Nonexistence of Global Weak Solutions for Evolution Equations with Fractional Laplacian

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Abstract—A nonlocal nonlinear parabolic equation with fractional Laplacian is considered. By means of the method of test functions, the nonexistence of nontrivial global weak solutions is demonstrated. Simultaneously, the nonexistence of nontrivial weak solutions for the corresponding elliptic case is established.

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1. INTRODUCTION

In the present paper, we study the Cauchy problem for the nonlocal nonlinear parabolic equation

\[
\begin{align*}
    u_t + (-\Delta)^{\beta/2}(|u|^p) &= |u|^q, & x \in \mathbb{R}^N, & t > 0, \\
    u(x,0) &= u_0(x), & x \in \mathbb{R}^N,
\end{align*}
\]

where \( u_0 \in L^1_{\text{loc}}(\mathbb{R}^N) \), \( N \geq 1 \), \( 0 < \beta < 2 \), \( p > 0 \), \( q > 1 \), and the nonlocal operator \((-\Delta)^{\beta/2}\) is defined as

\[ (-\Delta)^{\beta/2}v(x) := \mathcal{F}^{-1}(|\xi|^{\beta}\mathcal{F}(v)(\xi))(x) \]

for any \( v \in D((-\Delta)^{\beta/2}) = H^{\beta}(\mathbb{R}^N) \); here \( H^{\beta}(\mathbb{R}^N) \) is the homogeneous Sobolev space of order \( \beta \) defined by \( H^{\beta}(\mathbb{R}^N) = \{ u \in \mathcal{L}^2(\mathbb{R}^N); (-\Delta)^{\beta/2}u \in L^2(\mathbb{R}^N) \} \) for \( \beta \in \mathbb{N} \), where \( \mathcal{F} \) denotes the Fourier transform, \( \mathcal{F}^{-1} \) is the inverse Fourier transform, and \( \Gamma \) is the Euler gamma function.

Definition 1. Let \( u_0 \in L^1_{\text{loc}}(\mathbb{R}^N) \), \( 0 < \beta < 2 \), and let \( T > 0 \). We shall call \( u \) a weak solution of problem (1) if \( u \in L^p((0,T), L^{2p}(\mathbb{R}^N)) \cap L^q((0,T), L^q_{\text{loc}}(\mathbb{R}^N)) \) satisfies the equation

\[ \int_{\mathbb{R}^N} u_0(x) \varphi(x,0) \, dx + \int_0^T \int_{\mathbb{R}^N} |u|^q \varphi(x,t) \, dx \, dt = \int_0^T \int_{\mathbb{R}^N} u \varphi_t(x,t) \, dx \, dt \]

for all compactly supported functions \( \varphi \in C([0,T], H^{\beta}(\mathbb{R}^N)) \cap C^1([0,T], L^\infty(\mathbb{R}^N)) \)

for which \( \varphi(\cdot,T) = 0 \).

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Our main results are as follows.

**Theorem 1.** Let $0 < \beta < 2$, $p > 0$, and let $q > 1$. For $u_0 \in L^1_{\text{loc}}(\mathbb{R}^N)$, $u_0 \geq 0$, if $p < q \leq p + \beta/N$, then problem (1) has no nontrivial global weak solutions.

**Theorem 2.** Let $0 < \beta < 2$, $p > 0$, and let $q > 1$. For $u_0 \in L^1_{\text{loc}}(\mathbb{R}^N)$, we assume that there exists a constant $\varepsilon > 0$ such that, for all $0 < \gamma < N$, the initial function satisfies the following condition:

$$u_0(x) \geq \varepsilon (1 + |x|^2)^{-\gamma/2}.$$  

If $p < q < p + \beta/\gamma$, then problem (1) has no global weak solutions.

We prove these nonexistence theorems by using a modification of the method of test functions due to Zhang [1], Mitidieri and Pokhozhaev [2], [3], Kirane, etc. (see [4]–[6]); this method was also used by Baras and Kersner [7] for the study of necessary conditions for the existence of local solutions. However, it seems impossible to use the method directly, because it was first developed for other types of differential operators, such as integer powers of the Laplacian. But it is known [8] that many equations and inequalities containing such operators have comparatively small sets of solutions. For example, harmonic functions (the solutions of the Laplace equation) cannot approximate an arbitrary function and inequalities containing such operators have comparatively small sets of solutions. For example, nonlinear summands that exclude the existence of any nontrivial solutions in general.

Conversely, the sets of solutions for problems involving differential operators of noninteger order are usually much wider; sometimes they are even locally dense in the space $C(\mathbb{R}^N)$, as in the case of $s$-harmonic functions (i.e., $u$ such that $(-\Delta)^s u = 0$); see [9], [10] for the case $0 < s < 1$ and [8], [11] for $s > 1$. Therefore, in order to obtain results on the nonexistence of solutions in this situation, it is required to eliminate the existence of solutions from this wider set. Thus, the problem of the nonexistence of solutions to problems involving operators of noninteger order is more delicate and requires an essential modification of the known methods. Apparently, results of such type have only been obtained in a few particular cases, for example, in [12], for systems of elliptic equations containing fractional powers of the Laplacian and also by the second and third authors of the present paper in [13]–[15] for some (mostly, elliptic) inequalities and their systems with the same operators. Here we carry over these results to the evolution problem (1) not covered by previous results.

Note that, for the limit case $\beta \to 2$, Theorem 1 was proved in [2, Theorem 29.1], and the assertion of Theorem 2 can be obtained in a similar way.

In what follows, the positive constants will be denoted by the letter $C$ and may vary from one line to another.

The remaining part of the paper consists of two sections and a supplement. In Sec. 2, we prove Theorem 1, and in Sec. 3, Theorem 2. In the supplement, we give the proof of Ju’s inequality used in Sec. 2.

### 2. PROOF OF THEOREM 1

The proof is carried out by contradiction. We assume that $u$ is a global weak solution of (1); then, for all $T \gg 1$, we have

$$\int_{\mathbb{R}^N} u_0(x)\varphi(x,0) \, dx + \int_0^T \int_{\mathbb{R}^N} |u|^q \varphi(x,t) \, dx \, dt$$

$$= \int_0^T \int_{\mathbb{R}^N} |u|^p (-\Delta)^{\beta/2} \varphi \, dx \, dt - \int_0^T \int_{\mathbb{R}^N} u \varphi_t(x,t) \, dx \, dt$$

for all test functions $\varphi \in C([0,T], H^\beta(\mathbb{R}^N)) \cap C^1([0,T], L^\infty(\mathbb{R}^N))$ such that $\text{supp} \varphi$ is compact and $\varphi(\cdot, T) = 0$. Now we take

$$\varphi(x,t) := \varphi_1(x)\varphi_2(t), \quad \text{with} \quad \varphi_1(x) := \Phi\left(\frac{|x|}{T^\alpha}\right), \quad \varphi_2(t) := \Phi\left(\frac{t}{T}\right),$$

where

$$\Phi(s) := \int_s^{\infty} e^{-t} \, dt.$$
where \( \alpha = (q - p)/(\beta(q - 1)), \ell, \eta \gg 1 \), and \( \Phi \) is a smooth nonnegative nonincreasing function,

\[
\Phi(r) = \begin{cases} 
1 & \text{for } 0 \leq r \leq \frac{1}{2}, \\
0 & \text{for } r \geq 1.
\end{cases}
\]

We obtain

\[
\int_{\Omega_T} u_0(x)\varphi_1^\ell(x) \, dx + \int_0^T \int_{\Omega_T} |u|^q \varphi(x, t) \, dx \, dt \\
= \int_0^T \int_{\mathbb{R}^N} |u|^p \varphi_2^\eta(t)(-\Delta)^{\beta/2}(\varphi_1^\ell(x)) \, dx \, dt - \int_0^T \int_{\Omega_T} u \varphi_1^\ell(x) \frac{d}{dt} \varphi_2^\eta(t) \, dx \, dt,
\]

where

\[
\Omega_T := \{ x \in \mathbb{R}^N; |x| \leq T^\alpha \}. \tag{3}
\]

Using Ju's inequality \((-\Delta)^{\beta/2}(\varphi_1^\ell) \leq \ell \varphi_1^{\ell-1}(-\Delta)^{\beta/2}(\varphi_1)\) (see the supplement) and the inequality \( u_0 \geq 0 \), we can write

\[
\int_0^T \int_{\Omega_T} |u|^q \varphi(x, t) \, dx \, dt \leq C \int_0^T \int_{\Omega_T} |u|^p \varphi_2^\eta(t) \varphi_1^{\ell-1}(x)(-\Delta)^{\beta/2}(\varphi_1(x)) \, dx \, dt \\
+ C \int_0^T \int_{\Omega_T} |u| \varphi_1^\ell(x) \varphi_2^{\eta-1}(t) \left| \frac{d}{dt} \varphi_2(t) \right| \, dx \, dt \\
\leq C \int_0^T \int_{\Omega_T} |u|^p \varphi^{p/q} \varphi^{q/p}_2(t) \varphi_1^{\ell-1}(x)(-\Delta)^{\beta/2}(\varphi_1(x)) \, dx \, dt \\
+ C \int_0^T \int_{\Omega_T} |u| \varphi_1^{1/q} \varphi^{-1/q}_2(t) \varphi_1^{\eta-1}(x) \varphi_2^{\eta-1}(t) \left| \frac{d}{dt} \varphi_2(t) \right| \, dx \, dt. \tag{4}
\]

Therefore, applying Young's inequality

\[
ab \leq \frac{1}{4} a^{q/p} + Ct^{\beta/(q-p)} \tag{5}
\]

with

\[
\begin{align*}
a &= |u|^p \varphi^{p/q}, \\
b &= C \varphi^{p/q} \varphi_2^\eta(t) \varphi_1^{\ell-1}(x)(-\Delta)^{\beta/2}(\varphi_1(x))\end{align*} \tag{6}
\]

to the first integral on the right-hand side of (4) and the Young inequality

\[
ab \leq \frac{1}{4} a^q + Ct^{\bar{q}}, \quad \text{where } \bar{q} = \frac{q}{q-1}, \tag{7}
\]

with

\[
\begin{align*}
a &= |u| \varphi^{1/q}, \\
b &= C \varphi^{-1/q} \varphi_1^{\ell}(x) \varphi_2^{\eta-1}(t) \left| \frac{d}{dt} \varphi_2(t) \right|
\end{align*} \tag{8}
\]

to the second integral on the right-hand side of (4), we obtain

\[
\frac{1}{2} \int_0^T \int_{\Omega_T} |u|^q \varphi(x, t) \, dx \, dt \leq C \int_0^T \int_{\Omega_T} \varphi_2^\eta(t) \varphi_1^{\ell-q/(q-p)}(x)(-\Delta)^{\beta/2}(\varphi_1(x)) |\varphi^{q/(q-p)}(x)| \, dx \, dt \\
+ C \int_0^T \int_{\Omega_T} \varphi_1^\ell(x) \varphi_2^{\eta-\bar{q}}(t) \left| \frac{d}{dt} \varphi_2(t) \right|^{\bar{q}} \, dx \, dt. \tag{9}
\]
Now, scaling the variables $s = T^{-1}t$, $y = T^{-\alpha}x$ on the right-hand side of (9), we conclude that

$$
\int_0^T \int_{\Omega_T} |u|^q \varphi(x, t) \, dx \, dt \leq CT^{-\delta},
$$

(10)

where $\delta := q/(q - 1) - N\alpha - 1$. Further, noting that

$$
q \leq q^* := p + \frac{\beta}{N} \iff \delta \geq 0,
$$

we must distinguish the following two cases.

**Case 1:** $q < q^*$ (i.e., $\delta > 0$). Passing to the limit as $T \to \infty$ in (10), we obtain

$$
\lim_{T \to \infty} \int_0^T \int_{|x| \leq (BT)^\alpha} |u|^q \varphi(x, t) \, dx \, dt = 0.
$$

Using Lebesgue’s dominated convergence theorem and the fact that $\varphi(x, t) \to 1$ as $T \to \infty$, we conclude that

$$
\int_0^\infty \int_{\mathbb{R}^N} |u|^q (x) \, dx \, dt = 0,
$$

and, by virtue of the continuity of $u$ with respect to the temporal and spatial variables, we see that $u \equiv 0$.

**Case 2:** $q = q^*$ (i.e., $\delta = 0$). Applying inequality (10) with $T \to \infty$ and taking into account the equality $q = q^*$, we obtain $u \in L^q((0, \infty), L^q(\mathbb{R}^N))$, whence we have

$$
\lim_{T \to \infty} \int_{T/2}^T \int_{|x| \leq (BT)^\alpha} |u|^q \varphi(x, t) \, dx \, dt
$$

$$
\begin{align*}
&= \lim_{T \to \infty} \int_0^T \int_{|x| \leq (BT)^\alpha} |u|^q \varphi(x, t) \, dx \, dt - \lim_{T \to \infty} \int_0^{T/2} \int_{|x| \leq (BT)^\alpha} |u|^q \varphi(x, t) \, dx \, dt \\
&= \int_0^\infty \int_{\mathbb{R}^N} |u|^q \varphi(x, t) \, dx \, dt - \int_0^\infty \int_{\mathbb{R}^N} |u|^q \varphi(x, t) \, dx \, dt = 0.
\end{align*}
$$

(11)

On the other hand, repeating the same calculations as above, but also additionally using the equality $\varphi_1(x) := \Phi(|x|/(B^{\alpha}T))$, where $1 \leq B < T$ is sufficiently large and $B$ does not tend to $\infty$ as $T \to \infty$, and applying Hölder’s inequality

$$
\int ab \leq \left( \int a^q \right)^{1/q} \left( \int b^\bar{q} \right)^{1/\bar{q}}
$$

with

$$
\begin{align*}
a &= |u|^q \varphi_1^{1/q}, \\
b &= \varphi_2^{-1/q}(x) \varphi_1^\eta \varphi_2^{|q-1|}(t) \left| \frac{d}{dt} \varphi_2(t) \right|
\end{align*}
$$

to the second integral on the right-hand side of (4) instead of Young’s inequality, we obtain

$$
\begin{align*}
\int_0^T \int_{\Omega_{BT}} |u|^q \varphi(x, t) \, dx \, dt \\
&\leq \frac{1}{4} \int_0^T \int_{\Omega_{BT}} |u|^q \varphi(x, t) \, dx \, dt \\
&\quad + C \int_0^T \int_{\Omega_{BT}} \varphi_2^{-\eta}(t) \varphi_1^\ell (x) \varphi_2^{\eta-q/p}(x) \Delta (\varphi_1(x))^{q/(q-p)} \, dx \, dt \\
&\quad + C \left( \int_{T/2}^T \int_{\Omega_{BT}} |u|^q \varphi(x, t) \, dx \, dt \right)^{1/q} \left( \int_0^T \int_{\Omega_{BT}} \varphi_1^\ell (x) \varphi_2^{-\eta}(t) \left| \frac{d}{dt} \varphi_2(t) \right|^\bar{q} \, dx \, dt \right)^{1/\bar{q}}.
\end{align*}
$$
where \( \Omega_{BT} := \{ x \in \mathbb{R}^N; |x| \leq (BT)\alpha \} \); therefore, 

\[
\int_0^T \int_{\Omega_{BT}} |u|^q \varphi(x,t) \, dx \, dt 
\leq C \int_0^T \int_{\Omega_{BT}} \varphi_2^\eta(t)\varphi_1^{\frac{\ell-q/(q-p)}{\eta}}(x)(-\Delta)^{\beta/2}(\varphi_1(x))|^{q/(q-p)} \, dx \, dt 
\]

\[
+ C \left( \int_{T/2}^T \int_{\Omega_{BT}} |u|^q \varphi(x,t) \, dx \, dt \right)^{1/q} \left( \int_0^T \int_{\Omega_{BT}} \varphi_1^{\frac{\eta}{\eta-\tilde{q}}}(t) \left| \frac{d}{dt} \varphi_2(t) \right| \, dx \, dt \right)^{1/\tilde{q}}.
\]

Scaling as follows: \( s = T^{-1}t, y = (TB)^{-\alpha}x \) and taking into account the equality \( q = q^* \), we easily conclude that 

\[
\int_0^T \int_{\Omega_{BT}} |u|^q \varphi(x,t) \, dx \, dt \leq CB^{-1} + C(B^N\alpha)^{1/\tilde{q}} \left( \int_{T/2}^T \int_{\Omega_{BT}} |u|^q \varphi(x,t) \, dx \, dt \right)^{1/q}.
\]

Therefore, passing to the limit as \( T \to \infty \) and then as \( B \to \infty \), and using (11), we obtain 

\[
\int_0^\infty \int_{\mathbb{R}^N} |u|^q(x,t) \, dx \, dt = 0,
\]

i.e., \( u \equiv 0 \). This concludes the proof.

3. PROOF OF THEOREM 2

The proof is also carried out by contradiction and uses the same arguments as in Sec. 2. Suppose that \( u \) is a global weak solution of (1); then, for \( T \gg 1 \), we have 

\[
\int_{\mathbb{R}^N} u_0(x)\varphi(x,0) \, dx + \int_0^T \int_{\mathbb{R}^N} |u|^q \varphi(x,t) \, dx \, dt 
= \int_0^T \int_{\mathbb{R}^N} |u|^p(-\Delta)^{\beta/2} \varphi \, dx \, dt - \int_0^T \int_{\mathbb{R}^N} u \varphi_t(x,t) \, dx \, dt \tag{12}
\]

for all test functions \( \varphi \in C((0,T], H^\beta(\mathbb{R}^N)) \cap C^1([0,T], L^\infty(\mathbb{R}^N)) \) such that \( \text{supp} \varphi \) is compact and \( \varphi(\cdot,T) = 0 \).

Using the same test functions \( \varphi(x,t) \) as above, we obtain 

\[
\int_{\Omega_T} u_0(x)\varphi_1^\ell(x) \, dx + \int_0^T \int_{\Omega_T} |u|^q \varphi(x,t) \, dx \, dt 
= \int_0^T \int_{\mathbb{R}^N} |u|^p\varphi_2^\eta(t)(-\Delta)^{\beta/2}(\varphi_1(x)) \, dx \, dt - \int_0^T \int_{\Omega_T} u \varphi_1^\ell(x) \frac{d}{dt} \varphi_2^\eta \, dx \, dt,
\]

where \( \Omega_T \) is given by (3). Using Ju’s inequality \((-\Delta)^{\beta/2}(\varphi_1^\ell) \leq \ell\varphi_1^{\ell-1}(-\Delta)^{\beta/2}(\varphi_1)\), we can write 

\[
\int_0^T \int_{\Omega_T} |u|^q \varphi(x,t) \, dx \, dt + \int_{\Omega_T} u_0(x)\varphi_1^\ell(x) \, dx 
\leq C \int_0^T \int_{\Omega_T} |u|^p\varphi_2^\eta(t)\varphi_1^{\ell-1}(x)(-\Delta)^{\beta/2}(\varphi_1(x)) \, dx \, dt 
\]

\[
+ C \int_0^T \int_{\Omega_T} |u|\varphi_1^\ell(x)\varphi_2^{\eta-1}(t) \left| \frac{d}{dt} \varphi_2(t) \right| \, dx \, dt 
\leq C \int_0^T \int_{\Omega_T} |u|^p\varphi_2^\eta\varphi_1^{\ell-1}(x)(-\Delta)^{\beta/2}(\varphi_1(x)) \, dx \, dt 
\]

\[
+ C \int_0^T \int_{\Omega_T} |u|\varphi_1^{\ell-1}\varphi_2(\varphi_1(x))\varphi_2^{\eta-1}(t) \left| \frac{d}{dt} \varphi_2(t) \right| \, dx \, dt. \tag{13}
\]
Since \( u_0(x) \geq \varepsilon (1 + |x|^2)^{-\gamma/2} \), we have
\[
\int_{\Omega_T} u_0(x)\varphi_1^p(x) \, dx \geq \int_{|x| \leq T_{\alpha}^\gamma} u_0(x) \, dx \geq \varepsilon \int_{|x| \leq T_{\alpha}^\gamma} (1 + |x|^2)^{-\gamma/2} \, dx \geq C\varepsilon T^{\alpha(N-\gamma)}.
\]
Therefore, estimating the right-hand side of inequality (13) exactly in the same way as for the right-hand side of inequality (4) (see formulas (5)–(8)), we obtain
\[
C\varepsilon T^{\alpha(N-\gamma)} + \frac{1}{2} \int_0^T \int_{\Omega_T} |u|^q \varphi(x, t) \, dx \, dt
\leq C \int_0^T \int_{\Omega_T} \varphi_2^p(t) \varphi_1^{\frac{\varepsilon - q}{(q-p)}(x)|(-\Delta)^{\beta/2}\varphi_1(x)|^{\frac{q}{(q-p)}} \, dx \, dt
\]
\[
+ C \int_0^T \int_{\Omega_T} \varphi_2^p(x)\varphi_2^{\frac{\varepsilon}{q}}(t) \left| \frac{d}{dt}\varphi_2(t) \right| \frac{q}{\delta} \, dx \, dt.
\]
(14)

Again introducing the scaled variables \( s = T^{-1}t, y = T^{-\alpha}x \) on the right-hand side of (14) and taking into account the nonnegativity of the second summand on the left-hand side of (14), we conclude that
\[
C\varepsilon T^{\alpha(N-\gamma)} \leq C\varepsilon T^{-q/(q-1)+N\alpha+1},
\]
i.e.,
\[
\varepsilon \leq CT^{\delta^*}, \quad \text{where} \quad \delta^* := q/(q-1) - N\alpha - 1 + \alpha(N-\gamma).
\]
(15)

Further, noting that
\[
q < q^{**} := p + \frac{\beta}{\gamma} \quad \iff \quad \delta^* > 0
\]
and passing to the limit as \( T \to \infty \) in (15), we obtain a contradiction.

**Remark 1.** Applying the same calculations to the corresponding nonlocal elliptic equation
\[
(-\Delta)^{\beta/2}(|u|^p) = |u|^q, \quad x \in \mathbb{R}^N,
\]
(16)

where \( N \geq 1, 0 < \beta \leq 2, p > 0, \) and \( q > 1, \) we can obtain the following result: if
\[
p < q < \frac{Np}{(N-\beta)^+},
\]
then problem (16) has no nontrivial weak solutions.

**Remark 2.** Using the test function from the recent paper [16], we can obtain a similar result for the following equation:
\[
\begin{aligned}
\begin{cases}
u + (-\Delta)^{\beta/2}(|u|^{p-1}u) = |u|^q, & x \in \mathbb{R}^N, \quad t > 0, \\
u(x, 0) = u_0(x), & x \in \mathbb{R}^N,
\end{cases}
\end{aligned}
\]

where \( p, q > 1. \)

**SUPPLEMENT**

In this supplement, we give the proof of Ju’s inequality (see [17, Proposition 3.3]) for the dimension \( N \geq 1, \) where \( \delta \in [0, 2] \) and \( q \geq 1, \) for all nonnegative Schwartz distributions \( \psi \) (in the general case)
\[
(-\Delta)^{\delta/2}\psi^q \leq q\psi^{q-1}(-\Delta)^{\delta/2}\psi.
\]
The cases \( \delta = 0 \) and \( \delta = 2, \) as well as \( q = 1, \) are obvious. If \( \delta \in (0, 2) \) and \( q > 1, \) then, using [18, Definition 3.2], we obtain
\[
(-\Delta)^{\delta/2}\psi(x) = -c_N(\delta) \int_{\mathbb{R}^N} \frac{\psi(x + z) - \psi(x)}{|z|^{N+\delta}} \, dz \quad \text{for all} \quad x \in \mathbb{R}^N,
\]
where \( c_N(\delta) = 2^q \Gamma((N + \delta)/2)/(\pi^{N/2} \Gamma(1 - \delta/2)) \). Then

\[
(\psi(x))^{q-1}(-\Delta)^{\delta/2}\psi(x) = -c_N(\delta) \int_{\mathbb{R}^N} \frac{(\psi(x))^q - (\psi(x+z))^q}{|z|^{N+\delta}} \, dz.
\]

By Young’s inequality, we have

\[
(\psi(x))^{q-1}\psi(x+z) \leq \frac{q-1}{q}(\psi(x))^q + \frac{1}{q}(\psi(x+z))^q.
\]

Hence

\[
(\psi(x))^{q-1}(-\Delta)^{\delta/2}\psi(x) \geq -c_N(\delta) \int_{\mathbb{R}^N} \frac{(\psi(x+z))^q - (\psi(x))^q}{|z|^{N+\delta}} \, dz = \frac{1}{q}(-\Delta)^{\delta/2}(\psi(x))^q.
\]

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