The hadron-quark transition with a lattice of nonlocal confining solitons

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Abstract

We use a lattice of nonlocal confining solitons to describe nuclear matter in the Wigner-Seitz approximation. The average density is varied by changing the size of the Wigner-Seitz cell. At sufficiently large density quark energy bands develop. The intersection of the filled valence band with the next empty band at a few times standard nuclear density signals a transition from a color insulator to a color conductor and is identified with the critical density for quark deconfinement.
The search for the hadron-quark transition in nuclear collisions has provided powerful motivation for experimental and theoretical research in high-energy nuclear physics. Experimental results are now emerging from Brookhaven’s Alternating Gradient Synchrotron (AGS) and CERN’s Superconducting Proton Synchrotron (SPS), while the planned Relativistic Heavy Ion Collider (RHIC) at Brookhaven represents the first major construction priority of the US nuclear physics community and is expected to be operational before the turn of the century. Work on the theory side can be broadly categorized into studying the fundamental properties of the strong interaction and exploring phenomenological aspects such as the signatures and characterization of the quark-gluon plasma (QGP). While phenomenological support for experiment is crucial for the success of the overall effort, in the present paper we address the more fundamental question of modeling the hadron-quark transition.

Modeling the expected transition will remain important as long as a complete solution of quantum chromodynamics (QCD) in a reasonably large space-time volume is out of reach. Soliton matter was proposed as a model of dense nuclear matter by several authors [1–3]. In the following we focus attention on the behavior of soliton matter as a function of density at zero temperature. The simplest approach in this context is to think about nuclear matter as a crystal of MIT bags [4]. In more elaborate many-soliton models the rigid walls of the MIT bags are replaced by external fields which are supposed to represent the collective nonperturbative effects of QCD. [5] Several aspects of this picture were developed further for the case of a single soliton in the framework of the Color Dielectric Model (CDM) [3].

The alternative approach taken here is based on the hadronization [7,8] of a model truncation of QCD, known as the Global Color Model (GCM) [9]. The GCM is defined by the action

$$S_{GCM}[\bar{q}, q] = \int d^4 x d^4 y \left\{ \bar{q}(x) \left[ (\gamma \cdot \partial + m_q) \delta(x - y) \right] q(y) + \frac{g^2}{2} j^a_{\nu}(x) D(x - y) j^a_{\nu}(y) \right\}, \quad (1)$$

where $j^a_{\nu}(x) = \bar{q}(x) \gamma_{\nu} \frac{\lambda^a}{2} q(x)$ is the color quark current. QCD phenomenology such as confine-
ment is incorporated into the quark-quark interaction, \( D \). This model has proven successful in the description of low-energy QCD through the reproduction of chiral perturbation theory, meson form factors and spectra, and the soliton and Fadeev descriptions of the nucleon. The relationship of this model to other models of QCD has recently been summarized.

The formation of solitons is achieved by minimizing the energy functional for the system at the mean field level with the baryon number constrained by a chemical potential. For simplicity, as with the previous treatment of this model, we restrict our attention here to a scalar-isoscalar mean field. The self-consistent solution of the mean-field equations entails iterating to convergence the Dirac equation for the vacuum-dressed quarks and a nonlinear Klein-Gordon equation for the scalar field.

There are two novel features of this model that are particularly relevant to the present discussion. First, the meson fields and their self interactions are generated through the hadronization of Eq. (1). This implies in particular that the “macroscopic” behavior is determined entirely from the underlying quark-quark interaction and parameters which enter at the quark level. Second, the confinement mechanism that arises here is due to vacuum repulsion as supplied by the quark self-energy dressing. This point is demonstrated as follows.

The quark-quark interaction, \( D \) in Eq. (1), can be chosen to produce a quark self energy, \( \Sigma(p) = i\gamma \cdot p [A(p^2) - 1] + B(p^2) \), which prohibits the propagation of quarks in the vacuum. Here we take for example in momentum space \( g^2D(q) = 3\pi^4\alpha^2\delta^{(4)}(q) \), which in the rainbow approximation to the Dyson-Schwinger equation gives the result

\[
A(p^2) = \begin{cases} 
2 & p^2 \leq \frac{\alpha^2}{4} \\
\frac{1}{2}[1 + (1 + \frac{2\alpha^2}{p^2})^{\frac{1}{2}}] & p^2 > \frac{\alpha^2}{4}
\end{cases},
B(p^2) = \begin{cases} 
(\alpha^2 - 4p^2)^{\frac{1}{2}} & p^2 \leq \frac{\alpha^2}{4} \\
0 & p^2 > \frac{\alpha^2}{4}
\end{cases}.
\]

\( ^1 \)For convenience a gauge is chosen here for which the effective gluon two-point function is diagonal in Lorentz indices, \( D_{\mu\nu}(x - y) = \delta_{\mu\nu}D(x - y) \), and the Euclidean metric is employed.
That the forms (2) produce a model of confinement as described above can be seen by the absence of a solution to the equation $p^2 + M^2(p^2) = 0$, with $M = B/A$ (no on-mass-shell point).

The mean field produces a cavity in the vacuum in which the quarks can propagate. For example, in the case of quarks coupled to a constant scalar mean field by the scalar self-energy function $B$, the quark inverse Green’s function obtains the following form:

$$G^{-1}(p) = i \hat{p}A(p^2) + B(p^2)(1 + \chi),$$

where $0 \geq \chi \geq -2$ characterizes the strength of the mean field. In this case a continuous single-particle energy spectrum, $E^2(p^2) = p^2 + M^2_C$, is obtained because of the constant potential, with the constituent mass given by

$$M^2_C = \frac{\alpha^2}{4} \frac{(1 + \chi)^2}{1 - (1 + \chi)^2}.$$  \hfill (3)

From (3) it is evident that the increase of the constituent mass as the strength, $\chi$, of the mean field decreases toward its vacuum value ($\chi = 0$), is due to the repulsive interaction with the vacuum. For the case of the constant mean field, the quarks are allowed to propagate throughout space. However, due to the energy stored in the mean field, the self-consistent (minimum-energy) solution acquires a finite range. In that case, as a quark in the system is separated from the others, the influence of the mean field on the quark diminishes and the mass rises as in (3). The quarks are thus confined to the region of nonzero mean field by virtue of their interaction with the vacuum.

In the description of matter, as the density increases the space between isolated solitons decreases, and the mean field can acquire nonzero values throughout space. One can conclude based on Eq.(3) that as the vacuum is filled by the nonzero mean field the quarks can sample a larger region of space. Finally, in sufficiently dense matter, localized clusters of quarks (nucleons) cease to be energetically favored, and a conducting phase is established.

To discuss the many-soliton problem in more detail and to identify the transition density, we now consider a lattice of static solitons as described above. The relative motion of the

\footnote{This coupling actually arises from chiral symmetry considerations \cite{9}.}
nucleons is thus neglected in the present treatment and in similar studies. Here we are not interested in reproducing the saturation properties of nuclear matter. Rather, we focus on qualitative changes in the behavior of the system (as a function of the lattice spacing) that can be identified with the hadron-quark transition in the framework of the model. A more complete treatment should include the kinetic energy of moving solitons.

As opposed to the single soliton situation, where the boundary conditions require that the self-consistent solutions of the Dirac and Klein-Gordon equations vanish as $r \to \infty$, we have periodic boundary conditions on the lattice, $\sigma(r) = \sigma(r + a)$ for the meson field, and similarly for the quark spinors, where $a$ is the lattice vector. As is well known from solid state physics [17], for sufficiently small values of the lattice constant $a = |a|$, the sharp energy eigenvalues of the isolated soliton are replaced by sets of closely spaced eigenvalues (energy bands), each soliton contributing one energy level to each band. As the lattice constant goes to infinity, the bands converge to the single-soliton solutions.

Following earlier calculations for soliton matter [1,5], we address the problem in the Wigner-Seitz approximation [18]. Introducing a spherical Wigner-Seitz cell of radius $R$ permits the decomposition of the quark spinors as

$$u_{n\kappa}(r) = \begin{pmatrix} g_{n\kappa}(r) \mathcal{Y}_{\ell J}^{m J} (\hat{r}) \\ f_{n\kappa}(r) \mathcal{Y}_{\ell J}^{m J} (\hat{r}) \end{pmatrix},$$

where the $\mathcal{Y}$-s are vector spherical harmonics, and prescribes the following boundary conditions at the edge of the Wigner-Seitz cell:

$$\left. \frac{d}{dr} \sigma(r) \right|_{r=R} = 0 \quad (5)$$

for the sigma field, and

$$\left. \frac{d}{dr} g_{n\kappa}(r) \right|_{r=R} = \left. f_{n\kappa}(r) \right|_{r=R} = 0 \quad (6)$$

for the upper and lower components of the quark spinors corresponding to the lowest-energy state in each energy band [1,5].
To solve the coupled equations for the many-soliton problem, it is advantageous to work with the Fourier expansions of the radial quark wave functions $g_{nk}(r)$ and $f_{nk}(r)$. The Dirac equation then reduces to a matrix equation in momentum space and the boundary conditions can be easily incorporated. We start by solving for the quark energy eigenvalues in an arbitrary sigma field. The quark wave functions of the lowest-energy state are then generated, which (after Fourier transformation) provide the source of the sigma field as input to the nonlinear Klein-Gordon equation in coordinate space. The Dirac equation is solved again in the new sigma field and the quark wave functions are compared (in quadrature) to the ones obtained in the previous iteration. Dependent on the radius of the Wigner-Seitz cell, $R$, and on the required tolerance, the procedure typically converges in 3-6 iterations. Once the final sigma field has been obtained at a given $R$, the Dirac equation in this field is solved for the energies of the higher excited states. Each Dirac spinor carries two different orbital quantum numbers ($\ell$ and $\ell'$) and only the total angular momentum $j$ is a good quantum number. In particular, the lowest-energy state has $\ell = 0$, $\ell' = 1$, $j = 1/2$, and the next state is characterized by $\ell = 1$, $\ell' = 2$, $j = 3/2$. We will refer to these states as the $1s1/2$ and $1p3/2$ states, respectively.

The bottom and top energies of the lowest energy bands as a function of the radius of the Wigner-Seitz cell for the parameter value $\alpha = 1.35$ GeV in (2) are shown in the Figure. The different symbols represent the calculated energies of the bottom of the three lowest energy bands. On one point we indicate a typical uncertainty we associate with our calculation. This uncertainty mainly arises from the freedom in prescribed tolerances at different stages of the calculation. The calculated points are fitted to smooth curves to guide the eye and to approximately determine the intersection point between the top of the lowest energy band ($1s1/2$) and the bottom of the next two bands ($1p3/2$ and $2s1/2$). For the top of the energy bands $\epsilon_{top}$ as a function of the cell radius $R$ we use the simple approximation

$$\epsilon_{top} = (\epsilon_{bot}^2 + \left(\frac{\pi}{2R}\right)^2)^{1/2},$$

where $\epsilon_{bot}$ is the bottom of the energy band obtained with the boundary conditions (6).
More elaborate prescriptions for the top of the bands are also used in the literature \[5\], but the above simple estimate is consistent with the qualitative nature of the present study. For large \( R \), \( \epsilon_{\text{top}} \longrightarrow \epsilon_{\text{bot}} \), and the eigenvalues approach those of the isolated soliton. As \( R \) is decreased, we first observe a decrease in the energy of the lowest band. This attraction is a consequence of the boundary condition on the large component of the quark wave function (5), which allows less curvature and therefore less quark kinetic energy than in the case of the isolated soliton. For higher densities, when the solitons and the quark wave functions start to significantly overlap, the resulting repulsion overcomes the attraction, and a minimum develops. Since we have a mean-field model lacking the details of the nucleon-nucleon interaction, we do not expect the minimum to be at the saturation density of nuclear matter. Upon further increase of the density, the energy of the \( 1p3/2 \) band does not increase any longer; this is attributable to the increase of the sigma field at the cell boundary, which makes it advantageous for the system to arrange itself in a configuration with nonzero orbital angular momentum and with quark density peaked away from the origin.

The intersection of different bands can be interpreted to signal a transition to a qualitatively different high-energy phase. For moderate values of \( R \) no bands intersect, and matter is an insulator. However, as \( R \) decreases, the lowest, occupied band intersects the \( 1p3/2 \) and \( 2s1/2 \) bands. When the energy of an unfilled level falls below that of the highest occupied state, quarks with this energy become free to migrate throughout the crystal as electrons in a metal, and color conductivity sets in. We follow Ref. \[1\] and identify the onset of color conductivity with the deconfinement transition. The value of the critical density \( \rho_c \) in the model depends on the assumptions on band filling. \[5\] Taking the simple approximation of uniformly filled bands and considering the \( 1s1/2 \rightarrow 1p3/2 \) band crossing, with \( \alpha = 1.35 \) GeV we get \( \rho_c = (2.6 \pm 0.2) \rho_0 \), where \( \rho_0 = 0.17 \text{ fm}^{-1} \) is the standard nuclear matter density. Assuming that the lowest energy band is partially filled will increase the critical density obtained in the model.

In the Table we show variations corresponding to changes in the value of the parameter \( \alpha \) in Eq. \[2\]. The root mean square radius of the soliton for large \( R \) (isolated nucleon), the
Wigner-Seitz radius and density belonging to the minimum energy, the critical Wigner-Seitz radius, and the critical density are displayed for selected values of $\alpha$. The present model does not include any meson dressing, and thus care should be taken when the calculated $\langle r^2 \rangle^{1/2}$ is compared to the experimental value of the root mean square charge radius of the proton, $\langle r^2 \rangle_{\text{exp}}^{1/2} \approx 0.83$ fm. We feel that in this schematic model the range selected represents a reasonable interval of variation for the single free parameter $\alpha$.

In summary, we discussed the properties of an infinite system of nonlocal, confining solitons in the Wigner-Seitz approximation. Due to the internally-generated meson fields and the specific confinement mechanism associated with the model, our study represents a step forward in the description of nuclear matter as a system of solitons. The quark energy bands that develop in the system with decreasing cell size were calculated. We identified the insulator-conductor transition in the crystal with the deconfinement transition expected in dense hadronic matter. In studies of the nuclear equation of state it is customary to use a hadronic equation of state at low energy densities and a bag-model equation of state at high energy densities; the critical parameters are than determined by extrapolating both descriptions to the phase transition region and solving the Gibbs conditions for phase coexistence [20]. The present model is much more satisfactory as it uses the same basic degrees of freedom in both phases and across the phase transition. In the future, the model can also be used to examine in-medium properties such as the dependence of the nucleon mass on the density of the medium.

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Figure caption

The bottom and top energies of the lowest energy bands of the soliton lattice as a function of the radius of the Wigner-Seitz cell for $\alpha =1.35$ GeV. The symbols represent the calculated energies of the bottom of the three lowest energy bands. For the top of the energy bands the approximation (7) is used.

Table caption

Variations of the root mean square radius of a single soliton, the Wigner-Seitz radius and density belonging to the minimum energy, the critical Wigner-Seitz radius, and the critical density with the single model parameter $\alpha$. 


| α (GeV)  | 1.25 | 1.35 | 1.45 |
|---------|------|------|------|
| $\langle r^2 \rangle^{1/2}$ (fm) | .71  | .67  | .64  |
| $R_{\text{min}}$ (fm)       | 1.33± .02  | 1.26± .02  | 1.17± .02 |
| $\rho_{\text{min}}/\rho_0$       | .60 ± .03  | .70 ± .04  | .88 ± .05 |
| $R_c$ (fm)       | .86 ± .02  | .82 ± .02  | .77 ± .02 |
| $\rho_c/\rho_0$       | 2.2 ± .2  | 2.6 ± .2  | 3.1 ± .3 |
