On holomorphic functions on negatively curved manifolds

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Abstract
Based on a well known Sh.-T. Yau theorem we obtain that the real part of a holomorphic function on a Kähler manifold with the Ricci curvature bounded from below by $-1$ is contractive with respect to the distance on the manifold and the hyperbolic distance on $(-1, 1)$ inhered from the domain $(-1, 1) \times \mathbb{R}$. Moreover, in the case of bounded holomorphic functions we prove that the modulus is contractive with respect to the distance on the manifold and the hyperbolic distance on the unit disk.

Keywords Holomorphic mappings on complex manifolds · Modulus and the real part of a holomorphic function · Hyperbolic distance · Bergman distance · Negatively curved manifolds · Ricci curvature

Mathematics Subject Classification Primary 32Q05 · 32Q15; Secondary 31C05 · 30A10

1 Introduction

1.1 The Yau generalization of the Schwarz–Pick lemma

The result concerning holomorphic mappings on Kähler manifolds given in the following proposition is well known and proved by Yau [19], in 1978. We refer also to the work of Royden [18] for certain improvements of this result and the Kobayashi book [8] as a general reference for holomorphic mappings between complex manifolds where the questions on distance, area and volume decreasing properties are considered.
Proposition 1.1. (Cf. [19]) Let $M$ be a complete Kähler manifold with the Ricci curvature bounded from below by a negative constant $K_M$. Let $N$ be another Hermitian manifold with holomorphic sectional curvature bounded from above by a negative constant $K_N$. Then any holomorphic mapping from $M$ into $N$ does not increase distances more than a factor depending only on the curvatures of $M$ and $N$. The factor is $K_M/K_N$.

In the rest of the paper we denote by $M$ the kind of the Hermitian manifold as in the above proposition. On the other hand, the manifold $N$ may be, for example, a bounded homogenous domain in $\mathbb{C}^n$ equipped with the Bergman metric, since a such one domain has constant holomorphic sectional curvature equal to $-1$ [3]; in particular it may be the unit ball in $\mathbb{C}^n$.

From Proposition 1.1 it follows that if the Ricci curvature of $M$ is bounded from below by $-1$, and if the holomorphic section curvature of $N$ is bounded from above by $-1$, then any holomorphic mapping $f : M \to N$ is distance decreasing, i.e.,

$$d_N(f(z), f(w)) \leq d_M(z, w), \quad z, w \in M,$$

where $d_M$ is the distance on $M$, and $d_N$ is the distance on $N$.

Recall that the distance on a Hermitian manifold $N$ is given by

$$d_N(z, w) = \inf_{\gamma} \ell(\gamma),$$

where $\gamma : [a, b] \to N$ is among partially $C^1$-smooth paths on $N$ with endpoints at $z$ and $w$, i.e., $\gamma(a) = z$ and $\gamma(b) = w$. We have denoted by $\ell(\gamma)$ the length of the path $\gamma$ with respect to the Hermitian form $H_z = H^N_z, z \in N$ on $N$ which is

$$\ell(\gamma) = \int_a^b \sqrt{H_{\gamma(t)}(\gamma'(t), \gamma'(t))} dt.$$

Let us note that if $\gamma : [a, b] \to M$ is a partially $C^1$-smooth path parameterized by arc length, then since

$$\ell(\gamma|_{[a,t]}) = \int_a^t \sqrt{H_{\gamma(t)}(\gamma'(t), \gamma'(t))} dt = t - a, \quad t \in [a, b],$$

we have $H_{\gamma(t)}(\gamma'(t), \gamma'(t)) = 1, t \in [a, b]$.

If $N \subseteq \mathbb{C}$ is a surface (that is 1-dimensional Hermitian manifold), the holomorphic section curvature are equal to the Gauss curvature of $N$, which is

$$-\frac{\Delta \log h(z)}{h(z)^2},$$

where the Hermitian form $H^N_z$ on $N$ is represented as $H^N_z(u, v) = h(z)^2 u \overline{v}, z \in N, u, v \in \mathbb{C}, h(z) > 0$. 

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The result stated in the above proposition is a generalization of the classical Schwarz–Pick lemma which says that a holomorphic function which maps the unit disk \( D \subseteq \mathbb{C} \) into itself does not increase the hyperbolic distance \( \sigma \) on \( D \), i.e.,

\[
\sigma(f(z), f(w)) \leq \sigma(z, w), \quad z, w \in D;
\]

recall that the hyperbolic metric on \( D \) is given by the following Hermitian form

\[
H^D_z(u, v) = \frac{4uv}{(1 - |z|^2)^2}, \quad z \in D, \ u, v \in \mathbb{C},
\]

with the Gauss curvature equal to \(-1\).

Proposition 1.1 is also generalization of the Ahlfors result for surfaces with non-positive Gauss curvature \([1,2]\).

1.2 The main results of this work

Thought the rest of this paper let \( J = (a, b) \subseteq \mathbb{R} \) be an interval, bounded or unbounded but not equal to \( \mathbb{R} \). The weight on \( J \) is any positive and continuous function on this interval. If \( \omega \) is a weight on \( J \), we define the \( \omega \)-distance between \( a, b \in J \) in the following way

\[
d_\omega(a, b) = \int_a^b \omega(t)dt.
\]

For a weight \( \omega \in C^2(J) \) we introduce the following quantity

\[
k_\omega(t) = \frac{\omega'(t)^2 - \omega(t)\omega''(t)}{\omega(t)^4}, \quad t \in J.
\]

The first main result of this paper is the distance decreasing property of the real part of a holomorphic function on the manifold \( M \) with respect to the distance on \( M \) and the \( \omega \)-distance on \( J = (a, b) \) provided that \( k_\omega \leq -1 \). Therefore, if \( J \) is equipped with a weight \( \omega \) with \( k_\omega \leq -1 \) (of special interest for us are weights with \( k_\omega \equiv -1 \) on \( J \)), the result says that the real part of a holomorphic function \( f : M \rightarrow J \) is distance decreasing, i.e., we have

\[
d_\omega(\text{Re}f(z), \text{Re}f(w)) \leq d_M(z, w), \quad z, w \in M.
\]

This improves some recent results \([4,7,13]\) obtained for real–valued harmonic functions on proper simply connected domains in \( \mathbb{C} \), since on a such type domain a harmonic function is a real part of a holomorphic one. See also the recent papers \([9,12,14]\).

In the second part of the next section we prove that the modulus of a bounded holomorphic function is distance decreasing with respect to the distance on \( M \) and the
hyperbolic distance $\sigma$ on $\mathbb{D}$. More precisely, this result says that

$$\sigma(|f|(z), |f|(w)) \leq d_M(z, w), \quad z, w \in M,$$

provided that $f$ is bounded by 1.

2 Distance decreasing property of the real part and the modulus

2.1 Distance decreasing property of the real part of a holomorphic function

Our first main result follow from the following theorem.

Theorem 2.1 Let $f$ be a holomorphic function on $M$ such that $\text{Re} f \in J$. Let $\omega$ be a weight on $J$ which satisfies $k_\omega \leq -1$. Then $\text{Re} f$ is distance decreasing, i.e., we have

$$d_\omega(\text{Re} f(z), \text{Re} f(w)) \leq d_M(z, w), \quad z, w \in M. \quad (2.1)$$

Proof Note that $f$ maps $M$ into the vertical strip domain $N = J \times \mathbb{R} \subseteq \mathbb{C}$. Introduce the Hermitian form on $N$ in the following way

$$H_z^N(u, v) = \tilde{\omega}(z)^2 u \bar{v}, \quad u, v \in \mathbb{C}, \quad \tilde{\omega}(z) = \omega(\text{Re} z), \quad z \in N.$$

Let $d_N$ be the distance on $N$. According to our assumption, the Gauss curvature of $N$ is bounded by $-1$ form above. Indeed, for $z \in N$ we have

$$-\frac{\Delta \log \tilde{\omega}(z)}{\tilde{\omega}^2(z)} = \frac{\omega'(\text{Re} z)^2 - \omega(\text{Re} z)\omega''(\text{Re} z)}{\omega(\text{Re} z)^2} = k_\omega(\text{Re} z) \leq -1.$$

In view of Proposition 1.1 the function $f : M \rightarrow N$ is distance decreasing, which implies that

$$\sup_{\xi \in \mathbb{C}^m, H_z(\xi, \xi) = 1} |(\nabla f(z), \xi)| \leq \frac{1}{\omega(f(z))}, \quad z \in M \quad (2.2)$$

in local coordinates $z_1, z_2, \ldots, z_m$ around $z$, where $\nabla f = (\partial_{z_1}, \partial_{z_2}, \ldots, \partial_{z_m})$ and $m = \text{dim} M$. Indeed, let $\gamma : [a, b] \rightarrow M$ be a smooth path parameterized by arc length, and let $\gamma'(a) = \xi$. Since $f : M \rightarrow N$ is distance decreasing, we have

$$d_N(f(\gamma(t)), f(\gamma(a))) \leq d_M(\gamma(t), \gamma(a)) \leq \ell(\gamma|_{[a, t]}), \quad t \in [a, b].$$

If $\nabla f(z) \neq 0$, then for $t$ sufficiently close to $a$ there holds $f(\gamma(t)) \neq f(\gamma(a))$. It follows that

$$\frac{d_N(f(\gamma(t)), f(\gamma(a)))}{|f(\gamma(t)) - f(\gamma(a))|} \frac{|f(\gamma(t)) - f(\gamma(a))|}{t - a} \frac{t - a}{\ell(\gamma|_{[a, t]})} \leq 1.$$
Having in mind that \( \lim_{a \to \zeta} \frac{d_{\mathcal{H}}(\zeta, \eta)}{|\zeta - \eta|} = \tilde{\omega}(\zeta) \) (for this equality in a general setting, i.e., for vector spaces with norm, we refer to \([10]\)), from the last inequality we obtain (2.2).

Let now \( z, w \in M \) be arbitrary, and let \( \gamma : [a, b] \to M \) be a path. Assume that \( \gamma \) is \( C^1 \)-smooth parameterized by arc length, so that we have \( \ell(\gamma) = b - a \), and contained in a single chart of \( M \) (in general case we should consider pieces of \( \gamma \) and apply the following rezoning on the each part of the path). Denote \( t(\gamma) = \gamma'(t) \). Then we have \( H_{\gamma(t)}(t_{\gamma}(t), t'_{\gamma}(t)) = H_{\gamma(t)}(\gamma'(t), \gamma'(t)) = 1 \). Note that \( \delta = \text{Re} f \circ \gamma \) may be considered as a \( C^1 \)-smooth path in \( J \) with endpoints \( \text{Re} f(\gamma) \in J \) and \( \text{Re} f(w) \in J \). Therefore, using (2.2) in the second inequality below, for the \( \omega \)-distance between \( \text{Re} f(z) \) and \( \text{Re} f(w) \) we obtain

\[
d_{\omega}(\text{Re} f(z), \text{Re} f(w)) \leq \int_{a}^{b} \omega(\delta(t))|\delta'(t)|dt = \int_{a}^{b} \omega(\text{Re} f(\gamma(t))) \left| \frac{d}{dt}\text{Re} f(\gamma(t)) \right| dt
\]

\[
= \int_{a}^{b} \omega(\text{Re} f(\gamma(t))) \left| \text{Re} \nabla f(\gamma(t)), t_{\gamma}(t) \right| dt
\]

\[
\leq \int_{a}^{b} \omega(\text{Re} f(\gamma(t))) \left| \nabla f(\gamma(t)), t_{\gamma}(t) \right| dt
\]

\[
\leq \int_{a}^{b} \omega(\text{Re} f(\gamma(t))) \frac{d}{dt} \text{Re} f(\gamma(t)) dt = b - a = \ell(\gamma),
\]

since \( \tilde{\omega}(z) = \omega(\text{Re} z) \). If we take infimum over all \( \gamma \), we obtain the distance decreasing property of the real part of \( f \), i.e., (2.1).

\[\square\]

**Remark 2.2** Having in mind the preceding theorem we should find positive solutions of the differential equation

\[
\frac{\omega^2 - \omega \omega''}{\omega^4} = -k^2, \quad k \geq 1.
\]

This differential equation may be rewritten as \( \frac{(\log \omega)''}{\omega} = k^2 \). If we introduce \( \lambda = \log(2k^2 \omega^2) \), then we obtain the following equation \( \lambda'' = e^\lambda \). The last equation may be solved in the standard way. One obtains the general solutions:

\[
e^{\lambda(t)} \sin^2(C_1 t + C_2) = 2C_1^2, \quad e^{\lambda(t)} \sin^2(C_1 t + C_2) = 2C_1^2, \quad e^{\lambda(t)} (t + C)^2 = 2,
\]

where \( C_1, C_2 \) and \( C \) are constants.

Therefore, the positive solutions of the equation \( \frac{\omega^2 - \omega \omega''}{\omega^4} = -k^2 \) on \( J \) are:

- \( \omega(t) = \frac{C_1}{k|\sin(C_1 t + C_2)|} \);
- \( \omega(t) = \frac{C_1}{k|\sin(C_1 t + C_2)|} \);
- \( \omega(t) = \frac{C_1}{k|t + C|} \).
where the constants $C_1 > 0$, $C_2$ and $C$ should be adjusted in a such way that the interval $J$ is contained in the domain of $\omega$.

An immediate consequence of the preceding theorem and the remark is the following theorem.

**Theorem 2.3** (i) Let $f$ be a holomorphic function on the manifold $M$ with the real part bounded by 1. Then $\text{Re} f$ is contractive with respect to the distance on $M$ and the hyperbolic distance on $(-1, 1)$ inhered from the domain $(-1, 1) \times \mathbb{R}$.

(ii) Let $f$ be a holomorphic function on the manifold $M$ such that the real part of $f$ is positive. Then $\text{Re} f$ is contractive with respect to the distance on $M$ and the hyperbolic distance on $(0, \infty)$ inhered from $(0, \infty) \times \mathbb{R}$.

**Proof** (i) The hyperbolic metric on $(-1, 1) \times \mathbb{R}$ is given by the Hermitian form

$$H_z(u, v) = \left( \frac{\pi}{2 \cos(\frac{\pi}{2} \text{Re} z)} \right)^2 u \overline{v}, \quad z \in (-1, 1) \times \mathbb{R}, \ u, v \in \mathbb{C}.$$ 

The corresponding weight on $(-1, 1)$ is $\omega(t) = \frac{\pi}{2 \cos(\frac{\pi}{2} t)}$ which is the first solution given in the preceding remark for $k = 1$, $C_1 = \frac{\pi}{2}$, and $C_2 = -\frac{\pi}{2}$.

(ii) The hyperbolic metric on the half–plane $\{ z \in \mathbb{C} : \text{Re} z > 0 \}$ is

$$H_z(u, v) = \frac{1}{(\text{Re} z)^2} u \overline{v}, \quad u, v \in \mathbb{C},$$

so the weight on $J = (0, \infty)$ is given by $\omega(t) = \frac{1}{t}$, which is the third solution for $k = 1$ and $C = 0$.

$\square$

**Remark 2.4** We would like here to compare the result given in above theorem with some recent results for real–valued harmonic functions. Let us replace the manifold $M$ with the unit disk $\mathbb{D}$ equipped with the hyperbolic metric (with the Gauss curvature equal to $-1$). Then the first part of the above theorem recovers the result obtained by Chen [4,17].

Note that the weight $\omega(t) = \frac{\pi}{2} \frac{1}{\cos(\frac{\pi}{2} t)}$ on $(-1, 1)$ is comparable to the following one $\tilde{\omega}(t) = \frac{2}{1-t^2}$ on the same interval in the sense that $\frac{\pi}{4} \tilde{\omega} \leq \omega$ on $(-1, 1)$. Indeed, this inequality is elementary and it may be found in [4] or [13]. This means that the distance $d_\omega(t, t')$ is greater then $\frac{\pi}{4} d_{\tilde{\omega}}(t, t')$ for $t, t' \in (-1, 1)$. Thus we have just recovered the result of Kalaj and Vuorinen [7] which says that for a harmonic function $U : \mathbb{D} \rightarrow (-1, 1)$ there holds

$$\sigma(U(z), U(w)) \leq \frac{4}{\pi} \sigma(z, w), \quad z, w \in \mathbb{D},$$

since $\tilde{\omega}$-distance on $(-1, 1)$ is equal to the distance on $(-1, 1)$ inhered from the hyperbolic distance $\sigma$ on $\mathbb{D}$. This improves also [6].

On the other hand, the part (ii) gives the result of Marković [11] for positive harmonic functions on the unit disk. See also [15].
2.2 Distance decreasing property of the modulus of a holomorphic function

Having in mind the preceding theorem on the contraction of the real part of a holomorphic function, one may pose the natural question what can be said about the modulus $|f|$ of a bounded holomorphic function $f$ on the manifold $M$. In the rest of this paper we will consider the question concerning the distance contraction of the modulus of a bounded holomorphic function on the manifold $M$ with respect to the distance on $M$ and the hyperbolic distance on $D$.

This question is also motivated by the Pavlović work [16] on the Schwarz lemma for the modulus of a holomorphic mapping on the disk $D$ with values in the unit ball $B^n$. Note that $|f|$ is not necessary (real) differentiable on $D$, so the Schwarz lemma for the modulus of a holomorphic function has to be formulated in the following way

$$\nabla^* |f|(z) \leq \frac{1 - |f(z)|^2}{1 - |z|^2}, \quad z \in D, \quad (2.3)$$

where for a real–valued function $g$ on $D$ we have denoted

$$\nabla^* g(z) = \lim_{h \to 0} \sup \frac{|g(z + h)|}{|h|}.$$ 

If $f(z) \neq 0$, then $|f|$ is differentiable at $z \in D$, and we may replace $\nabla^* |f|(z)$ with the modulus of the ordinary gradient $|\nabla |f|(z)|$.

The Pavlović result is stated and proved in the form we have just mentioned. The lemma is used to extend some results of K. Dyakonov on modulus of continuity of holomorphic functions on the disk. However, it may be rewritten in the manner which does not involve any kind of derivative by saying that $|f|$ is contractive in the hyperbolic distance $\sigma$ on $D$, i.e.,

$$\sigma (|f|(z), |f|(w)) \leq \sigma (z, w), \quad z, w \in D.$$ 

Indeed, this follows since (2.3) may be rewritten as

$$\frac{2\nabla^* |f|(z)}{1 - |f(z)|^2} \leq \frac{2}{1 - |z|^2}$$

(see [10] for a similar result in a general setting, i.e., for Fréchet differentiable mappings between vector spaces with norm). In the sequel we will consider a generalization of the Pavlović result for holomorphic functions on the manifold $M$.

Recall that the pseudo–hyperbolic distance $\rho$ on $D$ is given by

$$\rho (z, w) = |\varphi_z(w)|, \quad z, w \in D,$$

where

$$\varphi_z(w) = \frac{z - w}{1 - \bar{z}w}, \quad w \in D,$$
is a conformal transformation of the disk $\mathbb{D}$ onto itself.

Duren and Weir [5] showed that the pseudo-hyperbolic distance (even on the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$) satisfies the inequality

$$\rho(|z|,|w|) \leq \rho(z,w), \quad z, w \in \mathbb{D}. \quad (2.4)$$

They gave an interesting geometric proof (in any dimension). For the sake of completeness we will prove this inequality directly having in mind a well known identity

$$1 - |z|^2 = (1 - |w|^2)(1 - |z|^2)\frac{1 - \bar{z}w}{|1 - \bar{z}w|}, \quad z, w \in \mathbb{D}$$

(this proof may be adapted for the unit ball). Using the simple inequality $|1 - \bar{z}w| \geq |1 - |w|| = 1 - |w|$, we obtain (2.4).

Since $t \to \log \frac{1 + t}{1 - t}$ is an increasing function on $(-1, 1)$, and since

$$\sigma(z,w) = \log \frac{1 + \rho(z,w)}{1 - \rho(z,w)},$$

we may conclude that the hyperbolic distance on the unit disk satisfies the inequality

$$\sigma(|z|,|w|) \leq \sigma(z,w), \quad z, w \in \mathbb{D}. \quad (2.5)$$

**Theorem 2.5** Let $f$ be a holomorphic function on the manifold $M$ such that $|f(z)| < 1$, $z \in M$. Then we have

$$\sigma(|f(z)|,|f(w)|) \leq d_M(z,w), \quad z, w \in M.$$  

In other words, the modulus of $f$ is contractive with respect to the distance on $M$ and the hyperbolic distance on $\mathbb{D}$.

**Proof** Using (2.5) we obtain

$$\sigma(|f(z)|,|f(w)|) \leq \sigma(f(z),f(w)) \leq d_M(z,w), \quad z, w \in M,$$

which we aimed to prove. \qed

**Remark 2.6** That the modulus of a holomorphic mapping on $M$ with values in the unit ball $\mathbb{B}^n \subseteq \mathbb{C}^n$ is contractive (the above mentioned Pavlović result) one may see in the same way as in the proof of the preceding theorem. Recall that the Bergman metric on $\mathbb{B}^n$ is a Hermitian form given by

$$H_z(u,v) = 2\frac{(1 - |z|^2) \langle u, \overline{v} \rangle + \langle u, \overline{z} \rangle \langle v, \overline{z} \rangle}{(1 - |z|^2)^2}, \quad z \in \mathbb{B}^n, u, v \in \mathbb{C}^n.$$

The holomorphic sectional curvature of the Bergman metric is constant and equal to $-1$. Therefore, by the Yau theorem any holomorphic mapping $f : M \to \mathbb{B}^n$ is ...
distance decreasing. On the other hand, the explicit form of the Bergman distance on $\mathbb{B}^n$ is
\[
\beta(z, w) = \log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|} = \log \frac{1 + \rho(z, w)}{1 - \rho(z, w)}, \quad z, w \in \mathbb{B}^n,
\]
where
\[
\varphi_z(w) = \frac{z - \langle w, z \rangle |z|^2 - \sqrt{1 - |z|^2} (w - \langle w, z \rangle |z|^2)}{1 - \langle w, z \rangle}
\]
is a bi-holomorphic transformation of $\mathbb{B}^n$ onto itself, and $\rho(z, w) = |\varphi_z(w)|$ is the pseudo–hyperbolic distance on $\mathbb{B}^n$.

For the unit disk $\mathbb{D} \subseteq \mathbb{B}^n$ the restricted Bergman distance $\beta|_\mathbb{D}$ coincides with the hyperbolic distance $\sigma$ on $\mathbb{D}$. Therefore, from the Yau theorem and the Duren – Weir result
\[
\beta(|z|, |w|) \leq \beta(z, w), \quad z, w \in \mathbb{B}^n,
\]
we may deduce the Schwarz lemma for the modulus od a vector–valued holomorphic mapping on $M$. In particular, if $M$ is the unit disc $\mathbb{D}$ equipped with the hyperbolic metric, we have the Pavlović result for holomorphic mappings of $\mathbb{D}$ into $\mathbb{B}^n$ in the derivative–free formulation.

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