ON THE MASS OF THE EXTERIOR BLOW-UP POINTS.

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ABSTRACT. We consider the following problem on open set $\Omega$ of $\mathbb{R}^2$:
\[
\begin{cases}
-\Delta u_i = V_i e^{u_i} & \text{in } \Omega \subset \mathbb{R}^2, \\
u_i = 0 & \text{in } \partial \Omega.
\end{cases}
\]
We assume that:
\[
\int_{\Omega} e^{u_i} dy \leq C,
\]
and,
\[
0 \leq V_i \leq b < +\infty.
\]
On the other hand, if we assume that $V_i$ is $s$-holderian with $1/2 < s \leq 1$, then, each exterior blow-up point is simple. As application, we have a compactness result for the case when:
\[
\int_{\Omega} V_i e^{u_i} dy \leq 40\pi - \epsilon, \epsilon > 0.
\]

1. INTRODUCTION AND MAIN RESULTS

We set $\Delta = \partial_{11} + \partial_{22}$ on open set $\Omega$ of $\mathbb{R}^2$ with a smooth boundary.

We consider the following problem on $\Omega \subset \mathbb{R}^2$:
\[
(P) \begin{cases}
-\Delta u_i = V_i e^{u_i} & \text{in } \Omega \subset \mathbb{R}^2, \\
u_i = 0 & \text{in } \partial \Omega.
\end{cases}
\]
We assume that:
\[
\int_{\Omega} e^{u_i} dy \leq C,
\]
and,
\[
0 \leq V_i \leq b < +\infty.
\]

The previous equation is called, the Prescribed Scalar Curvature equation, in relation with conformal change of metrics. The function $V_i$ is the prescribed curvature.

Here, we try to find some a priori estimates for sequences of the previous problem.

Equations of this type were studied by many authors, see [5-8, 10-15]. We can see in [5], different results for the solutions of those type of equations with or without boundaries conditions and, with minimal conditions on $V$, for example we suppose $V_i \geq 0$ and $V_i \in L^p(\Omega)$ or $V_i e^{u_i} \in L^p(\Omega)$ with $p \in [1, +\infty]$.

Among other results, we can see in [5], the following important Theorem,

**Theorem A (Brezis-Merle [5]):** If $(u_i)$ and $(V_i)$, are two sequences of functions relatively to the previous problem $(P)$ with, $0 < a \leq V_i \leq b < +\infty$, then, for all compact set $K$ of $\Omega$,
\[
\sup_K u_i \leq c = c(a, b, m, K, \Omega) \text{ if } \inf_{\Omega} u_i \geq m.
\]

A simple consequence of this theorem is that, if we assume $u_i = 0$ on $\partial \Omega$ then, the sequence $(u_i)$ is locally uniformly bounded. We can find in [5] an interior estimate if we assume $a = 0$, but we need an assumption on the integral of $e^{u_i}$, precisely, we have in [5]:

\[
\int_{\Omega} e^{u_i} dy \leq C
\]
Theorem B (Brezis-Merle [5]). If \((u_i), \text{ and } (V_i)\) are two sequences of functions relatively to the previous problem \((P)\) with, \(0 \leq V_i \leq b < +\infty\), and,

\[
\int_{\Omega} e^{u_i} \, dy \leq C,
\]

then, for all compact set \(K\) of \(\Omega\),

\[
\sup_K u_i \leq c = c(b, C, K, \Omega).
\]

If, we assume \(V\) with more regularity, we can have another type of estimates, \(\sup + \inf\). It was proved, by Shafrir, see [13], that, if \((u_i), \text{ and } (V_i)\), are two sequences of functions solutions of the previous equation without assumption on the boundary and, \(0 < a \leq V_i \leq b < +\infty\), then we have the following interior estimate:

\[
C \left(\frac{a}{b}\right) \sup_K u_i + \inf_{\Omega} u_i \leq c = c(a, b, K, \Omega).
\]

We can see in [7], an explicit value of \(C \left(\frac{a}{b}\right) = \sqrt{\frac{a}{b}}\). In his proof, Shafrir has used the Stokes formula and an isoperimetric inequality, see [3]. For Chen-Lin, they have used the blow-up analysis combined with some geometric type inequality for the integral curvature.

Now, if we suppose \((V_i)\), uniformly Lipschitzian with \(A\) the Lipschitz constant, then, \(C(a/b) = 1\) and \(c = c(a, b, A, K, \Omega)\), see Brézis-Li-Shafrir [4]. This result was extended for Hölderian sequences \((V_i)\), by Chen-Lin, see [7]. Also, we can see in [10], an extension of the Brezis-Li-Shafrir to compact Riemann surface without boundary. We can see in [11] explicit form, \((8\pi m, m \in \mathbb{N}^* \text{ exactly})\), for the numbers in front of the Dirac masses, when the solutions blow-up. Here, the notion of isolated blow-up point is used. Also, we can see in [14] refined estimates near the isolated blow-up points and the bubbling behavior of the blow-up sequences.

We have in [15]:

Theorem C (Wolansky.G.,[15]). If \((u_i)\) and \((V_i)\) are two sequences of functions solutions of the problem \((P)\) without the boundary condition, with,

\[
0 \leq V_i \leq b < +\infty,
\]

\[
||\nabla V_i||_{L^\infty(\Omega)} \leq C_1,
\]

\[
\int_{\Omega} e^{u_i} \, dy \leq C_2,
\]

and,

\[
\sup_{\partial \Omega} u_i - \inf_{\partial \Omega} u_i \leq C_3,
\]

the last condition replace the boundary condition.

We assume that \((iii)\) holds in the theorem 3 of [5], then, in the sense of the distributions:

\[
V_i e^{u_i} \to m \sum_{j=0}^{m} 8\pi \delta_{x_j},
\]

in other words, we have:

\[
\alpha_j = 8\pi, \quad j = 0 \ldots m,
\]

in \((iii)\) of the theorem 3 of [5].

To understand the notations, it is interrestant to take a look to a previous prints on arXiv, see [1] and [2].

Our main results are:
Theorem 1. Assume that, \( V_i \) is uniformly \( s \)-holderian with \( 1/2 < s \leq 1 \), and that:

\[
\max_{\Omega} u_i \to +\infty.
\]

Then, each exterior blow-up point is simple.

There are \( m \) blow-up points on the boundary (perhaps the same) such that:

\[
\int_{B(x_i, \delta_i; \epsilon)} V_i(x_i^j + \delta_i^j y) e^{u_i} \to 8\pi.
\]

and,

\[
\int_{\Omega} V_i e^{u_i} \to \int_{\Omega} V e^u + \sum_{j=1}^{m} 8\pi \delta_{x_j}.
\]

Theorem 2. Assume that, \( V_i \) is uniformly \( s \)-holderian with \( 1/2 < s \leq 1 \), and,

\[
\int_{B(0)} V_i e^{u_i} dy \leq 40\pi - \epsilon, \quad \epsilon > 0,
\]

then we have:

\[
\sup_{\Omega} u_i \leq c = c(b, C, A, s, \Omega).
\]

where \( A \) is the holderian constant of \( V_i \).

2. PROOF OF THE RESULT:

Proof of the theorem 1:

Let’s consider the following function on the ball of center 0 and radius \( 1/2 \); And let us consider \( \epsilon > 0 \)

\[
v_i(y) = u_i(x_i + \delta_i y) + 2 \log \delta_i, \quad y \in B(0, 1/2)
\]

This function is solution of the following equation:

\[
-\Delta v_i = V_i(x_i + \delta_i y) e^{u_i}, \quad y \in B(0, 1/2)
\]

The function \( v_i \) satisfy the following inequality (without loss of generality):

\[
\sup_{\partial B(0,1/4)} v_i - \inf_{\partial B(0,1/4)} v_i \leq C,
\]

Let us consider the following functions:

\[
\begin{cases}
-\Delta v_0^i = 0 & \text{in } B(0, 1/4) \\
v_0^i = u_i(x_i + \delta_i y) & \text{on } \partial B(0, 1/4).
\end{cases}
\]

By the elliptic estimates we have:

\[
v_0^i \in C^2(\bar{B}(0, 1/4)).
\]

We can write:

\[
-\Delta (v_i - v_0^i) = V_i(x_i + \delta_i y) e^{u_i} e^{v_i - v_0^i} = K_1 K_2 e^{v_i - v_0^i},
\]

With this notations, we have:

\[
||\nabla (v_i - v_0^i)||_{L^q(B(0, \epsilon))} \leq C_q.
\]

\[
v_i - v_0^i \to G \text{ in } W^{1,q}_0,
\]

And, because, for \( \epsilon > 0 \) small enough:
\[
||\nabla G||_{L^s(B(0, \epsilon))} \leq \epsilon' << 1,
\]
We have, for \(\epsilon > 0\) small enough:
\[
||\nabla (v_i - v_i^0)||_{L^s(B(0, \epsilon))} \leq 2\epsilon' << 1.
\]
and,
\[
||\nabla v_i||_{L^s(B(0, \epsilon))} \leq 3\epsilon' << 1.
\]
Set,
\[
u = v_i - v_i^0, \quad z_1 = 0,
\]
Then,
\[-\Delta u = K_1 K_2 e^u, \quad \text{in } B(0, 1/4),\]
and,
\[\text{osc}(u) = 0.\]
We use Woalnsky's theorem, see [15]. In fact \(K_2\) is a \(C^1\) function uniformly bounded and \(K_1\) is \(s\)-holderian with \(1/2 < s \leq 1\). Because we take the logarithm in \(K\), the part which contain \(K_2\) have similar proof as in this paper we use the Stokes formula. Only the case of \(K_1\) \(s\)-holderian is difficult. For this and without loss of generality, we can assume the \(K = K_1 = V_i(x_i + \delta_i y)\).
We set:
\[
\Delta \tilde{u} = \Delta v_i = \rho = -Ke^{\tilde{u}} = -K_1 e^v_i
\]
Let us consider the following term of Wolansky computations:
\[
\int_{B^*} \int_{B^*} \text{div} \left( (z - z_1) \rho \right) \log K + \int_{\partial B^*} (\nu > \rho) \log K,
\]
First, we write:
\[
\int_{B^*} \int_{B^*} \text{div} \left( (z - z_1) \rho \right) \log K = 2 \int_{B^*} \rho \log K + \int_{B^*} (z - z_1) \nabla \rho \log K
\]
which we can write as:
\[
- \int_{B^*} \int_{B^*} \text{div} \left( (z - z_1) \rho \right) \log K = 2 \int_{B^*} K \log Ke^u + \int_{B^*} (z - z_1) \nabla u > K \log Ke^u + \int_{B^*} (z - z_1) (\nabla K) \log K > e^u,
\]
We can write:
\[
\nabla (K \log K - K) = (\nabla K) (\log K)
\]
Thus, by integration by part we have:
\[
\int_{B^*} (z - z_1) (\nabla K) \log K > e^u = \int_{B^*} (z - z_1) (\nabla (K \log K - K)) > e^u = \int_{\partial B^*} (z - z_1) \nu > (K \log K - K) e^u - 2 \int_{B^*} (K \log K - K) e^u - \int_{B^*} (z - z_1) \nabla u > (K \log K - K) e^u
\]
Thus,
\[
-(\int_{B^*} \int_{B^*} \text{div} \left( (z - z_1) \rho \right) \log K + \int_{\partial B^*} (\nu > \rho) \log K) = - \int_{\partial B^*} (z - z_1) \nu > Ke^u + \int_{B^*} (z - z_1) \nabla u > Ke^u + 2 \int_{B^*} Ke^u
\]
But, we can write the following.
\[ \int_{B^*} <(z-z_1)|\nabla u > Ke^u = \int_{B^*} <(z-z_1)|\nabla u > (K-K(z_1))e^u + K(z_1) \int_{B^*} <(z-z_1)|\nabla u > e^u, \]

and, after integration by parts:

\[ K(z_1) \int_{B^*} <(z-z_1)|\nabla u > e^u = K(z_1) \int_{\partial B^*} <(z-z_1)|\nu > e^u - 2K(z_1) \int_{B^*} e^u, \]

Finally, we have, for the Wolansky term:

\[ \int_{B^*} \text{div}((z-z_1)\rho)\log K + \int_{\partial B^*} (<(z-z_1)|\nu > \rho)\log K = \]

\[ = \int_{B^*} <(z-z_1)|\nabla u > (K-K(z_1))e^u + \left(2\int_{B^*} (K-K(z_1))e^u\right) + \]

\[ + \left(\int_{\partial B^*} <(z-z_1)|\nu > (K(z_1) - K)e^u\right) \]

But, we have soon that if \( K \) is \( s \)-holderian with \( 1 \geq s > 1/2 \), around each exterior blow-up point we have, the following estimate:

\[ \int_{B^*} <(z-z_1)|\nabla u > (K-K(z_1))e^u = \]

\[ = \int_{\{0,1\}} <(y-z_1)|\nabla v_i > (V_i(x_i + \delta_i y) - V_i(x_i))e^{v_i} \, dy = \]

\[ = \int_{B(x_i+\delta_i c)} <(x-x_i)|\nabla u_i > (V_i(x) - V_i(x_i))e^{u_i} \, dy = o(1)M_\varepsilon \]

Thus, \[ \int_{B^*} \text{div}((z-z_1)\rho)\log K + \int_{\partial B^*} (<(z-z_1)|\nu > \rho)\log K = o(1)M_\varepsilon = o(1) \int_{B^*} Ke^u \]

We argue by contradiction and we suppose that we have around the exterior blow-up point 2 or 3 blow-up points, for example. We prove, as in a previous paper, that, the last quantity tends to 0. But according to Wolansky paper, see [15]:

\[ \int_{B^*} V_i(x_i + \delta_i y)e^{v_i} \to 8\pi. \]

Around each exterior blow-up points, there is one blow-up point.

Consider the following quantity:

\[ B_i = \int_{B(x_i+\delta_i c)} <(x-x_i)|\nabla u_i > (V_i(x) - V_i(x_i))e^{u_i} \, dy. \]

Suppose that, we have \( m > 0 \) interior blow-up points. Consider the blow-up point \( t^k_i \) and the associated set \( \Omega_k \) defined as the set of the points nearest \( t^k_i \) we use step by step triangles which are nearest \( x_i \) and we take the mediatrices of those triangles.

\[ \Omega_k = \{ x \in B(x_i,\delta_i c), |x-t^k_i| \leq |x-t^l_i|, j \neq k \}, \]

we write:

\[ B_i = \sum_{k=1}^{m} \int_{\Omega_k} <(x-x_i)|\nabla u_i > (V_i(x) - V_i(x_i))e^{u_i} \, dy. \]

We set, \[ B^k_i = \int_{\Omega_k} <(x-x_i)|\nabla u_i > (V_i(x) - V_i(x_i))e^{u_i} \, dy, \]
We divide this integral in 4 integrals:

\[
B_i^k = \int_{\Omega_k} (x-t_i^k)|\nabla u_i| > (V_i(x) - V_i(t_i^k))e^{u_i} dy + \int_{\Omega_k} < (t_i^k-x_i)|\nabla u_i| > (V_i(x) - V_i(x_i))e^{u_i} dy =
\]

\[
= \int_{\Omega_k} (x-t_i^k)|\nabla u_i| > (V_i(x) - V_i(t_i^k))e^{u_i} dy + \int_{\Omega_k} < (x-t_i^k)|\nabla u_i| > (V_i(x) - V_i(x_i))e^{u_i} dy +
\]

\[
+ \int_{\Omega_k} < (t_i^k-x_i)|\nabla u_i| > (V_i(x) - V_i(t_i^k))e^{u_i} dy + \int_{\Omega_k} < (t_i^k-x_i)|\nabla u_i| > (V_i(x) - V_i(x_i))e^{u_i} dy.
\]

We set:

\[
A_1 = \int_{\Omega_k} (x-t_i^k)|\nabla u_i| > (V_i(x) - V_i(t_i^k))e^{u_i} dy,
\]

\[
A_2 = \int_{\Omega_k} < (x-t_i^k)|\nabla u_i| > (V_i(t_i^k) - V_i(x_i))e^{u_i} dy,
\]

\[
A_3 = \int_{\Omega_k} < (t_i^k-x_i)|\nabla u_i| > (V_i(x) - V_i(t_i^k))e^{u_i} dy,
\]

\[
A_4 = \int_{\Omega_k} < (t_i^k-x_i)|\nabla u_i| > (V_i(t_i^k) - V_i(x_i))e^{u_i} dy.
\]

For \(A_1\) and \(A_2\) we use the fact that in \(\Omega_k\) we have:

\[
u(x) + 2 \log|x - t_i^k| \leq C,
\]

to conclude that for \(0 < s \leq 1\):

\[
A_1 = A_2 = o(1),
\]

we have integrals of the form:

\[
A'_1 = \int_{\Omega_k} |\nabla u_i|e^{(1/2-s/2)u_i} dy = o(1),
\]

and,

\[
A'_2 = \int_{\Omega_k} |\nabla u_i|e^{(1/2-s/4)u_i} dy = o(1).
\]

For \(A_3\) we use the previous fact and the \(\sup + \inf\) inequality to conclude that for \(1/2 < s \leq 1\):

\[
A_3 = o(1)
\]

because we have an integral of the form:

\[
A'_3 = \int_{\Omega_k} |\nabla u_i|e^{(3/4-s/2)u_i} dy = o(1).
\]

For \(A_4\) we use integration by part to have:

\[
A_4 = \int_{\partial\Omega_k} < (t_i^k - x_i)u > (V_i(t_i^k) - V_i(x_i))e^{u_i} dy.
\]

But, the boundary of \(\Omega_k\) is the union of parts of mediatrices of segments linked to \(t_i^k\). Let’s consider a point \(t_j^l\) linked to \(t_i^k\) and denote \(D_{i,j,k}\) the mediatrice of the segment \((t_i^k, t_j^l)\), which is in the boundary of \(\Omega_k\). Note that this mediatrice is in the boundary of \(\Omega_j\) and the same decomposition for \(\Omega_j\) gives us the following term:

\[
A'_4 = \int_{D_{i,j,k}} < (t_j^l - x_i)u > (V_i(t_j^l) - V_i(x_i))e^{u_i} dy.
\]

Thus, we have to estimate the sum of the 2 following terms:
\[ A_5 = \int_{D_{i,j,k}} <(t_i^k - x_i)|\nu > (V_i(t_i^k) - V_i(x_i))e^{nu}dy. \]

and,

\[ A_6 = A_4' = - \int_{D_{i,j,k}} <(t_j^i - x_i)|\nu > (V_j(t_j^i) - V_i(x_i))e^{nu}dy. \]

We can write them as follows:

\[ A_5 = \int_{D_{i,j,k}} <(x - x_i)|\nu > (V_i(t_i^k) - V_i(x_i))e^{nu}dy + \int_{D_{i,j,k}} <(t_i^k - x)|\nu > (V_i(t_i^k) - V_i(x_i))e^{nu}dy. \]

and,

\[ A_6 = - \int_{D_{i,j,k}} <(x - x_i)|\nu > (V_i(t_i^k) - V_i(x_i))e^{nu}dy - \int_{D_{i,j,k}} <(t_i^k - x)|\nu > (V_i(t_i^k) - V_i(x_i))e^{nu}dy. \]

We can write:

\[ \int_{D_{i,j,k}} <(x - x_i)|\nu > (V_i(t_i^k) - V_i(x_i))e^{nu}dy - \int_{D_{i,j,k}} <(x - x_i)|\nu > (V_i(t_i^k) - V_i(x_i))e^{nu}dy = 0(1), \]

for \(1/2 < s \leq 1\). Because, we do integration on the mediatrice of \((t_i^k, t_i^k)\), \(|x - t_i^k| = |x - t_i^k|\), and:

\[ |V_i(t_i^k) - V_i(x_i)| \leq 2A|x - t_i^k|^s \]

and,

\[ u_i(x) + 2\log|x - t_i^k| \leq C, \]

To estimate the integral of the following term:

\[ e^{(3/4 - s/2)u_i} \leq C_{t}(-3/2 + s), \]

which is integrable and tends to 0, for \(1/2 < s \leq 1\), because we are on the ball \(B(x_i, \delta_i\epsilon)\).

In other part, for the term:

\[ \int_{D_{i,j,k}} <(t_i^k - x)|\nu > (V_i(t_i^k) - V_i(x_i))e^{nu}dy - \int_{D_{i,j,k}} <(t_i^k - x)|\nu > (V_i(t_i^k) - V_i(x_i))e^{nu}dy. \]

We use the fact that, on \(D_{i,j,k}\):

\[ |x - t_i^k| = |x - t_i^k|, \]

\[ u_i(x) + 2\log|x - t_i^k| \leq C, \]

\[ |V_i(t_i^k) - V_i(x_i)| \leq 2A|x_i - t_i^k|^s \leq \delta_i^s, \]

and,

\[ |V_i(t_j^i) - V_i(x_i)| \leq 2A|x_i - t_j^i|^s \leq \delta_i^s, \]

To estimate the integral of the following term:

\[ e^{(1/2 - s/4)u_i} \leq C_{t}(-1 + s/2), \]

which is integrable and tends to 0, because we are on the ball \(B(x_i, \delta_i\epsilon)\).
Thus,

\[ B_i = o(1), \]

**Proof of the theorem 2:**

Next, we use the formulation of the case of three blow-up points, see [2]. Because the blow-ups points are simple, we can consider the following function:

\[ v_i(\theta) = u_i(x_i + r_i\theta) - u_i(x_i), \]

where \( r_i \) is such that:

\[ r_i = e^{-u_i(x_i)/2}, \]

\[ \int_{B_i} V_i(x_i + \delta_i y) e^{u_i} \to 8\pi. \]

\[ u_i(x_i + r_i\theta) = \int_{\Omega} G(x_i + r_i\theta, y) V_i(y) e^{u_i(y)} dx = \]

\[ = \int_{\Omega - B(x_i, 2\delta_i \epsilon')} G(x_i, y) V_i e^{u_i(y)} dy + \int_{B(x_i, 2\delta_i \epsilon')} G(x_i + r_i\theta, y) V_i e^{u_i(y)} dy \]

We write, \( y = x_i + r_i\theta, \) with \( \tilde{\theta} \leq \frac{\delta_i}{r_i} \).

\[ u_i(x_i + r_i\theta) = \int_{B(0, 2\delta_i \epsilon')} \frac{1}{2\pi} \log \frac{|1 - (x_i + r_i\theta)/(x_i + r_i\theta)|}{r_i|\theta - \theta|} V_i e^{u_i(y)} r_i^2 dy + \]

\[ + \int_{\Omega - B(x_i, 2\delta_i \epsilon')} G(x_i, y) V_i e^{u_i(y)} dy \]

\[ u_i(x_i) = \int_{\Omega - B(x_i, 2\delta_i \epsilon')} G(x_i, y) V_i e^{u_i(y)} dy + \int_{B(x_i, 2\delta_i \epsilon')} G(x_i + r_i\theta, y) V_i e^{u_i(y)} dy \]

Hence,

\[ u_i(x_i) = \int_{B(0, 2\delta_i \epsilon')} \frac{1}{2\pi} \log \frac{|1 - x_i(x_i + r_i\theta)|}{r_i|\theta - \theta|} V_i e^{u_i(y)} r_i^2 dy + \]

\[ + \int_{\Omega - B(x_i, 2\delta_i \epsilon')} G(x_i, y) V_i e^{u_i(y)} dy \]

We look to the difference,

\[ v_i(\theta) = u_i(x_i + r_i\theta) - u_i(x_i) = \int_{B(0, 2\delta_i \epsilon')} \frac{1}{2\pi} \log \frac{|\tilde{\theta}|}{|\theta - \theta|} V_i e^{u_i(y)} r_i^2 dy + h_1 + h_2, \]

where,

\[ h_1(\theta) = \int_{\Omega - B(x_i, 2\delta_i \epsilon')} G(x_i + r_i\theta, y) V_i e^{u_i(y)} dy - \int_{\Omega - B(x_i, 2\delta_i \epsilon')} G(x_i, y) V_i e^{u_i(y)} dy, \]

and,

\[ h_2(\theta) = \int_{B(0, 2\delta_i \epsilon')} \frac{1}{2\pi} \log \frac{|1 - (x_i + r_i\theta)y|}{|1 - x_i y|} V_i e^{u_i(y)} dy. \]

Remark that, \( h_1 \) and \( h_2 \) are two harmonic functions, uniformly bounded.

According to the maximum principle, the harmonic function \( G(x_i + r_i\theta, \cdot) \) on \( \Omega - B(x_i, 2\delta_i \epsilon') \) take its maximum on the boundary of \( B(x_i, 2\delta_i \epsilon') \), we can compute this maximum:

\[ G(x_i + r_i\theta, y_i) = \frac{1}{2\pi} \log \frac{|1 - (x_i + r_i\theta)y_i|}{|x_i + r_i\theta - y_i|} \approx \frac{1}{2\pi} \log \frac{|1 + |x_i||\delta_i - \delta_i(3\epsilon' + o(1))|}{\delta_i \epsilon'} \leq C_{\epsilon'} < +\infty \]
with \( y_i = x_i + 2\delta_i \theta_i e', |\theta_i| = 1 \), and \(|r_i \theta| \leq \delta_i e'\).

We can remark, for \(|\theta| \leq \frac{\delta_i e'}{r_i}\), that \( v_i \) is such that:

\[
v_i = h_1 + h_2 + \frac{1}{2\pi} \int_{B(0,2d e')} \frac{1}{|\theta - \tilde{\theta}|} V_i e^{u_i(y)} r_i^2 dy,
\]

\[
v_i = h_1 + h_2 + \frac{1}{2\pi} \int_{B(0,2d e')} \frac{1}{|\theta - \tilde{\theta}|} V(x_i + r_i \tilde{\theta}) e^{v_i(\theta)} d\tilde{\theta},
\]

with \( h_1 \) and \( h_2 \), the two uniformly bounded harmonic functions.

**Remark:** In the case of 2 or 3 or 4 blow-up points, and if we consider the half ball, we have supplementary terms, around the 2 other blow-up terms. Note that the Green function of the half ball is quasi-similar to the one of the unit ball and our computations are the same if we consider the half ball.

By the asymptotic estimates of Cheng-Lin, we can see that, we have the following uniform estimates at infinity. We have, after considering the half ball and its Green function, the following estimates:

\[
|h| \leq \delta_i e', \quad \text{here,}
\]

\[
|\theta| \leq \frac{\delta_i e'}{r_i}, \quad \text{here,}
\]

\[
\text{if, for } \epsilon > 0, \quad \text{we have:}
\]

\[
-4 - \epsilon \log |\theta| - C_{\epsilon, e'} \leq v_i(\theta) \leq (-4 + \epsilon) \log |\theta| + C_{\epsilon, e'},
\]

and,

\[
\partial_j v_i \simeq \partial_j u_0(\theta) \pm \frac{\epsilon}{|\theta|} + C \left( \frac{r_i}{d_i} \right)^2 |\theta| + m \times \left( \frac{r_i}{d_i} \right) + \sum_{k=2}^{m} C_1 \left( \frac{r_i}{d(x_i, x_i^k)} \right),
\]

In the case, we have:

\[
d(x_i, x_i^k) \to +\infty \quad \text{for } k = 2 \ldots m,
\]

We have after using the previous term of the Pohozaev identity, for \( 1/2 < s \leq 1 \):

\[
o(1) = J_i' = m' + \sum_{k=1}^{m} C_k o(1),
\]

\[
0 = \lim_{\epsilon \to 0} \lim_{i \to \epsilon} J_i' = m',
\]

which contradict the fact that \( m' > 0 \). Here,

\[
J_i = B_i = \int_{B(x_i, d_i e')} < x_i^j \nabla (u_i - u) > (V_i - V_i(x_i)) e^{u_i} dy.
\]

We use the previous formulation around each blow-up point. If, for \( x_i^j \), we have:

\[
d(x_i^j, x_i^k) \to +\infty \quad \text{for } k \neq j, k = 1 \ldots m,
\]

We use the previous formulation around this blow-up point. We consider the following quantity:

\[
J_i' = B_i' = \int_{B(x_i^j, d_i e')} < x_i^j \nabla (u_i - u) > (V_i - V_i(x_i^j)) e^{u_i} dy.
\]

with,
\[ x_i^j = (\delta_i^j, 0), \]

In this case, we set:

\[ v_i^j(\theta) = u_i(x_i^j + r_i^j \theta) - u_i(x_i^j), \]

where \( r_i^j \) is such that:

\[ r_i^j = e^{-u_i(x_i^j)/2}, \]

\[ \int_{B(x_i^j, \delta_i^j \epsilon)} V_i(x_i^j + \delta_i^j \epsilon y) e^{u_i} \to 8\pi. \]

We have, after considering the half ball and its Green function, the following estimates:

\[ \forall \epsilon > 0, \epsilon' > 0 \exists k_{\epsilon, \epsilon'} \in \mathbb{R}_+, \ i_{\epsilon, \epsilon'} \in \mathbb{N} \text{ and } C_{\epsilon, \epsilon'} > 0, \text{ such that, for } i \geq i_{\epsilon, \epsilon'} \text{ and } k_{\epsilon, \epsilon'} \leq |\theta| \leq \delta_{i_{\epsilon, \epsilon'}}^j, \]

\[ (-4 + \epsilon) \log |\theta| - C_{\epsilon, \epsilon'} \leq v_i^j(\theta) \leq (-4 + \epsilon) \log |\theta| + C_{\epsilon, \epsilon'}, \]

and,

\[ \partial_k v_i^j \simeq \partial_k u_i^0(\theta) \pm \frac{\epsilon}{|\theta|} + C \left( \frac{r_i^j}{\delta_i^j} \right)^2 |\theta| + m \times \left( \frac{r_i^j}{\delta_i^j} \right) + \sum_{l \neq j}^m C_l \left( \frac{r_i^j}{d(x_i^j, x_i^k)} \right), \]

We have after using the previous term of the Pohozaev identity, for \( 1/2 < s \leq 1 \):

\[ o(1) = J_i^j = B_i^j = m' + \sum_{l \neq j}^m C_l o(1), \]

\[ 0 = \lim_{\epsilon \to 0} \lim_{\epsilon' \to 0} \lim_{i \to 1} J_i^j = m', \]

which contradict the fact that \( m' > 0 \).

If, for \( x_i^j \), we have:

\[ \frac{d(x_i^j, x_i^k)}{\delta_i^j} \leq C_{j, k} \text{ for some } k = k_j \neq j, 1 \leq k \leq m, \]

All the distances \( d(x_i^j, x_i^k) \) are comparable with some \( \delta_i^j \). This means that we can use the Pohozaev identity directly. We can do this for example, for 4 blow-ups points.

We have many cases:

Case 1: the blow-up points are "equivalents", it seems that we have the same radius for the blow-up points.

Case 2: 3 points are "equivalents" and another blow-up point linked to the 3 blow-up points. We apply the Pohozaev identity directly with central point which link the 3 blow-up to the last.

Case 3: 2 pair of blow-up points separated.

Case 3.1: the 2 pair are linked: we apply the Pohozaev identity.

Case 3.2: the two pair are separated. It is the case of two separated blow-up points, see [1]

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