GÖLLNITZ-GORDON PARTITIONS WITH WEIGHTS AND PARITY CONDITIONS

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Abstract. A Göllnitz-Gordon partition is one in which the parts differ by at least 2, and where the inequality is strict if a part is even. Let \( Q_i(n) \) denote the number of partitions of \( n \) into distinct parts \( \not\equiv i \) (mod 4). By attaching weights which are powers of 2 and imposing certain parity conditions on Göllnitz-Gordon partitions, we show that these are equinumerous with \( Q_i(n) \) for \( i = 0, 2 \). These complement results of Göllnitz on \( Q_i(n) \) for \( i = 1, 3 \), and of Alladi who provided a uniform treatment of all four \( Q_i(n) \), \( i = 0, 1, 2, 3 \), in terms of weighted partitions into parts differing by \( \geq 4 \). Our approach here provides a uniform treatment of all four \( Q_i(n) \) in terms of certain double series representations. These double series identities are part of a new infinite hierarchy of multiple series identities.

1. Introduction

For \( i = 0, 1, 2, 3 \), let \( Q_i(n) \) denote the number of partitions of \( n \) into distinct parts \( \not\equiv i \) (mod 4). The well known (Little) Theorem of Göllnitz \([3]\) is:

**Theorem 1.** For \( i = 1, 3 \), \( Q_i(n) \) equals the number of partitions of \( n \) into parts differing by \( \geq 2 \), where the inequality is strict if a part is odd, and the smallest part is \( > \frac{(4-i)}{2} \).

The analytic representation of Theorem 1 is

\[
\sum_{n=0}^{\infty} \frac{q^{n^2+n}(-q^2;q^2)_n}{(q^2;q^2)_n} = (-q^2;q^4)_\infty(-q^3;q^4)_\infty(-q^4;q^4)_\infty
\]

when \( i = 1 \), and

\[
\sum_{n=0}^{\infty} \frac{q^{n^2+n}(-q^{-1};q^2)_n}{(q^2;q^2)_n} = (-q^4;q_\infty)(-q^2;q^4)_\infty(-q^4;q^4)_\infty,
\]

when \( i = 3 \). In \([14], [15]\), and in what follows, we have used the standard notation

\[(a)_n = (a;q)_n = \prod_{j=0}^{n-1} (1 - aq^j)\]

for any complex number \( a \), and

\[(a)_\infty = \lim_{n \to \infty} (a)_n = \prod_{j=0}^{\infty} (1 - aq^j),\]

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for \(|q| < 1\). The products on the right in (1.1), (1.2) are also equal to

\[
\frac{1}{(q^2;q^8)_\infty(q^4;q^8)_\infty(q^7;q^8)_\infty}
\]

and

\[
\frac{1}{(q;q^8)_\infty(q^3;q^8)_\infty(q^6;q^8)_\infty},
\]

respectively, which have obvious interpretations as generating functions of partitions into parts in certain residue classes \((\text{mod } 8)\), repetition allowed. The equally well known Göllnitz-Gordon partition theorem is

**Theorem 2.** For \(i = 1, 3\), the number of partitions into parts \(\equiv \pm i, 4 \pmod{8}\) equals the number of partitions into parts differing by \(\geq 2\), where the inequality is strict if a part is even, and the smallest part is \(\geq i\).

The analytic representation of Theorem 2 is

\[
\sum_{n=0}^{\infty} q^{n^2} (-q;q^2)_n = \frac{1}{(q;q^8)_\infty(q^4;q^8)_\infty(q^7;q^8)_\infty}
\]

when \(i = 1\), and

\[
\sum_{n=0}^{\infty} q^{n^2+2n} (-q;q^2)_n = \frac{1}{(q^3;q^8)_\infty(q^4;q^8)_\infty(q^7;q^8)_\infty}
\]

when \(i = 3\). Actually (1.3) and (1.4) are equations (36) and (37) in Slater’s famous list [9], but it was Göllnitz [6] and Gordon [7] who independently realized their combinatorial interpretation.

By a reformulation of the (Big) Theorem of Göllnitz [6] (not Theorem 1) using certain quartic transformations, Alladi [1] provided a uniform treatment of all four partition functions \(Q_i(n), i = 0, 1, 2, 3\) in terms of partitions into parts differing by \(\geq 4\), and with certain powers of 2 as weights attached. As a consequence, it was noticed in [1] that \(Q_2(n)\) and \(Q_0(n)\) possess certain more interesting properties than their well known counterparts \(Q_1(n)\) and \(Q_3(n)\). In particular, \(Q_2(n)\) alone among the four functions satisfies the property that for every positive integer \(k\), \(Q_2(n)\) is a multiple of \(2^k\) for almost all \(n\) which was proved by Gordon in an Appendix to [1].

Our goal is to prove Theorem 3 in §2 which shows that by attaching weights which are powers of 2 to the Göllnitz-Gordon partitions of \(n\), and by imposing certain parity conditions, this is made equal to \(Q_2(n)\). Here by a Göllnitz-Gordon partition we mean a partition into parts differing by \(\geq 2\), where the inequality is strict if a part is even. There is a similar result for \(Q_0(n)\), and this is stated as Theorem 4 at the end of §2.

Theorems 3 and 4 are nice complements to Theorem 1 and to results of Alladi [1].

A combinatorial proof of Theorem 3 is given in full in the next section. Theorem 4 is only stated, and its proof which is similar, is omitted.

In proving Theorem 3 we are able to cast it as an analytic identity (see (3.2) in §3) which equates a double series with the product which is the generating function of \(Q_2(n)\). It turns out that there is a two parameter
refinement of (3.2) (see §3) which leads to similar double series representations for all four products

\[ \prod_{m > 0, m \not\equiv i \pmod{4}} (1 + q^m) \]

for \( i = 0, 1, 2, 3 \). It will be shown in §3 that only in the cases \( i = 1, 3 \) do these double series reduce to the single series in (1.1) and (1.2).

Actually, the double series identity (3.2) is the case \( k = 2 \) of a new infinite hierarchy of identities valid for every \( k \geq 1 \). In §4 we use a limiting case of Bailey’s lemma to derive this hierarchy. We give a partition theoretic interpretation of the case \( k = 1 \) and state without proof a doubly bounded polynomial identity which yields our new hierarchy as a limiting case. This polynomial identity will be investigated in detail elsewhere.

2. A NEW WEIGHTED PARTITION THEOREM

Normally, by the parity of an integer we mean its residue class \( \pmod{2} \). Here by the parity of an odd (or even) integer we mean its residue class \( \pmod{4} \).

Next, given a partition \( \pi \) into parts differing by \( \geq 2 \), by a chain \( \chi \) in \( \pi \) we mean a maximal string of parts differing by exactly 2. Thus every partition into parts differing by \( \geq 2 \) can be decomposed into chains. Note that if one part of a chain is odd (resp. even), then all parts of the chain are odd (resp. even). Hence we may refer to a chain as an odd chain or an even chain. Also let \( \lambda(\chi) \) denote the least part of a chain \( \chi \) and \( \lambda(\pi) \) the least part of \( \pi \).

Note that in a Göllnitz-Gordon partition, since the gap between even parts is \( > 2 \), this is the same as saying that every even chain is of length 1, that is, it has only one element.

Finally, given part \( b \) of partition \( \pi \), by \( t(b; \pi) = t(b) \) we denote the number of odd parts of \( \pi \) that are \( < b \).

With this new statistic \( t \) we now have

**Theorem 3.** Let \( S \) denote the set of all special Göllnitz-Gordon partitions, namely, Göllnitz-Gordon partitions \( \pi \) satisfying the parity condition that for every even part \( b \) of \( \pi \)

\[ b \equiv 2t(b) \pmod{4}. \] (2.1)

Decompose each \( \pi \in S \) into chains \( \chi \) and define the weight \( \omega(\chi) \) as

\[
\omega(\chi) = \begin{cases} 
2, & \text{if } \chi \text{ is an odd chain, } \lambda(\chi) \geq 5, \text{ and } \lambda(\chi) \equiv 1 + 2t(\lambda(\chi)) \pmod{4}, \\
1, & \text{otherwise}.
\end{cases}
\] (2.2)

The weight \( \omega(\pi) \) of the partition \( \pi \) is defined multiplicatively as

\[ \omega(\pi) = \prod_{\chi} \omega(\chi), \]
the product over all chains \( \chi \) of \( \pi \). We then have

\[
Q_2(n) = \sum_{\pi \in S, \sigma(\pi) = n} \omega(\pi),
\]

where \( \sigma(\pi) \) is the sum of the parts of \( \pi \).

**Proof:** Consider the partition \( \pi : b_1 + b_2 + \ldots + b_N, \pi \in S \), where contrary to the standard practice of writing parts in descending order, we now have \( b_1 < b_2 < \ldots < b_N \). Subtract 0 from \( b_1 \), 2 from \( b_2 \), \ldots, \( 2N - 2 \) from \( b_N \), to get a partition \( \pi^* \). We call this process the *Euler subtraction*. Note that in \( \pi^* \) the even parts cannot repeat, but the odd parts can. Let the parts of \( \pi^* \) be \( b_1^* \leq b_2^* \leq \ldots \leq b_N^* \).

Now identify the parts of \( \pi \) which are odd, and which are the smallest parts of chains and satisfy both the parity and low bound conditions in (2.2). Mark such parts with a tilde. In this decomposition we adopt the following rule:

(a) the odd parts of \( \pi^* \) which are not identified as above are put in \( \pi_1^* \).

(b) the odd parts of \( \pi^* \) which have been identified could be put in either \( \pi_1^* \) or \( \pi_2^* \).

Thus we have two choices for each identified part.

Let us say, in a certain given situation, after making the choices, we have \( n_1 \) parts in \( \pi_1^* \) and \( n_2 \) parts in \( \pi_2^* \).

We now add 0 to the smallest part of \( \pi_1^* \), 2 to the second smallest part of \( \pi_1^* \), \ldots, \( 2n_2 - 2 \) to the largest part of \( \pi_2^* \), 2\( n_2 \) to the smallest part of \( \pi_1^* \), \( 2n_2 + 2 \) to the second smallest part of \( \pi_1^* \), \ldots, \( 2(n_1 + n_2) - 2 = 2N - 2 \) to the largest part of \( \pi_1^* \). We call this the *Bressoud redistribution* process. As a consequence of this redistribution, we have created two partitions \( \pi_1 \) (out of \( \pi_1^* \)) and \( \pi_2 \) (out of \( \pi_2^* \)) satisfying the following conditions:

(i) \( \pi_1 \) consists only of distinct odd parts, with each odd part being greater than twice the number of parts of \( \pi_2 \).

(ii) Since both the even and odd parts of \( \pi_2^* \) are distinct, the parts of \( \pi_2 \) differ by \( \geq 4 \). Also since the odd parts of \( \pi_2^* \) are chosen from the smallest of parts of certain chains in \( \pi \), the odd parts of \( \pi_2 \) actually differ by \( \geq 6 \), and each such odd part is \( \geq 5 \).

In transforming the original partition \( \pi \) into the pair \((\pi_1, \pi_2)\), we need to see how the parity conditions of \( \pi \) given by \( \ref{21} \) and \( \ref{22} \) transform to parity conditions in \( \pi_1 \) and \( \pi_2 \).

First observe that since the parity conditions on \( \pi \) are imposed only on the even parts of \( \pi \) and the identified odd parts of \( \pi \), the transformed parity conditions (to be determined below) will be imposed only on \( \pi_2 \) and not on \( \pi_1 \). Thus \( \pi_1 \) will satisfy only condition (i) above.

Suppose \( b_k \) is an even part of \( \pi \) and that \( t(b_k; \pi) = t \), that is there are \( t \) odd parts of \( \pi \) which are less than \( b_k \). Now \( b_k \) becomes

\[
b_k^* = b_k - (2k - 2)
\]
after the Euler subtraction. Notice that 
\[ t(b_k^*; \pi^*) = t(b_k; \pi) = t. \]
Now suppose that from among the \( t \) odd parts of \( \pi^* \), \( r \) of them are put in \( \pi_1^* \) and the remaining \( t - r \) odd parts are put in \( \pi_2^* \). Then \( b_k^* \) becomes the \( (k - r) \)-th smallest part in \( \pi_2^* \). So in the Bressoud redistribution process, \( 2(k - r) - 2 \) is added to \( b_k^* \) making it a new even part \( e_{k-r} \) in \( \pi_2 \). Thus
\[
e_{k-r} = b_k^* + 2(k - r) - 2 = b_k - (2k - 2) + 2(k - r) - 2 = b_k - 2r.
\tag{2.3}\]
We see from (2.1) and (2.3) that
\[
e_{k-r} \equiv 2t - 2r = 2t(e_{k-r}; \pi_2) \pmod{4}
\tag{2.4}\]
and so the parity condition (2.1) on the even parts does not change when going to \( \pi_2 \). Thus we may write (2.4) in short as
\[
e \equiv 2t(e) \pmod{4}
\tag{2.5}\]
for any even part in \( \pi_2 \).

Now we need to determine the parity conditions on the odd parts in \( \pi_2 \) which are derived from some of the identified odd parts of \( \pi \). To this end suppose that \( \tilde{b}_k \) is an identified odd part of \( \pi \) which becomes \( \tilde{b}_k^* = \tilde{b}_k - (2k - 2) \) in \( \pi^* \) due to the Euler subtraction, and that \( \tilde{b}_k^* \) is placed in \( \pi_2^* \). Let \( t(\tilde{b}_k; \pi) = t \). Notice that
\[
t(\tilde{b}_k; \pi) = t(\tilde{b}_k^*; \pi^*) = t.
\]
Suppose that from among the \( t \) odd parts of \( \pi^* \) which are \( \tilde{b}_k^* \), \( r \) of them are placed in \( \pi_1^* \) and the remaining \( t - r \) are placed in \( \pi_2^* \). Then \( \tilde{b}_k^* \) becomes the \( (k - r) \)-th smallest part in \( \pi_2^* \). Thus under the Bressoud redistribution, \( 2(k - r) - 2 \) is added to it to yield the part \( f_k \) given by
\[
f_k = \tilde{b}_k^* + 2(k - r) - 2 = \tilde{b}_k - (2k - 2) + (2(k - r) - 2) = \tilde{b}_k - 2r
\]
as in (2.3). Therefore the parity condition (2.2) yields
\[
f_k \equiv 1 + 2t - 2r = 1 + 2(t - r) \pmod{4}.
\]
But \( t(f_k; \pi_2) = t - r \). So this could be expressed in short as
\[
f \equiv 1 + 2t(f) \pmod{4}
\tag{2.6}\]
for any odd part of \( \pi_2 \). Thus the pair of partitions \( (\pi_1, \pi_2) \) is determined by condition (i) on \( \pi_1 \), and conditions (ii) and the parity conditions (2.5) and (2.6) on \( \pi_2 \).

In going from \( \pi \) to the pair \( (\pi_1, \pi_2) \) we had a choice of deciding whether an identified part of \( \pi \) would end up in \( \pi_1 \) or \( \pi_2 \). This choice is precisely the weight \( \omega(\chi) = 2 \) associated with certain chains \( \chi \). The weight of the partition \( \pi \) is computed multiplicatively because these choices are independent. So what we have established up to now is:
Lemma 1. The weighted count of the special Göllnitz-Gordon partitions of \( n \) equals the number of bipartitions \((\pi_1, \pi_2)\) of \( n \) satisfying conditions (i), (ii), (2.5) and (2.6).

Next, we discuss a bijective map

\[
\pi_2 \mapsto (\pi_3, \pi_4),
\]

where \( \pi_3 \) is a partition into distinct multiples of 4 and \( \pi_4 \) is a partition into distinct odd parts such that

\[
\nu(\pi_2) = \nu(\pi_3)
\]

and

\[
2\nu(\pi_2) > \Lambda(\pi_4).
\]

Here by \( \nu(\pi) \) we mean the number of parts of a partition \( \pi \) and by \( \Lambda(\pi) \) the largest part of \( \pi \).

To describe the map (2.7) we represent \( \pi_2 \) as a Ferrers graph with weights 1, 2 or 4, at each node. We construct the graph as follows:

1) With each odd (resp. even) part \( f \) (resp. \( e \)) of \( \pi_2 \) we associate a row of \( \frac{3f+2f(f)}{4} \) (resp. \( \frac{e+2e(e)}{4} \)) nodes.

2) We place a 1 at end of any row that represents an odd part of \( \pi_2 \).

3) Every node in the column directly above each 1 is given weight 2.

4) Each remaining node is given weight 4.

Every part of \( \pi_2 \) is given by the sum of weights in an associated row. It is clear from these weights, that the partition represented by this weighted Ferrers graph satisfies precisely the conditions (ii), (2.5) and (2.6) that characterize \( \pi_2 \).

\[\pi_2: \begin{array}{cccccccc}
4 & 2 & 4 & 4 & 2 & 4 & 4 & 1 \\
4 & 2 & 4 & 4 & 2 & 4 & 4 \\
4 & 2 & 4 & 4 & 1 \\
4 & 1 
\end{array}\]

Next we extract from this weighted Ferrers graph all columns with a 1 at the bottom, and assemble these columns as rows to form a 2-modular Ferrers graph as shown below.

\[\pi_4: \begin{array}{cccc}
2 & 2 & 2 & 1 \\
2 & 2 \\
1 
\end{array}\]

Clearly this 2-modular graph represents a partition \( \pi_4 \) that satisfies condition (2.9).

After this extraction, the decorated graph of \( \pi_2 \) becomes a 4-modular graph (in this case a graph with weight 4 at every node). This graph \( \pi_3 \) clearly satisfies (2.8).
\[
\pi_3 : \quad 4 \quad 4 \quad 4 \quad 4 \quad 4 \\
4 \quad 4 \quad 4 \quad 4 \\
4 \quad 4 \quad 4 \\
4
\]

It is easy to check that (2.7) is a bijection. Thus Lemma 1 can be recasted in the form

Lemma 2. The weighted count of the special Göllnitz-Gordon partitions of \(n\) as in Theorem 3 is equal to the number of partitions of \(n\) in the form \((\pi_1, \pi_3, \pi_4)\) where

(iii) \(\pi_3\) consists only of distinct multiples of 4,
(iv) \(\pi_4\) has distinct odd parts and \(\Lambda(\pi_4) < 2\nu(\pi_3)\),
(v) \(\pi_1\) has distinct odd parts and \(\lambda(\pi_1) > 2\nu(\pi_3)\),

Finally, observe that conditions (iv) and (v) above yield partitions into distinct odd parts (without any other conditions). This together with (iii) yields partitions counted by \(Q_2(n)\), thereby completing the combinatorial proof of Theorem 3.

In a similar fashion, we can obtain the following representation for \(Q_0(n)\) with weights and parity conditions imposed on the Göllnitz-Gordon partitions:

Theorem 4. Let \(S^*\) denote the set of all special Göllnitz-Gordon partitions, namely, Göllnitz-Gordon partitions \(\pi\) satisfying the parity condition that for every even part \(b\) of \(\pi\)

\[b \equiv 2(t(b) - 1) \pmod{4}.\]  

(2.10)

Decompose each \(\pi \in S^*\) into chains \(\chi\) and define the weight \(\omega(\chi)\) as

\[\omega(\chi) = \begin{cases} 
2, & \text{if } \chi \text{ is an odd chain, } \lambda(\chi) \geq 3, \text{ and } \lambda(\chi) \equiv 2t(\lambda(\chi)) - 1 \pmod{4}, \\
1, & \text{otherwise.}
\end{cases}\]

(2.11)

The weight \(\omega(\pi)\) of the partition \(\pi\) is defined multiplicatively as

\[\omega(\pi) = \prod_{\chi} \omega(\chi),\]

the product over all chains \(\chi\) of \(\pi\). We then have

\[Q_0(n) = \sum_{\pi \in S^*, \sigma(\pi) = n} \omega(\pi),\]

where \(\sigma(\pi)\) is the sum of the parts of \(\pi\).
3. SERIES REPRESENTATIONS

If we let \( \nu(\pi_1) = n_1 \) and \( \nu(\pi_2) = n_2 \), then (2.4) and conditions (iii), (iv), and (v) of Lemma 2 imply that the generating function of all such triples of partitions \( (\pi_1, \pi_3, \pi_4) \) is

\[
q^{n_1^2 + 2n_1n_2} q^{2n_2^2 + 2n_2} \frac{q^{2n_2^2 + 2n_2}}{(q^2; q^2)_{n_1} (q^4; q^4)_{n_2}} (-q; q^2)_{n_2}.
\]  

(3.1)

If the expression in (3.1) is summed over all non-negative integers \( n_1 \) and \( n_2 \), it yields

\[
\sum_{n_1} \sum_{n_2} q^{n_1^2 + 2n_1n_2 + 2n_2^2 + 2n_2} (-q; q^2)_{n_2} = \sum_{n_2} q^{2n_2^2 + 2n_2} (-q; q^2)_{n_2} \sum_{n_1} q^{n_1^2 + 2n_1n_2} (q^2; q^2)_{n_1} = (-q; q^2)_{n_2} \sum_{n_2} \frac{q^{2n_2^2 + 2n_2}}{(q^4; q^4)_{n_2}} (-q; q^2)_{n_2} \sum_{n_1} \frac{q^{n_1^2 + 2n_1n_2}}{(q^2; q^2)_{n_1}}
\]

\[
= (-q; q^2)_\infty (-q; q^4)_\infty = (-q; q^4)_\infty (-q^3; q^4)_\infty (-q^4; q^4)_\infty = \sum_n Q_2(n) q^n.
\]  

(3.2)

By just following the above steps we can actually get a two parameter refinement of (3.2), namely,

\[
\sum_{n_1} \sum_{n_2} \frac{z^{n_1}\omega^{n_1}q^{n_1^2 + 2n_1n_2 + 2n_2^2 + 2n_2} (-zq; q^2)_{n_2}}{(q^2; q^2)_{n_1} (q^4; q^4)_{n_2}} = (-zq; q^4)_\infty (-zq^3; q^4)_\infty (-\omega q^2; q^4)_\infty.
\]  

(3.3)

One may view (3.2) as the analytic version of Theorem 3. In reality, the correct way to view (3.2) is that, if the summand on the left is decomposed into three factors as (3.1), then (3.2) is the analytic version of the statement that the number of partitions of an integer \( n \) into the triple of partitions \( (\pi_1, \pi_3, \pi_4) \) is equal to \( Q_2(n) \). This is of course only the final step of the proof given above. and (3.2), which is quite simple, is equivalent to it.

The advantage in the two parameter refinement (3.2) is that by suitable choice of the parameters we get similar representations involving \( Q_i(n) \) for \( i = 0, 1, 3 \). For example, if we replace \( \omega \) by \( \omega q^{-2} \) in (3.2) we get

\[
\sum_{n_1} \sum_{n_2} \frac{z^{n_1}\omega^{n_2}q^{n_1^2 + 2n_1n_2 + 2n_2^2 + 2n_2} (-zq; q^2)_{n_2}}{(q^2; q^2)_{n_1} (q^4; q^4)_{n_2}} = (-zq; q^4)_\infty (-zq^3; q^4)_\infty (-\omega q^2; q^4)_\infty,
\]  

(3.4)

which is the analytic representation of Theorem 4 above.

Next, replacing \( z \) by \( zq \) and \( \omega \) by \( \omega^{-1} \) in (3.2) we get

\[
\sum_{n_1} \sum_{n_2} \frac{z^{n_1}\omega^{n_2}q^{n_1^2 + 2n_1n_2 + 2n_2^2 + 2n_1n_2 + n_2} (-zq^2; q^2)_{n_2}}{(q^2; q^2)_{n_1} (q^4; q^4)_{n_2}} = (-zq^2; q^4)_\infty (-zq^3; q^4)_\infty (-zq^4; q^4)_\infty.
\]  

(3.5)

Now choose \( z = 1 \) in (3.5). Then the double series on the left becomes

\[
\sum_{n_1} \sum_{n_2} \omega^{n_2}q^{n_1^2 + 2n_1n_2 + 2n_2^2 + n_1 + n_2} = \sum_{n_1} \sum_{n_2} \omega^{n_2}q^{n_1 + n_2} (q^2; q^2)_{n_1} (q^4; q^4)_{n_2}.
\]  

(3.6)

If we now put \( n = n_1 + n_2 \) and \( j = n_2 \), then (3.6) could be rewritten in the form

\[
\sum_n \frac{q^{n^2 + n}}{(q^2; q^2)_n} \sum_{j=0}^n \omega^j q^2 (q^2; q^2)_n.
\]
Once again, putting \( n \) following limiting case of Bailey's lemma:

\[
\sum_{n=0}^{\infty} \frac{q^{n^2+n}(-\omega q^2; q^2)_n}{(q^2; q^2)_n} = (-q^2; q^4)_\infty(-\omega q^3; q^4)_\infty(-q^4; q^4)_\infty,
\]

which is the single series identity (1.1) in a refined form.

Similarly, replacing \( \omega \) by \( \omega q^{-3} \) and \( z \) by \( z q \) in (3.8), we get

\[
\sum_{n_1} \sum_{n_2} \frac{z^{n_1} \omega^{n_2} q^{n_1^2+2n_1 n_2+2n_2^2+n_1-n_2}(-z q^2; q^2)_{n_2}}{(q^2; q^2)_{n_1}(q^2; q^2)_{n_2}} = (-z q^2; q^4)_\infty(-\omega q^3; q^4)_\infty(-z q^4; q^4)_\infty.
\]

Now the choice \( z = 1 \) makes the double series in (3.8) as

\[
\sum_{n_1} \sum_{n_2} \frac{\omega^{n_2} q^{n_1^2+2n_1 n_2+2n_2^2+n_1-n_2}}{(q^2; q^2)_{n_1}(q^2; q^2)_{n_2}} = \sum_{n_1} \sum_{n_2} \frac{\omega^{n_2} q^{(n_1+n_2)^2+n_1^2+n_2+n_1-n_2}}{(q^2; q^2)_{n_1}(q^2; q^2)_{n_2}}.
\]

Once again, putting \( n = n_1 + n_2 \) and \( j = n_2 \) makes (3.9) into

\[
\sum \frac{q^{n^2+n}}{(q^2; q^2)_n} \sum \frac{\omega^j q^{2j-2j} (q^2; q^2)_n}{(q^2; q^2)_{n-j}} = \sum_{n=0}^{\infty} \frac{q^{n^2+n}(-\omega q^{-1}; q^2)_n}{(q^2; q^2)_n} = (-q^2; q^4)_\infty(-\omega q^3; q^4)_\infty(-q^4; q^4)_\infty,
\]

which is a refinement of the single series identity (1.2). Thus precisely in the cases \( i = 1, 3 \), can the double series be reduced to single series by setting one of the parameters \( z = 1 \).

4. A NEW INFINITE HIERARCHY

Identity (3.2) given above is just the case \( k = 2 \) of a new infinite hierarchy of multiple series identities (4.12) given below.

To derive this hierarchy, we will need the definition of a Bailey pair, and a special case of Bailey’s lemma which produces a new Bailey pair from a given Bailey pair [2].

**Definition:** A pair of sequences \( \alpha_n(q), \beta_n(q) \) is called a Bailey pair (relative to 1) if for all \( n \geq 0 \)

\[
\beta_n(q) = \sum_{i=0}^{n} \frac{\alpha_i(q)}{(q)_{n-i}(q)_{n+i}}.
\]

By setting \( a = 1, \rho_1 = -q^\frac{1}{2} \), and letting \( \rho_2 \to \infty \) in the formulas (3.29) and (3.30) of [2], we obtain the following limiting case of Bailey’s lemma:

**Lemma 3.** Suppose \( (\alpha_n(q), \beta_n(q)) \) is a Bailey pair. Then \( (\alpha_n^{(1)}(q), \beta_n^{(1)}(q)) \) is another Bailey pair, where

\[
\alpha_n^{(1)}(q) = q^{n^2} \alpha_n(q),
\]

\[
\beta_n^{(1)}(q) = \sum_{i=0}^{n} \frac{(-\sqrt{q})^i}{(q)_{n-i}(-\sqrt{q})_n} q^{\frac{n}{2}} \beta_i(q).
\]
From \((\alpha_n^{(1)}(q), \beta_n^{(1)}(q))\) one can produce next Bailey pair \((\alpha_n^{(2)}(q), \beta_n^{(2)}(q))\) simply using \((\alpha_n^{(1)}(q), \beta_n^{(1)}(q))\) as the initial Bailey pair. It is easy to check that the \(k\)-fold iteration of (the limiting case of) Bailey’s Lemma yields

\[
\alpha_n^{(k)}(q) = q^{\frac{k^2}{2}} \alpha_n(q),
\]

\[
\beta_n^{(k)}(q) = \sum_{\vec{n}} \frac{q^{2 N_1^2 + \cdots + N_k^2}}{(q)_{n-N_1}(q)_{n_1}(q)_{n_2} \cdots (q)_{n_{k-1}}(q^2; q^2)_n} \beta_{n_k}(q),
\]

where \(\vec{n} = (n_1, n_2, \ldots, n_k)\) and \(N_i = n_i + n_{i+1} + \cdots + n_k\), with \(i = 1, 2, \ldots, k\). In [8], [9] Slater derived A-M families of Bailey pairs to produce the celebrated list of 130 identities of the Rogers-Ramanujan type. We shall need her \(E(4)\) pair:

\[
\alpha_n = \begin{cases} 
(-1)^n q^{\frac{n^2}{2}} (q^n + q^{-n}), & \text{if } n > 0, \\
1, & \text{if } n = 0,
\end{cases}
\]

\[
\beta_n = \frac{q^n}{(q^2; q^2)_n}.
\]

It follows from (4.1) and (4.4) - (4.6) that

\[
\sum_{\vec{n}} \frac{q^{\frac{1}{2} (N_1^2 + N_2^2 + \cdots + N_k^2)} \beta_{n_k}(q)}{(q)_{n-N_1}(q)_{n_1}(q)_{n_2} \cdots (q)_{n_{k-1}}(q^2; q^2)_n} = (-\sqrt{q})_n \sum_{j=-n}^{n} (-1)^j q^{\frac{j^2}{2} + j} \left[\frac{2n}{n+j}q\right],
\]

where \(q\)-binomial coefficients are defined as

\[
\left[\frac{n+m}{n}\right]_q = \frac{(q^{m+1})_n}{(q)_n}.
\]

It is easy to check that

\[
\lim_{n \to \infty} \left[\frac{n}{m}\right]_q = \frac{1}{(q)_m},
\]

and

\[
\lim_{n \to \infty} \left[\frac{2n}{n+j}\right]_q = \frac{1}{(q)_{\infty}}.
\]

Next, we recall Jacobi’s triple product identity

\[
\sum_{n=-\infty}^{\infty} q^{n^2} z^n = (q^2, -qz, -\frac{q}{z}; q^2)_{\infty},
\]

where \((a_1, a_2, \ldots, a_m; q)_\infty = (a_1)_{\infty}(a_2)_{\infty} \cdots (a_m)_{\infty}\).

If we let \(n\) tend to infinity in (4.7) with \(q \to q^2\), we obtain with the aid of (4.10) and (4.11) the desired
identity

\[
\sum_{n} \frac{q^{N_1^2 + \cdots + N_k^2 + 2N_k} (-q; q^2)_{N_k}}{(q^2; q^2)_{n_1} \cdots (q^2; q^2)_{n_{k-1}} (q^4; q^4)_{n_k}} = \frac{(-q; q^2)_{\infty}(q^{2k+4}, q^k, q^{k+4}; q^{2k+4})_{\infty}}{(q^2; q^2)_{\infty}} = \frac{(q^2; q^4)_{\infty}(q^{2k+4}, q^k, q^{k+4}; q^{2k+4})_{\infty}}{(q)_{\infty}}.
\]

(4.12)

Here we used the simple relation

\[
(q^2; q^4)_{\infty} = (q, -q; q^2)_{\infty} = (-q; q^2)_{\infty}.
\]

Making use of

\[
\frac{(-q; q^2)_{\infty}(q^8, q^2, q^6; q^8)}{(q^2; q^2)_{\infty}} = \frac{(-q; q^2)_{\infty}(q^4, q^3)_{\infty}}{(q^2; q^2)_{\infty}} = (-q; q^2)_{\infty}(-q^4; q^4)_{\infty},
\]

(4.13)

it is straightforward to verify that (4.12) with \(k = 2\) yields (3.2), as claimed.

When \(k = 1\), (4.12) becomes

\[
\sum_{n \geq 0} \frac{q^{n^2+2n}(-q; q^2)_n}{(q^4; q^4)_n} = \frac{(q^2; q^4)_{\infty}(q^6, q^1, q^5; q^6)_{\infty}}{(q)_{\infty}} = \frac{(-q^3; q^6)_{\infty}}{(q^4, q^3; q^{12})_{\infty}}.
\]

(4.14)

Surprisingly, (4.14) is missing from the Slater list. It was given by Andrews in [3].

By using the statistic \(s(b; \pi) = \text{number of even parts of the partition } \pi \text{ which are less than the part } b\), it can be shown that the following partition theorem is a combinatorial interpretation of (4.14):

**Theorem 5.** Let \(G(N)\) denote the number of partitions \(\pi\) of \(N\) into distinct parts such that no gap between consecutive parts is \(\equiv 1 \pmod 4\), and where the \(k\)-th smallest part \(b\) is \(\equiv 1 + 2k + 2s(b; \pi) \pmod 4\) if \(b\) is odd, and \(\equiv 2 + 2k + 2s(b; \pi) \pmod 4\), if \(b\) is even.

Let \(P(N)\) denote the number of partitions of \(N\) into parts \(\equiv \pm 3, \pm 4 \pmod {12}\), such that parts \(\equiv 3 \pmod 6\) are distinct. Then,

\[G(N) = P(N).\]

**Remark:** Theorem 5 can be stated without appeal to the statistic \(s(b; \pi)\), but we preferred to state it this way to emphasise a different parity condition and to show similarity with Theorems 3 and 4.

It would be interesting to find partition theoretical interpretation of (4.12) with \(k > 2\). To this end we observe that the product on the right of (4.12) with \(k \equiv 0 \pmod 4\) can be interpreted as a generating function for partitions into parts \(\not\equiv 2 \pmod 4\), \(\not\equiv 0, \pm k \pmod {2k+4}\).
It is instructive to compare this product
\[
\prod_{\substack{n \geq 1 \mod 4 \\ n \equiv 0, \pm (2K-2) \mod 4K}} (1 - q^n)^{-1}
\]
and the generalized Göllnitz-Gordon product \((7.4.4); [4]\)
\[
\prod_{\substack{n \geq 1 \mod 4 \\ n \equiv 0, \pm (2k-1) \mod 4k}} (1 - q^n)^{-1}.
\]
Here \(K = 1 + \frac{k}{2}\) with \(k \equiv 0 \mod 4\) and \(k\) is a positive integer.

The right hand side of (4.12) can be rewritten as
\[
\left(q^2; q^4\right)_\infty (q^k, q^{k+4}, q^{k+2}, -q^{k+2}; q^{2k+4})_\infty (q^{4k+8}; q^{4k+8})_\infty \frac{(q)}{(q^\infty)}
\]
if \(k\) is odd, and
\[
\left(q^2; q^4\right)_\infty (q^{2k+4}, q^{2k+2})_\infty (q^\frac{k}{2}, q^\frac{k-2}{2}, q^{2+\frac{k}{2}}, q^{k+2})_\infty \frac{(q)}{(q^\infty)}
\]
if \(k \equiv 2 \mod 4\).

This enables us to interprete the right hand side of (4.12) as:
A. \(k \equiv 1 \mod 2\). RHS (4.12) is the generating function for partitions into parts \(\not\equiv 2 \mod 4\), \(\not\equiv \pm k \mod (2k+4), \not\equiv 0 \mod (4k+8)\), such that parts \(\equiv k \pm 2 \mod (2k+4)\) are distinct.

B. \(k \equiv 2 \mod 4\). RHS (4.12) is the generating function for partitions into parts \(\not\equiv 2 \mod 4\), \(\not\equiv 0 \mod (2k+4)\), such that parts \(\not\equiv \pm \frac{k}{2} \mod (k+2)\) are distinct.

We would like to conclude with the following observation. The hierarchy (4.12) follows in the limit \(l, m \to \infty\) from the doubly bounded polynomial identity
\[
\sum_{\pi, s} q^{N_1^2 + \cdots + N_s^2 + s + 2N_k} \left[ l + m - N_l \atop m - N_1 \right] (l - \sum_{j=1}^{k-1} N_i + n_j) \left[ n_k + [l - \sum_{j=1}^{k-1} N_i - s] \atop n_k \right] \left[ n_q \right] \left[ n_q \right] =
\sum_{j=-\infty}^{\infty} \left\{ q^{(4k+8)j^2 + 4j} U(l, m, 2(k+2)j + 1, 2j, q^2) - q^{(4k+8)(j^2 + 4(k+1)j + k)} U(l, m, 2(k+2)j + k + 1, 2j + 1, q^2) \right\},
\]
where \([x]\) is the largest integer \(\leq x\), \(U(l, m, a, b, q) = T_w(l, m, a, b, q) + T_w(l, m, a + 1, b, q)\), and the refined \(q\)-trinomial coefficients \([10]\) are defined as
\[
T_w(l, m, a, b, q) := \sum_{n+l \equiv a \mod 2} q^{\frac{m^2}{2}} \left[ m + b + \frac{t-a-n}{2} \atop m - b \right] \left[ m - b + \frac{t+q-a-n}{2} \atop m - b \right].
\]
Using (14) together with Warnaar’s limiting formula \((2.26); [10]\)
\[
\lim_{l \to \infty} U(l, m, a, b, q) = \left( \frac{\sqrt{q}}{(q)_{2m}} \right) m \left[ m + b \atop q \right].
\]
we obtain (4.14) with \( q \to q^2 \) and \( n \to m \) as \( l \to \infty \) in (4.12). On the other hand, if we let \( m \to \infty \) in (4.16) we find that

\[
\lim_{m \to \infty} T_w(l, m, a, b, q) = \frac{1}{(q)_l} T_{AB}(l, a, q),
\]

where the Andrews-Baxter \( q \)-trinomial coefficients [5] are defined as

\[
T_{AB}(l, a, q) := \sum_{n \equiv a \pmod{2}} q^{\frac{n^2}{2}} \left[ \binom{l}{n} q^\frac{l}{2} \right] q^{\left[ \frac{l-n}{2} \right]}.
\]

(4.19)

And so, (4.15) becomes in the limit \( m \to \infty \)

\[
\sum_{n,s} q^{N_1^2 + \cdots + N_k^2 + 2N_k} \prod_{j=1}^{k-1} \left[ l - \sum_{i=1}^{j} N_i + n_j \right] q^{n_k + \left[ \frac{l-\sum_{i=1}^{k} N_i}{2} \right]} \left[ \frac{n_k}{q} \right] q^{\left[ \frac{n_k}{2} \right]} q^2 \sum_{j=-\infty}^{\infty} \left\{ q^{(4k+8)j^2 + 4j} U(l, 2(k+2)j + 1, q^2) - q^{(4k+8)j^2 + 4(k+1)j + k} U(l, 2(k+2)j + k + 1, q^2) \right\},
\]

(4.20)

where

\[
U(l, a, q) = T_{AB}(l, a, q) + T_{AB}(l, a + 1, q).
\]

(4.21)

The proof of (4.15) will be given elsewhere.

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