Q} \tilde{u}_{e,Q,3}) dy \right| \leq C \epsilon e^{-2d(1+o(1))},
\]
from which we deduce that
\[
\int_{R^n_+} (\nabla \tilde{u}_{e,Q} \nabla \tilde{u}_{e,Q} + \tilde{u}_{e,Q} \tilde{u}_{e,Q}) dy = \int_{R^n_+} (\nabla \tilde{u}_{e,Q} \nabla \tilde{u}_{e,Q,2} + \tilde{u}_{e,Q} \tilde{u}_{e,Q,2}) dy
\]
\[
+ o \left( \epsilon \left( e^{-\frac{4d}{\sqrt{p}}(1+o(1))} + e^{-2d(1+o(1))} \right) \right).
\]
Similar estimates also yield
\[
\int_{R^n_+} (|\nabla \tilde{u}_{e,Q}|^2 + \tilde{u}_{e,Q}^2) dy = \int_{R^n_+} (|\nabla \tilde{u}_{e,Q,2}|^2 + \tilde{u}_{e,Q,2}^2) dy + o \left( \epsilon \left( e^{-\frac{4d}{\sqrt{p}}-d(1+o(1))} + e^{-2d(1+o(1))} \right) \right).
\]
From a straightforward computation one finds that for any function \( v \)
\[ \nabla \tilde{u}_{e,Q,2} \nabla v + \tilde{u}_{e,Q,2}^2 v = \nabla \Xi_d \cdot \nabla (\chi_{\mu_0} (\epsilon) \chi_D v) + \Xi_d \chi_{\mu_0} (\epsilon) \chi_D v + \nabla (\chi_{\mu_0} (\epsilon) \chi_D) (\Xi_d \nabla v - v \nabla \Xi_d). \]
Applying this relation for \( v = \tilde{u}_{e,Q} \) and \( v = \tilde{u}_{e,Q,2} \) respectively, and using (8), (13), (46), (47) and (48) we find that
\[
\int_{R^n_+} (\nabla \tilde{u}_{e,Q} \nabla \tilde{u}_{e,Q,2} + \tilde{u}_{e,Q} \tilde{u}_{e,Q,2}) dy = \int_{R^n_+} (\nabla (\chi_{\mu_0} (\epsilon) \chi_D \tilde{u}_{e,Q}) \nabla \Xi_d + \nabla (\chi_{\mu_0} (\epsilon) \chi_D \tilde{u}_{e,Q} \Xi_d) dy
\]
\[
+ o \left( \epsilon \left( e^{-\frac{2d}{\sqrt{p}}(1+o(1))} + e^{-2d(1+o(1))} + e^{-\frac{4d}{\sqrt{p}}(1+o(1))} \right) \right).
\]
\[
\int_{\Omega^n_+} \left( |\nabla \bar{u}_{e,Q}|^2 + \bar{u}_{e,Q}^2 \right) dy = \int_{\Omega^n_+} \left( |\nabla (\chi_{\mu_0} (\cdot) \chi_D \Xi_d) |^2 + (\chi_{\mu_0} (\cdot) \chi_D \Xi_d)^2 \right) dy.
\]

Using now the fact that, by our construction, the function \( \chi_{\mu_0} (\cdot) \chi_D u_{e,Q} = \chi_{\mu_0} (\cdot) \chi_D (\bar{u}_{e,Q} + \bar{u}_{e,Q}) \) vanishes on \( d(\partial \Sigma_D - Q_0) \), from (44) we obtain
\[
\int_{\Omega^n_+} \left( \nabla (\chi_{\mu_0} (\cdot) \chi_D \bar{u}_{e,Q}) \nabla \Xi_d + \chi_{\mu_0} (\cdot) \chi_D \bar{u}_{e,Q} \Xi_d \right) dy \\
+ \frac{1}{2} \int_{\Omega^n_+} \left( |\nabla (\chi_{\mu_0} (\cdot) \chi_D \Xi_d) |^2 + (\chi_{\mu_0} (\cdot) \chi_D \Xi_d)^2 \right) dy \\
= \frac{1}{2} \int_{\Omega^n_+} \left( \nabla (\chi_{\mu_0} (\cdot) \chi_D \bar{u}_{e,Q}) \nabla \Xi_d + \chi_{\mu_0} (\cdot) \chi_D \bar{u}_{e,Q} \Xi_d \right) dy.
\]

From (86) and the last eight formulas we find
\[
I_e (u_{e,Q}) = I_e (\bar{u}_{e,Q}) + \frac{1}{2} \int_{\Omega^n_+} \left( \nabla \bar{u}_{e,Q} \nabla \bar{u}_{e,Q} + \bar{u}_{e,Q} \bar{u}_{e,Q} \right) dy + \frac{1}{p + 1} \int_{\Omega_e} |[\bar{u}_{e,Q}]^{p+1} - |u_{e,Q}|^{p+1} dy \\
+ o \left( e^{-\frac{d(1+\alpha)}{2} (1+1)} + e^{-\frac{\sqrt{d\tan \alpha}}{\sqrt{\tan^2 \alpha + 1}} (1+1)} + e^{-\frac{d(1+\alpha)}{2} (1+1)} \right).
\]

From (8), (13), (14) and (46) we have that
\[
\int_{\Omega^n_+} \left( \nabla \bar{u}_{e,Q} \nabla \bar{u}_{e,Q} + \bar{u}_{e,Q} \bar{u}_{e,Q} \right) dy = I'_e (\bar{u}_{e,Q}) \left[ \bar{u}_{e,Q} \right] + \int_{\Omega_e} |\bar{u}_{e,Q}|^p \bar{u}_{e,Q} dy \\
\leq C e^2 e^{-d(1+1)} + \int_{\Omega_e} |\bar{u}_{e,Q}|^p \bar{u}_{e,Q} dy,
\]

and then
\[
I_e (u_{e,Q}) = I_e (\bar{u}_{e,Q}) + \frac{1}{2} \int_{\Omega_e} |\bar{u}_{e,Q}|^p \bar{u}_{e,Q} dy + \frac{1}{p + 1} \int_{\Omega_e} |[\bar{u}_{e,Q}]^{p+1} - |u_{e,Q}|^{p+1} dy \\
+ o \left( e^{-\frac{d(1+\alpha)}{2} (1+1)} + e^{-\frac{\sqrt{d\tan \alpha}}{\sqrt{\tan^2 \alpha + 1}} (1+1)} + e^{-\frac{d(1+\alpha)}{2} (1+1)} \right) \\
+ o \left( e^2 e^{-d(1+1)} \right).
\]

Using a Taylor expansion we can write that
\[
|\bar{u}_{e,Q}|^{p+1} - |u_{e,Q}|^{p+1} = \left\{ \begin{array}{ll}
-(p+1) |\bar{u}_{e,Q}|^p \bar{u}_{e,Q} + o \left( |\bar{u}_{e,Q}|^{p-1} \bar{u}_{e,Q} \right) & \text{for } \bar{u}_{e,Q} \in \left( 0, \frac{1}{2} \bar{u}_{e,Q} \right), \\
0 & \text{otherwise},
\end{array} \right.
\]

As for the estimate of \( A_2 \) in (70), we divide the domain into the two regions \( B_1, B_2 \), and deduce that
\[
\frac{1}{p + 1} \int_{\Omega_e} |[\bar{u}_{e,Q}]^{p+1} - |u_{e,Q}|^{p+1} dy = - \int_{\Omega_e} |\bar{u}_{e,Q}|^p \bar{u}_{e,Q} dy \\
+ o \left( e^{-\frac{d(1+1)}{2}} - \frac{2 d\tan \alpha}{\sqrt{\tan^2 \alpha + 1}} (1+1) \right) + e^{-\frac{d(1+1)}{2}} (1+1) + e^{-d(1+1)} \left( 1 - \frac{1}{e^{1/2}} \right).
\]

Therefore using (87) the energy becomes
\[
I_e (u_{e,Q}) = I_e (\bar{u}_{e,Q}) - \frac{1}{2} \int_{\Omega_e} |\bar{u}_{e,Q}|^p \bar{u}_{e,Q} dy + o \left( e^{-\frac{d(1+1)}{2}} - \frac{2 d\tan \alpha}{\sqrt{\tan^2 \alpha + 1}} (1+1) \right) + e^{-\frac{d(1+1)}{2}} (1+1) \\
+ o \left( e^{-\frac{d(1+1)}{2}} + e^{-\frac{2 d\tan \alpha}{\sqrt{\tan^2 \alpha + 1}} (1+1)} + e^{-\frac{d(1+1)}{2}} (1+1) + e^{-d(1+1)} \right).
\]
From (47), the expression of \( \tilde{u}_{r,Q} \) and the estimates in the same spirit as above one finds that

\[
\int_{\Omega_{r}} |\tilde{u}_{r,Q}|^{p} \tilde{u}_{r,Q} dy = - \left( e^{-2d(1+o(1))} + e^{-d - \frac{2\sqrt{d} \tan \alpha}{\sqrt{\tan^{2} \alpha + 1}} (1+o(1))} \right),
\]

and hence from Proposition 2.4 we finally find

\[
I_{e} (u_{r,Q}) = \tilde{C}_{0} - \tilde{C}_{1} e H (eQ) + O (\epsilon^{2}) + e^{-2d(1+o(1))} + e^{-d - \frac{2\sqrt{d} \tan \alpha}{\sqrt{\tan^{2} \alpha + 1}} (1+o(1))}
\]

\[
+ o \left( e^{-d - \frac{2\sqrt{d} \tan \alpha}{\sqrt{\tan^{2} \alpha + 1}} (1+o(1))} \right)
\]

\[
+ o \left( e^{-d - \frac{2\sqrt{d} \tan \alpha}{\sqrt{\tan^{2} \alpha + 1}} (1+o(1))} \right) + e^{2} e^{-d(1+o(1))}. \tag{89}
\]

The conclusion follows from the Schwartz inequality.

We give also a related result about the computation of the derivative of the energy with respect to \( Q \). Again, we will be rather sketchy in the proof since the arguments are quite similar to the previous ones.

**Proposition 4.2.** For \( \epsilon \to 0 \) and \( d = d(Q) \to +\infty \), the following expansions hold

\[
\frac{\partial}{\partial Q_{T}} I_{e} (u_{r,Q}) = - \tilde{C}_{1} e^{2} \nabla_{T} H (eQ) + o (\epsilon^{2}); \tag{90}
\]

\[
\frac{\partial}{\partial Q_{d}} I_{e} (u_{r,Q}) = - \tilde{C}_{1} e^{2} \nabla_{d} H (eQ) - e^{-d - \frac{2\sqrt{d} \tan \alpha}{\sqrt{\tan^{2} \alpha + 1}} (1+o(1))} + o (\epsilon^{2}), \tag{91}
\]

where \( \tilde{C}_{0} \) and \( \tilde{C}_{1} \) are the constants in Proposition 2.4.

**Proof.** After some elementary calculations, recalling the definition of \( \tilde{u}_{r,Q} \) in (12), we can write

\[
I'_{e} (u_{r,Q}) \left[ \frac{\partial u_{r,Q}}{\partial Q} \right] = \frac{\partial}{\partial Q} I_{e} (u_{r,Q}) + \int_{\Omega} \left( \nabla_{g} \tilde{u}_{r,Q} \nabla g \frac{\partial \tilde{u}_{r,Q}}{\partial Q} + \tilde{u}_{r,Q} \frac{\partial \tilde{u}_{r,Q}}{\partial Q} \right) dy - \int_{\Omega} \tilde{u}_{r,Q}^{p} \frac{\partial \tilde{u}_{r,Q}}{\partial Q} dy
\]

\[
+ \int_{\Omega} \left( \nabla_{g} \tilde{u}_{r,Q} \nabla g \frac{\partial \tilde{u}_{r,Q}}{\partial Q} + \tilde{u}_{r,Q} \frac{\partial \tilde{u}_{r,Q}}{\partial Q} \right) dy + \int_{\Omega} \left( \tilde{u}_{r,Q} - u_{r,Q}^{p} \right) \frac{\partial u_{r,Q}}{\partial Q} dy. \tag{92}
\]

where \( \tilde{u}_{r,Q} = u_{r,Q} - \tilde{u}_{r,Q} \) was defined in (63). The first term on the right hand side is estimated in Proposition 2.4. The next two, integrating by parts and using Proposition 2.4 can be estimated in terms of a quantity like

\[
C \epsilon^{2} \int_{\Omega_{r}} (1 + |y|^{K}) \frac{\partial \tilde{u}_{r,Q}}{\partial Q} dy.
\]

From the same arguments as in the proof of Proposition 3.13 one deduces that the latter integral is of order \( \epsilon^{2} \left( e^{-2d(1+o(1))} + e^{-d - \frac{2\sqrt{d} \tan \alpha}{\sqrt{\tan^{2} \alpha + 1}} (1+o(1))} \right) \). To control the first integral in the last line of (92) we can reason as for the estimate of \( A_{1,2} \) in the proof of Proposition 3.12 to see that this is of order \( e^{-d(1+o(1))} (\epsilon + e^{-d(1+o(1))}) \left\| \frac{\partial u_{r,Q}}{\partial Q} \right\|_{H^{1}_{0}(\Omega_{r})} \). From the proof of Proposition 3.13 one can deduce that \( \left\| \frac{\partial u_{r,Q}}{\partial Q} \right\|_{H^{1}_{0}(\Omega_{r})} \leq C (\epsilon^{2} + e^{-d(1+o(1))}) \), and hence the integral under interest is controlled by

\[
o (\epsilon^{2}) + e^{-3d(1+o(1))}.
\]
Finally, the last term in (92) can be estimated using a Taylor expansion as for the term $A_2$ in the proof of Proposition 3.12 and up to higher order is given by

$$p \int_{\mathbb{R}^n} U_Q^{p-1}(y) \frac{\partial}{\partial q} U_Q(y) \cdot q \, dy,$$

where $q$ stands either for the variation of $Q$ in the coordinates $y$. If $q$ preserves $d$, the latter integral gives a negligible contribution, and we find (90). If instead $q$ is directed toward the gradient of $d$ the above estimates (and in particular (47)) allow to deduce (91).

\[\square\]

### 4.2 Finite-dimensional reduction and study of the constrained functional

We apply now the abstract setting described in Subsection 2.1. In fact, the following two Lemmas hold.

**Lemma 4.3.** If $C_\Omega$ is as in the previous section and if we choose

$$Z_\epsilon = \left\{ u_{\epsilon,Q} : C_\Omega < d < \frac{1}{\epsilon C_\Omega} \right\},$$

then the properties i), iii) and iv) in Subsection 2.1 hold true, with $\gamma = \min \{1, p - 1\}$.

**Proof.** It is immediate to prove that i) and iii) hold; in particular, the value of $\gamma$ comes from the standard properties of Nemitski operators. Property iv) can be easily deduced from the fact that the kernel of the linearization of (5) in the half space is spanned by $\partial U / \partial x_1, \ldots, \partial U / \partial x_{n-1}$, as proved in [Oh], and from some localization arguments which can be found in Subsections 4.2, 9.2 and 9.3 of [AM].

**Lemma 4.4.** For any small positive constant $\delta$, if we take

$$Z_\epsilon = \left\{ u_{\epsilon,Q} : (2 - \delta) |\log \epsilon| < d < \frac{1}{\epsilon C_\Omega} \right\},$$

then also property ii) in Subsection 2.1 holds true, with

$$f(\epsilon) = \epsilon \min \left\{ 3 - \delta, \frac{2\delta}{2 - \delta}, (2 - \delta) \left( \frac{1}{2} \sqrt{\frac{\tan \alpha (\tan^2 \alpha - 1)}{\tan^2 \alpha + 1}} + \frac{2\tan \alpha}{\sqrt{\tan^2 \alpha + 1}} \right) (2 - \delta) \left( \frac{1}{2} + \frac{2\tan \alpha}{\sqrt{\tan^2 \alpha + 1}} \right) \right\}.$$ 

**Proof.** This lemma simply follows from Propositions 3.12 and 3.13.

As a corollary of the above two lemmas we can apply Proposition 2.1 and Theorem 2.2, so we expand next the reduced functional and its gradient on the natural constraint $\tilde{Z}_\epsilon$.

**Proposition 4.5.** With the choice of $\tilde{Z}_\epsilon$ in Lemma 4.4, if $w_\epsilon$ is given by Proposition 2.1, then we have

$$I_\epsilon (u_{\epsilon,Q}) := I_\epsilon (u_{\epsilon,Q} + w_\epsilon (u_{\epsilon,Q})) = \tilde{C}_0 - \tilde{C}_1 \epsilon H (\epsilon Q) + e^{-2d(1+o(1))} + e^{-d \left( 1 + \frac{2\tan \alpha}{\sqrt{\tan^2 \alpha + 1}} \right) (1+o(1))} + o(\epsilon^2);$$

(93)
\[
\frac{\partial}{\partial Q_T} I_e (u_e, Q) = -\tilde{C}_1 \epsilon^2 \nabla_T H (\epsilon Q) + o (\epsilon^2) ; \\
\frac{\partial}{\partial Q_d} I_e (u_e, Q) = -\tilde{C}_1 \epsilon^2 \nabla_d H (\epsilon Q) + e^{-d\left(\frac{\sqrt{2 \tan \alpha}}{\tan^2 \alpha + 1}\right)/(1+o(1))} + o (\epsilon^2),
\]

as \(\epsilon \to 0\), where \(\tilde{C}_0\) and \(\tilde{C}_1\) are as in Proposition 4.1 and where \(Q_T, Q_d\) are as in the proof of Proposition 3.13.

Proof. By Propositions 2.1 and 3.12 we have that

\[
\|w_e (u_e, Q)\| \leq C_1 \|I_e' (u_e, Q)\| \leq C \left( \epsilon^2 + \epsilon e^{-d(1+o(1))} \right) + C \left( e^{-d\left(\frac{\sqrt{2 \tan \alpha}}{\tan^2 \alpha + 1}\right)/(1+o(1))} + e^{-d\left(\frac{2 \sqrt{2 \tan \alpha}}{\tan^2 \alpha + 1}\right)/(1+o(1))} \right).
\]

From the regularity of \(I_e\) and Proposition 4.1 we then have

\[
I_e (u_e, Q + w_e (u_e, Q)) = I_e (u_e, Q) + I_e' (u_e, Q) [w_e (u_e, Q)] + o \left( \|w_e (u_e, Q)\|^2 \right)
\]

\[
= \tilde{C}_0 - \tilde{C}_1 \epsilon H (\epsilon Q) + e^{-2d(1+o(1))} + e^{-d\left(\frac{\sqrt{2 \tan \alpha}}{\tan^2 \alpha + 1}\right)/(1+o(1))} + o (\epsilon^2)
\]

\[
+ o \left( \epsilon^{6-2\delta} + \epsilon^{(p+1)(2-\delta)} + \epsilon^{2-\delta} \left( \frac{\sqrt{2 \tan \alpha}}{\tan^2 \alpha + 1} \right) + \epsilon^{2-\delta} \left( \frac{2 \sqrt{2 \tan \alpha}}{\tan^2 \alpha + 1} \right) \right).
\]

This immediately gives (93), since \(p > 1\) and \(\delta\) is small.

The remaining two estimates are also rather immediate for \(p \geq 2\) in fact in this case property iii) in Subsection 2.1 holds true for \(\gamma = 1\), so we also have \(\|\partial Q w_e\| \leq C f (\epsilon)\) by the last statement in Proposition 2.1. This, together with the Lipschitzianity of \(I_e'\) implies that

\[
\frac{\partial}{\partial Q} I_e (u_e, Q) = I_e' (u_e, Q) [\partial Q u_e, Q + \partial Q w_e] = \frac{\partial}{\partial Q} I_e (u_e, Q)
\]

\[
+ I_e'' (u_e, Q) [w_e, \partial Q w_e] + I_e'' (u_e, Q) [w_e, \partial Q w_e] + \|w_e\|^{-1} (\|\partial Q u_e\| + \|\partial Q w_e\|)
\]

\[
= \frac{\partial}{\partial Q} I_e (u_e, Q) + o \left( \epsilon^2 \right) + \frac{\partial}{\partial Q} I_e (u_e, Q)
\]

\[
+ o \left( \epsilon^{6-2\delta} + \epsilon^{(p+1)(2-\delta)} + \epsilon^{2-\delta} \left( \frac{\sqrt{2 \tan \alpha}}{\tan^2 \alpha + 1} \right) + \epsilon^{2-\delta} \left( \frac{2 \sqrt{2 \tan \alpha}}{\tan^2 \alpha + 1} \right) \right),
\]

since \(\gamma = 1\). The last two estimates then follow from Proposition 4.2.

For the case \(1 < p < 2\), we reason as in the proof of Proposition 4.5 in [GMMP1] to obtain the estimates. This concludes the proof.

4.3 Proof of Theorem 1.1

We use degree theory and the previous expansions. First of all, since \(Q\) is non degenerate for \(H |\Gamma\), we can find a small neighborhood \(V\) of \(Q\) in \(\Gamma\) such that \(\nabla H |\Gamma| \neq 0\) on \(\partial V\) and such that in some set of coordinates

\[
\text{deg} (\nabla H |\Gamma, V, 0) \neq 0.
\]

Then, if \(\delta\) is as in Lemma 4.4 we choose \(0 < \beta < \frac{\delta}{2}\), and consider the set

\[
Y = \{(d, Q) : d \in \{(2 - \beta) \log \epsilon, (2 + \beta) \log \epsilon\}, \epsilon Q \in V\}.
\]
Since $\nabla H \mid_\Gamma (Q)$ corresponds to $\nabla_T H (\epsilon Q)$ in the scaled domain $\Omega_\epsilon$, by using (94) and our choice of $V$ we know that, as $\epsilon \to 0$

$$\nabla Q_T I_\epsilon (u_\epsilon, Q) = -\tilde{C}_1 \epsilon^2 \nabla_T H (\epsilon Q) + o (\epsilon^2) \neq 0 \quad \text{on } \frac{1}{\epsilon} \partial V. \quad (97)$$

On the other hand, by (95) we also have

$$\nabla Q_d I_\epsilon (u_\epsilon, Q) = -\epsilon (2 - \beta) \left( 1 + \frac{\sqrt{2} \tan \alpha}{\sqrt{\tan^2 \alpha + 1}} \right) \quad \text{for } d = (2 - \beta) \left| \log \epsilon \right|, \quad (98)$$

and

$$\nabla Q_d I_\epsilon (u_\epsilon, Q) = -\tilde{C}_1 \epsilon^2 \nabla d H (\epsilon Q) + o (\epsilon^2), \quad \text{for } d = (2 + \beta) \left| \log \epsilon \right|. \quad (99)$$

Since we are assuming that the gradient of $H$ points toward $\partial D \Omega$ near the interface $\Gamma$, $\nabla d H (\epsilon Q)$ is negative and therefore the two $d$-derivatives in the last two formulas have opposite signs. It follows from the product formula for the degree and (97)-(99) that

$$\deg (\nabla I_\epsilon, Y, 0) = -\deg (\nabla H \mid_\Gamma, V, 0) \neq 0,$$

which proves the existence of a critical point for $I_\epsilon$ in $Y$. Since we can choose $V$ and $\beta$ arbitrarily small, the solution has the asymptotic behavior required by the theorem, and more precisely by Remark 1.2(b): the uniqueness of the global maximum follows from the asymptotics of the solution and standard elliptic regularity estimates.

**Remark 4.6.** To prove also the assertion in Remark 1.2(a), using (93) in the case of local maximum it is easy to construct an open set of $Z_\epsilon$ where the maximum of $I_\epsilon$ at the interior is strictly larger than the maximum at the boundary. On the other hand, when we have a local minimum, one can construct a mountain-pass path connecting the two points parametrized by $\left( \frac{1}{\epsilon} Q, (2 - \beta) \left| \log \epsilon \right| \right)$ and $\left( \frac{1}{\epsilon} Q, (2 + \beta) \left| \log \epsilon \right| \right)$. Using a suitably truncated pseudo-gradient flow, one can prove that the evolution of the path remains inside $\frac{1}{\epsilon} V \times \left( (2 - \beta) \left| \log \epsilon \right|, (2 + \beta) \left| \log \epsilon \right| \right)$, and still find a critical point of $I_\epsilon$.

**Acknowledgements**

The author heartily thanks Andrea Malchiodi for helpful discussions and for having proposed her the topic. Moreover she wants to thank Carlo Sinestrari for his useful suggestions. The author has been supported by the project FIRB-Ideas Analysis and Beyond.

**References**

[ABC] A. Ambrosetti, M. Badiale, S. Cingolani, *Semiclassical states of nonlinear Schrödinger equations*, Arch. Rat. Mech. Anal. 140 (1997), 285-300.

[AM] A. Ambrosetti, A. Malchiodi, *Perturbation Methods and Semilinear Elliptic Problems on $\mathbb{R}^n$*, Birkhäuser, Progr. in Math. 240 (2005).

[AMS] A. Ambrosetti, A. Malchiodi, S. Secchi, *Multiplicity results for some nonlinear singularly perturbed elliptic problems on $\mathbb{R}^n$*, Arch. Rat. Mech. Anal. 159 (2001), no. 3, 253-271.

[BL] H. Berestycki, P.L. Lions, *Nonlinear scalar field equations (Part I and Part II)*, Arch. Rat. Mech. Anal. 82 (1983), 313-376.
[CS] P. Cannarsa, C. Sinestrari, *Semiconcave Functions, Hamilton-Jacobi Equations, and Optimal Control*, Progress in Nonlinear Differential Equations and their Applications 58, Birkhäuser, Boston (2004).

[CH] R.G. Casten, C.J. Holland, *Instability results for reaction diffusion equations with Neumann boundary conditions*, J. Diff. Eq. 27 (1978), no. 2, 266-273.

[DFW] M. Del Pino, P. Felmer, J. Wei, *On the role of the mean curvature in some singularly perturbed Neumann problems*, S.I.A.M. J. Math. Anal. 31 (1999), 63-79.

[Di] S. Dipierro, *Concentration of solutions for a singularly perturbed Neumann problem in non-smooth domains*, Ann. Inst. H. Poincaré (C) Anal. Non Linéaire 28 (2011), no. 1, 107-126.

[FW] A. Floer, A. Weinstein, *Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential*, J. Funct. Anal. 69 (1986), 397-408.

[Gri] P. Grisvard, *Elliptic problems in nonsmooth domains*, Pitman, London (1985).

[GMMP1] J. Garcia Azorero, A. Malchiodi, L. Montoro, I. Peral *Concentration of solutions for some singularly perturbed mixed problems. Part I: existence results*, Archive Rat. Mech. Anal. 196 (2010), no. 3, 907-950.

[GMMP2] J. Garcia Azorero, A. Malchiodi, L. Montoro, I. Peral *Concentration of solutions for some singularly perturbed mixed problems. Part II: asymptotic of minimal energy solutions*, Ann. Inst. H. Poincaré Anal. Non Linéaire 27 (2010), 37-56.

[GM] A. Gierer, H. Meinhardt, *A theory of biological pattern formation*, Kybernetik (Berlin), 12 (1972), 30-39.

[Gri] P. Grisvard, *Elliptic problems in nonsmooth domains*, Pitman, London (1985).

[Gr] M. Grossi, *Some results on a class of nonlinear Schrödinger equations*, Math. Z. 235 (2000), no. 4, 687-705.

[GPW] M. Grossi, A. Pistoia, J. Wei, *Existence of multipeak solutions for a semilinear Neumann problem via non smooth critical point theory*, Calc. Var. Partial Differential Equations 11 (2000), no. 2, 143-175.

[Gu] C. Gui, *Multipeak solutions for a semilinear Neumann problem*, Duke Math. J. 84 (1996), no. 3, 739-769.

[Li] Y.Y. Li, *On a singularly perturbed equation with Neumann boundary conditions*, Comm. Partial Differential Equations 23 (1998), no. 3-4, 487-545.

[LN] Y.Y. Li, L. Nirenberg *The Dirichlet problem for singularly perturbed elliptic equation*, Comm. Pure Appl. Math. 51 (1998), 1445-1490.

[LNT] C.S. Lin, W.M. Ni, I. Takagi, *Large amplitude stationary solutions to a chemotaxis systems*, J. Differential Equations 72 (1988), 1-27.

[Ma] A. Malchiodi, *Concentration of solutions for some singularly perturbed Neumann problems*, Geometric analysis and PDEs, 63–115, Lecture Notes in Math., 1977, Springer, Dordrecht (2009).
[Mat] H. Matano, *Asymptotic behavior and stability of solutions of semilinear diffusion equations*, Publ. Res. Inst. Math. Sci. 15 (1979), 401-454.

[Ni] W.M. Ni, *Diffusion, cross-diffusion, and their spike-layer steady states*, Notices Amer. Math. Soc. 45 (1998), no. 1, 9-18.

[NPT] W.M. Ni, X.B. Pan, I. Takagi, *Singular behavior of least-energy solutions of a semilinear Neumann problem involving critical Sobolev exponents*, Duke Math. J. 67 (1992), no. 1, 1-20.

[NT1] W.M. Ni, I. Takagi, *On the shape of least-energy solution to a semilinear Neumann problem*, Comm. Pure Appl. Math. 41 (1991), 819-851.

[NT2] W.M. Ni, I. Takagi, *Locating the peaks of least-energy solutions to a semilinear Neumann problem*, Duke Math. J. 70 (1993), 247-281.

[NTY] W.M. Ni, I. Takagi, E. Yanagida, *Stability of least energy patterns of the shadow system for an activator-inhibitor model. Recent topics in mathematics moving toward science and engineering*, Japan J. Indust. Appl. Math. 18 (2001), no. 2, 259-272.

[NW] W.M. Ni, J. Wei, *On the location and profile of spike-layer solutions to singularly perturbed semilinear Dirichlet problems*, Comm. Pure Appl. Math. 48 (1995), 731-768.

[Oh] Y. G. Oh, *On positive Multi-lump bound states of nonlinear Schrödinger equations under multiple well potentials*, Comm. Math. Phys. 131 (1990), 223-253.

[St] W.A. Strauss, *Existence of solitary waves in higher dimensions*, Comm. Math. Phys. 55 (1977), 149-162.

[Tu] A.M. Turing, *The chemical basis of morphogenesis*, Phil. Trans. Royal Soc. London, Series B, Biological Sciences, 237 (1952), 37-72.

[We] J. Wei, *On the boundary spike layer solutions of a singularly perturbed semilinear Neumann problem*, J. Differential Equations 134 (1997), no. 1, 104-133.

[We2] J. Wei, *On the effect of domain geometry in singular perturbation problems*, Differential Integral Equations 13 (2000), no. 1-3, 15-45.